Concurrence as a Relative Entropy with Hilbert-Schmidt Distance in Bell Decomposable States

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Abstract

Hilbert-Schmidt distance reduces to Euclidean distance in Bell decomposable states. Based on this, entanglement of these states are obtained according to the protocol proposed in Ref. [V. Vedral et al, Phys. Rev. Lett. 78, 2275 (1995)] with Hilbert-Schmidt distance. It is shown that this measure is equal to the concurrence and thus can be used to generate entanglement of formation. We also introduce a new measure of distance and show that under the action of restricted LQCC operations, the associated measure of entanglement transforms in the same way as the concurrence transforms.

Keywords: Quantum entanglement, Bell decomposable states, Concurrence, Relative entropy, Hilbert-Schmidt distance,

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1 Introduction

Perhaps, quantum entanglement is the most non classical features of quantum mechanics [1, 2] which has recently been attracted much attention. It plays a central role in quantum information theory [3, 4, 5]. Entanglement is usually arise from quantum correlations between separated subsystems which can not be created by local actions on each subsystem. By definition, a mixed state $\rho$ of a bipartite system is said to be separable (non entangled) if it can be written as a convex combination of pure product states

$$\rho = \sum_i p_i |\phi^A_i\rangle \langle \phi^A_i | \otimes |\psi^B_i\rangle \langle \psi^B_i | , \quad (1-1)$$

where $|\phi^A_i\rangle$ and $|\psi^B_i\rangle$ are pure states of subsystems $A$ and $B$, respectively. Although, in a pure state of bipartite systems it is easy to check whether a given state is, or is not entangled, the question is yet an open problem in the case of mixed states.

There is also an increasing attention in quantifying entanglement, particularly for mixed states of a bipartite system, and a number of measures have been proposed [5, 6, 7, 11]. Among them the entanglement of formation has more importance, since it intends to quantify the resources needed to create a given entangled state. Vedral et al. in [6, 7] introduced a class of distance measures suitable for entanglement measures. They also showed that the quantum relative entropy and the Bures metric satisfy three conditions that a good measure of entanglement must satisfy and can therefore be used as generators of measures of entanglement.

Hilbert-Schmidt distance have been used as a measure of distance in [8]. They obtained the entanglement of Bell decomposable states and part of pure states for $2 \otimes 2$ systems according to H-S distances.

In this paper we show that H-S distance reduces to Euclidean distance for special kind of $2 \otimes 2$
states, called Bell decomposable states. Based on this, we can rather easily calculate entanglement measure associated with H-S distance of these states. We also show that thus obtained quantity is equal to the concurrence and thus can be used to generate entanglement of formation.

Finally, we present a new measure of distance in operators space and show that the corresponding entanglement measure reduces to concurrence. Starting from BD sates, we perform quantum operations and classical communications (LQCC) [9, 10] and as consequence one obtains new entangled mixed density matrices with entanglement measure with a functionality analogous to the concurrence.

The paper is organized as follows. In section 2 we review BD states and present a perspective of their geometry. In section 3 we show that H-S distance of these states reduces to Euclidean distance. In section 4 we give a brief review of Wootters’ concurrence, then we evaluate entanglement measure associated with H-S distance and show that it is equal to the concurrence. In section 5 we introduce a new measure of distance and show that its corresponding entanglement measure is also equal to the concurrence for BD states. We perform LQCC action on these states and show that under LQCC, the corresponding entanglement measure transforms in the same way as the concurrence transforms. The paper is ended with a brief conclusion.

2 Bell Decomposable States

In this section we briefly review Bell decomposable (BD) states and some of their properties. A BD state is defined by

$$\rho = \sum_{i=1}^{4} p_i |\psi_i\rangle \langle \psi_i|, \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^{4} p_i = 1, \quad (2.2)$$
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where $|\psi_i\rangle$ is Bell state given by

$$|\psi_1\rangle = |\phi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle),$$  \hspace{1cm} (2-3) \\
$$|\psi_2\rangle = |\phi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle),$$  \hspace{1cm} (2-4) \\
$$|\psi_3\rangle = |\psi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle),$$  \hspace{1cm} (2-5) \\
$$|\psi_4\rangle = |\psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).$$  \hspace{1cm} (2-6)

In terms of Pauli’s matrices, $\rho$ can be written as

$$\rho = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} t_i \sigma_i \otimes \sigma_i),$$ \hspace{1cm} (2-7) \\

where

$$t_1 = p_1 - p_2 + p_3 - p_4,$$  \\
$$t_2 = -p_1 + p_2 + p_3 - p_4,$$  \hspace{1cm} (2-8) \\
$$t_3 = p_1 + p_2 - p_3 - p_4.$$

From positivity of $\rho$ we get

$$1 + t_1 - t_2 + t_3 \geq 0,$$
$$1 - t_1 + t_2 + t_3 \geq 0,$$  \hspace{1cm} (2-9) \\
$$1 + t_1 + t_2 - t_3 \geq 0,$$
$$1 - t_1 - t_2 - t_3 \geq 0.$$

These equations form a tetrahedral with its vertices located at (1, −1, 1), (−1, 1, 1), (1, 1, −1), (−1, −1, −1) [16]. In fact these vertices are Bell states given in Eqs. (2-3) to (2-6), respectively.

According to the Peres and Horodecki’s condition for separability [14, 15], a 2-qubit state is separable if and only if its partial transpose is positive. This implies that $\rho$ given in Eq. (2-7) is
separable if and only if $t_i$ satisfy Eqs. (2-9) and

$$
1 + t_1 + t_2 + t_3 \geq 0,
1 - t_1 - t_2 + t_3 \geq 0,
1 + t_1 - t_2 - t_3 \geq 0,
1 - t_1 + t_2 - t_3 \geq 0.
$$

(2-10)

Inequalities (2-9) and (2-10) form an octahedral with its vertices located at $O_1^\pm = (\pm 1, 0, 0)$, $O_2^\pm = (0, \pm 1, 0)$ and $O_3^\pm = (0, 0, \pm 1)$. Hence, tetrahedral of Eqs. (2-9) is divided into five regions. Central regions, defined by octahedral, are separable states. There are also four smaller equivalent tetrahedral corresponding to entangled states. Each tetrahedral takes one Bell state as one of its vertices. Three other vertices of each tetrahedral form a triangle which is its common face with the octahedral (See Fig. 1).

3 Hilbert-Schmidt distance for BD states

In the Hilbert-Schmidt (H-S) space density matrices are regarded as vectors rather than operators in the conventional quantum mechanics. The inner product between two operators $A$ and $B$ in H-S space is defined as

$$
\langle A, B \rangle = \text{tr}(A^\dagger B),
$$

(3-11)

where $A$ and $B$ are $4 \times 4$ matrices.

Using definition of inner product, we can define the norm of a given vector $A$ in H-S space as

$$
\|A\| = \sqrt{\langle A, A \rangle} = \sqrt{\text{tr}(A^\dagger A)},
$$

(3-12)

which is called trace norm. Hence it is natural to obtain the distance between two vectors $A$ and
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$B$ in H-S space as

$$d = ||A - B||. \quad (3-13)$$

Now, we show that for BD states H-S distance is equivalent to Euclidean distance. Let us consider two density matrix $\rho$ and $\rho'$. Using Eq. (2-7) we can expand them in terms of Pauli’s matrices

$$\rho = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} t_i \sigma_i \otimes \sigma_i), \quad (3-14)$$

$$\rho' = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} t'_i \sigma_i \otimes \sigma_i). \quad (3-15)$$

Using the above equations one can straightforwardly evaluate their H-S distance, where we have

$$||\rho - \rho'|| = \sqrt{\text{tr}(\rho - \rho')^2} = \frac{1}{2} \sqrt{\sum_{i=1}^{3} (t_i - t'_i)^2}. \quad (3-16)$$

Now we can to use the above results to obtain concurrence in the next section.

4 Concurrence as relative entropy with H-S distance

From the various measures proposed to quantify entanglement, the entanglement of formation has a special position which in fact intends to quantify the resources needed to create a given entangled state \[5\]. Wootters in \[11\] has shown that for a 2-qubit system entanglement of formation of a mixed state $\rho$ can be defined as

$$E(\rho) = H(\frac{1}{2} + \frac{1}{2}\sqrt{1-C^2}), \quad (4-17)$$

where $H(x) = -x \ln x - (1-x) \ln (1-x)$ is binary entropy and $C(\rho)$, called concurrence, is defined by

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \quad (4-18)$$
where the $\lambda_i$ are the non-negative eigenvalues, in decreasing order, of the Hermitian matrix $R \equiv \sqrt[\rho]{\rho \rho} \sqrt[\rho]{\rho}$ and

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^*(\sigma_y \otimes \sigma_y), \quad (4-19)$$

where $\rho^*$ is the complex conjugate of $\rho$ when it is expressed in a fixed basis such as $\{|\uparrow\rangle, |\downarrow\rangle\}$, and $\sigma_y$ is

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
on the same basis.

Now, let us consider a Bell decomposable state given by (2-7). One can show that for these states $\tilde{\rho} = \rho$ and thus $R = \sqrt{\tilde{\rho} \rho} \sqrt{\rho} = \rho$. Calculating the eigenvalues of $R$ we get

$$C = \max\{0, -\frac{1}{2}(1 + t_1 + t_2 + t_3)\} = \max\{0, 2 \rho_4 - 1\}. \quad (4-20)$$

In the sequel we obtain concurrence given in Eq. (4-20) from an entirely different approach. Vedral et al. in [6, 7] introduced a class of distance measures suitable for entanglement measures. According to their methods, entanglement measure for a given state $\rho$ is defined as

$$E(\rho) = \min_{\sigma \in D} D(\rho \parallel \sigma), \quad (4-21)$$

where $D$ is any measure of distance (not necessarily a metric) between two density matrix $\rho$ and $\sigma$ and $D$ is the set of all separable states. They have also shown that quantum entropy and Bures metric satisfy three conditions that a good measure of entanglement must satisfy and can therefore be used as generators of measures of entanglement.

Witte et al. used H-S distance as a candidate for Eq. (4-21). They obtained entanglement of the BD states based on H-S distance from a rigorous method.

Now entanglement of BD states can be easily evaluated by using Eq. (3-16), where H-S distance is equal to Euclidean distance.
Let us consider state $\rho$ in the entangled tetrahedral corresponding to singlet state ($p_4 \geq \frac{1}{2}$).

It can be easily seen that the nearest separable surface to this state is $x_1 + x_2 + x_3 + 1$ which is its common face with octahedral (See Fig. 2). If $\rho_s$ denotes nearest separable density matrix to $\rho$, then it must lie on this separable surface. This means that $p_4' = \frac{1}{2}$. Minimizing the Euclidean distance between $\rho$ and $\rho_s$ with the constraints $p_4' = \frac{1}{2}$ and $\sum_{i=1}^{4} p_i' = 1$, we get

$$p_i' = p_i + \frac{1}{3} \left( p_4 - \frac{1}{2} \right) \quad \text{for} \quad i = 1, 2, 3 \quad \text{and} \quad p_4' = \frac{1}{2}. \quad (4-22)$$

In terms of parameters $t_i$, Eq. (4-22) takes the following form

$$t_i' = t_i - \frac{1 + t_1 + t_2 + t_3}{3}. \quad (4-23)$$

Using the above result and Eq. (3-16) we obtain

$$D(\rho \parallel \sigma) = -\frac{1 + t_1 + t_2 + t_3}{2\sqrt{3}} = \frac{C}{\sqrt{3}}. \quad (4-24)$$

Now we can define entanglement of $\rho$ as

$$E(\rho) = \sqrt{3}D(\rho \parallel \sigma) = C \quad (4-25)$$

Right hand side of Eq. (4-25) is the concurrence given in Eq. (4-20).

5 Tilde norm

Here in this section, we introduce a new norm defined as

$$\|\tilde{A}\| := \sqrt{tr(\tilde{A} \tilde{A})}, \quad (5-26)$$

where $\tilde{A}$ is defined according to Eq. (4-19). With respect to this norm the distance between two density matrices $\rho_1$ and $\rho_2$ is defined by

$$d = \|\rho_1 - \rho_2\| = \sqrt{tr((\rho_1 - \rho_2)(\tilde{\rho}_1 - \tilde{\rho}_2))}. \quad (5-27)$$
In the sequel we use Eq. (5-27) as a measure of distance. We define entanglement of a state \( \rho \) by

\[
E(\rho) := \min_{\sigma \in D} \| \rho - \sigma \| .
\]  

(5-28)

It is straightforward to see that for BD states the above distance reduces to H-S distance. Hence the separable state \( \sigma \) which minimize expression (5-28) is the same as \( \rho_s \) given in Eq. (4-23).

In the sequel we perform local quantum operations and classical communications (LQCC) on BD states to study the change of entanglement.

A general LQCC transformation is defined by

\[
\rho' = \frac{(A \otimes B) \rho (A \otimes B)^\dagger}{\text{tr}((A \otimes B) \rho (A \otimes B)^\dagger)},
\]  

(5-29)

where operators \( A \) and \( B \) can be written as

\[
A \otimes B = U_A f^{\mu,a,m} \otimes U_B f^{\nu,b,n},
\]  

(5-30)

where \( U_A \) and \( U_B \) are unitary operators acting on subsystems \( A \) and \( B \), respectively and the filters \( f^{\mu,a,m} \) and \( f^{\nu,b,n} \) are defined by

\[
f^{\mu,a,m} = \mu(I_2 + a m \sigma),
\]  

(5-31)

\[
f^{\nu,b,n} = \nu(I_2 + b n \sigma).
\]  

As it is shown in Refs. [9, 10], the concurrence of the state \( \rho \) transforms under LQCC of the form given in Eq. (5-29) as

\[
C(\rho') = \frac{\mu^2 \nu^2 (1 - a^2)(1 - b^2)}{t(\rho; \mu, a, m, \nu, b, n)} C(\rho),
\]  

(5-32)

where

\[
t(\rho; \mu, a, m, \nu, b, n) = \text{tr}((A \otimes B) \rho (A \otimes B)^\dagger).
\]  

(5-33)

Now we perform LQCC transformation on states \( \rho \) and \( \rho_s \)

\[
\rho' = \frac{U_A f^{\mu,a,m} \otimes U_B f^{\nu,b,n} \rho f^{\mu,a,m} U_A^\dagger \otimes f^{\nu,b,n} U_B^\dagger}{t(\rho; \mu, a, m, \nu, b, n)},
\]  

(5-34)

\[
\rho_s' = \frac{U_A f^{\mu,a,m} \otimes U_B f^{\nu,b,n} \rho_s f^{\mu,a,m} U_A^\dagger \otimes f^{\nu,b,n} U_B^\dagger}{t(\rho_s; \mu, a, m, \nu, b, n)}.
\]  

(5-35)
Under LQCC transformation $\tilde{\rho}$ and $\tilde{\rho}_s$ change as:

$$\tilde{\rho}' = \frac{U_A f^{\mu,a,-m} U_B f^{\nu,b,-n} \tilde{\rho} f^{\mu,a,-m} U_A^\dagger \otimes f^{\nu,b,-n} U_B^\dagger}{t(\rho; \mu, a, m, \nu, b, n)},$$

(5-35)

$$\tilde{\rho}'_s = \frac{U_A f^{\mu,a,-m} U_B f^{\nu,b,-n} \tilde{\rho}_s f^{\mu,a,-m} U_A^\dagger \otimes f^{\nu,b,-n} U_B^\dagger}{t(\rho_s; \mu, a, m, \nu, b, n)}.$$  

(5-36)

Now, taking into account that $f^{\mu,a,m} f^{\mu,a,-m} = \mu^2 (1 - a^2) I_2$ and $f^{\nu,b,n} f^{\nu,b,-n} = \nu^2 (1 - b^2) I_2$, and using Eq. (5-27) we can evaluate the distance between two states $\rho'$ and $\rho'_s$, where we have

$$\|\rho' - \rho'_s\| = \mu^4 \nu^4 (1 - a^2)^2 (1 - b^2)^2 tr \left( \left( \frac{\rho}{t(\rho)} - \frac{\rho_s}{t(\rho_s)} \right) \left( \frac{\tilde{\rho}}{t(\rho)} - \frac{\tilde{\rho}_s}{t(\rho_s)} \right) \right)$$

(5-37)

where $t(\rho)$ and $t(\rho_s)$ are defined according to Eqs. (2-7) and (4-23) as

$$t(\rho) = t(\rho; \mu, a, m, \nu, b, n) = \mu^2 \nu^2 \left( (1 + a^2)(1 + b^2) + 4ab \sum_{i=1}^3 m_i t_i n_i \right)$$

$$t(\rho_s) = t(\rho_s; \mu, a, m, \nu, b, m) = \mu^2 \nu^2 \left( (1 + a^2)(1 + b^2) + 4ab \sum_{i=1}^3 m_i t'_i n_i \right)$$

$$= t(\rho) - \frac{4}{3} \mu^2 \nu^2 a b m.n (1 + t_1 + t_2 + t_3)$$

$$= t(\rho) - \frac{4}{3} \mu^2 \nu^2 a b m.n C,$$

(5-38)

where in the last line we used concurrence $C$ given in (4-20). In the sequel we make special choices for LQCC transformations. Let us consider cases that either $m.n = 0$ or $a b = 0$, that is, $t(\rho) = t(\rho_s)$. In this case we get

$$E(\rho') = \frac{\mu^2 \nu^2 (1 - a^2)(1 - b^2)}{t(\rho; \mu, a, m, \nu, b, m)} E(\rho).$$

(5-39)

Comparison of the above result with Eq. (5-32) shows that under the action of LQCC, the newly defined measure of entanglement changes in the same way as the concurrence changes, therefore it is the same as the concurrence.

**Conclusion**

We have shown that H-S distance is equivalent to Euclidean distance for BD states. Based on
this, H-S entanglement measure of these states are easily obtained and it has been shown that, H-S entanglement measure is equal to the concurrence. Based on spin flipping transformation of density matrices, we have introduced a new measure of distance, and we have shown that its corresponding measure of entanglement is equal to the concurrence for BD states. Starting from BD states together by performing restricted LQCC action, we have shown that the transformed entanglement measure is equal to the concurrence.

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Figure Captions

Figure 1: All BD states correspond to the interior points of tetrahedral. Vertices $P_1$, $P_2$, $P_3$ and $P_4$ denote projectors corresponding to Bell states defined by Eqs. (2-3) to (2-6), respectively. The interior points of octahedral correspond to separable states.

Figure 2: Entangled tetrahedral corresponding to singlet state. The points of line $P_4 C$ correspond to entangled Werner states. Points $t$ and $t'$ correspond to a generic BD state $\rho$ and associated nearest separable state $\rho_s$, respectively.
