Good Integers and some Applications in Coding Theory

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Abstract

A class of good integers has been introduced by P. Moree in 1997 together with the characterization of good odd integers. Such integers have shown to have nice number theoretical properties and wide applications. In this paper, a complete characterization of all good integers is given. Two subclasses of good integers are introduced, namely, oddly-good and evenly-good integers. The characterization and properties of good integers in these two subclasses are determined. As applications, good integers and oddly-good integers are applied in the study of the hulls of abelian codes. The average dimension of the hulls of abelian codes is given together with some upper and lower bounds.

Keywords: Good integers; Abelian codes; Hull of abelian codes; Euclidean inner product; Hermitian inner product.
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1 Introduction

The concept of good integers has been introduced in [13] by P. Moree. For fixed coprime nonzero integers $a$ and $b$, a positive integer $d$ is said to be good (with respect to $a$ and $b$) if it is a divisor of $a^k+b^k$ for some integer $k \geq 1$. Otherwise, $d$ is said to be bad. Denote by $G_{(a,b)}$ the set of good integers defined with respect to $a$ and $b$. The characterization of odd integers in $G_{(a,b)}$ has been given in [13]. In [9], the set $G_{(q,1)}$ has been studied and applied in constructing BCH codes with good design distances, where $q$ is a prime power. The set $G_{(2^r,1)}$ has been applied in counting the Euclidean self-dual cyclic and abelian codes over finite fields in [5] and [6], respectively.

In this paper, two subclasses of good integers are introduced as follows. For given nonzero coprime integers $a$ and $b$, a positive integer $d$ is said to be oddly-good (with respect to $a$ and $b$) if $d$ divides $a^k+b^k$ for some odd integer $k \geq 1$, and evenly-good (with respect to $a$ and $b$) if $d$ divides $a^k+b^k$ for some even integer $k \geq 2$. 

1.1 Oddly-good integers

1.2 Evenly-good integers
Therefore, \( d \) is good if it is oddly-good or evenly-good. Denote by \( OG_{(a,b)} \) (resp., \( EG_{(a,b)} \)) the set of oddly-good (resp., evenly-good) integers defined with respect to \( a \) and \( b \). Clearly, \( G_{(a,b)} = G_{(b,a)} \), \( OG_{(a,b)} = OG_{(b,a)} \) and \( EG_{(a,b)} = EG_{(b,a)} \). In [17], some basic properties of \( OG_{(2^\nu,1)} \) and \( EG_{(2^\nu,1)} \) have been studied and applied in enumerating Hermitian self-dual abelian codes over finite fields.

The hull of a linear code, the intersection of a code and its dual, is key to determining the complexity of algorithms in determining the automorphism group of a linear code and testing the permutation equivalence of two codes (see [16], [10], [11], [12], [18] and [19]). Precisely, most of the algorithms do not work if the size of the hull is large. In [17], the number of distinct linear codes of length \( n \) over \( \mathbb{F}_q \) which have hulls of a given dimension has been established. In [20], some additional properties of \( G_{(q,1)} \) have been studied and applied in the determination of the average dimension of the hulls of cyclic codes. Later, in [15], \( G_{(q,1)} \) has been applied in determining the dimensions of the hulls of cyclic and negacyclic codes over finite fields. To the best of our knowledge, properties of the hulls of abelian codes have not been well studied.

In this paper, we aim to characterize the classes of good integers, oddly-good integers and evenly-good integers defined with respect to arbitrary coprime nonzero integers \( a \) and \( b \). As applications, the hulls of abelian codes are studied using good and oddly-good integers. The average dimension of the hulls of abelian codes is determined under both the Euclidean and Hermitian inner products.

The paper is organized as follows. In Section 2, some properties of good odd integers are recalled and the characterization of all good integers is given. The characterizations of oddly-good and evenly-good integers are given in Section 3. In Section 4, applications of good integers in determining the average dimension of the hulls of abelian codes are given.

## 2 Good Integers

In this section, some basic properties of good odd integers in [13] are recalled and the characterization of arbitrary good integers is given.

For pairwise coprime nonzero integers \( a, b \) and \( n > 0 \), let \( \text{ord}_n(a) \) denote the multiplicative order of \( a \) modulo \( n \). In this case, \( b^{-1} \) exists in the multiplicative group \( \mathbb{Z}_n^\times \). Let \( \text{ord}_n\left(\frac{a}{b}\right) \) denote the multiplicative order of \( ab^{-1} \) modulo \( n \). Denote by \( 2^\gamma|n \) if \( \gamma \) is the largest integer such that \( 2^\gamma|n \).

From the definition of a good integer, \( 1 \) is always good and we have the following property.

**Lemma 2.1.** Let \( a \) and \( b \) be nonzero coprime integers and let \( d \) be a positive integer. If \( d \in G_{(a,b)} \), then \( \gcd(a,d) = 1 = \gcd(b,d) \).

**Proof.** Assume that \( d \in G_{(a,b)} \). Suppose that \( \gcd(a,d) \neq 1 \) or \( \gcd(b,d) \neq 1 \). Since \( G_{(a,b)} = G_{(b,a)} \), we may assume that \( p|\gcd(a,d) \) for some prime \( p \). Then there exists a positive integer \( k \) such that \( d|(a^k + b^k) \) which implies that \( p|b^k \). It follows that \( p|\gcd(a,b) \), a contradiction. \( \Box \)

Properties of good odd integers have been studied in [13]. Some results used in this paper are summarized as follows.
Lemma 2.2 ([13 Proposition 2]). Let $p$ be an odd prime and let $r$ be a positive integer. If $p^r$ is good, then $\text{ord}_p\left(\frac{a}{b}\right) = 2s$, where $s$ is the smallest positive integer such that $(ab^{-1})^s \equiv -1 (\text{mod } p^r)$.

Lemma 2.3 ([13 Proposition 4]). Let $p$ be an odd prime and let $r$ be a positive integer. Then $\text{ord}_p\left(\frac{a}{b}\right) = \text{ord}_p\left(\frac{a}{b}\right)p^i$ for some $i \geq 0$.

Lemma 2.4 ([13 Theorem 1]). Let $d > 1$ be an odd integer. Then $d \in G(ab)$ if and only if there exists $s \geq 1$ such that $2^s|\text{ord}_p\left(\frac{a}{b}\right)$ for every prime $p$ dividing $d$.

It has been briefly discussed in [13] that a good even integer exists if and only if $ab$ is odd. For completeness, the characterization and properties of all good (odd and even) integers are given as follows.

Proposition 2.5. Let $a$ and $b$ be nonzero coprime odd integers and let $\beta \geq 1$ be an integer. Then the following statements are equivalent.

i) $2^\beta|(a + b)$.

ii) $2^\beta \in G(ab)$.

iii) $\beta = 1$ or $\text{ord}_{2^\beta}\left(\frac{a}{b}\right) = 2$.

Proof. Clearly, i) $\Rightarrow$ ii) holds true.

To prove ii) $\Rightarrow$ iii), assume that $2^\beta \in G(ab)$. Then $2^\beta|(a^k + b^k)$ for some integer $k \geq 1$. Assume that $\beta > 1$. Then $4|(a^k + b^k)$. If $k$ is even, then $a^k \equiv 1 \text{ mod } 4$ and $b^k \equiv 1 \text{ mod } 4$ which implies that $(a^k + b^k) \equiv 2 \text{ mod } 4$, a contradiction. It follows that $k$ is odd. Since $a^k + b^k = (a + b)\left(\sum_{i=0}^{k-1} (-1)^i a^{k-1-i} b^i\right)$ and $k-1 \equiv -1 \text{ mod } 2^\beta$, we have that $2^\beta|(a + b)$. Hence, $ab^{-1} \equiv -1 \text{ mod } 2^\beta$ which implies that $\text{ord}_{2^\beta}\left(\frac{a}{b}\right) = 2$ as desired.

To prove iii) $\Rightarrow$ i), assume that $\beta = 1$ or $\text{ord}_{2^\beta}\left(\frac{a}{b}\right) = 2$.

Case 1 $\beta = 1$. Since $a + b$ is even, we have $2^\beta|(a + b)$.

Case 2 $\text{ord}_{2^\beta}\left(\frac{a}{b}\right) = 2$. We have $(ab^{-1})^2 \equiv 1 \text{ mod } 2^\beta$ and $ab^{-1} \not\equiv 1 \text{ mod } 2^\beta$. It follows that $ab^{-1} \equiv -1 \text{ mod } 2^\beta$. Hence, $2^\beta|(a + b)$ as desired. \hfill $\Box$

Proposition 2.6. Let $a$, $b$ and $d > 1$ be pairwise coprime odd integers. Then $d \in G(ab)$ if and only if $2d \in G(ab)$. In this case, $\text{ord}_{2^d}\left(\frac{a}{b}\right) = \text{ord}_d\left(\frac{a}{b}\right)$ is even.

Proof. Since $d$ is odd and $a^k + b^k$ is even for all positive integers $k$, $d|(a^k + b^k)$ if and only if $2d|(a^k + b^k)$. The characterization follows immediately.

Assume that $d \in G(ab)$. Let $k$ be the smallest positive integer such that $d|(a^k + b^k)$. It follows that $(ab^{-1})^k \equiv -1 \text{ mod } d$ and $\text{ord}_d\left(\frac{a}{b}\right) \nmid k$. Hence, $\text{ord}_d\left(\frac{a}{b}\right) = 2k$. Since $a^k + b^k$ is even and $d$ is odd, we have $2d|(a^k + b^k)$. Hence, $(ab^{-1})^k \equiv -1 \text{ mod } 2d$ which implies that $\text{ord}_{2d}\left(\frac{a}{b}\right)|2k$. Since $\text{ord}_d\left(\frac{a}{b}\right) \leq \text{ord}_{2d}\left(\frac{a}{b}\right)$, we have that $\text{ord}_{2d}\left(\frac{a}{b}\right) = \text{ord}_{2d}\left(\frac{a}{b}\right) = 2k$ is even. \hfill $\Box$

Proposition 2.7. Let $a, b$ and $d > 1$ be pairwise coprime odd positive integers and let $\beta \geq 2$ be an integer. Then $2^\beta d \in G(ab)$ if and only if $\text{ord}_{2^\beta}\left(\frac{a}{b}\right) = 2$ and $2||\text{ord}_d\left(\frac{a}{b}\right)$. In this case, $2||\text{ord}_{2^\beta d}\left(\frac{a}{b}\right)$.
Proof. Assume that $2^3d \in G_{(a,b)}$. Let $k$ be the smallest positive integer such that $2^3d|(a^k + b^k)$. Then $2^3|(a^k + b^k)$ and $d|(a^k + b^k)$. From the proof of Proposition 2.5, $\text{ord}_{2^3}(\frac{a}{b}) = 2$ and $k$ must be odd. It follows that $(ab^{-1})^{2k} \equiv 1 \mod d$ but $(ab^{-1})^{k} \equiv -1 \not\equiv 1 \mod d$. Hence, $\text{ord}_{d}(\frac{a}{b}) = 2k$ which means that $2|\text{ord}_{d}(\frac{a}{b})$.

Conversely, assume that $\text{ord}_{d}(\frac{a}{b}) = 2k$. Let $k$ be the odd positive integer $k$ such that $\text{ord}_{d}(\frac{a}{b}) = 2k$. Then $(ab^{-1})^{2k} \equiv -1 \mod d$ and $(ab^{-1})^{k} \equiv ab^{-1} \equiv -1 \mod 2^3$. Since $d$ is odd, $(ab^{-1})^{k} \equiv -1 \mod 2^3d$. Hence, $2^3d|(a^k + b^k)$ which means $2^3d \in G_{(a,b)}$ as desired.

In this case, the smallest positive integer $k$ such that $2^3d|(a^k + b^k)$ is odd. Then $(ab^{-1})^{k} \equiv -1 \mod 2^3d$, and hence, $\text{ord}_{2^3d}(\frac{a}{b}) = 2k$. Therefore, $2|\text{ord}_{2^3d}(\frac{a}{b})$. \qed

From Propositions 2.5, 2.7, the characterization of good integers can be summarized based on $\beta$ and the parity of $ab$ as follows.

**Theorem 2.8.** Let $a$ and $b$ be coprime nonzero integers and let $\ell = 2^3d$ be a positive integer such that $d$ is odd and $\beta \geq 0$. Then one of the following statements holds.

1. If $ab$ is odd, then $\ell = 2^3d \in G_{(a,b)}$ if and only if one of the following statements holds.
   
   (a) $\beta \in \{0, 1\}$ and $d = 1$.
   
   (b) $\beta \in \{0, 1\}$, $d \geq 3$ and there exists $s \geq 1$ such that $2^s|\text{ord}_p(\frac{a}{b})$ for every prime $p$ dividing $d$.
   
   (c) $\beta \geq 2$, $d = 1$ and $\text{ord}_{2^3}(\frac{a}{b}) = 2$.
   
   (d) $\beta \geq 2$, $d \geq 3$, $\text{ord}_{2^3}(\frac{a}{b}) = 2$ and $2|\text{ord}_{d}(\frac{a}{b})$.

2. If $ab$ is even, then $\ell = 2^3d \in G_{(a,b)}$ if and only if one of the following statements holds.
   
   (a) $\beta = 0$ and $d = 1$.
   
   (b) $\beta = 0$, $d \geq 3$, and there exists $s \geq 1$ such that $2^s|\text{ord}_p(\frac{a}{b})$ for every prime $p$ dividing $d$.

We note that the condition $\text{ord}_{2^3}(\frac{a}{b}) = 2$ in Theorem 2.8 is equivalent to $2^3|(a + b)$ by Proposition 2.5.

From Propositions 2.5, 2.7, and Theorem 2.8, other necessary conditions for a positive integer to be good can be given in the following corollary. These are sometime useful in applications.

**Corollary 2.9.** Let $a$ and $b$ be coprime nonzero integers and let $\ell = 2^3d$ be a positive integer such that $d$ is odd and $\beta \geq 0$. Let $\gamma \geq 0$ be an integer such that $2^\gamma|(a + b)$. If $\ell \in G_{(a,b)}$, then one of the following statements holds.

1. $\ell = 1$ and $ab$ is even.

2. $\ell \in \{1, 2\}$ and $ab$ is odd.

3. $2|\text{ord}_{\ell}(\frac{a}{b})$ if and only if one of the following statements holds.
(a) $d = 1$ and $2 \leq \beta \leq \gamma$.
(b) $d \geq 3$, $2|\text{ord}_p(\frac{a}{b})$ for every prime $p$ dividing $d$, and $0 \leq \beta \leq \gamma$.

4. There exists an integer $s \geq 2$ such that $2^s|\text{ord}_p(\frac{a}{b})$ if and only if $d \geq 3$, $2^s|\text{ord}_p(\frac{a}{b})$ for every prime $p$ dividing $d$, and $0 \leq \beta \leq 1$.

3 Oddly-Good and Evenly-Good Integers

For given pairwise coprime nonzero integers $a$, $b$ and $\ell > 0$, recall that $\ell$ is said to be oddly-good if $\ell$ divides $a^k + b^k$ for some odd integer $k \geq 1$, and evenly-good if $\ell$ divides $a^k + b^k$ for some even integer $k \geq 2$. In this section, the characterizations and properties of oddly-good and evenly-good integers are determined.

We note that 1 is always good. Since $1|(a + b)$ and $1|(a^2 + b^2)$, we have that 1 is both oddly-good and evenly-good. The integer 2 is good if and only if $ab$ is odd. In this case, $a + b$ is even and hence, $2|(a + b)$ and $2|(a^2 + b^2)$ which imply that 2 is both oddly-good and evenly-good. However, for an integer $\ell > 2$, $\ell$ can be either oddly-good or evenly-good, but not both.

**Proposition 3.1.** Let $a$, $b$ and $\ell > 2$ be pairwise coprime nonzero integers. If $\ell \notin G_{(a,b)}$, then either $\ell \in OG_{(a,b)}$ or $\ell \in EG_{(a,b)}$, but not both.

**Proof.** We distinguish the proof into four cases.

**Case 1** $\ell$ is odd. Assume that $s$ and $t$ are positive integers such that $s \geq t$, $\ell|(a^s+b^s)$, and $\ell|(a^t+b^t)$. It is sufficient to show that $s$ and $t$ have the same parity. Let $p$ be a prime divisor of $\ell$. Then $p|(a^s + b^s)$ and $p|(a^t + b^t)$, or equivalently, $a^s \equiv -b^s \mod p$ and $a^t \equiv -b^t \mod p$. These hold true if and only if $(ab^{-1})^s \equiv -1 \mod p$ and $(ab^{-1})^t \equiv -1 \mod p$. It follows that $(ab^{-1})^s - (ab^{-1})^t \equiv 0 \mod p$, and hence, $(ab^{-1})^t((ab^{-1})^{s-t} - 1) \equiv 0 \mod p$. This means $(ab^{-1})^{s-t} \equiv 1 \mod p$ which implies that $\text{ord}_p(\frac{a}{b}) | (s-t)$. Since $d$ is good, by Lemma 2.3, $\text{ord}_p(\frac{a}{b})$ is even which implies $s-t$ is even. Therefore, $s$ and $t$ have the same parity.

**Case 2** $\ell = 2^\beta$, where $\beta \geq 2$. In this case, $\ell$ is oddly-good by the proof of Proposition 2.5.

**Case 3** $\ell = 2d$, where $d$ is odd. By Proposition 2.6, $d \notin G_{(a,b)}$. Moreover, $d$ is either oddly-good or evenly-good by Case 1. For each positive integer $k$, $d|(a^k + b^k)$ if and only if $2d|(a^k + b^k)$. Therefore, $2d$ is oddly (resp., evenly) good if and only if $\ell$ is oddly (resp., evenly) good.

**Case 4** $\ell = 2^\beta d$, where $d$ is odd and $\beta \geq 2$. By the proof of Proposition 2.7, $\ell$ is oddly-good.

From Proposition 3.1 it follows that $G_{(a,b)} = OG_{(a,b)} \cup EG_{(a,b)}$. Moreover, we have $OG_{(a,b)} \cap EG_{(a,b)} = \{1\}$ if $ab$ is even and $OG_{(a,b)} \cap EG_{(a,b)} = \{1, 2\}$ if $ab$ is odd. Properties of $OG_{(a,b)}$ and $EG_{(a,b)}$ are determined in the next subsections.

3.1 Oddly-Good Integers

In this subsection, we focus on the characterization and properties of oddly-good integers.

The following lemma from [13] is useful.
Lemma 3.2 ([13, Lemma 1]). Let $a_1, a_2, \ldots, a_t$ be positive integers. Then the system of congruences
\[ x \equiv a_1 \pmod{2^{a_1}}, \quad x \equiv a_2 \pmod{2^{a_2}}, \quad \ldots, \quad x \equiv a_t \pmod{2^{a_t}} \]
has a solution $x$ if and only if there exists $s \geq 0$ such that $2^s \mid |a_i| \text{ for all } i = 1, 2, \ldots, t$.

Proposition 3.3. Let $a$ and $b$ be coprime nonzero integers and let $d > 1$ be an odd integer. Then $d \in OG_{(a,b)}$ if and only if $2 \mid \ord_p \left( \frac{a}{b} \right)$ for every prime $p$ dividing $d$.

Proof. Let $p_1, p_2, \ldots, p_t$ be the distinct prime divisors of $d$. For each $1 \leq i \leq t$, let $r_i$ be the positive integer such that $p_i^{r_i} \mid d$. To prove the necessity, assume that $d \in OG_{(a,b)}$. There exists an odd positive integer $c$ such that $\left( \frac{a}{b} \right)^c \equiv -1 \pmod{d}$ and, hence, $\left( \frac{a}{b} \right)^c \equiv -1 \pmod{p_i^{r_i}}$ for all $i = 1, 2, \ldots, t$. By Lemma 2.2, it follows that $\ord_{p_i^{r_i}} \left( \frac{a}{b} \right)$ is even and
\[ c \equiv \frac{\ord_{p_i^{r_i}} \left( \frac{a}{b} \right)}{2} \pmod{\ord_{p_i^{r_i}} \left( \frac{a}{b} \right)} \]
for all $i = 1, 2, \ldots, t$. Since $c$ is an odd integer, $\frac{\ord_{p_i^{r_i}} \left( \frac{a}{b} \right)}{2}$ is odd for all $i = 1, 2, \ldots, t$. Therefore, by Lemma 2.3, $2 \mid \ord_p \left( \frac{a}{b} \right)$ for every prime $p$ dividing $d$.

Conversely, assume that $2 \mid \ord_p \left( \frac{a}{b} \right)$ for every prime $p$ dividing $d$. Then, by Lemma 2.3 we have $2 \mid \ord_{p_i^{r_i}} \left( \frac{a}{b} \right)$ for all $i = 1, 2, \ldots, t$. By Lemma 3.2 there exists an integer $c$ such that
\[ c \equiv \frac{\ord_{p_i^{r_i}} \left( \frac{a}{b} \right)}{2} \pmod{\ord_{p_i^{r_i}} \left( \frac{a}{b} \right)} \]
for all $i = 1, 2, \ldots, t$. Since $\frac{\ord_{p_i^{r_i}} \left( \frac{a}{b} \right)}{2}$ is odd, $c$ is an odd integer. Thus $\left( \frac{a}{b} \right)^c \equiv -1 \pmod{p_i^{r_i}}$ for all $i = 1, 2, \ldots, t$, and hence, $\left( \frac{a}{b} \right)^c \equiv -1 \pmod{d}$. Therefore, $d$ is oddly-good as desired. \qed

From Proposition 3.3 we have the following characterization of oddly-good integers.

Corollary 3.4. Let $a$ and $b$ be coprime nonzero integers and let $d > 1$ be an odd integer. Then the following statements are equivalent.

1. $d \in OG_{(a,b)}$.
2. $j \in OG_{(a,b)}$ for all divisors $j$ of $d$.
3. $p \in OG_{(a,b)}$ for all prime divisors $p$ of $d$.

Proof. The statements (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious. By Proposition 3.3, the statement (3) $\Rightarrow$ (1) follows. \qed

From the proof of Propositions 2.6, 2.7 and 3.1, we have the following characterizations.
Corollary 3.5. Let $a$ and $b$ be coprime nonzero integers and let $d > 1$ be an odd integer.

1. The following statements are equivalent.
   \begin{enumerate}
   \item[(a)] $d \in OG_{(a,b)}$.
   \item[(b)] $2d \in OG_{(a,b)}$.
   \end{enumerate}

2. For each $\beta \geq 2$, $2^\beta d \in OG_{(a,b)}$ if and only if $2^\beta d \in G_{(a,b)}$.

The characterization of oddly-good integers can be summarized in the following theorem.

Theorem 3.6. Let $a$ and $b$ be coprime nonzero integers and let $\ell = 2^\beta d$ be an integer such that $d$ is odd and $\beta \geq 0$. Then one of the following statements holds.

1. If $ab$ is odd, then $\ell = 2^\beta d \in OG_{(a,b)}$ if and only if one of the following statements holds.
   \begin{enumerate}
   \item[(a)] $\beta \in \{0, 1\}$ and $d = 1$.
   \item[(b)] $\beta \in \{0, 1\}$, $d \geq 3$, and $2|\ord_p(\frac{a}{b})$ for every prime $p$ dividing $d$.
   \item[(c)] $\beta \geq 2$, $d = 1$ and $\ord_{2^\beta}(\frac{a}{b}) = 2$.
   \item[(d)] $\beta \geq 2$, $d \geq 3$, $\ord_{2^\beta}(\frac{a}{b}) = 2$ and $2|\ord_d(\frac{a}{b})$.
   \end{enumerate}

2. If $ab$ is even, then $\ell = 2^\beta d \in OG_{(a,b)}$ if and only if one of the following statements holds.
   \begin{enumerate}
   \item[(a)] $\beta = 0$ and $d = 1$.
   \item[(b)] $\beta = 0$, $d \geq 3$, and $2|\ord_p(\frac{a}{b})$ for every prime $p$ dividing $d$.
   \end{enumerate}

Note that the condition $\ord_{2^\beta}(\frac{a}{b}) = 2$ in Theorem 3.6 is equivalent to $2^\beta|(a+b)$ by Proposition 2.5.

The following necessary conditions for a positive integer to be oddly-good can be concluded from Proposition 3.3 and Theorem 3.6.

Corollary 3.7. Let $a$ and $b$ be coprime nonzero integers and let $\ell = 2^\beta d$ be a positive integer such that $d$ is odd and $\beta \geq 0$. Let $\gamma \geq 0$ be an integer such that $2^\gamma|\gamma|(a+b)$. If $\ell \in OG_{(a,b)}$, then one of the following statements holds.

1. $\ell = 1$ and $ab$ is even.

2. $\ell \in \{1, 2\}$ and $ab$ is odd.

3. $2|\ord_\ell(\frac{a}{b})$ if and only if one of the following statements holds.
   \begin{enumerate}
   \item[(a)] $d = 1$ and $2 \leq \beta \leq \gamma$.
   \item[(b)] $d \geq 3$, $2|\ord_p(\frac{a}{b})$ for every prime $p$ dividing $d$, and $0 \leq \beta \leq \gamma$.
   \end{enumerate}
3.2 Evenly-Good Integers

In this subsection, we focus on properties of evenly-good integers. From Proposition 3.1, an integer greater than 2 can be either oddly-good or evenly-good. Then, by Lemma 2.4 and Proposition 3.3, we have the following characterization of evenly-good integers.

**Proposition 3.8.** Let \(a\) and \(b\) be coprime nonzero integers and let \(d > 1\) be an odd integer. Then \(d \in \text{EG}_{(a,b)}\) if and only if there exists \(s \geq 2\) such that \(2^s \| \text{ord}_p(a)\) for every prime \(p\) dividing \(d\).

**Corollary 3.9.** Let \(a\) and \(b\) be coprime nonzero integers and let \(d > 1\) be an odd integer. If \(d \in \text{EG}_{(a,b)}\) (resp. \(d \in \text{G}_{(a,b)}\)), then \(j \in \text{EG}_{(a,b)}\) (resp. \(j \in \text{G}_{(a,b)}\)) for all divisors \(j\) of \(d\).

The above properties of evenly-good integers can be summarized as follows.

**Theorem 3.10.** Let \(a\) and \(b\) be coprime nonzero integers and let \(\ell = 2^\beta d\) be an integer such that \(d\) is odd and \(\beta \geq 0\). Then one of the following statements holds.

1. If \(ab\) is odd, then \(\ell = 2^\beta d \in \text{EG}_{(a,b)}\) if and only if one of the following statements holds.
   
   (a) \(\beta \in \{0, 1\}\) and \(d = 1\).
   
   (b) \(\beta \in \{0, 1\}, d \geq 3\), and there exists \(s \geq 2\) such that \(2^s \| \text{ord}_p(a)\) for every prime \(p\) dividing \(d\).

2. If \(ab\) is even, then \(\ell = 2^\beta d \in \text{EG}_{(a,b)}\) if and only if one of the following statements holds.
   
   (a) \(\beta = 0\) and \(d = 1\).
   
   (b) \(\beta = 0, d \geq 3\), and there exists \(s \geq 2\) such that \(2^s \| \text{ord}_p(a)\) for every prime \(p\) dividing \(d\).

The following necessary conditions for a positive integer to be evenly-good integers can be obtained from Proposition 3.8 and Theorem 3.10.

**Corollary 3.11.** Let \(a\) and \(b\) be coprime nonzero integers and let \(\ell = 2^\beta d\) be a positive integer such that \(d\) is odd and \(\beta \geq 0\). Let \(\gamma \geq 0\) be an integer such that \(2^\gamma \| (a + b)\). If \(\ell \in \text{EG}_{(a,b)}\), then one of the following statements holds.

1. \(\ell = 1\) and \(ab\) is even.

2. \(\ell \in \{1, 2\}\) and \(ab\) is odd.

3. There exists an integer \(s \geq 2\) such that \(2^s \| \text{ord}_p(a)\) if and only if \(d \geq 3, 2^s \| \text{ord}_p(a)\) for every prime \(p\) dividing \(d\), and \(0 \leq \beta \leq 1\).
4 Applications

This section is devoted to applications of good integers in coding theory. As discussed in the introduction, good and oddly-good integers have wide applications in coding theory. The important ones are the construction of good BCH codes in [3], the enumerations of the Euclidean self-dual cyclic and abelian codes in [5] and [6], the enumeration of Hermitian self-dual abelian codes in [7], the study of the average dimension of the hulls of cyclic codes in [20], and the determinations of the dimension of the hulls of cyclic and negacyclic codes in [15].

The hull of a linear code is known to be key in determining the complexity of algorithms in finding the automorphism group of a linear code and testing the permutation equivalence of two codes [16], [10], [11], [12], [18], and [19]. Therefore, the study of the hulls is one of the interesting problems in coding theory. Some properties of the hulls of linear and cyclic codes have been further studied in [17], [20] and [15].

In this section, we focus on applications of \(G_{(p,\nu)}\) and \(OG_{(p,\nu)}\) in determining the average dimension of the hulls of abelian codes under both the Euclidean and Hermitian inner products which have not been well studied. Specifically, we are mainly focused on the hulls of abelian codes in principal ideal group algebras. As a special case, the results on the hulls of cyclic codes in [20] can be viewed as corollaries.

4.1 Abelian Codes in Principal Ideal Group Algebras

For a prime \(p\) and a positive integer \(\nu\), let \(F_{p^\nu}\) denote the finite field of \(p^\nu\). Let \(G\) be a finite abelian group, written additively. Denote by \(F_{p^\nu}[G]\) the group algebra of \(G\) over \(F_{p^\nu}\). The elements in \(F_{p^\nu}[G]\) will be written as \(\sum_{g \in G} \alpha_g Y^g\), where \(\alpha_g \in F_{p^\nu}\). The addition and the multiplication in \(F_{p^\nu}[G]\) are given as in the usual polynomial rings over \(F_{p^\nu}\) with the indeterminate \(Y\), where the indices are computed additively in \(G\). A group algebra \(F_{p^\nu}[G]\) is said to be a principal ideal group algebra (PIGA) if every ideal in \(F_{p^\nu}[G]\) is generated by a single element. In [3], it has been shown that \(F_{p^\nu}[G]\) is a PIGA if and only if the Sylow \(p\)-subgroup of \(G\) is cyclic.

An abelian code in \(F_{p^\nu}[G]\) is an ideal in \(F_{p^\nu}[G]\) (see [6] and [7]). The Euclidean inner product between \(u = \sum_{g \in G} u_g Y^g\) and \(v = \sum_{g \in G} v_g Y^g\) \(F_{p^\nu}[G]\) is defined to be \(\langle u, v \rangle_E := \sum_{g \in G} u_g v_g\). The Euclidean dual of an abelian code \(C\) in \(F_{p^\nu}[G]\) is defined to be \(C_E := \{v \in F_{p^\nu}[G] \mid \langle c, v \rangle_E = 0 \text{ for all } c \in C\}\). The Euclidean hull of a code \(C\) is defined to be \(H_E(C) := C \cap C_E\). In \(F_{p^\nu}[G]\), the Hermitian inner product between \(u = \sum_{g \in G} u_g Y^g\) and \(v = \sum_{g \in G} v_g Y^g\) \(F_{p^\nu}[G]\) is defined to be \(\langle u, v \rangle_H := \sum_{g \in G} u_g \bar{v}_g\). The Hermitian dual of an abelian code \(C\) in \(F_{p^\nu}[G]\) is defined to be \(C_H := \{v \in F_{p^\nu}[G] \mid \langle c, v \rangle_H = 0 \text{ for all } c \in C\}\). The Hermitian hull of a code \(C\) is defined to be \(H_H(C) := C \cap C_H\).

The average dimension of the Euclidean (resp. Hermitian) hulls of abelian codes in \(F_{p^\nu}[G]\) (resp., in \(F_{p^\nu}[G]\)) is defined to be

\[
\text{avg}^E_{p^\nu}(G) := \frac{\sum_{C \in C(p, G)} \dim(H_E(C))}{|C(p, G)|} \quad \text{(resp., avg}^H_{p^\nu}(G) := \frac{\sum_{C \in C(p, G)} \dim(H_H(C))}{|C(p, G)|}),
\]
where \( C(p^r, G) \) (resp., in \( C(p^{2r}, G) \)) is the set of all abelian codes in \( \mathbb{F}_{p^r}[G] \) (resp., in \( \mathbb{F}_{p^{2r}}[G] \)).

The rest of this section, we focus on abelian codes in PIGAs. It therefore suffices to restrict the study to a group algebra \( \mathbb{F}_{p^r}[A \times \mathbb{Z}_{p^k}] \), where \( A \) is a finite abelian group such that \( p \nmid \vert A \vert \) and \( k \geq 0 \) is an integer.

For positive integers \( i \) and \( j \) with \( \gcd(i, j) = 1 \), let \( \text{ord}_j(i) \) denote the multiplicative order of \( i \) modulo \( j \). For each \( a \in A \), denote by \( \text{ord}(a) \) the additive order of \( a \) in \( A \). For a positive integer \( q \) with \( \gcd(\vert A \vert, q) = 1 \), a \( q \)-cyclotomic class of \( A \) containing \( a \in A \), denoted by \( S_q(a) \), is defined to be the set

\[
S_q(a) := \{q^i \cdot a \mid i = 0, 1, \ldots\} = \{q^i \cdot a \mid 0 \leq i < \text{ord}(a)\},
\]

where \( q^i \cdot a := \sum_{j=1}^{q^i} a \) in \( A \).

First, we consider the decomposition of \( \mathcal{R} := \mathbb{F}_{p^r}[A] \). In this case, \( \mathcal{R} \) is semisimple (see [2]) which can be decomposed using the Discrete Fourier Transform in [14] (see [7] and [6] for more details). For completeness, the decomposition used in this paper is summarized as follows.

An idempotent in \( \mathcal{R} \) is a nonzero element \( e \) such that \( e^2 = e \). It is called primitive if for every other idempotent \( f \), either \( ef = e \) or \( ef = 0 \). The primitive idempotents in \( \mathcal{R} \) are induced by the \( p^r \)-cyclotomic classes of \( A \) (see [3, Proposition II.4]). Let \( \{a_1, a_2, \ldots, a_t\} \) be a complete set of representatives of \( p^r \)-cyclotomic classes of \( A \) and let \( e_i \) be the primitive idempotent induced by \( S_{p^r}(a_i) \) for all \( 1 \leq i \leq t \). From [14], \( \mathcal{R} \) can be decomposed as

\[
\mathcal{R} = \bigoplus_{i=1}^{t} \mathcal{R} e_i. \tag{4.1}
\]

It is well known (see [3] and [7]) that \( \mathcal{R} e_i \cong \mathbb{F}_{p^{r s_i}} \), where \( s_i = \vert S_{p^r}(a_i) \vert \) provided that \( e_i \) is induced by \( S_{p^r}(a_i) \), and hence,

\[
\mathbb{F}_{p^r}[A \times \mathbb{Z}_{p^k}] \cong \bigoplus_{i=1}^{t} (\mathcal{R} e_i) [\mathbb{Z}_{p^k}] \cong \prod_{i=1}^{t} \mathbb{F}_{p^{r s_i}}[\mathbb{Z}_{p^k}] \cong \prod_{i=1}^{t} \mathbb{F}_{p^{r s_i}}[x]/(x^{p^k} - 1). \tag{4.2}
\]

Therefore, every abelian code \( C \) in \( \mathbb{F}_{p^r}[A \times \mathbb{Z}_{p^k}] \) can be viewed as

\[
C \cong \prod_{i=1}^{t} C_i, \tag{4.3}
\]

where \( C_i \) is a cyclic code of length \( p^k \) over \( \mathbb{F}_{p^{r s_i}} \) for all \( i = 1, 2, \ldots, t \).

**Remark 1.** It is well known that every cyclic code \( D \) of length \( p^k \) over \( \mathbb{F}_{p^r} \) is uniquely generated as ideal in \( \mathbb{F}_{p^r}[x]/(x^{p^k} - 1) \) by \( (x - 1)^j \) for some \( 0 \leq j \leq p^k \). Note that the such code has \( \mathbb{F}_{p^r} \)-dimension \( p^k - j \). The Euclidean and Hermitian duals of \( D \) are of the same form \( D^{\perp_E} = D^{\perp_H} \) generated by \( (x - 1)^{p^k-j} \).

In order to study properties of the hulls abelian codes in PIGAs under the Euclidean and Hermitian inner products, two rearrangements of the ideas \( \mathcal{R} e_i \) in the decomposition [12] will be discussed in the following subsections.
4.2 The Average Dimension of the Euclidean Hull of Abelian Codes in PIGAs

In this section, we focus on an application of good integers in determining of the average dimension of the Euclidean hulls of abelian codes in \( \mathbb{F}_{p^\nu}[A \times \mathbb{Z}_{p^k}] \), where \( \nu > 0 \) and \( k \geq 0 \) are integers and \( p \nmid |A| \).

A \( p^\nu \)-cyclotomic class \( S_{p^\nu}(a) \) is said to be of type I if \( S_{p^\nu}(a) = S_{p^\nu}(-a) \), or type II if \( S_{p^\nu}(-a) \neq S_{p^\nu}(a) \).

Without loss of generality, the representatives \( a_1, a_2, \ldots, a_t \) of \( p^\nu \)-cyclotomic classes of \( A \) can be chosen such that \( \{a_j | j = 1, 2, \ldots, r_1 \} \) and \( \{a_{r_1+i}, a_{r_1+rg+t} = -a_{r_1+t} | l = 1, 2, \ldots, r_\# \} \) are sets of representatives of \( p^\nu \)-cyclotomic classes of \( A \) of types I and II, respectively, where \( t = r_1 + 2r_\# \). As assumed above, \( s_i = |S_{p^\nu}(a_i)| \) for all \( 1 \leq i \leq t \). Clearly, \( s_{r_1+t} = s_{r_1+rg+t} \) for all \( 1 \leq l \leq r_\# \).

Rearranging the terms in the decomposition of \( \mathcal{R} \) (see (4.1)) based on these 2 types of cyclotomic classes in [6], we have

\[
\mathbb{F}_{p^\nu}[A \times \mathbb{Z}_{p^k}] \cong \left( \prod_{j=1}^{r_1} \mathbb{K}_j[\mathbb{Z}_{p^k}] \right) \times \left( \prod_{l=1}^{r_\#} \left( \mathbb{L}_l[\mathbb{Z}_{p^k}] \times \mathbb{L}_l[\mathbb{Z}_{p^k}] \right) \right),
\]

(4.4)

where \( \mathbb{K}_j \cong \mathbb{F}_{p^{\nu_j}} \) for all \( j = 1, 2, \ldots, r_1 \) and \( \mathbb{L}_l \cong \mathbb{F}_{p^{\nu_{r_1+l}}} \) for all \( l = 1, 2, \ldots, r_\# \).

From (4.4), it follows that an abelian code \( C \) in \( \mathbb{F}_{p^\nu}[A \times \mathbb{Z}_{p^k}] \) can be viewed as

\[
C \cong \left( \prod_{j=1}^{r_1} C_j \right) \times \left( \prod_{l=1}^{r_\#} (D_l \times D'_l) \right),
\]

(4.5)

where \( C_j, D_s \) and \( D'_s \) are cyclic codes in \( \mathbb{K}_j[\mathbb{Z}_{p^k}], \mathbb{L}_l[\mathbb{Z}_{p^k}] \) and \( \mathbb{L}_l[\mathbb{Z}_{p^k}] \), respectively, for all \( j = 1, 2, \ldots, r_1 \) and \( l = 1, 2, \ldots, r_\# \).

From [6] Section II.D and Remark 1 the Euclidean dual of \( C \) in (4.5) is of the form

\[
C^\perp_E \cong \left( \prod_{j=1}^{r_1} C^\perp_{j,E} \right) \times \left( \prod_{l=1}^{r_\#} (D'_l)^\perp \times (D'_l)^\perp \right),
\]

(4.6)

**Lemma 4.1.** There is a one-to-one correspondence between \( C(p^\nu, A \times \mathbb{Z}_{p^k}) \) and \( \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_t) | 0 \leq \epsilon_i \leq p^k \text{ for all } 1 \leq i \leq t\} \), where \( t = r_1 + 2r_\# \).

**Proof.** From (4.5) and the discussion above, it is not difficult to see that the map

\[
C \mapsto (\dim_{\mathbb{F}_{p^{\nu_1}}}(C_1), \ldots, \dim_{\mathbb{F}_{p^{\nu_{r_1}}}}(C_{r_1}),
\dim_{\mathbb{F}_{p^{\nu_{r_1+1}}}}(D_1), \ldots, \dim_{\mathbb{F}_{p^{\nu_{r_1+rg}}}}(D_{r_\#}), \dim_{\mathbb{F}_{p^{\nu_{r_1+1+r_\#}}}}(D'_1), \ldots, \dim_{\mathbb{F}_{p^{\nu_{r_1+rg+1}}}}(D'_{r_\#}))
\]

is a one-to-one correspondence from \( C(p^\nu, A \times \mathbb{Z}_{p^k}) \) to \( \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_t) | 0 \leq \epsilon_i \leq p^k \text{ for all } 1 \leq i \leq t\} \).

Form the above discussion, Remark 1 and Lemma 4.1 the next lemma follows.
Lemma 4.2. Let $C$ be an abelian code in $\mathbb{F}_{p^\nu}[A \times \mathbb{Z}_{p^k}]$ decomposed as in \((4.5)\). Then the following statements hold.

1. The Euclidean hull of $C$ is
   \[
   \mathcal{H}_E(C) \cong \left( \prod_{j=1}^{r_1} (C_j \cap C_j^{\perp_E}) \right) \times \left( \prod_{l=1}^{r_2} ((D_l \cap (D_l')^{\perp_E}) \times (D_l' \cap D_l^{\perp_E})) \right).
   \]

2. If $C$ corresponds to $(\epsilon_1, \epsilon_2, \ldots, \epsilon_t)$ with $t = r_1 + 2r_2$ as in Lemma 4.1, then
   \[
   \dim(\mathcal{H}_E(C)) = \sum_{j=1}^{r_1} s_j \min(\epsilon_j, p^k - \epsilon_j) + \sum_{i=1}^{r_2} s_{r_1+i} \min(\epsilon_{r_1+i}, p^k - \epsilon_{r_1+i})
   \]
   \[
   + \sum_{l=1}^{r_2} s_{r_1+l} \min(\epsilon_{r_1+r_2+l}, p^k - \epsilon_{r_1+l}).
   \]

Let $Q_{p^\nu}(A) = \{ a \in A \mid -a \in S_{p^\nu}(a) \}$. It is not difficult to see that $Q_{p^\nu}(A)$ is the union of all $p^\nu$-cyclo
tomic classes of $A$ of types I and $|Q_{p^\nu}(A)| = \sum_{j=1}^{r_1} s_j$, where $s_j = |S_{p^\nu}(a_j)|$ for all $1 \leq j \leq r_1$.

The average dimension of the Euclidean hull of abelian codes in $\mathbb{F}_{p^\nu}[A \times \mathbb{Z}_{p^k}]$ is determined using the technique in [20] as follows.

Theorem 4.3. Let $p$ be a prime and let $\nu$ be a positive integer. Let $A$ be a finite abelian group of order $m$ such that $p \nmid m$. Then

\[
\text{avg}_{p^\nu}^E(A \times \mathbb{Z}_{p^k}) = mp^k \left( \frac{1}{3} - \frac{1}{6(p^k+1)} \right) - |Q_{p^\nu}(A)| \left( \frac{p^k+1}{12} + \frac{2-3\delta_p}{12(p^k+1)} \right),
\]

where

$\delta_p = \begin{cases} 
1 & \text{if } p = 2, \\
0 & \text{if } p \text{ is odd}.
\end{cases}$

Proof. Let $X$ be the random variable of the dimension $\dim(\mathcal{H}_E(C))$, where $C$ is chosen randomly from $\mathcal{C}(p^\nu, A \times \mathbb{Z}_{p^k})$ with uniform probability. Then $\text{avg}_{p^\nu}^E(A \times \mathbb{Z}_{p^k})$ is the expectation $E(X)$ of $X$. Using Lemma 4.2 and the arguments similar to those in the proof of [20, Proposition 22], $E(X)$ can be determined. Hence, the formula for $\text{avg}_{p^\nu}^E(A \times \mathbb{Z}_{p^k})$ follows. \qed

We not that if $A$ is the cyclic group of order $m$, then the average dimension of the hulls of cyclic codes of length $mp^k$ can be obtained as a special case of Theorem 4.3.

Since $0 \in Q_{p^\nu}(A)$, we have $|Q_{p^\nu}(A)| \geq 1$. Hence, by Theorem 4.3, the next corollary follows.

Corollary 4.4. Let $p$ be a prime and let $\nu$ be a positive integer. Let $A$ be a finite abelian group of order $m$ such that $p \nmid m$. Then the following statements hold.

1. $\text{avg}_{p^\nu}^E(A \times \mathbb{Z}_{p^k}) < \frac{mp^k}{3}$. 

2. \( \text{avg}_{E_{p^\nu}}^E(A) = \frac{m - |Q_{p^\nu}(A)|}{4} \)

3. \( \text{avg}_{E_{p^\nu}}^E(A) \leq \frac{m - 1}{4} \).

Let \( \chi \) be a function defined by

\[
\chi(d, p^\nu) = \begin{cases} 
1 & \text{if } d \in G_{(p^\nu, 1)}, \\
0 & \text{otherwise}.
\end{cases}
\]

Good integers play an important role in determining \( \text{avg}_{E_{p^\nu}}^E(A \times \mathbb{Z}_{p^k}) \) in the following results.

**Lemma 4.5** (Lemma 4.5). Let \( p \) be a prime and let \( \nu \) be a positive integer. Let \( A \) be a finite abelian group such that \( p \nmid |A| \) and let \( a \in A \). Then \( S_{p^\nu}(a) \) is of type I if and only if \( \text{ord}(a) \in G_{(p^\nu, 1)} \).

**Lemma 4.6.** Let \( p \) be a prime and let \( \nu \) be a positive integer. Let \( A \) be a finite abelian group of order \( m \) and exponent \( M \) with \( p \nmid M \). Then

\[
|Q_{p^\nu}(A)| = \prod_{d \mid M} \chi(d, p^\nu)N_A(d),
\]

where \( N_A(d) \) is the number of elements of order \( d \) in \( A \) determined in [7].

In particular, \( |Q_{p^\nu}(A)| = m \) if and only if \( m \in G_{(p^\nu, 1)} \).

**Proof.** The statements follow immediately from Lemma 4.5.

For each integer \( \alpha \geq 0 \), let \( P_{p^\nu, \alpha} \) denote the set of primes \( p \) such that \( 2^\alpha || \text{ord}_p(q) \). For a subset \( T \) of \( \mathbb{N} \), denote by \( \langle\langle T \rangle\rangle \) the multiplicative semigroup generated by \( T \).

For each positive integer \( m \), we have the following presentation

\[
m = 2^\beta m_0 m_1 m_2 m_3 \ldots,
\]

where \( m_\alpha \in \langle\langle P_{p^\nu, \alpha} \rangle\rangle \) for all \( \alpha \geq 0 \).

**Proposition 4.7.** Let \( p \) be a prime and let \( \nu \) be a positive integer. Let \( A \) be a finite abelian group of order \( m = 2^\beta m_0 m_1 m_2 m_3 \ldots \) and exponent \( M = 2^B M_0 M_1 M_2 M_3 \ldots \), where \( m_\alpha \) and \( M_\alpha \) are in \( \langle\langle P_{p^\nu, \alpha} \rangle\rangle \) for all \( \alpha \geq 0 \). Let \( \gamma \geq 0 \) be an integer such that \( 2^\gamma || (p^\nu + 1) \). Then

\[
|Q_{p^\nu}(A)| = m_1 \sum_{i=0}^{\gamma} N_A(2^i) + (1 + N_A(2))^{\min\{1, \beta\}} \sum_{\alpha \geq 2} (m_\alpha - 1). \quad (4.7)
\]

(Note that \( N_A(2^i) \) will be regarded as 0 if \( i > B \).)

**Proof.** Observe that \( M' | m' \) and \( M_\alpha | m_\alpha \) for all \( \alpha \geq 0 \) and \( \beta \geq B \).

**Case 1** \( B = 0 \). Then \( \beta = 0 \). By Lemma 4.6, we have
\[ |Q_{p''}(A)| = \sum_{d \mid M} \chi(d, p'')N_A(d) \]
\[ = \sum_{d \mid M, d \in G(p'', 1)} N_A(d) \]
\[ = N_A(1) + \sum_{\alpha \geq 1} \sum_{d \mid M, d > 1} N_A(d) \]
\[ = N_A(1) + \sum_{\alpha \geq 1} (m_\alpha - 1) \]
\[ = m_1 + \sum_{\alpha \geq 2} (m_\alpha - 1). \]

**Case 2** \( B \neq 0 \). Then \( \beta \neq 0 \). By Lemma 4.6 and Corollary 2.9, we have
\[ |Q_{p''}(A)| = \sum_{d \mid M} \chi(d, p'')N_A(d) \]
\[ = \sum_{d \mid M, d \in G(p'', 1)} N_A(d) \]
\[ = N_A(1) + N_A(2) + \sum_{d \mid 2^{\min\{\beta, \gamma\}} M_1, d \notin \{1, 2\}} N_A(d) \]
\[ + \sum_{\alpha \geq 2} \sum_{d \mid 2^{\min\{\beta, \gamma\}} M_1, d \notin \{1, 2\}} N_A(d) \]
\[ = 1 + N_A(2) + \left( \min\{2^{\beta}, \sum_{i=0}^{\gamma} N_A(2^i)\} m_1 - 1 - N_A(2) \right) \]
\[ + \sum_{\alpha \geq 2} ((1 + N_A(2)) m_\alpha - 1 - N_A(2)) \]
\[ = m_1 \sum_{i=0}^{\gamma} N_A(2^i) + \sum_{\alpha \geq 2} ((1 + N_A(2)) m_\alpha - 1 - N_A(2)) \]
\[ = m_1 \sum_{i=0}^{\gamma} N_A(2^i) + (1 + N_A(2)) \sum_{\alpha \geq 2} (m_\alpha - 1), \]

Combining the two cases, we conclude the proposition.

\[ \square \]

**Remark 2.** From Proposition 4.7, we have the following observations.

1. If \( A = \mathbb{Z}_{2^\beta} \times H \) is a group of order \( m \), then
\[ \sum_{i=0}^{\gamma} N_A(2^i) = \begin{cases} 2^\beta & \text{if } \beta \leq \gamma \\ 2^\gamma & \text{if } \gamma \leq \beta \end{cases} \]
\[ = 2^{\min\{\gamma, \beta\}}, \]

which coincides with [20, Proposition 24].
2. If $A = (Z_2)^\beta \times H$ is a group of order $m$, then

$$\sum_{i=0}^{\gamma} N_A(2^i) = \begin{cases} 2^\beta & \text{if } \gamma \geq 1 \\ 1 & \text{if } \gamma = 0 \end{cases} = 2^{\min\{\beta, \gamma\}}.$$ 

For a prime $q$, let $A_1$ and $A_2$ be a finite abelian $q$-groups of the same order $q^r$ with primary decompositions $A_1 = \prod_{i=1}^{s} Z_{q^{a_i}}$ and $A_2 = \prod_{i=1}^{t} Z_{q^{b_i}}$. The group $A_1$ is said to be finer than $A_2$, denoted by $A_1 \preceq A_2$, if there exist a sequence $s_0 = 0 < s_1 < s_2 < \cdots < s_t = t$ and a permutation $\rho$ on $\{1, 2, \ldots, s\}$ such that $(\sum_{j=s_{i-1}+1}^{s_i} \rho(a_j)) = b_i$ for all $i = 1, 2, \ldots, t$. Note that $Z_{q^r} \preceq A \preceq Z_{q^r}$ for all $q$-groups $A$ of size $q^r$.

Using this concept, we have a sufficient condition to compare the values $|Q_{p^\nu}(A)|$ for some finite abelian groups $A$ of the same size in terms of their Sylow 2-subgroups.

**Lemma 4.8.** Let $p$ be a prime and let $\nu$ be a positive integer. Let $A$ and $B$ be finite abelian groups of the same order $m$ with $p \nmid m$. If the Sylow 2-subgroup of $A$ is finer than the Sylow 2-subgroup of $B$, then

$$|Q_{p^\nu}(A)| \geq |Q_{p^\nu}(B)|.$$ 

In particular, if the 2-subgroup of $A$ and the 2-subgroup of $B$ are isomorphic, then $|Q_{p^\nu}(A)| = |Q_{p^\nu}(B)|$.

**Proof.** From Proposition 4.7 it suffices to show that

$$\sum_{i=0}^{\gamma} N_A(2^i) \geq \sum_{i=0}^{\gamma} N_B(2^i) \tag{4.8}$$

and

$$(1 + N_A(2))^{\min\{1, \beta\}} \geq (1 + N_B(2))^{\min\{1, \beta\}}. \tag{4.9}$$

Let $B_A$ be the exponent of the Sylow 2-subgroup of $A$. Then $B_A$ is less than or equal to the exponent of the Sylow 2-subgroup of $B$. Hence, $N_A(2^i) \geq N_B(2^i)$ for all $i = 0, 1, \ldots, B_A$. If $\gamma \leq B_A$, we are done. Assume that $B_A < \gamma$. Then

$$\sum_{i=0}^{\gamma} N_A(2^i) = \sum_{i=0}^{B_A} N_A(2^i) \geq \sum_{i=0}^{\gamma} N_B(2^i).$$

Therefore, the results follow. $\square$
Corollary 4.9. Let $p$ be a prime and let $\nu$ be a positive integer. Let $A$ be a finite abelian group of order $m = 2^\beta m_0 m_1 m_2 m_3 \ldots$, where $m_\alpha \in \langle \langle p^\nu, \alpha \rangle \rangle$ for all $\alpha \geq 0$. Let $\gamma \geq 0$ be an integer such that $2^\gamma | (p^\nu + 1)$. Then we have

\[ 2^{\min\{\beta, \gamma\}} m_1 + 2^{\min\{1, \beta\}} \sum_{\alpha \geq 2} (m_\alpha - 1) \leq |Q_{p^\nu}(A)| \leq 2^{\min\{\beta, \gamma, \beta\}} m_1 + 2^{\min\{1, \beta\}} \sum_{\alpha \geq 2} (m_\alpha - 1). \]

Proof. First, we note that $(\mathbb{Z}_2)^\beta \leq B \leq \mathbb{Z}_{2^\beta}$ for every abelian 2-group $B$ of size $2^\beta$. The result is therefore follows from Remark 2 and Corollary 4.8. □

From Theorem 4.3, Proposition 4.7, and Corollary 4.9, we have the following remark.

Remark 3. Let $p$ be a prime and let $\nu$ be a positive integer. Let $A$ be a finite abelian group of order $m$ such that $p \nmid m$. Then we have the following observations

1. The value $\text{avg}_{\mathbb{F}_{p^\nu}}(A \times \mathbb{Z}_{p^k})$ can be determined by substituting the value of $|Q_{p^\nu}(A)|$ from Proposition 4.7 in to Theorem 4.3.

2. Some lower and upper bounds of $\text{avg}_{\mathbb{F}_{p^\nu}}(A \times \mathbb{Z}_{p^k})$ can be computed by substituting the bounds of $|Q_{p^\nu}(A)|$ from Corollary 4.9 in to Theorem 4.3.

3. If the Sylow 2-subgroup of $A$ is trivial (or equivalently, $m$ is odd), then $\text{avg}_{\mathbb{F}_{p^\nu}}(A \times \mathbb{Z}_{p^k})$ is independent of $A$. Precisely, $\text{avg}_{\mathbb{F}_{p^\nu}}(A \times \mathbb{Z}_{p^k})$ depends only on the cardinality $m$ of $A$.

Using Theorem 4.3, Corollary 4.9, and the arguments similar to those in the proof of [20, Theorem 25], we conclude the following bounds.

Corollary 4.10. Let $p$ be a prime and let $\nu$ be a positive integer. Let $A$ be a finite abelian group of order $m$ such that $p \nmid m$. Then one of the following statements holds.

1. $\text{avg}_{\mathbb{F}_{p^\nu}}(A \times \mathbb{Z}_{p^k}) = 0$ if and only if $k = 0$ and $m \in G_{(p^\nu, 1)}$.

2. If $k > 0$ or $m \notin G_{(p^\nu, 1)}$, then $mp^k \leq \text{avg}_{\mathbb{F}_{p^\nu}}(A \times \mathbb{Z}_{p^k}) < \frac{mp^k}{3}$.

The above results imply that $\text{avg}_{\mathbb{F}_{p^\nu}}(A \times \mathbb{Z}_{p^k})$ is zero or grows as the same rate with $mp^k$. Note that if $A$ is a cyclic group, these results coincide with [20, Theorem 25].

4.3 The Average Dimension of the Hermitian Hull of Abelian Codes in PIGAs

As the Hermitian inner product is defined over a finite field of square order. Here we focus on abelian codes in $\mathbb{F}_{p^{2\nu}}[A \times \mathbb{Z}_{2^k}]$, where $k \geq 0$ and $\nu \geq 1$ are integers and $p$ is a prime. We note that the results in this subsection can be obtained using the arguments analogous to those in Subsection 4.2. The different is the
decomposition of the group algebra $\mathcal{R} := \mathbb{F}_{p^{2\nu}}[A]$. Therefore, some of the proof will be omitted. For convenience, the theorem numbers are given in the form $3.a'$ if it corresponds to $3.a$ in Subsection 4.2.

For each $a \in A$, the $p^{2\nu}$-cyclotomic class $S_{p^{2\nu}}(a)$ is said to be of type $I$ if $S_{p^{2\nu}}(a) = S_{p^{\nu}}(-p^{\nu}a)$ or type $II$ if $S_{p^{2\nu}}(a) \neq S_{p^{2\nu}}(-p^{\nu}a)$.

Without loss of generality, assume that $b_1, b_2, \ldots, b_t$ are representatives of the $p^{2\nu}$-cyclotomic classes such that \{\textit{for all } $j = 1, 2, \ldots, r_F$\} and \{\textit{for all } $j = 1, 2, \ldots, r_W$\} are sets of representatives of $p^{2\nu}$-cyclotomic classes of $A$ of types $I$ and $II$, respectively, where $t = r_F + 2r_W$. Further, assume that $|S_{p^{2\nu}}(b_i)| = t_i$ for all $1 \leq i \leq t$. Clearly, $t_{r_F+i} = t_{r_F+r_W+i}$ for all $1 \leq l \leq r_W$.

Rearranging the terms in the decomposition of $\mathcal{R}$ in (4.1) based on the above 2 types of $p^{2\nu}$-cyclotomic classes (see [7]), we have

$$\mathbb{F}_{p^{2\nu}}[A \times \mathbb{Z}_{p^k}] \cong \left( \prod_{j=1}^{r_F} K_j[\mathbb{Z}_{p^k}] \right) \times \left( \prod_{l=1}^{r_W} (L_l[\mathbb{Z}_{p^k}] \times L_l[\mathbb{Z}_{p^k}]) \right), \tag{4.10}$$

where $K_j \cong \mathbb{F}_{p^{2\nu}}$ for all $j = 1, 2, \ldots, r_F$ and $L_l \cong \mathbb{F}_{p^{2\nu+r_W}}$ for all $l = 1, 2, \ldots, r_W$.

From (4.10), an abelian code $C$ in $\mathbb{F}_{p^{2\nu}}[A \times \mathbb{Z}_{p^k}]$ can be viewed as

$$C \cong \left( \prod_{j=1}^{r_F} C_j \right) \times \left( \prod_{l=1}^{r_W} (D_l \times D'_l) \right). \tag{4.11}$$

where $C_j, D_l$ and $D'_l$ are cyclic codes in $K_j[\mathbb{Z}_{p^k}], L_l[\mathbb{Z}_{p^k}]$ and $L_l[\mathbb{Z}_{p^k}]$, respectively, for all $j = 1, 2, \ldots, r_F$ and $l = 1, 2, \ldots, r_W$.

In [7] Section II.D] and Remark 1 the Hermitian dual of $C$ in (4.11) is of the form

$$C^{\perp H} \cong \left( \prod_{j=1}^{r_F} C_j^{\perp E} \right) \times \left( \prod_{l=1}^{r_W} ((D'_l \cap (D'_l)^{\perp E}) \times (D'_l \cap (D'_l)^{\perp E})) \right). \tag{4.12}$$

Lemma 4.1. There is a one-to-one correspondence between $C(p^{2\nu}, A \times \mathbb{Z}_{p^k})$ and \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_t) \mid 0 \leq \epsilon_i \leq p^k \text{ for all } 1 \leq i \leq t\}, \text{where } t = r_F + 2r_W.$$

The following corollary can be obtained using Remark 1 Lemma 4.1 and the above discussion.

Lemma 4.2. Let $C$ be an abelian code in $\mathbb{F}_{p^{2\nu}}[A \times \mathbb{Z}_{p^k}]$ decomposed as (4.11). Then the following statements hold.

1. The Hermitian hull of $C$

$$\mathcal{H}_H(C) \cong \left( \prod_{j=1}^{r_F} (C_j \cap C_j^{\perp E}) \right) \times \left( \prod_{s=1}^{r_W} ((D_s \cap (D'_s)^{\perp E}) \times (D'_s \cap (D'_s)^{\perp E})) \right). \tag{4.13}$$
2. If $C$ corresponds to $(\epsilon_1, \epsilon_2, \ldots, \epsilon_t)$ with $t = r_\Upsilon + 2r_\Upsilon$ as in Lemma 4.7, then

$$
\dim(\mathcal{H}_H(C)) = \sum_{j=1}^{r_\Upsilon} t_j \min(\epsilon_j, p^k - \epsilon_j) + \sum_{l=1}^{r_\Upsilon} t_{r_\Upsilon+l} \min(\epsilon_{r_\Upsilon+l}, p^k - \epsilon_{r_\Upsilon+l}) + \sum_{l=1}^{r_\Upsilon} t_{r_\Upsilon+l} \min(\epsilon_{r_\Upsilon+r_\Upsilon+l}, p^k - \epsilon_{r_\Upsilon+l}).
$$

Let $R_{p^2
u}(A) = \{a \in A \mid -p^\nu \cdot a \in S_{p^2
u}(a)\}$. It is not difficult to see that $R_{p^2
u}(A)$ is the union of all $p^{2\nu}$-cyclotomic classes of $A$ of type $I'$ and $|R_{p^2
u}(A)| = \sum_{j=1}^{r_I} t_j$, where $t_j = S_{p^2
u}(b_j)$ for all $1 \leq j \leq r_I$.

**Theorem 4.3.** Let $p$ be a prime and let $\nu$ be a positive integer. Let $A$ be a finite abelian group of order $m$ such that $p \nmid m$. Then

$$
\text{avg}_H^{p^2
u}(A \times \mathbb{Z}_{p^k}) = mp^k \left(\frac{1}{3} - \frac{1}{6(p^k + 1)}\right) - |R_{p^2
u}(A)| \left(\frac{p^k + 1}{12} + \frac{2 - 3\delta_p}{12(p^k + 1)}\right),
$$

where

$$
\delta_p = \begin{cases} 
1 & \text{if } p = 2, \\
0 & \text{if } p \text{ is odd.}
\end{cases}
$$

**Corollary 4.4.** Let $p$ be a prime and let $\nu$ be a positive integer. Let $A$ be a finite abelian group of order $m$ such that $p \nmid m$. Then the following statements hold.

1. $\text{avg}_H^{p^2
u}(A \times \mathbb{Z}_{p^k}) < mp^k / 3$.
2. $\text{avg}_H^{p^2
u}(A) = \frac{m - |R_{p^2
u}(A)|}{4}$.
3. $\text{avg}_H^{p^2
u}(A) \leq \frac{m-1}{4}$.

Let $\lambda$ be the function defined by

$$
\lambda(d, q) = \begin{cases} 
1 & \text{if } d \in OG(q, 1), \\
0 & \text{otherwise.}
\end{cases}
$$

Oddly-good integers play a role in determining $\text{avg}_H^{p^2
u}(A \times \mathbb{Z}_{p^k})$ in the following results.

**Lemma 4.5 (Lemma 3.5).** Let $p$ be a prime and let $\nu$ be a positive integer. Let $A$ be a finite abelian group with $p \nmid |A|$ and let $a \in A$. Then $S_{p^2
u}(a)$ is of type $I'$ if and only if $\text{ord}(a) \in OG(p^\nu, 1)$.

From Lemma 4.5, we have the following result.
Lemma 4.6. Let \( p \) be a prime and let \( \nu \) be a positive integer. Let \( A \) be a finite abelian group of order \( m \) and exponent \( M \) with \( p \nmid m \). Then
\[
|R_{p,\nu}(A)| = \prod_{d \mid M} \lambda(d, p^\nu)N_A(d),
\]
where \( N_A(d) \) is the number of elements of order \( d \) in \( A \) determined in [1].

Proposition 4.7. Let \( p \) be a prime and let \( \nu \) be a positive integer. Let \( A \) be a finite abelian group of order \( m = 2^\beta m_0 m_1 m_2 m_3 \ldots \), where \( m_\alpha \in \langle \langle P_{p^\nu, \alpha} \rangle \rangle \) for all \( \alpha \geq 0 \). Let \( \gamma \geq 0 \) be an integer such that \( 2^\gamma \mid (p^\nu + 1) \). Then
\[
|R_{p,\nu}(A)| = m_1 \sum_{i=0}^{\gamma} N_A(2^i).
\]

Proof. Let \( M = 2^\beta M_0 M_1 M_2 M_3 \ldots \) be the exponent of \( A \), where \( M_\alpha \) is in \( \langle \langle P_{p^\nu, \alpha} \rangle \rangle \) for all \( \alpha \geq 0 \). Observe that \( M' \mid m' \) and \( M_\alpha \mid m_\alpha \) for all \( \alpha \geq 0 \) and \( \beta \geq B \). It follows that
\[
|R_{p,\nu}(A)| = \sum_{d \mid M} \lambda(d, q)N_A(d)
\]
\[
= \sum_{d \mid M, d \in OG(q,1)} N_A(d)
\]
\[
= \sum_{d \mid 2^{\min\{\beta, \gamma\}} M_1} N_A(d)
\]
\[
= m_1 \sum_{i=0}^{\gamma} N_A(2^i).
\]
as desired. \( \square \)

Lemma 4.8. Let \( p \) be a prime and let \( \nu \) be a positive integer. Let \( A \) and \( B \) be abelian groups of the same order \( m \) and \( p \nmid m \). If the Sylow 2-subgroup of \( A \) is finer than the Sylow 2-subgroup of \( B \), then
\[
|R_{p,\nu}(A)| \geq |R_{p,\nu}(B)|.
\]

From Lemma 4.8 and Remark 2 we have the following bounds for \( |R_{p,\nu}(A)| \).

Corollary 4.9. Let \( p \) be a prime and let \( \nu \) be a positive integer. Let \( A \) be a finite abelian group of order \( m = 2^\beta m_0 m_1 m_2 m_3 \ldots \), where \( m_\alpha \) is in \( \langle \langle P_{p^\nu, \alpha} \rangle \rangle \) for all \( \alpha \geq 0 \). Let \( \gamma \geq 0 \) be an integer such that \( 2^\gamma \mid (p^\nu + 1) \). Then we have
\[
2^{\min\{\beta, \gamma\}} m_1 \leq |R_{p,\nu}(A)| \leq 2^{\min\{\beta, \gamma \beta\}} m_1.
\]

Remark 4. Let \( p \) be a prime and let \( \nu \) be a positive integer. Let \( A \) be a finite abelian group of order \( m \) such that \( p \nmid m \). Then we have the following observations

1. The value \( \text{avg}_{E_p}(A \times \mathbb{Z}_{p^\nu}) \) can be determined by substituting the value of \( |Q_{p^\nu}(A)| \) from Proposition 4.7 in to Theorem 4.3.
2. Some lower and upper bounds of $\text{avg}^H_{p^{2\nu}}(A \times \mathbb{Z}_{p^k})$ can be computed by substituting the bounds of $|R_{p^{2\nu}}(A)|$ from Corollary 4.9 in to Theorem 4.3.

3. If the Sylow 2-subgroup of $A$ is trivial (or equivalently, $m$ is odd), then $\text{avg}^H_{p^{2\nu}}(A \times \mathbb{Z}_{p^k})$ is independent of $A$. Precisely, $\text{avg}^H_{p^{2\nu}}(A \times \mathbb{Z}_{p^k})$ depends only on the cardinality $m$ of $A$.

Using Theorem 4.3, Corollary 4.9, and the arguments similar to those in the proof of [20, Theorem 25], we deduce the following bounds.

**Corollary 4.10.** Let $p$ be a prime and let $\nu$ be a positive integer. Let $A$ be a finite abelian group of order $m$ such that $p \nmid m$. Then one of the following statements hold.

1. $\text{avg}^H_{p^{2\nu}}(A \times \mathbb{Z}_{p^k}) = 0$ if and only if $k = 0$ and $m \in OG(q,1)$.

2. If $k > 0$ or $m \not\in OG(p^{\nu},1)$, then \[
\frac{mp^k}{8} \leq \text{avg}^H_{p^{2\nu}}(A \times \mathbb{Z}_{p^k}) < \frac{mp^k}{3}.
\]

The above results imply that $\text{avg}^H_{p^{2\nu}}(A \times \mathbb{Z}_{p^k})$ is zero or grows as the same rate with $mp^k$. Note that if $A$ is a cyclic group, these results coincide with [8, Theorem 4.9].

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