Prospects for gravitational-wave observations of neutron-star tidal disruption in neutron-star/black-hole binaries

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For an inspiraling neutron-star/black-hole binary (NS/BH), we estimate the gravity-wave frequency \( f_{\text{td}} \) at the onset of NS tidal disruption. We model the NS as a tidally distorted, homogeneous, Newtonian ellipsoid on a circular, equatorial geodesic around a Kerr BH. We find that \( f_{\text{td}} \) depends strongly on the NS radius \( R \), and estimate that LIGO-II (ca. 2006–2008) might measure \( R \) to 15% precision at 140 Mpc (\( \sim 1 \) event/yr under current estimates). This suggests that LIGO-II might extract valuable information about the NS equation of state from tidal-disruption waves.

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The equation of state of the bulk nuclear matter inside a neutron star (NS) is poorly understood [1]. For example, candidate equations of state that are compatible with nuclear physics experiments and theory predict, for a 1.4 \( M_\odot \) NS, a radius anywhere from about 8 km to 16 km [2]. Thorne has conjectured that insights into the equation of state might come from measurements of the gravitational waveform emitted by merging NS/NS binaries and/or tidally disrupting NS’s in neutron-star/black-hole (NS/BH) binaries [3,4]. More recently, Newtonian models of NS/NS mergers have given strong evidence that the merger waves do carry equation-of-state information, but for NS/NS are emitted at frequencies (\( \sim 1400\text{–}2800 \) Hz) too high for measurement by LIGO-type gravity-wave interferometers [5,6]. In this paper, we show that the prospects for NS/BH measurements are much brighter.

Central to these prospects is the question of whether NS tidal-disruption waves lie in the band of good interferometer sensitivity (for LIGO-II, \( \sim 30\text{–}1000 \) Hz [7]; see Fig. 1). Numerical modeling of NS tidal disruption in NS/BH binaries is only now getting underway [8] and has not yet included computations of the emitted gravity waves or even their frequency bands. As a result, the best frameworks now available for estimating the tidal-disruption gravity-wave band are highly simplified, quasi-analytic models by Shibata [9] and by Wiggins and Lai [10], which represent the inspiraling NS as an irrotational, incompressible or polytropic Newtonian ellipsoid, moving on a circular, equatorial geodesic orbit around a Kerr BH, and being tidally distorted by the Kerr Riemann tensor. For simplicity we focus on Shibata’s homogeneous models, and then appeal to the polytropic models for evidence that compressibility has only small effects.

In Shibata’s analysis, the NS gravitational field, its centrifugal potential, and the Newtonian tidal potential constructed from the Kerr Riemann tensor are all quadratic functions of position. As a result, a class of equilibrium solutions are the classic irrotational, homogeneously Roche-Riemann ellipsoids [12]. Given a choice of the binary parameters \( M, a \) and \( r \) (the BH mass and angular momentum per unit mass, and the orbital separation, i. e., the Boyer-Lindquist radius of the geodesic), there is a one-parameter family of such NS models with density \( \rho \) ranging downward through the family to a minimum \( \rho_{\text{crit}}(M, a, r) \).

We model the inspiraling NS as one of Shibata’s irrotational ellipsoids, identified by its mass \( m \), and its density \( \rho \) or mean radius \( R = (3m/4\pi\rho)^{1/3} \). In our simple framework, the uncertainty in \( m(R) \) embodies the uncertainty about the NS equation of state. We describe the inspiral as a sequence of circular, equatorial Kerr geodesics that shrink inward until the NS reaches the innermost stable circular orbit, \( r = r_{\text{isco}} \), or begins to tidally disrupt [which happens at the radius \( r_{\text{td}} \) where the star’s density \( \rho \) matches the critical density \( \rho_{\text{crit}}(M, a, r_{\text{td}}) \)].

The Kerr geometry provides a one-to-one correspondence between the orbital radius \( r_{\text{td}} \) and the gravity-wave
frequency $f_{td}$ at which tidal disruption begins:

$$f_{td}(M, a, r_{td}) = \frac{1}{\pi(a + \sqrt{r_{td}^2/M})} \quad \text{(1)}$$

(here and below we set $G = c = 1$). It is this $f_{td}$ that LIGO-II can measure. Having measured $f_{td}$ and determined the masses $M$ and $m$ from the observed inspiral waveforms [13], one can compute $r_{td}$ and then the NS density $\rho = \rho_{eq}(M, a, r_{td})$ and the mean NS radius $R$. Thereby, the LIGO-II observations can determine a point on the NS mass-radius curve $m(R)$, which represents the NS equation of state in our simplified analysis. Even one such point could give valuable information about the real NS equation of state, and several such points could determine it remarkably well [14].

To estimate the accuracy with which LIGO-II might determine the NS radius $R$, we need the explicit relationship between $R$ and the disruption-onset frequency $f_{td}$. More precisely, we need $R(m, M, a, f_{td})$, which can be derived as follows: (i) $r_{td}(M, a, f_{td})$ is obtained by inverting Eq. (1); (ii) $\rho_{eq}(M, a, r_{td})$ is obtained by solving Eq. (3.9) of [9] for the ratios of semiaxes of the equilibrium configurations, and then extremizing Eq. (3.10) of [9], in which $\tilde{\Omega}^2 = M/(\pi pr^3)$; (iii) then $R$ is obtained as $R = [3m/4\pi \rho_{eq}(M, a, r_{td})]^1/3$. The result has the form

$$R(m, M, a, f_{td}) = m^{1/3}M^{2/3} \tilde{D} \left[ \frac{a}{M} ; f_{td}M \right], \quad \text{(2)}$$

where $\tilde{D}$ is a dimensionless function with remarkably weak dependence on $a/M$ [15]. This $R(f_{td})$ is shown in Fig. 2 for various $M$, for $a/M = 0.998$ (the curves for other $a/M$ are almost identical to these), and for $m = 1.4M_\odot$ [16]. The radii shown, $R = 8-16$ km for $m = 1.4M_\odot$, correspond to the range of predictions by plausible NS equations of state [9]. The curves in Fig. 2 are well approximated by the formula (with $G = c = 1$)

$$\frac{R}{m^{1/3}M^{2/3}} \approx \left\{ \begin{array}{ll} 0.145 (f_{td}M)^{-0.71} & \text{for } f_{td}M \lesssim 0.045, \\ 0.069 (f_{td}M)^{-0.95} & \text{for } f_{td}M \gtrsim 0.045. \end{array} \right. \quad \text{(3)}$$

Although the BH spin parameter $a$ has negligible influence on the function $R(f_{td})$, it strongly influences the radius $r_{isco}$ of the innermost stable circular orbit [17]. If the NS is still intact when it reaches $r_{isco}$, it then will plunge rapidly into the BH and the tidal-disruption waves, if any, are likely to be so weak and short-lived as to be useless for measuring NS properties. Thus, there is not much hope of measuring tidal disruption unless $f_{td} < f_{\text{plunge}} = [\text{Eq. (1)},$ with $r_{td}$ replaced by $r_{isco}(M, a)]$; i. e., unless $f_{td}$ is left of the relevant big dot in Fig. 2.

Figure 2 and the above discussion show that (i) for a wide range of realistic parameters, tidal disruption occurs before the plunge begins, and (ii) for all realistic parameters except a very narrow range ($M \lesssim 10M_\odot$ and $R \lesssim 10$ km), the tidal-disruption waves fall in the range of good LIGO sensitivity, $f \lesssim 1000$ Hz. The Lai-Wiggins polytropic NS models [14] give similar curves and conclusions: for polytropic indices $n = 0.5$ and 1.0, which approximate NS equations of state, the $R(f_{td})$ curves are displaced upward in frequency from those of Fig. 2 by a mere $\sim 50$ and $\sim 100$ Hz.

Turn now to an estimate of the accuracy to which LIGO-II could measure $f_{td}$ (and then $R$) using Wiener optimal filtering [18]. The measured gravity-wave data stream $g(t)$ is compared to a set of theoretical inspiral templates $h(\tilde{\theta}; t)$, indexed by the parameters $\tilde{\theta}$ of the binary; a “best fit” $\tilde{\theta}$ is found which maximizes the likelihood of observing $g(t)$ given a “true” signal $h(\tilde{\theta}^\ast; t)$, and given a statistical model of the detector noise [a Gaussian $\tilde{\theta}$ random process with zero mean and spectral density $S_\theta(f)$]. For strong enough signals, $\tilde{\theta}$ will have a gaussian distribution centered around its “true value” $\bar{\theta}$, with covariance matrix $\Gamma$:

$$C_{ij} = (\Gamma^{-1})_{ij}, \quad \Gamma_{ij} \approx 2 \left[ \frac{\partial h}{\partial \tilde{\theta}^i} (\tilde{\theta}^\ast) \right] \left[ \frac{\partial h}{\partial \tilde{\theta}^j} (\tilde{\theta}^\ast) \right], \quad \text{(4)}$$

where the “inner product” $(\ldots)$ is defined for any two real data streams $g(t)$, $h(t)$ in terms of their Fourier transforms $\tilde{g}(f)$, $\tilde{h}(f)$ by

$$\langle g, h \rangle = \int_{-\infty}^{\infty} df \frac{\tilde{g}(f) \tilde{h}^\ast(f)}{S_\theta(|f|)}. \quad \text{(5)}$$

Because so little is known about the tidal disruption and our NS models are so crude, we use the simplest of templates in our analysis: slow-motion, quadrupolar waveforms for point particles in circular, Keplerian
orbits with quadrupole-governed inspiral. The Fourier-transformed waveform, squared and averaged over binary directions and orientations, is given by

$$\langle |\hat{h}_b|^2 \rangle = \frac{\pi \mu M_T^3}{30 \delta^2} \frac{1}{(\pi M_T f)^{7/3}} \theta(f_{\text{plunge}} - f),$$

where \(\mu\) and \(M_T\) are the reduced and total masses, \(d\) is the distance to the binary, and the step function shuts off the signal at the onset of plunge.

For typical observations, optimal filtering of the inspiral signal should give good estimates of \(M\) and \(m\) [13]. We therefore assume that the accuracy in measuring the emission within a few dynamical time-scales after \(f_{\text{td}}\) has been reached [23]. Correspondingly, we employ a toy model where the inspiral waveform of Eq. (6) dies out over a few dynamical time-scales after \(f_{\text{plunge}}\). The estimation of \(f_{\text{td}}\) depends heavily on the details of the tidal-disruption waveforms, which are largely unknown. However, it is reasonable to expect tidal disruption to be a sudden event that significantly weakens gravity-wave emission within a few dynamical time-scales after \(f_{\text{td}}\) has been reached [24].

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not only distinguish it from a plunge shutoff, but that also carry equation-of-state information which is richer than in our crude model. For example, simulations of tidal disruption in NS/NS binaries show a spectrum with an inspiral cutoff followed by a valley, a moderately sharp peak, and a cliff; however, the NS/BH case is likely to be different, and the issue will ultimately be settled only by detailed numerical simulations.

Given these large uncertainties, our results can only be rough indications of the prospects for learning about NS’s from tidal-disruption waveforms. They do, however, suggest that observations of tidal disruption in NS/BH binaries might be possible in ~ 2006–2008 with LIGO-II, and might yield useful insights into the NS equation of state. The success of this endeavor will require the development of better theoretical and numerical techniques for modeling NS tidal disruption and computing the dependence of the disruption waveforms on the NS equation of state; we strongly advocate such an effort.

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[22] For a signal to noise ratio ≥ 10 (fairly typical of the observations examined in this paper), and if spins can be treated as negligible, ∆m/m, ∆M/M ≤ 0.02 [10], and from Eq. (10) the influence of ∆m and ∆M on ∆R gives ∆R/R ∼ 0.005. If spins are important these errors increase tenfold, but might be considerably reduced if the *a priori* knowledge of m from known NS/NS binaries [10] can be applied to NS/BH systems.

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