A CONVERGENT LINEARIZED LAGRANGE FINITE ELEMENT METHOD FOR THE MAGNETO-HYDRODYNAMIC EQUATIONS IN 2D NONSMOOTH AND NONCONVEX DOMAINS

BUYANG LI*, JILU WANG†, AND LIWEI XU‡

Abstract. A new fully discrete linearized $H^1$-conforming Lagrange finite element method is proposed for solving the two-dimensional magneto-hydrodynamics equations based on a magnetic potential formulation. The proposed method yields numerical solutions that converge in general domains that may be nonconvex, nonsmooth and multi-connected. The convergence of subsequences of the numerical solutions is proved only based on the regularity of the initial conditions and source terms, without extra assumptions on the regularity of the solution. Strong convergence in $L^2(0,T;L^2(\Omega))$ was proved for the numerical solutions of both $u$ and $H$ without any mesh restriction.

Key words. MHD, $H^1$-conforming, finite element, nonsmooth, nonconvex, convergence

AMS subject classifications. 65M12, 65N30, 85A30

1. Introduction. This article is concerned with numerical approximation of the incompressible magneto-hydrodynamics (MHD) equations

\begin{align}
\mu \partial_t \mathbf{H} + \sigma^{-1} \nabla \times (\nabla \times \mathbf{H}) - \mu \nabla \times (\mathbf{u} \times \mathbf{H}) &= \sigma^{-1} \nabla \times \mathbf{J} \\
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} - \mu \mathbf{H} \times (\nabla \times \mathbf{H}) \tag{1.1}
\end{align}

\begin{align}
\nabla \cdot \mathbf{u} &= 0 \tag{1.2}
\end{align}

\begin{align}
\nabla \cdot \mathbf{B} = \partial B_1 \overline{\partial x_1} - \partial B_2 \overline{\partial x_2} \quad \nabla \times \mathbf{B} = \partial B_1 \overline{\partial x_1} + \partial B_2 \overline{\partial x_2} \tag{1.3}
\end{align}

in a polygonal type domain $\Omega = \Omega_0 \setminus \bigcup_{j=1}^m \Omega_j \subset \mathbb{R}^2$, where both $\Omega_0$ and $\Omega_j \subset \Omega_0$, $j = 1, \ldots, m$, are polygons (thus the domain is possibly nonconvex and multi-connected), and the following 2D notations for the curl, divergence, and gradient operators are used for a vector field $\mathbf{B} = (B_1, B_2)$ and a scalar field $\psi$:

\begin{align}
\nabla \times \mathbf{B} &= \frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} \\
\nabla \cdot \mathbf{B} &= \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} \\
\nabla \times \psi &= \left( \frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1} \right) \\
\nabla \cdot \psi &= \frac{\partial \psi}{\partial x_1} + \frac{\partial \psi}{\partial x_2} \\
\mathbf{B} \times \psi &= (B_2 \psi, -B_1 \psi).
\end{align}

In the system (1.1)- (1.3), $\mathbf{u}$ denotes the velocity field, $\mathbf{H}$ the magnetic field, $p$ the pressure, $\mathbf{f}$ and $\mathbf{J}$ the given source terms, $\nu$ the viscosity of the fluid, $\sigma$ the magnetic Reynolds number, and $\mu = M^2 \nu \sigma^{-1}$, where $M$ denotes the Hartman number.

We consider (1.1)- (1.3) under the perfectly conducting and no-slip boundary conditions

\begin{align}
\mathbf{H} \cdot \mathbf{n} &= 0, \quad \nabla \times \mathbf{H} = \mathbf{J}, \quad \text{and} \quad \mathbf{u} = 0 \quad \text{on} \quad \partial \Omega \times (0,T), \tag{1.4}
\end{align}

and the initial conditions

\begin{align}
\mathbf{H}|_{t=0} = \mathbf{H}_0 \quad \text{and} \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in} \quad \Omega. \tag{1.5}
\end{align}

The given source terms and the initial data for $\mathbf{H}$ and $\mathbf{u}$ are assumed to satisfy

\begin{align}
\mathbf{J} \in C([0,T]; L^2(\Omega)) \quad \mathbf{f} \in C([0,T]; L^2(\Omega)) \quad \mathbf{H}_0, \mathbf{u}_0 \in L^2(\Omega) \quad \nabla \cdot \mathbf{H}_0 = \nabla \cdot \mathbf{u}_0 = 0. \tag{1.6}
\end{align}

The MHD equations (1.1)- (1.3) describe the interaction between a magnetic field and a viscous incompressible conducting fluid flow. The mathematical theory of existence and uniqueness of solutions for the initial-boundary value problem (1.1)- (1.5) was established in [28] for a smooth or convex domain $\Omega$. In particular, under the regularity assumption (1.0) for the source terms and initial data, the problem has a unique weak solution (for any given
Numerical methods and analysis for the MHD equations have been done from many different point of views. For the stationary MHD equations, existence, uniqueness, and finite element approximations were studied in [16] and [32] for small data and general data (for every regular branch of the solutions), respectively. To overcome the numerical instability caused by possibly small hydrodynamic diffusion, a stabilized finite element method (FEM) was introduced in [11]. These articles are concerned with $H^1$-conforming FEMs for the magnetic field $H$ and the error estimates are based on the $H^1(\Omega)$-regularity of the magnetic field $H$. Such regularity holds in convex or smooth domains. However, in nonconvex and nonsmooth domains, the solution of the magnetic field is generally in $H(\text{curl}, \Omega)$ instead of $H^1(\Omega)$.

In more general domains, possibly nonconvex and nonsmooth, a mixed FEM with curl-conforming Nédélec edge elements was proposed for solving the magnetic field in the stationary MHD equations in [27], where an additional gradient term $\nabla q$ was added to the magnetic potential equation to enforce the divergence-free condition for the magnetic field in a weak sense; the $H^1$-conforming FEM was used for the velocity. An error estimate for this numerical method was proved under the regularity assumptions

$$(u, p) \in H^{s+1}(\Omega) \times H^s(\Omega) \quad \text{and} \quad (H, \nabla \times H) \in H^s(\Omega) \times H^s(\Omega) \quad \text{for some} \ s > \frac{1}{2} \ (1.7)$$

which hold when the source term of the stationary MHD equations is sufficiently small. The same method for the magnetic field was also used in [13] for solving the MHD equations, where a divergence-conforming FEM was used for the velocity.

In the case of low magnetic Reynolds numbers, the MHD model usually consists of a time-dependent Navier-Stokes equation and a stationary electric potential equation with given magnetic field. In [20], two implicit-explicit methods (of first and second order) decoupling the velocity from electric potential were proposed and analyzed. In [25], the method proposed in [20] was further combined with the technique of [30] to decouple the pressure from velocity. For the model with low magnetic Reynolds numbers considered in [20, 25], the $H^1$-conforming FEMs were proved to be convergent.

For time-dependent MHD equations, many different numerical methods have been developed and analyzed:

- $H^1$-conforming FEMs for the velocity and magnetic fields were used in [2, 14, 17, 33] with several different time discretization methods. Error estimates were carried out for several of these $H^1$-conforming FEMs in smooth or convex domains. Similarly as for the stationary problem, in a nonconvex and nonsmooth domain, the solution of the magnetic field of the time-dependent MHD equations is generally in $H(\text{curl}, \Omega)$ in the spatial direction instead of $H^1(\Omega)$. In this case the $H^1(\Omega)$ or $H^2(\Omega)$ regularity assumptions used in the existing analyses for the $H^1$-conforming FEMs generally do not hold.

- A Galerkin least square FEM was proposed for solving the augmented MHD equations by adding an additional gradient term $\nabla q$ to the equation of magnetic field [26], where numerical simulations were shown to illustrate the performance of the numerical methods. Rigorous proof for the convergence of numerical solutions remains an open.

- The magnetic potential formulation was used in [29], where the equivalent formulation of the two-dimensional MHD equations

$$\mu \partial_t A - \sigma^{-1} \Delta A - \mu u \times (\nabla \times A) = \sigma^{-1} J \quad (1.8)$$

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = f + \mu \nabla \cdot [(\nabla \times A) \otimes (\nabla \times A) - \frac{1}{2} \nabla \times A]^2] \quad (1.9)$$

$$\nabla \cdot u = 0 \quad (1.10)$$

was solved by a fully implicit $H^1$-conforming FEM, where $\mathbb{I}$ denotes the identity matrix. Numerical results were given without proof for the convergence of numerical solutions.

- Divergence-free preserving methods for MHD and ideal MHD have been developed in many articles. In particular, the locally divergence-free subspace was used in the discontinuous Galerkin (DG) methods for the MHD equations [21]. The divergence-free
The rest of the paper is organized as follows. In Section 2, we present an equivalent mag-
Since the matrix $M$ (see Remark 2.1), the solution of the reformulated problem is also a weak solution of the original MHD equations. Then we define weak solutions of the problem in section 2.2. It is easy to verify that a weak solution of the two-dimensional MHD equations (1.1)-(1.3) in terms of the magnetic potential.

In section 2.1 we first formally derive an equivalent formulation of the two-dimensional MHD equations (1.1)-(1.3). In Section 3, we propose a fully discrete linearized $H^1$-conforming Lagrange finite element method for solving the problem, and present the main theoretical result about the convergence of the numerical solutions. Rigorous proof of the main theoretical result is presented in Section 4. Numerical experiments are given in Section 5 to support the theoretical analyses.

2. Equivalent formulation. In section 2.1 we first formally derive an equivalent formulation of the two-dimensional MHD equations (1.1)-(1.3) in terms of the magnetic potential. Then we define weak solutions of the problem in section 2.2. It is easy to verify that a weak solution of the reformulated problem is also a weak solution of the original MHD equations (see Remark 2.1).

2.1. Formal derivation. By taking the divergence of (1.1) we obtain
\[
\mu \partial_t \nabla \cdot H = 0,
\]
which together with the divergence-free initial condition $\nabla \cdot H_0 = 0$ in (1.6) implies
\[
\nabla \cdot H = 0. \tag{2.1}
\]

Let $m$ denote the number of holes of the domain $\Omega$, and let $\Gamma_j = \partial \Omega_j$ denote the boundary of the $j$th hole. Then the divergence-free vector field $H$ can be decomposed as (cf. [4], with slightly different boundary conditions)
\[
H = \nabla \times A + \sum_{j=1}^m \beta_j \nabla \times \varphi_j, \tag{2.2}
\]
where $\beta_j$ for $j = 1, \ldots, m$, are constants independent of the spatial variables, $A$ is the solution of
\[
\begin{cases}
- \Delta A = \nabla \times H \quad &\text{in } \Omega \\
A = 0 \quad &\text{on } \partial \Omega,
\end{cases} \tag{2.3}
\]
and $\varphi_j$ is the solution of
\[
\begin{cases}
\Delta \varphi_j = 0 \quad &\text{in } \Omega \\
\varphi_j = 1 \quad &\text{on } \Gamma_j \\
\varphi_j = 0 \quad &\text{on } \partial \Omega \setminus \Gamma_j.
\end{cases} \tag{2.4}
\]

Integrating the time derivative of (2.2) against $\nabla \times \varphi_i$ yields
\[
(\partial_t H, \nabla \times \varphi_i) = \sum_{j=1}^m \frac{d\beta_j}{dt} (\nabla \times \varphi_j, \nabla \times \varphi_i),
\]
where we have used (2.3)-(2.4) and
\[
(\nabla \times A, \nabla \times \varphi) = -(A, \nabla \times (\nabla \times \varphi)) = (A, \Delta \varphi) = 0.
\]

Furthermore, integrating (1.1) against $\nabla \times \varphi_i$ and using integration by parts, and with the boundary condition (1.4), we obtain
\[
(\partial_t H, \nabla \times \varphi_i) = 0 \quad \text{for } i = 1, \ldots, m.
\]

Since the matrix $M_{ij} = (\nabla \times \varphi_j, \nabla \times \varphi_i)$ is positive definite, the two identities above imply
\[
\frac{d\beta_j}{dt} = 0 \quad \text{for } i = 1, \ldots, m. \tag{2.5}
\]

Therefore, $\beta_j$ for $j = 1, \ldots, m$, are constants independent of time.

Now we substitute $H = \nabla \times A + \sum_{j=1}^m \beta_j \nabla \times \varphi_j$ into (1.1). This yields
\[
\nabla \times (\mu \partial_t A - \sigma^{-1} \Delta A - \sigma^{-1} J - \mu \mathbf{u} \times (\nabla \times A + \sum_{j=1}^m \beta_j \nabla \times \varphi_j)) = 0
\]
which implies
\[
\mu \partial_t A - \sigma^{-1} \Delta A - \sigma^{-1} J - \mu \mathbf{u} \times (\nabla \times A + \sum_{j=1}^m \beta_j \nabla \times \varphi_j) = \text{const.} \tag{2.6}
\]

With the boundary condition (1.4), it is easy to derive that the constant on the right-hand side of the above equation equals zero.

Thus, instead of solving (1.1)-(1.2) directly, we propose to solve (2.4) and the following equations:
\[
\mu \partial_t A - \sigma^{-1} \Delta A - \mu \mathbf{u} \times (\nabla \times A + \sum_{j=1}^m \beta_j \nabla \times \varphi_j) = \sigma^{-1} J \tag{2.7}
\]
\[
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} + \mu (\nabla \times A + \sum_{j=1}^m \beta_j \nabla \times \varphi_j) \times \Delta A \tag{2.8}
\]
\[ \nabla \cdot \mathbf{u} = 0, \]  
which can be obtained by integrating (2.2) against \( \nabla \times \mathbf{u} \) at time \( t = 0 \).

Remark 2.1.

The boundary and initial conditions for (2.7)-(2.9) are given by

\[ \mathbf{u} = 0, \quad A = 0 \quad \text{on} \quad \partial \Omega \times (0, T) \]  
and

\[ \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad A|_{t=0} = A_0 \quad \text{in} \quad \Omega, \]

where \( A_0 \) is the solution of

\[ \begin{cases} -\Delta A_0 = \nabla \times \mathbf{H}_0 & \text{in} \ \Omega \\ A_0 = 0 & \text{on} \ \partial \Omega. \end{cases} \]  

After solving of (2.4) and (2.7)-(2.13), we can obtain the magnetic field \( \mathbf{H} = \nabla \times \mathbf{A} + \sum_{j=1}^{m} \beta_j \nabla \times \varphi_j \).

2.2. Weak solution. For \( k \geq 0 \) and \( p \in [1, \infty] \), we denote by \( W^{k,p}(\Omega) \) the conventional Sobolev space of functions defined on \( \Omega \), with abbreviations \( L^p(\Omega) = W^{0,p}(\Omega) \) and \( H^k(\Omega) = W^{k,2}(\Omega) \). Let \( W_0^{1,p}(\Omega) \) be the space of functions in \( W^{1,p}(\Omega) \) with zero traces on the boundary \( \partial \Omega \), and denote \( H_0^k(\Omega) = W_0^{k,2}(\Omega) \). The corresponding vector-valued spaces are

\[ \begin{align*}
\mathbf{L}^p(\Omega) &= L^p(\Omega) \times L^p(\Omega) \\
\mathbf{W}^{k,p}(\Omega) &= W^{k,p}(\Omega) \times W^{k,p}(\Omega) \\
\mathbf{W}_0^{1,p}(\Omega) &= W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \\
\mathbf{H}_0^{k}(\Omega) &= \{ \mathbf{v} \in \mathbf{H}_0^{k}(\Omega) : \nabla \cdot \mathbf{v} = 0 \}. 
\end{align*} \]

The inner product of \( L^2(\Omega) \) is denoted by \( \langle \cdot, \cdot \rangle \).

A quadruple \( (A, \mathbf{u}, (\varphi_j)_{j=1}^{m}, (\beta_j)_{j=1}^{m}) \) is called a weak solution of (2.4) and (2.7)-(2.13) if

\[ \begin{align*}
A &\in L^\infty(0, T; H^1(\Omega)), \quad \Delta A \in L^2(0, T; L^2(\Omega)), \quad \partial_t A \in L^s(0, T; L^2(\Omega)) \quad \forall s \in (1, 2) \\
\mathbf{u} &\in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^{1,\text{div}}(\Omega)), \quad \partial_t \mathbf{u} \in L^s(0, T; H_0^{1,\text{div}}(\Omega)) \quad \forall s \in (1, 2) \\
\varphi_j &\in H^{\frac{3}{2}+\delta}(\Omega), \quad j = 1, \ldots, m, \quad \text{are solutions of (2.4), where } \delta \in (0, 1) \end{align*} \]

with \( A|_{t=0} = A_0 \) and \( \mathbf{u}|_{t=0} = \mathbf{u}_0 \), and the following equations hold for all test functions \( a \in L^\infty(0, T; H_0^{1}(\Omega)) \) and \( \mathbf{v} \in L^\infty(0, T; H_0^{1,\text{div}}(\Omega)) \),

\[ \begin{align*}
\int_0^T \left[ \mu (\partial_t A, a) - (\sigma^{-1} \Delta A, a) - (\mu \mathbf{u} \times (\nabla \times A + \sum_{j=1}^{m} \beta_j \nabla \times \varphi_j), a) \right] dt \\
= \int_0^T (\sigma^{-1} J, a) dt \\
\int_0^T \left[ \left( (\partial_t \mathbf{u}, \mathbf{v}) + \frac{1}{2} (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) - \frac{1}{2} (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{u}) + (\nu \nabla \mathbf{u}, \nabla \mathbf{v}) \right) dt \\
= \int_0^T \left[ (f, \mathbf{v}) + (\mu (\nabla \times A + \sum_{j=1}^{m} \beta_j \nabla \times \varphi_j) \times \Delta A, \mathbf{v}) \right] dt. \end{align*} \]

Remark 2.1.

1. The pressure \( p \) does not appear in the definition of weak solutions as we have restricted both \( \mathbf{u} \) and the test function \( \mathbf{v} \) to \( H_0^{1,\text{div}}(\Omega) \).

2. If the domain is simply connected (without any holes) then \( m = 0 \). In this case, a pair \( (A, \mathbf{u}) \) is called a weak solution if (2.14)-(2.15) and (2.18)-(2.19) hold.

3. By substituting \( a = \nabla \times b \) into (2.18), we see that if \( (A, \mathbf{u}, (\varphi_j)_{j=1}^{m}, (\beta_j)_{j=1}^{m}) \) is a weak solution of (2.4) and (2.7)-(2.13) then \( (\mathbf{H}, \mathbf{u}) \) is a weak solution of (1.1)-(1.3) in the sense
that
\( H = \nabla \times A + \sum_{j=1}^{m} \beta_j \nabla \times \varphi_j \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H(\Omega; \text{curl})) \),

\( \partial_t H \in L^s(0, T; H(\Omega; \text{curl}))' \), \( \forall s \in (1, 2) \),

\( H|_{t=0} = H_0 \) and \( u|_{t=0} = u_0 \),

and the following equations hold for all test functions \( b \in L^\infty(0, T; \text{H}(\Omega; \text{curl})) \) and \( v \in L^\infty(0, T; \text{H}_{\text{div}}(\Omega)) \),

\[
\int_0^T \left[ (\mu \partial_t H, b) + (\sigma^{-1} \nabla \times H, \nabla \times b) - (\mu u \times H, \nabla \times b) \right] dt = \int_0^T (\sigma^{-1} J, \nabla \times b) dt,
\]

\[
\int_0^T \left[ (\partial_t u, v) + \frac{1}{2} (u \cdot \nabla u, v) - \frac{1}{2} (u \cdot \nabla v, u) + (\nu \nabla u, \nabla v) \right] dt = \int_0^T (f, v) - (\mu H \times (\nabla \times H), v) dt.
\]

3. **Numerical method.** In this section, we introduce a fully discrete numerical method for solving \((2.21)\) and \((2.22)-(2.23)\), and then present the main theoretical results about the convergence of the numerical solutions.

Let \( \Delta \) be a quasi-uniform partition of \( \Omega \) into triangles \( K_j \), \( j = 1, \ldots, M \), and denote by \( h = \max_{1 \leq j \leq M} \{ \text{diam} K_j \} \) the mesh size. For any integer \( r \geq 1 \), we define the Taylor–Hood finite element space \( \mathcal{S}_h^{1+r} \times \mathcal{S}_h^r \) with

\[
\mathcal{S}_h^{1+r} = \{ \chi_h \in H^1(\Omega) : \chi_h|_{K_j} \in P_r(K_j), \forall K_j \in \Delta_h \}
\]

\[
\mathcal{S}_h^r = \mathcal{S}_h^{1+r} \cap H_0^0(\Omega)
\]

\[
\hat{\mathcal{S}}_h^{1+r} = \mathcal{S}_h^{1+r} \times \hat{\mathcal{S}}_h^r,
\]

where \( P_r(K_j) \) is the space of polynomials of degree \( r \) on the triangle \( K_j \).

For any given \( j = 1, \ldots, m \), let \( \varphi_{j,h} \in \mathcal{S}_h^{1+r} \) be the finite element solution of \((2.24)\), i.e.,

\[
(\nabla \varphi_{j,h}, \nabla v_h) = 0 \quad \forall v_h \in \hat{\mathcal{S}}_h^r
\]

such that \( \varphi_{j,h} = 1 \) on \( \Gamma_j \) and \( \varphi_{j,h} = 0 \) on \( \partial \Omega \setminus \Gamma_j \). Let \( \beta_{j,h}, j = 1, \ldots, m \), be the constants (independent of space and time) determined by the equations

\[
\sum_{j=1}^{m} \beta_{j,h} (\nabla \varphi_{j,h}, \nabla \varphi_{i,h}) = (H_0, \nabla \times \varphi_{i,h}) \quad \text{for } i = 1, \ldots, m.
\]

Let \( \{ t_n = n\tau \}_{n=0}^N \) denote a uniform partition of the time interval \( [0, T] \), with a step size \( \tau = T/N \), and \( u^n = u(x, t_n) \). A fully discrete numerical scheme for the system \((2.21)-(2.23)\) is to find \( A^n_h \in \mathcal{S}_h^{1+r} \), \( u^n_h \in \mathcal{S}_h^{r+1} \), and \( p^n_h \in \mathcal{S}_h^r \) such that

\[
\left( \frac{A^n_h - A_{n-1}^h}{\tau}, a_h \right) + (\sigma^{-1} \nabla A^n_h, \nabla a_h) - (\mu u^n_h \times (\nabla \times A^n_{h-1} + \sum_{j=1}^{m} \beta_{j,h} \nabla \times \varphi_j), a_h)
\]

\[
= (\sigma^{-1} J^n, a_h)
\]

\[
\left( \frac{u^n_h - u_{n-1}^h}{\tau}, v_h \right) + \frac{1}{2} (u_{n-1}^h \cdot \nabla u^n_h, v_h) - \frac{1}{2} (u^n_h \cdot \nabla u_{n-1}^h, u^n_h) + (\nu \nabla u^n_h, \nabla v_h) - (p^n_h, \nabla \cdot v_h)
\]

\[
= (f^n, v_h) + (\mu (\nabla \times A_{h-1}^{n-1} + \sum_{j=1}^{m} \beta_{j,h} \nabla \times \varphi_{j,h}) \times \Delta_h A^n_h, v_h)
\]

\[
(\nabla \cdot u^n_h, q_h) = 0
\]

hold for all test functions \( a_h \in \hat{\mathcal{S}}_h^{r+1}, v_h \in \mathcal{S}_h^{r+1}, q_h \in \mathcal{S}_h^r \), and \( n = 1, 2, \ldots, N \). The operator \( \Delta_h : \hat{\mathcal{S}}_h^{r+1} \to \hat{\mathcal{S}}_h^{r+1} \) is defined via the duality:

\[
(\Delta_h A^n_h, a_h) = - (\nabla A^n_h, \nabla a_h) \quad \forall a_h \in \hat{\mathcal{S}}_h^{r+1}.
\]

The initial condition \( A^0_h \in \mathcal{S}_h^{1+r} \) can be determined by

\[
(\nabla A^0_h, \nabla a_h) = (H_0, \nabla \times a_h) \quad \forall a_h \in \hat{\mathcal{S}}_h^{r+1}.
\]
The initial condition for velocity is given by $u_h^0 = P_h u_0$, where $P_h$ denotes the $L^2$ projection from $L^2(\Omega)$ onto $\tilde{S}_h^{\tau+1}$, defined by

$$
(P_h \varphi, \chi_h) = (\varphi, \chi_h) \quad \forall \varphi \in L^2(\Omega), \forall \chi_h \in \tilde{S}_h^{\tau+1},
$$
which automatically extends to the vector-valued space $\tilde{S}_h^{\tau+1}$, i.e.,

$$
(P_h \varphi, \chi_h) = (\varphi, \chi_h) \quad \forall \varphi \in L^2(\Omega), \forall \chi_h \in \tilde{S}_h^{\tau+1}.
$$

For any sequence $\omega_{h,n}^n$, $n = 1, 2, \ldots$, we define the piecewise constant functions $\omega_{h,\tau}^+ = \omega_{h,n}^n$ and $\omega_{h,\tau}^- = \omega_{h,n}^{n-1}$ for $t \in (t_{n-1}, t_n)$ and $n = 1, 2, \ldots, N$. Then we have the following result on the convergence of the numerical solutions.

**Theorem 3.1.** Under assumption (1.1), the fully discrete finite element scheme (3.1) has a unique solution. For any sequence $(\tau_n, h_n) \to (0, 0)$ such that the corresponding numerical solutions converge to a weak solution $(A, u, (\varphi_j)^m_{j=1}, (\beta_j)^m_{j=1})$ of (4.1) and (4.2) in the following sense:

$$
A_{\tau_{n_k}, h_{n_k}}^+ \text{ converges to } A \text{ in } L^2(0, T; H^1(\Omega)),
$$

$$
u_{\tau_{n_k}, h_{n_k}}^+ \text{ converges to } \nu \text{ in } L^2(0, T; \mathbf{L}^2(\Omega)),
$$

$$
\varphi_{j,h} \text{ converges to } \varphi_j \text{ in } H^1(\Omega),
$$

$$
\beta_{j,h} \text{ converges to } \beta_j \text{ in } \mathbb{R},
$$

which also imply that $H_{\tau_{n_k}, h_{n_k}}^+ = \nabla + A_{\tau_{n_k}, h_{n_k}}^+ + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}$ converges to $\mathbf{H} = \nabla + A + \sum_{j=1}^m \beta_j \nabla \times \varphi_j$ in $L^2(0, T; \mathbf{L}^2(\Omega))$.

**Remark 3.1.** The uniqueness of weak solutions was proved in [28] for convex and smooth domains, but remains open for nonconvex and nonsmooth domains. Our proof shows that every sequence of numerical solutions contains a subsequence converging to a weak solution of the PDE problem. If the weak solution is unique, then Theorem 3.1 implies that the numerical solutions converge to the unique weak solution as $(\tau, h) \to (0, 0)$ (without taking a subsequence).

**4. Proof of Theorem 3.1.** In this section, we prove Theorem 3.1 by using a compactness argument. We first introduce some standard notations of finite element spaces in Section 4.1 and then present energy estimates for the numerical solutions in Section 4.2. In Section 4.3, we utilize the compactness of the numerical solutions to prove the existence of a subsequence (in every sequence of numerical solutions) that converges to a weak solution of the PDE problem.

Throughout this paper, we denote by $C$ a generic positive constant which could be different at different places but would be independent of $n$, $h$, and $\tau$. To simplify notation, we use the abbreviations $W^{k,p} = W^{k,p}(\Omega)$, $L^p = L^p(\Omega)$ and $H^k = H^k(\Omega)$ for $k \geq 0$ and $1 \leq p \leq \infty$.

**4.1. Preliminaries.** It is known that the Taylor–Hood finite element space $\tilde{S}_h^{\tau+1} \times S_h^{r}$ ($r \geq 1$) satisfies the following discrete LBB condition for some constant $\gamma > 0$:

$$
\sup_{\chi_h \in S_h^{r}} \frac{(\varphi_h, \nabla \cdot \chi_h)}{\|\nabla \chi_h\|_{L^2(\Omega)}} \geq \gamma \|\varphi_h\|_{L^2(\Omega)} \quad \forall \varphi_h \in S_h^{r}.
$$

Over the finite element spaces $S_h^{\tau+1}$ and $\tilde{S}_h^{\tau+1}$, we define the $L^2$ projection $P_h$ in (3.2)-(3.9). Besides, we also define the $L^2$ projection $\bar{P}_h : L^2(\Omega) \to S_h^{r}$ (without enforcing boundary conditions), i.e.,

$$
(\bar{P}_h \varphi, \chi_h) = (\varphi, \chi_h) \quad \forall \varphi \in L^2(\Omega), \forall \chi_h \in S_h^{r}.
$$

These $L^2$ projections satisfy the following standard estimates for $1 \leq p \leq \infty$:

$$
\|P_h \varphi\|_{W^{1,p}(\Omega)} \leq C \|\varphi\|_{W^{1,p}(\Omega)} \quad \forall \varphi \in W^{1,p}_0(\Omega)
$$

$$
\|P_h \varphi\|_{L^p(\Omega)} \leq C \|\varphi\|_{L^p(\Omega)} \quad \forall \varphi \in L^p(\Omega)
$$

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\[ \| P_h \varphi \|_{W^{-1,p}(\Omega)} \leq C \| \varphi \|_{W^{-1,p}(\Omega)} \quad \forall \varphi \in W^{-1,p}(\Omega) \quad (4.4) \]
\[ \| P_h \varphi \|_{L^p(\Omega)} \leq C \| \varphi \|_{L^p(\Omega)} \quad \forall \varphi \in L^p(\Omega) \quad (4.5) \]
\[ \lim_{h \to 0} \| P_h \varphi - \varphi \|_{L^p(\Omega)} = 0 \quad \forall \varphi \in L^p(\Omega) \quad (4.6) \]
\[ \lim_{h \to 0} \| P_h \varphi - \varphi \|_{W^{1,p}(\Omega)} = 0 \quad \forall \varphi \in W_0^{1,p}(\Omega) \quad (4.7) \]
\[ \lim_{h \to 0} \| P_h \varphi - \varphi \|_{L^p(\Omega)} = 0 \quad \forall \varphi \in L^p(\Omega). \quad (4.8) \]

For any \( v \in H_{0,\text{div}}^1(\Omega) \) we define \( Q_h v \) to be the Fortin projection of \( v \) onto \( S_h^{r+1} \), satisfying
\[ \langle \nabla \cdot (v - Q_h v), q_h \rangle = 0 \quad \forall q_h \in S_h^{r+1} \quad (4.9) \]
which has the following property (cf. [12] and [13] Lemma 3.4)
\[ \| Q_h v \|_{H^1} \leq C \| v \|_{H^1} \quad \forall v \in H_{0,\text{div}}^1(\Omega) \quad (4.10) \]
\[ \| v - Q_h v \|_{L^2} \leq C \| v \|_{H^1} \quad \forall v \in H_{0,\text{div}}^1(\Omega). \quad (4.11) \]

Furthermore, over the finite element spaces \( \tilde{S}_h^{r+1} \), the following inverse inequality holds; see [10, 31].
\[ \| \chi_h \|_{W^{1,q}} \leq C h^{-1} \| \chi_h \|_{L^q} \quad (4.12) \]
\[ \| \chi_h \|_{W^{m,q}} \leq C h^{2/q - 2/q} \| \chi_h \|_{W^{m,1}} \quad (4.13) \]
for all \( \chi_h \in \tilde{S}_h^{r+1} \), and \( 1 \leq q \leq \infty, m = 0, 1 \).

4.2. **Energy estimate.** For any sequence of functions \( \omega_n(\cdot), n = 0, 1 \ldots, N \), we let \( \omega_{n,\tau} \) denote a piecewise linear function in time defined by
\[ \omega_{n,\tau}(t) := \frac{t_n - t}{\tau} \omega_{n-1} + \frac{t - t_n-1}{\tau} \omega_n \quad (4.14) \]
for \( t \in (t_{n-1}, t_n) \) and \( n = 1, 2, \ldots, N \). Then we have the following energy estimate for the numerical solutions.

**PROPOSITION 4.1.** The numerical scheme \([\Omega, T], [\mathbb{P}_n] \) admits a unique solution \((A^n_{h,\tau}, u^+_{h,\tau}, p^n_{h,\tau})\), which satisfies the following estimates for \( s \in (1, 2) \):
\[ \| A_h \|_{C(0,T;H^s(\Omega))} + \| \Delta_h A_h \|_{L^2(0,T;L^2(\Omega))} + \| \partial_t A_h \|_{L^4(0,T;L^4(\Omega))} \leq C. \quad (4.15) \]

Furthermore, the finite element solution \((A^+_{h,\tau}, u^+_{h,\tau})\) satisfies the estimate
\[ \| A^+_{h,\tau} \|_{L^\infty(0,T;H^s(\Omega))} + \| \Delta_h A^+_{h,\tau} \|_{L^2(0,T;L^2(\Omega))} + \| u^+_{h,\tau} \|_{L^\infty(0,T;L^2(\Omega))} \leq C. \quad (4.16) \]

To prove Proposition 4.1 we need to use the following two lemmas.

**LEMMA 4.2.** There exists \( \alpha > 0 \) (depending only on the domain \( \Omega \)) such that
\[ \| \chi_h \|_{W^{1,4+\alpha}} \leq C \| \Delta_h \chi_h \|_{L^2} \quad (4.17) \]
for any \( \chi_h \in \tilde{S}_h^{r+1} \). Furthermore, if \( h_n \to 0 \) as \( n \to \infty \), and \( \Delta_h \chi_h \) is bounded in \( L^2(\Omega) \), then \( \chi_h \) is compact in \( W^{1,4+\alpha}(\Omega) \).

**Proof.** For any given \( \chi_h \in \tilde{S}_h^{r+1} \), let \( \chi_n \in H_0^1(\Omega) \) be the weak solution of the PDE problem
\[ \Delta \chi_n = \Delta_h \chi_h \quad (4.18) \]
with homogeneous Dirichlet boundary condition. Thus \( \chi_n \) satisfies the weak formulation
\[ \langle \nabla \chi_n, \nabla \chi \rangle = -\langle \Delta_h \chi_h, \chi \rangle = \langle \nabla \chi_h, \nabla \chi \rangle \]
for any \( \chi \in \tilde{S}_h^{r+1} \), which implies \( \chi_h \) is the Ritz projection of \( \chi_n \).

On the one hand, as the solution of the PDE problem (4.18), \( \chi_n \) satisfies the standard PDE estimate
\[ \| \chi_n \|_{H^{\frac{1}{2} + \delta}} \leq C \| \Delta_h \chi_h \|_{L^2} \quad (4.19) \]
for some constant \( \delta \in (0, \frac{1}{4}] \) depending on the maximal interior angle of the domain \( \Omega \). The estimate (4.19) is a consequence of [3] Corollary 3.9, with fractional \( k \); also see [9, p. 30] and [2] (23.3)]. Since \( H^{\frac{1}{2} + \delta} \) is compactly embedded into \( W^{1,4+\alpha} \) for \( \alpha \in (0, \frac{1}{2(4+\alpha)}) \) in
a polygon (cf. [1, Theorem 7.34]), with \( p = 2 \) and \( s = \frac{1}{2} + \delta \), it follows that
\[
\|\chi_n\|_{W^{1,4+\alpha}} \leq C\|\chi_n\|_{H^{\frac{1}{2}+\delta}} \leq C\|\Delta h\chi_n\|_{L^2}
\] (4.20)
and the set of functions \( \{\chi_n : n = 1, 2, \ldots\} \) is compact in \( W^{1,4+\alpha}(\Omega) \).

On the other hand, as the Ritz projection of \( \chi_n \), the finite element function \( \chi_{h_n} \) satisfies the standard error estimate
\[
\|\chi_{h_n} - \chi_n\|_{H^1} \leq C\|\chi_n\|_{H^{\frac{1}{2}+\delta}} \leq C\|\Delta h\chi_n\|_{L^2}h_n^{\frac{1}{2}+\delta} \rightarrow 0
\]
as \( n \rightarrow \infty \). By using the triangle inequality and the inverse inequality of the finite element space, we have
\[
\|\chi_{h_n} - \chi_n\|_{W^{1,4+\alpha}} \leq \|\chi_{h_n} - P_h\chi_n\|_{W^{1,4+\alpha}} + \|P_h\chi_n - \chi_n\|_{W^{1,4+\alpha}}
\]
\[
\leq C_1h_n^{\frac{1}{2}+\delta}\|\chi_{h_n} - P_h\chi_n\|_{H^1} + \|P_h\chi_n - \chi_n\|_{W^{1,4+\alpha}}
\]
\[
\leq C_2h_n^{\frac{1}{2}+\delta}\|\chi_{h_n} - \chi_n\|_{H^1} + h_n^{\frac{1}{2}+\delta}\|\chi_{h_n} - P_h\chi_n\|_{H^1} + \|P_h\chi_n - \chi_n\|_{W^{1,4+\alpha}}
\]
\[
\leq C\|\chi_n\|_{H^1}h_n^{\frac{3}{2}+\delta} + C\|\chi_n\|_{H^{\frac{1}{2}+\delta}}h_n^{\frac{1}{2}+\delta} - \frac{2+\alpha}{2+\alpha}
\]
\[
\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]
where we have used the fact \( \frac{1}{2} + \delta - \frac{2+\alpha}{2+\alpha} > 0 \) for \( \alpha \in (0, \frac{84}{129}) \) in the last inequality. Thus, by (4.20), we obtain
\[
\|\chi_{h_n}\|_{W^{1,4+\alpha}} \leq \|\chi_n\|_{W^{1,4+\alpha}} + C\|\Delta h\chi_n\|_{L^2} \leq C\|\Delta h\chi_n\|_{L^2}.
\]
Since \( \{\chi_n : n = 1, 2, \ldots\} \) is compact in \( W^{1,4+\alpha}(\Omega) \) and with the above result, the set of functions \( \{\chi_n : n = 1, 2, \ldots\} \) is also compact in \( W^{1,4+\alpha}(\Omega) \). This completes the proof of Lemma 4.2. \( \square \)

**Lemma 4.3.** There exists \( \alpha > 0 \) (depending only on the domain \( \Omega \)) such that the function \( \varphi_{j,h} \) determined by (3.1) converges to \( \varphi_j \) in the following sense:
\[
\lim_{h \to 0} \|\varphi_{j,h} - \varphi_j\|_{W^{1,4+\alpha}(\Omega)} = 0.
\] (4.21)
Furthermore,
\[
\lim_{h \to 0} |\beta_{j,h} - \beta_j| = 0.
\] (4.22)

**Proof.** Let \( \chi \) be a smooth cut-off function such that \( \chi = 1 \) in a neighborhood of \( \Omega_j \) and \( \chi = 0 \) on \( \cup_i \neq j \partial \Omega_i \). Then \( \phi_j - \chi \) is the solution of
\[
\begin{cases}
\Delta(\phi_j - \chi) = -\Delta \chi & \text{in } \Omega \\
\phi_j - \chi = 0 & \text{on } \partial \Omega
\end{cases}
\] (4.23)
which implies (similar as (4.19))
\[
\|\phi_j - \chi\|_{H^{\frac{1}{2}+\delta}} \leq C\|\Delta \chi\|_{L^2} \leq C \quad \text{for some } \delta > 0 \text{ depending only on the domain } \Omega.
\]
Therefore, \( \|\phi_j\|_{H^{\frac{1}{2}+\delta}} \leq C \). In view of the compact embedding \( H^{\frac{1}{2}+\delta}(\Omega) \hookrightarrow W^{1,4+\alpha}(\Omega) \) for \( \alpha \in (0, \frac{84}{129}) \), we have \( \nabla \times \phi_j \in L^{4+\alpha}(\Omega) \). Since \( \phi_{j,h} \) is the finite element solution of \( \phi_j \), it follows that
\[
\|P_h\phi_j - \phi_{j,h}\|_{H^1} \leq C\|\phi_j\|_{H^{\frac{1}{2}+\delta}} - \frac{2+\alpha}{2+\alpha}.
\] (4.24)
where \( P_h \) is the \( L^2 \)-projection operator, satisfying (4.2). By using the inverse inequality we obtain
\[
\|P_h\phi_j - \phi_{j,h}\|_{W^{1,4+\alpha}} \leq C\|\phi_j\|_{H^{\frac{1}{2}+\delta}} - \frac{2+\alpha}{2+\alpha}. \|P_h\phi_j - \phi_{j,h}\|_{H^1} \leq C\|\phi_j\|_{H^{\frac{1}{2}+\delta}} - \frac{2+\alpha}{2+\alpha}.
\] (4.25)
For \( \alpha \in (0, \frac{84}{129}) \) we have \( \frac{1}{2} + \delta - \frac{2+\alpha}{2+\alpha} > 0 \). Since \( H^{\frac{1}{2}+\delta}(\Omega) \hookrightarrow W^{\frac{1}{2}+\delta - \frac{2+\alpha}{2+\alpha},4+\alpha}(\Omega) \), it follows that
\[
\|\phi_j - P_h\phi_j\|_{W^{1,4+\alpha}} \leq C\|\phi_j\|_{W^{\frac{1}{2}+\delta - \frac{2+\alpha}{2+\alpha},4+\alpha}}\leq C\|\phi_j\|_{H^{\frac{1}{2}+\delta}}h^{\frac{1}{2}+\delta - \frac{2+\alpha}{2+\alpha}} \leq C\|\phi_j\|_{H^{\frac{1}{2}+\delta}}h^{\frac{1}{2}+\delta - \frac{2+\alpha}{2+\alpha}} \leq C\|\phi_j\|_{H^{\frac{1}{2}+\delta}}h^{\frac{1}{2}+\delta - \frac{2+\alpha}{2+\alpha}}.
\] (4.26)
Therefore we have

\[ \| \varphi_j - \varphi_{j,h} \|_{W^{1,\infty}} \leq \| \varphi_j - P_h \varphi_j \|_{W^{1,\infty}} + \| P_h \varphi_j - \varphi_{j,h} \|_{W^{1,\infty}} \leq C h^{\frac{1}{2} + \delta - \frac{2}{p+1}} \]  

(4.27)

which implies (4.21).

Let \( M = (M_{ij}) \) and \( M_h = (M_{ij,h}) \) be two \( m \times m \) matrices, with \( M_{ij} = (\nabla \cdot \varphi_j, \nabla \cdot \varphi_i) \) and \( M_{ij,h} = (\nabla \cdot \varphi_{j,h}, \nabla \cdot \varphi_{i,h}) \). Since \( M = (M_{ij}) \) is positive definite and (4.21) implies \( \lim_{h \to 0} M_h = M \), it follows that \( \lim_{n \to 0} M_h^{-1} = M^{-1} \). Then (3.2) and (2.10) imply

\[
\lim_{h \to 0} \beta_{i,h} = \lim_{h \to 0} \sum_{j=1}^m (M_h^{-1})_{ij} (H_0, \nabla \times \varphi_{j,h})
\]

\[
= \sum_{j=1}^m (M^{-1})_{ij} \lim_{h \to 0} (H_0, \nabla \times \varphi_{j,h})
\]

\[
= \sum_{j=1}^m (M^{-1})_{ij} (H_0, \nabla \times \varphi_j)
\]

\[
= \beta_i.
\]

With above results, we start to prove Proposition 4.1.

**Proof of Proposition 4.1** First, we start with estimating \( \| \nabla A_h^n \|_{L^2} \) and \( \| u_h^n \|_{L^2} \). To this end, we substitute \( a_h^n = -\Delta_h A_h^n \), \( v_h = u_h^n \) and \( q_h = p_h^n \) into (3.3)-(3.4). Then we obtain

\[
\left( \mu \left( \nabla A_h^n - \nabla A_h^{n-1} \right), \nabla A_h^n \right) + \sigma^{-1} \| \Delta_h A_h^n \|_{L^2}^2 + (\mu u_h^n \times \left( \nabla \times A_h^{n-1} + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h} \right), \Delta_h A_h^n) = (f^n, A_h^n) \]  

(4.28)

\[
\left( \frac{u_h^n - u_h^{n-1}}{\tau}, u_h^n \right) + \nu \| \nabla u_h^n \|_{L^2}^2 = (f^n, u_h^n) + (\mu \left( \nabla \times A_h^{n-1} + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h} \right)x \Delta_h A_h^n, u_h^n) = (f^n, u_h^n) + (\mu u_h^n \times \left( \nabla \times A_h^{n-1} + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h} \right), \Delta_h A_h^n). \]  

(4.29)

Summing up the two equations above yields

\[
\frac{\mu \| \nabla A_h^n \|_{L^2}^2 - \mu \| \nabla A_h^{n-1} \|_{L^2}^2}{2\tau} + \frac{\| u_h^n \|_{L^2}^2 - \| u_h^{n-1} \|_{L^2}^2}{2\tau} + \sigma^{-1} \| \Delta_h A_h^n \|_{L^2}^2 + \nu \| \nabla u_h^n \|_{L^2}^2
\]

\[
\leq (\sigma^{-1} f^n, A_h^n) + (f^n, u_h^n)
\]

\[
\leq \frac{\sigma^{-1}}{2} \| f^n \|_{L^2}^2 + \frac{\sigma^{-1}}{2} \| \Delta_h A_h^n \|_{L^2}^2 + \frac{1}{4\epsilon} \| f^n \|_{L^2}^2 + \epsilon \| u_h^n \|_{L^2}^2
\]

\[
\leq \frac{\sigma^{-1}}{2} \| f^n \|_{L^2}^2 + \frac{1}{4\epsilon} \| f^n \|_{L^2}^2 + C \epsilon \| \nabla u_h^n \|_{L^2}^2. \]  

(4.30)

By choosing a sufficiently small \( \epsilon \), the inequality above implies

\[
\frac{\mu \| \nabla A_h^n \|_{L^2}^2 - \mu \| \nabla A_h^{n-1} \|_{L^2}^2}{2\tau} + \frac{\| u_h^n \|_{L^2}^2 - \| u_h^{n-1} \|_{L^2}^2}{2\tau} + \sigma^{-1} \| \Delta_h A_h^n \|_{L^2}^2 + \frac{\nu}{2} \| \nabla u_h^n \|_{L^2}^2
\]

\[
\leq C (\| f^n \|_{L^2}^2 + \| f^n \|_{L^2}^2). \]  

(4.31)

By summing up the inequality above for \( n = 1, \ldots, m \), with \( 1 \leq m \leq N \), we obtain

\[
\max_{1 \leq n \leq N} \left( \| \nabla A_h^n \|_{L^2}^2 + \| u_h^n \|_{L^2}^2 \right) + \tau \sum_{n=1}^N (\| \Delta_h A_h^n \|_{L^2}^2 + \| \nabla u_h^n \|_{L^2}^2)
\]

\[
\leq C \tau \sum_{n=1}^N (\| f^n \|_{L^2}^2 + \| f^n \|_{L^2}^2) + C (\| \nabla A_h^n \|_{L^2}^2 + \| u_h^n \|_{L^2}^2)
\]

\[
\leq C, \]  

(4.32)

where we have used the boundedness of \( \| \nabla A_h^n \|_{L^2}^2 \) and \( \| u_h^n \|_{L^2}^2 \), which are direct consequences of the definitions of \( u_h^n = P_h u_0 \) and \( A_h^n \) in (3.7). The estimate (4.31) also implies that, by setting \( A_h^{n-1} = J^n = 0 \) and \( u_h^{n-1} = f^n = 0 \), the homogeneous linear system given by (3.1)-
which holds for all \(3.5\) has only zero solution. This implies the existence and uniqueness of solutions of the linear system \((3.11)-(3.13)\).

Second, we estimate \(q, \bar{q} > 2\) satisfying \(\frac{q}{\bar{q}} = \frac{1}{2}\) for any \(q\), \(\bar{q} > 2\) satisfying \(\frac{q}{\bar{q}} = \frac{1}{2}\).

\[
\left( \frac{A_n - A_{n-1}}{\tau} \right) = \left( \sigma^{-1} \Delta_h A_{h,n}, a_h \right) + (\mu u_h^n \times (\nabla \times A_{h,n} + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}), a_h) + (\sigma^{-1} J^n, a_h) 
\leq (\sigma^{-1}\|\Delta_h A_{h,n}\|_{L^2} + \|u_h^n \times (\nabla \times A_{h,n} + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h})\|_{L^2} + \sigma^{-1}\|J^n\|_{L^2})\|a_h\|_{L^2} 
\leq C(\|\Delta_h A_{h,n}\|_{L^2} + \|\Delta_h A_{h,n}\|_{L^2} + \|\nabla \times A_{h,n} + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}\|_{L^2} + \|J^n\|_{L^2})\|a_h\|_{L^2} 
\leq C(\|\Delta_h A_{h,n}\|_{L^2} + \|\nabla \times A_{h,n} - 1 + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}\|_{L^2} + \|J^n\|_{L^2})\|a_h\|_{L^2} 
\leq C(\|\Delta_h A_{h,n}\|_{L^2} + \|\nabla \times A_{h,n} - 1 + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}\|_{L^2} + \|J^n\|_{L^2})\|a_h\|_{L^2}.
\]

The inequality above implies (via the duality argument)

\[
\left( \frac{A_n - A_{n-1}}{\tau} \right) \leq C(\|\Delta_h A_{h,n}\|_{L^2} + \|\nabla \times A_{h,n} - 1 + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}\|_{L^2} + \|J^n\|_{L^2}) 
\]

which holds for all \(q \in (2, 4)\) with \(\bar{q} = \frac{2q}{q-2}\). By using Lemma \(4.32\) and \(4.32\), we have

\[
\|\nabla \times A_{h,n} - 1\|_{L^2} \leq C\|\nabla \times A_{h,n} - 1\|_{L^2}^{\frac{1}{2}} \|\nabla \times A_{h,n} - 1\|_{L^2}^{\frac{1}{2}} \leq C\|\Delta_h A_{h,n}\|_{L^2}^{\frac{1}{2}}.
\]

Therefore,

\[
(\tau \sum_{n=1}^{N} \|\nabla \times A_{h,n} - 1\|_{L^2}^2 + \|\nabla \times A_{h,n} - 1\|_{L^2} + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}\|_{L^2})^{\frac{1}{2}} \leq C \left( \tau \sum_{n=1}^{N} \|\nabla \times A_{h,n} - 1\|_{L^2}^2 + \|\nabla \times A_{h,n} - 1\|_{L^2} + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}\|_{L^2} \right)^{\frac{1}{2}} \leq C \left( \tau \sum_{n=1}^{N} \|\nabla \times A_{h,n} - 1\|_{L^2}^2 + \|\nabla \times A_{h,n} - 1\|_{L^2} + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}\|_{L^2} \right)^{\frac{1}{2}} \leq C,
\]

where we have used Lemma \(4.33\) in the last inequality. For any given \(1 < s < 2\) one can choose \(q = \frac{2s}{3s-2} \in (2, 4)\) so that \(2 - \frac{1}{q} = \frac{2s}{3s-2}\) and

\[
\left( \tau \sum_{n=1}^{N} \|\nabla \times A_{h,n} - 1\|_{L^2}^2 + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}\|_{L^2} \right)^{\frac{1}{2}} \leq C \left( \tau \sum_{n=1}^{N} \|\nabla \times A_{h,n} - 1\|_{L^2}^2 + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}\|_{L^2} \right)^{\frac{1}{2}} \leq C,
\]

Third, we present the estimate for \(\|\frac{u^n_h - u^{n-1}_h}{\tau}\|_{H_{h, div}(\Omega)}^2\). From \(3.5\) we can derive that

\[
\left( \frac{u^n_h - u^{n-1}_h}{\tau}, v_h \right) = \left( P_h g^n, v_h \right) + \left( p^n_h, \nabla \cdot v_h \right),
\]

where

\[
g^n = -\frac{1}{2} u^{n-1}_h \cdot \nabla u^n_h - \frac{1}{2} \nabla \cdot u^n_h \cdot u^n_h - \frac{1}{2} u^{n-1}_h \cdot \nabla u^n_h - \nabla \Delta_h u^n_h + f^n + \mu (\nabla \times A_{h,n} - 1 + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}) \cdot \Delta_h A_{h,n}.
\]
Substituting $v_h = \frac{u^n_h - u_h^{n-1}}{\tau}$ into (4.35) yields
\[
\left\| \frac{u^n_h - u_h^{n-1}}{\tau} \right\|_{L^2}^2 \leq C \|P_h g^n\|_{L^2} \left\| \frac{u^n_h - u_h^{n-1}}{\tau} \right\|_{L^2} + \left( p^n_h, \frac{\nabla \cdot u^n_h - \nabla \cdot u_h^{n-1}}{\tau} \right)
\]
where we have used (4.35). Then,
\[
\left\| \frac{u^n_h - u_h^{n-1}}{\tau} \right\|_{L^2} \leq C \|P_h g^n\|_{L^2}. \tag{4.36}
\]
From (4.39), we have $(\nabla \cdot v, p^n_h) = (\nabla \cdot Q_h v, p^n_h) = 0$ for any $v \in H^1_{0, \text{div}}(\Omega)$ and thus
\[
\left( \frac{u^n_h - u_h^{n-1}}{\tau}, v \right) = \left( \frac{u^n_h - u_h^{n-1}}{\tau}, Q_h v \right) + \left( \frac{u^n_h - u_h^{n-1}}{\tau}, v - Q_h v \right)
\]
\[
= (P_h g^n, Q_h v) + \left( \frac{u^n_h - u_h^{n-1}}{\tau}, v - Q_h v \right)
\]
\[
\leq C \|g^n\|_{H^{-1}(\Omega)} \|v\|_{H^1} + C \left\| \frac{u^n_h - u_h^{n-1}}{\tau} \right\|_{L^2} \|v - Q_h v\|_{L^2}, \quad \forall v \in H^1_{0, \text{div}}(\Omega). \tag{4.37}
\]
By using Hölder’s inequality and (4.38), we have
\[
\left\| \frac{u^n_h - u_h^{n-1}}{\tau} \right\|_{L^2} \|v - Q_h v\|_{L^2} \leq C \|u^n_h - u_h^{n-1}\|_{L^2} \|v\|_{H^1}
\]
\[
\leq C \|P_h g^n\|_{L^2} \|v\|_{H^1}
\]
\[
\leq C \|P_h g^n\|_{H^{-1}} \|v\|_{H^1}
\]
\[
\leq C \|g^n\|_{H^{-1}} \|v\|_{H^1},
\]
where we have used the inverse inequality in the second to last inequality, and the $H^{-1}(\Omega)$ stability of $P_h$ in the last inequality. The two estimates above imply
\[
\left\| \frac{u^n_h - u_h^{n-1}}{\tau} \right\|_{H^1_{0, \text{div}}(\Omega)} \leq C \|g^n\|_{H^{-1}}. \tag{4.38}
\]
From (4.35) we obtain for any $q > 1$
\[
\|g^n\|_{H^{-1}} \leq C \|u^n_h \cdot \nabla u^n_h\|_{H^{-1}} + C \sup_{v_h, \neq 0} \frac{(\nabla \cdot u^n_h) \cdot u_h^{n-1} + \nabla u^n_h \cdot \nabla v_h}{\|v_h\|_{H^1_0}}
\]
\[
+ C \sup_{v_h, \neq 0} \frac{(\Delta_h u^n_h, v_h)}{\|v_h\|_{H^1_0}} + C \|f^n\|_{H^{-1}} + C \|\nabla \times A_h^{n-1} + \sum_{j=1}^{m} \beta_{j,h} \nabla \times \varphi_{j,h} \nabla \times A_h^n\|_{H^{-1}}
\]
\[
= C \|u^n_h \cdot \nabla u^n_h\|_{H^{-1}} + C \sup_{v_h, \neq 0} \frac{(u^n_h, u_h^{n-1} \cdot \nabla v_h)}{\|v_h\|_{H^1_0}}
\]
\[
+ C \sup_{v_h, \neq 0} \frac{(\nabla u^n_h, \nabla v_h)}{\|v_h\|_{H^1_0}} + C \|f^n\|_{H^{-1}} + C \|\nabla \times A_h^{n-1} + \sum_{j=1}^{m} \beta_{j,h} \nabla \times \varphi_{j,h} \nabla \times A_h^n\|_{L^q}
\]
\[
\leq C \|u^n_h \cdot \nabla u^n_h\|_{L^q} + C \|u_h^{n-1}\|_{L^q} \|u^n_h\|_{L^4}
\]
\[
+ C \|\nabla u^n_h\|_{L^2} + C \|f^n\|_{H^{-1}} + C \|\nabla \times A_h^{n-1} + \sum_{j=1}^{m} \beta_{j,h} \nabla \times \varphi_{j,h} \nabla \times A_h^n\|_{L^q}
\]
\[
\leq C \|u^n_h\|_{L^q} \|\nabla u^n_h\|_{L^2} + C \|u_h^{n-1}\|_{L^q} \|u^n_h\|_{L^4}
\]
\[
+ C \|u^n_h\|_{H^1} + C \|f^n\|_{H^{-1}} + C \|\nabla A_h^{n-1}\|_{L^q} + C \|\nabla \varphi_{j,h}\|_{L^q} \|\Delta_h A_h^n\|_{L^2}, \tag{4.39}
\]
where we have used Sobolev embedding $L^q(\Omega) \hookrightarrow H^{-1}(\Omega)$ $(\forall q > 1)$ in the second to last
inequality, and Hölder’s inequality with $\frac{1}{q} = \frac{1}{2} + \frac{1}{2q/(2-q)}$ in the last inequality. In particular, for $1 < q < \frac{4}{3}$ we have $\frac{2q}{2q-4} < 4$ and so

$$\|u_h^{n-1}\|_{L^{\frac{2q}{2q-4}}} \leq C\|u_h^{n-1}\|_{L^2}^{\frac{2q}{2q-4}} \|\nabla u_h^{n-1}\|_{L^2}$$

$$\|\nabla A_h^{n-1}\|_{L^{\frac{2q}{2q-4}}} \leq C\|\nabla A_h^{n-1}\|_{L^2} \|\nabla A_h^{n-1}\|_{L^4}$$

$$\|\nabla \varphi_{j,h}^{n-1}\|_{L^{\frac{2q}{2q-4}}} \leq C.$$ 

Substituting the three inequalities above into (4.39) yields

$$\|g^n\|_{H^{-1}}$$

$$\leq C\|u_h^{n-1}\|_{L^2}^{\frac{2}{2q-4}} \|\nabla u_h^{n-1}\|_{L^2} + C\|u_h^{n-1}\|_{L^2} \|\nabla u_h^{n-1}\|_{L^2} + C\|\nabla u_h^{n-1}\|_{L^2} + C\|\nabla u_h^{n-1}\|_{L^2} + C\|\nabla u_h^{n-1}\|_{L^2}$$

$$\leq C\|u_h^{n-1}\|_{L^2} + C\|f^n\|_{H^{-1}} + (C\|\nabla A_h^{n-1}\|_{L^2} + C\|\nabla A_h^{n-1}\|_{L^2}) + C\|\nabla A_h^{n-1}\|_{L^2}$$

$$\leq C\|u_h^{n-1}\|_{L^2} + C\|f^n\|_{H^{-1}} + (C\|\nabla A_h^{n-1}\|_{L^2} + C\|\nabla A_h^{n-1}\|_{L^2}) + C\|\nabla A_h^{n-1}\|_{L^2}$$

(4.40)

for all $q \in (1, 4/3)$, where we have used (4.32) in the second to last inequality and Young’s inequality at the last step. For any $s \in (1, 2)$ we can choose $q$ to be sufficiently close to 1 so that $s(3 - \frac{2}{q}) \leq 2$ and $s(5 - \frac{2}{q}) \leq 2$, and therefore

$$\tau \sum_{n=1}^N \|g^n\|_{H^{-1}} \leq C \tau \sum_{n=1}^N \left( \|\nabla u_h^{n-1}\|_{L^2}^{s(2 - \frac{2}{q})} \|\nabla u_h^{n-1}\|_{L^2} + \|\nabla u_h^{n-1}\|_{L^2} \|\nabla u_h^{n-1}\|_{L^2} \right)$$

$$+ C \tau \sum_{n=1}^N \left( \|u_h^{n-1}\|_{H^1} + ||f^n||_{H^{-1}} + \|\Delta_h A_h^{n-1}\|_{L^2} \right) + C$$

$$\leq C \tau \sum_{n=1}^N \left( \|\nabla u_h^{n-1}\|_{L^2}^{s(3 - \frac{2}{q})} + \|\nabla u_h^{n-1}\|_{L^2}^{s(3 - \frac{2}{q})} + \|\nabla u_h^{n-1}\|_{L^2}^{3 - \frac{2}{q}} + \|\nabla u_h^{n-1}\|_{L^2}^{3 - \frac{2}{q}} \right)$$

$$+ C \tau \sum_{n=1}^N \left( \|u_h^{n-1}\|_{H^1} + ||f^n||_{H^{-1}} + \|\Delta_h A_h^{n-1}\|_{L^2} \right) + C$$

$$\leq C \tau \sum_{n=1}^N \left( \|\nabla u_h^{n-1}\|_{L^2} + \|\nabla u_h^{n-1}\|_{L^2}^{3 - \frac{2}{q}} + \|\nabla u_h^{n-1}\|_{L^2}^{3 - \frac{2}{q}} + \|\nabla u_h^{n-1}\|_{L^2}^{3 - \frac{2}{q}} \right)$$

$$+ C \tau \sum_{n=1}^N \left( \|u_h^{n-1}\|_{H^1} + ||f^n||_{H^{-1}} + \|\Delta_h A_h^{n-1}\|_{L^2} \right) + C$$

$$\leq C \quad \forall \; s \in (1, 2).$$

Substituting (4.41) into (4.38) yields

$$\tau \sum_{n=1}^N \left\| \frac{u_h^n - u_h^{n-1}}{\tau} \right\|_{H_{0,div}^s(\Omega)} \leq C \quad \forall \; s \in (1, 2).$$

(4.42)

Thus, (4.32), (4.34), and (4.42) imply the estimates (4.15) and (4.16). The proof of Proposition 4.11 is complete.

It is easy to see that the numerical scheme (4.31), (4.35) implies that the following equations hold

$$\int_0^T \left[ \left( \mu \delta_t A_h + (\sigma^{-1} \nabla A_h + \nabla a_h) - \left( \mu u_h^+ + (\nabla \times A_h^+ + \sum_{j=1}^m \beta_j \nabla \times \varphi_{j,h}) \right) \right) \right] dt.$$
To see this, we test where (4.47) is an obvious consequence of (4.46). Furthermore, we have
\(|\nabla \cdot \mathbf{u}_h| = 0\) where we have used (4.47). Since
\(h\) implies the existence of functions given in Proposition 4.1.

Let
\[4.3. \text{Compactness and convergence.} \]
In the next subsection we pass to the limit in (4.43)-(4.45) by using the energy estimates given in Proposition 4.1.

\[\int_0^T T \int_0^t (F, \mathbf{v}_h) + (\mu (\nabla \times A_h^+ + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}) \times \Delta_h A_h^+ , \mathbf{v}_h) dt = 0\]
for all \(a_h \in L^2(0, T; S_h^{r+1}), \mathbf{v}_h \in L^2(0, T; S_h^{r+1})\) and \(q_h \in L^2(0, T; S_h^r)\).

In the next subsection we pass to the limit in (4.43)-(4.45) by using the energy estimates given in Proposition 4.1.

\[\int_0^T (T \int_0^t (F, \mathbf{v}_h) dt + (\mu (\nabla \times A_h^+ + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}) \times \Delta_h A_h^+ , \mathbf{v}_h) dt = 0\]

For all \(a_h \in L^2(0, T; S_h^{r+1}), \mathbf{v}_h \in L^2(0, T; S_h^{r+1})\) and \(q_h \in L^2(0, T; S_h^r)\).

In the next subsection we pass to the limit in (4.43)-(4.45) by using the energy estimates given in Proposition 4.1.

\[\int_0^T (T \int_0^t (F, \mathbf{v}_h) dt + (\mu (\nabla \times A_h^+ + \sum_{j=1}^m \beta_{j,h} \nabla \times \varphi_{j,h}) \times \Delta_h A_h^+ , \mathbf{v}_h) dt = 0\]

For all \(a_h \in L^2(0, T; S_h^{r+1}), \mathbf{v}_h \in L^2(0, T; S_h^{r+1})\) and \(q_h \in L^2(0, T; S_h^r)\).
on the other hand, where we have used the convergence of $A$ for $A_{nm}$, it follows that the function $F \in L^2(0, T; L^2(\Omega))$ satisfies

$$
\int_0^T (F, v) dt = - \int_0^T (\nabla A, \nabla v) dt \quad \forall v \in L^2(0, T; H^1_0(\Omega)).
$$

This implies $\Delta A = F \in L^2(0, T; L^2(\Omega))$ and therefore $\Delta_h A_{nm} \rightarrow \Delta A$ weakly in $L^2(0, T; L^2(\Omega))$.

Moreover, the initial value of the limit function $A$ must be equal to $A_0$. This can be proved in the following way: on the one hand, for any given smooth function $a$ such that $a|_{t=T} = 0$,

$$
\int_0^T (\partial_h A_{nm, \tau_{nm}}, a) dt \rightarrow - \int_0^T (\partial_t A, a) dt \quad \text{as } m \rightarrow \infty;
$$

on the other hand,

$$
\int_0^T (\partial_h A_{nm, \tau_{nm}}, a) dt = - \int_0^T (A_{nm, \tau_{nm}}, \partial_t a) dt + (A_{nm}^0, a|_{t=0})
$$

$$
\rightarrow - \int_0^T (A, \partial_t a) dt + (A_0, a|_{t=0}) \quad \text{as } m \rightarrow \infty.
$$

where we have used the convergence of $A_{nm, \tau_{nm}}$ in (4.46) and the convergence of $A_{nm}^0$ to $A_0$ in $H^2_0(\Omega)$. The latter is a simple consequence of the convergence theory for the elliptic problem (2.13) and (3.7). Therefore,

$$
- \int_0^T (\partial_t A, a) dt = - \int_0^T (A_{nm, \tau_{nm}}, \partial_t a) dt + (A_{nm}^0, a|_{t=0}),
$$

which implies $A|_{t=0} = A_0$ in $L^2(\Omega)$. Similarly, we also have $u|_{t=0} = u_0$ in $H^1_{0, \text{div}}(\Omega)'$. This proves

$$
A|_{t=0} = A_0 \quad \text{and } u|_{t=0} = u_0. \quad (4.54)
$$

In Proposition 4.1 we have proved

$$
\|\Delta_h A_{nm, \tau_{nm}}\|_{L^2(0, T; L^2(\Omega))} + \|\partial_t A_{nm, \tau_{nm}}\|_{L^2(0, T; L^2(\Omega))} \leq C.
$$

Then Lemma 4.2 and the Aubin–Lions–Simon lemma (cf. [3, Theorem II.5.16]) imply that $A_{nm, \tau_{nm}}$ is compact in $L^2(0, T; W^{1,1+t}(\Omega))$, and thus one can choose the sequence to have the following property:

$$
A_{nm, \tau_{nm}} \rightarrow A \text{ strongly in } L^2(0, T; W^{1,1+t}(\Omega)). \quad (4.55)
$$

Since $A_{nm, \tau_{nm}} - A$ is bounded in $L^\infty(0, T; H^1(\Omega))$ and convergent to zero in $L^2(0, T; W^{1,1+t}(\Omega))$, and (interpolation inequality)

$$
\|A_{nm, \tau_{nm}} - A\|_{L^{q_0}(0, T; W^{1,1+t}(\Omega))} \leq C\|A_{nm, \tau_{nm}} - A\|_{L^\infty(0, T; H^1(\Omega))} \|A_{nm, \tau_{nm}} - A\|_{L^2(0, T; W^{1,1+t}(\Omega))}
$$

for $\frac{1}{2} + \frac{\theta}{4+t} = \frac{1}{q}$ and $\frac{1}{q_0} = \frac{1}{q} + \frac{\theta}{4+t}$, it follows that

$$
A_{nm, \tau_{nm}} \rightarrow A \text{ strongly in } L^{q_0}(0, T; W^{1,1+t}(\Omega)) \text{ for some } q_0 > 2. \quad (4.56)
$$

Since $H^1(\Omega) \hookrightarrow L^q(\Omega) \hookrightarrow H^1_{0, \text{div}}(\Omega)'$ for all $2 \leq q < \infty$ and

$$
\|u_{nm, \tau_{nm}}\|_{L^2(0, T; H^1(\Omega))} + \|\partial_t u_{nm, \tau_{nm}}\|_{L^2(0, T; H^1_{0, \text{div}}(\Omega)')} \leq C,
$$

the Aubin–Lions–Simon lemma (cf. [3, Theorem II.5.16]) implies that $u_{nm, \tau_{nm}}$ is compact in $L^2(0, T; L^2(\Omega))$ for all $2 \leq q < \infty$, and thus one can choose the subsequence to have the following property:

$$
\|u_{nm, \tau_{nm}}\|_{L^2(0, T; L^2(\Omega))} \leq C. \quad (4.57)
$$

The estimates above are for the piecewise linear functions $A_{nm, \tau_{nm}}^\pm$ and $u_{nm, \tau_{nm}}^\pm$. For the piecewise constant functions $A_{nm, \tau_{nm}}^\pm$ and $u_{nm, \tau_{nm}}^\pm$ we have similar estimates:

$$
A_{nm, \tau_{nm}}^\pm \rightarrow A \text{ weakly* in } L^\infty(0, T; H^1(\Omega)), \quad (4.58)
$$
\[ \nabla A^\pm \rightarrow \nabla A \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)), \quad (4.59) \]

\[ \Delta h A^\pm \rightarrow \Delta A \text{ weakly in } L^2(0, T; L^2(\Omega)), \quad (4.60) \]

\[ u^\pm \rightarrow u \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)), \quad (4.61) \]

\[ u^\pm \rightarrow u \text{ weakly in } L^2(0, T; H^1_0(\Omega)), \quad (4.62) \]

\[ A_{h,n,m} \rightarrow A \text{ strongly in } L^q(0, T; W^{1, q}(\Omega)) \text{ for some } q > 2, \quad (4.63) \]

\[ u_{h,n,m} \rightarrow u \text{ strongly in } L^2(0, T; L^4(\Omega)). \quad (4.64) \]

Since \( u_{h,n,m} - u \) is bounded in \( L^\infty(0, T; L^2(\Omega)) \) and convergent to zero in \( L^2(0, T; L^4(\Omega)) \), and (interpolation inequality)

\[ \| u_{h,n,m} - u \|_{L^4(0, T; L^\infty(\Omega))} \leq C \| u_{h,n,m} - u \|_{L^2(0, T; L^2(\Omega))} \| u_{h,n,m} - u \|_{L^2(0, T; L^4(\Omega))} \]

it follows that

\[ u_{h,n,m} \rightarrow u \text{ strongly in } L^2(0, T; L^\infty(\Omega)). \quad (4.65) \]

Then Lemma 4.3 and (4.62) and (4.63) imply

\[ u^+ \rightarrow (\nabla \times A^+_h + \sum_{j=1}^m \beta_j \nabla \times \varphi_j, h) \text{ converges to } u \times (\nabla \times A + \sum_{j=1}^m \beta_j \nabla \times \varphi_j) \text{ weakly in } L^p(0, T; L^2(\Omega)) \text{ for some } p > 1, \quad (4.66) \]

(4.62) and (4.63) imply

\[ u^- \cdot \nabla u^+ \rightarrow u \cdot \nabla u \text{ weakly in } L^\frac{4}{3}(0, T; L^\frac{2}{3}(\Omega)^2), \quad (4.67) \]

(4.61) and (4.64) imply

\[ u^- \otimes u^+ \rightarrow u \otimes u \text{ weakly in } L^2(0, T; L^\frac{4}{3}(\Omega)^2), \quad (4.68) \]

Lemma 4.3, (4.60) and (4.63) imply

\[ (\nabla \times A^+_h + \sum_{j=1}^m \beta_j \nabla \times \varphi_j, h) \times \Delta h A^+_h \rightarrow (\nabla \times A + \sum_{j=1}^m \beta_j \nabla \times \varphi_j) \times \Delta A \text{ weakly in } L^\frac{2}{3+p}(0, T; L^\frac{2}{3}(\Omega)^2). \quad (4.69) \]

For any given \( a \in L^\infty(0, T; L^2(\Omega)) \), its \( L^2 \) projection \( a_h = P_h a \) converges to \( a \) strongly in \( L^p(0, T; L^2(\Omega)) \) for the number \( p > 1 \) in (4.66).

Similarly, for any given \( v \in L^\infty(0, T; H^1_0(\Omega)) \), its Fortin projection \( v_h = Q_h v \) converges to \( v \) strongly in \( L^\infty(0, T; H^1_0(\Omega)) \) and satisfies

\[ (q_h, \nabla \cdot v_h) = 0 \text{ for all } q_h \in S^* \]

Then, by taking the limit of a subsequence in (4.63), we obtain

\[ \int_0^T \left[ (\mu \partial_t A, a) + \left( \sigma^{-1} \Delta A, a \right) - \left( \mu u \times (\nabla \times A + \sum_{j=1}^m \beta_j \nabla \times \varphi_j), a \right) \right] dt \]

\[ = \int_0^T \left( \sigma^{-1} J, a \right) dt \quad (4.70) \]

\[ \int_0^T \left[ (\partial_t u, v) + \frac{1}{2} (u \cdot \nabla u, v) - \frac{1}{2} (u \cdot \nabla v, u) + (v \nabla u, \nabla v) \right] dt \]

\[ = \int_0^T \left[ (f, v) + (\mu (\nabla \times A + \sum_{j=1}^m \beta_j \nabla \times \varphi_j) \times \Delta A, v) \right] dt \quad (4.71) \]

for all \( a \in L^\infty(0, T; L^2(\Omega)) \) and \( v \in L^\infty(0, T; H^1_0(\Omega)) \).

The convergence results (4.60) and (4.62) show that \( A \) and \( u \) possess the regularity in (2.13) - (2.15), except the regularity \( u \in L^2(0, T; H^1_0(\Omega)) \).

Finally, we show that \( u \in L^2(0, T; H^1_0(\Omega)) \). In fact, for any given \( q \in L^2(0, T; L^2(\Omega)) \), its \( L^2 \) projection \( q_h = \bar{P}_h q \) converges to \( q \) strongly in \( L^2(0, T; L^2(\Omega)) \). By taking limit in (4.25) we obtain

\[ \int_0^T (\nabla \cdot u, q) dt = 0 \]

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where \((r, \theta)\) denotes the polar coordinates and \(\Phi(r)\) is the unique \(7^{th}\) order polynomial satisfying the conditions \(\Phi'(0.1) = \Phi''(0.1) = \Phi''(0.4) = \Phi'(0.4) = 0\) and \(\Phi(0.1) = 1\). The constructed solutions in (5.4)-(5.5) satisfy the boundary conditions
\[
\mathbf{H} \cdot \mathbf{n} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}, \quad \text{and} \quad \mathbf{u} = 0 \quad \text{on} \quad \Omega \times (0, T) .
\]

We solve the equations (5.1)-(5.3) up to time \(T = 1\) by the proposed numerical method (3.1)-(3.5) with \(r = 1\), i.e., with the P2 element for \(A\) and P2-P1 elements for \((u, p)\). The numerical solution of magnetic field is given by \(\mathbf{H}_h^n = \nabla \times \mathbf{A}_h^n\). Since the L-shape domain is simply connected, it follows that \(m = 0\) (the constants \(\beta_j, j = 1, \ldots, m\), are not needed). The L-shape domain is triangulated quasi-uniformly with \(M\) nodes per unit length on each side, and we denote by \(h = 1/M\) for simplicity.

We compare the numerical solutions with the exact solution given by (5.4)-(5.5) and present the errors of the numerical solutions in Table 5.1. For comparison, we also present

| Example 5.1 | In this section we present two numerical examples to illustrate the convergence of the proposed numerical method in nonconvex and nonsmooth domains.

Example 5.1. In the first example, we consider the MHD equations
\[
\begin{align*}
\partial_t \mathbf{H} + \nabla \times (\nabla \times \mathbf{H}) - \nabla \times (\mathbf{u} \times \mathbf{H}) &= \nabla \times \mathbf{J} \quad (5.1) \\
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p &= \mathbf{f} - \mu \mathbf{H} \times (\nabla \times \mathbf{H}) \quad (5.2) \\
\nabla \cdot \mathbf{u} &= g \quad (5.3)
\end{align*}
\]
in a simply connected L-shape domain \(\Omega\) whose longest side has unit length, centered at the origin; see Figure 5.1 (left). The source terms \(f, g\) and
\[J = \partial_t u_1 + \nabla \times \mathbf{H} - \mathbf{u} \times \mathbf{H}\]
are calculated by substituting the following exact solution into (5.1)-(5.3):
\[
\mathbf{u} = \begin{pmatrix}
  u_1 \\
  u_2 
\end{pmatrix} = \begin{pmatrix}
  t^2 \Phi(r)r^{2/3} \sin(2\theta/3) \\
  t^2 \Phi(r)r^{2/3} \sin(2\theta/3)
\end{pmatrix}, \quad p = 0, \quad (5.4)
\]
\[
\mathbf{H} = \nabla \times \mathbf{u}_1 = \begin{pmatrix}
  2t^2 \Phi(r)r^{-1/3} \cos(\theta/3) + t^2 \Phi'(r)r^{2/3} \sin(2\theta/3) \sin(\theta) \\
  2t^2 \Phi(r)r^{-1/3} \sin(\theta/3) - t^2 \Phi'(r)r^{2/3} \sin(2\theta/3) \cos(\theta)
\end{pmatrix}, \quad (5.5)
\]
where \((r, \theta)\) denotes the polar coordinates and \(\Phi(r)\) is a \(C^3(\Omega)\) cut-off function. Here,
\[
\Phi(r) = \begin{cases}
  0.1 & \text{if } r < 0.1 \\
  \Upsilon(\tau) & \text{if } 0.1 \leq r \leq 0.4 \\
  0 & \text{if } r > 0.4
\end{cases}
\]
and \(\Upsilon(\tau)\) is the unique \(7^{th}\) order polynomial satisfying the conditions \(\Upsilon'(0.1) = \Upsilon''(0.1) = \Upsilon''(0.4) = \Upsilon'(0.4) = \Upsilon''(0.4) = 0\) and \(\Upsilon(0.1) = 1\). The constructed solutions in (5.4)-(5.5) satisfy the boundary conditions
\[
\mathbf{H} \cdot \mathbf{n} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}, \quad \text{and} \quad \mathbf{u} = 0 \quad \text{on} \quad \Omega \times (0, T). \quad (5.6)
\]
the numerical results of a “direct $H^1$-conforming FEM” in Table 5.2 with the same time-stepping scheme as the proposed method (3.1)-(3.5). In particular, the direct $H^1$-conforming FEM seeks $H^n_h \in S^2_h, u^n_h \in S^2_h$ and $p^n_h \in S^1_h$, with $H^n_h \cdot n = 0$ on $\partial \Omega$, such that the following equations hold for all test functions $a_h \in S^2_h, v_h \in S^2_h$ and $q_h \in S^1_h$ with $a_h \cdot n = 0$ on $\partial \Omega$:

$$
\begin{aligned}
\left( \frac{H^n_h - H^{n-1}_h}{\tau}, a_h \right) + (\sigma^{-1} \nabla \times H^n_h, \nabla \times a_h) - (\mu u^n_h \times H^{n-1}_h, a_h) \\
= (\sigma^{-1} J^n, \nabla \times a_h)
\end{aligned}
$$

(5.7)

$$
\begin{aligned}
\left( \frac{u^n_h - u^{n-1}_h}{\tau}, v_h \right) + \frac{1}{2}(u^{n-1}_h \cdot \nabla u^n_h, v_h) - \frac{1}{2}(u^{n-1}_h \cdot \nabla v^n_h, u^n_h) + (\nu \nabla u^n_h, \nabla v^n_h) - (p^n_h, \nabla \cdot v_h) \\
= (f^n, v_h) + (\mu H^{n-1}_h \times (\nabla \times H^n_h), v_h)
\end{aligned}
$$

(5.8)

$$
(\nabla \cdot u^n_h, q_h) = (g^n, q_h).
$$

(5.9)

The numerical results in Tables 5.1 and 5.2 show that the proposed method has slightly higher accuracy than the $H^1$-conforming FEM in computing the magnetic field $H$ (with the same degree of finite elements and similar computational complexity). The convergence of the proposed numerical method is proved in Theorem 3.1 (also see Remark 3.1), while the convergence of the direct $H^1$-conforming FEM remains open in nonconvex and nonsmooth domains.

**Table 5.1**

| $h$   | $\| H^n_h - H(\cdot, t_N) \|_{L^2}$ | $\| u^n_h - u(\cdot, t_N) \|_{L^2}$ |
|-------|-----------------------------------|-----------------------------------|
| 1/16  | 4.265E-03                         | 6.272E-05                         |
| 1/32  | 1.949E-03                         | 2.126E-05                         |
| 1/64  | 1.106E-03                         | 8.987E-06                         |
| 1/128 | 7.043E-04                         | 3.801E-06                         |
| convergence rate | $O(h^{1/2})$ | $O(h^{1/2})$ |

**Table 5.2**

| $h$   | $\| H^n_h - H(\cdot, t_N) \|_{L^2}$ | $\| u^n_h - u(\cdot, t_N) \|_{L^2}$ |
|-------|-----------------------------------|-----------------------------------|
| 1/16  | 1.302E-02                         | 6.058E-05                         |
| 1/32  | 7.704E-03                         | 2.133E-05                         |
| 1/64  | 4.860E-03                         | 8.371E-06                         |
| 1/128 | 3.032E-03                         | 3.730E-06                         |
| convergence rate | $O(h^{1/2})$ | $O(h^{1/2})$ |

**Example 5.2.** In the second example, we consider the MHD equations (5.1)-(5.3) in a multi-connected domain shown in Figure 5.1 (right), with the source terms $f, g$ and $J = \partial_t u_1 + \nabla \times H - u \times H$
determined by the exact solution

$$
\begin{aligned}
\mathbf{u} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (r^2 \Phi(r))^{2/3} \sin(2\theta/3) \\ (r^2 \Phi(r))^{2/3} \sin(2\theta/3) \end{pmatrix}, \\
p &= 0,
\end{aligned}
$$

(5.10)

$$
\begin{aligned}
\mathbf{H} &= \nabla \times u_1 + \nabla \times \varphi,
\end{aligned}
$$

(5.11)
where $\varphi$ is the harmonic function satisfying
\[
\begin{align*}
\Delta \varphi &= 0 \quad \text{in } \Omega \\
\varphi &= 1 \quad \text{on the inner boundary} \\
\varphi &= 0 \quad \text{on the outer boundary.}
\end{align*}
\]
(5.12)

In this case $m = 1$ and $\beta_1 = 1$.

We solve the MHD equations up to time $T = 1$ by the proposed numerical method (3.1)-(3.5) with the P2 element for $A$ and P2-P1 elements for $(u, p)$, and compare the numerical solutions with the exact solution given by (5.10)-(5.11) (where $\varphi$ is approximated numerically). The domain is triangulated quasi-uniformly, with $M$ nodes per unit length on each side, and we denote by $h = 1/M$ for simplicity. The errors of the numerical solutions are presented in Table 5.3. For comparison, we also present the numerical results of the direct $H^1$-conforming FEM (5.7)-(5.9) with the P2 element for $H$ and P2-P1 elements for $(u, p)$ in Table 5.4. Numerical results in Tables 5.3-5.4 show that the proposed method is much more accurate than the direct $H^1$-conforming FEM in such a multi-connected nonsmooth domain. The reason may be that the proposed method approximates the harmonic part $\nabla \times \varphi$ at the initial time in a more accurate way than the direct $H^1$-conforming FEM, which cannot separate $\nabla \times \varphi$ from $\mathbf{H}$.

| $h$   | $\| \mathbf{H}_h^N - \mathbf{H}(\cdot, t_N) \|_{L^2}$ | $\| u_h^N - u(\cdot, t_N) \|_{L^2}$ |
|-------|-------------------------------------------------|---------------------------------|
| 1/32  | 1.911E-03                                       | 2.146E-05                       |
| 1/64  | 1.114E-03                                       | 8.857E-06                       |
| 1/128 | 6.806E-04                                       | 3.863E-06                       |
| 1/256 | 4.226E-04                                       | 1.935E-06                       |
| convergence rate | $O(h^{0.68})$ | $O(h^{0.99})$ |

| $h$   | $\| \mathbf{H}_h^N - \mathbf{H}(\cdot, t_N) \|_{L^2}$ | $\| u_h^N - u(\cdot, t_N) \|_{L^2}$ |
|-------|-------------------------------------------------|---------------------------------|
| 1/32  | 2.932E-01                                       | 1.260E-04                       |
| 1/64  | 1.790E-01                                       | 1.218E-04                       |
| 1/128 | 1.113E-01                                       | 1.227E-04                       |
| 1/256 | 7.386E-02                                       | 1.237E-04                       |
| convergence rate | $O(h^{0.39})$ | $O(h^{0.99})$ |

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