New Stabilization of the Burnett Equations when Entropy Change to $Kn^0$ Vanishes

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Abstract

We assume that to zero order in the Knudsen number the deviation of the entropy from a background value vanishes. We then show that adding a super-Burnett term we obtain a stable state of rest. The resulting equations have the same form as the Burnett equations but with the value of some coefficients changed. In particular the result applies to nonlinear acoustics.

We consider a slightly rarefied gas. To first order in the Knudsen number, $Kn$, the Navier-Stokes equations are valid. Burnett [1] derived the corresponding equations to second order in $Kn$. Bobylev [2] showed that the state of rest is unstable for the Burnett equations, see also Uribe et al. [3]. In this contribution we make the assumption that the deviation of entropy from a background value is of the order of $KnMa$, where $Ma$ is the Mach number. This is the case for nonlinear acoustics, where $Ma \sim Kn$. We show that with an error $Kn^3$ the Burnett equations in this case can be replace by equations which are linearly stable.

In the one-dimensional case, the Burnett expressions for the $xx$ component of the pressure tensor $P$ and the heat current $q$ are, see Chapman & Cowling [4] (the dots indicate nonlinear Burnett terms)

\[
P = \frac{\rho T}{m} - \frac{4\mu}{3}v_x - \frac{2}{3} \frac{\mu^2}{\rho} \left[ \frac{\omega_2}{\rho} \frac{\partial_x p}{\rho} + (\omega_2 - \omega_3) \frac{T_{xx}}{T} \right] + \ldots,
\]

\[
q = -\kappa T_x - \frac{2}{3} \frac{\mu^2}{\rho} (\theta_2 - \theta_4) v_{xx} + \ldots
\]

We now linearize around a state at rest and uniform temperature and density, writing

\[
T = T_0[1 + \tilde{T}], \quad \rho = \rho_0[1 + \tilde{\rho}], \quad v = \sqrt{\frac{k_B T_0}{m}} \tilde{v}.
\]

We introduce dimensionless variables, where the unit of length is of the order of the mean free path.
\[
x = x^* \frac{\mu_0}{\rho_0} \sqrt{\frac{m}{k_B T_0}}, \quad t = t^* \frac{\mu_0}{\rho_0} \frac{m}{k_B T_0}.
\]

In the sequel stars and tildes are omitted. We obtain the linearized one-dimensional Burnett equations

\[
\begin{align*}
\rho_t + v_x &= 0, \quad (1) \\
v_t &= -(\rho + T)_x + \frac{4}{3} v_{xx} + \frac{2}{3} \omega_2 \rho_{xxx} - \frac{2}{3} (\omega_3 - \omega_2) T_{xxx}, \quad (2) \\
3T_t &= -v_x + \frac{3}{2} fT_{xx} - \frac{2}{3} (\theta_4 - \theta_2) v_{xxx}. \quad (3)
\end{align*}
\]

\[f = \frac{2m\kappa}{3k_B \mu}\] is the Eucken number. In the calculations we use the value 
\[f = 5/2.\] This is the lowest approximation in terms of Sonine polynomial expansion for any interatomic potential and is experimentally found to be a good approximation, see [3].

Now we assume that the entropy change is of \(0(Ku Ma)\). We then have

\[\frac{dT}{T} = (\gamma - 1) \frac{d\rho}{\rho} + 0(Ku Ma).\]

\((\gamma = c_p/c_v)\). Linearizing and using dimensionless units we find

\[(\gamma - 1) \rho_{xxx} - T_{xxx} = 0(Ku Ma).\]

Hence, to within terms \(0(Ku)\) we have for any value \(\alpha\) which is \(0(1)\)

\[\omega_2 \rho_{xxx} + (\omega_2 - \omega_3) T_{xxx} = [\omega_2 + \alpha(\gamma - 1)] \rho_{xxx} + \frac{2}{3} (\omega_2 - \omega_3 - \alpha) T_{xxx} + 0(Ku Ma).
\]

Thus, we can change the values of \(\omega_2, \omega_3\) to \(\bar{\omega}_2\) och \(\bar{\omega}_3\) in the linear part of the Burnett contribution.

\[
\begin{align*}
\bar{\omega}_2 &= \omega_2 + \alpha(\gamma - 1), \quad (4) \\
\bar{\omega}_3 &= \omega_3 + \alpha\gamma. \quad (5)
\end{align*}
\]

Let us now choose \(\alpha\) so that the coefficient of \(\rho_{xxx}\) vanishes, or \(\bar{\omega}_2 = 0\). This gives for a monatomic gas

\[\bar{\omega}_2 = 0, \quad \bar{\omega}_3 = \omega_3 - \frac{5}{2} \omega_2.
\]

For Maxwell molecules

\[\omega_2 = 2, \omega_3 = 3; \ \bar{\omega}_2 = 0, \ \bar{\omega}_3 = -2.
\]

For hard spheres

\[\omega_2 = 2.028, \omega_3 = 2.418, \ \bar{\omega}_2 = 0, \ \bar{\omega}_3 = -2.652.
\]
Figure 1: Complex growth factor $\Lambda$ for hard spheres, $0 < k < 6$. Rings Burnett, crosses our equations.

As a consequence, the $\rho_{xxx}$ term disappears. The sign of the $T_{xxx}$ term changes. For solutions proportional to $\exp[ikx + \Lambda t]$ we find

Asymptotically, for $k \to \infty$ we have, when $(\theta_2 - \theta_4)(\tilde{\omega}_2 - \tilde{\omega}_3) > 0$,

$$
\Lambda = -\frac{27}{8(\theta_2 - \theta_4)(\tilde{\omega}_2 - \tilde{\omega}_3)}\left(\frac{2}{3}\tilde{\omega}_2 + \frac{1}{k^2}\right)f,
$$

$$
\Lambda = \pm i\sqrt{\frac{8}{27}(\theta_2 - \theta_4)(\tilde{\omega}_2 - \tilde{\omega}_3)k^3 - \frac{(3f + 4)}{6}k^2}.
$$

Clearly, there is one mode that is nonpropagating and damped and there are two propagating, damped modes. One entropy mode and two sound wave modes. It is really not necessary to have $\tilde{\omega}_2 = 0$, but just to have $\tilde{\omega}_2 - \tilde{\omega}_3 > 0$.

Let us write down the resulting equations, first in the one-dimensional case. We neglect the nonlinear Burnett terms.

$$
\rho_t + (\rho v)_x = 0
$$

$$
\rho(v_t + vv_x) = -\frac{1}{m}(\rho T)_x + \frac{4}{3}(\mu v_x)_x + \frac{2}{3}(\omega_3 - \frac{5}{2}\omega_2)\frac{\mu^2}{\rho T}T_{xxx}
$$

$$
\frac{3}{2m}\rho(T_t + vT_x) = -\frac{1}{m}\rho T v_x + (\kappa T)_x + \frac{2}{3}(\theta_4 - \theta_2)\frac{\mu^2}{\rho}v_{xxx}
$$

Here, the coefficients of the Burnett terms can be taken at the background value, but the variations of $\mu$, $\kappa$ in the Navier-Stokes terms have to be taken into account.
We now give the phase velocity. Note that the phase velocity is constant plus a term to order $Kn^2$. Hence the deviation from straight lines for the Navier-Stokes equations is not physically relevant but that the deviation for the Burnett equations (and our equations) is. Note that the difference between the Burnett equations and our equations is for larger $k$ than those shown in Fig. 2.

The three-dimensional equations are

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \mathbf{v})_x &= 0, \\
\rho (\mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v}) &= -\frac{1}{m} \nabla (\rho T), \\
+ \nabla \cdot \{ \mu (\nabla \mathbf{v} + (\nabla \mathbf{v})^T - \frac{2}{3} (\nabla \cdot \mathbf{v}) \mathbf{1}) \} + \frac{2}{3} (\omega_1 - \frac{5}{2} \omega_2) \frac{\mu^2}{\rho T} \Delta (\nabla T) \\
\frac{3}{2m} \rho (T_t + \mathbf{v} \cdot \nabla T) &= -\frac{1}{m} \rho T (\nabla \cdot \mathbf{v}) + \nabla \cdot (\kappa \nabla T) + \frac{2}{3} (\theta_1 - \theta_2) \frac{\mu^2}{\rho} \Delta (\nabla \cdot \mathbf{v}).
\end{align*}
\]

In earlier contributions by Jin and Slemrod, see also S and Z the Burnett equations were regularized to a set of 13 first order equations generally valid.

The present regularization applies when the deviations of entropy is $0(KnMa)$ but gives equations which can more easily be applied for small Knudsen numbers. The condition on the entropy applies for nonlinear sound propagation. The same assumption that $Ma \sim Kn$ is called the weakly nonlinear case in Sone.
where stationarity is assumed, but here sound waves are included as well.

Recently, the present author has also obtained another set of regularized equations \[10\], which like the Burnett equations and the equations in this work are equations for \(\rho, v, T\). They are, however generally valid, with no limitation on entropy or Mach number. They are first order in time and third order in space, but also contain mixed derivatives first order in time and up to second order in space.

References

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