Classification of spacetimes according to conformal Killing vectors

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Abstract: Conformal Killing vectors (CKVs) preserve the spacetime metric up to a factor. Homothetic vectors and Killing vectors are special cases of CKVs. Classification of some classes of spacetimes on the basis of CKVs give interesting results showing how homothetic and Killing vectors which form subsets of the set of CKVs can be recovered as a result of the above classification.

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Einstein’s theory of general relativity is based on the realization that geometry, represented by the Riemann curvature tensor $R_{abcd}$ of the spacetime can be related to the distribution and motion of matter, denoted by the stress-energy tensor $T_{ab}$. This relation is explained by Einstein’s field equations (EFEs),

$$R_{ab} - \frac{1}{2} R g_{ab} = \kappa T_{ab} \quad (a, b = 0, 1, 2, 3).$$

Here $g_{ab}$ is the metric tensor, $R_{ab}$ the Ricci tensor, $R$ the Ricci scalar and $\kappa = \frac{8 \pi G}{c^2}$, where $G$ and $c$ are the gravitational constant and the speed of light respectively. (We have ignored the term with the cosmological constant.) Metric, $g_{ab}$, is the dynamical quantity in EFEs which varies over the spacetime. EFEs (1) break down into ten highly non-linear differential equations and so far very few exact solutions have been discovered by imposing certain restrictions [1]. One of such restrictions could be to allow a spacetime to admit certain symmetry properties. For example, the isometry group $G_m$ of $(M, g)$ is the Lie group of smooth maps of manifold $M$ onto itself leaving $g$ invariant. The subscript “$m$” is equal to the number of generators or isometries of the group. It is the Lie algebra of continuously differentiable transformations $K^a \partial / \partial x^a$ where $K^a = K^a(x^b)$ are the components of the vector field $K$ known as a Killing vector (KV) field. In other words, a KV field $K$ is a field along which the Lie derivative of the metric tensor $g$ is zero i.e. $\mathcal{L}_K (g_{ab}) = 0$.

In addition to isometries there are other types of motions which are even more restrictive and therefore could be more useful as far as the solution of Eqs.(1) and their properties are concerned. For example, the study of homothetic vectors (HVs) and conformal Killing vectors (CKVs) are significant in general relativity [2]. CKVs are motions along which the metric tensor of a spacetime remains invariant up to a scale i.e.

$$\mathcal{L}_\xi g_{ab} = g_{ab,c} \xi^c + g_{ac} \xi^c_{,b} + g_{bc} \xi^c_{,a} = 2 \phi g_{ab} .$$

Conformal motions are determined by the arbitrary constants appearing in the vector field $\xi = \xi^a \partial / \partial x^a$ when $\phi = \phi (t, x, y, z)$. In the above equation, “,” represents derivative with respect to coordinates $x^a$. If $\phi$ is constant $\xi$ represents HVs and if it is zero, we simply get the KVs. It is clear from the definition that HVs and KVs are special cases of CKVs. The study of the symmetry groups of a spacetime is a useful tool not only in constructing spacetime solutions of EFEs but also for classifying the known solutions according to the Lie algebras, or structure generated by these symmetries. Previously, CKVs have been studied for various spacetimes like Minkowski [3], Robertson-Walker [4] and pp-waves [5].

Important results regarding the dimensionality of these symmetries include (see, for example, Refs. 2, 6):

1. Riemannian space $V_n$ admits a group of motions $G_m$ where $m \leq n (n + 1) / 2$. 

2. A Riemannian space $V_n$ cannot admit a maximal group of motions $G_m$ where
$m = n(n + 1)/2 - 1$. If a spacetime admits a $G_m$ as the maximal group of isometries
then the HVs group $H_r$ is at the most of order $r = m + 1$.

3. The set of conformal vector fields on $M$ is finite-dimensional and its dimension
is less then or equal to 15. If this maximum number is attained, the spacetime is
conformally flat. If it is not conformally flat then the maximal dimension is 7.

Let us consider, for example, the class of spherically symmetric spacetimes which,
in the usual coordinates, with $\nu(t,r)$, $\lambda(t,r)$ and $\mu(t,r)$ as arbitrary functions, can
be written as

$$ds^2 = -e^{\nu(t,r)}dt^2 + e^{\lambda(t,r)}dr^2 + e^{\mu(t,r)}(d\theta^2 + \sin^2 \theta d\phi^2).$$

These spacetimes admit 3 KVs

$$K^1 = \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \cot \theta \frac{\partial}{\partial \phi},$$

$$K^2 = \cos \phi \frac{\partial}{\partial \theta} - \sin \phi \cot \theta \frac{\partial}{\partial \phi},$$

$$K^3 = \frac{\partial}{\partial \phi}.$$

In the static case these admit a timelike KV, $K^4 = \partial/\partial t$, also. The classification of
HV$s of spherically symmetric spacetimes admitting maximal isometry groups larger
than $SO(3)$ was obtained along with their metrics [6] by using the homothety equations
and without imposing any restriction on the stress-energy tensor. The possible
maximal homothety groups $H_r$ for these spacetimes are of the order $r = 4, 5, 7, 11$;
for $r = 11$, the only spacetime is Minkowski. The general solution and classification
of conformal motions for these spacetimes [7] shows that the group of CKVs is $G_{4+n}$
where $n$, the number of CKVs, is either 2 or 11. In the case $n = 2$, both CKVs are
necessarily proper. For the conformally flat case, up to 6 of the 11 CKVs may be
improper.

For the plane symmetric metric

$$ds^2 = -e^{\nu(t,x)}dt^2 + e^{\lambda(t,x)}dx^2 + e^{\mu(t,x)}(dy^2 + dz^2),$$

the minimal symmetry is given by

$$K^1 = \frac{\partial}{\partial y}, K^2 = \frac{\partial}{\partial z}, K^3 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}.$$
HVs [9] are of the order 5, 6, 7 or 11. Classification of these spacetimes according to CKVs [10] is also in accordance with the established results.

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