Gamma-convergence of nonlocal perimeter functionals

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1 Introduction

For a measurable set $E \subset \mathbb{R}^n$, $n \geq 1$, $0 < s < 1$, and a connected open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary (or simply $\Omega = (a, b) \subset \mathbb{R}$ if $n = 1$), we consider the functional

$$J_s(E, \Omega) := J^1_s(E, \Omega) + J^2_s(E, \Omega),$$

where

$$J^1_s(E, \Omega) := \int_{E \cap \Omega} \int_{E^c \cap \Omega} \frac{1}{|x - y|^{n+s}} dxdy,$$

$$J^2_s(E, \Omega) := \int_{E \cap \Omega} \int_{E^c \cap \Omega} \frac{1}{|x - y|^{n+s}} dxdy + \int_{E \cap \Omega^c} \int_{E^c \cap \Omega} \frac{1}{|x - y|^{n+s}} dxdy.$$

The functional $J_s(E, \Omega)$ can be thought of as a fractional perimeter of $E$ in $\Omega$ which is non-local in the sense that it is not determined by the behaviour of $E$ in a neighbourhood of $\partial E \cap \Omega$, and which can be finite even if the Hausdorff dimension of $\partial E$ is $n - s > n - 1$. Notice that the term $J^1_s(E, \Omega)$ is simply half of the fractional Sobolev space seminorm $|\chi_E|_{W^{s,1}(\Omega)}$, where $\chi_E$ denotes the characteristic function of $E$. Roughly speaking this term represents the $(n - s)$-dimensional fractional perimeter of $E$ inside $\Omega$, while $J^2_s$ is the contribution near $\partial \Omega$. This can be made precise when letting $s \uparrow 1$. We also recall the following elementary scaling property:

$$J^i_s(\lambda E, \lambda \Omega) = \lambda^{n-s} J^i_s(E, \Omega) \quad \text{for } \lambda > 0, \ i = 1, 2. \quad (1)$$

This functional has already been investigated by several authors. In [15] Visintin studied some basic properties of $J_s$, and in particular he showed that $J_s$ satisfies a suitable co-area formula, see Lemma [10] below. Caffarelli, Roquejoffre and Savin [4] studied the behavior of minimizers of $J_s$, proving that if $E$ is a local minimizer of $J_s(\cdot, \Omega)$, i.e.

$$J_s(E, \Omega) \leq J_s(F, \Omega) \quad \text{whenever } E \Delta F \subset \Omega,$$

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then \((\partial E) \cap \Omega\) is of class \(C^{1,\alpha}\) up to a set of Hausdorff codimension in \(\mathbb{R}^n\) at least 2.

As it is well-known (see for instance [10] and the references therein), for minimizers \(E\) of the classical De Giorgi’s perimeter, which we shall denote \(P(E, \Omega)\), the regularity results are stronger. The boundary of a local minimizer \(E\) of \(P(\cdot, \Omega)\) is analytic if \(n \leq 7\), it has (at most) isolated singularities when \(n = 8\) and it is analytic up to a set of codimension at least 8 in \(\mathbb{R}^n\) if \(n \geq 9\). This suggests that the results of [4] might not be optimal for \(s\) close to 1.

Motivated by this, Caffarelli and Valdinoci [5] studied the limiting properties of minimal sets for the \(s\)-perimeter as \(s \to 1^-\).

Partly motivated by their work, we make a complete analysis of the limiting properties, in the sense of \(\Gamma\)-convergence, of \(J_s\) as \(s \to 1^-\), under no other assumption than the measurability of the sets considered. Our proofs differ in particular from those in [5] because they do not rely on uniform (as \(s \to 1^-\)) regularity estimates on \(s\)-minimal boundaries borrowed from [5]. The only result we need from [4], in the proof of our Lemma 14, is the local minimality of halfspaces, whose proof is reproduced in the appendix.

We start by proving a coercivity result.

**Theorem 1 (Equi-coercivity)** Assume that \(s_i \uparrow 1\) and that \(E_i\) are measurable sets satisfying
\[
\sup_{i \in \mathbb{N}} (1 - s_i) J_{s_i}^1(E_i, \Omega') < \infty \quad \forall \Omega' \subseteq \Omega.
\]

Then \((E_i)\) is relatively compact in \(L^1_{\text{loc}}(\Omega)\), any limit point \(E\) has locally finite perimeter in \(\Omega\).

Notice the scaling factor \((1 - s)\), which accounts for the fact that \(J_1^1(E, \Omega) = +\infty\) unless \(E \subseteq \Omega\), or \(\Omega \subseteq E\), as already shown by Brézis [3].

Let \(\omega_k\) denote the volume of the unit ball in \(\mathbb{R}^k\) for \(k \geq 1\), and set \(\omega_0 := 1\).

**Theorem 2 (\(\Gamma\)-convergence)** For every measurable set \(E \subseteq \mathbb{R}^n\) we have
\[
\begin{align*}
\Gamma - \liminf_{s_i \uparrow 1} (1 - s_i) J_{s_i}^1(E_i, \Omega) &\geq \omega_{n-1} P(E, \Omega), \\
\Gamma - \limsup_{s_i \uparrow 1} (1 - s_i) J_{s_i}^1(E_i, \Omega) &\leq \omega_{n-1} P(E, \Omega),
\end{align*}
\]
with respect to the local convergence in measure, i.e. the \(L^1_{\text{loc}}\) convergence of the corresponding characteristic functions in \(\mathbb{R}^n\).

We recall that (2) means that
\[
\liminf_{i \to \infty} (1 - s_i) J_{s_i}^1(E_i, \Omega) \geq \omega_{n-1} P(E, \Omega) \quad \text{whenever } \chi_{E_i} \to \chi_E \text{ in } L^1_{\text{loc}}(\mathbb{R}^n), \ s_i \uparrow 1,
\]
and that for every measurable set \(E\) and sequence \(s_i \uparrow 1\) there exists a sequence \(E_i\) with \(\chi_{E_i} \to \chi_E\) in \(L^1_{\text{loc}}(\mathbb{R}^n)\) such that
\[
\limsup_{i \to \infty} (1 - s_i) J_{s_i}(E_i, \Omega) \leq \omega_{n-1} P(E, \Omega).
\]

We finally show that as \(s \uparrow 1\) local minimizers converge to local minimizers, where by a local minimizer of \(J_s(\cdot, \Omega)\) we mean a Borel set \(E \subseteq \mathbb{R}^n\) such that \(J_s(E, \Omega) \leq J_s(F, \Omega)\) whenever \(E \Delta F \subseteq \Omega\). Notice that if \(E\) is a local minimizer of \(J_s(\cdot, \Omega)\) and \(\Omega' \subset \Omega\), then \(E\) is also a local minimizer of \(J_s(\cdot, \Omega')\). A similar definition holds for \(P(\cdot, \Omega)\).
Theorem 3 (Convergence of local minimizers) Assume that $s_i \uparrow 1$, $E_i$ are local minimizer of $J_{s_i}(\cdot, \Omega)$, and $\chi_{E_i} \to \chi_E$ in $L^1_{\text{loc}}(\mathbb{R}^n)$. Then
\[
\limsup_{i \to \infty} (1 - s_i)J_{s_i}(E_i, \Omega') < +\infty \quad \forall \Omega' \Subset \Omega,
\]
$E$ is a local minimizer of $P(\cdot, \Omega)$ and $(1 - s_i)J_{s_i}(E_i, \Omega') \to \omega_{n-1}P(E, \Omega')$ whenever $\Omega' \Subset \Omega$ and $P(E, \partial \Omega') = 0$.

We point out that $\Gamma$-convergence results for functionals reminiscent of $J^1_s(\cdot, \mathbb{R}^n)$ have been proven in [13], [14].

We fix some notation used throughout the paper:
- we write $x \in \mathbb{R}^n$ as $(x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$;
- we denote by $H$ the halfspace $\{x : x_n \leq 0\}$ and by $Q = (-1/2, 1/2)^n$ the canonical unit cube;
- we denote by $B_r(x)$ the ball of radius $r$ centered at $x$ and, unless otherwise specified, $B_r := B_r(0)$;
- for every $h \in \mathbb{R}^n$ and function $u$ defined on $U \subset \mathbb{R}^n$ we set $\tau_h u(x) := u(x + h)$ for all $x \in U - h$.

For the definition and basic properties of the perimeter $P(E, \Omega)$ in the sense of De Giorgi we refer to the monographs [1] and [10].

2 Proof of Theorem 1

The proof is a direct consequence of the Frechet-Kolmogorov compactness criterion in $L^p_{\text{loc}}$ (applied with $p = 1$), ensuring pre-compactness of any family $\mathcal{G} \subset L^1_{\text{loc}}(\Omega)$ satisfying
\[
\limsup_{h \to 0} \|\tau_h u - u\|_{L^1(\Omega')} = 0 \quad \forall \Omega' \Subset \Omega,
\]
and of the following pointwise upper bound on $\|\tau_h u - u\|_{L^1}$: for all $u \in L^1(\Omega)$, $A \Subset \Omega$, $h \in \mathbb{R}^n$ with $|h| < \text{dist}(A, \partial \Omega)/2$ and $s \in (0, 1)$ we have
\[
\|\tau_h u - u\|_{L^1(A)} \leq C(n)|h|^s(1 - s)\mathcal{F}_s(u, \Omega),
\]
where
\[
\mathcal{F}_s(u, \Omega) := \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} \, dx \, dy.
\]
The functional $\mathcal{F}_s$ is obviously related to $J^1_s$ by
\[
\mathcal{F}_s(\chi_E, \Omega) = 2J^1_s(E, \Omega).
\]
The upper bound (1) is a direct consequence of Proposition 4 below, whose proof can be found in [11]. Since the inequality is not explicitly stated in [11], we repeat it for the reader’s convenience.

Proposition 4 For all $u \in L^1(\Omega)$, $A \Subset \Omega$ and $s \in (0, 1)$ we have
\[
\frac{\|\tau_h u - u\|_{L^1(A)}}{|h|^s} \leq C(n)(1 - s) \int_{B(|h|)} \frac{\|\tau_\xi u - u\|_{L^1(A_{|\xi|})}}{|\xi|^{n+s}} \, d\xi
\]
whenever $0 < |h| < \text{dist}(A, \partial \Omega)/2$, and $A_{|\xi|} := \{x \in \mathbb{R}^n : \text{dist}(x, A) < |\xi|\}$.
We start with two preliminary results.

**Proposition 5** Let \( u \in L^1(\Omega) \), \( h \in \mathbb{R}^n \) and \( A \Subset \Omega \) open with \( |h| < \text{dist}(A, \partial \Omega)/2 \). Then for any \( z \in (0, |h|) \) we have:

\[
\| \tau_h u - u \|_{L^1(A)} \leq C(n) \frac{|h|}{z^{n+1}} \int_{B_z} \| \tau_{|h|} u - u \|_{L^1(A_{|h|})} \, d\xi,
\]

where \( A_{|h|} \) is as in Proposition 4.

**Proof.** Fix a non-negative function \( \varphi \in C^1_c(B_1) \) with \( \int_{B_1} \varphi \, dx = 1 \). For \( x \in A \) and \( z \in (0, |h|) \) we write

\[
u(x) = \frac{1}{2^n} \int_{B_z} u(x+y) \varphi\left(\frac{y}{z}\right) \, dy + \frac{1}{2^n} \int_{B_z} (u(x) - u(x+y)) \varphi\left(\frac{y}{z}\right) \, dy
\]

\[
=: U(x, z) + V(x, z).
\]

Then we have

\[
|u(x + h) - u(x)| \leq |U(x + h, z) - U(x, z)| + |V(x + h, z)| + |V(x, z)|. \tag{8}
\]

The second and third terms can be easily estimated as follows:

\[
|V(x + h, z)| + |V(x, z)| \leq \sup_{B_z} |\varphi| \int_{B_z} \{ |\tau_y u(x) - u(x)| + |\tau_y u(x + h) - u(x + h)| \} \, dy.
\]

For the first one instead notice that

\[
\nabla_x U(x, z) = -\frac{1}{2^n} \int_{B_z(x)} u(y) \nabla \varphi\left(\frac{y - x}{z}\right) \, dy
\]

\[
= -\frac{1}{2^n} \int_{B_z(x)} (u(y) - u(x)) \nabla \varphi\left(\frac{y - x}{z}\right) \, dy
\]

and so

\[
|U(x + h, z) - U(x, z)| \leq |h| \int_0^1 |\nabla_x U(x + sh, z)| \, ds
\]

\[
\leq \sup |\nabla \varphi| \frac{|h|}{z^{n+1}} \int_0^1 \int_{B_z} |u(y + x + sh) - u(x + sh)| \, dy ds.
\]

Notice now that \( z \leq |h| \) and so \( 1 \leq |h|/z \), hence from (8) we have:

\[
|u(x + h) - u(x)| \leq C \left\{ \frac{1}{2^n} \int_{B_z} |\tau_y u(x) - u(x)| + |\tau_y u(x + h) - u(x + h)| \, dy
\]

\[
+ \frac{|h|}{z^{n+1}} \int_0^1 \int_{B_z} |u(y + x + sh) - u(x + sh)| \, dy ds \right\}
\]

\[
\leq C \left\{ \frac{|h|}{z^{n+1}} \left\{ \int_{B_z} |\tau_y u(x) - u(x)| + |\tau_y u(x + h) - u(x + h)| \, dy
\]

\[
+ \int_0^1 \int_{B_z} |\tau_y u(x + sh) - u(x + sh)| \, dy ds \right\}
\]
with \( C = \sup |\varphi| + \sup |\nabla \varphi| \). Integrating both sides over \( A \) we infer (7) with \( C(n) = 3C \). □

Recall now the following version of Hardy’s inequality:

**Proposition 6** Let \( g : \mathbb{R} \to [0, \infty) \) be a Borel function, then for every \( s > 0 \) we have

\[
\int_0^r \frac{1}{\xi^{n+s+1}} \int_0^\xi g(t) dt d\xi \leq \frac{1}{n+s} \int_0^r \frac{g(t)}{t^{n+s}} dt \quad \forall r \geq 0.
\]

(9)

**Proof.** We have

\[
\int_0^r \frac{1}{\xi^{n+s+1}} \int_0^\xi g(t) dt d\xi = \int_0^r g(t) \int_t^r \frac{1}{\xi^{n+s+1}} d\xi dt
\]

\[
= \frac{1}{n+s} \int_0^r g(t) \left( \frac{1}{t^{n+s}} - \frac{1}{r^{n+s}} \right) dt \leq \frac{1}{n+s} \int_0^r g(t) \frac{1}{t^{n+s}} dt.
\]

□

**Proof of Proposition 6.** Multiply both sides of (7) by \( z^{-(1-s)} \) and integrate with respect to \( z \) between 0 and \( |h| \) to obtain

\[
\frac{|h|^{1-s}}{1-s} \| \tau_h u - u \|_{L^1(A)} \leq C(n)|h| \int_0^{|h|} \frac{1}{z^{n+s+1}} \int_{B_z} \| \tau_\xi u - u \|_{L^1(A|\xi|)} d\xi dz.
\]

Now apply inequality (9) with

\[
g(t) := \int_{\partial B_t} \| \tau_\xi u - u \|_{L^1(A|\xi|)} d\mathcal{H}^{n-1}(\xi)
\]

and obtain

\[
\int_0^{|h|} \frac{1}{z^{n+s+1}} \int_{B_z} \| \tau_\xi u - u \|_{L^1(A|\xi|)} d\xi dz = \int_0^{|h|} \frac{1}{z^{n+s+1}} \int_0^z g(t) dt dz
\]

\[
\leq C(n) \int_0^{|h|} \frac{1}{t^{n+s}} g(t) dt = C(n) \int_{B_{|h|}} \| \tau_\xi u - u \|_{L^1(A|\xi|)} d\xi.
\]

(10)

Putting all together

\[
\frac{\| \tau_h u - u \|_{L^1(A)}}{(1-s)} \leq C(n)|h|^{s} \int_{B_{|h|}} \frac{\| \tau_\xi u - u \|_{L^1(A|\xi|)}}{|\xi|^{n+s}} d\xi
\]

and the thesis follows. □
3 Proof of Theorem 2

In the proof of the lim inf inequality we shall adapt to this framework the blow-up technique introduced, for the first time in the context of lower semicontinuity, by Fonseca and Müller in [9]. The proof of the lim sup inequality, which is typically constructive and by density, is slightly different from the analogous results in [5], since we approximate with polyhedra, rather than $C^{1,\alpha}$ sets. Notice also that the natural strategies in the proof of the lim inf and lim sup inequalities produce constants $\Gamma_n$, see (11), and $\Gamma^*_n \geq \Gamma_n$, see (17); our final task will be to show that they both coincide with $\omega_{n-1}$.

3.1 The $\Gamma - \lim inf$ inequality

Let us define

$$\Gamma_n := \inf \left\{ \liminf_{s_i \uparrow 1} (1 - s_i) J^1_s(E_i, Q) \mid \chi_{E_i} \rightarrow \chi_H \text{ in } L^1(Q) \right\}. \tag{11}$$

We denote by $\mathcal{C}$ the family of all $n$-cubes in $\mathbb{R}^n$

$$\mathcal{C} := \{ R(x + rQ) : x \in \mathbb{R}^n, r > 0, R \in SO(n) \}.$$

Lemma 7 Given $s_i \uparrow 1$ and sets $E_i \subset \mathbb{R}^n$ with $\chi_{E_i} \rightarrow \chi_E$ in $L^1_{loc}(\mathbb{R}^n)$ as $i \to \infty$, one has

$$\liminf_{i \to \infty} (1 - s_i) J^1_s(E_i, \Omega) \geq \Gamma_n P(E, \Omega). \tag{12}$$

We can assume that the left-hand side of (12) is finite, otherwise the inequality is trivial. Then, passing to the limit as $i \to \infty$ in (6) with $s = s_i$ we get

$$\| \tau_h \chi_E - \chi_{E_i} \|_{L^1(\Omega')} \leq C(n) |h| \liminf_{i \to \infty} (1 - s_i) J^1_s(E_i, \Omega) \quad \forall \Omega' \subset \Omega$$

whenever $|h| < \text{dist}(\Omega', \partial \Omega)/2$, hence $E$ has finite perimeter in $\Omega$.

We shall denote by $\mu$ the perimeter measure of $E$, i.e. $\mu(A) = |D\chi_E|(A)$ for any Borel set $A \subset \Omega$, and we shall use the following property of sets of finite perimeter: for $\mu$-a.e. $x \in \Omega$ there exists $R_x \in SO(n)$ such that $(E - x)/r$ locally converge in measure to $R_x H$ as $r \to 0$. In addition,

$$\lim_{r \to 0} \frac{\mu(x + rR_x Q)}{r^{n-1}} = 1, \quad \text{for } \mu\text{-a.e. } x. \tag{13}$$

Indeed this property holds for every $x \in \mathcal{F}E$, where $\mathcal{F}E$ denotes the reduced boundary of $E$, see Theorem 3.59(b) in [1].

Now, given a cube $C \in \mathcal{C}$ contained in $\Omega$ we set

$$\alpha_i(C) := (1 - s_i) J^1_s(E_i, C)$$

and

$$\alpha(C) := \liminf_{i \to \infty} \alpha_i(C).$$
We claim that, setting $C_r(x) := x + rR_xQ$, where $R_x$ is as in (13), for $\mu$-a.e. $x$ we have
\[
\liminf_{r \to 0} \frac{\alpha(C_r(x))}{\mu(C_r(x))} \geq \Gamma_n \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n.
\] (14)

Then observing that for all $\varepsilon > 0$ the family
\[
A := \left\{ C_r(x) \subset \Omega : (1 + \varepsilon) \alpha(C_r(x)) \geq \Gamma_n \mu(C_r(x)) \right\}
\]
is a fine covering of $\mu$-almost all of $\Omega$, by a suitable variant of Vitali’s theorem (see [12]) we can extract a countable subfamily of disjoint cubes
\[
\{ C_j \subset \Omega : j \in J \}
\]
such that $\mu(\Omega \setminus \bigcup_{j \in J} C_j) = 0$, whence
\[
\Gamma_n P(E, \Omega) = \Gamma_n \mu\left( \bigcup_{j \in J} C_j \right) = \Gamma_n \sum_{j \in J} \mu(C_j)
\]
\[
\leq (1 + \varepsilon) \sum_{j \in J} \alpha(C_j) \leq (1 + \varepsilon) \liminf_{i \to \infty} \sum_{j \in J} \alpha_i(C_j)
\]
\[
\leq (1 + \varepsilon) \liminf_{i \to \infty} (1 - s_i) J_{s_i}^1(E_i, \Omega).
\]

In the last inequality we used that $J_{s_i}^1$ is superadditive and positive for every $s \in (0, 1)$. Since $\varepsilon > 0$ is arbitrary we get the $\Gamma - \liminf$ estimate.

We now prove the inequality in (13) at any point $x$ such that $(E - x)/r$ converges locally in measure as $r \to 0$ to $R_xH$ and (13) holds. Because of (13), we need to show that
\[
\liminf_{r \to 0} \frac{\alpha(C_r(x))}{r^{n-1}} \geq \Gamma_n.
\] (15)

Since from now on $x$ is fixed, we can assume with no loss of generality (by rotation invariance) that $R_x = I$, so that the limit hyperplane is $H$ and the cubes $C_r(x)$ are the standard ones $x + rQ$. Let us choose a sequence $r_k \to 0$ such that
\[
\liminf_{r \to 0} \frac{\alpha(C_r(x))}{r^{n-1}} = \lim_{k \to \infty} \frac{\alpha(C_{r_k}(x))}{r_k^{n-1}}.
\]

For $k > 0$ we can choose $i(k)$ so large that the following conditions hold:
\[
\begin{cases}
\alpha_{i(k)}(C_{r_k}(x)) \leq \alpha(C_{r_k}(x)) + r_k^n, \\
r_k^{1 - s_i(k)} \geq 1 - \frac{1}{k}, \\
\int_{C_{r_k}(x)} |\chi_{E_{i(k)}} - \chi_E| dx < \frac{1}{k}.
\end{cases}
\]
Then we infer

$$\frac{\alpha(Cr_k(x))}{r_{k}^{n-1}} \geq \alpha_i(k)\frac{C_r(x)}{r_k} - r_k$$

$$= \frac{(1 - s_i(k))J_{s_i(k)}^1((E_i(k) - x) / r_k, Q) r_k^{n-s_i(k)}}{r_k^{n-1}} - r_k$$

$$\geq \left(1 - \frac{1}{k}\right)(1 - s_i(k))J_{s_i(k)}^1((E_i(k) - x) / r_k, Q) - r_k,$$

i.e.

$$\lim_{k \to \infty} \frac{\alpha(Cr_k(x))}{r_k^{n-1}} \geq \lim inf_{k \to \infty} (1 - s_i(k))J_{s_i(k)}^1((E_i(k) - x) / r_k, Q).$$

On the other hand we have

$$\lim_{k \to \infty} \int_Q |\chi(E_i(k) - x) / r_k - \chi(E-x) / r_k| dx = 0,$$

and

$$\lim_{k \to \infty} \int_Q |\chi(E-x) / r_k - \chi H| dx = 0.$$

It follows that \((E_i(k) - x) / r_k \to H\) in \(L^1(Q)\). Recalling the definition of \(\Gamma_n\) we conclude the proof of (15) and of Lemma 7.

### 3.2 The \(\Gamma - \lim sup\) inequality

It is enough to prove the \(\Gamma - \lim sup\) inequality for a collection \(B\) of sets of finite perimeter which is dense in energy, i.e. such that for every set \(E\) of finite perimeter there exists \(E_k \in B\) with \(\chi_{E_k} \to \chi_E\) in \(L^1_{\text{loc}}(\mathbb{R}^n)\) as \(k \to \infty\) and \(\lim sup_k P(E_k, \Omega) = P(E, \Omega)\). Indeed, let \(d\) be a distance inducing the \(L^1_{\text{loc}}\) convergence and, for a set \(E\) of finite perimeter, let \(E_k\) be as above. Given \(s_k \uparrow 1\), we can find sets \(\hat{E}_k\) with \(d(\chi_{\hat{E}_k}, \chi_{E_k}) < 1/k\) and

$$\lim_{k \to \infty} (1 - s_k)J_{s_k}((\hat{E}_k, \Omega) \leq \Gamma_n^* P(E_k, \Omega) + \frac{1}{k}.$$ 

Then we have \(\chi_{\hat{E}_k} \to \chi_E\) in \(L^1_{\text{loc}}(\mathbb{R}^n)\) and

$$\lim sup_{k \to \infty} (1 - s_k)J_{s_k}((\hat{E}_k, \Omega) \leq \lim sup_{k \to \infty} \Gamma_n^* P(E_k, \Omega) = \Gamma_n^* P(E, \Omega).$$

We shall take \(B\) to be the collection of polyhedra \(\Pi\) which satisfy \(P(\Pi, \partial \Omega) = 0\) (i.e. with faces transversal to \(\partial \Omega\), see Proposition 15). Equivalently,

$$\lim_{\delta \to 0} P(\Pi, \Omega_{\delta}^{+} \cup \Omega_{\delta}^{-}) = 0,$$

where

$$\Omega_{\delta}^{+} := \{x \in \Omega^c \mid d(x, \Omega) < \delta\}$$

$$\Omega_{\delta}^{-} := \{x \in \Omega \mid d(x, \Omega^c) < \delta\}.$$  \(16\)

In fact, we have:
**Lemma 8** For a polyhedron $\Pi \subset \mathbb{R}^n$ there holds

$$\limsup_{s \to 1}(1-s)\mathcal{J}_s(\Pi; \Omega) \leq \Gamma_1^* P(\Pi, \Omega) + 2\Gamma_1^* \lim_{s \to 0} P(\Pi, \Omega_\delta^+ \cup \Omega_\delta^-),$$

where

$$\Gamma_1^* := \limsup_{s \to 1}(1-s)\mathcal{J}_s^1(H, Q). \quad (17)$$

**Proof.** Step 1. We first estimate $\mathcal{J}_s^1(\Pi, \Omega)$. For a fixed $\varepsilon > 0$ set 

$$\partial \Pi_\varepsilon := \{x \in \Omega \mid d(x, \partial \Pi) < \varepsilon\}, \quad (\partial \Pi)_{\varepsilon}^{-} := (\partial \Pi)_\varepsilon \cap \Pi.$$

We can find $N_{\varepsilon}$ disjoint cubes $Q_i^\varepsilon \subset \Omega$, $1 \leq i \leq N_{\varepsilon}$, of side length $\varepsilon$ satisfying the following properties:

(i) if $\tilde{Q}_i^\varepsilon$ denotes the dilation of $Q_i^\varepsilon$ by a factor $(1 + \varepsilon)$, then each cube $\tilde{Q}_i^\varepsilon$ intersects exactly one face $\Sigma$ of $\partial \Pi$, its barycenter belongs to $\Sigma$ and each of its sides is either parallel or orthogonal to $\Sigma$;

(ii) $\mathcal{H}^{n-1}\left((\partial \Pi) \cap \Omega \right) \cup \bigcup_{i=1}^{N_{\varepsilon}} Q_i^\varepsilon = |P(\Pi, \Omega) - N_{\varepsilon} \varepsilon^{n-1}| \to 0$ as $\varepsilon \to 0$.

For $x \in \mathbb{R}^n$ set

$$I_s(x) := \int_{\Pi \cap \Omega} \frac{dy}{|x - y|^{n+s}}.$$ 

We consider several cases.

**Case 1:** $x \in (\Pi \cap \Omega) \setminus (\partial \Pi)_{\varepsilon}^-$. Then for $y \in \Pi^c \cap \Omega$ we have $|x - y| \geq \varepsilon$, hence

$$I_s(x) \leq \int_{(B_s(x))^c} \frac{1}{|x - y|^{n+s}}dy = n\omega_n \int_{\varepsilon}^{\infty} \frac{1}{\rho^{s+1}}d\rho = \frac{n\omega_n}{s\varepsilon^s},$$

since $n\omega_n = \mathcal{H}^{n-1}(S^{n-1})$. Therefore

$$\int_{(\Pi \cap \Omega) \setminus (\partial \Pi)_{\varepsilon}^-} I_s(x)dx \leq \frac{n\omega_n \mathcal{L}^n(\Pi \cap \Omega)}{s\varepsilon^s}. \quad (18)$$

**Case 2:** $x \in (\partial \Pi)_{\varepsilon}^- \cup \bigcup_{i=1}^{N_{\varepsilon}} Q_i^\varepsilon$. Then

$$I_s(x) \leq \int_{(B_s(x))^c} \frac{1}{|x - y|^{n+s}}dy = n\omega_n \int_{\varepsilon}^{\infty} \frac{1}{\rho^{s+1}}d\rho = \frac{n\omega_n}{s(d(x, \Pi^c \cap \Omega))}, \quad (19)$$

Now write $(\partial \Pi) \cap \Omega = \bigcup_{j=1}^J \Sigma_j$, where each $\Sigma_j$ is the intersection of a face of $\partial \Pi$ with $\Omega$, and define

$$(\partial \Pi)_{\varepsilon,j}^- := \{x \in (\partial \Pi)^c_{\varepsilon} : \text{dist}(x, \Pi^c \cap \Omega) = \text{dist}(x, \Sigma_j)\}.$$ 

Clearly $(\partial \Pi)^c_{\varepsilon,j} = \bigcup_{j=1}^J (\partial \Pi)^c_{\varepsilon,j}$. Moreover we have

$$(\partial \Pi)_{\varepsilon,j}^- \subset \{x + t\nu : x \in \Sigma_{\varepsilon,j}, t \in (0, \varepsilon), \nu \text{ is the interior unit normal to } \Sigma_{\varepsilon,j}\},$$

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and $\Sigma_{\varepsilon,j}$ is the set of points $x$ belonging to the same hyperplane as $\Sigma_j$ and with $\text{dist}(x, \Sigma_j) \leq \varepsilon$. Clearly $\mathcal{H}^{n-1}(\Sigma_{\varepsilon,j}) \leq \mathcal{H}^{n-1}(\Sigma_j) + C\varepsilon$ as $\varepsilon \to 0$. Then from (19) we infer

$$
\int_{(\partial \Pi)_{\varepsilon} \setminus \bigcup_{i=1}^{N_{\varepsilon}} Q_i} I_s(x) \, dx \leq \frac{n \omega_n}{s} \sum_{j=1}^{J} \int_{(\partial \Pi)_{\varepsilon} \setminus \bigcup_{i=1}^{N_{\varepsilon}} Q_i} \frac{1}{(d(x, \Pi^c))^s} \, dx
$$

$$
\leq \frac{n \omega_n}{s} \sum_{j=1}^{J} \int_{(\Sigma_{\varepsilon,j}) \setminus \bigcup_{i=1}^{N_{\varepsilon}} Q_i} \left( \int_{0}^{\varepsilon} \frac{dt}{t^s} \right) d\mathcal{H}^{n-1}
$$

$$
= \frac{n \omega_n \varepsilon^{1-s}}{s(1-s)} \mathcal{H}^{n-1} \left( \left( \bigcup_{j=1}^{J} \Sigma_{\varepsilon,j} \right) \setminus \bigcup_{i=1}^{N_{\varepsilon}} Q_i \right) = \frac{\varepsilon^{1-s} o(1)}{s(1-s)},
$$

with error $o(1) \to 0$ as $\varepsilon \to 0$ and independent of $s$.

**Case 3:** $x \in \Pi \cap \bigcup_{i=1}^{N_{\varepsilon}} Q_i$. In this case we write

$$
I_s(x) = \int_{(\Pi \cap \Omega) \cap \{ |y| \leq \varepsilon^2 \}} \frac{dy}{|x-y|^{n+s}} + \int_{(\Pi \cap \Omega) \cap \{ |y| < \varepsilon^2 \}} \frac{dy}{|x-y|^{n+s}}
$$

$$
=: I_s^1(x) + I_s^2(x).
$$

Then, similar to the case 1,

$$
I_s^1(x) \leq n \omega_n \int_{\varepsilon^2}^{\infty} \frac{1}{\rho^{s+1}} \, d\rho = \frac{n \omega_n}{s \varepsilon^{2s}}.
$$

hence (since all cubes are contained in $\Omega$)

$$
\int_{\Pi \cap \bigcup_{i=1}^{N_{\varepsilon}} Q_i} I_s^1(x) \, dx \leq \frac{\mathcal{L}^n(\Omega) n \omega_n}{s \varepsilon^{2s}}.
$$

As for $I_s^2(x)$ observe that if $x \in Q_i^c$ and $|x-y| \leq \varepsilon^2$, then $y \in \tilde{Q}_i^c$, where $\tilde{Q}_i^c$ is the cube obtained by dilating $Q_i^c$ by a factor $1 + \varepsilon$ (hence the side length of $\tilde{Q}_i^c$ is $\varepsilon + \varepsilon^2$). Then

$$
\int_{\Pi \cap \bigcup_{i=1}^{N_{\varepsilon}} Q_i^c} I_s^2(x) \, dx \leq \sum_{i=1}^{N_{\varepsilon}} \int_{\Pi \cap \tilde{Q}_i^c} \int_{\Pi \cap \tilde{Q}_i^c} \frac{1}{|x-y|^{n+s}} \, dy \, dx \leq \sum_{i=1}^{N_{\varepsilon}} \int_{\Pi \cap \tilde{Q}_i^c} \int_{\Pi \cap \tilde{Q}_i^c} \frac{1}{|x-y|^{n+s}} \, dy \, dx
$$

$$
= N_{\varepsilon} J_s^1(H, (\varepsilon + \varepsilon^2)Q) = N_{\varepsilon}(\varepsilon + \varepsilon^2)^{n-s} J_s^1(H, Q),
$$

(22)

where in the last identity we used the scaling property (11). Keeping $\varepsilon > 0$ fixed, letting $s$ go to 1 and putting (18), (22) together we infer

$$
\limsup_{s \uparrow 1} (1-s) J_s^1(\Pi, \Omega) \leq o(1) + \limsup_{s \uparrow 1} (1-s) J_s^1(H, Q) N_{\varepsilon}(\varepsilon + \varepsilon^2)^{n-1}
$$

$$
= o(1) + \Gamma_n^* P(\Pi, \Omega),
$$

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with error $o(1) \to 0$ as $\varepsilon \to 0$ uniformly in $s$. Since $\varepsilon > 0$ is arbitrary, we conclude

$$\limsup_{s \uparrow 1} (1 - s) \mathcal{J}_s^1(\Pi, \Omega) \leq \Gamma^*_n P(\Pi, \Omega).$$

**Step 2.** It now remains to estimate $\mathcal{J}_s^2$. Let us start by considering the term

$$\int_{\Pi \cap \Omega} \int_{\Pi^c \cap \Omega^c} \frac{1}{|x - y|^{n+s}} dy dx.$$

**Case 1:** $x \in \Pi \cap (\Omega \setminus \Omega^-)$. Then for $y \in \Pi^c \cap \Omega^c$ we have $|x - y| \geq \delta$, whence

$$I(x) := \int_{\Pi \cap \Omega^c} \frac{dy}{|x - y|^{n+s}} \leq n\omega_n \int_{\delta}^{\infty} \frac{d\rho}{\rho^{1+s}} = \frac{n\omega_n}{s\delta^s}.$$

**Case 2:** $x \in \Pi \cap \Omega^-$. In this case, using the same argument of case 1 for $y \in \Pi^c \cap (\Omega^c \setminus \Omega^+)\cup \Omega^+\delta$, we have

$$I(x) = \int_{\Pi \cap \Omega^c} \frac{dy}{|x - y|^{n+s}} \leq \int_{\Pi \cap \Omega^c} \frac{dy}{|x - y|^{n+s}} + \frac{n\omega_n}{s\delta^s}.$$

Therefore

$$\int_{\Pi \cap \Omega} \int_{\Pi^c \cap \Omega^c} \frac{dy dx}{|x - y|^{n+s}} \leq \frac{2n\omega_n|\Omega|}{s\delta^s} + \int_{\Pi \cap \Omega^-} \int_{\Pi \cap \Omega^+} \frac{dy dx}{|x - y|^{n+s}} \leq \frac{2n\omega_n|\Omega|}{s\delta^s} + \int_{\Pi \cap (\Omega^- \cup \Omega^+)} \int_{\Pi \cap (\Omega^- \cup \Omega^+)} \frac{dy dx}{|x - y|^{n+s}}.$$

An obvious similar estimate can be obtained by swapping $\Pi$ and $\Pi^c$, finally yielding

$$\mathcal{J}_s^2(\Pi, \Omega) \leq \frac{4n\omega_n|\Omega|}{s\delta^s} + 2 \int_{\Pi \cap (\Omega^- \cup \Omega^+)} \int_{\Pi \cap (\Omega^- \cup \Omega^+)} \frac{dy dx}{|x - y|^{n+s}} = \frac{4n\omega_n|\Omega|}{s\delta^s} + 2 \mathcal{J}_s^1(\Pi, \Omega^- \cup \Omega^+).$$

Using the result of step 1 we get

$$\limsup_{s \uparrow 1} (1 - s) \mathcal{J}_s^2(\Pi, \Omega) \leq 2\Gamma^*_n P(\Pi, \Omega^- \cup \Omega^+).$$

Since $\delta > 0$ is arbitrary, letting $\delta$ go to zero we conclude the proof of the lemma.

\[\square\]

**Lemma 9 (Characterization of $\Gamma^*_n$)** The $\limsup$ in (17) is a limit and $\Gamma^*_n = \omega_{n-1}$. 

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Proof. The proof is inspired from [5, Lemma 11]. We shall actually prove a slightly stronger statement. Set for $a > 0$

$$Q_a := \{ x : |x_i| \leq 1/2 \text{ for } 1 \leq i \leq n - 1, |x_n| \leq a \}.$$ 

Then we show that

$$\lim_{s \uparrow 1}(1 - s)J_s^1(H, Q_a) = \omega_{n-1}, \quad \forall a > 0.$$ 

Let us first consider the case $n \geq 2$. Fix $x \in Q_a \cap H$ and write as usual $x = (x', x_n)$, $y = (y', y_n)$. We consider

$$I_s(x) := \int_{Q_a \cap H} \frac{1}{|x - y|^{n+s}} dy = \int_0^a \int_{Q_a \cap \partial H} \frac{1}{|x - y|^{n+s}} dy dy_n.$$ 

With the change of variable $z' = (y' - x')/|y_n - x_n|$ and setting

$$\Sigma(x, y_n) := \left\{ z' \in \mathbb{R}^{n-1} : \left| z'_i + \frac{x_i'}{x_n - y_n} \right| \leq \frac{1}{2|x_n - y_n|} \text{ for } 1 \leq i \leq n - 1 \right\},$$

we get

$$I_s(x) = \int_0^a \int_{\Sigma(x, y_n)} \frac{1}{|x_n - y_n|^{s+1}(1 + |z'|^2)^{(n+s)/2}} dz' dy_n \leq \int_0^a \frac{1}{|x_n - y_n|^{s+1}} \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |z'|^2)^{(n+s)/2}} dz'$$

$$= \frac{(-x_n)^{-s} - (a - x_n)^{-s}}{s} \cdot (n - 1)\omega_{n-1} \int_0^{\infty} \frac{\rho^{n-2}}{(1 + \rho^2)^{(n+1)/2}} d\rho.$$ 

Now integrating $I$ with respect to $x$, observing that $\mathcal{H}^{n-1}(Q_a \cap \partial H) = 1$ and that by dominated convergence one has

$$\lim_{s \uparrow 1} \int_0^{\infty} \frac{\rho^{n-2}}{(1 + \rho^2)^{(n+1)/2}} d\rho = \int_0^{\infty} \frac{\rho^{n-2}}{(1 + \rho^2)^{(n+1)/2}} d\rho$$

$$= \left[ \frac{\rho^{n-1}}{(n - 1)(1 + \rho^2)^{(n-1)/2}} \right]_0^{\infty} = \frac{1}{n - 1},$$

we get

$$\int_{H \cap Q_a} I_s(x) dx \leq \mathcal{H}^{n-1}(Q_a \cap \partial H) \sup_{x' \in Q_a \cap \partial H} \int_{-a}^{0} I_s(x', x_n) dx_n$$

$$\leq \omega_{n-1}(1 + o(1)) \int_{-a}^{0} \frac{(-x_n)^{-s} - (a - x_n)^{-s}}{s} dx_n$$

$$= \omega_{n-1}(1 + o(1)) a^{1-s}(2 - 2^{1-s})$$

$$s(1 - s),$$

with error $o(1) \to 0$ as $s \uparrow 1$ dependent only on $s$. Therefore

$$\limsup_{s \uparrow 1} (1 - s)J_s^1(H, Q_a) = \limsup_{s \uparrow 1} (1 - s) \int_{H \cap Q_a} I_s(x) dx \leq \omega_{n-1}.$$ 

(25)
Now observing that for $\varepsilon$ small enough

$$|x_n| \leq \varepsilon^2, \quad |y_n| \leq \varepsilon^2, \quad |x_i| \leq \frac{1}{2} - \varepsilon \quad \text{for} \ 1 \leq i \leq n - 1$$

implies that $B_{1/(2\varepsilon)}(0) \subset \Sigma(x, y_n)$, similar to (23) we estimate

$$I_s(x) \geq \int_0^{\varepsilon^2} \int_{Q_0 \cap \partial H} \frac{1}{|x-y|^{n+s}} dy/2^s d_z dy_n,$$

whenever $x$ is as in (26). Integrating with respect to $x$ satisfying (26) one has

$$\int_{H \cap Q_a} I_s(x) dx \geq (1 - 2\varepsilon)^{n-1} \int_{-\varepsilon^2}^{0} \frac{(-x_n)^{-s} - (\varepsilon^2 - x_n)^{-s}}{s} dx_n \times (n-1)\omega_{n-1} \int_0^{\frac{1}{2\varepsilon}} \frac{\rho^{n-2}}{(1 + \rho^2)^{(n+s)/2}} d\rho,$$

where $n \geq 2$.

3.3 Gluing construction and characterization of the geometric constants

A key observation in [15], which we shall need, is that $F$ satisfies a generalized coarea formula, namely $F_s(u, \Omega) = \int_0^1 F_s(\chi_{\{u \geq t\}}, \Omega) dt$; we reproduce here the simple proof of this fact and we state the result in terms of $\mathcal{J}_s$. 

$$\lim_{s \uparrow 1} (1-s) \mathcal{J}_s(H, Q_a) = \omega_n,$$

which together with (25) completes the proof when $n \geq 2$.

When $n = 1$ one computes explicitly

$$\mathcal{J}_s^1(H, Q_a) = \int_{-a}^a \int_0^a \frac{1}{|x-y|^{1+s}} dy dx = \int_{-a}^a \frac{(-x)^{-s} - (a - x)^{-s}}{s} dx = \frac{a^{1-s}(2 - 2^{1-s})}{s(1-s)} ,$$

hence

$$\lim_{s \uparrow 1} (1-s) \mathcal{J}_s^1(H, Q_a) = 1 = \omega_0.$$
Lemma 10 (Coarea formula) For every measurable function \( u : \Omega \to [0, 1] \) we have
\[
\frac{1}{2} F_s(u, \Omega) = \int_0^1 J_s^1(\{u > t\}, \Omega) dt.
\]

Proof. Given \( x, y \in \Omega \), the function \( t \mapsto \chi_{\{u > t\}}(x) - \chi_{\{u > t\}}(y) \) takes its values in \( \{-1, 0, 1\} \) and it is nonzero precisely in the interval having \( u(x) \) and \( u(y) \) as extreme points, hence
\[
|u(x) - u(y)| = \int_0^1 |\chi_{\{u > t\}}(x) - \chi_{\{u > t\}}(y)| dt.
\]
Substituting into (5), using Fubini’s theorem and observing that
\[
|\chi_{\{u > t\}}(x) - \chi_{\{u > t\}}(y)| = \chi_{\{u > t\}}(x) \chi_{\Omega \setminus \{u > t\}}(y) + \chi_{\Omega \setminus \{u > t\}}(x) \chi_{\{u > t\}}(y),
\]
we infer
\[
F_s(u, \Omega) = \int_\Omega \int_\Omega \int_0^1 |\chi_{\{u > t\}}(x) - \chi_{\{u > t\}}(y)| \frac{1}{|x - y|^{n+s}} dx dy dt
\]
\[
= 2 \int_0^1 \int_{\{u > t\}} \int_{\Omega \setminus \{u > t\}} \frac{1}{|x - y|^{n+s}} dx dy dt
\]
\[
= 2 \int_0^1 J_s^1(\{u > t\}, \Omega) dt.
\]
\[\Box\]

Proposition 11 (Gluing) Given \( s \in (0, 1) \), measurable sets \( E_1, E_2 \) in \( \mathbb{R}^n \) with \( J_s^1(E_i, \Omega) < \infty \) for \( i = 1, 2 \) and given \( \delta_1 > \delta_2 > 0 \) we can find a measurable set \( F \) such that
\[
(a) \quad \|\chi_F - \chi_{E_1}\|_{L^1(\Omega)} \leq \|\chi_{E_1} - \chi_{E_2}\|_{L^1(\Omega)},
\]
\[
(b) \quad F \cap (\Omega \setminus \Omega_{\delta_1}) = E_1 \cap (\Omega \setminus \Omega_{\delta_1}), \quad F \cap \Omega_{\delta_2} = E_2 \cap \Omega_{\delta_2}, \quad \text{where}
\]
\[
\Omega_{\delta} := \{x \in \Omega : d(x, \Omega^c) \leq \delta\} \quad \text{for} \quad \delta > 0,
\]
\[
(c) \quad \text{for all} \quad \varepsilon > 0 \quad \text{we have}
\]
\[
J_s^1(F, \Omega) \leq J_s^1(E_1, \Omega) + J_s^1(E_2, \Omega_{\delta_1 + \varepsilon}) + \frac{C}{\varepsilon^{n+s}}
\]
\[
+ C(\Omega, \delta_1, \delta_2) \left[ \frac{\|\chi_{E_1} - \chi_{E_2}\|_{L^1(\Omega_{\delta_1} \setminus \Omega_{\delta_2})}}{(1 - s)} + \|\chi_{E_1} - \chi_{E_2}\|_{L^1(\Omega)} \right].
\]

Proof. Consider a function \( \varphi \in C^\infty(\mathbb{R}^n) \) such that \( 0 \leq \varphi \leq 1 \) in \( \Omega \), \( \varphi \equiv 0 \) in \( \Omega_{\delta_2} \), \( \varphi \equiv 1 \) in \( \Omega \setminus \Omega_{\delta_1} \), and \( |\nabla \varphi| \leq 2/(\delta_1 - \delta_2) \).
Given two measurable functions \( u, v : \Omega \to [0,1] \) such that \( \mathcal{F}_s(u, \Omega) < \infty, \mathcal{F}_s(v, \Omega) < \infty, \) define \( w : \Omega \to [0,1] \) as \( w := \varphi u + (1 - \varphi)v. \) For \( x, y \in \Omega \) we can write

\[
\begin{align*}
    w(x) - w(y) &= (\varphi(x) - \varphi(y))u(y) + \varphi(x)(u(x) - u(y)) \\
    &\quad + (1 - \varphi(x))(v(x) - v(y)) - v(y)(\varphi(x) - \varphi(y)) \\
    &= (\varphi(x) - \varphi(y))(u(y) - v(y)) + \varphi(x)(u(x) - u(y)) \\
    &\quad + (1 - \varphi(x))(v(x) - v(y)),
\end{align*}
\]

and infer

\[
|w(x) - w(y)| \leq |\varphi(x) - \varphi(y)||u(y) - v(y)| \\
    + \chi_{\{\varphi \neq 0\}}(x)|u(x) - u(y)| + \chi_{\{\varphi \neq 1\}}(x)|v(x) - v(y)|.
\]

Observing that \( \{ \varphi \neq 0 \} \subset \Omega \setminus \Omega_{\delta_2} \) and \( \{ \varphi \neq 1 \} \subset \Omega_{\delta_1} \) we get

\[
\mathcal{F}_s(w, \Omega) \leq \int_{\Omega} |u(y) - v(y)| \int_{\Omega} |\varphi(x) - \varphi(y)| \frac{dx \, dy}{|x - y|^{n+s}} \\
    + \int_{\Omega \setminus \Omega_{\delta_2}} \int_{\Omega} |u(x) - u(y)| \frac{dx \, dy}{|x - y|^{n+s}} + \int_{\Omega_{\delta_1}} \int_{\Omega} |v(x) - v(y)| \frac{dx \, dy}{|x - y|^{n+s}} \\
    =: I_1 + I_2 + I_3.
\]

From

\[
|\varphi(x) - \varphi(y)| \leq |\nabla \varphi(y)||x - y| + \frac{1}{2} \|\nabla^2 \varphi\|_{\infty} |x - y|^2
\]

and the inequalities \( \int_{\Omega} |x - y|^{-(n+s-\alpha)} dx \leq C(\Omega)/(\alpha - s) \) (with \( \alpha = 1, \alpha = 2 \)) we have

\[
I_1 \leq \int_{\Omega} |u(y) - v(y)| \int_{\Omega} \left( \frac{|\nabla \varphi(y)|}{|x - y|^{n+s-1}} + \frac{\|\nabla^2 \varphi\|_{\infty}}{2|x - y|^{n+s-2}} \right) \frac{dx \, dy}{|x - y|^{n+s}} \\
    \leq C(\Omega, \delta_1, \delta_2) \left( \frac{\|u - v\|_{L^1(\Omega \setminus \Omega_{\delta_2})}}{1-s} + \frac{\|u - v\|_{L^1(\Omega)}}{(2-s)} \right).
\]

Clearly \( I_2 \leq \mathcal{F}_s(u, \Omega). \) As for \( I_3, \) choosing \( \varepsilon > 0 \) we get

\[
I_3 \leq \int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_1+\varepsilon}} \frac{|v(x) - v(y)|}{|x - y|^{n+s}} \frac{dx \, dy}{|x - y|^{n+s}} + \int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_1+\varepsilon}} \frac{|v(x) - v(y)|}{|x - y|^{n+s}} \frac{dx \, dy}{|x - y|^{n+s}} \\
    \leq \mathcal{F}_s(v, \Omega_{\delta_1+\varepsilon}) + \frac{2C^n(\Omega_{\delta_1})C^n(\Omega \setminus \Omega_{\delta_1+\varepsilon})}{\varepsilon^{n+s}}.
\]

Summing up we obtain

\[
\mathcal{F}_s(w, \Omega) \leq \mathcal{F}_s(u, \Omega) + \mathcal{F}_s(v, \Omega_{\delta_1+\varepsilon}) + C(\Omega, \delta_1, \delta_2) \left( \frac{\|u - v\|_{L^1(\Omega \setminus \Omega_{\delta_2})}}{1-s} \right) \\
    + C(\Omega, \delta_1, \delta_2) \|u - v\|_{L^1(\Omega)} + \frac{C(\Omega)}{\varepsilon^{n+s}}.
\]

(27)
We now apply this with \( u = \chi_{E_1}, v = \chi_{E_2} \), so that (27) reads as
\[
\mathcal{F}_s(w, \Omega) \leq 2J^1_s(E_1, \Omega) + 2J^1_s(E_2, \Omega_{\delta_1+\varepsilon}) + C(\Omega, \delta_1, \delta_2) \frac{\|\chi_{E_1} - \chi_{E_2}\|_{L^1(\Omega_{\delta_1}\setminus\Omega_{\delta_2})}}{1-s} \\
+ C(\Omega, \delta_1, \delta_2)\|\chi_{E_1} - \chi_{E_2}\|_{L^1(\Omega)} + \frac{C(\Omega)}{\varepsilon^{n+s}},
\]
and by Lemma 10 there exists \( t \in (0, 1) \) such that \( F := \{w > t\} \) satisfies
\[
2J^1_s(F, \Omega) \leq \mathcal{F}_s(w, \Omega).
\]
By construction we see that \( \tilde{\Gamma} = \Gamma \) (by the gluing Proposition 11) and then use the local minimality of hyperplanes to show that \( \tilde{\Gamma} = \Gamma \) (by the gluing Proposition 11) and then use the local minimality of hyperplanes to show that \( \tilde{\Gamma} = \Gamma \).

The following corollary is an immediate consequence of Proposition 11.

**Corollary 12** Given measurable sets \( E_s \subset \mathbb{R}^n \) for \( s \in (0, 1) \), with \( \chi_{E_s} \to \chi_E \) in \( L^1(\Omega) \) as \( s \uparrow 1 \) and with \( J^1_s(E_s, \Omega) < \infty, J^1_s(E, \Omega) < \infty \), and given \( \delta_1 > \delta_2 > 0 \) we can find measurable sets \( F_s \subset \mathbb{R}^n \) such that
\[
\begin{align*}
(a) \ & \chi_{F_s} \to \chi_E \text{ in } L^1(\Omega) \text{ as } i \to \infty, \\
(b) \ & F_s \cap (\Omega \setminus \Omega_{\delta_1}) = E_s \cap (\Omega \setminus \Omega_{\delta_1}), \ F_s \cap \Omega_{\delta_2} = E \cap \Omega_{\delta_2}, \\
(c) \ & \liminf_{s \uparrow 1}(1-s)J^1_s(F_s, \Omega) \leq \liminf_{s \uparrow 1}(1-s)J^1_s(E_s, \Omega) + \limsup_{s \uparrow 1}(1-s)J^1_s(E, \Omega_{\delta_1+\varepsilon}).
\end{align*}
\]

We devote the rest of the section to the proof of the equality of the constants \( \Gamma_n \) and \( \Gamma^*_n \) appearing in the proof of the \( \Gamma \)-liminf and \( \Gamma \)-limsup respectively (we already proved that \( \Gamma^*_n = \omega_n^{-1} \)). We shall introduce an intermediate quantity \( \tilde{\Gamma}_n \in [\Gamma_n, \Gamma^*_n] \) and prove in two steps that \( \tilde{\Gamma}_n = \Gamma_n \) (by the gluing Proposition 11) and then use the local minimality of hyperplanes to show that \( \tilde{\Gamma}_n = \Gamma_n \).

**Lemma 13** We have \( \Gamma_n = \tilde{\Gamma}_n \), where
\[
\tilde{\Gamma}_n := \inf \{ \liminf_{s \uparrow 1}(1-s)J^1_s(E_s, Q) \}
\]
with the infimum taken over all families of measurable sets \( (E_s)_{0 < s < 1} \) with the property that \( \chi_{E_s} \to \chi_H \) in \( L^1(Q) \) as \( s \uparrow 1 \) and, for some \( \delta > 0 \), \( E_s \cap Q^\delta = H \cap Q^\delta \) for all \( s \in (0, 1) \), where \( Q^\delta = \{x \in Q : d(x, Q^\infty) < \delta\} \).

**Proof.** Clearly \( \tilde{\Gamma}_n \geq \Gamma_n \). In order to prove the converse consider sets \( E_s \subset \mathbb{R}^n \) for \( s \in (0, 1) \) with \( \chi_{E_s} \to \chi_H \) in \( L^1(Q) \) as \( s \uparrow 1 \). Without loss of generality we can assume that \( J^1_s(E_s, \Omega) < \infty \)
for all $s \in (0,1)$. Then according to Corollary 12 for any given $\delta > 0$ we can find a family of measurable sets $(F_s)_{0<s<1}$ such that $\chi_{F_s} \to \chi_{H}$ in $L^1(Q)$ as $s \uparrow 1$, $F_s \cap Q^\delta = H \cap Q^\delta$ and

$$\tilde{\Gamma}_n \leq \liminf_{s \uparrow 1}(1-s)J^1_s(F_s, \Omega) \leq \liminf_{s \uparrow 1}(1-s)J^1_s(E_s, Q) + \Gamma^{*}_{n} \inf_{\varepsilon > 0} P(H, Q^{\delta+\varepsilon}),$$

where we also used Lemma 8. Since $\delta > 0$ is arbitrary and $P(H, Q^\delta) \to 0$ as $\delta \to 0$ we infer

$$\tilde{\Gamma}_n \leq \liminf_{s \uparrow 1}(1-s)J^1_s(E_s, Q)$$

and, since $(E_s)_{0<s<1}$ is arbitrary, this proves that $\tilde{\Gamma}_n \leq \Gamma_n$. □

**Lemma 14** We have $\tilde{\Gamma}_n = \Gamma^{*}_n$.

**Proof.** Clearly $\tilde{\Gamma}_n \leq \Gamma^{*}_n$. In order to prove the converse we consider sets $(E_s)_{0<s<1}$ with $\chi_{E_s} \to \chi_{H}$ in $L^1(Q)$ as $s \uparrow 1$ and with $E_s \cap Q^\delta = H \cap Q^\delta$ for some $\delta > 0$ (here $Q^\delta$ is defined as in Lemma 13). Since our goal is to estimate $J^1_s(E_s, Q)$ from below, possibly modifying $E_s$ outside $Q$ we may assume that

$$E_s \cap (\mathbb{R}^n \setminus Q) = H \cap (\mathbb{R}^n \setminus Q).$$

This implies, according to Proposition 17 in the Appendix, that $J_s(H, Q) \leq J_s(E_s, Q)$ for $s \in (0,1)$. Then, in order to prove that

$$\lim_{s \uparrow 1}(1-s)J^1_s(H, Q) \leq \liminf_{s \to 1-}(1-s)J^1_s(E_s, Q),$$

it is enough to show that

$$\lim_{s \uparrow 1}(1-s)(J^2_s(H, Q) - J^2_s(E_s, Q)) = 0.$$  \hspace{1cm} (31)

One immediately sees that (29) implies

$$|J^2_s(H, Q) - J^2_s(E_s, Q)| \leq \int_{(E_s \Delta H) \cap Q} \int_{H \cap Q^\delta} \frac{1}{|x-y|^{n+s}} dx \, dy + \int_{(E_s \Delta H^c) \cap Q} \int_{H \cap Q^\delta} \frac{1}{|x-y|^{n+s}} dx \, dy =: I + II.$$

Observing that $(E_s \Delta H) \cap Q^\delta = \emptyset$ we can estimate for $y \in (E_s \Delta H) \cap Q$

$$\int_{H \cap Q^\delta} \frac{1}{|x-y|^{n+s}} dx \leq \int_{\mathbb{R}^n \setminus B_0(y)} \frac{1}{|x-y|^{n+s}} dx = \frac{n \omega_n}{s \delta^s},$$

hence $I \leq n \omega_n/(s \delta^s)$. One can bound from above $II$ in the same way. Now (31) follows at once upon multiplying by $1-s$ and letting $s \uparrow 1$. This shows (30), and taking the infimum in (30) over all families $(E_s)_{0<s<1}$ as above shows that $\Gamma^{*}_n \leq \tilde{\Gamma}_n$. □
4 Proof of Theorem 3

In order to prove (3) define $\Omega_\delta$ as in Proposition 11 for some small $\delta > 0$ and set $F_i := E_i \cap (\Omega^c \cup \Omega_\delta)$. By the minimality of $E_i$ we then have

$$\limsup_{i \to \infty} (1 - s_i) \mathcal{J}_{s_i}(E_i, \Omega \setminus \Omega_\delta) \leq \limsup_{i \to \infty} (1 - s_i) \left( \mathcal{J}_{s_i}(E_i, \Omega) - \mathcal{J}^1_{s_i}(E_i, \Omega_\delta) \right)$$

$$\leq \limsup_{i \to \infty} (1 - s_i) \left( \mathcal{J}_{s_i}(F_i, \Omega) - \mathcal{J}^1_{s_i}(E_i, \Omega_\delta) \right)$$

$$= \limsup_{i \to \infty} (1 - s_i) \left( (\mathcal{J}^1_{s_i}(F_i, \Omega) - \mathcal{J}^1_{s_i}(F_i, \Omega_\delta)) + \mathcal{J}^2_{s_i}(F_i, \Omega) \right).$$

Since $F_i \cap (\Omega \setminus \Omega_\delta) = \emptyset$ we have, using Proposition 16 in the appendix,

$$\limsup_{i \to \infty} (1 - s_i) \left( \mathcal{J}^1_{s_i}(F_i, \Omega) - \mathcal{J}^1_{s_i}(F_i, \Omega_\delta) \right) \leq \limsup_{i \to \infty} (1 - s_i) \mathcal{J}^1_{s_i}(\Omega \setminus \Omega_\delta, \Omega)$$

$$= \limsup_{i \to \infty} (1 - s_i) \frac{\mathcal{F}_{s_i}(\chi_{\Omega \setminus \Omega_\delta}, \Omega)}{2^2}$$

$$\leq \frac{n\omega_n P(\Omega \setminus \Omega_\delta, \mathbb{R}^n)}{2}.$$

Again using Proposition 16 in the appendix we get

$$\limsup_{i \to \infty} (1 - s_i) \mathcal{J}^2_{s_i}(F_i, \Omega) \leq \limsup_{i \to \infty} (1 - s_i) \mathcal{J}^1_{s_i}(\Omega, \mathbb{R}^n) \leq \frac{n\omega_n P(\Omega, \mathbb{R}^n)}{2},$$

whence (3) follows for $\Omega' \subset \Omega \setminus \Omega_\delta$, hence for every $\Omega' \in \Omega$.

For the sake of simplicity we first consider perturbations in compactly supported balls. The general case will require only minor modifications.

Consider the monotone set function $\alpha_i(A) := (1 - s_i) \mathcal{J}^1_{s_i}(E_i, A)$ for every open set $F \subset \Omega$ (see the appendix for the definition and some basic properties of monotone set functions), extended to

$$\alpha_i(F) := \inf \{ \alpha_i(A) : F \subset A \subset \Omega, \ A \text{ open} \}$$

for every $F \subset \Omega$. Clearly $\alpha_i$ is regular. Thanks to (3) and Theorem 21 up to extracting a subsequence, $\alpha_i$ weakly converges to a regular monotone set function $\alpha$, which is regular and super-additive. We shall now prove that if $B_R(x) \in \Omega$ and $\alpha(\partial B_R(x)) = 0$, then $E$ is a local minimum of the functional $P(\cdot, B_R(x))$, and

$$\lim_{i \to \infty} (1 - s_i) \mathcal{J}_{s_i}(E_i, B_R(x)) = P(E, B_R(x)).$$

Indeed consider a Borel set $F \subset \Omega$ such that $E \Delta F \Subset B_R$ (here and in the following $x$ is fixed and $B_r := B_r(x)$ for any $r > 0$). Then we can find $r < R$ such that $E \Delta F \subset B_r$. By Theorem 2 there exist sets $F_i$ such that

$$\lim_{i \to \infty} |(F_i \Delta F) \cap B_R| = 0, \quad \lim_{i \to \infty} (1 - s_i) \mathcal{J}_{s_i}(F_i, B_R) = \omega_{n-1} P(F, B_R).$$
According to Proposition [11] given $\rho$ and $t$ with $r < \rho < t < R$, we can find sets $G_i$ such that

$$G_i = E_i \cap B_t, \quad G_i = F_i \cap B_\rho,$$

and for all $\varepsilon > 0$ there holds

$$\mathcal{J}_s^1(G_i, B_R) \leq \mathcal{J}_s^1(F_i, B_R) + \mathcal{J}_s^1(E_i, B_R \setminus \overline{B}_{\rho - \varepsilon}) + \frac{C}{\varepsilon^{n+s_i}} + C(\mathcal{E}_i \Delta F_i) \leq 1 + C|\mathcal{(E}_i \Delta E_i) \cap B_R|.$$

By the local minimality of $E_i$ we infer

$$\mathcal{J}_s^1(E_i, B_R) \leq \mathcal{J}_s^1(G_i, B_R).$$

We shall now estimate

$$\mathcal{J}_s^2(G_i, B_R) = \int_{G_i \cap B_R} \int_{E_i \cap B_R} \frac{dxdy}{|x - y|^{n+s_i}} + \int_{G_i \cap B_R} \int_{F_i \cap B_R} \frac{dxdy}{|x - y|^{n+s_i}} =: I + II$$

We have

$$I = \int_{E_i \cap B_R} \int_{E_i \cap B_R} \frac{dxdy}{|x - y|^{n+s_i}} = \int_{E_i \cap B_R} \int_{E_i \cap B_R} \frac{dxdy}{|x - y|^{n+s_i}} + \int_{E_i \cap (B_R \setminus B_t)} \int_{E_i \cap (B_R \setminus B_t)} \frac{dxdy}{|x - y|^{n+s_i}}$$

$$\leq \frac{C(\mathcal{E}_i \cap B_R)}{s_i(R - t)^{s_i}} + \frac{C(\mathcal{E}_i \cap B_R)}{s_i(R - t)^{s_i}} + \frac{C(\mathcal{E}_i \cap B_R)}{s_i(R - t)^{s_i}} + \frac{C(\mathcal{E}_i \cap B_R)}{s_i(R - t)^{s_i}}$$

$$\leq \mathcal{J}_s^1(E_i, B_R \setminus \overline{B}_t) + \frac{C}{s_i} \left( \frac{1}{(R - t)^{s_i}} + \frac{1}{(R' - R)^{s_i}} \right),$$

for any $R' \in (R, \text{dist}(x, \partial \Omega))$. Since $II$ can be estimated in a similar way, we infer

$$\mathcal{J}_s^2(G_i, B_R) \leq 2\mathcal{J}_s^1(E_i, B_R \setminus \overline{B}_t) + \frac{C}{s_i} \left( \frac{1}{(R - t)^{s_i}} + \frac{1}{(R' - R)^{s_i}} \right),$$

hence,

$$\limsup_{i \to \infty} (1 - s_i)\mathcal{J}_s^2(G_i, B_R) \leq 2\limsup_{i \to \infty} (1 - s_i)\mathcal{J}_s^1(E_i, B_R \setminus \overline{B}_t).$$

Finally

$$\omega_{n-1} P(E, B_R) \leq \liminf_{i \to \infty} (1 - s_i)\mathcal{J}_s^1(E_i, B_R) \leq \liminf_{i \to \infty} (1 - s_i)\mathcal{J}_s^1(G_i, B_R)$$

$$\leq \liminf_{i \to \infty} (1 - s_i)\mathcal{J}_s^1(F_i, B_R) + \limsup_{i \to \infty} (1 - s_i)\mathcal{J}_s^2(G_i, B_R)$$

$$\leq \liminf_{i \to \infty} (1 - s_i)\mathcal{J}_s^1(F_i, B_R) + 3\limsup_{i \to \infty} (1 - s_i)\mathcal{J}_s^1(E_i, B_R \setminus \overline{B}_{\rho - \varepsilon})$$

$$+ C\lim_{i \to \infty} |(E_i \Delta F_i) \cap (B_1 \setminus B_\rho)|.$$
The last term is zero, since \( E = F \) in \( B_i \setminus B_{\rho} \) and \( |(E_i \Delta E) \cap B_R| \to 0, |(F_i \Delta F) \cap B_R| \to 0 \) as \( i \to \infty \). Using Proposition 22 from the appendix, and recalling that \( \alpha(\partial B_R) = 0 \), we infer

\[
\lim_{R' \downarrow R, \rho' \uparrow R, \varepsilon \downarrow 0} \limsup_{i \to \infty} (1 - s_i) J^1 s_i (E_i, B_{R'} \setminus B_{\rho}) = \lim_{\delta \to 0} \limsup_{i \to \infty} \alpha_i (B_{R+\delta} \setminus B_{R-\delta}) = 0,
\]

and (32) finally yields

\[
\omega_{n-1} P(E, B_R) \leq \lim_{i \to \infty} (1 - s_i) J^1 s_i (E_i, B_R) = \omega_{n-1} P(F, B_R),
\]

so \( E \) is a local minimizer of \( P(\cdot, B_R) \). Choosing \( F = E \) the chain of inequalities in (32) gives

\[
\lim_{i \to \infty} (1 - s_i) J^1 s_i (E_i, B_R) = \omega_{n-1} P(E, B_R), \tag{33}
\]

as wished. In order to complete the proof we first remark that the above arguments applies to any open set \( \Omega' \subset \Omega \) with Lipschitz boundary and \( \alpha(\partial \Omega') = 0 \), upon replacing \( B_R(x) \) by \( \Omega' \), \( B_{R+\delta} \) by \( N_\delta(\Omega') \) and \( B_{R-\delta} \) by \( N_{-\delta}(\Omega') \), where

\[
N_\delta(\Omega') := \{ x \in \Omega : d(x, \Omega') < \delta \}, \quad N_{-\delta}(\Omega') := \{ x \in \Omega' : d(x, \partial \Omega') > \delta \} \quad \text{for} \ \delta > 0 \ \text{small}.
\]

In particular \( \alpha(\Omega') = P(E, \Omega') \) for every open set \( \Omega' \subset \Omega \) with Lipschitz boundary and \( \alpha(\partial \Omega') = 0 \). Since for every \( \Omega' \subset \Omega \) and \( \varepsilon > 0 \) small enough the set

\[
\{ \delta \in (-\varepsilon, \varepsilon) : \alpha(\partial N_\delta(\Omega')) > 0 \}
\]

is at most countable (remember that \( \alpha \) is super-additive and locally finite), and since both \( \alpha \) and \( P(E, \cdot) \) are regular monotone set functions on \( \Omega \), it is not difficult to show that \( \alpha = P(E, \cdot) \), and the proof is complete. \( \square \)

5 Appendix. Some useful results

We list here some results which we used in the previous sections.

**Proposition 15** Let \( E \subset \mathbb{R}^n \) be a set with finite perimeter in \( \Omega \). Then for every \( \varepsilon > 0 \) there exists a polyhedral set \( \Pi \subset \mathbb{R}^n \) such that

(i) \( |(E \Delta \Pi) \cap \Omega| < \varepsilon \),

(ii) \( |P(E, \Omega) - P(\Pi, \Omega)| < \varepsilon \),

(iii) \( P(\Pi, \partial \Omega) = 0 \).

**Proof.** Classical theorems (see for example [1, 7]) imply that there exists a polyhedral set \( \Pi' \) satisfying (i) and (ii). In order to get (iii) first notice that

\[
P(\Pi', \partial \Omega) > 0 \text{ if and only if } \mathcal{H}^{n-1}(\partial \Pi' \cap \partial \Omega) > 0,
\]
and that the latter condition can be satisfied only if \( \partial \Omega \) contains a piece \( \Sigma \) with \( H^{n-1}(\Sigma) > 0 \) contained in a hyperplane and \( \nu_\Omega = \pm \nu_\Omega = \text{const on } \Sigma \) (here \( \nu_\Omega \) and \( \nu_\Omega \) denote the interior unit normal to \( \partial \Omega \) and \( \partial \Omega' \) respectively). Since the set
\[
\{ \nu \in S^{n-1} : H^{n-1}(\{ x \in \partial \Omega : \nu_\Omega(x) = \nu \}) > 0 \}
\]
is at most countable, it is easy to see that there exists a rotation \( R \in SO(n) \) close enough to the identity so that the polyhedron \( \Pi := R(\Pi') \) satisfies (i), (ii) and (iii).

**Proposition 16** Let \( u \in BV(\Omega) \) and let \( \Omega' \subseteq \Omega \) be open. Then we have
\[
\limsup_{s \uparrow 1} (1 - s) F_s(u, \Omega') \leq n \omega_n \limsup_{|h| \to 0} \int_{\Omega'} \frac{|u(x + h) - u(x)|}{|h|} dx \leq n \omega_n |Du|(\Omega). \tag{34}
\]

**Proof.** For \( h \in \mathbb{R}^n \) let us define
\[
g(h) = \int_{\Omega'} \frac{|u(x + h) - u(x)|}{|h|} dx
\]
and fix \( L > \limsup_{|h| \to 0} g(h) \). Then there exists \( \delta_L > 0 \) such that \( \Omega' + h \subseteq \Omega \) for all \( h \in B_{\delta_L} \) and \( L \geq g(h) \) for \( 0 < |h| \leq \delta_L \). Multiplying by \(|h|^{-n-s+1}\) and integrating with respect to \( h \) on \( B_{\delta_L} \) we obtain
\[
\frac{n \omega_n \delta_L^{1-s} L}{1 - s} \geq \int_{B_{\delta_L}} g(h) |h|^{n+s-1} dh = \int_{B_{\delta_L}} \int_{\Omega'} \frac{|u(x + h) - u(x)|}{|h|^{n+s}} dh dx + \int_{B_{\delta_L}} \int_{\Omega'} \frac{|u(x) - u(y)|}{|h|^{n+s}} dh dy. \tag{35}
\]
Now notice that
\[
\int_{\Omega'} \int_{\Omega'} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} dxdy = \int_{(\Omega' \times \Omega') \cap \{|x-y| \leq \delta_L\}} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} dxdy + \int_{(\Omega' \times \Omega') \cap \{|x-y| \geq \delta_L\}} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} dxdy
\]
\[
\leq \int_{B_{\delta_L}} \int_{\Omega'} \frac{|u(x + h) - u(x)|}{|h|^{n+s}} dh dx + \int_{B_{\delta_L}} \int_{\Omega'} \frac{|u(x + h) - u(x)|}{|h|^{n+s}} dh dx + \int_{B_{\delta_L}} \int_{\Omega'} \frac{|u(x + h) - u(x)|}{|h|^{n+s}} dh dx
\]
\[
\leq \frac{2 n \omega_n}{s \delta_L^n} \int_{\Omega'} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} dxdy + \frac{2 n \omega_n}{s \delta_L^n} \|u\|_{L^1(\Omega)}.
\]
Putting together (35) and (36) we obtain
\[
n \omega_n L \geq \limsup_{s \uparrow 1} (1 - s) \int_{\Omega'} \int_{\Omega'} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} dxdy,
\]
and for \( L \to \limsup_{|h| \to 0} g(h) \) the first inequality in (34). The second one is well-known. \( \Box \)
5.1 Minimality of $H$

**Proposition 17** For every $s \in (0,1)$, $H$ is the unique minimizer of $J_s(\cdot, Q)$, in the sense that $J_s(H, Q) \leq J_s(F, Q)$ for every set $F \subset \mathbb{R}^n$ with $F \cap Q^c = H \cap Q^c$, with strict inequality if $F \neq H$.

The proof of Proposition 17 easily follows from a couple of results of [4], which we give here for the sake of completeness.

**Proposition 18 (Existence of minimizers)** Given $E_0 \subset \Omega$ and $s \in (0,1)$ there exists $E \subset \mathbb{R}^n$ such that $E \cap \Omega^c = E_0$ and

$$\inf_{F \cap \Omega^c = E_0} J_s(F, \Omega) = J_s(E, \Omega).$$

**Proof.** This follows immediately from the lower semicontinuity of $J_s$ with respect to the $L^1_{\text{loc}}$ convergence (a simple consequence of Fatou’s lemma) and the coercivity estimate of Proposition 4.

In general a set $E$ satisfying (37) will be called a minimizer of $J_s(\cdot, \Omega)$. Following the notation of [4], we set $L(A, B) := \int_A \int_B |x - y|^{-n-s} dxdy$ for $s \in (0,1)$ and $A, B \subset \mathbb{R}^n$ measurable. Notice that $L(A \cup B, C) = L(A, C) + L(B, C)$ if $|A \cap B| = 0$ and $L(A, B) = L(B, A)$. Now we can write

$$J_s(E, \Omega) = L(E \cap \Omega, E^c) + L(E \cap \Omega^c, E^c \cap \Omega).$$

It is easy to check that a minimizer $E$ of $J_s(\cdot, \Omega)$ satisfies

$$L(A, E) \leq L(A, E^c \setminus A) \quad \text{for } A \subset E^c \cap \Omega \quad (38)$$

$$L(A, E^c) \leq L(A, E \setminus A) \quad \text{for } A \subset E \cap \Omega. \quad (39)$$

It suffices indeed to compare $E$ with $E \setminus A$ and with $E \cup A$.

**Proposition 19 (Comparison principle I)** Let $E$ satisfy (38) with $\Omega = Q$ and assume that $H \cap Q^c \subset E$. Then $H \subset E$ up to a set of measure zero (i.e. $|H \cap E^c| = 0$).

**Proof.** Let $T(x', x_n) := (x', -x_n)$ denote the reflection across $\partial H$ and set $A^- := H \cap E^c$, $A^+ := T(A^-) \cap E^c$, $A := A^- \cup A^+ \subset E^c \cap Q$, $A_1 := A^+ \cup T(A^+)$, $A_2 := A^- \setminus T(A^+)$ and $F := T(E^c \setminus A) \subset H$. Then, observing that $L(B, C) = L(T(B), T(C))$, from (38) we infer

$$0 \geq L(A, E) - L(A, E^c \setminus A) = L(A, E) - L(T(A), F) = L(A, E) - L(A_1, F) - L(T(A_2), F)$$

$$= L(A, E) - L(A, F) + L(A_2, F) - L(T(A_2), F) = L(A, E \setminus F) + L(A_2, F) - L(T(A_2), F)$$

$$= L(A_1, E \setminus F) + L(A_2, E \setminus F) + (L(A_2, F) - L(T(A_2), F)).$$

The first two terms on the right-hand side are clearly positive. We also have $L(A_2, F) > L(T(A_2), F)$ unless $|A_2| = 0$, since for $y \in F$ and $x \in A_2 \setminus \partial H$ one has $|x - y| < |T(x) - y|$. Therefore the right-hand side must be zero, $|A_2| = 0$ and either $|A_1| = 0$ (and the proof is complete), or $|E \setminus F| = 0$. In the latter case consider for a small $\varepsilon > 0$ the translated set $E_\varepsilon :=
\[ E + (0, \ldots, 0, \varepsilon), \text{which satisfies (38) in } \tilde{Q}_\varepsilon := Q + (0, \ldots, 0, \varepsilon), \text{hence also in } \tilde{Q}_\varepsilon := Q + T(Q_\varepsilon). \]

Repeating the above procedure for \( E_\varepsilon \in \tilde{Q}_\varepsilon \) we get \( |A_2, \varepsilon| = 0 \) (\( A, A^+ \varepsilon, \text{etc. are defined as above with respect to the set } E_\varepsilon \text{ in the domain } \tilde{Q}_\varepsilon \), still reflecting across \( \partial H \); we use also the fact since \( H \subset H_\varepsilon := H + (0, \ldots, 0, \varepsilon), \text{we have } H \cap \tilde{Q}_\varepsilon \subset E_\varepsilon \) and, since \( |E_\varepsilon \setminus F_\varepsilon| = \infty, \text{ and } |A_{1, \varepsilon}| = 0. \) This implies at once that \( |A_\varepsilon^-| = 0 \) and \( |L_\varepsilon \setminus E_\varepsilon| = 0. \) Since this is true for every small \( \varepsilon > 0, \) it follows that \( H \subset E \) (up to a set of measure 0). \( \square \)

By a similar argument, the proposition above also holds replacing \( H \) by \( H^c. \) Also, it is easy to see that if \( E \) satisfies (39), then \( E^c \) satisfies (38), hence by applying Proposition 19 to \( E^c \) and \( H^c \) one has the following corollary.

**Proposition 20 (Comparison principle II)** Let \( E \) satisfy (39) with \( \Omega = Q \) and assume that \( E \cap Q^c \subset H. \) Then \( E \subset H \) up to a set of measure zero (i.e. \( |H^c \cap E| = 0 \)).

**Proof of Proposition 17** According to Proposition 18 a minimizer \( E \) of \( J_\varepsilon(\cdot, Q) \) with \( E \cap Q^c = H \cap Q^c \) exists. Then \( E \) satisfies both (38) and (39), hence by Propositions 19 and 20 we have \( H \subset E \) and \( E \subset H \) (up to sets of measure 0), i.e. \( E = H. \) \( \square \)

### 5.2 Monotone set functions

We report some of the main results of [8], see also [6, Chapter 16] for more general and related results. In the sequel for an open set \( \Omega \subset \mathbb{R}^n, \) we denote by \( \mathcal{P}(\Omega) \) the set of subsets of \( \Omega \) and by \( \mathcal{A}(\Omega), \mathcal{K}(\Omega) \subset \mathcal{P}(\Omega), \) the collection of open and compact subset of \( \Omega \) respectively. We also define

\[ \mathcal{C}(\Omega) := \left\{ \bigcup_{i=1}^M Q_i : Q_i \in \mathcal{Q}, M \in \mathbb{N} \right\}, \]

where \( \mathcal{Q} \) is countable the set of open cubes \( Q_r(x) := x + rQ \in \Omega \text{ with } x \in \mathbb{Q}^n \) and \( 0 < r \in \mathcal{Q}. \)

The collections \( \mathcal{A}(\Omega), \mathcal{K}(\Omega) \) and \( \mathcal{C}(\Omega) \) satisfy the following property

\[ A \in \mathcal{A}(\Omega), K \in \mathcal{K}(\Omega), K \subset A \Rightarrow \text{ there exists } C \in \mathcal{C}(\Omega) \text{ with } K \subset C \subset A. \] (40)

We say that a set function \( \alpha : \mathcal{P}(\Omega) \to [0, \infty] \) is monotone if

\[ \alpha(E) \leq \alpha(F) \text{ wherever } E \subset F, \]

and that a monotone set function is regular if the following two conditions hold

\[ \alpha(A) = \sup \{ \alpha(K) : K \subset A, K \in \mathcal{K}(\Omega) \} \text{ for any } A \in \mathcal{A}(\Omega), \] (41)

\[ \alpha(E) = \inf \{ \alpha(A) : E \subset A, A \in \mathcal{A}(\Omega) \} \text{ for any } E \in \mathcal{P}(\Omega). \] (42)

Thanks to (40) it is clear that (41) is equivalent to

\[ \alpha(A) = \sup \{ \alpha(V) : V \in \mathcal{A}(\Omega) \} = \sup \{ \alpha(C) : C \in \mathcal{A}(\Omega), C \in \mathcal{C}(\Omega) \}. \] (43)
We also say that a monotone set function $\alpha$ is super-additive if
\[
\alpha(E \cup F) \geq \alpha(E) + \alpha(F), \text{ wherever } E, F \in \mathcal{P}(\Omega), \ E \cap F = \emptyset.
\]

We say that a sequence of regular monotone set functions $\alpha_i$ weakly converges to a monotone set function $\alpha$ if the following two conditions hold:
\[
\begin{align*}
\liminf_{i \to \infty} \alpha_i(A) &\geq \alpha(A) \text{ for every } A \in \mathcal{A}(\Omega), \quad (44) \\
\limsup_{i \to \infty} \alpha_i(K) &\leq \alpha(K) \text{ for every } K \in \mathcal{K}(\Omega). \quad (45)
\end{align*}
\]

The limit need not be unique, but it is easy to see that a sequence of regular monotone set functions admits at most one regular limit.

**Theorem 21 (De Giorgi-Letta)** Let $(\alpha_i)$ be a sequence of regular monotone set functions such that
\[
\limsup_{i \to \infty} \alpha_i(\Omega') < \infty \quad \text{for every open set } \Omega' \Subset \Omega.
\]

Then there exists a subsequence $(\alpha_i')$ weakly converging to a regular monotone set function $\alpha$. Moreover if each $\alpha_i$ is super-additive on disjoint open sets$^1$ (and hence on disjoint compact sets), then so is $\alpha$.

**Proof.** Since the proof is standard we only sketch it.

**Step 1.** Being $\mathcal{C}(\Omega)$ countable, we can easily extract a diagonal subsequence, still denoted by $(\alpha_i)$ such that,
\[
\beta(C) := \liminf_{i \to \infty} \alpha_i(C) < \infty \quad \text{for any } C \in \mathcal{C}(\Omega).
\]

**Step 2.** We define
\[
\begin{align*}
\alpha(A) &:= \sup \{ \beta(C) : C \Subset A, \ C \in \mathcal{C}(\Omega) \} \quad \text{for every } A \in \mathcal{A}(\Omega), \\
\alpha(E) &:= \inf \{ \alpha(A) : A \Supset E, \ A \in \mathcal{A}(\Omega) \} \quad \text{for every } E \in \mathcal{P}(\Omega).
\end{align*}
\]

Clearly for $C \in \mathcal{C}(\Omega)$ we have $\alpha(C) \leq \beta(C)$.

**Step 3.** The set function $\alpha$ is clearly monotone, and if every $\alpha_i$ is super-additive on disjoint open sets, then so is $\alpha$. It is also easy to see that (44) is satisfied. As for (45), it is an easy consequence of the identity
\[
\alpha(K) = \inf \{ \beta(C) : C \Supset K, \ C \in \mathcal{C}(\Omega) \}.
\]

which follows from (40). Then $\alpha_i$ converges weakly to $\alpha$.

**Step 4.** It remains to prove the regularity of $\alpha$. Identity (42) follows by the definition of $\alpha$. In order to prove (41) fix any $A \in \mathcal{A}(\Omega)$. Then for $C \in \mathcal{C}(\Omega)$ with $C \Subset A$, we have
\[
\beta(C) = \lim_{i \to \infty} \alpha_i(C) \leq \limsup_{i \to \infty} \alpha_i(\overline{C}) \leq \alpha(\overline{C}) \leq \alpha(C') \leq \beta(C').
\]

From this and the definition of $\alpha(A)$, (43) follows at once, hence also (41). \qed

---

$^1$This means that $\alpha_i(A \cup B) \geq \alpha_i(A) + \alpha_i(B)$ wherever $A, B \in \mathcal{A}(\Omega)$ are disjoint.
Proposition 22 Let \((\alpha_i)\) be a sequence of regular monotone set functions weakly converging to a regular monotone set function \(\alpha\), and let \(K_j \downarrow K\) be a decreasing sequence of compact sets such that \(\alpha(K) = 0\). Then
\[
\lim_{j \to \infty} \limsup_{i \to \infty} \alpha_i(K_j) = 0
\]

Proof. We have
\[
0 = \alpha(K) = \lim_{j \to \infty} \alpha(K_j) = \lim_{j \to \infty} \limsup_{i \to \infty} \alpha_i(K_j),
\]
where the second equality follows from the regularity of \(\alpha\). Indeed for \(A \in \mathcal{A}(\Omega)\) with \(A \supset K\), we have by compactness \(A \supset K_j\) for \(j\) large enough, hence
\[
\alpha(A) \geq \lim_{j \to \infty} \alpha(K_j) \geq \alpha(K) = 0,
\]
and the claim follows by taking the infimum over all \(A \in \mathcal{A}(\Omega)\) with \(A \supset K\). \(\square\)

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