A nested sequence of projectors and corresponding braid matrices $\hat{R}(\theta)$: (1) Odd dimensions.

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Abstract

A basis of $N^2$ projectors, each an $N^2 \times N^2$ matrix with constant elements, is implemented to construct a class of braid matrices $\hat{R}(\theta)$, $\theta$ being the spectral parameter. Only odd values of $N$ are considered here. Our ansatz for the projectors $P_\alpha$ appearing in the spectral decomposition of $\hat{R}(\theta)$ leads to exponentials $\exp(m_\alpha \theta)$ as the coefficient of $P_\alpha$. The sums and differences of such exponentials on the diagonal and the antidiagonal respectively provide the $(2N^2 - 1)$ nonzero elements of $\hat{R}(\theta)$. One element at the center is normalized to unity. A class of supplementary constraints imposed by the braid equation leaves $\frac{1}{2}(N + 3)(N - 1)$ free parameters $m_\alpha$. The diagonalizer of $\hat{R}(\theta)$ is presented for all $N$. Transfer matrices $t(\theta)$ and $L(\theta)$ operators corresponding to our $\hat{R}(\theta)$ are studied. Our diagonalizer signals specific combinations of the components of the operators that lead to a quadratic algebra of $N^2$ constant $N \times N$ matrices. The $\theta$-dependence factors out for such combinations. $\hat{R}(\theta)$ is developed in a power series in $\theta$. The basic difference arising for even dimensions is made explicit. Some special features of our $\hat{R}(\theta)$ are discussed in a concluding section.

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1 Introduction

In Sec.8 of Ref.1 a sequence of projectors with constant elements and particularly simple and convenient properties were introduced for arbitrary dimension $N$ (i.e., $N^2 \times N^2$ matrices). For the case $N = 2$ they provide the spectral resolutions of the 6-vertex and the 8-vertex models (Sec.6 and Sec.7 of Ref.1, citing other sources). Along with $N$ the set of projectors is enlarged in number and in dimension systematically at each step to give what we called a ”nested sequence”. The projectors were presented Ref.1 for all $N$ and their basic features were studied, including diagonalization, for arbitrary $N$. But no higher dimensional braid matrices were constructed on such bases. That was ”beyond the scope” of that paper. Here we enlarge the scope and present explicit constructions for all odd $N$. Even dimensions will be studied separately elsewhere. Such a separation corresponds to strikingly different features arising in the respective cases.

After obtaining explicitly $\hat{R}(\theta)$ the corresponding transfer matrices $t(\theta)$ and $L(\theta)$ operators are studied. They are found to lead to a remarkable class of quadratic algebra (Sec.4). Development of $\hat{R}(\theta)$ in powers of the spectral parameter $\theta$ is also studied (Sec.5). Basic differences arising for even dimensions are pointed out. (Sec.6). The special features of our class of solutions are discussed in conclusion (Sec.7). Construction of our solutions is presented in App.A and some basic results concerning $t(\theta)$ and $L(\theta)$ are collected together in App.B.
2 Braid matrices for odd dimensions (Ansatz and Solutions):

We start by specifying our notations and conventions in detail since they turn out to be crucial in successful construction of the solutions. Thus, rather than using the simple and elegant notation of Sec.8 of Ref.1 for our projectors we introduce below a structure better suited to our present purpose.

Let

\[ N = 2p - 1 \quad (p = 2, 3, \ldots) \]

and

\[ \bar{i} = N - i + 1 \quad (i + \bar{i} = 2p, \quad \bar{\bar{i}} = i) \]

so that for

\[ i = 1, 2, \ldots, (p - 1) \]

respectively

\[ \bar{i} = (2p - 1), (2p - 2), \ldots, (p + 1) \]

and

\[ \bar{p} = p \]

The \( N^2 \times N^2 \) braid matrix \( \hat{R}(\theta) \), with the spectral parameter \( \theta \), is given in terms of its components as

\[ \hat{R}(\theta) = (\hat{R}(\theta))_{ab,cd}(ab) \otimes (cd) \tag{2.1} \]

where \((a, b, c, d)\) take values in the domain \((i, \bar{i}, p)\) and \((ab)\) is the \( N \times N \) matrix with only one nonzero element, unity, at row \( a \) and column \( b \).

The basis of projectors is given by \(( \text{with } \epsilon = \pm)\) the set

\[ P_{\epsilon pp} = (pp) \otimes (\epsilon pp) \]
\[2P_{pi(\epsilon)} = (pp) \otimes \left( ((ii) + (\bar{i}\bar{i})) + \epsilon((i\bar{i}) + (\bar{i}i)) \right)\]
\[2P_{ip(\epsilon)} = \left( ((ii) + (\bar{i}\bar{i})) + \epsilon((i\bar{i}) + (\bar{i}i)) \right) \otimes (pp)\]
\[2P_{ij(\epsilon)} = \left( ((ii) \otimes (jj) + (i\bar{i}) \otimes (j\bar{j})) + \epsilon((i\bar{i}) \otimes (j\bar{j}) + (\bar{i}i) \otimes (j\bar{j})) \right)\]
\[2P_{ij(\epsilon)} = \left( ((ii) \otimes (jj) + (i\bar{i}) \otimes (j\bar{j})) + \epsilon((i\bar{i}) \otimes (j\bar{j}) + (\bar{i}i) \otimes (j\bar{j})) \right)\] (2.2)

Condensing the triplets \((i, j, \epsilon), (i, p, \epsilon), \ldots\) and also \((pp)\) into \((\alpha, \beta, \ldots)\) the basis (2.2) satisfies
\[P_{\alpha}P_{\beta} = \delta_{\alpha\beta}P_{\alpha}, \quad \sum_{\alpha} P_{\alpha} = I_{N^2 \times N^2}\] (2.3)

The total number of \(P_{\alpha}\) is
\[1 + 4(p - 1) + 4(p - 1)^2 = (2p - 1)^2 = N^2\]

They have, apart from the overall factor \(\frac{1}{2}\) for all projectors except \(P_{pp}\), only the constant elements \((\pm 1, 0)\). There is, for example, no \(q\) in our formalism.

The braid matrix is postulated in the spectrally resolved form
\[
\hat{R}(\theta) = P_{pp} + \sum_{i,\epsilon} \left( f_{pi(\epsilon)}^{(\epsilon)}(\theta) \right) P_{pi(\epsilon)} + f_{ip(\epsilon)}^{(\epsilon)}(\theta) P_{ip(\epsilon)} \\
+ \sum_{i,j,\epsilon} \left( f_{ij(\epsilon)}^{(\epsilon)}(\theta) \right) P_{ij(\epsilon)} + f_{i\bar{j}(\epsilon)}^{(\epsilon)}(\theta) P_{i\bar{j}(\epsilon)} \] (2.4)

The coefficient of \(P_{pp}\) is normalized to unity. The \((N^2 - 1)\) functions \(f_{ab}^{(\epsilon)}\) are to be extracted from the constraints imposed by the braid equation
\[\hat{R}_{12}(\theta) \hat{R}_{23}(\theta + \theta') \hat{R}_{12}(\theta') = \hat{R}_{23}(\theta') \hat{R}_{12}(\theta + \theta') \hat{R}_{23}(\theta)\] (2.5)

Here (suppressing \(\theta\))
\[ \hat{R}_{12} = \hat{R} \otimes I_{N \times N}, \quad \hat{R}_{23} = I_{N \times N} \otimes \hat{R} \]

In terms of the coefficients \((\hat{R}(\theta))_{ab,cd}\) defined in (2.1) one obtains (summing over the repeated indices \((l,m,n)\))

\[
(\hat{R}(\theta))_{al,cm} (\hat{R}(\theta + \theta'))_{mn,ef} (\hat{R}(\theta'))_{lb,nd} = (\hat{R}(\theta'))_{cl,em} (\hat{R}(\theta + \theta'))_{ab,ln} (\hat{R}(\theta))_{nd,mf} \tag{2.6}
\]

This corresponds to the point \((ab) \otimes (cd) \otimes (ef)\) of the base space \(V \otimes V \otimes V\). Our ansatz (2.4) along with (2.2) implies very strong constraints (typical of odd dimensions). The solutions are obtained in App. A. One has

\[
f^{(c)}_{ab}(\theta) = \exp(m^{(c)}_{ab}\theta); \quad (ab) = (pi), (ip), (ij), (i\bar{j}) \tag{2.7}
\]

where the parameters \(m^{(c)}_{ab}\) are all independent except that for each \(i\)

\[
m^{(c)}_{ij} = m^{(c)}_{i\bar{j}}, \quad (\bar{j} = 2p - j) \tag{2.8}
\]

The constraints (2.7), (2.8) are necessary and sufficient. Thus for \(N = 3\) one has

\[
\hat{R}(\theta) = \begin{pmatrix}
  a_+ & 0 & 0 & 0 & 0 & 0 & 0 & a_- \\
  0 & b_+ & 0 & 0 & 0 & 0 & 0 & b_- \\
  0 & 0 & a_+ & 0 & 0 & a_- & 0 & 0 \\
  0 & 0 & 0 & c_+ & 0 & c_- & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & c_- & 0 & c_+ & 0 & 0 \\
  0 & 0 & a_- & 0 & 0 & a_+ & 0 & 0 \\
  0 & b_- & 0 & 0 & 0 & 0 & b_+ & 0 \\
  a_- & 0 & 0 & 0 & 0 & 0 & 0 & a_+
\end{pmatrix} \tag{2.9}
\]

where

\[
a_\pm = \frac{1}{2}(e^{m^{(c)}_{i\bar{i}}\theta} \pm e^{m^{(c)}_{ii}\theta})
\]
\[b_\pm = \frac{1}{2}(e^{m_{12}^\pm \theta} \pm e^{m_{21}^\pm \theta})\]
\[c_\pm = \frac{1}{2}(e^{m_{21}^\pm \theta} \pm e^{m_{12}^\pm \theta})\] (2.10)

The six parameters remaining after application of (2.8) (which imposes the repetition of \(a_\pm\)) are all independent. For \(m_{ab}^{(-)} = m_{ab}^{(+)}\) one obtains hyperbolic functions as particular cases. For all \(N\), the nonzero elements are confined to the diagonal and the antidiagonal as above with a common element, unity, at the centre. Apart from the normalized element, the coefficients of the projectors are simply exponentials. The total number of independent parameters \(m_{ab}^\pm\) is

\[(2p - 1)^2 - 1 - 2(p - 1)^2 = 2(p^2 - 1) = \frac{1}{2}(N + 3)(N - 1)\] (2.11)

Note that the coefficient of \(P_{pp}\) in (2.4) has to be nonzero for \(\hat{R}(\theta)\) to be invertible and hence can safely be normalized to unity. Indeed, each coefficient in (2.4) has to be nonzero for \(\hat{R}(\theta)\) to be invertible. This is more evident after diagonalization (Sec.3). For even \(N\) there is no index \(p = \bar{p}\). In App.A the crucial role of the index \(p\) will be made more evident. The projectors \(P_{pi(\epsilon)}\) and \(P_{ip(\epsilon)}\) will be seen to impose the highly constrained solutions (2.7) with (2.8).

Implementing (2.8) one obtains from (2.4)

\[\hat{R}(\theta) = P_{pp} + \sum_{i,\epsilon}(f_{pi(\epsilon)}^{(\epsilon)}(\theta)P_{pi(\epsilon)} + f_{ip(\epsilon)}^{(\epsilon)}(\theta)P_{ip(\epsilon)}) + \sum_{i,j,\epsilon}f_{ij}^{(\epsilon)}(\theta)(P_{ij(\epsilon)} + P_{i\bar{j}(\epsilon)})\] (2.12)

Defining

\[\tilde{P}_{ij(\epsilon)} = P_{ij(\epsilon)} + P_{i\bar{j}(\epsilon)}\] (2.13)
and conserving all other projectors as before one obtains a basis of \((2p^2 - 1)\) projectors still satisfying (2.3) where now the indices summed over are \((i, j, \epsilon), (i, p, \epsilon), (p, i, \epsilon), (pp)\).

Now

\[
\hat{R}(\theta) = P_{pp} + \sum_{i, \epsilon} (f^{(\epsilon)}_{pi}(\theta)P_{pi(\epsilon)} + f^{(\epsilon)}_{ip}(\theta)P_{ip(\epsilon)}) + \sum_{i,j,\epsilon} f^{(\epsilon)}_{ij}(\theta)\tilde{P}_{ij(\epsilon)} \tag{2.14}
\]

In this basis all the \(\frac{1}{2}(N + 3)(N - 1)\) parameters are independent. When they are all chosen to be distinct (and different from 1) the polynomial equation (of \(\frac{1}{2}(N + 3)(N - 1)\) degree and with distinct roots) satisfied by \(\hat{R}(\theta)\) and the projectors in terms of \(\hat{R}(\theta)\) are obtained respectively as in (1.5) and (1.6) of Ref.1. The initial basis, due to the symmetry and simplicity of the projectors, is most convenient for certain purposes. The second one has the virtue of eliminating constraints. Each should be implemented according to the context.

If two or more of the \(\frac{1}{2}(N + 3)(N - 1)\) free parameters are allowed to coincide, then introducing the sum of the corresponding projectors (as in (2.13)) the basis can again be redefined (as in (2.14)). The degree of the minimal polynomial equation satisfied by \(\hat{R}(\theta)\) diminishes correspondingly.

Our matrices all satisfy

\[
\hat{R}(-\theta)\hat{R}(\theta) = I, \quad \hat{R}(0) = I \tag{2.15}
\]

3 Diagonalization:

Our general approach to diagonalization is presented step by step in Sec.9 of Ref.1.
The matrix \( M \) that diagonalizes each projector \( P_\alpha \) of (2.3) (namely, \( P_{pp}, P_{pi(\epsilon)}, P_{ij(\epsilon)} \) of (2.2) ) and hence \( \hat{R}(\theta) \) of (2.4) is given below. As compared to the the results of Sec.8 of Ref.1, \( M \) is presented here in our current notations.

Set

\[
\sqrt{2}M = \sqrt{2}M^{-1} = \sqrt{2}(pp) \otimes (pp) + \\
(pp) \otimes \left( \sum_i ((ii) - (\bar{i}i) + (\bar{i}i)) \right) + \left( \sum_i ((ii) - (i\bar{i}) + (i\bar{i})) \right) \otimes (pp) + \\
\sum_{i,j} \left( ((ii) - (\bar{i}i)) \otimes ((jj) + (\bar{j}j)) + ((ii) + (i\bar{i})) \otimes ((jj) + (\bar{j}j)) \right)
\]

One verifies in a straightforward fashion (with \( \epsilon = \pm 1 \) on the right) that

\[
MP_{pp}M^{-1} = (pp) \otimes (pp)
\]

\[
2MP_{pk(\epsilon)}M^{-1} = (pp) \otimes ((1 + \epsilon)(kk) + (1 - \epsilon)(\bar{k}\bar{k}))
\]

\[
2MP_{kp(\epsilon)}M^{-1} = ((1 + \epsilon)(kk) + (1 - \epsilon)(\bar{k}\bar{k})) \otimes (pp)
\]

\[
2MP_{kl(\epsilon)}M^{-1} = (1 + \epsilon)(kk) \otimes (ll) + (1 - \epsilon)(\bar{k}\bar{k}) \otimes (\bar{l}\bar{l})
\]

\[
2MP_{k\bar{l}(\epsilon)}M^{-1} = (1 + \epsilon)(kk) \otimes (\bar{l}\bar{l}) + (1 - \epsilon)(\bar{k}\bar{k}) \otimes (ll)
\]

Hence taking account of (2.8) (i.e, \( f_{ij}^{(\epsilon)}(\theta) = f_{ij}^{(\epsilon)}(\theta) \)) one obtains

\[
2M\hat{R}(\theta)M^{-1} = 2(pp) \otimes (pp) + \\
\sum_{i,\epsilon} \left( f_{pi}^{(\epsilon)}(\theta)((1 + \epsilon)(pp) \otimes (ii) + (1 - \epsilon)(pp) \otimes (\bar{i}\bar{i})) \right) + \\
\sum_{i,\epsilon} \left( f_{pi}^{(\epsilon)}(\theta)((1 + \epsilon)(ii) \otimes (jj) + (ii) \otimes (\bar{j}\bar{j})) \right)
\]
\[ + (1 - \epsilon) \left( \bar{e}_{ij} \otimes (jj) + \bar{e}_{ji} \otimes (jj) \right) \]  
\begin{equation} \tag{3.3} \end{equation}

For \( N = 3 \) this gives

\[ M \hat{R}(\theta) M^{-1} \equiv \hat{R}_d(\theta) \]

\[ = (e^{m_{11}} \theta, e^{m_{12}} \theta, e^{m_{21}} \theta, 1, e^{m_{22}} \theta, e^{m_{22}} \theta, e^{m_{11}} \theta, e^{m_{11}} \theta)_{\text{diag}} \]  
\begin{equation} \tag{3.4} \end{equation}

The diagonalizer is

\[ \sqrt{2}M = \sqrt{2}M^{-1} = \begin{pmatrix} 
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 
\end{pmatrix} \]  
\begin{equation} \tag{3.5} \end{equation}

The generalizations of (3.4) and (3.5) for all \( N \) are quite evident.

If an \( \hat{R}(\theta) \) satisfying the braid equation (2.5) is diagonalized the corresponding \( \hat{R}_d(\theta) \), in general, does not directly satisfy (2.5). This is evident from all the examples of Ref.1. The general explanation is simple. Interpolated factors of the type \( M_{12}M_{23}^{-1} \) will be lacking in the latter case as compared to the former. If \( \hat{R}(\theta) \) is diagonal to start with (2.6) reduces to

\[ (\hat{R}(\theta))_{aa,bb}(\hat{R}(\theta + \theta'))_{bb,cc}(\hat{R}(\theta'))_{aa,bb} \]

\[ = (\hat{R}(\theta'))_{bb,cc}(\hat{R}(\theta + \theta'))_{aa,bb}(\hat{R}(\theta))_{bb,cc} \]  
\begin{equation} \tag{3.6} \end{equation}

The braid equation is satisfied if, for each \((a, b)\),

\[ (\hat{R}(\theta))_{aa,bb}(\hat{R}(\theta'))_{aa,bb} = (\hat{R}(\theta + \theta'))_{aa,bb} \]  
\begin{equation} \tag{3.7} \end{equation}
(\hat{R}(\theta))_{a,a,b} = e^{m_{ab}\theta} \tag{3.8}

where the parameters $m_{ab}$ are mutually independent. Now, conversely, if $\hat{R}(\theta)$ is conjugated as

$$\hat{R}'(\theta) = A\hat{R}(\theta)A^{-1} \tag{3.9}$$

in general, $\hat{R}'(\theta)$ will no longer satisfy the braid equation since such products as $A_{12}^{-1}A_{23}$ will depend on the structure of $A$. The structure of our $M$ is such that for arbitrary odd $N$

$$M^{-1}\hat{R}_d(\theta)M$$

continues to satisfy the braid equation provided

$$m_{ij}^{(e)} = m_{ij}^{(\bar{e})}$$

Thus it is seen how the $2(p - 1)^2$ crucial constraints (2.8), the structure of our nested sequence of projectors and that of our $M$ are all linked.

The relevance of our $M$ to the algebra of the $L$-operators is pointed out at the end of Sec.4 after displaying the crucial algebraic structure arising there.

4 $L(\theta)$-operators and transfer matrices:

A general discussion, citing relevant sources, is presented in App.$B$. Here the basic results concerning the the $N \times N$ realizations of the $N^2$ blocks of the transfer matrix $t(\theta)$ and the operator $L^+(\theta)$ are used in the context of braid matrices constructed in Sec.2 and App.$A$.

In (B.22) and (B.23) we show in a transparent fashion why, unless (B.16) is generalized, say, by implementing central operators in the argument of
\( \hat{R}(\theta - \theta') \), one cannot obtain an \( L^-(\theta) \neq L^+(\theta) \). We do not study such general structures here and hence consider only the above-mentioned fundamental realizations of \( L^+(\theta) \) with the standard prescription for coproduct. This will, in any case provide a subalgebra in an appropriately generalized quasi-Hopf structure. This \( L^+(\theta) \) and \( t(\theta) \), as shown in (B.28), are related (for the fundamental \( N \times N \) representations of blocks) as

\[
t(\theta) = PL^+(\theta)P, \quad (t_{ab}(\theta))_{cd} = (L^+_{cd}(\theta))_{ab} \tag{4.1}
\]

In studying multistate statistical models corresponding to our \( \hat{R}(\theta) \) (see the comments and references in Sec.7) the algebra of the blocks of \( t(\theta) \) is particularly relevant. In our case this algebra is found (see below) to be very simply related to the corresponding one for \( L^+(\theta) \). So one can start either with \( L^+(\theta) \) or \( t(\theta) \) and then obtain the other easily. We choose to display the remarkable structure that emerges first in terms of \( L^+(\theta) \). We start with (B.14), i.e,

\[
L^+(\theta) = \hat{R}(\theta)P \tag{4.2}
\]

In terms of the matrices \((ab)\) defined below (2.1), one obtains

\[
L^+_{(pp)} = (pp) \equiv X_{pp} \\
e^{-m_{ip}(\epsilon)}(L^+_{ip}(\theta) + \epsilon L^+_{ip}(\theta)) = (p\bar{i}) + \epsilon(p\bar{i}) \equiv X_{p\bar{i}}^{(c)} \\
e^{-m_{pi}(\epsilon)}(L^+_{pi}(\theta) + \epsilon L^+_{pi}(\theta)) = (i\bar{p}) + \epsilon(i\bar{p}) \equiv X_{i\bar{p}}^{(c)} \\
e^{-m_{ij}(\epsilon)}(L^+_{ij}(\theta) + \epsilon L^+_{ij}(\theta)) = (ij) + \epsilon(i\bar{j}) \equiv X_{ij}^{(c)} \\
e^{-m_{ji}(\epsilon)}(L^+_{ji}(\theta) + \epsilon L^+_{ji}(\theta)) = (\bar{i}j) + \epsilon(j\bar{i}) \equiv X_{ji}^{(c)} \tag{4.3}
\]

In the last equation (2.8) has been implemented i.e,

\[
m_{ij}^{(c)} = m_{ji}^{(c)}
\]
From these one obtains

\[ 2L^+_{ij}(\theta) = (e^{m^{(+)}_{ij}\theta}X^{(+)}_{ij} + e^{m^{(-)}_{ij}\theta}X^{(-)}_{ij}) \]

\[ 2L^+_{ij}(\theta) = (e^{m^{(+)}_{ij}\theta}X^{(+)}_{ij} - e^{m^{(-)}_{ij}\theta}X^{(-)}_{ij}) \] (4.4)

and so on.

For \( N = 3 \) one obtains (with \( \epsilon = \pm 1 \) in the matrices on the right)

\[ L^+_{22}(\theta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ L^+_{12}(\theta) + \epsilon L^+_{12}(\theta) = e^{m^{(i)}_{12}\theta} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \epsilon \\ 0 & 0 & 0 \end{pmatrix} \]

\[ L^+_{21}(\theta) + \epsilon L^+_{21}(\theta) = e^{m^{(i)}_{21}\theta} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & \epsilon & 0 \end{pmatrix} \]

\[ L^+_{11}(\theta) + \epsilon L^+_{11}(\theta) = e^{m^{(i)}_{11}\theta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & 0 & \epsilon \end{pmatrix} \]

\[ L^+_{11}(\theta) + \epsilon L^+_{11}(\theta) = e^{m^{(i)}_{11}\theta} \begin{pmatrix} 0 & 0 & \epsilon \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \] (4.5)

The constant \( N \times N \) matrices \( X^{(\epsilon)}_{ab} \), where \( X_{pp} \) has only one nonzero element, unity, and all the others only two ((1, 1) or (1, -1)) specify a quadratic algebra. We give below only the nonzero bilinear products, all others vanishing. Further results, such as commutators, can be systematically obtained from those below:

\[ X_{pp}X_{pp} = X_{pp}, \quad X_{pp}X^{(\epsilon)}_{pi} = X^{(\epsilon)}_{pi}, \quad X^{\epsilon}_{ip}X_{pp} = X^{\epsilon}_{ip} \]
\[
X^{(e)}_{pi} X^{(e')}_{ip} = (1 + \epsilon \epsilon') X_{pp}, \quad X^{(e)}_{ip} X^{(e')}_{pj} = X^{(e')}_{ij} + \epsilon X^{(ee')}_{ij} \\
X^{(e)}_{pi} X^{(e')}_{ij} = X^{(ee')}_{pj}, \quad X^{(e)}_{ij} X^{(e')}_{jp} = X^{(ee')}_{ip} \\
X^{(e)}_{pi} X^{(e')}_{ij} = \epsilon X^{(ee')}_{pi}, \quad X^{(e)}_{ji} X^{(e')}_{ip} = \epsilon \epsilon' X^{(ee')}_{ij} \\
X^{(e)}_{ij} X^{(e')}_{jk} = X^{(ee')}_{ik}, \quad X^{(e)}_{ij} X^{(e')}_{jk} = \epsilon X^{(ee')}_{ik} \\
X^{(e)}_{ji} X^{(e')}_{ik} = X^{(ee')}_{jk}, \quad X^{(e)}_{ji} X^{(e')}_{ik} = \epsilon X^{(ee')}_{jk}
\] (4.6)

(No sum over repeated indices.)

Note that
\[
C_1 \equiv \frac{1}{N} (X_{pp} + \sum_i X^{(+)i}) = \frac{1}{N} I_{N \times N} \quad (4.7)
\]

Hence \((X_{pp} - C_1)\) and \((X^{(+)i} - 2C_1)\), along with the others give an algebra of \(N^2\) traceless matrices.

Higher dimensional realizations are given by the coproducts
\[
\Delta L = L \hat{\otimes} L \quad (4.8)
\]

Here \(\hat{\otimes}\) implies tensor product combined with matrix multiplication. The prescription can be implemented repeatedly in a straightforward fashion. But it leads, in general, to reducible structures. A systematic study of extraction of irreducible components is beyond the scope of this paper. Let us, however, take a closer look at the structure of the algebra (4.6) and the special role of the index \(p\).

The generators without \(p\) (i.e., \(X^{(e)}_{ij}, X^{(e)}_{ik}\)) form a closed subalgebra. The generators with a single \(p\) (i.e., \(X^{(e)}_{pi}, X^{(e)}_{ip}\)) provide a semidirect product structure with the preceding set. But now to close it one has to extend the first set to a direct product structure by including \(X_{pp}\).
From (4.1) and (4.3) it can be shown that \( t(\theta) \) and \( L^+(\theta) \) are essentially related through the interchange of the roles of \( X^{(e)}_{pi} \) and \( X^{(e)}_{ip} \). Thus for \( N = 3 \) there is an interchange of \( b_\pm \) and \( c_\pm \). One obtains for this case

\[
\begin{align*}
\begin{pmatrix}
an_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_- \\
0 & 0 & 0 & c_+ & 0 & c_- & 0 & 0 & 0 \\
0 & 0 & a_- & 0 & 0 & 0 & a_+ & 0 & 0 \\
0 & b_+ & 0 & 0 & 0 & 0 & 0 & b_- & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & b_- & 0 & 0 & 0 & 0 & 0 & b_+ & 0 \\
0 & 0 & a_+ & 0 & 0 & 0 & a_- & 0 & 0 \\
0 & 0 & 0 & c_- & 0 & c_+ & 0 & 0 & 0 \\
\end{pmatrix}
\end{align*}
\tag{4.9}
\]

\[
\begin{align*}
\begin{pmatrix}
an_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_- \\
0 & 0 & 0 & b_+ & 0 & b_- & 0 & 0 & 0 \\
0 & 0 & a_- & 0 & 0 & 0 & a_+ & 0 & 0 \\
0 & c_+ & 0 & 0 & 0 & 0 & 0 & c_- & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & c_- & 0 & 0 & 0 & 0 & 0 & c_+ & 0 \\
0 & 0 & a_+ & 0 & 0 & 0 & a_- & 0 & 0 \\
0 & 0 & 0 & b_- & 0 & b_+ & 0 & 0 & 0 \\
\end{pmatrix}
\end{align*}
\tag{4.10}
\]

From these the \( 3 \times 3 \) blocks can be read off.

We close this section by pointing out the relevance of our diagonalizer \( M \) of Sec.3 to the structure of \( L^+(\theta) \) (and hence of \( t(\theta) \)). If one constructs

\[
ML^+(\theta)M^{-1}
\tag{4.11}
\]

precisely the combinations on the left of the set (4.3) are seen to emerge. Thus our \( M \) leads directly to the remarkable structure (4.6).

5 \( \theta \)-expansion:

Let us start with the following notations and conventions:
(1): The condensed notation \((\alpha, \beta, \ldots)\) of (2.3) implies for each index \(\alpha\) either \((pp)\) or a triplet \((p, i, \epsilon), (i, j, \epsilon), \ldots\) and so on. We introduce sum over \(\alpha'\) where in \(\sum_{\alpha'}\) the index \((pp)\) is excluded. As for the other projectors one may consider alternatively either the basis given by (2.12) or that by (2.14).

(2): We also define

\[
H \equiv \sum_{\alpha'} m_{\alpha'} P_{\alpha'}
\]

(5.1)

when, using (2.3),

\[
H^n = (\sum_{\alpha'} m_{\alpha'} P_{\alpha'})^n = \sum_{\alpha'} m_{\alpha'}^n P_{\alpha'}
\]

(5.2)

Now one can expand as follows (with \(n \geq 1\))

\[
\hat{R}(\theta) = P_{pp} + \sum_{\alpha'} e^{m_{\alpha'} \theta} P_{\alpha'}
\]

\[
= P_{pp} + \sum_{\alpha'} \left(1 + \sum_n \frac{(m_{\alpha'} \theta)^n}{n!}\right) P_{\alpha'} = I + \sum_n \frac{\theta^n}{n!} (\sum_{\alpha'} m_{\alpha'}^n P_{\alpha'})
\]

\[
= I + \sum_n \frac{\theta^n}{n!} H^n = e^{\theta H}
\]

(5.3)

Addition of \(P_{pp} = (I - \sum_{\alpha'} P_{\alpha'})\) to \(H\) corresponds to a change of normalization of \(\hat{R}(\theta)\) along with an evident redefinition \(m_{\alpha'} \to (m_{\alpha'} - 1)\). None of the considerations below are affected by such a redefinition \((\sum_{\alpha'} \to \sum_{\alpha})\) of \(H\). More generally, say for \(q\)-deformed \((A, B, C, D)\)-type algebras, if \(\hat{R}(\theta)\) is spectrally resolved on a complete basis of projectors (Sec.2, Ref.1), setting \(l_i(\theta) = \ln k_i(\theta)\) and normalizing suitably one obtains, following the steps leading to (5.3),

\[
\hat{R}(\theta) = \sum_i k_i(\theta) P_i = \sum_i e^{l_i(\theta)} P_i = e^{(\sum_i l_i(\theta) P_i)}
\]

(5.4)
Here, in general, upon expansion in powers of $\theta$ the exponents $l_i(\theta)$ lead to fairly involved structures. In our present case $\theta$ is simply a factor in the exponent. Hence the situation is much simpler. Using (5.3) the braid equation becomes (with $H_{12} = H \otimes I$, $H_{23} = I \otimes H$)

$$e^{\theta H_{12}} e^{(\theta + \theta') H_{23}} e^{\theta' H_{12}} = e^{\theta' H_{23}} e^{(\theta + \theta') H_{12}} e^{\theta H_{23}}$$  \hspace{1cm} (5.5)

Setting, with $(n, n', n'') \geq 1$,

$$S = \sum_n \frac{\theta^n}{n!} H_{12}^n, \quad S' = \sum_{n'} \frac{\theta'^{n'}}{n'!} H_{12}^{n'}, \quad S'' = \sum_{n''} \frac{(\theta + \theta')^{n''}}{n''!} H_{23}^{n''}$$  \hspace{1cm} (5.6)

The left hand side of (5.5) is

$$(L.H.S.) = (I + S)(I + S'')(I + S')$$

$$= I + (S + S' + S'') + (SS' + SS'' + SS') + SSS'$$  \hspace{1cm} (5.7)

The (R.H.S.) is obtained from the (L.H.S.) via the following interchanges:

$$(12) \leftrightarrow (23), \quad \theta \leftrightarrow \theta'$$  \hspace{1cm} (5.8)

Now let us compare the coefficients of $\theta^r \theta^s$ for different pairs $(r, s)$ on both sides of (5.5).

The linear and the quadratic terms on both sides are found to be symmetric under (5.8) and hence cancel. Among the cubic terms only the coefficients of

$$\theta \theta' (\theta + \theta')$$

are found to lead to a nontrivial relation. One obtains, on regrouping terms,

$$[[H_{12}, H_{23}], H_{12}] = [[H_{23}, H_{12}], H_{23}]$$  \hspace{1cm} (5.9)

Compare this with (2.5). See also the remarks in Sec.7.
But from (5.2) one obtains

\[ H_{12}^2 = \sum_{\alpha'} m_{\alpha'}^2 (P_{\alpha'})_{12}, \quad H_{23}^2 = \sum_{\alpha'} m_{\alpha'}^2 (P_{\alpha'})_{23} \]  

(5.10)

Hence in terms of the projectors one obtains

\[ \sum_{\alpha',\beta',\gamma'} m_{\alpha'} m_{\beta'} m_{\gamma'} \left( (P_{\alpha'})_{12} (P_{\beta'})_{23} (P_{\gamma'})_{12} - (P_{\alpha'})_{23} (P_{\beta'})_{12} (P_{\gamma'})_{23} \right) \]

\[ = \frac{1}{2} \sum_{\alpha',\beta'} m_{\alpha'} m_{\beta'} (m_{\alpha'} - m_{\beta'}) \left( (P_{\alpha'})_{12} (P_{\beta'})_{23} - (P_{\alpha'})_{23} (P_{\beta'})_{12} \right) \]  

(5.11)

Since there are \( \frac{1}{2}(N + 3)(N - 1) \) independent parameters \( m_\alpha \), comparing coefficients of distinct triplets on each side one obtains a series of results. We will not display them explicitly. In Sec.3 of Ref.2 we have studied analogous reductions (from trilinear to bilinear forms) for \( q \)-deformed unitary, orthogonal and symplectic cases. There they were studied in the context of "modified braid equations" ([3], [4]) presented as a complementary facet of Baxterization (i.e., the introduction of a spectral parameter). Here we started from the \( \theta \)-dependent form (2.5) and implemented our \( \theta \)-expansion leading to the hierarchy starting with (5.9) and (5.11). Without attempting to analyse how the higher order members of the hierarchy can be reduced in order, in successive steps, we just mention the following point concerning (5.11).

In (1.18) of Ref.2, even for the orthogonal and the symplectic cases the modified braid equation could be expressed in terms of tensored \( (\hat{R}(\theta))^{\pm 1} \) by expressing the projectors in their terms using the minimal (cubic) polynomial equation satisfied by \( \hat{R}(\theta) \). For the unitary case (with a quadratic polynomial) the task was much more simple. In our present case, despite various particularly simple aspects, the order of the minimal polynomial increases as \( N^2 \) instead of remaining fixed as for the cases mentioned before.
Hence relations of the type (5.11) are best considered in terms of projectors themselves.

Expansions in terms of the spectral parameter has been considered in the context of Yangian Double and central extensions [5, 6]. We intend to study elsewhere analogous aspects generalizing our class of braid matrices.

6 Comparison with even dimensional cases:

The sequence of projectors presented in Sec.8 of Ref.1 is a direct generalization of the basis arising in the spectral resolution of the 6-vertex and the 8-vertex braid matrices. From the abundant literature on such models the most directly relevant sources are cited in Sec.6 and Sec.7 of Ref.1. These $4 \times 4$ projectors are (with $\epsilon = \pm 1$ in the matrices)

\[
2P_{1(\epsilon)} = \begin{pmatrix}
1 & 0 & 0 & \epsilon \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\epsilon & 0 & 0 & 1
\end{pmatrix}, \quad 2P_{2(\epsilon)} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & \epsilon & 0 \\
0 & \epsilon & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] (6.1)

But even for this simplest member ($N = 2$) of the hierarchy the coefficients in

\[
\hat{R}(\theta) = \sum_\epsilon (f_{1(\epsilon)}(\theta)P_{1(\epsilon)} + f_{2(\epsilon)}(\theta)P_{2(\epsilon)})
\] (6.2)

are not constrained to simple exponentials as for $N = (2p - 1)$. For the 8-vertex model (see sources cited in Sec.7 of Ref.1) one obtains

\[
f_{1(\pm)}(\theta) = \frac{g_{\pm}(\theta)}{g_{\pm}(-\theta)} \quad f_{2(\pm)}(\theta) = \frac{h_{\pm}(\theta)}{h_{\pm}(-\theta)}
\] (6.3)

where with $z = e^\theta$, two parameters $p$ and $q$ and

\[
(x; a)_\infty = \prod_{n \geq 0} (1 - xa^n)
\] (6.4)
\[ \begin{align*}
g_{\pm}(z) &= (\mp p^{\frac{1}{2}} q^{-1} z; p)_{\infty} (\mp p^{\frac{1}{2}} q z^{-1}; p)_{\infty} \\
h_{\pm}(z) &= (q^{\frac{1}{2}} z^{-\frac{1}{2}} \pm q^{-\frac{1}{2}} z^{\frac{1}{2}})(\mp pq^{-1} z; p)_{\infty} (\mp pq z^{-1}; p)_{\infty}
\end{align*} \] (6.5)

The question of normalization is discussed in Sec. 7 of Ref. 1. In the trigonometric 6-vertex limit one obtains (as in Sec. 6 of Ref. 1)

\[ f_{1}(\pm)(\theta) = 1, \quad f_{2}(+)(\theta) = \frac{cosh\frac{1}{2}(\gamma - \theta)}{cosh\frac{1}{2}(\gamma + \theta)}, \quad f_{2}(-)(\theta) = \frac{sinh\frac{1}{2}(\gamma - \theta)}{sinh\frac{1}{2}(\gamma + \theta)} \] (6.7)

The reason for such a scope is that ( unlike \( p = \bar{p} \) for \( N = (2p - 1) \) ) for even \( N \) there is no index \( i = \bar{i} \). The successive stages of the construction of solutions in App. A make it amply explicit how the presence of a \( p(= \bar{p}) \), along with the structure of the projectors in our nested sequence, constrains the coefficients to be simply exponentials. The generalization for \( N = 2n \quad (n > 1) \) of the hyperbolic and elliptic solutions displayed above will be explored elsewhere implementing our basis of projectors.

7 Discussion:

In Ref. 1 braid matrices were studied systematically via their spectral resolutions on appropriate bases of projectors. Such a study was already initiated in previous works ( Ref. 2 and Ref. 7 ) and led to canonical factorization and diagonalization in Ref. 1. In Sec. 8 of Ref. 1 this approach was taken to its limit. In the other sections almost all known braid matrices of interest were studied via spectral resolutions. In Sec. 8 a basis of projectors ( called a "nested sequence" ) with particularly simple, attractive properties was hopefully presented for constructing new classes of braid matrices in all dimensions. In such a basis, satisfying (2.3), one has \( N^2 \) matrices, each \( N^2 \times N^2 \) and with only constant elements ( see (2.2) and (6.1) ). They can be considered as
the most simple and symmetric generalizations of projectors appearing in the 6-vertex and the 8-vertex models. But the central question was not addressed in Ref.1. Can such a basis of projectors be dressed up with suitable coefficients to provide a braid matrix satisfying (2.5)? While the number of coefficients increases as $N^2$ the number of trilinear constraints on them corresponding to the products of $N^3 \times N^3$ matrices increases much faster. Hence the question. In this paper we we present an affirmative answer and explicit solutions for all odd $N$. The even-$N$ case will be studied elsewhere.

Let us note some basic features of our solutions in the context of the formulation in Ref.1. The canonically factorizable form of the coefficients [1] give

$$\hat{R}(\theta) = \sum_i \frac{f_i(\theta)}{f_i(-\theta)} P_i$$

(7.1)

This is evidently compatible with (2.7) since

$$e^{m\theta} = (e^{\frac{1}{2}m\theta})(e^{-\frac{1}{2}m\theta})^{-1}$$

But in Ref.1 we systematically extracted (see the relevant discussion in Ref.1) the standard (non-Baxterized) braid matrices satisfying

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$$

(7.2)

as the limits

$$\lim_{\theta \to \pm\infty} \hat{R}(\theta) = (\hat{R})^{\pm1}$$

(7.3)

For our present class of solutions however each coefficient $e^{m_\alpha}\theta$ either diverges or vanishes in the above limits. So rather than the Baxterization of a preexisting (7.2) to (2.5) this class can be considered (see Sec.5) to be an exponentiation of

$$[[H_{12}, H_{23}], H_{12}] = [[H_{23}, H_{12}], H_{23}]$$

(7.4)
$$\hat{R}_{12}(\theta)\hat{R}_{23}(\theta + \theta')\hat{R}_{12}(\theta') = \hat{R}_{23}(\theta')\hat{R}_{12}(\theta + \theta')\hat{R}_{23}(\theta) \quad (7.5)$$

since, as shown in Sec.5, the passage

$$H \equiv \sum_{\alpha'} m_{\alpha'} P_{\alpha'} \rightarrow \hat{R}(\theta) = e^{\theta H}$$

corresponds to one from (7.4) to (7.5).

One may compare this with the well-known so called "classical" $r$-matrix equation obtained by expanding the $q$-dependent YB matrix $R(\theta)(= P\hat{R}(\theta))$ satisfying

$$R_{12}(\theta)R_{13}(\theta + \theta')R_{23}(\theta') = R_{23}(\theta')R_{13}(\theta + \theta')R_{12}(\theta) \quad (7.6)$$

in powers of $h(= \ln q)$. One obtains for

$$R_q(\theta) = I + 2hr(\theta) + O(h^2)$$

$$[r_{12}(\theta), (r_{13}(\theta + \theta') + r_{23}(\theta'))] + [r_{13}(\theta + \theta'), r_{23}(\theta')] = 0 \quad (7.7)$$

This has only single commutators. In our case there is no $q$. Expanding in powers of $\theta$ we obtain as the first nontrivial relation the equation (7.4) with double commutators and with the two sides still directly related through the interchange (12) $\leftrightarrow$ (23). In the extensive literature concerning $r$-matrices one may note in particular a classification of solutions (Ref.8). Our projectors lead to a solution of (7.4) with $\frac{1}{2}(N + 3)(N - 1)$ parameters for $N = (2p - 1)$. A more general study, starting from (7.4) should be worthwhile.
We repeat a feature noted in Sec.5. Our class of solutions has many particularly simple aspects. But the number of projectors ($P_{\alpha}$) and that of the parameters ($m_{\alpha}$) increase as $N^2$ with the dimension. The degree of the minimal polynomial equation satisfied by $\hat{R}(\theta)$ increases with them. This is in sharp contrast with well-known cases corresponding to $q$-deformed unitary, orthogonal and symplectic cases. There the structures of the projectors are much less simple. But their number does not increase with the dimension. As noted below (2.14), the degree of the minimal polynomial can be lowered by allowing some of the free parameters to coincide, giving simpler subcases. But our solution is more general.

For $m_{ab}^+ > m_{ab}^-$ all the nonzero elements of our $\hat{R}(\theta)$ are positive and hence can be consistently interpreted as Boltzmann weights of a multistate statistical model. In Sec.11 of Ref.1 the possibility of a class of multistate model was briefly indicated and compared with one proposed in Ref.9. (See also Sec.4 of Ref.10.) In both cases $(2N^2 - N)$ elements out of $N^4$ ones of $\hat{R}(\theta)$ are nonzero. Here we have $(2N^2 - 1)$ nonzero weights. Moreover the explicit solution of Ref.9 (and Ref.10) restricts the number of parameters as in the 6-vertex model (Sec.6). For our present class there is more scope in this respect.

It is a pleasure to thank Daniel Arnaudon. Using a program, he verified for the first member of our hierarchy of solutions that the constraints obtained here are not only sufficient but also necessary. This was reassuring.

8 APPENDIX A. Solving the braid equation:

In (2.6), namely,
corresponding to the site \((ab) \otimes (cd) \otimes (ef)\) one has to implement the content of the ansatz (2.4). From (2.2) and (2.4) one obtains the following nonzero elements of \(\hat{R}(\theta)\). The arguments \(\theta\) is suppressed in (A.2) to simplify the notation and the subscripts correspond to the sites \((ab) \otimes (cd)\).

\[
\begin{align*}
\hat{R}_{pp,pp} &= 1 \\
\hat{R}_{pp,ii} &= \frac{1}{2}(f_{pi}^{(+)} + f_{pi}^{(-)}) = \hat{R}_{pp,\bar{i}} \\
\hat{R}_{pp,\bar{i}i} &= \frac{1}{2}(f_{pi}^{(+)} - f_{pi}^{(-)}) = \hat{R}_{pp,\bar{i}} \\
\hat{R}_{ii,pp} &= \frac{1}{2}(f_{ip}^{(+)} + f_{ip}^{(-)}) = \hat{R}_{ii,pp} \\
\hat{R}_{ii,\bar{i}p} &= \frac{1}{2}(f_{ip}^{(+)} - f_{ip}^{(-)}) = \hat{R}_{ii,pp} \\
\hat{R}_{ii,jj} &= \frac{1}{2}(f_{ij}^{(+)} + f_{ij}^{(-)}) = \hat{R}_{ii,jj} \\
\hat{R}_{i\bar{i},jj} &= \frac{1}{2}(f_{ij}^{(+)} - f_{ij}^{(-)}) = \hat{R}_{ii,jj} \\
\hat{R}_{ii,\bar{j}j} &= \frac{1}{2}(f_{ij}^{(+)} + f_{ij}^{(-)}) = \hat{R}_{ii,jj} \\
\hat{R}_{i\bar{i},j\bar{j}} &= \frac{1}{2}(f_{ij}^{(+)} - f_{ij}^{(-)}) = \hat{R}_{ii,jj}
\end{align*}
\] (A.2)

These are the only nonzero elements, the total number being

\[1 + 8(p - 1) + 8(p - 1)^2 = 2(2p - 1)^2 - 1 = 2N^2 - 1\]

Note the following points:
The elements above all being situated on the diagonal and the antidiagonal there are none of the type $\hat{R}_{i,i,j,j}$, $\hat{R}_{i,i,j,j}$ and so on.

- In the product $(ab) \otimes (cd) \otimes (ef)$ for a given $a, b$ can only be $a$ or $\bar{a}$ for the coefficient to be nonzero. This holds also for the other pairs.

- Among $(a, b, c, d, e, f)$ the number of with (or without) bar must be even for the coefficient to be nonzero. This is one consequence of (A.2). However, in such countings one must keep in mind that $p = \bar{p}$.

The preceding considerations simplify considerably the computations as we analyse systematically the different classes of $(ab) \otimes (cd) \otimes (ef)$ with nonvanishing coefficients, lowering the multiplicity of $(pp)$ in the triple product above by steps.

Case (1): The case $(pp) \otimes (pp) \otimes (pp)$ is trivial since (A.1) reduces to

$$1 = 1$$

Case (2): Next consider the classes (with $(ab) \neq (pp)$)

1. $(pp) \otimes (pp) \otimes (ab)$,
2. $(ab) \otimes (pp) \otimes (pp)$,
3. $(pp) \otimes (ab) \otimes (pp)$

From our previous remarks it follows that it is sufficient to consider the possibilities

$$(ab) = (ii), (\bar{i}\bar{i})$$

Note also that in (A.2) $\hat{R}_{pp,ii} = \hat{R}_{pp,\bar{i}\bar{i}}$ and so on.

For (1), (A.1) is easily seen to reduce to

$$\hat{R}(\theta + \theta')_{pp,ab} = \hat{R}(\theta')_{pp,ac} \hat{R}(\theta)_{pp,cb}$$  \hspace{1cm} (A.3)
Analogous treatments of the subcases (1), (2), (3) lead respectively (implementing (A.2) with $\epsilon = \pm$ and also both possibilities for $(ab)$ mentioned above) to the constraints

$$ f^{(\epsilon)}_{pi}(\theta + \theta') = f^{(\epsilon)}_{pi}(\theta) f^{(\epsilon)}_{pi}(\theta') $$ \hspace{1cm} (A.4)

$$ f^{(\epsilon)}_{ip}(\theta + \theta') = f^{(\epsilon)}_{ip}(\theta) f^{(\epsilon)}_{ip}(\theta') $$ \hspace{1cm} (A.5)

$$ f^{(+)}_{pi}(\theta) f^{(+)}_{pi}(\theta') + f^{(-)}_{pi}(\theta) f^{(-)}_{pi}(\theta') = f^{(+)}_{pi}(\theta + \theta') f^{(+)}_{ip}(\theta') + f^{(-)}_{pi}(\theta + \theta') f^{(-)}_{ip}(\theta') $$ \hspace{1cm} (A.6)

On implementing (A.4) and (A.5) one reduces (A.6) to an identity. Then from the first two one obtains the solutions

$$ f^{(\epsilon)}_{pi}(\theta) = e^{m^{(\epsilon)}_{pi}\theta} $$ \hspace{1cm} (A.7)

$$ f^{(\epsilon)}_{ip}(\theta) = e^{m^{(\epsilon)}_{ip}\theta} $$ \hspace{1cm} (A.8)

the indeterminates $m^{(\epsilon)}_{pi}$, $m^{(\epsilon)}_{ip}$ being independent parameters.

Continuing to reduce the multiplicity of $(pp)$ and remembering the restrictions implied by (A.2) we start by considering successively the cases

(4) : $(pp) \otimes (ii) \otimes (jj)$

(5) : $(pp) \otimes (i\bar{i}) \otimes (j\bar{j})$

(6) : $(pp) \otimes (i\bar{i}) \otimes (jj)$

The last one survives with nonzero coefficient since $p = \bar{p}$. We present directly the results, the derivations being straightforward.

Defining

$$ A_{ab}(\theta) \equiv f^{(+)}_{ab}(\theta) + f^{(-)}_{ab}(\theta), \quad B_{ab}(\theta) \equiv f^{(+)}_{ab}(\theta) - f^{(-)}_{ab}(\theta) $$
one obtains respectively from the above cases

\[ A_{ij}(\theta + \theta') = f_{ij}^{(+)}(\theta) f_{ij}^{(+)}(\theta') + f_{ij}^{(-)}(\theta) f_{ij}^{(-)}(\theta') \] (A.9)

\[ A_{pi}(\theta)A_{pi}(\theta')B_{ij}(\theta + \theta') + B_{pi}(\theta)B_{pi}(\theta')B_{ij}(\theta + \theta') = A_{pi}(\theta + \theta')(B_{ij}(\theta)A_{ij}(\theta') + A_{ij}(\theta)B_{ij}(\theta')) \] (A.10)

\[ A_{pi}(\theta)B_{pi}(\theta')A_{ij}(\theta + \theta') + B_{pi}(\theta)A_{pi}(\theta')A_{ij}(\theta + \theta') = B_{pi}(\theta + \theta')(A_{ij}(\theta')A_{ij}(\theta) + B_{ij}(\theta')B_{ij}(\theta)) \] (A.11)

Taking account of (A.4) and (A.5) (and hence of (A.7) and (A.8)) and noting that keeping \((\theta + \theta')\) fixed one can vary \(\psi\) in

\[ \theta = \phi + \psi, \quad \theta' = \phi - \psi \]

one finds that the last three equations are satisfied if

\[ f_{ij}^{(e)}(\theta) = f_{ij}^{(e)}(\theta) \] (A.12)

and

\[ f_{ij}^{(e)}(\theta) f_{ij}^{(e)}(\theta') = f_{ij}^{(e)}(\theta + \theta') \] (A.13)

These are found to be necessary and sufficient. Hence

\[ f_{ij}^{(e)}(\theta) = f_{ij}^{(e)}(\theta) = e^{\mu_{ij}\theta} \] (A.14)

Permutation of the factors of the cases \((4, 5, 6)\) above (such as \((ii) \otimes (pp) \otimes (jj)\) and so on) can be shown to lead to no supplementary constraints.

Finally one considers the cases

\( (ab) \otimes (cd) \otimes (ef) \)
where no factor is \((pp)\). For each subcase the constraints implied by \((A.1)\) along with \((A.2)\) are easily extracted. It is found that they are all satisfied by implementing \((A.12)\) and \((A.13)\). Since the subcases are treated quite similarly, it is sufficient to display two of them. We present again only the the final steps. For

\[(ii) \otimes (jj) \otimes (kk)\]

with no barred index, \((A.13)\) reduces \((A.1)\), in terms of \(A_{ab}\) defined above, to

\[
L.H.S. = \frac{1}{4} A_{ij}(\theta + \theta') A_{jk}(\theta + \theta') = R.H.S. \tag{A.15}
\]

Similarly, for

\[(\bar{i}i) \otimes (jj) \otimes (kk)\]

one obtains finally

\[
L.H.S. = \frac{1}{4} B_{ij}(\theta + \theta') B_{jk}(\theta + \theta') = R.H.S. \tag{A.16}
\]

In both cases, apart from the exponential form for each \(f\), \((A.14)\) is essential. Thus we have verified the solution announced in (2.7) and (2.8). It is instructive to compute explicitly the case (2.9) where one has only \((i, \bar{i}, p)\) with

\[i = 1, \quad p = 2\]

One finds that (2.10) is sufficient. Moreover, if one sets

\[m_{11}^{(e)} \neq m_{11}^{(e)} \tag{A.17}\]

so that \(a_{\pm}\) is not repeated as in (2.9), the braid equation is \textit{not} satisfied. This is an example of the necessity of \((A.14)\).

As a check, the solution for \(N = 3\) was also obtained (instead of directly using \((A.1)\) and \((A.2)\)) by computing the triple tensor products of the projectors in (2.5).
9 APPENDIX B. L-operators and transfer matrices ( fundamental representations ) :

Here we collect together some known results ( citing sources below ) coherently with our notations and conventions and emphasize certain aspects arising in the presence of the spectral parameter $\theta$.

For non-Baxterized braid matrices ( without $\theta$ ) satisfying

$$\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23} \quad (B.1)$$

the FRT equations for the $L$-operators ( eqn.(2.3) of Ref.11 ) can be expressed in our notations as

$$\hat{R} L^\pm_2 L^\pm_1 = L^\pm_2 L^\pm_1 \hat{R} \quad (B.2)$$

$$\hat{R} L^+_2 L^-_1 = L^-_2 L^+_1 \hat{R} \quad (B.3)$$

Here $\hat{R}$ is a $N^2 \times N^2$ matrix for any $N$ and

$$L_1 = L \otimes I_{N \times N}, \quad L_2 = I_{N \times N} \otimes L$$

Writing these in terms of components ( as will be done below for the $\theta$-dependent case ) it can be shown that the lowest dimensional realizations

of the $N^2$ blocks $L^\pm_{ab}$ ( each $N \times N$ ) can be obtained in our notations i.e, with

$$\hat{R} = \hat{R}_{ab,cd}(ab) \otimes (cd) \quad (B.4)$$

as

$$(L^+_a)_{cd} = \hat{R}_{ad,cb} \quad (B.5)$$

$$(L^-_a)_{cd} = \hat{R}^{-1}_{ad,cb} \quad (B.6)$$

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or
\begin{align*}
L^+ &= \hat{R}P = (PR)P = R_{21} \quad (B.7) \\
L^- &= \hat{R}^{-1}P = (R^{-1}P)P = R^{-1} \quad (B.8)
\end{align*}

Apart from differences of notations and conventions these correspond (to cite only one source) to eqns. (4.9) of Ref. 12. In the familiar $L^\pm$ of $Sl_q(2)$, implementing $2 \times 2$ realizations of $(q^\pm H, X_\pm)$ one obtains $(B.7)$ and $(B.8)$, which however hold for any $\hat{R}$ satisfying $(B.1)$.

Now let us introduce $\theta$. Corresponding to $(B.1)$ and $(B.2)$ one now has respectively
\begin{align*}
\hat{R}_{12}(\theta - \theta')\hat{R}_{23}(\theta)\hat{R}_{12}(\theta') &= \hat{R}_{23}(\theta')\hat{R}_{12}(\theta)\hat{R}_{23}(\theta - \theta') \quad (B.9) \\
\hat{R}(\theta - \theta')L^\pm_2(\theta)L^\pm_2(\theta') &= L^\pm_2(\theta')L^\pm_2(\theta)\hat{R}(\theta - \theta') \quad (B.10)
\end{align*}

( The corresponding situation for $(B.3)$ will be discussed below. )

In terms of components one writes
\begin{align*}
(\hat{R}(\theta - \theta'))_{al,cm}(\hat{R}(\theta))_{mn,ef}(\hat{R}(\theta'))_{lb,nd} &= (\hat{R}(\theta'))_{cl,em}(\hat{R}(\theta))_{ab,ln}(\hat{R}(\theta - \theta'))_{nd,mf} \quad (B.11) \\
(\hat{R}(\theta - \theta'))_{al,cm}(L^\pm(\theta))_{mn,ef}(L^\pm(\theta'))_{lb,nd} &= (L^\pm(\theta'))_{cl,em}(L^\pm(\theta))_{ab,ln}(\hat{R}(\theta - \theta'))_{nd,mf} \quad (B.12)
\end{align*}

One finds that (considering $L^+(\theta)$ to start with)
\begin{align*}
(L^+(\theta))_{ab,cd} &= (\hat{R}(\theta))_{ad,eb} \quad (B.13)
\end{align*}
or
\begin{align*}
L^+(\theta) &= \hat{R}(\theta)P = PR(\theta)P = R_{21}(\theta) \quad (B.14)
\end{align*}
is a solution. This is strictly analogous to (B.7). The same solution evidently holds for $L^-(\theta)$. But if one wants to avoid the degeneracy

$$L^-(\theta) = L^+(\theta) \quad (B.15)$$

can one obtain a different solution for $L^-(\theta)$ analogous to (B.8)? We show below in a particularly transparent fashion that there is an obstruction if one directly generalizes (B.3) as

$$\hat{R}(\theta - \theta')L^+_2(\theta)L^-_1(\theta') = L^-_2(\theta')L^+_1(\theta)\hat{R}(\theta - \theta') \quad (B.16)$$

When this is further generalized by introducing a central operator in the argument of $\hat{R}(\theta - \theta')$ on one side (or in a different fashion on each side) and thus distinguish the two arguments, there can be a way out. (Ref.13 is a review article citing numerous sources. Particularly relevant is Sec.2.1.4.) But let us consider the consequences of (B.16) combined with (B.13), (B.14) and the basic properties (2.15), i.e,

$$\hat{R}(-\theta) = \hat{R}^{-1}(\theta), \quad \hat{R}(0) = I \quad (B.17)$$

From (B.14) and (B.17),

$$L^+(0) = P, \quad L^+(0)_{ab} = (ba) \quad (B.18)$$

(This has no counterpart for (B.7).)

Hence setting $\theta = 0$ in (B.16), using (B.16) and (B.17) and then writing $\theta$ for $\theta'$ one obtains

$$\hat{R}^{-1}(\theta)P_2L^-_1(\theta) = L^-_2(\theta)P_1\hat{R}^{-1}(\theta) \quad (B.19)$$

Writing (B.19) in terms of components analogously to (B.12) one obtains

$$(\hat{R}(-\theta))_{al,cm}(\delta_{ma}\delta_{fe})(L^-(\theta)_{ld})_{nb} = (L^-(\theta)_{cm})_{et}(\delta_{ab}\delta_{nl})(\hat{R}(-\theta))_{nd,mf} \quad (B.20)$$
or

\[(\hat{R}^{-1}(\theta))_{al,cm}(L^{-}(\theta)_{ld})_{mb}\delta_{ef} = \delta_{ab}(L^{-}(\theta)_{cm})_{et}(\hat{R}^{-1}(\theta))_{ld,mf}\] (B.21)

Hence, finally,

\[(\hat{R}^{-1}(\theta)L^{-}(\theta)P) \otimes I = I \otimes (L^{-}(\theta)P\hat{R}^{-1}(\theta))\] (B.22)

For

\[L^{-}(\theta) = L^{+}(\theta) = \hat{R}(\theta)P, \quad L^{-}(\theta)P = \hat{R}(\theta)\] (B.23)

(B.22) is trivially satisfied (furnishing a convincing check). But a distinct solution for \(L^{-}(\theta)\) reducing (say, as \(\theta \rightarrow \infty\)) to (B.8) is no longer available in the general case if (B.16) is strictly maintained. One obtains (B.8) easily from the symmetry of (B.1) under inversion since, unlike for (B.16), the orders of \((\theta, \theta')\) on each side do not enter in that context. But even apart from that (B.18) now imposes the constraint (B.22), linear in \(L^{-}(\theta)\). We do not consider in this paper generalizations of (B.16) leading to quasi-Hopf structures for consistent coproducts.

We now consider transfer matrices and note how the lowest dimensional representations can be extracted from those of the \(L\)-operators. The transfer matrix \(t(\theta)\) has to satisfy

\[\hat{R}(\theta - \theta')(t(\theta) \hat{\otimes} t(\theta')) = (t(\theta') \hat{\otimes} t(\theta))\hat{R}(\theta - \theta')\] (B.24)

where \(\hat{\otimes}\), combining tensor and matrix products leads to

\[t(\theta) \hat{\otimes} t(\theta') = (t(\theta) \otimes I)(I \otimes t(\theta')) = t_1(\theta).t_2(\theta')\] (B.25)

Writing (B.24) as

\[(P\hat{R}(\theta - \theta')P)(P(t_1(\theta)P)(Pt_2(\theta')P)\]
\[ = (Pt_1(\theta')P)(Pt_2(\theta)P)(P\hat{R}(\theta - \theta')P) \quad (B.26) \]

or

\[ \hat{R}_{21}(\theta - \theta')t_2(\theta)t_1(\theta') = t_2(\theta')t_1(\theta)\hat{R}_{21}(\theta - \theta') \quad (B.27) \]

Now comparing (B.27) with (B.10) and (B.14) one finds the solution

\[ t(\theta) = (P\hat{R}(\theta)P)P = P\hat{R}(\theta) = R(\theta) \quad (B.28) \]

In absence of \( \theta \), i.e., for (B.1), this corresponds (with some notational differences) to the realization \( \rho_+ \) of of eqn. (4.5) of Ref. 12. But corresponding to (B.8), unavailable in our context, there is another realization \( \rho_- \) in Ref. 12. We are concerned only with (B.28). Products analogous to (4.8) of our Sec. 4 lead to higher dimensional transfer matrices corresponding to longer chains as successive sites are added.

**References**

[1] A.Chakrabarti, J.Math.Phys. **44**, 5320 (2003)

[2] A.Chakrabarti and R.Chakrabarti, J.Math.Phys. **44**, 785 (2003)

[3] M.Gerstenhaber, A.Giaquinto and S.D.Schack, Isr.Math.Conf.Pr. **7**, 45 (1993)

[4] M.Gerstenhaber and A.Giaquinto, Lett.Math.Phys. **44**, 131 (1998)

[5] S.M.Khoroshkin and V.Tolstoy, Lett.Math.Phys. **36**, 373 (2003)

[6] S.M.Khoroshkin, Central extension of the Yangian double, q-alg/9602031
[7] D. Arnaudon, A. Chakrabarti, V. K. Dobrev and S. Mihov, Int. J. Mod. Phys. A18, 4201 (2003)

[8] A. A. Belavin and V. G. Drinfeld, Funkt. anal. iego prolozh. 16 1 (1982)

[9] O. Babelon, H. J. de Vega and C. Viallet, Nucl. Phys. B190, 542 (1981)

[10] H. J. de Vega, Int. J. Mod. Phys. A 4, 2371 (1989)

[11] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtadzhyan, Leningrad Math. J. 1, 193 (1990)

[12] S. Majid, Foundations of Quantum Group Theory, C. U. P. (1995)

[13] L. Frappat, Quantum elliptic algebras and double Yangians, math.QA /0201245