An Iteratively Reweighted Method for Sparse Optimization on Nonconvex $\ell_p$ Ball

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Abstract

This paper is intended to solve the nonconvex $\ell_p$-ball constrained nonlinear optimization problems. An iteratively reweighted method is proposed, which solves a sequence of weighted $\ell_1$-ball projection subproblems. At each iteration, the next iterate is obtained by moving along the negative gradient with a stepsize and then projecting the resulted point onto the weighted $\ell_1$ ball to approximate the $\ell_p$ ball. Specifically, if the current iterate is in the interior of the feasible set, then the weighted $\ell_1$ ball is formed by linearizing the $\ell_p$ norm at the current iterate. If the current iterate is on the boundary of the feasible set, then the weighted $\ell_1$ ball is formed differently by keeping those zero components in the current iterate still zero. In our analysis, we prove that the generated iterates converge to a first-order stationary point. Numerical experiments demonstrate the effectiveness of the proposed method.

Keywords—nonconvex regularization, sparse optimization, iterative reweighting method, weighted $\ell_1$-ball projection

1 Introduction

The use of a bounded norm constraint has received considerable attention in constrained nonlinear programming because it covers a rich set of applications, in-
cluding machine learning, statistics, compressed sensing, and many other related fields [1, 2, 3, 4]. Enforcing a norm ball generally results in sparse solutions of interest, which can capture the low-dimensional structures of the high-dimensional problems [5, 6]. These benefits are evidenced in many learning tasks in which sparsity is usually the prior information for the desired solutions [7]. In the past decade, people have witnessed significant advances in various norm constrained optimization models and numerical algorithms for solving norm ball constrained optimization. For example, in deep learning applications, weight pruning is a widely used technique in model compression [8, 9], whose primary focus is to choose a set of representative weights to balance the compactness and accuracy of the huge-size deep neural networks. Toward this end, researchers often formulated this problem as the convex loss function constrained by a bounded norm (e.g., $\ell_1$ norm and $\ell_0$ norm) [10]. Likewise, in the context of sparse signal recovery with linear measurements, the unknown signals need to be estimated based on the given sensing matrix and noisy observations. This is usually achieved by solving a least-squares problem with norm balls [11].

Although the sparse optimization problems constrained by the $\ell_0$ ball or $\ell_1$ ball often arise in numerous contexts [2, 12, 13, 14, 15, 16], based on the empirical evidence presented in [3], the $\ell_p$ ball constrained problems with $p \in (0, 1)$ usually shows its superior performance in many areas including sparse recovery and phase transition compared with the $\ell_1$ ball and $\ell_0$ ball constraints. Moreover, the benefits of using nonconvex $\ell_p$ ball in Euclidean projection are revealed in [1] for image denoising and sparse principal component analysis and [17] for computer vision. However, until now, to our knowledge, the algorithms proposed for handling the problems constrained by $\ell_p$ ball with $p \in (0, 1)$ are limited. This is mainly due to the nonsmooth and nonconvex nature of the feasible set, which makes the analysis challenging and hinders the development of numerical algorithms.

Many research efforts are devoted to designing algorithms for solving nonconvex set constrained minimization problems. For example, projected gradient descent (PGD) is considered for solving general nonconvex constrained optimization problem [18], which requires the projection onto the feasible set for each iteration. As presented in [18], PGD usually needs the global optimal solution of the projection onto the nonconvex set at each iteration. However, this is generally a nontrivial task; as shown in [19], finding a first-order stationary solution of the $\ell_p$ ball projection problem is NP-hard, let alone compute the global optimal solution. The authors in [12] analyzed the convergence rates of the PGD algorithm for solving the $\ell_p$ ball constrained least-squares, for which they also assumed that a practical algorithm could be used to compute the global optimal solution of the $\ell_p$ ball projections for each iteration, making it impractical for real-world applications.
In this paper, we propose an iteratively reweighted $\ell_1$ ball method for solving the minimization problem constrained by the nonconvex $\ell_p$ norm ball. The main idea of our approach is to use the weighted $\ell_1$ balls to approximate the $\ell_p$ ball so that our approach iteratively solves a sequence of tractable projection subproblems. At an iterate in the interior of the feasible set, the weighted $\ell_1$ ball is constructed by linearizing the $\ell_p$ norm with respect to the nonzero components in the current iterate and setting the weights associated with the zero components according to the constraint residuals. At an iterate on the boundary of the feasible set, we form the weighted $\ell_1$ ball by linearizing the constraint set for the nonzero components while fixing the rest components as zero.

The idea of using iteratively reweighted norms to approximate $\ell_p$ norm is not novel. However, the existing research efforts were largely devoted to solving the $\ell_p$-norm regularization problems \[20, 21, 22\], meaning the $\ell_p$ norm presents in the objective instead of the constraint. To our knowledge, there has not been much work on designing iteratively reweighted methods for handling the case where the $\ell_p$-norm appears in the constraint. As a particular focus, the iteratively reweighted $\ell_1$ ball projection algorithm was proposed in \[19\] for solving the Euclidean projection of a vector onto the nonconvex $\ell_p$ ball. Their main idea is to use a relaxing vector to smooth the $\ell_p$ norm at each iteration, and then the weighted $\ell_1$ balls are obtained by linearizing the relaxed constraints. Their method requires carefully driving the relaxation parameter to zero to enforce convergence. Our method does not involve a relaxation parameter.

We summarize the contributions of this paper in the following.

1. The problem with $\ell_p$ ball constraint considered in our paper has general form. Investigating such a problem offers the possibility to solve a broader class of objective functions that can include various applications.

2. We propose an iteratively reweighted $\ell_1$ ball method for handling the $\ell_p$-ball constrained minimization, which is easy to implement and computationally efficient. At each iteration, our algorithm only needs to solve weighted $\ell_1$-ball projection subproblems. The effectiveness of the proposed method is exhibited in the numerical studies via sparse signal recovery and logistic regression problems.

3. We establish the global convergence analysis and show that the sequence generated by our method converges to the first-order optimal solutions from any feasible initial points.
2 Preliminaries

Throughout this paper, we use $\mathbb{R}^n$ to denote the Euclidean $n$-space with the dot product defined by $\langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i$. Moreover, $\circ$ defines a component-wise product between $x, y \in \mathbb{R}^n$, i.e., $z = x \circ y$ with $z_i = x_i y_i$ for $i \in [n]$. The nonnegative orthant of $\mathbb{R}^n$ is denoted as $\mathbb{R}_{n+}^n$ and the interior of $\mathbb{R}_{n+}^n$ is denoted as $\mathbb{R}_{n+}^n$. For $x \in \mathbb{R}^n$, the $i$th component of $x$ is represented by $x_i$ and $|x| = [|x_1|, \ldots, |x_n|]^T$. In particular, we use $e$ and $0$ to denote the vectors of all ones and zeros of appropriate size, respectively. Moreover, for $p > 0$, the $\ell_p$ (quasi)norm of $x \in \mathbb{R}^n$ is defined as $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$. Denote $\mathbb{N}$ as the set of natural numbers. We use $[n] \subset \mathbb{N}$ to represent a set containing the natural numbers from 0 to $n$, i.e., $[n] = \{0, 1, 2, \ldots, n\}$. The cardinality of an index set $S \subset \mathbb{N}$ is denoted by $|S|$. We annotate $x_S \in \mathbb{R}^{|S|}$ as the subvector of $x$ where the components are indexed by $S$. The support set of $x \in \mathbb{R}^n$ is defined as $\mathcal{I}(x) := \{i \in [n] \mid x_i \neq 0\}$, and its complementary set is denoted as $\mathcal{A}(x) := \{i \in [n] \mid x_i = 0\}$. The signum function of $x \in \mathbb{R}$ is defined as $\text{sgn}(x) = -1$ if $x < 0$, $\text{sgn}(x) = 0$ if $x = 0$ and $\text{sgn}(x) = 1$ if $x > 0$. In addition, for $x \in \mathbb{R}^n$, $\text{sgn}(x) = [\text{sgn}(x_1), \ldots, \text{sgn}(x_n)]^T$.

3 Algorithm description

In this paper, we focus on solving the nonlinear optimization problem involving the nonconvex $\ell_p$ ball constraint, which is formalized as

$$\min_x f(x) \quad \text{s.t.} \quad x \in \Omega := \{x \in \mathbb{R}^n \mid \|x\|_p \leq r\}, \quad (\mathcal{P})$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable, $p \in (0, 1)$ and $r > 0$ is referred to as the radius of the $\ell_p$ ball. We know $f$ is (locally) $L$-Lipschitz differentiable on $\Omega$.

Our algorithm alternatively solves two types of subproblems based on whether the current iterate is on the boundary of $\Omega$. At the $k$th iteration, we approximate the objective $f$ by

$$\phi(x; x^k) := f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{\beta}{2}\|x - x^k\|_2^2,$$

with $\beta > L$. Specifically, if $\|x^k\|_p = r$, the $\ell_p$ ball is linearized at $x^k$, so that we require the next iterate to satisfy

$$\|x^k\|_p + \sum_{i \in \mathcal{I}(x^k)} p|x^k|^{p-1}(|x_i| - |x_i^k|) \leq r \quad \text{and} \quad x_i = 0, \quad i \in \mathcal{A}(x^k),$$

$$4$$
or, equivalently, solving the following subproblem for $x^{k+1}$,

$$
\min_x \phi(x; x^k) \\
\text{s.t. } \sum_{i \in \mathcal{I}(x^k)} |x_i^k|^{p-1}|x_i| \leq r,
$$

$$
(P_1)
$$

If the current iterate is in the interior of $\Omega$, i.e., $\|x^k\|^p_p < r$, we use the constraint residual as the perturbation parameter to the zero components and then formulate a weighted $\ell_1$ ball by linearizing the $\ell_p$ ball at $x^k$. To achieve this, letting

$$
\epsilon^k = c \left( \frac{r - \|x^k\|^p_p}{|\mathcal{A}(x^k)|+1} \right)^{\frac{1}{p}},
$$

$$
r^k := \frac{1}{p} (r + (p - 1)\|x^k\|^p_p - |\mathcal{A}(x^k)|\epsilon^p) > 1 - \frac{\epsilon^p}{p} (r - \|x^k\|^p_p) > 0
$$

with $c \in (0, 1)$. We require the next iterate to satisfy

$$
\|x^k \nabla_{\mathcal{I}(x^k)}\|^p_p + |\mathcal{A}(x^k)|\epsilon^p + \sum_{i \in \mathcal{I}(x^k)} p|x_i^k|^{p-1}(|x_i| - |x_i^k|) + \sum_{i \in \mathcal{A}(x^k)} p(\epsilon^k)^{p-1}|x_i| \leq r.
$$

In other words, we solve the following subproblem for $x^{k+1}$,

$$
\min_x \phi(x; x^k) \\
\text{s.t. } \sum_{i \in \mathcal{I}(x^k)} |x_i^k|^{p-1}|x_i| + \sum_{i \in \mathcal{A}(x^k)} (\epsilon^k)^{p-1}|x_i| \leq r^k.
$$

$$
(P_2)
$$

We state our proposed Iterative Reweighted $\ell_1$ Ball Algorithm, hereinafter named IR1B, in Algorithm 1.

For each iteration of IR1B, a weighted $\ell_1$ ball projection subproblem needs to be solved. We mention that there have been many efficient $\ell_1$-ball projection algorithms proposed with observed complexity $O(n)$ in the past decades, e.g., [13, 23, 24, 25]. These algorithms can be modified easily and extended to solve the weighted $\ell_1$-ball projection subproblems here. We choose the algorithm proposed by [25], modify and use it as our subproblem solver. The details of this algorithm are therefore skipped. We are aware that the performance of the proposed algorithm can be further improved by carefully designing a more efficient subproblem solver, which, however, is not the primary focus of this paper.
Algorithm 1 Iterative Reweighted ℓ₁ Ball Algorithm (IR1B)

1: **Input:** Choose \( x^0 \in \Omega, \beta > \frac{L}{2}, c \in (0, 1) \) and tolerance \( \text{tol} \geq 0 \).
2: repeat
3: if \( \|x^k\|_p = r \) then
4: Solve \((P_1)\) for \( x^{k+1} \).
5: end if
6: if \( \|x^k\|_p < r \) then
7: Set \( \epsilon_k \) according to (1).
8: Solve \((P_2)\) for \( x^{k+1} \).
9: end if
10: Set \( k \leftarrow k + 1 \).
11: until \( \|x^{k+1} - x^k\|_2 \leq \text{tol} \)

4 Convergence analysis

The convergence properties of IR1B are the subject of this section. We first provide the first-order necessary optimality conditions to characterize the optimal solutions of \((P)\), and then prove the well-posedness of the algorithm along with global and local convergence results.

4.1 First-order optimality condition

To characterize the optimal solution of \((P)\), we introduce the concept of Fréchet normal cone \([26]\), which is a generalization of the normal cone for closed convex sets.

**Definition 4.1.** (Fréchet normal cone) Given a closed set \( \mathcal{X} \subset \mathbb{R}^n \) and \( x^* \in \mathcal{X} \), the Fréchet normal cone of \( \mathcal{X} \) at \( x^* \) can be defined as

\[
N_{\mathcal{X}}(x^*) := \{ \eta \in \mathbb{R}^n \mid \limsup_{x \to x^*} \frac{\langle \eta, x - x^* \rangle}{\|x - x^*\|} \leq 0 \}.
\]

We now investigate the Fréchet normal cone of the \( \ell_p \) ball at \( \bar{x} \in \mathbb{R}^n \). The following theorem characterizes the elements of \( N_{\Omega}(\bar{x}) \), and its proof can be found in Appendix [A].

**Lemma 4.2.** If \( \bar{x} \in \mathbb{R}^n \) satisfies \( \|\bar{x}\|_p < r \), then \( N_{\Omega}(\bar{x}) = \{0\} \). If \( \bar{x} \in \mathbb{R}^n \) satisfies \( \|\bar{x}\|_p = r \), then \( \eta \in N_{\Omega}(\bar{x}) \) with

\[
\eta_i = \begin{cases} 
  p|\bar{x}_i|^{p-1}sgn(\bar{x}_i), & \text{if } i \in I(\bar{x}), \\
  \eta_i \in \mathbb{R}, & \text{if } i \in A(\bar{x}). 
\end{cases}
\]
Using Fréchet normal cone and [27, Theorem 4.3], we have the following necessary optimality condition to \((P)\).

**Theorem 4.3** (Fermat’s rule). If \((P)\) has a local minimum at \(\bar{x}\), then it holds that

\[ 0 \in \nabla f(\bar{x}) + \mathcal{N}_\Omega(\bar{x}). \tag{3} \]

We call a point satisfying the necessary optimality condition (3) as a first-order optimal solution of \((P)\).

The optimal solution \(x^\ast\) of \((P)\) can be classified into two cases: \(x^\ast\) is on the boundary of the \(\ell_p\) ball or \(x^\ast\) is in the interior of the \(\ell_p\) ball. For the first case, since the \(\ell_p\) norm constraint is active, we have from Lemma 4.2 that any \(x\) satisfying the following conditions is first-order optimal

\[
\nabla_i f(x) + \lambda_p |x_i|^{p-1} \text{sgn}(x_i) = 0, \quad i \in I(x) \tag{4a}
\]

\[
||x||_p^p = r, \tag{4b}
\]

\[
\lambda \geq 0. \tag{4c}
\]

Here (4a) can be equivalently written as

\[
\nabla_i f(x)x_i + \lambda_p |x_i|^p = 0, \quad i \in I(x). \tag{5}
\]

For the second case, the first-order necessary optimal condition is simply

\[
\nabla f(x) = 0, \tag{6a}
\]

\[
||x||_p^p < r. \tag{6b}
\]

A primal-dual pair \((x, \lambda)\) satisfying (6a)-(6b) or (4a)-(4c) is called the first-order optimal solution.

4.2 Well-posedness

We prove that IR1B is well-posed in the sense that each iteration is well-defined. Our first lemma reveals that if \(x^k\) is feasible, then the \(k\)th subproblems \((P_1)\) and \((P_2)\) are all feasible and the next iterate \(x^{k+1}\) is also feasible.

**Lemma 4.4.** Given any \(x^k \in \Omega\). Subproblems \((P_1)\) and \((P_2)\) are all feasible. Moreover, \(||x^{k+1}||_p^p \leq r\), meaning \(\{x^k\} \subset \Omega\).

**Proof.** Obviously, \((P_1)\) and \((P_2)\) are all feasible. Suppose \((P_1)\) is solved at the \(k\)th iteration. By concavity of \((\cdot)^p\), we have

\[
||x^{k+1}||_p^p = ||x^{k+1}_{I(x^k)}||_p^p \leq ||x^k_{I(x^k)}||_p^p + \sum_{i \in I(x^k)} p|x_i^k|^{p-1}(|x_i^{k+1}| - |x_i^k|) \leq r.
\]
Suppose \([\mathcal{P}_2]\) is solved at the \(k\)th iteration. By concavity,

\[
\|x^{k+1}\|_p \leq \|x^{k+1}_{\mathcal{I}(x^k)}\|_p + \|x^{k+1}_{\mathcal{A}(x^k)}\| + \epsilon e\|_p
\]

\[
\leq \|x^k_{\mathcal{I}(x^k)}\|_p + \|x^k_{\mathcal{A}(x^k)}\| + \epsilon e\|_p + \sum_{i \in \mathcal{I}(x^k)} p|x^k_i|^{p-1}(|x^{k+1}_i| - |x^k_i|)
\]

\[
+ \sum_{i \in \mathcal{A}(x^k)} pe^{p-1}(|x^{k+1}_i| - |x^k_i|)
\]

\[
= \|x^k_{\mathcal{I}(x^k)}\|_p + |\mathcal{A}(x^k)| e^{p} + \sum_{i \in \mathcal{I}(x^k)} p|x^k_i|^{p-1}(|x^{k+1}_i| - |x^k_i|)
\]

\[
+ \sum_{i \in \mathcal{A}(x^k)} pe^{p-1}|x^{k+1}_i| \leq r.
\]

For ease of notation, denote \(\mathcal{I}^k := \mathcal{I}(x^k)\) and \(\mathcal{A}^k := \mathcal{A}(x^k)\). Notice that the Slater conditions holds for \([\mathcal{P}_1]\) and \([\mathcal{P}_2]\). For \([\mathcal{P}_1]\), the new iterate \(x^{k+1}\) satisfies the following Karush-Kuhn-Tucker (KKT) conditions of \([\mathcal{P}_1]\)

\[
\nabla_i f(x^k) + \beta(x^{k+1}_i - x^k_i) + \lambda^{k+1} p|x^k_i|^{p-1} \xi^{k+1}_i = 0, \ i \in \mathcal{I}^k
\]

\[
\lambda^{k+1} \left( \sum_{i \in \mathcal{I}(x^k)} p|x^k_i|^{p-1}|x^{k+1}_i| - r \right) = 0,
\]

\[
\sum_{i \in \mathcal{I}(x^k)} |x^k_i|^{p-1}|x^{k+1}_i| \leq r,
\]

\[
\lambda^{k+1} \geq 0, \ \xi^{k+1}_i \in \partial|x^{k+1}_i|,
\]

where \(\lambda^{k+1}\) is the multiplier associated with the weighted \(\ell_1\) ball constraint. Clearly, \(\lambda^{k+1}\) satisfies

\[
\lambda^{k+1} = -\frac{\sum_{i \in \mathcal{I}(x^k)} [x^{k+1}_i \nabla_i f(x^k) + \beta x^{k+1}_i(x^{k+1}_i - x^k_i) + \lambda^{k+1} p|x^k_i|^{p-1}]}{p \sum_{i \in \mathcal{I}(x^k)} |x^k_i|^{p-1}|x^{k+1}_i|}.
\]
As for \( \{P_2\} \), the new iterate \( x^{k+1} \) satisfies
\[
\nabla_i f(x^k) + \beta(x_i^{k+1} - x_i^k) + \lambda^{k+1}p|x_i^k|^{p-1}\xi_i = 0, \quad i \in I^k \quad (9a)
\]
\[
\nabla_i f(x^k) + \beta(x_i^{k+1} - x_i^k) + \lambda^{k+1}p(x_i^{k+1} - r_i) = 0, \quad i \in A^k \quad (9b)
\]
\[
\lambda^{k+1} \left( \sum_{i \in I(x^k)} |x_i^k|^{p-1}|x_i^{k+1}| + \sum_{i \in A(x^k)} (e^k)^{p-1}|x_i^{k+1}| - r_i \right) = 0, \quad (9c)
\]
\[
\sum_{i \in I(x^k)} |x_i^k|^{p-1}|x_i^{k+1}| + \sum_{i \in A(x^k)} (e^k)^{p-1}|x_i^{k+1}| \leq r, \quad (9d)
\]
\[
\lambda^{k+1} \geq 0, \quad \xi_i^{k+1} \in \partial|x_i^{k+1}|, \quad (9e)
\]
where \( \lambda^{k+1} \) is the multiplier associated with the weighted \( \ell_1 \) ball constraint. Clearly, \( \lambda^{k+1} \) satisfies
\[
\lambda^{k+1} = -\frac{\sum_{i=1}^n |x_i^{k+1}\nabla_i f(x^k) + \beta(x_i^{k+1} - x_i^k)|}{\sum_{i \in I(x^k)} |x_i^k|^{p-1}|x_i^{k+1}| + \sum_{i \in A(x^k)} (e^k)^{p-1}|x_i^{k+1}|}. \quad (10)
\]

The next lemma enumerates relevant properties of subproblem \( \{P_1\} \) and \( \{P_2\} \).

**Lemma 4.5.** During the \( k \)th iteration of IR1B, we have the following:

(a) If \( x^{k+1} = x^k \) after solving \( \{P_1\} \), then \( x^k \) satisfies conditions \( (4a)-(4c) \).

(b) If \( x^{k+1} = x^k \) after solving \( \{P_2\} \), then \( x^k \) satisfies conditions \( (6a)-(6b) \).

**Proof.** (a) If \( x^{k+1} = x^k \) after solving \( \{P_1\} \), plugging \( x^{k+1} = x^k \) into (7), we have \( x^k \) satisfies (4). Therefore, \( x^k \) is first-order optimal for \( \{P\} \).

(b) If \( x^{k+1} = x^k \) after solving \( \{P_2\} \), we plug \( x^{k+1} = x^k \) into (9). The constraint is inactive at \( x^k \), since
\[
\|x^k_I(x^k)\|^p + |A(x^k)|\epsilon^p + \sum_{i \in I(x^k)} p|x_i^k|^{p-1}(|x_i^k| - |x_i^k|) + \sum_{i \in A(x^k)} (p(x_i^{k+1})^{p-1}|x_i^k|)
\]
\[
= \|x^k\|^p + |A(x^k)|\epsilon^p = \|x^k\|^p + |A(x^k)|\epsilon^p \left( \frac{r - \|x^k\|^p}{|A(x^k)|+1} \right)
\]
\[
< \|x^k\|^p + (r - \|x^k\|^p) = r.
\]

Therefore, \( \lambda^{k+1} = 0 \) and (9) now reverts to \( \nabla f(x^k) = 0 \), meaning \( x^k \) is first-order optimal for \( \{P\} \).

Our main theorem in this subsection summarizes the well-posedness of IR1B, which can be derived trivially using Lemma 4.4 and Lemma 4.5.
Theorem 4.6. Assume the tolerance in IR1B is set as \( \text{tol} = 0 \). During the iteration of IR1B, one of the following must occur

(a) IR1B terminates after solving subproblem \((P_1)\) with \((x^{k+1}, \lambda^{k+1})\) satisfying (4).

(b) IR1B terminates after solving subproblem \((P_2)\) with \(x^{k+1}\) satisfying (6).

(c) IR1B generates an infinite sequence \(\{(x^k, \lambda^k)\}\), where, for all \(k\), \(\{x^k\} \subset \Omega\) and \(\{\lambda^k\} \subset \mathbb{R}_+\).

4.3 Global convergence

We now prove properties related to the global convergence of IR1B under the assumption that an infinite sequence of iterates is generated; i.e., we focus on the situation described in Theorem 4.6(c). We first show that the objective of \((P)\) is monotonically decreasing. For this purpose, we define the decrease in \(\phi(\cdot; x^k)\) caused by \(x^{k+1}\) from \(x^k\) as

\[
\Delta \phi(x^{k+1}; x^k) = \phi(x^k; x^k) - \phi(x^{k+1}; x^k).
\]

We first show this reduction vanishes as \(k \to \infty\).

Lemma 4.7. Suppose \(\{x^k\}\) is generated by IR1B. It holds true that

\[
f(x^k) - f(x^{k+1}) \geq \Delta \phi(x^{k+1}; x^k).
\]

Therefore, \(\{f(x^k)\}\) is monotonically decreasing and \(\lim_{k \to \infty} \Delta \phi(x^{k+1}; x^k) = 0\).

Proof. Since \(f\) is locally Lipschitz differentiable on \(\Omega\) and \(\beta > L\), we have

\[
f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2\]

\[
< f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{\beta}{2} \|x^{k+1} - x^k\|^2
\]

\[
= f(x^k) - \Delta \phi(x^{k+1}; x^k),
\]

then we have

\[
\Delta \phi(x^{k+1}; x^k) \leq f(x^k) - f(x^{k+1}).
\]

Rearranging, this leads to the desired result. On the other hand, summing both sides of (11) from \(t = 0\) to \(k\) gives

\[
\sum_{t=0}^{k} \Delta \phi(x^{t+1}; x^t) \leq \sum_{t=0}^{k} (f(x^t) - f(x^{t+1})) = f(x^0) - f(x^{t+1})
\]

\[
\leq f(x^0) - \frac{f}{+\infty}
\]
with \( f := \inf_{x \in \Omega} f(x) \). Letting \( t \to \infty \), we have
\[
\lim_{k \to \infty} \Delta \phi(x^{k+1}; x^k) = 0.
\]

The next lemma shows that the displacement of the iterates vanishes in the limit.

**Lemma 4.8.** Suppose \( \{x^k\} \) is generated by Algorithm 1. Then
\[
\lim_{k \to \infty} \|x^{k+1} - x^k\|_2 = 0.
\]  

**Proof.** By [25, Theorem 3.7], we have \( \|x^{k+1} - x^k\|_2 \leq (2/\beta)\Delta \phi(x^{k+1}; x^k) \). Therefore, we have \( \lim_{k \to \infty} \|x^{k+1} - x^k\|_2 = 0 \) by Lemma 4.7, as desired. \( \square \)

Define two subsequences based on whether \( (P_1) \) or \( (P_2) \) is solved,
\[
S_1 = \{ k \ | \ \|x^k\|_p = r \} \quad \text{and} \quad S_2 = \{ k \ | \ \|x^k\|_p < r \}.
\]

We are now ready to provide the global convergence result for IR1B.

**Theorem 4.9.** Let \( \{(x^k, \lambda^k)\} \) be generated by Algorithm 1. Then, every cluster point \( (x^*, \lambda^*) \) of \( \{(x^k, \lambda^k)\} \) is first-order optimal for \( (P) \).

**Proof.** By Lemma 4.5, if \( x^* \) is a limit point of \( S_1 \), it suffices to show that the subproblem \( (P_1) \) at \( x^* \) has a stationary point \( \hat{x} \neq x^* \) and that
\[
\Delta \phi(\hat{x}; x^*) = \phi(x^*; x^*) - \phi(\hat{x}; x^*) > \delta > 0.
\]  

Consider a subsequence \( \hat{S}_1 \subset S_1 \) such that \( \{x^k\}_{\hat{S}_1} \to x^* \). Notice by Lemma 4.7 there exists \( k_0 \in \mathbb{N} \) such that
\[
\Delta \phi(x^{k+1}; x^k) = \phi(x^k; x^k) - \phi(x^{k+1}; x^k) < \delta/4,
\]
or, equivalently,
\[
\phi(x^{k+1}; x^k) > \phi(x^k; x^k) - \delta/4
\]  

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for all \( k > k_0, k \in \hat{S}_1 \). To derive a conclusion contradicting \([15]\), first of all, notice that \( \phi(x^k; x^k) - \phi(x^*; x^*) = f(x^k) - f(x^*) \). By the continuity of \( f \), there exists \( k_1 \) such that for all \( k > k_1, k \in \hat{S}_1 \)

\[
|\phi(x^k; x^k) - \phi(x^*; x^*)| < \delta/4.
\]  

(16)

Denote \( \hat{x}^k \) as the projection of \( \hat{x} \) onto the feasible region of \((P_1)\) at \( x^k \). By the continuity of \( \phi \), there exists \( k_2 \) such that for all \( k > k_2, k \in \hat{S}_1 \),

\[
|\phi(\hat{x}^k; x^*) - \phi(x^k; x^k)| < \delta/4 \quad \text{and} \quad |\phi(x^k; x^*) - \phi(\hat{x}; x^*)| < \delta/4.
\]  

(17)

Combining \([14], [15], [16]\) and \([17]\), for any \( k > \max\{k_0, k_1, k_2\} \), we have

\[
\begin{align*}
\phi(x^k; x^k) - \phi(\hat{x}^k; x^k) &= \phi(x^k; x^k) - \phi(x^*; x^*) + \phi(x^*; x^*) - \phi(\hat{x}; x^*) \\
&\quad + \phi(\hat{x}; x^*) - \phi(\hat{x}^k; x^*) + \phi(\hat{x}^k; x^*) - \phi(\hat{x}^k; x^k) \\
&\geq -|\phi(x^k; x^k) - \phi(x^*; x^*)| + |\phi(\hat{x}; x^*) - \phi(\hat{x}^k; x^k)| \\
&\quad - |\phi(\hat{x}^k; x^*) - \phi(\hat{x}; x^*)| - \phi(\hat{x}^k; x^*) - \phi(\hat{x}^k; x^k) \\
&> -\delta/4 + \delta - \delta/4 - \delta/4 = \delta/4,
\end{align*}
\]

which contradicts \([15]\). By \([18]\), \( \hat{x}^k \) is feasible for the \( k \)th subproblem \((P_1)\) and has lower objective than \( x^{k+1} \), and this contradicts the fact that \( x^{k+1} \) is optimal to \((P_1)\). Therefore, by Lemma 4.5(a), \( x^* \) is first-order optimal for \((P)\).

Case (ii): we consider two subcases that \( \|x^*\|^p_p < r \) and \( \|x^*\|^p_p = r \).

(ii)-(a): Suppose \( \|x^*\|^p_p < r \) with \( \epsilon^* = c \left( \frac{r - \|x^*\|^p_p}{\|x^*\|^p_p + 1} \right)^{1/p} > 0 \). By Lemma 4.5(a), we can assume by contradiction that the subproblem \((P_2)\) at \( x^* \) has a stationary point \( \hat{x} \neq x^* \) satisfying

\[
\Delta \phi(\hat{x}; x^*) = \phi(x^*; x^*) - \phi(\hat{x}; x^*) \geq \delta > 0.
\]  

(19)

Now consider subsequence \( \hat{S}_2 \subset S_2 \) such that \( \{x^k\}_{k \in \hat{S}_2} \to x^* \) and \( \{\epsilon^k\}_{k \in \hat{S}_2} \to \epsilon^* \).

By Lemma 4.7 for sufficiently large \( k \in \hat{S}_2 \), we have

\[
\phi(x^{k+1}, x^k) > \phi(x^k; x^k) - \delta/4.
\]  

(20)

Notice that \( \phi(x^k; x^k) - \phi(x^*; x^*) = f(x^k) - f(x^*) \). By the continuity of \( f \), there exists sufficiently large \( k \in \hat{S}_2 \) such that

\[
|\phi(x^k; x^k) - \phi(x^*; x^*)| < \delta/4.
\]  

(21)

Now consider projecting \( \hat{x} \) onto the feasible region of \((P_2)\) at \( x^k \), and denote the projection point as \( \tilde{x}^k \). By the continuity of \( p \), there exists sufficiently large \( k \in \hat{S}_2 \) such that

\[
|\phi(\tilde{x}^k; x^*) - \phi(\tilde{x}^k; x^k)| < \delta/4 \quad \text{and} \quad |\phi(\tilde{x}^k; x^*) - \phi(\tilde{x}; x^*)| < \delta/4.
\]  

(22)
Combining (19), (20), (21) and (22), for sufficiently large $k \in \hat{S}_2$, we have
\[
\phi(x^k; x^k) - \phi(\tilde{x}^k; x^k) = \phi(x^k; x^k) - \phi(x^*; x^*) + \phi(x^*; x^*) - \phi(\tilde{x}^k; x^k) \\
+ \phi(\tilde{x}; x^k) - \phi(\tilde{x}^k; x^k) + \phi(\tilde{x}^k; x^k) - \phi(\tilde{x}; x^k) \\
\geq -|\phi(x^k; x^k) - \phi(x^*; x^*)| + |\phi(x^*; x^*) - \phi(\tilde{x}; x^*)| \\
- |\phi(\tilde{x}^k; x^k) - \phi(\tilde{x}; x^k)| - \phi(\tilde{x}^k; x^k) - \phi(\tilde{x}; x^k) \\
> -\delta/4 + \delta - \delta/4 - \delta/4 = \delta/4,
\]
contradicting (20). This indicates that $\tilde{x}^k$ is feasible for the $k$th subproblem (P2) and has lower objective than $x^{k+1}$. Obviously, this contradicts the optimality of $x^k$ for (P2) at the $k$th iteration. Therefore, by Lemma 4.5(b), $x^*$ is first-order optimal for (P).

(ii)-(b): Suppose $\|x^*\|_p^p = r$ with $e^* = c \left( \frac{r - \|x^*\|_p^p}{\|x^*\|_p^p} + 1 \right)^{1/p} = 0$. We know $\mathcal{I}(x^*) \neq \emptyset$.
Consider subsequence $\hat{S}_2$ such that $\{x^k\}_{k \in \hat{S}_2} \to x^*$. By (4.8), for any $i \in \mathcal{I}(x^*)$, there exists $\hat{k} \in \hat{S}_2$ such that $\{x^k\}_{k \geq \hat{k}, k \in \hat{S}_2}$ and $\{x^{k+1}\}_{k \geq \hat{k}, k \in \hat{S}_2}$ are bounded away from 0, meaning $\mathcal{I}(x^*) \subset \mathcal{I}(x^k)$ for sufficiently large $k \in \hat{S}_2$. By (9), we have
\[
\lambda^{k+1} = -\frac{x_i \nabla_i f(x^k) + \beta x_i (x^{k+1}_i - x^k_i)}{p|x^k_i|^{p-1}|x^{k+1}_i|}, \quad i \in \mathcal{I}(x^*).
\]
Obviously, $\{\lambda^{k+1}\}_{k \geq \hat{k}, k \in \hat{S}_2}$ are bounded above. Let $\lambda^*$ be a limit point of $\{\lambda^{k+1}\}$ with subsequence $\{\lambda^{k+1}\}_{\hat{S}_2} \to \lambda^*$. By Lemma 4.8, $\{x^{k+1}\}_{\hat{S}_2} \to x^*$. It follows that for any $i \in \mathcal{I}(x^*)$,
\[
0 = \nabla_i f(x^*) x^*_i + \lambda^* p|x^*_i|^p = \lim_{k \to \infty} \nabla_i f(x^k) x^{k+1}_i + \lambda^{k+1} p|x^{k+1}_i|^p.
\]
This proves that $x^*$ is first-order optimal for (P). \hfill \square

5 Numerical Experiments

In this section, we test the proposed algorithm IR1B on synthetic data and real-world data to demonstrate the practicability and effectiveness for solving (P). With this target, we test the sparse signal recovery problem that stems from compressive sensing and also test the logistic regression problem in statistical machine learning. Of all the tests, we choose the $\ell_0$ ball or the $\ell_1$ ball constraints to apply to the same testing problems as the benchmark to show the quality of the solutions of these problems, since the existing approach for solving (P) is limited.
We implement all codes using Python and run all experiments on a laptop under Ubuntu with 7.5 GB main memory and Intel Core i7-7500U processor (2.70GHz × 4).

5.1 $\ell_p$-constrained Least squares on synthetic data

Consider an underdetermined sensing matrix $A \in \mathbb{R}^{m \times n}$ and a noisy measurement vector $y \in \mathbb{R}^m$. The unknown signal $x^\dagger$ with $d$-nonzeros to be estimated satisfies $y = Ax^\dagger$. Then the $\ell_p$-constrained signal recovery problems can be formulated as

$$
\min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - y\|_2^2 \\
s.t. \quad \|x\|_p \leq r,
$$

where $p \in (0, 1]$. In particular, the plain Iterative Hard Thresholding (IHT) [12] algorithm and the method described in [25] (abbreviated as GPM) are well-known for solving (25) when $p = 0$ and $p = 1$, respectively.

We first state the way to generate simulation data. Concretely, $x_{\text{ori}}$ represents the original $d$-sparse vector. In this context, $m$ is the number of noisy measurements, and we set $m \in [50, 1000]$ and increase it by 50 for each comparison. Given each $m$, we randomly generate $d$ nonzero elements of $x_{\text{ori}}$ by assigning their value with $+1$ and $-1$ with equal probability, and each entry of $A$ is sampled from a standard Gaussian distribution. Then we form $b = Ax_{\text{ori}} + \eta$ with $\eta$ being Gaussian noise with 0 mean and $10^{-2}$ standard deviation.

In this experiment, we set $c = 0.95$, $r = s$ and initialize $x^0 = 0.9(\|\nu\|_1)^{1/p}$ such that $\|x\|_p \leq r$, where each entry of $\nu \in \mathbb{R}^n$ are uniformly sampled over the interval $[0, 1)$. The stepsize $1/\beta$ for IR1B and $\beta$ for IHT and GPM is used, where $\beta = 1.1L$, $L$ represents the Lipschitz constant of the objective in (25) and $\lambda_{\text{max}}(A^T A)$ denotes the largest eigenvalue of $A^T A$. For IR1B, we determine that a point is on the boundary of the $\ell_p$ ball if $|r - \|x\|_p| \leq 10^{-8}$. IR1B, IHT and GPM are terminated if $\|x^{k+1} - x^k\|_2 \leq \text{tol}$ with tol = $10^{-5}$. In addition, we declare a success for the test if $\|x^* - x^\dagger\|_2/\|x^\dagger\|_2 < 10^{-3}$ is satisfied, where $x^*$ denotes the optimal solution output by three algorithms. With the different values of $p$, we compare three algorithms with respect to the empirical probability of success defined by the ratio of the number of successes to the total 50 runs for each comparison. The curve is shown in Figure 1 and each presented result is the average of 50 independent runs.

As observed in Figure 1, with $p \in \{0.3, 0.5, 0.7\}$, IR1B successfully solves (25). In particular, we can see that $\ell_{0.7}$ ball has 5 successful recoveries and requires the minimal number of observations, i.e., $m = 400$. On the other hand, we observe that $\ell_{0.3}$ ball and $\ell_{0.5}$ ball yield more successful recoveries than $\ell_0$ ball and $\ell_1$ ball.
when \( m = 450 \). When \( m > 600 \), there are a few successful recoveries for \( \ell_0 \) ball while other \( \ell_p \) balls achieve success for each run. Overall, the \( \ell_p \) ball with \( p \in (0, 1) \) is superior to the \( \ell_0 \) ball and \( \ell_1 \) ball for recovering the original sparse signal if the number of observation elements is limited.

### 5.2 \( \ell_p \)-constrained logistic regression on real-world data

The focus of this test is to consider the logistic regression model which is a popular classification approach in the context of supervised learning. Given a collection of data pairs \( \{(x^{(i)}, y^{(i)})\}_{i=1}^m \), where \( x^{(i)} \in \mathbb{R}^n \) represents a feature vector and \( y^{(i)} \in \{-1, +1\} \) denotes a binary instance label. Then, the \( \ell_p \) ball constrained logistic regression problem is

\[
\min_{\theta} \sum_{i=1}^n \log \left( 1 + \exp(-\theta^T x^{(i)}) \right) \quad \text{s.t.} \quad \|\theta\|_p \leq r.
\]  

We evaluate the performance of IR1B on the Wisconsin breast cancer dataset \[28\], which contains 569 instances and 30 features. For compactness, we denote the data matrix as \( X = (x^{(1)}; \ldots; x^{(m)})^T \in \mathbb{R}^{569 \times 30} \). In this test, we randomly split \( X \) into the train subset \( X_{\text{train}} \) and test subset \( X_{\text{test}} \) where the test subset size accounts for 40\% of the total data set size.

Let \( \theta^\dagger \) be the optimal solution returned by IR1B and \( \hat{y} \) be the predicted class label, i.e., \( \hat{y}^{(i)} = 1 \) if \( (1 + \exp(-\theta^T x^{(i)}))^{-1} \geq 0.5 \); otherwise \( \hat{y}^{(i)} = 0 \). We initialize
$\theta^0 = 0$ and set $\beta = 1.1L$, where $L = 0.25\lambda_{\text{max}}(X^T X)$ represents the Lipschitz constant of the objective in (26), and all other parameters remain the same as the first experiment.

We plot the curve of two errors versus a range of values of $r$ varying from 2 to 35, as shown in Figure 2.

![Figure 2: The empirical probability of success versus $m$ for $\ell_{0.5}$ ball and $\ell_1$ ball constraints.](image)

Figure 2 illustrates the prediction accuracy on the test data for $\ell_{0.5}$ ball constraint and $\ell_1$ ball constraint. As observed, increasing $r$ gradually improves the prediction accuracy; we can see that $\ell_p$ ball constraint yields a larger prediction accuracy over the test data than that of $\ell_1$ ball constraint. These observations suggest that the $\ell_p$ ball constraint can achieve better generalization error than the $\ell_1$ ball constraint.

6 Conclusions

In this paper, we proposed and analyzed the $\ell_p$ ball constrained nonlinear optimization problems. The proposed iteratively reweighted $\ell$ ball method is simple to implement and only needs to solve a weighted $\ell_1$ ball projection problem. We proved the proposed the global convergence of our proposed algorithm to the first-order stationary point of the original problem from any feasible initial point. The effectiveness of the proposed algorithm was demonstrated on the sparse signal recovery problems and logistic regression problems.
A Appendix: Proof of lemma 4.2: Characterizing the elements of the Fréchet normal cone

Proof. This proof is an direct extension of [19, Proposition 2.4] for a general objective function $f$. For any $x \in \Omega$ and $x$ sufficiently close to $\bar{x}$, we have

$$0 \geq \sum_{i=1}^{n} |x_i|^p - \sum_{i=1}^{n} |\bar{x}_i|^p = \sum_{i \in A(x)} (|x_i|^p - 0) + \sum_{i \not\in I(x)} (|x_i|^p - |\bar{x}_i|^p)$$

$$= \sum_{i \in A(x)} |x_i|^p \text{sgn}(x_i)x_i + \sum_{i \not\in I(x)} p|\bar{x}_i|^{p-1}(|x_i| - |\bar{x}_i|) + o(||x_I(x)|| - ||\bar{x}_I(x)||_2)$$

$$\geq \sum_{i \in A(x)} (|x_i|^p \text{sgn}(x_i) - \eta_i)x_i + \sum_{i \not\in I(x)} \eta_i x_i$$

$$+ \sum_{i \not\in I(x)} p|\bar{x}_i|^{p-1} \text{sgn}(\bar{x}_i)(x_i - \bar{x}_i) + o(||x_I(x)|| - ||\bar{x}_I(x)||_2),$$

where the second equality is obtained by the Taylor series approximation of $|x_i|^p$ at $\bar{x}_i$ and the second inequality is by the convexity of $| \cdot |$. It then follows that

$$\frac{\langle \eta, x - \bar{x} \rangle}{||x - \bar{x}||_2} = \frac{\sum_{i \in A(x)} \eta_i x_i + \sum_{i \not\in I(x)} p|\bar{x}_i|^{p-1} \text{sgn}(\bar{x}_i)(x_i - \bar{x}_i)}{||x - \bar{x}||_2}$$

$$\leq \frac{\sum_{i \in A(x)} (\eta_i - |x_i|^p \text{sgn}(x_i))x_i + o(||x_I(x)|| - ||\bar{x}_I(x)||_2)}{||x - \bar{x}||_2}$$

$$\leq \frac{\sum_{i \in A(x)} (\eta_i \text{sgn}(x_i) - |x_i|^p \text{sgn}(x_i))x_i}{||x - \bar{x}||_2} + \frac{o(||x_I(x)|| - ||\bar{x}_I(x)||_2)}{||x_I(x) - \bar{x}_I(x)||_2}.$$ 

As $x \to \bar{x}$, we have $\frac{o(||x_I(x) - \bar{x}_I(x)||_2)}{||x_I(x) - \bar{x}_I(x)||_2} \to 0$. Furthermore, we have $\eta_i \text{sgn}(x_i) - |x_i|^p \text{sgn}(x_i) < 0$ for all $i \in A(x)$. It follows that $\limsup_{x \to \bar{x}, x \in \Omega} \frac{\langle \eta, x - \bar{x} \rangle}{||x - \bar{x}||_2} \leq 0$, completing the proof. 

\[\square\]

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