Conformal type of ends of revolution in space forms of constant sectional curvature

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Received: 8 June 2015 / Accepted: 6 November 2015 / Published online: 25 November 2015
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Abstract In this paper, we consider the conformal type (parabolicity or non-parabolicity) of complete ends of revolution immersed in simply connected space forms of constant sectional curvature. We show that any complete end of revolution in the 3-dimensional Euclidean space or in a 3-dimensional sphere is parabolic. In the case of ends of revolution in the hyperbolic 3-dimensional space, we find sufficient conditions to attain parabolicity for complete ends of revolution using their relative position to the complete flat surfaces of revolution.

Keywords End of revolution · Parabolicity · Stochastic completeness · Euclidean space · Sphere · Hyperbolic space

Mathematics Subject Classification Primary 53C20 · 53C40; Secondary 53C42

1 Introduction

Let \( \Sigma \) be a complete and non-compact surface. Let \( D \subset \Sigma \) be an open precompact subset of \( \Sigma \) with a smooth boundary. An end \( E \) of \( \Sigma \) with respect to \( D \) is a connected unbounded component of \( \Sigma \setminus D \). An end \( E \) is parabolic [13, 24, 27] if every bounded harmonic function on \( E \) is determined by its boundary value.

This paper is concerned with the study of the conformal type (parabolicity or non-parabolicity) of complete ends of revolution immersed in the 3-dimensional Euclidean space \( \mathbb{R}^3 \), in the 3-dimensional hyperbolic space \( \mathbb{H}^3 \), or in the 3-dimensional sphere \( S^3 \). Let us

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1 Work partially supported by the Universitat Jaume I Research Program Project P1-1B2012-18, and DGI-MINECO grant (FEDER) MTM2013-48371-C2-2-P.
denote by \( \mathbb{M}^3(\kappa) \) the simply connected space form of constant sectional curvature \( \kappa \in \mathbb{R} \). Hence, \( \mathbb{M}^3(1) = \mathbb{S}^3 \), \( \mathbb{M}^3(0) = \mathbb{R}^3 \), \( \mathbb{M}^3(-1) = \mathbb{H}^3 \). An end of a complete surface in \( \mathbb{M}^3(\kappa) \) is a **complete end of revolution** if there exists a geodesic in \( \mathbb{M}^3(\kappa) \) such that the end is invariant by the group of rotations of \( \mathbb{M}^3(\kappa) \) that leave this geodesic point-wise fixed. More precisely, see Definitions 2.1 and 2.2, an end of revolution will be the rotation along a geodesic ray \( \gamma \) of \( \mathbb{M}^3(\kappa) \) of a profile curve \( \beta : [0, \infty) \rightarrow \mathbb{M}^2(\kappa) \) contained in a totally geodesic hypersurface \( \mathbb{M}^2(\kappa) \) where the ray \( \gamma \) belongs. In order to guarantee smoothness and that the end is the end of a complete surface, we require that the generating curve \( \beta \) be regular, with infinite longitude and does not intersect the geodesic ray \( \gamma \).

The conformal type of a Riemannian manifold has been largely studied. In particular, in [34] sufficient and necessary conditions for the parabolicity of a manifold with a warped cylindrical end were provided, in [13] rotationally symmetric manifolds were analyzed, and in the examples in [19, 26] certain surfaces of revolution in \( \mathbb{R}^3 \) were studied from an extrinsic approach.

A complete end of revolution in \( \mathbb{M}^3(\kappa) \) is isometric to \( [0, \infty) \times \mathbb{S}^1 \) with warping metric

\[
g = ds^2 + w^2_\kappa(s)d\theta^2,
\]

where the warping function \( w_\kappa \) is determined by the profile curve of the end and the sectional curvature of the ambient space (see Proposition 2.3). In Proposition 2.5, we prove that an end of revolution is isometric to a rotationally symmetric model space where we have subtracted a geodesic ball centered at the center of the model. Hence, by using the general criterion for parabolicity of rotationally symmetric model spaces given in [13] (or the criterion for manifolds with only one cylindrical end given in [34]), we can study the conformal type of complete ends of revolution in \( \mathbb{M}^3(\kappa) \).

Our first result characterizes the conformal type of complete ends of revolution in \( \mathbb{R}^3 \) or in \( \mathbb{S}^3 \)

**Theorem A.** Any complete end of revolution in \( \mathbb{R}^3 \), or in \( \mathbb{S}^3 \), is a parabolic end.

The conformal classification of ends of revolution in \( \mathbb{H}^3 \) becomes more complicated. In hyperbolic space there is no restriction on the conformal type of ends of revolution. In fact, in Sect. 9 we will show examples of parabolic and non-parabolic ends in \( \mathbb{H}^3 \). In the half-space model of hyperbolic space,

\[
\mathbb{H}^3 : = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}, \quad g_{\mathbb{H}^3} = \frac{1}{x_3^3} (dx_1^2 + dx_2^2 + dx_3^2), \quad (1.1)
\]

we can provide a sufficient condition for parabolicity using a c-cone.

**Definition 1.1** Given \( c \in \mathbb{R}_+ \), a c-cone in \( \mathbb{H}^3 \) is the rotation around the \( x_3 \)-axis of curve

\[
\beta : (0, \infty) \rightarrow \mathbb{H}^3, \quad \beta(t) = (t, 0, ct). \quad (1.2)
\]

Observe that a c-cone is the set of points at a fixed distance from the \( x_3 \)-axis and any c-cone divides the hyperbolic space \( \mathbb{H}^3 \) into two parts (see Fig. 1): one part on the c-cone, and the other part under the c-cone. Using this partition property of the c-cones, we can state the following Theorem:

**Theorem B.** Let \( E \) be a complete end of revolution in \( \mathbb{H}^3 \). Suppose that the end \( E \) is contained in the upper half-space of \( \mathbb{H}^3 \) determined by the c-cone for some \( c > 0 \). Then, the end \( E \) is parabolic.
The $c$-cone (or the set of points at a fixed distance from a geodesic) and the horosphere are the only complete flat surfaces immersed in $\mathbb{H}^3$. Every one of these flat surfaces divides $\mathbb{H}^3$ into two parts. If an end of revolution is contained on a $c$-cone, or on (inside) a horosphere (horoball), the end is parabolic.

Another sufficient condition can be provided using horospheres.

**Theorem C.** Let $E$ be a complete end of revolution in $\mathbb{H}^3$. Suppose that the end $E$ is contained in the upper half-space of $\mathbb{H}^3$ determined by the horosphere $\{x_3 = z\}$ for some $z > 0$. Then, the end $E$ is parabolic.

By using the above Theorem we can characterize the conformal type of ends of revolution immersed inside a compact set of hyperbolic space.

**Corollary D.** Let $E$ be a complete end of revolution in $\mathbb{H}^3$ contained in a compact set of $\mathbb{H}^3$. Then, $E$ is a parabolic end.

Moreover, Theorem C allows us to know that complete non-parabolic ends of revolution in $\mathbb{H}^3$ approach the $\{x_3 = 0\}$ plane.

**Corollary E.** Let $E$ be a complete and non-parabolic end of revolution in $\mathbb{H}^3$. Then,

$$\inf_{p \in E} x_3(p) = 0.$$
Otherwise, the Brownian motion on $\Sigma$ is recurrent. Given a complete and non-compact surface $\Sigma$ and an open precompact set $D \subset \Sigma$, the Brownian motion is recurrent if and only if every end of $\Sigma$ is parabolic with respect to $D$.

Another property of Brownian motion to be considered in this paper is stochastic completeness. This is a property of a stochastic process having an infinite lifetime. In other words, a process is stochastically complete if the total probability of the particle being found in the space is constantly equal to 1. For Brownian motion, this means

$$\int_{\Sigma} p(t, x, y) dA(y) = 1,$$

(1.6)

for any $t > 0$. Therefore, the heat kernel is an authentic measure of probability. A geodesically complete manifold with recurrent Brownian motion is (see [13]) stochastically complete. Hence, taking into account Theorem A, any surface with only complete ends of revolution in $\mathbb{R}^3$ or $S^3$ is stochastically complete. For complete ends of revolution in $M^3(\kappa)$ we can state

**Theorem F.** Let $\Sigma$ be a complete and non-compact surface of finite topological type immersed in $M^3(\kappa)$ with $\kappa \in \mathbb{R}$. Suppose that there exists a compact subset $\Omega \subset \Sigma$ of $\Sigma$ such that every end of $\Sigma$ with respect to $\Omega$ is a complete end of revolution in $M^3(\kappa)$. Then, $\Sigma$ is stochastically complete.

In [17], Hoffman and Meeks proved that a properly immersed minimal surface in $\mathbb{R}^3$ disjoint from a plane is a plane. Otherwise stated, if $M$ is a minimal surface properly immersed in $\mathbb{R}^3$ and for some $c > 0$, $M \cap \{z > c\} \neq \emptyset$, then either $M \cap \{z = c\} \neq \emptyset$ or $M$ is a plane parallel to $\{z = c\}$.

For hyperbolic space, in [32] Rodriguez and Rosenberg proved that every constant mean curvature one surface $M$, properly embedded in a horoball $B \subset H^3$ such that $M \cap \partial B = \emptyset$, is a horosphere. As a surprising corollary of Theorem F, we obtain

**Theorem G.** Let $M$ be a complete non-compact surface of revolution properly immersed in $H^3$. Suppose $M \cap B \neq \emptyset$ for some horoball $B \subset H^3$ then,

1. If $M$ has negative sectional curvature, $M \cap \partial B \neq \emptyset$ (otherwise stated, $M$ touches the horosphere $\partial B$).
2. If $M$ has constant non-positive sectional curvature and $M \cap \partial B = \emptyset$, $M$ is a horosphere.
3. If $M$ has constant mean curvature with $\|H\| \leq 1$, and $M \cap \partial B = \emptyset$, $M$ is a horosphere.

### 1.1 Outline of the paper

The structure of this paper is as follows.

In Sect. 2, we introduce the definition of complete end of revolution and show that any complete end of revolution can be considered a submanifold smoothly immersed in $M^3(\kappa)$, and intrinsically each end of revolution is endowed with a warped product metric. Indeed, in Corollary 2.5 it is proved that any end of revolution is isometric to a rotationally symmetric 2-dimensional manifold where a geodesic ball is subtracted. That allows us, by using the well-known criteria for parabolicity of rotationally symmetric model manifolds, to obtain Theorem 2.8 and Corollary 2.9, where sufficient and necessary conditions for parabolicity in terms of the warped function of each end of revolution are provided. In Sect. 2.1, and making use of conformal models of $M^3(\kappa)$, we obtain the explicit expressions of such warping functions. With these techniques we can prove Theorems A, B, C, F and G in Sects. 3, 4, 5, 6, and 7, respectively. Finally, Sect. 9 deals with several examples of application of the main Theorems.
2 Preliminaries

2.1 Conformal models of $\mathbb{M}^3(\kappa)$ and ends of revolution

The real space forms $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ are the 3-dimensional simply connected real space forms $\mathbb{M}^3(\kappa)$ of constant sectional curvature $\kappa = 0$, 1 and $-1$, respectively. From now on we are going to work with conformal models of $\mathbb{M}^3(\kappa)$ (see [23]), namely

$$\mathbb{R}^3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : g = dx_1^2 + dx_2^2 + dx_3^2\}$$

$$\mathbb{H}^3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3, x_3 > 0 : g_{-1} = \frac{1}{x_3^2} (dx_1^2 + dx_2^2 + dx_3^2)\}$$

$$\mathbb{S}^3 - \{N\} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : g_1 = \frac{4 (dx_1^2 + dx_2^2 + dx_3^2)}{(1 + x_1^2 + x_2^2 + x_3^2)^2}\}$$

Indeed, $\mathbb{M}^3(\kappa)$ can be seen as $\mathbb{R}^3$ endowed with a conformal metric:

$$\mathbb{M}^3(\kappa) := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : g_\kappa = \eta_\kappa(x) \cdot g\}$$

with

$$\eta_\kappa(x) := \begin{cases} 
\frac{1}{x_3^2} & \text{if } \kappa = -1 \\
1 & \text{if } \kappa = 0 \\
\frac{4}{(1+x_1^2+x_2^2+x_3^2)} & \text{if } \kappa = 1
\end{cases}$$

Observe that the $x_3$-axis in such models is a geodesic curve. For any $\theta \in [0, 2\pi]$ we can define the rotation $R_\theta(x)$ of angle $\theta$ around the $x_3$ axis of the point $x = (x_1, x_2, x_3) \in \mathbb{M}^3(\kappa)$ as

$$R_\theta \begin{pmatrix} x_1 \\
x_2 \\
x_3 \end{pmatrix} := \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\
x_2 \\
x_3 \end{pmatrix}$$

The rotation $R_\theta$ leaves the points of the $x_3$-axis fixed. In fact for any point $x \in \mathbb{M}^3(\kappa)$ on the $x_3$-axis, the group

$$G := \{R_\theta : \theta \in [0, 2\pi]\}$$

with the multiplication law given by the composition, is the subgroup of the isotropy group of $x$ such that the geodesic curve $\gamma(t) = (0, 0, x_3 + t)$ remains fixed under the action of $G$. Observe moreover that the $(x_1, x_3)$-plane is a totally geodesic submanifold of $\mathbb{M}^3(\kappa)$. Now we can define a complete end of revolution around the $x_3$-axis and a complete end of revolution in $\mathbb{M}^3(\kappa)$.

**Definition 2.1** (Complete end of revolution around the $x_3$-axis) Let $\gamma : [0, \infty) \to \mathbb{M}^2(\kappa)$ be a regular curve of infinite length in the $(x_1, x_3)$-plane of $\mathbb{M}^3(\kappa)$ and such that $\gamma_1(t) > 0, \ \forall t \in [0, \infty)$. A complete end of revolution $E$ around the $x_3$-axis is the set of points

$$E := \{g(x) : x \in \gamma([0, \infty)) \text{ and } g \in G\}.$$

The above curve $\gamma$ will be called the *profile curve* of the end $E$. 
If we parameterize the curve $\gamma$ as 
\[ \gamma(s) = (\gamma_1(s), 0, \gamma_2(s)) \]
then the complete end of revolution $E$ around the $x_3$-axis can be parameterized as 
\[ f(s, \theta) = (\gamma_1(s) \cos \theta, \gamma_1(s) \sin \theta, \gamma_2(s)) \]
for $s \in [0, \infty)$ and $\theta \in [0, 2\pi]$. Since $G$ acts freely on $\mathbb{M}^3(\kappa) \setminus \{x_3 \text{-axis}\}$, $E$ is a smooth surface with boundary 
\[ \partial E = \{ (\gamma_1(0) \cos \theta, \gamma_1(0) \sin \theta, \gamma_2(0)) \mid \theta \in [0, 2\pi] \} \]

**Definition 2.2** (*Complete end of revolution*) A set of points $E$ of $\mathbb{M}^3(\kappa)$ is a complete end of revolution of $\mathbb{M}^3(\kappa)$ if and only if there exists an isometry $h$ of $\mathbb{M}^3(\kappa)$ such that the set $h(E)$ is a complete end of revolution around the $x_3$-axis.

Let us recall that every regular curve admits a reparameterization by the arc length. Then, from the metrics defined in (2.1) and considering that the parameterization given in (2.4) satisfies 
\[ \| f_s(s, \theta) \|^2 = 1, \]
it is easy to see that the metric inherited by the end from each ambient space $\mathbb{M}^3(\kappa)$ can be written as 
\[ g_E = ds^2 + w_\kappa^2(s) d\theta^2 \] 
(2.5)

where 
\[ w_\kappa(s) := \begin{cases} 
\frac{\gamma_1(s)}{\gamma_2(s)} & \text{if } \kappa = -1 \\
\gamma_1(s) & \text{if } \kappa = 0 \\
\frac{2\gamma_1(s)}{1 + \gamma_1^2(s) + \gamma_2^2(s)} & \text{if } \kappa = 1 
\end{cases} \] 
(2.6)

Indeed, taking $\eta_\kappa(s)$ from Definition (2.3), we can rewrite the inherited metric as 
\[ g_E = ds^2 + \eta_\kappa(s) w_0^2(s) d\theta^2 \] 
(2.7)

Hence, we can summarize with the following proposition.

**Proposition 2.3** Let $E$ be a complete end of revolution of $\mathbb{M}^3(\kappa)$ with profile curve $\gamma : [0, \infty) \to \mathbb{M}^3(\kappa)$, then $E$ is isometric to $[0, \infty) \times \mathbb{S}^1$ with warping metric 
\[ g_E = ds^2 + w_\kappa^2(s) d\theta^2, \] 
(2.8)

where $w_\kappa$ is given in (2.6).

### 2.2 Rotationally symmetric model spaces and complete ends of revolution in $\mathbb{M}^3(\kappa)$

Rotationally symmetric model spaces are generalized manifolds of revolution using warped products. Let us recall here the following definition of a model space.

**Definition 2.4** (See [12, 13, 15]) A $w$-model space $\mathbb{M}_w^n$ is a simply connected $n$-dimensional smooth manifold $\mathbb{M}_w^n$, with a point $o_w \in \mathbb{M}_w^n$ called the center point of the model space such that $\mathbb{M}_w^n \setminus \{o_w\}$ is isometric to a smooth warped product with base $B^1 = (0, \Lambda) \subset \mathbb{R}$ (where $0 < \Lambda \leq \infty$), fiber $F^{n-1} = \mathbb{S}_1^{n-1}$ (i.e., the unit $(n-1)$-sphere with standard metric), and positive warping function $w : (0, \Lambda) \to \mathbb{R}_+$. The metric tensor $g_{\mathbb{M}_w^n}$ is thus given by: 
\[ g_{\mathbb{M}_w^n} = r^* (g(0, \Lambda)) + (w \circ \pi)^2 \theta^* (g_{\mathbb{S}_1^{n-1}}) \] 
(2.9)
where \( r : \mathbb{M}^n_w \to (0, \Lambda) \) and \( \theta : \mathbb{M}^n_w \to S^{n-1} \) are the projections onto the factors of the warped product, and \( g(0, \Lambda) \) and \( g_{S^{n-1}} \) the standard metric tensor on the interval and the sphere, respectively.

Despite the freedom in the choice of the \( w \) function in the above definition, there are certain restrictions around \( r \to 0 \). In order to attain \( \mathbb{M}^n_w \), a smooth metric tensor around \( o_w \), the positive warping function \( w \) should hold the following equalities (see [12,31]):

\[

text{where } w^{(2k)}(r) \text{ are the even derivatives of } w.
\]

The parameter \( \Lambda \) in the above definition is called the \textit{radius of the model space}. If \( \Lambda = \infty \), then \( o_w \) is a pole of \( \mathbb{M}^n_w \) (i.e., the exponential map \( \exp_{o_w} : T_{o_w} \mathbb{M}^n_w \to \mathbb{M}^n_w \) is a diffeomorphism).

Observe that a rotationally symmetric model space \( \mathbb{M}^n_w \) is rotationally symmetric at \( o_w \in \mathbb{M}^n_w \) in the sense that the isotropy subgroup at \( o_w \) of the isometry group is \( O(n) \). More commonly, one regards the functions \((r, \theta)\) as global coordinate functions on \( \mathbb{M}^n_w - \{ o_w \} \) and the expression of the metric tensor (2.9) is written as \( g_{\mathbb{M}^n_w} = dr^2 + (w(r))^2 d\theta^2 \), where \( dr^2 \) denotes the standard metric on the interval and \( d\theta^2 \) denotes the standard metric on \( S^{n-1} \). In this context, \((r, \theta)\) are called \textit{geodesic polar coordinates} around \( o_w \).

In view of the definitions of ends of revolution in \( \mathbb{M}^3(\kappa) \) and the definition of a rotationally symmetric model manifold, we can state the following proposition.

**Proposition 2.5** Let \( E \) be a complete end of revolution in \( \mathbb{M}^3(\kappa) \), let \( w_k \) be the warping function given by (2.6). Then, for every positive radius \( \rho > 0 \) there exists a \( W \in C^\infty[0, \infty) \) with

\[
\begin{align*}
W(x) &= w_k(x - \rho), \quad \forall x \geq \rho \\
W(0) &= 0, \\
W'(0) &= 1, \\
W^{(2k)}(0) &= 0, \\
W(x) &> 0, \quad \forall x > 0,
\end{align*}
\]

such that \( E \) is therefore isometric to \( \mathbb{M}^2_w \backslash B_\rho(o_w) \), where \( B_\rho(o_w) \) is the geodesic ball of radius \( \rho \) centered at \( o_w \in \mathbb{M}^2_w \).

**Proof** We can choose \( \epsilon > 0 \) such that \( \epsilon < \rho \). We can define, moreover, the function \( F : [\frac{\epsilon}{2}, \epsilon] \cup [\rho, \infty) \to \mathbb{R}_+ \) given by

\[
F(x) := \begin{cases} 
\frac{x}{w_k(x - \rho)} & \text{if } \frac{\epsilon}{2} \leq x \leq \epsilon \\
1 & \text{if } x \geq \rho.
\end{cases}
\]

Since \( F \) is a \( C^\infty \) function from the closed set \( C := [\frac{\epsilon}{2}, \epsilon] \cup [\rho, \infty) \) of \( (0, \infty) \) to \( \mathbb{R}_+ \) and \( F \) has a continuous extension to \( (0, \infty) \), then by using the extension Lemma for smooth maps (see Corollary 6.27 of [22]), there exists a \( C^\infty \) function \( \tilde{F} : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \tilde{F}|_C = F \).

Finally we can define the function \( W : [0, \infty) \to [0, \infty) \) by

\[
W(x) := \begin{cases} 
x & \text{if } x \leq \rho \\
\tilde{F}(x) & \text{if } x \geq \rho.
\end{cases}
\]
Then,
\[ g_{M^2_w} = dr^2 + W^2(r)d\theta^2, \]
where we have used geodesic polar coordinates around \( o_W \). Now, given a complete end of revolution \( E \) with profile curve \( \gamma : [0, \infty) \to M^3(\kappa) \), the following map

\[ h : M^2_w \setminus B_\rho(o_W) \to E, \quad p \to h(p) = R_{\theta(p)}(\gamma(r(p) - \rho)) \]
defines an isometry between \( M^2_w \setminus B_\rho(o_W) \) and \( E \). \( \square \)

### 2.3 Recurrence and non-explosion of ends of revolution

Conditions for recurrence and non-explosion of the Brownian motion on a Riemannian manifold have been largely studied (see [1,13,20,21,30] for example). In the particular case of rotationally symmetric model manifolds

**Theorem 2.6** [13] Let \( M^n_w \) be a model manifold with \( \Lambda = \infty \) (so that \( M^n_w \) is geodesically complete and non-compact). Then \( M^n_w \) is recurrent if and only if

\[ \int_0^\infty \frac{dt}{w^{n-1}(t)} = \infty \]

(2.11)

**Theorem 2.7** [13] Let \( M^n_w \) be a model manifold with \( \Lambda = \infty \). Then \( M^n_w \) is stochastically complete if and only if

\[ \int_{\infty}^\infty \int_0^t \frac{w^{n-1}(s)ds}{w^{n-1}(t)}dt = \infty \]

(2.12)

In fact (see [14] proof of Theorem 1.5), if \( \int_{\infty}^\infty \int_0^t \frac{w^{n-1}(s)ds}{w^{n-1}(t)}dt = \infty \), then we can construct a 1-superharmonic and radial function \( v \) in \( M \setminus B^n_\rho(o_w) \) (that is \(-\Delta u + u \geq 0\)) such that \( v(\rho) = 1, v'(0) = 0, \) and \( v(x) \to \infty \) as \( x \to \infty \).

There are sufficient conditions to attain parabolicity for rotationally symmetric model manifolds as well

**Theorem 2.8** (see [13]) Let \( M^n_w \) be a rotationally symmetric model manifold. Suppose that

\[ \int_{\infty}^\infty \frac{tdt}{\int_0^t w(s)ds} = \infty, \]

then \( M^n_w \) is recurrent.

In view of Corollary 2.5 we can state

**Corollary 2.9** Let \( E \) be an end of revolution in \( M^3(\kappa) \) isometric to the rotationally symmetric model manifold \( M^2_w \setminus B_\rho(o_w) \) for some radius \( \rho > 0 \), then \( E \) is parabolic if and only if

\[ \int_\rho^\infty \frac{dt}{w(t)} = \infty \]

(2.13)

Moreover, if

\[ \int_\rho^\infty \frac{\int_0^t w(s)ds}{w(t)}dt = \infty \]

(2.14)
then there exist a compact \( K \subset E \) and 1-superharmonic function such that \( v(x) \to \infty \) when \( x \to \infty \).

To prove the main theorems, moreover, we will need the following proposition.

**Proposition 2.10** (see Theorem 1.3 of [14]) Let \( M \) be a connected manifold and \( K \subset M \) be a compact set. Assume that there exists a 1-superharmonic function in \( m \setminus K \) such that \( v(x) \to \infty \) as \( x \to \infty \). Then \( M \) is stochastically complete.

Hence, by using Proposition 2.5 for complete ends of revolution we can state the following two summarizing propositions.

**Proposition 2.11** Let \( E \) be a complete end of revolution of \( \mathbb{M}^3(\kappa) \) parameterized by
\[
f(s, \theta) = (\gamma_1(s) \cos \theta, \gamma_1(s) \sin \theta, \gamma_2(s))
\]
with \( s \in [0, \infty), \theta \in [0, 2\pi] \). Suppose \( \gamma(s) = (\gamma_1(s), 0, \gamma_2(s)) \) is a regular curve parameterized by the arc length and suppose moreover that \( \gamma_1 > 0 \). Then, for \( \rho > 0 \), \( E \) is isometric to \( \mathbb{M}^2_w \setminus B_{\rho}(o_w) \), where \( \mathbb{M}^2_w \) is the rotationally symmetric model space given by the warping function \( w \) satisfying
\[
\begin{align*}
&\quad w(t + \rho) = w_\kappa(t) \text{ for } t \geq 0 \\
w(0) = 0 \\
w'(0) = 0 \\
w^{(2k)}(0) = 0
\end{align*}
(2.15)
\]
with \( w_\kappa \) given in Definition (2.6).

Moreover, by applying Corollary 2.9

**Proposition 2.12** Under the assumptions of Proposition 2.11, \( E \) is parabolic if and only if
\[
\int_0^\infty \frac{ds}{w_\kappa(s)} = \infty
(2.16)
\]
and if (2.16) holds, or
\[
\int_0^\infty \frac{\int_0^t w_\kappa(s) ds}{w_\kappa(t)} dt = \infty
(2.17)
\]
then there exist a compact set \( K \subset E \) and 1-superharmonic function such that \( v(x) \to \infty \) when \( x \to \infty \).

**Proof** Applying Corollary 2.9 and Proposition 2.11, the end \( E \) is parabolic if and only if
\[
\int_0^\infty \frac{dt}{w(t)} = \int_0^\infty \frac{ds}{w_\kappa(s)} = \infty
(2.18)
\]
Moreover, if
\[
\int_0^\infty \frac{\int_0^t w(s) ds}{w(t)} dt = \int_0^\infty \frac{\int_0^t w(s) ds + \int_0^t w_\kappa(s) ds}{w_\kappa(t)} dt = \infty
(2.19)
\]
there exist a compact set \( K \subset E \) and 1-superharmonic function such that \( v(x) \to \infty \) when \( x \to \infty \). But if \( \int_0^\infty \frac{ds}{w_\kappa(s)} = \infty \) or if \( \int_0^\infty \frac{\int_0^t w_\kappa(s) ds}{w_\kappa(t)} dt = \infty \), condition (2.19) is fulfilled. \( \square \)

**Remark a.** In view of the above proposition, observe that if a complete end of revolution \( E \subset \mathbb{M}^3(\kappa) \) is parabolic, then there exist a compact set \( K \subset E \) and 1-superharmonic function such that \( v(x) \to \infty \) when \( x \to \infty \).
From Theorem 2.8 we can state

**Corollary 2.13** Let \( E \) be an end of revolution in \( \mathbb{M}^3(\kappa) \) if

\[
\int_{\infty}^{t} \frac{tdt}{\int_{0}^{t} w_\kappa(s)ds} = \infty,
\]

then \( E \) is parabolic.

### 3 Proof of Theorem A

Theorem A states that any end of revolution in \( \mathbb{R}^3 \) or in \( \mathbb{S}^3 \) is a parabolic end. Here we split the proof into these two settings.

#### 3.1 End immersed in \( \mathbb{R}^3 \)

**Proof** Let us recall that the generating curve \( \gamma(s) \) was parameterized by its arc length. This implies that \((\dot{\gamma}_1)^2(s) \leq 1\). Using Definition (2.6), we find this equivalent to

\[-1 \leq \dot{w}_0(t) \leq 1\]

Integrating the latter we obtain

\[-t \leq w_0(t) - w_0(0) \leq t\]

and thus \( w_0(t) \leq t + w_0(0) \). By using the criterion for parabolicity given by Proposition 2.12

\[
\int_{0}^{\infty} \frac{1}{w_0(s)} ds = \lim_{R \to \infty} \int_{0}^{R} \frac{1}{w_0(s)} ds \geq \lim_{R \to \infty} \int_{0}^{R} \frac{1}{t + w_0(0)} dt = \infty, \quad (3.1)
\]

which means that each complete end of revolution \( E \), when immersed in \( \mathbb{R}^3 \), is of the parabolic conformal type regardless of the profile curve. \( \square \)

#### 3.2 End immersed in \( \mathbb{S}^3 \)

**Proof** Applying the criterion for parabolicity in Proposition 2.12 and the expression (2.6) for the function \( w_1(s) \), we have to prove that the following integral is divergent

\[
\int_{0}^{\infty} \frac{1}{w_1(s)} ds = \int_{0}^{\infty} \frac{1 + \gamma_1^2(s) + \gamma_2^2(s)}{2\gamma_1(s)} ds.
\]

For any \( \varepsilon > 0 \), we can split the interval where we integrate \((I = [0, \infty)) \) into two parts: \( I_+ = \{ t \in I : \gamma_1(t) \geq \varepsilon \} \) and \( I_- = \{ t \in I : \gamma_1(t) < \varepsilon \} \), such that \( I_+ \cup I_- = I, \quad I_+ \cap I_- = \emptyset \) and since \( \int_I dx = \infty \), then \( \int_{I_+} dx + \int_{I_-} dx = \infty \). Thus we have two cases.

**Case I:** \( \int_{I_+} dx = \infty \), so as we have seen:

\[
\int_{I_+} \frac{1}{w_1(s)} ds \geq \int_{I_+} \frac{1 + \gamma_1^2(s) + \gamma_2^2(s)}{2\gamma_1(s)} ds \geq \frac{1}{2} \int_{I_+} \gamma_1(s) ds \geq \frac{1}{2} \varepsilon \int_{I_+} ds = \infty
\]
Case II: $\int_{I_{-}} dx < \infty$ ($\int_{I_{-}} dx = \infty$), then
\[
\int_{I_{-}} \frac{1}{w_{2}(s)} ds \geq \int_{I_{-}} \frac{1 + \gamma_{1}^{2}(s) + \gamma_{2}^{2}(s)}{2\gamma_{1}(s)} ds \geq \int_{I_{-}} \frac{1}{2\gamma_{1}(s)} ds
\]
\[
\geq \int_{I_{-}} \frac{1}{2\varepsilon} ds \geq \frac{1}{2\varepsilon} \int_{I_{-}} ds = \infty
\]
which again means that the end is parabolic regardless of the profile curve. \qed

### 4 Proof of Theorem B

Theorem B states that every end of revolution in the upper half-space of $\mathbb{H}^3$ determined by a $c$-cone is a parabolic end. Observe that if the end is on a $c$-cone, then the generating profile curve $\gamma(s) = (\gamma_1(s), 0, \gamma_2(s))$ satisfies
\[
\frac{\gamma_2(s)}{\gamma_1(s)} \geq c. \tag{4.1}
\]

Substituting the function $w_{-1}(s)$ given by (2.6) in the criterion for parabolicity given in Proposition (2.12), we get that the end of the surface will be parabolic because
\[
\int_{0}^{\infty} \frac{1}{w_{-1}(s)} ds = \int_{0}^{\infty} \frac{\gamma_2(s)}{\gamma_1(s)} ds \geq \int_{0}^{\infty} c \ ds = \infty.
\]

This finishes the proof of Theorem B. However, we can state something more general. Let us denote $I_{+} := \{ t \in I : \frac{\gamma_2(t)}{\gamma_1(t)} \geq c \}$ and $I_{-} := \{ t \in I : \frac{\gamma_2(t)}{\gamma_1(t)} < c \}$. Then,
\[
\int_{I_{-}} \frac{1}{w_{-1}(s)} ds = \int_{I_{+}} \gamma_2(s) \ ds + \int_{I_{-}} \frac{\gamma_2(s)}{\gamma_1(s)} ds \geq \int_{I_{+}} ds + \int_{I_{-}} \frac{\gamma_2(s)}{\gamma_1(s)} ds.
\]

Hence, if $\int_{I_{+}} ds = \infty$, the end will still be parabolic. Therefore,

**Theorem 4.1** Let $E$ be an end of revolution in $\mathbb{H}^3$. Suppose that the generating curve of $E$ satisfies
\[
\int_{I_{+}} ds = \infty.
\]

Then, the end is parabolic.

### 5 Proof of Theorem C

The first step to prove Theorem C is first to prove the following proposition.

**Proposition 5.1** Let $E$ be a complete end of revolution immersed in $\mathbb{H}^3$. Suppose that $E$ is a non-parabolic end, and $\gamma : [0, \infty) \to \mathbb{H}^3$ is the profile curve of $E$ parameterized by the arc length in the half-space model of the hyperbolic space given by
\[
\gamma(s) = (\gamma_1(s), 0, \gamma_2(s)),
\]
then
\[ \sup_{s \in [0, \infty)} \gamma_1(s) < \infty. \]

**Proof** Since \( \gamma_1 \) is a positive and smooth function, then
\[
\log \gamma_1(s) - \log \gamma_1(0) = \int_0^s \frac{d}{dt} (\log \gamma_1(t)) \, dt = \int_0^s \frac{\dot{\gamma}_1(t)}{\gamma_1(t)} \, dt. \tag{5.1}
\]

But taking into account that \( \gamma \) is parameterized by the arc length, namely,
\[
\frac{(\dot{\gamma}_1(s))^2 + (\dot{\gamma}_2(s))^2}{(\gamma_2(s))^2} = 1,
\]
then
\[ \dot{\gamma}_1(s) \leq |\dot{\gamma}_1(s)| \leq \gamma_2(s) \]
and hence, inequality (5.1) can be rewritten as
\[
\log \gamma_1(s) - \log \gamma_1(0) \leq \int_0^s \frac{\gamma_2(t)}{\gamma_1(t)} \, dt \tag{5.2}
\]

By using the function \( w_{-1}(s) \) given by (2.6) in the criterion for parabolicity given in Proposition 2.12
\[
\int_0^\infty \frac{1}{w_{-1}(s)} \, ds = \int_0^\infty \frac{\gamma_2(s)}{\gamma_1(s)} \, ds < \infty. \tag{5.3}
\]

And hence,
\[
\log \gamma_1(s) - \log \gamma_1(0) < \int_0^\infty \frac{\gamma_2(t)}{\gamma_1(t)} \, dt < \infty. \tag{5.4}
\]

**Proof of Theorem C** Theorem C states that any complete end of revolution contained on a horosphere \( \{ x_3 = z \} \) is a parabolic end. Since the end is on the horosphere \( \{ x_3 = z \} \) then
\[
\int_0^\infty \frac{1}{w_{-1}(s)} \, ds = \int_0^\infty \frac{\gamma_2(s)}{\gamma_1(s)} \, ds \geq z \int_0^\infty \frac{1}{\gamma_1(s)} \, ds. \tag{5.5}
\]

Hence, the end is parabolic because otherwise if we suppose that \( E \) is non-parabolic, by using Proposition 5.1,
\[
\int_0^\infty \frac{1}{w_{-1}(s)} \, ds \geq z \int_0^\infty \frac{1}{\sup_{s \in [0, \infty)} \gamma_1(s)} \, ds = \infty. \tag{5.6}
\]
Then by using the criterion for parabolicity given in Proposition 2.12, the end \( E \) is parabolic (contradiction).

\[ \square \]

**6 Proof of Theorem F**

Recall that Theorem F states

**Theorem.** Let \( \Sigma \) be a complete and non-compact surface of finite topological type immersed in \( M^3(\kappa) \) with \( \kappa \in \mathbb{R} \). Suppose that there exists a compact subset \( \Omega \subset \Sigma \) of \( \Sigma \) such that every end of \( \Sigma \) with respect to \( \Omega \) is an end of revolution in \( M^3(\kappa) \). Then, \( \Sigma \) is stochastically complete.
Lemma 6.1 Let $E$ be an end of revolution in $\mathbb{H}^3$, then, there exist a compact set $K \subset E$ and 1-superharmonic function such that $v(x) \to \infty$ when $x \to \infty$.

Proof From (2.6) we have that

$$w_{-1}(s) = \frac{\gamma_1(s)}{\gamma_2(s)}$$

hence,

$$\frac{\dot{w}_{-1}(s)}{w_{-1}(s)} = \frac{\dot{\gamma}_1(s)}{\gamma_1(s)} - \frac{\dot{\gamma}_2(s)}{\gamma_2(s)}$$

From the condition that $\gamma(s)$ is parameterized by the arc length, we have that $|\dot{\gamma}_1(s)| \leq \gamma_2(s)$ and $|\dot{\gamma}_2(s)| \leq \gamma_2(s)$. Then

$$\frac{\dot{w}_{-1}(s)}{w_{-1}(s)} \leq \frac{\gamma_2(s)}{\gamma_1(s)} + \frac{\gamma_2(s)}{\gamma_2(s)} = \frac{1}{w_{-1}(s)} + 1 \quad (6.1)$$

For any $c > 0$ we can now split the interval $I = [0, \infty)$ into two parts, $I = I_+ \cup I_-$ such that $I_+ := \{s \in \mathbb{R} : \frac{\gamma_2(s)}{\gamma_1(s)} \geq c\}$ and $I_- = \{s \in \mathbb{R} : \frac{\gamma_2(s)}{\gamma_1(s)} < c\}$. With $I_+ \cap I_- = \emptyset$, $\int_I ds = \int_{I_+} ds + \int_{I_-} ds = \infty$. We again have two cases:

Case I: $\int_{I_-} ds = \infty$. Then, by using Theorem 4.1 and taking into account remark a the Lemma follows.

Case II: $\int_{I_+} ds < \infty$ ($\int_{I_-} ds = \infty$). For $s > \rho$,

$$w(s) - w(\rho) = \int_{\rho}^{s} \dot{w}(r) dr = \int_{[\rho, s]\cap I_+} \dot{w}(r) dr + \int_{[\rho, s]\cap I_-} \dot{w}(r) dr \leq \int_{[\rho, s]\cap I_+} w'(r) dr + (1 + c) \int_{[\rho, s]\cap I_-} w(r) dr \quad (6.2)$$

where we have considered that in $I_-$, by using inequality (6.1), $\dot{w}(r) < (1 + c) w(r)$. Thus,

$$1 < \frac{w(\rho)}{w(s)} + \frac{\int_{[\rho, s]\cap I_+} w'(r) dr}{w(s)} + \frac{(1 + c) \int_{[\rho, s]\cap I_-} w(r) dr}{w(s)}$$

Integrating in $[\rho, R]$,

$$R - \rho < \int_{\rho}^{R} \frac{w(\rho)}{w(s)} ds + \int_{\rho}^{R} \frac{\int_{[\rho, s]\cap I_+} w'(r) dr}{w(s)} ds + \int_{\rho}^{R} \frac{(1 + c) \int_{[\rho, s]\cap I_-} w(r) dr}{w(s)} ds$$

$$\leq \int_{\rho}^{\infty} \frac{w(\rho)}{w(s)} ds + \int_{\rho}^{\infty} \frac{\int_{[\rho, s]\cap I_+} w'(r) dr}{w(s)} ds + (1 + c) \int_{\rho}^{\infty} \frac{\int_{\rho}^{s} w(r) dr}{w(s)} ds \quad (6.3)$$

$$\leq \int_{\rho}^{\infty} \frac{w(\rho)}{w(s)} ds + \int_{\rho}^{\infty} \frac{\int_{[\rho, s]\cap I_+} w'(r) dr}{w(s)} ds + (1 + c) \int_{\rho}^{\infty} \frac{\int_{\rho}^{s} w(r) dr}{w(s)} ds \quad (6.4)$$
taking into account that by using inequality (6.1), \( w'(r) \leq 1 + w(r) \), then for any \( R > \rho \),

\[
R - \rho < w(\rho) \int_\rho^\infty \frac{1}{w(s)} \, ds + \int_\rho^\infty \frac{\int_{[\rho, s] \cap I_+} (1 + w(r)) \, dr}{w(s)} \, ds
\]
\[
+ (1 + c) \int_\rho^\infty \frac{\int_{[\rho, s]} w(r) \, dr}{w(s)} \, ds
\]
\[
\leq \left( w(\rho) + \int_{I_+} \, ds \right) \int_0^\infty \frac{1}{w(s)} \, ds + (2 + c) \int_\rho^\infty \frac{\int_{[\rho, s]} w(r) \, dr}{w(s)} \, ds.
\]

(6.5)

Now letting \( R \) tend to infinity, we conclude that either

\[
\int_0^\infty \frac{1}{w(s)} \, ds = \infty
\]
or

\[
\int_\rho^\infty \frac{\int_{[\rho, s]} w(r) \, dr}{w(s)} \, ds = \infty.
\]

But in any case the Lemma follows by using Proposition 2.12.

Now, by applying the above proposition in each connected component of \( \Sigma \setminus \Omega \) (call them \( \{E_i\} \)), there exist a compact set \( K_i \subset E_i \) and a 1-superharmonic function \( v_i \) in \( E_i \setminus K_i \) such that \( v_i(x) \to \infty \) as \( x \to \infty \). Defining a compact \( C \) as

\[
C = \Omega \cup_i K_i
\]

and the function \( \tilde{v} : \Sigma \setminus C \to \mathbb{R} \) by

\[
\tilde{v}(x) := v_i(x), \quad \text{if} \ x \in E_i,
\]

we conclude that \( \tilde{v} \) is a 1-superharmonic function in \( \Sigma \setminus C \) and \( \tilde{v}(x) \to \infty \) as \( x \to \infty \). Hence, by applying Proposition 2.12, the Corollary is proved.

\[\square\]

7 Proof of Theorem G

Recently in [4] it is proved that any non-compact surface \( \Sigma \) which is stochastically and geodesically complete, and properly immersed in a horoball of the hyperbolic space \( B \subset \mathbb{H}^3 \), has

\[
\sup_{x \in \Sigma} \|H\| \geq 1,
\]

(7.1)

and

\[
\sup_{x \in \Sigma} K_G \geq 0.
\]

(7.2)

Hence, there are no surfaces of revolution which are negatively curved and properly immersed in a horoball (statement (1) of Theorem G). Moreover, if \( \Sigma \) is a cmc-surface with \( \|H\| \leq 1 \), then \( \Sigma \) is a cmc one surface and therefore by using [32] it is a horosphere. On the other hand, if \( \Sigma \) has constant non-positive sectional curvature, then \( K_G = 0 \). But the only complete flat surface contained in a horoball is the horosphere (see Theorem 3 of [9] for instance).
8 Movement of the centroid of a curve in $\mathbb{H}^2$ and its applications to the conformal type

Given a regular curve $\gamma : [0, \infty) \to \mathbb{H}^2 \subset \mathbb{H}^3$, parameterized by the arc length in the half-space model of the hyperbolic space

$$\gamma(s) = (\gamma_1(s), 0, \gamma_2(s)), \quad \text{with} \quad \gamma_1 > 0 \quad \text{and} \quad \gamma_2 > 0,$$

we shall say that the segment $\gamma([0, s])$ has centroid $x_g(\gamma([0, s]))$, given by

$$x_g(\gamma([0, s])) := \frac{\int_0^s w(t)dt}{s}.$$

**Theorem 8.1** Suppose that the centroid of a regular curve $\gamma : [0, \infty) \to \mathbb{H}^2$ satisfies

$$x_g(\gamma([0, s])) \leq Cs$$

for some $C > 0$ and any $s \geq s_0$ for some $s_0 > 0$. Then the end of revolution given by

$$f(\theta, s) := R_{\theta} \gamma$$

is a parabolic end.

**Proof** Observe that

$$\int_0^s \frac{t dt}{\int_0^t w(s)ds} \geq \int_{s_0}^\infty \frac{dt}{x_g(\gamma([0, t]))} \geq \int_{s_0}^\infty \frac{dt}{Ct} = \infty.$$  \hspace{1cm} (8.1)

And the Theorem is proved by using corollary 2.13. \hfill \Box

**Definition 8.2** We shall say that a regular curve $\gamma : [0, \infty) \to \mathbb{H}^2$, parameterized by the arc length $\gamma(s) = (\gamma_1(s), 0, \gamma_2(s))$ (where we have used the half-space model of the hyperbolic space and we have assumed $\gamma_1 > 0, \gamma_2 > 0$), has confined centroid if the limit

$$\lim_{s \to \infty} x_g(\gamma([0, s]))$$

exists and

$$\lim_{s \to \infty} x_g(\gamma([0, s])) < \infty.$$

**Theorem 8.3** Suppose that a regular curve $\gamma : [0, \infty) \to \mathbb{H}^2$ has a confined centroid. Then the end of revolution given by

$$E = \{ R_{\theta} (\gamma(s)) : \theta \in [0, 2\pi] \quad \text{and} \quad s \in [0, \infty) \}$$

is a parabolic end.

**Proof** Since $\gamma$ has a confined centroid, for each $\epsilon > 0$ there exist $R$ large enough such that for any $t > R$

$$\frac{\int_0^t w(s)ds}{t} \leq x_g(\gamma) + \epsilon.$$  \hspace{1cm} (8.2)

Therefore,

$$\int_R^\infty \frac{t dt}{\int_0^t w(s)ds} \geq \int_R^\infty \frac{t dt}{(x_g(\gamma) + \epsilon)t} = \infty.$$  \hspace{1cm} (8.3)

And the Theorem is proved by using Corollary 2.13. \hfill \Box
9 Examples of application

We would like to highlight some examples with relevant properties of complete ends of revolution in $M^3(\kappa)$.

9.1 Surfaces immersed in a ball of $\mathbb{R}^3$

The topic of complete immersions into geodesic balls of $\mathbb{R}^3$ has been largely studied from the Labyrinth example of Nadirashvili (cf. [29]). From the conformal point of view, the Brownian motion of any complete bounded minimal surface in $\mathbb{R}^3$ is transient (non-recurrent) (see [5] for instance). Moreover, the Brownian motion of a submanifold is transient (see [10]) if the submanifold admits a complete immersion within a geodesic ball of radius $R$ with a mean curvature vector field $H$ bounded by

$$\|H\| < \frac{1}{R}$$

Taking into account that by Theorem A any end of revolution in $\mathbb{R}^3$ is a parabolic end, we can state

**Corollary 9.1** Let $\Sigma$ be a surface isometrically immersed within a geodesic ball $B_R \subset \mathbb{R}^3$. Suppose that $\Sigma$ has at least one end of revolution in $\mathbb{R}^3$. Then, the mean curvature vector field $H$ satisfies

$$\sup_{x \in \Sigma} \|H(x)\| \geq \frac{1}{R}.$$  

**Remark b** In [6], Jorge and Xavier proved that every submanifold $M$ whose scalar curvature is bounded from below immersed in a geodesic ball $B_R \subset \mathbb{R}^n$ of radius $R$ satisfies

$$\sup_M \|H\| \geq \frac{1}{R}.$$  

**Proof of Corollary 9.1** Suppose, on the contrary, that

$$\sup_{x \in \Sigma} \|H(x)\| < \frac{1}{R}.$$  

Then, by using Corollary 2.7 of [10], $\Sigma$ has a positive Cheeger constant $h(\Sigma) > 0$, in particular the end of revolution $E$ also has a positive Cheeger constant $h(E) > 0$, and therefore $E$ has a positive fundamental tone $\lambda^*(E)$, which implies that $E$ is non-parabolic (see [13]) in contradiction with Theorem A.

In the particular case of minimal surfaces, Corollary 9.1 implies that bounded minimal surfaces of revolution do not exist. In fact, this is in agreement with the classical result of Bonnet, which states that, up to rigid motions, the only minimal surfaces of revolution in $\mathbb{R}^3$ are the catenoid and the plane.

A natural question is to ask for the existence of recurrent surfaces immersed in a geodesic ball of $\mathbb{R}^3$. We can guarantee the existence of such surfaces and this can be seen through the following self-made example.

**Example 9.2** Consider the curve $\Gamma: \mathbb{R} \to \mathbb{R}^3$ parameterized as (see also Fig. 2):

$$\alpha_{R,a}(t) = \left( \frac{(R-a)t^2}{(t^2+1)} + a, 0, \sin \left( \frac{(R-a)t^3}{(t^2+1)} + at \right) \right)$$
When it is rotated over the $x_3$-axis, it generates a complete surface of revolution in $\mathbb{R}^3$ which is bounded, i.e., it can be kept inside a cylinder with radius $R$ and height $h = 1$. By using Theorem A, this surface is recurrent. Note also that the mean curvature of this surface is unbounded.

9.2 Surfaces in $\mathbb{H}^3$ with transient Brownian motion

The spherical catenoids immersed in $\mathbb{H}^3$ are examples of surfaces of revolution in $\mathbb{H}^3$ with transient (non-recurrent) Brownian motion. Spherical catenoids have been studied in [7,28] or [33] and, specifically, using the Upper Half-space Model in [3].

Example 9.3 (Spherical catenoids) Spherical catenoids are the minimal complete surfaces of revolution generated by the rotation of the family of curves

$$\gamma_a(s) = \left(e^{\Lambda_a(s)} \tanh(y_a(s)), 0, \frac{e^{\Lambda_a(s)}}{\cosh(y_a(s))}\right)$$

where

$$y_a(s) := a + \int_0^s \frac{\cosh(2a) \sinh(2t)}{(\cosh(2a))^2 \cosh(2t)^2 - 1} \, dt.$$  

and

$$\Lambda_a(s) := \sqrt{2} \sinh(2a) \int_0^s \frac{\cosh(2a) \cosh(2t) - 1}{\cosh^2(2a) \cosh^2(2t) - 1} \, dt.$$  

The warping function is thus given by

$$w_{-1}(s) = \frac{\gamma_{a1}(s)}{\gamma_{a2}(s)} = \sinh(y_a(s))$$

and hence the following integral can be shown as a convergent integral,

$$\int_{-\infty}^{\infty} \frac{1}{w_{-1}(s)} \, ds = \int_{-\infty}^{\infty} \frac{1}{\sinh(y_a(s))} \, ds < \infty$$

thus proving that each of the ends of the surface is non-parabolic. However, the transience of spherical catenoids can be proved by taking into account that spherical catenoids are minimal surfaces, and every minimal surface of $\mathbb{H}^3$ is transient (see [25] for instance).

Observe (see Fig. 3) that by Theorem 4.1, the part of the curve $\gamma_a(s)$ lying over an arbitrary line $t \to (t, c \cdot t)$ has to be of finite length.
Fig. 3  Half Spherical Catenoids immersed in $\mathbb{H}^3$: $\Sigma_{0.05}$ and $\Sigma_{0.5}$. Observe that the catenoid lies under a horosphere, the part of the profile curve lying over an arbitrary line $t \to (t, 0, c \cdot t)$ has finite length (Theorem 4.1), each of the two ends approaches the plane $\{x_3 = 0\}$ (Corollary E), and $\sup x_1 < \infty$ (Proposition 5.1).

Example 9.4  Consider the following profile curves in $\mathbb{H}^3$

$$\gamma_k(s) = \left( k \sin \left( \frac{s}{k} \right), 0, k \sin \left( \frac{s}{k} \right) + 1 - k \right),$$

parameterized in the half-space model of $\mathbb{H}^3$ with $s \in [0, k \arccos ((k - 1)/k))$ for $k \geq 1/2$ and $s \in [0, 2\pi]$ for $0 < k < 1/2$. In [8], the curvature of the family of surfaces $S_k$, obtained by rotating such profile curves around the $x_3$-axis, is studied, and it is shown that they are surfaces of constant Gaussian curvature $K(k)$ depending on the value of $k$

$$K(k) = -1 + \frac{(k - 1)^2}{k^2}.$$

The parabolicity of such surfaces is studied in section 5 of [18], obtaining that $S_k$ is parabolic for $k \in (0, 1/2]$ and hyperbolic for $k > 1/2$. As can be shown for $k < 1/2$, the surfaces $S_k$ are contained in a compact set of $\mathbb{H}^3$ (in fact they are compact) and thus, by using corollary D, they are therefore recurrent surfaces. For $k = 1/2$, the surface $S_{1/2}$ is contained in a horosphere (see Fig. 4), and hence by using Theorem C, $S_{1/2}$ is a recurrent surface. In fact, $S_{1/2}$ is a complete flat surface (in fact a horosphere). For $k > 1/2$, the surface $S_k$ is a
Fig. 4  Profile curves of the surfaces of revolution given in example 9.4. From left to right profile curve of $S_{1/3}$, profile curve of $S_{1/2}$ with dashed horosphere profile curve, and the profile curve of $S_1$. Surfaces $S_{1/3}$ and $S_{1/2}$ are recurrent and $S_1$ is transient.

surface with only one end, and since the surface is transient, the end approaches the plane $\{x_3 = 0\}$ (Corollary E), and $\text{sup} \, x_1 < \infty$ (Proposition 5.1).

9.3 Surfaces in $\mathbb{H}^3$ with recurrent Brownian motion

Constructing a surface of revolution in $\mathbb{H}^3$ with recurrent Brownian motion can be achieved, using our Theorem B, if we construct a surface such that every end is of revolution and every end is on a $c$-cone for some $c > 0$. In our example we are using clothoids.

Example 9.5 Clothoids or spirals of Cornu are curves generated by pairs of functions of the form

$$\text{clothoid}[n, a][t] = a \left( \int_0^t \sin \left( \frac{s^{n+1}}{n+1} \right) \, ds, 0, \int_0^t \cos \left( \frac{s^{n+1}}{n+1} \right) \, ds \right)$$

and are commonly used in construction (cf. [11]). By changing $t \rightarrow e^t$, in case $a = n = 1$, we obtain a complete curve which can be easily immersed in $\mathbb{H}^2$. The surface of revolution obtained when rotating the curve over the vertical axes appears to have two parabolic ends (see Fig. 5). Note that the immersion of the surface is not proper.

Example 9.6 (horosphere) Another interesting example of a surface of revolution is the horosphere, which, in the upper half-space model of $\mathbb{H}^3$, is just the $x_3 = z$ plane. An end of revolution can be obtained by rotating the parameterized by arc length curve

$$\gamma : [0, \infty) \rightarrow \mathbb{H}^3, \quad \gamma(s) = (z \, s + 1, 0, z)$$

along the $x_3$-axis. Note that

$$w_{-1}(s) = \frac{\gamma_1(s)}{\gamma_2(s)} = s + \frac{1}{z}.$$ 

If we observe the motion of the centroid

$$x_g(\gamma([0, s])) = \frac{\int_0^s w_{-1}(s) \, ds}{s} = \frac{s}{2} + \frac{1}{z}.$$
then, given \( s_0 > 0 \), for any \( s \geq s_0 \) and denoting \( C := \left( \frac{1}{2} + \frac{1}{z s_0} \right) \),

\[ x_g(\gamma([0, s])) = s \left( \frac{1}{2} + \frac{1}{z s} \right) \leq C s \]

By Theorem 8.1, the horosphere is a recurrent surface. This result can be achieved directly by using Theorem C or by the fact that, since the horosphere is a flat surface, it has finite total curvature and hence (by using [20]) the Brownian motion is recurrent.

### 9.4 Surfaces of revolution in \( \mathbb{H}^3 \) with one parabolic end and one non-parabolic end

The following example uses vertical lines instead of horizontal lines as in the horospheres

**Example 9.7** (cylinders)

The family of parameterized curves in \( \mathbb{H}^3 \)

\[ \beta_{b,c} : (-\infty, \infty) \to \mathbb{H}^3, \quad \beta_{b,c}(t) = (b, 0, c \cdot e^t) \]

can be rotated over the vertical axes \((0, x_3)\) to obtain a cylinder (see Fig. 6). This surface of revolution in \( \mathbb{H}^3 \) has two ends of revolution in \( \mathbb{H}^3 \): one upper end \( E^+ \), given by the rotation of the parameterized by arc length curve

\[ \gamma_{E^+} : [0, \infty) \to \mathbb{H}^3, \quad \gamma_{b,c}(s) = (b, 0, c \cdot e^s) \]

and another end \( E^- \), obtained by the rotation of the parameterized by arc length curve

\[ \gamma_{E^-} : [0, \infty) \to \mathbb{H}^3, \quad \gamma_{b,c}(s) = (b, 0, c \cdot e^{-s}) \]

Observe that the end \( E^+ \) is on the \( \frac{c}{b} \)-cone, and hence by Theorem B it is a parabolic end. On the other hand, the end \( E^- \) has

\[ w_{-1}(s) = b e^s \]

and thus

\[ \int_0^\infty \frac{1}{w_{-1}(s)} ds = \int_0^\infty \frac{c}{b} e^{-s} ds = \frac{c}{b} < \infty \]
The cylinder immersed in $\mathbb{H}^3$ with parameters $b = 2$, $c = 1$. This surface has two ends, one parabolic and the other non-parabolic. The end $E_-$ is therefore non-parabolic by using Proposition 2.12.

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