Representations and BPS states of $10+2$ superalgebra

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Abstract

The 12d supersymmetry algebra is considered, and classification of BPS states for some canonical form of second-rank central charge is given. It is shown, that possible fractions of survived supersymmetry can be $1/16$, $1/8$, $3/16$, $1/4$, $5/16$ and $1/2$, the values $3/8$, $7/16$ cannot be achieved in this way. The consideration of a special case of non-zero sixth-rank tensor charge also is included.

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1 Introduction.

One direction of recent development of supersymmetric theories in higher dimensions is the consideration of 12-dimensional theories with signature 10+2. This dimension is the highest possible in a sense that corresponding Lorentz group permits Majorana-Weyl spinor $Q$, with 32 real components, which upon reduction to 4d gives 8 spinors, appropriate to N=8 bound in four dimensions. Corresponding 12d susy algebra has the following key relation

\[
\{\overline{Q}, Q\} = \Gamma^{MN} Z_{MN} + \Gamma^{MNPQRL} Z_{MNPQRL}^+ 
\]

where

\[
M, N... = 0, 0', 1, ... 10 
\]

which is of now common type, i.e. includes tensorial “central” charges, in the second one superscript (+) means the self-dual part of tensor. The non-usual feature is the absence of $\Gamma^M P_M$ term, which is not permitted by symmetry considerations. Algebra (1) may be considered as, rewritten in condensed notations, the 11d M-theory superalgebra, in a sense that changing a notation in a way appropriate for dimensional reduction by second time dimension to 11d:

\[
Z_{\mu0'} \Rightarrow P_\mu, Z_{\mu\nu\rho\sigma0'} \Rightarrow Z_{\mu\nu\rho\sigma} 
\]

we obtain the M-theory superalgebra.

So, any statement concerning (1) has a counterpart for (4). The 12d susy theories have been discussed in 1-9 where different aspects are revealed, such as a connection to F-theory, reduction to 11d, etc.

The important problem is the construction of representations of 12d algebra (1). This problem actually reduces to the problem of finding the rank of the matrix in the r.h.s at different values of charges $Z_{MN}$ and $Z_{MNPQRL}^+$. The diagonalization of that matrix brings (1) to the form of an algebra of $n$ pairs of fermionic creation-annihilation operators, where $n$ is half of the rank of matrix.

The main aim of the present paper is the construction and analysis of some of the representations of algebra (1) in the case, mainly, when sixth-rank tensor is equal to zero, and even in that case it is not possible to obtain
a complete answer, but we study some important cases. The problem lies in obtaining a canonical form of antisymmetric tensor in (2+10) dimensions. In addition, some cases with non-zero sixth rank charges are analyzed.

When results for (1) are interpreted for (4), we obtain what is called on the language of M-brane an intersection of branes, with some amount of supersymmetry maintained (and remaining part broken). When some part of supersymmetry is maintained, the multiplet is called shortened and states are called BPS. Many such states are presented by classical solutions of corresponding supergravities equations, and possibly all of them have such a representation, so our analysis shows prospects for the search of such solutions.

Let’s stress that such representation is possible only for 11d form (interpretation) of algebra (1). For 12d case it is not possible, since the very existence of full 12d theory is not established up to now.

A number of results on the problem of calculating of the rank of of the matrix in (1) are obtained in .

2 Membrane

In this section we consider the algebra (1) with non-zero membrane charge only. The analysis is convenient to carry on in a language of determinant of the matrix on r.h.s. of (1). More exactly, multiplying (1) by $\Gamma_{00}'$, we bring the problem to the calculation of determinant of the matrix in the r.h.s. of

$$\{Q, Q\} = \Gamma_{00}' \Gamma^{MN} Z_{MN}$$

(5)

Next step will be to bring the matrix $Z_{MN}$ to canonical form. In euclidean space-time the canonical form would be the well-known form with 2x2 antisymmetric blocks on a main diagonal with eigenvalues $\lambda_1 \lambda_2 ... \lambda_6$. In pseudoeuclidean space-time some matrixes cannot be brought to that form, and classification of canonical forms is more complicated. The complete classification, which is connected to classification of the orbits of coadjoint representation of corresponding Lorentz group, will be discussed elsewhere. Nevertheless, the mentioned quazidiagonal form is one of the canonical forms, and analysis in that case can be carried up to the end. In that case the matrix
has the following form:

\[ \det \Gamma^{00'} \Gamma^{MN} Z_{MN} = \det (\lambda_1 + \lambda_2 \Gamma^{00'} 12 + \ldots + \lambda_6 \Gamma^{00'} 9) \]  

(6)

(Notation \( \natural \) is for tenth \( \Gamma \)-matrix: \( \Gamma^\mu \) at \( \mu=10 \)). Gamma-matrixes in r.h.s of this relation are commuting with each other, hence diagonalizable. Each of them has eigenvalues \( \pm 1 \) only, since square of them is 1, and number of +1 is equal to number of -1, since trace of them is zero. So, determinant in (6) is equal to product of linear combinations of \( \lambda_i \) with coefficients \( \pm 1 \). It may be proved, that each combination enters in that product twice and nothing else is contained. The number of combinations is \( 2^5 = 32 \), which is exactly half of a power of determinant in units of \( \lambda \). So:

\[ \det \Gamma^{00'} \Gamma^{MN} Z_{MN} = \Pi(\lambda_1 \pm \lambda_2 \pm \ldots \pm \lambda_6)^2 \]  

(7)

Actually, for discussion of consequences of (7) on a representation of (5), we need an answer for determinants of operator in r.h.s in subspaces of chiral spinors. The answer is the following: the product in (7) can be divided into two classes, with even and odd numbers of sign changes in (7), and products in each classes are equal to determinants in chiral subspaces.

\[ \det(\Gamma^{00'} \Gamma^{MN} Z_{MN})_{\pm} = \Pi_{\pm}(\lambda_1 \pm \lambda_2 \pm \ldots \pm \lambda_6)^2 \]  

(8)

where \( \Pi_{\pm} \) denotes the product of combinations with even and odd numbers of negative signs, respectively. From now on we consider the chiral determinant with even number of sign changes:

\[ \Pi_+(\lambda_1 \pm \lambda_2 \pm \ldots \pm \lambda_6)^2 \]  

(9)

Before starting an analysis of representations of (7), it is important to take into account the positivity property of l.h.s of (3), which means that all eigenvalues of r.h.s have to be non-negative (when some of them are zero, this is just BPS cases). The requirement of non-negativeness of all eigenvalues in the product of (7) leads to the following inequalities, which have to be satisfied by \( \lambda_1, \lambda_2, \ldots, \lambda_6 \):

\[ \lambda_1 \geq | \lambda_2 | + \ldots + | \lambda_6 | \]  

(10)

in the case when even number of \( \lambda_2, \ldots, \lambda_6 \) is positive, and

\[ \lambda_1 \geq | \lambda_2 | + \ldots + | \lambda_5 | - | \lambda_6 | \]  

(11)
when odd number of $\lambda_2, ..., \lambda_6$ is positive, and for definiteness we assume that $\lambda_6$ has a minimal absolute value among $\lambda_2, ..., \lambda_6$. When one of the $\lambda$-s is zero, both inequalities coincide.

Now we turn to the discussion of representations of supersymmetry algebra (5), which, as mentioned above, is based actually on the calculation of a numbers of pairs of fermionic creation-annihilation operators, which, in turn, is equal to the half of the rank of matrix in the r.h.s. of (5). The relevant determinant is $\mathbf{8}$ and is already prepared for discussion of it’s rank, since is expressed as a product of eigenvalues.

First, for general set of $\lambda$ determinant is non-zero, and dimension of representation is $2^{16}$. These are not BPS states, since each supersymmetry generator acts non-trivially on some of them. Next, when one eigenvalue in (5) is zero, rank is lowered by 2. Dimension of representation in the case of two zeros is $2^{14}$. This is already a BPS state, and number of conserved supersymmetries is 2, i.e. 1/16 of initial number, which was 32. Without loss of generality, we can assume that zero eigenvalue is the sum of $\lambda$-s with positive sign:

$$\lambda_1 + \lambda_2 + ... + \lambda_6 = 0 \quad (12)$$

Now, the analysis of possibilities of appearing new zeros in (5) can be greatly simplified by taking into account the inequalities (10), (11). First, the case when one of $\lambda$ ($\lambda_6$, namely) is zero, gives, as a result of (11), (12), that new zeros of determinant $\mathbf{8}$ can appear only when some other $\lambda$ are zero. When total number of zeros among $\lambda_2, ..., \lambda_6$ is 1, 2, 3, 4, then, as can be easily established, number of zeros in product $\mathbf{8}$ is (counting (12) also), respectively, 2, 4, 8, 16. (All five $\lambda_2, ..., \lambda_6$ cannot be zero, since from (12) it follows that $\lambda_1 = 0$, which means that we are considering the vacuum state.) So, the fractions of survived supersymmetries, from the BPS point of view, are $1/16$, $1/8$, $1/4$, $1/2$, respectively.

When all $\lambda_2, ..., \lambda_6$ are non-zero, then, if (10) is applicable, then there is no new zero eigenvalues in $\mathbf{8}$, which easily follows from (12). When inequality (11) is applicable, then the number of zeros depends on the number of $\lambda$ among $\lambda_2, ..., \lambda_5$, with absolute values, equal to that of $\lambda_6$. When that number is 0, 1, 2, 3, 4, then the number of zeros in $\mathbf{8}$, not forgetting (12), is 2, 4, 6, 8, 10, respectively. Correspondingly, the fractions of survived supersymmetries are $1/16$, $1/8$, $3/16$, $1/4$, $5/16$, respectively.

One of the conclusions of this analysis is that the same fractions of sur-
vived supersymmetries can be achieved in two different ways. That is possible for the cases of fractions $1/16$, $1/8$, $1/4$. Other values - $3/16$, $5/16$ and $1/2$ can be achieved only by one of two mechanisms, presented above. The values $3/8$, $7/16$ cannot be obtained in these cases.

As mentioned in Introduction, these results can be interpreted from the point of view of 11d M-theory superalgebra.

3 Inclusion of 6-form charges

The number of Lorentz-invariant combinations grows strongly, when considering the last term in (1), i.e. the five-brane charge. Instead of six $\lambda$, now we have in addition 462 invariants. The above analysis can be extended in a simple way to the case of some non-zero 6-form charges, among these 462. That is the case, when corresponding gamma-matrixes of 6-form charges commute with matrixes of 2-form charges. There is 20 such matrixes, with 10 different coefficients, due to the self-duality constraint. These are all possible products of $\Gamma^{00}, \Gamma^{12}, ..., \Gamma^{95}$.

The answers for full determinant, and for chiral determinants are the following.

Let’s denote the values of tensor $Z^{+}_{MNPQRL}$ in a coordinate system where $Z_{MN}$ is already diagonal (in a sense of previous Section) by $z_1, z_2, ..., z_{10}$. Exactly, $z_1 = Z^{+}_{00'1234}$, $z_2 = Z^{+}_{00'1256}, ..., z_{10} = Z^{+}_{00'78910}$. Components, dual to these, are equal to them, due to self-duality, all other components of $Z^{+}_{MNPQRL}$ are equal to zero.

The all matrixes in r.h.s. of the (1) again are simultaneously diagonalizable, and signs in front of new $z$ elements are a factors of the signs in front of corresponding $\lambda$ (correspondence is evident - $z_1$ corresponds to $\lambda_1, \lambda_2$ and $\lambda_3$, etc.). So we can immediately write down the answer for corresponding determinant:

$$\det \Gamma^{00'} (\Gamma^{MN} Z_{MN} + \Gamma^{MNPQRL} Z^{+}_{MNPQRL}) =$$
$$\det(\lambda_1 + \lambda_2 \Gamma^{00'}12 + ... + \lambda_6 \Gamma^{00'}9,10 + \Gamma^{00'} \Gamma^{MNPQRL} Z^{+}_{MNPQRL}) =$$
$$\Pi_+(\lambda_1 \pm \lambda_2 \pm ... \pm \lambda_6 \pm z_1 \pm z_2 \pm ... \pm z_{10})^2$$  \hspace{1cm} (13)

where the signs of $z$ are chosen according to rule mentioned above, i.e., the sign of given $z$ is the product of the signs of three corresponding $\lambda$ (one
of which is always $\lambda_1$, so one of these signs is $+1$), and the product is, as above, over all even sign changes of $\lambda$.

The positivity requirement in this case leads to the conditions on the set of $\lambda$ and $z$, that eigenvalues in (13) all have to be non-negative. The cases with some eigenvalues equal to zero are, as above, the BPS cases. The condition of positivity in this case includes 16 inequalities on a 15 variables $\lambda, z$, and cannot be written in a simple form.

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5 References

References

1. C.Vafa, Evidence for F-theory, Nucl.Phys. B469 (1996) 403
2. H.Nishino and E.Sezgin, Supersymmetric Yang-Mills equations in 10+2 dimensions, Phys.Lett.B388, (1996) 569
3. I.Bars, Supersymmetry, p-brane duality and hidden space and time dimensions, Phys.Rev. D54 (1996) 5203
4. H.Nishino, N=2 chiral supergravity in 10+2 dimensions as consistent background to (2+2)-brane, hep-th/9706148
5. S.F.Hewson and M.J.Perry, Nucl.Phys. B492, (1997) 249
6. I.Bars and C.Kounnas, Theories with two times, Phys.Lett. B402 (1997) 25
7. A.Tseytlin, Type IIB instanton as a wave in twelve dimensions, Phys.Rev.Lett. 78 (1997) 1864
8. I.Rudichev, E.Sezgin and P.Sundell, Supersymmetry in dimensions beyond eleven, hep-th/9711127
9. S.F.Hewson, Threebranes in twelve dimensions, hep-th/9801029
10. S.F. Hewson, An approach to F-theory, [hep-th/9712017]
11. P.K. Townsend M-theory from its superalgebra, [hep-th/9712004]