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On differential operators for bivariate Chebyshev polynomials

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Abstract We construct the differential operators for which bivariate Chebyshev polynomials of the first kind, associated with simple Lie algebras $C_2$ and $G_2$, are eigenfunctions.

1. In these notes, we obtain differential operators for which bivariate Chebyshev polynomials of the first kind, associated with the root systems of the simple Lie algebras $C_2$ and $G_2$, are eigenfunctions. For the case of bivariate Chebyshev polynomials, associated with the Lie algebra $A_2$, such operators were obtained in the well known Koornwinder’s work \cite{1}.

Chebyshev polynomials in several variables are natural generalizations of the classical Chebyshev polynomials in one variable (see, for example \cite{2}). The polynomials of the first kind can be defined in the following manner.

Denote by $R$ a reducible system of roots for a simple Lie algebra $L$. A system of roots is a set of vectors in $d$-dimensional Euclidean space $E^d$ with a scalar product $(\ldots,\ldots)$. This system is completely determined by a basis of simple roots $\alpha_i$, $i = 1, \ldots, d$ and by a group of reflections of $R$ called a Weyl group $W(R)$. Generating elements of the Weyl group $w_i$, $i = 1, \ldots, d$ acts on any vector $x \in E^d$ according to the formula

$$w_i x = x - \frac{2(x, \alpha_i)}{\langle \alpha_i, \alpha_i \rangle} \alpha_i.$$  \hspace{1cm} (1)

In particular, if $x = \alpha_i$ we obtain from (1) $w_i \alpha_i = -\alpha_i$. A system of roots $R$ is closed under the action of related Weyl group $W(R)$.

To any root $\alpha$ from the system $R$ corresponds the coroot

$$\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}.$$  \hspace{1cm} (2)

For the basis of the simple coroots $\alpha_i^\vee$, $i = 1, \ldots, d$ one can define the dual basis of fundamental weights $\lambda_i$, $i = 1, \ldots, d$

$$\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$$

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(we identify the dual space $E^d^\ast$ with $E^d$). The bases of roots and weights are related by the linear transformation
\[ \alpha_i = \sum_j C_{ij} \lambda_j, \quad C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \] (2)
where $C$ is the Cartan matrix of the Lie algebra $L$.

For any Lie algebra $L$ with related sistem of roots $R$ and Weyl group $W(R)$, an orbit function $\Phi_n(\phi)$ is defined as
\[ T_L^L(\phi) = \frac{1}{|W(R)|} \sum_{w \in W(R)} e^{i(wn,\phi)}. \] (3)

In the formula (3) $|W(R)|$ is a number of elements in a group $W(R)$, $n$ is expressed in the basis of fundamental weights $\{\lambda_i\}$ and $\phi$ is expressed in the dual basis of coroots $\{\alpha_i^\vee\}$
\[ n = \sum_{i=1}^d n_i \lambda_i \quad n_i \in Z, \quad \phi = \sum_{i=1}^d \phi_i \alpha_i^\vee \quad \phi_i \in [0, 2\pi). \]

Obviously $T^L_n(\phi)$ is a $W(R)$-invariant function because of
\[ T^L_{\tilde{w}n}(\phi) = T^L_n(\phi), \quad \forall \tilde{w} \in W(R). \]

Then we define the new variables $x_i$ (generalized cosines) by the relations
\[ x_i = T_{e_i}(\phi), \quad e_i = (0,\ldots,0,1,0,\ldots,0). \] (4)

It is shown in the works [1, 3, 4, 5, 6, 7] that the function $T_n(\phi)$ defined by the formula (3) with non-negative integer $n_i$ from $n = (n_1,\ldots,n_d)$ can be expressed in the terms of $x_i$. This function gives us up to a normalization the multivariate Chebyshev polynomials $T_{n_1,\ldots,n_d}$ of the first kind.

2. The simplest example of the above construction is the classical Chebyshev polynomials associated with the Lie algebra $A_1$. The related Weyl group consists from the identical element $w_0$ and the reflection of the single positive root $w_1 \lambda = -\lambda$. In this case the definition (3) gives
\[ T_n(\phi) = \frac{1}{2}(e^{in\phi} + e^{-in\phi}) = \cos n\phi, \quad x = T_1(\phi) = \cos \phi. \] (5)

To derive the differential operator(s) for which the classical polynomials of the first kind $T_n(x)$ are eigenfunction we firstly write out the differential equation for $\cos n\phi$
\[ \frac{d^2 \cos n\phi}{d\phi^2} + n^2 \cos n\phi = 0. \] (6)

It follows from (3) that desired operator in terms of the angle variable $\phi$ has the form
\[ L^{(A_1)}(\phi) = \frac{d^2}{d\phi^2}. \] (7)
Changing the variable \( \cos \phi \to x \) in (7) we obtain the well known operator in terms of \( x \)

\[
L^{(A_1)}(x) = (1 - x^2) \frac{d^2}{dx^2} - x \frac{d}{dx}.
\]  

(8)

3. Now we turn to the generalized cosine associated with the Lie algebra \( A_2 \). At the first step we find the orbit function related to the algebra \( A_2 \). The root system of this algebra has two fundamental roots \( \alpha_1, \alpha_2 \) and includes the positive root \( \alpha_1 + \alpha_2 \) together with their reflections. The action of generating elements \( w_1, w_2 \) of the Weyl group \( W(A_2) \) on the fundamental roots are given by the formulas

\[
w_1 \alpha_1 = -\alpha_1, \quad w_1 \alpha_2 = \alpha_1 + \alpha_2, \quad w_2 \alpha_1 = \alpha_1 + \alpha_2, \quad w_2 \alpha_2 = -\alpha_2.
\]

Taking into account (2) and explicit form of the Cartan matrix \( C(A_2) \) (see, for example [8]) we obtain the action of \( w_1, w_2 \) on the fundamental weights

\[
w_1 \lambda_1 = \lambda_2 - \lambda_1, \quad w_1 \lambda_2 = \lambda_2, \quad w_2 \lambda_1 = \lambda_1, \quad w_2 \lambda_2 = \lambda_1 - \lambda_2.
\]  

(9)

The action of the other group elements on the fundamental weights is determined by their representation in terms of the generating elements

\[
w_3 = w_1 w_2, \quad w_4 = w_2 w_1, \quad w_5 = w_1 w_2 w_1, \quad w_0 = e.
\]  

(10)

Using these formulas, the definition (3) and the notation

\[
n = m \lambda_1 + n \lambda_2, \quad \phi = \phi \alpha_1^\vee + \psi \alpha_2^\vee
\]

we find the \( W(A_2) \)-invariant function of two variables

\[
T_{m,n}(\phi, \psi) = e^{im\phi} e^{in\psi} + e^{im(\psi-\phi)} e^{in(\phi-\psi)} + e^{im(\psi+\phi)} e^{-in\phi} + e^{-im\psi} e^{in(\phi-\psi)} + e^{-im\psi} e^{-in\phi}.
\]  

(11)

The normalization factor was omitted in (11) because it is not essential for our purpose.

At the second step we find differential operators for which the orbit functions \( T_{m,n}(\phi, \psi) \) for any \( m, n \) are the eigenfunctions

\[
L_N(T_{m,n}) = E_{m,n} T_{m,n}.
\]

The form of the orbit function implies that the action of the operator \( L_N \) on each exponent from (11) must gives us the same eigenvalues \( E_{m,n} \) for any \( m, n \). For this reason we search the operators of the form

\[
L_N^{(A_2)}(\phi, \psi) = \sum_{k=0}^{N} a_k \frac{\partial^N}{\partial \phi^{(N-k)} \partial \psi^k},
\]  

(12)

with real constant coefficients \( a_k, k = 0, \ldots, N \).

Let us act by the operator \( L_N^{(A_2)}(\phi, \psi) \) on \( T_{m,n} \) and write out the chain of equalities of coefficients at the each exponent of (11)

\[
\sum_{k=0}^{N} a_k m^{N-k} n^k = \sum_{k=0}^{N} a_k (-m)^{N-k} (m + n)^k = \sum_{k=0}^{N} a_k (m + n)^{N-k} (-n)^k =
\]
\[
\sum_{k=0}^{N} a_k (-m - n)^{N-k} (m)^k = \sum_{k=0}^{N} a_k n^{N-k} (-m - n)^k = \sum_{k=0}^{N} a_k (-n)^{N-k} (-m)^k.
\]

Some conclusions about the properties of the coefficients \(a_k\) can be made directly from the form of the sums. For example, changing the summation index in the last sum of the chain \(k \rightarrow N - k\) and compare this sum with the first one we conclude that \(a_k = a_{N-k}\) for the even \(N\), and \(a_k = -a_{N-k}\) for the odd \(N\).

To calculate the coefficients \(a_k\) in the explicit form it is necessary to solve some equation systems which arise from equalization of coefficients at the same monomials \(m^p n^q\) in the above chain. It is convenient to reformulate this problem as a problem of calculation of the vector

\[
V_{N+1} = (a_0, a_1, ..., a_N)
\]

which is a common eigenvector with the eigenvalue 1 of the matrices related to the equation systems under consideration.

Consider for example the first equality from the chain. We can write the following equation

\[
M_1 V_{N+1} = E_{N+1} V_{N+1} = V_{N+1},
\]

where \(E_{N+1}\) is the unit \((N+1) \times (N+1)\) matrix, \(M_1\) is the lower triangular matrix of the same degree with the nonzero matrix elements

\[
(M_1)_{ij} = (-1)^{j+1} \binom{N+1 - j}{N+1 - i}, \quad i, j = 1, ..N + 1,
\]

where \(\binom{j}{i}\) is the binomial coefficient. The equality of the first and third sums gives us the equation

\[
M_2 V_{N+1} = V_{N+1},
\]

where \(M_2\) is the upper triangular matrix with the nonzero matrix elements

\[
(M_2)_{ij} = (-1)^{j+1} \binom{j-1}{i-1}, \quad i, j = 1, ..N + 1.
\]

By the same manner we obtain the matrices \(M_i, i = 3, 4, 5\), from the above equalities. It can be easily checked that these matrices are connected \(M_1, M_2\) by the following formulas

\[
M_3 = M_1 M_2, \quad M_4 = M_2 M_1, \quad M_5 = M_1 M_2 M_1, \quad M_0 = E_{N+1}.
\]

Moreover, under the correspondence \(w_i \sim M_i\) we reproduce the multiplication table of the Weyl group \(W(A_2)\) including the equalities

\[
M_1^2 = M_2^2 = M_5^2 = M_3^2 = M_1 = E_{N+1}, \quad M_3^2 = M_4, \quad M_4^2 = M_3.
\]

It follows from the above that the homomorphism \(w_i \rightarrow M_i, i = 0, .., 5, M_0 = E_{N+1} = w_0\) realizes faithful \((N + 1)\)-dimensional representation of the Weyl group \(W(A_2)\). Since the matrices \(M_1\) and \(M_2\) are the images of the generators for the Weyl group \(W(A_2)\), we can calculate the joint eigenvectors only for these two matrices.
Joint solution of (13) and (15) in the cases $N = 2, 3$ gives us the following result

$$N = 2, \quad V_{3}^{A_{2}} = (1, 1, 1), \quad N = 3, \quad V_{4}^{A_{2}} = (2, 3, -3, -2).$$

(16)

The related independent operators in the angle variables with their spectrums have the forms

$$L_{A_{2}}^{3} = \partial_{\phi}^{2} + \partial_{\psi}^{2} + \partial_{\psi_{2}}^{2}, \quad E_{3}^{A_{2}}(m, n) = m^2 + mn + n^2,$$

(17)

$$L_{A_{2}}^{4} = 2\partial_{\phi}^{2} + 3\partial_{\phi_{2}}^{2} - 3\partial_{\phi_{2}^2}^{2} - 2\partial_{\psi}^{2}, \quad E_{4}^{A_{2}}(m, n) = 2m^3 + 3m^2n - 3mn^2 - 2n^2.$$  

(18)

High degree operators can be constructed as

$$L = P(L_{A_{2}}^{3}, L_{A_{2}}^{4})$$

where $P$ is any polynomial in two variables.

4. At the last step it is necessary to replace the angle variables $(\phi, \psi)$ by $(x, y)$ which are defined according to the relation (11) as

$$x = \frac{1}{2} T_{1,0} = e^{i\phi} + e^{i(\psi - \phi)} + e^{-i\psi},$$

(19)

$$y = \frac{1}{2} T_{0,1} = e^{i\psi} + e^{i(\phi - \psi)} + e^{-i\phi}.$$  

(20)

This routine procedure in the case $N = 2$ gives us the operator

$$L_{A_{2}}^{3} = (x^2 - 3y) \frac{\partial^2}{\partial x^2} + (xy - 9) \frac{\partial^2}{\partial x \partial y} + (y^2 - 3x) \frac{\partial^2}{\partial y^2} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.$$  

(21)

The bivariate Chebyshev polynomials of the first kind associated with the Lie algebra $A_{2}$ are eigenvectors of $L_{A_{2}}^{3}$ with eigenvalues defined by (17). The operator (21) was obtained for the first time by T. Koornwinder in the well known work [1]. Our calculation method, presented above, is different from the method used in [1].

5. Here we use the same calculation scheme as above for the case of the polynomials, associated with the Lie algebra $C_{2}$. The root system of the algebra $C_{2}$ has two fundamental roots $\alpha_{1}, \alpha_{2}$ and includes the positive root $\alpha_{1} + \alpha_{2}, 2\alpha_{1} + \alpha_{2}$ and their reflections. The action of generating elements $w_{1}, w_{2}$ of the Weyl group $W(A_{2})$ on the fundamental roots are given by the formulas

$$w_{1}\alpha_{1} = -\alpha_{1}, \quad w_{1}\alpha_{2} = 2\alpha_{1} + \alpha_{2}, \quad w_{2}\alpha_{1} = \alpha_{1} + \alpha_{2}, \quad w_{2}\alpha_{2} = -\alpha_{2},$$

$$w_{1}\lambda_{1} = \lambda_{2} - \lambda_{1}, \quad w_{1}\lambda_{2} = \lambda_{2}, \quad w_{2}\lambda_{1} = \lambda_{1}, \quad w_{2}\lambda_{2} = 2\lambda_{1} - \lambda_{2}.$$  

The action of the other group elements on the fundamental weights is determined by their representation in terms of the generating elements

$$w_{3} = w_{1}w_{2}, \quad w_{4} = w_{2}w_{1}, \quad w_{5} = w_{1}w_{2}w_{1}, \quad w_{6} = w_{2}w_{1}w_{2}, \quad w_{7} = (w_{1}w_{2})^{2}, \quad e = w_{0}.$$  

(22)
Using the above formulas we obtain the following $W(C_2)$-invariant orbit function
\[
T^{C_2}_{m,n}(\phi, \psi) = e^{2\pi i (m\phi + n\psi)} + e^{2\pi i (m(\phi - \psi) + n(\phi + \psi))} + e^{2\pi i (m(\psi - \phi) + n(-2\phi + \psi))} + e^{2\pi i (m\phi + n(2\phi - \psi))} + e^{2\pi i (2\phi - \psi)} + e^{2\pi i (m(\phi - \psi) + n(-2\phi + \psi))} + e^{2\pi i (m(\phi - \psi) - n\psi)} + e^{2\pi i (-m\phi - n\psi)}.
\]
(23)

The action of the operator (12) on $T^{C_2}_{m,n}(\phi, \psi)$ produces coefficients at each exponent of (23). The condition of equality of these coefficients gives us the following independent relations
\[
\sum_{k=0}^{N} a_k m^{N-k} n^k = \sum_{k=0}^{N} a_k (m)^{N-k} (-m - n)^k = \sum_{k=0}^{N} a_k (m + 2n)^{N-k} (-n)^k = \sum_{k=0}^{N} a_k (m + 2n)^{N-k} (-n)^k = \sum_{k=0}^{N} a_k (-m)^{N-k} (-n)^k.
\]

It follows from the equality of the first and last sums that the coefficients $a_k$ are nonzero only for the even $N$. In this case the matrix elements of the matrices $M_i, \ i = 1, 2$ have the form
\[
(M_1)_{ij} = (-1)^{j+1} \binom{N + 1 - j}{N + 1 - i}, \quad (M_2)_{ij} = (-1)^{j+1} 2^{j-i} \binom{j - 1}{i - 1}, \quad i, j = 1, \ldots N + 1.
\]

These matrices are commutative
\[
[M_1, M_2] = 0, \quad M_1^2 = M_2^2 = E_{N+1}.
\]

Besides $M_i, \ i = 1, 2$ there is only one independent matrix $M_3$
\[
M_3 = M_1 M_2.
\]

Coordinates $a_k$ of any joint eigenvectors with unit eigenvalues of the matrices $M_i, \ i = 1, 2$ give us the coefficients of the operator $L^{C_2}_N$ from (12). For the cases $N = 2, 4$ we obtain the following result
\[
N = 2, \quad V^{C_2}_3 = (1, 2, 2), \quad N = 4, \quad V^{C_2}_5 = (1, 4, 1, 0, 0), \quad V^{C_2}_6 = (0, 0, 1, 2, 1).
\]
(24)

The related independent operators in the angle variables with their spectrums have the forms
\[
L^{C_2}_3 = \partial^2_{\phi^2} + 2\partial^2_{\phi\psi} + 2\partial^2_{\psi^2}, \quad E^{C_2}_3(m, n) = m^2 + 2mn + 2n^2,
\]
(25)
\[
L^{C_2}_{5a} = \partial^4_{\phi^4} + 4\partial^4_{\phi^3\psi} + \partial^4_{\phi^2\psi^2}, \quad E^{C_2}_{5a}(m, n) = m^2(m^2 + 4mn + n^2),
\]
(26)
\[
L^{C_2}_{5b} = \partial^4_{\phi^2\psi^2} + 2\partial^4_{\phi\psi^3} + \partial^4_{\psi^4}, \quad E^{C_2}_{5b}(m, n) = n^2(m + n)^2.
\]
(27)

6. Transition from the angle coordinates to Descartes ones are given by the relations (see, for example, [9])
\[
x = \frac{1}{2} T^{C_2}_{1,0} = e^{2\pi i \phi} + e^{-2\pi i \phi} + e^{2\pi i (\phi - \psi)} + e^{-2\pi i (\phi - \psi)},
\]
(28)
\[
y = \frac{1}{2} T^{C_2}_{0,1} = e^{2\pi i \psi} + e^{-2\pi i \psi} + e^{2\pi i (2\phi - \psi)} + e^{-2\pi i (2\phi - \psi)}.
\]
(29)
For the case (24) we obtain
\[ L^{C_2}(x, y) = (x^2 - 2y - 8) \frac{\partial^2}{\partial x^2} + 2x(y - 4) \frac{\partial^2}{\partial x \partial y} + 2(y^2 + 4y - 2x^2) \frac{\partial^2}{\partial y^2} + x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}. \quad (30) \]

7. To finish these brief notes we consider the case of the polynomials, associated with the Lie algebra \(G_2\). The root system of the algebra \(G_2\) has two fundamental roots \(\alpha_1, \alpha_2\) and includes the positive roots \(\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\) and their reflections. The action of generating elements \(w_1, w_2\) of the Weyl group \(W(A_2)\) on the fundamental roots are given by the formulas
\[ w_1\alpha_1 = -\alpha_1, \quad w_1\alpha_2 = 3\alpha_1 + \alpha_2, \quad w_2\alpha_1 = \alpha_1 + \alpha_2, \quad w_2\alpha_2 = -\alpha_2, \]
\[ w_1\lambda_1 = \lambda_2 - \lambda_1, \quad w_1\lambda_2 = \lambda_2, \quad w_2\lambda_1 = \lambda_1, \quad w_2\lambda_2 = 2\lambda_1 - \lambda_2. \]
The action of the other group elements on the fundamental weights is determined by their representation in terms of the generating elements
\[ w_3 = w_1w_2, \quad w_4 = w_2w_1, \quad w_5 = w_2w_1w_2, \quad w_6 = w_1w_2w_1, \quad w_7 = (w_1w_2)^2, \]
\[ w_8 = (w_2w_1)^2, \quad w_9 = w_2(w_1w_2)^2, \quad w_{10} = w_1(w_2w_1)^2, \quad w_{11} = (w_1w_2)^3, \quad w_0 = e. \]
Using these formulas and definition (31) we obtain the following \(W(G_2)\)-invariant orbit function
\[ T_{m,n}^{G_2} = e^{2\pi i(m\phi+n\psi)} + e^{2\pi i(m(-\phi+\psi)+n(-3\phi+2\psi))} + e^{2\pi i(m(2\phi-\psi)+n(3\phi-\psi))} + \]
\[ e^{2\pi i(-m\phi+n\psi)} + e^{2\pi i(-m(-\phi+\psi)+n(-3\phi+2\psi))} + e^{2\pi i(-m(2\phi-\psi)+n(3\phi-2\psi))} + \]
\[ e^{2\pi i(m\phi+n(3\phi-\psi))} + e^{2\pi i(m(-\phi+\psi)+n\psi)} + e^{2\pi i(m(2\phi-\psi)+n(3\phi-2\psi))} + \]
\[ e^{2\pi i(-m\phi+n(3\phi-\psi))} + e^{2\pi i(-m(-\phi+\psi)+n\psi)} + e^{2\pi i(-m(2\phi-\psi)+n(3\phi-2\psi))}. \quad (31) \]
The action of the operator (12) on \(T_{m,n}^{G_2}(\phi, \psi)\) produces coefficients at each exponent of (31).

The condition of equality of these coefficients gives us the following independent relations
\[ \sum_{k=0}^{N} a_k m^{-k} n^k = \sum_{k=0}^{N} a_k (m)^{N-k} (-m-n)^k = \sum_{k=0}^{N} a_k (m+3n)^{N-k} (-n)^k = \]
\[ \sum_{k=0}^{N} a_k (2m+3n)^{N-k} (-m-n)^k = \sum_{k=0}^{N} a_k (m+3n)^{N-k} (-m-2n)^k = \sum_{k=0}^{N} a_k (2m+3n)^{N-k} (-m-2n)^k. \]

Equality of the first and the second sums gives us the matrix \(M_1\) which is the same as in the \(A_2\) and \(C_2\) cases (14). Equality of the first and the second sums gives us the matrix \(M_2\)
\[ (M_1)_{ij} = (-1)^{j+1} \binom{N+1-j}{N+1-i}, \quad (M_2)_{ij} = (-1)^{j+1} \frac{3^{j-i}}{i-1}, \quad i, j = 1, \ldots N + 1. \]
The remaining matrices are
\[ M_3 = M_1 M_2, \quad M_4 = M_2 M_1, \quad M_5 = M_1 M_2 M_1 = M_2 M_1 M_2. \]
Coordinates $a_k$ of any joint eigenvectors with unit eigenvalues of the matrices $M_i$, $i = 1, 2$ give us the coefficients of the operator $L_N^{(G_2)}$ from (12). For the cases $N = 2$ we obtain the following result (there are no solutions for the odd cases)

$$N = 2, \quad V_2^{G_2} = (1, 3, 3).$$

(32)

The related independent operator in the angle variables with its spectrum has the form

$$L_3^{G_2} = \partial_{\phi^2} + 3\partial_{\psi^2}, \quad E_3^{G_2}(m, n) = m^2 + 3mn + 3n^2,$$

(33)

Calculations in the cases $N = 4, 6$ give us only $L_5^{G_2} = (L_3^{G_2})^2$, $L_7^{G_2} = (L_3^{G_2})^3$.

8. Transition from the angle coordinates to Descartes ones is given by the relations

$$x = \frac{1}{2} T_{G_2}^{G_2} = e^{2\pi i(\phi)} + e^{2\pi i(-\phi + \psi)} + e^{2\pi i(2\phi - \psi)} + e^{2\pi i(-\phi)} + e^{2\pi i(\phi + \psi)},$$

(34)

$$y = \frac{1}{2} T_{G_2}^{G_2} + e^{2\pi i(-3\phi + 2\psi)} + e^{2\pi i(3\phi - \psi)} + e^{2\pi i(-\psi)} + e^{2\pi i(\phi - 2\psi)} + e^{2\pi i(-3\phi + \psi)}.$$  

(35)

For the case (33) we obtain

$$L^{G_2}(x, y) = (x^2 - 3x - y - 12) \frac{\partial^2}{\partial x^2} + (3xy - 6x^2 + 12y + 36) \frac{\partial^2}{\partial x \partial y} + (3y^2 + 9y - 3x^3 + 9xy + 27x) \frac{\partial^2}{\partial y^2} + x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y}.$$

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