A theoretical model of Lorentz force particle analyzer: a potential flow perspective

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Abstract

A Lorentz force particle analyzer (LFPA) is a device for the contactless measurement of micron-sized particles in electrically conducting fluids. The LFPA is based on the measurement of either force or torque changes acting upon a magnetic-generating system when the particle passes near a magnetic field. In this paper, we first formulate the theory of the LFPA using a magnetic dipole with a magnetic field penetrating a particle-laden thin fluid layer. The disturbed electromagnetic quantities in the presence of a particle are analytically solved using an approach analogous to that for potential flow, providing a relationship between the measured physical quantities, i.e., force, torque and induced magnetic field, and the particle size. A detailed investigation of physical properties and operating parameters provides a rational framework for predicting the sensitivity of LFPAs in laboratory experiments and in industrial practice.

1. Introduction

Maxwell’s classic monograph stated a problem in which insulating particles were present in an isotropic conductor imposing a uniform electric field and in which the electric lines were disturbed by these particles, inducing an electro-resistance change corresponding to a certain fraction of the particles [1]. Such elaboration opened an era of particle detection within conducting solids or flows by means of electromagnetism. The Coulter principle (CP), for example, is based on this deduction [2]. Furthermore, this work can be considered an extension of the case of an electric field to the case of an electromagnetic field; the present contactless technique also transitions from using mechanical contact to being contactless while maintaining physical contact.

We take the example of a liquid metal cleanliness analyzer (LiMCA) based on electromagnetism. Despite the long history of researchers developing inspection technologies [3, 4], traditional nondestructive detection methods, such as ultrasonic inspection, eddy current tests, and x-ray detection are not suitable for metallurgical cleanliness inspection because their use is limited to temperatures far below the melting points of practically relevant metals [4]. In the 1940s, Coulter investigated suspensions of particles in a viscous fluid by forcing particles to flow through a small aperture located in an insulating tube. Two electrodes were located on each side of the aperture and were connected to an external source of electric current. When a particle passed through the aperture with a diameter slightly larger than the particles, a voltage pulse signal was measured. This signal was proportional to the particle size. The concept behind this apparatus is known as the CP [2, 5, 6]. Thirty years later, researchers at McGill University made a breakthrough in the analysis of molten metal quality by improving this technique to develop a LiMCA [7]. Using high current and high amplification rates, they successfully converted micro-voltages into clear signals above the background noise level. LiMCA has since been used to monitor the quality of molten metals such as gallium [8], lead solders, magnesium [9–11], zinc and aluminum [12–14] at moderate temperatures. However, using LiMCA to detect inclusions in melts such as liquid steel remains challenging because of the high temperatures involved (1500 °C–1700 °C) and because of material problems such as thermal shock, corrosion, melting, dissolution and blockage of the small orifice [15]. Thus, a noncontact inspection method that can be used at high temperatures is needed.
Recently, a method involving the placement of a small permanent magnet near the flow has been proposed and is termed Lorentz force particle analyzer (LFPA) \[16, 17\], which is inspired by another exiting technique named Lorentz force velocimetry (LFV) \[18, 19\]. The LFPA is a contactless method for particle inspection via magnetic induction. The goal of the present work is to explain the physics underlying this method by formulating a theoretical model. Our previous papers introduced only the basic principles and experimental and numerical prototypes \[16, 17\] and a two-dimensional elemental theoretical model. Several essential questions remain unanswered:

1. The secondary magnetic field can also actually be a measuring quantity, how does its distribution varies with the presence of the particle?
2. How do physical and operating parameters influence the measurement?
3. Which is the favorable magnetization direction of the permanent magnet (or magnetic dipole)?
4. The magnetic and eddy current vector fields are non-uniform, so what steps should be taken if the experimental and numerical calibration is not successful? All these questions from theory to practice need to be answered clearly before such a technique is implemented.

To this end, a case where a conducting fluid flow layer containing an insulating particle flows past a permanent magnet, which is idealized as a magnetic dipole with arbitrary orientation, is presented. This case can be considered an extension of the work involving the theoretical analysis of the case without the particle \[20\].

The derivation of the eddy current redistribution arising from the particle disturbance is further solved analogously to the classic potential flow past a sphere. Furthermore, the considerable importance of the magnetic field uniformity and the relative electric conductance between the fluid and the particle are mathematically addressed with a simple numerical strategy in anticipation of practical implementation.

### 2. Basic principles

#### 2.1. Physical principle

As shown in figure 1, we consider a magnetic dipole with orientation in the \(-z\) direction and magnetic moment \(m = -me_z\) (later we extend this to any random direction); it provides a static magnetic field, which penetrates into an electrically conducting fluid,

\[
B(R) = \frac{\mu_0}{4\pi} \nabla \left( m \cdot \nabla \frac{1}{R_m} \right).
\]

where \(R_m = R_0 - R\) is the distance from the magnetic dipole to any spatial location and \(\mu_0\) is the vacuum permeability. The dipole is assumed to be arbitrarily oriented in space, i.e., \(m = me_\alpha\), where \(e_\alpha\) denotes the unit vector of the orientation. The presence of a particle does not influence the magnetic field distribution in the space.

Eddy currents are induced in this electrically conducting fluid and can be derived from Ohm’s law for a moving electrically conducting fluid:
where $\phi$ is the electrical potential and $u$ is the fluid velocity, which is assumed here to be constant for simplicity. In the particle-free case, eddy loops form in a clockwise direction. Clearly, when a particle is present in the fluid, this eddy current will be redistributed; the eddy current cannot penetrate the insulating particle, as shown in figure 2(b).

According to Faraday’s laws, an induced magnetic field will be exerted as a result of these eddy currents outside this fluid layer; we term this field the secondary magnetic field. The different eddy currents without and with a particle will form different secondary magnetic fields $b$, as shown schematically in figures 3(a) and (b), respectively.

The associated Lorentz force on any unit volume of the moving fluid can be expressed as

$$j = \sigma(-\nabla \phi + u \times B),$$

which tends to act opposite to the flow as explained in classic textbooks [18, 19]. A reaction force is also exerted on the magnet itself and tends to entrain this magnet in the flow direction. Figure 4 indicates that this global force ($F_0$) is constant in the case of fluid flow without insulating particles. This force can be measured with a high-resolution force sensor, such as a laser-cantilever apparatus [21–23].

As the conducting layer passes through the field provided by the permanent magnet, eddy current $j$ is induced in the flow and the current creates a secondary magnetic field $b$. The total Lorentz force can also be obtained by calculating the interaction of the induced magnetic field acting on the magnetic dipole [17]:

$$F = m \cdot \nabla b.$$

In the present paper, we calculate the Lorentz force using the second routine.

Figure 2. Eddy current induced by the interaction of a magnetic dipole with a conducting moving layer (a) without particle and (b) with a particle.

Figure 3. Secondary magnetic field $b$ and $b'$ exerted by the eddy current in the (a) absence and (b) presence of a particle.

\[ j = \sigma(-\nabla \phi + u \times B), \]
2.2. Particle inspecting principle

When an insulating particle is present within the electrically conducting fluid and passes by the magnet, the difference between the electrical conductivities of the two materials produces a spatiotemporal redistribution of the eddy currents $\mathbf{J}$ (different from $\mathbf{J}$) and induced a substantial change in the measured force $\mathbf{F}$. The graph of the difference $\Delta \mathbf{F} = \mathbf{F}_0 - \mathbf{F}_1$ versus time exhibits a negative pulse (figure 5(b)) and suggests that the presence of the particle can be derived from the functional dependence of $\Delta \mathbf{F}$ on the particle size.

By measuring the variation of the interaction with the magnet, the LFPA can count and size the inclusion. The feasibility of this method at various scales was verified in our previous work [16, 17], where several prototype experiments and a numerical model were presented. The results demonstrate that an LFPA can, in principle, detect micron-sized particles in a solid or liquid metal. This method has the merits of offering contactless and online quantitative measurements and can be used in an extensive range of applications. Thus, exploring the physical mechanism underlying this method is important.

3. Interaction of a magnetic dipole with a moving particle-laden conducting layer

3.1. Physical review of the magnetic dipole interacting with the particle-free conducting flow layer

The work of Votyakov and Thess [20] involved a moving conducting layer containing an insulating spherical particle flowing past a permanent magnet, as shown in figure 1(a) (replacing the solid with a liquid conductor). The conductive layer extends from $z = z_d$ to $z = z_u$ and flows from $-\infty$ to $\infty$ along the x-axis at a uniform
velocity $u = u e_z$; the indices $d$ and $u$ denote the downward and upward sides, respectively. The magnetic field provided by the permanent magnet is modeled by the well-known magnetic dipole, where the position is denoted by $R_0 = (x_0, y_0, z_0)$. The distribution of the magnetic field located at $R = (x, y, z)$ can be expressed as equation (1). Thus, the general case can be considered a linear combination of the three generic cases: $n = x, y, z$. Because of the double divergence operator in equation (1), the magnitude of the field is approximately the reciprocal of the cube of the distance: $B(R) \sim O(R_m^{-3})$.

Before analyzing the variation of the interaction acting on the magnet, we note that the distribution of the eddy current when the flow is free of particles has been derived by Votyakov and Thess. The distribution of the eddy current is [20]

$$j(x, y, z) = \nabla \times \left\{ -\sigma \mu_0 \partial_x \partial_{m_x} \nabla \ln \left[ R_m + \frac{(\Delta z)^2}{|\Delta z|} \right] \right\},$$  

(5)

where $\Delta z = (z - z_0)$, $\mu = \mu_0 m / 4$, and $\partial_{m_x} = m \cdot \nabla$. The streamlines of the eddy current are parallel to the layer, and this distribution obeys the conservation of charge as shown in figure 2. Simplification of the expressions of the induced magnetic field, Lorentz force and torque that refer to the work of Votyakov and Thess will not be repeated [20]. These fields are indicated by the base field, whereas the disturbed field caused by the insulated particle will be solved in the following section.

3.2. Disturbed field when a particle is present

If the conducting layer is particle-free, the Lorentz force and the reaction force acting upon the permanent magnet remain invariant. However, an insulating particle within the liquid conductor results in a change in the reaction force. Thus, micron-sized particles can be measured and counted analogously to the potential field when a particle is present [20]. Therefore, the disturbed eddy current is $j_{fl}$ free of particles has been derived by Votyakov and Thess. The distribution of the magnetic field disturbed by a particle is reasonably assumed to be analogous to the potential flow of an incompressible ideal fluid passing a spherical obstacle. Such a classical flow has a well-known exact solution (see, for instance, Landau and Lifshitz [24]), and the velocity potential $\phi_v$ of the flow is

$$\phi_v = u \cdot R - \frac{1}{2} \rho' u \cdot \nabla \frac{1}{R},$$  

(6)

where $u$ is the initial uniform incoming flow and $\rho$ is the radius of the spherical particle. The first term on the right-hand side of equation (6) denotes the velocity potential of the uniform flow; the second term is the disturbed velocity potential resulting from the spherical obstacle. This disturbed term is clearly in the form of a dipole potential. Analogously, the disturbed electric potential field arising from the insulating particle suspended in the layer is given as

$$\phi_p = -\frac{1}{2} \rho' j_{fl} \cdot \nabla \frac{1}{R},$$  

(7)

where $j_{fl}$ is the base current at the position of the particle present and the subscript $P$ denotes the disturbed field caused by the particle. Without loss of generality, the mathematical description of this problem is not altered if the particle is taken as the origin of the inertial reference system, relative to which the permanent magnet moves uniformly along the layer. Thus, the base current at the origin from equation (5) is

$$j_{fl}(x, y, z) = \nabla \times \left\{ \sigma \mu_0 \partial_x \partial_{m_x} \nabla \ln \left[ R_0 + |z_0| \right] \right\},$$  

(8)

where $\nabla_0 = \partial_{x_0} e_x + \partial_{y_0} e_y + \partial_{z_0} e_z$, $\partial_{m_x} = \partial_{m_0}$. In addition, the distance between the particle and the magnet is $R_0 = \sqrt{x^2_0 + y^2_0 + z^2}. An expansion of the above formula can be found in the appendix and is generally written as $j_{fl} = j_{flx} e_x + j_{fly} e_y$. Therefore, the disturbed eddy current is

$$j_p = -\frac{\rho'}{2} \nabla \left( j_{fl} \cdot \nabla \frac{1}{R} \right),$$  

(9)

Clearly, the normal component of the disturbed eddy current arising from the presence of the particle is not zero and charge conservation becomes invalid at both surfaces of the layers. To minimize this inconsistency, an additional two dipoles are virtually placed at $(0, 0, 2z_{ud})$, which are positions symmetrical to the particle with respect to the upper and lower surfaces of the layer (figure 6). Thus, the disturbed current caused by the insulating particle should be added to the influence exerted by the two virtual dipoles.
Here, \( R_{ud} = (0, 0, -2z_{ud}) \) are the displacement vectors away from the two virtual dipoles. Because of the symmetry of the dipoles, the inconsistency of the disturbed current on the interface contributed by the two nearer dipoles is eliminated in pairs, and the charge conservation at the layer surface is subject only to the most distant virtual dipole. Inside the layer, the current distribution is also affected by the two virtual dipoles. However, because the disturbed eddy current caused by each virtual dipole damps as rapidly as \( O(R)^{-2} \), the influence of the farthest dipole may be negligible. For example, for the same virtual dipole, the disturbed current at the farther layer surface is, at most, one eighth of the disturbed current at the nearer surface. Therefore, the disturbed current can reasonably be computed by considering only the two nearer dipoles:

\[
j_p = -\frac{\rho^3}{2} \nabla \left( \frac{1}{R} + \frac{1}{R_{ud}} \right) \cdot \mathbf{r}.
\]

(10)

where the distances \( R_u \) and \( R_d \) are applied separately in the sub-layers of \( [(z_u + z_d)/2, z_u] \) and \( [z_d, (z_u + z_d)/2] \), respectively. In turn, the expression of the eddy current can be written as

\[
j_p = \nabla \times \left[ -\frac{\rho^3}{2} \nabla \left( \frac{1}{R} + \frac{1}{R_{ud}} \right) \right].
\]

(12)

In contrast to Ampere’s law,

\[
j = \frac{1}{\mu_0} \nabla \times (\mathbf{B} + \mathbf{b} + \mathbf{b}_p),
\]

(13)

where \( \mathbf{B} \) is the field provided by the permanent magnet and is curl free \((\nabla \times \mathbf{B} = 0)\), whereas \( \mathbf{b} \) is the induced magnetic field (or secondary magnetic field), and its curl is the base current \((\nabla \times \mathbf{b} = \mu_0 \mathbf{j})\). Thus, the nonzero terms in equation (13) describe the disturbed eddy current and the induced magnetic field

\[
j_p = \frac{1}{\mu_0} \nabla \times \mathbf{b}_p.
\]

(14)

The disturbed induced magnetic field \( \mathbf{b}_p \), in comparison with equation (12), can be written as

\[
\mathbf{b}_p = -\frac{\rho^3 \mu_0}{2} \nabla \left( \frac{1}{R} + \frac{1}{R_{ud}} \right) + \nabla a.
\]

(15)

The last term of the magnetic scalar potential \( a(x, y, z) \) comes from equation (13), is accompanied by the particle-free induced field \( \mathbf{b} \) and is redundant. Thus, the disturbance of the \( z \)-component induces a magnetic field at the surface of the layer \( b_{pz} \), which is given as

\[
\frac{b_{pz}}{\rho^3 \mu_0} = -\frac{j_0 x - j_0 y}{(x^2 + y^2 + z_{ud}^2)^{3/2}} = -j_0 \cdot \left( \nabla \frac{1}{R} \right)_{z = z_{ud}} = j_0 \partial_z \left( \frac{1}{R} \tanh \frac{z}{R} \right)_{z = z_{ud}}^z = j_0 \partial_z \left( \frac{1}{R} \tanh \frac{z}{R} \right)_{z = z_{ud}}.
\]

(16)
where \( \mathbf{J}_0 = (-j_{0y}, j_{0x}, 0) \) and \( \partial_j = \frac{I_0}{I_0} \cdot \nabla = -\frac{j_{0y}}{I_0} \frac{\partial}{\partial x} + \frac{j_{0x}}{I_0} \frac{\partial}{\partial y} \).

The disturbed induced magnetic field arises from layer \( \delta \mathbf{B}_0^\delta \); its scalar potential is denoted by \( A(x, y, z) \) and needs to be specified. This scalar potential is automatically curl-free because of the expression
\[
\nabla \times \mathbf{b}_0^\delta = \nabla^2 A = 0.
\]

The induced magnetic field satisfies the flux conservation across the layer surface, which ensures that
\[
\partial_z A|_{z=a,d} = b_{p.z}.
\]

Upon substitution of equation (16), the scalar potential can be obtained by examination:
\[
A(x, y, z) = \rho^3 \mu_0 j_0 \partial_j \tanh{\frac{z}{R}} + A_\infty(x, y),
\]
where the term \( A_\infty(x, y) \) can be determined from the additional boundary condition specifying that the induced magnetic field of the particle-laden induced field vanishes at infinity: \( \lim_{z \to \infty} A(x, y, z) = 0 \). Thus, we have
\[
A_\infty(x, y) = -\lim_{z \to \infty} \rho^3 \mu_0 j_0 \partial_j \tanh{\frac{z}{R}} = \frac{z}{|z|} \rho^3 \mu_0 j_0 \partial_j \ln \sqrt{x^2 + y^2}.
\]

Thus, the scalar potential of the disturbed induced magnetic field of equation (19) becomes
\[
A(x, y, z) = \rho^3 \mu_0 j_0 \partial_j \ln (R + |z|),
\]
and the secondary magnetic field for a particle outside the conductive layer is
\[
\mathbf{b}_0^\delta(x, y, z) = \nabla A = \rho^3 \mu_0 j_0 \partial_j \nabla \ln (R + |z|),
\]
where the superscript (’’) denotes the disturbed field out of the layer. More generally, the potential energy \( U \) of a magnetic dipole \( \mathbf{m} \) in a magnetic field \( \mathbf{B} \) is
\[
U = -\mathbf{m} \cdot \mathbf{B}.
\]
The force acting upon the dipole can be calculated from the negative gradient of the potential energy as
\[
\mathbf{F} = -\nabla U.
\]
Thus, the variation of the reaction force acting on the magnet can then be given as
\[
\mathbf{F}^* = -(\mathbf{m} \cdot \nabla) \mathbf{b}_0^\delta = \rho^3 \mu_0 j_0 \mathbf{m} \partial_j \nabla \ln (R + |z|).
\]

In addition, the torque acting upon the magnet is
\[
\mathbf{T}^* = \mathbf{m} \times \mathbf{b}_0^\delta = \rho^3 \mu_0 j_0 \mathbf{m} \times \partial_j \nabla \ln (R + |z|).
\]

These interactions can be expressed as the inner product of the magnetic dipole moment \( \mathbf{m} \), the eddy current at the insulated particle \( \mathbf{J}_0 \) and the second-order tensor \( \mathbf{F} \) or third-order tensor \( \mathbf{T} \). Additionally, these tensors can be determined via linear combinations of several coefficients, as shown in equations (32)–(33) in the appendix.

3.3. Influence of the orientation of the magnetic dipole on the reaction force and the torque

The magnetic dipole orientation and the distance to the particle significantly influence the change in the reaction force (figure 7) and the torque (figure 8) acting upon it. We take the specific orientation of \( \mathbf{m}/m = (1, 0, 0), (0, 1, 0), (0, 0, 1) \). In these figures, only half of the process in which the magnet is on top of the particle is presented because of symmetry; for convenience, all the variations of the reaction force (figure 7) and torque (figure 8) have been nondimensionalized relative to the maximum values of \( F_0^\delta \) (figure 7) and \( z_0 F_0^\delta \) (figure 8), respectively. Specifically, the maximum variations of the interaction, corresponding to the magnetic dipole passing the particle, can be computed from equations (23) and (24); its expansion is shown in the appendix.

\[
F_\theta^\delta = \max \{ F^\delta \} = \frac{\mu_0 m j_{0y} \rho^3}{z_0^5} \mathbf{e}_z, \quad T_\theta^\delta = \max \{ T^\delta \} = \frac{\mu_0 m j_{0y} \rho^3}{2z_0^5} \mathbf{e}_y.
\]

Considering \( j_{0y} = -\sigma u_\mu/z_0^5 \), the maximum variations result in
\[
F_\theta^\delta = \frac{4\pi \mu_0 \rho^3}{z_0} \mathbf{e}_z, \quad T_\theta^\delta = \frac{2\pi \mu_0 \rho^3}{z_0} \mathbf{e}_y.
\]

When the particle is located at the origin of the coordinates, the distance from the origin is simply equal to \( R_0 \). When the dipole orientation of the magnet is horizontal, i.e., when \( \mathbf{m}/m = (1, 0, 0), (0, 1, 0), (0, 0, 1) \), the instant eddy current underneath the magnetic field is quite weak. The disturbed induced magnetic field and the variation of the interaction vanish when the particle is underneath the magnetic dipole. When the magnetic dipole is perpendicular to the flow layer \( \mathbf{m}/m = (0, 0, 1) \), the spatial eddy current in the particle-free case underneath the magnetic dipole is the strongest over the layer and the disturbed eddy current in the case of a particle is larger than the disturbed eddy currents of the above two aforementioned cases as well. Within these
subsets of figures 7 and 8, only the one of the first subset in figure 7(c) and the second subset in figure 8(c) where the reaction force and torque exhibit a simple single peak and yields the largest peak value, respectively. Therefore, they are the most favorable option for the measurement.

In addition, the disturbed reaction force $F^y$ and torque $T^y$ often display the most predominant response to the insulating particle and are the quantities most typically measured in practice. All these transients vary remarkably across the region in which the physical quantities such as the magnetic field and eddy current accumulate. This region is termed the electromagnetic sensing zone (EMSZ) [16]. Clearly, from the viewpoint of surveying, we expect this force and torque to be as large possible to be detectable and measurable. Additionally, the signal of the pulse should be sufficiently sharp to allow straightforward interpretation of particle characteristics such as size and number.

4. Summary and conclusions

We present a theoretical model for analyzing the variation of the interaction acting upon a magnetic dipole caused by the presence of an embedded insulating spherical particle suspended in the conducting layer. This theoretical investigation includes formulae describing the disturbed field of the eddy currents $j_p$, the induced magnetic fields $b_p$, and $b^p$ and the Lorentz force $F^y$.

The distribution of the variations in EMSZ indicates that the interactions $F^y$, $T^y$ are stronger than the others and are ideal choices for LFPA measurement. These variations, which are directly proportional to the particle size, obey a strong negative power law with respect to the distance between the particle and the magnet. However, this distance effect can be eliminated through careful treatment and a precise machining process. This theoretical model of the LFPA is a benchmark for laboratory experiments and industrial practice. Finally, the
present model can be applied in a straightforward manner to describe the inspection of a defect in solid conducting materials using the LFPA. The theory and practical investigation are unified.

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Appendix A. Expansion of the current expression at the origin

Here, all the symbols denoting the location of the external source \( R_0 = (x_0, y_0, z_0) \) are replaced with \( R = (x, y, z) \) for convenience. The current in the vicinity of the insulated particle can be written as:

\[
\frac{j_{(x,y,z)}}{\sigma mu} = \partial_x \partial_z \nabla \ln(R + |z|) \times \vec{e}_z = \frac{m}{m} \cdot \partial_x \nabla \ln(R + |z|) \times \vec{e}_z = \frac{m}{m} \vec{J}_0 \times \vec{e}_z.
\]  

Figure 8. Space distribution of the dimensionless torque variation acting upon the PM with the magnetic dipole orientation of (a) \( m/m = (1, 0, 0) \); (b) \( m/m = (0, 1, 0) \) and (c) \( m/m = (0, 0, 1) \) when the particle passes the EMSZ. The second subset of (c) is the most favorable circumstance.
The elements of the symmetrical matrix $\mathbf{J}_0 = \partial_s \nabla \ln (R + |z|)$ are as follows:

\[
\begin{align*}
J_{11} &= -3DX - \partial_s DX^2, \quad J_{22} = -DX - \partial_s DY^2, \quad J_{33} = -\partial_s E_Z^2, \\
J_{12} &= J_{21} = -DY - \partial_s DXY, \quad J_{13} = J_{31} = -EZ - \partial_s EXZ, \quad J_{23} = J_{32} = -\partial_s EYZ.
\end{align*}
\]  

(28)

Here, we introduce $C = \frac{1}{R (R + |z|)}$; accordingly, $D = -\frac{\partial C}{x} = \left(2 + \frac{|z|}{R}\right) C^2$, $E = -\frac{\partial C}{y} = \frac{1}{R^2}$.

\[
\partial_s D = -\frac{C_{12} x}{R} - 2CDX \left(2 + \frac{|z|}{R}\right), \quad \partial_s E = -\frac{3x}{R^2}.
\]

Appendix B. Expansion of the disturbed induced magnetic field and the interaction affecting an external source

The disturbed field of the induced magnetic field can be written as

\[
\mathbf{b}_\rho = \mathbf{J}_0 \cdot \nabla \ln (R + |z|) = \mathbf{J}_0 \cdot \mathbf{b}.
\]  

(29)

The elements of the symmetrical matrix $\mathbf{b} = \nabla \ln (R + |z|)$ are as follows:

\[
\begin{align*}
b_{11} &= C - DX^2, \quad b_{22} = C - DY^2, \quad b_{33} = E_Z^2, \\
b_{12} &= b_{21} = -DXY, \quad b_{13} = b_{31} = -EXZ, \quad b_{23} = b_{32} = -EYZ.
\end{align*}
\]  

(30)

The disturbed field of the interaction acting on the external source can be written as

\[
\mathbf{F}_\rho = \mathbf{m} \cdot \nabla \nabla \ln (R + |z|) \cdot \mathbf{J}_0 = \mathbf{m} \cdot \mathbf{F} \cdot \mathbf{J}_0.
\]  

(31)

The elements of the symmetrical matrix $\mathbf{F} = \nabla \nabla \ln (R + |z|)$ are as follows:

\[
\begin{align*}
F_{11} &= -3DX - \partial_s DX^2, \quad F_{22} = -3DY - \partial_s DY^2, \quad F_{33} = -2EZ - \partial_s E_Z^2, \\
F_{331} &= F_{313} = -\partial_s EXZ, \quad F_{332} = F_{323} = F_{233} = -\partial_s E_Z^2, \quad F_{112} = F_{211} = -DY - \partial_s DXY, \\
F_{222} &= F_{212} = -DX - \partial_s DX^2, \quad F_{113} = F_{311} = -EZ - \partial_s EXZ, \quad F_{113} = F_{313} = -EZ - \partial_s E_Z^2.
\end{align*}
\]  

(32)

Here, \[
\begin{align*}
\partial_s D &= -\frac{C_{12} |z|}{R^2} - 2CDY \left(2 + \frac{|z|}{R}\right), \\
\partial_s E &= -E
\end{align*}
\]

The disturbed field of the torque acting on the external source can be written as

\[
\mathbf{T}^* = \mathbf{m} \times (\mathbf{b} \cdot \mathbf{J}_0) = \mathbf{m} \times \left[ -j_y b_{11} + j_x b_{12} , -j_y b_{21} + j_x b_{22} , -j_y b_{31} + j_x b_{32} \right]
\]

\[
= \left[ -m_y j_x b_{31} + m_x j_y b_{32} + m_z j_y b_{21} - m_z j_x b_{22} , -m_x j_y b_{11} + m_y j_x b_{12} + m_z j_y b_{31} - m_z j_x b_{32} , -m_x j_y b_{31} + m_y j_x b_{32} + m_z j_y b_{21} - m_z j_x b_{22} \right]
\]  

(33)

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