Cosmological perturbations and the Weinberg theorem

Mohammad Akhshik, Hassan Firouzjahi and Sadra Jazayeri

School of Astronomy, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5531, Tehran, Iran
Department of Physics, Sharif University of Technology, Tehran, Iran
E-mail: m.akhshik@ipm.ir, firouz@ipm.ir, sadraj@ipm.ir

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Abstract. The celebrated Weinberg theorem in cosmological perturbation theory states that there always exist two adiabatic scalar modes in which the comoving curvature perturbation is conserved on super-horizon scales. In particular, when the perturbations are generated from a single source, such as in single field models of inflation, both of the two allowed independent solutions are adiabatic and conserved on super-horizon scales. There are few known examples in literature which violate this theorem. We revisit the theorem and specify the loopholes in some technical assumptions which violate the theorem in models of non-attractor inflation, fluid inflation, solid inflation and in the model of pseudo conformal universe.

Keywords: inflation, cosmological perturbation theory

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1 Introduction

Cosmological perturbations theory is the vital tool to connect the predictions of perturbations generated from seed quantum fluctuations in early universe, such as during inflation, to late time cosmological observations such as cosmic microwave background (CMB) or large scale structures (LSS). After inflation ends, the universe enters into the violent phase of reheating and the follow up radiation and matter dominated eras with different sources of energy and matter constituents. However, the fact that there exists adiabatic perturbations which are conserved on super-horizon scales is a powerful tool to connect the large scale fluctuations in CMB or LSS to the corresponding curvature perturbations generated during inflation when the mode of interest leaves the horizon.

It is well-known that the comoving curvature perturbation \( R \) or the curvature perturbations on surface of constant energy density \( \zeta \) are conserved on super-horizon scales in models of single field slow-roll inflation, for a review see [1–4]. Weinberg has generalized this conclusion to a broad class of cosmological perturbations in early universe [1, 5]. The celebrated Weinberg theorem states that whatever the content of the universe, the comoving curvature perturbations in Newtonian gauge always has two adiabatic modes which are frozen on super-horizon scales, corresponding to \( k/a \ll H \) in which \( a \) is the cosmic scale factor, \( H \) is the Hubble expansion rate and \( k \) is the comoving wave-number (in Fourier space). This theorem also states that in addition there is one tensor mode which is conserved on super-horizon scales. In our studies here, we shall concentrate on scalar perturbations.

In particular, Weinberg’s theorem has strong implications for models in which perturbations are generated from a single source, such as in models of single field inflation. In these models, the counting of independent degrees of freedom indicate that we have only two independent modes of curvature perturbations. Consequently, Weinberg’s theorem imply that both of these two modes should be conserved on super-horizon scales in single field models.
The theorem states that the dominant mode is the usual conserved mode in single field inflation models while the other adiabatic mode is actually $R_k = 0$. Of course, these conclusions conform with the known results in single field slow roll inflation models as mentioned above (more precisely, in single field slow roll models the decaying mode approaches $R_k = 0$). However, there are known examples in literature such as models of non-attractor inflation, fluid inflation, solid inflation, pseudo conformal universe and Galilean Genesis in which the curvature perturbation is not frozen on super-horizon scales. For example it is known that in models of non-attractor inflation the usual would-be decaying mode is actually the growing mode and $R_k$ grows like $a^3 [6-10]$. Logically, therefore, one is led to ask how these models evade Weinberg’s theorem. The goal of this work is to shed some light on this question. We revisit the mechanism in which this theorem is proved and specify the loopholes in some technical assumptions required in the theorem which are violated in these scenarios. There are some generic features on the violation of these technical assumptions which are shared in these models but we shall study each model independently to specify the exact nature of the violation of the theorem.

2 A brief review of Weinberg’s theorem

In this section we briefly review the Weinberg’s theorem which is independent of model (i.e. without assuming scalar fields etc.). For a more extensive review see [1, 5].

We are interested in scalar perturbations of the metric and matter sources. The scalar sector of metric perturbations in the Newtonian gauge has the following form

$$ds^2 = -(1 + 2\Phi(t, \mathbf{x})) dt^2 + a(t)^2 (1 - 2\Psi(t, \mathbf{x})) d\mathbf{x}^2,$$

in which $\Phi$ and $\Psi$ are the Bardeen potentials. The advantage in using the Newtonian gauge in the analysis of [1, 5] is that this gauge leaves no residual gauge symmetry except for the mode with the zero wavenumber, $k = 0$. This was crucially used in the proof of the theorem.

Let us start with the homogeneous FRW background and then consider the solutions of the perturbed Einstein field equations which are homogeneous but time-dependent: $\Phi = \Phi(t)$ and $\Psi = \Psi(t)$. Of course, they are not physical solutions by themselves as in general they may be removed by a coordinate transformation, $x^\mu \rightarrow x^\mu + \epsilon^\mu(t, \mathbf{x})$. The goal is to see under what conditions a subset of these solutions can be extended to non-zero wavenumber which satisfy all Einstein’s equations. If so, then these subset of solutions represent physical solutions.

As demonstrated in [1, 5] one concludes that there is always a spatially homogeneous solution to the set of perturbed Einstein equations in Newtonian gauge in which

$$\Psi(t) = H\epsilon(t) - \frac{\omega_{ij}}{3}, \quad \Phi(t) = -\dot{\epsilon}(t)$$

$$\delta p = -\dot{\epsilon} \epsilon(t), \quad \delta\rho = -\dot{\epsilon} \epsilon(t), \quad \delta u = \epsilon(t), \quad \pi^S = 0,$$

where $\delta \rho$ and $\delta p$ represent respectively the perturbed energy density and pressure, $\delta u$ is the perturbed velocity potential and $\pi^S$ is the anisotropic inertia (pressure) term. In addition, $\epsilon(t)$ is a function encoding the time-dependent part of $\epsilon_0(t, \mathbf{x})$ in the coordinate transformation $x^\mu \rightarrow x^\mu + \epsilon^\mu(t, \mathbf{x})$ and $\omega_{ij}$ is a constant matrix (note that $\omega_{ii}$ is the trace of $\omega_{ij}$).

As mentioned above, the solution given in eqs. (2.2) and (2.3) are not physical in general. They become physical if they can be promoted to non-zero wave-numbers. In other words,
the solutions in eqs. (2.2) and (2.3) become physical if they also satisfy the Einstein fields equations when \(k \neq 0\). Imposing that eqs. (2.2) and (2.3) also satisfy the inhomogeneous perturbed Einstein equations one obtain two sets of independent physical solutions. The first set of solution is given by

\[
\Psi = \Phi = R \left[ -1 + H(t) \int_{\tau}^{t} a(t') dt' \right]
\]

\[
\frac{\delta p}{\dot{\rho}} = \frac{\delta \rho}{\dot{\rho}} = \delta u = - \frac{R}{a(t)} \int_{\tau}^{t} a(t') dt',
\]

in which \(R\) is the comoving curvature perturbation which is also conserved, \(R = \omega_{ii}/3\).

The second class of the physical solution is obtained to be

\[
\Psi = \Phi = CH(t(a(t))
\]

\[
\frac{\delta p}{\dot{\rho}} = \frac{\delta \rho}{\dot{\rho}} = \delta u = - C a(t),
\]

in which \(C\) is a constant. Furthermore, for this mode \(R = 0\).

We note that in both classes of solutions all scalar quantity such as \(\rho\) or \(p\) have equal value for \(\delta s/\dot{s}\), i.e. \(\delta \rho/\dot{\rho} = \delta p/\dot{p}\). For this reason these solutions are called adiabatic. In addition, in order to simplify our presentation of the theorem, we implicitly assumed that there is no anisotropic stress, \(\pi^s = 0\) and consequently \(\Phi = \Psi\). However, as in [5], one can extend these analysis to more general case in which \(\pi^s \neq 0\).

This summarizes the statement of the theorem. The details of assumptions and the derivations employed in [1, 5] seem to leave no loophole. However, there are two technical assumptions which may not be justified in general. The first technical assumption is that the set of perturbed Einstein equations are regular at \(k = 0\) so the transition from the gauge mode \(k = 0\) to the physical mode with \(k \neq 0\) but with \(k \to 0\) can be made continuously. The necessity of this technical assumption was already mentioned in [5] (see also [11]) and the fact that this technical assumptions may be invalidated in some certain cases. As we shall see a particular example in which this technical assumption is violated is the model of solid inflation.

However, a more subtle and somewhat hidden point in the proof of [1, 5] is the extent to which one can take the limit \(k \to 0\) arbitrarily for the super-horizon mode, without causing difficulties. The super-horizon condition is \(k/aH \ll 1\). So whenever we take \(k \to 0\) when dealing with the Einstein equations we actually mean the extent to which \(k/aH\) goes to zero. Now suppose the fields equations or the constraints are written such that

\[
\alpha(t)y_1 + \frac{k^2}{a^2 H^2} y_2 = \beta(t)y_3,
\]

in which \(\alpha(t)\) and \(\beta(t)\) are functions of background quantities such as \(H, \dot{H}\) etc but independent of \(k\), and \(y_i\) collectively represents some physical fields. As we shall see, the Poisson equation is a constraint like the above equation, see eq. (3.7). Now when we take \(k \to 0\) as the definition of super-horizon limit we actually mean \(\frac{k^2}{a^2 H^2} \to 0\) to arbitrary extent. However, this should be compared with the coefficients \(\alpha(t)\) or \(\beta(t)\). For example, if the coefficient \(\alpha(t)\) approaches zero more faster than \(1/a^2\), then taking \(k \to 0\) as the criteria to turn on a physical super-horizon mode from a pure gauge mode \(k = 0\) is ill-defined. As we shall see this
is exactly what happens in models of non-attractor inflation in which $\alpha(t)$ falls off like $a^{-6}$, much faster then the combination $k^2 H^2$. In this situation, the proof of [1, 5] is not expected to go through and the results of [1, 5] are violated in one way or another.

3 Non-attractor inflation

In this section, we study in detail how Weinberg’s theorem is violated in models of non-attractor inflation.

Let us first briefly review the models of non-attractor inflation. These models are proposed as a counterexample which violate the Maldacena’s single field non-Gaussianity consistency condition [12, 13]. In its simplest realization [6] (see also [14]), the model consists of a scalar field $\phi$ rolling in a flat potential $V = V_0$. From the background field equations we obtain $\dot{\phi} \propto a^{-3}$ while the first slow-roll parameter $\epsilon \equiv -\dot{H}/H^2$ falls off like $a^{-6}$. As studies in [6] a scale-invariant curvature perturbation with $n_s = 1$ can be obtained with the second slow-roll parameter $\eta \equiv \dot{\epsilon}/\epsilon H \simeq -6$. The large deviation of $\eta$ from the usual slow-roll condition is a manifestation of the fact that the potential is exactly flat and $\epsilon$ falls off exponentially during inflation. The crucial effect in non-attractor model is that the dominant curvature perturbation is not frozen on super-horizon scales and $\mathcal{R}$ grows like $a^3$. However, note that we still have the constant mode solution for $\mathcal{R}$ which is now the sub-leading mode. Putting it another way, the would-be decaying mode in conventional slow-roll inflation is now actually the growing mode while the would-be dominant mode in slow-roll models, corresponding to $\mathcal{R} = \text{constant}$, is here the sub-leading mode. Now comparing with Weinberg’s theorem, we recover the mode $\mathcal{R} = \text{constant}$. However, we do not recover the other solution $\mathcal{R} = 0$ and instead we get $\mathcal{R} \propto a^3$. This obviously calls for an inspection as to how the Weinberg theorem is violated in this setup.

There are two important comments in order. The first comment is that the fact that $\mathcal{R}$ is not frozen on super-horizon scales is the key to violate the single field non-Gaussianity consistency condition. Indeed, if $\mathcal{R}$ was frozen on super-horizon scales then by a change of coordinate $x^i \rightarrow e^\mathcal{R} x^i$ one could eliminate $\mathcal{R}$ completely yielding a zero value for the non-Gaussianity parameter $f_{\text{NL}}$ in the squeezed limit. The second comment is that the model, as proposed, suffers from the graceful exit problem as there is no mechanism to terminate inflation. However, in a more realistic situation one can imagine that towards the end of inflation a mechanism like waterfall phase transition happens terminating inflation efficiently. This can be achieved by a heavy waterfall field which has no contribution in curvature perturbation as in models of hybrid inflation.

The above simple non-attractor model was extended to more interesting cases in the context of K-inflation in which the potential is not flat and the scalar perturbations have a non-trivial sound speed $c_s$ [7, 8], see also [15]. The non-Gaussianity parameter $f_{\text{NL}}$ in the squeezed limit is given by $f_{\text{NL}} = 5(1 + c_s^2)/4c_s^2$ which clearly violates Maldacena’s consistency condition.

For the later reference, it is helpful to calculate the relation between $\Phi$ and $\mathcal{R}$ given in eq. (2.4) for the first adiabatic mode in Weinberg’s theorem in non-attractor model. With an integration by parts the relation between $\Phi$ and $\mathcal{R}$ is obtained to be

$$\Phi = \mathcal{R} \left[ -1 + \frac{H}{a} \left( a \frac{H}{H} + \int \frac{a dH}{H^2} \right) \right]. \quad \text{(3.1)}$$
Now taking $\epsilon = -\dot{H}/H^2 \propto \tau^6$, and to leading order in $\epsilon$, $H\tau \simeq -1$, the above integral can be cast into an integral over $\tau$ in the form of $\int d\tau \tau^4$ yielding

$$\Phi = \frac{\epsilon}{5} \mathcal{R}. \quad (3.2)$$

We emphasis again that the above relation between $\Phi$ and $\mathcal{R}$ is valid only for the first mode in Weinberg’s theorem given in eq. (2.4) which will be used in subsequent analysis.

Below we demonstrate the violation of the theorem in simple model of non-attractor inflation \cite{6} with $V(\phi) = V_0$ in three different methods. In the first method, we obtain the second order differential equation for $R$ and specify how the theorem is violated. In the second method, we solve the sets of Einstein equations to obtain $\Phi$ directly and look at its super-horizon limit $k/aH \ll 1$ or alternatively $k\tau \rightarrow 0$ in which $\tau$ is the conformal time related to physical time via $d\tau = dt/a(t)$. In the third method, we construct the solution first in the comoving gauge and then calculate $\Phi$ in Newtonian gauge which enables us to view the violation of the theorem from a different perspective.

### 3.1 An equation for $R$

We work in the Newtonian gauge and set $\Psi = \Phi$ as there is no anisotropic inertia. Going to Fourier space, the set of perturbed Einstein equation to be solved are

$$\ddot{\Phi} + 3H \dot{\Phi} + \frac{k^2}{a^2} \delta \phi = 4\dot{\phi} \Phi, \quad (3.4)$$

supplemented with the constraint equation (the Poisson equation)

$$\left(\dot{H} + \frac{k^2}{a^2}\right)\Phi = 4\pi G \left(-\dot{\phi} \ddot{\phi} + \dddot{\phi}\right), \quad (3.5)$$

in which a dot indicates the derivative with respect to cosmic time $t$ and $G$ is the Newton constant.

It is more convenient to work with the velocity potential $\delta u = -\delta \phi/\dot{\phi}$ in which eqs. (3.3) and (3.5) are cast into

$$\ddot{\Phi} + H \dot{\Phi} = -\epsilon H^2 \delta u, \quad (3.6)$$

and

$$\left(\epsilon - \frac{k^2}{a^2 H^2}\right)\Phi = -\epsilon \delta \dot{u}, \quad (3.7)$$

in which $\epsilon = -\dot{H}/H^2 = 4\pi G \dot{\phi}^2/H^2$.

As promised before, eq. (3.7) has the form of eq. (2.6) and we can guess how the theorem in \cite{1, 5} may be violated. If we take the arbitrary mathematical limit $k \rightarrow 0$ then the second term in eq. (3.7) can be discarded and we obtain the relation $\Phi = -\delta \dot{u}$ which is the starting point in \cite{5} when proving the theorem for the scalar fields. In usual situations, such as in slow-roll models, in which $\epsilon$ is nearly constant, taking the super-horizon limit simply as $k \rightarrow 0$ is safe justifying neglecting the second term in eq. (3.7). However, in the non-attractor model we have $\epsilon \propto a^{-6}$ so the first term in eq. (3.7) falls off much faster than the second term. On the other hand, when we take $k \rightarrow 0$ we actually rely on the fact that $a(t)$ expands exponentially so $k/aH$ falls off quickly for a given $k$. This was the trick to turn on the physical
solution from the pure gauge mode $k = 0$ in [1, 5]. Now in the non-attractor models, with
the first term in eq. (3.7) falling much faster than the term containing $k^2$, then taking $k \to 0$
as the criteria for super-horizon mode is ill-defined. Surprisingly, the would be decaying term
in eq. (3.7) (the term containing $k^2$) now is the leading term. For this reason, we keep both
terms in bracket in eq. (3.7) without dropping the term containing $k^2$.

The comoving curvature perturbation $\mathcal{R}$ is given by

$$ R = H \delta u - \Phi. \quad (3.8) $$

Plugging this into the conservation equation (3.6) yields

$$ H \delta \dot{u} + H^2 \delta u = \dot{\mathcal{R}} + H \mathcal{R}. \quad (3.9) $$

Now we manipulate eqs. (3.7), (3.9) to obtain

$$ \delta u = \frac{\mathcal{R}}{H} + \frac{a^2 \epsilon}{k^2} \dot{\mathcal{R}} \quad (3.10) $$

and

$$ \Phi = \left( \frac{a^2 H^2}{k^2} \right) \frac{\dot{\mathcal{R}}}{H}. \quad (3.11) $$

The above equations show a non-trivial interplay between $\dot{\mathcal{R}}$ and $k^{-2}$. Indeed, taking the
mathematical limit $k^2 = 0$ requires that $\dot{\mathcal{R}} = 0$ for the equations to be consistent. This
brings us to the conclusion of [5].

Now, with $\delta u$ and $\Phi$ expressed in terms of $\mathcal{R}$ and $\dot{\mathcal{R}}$ in eqs. (3.10) and (3.11),
we can cast the remaining equation (3.4) into a second order differential equation for $\mathcal{R}$. With some
long but otherwise simple manipulations we obtain

$$ \partial_t (a^3 \epsilon \dot{\mathcal{R}}) + k^2 \epsilon a \mathcal{R} = 0. \quad (3.12) $$

This is a known equation for $\mathcal{R}$ which can easily be obtained in other gauges, such as comoving
gauge as employed in [6]. However, we went into long procedure of deriving eq. (3.12) in
Newtonian gauge in order to be on the same platform as in [1, 5] and in order to pin down
the loophole in the technical assumption employed in [1, 5] to prove the theorem.

Now, the super-horizon limit in eq. (3.12) can be taken without any problem. The
mathematical limit of taking $k \to 0$ as employed in [1, 5] makes sense only in eq. (3.12) in
which the coefficient of $\dot{\mathcal{R}}$, $a^3 \epsilon$, does not vanish faster than the coefficient of $k^2$. This is
opposite to the situation in eq. (3.7) in which the first term in eq. (3.7) falls off much faster
than the second term containing $k^2$.

Taking the super-horizon limit of eq. (3.12) we obtain

$$ \mathcal{R} = C_1 + C_2 \int \frac{dt}{a^3 \epsilon}, \quad (3.13) $$

in which $C_1$ and $C_2$ are two constants of integrations representing the two independent modes.
The mode represented by $C_1$ is the usual mode which also exists in [1, 5]. The difference now
is in the mode represented by $C_2$. In conventional slow-roll model in which $\epsilon$ is constant, this
mode decays and one approaches the other solution in [1, 5] labeled by $\mathcal{R} = 0$. However, in
non-attractor model in which $\epsilon \propto a^{-6}$, this solution is the growing mode yielding $\mathcal{R} \sim a(t)^3$
as observed in [6].
3.2 The equation for $\Phi$

Here we solve the Einstein equations in Newtonian gauge directly to obtain $\Phi$. The corresponding equations involving the (00) and (ii) components of Einstein’s equations, with $\Psi = \Phi$, are

\begin{align*}
\ddot{\Phi} + 7H\dot{\Phi} + (6H^2 + 2\dot{H})\Phi + \frac{k^2}{a^2}\Phi &= -4\pi G(\delta \rho - \delta P) \quad (3.14) \\
3\ddot{\Phi} + 9H\dot{\Psi} + 6(H^2 + \dot{H})\Phi - \frac{k^2}{a^2}\Phi &= 4\pi G(\delta \rho + 3\delta P) \quad (3.15)
\end{align*}

while the (0i) equation is as given in eq. (3.3).

The general forms of $\delta \rho$ and $\delta P$ are given by

\begin{align*}
\delta P &= \dot{\phi} \delta \dot{\phi} - \dot{\phi}^2 \Phi - V_{\phi} \delta \phi \quad (3.16) \\
\delta \rho &= \dot{\phi} \delta \dot{\phi} - \dot{\phi}^2 \Phi + V_{\phi} \delta \phi. \quad (3.17)
\end{align*}

Note the curious effect that in our simple non-attractor model with a constant potential, $V = V_0$, we obtain $\delta \rho = \delta P = \dot{\phi} \delta \dot{\phi} - \dot{\phi}^2 \Phi$. With $\delta \rho = \delta P$, eq. (3.14) can be solved directly without the need to solve for $\delta u$, $\delta \rho$ and $\delta P$ from other equations.

Our goal is to find the solution of $\Phi$ from eq. (3.14) and then use this value of $\Phi$ to calculate $\mathcal{R}$. Note that from eq. (3.8), and after eliminating $\delta u = -\dot{\phi} / \dot{\phi}$ using eq. (3.3), the relation between $\Phi$ and $\mathcal{R}$ is

\begin{equation}
\mathcal{R} = -\Phi + \frac{H}{H}(\Phi + H\Phi) = -\Phi - \frac{1}{\epsilon} \left( \frac{\Phi'}{aH} + \Phi \right), \quad (3.18)
\end{equation}

in which a prime indicates the derivative with respect to the conformal time $\tau$ where $d\tau = dt/a(t)$.

In general, eq. (3.14) can not be solved exactly because of the slow-roll correction coming from $\dot{H}$. Here, we solve it to leading order in $\epsilon = -\dot{H}/H^2$. Note that because of the $1/\epsilon$ factor in eq. (3.18), we need to solve eq. (3.14) to first order in $\epsilon$ to find the sub-leading corrections in $\mathcal{R}$.

At zeroth order in $\epsilon$ and taking $aH = -1/\tau$, eq. (3.14) is cast into the simple form

\begin{equation}
\Phi'' - \frac{6}{\tau} \Phi' + \frac{6}{\tau^2} \Phi + k^2 \Phi = 0. \quad (3.19)
\end{equation}

The general solution is represented in terms of two independent solutions $\Phi_1^{(0)}$ and $\Phi_2^{(0)}$ in which

\begin{align*}
\Phi_1^{(0)}(k, \tau) &= k\tau(k^2\tau^2 - 3) \sin k\tau + 3k^2\tau^2 \cos(k\tau) \quad (3.20) \\
\Phi_2^{(0)}(k, \tau) &= k^6\tau(k^2\tau^2 - 3) \cos(k\tau) - 3k^7\tau^2 \sin(k\tau). \quad (3.21)
\end{align*}

Note that the superscript (0) above indicates that we have calculated $\Phi$ to zeroth order of $\epsilon$. Note also the overall power of $k$ which is different for $\Phi_1^{(0)}(k, \tau)$ and $\Phi_2^{(0)}(k, \tau)$. This is chosen for convenience in follow up calculations, as an overall power of $k$ can be absorbed into constants of integration $C_i(k)$ and $C_i(\tau)$ as we shall see below. However, it is important to note that for each $i = 1, 2$ it is the relative $k$-dependence of $\Phi_i$ and $\mathcal{R}_i$ (obtained from $\Phi_i$ below) which matters.
Having calculated the zeroth order solution of eq. (3.14) now we calculate the next leading term \( \Phi_1^{(1)}(k, \tau) \) for both modes \( i = 1, 2 \). For this we also should take into account that to next slow-roll correction in non-attractor model we have \( aH \approx -(1 + \epsilon/7)\tau^{-1} \). The corresponding differential equation for \( \Phi_1^{(1)}(k, \tau) \) obtained from perturbing eq. (3.14) is

\[
\Phi_1^{(1)''} - \frac{6}{\tau} \Phi_1^{(1)'} + \frac{6}{\tau^2} \Phi_1^{(1)} + k^2 \Phi_1^{(1)} = \frac{6\epsilon}{7\tau} \Phi_1^{(0)'} + \frac{2\epsilon}{7\tau^2} \Phi_1^{(0)} \quad i = 1, 2. \tag{3.22}
\]

The above equation for \( i = 1, 2 \) can be solved separately yielding

\[
\Phi_1^{(1)} = \frac{\epsilon}{28} \left[ \cos(k\tau)(21 + 4k^2\tau^2 + k\tau \sin(k\tau)(5 + 2k^2\tau^2)) \right], \tag{3.23}
\]

and

\[
\Phi_2^{(1)} = \frac{\epsilon}{28\tau^5} \left[ \cos(k\tau)(-945 + 315k^2\tau^2 + 5k^6\tau^6 + 2k^8\tau^8)
- k\tau \sin(k\tau)(945 + 21k^4\tau^4 + 4k^6\tau^6) \right]. \tag{3.24}
\]

Having calculated \( \Phi_i = \Phi_i^{(0)} + \Phi_i^{(1)} \) we can calculate \( \mathcal{R} \) from eq. (3.18), yielding to leading order

\[
\mathcal{R}_1 = \frac{k^3\tau^3}{\epsilon} \left( - \sin(k\tau) + k\tau \cos(k\tau) \right)
+ \frac{1}{28} \left[ \cos(k\tau)(105 - 63k^2\tau^2 - 2k^4\tau^4) + k\tau \sin(k\tau)(105 - 16k^2\tau^2) \right] + \mathcal{O}(\epsilon), \tag{3.25}
\]

and

\[
\mathcal{R}_2 = -\frac{k^3\tau^3}{\epsilon} \left( k\tau \sin(k\tau) + \cos(k\tau) \right) - \frac{k^2\cos(k\tau)}{28\tau^3} (315 - 105k^4\tau^4 + 16k^6\tau^6)
- \frac{k^2\sin(k\tau)k\tau}{28\tau^3} (315 + 105k^2\tau^2 - 63k^4\tau^4 - 2k^6\tau^6) + \mathcal{O}(\epsilon). \tag{3.26}
\]

Note that the general solution for \( \mathcal{R} \) is given in terms of two independent solutions \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) via \( \mathcal{R} = C_1(k)\mathcal{R}_1 + C_2(k)\mathcal{R}_2 \) in which \( C_1(k) \) and \( C_2(k) \) are two constants of integrations. As mentioned before, \( C_1(k) \) are \( k \)-dependent so an overall power of \( k \) can be absorbed in both \( \Phi_i \) and \( \mathcal{R}_i \). However, for each \( i \), it is the relative \( k \)-dependence of \( \Phi_i \) and \( \mathcal{R}_i \) which is important.

The above expressions for \( (\Phi_1, \mathcal{R}_1) \) and \( (\Phi_2, \mathcal{R}_2) \) are valid for both sub-horizon and super-horizon limits. Now, in order to make contact with Weinberg’s theorem, let us look at the super-horizon limits of the above solutions corresponding to \( \frac{k}{\sqrt{\tau}} = -k\tau \to 0 \). In this limit for the first mode we obtain

\[
\Phi_1 \simeq -\frac{k^6\tau^6}{15} + \frac{3\epsilon}{4} \quad (k\tau \to 0), \tag{3.27}
\]

and

\[
\mathcal{R}_1 \simeq -\frac{k^6\tau^6}{3\epsilon} + \frac{15\epsilon}{4} \quad (k\tau \to 0). \tag{3.28}
\]

From the above solutions we observe that \( \Phi_1 = \frac{\epsilon}{5}\mathcal{R}_1 \) in exact agreement with Weinberg’s theorem as given in eq. (3.2). Also note that in the mathematical limit \( k = 0 \) we see that \( \mathcal{R}_1 \) becomes constant as was expected. However, as we discussed in previous sub-section, we
have to be careful when taking the super-horizon limit \( k \tau \to 0 \) while \( k \) is held fixed. In this limit \( \epsilon \propto \tau^6 \) so the first term in eq. (3.28) is a constant too. To compare the two contributions in eq. (3.28), let us parameterize \( \epsilon \) as

\[
\epsilon(\tau) = \epsilon_*(\tau/\tau_*)^6, \tag{3.29}
\]

in which \( \tau_* \) indicates the time when the mode \( k \) leaves the horizon corresponding to \( k \tau_* = -1 \).

Plugging this in eq. (3.28) we obtain

\[
\mathcal{R}_1 \simeq -\frac{1}{3\epsilon_*} + \frac{15}{4} \quad (k \tau \to 0). \tag{3.30}
\]

From this expression we see that the first term in eq. (3.28) typically dominates over the second term.

Now let us look at the second mode in super-horizon limit in which we obtain

\[
\Phi_2 \simeq -\frac{135\epsilon}{4\tau^5} \left( 1 + \frac{1}{6} k^2 \tau^2 \right) \quad (k \tau \to 0), \tag{3.31}
\]

and

\[
\mathcal{R}_2 \simeq -\frac{k^8 \tau^3}{\epsilon} - \frac{45k^2}{4\tau^3} \quad (k \tau \to 0). \tag{3.32}
\]

In the mathematical limit \( k = 0 \), from the above solutions we find \( \mathcal{R} = 0 \) while \( \Phi_2 \propto \epsilon/\tau^5 \propto H/a \) in agreement with the findings of [1, 5] for the second mode. However, in the physical super-horizon limit in which \( k \tau \to 0 \) while \( k \) is held fixed, and with \( \epsilon \) given in eq. (3.29), we obtain

\[
\mathcal{R}_2 \simeq -\frac{k^2}{\tau^3} \left( \frac{1}{\epsilon_*} + \frac{45}{4} \right). \tag{3.33}
\]

The above result indicates the \( 1/\tau^3 \) growth of \( \mathcal{R} \) on super-horizon as observed in [6]. Note that the \( 1/\tau^3 \) growth in \( \mathcal{R}_2 \) is specific to non-attractor model in which \( \epsilon \) falls off exponentially.

Now we can see how the non-attractor solution evades Weinberg’s theorem. As just mentioned above, our results in the mathematical limit \( k = 0 \) agree with the second mode of Weinberg. However, the physical super-horizon limit is when \( k \tau \to 0 \) for a given \( k \). In this limit, and very similar to discussions after eq. (3.7), the singular \( 1/\tau^3 \) pre-factor accompanying \( k^2 \) in \( \mathcal{R}_2 \) determines the structure of the physical solution. As we argued before, the mathematical super-horizon limit \( k \to 0 \) employed in [1, 5], without taking into account the strong time-dependence of \( \epsilon \), can not capture this solution.

### 3.3 From comoving gauge to Newtonian gauge

In this sub-section we present the equations in comoving gauge which is more convenient for models containing scalar fields. Then we move from comoving gauge to Newtonian gauge which provides us with yet another insight as how the theorem in [1, 5] is violated.

Let us start with the ADM formalism in comoving gauge \( \delta \phi = 0 \), in which the metric perturbations has the following form

\[
ds^2 = -N^2 dt^2 + g_{ij}(N^i dt + dx^i)(N^j dt + dx^j). \tag{3.34}
\]

Here \( N \) and \( N^i \) are the lapse function and the shift vectors which are obtained algebraically from the constraint equations.
In comoving gauge, the spatial metric take the following simple form (neglecting transverse and traceless part)

$$g_{ij} = a^2 (1 + 2\mathcal{R}) \delta_{ij}.$$  \hfill (3.35)

As usual, we may write down the quadratic action and solve for the lapse function and the shift vector. Defining the lapse function and the shift vector via

$$g_{0i} = N_i \equiv \partial_i \psi, \quad g_{00} \equiv -(1 + 2N_1),$$  \hfill (3.36)

from the constraint equations we obtain

$$N_1 = \frac{2\dot{\mathcal{R}}}{H},$$  \hfill (3.37)

and,

$$\psi = -\frac{\mathcal{R}}{H} + \chi, \quad \chi \equiv \partial^{-2}(a^2 \epsilon \dot{\mathcal{R}}).$$  \hfill (3.38)

Note that in usual attractor case in which $\mathcal{R}$ is conserved outside horizon we have

$$\dot{\mathcal{R}} \sim k^2 \frac{r}{a^2 H^2},$$  \hfill (3.39)

so $\chi$ is analytic in $k$. However, in non-attractor case in which [6]

$$\dot{\mathcal{R}} = -3HR + \mathcal{O}(k^2/a^2 H^2),$$  \hfill (3.40)

then $\chi$ is non-analytic in $k$. This is another sign that the prescription of taking $k \to 0$ employed in [1, 5] as the definition of super-horizon limit is problematic.

Now we perform the coordinate transformation from the comoving gauge to the Newtonian gauge. Consider the coordinate transformation

$$x^i \to x^i + \xi^i, \quad \xi^i = \partial_i \epsilon^S,$$  \hfill (3.41)

in which $\epsilon^S$ is the scalar part of spatial coordinate transformation.

If we split the metric as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, under the above coordinate transformation we have,

$$\Delta h_{i0} = \partial_i(-\dot{\epsilon}^S - \epsilon^0 + 2He^S),$$  \hfill (3.42)

$$\Delta h_{ij} = -2\partial_i \partial_j e^S + 2a^2 H e^0 \delta_{ij},$$  \hfill (3.43)

$$\Delta h_{00} = -2\epsilon^0,$$  \hfill (3.44)

in which $\Delta h_{\mu\nu}$ indicates the change in $h_{\mu\nu}$ in transforming from the comoving gauge to the Newtonian gauge.

In the Newtonian gauge we should keep the spatial metric diagonal so from eq. (3.43) we require\footnote{Note that in general $\epsilon^S = f(t)$ will keep the spatial metric diagonal too. However, this choice gives rise to pure gauge mode which has been already taken care of in Weinberg’s theorem.}

$$e^S = 0.$$  \hfill (3.45)
In addition, in Newtonian gauge $h_{0i} = 0$ and taking into account that in comoving gauge $h_{0i} = N_i$ is given in eqs. (3.36) (3.38), from eq. (3.42) we obtain

$$\partial_i \left[ - \frac{\mathcal{R}}{H} + \chi - e^0 \right] = 0.$$  \hspace{1cm} (3.46)

Therefore, neglecting pure gauge mode, from this equation we obtain

$$e^0 = -\frac{\mathcal{R}}{H} + \chi = -\frac{\mathcal{R}}{H} + \partial^{-2}(a^2 \epsilon \mathcal{R}).$$  \hspace{1cm} (3.47)

Now, plugging this value of $e^0$ into eqs. (3.44) and (3.43) the components of metric in Newtonian gauge is obtained to be

$$g_{00} = -1 - \frac{2\mathcal{R}}{H} - 2e^0 = -1 + 2\epsilon \mathcal{R} - 2\partial_i \partial^{-2}(a^2 \epsilon \mathcal{R}),$$  \hspace{1cm} (3.48)

$$g_{ij} = a^2 \left[ 1 + 2H\partial^{-2}(a^2 \epsilon \mathcal{R}) \right] \delta_{ij}.$$  \hspace{1cm} (3.49)

The above expressions for $g_{00}$ and $g_{ij}$ give two independent formulas for $\Phi$ and $\Psi$. Now imposing the constraint $\Phi = \Psi$ in Newtonian gauge, we readily obtain the second order differential equation for $\mathcal{R}$ as given in eq. (3.12). In addition, once $\mathcal{R}$ is solved this way, we can plug it into eq. (3.49) to obtain $\Phi$ as follows

$$\Phi = \Psi = -H \partial^{-2}(a^2 \epsilon \mathcal{R}) = -H \chi.$$  \hspace{1cm} (3.50)

Note that the above solution works for both attractor and non-attractor phases, and it is physical because we obtained it from coordinate transformation of a physical solution in comoving gauge.

Now, as it is stressed earlier, in attractor case $\chi$ is analytic in $k$ i.e. it is well defined in $k \to 0$ limit. Therefore both of Weinberg’s adiabatic modes are physical and the theorem works well. This is also seen from the explicit solutions of $\mathcal{R}$ in eq. (3.12) as discussed in previous sub-section. However, in the non-attractor case that $\mathcal{R}$ evolves on super-horizon scales $\chi$ is non-analytic in $k$ so the limit $k \to 0$ is not well defined mathematically. This is also seen from the structure of eq. (3.50) in which $\Phi = (a^2 \epsilon / k^2) \mathcal{R}$. The analyticity of the results for the limit $k \to 0$ requires that $\mathcal{R} = 0$. Conversely, if we do not know $\mathcal{R} = 0$ a priori then we can not assume the analyticity of the solutions in the limit $k \to 0$ which is taken as the guiding principle to distinguish the physical solution from the pure gauge mode.

Before we conclude this section we comment that the violation of Weinberg’s theorem in non-attractor model leads to the evolution of curvature perturbation outside the horizon. This evolution in turn leads to the violation of single field consistency condition. Consequently, large non-Gaussianity in squeezed limit is generated in this model. This is because in squeezed limit one of the modes is much longer than the other two modes and therefore it leaves the horizon much earlier. If curvature perturbation is constant outside the horizon this constant mode will only change the background for the other two modes, generating no local-type non-Gaussianity. However this simple intuitive argument fails to hold when curvature perturbations evolve outside the horizon. Specifically, in the most simplest non-attractor model considered here we have $\mathcal{f}_{NL} = \frac{5}{2}$.
4 Fluid inflation

Fluid inflation, presented originally in [16], is another example in which Weinberg’s theorem is violated. Here we briefly review the setup of fluid inflation and present the reasons why it violates Weinberg’s theorem in close analogy with non-attractor scenarios.

The fluid setup is given by the following Lagrangian density [17, 18]

\[
L = \frac{1}{2} M_P^2 \sqrt{-g} R - \sqrt{-g} \rho (1 + e(\rho)) + \sqrt{-g} \lambda_1 (g_{\mu\nu} U^\mu U^\nu + 1) + \sqrt{-g} \lambda_2 (\rho U^\mu)_{;\mu},
\]

(4.1)
in which \(M_P\) is the reduced Planck mass, \(\rho\) is the rest mass density, \(e(\rho)\) is the specific internal energy and \(U^\mu\) is the 4-velocity. In addition, \(\lambda_1\) and \(\lambda_2\) are two Lagrange multipliers to enforce the normalization of the 4-velocity and the conservation of the rest mass density.

With this prescription, the total energy density, \(E\), is given by

\[
E = \rho (1 + e).
\]

(4.2)

As in [16] we concentrate on an isentropic or barotropic fluid for which \(e = e(\rho)\). Having this said, there is no restriction to consider more general situations in which \(e\) can also be a function of other thermodynamic variables such as entropy.

Varying the action with respect to the Lagrange multipliers \(\lambda_1\) and \(\lambda_2\) and dynamical fields \(\rho\) and \(g_{\mu\nu}\) we recover the Einstein’s fields equation in which now the stress energy tensor \(T^{\mu\nu}\) takes the form of a perfect fluid

\[
T^{\mu\nu} = (E + P) U^\mu U^\nu + P g^{\mu\nu}.
\]

(4.3)

Here \(P\) plays the role of pressure in which for an isentropic fluid is represented by

\[
\frac{d e(\rho)}{d \rho} = \frac{P}{\rho^2}.
\]

(4.4)

Knowing that \(e = e(\rho)\), from the above equation we conclude that \(P\) is a function of \(\rho\). Alternatively, from eq. (4.2) we also conclude

\[
\frac{d E}{d \rho} = \frac{E + P}{\rho}.
\]

(4.5)

We note that eqs. (4.4) and (4.5) imply that \(P\) is a function of \(E\), \(P = P(E)\), which is expected for a barotropic fluid.

An important parameter of the fluid is the sound speed of perturbations \(c_s\) which is given by

\[
c_s^2 \equiv \frac{\dot{P}}{E}.
\]

(4.6)

For a small perturbation, and using the conservation equation \(\dot{E} + 3H(E + P) = 0\), this implies

\[
\delta P = c_s^2 \delta E = c_s^2 (E + P) \frac{\delta \rho}{\rho}.
\]

(4.7)

Note that the definition (4.6) makes sense as we consider a barotropic fluid. In order for the perturbations to be stable we require \(c_s^2 > 0\), while for the perturbations to be sub-luminal we also require \(c_s^2 \leq 1\).
The cosmological dynamics of the system has the usual FRW form. However, as compared to inflation based on scalar field dynamics, we note that for fluid setup the total energy density \( E \) and the pressure \( P \) internally are functions of the rest mass density \( \rho \). This yields a non-trivial equation of state \( P = P(E) \) for a barotropic fluid in which \( c_s \) plays non-trivial roles in perturbation analysis.

The first and second slow-roll parameters \( \epsilon \) and \( \eta \) respectively are

\[
\epsilon = -\frac{\dot{H}}{H^2} = \frac{E + P}{2M_p^2 H^2},
\]

and

\[
\eta \equiv \frac{\dot{\epsilon}}{H \epsilon} = 2 \epsilon - 3(1 + c_s^2).
\]

From the form of \( \eta \) we see the important difference compared to conventional slow-roll models of scalar field theories. Requiring that \( 0 < c_s^2 \leq 1 \), and taking \( \epsilon \ll 1 \) in order to sustain a long enough period of inflation, we conclude that \(-6 \lesssim \eta \lesssim -3\). At this stage we can not pin down the exact value of \( \eta \), this should be fixed from the scale-invariance of the curvature perturbation power spectrum. However, for \( \eta \) given in the above range, we readily conclude that \( \epsilon \) falls off exponentially which, as we shall see below, closely resembles the non-attractor scenario.

To perform the cosmological perturbation analysis we go to comoving gauge defined on a time-slicing in which the fluid’s 4-velocity is orthogonal to the hypersurface \( t = \text{constant} \) and the three-dimensional spatial metric is conformally flat [16]. Calculating the quadratic action in comoving gauge we obtain

\[
(z^2 \mathcal{R}')' + c_s^2 k^2 z^2 \mathcal{R} = 0,
\]

in which a prime denotes the derivative with respect to conformal time and \( z \) is defined via \( z^2 = 2\epsilon a^2/c_s^2 \). We note that the above equation for \( \mathcal{R} \) is similar to eq. (3.12) obtained for scalar field theory. Now quantizing the system and calculating the power spectrum, the spectral index is obtained to be \( n_s \simeq 3(1 - c_s^2) \) [16]. We see that to obtain a scale invariant power spectrum we require \( c_s^2 = 1 \). Consequently, from eq. (4.9) we conclude that \( \eta \simeq -6 \) and hence \( \epsilon \propto a(t)^{-6} \). Very interestingly, we see that fluid inflation is a non-trivial realization of non-attractor setup, completely independent of scalar field dynamics. Now it should not be surprising that in fluid setup, \( \mathcal{R} \) is not frozen on super-horizon scales and indeed we readily conclude that \( \mathcal{R} \propto a(t)^3 \) [16].

Having established the direct link between the fluid setup and the non-attractor setup, we can use any of the arguments presented in sub-sections 3.1, 3.2 or 3.3 to understand why Weinberg’s theorem is violated in the model of fluid inflation. For example in the method of sub-section 3.1, in Poisson constraint eq. (3.7) we find that \( \epsilon \) falls off much stronger than the combination \( k^2/a^2 H^2 \) so one can not take \( k \to 0 \) arbitrarily for a given \( k \) to define the super-horizon limit. Or in the method of subsection 3.2, with \( c_s = 1 \) we conclude that \( \delta P \propto \delta E \), and similar to non-attractor case, eq. (3.14) can be solved directly to find \( \Phi \). The rest of the argument as how the theorem is violated in fluid inflation setup goes parallel to the discussions after eq. (3.33).

5 Solid inflation

In this section we study the model of solid inflation [19] which is another known example in literature which violates Weinberg’s theorem; for other works on solid inflation see [20–25].
As the name indicates, in this model inflation is driven by a configuration resembling a solid. In this setup the three-dimensional space is divided into small cells such that the location of each cell is defined by the value of scalar fields $\phi^I$ for $I = 1, 2$ and 3. More specifically, at the background level the position of each cell is represented by

$$\langle \phi^I \rangle = x^I, \quad I = 1, 2, 3.$$  \hspace{1cm} (5.1)

At this stage the ansatz (5.1) naively seems to violate the isotropy and the homogeneity of the cosmological background as the scalar fields $\phi^I$ are time-independent and depend explicitly on $x^I$. However, on the physical grounds, one should impose the following internal symmetries to keep the background isotropic and homogeneous

$$\phi^I \to \phi^I + C^I, \hspace{1cm} (5.2)$$

and

$$\phi^I \to O^I_J \phi^J, \quad O^I_J \in \text{SO}(3). \hspace{1cm} (5.3)$$

We note that $C^I$ are constants while $O^I_J$ belong to SO(3) rotation group. The symmetry under translation in field space imposed by eq. (5.2) enforces that the dynamical quantities in the Lagrangian are constructed from derivatives of the scalar fields $\partial \phi^I$. Consequently, the background eq. (5.2) becomes invariant under translation. Furthermore, the internal SO(3) rotation invariance guarantees the isotropy of the background. In conclusion, with the internal symmetries (5.2) and (5.3) enforced, the background is consistent with the cosmological principles.

The most general action consistent with the above internal symmetries which is minimally coupled to gravity is given by

$$S = \int d^4x \sqrt{-g}\left\{ \frac{M_P^2}{2} R + F[X, Y, Z] \right\}, \hspace{1cm} (5.4)$$

in which $M_P$ is the reduced Planck mass related to Newton constant via $M_P^2 = 1/8\pi G$ and $F$ is a function incorporating the properties of the solid. The condition that the action is invariant under the internal symmetries (5.2) and (5.3) requires that the variables $X$, $Y$ and $Z$ are functions of the derivatives of $\phi^I$ which in turn are given in terms of the SO(3) invariant matrix $B^{IJ}$ via

$$X \equiv [B], \quad Y \equiv \frac{[B^2]}{[B]^2}, \quad Z \equiv \frac{[B^3]}{[B]^3}, \hspace{1cm} (5.5)$$

in which $[B] \equiv \text{Tr}(B)$ and

$$B^{IJ} \equiv g^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J. \hspace{1cm} (5.6)$$

Our convention is that the Greek indices $\mu, \nu, \ldots$ indicate the four-dimensional spatial coordinates while the capital Latin indices $I, J, \ldots$ represent the three-dimensional internal matter field space.

At the background level we can check that

$$X = \frac{3}{a(t)^2}, \quad Y = \frac{1}{3}, \quad Z = \frac{1}{9}. \hspace{1cm} (5.7)$$

Note that the variables $Y$ and $Z$ are constructed such that they are insensitive to the volume of 3-space while the information about the background volume is entirely encoded in $X$. 

- 14 -
The energy momentum-tensor is given by
\[ T^{\mu}_{\nu} = \delta^{\mu}_{\nu}F - 2g^{\mu\alpha}\partial_{\alpha}\phi^{I}\partial_{\nu}\phi^{J}M^{IJ}, \] (5.8)
in which we have defined \( M^{IJ} \) via
\[ M^{IJ} \equiv \left( F_{X} - \frac{2F_{Y}Y}{X} - \frac{3F_{Z}Z}{X} \right)\delta^{IJ} + \frac{2F_{Y}B^{IJ}}{X^{2}} + \frac{3F_{Z}B^{IK}B^{KJ}}{X^{3}}, \] (5.9)
where \( F_{X} \equiv \partial F/\partial X \) and so on.

With the above form of \( T^{\mu}_{\nu} \), the energy density \( \rho \) and the pressure \( P \) at the background level are given by
\[ \rho = -F, \quad P = F - \frac{2}{a^{2}}F_{X}, \] (5.10)
yielding the expected cosmological equations
\[ 3M_{p}^{2}H^{2} = \rho, \quad \dot{H} = -\frac{1}{2M_{p}^{2}}(\rho + P). \] (5.11)

On the other hand, by varying the action with respect to \( \phi^{I} \), the scalar fields equations is obtained to be
\[ \partial_{\mu}\left( \sqrt{-g} \partial F / \partial B^{ab} \partial_{\mu} \phi^{I} \right) = 0. \] (5.12)
We note the curious effect that at the background level \( \phi^{I} \) are independent of \( t \) and eq. (5.12) is automatically satisfied so we do not get any information from eq. (5.12) at the background level.

At this level it may look that the solid scenario is a model with three inflationary fields \( \phi^{I} \) which can generate entropy perturbations which can naturally bypass Weinberg’s theorem. However, as studied in [19], the scalar perturbations are generated effectively by one degree of freedom. This scalar perturbation is described by the single field \( \pi^{L} \) corresponding to the longitudinal component of the fluid excitations, which are dubbed as “phonons” in [19]. More specifically, suppose
\[ \phi^{I} = x^{I} + \pi^{I}(t, \mathbf{x}), \] (5.13)
and decompose the filed \( \pi^{I} \) into its transverse and longitudinal parts as
\[ \pi^{I}(t, \mathbf{x}) = \frac{\partial_{i}}{\sqrt{-g}}\pi_{L}(t, \mathbf{x}) + \pi_{T}^{I}(t, \mathbf{x}), \] (5.14)
in which \( \partial_{i}\pi_{T}^{I} = 0 \). In this decomposition, \( \pi_{L} \) sources the curvature perturbations while \( \pi_{T}^{I} \) sources the vector perturbations. Note that we do not pursue the \( \pi_{T}^{I} \) excitations any further because the vector perturbations are damped after inflation.

Going to flat gauge, the curvature perturbations is given by \( \zeta = -\frac{k}{3}\pi_{L} \) which on super-horizon scales is obtained to be
\[ \zeta(\tau) \propto \left( -c_{L}k\tau \right)^{-A}(1 + B\ln(-c_{L}k\tau)), \] (5.15)
in which \( A \) and \( B \) are constants of order slow roll parameters \( \epsilon \) and \( c_{L} \simeq 1/\sqrt{3} \) is the sound speed of phonons. From the above expression we observe a mild running of curvature perturbation on super-horizon scales varying like \( \epsilon N \) in which \( N = \ln(-k\tau) \) is the number of
e-folds before the end of inflation. Our goal in this section is to understand how this happens, bypassing the theorem in [1, 5].

To address this question, we obtain the perturbed Einstein equations in Newtonian gauge. For this purpose, first we need the components of the perturbed energy momentum tensor $\delta T_{\nu}^\mu$. Using eq. (5.8) we have

$$\delta T_{\tau}^{\tau} = \delta F = F_X \delta X + F_Y \delta Y + F_Z \delta Z.$$ (5.16)

However, with some efforts one can show that $\delta Y$ and $\delta Z$ vanish up to linear order in perturbations so we can neglect their contributions and $\delta T_{\tau}^{\tau} = F_X \delta X$. On the other hand for $\delta X$ we have

$$\delta X = \delta \phi + \frac{2}{a^2} \partial_i \delta \phi^i = \frac{2X}{3} (3\Psi - k\pi_L),$$ (5.17)

in which the relation $\pi^i = \frac{1}{k} \partial_i \pi_L$ has been used. As a result, for $\delta T_{\tau}^{\tau}$ component we obtain

$$\delta T_{\tau}^{\tau} = 2XF_X \left( \Psi - \frac{k}{3} \pi_L \right).$$ (5.18)

Similarly, for $\delta T_{i}^{\tau}$ component we have

$$\delta T_{i}^{\tau} = \frac{2X}{3} F_X \delta \phi^i,$$ (5.19)

in which a prime indicates the derivative with respect to conformal time $\tau$.

On the other hand, the calculation of $\delta T_{i}^{j}$ is more non-trivial. We have

$$\delta T_{j}^{i} = F_X \delta X \delta^i_j - \frac{4}{a^2} \Psi M_{ij} - \frac{2}{a^2} F_X \Pi^{ij} - \frac{2}{a^2} \delta M^{ij},$$ (5.20)

in which $\Pi^{ij}$ is defined via

$$\Pi^{ij} \equiv \partial_i \pi^j + \partial_j \pi^i.$$ (5.21)

On the other hand, one can show that

$$\delta M^{ij} = F_{XX} \delta X \delta^i_j - \frac{2(F_Y + F_Z)}{3X^2} \delta X \delta^{ij} - \frac{2(F_Y + F_Z)}{3X} (2\Psi \delta_{ij} + \Pi^{ij}).$$ (5.22)

Plugging this expression in eq. (5.20) we obtain

$$\delta T_{j}^{i} = \left( F_X - \frac{2X}{3} F_{XX} \right) \delta X \delta^i_j - \frac{4XF_X \Psi}{3} \delta^i_j - \frac{2X}{3} F_X \Pi^{ij} + \frac{4}{9} (F_Y + F_Z) \left[ \left( \frac{\delta X}{X} - 2\Psi \right) \delta^i_j - \Pi^{ij} \right].$$ (5.23)

So far no assumption was made beyond the linear perturbation theory. To simplify the analysis we impose the slow-roll assumptions and ignore terms higher in powers of the slow-roll parameter $\epsilon$. To leading order in $\epsilon$ one can show that $c_L^2 \simeq \frac{1}{4}$, $F_Y \simeq -F_Z$ and $F_{XX} \simeq -\frac{F_X}{X}$ [19]. Putting the above results together we obtain the following set of perturbed
Einstein equations

\[ \Phi - \Psi = \frac{4(\mathcal{H}' - \mathcal{H}^2)}{k} \pi_L, \]  
\[ \Psi' + \mathcal{H} \Phi = -\frac{(\mathcal{H}' - \mathcal{H}^2)}{k} \pi'_L, \]  
\[ 12\mathcal{H}' \Psi' + 3k^2 \Psi + (k^2 + 12\mathcal{H}^2)\Phi = 12(\mathcal{H}' - \mathcal{H}^2)\Psi. \] (5.24)

\[ \Psi'' + k^2 \Psi + 5 \mathcal{H} \Psi' + \mathcal{H} \Phi' + (2 \mathcal{H}' + 4 \mathcal{H}^2) \Phi = (\mathcal{H}' - \mathcal{H}^2) \left( 6 \Psi - \frac{2}{3} k \pi_L \right), \] (5.25)

\[ \Phi = 12(\mathcal{H}' - \mathcal{H}^2) \Psi - 3k^2 \Psi - \frac{12}{3} k \pi_L. \] (5.26)

in which \( \mathcal{H} \equiv aH \).

Note the interesting conclusion from eq. (5.24) that, unlike conventional models of inflation, \( \Psi \neq \Phi \). This is because in the model of solid inflation the longitudinal mode \( \pi_L \) sources the anisotropic stress \( \pi^S \) and therefore we have \( \phi \neq \Psi \). To see this explicitly, note that \( \pi^S \) is related to \( \delta T^i_j \) via \( \delta T^i_j = \delta P \delta^i_j + \partial_i \partial_j \pi^S \) [1]. Now with \( \delta T^i_j \) given in eq. (5.23), in the slow-roll limit, we obtain

\[ \pi^S = -\frac{4 \epsilon F^*}{3k} \pi_L. \] (5.28)

On the other hand, the \( i \neq j \) component of the perturbed Einstein equation in general is written as [1]

\[ \partial_i \partial_j (\Phi - \Psi) = -\frac{a^2}{M_P^2} \partial_i \partial_j \pi^S. \] (5.29)

Now with the form of \( \pi^S \) given in eq. (5.28) we obtain eq. (5.24). Also note that eq. (5.28) shows the \( 1/k \) non-analytic relation between \( \pi^S \) and \( \pi_L \) which directly violates the analyticity assumption employed in the proof [1, 5]. Consequently, it should not be surprising that the conclusion in [1, 5] is violated in solid inflation.

As another sign of non-analytic structure of solid model, note that eq. (5.25) represents the momentum conservation equation, i.e. the \((0i)\) component of Einstein equation, in which the scalar velocity potential (in convention of [1]) is obtained to be

\[ \delta u = -\frac{a}{k} \pi'_L. \] (5.30)

Again, we see the non-analytic \( 1/k \) behavior in fields’ equations as discussed above.

One can eliminate \( \pi_L \) and \( \Phi \) in favors of \( \Psi \) and obtain a closed second order differential equation for \( \Psi \). For this purpose from eqs. (5.24) and (5.26) we obtain

\[ \pi_L = \frac{k[(3\mathcal{H}' - 6 \mathcal{H}^2 - k^2)\Psi - 3 \mathcal{H} \Psi']}{(\mathcal{H}' - \mathcal{H}^2)(k^2 + 12 \mathcal{H}^2)}, \] (5.31)

\[ \Phi = 12(\mathcal{H}' - \mathcal{H}^2) \Psi - 3k^2 \Psi - 12 \mathcal{H} \Psi'. \] (5.32)

Now plugging the above expressions for \( \pi_L \) and \( \Phi \) in eq. (5.25), and using the following relations which is valid in slow-roll limit

\[ \mathcal{H}' - \mathcal{H}^2 \approx \frac{c}{\tau^2}, \quad \mathcal{H}'' \approx 2 \mathcal{H}^2, \] (5.33)

we obtain our desired equation for \( \Psi \)

\[ 3(k^2 + 12 \mathcal{H}^2) \Psi'' - 72 \mathcal{H}^3 \Psi' + (k^4 - 12 \mathcal{H}^2(k^2 + 6 \mathcal{H}^2)) \Psi = 0. \] (5.34)
Happily eq. (5.34) can be solved analytically. Imposing the Minkowski initial condition for the modes inside the horizon (corresponding to \( k|\tau| \gg 1 \)), we obtain
\[
\Psi(x) = -\sqrt{\frac{3}{2k}} e^{\frac{ix}{\sqrt{3}}},
\]
in which we have defined \( x \equiv k\tau \). Note that the factor \( 1/\sqrt{3} \) in the exponent appears because the modes deep inside the horizon propagate with the sound speed \( c_s^L \approx 1/3 \).

Now let us look at the above solution in the super-horizon limit \( x \to 0 \)
\[
\Psi \propto \frac{1}{(k\tau)^2} e^{2N} \quad (k\tau \to 0),
\]
in which \( N \) is the number of e-fold towards the end of inflation with the convention \( N > 0 \). The above equation clearly demonstrates that on super-horizon scales the gravitational potential grows exponentially. This non-perturbative growth of \( \Psi \) implies that the Newtonian gauge is not a reliable gauge to study perturbations in solid inflation.

Now with \( \Psi \) calculated in eq. (5.35) we can calculate \( \zeta \). Knowing that \( \zeta \) is given by \( \zeta = -\frac{1}{3} \pi_L \), from eq. (5.35) we can calculate \( \zeta \) yielding eq. (5.15) to leading order in slow-roll corrections.

It is important to note that because of the non-zero anisotropic stress \( \pi^S \), we have \( \Psi \neq \Phi \). However, this by itself is not the source of violation of the Weinberg’s theorem. Instead, the non-analytic relation between \( \pi^S \) and \( \pi_L \), as given in eq. (5.28), is the key reason for the violation of this theorem in solid inflation. Note that because \( \zeta = -\frac{1}{3} \pi_L \), eq. (5.28) also implies the non-analytic relation
\[
\pi^S \sim \frac{\zeta}{k^2}.
\]
In addition, from eq. (5.30) we also have the non-analytic relation between \( \delta u \) and \( \zeta \). These non-analytic behaviors between \( \pi^S \), \( \delta u \) and \( \zeta \) are in direct conflicts with the analyticity assumption employed in the proof of [1, 5], as also mentioned in [19] (see also [11, 26, 27]).

We also comment that in solid model \( R \neq -\zeta \), even on super-horizon scales. This is because \( \zeta \) is not frozen on super-horizon scales yielding \( R \simeq -c_s^L \zeta \) on these scales.

Finally, we comment on the violation of non-Gaussianity consistency condition in solid inflation. As we have seen, in solid inflation the presence of non-analytic anisotropic stress causes the violation of Weinberg’s theorem. Consequently the curvature perturbation grows outside the horizon so, as in non-attractor inflation, the consistency condition does not hold here. However, the squeezed limit non-Gaussianity in solid inflation is very different from non-attractor model in which the bispectrum has a non-trivial shape, i.e. it depends on the direction from which we reach the squeezed limit \( k_3 \ll k_1 \sim k_2 \) [19, 22]. This direction-dependency in momentum space is a novel feature of solid inflation which is a consequence of the peculiarities caused by the exotic symmetry breaking \( \langle \phi^I \rangle = x^I \).

6 Pseudo-conformal universe

In this section we study yet another example in literature which is known to violate the theorem in [1, 5], the pseudo-conformal universe. This model was proposed in [28] as an alternative to inflation which relies on conformal symmetries capable of generating nearly
scale invariant power spectrum while solving the flatness and the horizon problems. The model shares similarities to the U(1) model \cite{29, 31, 31} and the Galilean Genesis scenario \cite{32}.

In the model of pseudo-conformal universe it is assumed that the early universe (before the big bang) enjoys an approximate conformal symmetry in a near flat background. At this early stage one or more of the conformal fields develop time-dependent expectation values which break the conformal symmetry. In addition, it is assumed that there are other fields with zero conformal weight (i.e. isocurvature fields) which acquire a nearly scale-invariant power spectrum generating the observed curvature perturbations.

To be specific, and following \cite{28}, we consider a simple model containing the negative quartic potential \( V = -\frac{1}{4} \lambda \phi^4 \) with \( \lambda > 0 \) which is minimally coupled to gravity. The model is classically conformal invariant. It is assumed that there are sub-leading corrections that can uplift the potential making the potential bounded from below. One mode of \( \delta \phi \) perturbations is freezing while the other mode grows on super-horizon scales. The latter is the mode of interest which violates the theorem in \cite{1, 5}.

However, as noted above, the observed curvature perturbations are generated by the additional field \( \chi \) which has the conformal weight zero and at the background level has no expectation values, \( \chi = 0 \). However, we will not study this field as we are interested to see how the growing mode of the conformal field fluctuation \( \delta \phi \) violates Weinberg’s theorem.

In the past infinity \( t = -\infty \), the scalar field starts rolling from \( \phi = 0 \). As the scalar field develops an expectation value and the conformal invariance is broken the universe starts a slow phase of contraction in which gravity is very weak, corresponding to \( \lambda M^2 P t^2 \gg 1 \), and calculations can be accurately approximated to leading orders of \( 1/M^2 P \).

The leading order \( 1/M^2 P \) corrections to the slowly-contracting scale factor \( a(t) \), the Hubble expansion rate \( H \) and the zeroth order evolution of \( \phi(t) \) were presented in \cite{28}. Here, we extend these results to next leading order \( 1/M^4 P \) in order to consistently calculate the next order corrections in \( \Phi \) and \( R \). To order \( 1/M^4 P \) we have

\[
\begin{align*}
    a(t) &= 1 - \frac{1}{6 \lambda M^2 P t^2} - \frac{13}{360 \lambda^2 M^4 P t^4} + \ldots \quad (6.1) \\
    H(t) &= \frac{1}{3 \lambda M^2 P t^3} + \frac{1}{5 \lambda^2 M^4 P t^5} + \ldots \quad (6.2) \\
    \phi(t) &= \sqrt{\frac{2}{\lambda}} t + \sqrt{\frac{2}{\lambda}} \frac{1}{6 \lambda M^2 P t^3} + \frac{19}{360} \sqrt{\frac{2}{\lambda}} \frac{1}{\lambda^2 M^4 P t^5} + \ldots \quad (6.3)
\end{align*}
\]

Note that in this model universe is in a phase of slow contraction so modes leave the horizon smoothly similar to an inflationary background. The criteria for the mode to be super-horizon is \( k|t| \ll \frac{1}{\sqrt{\lambda M P |t|}} [28] \). Note that \( t < 0 \) so that is why we have used \( |t| \). On the other hand, in order for the gravitational back-reaction to be small we require \( M_P |t| \gg 1 \). Combining these two conditions we have

\[
    k|t| < \frac{1}{\sqrt{\lambda M_P |t|}} \ll 1. \quad (6.4)
\]

From the background solutions we can calculate \( \epsilon = -\dot{H}/H^2 = 9\lambda M^2 P t^2 \). From the weak gravity condition this implies that \( \epsilon \gg 1 \). As we shall see below, the strong time-dependence of \( \epsilon \) plays crucial roles in violating Weinberg’s theorem.

Our strategy here is very similar to the strategy employed in sub-section 3.2. We would like to calculate \( \Phi \) to leading orders in \( 1/M^2 P \) and then calculate \( R \) and see how the theorem
in [1, 5] is violated. The corresponding equations for $\delta \phi$ and $\Phi$ are as in eqs. (3.14) and (3.3) in which now $\delta \rho - \delta P = -2\lambda \phi^3 \delta \phi$. Using eq. (3.3) to eliminate $\delta \phi$, from from eq. (3.14) we obtain

$$\ddot{\Phi} + \left(7 - \frac{2\lambda \phi^3}{H\phi}\right) H \dot{\Phi} + \left(6H^2 + 2\dot{H} + \frac{k^2}{a^2} - \frac{2\lambda \phi^3}{\phi} H\right) \Phi = 0. \quad (6.5)$$

Plugging the background values of $a(t)$, $H(t)$ and $\phi(t)$ into the above equation, to leading order of $1/M_P^2$ we obtain

$$\ddot{\Phi}_k + \left(\frac{4}{t^3} + \frac{7}{3\lambda M_P^2 t^3}\right) \dot{\Phi}_k + \left(k^2 + \frac{k^2}{3\lambda M_P^2 t^2} - \frac{2}{3\lambda M_P^2 t^4}\right) \Phi_k = 0. \quad (6.6)$$

Now we solve eq. (6.6) order by order in powers of $1/M_P^2$. At the zeroth order the solutions are given by

$$\Phi_1^{(0)} = \frac{1}{t^3} \left((kt) \cos(kt) - \sin(kt))\right) \quad (6.7)$$

$$\Phi_2^{(0)} = \frac{1}{t^3} \left(\cos(kt) + (kt) \sin(kt)\right). \quad (6.8)$$

Now if we take the mathematical limit $k \to 0$ it is easy to check that

$$\Phi_1^{(0)} \to \frac{-k^3}{3}, \quad R_1^{(0)} \to \frac{k^3}{3},$$

and

$$\Phi_2^{(0)} \to \frac{1}{t^3}, \quad R_2^{(0)} \to -\frac{k^2}{3t}. \quad (6.10)$$

In particular, the above expressions yields $\Phi_1^{(0)} = -R_1^{(0)}$ in agreement with eq. (2.4) while $\Phi_2^{(0)} \propto \frac{H}{a}$ and $R = 0$ in agreement with eq. (2.5) to zeroth order of $1/M_P^2$.

Now we calculate the next correction in $\Phi$. The corrections after solving eq. (6.6) to leading order in $1/M_P^2$ is obtained to be

$$\Phi_1^{(1)} = \frac{1}{30\lambda M_P^2 t^5} \left[4k^2 t^2 (kt \cos(kt) - \sin(kt)) \text{Ci}(2kt) + 4k^2 t^2 (\cos(kt) + kt \sin(kt)) \text{Si}(2kt) + 3k^2 t^2 \sin(kt) + 23kt \sin(kt) - 23\sin(kt)\right] \quad (6.11)$$

and

$$\Phi_2^{(1)} = \frac{1}{30\lambda M_P^2 t^5} \left[-4k^2 t^2 (\cos(kt) + kt \sin(kt)) \text{Ci}(2kt) + 4k^2 t^2 (kt \cos(kt) - \sin(kt)) \text{Si}(2kt) - 3k^2 t^2 \cos(kt) + 23kt \sin(kt) + 23\cos(kt)\right] \quad (6.12)$$

in which $\text{Si}(x) \equiv \int_0^x dy \sin(y)/y$, $\text{Ci}(x) \equiv \gamma + \ln(x) + \int_0^x dy (\cos(y - 1)/y)$ and $\gamma$ is the Euler number. Having obtained $\Phi_1 = \Phi_1^{(0)} + \Phi_1^{(1)}$ we can also calculate $R_1$ using eq. (3.18). However, it is more instructive to look at the super-horizon limit of these solutions, $\lambda \lambda M_P^2 |t|^3 \ll 1$. For the first mode we obtain

$$\Phi_1 \simeq \left(-\frac{1}{3} + \frac{1}{9\lambda M_P^2 t^2}\right)k^3 + \left(\frac{112 - 60(\gamma + \ln(2kt))}{1530\lambda M_P^2 t^2}\right)k^5 \quad (\sqrt{\lambda k M_P^2 t^2} \ll 1), \quad (6.13)$$
and
\[ R_1 \simeq \frac{k^3}{3} + \left( -\frac{t^2}{18} + \frac{-17 + 12(\gamma + \ln(2kt))}{270\Lambda M_P^2} \right) k^5 \] \quad \left( \sqrt{\Lambda} k M_P t^2 \ll 1 \right). \tag{6.14}

In particular note that \( \Phi_1 \simeq (-1 + \frac{1}{\Lambda k M_P^2 t^2})R_1 \) as anticipated from eq. (2.4). As expected, this mode satisfies the results of [1, 5].

Now, let us look at the second mode in the super-horizon limit obtaining
\[ \Phi_2 \simeq \left( \frac{1}{t^3} + \frac{23}{30\lambda M_P^2 t^5} \right) + \left( \frac{1}{2t} + \frac{17 - 8(\gamma + \ln(2kt))}{60\Lambda M_P^2 t^3} \right) k^2 \] \quad \left( \sqrt{\Lambda} k M_P t^2 \ll 1 \right) \tag{6.15}

and
\[ R_2 \simeq -\left( \frac{1}{3t} + \frac{7}{90\lambda M_P^2 t^3} \right) k^2 \] \quad \left( \sqrt{\Lambda} k M_P t^2 \ll 1 \right). \tag{6.16}

In the mathematical limit in which \( k = 0 \), we obtain \( R_2 = 0 \) and \( \Phi_2 \propto \frac{M_P}{t} \) in exact agreement with the results of [1, 5]. In the physical super-horizon limit in which \( \sqrt{\Lambda} k M_P t^2 \ll 1 \) while \( k \) is held fixed we observe the \( 1/t \) grows of \( R_2 \) in super-horizon limit. We see that the situation here is very similar to discussions in sub-section 3.2. We also comment that the \( 1/t \) growth of \( R \) on super-horizon scales was also observed in the model of Galilean Genesis [32].

It is also instructive to understand how the proof [1, 5] is violated in pseudo conformal universe in the method discussed in sub-section 3.1. As we noticed there, the key place to look for is the Poisson equation. Let us start with the original Poisson equation (3.5) yielding for pseudo conformal model
\[ \left( -\frac{1}{\lambda t^4} + M_P^2 k^2 \right) \Phi = \frac{1}{2} \left( -\dot{\phi} \delta \dot{\phi} + \ddot{\phi} \delta \phi \right). \tag{6.17} \]

In the proof of [1, 5] the mathematical super-horizon limit corresponds to \( k = 0 \) independent of how large \( M_P \) is. However, similar to argument mentioned after eqs. (2.6) and (3.7), this limit is ambiguous here. This is because in this model gravity is assumed to be very weak so we work in the limit \( M_P \rightarrow \infty \). Therefore, in order to be safe, we shall keep both terms in big bracket in eq. (6.17). The rest of analysis go exactly as in sub-section 3.1 and we obtain the second order differential equation for \( R \) given in eq. (3.12). Note the interesting fact that in eq. (3.12) no factor of \( M_P \) appears so no ambiguity in taking \( k \rightarrow 0 \) while \( M_P \rightarrow \infty \) arises now. In addition \( a(t) \) is very slow-changing and the \( \epsilon \)-dependence is the same for both terms in eq. (3.12). Therefore, the mathematical super-horizon limit \( k \rightarrow 0 \) is justified in eq. (3.12). In this limit, the two independent solutions are given as in eq. (3.13) represented by constants \( C_1 \) and \( C_2 \). The first mode is the constant mode as expected. Now for the second mode we obtain
\[ R_2 = C_2 \int \frac{dt}{a^3 \epsilon} \simeq \frac{-C_2}{9\lambda M_P^2 t}. \tag{6.18} \]

Interestingly, we see again that \( R_2 \propto \frac{1}{t} \) as obtained in eq. (6.16).

7 Summary and discussions

In this work we have revisited the celebrated Weinberg theorem in cosmological perturbation theory. The theorem states that there always exists two adiabatic scalar modes which are
constant on super-horizon scales. Despite its wide applicability, however there are known examples in literature which violate this theorem. We have concentrated on loopholes in some technical assumptions which are violated in models of non-attractor inflation, fluid inflation, solid inflation and pseudo conformal universe.

We have seen that the theorem in [1, 5] can be violated in two different ways. The obvious way is when there is non-analytic relation in terms of the wave-number $k$ in Einstein fields equations. This situation was already anticipated in [1, 5]. The case of solid inflation is a specific example in which $\pi^S$ is non-analytically related to $\zeta$ via $\pi^S \propto \zeta/k^2$. However, the more non-trivial examples are the cases in which some parameters of the background, like the slow-roll parameter $\epsilon$, show strong time-dependence in which the mathematical treatment of the super-horizon limit $k \to 0$ is ambiguous as we discussed after eqs. (2.6) and (3.7). This is the case in non-attractor inflation, fluid inflation and in pseudo conformal model. In the first two examples $\epsilon$ falls off like $1/a^6$ and the combination $k^2/a^2\epsilon$ appearing in Poisson equation diverges even on super-horizon scales. In the latter example $\epsilon \sim M_p^2 t^2 \gg 1$ showing a strong time-dependence.

Weinberg’s theorem can be easily extended to the case that includes gradient modes containing $O(k)$ corrections. This extension is studied in [33–35]. Even with the extension of Weinberg’s theorem to include the gradient modes we do not expect our previous conclusion on the violations of the theorem to change. For example in non-attractor model it was the coefficient of the term at the order $O(k^0)$ which was rapidly decaying during inflation as in eqs. (2.6) or (3.7), so the inclusion of corrections of $O(k)$ does not change the conclusion. On the other hand in solid inflation the anisotropic stress is non-analytic as $k^{-2}$. In addition the assumption $\Phi = \Psi$ fails to hold. Therefore, Weinberg’s theorem is violated both in zeroth and first order of $k$.

In this work we have studied the violation of Weinberg’s theorem in very early Universe. Having this said, Weinberg’s theorem can have important implications in late Universe too. One immediate place which comes to mind is when the Universe is matter dominated in which sound horizon decays to (approximately) zero. In this circumstance, we have many modes which are outside the sound horizon yet inside Hubble horizon and hence they are observable. As one can easily check, for these modes, the Bardeen potential freezes as one expects from Weinberg’s theorem. Another important place is the time of recombination in which the sound speed drops from nearly $1/\sqrt{3}$ to nearly zero. In both of these examples it is the sound horizon, i.e. the coefficient of the term containing $O(k^2)$, which shrinks so the situations are in favor of Weinberg’s theorem. This is opposite to non-attractor example in which it was the coefficients of $O(k^0)$ term which was falling off as time goes by.

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