A note on the $(\infty, n)$-category of cobordisms

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In this note we give a precise definition of fully extended topological field theories à la Lurie. Using complete $n$-fold Segal spaces as a model, we construct an $(\infty, n)$-category of $n$-dimensional cobordisms, possibly with tangential structure. We endow it with a symmetric monoidal structure and show that we can recover the usual category of cobordisms $n\text{Cob}$ and the cobordism bicategory $n\text{Cob}^{\text{ext}}$ from it.

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Introduction

Topological field theories (TFTs) arose as toy models for physical quantum field theories and have proven to be of mathematical interest, notably because they are a fruitful tool for studying topology.
An \( n \)-dimensional TFT is a symmetric monoidal functor from the category of bordisms, which has closed \((n-1)\)-dimensional manifolds as objects and \( n \)-dimensional bordisms as morphisms, to any other symmetric monoidal category, which classically is taken to be the category of vector spaces or chain complexes.

A classification of \( 1 \)- and \( 2 \)-dimensional TFTs follows from classification theorems for \( 1 \)- and \( 2 \)-dimensional compact manifolds with boundary, cf. [Abr99]. In order to obtain a classification result for larger values of \( n \) one needs a suitable replacement of the classification of compact \( n \)-manifolds with boundary used in the low-dimensional cases. Moreover, as explained in [BD95], this approach requires passing to “extended” topological field theories. Here extended means that we need to be able to evaluate the \( n \)-TFT not only at \( n \)- and \((n-1)\)-dimensional manifolds, but also at \((n-2)\)-, \(1\)-, and \(0\)-dimensional manifolds. Thus, an extended \( n \)-TFT is a symmetric monoidal functor out of a higher category of bordisms. In light of the hope of computability of the invariants determined by an \( n \)-TFT, e.g. by a triangulation, it is natural to include this data. Furthermore, Baez and Dolan conjectured that, analogously to the \( 1 \)-dimensional case, extended \( n \)-TFTs are fully determined by their value at a point, calling this the cobordism hypothesis. A definition of a suitable bicategory of \( n \)-cobordisms and a proof of a classification theorem of extended TFTs for dimension 2 was given in [SP09].

In his expository manuscript [Lur09c], Lurie suggested passing to \((x, n)\)-categories for a proof of the cobordism hypothesis in arbitrary dimension \( n \). He gave a detailed sketch of such a proof using a suitable higher category of cobordisms, which, informally speaking, has zero-dimensional manifolds as objects, bordisms between objects as 1-morphisms, bordisms between bordisms as 2-morphisms, etc., and for \( k > n \) there are only invertible \( k \)-morphisms given by diffeomorphisms and their isotopies. However, finding an explicit model for such a higher category poses one of the difficulties in rigorously defining these \( n \)-dimensional TFTs, which are called “fully extended”.

In [Lur09c], Lurie gave a short sketch of a definition of this \((x, n)\)-category using complete \( n \)-fold Segal spaces as a model. Instead of using manifolds with corners and gluing them, his approach was to conversely use embedded closed (not necessarily compact) manifolds, following along the lines of [GTMW09, Gal11], and to specify points where they are cut into bordisms of which the embedded manifold is a composition. Whitney’s embedding theorem ensures that every \( n \)-dimensional manifold \( M \) can be embedded into some large enough vector space and suitable versions for manifolds with boundary can be adapted to obtain an embedding theorem for bordisms, see Section 5. Moreover, the rough idea behind the definition of the levels of \( \text{P Bord}_n \) is that the \((k_1, \ldots, k_n)\)-level of our \( n \)-fold Segal space \( \text{P Bord}_n \) should be a classifying space for diffeomorphisms of in the \( i \)th direction \( k_i \)-fold composable \( n \)-bordisms. Lurie’s idea was to use the fact that the space of embeddings of \( M \) into \( \mathbb{R}^\infty \) is contractible to justify the construction.

Modifying this approach, the main goal of this note is to provide a detailed construction of such a \((x, n)\)-category of bordisms, suitable for explicitly constructing an example of a fully extended \( n \)TFT, which will be the content of a subsequent paper [CS15]. As we explain in Section 6.3 Lurie’s sketch does not lead to an \( n \)-fold Segal space, as the essential constancy condition is violated. In our Definition 5.1 we propose a stronger condition on elements in the levels of the Segal space. We show that this indeed yields a \( n \)-fold Segal space \( \text{P Bord}_n \). Its completion \( \text{Bord}_n \) defines an \((x, n)\)-category of \( n \)-cobordisms and thus is a corrigendum to Lurie’s \( n \)-fold simplicial space of bordisms from [Lur09c].

Furthermore, we endow it with a symmetric monoidal structure and also consider bordism categories with additional structure, e.g. orientations and framings, which allows us, in Section 10, to rigorously define fully extended topological field theories.

Our main motivation for giving a precise definition of the \((x, n)\)-category of bordisms was a different goal: namely, in the subsequent paper [CS15] we explicitly construct an example of a fully extended topological field theory. Given an \( E_n \)-algebra \( A \) we show that factorization homology with coefficients in \( A \) leads to a fully extended \( n \)-dimensional topological field theory with target category a suitable \((x, n)\)-Morita category with \( E_n \)-algebras as objects, bimodules as 1-morphisms, bimodules between bimodules as 2-morphisms, etc.
Organization of the paper In Part I, consisting of the first three sections, we recall the necessary tools from higher category theory needed to define fully extended TFTs.

Section 1 reviews the model for $(\infty, 1)$-categories given by complete Segal spaces and recalls some useful information about other models. In Section 2 we explain the model for $(\infty, n)$-categories given by complete $n$-fold Segal spaces and introduce a model which is a hybrid between complete $n$-fold Segal spaces and $n$-fold Segal categories.

We propose two possible definitions of symmetric monoidal structures on complete $n$-fold Segal spaces in Section 3; one as a $\Gamma$-object in complete $n$-fold Segal spaces following [TV09] and one as a tower of suitable $(n + k)$-fold Segal spaces with one object, 1-morphism,..., $(k − 1)$-morphism for $k \geq 0$ following the Delooping Hypothesis.

Part II is devoted to the construction of $\text{Bord}_n$.

Our construction of the $(\infty, n)$-category $\text{Bord}_n$ of higher cobordisms is based on a simpler complete Segal space Int of closed intervals, which we introduce in Section 4. The closed intervals correspond to places where we are allowed to cut the manifold into the bordisms it composes. The fact that we prescribe closed intervals instead of just a point corresponds to fixing collars of the bordisms.

Section 5 is the central part of this article and consists of the construction of the complete $n$-fold Segal space $\text{Bord}_n$ of cobordisms. We discuss variants of $\text{Bord}_n$, including $(\infty, d)$-categories of bordisms for arbitrary $d$, and compare our definition to Lurie’s sketch in Section 6.

In Section 7 we endow $\text{Bord}_n$ with a symmetric monoidal structure, both as a $\Gamma$-object and as a tower.

In Section 8 we elaborate on the interpretation of the objects in $\text{Bord}_n$ as $n$-bordisms. Furthermore we show that the homotopy (bi)category of $\text{Bord}_n$ is what one should expect, namely the homotopy category of the $(\infty, 1)$-category of $n$-bordisms $\text{Bord}_n^{(\infty, 1)}$ gives back the classical cobordism category $n\text{Cob}$ and the homotopy bicategory of the $(\infty, 2)$-category of $n$-bordisms $\text{Bord}_n^{(\infty, 2)}$ is Schommer-Pries’ bicategory $n\text{Cob}^{extr}$ from [SP09].

Finally, in Section 9 we consider bordism categories with additional structure such as orientations, denoted by $\text{Bord}_n^{or}$, and framings, denoted by $\text{Bord}_n^{fr}$, which allows us to define fully extended $n$-dimensional topological field theories in Section 10.

Conventions

1. Let $\text{Space}$ denote the category of simplicial sets with its usual model structure. By $\text{space}$ we mean a fibrant object in $\text{Space}$, i.e. a Kan complex.

2. We denote the simplex category by $\Delta$. Objects are denoted by $[m] = (0 < \cdots < m)$ and morphisms are monotone maps.

3. We denote the extended simplex $\{(x_0, \ldots, x_l) \in \mathbb{R}^{l+1} : \sum x_i = 1\}$ by $|\Delta^l|_e$.

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Part I.
Symmetric monoidal $\left(\infty, n\right)$-categories

A higher category, say, an $n$-category for $n \geq 0$, has not only objects and (1-)morphisms, but also $k$-morphisms between $(k-1)$-morphisms for $1 \leq k \leq n$. Strict higher categories can be rigorously defined, however, most higher categories which occur in nature are not strict. Thus, we need to weaken some axioms and coherence between the weakenings become rather involved to formulate explicitly. Things turn out to become somewhat easier when using a geometric definition, in particular when furthermore allowing to have $k$-morphisms for all $k \geq 1$, which for $k \geq n$ are invertible. Such a higher category is called an $(\infty, n)$-category. There are several models for such $(\infty, n)$-categories, e.g. Segal $n$-categories (cf. [HS98]), $\Theta_n$-spaces (cf. [Rez10]), and complete $n$-fold Segal spaces, which all are equivalent in an appropriate sense (cf. [Toe05, BSI]). For our purposes, the latter model turns out to be well-suited and in this Part we recall some basic facts about complete $n$-fold Segal spaces as higher categories. This is not at all exhaustive, and more details can be found in e.g. [BR13, Zha13]. We also refer to [Ber11], especially for their role in the proof of the Cobordism Hypothesis in [Lur09c].

Symmetric monoidal structures on $(\infty, n)$-categories for $n > 1$) are not very much studied in the literature. We provide a brief review of these and describe them in two different ways: as $\Gamma$-objects on the one hand and using the Delooping Hypothesis on the other hand.

1. Models for $(\infty, 1)$-categories

A higher category, say, an $n$-category for $n \geq 0$, has not only objects and (1-)morphisms, but also $k$-morphisms between $(k-1)$-morphisms for $1 \leq k \leq n$. Strict higher categories can be rigorously defined, however, most higher categories which occur in nature are not strict. Thus, we need to weaken some axioms and coherence between the weakenings become rather involved to formulate explicitly. Things turn out to become somewhat easier when using a geometric definition, in particular when furthermore allowing to have $k$-morphisms for all $k \geq 1$, which for $k \geq n$ are invertible. Such a higher category is called an $(\infty, n)$-category.

For $n = 1$, an $(\infty, 1)$-category should be a category up to coherent homotopy which is encoded in the invertible higher morphisms. In this section, we will mention and give references for several models for $(\infty, 1)$-categories we will use in the later sections. A good overview on different models for $(\infty, 1)$-categories can be found in [Ber10]. It should be mentioned that by [Lur09a] up to equivalence, there is essentially only one theory of $(\infty, 1)$-categories, explicit equivalences between the models mentioned here have been proved e.g. in [DKS99, Ber07, BK12, Hor14]. One additional model which should be mentioned is that of Joyal’s quasi-categories. It has been intensively studied, most prominently in [Lur09a].

1.1. The homotopy hypothesis and $(\infty, 0)$-categories

The basic hypothesis upon which $\infty$-category theory is based is the following

Hypothesis 1.1 (Homotopy hypothesis). Spaces are models for $\infty$-groupoids, also referred to as $(\infty, 0)$-categories.

Given a space $X$, its points are thought of as objects of the $(\infty, 0)$-category, 1-morphisms as paths between points, 2-morphisms as homotopies between paths, 3-morphisms as homotopies between homotopies, and
so forth. With this interpretation, it is clear that all \( n \)-morphisms are invertible up to homotopies, which are higher morphisms.

We take this hypothesis as the basic definition, and model spaces with simplicial sets rather than with topological spaces.

**Definition 1.2.** An \((\infty, 0)\)-category, or \(\infty\)-groupoid, is a space, i.e. according to our conventions, a fibrant simplicial set, i.e. a Kan complex.

### 1.2. Topologically enriched categories

One particularly simple, but quite rigid model is that of topologically, or simplicially, enriched categories.

**Definition 1.3.** A topological category is a category enriched in topological spaces (or simplicial sets, depending on the purpose).

Topological categories are discussed and used in [Lur09a, TV05]. However, for our applications they turn out to be too rigid. We would also allow some flexibility for objects, not only morphisms, thus also requiring spaces of objects.

### 1.3. Segal spaces

Complete Segal spaces, first introduced by Rezk in [Rez01] as a model for \((\infty, 1)\)-categories, turn out to be very well-suited for geometric applications. We recall the definition in this section.

**Definition 1.4.** A \((1\text{-fold})\) Segal space is a simplicial space \(X = X_\bullet\) which satisfies the Segal condition: for any \( n, m \geq 0 \) the commuting square

\[
\begin{array}{ccc}
X_{m+n} & \to & X_m \\
\downarrow & & \downarrow \\
X_n & \to & X_0 \\
\end{array}
\]

induced by the maps \([m] \to [m+n], (0 < \cdots < m) \mapsto (0 < \cdots < m),\) and \([n] \to [m+n], (0 < \cdots < m+n) \mapsto (m+1 < \cdots < m+n),\) is a homotopy pullback square. In other words,

\[
X_{m+n} \to X_m \underset{X_0}{\times} X_n,
\]

is a weak equivalence.

Defining a map of Segal spaces to be a map of the underlying simplicial spaces gives a category of Segal spaces, \(\text{SeSp} = \text{SeSp}_1\).

**Remark 1.5.** For any \( m \geq 1 \), consider the maps \(g_\beta : [1] \to [m], (0 < 1) \mapsto (\beta - 1 < \beta)\) for \(1 \leq \beta \leq m\). Note that the Segal condition is equivalent to the condition that the maps induced by \(g_1, \ldots, g_m\),

\[
X_m \to X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1
\]

is a weak equivalence.

**Remark 1.6.** Following [Lur09c] we omit the Reedy fibrant condition which often appears in the literature. In particular, this condition would guarantee in particular that the canonical map

\[
X_m \underset{X_0}{\times} X_n \to X_m \underset{X_0}{\times} X_n
\]
is a weak equivalence. Our definition corresponds to the choice of the projective model structure instead of the injective (Reedy) model structure, which is slightly different (though Quillen equivalent) compared to [Rez01]. We will explain this in more detail in Section 1.3.1.

**Definition 1.7.** We will refer to the spaces $X_n$ as the *levels* of the Segal space $X$.

**Example 1.8.** Let $\mathcal{C}$ be a small topological category. Then its nerve $N(\mathcal{C})$ is is a Segal space. Recall that it is defined by

$$N(\mathcal{C})_n = \bigsqcup_{x_0, \ldots, x_n \in \text{Ob} \mathcal{C}} \text{Hom}_\mathcal{C}(x_0, x_1) \times \cdots \times \text{Hom}_\mathcal{C}(x_n-1, x_n),$$

its face maps are given by composition of morphisms, and degeneracies by insertions of identities. Moreover, a simplicial set, viewed as a simplicial space with discrete levels, satisfies the Segal condition if and only if it is the nerve of an (ordinary) category.

### 1.3.1. Segal spaces as $(\infty, 1)$-categories

The above example motivates the following interpretation of Segal spaces as models for $(\infty, 1)$-categories. If $X_\bullet$ is a Segal space then we view the set of 0-simplices of the space $X_0$ as the set of objects. For $x, y \in X_0$ we view

$$\text{Hom}_X(x, y) = \{x\} \times_{X_0} hX_1 \times_{X_0} \{y\}$$

as the $(\infty, 0)$-category, i.e. the space, of arrows from $x$ to $y$. More generally, we view $X_n$ as the $(\infty, 0)$-category, i.e. the space, of $n$-tuples of composable arrows together with a composition. Note that given an $n$-tuple of composable arrows, there is a contractible space of compositions. Moreover, one can interpret paths in the space $X_1$ of 1-morphisms as 2-morphisms, which thus are invertible up to homotopies, which in turn are 3-morphisms, and so forth.

### 1.3.2. The homotopy category of a Segal space

To a higher category one can intuitively associate an ordinary category, its *homotopy category*, which has the same objects and whose morphisms are 2-isomorphism classes of 1-morphisms. For Segal spaces, one can realize this idea as follows.

**Definition 1.9.** The *homotopy category* $h_1(X)$ of a Segal space $X = X_\bullet$ is the (ordinary) category whose objects are the 0-simplices of the space $X_0$ and whose morphisms between objects $x, y \in X_0$ are,

$$\text{Hom}_{h_1(X)}(x, y) = \pi_0(\text{Hom}_X(x, y))$$

$$= \pi_0 \left( \{x\} \times_{X_0} hX_1 \times_{X_0} \{y\} \right).$$

For $x, y, z \in X_0$, the following diagram induces the composition of morphisms, as weak equivalences induce bijections on $\pi_0$.

$$\left( \{x\} \times_{X_0} hX_1 \times_{X_0} \{y\} \right) \times \left( \{y\} \times_{X_0} hX_1 \times_{X_0} \{z\} \right) \quad \longrightarrow \quad \{x\} \times_{X_0} hX_1 \times_{X_0} hX_1 \times_{X_0} \{z\}$$

$$\begin{array}{rcl}
\cong & \{x\} \times_{X_0} hX_2 \times_{X_0} \{z\} \\
\longrightarrow & \{x\} \times_{X_0} hX_1 \times_{X_0} \{z\}. 
\end{array}$$

**Example 1.10.** Given a small (ordinary) category $\mathcal{C}$, the homotopy category of its nerve, viewed as a simplicial space with discrete levels, is equivalent to $\mathcal{C}$,

$$h_1(N(\mathcal{C})) \simeq \mathcal{C}.$$
The above example motivates the following definition of equivalences of Segal spaces.

**Definition 1.11.** A map \( f : X \to Y \) of Segal spaces is a Dwyer-Kan equivalence, or categorical equivalence, if

1. the induced map \( h_1(f) : h_1(X) \to h_1(Y) \) on homotopy categories is essentially surjective, and
2. for each pair of objects \( x, y \in X_0 \) the induced map \( \text{Hom}_X(x, y) \to \text{Hom}_Y(f(x), f(y)) \) is a weak equivalence.

### 1.4. Complete Segal spaces

**Definition 1.12.** Let \( f \) be an element in \( X_1 \) with source and target \( x \) and \( y \), i.e. its images under the two face maps \( X_1 \rightrightarrows X_0 \) are \( x \) and \( y \). It is called invertible if its image under

\[
\{x\} \times X_1 \times \{y\} \to \{x\} \times h_{X_0} X_1 \times h_{X_0} \{y\} \to \pi_0 \left( \{x\} \times h_{X_0} X_1 \times h_{X_0} \{y\} \right) = \text{Hom}_{h_1(X)}(x, y),
\]

is an invertible morphism in \( h_1(X) \).

Denote by \( X_1^{\text{inv}} \) the subspace of invertible arrows and observe that the map \( X_0 \to X_1 \) factors through \( X_1^{\text{inv}} \), since the image of \( x \in X_0 \) under the degeneracy \( X_0 \to \{x\} \times_{X_0} X_1 \times_{X_0} \{x\} \to \text{Hom}_{h_1(X)}(x, x) \) is \( \text{id}_x \), which is invertible.

**Definition 1.13.** A Segal space \( X_\bullet \) is complete if the map \( X_0 \to X_1^{\text{inv}} \) is a weak equivalence. We denote the full subcategory of \( \mathcal{SSp} \) whose objects are complete Segal spaces by \( \mathcal{CSSp} = \mathcal{CSSp}_1 \).

**Example 1.14.** Let \( \mathcal{C} \) be a category. Then \( \mathcal{N}(\mathcal{C}) \) is a complete Segal space if and only if \( \mathcal{C} \) is skeletal, i.e. objects which are isomorphic are equal.

### 1.4.1. Complete Segal spaces as fibrant objects

We describe various model structures on the category \( s\text{Space} \) of simplicial spaces in this section. We refer to \([\text{Rez01}]\) and \([\text{Hor14}]\) for more details.

Let us first consider the injective and projective model structures on the category of simplicial spaces, denoted by \( s\text{Space}_i \) and \( s\text{Space}_f \), respectively. Note that all objects are fibrant in \( s\text{Space}_f \), while fibrant objects of \( s\text{Space}_i \) are Reedy fibrant simplicial spaces. Conversely, all objects are cofibrant in \( s\text{Space}_e \). These model categories are Quillen equivalent (via the identity functor).

First we perform Bousfield localizations of the previous model structures \( s\text{Space}_e \) and \( s\text{Space}_f \) with respect to the morphisms

\[
\Delta[1] \coprod_{\Delta[0]} \cdots \coprod_{\Delta[0]} \Delta[1] \to \Delta[n]
\]

which represent the maps \( X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1 \), respectively their cofibrant replacement \( X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1 \). This provides two Quillen equivalent model categories, denoted \( s\text{Space}^{\text{Se}}_e \) and \( s\text{Space}^{\text{Se}}_f \). Fibrant objects in \( s\text{Space}^{\text{Se}}_e \), respectively \( s\text{Space}^{\text{Se}}_f \), are Reedy fibrant Segal spaces, respectively Segal spaces.

To include completeness, one can further Bousfield localize with respect to the morphism

\[
\Delta[0] \to \mathcal{N}(I[1]),
\]
where $I_1$ is the walking isomorphism, which is the category with two objects and one invertible morphism between them. This provides two Quillen equivalent model categories, denoted $\text{SSpace}^{CSe}_e$ and $\text{SSpace}^{CSe}_f$. Fibrant objects in $\text{SSpace}^{CSe}_e$, respectively $\text{SSpace}^{CSe}_f$, are Reedy fibrant complete Segal spaces, respectively complete Segal spaces.

Summarizing, we have the following diagram

$$\begin{align*}
\text{SSpace}_e & \to \text{SSpace}_f \\
\text{SSpace}^{Se}_e & \to \text{SSpace}^{Se}_f \\
\text{SSpace}^{CSe}_e & \to \text{SSpace}^{CSe}_f
\end{align*}$$

where the horizontal arrows are Quillen equivalences induced by the identity and the vertical arrows are localizations.

**Proposition 1.15.** Let $X$ and $Y$ be Segal spaces. A morphism $f : X \to Y$ is a weak equivalence in $\text{SSpace}^{CSe}_f$ if and only if it is a Dwyer-Kan equivalence.

We refer to [Hor14, Theorem 5.15] for a proof, which makes use of the analogous result for Reedy fibrant Segal spaces in $\text{SSpace}^{CSe}_e$ from [Rez01, Theorem 7.7].

As a consequence the obvious inclusions induce the following equivalences of categories:

$$
\text{CSSp}[\text{lwe}^{-1}] \longrightarrow \text{SeSp}[\text{DK}^{-1}] \longrightarrow \text{Ho}(\text{SSpace}^{CSe}_f),
$$

where $\text{DK}$ and $\text{lwe}$ stand for the subcategory of Dwyer-Kan and levelwise weak equivalences, respectively.

This justifies the following definition.

**Definition 1.16.** An $(\infty, 1)$-category is a complete Segal space.

**Remark 1.17.** The completeness condition says that all invertible morphisms essentially are just identities up to the choice of a path. So strictly speaking, complete Segal spaces should be called skeletal, or, according to [Joy], reduced $(\infty, 1)$-categories. However, any category is equivalent to a reduced one, so this is not a strong restriction.

**Remark 1.18.** Let denote by $\text{CSSp}$ the category of Reedy fibrant complete Segal spaces, that is to say the fibrant objects in $\text{SSpace}^{CSe}_e$. Remember that $\text{SSpace}^{CSe}_e$ and $\text{SSpace}^{CSe}_f$ are Quillen equivalent, so that the embedding $\text{CSSp}_e \subset \text{CSSp}$ induces an equivalence $\text{CSSp}_e[\text{lwe}^{-1}] \to \text{CSSp}[\text{lwe}^{-1}]$ of which an inverse is given by the Reedy fibrant replacement functor $(-)^R$. Sometimes it turns out to be more useful to work in the model category $\text{SSpace}^{CSe}_e$ as every object is cofibrant. Note that, as the Reedy fibrant replacement functor does not change the homotopy type of the levels, Reedy fibrant replacement from $\text{SSpace}^{CSe}_f$ to $\text{SSpace}^{CSe}_e$ does not lose any information.

**Definition 1.19.** The fibrant replacement functor in the model category $\text{SSpace}^{CSe}_f$ sending a Segal space to its fibrant replacement is called completion. In [Rez01] Rezk gave a rather explicit construction of the completion of Segal spaces. He showed that there is a completion functor which to every Segal space $X$ associates a complete Segal space $\hat{X}$ together with a map $i_X : X \to \hat{X}$, which is a Dwyer-Kan equivalence.

**Remark 1.20.** It is worth noticing that $\text{SSpace}_f$, $\text{SSpace}_e$, $\text{SSpace}^{Se}_e$, $\text{SSpace}^{Se}_f$, $\text{SSpace}^{CSe}_f$ and $\text{SSpace}^{CSe}_e$ are all Cartesian closed simplicial model categories. In particular, for any simplicial space $X$ and any complete Segal space $Y$, the simplicial space $Y^X$ is a complete Segal space.
1.4.2. The classification diagram – the Rezk or relative nerve

Many examples of (complete) Segal spaces arise by a construction due to Rezk [Rez01] which produces a (complete) Segal space from a simplicial closed model category. More generally, several authors [BK11, LM14] proved that this construction also gives a complete Segal space for far-reaching generalizations of model categories, namely, for relative categories with certain weak conditions. For instance, categories of fibrant objects in the sense of Brown satisfy the conditions to obtain a Segal space; if they additionally are saturated, they lead to complete Segal spaces.

**Definition 1.21.** Let \((\mathcal{C}, \mathcal{W})\) be a relative category. Consider the simplicial object in categories \(\mathcal{C}_\bullet\) given by \(\mathcal{C}_n := \text{Fun}([n], \mathcal{C})\). It has a subobject \(\mathcal{C}_n^W\), where \(\mathcal{C}_n^W \subset \mathcal{C}_n\) is the subcategory which has the same objects and whose morphisms consist only of those composed of those in \(\mathcal{W}\). Taking its nerve we obtain a simplicial space \(N(\mathcal{C}, \mathcal{W})_\bullet\) with
\[
N(\mathcal{C}, \mathcal{W})_n = N(\mathcal{C}_n^W).
\]

It is proved in [LM14] that this simplicial space satisfies the Segal condition if \((\mathcal{C}, \mathcal{W})\) admits a suitable homotopical three-arrow calculus. Moreover, it is complete if it additionally is saturated, i.e. a morphism is a weak equivalence if and only if it is an isomorphism in the homotopy category. However, it is not level-wise fibrant unless we started with an \(\infty\)-groupoid. Its level-wise fibrant replacement is called the Rezk or relative nerve or the classification diagram, which, by abuse of notation, we again denote by \(N(\mathcal{C}, \mathcal{W})\).

**Example 1.22.** Let \(\mathcal{C}\) be a small category. Then it is straightforward to see that \(N(\mathcal{C}, \text{Iso}\mathcal{C})\) is a complete Segal space.

This construction applied to the model category of complete Segal spaces from the previous section gives a construction of the \((\infty, 1)\)-category of \((\infty, 1)\)-categories:

\[N(\mathcal{CSSp}, \text{fwe})\]

1.5. Segal categories

Let us mention another model for \((\infty, 1)\)-categories given by certain Segal spaces, which avoids completeness and allows non-reduced \((\infty, 1)\)-categories.

**Definition 1.23.** A Segal (1-)category is a Segal space \(X = X_\bullet\) such that \(X_0\) is discrete. We denote \(\text{SegCat}\) the full subcategory of \(\text{SegSp}\) consisting of Segal categories.

Segal categories also are the fibrant objects in a certain model category that is Quillen equivalent to \(s\text{Space}^{\text{Seg}}\), see the above mentioned [Ber10] or [Lur09b] for more details and references. In particular, the embedding \(\text{SegCat} \subset \text{SegSp}\) induces an equivalence of complete Segal spaces

\[N(\text{SegCat}, \mathcal{DK}) \longrightarrow N(\text{SegSp}, \mathcal{DK}).\]

1.6. Relative categories

Following [BK12] a rather weak notion of \((\infty, 1)\)-category is given by relative categories.

**Definition 1.24.** A relative category is a (complete) Segal simplicial set, since we defined a space to be fibrant.

\[\text{SegCat} \subset \text{SegSp}\]
Example 1.25. Let $\mathcal{C} = \text{Ch}_R$ be the category of chain complexes over a ring $R$ and let $\mathcal{W} \subseteq \mathcal{C}$ be the subcategory of chain complexes and quasi-isomorphisms.

Remark 1.26. One should think of the weak equivalences as being “formally inverted”.

$\mathcal{RelCat}$ admits a model structure exhibiting it as a model for $(\infty, 1)$-categories: in $\lfloor \text{BK11} \rfloor$ the model structure of $s\text{Space}^{CSe}_\xi$ is transferred along a slight modification of the relative nerve, 

$$N_\xi : \mathcal{RelCat} \longrightarrow s\text{Space}^{CSe}_\xi : K_\xi$$

thus making the above adjunction into a Quillen equivalence.

### 1.7. Categories internal to simplicial sets

Instead of enriching categories in a category of spaces as in Section 1.2, for certain applications it turns out to be useful to also have a space of objects (thus allowing more flexibility than in topological categories), but keeping strict composition (and thus having more rigidity than in Segal spaces). This philosophy is implemented when considering categories internal to spaces.

In $\lfloor \text{Hor14} \rfloor$ Section 3] Horel introduces the following notion.

**Definition 1.27.** A category internal to spaces or for short, an internal category, consists of simplicial sets $C_0, C_1$ together with source and target morphisms $s, t : C_1 \to C_0$, a degeneracy morphism $d : C_0 \to C_1$ satisfying $s \circ d = t \circ d = id_{C_0}$, and a composition morphism $\circ : C_1 \times_{C_0} C_1 \to C_1$ such that for any $x \in C_0$, the maps $- \circ d(x)$ and $d(x) \circ -$ are the identity. Let $\mathcal{ICat}$ denote the category of categories internal to simplicial sets, where morphisms from $(C_0, C_1)$ to $(D_0, D_1)$ are pairs of morphisms $C_i \to D_i$ for $i = 0, 1$ which are compatible with the additional structure in the obvious way.

Note that there is an equivalence of categories $\mathcal{ICat} \to \text{Cat}^{\Delta^{op}}$. Composition with the level-wise nerve and swapping the simplicial directions gives a functor

$$N : \mathcal{ICat} \longrightarrow \text{Cat}^{\Delta^{op}} \xrightarrow{N(-)} \text{Space}^{\Delta^{op}} \xrightarrow{\text{swap}} s\text{Space} .$$

Similarly to $\mathcal{RelCat}$, the model structure of $s\text{Space}^{Se}_f$ and $s\text{Space}^{CSe}_f$ can be transferred along $N$ to endow $\mathcal{ICat}$ with a model structure, the latter exhibiting it as a model for $(\infty, 1)$-categories. Examples of fibrant objects in the former model category are given by the following strongly Segal internal categories ($\lfloor \text{Hor14} \rfloor$ Proposition 5.13), i.e. their nerve is a Segal space.

**Definition 1.28.** A strongly Segal internal category is a category $C = (C_0, C_1)$ internal to spaces such that the space of objects is fibrant, i.e. a Kan complex, and the source and target maps $s, t : C_1 \to C_0$ are fibrations of simplicial sets. We denote by $\mathcal{ICat}^{Se}$ the category of strongly Segal internal categories.

Since the model structure was transferred, we have a Quillen equivalence given by the nerve,

$$N : \mathcal{ICat} \longrightarrow s\text{Space}^{Se}_f .$$

Moreover, categorical equivalences of strongly Segal internal categories are (by definition) precisely the morphisms that are sent to Dwyer-Kan equivalences by the nerve. Thus, the induced morphism

$$N(\mathcal{ICat}^{Se}, \text{cat. eq.}) \longrightarrow N(\text{ScSp}, DK)$$

is an equivalence of complete Segal spaces.
1.8. Adjunctions between \((\infty, 1)\)-categories

Let \(X\) and \(Y\) be \((\infty, 1)\)-categories.

**Definition 1.29.** An adjunction between \(X\) and \(Y\) is a pair of \(\infty\)-functors \(R : X \to Y\) and \(L : Y \to X\) together with a natural transformation

\[
\eta \in \text{Hom}_{\text{Fun}(Y,Y)}(Id_Y, R \circ L),
\]

which we write \(\eta : Id_Y \Rightarrow R \circ L\), such that for any \(x \in X_0\) and any \(y \in Y_0\) the composition

\[
\text{Hom}_X(L(y), x) \to \text{Hom}_Y(R(L(y)), R(x)) \cong \text{Hom}_Y(R \circ L(y), R(x)) \to \text{Hom}_Y(y, R(x)) \tag{1}
\]

is a weak equivalence of spaces.

As usual, we say that \(R\), resp. \(L\), is the right adjoint, resp. left adjoint.

**Explanations.** Let us comment on the above definition. Recall that

\[
R \in \text{Hom}_{N(\text{CSSp, lwe})}(X, Y) \cong \text{Fun}(X, Y)_0, \quad L \in \text{Hom}_{N(\text{CSSp, lwe})}(Y, X) \cong \text{Fun}(Y, X)_0,
\]

and \(Id_Y\) is the constant path in \(\text{Hom}_{N(\text{CSSp, lwe})}(Y, Y) \cong \text{Fun}(Y, Y)_0\). Note that as usual we slightly abuse language as only the connected component of

\[
R \circ L \in \text{Hom}_{N(\text{CSSp, lwe})}(Y, Y) \cong \text{Fun}(Y, Y)_0
\]

is defined (through the composition in \(h_1(N(\text{CSSp, lwe}))\)). Of course, one can easily check that for any \(x \in X_0\), \(R(L(x))\) and \(R \circ L(x)\) lie in the same connected component in \(Y_0\). It remains to understand the last arrow in (1).

Note that for any two functors \(F, G : X \to Y\) the space of natural transformations \(\text{Hom}_{\text{Fun}(X,Y)}(F,G)\) is equivalent to

\[
\{F\} \underset{\text{Fun}(X,Y)_0}{\times} \text{Fun}(X,Y)_1 \underset{\text{Fun}(X,Y)_0}{\times} \{G\}
\]

For any \(x \in X_0\), \(\pi_0(\text{Hom}_{\text{Fun}(X,Y)}(F,G))\) thus has a map to

\[
\pi_0 \left(\{F(x)\} \underset{Y_0}{\times} \{G(x)\} \right) = \text{Hom}_{h_1(Y)}(F(x), G(x)).
\]

Hence with any natural transformation \(\alpha : F \Rightarrow G\) and any \(x \in X_0\) we can associate a morphism \(\alpha_x \in \text{Hom}_Y(F(x), G(x))\), which in turn defines (up to homotopy) a map

\[
\text{Hom}_Y(G(x), y) \xrightarrow{\text{const}} \text{Hom}_Y(F(x), y).
\]

**Example 1.30.** Assume we are given two model categories \(\mathcal{M}\) and \(\mathcal{N}\) together with a Quillen adjunction \(\mathcal{N} \xrightarrow{L} \mathcal{M}\). Let \(X = N(\mathcal{M}, \mathcal{W}_\mathcal{M})\) and \(Y = N(\mathcal{N}, \mathcal{W}_\mathcal{N})\). We wish to explain that the above Quillen adjunction induces an adjunction in the sense of Definition 1.29 between the \((\infty, 1)\)-categories \(X\) and \(Y\). This has already been done in the context of quasi-categories in [Hin, Proposition 1.5.1] (see also [Maz13] for a similar but more general result), therefore we will be brief.

Since \(L\) preserves weak equivalences between cofibrant objects then its left derived functor \(L' = L \circ Q\) (\(Q\) is a colibrant replacement functor) preserves weak equivalences and thus induces a morphism \(L : Y \to X\). Similarly, the right derived functor \(R' = R \circ P\) (\(P\) is a fibrant replacement functor) induces a morphism \(R : X \to Y\). Note that \(Q\) preserves weak equivalences as well and thus induces a morphism \(Q : Y \to Y\), which lies in the connected component of \(Id_Y \in \text{Hom}_{N(\text{CSSp, lwe})}(Y, Y) \cong \text{Fun}(Y, Y)_0\).
Let $\eta_0 : Id_{\mathcal{N}} \Rightarrow R \circ L$ be the unit of the adjunction $L \dashv R$. It induces a natural transformation $\eta : Q \rightarrow R \circ P \circ L \circ Q = \mathcal{R} R \circ LL$, and thus defines an element in

$$\text{Hom}_{\text{Fun}}(Q, R \circ L) \cong \text{Hom}_{\text{Fun}}(Id, R \circ L),$$

which we still denote $\eta$.

It remains to prove that the induced map (1) is indeed a weak equivalence, which one can prove by representing the sequence of maps (1) is represented by the following one:

$$\text{Map}_{\mathcal{M}}(LL(y), P(x)) \rightarrow \text{Map}_{\mathcal{N}}(R \circ R(L(y)), R \circ R(P(x))) \cong \text{Map}_{\mathcal{N}}(R \circ P \circ L \circ Q(y), R \circ P(x)) \rightarrow \text{Map}_{\mathcal{N}}(Q(y), R \circ R(x)), \text{ which we know is an equivalence because } L \dashv R \text{ is a Quillen adjunction.}$$

**Remark 1.31.** Similarly, from a Quillen equivalence between $\mathcal{M}$ and $\mathcal{N}$ one gets a categorical equivalence, and thus a weak equivalence, between $N(\mathcal{M}, \mathcal{W}_{\mathcal{M}})$ and $N(\mathcal{N}, \mathcal{W}_{\mathcal{N}})$. We have already used this observation above.

## 2. Models for $(\infty, n)$-categories

As a model for $(\infty, n)$-categories, we will use complete $n$-fold Segal spaces, which were first introduced by Barwick in his thesis and appeared prominently in Lurie’s [Lur09c]. Our examples will arise from a more rigid model, namely from internal $n$-uple categories, with are $n$-uple categories (the higher analog of a double category) internal to simplicial sets.

### 2.1. Internal $n$-uple categories

Iterating the approach in [Hor14], one obtains a model for $(\infty, n)$-categories given by $n$-uple categories internal to simplicial sets, i.e. categories internal to the category of $(n - 1)$-uple categories internal to simplicial sets. Unravelling the definition for $n = 2$, there is a space of objects, a space of “horizontal” 1-morphisms, a space of “vertical” 1-morphisms, and a space of 2-morphisms, together with unit maps and composition maps. For larger $n$, there is a space of objects and suitable spaces of higher morphisms “in all directions”, again together with unit maps and composition maps. Equivalently, an $n$-uple category internal to simplicial sets is a simplicial object in (strict) $n$-fold categories.

Just as one can extract a bicategory from a double category, one can consider the maximal underlying internal $n$-category of an internal $n$-tuple category.

**Remark 2.1.** Note that composition is well-defined on the nose, as opposed to the models we will consider in the next sections.

**Remark 2.2.** It would be interesting to set up the model categorical details of internal $n$-fold categories and internal $n$-categories similarly to [Hor14]. This is work in progress by Giovanni Caviglia and Geoffroy Horel, [CH].

### 2.2. $n$-uple and $n$-fold Segal spaces

An $n$-uple Segal space is an $n$-fold simplicial space with an extra condition ensuring it is the $\infty$-analog of an $n$-uple category.

**Definition 2.3.** An $n$-uple Segal space is an $n$-fold simplicial space $X = X_{\bullet, \ldots, \bullet}$ such that for every $1 \leq i \leq n$, and every $k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n \geq 0$, $X_{k_1, \ldots, k_{i-1}, \bullet, k_{i+1}, \ldots, k_n}$. 

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is a Segal space.

Defining a map of \( n \)-uple Segal spaces to be a map of the underlying \( n \)-fold simplicial spaces gives a category of \( n \)-uple Segal spaces, \( \mathcal{S}_n \).

Imposing an extra globularity condition leads to a model for \( \infty \)-analog of \( n \)-categories:

**Definition 2.4.** An \( n \)-fold simplicial space \( X_\bullet, \ldots, \bullet \) is essentially constant if there is a weak homotopy equivalence of \( n \)-fold simplicial spaces \( Y \rightarrow X \), where \( Y \) is constant.

**Definition 2.5.** An \( n \)-fold Segal space is an \( n \)-uple Segal space \( X_\bullet, \ldots, \bullet \) such that for every \( 1 \leq i \leq n \), and every \( k_1, \ldots, k_{i-1} \geq 0 \),

\[
X_{k_1, \ldots, k_{i-1}, 0, \bullet, \ldots, \bullet}
\]

is essentially constant.

Defining a map of \( n \)-fold Segal spaces to be a map of the underlying \( n \)-fold simplicial spaces gives a category of \( n \)-fold Segal spaces, \( \mathcal{S}_n \).

**Remark 2.6.** Alternatively, one can formulate the conditions iteratively. First, an \( n \)-uple Segal space is a simplicial object \( Y_\bullet \) in \( (n-1) \)-fold Segal spaces which satisfies the Segal condition. Then, an \( n \)-fold Segal space is an \( n \)-uple Segal space such that \( Y_0 \) is essentially constant (as an \( (n-1) \)-fold Segal space).

To get back the above definition, the ordering of the indices is crucial: \( X_{k_1, \ldots, k_n} = (Y_{k_i})_{k_2, \ldots, k_n} \).

### 2.2.1. Interpretation as higher categories

An \( n \)-fold Segal space can be thought of as a higher category in the following way.

The first condition means that this is an \( n \)-fold category, i.e. there are \( n \) different “directions” in which we can “compose”. An element of \( X_{k_1, \ldots, k_n} \) should be thought of as a composition consisting of \( k_i \) composed morphisms in the \( i \)th direction.

The second condition imposes that we indeed have a higher \( n \)-category, i.e. an \( n \)-morphism has as source and target two \((n-1)\)-morphisms which themselves have the “same” (in the sense that they are homotopic) source and target.

For \( n = 2 \) one can think of this second condition as “fattening” the objects in a bicategory. A 2-morphism in a bicategory can be depicted as

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\]

The top and bottom arrows are the source and target, which are 1-morphisms between the same objects.

In a 2-fold Segal space \( X_{\bullet, \bullet} \), an element in \( X_{1,1} \) can be depicted as

\[
\begin{array}{c}
x_{0,0} \\
\downarrow \\
x_{1,0} \\
\downarrow \\
x_{0,1} = x_{0,0}
\end{array}
\]

The images under the source and target maps in the first direction \( X_{1,1} \Rightarrow X_{1,0} \) are 1-morphisms which are depicted by the horizontal arrows. The images under the source and target maps in the second direction \( X_{1,1} \Rightarrow X_{0,1} \) are 1-morphisms, depicted by the dashed vertical arrows, which are essentially
just identity maps, up to homotopy, since \( X_{0,1} \cong X_{0,0} \). Thus, here the source and target 1-morphisms (the horizontal ones) themselves do not have the same source and target anymore, but up to homotopy they do.

The same idea works with higher morphisms, in particular one can imagine the corresponding diagrams for \( n = 3 \). A 3-morphism in a tricategory can be depicted as

\[
\begin{array}{c}
\circ \\
\downarrow \\
\circ \\
\end{array}
\]

whereas a 3-morphism, i.e. an element in \( X_{1,1,1} \) in a 3-fold Segal space \( X \) can be depicted as

\[
\begin{array}{c}
\circ \\
\downarrow \\
\circ \\
\end{array}
\begin{array}{c}
\circ \\
\downarrow \\
\circ \\
\end{array}
\]

Here the dotted arrows are those in \( X_{0,1,1} \cong X_{0,0,1} \cong X_{0,0,0} \) and the dashed ones are those in \( X_{1,0,1} \cong X_{1,0,0} \).

Thus, we should think of the set of 0-simplices of the space \( X_{0,...,0} \) as the objects of our category, and elements of \( X_{1,...,1,0,...,0} \) as \( i \)-morphisms, where \( 0 < i \leq n \) is the number of 1’s. Pictorially, they are the \( i \)-th “horizontal” arrows. Moreover, the other “vertical” arrows are essentially just identities of lower morphisms. Similarly to before, paths in \( X_{1,...,1} \) should be thought of as \( (n+1) \)-morphisms, which therefore are invertible up to a homotopy, which itself is an \( (n+2) \)-morphism, and so forth.

### 2.2.2. The homotopy bicategory of a 2-fold Segal space

To any higher category one can intuitively associate a bicategory having the same objects and 1-morphisms, and with 2-morphisms being 3-isomorphism classes of the original 2-morphisms.

**Definition 2.7.** The *homotopy bicategory* \( h_2(X) \) of a 2-fold Segal space \( X = X_{\bullet, \bullet} \) is defined as follows: objects are the points of the space \( X_{0,0} \) and

\[
\text{Hom}_{h_2(X)}(x, y) = h_1(\text{Hom}_X(x, y)) = h_1 \left( \left\{ \begin{array}{c} x \\ X_{0,0} \end{array} \right\} \times \left\{ \begin{array}{c} y \\ X_{0,0} \end{array} \right\} \right)
\]

as Hom categories. Horizontal composition is defined as follows:

\[
\left( \left\{ \begin{array}{c} h \\ X_{0,0} \end{array} \right\} \times \left\{ \begin{array}{c} h \\ X_{0,0} \end{array} \right\} \right) \times \left( \left\{ \begin{array}{c} h \\ X_{0,0} \end{array} \right\} \times \left\{ \begin{array}{c} h \\ X_{0,0} \end{array} \right\} \right) \times \left\{ \begin{array}{c} h \\ X_{0,0} \end{array} \right\} \times \left\{ \begin{array}{c} h \\ X_{0,0} \end{array} \right\} \times \left\{ \begin{array}{c} h \\ X_{0,0} \end{array} \right\} \times \left\{ \begin{array}{c} h \\ X_{0,0} \end{array} \right\} \times \left\{ \begin{array}{c} h \\ X_{0,0} \end{array} \right\} \times \left\{ \begin{array}{c} h \\ X_{0,0} \end{array} \right\} \times \left\{ \begin{array}{c} h \\ X_{0,0} \end{array} \right\} \times \left\{ \begin{array}{c} h \\ X_{0,0} \end{array} \right\} \times \left\{ \begin{array}{c} h \\ X_{0,0} \end{array} \right\} \right)
\]

The second arrow happens to go in the wrong way but it is a weak equivalence. Therefore after taking \( h_1 \) it turns out to be an equivalence of categories, and thus to have an inverse (assuming the axiom of choice).
2.3. Complete and hybrid \( n \)-fold Segal spaces

As with (1-fold) Segal spaces, we need to impose an extra condition to ensure that invertible \( k \)-morphisms are paths in the space of \((k - 1)\)-morphisms. Again, there are several ways to include its information.

**Definition 2.8.** Let \( X \) be an \( n \)-fold Segal space and \( 1 \leq i, j \leq n \). It is said to satisfy

- \( CSS^i \) if for every \( k_1, \ldots, k_{i-1} \geq 0 \),
  \[
  X_{k_1, \ldots, k_{i-1}, \bullet, \ldots, \bullet}
  \]
  is a complete Segal space.

- \( SC^j \) if for every \( k_1, \ldots, k_{j-1} \geq 0 \),
  \[
  X_{k_1, \ldots, k_{j-1}, 0, \ldots, \bullet}
  \]
  is discrete, i.e. a discrete space viewed as a constant \( (n - j + 1) \)-fold Segal space.

**Definition 2.9.** An \( n \)-fold Segal space is

1. **complete**, if for every \( 1 \leq i \leq n \), \( X \) satisfies \( CSS^i \).

2. a **Segal \( n \)-category** if for every \( 1 \leq j \leq n \), \( X \) satisfies \( SC^j \).

3. **\( m \)-hybrid** for \( m \geq 0 \) if condition \( CSS^i \) is satisfied for \( i > m \) and condition \( SC^j \) is satisfied for \( j \leq m \).

Denote the full subcategory of \( \mathcal{S}e\mathcal{S}p_n \) of complete \( n \)-fold Segal spaces by \( CSSp_n \).

**Remark 2.10.** Note that an \( n \)-hybrid \( n \)-fold Segal space is a Segal \( n \)-category, while an \( n \)-fold Segal space is 0-hybrid if and only if it is complete.

For our purposes, the model of complete \( n \)-fold Segal spaces is well-suited, so we define

**Definition 2.11.** An \((\infty, n)\)-category is an complete \( n \)-fold Segal space.

2.3.1. The underlying model categories

Similarly to subsection 1.4.1 there are model categories running in the background. We can consider either the injective or projective model structure on the category of \( n \)-fold simplicial spaces \( s\mathcal{S}pace_n \), which we denote by \( s\mathcal{S}pace_{n,c} \) respectively \( s\mathcal{S}pace_{n,f} \). Bousfield localizations at the analogs of the Segal maps give model structures whose fibrant objects are (Reedy fibrant) \( n \)-uple Segal spaces, further localizing at maps governing essential constancy, the fibrant objects become (Reedy fibrant) \( n \)-fold Segal spaces, and a third localization at a map imposing completeness gives model structures whose fibrant objects which are (Reedy fibrant) \( n \)-fold Segal spaces, see [Lur09b, BS11] and [JS15, Appendix]. Note that again, the identity map induces a Quillen equivalence between \( s\mathcal{S}pace_{n,c} \) and \( s\mathcal{S}pace_{n,f} \) which descends to the localizations.

Alternatively, and by [JS15, Appendix, Proposition A.9] equivalently, the construction of complete Segal objects for absolute distributors from [Lur09b] provides an iterative definition of these model categories by considering simplicial objects in a suitable model category (which is taken to be the appropriate localization of \( s\mathcal{S}pace_{n-1,c} \) respectively \( s\mathcal{S}pace_{n-1,f} \)) and localizes at the maps governing the Segal condition, essential constancy, and/or completeness in the new simplicial direction.

[Lur09b] also provides a model category whose fibrant objects are Segal category objects in some suitable underlying model category, thus allowing to iterate the construction of Segal categories as well. Applying
this construction \( m \) times to the above one for complete \((n - m)\)-fold Segal spaces provides a model category whose fibrant objects are \( m \)-hybrid \( n \)-fold Segal spaces.

One can show (see e.g. in [Zha13] for the Reedy model structure version) that equivalences between (possibly non-complete) \( n \)-fold Segal spaces for this model structure are exactly the Dwyer-Kan equivalences, which are defined inductively:

**Definition 2.12.** A morphism \( f : X \to Y \) of \( n \)-fold Segal spaces is a Dwyer-Kan equivalence if

1. the induced functor \( h_1(f) : h_1(X) \to h_1(Y) \) is essentially surjective.

2. for each pair of objects \( x, y \in X_{0, \ldots, 0} \), the induced morphism \( \text{Hom}_X(x, y) \to \text{Hom}_Y(f(x), f(y)) \) is a Dwyer-Kan equivalence of \((n - 1)\)-fold Segal spaces.

Note that by definition, \( h_1(X) := h_1(X_{0, \ldots, 0}) \), and

\[
\text{Hom}_X(x, y)_{0, \ldots, 0} := \{ x \} \times_{X_{0, \ldots, 0}} X_{1, \ldots, n} \times_{X_{0, \ldots, 0}} \{ y \}.
\]

Again we obtain equivalences of complete Segal spaces

\[
N(\text{CSS}_n, \text{fwe}) \longrightarrow N(\text{SeSp}_n, \text{DK}) \longrightarrow N(\text{Space}_n, \mathcal{W}_f)^{\text{CSS}_n},
\]

where \( \mathcal{W}^\text{CSS}_f \) is the subcategory of weak equivalences in the localization \( \text{Space}_n^{\text{CSS}_f} \).

**Remark 2.13.** Note that \( \text{CSS}_n \) is the subcategory of fibrant objects for a left Bousfield localization of \( \text{Space}_n, f \), so weak equivalences of complete \( n \)-fold Segal spaces are levelwise weak equivalences. Again the Quillen equivalence between \( \text{SeSp}_n, c \) and \( \text{Space}_n, f \) induces an equivalence \( N(\text{CSS}_n, \text{fwe}) \longrightarrow N(\text{CSS}_n, \text{fwe}) \), whose inverse is given by Reedy fibrant replacement \((-)^R\).

**Definition 2.14.** In light of the iterative definition of an \( n \)-fold Segal space, i.e. viewing an \( n \)-fold Segal space as an \((n - 1)\)-fold Segal space, condition \((\text{CSS}^i)\) above means that the \( i \)th iteration is a complete Segal space object. Thus, given an \( n \)-fold Segal space \( X_{\ldots, \ldots, 0} \), one can apply the completion functor iteratively to obtain a complete \( n \)-fold Segal space \( \bar{X}_{\ldots, \ldots, 0} \), its \((n\text{-fold}) \text{ completion}\). This yields a map \( X \to \bar{X} \), the completion map, which is universal among all maps (in the homotopy category) to complete \( n \)-fold Segal spaces. It is a left adjoint to the embedding of \( \text{CSS}_n, \text{fwe}^{-1} \) into \( \text{SeSp}_n, \text{fwe}^{-1} \).

If an \( n \)-fold Segal space \( X_{\ldots, \ldots, 0} \) satisfies \((\text{SC}^j)\) for \( j \leq m \), we can apply the completion functor just to the last \((n - m)\) indices to obtain an \( m \)-hybrid \( n \)-fold Segal space \( \bar{X}_{\ldots, \ldots}^m \), its \( m \)-hybrid completion.

### 2.4. Constructions of \( n \)-fold Segal spaces

We describe several intuitive constructions of \((\infty, n)\)-categories in terms of \((\text{complete})\) \( n \)-fold Segal spaces.

#### 2.4.1. Truncation

Given an \((\infty, n)\)-category, for \( k \leq n \) its \((\infty, k)\)-truncation, or \( k\text{-truncation} \), is the \((\infty, k)\)-category obtained by discarding the non-invertible \( m \)-morphisms for \( k < m \leq n \).

In terms of \( n \)-fold Segal spaces, there is a functor \( \text{SeSp}_n \to \text{SeSp}_k \) sending \( X = X_{\ldots, \ldots} \) to its \( k \)-truncation, the \( k \)-fold Segal space

\[
\tau_k X = X_{\underbrace{\ldots, \ldots}_{k \text{ times}}}^{0, \ldots, 0} \underbrace{0, \ldots, 0}_{n-k \text{ times}}.
\]

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If $X$ is $m$-hybrid then so is $\tau_k X$ by the definition of the conditions $CSS_1^{i}$ and $SC_1^{j}$. In particular, if $X$ is complete, then $\tau_k X$ is as well, and thus, the truncation of an $(\infty, n)$-category is an $(\infty, k)$-category.

**Warning.** Truncation does not behave well with completion, i.e. the truncation of the completion is not the completion of the truncation. However, we get a map in one direction.

\[
\tau_k(X) \xrightarrow{\text{map}} \tau_k(\tilde{X})
\]

In general, this map is not an equivalence. So in general one should always complete an $n$-fold Segal space before truncating it.

**Remark 2.15.** As explained above, the $k$-truncation of an $(\infty, n)$-category $X$ should be the maximal $(\infty, k)$-category contained in $X$. However, the image of the degeneracy $X_{1, \ldots, 1, 0, \ldots, 0} \xrightarrow{k} X_{1, \ldots, 1, 0, \ldots, 0}$ consists exactly of the invertible $m$-morphisms for $k < m \leq n$ if and only if $X$ satisfies $CSS_1^{i}$ for $k < i \leq n$. For example, if $X = X_\bullet$ is a (1-fold) Segal space then $X_0$ is the underlying $\infty$-groupoid of invertible morphisms if and only if $X$ is complete.

**2.4.2. Extension**

Any $(\infty, n)$-category can be viewed as an $(\infty, n + 1)$-category with only identities as $(n + 1)$-morphisms.

In terms of $n$-fold Segal spaces, any $n$-fold Segal space can be viewed as a constant simplicial object in $n$-fold Segal spaces, i.e. an $(n + 1)$-fold Segal space which is constant in the first index. Explicitly, if $X_{\bullet, \ldots, \bullet}$ is an $n$-fold Segal space, then $\varepsilon(X)_{\bullet, \ldots, \bullet}$ is the constant simplicial object in the category of Segal spaces given by $X$, i.e. it is the $(n + 1)$-fold Segal space such that for every $k \geq 0$,

\[
\varepsilon(X)_k, \ldots, \bullet = X_{\bullet, \ldots, \bullet}
\]

and the face and degeneracy maps in the first index are identity maps.

**Lemma 2.16.** If $X$ is complete, then $\varepsilon(X)$ is complete.

**Proof.** Since $X$ is complete, it satisfies $CSS_1^{i}$ for $i > 1$. For $i = 0$, we have to show that $\varepsilon(X)_{\bullet, 0, \ldots, 0}$ is complete. This is satisfied because

\[
(\varepsilon(X)_{1, 0, \ldots, 0})^{inv} = \varepsilon(X)_{1, 0, \ldots, 0} = X_{0, \ldots, 0} = \varepsilon(X)_{0, 0, \ldots, 0},
\]

since morphisms between two elements $x, y$ in the homotopy category of $\varepsilon(X)_{\bullet, k_2, \ldots, k_n}$ are just connected components of the space of paths in $X_{k_2, \ldots, k_n}$, and thus are always invertible.

We call $\varepsilon$ the extension functor, which is left adjoint to $\tau_{n-1}$. Moreover, the unit $id \rightarrow \tau_1 \circ \varepsilon$ of the adjunction is the identity.

**2.4.3. The higher category of morphisms and loopings**

Given two objects $x, y$ in an $(\infty, n)$-category, morphisms from $x$ to $y$ should form an $(\infty, n - 1)$-category. This can be realized for $n$-fold Segal spaces, which is one of the main advantages of this model for $(\infty, n)$-categories.
Definition 2.17. Let \( X = X_\bullet, \ldots, \bullet \) be an \( n \)-fold Segal space. As we have seen above one should think
of objects as vertices of the space \( X_{0, \ldots, 0} \). Let \( x, y \in X_{0, \ldots, 0} \). The \((n-1)\)-fold Segal space of morphisms from \( x \) to \( y \) is
\[
\text{Hom}_X(x,y)_{\bullet, \ldots, \bullet} = \{ x \} \times_{X_0, \ldots, \bullet} X_1, \ldots, \bullet \times_{X_{0, \bullet, \ldots, \bullet}} \{ y \}.
\]

Remark 2.18. Note that if \( X \) is \( m \)-hybrid, then \( \text{Hom}_X(x,y) \) is \((m-1)\)-hybrid.

Example 2.19 (Compatibility with extension). Let \( X \) be an \((\infty,0)\)-category, i.e. a space, viewed as
an an \((\infty,1)\)-category, i.e. a constant (complete) Segal space \( \varepsilon(X)_\bullet, \varepsilon(X)_k = X \). For any two objects
\( x, y \in \varepsilon(X)_0 = X \) the \((\infty,0)\)-category, i.e. the topological space, of morphisms from \( x \) to \( y \) is
\[
\text{Hom}_{\varepsilon(X)}(x,y) = \{ x \} \times_{\varepsilon(X)_0} \varepsilon(X)_1 \times_{\varepsilon(X)_0} \{ y \} = \{ x \} \times_{X} \{ y \} = \text{Path}_X(x,y),
\]
the path space in \( X \), which coincides with what one expects by the interpretation of paths, homotopies,
homotopies between homotopies, etc. being higher invertible morphisms.

Definition 2.20. Let \( X \) be an \( n \)-fold Segal space, and \( x \in X_0 \) an object in \( X \). Then the \textit{looping of} \( X 
\) at \( x \) is the \((n-1)\)-fold Segal space
\[
L(X,x)_{\bullet, \ldots, \bullet} = \text{Hom}_X(x,x)_{\bullet, \ldots, \bullet} = \{ x \} \times_{X_0, \bullet, \ldots, \bullet} X_1, \bullet, \ldots, \bullet \times_{X_0, \bullet, \ldots, \bullet} \{ x \}.
\]
In the following, it will often be clear at which element we are looping, e.g. if there essentially is only
one element, or at a unit for a monoidal structure, which we define in the next section. Then we omit
the \( x \) from the notation and just write
\[
LX = L(X) = L(X,x).
\]
We can iterate this procedure as follows.

Definition 2.21. Let \( L_0(X,x) = X \). For \( 1 \leq k \leq n \), let the \textit{k-fold iterated looping} be the \((n-k)\)-fold Segal space
\[
L_k(X,x) = L(L_{k-1}(X,x),x),
\]
where we view \( x \) as a trivial \( k \)-morphism via the degeneracy maps, i.e. an element in \( L_{k-1}(X,x)_{0,\ldots,0} \subset \)
\( X_{1,\ldots,1,0,\ldots,0} \) with \( k \) 1's.

Remark 2.22. We remark that looping commutes with taking the ordinary or the \( m \)-hybrid completion,
since completion is taken index per index.

2.4.4. (Complete) \( n \)-fold from (complete) \( n \)-uple Segal spaces

We can extract the maximal \( n \)-fold (complete) Segal space from an \( n \)-uple one by the following procedure.
Let us recall and introduce some notation for various model structures on the category of \( n \)-fold simplicial
sets.

- \textbf{sSpace}^{(C)\text{Se}}_{0,f} \), where fibrant objects are (complete) \( n \)-fold Segal spaces.
- \textbf{sSpace}^{(C)\text{Se}}_{n,c} \), where fibrant objects are Reedy fibrant (complete) \( n \)-fold Segal spaces.
- \textbf{sSpace}^{(C)\text{Se}}_{n,f} \), where fibrant objects are (complete) \( n \)-uple Segal spaces.
- \textbf{sSpace}^{(C)\text{Se}}_{n,c} \), where fibrant objects are Reedy fibrant (complete) \( n \)-uple Segal spaces.
From now, $\ast \in \{c, f\}$. There are (four) Quillen adjunctions

$$s\text{Space}_{n, \ast}^{(C)Sc} \xrightarrow{id} s\text{Space}_{n, \ast}^{(C)Sc}.$$  

Let us denote (in a rather unusual way) $L := \text{Rid} : N(s\text{Space}_{n, \ast}^{(C)Sc}, w.e.) \rightarrow N(s\text{Space}_{n, \ast}^{(C)Sc}, w.e.)$. Observe that on fibrant objects, $L$ is nothing but the inclusion of (possibly Reedy fibrant or complete) $n$-fold Segal spaces into (eventually Reedy fibrant or complete) $n$-uple Segal spaces. After [Hau14], we know it has a right adjoint $R$. For we given (possibly Reedy fibrant or complete) $n$-uple Segal space $X$, we wish to compute $R(X)$. By adjunction, we know that

$$R(X)_{\ast, \ldots, \ast} \cong \text{Hom}(\Delta^\otimes \bar{\ast}, R(X)) \cong \text{Hom}(L(\Delta^\otimes \bar{\ast}), X),$$

where $\Delta^\otimes \bar{\ast}$ is the $n$-fold simplicial set represented by $[k_1] \times \cdots \times [k_n]$.

Let $\Theta^\otimes \bar{\ast}$ be the walking $\bar{\ast}$-tuple of $n$-morphisms strict $n$-category as in [JS15, Definition 5.1]. The $n$-fold nerve of $\Theta^\otimes \bar{\ast}$ is

- levelwise fibrant (because $\Theta^\otimes \bar{\ast}$ is discrete).
- a Segal space (because $\Theta^\otimes \bar{\ast}$ is a strict $n$-category).
- complete (because $\Theta^\otimes \bar{\ast}$ is reduced).

Let us thus abuse notation and still write $\Theta^\otimes \bar{\ast}$ for this complete $n$-fold Segal space.

**Lemma 2.23.** The natural map $\Delta^\otimes \bar{\ast} \rightarrow \Theta^\otimes \bar{\ast}$ is a weak equivalence in $s\text{Space}_{n, \ast}^{(C)Sc}$.

**Proof.** We need to show that for any fibrant object $Y$ in $s\text{Space}_{jF-S_n, \ast}^{(C)Sc}$ the induced map $\text{Map}^h(\Delta^\otimes \bar{\ast}, Y) \rightarrow \text{Map}^h(\Theta^\otimes \bar{\ast}, Y)$ is a weak equivalence of simplicial sets. Following [JS15, Lemma 2.10] it is sufficient to prove it when $\ast = c$, in which case

$$\text{Map}^h(\Delta^\otimes \bar{\ast}, Y) \cong \text{Map}(\Delta^\otimes \bar{\ast}, Y) \cong Y_{k_1, \ldots, k_n}$$

and

$$\text{Map}^h(\Theta^\otimes \bar{\ast}, Y) \cong \text{Map}(\Theta^\otimes \bar{\ast}, Y) \rightarrow Y_{k_1, \ldots, k_n},$$

where the last arrow is a weak equivalence because of condition (ii) in Definition [20].

**Remark 2.24.** The above Lemma is actually tautological if one observes that the model structure $s\text{Space}_{n, \ast}^{(C)Sc}$ can be obtained as the left Bousfield localization of $s\text{Space}_{n, \ast}^{(C)Sc}$ along $\Delta^\otimes \bar{\ast} \rightarrow \Theta^\otimes \bar{\ast}$.

**Corollary 2.25.** Given a (complete) $n$-uple Segal space $X$ its maximal underlying (complete) $n$-fold Segal space is

$$R(X)_{k_1, \ldots, k_n} = \text{Map}^h(\Theta^\otimes \bar{\ast}, X).$$

### 3. Symmetric monoidal structures

#### 3.1. Definition via $\Gamma$-object

Following [Toe, TV09], we define a symmetric monoidal $n$-fold Segal space in analogy to so-called $\Gamma$-spaces.
Definition 3.1. Segal’s category $\Gamma$ is the category whose objects are the finite sets 
$$\langle m \rangle = \{0, \ldots, m\},$$
for $m \geq 0$ which are pointed at 0. Morphisms are pointed functions, i.e. for $k, m \geq 0$, functions
$$f : \langle m \rangle \rightarrow \langle k \rangle, \quad f(0) = 0.$$
For every $m \geq 0$, there are $m$ canonical morphisms
$$\gamma_\beta : \langle m \rangle \rightarrow \langle 1 \rangle, \quad j \mapsto \delta_{\beta j}$$
for $1 \leq \beta \leq m$, called the Segal morphisms.

Remark 3.2. Note that $\Gamma$ is the skeleton of the category of finite pointed sets $\text{Fin}_a$. In his original paper [Seg74], Segal defined $\Gamma$ to be the opposite category of $\text{Fin}_a$. However, in the literature, $\Gamma$ has often appeared in the above convention.

We would now like to define a symmetric monoidal $(\infty, n)$-category to be an $(\infty, 1)$-functor from $\Gamma$ to the $(\infty, 1)$-category of $(\infty, n)$-categories which satisfies certain properties. Recall from Section 2.3.1 that the $(\infty, 1)$-category of $(\infty, n)$-categories is presented by a model category in which the fibrant objects are complete $n$-fold Segal spaces. More precisely, the $(\infty, 1)$-category of $(\infty, n)$-categories is defined to be the complete Segal space $N(\text{CSS}^n_{\text{Sp}}, \text{fwe}) \cong N(\text{CSS}^n_{\text{Sp}}, \text{DK})$.

Using the strictification theorem of Toën-Vezzosi from [TV02] every such functor can be represented by a strict functor from $\Gamma$ to $\text{CSS}^n_{\text{Sp}}$. Moreover, the $(\infty, 1)$-category of $(\infty, 1)$-functors can be computed using the model category $(\text{CSS}^n_{\text{Sp}})^\Gamma$ of $\Gamma$-diagrams in $\text{CSS}^n_{\text{Sp}}$. Thus the following definition suffices.

Definition 3.3. A symmetric monoidal complete $n$-fold Segal space is a (strict) functor from $\Gamma$ to the (strict) category of complete $n$-fold Segal spaces $\text{CSS}^n_{\text{Sp}}$,
$$A : \Gamma \rightarrow \text{CSS}^n_{\text{Sp}}$$
such that for every $m \geq 0$, the induced map
$$A \left( \prod_{1 \leq \beta \leq m} \gamma_\beta \right) : A(\langle m \rangle) \rightarrow (A(\langle 1 \rangle))^m$$
is an equivalence of complete $n$-fold Segal spaces.

The complete $n$-fold Segal space $X = A(\langle 1 \rangle)$ is called the complete $n$-fold Segal space underlying $A$, and by abuse of language we will sometimes call a complete $n$-fold Segal space $X$ symmetric monoidal, if there is a symmetric monoidal complete $n$-fold Segal space $A$ such that $A(\langle 1 \rangle) = X$.

Note that we can define symmetric monoidal $n$-fold Segal spaces in a similar way, by replacing $\text{CSS}^n_{\text{Sp}}$ be $\text{CSS}^n_{\text{Sp}}$. 

Remark 3.4. Note that in particular, for $m = 0$, this implies that $A(\langle 0 \rangle)$ is levelwise equivalent to a point, viewed as a constant $n$-fold Segal space, which we will denote by $\mathbb{1}$.

Definition 3.5. The $(\infty, 1)$-category, i.e. complete Segal space, of functors from $\Gamma$ to $\text{CSS}^n_{\text{Sp}}$, which as mentioned above can be computed using the model category of $\Gamma$-diagrams in $\text{CSS}^n_{\text{Sp}}$, has a full sub-$(\infty, 1)$-category of symmetric monoidal complete $n$-fold Segal spaces. Similarly to Section 2.3.1 this $(\infty, 1)$-category can be realized as the localization of the projective model structure on $\text{CSS}^n_{\text{Sp}}$, localizing with respect to the Segal morphisms, see [JS15, Example A.11]. A 1-morphism in this category is called a symmetric monoidal functor of $(\infty, n)$-categories.
Since the completion map \(X \to \hat{X}\) is a weak equivalence and preserves finite products of Segal spaces up to weak equivalence, we obtain the following

**Lemma 3.6.** If \(A : \Gamma \to \mathcal{S}
\text{Sp}_n\) is a symmetric monoidal \(n\)-fold Segal space, then

\[
\hat{A} : \Gamma \to CSSp_n, \quad \langle m \rangle \mapsto \hat{A}(\langle m \rangle)
\]

is a symmetric monoidal complete \(n\)-fold Segal space.

**Example 3.7.** Let \(A : \Gamma \to \mathcal{S}
\text{Sp}\) be a symmetric monoidal Segal space. Consider the product of maps \(\gamma_1 \times \gamma_2\) and the map induced by the map \(\gamma : \langle 2 \rangle \to \langle 1 \rangle; 1, 2 \mapsto 1\),

\[
A(1) \times A(1) \xleftarrow{\cong} A(\langle \gamma \rangle) \xrightarrow{A(\langle \gamma \rangle)} A(1).
\]

Passing to the homotopy category, we obtain a map

\[
h_1(A(1)) \times h_1(A(1)) \to h_1(A(1)).
\]

Toën and Vezzosi showed in [TV09] that this is a symmetric monoidal structure on the category \(h_1(A(1))\). Roughly speaking, this uses functoriality of \(A\). Associativity uses the Segal space \(A(\langle 3 \rangle, A(0)\) corresponds to the unit, and the map \(e : \langle 2 \rangle \to \langle 1 \rangle; 1, 2 \mapsto 1\) induces the commutativity constraint.

**Example 3.8.** Truncations and extensions of symmetric monoidal \((\infty, n)\)-categories again are symmetric monoidal. Let \(A\) be a symmetric monoidal \(n\)-fold (complete) Segal space. Since \(\tau_k\) and \(\varepsilon\) are functorial and preserve weak equivalences, the assignments

\[
\tau_k(A)\langle m \rangle = \tau_k(A\langle m \rangle), \quad \varepsilon(A)\langle m \rangle = \varepsilon(A\langle m \rangle)
\]

can be extended to functors \(\tau_k(A)\) and \(\varepsilon(A)\), and the images of \(A(\prod_{1 \leq i \leq m} \langle \gamma_i \rangle)\) are again weak equivalences. Thus, they endow the \(k\)-truncation and extension with a symmetric monoidal structure.

**Example 3.9.** Recall that \(A(\langle 0 \rangle)\) is weakly equivalent to the point \(\mathbb{1}\) viewed as a constant \(n\)-fold Segal space. For every \(m \geq 0\) there is a unique map \(\langle 0 \rangle \to \langle m \rangle\), which induces a map \(\mathbb{1} \simeq A(\langle 0 \rangle) \to A(\langle m \rangle)\) which picks out a distinguished object \(\mathbb{1}_{\langle m \rangle} \in A(\langle m \rangle)\). The looping of a symmetric monoidal \(n\)-fold Segal space \(A\) with respect this object also is symmetric monoidal, with

\[
L(A)\langle m \rangle = L(A(\langle m \rangle), \mathbb{1}_{\langle m \rangle}),
\]

which extends to a symmetric monoidal structure similarly to in the previous example. Note that since there is the choice for the unit \(\mathbb{1}_{\langle m \rangle}\) is contractible, different choices lead to equivalent loopings.

**Example 3.10.** Important examples come from the classification diagram construction. Let \(C\) be a small symmetric monoidal category and let \(W = \text{Iso}C\). As we saw in Section 1.23 this gives a complete Segal space \(C_* = N(C, W)\). The symmetric monoidal structure of \(C\) endows \(C_*\) with the structure of a symmetric monoidal complete Segal space:

First note that \(W^{\times m} = \text{Iso}(C^{\times m})\) for every \(m\). On objects, let \(A : \Gamma \to CSSp\) be given by \(A(\langle m \rangle) = N(C^{\times m}, W^{\times m})\). We explain the image of the map \(\langle 2 \rangle \to \langle 1 \rangle; 1, 2 \mapsto 1\), which should be a map \(A(\langle 2 \rangle) \to A(\langle 1 \rangle)\). The image of an arbitrary map \(\langle m \rangle \to \langle l \rangle\) can be defined similarly.

An \(l\)-simplex in \(A(\langle 2 \rangle) = N(C \times C, W \times W)\) is a pair

\[
C_0 \xrightarrow{w_1} \cdots \xrightarrow{w_l} C_l, \quad D_0 \xrightarrow{w_1'} \cdots \xrightarrow{w_l'} D_l,
\]

and is sent to

\[
C_0 \otimes D_0 \xrightarrow{w_1''} \cdots \xrightarrow{w_l''} C_l \otimes D_l,
\]

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More generally, if we start with a symmetric monoidal relative category

\[ A(2)_k = N(C \times C, W \times W)_k \]

is a pair of diagrams

\[
\begin{array}{cccccc}
  C_{0,0} & \rightarrow & C_{1,0} & \rightarrow & \cdots & \rightarrow & C_{k,0} \\
  w_{01} & \downarrow & w_{11} & \downarrow & w_{k1} & \downarrow & \vdots \\
  C_{0,1} & \rightarrow & C_{1,1} & \rightarrow & \cdots & \rightarrow & C_{k,1} \\
  w_{02} & \downarrow & w_{12} & \downarrow & w_{k2} & \downarrow & \vdots \\
  & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  C_{0,l} & \rightarrow & C_{1,l} & \rightarrow & \cdots & \rightarrow & C_{k,l} \\
  w_{0l} & \downarrow & w_{1l} & \downarrow & w_{kl} & \downarrow & \vdots \\
  D_{0,0} & \rightarrow & D_{1,0} & \rightarrow & \cdots & \rightarrow & D_{k,0} \\
  v_{01} & \downarrow & v_{11} & \downarrow & v_{k1} & \downarrow & \vdots \\
  D_{0,1} & \rightarrow & D_{1,1} & \rightarrow & \cdots & \rightarrow & D_{k,1} \\
  v_{02} & \downarrow & v_{12} & \downarrow & v_{k2} & \downarrow & \vdots \\
  & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  D_{0,l} & \rightarrow & D_{1,l} & \rightarrow & \cdots & \rightarrow & D_{k,l} \\
  v_{0l} & \downarrow & v_{1l} & \downarrow & v_{kl} & \downarrow & \vdots \\
\end{array}
\]

which is sent to the diagram

\[
\begin{array}{cccccc}
  C_{0,0} \otimes D_{0,0} & \rightarrow & C_{1,0} \otimes D_{1,0} & \rightarrow & \cdots & \\
  f_{01} \otimes g_{01} & \downarrow & f_{11} \otimes g_{11} & \downarrow & \vdots & \\
  C_{0,1} \otimes D_{0,1} & \rightarrow & C_{1,1} \otimes D_{1,1} & \rightarrow & \cdots & \\
  f_{02} \otimes g_{02} & \downarrow & f_{12} \otimes g_{11} & \downarrow & \vdots & \\
  & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\]

where the vertical maps are defined as for the objects.

Finally, we need to check that \( A(\prod_{1 \leq \beta \leq m} \alpha_\beta) \) is a weak equivalence. This follows from the fact that

\[
(A(m)_k) = N(C^{\times m}, W^{\times m})_k = (N(C, W)_k)^{\times m} = (A(1)_k)^m.
\]

**Remark 3.11.** More generally, if we start with a symmetric monoidal relative category \((C, W)\) (a definition can e.g. be found in [Cam14]) such that all \(N(C^{\times m}, W^{\times m})\) are (complete) Segal spaces, then the above construction for \((C, W)\) yields a symmetric monoidal (complete) Segal space \(N(C, W)\).

### 3.2. Definition via towers of \((n + i)\)-fold Segal spaces

Our motivation for the following definition of a \((k)\)-monoidal complete \(n\)-fold Segal space comes from the Delooping Hypothesis, which is inspired by the fact that a monoidal category can be seen as a bicategory with just one object. Similarly, a \(k\)-monoidal \(n\)-category should be a \((k + n)\)-category (whatever that is) with only one object, one 1-morphism, one 2-morphism, and so on up to one \((k - 1)\)-morphism.

**Hypothesis 3.12** (Delooping Hypothesis). \(k\)-monoidal \((\infty, n)\)-categories can be identified with \((k - j)\)-monoidal, \((j - 1)\)-simply connected \((\infty, n + j)\)-categories for any \(0 \leq j \leq k\), where \((j - 1)\)-simply connected means that any two parallel \(i\)-morphisms are equivalent for \(i < j\). In particular, monoidal \((\infty, n)\)-categories can be identified with \((\infty, n + 1)\)-categories with (essentially) one object.

**3.2.1. Monoidal \(n\)-fold complete Segal spaces**

We use the last statement in the delooping hypothesis as the motivation for the following definition. However, first we need to explain what “having (essentially) one object” means.
Definition 3.13. An $n$-fold Segal space $X$ is called pointed or $\theta$-connected, if

$$X_{\bullet, \ldots, \bullet},$$

is weakly equivalent to the point viewed as a constant $n$-fold Segal space. In particular, a pointed $n$-fold Segal space has a contractible space of objects.

Definition 3.14. A monoidal complete $n$-fold Segal space is a 1-hybrid $(n+1)$-fold Segal space $X^{(1)}$ which is pointed. Note that as $X^{(1)}$ is 1-hybrid, $X_{\bullet, \ldots, \bullet}^{(1)}$ is discrete. Thus, to be pointed implies that $X_{\bullet, \ldots, \bullet}$ is equal to the point viewed as a constant $n$-fold Segal space; we denote the unique object by $\ast$. We say that this endows the $n$-fold complete Segal space

$$X = L(X^{(1)}) = L(X^{(1)}, \ast)$$

with a monoidal structure and that $X^{(1)}$ is a delooping of $X$.

Remark 3.15. Without the completeness condition, we could define a monoidal $n$-fold Segal space to be an $(n+1)$-fold Segal space $X^{(1)}$ which is pointed. Then $L(X^{(1)}, \ast) = \text{Hom}_{X^{(1)}}(\ast, \ast)$ is independent of the choice of point $\ast \in X_{\ast, \ldots, \ast}$ and we can say that this endows the $n$-fold Segal space $X = L(X^{(1)}) = L(X^{(1)}, \ast)$ with a monoidal structure. However, a complete Segal space will not have a contractible space as $X_{\bullet, \ldots, \bullet}$ (unless it is trivial). Thus, we need to introduce a model for $(\infty, n+k)$-categories which can have a point as the set of objects, 1-morphisms, etcetera, which motivates our use of hybrid Segal spaces.

Remark 3.16. Let $X$ be an $m$-hybrid $n$-fold Segal space with $m > 0$ which is pointed. Then $X_{\bullet, \ldots, \bullet} = \ast$, and therefore the looping just is

$$L(X)_{\bullet, \ldots, \bullet} = \{\ast\} \times X_{\bullet, \ldots, \bullet} \times \{\ast\} = X_{\bullet, \ldots, \bullet}.$$

A similar definition works for hybrid Segal spaces.

Definition 3.17. A monoidal $m$-hybrid $n$-fold Segal space is an $(m+1)$-hybrid $(n+1)$-fold Segal space $X^{(1)}$ which is pointed. We say that this endows the $m$-hybrid $n$-fold Segal space

$$X = L(X^{(1)})$$

with a monoidal structure and that $X^{(1)}$ is a delooping of $X$.

Remark 3.18. Definitions 3.14 and 3.17 are special cases of the following more general construction of monoids in a model category. Let $\mathcal{M}$ be a left proper cellular model category, and consider the projective model structure on the category $\mathcal{M}^{\Delta^{op}}$ of simplicial objects in $\mathcal{M}$.

By the strictification theorem by Toën and Vezzosi from [TV02], the $(\infty, 1)$-category of $(\infty, 1)$-functors between the $(\infty, 1)$-categories represented by $\Delta^{op}$ and $\mathcal{M}$ is equivalent to $N(M^{\Delta^{op}})$. We say that a object $X_\bullet \in \mathcal{M}^{\Delta^{op}}$ is a weak monoid if the Segal maps

$$X_n \longrightarrow X^\bullet_1$$

are weak equivalences. One can show that the $(\infty, 1)$-category of monoids in $N(\mathcal{M}, \text{we})$, which is, as usual, obtained by a localization of the model structure on $\mathcal{M}^{\Delta^{op}}$ with respect to the maps governing the Segal morphisms, is equivalent to the relative nerve of the relative category of weak monoids in $\mathcal{M}$ and levelwise weak equivalences.

Monoidal $m$-hybrid $n$-Segal spaces are exactly the weak monoids in $m$-hybrid $n$-Segal spaces.

Example 3.19. Let $\mathcal{C}$ be a small monoidal category and let $\mathcal{W} = \text{Iso} \mathcal{C}$. As we saw in Section 3.21 this gives a complete Segal space $\mathcal{C}_\bullet = N(\mathcal{C}, \mathcal{W})$. The monoidal structure of $\mathcal{C}$ endows $\mathcal{C}_\bullet$ with the structure of a monoidal complete Segal space:
Let \( C_{m,n} = C_n^\otimes m \) be the category which has objects of the form

\[
C_{01} \otimes \cdots \otimes C_{0m} \xrightarrow{e_1} \cdots \xrightarrow{e_n} C_{n0} \otimes \cdots \otimes C_{nm}
\]

and morphisms of the form

\[
C_{01} \otimes \cdots \otimes C_{0m} \xrightarrow{c_1} \cdots \xrightarrow{c_n} C_{n0} \otimes \cdots \otimes C_{nm} \\
D_{01} \otimes \cdots \otimes D_{0m} \xrightarrow{d_1} \cdots \xrightarrow{d_n} D_{n0} \otimes \cdots \otimes D_{nm},
\]

where \( c_1, \ldots, c_n, d_1, \ldots, d_n \), and \( f^0, \ldots, f^n \) are morphisms in \( C \).

Consider its subcategory \( C_{m,n}^W \subset C_{m,n} \) which has the same objects, and vertical morphisms involving only the ones in \( W = \text{Iso} C \), i.e. \( f^0, \ldots, f^n \) are morphisms in \( W \).

Now let \( C_{m,n}^{(1)} = N(C_{m,n}^W) \), the (ordinary) nerve. By a direct verification one sees that the collection \( C_{m,n}^{(1)} \) is a 2-fold Segal space. Moreover,

1. \( C_{0,n}^{(1)} = N(C_n^\otimes 0) = \ast \), so \( C_{0, \bullet}^{(1)} \) is discrete and equal to the point viewed as a constant Segal space, and
2. for every \( m \geq 0 \), \( C_{m, \bullet}^{(1)} = N(C_{m,n}^W) = N((C_{n}^\otimes m)^W) \), which is a complete Segal space.

Summarizing, \( C^{(1)} \) is a 1-hybrid 2-fold Segal space which is pointed and endows \( L(C^{(1)}_{\bullet}) = C_{\bullet} \) with the structure of a monoidal complete Segal space.

### 3.2.2. \( k \)-monoidal \( n \)-fold complete Segal spaces

To encode braided or symmetric monoidal structures, we can push this definition even further.

**Definition 3.20.** An \( n \)-fold Segal space \( X \) is called \( j \)-connected,

\[
X_{1, \ldots, 1, \, 0, \ldots, \bullet}
\]

is weakly equivalent to the point viewed as a constant \( n \)-fold Segal space.

**Remark 3.21.** Note that being \( j \)-connected implies being \( i \)-connected for every \( 0 \leq i < j \).

**Definition 3.22.** A \( k \)-monoidal \( m \)-hybrid \( n \)-fold Segal space is an \((m + k)\)-hybrid \((n + k)\)-fold Segal space \( X^{(k)} \) which is \((k - 1)\)-connected.

**Remark 3.23.** Note that as \( X^{(k)} \) is \((m + k)\)-hybrid, \( X_{1, \ldots, 1, \, 0, \ldots, \bullet}^{(k)} \) is discrete for every \( 0 \leq i < k \).

Thus, being \((k - 1)\)-connected implies that \( X_{1, \ldots, 1, \, 0, \ldots, \bullet}^{(k)} \) is equal to the point viewed as a constant \((n - i + 1)\)-fold Segal space for every \( 0 \leq i < k \).

By the following proposition this definition satisfies the delooping hypothesis. In practice this allows to define a \( k \)-monoidal \( n \)-fold complete Segal space step-by-step by defining a tower of monoidal \( i \)-hybrid \((n + i)\)-fold Segal spaces for \( 0 \leq i < k \).
Proposition 3.24. The data of a $k$-monoidal $n$-fold complete Segal space is the same as a tower of monoidal $i$-hybrid $(n+i)$-fold Segal spaces $X^{(i+1)}$ for $0 \leq i < k$ together with weak equivalences

$$X^{(j)} \approx L(X^{(j+1)})$$

for every $0 \leq j < k - 1$.

Definition 3.25. We say that these equivalent data endow the complete $n$-fold Segal space

$$X = X^{(0)} \approx L(X^{(1)})$$

with a $k$-monoidal structure. The $(n+i+1)$-fold Segal space $X^{(i+1)}$ is called an $i$-fold delooping of $X$.

Before proving the proposition, we need the following lemmas.

Lemma 3.26. If $X$ is a $k$-monoidal $m$-hybrid $n$-fold Segal space, and $0 \leq l \leq k$, then $X$ is an $l$-monoidal $(m+k-l)$-hybrid $(n+k-l)$-fold Segal space.

Proof. Since $X$ is a $k$-monoidal $m$-hybrid $n$-fold Segal space, $X$ is a $(m+k)$-hybrid $(n+k)$-fold Segal space such that

$$X_{1,\ldots,1,0,\ldots,0} = \ast.$$

This implies that $X_{1,\ldots,1,0,\ldots,0} = \ast$. \qed

Lemma 3.27. Let $X$ be a $k$-monoidal $m$-hybrid $n$-fold Segal space. Then $L(X) = L(X, \ast)$ is a $(k-1)$-monoidal $(m-1)$-hybrid $n$-fold Segal space.

Proof. This follows from

$$L(X) = \text{Hom}(\ast, \ast) = \{\ast\} \times_{X_{0,\ldots,\ast}} X_{1,\ldots,\ast},$$

since $X_{0,\ldots,\ast} = \{\ast\}$ is a point. \qed

Proof of Proposition 3.24. Let $Y$ be a $k$-monoidal $n$-fold complete Segal space. By Lemma 3.26 it is a monoidal $(k-1)$-hybrid $(n+k-1)$-fold Segal space and we define the top layer of our tower to be $X^{(k)} = Y$.

Now let $X^{(k-1)} = L(X^{(k)})$. By Lemmas 3.27 and 3.26 this is a monoidal $(k-2)$-hybrid $(n+k-2)$-fold Segal space.

Inductively, define $X^{(i)} = L(X^{(i+1)})$ for $1 \leq i \leq k-1$. Similarly to above, by Lemmas 3.27 and 3.26 this is a monoidal $(i-1)$-hybrid $(n+i-1)$-fold Segal space.

Conversely, assume we are given a tower $X^{(i)}$ as in the proposition. Since $Y = X^{(k)}$ is a monoidal $(k-1)$-hybrid $(n+k-1)$-fold Segal space,

$$Y_{0,\ldots,\ast} = X^{(k)}_{0,\ldots,\ast} = \ast. \quad (2)$$

Since $X^{(k-1)}$ is a monoidal $(k-2)$-hybrid $(n+k-2)$-fold Segal space and by (2),

$$Y_{1,0,\ldots,\ast} = X^{(k)}_{1,0,\ldots,\ast} = \{\ast\} \times_{X^{(k)}_{0,0,\ldots,\ast}} X^{(k)}_{1,0,\ldots,\ast} \times_{X^{(k)}_{0,0,\ldots,\ast}} \{\ast\} = L(X^{(k)})_{0,\ldots,\ast} \approx X^{(k-1)}_{0,\ldots,\ast} = \ast. \quad (3)$$
Since $X^{(k)}$ is $k$-hybrid, $Y_{1,0,0,\ldots} = X_{1,0,0,\ldots}^{(k)}$ is discrete and so $Y_{1,0,0,\ldots} = \ast$.

Inductively, for $0 \leq i < k$, since $X^{(k-i)}$ is a monoidal $(k-i-1)$-hybrid $(n+k-i-1)$-fold Segal space and by (2), (3),...

\[
Y_{1,\ldots,1}^{(k)} \ast \ldots = X_{1,\ldots,1}^{(k)} \ast \ldots = \{\ast\} \times_{X_{1,\ldots,1}^{(k)}} \ast \times_{X_{1,\ldots,1}^{(k)}} \ast \ldots = L(X^{(k)})_{1,\ldots,1} \ast \ldots = \ast.
\]

Again, since $X^{(k)}$ is $k$-hybrid, $Y_{1,\ldots,1} = X_{1,\ldots,1}$ is discrete and so $Y_{1,0,0,\ldots} = \ast$. \(\square\)

We now provide the analog of Example 3.9 in this setting. It illustrates the idea that given a bicategory $\mathcal{C}$ and an object $x$ in $\mathcal{C}$, the endomorphism, or loop category $\text{End}_\mathcal{C}(x) = L(\mathcal{C}, x)$ is monoidal, with the monoidal structure coming from composition of endomorphisms. We now prove that a similar statement holds for $n$-fold Segal spaces.

**Definition 3.28.** Fix $1 \leq l \leq n$. Let $X_{k,\ldots}$ be an $n$-fold Segal space and $x \in X_{0,\ldots,0}$. We define a new $n$-fold Segal space $X / x$ as follows. If $k_i = 0$ for $1 \leq i \leq l$, let

\[
(X / x)_{k_1,\ldots,k_i,0,\ldots,k_n} = \ast \equiv \{x\}.
\]

The $(n-l)$-fold Segal space $(X / x)_{k_1,\ldots,k_l,\ldots}$ is a homotopy fiber:

\[
(X / x)_{k_1,\ldots,k_l,\ldots} \xrightarrow{S} X_{0,\ldots,0,\ldots} \xrightarrow{\times \binom{(k_1+1)\ldots(k_n+1)}}
\]

where $S : X_{k_1,\ldots,k_l,\ldots} \to X_{0,\ldots,0,\ldots}$ is the product of all maps arising from the maps $f_i : [0] \to [k_i]$.

The remaining face maps send everything to the point $\ast$, which we identify with $x$, or, more precisely, its image under the appropriate composition of degeneracy maps. The remaining degeneracy maps $d_{\ldots,\ast} : (X / x)_{k_1,\ldots,k_{i-1},\ldots} \to (X / x)_{k_1,\ldots,k_{i-1},\ldots}$ satisfy $d_{\ldots,\ast}(\ast) = d_{\ldots,\ast}(x)$, where again we identify $x$ with its image under the appropriate composition of degeneracy maps. Since $X$ is an $n$-fold simplicial set, $X / x$ is well-defined as an $n$-fold simplicial set. The Segal condition is preserved, and, if $X$ satisfied condition [CSS] for some $i > l$, then $X / x$ does too.

It is called the $l$th level fiber $X / x$ of $X$ over $x$.

**Lemma 3.29.** Let $1 \leq l \leq n$ and let $X_{k,\ldots}$ be an $n$-fold Segal space which satisfies [CSS] for $i > l$. Then for any $x \in X_{0,\ldots,0}$, the $l$th level fiber $X / x$ is an $l$-monoidal complete $(n-l)$-fold Segal space which endows $L_l(X, x)$ with an $l$-monoidal structure.
Definition 3.32. A symmetric monoidal structure on a complete n-fold Segal space $X$ is a tower of monoidal $i$-hybrid $(n+i)$-fold Segal spaces $X^{(i+1)}$ for $i \geq 0$ such that if we set $X = X^{(0)}$, for every $i \geq 0$,

$$X^{(i)} \simeq L(X^{(i+1)}).$$

3.3. Comparing the two definitions

In this section we show that every symmetric monoidal $(x,n)$-category defined as in Section 3.2 gives one as defined in Section 3.3. The converse also is true, but we do not want to go into the details here. See [GH15, Corollary 6.3.13].

Consider the following map:

$$f : \Delta^{op} \longrightarrow \Gamma,$$

$$[m] \longmapsto \langle m \rangle,$$

and a map $(f : [n] \to [m])$ in $\Delta$ is sent to $\tilde{f} : \langle m \rangle \to \langle n \rangle$, where $\tilde{f}(0) = 0$ and for $j \neq 0$,

$$\tilde{f}(j) = \begin{cases} \min\{i : f(i) = j\}, & \text{if it exists}, \\ 0, & \text{otherwise}. \end{cases}$$
Note that it sends the maps $g_\beta$ from remark 1.5 to the Segal morphisms $\gamma_\beta$ from Definition 3.1.

Let $A : \Gamma \to CSS_p$ be a symmetric monoidal complete $n$-fold Segal space. Then composition with the above map

$$\tilde{X}^{(1)} : \Delta^{op} \xrightarrow{f} \Gamma \xrightarrow{A} CSS_p$$

is an $(n + 1)$-fold simplicial set. Moreover, since $f$ sends the maps $g_\beta$ from remark 1.5 to the Segal morphisms $\gamma_\beta$ from Definition 3.1, $\tilde{X}^{(1)}$ is an $(n + 1)$-fold Segal space. It satisfies $(CSS^i)$ for $i \geq 1$. However, it does not satisfy $(SC^j)$ for $j \leq k$ since $X_{0,\ldots,0}$ may not be discrete. We can easily remedy this problem: choose an object $x \in X_{0,\ldots,0}$ and consider the $(n + 1)$-fold Segal space $X^{(1)} = \tilde{X}^{(1)}/x$. Unravelling the definition,

$$X^{(1)}_{k_1,\ldots,\ldots} = \begin{cases} \ast & k_1 = 0 \\ \tilde{X}^{(1)}_{k_1,\ldots} & k_1 \neq 0 \end{cases}$$

as complete $n$-fold Segal spaces. Lemma 3.29 implies that $X^{(1)}$ is a monoidal complete $n$-fold Segal space.

The higher layers of the tower are obtained from the maps $\Gamma \to CSS_p$ coming from taking disjoint union of finite pointed sets and identifying their base points. Then, composing with $f^k$ we obtain

$$\tilde{X}^{(k)} : (\Delta^{op})^k \to \Gamma^k \to \Gamma \to CSS_p$$

Similarly, $\tilde{X}^{(k)}$ is $(k - 1)$-connected, but might not satisfy $(SC^j)$ for $j \leq k$. Choosing any object $x \in X^{(k)}_{0,\ldots,0}$, the $k$th level fiber $X^{(k)} = \tilde{X}^{(k)}/x$ is the desired $k$-monoidal complete $n$-fold Segal space.

Part II.

The $(\infty, n)$-category of cobordisms

To rigorously define fully extended topological field theories we need a suitable $(\infty, n)$-category of cobordisms, which, informally speaking, has zero-dimensional manifolds as objects, bordisms between objects as 1-morphisms, bordisms between bordisms as 2-morphisms, etc., and for $k \geq n$ there are only invertible $k$-morphisms. Finding an explicit model for such a higher category, i.e. defining a complete $n$-fold Segal space of bordisms, is the main goal of this chapter. We endow it with a symmetric monoidal structure and also consider bordism categories with additional structure, e.g. orientations and framings, which allows us, in Section 10, to rigorously define fully extended topological field theories.

4. The complete $n$-fold Segal space of closed intervals

In this section we define a complete Segal space $Int^c$ of closed intervals in $\mathbb{R}$ which will form the basis of the $n$-fold Segal space of cobordisms.

4.1. $Int^c$ as an internal category

We first define a strongly Segal internal category $Int^c$ of closed intervals in $\mathbb{R}$.
Its space of objects is

$$\text{Int}^0 = \{(a, b) : a < b\} \subset \mathbb{R}^2 \quad (*)$$

with the standard topology from $\mathbb{R}^2$. We interpret an element $(a, b) \in \text{Int}^0$ as the closed interval $I = [a, b]$. Note that this interpretation gives a bijection which we use as an identification

$$\text{Int}^0 \leftrightarrow \{\text{closed bounded intervals } I = [a, b] \text{ in } \mathbb{R} \text{ with non-empty interior}\}.$$

Taking smooth singular chains we obtain a Kan complex whose $l$-simplices are pairs of smooth maps $a, b : [\Delta^1]_{|e} \to \mathbb{R}$ such that $a(s) < b(s)$ for every $s \in [\Delta^1]_{|e}$. Faces and degeneracies are the usual ones. We view such an $l$-simplex as a \textit{closed interval bundle} and denote it by $[a, b] \to [\Delta^1]_{|e}$ or $(a(s), b(s))_{s \in [\Delta^1]_{|e}}$.

The space of morphisms is

$$\text{Int}^1_c = \{(a_0, a_1, b_0, b_1) : a_j < b_j \text{ for } j = 0, 1, \text{ and } a_0 \leq a_1, b_0 \leq b_1\} \subset \mathbb{R}^4 \quad (\star)$$

again with the standard topology from $\mathbb{R}^4$. Now we interpret an element $(a_0, a_1, b_0, b_1) \in \text{Int}^1_c$ as a pair of ordered closed intervals $I_0 \leq I_1$, where $I_0 = [a_0, b_0]$ and $I_1 = [a_1, b_1]$, where “ordered” means that $a_0 \leq a_1$ and $b_0 \leq b_1$. This gives an identification

$$\text{Int}^1_c \leftrightarrow \{I_0 \leq I_1 : I_j = [a_j, b_j] \text{ with } a_j < b_j \text{ for } j = 0, 1, \text{ and } a_0 \leq a_1, b_0 \leq b_1\}.$$

As above we consider $\text{Int}^1_c$ as a Kan complex whose $l$-simplices now are quadruples of smooth maps $a_0, a_1, b_0, b_1 : [\Delta^1]_{|e} \to \mathbb{R}$ such that $a_j(s) < b_j(s)$ for $j = 0, 1$, $a_0(s) \leq a_1(s)$, and $b_0(s) \leq b_1(s)$ for every $s \in [\Delta^1]_{|e}$. We view such an $l$-simplex as an \textit{closed interval bundle with two closed subintervals} and denote it by $[a_0, b_0] \leq [a_1, b_1] \to [\Delta^1]_{|e}$ or $(I_0(s) \leq I_1(s))_{s \in [\Delta^1]_{|e}}$.

The face and degeneracy maps

$$\begin{align*}
\text{Int}^0 \xrightarrow{s} & \xrightarrow{t} \text{Int}^1_c \\
\text{Int}^0 \xrightarrow{d} & \xrightarrow{t} \text{Int}^1_c
\end{align*}$$

arise from forgetting respectively repeating an interval:

$$s : [a_0, b_0] \leq [a_1, b_1] \mapsto [a_0, b_0],$$

$$t : [a_0, b_0] \leq [a_1, b_1] \mapsto [a_1, b_1],$$

and

$$d : [a, b] \mapsto [a, b].$$

Composition is given by remembering the outer intervals:

$$(a_0, b_0] \leq [a_1, b_1]) \circ ([a_1, b_1] \leq [a_2, b_2]) = ([a_0, b_0] \leq [a_2, b_2]).$$

Note that all above assignments are well-defined for $l$-simplices as well and commute with the faces and degeneracies. Moreover, $s$ and $t$ are fibrations since they are restrictions of projections.

Summarizing, we obtain

\textbf{Lemma 4.1.} $\text{Int}^c$ is a strongly Segal internal category.

Moreover, the spaces of objects and morphisms are contractible:

\textbf{Lemma 4.2.} $\text{Int}^0 \simeq \text{Int}^1_c \simeq \ast$.

\textbf{Proof.} The underlying topological space is contractible as a subspace of $\mathbb{R}^{2k}$, so the associated Kan complex given by taking smooth chains also is contractible. \hfill $\blacksquare$
4.2. \( \text{Int}^c \) as a complete Segal space

We defined \( \text{Int}^c \) as a strongly Segal internal category in the previous section. Its nerve, constructed in Section 1.7, is a Segal space \( \text{Int}^c_\bullet = N(\text{Int}^c)_\bullet \). Let us spell out this Segal space in more detail to become more familiar with it.

For an integer \( k \geq 0 \),

\[
\text{Int}^c_k = \left\{(a, b) = (a_0, \ldots, a_k, b_0, \ldots, b_k) : a_j < b_j \text{ for } 0 \leq j \leq k, \text{ and } a_{j-1} \leq a_j \text{ and } b_{j-1} \leq b_j \text{ for } 1 \leq j \leq k \right\} \subset \mathbb{R}^{2k}
\]

(\ast)

As before, we interpret an element \((a, b)\) as an ordered \((k + 1)\)-tuple of closed intervals \( \underline{I} = I_0 \leq \cdots \leq I_k \) with left endpoints \( a_j \) and right endpoints \( b_j \), such that \( I_j \) has non-empty interior. By “ordered”, i.e. \( I_j \leq I_{j'} \), we mean that the endpoints are ordered, i.e. \( a_j \leq a_{j'} \) and \( b_j \leq b_{j'} \) for \( j \leq j' \).

**Spatial structure of the levels**  The spatial structure of a level \( \text{Int}^c_k \) comes from taking smooth singular chains of the topology induced by the standard topology on \( \mathbb{R}^{2k} \). Thus, an \( l \)-simplex consists of smooth maps

\[
|\Delta^l|_e \rightarrow \mathbb{R}, \quad s \mapsto a_j(s), b_j(s)
\]

for \( j = 0, \ldots, k \) such that for every \( s \in |\Delta^l|_e \), the following inequalities hold:

\[
\begin{align*}
& a_i(s) < b_i(s), \quad \text{for } i = 0, \ldots, k \\
& a_{i-1}(s) \leq a_i(s), \quad \text{and} \\
& b_{i-1}(s) \leq b_i(s) \quad \text{for } i = 1, \ldots, k.
\end{align*}
\]

We denote an \( l \)-simplex by \( \langle I_0 \leq \cdots \leq I_k \rangle \rightarrow |\Delta^l|_e \) or \( \langle I_0(s) \leq \cdots \leq I_k(s) \rangle_{s \in |\Delta^l|_e} \) and call it a **closed interval bundle with \( k \) subintervals**.

For a morphism \( f : [m] \rightarrow [l] \) in the simplex category \( \Delta \), i.e. a (weakly) order-preserving map, let \( |f| : |\Delta^m|_e \rightarrow |\Delta^l|_e \) be the induced map between standard simplices. Let \( f^\Delta \) be the map sending an \( l \)-simplex in \( \text{Int}^c_k \) to the \( m \)-simplex in \( \text{Int}^c_k \) given by precomposing with \( |f| \),

\[
\langle I_0(s) \leq \cdots \leq I_k(s) \rangle_{s \in |\Delta^l|_e} \mapsto \langle I_0(|f|(s)) \leq \cdots \leq I_k(|f|(s)) \rangle_{s \in |\Delta^m|_e}.
\]

**Notation 4.3.** We denote the spatial face and degeneracy maps of \( \text{Int}^c_k \) by \( d_j^\Delta \) and \( s_j^\Delta \) for \( 0 \leq j \leq l \).

We will need the following lemma later for the Segal condition. Its proof is analogous to that of Lemma 4.2.

**Lemma 4.4.** Each level \( \text{Int}^c_k \) is a contractible Kan complex.

**Proof.** As the spatial structure arises by taking smooth singular chains of a topological space, it is a Kan complex. Moreover, the underlying topological space is contractible as a subspace of \( \mathbb{R}^{2k} \), so the Kan complex also is contractible. \( \square \)

**Simplicial structure – the simplicial space \( \text{Int}^c \)**

**Lemma 4.5.** For a morphism \( g : [m] \rightarrow [k] \) be a morphism in \( \Delta \), the following assignment defines a functor \( \text{Int}^c : \Delta^{op} \rightarrow \text{Space} \), i.e. a simplicial space.

\[
\text{Int}_k \xrightarrow{g^\Delta} \text{Int}_m, \\
\langle I_0(s) \leq \cdots \leq I_k(s) \rangle_{s \in |\Delta^l|_e} \mapsto \langle I_{g(0)}(s) \leq \cdots \leq I_{g(m)}(s) \rangle_{s \in |\Delta^l|_e}.
\]
Proof. The assignment is clearly functorial and \( f^\Delta \) and \( g^* \) commute for all morphisms \( f, g \) in \( \Delta \). \( \square \)

**Notation 4.6.** We denote the **simplicial face and degeneracy maps** by \( d_j \) and \( s_j \) for \( 0 \leq j \leq k \).

Explicitly, they are given by the following formulas. The \( j \)th degeneracy map is given by doubling the \( j \)th interval, and the \( j \)th face map is given by deleting the \( j \)th interval,

\[
\begin{align*}
\text{Int}_k & \xrightarrow{s_j} \text{Int}_{k+1}, \\
I_0 \leq \cdots \leq I_k & \quad \mapsto \quad I_0 \leq \cdots \leq I_j \leq \cdots \leq I_k, \\
\text{Int}_k & \xrightarrow{d_j} \text{Int}_{k-1}, \\
I_0 \leq \cdots \leq I_k & \quad \mapsto \quad \hat{I}_0 \leq \cdots \leq \hat{I}_j \leq \cdots \leq I_k.
\end{align*}
\]

**The complete Segal space \( \text{Int}^c \)**

**Proposition 4.7.** \( \text{Int}^c \) is a complete Segal space. Moreover, the inclusion \( * \hookrightarrow \text{Int}^c \) given by degeneracies, where \( * \) is seen as a constant complete Segal space, is an equivalence of complete Segal spaces.

Proof. We have seen in Lemma 4.4 that every \( \text{Int}^c_k \) is contractible. This ensures the Segal condition, namely that

\[
\begin{align*}
\text{Int}^c_k & \xrightarrow{h} \text{Int}^c_1 \times \cdots \times \text{Int}^c_1, \\
\text{Int}^c_k & \xrightarrow{h} \text{Int}^c_1 \xrightarrow{h} \cdots \xrightarrow{h} \text{Int}^c_1,
\end{align*}
\]

completeness, and ensures that the given inclusion is a level-wise equivalence. \( \square \)

4.3. The internal category or complete Segal space \( \text{Int} \) of ordered closed intervals in an open one

We now change our interpretation of the spaces \( \text{Int} \): we do not identify them with the spaces of ordered closed bounded intervals \( I_0 \leq \cdots \leq I_k \) anymore, but as ordered intervals which are closed in \( a_0, b_k \), i.e. we interpret the elements as

\[
\hat{I}_0 \leq \cdots \leq \hat{I}_k,
\]

where \( \hat{I}_j = I_j \cap (a_0, b_k) \) for \( 0 \leq j \leq k \). Thus, in the generic case when \( a_j \neq a_0 \) for \( 0 < j \leq k \) and \( b_j \neq b_k \) for \( 0 \leq j < k \), then \( \hat{I}_0 \leq \cdots \leq \hat{I}_k \) are the half-open or closed intervals

\[
(a_0, b_1] \leq [a_1, b_1] \leq \cdots \leq [a_{k-1}, b_{k-1}] \leq [a_k, b_k).
\]

If we view the elements in \( \text{Int} \) in this way, we will denote the internal category (or analogously the Segal space) by \( \text{Int} \).

Note that the identity gives an isomorphism of complete Segal spaces describing the change of interpretation:

\[
\text{Int}^c_k \xrightarrow{\approx} \text{Int}_k \\
(I_0 \leq \cdots \leq I_k) \quad \mapsto \quad (\hat{I}_0 \leq \cdots \leq \hat{I}_k),
\]

where \( \hat{I}_j = I_j \cap (a_0, b_k) \) for \( j = 0, \ldots, k \). Conversely, \( I_j = \text{cl}_R(\hat{I}_j) \), the closure of \( \hat{I}_j \) in \( \mathbb{R} \).

**Definition 4.8.** Let

\[
\text{Int}^n_{\bullet, \ldots, \bullet} = (\text{Int}_{\bullet})^n.
\]

We denote an element in \( \text{Int}^n_{k_1, \ldots, k_n} \) by

\[
\overline{I} = (\overline{I}_1 \overline{I}_2) = (I_0^1 \leq \cdots \leq I_{k_1}^1)_{1 \leq i \leq n}.
\]

**Lemma 4.9.** The \( n \)-fold simplicial space \( \text{Int}^n_{\bullet, \ldots, \bullet} \) is a complete \( n \)-fold Segal space. Moreover, the inclusion \( * \hookrightarrow \text{Int}^n_{\bullet, \ldots, \bullet} \) is an equivalence of complete \( n \)-fold Segal spaces.
Definition 4.11. We will also require the following rescaling maps.

Lemma 4.12. Int

 Definition 4.10. Fix \( k \geq 0 \). The map of spaces

\[
\rho : \text{Int}_k \longrightarrow \text{Int}_0
\]

is called the boxing map. By abuse of notation we denote the total space of \( B(\mathcal{L}) \to |\Delta^1|_e \times |\Delta^1|_e \).

Its \( n \)-fold product gives, for every \( k_1, \ldots, k_n \geq 0 \), a map \( \text{Int}_{k_1, \ldots, k_n} \to \text{Int}_0 \) with sends an element to the smallest open box containing all intervals,

\[
\mathcal{T} = (I_0 \leq \cdots \leq I_k)_{1 \leq i \leq n} \longrightarrow B(\mathcal{T}) = B(\mathcal{T}|_e) = (a^1_{k_1}, b^1_{k_1}) \times \cdots \times (a^n_{k_n}, b^n_{k_n}).
\]

We will also require the following rescaling maps.

Definition 4.11. For an element \( \mathcal{T} \in \text{Int}_{k_1, \ldots, k_n} \), let \( \rho(\mathcal{T}) : B(\mathcal{T}) \to (0,1)^n \) be the restriction of the product of the affine maps \( \mathbb{R} \to \mathbb{R} \) sending \( a^i_0 \) to 0 and \( b^i_k \) to 1. We call it the box rescaling map.

4.5. A variant: closed intervals in \((0,1)\)

One might prefer to restrict to intervals which lie in \((0,1)\), modifying the definition to

\[
\text{Int}_{k}^{(0,1)} = \{(a,b) = (a_0, \ldots, a_k, b_0, \ldots, b_k) : a_j < b_j \text{ for } 0 \leq j \leq k, 0 = a_0 \leq a_1 \leq \cdots \leq a_k \quad \text{and} \quad b_0 \leq \cdots \leq b_{k-1} \leq b_k = 1\} \subset \text{Int}_k
\]

The simplicial structure now has to be modified to ensure that the outer endpoints again are 0 and 1. This is provided by composition with an affine rescaling map: Let \( g : [m] \to [k] \) be a morphism in \( \Delta \). Then, let

\[
\text{Int}_{k}^{(0,1)} \xrightarrow{g^*} \text{Int}_{m}^{(0,1)},
\]

\[
(I_0 \leq \cdots \leq I_k) \to |\Delta^1|_e \xrightarrow{\rho_g(I_{g(0)}) \leq \cdots \leq I_{g(m)}} |\Delta^1|_e,
\]

where the rescaling map \( \rho_g \) is the unique affine transformation \( \mathbb{R} \to \mathbb{R} \) sending \( a_{g(0)} \) to 0 and \( b_{g(m)} \) to 1.

Lemma 4.12. \( \text{Int}_{(0,1)} \) is a complete Segal space.

Proof. The only thing which is not completely analogous to \( \text{Int}^{e} \) is checking that it is a simplicial space. Given two maps \( [m] \xrightarrow{g} [k] \xrightarrow{h} [p] \), and \( I_0 \leq \cdots \leq I_p \), the rescaling map \( \rho_{g \circ h} \) and the composition of the rescaling maps \( \rho_{g} \circ \rho_{h} \) both send \( a_{g(0)} \) to 0 and \( b_{g(m)} \) to 1 and, since affine transformations \( \mathbb{R} \to \mathbb{R} \) are uniquely determined by the image of two points, this implies that they coincide. Thus, this gives a functor \( \Delta^{op} \to \text{Space} \).
Note that the degeneracy maps are the same ones, given by repeating an interval. However, the face maps need to modified: after deleting an end interval we have to rescale the remaining intervals linearly to \((0,1)\). Explicitly, for \(j = 0\), the rescaling map is the affine map \(p_{0} : (a_{0}, b_{0}) = \frac{a-b}{b_{0}}\) and for \(j = k\), it is the affine map \(p_{k} : (0, b_{k-1}) = \frac{a-b}{b_{k-1}}\). Then,

\[
\text{Int}_{k}^{(0,1)} \xrightarrow{d_{j}} \text{Int}_{k-1}^{(0,1)},
\]

\[
I_{0} \leq \cdots \leq I_{k} \quad \mapsto \quad \begin{cases} I_{0} \leq \cdots \leq \hat{I}_{j} \leq \cdots \leq I_{k}, & j \neq 0, k, \\ (0, b_{k-1}) \leq \cdots \leq \frac{a_{k}-a_{k-1}}{1}, & j = 0, \\ (0, \frac{a_{k}-a_{k-1}}{1}) \leq \cdots \leq \frac{a_{k}-a_{k-1}}{1}, & j = k. \end{cases}
\]

**Remark 4.13.** An advantage of this “reduced” version is that the space of objects is just a point: for \(k = 0\), the condition on the endpoints of the intervals becomes \(a_{0} = 0\) and \(b_{0} = 1\), so the only element is \((0,1) \in \text{Int}_{0}\). In particular, \(\text{Int}_{0}\) is discrete.

**Remark 4.14.** Note that the boxing maps applied to \(\text{Int}_{k}^{(0,1)}\) are trivial: for \(I = I_{0} \leq \cdots \leq I_{k}\), we always have that \(B(I) = (0,1)\). Moreover, \(\text{Int}_{k}^{(0,1)}\) is the preimage of \((0,1)\) under the boxing maps.

## 5. The \((\infty, n)\)-category of bordisms \(\text{Bord}_{n}\)

In this section we define an \(n\)-fold Segal space \(\text{P Bord}_{n}\) in several steps. However, it will turn out not to be complete in general. By applying the completion functor we obtain a complete \(n\)-fold Segal space, the \((\infty, n)\)-category of bordisms \(\text{Bord}_{n}\).

Let \(V\) be a finite dimensional vector space. We first define the levels relative to \(V\) with elements being certain submanifolds of the (finite dimensional) vector space \(V \times \mathbb{R}^{n} \cong V \times B\), where \(B\) is an open box, i.e. a product of \(n\) bounded open intervals in \(\mathbb{R}\). Then we vary \(V\), i.e. we take the limit over all finite dimensional vector spaces lying in some fixed infinite dimensional vector space, e.g. \(\mathbb{R}^{\infty}\). The idea behind this process is that by Whitney’s embedding theorem, every manifold can be embedded in some large enough vector space, so in the limit, we include representatives of every \(n\)-dimensional manifold. We use \(V \times B\) instead of \(V \times \mathbb{R}^{n}\) as in this case the spatial structure is easier to write down explicitly.

### 5.1. The level sets \((\text{P Bord}_{n})_{k_{1}, \ldots, k_{n}}\)

For \(S \subseteq \{1, \ldots, n\}\) denote the projection from \(\mathbb{R}^{n}\) onto the coordinates indexed by \(S\) by \(\pi_{S} : \mathbb{R}^{n} \to \mathbb{R}^{S}\).

**Definition 5.1.** Let \(V\) be a finite dimensional \(\mathbb{R}\)-vector space, which we identify with some \(\mathbb{R}^{\infty}\) with some metric. For every \(n\)-tuple \(k_{1}, \ldots, k_{n} \geq 0\), let \((\text{P Bord}_{n})_{k_{1}, \ldots, k_{n}}\) be the collection of tuples \((M, \mathcal{I}) = (I_{0} \leq \cdots \leq I_{k_{i}}, 1 \leq i \leq n)\), satisfying the following conditions:

1. For \(1 \leq i \leq n\),

\[
(I_{0} \leq \cdots \leq I_{k_{i}}) \in \text{Int}_{k_{i}}.
\]

2. \(M\) is a closed and bounded \(n\)-dimensional submanifold of \(V \times B(\mathcal{I})\) and the composition \(\pi : M \hookrightarrow V \times B(\mathcal{I}) \to B(\mathcal{I})\) is a proper map.\(^{2}\)

3. For every \(S \subseteq \{1, \ldots, n\}\), let \(p_{S} : M \xrightarrow{\mathcal{I}} B(\mathcal{I}) \xrightarrow{\pi_{S}} \mathbb{R}^{S}\) be the composition of \(\pi\) with the projection \(\pi_{S}\) onto the \(S\)-coordinates. Then for every \(1 \leq i \leq n\) and \(0 \leq j_{i} \leq k_{i}\), at every \(x \in p_{\{i\}}^{-1}(I_{j_{i}}^{i})\), the map \(p_{\{i, \ldots, n\}}\) is submersive.

\(^{2}\)Recall the boxing map from Section 4.4.
Remark 5.2. For $k_1, \ldots, k_n \geq 0$, one should think of an element in $(\text{PBord}_n)_{k_1, \ldots, k_n}$ as a collection of $k_1 \cdots k_n$ composed bordisms, with $k_i$ composed bordisms with collars in the $i$th direction. They can be understood as follows.

- Condition 3 in particular implies that at every $x \in p_{[n]}^{-1}(I_j^n)$, the map $p_{[n]}$ is submersive, so if we choose $t_{j}^{0} \in I_j^n$, it is a regular value of $p_{[n]}$, and so $p_{[n]}^{-1}(t_{j}^{0})$ is an $(n-1)$-dimensional manifold. The embedded manifold $M$ should be thought of as a composition of $n$-bordisms and $p_{[n]}^{-1}(t_{j}^{0})$ is one of the $(n-1)$-bordisms in the composition.

- At $x \in p_{[n-1]}^{-1}(I_j^{n-1})$, the map $p_{[n-1]}$ is submersive, so for $t_i^{n-1} \in I_i^{n-1}$, the preimage
  
  $p_{[n-1]}^{-1}(I_j^{n-1}, t_i^{n-1})$

  is an $(n-2)$-dimensional manifold, which should be thought of as one of the $(n-2)$-bordisms which are connected by the composition of $n$-bordisms $M$. Moreover, again since $p_{[n-1]}$ is submersive at $p_{[n-1]}^{-1}(I_j^{n-1})$, the preimage $p_{[n-1]}^{-1}(t_i^{n-1})$ is a trivial $(n-1)$-bordism between the $(n-2)$-bordisms it connects.

- Similarly, for $(t_j^{k}, \ldots, t_j^{n}) \in I_{j_k}^k \times \cdots \times I_{j_n}^n$, the preimage
  
  $p_{[n]}^{-1}(I_{j}^{k}, \cdots, t_{j}^{n})$

  is a $(k-1)$-dimensional manifold, which should be thought of as one of the $(k-1)$-bordisms which is connected by the composition of $n$-bordisms $M$.

- Moreover, the following proposition shows that different choices of “cutting points” $t_i^j \in I_{i}^j$ lead to diffeomorphic bordisms. In the case when $b_{j}^i < a_{j+1}^i$ one should thus think of the $n$-bordisms we compose as $p_{-1}(\prod_{i=1}^{n} [b_{j}^i, a_{j+1}^i])$, and the preimages of the specified intervals as collars of the bordisms along which they are composed. Otherwise, one should think of that $n$-bordism in the composition as being “degenerate”, i.e. of being a trivial $n$-bordism.

We will come back to this interpretation in Section 8.10 when we compute homotopy (bi)categories.

Example 5.3. An example of an element in $(\text{PBord}_3^3)$ is depicted below. It represents a composition of three 1-bordisms, the first one of which is “degenerate”, i.e. a trivial 1-bordism between two points.

![Diagram of three 1-bordisms](image)

**Proposition 5.4.** Let $(M, \overline{b}) \in (\text{PBord}_n^k)_{k_1, \ldots, k_n}$. Fix $1 \leq i \leq n$ and $0 \leq j \leq j' \leq k_i$. Then for any $u_{j}^i, v_{j}^i \in I_{j}^i$ and $u_{j'}^i, v_{j'}^i \in I_{j'}^i$, such that $u_{j}^i < u_{j'}^i$ and $v_{j}^i < v_{j'}^i$, there is a diffeomorphism

$$p_{[i]}^{-1}([u_{j}^i, u_{j'}^i]) \rightarrow p_{[i]}^{-1}([v_{j}^i, v_{j'}^i]).$$

**Proof.** Since the map $p_{[i]}$ is submersive in $I_{j}^i$ and $I_{j'}^i$, we can apply the Morse lemma, which we recall in Section 8.10 to $p_{[i]}$ twice to obtain diffeomorphisms

$$p_{[i]}^{-1}([u_{j}^i, u_{j'}^i]) \rightarrow p_{[i]}^{-1}([v_{j}^i, v_{j'}^i]) \rightarrow p_{[i]}^{-1}([v_{j}^i, v_{j'}^i]).$$
Applying the proposition successively for \( i = 1, \ldots, n \) yields

**Corollary 5.5.** Let \((M, \overline{I}) \in (\text{PBord}^V_n)_{k_1, \ldots, k_n}\) and let \(B_1, B_2 \subseteq \mathbb{R}^n\) be products of non-empty closed bounded intervals with endpoints lying in the same specified intervals, i.e. \(B_1 = \prod [u_i^j, v_i^j]\) and \(B_2 = \prod [v_i^{j'}, v_i^{j''}]\), where \(0 \leq j < j' \leq k_i\) and \(u_i^j, v_i^j \in I_j^i\) and \(u_i^{j'}, v_i^{j'} \in I_j^{i'}\) such that \(u_j^j < u_i^{j'}\) and \(v_j^{j'} < v_i^{j'}\) for every \(1 \leq i \leq n\). Then there is a diffeomorphism

\[
\pi^{-1}(B_1) \longrightarrow \pi^{-1}(B_2).
\]

### 5.2. The spaces \((\text{PBord}^V_n)_{k_1, \ldots, k_n}\)

The level sets \((\text{PBord}^V_n)_{k_1, \ldots, k_n}\) form the underlying set of 0-simplices of a space which we construct in this subsection.

#### 5.2.1. The topological space \((\text{PBord}^V_n)_{k_1, \ldots, k_n}\)

We endow the set \((\text{PBord}^V_n)_{k_1, \ldots, k_n}\) with the following topology coming from modifications of the Whitney \(C^\infty\)-topology on \(\text{Emb}(M, V \times (0, 1)^n)\).

In [Gal11], spelled out in more detail in [GRW10], a topology is constructed on the set of closed (not necessarily compact) \(n\)-dimensional submanifolds \(M \subseteq V \times (0, 1)^n\), which we identify with the quotient

\[
\text{Sub}(V \times (0, 1)^n) \cong \bigsqcup_{[M]} \text{Emb}(M, V \times (0, 1)^n)/\text{Diff}(M),
\]

where the coproduct is taken over diffeomorphisms classes on \(n\)-manifolds. It is given by defining the neighborhood basis at \(M\) to be

\[
\{N \subset V \times (0, 1)^n : N \cap K = j(M) \cap K, j \in W\},
\]

where \(K \subset V \times (0, 1)^n\) is compact and \(W \subset \text{Emb}(M, V \times (0, 1)^n)\) is a neighborhood of the inclusion \(M \hookrightarrow V \times (0, 1)^n\) in the Whitney \(C^\infty\)-topology. Thus we obtain a topology on

\[
\text{Sub}(V \times (0, 1)^n) \times \text{Int}^n_{k_1, \ldots, k_n},
\]

where we view \(\text{Int}^n_{k_1, \ldots, k_n}\) as a (topological) subspace of \(\mathbb{R}^{2k}\).

For an element \(\overline{I} \in \text{Int}^n_{k_1, \ldots, k_n}\), recall from Definition 4.11 the box rescaling map \(\rho(\overline{I}) : \text{B}(\overline{I}) \rightarrow (0, 1)^n\).

Then we identify an element \((M, \overline{I}) \in (\text{PBord}^V_n)_{k_1, \ldots, k_n}\) whose underlying submanifold is the image of an embedding \(\iota : M \hookrightarrow V \times \text{B}(\overline{I})\) with the element \([\overline{I} \circ \iota, \overline{I}]\) in the above space. This identification gives an inclusion

\[
(\text{PBord}^V_n)_{k_1, \ldots, k_n} \subseteq \text{Sub}(V \times (0, 1)^n) \times \text{Int}^n_{k_1, \ldots, k_n},
\]

which we use to topologize the left hand side.

#### 5.2.2. The space \((\text{PBord}^V_n)_{k_1, \ldots, k_n}\)

To model the levels of the bordism category as spaces, i.e. as Kan complexes, we can start with the above version as a topological space and take singular chains of this topological space. However, smooth maps

\[\text{GRW10}\] use the notation \(\Psi(V \times (0, 1)^n) = \text{Sub}(V \times (0, 1)^n)\).
from a smooth manifold \( X \) to \( \text{Sub}(V \times (0,1)^n) \) as defined in [GRW10]. Definition 2.16, Lemma 2.17] are easier to handle. By Lemma 2.18 in the same paper, every continuous map from a smooth manifold, in particular the \( |\Delta_l|_c \), to the topological space \( (\text{PBord}_n^Y)_{k_1,\ldots,k_n} \) can be perturbed to a smooth one, so the homotopy type when considering smooth singular chains does not change. This leads to the following unravelled version of the space \( (\text{PBord}_n^Y)_{k_1,\ldots,k_n} \).

**Definition 5.6.** An \( l \)-simplex of \( (\text{PBord}_n^Y)_{k_1,\ldots,k_n} \) consists of tuples \( (M, \overline{T}(s) = (I_0(s) \leq \cdots \leq I_{k_l}(s))_{s \in |\Delta_l|_c} \) such that

1. \( \overline{T} = (I_0 \leq \cdots \leq I_{k_l})_{1 \leq i \leq n} \to |\Delta_l|_c \) is an \( l \)-simplex in \( \text{Int}^n_{k_1,\ldots,k_n} \),
2. \( M \) is a closed and bounded \((n+l)\)-dimensional submanifold of \( V \times B(\overline{T}(s))_{s \in |\Delta_l|_c} \times |\Delta_l|_c \) such that
   a) the composition \( \pi : M \to V \times B(\overline{T}(s))_{s \in |\Delta_l|_c} \times |\Delta_l|_c \to B(\overline{T}(s))_{s \in |\Delta_l|_c} \times |\Delta_l|_c \) of the inclusion with the projection is proper,
   b) its composition with the projection onto \( |\Delta_l|_c \) is a submersion \( M \to |\Delta_l|_c \), and
3. for every \( S \subseteq \{1,\ldots,n\} \), let \( p_S : M \to B(\overline{T}(s))_{s \in |\Delta_l|_c} \times |\Delta_l|_c \to \mathbb{R}^S \times |\Delta_l|_c \) be the composition of \( \pi \) with the projection \( \pi_S \) onto the \( S \)-coordinates. Then for every \( 1 \leq i \leq n \) and \( 0 \leq j_i \leq k_i \), at every \( x \in p_{\{i\}}^{-1}(I_j) \times |\Delta_i|_c \), the map \( p_{\{i\}} \) is submersive.

**Remark 5.7.** Note that for \( l = 0 \) we recover Definition 5.1. Moreover, for every \( s \in |\Delta_l|_c \) the fiber \( M_s \) of \( M \to |\Delta_l|_c \) determines an element in \( (\text{PBord}_n^Y)_{k_1,\ldots,k_n} \)

\[
(M_s) = (M_s \in V \times B(\overline{T}(s)), \overline{T}(s)).
\]

We will use the notation \( \pi_s : M_s \to B(\overline{T}(s)) \) for the composition of the embedding and the projection.

**Remark 5.8.** The conditions (2a), (2b), and (3) imply that \( M \to |\Delta_l|_c \) is a smooth fiber bundle, and, since \( |\Delta_l|_c \) is contractible, even a trivial fiber bundle. The proof is a more elaborate version of the argument after Definition 2.6 in [GTMW09].

We now lift the spatial structure of \( \text{Int}^n_{k_1,\ldots,k_n} \) to \( (\text{PBord}_n^Y)_{k_1,\ldots,k_n} \).

Fix \( k \geq 0 \) and let \( f : [m] \to [l] \) be a morphism in the simplex category \( \Delta \), i.e. a (weakly) order-preserving map. Then let \( |f| : |\Delta^m|_c \to |\Delta^l|_c \) be the induced map between standard simplices.

Let \( f^\Delta \) be the map sending an \( l \)-simplex in \( (\text{PBord}_n^Y)_{k_1,\ldots,k_n} \) to the \( m \)-simplex which consists of

1. for \( 1 \leq i \leq n \), the \( m \)-simplex in \( \text{Int}^n_{k_i} \) obtained by applying \( f^\Delta \),
\[
f^\Delta(I_0(s) \leq \cdots \leq I_{k_l}(s)) = (I_0(|f|(s)) \leq \cdots \leq I_{k_l}(|f|(s))_{s \in |\Delta^m|_c};
\]
2. The \((n+m)\)-dimensional submanifold \( f^\Delta M \subseteq V \times B(\overline{T}(s))_{s \in |\Delta^m|_c} \times |\Delta^m|_c \) obtained by the pullback of \( M \to |\Delta_l|_c \) along \(|f|\). Note that its fiber at \( s \in |\Delta^m|_c \) is \( f^\Delta M_s = M_{|f|(s)} \) and
\[
f^\Delta M = \bigcup_{s \in |\Delta^m|_c} M_{|f|(s)}.
\]

**Proposition 5.9.** \( (\text{PBord}_n^Y)_{k_1,\ldots,k_n} \) is a space. Moreover, it is a Kan complex.

**Proof.** The above assignment indeed is well-defined since the underlying assignment for the underlying intervals is well-defined and since the map \(|f|\) is a submersion, the pullback of \( M \to |\Delta_l|_c \) along \(|f|\) also

\[\text{Recall from Section [4] that by abuse of notation, } B(\overline{T}(s))_{s \in |\Delta_l|_c} \times |\Delta_l|_c \text{ denotes the total space of } B(\overline{T}) \to |\Delta_l|_c.\]
is a submersion. Moreover, the assignment is functorial, since pullback commutes contravariantly with composition.

It remains to show that this space is a Kan complex. This follows from the fact that smooth singular chains of a topological space form a Kan complex and the above space arises this way when endowing the underlying set with the topology coming from the space of submanifolds.

**Notation 5.10.** We denote the spatial face and degeneracy maps of \( (\text{P Bord}_n^V)_{k_1,\ldots,k_n} \) by \( d_j^\delta \) and \( s_j^\delta \) for \( 0 \leq j \leq l \).

**Example 5.11.** We now construct an example of a path. It shows that cutting off part of the collar of a bordism yields an element which is connected to the original one by a path.

Let \( (M,\Gamma) = (I_0^i \leq \cdots \leq I_{k_i}^i, i=1,\ldots,n) \in (\text{P Bord}_n^V)_{k_1,\ldots,k_n} \) and fix \( 1 \leq i \leq n \). We show that cutting off a short enough piece in the \( i \)-th direction at an end of an element of \( (\text{P Bord}_n^V)_{k_1,\ldots,k_n} \) leads to an element which is connected by a path to the original one. Fix \( 1 \leq i \leq n \) and let \( \varepsilon < b_i^j - a_i^j \).

For \( 0 \leq j \leq k_i \) and \( s \in [0,1] \) let

\[
I_j^i(s) = (a_i^j + s\varepsilon, b_i^j) \cap I_j^i,
\]

and then \( B(\Gamma(s)) = (a_i^j + s\varepsilon, b_i^j) \). Then let \( M(\varepsilon) \) be the preimage of the subset \( B(\Gamma(s))_{s\in|\Delta'|_s} \subseteq B(\Gamma) \times [0,1] \) of \( M \times [0,1] \to B(\Gamma) \times [0,1] \), i.e.

\[
M(\varepsilon) \xrightarrow{n} M \xrightarrow{\downarrow} B(\Gamma(s))_{s\in|\Delta'|_s} \xrightarrow{\downarrow} B(\Gamma) \times [0,1]
\]

Then \( (M(\varepsilon),\Gamma(s)) \) is a 1-simplex in \( (\text{P Bord}_n^V)_{k_1,\ldots,k_n} \) with fibers \( M(\varepsilon)_s = p_{[s]}^{-1}(a_i^j + s\varepsilon, b_i^j)) \).

**Remark 5.12.** In the above example we constructed a path from an element in \( (\text{P Bord}_n^V)_{k_1,\ldots,k_n} \) to its cutoff, where we cut off the preimage of \( p_i^{-1}(a_i^j, \varepsilon) \) for suitably small \( \varepsilon \). Note that the same argument holds for cutting off the preimage of \( p_i^{-1}(b_i^j - \delta, b_i^j) \) for suitably small \( \delta \). Moreover, we can iterate the process and cut off \( \varepsilon, \delta \) strips in all \( i \) directions. Choosing \( \varepsilon_i = \frac{b_i^j - a_i^j}{2}, \delta_i = \frac{b_i^j - a_i^j}{2} \) yields a path to its cutoff with underlying submanifold

\[
cut(M) = \pi^{-1}\left( \prod_{i=1}^n \left( \frac{a_i^j + b_i^j}{2}, \frac{a_i^j + b_i^j}{2} \right) \right).
\]

So far the definition depends on the choice of the vector space \( V \). However, in the bordism category we need to consider all (not necessarily compact) \( n \)-dimensional manifolds. By Whitney’s embedding theorem any such manifold can be embedded into some finite (large) dimensional vector space \( V \), so we need to allow big enough vector spaces.

**Definition 5.13.** Fix some (countably) infinite dimensional vector space, e.g. \( \mathbb{R}^\infty \). Then

\[
\text{P Bord}_n = \lim_{V \subseteq \mathbb{R}^\infty} \text{P Bord}_n^V.
\]

### 5.3. The \( n \)-fold simplicial space \( (\text{P Bord}_n)_{\ast,\ldots,\ast} \)

We make the collection of spaces \( (\text{P Bord}_n)_{\ast,\ldots,\ast} \) into an \( n \)-fold simplicial space by lifting the simplicial structure of \( \text{Int}_{\ast,\ldots,\ast} \). We first define the structure on 0-simplices, which makes \( (\text{P Bord}_n)_{\ast,\ldots,\ast} \) into an
their product in $\Delta^n$. Then

$$ \text{Definition 5.14.} \text{ For every } 1 \leq i \leq n, \text{ let } g_i : [m_i] \to [k_i] \text{ be a morphism in } \Delta, \text{ and denote by } g = \prod g_i \text{ their product in } \Delta^n. \text{ Then } (PBord_n)_{k_1, \ldots, k_n} \xrightarrow{g^*} (PBord_n)_{m_1, \ldots, m_n}. $$

applies $g_i^*$ to the $i$th tuple of intervals and perhaps cuts the manifold. Explicitly, it sends an element

$$ (M \subset V \times B(\bar{T}), \bar{T} = (I_{0i} \leq \cdots \leq I_{ki_i})^n_{i=1}) $$

to

$$ \left( g^*M = \pi^{-1}(B(g^*(\bar{T}))) \subset V \times B(g^*(\bar{T})), g^*(\bar{T}) = (I_{0i}^i \leq \cdots \leq I_{ki_i}^i)^n_{i=1} \right). $$

Note that as the manifold $g^*M$ is the preimage of the new box, we just cut off the part of the manifold outside the new box.

**Notation 5.15.** We denote the (simplicial) face and degeneracy maps by $d^i_j : (PBord_n)_{k_1, \ldots, k_n} \to (PBord_n)_{k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n}$ and $s^i_j : (PBord_n)_{k_1, \ldots, k_n} \to (PBord_n)_{k_1, \ldots, k_{i-1}, k_i, \ldots, k_n}$ for $0 \leq j \leq k_i$.

**Remark 5.16.** Recall from remark 5.2 that for $k_1, \ldots, k_n \geq 0$, one should think of an element in the set $(PBord_n)_{k_1, \ldots, k_n}$ as a collection of $k_1 \cdots k_n$ composed bordisms with $k_i$ composed bordisms with collars in the $i$th direction. These composed collared bordisms are the images under the maps

$$ D(j_1, \ldots, j_k) : (PBord_n)_{k_1, \ldots, k_n} \to (PBord_n)_{1, \ldots, 1} $$

for $(1 \leq j_i \leq k_i)_{1 \leq i \leq n}$ arising as compositions of face maps, i.e. $D(j_1, \ldots, j_k)$ is the map determined by the maps

$$ [1] \to [k_i], \quad (0 < 1) \to (j_i - 1 < j_i) $$

in the category $\Delta$ of finite ordered sets. This should be thought of as sending an element to the $(j_1, \ldots, j_k)$-th collared bordism in the composition.

**Definition 5.17.** For every $1 \leq i \leq n$, let $g_i : [m_i] \to [k_i]$ be a morphism in $\Delta$, and denote by $g = \prod g_i$ their product in $(\Delta^op)^n$. On $l$-simplices, $g^*$ sends an element

$$ (M \subset V \times B(\bar{T}(s))_{s \in |\Delta^l|}, \bar{T}(s) = (I_{0i}^l \leq \cdots \leq I_{ki_i}^l)^n_{i=1}(s)) $$

to

$$ (g^*M = \pi^{-1}(B(g^*(\bar{T}(s)))_{s \in |\Delta^l|}, \bar{T}(s) = (I_{0i}^g(s) \leq \cdots \leq I_{ki_i}^g(s))^n_{i=1}(s)), g^*(\bar{T}(s)) = (I_{0i}^g(s) \leq \cdots \leq I_{ki_i}^g(s))^n_{i=1}(s)), $$

where $\pi : M \subset V \times B(\bar{T}(s))_{s \in |\Delta^l|} \times |\Delta^l| \to B(\bar{T}(s))_{s \in |\Delta^l|} \times |\Delta^l|$. Note that $(g^*M)_s = g^*M_s$.

**Proposition 5.18.** The spatial and simplicial structures of $(PBord_n)_{\bullet, \ldots, \bullet}$ are compatible, i.e. for $f : [l] \to [p], g_i : [m_i] \to [k_i]$ for $1 \leq i \leq n$, the induced maps

$$ f^\Delta \text{ and } g^* $$

commute. We thus obtain an $n$-fold simplicial space $(PBord_n)_{\bullet, \ldots, \bullet}$. 

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**Proof.** Since \( \text{Int}^n \) is a simplicial space, it is enough to show that the maps commute on the manifold part, i.e.

\[
g^* f \Delta \simeq f \Delta g^* M.
\]

This follows from the commuting of the following diagram, in which all sides are pullback squares.

\[
\begin{array}{ccc}
B(\overline{\mathcal{T}}) \times |\Delta^n|_e & \rightarrow & B(g(\overline{\mathcal{T}}) \times |\Delta^n|_e \\
\text{id} \times |f| & & \text{id} \times |f|
\end{array}
\]

\[
\begin{array}{ccc}
f \Delta M & \rightarrow & g \Delta g^* M \approx f \Delta g^* M \\
\downarrow & & \downarrow
\end{array}
\]

\[
\begin{array}{ccc}
M & \leftarrow & g^* M
\end{array}
\]

\[
\begin{array}{ccc}
B(\overline{\mathcal{T}}) \times |\Delta^1|_e & \rightarrow & B(g(\overline{\mathcal{T}}) \times |\Delta^1|_e \\
\text{id} \times |f| & & \text{id} \times |f|
\end{array}
\]

**Lemma 5.19.** The face maps \((\text{Bord}_n)_*, \ldots, [n], \ldots, * \rightarrow (\text{Bord}_n)_*, \ldots, [n], \ldots, *\) are level-wise fibrations, i.e. fibrations of \(n\)-fold simplicial spaces.

**Proof.** We need to show that for any \(1 \leq i \leq n\) and \(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n\) the source and target maps \((\text{Bord}_n)_{k_1, \ldots, k_{i-1}, 1, k_{i+1}, \ldots, k_n} \rightarrow (\text{Bord}_n)_{k_1, \ldots, k_{i-1}, 0, k_{i+1}, \ldots, k_n}\) are fibrations. We will omit the indices and corresponding intervals for \(\alpha \neq i\) in the notation for clarity. Moreover, we explain the argument for the source map for lifting a path, the argument for lifting a horn inclusion and both for the target map are similar.

Let \((M \subset V \times (a, b) \times [0, 1], (a, b))\) be a 1-simplex in \((\text{Bord}_n)_0\) and \((N \subset V \times (a, b_1), I \subset I_1)\) an object in \((\text{Bord}_n)_1\) which is sent to \(M_0\) under the source map \(d_1\).

Consider the closure \(\overline{M} \subset V \times [a, b] \times [0, 1]\). Since \(M\) was bounded, \(\overline{M}\) is compact and the projection \(p_{[0,1]}\) still is a submersion. It induces a nowhere vanishing vector field on \(\overline{M}\) which as \(M\) is closed in \(V \times [a, b + \epsilon] \times [0, 1]\) as well can be extended to a vector field on a neighborhood \(U\) of \(\overline{M}\) in \(V \times [a, b + \epsilon] \times [0, 1]\). Now choose a partition of unity subordinate to the cover given by \(U\) and the complement of \(\overline{M}\) to extend the vector field (by 0) to \(V \times [a, b_1] \times [0, 1]\). Now flow the closure of \(N\) in \(V \times [a, b_1] \times [0, 1]\) (which also is compact) along this vector field to obtain the desired path. 

**5.4. The complete \(n\)-fold Segal space** \(\text{Bord}_n\)

**Proposition 5.20.** \((\text{Bord}_n)_*, \ldots, *\) is an \(n\)-fold Segal space.

**Proof.** It suffices to prove the following conditions:

1. The Segal condition is satisfied. By Lemma 5.19 it is enough to show that the “strict” Segal condition holds, i.e. that

\[
(\text{Bord}_n)_{k_1, \ldots, k_{i+1}, \ldots, k_n} \rightarrow (\text{Bord}_n)_{k_1, \ldots, k_i, \ldots, k_n} \times (\text{Bord}_n)_{k_1, \ldots, k_i, \ldots, k_n}.
\]
But an element in the right hand side is a pair of submanifolds $M \subset V \times (a_0, b_k)$ and $N \subset V \times (\tilde{a}_0, \tilde{b}_k)$ which coincide on the intersection $V \times (a_k, b_k)$ together with intervals $I_0 \leq \cdots \leq I_k$ and $I_0 \leq \cdots \leq \tilde{I}_k$ such that $I_k = \tilde{I}_0$. So we can glue them together to form a submanifold $M \cup N \subset V \times (a_0, \tilde{b}_k)$, and concatenate the intervals $I_0 \leq \cdots \leq I_k \leq \tilde{I}_1 \leq \cdots \leq \tilde{I}_k$. Thus the above strict Segal map even is a homeomorphism.

Note that this construction extends to $l$-simplices: Now we glue submanifolds of $V \times (a_0, b_k) \times |\Delta^l|_e$ and $V \times (\tilde{a}_0, \tilde{b}_k) \times |\Delta^l|_e$ to form one of $V \times (0, \tilde{b}_k) \times |\Delta^l|_e$.

2. For every $i$ and every $k_1, \ldots, k_{i-1}$, the $(n-i)$-fold Segal space $(\text{PBord}_n)_{k_1, \ldots, k_{i-1}, 0, \bullet, \ldots, \bullet}$ is essentially constant.

We show that the degeneracy inclusion map

$$(\text{PBord}_n)_{k_1, \ldots, k_{i-1}, 0, 0, \ldots, 0} \hookrightarrow (\text{PBord}_n)_{k_1, \ldots, k_{i-1}, 0, k_{i+1}, \ldots, k_n}$$

admits a deformation retraction and thus is a weak equivalence.

We claim that the assignment sending an pair consisting of $t \in [0, 1]$ and an $l$-simplex

$$(M \subset V \times B(I(s)) \times |\Delta^l|_e, ((I^\beta(s) \leq \cdots \leq I^\gamma_k(s))_{1 \leq \beta < \gamma, i}, (a_\alpha^0(s), b_\alpha^0(s)), (I^\alpha_0(s) \leq \cdots \leq I^\gamma_k(s))_{1 \leq \alpha < n}, x \in |\Delta^l|_e),$$

in $(\text{PBord}_n)_{k_1, \ldots, k_{i-1}, 0, k_{i+1}, \ldots, k_n}$ to

$$(M \subset V \times B(I(s)) \times |\Delta^l|_e, ((I^\beta(s) \leq \cdots \leq I^\gamma_k(s))_{1 \leq \beta < \gamma, i}, (a_\alpha^0(s), b_\alpha^0(s)), (I^\alpha_0(s, t) \leq \cdots \leq I^\gamma_k(s, t))_{1 \leq \alpha < n}, (s, t) \in |\Delta^l|_e \times [0, 1])$$

where for $\alpha > i$ and ever $0 \leq j \leq k,$

$$a_\alpha^j(s, t) = (1 - t)a_\alpha^0(s) + ta_\alpha^0(s),$$

$$b_\alpha^j(s, t) = (1 - t)b_\alpha^0(s) + tb_\alpha^0(s).$$

is a homotopy $H: [0, 1] \times (\text{PBord}_n)_{k_1, \ldots, k_{i-1}, 0, k_{i+1}, \ldots, k_n} \to (\text{PBord}_n)_{k_1, \ldots, k_{i-1}, 0, k_{i+1}, \ldots, k_n}$ exhibiting the deformation retract. Note that $B(I(s), t) = B(I(s))$ for every $t \in [0, 1]$. Moreover, for $t = 0$ we have that $I^\alpha_0(s, 0) = I^\alpha_0(s)$ and the $l$-simplex is sent to itself. For $t = 1$ we have $I^\alpha_0(s, 1) = (a_\alpha^0(s), b_\alpha^0(s))$, so the image lies in $(\text{PBord}_n)_{k_1, \ldots, k_{i-1}, 0, 0, \ldots, 0}$.

It suffices to check that for every $t \in [0, 1]$ the image indeed is an $l$-simplex in $(\text{PBord}_n)_{k_1, \ldots, k_{i-1}, 0, k_{i+1}, \ldots, k_n}$. Since $(M, I(s)) \in (\text{PBord}_n)_{k_1, \ldots, k_{i-1}, 0, k_{i+1}, \ldots, k_n}$, this reduces to checking
Remark 5.21. This proposition together with Lemma 5.19 actually show that PBord\textsuperscript{n} is a strongly Segal internal n-fold category in the sense of [224].

The last condition necessary to be a good model for the (\infty, n)-category of bordisms is completeness, which PBord\textsubscript{n} in general does not satisfy. However, we can pass to its completion Bord\textsubscript{n}.

Definition 5.22. The (\infty, n)-category of cobordisms Bord\textsubscript{n} is the n-fold completion PBord\textsubscript{n} of PBord\textsubscript{n}, which is a complete n-fold Segal space.

Remark 5.23. For n \geq 6, PBord\textsubscript{n} is not complete, see the full explanation in [Lur09c]. 2.2.8. For n = 1 and n = 2, by the classification theorems of one- and two-dimensional manifolds, PBord\textsubscript{n} is complete, and therefore Bord\textsubscript{n} = PBord\textsubscript{n}.

6. Variants of Bord\textsubscript{n} and comparison with Lurie’s definition

6.1. The (\infty, d)-category of n-bordisms for any d \geq 0

We can define a d-fold Segal space whose d-morphisms are n-bordisms for d \geq 0. For d < n this amounts to extending the category of n-dimensional cobordisms down to (n - d)-dimensional objects.

Definition 6.1. Let V be a finite dimensional \mathbb{R}-vector space, which we identify with some \mathbb{R}^r with some metric. Let n \geq 0, d = n + l \geq 0. For every d-tuple k_1, \ldots, k_d \geq 0, we let (PBord\textsuperscript{\scriptscriptstyle V\textsubscript{k_1},...,k_d})_n be the collection of tuples (M, T = (I_0, \ldots, I_{k_1})_{1 \leq i \leq d}) satisfying conditions analogous to (1)-(3) in Definition 5.21 i.e.

1. For 1 \leq i \leq d, 
   \[ (I_0, \ldots, I_{k_i}) \in \text{Int}_{k_i}. \]

2. M is a closed and bounded n-dimensional submanifold of V \times B(T) and the composition \pi : M \hookrightarrow V \times B(T) \twoheadrightarrow B(T) is a proper map.

3. For every S \subseteq \{1, \ldots, d\}, let p_S : M \twoheadrightarrow B(T) \xrightarrow{\pi_S} \mathbb{R}^S be the composition of \pi with the projection \pi_S onto the S-coordinates. Then for every 1 \leq i \leq d and 0 \leq j_i \leq k_i, at every x \in p^{-1}_i(I_{j_i}^i), the map p(\cdots, d) is submersive.

We make (PBord\textsuperscript{\scriptscriptstyle V\textsubscript{k_1},...,k_d})_n into a space similarly to (PBord\textsuperscript{\scriptscriptstyle V\textsubscript{k_1},...,k_d})_n, and again we take the limit over all finite dimensional vector spaces in a given infinite dimensional vector space, say \mathbb{R}^\infty:

\[
\text{PBord}\textsuperscript{\scriptscriptstyle n} = \lim_{V \in \mathbb{R}^\infty} \text{PBord}_{n}^V.
\]

Proposition 6.2. (PBord\textsuperscript{\scriptscriptstyle n})^{\bullet, \ldots, \bullet} is a d = (n + l)-fold Segal space.

Proof. The proof is completely analogous to the proof of Proposition 5.20. \[\square\]
Definition 6.3. For \( l \leq 0 \) let \( d = n + l \leq n \) and let \( \text{Bord}^l_n \), which we will also denote by \( \text{Bord}^{(x,d)}_n \), be the \( d \)-fold completion of \( \text{PBord}^l_n \), the \((x,d)\)-category of \( n \)-bordisms.

Remark 6.4. For \( l > 0 \), the underlying submanifold of objects of \( \text{PBord}^l_n \), i.e. elements in \( (\text{PBord}^l_n)^{0,\ldots,0} \), are \( n \)-dimensional manifolds which have a submersion onto \( \mathbb{R}^{n+l} \). This implies that \( M = \emptyset \). Thus, the only objects are \((\emptyset, I_0^1, \ldots, I_0^{n+l})\) and \((\text{PBord}^l_n)^{0,\ldots,0} \simeq \text{Int}_0^0 \simeq \ast \). Similarly, \((\text{PBord}^l_n)^{0,1,\ldots,k_n+1} \simeq \text{Int}_0^{n,k_2,\ldots,k_n+1} \simeq \ast \). Thus, \((\text{PBord}^l_n)^{0,\ldots,\ast} \) is equivalent to the point viewed as a constant \((n-1)\)-fold Segal space. Similarly, \((\text{PBord}^l_n)^{1,\ldots,0,\ast} \) with \((l-1)\)’s, is equivalent to the point viewed as a constant \((n-l)\)-fold Segal space.

6.2. Unbounded submanifolds, \((0,1)^n\) as a parameter space, and cutting points

6.2.1. Unbounded submanifolds

We could have omitted the condition that \( M \) be bounded in condition (2) in Definitions 5.1 and 5.6, requiring it only to be closed. This modification leads to an \( n \)-fold simplicial space \( \text{PBord}^{nb}_n \) which does not satisfy the property in Lemma 5.13.

However, recall from Section 4.12 that for every element in \((\text{PBord}_n)^{k_1,\ldots,k_n}\), we constructed a path to its cutoff. There is a similar cutoff path for every element in \((\text{PBord}^{nb}_n)^{k_1,\ldots,k_n}\) to an element whose underlying submanifold

\[
\text{cut}(M) = \pi^{-1}\left( \prod_{i=1}^{n} \left( \frac{k_i}{2}, \frac{a_i}{2} \right) \right)
\]

is bounded in the \( V \)-direction. This construction extends to \( l \)-simplices and yields a map of \( n \)-fold simplicial spaces

\[
\text{cut} : \text{PBord}^{nb}_n \longrightarrow \text{PBord}_n,
\]

sending an element to its “cutoff”. It is a level-wise weak equivalence of \( n \)-fold simplicial spaces, and since \( \text{PBord}_n \) is an \( n \)-fold Segal space, \( \text{PBord}^{nb}_n \) is too.

6.2.2. Restricting the boxing to \((0,1)^n\)

Instead of basing \( \text{PBord}_n \) on \( \text{Int} \), we could instead use \( \text{Int}^{(0,1)} \) from Section 4.5. This approach leads to an \( n \)-fold Segal space \( \text{PBord}^{(0,1)}_n \) which is equivalent to \( \text{PBord}_n \) via a rescaling map

\[
\text{PBord}_n \overset{\rho}{\longrightarrow} \text{PBord}^{(0,1)}_n,
\]

using the box rescaling maps \( \rho : B(I) \rightarrow (0,1)^n \) from Definition 4.11. An element

\[
(M \subset V \times B(I), \overrightarrow{I}) = (I_0^1 \leq \cdots \leq I_0^{k_n})_{1 \leq i \leq n}
\]

is sent to

\[
(M \subset V \times B(I), \rho \overrightarrow{I}, V \times (0,1)^n, (\rho(I_0^1) \leq \cdots \leq \rho(I_0^{k_n}))_{1 \leq i \leq n}),
\]

and similarly for the \( l \)-simplices. On a fixed level, i.e. for fixed \( k_1, \ldots, k_n \), there is an inclusion of spaces \( t : (\text{PBord}^{(0,1)}_n)^{k_1,\ldots,k_n} \hookrightarrow (\text{PBord}_n)^{k_1,\ldots,k_n} \) and the above map is a retract of the inclusion. The \( n \)-fold simplicial structure needs to be modified by rescaling maps to ensure that the boxing stays \((0,1)^n\): For a morphism \( g = \prod g_i \) in \( \Delta^n \), the associated morphism of spaces is \( \rho \circ g \circ t \). We leave it to the reader to fill in the details.
6.2.3. Cutting points

Another variant of an \(n\)-fold Segal space of cobordisms can be obtained by replacing the intervals \(I^i\) in Definition \[6.1\] of \(\text{PBord}_n\) by specified “cutting points” \(t_j\), which correspond to where we cut our composition of bordisms. Equivalently, we can say that in this case the intervals are replaced by intervals consisting of just one point, i.e. \(a_j^i = b_j^i = t_j^i\). The levels of this \(n\)-fold Segal space \(\text{PBord}^t_n\) can be made into spaces as we did for \(\text{PBord}_n\), but we now need to impose the extra condition that elements of the levels are connected by a path if they coincide inside the boxing of \(t_j\)s, i.e. over \([t_0^1, t_{k_1}^1] \times \cdots \times [t_0^n, t_{k_n}^n]\).

However, for \(\text{PBord}^t_n\) the Segal condition is more difficult to prove, as in this case we do not specify the collar along which we glue. Since the space of collars is contractible, sending an interval \(I\) with endpoints \(a\) and \(b\) to its midpoint \(t = \frac{1}{2}(a+b)\) induces an equivalence of \(n\)-fold Segal spaces from \(\text{PBord}_n\) to \(\text{PBord}^t_n\).

6.3. Comparison with Lurie’s definition of cobordisms

In [Lur09b], Lurie defined the \(n\)-fold Segal space of cobordisms as follows:

**Definition 6.5.** Let \(V\) be a finite dimensional vector space. For every \(n\)-tuple \(k_1, \ldots, k_n \geq 0\), let \((\text{PBord}^{V,L}_n)_{k_1, \ldots, k_n}\) be the collection of tuples \((M, (t_0^i, \ldots, t_k^i))_{i=1,\ldots,n}\), where

1. For \(1 \leq i \leq n\),
   \[t_0^i \leq \cdots \leq t_{k_i}^i\]
   is an ordered \((k_i + 1)\)-tuple of elements in \(\mathbb{R}\).
2. \(M\) is a closed \(n\)-dimensional submanifold of \(V \times \mathbb{R}^n\) and the composition \(\pi : M \hookrightarrow V \times \mathbb{R}^n \to \mathbb{R}^n\) is a proper map.
3. For every \(S \subseteq \{1, \ldots, n\}\) and for every collection \(\{j_i\}_{i \in S}\), where \(0 \leq j_i \leq k_i\), the composition \(p_S : M \xrightarrow{\pi} \mathbb{R}^n \to \mathbb{R}^S\) does not have \((t_{j_i})_{i \in S}\) as a critical value.
4. For every \(x \in M\) such that \(p_{\{i\}}(x) \in \{t_0^i, \ldots, t_{k_i}^i\}\), the map \(p_{\{i+1, \ldots, n\}}\) is submersive at \(x\).

It is endowed with a topology coming from the Whitney topology similar to what we described in Section 5.2.1. Similarly to before, let

\[
\text{PBord}^{t,L}_n = \lim_{V \subset \mathbb{R}^n} \text{PBord}^{V,L}_n
\]

Comparing this definition with Definition \[5.1\] and \(\text{PBord}^r_n\) from 6.2.3 above, our condition \(3\) on \(\text{PBord}^t_n\) is replaced by the two strictly weaker conditions \((3)\) and \((4)\) on \(\text{PBord}^r_n\), which are both implied by \(3\):

**Lemma 6.6.** Let \(M\) be a closed \(n\)-dimensional manifold and \(\pi : M \to \mathbb{R}^n\). Moreover, for \(1 \leq i \leq n\) let \((t_0^i \leq \cdots \leq t_{k_i}^i)\) be an ordered \((k_i + 1)\)-tuple of elements in \(\mathbb{R}\). Denote for \(S \subseteq \{1, \ldots, n\}\) the composition \(M \xrightarrow{\pi} \mathbb{R}^n \to \mathbb{R}^S\) by \(p_S\). Assume that condition \(3\) from Definition \[5.1\] holds, i.e. for every \(1 \leq i \leq n\) and \(0 \leq j_i \leq k_i\), for \(x \in M\) such that \(p_{\{i\}}(x) = t_{j_i}^i\), the map \(p_{\{i, \ldots, n\}}\) is submersive at \(x\). Then,

3. For every \(S \subseteq \{1, \ldots, n\}\) and for every collection \(\{j_i\}_{i \in S}\), where \(0 \leq j_i \leq k_i\), the composition \(p_S : M \xrightarrow{\pi} \mathbb{R}^n \to \mathbb{R}^S\) does not have \((t_{j_i})_{i \in S}\) as a critical value.
4. For every \(x \in M\) such that \(p_{\{i\}}(x) \in \{t_0^i, \ldots, t_{k_i}^i\}\), the map \(p_{\{i+1, \ldots, n\}}\) is submersive at \(x\).
Proof. Let $i_0 = \inf S$. Consider the following diagram

\[
\begin{array}{ccc}
M & \xrightarrow{p_{(i_0,\ldots,n)}} & \mathbb{R}^{(i_0,\ldots,n)} \\
\downarrow{p_S} & & \downarrow{\text{proj}} \\
\mathbb{R}^5 & & 
\end{array}
\]

For 3, let $x \in p_S^{-1}((t_j)_{j \in S})$. Then $p_{(i_0)}(x) = t_{i_0}^0$, so by assumption the map $p_{(i_0,\ldots,n)}$ is submersive at $x$. Since $\text{proj}$ is submersive, $p_S = \text{proj} \circ p_{(i_0,\ldots,n)}$ also is submersive at $x$.

For 4, note that if $p_{(i,\ldots,n)}$ is submersive at $x$ then $p_{(i+1,\ldots,n)}$ is submersive at $x$. 

However, Lurie’s $n$-fold simplicial space $\text{PBord}_n^L$ is not an $n$-fold Segal space as we will see in the example below. Thus, our $\text{PBord}_n^L$ is a corrigendum of Lurie’s $\text{PBord}_n^L$ from [Lur09c].

Example 6.7.

Consider the 2 dimensional torus $T$ in $\mathbb{R} \times \mathbb{R}^2$, embedded such that the projection onto $\mathbb{R}^2$ is an annulus, and consider the tuple $(T \subset \mathbb{R} \times \mathbb{R}^2, t_1^0, t_2^0, \ldots, t_k^2)$, where $t_i^0$ is indicated in the picture of the projection plane $\mathbb{R}^2$ on the left. Then, because of condition (3), $t_0^2 \leq t_1^2 \leq \ldots \leq t_k^2$ can be chosen everywhere such that any $(t_0^0, t_1^2)$ is not a point where the vertical $(t_0^0)$-line intersects the two circles in the picture. Thus, if $t_j^2$ and $t_j^2$ are in two different connected components of this line minus these forbidden points, there is no path connecting this point to an element in the image of the degeneracy map. However, it satisfies the conditions 1., 2., 3., and 4. in the definition of $(\text{PBord}_2^L)_{0,k^2}$, so $(\text{PBord}_2^L)_{0,\bullet}$ is not essentially constant.

6.4. The $n$-fold category $\text{Bord}^{\text{uple}}_n$

Condition 3. in Definition [6.5] ensures that the fibers $p_S^{-1}((t_j)_{j \in S})$ are $(n - |S|)$-dimensional smooth manifolds, i.e. that a $k$-morphism, which is a $k$-dimensional cobordism, indeed goes from a $(k-1)$-dimensional cobordism to another one.

Our condition 4. ensures in addition the globularity condition, i.e. essential constancy, namely that we have an “$n$-category” instead of an “$n$-uple category”. This difference for $n = 2$ is the same as the difference between a “bicategory” and a “double category”.

Consider the following interval version of condition 3.

3. For every $S \subseteq \{1, \ldots, n\}$ and for every collection $\{j_i\}_{i \in S}$, where $0 \leq j_i \leq k_i$, the composition $p_S: M \xrightarrow{\gamma} \mathbb{R}^n \xrightarrow{\text{proj}} \mathbb{R}^2$ does not have any critical value in $(I_{j_i})_{i \in S}$.

Relaxing our condition [3] in Definition [6.4] to this (3) gives an $n$-uple Segal space $\text{PBord}^{\text{uple}}_n$. Completing gives a complete $n$-uple Segal space $\text{Bord}^{\text{uple}}_n$.

Remark 6.8. Using the construction in Section [2.4.4] one can see that $\text{Bord}_n$ indeed is the maximal complete $n$-fold Segal space underlying the complete $n$-uple Segal space $\text{Bord}^{\text{uple}}_n$.

Example 6.9 (The torus as a composition). The difference between the $n$-fold and the $n$-uple Segal spaces can be seen when decomposing the torus, viewed as a 2-morphism in the respective $n$-(uple) categories. We will omit drawing the intervals outside of the torus and just draw the “cutting lines”, which should be understood as actually extending to small closed intervals around them.

The torus as a 2-morphism in $\text{Bord}_2^{\text{uple}}$ can be decomposed simultaneously in both directions. One
possible decomposition into in some sense elementary pieces is the following:

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{torus_decomposition.png}
\end{array}
\]

However, similar to the argument in Example 6.7, this decomposition is not a valid decomposition in \( \text{Bord}_2 \), as condition 5 in Definition 5.1 is violated.

The torus as a 2-morphism in \( \text{Bord}_2 \) can only be decomposed “successively”, so we first decompose it in the first direction, i.e. the first coordinate, e.g. as

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{torus_decomposition_successive.png}
\end{array}
\]

which is an element in \((\text{Bord}_2)_{4,1}\) and then decompose the two middle pieces, the images under the compositions of face maps \((\text{Bord}_2)_{4,1} \Rightarrow (\text{Bord}_2)_{1,1}\), as

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{torus_decomposition_successive_2.png}
\end{array}
\]

Altogether a possible decomposition of the torus into elementary pieces in \( \text{Bord}_2 \) is

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{torus_double_decomposition.png}
\end{array}
\]

This of course also is a valid decomposition in the 2-fold category \( \text{Bord}^{\text{inde}}_2 \).

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7. The symmetric monoidal structure on $\text{Bord}_n$

The $(x,n)$-category $\text{Bord}_n$ is symmetric monoidal with its symmetric monoidal structure essentially arising from taking disjoint unions. In this section we endow $\text{Bord}_n$ with a symmetric monoidal structure in two ways. In Section 7.1 the symmetric monoidal structure arises from a $\Gamma$-object. In Section 7.2 a symmetric monoidal structure is defined using a tower of monoidal $i$-hybrid $(n+i)$-fold Segal spaces.

7.1. The symmetric monoidal structure arising as a $\Gamma$-object

We construct a sequence of $n$-fold Segal spaces $\text{Bord}_V^{\Gamma}$, which form a $\Gamma$-object in $n$-fold Segal spaces which in turn endows $\text{Bord}_n$ with a symmetric monoidal structure as defined in Section 3.1.

**Definition 7.1.** Let $V$ be a finite dimensional vector space. For every $k_1, \ldots, k_n$, let $(\text{PBord}_V^{\Gamma})_{k_1,...,k_n}$ be the collection of tuples

$$(M_1, \ldots, M_m, (I^0_1 \leq \ldots \leq I^0_{k_1}), \ldots, (I^0_{k_n} = \ldots = I^0_{m})),$$

where each $(M_\beta, (I^0_\beta \leq \ldots \leq I^0_{k_\beta})_{1=1,...,n})$ is an element of $(\text{PBord}_V^{\Gamma})_{k_1,...,k_n}$ and $M_1, \ldots, M_m$ are disjoint. It can be made into an $n$-fold simplicial space similarly to $\text{PBord}_V^{\Gamma}$. Moreover, similarly to the definition of $\text{Bord}_n$, we take the limit over all $V \subseteq \mathbb{R}^\infty$ and complete to get an $n$-fold complete Segal space $\text{Bord}_n$.

**Proposition 7.2.** The assignment

$$\Gamma \longrightarrow \text{CSS}_p_n,$$

$$[m] \longrightarrow \text{Bord}_n[m]$$

extends to a functor and endows $\text{Bord}_n$ with a symmetric monoidal structure.

**Proof.** By Lemma 3.6 it is enough to show that the functor sending $[m]$ to $\text{PBord}_n[m]$ and a morphism $f : [m] \rightarrow [k]$ in $\Gamma$ to the morphism

$$\text{PBord}_n[m] \rightarrow \text{PBord}_n[k],$$

$$(M_1, \ldots, M_m, I') \mapsto (\prod_{\beta \in f^{-1}(1)} M_\beta, \ldots, \prod_{\beta \in f^{-1}(k)} M_\beta, I'),$$

is a functor $\Gamma \rightarrow \text{CSS}_p_n$ with the property that

$$\prod_{1 \leq \beta \leq n} \gamma_\beta : \text{Bord}_n[m] \rightarrow (\text{PBord}_n[1])^m$$

is an equivalence of $n$-fold Segal spaces.

The map $\prod_{1 \leq \beta \leq n} \gamma_\beta$ is a level-wise inclusion and we show that level-wise it is a weak equivalence. Let $((M_1), \ldots, (M_n)) \in (\text{PBord}_n[1])^m$. We construct a path to an element in the image of $\prod_{1 \leq \beta \leq n} \gamma_\beta$ which induces a strong homotopy equivalence between the above spaces. First, there is a path to an element for which all $(M_\alpha)$ have the same specified intervals by composing all except one with a suitable smooth rescaling. Secondly, there is a path with parameter $s \in [0,1]$ given by composing the embedding $M_\alpha \rightarrow V \times B(\overline{I})$ with the embedding into $\mathbb{R} \times V \times B(\overline{I})$ given by the map $V \rightarrow \mathbb{R} \times V, v \mapsto (s\alpha, v)$.

**Remark 7.3.** More generally, the same construction works for $\text{Bord}_n^{(x,d)}$ for $d \leq n$ using a sequence of $d$-fold Segal spaces $\text{PBord}_n^{(x,d)}[m]$ for $l = n - d$. 

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7.2. Looping, the monoidal structure and the tower

Our goal for this section is to endow \( \text{Bord}_n \) with a symmetric monoidal structure arising from a tower of monoidal \( l \)-hybrid \((n + l)\)-fold Segal spaces \( \text{Bord}^l_n \) for \( l \geq 0 \).

Recall from Section \(6.1\) the \((n + l)\)-folds Segal spaces \( \text{PBord}^l_n \) of \( n \)-dimensional bordisms. We will use these to define the symmetric monoidal structure on \( \text{Bord}_n \), and, more generally, on \( \text{Bord}^{(x,d)}_n \) for \( d \leq n \).

We saw in Remark \(6.4\) that \( \text{PBord}^l_n \) is \( p_{l-1}q \)-connected if \( l \geq 0 \). However, it does not have a discrete space of objects, 1-morphisms, \ldots, \((l - 1)\)-morphisms. For \( l \leq 0 \) it is even worse as \( \text{PBord}^l_n \) is not even connected. However, in any \( \text{PBord}^l_n \), there is the distinguished object

\[ \emptyset = (\emptyset, (0,1)), \]

and by Proposition \(3.31\) it suffices to prove the following theorem.

**Theorem 7.4.** For \( n + l \geq k \geq 0 \), there are weak equivalences

\[ L_k(\text{PBord}^l_n, \emptyset) \xrightarrow{u} \text{PBord}^{l-k}_n. \]

Since looping and completion commute by Remark \(2.22\) the following corollary is immediate.

**Corollary 7.5.** \( \text{Bord}_n \) and \( \text{Bord}^{(x,d)}_n \) are symmetric monoidal.

We can extract the \( k \)-monoidal \((n + l)\)-fold complete Segal spaces which form the tower:

**Definition 7.6.** For \( k > 0 \) and \( d \geq 0 \), consider the \( k \)th level fiber

\[ \text{PBord}^{k-(n-d)}_n / \emptyset. \]

It is a \((k - 1)\)-connected \((n + l)\)-fold Segal space which satisfies

\[ L_k \left( \text{PBord}^{k-(n-d)}_n / \emptyset \right) \simeq L_k(\text{PBord}^{k-(n-d)}_n, \emptyset) \simeq \text{PBord}^{-(n-d)}_n \]

by the above Theorem. Its \( k \)-hybrid completion thus is a \( k \)-monoidal complete \((d + k)\)-fold Segal space. Since looping and completion commute by Remark \(2.22\) for \( k \leq 0 \) the collection thereof endows the complete \( d \)-fold Segal space \( \text{Bord}^{(x,d)}_n \) = \( \text{Bord}_n \) for \( d = n + l \) with a symmetric monoidal structure. For \( d = n \), we obtain the symmetric monoidal structure on the complete \( n \)-fold Segal space \( \text{Bord}_n \).

**Definition 7.7.** The maps \( \ell \) and \( u \) constructed in the proof below are called the *looping* and *delooping* maps.

Another interesting consequence is the following corollary.

**Corollary 7.8.** There is an equivalence of symmetric monoidal \((x,d)\)-categories

\[ L_{n-d}(\text{Bord}_n) \simeq \text{Bord}^{(x,d)}_n. \]

**Proof of Theorem 7.4.** It is enough to show that

\[ L(\text{PBord}^l_n, \emptyset) = \text{Hom}_{\text{PBord}^l_n}(\emptyset, \emptyset) \simeq \text{PBord}^{l-1}_n. \]
The statement for general \( k \) follows by induction.

We define a map

\[ u : L(\text{PBord}_n^l) \rightarrow \text{PBord}_n^{l-1} \]

by sending an element in \( \text{Hom}_{\text{PBord}_n^l}(\mathcal{O}, \mathcal{O})_{k_2, \ldots, k_{n+1}} \)

\[ (M_l) = (M \subseteq V \times (a_0^1, b_1^1) \times B(\mathcal{I}), I_0^1 \leq I_1^1, \mathcal{I} = (I_0^1 \leq \cdots \leq I_{k_1}^1)_{i=2}^{n+1} \in (\text{PBord}_n^l)_{1, k_2, \ldots, k_{n+1}} \]

to

\[ (M_{l-1}) = (M \subseteq V \times (a_0^1, b_1^1) \times B(\mathcal{I}), I_0^1 \leq \cdots \leq I_{k_1}^1, \mathcal{I} = (I_0^1 \leq \cdots \leq I_{k_1}^1)_{i=2}^{n+1} \)

so it “forgets” the first specified intervals. Note that in the above, we view \( \sim \)
space using a diffeomorphism \( \sim \). We know that \( \sim \) space using a diffeomorphism \( \sim \). Moreover, we know that \( \sim \) is bounded in the \( \sim \)

Start with an element \( \sim \) be the infimum of such \( \sim \). Let \( \sim \) be the supremum of such \( \sim \). Choose \( \sim \) vectors \( \sim \) vary within a fixed countably infinite dimensional space. Choose \( \sim \) with a countable basis consisting of vectors \( \sim \). In taking the limit is enough to consider the finite dimensional subspaces \( \sim \) spanned by the first \( \sim \) vectors \( \sim \). Then the map \( \sim \) we constructed above was defined as an inductive system of maps

\[ u_V : L(\text{PBord}_n^V) \rightarrow \text{PBord}_n^{V-1}, \]

where \( \sim \) is proper. Since \( \sim \) is a vector

It suffices to show that these maps induce an inductive system of equivalences of \( \sim \)-fold Segal spaces, with homotopy inverses given by the following inductive system of maps

\[ \ell_r : \text{PBord}_n^{l-1, V_r+1} \rightarrow L(\text{PBord}_n^{V_r}). \]

Start with an element \( (M_{l-1}) = (M \subseteq V_{r+1} \times B(\mathcal{I}), \mathcal{I} = (I_0^1 \leq \cdots \leq I_{k_1}^1)_{i=2}^{n+1} \in (\text{PBord}_n^{l-1, V_{r+1}}). \) Since it is bounded in the \( \sim \)-direction, there are \( A, B \) such that

\[ B < \pi_{v_1}(M) < A, \]

where \( \pi_{v_1} : M \leq ((v_1) \oplus (v_2, \ldots, v_{r+1})) \times B(\mathcal{I}) \rightarrow (v_1) \oplus \mathcal{V}_1. \) Let \( \hat{B} \) be the supremum of such \( B \) and let \( \hat{A} \) be the infimum of such \( A. \) Let \( a_0^1 = \hat{A} - 2, b_0^1 = \hat{A} - 1, a_1^1 = \hat{B} + 1 \) and \( b_1^1 = \hat{B} + 2. \) Finally, we send \( (M_{l-1}) \) to

\[ (M_l) = (M \subseteq (v_2, \ldots, v_{r+1}) \times (a_0^1, b_1^1) \times B(\mathcal{I}), (a_0^1, b_0^1) \leq (a_1^1, b_1^1), \mathcal{I} = (I_0^1 \leq \cdots \leq I_{k_1}^1)_{i=2}^{n+1}. \]
By construction, $\ell_r \circ u_r \sim \text{id}$, $u_r \circ \ell_r = \text{id}$, where $\ell_r \circ u_r$ just changes the first two intervals $I_0 \preceq I_1$ and thus is homotopy equivalent to the identity.

8. Interpretation of bordisms as manifolds with corners and the homotopy (bi)category

8.1. Bordisms as manifolds with corners and embeddings thereof

For the definition and notation for $\langle k \rangle$-manifolds used in this section we refer to [Lau00]. They have been used in bordism theory, e.g. in [SP09] and [LP08].

**Definition 8.1.**
- A (cubical) 0-bordism is a closed manifold.
- An $n$-dimensional cubical $k$-bordism is an $n$-dimensional $\langle k \rangle$-manifold whose tuple of faces are denoted by $(\partial_1 M, \ldots, \partial_k M)$ together with decompositions
  $$\partial_i M = \partial_{i,\text{in}} M \sqcup \partial_{i,\text{out}} M,$$
  such that $\partial_{i,\text{in}} M$ and $\partial_{i,\text{out}} M$ are $(n-1)$-dimensional cubical $(k-1)$-bordisms.
- An $n$-dimensional $k$-bordism is an $n$-dimensional cubical $k$-bordism such that $\partial_{i,\text{in}} M$ and $\partial_{i,\text{out}} M$ are trivial in the sense that there are $(n-k-1+i)$-dimensional $(i-1)$-bordisms $M_{i,\text{in}}$ and $M_{i,\text{out}}$ such that there are homeomorphisms
  $$\partial_{i,\text{in}} M \simeq M_{i,\text{in}} \times [0,1]^{k-i} \quad \text{and} \quad \partial_{i,\text{out}} M \simeq M_{i,\text{out}} \times [0,1]^{k-i}$$
  for $1 \leq i \leq n-1$.

**Remark 8.2.** For $k = 2$ our definition of 2-bordism agrees with that in [SP09]. One should think of $M_{i,\text{in}}$ and $M_{i,\text{out}}$ as the $i$-source and $i$-target of $M$.

**Example 8.3.** An example of a 2-dimensional 2-bordism is illustrated in the following picture.

![Diagram of a 2-dimensional 2-bordism]

Its tuple $(\partial_1 M, \partial_2 M)$ of faces is given by the the vertical and the horizontal faces, respectively.

**Example 8.4.** Let $M = [0,1]^k$. It is a $k$-bordism with

$$\partial_{i,\text{in}} M = [0,1]^{i-1} \times \{0\} \times [0,1]^{k-i} \quad \text{and} \quad \partial_{i,\text{out}} M = [0,1]^{i-1} \times \{1\} \times [0,1]^{k-i}.$$

Recall the following embedding theorem for $\langle k \rangle$-manifolds.

**Theorem 8.5** (Proposition 2.1.7 in [Lau00]). Any compact $\langle k \rangle$-manifold admits a neat embedding in $\mathbb{R}^k_+ \times \mathbb{R}^m$ for some $m$.

Any $\langle k \rangle$-manifold admits a compatible collaring:
Lemma 8.6 (Lemma 2.1.6 in [Lau00]). For $a \in 2^k$ we write $1 - a = (1, \ldots, 1) - a$. Any \langle k \rangle -manifold $M$ admits a \langle k \rangle -diagram $C$ of embeddings
\[ C(a < b) : \mathbb{R}_+^k(1 - a) \times M(a) \hookrightarrow \mathbb{R}_+^k(1 - b) \times M(b) \]
with the property that $C(a < b)$ restricted to $\mathbb{R}_+^k(1 - b) \times M(a)$ is the inclusion map $\text{id} \times M(a < b)$.

We need to adapt the definition of a neat embedding for \langle k \rangle -manifolds for bordisms.

Definition 8.7. A neat embedding $\iota$ of a (cubical) $k$-bordism $M$ is a natural transformation of $M$ to $\mathbb{R}^m \times [0, 1]^k$ for some $m$, both viewed as functors $2^k \to \mathcal{T}$, such that

1. $\iota(a)$ is an inclusion of a submanifold for all $a \in 2^k$ respecting the prescribed decomposition of the faces of the bordism

2. the intersections $M(a) \cap (\mathbb{R}^m \times [0, 1]^k(b)) = M(b)$ are transversal for all $b < a$.

Now the embedding theorem for \langle k \rangle -manifolds leads to an embedding theorem for (cubical) $k$-bordisms.

Theorem 8.8. Any $n$-dimensional (cubical) $k$-bordism $M$ admits a neat embedding into $\mathbb{R}^m \times [0, 1]^k$.

Proof. Let $M$ be an $n$-dimensional $k$-bordism. We will use that the product of an embedding with any smooth map still is an embedding.

By the above theorem, there is a (neat) embedding $\iota : M \hookrightarrow \mathbb{R}_+^k \times \mathbb{R}^m' \subset \mathbb{R}^{k+m'} = \mathbb{R}^m$ for some $m'$ and $m = m' + k$. We will construct a smooth map $h : M \to [0, 1]^k$ such that its product with $\iota$ is a neat embedding.

Note that $[0, 1]^k$ is a $k$-bordism. We fix a collaring, e.g. the one given by the embeddings determined by diffeomorphisms
\[ \mathbb{R}_+^\alpha \times [0, 1]^{k-\alpha} \cong \left[ 0, \frac{1}{6} \right] \alpha \times [0, 1]^{k-\alpha} \]

Let $a = (a_i) \in 2^k$. Denote by $|a| = \sum a_i$ and $S(a) = \{ i : a_i = 0 \} \subset \{ 1, \ldots, k \}$. Note that $\lvert S(a) \rvert = k - |a|$. By the above lemma, there is a collaring of the \langle k \rangle -manifold $M$. The collaring gives an embedding $C(a < 1) : \mathbb{R}_+^{|S(a)|} \times M(a) \hookrightarrow M(1 - 0) = M$ whose image is a neighborhood $U(a)$ of the corner $M(a)$. The decompositions $\hat{\partial}_i M = \hat{\partial}_{i, \text{in}} M \sqcup \hat{\partial}_{i, \text{out}} M$ give a decomposition of $M(a)$ into $2^{k-|a|}$ disjoint components:
\[ M(a) = \bigsqcup_{i : a_i \neq 0} \hat{\partial}_i M = \bigsqcup_{i : a_i \neq 0} \hat{\partial}_{i, \text{in}} M \sqcup \hat{\partial}_{i, \text{out}} M = \bigsqcup_{a \in 2^{|S(a)|}} M(a, \alpha), \]
i.e. an element $c \in M(a)$ lies in $M(a, \alpha)$ if and only if
\[ \alpha_i = \begin{cases} 0, & c \in \hat{\partial}_{i, \text{in}} M \\ 1, & c \in \hat{\partial}_{i, \text{out}} M \end{cases} \]

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This decomposition also determines a decomposition of $U(a)$ into $2^{k-|a|}$ disjoint components $U(a, \alpha)$ for $\alpha \in 2^S(a)$ such that $U(a, \alpha)$ is the image of $\mathbb{R}^*_+ \times M(a, \alpha)$ under $C(a < 1)$. The chosen collarling of $[0, 1]^k$ induces one on $[0, 1]^S(a)$, which in turn determines an embedding $t_\alpha : \mathbb{R}^*_+ \to [0, 1]^S(a)$ for any particular corner $\alpha \in 2^S(a) \subset [0, 1]^S(a)$. Note that the images of these embeddings for varying corners are disjoint. We define $h_\alpha$ on $U(a, \alpha)$ to be the composition

$$U(a, \alpha) \cong \mathbb{R}^*_+ \times M(a, \alpha) \xrightarrow{pr} \mathbb{R}^*_+ \xrightarrow{t_\alpha} [0, 1]^S(a).$$

For $a = 0 \in 2^k$, i.e the lowest $(=n-k)$-dimensional corners of $M$, the function $h_0 : U(0) \to [0, 1]^k$ is the restriction $h_{U(0)}$ to $U(0)$ of the desired function $h$.

For $a > 0$, assume $h$ is already defined on $U(b)$ for $b < a$ with $|b| = |a| - 1$. Fix $\alpha \in 2^S(a)$ and let $\beta \in 2^S(b)$ such that $\beta_i = \alpha_i$ for every $i \in S(a) \subset S(b)$. Then $U(b, \beta) \subset U(a, \alpha)$. Since $U(b, \beta)$ are disjoint for varying $\beta$ and the collarings restrict compatibly, we can choose a smooth function $h_{\beta, \alpha} : U(a, \alpha) \to [0, 1]^{(1, \ldots, k)}(a)$ such that the product with $h_\alpha$ agrees with $h$ on $U(b, \beta)$ for all such $\beta$. This defines a smooth map $h : M \to [0, 1]^k$.

We claim that the product $\iota \times h : M \hookrightarrow \mathbb{R}^m \times [0, 1]^k$ is a neat embedding of $k$-bordisms. The first condition is fulfilled by construction, as $h$ is defined so that $M(a, \alpha)$ is sent to $\alpha(c) \in [0, 1]^S(a)$. For the second condition note that by construction, $M(a, \alpha) = h_{\alpha, 1}^{-1}(\alpha)$, $M(b, \beta) = h_{\beta, 1}^{-1}(\beta)$, and $h_0 = h_\alpha \times h_{\beta, \alpha}$ on $U(b, \beta)$. But on $U(b, \beta)$ the function $h_{\beta, \alpha}$ is a projection onto the extra collar coordinate, with $M(b, \beta)$ the preimage of $0 \in \mathbb{R}^*_+$. Thus the intersection is transversal.

Conversely,

**Proposition 8.9.** For $l \geq -n$ and $d = n + l$ any element in $(\text{PBord}^l_k)_{k_1, \ldots, k_d}$ leads to a $(k_1, \ldots, k_d)$-fold composition of $n$-dimensional d-bordisms.

**Proof.** Let $(M \subset V \times B(\overline{\mathcal{I}}), \overline{\mathcal{I}})$ be an element in $(\text{PBord}^l_k)_{k_1, \ldots, k_d}$. As usual, we use the notation $\pi : M \hookrightarrow V \times B(\overline{\mathcal{I}}) \to B(\overline{\mathcal{I}})$. Then for $(1 \leq j_i \leq k_i)_{1 \leq i \leq d}$ define

$$M_{j_1, \ldots, j_d} = \pi^{-1}\left(\prod_{i} \left[\frac{2a_{j_i}^i - b_{j_i - 1}^i}{3}, \frac{2a_{j_i}^i + b_{j_i}^i}{3}\right]\right).$$

They are $n$-dimensional cubical d-bordisms since they are manifolds with corners with a decomposition of the boundary given by the preimages of the faces of the cube, similarly to Example [53].

$$\partial_{\text{in}, \text{in}} M_{j_1, \ldots, j_d} = \pi^{-1}\left(\prod_{i} \left[\frac{2a_{j_i}^i - b_{j_i - 1}^i}{3}, \frac{2a_{j_i}^i + b_{j_i}^i}{3}\right] \times \left(\frac{2a_{j_i}^i - b_{j_i - 1}^i}{3}, \frac{2a_{j_i}^i + b_{j_i}^i}{3}\right)\right)$$

and

$$\partial_{\text{out}, \text{in}} M_{j_1, \ldots, j_d} = \pi^{-1}\left(\prod_{i} \left[\frac{2a_{j_i}^i - b_{j_i - 1}^i}{3}, \frac{2a_{j_i}^i + b_{j_i}^i}{3}\right] \times \left(\frac{a_{j_i}^i + 2b_{j_i}^i}{3}, \frac{a_{j_i}^i + 2b_{j_i}^i}{3}\right)\right).$$

The triviality condition to be a d-bordism follows from condition [3] in Definition [61]. Note that we essentially extracted the underlying $k$-bordisms from Remarks [5,2] and [5,16].

Moreover, they are composable along the faces in the sense that $\partial_{\text{in}, \text{out}} M_{j_1, \ldots, j_{i-1}, j_i, \ldots, j_d}$ and $\partial_{\text{in}, \text{in}} M_{j_1, \ldots, j_d}$ can be glued along their collar to form a new $k$-bordism given by

$$\pi^{-1}\left(\prod_{i'} \left[\frac{2a_{j_i}^i - b_{j_i - 1}^i}{3}, \frac{2a_{j_i}^i + b_{j_i}^i}{3}\right] \times \left(\frac{2a_{j_i}^i - b_{j_i - 1}^i}{3}, \frac{2a_{j_i}^i + b_{j_i}^i}{3}\right)\right).$$

□
8.2. A time-dependent Morse lemma

We have already seen in Remarks 5.2 and 5.16, and in Corollary 5.5 and Proposition 8.9, that the Morse lemma allows interpreting an element in \( PBord_{1}^{k_1} \cdots k_n \) as a composition of \( k_1 \cdots k_n \) bordisms. In this section we will see that paths in that space lead to diffeomorphisms of the composed bordisms and remark on why this space should be thought of as the classifying space thereof.

The following theorem is classical Morse lemma, as can be found e.g. in [Mil63].

**Theorem 8.10 (Morse lemma).** Let \( f \) be a smooth proper real-valued function on a manifold \( M \). Let \( a < b \) and suppose that the interval \( [a, b] \) contains no critical values of \( f \). Then \( M^a = f^{-1}((-\infty, a]) \) is diffeomorphic to \( M^b = f^{-1}((-\infty, b]] \).

We repeat the proof here since later on in this section we will adapt it to the situation we need.

**Proof.** Choose a metric on \( M \), and consider the vector field

\[ V = \frac{\nabla_y f}{|\nabla_y f|^2}, \]

where \( \nabla_y \) is the gradient on \( M \). Since \( f \) has no critical value in \( [a, b] \), \( V \) is defined in \( f^{-1}((-\infty, a - \epsilon]) \), for suitable \( \epsilon \).

Choose a smooth function \( g : \mathbb{R} \to \mathbb{R} \) which is 1 on \( (a - \epsilon, b + \epsilon) \) and compactly supported in \( (a - \epsilon, b + \epsilon) \). Extend \( g \) to a function \( g : M \to \mathbb{R} \) by setting \( g(y) = g(f(y)) \). Then

\[ V = g \frac{\nabla_y f}{|\nabla_y f|^2} \]

is a compactly supported vector field on \( M \) and hence generates a 1-parameter group of diffeomorphisms

\[ \psi_t : M \to M. \]

Viewing \( f - (a + t) \) as a function on \( \mathbb{R} \times M, (t, y) \mapsto f(y) - (a + t), \) we find that in \( f^{-1}((-\infty, b + \epsilon)) \),

\[ \psi_t(f - (a + t)) = 1 = \frac{\nabla_y f}{|\nabla_y f|^2} \cdot (f - (a + t)) = V \cdot (f - (a + t)), \]

and so the flow preserves the set

\[ \{(t, y) : f(y) = a + t\}. \]

Thus, the diffeomorphism \( \psi_{b-a} \) restricts to a diffeomorphism

\[ \psi_{b-a}|_{M^a} : M^a \to M^b. \]

\( \square \)

In Lemma 3.1 in [GWW] Gay, Wehrheim, and Woodward prove a time-dependent Morse lemma which shows that a smooth family of composed cobordisms in their (ordinary) category of (connected) cobordisms gives rise to a diffeomorphism which intertwines with the cobordisms. We adapt this lemma to a variant which will be suitable for our situation in the higher categorical setting.

We start by defining some rescaling data to compare bordisms with different families of underlying intervals.

**Definition 8.11.** Let \( (I_0(s) \leq \cdots \leq I_k(s)) \to [0, 1] \) be a 1-simplex in \( \text{Int}_k \). A smooth family of strictly monotonically increasing diffeomorphisms

\[ (\varphi_{s,t} : (a_0(s), b_k(s)) \to (a_0(t), b_k(t)))_{s, t \in [\Delta^k]} \]

is said to intertwine with the composed intervals if the following condition is satisfied for every morphism \( f : [m] \to [l] \) in the simplex category \( \Delta \): Let \( |f| : |\Delta^m| \to |\Delta^l| \) be the induced map between standard simplices. For every \( 0 \leq j < k \) such that
• either for every $s \in |f|(|\Delta^m|)$ the intersection $I_j(s) \cap I_{j+1}(s)$ is empty
• or for every $s \in |f|(|\Delta^m|)$ the intersection $I_j(s) \cap I_{j+1}(s)$ contains only one element,

we require that for every $s \in |f|(|\Delta^m|)$,

$$b_j(s) \xrightarrow{\varphi_{s,t}} b_j(t), \quad a_{j+1}(s) \xrightarrow{\varphi_{s,t}} a_{j+1}(t);$$

**Remark 8.12.** Note that it is enough to check this condition for $m \leq l$.

**Theorem 8.13.** Let $(M \subset \mathbb{R}^r \times B(\mathbf{T}) \times [0, 1], \mathbf{T})$ be a 1-simplex in $(\text{PBord}_1^\circ)_{k_1, \ldots, k_n}$. Then

1. for every $1 \leq i \leq n$, there is a smooth family of strictly monotonically increasing diffeomorphisms $\varphi_{s,t}^i : \left(a_0^i(s), b_0^i(s)\right) \to \left(a_0^i(t), b_0^i(t)\right)$ for $s, t \in [0, 1]$ called a rescaling datum such that

   a) $\varphi_{s,s}^i = \text{id}$ for every $s \in [0, 1]$,

   b) $\varphi_{s,t}^i = \varphi_{t,s}^{-1}$ for every $s, t \in [0, 1]$,

   c) $(\varphi_{s,t})_{s,t\in|\Delta^m|}$ intertwines with the composed intervals.

2. there is a smooth family of diffeomorphisms

   $$(\psi_{s,t} : M_s \to M_t)_{s, t \in [0, 1]},$$

such that $\psi_{s,s} = \text{id}_{M_s}$ and $\psi_{s,t} = \psi_{t,s}^{-1}$, which intertwine with the composed bordisms with respect to the product of the rescaling data $\varphi_{s,t} = \left(\varphi_{s,t}^i\right)_{i=1}^n : B(\mathbf{T}(s)) \to B(\mathbf{T}(t))$. By this we mean the following: denoting by $\pi_s$ the composition $M_s \to V \times B(\mathbf{T}(s)) \to B(\mathbf{T}(s))$, for $1 \leq i \leq n$ and $0 \leq j_i, l_i \leq k_i$ let

   $$t_{j_i}^i \in I_{j_i}^i(s) \text{ such that } \varphi_{s,t}(t_{j_i}^i) \in I_{j_i}^i(t), \quad \text{and} \quad t_{l_i}^i \in I_{l_i}^i(s) \text{ such that } \varphi_{s,t}(t_{l_i}^i) \in I_{l_i}^i(t).$$

Then $\psi_{s,t}$ restricts to a diffeomorphism

$$\pi_s^{-1} \left( \prod_{i=1}^n [t_{j_i}^i, t_{l_i}^i] \right) \xrightarrow{\psi_{s,t}} \pi_s^{-1} \left( \prod_{i=1}^n [\varphi_{s,t}(t_{j_i}^i), \varphi_{s,t}(t_{l_i}^i)] \right),$$

i.e. denoting $B = \prod_{i=1}^n [t_{j_i}^i, t_{l_i}^i]$.

\[ \text{description} \]
\[ \psi_{s,t} \pi_s \]
\[ \pi_s^{-1} (\varphi_{s,t}(B)) \pi_s \]
\[ M_s \]
\[ \pi_s^{-1} (\varphi_{s,t}(B)) \pi_s \]
\[ M_t \]
\[ \varphi_{s,t}(B) \]
\[ B(\mathbf{T}(s)) \]
\[ B[\text{description}] \]
\[ \psi_{s,t} \]
Proof. The main strategy of the proof is the same as for the classical Morse lemma. Namely, we will construct a suitable vector field whose flow gives the desired diffeomorphisms. First, we fix the metric on $M$ induced by the restriction of the standard metric on $\mathbb{R}^r \times B(\bar{T}) \times [0, 1]$. Recall from Remark \ref{rem:transversality} the map $M \to [0, 1]$ exhibits $M$ as a trivial fiber bundle, so there is a diffeomorphism $M \cong [0, 1] \times N$ as abstract manifolds. For every $s \in [0, 1]$, the fiber $M_s$ is diffeomorphic to $N$ as abstract manifolds. We fix the metric on $N$ induced by the diffeomorphism $N \cong M_0$, and use the notation $f_s : N \cong M_s \mapsto V \times B(\bar{T}(s)) \to B(\bar{T}(s))$.

For steps 1-3 assume that $l = -(n - 1)$.

**Step 1: disjoint intervals** First assume that for all $0 \leq j \leq k$ and for every $s \in [0, 1]$ we have

$$I_j(s) \cap I_{j+1}(s) = \emptyset.$$ 

Let

$$A_j = \bigcup_{s \in [0, 1]} \{s\} \times f_s^{-1}(a_j(s)) \subset [0, 1] \times N, \quad B_j = \bigcup_{s \in [0, 1]} \{s\} \times f_s^{-1}(b_j(s)) \subset [0, 1] \times N.$$

Now for $0 \leq j \leq k$ consider the vector fields

$$V_j = \left( \partial_s, \partial_x(a_j(s) - f_s) \frac{\nabla_y f_s}{|\nabla_y f_s|^2} \right), \quad W_j = \left( \partial_s, \partial_x(b_j(s) - f_s) \frac{\nabla_y f_s}{|\nabla_y f_s|^2} \right),$$

where $\nabla_y$ is the gradient on $N$. Since $f_s$ has no critical value in $I_j(s)$, the vector fields $V_j$ and $W_j$ are defined on $f^{-1}(U_j)$, where $U_j$ is a neighborhood of $\bigcup_{s \in [0, 1]} \{s\} \times I_j(s)$. Moreover, viewing $a_j : (s, y) \mapsto a_j(s)$ as a function on $[0, 1] \times N$,

$$V_j(f - a_j) = \partial_s(f - a_j) + \partial_x(a_j - f) \frac{\nabla_y f}{|\nabla_y f|^2}(f - a_j) = \partial_s(f - a_j) + \partial_x(a_j - f) = 0,$$

so the vector field $V_j$ is tangent to $A_j$ and similarly, $W_j$ is tangent to $B_j$.

We would now like to construct a vector field $\mathcal{V}$ on $N$ which for every $0 \leq j \leq k$, at $A_j$ restricts to $V_j$ and at $B_j$ restricts to $W_j$, and such that there exists a family of functions $\{c_x : [0, 1] \to \mathbb{R}\}_{x \in I_j(0)}$ such that

- $c_x(0) = x$, $c_x(s) \in I_j(s)$,
- the graphs of $c_x$ for varying $x$ partition $\bigcup_{s \in [0, 1]} \{s\} \times [a_j(s), b_j(s)]$, and
- $\mathcal{V}$ is tangent to $C_x = \bigcup_{s \in [0, 1]} \{s\} \times f_s^{-1}(c_x(s))$.

We will use $c_x$ to define $\varphi_0, s(x) = c_x(s)$ and $\varphi_{s,t} = \varphi_{0,t} \circ \varphi_{0,s}^{-1}$. Moreover, the diffeomorphisms $\psi_{s,t}$ will arise as the flow along $\mathcal{V}$.

Fix smooth functions $g_j, h_j : B(\bar{T}(s))_{s \in [0, 1]} \times [0, 1] \to \mathbb{R}_{\geq 0}$ which satisfy the following conditions:

1. $g_j, h_j$ are compactly supported in $U_j$,
2. $g_j = 1$ in a neighborhood of graph $a_j = \{(s, a_j(s)) : s \in [0, 1]\}$, $h_j = 1$ in a neighborhood of graph $b_j$,
3. $g_j + h_j = 1$ in $\bigcup_{s \in [0, 1]} \{s\} \times I_j(s)$, and the supports of the $g_j + h_j$ are disjoint.

By abuse of notation, extend the functions $g_j, h_j$ to functions $g_j, h_j : [0, 1] \times N \to \mathbb{R}$ by setting $g_j(s, y) := g_j(s, f_s(y))$ and $h_j(s, y) := h_j(s, f_s(y))$. Then consider the following vector field on $f^{-1}(U_j)$:

$$\mathcal{V}_j = \left( \partial_s, (g_j \partial_x(a_j) + h_j \partial_x(b_j) - \partial_x(f)) \frac{\nabla_y f}{|\nabla_y f|^2} \right).$$
This vector field is supported on the support of \( g_j + h_j \) and thus extends to a vector field on \( N \). Note that for \((s, y) \in A_j, V_j(s, y) = V_j(s, y)\), and for \((s, y) \in B_j, V_j(s, y) = W_j(s, y)\).

Now let \( V \) be the vector field on \([0, 1] \times N\) constructed by combining the above vector fields as follows:

\[
V = \left( \hat{c}_s, \sum_{0 \leq j \leq k} (g_j \hat{c}_s(a_j) + h_j \hat{c}_s(b_j) - \hat{c}_s(f)) \frac{\nabla f_s}{|\nabla f_s|^2} \right).
\]

Note that in \( \bigcup_{s \in [0,1]} \{s \} \times f_s^{-1}(I_j(s)) \), it restricts to \( V_j \).

In order for \( V \) to be tangent to \( C_x \), the functions \( c_x \) must satisfy the following equation at points in \( C_x \).

\[
0 = V_j(f - c_x) = \hat{c}_s(f - c_x) + (g_j \hat{c}_s(a_j) + h_j \hat{c}_s(b_j) - \hat{c}_s(f)) \frac{\nabla f_s}{|\nabla f_s|^2} (f - c_x) = -\hat{c}_s(c_x) + g_j \hat{c}_s(a_j) + h_j \hat{c}_s(b_j).
\]

This leads to the ordinary differential equation with smooth coefficients on \([0, 1],\]

\[
\hat{c}_s(c_x)(s) = g_j(s, c_x(s)) \hat{c}_s(a_j)(s) + h_j(s, c_x(s)) \hat{c}_s(b_j)(s),
\]

\[
c_x(0) = x.
\]

By Picard-Lindelöf, it has a unique a priori local solution. To see that it extends to \( s \in [0, 1],\) consider the smooth function \( F : [0, 1] \times N \to [0, 1] \times B(\ell(s)) \), defined to be \( \pi \) under the diffeomorphism \( M \cong [0, 1] \times N, \) so \( F(s, y) = (s, f(s, y)) = (s, f_s(y)). \) Since \( \pi \) is proper, so is \( F. \) Moreover, \( C_x = F^{-1}(\text{graph } c_x). \) For fixed \( x, \) graph \( c_x \) sits inside the support of \( g_j + h_j, \) for some \( j, \) and thus is compact in \([0, 1] \times B(\ell(s)) \). Therefore \( C_x \) lies in a compact part of \([0, 1] \times N \) and thus the local solution exists for all \( s \in [0, 1].\)

We now define our rescaling data essentially by following the curve \( c_x. \) Explicitly, let \( \varphi_{0,s} : B(\ell(0)) \to B(\ell(s)) \) be defined on \([a_j(0), b_j(0)]\) by sending \( x_0 \) to \( c_{\varphi_{0}}(s). \) Note that by construction, it sends \( a_j(0), b_j(0) \) to \( a_j(s), b_j(s). \) Since the solution \( c_x \) of the ODE varies smoothly with respect to the initial value \( x \) this map is a diffeomorphism. So we can define \( \varphi_{s,t} : B(\ell(s)) \to B(\ell(t)) \) on \([a_j(s), b_j(s)]\) by sending \( x_s = c_{\varphi_{0}}(s) \) to \( c_{\varphi_{s,t}}(t). \) We extend \( \varphi_{s,t} \) to a diffeomorphism in between these intervals in the following way. Let \( \bar{g}_j, \bar{h}_j : [b_j(0), a_{j+1}(0)] \to \mathbb{R} \) be a partition of unity such that \( \bar{g}_j \) is strictly decreasing, \( \bar{g}_j(b_j(s)) = 1, \) and \( \bar{h}_j(a_{j+1}(s)) = 1. \) Then, for \( x_0 \in [b_j(0), a_{j+1}(0)] \) set

\[
c_{\varphi_{s,t}}(s) = \bar{g}_j(x_0)c_{\bar{b}_j(0)}(s) + \bar{h}_j(x_0)c_{\bar{a}_{j+1}(0)}(s) \quad \text{and} \quad \varphi_{s,t}(c_{\varphi_{0}}(s)) = c_{\varphi_{s,t}}(t).
\]

As mentioned above, we obtain the diffeomorphisms \( \psi_{s,t} \) by flowing along the vector field \( V. \) Since \( V \) is tangent to the sets \( C_x = \bigcup_{s \in [0,1]} \{s \} \times f_s^{-1}(c_x(s)) \) for \( x \in I_0(0) \cup \cdots \cup I_k(0), \) the flow preserves \( C_x, \) and \( \bigcup_{s \in [0,1]} \{s \} \times f_s^{-1}([b_j(s), a_{j+1}(s)]) \) in between. Again, this implies that the flow exists for all \( s \in [0, 1]. \)

It is of the form \( \Psi(t - s, (s, y)) = (t, \psi_{s,t}(y)) \) for \( 0 \leq s \leq t \leq 1, \) where \( (\psi_{s,t})_{s,t \in [0,1]} \) is a family of diffeomorphisms on \( N. \) We transport them under the diffeomorphism \( M \cong [0, 1] \times N \) to diffeomorphisms \( (\psi_{s,t} : M_s \to M_t)_{s,t \in [0,1]}, \) which by construction intertwine with the composed bordisms with respect to the rescaling data \( \varphi_{s,t}. \)

**Step 2: common endpoints** Now consider the case that for \( 0 \leq j \leq k \) we have that either for every \( s \in [0, 1], \) \( I_j(s) \cap I_{j+1}(s) = \emptyset \) as in the previous case or for every \( s \in [0, 1], \) we have

\[
|I_j(s) \cap I_{j+1}(s)| = 1.
\]

In this case, one can modify the above argument. We explain for the case of two intervals with one common endpoint, i.e. \( b_j(s) = a_{j+1}(s). \)
Instead of choosing smooth functions \( g_j, h_j, g_{j+1}, h_{j+1} : [0, 1] \times B(I(s))_{s \in [0, 1]} \to \mathbb{R} \) such that the supports of \( g_j + h_j \) and \( g_{j+1} + h_{j+1} \) are disjoint (which now is not possible), we fix three smooth functions \( f_j, g_j, h_j : [0, 1] \times B(I(s))_{s \in [0, 1]} \to \mathbb{R} \) which satisfy the following conditions:

1. \( f_j, g_j, h_j \) are compactly supported in \( U_j \cup U_{j+1} \),
2. \( f_j = 1 \) in a neighborhood of graph \( a_j = \{(s, a_j(s)) : s \in [0, 1]\} \),
   \( g_j = 1 \) in a neighborhood of graph \( b_j \) of \( a_j \),
   \( h_j = 1 \) in a neighborhood of graph \( b_{j+1} \),
3. \( f_j + g_j + h_j = 1 \) in \( \bigcup_{s \in [0, 1]} (s \times (I_j(s) \cup I_{j+1}(s))) \), and the support of the \( f_j + g_j + h_j \) is disjoint to the sums associated to the other intervals.

Now continue the proof similarly to above.

**Step 3: overlapping intervals**  It remains to consider the case when for some \( 0 \leq j \leq k \) and some \( s \in [0, 1] \),

\[ I_j(s) \cap I_{j+1}(s) \]

has non-empty interior.

*Intervals always overlap.* First, if \( I_j(s) \cap I_{j+1}(s) \) has non-empty interior for every \( s \in [0, 1] \), then one can do the above construction with the intervals \( I_j(s), I_{j+1}(s) \) replaced by the interval \( I_j(s) \cup I_{j+1}(s) \).

*Intervals do not always overlap.* If \( I_j(s) \cap I_{j+1}(s) \) sometimes has non-empty interior, but not for every \( s \in [0, 1] \), we can combine the cases treated so far.

We explain the process in the case that there is an \( \tilde{s} \) such that for \( s < \tilde{s} \), \( I_j(s) \cap I_{j+1}(s) = \emptyset \) and for \( s \geq \tilde{s} \), \( I_j(s) \cap I_{j+1}(s) \neq \emptyset \). In this case, \( \tilde{x} = b_j(\tilde{s}) = a_{j+1}(\tilde{s}) \), which is a regular value of \( f \). Since \( f \) is smooth, there is an open ball \( U_j \) centered at \( (\tilde{s}, \tilde{x}) \) in \( [0, 1] \times B(I(s))_{s \in [0, 1]} \) such that for \( (s, x) \in U \), \( x \) is a regular value of \( f \). Let \( \tilde{s} < \tilde{s} \) be such that for every \( s \leq \tilde{s} \), the set \( \{s\} \times [a_j(s), b_{j+1}(s)] \) is covered by \( U \cup \{(s) \times (I_j(s) \cup I_{j+1}(s))\} \). Choose \( s_0 \) and \( t_0 \) such that \( \tilde{s} < s_0 < t_0 \).

\[
\begin{array}{c}
\tilde{s}_j(0) & \tilde{a}_{j+1}(0) \\
\tilde{a}_j(0) & \tilde{a}_{j+1}(0) \\
\tilde{a}_j(1) & \tilde{a}_{j+1}(1) \\
\end{array}
\]

In \([0, t_0]\), we are in the situation of disjoint intervals and can use the first construction to obtain \( c^{(2)}_x(s) \) and \( \mathcal{V}^{(2)}(s, y) \) for \( s \leq t_0 \).

In \([s_0, 1]\), we apply the construction from step 1 to the intervals \( I_j(s) \) and \( I_{j+1}(s) \) replaced by the interval \([a_j(s), b_{j+1}(s)]\) to obtain \( c^{(2)}_x(s) \) and \( \mathcal{V}^{(2)}(s, y) \) for \( s \geq s_0 \).

Now choose a partition of unity \( G, H : [0, 1] \to \mathbb{R} \) such that \( G|_{[0, s_0]} = 1, H|_{[t_0, 1]} = 1 \), and \( G \) is strictly decreasing on \([s_0, t_0]\). For \( s < t \) define

\[
c_x(s) = G(s)c^{(1)}_x(s) + H(s)c^{(2)}_x(s), \quad \mathcal{V}(s, y) = G(s)\mathcal{V}^{(1)}(s, y) + H(s)\mathcal{V}^{(2)}(s, y).
\]

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Then define $\varphi_{s,t}$ and $\psi_{s,t}$ as before.

**Step 4: several directions** Assume now that $l > -(n - 1)$. Let

$$\pi_s : N \cong M_s \hookrightarrow V \times B(T(s)) \rightarrow B(T(s))$$

and or $1 \leq i \leq n$ denote by $(p_i)_s : N \rightarrow B(I^s(s))$ the composition of $\pi_s$ with the projection to the $i$th coordinate. Note that by condition 3 in Definition 5.11 the function $(p_i)_s$ does not have a critical point in $I^s(s) \cup \ldots \cup I^s_k(s)$.

By steps 1-3 for each $i$ we got a vector field

$$\mathcal{V} = \left( \xi_i, \Pi_i(s, y) \frac{\nabla_y (p_i)_s}{|\nabla_y (p_i)_s|^2} \right),$$

e.g. see [4]. We combine them to obtain a new vector field on $[0, 1] \times N$ given by

$$\tilde{\mathcal{V}} = \left( \xi_i, \sum_{i=1}^n \Pi_i(s, y) \frac{\nabla_y (p_i)_s}{|\nabla_y (p_i)_s|^2} \right).$$

For $i \neq j$ the projections $(p_i)_0$ and $(p_j)_0$ are orthogonal with respect to the metric on $N$ and moreover, $(p_i)_s$, $(p_j)_s$ stay orthogonal along the path, because the change of metric on $N \cong M_s$ induced by the embedding of $M_s$ respects orthogonality on $B(T)$. This implies that

$$\frac{\nabla_y (p_i)_s}{|\nabla_y (p_i)_s|^2} p_j = \delta_{ij},$$

and so $\tilde{\mathcal{V}}$ still is tangent to the respective $C^1$ in each direction and thus its flow, if it exists globally, will give rise to the desired diffeomorphisms and rescaling data.

The global existence follows from the special form of the vector field. Given a point $(t, y_t) \in N$, the flow will preserve a set of the form

$$\{ (s, y) : \pi_s(y_s) = (c^1_{x_0}(s), \ldots, c^n_{x_0}(s)) = (\xi_1(s), \ldots, \xi_n(s)) \},$$

where the right hand side is in the notation of Example 5.11 and $\xi_{x_0}(t) = \xi(t) = y_t$. One can show, as in the example, that this set lies in a compact part of $N$ and thus the flow exists globally.

**Remark 8.14.** Building upon the previous section, in particular Proposition 8.9 we could define diffeomorphisms of a $(k_1, \ldots, k_d)$-fold composition of $n$-dimensional $d$-bordisms to be diffeomorphisms of the composition which “intertwine with”, i.e. they restrict to, the composed bordisms. Then the above theorem shows that a path in $(\text{P Bord}^n_{\text{d}})_{k_1, \ldots, k_d}$ leads to such an intertwining diffeomorphism of the compositions at the start at the end of the path.

Actually, much more is true: for fixed $k_1, \ldots, k_d$ and a $(k_1, \ldots, k_d)$-fold composition $M$ of $n$-dimensional $d$-bordisms $M_{j_1, \ldots, j_d}$ which we denote by $(M, (M_{j_1, \ldots, j_d}))$, consider the group of such intertwining diffeomorphisms $\text{Diff}(M, (M_{j_1, \ldots, j_d}))$. Then $(\text{P Bord}^n_{\text{d}})_{k_1, \ldots, k_n}$ is the classifying space of $\text{Diff}(M, (M_{j_1, \ldots, j_d}))$.

We sketch the argument, essentially following the one for showing that $\text{Sub}(M, \mathbb{R}^\infty)$ is a classifying space for the group of diffeomorphisms of $M$, and its modifications in [GTMW09] and [Lur09c]. Consider the space $\text{Emb}((M, (M_{j_1, \ldots, j_d})), \mathbb{R}^\infty \times [0, 1]^d)$ of neat embeddings of the composition which restricts to neat embeddings of the composed bordisms. It is non-empty by the embedding theorem for $d$-bordisms (Theorem 8.8) and contractible, which can be seen similarly to $\text{Emb}(M, \mathbb{R}^\infty)$ being contractible. We get a principle $\text{Diff}(M, (M_{j_1, \ldots, j_d}))$-bundle

$$\text{Emb}((M, (M_{j_1, \ldots, j_d})), \mathbb{R}^\infty \times [0, 1]^d) \rightarrow \text{Emb}((M, (M_{j_1, \ldots, j_d})), \mathbb{R}^\infty \times [0, 1]^d)/\text{Diff}(M, (M_{j_1, \ldots, j_d}))$$

where the right hand side is equivalent to $(\text{P Bord}^n_{\text{d}})_{k_1, \ldots, k_n}$.
8.3. The homotopy category \( h_1(\text{Bord}_n^{(\infty,1)}) \)

8.3.1. The symmetric monoidal structure on \( h_1(\text{Bord}_n^{(\infty,1)}) \)

The \( \text{Bord}_n^{(\infty,1)} \) is an \((\infty,1)\)-category with a symmetric monoidal structure defined in two ways similarly to that of Bord_1. Both induce a symmetric monoidal structure on the homotopy category \( h_1(\text{Bord}_n^{(\infty,1)}) \). We now make this symmetric monoidal structure more explicit for later purposes.

**coming from a \( \Gamma \)-object** We can either obtain the symmetric monoidal structure as a \( \Gamma \)-object on \( \text{Bord}_n^{(\infty,1)} \approx L_{n-1}(\text{Bord}_n) \) by iterating the construction of the symmetric monoidal structure on the looping from Example 3.3 or by constructing a functor from an assignment \([m] \mapsto \text{Bord}_{n}^{(-1)}[m] \) as mentioned in Remark 7.3. In the second case, \( \text{Bord}_{n}^{(-1)}[m] \) arises, similarly to \( \text{Bord}_n[m] \), from the spaces \( (\text{PBord}_n^{(-1)}[m])_{k_1,\ldots,k_n} \), which as a set is the collection of tuples

\[
(M_1, \ldots, M_m, (I_0 \leq \ldots \leq I_k)),
\]

where \( M_1, \ldots, M_m \) are disjoint \( n \)-dimensional submanifolds of \( V \times B(\mathcal{L}) = (a_0, b_k) \), and each \( (M_\beta, (I_0 \leq \ldots \leq I_k)) \in (\text{PBord}_n^{(-1)})_{k_1,\ldots,k_n} \).

We saw in Example 8.7 that a \( \Gamma \)-object endows the homotopy category of its underlying Segal space with a symmetric monoidal structure. Explicitly, in the second case, it comes from the following maps.

\[
\begin{array}{c}
\text{Bord}_n^{(-1)}[1] \times \text{Bord}_n^{(-1)}[1] & \xleftarrow{\simeq} & \text{Bord}_n^{(-1)}[2] \\
(M_1, \mathcal{L}, (M_2, \mathcal{L})) & \xleftarrow{\gamma_1 \times \gamma_2} & (M_1, M_2, \mathcal{L}) & \xrightarrow{\gamma} & (M_1 \amalg M_2, \mathcal{L}),
\end{array}
\]

**coming from a tower** To understand the symmetric monoidal structure on \( h_1(\text{Bord}_n^{(\infty,1)}) \) coming from a symmetric monoidal structure as a tower, we use that \( \text{Bord}_n^{(\infty,1)} = \text{Bord}_n^{(-1)} \) has a symmetric monoidal structure coming from the collection of \( k \)-hybrid \((k+1)\)-fold Segal spaces given (essentially) by the \( k \)-hybrid completion of

\[
\text{PBord}_n^{(-1)} / \emptyset
\]

This symmetric monoidal structure induces one on the homotopy category \( h_1(\text{Bord}_n^{(\infty,1)}) \approx h_1(\text{Bord}_n^{(-1)}) \).

Since completion is a Dwyer-Kan equivalence, see 1.19, it is enough to understand the symmetric monoidal structure on \( h_1(\text{PBord}_n^{(-1)}) \).

The monoidal structure arises from composition in \( \text{PBord}_n^{(-1)} \), the next layer of the tower \( \text{PBord}_n^{(-1)} \) gives a braiding and the higher layers show that it is symmetric monoidal, see Section 8.3. Consider the diagram

\[
\begin{array}{c}
(\text{PBord}_n^{(-1)})_1, \bullet & \times \hspace{1cm} & (\text{PBord}_n^{(-1)})_1, \bullet \\
(\text{PBord}_n^{(-1)})_0, \bullet & \xleftarrow{h} & (\text{PBord}_n^{(-1)})_0, \bullet \\
& \xleftarrow{\simeq} & (\text{PBord}_n^{(-1)})_2, \bullet \\
& \xrightarrow{d_1} & (\text{PBord}_n^{(-1)})_1, \bullet,
\end{array}
\]

which induces

Similarly to Remark 8.16 we find that

\[
(\text{PBord}_n^{(-1)} / \emptyset)_1, \bullet = L(\text{PBord}_n^{(-1)}), \bullet \approx (\text{PBord}_n^{(-1)}), \bullet
\]

and together with the maps above this gives a monoidal structure

\[
\begin{align*}
\text{h}_1(\text{PBord}_n^{(-1)}) \times \text{h}_1(\text{PBord}_n^{(-1)}) & \longrightarrow \text{h}_1(\text{PBord}_n^{(-1)}).
\end{align*}
\]
on \( h_1(\text{PBord}_n^{(n-1)}) \). We spell this structure out explicitly. Consider two objects or 1-morphisms represented by elements
\[
(M) = (M \subseteq V \times B(\underline{L}), \underline{L}), \quad (N) = (N \subseteq W \times B(\tilde{L}), \tilde{L})
\]
in \((\text{PBord}_n^{(n-1)})_k\) for \( k = 0 \) or \( k = 1 \). Without loss of generality we can assume that \( V = W = V_d \), that \((M), (N) \in (\text{PBord}_n^{n+1, bd})_k\), and that (perhaps after rescaling) \( \underline{L} = \tilde{L} \).

Under the map \( \ell^d : (\text{PBord}_n^{(n-1), bd}) \to L((\text{PBord}_n^{(n-1), bd}) \times (\text{PBord}_n^{(n-1)})) \) from Proposition 5.20 \((M)\) and \((N)\) are sent to
\[
(M_1) = (M \subseteq V_{d-1} \times (a_0^1, b_1) \times B(\underline{L}), (a_0^1, b_1) \leq [a_1^1, b_1^1), \underline{L}),
\]
\[
(N_1) = (N \subseteq V_{d-1} \times (\tilde{a}_0^1, \tilde{b}_1) \times B(\tilde{L}), (\tilde{a}_0^1, \tilde{b}_1) \leq [\tilde{a}_1^1, \tilde{b}_1^1), \tilde{L}).
\]

We now choose a path from \((N_1)\) to another element \((N_2)\) by moving and rescaling the first coordinate such that the pair \(((M), (N_2))\) lies in
\[
(\text{PBord}_n^{(n-1)}_{1, *}) \times (\text{PBord}_n^{(n-1)}_{1, *})
\]
i.e. such that
\[
(N_2) = (N \subseteq V_{d-1} \times (a_0^1, \beta) \times B(\underline{L}), (a_0^1, b_1) \leq [a_1^1, b_1^1) \leq [\alpha, \beta), \underline{L}).
\]

Now we can use the gluing map analogous to the one from the proof of the Segal condition for \( \text{PBord}_n \) in Proposition 5.20. Since \( d_1^1(\underline{M_1}) = d_0^1((\tilde{N}_1)) = \emptyset \) in this case \( M \) and \( N_2 \) are disjoint as submanifolds of \( V_{d-1} \times (a_0^1, \beta) \times B(\underline{L}) \). Thus its image is
\[
(M \sqcup N \subseteq V_{d-1} \times (a_0^1, \beta) \times B(\underline{L}), (a_0^1, b_1) \leq [a_1^1, b_1^1) \leq [\alpha, \beta), \underline{L})
\]
The third face map \( d_1^1 \) sends it to
\[
(M \sqcup N \subseteq V_{d-1} \times (a_0^1, \beta) \times B(\underline{L}), (a_0^1, b_1) \leq [\alpha, \beta), \underline{L})
\]
which by \( u^d_{bd} : L(\text{PBord}_n^{1, bd}) \to \text{PBord}_n^{bd} \) is sent to
\[
(M \sqcup N \subseteq V_d \times B(\underline{L}), \underline{L})
\]

### 8.3.2. The homotopy category and \( n\text{Cob} \)

Our higher categories of cobordisms give back the ordinary categories of \( n\)-cobordisms, as we see in the following proposition.

**Proposition 8.15.** There is an equivalence of symmetric monoidal categories between the homotopy category of the \((\infty, 1)\)-category \( \text{Bord}_n^{(\infty, 1)} \) and the category of \( n\)-cobordisms,
\[
h_1(\text{Bord}_n^{(\infty, 1)}) \simeq n\text{Cob}.
\]

**Proof.** We first show that there is an equivalence of categories \( h_1(\text{Bord}_n^{(\infty, 1)}) \simeq n\text{Cob} \) and then show that it respects the symmetric monoidal structures.

Rezk’s completion functor is a Dwyer-Kan equivalence of Segal spaces, and thus by definition induces an equivalence of the homotopy categories. So it is enough to show that
\[
h_1(\text{PBord}_n^{(n-1)}) \simeq n\text{Cob}.
\]
We define a functor
\[
F : h_1(\text{PBord}_n^{(n-1)}) \to n\text{Cob}
\]
and show that it is essentially surjective and fully faithful.
We let $F^pV$ be a morphism in $\mathcal{V}$, to $\mathcal{V}$. From this path we can obtain another one which has “shorter” intervals, namely just by shrinking them $p$. We need to check that this assignment is well-defined, i.e. independent of the choice of the representative $F^pV$.

It remains to show that $\phi$ induces a 1-simplex $\tilde{N} = \pi^{-1}\left(\frac{2a_0 + b_0}{3}, \frac{a_1 + 2b_1}{3}\right)$.

This is an $n$-dimensional manifold with boundary $\pi^{-1}\left(\frac{2a_0 + b_0}{3}\right) \cup \pi^{-1}\left(\frac{a_1 + 2b_1}{3}\right)$.

Since $\pi$ only has regular values in $I_0$ and $I_1$, the Morse lemma gives diffeomorphisms

$$\pi^{-1}\left(\frac{2a_0 + b_0}{3}\right) \cong \pi^{-1}\left(\frac{a_0 + b_0}{2}\right) \quad \text{and} \quad \pi^{-1}\left(\frac{a_1 + 2b_1}{3}\right) \cong \pi^{-1}\left(\frac{a_1 + b_1}{2}\right),$$

Thus $F((N))$ is an $n$-dimensional cobordism from the image of the source $F(d_0(N))$ to the image of the target $F(d_1(N))$.

We need to check that this assignment is well-defined, i.e. independent of the choice of the representative of the isomorphism class. Any two representatives $(N_0), (N_1)$ are connected by a path in $(\text{PBord}_n^{(n-1)})_1$. From this path we can obtain another one which has “shorter” intervals, namely just by shrinking them to $(a_0(s), \frac{2a_0(s) + b_0(s)}{3}, \frac{a_1(s) + 2b_1(s)}{3}, b_1(s))$. Now Theorem 13 gives a diffeomorphism $\psi_{0,1} : N_0 \to N_1$ which restricts to a diffeomorphism $\tilde{\psi}_{0,1} : \tilde{N}_0 \to \tilde{N}_1$.

Note that the Morse lemma implies that any image of the degeneracy map in $(\text{PBord}_n^{(n-1)})_1$ is sent to an identity morphism in $\text{nCob}$ and that $F$ behaves well with composition.

The functor is an equivalence of categories Whitney’s embedding theorem shows that $F$ is essentially surjective. Moreover, it is injective on morphisms: Let $t_0 : N_0 \hookrightarrow V \times B(\bar{L})$ and $t_1 : N_1 \hookrightarrow W \times B(\bar{L})$ be embeddings which are representatives of two 1-morphisms $(N_0 \subset V \times B(\bar{L}), \bar{I})$ and $(N_1 \subset W \times B(\bar{L}), \bar{I})$ which have diffeomorphic images. W.l.o.g. we can assume that $V = W$ and $\bar{I} = \bar{I}$. Then there is a diffeomorphism $\psi : N_0 \to N_1$, which can be extended to the rest of the collars, i.e. we get a diffeomorphism $\psi : N_0 \to N_1$. Since $\text{Emb}(N_0, \mathbb{R}^\infty \times B(\bar{L}))$ is contractible, there is a path from $t_0$ to $t_1 \circ \psi$, which induces a 1-simplex $(N \subset V \times B(\bar{L}) \times [0,1], B(\bar{L}))$ in $(\text{PBord}_n^{(n-1)})_1$ such that the fiber at $s = 0$ is $(N_0 \subset V \times B(\bar{L}), \bar{I})$ and the fiber at 1 is $\text{im}(t_1 \circ \psi)(N_0) = N_1 \subset V \times B(\bar{L})$.

It remains to show that $F$ is full. In the case $n = 1, 2$ this is easy to show, as we have a classification theorem for 1- and 2-dimensional manifolds with boundary. In the 1-dimensional case it is enough to show that an open line, the circle and the half-circle, once as a bordism from 2 points to the empty set and once vice versa, lie in the image of the map, which is straightforward. In the two dimensional case, the pair-of-pants decomposition tells us how to embed the manifold.

For general $n$ we first embed the manifold with boundary into $\mathbb{R}^+ \times \mathbb{R}^{2n}$ using a variant of Whitney’s embedding theorem for manifolds with boundary, cf. [Lau00]. Then the boundary of the halfspace is...
Finally, in both cases, any element represented by $\text{pH}$ the element hyperplane, without loss of generality given by the equation $t$ where the images of $M$ we first show that the boundary components can be separated by a hyperplane in $\mathbb{R}^{2n}$. The boundary components are compact, so they can be embedded into (large enough) balls $B^{2n}$. By perhaps first applying a suitable “stretching” transformation, one can assume that these balls do not intersect. Now, since $2n > 1$ we have that the configuration space of these balls $\pi_0(\text{Conf}(B^{2n}, \mathbb{R}^{2n})) \cong *$ is contractible, there is a transformation to a configuration in which the boundary components are separated by a hyperplane, without loss of generality given by the equation $\{x_1 = 0\} \subset \mathbb{R}^{2n}$.

Consider the restriction of the (holomorphic) logarithm function with branch cut $-i\mathbb{R}^+ \to (\mathbb{R}^+ \times \mathbb{R}) \setminus (0, 0)^2 \cong \mathbb{H} : 0 \leq y \leq \pi$. It is a homeomorphism to $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \pi\}$. We can apply $\log \times id_{\mathbb{R}^{2n-1}}$ to $(\mathbb{R}^+ \times \mathbb{R}^1) \times \mathbb{R}^{2n-1}$ and, composing this with a suitable rescaling, obtain an embedding into $(\epsilon, 1-\epsilon) \times \mathbb{R}^{2n}$. Now choose a collaring of the bordism to extend the embedding to $(0, 1) \times \mathbb{R}^{2n}$.

**The functor is a symmetric monoidal equivalence** In the case of the structure coming from a $\Gamma$-object, one can similarly to the previous paragraph define an equivalence of categories $F[m] : h_1\text{Bord}^n_{\text{ext}}(n-1)[m] \longrightarrow n\text{Cob}^m$.

Then one can easily check that the following diagram commutes.

$$
\begin{array}{ccc}
\text{nCob} \times \text{nCob} & \xrightarrow{F \times F} & \text{nCob} \times \text{nCob} \\
\downarrow{\text{F}[m]} & & \downarrow{\text{F}[m]} \\
\text{nCob} & \xrightarrow{\text{F}} & \text{nCob}
\end{array}
$$

For the case of the structure coming from a tower, we explicitly saw that the symmetric structure on $h_1(\text{Bord}^{(\alpha, 1)}_n)$ sends two objects or 1-morphisms determined by $(M) = (M \subseteq V_d \times B(\underline{1}), \underline{1})$, $(N) = (N \subseteq V_d \times B(\underline{1}), \underline{1})$ to $(M \sqcup N) = (M \sqcup N \leftrightarrow V \times B(\underline{1}), \underline{1})$, where the images of $M$ and $N$ lie in disjoint “heights” in the $v_1$-direction in $V_d$. Thus, under the functor $F$ the element $(M \sqcup N)$ is sent to $F((M)) \sqcup F((N))$.

Finally, in both cases, any element represented by $\langle \emptyset, \underline{1} \rangle$ is sent to $\emptyset$. $\square$

**8.4. The homotopy bicategory $h_2(\text{Bord}^{(\alpha, 2)}_n)$ and comparison with $n\text{Cob}^{ext}$**

C. Schommer-Pries defined a symmetric monoidal bicategory $n\text{Cob}^{ext}$ of $n$-dimensional cobordisms in his thesis [SP09]. In this section we show that the homotopy bicategory of our $(\alpha, 2)$-category of $n$-dimensional bordisms is symmetric monoidally equivalent to this bicategory.

**8.4.1. The bicategory $n\text{Cob}^{ext}$**

We first briefly recall the definition of $n\text{Cob}^{ext}$. For details we refer to [SP09].

**Definition 8.16.** The bicategory $n\text{Cob}^{ext}$ has

- (n-2)-dimensional smooth closed manifolds as objects,
• 1-morphisms are \((n - 1)\)-dimensional 1-bordisms between objects, and
• 2-morphisms are isomorphism classes of \(n\)-dimensional 2-bordisms between 1-morphisms,

where

1. a 1-bordism from an object \(Y_0\) to an object \(Y_1\) is an \((n - 1)\)-dimensional 1-bordism, i.e. a smooth compact \((n - 1)\)-dimensional manifold with boundary \(W\), together with a decomposition and isomorphism

\[
\partial W = \partial_{in} W \sqcup \partial_{out} W \cong Y_0 \sqcup Y_1;
\]

2. for two 1-bordisms \(W_0\) and \(W_1\) from \(Y_0\) to \(Y_1\), a 2-bordism from \(W_0\) to \(W_1\) is an \(n\)-dimensional 2-bordism equipped with
   - a decomposition and isomorphism
     \[
     \partial_0 S = \partial_{0,in} S \sqcup \partial_{0,out} S \to W_0 \sqcup W_1,
     \]
   - a decomposition and isomorphism
     \[
     \partial_1 S = \partial_{1,in} S \sqcup \partial_{1,out} S \to Y_0 \times [0, 1] \sqcup Y_1 \times [0, 1].
     \]

3. Two 2-bordisms \(S, S'\) are isomorphic if there is a diffeomorphism \(h : S \to S'\) compatible with the boundary data.

Vertical and horizontal compositions of 2-morphisms are defined by choosing collars and gluing. This is well-defined because 2-morphisms are isomorphism classes of 2-bordisms, and thus the composition does not depend on the choice of the collar. However, composition of 1-morphisms requires the use of a choice of a collar, which requires the axiom of choice, and then composition is defined by the unique gluing. However, this gluing is associative only up to non-canonical isomorphism of 1-bordisms which gives a canonical isomorphism class of 2-bordisms realizing the associativity of horizontal composition in the axioms of a bicategory.

It is symmetric monoidal, with symmetric monoidal structure given by taking disjoint unions.

### 8.4.2. The symmetric monoidal structure on \(h_2(Bord^{(x,2)}_n)\)

The symmetric monoidal structure on \(Bord^{(x,2)}_n\) arising as a \(\Gamma\)-object gives us

\[
Bord^{(x,2)}_n[1] \times Bord^{(x,2)}_n[1] \xrightarrow{\cong} Bord^{(x,2)}_n[2] \to Bord^{(x,2)}_n[1]
\]

which induces

\[
h_2(Bord^{(x,2)}_n) \times h_2(Bord^{(x,2)}_n) \to h_2(Bord^{(x,2)}_n).
\]

This makes \(h_2(Bord^{(x,2)}_n)\) into a symmetric monoidal bicategory, where the associativity follows from the equivalence \(Bord^{(x,2)}_n[3] \to Bord^{(x,2)}_n[1]^x\).

The symmetric monoidal structure on \(h_2(Bord^{(x,2)}_n)\) arising from the tower can be unravelled similarly to in Section 8.3.1 any two objects, 1-morphisms, or 2-morphisms coming from elements

\[
(M \subset V \times B(\overline{T}), I_0^n(\leq I_1^n), I_0^n(\leq I_1^n)), \quad (N \subset V \times B(\overline{T}), I_0^n(\leq I_1^n), I_0^n(\leq I_1^n))
\]

are sent to, essentially,

\[
(M \sqcup V \times B(\overline{T}), I_0^n(\leq I_1^n), I_0^n(\leq I_1^n)),
\]

where \(M\) and \(N\) are embedded into different heights in the \(V\)-direction.

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8.4.3. The homotopy bicategory and $2\text{Cob}^{\text{ext}}$

In this section we show that our $(\infty, 2)$-category of $n$-dimensional cobordisms indeed gives back the bicategory $n\text{Cob}^{\text{ext}}$ as its homotopy bicategory.

**Proposition 8.17.** There is an equivalence of symmetric monoidal bicategories between $h_2(Bord_n^{(\infty, 2)})$ and $n\text{Cob}^{\text{ext}}$.

**Proof.** By Whitehead’s theorem for symmetric monoidal bicategories, see [SP09], theorem 2.21, it is enough to find a functor $F$ which is

1. essentially surjective on objects, i.e. $F$ induces an isomorphism $\pi_0(h_2(Bord_n^{(\infty, 2)})) \cong \pi_0(n\text{Cob}^{\text{ext}})$,

2. essentially full on 1-morphisms, i.e. for every $x, y \in \text{Ob } h_2(Bord_n^{(\infty, 2)})$, the induced functor $F_{x,y} : h_2(Bord_n^{(\infty, 2)})(x, y) \to n\text{Cob}^{\text{ext}}(Fx, Fy)$ is essentially surjective, and

3. fully-faithful on 2-morphisms, i.e. for every $x, y \in \text{Ob } h_2(Bord_n^{(\infty, 2)})$, the induced functor $F_{x,y} : h_2(Bord_n^{(\infty, 2)})(x, y) \to n\text{Cob}^{\text{ext}}(Fx, Fy)$ is fully-faithful.

First of all, recall from [149] that Rezk’s completion functor is a Dwyer-Kan equivalence of Segal spaces, and thus by definition induces an equivalence of the homotopy bicategories. So it is enough to show that $h_2(PBord_n^{(n-2)}) \cong n\text{Cob}^{\text{ext}}$.

**Definition of the functor**

Let $F : h_2(PBord_n^{(n-2)}) \to n\text{Cob}^{\text{ext}}$ be the functor defined as follows:

Consider an object

$$(M) = (M \subseteq V \times (a^1, b^1) \times (a^2, b^2), (a^1, b^1), (a^2, b^2)) \in (PBord_n^{(n-2)})_{0,0}.$$  

Since $\pi : M \to (a^1, b^1) \times (a^2, b^2)$ is proper and submersive, in particular $(\frac{a^1 + b^1}{2}, \frac{a^2 + b^2}{2})$ is a regular value of $\pi$. We define

$$F((M)) = \pi^{-1}\left(\frac{a^1 + b^1}{2}, \frac{a^2 + b^2}{2}\right),$$

which is a closed $(n-2)$-dimensional abstract manifold, i.e. an object in $n\text{Cob}^{\text{ext}}$. Note that because of condition (3) in the definition of $PBord_n^{(n-2)}$, we could have taken the fiber over any other point and would have gotten a diffeomorphic image.

For a 1-morphism

$$(M) = (M \subseteq V \times (a^1_0, b^1_0) \times (a^2, b^2), I_0^1, a^1, (a^2, b^2)) \in (PBord_n^{(n-2)})_{1,0}$$

the map $\pi$ is proper and $p_2 : M \to (a^2, b^2)$ is submersive, so

$$F((M)) = \pi^{-1}\left(\frac{2a^1_0 + b^1_0}{3}, \frac{a^1 + 2b^1_0}{3}\times \frac{a^2 + b^2}{2}\right),$$

is a compact $(n-1)$-dimensional 1-bordism, i.e. a manifold with boundary. Note that $2a^1_0 + b^1_0 < a^1_0 + 2b^1_0 < a^1_0 + 2b^1_0$.  

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Moreover, the decomposition of the boundary of the image is given by
\[
\pi^{-1}\left(\left[\frac{2a_0^1 + b_0^1}{3}, \frac{a_0^1 + b_0^1}{3}\right]\times \left[\frac{a_0^2 + b_0^2}{3}, \frac{a_0^2 + b_0^2}{3}\right]\right)
\cong \pi^{-1}\left(\left[\frac{2a_0^1 + b_0^1}{3}, \frac{a_0^1 + b_0^1}{3}\right]\times \left[\frac{a_0^2 + b_0^2}{3}, \frac{a_0^2 + b_0^2}{3}\right]\right).
\]

where by condition (3) in Definition 5.1 and the Morse lemma we have diffeomorphisms
\[
\pi^{-1}\left(\left[\frac{a_0^1 + b_0^1}{3}, \frac{a_0^1 + b_0^1}{3}\right]\times \left[\frac{a_0^2 + b_0^2}{3}, \frac{a_0^2 + b_0^2}{3}\right]\right),
\]

which are the diffeomorphisms come from applying the Morse lemma to \(p_1\) on \(I_0^1\) and \(I_1^1\).

We send a 2-morphism, which is an element in \(\pi_0(\text{P Bord}_{n}^{(n-2)})\) represented by an element
\[
(M) = (M \subseteq V \times (a_0^1, b_1^1) \times (a_0^2, b_1^2), I_0^1 \subseteq I_1^1, I_0^2 \subseteq I_1^2)
\]
to
\[
F((M)) = \pi^{-1}\left(\left[\frac{2a_0^1 + b_0^1}{3}, \frac{a_0^1 + b_0^1}{3}\right]\times \left[\frac{a_0^2 + b_0^2}{3}, \frac{a_0^2 + b_0^2}{3}\right]\right),
\]

which is a compact \(n\)-dimensional manifold with corners since \(\pi\) is proper and \(p_1\) and \(p_2\) are submersive
over \(I_0^1, I_1^1\) respectively

\[I_0^2, I_1^2.\]

It is a 2-bordism \(S\) with decomposition of the boundary coming from the inverse images under \(\pi\)
of the sides of the rectangle \(\left[\frac{2a_0^1 + b_0^1}{3}, \frac{a_0^1 + b_0^1}{3}\right]\times \left[\frac{a_0^2 + b_0^2}{3}, \frac{a_0^2 + b_0^2}{3}\right]\):
\[
\tilde{c}_0S = \pi^{-1}\left(\left[\frac{2a_0^1 + b_0^1}{3}, \frac{a_0^1 + b_0^1}{3}\right]\times \left[\frac{a_0^2 + b_0^2}{3}, \frac{a_0^2 + b_0^2}{3}\right]\right)
\]

where similarly to above, the Morse lemma provides diffeomorphisms
\[
\tilde{c}_{0, in}S \cong \pi^{-1}\left(\left[\frac{2a_0^1 + b_0^1}{3}, \frac{a_0^1 + b_0^1}{3}\right]\times \left(\frac{a_0^2 + b_0^2}{3}\right)\right) = F(d^1(M))
\]

and
\[
\tilde{c}_{0, out}S \cong \pi^{-1}\left(\left(\frac{2a_0^1 + b_0^1}{3}, \frac{a_0^1 + b_0^1}{3}\right)\times \left(\frac{a_0^2 + b_0^2}{3}\right)\right) = F(d^0(M)).
\]

Moreover,
\[
\tilde{c}_1S = \pi^{-1}\left(\left[\frac{2a_0^1 + b_0^1}{3}, \frac{a_0^1 + b_0^1}{3}\right]\times \left[\frac{a_0^2 + b_0^2}{3}, \frac{a_0^2 + b_0^2}{3}\right]\right)
\]

where by condition (4) in Definition 5.1 and the Morse lemma we have diffeomorphisms
\[
\tilde{c}_{1, in}S \cong \pi^{-1}\left(\left[\frac{2a_0^1 + b_0^1}{3}, \frac{a_0^1 + b_0^1}{3}\right] \times [0, 1]\right) = F(d^1d^1(M)) \times [0, 1] = F(d^1d^1(M)) \times [0, 1]
\]

and
\[
\tilde{c}_{1, out}S \cong \pi^{-1}\left(\left[\frac{2a_0^1 + b_0^1}{3}, \frac{a_0^1 + b_0^1}{3}\right] \times [0, 1]\right) = F(d^0d^1(M)) \times [0, 1] = F(d^0d^1(M)) \times [0, 1].
\]

This makes \(S\) into a 2-bordism between the images under \(F\) of the source and target of our 2-bordism.

It remains to show that this assignment was independent of the choice of representative. Any path
in \((\text{PBord}_{n}^{(n-1)})_{1, 1}\) induces a diffeomorphism from the start to the end of the path intertwining with
the composed bordisms by Theorem 8.13. More precisely, from any path we can obtain another one which
has “shorter” intervals, namely just by shrinking them to \((a_0^1(s), \frac{2a_0^1(s) + b_0^1(s)}{3})\) and \(\left[\frac{a_0^2(s) + b_0^2(s)}{3}, b_1^1(s)\right]\).

Now Theorem 8.13 gives a diffeomorphism of the images.

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The functor is an equivalence of bicategories. We check (1)-(3) of Whitehead’s theorem.

For (1), the point is the image of the plane $M = (0,1)^2 \xrightarrow{\text{id}} (0,1)^2, (0,1), (0,1))$. For $k$ points, we can take $k$ disjoint parallel planes in $\mathbb{R} \times (0,1)^2$ which intersect $\mathbb{R}$ in $k$ different points, e.g. $0, \ldots, k-1$, and the intervals $I_0^k = I_0^k = (0,1)$.

Next we show that $F$ is essentially full on 1-morphisms and 2-morphisms.

For $n = 2$ we can use the classification of 1-dimensional manifolds with boundary and the classification theorem 3.33 of Schommer-Pries in [SP09]: any connected component of a 1-dimensional manifold can be cut into pieces diffeomorphic to straight lines and left and right half circles. These all lie in the image of $F$ in a very simple way, e.g. a straight line is the image of

$$\left(M = (0,1)^2 \xrightarrow{\text{id}} (0,1)^2, (0,\frac{1}{3}] \leq \left(\frac{2}{3},1\right), (0,1)\right),$$

and the right and left half circles are the images of the following embeddings $(0,1)^2 \hookrightarrow \mathbb{R} \times (0,1)^2$ with suitable choices of intervals.

The classification theorem for 2-bordisms gives a set of generating 2-morphisms of $2\text{Cob}^\text{vert}$ for which one easily sees that they all are the image of an element in $(\text{Bord}_2)_1$. By gluing these preimages in a suitable way, we get an element whose image is diffeomorphic to the (connected component of) the element we started with.

For the general case $n \geq 2$ we cannot use a classification theorem anymore. However, the embedding theorem 8.35 gives a suitable replacement. Given a $k$-bordism $M$, it gives a neat embedding

$$\iota : M \hookrightarrow \mathbb{R}^m \times [0,1]^2.$$

Since the embedding is neat, the composition $\pi : M \hookrightarrow \mathbb{R}^m \times [0,1]^2 \rightarrow [0,1]^2$ satisfies certain submersivity conditions over a neighborhood of the boundary of $[0,1]^2$. Choose intervals $(0,b_0^1] \leq [a_1^1,1]$ and $(0,b_0^2] \leq [a_1^2,1)$ contained in those neighborhoods. Then the submersivity conditions translate to those in $[3]$ in Definition 6.11 for

$$\left(M = \iota(\pi^{-1}((0,1)^2)) \in \mathbb{R}^m \times (0,1)^2, (0,b_0^1] \leq [a_1^1,1], (0,b_0^2] \leq [a_1^2,1)\right).$$

Note that $F((\tilde{M}))$ is obtained from $M$ by cutting off part of its collaring, so they are diffeomorphic, as can be seen by applying the Morse lemma to the appropriate projections $p_2$ repeatedly.

For faithfulness, a similar argument as in the proof of Proposition 8.35 works: let $i_0 : N_0 \hookrightarrow V \times B(\mathcal{L})$ and $i_1 : N_1 \hookrightarrow W \times B(\mathcal{L})$ be embeddings which are representatives of two 2-morphisms $(N_0 \subset V \times B(\mathcal{L}), \mathcal{L})$ and $(N_1 \subset W \times B(\mathcal{L}), \mathcal{L})$ which have diffeomorphic images. W.l.o.g. we can assume that $V = W$ and $L = L$. Then there is a diffeomorphism $\psi : N_0 \rightarrow N_1$, which can be extended to the rest of the collars, i.e. we get a diffeomorphism $\psi : N_0 \rightarrow N_1$. Since $\text{Emb}(N_0, \mathbb{R}^m \times B(\mathcal{L}))$ is contractible, there is a path from $i_0$ to $i_1 \circ \psi$, which induces a 1-simplex $(N \subset V \times B(\mathcal{L}) \times [0,1], B(\mathcal{L}))$ in $(\text{PBord}_n^{-1})_{1,1}$ such that the fiber at $s = 0$ is $(N_0 \subset V \times B(\mathcal{L}), L)$ and the fiber at 1 is $(\text{im}(i_1 \circ \psi)(N_0) = N_1 \subset V \times B(\mathcal{L}), L)$.
The functor is a symmetric monoidal equivalence  That the equivalence of bicategories

\[ F : h_2(\text{Bord}_n^{(\infty,2)}) \cong \text{nCob}_{\text{ext}} \]

respects the symmetric monoidal structures can been seen by explicitly writing out the symmetric monoidal structures similarly to in Proposition 8.15.

9. Cobordisms with additional structure: orientations and framings

In the study of fully extended topological field theories, one is particularly interested in manifolds with extra structure, especially that of a framing. In this section we explain how to define the \( (\infty, n) \)-category of structured \( n \)-bordisms, in particular for the structure of an orientation or a framing.

9.1. Structured manifolds

We first recall the definition of structured manifolds and the topology on their morphism spaces making them into a topological category. In the next subsection we will see that the smooth singular chains on these topological spaces essentially will give rise to the spatial structure of the levels of the \( n \)-fold Segal space of structured bordisms similarly to the construction in Section 5.2.

Throughout this subsection, let \( M \) be an \( n \)-dimensional smooth manifold.

**Definition 9.1.** Let \( X \) be a topological space and \( E \to X \) a topological \( n \)-dimensional vector bundle which corresponds to a (homotopy class of) map(s) \( e : X \to B\text{GL}(\mathbb{R}^n) \) from \( X \) to the classifying space of the topological group \( \text{GL}(\mathbb{R}^n) \). More generally, we could also consider a map \( e : X \to B\text{Homeo}(\mathbb{R}^n) \) to the classifying space of the topological group of homeomorphisms of \( \mathbb{R}^n \), but for our purposes vector bundles are enough. An \((X, E)\)-structure or, equivalently, an \((X, e)\)-structure on an \( n \)-dimensional manifold \( M \) consists of the following data:

1. a map \( f : M \to X \), and
2. an isomorphism of vector bundles \( \text{triv} : TM \cong f^*(E) \).

Denote the set of \((X, E)\)-structured \( n \)-dimensional manifolds by \( \text{Man}^{(X, E)}_n \).

An interesting class of such structures arises from topological groups with a morphism to \( O(n) \).

**Definition 9.2.** Let \( G \) be a topological group together with a continuous homomorphism \( e : G \to O(n) \), which induces \( e : BG \to B\text{GL}(\mathbb{R}^n) \). As usual, let \( BG = EG/G \) be the classifying space of \( G \), where \( EG \) is total space of its universal bundle, which is a weakly contractible space on which \( G \) acts freely. Then consider the vector bundle \( E = (\mathbb{R}^n \times EG)/G \) on \( BG \). A \((BG, E)\)-structure or, equivalently, a \((BG, e)\)-structure on an \( n \)-dimensional manifold \( M \) is called a \( G \)-structure on \( M \). The set of \( G \)-structured \( n \)-dimensional manifolds is denoted by \( \text{Man}_n^G \).

For us, the most important examples will be the following three examples.

**Example 9.3.** If \( G \) is the trivial group, \( X = BG = * \) and \( E \) is trivial. Then a \( G \)-structure on \( M \) is a trivialization of \( TM \), i.e. a framing.

**Example 9.4.** Let \( G = O(n) \) and \( e = id_{O(n)} \). Then, since the inclusion \( O(n) \to \text{Diff}(\mathbb{R}^n) \) is a deformation retract, an \( O(n) \)-structured manifold is just a smooth manifold.
Example 9.5. Let $G = SO(n)$ and $e : SO(n) \to O(n)$ is the inclusion. Then an $SO(n)$-structured manifold is an oriented manifold.

Definition 9.6. Let $M$ and $N$ be $(X, E)$-structured manifolds. Then let the space of morphisms from $M$ to $N$ be

$$\text{Map}^{(X,E)}(M, N) = \text{Emb}(M, N) \times_{\text{Map}_{/\text{Bord}_{\text{tm}}(\mathbb{R}^n)}} \text{Map}_X(M, N).$$

Taking (singular or differentiable) chains leads to a space, i.e. a simplicial set of morphisms from $M$ to $N$. Thus we get a topological (or simplicial) category $\text{Map}^{(X,E)}_n$ of $(X, E)$-structured manifolds. Disjoint union gives $\text{Man}^{(X,E)}_n$ a symmetric monoidal structure.

Remark 9.7. For $G = O(n)$ we recover $\text{Emb}(M, N)$, and for $G = SO(n)$, the space of orientations on a manifold is discrete, so an element in $\text{Map}^{SO(n)}(M, N)$ is an orientation preserving map.

If $G$ is the trivial group we saw above that a $G$-structure is a framing. In this case, the above homotopy fiber product reduces to

$$\text{Map}^{(X,E)}(M, N) = \text{Emb}(M, N) \times_{\text{Map}_{/\text{Bord}_{\text{tm}}(\mathbb{R}^n)}} \text{Map}(M, N).$$

Thus, a framed embedding is a pair $(f, h)$, where $f : M \to N$ lies in $\text{Emb}(M, N)$ and $h$ is a homotopy between between the trivialization of $TM$ induced by the framing of $M$ and that induced by the pullback of the framing on $N$.

### 9.2. The $(\infty, n)$-category of structured cobordisms

Fix a type of structure given by the pair $(X, E)$. In this subsection we define the $n$-fold (complete) Segal space of $(X, E)$-structured cobordisms $\text{Bord}^{(X,E)}_n$.

Compared to Definition 9.1, we add an $(X, E)$-structure to the data of an element in a level set.

Definition 9.8. Let $V$ be a finite dimensional vector space. For every $n$-tuple $k_1, \ldots, k_n \geq 0$, let $(\text{PBord}^{(X,E),V}_n)_{k_1,\ldots,k_n}$ be the collection of tuples $(M, f, \text{triv}, (I_0^i \leq \cdots \leq I_n^i)^i_{i=1,\ldots,n})$, where

1. $(M, (I_0^i \leq \cdots \leq I_n^i)^i_{i=1,\ldots,n})$ is an element in the set $(\text{PBord}^V_n)_{k_1,\ldots,k_n}$, and
2. $(f, \text{triv})$ is an $(X, E)$-structure on the (abstract) manifold $M$.

Remark 9.9. Note that there is a forgetful map

$$U : (\text{PBord}^{(X,E),V}_n)_{k_1,\ldots,k_n} \to (\text{PBord}^V_n)_{k_1,\ldots,k_n}$$

forgetting the $(X, E)$-structure.

Definition 9.10. An $l$-simplex of $(\text{PBord}^{(X,E),V}_n)_{k_1,\ldots,k_n}$ consists of tuples $(M, f, \text{triv}, \bar{T}(s) = (I_0(s) \leq \cdots \leq I_n(s))^s_{s=\Delta|\bar{T}|})$ such that

1. $\bar{T} = (I_0 \leq \cdots \leq I_n)^i_{i=1,\ldots,n} \to |\Delta|^\ast$ is an $l$-simplex in $\text{Int}^{n}_{k_1,\ldots,k_n}$,
2. $M$ is a closed and bounded $(n + l)$-dimensional submanifold of $V \times B(\bar{T}(s))_{s=|\Delta|^\ast} \times |\Delta|^\ast$ such that we have

   a) the composition $\pi : M \hookrightarrow V \times B(\bar{T}(s))_{s=|\Delta|^\ast} \times |\Delta|^\ast \xrightarrow{\text{incl}} B(\bar{T}(s))_{s=|\Delta|^\ast} \times |\Delta|^\ast$ of the inclusion with the projection is proper,

\footnote{Recall from Section 9.3 that by abuse of notation, $B(\bar{T}(s))_{s=|\Delta|^\ast} \times |\Delta|^\ast$ denotes the total space of $B(\bar{T}) \to |\Delta|^\ast$.}
b) its composition with the projection onto $|Δ^l|_e$ is a submersion $π_1 : M \to |Δ^l|_e$,

c) $(f, \text{triv}) : \ker(Dπ_1 : TM \to T|Δ^l|_e) \to f^*E$ is a fiberwise linear isomorphism.

3. for every $S \subseteq \{1, \ldots, n\}$, let $p_S : M \xrightarrow{\pi} B(\mathbb{Z}(s))_{s \in |Δ^l|_e} \times |Δ^l|_e$ be the composition of $π$ with the projection $π_S$ onto the $S$-coordinates. Then for every $1 \leq i \leq n$ and $0 \leq j_i \leq k_i$, at every $x \in p_{(i)}^{-1}(I_{j_i} \times |Δ^l|_e)$, the map $p_{(i, \ldots, n)}$ is submersive.

Similarly as for PBord$^n$, the levels can be given a spatial structure with the above $l$-simplices and then the collection of levels can be made into a complete $n$-fold Segal space Bord$^n(X,E)$.

Moreover, Bord$^n(X,E)$ has a symmetric monoidal structure given by $(X, E)$-structured versions of the $Γ$-object and of the tower giving Bord$^n$ a symmetric monoidal structure.

**9.3. Example: Objects in Bord$^fr_2$ are 2-dualizable**

In dimension one, a framing is the same as an orientation. Thus the first interesting case is the two-dimensional one. However, we will see that nevertheless, any object in Bord$^fr_2$ is 2-dualizable. Being 2-dualizable means that it is dualizable with evaluation and coevaluation maps themselves have adjoints, see [Lur09c].

Consider an object in Bord$^fr_2$, which, since in this case Bord$^fr_2 = \text{PBord}^fr_2$ by remark 5.23, is an element of the form

$$\{M \subseteq V \times (a^1, b^1) \times (a^2, b^2), F, (a^1, b^1), (a^2, b^2)\},$$

where $F$ is a framing of $M$. By the submersivity condition 3 in the Definition 5.4 of PBord$^fr_2$, $M$ is a disjoint union of manifolds which are diffeomorphic to $(0, 1)^2$. Thus, it suffices to consider an element of the form

$$\{(0, 1)^2 \subseteq (0, 1)^2, F, (0, 1), (0, 1)\},$$

where $F$ is a framing of $(0, 1)^2$. Depict this element by

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One should think of this as a point together with a 2-framing.

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We claim that its dual is the same underlying unstructured manifold together with the opposite framing

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An evaluation 1-morphism $ev$ between them is given by the element in $(\text{Bord}^fr_2)_{1, 0}$ which is a strip, i.e. $(0, 1)^2$, with the framing given by slowly rotating the framing by 180°, and is embedded into $\mathbb{R} \times (0, 1)^2$ by folding it over once as depicted further down.
A coevaluation $\text{coev}$ is given similarly by rotating the framing along the strip in the other direction, by $-180^\circ$.

The composition

is connected by a path to the flat strip with the following framing given by pulling at the ends of the strip to flatten it.

This strip is homotopic to the same strip with the trivial framing. Thus the composition is connected by a path to the identity and thus is the identity in the homotopy category. Similarly,

$$\left(\text{ev} \otimes \text{id} \right) \circ \left(\text{id} \otimes \text{coev} \right) \simeq \text{id}.$$ 

In the above construction, we used $\text{ev}$ and $\text{coev}$ which arose from strips with framing rotating by $\pm 180^\circ$. A similar argument holds if you use for the evaluation any strip with the framing rotating by $\alpha \pi$ for any odd integer $\alpha$ and for the coevaluation rotation by $\beta \pi$ for any odd $\beta$. Denoting these by $\text{ev}(\alpha)$ and $\text{coev}(\beta)$, they will be adjoints to each other if $\alpha + \beta = 2$.

The counit of the adjunction is given by the cap with the framing coming from the trivial framing on the (flat) disk.
Similarly, the unit of the adjunction is given by a saddle with the framing coming from the one of the torus which turns by $2\pi$ along each of the fundamental loops.

Then the following 2-bordism also is framed and exhibits the adjunction.

10. Fully extended topological field theories

Now that we have a good definition of a symmetric monoidal $(\infty,n)$-category of bordisms modelled as a symmetric monoidal complete $n$-fold Segal space, we can define fully extended topological field theories à la Lurie.

10.1. Definition

**Definition 10.1.** A fully extended unoriented $n$-dimensional topological field theory is a symmetric monoidal functor of $(\infty,n)$-categories with source $\text{Bord}_n$.

**Remark 10.2.** Consider a fully extended unoriented $n$-dimensional topological field theory

$$Z : \text{Bord}_n \to \mathcal{C},$$
where $\mathcal{C}$ is a symmetric monoidal complete $n$-fold Segal space. We have seen in Corollary 7.8 and Section 8 that $h_1(L_{n-1}(\text{Bord}_n)) \cong h_1(\text{Bord}_n^{10}) \simeq n\text{Cob}$. Thus $Z$ induces a symmetric monoidal functor
\[ n\text{Cob} \cong h_1(L_{n-1}(\text{Bord}_n)) \longrightarrow h_1(L_{n-1}(\mathcal{C}(\ast))), \]

i.e. an ordinary $n$-dimensional topological field theory.

Similarly, a fully extended unoriented $n$-dimensional topological field theory with target $\mathcal{C}$ yields an extended 2TFT
\[ n\text{Cob}^{\text{ext}} \cong h_2(L_{n-2}(\text{Bord}_n)) \longrightarrow h_2(L_{n-2}(\mathcal{C}, Z(\ast))). \]

### Additional structure

Recall from the previous section that there are variants of Bord$_n$ which require that the underlying manifolds of their elements to be endowed with some additional structure, e.g. an orientation or a framing. These variants lead to the following definitions.

**Definition 10.3.** Fix a type of structure given by the pair $(X, E)$. A *fully extended $n$-dimensional $(X, E)$-topological field theory* is a symmetric monoidal functor of $(\infty, n)$-categories with source Bord$_n^{(X,E)}$.

In particular, the most interesting cases are the following:

- **Definition 10.4.** A *fully extended $n$-dimensional framed topological field theory* is a symmetric monoidal functor of $(\infty, n)$-categories with source Bord$_n^{fr}$.

- **Definition 10.5.** A *fully extended $n$-dimensional oriented topological field theory* is a symmetric monoidal functor of $(\infty, n)$-categories with source Bord$_n^{or}$.

### 10.2. $n$-TFT yields $k$-TFT

Every fully extended $n$-dimensional (unoriented, oriented, framed) TFT yields a fully extended $k$-dimensional (unoriented, oriented, framed) TFT for any $k \leq n$ by truncation from subsection 2.4.1.

Note that for $k < n$, we have a map of $k$-fold Segal spaces
\[ \text{PBord}_k \longrightarrow \tau_k(\text{PBord}_n) = (\text{PBord}_n)_{\bullet, \ldots, \cdot 0, \ldots, 0}, \]

induced by sending $(M \hookrightarrow V \times B(\vec{1}, \vec{1}) = (I^0_1 \subseteq \cdots \subseteq I^k_{j_1}) \in (\text{PBord}_k)_{j_1, \ldots, j_k}$ to
\[ (M \times (0,1)^{n-k} \hookrightarrow V \times (0,1)^{n-k} \times B(\vec{1}, \vec{1}) = (0,1), \ldots, (0,1))_{n-k}. \]

The completion map $\text{PBord}_k \rightarrow \text{Bord}_n$ induces a map on the truncations. Precomposition with the above map yields a map of (in general non-complete) $n$-fold Segal spaces
\[ \text{PBord}_k \longrightarrow \tau_k(\text{PBord}_n) \longrightarrow \tau_k(\text{Bord}_n). \]

Recall from 2.4.1 that since $\tau_k(\text{Bord}_n)$ is complete, by the universal property of the completion we obtain a map $\text{Bord}_k \rightarrow \tau_k(\text{Bord}_n)$, which is compatible with the symmetric monoidal structure (for both approaches).

This ensures that any fully extended $n$-dimensional (unoriented, oriented, framed) TFT with values in a complete $n$-fold Segal space $\mathcal{C}$, Bord$_n \rightarrow \mathcal{C}$ leads to a $k$-dimensional (unoriented, oriented, framed) TFT given by the composition
\[ \text{Bord}_k \longrightarrow \tau_k(\text{Bord}_n) \longrightarrow \tau_k(\mathcal{C}) \]

with values in the complete $k$-fold Segal space $\tau_k(\mathcal{C})$. 

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