HANDLE OPERATORS IN R.C.F.T.

M. CRESCIMANNO

Center for Theoretical Physics, M.I.T., 77 Mass. Ave.
Cambridge, MA. 02139-4307

ABSTRACT

For the series associated to a group or coset R.C.F.T. there is a simple universal form for the inverse of the handle operator in the ring of fusions. These formulae may be easily understood from the quantization of the associated Chern-Simons theory.

1. Introduction

Among the data that defines a rational conformal field theory (RCFT), the fusion algebra plays a central role\(^1\). It restricts the form of the modular representations and fixes a convenient basis for discussing the dimension of the space of blocks on higher genus \((g > 0)\). Let roman letters label the integrable representations of a group or coset RCFT and \(\mathcal{O}_j\) be the operators of fusion

\[
\mathcal{O}_i \times \mathcal{O}_j = N_{ij}^k \mathcal{O}_k.
\]

(1.1)

As described in Ref.[2,3], the dimension of the space of conformal blocks in genus \(g > 0\) is given by

\[
\dim \mathcal{H}^g = Tr(K^g - 1),
\]

(1.2)

where \(K\) is a matrix in the space of integrable representations and is a particular linear combination of the operators \(\mathcal{O}_j\)

\[
K = \sum_i Tr(\mathcal{O}_i) \mathcal{O}_i.
\]

(1.3)

This is a very general characterization of \(K\). For a gaussian model (i.e. one composed of any number of non-interacting massless bosons) \(K\) is the unit matrix scaled by the number of integrable representations (a general property of \(K^{-1}\) is \(Tr(K^{-1}) = 1\).)

For the case of group or coset RCFT, \(K\) has no obvious classical group-theoretic interpretation. That is, for example, given \(G_k\) and forming \(K\) of Eq.(1.3) for this theory it is clear that \(K\) depends on the level \(k\) and on the group theory of \(G\) in a complicated way.

The purpose of this note is to demonstrate that the inverse matrix, \(K^{-1}\) does admit a simple interpretation in terms of classical group-theoretic ideas. Indeed, we show that \(K^{-1}\) for a given group or coset RCFT is a ratio

\[
K^{-1} = w^2 / \text{vol}(k),
\]

(1.4)

where \(w^2\) is a fixed linear combination of representations (depending only on the group theory of \(G\)) and \(\text{vol}(k)\) is the naive volume of the moduli space of flat \(G\)-connections over the torus and is a simple combinatorial factor that depends on the
level $k$. For example, for $SU(2)_k$ one finds

$$K^{-1} = \frac{1}{2(k+2)} [3O_1 - O_3]$$

(1.5)

for all level $k$. Here the subscripts refer to the dimensions of the representations.

2. Chern-Simons and the Handle Operator

Perhaps the simplest and most revealing way of understanding formulae of this type is via the quantization of Chern-Simons (CS) theory\(^4\). Here, we sketch derivation of Eq.(1.4). For more details see Ref.[5,6,7]. Recall that the Hilbert space of CS theory is naturally isomorphic to the space of conformal blocks of a CFT, and that one convenient way to construct the Hilbert space of CS theory is by quantizing the space, $\mathcal{M}$ of flat gauge connections over a given surface. The moduli space $\mathcal{M}$ is not quite a manifold; in general it contains a set of measure zero where there are cusps, self-intersections, disconnected points, and other pathologies. Fortunately, we will only concern ourselves here with the moduli space of flat gauge connections over the torus and this, for the classical Lie groups, has only cusp-type singularities. Thus, for that case it is enough to study the quantization of the covering space $\hat{\mathcal{M}}$ of the moduli space $\mathcal{M} = \hat{\mathcal{M}}/W$, where the group $W$ is essentially a realization of the Weyl group. As usual, the quantization on $\hat{\mathcal{M}}$ proceeds via the symplectic form $\Omega = (k + c)\text{Tr}$ where $c$ is the quadratic casimir of the adjoint representation and $\text{Tr}$ is the matrix through which the inner products in the Cartan subalgebra are taken (for the simply laced Lie algebras it is just the Cartan matrix; for the non-simply laced case it is an appropriate symmetrization of the Cartan matrix.) Call $\hat{\mathcal{H}}$ the quantum Hilbert space that results from the quantization of covering space $\hat{\mathcal{M}}$. The Hilbert space $\hat{\mathcal{H}}$ is isomorphic to the space of conformal blocks of a gaussian model, and admits an action of the covering group $W$. Call that group action $W_{\hat{\mathcal{H}}}$. As discussed in the literature, the Hilbert space $\mathcal{H}$ of the CS theory is found by modding $\hat{\mathcal{H}}$ by this group action

$$\begin{array}{ccc}
\hat{\mathcal{M}} & \xrightarrow{\Omega} & \hat{\mathcal{H}} \\
W & \downarrow & W_{\hat{\mathcal{H}}} \\
\mathcal{M} & \xrightarrow{\mathcal{H}} &
\end{array}$$

The "handle-squashing" operator $K^{-1}$ is then just the push-forward of the handle-squashing operator in the $\mathcal{H}$ theory. Since the $\mathcal{H}$ theory is gaussian, we know that on $\mathcal{H}$ the $K^{-1}$ is the unit matrix divided by $\text{dim}\mathcal{H}$. For example, in $G_k$ CS theory the $K^{-1}$ of the Hilbert space associated to the cover of moduli space over the torus is

$$K^{-1}_{\mathcal{H}} = \left(\frac{\Lambda_w}{(k + c)\Lambda_r}\right)^{-1} 1,$$

(2.1)

where $\Lambda_w$ is the co-root lattice and $\Lambda_r$ is the root lattice of $G$. 
Now, just as all the points in $\mathcal{M}$ are invariant under the action of $W$, all the states of $\mathcal{H}$ form a covariant multiplets under the action of $W_\mathcal{H}$. As described in the literature, gauge invariance of the entire partition function requires that the states in $ψ \in \mathcal{H}$ are alternating states, that is, they satisfy $wψ = det(w)ψ \quad \forall w \in W_\mathcal{H}$.

There is a fundamental alternating operator $Γ$ which "projects" all the states in $\hat{\mathcal{H}}$ onto the states in $\mathcal{H}$

$$Γ = \frac{1}{\sqrt{|W|}} \sum_{w \in W} det(w) Π_i B_i^{(ω_i,wρ)}$$

where $ρ = \frac{1}{2} Σ_{α>0} α$ and the inner product in the exponent of the raising operators $B_i$ are in terms of the co-root basis, $\{ω_i\}$. $|W|$ is the order of the Weyl group. To compute modular representations, fusions, etc. in the ring of operators on $\mathcal{H}$ one simply conjugates the corresponding quantity in the $\hat{\mathcal{H}}$ by the operator $Γ$. Thus, by pushing-forward Eq.(2.1) we find the handle-squashing operator for the case of $G_k$ to be $K^{-1} = \frac{|W|}{\dim \hat{\mathcal{H}}} Γ^\dagger Γ$ which we may write (using the consequence of symmetry $Γ^\dagger = -Γ$) as

$$K^{-1}_G = -\frac{1}{\text{vol}(G_k)} Γ^2 ,$$

where

$$\text{vol}(G_k) = \frac{|Λ_w^{(G_k)}|}{|W|}$$

may be thought of as the naive symplectic volume of the $\mathcal{M}$ and $Γ^2$ is, of course, invariant under the action of the Weyl group and, for a given $G$, is a fixed linear combination of representations and is thus independent of the level.

For cosets "without fixed points" there is also a simple formula for the handle-squashing operator $K^{-1}$ of the form Eq.(2.3). For a simple coset $G_k/H_k$, where the bonus currents act without fixed points

$$K^{-1}_{G/H} = |Z|^2 (K^{-1}_G \otimes K^{-1}_H)|_{G/H}$$

where $Z = Z_G \cap H$ is the common center and $|Z|$ is the number of elements it has. Since the center action may be thought of as acting via $Z \times Z$ as a further identification in $\hat{\mathcal{M}}_G \times \hat{\mathcal{M}}_H$ and since it acts freely, we see that collecting the factors we may represent $K^{-1}_{G/H}$ again as a ratio of a particular group theoretic part and a naive symplectic volume of the coset’s moduli space of flat connections $\text{vol}(G_k/H_k) = \text{vol}(G_k)\text{vol}(H_k)/|Z|^2$.

3. Examples and Summary

Using the above ideas we now list a few examples of these explicit handle operator formulae

$$K^{-1}_{U(1)} = \frac{1}{2k} O_1$$

$$K^{-1}_{SU(2)} = \frac{1}{2(k+2)} (3O_1 - O_3)$$

$$K^{-1}_{SU(3)} = \frac{1}{3(k+3)^2} (9O_1 - 6O_8 + 3[O_{10} + \overline{O_{10}}] - O_{27})$$
\[ K_{SO(5)}^{-1} = \frac{1}{4(k+3)^2}(22\mathcal{O}_{(0,0)} - 4\mathcal{O}_{(1,0)} - 7\mathcal{O}_{(0,2)} - 2\mathcal{O}_{(2,0)} + 6\mathcal{O}_{(1,2)} - 3\mathcal{O}_{(3,0)} - 3\mathcal{O}_{(0,4)} + \mathcal{O}_{(2,2)}) \]

\[ K_{SU(2)/U(1)}^{-1} = \frac{1}{k(k+2)}(3\mathcal{O}_1 \otimes \mathcal{O}_1 - \mathcal{O}_3 \otimes \mathcal{O}_1) \]

Note that for low level \( k \) the operators in the above expressions are to be understood as being related to the integral representations modulo the action of the 'Weyl reflection about the \( k \)-line'\(^5\).

We have described in general for group and simple coset type RCFT's how formulae for \( K^{-1} \) of this type arise. The formulae have a natural interpretation in CS theory as a ratio of a group-theoretic part and a naive volume of moduli space.

There is an intriguing mathematical connection between these ideas in the RCFT context and \( N = 2 \) theories. Also it seems possible to explore properties of the space of blocks in higher genus with some of these techniques. Finally, these formulae may be useful for better understanding the classical \( (k \to \infty) \) limit of CS theory.

4. Acknowledgements

We wish to thank S. Axelrod, K. Bardakci, S. A. Hotes, H. J. Schnitzer and I. M. Singer for conversations, and the organizers of this conference. This work was supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under contract #DE-AC02-76ER03069, and by the Division of Applied Mathematics of the U.S. Department of Energy under contract #DE-FG02-88ER25066.

5. References

1. E. Verlinde, *Nucl. Phys.* **B300** (1988) 360.
2. H. Verlinde and E. Verlinde, ”Conformal Field Theory and Geometric Quantization,” published in *Trieste Superstrings* (1989), 422.
3. R. Bott, *Surveys in Diff. Geom.* 1 (1991) 1.
4. E. Witten, *Commun. Math. Phys.* **121** (1989) 351.
5. M. Crescimanno and S. A. Hotes, *Nucl. Phys.* **B372** (1992) 683.
6. M. Crescimanno, *Nucl. Phys.* **B393** (1993) 361.
7. M. Crescimanno, *Mod. Phys. Lett.* **A8** (1993) 1877.