Covariant geometric quantization of non-relativistic Hamiltonian mechanics

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Abstract. We provide geometric quantization of the vertical cotangent bundle \( V^*Q \) equipped with the canonical Poisson structure. This is a momentum phase space of non-relativistic mechanics with the configuration bundle \( Q \to \mathbb{R} \). The goal is the Schrödinger representation of \( V^*Q \). We show that this quantization is equivalent to the fibrewise quantization of symplectic fibres of \( V^*Q \to \mathbb{R} \) that makes the quantum algebra of non-relativistic mechanics an instantwise algebra. Quantization of the classical evolution equation defines a connection on this instantwise algebra, which provides quantum evolution in non-relativistic mechanics as a parallel transport along time.

1 Introduction

We study covariant geometric quantization of non-relativistic Hamiltonian mechanics subject to time-dependent transformations.

Its configuration space is a fibre bundle \( Q \to \mathbb{R} \) equipped with bundle coordinates \((t, q^k)\), \(k = 1, \ldots, m\), where \(t\) is the Cartesian coordinate on the time axis \(\mathbb{R}\) with affine transition functions \(t' = t + \text{const}\). Different trivializations \(Q \cong \mathbb{R} \times M\) of \(Q\) correspond to different non-relativistic reference frames. In contrary to all the existent quantizations of non-relativistic mechanics (e.g., [14, 13]), we do not fix a trivialization of \(Q\).

The momentum phase space of non-relativistic mechanics is the vertical cotangent bundle \(V^*Q\) of \(Q \to \mathbb{R}\), endowed with the holonomic coordinates \((t, q^k, p_k)\). It is provided with the canonical Poisson structure

\[
\{f, f'\}_V = \partial_k f \partial_k f' - \partial_k f \partial_k f', \quad f, f' \in C^\infty(V^*Q),
\]  

\(1\)
whose symplectic foliation coincides with the fibration $V^*Q \to \mathbb{R}$ \cite{8,12}. Given a trivialization

$$V^*Q \cong \mathbb{R} \times T^*M,$$  \hspace{1cm} (2)

the Poisson manifold $(V^*Q, \{ , \} )$ is isomorphic to the direct product of the Poisson manifold $\mathbb{R}$ with the zero Poisson structure and the symplectic manifold $T^*M$. An important peculiarity of the Poisson structure \cite{11} is that the Poisson algebra $C^\infty(V^*Q)$ of smooth real functions on $V^*Q$ is a Lie algebra over the ring $C^\infty(\mathbb{R})$ of functions of time alone.

Our goal is the geometric quantization of the Poisson bundle $V^*Q \to \mathbb{R}$, but it is not sufficient for quantization of non-relativistic mechanics.

The problem is that non-relativistic mechanics can not be described as a Poisson Hamiltonian system on the momentum phase space $V^*Q$. Indeed, a non-relativistic Hamiltonian $H$ is not an element of the Poisson algebra $C^\infty(V^*Q)$. Its definition involves the cotangent bundle $T^*Q$ of $Q$. Coordinated by $(q^0 = t, q^k, p_0 = p, p_k)$, the cotangent bundle $T^*Q$ plays the role of the homogeneous momentum phase space of non-relativistic mechanics. It is equipped with the canonical Liouville form $\Xi = p_\lambda dq^\lambda$, the symplectic form $\Omega = d\Xi$, and the corresponding Poisson bracket

$$\{f, g\}_T = \partial^\lambda f \partial_\lambda g - \partial_\lambda f \partial^\lambda g', \hspace{1cm} f, f' \in C^\infty(T^*Q).$$

Due to the one-dimensional canonical fibration

$$\zeta : T^*Q \to V^*Q,$$  \hspace{1cm} (3)

the cotangent bundle $T^*Q$ provides the symplectic realization of the Poisson manifold $V^*Q$, i.e.,

$$\zeta^*\{f, f'\}_V = \{\zeta^* f, \zeta^* f'\}_T$$

for all $f, f' \in C^\infty(V^*Q)$. A Hamiltonian on $V^*Q$ is defined as a global section

$$h : V^*Q \to T^*Q, \hspace{1cm} p \circ h = -H(t, q^j, p_j),$$  \hspace{1cm} (4)

of the one-dimensional affine bundle \cite{8,11,12}. As a consequence (see Section 6), the evolution equation of non-relativistic mechanics is expressed into the Poisson bracket $\{ , \}_T$ on $T^*Q$. It reads

$$\partial_{H^*}(\zeta^* f) = \{H^*, \zeta^* f\}_T,$$  \hspace{1cm} (5)
where $\vartheta_{H^*}$ is the Hamiltonian vector field of the function
\begin{equation}
H^* = \partial_t (\Xi - \zeta^* h^* \Xi) = p + H
\end{equation}
on $T^*Q$.

Therefore, we need the compatible geometric quantizations both of the cotangent bundle $T^*Q$ and the vertical cotangent bundle $V^*Q$ such that the monomorphism
\begin{equation}
\zeta^* : (C^\infty(V^*Q), \{ \}, \nu) \to (C^\infty(T^*Q), \{ , \} )
\end{equation}
of the Poisson algebra on $V^*Q$ to that on $T^*Q$ is prolonged to a monomorphism of quantum algebras of $V^*Q$ and $T^*Q$.

Recall that the geometric quantization procedure falls into three steps: prequantization, polarization and metaplectic correction (e.g., [3, 14, 19]). Given a symplectic manifold $(Z, \Omega)$ and the corresponding Poisson bracket $\{ , \}$, prequantization associates to each element $f$ of the Poisson algebra $C^\infty(Z)$ on $Z$ a first order differential operator $\hat{f}$ in the space of sections of a complex line bundle $C$ over $Z$ such that the Dirac condition
\begin{equation}
[\hat{f}, \hat{f}'] = -i \{ \hat{f}, f' \}
\end{equation}
holds. Polarization of a symplectic manifold $(Z, \Omega)$ is defined as a maximal involutive distribution $T \subset TZ$ such that $\text{Orth}_\Omega T = T$, i.e.,
\begin{equation}
\Omega(\vartheta, v) = 0, \quad \forall \vartheta, v \in T_z, \quad z \in Z.
\end{equation}
Given the Lie algebra $T(Z)$ of global sections of $T \to Z$, let $\mathcal{A}_T \subset C^\infty(Z)$ denote the subalgebra of functions $f$ whose Hamiltonian vector fields $\vartheta_f$ fulfill the condition
\begin{equation}
[\vartheta_f, T(Z)] \subset T(Z).
\end{equation}
Elements of this subalgebra are only quantized. Metaplectic correction provides the pre-Hilbert space $E_T$ where the quantum algebra $\mathcal{A}_T$ acts by symmetric operators. This is a certain subspace of sections of the tensor product $C \otimes D_{1/2}$ of the prequantization line bundle $C \to Z$ and a bundle $D_{1/2} \to Z$ of half-densities on $Z$. The geometric quantization procedure has been extended to Poisson manifolds [16, 17] and to Jacobi manifolds [7].

We show that standard prequantization of the cotangent bundle $T^*Q$ (e.g., [3, 14, 19]) provides the compatible prequantization of the Poisson manifold $V^*Q$ such that the monomorphism $\zeta^*$ ([8]) is prolonged to a monomorphism of prequantum algebras.

In contrast with the prequantization procedure, polarization of $T^*Q$ need not imply a compatible polarization of $V^*Q$, unless it includes the vertical cotangent bundle $V_\zeta T^*Q$. 

of the fibre bundle $\zeta$ (3), i.e., spans over vectors $\partial^0$. The canonical real polarization of $T^*Q$, satisfying the condition

$$V_\zeta T^*Q \subset T,$$

is the vertical polarization. It coincides with the vertical tangent bundle $VT^*Q$ of $T^*Q$, i.e., spans over all the vectors $\partial^\lambda$. We show that this polarization and the corresponding metaplectic correction of $T^*Q$ induces the compatible quantization of the Poisson manifold $V^*Q$ such that the monomorphism of Poisson algebras $\zeta^*$ (7) is prolonged to a monomorphism of quantum algebras of $V^*Q$ and $T^*Q$. The quantum algebra $A_V$ of $V^*Q$ consists of functions on $V^*Q$ which are at most affine in momenta $p_k$ and have the Schrödinger representation in the space of half-densities on $Q$. It is essential that, since these operators does not contain the derivative with respect to time, the quantum algebra $A_V$ is a $C^\infty(\mathbb{R})$-algebra.

We prove that the Schrödinger quantization of $V^*Q$ yields geometric quantization of symplectic fibres of the Poisson bundle $V^*Q \to \mathbb{R}$ such that any quantum operator $\hat{f}$ on $V^*Q$, restricted to the fibre $V^*_tQ$, $t \in \mathbb{R}$, coincides with the quantum operator on the symplectic manifold $V^*_tQ$ of the function $f|_{V^*_tQ}$. Thus, the quantum algebra $A_V$ of the Poisson bundle $V^*Q \to \mathbb{R}$ can be seen as the instantwise $C^\infty(\mathbb{R})$-algebra of its symplectic fibres. This agrees with the instantwise quantization of symplectic fibres $\{t\} \times T^*M$ of the direct product (2) in [14].

A fault of the Schrödinger representation on half-densities is that the Hamiltonian function $H^*$ (6) does not belong to the quantum algebra $A_T$ of $T^*Q$ in general. A rather sophisticated solution of this problem for quadratic Hamiltonians has been suggested in [14]. However, the Laplace operator constructed in [14] does not fulfill the Dirac condition (8). If a Hamiltonian $H$ is a polynomial of momenta $p_k$, one can represent it as an element of the universal enveloping algebra of the Lie algebra $A_T$, but this representation is not necessarily globally defined.

In order to include a Hamiltonian function $H^*$ (3) to the quantum algebra, one can choose the Hamiltonian polarization of $T^*Q$ which contains the Hamiltonian vector field $\partial_{H^*}$ of $H^*$. However, it does not satisfy the condition (11) and does not define any polarization of the Poisson manifold $V^*Q$. This polarization is the necessary ingredient in a different variant of geometric quantization of $V^*Q$ which is seen as a presymplectic manifold $(V^*Q, h^*\Omega)$. Given a trivialization (2), this quantization has been studied in [19]. In Section 5, its frame-covariant form is discussed.

Finally, since the quantum algebra $A_V$ of the Poisson manifold $V^*Q$ is a $C^\infty(\mathbb{R})$-algebra and since $\hat{p} = -i\partial/\partial t$, quantization of the classical evolution equation (5) defines
on the enveloping quantum algebra $\mathcal{A}_V$ and describes quantum evolution in non-relativistic mechanics as a parallel transport along time.

2 Prequantization

Basing on the standard prequantization of the cotangent bundle $T^*Q$, we here construct the compatible prequantizations of the Poisson bundle $V^*Q \rightarrow \mathbb{R}$ and its symplectic leaves.

Recall the prequantization of $T^*Q$ (e.g., [3, 14, 19]). Since its symplectic form $\Omega$ is exact and belongs to the zero de Rham cohomology class, the prequantization bundle is the trivial complex line bundle

$$C = T^*Q \times \mathbb{C} \rightarrow T^*Q,$$

whose Chern class $c_1$ is zero. Coordinated by $(q^\lambda, p_\lambda, c)$, it is provided with the admissible linear connection

$$A = dp_\lambda \otimes \partial^\lambda + dq^\lambda \otimes (\partial_\lambda + ip_\lambda c_\partial c)$$

with the strength form $F = -i\Omega$ and the Chern form

$$c_1 = \frac{i}{2\pi} F = \frac{1}{2\pi} \Omega.$$ 

The $A$-invariant Hermitian fibre metric on $C$ is $g(c, c) = c\overline{c}$. The covariant derivative of sections $s$ of the prequantization bundle $C$ (13) relative to the connection $A$ (14) along the vector field $u$ on $T^*Q$ takes the form

$$\nabla_u(s) = (u^\lambda \partial_\lambda - iw^\lambda p_\lambda)s.$$ 

Given a function $f \in C^\infty(T^*Q)$, the covariant derivative (15) along the Hamiltonian vector field

$$\vartheta_f = \partial^\lambda f \partial_\lambda - \partial_\lambda f \partial^\lambda, \quad \vartheta_f \mid \Omega = -df$$

of $f$ reads

$$\nabla_{\vartheta_f} = \partial^\lambda f (\partial_\lambda - ip_\lambda) - \partial_\lambda f \partial^\lambda.$$
Then, in order to satisfy the Dirac condition \( (8) \), one assigns to each element \( f \) of the Poisson algebra \( C^\infty(T^*Q) \) the first order differential operator

\[
\hat{f}(s) = -i(\nabla_{\partial_f} + if)s = [-i\partial_f + (f - p_\lambda \partial^\lambda f)]s \tag{16}
\]
on sections \( s \in C(T^*Q) \) of the prequantization bundle \( C \). For instance, the prequantum operators \( (16) \) for local functions \( f = p_\lambda, f = q^k, \) a global function \( f = t, \) and the constant function \( f = 1 \) read

\[
\hat{p}_\lambda = -i\partial_\lambda, \quad \hat{q}^\lambda = i\partial^\lambda + q^\lambda, \quad \hat{1} = 1.
\]

For elements \( f \) of the Poisson subalgebra \( C^\infty(V^*Q) \subset C^\infty(T^*Q) \), the Kostant–Souriau formula \( (16) \) takes the form

\[
\hat{f}(s) = [-i(\partial^k f\partial_k - \partial_\lambda f\partial^\lambda) + (f - p_k \partial^k f)]s. \tag{17}
\]

Turn now to prequantization of the Poisson manifold \( (V^*Q, \{,\}_V) \). The Poisson bivector \( w \) of the Poisson structure on \( V^*Q \) reads

\[
w = \partial^k \wedge \partial_k = -[w, u]_{SN}, \tag{18}
\]
where \([,]_{SN}\) is the Schouten–Nijenhuis bracket and \( u = p_k \partial^k \) is the Liouville vector field on the vertical cotangent bundle \( V^*Q \to Q \). The relation \( (18) \) shows that the Poisson bivector \( w \) is exact and, consequently, has the zero Lichnerowicz–Poisson cohomology class \([8, 17]\). Therefore, let us consider the trivial complex line bundle

\[
C_V = V^*Q \times C \to V^*Q \tag{19}
\]
such that the zero Lichnerowicz–Poisson cohomology class of \( w \) is the image of the zero Chern class \( c_1 \) of \( C_V \) under the cohomology homomorphisms

\[
H^*(V^*Q, \mathbb{Z}) \to H_{deRh}^*(V^*Q) \to H_{LP}^*(V^*Q).
\]

Since the line bundles \( C \) \([13]\) and \( C_V \) \([19]\) are trivial, \( C \) can be seen as the pull-back \( \zeta^*C_V \) of \( C_V \), while \( C_V \) is isomorphic to the pull-back \( h^*C \) of \( C \) with respect to a section \( h \) \([1]\) of the affine bundle \([3]\). Since \( C_V = h^*C \) and since the covariant derivative of the connection \( A \) \([14]\) along the fibres of \( \zeta \) \([3]\) is trivial, let us consider the pull-back

\[
h^*A = dp_k \otimes \partial^k + dq^k \otimes (\partial_k + ip_k c\partial_c) + dt \otimes (\partial_t - iHc\partial_c) \tag{20}
\]
of the connection $A$ onto $C_V \to V^*Q$. This connection defines the contravariant derivative
\[ \nabla_\phi s_V = \nabla_{w\phi} s_V \] (21)
of sections $s_V$ of $C_V \to V^*Q$ along one-forms $\phi$ on $V^*Q$, which corresponds to a contravariant connection $A_V$ on the line bundle $C_V \to V^*Q$. It is readily observed that this contravariant connection does not depend on the choice of a section $h$. By virtue of the relation (21), the curvature bivector of $A_V$ equals to $-iw$, i.e., $A_V$ is an admissible connection for the canonical Poisson structure on $V^*Q$. Then the Kostant–Souriau formula
\[ \hat{f}_V(s_V) = (-i\nabla_{\partial^f} + f)s_V = [-i(\partial^f \partial_k - \partial_k \partial^f) + (f - p_k \partial^k f)]s_V \] (22)
defines prequantization of the Poisson manifold $V^*Q$.

In particular, the prequantum operators of functions $f \in C^\infty(R)$ of time alone reduces simply to multiplication $\hat{f}_V s_V = f s_V$ by these functions. Consequently, the prequantum algebra $\hat{C}^\infty(V^*Q)$ inherits the structure of a $C^\infty(R)$-algebra.

It is immediately observed that the prequantum operator $\hat{f}_V$ (22) coincides with the prequantum operator $\zeta^* \hat{f}$ (17) restricted to the pull-back sections $s = \zeta^* s_V$ of the line bundle $C$. Thus, prequantization of the Poisson algebra $C^\infty(V^*Q)$ on the Poisson manifold $(V^*Q, \{\},)$ is equivalent to its prequantization as a subalgebra of the Poisson algebra $C^\infty(T^*Q)$ on the symplectic manifold $T^*Q$.

The above prequantization of the Poisson manifold $V^*Q$ yields prequantization of its symplectic leaves as follows.

Since $w^* \phi = \phi^k \partial_k - \phi_k \partial^k$ is a vertical vector field on $V^*Q \to R$ for any one-form $\phi$ on $V^*Q$, the contravariant derivative (21) defines a connection along each fibre $V_{t}^*Q$, $t \in R$, of the Poisson bundle $V^*Q \to R$. This is the pull-back
\[ A_t = i_t^* h^* A = dp_k \otimes \partial^k + dq_k \otimes (\partial_\lambda + ip_k c \partial_c) \]
of the connection $h^* A$ (20) on the pull-back bundle $i_t^* C_V \to V_{t}^*Q$ with respect to the imbedding $i_t : V_{t}^*Q \to V^*Q$. It is readily observed that this connection is admissible for the symplectic structure
\[ \Omega_t = dp_k \wedge dq^k \]
on $V_{t}^*Q$, and provides prequantization of the symplectic manifold $(V_{t}^*Q, \Omega_t)$. The corresponding prequantization formula is given by the expression (22) where functions $f$ and sections $s_V$ are restricted to $V_{t}^*Q$. Thus, the prequantization (22) of the Poisson manifold $V^*Q$ is a leafwise prequantization (18).
3  Polarization

Given compatible prequantizations of the cotangent bundle $T^*Q$, the Poisson bundle $V^*Q \to \mathbb{R}$ and its simplectic fibres, let us now construct their compatible polarizations.

Recall that, given a polarization $\mathbf{T}$ of a prequantum symplectic manifold $(Z, \Omega)$, the subalgebra $\mathcal{A}_T \subset C^\infty(Z)$ of functions $f$ obeying the condition (11) is only quantized. Moreover, after further metaplectic correction, one consider a representation of this algebra in a quantum space $E_T$ such that

$$\nabla u e = 0, \quad \forall u \in \mathbf{T}(Z), \quad e \in E_T.$$  \hfill (23)

Recall that by a polarization of a Poisson manifold $(Z, \{ \cdot, \cdot \})$ is meant a sheaf $\mathbf{T}^*$ of germs of complex functions on $Z$ whose stalks $\mathbf{T}_z^*$, $z \in Z$, are Abelian algebras with respect to the Poisson bracket $\{ \cdot, \cdot \}$ \[18\]. One can also require that the algebras $\mathbf{T}_z^*$ are maximal, but this condition need not hold under pull-back and push-forward operations. Let $\mathbf{T}^*(Z)$ be the structure algebra of global sections of the sheaf $\mathbf{T}^*$; it is also called a Poisson polarization \[13, 17\]. A quantum algebra $\mathcal{A}_T$ associated to the Poisson polarization $\mathbf{T}^*$ is defined as a subalgebra of the Poisson algebra $C^\infty(Z)$ which consists of functions $f$ such that

$$\{ f, \mathbf{T}^*(Z) \} \subset \mathbf{T}^*(Z).$$

Polarization of a symplectic manifold yields its maximal Poisson polarization, and vice versa.

There are different polarizations of the cotangent bundle $T^*Q$. We will consider those polarizations of $T^*Q$ whose direct image as Poisson polarizations onto $V^*Q$ with respect to the morphism $\zeta$ \[3\] are polarizations of the Poisson manifold $V^*Q$. This takes place if the germs of polarization $\mathbf{T}^*$ of the Poisson manifold $(T^*Q, \{ \cdot, \cdot \}_T)$ are constant along the fibres of the fibration $\zeta$ \[3\] \[18\], i.e., are germs of functions independent of the momentum coordinate $p_0 = p$. It means that the corresponding polarization $\mathbf{T}$ of the symplectic manifold $T^*Q$ is vertical with respect to the fibration $T^*Q \to \mathbf{R}$, i.e., obeys the condition $\{ \zeta \}$. A short calculation shows that, in this case, the associated quantum algebra $\mathcal{A}_T$ consists of functions $f \in C^\infty(T^*Q)$ which are at most affine in the momentum coordinate $p_0$. Moreover, given such a polarization, the equality (23) implies the equality

$$\nabla u_0 e = 0, \quad e \in E_T,$$ for any vertical vector field $u_0 \phi^0$ on the fibre bundle $T^*Q \to V^*Q$. Then the prequantization formulas (17) and (22) for the Poisson algebra $C^\infty(V^*Q)$ coincide on quantum
spaces, i.e., the monomorphism $\zeta^*$ is prolonged to monomorphism of quantum algebras of $V^*Q$ and $T^*Q$.

The vertical polarization $VT^*Q$ of $T^*Q$ obeys the condition (11). It is a strongly admissible polarization, and its integral manifolds are fibres of the cotangent bundle $T^*Q \to Q$. One can verify easily that the associated quantum algebra $A_T$ consists of functions on $T^*Q$ which are at most affine in momenta $p_\lambda$. The quantum space $E_T$ associated to the vertical polarization obeys the condition

$$\nabla_{u^\lambda \partial_\lambda} e = 0, \quad \forall e \in E_T. \quad (24)$$

Therefore, the operators of the quantum algebra $A_T$ on this quantum space read

$$f = a^\lambda(q^\mu)p_\lambda + b(q^\mu), \quad \hat{f} = -i\nabla_{a^\lambda \partial_\lambda} + b. \quad (25)$$

This is the Schrödinger representation of $T^*Q$.

The vertical polarization of $T^*Q$ defines the maximal polarization $T^*$ of the Poisson manifold $V^*Q$ which consists of germs of functions constant on the fibres of $V^*Q \to Q$. The associated quantum space $E_V$ obeys the condition

$$\nabla_{u^k \partial_k} e = 0, \quad \forall e \in E_V. \quad (26)$$

The quantum algebra $A_V$ corresponding to this polarization of $V^*Q$ consists of functions on $V^*Q$ which are at most affine in momenta $p_k$. Their quantum operators read

$$f = a^\lambda(q^\mu)p_k + b(q^\mu), \quad \hat{f} = -i\nabla_{a^\lambda \partial_\lambda} + b. \quad (27)$$

This is the Schrödinger representation of $V^*Q$.

In turn, each symplectic fibre $V_t^*Q$, $t \in \mathbb{R}$, of the Poisson bundle $V^*Q \to \mathbb{R}$ is provided with the pull-back polarization $T_t^* = i_t^* T^*$ with respect to the Poisson morphism $i_t : V_t^*Q \to V^*Q$. The corresponding distribution $T_t$ coincides with the vertical tangent bundle of the fibre bundle $V_t^*Q \to Q_t$. The associated quantum algebra $A_t$ consists of functions on $V_t^*Q$ which are at most affine in momenta $p_k$, while the quantum space $E_t$ obeys the condition similar to (26). Therefore, the representation of the quantum algebra $A_t$ takes the form (27). It follows that the Schrödinger representation the Poisson bundle $V^*Q \to \mathbb{R}$ is a fibrewise representation.
4 Metaplectic correction

To complete the geometric quantization procedure of $V^*Q$, let us consider the metaplectic correction of the Schrödinger representations of $T^*Q$ and $V^*Q$.

The representation of the quantum algebra $A_T$ can be defined in the subspace of sections of the line bundle $C \to T^*Q$ which fulfill the relation (24). This representation reads

$$f = a^\lambda (q^\mu) \partial_\lambda + b(q^\mu), \quad \hat{f} = -i a^\lambda \partial_\lambda + b.$$  \hfill (28)

Therefore, it can be restricted to the sections $s$ of the pull-back line bundle $C_Q = \hat{0}^* C \to Q$ where $\hat{0}$ is the canonical zero section of the cotangent bundle $T^*Q \to Q$. However, this is not yet a representation in a Hilbert space.

Let $Q$ be an oriented manifold. Applying the general metaplectic technique, we come to the vector bundle $D_{1/2} \to Q$ of complex half-densities on $Q$ with the transition functions $\rho' = J^{-1/2} \rho$, where $J$ is the Jacobian of the coordinate transition functions on $Q$. Since $C_Q \to Q$ is a trivial bundle, the tensor product $C_Q \otimes D_{1/2}$ is isomorphic to $D_{1/2}$. Therefore, the quantization formula (28) can be extended to sections of the half-density bundle $D_{1/2} \to Q$ as

$$f = a^\lambda (q^\mu) \partial_\lambda + b(q^\mu), \quad \hat{f}\rho = (-i L a^\lambda \partial_\lambda + b) \rho = (-i a^\lambda \partial_\lambda - \frac{i}{2} \partial_\lambda (a^\lambda) + b) \rho,$$  \hfill (29)

where $L$ denotes the Lie derivative. The second term in the right-hand side of this formula is a metaplectic correction. It makes the operator $\hat{f}$ (29) symmetric with respect to the Hermitian form

$$\langle \rho_1 | \rho_2 \rangle = \left( \frac{1}{2\pi} \right)^m \int_Q \rho_1 \rho_2$$

on the pre-Hilbert space $E_T$ of sections of $D_{1/2}$ with compact support. The completion $\overline{E}_T$ of $E_T$ provides a Hilbert space of the Schrödinger representation of the quantum algebra $A_T$, where the operators (29) are essentially self-adjoint, but not necessarily bounded. Of course, functions of compact support on the time axis $R$ have a limited physical application, but we can always restrict our consideration to some bounded interval of $R$.

Since, in the case of the vertical polarization, there is a monomorphism of the quantum algebra $A_V$ to the quantum algebra $A_T$, one can define the Schrödinger representation of $A_V$ by the operators

$$f = a^k (q^\mu) \partial_k + b(q^\mu), \quad \hat{f}\rho = (-i a^k \partial_k - \frac{i}{2} \partial_k (a^k) + b) \rho.$$  \hfill (30)
in the same space of complex half-densities on $Q$ as that of $\mathcal{A}_T$. Moreover, this representation preserves the structure of $\mathcal{A}_V$ as a $C^\infty(\mathbb{R})$-algebra.

The metaplectic correction of the Poisson bundle $V^*Q \to \mathbb{R}$ also provides the metaplectic corrections of its symplectic leaves as follows. It is readily observed that the Jacobian $J$ restricted to a fibre $Q_t$, $t \in \mathbb{R}$, is a Jacobian of coordinate transformations on $Q_t$. Therefore, any half-density $\rho$ on $Q$, restricted to $Q_t$, is a half-density on $Q_t$. Then the representation (30) restricted to a fibre $V_t^*Q$ is exactly the metaplectic correction of the Schrödinger representation of the symplectic manifold $V_t^*Q$ on half-densities on $Q_t$.

Thus, the quantum algebra $\mathcal{A}_V$ of the Poisson bundle $V^*Q \to \mathbb{R}$, given by the operators (30), can be seen as the instantwise algebra of operators on its symplectic fibres.

The representation (30) can be extended locally to functions on $T^*Q$ which are polynomials of momenta $p_\lambda$. These functions can be represented by elements of the universal enveloping algebra $\overline{\mathcal{A}}_T$ of the Lie algebra $\mathcal{A}_T$, but this representation is not necessarily globally defined. For instance, a generic quadratic Hamiltonian

$$H = a^{jk}(q^\lambda)p_jp_k + b^k(q^\lambda)p_k + c(q^\lambda)$$

leads to the Hamiltonian function $H^* = p + H$ which is not an element of $\overline{\mathcal{A}}_T$ because of the quadratic term. This term can be quantized only locally, unless the Jacobian of the coordinate transition functions on $Q$ is independent of fibre coordinates $q^k$ on $Q$.

5 Presymplectic quantization

As was mentioned above, to include a Hamiltonian function $H^*$ (8) to the quantum algebra, one can choose the Hamiltonian polarization of $T^*Q$. This polarization accompanies a different approach to geometric quantization of the momentum phase space of non-relativistic mechanics $V^*Q$ which is considered as a presymplectic manifold. In comparison with the above one, this quantization does not lead to quantization of the Poisson algebra of functions on $V^*Q$ as follows.

Every global section $h$ (4) of the affine bundle $\zeta$ (3) yields the pull-back Hamiltonian form

$$H = h^*\Xi = p_k dq^k - \mathcal{H}dt$$

on $V^*Q$. With respect to a trivialization (4), the form $H$ is the well-known integral invariant of Poincaré–Cartan [1]. Given a Hamiltonian form $H$ (32), there exists a unique Hamiltonian connection

$$\gamma_H = \partial_t + \partial^k H \partial_k - \partial_k H \partial^k$$
on the fibre bundle $V^*Q \to \mathbb{R}$ such that

$$\gamma_H|dH = 0$$

(34). It defines the Hamilton equations on $V^*Q$.

A glance at the equation (34) shows that one can think of the Hamiltonian connection $\gamma_H$ as being the Hamiltonian vector field of a zero Hamiltonian with respect to the presymplectic form $dH$ on $V^*Q$. Therefore, one can study geometric quantization of the presymplectic manifold $(V^*Q, dH)$.

Usually, geometric quantization is not applied directly to a presymplectic manifold $(Z, \omega)$, but to a symplectic manifold $(Z', \omega')$ such that the presymplectic form $\omega$ is a pull-back of the symplectic form $\omega'$. Such a symplectic manifold always exists. The following two possibilities are usually considered: (i) $(Z', \omega')$ is a reduction of $(Z, \omega)$ along the leaves of the characteristic distribution of the presymplectic form $\omega$ of constant rank [6, 15], and (ii) there is a coisotropic imbedding of $(Z, \omega)$ to $(Z', \omega')$ [4, 5].

In application to $(V^*Q, dH)$, the reduction procedure however meets difficulties. Since the kernel of $dH$ is generated by the vectors $(\partial_i, \partial_k \mathcal{H} \partial^k - \partial^k \mathcal{H} \partial_k, \ k = 1, \ldots, m)$, the presymplectic form $dH$ in physical models is almost never of constant rank. Therefore, one has to provide an exclusive analysis of each physical model, and to cut out a certain subset of $V^*Q$ in order to use the reduction procedure.

The second variant of geometric quantization of the presymplectic manifold $(V^*Q, dH)$ seems more attractive because the section $h$ [13] is a coisotropic imbedding. Indeed, the tangent bundle $TN_h$ of the closed imbedded submanifold $N_h = h(V^*Q)$ of $T^*Q$ consists of the vectors

$$u = -(u^\mu \partial_{\mu} \mathcal{H} + u_j \partial^j \mathcal{H}) \partial^0 + u_k \partial^k + u^\mu \partial_{\mu}.$$  

(35)

Let us prove that the orthogonal distribution $\text{Orth}_\Omega TN_h$ of $TN_h$ with respect to the symplectic form $\Omega$ belongs to $TN_h$. By definition, it consists of the vectors $u \in T_z T^*Q$, $z \in T^*Q$, such that

$$u|v|\Omega = 0, \quad \forall v \in T_z N_h.$$ 

A simple calculation shows that these vectors obey the conditions

$$-u^0 \partial^0 \mathcal{H} + u^i = 0, \quad u^0 \partial_{\mu} - u_{\mu} = 0$$

and, consequently, take the form (35).
The image $N_h = h(V^*Q)$ of the coisotropic imbedding $h$ is given by the constraint

$$H^* = p + H(t, q^k, p_k) = 0.$$ 

Then the geometric quantization of the presymplectic manifold $(V^*Q, dH)$ consists in geometric quantization of the cotangent bundle $T^*Q$ and setting the quantum constraint condition

$$\hat{H}^* \psi = 0$$

on physically admissible quantum states. This condition implies that $\hat{H}^*$ belongs to the quantum algebra of $T^*Q$. It takes place if the above mentioned Hamiltonian polarization of $T^*Q$, which contains the Hamiltonian vector field

$$\vartheta_{H^*} = \partial_0 + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k,$$ 

is used.

Such a polarization of $T^*Q$ always exists. Indeed, any section $h$ of the affine bundle $T^*Q \to V^*Q$ defines the splitting

$$a_\lambda \partial^\lambda = a_k (\partial^k - \partial^k \mathcal{H} \partial^0) + (a_0 + a_k \partial^k \mathcal{H}) \partial^0$$

of the vertical tangent bundle $VT^*Q$ of $T^*Q \to Q$. One can justify this fact, e.g., by inspection of the coordinate transformation law. Then elements $(\partial^k - \partial^k \mathcal{H} \partial^0, k = 1, \ldots, m)$, and the values of the Hamiltonian vector field $\vartheta_{H^*}$ (36) obey the polarization condition (4) and generates a polarization of $T^*Q$. It is clear that the Hamiltonian polarization does not satisfy the condition (11), and does not define any polarization of the Poisson manifold $V^*Q$.

Nevertheless, given a trivialization (2), symplectic fibres $V^*_tQ, t \in \mathbb{R}$, of the Poisson bundle $V^*Q \to \mathbb{R}$ can be provided with the Hamiltonian polarization $\mathbf{T}_t$ generated by vectors $\partial_k \mathcal{H} \partial^k - \partial^k \mathcal{H} \partial_k, k = 1, \ldots, m$, except the points where

$$d\mathcal{H} = \partial_k \mathcal{H} dq^k + \partial^k \mathcal{H} dp_k = 0.$$ 

This is a standard polarization in conservative Hamiltonian mechanics of one-dimensional systems, but it requires an exclusive analysis of each model.
6 Classical and quantum evolution equations

Turn now to the evolution equation in classical and quantum non-relativistic mechanics.

Given a Hamiltonian connection $\gamma_H$ on the momentum phase space $V^*Q$, let us consider the Lie derivative

$$L_{\gamma_H} f = \gamma_H] df = (\partial_t + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k) f$$

of a function $f \in C^\infty(V^*Q)$ along $\gamma_H$. This equality is the evolution equation in classical non-relativistic Hamiltonian mechanics. Substituting a solution of the Hamilton equations in its right-hand side, one obtains the time evolution of $f$ along this solution. Given a trivialization (2), the evolution equation (37) can be written as

$$L_{\gamma} f = \partial_t f + \{\mathcal{H}, f\}_V.$$ 

However, taken separately, the terms in its right-hand side are ill-behaved under time-dependent transformations. Let us bring the evolution equation into the frame-covariant form.

The affine bundle $\zeta$ (3) is modelled over the trivial line bundle $V^*Q \times \mathbb{R} \to V^*Q$. Therefore, Hamiltonian forms $H$ constitute an affine space modelled over the vector space $C^\infty(V^*Q)$. To choose a centre of this affine space, let us consider a connection

$$\Gamma = \partial_t + \Gamma^k \partial_k$$

on the configuration bundle $Q \to \mathbb{R}$. By definition, it is a section of the affine bundle (3), and yields the Hamiltonian form

$$H_\Gamma = \Gamma^* \Xi = p_k dq^k - \mathcal{H}_\Gamma dt, \quad \mathcal{H}_\Gamma = p_k \Gamma^k dt.$$ 

Its Hamiltonian connection is the canonical lift

$$V^*\Gamma = \partial_t + \Gamma^i \partial_i - p_i \partial_j \Gamma^i \partial^j$$

of the connection $\Gamma$ onto $V^*Q \to \mathbb{R}$. Then any Hamiltonian form $H$ (32) on the momentum phase space $V^*Q$ admits the splittings

$$H = H_\Gamma - \tilde{\mathcal{H}}_\Gamma dt, \quad \tilde{\mathcal{H}}_\Gamma = \mathcal{H} - p_k \Gamma^k,$$

where $\tilde{\mathcal{H}}_\Gamma$ is a function on $V^*Q$. The physical meaning of this splitting becomes clear due to the fact that every trivialization of $Q \to \mathbb{R}$ yields a complete connection $\Gamma$ on $Q$, and vice versa [8, 9]. From the physical viewpoint, the vertical part of this connection $\Gamma$
\( \text{(38)} \) can be seen as a velocity of an "observer", and \( \Gamma \) characterizes a reference frame in non-relativistic time-dependent mechanics [8, 11, 12]. Then one can show that \( \tilde{\mathcal{H}}_\Gamma \) in the splitting \( \text{(39)} \) is the energy function with respect to this reference frame [2, 8, 12].

Given the splitting \( \text{(39)} \), the evolution equation can be written in the frame-covariant form

\[
L_{\gamma\hbar} f = V^* \Gamma \{ H, f \}.
\]

However, the first term in its right-hand side is not reduced to the Poisson bracket on \( V^* Q \), and is not quantized in the framework of geometric quantization of the Poisson manifold \( V^* Q \). To bring the right-hand side of the evolution equation into a Poisson bracket alone, let us consider the pull-back \( \zeta^* H \) of the Hamiltonian form \( H = h^* \Xi \) onto the cotangent bundle \( T^* Q \). It is readily observed that the difference \( \Xi - \zeta^* H \) is a horizontal 1-form on \( T^* Q \rightarrow \mathbb{R} \), and we obtain the function \( \text{(6)} \) on \( T^* Q \). Then the relation

\[
\zeta^* (L_{\gamma\hbar} f) = \partial_{H^*} (\zeta^* f) = \{ \mathcal{H}^*, \zeta^* f \}_T,
\]

holds for any element \( f \) of the Poisson algebra \( C^\infty(V^* Q) \).

Since the quantum algebra \( \mathcal{A}_V \) of the Poisson manifold \( V^* Q \) can be seen as the instantaneous algebra, one can quantize the evolution equation \( \text{(3)} \) as follows.

Given the Hamiltonian function \( \mathcal{H}^* \text{(5)} \), let \( \hat{\mathcal{H}}^* \) be the corresponding quantum operator written, e.g., as an element of universal enveloping algebra of the quantum algebra \( \mathcal{A}_T \). The bracket \( \text{(12)} \) defines a derivation of the enveloping algebra \( \mathcal{A}_V \) of the quantum algebra \( \mathcal{A}_V \), which is also a \( C^\infty(\mathbb{R}) \)-algebra. Moreover, since \( \hat{p} = -i \partial / \partial t \), the derivation \( \text{(12)} \) obeys the Leibniz rule

\[
\nabla (r(t) \hat{f}) = \partial_t r(t) \hat{f} + r(t) \nabla \hat{f}.
\]

Therefore, it is a connection on the \( C^\infty(\mathbb{R}) \)-algebra \( \mathcal{A}_V \), and defines quantum evolution of \( \mathcal{A}_V \) as a parallel transport along time \( \text{[1, 3]} \).

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