The γ-Dimension of Images of Bi-Lipschitz Function

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Abstract. The integral staircase function defined on a fractal set is special case of the bi-Lipschitz function of order \( \alpha \in (0,1) \) and the image of this function does not preserve \( \gamma \)-dimensions. In this paper, we show the sufficient condition for any function defined on a fractal set satisfies the bi-Lipschitz condition of order \( \alpha \in (0,1) \). Moreover, the image of the bi-Lipschitz function does not preserve \( \gamma \)-dimensions of its domain.

1. Introduction

The triadic Cantor set (Cantor ternary set) \( C \) is an interesting example of an uncountable set of measure zero and has many interesting properties and consequences in the fields of set theory, topology, and fractal theory. The triadic Cantor set \( C \) is created by repeatedly deleting the open middle thirds of a set of line segments. Parvate and Gangal [5] showed if \( C \) is the triadic Cantor set then the Lebesgue-Cantor staircase function \( S_C^{ln 2/ln 3} : C \to R \) is bounded by \( |y - x|^{ln 2/ln 3} \) from below and above as follows

\[
c_1 |y - x|^{ln 2/ln 3} \leq |S_C^{ln 2/ln 3} (y) - S_C^{ln 2/ln 3} (x)| \leq c_2 |y - x|^{ln 2/ln 3}
\]

for all \( x, y \in C \) where \( c_1, c_2 \in R^+ \) with \( c_1 \leq c_2 \). The functions that satisfies condition (1) is known as a bi-Lipschitz condition (bi-Lipschitz transformation) of order \( ln 2/ln 3 \). It was also shown that the measure of the triadic Cantor set \( C \) is zero and the \( \gamma \)--dimension of \( C \) is \( ln 2/ln 3 \), but the image of the Lebesgue-Cantor staircase function \( S_C^{ln 2/ln 3} (C) = \{ S_C^{ln 2/ln 3} (x) : x \in C \} \) has measure 1 and \( \gamma \)--dimension that is 1 (see [1] and [4]). This means that Lebesgue-Cantor staircase function does not preserve \( \gamma \)--dimensions of \( C \).

We note the integral staircase function is a generalization of the Lebesgue-Cantor staircase function. This function plays a key role in calculus on fractal or \( F^\alpha \)--calculus [5].

In [6], Wibowo et.al have discussed the integral staircase function that is defined on the \( \gamma \)--dimensional compact and \( F^\alpha \)--perfect sets \( F \) which satisfies the bi-Lipschitz condition of order \( \alpha \in (0,1) \). Furthermore, it is shown with using \( F^\alpha \)--calculus that the images of the integral staircase function does not preserve \( \gamma \)--dimensions of \( F \). In the words, the \( \gamma \)--dimension of triadic Cantor set is not invariant under bi-Lipschitz transformations of order \( ln 2/ln 3 \).

The fundamental property of \( \gamma \)--dimension is that, it is invariant under bi-Lipschitz transformations of order 1 if define on compact set. Thus if two sets have different dimensions there cannot be a bi-Lipschitz mapping of order 1 from one onto the other. One approach to fractal geometry
is to regard two sets as ‘the same’ if there is a bi-Lipschitz mapping of order 1 between them (see [2] and [5]).

In this paper, we will show to the sufficient conditions that any function defined on a fractal set satisfies the bi-Lipschitz condition with the order \( \alpha \in (0,1) \). From the characterization of the bi-Lipschitz condition with order \( \alpha \in (0,1) \), it will be shown that the relationship of the \( \gamma \)-dimension of domain \( F \) and its image under the bi-Lipschitz mapping will be done by using the \( F^\alpha \)-calculus.

### 2. Basic Tools in the \( F^\alpha \)-Calculus

In this section, we summarize \( F^\alpha \)-calculus without proofs, especially about coarse-grained mass, \( \gamma \)-dimension, the integral staircase function and their properties [5], and the bi-Lipschitz condition with order \( \alpha \in (0,1) \) [6].

**Definition 2.1** Given \( \delta > 0 \) and \( a \leq b \), the mass function \( \gamma^\alpha(F, a, b) \) of \( F \cap [a, b] \) is given by

\[
\gamma^\alpha(F, a, b) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha + 1)} \mid \sum_{i=0}^{n-1} (x_{i+1} - x_i) \leq \delta, \Gamma(\cdot) \text{ is gamma function} \right\}
\]

where \( |P| = \max_{0 \leq i < n} (x_{i+1} - x_i) \), \( \Gamma(\cdot) \) is gamma function, and \( \theta(F, [x_i, x_{i+1}]) = 1 \) if \( F \cap [x_i, x_{i+1}] \neq \emptyset \).

The similarity of the definitions of the mass function and the Hausdorff outer measure is that, the mass function can be used to define a fractal dimension. We call this number the \( \gamma \)-dimension of \( F \).

**Definition 2.2** The \( \gamma \)-dimension of \( F \cap [a, b] \), denoted by \( \text{dim}_\gamma(F \cap [a, b]) \), is

\[
\text{dim}_\gamma(F \cap [a, b]) = \inf \left\{ \beta : \gamma^\beta(F, a, b) = 0 \right\} = \sup \left\{ \beta : \gamma^\beta(F, a, b) = \infty \right\}.
\]

The integral staircase function \( S_\beta^\alpha(x) \) of order \( \alpha \in (0,1) \) for a fractal set \( F \) is defined as follows.

**Definition 2.3** Let \( a_0 \) be an arbitrary but fixed real number. The integral staircase function \( S_\beta^\alpha(x) \) of order \( \alpha \in (0,1) \) for a set \( F \) is given by

\[
S_\beta^\alpha(x) = \begin{cases} 
\gamma^\alpha(F, a_0, x), & x \geq a_0 \\
\gamma^\alpha(F, a, a_0), & x < a_0.
\end{cases}
\]

The number \( a_0 \) can be chosen according to convenience. We also define the concepts of \( F \)-limits and \( F \)-continuity, which will be used in the next section.

**Definition 2.4** Let \( F \subset R, f : R \to R \) and \( x \in F \). A number \( l \) is said to be the limit of \( f \) through the points of \( F \), or simply \( F \)-limit, as \( y \to x \), if given any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( y \in F \) and \( |y - x| < \delta \Rightarrow |f(y) - l| < \varepsilon \).

If such a number exists, then it is denoted by

\[
l = F = \lim_{y \to x} f(y).
\]

This definition does not involve values of the function at \( y \) if \( y \notin F \). Also, \( F \)-limit is not defined at points \( x \notin F \). All points of change of \( x \) is named these to \( f \) change of \( f(x) \) and is denoted by \( \text{Sch}(f) \). If \( \text{Sch}(S_\beta^\alpha) \) is a closed set [2] and every point in it is a limit point, then \( \text{Sch}(S_\beta^\alpha) \) is called \( \alpha \)-perfect.

**Definition 2.5** A function \( f : R \to R \) is said to be \( F \)-continuous at \( x \in F \) if

\[
f(x) = F = \lim_{y \to x} f(y).
\]

We note that the notion of \( F \)-continuity is not defined at \( x \notin F \).

If \( F \) is an \( \alpha \)-perfect set, then the \( F^\alpha \)-derivative of a function \( f \) defined on \( F \) at a point \( x \) is defined to be the following, assuming the \( F \)-limit exists

**Definition 2.6** If \( F \) is an \( \alpha \)-perfect set then the \( F^\alpha \)-derivative of \( f \) at \( x \) is

\[
D_\beta^\alpha(f(x)) = \begin{cases} 
F - \lim_{y \to x} \frac{f(y) - f(x)}{S_\beta^\alpha(y) - S_\beta^\alpha(x)}, & x \in F \\
0, & x \notin F
\end{cases}
\]

if the \( F \)-limit exists.

Now we state the analogue of the law of the mean in \( F^\alpha \)-calculus.
\textbf{Theorem 2.7} Let \( f: F \subset R \rightarrow R \) be a continuous function such that its set of change \( \text{Sch}(f) \) is contained in an \( \alpha \)-perfect set \( F \subset R \). \( D_{F}^\alpha (f(x)) \) exists at all points \( x \in F \cap [x, y] \) and \( S_{F}^\alpha (x) \neq S_{F}^\alpha (y) \). Then there exists a point \( c \in F \) such that
\[ D_{F}^\alpha (f(c)) \leq \frac{f(y)-f(x)}{S_{F}^\alpha (y)-S_{F}^\alpha (x)} \]
and a point \( d \in F \) such that
\[ \frac{f(y)-f(x)}{S_{F}^\alpha (y)-S_{F}^\alpha (x)} \leq D_{F}^\alpha (f(d)). \]

\textit{Proof.} See Corollary 51 of [5].

Based on the definition of the Holder condition with order \( \alpha \in (0,1) \) and the definition of bi-Lipschitz condition with \( \alpha = 1 \) in the section 1.2 of [2], can be defined as the bi-Lipschitz (bi-Holder) condition with the order \( \alpha \in (0,1) \) is defined as follows.

\textbf{Definition 2.8} If \( F \subset R \) and \( f:F \rightarrow R \) satisfies a bi-Lipschitz condition order \( \alpha \in (0,1) \), or \( f \) is bi-Lipschitz continuous order \( \alpha \in (0,1) \), then there exists real numbers \( c_{1} \) and \( c_{2} \) with \( c_{1} \leq c_{2} \) such that
\[ c_{1}|y-x|^\alpha \leq |f(y) - f(x)| \leq c_{2}|y-x|^\alpha \]
for all \( x, y \in F \).

From Theorem 3.1 of [6], we have that the integral staircase function satisfies the bi-Lipschitz condition with order \( \alpha \in (0,1) \), i.e. \( |S_{F}^\alpha (y) - S_{F}^\alpha (x)| \) is bounded by \( |y-x|^\alpha \) from below and above for all \( x, y \in F \) where \( F \) are compact and \( \alpha \)-perfect set.

\textbf{Theorem 2.9} If \( F \) be a compact and \( \alpha \)-perfect sets and let \( S_{F}^\alpha : F \subset [a, b] \rightarrow R, \alpha \in (0,1) \) be an integral staircase function, then there exists positive real numbers \( c_{1} \) and \( c_{2} \) with \( c_{1} \leq c_{2} \) such that
\[ c_{1}|y-x|^\alpha \leq |S_{F}^\alpha (y) - S_{F}^\alpha (x)| \leq c_{2}|y-x|^\alpha \]
for all \( x, y \in F \cap [a, b] \).

\textit{Proof.} See Theorem 3.1 of [6].

3. Main Results

It can be seen in Theorem 2.9 that the integral staircase function satisfies the bi-Lipschitz condition with the order \( \alpha \in (0,1) \). The following theorem show the sufficient condition for any function defined on a fractal set satisfies the bi-Lipschitz condition with the order \( \alpha \in (0,1) \).

\textbf{Theorem 3.1} If \( F \subset R \) be a compact and \( \alpha \)-perfect sets and \( \text{Sch}(f) \subset F \). If \( f \) is \( F^\alpha \) -differentiable and \( D_{F}^\alpha (f(x)) > 0 \) for all \( x \in [a, b] \) with \( \alpha, 0 < \alpha < 1 \), then there exists positive real numbers \( c_{1} \) and \( c_{2} \) with \( c_{1} \leq c_{2} \) such that
\[ k_{1}|y-x|^\alpha \leq |f(y) - f(x)| \leq k_{2}|y-x|^\alpha \]
for all \( x, y \in F \cap [a, b] \).

\textit{Proof.} Let \( f \) be a \( F^\alpha \) -differentiable and \( D_{F}^\alpha (f(x)) > 0 \) for all \( x \in [a, b] \) with \( \alpha, 0 < \alpha < 1 \) and \( S_{F}^\alpha (x) \neq S_{F}^\alpha (y) \). By Theorem 2.7, there exists a points \( c, d \in F \) such that
\[ 0 < D_{F}^\alpha (f(c)) \leq \frac{f(y)-f(x)}{S_{F}^\alpha (y)-S_{F}^\alpha (x)} \quad \text{and} \quad 0 < \frac{f(y)-f(x)}{S_{F}^\alpha (y)-S_{F}^\alpha (x)} \leq D_{F}^\alpha (f(d)). \]

From (2), we get
\[ |D_{F}^\alpha (f(c))| \leq \frac{|f(y)-f(x)|}{S_{F}^\alpha (y)-S_{F}^\alpha (x)} \quad \text{(3)} \]
and
\[ \left| \frac{f(y)-f(x)}{S_{F}^\alpha (y)-S_{F}^\alpha (x)} \right| \leq |D_{F}^\alpha (f(d))|. \quad \text{(4)} \]

Combining the results (3) and (4), we obtained
\[ |D_{F}^\alpha (f(c))| \leq \left| \frac{f(y)-f(x)}{S_{F}^\alpha (y)-S_{F}^\alpha (x)} \right| \leq |D_{F}^\alpha (f(d))| \]
\[ |D_{F}^\alpha (f(c))||S_{F}^\alpha (y) - S_{F}^\alpha (x)| \leq |f(y) - f(x)| \leq |D_{F}^\alpha (f(d))||S_{F}^\alpha (y) - S_{F}^\alpha (x)|. \]
If we take \( k'_{1} = \min \{ |D_{F}^\alpha (f(c))|, |D_{F}^\alpha (f(d))| \} \) and \( k'_{2} = \max \{ |D_{F}^\alpha (f(c))|, |D_{F}^\alpha (f(d))| \} \)
, then we have
\[ k'_1 |S_γ^f(y) - S_γ^f(x)| \leq |f(y) - f(x)| \leq k'_2 |S_γ^f(y) - S_γ^f(x)|. \]

If \( F \) be a compact set and \( \alpha \)-perfect set and \( \text{Sch}(f) \subset F \) by using Theorem 2.9 are substituted into (5), then there exists a real numbers \( c_1, c_2, 0 < c_1 \leq c_2 < \infty \) such that
\[
\frac{k'_1 c_1}{\Gamma(\alpha+1)} |y - x|^\alpha \leq |f(y) - f(x)| \leq \frac{k'_2 c_2}{\Gamma(\alpha+1)} |y - x|^\alpha
\]
for all \( x, y \in F \cap [a, b] \).

If we take \( \frac{k'_1 c_1}{\Gamma(\alpha+1)} = k_1 \) and \( \frac{k'_2 c_2}{\Gamma(\alpha+1)} = k_2 \) are substituted into (6), we find
\[
k_1 |y - x|^\alpha \leq |f(y) - f(x)| \leq k_2 |y - x|^\alpha.
\]

The proof is complete.

The following theorem shows the relationship between the bi-Lipschitz condition and the mass function of \( f(F) = \{ f(x) : x \in F \} \).

**Theorem 3.2** Let \( F \) be a compact set, \( F \)-perfect set and \( \text{Sch}(f) \subset F \). If \( f: F \subset [a, b] \rightarrow R, \alpha \in (0,1) \) satisfies
\[
k_1 |y - x|^\alpha \leq |f(y) - f(x)| \leq k_2 |y - x|^\alpha
\]
, then for any real number \( s \) we get
\[
k'_1 \frac{s}{\alpha} \gamma^s(F, x, y) \leq \gamma^s(F, f(x), f(y)) \leq k'_2 \frac{s}{\alpha} \gamma^s(F, x, y).
\]

**Proof.** Let \( P_{[a,b]} = \{ [a = x_0, x_1, x_2, \ldots, b = x_n], a < b, x_i < x_{i+1}, i = 0, 1, 2, \ldots, n-1 \} \) be any sub-division of \([a, b] \). Therefore
\[
f(F \cap [x_i, x_{i+1}]) \subset \{ f(x_i), f(x_{i+1}) \}, i = 0, 1, 2, \ldots, n - 1
\]
and with Theorem 3.1, we obtained
\[
k'_1 |x_{i+1} - x_i|^\alpha \leq |f(x_{i+1}) - f(x_i)| \leq k'_2 |x_{i+1} - x_i|^\alpha, i = 0, 1, 2, \ldots, n - 1.
\]
Let \( P'_{[f(a), f(b)]} \) be any subdivision of \([f(a), f(b)]\) i.e.
\[
P'_{[f(a), f(b)]} = \{ f(a) = f(x_0), f(x_1), \ldots, f(b) = f(x_n) \},
\]
where \( f(a) < f(b) \) and \( f(x_i) < f(x_{i+1}) \) for \( i = 0, 1, 2, \ldots, n - 1 \).

We have
\[
k'_1 \frac{s}{\alpha} \frac{1}{\Gamma(\alpha+1)} \sum_{i=0}^{n-1} |x_{i+1} - x_i|^\alpha \theta(F, [x_i, x_{i+1}]) \leq
\frac{1}{\Gamma(\alpha+1)} \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|^s \theta(F, f(F \cap [x_i, x_{i+1}]) \leq
\leq k'_2 \frac{s}{\alpha} \frac{1}{\Gamma(\alpha+1)} \sum_{i=0}^{n-1} |x_{i+1} - x_i|^\alpha \theta(F, [x_i, x_{i+1}])
\]
\[
k'_1 \frac{s}{\alpha} \inf_{P_{[a,b]}:|P| \leq \delta} \sigma^s[F, P] \leq \inf_{P'_{[f(a), f(b)]}:|P'| \leq \delta'} \sigma^s[F, P'] \leq k'_2 \frac{s}{\alpha} \inf_{P_{[a,b]}:|P| \leq \delta} \sigma^s[F, P']
\]
where \( \delta' = k_2 \delta^\alpha \).

If we take infimum over all subdivisions \( P \) and \( P' \) respectively such that \( |P| \leq \delta \) and \( |P'| \leq \delta' = k_2 \delta^\alpha \), we get
\[
k'_1 \frac{s}{\alpha} \gamma^s(F, a, b) \leq \gamma^s(F, f(a), f(b)) \leq k'_2 \frac{s}{\alpha} \gamma^s(F, a, b).
\]

Taking the limit respectively as \( \delta \rightarrow 0 \) and \( \delta' \rightarrow 0 \), we obtained
\[
k'_1 \frac{s}{\alpha} F - \lim_{\delta \rightarrow 0} \gamma^s(F, a, b) \leq F - \lim_{\delta' \rightarrow 0} \gamma^s(F, a, b) \leq k'_2 \frac{s}{\alpha} F - \lim_{\delta' \rightarrow 0} \gamma^s(F, a, b)
\]
\[
k'_1 \frac{s}{\alpha} \gamma^s(F, x, y) \leq \gamma^s(F, f(x), f(y)) \leq k'_2 \frac{s}{\alpha} \gamma^s(F, x, y)
\]
and the proof is complete.

The following result shows relationship between \( \gamma \)-dimension of fractal set \( F \) with \( \gamma \)-dimension of the image of the function \( f \) satisfying bi-Lipschitz condition.
Theorem 3.3 If $F$ is a compact set, $F$-perfect set and $\text{Sch}(f) \subset F$ with $\dim_\gamma(F) = \alpha, \alpha \in (0,1)$. If $f: F \subset [a,b] \to \mathbb{R}$ satisfy
\[
k_1 |y - x|^{\alpha} \leq |f(y) - f(x)| \leq k_2 |y - x|^{\alpha}
\]
, then
\[
dim_\gamma(f(F)) = 1.
\]
Proof. By using Theorem 3.2 for case $s > \dim_\gamma(F)$, we get
\[
k_1^{s/\alpha} \gamma^s(F, x, y) \leq \gamma^{s/\alpha}(f(F), f(x), S_{k_1}^a(y)) \leq k_2^{s/\alpha} \gamma^s(F, x, y)
\]
so that results
\[
dim_\gamma(f(F)) \leq \frac{s}{\alpha} \quad \text{for} \quad s > \dim_\gamma(F).
\]
Thus
\[
dim_\gamma(f(F)) \leq \frac{1}{\alpha} \dim_\gamma(F). \tag{7}
\]
Conversely, for case $s < \dim_\gamma(F)$
\[
\infty = k_2^{s/\alpha} \gamma^s(F, x, y) \leq \gamma^{s/\alpha}(f(F), f(x), f(y))
\]
we have
\[
\frac{s}{\alpha} \leq \dim_\gamma(f(F)) \quad \text{untuk} \quad s < \dim_\gamma(F).
\]
Hence
\[
\frac{1}{\alpha} \dim_\gamma(F) \leq \dim_\gamma(f(F)). \tag{8}
\]
From (7) and (8) we get result
\[
\dim_\gamma(f(F)) = \frac{1}{\alpha} \dim_\gamma(F) = 1
\]
and the proof is complete. ■

Based on Theorem 3.3 we obtained $\dim_\gamma(f(F)) = 1$ for any $\in (0,1)$, this value not dependent on $\gamma$ – dimension of $F$. This means the image of the bi-Lipschitz function does not preserve $\gamma$ – dimensions of $F$.

4. Conclusions
In the Theorem 3.1, sufficient conditions have been proven so that the function satisfy the bi-Lipschitz condition of order $\alpha \in (0,1)$. Then it is also shown that using $F^\alpha$ – calculus the image of the function $f$ satisfy bi-Lipschitz condition does not preserve $\gamma$ – dimensions of $F$.

Open Problem. Based on previous work by Indrati and Aryati (see [3] and [4]), on the countably Lipschitz condition and constructed an integral involving countably Lipschitz condition, the research can be continued to the countably bi-Lipschitz condition which is a generalization of the bi-Lipschitz condition and constructed an integral involving countably bi-Lipschitz condition of order $\alpha \in (0,1)$.

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