Derived equivalence classification of one-parametric selfinjective algebras

Rafał Bocian\textsuperscript{a}, Thorsten Holm\textsuperscript{b} and Andrzej Skowroński\textsuperscript{a,∗}

\textsuperscript{a}Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

\textsuperscript{b}Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom

\textit{Dedicated to Claus Michael Ringel on the occasion of his sixtieth birthday}

Abstract

In continuation of our paper [7] we complete the description of the derived equivalence normal forms of all one-parametric selfinjective algebras over algebraically closed fields which admit simply connected Galois coverings. As a consequence, a description of the stable equivalence normal forms of these algebras is obtained.

\textit{2000 MSC:} Primary: 16G10, 18E30; Secondary: 16D50, 16G60, 16G70.

\textit{Key Words and Phrases.} Selfinjective algebras, derived equivalence, stable equivalence, tame representation type.

0. Introduction and the main result.

Throughout the paper $K$ will denote a fixed algebraically closed field. By an algebra we mean a finite dimensional $K$-algebra (associative, with an identity), which we shall assume (without loss of generality) to be basic and connected. An algebra $A$ can be written as a bound quiver algebra $A \cong KQ/I$, where $Q = Q_A$ is the Gabriel quiver of $A$ and $I$ is an admissible ideal in the path algebra $KQ$ of $Q$. An algebra $A$ is called selfinjective if the projective $A$-modules are injective.

From Drozd’s Tame and Wild Theorem [14] the class of algebras may be divided into two disjoint classes. One class consists of the tame algebras for which the indecomposable
modules occur, in each dimension \(d\), in a finite number of discrete and a finite number of one-parametric families. The second class is formed by the wild algebras whose representation theory comprises the representation theories of all finite dimensional \(K\)-algebras. Accordingly we may realistically hope to classify the indecomposable finite dimensional modules only for the tame algebras. A special class of tame algebras is formed by the algebras of finite representation type having only finitely many isomorphism classes of indecomposable finite dimensional modules. The representation theory of algebras of finite representation type is presently well understood, and in particular the Morita equivalence classes of all selfinjective algebras of finite representation type are classified (see [12], [22], [30], [31], [37]). The representation theory of arbitrary tame algebras is still only emerging.

For an algebra \(A\), we denote by \(\text{mod} \ A\) the category of finite dimensional left \(A\)-modules and by \(\underline{\text{mod}} \ A\) its stable category. Recall that the objects of \(\underline{\text{mod}} \ A\) are the objects of \(\text{mod} \ A\) without nonzero projective direct summands, and for any two objects \(M\) and \(N\) in \(\text{mod} \ A\) the \(K\)-module \(\text{Hom}_A (M, N)\) of morphisms from \(M\) to \(N\) is the quotient \(\text{Hom}_A (M, N) / P(M, N)\), where \(P(M, N)\) is the submodule of \(\text{Hom}_A (M, N)\) consisting of all \(A\)-homomorphisms which factorize through a projective \(A\)-module. Moreover, we denote by \(D^b (\text{mod} \ A)\) the derived category of bounded complexes of modules from \(\text{mod} \ A\) [20]. Two algebras \(A\) and \(B\) are said to be \textit{stably equivalent} if the stable module categories \(\underline{\text{mod}} \ A\) and \(\underline{\text{mod}} \ B\) are equivalent. Finally, two algebras \(A\) and \(B\) are said to be \textit{derived equivalent} if the derived categories \(D^b (\text{mod} \ A)\) and \(D^b (\text{mod} \ B)\) are equivalent as triangulated categories. Since Happel’s work [19] interpreting tilting theory in terms of equivalences of derived categories, the machinery of derived categories has been of interest to representation theorists. In [28] J. Rickard proved his celebrated criterion: two algebras \(A\) and \(B\) are derived equivalent if and only if \(B\) is the endomorphism algebra of a tilting complex over \(A\). Since a lot of interesting properties are preserved by derived equivalences of algebras, it is for many purposes important to classify classes of algebras up to derived equivalence, instead of Morita equivalence. For instance, for selfinjective algebras the representation type is an invariant of the derived category. In fact, derived equivalent selfinjective algebras are stably equivalent [29], and for any algebra \(A\) a stable equivalence preserves the representation type [24], [25]. Finally, we note that derived equivalent algebras have the same number of pairwise nonisomorphic simple modules. The derived equivalence classification has been established for some distinguished classes of tame selfinjective algebras (see [2], [6], [7], [21], [23], [27], [29] for some work in this direction).

We are concerned with the problem of the derived equivalence classification of one-parametric selfinjective algebras. Recall that an algebra \(A\) of infinite representation type is called \textit{one-parametric} if there exists a \(A\)-\(K[x]\)-bimodule \(M\) which is finitely generated and free as right \(K[x]\)-module and, for any dimension \(d\), all but a finite number of isomorphism classes of indecomposable (left) \(A\)-modules of dimensional \(d\) are of the form \(M \otimes K[x] / (x - \lambda)^m\) for some \(\lambda \in K\) and some \(m \geq 1\). We also mention that the class of one-parametric algebras coincides with the class of algebras having exactly one generic module [13]. By general theory, the class of one-parametric selfinjective algebras splits into two classes: the \textit{standard algebras}, having simply connected Galois coverings, and the remaining \textit{nonstandard algebras}. It is expected that the nonstandard one-parametric (even
the representation-infinite domestic) selfinjective algebras occur only in characteristic 2 and are geometric and socle deformations of standard one-parametric selfinjective algebras.

By general theory \cite{35}, \cite{26}, the class of standard one-parametric selfinjective algebras coincides with the class of selfinjective algebras of the form $\hat{B}/G$, where $\hat{B}$ is the repetitive algebra of a tilted algebra $B$ of Euclidean type $\tilde{A}_m$ or $\tilde{D}_n$, and $G$ is an infinite cyclic group generated by an automorphism $\varphi$ of $\hat{B}$ whose square $\varphi^2$ is a twist $\rho\nu_{\hat{B}}$ of the Nakayama automorphism $\nu_{\hat{B}}$ by a rigid automorphism $\rho$ of $\hat{B}$. We also note that the stable Auslander-Reiten quiver of a standard one-parametric selfinjective algebra consists of one Euclidean component $\mathbb{Z}\Delta$ of type $\Delta \in \{\tilde{A}_m, \tilde{D}_n\}$ and a $\mathbb{P}_1(K)$-family of stable tubes. In \cite{9}, \cite{10}, \cite{11} we classified the Morita equivalence classes of all standard one-parametric selfinjective algebras by algebras arising from Brauer graphs. Moreover, the derived (respectively, stable) equivalence normal forms of all standard one-parametric weakly symmetric algebras has been established in our paper \cite{7}. Recall that a selfinjective algebra $A$ is called weakly symmetric if the socle $\text{soc} \ P$ of any indecomposable projective $A$-module $P$ is isomorphic to its top $P/\text{rad} \ P$.

The main aim of the paper is to establish the derived (respectively, stable) equivalence normal forms of all standard one-parametric algebras which are not weakly symmetric, using the Morita equivalence classification of this class of algebras established in \cite{11}. For $\theta \in K \setminus \{0\}$, we associate to a Brauer graph $T_{p,q,k}$ of the form

![Brauer graph](image_url)

with $p, q \geq 0$ and $k \geq 2$, and a positive integer $s$ with $1 \leq s \leq k-1$, $\gcd(s+2,k) = 1$, $\gcd(s,k) = 1$, a standard one-parametric selfinjective algebra $\Lambda(p,q,k,s,\theta)$ of Euclidean type $\tilde{A}_m$ (with $m = 2(p + q + 1)k - 1$). Furthermore, to a Brauer tree $T_n$ of the form

![Brauer tree](image_url)
we associate a standard one-parametric selfinjective algebra $\Gamma^*(n)$ of Euclidean type $\tilde{D}_n$. See Section 1 below for the precise definitions.

The following theorem is the main result of the paper.

**Theorem 1.** For a standard selfinjective algebra $A$ the following statements are equivalent:

1. $A$ is one-parametric but not weakly symmetric.
2. $A$ is derived equivalent to an algebra of the form $\Lambda(p,q,k,s,\theta)$ or $\Gamma^*(n)$.
3. $A$ is stably equivalent to an algebra of the form $\Lambda(p,q,k,s,\theta)$ or $\Gamma^*(n)$.

We note that the algebras $\Lambda(p,q,k,s,\theta)$ and $\Gamma^*(n)$ are never stably equivalent, because their stable Auslander-Reiten quivers are not isomorphic (see Propositions 7.1 and 7.2). Moreover, by Proposition 7.1 the algebras $\Gamma^*(m)$ and $\Gamma^*(n)$ are stably equivalent only for $m = n$. Finally, we refer to Proposition 10.1 for a necessity criterion for the stable equivalence of algebras of the form $\Lambda(p,q,k,s,\theta)$.

As an application of Theorem 1 and results of our earlier papers [7] and [8] we obtain the following description of Auslander’s representation dimension $\text{repdim}(A)$ and Rouquier’s dimension $\dim\text{mod}\ A$ of the triangulated category $\text{mod}\ A$ (see Section 8 for the relevant definitions) of arbitrary standard one-parametric selfinjective algebras $A$.

**Theorem 2.** Let $A$ be a standard one-parametric selfinjective algebra. Then $\text{repdim}(A) = 3$ and $\dim\text{mod}\ A = 1$.

The paper is organized as follows. In Section 1 we recall the Morita equivalence classification of all standard one-parametric selfinjective but not weakly symmetric algebras. Sections 2, 3, 4, and 5 are devoted to the proof of the implication (1) $\implies$ (2) of Theorem 1 in the Euclidean case $\tilde{A}_m$, and Section 6 to the proof of this implication in the Euclidean case $\tilde{D}_n$. In Section 7 we describe the stable Auslander-Reiten quivers of algebras occurring in Theorem 1. Section 8 is devoted to a class of weakly symmetric algebras needed in the proof of implication (3) $\implies$ (1) of Theorem 1. In Section 9 we complete the proof of Theorem 1. In Section 10 we discuss the stable equivalence of algebras occurring in Theorem 1. The final Section 11 is devoted to the proof of Theorem 2.

For basic background on the representation theory of algebras we refer to [5], [32], and for background on selfinjective algebras to [15], [39].
1. One-parametric nonweakly symmetric selfinjective algebras

We describe in this section the Morita equivalence classification of all standard one-parametric but not weakly symmetric selfinjective algebras established in [11].

A Brauer graph $T$ is a finite connected undirected graph, where for each vertex there is a fixed circular order on the edges adjacent to it (see [1], [23], [27], [33]). In our context we assume that $T$ has at most one cycle (which may be or may not be a loop). We draw $T$ in a plane and agree that the edges adjacent to a given vertex are clockwise ordered. Given a Brauer graph $T$, this defines a Brauer quiver $Q_T$ as follows. The vertices of $Q_T$ are the edges of $T$ and there is an arrow $i \to j$ in $Q_T$ if and only if $j$ is the direct successor of $i$ in the order around some vertex (to which $i$ and $j$ are both adjacent). We require that every vertex of $Q_T$ belongs to exactly two cycles. Note that this implicitly means that, for every end vertex of $T$, there is a loop in $Q_T$.

Let $T$ be a Brauer graph with exactly one cycle $\mathcal{R}_k$, having $k \geq 2$ edges. Let $v_1, v_2, \ldots, v_k$ be the vertices of $\mathcal{R}_k$ and $e_i = \{v_i, v_{i+1}\}$, $i = 1, 2, \ldots, k$, where $v_{k+1} = v_1$, the edges of $\mathcal{R}_k$. If $v$ is a vertex of the Brauer graph $T$ which is not a vertex of the cycle $\mathcal{R}_k$ then by $n(v)$ we denote the edge incident to $v$ on the unique walk in $T$ from $v$ to the cycle $\mathcal{R}_k$. Moreover, for $i = 1, 2, \ldots, k$, we denote by $n(v_i)$ the edge $e_i$. For a vertex $v$ of the graph $T$, we denote by $l(\mathcal{R}_k, v)$ the distance of $v$ to the cycle $\mathcal{R}_k$. Hence $l(\mathcal{R}_k, v) = 0$ if and only if $v$ belongs to $\mathcal{R}_k$. By an automorphism of the Brauer graph $T$ we mean an automorphism of the graph $T$ which preserves the fixed circular order on the edges adjacent to any vertex. A rotation of the Brauer graph $T$ is an automorphism $\sigma$ of the Brauer graph $T$ such that, for some integer $s$ with $1 \leq s \leq k - 1$, we have $\sigma(v_i) = v_{i+s}$ for all $i = 1, 2, \ldots, k$ (where $k + r = r$ for $r \geq 1$), and then we set $\sigma = \sigma_s$. For $k = 2$, we set $\sigma_1(e_1) = e_2$ and $\sigma_1(e_2) = e_1$.

Assume that $s$ is a positive integer such that $1 \leq s \leq k - 1$ and $\gcd(s+2,k) = 1$. We shall define a generalized Brauer quiver $Q(T, \sigma_s)$, obtained from the usual Brauer quiver $Q_T$ of the Brauer graph $T$ by shifting some arrows of $Q_T$ using the rotation $\sigma_s$ of $T$. By a $\sigma_s$-orbit of a vertex $v$ of $T$ we mean the orbit of $v$ with respect to the action of the cyclic group generated by $\sigma_s$ on the vertices of $T$. We note that if two vertices $v$ and $w$ of $T$ belong to the same $\sigma_s$-orbit then $l(\mathcal{R}_k, v) = l(\mathcal{R}_k, w)$. Moreover, all $\sigma_s$-orbits of vertices of $T$ have the same number of elements, namely $k/d$, where $d = \gcd(s, k)$. For $m \geq 0$, denote by $V_m$ the set of all vertices of $T$ with $l(\mathcal{R}_k, v) = m$. Observe that $V_m$ is a disjoint union of $\frac{dV_m}{k}$ $\sigma_s$-orbits.

In order to define the generalized Brauer quiver $Q(T, \sigma_s)$, we introduce an order $p(T, \sigma_s)$ of the edges of the Brauer graph $T$, as the union of $\sum_{m=0}^{\infty} \frac{dV_m}{k}$ cyclic orders $p(T, \sigma_s, v)$ defined for the representatives $v$ of all pairwise different $\sigma_s$-orbits of vertices of $T$. Let $v$ be a vertex of $T$. We define the cyclic order $p(T, \sigma_s, v)$ by invoking the cyclic orders of edges around the vertices $v, \sigma_s(v), \ldots, \sigma_s^{k/d-1}(v)$ in the Brauer graph $T$. Let $r \in \{0, 1, \ldots, k/d - 1\}$, let $i$ be an edge of $T$ adjacent to the vertex $\sigma_s^r(v)$, and let $j$ be the direct successor of $i$ in the cyclic order in $T$ around $\sigma_s^r(v)$. If $j \neq n(\sigma_s^r(v))$, then $j$ is defined to be the direct successor of $i$ in the cyclic order $p(T, \sigma_s, v)$. If $j = n(\sigma_s^r(v))$
then \((\sigma_s^{r+1}(v)) = \sigma_s(n(\sigma_s^r(v)))\) is said to be the direct successor of \(i\) in the cyclic order \(p(T, \sigma_s, v)\). Therefore, we replaced the cyclic orders around the vertices \(\sigma_s^r(v)\), \(0 \leq r \leq k/d - 1\), by one (bigger) cyclic order \(p(T, \sigma_s, v)\). Observe also that if \(e = \{v, w\}\) is an edge of \(T\) which is not on the cycle \(R_k\), or if \(e\) is on the cycle \(R_k\) and \(d > 1\), then \(e\) belongs to exactly two cyclic orders, namely \(p(T, \sigma_s, v)\) and \(p(T, \sigma_s, w)\). On the other hand, if \(e = \{v, w\}\) is an edge of the cycle \(R_k\) and \(d = 1\), then \(e\) occurs twice in the cyclic order \(p(T, \sigma_s, v) = p(T, \sigma_s, w)\).

We define the generalized Brauer quiver \(Q(T, \sigma_s)\) as follows. The vertices of \(Q(T, \sigma_s)\) are the edges of \(T\) and there is an arrow \(i \rightarrow j\) in \(Q(T, \sigma_s)\) if and only if \(j\) is the direct successor of \(i\) in the order \(p(T, \sigma_s)\).

For \(\theta \in K \setminus \{0\}\), we define the algebra \(\Omega^{(1)}(T, \sigma_s, \theta)\) as the bound quiver algebra \(KQ(T, \sigma_s)/I^{(1)}(T, \sigma_s, \theta)\), where \(KQ(T, \sigma_s)\) is the path algebra of the quiver \(Q(T, \sigma_s)\) and \(I^{(1)}(T, \sigma_s, \theta)\) is the ideal in \(KQ(T, \sigma_s)\) generated by the elements:

1. \(\alpha \beta\) where \(\alpha = i_1 \rightarrow i_2\), \(\beta = i_2 \rightarrow i_3\) and \(i_1, i_2, i_3\) are not consecutive elements in the cyclic order \(p(T, \sigma_s)\).
2. \(C(i, p(T, \sigma_s, v)) - C(i, p(T, \sigma_s, w)), \) for \(i \neq e_1\) and \(C(e_1, p(T, \sigma_s, v)) - \theta C(e_1, p(T, \sigma_s, w)), \)

where \(i = \{v, w\}\) is an edge of \(T\), \(C(i, p(T, \sigma_s, v))\) and \(C(i, p(T, \sigma_s, w))\) are the paths from \(i\) to \(\sigma_s(i)\) in the quiver \(Q(T, \sigma_s)\), corresponding to the consecutive elements \(i, \ldots, \sigma_s(i)\) of the cyclic orders \(p(T, \sigma_s, v)\) and \(p(T, \sigma_s, w)\), respectively.

If \(T = T_{p,q,k}\) is of the form \((\ast)\) and \(\gcd(s, k) = 1\), then we denote \(\Omega^{(1)}(T, \sigma_s, \theta)\) by \(\Lambda(p, q, k, \theta)\).

**Example 1.1.** Let \(s = 2\) and \(T_{2,1,3}\) be the following Brauer graph

```
Then the order \(p(T_{2,1,3}, \sigma_2)\) is the union of the four cycles:
(1) 4, 5, 3, 12, 2, 8, 9, 2, 11, 1, 6, 7, 1, 10, 3,
(2) 4, 8, 6,
(3) 5, 9, 7,
(4) 10, 12, 11.
```
Then the generalized Brauer quiver $Q(T_{2,1,3}, \sigma_2)$ of the graph $T_{2,1,3}$ is of the form

![Diagram of the generalized Brauer quiver](image)

and $\Lambda(2, 1, 3, 2, \theta)$ is given by the above quiver and the ideal $T^{(i)}(T_{2,1,3}, \sigma_2, \theta)$ in $KQ(T_{2,1,3}, \sigma_2)$ generated by the elements: $\alpha_1\beta_{10}, \alpha_2\beta_{11}, \alpha_3\beta_{12}, \beta_{10}\alpha_{12}, \beta_{12}\alpha_{11}, \gamma_1\beta_6, \beta_6\gamma_4, \gamma_3\beta_4, \beta_4\gamma_8, \gamma_2\beta_8, \beta_8\gamma_6, \gamma_4\beta_5, \beta_5\gamma_9, \gamma_6\beta_7, \beta_7\gamma_5, \gamma_8\beta_9, \beta_9\gamma_7, \alpha_{11}\alpha_1, \alpha_{10}\alpha_3, \alpha_{12}\alpha_2, \gamma_6\gamma_2, \gamma_5\gamma_3, \gamma_7\gamma_1, \alpha_1\alpha_{10}\gamma_3\gamma_5 - \theta \gamma_7 \gamma_6 \gamma_7 \alpha_1 \alpha_{10}, \alpha_2 \alpha_{11} \gamma_1 \gamma_6 \gamma_7 - \gamma_2 \gamma_8 \gamma_9 \alpha_2 \alpha_{11}, \alpha_3 \alpha_{12} \gamma_7 \gamma_8 \gamma_9 - \gamma_3 \gamma_4 \gamma_5 \alpha_3 \alpha_{12}, \alpha_{10} \gamma_3 \gamma_4 \gamma_5 \alpha_3 - \beta_{10}, \alpha_{11} \gamma_1 \gamma_6 \gamma_7 \alpha_1 - \beta_{11}, \alpha_{12} \gamma_2 \gamma_8 \gamma_9 \alpha_2 - \beta_{12}, \gamma_4 \gamma_5 \alpha_3 \alpha_{12} \gamma_2 \gamma_8 \gamma_9 - \beta_5, \gamma_6 \gamma_7 \alpha_1 \alpha_{10} \gamma_3 - \beta_6, \gamma_7 \alpha_1 \alpha_{10} \gamma_3 \gamma_4 - \beta_7, \gamma_8 \gamma_9 \alpha_2 \alpha_{11} \gamma_1 - \beta_8, \gamma_9 \alpha_2 \alpha_{11} \gamma_1 \gamma_6 - \beta_9.$

Let $T$ be a Brauer tree. Then the simple cycles of the Brauer quiver $Q_T$ may be divided into two camps, $\alpha$-camps and $\beta$-camps, in such a way that any two cycles which intersect nontrivially belong to different camps. We denote by $\alpha_i$ (respectively, $\beta_i$) the arrow of the $\alpha$-camp (respectively, $\beta$-camp) of $Q_T$ starting at a vertex $i$, and by $\alpha(i)$ (respectively, $\beta(i)$) the end vertex of $\alpha_i$ (respectively, $\beta_i$). We also denote by $A_i$ (respectively, $B_i$) the cycle from $i$ to $i$ going once around the $\alpha$-cycle (respectively, $\beta$-cycle) through $i$.

Let $T$ be a Brauer tree with two (different) distinguished vertices $v_1$ and $v_2$ such that $v_1$ is the end of exactly one edge $a$. Let the edge $b$ be the direct successor of the edge $a$ and let $c$ be the direct predecessor of the edge $a$ in the cyclic order of edges at the end vertex $u$ of $a$ different from $v_1$. The vertices $v_1, v_2$ and edges $b, c$ determine a subtree

![Diagram of the subtree](image)

of the Brauer tree $T$, where possibly $u = v_2$, $v_2 = v_3$, $b = e$, $c = e$, $b = c = e$, but every time $a \neq b$ and $a \neq c$. We assume that the Brauer quiver $Q_T$ has exactly one exceptional
cycle (with multiplicity two) given by the edges of $T$ converging at the exceptional vertex $v_2$. Moreover, we assume that the cycle in $Q_T$ corresponding to the vertex $u$ is an $\alpha$-cycle.

We define the algebra $\Omega^{(2)}(T, v_1, v_2)$ as the bound quiver algebra $KQ_T^2 / \mathcal{I}^{(2)}(T, v_1, v_2)$, where $KQ_T^2$ is the path algebra of the quiver

$$Q_T^2 = ((Q_T)_0 \cup \{w\}, (Q_T)_1 \cup \{\gamma_1 : c \rightarrow w, \gamma_2 : w \rightarrow b\} \setminus \{\beta_a : a \rightarrow a\})$$

and $\mathcal{I}^{(2)}(T, v_1, v_2)$ is the ideal in $KQ_T^2$ generated by the elements:

1. $\alpha_i \beta_{\alpha(i)}$, for all vertices $i$ of $Q_T$ different from $c$,
2. $\beta_i \alpha_{\beta(i)}$, for all vertices $i$ of $Q_T$ different from $a$,
3. $A_j - B_j$, if the both $\alpha$-cycle and $\beta$-cycle through the vertex $j$ are not exceptional,
4. $A_j^2 - B_j$, if the $\alpha$-cycle through the vertex $j$ is exceptional but the $\beta$-cycle through $j$ is not exceptional,
5. $A_j - B_j^2$, if the $\alpha$-cycle through the vertex $j$ is not exceptional but the $\beta$-cycle through the vertex $j$ is exceptional,
6. $\gamma_2 \beta_b, \beta_{\beta_1(c)} \gamma_1$,
7. $\gamma_2 \alpha_b \ldots \alpha_{\alpha^{-1}(c)} \gamma_1, A_a (\gamma_2 \gamma_1, A_a, i f b = c = e)$, if the $\alpha$-cycle through the vertex $a$ is not exceptional,
8. $\gamma_2 A_b \alpha_b \ldots \alpha_{\alpha^{-1}(c)} \gamma_1, A_a^2 (\gamma_2 A_b \gamma_1, A_a^2, i f b = c = e)$, if the $\alpha$-cycle through the vertex $a$ is exceptional,
9. $\alpha_c \alpha_a - \gamma_1 \gamma_2$.

If $Q_T^2$ is of the form $(**)$ then we denote $\Omega^{(2)}(T, v_1, v_2)$ by $\Gamma^*(n)$.

**Example 1.2.** Let $T_6$ be the following Brauer tree

Then the Brauer quiver $Q_{T_6}^2$ of the tree $T_6$ is of the form
and \( \Gamma^*(6) \) is given by the above quiver and the ideal \( T^{(2)}(T_6, v_1, v_2) \) in \( KQ_{T_6}^2 \) generated by the elements: \( \beta_1 \alpha_2, \alpha_2 \beta_2, \beta_2 \alpha_3, \alpha_3 \beta_3, \beta_3 \alpha_4, \alpha_4 \beta_4, \beta_4 \alpha_5, \alpha_5 \beta_5, \beta_5 \alpha_6, \alpha_6 \beta_6, \beta_6 \alpha_1, \gamma_1 \beta_1, \beta_6 \gamma_1, \gamma_2 \gamma_1, \alpha_7 \alpha_1, (\beta_1 \beta_2 \beta_3 \beta_4 \beta_5 \beta_6)^2 - \alpha_1 \alpha_7, (\beta_2 \beta_3 \beta_4 \beta_5 \beta_6 \beta_1)^2 - \alpha_2, (\beta_3 \beta_4 \beta_5 \beta_6 \beta_1 \beta_2)^2 - \alpha_3, (\beta_4 \beta_5 \beta_6 \beta_1 \beta_2 \beta_3)^2 - \alpha_4, (\beta_5 \beta_6 \beta_1 \beta_2 \beta_3 \beta_4)^2 - \alpha_5, (\beta_6 \beta_1 \beta_2 \beta_3 \beta_4)^2 - \alpha_6, \alpha_1 \alpha_7 - \gamma_1 \gamma_2. 

The following theorem is the main result of [11].

**Theorem 1.3.** Let \( A \) be a basic connected standard selfinjective algebra. Then \( A \) is one-parametric but not weakly symmetric if and only if \( A \) is isomorphic to one of the algebras of the forms \( \Omega^{(1)}(T, \sigma_s, \theta) \) or \( \Omega^{(2)}(T, v_1, v_2) \).

For dealing with the algebras \( \Omega^{(1)}(T, \sigma_s, \theta) \), the following definitions will be convenient in the sequel. Let \( i \) be an edge in \( T \). We define the level of \( i \) to be the distance of \( i \) from the cycle of \( T \). In particular, the edges of level 0 are precisely the edges on the cycle of \( T \). For any edge \( i \) of level \( r \geq 2 \) we denote by \( n(i) \) the unique edge of level \( r - 1 \) adjacent to \( i \). At each vertex \( v_1, \ldots, v_k \) on the cycle of \( T \) there are two Brauer trees attached, an inner one and an outer one. For \( u = 1, \ldots, k \), we denote these Brauer trees by \( T_u^{inn} \) and \( T_u^{out} \), respectively, with \( T_u^{inn} \cap T_u^{out} = \{ v_u \} \). Recall that we denote the edges of the cycle of \( T \) by \( e_u \), for \( 1 \leq u \leq k \), where \( e_u \) has adjacent vertices \( v_u \) and \( v_{u+1} \).

The following result completely describes the Cartan matrices of the algebras \( \Omega^{(1)}(T, \sigma_s, \theta) \). We state it for the convenience of the reader since it might be useful to recall it when we are dealing with tilting complexes later. The result follows directly from the defining relations as listed in the definition above. Therefore the details of the proof are left to the reader.

**Proposition 1.4.** Given the algebra \( \Omega^{(1)}(T, \sigma_s, \theta) \), and edges \( i \) of level \( r \) and \( j \) of level \( t \) of the Brauer graph \( T \), the following holds for the Cartan entries \( \dim \text{Hom}(P(i), P(j)) \).

1. If \( |r-t| \geq 2 \), then \( \dim \text{Hom}(P(i), P(j)) = 0 \).
2. Let \( |r-t| = 1 \).
   a. For \( t = r - 1 \), the following holds.
      a. Let \( r \geq 2 \). Then \( \dim \text{Hom}(P(i), P(j)) = 0 \), unless \( j = \sigma(n(i)) \), in which case \( \text{Hom}(P(i), P(j)) \) is one-dimensional.
• Let \( r = 1 \) and \( i \in T_u^{\text{out}} \) for some \( 1 \leq u \leq k \). Then \( \dim \text{Hom} (P(i), P(j)) = 0 \), unless \( j = \sigma(e_u) \) or \( j = \sigma(e_{u-1}) \), in which case \( \dim \text{Hom} (P(i), P(j)) \) is one-dimensional.

• Let \( r = 1 \) and \( i \in T_u^{\text{inn}} \) for some \( 1 \leq u \leq k \). Then

\[
\dim \text{Hom}(P(i), P(j)) = \begin{cases} 
2 & \text{if } s = k-1 \text{ and } j = e_{u-1} \\
1 & \text{if } s \neq k-1 \text{ and } (j = e_{u-1} \text{ or } j = \sigma(e_u)) \\
0 & \text{if } j \neq e_{u-1} \text{ and } j \neq \sigma(e_u) 
\end{cases}
\]

(b) For \( t = r+1 \), the following holds.

• Let \( r \geq 1 \). Then \( \dim \text{Hom} (P(i), P(j)) = 0 \), unless \( j \) is an edge adjacent to \( i \), in which case \( \dim \text{Hom} (P(i), P(j)) \) is one-dimensional.

• Let \( r = 0 \) and \( i = e_u \) for some \( 1 \leq u \leq k \). Then

\[
\dim \text{Hom}(P(i), P(j)) = \begin{cases} 
2 & \text{if } s = k-1 \text{ and } j \in T_u^{\text{inn}} \\
1 & \text{if } (s \neq k-1 \text{ and } j \notin T_u^{\text{inn}}) \text{ or } (j \in T_u^{\text{out}} \cup T_{u+1}^{\text{out}} \cup T_{\sigma(u+1)}^{\text{inn}}) \\
0 & \text{if } j \notin T_u^{\text{out}} \cup T_{u+1}^{\text{out}} \cup T_{\sigma(u+1)}^{\text{inn}} 
\end{cases}
\]

(3) Let \( |r-t| = 0 \).

• Let \( r \geq 1 \) and \( i = j \). Then \( \dim \text{Hom} (P(i), P(j)) = 1 \).

• Let \( r \geq 2 \) and \( i \in T_u^{\text{out}} \cup T_u^{\text{inn}} \) for some \( 1 \leq u \leq k \). If \( j \notin T_u^{\text{out}} \cup T_{\sigma(u)}^{\text{out}} \cup T_{\sigma(u)}^{\text{inn}} \), then \( \dim \text{Hom} (P(i), P(j)) = 0 \). Moreover, \( \dim \text{Hom} (P(i), P(j)) = 0 \), unless one of the following cases holds, in which \( \dim \text{Hom} (P(i), P(j)) \) is one-dimensional.

(i) \( j \) is a successor of \( i \) and a predecessor of \( n(i) \) in the cyclic order in the graph \( T \) around the common vertex of \( i \) and \( n(i) \).

(ii) \( j \) is a predecessor of \( \sigma(i) \) and a successor of \( n(\sigma(i)) = n(i) \) in the cyclic order in the graph \( T \) around the common vertex of \( \sigma(i) \) and \( n(\sigma(i)) \).

• Let \( r = 1 \) and \( i \in T_u^{\text{out}} \cup T_u^{\text{inn}} \) for some \( 1 \leq u \leq k \). If \( j \notin T_u^{\text{out}} \cup T_{\sigma(u)}^{\text{out}} \cup T_{\sigma(u)}^{\text{inn}} \), then \( \dim \text{Hom} (P(i), P(j)) = 0 \). Moreover, if \( i \in T_u^{\text{out}} \) (respectively, \( i \in T_u^{\text{inn}} \)), then \( \dim \text{Hom} (P(i), P(j)) = 0 \), unless one of the following cases holds, in which \( \dim \text{Hom} (P(i), P(j)) \) is one-dimensional.

(i) \( j \in T_{\sigma(u)}^{\text{inn}} \) (respectively, \( j \in T_u^{\text{out}} \))

(ii) \( j \in T_u^{\text{out}} \) (respectively, \( j \in T_u^{\text{inn}} \)) and \( j \) is a successor of \( i \) in the cyclic order in the graph \( T \) around \( v_u \).

(iii) \( j \in T_{\sigma(u)}^{\text{out}} \) (respectively, \( j \in T_{\sigma(u)}^{\text{inn}} \)) and \( j \) is a predecessor of \( \sigma(i) \) in the cyclic order in the graph \( T \) around \( \sigma(v_u) \).

• Let \( r = 0 \) and \( i = e_u \). Then

\[
\dim \text{Hom}(P(i), P(j)) = \begin{cases} 
2 & \text{if } s = k-1 \text{ and } (j = i \text{ or } j = e_{u-1}) \\
1 & \text{if } s \neq k-1 \\
& \text{and } (j = i \text{ or } j = e_{u-1} \text{ or } j = \sigma(e_u) \text{ or } j = \sigma(e_{u+1})) \\
0 & \text{if } j \neq i \text{ and } j \neq e_{u-1} \text{ and } j \neq \sigma(e_u) \text{ and } j \neq \sigma(e_{u+1}) 
\end{cases}
\]
(4) Let \( i \in T^\text{out}_u \) of level \( \geq 2 \), and let \( j \in T^\text{inn}_w \) of level \( \geq 1 \) for any \( 1 \leq u, w \leq k \). Then \( \dim \text{Hom} (P(i), P(j)) = 0 = \dim \text{Hom} (P(j), P(i)) \).

(5) Let \( i \in T^\text{inn}_u \) of level \( \geq 2 \), and let \( j \in T^\text{out}_w \) of level \( \geq 1 \) for any \( 1 \leq u, w \leq k \). Then \( \dim \text{Hom} (P(i), P(j)) = 0 = \dim \text{Hom} (P(j), P(i)) \).

2. Tilting complexes for Brauer graph algebras with a cycle

In this section we are going to prove a first reduction step towards the derived equivalence classification of the algebras \( \Omega^{(1)} (T, \sigma_s, \theta) \). We show that, up to derived equivalence, we can assume the Brauer trees attached to the vertices of the cycle to be stars, i.e., all edges have level at most 1. The proof will be based on the well-known construction of derived equivalences for Brauer tree algebras \([29]\). But of course, the crucial new aspects are that for the algebras \( \Omega^{(1)} (T, \sigma_s, \theta) \) the Brauer graph contains a cycle and that we have to take the automorphism \( \sigma_s \) into account.

We shall need some notation. Let \( \Omega^{(1)} (T, \sigma_s, \theta) \) be as defined above, with Brauer graph \( T \) having exactly one cycle of length \( k \), and rotation automorphism \( \sigma_s \). For abbreviation, we usually omit the index and just write \( \sigma \) for \( \sigma_s \).

In a slight abuse of notation we denote the successive edges on the cycle of \( T \) just by \( 1, 2, \ldots, k \), instead of \( e_1, \ldots, e_k \). Then the vertex \( v_u \) of \( T \) lying on the cycle has adjacent edges \( u-1 \) and \( u \).

Recall that the edges of the Brauer graph \( T \) correspond to the simple modules of \( \Omega^{(1)} (T, \sigma_s, \theta) \). For an edge \( i \) we denote the corresponding projective indecomposable module by \( P(i) \).

The following notation will be very convenient and will be used frequently throughout the paper. Let \( i \) and \( j \) be edges in \( T \). We denote by \([i, j]\) the 'interval' from \( i \) to \( j \) in the order \( p(T, \sigma) \) of edges of \( T \), that is, the set of edges \( \{ f \mid i \leq f \leq j \} \) (which might be empty). Note that the terminology 'interval' is slightly misleading since the set \([i, j]\) is not necessarily linearly ordered. We shall only use this notation as a convenient way of denoting homomorphisms. In fact, a sequence of consecutive edges between \( i \) and \( j \) describes a path in the Brauer quiver; multiplication with any such path gives a homomorphism \( P(i) \to P(j) \) between the corresponding projective indecomposable modules, which we also denote by \([i, j]: P(i) \to P(j)\). Whenever we use this notation, it will always be clear from the context which sequence of edges is meant. As described in detail in Proposition 1.4, many of the Cartan invariants are 1; in these cases the above notation for homomorphisms is unambiguous.

Our main result in this section is the following.

**Proposition 2.1.** Let \( \Omega^{(1)} (T, \sigma, \theta) \) be as defined above. Then \( \Omega^{(1)} (T, \sigma, \theta) \) is derived equivalent to \( \Omega^{(1)} (\tilde{T}, \sigma, \theta) \) with Brauer graph \( \tilde{T} \) of the following shape
where \( p_u \) is the number of edges of \( T^\text{out}_{\sigma(u)+1} \) and \( q_u \) is the number of edges of \( T^\text{inn}_{u-1} \).

In particular, in the Brauer graph \( \tilde{T} \), all edges are of level at most 1.

Note the difference between these algebras \( \Omega^{(1)} \left( \tilde{T}, \sigma, \theta \right) \) and the normal forms \( \Lambda(p, q, k, s, \theta) \) given in our main result Theorem 1. In this proposition we still have different numbers of edges attached to the vertices of the cycle.

The rest of this section will be devoted to a proof of Proposition 2.1.

Construction of the tilting complex. For each edge \( i \) of \( T \) we consider the following path from an edge on the cycle of \( T \) to \( i \). If \( i \in T^\text{out}_u \) then we take the unique shortest path from \( u-1 \) to \( i \). If \( i \in T^\text{inn}_u \) we take the unique shortest path from \( u \) to \( i \). Denote the edges on this path by \( i_0, i_1, \ldots, i_l = i \). By construction, there is a unique (up to scalar multiplication) homomorphism \([i_j, i_{j+1}] : P(i_j) \to P(i_{j+1})\) between the corresponding projective indecomposable modules (see Proposition 1.4).

We then consider the following bounded complex of projective \( \Omega^{(1)}(T, \sigma, \theta) \)-modules

\[
Q(i) : 0 \to P(i_0) \xrightarrow{[i_0, i_1]} P(i_1) \to \ldots \to P(i_{l-1}) \xrightarrow{[i_{l-1}, i]} P(i) \to 0
\]

in which all maps are non-zero and where \( P(i_0) \) is in degree 0. The entries in degree \(-r\) are corresponding to edges of level \( r \).

Note that the composition of any two maps is zero by the usual Brauer tree relations which also hold in \( \Omega^{(1)}(T, \sigma_s, \theta) \). For the edges on the cycle of \( T \) this construction just gives stalk complexes concentrated in degree 0.

**Proposition 2.2.** The complex \( Q := \bigoplus_{i \in T} Q(i) \) is a tilting complex for \( \Omega^{(1)}(T, \sigma, \theta) \).

**Proof.** Exactly as in Rickard’s construction for Brauer tree algebras [29] one sees that \( \text{add}(Q) \) generates the homotopy category \( K^b \left( \Omega^{(1)}(T, \sigma, \theta) \right) \) as a triangulated category.
We have to show that in the homotopy category, Hom(Q, Q[n]) = 0 for all n ≠ 0.

By Proposition 1.4 (1), it is clear that Hom(Q, Q[n]) = 0 unless n ∈ {−1, 0, 1}.

We first consider a map α of complexes from Q(i) to Q(j)[−1]. Hence we have a commutative diagram of the form

\[
\begin{array}{cccccccc}
0 & \to & P(i_0) & \to & P(i_1) & \to & \ldots & \to & P(i_r) & \to & P(i_{r+1}) & \to & P(i_{r+2}) & \to & \ldots \\
\downarrow^{\alpha_0} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P(j_0) & \to & P(j_1) & \to & \ldots & \to & P(j_r) & \to & P(j_{r+1}) & \to & P(j_{r+2}) & \to & \ldots
\end{array}
\]

Let r be minimal such that \(\alpha_r \neq 0\). By Proposition 1.4, the composition \(P(i_r) \to P(i_{r+1}) \to P(j_r)\) is non-zero. In fact, if the level \(r \geq 1\) then this composition is the socle map; if \(r = 0\) then there are two possibilities for \(j_0\), but in both cases the composition is non-zero (recall that the map \(P(i_0) \to P(i_1)\) is given by multiplication with the shortest path from \(i_0\) to \(i_1\)). This contradicts the fact that \(\alpha\) is a map of complexes, which proves Hom(Q, Q[−1]) = 0, as desired.

Secondly, we have to consider maps α of complexes from Q(i) to Q(j)[1]. We then have a commutative diagram of the form

\[
\begin{array}{cccccccc}
0 & \to & P(i_0) & \to & P(i_1) & \to & \ldots & \to & P(i_{r-1}) & \to & P(i_r) & \to & P(i_{r+1}) & \to & \ldots \\
\downarrow^{\alpha_0} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P(j_0) & \to & P(j_1) & \to & \ldots & \to & P(j_r) & \to & P(j_{r+1}) & \to & P(j_{r+2}) & \to & \ldots
\end{array}
\]

Let r be maximal with the property \(\alpha_r \neq 0\). Moreover, we choose α from its homotopy class such that this value of r is minimal. Assume first that \(r \geq 1\). The existence of a non-zero map \(\alpha_r\) from level \(r \geq 1\) to level \(r + 1\) implies that \(j_{r+1}\) is adjacent to \(i_r\) (see Proposition 1.4 (2b)). But this implies \(j_r = i_r\), which means that \(\alpha_r\) factors through \(P(j_r)\). This factorisation gives a homotopy from \(\alpha\) to a map \(\beta\) for which the minimal non-zero map occurs in a degree strictly smaller than for \(\alpha\). This contradicts the choice of \(\alpha\) as being of minimal such degree in its homotopy class.

Now assume that \(r = 0\). There are four possibilities for a non-zero map from level 0 to level 1: either \(j_1\) is in \(T_{\text{out}}^{i_0+1}\), in \(T_{\text{inn}}^{i_0}\), in \(T_{\text{out}}^{i_0}\) or in \(T_{\sigma(i_0)+1}\). In the first two cases we deduce that \(j_0 = i_0\), in the third case that \(j_0 = i_0 - 1\), and in the latter case that \(j_0 = \sigma(i_0) + 1\). In all cases, \(\alpha_0\) factors through \(P(j_0)\), implying that \(\alpha\) is homotopic to zero.

Hence, Hom(Q, Q[1]) = 0 in the homotopy category, as desired.

This completes the proof that Q is a tilting complex for \(\Omega(1) (T, \sigma, \theta)\).  

By Rickard’s criterion, the endomorphism ring of the tilting complex Q for \(\Omega(1) (T, \sigma, \theta)\) is derived equivalent to the algebra \(\Omega(1) (T, \sigma, \theta)\). So it remains to prove that this endomorphism ring has the structure described in Proposition 2.1.

**Proposition 2.3.** Let \(\Omega(1) (T, \sigma, \theta)\) be as above, with Brauer trees attached to the vertices of the cycle denoted by \(T_1^{\text{out}}, \ldots, T_k^{\text{out}}\) and \(T_1^{\text{inn}}, \ldots, T_k^{\text{inn}}\), where \(T_{\text{out}}^u\) and \(T_{\text{inn}}^u\) are attached to the vertex between the edges \(u - 1\) and \(u\) on the cycle.
Let \( Q = \oplus_{i \in T} Q(i) \) be the tilting complex for \( \Omega^{(1)} (T, \sigma, \theta) \) as defined above. Then the endomorphism ring of \( Q \) is isomorphic to the algebra \( \Omega^{(1)} (\tilde{T}, \sigma, \theta) \), where the Brauer graph \( \tilde{T} \) can be described by the following properties.

(i) The cycle of \( \tilde{T} \) is the same as the cycle of \( T \).

(ii) All edges of \( \tilde{T} \) have level at most 1.

(iii) Denote by \( E_{u_{\text{out}}} \) and \( E_{u_{\text{inn}}} \) the numbers of edges of the Brauer trees \( T_{u_{\text{out}}} \) and \( T_{u_{\text{inn}}} \), respectively. In \( \tilde{T} \), attached to the vertex adjacent to the edges \( u - 1 \) and \( u \), we have \( E_{\sigma(u + 1)} \) outer edges and \( E_{u_{\text{inn}}} \) inner edges.

Note that compared with the algebra \( \Omega^{(1)} (T, \sigma, \theta) \) the inner Brauer trees have been shifted by one vertex on the cycle, and the outer Brauer trees have been shifted by \( -(s + 1) \) where \( \sigma = \sigma_s \). This means that the order \( p(\tilde{T}, \sigma) \) on the edges of End(\( Q \)) is different from the order of edges of the original algebra \( \Omega^{(1)} (T, \sigma, \theta) \). But note that the automorphism \( \sigma \) is the same for both \( \Omega^{(1)} (T, \sigma, \theta) \) and the endomorphism ring of \( Q \).

The rest of this section is devoted to a proof of Proposition 2.3. The proof consists of three steps. We first describe morphisms between direct summands of \( Q \), which correspond to the arrows of the algebra \( \Omega^{(1)} (\tilde{T}, \sigma, \theta) \). Secondly, we show that these maps satisfy the defining relations of \( \Omega^{(1)} (\tilde{T}, \sigma, \theta) \). In the third step we show that End(\( Q \)) is actually isomorphic to \( \Omega^{(1)} (\tilde{T}, \sigma, \theta) \). To this end, we need to show that there are no further relations in End(\( Q \)).

I. The morphisms between summands of \( Q \). The direct summands of the tilting complex \( Q \) are complexes defined completely analogous to Rickard’s construction for the classical Brauer tree algebras. Recall that for a Brauer tree algebra, the endomorphism ring was a star; more precisely, the cyclic ordering of the edges for the endomorphism ring can be obtained by the famous ’walk around the Brauer tree’ (see [18]). Note that in contrast to the usual Brauer tree setting, in our case one can not walk completely around the Brauer tree since our algebras are not symmetric. Still, one gets for the endomorphism ring of the complexes corresponding to any Brauer tree \( T_{u_{\text{out}}} \) and \( T_{u_{\text{inn}}} \) a cyclic ordering of the edges. In other words, in End(\( Q \)), all edges will have level at most 1.

Let us now define explicitly the morphisms between direct summands of \( Q \) corresponding to successive edges in the order \( p(\tilde{T}, \sigma) \).

(1) The maps inside the Brauer trees.

Within any of the Brauer trees \( T_{u_{\text{out}}} \) and \( T_{u_{\text{inn}}} \) of End(\( Q \)) these morphisms are given as in Rickard’s construction by a walk around the Brauer tree. Actually, for our proof we don’t need to know these maps in detail; for checking relations it will only be important that their composition in degree 0 is the identity (which is immediate from Rickard’s classical construction).

Next, we show that we actually have non-zero sequences of morphisms starting from \( Q(u) \) through all complexes corresponding to vertices of \( T_{u-1} \) to \( Q(u - 1) \) and further on through all complexes corresponding to edges of \( T_{\sigma(u) + 1} \) and finally to \( Q(\sigma(u)) \).
The maps \( Q(u) \to Q(T_{u-1}^{\text{inn}}) \).

For any edge \( u \) on the cycle consider \([u, u - 1]\), that is, the sequence of edges \( u \), consecutive edges of level 1 in \( T_{u-1}^{\text{inn}} \) and \( u - 1 \). Then multiplication by \([u, u - 1]\) gives a homomorphism \( P(u) \to P(u - 1) \).

Then for any \( f \) in \( T_{u-1}^{\text{inn}} \) we have a homomorphism of complexes \( Q(u) \to Q(f) \) given by multiplication with \([u, u - 1]\) in degree 0.

In fact, this is a map of complexes since \( Q(f) \) starts with a map from \( P(u - 1) \) given by walking around the endpoint of \( u - 1 \) not adjacent to \( u \). Since the edges are not consecutive in the order \( p(T, \sigma) \), the composition with \([u, u - 1]\) is zero.

The maps \( Q(T_{u-1}^{\text{inn}}) \to Q(u - 1) \).

Let \( f \) be in \( T_{u-1}^{\text{inn}} \). There is an obvious map from \( Q(f) \) to the stalk complex \( Q(u - 1) \) given by the identity in degree 0.

The maps \( Q(u - 1) \to Q(T_{\sigma(u)+1}^{\text{out}}) \).

Let \( q_u \) denote the map \( P(u - 1) \to P(\sigma(u)) \) given by multiplication with \([u - 1, \sigma(u)]\), i.e., the sequence of edges \( u \), all edges of level 1 in \( T_{u}^{\text{out}} \), and \( \sigma(u) \).

Let \( g \) be an edge in \( T_{\sigma(u)+1}^{\text{out}} \). We have a homomorphism of complexes \( Q(u - 1) \to Q(g) \) given by multiplication with \( q_u \) in degree 0.

In fact, this is a map of complexes since \( Q(g) \) starts with a map from \( P(\sigma(u)) \) given by an edge not consecutive to the last edge of \( q_u \) in the order \( p(T, \sigma) \).

The maps \( Q(T_{\sigma(u)+1}^{\text{out}}) \to Q(\sigma(u)) \).

Finally, for any \( g \) in \( T_{\sigma(u)+1}^{\text{out}} \) there is an obvious map from \( Q(g) \) to the stalk complex \( Q(\sigma(u)) \) given by the identity in degree 0.

The composition of the maps just described is the map from \( Q(u) \) to \( Q(\sigma(u)) \) given by the socle map. In particular, this composition is non-zero in \( \text{End}(Q) \), as desired.

The maps around the endpoints of \( \tilde{T} \).

So far, we have described maps for all consecutive elements in the ordering \( p(\tilde{T}, \sigma) \) around the vertices on the cycle. It remains to define maps corresponding to the endpoints of the edges of level 1 of the Brauer graph \( \tilde{T} \). The edges of level 1 in \( \tilde{T} \) correspond to complexes \( Q(f) \) where \( f \) is any edge in \( T \) not on the cycle.

We define the map \( Q(f) \to Q(\sigma(f)) \) to be multiplication with the socle map \([f, \sigma(f)]\) in degree 0 and the zero map in all other degrees.

We shall show below that this map of complexes is not homotopic to zero, that is, it is non-zero in \( \text{End}(Q) \).

We have thus described maps of complexes corresponding to all consecutive edges in the order \( p(\tilde{T}, \sigma) \). In other words, we have given maps corresponding to all arrows in the quiver of \( \Omega^{(1)}(\tilde{T}, \sigma, \theta) \).

II. Checking the defining relations.

We have to show that the maps defined in (1)-(6) above actually satisfy the defining relations of \( \Omega^{(1)}(\tilde{T}, \sigma, \theta) \) (see Section II for the definition). The Brauer trees attached to
vertices on the cycle of $\tilde{T}$ are denoted by $\tilde{T}_u^{\text{out}} = Q(T_u^{\text{out}})$ and $\tilde{T}_u^{\text{inn}} = Q(T_u^{\text{inn}})$, respectively. Hence, attached to the vertex in $\tilde{T}$ adjacent to $u-1$ and $u$ we have $\tilde{T}_u^{\text{out}}$ and $\tilde{T}_u^{\text{inn}}$.

(i) Products of non-consecutive edges.

Let $f \in \tilde{T}_u^{\text{out}}$ and $g \in \tilde{T}_v^{\text{out}}$. Then $f, u, g$ are not consecutive in the order of edges of $\tilde{T}$. We have to show that the corresponding composition of the maps defined above in (5) and (4) is homotopic to zero. The composition is given in degree 0 by

$$Q(f) \xrightarrow{id} Q(u) \xrightarrow{[u,\sigma u+1]} Q(g)$$

and by the zero map in all other degrees. Write $Q(f) : 0 \to P(u) \to P(i_1) \to \ldots$ and $Q(g) : 0 \to P(\sigma(u)+1) \to P(j_1) \to \ldots$.

A suitable homotopy map from $Q(f)$ to $Q(g)$ is given by $[i_1, \sigma(u)+1] : P(i_1) \to P(\sigma(u)+1)$ from degree $-1$ to degree 0 and the zero maps in all other degrees. In fact, in degree 0 we clearly get $[u, \sigma(u)+1]$, and in degree $-1$ we indeed get the zero map since $i_1, \sigma(u)+1, j_1$ are not consecutive in the order of edges of $T$ ($i_1$ and $j_1$ are both outer edges).

The argument for inner edges works similarly. If $f \in \tilde{T}_u^{\text{inn}}$ and $g \in \tilde{T}_v^{\text{inn}}$ then the composition $Q(f) \to Q(u) \to Q(g)$ is given by $[u, -u-1]$ in degree 0 and the zero maps in all other degrees. This composition is homotopic to zero via the homotopy map $[i_1, -u-1] : P(i_1) \to P(u-1)$ from degree $-1$ to degree 0, and zero in all other degrees. (Use that $i_1, u, j_1$ are not consecutive since $i_1$ and $j_1$ are both inner edges.)

(ii) Socle relations.

Let $f$ be any edge not on the cycle. The edge in $\tilde{T}$ corresponding to the complex $Q(f)$ is of level 1. So there are two maps $Q(f) \to Q(\sigma(f))$, given by the orders around the two endpoints of this edge. Using the definitions in (1)-(6) above one directly sees that both compositions are given by the socle map $[f, \sigma(f)]$ in degree 0 and the zero maps in all other degrees.

Now consider an edge $u$ on the cycle, with endpoints $v$ and $w$. Then by the above definitions, the compositions of maps $Q(u) \to Q(\sigma(u))$ in $\text{End}(Q)$ around the two endpoints are given by multiplication with $C(u, p(T, \sigma, v))$ and $C(u, p(T, \sigma, w))$ (i.e. by the socle maps $P(u) \to P(\sigma(u))$). By definition of the relations in $\Omega^{(1)}(T, \sigma, \theta)$ these maps are equal for all $u \neq 1$ and for the distinguished vertex $u=1$ they are equal up to multiplication by the scalar $\theta$. Hence we get exactly the desired relations for $\Omega^{(1)}(\tilde{T}, \sigma, \theta)$.

III. No further relations.

In the final step, we have to prove that there are no further relations in $\text{End}(Q)$. To this end it suffices to show that for all edges $f$ the socle map $Q(f) \to Q(\sigma(f))$ is not homotopic to zero, that is, non-zero in $\text{End}(Q)$. This is clear for the edges on the cycle, since the complexes are stalk complexes and the socle map is just the socle map on the projective indecomposable modules.

Let $f$ be any edge in $\tilde{T}_u^{\text{out}}$ for some $u$. The corresponding complex $Q(f)$ has the form

$$0 \to P(u) \xrightarrow{[u,i_1]} P(i_1) \xrightarrow{[i_1,i_2]} P(i_2) \to \ldots \to P(i_{l-1}) \xrightarrow{[i_{l-1},f]} P(f) \to 0$$
where $u, i_1, i_2, \ldots, i_{l-1}, f$ is the unique shortest walk in $T$ from the edge $u$ on the cycle to $f$. The socle map $\alpha_f : Q(f) \to Q(\sigma(f))$ is given by the zero map in all non-zero degrees and in degree 0 it is given by the composition $Q(f) \xrightarrow{id} Q(u) \xrightarrow{[u,u-1]} Q(u-1) \xrightarrow{[u,\sigma u]} Q(\sigma(f))$, that is, by the socle map $[u, \sigma(u)] : P(u) \to P(\sigma(u))$. We claim that this map of complexes $Q(f) \to Q(\sigma(f))$ is not homotopic to zero. In fact, assuming there were a homotopy map, then for degree 0 this map has to be $[i_1, \sigma(u)] : P(i_1) \to P(\sigma(u))$. Since $\alpha_f$ is 0 in degree -1, the homotopy map has to be given by $-\|[i_2, \sigma(i_1)] : P(i_2) \to P(\sigma(i_1))$. Inductively, the final homotopy map $P(f) \to P(i_{l-1})$ must be given by $\pm [f, \sigma(i_{l-1})]$. But then in degree $-l$ the composition $P(f) \to P(\sigma(i_{l-1})) \to P(\sigma(f))$ is the non-zero socle map, a contradiction, since $\alpha_f$ is zero in degree $-l$. Hence, $\alpha_f$ is not homotopic to zero, as desired.

A completely analogous argument shows that, for any inner edge $g$, the socle map $Q(g) \to Q(\sigma(g))$ is not homotopic to zero. We leave the details to the reader.

This completes the proof of Proposition 2.3.

3. Moving edges

In the previous section we have proved that any algebra $\Omega^{(1)}(T, \sigma, \theta)$ is derived equivalent to an algebra $\Omega^{(1)}(\tilde{T}, \sigma, \theta)$ where all edges in $\tilde{T}$ are of level at most 1 (i.e., the Brauer trees attached to the vertices on the cycle are stars).

In this section we shall prove a further reduction step by showing that, up to derived equivalence, we can concentrate all edges of level 1 around the vertices of a single $\sigma$-orbit.

There is nothing to prove if the $\sigma$-action is already transitive. So in this section we can assume that the action of $\sigma$ on the vertices of the cycle is not transitive.

The main result of this section is the following.

**Proposition 3.1.** Let $A = \Omega^{(1)}(\tilde{T}, \sigma, \theta)$ be as in Proposition 2.3 where all edges are of level at most 1. Fix any vertex $v$ on the cycle. Then $A$ is derived equivalent to the algebra $\Omega^{(1)}(\bar{T}, \sigma, \theta)$ where the Brauer graph $\bar{T}$ has the following properties:

1. The cycle of $\bar{T}$ is the same as the cycle of $\tilde{T}$. All edges of $\bar{T}$ are of level at most 1.
2. All edges of level 1 are attached to some vertex in the $\sigma$-orbit of the fixed vertex $v$.

This result will follow immediately from the following two lemmas. The first lemma asserts that, up to derived equivalence, we can move inner edges by one vertex on the cycle. The analogous second lemma states that outer edges can be moved by $-(s+1)$ steps on the cycle, where $\sigma = \sigma_s$. By applying these two results about moving inner and outer edges inductively, we can clearly move all edges of level 1 in $\tilde{T}$ to some vertex in the $\sigma$-orbit of the fixed vertex $v$.

**Lemma 3.2.** Let $A = \Omega^{(1)}(\tilde{T}, \sigma, \theta)$ be as above with all edges of level at most 1. Fix any edge $u$ on the cycle, and denote the subsequent inner edges (in the order $p(\tilde{T}, \sigma)$) by $u_1, \ldots, u_{q_u}$, and the inner edges subsequent to $u+1$ by $(u+1)_1, \ldots, (u+1)_{q_{u+1}}$. Then $A$ is derived equivalent to an algebra $\Omega^{(1)}(\bar{T}, \sigma, \theta)$, where the order $p(\bar{T}, \sigma)$ is given as
follows. For any integer $j$ the order around the common vertex of $\sigma^j(u+1)$ and $\sigma^j(u)$ is given by $..., \sigma^j(u+1), \sigma^j(u+1), ..., \sigma^j((u+1)_{q_{u+1}}), \sigma^j(u_1), ..., \sigma^j(u_{q_u}), \sigma^j(u), ...,$ and around the other endpoint of $\sigma^j(u)$ by $\sigma^j(u), \sigma^j(u-1)$.

In other words, the inner edges $\{\sigma^j(u_1), ..., \sigma^j(u_{q_u}) \mid j \in \mathbb{Z}\}$ can be moved to the next vertex on the cycle, up to derived equivalence (and all other edges remain unchanged).

**Proof.** We consider the following bounded complexes of projective $A$-modules. For any edge $z \notin S := \{\sigma^j(u_i) \mid j \in \mathbb{Z}, 1 \leq i \leq q_u\}$ let $Q(z)$ be the stalk complex with the corresponding projective indecomposable module $P(z)$ in degree 0. For the edges in the $\sigma$-orbit of $u_1, \ldots, u_{q_u}$ we define the following complexes

$$
Q(\sigma^j(u_i)) : 0 \to P(\sigma^j(u)) \overset{[\sigma^j(u), \sigma^j(u_i)]}{\longrightarrow} P(\sigma^j(u_i)) \to 0
$$

in degrees 0 and $-1$.

**Claim:** The direct sum $Q := \oplus_{z \in \mathbb{T}} Q_z$ is a tilting complex for $\Omega^{(1)}(\mathbb{T}, \sigma, \theta)$.

**Proof of the claim.** From the construction it is clear that $add(Q)$ generates the homotopy category $K^b\left(\Omega^{(1)}\left(\mathbb{T}, \sigma, \theta\right)\right)$ as triangulated category. It remains to show that, in the homotopy category, $\text{Hom}(Q, Q[n]) = 0$ for all $n \neq 0$. This is clear for $|n| \geq 2$, since we are only dealing with two-term complexes. From the relations of $A$ (cf. also Proposition [1.4]) we can directly deduce that all non-zero maps $P(z) \to P(\sigma^j(u_i))$ with $z \notin S$ factor through $P(\sigma^j(u))$. Hence, $\text{Hom}(Q, Q[1]) = 0$. Similarly, one sees that any non-zero map $P(\sigma^j(u_i)) \to P(z)$ with $z \notin S$ remains non-zero when composed with $[\sigma^j(u), \sigma^j(u_i)]$. Hence, $\text{Hom}(Q, Q[-1]) = 0$, which proves the claim.

The endomorphism ring of $Q$. By Rickard’s theorem, the endomorphism ring of $Q$ (in the homotopy category) is derived equivalent to $A$. So in order to complete the proof we have to show that $\text{End}(Q)$ has the structure described above, where the edges in the $\sigma$-orbit of $u_1, \ldots, u_{q_u}$ are moved by one step to the next vertex on the cycle (and all other edges remain unchanged). Since 'most’ summands of $Q$ are stalk complexes, it is clear that the Brauer graph for $\text{End}(Q)$ will only change as far as the edges in the $\sigma$-orbit of $u_1, \ldots, u_{q_u}$ are concerned.

We define the following maps of complexes between summands of $Q$ involving some $Q(\sigma^j(u_i))$. For simplicity we describe them for the $Q(u_i)$; the other maps are obtained by applying $\sigma^j$. Let $Q((u+1)_{q_{u+1}}, u_1) \to Q(u_1)$ be defined by multiplication with $[(u+1)_{q_{u+1}}, u]$ in degree 0. Note that this is indeed a map of complexes since $(u+1)_{q_{u+1}}, u, u_1$ are not consecutive in the ordering of edges of $\mathbb{T}$. For any $i \in \{1, \ldots, q_u-1\}$ we define a map of complexes $Q(u_i) \to Q(u_{i+1})$ by the identity in degree 0 and $[u_i, u_{i+1}]$ in degree $-1$. Finally, we define a map $Q(u_{q_u}) \to Q(u)$ by the identity in degree 0 and a map $Q(u) \to Q(u-1)$ by $[u, u-1]$. We denote the edges of the Brauer graph for $\text{End}(Q)$ corresponding to $Q(\sigma^j(u_i))$ by $\sigma^j(u_i)$, and the 'old' edges corresponding to the stalk complexes as before. All relations of $\Omega^{(1)}(\mathbb{T}, \sigma, \theta)$ not involving these new complexes immediately follow from the corresponding same relations in $\Omega^{(1)}(\mathbb{T}, \sigma, \theta)$. For the other relations, by applying $\sigma^j$,
it suffices to check the relations involving \( \overline{u}_i \). The edges \( \overline{u}_q, u, u-1 \) are claimed not to be consecutive in the order of \( \text{End}(Q) \). In fact, the corresponding map \( Q(u_q) \to P(u) \to P(u-1) \) is given by \([u, u-1]\) in degree 0. It is homotopic to zero via the homotopy map \([u_q, u-1]\), giving the desired zero relation in \( \text{End}(Q) \). Since the maps \( Q(u_i) \to Q(u_{i+1}) \), for \( i \in \{1, \ldots, q_u-1\} \), are in degree 0 just the identity, all relations starting at any of the edges \((u+1), i\) still hold in \( \text{End}(Q) \) with the new order of edges. Moreover, the composition of the socle map \( Q(\sigma^{-1}((u+1)_{q_u+1})) \to Q((u+1)_{q_u+1}) \) with the map \( Q((u+1)_{q_u+1}) \to Q(u_1) \) is zero, since the latter is given by \( [(u+1)_{q_u+1}, u] \).

It remains to show that the socle relations hold for the edges in \( \text{End}(Q) \) corresponding to the complexes \( Q(\sigma^j(u_i)) \). Again, for simplicity we only consider \( \overline{u}_i = Q(u_i) \). Note that the new order of edges on \( \text{End}(Q) \) is claimed to have the following consecutive edges:

\[
\overline{u}_i, \ldots, \overline{u}_q, [u, \sigma(u+1)], \sigma((u+1)_1), \ldots, \sigma((u+1)_{q_u+1}), (\overline{u}_1), \ldots, (\overline{u}_{i-1}), (\overline{u}_i).
\]

The corresponding composition \( Q(u_i) \to Q(\sigma(u_i)) \) is given by the socle map \( P(u) \to P(\sigma(u)) \) in degree 0 and the zero map in degree \(-1\). We claim that this composition is not homotopic to zero. In fact, by the relations of \( \Omega(1)^{(1)}(\tilde{T}, \sigma, \theta) \), this socle map is given by the order around one of the endpoints of the edge \( u \). So in degree 0 it could only be factored by the homotopy map \([u_i, \sigma(u)]\), but then in degree \(-1\) this gives the non-zero socle map \( P(u_i) \to P(\sigma(u_i)) \), which proves the claim. Note that if we compose this socle map \( Q(u_i) \to Q(\sigma(u_i)) \) with the next map \( Q(\sigma(u_i)) \to Q(\sigma(u_{i+1})) \), then this composition becomes homotopic to zero via the homotopy map \([u_i, \sigma(u)]\).

Note that for the socle relation on \( P(e_1) \), where \( e_1 \) is the distinguished vertex on the cycle, involving the scalar \( \theta \), the corresponding relation on \( Q(e_1) \) in \( \text{End}(Q) \) involves the same scalar \( \theta \). In fact, the above defined maps on the \( Q(u_i) \)'s are the identity in degree 0, so nothing changes for the stalk complexes.

This completes the proof of the lemma \( \square \).

There is the following analogous result for moving outer edges. Since the proof is very similar to the proof of Lemma 3.2 we leave the details to the reader.

**Lemma 3.3.** Let \( A = \Omega(1)^{(1)}(\tilde{T}, \sigma, \theta) \) be as above with all edges of level at most 1. Fix any edge \( u \) on the cycle, and denote the outer edges succeeding \( u-1 \) (in the order \( p(\tilde{T}, \sigma) \)) by \( u^1, \ldots, u^p_u \), and the outer edges succeeding \( u-2 \) by \( (u-1)^1, \ldots, (u-1)^{p_u-1} \). Then \( A \) is derived equivalent to an algebra \( \Omega(1)^{(1)}(\tilde{T}, \sigma, \theta) \) where the order \( p(\tilde{T}, \sigma) \) of edges is given as follows. For any integer \( j \) the order around the common vertex of \( \sigma^j(u-2) \) and \( \sigma^j(u-1) \) is given by

\[
\ldots, \sigma^j(u-2), \sigma^j((u-1)^1), \ldots, \sigma^j((u-1)^{p_u-1}), \sigma^{j+1}(u^1), \ldots, \sigma^{j+1}(u^p_u), \sigma^{j+1}(u-1), \ldots
\]

and around the other endpoint of \( \sigma^j(u-1) \) by \( \ldots, \sigma^j(u-1), \sigma^{j+1}(u), \ldots \).

In other words, all outer edges in the \( \sigma \)-orbit of \( u^1, \ldots, u^p_u \) have been shifted by \(-(s+1)\) vertices on the cycle (and all other edges remain unchanged). \( \square \)
4. Reduction to the transitive case

In the last two sections we proved that any algebra \( \Omega^{(1)}(T, \sigma, \theta) \) is derived equivalent to an algebra \( \Omega^{(1)}(\overline{T}, \sigma, \theta) \) where all edges of the Brauer graph \( \overline{T} \) are of level at most 1, and all edges of level 1 are attached to vertices in a single \( \sigma \)-orbit.

The main aim of this section is to prove that, up to derived equivalence, we can even assume that the \( \sigma \)-action is transitive. If the \( \sigma \)-action on \( \Omega^{(1)}(\overline{T}, \sigma, \theta) \) is not already transitive then we will have vertices on the cycle of \( \overline{T} \) with no edges of level 1 attached. Derived equivalences for this situation are provided by the following result. The crucial aspect of this result is that if the \( \sigma \)-action is not transitive, then one can reduce the number of edges on the cycle.

Recall that the edge \( e_1 \) on the cycle plays a special role in that the socle relation involves the scalar \( \theta \). The results in the preceding section allow us to concentrate the edges in the \( \sigma \)-orbit of any vertex on the cycle. So for our purposes we can assume that the distinguished edge \( e := 1 \) has no edges attached to it, i.e. the technical assumption in the following result can always be guaranteed in our context.

**Proposition 4.1.** Let \( A = \Omega^{(1)}(\overline{T}, \sigma, \theta) \) be as above, with all edges of \( \overline{T} \) of level at most 1. Assume that for some edge \( u \) on the cycle there are no edges of level 1 attached to the vertices between \( u - 1 \) and \( u \) and between \( u \) and \( u + 1 \) (that is, in the ordering of edges we have that \( u, u - 1, \sigma(u) \) are consecutive and \( u + 1, u, \sigma(u + 1) \) are consecutive). Moreover, assume that \( u - 1 \) and \( u + 1 \) are not in the \( \sigma \)-orbit of the distinguished edge.

Then \( A \) is derived equivalent to an algebra \( \Omega^{(1)}(T^*, \sigma^*, \theta) \) where all edges of \( T^* \) are of level at most 1, and where all edges in the \( \sigma \)-orbit of \( u - 1 \) become inner edges of level 1 and all edges in the \( \sigma \)-orbit of \( u + 1 \) become outer edges of level 1 in \( T^* \).

In particular, the number of edges on the cycle of \( \overline{T} \) can be reduced, up to derived equivalence. The automorphism \( \sigma^* \) is again given by \( f \mapsto \sigma(f) \) for all edges \( f \). But since the number of edges decreases, the rotation parameter changes.

**Proof.** We consider the following two-term complex of projective \( A \)-modules. For any edge \( z \notin \{ \sigma^j(u) \mid j \in \mathbb{Z} \} \) we let \( Q(z) : 0 \to P(z) \to 0 \) be the stalk complex in degree 0. By assumption, for any \( j \in \mathbb{Z} \), there are no edges between \( \sigma^j(u) \) and \( \sigma^j(u - 1) \) and between \( \sigma^j(u) \) and \( \sigma^j(u + 1) \). Hence, every map from \( P(z) \) to \( P(\sigma^j(u)) \) factors through \( P(\sigma^{j-1}(u - 1)) \) or \( P(\sigma^j(u + 1)) \). Thus we are led to define the complexes

\[
Q(\sigma^j(u)) : 0 \to P(\sigma^{j-1}(u - 1)) \oplus P(\sigma^j(u + 1)) \to P(\sigma^j(u)) \to 0,
\]

in degrees 0 and \(-1\). Then the direct sum \( Q := \bigoplus_{z \in T} Q_z \) is a tilting complex for \( A \). We leave the straightforward verification to the reader.

We use the same notation as before for edges in the Brauer graph \( \overline{T} \) of \( A \). We denote by \((u - 1)_1, \ldots, (u - 1)_{q_{u - 1}}\) the inner edges and by \((u - 1)^1, \ldots, (u - 1)^{p_{u - 1}}\) the outer edges between \( u - 1 \) and \( u - 2 \), respectively. Moreover, let \((u + 2)_1, \ldots, (u + 2)_{q_{u + 2}}\) be the inner edges and \((u + 2)^1, \ldots, (u + 2)^{p_{u + 2}}\) the outer edges between \( u + 1 \) and \( u + 2 \), respectively. Note that any of the cases \( q_{u - 1} = 0, p_{u - 1} = 0, q_{u + 2} = 0 \) or \( p_{u + 2} = 0 \) is allowed.
We have to determine the endomorphism ring of \( Q \) in the homotopy category \( K^b(A) \) (it is derived equivalent to \( A \) by Rickard’s theorem). For describing the Brauer graph of the endomorphism ring \( \text{End}(Q) \), we denote the ‘new’ two-term complexes by \( \tilde{\sigma}^j(u) := Q(\sigma^j(u)) \). The edges corresponding to the stalk complexes are denoted as before.

We claim that \( \text{End}(Q) \) is of the form \( \Omega^{(1)}(T^*, \sigma^*, \theta) \) where the Brauer graph \( T^* \) is given as follows. Note that the Brauer graph will only change around the new complexes \( Q \) of the endomorphism ring \( \text{End}(Q) \) (since the others are stalk complexes). It suffices to describe the structure around one of the new complexes, say \( \tilde{\sigma}(u) \).

**Claim:** The ordering \( p(T^*, \sigma^*) \) of edges around the one endpoint of \( \tilde{\sigma}(u) \) is given by

\[
..., \tilde{\sigma}(u), u-1, (u-1)_1, \ldots, (u-1)_{q_u-1}, (\tilde{\sigma}^0(u-3)), u-2, (u-1)^1, \ldots, (u-1)^{p-u-1}, \tilde{\sigma}^2(u), ...
\]

where \( \tilde{\sigma}^0(u-3) \) occurs exactly if \( u-3 \) is contained in the \( \sigma \)-orbit of \( u \). In this case, \( \tilde{\sigma}^0(u-3) \) is on the cycle of \( T^* \); otherwise, \( u-2 \) is on the cycle.

The order \( p(T^*, \sigma^*) \) around the other endpoint of \( \tilde{\sigma}(u) \) is given by

\[
..., \tilde{\sigma}(u), \sigma(u+1), \sigma((u+2)^1), \ldots, \sigma((u+2)^{p-u+2}), (\tilde{\sigma}^3(u+3)), \sigma^2(u+2), \sigma^2((u+2)_1), \ldots, \sigma^2((u+2)_{q_{u+2}}), \tilde{\sigma}^2(u), ...
\]

where \( \tilde{\sigma}^3(u+3) \) occurs precisely if \( u-3 \) is in the \( \sigma \)-orbit of \( u \). In this case, \( \tilde{\sigma}^3(u+3) \) is on the cycle of \( T^* \); otherwise \( \sigma^2(u+2) \) is on the cycle.

The adjacent edges on the cycle of \( T^* \) are \( ..., \tilde{\sigma}^0(u-3), \tilde{\sigma}(u), \tilde{\sigma}^2(u+3), ..., \) if \( u-3 \) is in the \( \sigma \)-orbit of \( u \); otherwise the edges on the cycle are given by \( ..., u-2, \tilde{\sigma}(u), \sigma(u+2), ... \).

Note that the crucial aspect of this description is that the edges \( \sigma^j(u-1) \) and \( \sigma^j(u+1) \), which were on the cycle in \( T \), become inner and outer edges of level 1 in the Brauer graph for \( \text{End}(Q) \). In particular, the cycle for \( \text{End}(Q) \) has fewer edges than the one for \( A \).

**Proof of the claim.** We have to define maps of complexes between summands of \( Q \) corresponding to the consecutive edges as given in the claim.

We define a map \( Q(\sigma(u)) \rightarrow Q(u-1) \) by the projection onto \( P(u-1) \) in degree 0.

The maps between the following stalk complexes are defined by \([u-1, (u-1)_1] : Q(u-1) \rightarrow Q((u-1)_1) \), and, for \( i = 1, \ldots, q_u-1 \) by \([u-1, (u-1)_{i+1}] : Q((u-1)_i) \rightarrow Q((u-1)_{i+1}) \).

Moreover, there is the map \([u-1, u-2] : Q((u-1)_{q_u-1}) \rightarrow Q(u-2) \). If \( u-3 \) is contained in the \( \sigma \)-orbit of \( u \) then this map factors through the complex

\[
Q(u-3) : 0 \rightarrow P(\sigma^{-1}(u-4)) \oplus P(u-2) \rightarrow P(u-3) \rightarrow 0
\]

(Indeed a map of complexes since \( (u-1)_{q_u-1}, u-2, u-3 \) are not consecutive edges). This factorization then makes \( u-2 \) an edge of level 1 in \( T^* \). Otherwise, \( u-2 \) is on the cycle of \( T^* \), between \( Q((u-1)_{q_u-1}) \) and \( Q((u-1)_1) \).

The maps \([u-1, (u-1)^{i+1}] : Q((u-1)^i) \rightarrow Q((u-1)^{i+1}) \) are as before for \( A \).
Finally, we have a map $Q((u - 1)^{p_u - 1}) \to Q(\sigma^2(u))$ given by $((u - 1)^{p_u - 1}, \sigma(u - 1), 0)$ in degree 0. Note that this indeed is a map of complexes since $(u - 1)^{p_u - 1}, \sigma(u - 1), \sigma^2(u)$ are not consecutive edges.

Next, we consider the edges around the other endpoint of the edge $\tilde{\sigma}(u) = Q(\sigma(u))$.

Let the map $Q(\sigma(u)) \to Q(\sigma(u + 1))$ be defined by the projection onto $P(\sigma(u + 1))$ in degree 0. Moreover, we define the following maps between stalk complexes. We set $[\sigma(u + 1), \sigma((u + 2)_i^1)] : Q(\sigma(u + 1)) \to Q(\sigma((u + 2)_i^1))$ and, for all $i = 1, \ldots, p_{u+2} - 1$, we set $[\sigma((u + 2)_i^1), \sigma((u + 2)_{i+1}^1)] : Q(\sigma((u + 2)_i^1)) \to Q(\sigma((u + 2)^{i+1}_{i+1}))$. As in $A$, there is a map $[\sigma((u + 2)^{p_{u+2}}), \sigma^2(u + 2)] : Q(\sigma((u + 2)^{p_{u+2}})) \to Q(\sigma^2(u + 2))$.

If $u - 3$ is in the $\sigma$-orbit of $u$, then also $\sigma^3(u + 3)$ is in the $\sigma$-orbit of $u$. Then the above map $[\sigma((u + 2)^{p_{u+2}}), \sigma^2(u + 2)]$ factors through (the first summand of) the complex

$$Q(\sigma^3(u + 3)) : 0 \to P(\sigma^2(u + 2)) \oplus P(\sigma^3(u + 4)) \to P(\sigma^3(u + 3)) \to 0.$$

Note that this indeed gives a map of complexes since $\sigma((u + 2)^{p_{u+2}}), \sigma^2(u + 2), \sigma^3(u + 3)$ are not consecutive edges. In this case, $Q(\sigma^2(u + 3))$ is the next edge on the cycle adjacent to $Q(\sigma(u))$. Otherwise, $\sigma(u + 2)$ is the next edge on the cycle.

The maps between the successive inner edges come from stalk complexes and are given exactly as in $A$, that is, we consider $[\sigma^2((u + 2)_i), \sigma^2((u + 2)_{i+1}^1)]$, and, for $i = 1, \ldots, p_{u+2} - 1$, we have $[\sigma^2((u + 2)_i), \sigma^2((u + 2)_{i+1}^1)]$.

Finally, we need a map $Q(\sigma^2((u + 2)_{q_{u+2}})) \to Q(\sigma^2(u))$ which we define by

$$(0, [\sigma^2((u + 2)_{q_{u+2}}), \sigma^2(u + 1)]) : P(\sigma^2((u + 2)_{q_{u+2}})) \to P(\sigma(u - 1)) \oplus P(\sigma^2(u + 1))$$

in degree 0. Note that this is a map of complexes since $\sigma^2((u + 2)_{q_{u+2}}), \sigma^2(u + 1), \sigma^2(u)$ are not consecutive edges.

Clearly, all maps defined above are not homotopic to zero.

In order to complete the proof we have to check that they satisfy the defining relations of $\Omega(T^*, \sigma^*, \theta)$, up to homotopy.

1) **Non-consecutive edges.** We only have to check sequences of edges where one of the new complexes occurs. (For the stalk complexes it directly follows from the same relations in $A$ that non-consecutive edges give zero relations.)

The edges $(u - 1)^{p_u - 1}, \tilde{\sigma}^2(u), \sigma^2(u + 1)$ are not consecutive in the order $p(T^*, \sigma^*)$. The corresponding composition of maps of complexes, as defined above, is given by first mapping to the first summand in degree 0, then projecting onto the second summand. Clearly, this composition is zero. Similarly, the inner edges $\sigma((u + 2)_{q_{u+2}}), \tilde{\sigma}(u), u - 1$ are not consecutive. The corresponding composition of maps of complexes first maps into the second summand in degree 0 and then projects onto the first summand, which is obviously zero.

2) **Socle relations.** From the above definitions, it is easy to check that for all edges $f$ corresponding to stalk complexes the socle map $Q(f) \to Q(\sigma^f(u))$ is just given by the socle map $P(f) \to P(\sigma(f))$. Then the socle relations immediately follow from the socle relations of $A$. (Clearly, around the endpoint of $f$ not attached to the cycle, we also define
$Q(f) \to Q(\sigma^*(f))$ by the same socle map.) So it only remains to prove the socle relations for the new complexes $Q(\sigma^i(u))$. Again, it suffices to consider

$$Q(\sigma(u)) : 0 \to P(u - 1) \oplus P(\sigma(u + 1)) \to P(\sigma(u)) \to 0.$$ 

Around the one endpoint of $\tilde{\sigma}(u)$ we obtain the map $C_1^* : Q(\sigma(u)) \to Q(\sigma^2(u))$ which is given by the socle map $P(u - 1) \to P(\sigma(u - 1))$ on the first summand in degree 0. Around the other endpoint we obtain the map $C_2^*$ given by the socle map $P(\sigma(u + 1)) \to P(\sigma^2(u + 1))$ on the second summand in degree 0. Both maps are zero in degree $-1$.

In order to satisfy the relations of $\Omega^{(1)}(T^*, \sigma^*, \theta)$ these maps have to be homotopic (up to the scalar $\theta$ in case of the distinguished edge involved). By assumption, $u - 1$ and $u + 1$ are not in the $\sigma$-orbit of the distinguished edge 1. In particular, the distinguished edge remains on the cycle of $\text{End}(Q)$. For checking the relation involving the scalar $\theta$, we can assume that $\sigma(u) = 1$ is the distinguished edge (otherwise, set $\theta = 1$ in the sequel). Then the map $C_1^* - \theta C_2^*$ is homotopic to zero via the homotopy map

$$([\sigma(u), \sigma(u - 1)], -\theta[\sigma(u), \sigma^2(u + 1)]) : P(\sigma(u)) \to P(\sigma(u - 1)) \oplus P(\sigma^2(u + 1)).$$

(For checking the details, one also repeatedly uses the fact that non-consecutive edges in the order of $T$ give zero relations.)

This completes the proof of Proposition 4.1.

\section{Derived normal forms for the algebras $\Omega^{(1)}$}

In this section we indicate how one can combine the reduction steps in the previous sections to obtain a proof of the fact that any algebra $\Omega^{(1)}(T, \sigma, \theta)$ is derived equivalent to one of the normal forms $\Lambda(p, q, k, s, \theta)$.

We can actually prove the following more precise result.

\textbf{Theorem 5.1.} Let $A = \Omega^{(1)}(T, \sigma, \theta)$ be an algebra as defined in Section 7, where the cycle of $T$ has length $k$. Denote by $o(T)$ the (total) number of outer edges in $T$, and by $i(T)$ the (total) number of inner edges in $T$. Then $A$ is derived equivalent to the normal form $\Lambda(p', q', k', s', \theta)$, where the parameters $p', q', k', s'$ are given as follows

$$p' = \frac{\gcd(s, k)}{k} \cdot o(T) + \frac{\gcd(s, k) - 1}{2},$$
$$q' = \frac{\gcd(s, k)}{k} \cdot i(T) + \frac{\gcd(s, k) - 1}{2},$$
$$k' = \frac{k}{\gcd(s, k)},$$
$$s' = \frac{s}{\gcd(s, k)}.$$ 

\textbf{Remark.} It might not be obvious that the parameters $p'$ and $q'$ as given in the theorem are actually natural numbers. Let us recall the assumptions on the parameters $s, k$ in the definition of $\Omega^{(1)}(T, \sigma, \theta)$, where $T$ has a cycle of length $k \geq 2$ (see Section 4).

The assumption for the one-parametric algebras has been that $1 \leq s \leq k - 1$ and $\gcd(s + 2, k) = 1$. In particular, this implies that $\gcd(s, k)$ is odd, that is, the second summand in the formulae for $p'$ and $q'$ is an integer.
The automorphism $\sigma_s$ is just rotation of $T$ by $s$ steps on the cycle of length $k$. Hence, the order of $\sigma_s$ (as an automorphism) is $\frac{k}{\gcd(s,k)}$. In particular, the $\sigma_s$-orbit of any edge (or any vertex) consists of exactly $\frac{k}{\gcd(s,k)}$ edges (or vertices). The group generated by $\sigma_s$ acts on the outer (and inner) edges of $T$. Hence, the orbit length $\frac{k}{\gcd(s,k)}$ divides the numbers $o(T)$ and $i(T)$. In particular, the formulae for $p'$ and $q'$ actually give natural numbers.

**Proof of Theorem 5.1.** Let $A = \Omega^{(1)}(T, \sigma_s, \theta)$ be an algebra as defined in Section 1 with a cycle of length $k$. For the proof of the theorem we need to have a very detailed look at the reduction steps of the previous sections. In particular, we need to know how the parameters $s, k, o(T), i(T)$ occurring on the right hand side are changed in each step.

The first reduction step Proposition 2.1 states that, up to derived equivalence, the inner and outer Brauer trees can be made into stars, that is, all edges have level at most 1. The cycle of $T$ remains unchanged, as does the automorphism $\sigma_s$. So, the parameters considered in the statement of the theorem are not affected by this first reduction step.

The second reduction step Proposition 3.1 states that $A$ is derived equivalent to an algebra $\Omega^{(1)}(\overline{T}, \sigma_s, \theta)$ where all edges of level 1 are attached to some vertex in the $\sigma_s$-orbit of one fixed vertex $v$. Recall that in this second reduction step inner (resp. outer) edges remain inner (resp. outer) edges. In particular, the numbers $o(T)$ and $i(T)$ are not changed by this reduction step. Note also that again the cycle and the automorphism $\sigma_s$ are not changed.

We can determine the number of outer and inner edges attached to each vertex in the $\sigma_s$-orbit of the fixed vertex $v$ of $\overline{T}$. There are $o(T)$ outer edges in total; they are equally distributed to the $\frac{k}{\gcd(s,k)}$ vertices in the $\sigma_s$-orbit. Hence, at each such a vertex we have $\frac{\gcd(s,k)}{k} \cdot o(T)$ outer and $\frac{\gcd(s,k)}{k} \cdot i(T)$ inner edges attached. Note that these terms are exactly the first summands in the formulae for $p'$ and $q'$.

In the third reduction step Proposition 4.1 the length $k$ of the cycle and the rotation parameter $s$ are changed. Recall the situation of Proposition 4.1. We assume that there are two consecutive vertices on the cycle of $\overline{T}$ with no edges of level 1 attached to them (i.e., these vertices are not in the $\sigma_s$-orbit of the distinguished vertex $v$). The adjacent edges were denoted $u - 1, u, u + 1$. Then, up to derived equivalence, we can change the Brauer graph $\overline{T}$ to the Brauer graph $T^*$ where all edges in the $\sigma_s$-orbit of $u - 1$ become inner edges of level 1, and all edges in the $\sigma_s$-orbit of $u + 1$ become outer edges of level 1. By applying again our second reduction step (moving edges), if necessary, we can assume that the new inner and outer edges are also attached to vertices in the $\sigma_s$-orbit of the fixed vertex $v$ (recall that this second reduction step does not affect the parameters).

We have to determine how the parameters $s$ and $k$ are changed. The $\frac{k}{\gcd(s,k)}$ vertices in the $\sigma_s$-orbit of the fixed vertex $v$ of $\overline{T}$ are at equal distance apart from each other, where this distance is $\gcd(s,k)$ (the length of the cycle divided by the length of the orbit). By Proposition 4.1 in each of the intervals between two vertices of the $\sigma_s$-orbit of $v$, two edges on the cycle become edges of level 1. Hence, the length of the cycle of $T^*$ is given by $k_1 := k - 2 \cdot \frac{k}{\gcd(s,k)}$. Concerning the rotation parameter, there are exactly $\frac{s}{\gcd(s,k)}$ such intervals between the fixed vertex $v$ and $\sigma_s(v)$. Hence, the automorphism $\sigma^*$ rotates $T^*$ by
\[ s_1 := s - 2 \cdot \frac{s}{\gcd(s, k)}. \]

We can now apply the third reduction step inductively. We set \( k_0 = k \) and \( s_0 = s \). After \( i \) iterations we obtain a Brauer graph \( T^*_i \) with a cycle of length \( k_i := k_{i-1} - 2 \cdot \frac{k_{i-1}}{\gcd(s_{i-1}, k_{i-1})} \) and rotation parameter \( s_i := s_{i-1} - 2 \cdot \frac{s_{i-1}}{\gcd(s_{i-1}, k_{i-1})} \).

By our assumptions we know that the length \( \gcd(s, k) \) of the intervals is odd (see the preceding remark). So, after \( \frac{\gcd(s, k) - 1}{2} \) iterations we get a Brauer graph for which the intervals are of length 1, that is, the action of the rotation is transitive. The corresponding algebra is one of the normal forms \( \Lambda(p', q', k', s', \theta) \), that is, we completed the proof of the implication \( (1) \Rightarrow (2) \) in our main result Theorem 1.

It remains to determine the parameters for the normal forms.

After the second reduction step, in the Brauer graph \( T \) we have \( \frac{\gcd(s, k)}{k} \cdot \alpha(T) \) outer edges and \( \frac{\gcd(s, k)}{k} \cdot \beta(T) \) inner edges attached to each vertex in the \( \sigma_{s'} \)-orbit of the fixed vertex \( v \).

After \( \frac{\gcd(s, k) - 1}{2} \) iterations of the third reduction step we get at each of these vertices exactly \( \frac{\gcd(s, k) - 1}{2} \) additional outer and inner edges. This proves the formulae for \( p' \) and \( q' \) given in Theorem 5.1.

Closed formulae for the number \( k' \) of edges on the cycle and the rotation parameter \( s' \) are provided by the following observation on the numbers \( k_i \) and \( s_i \) which were defined inductively above.

**Claim:** For any \( i = 0, \ldots, \frac{\gcd(s, k) - 1}{2} \), we have \( k_i = k \cdot \frac{\gcd(s, k) - 2i}{\gcd(s, k)} \) and \( s_i = s \cdot \frac{\gcd(s, k) - 2i}{\gcd(s, k)} \).

**Proof of the claim.** We first prove by induction that \( \gcd(s_i, k_i) = \gcd(s, k) - 2i \).

By definition, this holds for \( i = 0 \). Let \( i > 0 \). Then

\[
\gcd(s_i, k_i) = \gcd(s_{i-1}(1 - \frac{2}{\gcd(s_{i-1}, k_{i-1})}), k_{i-1}(1 - \frac{2}{\gcd(s_{i-1}, k_{i-1})})) \\
= (1 - \frac{2}{\gcd(s_{i-1}, k_{i-1})}) \gcd(s_{i-1}, k_{i-1}) \\
= \gcd(s_{i-1}, k_{i-1}) - 2 \\
= \gcd(s, k) - 2i,
\]

by induction.

Using this formula for the \( \gcd(s_i, k_i) \), the formulae for \( k_i \) and \( s_i \) now follow easily by induction. This completes the proof of the claim.

Recall that we got to our normal form \( \Lambda(p', q', k', s', \theta) \) after exactly \( i = \frac{\gcd(s, k) - 1}{2} \) iterations of our third reduction step. Using the above claim, the length of the cycle is easily computed to be \( k' = k' = \frac{k}{2} = \frac{k}{\gcd(s, k)} \) and the rotation parameter is equal to \( s' = \frac{s}{\gcd(s, k)} \). This completes the proof of Theorem 5.1. \( \square \)

The following example illustrates the above considerations.

**Example 5.2.** Let \( T \) be the Brauer graph of the form
and $s = 3$. Then, for $k = 6$, we have $\gcd(s + 2, k) = 1$ but $\gcd(s, k) = 3 \neq 1$. Furthermore, using the notation of Theorem 5.1, we have $o(T) = 14$, $i(T) = 6$ and thus obtain $p' = 8$, $q' = 4$, $k' = 2$, and $s' = 1$. Therefore, for $\theta \in K \setminus \{0\}$, the algebra $\Omega^{(1)}(T, \sigma_3, \theta)$ is derived equivalent to $\Lambda(8, 4, 2, 1, \theta) = \Omega^{(1)}(T_{8,4,2}, \sigma_1, \theta)$, where the Brauer graph $T_{8,4,2}$ is of the form

6. Derived normal forms for the algebras $\Omega^{(2)}$

The aim of this section is to indicate a proof of the following derived equivalence classification for the algebras $\Omega^{(2)}(T, v_1, v_2)$ as defined in Section 4.

**Proposition 6.1.** Any algebra $\Omega^{(2)}(T, v_1, v_2)$ is derived equivalent to the normal form $\Gamma^*(n)$ where $n + 2$ is the number of simple modules of $\Omega^{(2)}(T, v_1, v_2)$.

In particular, two algebras in the family $\Omega^{(2)}$ are derived equivalent if and only if they have the same number of simple modules.

The algebras $\Omega^{(2)}(T, v_1, v_2)$ are very similar to weakly symmetric algebras of Euclidean type studied in [7]. The difference only consists of a few relations which interchange the socles of two projective indecomposable modules making $\Omega^{(2)}(T, v_1, v_2)$ selfinjective but not weakly symmetric. Apart from that, all remaining relations are the same.
In [7, Section 4] we established the derived normal forms $\Gamma (n)$ for the weakly symmetric algebras $\Gamma^{(2)}(T, v_1, v_2)$. It turned out that the normal forms $\Gamma (n)$ are very similar to our new normal forms $\Gamma^*(n)$, up to a twist in the relations, exactly corresponding to the twist in the relations for going from $\Gamma^{(2)}(T, v_1, v_2)$ to $\Omega^{(2)}(T, v_1, v_2)$. In fact, the algebra $\Gamma^*(n)$ is given by the following quiver and relations

$$
\begin{align*}
\alpha_1 \alpha_2 &= (\beta_1 \beta_2 \ldots \beta_n)^2 = \gamma_1 \gamma_2, \\
\alpha_2 \beta_1 &= 0, \quad \gamma_2 \beta_1 = 0, \\
\beta_n \gamma_1 &= 0, \quad \alpha_2 \alpha_1 = 0, \quad \gamma_2 \gamma_1 = 0, \\
\beta_j \beta_{j+1} \ldots \beta_n \beta_1 \ldots \beta_{j-1} \beta_j &= 0, \quad 2 \leq j \leq n,
\end{align*}
$$

and $\Gamma (n)$ is obtained from $\Gamma^*(n)$ by replacing the relations $\alpha_2 \alpha_1 = 0$ and $\gamma_2 \gamma_1 = 0$ by the relations $\alpha_2 \gamma_1 = 0$ and $\gamma_2 \alpha_1 = 0$.

Because of the similarities described above it is therefore not surprising that the proof for the derived equivalence classification of the weakly symmetric algebras carries over almost verbatim to our new situation. We refrain from repeating the proof here, and refer for details to our previous paper [7, Section 4].

**7. Auslander-Reiten sequences**

In order to prove Theorem 1, we need more detailed information on the Auslander-Reiten quivers of standard one-parametric selfinjective algebras and the action of Heller’s syzygy operator on their stable module categories.

Let $A$ be a selfinjective algebra. We denote by $\Gamma_A$ the Auslander-Reiten quiver of $A$ and by $\tau_A$ and $\tau_A^{-1}$ the Auslander-Reiten translations $D \operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. We will identify an indecomposable module $M$ from $\operatorname{mod} A$ with the vertex $[M]$ of $\Gamma_A$ corresponding to it. Further, we denote by $\Gamma^*_A$ the stable Auslander-Reiten quiver of $A$, obtained from $\Gamma_A$ by removing the projective modules and the arrows attached to them. Recall also that for an indecomposable projective $A$-module $P$ we have in $\operatorname{mod} A$ an Auslander-Reiten sequence of the form

$$
0 \rightarrow \operatorname{rad} P \rightarrow \operatorname{rad} P/S \oplus P \rightarrow P/S \rightarrow 0,
$$

where $S$ is the socle of $P$ [5, Proposition V.5.5]. Finally, we denote by $\Omega_A$ Heller’s syzygy
operator on mod\(A\) which assigns to any module \(M\) the kernel of its projective cover \(P(M) \rightarrow M\).

The first result of this section describes the shape of the stable Auslander-Reiten quivers of the algebras \(\Gamma^* (n)\).

**Proposition 7.1.** \(\Gamma^*_n\) consists of one Euclidean component \(\mathbb{Z}\tilde{\mathbb{D}}_{2n+3}\), two stable tubes of rank 2, one stable tube of rank \(2n+1\), and a family of stable tubes of rank 1 indexed by \(K\{\}\).

**Proof.** It follows from \([11, \text{Section 2}]\) and \([7, \text{Proposition 5.3}]\) that \(\Gamma^* (n) \cong \tilde{C} (n) / (\rho_n \psi_n)\), where \(C (n)\) is the bound quiver algebra \(K\Delta (n) / I (n)\) given by the quiver

![Quiver diagram](image)

and the ideal \(I (n)\) in the path algebra \(K\Delta (n)\) generated by the elements \(\alpha_{n-1} \gamma\) and \(\alpha_{n-1} \delta\), \(\psi_n\) is a canonical automorphism of the repetitive algebra \(\tilde{C} (n)\) such that \(\psi_n^2 = \nu_{\tilde{C} (n)}\), and \(\rho_n\) is a rigid automorphism of \(\tilde{C} (n)\) with \(\rho_n^2 = id_{\tilde{C} (n)}\). Then \(C (n)\) is a tubular extension of the path algebra of the unique convex subquiver of \(\Delta (n)\) of Euclidean type \(\tilde{\mathbb{D}}_{n+4}\) of tubular type \((2, 2, 2n+1)\), and so \(C (n)\) is a representation-infinite tilted algebra of Euclidean type \(\tilde{\mathbb{D}}_{2n+3}\) (see \([32, (4.9)]\)). Hence the required shape of \(\Gamma^*_n\) follows from \([3, \text{Section 4}]\) and \([35, \text{Section 2}]\). \(\square\)

Our next aim is to describe the shape of the stable Auslander-Reiten quivers of the algebras \(\Lambda (p, q, k, s, \theta)\) and the structure of modules lying on the mouth of their large stable tubes.

Let \(p, q, k, s\) be integers such that \(p, q \geq 0, k \geq 2, 1 \leq s \leq k - 1, \gcd (s, k) = 1,\) and \(\gcd (s + 2, k) = 1\). Since \(\gcd (s + 2, k) = 1\), there is exactly one integer \(m_{k,s}\) such that \(1 \leq m_{k,s} \leq k - 1,\) and

\[
m_{k,s} (s + 2) + 1 \equiv 0 \pmod{k}.
\]

By definition, \(\Lambda (p, q, k, s, \theta)\) is the bound quiver algebra

\[
\Omega^{(1)} (T_{p,q,k}, \sigma_s, \theta) = KQ (T_{p,q,k}, \sigma_s) / T^{(1)} (T_{p,q,k}, \sigma_s, \theta),
\]

where the vertices of the quiver \(Q (T_{p,q,k}, \sigma_s)\) are the edges of the Brauer graph \(T_{p,q,k}\), and there is an arrow \(i \rightarrow j\) in \(Q (T_{p,q,k}, \sigma_s)\) if \(j\) is the direct successor of \(i\) in the partial order \(p (T_{p,q,k}, \sigma_s)\), as defined in Section \([11]\). We note that \(\Lambda (p, q, k, s, \theta)\) is a special biserial algebra.
Moreover, since we compose the arrows in $Q(T_{p,q,k}, \sigma_s)$ from left to right, the category mod $\Lambda (p, q, k, s, \theta)$ of finite dimensional (left) $\Lambda (p, q, k, s, \theta)$-modules is identified with the category of finite dimensional representations (over $K$) of the bound quiver opposite to the bound quiver $(Q(T_{p,q,k}, \sigma_s), \mathcal{T}^{1}(T_{p,q,k}, \sigma_s, \theta))$. For each edge $i$ of $T_{p,q,k}$, we denote by $S (i)$ the simple $\Lambda (p, q, k, s, \theta)$-module at $i$, and by $P (i)$ the projective cover of $S (i)$ in mod $\Lambda (p, q, k, s, \theta)$. The edges of $T_{p,q,k}$ of the form $i^r$, $1 \leq i \leq k$, $1 \leq r \leq p$, are said to be outer edges of $T_{p,q,k}$, while the edges of the form $i_l$, $1 \leq i \leq k$, $1 \leq l \leq q$, are said to be the inner edges of $T_{p,q,k}$. We call the simple modules $S (i^r)$ the outer simple modules and the simple modules $S (i_l)$ the inner simple modules. We describe now the structure of indecomposable projective $\Lambda (p, q, k, s, \theta)$-modules. Since $\Lambda (p, q, k, s, \theta)$ is a special biserial algebra, every indecomposable projective $\Lambda (p, q, k, s, \theta)$-module $P$ is either serial (has a unique composition series) or biserial (rad $P$/soc $P$ is a direct sum of two serial modules). Observe also, that for each vertex $i$ of $Q(T_{p,q,k}, \sigma_s)$, the vertex $\sigma_s(i)$ is the target of maximal paths in $Q(T_{p,q,k}, \sigma_s)$ with source $i$ which are not in $\mathcal{T}^{1}(T_{p,q,k}, \sigma_s, \theta)$, and hence $S (i)$ is the socle of the projective module $P (\sigma_s (i))$. This also shows that the action of $\sigma_s$ on the vertices of $Q(T_{p,q,k}, \sigma_s)$ is exactly the Nakayama permutation of the algebra $\Lambda (p, q, k, s, \theta)$. For each outer edge $i^r$ of $T_{p,q,k}$, the path

$$i^r \to \ldots \to i^p \to i+s \to (i+s)_1 \to \ldots \to (i+s)_q \to i+s-1 \to (i+s)^1 \to \ldots \to (i+s)^r = \sigma_s (i^r)$$

is the unique path in $Q(T_{p,q,k}, \sigma_s)$ from $i^r$ to $\sigma_s (i^r)$ of length at least 2 which is not in $\mathcal{T}^{1}(T_{p,q,k}, \sigma_s, \theta)$, and consequently $P ((i+s)^r)$ is the serial module with socle $S (i^r)$, and the simple composition factors given by the simple modules at the vertices of this path. Similarly, for each inner edge $i_l$ of $T_{p,q,k}$, the path

$$i_l \to \ldots \to i_q \to i-1 \to i^1 \to \ldots \to i^p \to i+s \to (i+s)_1 \to \ldots \to (i+s)_l = \sigma_s (i_l)$$

is the unique path in $Q(T_{p,q,k}, \sigma_s)$ from $i_l$ to $\sigma_s (i_l)$ of length at least 2 which is not in $\mathcal{T}^{1}(T_{p,q,k}, \sigma_s, \theta)$, and consequently $P ((i+s)_l)$ is the serial module with socle $S (i_l)$, and the simple composition factors given by the simple modules at the vertices of this path. Finally, for an edge $i$ of the cycle of $T_{p,q,k}$, we have two paths in $Q(T_{p,q,k}, \sigma_s)$ from $i$ to $\sigma_s (i) = i+s$ which are not in $\mathcal{T}^{1}(T_{p,q,k}, \sigma_s)$, and they are of the forms

$$i \to (i+1)^1 \to \ldots \to (i+1)^p \to i+1+s \to (i+1+s)_1 \to \ldots \to (i+1+s)_q \to i+s,$$

$$i \to i_1 \to \ldots \to i_q \to i-1 \to i^1 \to \ldots \to i^p \to i+s.$$

Hence, $P (i+s)$ is the biserial module with socle $S (i)$, and rad $P (i+s)$/soc $P (i+s)$ is the direct sum of two serial modules: one with the top $S ((i+1+s)_q)$, the socle $S ((i+1)^1)$, and the simple composition factors formed by the simple modules at the vertices of the first path, and the second with the top $S (i^p)$, the socle $S (i_1)$, and the simple composition factors formed by the simple modules at the vertices of the second path. We also note that, for $p = 0$, any path of the form $i \to (i+1)^1 \to \ldots \to (i+1)^p \to i+1+s$ is reduced to the...
arrow \( i \rightarrow i+1+s \). Similarly, for \( q = 0 \), any path of the form \( i \rightarrow i_1 \rightarrow \ldots \rightarrow i_q \rightarrow i-1 \) is reduced to the arrow \( i \rightarrow i-1 \). For \( i \in \{1,2,\ldots,k\} \), we denote by \( U(i) \) the factor module of \( P(i) \) by the unique serial submodule with the top \( S((i-s)p) \), if \( p \geq 1 \), and \( S(i-s-1) \) if \( p = 0 \). Similarly, for \( i \in \{1,2,\ldots,k\} \), we denote by \( V(i) \) the factor module of \( P(i) \) by the unique serial submodule with the top \( S((i+1)_q) \), if \( q \geq 1 \), and \( S(i+1) \) if \( q = 0 \).

The following proposition describes the stable Auslander-Reiten quivers of the algebras \( \Lambda(p,q,k,s,\theta) \).

**Proposition 7.2.** \( \Gamma^s_{\Lambda(p,q,k,s,\theta)} \) consists of one Euclidean component \( \mathbb{Z} \tilde{A}_{2(p+q+1)k-1} \), a stable tube \( \mathcal{T}^\text{out}_{p,q,k,s} \) of rank \( (2p+1)k \), a stable tube \( \mathcal{T}^\text{inn}_{p,q,k,s} \) of rank \( (2q+1)k \), and a family of stable tubes of rank 1 indexed by \( K \setminus \{0\} \). Moreover,

1. the mouth of the tube \( \mathcal{T}^\text{out}_{p,q,k,s} \) consists of the modules \( U(i), S(i^r), P(i^r)/\text{soc} \ P(i^r) \), \( 1 \leq i \leq k, 1 \leq r \leq p \);
2. the mouth of the tube \( \mathcal{T}^\text{inn}_{p,q,k,s} \) consists of the modules \( V(i), S(i_l), P(i_l)/\text{soc} \ P(i_l) \), \( 1 \leq i \leq k, 1 \leq l \leq q \).

**Proof.** Let \( \Lambda = \Lambda(p,q,k,s,\theta) \), \( Q = Q(T_{p,q,k},\sigma_s) \) and \( \mathcal{T} = \mathcal{T}^{(1)}(T_{p,q,k},\sigma_s,\theta) \). We know that \( \Lambda \) is a special biserial one-parametric selfinjective algebra (see Theorem 13). Then it follows from [17, Theorem 2.1] that \( \Gamma^s_{\Lambda} \) consists of one Euclidean component \( \mathbb{Z} \tilde{A}_r \) and a family of stable tubes indexed by \( \mathbb{P}_1(K) = K \cup \{\infty\} \), containing a family of stable tubes of rank 1 indexed (say) by \( K \setminus \{0\} \). Moreover, if \( \Gamma^s_{\Lambda} \) admits two stable tubes of ranks \( m \) and \( n \) greater than 1 then \( r = m+n-1 \). Therefore, since \( k \geq 2 \), it remains to prove that \( \Gamma^s_{\Lambda} \) admits the required stable tubes \( \mathcal{T}^\text{out}_{p,q,k,s} \) of rank \( (2p+1)k \) and \( \mathcal{T}^\text{inn}_{p,q,k,s} \) of rank \( (2q+1)k \). We will apply the known formulae for the Auslander-Reiten sequences of string modules over special biserial algebras (see [15, Chapter II]). We also note that the ideal \( \mathcal{T} \) is not an admissible ideal of the path algebra \( KQ \). Indeed, for any outer edge \( i^r \) of \( T_{p,q,k} \) we have \( C(i^r,p(T_{p,q,k},\sigma_s,v_i)) = C(i^r,p(T_{p,q,k},\sigma_s,w(i^r))) \in \mathcal{T} \), where \( v(i) \) is the vertex of \( i^r \) on the unique cycle of \( T_{p,q,k} \) and \( w(i^r) \) the second (outer) vertex of \( i^r \), and \( C(i^r,p(T_{p,q,k},\sigma_s,w(i^r))) \) is the arrow \( i^r \rightarrow \sigma_s(i^r) \). We call the arrow \( i^r \rightarrow \sigma_s(i^r) \) a superfluous arrow of the bound quiver \( (Q,\mathcal{T}) \).

Similarly, for any inner edge \( i_l \) of \( T_{p,q,k} \) we have \( C(i_l,p(T_{p,q,k},\sigma_s,v_i)) = C(i_l,p(T_{p,q,k},\sigma_s,w(i_l))) \in \mathcal{T} \), where \( v(i) \) is the vertex of \( i_l \) on the unique cycle of \( T_{p,q,k} \) and \( w(i_l) \) the second (inner) vertex of \( i_l \), and \( C(i_l,p(T_{p,q,k},\sigma_s,w(i_l))) \) is the arrow \( i_l \rightarrow \sigma_s(i_l) \). We call the arrow \( i_l \rightarrow \sigma_s(i_l) \) a superfluous arrow of the bound quiver \( (Q,\mathcal{T}) \).

1. The tube \( \mathcal{T}^\text{out}_{p,q,k,s} \). Consider first the case \( p = 0 \). Observe that, for each \( i \in \{1,2,\ldots,k\} \), the quiver \( Q \) contains a string of the form

\[
i \leftarrow (i+1)_q \leftarrow \ldots \leftarrow (i+1)_1 \leftarrow i+1 \rightarrow i+s+2 \leftarrow (i+s+3)_q \leftarrow \ldots \leftarrow (i+s+3)_1,
\]

and denote by \( L(i) \) the associated string \( \Lambda \)-module. Then we have in \( \text{mod} \ \Lambda \) an Auslander-Reiten sequence of the form

\[
0 \rightarrow U(i) \rightarrow L(i) \rightarrow U(i+s+2) \rightarrow 0,
\]
and hence $U(i) = \tau_A U(i+s+2)$. Since $\gcd(s+2, k) = 1$, we infer that the modules $U(1), U(2), \ldots, U(k)$ form a periodic orbit of a component $T_{0,q,k,s}^{out}$ of $\Gamma_A$. Clearly, $T_{0,q,k,s}^{out}$ is a stable tube of rank $k$ and $U(1), U(2), \ldots, U(k)$ lie on the mouth of $T_{0,q,k,s}^{out}$, because the modules $L(1), L(2), \ldots, L(k)$ are indecomposable.

Assume now $p \geq 1$. Fix $i \in \{1, 2, \ldots, k\}$. Note that the quiver $Q$ contains a string of the form

$$i^p \to i+s \leftarrow (i+s+1)_q \leftarrow \ldots \leftarrow (i+s+1)_1 \leftarrow i+s+1 \leftarrow (i+1)^p \leftarrow \ldots \leftarrow (i+1)^1,$$

and denote by $L(i)$ the associated string $\Lambda$-module. Then we have in $\text{mod} \Lambda$ an Auslander-Reiten sequence of the form

$$0 \to S(i^p) \to L(i) \to U(i+s) \to 0,$$

and hence $S(i^p) = \tau_A U(i+s)$. Moreover, we have $U(i+s) = \text{rad} P((i+s+1)^1)$. Further, $Q$ contains also a string of the form

$$i^1 \leftarrow i-1 \to (i-1)_1 \to \ldots \to (i-1)_q \to i-2 \to (i-1)^1 \to \ldots \to (i-1)^p,$$

and denote by $M(i)$ the associated string $\Lambda$-module. Then we have in $\text{mod} \Lambda$ an Auslander-Reiten sequence of the form

$$0 \to P((i-1)^p) / S((i-1-s)^p) \to M(i) \to S(i^1) \to 0,$$

and hence $P((i-1)^p) / S((i-1-s)^p) = \tau_A S(i^1)$. Observe that $S((i-1-s)^p) = \text{soc} P((i-1)^p)$.

Finally, if $p \geq 2$, we have in $Q$ a path

$$i^1 \to i^2 \to \ldots \to i^p,$$

and, for $r \in \{1, \ldots, p-1\}$, $i^r \to i^{r+1}$ is the unique nonsuperfluous arrow in $Q$ starting at $i^r$ (respectively, ending in $i^{r+1}$). Then we have in $\text{mod} \Lambda$ Auslander-Reiten sequences of the forms

$$0 \to S(i^r) \to L(i^r) \to S(i^{r+1}) \to 0,$$

$1 \leq r \leq p-1$, where $L(i^r)$ is the serial $\Lambda$-module given by the string $i^r \to i^{r+1}$, and consequently $S(i^r) = \tau_A S(i^{r+1})$. Observe also that $\Omega_A S(i^r) = \text{rad} P(i^r)$. Therefore, $S(i^r) = \tau_A S(i^{r+1})$ implies $\text{rad} P(i^r) = \tau_A \text{rad} P(i^{r+1})$, for $r \in \{1, \ldots, p-1\}$, $p \geq 2$.

Combining the calculations above and invoking again our assumption $\gcd(s+2, k) = 1$, we conclude that the modules $U(i), S(i^r), P(i^r) / \text{soc} P(i^r), 1 \leq i \leq k, 1 \leq r \leq p$, form the mouth of a stable tube $T_{p,q,k,s}^{out}$ of $\Gamma_A$ of rank $(2p+1)k = k+pk+pk$.

(2) The tube $T_{p,q,k,s}^{min}$. Consider first the case $q = 0$. For $i \in \{1, 2, \ldots, k\}$, the quiver $Q$ contains a string of the form

$$i \leftarrow (i-s)^p \leftarrow \ldots \leftarrow (i-s)^1 \leftarrow i-s-1 \to i-s-2 \leftarrow (i-2s-2)^p \leftarrow \ldots \leftarrow (i-2s-2)^1,$$

and denote by $R(i)$ the associated string $\Lambda$-module. Then we have in $\text{mod} \Lambda$ an Auslander-Reiten sequence of the form

$$0 \to V(i) \to R(i) \to V(i-s-2) \to 0,$$
and hence \( V(i) = \tau_\Lambda V(i-s-2) \). Since \( \gcd(s+2,k) = 1 \), we infer as above that the modules \( V(1), V(2), \ldots, V(k) \) form the mouth of a stable tube \( T_{p,k,s}^{\text{inn}} \) of \( \Gamma^s_\Lambda \) of rank \( k \).

Assume now \( q \geq 1 \). Fix \( i \in \{1,2,\ldots,k\} \). Observe that \( Q \) contains a string of the form
\[
i_p \to i-1 \leftarrow (i-1-s)^p \leftarrow \ldots \leftarrow (i-1-s)^1 \leftarrow i-2-s \leftarrow (i-1-s)_{q+1} \leftarrow \ldots \leftarrow (i-1-s)_1,
\]
and denote by \( R(i) \) the associated string \( \Lambda \)-module. Then we have in \( \text{mod } \Lambda \) an Auslander-Reiten sequence of the form
\[
0 \to S(i_q) \to R(i) \to V(i-1) \to 0,
\]
and hence \( S(i_q) = \tau_\Lambda V(i-1) \). Moreover, we have \( V(i-1) = \text{rad } P((i-1)_1) \). Further, \( Q \) contains also a string of the form
\[
i_1 \leftarrow i \to (i+1)^1 \to \ldots \to (i+1)^p \to i+1+s \to (i+1+s)_1 \to \ldots \to (i+1+s)_q,
\]
and denote by \( N(i) \) the associated string \( \Lambda \)-module. Then we have in \( \text{mod } \Lambda \) an Auslander-Reiten sequence of the form
\[
0 \to P((i+1+s)_{q+1}) \to S((i+1)_q) \to N(i) \to S(i_1) \to 0,
\]
and hence \( P((i+1+s)_{q+1}) \to S((i+1)_q) \to \tau_\Lambda S(i_1) \). Observe also that \( S((i+1)_q) = \text{soc } P((i+1+s)_{q+1}) \). Finally, if \( q \geq 2 \), we have in \( Q \) a path
\[
i_1 \to i_2 \to \ldots \to i_q,
\]
and, for \( l \in \{1,\ldots,q-1\} \), \( i_l \to i_{l+1} \) is the unique nonsuperfluous arrow of \( Q \) starting at \( i_l \) (respectively, ending in \( i_{l+1} \)). Then we have in \( \text{mod } \Lambda \) Auslander-Reiten sequences of the forms
\[
0 \to S(i_l) \to W(i_l) \to S(i_{l+1}) \to 0,
\]
where \( k \leq l \leq q-1 \), \( W(i_l) \) is the serial \( \Lambda \)-module given by the string \( i_l \to i_{l+1} \), and consequently \( S(i_l) = \tau_\Lambda S(i_{l+1}) \). Since \( \Omega_\Lambda S(i_l) = \text{rad } P(i_l) \), \( S(i_l) = \tau_\Lambda S(i_{l+1}) \) implies \( \text{rad } P(i_l) = \tau_\Lambda \text{rad } P(i_{l+1}) \). Moreover, observe that \( \tau_\Lambda^{2q+1} S(i_1) = S((i+s)_{1}) \) for any \( i \in \{1,2,\ldots,k\} \). Invoking now our assumption \( \gcd(s,k) = 1 \), we conclude that the modules \( V(i), S(i_1), P(i_l) / \text{soc } P(i_l), 1 \leq i \leq k, 1 \leq l \leq q \), form the mouth of a stable tube \( T_{p,k,s}^{\text{inn}} \) of \( \Gamma^s_\Lambda \) of rank \((2q+1)k = k + qk + qk\).

We call \( T_{p,k,s}^{\text{out}} \) the outer stable tube of \( \Gamma^s_\Lambda(p,q,k,s,\theta) \) and \( T_{p,k,s}^{\text{inn}} \) the inner stable tube of \( \Gamma^s_\Lambda(p,q,k,s,\theta) \).

The following proposition describes the action of the syzygy operator on the outer and inner stable tubes of \( \Gamma^s_\Lambda(p,q,k,s,\theta) \).
Proposition 7.3. \( (1) \) \(\Omega_{\Lambda(p,q,k,s,\theta)}\) fixes the outer stable tube \(T_{p,q,k,s}^\text{out}\) of \(\Gamma_{p,q,k,s,\theta}^s\), and \(\Omega_{\Lambda(p,q,k,s,\theta)} = \tau_{\Lambda(p,q,k,s,\theta)}^{m_k,s(2p+1)+p+1}\) on \(T_{p,q,k,s}^\text{out}\).

\( (2) \) \(\Omega_{\Lambda(p,q,k,s,\theta)}\) fixes the inner stable tube \(T_{p,q,k,s}^\text{inn}\) of \(\Gamma_{p,q,k,s,\theta}^s\), and \(\Omega_{\Lambda(p,q,k,s,\theta)} = \tau_{\Lambda(p,q,k,s,\theta)}^{(k-m_k,s)(2q+1)-q}\) on \(T_{p,q,k,s}^\text{inn}\).

Proof. Let \(\Lambda = \Lambda(p,q,k,s,\theta)\), \(T^\text{out} = T_{p,q,k,s}^\text{out}\), \(T^\text{inn} = T_{p,q,k,s}^\text{inn}\), and \(m = m_k,s\). We will apply the formulae on the actions of \(\tau_{\Lambda}\) on \(T^\text{out}\) and \(T^\text{inn}\) established in the proof of Proposition 7.2.

\( (1) \) It is enough to prove that

\[\Omega_{\Lambda}U(1) = \tau_{\Lambda}^{m_k,s(2p+1)+p+1}U(1)\.

Assume \(p = 0\). Observe that \(\Omega_{\Lambda}U(1) = U(s+2)\). Moreover, we have \(\tau_{\Lambda}^{-m-1}U(1) = U(1+(m+1)(s+2)) = U(s+3+m(s+2)) = U(s+2)\), since \(m(s+2)+1 \equiv 0 \pmod{k}\) implies \(s+3+m(s+2) \equiv s+2 \pmod{k}\). Hence, \(\Omega_{\Lambda}U(1) = \tau_{\Lambda}^{(m+1)}U(1)\), or equivalently \(\Omega_{\Lambda}U(1) = \tau_{\Lambda}^{m+1}U(1)\).

Assume \(p \geq 1\). In this case, we have \(U(1) = \text{rad} P(2^1)\), and then \(\Omega_{\Lambda}U(1) = S(2^1)\). We have \(\tau_{\Lambda}^{-p-1}U(1) = \tau_{\Lambda}^{-p-1}\text{rad} P(2^1) = \tau_{\Lambda}^{-p}P(2^p)/\text{soc} P(2^p) = S(3^1)\) and \(\tau_{\Lambda}^{-m(2p+1)}S(3^1) = S((3+m(s+2))^1) = S(2^1)\), because \(m(s+2)+1 \equiv 0 \pmod{k}\) implies \(3+m(s+2) \equiv 2 \pmod{k}\). Therefore, we obtain \(\Omega_{\Lambda}U(1) = \tau_{\Lambda}^{m(2p+1)+p+1}U(1)\), or equivalently \(\Omega_{\Lambda}U(1) = \tau_{\Lambda}^{m(2p+1)+p+1}U(1)\).

\( (2) \) It is enough to prove that

\[\Omega_{\Lambda}V(1) = \tau_{\Lambda}^{(k-m)(2q+1)-q}V(1)\.

Assume \(q = 0\). Observe that \(\Omega_{\Lambda}V(1) = V(2)\) and \(\tau_{\Lambda}^{k-m}V(1) = V(1+(k-m)(s+2))\). Then \(\Omega_{\Lambda}V(1) = \tau_{\Lambda}^{k-m}V(1)\), because \(m(s+2)+1 \equiv 0 \pmod{k}\) implies \(1+(k-m)(s+2) \equiv 2 \pmod{k}\).

Let \(q \geq 1\). In this case, we have \(V(1) = \text{rad} P(1) = \Omega_{\Lambda}S(1)\). Hence \(\Omega_{\Lambda}V(1) = \Omega_{\Lambda}^2S(1)\). Thus \(\Omega_{\Lambda}V(1) = \tau_{\Lambda}^{(k-m)(2q+1)-q}V(1)\) is equivalent to \(V(1) = \tau_{\Lambda}^{(k-m)(2q+1)-q}S(1)\).

We have \(\tau_{\Lambda}^{(k-m)(2q+1)-q}S(1) = S((1+(k-m)(s+2))_1) = S(2_1)\), because \(m(s+2)+1 \equiv 0 \pmod{k}\) implies \(1+(k-m)(s+2) \equiv 2 \pmod{k}\). Therefore, we obtain \(\tau_{\Lambda}^{(k-m)(2q+1)-q}S(1) = \tau_{\Lambda}^qS(2_1) = \tau_{\Lambda}^qS(2_q) = V(1)\), as required.

\(\square\)

8. Weakly symmetric algebras

For the proof of Theorem 1 we need also a family of one-parametric weakly symmetric algebras of Euclidean type \(\widetilde{A}_m\).

Let \(T\) be a Brauer graph with exactly one cycle \(\mathcal{R}_k\), having an odd number \(k\) of edges. Take the trivial rotation \(\sigma_\tau = \sigma_0\) (with \(s = 0\)) of \(T\). Observe that \(\text{gcd} (s+2,k) = \text{gcd} (2,k) = 1\). Moreover, for each vertex \(v\) of \(T\), the cyclic order \(p(T,\sigma_0,v)\), as defined
in Section II is the cyclic order of edges of the Brauer graph $T$ adjacent to $v$. Then, for $\theta \in \mathbb{K} \setminus \{0\}$, consider the algebra $\Omega^{(1)}(T, \theta) = \Omega^{(1)}(T, \sigma_0, \theta)$. For $\theta = 1$, the algebra $\Omega^{(1)}(T, 1)$ is just the one-parametric special biserial algebra $\Lambda(T)$ considered in [7], [9] and [10]. In general, $\Omega^{(1)}(T, \theta)$ is a one-parametric weakly symmetric special biserial algebra which is socle equivalent to $\Omega^{(1)}(T, 1) = \Lambda(T)$.

For $p, q \geq 1$ and $k = 1$, consider the Brauer graph $T_{p,q} = T_{p,q,1}$ of the form

![Brauer graph](image)

Further, for $\theta \in \mathbb{K} \setminus \{0\}$, consider the algebras $\Lambda(p, q, \theta) = \Lambda(p, q, 1, 0, \theta)$. Then $\Lambda(p, q, \theta)$ is a one-parametric weakly symmetric special biserial algebra given by the quiver

![Quiver](image)

and the relations:

\[
\alpha_1 \alpha_2 \ldots \alpha_p \alpha_{p+1} \beta_1 \beta_2 \ldots \beta_q \beta_{q+1} = \theta \beta_1 \beta_2 \ldots \beta_q \beta_{q+1} \alpha_1 \alpha_2 \ldots \alpha_p \alpha_{p+1}, \\
\alpha_{p+1} \alpha_1 = 0, \ \beta_{q+1} \beta_1 = 0, \\
\alpha_i \alpha_{i+1} \ldots \alpha_p \alpha_{p+1} \beta_1 \beta_2 \ldots \beta_q \beta_{q+1} \alpha_1 \ldots \alpha_i - 1 \alpha_i = 0, \ 2 \leq i \leq p + 1, \\
\beta_j \beta_{j+1} \ldots \beta_q \beta_{q+1} \alpha_1 \ldots \alpha_p \alpha_{p+1} \beta_1 \ldots \beta_{j-1} \beta_j = 0, \ 2 \leq j \leq q + 1.
\]

We also note that for $p = q = 0$, $\Lambda(0, 0, \theta)$ is the local 4-dimensional algebra $A(\theta) = K \langle \alpha, \beta \rangle / (\alpha^2, \beta^2, \alpha \beta - \theta \beta \alpha)$, and for $p + q \geq 1$ and $\theta = 1$, $\Lambda(p, q, 1)$ is the symmetric algebra $A(p+1, q+1)$ considered in [7].

The following proposition is then a direct consequence of arguments applied in the proofs of Proposition 2.1 and Theorem 5.1.

**Theorem 8.1.** Let $T$ be a Brauer graph with exactly one cycle, having an odd number $k$ of edges. Denote by $p$ the total number of outer edges of $T$ and by $q$ the total number of inner edges of $T$. Then, for any $\theta \in \mathbb{K} \setminus \{0\}$, the algebra $\Omega^{(1)}(T, \theta)$ is derived equivalent to $\Lambda(p’, q’, \theta)$, where $p’ = p + \frac{k - 1}{2}$ and $q’ = q + \frac{k - 1}{2}$. 
We also note that, in the symmetric case \( \theta = 1 \), the above proposition is related with \cite{23} Theorem 2.3.

As we noticed above, for \( p, q \geq 1 \) and \( \theta \in K \setminus \{0\} \), the algebras \( \Lambda(p, q, \theta) \) and \( \Lambda(p, q, 1) \) are socle equivalent (the factor algebras by the socles are isomorphic). Then the following two propositions follow directly from \cite{7} Propositions 5.3 and 5.4 and their proofs.

**Proposition 8.2.** For \( p, q \geq 0 \) and \( \theta \in K \setminus \{0\} \), the stable Auslander-Reiten quiver \( \Gamma^s_{\Lambda(p,q,\theta)} \) consists of one Euclidean component \( \mathcal{Z} \tilde{\Lambda}_2(\tilde{p}, \tilde{q}+\tilde{1}) \), an outer stable tube \( \mathcal{T}^\text{out}_{p,q,\theta} \) of rank \( 2p+1 \), an inner stable tube \( \mathcal{T}^\text{inn}_{p,q,\theta} \) of rank \( 2q+1 \), and a family of stable tubes of rank 1.

**Proposition 8.3.** For \( p, q \geq 0 \) and \( \theta \in K \setminus \{0\} \), we have

1. \( \Omega_{\Lambda(p,q,\theta)} \) fixes the outer stable tube \( \mathcal{T}^\text{out}_{p,q,\theta} \) and \( \Omega_{\Lambda(p,q,\theta)} = \tau_{p+1} \mathcal{T}^\text{out}_{p,q,\theta} \) on \( \mathcal{T}^\text{out}_{p,q,\theta} \).
2. \( \Omega_{\Lambda(p,q,\theta)} \) fixes the inner stable tube \( \mathcal{T}^\text{inn}_{p,q,\theta} \) and \( \Omega_{\Lambda(p,q,\theta)} = \tau_{p+1} \mathcal{T}^\text{inn}_{p,q,\theta} \) on \( \mathcal{T}^\text{inn}_{p,q,\theta} \).

For the proof of Theorem \cite{24} we need the following result.

**Proposition 8.4.** Let \( p, p', q, q' \geq 0 \), \( k \geq 2 \) and \( s \) be integers with \( 1 \leq s \leq k-1 \), \( \gcd(s+2, k) = 1 \), \( \gcd(s, k) = 1 \), and \( \theta, \theta' \in K \setminus \{0\} \). Then the algebras \( \Lambda(p, q, k, s, \theta) \) and \( \Lambda(p', q', k, s, \theta) \) are not stably equivalent.

**Proof.** Suppose that the algebras \( \Lambda = \Lambda(p, q, k, s, \theta) \) and \( \Lambda' = \Lambda(p', q', k, s, \theta) \) are stably equivalent, and let \( F: \text{mod} \Lambda \rightarrow \text{mod} \Lambda' \) be a functor inducing a stable equivalence. Since \( \Lambda \) and \( \Lambda' \) are selfinjective algebras of Loewy length at least 3, applying \cite{35} (X.1.9), \( (X.1.12) \), we have \( F \tau_{\Lambda} = \tau_{\Lambda'} F \) and \( F \Omega_{\Lambda} = \Omega_{\Lambda'} F \). In particular, the stable Auslander-Reiten quivers \( \Gamma^s_{\Lambda} \) and \( \Gamma^s_{\Lambda'} \), are isomorphic translation quivers. Further, since \( k \geq 2 \), it follows from Proposition \cite{22} that the stable tubes \( \mathcal{T}^\text{out}_{\Lambda} \) and \( \mathcal{T}^\text{inn}_{\Lambda} \) have ranks at least 2. Then, applying Proposition \cite{8.2} we conclude that \( \Gamma^s_{\Lambda} \) has two stable tubes of ranks at least 2, and hence the stable tubes \( \mathcal{T}^\text{out}_{\Lambda} \) and \( \mathcal{T}^\text{inn}_{\Lambda} \) are unique stable tubes of ranks at least 2 in \( \Gamma^s_{\Lambda} \). In particular, we have \( p', q' \geq 1 \) and

\[
\{ F(\mathcal{T}^\text{out}_{\Lambda}), F(\mathcal{T}^\text{inn}_{\Lambda}) \} = \{ \mathcal{T}^\text{out}_{\Lambda'}, \mathcal{T}^\text{inn}_{\Lambda'} \}.
\]

If \( F(\mathcal{T}^\text{out}_{\Lambda}) = \mathcal{T}^\text{out}_{\Lambda'} \) then, applying Propositions \cite{22} \( (X.3) \) \( (X.2) \) and \( (X.3) \), we get the equalities

\[
(2p+1)k = 2p'+1 \quad \text{and} \quad m(2p+1)+p+1 = p'+1,
\]

where \( m = m_{k,s} \) is such that \( m(s+2) + 1 \equiv 0 \pmod{k} \). Similarly, if \( F(\mathcal{T}^\text{out}_{\Lambda}) = \mathcal{T}^\text{inn}_{\Lambda'} \), we get the equalities

\[
(2p+1)k = 2q'+1 \quad \text{and} \quad m(2p+1)+p+1 = q'+1.
\]

The both cases force \( k = 2m+1 \). On the other hand, \( m(s+2) + 1 \equiv 0 \pmod{k} \) is equivalent to \( (2m+1)+sm \equiv 0 \pmod{k} \). Hence \( k = 2m+1 \) implies \( 2m+1 | sm \), a contradiction, because \( 1 \leq s \leq k-1 = 2m \) and \( \gcd(2m+1, s) = \gcd(k, s) = 1 \).
9. Proof of Theorem \[1\]

The aim of this section is to complete the proof of Theorem \[1\]. The implication (1) \(\implies\) (2) follows from Theorem \[5.1\] and Proposition \[6.1\]. The implication (2) \(\implies\) (3) is the direct consequence of the following general result proved by J. Rickard.

Proposition 9.1. Let \(A\) and \(\Lambda\) be derived equivalent selfinjective algebras. Then \(A\) and \(\Lambda\) are stably equivalent.

**Proof.** See [29, Corollary 2.2]. \(\square\)

We will need also the following general result.

Proposition 9.2. Let \(A\) and \(\Lambda\) be stably equivalent selfinjective algebras, and assume that \(\Lambda\) is symmetric of Loewy length at least 3. Then \(A\) is weakly symmetric.

**Proof.** See [7, Proposition 5.2]. \(\square\)

We are now in position to prove the implication (3) \(\implies\) (1) of Theorem \[1\]. Let \(A\) be a standard selfinjective algebra which is stably equivalent to an algebra of the form \(\Lambda(p,q,k,s,\theta)\) or \(\Gamma^*_n\). Then the stable Auslander-Reiten quiver \(\Gamma^*_s\) of \(A\) is isomorphic either to \(\Gamma^*_a(p,q,k,s,\theta)\) or to \(\Gamma^*_a(n)\). In particular, \(A\) is a one-parametric selfinjective algebra of an Euclidean type \(\tilde{A}(p+q+1)\) or \(\tilde{D}_{2n+3}\) (see [35] and Propositions \[7.1\] and \[7.2\]). Moreover, since the algebras \(\Lambda(p,q,k,s,\theta)\) and \(\Gamma^*(n)\) are not weakly symmetric, applying Proposition \[9.2\] we conclude that \(A\) is not symmetric. Then applying the main results of [9], [10] and [11] we obtain that \(A\) is isomorphic to an algebra of one of the forms \(\Omega^{(1)}(T,\sigma_1,\theta)\), \(\Omega^{(1)}(T,\theta)\), or \(\Omega^{(2)}(T,v_1,v_2)\). Suppose \(A\) is isomorphic to an algebra \(\Omega^{(1)}(T,\theta)\). Then it follows from Theorem \[8.1\] and Proposition \[8.1\] that \(A\) is stably equivalent to an algebra of the form \(\Lambda(p',q',\theta')\), for some \(p',q'\geq 0\) and \(\theta'\in K\setminus\{0\}\). Moreover, by our assumption (3), \(A\) is stably equivalent to an algebra \(\Lambda(p,q,k,s,\theta)\) or \(\Gamma^*(n)\). Since the stableAuslander-Reiten quivers \(\Gamma^*_a(p',q',\theta')\) and \(\Gamma^*_a(n)\) have, by Propositions \[8.1\] and \[8.2\] the unique Euclidean components of different types, we conclude that \(A\) is stably equivalent simultaneously to an algebra \(\Lambda(p,q,k,s,\theta)\) and to an algebra \(\Lambda(p',q',\theta')\). But this contradicts Proposition \[8.1\]. Therefore, the implication (3) \(\implies\) (1) holds. This finishes the proof of Theorem \[1\].

10. Stable equivalences

It follows from Proposition \[8.1\] that two algebras \(\Gamma^*(m)\) and \(\Gamma^*(n)\) are stably equivalent if and only if \(m = n\), or equivalently \(\Gamma^*(m)\) and \(\Gamma^*(n)\) are isomorphic. Clearly, two algebras \(\Lambda(p,q,k,s,\theta)\) and \(\Gamma^*(n)\) are never stably equivalent, because their stable Auslander-Reiten quivers are not isomorphic. The following proposition gives necessary conditions for two algebras \(\Lambda(p,q,k,s,\theta)\) and \(\Lambda(p',q',k',s',\theta')\) to be stably equivalent.

Proposition 10.1. Let \(\Lambda(p,q,k,s,\theta)\) and \(\Lambda(p',q',k',s',\theta')\) be stably equivalent algebras. Then one of two cases holds:
Hence, we get
inner tube $T$ and outer tube $T$, $s = s'$;

(2) $p = q'$, $q = p'$, $k = k'$, $s \neq s'$, $m_{k,s} + m_{k',s'} = k - 1$.

**Proof.** We abbreviate $\Lambda = \Lambda (p, q, k, s, \theta)$ and $\Lambda' = \Lambda (p', q', k', s', \theta')$. We have

$$p, q \geq 0, k \geq 2, 1 \leq s \leq k - 1, \gcd (s, k) = 1, \gcd (s + 2, k) = 1,$$

$$p', q' \geq 0, k' \geq 2, 1 \leq s' \leq k' - 1, \gcd (s', k') = 1, \gcd (s' + 2, k') = 1.$$

We also note that since $k \geq 2$ and $k' \geq 2$, $\Lambda$ and $\Lambda'$ are of Loewy length at least 3, and then for any stable equivalence $F : \text{mod} \Lambda \rightarrow \text{mod} \Lambda'$ we have $F \tau_{\Lambda} = \tau_{\Lambda'} F$ and $F \Omega_{\Lambda} = \Omega_{\Lambda'} F$ (see [3] (X.1.9), (X.1.12)). Recall also that $\Gamma_{\Lambda}$ has exactly two large stable tubes: the outer tube $T_{p,q,k,s}^{\text{out}}$ of rank $(2p+1)k$ and the inner tube $T_{p,q,k,s}^{\text{inn}}$ of rank $(2q+1)k$. Similarly, $\Gamma_{\Lambda'}$ has exactly two large stable tubes: the outer tube $T_{p',q',k',s'}^{\text{out}}$ of rank $(2p'+1)k'$ and the inner tube $T_{p',q',k',s'}^{\text{inn}}$ of rank $(2q'+1)k'$. Since the stable Auslander-Reiten quivers of $\Lambda$ and $\Lambda'$ are isomorphic, we obtain

$$(2p+1)k, (2q+1)k = \{(2p'+1)k', (2q'+1)k'\}.$$

Hence, we get

$$k (p+q+1) = k' (p' + q' + 1).$$

Let $m = m_{k,s}$ and $m' = m_{k',s'}$. We have the congruences

$$m (s+2)+1 \equiv 0 \pmod{k} \quad \text{and} \quad m' (s'+2)+1 \equiv 0 \pmod{k'},$$

or equivalently

$$(2m+1)+sm \equiv 0 \pmod{k} \quad \text{and} \quad (2m'+1)+s'm' \equiv 0 \pmod{k'}.$$

Hence $\gcd (m, k) = 1$ and $\gcd (m', k') = 1$. Then $\gcd (s, k) = 1$ and $\gcd (s', k') = 1$ imply $\gcd (2m+1, k) = 1$ and $\gcd (2m'+1, k') = 1$.

We have two cases to consider. Assume first that $T_{p,q,k,s}^{\text{out}} = T_{p',q',k',s'}^{\text{out}}$ and $T_{p,q,k,s}^{\text{inn}} = T_{p',q',k',s'}^{\text{inn}}$. Comparing the ranks of large stable tubes of $\Gamma_{\Lambda}$ and $\Gamma_{\Lambda'}$, we then have

$$(2p+1)k = (2p'+1)k' \quad \text{and} \quad (2q+1)k = (2q'+1)k'.$$

Further, comparing the actions of $\Omega_{\Lambda}$ and $\Omega_{\Lambda'}$ on the large stable tubes, we obtain from Proposition [7.3] the equalities

$$m (2p+1)+p+1 = m' (2p'+1)+p'+1 \quad \text{and} \quad (k-m) (2q+1)−q = (k′−m′) (2q′+1)−q'.$$
Moreover, the second equality and \((2q+1)k = (2q'+1)k'\) imply \(m (2q+1)+q = m' (2q'+1)+q'\). Summing up both equalities we obtain \((2m+1) (p+q+1) = (2m'+1) (p'+q'+1)\). Finally, invoking the equality \(k (p+q+1) = k' (p'+q'+1)\), we obtain

\[(2m+1)k' = (2m'+1)k.\]

But then \(\gcd (2m+1, k) = 1\) and \(\gcd (2m'+1, k') = 1\) imply \(k|k'\) and \(k'|k\). Hence \(k = k'\) and consequently \(p = p', q = q', m = m'\).

Assume that \(\mathcal{T}_{p,q,k,s}^{\text{out}} = \mathcal{T}_{p',q',k',s'}^{\text{inn}}\) and \(\mathcal{T}_{p,q,k,s}^{\text{inn}} = \mathcal{T}_{p',q',k',s'}^{\text{out}}\). Then comparing the ranks of these tubes, we obtain

\[(2p+1)k = (2q'+1)k'\] and \((2q+1)k = (2p'+1)k'\).

Further, comparing the actions of \(\Omega_{\Lambda}\) and \(\Omega_{\Lambda'}\) on these tubes, we obtain from Proposition 7.3

the equalities

\[m (2p+1)+p+1 = (k' - m') (2q'+1) - q'\] and \((k-m) (2q+1) - q = m' (2p'+1)+p'+1\).

Hence, we obtain

\[m' (2q'+1)+q' = k' (2q'+1) - m (2p+1) - p - 1\]
\[= k (2p+1) - m (2p+1) - p - 1\]
\[= (k-m) (2p+1) - p - 1.\]

Summing up, we get the equality

\[(2m'+1) (p'+q'+1) = 2 (k-m) - 1 (p+q+1).\]

Invoking again the equality \(k (p+q+1) = k' (p'+q'+1)\), we obtain

\[(2m'+1)k = (2 (k-m) - 1) k'.\]

We have \(\gcd (2m'+1, k') = 1\) and \(\gcd (2m+1, k) = 1\). Moreover, \(\gcd (2m+1, k) = 1\) implies \(\gcd (2 (k-m) - 1, k) = 1\). Therefore, \(k|k'\) and \(k'|k\), and so \(k = k'\). Then \(p = q', q = p'\).

Finally, \(2m'+1 = 2 (k-m) - 1\), and consequently \(m+m' = k-1\).

We note that the condition (2) in the above proposition forces \(k\) to be odd. Indeed, if \(k\) is even then \(m_{k,s}\) and \(m_{k',s'}\) have different parity, and this contradicts the congruence \(m_{k,s} (s+2) + m_{k',s'} (s'+2) + 2 \equiv 0 \pmod{k}\), because \(s+2\) and \(s'+2\) are odd by \(\gcd (s,k) = 1\) and \(\gcd (s',k') = \gcd (s',k) = 1\).

We also have the following fact.

**Proposition 10.2.** Two algebras \(\Lambda (p,q,k,s,\theta)\) and \(\Lambda (p',q',k',s',\theta')\) are isomorphic if and only if one of the following cases holds:

1. \(p = p', q = q', k = k', s = s', \theta = \theta;\)
2. \(p = q', q = p', k = k', s \neq s', m_{k,s} + m_{k',s'} = k - 1, \theta' = \theta^{-1}.\)
Proof. Let
\[ \Lambda(p, q, k, s, \theta) = KQ(T_{p,q,k}, \sigma_s) / T^{(1)}(T_{p,q,k}, \sigma_s, \theta) \]
and
\[ \Lambda(p', q', k', s', \theta') = KQ(T'_{p',q',k'}, \sigma_{s'}) / T^{(1)}(T'_{p',q',k'}, \sigma_{s'}, \theta'). \]

We denote by \( i_j \), for \( j = 1, 2, \ldots q \), \( i = 1, 2, \ldots, k \), the vertices of the quiver \( Q(T_{p,q,k}, \sigma_s) \) (respectively, \( Q(T'_{p',q',k'}, \sigma_{s'}) \)) corresponding to the inner edges of the graph \( T_{p,q,k} \) (respectively, \( T'_{p',q',k'} \)) and by \( i^j \), for \( j = 1, 2, \ldots p \), \( i = 1, 2, \ldots, k \), the vertices of the quiver \( Q(T_{p,q,k}, \sigma_s) \) (respectively, \( Q(T'_{p',q',k'}, \sigma_{s'}) \)) corresponding to the outer edges of the graph \( T_{p,q,k} \) (respectively, \( T'_{p',q',k'} \)). Moreover, we denote by \( \alpha_{i,j} : i \rightarrow i_{j+1} \), for \( j = 1, 2, \ldots q - 1 \), \( \beta_{i,j} : i \rightarrow i + s \), for \( i = 1, 2, \ldots, k \), the arrows of the quiver \( Q(T_{p,q,k}, \sigma_s) \) (respectively, \( Q(T'_{p',q',k'}, \sigma_{s'}) \)). The ideal \( T^{(1)}(T_{p,q,k}, \sigma_s, \theta) \) contains the following essential commutativity generators:
\[ \alpha_{i_0} \alpha_{i_1} \cdots \alpha_{i_q} \beta_{i_0} \beta_{i_1} \cdots \beta_{i_p} - \beta_{(i+1)^0} \beta_{(i+1)^1} \cdots \beta_{(i+1)^p} \alpha_{(i+1+s)^0} \alpha_{(i+1+s)^1} \cdots \alpha_{(i+1+s)^q}, \]
for \( i = 2, 3, \ldots, k \), and
\[ \alpha_{i_0} \alpha_{i_1} \cdots \alpha_{i_q} \beta_{i_0} \beta_{i_1} \cdots \beta_{i_p} - \beta_{i_0} \beta_{i_1} \cdots \beta_{i_p} \theta \beta_{2p} \beta_{2^1} \cdots \beta_{2p} \alpha_{(s+2)^0} \alpha_{(s+2)^1} \cdots \alpha_{(s+2)^q}, \]
for \( i = 1, 2, \ldots, k \). Further, the ideal \( T^{(1)}(T'_{p',q',k'}, \sigma_{s'}, \theta') \) contains the following essential commutativity generators:
\[ \alpha_{i_0} \alpha_{i_1} \cdots \alpha_{i_q} \beta_{i_0} \beta_{i_1} \cdots \beta_{i_p} - \beta_{(i+1)^0} \beta_{(i+1)^1} \cdots \beta_{(i+1)^p} \alpha_{(i+1+s')^0} \alpha_{(i+1+s')^1} \cdots \alpha_{(i+1+s')^q}, \]
for \( i = 2, 3, \ldots, k \), and
\[ \alpha_{i_0} \alpha_{i_1} \cdots \alpha_{i_q} \beta_{i_0} \beta_{i_1} \cdots \beta_{i_p} - \beta_{i_0} \beta_{i_1} \cdots \beta_{i_p} \theta' \beta_{2p} \beta_{2^1} \cdots \beta_{2p} \alpha_{(s+2)^0} \alpha_{(s+2)^1} \cdots \alpha_{(s+2)^q}, \]
Assume that the algebras \( \Lambda(p, q, k, s, \theta) \) and \( \Lambda(p', q', k', s', \theta') \) are isomorphic and let \( f: \Lambda(p, q, k, s, \theta) \rightarrow \Lambda(p', q', k', s', \theta') \) be a \( K \)-algebra isomorphism. In particular, the algebras \( \Lambda(p, q, k, s, \theta) \) and \( \Lambda(p', q', k', s', \theta') \) are stably equivalent. Then it follows from Proposition 10.1 that one of the following cases holds:
(a) \( p = p', q = q', k = k', s = s' \);
(b) \( p = q', q = p', k = k', s \neq s', m_{k,s} + m_{k',s'} = k - 1. \)
Assume (a) holds. Then we may assume (without loss of generality) that \( f(\alpha_{i_j}) = \alpha_{i_j} \), for \( j = 0, 1, \ldots, q \), \( i = 1, s + 2 \), and \( f(\beta_{ij}) = \beta_{ij} \), for \( j = 0, 1, \ldots, p \), \( i = 1, 2 \). Moreover, we
have the equalities:
\[
\theta' \beta_0 \beta_1 \ldots \beta_{2p} \alpha_{(s+2)_0} \alpha_{(s+2)_1} \ldots \alpha_{(s+2)_q} = \\
= \alpha_{1_0} \alpha_{1_1} \ldots \alpha_{1_q} \beta_{1_1} \ldots \beta_{1_p} \\
= f (\alpha_{1_0}) f (\alpha_{1_1}) \ldots f (\alpha_{1_q}) f (\beta_{1_1}) \ldots f (\beta_{1_p}) \\
= f (\alpha_{1_0} \alpha_{1_1} \ldots \alpha_{1_q} \beta_{1_1} \ldots \beta_{1_p}) \\
= f \left( \theta \beta_{2_0} \beta_{2_1} \ldots \beta_{2p} \alpha_{(s+2)_0} \alpha_{(s+2)_1} \ldots \alpha_{(s+2)_q} \right) \\
= \theta f (\beta_{2_0}) f (\beta_{2_1}) \ldots f (\beta_{2p}) f (\alpha_{(s+2)_0}) f (\alpha_{(s+2)_1}) \ldots f (\alpha_{(s+2)_q}) \\
= \theta \beta_{2_0} \beta_{2_1} \ldots \beta_{2p} \alpha_{(s+2)_0} \alpha_{(s+2)_1} \ldots \alpha_{(s+2)_q} .
\]

This implies \( \theta' = \theta \).

Assume (b) holds. Then we may assume (without loss of generality) that \( f (\alpha_{1_j}) = \beta_{2j} \), \( f \left( \alpha_{(s+2)_j} \right) = \beta_{1j} \), for \( j = 0, 1, \ldots, q \), and \( f (\beta_{2j}) = \alpha_{1_j} \), \( f (\beta_{1j}) = \alpha_{(s'+2)_j} \), for \( j = 0, 1, \ldots, p \). Moreover, we have the equalities:
\[
\theta' \beta_{2_0} \beta_{2_1} \ldots \beta_{2p} \alpha_{(s'+2)_0} \alpha_{(s'+2)_1} \ldots \alpha_{(s'+2)_q} = \\
= \alpha_{1_0} \alpha_{1_1} \ldots \alpha_{1_q} \beta_{1_1} \ldots \beta_{1_p} \\
= f (\beta_{2_0}) f (\beta_{2_1}) \ldots f (\beta_{2p}) f (\alpha_{(s'+2)_0}) f (\alpha_{(s'+2)_1}) \ldots f (\alpha_{(s'+2)_q}) \\
= f \left( \theta^{-1} \alpha_{1_0} \alpha_{1_1} \ldots \alpha_{1_q} \beta_{1_1} \ldots \beta_{1_p} \right) \\
= \theta^{-1} f (\alpha_{1_0}) f (\alpha_{1_1}) \ldots f (\alpha_{1_q}) f (\beta_{1_0}) f (\beta_{1_1}) \ldots f (\beta_{1_p}) \\
= \theta^{-1} \beta_{2_0} \beta_{2_1} \ldots \beta_{2p} \alpha_{(s'+2)_0} \alpha_{(s'+2)_1} \ldots \alpha_{(s'+2)_q} .
\]

This implies \( \theta' = \theta^{-1} \).

If (1) holds then obviously the algebras \( \Lambda (p, q, k, s, \theta) \) and \( \Lambda (p', q', k', s', \theta') \) are isomorphic. Assume (2) holds. We define a homomorphism \( f_{k,s,s'} : \Lambda (p, q, k, s, \theta) \rightarrow \Lambda (q, p, k, s', \theta') \) of \( K \)-algebras as follows. Since \( \gcd (s, k) = 1 \), for each \( i = 1, 2, \ldots, k \), there exists exactly one \( 0 \leq j (i) \leq k - 1 \) such that \( i \equiv 1 + j (i) s \mod k \). Let \( t (i) \) be the unique number such that \( 1 \leq t (i) \leq k \) and \( t (i) \equiv 1 + j (i) s \mod k \). We define \( f_{k,s,s'} (e_i) = e_{t (i)}, f_{k,s,s'} (\alpha_i) = \beta_{(t (i)+1)j}, \) for \( j = 0, 1, \ldots, q \), and \( f_{k,s,s'} (\beta_i) = \alpha_{2 (t (i)-1)}, \) for \( j = 0, 1, \ldots, p \). Let \( i \in \{ 1, 2, \ldots, k \} \). In order to prove that, for any element \( \omega \) of the ideal \( \bar{T}^{(1)} (T_{p,q,k,s}, \theta) \), \( f_{k,s,s'} (\omega) \) is an element of the ideal \( \bar{T}^{(1)} (T_{p',q',k', \sigma, s', \theta'}) \), we need the following four congruences:

(i) \( t (i - 1) \equiv t (i) + s' + 1 \mod k \),
(ii) \( t (i + s) \equiv t (i + s + 1) + s' + 1 \mod k \),
(iii) \( t (i + s + 1) \equiv t (i) - 1 \mod k \),
(iv) \( t (i + s) \equiv t (i - 1) - 1 \mod k \).
Observe that it is enough prove the congruences (i) and (iii).

Let \( m = m_{k,s} \) and \( m' = m_{k,s'} \). We first prove that

\[
ss' + s + s' \equiv 0 \pmod{k}.
\]

The congruences

\[
m(s+2)+1 \equiv 0 \pmod{k}, \quad m'(s'+2)+1 \equiv 0 \pmod{k}
\]

are the congruences

\[
ms + 2m + 1 \equiv 0 \pmod{k}\quad,\quad m's' + 2m' + 1 \equiv 0 \pmod{k}.
\] (A)

The conditions \( \gcd(s,k) = 1 \) and \( \gcd(s',k) = 1 \) imply the congruences

\[
mss' + 2ms' + s' \equiv 0 \pmod{k}, \quad m's's + 2m's + s \equiv 0 \pmod{k}.
\] (B)

Moreover, \( m + m' = k - 1 \) implies \( m + m' \equiv -1 \pmod{k} \), and hence

\[
m' \equiv -m - 1 \pmod{k}.
\]

From (A) we then obtain

\[
ms + 2m + 1 \equiv 0 \pmod{k}, \quad (-m - 1)s' + 2(-m - 1) + 1 \equiv 0 \pmod{k},
\]

and hence

\[
m(s - s') \equiv s' \pmod{k}.
\] (C)

From (B) we obtain

\[
mss' + 2ms' + s' \equiv 0 \pmod{k}, \quad (-m - 1)s' + 2(-m - 1) + 1 \equiv 0 \pmod{k},
\]

and hence

\[
ss' + s - s' + 2m(s - s') \equiv 0 \pmod{k}.
\] (D)

Combining (C) and (D) we get the required congruence

\[
ss' + s + s' \equiv 0 \pmod{k}.
\]

Further, \( i \equiv 1 + j\,(i) \pmod{k} \) and \( i - 1 \equiv 1 + j\,(i - 1) \pmod{k} \) imply

\[
(j\,(i) - j\,(i - 1))s \equiv 1 \pmod{k}.
\]

Since \( \gcd(s',k) = 1 \), this is equivalent to

\[
(j\,(i) - j\,(i - 1))s' \equiv s' \pmod{k}.
\]

Applying \( ss' + s + s' \equiv 0 \pmod{k} \), we then get

\[
(j\,(i) - j\,(i - 1))ss' \equiv -ss' - s \pmod{k}.
\]
Since \( \gcd(s, k) = 1 \), this implies that
\[
(j (i) - j (i-1)) s' \equiv -s' - 1 \pmod{k}.
\]
Finally, \( t (i) \equiv 1 + j (i) s' \pmod{k} \) and \( t (i-1) \equiv 1 + j (i-1) s' \pmod{k} \) imply
\[
t (i) - t (i-1) \equiv (j (i) - j (i-1)) s' \pmod{k},
\]
and consequently \( t (i) - t (i-1) \equiv -s' - 1 \pmod{k} \). Therefore, we obtain
\[
t (i-1) \equiv t (i) + s' + 1 \pmod{k},
\]
as required in (i).

Let \( i \equiv 1 + j (i) \pmod{k} \) and \( i + s + 1 \equiv 1 + j (i + s + 1) \pmod{k} \). This implies
\[
(j (i) - j (i+s+1)) s \equiv -s - 1 \pmod{k}.
\]
Since \( \gcd(s', k) = 1 \), this is equivalent to
\[
(j (i) - j (i+s+1)) ss' \equiv -ss' - s' \pmod{k}.
\]
Applying \( ss' + s + s' \equiv 0 \pmod{k} \), we then get
\[
(j (i) - j (i+s+1)) ss' \equiv s \pmod{k}.
\]
Since \( \gcd(s, k) = 1 \), this implies that
\[
(j (i) - j (i+s+1)) s' \equiv 1 \pmod{k}.
\]
Finally, \( t (i) \equiv 1 + j (i) s' \pmod{k} \) and \( t (i+s+1) \equiv 1 + j (i+s+1) s' \pmod{k} \) imply
\[
t (i) - t (i+s+1) \equiv (j (i) - j (i+s+1)) s' \pmod{k},
\]
and consequently \( t (i) - t (i+s+1) \equiv 1 \pmod{k} \). Therefore, we obtain
\[
t (i+s+1) \equiv t (i) - 1 \pmod{k},
\]
as required in (iii).

We note that \( f_{s,s',k} \circ f_{s',s,k} \equiv \text{id}_{\Lambda(p',q',k',s',\theta')} \) and \( f_{s',s,k} \circ f_{s,s',k} \equiv \text{id}_{\Lambda(p,q,k,s,\theta)} \). Hence the algebras \( \Lambda(p, q, k, s, \theta) \) and \( \Lambda(p', q', k', s', \theta') \) are isomorphic.

We do not know yet if two algebras of the forms \( \Lambda(p, q, k, s, \theta) \) and \( \Lambda(p', q', k', s', \theta') \) are stably equivalent if and only if they are isomorphic.

We end this section with two examples illustrating the above considerations.

**Example 10.3.** Let \( k = 8 \). Then the set of all \( s \in \{1, 2, \ldots, 7\} \) satisfying \( \gcd(s, 8) = 1 \) and \( \gcd(s+2, 8) = 1 \) consists of \( 1, 3, 5 \) and \( 7 \). Moreover, we have \( m_{8,1} = 5 \), \( m_{8,3} = 3 \), \( m_{8,5} = 1 \), \( m_{8,7} = 7 \). Observe that for any \( s, s' \in \{1, 3, 5, 7\} \) we have \( m_{8,s} + m_{8,s'} \neq k-1 = 7 \). Therefore, for any \( p, q, p', q' \geq 0 \), \( s, s' \in \{1, 3, 5, 7\} \) with \( s \neq s' \) and \( \theta \in K \setminus \{0\} \), we have \( \Lambda(p, q, s, \theta) \not\cong \Lambda(p', q', s', \theta') \).

**Example 10.4.** Let \( k = 5 \), \( s = 1 \) and \( s' = 2 \). Then \( \gcd(s, k) = 1 \), \( \gcd(s+2, k) = 1 \), \( \gcd(s', k) = 1 \), \( \gcd(s'+2, k) = 1 \). Moreover, \( m_{5,1} = 3 \), \( m_{5,2} = 1 \), and so \( m_{5,1} + m_{5,2} = 4 = k-1 \). Hence, for any \( p, q, p', q' \geq 0 \) and \( \theta \in K \setminus \{0\} \), we have \( \Lambda(p, q, 5, 1, \theta) \cong \Lambda(p', q', 5, 2, \theta) \).
11. Proof of Theorem

As a consequence of our main theorem we deduce in this section some results on homological invariants of one-parametric selfinjective algebras.

Recall that the representation dimension of an algebra $A$ was defined by Auslander [4] as the number

$$\text{repdim}(A) := \inf \{ \text{gldim } \text{End}_A(N) \mid N \text{ generator-cogenerator} \}.$$ 

A finitely generated $A$-module $N$ is a generator-cogenerator for $A$ if it contains all projective indecomposable and all injective indecomposable $A$-modules as direct summands. This invariant gives a way of measuring how far an algebra is from being of finite representation type. In fact, $\text{repdim}(A) \leq 2$ if and only if $A$ is of finite representation type [4].

We shall prove the following result which contributes further to the long-term project of determining the representation dimension for all (selfinjective) algebras of tame representation type.

**Theorem 11.1.** Let $A$ be a standard one-parametric selfinjective algebra. Then $A$ has representation dimension 3.

**Proof.** We shall use the main results of [16]. Firstly, if an algebra $A$ can be embedded into an algebra $B$, preserving radicals, and if $B$ is of finite representation type, then $A$ has representation dimension at most 3 ([16 Theorem 1.1]). As an application, we get secondly that every special biserial algebra has representation dimension at most 3 ([16 (1.3)]). We also use that for selfinjective algebras, the representation dimension is invariant under derived equivalences [38].

For $A$ weakly symmetric, the theorem is proved in our paper [8]. Therefore, it suffices to prove the theorem for the normal forms of our derived equivalence classification for standard one-parametric selfinjective algebras obtained in this paper.

The normal forms $\Lambda(p, q, k, s, \theta)$ are special biserial, hence they have representation dimension at most 3. Since they are not of finite representation type, the representation dimension is actually 3. The normal forms $\Gamma^*(n)$ are not special biserial. In order to show that $\text{repdim}(\Gamma^*(n)) = 3$, we shall construct a radical embedding of a certain factor algebra of $\Gamma^*(n)$ into an algebra of finite representation type. This proof is similar to the one for algebras of Euclidean type carried out in [8]. Since this paper might not be easily available, we include a complete proof here for the convenience of the reader.

Suitable factor algebras for determining representation dimensions are suggested by the following useful result ([16 (1.2)]): Let $A$ be a basic algebra, and let $P$ be an indecomposable projective-injective $A$-module. Consider the factor algebra $B := A/\text{soc}(P)$ modulo the socle of $P$. If $\text{repdim}(B) \leq 3$, then also $\text{repdim}(A) \leq 3$.

We now consider the algebras $\Gamma^*(n)$. Let $v$ be the unique vertex of the quiver of $\Gamma^*(n)$ with three incoming and outgoing arrows, with corresponding projective indecomposable module $P(v)$. We consider the factor algebra $\widetilde{\Gamma}^*(n) := \Gamma^*(n)/\text{soc}(P_v)$. We get additional
relations $\alpha_1 \alpha_2 = 0$, $\gamma_1 \gamma_2 = 0$ and $(\beta_1 \ldots \beta_n)^2 = 0$. We have to show that $\tilde{\Gamma}^*(n)$ has representation dimension 3.

In order to construct a radical embedding of $\tilde{\Gamma}^*(n)$ into an algebra of finite representation type we use the technique of splitting datum as introduced in [16]. We can distribute the edges adjacent to $v$ as follows: $E_1 \cup E_2 := \{\beta_n\} \cup \{\alpha_2, \gamma_2\}$ and $S_1 \cup S_2 := \{\beta_1\} \cup \{\alpha_1, \gamma_1\}$. The conditions for a splitting datum are satisfied: all products in $E_i S_j$ are zero for $i \neq j$ and all relations of $\tilde{\Gamma}^*(n)$ are monomial. Any splitting datum gives rise to a radical embedding into an algebra whose quiver is obtained by splitting the vertex under consideration. In our case, we get a radical embedding of $\tilde{\Gamma}^*(n)$ into the algebra $\tilde{\Gamma}^*(n)^{sp}$ given by the disjoint union of a cyclic quiver with edges $\beta_1, \ldots, \beta_n$ and relations $(\beta_1 \ldots \beta_n)^2 = 0$, $\beta_j \beta_{j+1} \ldots \beta_n \beta_1 \beta_2 \ldots \beta_{j-1} \beta_j = 0$, for $2 \leq j \leq n$, and the quiver

\[
\begin{array}{c}
\odot & \stackrel{\alpha_2}{\longrightarrow} & \odot & \stackrel{\gamma_1}{\longrightarrow} & \odot \\
\odot & \stackrel{\alpha_1}{\longrightarrow} & \odot & \stackrel{\gamma_2}{\longrightarrow} & \odot
\end{array}
\]

with relations $\alpha_1 \alpha_2 = 0$, $\gamma_1 \gamma_2 = 0$, $\alpha_2 \gamma_1 = 0$, $\gamma_2 \alpha_1 = 0$. Note that this new algebra $\tilde{\Gamma}^*(n)^{sp}$ is special biserial and of finite representation type [36].

By the radical embedding theorem from [16] we can deduce that $\tilde{\Gamma}^*(n)$ has representation dimension 3, as desired.

For selfinjective algebras, the representation dimension is intimately related to the notion of the dimension of a triangulated category as recently defined by R. Rouquier [34]. Let us briefly recall the main definitions. Let $A$ be a selfinjective algebra with stable module category $\text{mod }A$ of finitely generated $A$-modules. Recall that this is a triangulated category with shift $\Omega^{-1}$ (see [20]). For any $A$-module $M$ denote by $\langle M \rangle$ the full subcategory of $\text{mod }A$ with objects the direct summands of the modules obtained from $M$ by taking $i$-fold extensions of finite direct sums of shifts. Then following R. Rouquier [34], the dimension of $\text{mod }A$, denoted $\dim \text{mod }A$, is the minimal integer $d \geq 0$ such that there is an $A$-module $M$ with $\text{mod }A = \langle M \rangle_{d+1}$. In particular, we have $\dim \text{mod }A = 0$ if and only if $A$ has finite representation type. As with the representation dimension, it seems to be very hard to determine this dimension of the stable module category, even for examples of fairly small and well-known algebras.

The representation dimension of a selfinjective algebra is closely related to the dimension of its stable module category, as observed by R. Rouquier.

**Proposition 11.2.** Let $A$ be a selfinjective algebra. Then $\text{repdim}(A) \geq 2 + \dim \text{mod }A$.

**Proof.** See [34, Proposition 6.9].

Using this connection we can deduce the dimension of the stable module category of standard selfinjective one-parametric algebras from our Theorem 11.1.

**Corollary 11.3.** Let $A$ be a standard one-parametric selfinjective algebra. Then the stable module category $\text{mod }A$ has dimension 1.
It does not seem to be clear how one can determine the dimensions of stable module categories directly, without referring to the representation dimension.

The following problem would be interesting to consider.

**Question 1:** Is \( \dim \mod A \leq 1 \) for all (selfinjective) algebras of tame representation type?

This is a weaker version of the following long standing open problem.

**Question 2:** Is \( \text{repdim}(A) \leq 3 \) for all (selfinjective) algebras of tame representation type?

Note that the only known examples of algebras of representation dimension greater 3 were only recently discovered by Rouquier [34], and they are all of wild representation type.

**Acknowledgements.** The first and the third named authors gratefully acknowledge support from the Polish Scientific Grant KBN No. 1 P03A 018 27.

**References**

[1] J. L. Alperin, Local Representation Theory, Cambridge Studies in Advanced Mathematics 11, Cambridge Univ. Press, 1986.

[2] H. Asashiba, The derived equivalence classification of representation-finite selfinjective algebras, J. Algebra 214 (1999), 182–221.

[3] I. Assem, J. Nehring and A. Skowroński, Domestic trivial extensions of simply connected algebras, Tsukuba J. Math. 13 (1989), 31–72.

[4] M. Auslander, Representation dimension for Artin algebras, Queen Mary College, Mathematics Notes, 1971, pp.1-179; also in I. Reiten, S. Smalø, O. Solberg (Eds.), Selected works of Maurice Auslander Part I, AMS, Providence, 1999, pp.505-574.

[5] M. Auslander, I. Reiten and S. O. Smalø, Representation theory of artin algebras, Cambridge Studies in Advanced Mathematics 36, Cambridge University Press (1995).

[6] J. Białkowski, T. Holm and A. Skowroński, Derived equivalences for tame weakly symmetric algebras having only periodic modules, J. Algebra 269 (2003), 652–668.

[7] R. Bocián, T. Holm and A. Skowroński: Derived equivalence classification of weakly symmetric algebras of Euclidean type, J. Pure Appl. Algebra 191 (2004), 43–74.

[8] R. Bocián, T. Holm and A. Skowroński: The representation dimension of domestic weakly symmetric algebras, Cent. Eur. J. Math. 2, No.1 (2004), 67-75.

[9] R. Bocián and A. Skowroński, Symmetric special biserial algebras of Euclidean type, Colloq. Math., Vol. 96, (2003), 121–148.

[10] R. Bocián and A. Skowroński, Weakly symmetric algebras of Euclidean type, J. reine angew. Math. 580 (2005), 157–199.

[11] R. Bocián and A. Skowroński, One-parametric selfinjective algebras, J. Math. Soc. Japan 57 (2005), 491–512.

[12] O. Bretscher, C. Läser and C. Riedtmann, Selfinjective and simply connected algebras, Manuscripta Math. 36 (1982), 253–307.

[13] W. Crawley-Boevey, Tame algebras and generic modules, Proc. London Math. Soc. 63 (1991), 241–265.
[14] Yu. Drozd, Tame and wild matrix problems, In: Representation Theory II, Lecture Notes in Math. 832, Springer-Verlag (1980), 242–258.
[15] K. Erdmann, Blocks of tame representation type and related algebras, Lecture Notes in Math. 1428 (Springer Verlag, 1999).
[16] K. Erdmann, T. Holm, O. Iyama and J. Schröer, Radical embeddings and representation dimension, Advances Math. 185 (2004), 159–177.
[17] K. Erdmann and A. Skowroński, On Auslander-Reiten components of blocks and selfinjective biserial algebras, Trans. Amer. Math. Soc. 330 (1992), 165–189.
[18] J. A. Green, Walking around the Brauer tree. J. Austral. Math. Soc. 17 (1974), 197–213.
[19] D. Happel, On the derived category of a finite-dimensional algebra, Comment. Math. Helv. 62 (1987), 339–389.
[20] D. Happel, Triangulated categories in the representation theory of finite-dimensional algebras, London Mathematical Society Lecture Note Series 119 (Cambridge University Press, 1988).
[21] T. Holm, Derived equivalence classification of algebras of dihedral, semidihedral and quaternion type, J. Algebra 211 (1999), 159–205.
[22] D. Hughes and J. Waschbüsch, Trivial extensions of tilted algebras, Proc. London Math. Soc. 46 (1983), 347–364.
[23] M. Kauer, Derived equivalence of graph algebras, in: Trends in Representation Theory of Finite Dimensional Algebras, Contemp. Math 229 (1998), pp. 201–213.
[24] H. Krause, Stable equivalence preserves representation type, Comment. Math. Helv. 72 (1997), 266–284.
[25] H. Krause and G. Zwara, Stable equivalence and generic modules, Bull. London Math. Soc. 32 (2000), 615–618.
[26] H. Lenzing and A. Skowroński, On selfinjective algebras of Euclidean type, Colloq. Math. 79 (1999), 71–76.
[27] F. H.Membrillo-Hernández, Brauer tree algebras and derived equivalence, J. Pure Appl. Algebra 114 (1997), 231–258.
[28] J. Rickard, Morita theory for derived categories, J. London Math. Soc. 39 (1989), 436–456.
[29] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra 61 (1989), 303–317.
[30] C. Riedtmann, Representation-finite selfinjective algebras of type $\mathbb{A}_n$, In: Representation Theory II, Lecture Notes in Math. 831, Springer-Verlag, Berlin (1980), 449–520.
[31] C. Riedtmann, Representation-finite algebras of type $\mathbb{D}_n$, Compositio Math. 49, (1983), 231–282.
[32] C. M. Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Math. 1099 (Springer Verlag, 1984).
[33] K. W. Roggenkamp, Biserial algebras and graphs, In: Algebras and Modules II, CMS Conf. Proc. 24 (CMS/AMS 1998), 481–496.
[34] R. Rouquier, Dimensions of triangulated categories, Preprint (2003), arXiv:math.CT/0310134.
[35] A. Skowroński, Selfinjective algebras of polynomial growth, Math. Ann. 285 (1989), 177–199.
[36] A. Skowroński and J. Waschbüsch, Representation-finite biserial algebras, J. reine angew. Math., Vol. 345, (1983), 172–181.
[37] J. Waschbüsch, Symmetrische Algebren vom endlichen Modultyp, J. reine angew. Math. 321
[38] C. Xi, Representation dimension and quasi-hereditary algebras, Advances Math. 168 (2002), 193–212.

[39] K. Yamagata, Frobenius algebras, in: Handbook of Algebra Vol. 1 (Elsevier Science B. V., 1996), 841–887.