Discrete model of spacetime in terms of inverse spectra of the $T_0$ Alexandroff topological spaces

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Running title: $T_0$-discrete spacetime and inverse spectra
Abstract

The theory of inverse spectra of $T_0$ Alexandroff topological spaces is used to construct a model of $T_0$-discrete four-dimensional spacetime. The universe evolution is interpreted in terms of a sequence of topology changes in the set of $T_0$-discrete spaces realized as nerves of the canonical partitions of three-dimensional compact manifolds. The cosmological time arrow arises being connected with the refinement of the canonical partitions, and it is defined by the action of homomorphisms in the proper inverse spectrum of three-dimensional $T_0$-discrete spaces. A new causal order relation in this spectrum is postulated having the basic properties of the causal order in the pseudo-Riemannian spacetime however also bearing certain quasi-quantum features. An attempt is made to describe topological changes between compact manifolds in terms of bifurcations of proper inverse spectra; this led us to the concept of bispectrum. As a generalization of this concept, inverse multispectra and superspectrum are introduced. The last one enables us to introduce the discrete superspace, a discrete counterpart of the Wheeler–DeWitt superspace.

Key words: $T_0$ Alexandroff space, inverse spectrum, superspectrum
1 Introduction

Absolutization of any concept (here: in physics), although it is inevitable at certain stages of the development of the theory, later always leads to a contradiction (Bohm, 1965). Exactly this occurred with the concept of a smooth spacetime manifold. This concept, being successful in the classical relativistic physics as a model of the causally ordered set at large scales, under extrapolation to the quantum theory leads to appearance of the well known singularities and divergences. This is primarily connected with the ideal character of pointlike events and objects which can be recorded by classical observers (by their definition). The standard quantum theory, from its very beginning, has changed the approach to the definition of an observer and observables, however leaving as untouchable the concept of smooth spacetime manifold. But it seems to be natural and aesthetically attractive to accompany the quantization of physical fields by quantifying the spacetime arena, or even perform the latter in anticipation, the arena on (or together with) which evolve these fields. The idea to use discrete (finitary) structures as fundamental and really existing, but not approximational ones, in the description of the quantum spacetime relations, dates back to the works of Finkelstein (1969, 1988) and Isham (1989). Sorkin and co-authors have proposed both finitary substitutes to model causal relations between events (spacetime causal sets) in realistic measurements (Bombelli et al., 1987) and finitary topological structures to model the quantized spacetime (Sorkin, 1991). Recently these ideas were extensively developed in (Rideout and Sorkin, 2000; Raptis, 2000a; Raptis and Zapatrin, 2001; Mallios and Raptis, 2001).

Nor one has to absolutize the concrete discrete spacetime relations, not only since at the classical level (some ‘large scales’) the smooth spacetime manifold does describe the corresponding physical reality adequately, but, more importantly, since the concrete discreteness differs drastically at different levels. When passing to a deeper structural level, one has to be ready to discover that objects previously treated as ‘elementary’, should be considered as compound ones, ‘built’ of the next-level ‘elementary’ objects (remember, for example, the fate of hadrons later interpreted via quarks). From our point of view, the sound mathematical concept which describes both the discreteness and continuity ideas, as well as their interconnection, is the inverse spectrum of three-dimensional $T_0$-discrete spaces, also known as the $T_0$ Alexandroff spaces (Alexandroff, 1937, 1947; Arenas, 1997, 1999). This inverse spectrum has as its limit the continuous three-dimensional space (usu-
ally, image of a standard spacelike section of the spacetime), but this continuous space is never reached in the spectral evolution process. In this connection note that our approach differs from that of Sorkin and his co-authors as well as his followers; it is more similar to the approach of Isham (1989), namely to the canonical Hamiltonian description where the discretization is immediately applied to the three-dimensional space, but not to the full four-dimensional spacetime. However we do not take the 3-space as a section of the latter, but consider its spectral evolution in the course of acts of refinement which are inevitably also discrete, thus giving birth to a new (discrete) parameter, the ‘global time’. Thus, in our opinion, the spacetime is modelled by the proper inverse spectrum of three-dimensional $T_0$ Alexandroff spaces, while the global discrete evolution (the time arrow in the expanding universe) is related to a sequential refinement of the canonical partitions of a three-dimensional compact. Thus the global time automatically acquires the $T_0$-discrete topology, since the family of canonical partitions is a partially ordered infinite set (Alexandroff, 1937). This spectral evolution parameter yields only one additional dimension (timelike, see Subsection 2.5 where the concept of light cone is introduced without metrization) to the $n$ spatial dimensions (here, three) postulated from the very beginning. See also the $(n+1)$-argumentation in the framework of the conventional quantum theory given by van Dam and Ng (2001).

Our model can be related to the (3+1)-splitting of spacetime into a family of three-dimensional spacelike hypersurfaces (equivalently, to introduction of a normal congruence of timelike worldlines of local observers). This representation of the four-dimensional spacetime continuum is used in the canonical formulation of general relativity in terms of observables [see (Misner et al., 1973; Ashtekar, 1991) and references therein] since it presumes introduction of a reference frame as a continual system of observers situated at all points of the three-dimensional spacelike hypersurface and moving along the respective lines of the congruence. This method is also known as the monad formalism (giving a covariant description of global reference frames), see (Mitskievich, 1996), and its application to the canonical formulation of general relativity, (Antonov et al., 1978). Thus the discretization of three-dimensional hypersurfaces via a transition to the nerves of finite or locally finite coverings (in particular, partitions), automatically leads to a finite (for compacts or for compact regions of paracompacts) set of observers, and to a denumerable set of events they can detect in the evolving universe, i.e. in the course of shifting along the inverse spectrum in the direction of progressive refinement.
of the coverings.

In Section 2, we give a review of the basic mathematical concepts such as the $T_0$-discrete space ($T_0$ Alexandroff space), nerves of coverings (partitions) and of inverse spectra of topological spaces associated with the nerves. Moreover, Alexandroff’s procedure of discretization of compacts (construction of the proper inverse spectrum of any compact) is described, the procedure which is also applicable to paracompacts. These items are fairly well known to theoretical physicists after the paper (Sorkin, 1991) [see also (Raptis and Zapatrin, 2001; Mallios and Raptis, 2001)], but we present them in a style closer to the original papers of Alexandroff (1929, 1937, 1947) less accessible to the English-speaking reader; furthermore, this style we use in the physical interpretation of our results in Subsections 2.4, 2.5. In this connection it is worth mentioning that our starting attitude (discretization of three-dimensional spacelike sections) does not give us the possibility to reinterpret the obtained $T_0$-discrete spaces in terms of causal sets as this was done in (Sorkin, 1991; Raptis, 2000a) where the four-dimensional spacetime manifolds were discretized. Therefore in Subsection 2.5 we postulate a new causal order relation in the proper inverse spectrum $S_{pr}$ defining two sets, those of the causal past and causal future of any element of $S_{pr}$. Further we prove two Propositions justifying this postulate.

In Section 3 there is made an attempt to describe topological changes between compact manifolds in terms of bifurcations of proper inverse spectra. This led us in Subsection 3.2 to the concept of bispectrum. In Subsection 3.3 the concepts of inverse multispectra and superspectrum are introduced. In our opinion, this last concept should be the discrete counterpart of the superspace of the Wheeler–DeWitt quantum geometrodynamics. The introduction of these concepts makes it possible to set the problem of formulation of the quantum theoretical approach to the topodynamics (an analogue of geometrodynamics), and to propose the topological version of the many-worlds interpretation as well as a qualitative discrete-space analogue of Heisenberg’s uncertainty relation.
2 Inverse spectra of the $T_0$ Alexandroff spaces and their physical interpretation

2.1 $T_0$ Alexandroff spaces, partially ordered sets and simplicial complexes

By the Alexandroff space we mean a topological space $D$ every point of which has a minimal neighborhood or, equivalently, the space has a unique minimal base (Alexandroff, 1937) (the minimal neighborhood of a point $p \in D$ is denoted by $O(p)$ being the intersection of all open sets containing $p$). This is also equivalent to the fact that intersection of any family of open sets is open, and union of any number of closed sets is closed. Therefore for each Alexandroff space $D$, there is a dual space $D^*$ in which open sets are by a definition the closed sets of $D$, and vice versa.

We consider here only the Alexandroff spaces with the $T_0$ separability axiom [of any given two points of a topological space $D$, at least one is contained in an open set not containing the other point (Hocking and Young, 1988)]. Note that an Alexandroff space $D$ is $T_1$ iff $O(p) = p$ for any $p \in D$; in this case the space $D$ is trivially discrete (discrete in the common sense). But if we accept the $T_0$ axiom, a richer concept of discreteness arises for which there exists a functorial equivalence between the categories of $T_0$ Alexandroff spaces and partially ordered sets (hereafter referred to as posets). We shall use as synonyms ‘$T_0$ Alexandroff space’ and ‘$T_0$-discrete space’ [following Alexandroff (1937): “Diskrete Räume”], while the discrete spaces in the common sense will be called ‘$T_1$-discrete spaces’ as well. Given a $T_0$ Alexandroff space $D$, we construct a poset $P(D)$ with the order $p' \leq p$ iff $p \in O(p')$. Conversely, given a poset $P$, we construct $T_0$ Alexandroff space $D(P)$ with the topology generated by the minimal neighborhoods

$$O(p') = \{p \in P | p \geq p'\}. \quad (2.1)$$

It is straightforward to see that $D(P(D)) = D$ and $P(D(P)) = P$ and that under the functors, continuous mappings become order preserving mapping and conversely (Arenas, 1997). Note that the order can be also defined in the reversed way and we obtain the $T_0$-discrete space $D^*$ dual to $D$:

$$O^*(p') = \{p \in P | p' \geq p\} \quad (2.2)$$

is the minimal neighborhood of the point $p'$ in $D^*(P)$. 


A $T_0$ Alexandroff space $D$ is locally finite if for any point $p \in D$ the number of elements in $O(p)$ and in $\bar{p}$ is finite. We denote as $\bar{p}$ the closure of the point $p \in D$. The points with the property $\bar{p} = p$ are called $c$-vertices, and those with the property $O(p) = p$, $o$-vertices.

Now let $V$ be a set of (abstract) elements called vertices. An abstract simplicial complex $K$ is a collection of finite subsets of $V$ with the property that each element of $V$ lies in some element of $K$, and if $s$ is any element of $K$ (called simplex of $K$), then any subset $s'$ of $s$ is again a simplex of $K$ ($s'$ is said to be a face of $s$). In this case, if we suppose that $s' \subseteq s$, the simplicial complex $K$ turns into the poset $P(K)$, and therefore into the $T_0$-discrete space $D(P(K))$, or it turns into the dual one, $D^*(P(K))$ [see (2.1) or (2.2)].

2.2 Nerves of partitions and nerves’ inverse spectra

Nerves of coverings (in particular, canonical partitions) of normal spaces represent an important example of (abstract) simplicial complexes and $T_0$-discrete spaces.

Let $X$ be a normal topological space, i.e. a Hausdorff space satisfying the $T_4$ separability axiom (Hocking and Young,1988). A subset $A$ of the space $X$ is canonically closed if $A$ is a closure of its interior $\dot{A}$, i.e.

$$A = \overline{\dot{A}}.$$  

A canonical partition of the space $X$ is defined as a finite covering consisting of canonically closed sets,  

$$\alpha = \{A_1, \ldots, A_s\},$$  

(2.3)

with disjoint interiors, i.e. $\dot{A}_i \cap \dot{A}_j = \emptyset$ for $\forall i, j = 1, \ldots s; i \neq j$.

A canonical partition $\beta = \{B_1, \ldots, B_r\}$ is called a refinement of $\alpha$ if for any element $B_j \in \beta$ there is a unique element $A_i \in \alpha$ such that $A_i$ contains $B_j$ ($B_j \subseteq A_i$). It is worth being emphasized that, in the case of partition, if such an element $A_i$ exists, it is necessarily unique. It is also said that the partition $\beta$ follows $\alpha$ ($\beta \succ \alpha$).

For any pair $\alpha$, $\beta$ of canonical partitions, there exists a canonical partition $\gamma$ being a refinement of the both $\alpha$ and $\beta$. The sets having this property are called directed ones. Such a partition $\gamma$ may be obtained, for example, as a product $\alpha \wedge \beta$ of the partitions $\alpha$ and $\beta$ which consists of $\overline{\dot{A}_i \cap \dot{B}_j}$ for all elements $A_i$ and $B_j$ for which $\dot{A}_i \cap \dot{B}_j \neq \emptyset$. It is obvious that the collection
of all canonical partitions \( \{ \alpha \} \) of a normal space \( X \) is a partially ordered set, therefore \( \{ \alpha \} \) is a directed poset.

Now, following Alexandroff (1937), we introduce a special case of the \( T_0 \)-discrete spaces which are realized as nerves of the coverings of a normal space \( X \).

Let \( \alpha = \{A_1, \ldots, A_s\} \) be a covering (in particular, a canonical partition) of the normal space \( X \). As a nerve of the covering \( \alpha \), we call the simplicial complex \( N_\alpha \) consisting of simplices defined as sets \( \{A_{i_0}, \ldots, A_{i_q}\} \) of elements of the covering \( \alpha \) for which

\[
A_{i_0} \cap \ldots \cap A_{i_q} \neq \emptyset.  \tag{2.4}
\]

It is said that the simplex \( s^q_\alpha = \{A_{i_q}, \ldots, A_{i_q}\} \) has the dimension \( q \).

In accordance with the general procedure of determination of the topology on a simplicial complex, one has to consider the simplices \( s^q_\alpha \) as points of the topological space and define the minimal neighborhood of the simplex \( s^q_\alpha = \{A_{i_q}, \ldots, A_{i_q}\} \) as the set of simplices \( s^p_\alpha = \{A_{j_p}, \ldots, A_{j_p}\} \) such that

\[
A_{i_0} \cap \ldots \cap A_{i_q} \subseteq A_{j_0} \cap \ldots \cap A_{j_p}.  \tag{2.5}
\]

In other words, the minimal neighborhood \( O^*(s^q_\alpha) \) of the simplex \( s^q_\alpha \) form all its faces \( s^p_\alpha \), i.e.

\[
O^*(s^q_\alpha) = \{s^p_\alpha \in N_\alpha | s^p_\alpha \geq s^q_\alpha\}.  \tag{2.6}
\]

Thus the \( T_0 \)-discrete dual topology has been defined on the nerve \( N_\alpha \).

Exactly the nerves of canonical partitions with the \( T_0 \)-discrete dual topology are usually employed to the end of definition of spectra of \( T_0 \)-discrete spaces.

Let \( \{\alpha\} \) be a set of coverings (canonical partitions) of a normal space \( X \), and \( N_\alpha \), a nerve corresponding to a covering \( \alpha \in \{\alpha\} \); moreover, let \( X_\alpha \) be a \( T_0 \)-discrete space defined on the basis of the nerve \( N_\alpha \) via (2.5) or (2.6). The inverse spectrum of the nerves \( N_\alpha \) is defined as the set \( S = \{N_\alpha, \omega^\alpha_{\alpha'}\} \) where \( \omega^\alpha_{\alpha'} \) are simplicial mappings

\[
\omega^\alpha_{\alpha'} : N_{\alpha'} \rightarrow N_\alpha \tag{2.7}
\]

which are well defined only when \( \alpha' \) is a refinement of \( \alpha \) (\( \alpha' \succ \alpha \)), while for \( \alpha'' \succ \alpha' \succ \alpha \) the transitivity condition

\[
\omega^\alpha_{\alpha''} = \omega^\alpha_{\alpha'} \omega^\alpha_{\alpha''} \tag{2.8}
\]
should be fulfilled. (By the definition, a simplicial mapping \( \omega' \alpha \) maps any simplex from \( N' \alpha \) in a simplex in \( N \alpha \).) The inverse spectrum of \( T_0 \)-discrete spaces \( S = \{ X_\alpha, \omega' \alpha \} \) is introduced in the same manner, only the mappings \( \omega' \alpha \) now should be continuous. Since we shall consider below nerves with the dual topology, the pairs of objects, \( N \alpha \) and \( X_\alpha \), will be identified (\( N \alpha \iff X_\alpha \)).

A point \( \{ s_\alpha \} \) of the direct product \( \prod X_\alpha \) of the \( T_0 \)-discrete spaces corresponding to all coverings of the set \( \{ \alpha \} \), is called a coherent system of elements \( s_\alpha \) (thread, Alexandroff’s term) of the inverse spectrum \( S = \{ X_\alpha, \omega' \alpha \} \), if \( s_\alpha = \omega' \alpha s_\alpha' \) whenever \( \alpha' \succ \alpha \). The set of threads of a spectrum \( S \) represents a subspace \( \bar{S} \) of the topological space \( \prod X_\alpha \). The subspace \( \bar{S} \) endowed with the induced topology is called the total inverse limit of the spectrum \( S \),

\[
\bar{S} = \lim \{ X_\alpha, \omega' \alpha \} .
\] (2.9)

For us however of greater importance will be the concept of the upper inverse limit. First observe that a thread \( s = \{ s_\alpha \} \) is larger than \( \tilde{s} = \{ \tilde{s}_\alpha \} \), if for any \( \alpha \in \{ \alpha \} \), \( s_\alpha \geq \tilde{s}_\alpha \) (\( \tilde{s}_\alpha \) is a face of \( s_\alpha \)). A thread \( s \) is called the maximum one, if no thread greater than \( s \) exists. The subspace \( \hat{S} \) of the space \( \bar{S} \) consisting of all maximum threads, is called the upper inverse limit of the spectrum \( S \),

\[
\hat{S} = \operatorname{uplim} \{ X_\alpha, \omega' \alpha \} .
\] (2.10)

### 2.3 Discretization of compacts

For all compacts there is a standard procedure how to construct the upper inverse limit of the spectrum of \( T_0 \)-discrete spaces or of the nerves of all canonical partitions (finite by the definition, see Subsection 2.2) (Alexandroff, 1947). (For paracompacts the situation is fairly similar, but finite partitions should be changed to locally finite ones.)

Let \( \{ \alpha \} \) be a set of all canonical partitions of a compact \( X \). [First let us remark that the set \( \{ \alpha \} \) is cofinal to the set of all open coverings of the compact \( X \). This means that for any open covering \( \omega \) of \( X \) there is a canonical partition \( \alpha_\omega \in \{ \alpha \} \) being the refinement of \( \omega \) \((\alpha_\omega \succ \omega)\).] Let us construct for a canonical partition \( \alpha = \{ A_1, \ldots, A_s \} \) the respective nerve \( N_\alpha \) and introduce on it the dual topology. This yields a \( T_0 \)-discrete space which we denote by \( X_\alpha \). For any partition \( \alpha' = \{ A'_1, \ldots, A'_{r'} \} \) being a refinement
of \( \alpha \), the mapping \( \omega^\alpha_{\alpha'} : X_{\alpha'} \to X_\alpha \) is determined as follows: to any \( A'_j \in \alpha' \) there corresponds only one \( A_i \in \alpha \) which contains \( A'_j \). This element \( A_i \) of the partition \( \alpha \) is by the definition the image of \( A'_j \) under the mapping \( \omega^\alpha_{\alpha'} \), i.e.

\[
\omega^\alpha_{\alpha'} A'_j = A_i. \tag{2.11}
\]

Now for any point \( s^p_\alpha = \{A'_{j_0}, \ldots, A'_{j_p}\} \in X_{\alpha'} \) we have

\[
\omega^\alpha_{\alpha'} s^p_\alpha = \{\omega^\alpha_{\alpha'} A'_{j_0}, \ldots, \omega^\alpha_{\alpha'} A'_{j_p}\} = \{A_{i_{j_0}}, \ldots, A_{i_{j_p}}\} = s^q_\alpha \in X_\alpha. \tag{2.12}
\]

We see that \( q \leq p \) since among the sets \( \omega^\alpha_{\alpha'} A'_{j_0}, \ldots, \omega^\alpha_{\alpha'} A'_{j_p} \) some sets may be the same. This construction completes the deduction of the spectrum of \( T_0 \)-discrete spaces; it is called the \textit{proper inverse spectrum} of the compact \( X \) (Alexandroff, 1947),

\[
S_{pr} = \{X_\alpha, \omega^\alpha_{\alpha'}\}. \tag{2.13}
\]

Alexandroff (1947) has shown that any compact \( X \) is homeomorphic to the upper inverse limit of its proper spectrum \( S_{pr} \). To prove this theorem he used the following realization of the upper inverse limit of \( S_{pr} \).

For any point \( x \in X \) there is a unique point

\[
s^q_\alpha = \{A_{i_{j}}, \ldots, A_{i_{q}}\} \tag{2.14}
\]

of the \( T_0 \)-discrete space \( X_\alpha \) (the simplex of \( N_\alpha \)) such that \( x \in A_{i_{j_0}} \cap \ldots \cap A_{i_{j_q}} \), but in the canonical partition \( \alpha \) there are no more sets which contain the point \( x \). The point \( s^q_\alpha(x) \in X_\alpha \) is called the carrier of the point \( x \) in the discrete space \( X_\alpha \).

The set \( \{s^q_\alpha(x)\} \) of carriers of any point \( x \) in all spaces \( X_\alpha \) satisfies the conditions \( \omega^\alpha_{\alpha'} s^q_\alpha(x) = s^q_\alpha(x') \) whenever \( \alpha' \succ \alpha \), i.e. \( \{s^q_\alpha(x)\} \) is a thread of the proper spectrum \( S_{pr} \). It is easy to show that any thread of the carriers \( \{s^q_\alpha(x)\} \) of any point \( x \in X \) forms a maximum thread. Therefore to any point \( x \in X \) corresponds a unique maximum thread \( \{s^q_\alpha(x)\} \) pertaining to the upper inverse limit \( \hat{S}_{pr} \). The inverse assertion is also true, namely that any maximum thread \( \{s_\alpha\} \in \hat{S}_{pr} \) is a thread of carriers of a certain point \( x \in X \). Hence there exists a bijective mapping \( f : X \to \hat{S}_{pr} \) between the compact \( X \) and the upper inverse limit \( \hat{S}_{pr} \) of its proper spectrum \( S_{pr} \) since \( s^q_\alpha(x) = s^q_\alpha(x') \) for all canonical partitions \( \alpha \) of the compact \( X \) yields \( x = x' \). If in the space \( \hat{S}_{pr} \) has been introduced the topology induced by the inverse total limit \( \hat{S}_{pr} \) (see Subsection 2.2), thus the bijective mapping \( f \) becomes a homeomorphism between \( X \) and \( \hat{S}_{pr} \).
2.4 Physical interpretation of inverse spectra of $T_0$ Alexandroff spaces

Our model of spacetime is based on the assumption that the fundamental (and existing as a reality) is considered the $T_0$-discrete three-dimensional space, whose topology is evolving in the induced $T_0$-discrete time, while a continuous spatial section of the spacetime is treated as a limiting three-dimensional manifold which never is realized in the course of this discrete evolution. From the results of Alexandroff described in the preceding Subsection, it follows that for any three-dimensional compact $X$ there exists at least one inverse spectrum $S_{pr} = \{X_\alpha, \omega_\alpha\}$ whose upper limit is homeomorphic to the compact $X$. We may treat this spectrum as a primary object describing the discrete spacetime manifold (in the terminology of Riemann), while the compact $X$ is merely a result of the limiting process. The set of $T_0$ Alexandroff’s spaces $X_\alpha$ in the inverse spectrum $S_{pr}$ then is interpreted as a family of $T_0$-discrete analogues of three-dimensional sections of the four-dimensional continuum $M$.

From the canonical approach to general relativity it is known that to the end of description of the gravitational field in terms of observables, it is necessary to split the spacetime manifold $M$ into a complete family of three-dimensional spacelike hypersurfaces, introducing at the same time the congruence of timelike worldlines of local observers orthogonal to this family. The completeness of the family of spacelike hypersurfaces means that through any event (worldpoint) $p \in M$ passes one and only one hypersurface. Hence this family represents a linearly ordered (one-parametric) set of three-dimensional spacelike sections. From the viewpoint of the monad method (Mitskievich, 1996) (see more references therein) this procedure of splitting spacetime manifold $M$ is nothing but a choice of a (classical) reference frame, i.e. of a multitude of local test observers whose worldlines are identified with lines of the non-rotating congruence, while the spacelike sections orthogonal to the congruence, are the three-dimensional simultaneity hypersurfaces.

Returning to the construction of a discrete model of the spacetime manifold $M$ we suppose that the role of three-dimensional hypersurfaces of simultaneity is played namely by the $T_0$ Alexandroff spaces $X_\alpha$. Any two points $s^\alpha_\mu$ and $s^\alpha_\nu$ of $T_0$-discrete space $X_\alpha$ (i.e. simplices $s^\alpha_\mu$ and $s^\alpha_\nu$ of the nerve $N_\alpha$ of the canonical partition $\alpha$) are interpreted as two simultaneous events occurring on the $T_0$-discrete hypersurface $X_\alpha$ at the instant of the $T_0$-discrete
Observation 2.1. The proposed interpretation suggests that the set of canonical partitions \( \{ \alpha \} \) should be considered as the many-arrow time ["many-fingered" time in (Misner et al., 1973) §21.8]. This time-labelling set (by virtue of its partial orderedness) possesses a \( T_0 \)-discrete topology (see [2.1]). Now we accept the hypothesis: If \( \alpha' \succ \alpha \), the \( T_0 \)-discrete space \( X_{\alpha'} \) is in the future with respect to \( X_{\alpha} \). Then the homomorphism \( \omega_{\alpha}^{\alpha'} : X_{\alpha'} \rightarrow X_{\alpha} \) (defined in the spectrum \( S_{pr} \) iff \( \alpha' \succ \alpha \)) plays the role of a shift in the \( T_0 \)-discrete time \( \{ \alpha \} \) from the future to the past. It is worth emphasizing that the homomorphism \( \omega_{\alpha}^{\alpha'} \) determines a topological transition between two non-homeomorphic \( T_0 \) Alexandroff spaces \( X_{\alpha} \) and \( X_{\alpha'} \). Thus in this model a \( T_0 \)-discrete time shift always is a topology change.

In order to model the splitting of a spacetime manifold into a complete family of spacelike hypersurfaces, i.e. to construct the discrete analogue of the classical reference frame, it is necessary to single out of a partially ordered set (poset) \( \{ \alpha \} \), a complete linearly ordered subset (closset) of canonical partitions \( \{ \alpha_i | i \in I \} \). Remember that the set \( \{ \alpha_i | i \in I \} \) is called linearly ordered if for any two different elements \( \alpha_i \) and \( \alpha_j \) is true either \( \alpha_i \succ \alpha_j \) or \( \alpha_j \succ \alpha_i \). The completeness of the subset \( \{ \alpha_i | i \in I \} \) we define as follows: any element \( \alpha^* \in \{ \alpha \} \) such that
(a) \( \alpha_i \succ \alpha^* \succ \alpha_j \) for some pair of elements \( \alpha_i, \alpha_j \in \{ \alpha_i | i \in I \} \),
(b) for any \( \alpha_i \in \{ \alpha_i | i \in I \} \) is true either \( \alpha_i \succ \alpha^* \) or \( \alpha^* \succ \alpha_i \) pertains to \( \{ \alpha_i | i \in I \} \).

Now we shall prove that any closset \( \{ \alpha_i | i \in I \} \) of a poset \( \{ \alpha \} \), which is cofinal to this poset, is denumerable.

To this end let us consider an arbitrary closset \( \{ \alpha_i | i \in I \} \) cofinal to the poset \( \{ \alpha \} \). Due to its cofinality the closset \( \{ \alpha_i | i \in I \} \) is infinite. If it is denumerable, this is the end of the proof. Suppose that \( \{ \alpha_i | i \in I \} \) is non-denumerable and see that this leads to a contradiction. Select out of the closset \( \{ \alpha_i | i \in I \} \) a denumerable subset \( \{ \alpha_i | k \in \mathbb{Z}^+ \} \) (\( \mathbb{Z}^+ \) is the set of nonnegative integer numbers) being cofinal to the closset \( \{ \alpha_i | i \in I \} \). Such a subset always exists for any compact (Alexandroff, 1947). This subset is linearly ordered due to the linear orderedness of \( \{ \alpha_i | i \in I \} \). Consider any pair of consecutive partitions \( \alpha_{i_k} \) and \( \alpha_{i_{k+1}} \). Since \( \alpha_{i_k} \prec \alpha_{i_{k+1}} \), and the both partitions are finite (by the definition of canonical partitions, Subsection [2.1]), there exists only a finite set of partitions \( \alpha_{i_k}(1), \ldots, \alpha_{i_k}(C_k) \) out of the closset.
\{α_i \mid i ∈ I\}, such that
\[ α_{i_k} \prec α_{i_k}(1) \prec \ldots \prec α_{i_k}(C_k) \prec α_{i_k+1}. \]

It is clear that the obtained set of partitions
\[ \{α_{i_k}, α_{i_k}(1), \ldots, α_{i_k}(C_k) \mid k ∈ Z^+\} \] (2.15)
is equivalent to the initial set \{α_i \mid i ∈ I\}. But the set (2.15) is denumerable due to the fact that a set consisting of a denumerable set of finite sets, is denumerable. Thus the closset \{α_i \mid i ∈ I\} is denumerable, i.e. \( I \cong Z^+ \) (\( I \) is equivalent to \( Z^+ \) in the sense that the both have one and the same cardinality).

Now we consider three particular cases of clossets of poset \{α\}.

(1) If a closset \{α_i \mid i ∈ I\} is cofinal to the poset \{α\} and it includes the trivial partition \( α_0 = \{X\} \) consisting of one element (the compact \( X \) proper), then the inverse subspectrum
\[ S_{pr}(α_i, i ∈ I) = \{X_{α_i}, ω_{α_i}^α_j \mid i, j ∈ I\} \quad (2.16) \]
of the inverse spectrum \( S_{pr} \), describes a linearly ordered family of \( T_0 \)-discrete sections from the \( T_0 \)-discrete spacetime corresponding to \( S_{pr} \). This means that the inverse spectrum \( S_{pr}(α_i, i ∈ I) \) should be considered as a model of the \( T_0 \)-discrete spacetime in a fixed reference frame. To reiterate, the just introduced concept of discrete reference frame includes a complete family of linearly ordered (with a discrete time parameter \{α_i \mid i ∈ I\}) three-dimensional \( T_0 \) Alexandroff spaces and a system of homomorphisms between any two \( T_0 \)-discrete spaces in this family,
\[ ω_{α_i}^α_j : X_{α_j} → X_{α_i} \quad \text{whenever} \quad α_j \succ α_i. \]

**Observation 2.2.** Subspectrum \( S_{pr}(α_i, i ∈ I) \), like the spectrum \( S_{pr} \), represents a proper inverse spectrum whose upper limit is homeomorphic to the upper limit of \( S_{pr} \)
\[ \text{uplim} \{X_{α_i}, ω_{α_i}^α\} \cong \text{uplim} \{X_α, ω_α^α'\} \cong X. \]

Thus the inverse spectrum \( S_{pr}(α_i, i ∈ I) \) is modelling the same four-dimensional spacetime continuum \( M \), as does the spectrum \( S_{pr} \), but now in a fixed reference frame.
Observation 2.3. It was observed by Sorkin (Sorkin, 1995) that macroscopic space volume is supposed to be measured by the number of points on the $T_0$-discrete section $X_{\alpha_i}$ (the proposition adapted to our spacetime model). Then the discrete evolution of the universe described by a complete linearly ordered spectrum $S_{pr}(\alpha_i, i \in I)$, represents creation of the universe beginning with only one-point space $X_{\alpha_0}$ ($\alpha_0 = \{X\}$ is the trivial partition of the three-space $X$), with the subsequent expansion (in the sense of the growth of the number of points) in the course of transitions between $T_0$-discrete hypersurfaces from $X_{\alpha_i}$ to $X_{\alpha_j}$ where $\alpha_j \succ \alpha_i$. The homomorphism $\omega^{\alpha_j}_{\alpha_i} : X_{\alpha_j} \rightarrow X_{\alpha_i}$ describes a topology change with a decrease of the volume (in the above sense), being an operator acting in the direction opposite to the flow of the global (cosmological) time in the observed expanding universe.

(2) If a closset $\{\alpha_i | i \in I\}$ is cofinal to the poset $\{\alpha\}$, but contains a minimal non-trivial partition $\alpha_{\min} = \{A_1, \ldots, A_s\}$, $s > 1$, i.e. such that for any $i \in I$, $\alpha_i \succ \alpha_{\min}$, we shall say that the related inverse subspectrum

$$S_{pr}(\alpha_{\min}) = \{X_{\alpha_i}, \omega^{\alpha_j}_{\alpha_i} | \alpha_i \succ \alpha_{\min}, i, j \in I\}$$

is modelling $T_0$-discrete spacetime (in a fixed reference frame), but only beginning with the $T_0$-discrete time instant labelled by $\alpha_{\min}$. In this case, like in (1), the set of indices $I$ is equivalent to $\mathbb{Z}^+$. 

(3) Alternatively, if the closset $\{\alpha_i | i \in I\}$ is not cofinal to the poset $\{\alpha\}$, but contains both minimal and maximal partitions $\alpha_{\min}$ and $\alpha_{\max}$, such that $\alpha_{\min} \prec \alpha_i \preceq \alpha_{\max}$ for any $i \in I$, then the inverse subspectrum

$$S_{pr}(\alpha_{\min}, \alpha_{\max}) = \{X_{\alpha_i}, \omega^{\alpha_j}_{\alpha_i} | \alpha_{\min} \preceq \alpha_i \preceq \alpha_{\max}, i, j \in I\}$$

of the spectrum $S_{pr}$, describes a $T_0$-discrete spacetime sandwich. This sandwich contains a finite number of $T_0$-discrete hypersurfaces $X_{\alpha_i}$, since $\alpha_{\min}$ and $\alpha_{\max}$ are finite partitions. This $T_0$-discrete spacetime sandwich is an analogue of a finite region of the spacetime continuum between two non-intersecting spacelike hypersurfaces. Such sandwiches are used in the canonical approach to general relativity when the Cauchy problem is considered [see (Baierline et al., 1962; Misner et al., 1973)].

Returning to the case (1), observe that from the poset $\{\alpha\}$ of all canonical partitions of the space $X$, it is possible to single out an infinite set of clossets everyone of which corresponds to a certain discrete reference frame. For example, let us consider closset $\{\alpha'_i | i \in I\}$ of poset $\{\alpha\}$ cofinal to $\alpha$ and including the trivial partition $\alpha'_0 = \{X\}$. Let us also suppose that for all
indices \( i \in I \), with the exception of \( i = 0 \), the canonical partitions \( \alpha_i \) and \( \alpha'_i \) are not ordered with respect to \( \succ \). In this case we shall say that the inverse spectra

\[
S_{pr}(\alpha_i, i \in I) = \left\{ X_{\alpha_i}, \omega_{\alpha_i}^{\alpha_j} | i, j \in I \right\}, \\
S_{pr}(\alpha'_i, i \in I) = \left\{ X_{\alpha'_i}, \omega_{\alpha'_i}^{\alpha'_j} | i, j \in I \right\}
\]

(2.17)
describe one and the same \( T_0 \)-discrete spacetime, but in different discrete reference frames due to the unorderedness of the \( T_0 \)-discrete sections \( X_{\alpha_i} \) and \( X_{\alpha'_i} \) for all \( i \in I \).

One can describe a transition between these reference frames, taking a partition \( \alpha''_i \) for any pair of unordered partitions \( \alpha_i \) and \( \alpha'_i \), such that \( \alpha''_i \succ \alpha_i \), \( \alpha''_i \succ \alpha'_i \). (Such a partition \( \alpha''_i \) does exist, e.g., \( \alpha''_i = \alpha_i \wedge \alpha'_i \), due to the directedness of the set \( \{ \alpha \} \); however, \( \alpha''_i \) may not pertain to any of the clossets \( \{ \alpha_i | i \in I \} \) and \( \{ \alpha'_i | i \in I \} \).) Then we have two homomorphisms

\[
\omega_{\alpha''_i}^{\alpha_i} : X_{\alpha''_i} \rightarrow X_{\alpha_i} \quad \text{and} \quad \omega_{\alpha''_i}^{\alpha'_i} : X_{\alpha''_i} \rightarrow X_{\alpha'_i},
\]

and the transition from the first discrete reference frame, \( S_{pr}(\alpha_i, i \in I) \), to the second one, \( S_{pr}(\alpha'_i, i \in I) \), is described as a mapping

\[
X_{\alpha'_i} = \omega_{\alpha'_i}^{\alpha''_i} \left( \omega_{\alpha_i}^{\alpha''_i} \right)^{-1} X_{\alpha_i}.
\]

(2.18)

Here \( \left( \omega_{\alpha_i}^{\alpha''_i} \right)^{-1} \) is the many-valued mapping inverse to the homomorphism \( \omega_{\alpha'_i}^{\alpha''_i} \). The many-valuedness of this mapping is due to the property \( \alpha''_i \succ \alpha_i \) which is related to the very idea of description of the \( T_0 \)-discrete spacetime in terms of inverse spectra of the three-dimensional \( T_0 \)-Alexandroff spaces. This probably reflects the fact that the concept of discrete reference frame introduced in our model, acquires, due to spacetime discretization, certain quasi-quantum properties. With an exceptional sharpness these features are revealed in the construction of an analogue of the monad description of the global reference frame (in classical general relativity, as a congruence of worldlines of test observers). Now instead of the congruence of worldlines of classical observers we take the complete system of maximal threads being the upper limit of the inverse spectrum.

If we associate the worldline of the observer with the maximal thread \( \{ s_\alpha \} = \{ s_\alpha \in X_\alpha | \alpha \in \{ \alpha \} \} \) of the proper inverse spectrum \( S_{pr} \), then the events \( s_\alpha \) on the discrete worldline of the ‘observer’ \( \{ s_\alpha \} \) will be partially
ordered due to the partial orderedness of the set \( \{\alpha\} \) of all canonical partitions of the compact \( X \). In other words, the ‘observer’ in this interpretation exists in an infinite multitude of reference frames at once, and the proper time of such an ‘observer’ is actually many-arrow time. However an observer in the classical relativistic mechanics is fixing only one local reference frame (a single-arrow time). Therefore a more adequate counterpart of observer’s worldline should be a subthread

\[
\{s_{\alpha_i}\} = \{s_{\alpha_i} \in X_{\alpha_i} | \alpha_i \in \{\alpha_i | i \in I\}\}
\]  

(2.19)
corresponding to the subspectrum \( S_{pr}(\alpha_i, i \in I) \). (This exactly corresponds to the reference frame concept introduced via a family of \( T_0\)-discrete hypersurfaces \( X_{\alpha_i} \).) In this case there is a linear orderedness of the events \( s_{\alpha_i} \) on the observer’s discrete worldline \( \{s_{\alpha_i}\} \), meaning that any two partitions \( \alpha_i, \alpha_j \in \{\alpha_i | i \in I\} \) are ordered (for example, \( \alpha_j \succ \alpha_i \)), thus \( s_{\alpha_i} = \omega_{\alpha_i}^{\alpha_j} s_{\alpha_j} \). However in the general case through one point \( s_{\alpha_i} \in X_{\alpha_i} \) goes not one, but a finite or denumerable set of maximal threads of the spectrum \( S_{pr}(\alpha_i, i \in I) \). Thus the upper limit \( \hat{S}_{pr}(\alpha_i, i \in I) = \uplim \{X_{\alpha_i}, \omega_{\alpha_i}^{\alpha_j}\} \) defined as a complete system of maximal threads, describes a set of “multifurcating” observers.

[Observe that the cardinality of a set of maximal threads (observers) is equal to the cardinality of a three-dimensional compact \( X \), the same which is known for points on a spacelike hypersurface in general relativity.] It is worth being emphasized that the furcations of discrete worldlines of observers (threads) only occur in the future direction, \( i.e. \) with transitions to more refined partitions. This is the alternative expression of the fact that in this model evolution of the expanding universe is described by a sequence of topology changes (homomorphisms) within the class of \( T_0 \) Alexandroff spaces

\[
\omega_{\alpha_i}^{\alpha_i+1} : X_{\alpha_i+1} \rightarrow X_{\alpha_i},
\]  

(2.20)
as well as that the “evolution operator” \( (\omega_{\alpha_i}^{\alpha_i+1})^{-1} \) (inverse to the homomorphism \( \omega_{\alpha_i}^{\alpha_i+1} \)) is many-valued. These topology changes are in certain sense quasi-quantum processes with respect to the smooth classical evolution of three-geometries, \( e.g., \) in the framework of the canonical approach to general relativity.
2.5 Establishment of the causal order in the proper inverse spectrum

In the Observation 2.1 a hypothesis was accepted which established the partial ordering of three-dimensional $T_0$-discrete spatial sections (if $\alpha' \succ \alpha$, the $T_0$-discrete space $X_{\alpha'}$ is in future with respect to $X_\alpha$). This brought us to the concept of a discrete reference frame (see Observation 2.2). However the fact that two events $s_\alpha$ and $s_{\alpha'}$ pertain to spaces $X_\alpha$ and $X_{\alpha'}$, does not yet mean that the event $s_{\alpha'}$ is in the causal future of the event $s_\alpha$. Just this situation takes place for the 3+1-splitting of the continuous spacetime in a family of three-dimensional spacelike sections in the standard relativity theory, and it would be natural to reproduce it in the discrete case. This leads to the necessity to define in the proper inverse spectrum $S_{pr}$ (as a model of the $T_0$-discrete spacetime) such a relation of partial ordering of events, which would permit a causal interpretation, i.e. to define a causal order in $S_{pr}$.

To this end we first remark that from the definition of a complete linearly ordered subset (closset) $\{\alpha_i|i \in I\}$ (see Subsection 2.4) it follows that if a partition $\alpha_i$ contains $N$ canonically closed sets, then $\alpha_i+1$ consists of $N+1$ sets, since otherwise a partition $\alpha^* \in \{\alpha\}$ should exist such that $\alpha_i < \alpha^* < \alpha_i+1$, which would not pertain to the closset $\{\alpha_i|i \in I\}$. This fact contradicts to the supposition of completeness of $\{\alpha_i|i \in I\}$. Thus we can define a discrete time quantum as a transition from the $T_0$-discrete space $X_{\alpha_i}$ to $X_{\alpha_i+1}$ for any $i \in I$.

Just for the three closest (in the discrete time sense) spaces $X_{\alpha_i-1}$, $X_\alpha$, and $X_{\alpha_i+1}$, we introduce the causal order between events, then we extend it inductively to the inverse spectrum $S_{pr}(\alpha_i, i \in I)$ (2.16), and finally to the whole proper inverse spectrum $S_{pr}$ (2.13).

The causal past of an event $s_{\alpha_i}$ in the nearest past space $X_{\alpha_i-1}$ we define as the set

$$\text{CP}(s_{\alpha_i}) \cap X_{\alpha_i-1} \equiv \Lambda(s_{\alpha_i}) \cap X_{\alpha_i-1} := \overline{O(\omega_{\alpha_i-1}s_{\alpha_i})} \equiv \Lambda_{i-1}^i s_{\alpha_i}. \quad (2.21)$$

The just defined operator $\Lambda_{i-1}^i$ can act both on a separate point and on a subset of $X_{\alpha_i}$.

The verbal justification of this definition is that it is natural to assume that with an event $s_{\alpha_i}$ are causally related only such events $s_{\alpha_{i-1}}$ which pertain to the closure of the minimal neighborhood $O(\omega_{\alpha_{i-1}}s_{\alpha_i})$ of the event $\omega_{\alpha_{i-1}}s_{\alpha_i}$ being the spectral projection of the event $s_{\alpha_i}$ onto the $T_0$-discrete space $X_{\alpha_{i-1}}$ which is the nearest past of $X_{\alpha_i}$. This corresponds to some
extent to the idea of Finkelstein (1988) that the physical causal connection between events in a discretized spacetime should be such that connects the nearest-neighborhood events [see also (Raptis, 2000a)]. The closure is taken in order to include the intersection of the space $X_{\alpha_{i-1}}$ with the analogue of the past light cone of the event $s_{\alpha_i}$.

For an event $s_{\alpha_i}$, its causal past on the space $X_{\alpha_{i-k}}$ ($k$ is any positive integer) is obtained by the $k$-fold action of operators of type $\Lambda_{i-1}^i$ with successively decreasing indices

$$\text{CP}(s_{\alpha_i}) \cap X_{\alpha_{i-k}} \equiv \Lambda(s_{\alpha_i}) \cap X_{\alpha_{i-k}} := \Lambda_{i-1}^{i-k+1} \cdots \Lambda_{i-1}^i s_{\alpha_i}. \quad (2.22)$$

Similarly to (2.21), the causal future in the nearest future space $X_{\alpha_{i+1}}$ of an event $s_{\alpha_i}$ is

$$\text{CF}(s_{\alpha_i}) \cap X_{\alpha_{i+1}} \equiv V(s_{\alpha_i}) \cap X_{\alpha_{i+1}} := (\omega_{\alpha_i+1})^{-1}O s_{\alpha_i} \equiv V_{i-1}^i s_{\alpha_i}. \quad (2.23)$$

For an event $s_{\alpha_i}$, its causal future on the space $X_{\alpha_{i+k}}$ ($k$ is any positive integer) is

$$\text{CF}(s_{\alpha_i}) \cap X_{\alpha_{i+k}} \equiv V(s_{\alpha_i}) \cap X_{\alpha_{i+k}} := V_{i+k}^{i+k-1} \cdots V_{i-1}^i s_{\alpha_i}. \quad (2.24)$$

This definition together with (2.22) ensures the transitivity of the introduced causal order in the proper inverse spectrum $S_{\text{pr}}(\alpha, i \in I)$.

Passing to the introduction of the causal order in the inverse spectrum $S_{\text{pr}}$, we take two arbitrary canonical partitions $\alpha$ and $\alpha' \in \{\alpha\}$ with only one condition $\alpha' \succ \alpha$. Now consider the intersection of the causal past $\text{CP}(s_{\alpha'})$ of any event $s_{\alpha'} \in X'_{\alpha}$ with the space $X_{\alpha}$ and the intersection of the causal future $\text{CF}(s_{\alpha})$ of any event $s_{\alpha} \in X_{\alpha}$ with the space $X'_{\alpha}$. To this end we define a closset of canonical partitions

$$\text{Cl} = \{\alpha_i | i \in \overline{0, f}, f \in \mathbb{Z}^+\} \quad (2.25)$$

such that $\alpha_0 = \alpha, \alpha_f = \alpha'$ and $\alpha_i \succ \alpha_{i-1}$ for $i \in \overline{1, f}$. Then the definitions (2.22) and (2.24) yield

$$\text{CP}(s_{\alpha'}) \cap X_{\alpha} \equiv \Lambda(s_{\alpha'}) \cap X_{\alpha} := \Lambda_{0}^{f} \cdots \Lambda_{f-1}^i s_{\alpha'}. \quad (2.26)$$

and

$$\text{CF}(s_{\alpha}) \cap X'_{\alpha'} \equiv V(s_{\alpha}) \cap X'_{\alpha'} := V_{f}^{i+k} \cdots V_{1}^i s_{\alpha'}. \quad (2.27)$$

Now we say that the event $s_{\alpha}$ is in the causal past with respect to the event $s_{\alpha'}$ if $s_{\alpha} \in \Lambda(s_{\alpha'}) \cap X_{\alpha}$ and we write $s_{\alpha} \prec s_{\alpha'}$. Also we consider the event $s_{\alpha'}$.
to be in the causal future with respect to the event \( s_\alpha \) if \( s_\alpha' \in V(s_\alpha) \cap X_{\alpha'} \) and we write \( s_\alpha' \succ s_\alpha \). Immediately from these definitions follows the transitivity of these relations, \( i.e. \)

- if \( s_\alpha \prec s_\alpha' \) and \( s_\alpha' \prec s_\alpha'' \), then \( s_\alpha \prec s_\alpha'' \);
- if \( s_\alpha'' \succ s_\alpha' \) and \( s_\alpha' \succ s_\alpha \), then \( s_\alpha'' \succ s_\alpha \).

In the continuous pseudo-Riemannian spacetime \( M^4 \) these two relations are equivalent, \( i.e. \) \( s_\alpha' \succ s_\alpha \) implies \( s_\alpha \prec s_\alpha' \) and \( \text{vice versa} \). We prove this equivalence for the discrete case in the framework of our definitions. By virtue of the transitivity it is sufficient to prove it for only one step, \( i.e. \) for the transition from \( \alpha_i \) to \( \alpha_{i+1} \).

Proposition 2.1. Let

\[
\tilde{s}_{\alpha_{i+1}} \forall \in V(s_{\alpha_i}) \cap X_{\alpha_{i+1}} = \left(\omega_{\alpha_{i+1}}^{\alpha_{i+1}}\right)^{-1} Os_{\alpha_i},
\]

then

\[
s_{\alpha_i} \in \Lambda(\tilde{s}_{\alpha_{i+1}}) \cap X_{\alpha_i} = O(\omega_{\alpha_{i+1}}^{\alpha_{i+1}} \tilde{s}_{\alpha_{i+1}}),
\]

for any \( X_{\alpha_i} \) and \( X_{\alpha_{i+1}} \) such that \( \alpha_{i+1} \succ \alpha_i \).

Proof. From (2.28) it follows that \( \omega_{\alpha_{i+1}}^{\alpha_{i+1}} \tilde{s}_{\alpha_{i+1}} \in \omega_{\alpha_{i+1}}^{\alpha_{i+1}} \left(\omega_{\alpha_{i+1}}^{\alpha_{i+1}}\right)^{-1} Os_{\alpha_i} \subseteq Os_{\alpha_i} \). Setting \( \tilde{s}_{\alpha_i} = \omega_{\alpha_{i+1}}^{\alpha_{i+1}} \tilde{s}_{\alpha_{i+1}} \), we get \( \tilde{s}_{\alpha_i} \in Os_{\alpha_i} \). We now have to prove that \( s_{\alpha_i} \in Os_{\alpha_i} \). For a \( T_0 \) Alexandroff space, it follows from \( \tilde{s}_{\alpha_i} \in Os_{\alpha_i} \) that \( \exists s_{\alpha_i}^* \in Os_{\alpha_i} \) such that \( \tilde{s}_{\alpha_i} \in s_{\alpha_i}^* \), then \( s_{\alpha_i}^* \in Os_{\alpha_i} \) [see (Alexandroff, 1937)], so that \( s_{\alpha_i}^* \subseteq Os_{\alpha_i} \). But from \( s_{\alpha_i}^* \in Os_{\alpha_i} \), it also follows that \( s_{\alpha_i} \in s_{\alpha_i}^* \) (Alexandroff, 1937). Finally, \( s_{\alpha_i} \in s_{\alpha_i}^* \subseteq Os_{\alpha_i} \). Q.E.D.

Proposition 2.2. Let

\[
s_{\alpha_i} \forall \in \Lambda(\tilde{s}_{\alpha_{i+1}}) \cap X_{\alpha_i} = O(\omega_{\alpha_{i+1}}^{\alpha_{i+1}} \tilde{s}_{\alpha_{i+1}}),
\]

then

\[
\tilde{s}_{\alpha_{i+1}} \in V(s_{\alpha_i}) \cap X_{\alpha_{i+1}} = \left(\omega_{\alpha_{i+1}}^{\alpha_{i+1}}\right)^{-1} Os_{\alpha_i}
\]

for any \( X_{\alpha_i} \) and \( X_{\alpha_{i+1}} \) such that \( \alpha_{i+1} \succ \alpha_i \).

Proof. With \( \tilde{s}_{\alpha_i} = \omega_{\alpha_{i+1}}^{\alpha_{i+1}} \tilde{s}_{\alpha_{i+1}} \) (2.30) takes the form \( s_{\alpha_i} \in Os_{\alpha_i} \) yielding \( \tilde{s}_{\alpha_i} \in Os_{\alpha_i} \) (see the proof of the Proposition 2.1), \( i.e. \) \( \omega_{\alpha_{i+1}}^{\alpha_{i+1}} \tilde{s}_{\alpha_{i+1}} \in Os_{\alpha_i} \). Applying the operator \( \left(\omega_{\alpha_{i+1}}^{\alpha_{i+1}}\right)^{-1} \) to the last relation, we get \( \left(\omega_{\alpha_{i+1}}^{\alpha_{i+1}}\right)^{-1} \omega_{\alpha_{i+1}}^{\alpha_{i+1}} \tilde{s}_{\alpha_{i+1}} \subseteq \left(\omega_{\alpha_{i+1}}^{\alpha_{i+1}}\right)^{-1} Os_{\alpha_i} \). Since \( \tilde{s}_{\alpha_{i+1}} \in \left(\omega_{\alpha_{i+1}}^{\alpha_{i+1}}\right)^{-1} \omega_{\alpha_{i+1}}^{\alpha_{i+1}} \tilde{s}_{\alpha_{i+1}} \), we come to (2.31). This completes the proof.

The Propositions 2.1 and 2.2 show that the postulated causal structure in the \( T_0 \)-discrete spacetime modelled by the proper inverse spectrum \( S_{pr} \),
reveals the crucial properties of the causal order in pseudo-Riemannian manifolds. In other words, we have introduced the partial order corresponding to the classical relativistic definition of the past and future.

Observation 2.4. It is worth noting that the complete linearly ordered subset (closet) of canonical partitions \( (2.25) \) between \( \alpha \) and \( \alpha' \) is not uniquely defined. This is the result of the possibility to perform the subdivision of sets of the partition \( \alpha \) in different successions, but finally yielding one and the same partition \( \alpha' \). [In the continuous spacetime, this corresponds to consideration in a spacetime sandwich of different intermediate hypersurfaces (different reference frames), while the initial and final hypersurfaces remain fixed.] Due to the finiteness of the partitions \( \alpha \) and \( \alpha' \), only a finite number of such clossets \( \text{Cl}_n, \ n \in 1, N \) exists. It is easy to show on simple examples that different causal pasts \( \Lambda(s_{\alpha'}) \cap X_\alpha \) and futures \( V(s_\alpha) \cap X_{\alpha'} \) correspond to different choices of these \( \text{Cl}_n \), cf. \( (2.26) \) and \( (2.27) \). In the framework of the concepts introduced above (see Subsection 2.4), this means that the causal past and future do depend on the choice of discrete reference frame. This ambiguity of the discrete light cone could be a manifestation of the quasi-quantum nature of the proposed spacetime model. In the contemporary universe such an ambiguity in the causal ordering for different reference frames should be expected (according to our model) at the near-discreteness (say, Planck) scales. Thus at the resolution power of the present measurements the light cone should be invariant under changes of reference frames. However, at the near-discreteness scale resolution power, the concrete choice of reference frame [which is an idealization of measurements, in this case performed on essentially quantum objects and by quantum observers, thus the act of measurement (see the first treatment of this problem by Bohr and Rosenfeld (1933) in the framework of quantum electrodynamics) merely becomes a specific kind of interaction between two quantum objects] should influence these objects so much that the classical concept of the test reference body has to be abandoned. The discrete reference frame is itself a set of intrinsically quantum objects, thus observer(s) and observable(s) have now equal footing (the quasi-quantum nature mentioned above).
3 Multispectrum of an ensemble of compact manifolds

3.1 Prime summands and nice canonical partitions of three-dimensional compact manifolds

In Section 2 while constructing the proper inverse spectrum of the compact $X$, we used canonical partitions of $X$, i.e. such finite closed partitions $\alpha$ whose elements have disjoint interiors. However we did not suppose the elements of canonical partitions to be homeomorphic to three-dimensional closed discs $D^3$. This means that elements of a canonical partition could have a rather complicated topological structure. But in this case the nerve of a given canonical partition does not fully reflect the topological complexity of the compact $X$. Moreover, by a refinement of canonical partitions their nerves yield a more exhaustive description of the compact’s topological structure. In this Section we are trying to construct such a representation of any compact manifold $M$ and such a sequence of its canonical partitions that the effect of manifestation (in the corresponding sequence of nerves) of topological structures at finer scales would become more explicit.

To this end we shall need some additional concepts [see e.g. (Fomenko and Matveev, 1991)]

From each of two three-dimensional manifolds $M_1$ and $M_2$, an open three-dimensional disk is removed, and the remainders are glued together by means of a certain homeomorphism of the boundary spheres. The obtained manifold $M_1 \# M_2$ is called the connected sum of $M_1$ and $M_2$. Note that the sphere $S^3$ is a neutral element with respect to the connected sum operation in the sense that $M \# S^3 = S^3 \# M = M$ for any three-manifold $M$. A three-manifold is called prime if it cannot be represented as a connected sum of two manifolds neither of which is a three-sphere. Note that here, as in the number theory, neither $S^3$ nor the number 1 are respectively considered as primes.

Our basic construction of the $h$-levels compact manifold (hc-manifold) $M_h$ will follow from the Prime Decomposition Theorem of Kneser (1929) which states that any three-dimensional manifold $M$ can be represented as a finite connected sum of prime manifolds. Milnor (1962) has proven that the prime summands of $M$ are uniquely determined by $M$ up to a homeomorphism.

Moreover, we shall use the so-called nice coverings and nice partitions of compact manifolds. Remember that an open covering $\omega = \{O_i | i \in I_\omega\}$ of $M$
is called nice, if all nonempty intersections \(O_{i_0} \cap \ldots \cap O_{i_p}\) are homeomorphic to the open disk \(D^3\) (Bott and Tu, 1982). It is known that any manifold possesses a nice covering. If the manifold is compact, its nice covering can be chosen to be finite.

In order to define a nice canonical partition the Lebesgue lemma would be helpful; it will be important also in determination of the scales hierarchy on the hc-manifold \(M_h\). Denoting an open neighborhood of the set \(A_i \subset M\) as \(O(A_i)\), we may define the Lebesgue blowing up of the partition \(\alpha = \{A_i|i \in I_\alpha\}\) as an open covering \(\omega_\alpha = \{O(A_i)|i \in I_\alpha\}\) having the nerve \(N_{\omega_\alpha}\) isomorphic to the nerve \(N_\alpha\) of the partition \(\alpha\). The Lebesgue lemma states that the Lebesgue blowing up of any finite partition of a compact always exists. We shall give this lemma in Alexandroff’s formulation (Alexandroff, 1998):

Lemma. For any canonical partition \(\alpha = \{A_i|i \in I_\alpha\}\) of the compact \(X\), one can find such a number \(l_\alpha > 0\) called the Lebesgue number of the partition \(\alpha\), that the nerve \(N_{\omega_\alpha}\) of the open covering \(\omega_\alpha = \{O(A_i,l_\alpha)|i \in I_\alpha\}\), is isomorphic to the nerve \(N_\alpha\) of the initial partition \(\alpha\). Here \(O(A_i,l_\alpha)\) is the \(l_\alpha\)-neighborhood (the Lebesgue neighborhood) of the set \(A_i\), i.e.

\[
O(A_i,l_\alpha) = \{x \in X|\rho(x,y) < l_\alpha, \forall y \in A_i\},
\]

\(\rho(x,y)\) being the distance between the points \(x\) and \(y\) of the compact \(X\). (Compacts are normal spaces with denumerable base, thus they are always metrizable.)

Let us call a nice canonical partition of a compact manifold \(M\), such one that has at least one Lebesgue blowing up which is a nice open covering of \(M\). Note that any set \(A_i\) of a nice canonical partition is homeomorphic to a closed disk \(\overline{D}^3\). Nice canonical partitions (in particular, triangulations) describe topology of compact manifolds better than all other coverings and partitions. More precisely, the Čech homologies (and cohomologies) over nerves of the nice canonical partitions of a compact manifold \(M\) with coefficients in \(\mathbb{Z}\) are mutually isomorphic, being as well isomorphic to the Čech homologies (cohomologies) of the very manifold \(M\) (Bott and Tu, 1982):

\[
\tilde{H}_*(N_\alpha,\mathbb{Z}) \cong \check{H}_*(M,\mathbb{Z}), \quad \check{H}^*(N_\alpha,\mathbb{Z}) \cong \check{H}^*(M,\mathbb{Z}).
\]

In other words, \(T_0\) Alexandroff space \(M_\alpha\) corresponding to any nice canonical partition \(\alpha\), fully describes homological properties of the compact manifold \(M\).
3.2 Many-level compact manifolds and description of topology changes

In this Subsection we construct a family of $h$-levels compact manifolds $M_h$ for any $h \in \mathbb{Z}^+$, and describe the sequence of topology changes between manifolds of different levels, in terms of homomorphisms between the respective $T_0$ Alexandroff spaces.

We begin constructing with some closed compact connected manifold $M_0$ which is supposed to be either prime or a sphere $S^3$. First consider its trivial partition $\alpha_{-1} = \{M_0\}$ consisting only of the manifold $M_0$ itself. The nerve $N_{\alpha_{-1}}$ of this partition consists of only one point. This nerve by no means reflects the topology of $M_0$. Therefore one has to introduce a nice canonical partition

$$\alpha_0 = \{A_{i_0} | i_0 \in I_0\}$$

(3.1)

where $I_0$ is a finite set of indices. We shall call $M_0$ the 0-level compact manifold (0c-manifold); let us for the convenience represent it as

$$M_0 = \bigcup_{i_0 \in I_0} A_{i_0}$$

(3.2)

and characterize it by the nerve $N_{\alpha_0}$ with a $T_0$-discrete topology (i.e. by a $T_0$ Alexandroff space $M_{0,\alpha_0}$).

In order to construct a 1c-manifold $M_1$ we introduce a collection of manifolds

$$\sigma_1 = \{\Sigma_k^1 | k \in K_1\}$$

(3.3)

which includes the sphere $S^3$ ($\Sigma_0^1 = S^3$) as well as the prime compact closed manifolds. ($K_1$ is a finite set of indices.) The question which prime closed manifolds are included in this collection, still remains open. Next we choose for each $i_0$ some $\Sigma_{k(i_0)}^1 \in \sigma_1$ and form the connected sum $A_{i_0} \# \Sigma_{k(i_0)}^1$. Then the 1c-manifold $M_1$ is determined by analogy with $M_0$ as

$$M_1 = \bigcup_{i_0 \in I_0} \left(A_{i_0} \# \Sigma_{k(i_0)}^1\right), \ k(i_0) \in K_1.$$  

(3.4)

Observation 3.1. Actually, this definition gives a family of 1c-manifolds which depend on the choice of the integer-valued function $k = k(i_0)$, so that the exact notation for the manifold (3.4) should be $M_{1,k(i_0)}$. This family contains $|K_1||I_0|$ $T_0$-discrete manifolds where $|K_1|$ and $|I_0|$ are numbers of elements in the sets $K_1$ and $I_0$ respectively. Thus when the function $k = k(i_0)$
is a fixed one, we shall for the sake of conciseness adhere to the notation \( k(0) \). If \( k(i_0) = 0 \) for all \( i_0 \in I_0 \), the manifold remains unchanged, \( M_1 = M_0 \).

Thus on \( M_1 \) is induced a canonical (though, in general, not nice) partition

\[
\alpha_0' = \{ A_{i_0} = A_{i_0} \# \Sigma_{k(i_0)} I_0 \mid i_0 \in I_0, k(i_0) \in K_1 \}. \tag{3.5}
\]

By virtue of the definition of the partition \( \alpha_0' \) its nerve is isomorphic to the nerve of partition \( \alpha_0 \), \( N_{\alpha_0'} \cong N_{\alpha_0} \), though these nerves correspond to partitions of different manifolds, \( M_1 \) and \( M_0 \) respectively. (This construction can be considered as an example of the fact that \( T_0 \) Alexandroff spaces related to different manifolds, \( M_0 \) and \( M_1 \), may be mutually homeomorphic, \( M_0, \alpha_0 \cong M_1, \alpha'_0 \).) It is worth emphasizing that \( \alpha_0 \) is a nice partition, while \( \alpha_0' \) is not.

To the end of a better representation of the topological structure of the manifold \( M_1 \), let us introduce a nice partition \( \alpha_1 \) of the manifold \( M_1 \)

\[
\alpha_1 = \{ A_{i_0} = A_{i_0} \mid i_0 \in I_0, i_1 \in I_1(i_0) \}, \tag{3.6}
\]

more refined than \( \alpha_0' \). Here \( I_1(i_0) \) is a finite set of indices which depends on \( i_0 \), such that

\[
A_{i_0} = \bigcup_{i_1 \in I_1(i_0)} A_{i_0,i_1}. \tag{3.7}
\]

**Observation 3.2.** We impose an additional restriction on diameters of the partitions \( \alpha_0 \), \( \alpha_0' \), and \( \alpha_1 \) which enables to compare diameters of partitions of different manifolds (here, \( M_0 \) and \( M_1 \)), and to speak whether \( \alpha_1 \) really is a substantial refinement of \( \alpha_0' \). First note that the construction process of he-manifolds can be realized in the seven-dimensional Euclidean space \( \mathbb{R}^7 \), therefore the diameters of partitions of all manifolds \( M_h \) \((h \in \mathbb{Z}^+)\) can be compared in this universal embedding space (Kodama, 1958). Suppose that pasting prime manifolds into sets of the partition \( \alpha_0 \) does not alter their diameters, i.e. \( d(A_{i_0}) = d(A_{i_0} \# \Sigma_{k(i_0)} I_0) \). This means that diameters of the partitions \( \alpha_0 \) and \( \alpha_0' \) in the embedding space \( \mathbb{R}^7 \) are equal,

\[
d(\alpha_0) = d(\alpha_0'). \tag{3.8}
\]

On the other hand, we suppose that

\[
d(\alpha_1) \leq \min \left( \frac{1}{2}d(\alpha_0'), \frac{1}{2}l(\alpha_0') \right) \tag{3.9}
\]

where \( l(\alpha_0') \) is the Lebesgue number of the partition \( \alpha_0' \). In this case we shall call the partition \( \alpha_1 \), a *Lebesgue refinement* of the partition \( \alpha_0' \).
requirement [usual in the description of refinement of partitions to the end of creation of the proper inverse spectrum (Alexandroff, 1929; Boltyansky, 1951)] establishes a hierarchy between different levels of the manifold $M_1$, which are described (by our definition) by the partitions $\alpha'_0$ and $\alpha_1$ (or, similarly, between manifolds of different levels, $M_0$ and $M_1$). Really, let us blow up the sets of the partition $\alpha'_0$ by the diameter of the partition $\alpha_1$, i.e. determine the open covering

$$\omega'_0 = \left\{ O\left(A'_i, d(\alpha_1)\right) | i_0 \in I_0 \right\}.$$  

(3.10)

Then due to the condition (3.9) we have the homeomorphism of $T_0$-discrete spaces $M_1,\alpha'_0 \cong M_1,\omega'_0$. In other words, the $T_0$-discrete space $M_1,\omega'_0$ is invariant with respect to fluctuations of the diameter of the covering $\omega'_0$ in the limits determined by the diameter $d(\alpha_1)$ of the next-level partition $\alpha_1$ of the manifold $M_1$. This means that any level of the refinement in fact is represented by a whole class of coverings to which correspond mutually isomorphic nerves. Thus there is a hierarchy each level of which is exactly the above mentioned class; the first step of the establishment of this hierarchy was described by the condition (3.9).

Returning to the expression (3.7), we can determine the homomorphism of the nerves $\omega^\alpha_{\omega_0} : N_\alpha \to N_{\alpha'_0}$ and the respective $T_0$ Alexandroff spaces

$$\omega^\alpha_{\omega_0} : M_1,\alpha_1 \to M_1,\alpha'_0.$$  

(3.11)

Remembering the homeomorphism $M_0,\alpha_0 \cong M_1,\alpha'_0$, one can say that the homomorphism $\omega^\alpha_{\omega_0}$ describes the topology change between the discrete versions of the manifolds $M_1$ and $M_0$,

$$\omega^\alpha_{\omega_0} : M_1,\alpha_1 \to M_1,\alpha'_0 \cong M_0,\alpha_0.$$  

(3.12)

$T_0$-discrete space $M_{01}^\alpha := M_{0,\alpha_0} \cong M_1,\alpha'_0$ may be called the critical level in the proper inverse spectra of manifolds $M_0$ and $M_1$. One can say that at the level of the $T_0$-discrete space $M_{01}^\alpha$ occurs a bifurcation of the inverse spectra of manifolds $M_0$ and $M_1$ in the following sense: Introduce two sequences of partitions $\{\alpha_{0,i}\}$ and $\{\alpha_{1,i}\}$ of the manifolds $M_0$ and $M_1$ respectively. Let the index $i$ take integer values from some $p_0$, to $+\infty$ ($i \in [p_0, \infty)$). For $i = p_0$ let $\alpha_{0,p_0} = \{M_0\}$, $\alpha_{1,p_0} = \{M_1\}$, i.e. they are trivial partitions to which correspond trivial homeomorphic $T_0$-discrete spaces $M_{0,\alpha_0,p_0} \cong M_{1,\alpha_{1,p_0}}$. For $i = p_1 > p_0$ it is natural to take $\alpha_{0,p_1} = \alpha_0$ and $\alpha_{1,p_1} = \alpha'_0$, thus the $T_0$-discrete
spaces corresponding to these partitions are also homeomorphic, \( M_{0,\alpha_0,p_1} \cong M_{1,\alpha_1,p_1} \), by virtue of (3.12). One may say that topology of the manifold \( M_0 \) is equivalent to topology of \( M_1 \) at the scales \( d > d(\alpha_0) = d(\alpha_0') \), see (3.8).

More precisely, there exist two sequences of partitions \( \{\text{enlargements} \} \) of the partitions \( \alpha_{0,p_1} \) and \( \alpha_{1,p_1} \) respectively

\[
\begin{align*}
\alpha_{0,p_0} & \prec \alpha_{0,p_0+1} \prec \ldots \prec \alpha_{0,p_1} = \alpha_0, \\
\alpha_{1,p_0} & \prec \alpha_{1,p_0+1} \prec \ldots \prec \alpha_{1,p_1} = \alpha_0'
\end{align*}
\]

(3.13)

of the manifolds \( M_0 \) and \( M_1 \), for which the \( T_0 \)-discrete spaces are pairwise homeomorphic,

\[
M_{0,\alpha_{0,p_0}} \cong M_{1,\alpha_{1,p_0}}, \quad M_{0,\alpha_{0,p_0+1}} \cong M_{1,\alpha_{1,p_0+1}}, \ldots, \quad M_{0,\alpha_{0,p_1}} \cong M_{1,\alpha_{1,p_1}}.
\]

(3.14)

It is however obvious that for \( i > p_1 \) the homeomorphism does not hold due to the definition of the manifold \( M_1 \) through \( M_0 \) (3.4). Thus the inverse spectra \( \{\)contain homeomorphic \( T_0 \)-discrete spaces for \( i \in \mathbb{P}_0, \mathbb{P}_1 \) and no homeomorphic spaces for \( i \in \mathbb{P}_1 + 1, \infty \). Just due to the mentioned homeomorphism of the initial \( T_0 \)-discrete spaces in the spectra (3.15), as well as due to the fact that the manifold \( M_1 \) appears only at the level of the partition \( \alpha_{1,p_1} \), one may initiate the second spectrum in (3.15) with \( i = p_1 \). This yields the pair of spectra

\[
\begin{align*}
\{ M_{0,\alpha_{0,i}}, \omega_{\alpha_{0,i}}^{\alpha_{0,i+1}} | i \in \mathbb{P}_0, \infty \}, & \quad \{ M_{1,\alpha_{1,i}}, \omega_{\alpha_{1,i}}^{\alpha_{1,i+1}} | i \in \mathbb{P}_0, \infty \} \\
\{ M_{0,\alpha_{0,i}}, \omega_{\alpha_{0,i}}^{\alpha_{0,i+1}} | i \in \mathbb{P}_0, \infty \}, & \quad \{ M_{1,\alpha_{1,i}}, \omega_{\alpha_{1,i}}^{\alpha_{1,i+1}} | i \in \mathbb{P}_1, \infty \}
\end{align*}
\]

(3.15)

which we shall call a \textit{bispectrum}. We also shall say that the \( T_0 \)-discrete space \( M_{01}^{cr} := M_{0,\alpha_{0,p_1}} \cong M_{1,\alpha_{1,p_1}} \) represents the critical level (the bifurcation level) of the bispectrum (3.10).

When the function \( k(i_0) \) is not fixed (\textit{i.e.} when all such functions are taken into account), see the Observation 3.1, there is a multifurcation at the critical level (see Subsection 3.3).

To construct the 2nd-level compact manifold (2c-manifold) \( M_2 \), we take, generally speaking, another \( \{\)collection of manifolds

\[
\sigma_2 = \{ \Sigma_k^2 | k \in K_2 \}
\]

(3.17)

\footnote{A partition \( \beta = \{ B_j \} \) is called enlargement of a partition \( \alpha = \{ A_i \} \) if each \( B_j \) is a union of some sets \( A_i \in \alpha \) and each \( A_i \) enters only one union as a summand.}
containing prime compact closed three-manifolds and the sphere $S^3$ (as a neutral element). ($K_2$ is a finite set of indices.) Using this collection and the partition (3.6), we construct $M_2$ in an analogy with (3.4) as a union of connected sums,

$$M_2 = \bigcup_{i_0 \in I_0, i_1 \in I_1(i_0)} \left( A_{i_0,i_1} \# \Sigma^2_{k(i_0,i_1)} \right), \quad k(i_0,i_1) \in K_2.$$ 

**Observation 3.3.** Note again that in fact we define a family of $2c$-manifolds depending on two integer-valued functions, $k(i_0)$ (cf. Observation 3.1) and $k(i_0,i_1)$. Thus instead of $M_2$ one should write $M_2,k(i_0),k(i_0,i_1)$. When both functions are fixed, we shall use the abbreviation $M_2$.

Applying this procedure repeatedly, we come to the hc-manifold $M_h$ (more exactly, to a family of hc-manifolds, see the Observation 3.3). We give here the respective expressions.

Introduce a collection

$$\sigma_h = \left\{ \Sigma^h_k \mid k \in K_h \right\} \quad (3.18)$$

which includes prime compact closed manifolds, as well as $S^3$. Then $M_h$ is constructed with help of the connected sum operation applied to each set of the partition $\alpha_{h-1}$,

$$M_h = \bigcup_{i_p \in I_p(i_0, \ldots, i_{p-1})} \left( A_{i_0,\ldots,i_{h-1}} \# \Sigma^h_{k(i_0,\ldots,i_{h-1})} \right) \quad (3.19)$$

where $k(i_0, \ldots, i_{h-1})$ is a fixed integer-valued function with values in $K_h$. We automatically come to the canonical (though not nice) partition

$$\alpha'_{h-1} = \left\{ A'_{i_0,\ldots,i_{h-1}} = A_{i_0,\ldots,i_{h-1}} \# \Sigma^h_{k(i_0,\ldots,i_{h-1})} \mid i_p \in I_p(i_0, \ldots, i_{p-1}), \ p = 0, \ldots, h-1, k(i_0, \ldots, i_{h-1}) \in K_h \right\}. \quad (3.20)$$

Here $I_p(i_0, \ldots, i_{p-1})$ is a finite collection of indices which enumerates the sets $A_{i_0,\ldots,i_p} \in \alpha_p$ being subsets of $A'_{i_0,\ldots,i_{p-1}}$, i.e.

$$A'_{i_0,\ldots,i_{p-1}} = \bigcup_{i_p \in I_p(i_0, \ldots, i_{p-1})} A_{i_0,\ldots,i_p}. \quad (3.21)$$

We have again the isomorphism $M_{h-1,\alpha_{h-1}} \cong M_h,\alpha'_{h-1}$ of $T_0$-discrete spaces corresponding to different c-manifolds $M_{h-1}$ and $M_h$. 27
To complete the construction process, one has to introduce a nice canonical partition
\[
\alpha_h = \{ A_{i_0, \ldots, i_h} | i_p \in I_p(i_0, \ldots, i_{p-1}), \ p = 0, \ldots, h \} \tag{3.22}
\]
of the manifold \(M_h\); \(\alpha_h\) is a Lebesgue refinement of the partition \(\alpha'_{h-1}\), which satisfy the condition
\[
d(\alpha_h) \leq \min\left(\frac{1}{2}d(\alpha'_{h-1}), \frac{1}{2}l(\alpha'_{h-1})\right) \tag{3.23}
\]
(cf. the Observation 3.2).

The homomorphism
\[
\omega_{\alpha_{h-1}}^{\alpha_h} := \omega_{\alpha_{h-1}}^{\alpha_h} : M_{h,\alpha_h} \to M_{h,\alpha_{h-1}} \cong M_{h-1,\alpha_{h-1}} \tag{3.24}
\]
again gives the discrete description of the topology change between the manifolds \(M_h\) and \(M_{h-1}\). \(T_0\)-discrete space
\[
M_{h-1,1,h} := M_{h,\alpha_{h-1}} \cong M_{h-1,\alpha_{h-1}} \tag{3.25}
\]
represents the critical level at which occurs the bifurcation of bispectrum
\[
\left\{M_{h-1,\alpha_{h-1}}, \omega_{\alpha_{h-1}}^{\alpha_h, i+1} | i \in P_{h-1}, \infty \right\},
\left\{M_{h,\alpha_h}, \omega_{\alpha_h, i}^{\alpha_{h-1}, i+1} | i \in P_h, \infty \right\} \tag{3.26}
\]
in the same sense as for the bispectra (3.16).

### 3.3 Multispectra, superspectra, and their many-world interpretation

In the preceding Subsection there has been constructed a sequence of topology changes between compact manifolds \(M_{n+1} \to M_n\) \((n = 0, 1, \ldots, h - 1)\) in terms of a sequence of homomorphisms between \(T_0\) Alexandroff spaces
\[
\omega_{\alpha_n}^{\alpha_{n+1}} : M_{n+1,\alpha_{n+1}} \to M_{n,\alpha_n} \tag{3.27}
\]
[see, e.g., (3.21)] where \(\alpha_n\) and \(\alpha_{n+1}\) are nice canonical partitions of manifolds \(M_n\) and \(M_{n+1}\) respectively, which satisfy the Lebesgue refinement condition
\[
d(\alpha_{n+1}) \leq \min\left(\frac{1}{2}d(\alpha'_n), \frac{1}{2}l(\alpha'_n)\right) \tag{3.28}
\]
(see Observation 3.2). We have supposed herewith that all integer-valued functions \( k(i_0, \ldots, i_n) \) are fixed. Then it was possible to describe these topology changes in terms of bifurcations of \((n, n + 1)\)-level bispectra,

\[
\begin{align*}
\{ M_{n, \alpha_n, i}, \omega_{\alpha_n, i}^n | i \in p_n, \infty \}, \\
\{ M_{n+1, \alpha_{n+1}, i}, \omega_{\alpha_{n+1}, i}^{n+1} | i \in p_{n+1}, \infty \},
\end{align*}
\]

(3.29)

where \( p_{n+1} > p_n \) for all \( n = 0, 1, \ldots, h - 1 \).

In this Subsection we lift the restriction concerning the fixedness of functions \( k(i_0, \ldots, i_n) \) in the definition of manifolds of all levels, thus coming to the concepts of multispectrum and superspectrum of compact manifolds.

Passing to the construction of a multispectrum it is worth remembering that (see the Observation 3.1) the set \( J_1 \) of the mappings \( k : I_0 \to K_1 \) has the cardinality

\[ |J_1| = |K_1|^{|I_0|} \]

where \( |I_0| \) is the number of sets in the nice canonical partition \( \alpha_0 = \{ A_{i_0} | i_0 \in I_0 \} \) of the connected 0c-manifold \( M_0 \), while \( |K_1| \) is the number of elements in the collection \( \sigma_1 \) (3.3). Thus when all possible integer-valued functions \( k(i_0) \in J_1 \) are admissible, the compact manifold of the first level (1c-manifold) becomes multicomponent one,

\[
M_{1,mc} = \bigsqcup \bigcup_{k(i_0) \in J_1} \bigcup_{i_0 \in I_0} \left( A_{i_0} \# \Sigma_{k(i_0)}^1 \right)
\]

(3.30)

(\( \bigsqcup \) is a disjoint union) with the number \( |J_1| \) of connected components such as

\[
M_{1, k(i_0)} = \bigcup_{i_0 \in I_0} \left( A_{i_0} \# \Sigma_{k(i_0)}^1 \right) \equiv \bigcup_{i_0 \in I_0} A'_{i_0, k(i_0)}
\]

(3.31)

[\text{cf. (3.4)}]. We automatically come to the canonical (though not nice) partition

\[
\alpha'_{0, k(i_0)} = \left\{ A'_{i_0, k(i_0)} | i_0 \in I_0 \right\}
\]

(3.32)

of any component \( M_{1, k(i_0)} \) of the manifold \( M_{1,mc} \). Then we have the homeomorphism

\[
M_{0, \alpha_0} \simeq M_{1, k(i_0)}, \alpha'_{0}
\]

(3.33)

of \( T_0 \)-discrete spaces, like in the Subsection 3.2. Here \( M_{0, \alpha_0} \) is a \( T_0 \)-discrete space corresponding to the partition \( \alpha_0 \) of the manifold \( M_0 \), while \( M_{1, k(i_0)}, \alpha'_{0} \) is a \( T_0 \)-discrete space corresponding to the partition

\[
\alpha'_{0} = \left\{ A'_{i_0, k(i_0)} | i_0 \in I_0, k(i_0) \in J_1 \right\}
\]

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of the manifold $M^\text{mc}_0$ restricted onto the component $M_{1,k(i_0)}$. Due to this homeomorphism there is the covering mapping

$$\pi_0 : \bigcup_{k(i_0) \in J_1} M_{1,k(i_0),\alpha'_0} \to M_{0,\alpha_0}$$

such that its restriction onto one component $M_{1,k(i_0),\alpha'_0}$ is an identical homeomorphism. This covering mapping in fact involves a possibility to describe the topology change between the manifolds $M_0$ and $M^\text{mc}_1$. In order to give this description strictly in terms of inverse spectra, we introduce the following definitions analogous to those given in the Subsection 3.2.

Consider two sequences of canonical partitions $\{\alpha_{0,i}\}$ and $\{\alpha_{1,i}\}$ of the manifolds $M_0$ and $M^\text{mc}_1$ respectively. Here $i \in \mathbb{P}_0, \infty$. For $i = p_0$ we put $\alpha_{0,p_0} = \{M_0\}$ and $\alpha_{1,p_0} = \{M^\text{mc}_1\}$, thus the respective one-point spaces are homeomorphic,

$$M_{0,\alpha_{0,p_0}} \cong M_{1,\alpha_{1,p_0}}.$$  \hfill (3.35)

For some $i = p_1 > p_0$, set $\alpha_{0,p_1} = \alpha_0$ and $\alpha_{1,p_1} = \alpha'_0$. Thus due to (3.33) the homeomorphism

$$M_{0,\alpha_{0,p_1}} \cong M_{1,k(i_0),\alpha_{1,p_1}}$$ \hfill (3.36)

holds for any $k(i_0) \in J_1$. Then similar to (3.15) one can introduce the pair of inverse spectra

$$\left\{ M_{0,\alpha_{0,i},\omega_{\alpha_{0,i}}}^{\alpha_{0,i}+1} \mid i \in \mathbb{P}_0, \infty \right\}, \left\{ M^\text{mc}_{1,\alpha_{1,i}} \omega_{\alpha_{1,i}}^{\alpha_{1,i}+1} \mid i \in \mathbb{P}_0, \infty \right\}. \hfill (3.37)$$

It is worth emphasizing that for $i \in \mathbb{P}_0, p_1$, all components $M_{1,k(i_0),\alpha_{1,i}}$ of $M^\text{mc}_{1,\alpha_{1,i}}$ may be defined in such a manner that they will be homeomorphic to $M_{0,\alpha_{0,i}}$, so that they can be identified among themselves as well as with $M_{0,\alpha_{0,i}}$ [see (3.36) and the reasonings concerning the enlargement of partitions in the Subsection 3.2 with respect to (3.13) and (3.14)]. Besides, $M_{1,k(i_0)} = M_0$ for $k(i_0) \equiv 0$ since in this case the expression (3.31) takes the form

$$M_{1,k(i_0)} \mid k(i_0) \equiv 0 = \bigcup_{i_0 \in I_0} \left( A_{10} \# S^3 \right) = \bigcup_{i_0 \in I_0} A_{i_0} = M_0.$$  

Thus $M_0$ enters $M^\text{mc}_1$ as one of its components. Consequently, the first spectrum in the pair (3.37) can be restricted to the interval $i \in \mathbb{P}_0, p_1$, while the second one will begin at $i = p_1$, giving the following definition to the $(0,1)$-level multispectrum:

$$\left\{ M_{0,\alpha_{0,i},\omega_{\alpha_{0,i}}}^{\alpha_{0,i}+1} \mid i \in \mathbb{P}_0, p_1 \right\}, \left\{ M^\text{mc}_{1,\alpha_{1,i}} \omega_{\alpha_{1,i}}^{\alpha_{1,i}+1} \mid i \in \mathbb{P}_1, \infty \right\}. \hfill (3.38)$$
This term is justified by the fact that (3.38) describes the multifurcation

$$M_{0,\alpha_0} \to \bigcup_{k(i_0) \in J_1} M_{1,k(i_0),\alpha_1,p_1}$$  (3.39)

at the critical level (3.36), i.e. for \(i = p_1\), \(M_{01}^\ast := M_{0,\alpha_0,p_1}\).

Further we introduce the nice canonical partition

$$\alpha_1 = \{ A_{i_0,i_1,k(i_0)} | i_0 \in I_0, i_1 \in I_1(i_0,k(i_0)), k(i_0) \in J_1 \}$$

of the multicomponent 1c-manifold \(M_{1mc}\). Note that in this case one ought to explicitly write out the collection of indices \(I_1(i_0,k(i_0))\) as a function not only of \(i_0\) but also of the integer-valued function \(k(i_0) \in J_1\) which enumerates the components of \(M_{1mc}\). Thus instead of (3.40) we have

$$A'_{i_0,k(i_0)} = \bigcup_{i_1 \in I_1(i_0,k(i_0))} A_{i_0,i_1,k(i_0)}.$$  (3.40)

Note that by the transition to the manifolds of the subsequent levels, the number of connected components increases dramatically. Since bulkiness of the expressions increases similarly, we shall confine ourselves to the second-level manifold \(M_{2mc}\) being again defined via the collection of the compacts \(\sigma_2\) (3.17):

$$M_{2mc} = \bigcup_{k(i_0,i_1) \in J_2(k(i_0))} \bigcup_{i_1 \in I_1(i_0)} \left( A_{i_0,i_1,k(i_0)} \# \Sigma_{k(i_0,i_1)}^2 \right)$$

$$= \bigcup_{k(i_0,i_1) \in J_2(k(i_0))} M_{2,k(i_0),k(i_0,i_1)}.$$  (3.41)

Here \(J_2(k(i_0))\) is the set of components of the manifold \(M_{2mc}\) arising from one component \(M_{1,k(i_0)}\) of the manifold \(M_{1mc}\) due to the arbitrariness of the functions \(k(i_0,i_1)\) [for any fixed function \(k(i_0)\)]. In other words, \(J_2(k(i_0))\) is a set of mappings

$$k : I_1(k(i_0)) \to K_2$$  (3.42)

where \(I_1(k(i_0))\) is the set of elements of the partition \(\alpha_1\) pertaining to component \(M_{1,k(i_0)}\) of the manifold \(M_{1mc}\), i.e.

$$I_1(k(i_0)) = \bigcup_{i_0 \in I_0} I_1(i_0,k(i_0))$$  (3.43)
Then the number of components of the manifold $M_{2mc}$ arising from one component $M_{1,k(i_0)}$, is

$$|J_2(k(i_0))| = |K_2||I_1(k(i_0))|,$$

while the total amount of components of $M_{2mc}$ is

$$|J_2| = \sum_{k(i_0) \in J_1} |J_2(k(i_0))|.$$  

(3.44)
(3.45)

(Remember that $|A|$ means the cardinality of the set $A$.)

The expression (3.41) contains the definition of a canonical (but not nice) partition

$$\alpha'_1 = \left\{ A'_{i_0,i_1,k(i_0),k(i_0,i_1)} = A_{i_0,i_1,k(i_0)} \# \Sigma S_{k(i_0,i_1)} \right\}$$

$$i_0 \in I_0, i_1 \in I_1(k(i_0)), k(i_0) \in J_1, k(i_0,i_1) \in J_2(k(i_0)) \}$$

of $M_{2mc}$; by analogy with (3.33), exist homeomorphisms

$$M_{1,k(i_0),\alpha_1} \cong M_{2,k(i_0),k(i_0,i_1),\alpha'_1}, \ k(i_0) \forall \in J_1, k(i_0,i_1) \forall \in J_2(k(i_0))$$

between connected components of $T_0$-discrete spaces $M_{1mc}$ and $M_{2mc}$, respectively. Then, using the prescription given in (3.38) for the $(0,1)$-level multispectrum, one can determine the $(1,2)$-level multispectrum as

$$\left\{ M_{1,\alpha_1,i}, \omega_{\alpha_1,i}^{\alpha_1+1} \mid i \in p_1 \cup p_2 \right\}, \ \left\{ M_{2,\alpha_2,i}, \omega_{\alpha_2,i}^{\alpha_2+1} \mid i \in p_2 \cup \infty \right\},$$

(3.47)

as well as $(n, n + 1)$-level multispectra for any $n = 0, 1, \ldots, h - 1$ as

$$\left\{ M_{n,\alpha_n,i}, \omega_{\alpha_n,i}^{\alpha_n+1} \mid i \in p_n \cup p_{n+1} \right\}, \ \left\{ M_{n+1,\alpha_{n+1},i}, \omega_{\alpha_{n+1},i}^{\alpha_{n+1}+1} \mid i \in p_{n+1} \cup \infty \right\}.$$  

(3.48)

Here $p_{n+1} > p_n$ and in contrast to the bispectrum (3.29), the first inverse spectrum in (3.48) can be taken as finite, since for $i > p_{n+1}$ all $T_0$-discrete spaces of the first spectrum are contained in the $T_0$-discrete spaces of the second one,

$$M_{n,\alpha_n,i} \subset M_{n+1,\alpha_{n+1},i}.$$  

(3.49)

The upper limit of the multispectrum (3.48) is the multicomponent compact manifold $M_{n+1}$ which contains subspaces forming a chain of inclusions,

$$M_{n+1} \supset M_{n} \supset \cdots \supset M_{1} \supset M_{0}.$$
The multispectrum (3.48) describes multifurcations
\[ M_{n,k(i_0),...,k(i_0,...,i_{n-1}),\alpha_{n,p_{n+1}}} \rightarrow \]
\[ \bigcup_{k(i_0),...,i_n) \in J_{n+1}(k(i_0),...,k(i_0,...,i_{n-1}))} M_{n+1,k(i_0),...,k(i_0,...,i_{n-1}),\alpha_{n+1,p_{n+1}}} \] (3.50)
at the \((n+1)\)th critical level determined by the multicomponent \(T_0\)-discrete space
\[ M_{n,n+1}^{cr} := M_{n,\alpha_{n,p_{n+1}}}^{mc}. \] (3.51)

The union of all multispectra (3.48) for \(n = 0, \ldots, h-1\) we shall call the \(h\)-level *superspectrum* in an analogy with the Wheeler–DeWitt superspace. This analogy will follow from the interpretation of multi- and superspectra to which we turn now.

On a multicomponent manifold \(M_{n,mc}^{mc}\) being one of subspaces of the upper inverse limit of the multispectrum (3.48), let us introduce a partition \(\alpha_{n,i}\) where \(i \in p_n, p_{n+1}\). The diameter of this partition we denote as \(d_{n,i}\). One may now say that to this partition corresponds a system of observers (forming a discrete reference frame introduced in the Subsection 2.4) which realize measurements at the scales \(d_{n,i}\) in the interval
\[ d(\alpha_{n+1,p_{n+1}}) < d_{n,i} < d(\alpha_{n,p_n}). \] (3.52)

In this case we shall say that a subsystem of mutually interconnected (via chains of the sets of \(\alpha_{n,i}\)) observers may identify itself in one of the connected components of the \(T_0\)-discrete space \(M_{n,mc}^{mc}\) of the \((n, n+1)\)-level multispectrum (3.48), e.g., in the component
\[ M_{n,k(i_0),...,k(i_0,...,i_{n-1}),\alpha_{n,i}}. \] (3.53)

It is natural to question of the probability to occur in one or another component of the space \(M_{n,mc}^{mc}\). From the viewpoint of the many-world interpretation of quantum cosmology, the probability amplitude to occur in the universe (3.53) should be determined by a certain wave function
\[ \Psi(M_{n,k(i_0),...,k(i_0,...,i_{n-1}),\alpha_{n,i}}). \] (3.54)
The domain of definition of this probability amplitude function is the set of connected components of $T_0$-discrete spaces forming the $h$-level superspectrum

$$DSS = \left\{ M_{n,k(i_0),\ldots,k(i_{m-1}),\alpha_{n,i}} \mid k(i_0, \ldots, i_s) \in J_{s+1}(k(i_0), \ldots, k(i_{s-1})), \quad s \in 0, n-1, i \in p_n, p_{n+1}, n \in 0, h-1 \right\}. \quad (3.55)$$

Namely this set we shall call the *discrete superspace* (DSS) which we will treat as an analogue (in our model) of the Wheeler–DeWitt superspace (Wheeler, 1964; DeWitt, 1967). Performing measurements in a discrete reference frame, say, that which corresponds to the partition $\alpha_{n,i}$ [i.e., on the scales (3.52)], the observers merely find out in which connected component (3.53) (i.e., at which point of the DSS) they are localized. This represents just an example of the many-world interpretation of the wave function (Everett, 1957; DeWitt and Graham, 1973). Executing more refined measurements, e.g., on the scales (3.56), the observers occur in a wider set of connected components forming $T_0$-discrete space $M_{n+1,\alpha_{n+1,i}}$. This is related to the presence of the critical level (3.51) at which the phase transition of an increase of the number of the multiconnected-space components, (3.50), occurs. Hence in our model a certain kind of Heisenberg’s uncertainty principle holds: the finer scales at which observers perform the measurements (i.e., the more refined the canonical partition), the wider the ensemble of the connected components of $T_0$-discrete spaces in which they can detect themselves. However due to validity of the inclusion condition (3.49), there is a possibility for observers to find themselves in the former connected component while performing measurements with the accuracy (3.50) greater than that of (3.52).

Of course, the problem of evaluation of the wave function (3.54) depends on construction of some topological analogue of the Wheeler–DeWitt quantum geometrodynamics; it could be provisionally called discrete quantum topodynamics. The topodynamical analogue of the Wheeler–DeWitt equation is still unknown, but for the role of an analogue of the superspace, we propose the discrete superspace DSS (3.55) as the domain of definition of topodynamical wave functions.
4 Concluding remarks

In this paper the following three results may be emphasized.

1. On the basis of Alexandroff’s mathematical constructions and under the influence of Sorkin’s physical ideas, we have modelled the discrete spacetime via the proper inverse spectrum $S_{pr}$ relating the cosmological time arrow in the expanding universe to the refinement of canonical partitions $\{\alpha\}$ of the continuous compact three-dimensional space $X$ homeomorphic to the upper inverse limit $\hat{S}_{pr}$ of this spectrum. However this limiting compact space $X \cong \hat{S}_{pr}$ is never realized in the course of the discrete cosmological evolution although at a sufficiently fine partition $\alpha$, the corresponding $T_0$-discrete space $X_{\alpha}$ is a good approximation of the compact $X$. [When the measurements are performed with an uncertainty $\Delta d$ much greater than the diameter $d(\alpha)$ of the partition $\alpha$ ($\Delta d \gg d(\alpha)$), the discrete space $X_{\alpha}$ will be experimentally indistinguishable from the continuous space $X \cong \hat{S}_{pr}$.] Moreover the spectral evolution of $T_0$-discrete spaces automatically suggests the existence of $T_0$-discrete one-dimensional time being not specially postulated from the beginning.

It is obvious that in our model the problem of initial cosmological singularity is automatically lifted, since the initial one-point space naturally pertains to the same class of $T_0$-discrete spaces to which all other spatial discrete sections pertain (in contrast to the Friedmann–Robertson–Walker cosmological models), see Subsection 2.4 (2.16) and Subsection 3.2.

Evolution of the universe is described by the sequence of topological changes (2.20); this corresponds to the transition from the partition $\alpha_i$ to the next more refined one, $\alpha_{i+1}$. It is natural to associate the ‘discrete time quantum’ or ‘elementary topological change’ to the act of subdivision of only one set (element) of the partition $\alpha_i$ into two sets. When we have, say, $10^{360}$ elements of the partition $\alpha_i$ [cf. (Rideout and Sorkin, 2001)], the evolution described by the sequence of elementary topological changes $(\omega^{\alpha_{i+1}}_{\alpha_i})^{-1} : X_{\alpha_i} \rightarrow X_{\alpha_{i+1}}$ can be considered as continuous from the viewpoint of an observer with measuring devices having the resolving power much more coarse than $10^{-360}$ (in the sense that in the whole discrete universe the number of elements has increased from $10^{360}$ to $10^{360} + 1$). Moreover, on the proper inverse spectrum $S_{pr}$ it turned out to be possible to define such a relation of partial ordering (Subsection 2.5) that has the basic properties of the causal order in the pseudo-Riemannian spacetime [transitivity and mutual
compatibility of the sets pertaining to causal past and causal future (Proposi-
tions 2.1 and 2.2), however also bearing certain quasi-quantum features \( e.g. \) dependence of the causal past and future on the choice of discrete reference
frame at the near-discreteness, say, Planck, scales (Observation 2.4)].

2. Our interpretation of inverse spectra of the three-dimensional Alexan-
droff spaces permitted to represent topological changes between three-mani-
folds in terms of bispectra \( (3.29) \). Note that in accordance with the results of
Kneser (1929) and Milnor (1962) this approach permits to describe creation
of any 3-manifold \( M \) as a result of a succession of topological transitions from
an initial manifold which is homeomorphic to the sphere \( S^3 \).

3. Both our the considerations and results obtained here, are bearing
essentially classical and kinematical character (though they possess certain
quasi-quantum features, see, \( e.g. \), Subsections 2.4 and 2.5). However, having
in mind quantization and the necessity to introduce dynamics, we have
described a discrete analogue of the Wheeler–DeWitt superspace, just having
defined the discrete superspace DSS \( (3.55) \) as the arena of the future topody-
namics (an analogue of the Wheeler–DeWitt geometrodynamics). Already
at this qualitative stage we were able to formulate a topological counterpart
of the many-worlds interpretation of the quantum wave function \( (3.54) \), as
well as an analogue of Heisenberg’s uncertainty relation.

Alternatively, it is possible that introduction of dynamics in our model
could occur similarly to the sequential growth dynamics (Rideout and Sorkin,
2000). This approach may be adapted to our model [in the \( T_0 \)-discrete space-
time sandwich construction described in Subsection 2.4 (3)].

The algebraic approach to quantum topodynamics based on constructing
the direct spectrum of the Čech cohomology groups (Bott and Tu, 1982; Rap-
tis, 2000b; Mallios and Raptis 2001), also seems to be promising. Then, using
operators of coboundary and homotopy [possessing interesting anticommuta-
tion properties (Teleman, 1964)] which act on Čech cochain complexes (over
nerves of coverings with values in presheaves of certain algebraic structures),
one could expect to come to an analogue of the many-particle Fock represen-
tation. Here the part of independent (mutually non-interacting) topological
excitations (excitops) of the \((n + 1)\)st level, would be played by the prime
closed manifolds from the collection \( \sigma_{n+1} = \{ \Sigma_k^{n+1} | k \in K_{n+1} \} \), see Subsec-
tions 3.2, 3.3 (The question still remains open which prime closed manifolds
form this collection.) These excitations are determined on the \( n \)th level mani-
ifold \( M_n \) as a classical background. Here the hierarchy of scales \( (3.28) \) may
be of importance, the hierarchy being related to the fundamental role of the Lebesgue numbers in constructing the being ever refined systems of open coverings and canonical partitions.

For us it was a source of inspiration and moral support to read and reread the wonderful Richard Courant Lecture in Mathematical Sciences by Eugene P. Wigner (1960) so masterfully summarized in his own words: “The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure even though perhaps also to our bafflement, to wide branches of learning.” In any case, with all future corrections which life will give to our conclusions in this article, we are sure that the most abstract branch of mathematics touched by physicists until now, the topology, is becoming more and more accessible for a fruitful application in fundamental physics, opening new horizons in the organic synthesis of relativity and quantum.

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