Influence of Fluid in Dynamic Calculation of Thin-Walled Cylindrical Tanks by Means of Calculating Added Masses of Fluid

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Abstract. This paper presents the exact solution for the problem of the added masses of fluid, oscillating inside a cylindrical tank. A thin smooth cylindrical shell of finite length, filled with the ideal incompressible fluid that is subjected to transient dynamic load (it may be an earthquake, air explosion impact, etc.), is taken as a design model. In this case, the shell of a tank becomes a part of the oscillating process along with the fluid stored inside. The shell bending equation (stipulated by the membrane theory of shells) is taken as a design model for this case. In this equation, the shell deflection is expanded into Fourier series by oscillation modes, and the velocity potential takes the form of the Laplace's equation. Fluid's inertial reaction to the shell motion is covered by the shallow water theory, which is specified by the boundary free surface conditions and the influence of the tank hard bottom. These mathematical expressions of the shell response to the external transient impact are solved by the Bubnov–Galerkin method with the transformation of the initial relations into a system of ordinary differential equations, which include expressions for determining added masses of fluid by oscillation modes of the elastic shell.

1. Introduction

Thin-walled cylindrical tanks filled with various fluid are widely used at industrial sites. Accident analysis during operation of such facilities indicates that they are very sensitive to dynamic loads. This happens due to the fluid presence in a tank; that fluid becomes a part of the oscillatory process together with the tank shell, which increases the overall load.

To account for the joint oscillation of the tank with fluid, the problem of hydroelasticity should be solved. These solutions are frequently associated with significant mathematical difficulties and are rarely used in practical terms. Within this context, an issue of determining simple and straightforward ways of fluid tank dynamic calculation which take into account all particularities of such system, becomes very acute. This is the focus of this paper.
2. Topicality, scientific importance with brief review of relevant literature
The paper [1] presents a brief history study of tank reaction to dynamic loads, also the basic research methods, included in the specifications of various countries, are discussed there. Most of specifications apply a simplified model with two degrees of freedom: the impulse one (tank shell motion with the most part of the contents) and the impulse one (surface layer splashing). This approach is described in many sources (e.g., [2, 3]), including the design standard NP-031-01 [4].

The authors of the company standard STO-SA-03-002-2009 [5] followed a different path. The tank stability is defined by a tilting moment that shall not exceed the restraining forces of the tank and the fluid stored inside. The design formulas for this methodology are presented in the papers [6, 7]. With such approach, the presence of fluid increases the restraining forces [8] and reduces the chance of an accident.

The main antiseismic construction reference document of the Russian Federation - SP 14.13330.2014 [9], however, refers to the need of taking into account the influence of inertial motion of fluid, oscillating along with the tank. In this regard, the mass of oscillating water (the added mass of fluid) is added to the construction mass. This added mass is usually considered when a body is oscillating in some fluid; in this purview, it becomes an acceleration multiplier. The resulting hydrodynamic force, similar in its nature to inertial force, depends on acceleration of particles on the body surface during motion in fluid [10, 11]. The same forces are generated during fluid oscillation inside a tank [12].

The paper [13] presents the exact values of added masses of fluid, oscillating inside a round cylinder as per the complex variable theory. However, a boundary value problem is not established (it can be Laplace's equation [14]). The paper [15] defines added masses of a tube, which is modelled as an infinite cylinder; a tank, however, has the length that is finite, because of a hard bottom and a free surface, they must be taken into account during mathematical modelling.

More and more studies dedicated to dynamic calculation of tanks are carried out in various software: [16] - in Matlab [17, 18] - ANSYS, [19] - SCAD Office. However, the issue of coupled oscillation consideration of a tank with fluid by calculating added masses of fluid remains unsolved and requires further clarification.

3. Research objective
In connection with the above, following focal points of the study can be determined:
- The object of study: dynamic calculation of thin-walled cylindrical tanks filled with fluid.
- The subject of study: influence of fluid in dynamic calculation of thin-walled cylindrical tanks.
- The study objective is to obtain a mathematical model that determines the reaction of a thin-walled cylindrical tank filled with fluid to the dynamic load impact.
- The main task of the study is to find the exact value of the added masses of fluid that emerge during oscillations of a tank after dynamic load exposure.

The solution is based on a reliable method of elasticity theory, shell calculation as per membrane theory of shells. Inertial reaction of fluid to tank shell oscillations is taken into account by way of calculating added masses of fluid as per the shallow water theory. Such approach hasscientific background and has been used for the calculation of shells of infinite length extensively. As for an existing tank, the mathematical model shall be improved by introducing boundary conditions at the free surface and at the hard bottom. As a result, simple expressions will be derived to calculate the add masses of fluid; they can be used to provide calculation for thin-walled tanks during dynamic loads.

4. Theoretical part
This paper’s purview considers a design case in which a thin-walled cylindrical tank filled with fluid to the brim is exposed to external transient load (earthquake, air explosion impact, etc.) A shell with fluid limited by the free surface from the top and by a hard bottom is taken as a design model for the study. The study is performed under the following simplifying assumptions:
- The shell is considered smooth and thin, so that any values with degrees of smallness could be neglected \( h/R \) (where \( h \) is the thickness of the shell, and \( R \) is the radius of the shell);
- The linear shell element that is plane to the shell median surface remains so during deformation (Timoshenko beam theory);
- The shell material is considered isotropic; the fluid is ideal and incompressible;
- Any damping is ignored, as the main focus is to find the maximum angular displacement in relation to the axis of symmetry;
- Inertial effect of that part of the fluid that oscillates along with the tank shell is taken into account, as it is considered to be the added mass per unit area of construction (by the shallow water theory).

Let us consider the oscillation of the tank as a shell with fluid in the cylindrical coordinate system \((x, r, \theta)\). The axis \( x \) is aligned with the shell axis ‘figure 1 (a)’:

![Figure 1](image)

**Figure 1.** Design model for a shell with fluid in the cylindrical coordinate system \((x, r, \theta)\) in which:
(a) the origin of the coordinates coincides with the tank’s hard bottom; (b) the origin of the coordinates coincides with the centre of the shell.

Where \( h \) is the shell thickness, \( R \) the shell radius, \( L \) the shell length, \( \theta \) is the angular coordinate.

Let us write the shell bending equation as per the membrane theory, neglecting shell’s inertance (in comparison with the inertance of the fluid, as the fluid’s mass exceeds the shell’s mass by a large margin):

\[
Eh / [R^2(1 - \mu^2)] \cdot w = -\rho \cdot \partial \phi / \partial t \bigg|_{r=R} + F,
\]

where \( E \) is the elastic modulus of the shell’s material; \( w \) is the radial displacement of the shell (\( w > 0 \) with the inside displacement); \( \mu \) is the Poisson's ratio; \( \rho \) is the fluid density; \( \phi \) is the fluid velocity potential; \( t \) is the time; \( F \) is the external dynamic impact that causes shell oscillations.

In this equation, the expression before \( w \) is the bending stiffness of the shell, where the first term from the right-hand side constitutes the internal pressure from the fluid to the shell walls during oscillation. Velocity potential \( \phi \) can be expressed as a Laplace's equation:

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \alpha^2} = 0,
\]

with the following boundary conditions:

\[
\partial \phi / \partial r = -\dot{w} at r = R, \bar{\alpha} \in [-l, l], \partial \phi / \partial r = 0 at \bar{\alpha} = -l, \phi = 0 \bar{\alpha} = l
\]

where \( l = L/(2R), \bar{\alpha} = x/R \). In this connection, the origin of the coordinates is transferred to the shell center ‘figure 1 (b)’:

Let us assume the shell deflection in Fourier series form:
\[ w = a_0 + \sum_{i=1}^{\infty} a_i \cos(i \pi \xi / l) + \sum_{i=1}^{\infty} b_i \sin(i \pi \xi / l). \]  

(4)

Let us consider the individual components of the shell motion in terms of oscillations caused by inertial motion of fuel.

The first form of oscillation is the uniform shell compression. To find the inertial fluid reaction to such deformations, let us solve the simultaneous equations of through flow (5) and Euler equation (6):

\[ \pi R^2 \left( v + (\partial v / \partial x)dx - v \right) = 2\pi R \dot{w}dx, \]  

(5)  
\[ dp / dx = \rho \dot{v}, \]  

(6)  

where \( v \) is the fluid velocity; \( p \) is the fluid pressure.

After transformation, the equation of through flow (5) can be expressed as:

\[ dv = 2(\dot{w} / R)dx \]  

or \[ d\bar{v} = 2(\dot{\bar{w}} / R)dx. \]  

(7)

After the integration of the equation (7), we shall obtain:

\[ \bar{v} = 2(\dot{\bar{w}} / R)x. \]  

(8)

The result (8) is then inserted into the Euler equation (6):

\[ p = p_0 2(\dot{\bar{w}} / R)x dx \]  

or \[ p = 2p_0(\dot{\bar{w}} / R)(x^2 / 2) + C, \]  

(9)  

where \( C \) is the integration constant. To find it, we shall use the boundary condition: at \( x = L \) \( p = 0 \). Hence:

\[ C = -2p_0(L^2 / 2), p = p_0(\dot{\bar{w}} / R)(x^2 - L^2). \]  

Let us move to the new coordinates, shown in ‘figure 1 (b)’. To do this, the current coordinate \( x \) is to be replaced with \( (x + L / 2) \). In this case:

\[ (x^2 - L^2) = (x^2 + xL - 3 / 4 \cdot L^2). \]  

To simplify the solution, let us move to the non-dimensional values. Set \( l = L / (2R) \), \( \bar{x} = x / R \):

\[ p = \rho R \dot{\bar{w}}(\bar{x}^2 + 2\bar{x}l - 3l^2). \]  

(10)

Let us assume the shell radius \( R \) as the main physical value with independent dimension, in this case we shall obtain:

\[ p_0 = \bar{a}_0 \rho (\bar{x}^2 + 2\bar{x}l - 3l^2), \]  

(11)  

where \( p_0 \) is the non-dimensional pressure, \( \bar{a}_0 \) the non-dimensional acceleration, relevant to the uniform shell compression.

Let us consider other forms of shell oscillation. Inertial fluid reaction to the shell motion can be found using the shallow water theory [20, 21], according to which the added mass of fluid is proportional to the wavenumber \( \omega = i \pi / l \):

\[ m_{ad} = \rho /[f(\omega \beta)], \]  

(12)  

where \( f(\omega \beta) \) is the function; for a cylinder filled with fluid it is equal to:

\[ f(\omega \beta) = I_1(\omega R \beta) / I_0(\omega R \beta), \]  

(13)
where \( I_{1}(oR\beta) \) and \( I_{0}(oR\beta) \) are Bessel functions.

For incompressible fluid (the speed of sound is equal to infinity) the value \( \beta = 1 \). Thus, at \( r = R \):

\[
m_{ad} = \rho(l/i\pi) \cdot I_{0}(i\pi/l) / I_{1}(i\pi/l) \tag{14}
\]

Hence, the pressure by oscillation forms can be expressed as:

\[
\begin{align*}
p_{\text{cos}} &= \sum_{i=1}^{\infty} \ddot{a}_{i} m_{ad} \frac{i\pi x}{l} \cos \frac{i\pi x}{l} = \sum_{i=1}^{\infty} \ddot{a}_{i} \rho \frac{l}{i\pi} \frac{I_{0}(i\pi/l)}{I_{1}(i\pi/l)} \cos \frac{i\pi x}{l} \\
p_{\text{sin}} &= \sum_{i=1}^{\infty} \ddot{b}_{i} m_{ad} \frac{i\pi x}{l} \sin \frac{i\pi x}{l} = \sum_{i=1}^{\infty} \ddot{b}_{i} \rho \frac{l}{i\pi} \frac{I_{0}(i\pi/l)}{I_{1}(i\pi/l)} \sin \frac{i\pi x}{l}
\end{align*}
\tag{15}
\]

Where \( \ddot{a}_{i} \) and \( \ddot{b}_{i} \) are acceleration, relevant to symmetrical and asymmetrical oscillation forms.

The solution (15) is obtained for a cylindrical shell of infinite length. As for an existing shell, the mathematical model shall be improved by boundary conditions at the free surface of fluid and hard bottom influence.

Let us define the medium pressure at the free surface of fluid at symmetrical oscillation form.

\[
\rho_{f} = \frac{\Delta \pi}{\rho} \rho(r) d(r) / (\pi R^{2}), \tag{16}
\]

where \( \int_{0}^{2\pi} \frac{\pi}{\rho} \rho(r) d(r) \) is the total pressure throughout the free surface:

\[
p(r)_{\text{T}=1} = \sum_{i=1}^{\infty} \ddot{a}_{i} (-1)^{i} \rho(l/i\pi) \frac{I_{0}(i\pi/l)}{I_{1}(i\pi/l)} \cos(i\pi x/l) \tag{17}
\]

After equation integration (16) using expressions (17), the expression to define the pressure at the free surface is obtained:

\[
p_{f} = \sum_{i=1}^{\infty} \ddot{a}_{i} (-1)^{i} \rho \frac{2\pi l}{i\pi} \frac{l}{i\pi} \frac{1}{\pi} \sum_{i=1}^{\infty} \ddot{b}_{i} (-1)^{i} \rho \frac{2l^{2}}{i^{2} \pi^{2}} = \Delta p'. \tag{18}
\]

Fluid compliance in relation to the shell motion is due to the fluid flowing as an incompressible medium. During asymmetrical oscillation, the free flow of fluid affects the fixed bottom. Such reaction also makes further impact on the shell oscillation.

Volumetric flow of fluid.

\[
\int_{0}^{l/2l} \dot{b}_{i} 2\pi R \sin(i\pi x/l) dx = \dot{b}_{i} 2\pi R (l/i\pi) \cos(i\pi x/l)_{0}^{l/2l} = \dot{b}_{i} 2\pi R (l/i\pi). \tag{19}
\]

Mass flow:

\[
\dot{b}_{i} \rho 2\pi R (l/i\pi) 2l. \tag{20}
\]

Reaction force can be expressed in terms of acceleration (pressure):

\[
p_{b} = \ddot{b}_{i} 2\pi R (l/i\pi) 2l(l/i\pi R^{2}) = \ddot{b}_{i} \rho (4l^{2}/i\pi R^{2}), \tag{20}
\]

here \( 2l \) is the fluid column. Given the changes in the fluid column along the shell length, this component (in non-dimensional form) becomes:

\[
p_{b} = \ddot{b}_{i} \rho (4l^{2}/i\pi) (1/2 - \bar{x}/2l)(-1)^{i} \tag{21}
\]

Hence, the internal pressure from the fluid to the tank walls (the first right-hand term of the equation (1) with due regard for the shell motion as per oscillation forms (4) and obtained expressions (10, 15, 18, and 21) becomes:
\[ \rho \frac{\ddot{w}}{c_t} = p_0 + p_\text{cos} + p_\text{sin} + p_\beta = \rho a_0(3l^2 - 2\bar{x}l - \bar{x}^2) + \rho \sum_{i=1}^{\infty} \frac{l}{\pi} I_0(i \pi / l) \frac{\sin i \pi \bar{x}}{i \pi l} + (-1)^i 2l^2 \frac{1}{i^2 \pi^2} \]

Further, let us transform (1) with consideration of (22) by the Bubnov–Galerkin method to obtain the system of ordinary differential equations for defining the displacement column-vector \( \bar{\omega} \). Reduce the equation to the non-dimensional form, having taken the basic physical values as the ones with independent dimension \( E h / [R^2 (1 - \mu^2)] \), \( R, \rho \):

\[ A \bar{\omega} + C \bar{\omega} = \bar{P}, \]

where \( \bar{\omega} = (a_0, a_1, \ldots, a_n, b_1, \ldots, b_n)^T \) is the column-vector of displacement;

\( \bar{P} \) the column-vector of external pertubances;

\( C \) the stiffness matrix;

\( A \) the matrix of added masses, that takes the following form:

\[ A = 2l^2 \cdot \Delta, \text{where:} \]

\[ \Delta = \begin{bmatrix}
  \frac{4}{3} & 1/\pi^2 & -1/(2\pi)^2 & \cdots & (-1)^n \frac{1}{(i \pi)^2}
  \\
  1/\pi^2 & \frac{1}{\pi^2} & \cdots & \cdots & \cdots
  \\
  -1/(2\pi)^2 & \cdots & \frac{1}{\pi^2} & \cdots & \cdots
  \\
  (-1)^n \frac{1}{(i \pi)^2} & \cdots & \cdots & \frac{1}{(i \pi^2)^2} & \cdots
  \\
  \frac{1}{2\pi} & \cdots & \cdots & \frac{1}{2\pi} & \cdots
  \\
  (-1)^n \frac{1}{i \pi} & \cdots & \cdots & \frac{1}{i \pi} & \cdots
  \\
  \end{bmatrix} \]

\[ \gamma_{ii} = 1/(4il \pi) \cdot I_0(i \pi / l) / I_1(i \pi / l) \]

In the reduced formulas \( I_0(\bullet) \) и \( I_1(\bullet) \) are the modified Bessel formulas of a purely imaginary argument.

The stiffness matrix becomes:

\[ C = \begin{bmatrix}
  1 & 0 & 0 & \cdots & 0 \\
  0 & 1/2 & 0 & \cdots & 0 \\
  0 & 0 & 1/2 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1/2 \\
  \end{bmatrix} \]

During subsequent fuel tank shell durability assessment at dynamic loading, it is advisable to take into account only one term during deflection expansion in advance \( w = a_0(t) \). Other terms are quick to decrease in absolute value.

In this formulation, the equation of shell oscillations (1) can be reduced to:

\[ (\bar{\rho}_0 \cdot \bar{h} + \bar{m}^*) \ddot{\bar{w}} + \bar{w} = P \]

Where \( \bar{\rho}_0 = \rho_0 / \rho, \bar{h} = h / R \).
\( \rho_0 \) is the shell material density.

\[
\bar{m}^* = 2l^2 \cdot \Delta = 2l^2 \cdot \frac{4}{3} = \left( \frac{8}{3} \right) l^2.
\]

5. Practical significance, further proposals, implementation results and experimental research results

The introduced model of shell oscillations provided the means to give a reasonably accurate estimation of added masses of a liquid filler that can be used to assess the strength and stability of the system under external dynamic influence.

The main term of the added mass in dimensional values is of the following form:

\[
m^* = \bar{m}^* \rho R = \left( \frac{8}{3} \right) \rho \left( \frac{L^2}{R} \right)
\]

Analysis of the matrix \( \Delta \) (24) shows that the main part of the added mass of fluid, restricted by the elastic tank wall, corresponds to the first term of the expression (22), which in turn corresponds to a symmetric shell motion.

Other added masses of the series (22) decrease promptly. This allows one to compose simple mathematical models of oscillation with opportunities to validate the strength and stability of the shell exposed to dynamic load.

6. Conclusions

The established mathematical model (27) presented in the paper describe the reaction of a thin-walled cylindrical tank filled with fluid to dynamic load. The established equation takes account of shell mass increase due to the dynamic added masses, the exact value of which is defined in the expression (28).

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