EXISTENCE OF LOCALLY MAXIMALLY ENTANGLED QUANTUM STATES
VIA GEOMETRIC INVARIANT THEORY

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ABSTRACT. We study a question which has natural interpretations both in quantum mechanics and in geometry. Let $V_1, \ldots, V_n$ be complex vector spaces of dimension $d_1, \ldots, d_n$ and let $G = \text{SL}_{d_1} \times \cdots \times \text{SL}_{d_n}$. Geometrically, we ask: Given $(d_1, \ldots, d_n)$, when is the geometric invariant theory quotient $\mathbb{P}(V_1 \otimes \cdots \otimes V_n)//G$ non-empty? This is equivalent to the quantum mechanical question of whether the multipart quantum system with Hilbert space $V_1 \otimes \cdots \otimes V_n$ has a locally maximally entangled state, i.e. a state such that the density matrix for each elementary subsystem is a multiple of the identity. We show that the answer to this question is yes if and only if $R(d_1, \ldots, d_n) \geq 0$ where

$$R(d_1, \ldots, d_n) = \prod_{i} d_i + \sum_{k=1}^{n} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq n} (\gcd(d_{i_1}, \ldots, d_{i_k}))^2.$$ 

We also provide a simple recursive algorithm which determines the answer to the question, and we compute the dimension of the resulting quotient in the non-empty cases.

1. INTRODUCTION

In a multipart quantum system, the space of pure states is described by a tensor product Hilbert space

$$V = V_1 \otimes \cdots \otimes V_n,$$

where $V_i$ are $d_i$-dimensional Hilbert spaces describing the elementary subsystems in isolation\(^\text{1}\). Given a pure state $\psi \in V$, the associated state of the $i$th elementary subsystem is described by the reduced density operator $\rho_i : V_i \to V_i$, a nonnegative unit-trace Hermitian operator defined by the action of the contraction map $V_1 \otimes \cdots \otimes V_n \otimes V_i^* \otimes \cdots \otimes V_n^* \to V_i^* \otimes V_i^*$ on the operator $\rho_\psi = \psi \otimes \psi^* \in V \otimes V^*$. Equivalently, we have $\rho_i = \text{tr}_{V_i \otimes \cdots \otimes \hat{V_i} \otimes \cdots \otimes V_n} \rho_\psi$.

In general, the structure of entanglement in a multipart quantum system is related to the eigenvalue spectra of the reduced density matrices for subsystems. A subsystem $i$ is entangled with the rest of the system if its spectrum is different from $\{1, 0, 0, \ldots\}$ (i.e. if its density operator is not a projection operator associated with a single state). A subsystem $i$ is maximally entangled with the rest of the system if all eigenvalues of $\rho_i$ are equal i.e. $\rho_i = \mathbb{I}/d_i$. For

\(^\text{1}\)As usual, we require the states $\psi$ to be normalized $(\psi, \psi) = 1$ and identify states which are related by an overall phase $\psi \sim e^{i\theta} \psi$. Thus, we can identify states $\psi$ with points in the projective space $\mathbb{P}(V)$.

\(^\text{2}\)It is natural to assume that $d_i \geq 2$ for every $i$; however, we will allow trivial subsystems (with $d_i = 1$), as long as there at least two subsystems of dimension $d_i \geq 2$. 

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some \( \{d_1, \cdots, d_n\} \) it is possible to find states where each elementary subsystem has maximal entanglement:

**Definition 1.1.** A pure state \( \psi \) in a multipart quantum system described by Hilbert space \( V_1 \otimes \cdots \otimes V_n \) is said to be **locally maximally entangled** if for each elementary subsystem \( i \), the density operator \( \rho_i \) is a multiple of the identity operator, \( \rho_i = \mathbb{1}/d_i \).

States that are locally maximally entangled (LME) have many applications in the field of quantum information theory and quantum computing. For example, this property is present in Bell states, GHZ states, quantum error correcting code states, cluster states, and graph states.

In this paper, our goal is to understand for which \( \{d_1, \cdots, d_n\} \) such LME states exist, and in those cases to characterize the subset \( V_{LME} \subset V \) of such states in the full Hilbert space. While the necessary conditions \( d_i \leq \prod_{j \neq i} d_j \) are well known and have been suggested to be sufficient, we will see that the necessary and sufficient condition is significantly more complicated.

**Relation to geometry:** Remarkably, the quantum mechanics problem we have described is equivalent to two very natural problems in geometry, the first related to symplectic geometry and the second related to algebraic geometry and geometric invariant theory; see A. A. Klyachko [Kly02], [Kly07, § 3], and N. R. Wallach [Wall08, § 4].

Geometrically, the full space of pure states \( \mathbb{P}(V) \) admits a natural symplectic structure defined by the Fubini-Study symplectic form. Consider the action of \( K = SU_{d_1} \times \cdots \times SU_{d_n} \) on \( \mathbb{P}(V) \), where \( SU_{d_i} \) acts via the fundamental representation on \( V_i \). The spectra of the density operators \( \rho_i \) are invariant under the action of \( K \), so we can group LME states into equivalence classes defined by the \( K \) orbits. Since the symplectic form is invariant under the action of \( K \), we can define a moment map \( \mu : \mathbb{P}(V) \to \mathfrak{k}^* \) from our space to the dual of the Lie algebra of \( K \). The relation to symplectic geometry is provided by the result that \( \psi \in V_{LME} \) if and only if \( \mu(\psi) = 0 \) (see section 2). Thus, the space of equivalence classes of LME states under \( K \) is precisely the symplectic quotient \( \mu^{-1}(0)/K \).

By the Kempf-Ness theorem, this symplectic quotient is equivalent to the geometric invariant theory quotient \( \mathbb{P}(V)/G \), where

\[
G = SL_{d_1} \times \cdots \times SL_{d_n}
\]

is the complexification of \( K \). Thus for dimensions \( \{d_1, \cdots, d_n\} \), we arrive at the central observation:

**There exist LME states if and only if the quotient \( \mathbb{P}(V)/G \) is non-empty.**

Our main result provides a simple numerical criterion for when the quotient \( \mathbb{P}(V)/G \) is non-empty, and we give a formula for the dimension of the non-empty quotients. For

\[
d = (d_1, \ldots, d_n)
\]
we define the expected dimension
\[ \Delta(d) = \dim \mathbb{P}(V) - \dim G \]
\[ = \prod d_i - 1 - \sum (d_i^2 - 1) \]
as well as the arithmetic functions
\[ g_{\text{max}}(d) = \max_{1 \leq i < j \leq n} (\gcd(d_i, d_j)) \]
and
\[ R(d) = \prod d_i + \sum_{k=1}^{n} (-1)^k G_k(d), \]
where
\[ G_k(d) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} (\gcd(d_{i_1}, \ldots, d_{i_k}))^2. \]

Our main theorem is the following:

**Theorem 1.2.** The GIT quotient \( \mathbb{P}(V) // G \) is non-empty if and only if \( R(d) \geq 0 \). Moreover:

- If \( \Delta(d) > -2 \), then \( R > 0 \) and \( \dim \mathbb{P}(V) // G = \Delta(d) > 0 \).
- If \( \Delta(d) = -2 \), then \( R > 0 \) and \( \dim \mathbb{P}(V) // G = \max(g_{\text{max}}(d) - 3, 0) \).
- If \( \Delta(d) < -2 \), then \( R \leq 0 \) and the quotient is a single point for \( R = 0 \) and empty for \( R < 0 \).

We remark that Theorem 1.2 implies that the quotient is of the expected dimension \( \Delta(d) \) whenever this is non-negative. If the expected dimension is negative, then the quotient can be empty, a single point, or positive dimensional.

Our proof of Theorem 1.2 is based on solving the following simple algorithm which computes the dimension of \( \mathbb{P}(V) // G \).

**Theorem 1.3.** The GIT quotient \( \mathbb{P}(V) // G \) has dimension \( D(d) \), where \( d = (d_1, \ldots, d_n) \) and \( D \) is a function defined on weakly increasing tuples of integers by the following cases, depending on the size of \( d_n \) relative to \( P = d_1 \cdots d_{n-1} \).

- (a) \( d_n > P \) \( D(d) = -1 \).
- (b) \( d_n = P \) \( D(d) = 0 \).
- (c) \( \frac{P}{2} < d_n < P \) \( D(d) = D(\text{sort}(d_1, \ldots, d_{n-1}, P - d_n)) \).
- (d) \( d_{n-1} \leq d_n \leq \frac{P}{2} \) \( D(d) = \begin{cases} 0 & d = (1, \ldots, 1, 2, 2, 2) \\ d - 3 & d = (1, \ldots, 1, 2, d, d), d \geq 3 \\ \Delta(d) & \text{otherwise.} \end{cases} \)

In particular, the GIT quotient \( \mathbb{P}(V) // G \) is empty if and only if \( D(d) = -1 \) and is a single point if and only if \( D(d) = 0 \).

In case (c), \( D(d) \) is defined recursively in terms of the value of \( D \) on another dimension vector \( \text{sort}(d_1, \ldots, d_{n-1}, P - d_n) \), obtained by arranging the positive integers \( d_1, \ldots, d_{n-1}, P - d_n \).
in weakly increasing order. Since the sum of these integers is strictly less than \( \sum d_i \), the recursion stops after at most \( \sum d_i \) steps. In particular, the function \( D(d) \) is well defined.

Note that the condition that \( \mathbb{P}(V)/G = \emptyset \) is equivalent to \( V \) being a pseudo-homogeneous space for \( G \); see Corollary 5.2. Recall that an \( H \)-representation \( W \) is called a pseudo-homogeneous space if \( H \) has a Zariski dense orbit in \( W \). Pseudo-homogeneous spaces for reductive groups were classified by M. Sato and T. Kimura [SK77]. The vectors \( d \) such that \( \mathbb{P}(V)/G = \emptyset \) can, in principle, be described by appealing to this classification. Note that the passage from \( V \) to \( V' \) in Theorem 1.3(c) is an example of what Sato and Kimura called a castling operation.

Alternatively, the vectors \( d \) such that \( \mathbb{P}(V)/G = \emptyset \) (or equivalently, \( V/G \) is a point) can also be described by appealing to the classification of coregular irreducible representations \( \phi: H \to \text{GL}(W) \) such that \( H \) is semisimple, due to P. Littelmann [Li90]. Here \( \phi \) is called coregular if \( W/H \) is an affine space \( \mathbb{A}^m \). (We are only interested in the cases where \( H = \text{SL}_{d_1} \times \cdots \times \text{SL}_{d_n} \), \( V = V_{d_1} \otimes \cdots \otimes V_{d_n} \) and \( m = 0 \).) Note that the castling operation plays a prominent role in [Li90] as well.

**Example 1.4.** For \( n = 2 \), the GIT quotient \( \mathbb{P}(V)/G \) is non-empty if and only if \( (d_1, d_2) = (d, d) \). Indeed, if \( d_2 > d_1 \), then part (a) of Theorem 1.3 applies, and \( \mathbb{P}(V)/G = \emptyset \). If \( d_2 = d_1 \), then part (b) applies, and \( \mathbb{P}(V)/G \) is single point. \( \square \)

For \( n \geq 3 \), the situation is considerably more complicated. For example, the quotient for dimension vectors of the form \( (2, d_1, d_2) \) is nonempty iff \( d_1 = d_2 \geq 2 \) or \( d_2/d_1 = (k+1)/k \) with integer \( k \geq 2 \); see Corollary 7.1. The general characterization of the set of dimension vectors \( (d_1, d_2, d_3) \) which admit LME states is arithmetically complicated and can be described in terms of generalized Fibonacci sequences, see [BLRV].

The remainder of this paper is structured as follows. In section 2, we provide some additional background on the quantum mechanics problem and its connection to symplectic geometry and geometric invariant theory. Theorem 1.3 is proved in sections 3 and 4. Our argument does use the above-mentioned classifications due to Sato-Kimura and Littelmann; however, the proof of part (d) relies on the work of A. G. Elashvili [El72]. Sections 5 and 6 are devoted to analyzing the recursive algorithm appearing in Theorem 1.3 and proving Theorem 1.2.

In a companion paper [BLRV], we discuss numerous explicit results and examples of locally maximally entangled states with a view to applications in quantum information theory.

2. **Background**

We begin with a short review of relevant background material on quantum mechanics, the connection to symplectic geometry, and basics of geometric invariant theory.

**Density operators and entanglement.** In a multipart quantum system with Hilbert space \( V = V_1 \otimes \cdots \otimes V_d \), physical observables associated with subsystem \( i \) correspond to Hermitian operators \( \mathcal{O} : V_i \to V_i \); the expectation value of the observable in a measurement on a state \( \psi_i \in V_i \) is given by the inner product \( \langle \psi_i, \mathcal{O}\psi_i \rangle \). Any such observable can be promoted to an observable in the full multipart system; the associated Hermitian operator \( \hat{\mathcal{O}} \) acting on \( V \) is defined by \( \hat{\mathcal{O}} = I \otimes \cdots \otimes \mathcal{O} \otimes \cdots \otimes I \).

A crucial feature of multipart quantum systems is that their states are generally *entangled*; that is, they cannot be written as products \( \psi_1 \otimes \cdots \otimes \psi_n \). Furthermore, for \( \psi \in V \), there does not generally exist \( \psi_i \in V_i \) for which \( \langle \psi_i, \mathcal{O}\psi_i \rangle = \langle \psi, \hat{\mathcal{O}}\psi \rangle \) for all \( \mathcal{O} \) acting on \( V_i \). Thus, in...
the context of a multipart system it is no longer possible to represent the state of an individual subsystem simply as a vector or pure state in Hilbert space $V_i$. Rather, the subsystem can be described by a density operator, defined as a non-negative Hermitian operator $\rho_i : V_i \to V_i$ with unit trace. The density operator $\rho_i = \text{tr}(V_1 \otimes \cdots \hat{V}_i \cdots \otimes V_n)$ defined in the introduction is the unique density operator $\rho_i$ such that $(\psi, \hat{O}\psi) = \text{tr}(\rho_i O)$ for all $O$ acting on $V_i$.

A subsystem described by a density operator with eigenvalues/eigenvectors $\{ (p_i, \psi_i) \}$ can be interpreted as being in a statistical ensemble or mixed state in which we have state $\psi_i$ with probability $p_i$. This subsystem is entangled with the rest of the system unless $\{ p_i \} = \{ 1, 0, \ldots, 0 \}$. When the density matrix for the subsystem is a multiple of the identity operator, $\rho_i = I / d_i$, the subsystem is in an equal mixture of all possible states for the subsystem and we say that the subsystem is maximally mixed or maximally entangled with the rest of the system. The locally maximally entangled states that we characterize in this paper are defined by the condition that all elementary subsystems have this property.

The quantum marginal problem. The existence question that we consider is a special case of the quantum marginal problem: which collections of density operators $\{ \rho_\alpha \}$ associated with subsystems $\alpha$ of a multipart system can arise from a quantum state of the entire system? In our case where the subsystems are non-overlapping and the state of the full system is assumed to be pure, a general answer to this question has been provided by Klyachko [Kly04] (see [Walt14] for a review) via a set of inequalities on the spectra for the density operators, or equivalently, in terms of a criterion expressed in the language of representation theory of the symmetric group. These results provide an in-principle method to answer our question, but one that quickly becomes computationally intractable as the subsystem dimensions increase.

The moment map. We now briefly review the connection to symplectic geometry. The Fubini-Study symplectic form on $\mathbb{P}(V)$ is fixed up to overall scaling by its invariance under $U(V)$ transformations and is thus invariant under $K = \text{SU}(V_1) \times \cdots \times \text{SU}(V_n)$. The associated moment map $\mu : \mathbb{P}(V) \to \mathfrak{k}^*$ is given explicitly by

\begin{equation}
\mu(\psi) : k \mapsto (\psi, k\psi) \quad k \in \mathfrak{k}.
\end{equation}

Any $k$ may be written as a linear combination of elements of the form $I \otimes \cdots \hat{k_i} \otimes \cdots I$ with $k_i$ a traceless, Hermitian operator acting on $V_i$. For an element of this form, we have

\begin{equation}
\mu(\psi)(I \otimes \cdots \hat{k_i} \otimes \cdots I) = \text{tr}(\rho_i k_i).
\end{equation}

For $\psi \in V_{LME}$, each $\rho_i$ is proportional to the identity operator; tracelessness of $k_i$ then implies that the moment map vanishes. Conversely, $\text{tr}(\rho_i k_i)$ vanishes for arbitrary traceless Hermitian $k_i$ if and only if $\rho_i$ is proportional to the identity operator. Thus we have that $V_{LME} = \mu^{-1}(0)$.

The Kempf-Ness theorem and geometric invariant theory. As discussed above, the space of LME states, up to equivalence, is given by the symplectic quotient

$$\mu^{-1}(0)/K.$$ 

The Kempf-Ness theorem identifies this space with an algebro-geometric quotient given by geometric invariant theory which we now briefly describe. Let

$$G = \text{SL}(V_1) \times \cdots \times \text{SL}(V_n).$$
Note that $K$ is a maximal compact subgroup of $G$. $G$ acts algebraically on $\mathbb{P}(V_1 \otimes \cdots \otimes V_n) = \mathbb{P}(V)$ and the geometric invariant theory (GIT) quotient $\mathbb{P}(V)//G$ is the projective variety defined by

$$\mathbb{P}(V)//G = \text{Proj} \left( \mathbb{C}[V]^G \right).$$

Here $\mathbb{C}[V]$ is the ring of polynomial functions on $V$ (graded by degree), and $\mathbb{C}[V]^G$ is the subring of invariant functions. Concretely, $\text{Proj} \left( \mathbb{C}[V]^G \right)$ can be constructed as follows (cf. [Reid02]). It is possible to choose homogeneous generators $f_0, \ldots, f_N$ for the ring of invariants $\mathbb{C}[V]^G$ (see [MFK94, Thm. 1.1]), where we define $w_i$ to be the degree of $f_i$. Consider the rational map

$$\pi : \mathbb{P}(V) \rightarrow \mathbb{P}^N_{w_0,\ldots,w_N}$$

given by $v \mapsto (f_0(v) : \cdots : f_N(v))$, where $\mathbb{P}^N_{w_0,\ldots,w_N}$ stands for the $N$-dimensional graded weighted projective space with weights $w_0, \ldots, w_N$. Then the closure of the image of $\pi$ is the GIT quotient $\mathbb{P}(V)//G$, and $\pi$ is the quotient map. Geometrically, the points of $\mathbb{P}(V)//G$ correspond to closed orbits $O$ of $G$ in $V \setminus \{0\}$, up to projective equivalence.

In this context, the Kempf-Ness theorem states that there is a homeomorphism

$$\mu^{-1}(0)/K \cong \mathbb{P}(V)//G$$

where $\mathbb{P}(V)//G$ is given the complex analytic topology. The Kempf-Ness theorem thus converts the problem of understanding $\mu^{-1}(0)/K$, the space of equivalence classes of LME states, into the purely algebraic problem of understanding the GIT quotient $\mathbb{P}(V)//G$. We study this quotient in depth in sections 3 and 4.

3. INVARIANT-THEORETIC PRELIMINARIES

Notational conventions. In the sequel we will denote the $\mathbb{C}$-algebra of regular functions on complex affine algebraic variety $X$ by $\mathbb{C}[X]$. If $X$ is irreducible (but not necessarily affine), then $\mathbb{C}(X)$ will denote the field of rational functions on $X$.

Recall $\dim(X) = \text{trdeg}_{\mathbb{C}}(\mathbb{C}(X))$. Here $\dim(X)$ denotes the dimension of $X$ and $\text{trdeg}_{\mathbb{C}}(\mathbb{C}(X))$ denotes the transcendence degree of $\mathbb{C}(X)$ over $\mathbb{C}$, i.e., the maximal number of elements $f_1, \ldots, f_n \in \mathbb{C}(X)$ which are algebraically independent over $\mathbb{C}$.

Finally, if $G$ is a complex algebraic group acting on $X$, we will denote the ring of $G$-invariant regular functions on $X$ by $\mathbb{C}[X]^G$ and the field of $G$-invariant rational functions by $\mathbb{C}(X)^G$.

Stabilizers in general position. Let $G$ be a reductive complex linear algebraic group and $\rho : G \rightarrow \text{GL}(V)$ be a linear representation. By a theorem of Richardson [Rich72], the action of $G$ on $V$ has a stabilizer $S$ in general position. That is, there exists a closed subgroup $S \subset G$ and a $G$-invariant dense open subset $U \subset V$ such that the stabilizer $G_x$ is conjugate to $S$ for any $x \in U$. Note that here $S$ is uniquely defined by $\rho$ up to conjugacy and $\dim(G \cdot x) = \dim(G) - \dim(S)$ for every $x \in U$.

Lemma 3.1. Let $G$ be a semisimple linear algebraic group and $G \rightarrow \text{GL}(V)$ be a finite-dimensional representation of $G$. Let $\mathbb{C}[V]$ be the ring of polynomial functions on $V$ and let $\mathbb{C}(V)$ be the field of rational functions. Then

\footnote{Here, rational indicates that $\pi$ is only defined on a dense open set $U^{ss} \subset \mathbb{P}^N$. The points of $U^{ss}$ are called semistable points.}
(a) $\mathbb{C}(V)^G$ is the field of fractions of $\mathbb{C}[V]^G$. In other words, every $G$-invariant rational function on $V$ is a ratio of two $G$-invariant polynomials.

(b) The dimension of the GIT quotient $\mathbb{P}(V)//G$ equals $\dim(V) - \dim(G) + \dim(S) - 1$. 

(c) If $S$ is reductive, then $\mathbb{P}(V)//G \neq \emptyset$.

Note that $\dim(\mathbb{P}(V)//G) = -1$ if and only if $\mathbb{P}(V)//G = \emptyset$.

**Proof.** Part (a) is an easy consequence of the fact that the polynomial ring $\mathbb{C}[V]$ is a unique factorization domain; see, e.g. [Po70, Lemma 1] or [PV94, Theorem 3.3].

(b) Recall that by definition, $\mathbb{P}(V)//G = \text{Proj} \mathbb{C}[V]^G$. Thus

$$\dim \mathbb{P}(V)//G = \operatorname{trdeg} \mathbb{C}[V]^G - 1 = \operatorname{trdeg} \mathbb{C}(V)^G - 1.$$ 

By Rosenlicht’s theorem [Ros56, Theorem 2], the transcendence degree of $\mathbb{C}(V)^G$ is given by

$$\operatorname{trdeg} \mathbb{C}(V)^G = \dim(V) - \dim(G \cdot x) = \dim(V) - (\dim(G) - \dim(G_x));$$

where $x \in V$ is a point in general position; see also [PV94, Corollary, p. 156]. Now recall that $G_x$ is conjugate to $S$, so $\dim(G_x) = \dim(S)$, and part (b) follows.

(c) Suppose $S$ is reductive. Then by a theorem of Popov [Po70, Theorem 1], there exists a $0 \neq x \in V$ such that the orbit $G \cdot x$ is closed. By a theorem of Hilbert, $G$-invariant polynomials separate closed orbits in $V$; see, e.g., [MFK94, Corollary 1.2]). In particular, there exists an $p \in \mathbb{C}[V]^G$ such that $p(x) \neq p(0)$. This shows that $\mathbb{C}[V]^G \neq \mathbb{C}$ and thus $\mathbb{P}(V)//G = \text{Proj} \mathbb{C}[V]^G \neq \emptyset$. □

**Corollary 3.2.** (cf. [SK77, §2, Propositions 2 and 3]) Let $G$ be a semisimple linear algebraic group and $G \to \text{GL}(V)$ be a linear representation of $G$. Then the following conditions are equivalent.

(a) $G$ has a dense orbit in $V$,

(b) $\mathbb{C}(X)^G = \mathbb{C}$,

(c) $\mathbb{C}[X]^G = \mathbb{C}$,

(d) The GIT quotient $\mathbb{P}(V)//G$ empty.

**Proof.** The implications (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) are obvious. 

(d) $\implies$ (a). Assume that the GIT quotient $\mathbb{P}(V)//G$ empty, i.e., its dimension is $-1$. Then by Lemma [3.1](b), $\dim(V) = \dim(G) - \dim(S)$. On the other hand, for $x \in V$ in general position, $\dim(G \cdot x) = \dim(V)$. Thus shows that $G \cdot x$ is dense in $V$, as desired. □

**Corollary 3.3.** ([SK77, §3, Proposition 1]) Let $G \to \text{GL}(V)$ be a finite-dimensional representation of a semisimple linear algebraic group $G$. If $\dim(V) > \dim(G)$, then $\mathbb{P}(V)//G \neq \emptyset$.

**Proof.** If $\dim(V) > \dim(G)$, then by Lemma [3.1](b), $\dim(\mathbb{P}(V)//G) \geq \dim(S) \geq 0$, i.e., $\mathbb{P}(V)//G \neq \emptyset$. □

**The index of a representation.** Let $\rho: H \to \text{GL}_d$ be a faithful finite-dimensional representation of a simple complex linear algebraic group $H$. Let $\text{Lie}(H)$ be the Lie algebra of $H$ and

$$\rho^* = d\rho|_e: \text{Lie}(H) \to \text{Lie}(\text{GL}_d) = M_d$$
be the induced map of Lie algebras. Following E. M. Andreev, E. B. Vinberg and A. G. Elashvili [AVE67], we define the index $l(\rho)$ by the formula

$$l(\rho) = \frac{\text{Tr}(\rho^*(h)^2)}{\text{Tr}((\text{ad}(h))^2)},$$

where $h \in \text{Lie}(H)$. Note that the right hand side of the formula is independent of the choice of $h$, as long as $\text{Tr}(\text{ad}(h)^2) \neq 0$. It is clear from this definition that the index is additive, i.e.,

$$l(\rho_1 \oplus \rho_2) = l(\rho_1) + l(\rho_2).$$

**Example 3.4.** Consider the natural representation $\rho_{\text{nat}}$ of $H = \text{SL}_d$ on $V = \mathbb{C}^d$. Take

$$h = \text{diag}(\lambda_1, \ldots, \lambda_d) \in \mathfrak{s}_d,$$

where $\lambda_1 + \cdots + \lambda_d = 0$.

Here $\mathfrak{s}_d = \text{Lie}(\text{SL}_d)$. Then $\rho_{\text{nat}}^*(h) = h$, $\text{Tr}(h^2) = \sum_{i=1}^d \lambda_i^2$ and

$$\text{Tr}(\text{ad}(h)^2) = \sum_{i \neq j} (\lambda_i - \lambda_j)^2 = \sum_{i \neq j} (\lambda_i^2 + \lambda_j^2 - 2\lambda_i \lambda_j) =$$

$$2(d-1) \sum_{i=1}^d \lambda_i^2 - 2\left(\sum_{i=1}^d \lambda_i\right)^2 = 2d \sum_{i=1}^d \lambda_i^2.$$

We conclude that $l(\rho_{\text{nat}}) = \frac{1}{2d}$.

4. **Proof of Theorem 1.3**

**Proof of case (a):** We claim for every $v \in V$ the closure of the orbit $\text{SL}(V_n) \cdot v$ in $V$ contains 0. If we can prove this claim, then clearly $\mathbb{C}[V]^G = \mathbb{C}$, and thus $\mathbb{P}(V)//G$ is empty; see Corollary 3.2.

To prove the claim, let $U = V_1 \otimes \cdots \otimes V_{n-1}$, let $W = V_n$, and write $\sum_{i=1}^m u_i \otimes w_i$, where $u_1, \ldots, u_m$ form a basis of $U$ and $w_i \in W$. By our assumption,

$$m = d_1 \cdots d_{n-1} < d_n = \dim(V_n).$$

Hence, we can choose a basis $f_1, \ldots, f_{d_n}$ of $V_n$ such that $w_1, \ldots, w_m \in \text{Span}(f_1, \ldots, f_{d_n-1})$. Now define the 1-parameter subgroup $\lambda: \mathbb{G}_m \to \text{SL}(V_n)$ by $\lambda(t)f_j = tf_j$ for $j = 1, \ldots, d_n - 1$ and $\lambda(t)f_n = t^{-d_n+1}f_n$. Then $\lambda(t)v = tv \to 0$, as $t \to 0$ and hence, 0 lies in the closure of $\text{SL}(V_n) \cdot v$ in $V$, as claimed.

**Proof of case (b):** Let $U = V_1 \otimes \cdots \otimes V_{n-1}$ and let $W = V_n$. By our assumption $\dim(U) = \dim(W)$. Note that

$$\text{SL}(W) \subset G \subset \text{SL}(U) \times \text{SL}(W).$$

Identify $U$ with $W^*$ and thus $V = U \otimes W$ with the space of $n \times n$ matrices $M_n$, where $\text{SL}(U) \simeq \text{SL}_n$ acts by multiplication on the left and $\text{SL}(W) \simeq \text{SL}_n$ acts by multiplication on the right. Let $f: U \otimes W = M_n \to \mathbb{C}$ be the determinant map. Then $f$ is invariant under $\text{SL}(U) \times \text{SL}(W)$ and hence, under $G$; see (4). Thus shows that $\mathbb{C}[f] \subset \mathbb{C}[V]^G$. On the other hand,

$$\mathbb{C}[V]^G \subset \mathbb{C}[V]^{\text{SL}(W)} = \mathbb{C}[M_n]^{\text{SL}_n} = \mathbb{C}[f].$$
Thus $\mathbb{C}[V]^G = \mathbb{C}[f]$ is a polynomial ring in one variable. Consequently,

$$\mathbb{P}(V)/G = \text{Proj } \mathbb{C}[f] = \mathbb{P}^0$$

is a single point, as claimed.

**Proof of case (c):** Let $W = V_1 \otimes \cdots \otimes V_{d_{n-1}}$, let $H = SL_{d_1} \times \cdots SL_{d_{n-1}}$, and $V' = V_{d_1} \otimes \cdots \otimes V_{d_n} \otimes V_{P-d_n}$, and let $G' = SL_{d_1} \times \cdots \times SL_{d_{n-1}} \times SL_{P-d_n}$, where $P = d_1 \ldots d_{n-1}$. By [Li90 Lemma 2(a)], $V'/G$ is isomorphic to $V'/G'$. Thus

$$\dim(\mathbb{P}(V)/G) = \dim(V'/G) - 1 = \dim(V'/G') - 1 = \dim(\mathbb{P}(V')/G'),$$

as claimed.

**Proof of case (d):** Our argument will rely on the description of stabilizers in general position in irreducible representations $\rho: G \to \text{GL}(V)$ of a semisimple group $G$, satisfying the condition that the index

$$l(\rho|_H) \geq 1 \text{ for every simple normal subgroup } H \text{ of } G,$$

due to Elashvili [E72]. In order to apply this description to our representation of $G = SL(V_1) \times \cdots \times SL(V_n)$ of $G$ on $V = V_1 \otimes \cdots \otimes V_n$ (which we will denote by $\rho$), we need to check that condition (5) is satisfied for this representation.

The simple normal subgroups of $G$ are $H = SL(V_i)$ for $i = 1, \ldots, n$. Clearly the restriction $\rho|_{SL(V_i)}$ is isomorphic to the direct sum of $\dim(V_2 \otimes \cdots \otimes V_n) = d_2 \ldots d_n$ copies of the natural representation $\rho_{\text{nat}}$ of $SL(V_1)$. As we saw in Example [5.1.4] $l(\rho_{\text{nat}}) = \frac{1}{2d_1}$. Hence, by (3),

$$l(\rho|_{SL(V_i)}) = \frac{d_2 \ldots d_n}{2d_1}. \text{ Similarly } l(\rho|_{SL(V_i)}) = \frac{d_1 \cdot d_i - 1 \cdot d_{i+1} \ldots d_n}{2d_i} \text{ for any } i = 1, 2, \ldots, n.$$

Since $d_1 \leq \cdots \leq d_n$, the smallest of these indices is $l(\rho_n) = \frac{d_1 \ldots d_{n-1}}{2d_n}$. The assumption of part (d), that $d_n \leq 1/2d_1 \ldots d_{n-1}$ is thus equivalent to (5).

Under this assumption [E72 Theorem 9] asserts that the connected component $S^0$ of the stabilizer $S$ in general position for the action of $G$ on $V$ is as follows:

$$S^0 \simeq \begin{cases} (\mathbb{G}_m)^{d-1}, & \text{if } n = 3, (d_1, d_2, d_3) = (2, d, d) \text{ and } d \geq 3, \\ (\mathbb{G}_m)^2, & \text{if } n = 3 \text{ and } (d_1, d_2, d_3) = (2, 2, 2), \\ \{1\}, & \text{otherwise}. \end{cases}$$

In all cases, $S^0$ is reductive, and hence, so is $S$. By Lemma [3.1.(c)], we conclude that $\mathbb{P}(V)/G$ is non-empty. Moreover, using the formula $\dim(\mathbb{P}(V)/G) = \dim(V) - \dim(G) + \dim(S) - 1$ of Lemma [3.1(c)] and remembering that $\dim(S) = \dim(S^0)$, we readily check that

$$\dim(\mathbb{P}(V)/G) = \begin{cases} d - 3, & \text{if } n = 3, (d_1, d_2, d_3) = (2, d, d) \text{ and } d \geq 3, \\ 0, & \text{if } n = 3 \text{ and } (d_1, d_2, d_3) = (2, 2, 2), \text{ and} \\ \dim(V) - \dim(G) - 1 & \text{in all other cases}. \end{cases}$$

\[\square\]

**Remark 4.1.** Suppose $1 \leq d_1 \leq \cdots \leq d_n$ and

$$d_n \leq d_1 \ldots d_{n-1}/2.$$
Theorem 1.3(c) tells us that then there exists a non-constant $G = \text{SL}(V_1) \times \cdots \times \text{SL}(V_n)$-invariant polynomial on $V = V_1 \otimes \cdots \otimes V_n$; see Corollary 5.2. Here, $\dim(V_i) = d_i$, as before. Equivalently, there exists a homogeneous $G$-invariant polynomial of degree $\geq 1$. It is natural to try to exhibit such a polynomial explicitly.

It is well known that the hyperdeterminant $h_{d_1, \ldots, d_n} : V \to \mathbb{C}$ of format $d_1 \times \cdots \times d_n$ is a homogeneous $G$-invariant polynomial. However, $h_{d_1, \ldots, d_n} \neq 0$ if and only if

$$d_n \leq d_1 + \cdots + d_{n-1} - (n-2);$$

see [GKZ94, p. 446, Theorems 1.3 and Proposition 1.4]. This inequality is considerably weaker than (6). In other words, for many dimension vectors $d$, the hyperdeterminant is identically zero.

We also note that for any dimension vector $d$ and an integer $k \geq 1$, a procedure due to G. Gour and N. R. Wallach [GW13] produces a basis for the vector space $(\mathbb{C}[V]^G)_k$ of homogeneous $G$-invariant polynomials of degree $k \geq 1$. However, when $\mathbb{C}[V]^G \neq (0)$, it is not a priori clear for which $k$, $(\mathbb{C}[V]^G)_k \neq (0)$.

5. PROOF OF THE FIRST PART OF THEOREM 1.2

For each dimension vector $d = (d_1, \ldots, d_n)$, with $d_1 \leq \cdots \leq d_n$, the recursive algorithm provided by Theorem 1.3 brings us to some terminal dimension vector $e = (e_1, \ldots, e_n)$, where $e_1 \leq \cdots \leq e_n$. The GIT quotient $\mathbb{P}(V)//G$ is empty if the algorithm terminates on case (a), and non-empty if the algorithm terminates on case (b) or (d).

**Remark 5.1.** Recall from the Introduction (see footnote 2) pg. 11 that our standing assumption is that $n \geq 2$ and $d_{n-1} \geq 2$. We now observe that the terminal vector $e$ also satisfies these conditions. Obviously, $n$ does not change; our claim is that there are at most $n-2$ ones among the integers $e_1, \ldots, e_n$. Indeed, in each recursion step, the number of 1s in the list of dimensions can increase by one, but with $n-2$ 1s, we will either be in case (a) or (b), and the recursion terminates.

Let us now define a new dimension vector $a = (a_1, \ldots, a_m)$ by removing all 1s from $(e_1, \ldots, e_n)$.

**Lemma 5.2.** Suppose that a terminal vector $e = (e_1, \ldots, e_n)$ satisfying the conditions for case (a), (b), or (d), is obtained from $d = (d_1, \ldots, d_n)$ by repeatedly performing the transformation of Theorem 1.3(c). Assume further that $a = (a_1, \ldots, a_m)$ is obtained from $e$ by removing all 1s, as above. Then

(a) $\Delta(d) = \Delta(e) = \Delta(a)$,

(b) $g_{\max}(d) = g_{\max}(e) = g_{\max}(a)$, and

(c) $R(d) = R(e) = R(a)$.

Note that $m \geq 2$ by Remark 5.1. In particular, $g_{\max}(a)$ is well defined.

**Proof:** Let $d' = (d'_1, \ldots, d'_{n})$ be the set of dimensions obtained from $d$ by a single application of the castling transformation of Theorem 1.3(c). Since $\Delta(d_1, \ldots, d_n), R(d_1, \ldots, d_n)$ and $g_{\max}(d_1, \ldots, d_n)$ are all symmetric in $d_1, \ldots, d_n$, we may reorder $d'_1, \ldots, d'_{n}$ and thus assume

---

5Gour and Wallach show that $(\mathbb{C}[V]^G)_k = (0)$ unless $k$ is divisible by the least common multiple $l = \text{lcm}(d_1, \ldots, d_n)$. Thus we only need to consider $k$ of the form $lq$, with $q = 1, 2, 3, \ldots$. 
that $d'_i = d_i$ for $i \leq n - 1$ and $d'_{n} = P - d_n$, where $P = d_1 \ldots d_{n-1}$. The lemma follows by showing that each of the three quantities is invariant under $d \to d'$ and under eliminating a 1.

(a) An easy calculation shows that $\Delta(d') = P(P - d_n) - (P - d_n)^2 - \sum_{i=1}^{n-1}(d_i^2 - 1)$ is equal to

$$\Delta(d) = P d_n - d_n^2 - \sum_{i=1}^{n-1}(d_i^2 - 1).$$

The identity $\Delta(1, d_2, \ldots, d_n) = \Delta(d_2, \ldots, d_n)$ is immediate from the definition.

(b) It is again obvious from the definition that $g_{\text{max}}(1, e_2, \ldots, e_n) = g_{\text{max}}(e_2, \ldots, e_n)$. Since $m \geq 2$, this implies that $g_{\text{max}}(d) = g_{\text{max}}(e)$. To show invariance under the transformation in Theorem 1.3(c), we will show more generally that $\gcd(d'_1, \ldots, d'_k) = \gcd(d_1, \ldots, d_k)$ for any $2 \leq k \leq n$ and $1 \leq i_1 < \cdots < i_k \leq n$. If $i_k \leq n - 1$, this is obvious, since $d'_{i_k} = d_{i_k}$ for each $k$. If $i_k = n$, then

$$\gcd(d'_1, \ldots, d'_{i_k-1}, d'_n) = \gcd(d_1, \ldots, d_{i_k-1}, -d_n + P) = \gcd(d_1, \ldots, d_{i_k-1}, d_n),$$

since $P$ is divisible by $d_{i_k}$ for $1 \leq k \leq n - 1$.

(c) We have that $R(d_1, \ldots, d_n) = \Delta(d_1, \ldots, d_n) - n + 1 + \sum_{k=2}^{n}(-1)^k G_k(d_1, \ldots, d_n)$. Since $\Delta$ and the set of $\text{GCD}$s for all $k$-tuples with $k \geq 2$ are invariant under $d \to d'$, $R$ also invariant. If $d_1 = 1$, then all $\text{GCD}$s involving $d_1$ are equal to 1, so we have that $\sum_{k=1}^{n}(-1)^k G_k(d_1, \ldots, d_n)$ is equal to $\sum_{k=1}^{n}(-1)^k G_k(d_2, \ldots, d_n) + \sum_{k=1}^{n}(-1)^{a_{k-1}} = \sum_{k=1}^{n}(-1)^k G_k(d_2, \ldots, d_n)$. Since also $\prod_{i=1}^{n} d_i - \prod_{i \neq 1} d_i$, $R$ is unchanged if we remove dimensions equal to 1.

We now show that the value of $R = R(d)$ predicts on which case the algorithm in Theorem 1.3 will terminate, and thus whether or not the quotient is empty.

**Proposition 5.3.** For the dimension vector $d = (d_1, \ldots, d_n)$, the algorithm in Theorem 1.3 terminates on case (a) if and only if $R < 0$, on case (b) if and only if $R = 0$ and on case (d) if and only if $R > 0$.

**Proof.** Since the algorithm always terminates on case (a), (b), or (d), we need only show that $R(d)$ is respectively negative, zero, and positive in these three cases. Let $e = (e_1, \ldots, e_n)$ be the terminal vector, with $e_1 \leq \cdots \leq e_n$, and $a = (a_1, \ldots, a_{m-1}, a_m)$ be obtained from $e$ by removing all 1s, as above. By Lemma 5.2 it suffices to show that $R = R(a)$ is negative, positive, and zero in cases (a), (b) and (d), i.e., if $e_n > e_1 \cdots e_{n-1}$ and $e_n \leq \frac{1}{2}e_1 \cdots e_{n-1}$, or equivalently, if $a_n > a_1 \cdots a_{m-1}$, $a_m = a_1 \cdots a_{m-1}$ and $a_m \leq \frac{1}{2}a_1 \cdots a_{m-1}$, respectively. For the proof below, it will be useful to define $B_k$ to be the sum of all the terms in $G_k$ with $k$-tuples involving $a_m$ and $A_k = G_k(a_1, \ldots, a_{m-1})$ to be the sum of the remaining terms. Then $G_k = A_k + B_k$, and

$$R = a_1 \cdots a_m - a_m^2 + \sum_{k=1}^{m-1}(-1)^k(A_k - B_{k+1}).$$

We consider the three cases in turn.

**Case (b):** $a_m = a_1 \cdots a_{m-1}$.

Here, each term in $B_{k+1}$ is equal to the corresponding term in $A_k$ obtained by omitting $a_m$ from the $\text{GCD}$, so we have $B_{k+1} = A_k$ for $k \geq 1$. Since $a_1 \cdots a_m - a_m^2 = 0$ in this case, equation (7) gives $R(d_1, \ldots, d_n) = 0$. 
Case (a): $a_m > a_1 \cdots a_{m-1}$.

In this case, we can write $a_m = a_1 \cdots a_{m-1} + \alpha$ for some $\alpha > 0$. From (7) we have

\[(8) \quad R(a_1, \ldots, a_{m-1}, a_m) = -\alpha^2 - \alpha a_1 \cdots a_{m-1} - \sum_{k \geq 1 \text{ odd}} (A_k - B_{k+1}) + \sum_{k \geq 2 \text{ even}} (A_k - B_{k+1})\]

For each term in $A_k$, there is a corresponding term in $B_{k+1}$ obtained by including $a_m$ in the $gcd$. Since $gcd(a_{i_1}, \ldots, a_{i_k}, a_m) \leq gcd(a_{i_1}, \ldots, a_{i_k})$, we have $A_k \geq B_{k+1}$. Making use of this for odd $k$ and assuming for now that $\alpha \geq 2$, we obtain from (8)

\[R(a_1, \ldots, a_{m-1}, a_m) < -2a_1 \cdots a_{m-1} + \sum_{k \geq 2 \text{ even}} A_k\]

\[\leq -2a_1 \cdots a_{m-1} + \sum_{k \geq 2 \text{ even}} \left(\frac{m-1}{k}\right) a_{m-2} a_{m-1}\]

\[= -2a_1 \cdots a_{m-1} + (2^{m-2} - 1) a_{m-2} a_{m-1}\]

\[< (2^{m-2} - 2 + a_1 \cdots a_{m-3}) a_{m-2} a_{m-1}\]

where to obtain the third line, we use that $A_k$ has $\binom{m-1}{k}$ terms and that $gcd(a_{i_1}, \ldots, a_{i_k})^2 \leq a_{m-k}^2 \leq a_{m-2} a_{m-1}$ for $k \geq 2$ since the GCD of $k$ integers chosen from $(a_1, \ldots, a_{m-1})$ cannot exceed $a_{m-k}$, the $k$th largest number in this set. Since $a_i \geq 2$, the term in brackets in the final expression is negative, and we conclude that $R < 0$.

Next consider the case where $\alpha = 1$. Here, all GCDs involving $a_m$ are equal to 1, so we have $B_k = \binom{m-1}{k-1}$ and the terms in (8) involving $B$ are

\[(10) \quad \sum_{k \geq 1} (-1)^{k-1} B_{k+1} = \sum_{k \geq 1} (-1)^{k-1} \binom{m-1}{k} = 1\]

Then from (8), we get

\[(11) \quad R(a_1, \ldots, a_{m-1}, a_m) = -a_1 \cdots a_{m-1} - \sum_{k \geq 1 \text{ odd}} A_k + \sum_{k \geq 2 \text{ even}} A_k\]

As above, we can now decompose $A_k = C_k + D_k$, where $D_k$ represents all the terms in $A_k$ with $k$-tuples involving $a_{m-1}$ and $C_k$ are the remaining terms. The same argument as before shows that $C_k \geq D_{k+1}$. Starting from (11) and eliminating negative terms $-D_{2l+1}$ and $-(C_{2l-1} - D_{2l})$ we then have

\[R(a_1, \ldots, a_{m-1}, a_m) < -a_1 \cdots a_{m-1} + \sum_{k \geq 2 \text{ even}} C_k\]

\[\leq (2^{m-3} - 1 - a_1 \cdots a_{m-3}) a_{m-2} a_{m-1}\]

where the calculation is the same as in (9) but we end up with $2^{m-3}$ instead of $2^{m-2}$ since $C_k$ involve only $m-2$ $a_k$s. Again, the term in brackets in the final expression is negative, and we conclude that $R < 0$.

Case (d): $a_m \leq \frac{1}{2} a_1, \ldots, a_{m-1}$.

Note that this is only possible if $m \geq 3$. Starting from equation (7), we have

\[R = \frac{1}{4} a_1^2 \cdots a_{m-1}^2 - \left(\frac{1}{2} a_1 \cdots a_{m-1} - a_m\right)^2 + \sum_{k \geq 1} (-1)^k (A_k - B_{k+1})\]
\[ a_{m-1}^2(a_1 \cdots a_{m-2} - 1) + \sum_{k \geq 1} (-1)^k(A_k - B_{k+1}) \]

where in the second line, we have used that the maximum value of \((\frac{1}{2}a_1 \cdots a_{m-1} - a_m)^2\) will be for \(a_m = a_{m-1}\), and in the third line, we have discarded non-negative terms \((A_k - B_{k+1})\) for \(k\) even and positive terms \(B_{k+1}\) for \(k\) odd. Since each of the \(\binom{m-1}{k}\) GCDs contributing to \(A_k\) is less than or equal to \(a_{k-1}\), we have that

\[
R > a_{m-1}^2(a_1 \cdots a_{m-2} - 1) - \sum_{k \geq 1 \text{ odd}} a_{m-1}^2 \binom{m-1}{k}
\]

Since \(a_1, \ldots, a_{m-2} \geq 2\), we see that \(R > 0\) unless \((a_1, \ldots, a_{m-2}) = (2, \ldots, 2)\).

For this case, with \((a_1, \cdots, a_m) = (2, \ldots, 2, a_{m-1}, a_m)\), we can calculate the second line of (12) directly. Consider two cases.

Case 1: \(a_m\) is even. Here \(A_1 = 4(m-2) + a_{m-1}^2, B_2 = 4(m-2) + \gcd(a_{m-1}, a_m)^2\), and \(A_k = B_{k+1}\) for any \(k \geq 2\). Thus

\[
R > a_{m-1}^2(2^{m-2} - 1) - (A_1 - B_2) = a_{m-1}^2(2^{m-2} - 2) + \gcd(a_{m-1}, a_m)^2 > 0.
\]

Case 2: \(a_m\) is odd. Here

\[
B_2 = (m-2) + \gcd(a_{m-1}, a_m)^2 = \gcd(a_{m-1}, a_m)^2 - \binom{m-1}{0} + \binom{m-1}{1}
\]

and \(B_{k+1} = \binom{m-1}{k}\) for any \(k \geq 2\). Thus

\[
\sum_{k \geq 1} (-1)^kB_{k+1} = -\gcd(a_{m-1}, a_m)^2 + \sum_{i \geq 0} (-1)^i\binom{m-1}{i} = -\gcd(a_{m-1}, a_m)^2.
\]

Moreover, \(A_1 = 4(m-2) + a_{m-1}^2 = a_{m-1}^2 - 4\binom{m-1}{0} + 4\binom{m-1}{1}\) and

\[
A_k = 4\binom{m-2}{k} + \gcd(2, a_{m-1})^2\binom{m-2}{k-1} = 4\binom{m-1}{k} + (\gcd(2, a_{m-1})^2 - 4)\binom{m-2}{k-1}
\]

for any \(k \geq 2\). Thus

\[
R \geq a_{m-1}^2(2^{m-2} - 1) - \sum_{k \geq 1} (-1)^kB_{k+1} - A_1 + \sum_{k \geq 2} (-1)^kA_k
\]

\[
= a_{m-1}^2(2^{m-2} - 1) + \gcd(a_{m-1}, a_m)^2 - a_{m-1}^2 + 4\binom{m-1}{0} - 4\binom{m-1}{1} + \sum_{k \geq 2} (-1)^kA_k
\]

\[
= a_{m-1}^2(2^{m-2} - 2) + \gcd(a_{m-1}, a_m)^2 + 4\sum_{k \geq 0} (-1)^k\binom{m-1}{k}
\]

\[
+ (\gcd(2, a_{m-1})^2 - 4)\sum_{k \geq 2} (-1)^k\binom{m-2}{k-1}
\]
If $m \geq 4$, then the first term is $\geq 8$ and thus $R > 0$. As we mentioned above, the inequality $a_m \leq a_1 \cdots a_{m-1}/2$ forces $m$ to be $\geq 3$. Thus we may assume that $m = 3$. Since $a_1 = 2$, we have $a_3 \leq 2 \cdot a_2/2 = a_2$ and thus $a_2 = a_3$. In this case $\gcd(a_{m-1}, a_m) = \gcd(a_2, a_3) = a_3$. Substituting $m = 3$ into the above inequality, and remembering that $a_1 = 2$ and $a_2 = a_3 \geq 2$ is odd, we obtain

$$R \geq a_3^2(2^3 - 2) + a_2^2 + (\gcd(2, a_2)^2 - 4) \geq 0 + 9 + (1 - 4) > 0,$$

as desired. \hfill \Box

**Proof of Theorem 1.2 first part.** Suppose $R(d) < 0$. By Proposition 5.3 the algorithm of Theorem 1.3 terminates on case (a) and the quotient $P(V)/G$ is empty.

On the other hand, $R(d) \geq 0$, Proposition 5.3 shows that the algorithm of Theorem 1.3 terminates on case (b) or (d). Theorem 1.3 now tells us that the quotient $P(V)/G$ is non-empty. This proves the first assertion of Theorem 1.2. \hfill \Box

### 6. Conclusion of the proof of Theorem 1.2

We will prove the remaining statements in Theorem 1.2 using the following proposition.

**Proposition 6.1.** Suppose that we perform the recursive procedure of Theorem 1.3 starting with the dimension vector $d = (d_1, \ldots, d_n)$. Here, as always, $d_1 \leq \cdots \leq d_n$, $n \geq 2$ and $d_{n-1} \geq 2$. Denote the terminal dimension vector by $e = (e_1, \ldots, e_n)$.

1. If $\Delta(d) < -5$, then $e_1 \cdots e_{n-1} \leq e_n$. That is, the recursion in Theorem 1.3 terminates on case (a) or (b). The quotient $P(V)/G$ is empty or a single point.
2. If $-5 \leq \Delta(d) < -2$, then $e_1 \cdots e_{n-1} = e_n$. That is, the recursion in Theorem 1.3 terminates on case (b). The quotient $P(V)/G$ is a single point.
3. If $\Delta(d) = -2$, then the recursion in Theorem 1.3 terminates on case (d) with $e = (1, \ldots, 1, 2, a, a)$ for some $a \geq 2$. Here $a = g_{\text{max}}(d) \geq 3$. The quotient $P(V)/G$ is a point if $a = 2$ and has dimension $a - 3$ if $a \geq 3$.
4. If $\Delta(d) > -2$, then the recursion in Theorem 1.3 terminates on case (d) with $e \neq (1, \ldots, 1, 2, a, a)$ for any $a \geq 2$. In this case $\Delta(d) \geq 2$, and the quotient $P(V)/G$ has dimension $\Delta(d)$.

**Proof.** Let $a = (a_1, \ldots, a_m)$ be obtained by removing all 1s from $e = (e_1, \ldots, e_n)$, as in the previous section. By Remark 5.1, $m \geq 2$. By Lemma 5.2, $\Delta(d) = \Delta(e) = \Delta(a)$ and $g_{\text{max}}(d) = g_{\text{max}}(e) = g_{\text{max}}(a)$. Moreover, one readily sees that $a$ is also a terminal vector for the recursive procedure of Theorem 1.3 and that $e$ and $a$ correspond to the same terminal case in Theorem 1.3, i.e., case (a), (b) or (d). Thus, for the purpose of proving Proposition 6.1 we may replace $d$ by $a$. That is, we may assume that $d = (d_1, \ldots, d_n)$ is terminal, $2 \leq d_1 \leq \cdots \leq d_n$, and $n \geq 2$.

Set $P = d_1 \cdots d_{n-1}$. To prove the proposition, we will show that

1. if $d$ corresponds to case (a), i.e. $d_n > P$, then $\Delta(d) < -5$,
2. if $d$ corresponds to case (b), i.e. $d_n = P$, then $\Delta(d_1, \ldots, d_n) < -2$,
3. if $d = (2, a, a)$, then $\Delta(d) = -2$ and $g_{\text{max}}(d) = a$. 
(iv) If \( d \) corresponds to case (d), i.e. \( d_n \leq P/2 \) and moreover \( d \neq (2, a, a) \) for any integer \( a \geq 2 \), then \( \Delta(d) \geq 2 \).

If we can establish (i) - (iv), Proposition 6.1 will follow directly from Theorem 1.3.

To prove (i), let us fix \( d_1, \ldots, d_{n-1} \) and view
\[
\phi(x) = \Delta(d_1, \ldots, d_{n-1}, x) = xP - d_1^2 - \ldots - d_{n-1}^2 - x^2 + n - 1
\]
as a quadratic polynomial in \( x \). Note that \( f'(x) = P - 2x \), so \( \phi(x) \) is increasing for \( x \leq P/2 \) and decreasing for \( x \geq P/2 \). In particular, if \( d_n \geq P + 1 \), then
\[
\Delta(d) = \phi(d_n) \leq \phi(P + 1) = -P - 1 - \sum_{i=1}^{n-1} (d_i^2 - 1).
\]
Since \( n \geq 2 \), each \( d_i \geq 2 \) and in particular, \( P \geq d_1 \geq 2 \). This yields
\[
\Delta(d) \leq -2 - 1 - (2^2 - 1) = -6,
\]
as claimed.

To prove (ii), assume \( d_n = P \). Then \( \Delta(d) = \phi(P) = -\sum_{i=1}^{n-1} (d_i^2 - 1) < -2 \), since \( n \geq 2 \) and each \( d_i \geq 2 \).

(iii) is easy: \( \Delta(2, a, a) = 2a^2 - 4 - a^2 - a^2 + 3 - 1 = -2 \) by the definition of \( \Delta \), and \( g_{\max}(2, a, a) = a \) by the definition of \( g_{\max} \).

To prove (iv), assume that \( 2 \leq d_1 \leq \cdots \leq d_{n-1} \leq d_n \leq P/2 \). We are interested in the value of \( \Delta(d) = \phi(d_n) \). Since \( f'(x) \leq 0 \) for any \( x \) in the interval \([d_{n-1}, P/2]\), we have
\[
\Delta(d) = \phi(d_n) \geq \phi(d_{n-1}) = d_n - 1 - \sum_{i=1}^{n-1} (d_i^2 - 1).
\]
Remembering that \( P = d_1 \cdots d_{n-1} \) and \( d_i \leq d_{n-1} \) for \( i = 1, \ldots, n-1 \), we conclude that
\[
\begin{equation}
(13) \quad \Delta(d) \geq d_n - 1 - \sum_{i=1}^{n-1} (d_i^2 - 1) = d_n^2 - d_{n-1}^2 - 2d_{n-2}^2 + n - 1.
\end{equation}
\]
By our assumption, \( n \geq 3 \) and \((d_1, \ldots, d_n) \neq (2, d, d)\) for any \( d \geq 2 \). Let us now consider two cases.

Case 1: \( n \geq 4 \). In this case \( d_1 \cdots d_{n-2} - n \geq 2^{n-2} - n \geq 0 \) and (13) tells us that \( \Delta(d) \geq n - 1 \geq 3 \).

Case 2: \( n = 3 \) but \((d_1, d_2, d_3) \neq (2, d, d)\) for any \( d \geq 2 \). Here our assumption that \( d_3 \leq P/2 = d_1 d_2/2 \) implies \( d_1 \geq 3 \). In this case (13) yields \( \Delta(d) \geq d_2^2 (d_1 - 3) + 2 \), and \( d_1 \geq 3 \) implies \( \Delta(d) \geq 2 \). This completes the proof of (iv) and thus of Proposition 6.1.

\textbf{Proof of \textit{Theorem 1.2} second part.} To prove the second assertion of \textit{Theorem 1.2}, let us consider three cases, where \( \Delta(d) < -2 \), \( \Delta(d) = -2 \), and \( \Delta(d) > -2 \), respectively.

When \( \Delta(d) > -2 \), Proposition 6.1 tells us that the recursion terminates on case (d) and the dimension of \( \mathbb{P}(V) // G \) is \( \Delta(d) \geq 2 \). In this case \( R > 0 \) By Proposition 5.3.

When \( \Delta(d) = -2 \), we are in case (3) of Proposition 6.1. Here the recursion terminates on case (d) and the dimension of \( \mathbb{P}(V) // G \) is \( (g_{\max}(d) - 3) \) for \( g_{\max}(d) \geq 3 \) and 0 otherwise.

Finally, when \( \Delta(d) < -2 \) we are in case (1) or (2) of Proposition 6.1. The proposition tells us that the recursion terminates on case (a) or (b) of \textit{Theorem 1.3}. By Proposition 5.3, the recursion terminates in case (a) if \( R < 0 \) and in case (b) if \( R = 0 \). Combining this with
Theorem 1.3, we see that $\mathbb{P}(V)/\!\!/G$ is a single point if $\Delta(d) < -2$ and $R = 0$ and is empty if $\Delta(d) < -2$ and $R = 0$, as desired. \hfill $\square$

7. Examples

To conclude, we describe a few explicit results implied by Theorems 1.2 and 1.3.

**Corollary 7.1.** For dimension vectors $(2, d_2, d_3)$, with $2 \leq d_2 \leq d_3$, the quotient $\mathbb{P}(V)/\!\!/G$ is non-empty if and only if

(i) $(d_1, d_2, d_3) = (2, b, b)$ for $b \geq 2$ or

(ii) $(d_1, d_2, d_3) = (2, kb, (k + 1)b)$ for positive integers $k, b$ with $kb > 1$.

In case (i), the quotient $\mathbb{P}(V)/\!\!/G$ is of dimension $\max(b - 3, 0)$. In case (ii), $\mathbb{P}(V)/\!\!/G$ is a single point.

**Proof.** First assume that $(d_1, d_2, d_3)$ is as in (i) and (ii). Note that $\Delta(2, d_2, d_3) = -(d_3 - d_2)^2 - 2$. In particular, in case (i), $\Delta(1, d_2, d_3) = -2$, and the desired conclusion follows from Proposition 6.1. In case (ii), the recursive procedure of Theorem 1.3 yields

\[(2, kb, (k + 1)b) \mapsto (2, (k - 1)b, kb) \mapsto \cdots \mapsto (2, 2b, 3b) \mapsto (2, b, 2b).\]

The terminal triple $(2, b, 2b)$ is covered by Theorem 1.3(b), for any $b \geq 1$, except that for $b = 1$, we should write it as $(1, 2, 2)$, rather than $(2, 1, 2)$. (Note that we can also check directly that $R(d) = b^2 - 4 + \gcd(2, b)^2 > 0$ in case (i) and $R(d) = 0$ in case (ii).)

Conversely, suppose $\mathbb{P}(V)/\!\!/G$ is non-empty for some dimension vector $(2, d_2, d_3)$. Denote the terminal triple by $e = (e_1, e_2, e_3)$. Then either $e_1 = 1$ and $e_2 = 2$, or $e_1 = 2$. Moreover, either $e_1 e_2 = e_3$, as in Theorem 1.3(b) or $e_3 \leq e_1 e_2 / 2$, as in Theorem 1.3(d). This leaves us with

1. $(e_1, e_2, e_3) = (2, b, b)$ or
2. $(e_1, e_2, e_3) = (2, b, 2b)$, where $b \geq 2$ or
3. $(e_1, e_2, e_3) = (1, 2, 2)$.

In cases (2) and (3), we recover $(d_1, d_2, d_3) = (2, kb, (k + 1)b)$, for $b \geq 2$ and $b = 1$, respectively, by reversing the recursive procedure 14.

**Remark 7.2.** For $\{n = 3, d_1 > 2\}$ or $n \geq 3$, the naive dimension $\Delta(d_1, \ldots, d_n)$ considered as a function of $d_n$ is a downwards parabola that is positive for $d_n = d_{n-1}$, increases to a maximum at $d_n = \frac{1}{2}d_1 \cdots d_{n-1}$, and then decreases to $-2$ at some $d_s \in (P/2, P)$ where $P = d_1 \cdots d_{n-1}$. Thus, by Proposition 6.1, the quotient is nonempty and has dimension $\Delta(d_1, \ldots, d_n)$ for all $d_n$ in the range $[d_{n-1}, d_s)$. If $d_s$ is an integer, the quotient is non-empty for $d_n = d_s$ and has dimension governed by case (3) of Proposition 6.1. The remaining values of $d_n$ for which the quotient is nonempty are a set of sporadic cases with $d_s < d_n \leq P$ satisfying $R(d) = 0$ for which the quotient is a point. We provide a more detailed analysis of these sporadic cases for $n = 3$ in [BLRV].

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