Where will a pen fall to?

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(Dated: October 18, 2021)

We propose a simple experiment that everybody can carry out just while reading this paper. The only thing we need is a pen or a pencil and a finger. The dynamics of the falling pen, which is in touch with the finger, depends essentially on the initial inclination angle of the long axis of the pen. We point out two different types of trajectories of the falling pen and we investigate initial conditions for each of them to be realized. We write differential equations describing the dynamics of the pen and analyze them both numerically and analytically.

I. SETUP AND EXPERIMENT

We would like to show in our paper how sometimes very usual things taken from our daily live can be formulated into an elegant physical problem. Let us carry out a simple experiment. We take an ordinary pen by one of its ends while keeping the other in touch with our finger as it is shown schematically in Fig. 1. At first, we allow for a large enough initial angle between the long axis of the pen and the vertical, say $\alpha_0 = \pi/4$. We let the pen fall down. Due to the contact with the finger, however, it is not a free fall. While falling down the pen slides across the finger and this makes it to rotate. Finally, it fell on one side from the finger (for large initial angles, it is a side where the center of mass of the pen was located initially). If then we decrease gradually the initial angle and repeat the experiment again, at a certain angle we notice that the center of mass of the pen starts to pass through the point of the "pen-finger" contact and the pen falls down on the other side from the finger.

The question is: What is the critical value of the initial inclination angle in the considered setup which does separate the two different types of motion of the pen in our experiment?

First, let us understand upon what this critical angle, $\alpha_{cr}$, could depend? After some trials we can see that it depends on the initial position of the contact point between the pen and the finger. More precisely, being dimensionless, the critical angle depends on the ratio of the initial distance between the center of mass of the pen and the contact point to its total length, $l_0/L$. Interestingly, this ratio together with the coefficient of kinetic friction, $\mu$, between the pen and the finger, which is also dimensionless, are the only parameters which determine the critical angle (we will consider a pen and a finger of zero thickness which is close to realistic conditions and simplifies essentially the problem). Indeed, it is not difficult to see that we cannot construct any other dimensionless parameter in our problem, for example, from the mass of the pen or the gravitational constant, which means that the critical angle does not depend upon them.

We would like to determine the critical angle for all values of the parameter $l_0$. However, this proves to be rather a hard task. We will solve it only numerically. Two limits are of particular interest in this problem that correspond to the case of $l_0 = 0$ and $l_0 = L/2$. It is interesting to test our intuition and to try to guess the value of the critical angle in these limits. Is it $\pi/2$ and 0 respectively or not? We will try to exploit both of these cases analytically.

II. EQUATIONS OF MOTION

Differential equations which describe the evolution of the solid body momentum, $\mathbf{P}$, and angular momentum,
can be written in the general form:

\[
\frac{d\mathbf{P}}{dt} = \mathbf{F}, \quad \frac{d\mathbf{M}}{dt} = \mathbf{K},
\]

where \( \mathbf{F} \) is the total external force applied to the body and \( \mathbf{K} \) is the total torque.

There are two independent variables in our problem: the angle \( \alpha \) and the length \( l \), see Fig. 1. However, the normal force, \( \mathbf{N} \), which acts on the pen at the pen-finger contact point and which is perpendicular to the pen long axis will also enter our equations of motion. So we will need three equations in total to describe the dynamics of the pen. It is convenient to derive two of these equations from the angular momentum evolution equation written with respect to the axis passing through the center of mass of the pen perpendicular to the plane of Fig. 1 (in order to exclude the weight force from consideration):

\[
\frac{d}{dt} [l_0 \omega(t)] = N(t)l(t), \quad (2)
\]

where \( l_0 \) is the proper moment of inertia of the pen, \( \omega(t) \) its angular velocity, \( N(t) \) the absolute value of the normal force, \( l(t) \) the lever of this force. The dependence on time for quantities we indicate by the parentheses on the right with the time variable, \( t \), inside.

Now, we consider the evolution of the angular momentum defined with respect to the axis passing through the contact point \( O \) perpendicular to the plane of Fig. 1 (we exclude the normal force from the equation in this case):

\[
\frac{d}{dt} [(ml^2(t) + l_0)\omega(t)] = mgl(t) \sin \alpha(t), \quad (3)
\]

where \( m \) is the mass of the pen, \( g \) the gravitational constant.

The last equation we will obtain from the evolution equation for the total momentum of the pen. For this purpose we choose the frame of reference which rotates together with the pen, contact point \( O \) being in its origin. In this frame of reference the center of mass moves along one of two coordinate axes which coincides with the pen’s long axis. Since this frame of reference is not inertial we should take into account the centrifugal force acting on the pen as well. Finally, we have:

\[
\frac{d}{dt}[mv(t)] = ml(t)\omega^2(t) - mg \cos \alpha(t) + F_N(t), \quad (4)
\]

where \( v(t) \) is the velocity of the center of mass along the pen’s long axis, \( F_N(t) = \mu N(t) \) the friction force.

We will now simplify the above equations. First, we transform Eq. (3) to obtain:

\[
\frac{d\omega(t)}{dt} = \frac{ml(t)}{ml^2(t) + l_0} \left[ g \sin \alpha(t) - 2 \frac{dl(t)}{dt} \omega(t) \right]. \quad (5)
\]

Then, we find from Eq. (2) the normal force:

\[
N(t) = \frac{l_0}{l(t)} \frac{d\omega(t)}{dt} \quad (6)
\]

and insert it in Eq. (4) which takes now the form:

\[
\frac{dv(t)}{dt} = l(t)\omega^2(t) - g \cos \alpha(t) + \mu I_0 \left[ g \sin \alpha(t) - 2 \frac{dl(t)}{dt} \omega(t) \right]. \quad (7)
\]

We note that by definition \( \omega(t) = d\alpha/dt \) and \( v(t) = dl/dt \). Thus, we rewrite Eqs. (4) and (7) and we obtain the system of two coupled nonlinear second order differential equations which describes the evolution of the angle and the center of mass position in time:

\[
\alpha'' = \frac{l}{l^2 + l_0} (\sin \alpha - 2l' \alpha'), \quad (8)
\]

\[
l'' = l\alpha'^2 - \cos \alpha + \frac{\mu I_0}{l^2 + l_0} (\sin \alpha - 2l' \alpha'). \quad (9)
\]

Here we have made all quantities dimensionless upon the substitutions: \( t := t\sqrt{g/L}, l := l/L, \) and \( I_0 := I_0/(mL^2) \). We use a prime instead of \( d/dt \) from now on in our formulas in order to make them more compact.

We require that the solution of this system would satisfy four initial conditions: \( \alpha(0) = \alpha_0 \), \( l(0) = l_0 \), \( \alpha'(0) = 0 \), and \( l'(0) = 0 \). The last two conditions correspond to the zero initial velocity of the pen.

To find a solution of the above system is not a simple task. One way consists in employing numerical methods. In the next Section we will show results which gives this approach. After that we will consider two cases when the system can be simplified and investigated analytically.

In Appendix A we give an alternative way of derivation of the above system of equations specifically for those vigilant readers who could feel themselves uncertain about the rotating frame of reference.

### III. NUMERICAL SIMULATION

In order to solve numerically the system of Eqs. (8) and (9) we first come back to variables \( \omega \) and \( v \) and we rewrite it as a system of four first order differential equations with four independent variables: \( \alpha, \omega, l, v := v(t) \). Then, we use the finite difference method, namely Euler’s method, to solve it. For this we divide the time interval \([0,T]\) into \( N \) small intervals, each of a length \( \Delta t = T/N \), and substitute, e.g., the derivative \( d\alpha(t)/dt \) by the ratio \( [\alpha(t_{i+1}) - \alpha(t_i)]/\Delta t \), where \( t_{i+1} = t_i + \Delta t \) and \( \alpha(t_0) = \alpha_0 \).
We use the value of the proper moment of inertia for a pen’s center of mass for three different initial positions. The results of our calculation for the trajectories of the pen are shown in Fig. 2. In order to determine the critical angle we simulate the fall of the pen for a given initial length $l_0$. We have carried out the calculation for three values of the coefficient of kinetic friction $\mu$. In the case of non-zero value of the coefficient we see that $\alpha_{cr}$ is smaller than $\pi/2$ when $l_0 = 0$. This is due to the fact that there is no sliding for the angles $\alpha > \pi/2 - \arctan \mu$, when the friction is available, which follows from the couple of equations at equilibrium: $mg \cos \alpha = \mu N$ and $mg \sin \alpha = N$. Thus, when $\alpha > \pi/2 - \arctan \mu$ the pen first rotates without sliding, its center of mass falls below the finger and then (when $\alpha > \pi/2 + \arctan \mu$) it starts to slide. In contrast, when $\alpha < \pi/2 - \arctan \mu$ the pen starts to slide from the very beginning. The numerical result for $l_0 = 0$, see Fig. 3, agree well with the critical angle value $\alpha_{cr} = \pi/2 - \arctan \mu$.

A peculiar feature of the dependence of $\alpha_{cr}$ on $l_0$ is its finite value for $l_0 = 1/2$, i.e. when initially the pen touches the finger by its very end. This angle is about $27^\circ$ for $\mu = 0$.

We show in Fig. 3 the calculated critical angle values for the whole range of the initial length $l_0$. We have carried out the calculation for three values of the coefficient of kinetic friction $\mu$. The value of the coefficient we see that $\alpha_{cr}$ is smaller than $\pi/2$ when $l_0 = 0$. This is due to the fact that there is no sliding for the angles $\alpha > \pi/2 - \arctan \mu$, when the friction is available, which follows from the couple of equations at equilibrium: $mg \cos \alpha = \mu N$ and $mg \sin \alpha = N$. Thus, when $\alpha > \pi/2 - \arctan \mu$ the pen first rotates without sliding, its center of mass falls below the finger and then (when $\alpha > \pi/2 + \arctan \mu$) it starts to slide. In contrast, when $\alpha < \pi/2 - \arctan \mu$ the pen starts to slide from the very beginning. The numerical result for $l_0 = 0$, see Fig. 3, agree well with the critical angle value $\alpha_{cr} = \pi/2 - \arctan \mu$.

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### IV. ANALYTICAL SOLUTION

We consider now two cases when the system of equations (8) and (9), which describes the evolution of the inclination angle and the center of mass position of the pen in time, can be substantially simplified.
A. Angles close to \( \pi/2 \)

It should be clear that the critical angle will be about \( \pi/2 \) when \( l_0 \) is close to zero. In the limit \( |l| \ll \sqrt{l_0} \) and \( |\gamma| \ll 1 \), where \( \gamma = \pi/2 - \alpha \), Eqs. (5) and (7) simplify to the system (we consider only the case \( \mu = 0 \)):

\[
\begin{align*}
I_0 \gamma'' &= -l, \\
\gamma'' &= -\gamma.
\end{align*}
\]

(11)

(12)

We have supposed the derivatives \( \gamma' \) and \( \gamma'' \) to be negligibly small compared to unity as well. The solution of this system is a combination of trigonometric and hyperbolic sines and cosines. However, due to the zero initial condition for the velocity the sines drop out of the solution. Thus, we have

\[
\begin{align*}
\gamma(t) &= A \cos \omega t + B \cosh \omega t, \\
l(t) &= A \sqrt{l_0} \cos \omega t - B \sqrt{l_0} \cosh \omega t,
\end{align*}
\]

(13)

(14)

where \( A \) and \( B \) are constants and \( \omega = I_0^{-1/4} \).

We suppose that the critical trajectory corresponds to the case when \( l(T) = 0 \) and \( \gamma(T) = 0 \) for a particular moment \( T \). This could probably contradict the observation that the center of mass can pass twice through the contact point (see Fig. 2). However, we can hope that our assumption is valid at least for the limit \( \gamma \ll 1 \). It results in \( B = 0 \) and the oscillatory solution for \( \gamma \) and \( l \):

\[
\begin{align*}
\gamma(t) &= \gamma_0 \cos \omega t \quad \text{and} \quad l(t) = l_0 \cos \omega t,
\end{align*}
\]

(15)

where \( \gamma_0 = l_0 / \sqrt{l_0} \), which is exactly the condition we searched for. Finally, the critical angle in the considered limit equals to \( \alpha_{cr} = \pi/2 - l_0 / \sqrt{l_0} \). We trace this dependence of \( \alpha_{cr} \) upon \( l_0 \) in Fig. 3. The agreement between our theoretical result and numerical data for angles \( \alpha_{cr} \) close to \( \pi/2 \) is quite remarkable.

B. Small angles

Another limit of interest corresponds to small angles \( \alpha \). The principal question is whether it is possible to prove analytically that for \( l_0 = 1/2 \) the critical angle \( \alpha_{cr} \) has a finite non-zero value. So, we consider small angles \( \alpha \ll 1 \) and \( l_0 \) close to \( 1/2 \) and we rewrite Eqs. (8) and (9) in this limit as:

\[
\begin{align*}
(l^2 + l_0)\alpha'' + 2l' \alpha' - l \alpha &= 0, \\
l'' &= -1.
\end{align*}
\]

(16)

(17)

The solution of Eq. (17) is \( l = -t^2/2 + l_0 \) which corresponds to a free fall of the center of mass. In Eq. (18) we switch from the time variable, \( t \), to the length variable, \( l \), arriving at the equation:

\[
\frac{d^2 \alpha}{dl^2} + \left[ \frac{2l}{l^2 + l_0} - \frac{1}{2(l - l_0)} \right] \frac{d\alpha}{dl} - \frac{l \alpha}{2(l^2 + l_0)(l_0 - l)} = 0.
\]

(18)

We are not able to fully solve this equation. However, if we consider only the very beginning of the movement, when \( (l_0 - l) \ll 1 \), we need to keep only third and fourth terms in Eq. (18) which simplifies it essentially:

\[
\frac{d\alpha}{dl} + \frac{l \alpha}{l^2 + l_0} = 0.
\]

(19)

The solution of the above equation is

\[
\alpha(l) = \alpha_0 \sqrt{l_0^2 + l_0},
\]

(20)

which fulfills initial conditions \( \alpha = \alpha_0 \) and \( \alpha' = 0 \) at \( t = 0 \).

In order for our approximation to be self-consistent we should have \( \alpha(l) \ll 1 \) for all values of \( l \). While the solution (20) is valid only for \( (l_0 - l) \ll 1 \) we assume it does not differ much from an exact solution even for \( l \) close to zero. We cannot prove this assumption analytically, though a numerical simulation of Eq. (10) supports it. Thus, from the condition of self-consistency for \( l = 0 \) we have \( \alpha_0 \sqrt{l_0^2/2} + 1 \ll 1 \). This means that, in the case of \( l_0 = 1/2 \), the angle \( \alpha \) is always small if \( \alpha_0 \ll 1/\sqrt{l_0} = 1 \) (we note that this limit is close to \( \pi/6 = 30^{\circ} \)). For these small initial angles the center of mass of the pen will pass across the contact point with the finger. In contrast, for larger initial angles the pen will fall on the side where its center of mass was located initially. So, the critical angle is in between and it is larger than zero.

V. SUMMARY

We have proposed a simple experiment, which needs a pen and a finger all in all, and we have formulated it into a mathematical problem. The key question of the problem is the value of the critical angle which separates two different types of trajectories of the falling pen. Numerical methods allow to solve this problem and to find critical angles for any initial center of mass position \( l_0 \). Theoretical analysis of the equations provides us the critical angle value in the limit of small \( l_0 \) and allows us to say that for \( l_0 \) close to \( 1/2 \) the critical angle should have a finite non-zero value.

We would like to refer the reader to several problems which are also about the dynamics of the falling pen, however, in a totally different setup. We have encountered them while studying our problem. They are quite nice!

VI. ACKNOWLEDGEMENTS

We would like to thank Maxim Kirillov for discussions of a numerical solution, Florian Marquardt for remarks on the manuscript and many other people for their interest to the problem.
APPENDIX A: AN ALTERNATIVE DERIVATION
OF EQUATIONS OF MOTION

We keep the equation which describes the evolution of
the angular momentum in the center of mass frame of
reference:
\[
\frac{d}{dt} [I_0 \omega(t)] = N(t)l(t). \quad (A1)
\]
It permits to express the normal force through \( \alpha \) and \( l \)
as \( N = I_0 \alpha''/l \).
We write then two equations for the evolution of the
pen momentum in projection on \( x \) and \( y \) axes:
\[
m x'' = N \cos \alpha - \mu N \sin \alpha, \quad (A2)
\]
\[
m y'' = mg - N \sin \alpha - \mu N \cos \alpha. \quad (A3)
\]
Next, we introduce polar coordinates according to: \( x = -l \sin \alpha \) and \( y = -l \cos \alpha \). We note the expressions for
the second order derivatives in these coordinates:
\[
-x'' = l'' \sin \alpha + 2l' \alpha' \cos \alpha + l \left( \alpha'' \cos \alpha - (\alpha')^2 \sin \alpha \right),
\]
\[
y'' = l'' \cos \alpha - 2l' \alpha' \sin \alpha - l \left( \alpha'' \sin \alpha + (\alpha')^2 \cos \alpha \right).
\]
We make variables dimensionless as we did in Sec. II
and finally we obtain Eqs. (8) and (9) from Eqs. (A2)
and (A3) rewritten in polar coordinates.

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