A Bangert–Hingston Theorem for Starshaped Hypersurfaces

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Abstract

Let $Q$ be a closed manifold with non-trivial first Betti number that admits a non-trivial $S^1$-action, and $\Sigma \subseteq T^*Q$ a non-degenerate starshaped hypersurface. We prove that the number of geometrically distinct Reeb orbits of period at most $T$ on $\Sigma$ grows at least logarithmically in $T$.

1 Introduction

In a celebrated paper of Bangert and Hingston [BH84] it is shown that the number of geometrically distinct closed geodesics on a closed connected manifold $Q$ grows like the prime numbers whenever $\pi_1(Q) \cong \mathbb{Z}$ and $\dim Q \geq 2$. More precisely, if $N(T)$ denotes the number of geometrically distinct geodesics of period at most $T$, then

$$\liminf_{T \to \infty} N(T) \cdot \frac{\log(T)}{T} > 0.$$ 

Bangert and Hingston’s proof makes use of minimax values associated to the energy functional

$$E: \mathcal{L}Q \to \mathbb{R}, \ E(q) = \int_0^1 \frac{1}{2} \|\dot{q}(t)\|^2 \, dt$$

and two sequences of homotopy classes on the free loop space $\mathcal{L}Q$. Thanks to the Legendre transform, critical points of $E$, i.e. geodesics, are in one to one correspondence with the critical points of the Hamiltonian action functional $A_H: \mathcal{L}T^*Q \to \mathbb{R}$ associated to the kinetic Hamiltonian $H(q,p) = \frac{1}{2} \|p\|^2$. In particular, the quest for closed geodesics can be rephrased as a Hamiltonian problem, and if one wishes to fix the constant speed of the geodesics to be 1, the problem becomes a quest for periodic Reeb orbits on the unit sphere bundle

$$H^{-1}(1/2) = S^*Q.$$
The Hamiltonian, Lagrangian, and Reeb approach are essentially equivalent and are still interchangeable for general convex hypersurfaces in $T^*Q$. However, if $\Sigma \subseteq T^*Q$ is starshaped, there is no Legendre transform in general and therefore the notion of Reeb orbits does not admit a Lagrangian reformulation. Nevertheless, the attempt to carry over well-known geodesic growth type results to the realm of contact geometry on $\Sigma$ has been quite successful. Indeed, many striking results where the topology of $Q$ (or its loop space $LQ$) forces certain geodesic growths have found an analogous formulation for growth rates of Reeb orbits/chords/leaf-wise intersections on $\Sigma$ [MS11; Hei11; MMP12; Wul14].

A common theme throughout these results is the use of some flavour of Floer theory and its relation to the singular homology of the loop space $LQ$.

Prior to the work of Bangert and Hingston, several geodesic growth results had been established in the finite fundamental group case [GM69; Gro78; BTZ81; BZ82], while in the infinite abelian case very little was known, especially for the "smallest" case, i.e. $\pi_1(Q) \cong \mathbb{Z}$. The starshaped setting encounters the same issue, as for $\pi_1(Q) = \mathbb{Z}$ there is no known result, to the best of the author’s knowledge, on the growth of periodic Reeb orbits of $\Sigma \subseteq T^*Q$. More precisely, if $N_\Sigma(T)$ denotes the number of geometrically distinct periodic Reeb orbits of period at most $T$, then the behaviour of $N_\Sigma(T)$, as $T$ goes to infinity, is currently unknown. The following result partially fills this gap.

**Theorem** (Main Theorem). Let $Q$ be a $n$-dimensional closed connected manifold with $n \geq 2$, whose first Betti number is non-trivial. Assume that $Q$ admits a non-trivial $S^1$-action. Then for any non-degenerate starshaped hypersurface $\Sigma \subseteq T^*Q$ it holds:

$$\liminf_{T \to \infty} \frac{N_\Sigma(T)}{\log(T)} > 0.$$ 

In particular, $\Sigma$ admits infinitely many geometrically distinct Reeb orbits.

The conclusion of the Main Theorem holds for a number of examples, e.g. products $Q = S^1 \times M$ with $M$ a closed manifold, $H$-spaces $Q$ with infinite fundamental group, and principal $S^1$-bundles $Q$ over a closed connected base manifold $B$ with $\pi_2(B) = 0$ — see Theorem 6.5 and Corollary 6.7 for more details.

We will now present an outline of the proof of the Main Theorem and explain the various assumptions as we go along. The main ingredient is the use of spectral invariants $c_\alpha(H) \in \mathbb{R}, \quad \alpha \in H_*(\mathcal{L}Q) \setminus \{0\}$, where $H: T^*Q \to \mathbb{R}$ is an autonomous asymptotic quadratic Hamiltonian in the sense of Abbondandolo–Schwarz [AS06]. Let $F: T^*Q \to \mathbb{R}$ be a quadratic Hamiltonian with $\Sigma = F^{-1}(1)$, and let $G$ be a kinetic Hamiltonian together with a constant $\sigma > 1$ such that

$$G \leq F \leq \sigma G.$$ 

Then the spectral invariants are related to each other via

$$\frac{1}{\sigma} \cdot c_\alpha(G) \leq c_\alpha(F) \leq c_\alpha(G).$$ 

2
This is based on a pinching argument due to Macarini and Schlenk [MS11], which also plays a key role in the arguments of the aforementioned papers [Hei11; MMP12; Wul14].

The Betti number assumption ensures the existence of a homotopy class \( \eta \in \pi_1(Q) \), whose image in \( H_1(Q;\mathbb{Z}) \) is of infinite order. Denote by \( \mathcal{L}_mQ \) the connected component of the loop space associated to \( \eta^m \) for \( m \in \mathbb{Z} \). We say that a smooth \( S^1 \)-action \( \phi: S^1 \times Q \to Q \) is non-trivial with respect to \( \eta \), if the loop \( \gamma_q(t) := \phi(t, q) \) belongs to \( \mathcal{L}_kQ \) for some integer \( k \neq 0 \). In particular, the map

\[
s: Q \to \mathcal{L}_kQ, \quad s(q) := \gamma_q
\]

defines a section of the evaluation map \( ev_0: \mathcal{L}Q \to Q \). The existence of the section enables us to define a sequence of non-zero homology classes \( \alpha_m \) on pairwise distinct components of \( \mathcal{L}Q \) as follows: define the \textbf{m-iteration map}

\[
\mathcal{I}_m: \mathcal{L}Q \to \mathcal{L}Q, \quad \gamma \mapsto \gamma \cdots \gamma \quad \text{m-times}
\]

Observe that the map

\[
\mathcal{I}_m \circ s: Q \to \mathcal{L}Q
\]

defines a sequence of non-zero cohomology classes \( \alpha_1 \in H_n(\mathcal{L}_kQ) \), together with a sequence of non-zero homology classes

\[
\alpha_m := (\mathcal{I}_m)_s \alpha_1 \in H_n(\mathcal{L}_kQ) \setminus \{0\}, \quad \forall m \in \mathbb{N}.
\]

By spectrality we know that there exists a sequence of Hamiltonian orbits \( x_m \) of \( F \) satisfying

\[
ce_{\alpha_m}(F) = A_F(x_m).
\]

These orbits \( x_m \) correspond to periodic Reeb orbits on \( \Sigma \) by choice of \( F \). Using a Robbin–Salamon \textbf{index growth result} due to M. de Gosson, S. de Gosson and Piccione [GGP08] we can extract a subsequence, still denoted by \( x_m \), such that the \( x_m \) are not iterates of each other — this is the only part that uses the non-degeneracy of \( \Sigma \). The precise growth rate of the Reeb orbits \( x_m \) in terms of their action (and thus their period) is determined by using the spectral invariant inequality from above. This concludes the outline of the proof.

\[1\text{Note that any two orbits of } \phi \text{ have the same free homotopy class since } Q \text{ is assumed to be connected.}
\[2\text{We are implicitly working with } \mathbb{Z}_2\text{-coefficients.} \]
Under the \( S^1 \)-action assumption, Irie’s Main Theorem \cite{Iri14} on the finiteness of the Hofer–Zehnder capacity of disk cotangent bundles \( D^*Q \) is applicable, however this is not particularly fruitful when applied to the function \( F \) due to homogeneity: the finiteness of the Hofer–Zehnder capacity of \( D^*Q \) implies dense existence of orbits nearby \( \Sigma = F^{-1}(1) \) \cite{HZ12}, but it could happen that these orbits all correspond to a single Reeb orbit on \( \Sigma \). To make matters worse, no growth rate control can be deduced even if we knew that there are infinitely many Reeb orbits. In this sense, the Main Theorem can be viewed as a strengthening to the dynamical consequences of Irie’s Theorem in the case of starshaped hypersurfaces.

Let us mention that there is no need to bound the pinching factor \( \sigma \) in our arguments. This seems to be a reoccurring dichotomy between starshaped hypersurfaces in \( T^*Q \) (with \( Q \) closed) and (compact) starshaped hypersurfaces in \( \mathbb{R}^{2n} \), where for the latter the pinching factor \( \sigma \) plays a more crucial role \cite{Gir84, Ber+85, Vit89, Eke12, AGH16, Wan16, DL17, AM17}.

The methods in BH (Bangert and Hingston) \cite{BH84} are not well suited to our setting since spectral invariants can only be assigned to homology classes, while in the Lagrangian case, as done in BH, minimax values can be assigned to homotopy classes as well. This issue can be ascribed to the lack of a global flow for the negative gradient of the Hamiltonian action functional, which is in contrast to the existence of a global flow of the negative gradient of the Lagrangian action functional. Even under further assumptions, e.g. that the relevant homotopy groups inject into the free loop space homology via the Hurewicz homomorphism, we have not been able to adapt BH’s arguments.

\textbf{Remark 1.1.} Let us emphasise that non-degeneracy of \( \Sigma \) is a \( C^\infty \)-generic property; cf. \cite[Theorem 2.5]{MMP12}. While in the \( C^1 \)-generic case stronger dynamical conclusions than Theorem 1 can be drawn without any assumptions on \( \Sigma \), for instance using results due to Newhouse \cite[Section 5 and 6]{New20} and Smale \cite{Sma15}, yielding a horseshoe and hence exponential growth of the closed orbits, the existence of infinitely many orbits seems to be unknown in the general \( C^\infty \)-generic case.

\textbf{Remark 1.2.} In the case that \( \Sigma \) is convex, the Lagrangian formulation is sufficient to run the proof strategy of \cite{BH84}, also see \cite{BJ08} for the Finsler case. This gives a better growth rate, namely \( T/\log(T) \), without any non-degeneracy assumptions. The present work is part of the author’s PhD thesis, in which we also address the case of \( \Sigma \) convex without the non-degeneracy assumption — this gives some further insights into the key differences between starshapedness and convexity.

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2 Reeb Orbits and Starshaped Domains

Let $Q$ be a closed connected $n$-dimensional manifold. The $2n$-dimensional cotangent bundle $T^*Q$ is equipped with the symplectic form $\omega = d\lambda$ where $\lambda$ is the Liouville form that is locally given by $pdq$. For any Hamiltonian

$$H_t: T^*Q \longrightarrow \mathbb{R}$$

we define the Hamiltonian vector field $X_{H_t}$ via

$$\omega(\cdot, X_{H_t}) = dH_t.$$ 

Moreover, an almost complex structure $J$ on the manifold $T^*Q$ is said to be compatible if $g_J := \omega(\cdot, J\cdot)$ defines a Riemannian metric. We define the Hamiltonian action functional $A_H$ on loops $x: S^1 \rightarrow T^*Q$:

$$A_H(x) := \int_{S^1} x^*\lambda - \int_0^1 H_t(x(t)) dt.$$ 

The set of 1-periodic Hamiltonian orbits of $H_t$ is denoted by $P(H_t)$, and $P^I(H)$ denotes the subset of $P(H)$ of orbits $x$ with action $A_H(x) \in I \subseteq \mathbb{R}$. For $(-\infty, d]$ we abbreviate $P^d(H) = P(-\infty, d](H)$.

Let $\Sigma \subseteq T^*Q$ be a smooth, connected hypersurface with $\Sigma = \partial D$, where $D \subseteq T^*Q$ is a bounded domain $D$ that contains the zero section. Such a $\Sigma$ is called starshaped, if in each fibre $T^*_qQ$ the set $\Sigma_q := T^*_qQ \cap \Sigma$ is starshaped with respect to the origin $0_q \in T^*_qQ$. Observe that $\Sigma$ is necessarily compact.

Any starshaped $\Sigma$ is of restricted contact type with a contact form $\alpha_\Sigma$ defined as follows: for each point $q$ and $p \in T^*_qQ$ one can consider the path $\gamma_{(q,p)}(t) = t \cdot p \in T^*_qQ$ and define the Liouville vector field via

$$Y(q, p) := \dot{\gamma}_{(q,p)}(1) \in T_{(q, p)}T^*Q.$$ 

By definition, $Y|_{\Sigma}$ is outward pointing and transverse to $\Sigma$, thus implying that $\Sigma$ is of restricted contact type with contact form

$$\alpha_\Sigma := \omega(Y, \cdot|_{T\Sigma}) = \lambda|_{T\Sigma}$$

and contact structure

$$\xi := \text{ker} \alpha_\Sigma.$$ 

Let us translate the quest of finding Reeb orbits on $(\Sigma, \xi)$ into a Hamiltonian problem. Since $\Sigma$ is assumed to be starshaped, there exists a 2-homogeneous Hamiltonian

$$F: T^*Q \longrightarrow \mathbb{R}$$

\footnote{The whole sign convention agrees with the one of [AS06].}

\footnote{Indeed, one can check that locally $Y = \frac{\partial}{\partial p}$, thus $\omega(Y, \cdot) = pdq = \lambda.$}
that is uniquely defined by requiring
\[ F^{-1}(1) = \Sigma, \quad F(q, sp) = s^2 F(q, p), \quad \forall s \geq 0, \quad \forall (q, p) \in T^*Q. \]

Furthermore,
\[ \iota_{X_F|\Sigma} d\alpha_{\Sigma} = \iota_{X_F|\Sigma} d(\lambda|_{T\Sigma}) = -dF|_{T\Sigma} \]
and since \( F \) is constant along \( \Sigma \) we have \( \iota_{X_F|\Sigma} d\alpha_{\Sigma} = 0 \). Similarly, we know that \( X_F(x) \) is a vector in \( T_x \Sigma \) for all \( x \in \Sigma \), in particular using homogeneity of \( F \) we deduce
\[ \alpha_{\Sigma}(X_F(x)) = \lambda(X_F(x)) = 2 \cdot F(x) \equiv 2, \quad \forall x \in \Sigma, \]
and with the above we therefore obtain
\[ X_F|_{\Sigma} = 2 \cdot R. \]

In particular, for \( \varphi^t_F \) (resp. \( \varphi^t_R \)) the time-\( t \) Hamiltonian flow of \( F \) (resp. Reeb flow) we get
\[ \varphi^t_F(x) = \varphi^{2t}_R(x), \quad \forall x \in \Sigma, \quad t \in \mathbb{R}. \]

**Definition 2.1.** Denote \( O_R(t) \) the set of Reeb orbits of period less or equal \( t \). Similarly, \( O_{X_F|\Sigma}(t) \) (resp. \( O_{X_F}(t) \)) denotes the set of Hamiltonian orbits of \( X_F|\Sigma \) (resp. \( X_F \)) of period less or equal \( t \) (resp. action less or equal \( t \)).

The above flow relation implies
\[ \#O_{X_F|\Sigma}(t) = \#O_R(2t). \]

When doing Floer homology, we will consider only 1-periodic orbits, but across the whole cotangent bundle. The next proposition shows that fixing the level and letting the period vary has the same effect as fixing the period and varying the level. First we need some notation. Let \( x = (q, p) \in T^*Q \). Then for every \( s \in \mathbb{R} \) define
\[ s \cdot x = (q, sp) \in T^*Q. \]

For a loop \( x = (q, p) \) on \( T^*Q \) define
\[ x_s(t) := s \cdot x(st) = (q(st), s \cdot p(st)). \]

The following result is contained in [Hei11].

**Proposition 2.2.** There is a one to one correspondence between 1-periodic orbits \( x:S^1 \to T^*Q \) of \( X_F \) with action \( A_F(x) = a \), and \( \sqrt{a} \)-periodic orbits \( y \) of \( X_F|\Sigma \). The correspondence is given by
\[ x \mapsto x_{1/\sqrt{a}}, \quad y \mapsto y_{\sqrt{a}}. \]

\footnote{While \( F \) is smooth away from the zero section, we can only expect \( C^1 \) regularity near 0 — we will remedy this issue later by composing \( F \) with an auxiliary function \( f: \mathbb{R} \to \mathbb{R} \) that smooths out \( F \) without affecting the set of non-constant 1-periodic orbits; cf. Proposition \ref{prop:smoothness}.}

\footnote{We identify two orbits that are equal up to a time-shift.}
Proposition 2.2 implies
\[ \#O_{\mathcal{A}_F}(a) = \#O_{\mathcal{X}_{\mathcal{F}}}((\sqrt{a}) \] 
for all \( a > 0 \). All in all we thus obtain
\[ \#O_R(2t) = \#O_{\mathcal{X}_{\mathcal{F}}}((t^2) \rightleftharpoons \#O_{\mathcal{A}_F}(t^2), \quad \forall t > 0. \]
In particular, we can bound the number of geometrically distinct Reeb \( T \)-periodic orbits
\[ N_\Sigma(T) := \#O_R(T) \]
from below by bounding \( \#O_{\mathcal{A}_F}((T/2)^2) \) from below instead.

## 3 Spectral Invariants and Minimax Values

### 3.1 Spectral Invariants of Quadratic Hamiltonians

In this section we will introduce spectral invariants and compare them to minimax values on \( T^*Q \). Many of the results about spectral invariants here are very analogous to those in [Iri14]. The main difference is the choice of Hamiltonians — Irie works with Hamiltonians that are linear at infinity, while we opt for quadratic Hamiltonians. The main advantage of quadratic Hamiltonians is that their Floer homology computes the singular homology of the loop space thanks to the work of Abbondandolo, Majer and Schwarz [AS06, AM06, AS15]. The main reason this works so well is that the relevant chain isomorphisms are action preserving.

**Convention.** All homologies are implicitly understood to be over the field \( \mathbb{Z}_2 \) in order to avoid the need of local coefficients whenever the second Stiefel-Whitney class does not vanish over tori [AS14, Abo15].

Denote by \( H : S^1 \times T^*Q \rightarrow \mathbb{R} \) a Hamiltonian satisfying

**H0** every \( x \in \mathcal{P}(H) \) is non-degenerate,

**H1** \( dH(t,q,p)[Y] - H_t(q,p) \geq h_0\|p\|^2 - h_1 \), for some constants \( h_0 > 0 \) and \( h_2 \geq 0 \),

**H2** \( \|\nabla_q H(t,q,p)\| \leq h_2(1 + \|p\|^2) \) and \( \|\nabla_p H(t,q,p)\| \leq h_2(1 + \|p\|) \), for some constant \( h_2 \geq 0 \).

These are the conditions used by AS (Abbondandolo and Schwarz) to define Floer homology on cotangent bundles. Additionally, assume that \( H \) admits a Legendre dual Lagrangian

\[ L : S^1 \times TQ \rightarrow \mathbb{R} \]

(e.g. whenever \( H \) is strictly convex). Let \( \mathcal{L}Q \) be the free loop space of \( Q \). Denote by

\[ \mathcal{E}_L : \mathcal{L}Q \rightarrow \mathbb{R}, \quad \mathcal{E}_L(q) = \int_0^1 L_t(q(t), \dot{q}(t)) \, dt \]
the corresponding Lagrangian action functional and define
\[ CM_\bullet(\mathcal{E}_L) \]
as the Morse chain complex of \( \mathcal{E}_L \). Denote by
\[ CM^d_\bullet(\mathcal{E}_L) = CM_\bullet(-\infty,d](\mathcal{E}_L) \]
the filtered Morse chain complex of \( \mathcal{E}_L \) associated to a regular value \( d \in \mathbb{R} \) (similarly \( CF^d_\bullet(H) \) using the action functional \( A_H \)),\(^7\) and write
\[ i^d: HM^d_\bullet(\mathcal{E}_L) \to HM_\bullet(\mathcal{E}_L) \]
for the induced map.\(^8\)

**Definition 3.1.** Let \( \alpha \in HM_\bullet(L) \setminus \{0\} \) (resp. \( \beta \in HF_\bullet(H) \setminus \{0\} \)). The **minimax value** associated to \( \alpha \) and \( \mathcal{E}_L \) (resp. the **spectral invariant** associated to \( \beta \) and \( H \)) is defined as
\[ c_\alpha(L) = \inf_{\xi = \sum, \xi_i q_i \in \alpha} \mathcal{E}_L(\xi), \text{ where } \mathcal{E}_L(\xi) = \max_{\xi_i \neq 0} \mathcal{E}_L(q_i), \]
resp.
\[ c_\beta(H) = \inf_{\zeta = \sum, \zeta_i x_i \in \beta} A_H(\zeta), \text{ where } A_H(\zeta) = \max_{\zeta_i \neq 0} A_H(x_i). \]

An equivalent definition is given by
\[ c_\alpha(L) = \inf \{ c \mid \alpha \in \text{im}(i^c) \}, \]
where the infimum runs over \( \mathbb{R} \) minus the set of critical values of \( \mathcal{E}_L \). The definitions above beg the question as to whether there is a relation between minimax values and spectral invariants. The answer is yes, but this needs a further digression to the work of [AS06].

Therein, a chain map isomorphism
\[ \Theta: CM_\bullet(\mathcal{E}_L) \to CF_\bullet(H), \]
is constructed and shown to respect the action filtration, i.e. for every \( d \) the map \( \Theta \) induces a chain map isomorphism, still denoted by \( \Theta \):
\[ \Theta: CM^d_\bullet(\mathcal{E}_L) \to CF^d_\bullet(H). \]
In [AS15], AS came up with another construction yielding a chain isomorphism
\[ \Psi: CF_\bullet(H) \to CM_\bullet(\mathcal{E}_L), \]
which is action preserving as well and defines a chain homotopy inverse to \( \Theta \), i.e. \( \Psi \circ \Theta \) and \( \Theta \circ \Psi \) are chain homotopy equivalent to the corresponding identity on the chain level. With these at hand we have the following easy, but absolutely crucial proposition.

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\(^7\)The Floer chain complex \( CF_\bullet(H) \) is graded using the Conley–Zehnder index and “vertically preserving” trivializations as used in [AS06].

\(^8\)In general the induced map in homology is not an inclusion.
Proposition 3.2. Let $H$ and $L$ be dual and assume that they both satisfy the AS conditions $(H0)$, $(H1)$, $(H2)$. Then for all $\alpha \in \text{H}_\bullet(E_L) \setminus \{0\}$ and $\beta \in \text{H}_\bullet(H) \setminus \{0\}$ we have

$$c_\alpha(L) = c_{\Theta(\alpha)}(H), \quad c_\beta(H) = c_{\Psi(\beta)}(L).$$

Proof. Let $d > 0$ be such that $\alpha \in \text{im}(i^d)$. By properties of $\Theta$, we have the following commutative diagram with horizontal isomorphisms:

$$
\begin{array}{ccc}
\text{H}_\bullet(E_L) & \xrightarrow{\Theta} & \text{H}_\bullet(H) \\
\downarrow{i^d} & & \downarrow{i^d} \\
\text{H}^d_\bullet(E_L) & \xrightarrow{\Theta} & \text{H}^d_\bullet(H)
\end{array}
$$

This implies that $\Theta(\alpha)$ lies in the image of the corresponding $i^d$ too. Taking the infimum of $d$’s with $\alpha \in \text{im}(i^d)$ implies

$$c_{\Theta(\alpha)}(H) \leq c_\alpha(L).$$

The inequality above, however, is an equality: assume, by contradiction, that there is a value $e < c_\alpha(L)$ for which $\Theta(\alpha)$ lies in the image of $i^e$. Then the fact that $\Theta^{-1}$ is well defined and action preserving implies that $\alpha$ is in the image of $i^e$, contradicting the definition of $c_\alpha(L)$.

So we have shown

$$c_{\Theta(\alpha)}(H) = c_\alpha(L).$$

The other equality can be shown analogously, or directly deduced from the first one by setting $\alpha = \Psi(\beta)$ and observing that $\Theta(\alpha) = \Theta(\Psi(\beta)) = \beta$. \qed

Another result due to Abbondandolo and Majer \cite{AM06} asserts that the Morse homology of $E_L$, with Lagrangians $L$ as above, is isomorphic to the free loop space homology, i.e. there exists an isomorphism

$$\Upsilon : \text{H}_\bullet(LQ) \xrightarrow{\cong} \text{H}_\bullet(E_L).$$

This isomorphism also preserves the action filtration, i.e. it descends to yet another isomorphism:

$$\Upsilon : \text{H}_\bullet(\{E_L \leq d\}) \xrightarrow{\cong} \text{H}^d_\bullet(L),$$

for all $d \in \mathbb{R}$; cf. \cite[Section 2.4]{AS06}. Now recall that minimax values of $E_L$ can be defined over singular homology classes as well: for $\alpha \in \text{H}_\bullet(LQ) \setminus \{0\}$ we have

$$c_\alpha(L) = \inf_{\eta \in \alpha} \max_{n_\sigma \neq 0} E_L(\sigma),$$

where $\eta = \sum n_\sigma \sigma$ is a formal finite sum of simplices $\sigma : \Delta_n \to LQ$ and $|\eta|$ is the union of the image of those $\sigma$ with non-trivial coefficient $n_\sigma$. We adopt the notation

$$E_L(\eta) := \max_{n_\sigma \neq 0} E_L(\sigma).$$

The same standard argument as in Proposition 3.2 implies:

\footnote{One could also use $\Psi$ here, which actually is equal to $\Theta^{-1}$ on the homology level.}
Proposition 3.3. If $\alpha \in H_\bullet(\mathcal{L}Q) \setminus \{0\}$, then
\[ c_\alpha(L) = c_\Upsilon(\alpha)(L). \]

Once we have fixed $H$ and $L$, we can unambiguously talk about spectral invariants/minimax values associated to a singular/Morse homology class $\alpha$ thanks to Proposition 3.2 and Proposition 3.3 and therefore we shall often simply write
\[ c_\alpha(H) = c_\alpha(L) \]
without specifying $\alpha$. Of course, the Abbondandolo–Schwarz isomorphisms applied to $\alpha$ depend on the choice of $H$ and $L$.

3.2 Stability and Spectrality

We continue by establishing some expected properties of spectral invariants such as $C^0$-Lipschitz continuity and spectrality. Both are slightly non-standard since we are in a non-compact setting. We start with the former.

Proposition 3.4. Let $H^0, H^1$ be Hamiltonians satisfying (H0), (H1), (H2), and $\|H^0 - H^1\|_{C^0} < \infty$.

Then for every $\beta \in HF_\bullet \setminus \{0\}$ we have
\[ |c_\beta(H^0) - c_\beta(H^1)| \leq \|H^0 - H^1\|_{C^0}. \]

Proof. This follows verbatim from [Sch00, page 431] and/or [Iri14, page 2491] — the non-compactness does not affect the proof due to the imposed $C^0$-bound on the difference $H^0 - H^1$. For convenience of the reader we go through the argument anyway. First of all observe
\[ \Delta_{0,1} : = \int_0^1 \sup_{(p,q) \in T^*Q} (H^0_t - H^1_t) \, dt = \|H^0 - H^1\|_{C^0} < +\infty, \]
and by symmetry $\Delta_{1,0} < +\infty$. The $C^0$-bound also implies that the two Hamiltonians are close in the AS sense and therefore there exists a direct continuation isomorphism
\[ HF_\bullet(H^0) \congto HF_\bullet(H^1). \]

This map is defined by the count of $s$-dependent Floer solutions $u: \mathbb{R} \to \mathcal{L}T^*Q$ connecting critical points $x \in \mathcal{P}(H^0)$ and $y \in \mathcal{P}(H^1)$. We claim that the continuation descends to the filtered version
\[ HF^<_\bullet(H^0) \congto HF^<_{\bullet+\Delta_{0,1}}(H^1). \]

Indeed, for the usual connecting homotopy $H^s_t : = H^0_t + \beta(s) \cdot (H^1_t - H^0_t)$ with smooth
\[ \beta: \mathbb{R} \longrightarrow \mathbb{R}, \quad \beta(s) = \begin{cases} 0, & \text{if } s \leq 0, \\ 1, & \text{if } s \geq 1, \end{cases} \quad \beta'(s) \geq 0, \]

10See [AS06, Lemma 1.21] equation (1.39), (1.44) with $\varepsilon = 0$. 

10
we get
\[
0 \leq E(u) = \int_{\mathbb{R}} \|\partial_s u\|_2^2 \, ds \\
= \int_{\mathbb{R}} - dA_H[s] \, ds \\
= - \int_{\mathbb{R}} \frac{\partial}{\partial s} (A_H(u(s))) \, ds + \int_{\mathbb{R}} \frac{\partial A_H}{\partial s}(u(s)) \, ds \\
= A_{H^0}(x) - A_{H^1}(y) - \int_0^1 \beta(s) \int_0^1 (H^1_t(u(s)) - H^0_t(u(s))) \, dt \, ds \\
\leq A_{H^0}(x) - A_{H^1}(y) + \int_0^1 \sup (H^0_t - H^1_t) \, dt.
\]

From the “filtered” homomorphism above we can readily deduce that
\[
c_\beta(H^1) \leq c_\beta(H^0) + \Delta_{0,1}.
\]

Using the symmetry of the argument we also get
\[
c_\beta(H^0) \leq c_\beta(H^1) + \Delta_{1,0}
\]
and with the final observation that
\[
\max\{\Delta_{0,1}, \Delta_{1,0}\} \leq \|H^0 - H^1\|_{C^0(S^1 \times T^*Q)},
\]
which finishes the proof.

**Remark 3.5.** The $C^0$-bound in Proposition 3.4 is implicitly used to ensure that the needed $L^\infty$-estimates on the $s$-dependent Floer solutions hold. If $\|H^0 - H^1\|$ satisfies a certain quadratic bound \[AS06, Lemma 1.21\], then the $s$-dependent Floer solutions lie in a compact set $B \subseteq T^*Q$. In particular, under the assumption of \[AS06, Lemma 1.21\] the conclusion of Proposition 3.4 still holds after replacing $\|H^0 - H^1\|_{C^0}$ with $\|(H^0 - H^1)\|_B$.

Now we prove spectrality:

**Lemma 3.6.** Let $H$ be a Hamiltonian satisfying (H0), (H1), and (H2). Then the spectrum
\[
\mathcal{S}(H) := \{A_H(x) \mid x \in \mathcal{P}(H)\}
\]
is closed and discrete. Moreover, for every $\alpha \in HF(H) \setminus \{0\}$ there exists $x \in \mathcal{P}(H)$ such that
\[
c_\alpha(H) = A_H(x) \in \mathcal{S}(H).
\]
Proof. Let \((x_n)\) be a sequence of Hamiltonian orbits such that \(A_H(x_n)\) converges. In particular, \((x_n) \in \mathcal{P}^d(H)\) for some \(d \in \mathbb{R}\). But [AS06, Lemma 1.10] tells us that \(\mathcal{P}^d(H)\) is finite. This proves that \(S(H)\) is closed. To see that \(S(H)\) is discrete, observe that
\[
(-\infty, d] \cap S(H) = A_H(\mathcal{P}^d(H)).
\]
The proof of spectrality is the same as the one presented in [Iri14, Lemma 3.1]: assume by contradiction that \(c_\alpha(H) \notin S(H)\).

First of all, observe that \(-\infty < c_\alpha(H)\). If not there exists a sequence \((x_n) \subset \mathcal{P}(H)\) with \(A_H(x_n) \to -\infty\), contradicting finiteness of \(\mathcal{P}^d(H)\). Since \(S(H)\) is closed (as seen above), there exists a \(\varepsilon > 0\) such that
\[
[c_\alpha(H) - \varepsilon, c_\alpha(H) + \varepsilon] \cap S(H) = \emptyset.
\]
In particular
\[
HF^c_\alpha(H) - \varepsilon, c_\alpha(H) + \varepsilon\rangle = 0,
\]
and hence the homomorphism
\[
HF^c_\alpha(H) - \varepsilon \to HF^c_\alpha(H) + \varepsilon
\]
is an isomorphism. This however readily contradicts the infimum definition of \(c_\alpha(H)\). Therefore we have shown that \(c_\alpha(H) \in S(H)\).

The \(x \in \mathcal{P}(H)\) produced by Lemma 3.6 will be referred to as a carrier of the spectral invariant \(c_\alpha(H)\). Note that a carrier does not have to be unique. Moreover, a carrier inherits the homotopy class of the corresponding connected component of \(L\) on which \(\alpha\) is defined. More precisely, if the Floer class \(\alpha\) corresponds to a singular homology class on the component \(L_\eta Q\), where \(\eta\) is a conjugacy class in \(\pi_1(Q)\), then the free homotopy class of the carrier \(x\) is \(\eta\). Indeed, the AS isomorphism respects the homotopy class of the generators, i.e.
\[
H_\bullet(L_\eta Q) \cong HM_\bullet(E|_{L_\eta Q}) \cong HF_\bullet(H; \eta),
\]
where the latter denotes the Floer homology of \(H\) generated by Hamiltonian orbits with fixed free homotopy class \(\eta\).

Until now we have only considered 1-periodic Hamiltonians \(H\) satisfying all three AS conditions. However, we would like to work with spectral invariants for autonomous Hamiltonians which at most satisfy (H1) and (H2), e.g. \(F\) in Section 2. There are multiple ways to deal with this and we opt for the following: let \(H\) be an autonomous Hamiltonian satisfying (H1) and (H2) and consider
\[
H_n: S^1 \times T^*Q \to \mathbb{R}, H_n(t, q, p) = H(q, p) + W_n(t, q),
\]
a sequence of Hamiltonians satisfying (H0), (H1), and (H2), where \(W_n: S^1 \times Q \to \mathbb{R}\) are potentials tending to 0 in \(C^2\), cf. [Web02].
Definition 3.7. Let $\alpha \in H_*(\mathcal{L}Q) \setminus \{0\}$ and $H: T^*Q \to \mathbb{R}$ a Hamiltonians satisfying (H1) and (H2). Define

$$c_\alpha(H) := \lim_{n \to \infty} c_\alpha(H_n),$$

where $H_n = H + W_n$ are non-degenerate Hamiltonians with 1-periodic potentials $W_n$ that tend to 0 in $C^2$.

With Proposition 3.4 at hand we can show that Definition 3.7 works and that spectrality also holds in the degenerate case:

**Proposition 3.8.** Let $\alpha$ and $H$ as in Definition 3.7. Then $c_\alpha(H)$ is well defined and there exists $x \in \mathcal{P}(H)$ such that

$$c_\alpha(H) = \mathcal{A}_H(x) \in \mathcal{S}(H).$$

**Proof.** The proof strategy is a refinement of standard arguments to the non-compact setting [FS07, Proposition 5.1]. Let $H_n$ be a sequence of non-degenerate Hamiltonians converging to $H$ as in Definition 3.7. From Lemma 3.6 we get the existence of a sequence $x_n \in \mathcal{P}(H_n)$ such that

$$c_\alpha(H_n) = \mathcal{A}_{H_n}(x_n).$$

Additionally, a thorough inspection of [AS06, Lemma 1.10] reveals that the constants therein depend continuously on $H_n$, thus showing that $(x_n)$ lies in a compact set $B$ in $T^*Q$ — see also [BB11, Theorem 8.8]. Hence, by means of Arzelà-Ascoli we can apply Proposition 3.4:

$$|c_\alpha(\tilde{H}_n) - c_\alpha(H_n)| \leq \|\tilde{H}_n - H_n\|_{C^0} = \|V_n - W_n\|_{C^0} \to 0, \text{ for } n \to \infty.$$

**Remark 3.9.** Since the carrier $x$ of $c_\alpha(H)$ in Proposition 3.8 is constructed as a limit of carriers $x_n$, the homotopy property still holds for degenerate $H$, i.e. if $\alpha$ is a singular homology class on $\mathcal{L}Q$ and $x$ is a carrier of $c_\alpha(H)$, then $[x] = \eta$.

For degenerate Lagrangians we can exploit the results on the dual Hamiltonian side to recover the fact that the minimax values are attained: let $L: TQ \to \mathbb{R}$ be a Lagrangian in the sense of AS modulo the non-degeneracy condition. Denote by $H$ its dual Hamiltonian, which then satisfies (H1) and (H2), but is degenerate as well. Subtract a small potential $W_n: S^1 \times Q \to \mathbb{R}$ from $L$ so that the resulting Lagrangian $L_n(t, q, v) := L(q, v) - W_n(t, q, v)$ is non-degenerate. These Lagrangians admit dual Hamiltonians $H_n$, which are of the form

\[d(x_n(t_0), x_n(t_1)) \leq \int_{t_0}^{t_1} \|\dot{x}_n(t)\| \, dt \leq \|X_{H_n}|_B\| \cdot |t_0 - t_1| \leq \|H_n\|_{C^2(B)} |t_0 - t_1|,\]

bounded in $n$.\[\]
$H_n = H + W_n$ and satisfy (H0), (H1), and (H2). In particular, using Proposition 3.2 and Proposition 3.3 we get a unique limit

$$\lim_{n \to \infty} c_\alpha(L_n) = \lim_{n \to \infty} c_\alpha(H_n) = c_\alpha(H).$$

If the gradient flow of $\mathcal{E}_L$ is already sufficiently nice, e.g. whenever $L$ is a purely kinetic Lagrangian, Lusternik–Schirelmann theory (LS theory) is applicable and produces minimax values $c_\alpha(L)$ that are attained as critical values of $\mathcal{E}_L$ [Kli12, Chapter 2]. We show that the limit of the sequence $c_\alpha(L_n)$ above produces the same value as LS theory for $c_\alpha(L)$. This is crucial in order to relate the spectral invariants to the minimax values, and we will apply this to the kinetic Lagrangian action functional in the proof of the Main Theorem 6.5.

**Proposition 3.10.** Let $L$ and $L_n = L - W_n$ be as above and assume that the gradient flow of $\mathcal{E}_L$ is subject to LS theory. Let $\alpha \in H_\bullet(LQ) \setminus \{0\}$. Then the limit of the sequence $c_\alpha(L_n)$ agrees with the minimax value $c_\alpha(L)$ produced by LS theory, i.e.

$$c_\alpha(L) = \lim_{n \to \infty} c_\alpha(L_n).$$

**Proof.** First of all observe that $c_\alpha(L_n)$ can also be computed by viewing $\alpha$ as a singular homology class; cf. Proposition 3.3. Hence

$$|c_\alpha(L_n) - c_\alpha(L)| \leq \inf_{\tau \in ct} \sup_{\eta \in ct} |\mathcal{E}_{L_n}(\tau) - \mathcal{E}_{L_n}(\eta)| = |\mathcal{E}_{L_n}(q_n) - \mathcal{E}_L(q)|,$$

for some $q \in \text{Crit}(L)$ and $q_n \in \text{Crit}(L_n)$ — such $q_n$’s (resp. $q$) exist because

$$c_\alpha(L_n) = c_\alpha(H_n) = A_H(x_n) = \mathcal{E}_{L_n}(q_n),$$

by Proposition 3.2 and Lemma 3.6 (resp. standard LS theory).

We proceed with a case distinction first consider the case where there exists a subsequence, still denoted by $\mathcal{E}_{L_n}(q_n)$, such that

$$\mathcal{E}_L(q) \leq \mathcal{E}_{L_n}(q_n).$$

We claim that

$$|c_\alpha(L_n) - c_\alpha(L)| \leq \|W_n\|_{C^0} \to 0.$$

The LHS is equal to $\mathcal{E}_{L_n}(q_n) - \mathcal{E}_L(q)$. Denote by $\eta$ the singular chain with $\mathcal{E}_{L}(\eta) = \mathcal{E}_L(q)$. Since $\eta$ represents $\alpha$, the definition of $c_\alpha(L_n)$ and the identity $L_n = L - W_n$ imply

$$\mathcal{E}_{L_n}(q_n) = c_\alpha(L_n) \leq \mathcal{E}_{L_n}(\eta) \leq \mathcal{E}_L(\eta) + \|W_n\|_{C^0} = \mathcal{E}_L(q) + \|W_n\|_{C^0}.\text{[12]}

For the other case, denote by $\tau_n$ a representative of $\alpha$ with $\mathcal{E}_{L_n}(\tau_n) = \mathcal{E}_{L_n}(q_n)$ and use the same logic to deduce

$$\mathcal{E}_L(q) = c_\alpha(L) \leq \mathcal{E}_L(\tau_n) \leq \mathcal{E}_{L_n}(\tau_n) + \|W_n\|_{C^0} = \mathcal{E}_{L_n}(q_n) + \|W_n\|_{C^0}.$$

Combining the two cases grants

$$|c_\alpha(L_n) - c_\alpha(L)| \leq \|W_n\|_{C^0} \to 0.$$

This concludes the proof. \hfill \square

---

[12] The supremum of a sum is smaller than the sum of the individual suprema.
4 Pinching and Floer Homologies

4.1 Preliminaries

We closely follow [MS11; Hei11; Wul14] and construct three sequences of non-degenerate Hamiltonians. Let \( F : T^*Q \to \mathbb{R} \) be the Hamiltonian that realizes \( \Sigma \) as in Section 2. In particular, \( \Sigma \) is not assumed to be non-degenerate throughout this section.\(^{13}\) Up to rescaling the Riemannian metric we can assume that the kinetic Hamiltonian

\[
G(q,p) = \frac{1}{2} \|p\|^2
\]

is pointwise smaller than \( F \), i.e.

\[
G \leq F.
\]

Since \( Q \) is compact, there exists a constant \( \sigma > 1 \) such that

\[
\sigma G \geq F.
\]

Define a smooth auxiliary function

\[
f(r) = \begin{cases} 
0, & r \in (-\infty, \varepsilon^2], \\
r, & r \geq \varepsilon,
\end{cases}
\]

with \( 0 \leq f'(r) \leq 2 \). The inequality

\[
f \circ G \leq f \circ F \leq \sigma G
\]

still holds.

We will need the three Hamiltonians to agree at infinity to exploit the full power of the pinching. For this purpose we define, for fixed \( d \geq 0 \), a function

\[
\tau_d : \mathbb{R} \to \mathbb{R}, \quad \tau_d(r) = \begin{cases} 
0, & r \in (-\infty, \sqrt{2d}], \\
1, & r \in [2\sqrt{2d}, +\infty),
\end{cases}
\]

with \( \tau'_d \geq 0 \). The square root comes from the fact that we will feed \( \|p\| \) to \( \tau_d \) and use the following relations

\[
\sqrt{2d} = \|p\| \iff d = \frac{1}{2} \|p\|^2 \iff d = G(q,p).
\]

Let

\[
W_n : S^1 \times Q \to \mathbb{R}
\]

be a sequence of 1-periodic potentials, \( n \in \mathbb{N} \). Pick

\[
c, c_n \in (0, 1/4) \text{ with } c \geq c_n \text{ and } \lim_{n \to \infty} c_n = 0,
\]

\(^{13}\)The non-degeneracy of the relevant data to define Floer homology will be achieved by adding small time-dependent potentials as we will see shortly.
and define

\[ c'_n = \min \left\{ c_n, \frac{c_n}{\|X_{fF}\|_{C^0}}, \frac{c_n}{\|X_{\sigma G}\|_{C^0}}, \frac{c_n}{\|X_{fG}\|_{C^0}} \right\}. \]

Assume

\[ \|W_n\|_{C^1} < c'_n. \]

In particular, by choice of \( c_n \), we obtain

\[ \|W_n\|_{C^0} \to 0 \text{ as } n \to \infty. \]

We finally define the 3 Hamiltonians and suppress the dependence on \( d > 0 \) in the notation:

\[
\begin{align*}
G^+_n(t, q, p) &= \sigma \cdot G(q, p) + W_n(t, q), \\
K_n(t, q, p) &= (1 - \tau_{c+d})(||p||) \cdot [f \circ F(q, p) + W_n(t, q)] + \tau_{c+d}(||p||)G^+_n(t, q, p), \\
G^-_n(t, q, p) &= (1 - \tau_{c+d})(||p||) \cdot [f \circ G(q, p) + W_n(t, q)] + \tau_{c+d}(||p||)G^-_n(t, q, p), \\
\end{align*}
\]

By definition, \( G^+_n, K_n \) and \( G^-_n \) agree on \( \{G \geq 4d\} \). It is well known that for each Hamiltonian \( H \in \{f \circ G, f \circ F, \sigma G\} \), the set of potentials

\[ W \in V_H \subseteq C^\infty(S^1 \times Q) \]

such that \( H + W \) is non-degenerate, is open and dense. The intersection of these three sets of potentials is a residual set, and thus dense by the Baire Category Theorem. This allows us to pick the sequence \( W_n \) such that the above estimates are satisfied and all Hamiltonians \( H + W_n \) are non-degenerate; cf. [Web02]. Towards the end of the present section we will pass to the degenerate case, i.e. send \( W_n \) to 0 for \( n \to \infty \). We introduce the needed notation in advance:

\[
\begin{align*}
K &= (1 - \tau_{c+d}) \cdot (f \circ F) + \tau_{c+d} \cdot \sigma G, \\
G^- &= (1 - \tau_{c+d}) \cdot (f \circ G) + \tau_{c+d} \cdot \sigma G.
\end{align*}
\]

The following is a collection of standard results taken from [Hei11], which will be used throughout the whole subsection.

**Lemma 4.1.** Let \( H: T^*Q \to \mathbb{R} \) be a \( C^1 \)-map which is homogeneous of degree 2 on every fibre and \( c > 0 \). Let \( V: S^1 \times T^*Q \to \mathbb{R} \) smooth with

\[ \|V\|_{C^1} < \frac{c}{\|X_H\|_{C^0}}. \]

Let \( x \in \mathcal{P}(H + V) \), \( a = H(x(t_0)) \) for some fixed \( t_0 \in S^1 \). Then

---

\[ ^\text{14} \text{Any } c > 0, \text{ not necessarily the one chosen previously.} \]
Proposition 4.2. Let \( f \in C^1 < c, h : \mathbb{R} \to \mathbb{R} \) smooth and \( r > 0 \). Then for \( x \in \mathcal{P}(h \circ H + W) \):

(i) \( a - c < H(x(t)) < a + c, \quad \forall t \in S^1 \).

Let \( W : S^1 \times Q \to \mathbb{R} \) be smooth with \( \|W\|_{C^1} < c \). In the first situation we get

\[
\frac{d}{dt} \left( h(H(x(t))) \cdot H(x(t)) - h(H(x(t))) - W_t(x(t)) \right) dt.
\]

Proof. We will need this in a second. Observe

But

\[
S(rH + W) \subseteq \frac{1}{c} S(H + W) + [-c, c].
\]

The following is an adaptation of [Hei11, Proposition 2.2.2].

**Proposition 4.2.** Let \( x \in \mathcal{P}(K_n) \), \( d > 4\varepsilon + c \). Then

- If there is a \( t_0 \) with \( F(x(t_0)) > d \), then \( A_{K_n}(x) > d - 2c \).
- If \( F(x(t_0)) \leq d \) instead, then \( A_{K_n}(x) \leq d + c \).

The same result remains true after swapping \( K_n \) and \( F \) with \( G^- \) and \( G \).

**Proof.** In the first situation we get \( d < F(x(t_0)) = f \circ F(x(t_0)) \) because \( d > \varepsilon \) and the definition of \( f \). In particular, using \( \sigma G \geq f \circ F \):

\[
a := K_n(x(t_0)) - W_n(x(t_0)) \geq f \circ F(x(t_0)) > d.
\]

Thus item (i) in Lemma [III] applied to \( H := K_n - W_n, V = W_n \), and the choice of \( c \), tells us

\[
K_n(x) - W_n(x) > a - c > d - c.
\]

At the same time we have \( F(x) > \varepsilon \) — indeed, if we assume \( F(x(s)) \leq \varepsilon \) for some \( s \in S^1 \), we reach a contradiction: the inequality \( G(x(s)) \leq F(x(s)) \leq \varepsilon < d \) implies

\[
K_n(x(s)) = f \circ F(x(s)) + W_n(x(s))
\]

and therefore

\[
\varepsilon \geq F(x(s)) \geq f \circ F(x(s)) = K_n(x(s)) - W_n(x(s)) > d - c,
\]

which contradicts the choice of \( \varepsilon \).

Since \( F(x) \geq \varepsilon \), we have \( f \circ F(x) = F(x) \), in particular

\[
d(f \circ F)(x)[Y] = f'(F(x)) \cdot dF(x)[Y] = dF(x)[Y].
\]

We will need this in a second. Observe

\[
A_{K_n}(x) = \int x^* \lambda - \int_0^1 K_n(x) dt = \int_0^1 dK_n(x)[Y] - K_n(x) dt.
\]

But

\[
dK_n(x)[Y] = -\|p\| \cdot (\tau_{d+c})(\|p\|)(f \circ F)(x) + \|p\| \cdot (\tau_{d+c})(\|p\|) \sigma G(x)
\]

\[
+ (1 - \tau_{d+c})(\|p\| \cdot d(f \circ F)(x)[Y] + \tau_{d+c}(\|p\|) \cdot \sigma dG(x)[Y] + dW_n(x)[Y]
\]

\[
\geq 2(1 - \tau_{d+c})(\|p\|) F(x) + \tau_{d+c}(\|p\|) 2 \sigma G(x)
\]

\[
= 2 \cdot K_n(x) - 2W_n(x)
\]
Here we used the previously derived identity $d(f \circ F)(x)[Y] = 2F(x)$, the analogous Euler identity $dG(x)[Y] = 2G(x)$, and $dW_n(x)[Y] = 0$ (note that $W_n$ does not depend on the fibre variable $p$). Thus we can finally bound

$$A_{K_n}(x) = \int dK_n(x)[Y] - K_n(x) dt$$
$$\geq \int (K_n(x) - W_n(x)) - W_n(x) dt$$
$$\geq d - c - c$$
$$= d - 2c.$$

This concludes the proof of the first bullet point.

For the other item we observe that $F(x(t_0)) \leq d$ implies $G(x(t_0)) \leq d$, thus $\tau_{c+d}$ does vanish at $\|p(t_0)\|$. In particular

$$a = K_n(x(t_0)) - W_n(x(t_0)) = f \circ F(x(t_0)) \leq F(x(t_0)) \leq d.$$

Applying $(i)$ from Lemma [4.1] again tells us that

$$K_n(x) - W_n(x) < a + c \leq d + c,$$

but

$$f \circ F \leq K_n - W_n,$$

therefore implying with the above that $G(x) \leq d + c$ and thus that $\tau_{c+d}$ vanishes on the whole interval $\|p(t)\|$. This proves

$$K_n = f \circ F(x) + W_n(x).$$

Invoking Lemma [4.1] item $(ii)$ gives

$$A_{K_n}(x) = A_{f \circ F + W_n}(x) = \int 2f'(F(x)) - f(F(x)) dt - \int W_n(x) dt.$$

We make a case distinction to bound the left integral: if $F(x(s)) < \varepsilon^2$, then

$$2f'(F(x(s))) - f(F(x(s))) = 0.$$

For $F(x(s)) > \varepsilon$ we have

$$2f'(F(x(s))) - f(F(x(s))) = 2F(x(s)) - F(x(s)) \leq d + c,$$

and in the case $F(x(s)) \in [\varepsilon^2, \varepsilon]$ we get

$$2f'(F(x(s))) - f(F(x(s))) \leq 4F(x(s)) < 4\varepsilon \leq d$$

since $f' \leq 2$ and by assumption on $d$. All in all this proves

$$A_{K_n}(x) \leq d + c.$$

The case for $G_n$ is verbatim the same after swapping $F$ with $G$. \qed
Notation. We write $\mathcal{P}(H)$ to denote the non-constant orbits of the Hamiltonian $H$. The notation for action windows $\mathcal{P}^b$ is adopted.

**Proposition 4.3.** Let $d > \max\{4\varepsilon + c, 2c\}$ and $f$, $K$, $G^-$ be as above. Then for any $b \in (0, d - 2c)$ we have

$$\mathcal{P}^b(K) \subseteq \mathcal{P}^b(f \circ F) = \mathcal{P}^b(F)$$

and

$$\mathcal{P}^b(G^-) \subseteq \mathcal{P}^b(f \circ G) = \mathcal{P}^b(G).$$

**Proof.** The function $f$ does implicitly depend on $\varepsilon$. We may assume that $\varepsilon > 0$ has been chosen so small that the non-constant 1-periodic orbits of $f \circ G$ (resp. $f \circ F$) agree with those of $G$ (resp. $F$) and lie in the set $\{f \circ G \geq \varepsilon\}$ (resp. $\{f \circ F \geq \varepsilon\}$); cf. [Hei11, page 52]. In particular, for any $x \in \mathcal{P}^b(f \circ F)$ we have

$$A_{f \circ F}(x) = f \circ F(x) = F(x) = A_F(x),$$

with the analogous statement for $G$. Therefore we have just shown that for any $b > 0$ we have

$$\mathcal{P}^b(f \circ G) = \mathcal{P}^b(G) \text{ and } \mathcal{P}^b(f \circ F) = \mathcal{P}^b(F).$$

Now $x \in \mathcal{P}^b(G^-)$ with $b$ chosen as in the statement. In particular, $A_{G^-}(x) \leq b < d - 2c$ and thus the contrapositive of Proposition 2.2 applied to $G^-$ (i.e. $G^\sigma_n$ with $W_n = 0$) says $G(x) \leq d$. Therefore $G^- (x) = f \circ G$, hence

$$\mathcal{P}^b(G^-) \subseteq \mathcal{P}^b(f \circ G).$$

The analogous inclusion holds for $K$ and $F$. Restricting to the set of non-constant orbits and combining this with the above then concludes the proof.

We are finally able to state and prove our own version of the “Non-Crossing Lemma”; cf. [MS11, Lemma 3.3] and [Hei11, Lemma 2.2.3]. This will enable us to get a filtered continuation isomorphism between the Floer homologies of $G^- \sigma$ and $G^\sigma_n$. To this end, define the usual homotopy

$$G^s \sigma_n := (1 - \beta(s)) G^\sigma_n + \beta(s) G^\sigma_n$$

with $\beta$ chosen as in the proof of Proposition 3.2. We also define, for any $a \in \mathbb{R}$:

$$a(s) := \frac{a}{1 + \beta(s)(\sigma - 1)}.$$

The function $a(s)$ is strictly decreasing, starts at $a(0) = a$ and ends at $a(1) = \frac{a}{\sigma}$. In particular

$$\frac{a(0)}{a(1)} = \sigma.$$

19
Lemma 4.4 (Non-Crossing Lemma). Fix $d > \max\{4\varepsilon + c, 2c\}$ as in Proposition 4.3. Up to shrinking $\varepsilon = \varepsilon(c, \sigma, G) > 0$, the following holds true:

For any $a \in \mathbb{R}$ with

\[4(\sigma + 4)c < a < d - 2c\]

the relation

\[[a - c_n, a + c_n] \cap S(G + W_n) = \emptyset\]

implies

\[a(s) \notin S(G^*_n), \quad \forall s \in [0, 1].\]

Proof. We will make a case distinction and show that for every $x \in \mathcal{P}(G^*_n)$ we have:

1. if $x$ enters $\{G > d\}$, then $A_{G^*_n}(x) > a(s)$,
2. if $x$ enters $\{G \leq \varepsilon\}$, then $A_{G^*_n}(x) < a(s)$ (if $\varepsilon$ is chosen accordingly),
3. if $x$ stays in $\{\varepsilon < G \leq d\}$, then $[a - c_n, a + c_n]$ intersects $S(G + W_n)$, contradicting our assumption.

Showing 1, 2 and 3 above immediately implies our desired result.

Ad 1: the same computation as in Proposition 4.2 implies

\[A_{G^*_n}(x) > d - 2c > a \geq a(s).\]

This proves 1.

Ad 2: By part 1. we may assume that $x$ stays in $\{G \leq d\}$, in particular we can ignore $\tau_{c+d}$ and deduce

\[G^*_n(x) = (1 - \beta(s))(f \circ G)(x) + \beta(s) \cdot \sigma G(x) + W_n(x).\]

This readily implies

\[f \circ G(x) \leq G^*_n(x) - W_n(x).\]

For simplicity, define

\[\bar{h}(r) := \beta(s)\sigma \cdot r + (1 - \beta(s)) \cdot f(r)\]

and rewrite the above as

\[\bar{h} \circ G(x) + W_n(x) = G^*_n(x).\]

By the assumption in 2 there is some $t_0$ with $G(x(t_0)) \leq \varepsilon$. We apply Lemma 4.1 (i) to

\[G^*_n(x(t_0)) - W_n(x(t_0)) = \bar{h}(G(x(t_0)))\]

and use the choice of $c'_n > \|W_n\|_{C^1}$ to obtain

\[\bar{h}(G(x)) \leq \bar{h}(G(x(t_0))) + \|W_n\|_{C^1} \cdot (1 - \beta(s))X_{G \circ G} \|_{C^0} + \beta(s) \|X_{G \circ G} \|_{C^0}) \leq (\sigma + 1)\varepsilon + 2c_n.\]
We claim that
\[ G(x) \leq (\sigma + 1)\varepsilon + 2c_n \]
holds. Indeed, if not, then there is some \( r_0 \in S^1 \) with \( G(x(r_0)) > (\sigma + 1)\varepsilon + 2c_n \). In particular, \( G(x(r_0)) > \varepsilon \) which implies \( f \circ G(x(r_0)) = G(x(r_0)) \). However, as seen above:
\[ f \circ G(x) \leq G_n^s(x) - W_n(x) = \bar{h}(G(x)) \leq (\sigma + 1)\varepsilon + 2c_n, \]
which altogether leads to the following contradiction:
\[ (\sigma + 1)\varepsilon + 2c_n < G(x(r_0)) = f \circ G(x(r_0)) \leq G_n^s(x(r_0)) - W_n(x(r_0)) \leq (\sigma + 1)\varepsilon + 2c_n. \]
The claim follows.

Next we bound the action \( \mathcal{A}_{G_n^s} \): Observe \( \bar{h}'(r) = \beta(s)\sigma + (1 - \beta(s)) \cdot f'(r) \). Applying Lemma 4.1 item \( (ii) \) to \( \bar{h} \circ G + W_n \) gives
\[
\mathcal{A}_{G_n^s}(x) = \int 2 \left[ \sigma \beta(s) + (1 - \beta(s)) \cdot f'(G(x)) \right] \cdot G(x) - \left( \bar{h}(G(x) + W_n(x)) \right) dt \\
\leq \int (2\sigma + 4)G(x) dt + c_n.
\]
Together with the claim from above we obtain
\[ \mathcal{A}_{G_n^s}(x) \leq 2(\sigma + 1)^2\varepsilon + 4(\sigma + 2)c_n + c_n \leq 2(\sigma + 1)^2\varepsilon + 4 \cdot (\sigma + 3)c. \]
We choose \( \varepsilon \) now: since \( a/\sigma \leq a(s) \), it suffices to have
\[
0 < \varepsilon < \frac{2 - 4(\sigma + 3)c}{2(\sigma + 1)^2}
\]
to obtain \( \mathcal{A}_{G_n^s}(x) < a(s) \). By choice of \( a \) we have \( a > 4(\sigma + 4)c \), so we might as well choose
\[
0 < \varepsilon(c, \sigma, G) < \frac{2c}{(\sigma + 1)^2}.
\]
This takes care of case 2.

Ad 3: By contradiction assume that that \( \mathcal{A}_{G_n^s}(x) = a(s) \) for some \( s \in [0, 1] \). Since \( x \) lies in \( \{ \varepsilon < G \leq d \} \) we have
\[
G_n^s(x) = (1 - \beta(s)) \cdot G(x) + \beta(s)\sigma \cdot G(x) + W_n(x) \\
= [(\sigma - 1)\beta(s) + 1] \cdot G(x) + W_n(x).
\]
In particular \( a(s) \in S\left( [(\sigma - 1)\beta(s) + 1] \cdot G + W_n \right) \), thus by Lemma 4.1 \( (iii) \):
\[
[a - c_n, a + c_n] \cap S(G + W_n) \neq \emptyset,
\]
contradicting our assumption. This finishes the proof. \( \square \)
Notation. For better readability we set

\[ e = 4(\sigma + 4)c. \]

The condition in the Non-Crossing Lemma is met for almost all \( n \in \mathbb{N} \) as the next result shows:

**Proposition 4.5.** Let \( a \notin \mathcal{S}(G) \). Then there exists a positive integer \( N = N(a) \) such that:

\[ \forall n \geq N(a): \quad [a - c_n, a + c_n] \cap \mathcal{S}(G + W_n) = \emptyset. \]

**Proof.** Assume by contradiction that for all positive integers \( N \) there exists a \( n \geq N \) such that \( [a - c_n, a + c_n] \cap \mathcal{S}(G + W_n) \neq \emptyset \). In particular, we can extract a strictly increasing subsequence \((n_k)_{k \in \mathbb{N}}\) such that \([a - c_{n_k}, a + c_{n_k}] \cap \mathcal{S}(G + W_{n_k}) \neq \emptyset\). Any fixed sequence \( a_k \in [a - c_{n_k}, a + c_{n_k}] \cap \mathcal{S}(G + W_{n_k}) \) is bounded since \( 0 < c_n \leq c \) and thus admits a convergent subsequence, still denoted by \( a_k \). But \( c_{n_k} \to 0 \) for \( k \to \infty \), which then forces \( a_k \) to converge to \( a \) with

\[ a \in \mathcal{S}(G), \]

since \( W_{n_k} \to 0 \). This contradicts the assumption on \( a \). \( \square \)

Now we extract a whole interval

\[ [a, b] \subseteq (e, d - 2c] \]

for which Proposition 4.5 then holds. Pick any \( b \in (e, d - 2c] \) with \( b \notin \mathcal{S}(G) \), and apply Proposition 4.5 to get a \( N = N(b) \). Up to increasing \( N \) we can assume \( e < b - c_n \) for all \( n \geq N \). We pick \( a \in [b - c_N, b) \) with \( a \notin \mathcal{S}(G) \). In particular \( e < a \), thus

\[ [a, b] \subseteq (e, d - 2c]. \]

**Claim.** There exists \( N(a, b) \geq N(b) \) such that

\[ \forall n \geq N(a, b): \quad [a - c_n, b + c_n] \cap \mathcal{S}(G + W_n) = \emptyset. \]

Indeed, since \( a \notin \mathcal{S}(G) \), there exists a \( N(a) \) such that \([a - c_n, a + c_n] \) does not intersect \( \mathcal{S}(G + W_n) \) for all \( n \geq N(a) \); cf. Proposition 4.5. Taking \( N(a, b) = \max\{N(a), N(b)\} \) and \( n \geq N(a, b) \) we get

\[ \emptyset = ([a - c_n, a + c_n] \cup [b - c_n, b + c_n]) \cap \mathcal{S}(G + W_n) = [a - c_n, b + c_n] \cap \mathcal{S}(G + W_n), \]

which proves the claim.

**Lemma 4.6.** Let \( b \in (e, d - 2c] \setminus \mathcal{S}(G) \).

Then there exists \( N = N(b) \in \mathbb{N} \) with the following property:

\[ \text{HF}^b_{\sigma}(G_n^-) \xrightarrow{\cong} \text{HF}^b_{\sigma}(G_n^+), \quad \forall n \geq N(b), \]

induced by a concatenation of continuation morphisms.
Proof. Let \([a, b] \subseteq (e, d - 2c]\) and \(N = N(a, b)\) be as in the above claim and the discussion preceding the claim. Then for every \(v \in [a, b] \setminus S(G)\) we can apply Lemma 4.4 since \([v - c_n, v + c_n] \subseteq [a - c_n, b + c_n]\). This has the following implication: the whole strip \(S \subseteq \mathbb{R}^2\) bounded by the graphs \((s, b(s))\) and \((s, a(s))\) does not belong to the spectrum of \(S(G_n^a)\), i.e.

\[
v(s) \notin S(G_n^a), \quad \forall s \in [0, 1], \, v \in [a, b].\]

In particular, we can perform a “zig-zag” adiabatic argument: for \(s = 0\) we have that all \(v \in [a, b]\) satisfy

\[
v = v(0) \notin S(G_0^a) = S(G_0).
\]

That is, \([a, b]\) does not contain any critical value of \(A_{G_0}\), which implies

\[
HF^b_b(G_n) = HF^b_a(G_n).
\]

Now we move from \((0, a(0))\) horizontally within our strip \(S\) until we hit the upper graph of \(b(s)\), i.e. \((s_1, a(0)) = (s_1, b(s_1))\). Note that \(a(0) = a = b(s_1)\). By definition of our strip \(S\) we have \(a(0) \notin S(G_n^a)\) for \(s \in [0, s_1]\), hence by standard Floer theory

\[
HF^a_a(G_n) \cong HF^a_a(G_{s_1}^a) = HF^b_b(G_{s_1}^a).
\]

As before, the choice of our strip \(S\) implies that \([a(s_1), b(s_1)]\) does not intersect the spectrum of \(G_{s_1}^a\) and thus we can “drop” again:

\[
HF^b_b(G_{s_1}^a) = HF^a_a(G_{s_1}^a).
\]

Repeating the previous “horizontal move” we see that the whole algorithm can be performed a finite amount of times until we reach \((1, w(1))\) for some \(w \in [a, b]\) that is not necessarily equal to \(b\). At that point we “jump up” (analogously to the “dropping”) to \((1, b(1)) = (1, b/\sigma)\) without affecting the Floer homology. This finishes the proof.

\[\square\]

Remark 4.7. The size of the interval \([a, b]\) used in the proof above does not matter — we only needed a strictly smaller \(a\) that allowed us to bounce in a “zig-zag” fashion towards \(b(1) = b/\sigma\).

Combining this with the isomorphisms described in Section 3 we obtain:

**Corollary 4.8.** Let \(b \in (e, d - 2c) \setminus S(G)\) and \(N = N(b)\) as in Lemma 4.6. Denote by \(L_n^+\) the Lagrangian dual to \(G_n^+ = \sigma G + W_n\). Then for all \(n \geq N\) we have the following commutative diagram:

\[\text{Diagram 3.}\]

\[^{15}\text{This is nicely pictured in [MS11, figure 3].}\]
Denote by $L$ the dual Lagrangian to $G$, i.e.

$$L(q,v) = \frac{1}{2} \|v\|^2,$$

and also observe that $L^+_n$ defined as in Corollary 4.8 takes the form

$$L^+_n(t,q,v) = \frac{1}{2\sigma} \|v\|^2 - W_n(t,q).$$

We define $L^+ := \lim_{n \to \infty} L^+_n$ and observe

$$L^+ = \frac{1}{\sigma} L.$$

4.2 The Pinching Inequality

The next theorem can be viewed as the main building block of our main result.

**Theorem 4.9.** Let $\alpha \in H_*(\mathcal{L}Q) \setminus \{0\}$. Then

$$c_\alpha(G^-) = \sigma \cdot c_\alpha(\sigma G),$$

$$c_\alpha(\sigma G) \leq c_\alpha(K) \leq c_\alpha(G^-).$$

**Proof.** Let $b > 0$ big enough such that $\alpha$ is in the image of $H_*(\{\mathcal{E}_L \leq b - c\sigma\}) \to H_*(\mathcal{L}Q)^{10}$

In particular, if $q \in \{\mathcal{E}_L \leq b - c\sigma\}$, then

$$\mathcal{E}_{L_n^+}(t,q) \leq \frac{1}{\sigma} \mathcal{E}_L(q) + c \leq \frac{b}{\sigma}, \quad \forall n \in \mathbb{N}.$$

Thus $\alpha$ can be viewed as an element in $H_*(\{\mathcal{E}_L^+ \leq b/\sigma\})$. Accordingly, we take $d$ so big and $c$ so small that

$$b + \sigma c < d - 2c.$$

---

10 Note that by choice of $L$ we have $\mathcal{E}_L = \mathcal{E}$. 

24
Up to slightly perturbing $b$ we may also assume $b \notin S(G)$ and $b > e$ (if not, increase $b$ and $d$). By Corollary 4.8, we can send $\alpha$ through the commutative diagram and (abusively) still denote by $\alpha$ the Floer/Morse class in the corresponding group for each $n \geq N(b)$. Proposition 3.8 and the choice of $b$ tells us that

$$
\lim_{n \to \infty} c_{\alpha}(G_n^-) = c_{\alpha}(G^-) \leq b,
\lim_{n \to \infty} c_{\alpha}(K_n) = c_{\alpha}(K) \leq b,
\lim_{n \to \infty} c_{\alpha}(G_n^+) = c_{\alpha}(\sigma G) \leq \frac{b}{\sigma}.
$$

These limits allude to the proof strategy: let the action filtration $b$ run towards $c_{\alpha}(G^-)$ and use the isomorphism $HF^{b\star}(G_n^-) \cong HF^{b/\sigma}(G_n^+)$ in order to conclude $c_{\alpha}(G^+) \leq \sigma c_{\alpha}(G^-)$ and then use symmetry of the argument for the other inequality. This however, also requires to let $n$ go to infinity, while $n \geq N(b)$ has to be satisfied. Define

$$
B_m = c_{\alpha}(G^-) + c_m, \quad C_m = c_{\alpha}(K) + c_m, \quad D_m = \sigma \cdot (c_{\alpha}(\sigma G) + c_m).
$$

We claim that

$$
B_m \in (e, d - 2c], \quad \forall m \in \mathbb{N}
$$

holds so that we can apply Corollary 4.8 with action filtration $B_m$. By definition of $B$ and $c_m < c$ we get

$$
B_m \leq b + c < d - 2c \quad \forall m \in \mathbb{N}.
$$

By spectrality, i.e. Proposition 4.8

$$
c_{\alpha}(G^-) = A_{G^-}(x) \in S(G^-).
$$

The above bound and the first bullet point in Proposition 4.2 for $G^-$ imply

$$
G(x) \leq d.
$$

In particular, $\tau_{c+d}$ vanishes and hence $G(x) = G^-(x)$ with $x \in \mathcal{P}(G)$, but for $c$ sufficiently small\footnote{Choose $c$ so small that $e = 4(\sigma + 4)c$ lies below the non-zero spectrum of $G$.} the spectrum of $G$ is strictly greater than $e$, thus

$$
e < G(x) = G^-(x) = c_{\alpha}(G^-) \leq B_m,
$$

which proves the lower bound and thus that the sequence $B_m$ is contained in $(e, d - 2c]$. Up to perturbing $c_m$ we may thus assume

$$
B_m \in (e, d - 2c] \setminus S(G)
$$

and can therefore apply Lemma 4.6 to every single $B_m$:

$$
HF^{B_m\star}(G^-_{n_m}) \xrightarrow{\approx} HF^{B_m/\sigma}(G^+_{n_m}), \quad \forall n_m \geq N(B_m).
$$
Pick $(n_m)_{m \in \mathbb{N}}$ a strictly increasing sequence with $n_m \geq m$. Using Proposition 3.4 and Proposition 3.8 we get

$$|c_\alpha(G_{n_m}^-) - c_\alpha(G^-)| = \lim_{k \to \infty} |c_\alpha(G_{n_m}^-) - c_\alpha(G_k^-)|$$

$$\leq \lim_{k \to \infty} \|W_{n_m} - W_k\|_{C^0}$$

$$\leq \|W_{n_m}\|_{C^0}$$

$$< c_{n_m},$$

thus

$$c_\alpha(G_{n_m}^-) \leq c_\alpha(G^-) + c_{n_m} \leq c_\alpha(G^-) + c_m = B_m.$$ 

This means that the LHS of the morphism above “sees” $\alpha$, and so does the RHS. More precisely, we have the commutative diagram

$$
\begin{array}{ccc}
HF^B_m(G_{n_m}^-) & \xrightarrow{\cong} & HF^B_m/\sigma(G_{n_m}^+) \\
\downarrow i & & \downarrow i \\
HF^i(G_{n_m}^-) & \xrightarrow{\cong} & HF^i(\sigma G_{n_m}^+),
\end{array}
$$

where $\alpha$ belongs to the image of the left vertical map. By commutativity of the diagram and the definition of spectral invariants we obtain

$$c_\alpha(G_{n_m}^-) \leq \frac{B_m}{\sigma} = \frac{c_\alpha(G^-) + c_m}{\sigma}.$$ 

Taking the limit as $m$ goes to infinity then proves:

$$\sigma \cdot c_\alpha(\sigma G) \leq c_\alpha(G^-).$$

For the other inequality we consider $D_m$ defined as above and claim again, as before, that

$$D_m \in (e, d - 2c], \quad \forall m \in \mathbb{N}.$$ 

Proposition 3.2 and the choice of $b$ imply $c_\alpha(G_n^+) = c_\alpha(L_n^+) \leq b/\sigma$ and thus we get

$$c_\alpha(\sigma G) = c_\alpha(L^+) \leq b/\sigma$$

by Proposition 3.8. Therefore

$$D_m \leq b + \sigma c_m \leq b + \sigma c < d - 2c, \quad \forall m \in \mathbb{N}.$$ 

For the lower bound observe $\sigma c_\alpha(\sigma G) \leq D_m$, but

$$c_\alpha(\sigma G) \in S(\sigma G) = \frac{1}{\sigma}S(G),$$

by Lemma 4.1. As before, we may assume that $e$ bounds $S(G) \cap (0, +\infty)$ from below. In particular we end up with

$$e < \sigma c_\alpha(\sigma G) \leq D_m,$$
which proves the lower bound and hence the claim. We are now in a position to apply Lemma 4.6 again (with reversed arrow and after perturbing $c_m$):

$$\text{HF}^{D_m}_\cdot(G^+_{n_m}) \xrightarrow{\cong} \text{HF}^{D_m}_\cdot(G^-_{n_m}), \forall n_m \geq N(D_m).$$

As before we obtain

$$c_\alpha(G^+_{n_m}) \leq c_\alpha(\sigma G) + c_{n_m} \leq c_\alpha(\sigma G) + c_m = D_m.$$

Thus the LHS above sees $\alpha$, and thus the analogous commutative diagram grants

$$c_\alpha(G^-_{n_m}) \leq D_m = \sigma(c_\alpha(\sigma G) + c_m).$$

Taking the limit and also applying the previous inequality gives:

$$c_\alpha(G^-) \leq \sigma c_\alpha(\sigma G) \leq c_\alpha(G^-).$$

This proves the desired first equality.

The inequalities are less involved: we use yet another commutative diagram:

$$\begin{array}{ccc}
\text{HF}^a_\cdot(G^-_n) & \xrightarrow{i} & \text{HF}^a_\cdot(K_n) \\
\downarrow & & \downarrow \\
\text{HF}_\cdot(G^-_n) & \xrightarrow{\cong} & \text{HF}_\cdot(K_n). \\
\end{array}$$

The same reasoning as in the previous two cases above gives us

$$c_\alpha(K_n) \leq c_\alpha(G^-_n), \forall n \in \mathbb{N}.$$

Taking the limit proves

$$c_\alpha(K) \leq c_\alpha(G^-).$$

The other inequality

$$c_\alpha(\sigma G) \leq c_\alpha(K)$$

is proved the same way and thus the proof is complete. 

**Remark 4.10.** The crux of the proof of Theorem 4.9 comes from the fact that the filtration isomorphism from Lemma 4.6 does not hold for all $n$, i.e. changing level $b$ might increase $N(b)$. This forces us to use the somewhat convoluted sequence argument instead of just reading off the spectral invariant equality as done in Proposition 3.2.

Using Proposition 3.2 and Proposition 3.10 we obtain

$$\sigma c_\alpha(\sigma G) = \sigma c_\alpha(L^+) = \sigma c_\alpha(L/\sigma).$$

It readily follows from the definition of minimax values that

$$\sigma c_\alpha(L/\sigma) = c_\alpha(L), \forall \alpha \in \text{H}_*\cdot(\mathcal{L}Q) \setminus \{0\}.$$

Combining these with Theorem 4.9 grants:
Corollary 4.11. Let $\alpha \in H_\bullet(LQ) \setminus \{0\}$. Then the spectral invariants $c_\alpha(G^-)$ do not depend on $d$ and satisfy:

$$c_\alpha(G^-) = \sigma \cdot c_\alpha(\sigma G) = c_\alpha(L).$$

In particular

$$c_\alpha(K) \leq c_\alpha(L).$$

5 Index Relations

5.1 The Robbin–Salamon Index

One of the key steps in the proof of Theorem 6.5 is the use of an index iteration formula due to M. de Gosson, S. de Gosson and Piccione (GGP) [GGP08]. In order to correctly apply the latter, one needs some a priori control on the Conley–Zehnder/Robbin–Salamon index of the carriers of some carefully chosen spectral invariants. In this section we mainly recall some main properties of the Robbin–Salamon index [RS 93] and describe how the index of carriers of $c_\alpha(K)$ behave for any non-zero cohomology class $\alpha \in H_i(LQ)$ with respect to the degree $i \in \mathbb{N}$.

The Robbin–Salamon index picks two Lagrangian paths on a fixed $2n$-dimensional symplectic vector space, and associates to it a value in $\frac{1}{2}\mathbb{Z}$. For a path $\Gamma \in S(2n)$, i.e.

$$\Gamma : [0,1] \rightarrow \text{Sp}(2n), \quad \Gamma(0) = \text{id},$$

one defines

$$\mu_{RS}(\Gamma) = \mu_{RS}(\text{gr}(\Gamma), \Delta),$$

where $\text{gr}(\Gamma)$ is the graph of $\Gamma$ in the symplectic vector space $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\omega \otimes \omega)$ and $\Delta$ is the diagonal in the latter. With this definition, one obtains

$$\mu_{CZ}(\Gamma) = \mu_{RS}(\Gamma), \quad \forall \Gamma \in S^*(2n) := \{ \Gamma \in S(2n), \ | \det(\Gamma(1) - \text{id}) \neq 0 \}.$$ 

Many authors, GGP included, define the Conley–Zehnder index like this in the first place. There are several advantages of the Robbin–Salamon index, one being that there is no ambiguity whenever $\Gamma$ is a degenerate symplectic path, i.e. $\det(\Gamma(1) - \text{id}) = 0$.

The following proposition is a standard property of the Robbin–Salamon index, which we will use:

**Proposition 5.1.** Let $\Gamma \in S(2n)$ and $\Theta : S^1 \rightarrow \text{Sp}(2n)$ a loop. Then

$$\mu_{RS}(\Theta \cdot \Gamma) = \mu_{RS}(\Gamma) + 2\mu_{\text{Maslov}}(\Theta).$$

In particular, whenever $\Theta$ is homotopic to the constant loop $\text{id}$, it holds

$$\mu_{RS}(\Theta \cdot \Gamma) = \mu_{RS}(\Gamma).$$
Now for $T$-periodic Hamiltonian orbits $x$ of $F: T^*Q \to \mathbb{R}$ we define

$$\mu_{\text{RS}}(x) := \mu_{\text{RS}}(\Gamma_x), \quad \Gamma_x(t) = \Phi_x(t)^{-1} \circ d\varphi^t_H(x(0)) \circ \Phi_x(0), \quad t \in [0, T].$$

where

$$\Phi_x: \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2n} \to x^* T^*Q$$

is any vertically preserving symplectic trivialization of $x$. The argument in [AS06, Lemma 1.3] carries over to the Robbin–Salamon index and shows that the choice of vertically preserving symplectic trivialization does not affect the index, hence $\mu_{\text{RS}}(x)$ as above is well defined.

To apply the results of GGP we will view every 1-periodic Hamiltonian orbit $x = (q, p) \in \mathcal{P}(F)$ as a periodic Reeb/Hamiltonian orbit $x_{1/\sqrt{\alpha}}$ on $\Sigma$ (here $\alpha = F(x)$, see Section 2). We show that this shift does not affect the Robbin–Salamon index:

Let $x \in \mathcal{P}(F)$. For every $s > 0$, the orbit $x_s(t) = (q(st), s \cdot p(st))$ is a $1/s$-periodic Hamiltonian orbit of $F$; cf. Proposition 2.2. Define the map

$$c_s: T^*Q \to T^*Q, \quad (q, p) \mapsto (q, s \cdot p).$$

For $x_s$ we define the vertical trivialisation

$$\Phi_{x_s}: \mathbb{R}/s^{-1} \mathbb{Z} \times \mathbb{R}^{2n} \to (x_s)^* T^*Q, \quad \Phi_{x_s}(t, \cdot) := dc_s(x(st)) \circ \Phi_x(st, \cdot).$$

In particular, $\mu_{\text{RS}}(x_s)$ can be computed as the Robbin–Salamon index of the path $\Gamma_{x_s}$ defined as

$$\Gamma_{x_s}: [0, 1/s] \to \text{Sp}(2n), \quad \Gamma_{x_s}(t) := \Phi_{x_s}(t)^{-1} \circ d\varphi^t_{F}(x_s(0)) \circ \Phi_{x_s}(0).$$

**Proposition 5.2.** For all $s > 0$, $x \in \mathcal{P}(F)$ and $\Gamma_x, \Gamma_{x_s}$ as above, it holds:

$$\Gamma_{x_s}(t) = \Gamma_x(st), \quad \forall t \in \mathbb{R}/s^{-1} \mathbb{Z}.$$

In particular,

$$\mu_{\text{RS}}(x_s) = \mu_{\text{RS}}(x), \quad \forall s > 0.$$

**Proof.** The first identity follows from the definitions and the relation

$$c_{1/s} \circ \varphi^t_F \circ c_s = \varphi^{st}_F.$$

Indeed, $c_{1/s}$ is the inverse of $c_s$ and hence with the chain rule we are able to compute

$$\Gamma_{x_s}(t) = \Phi_{x_s}(t)^{-1} \circ d\varphi^t_{F}(x_s(0)) \circ \Phi_{x_s}(0)$$

$$= (\Phi_x(st)^{-1} \circ dc_s(x(st))^{-1}) \circ d\varphi^t_{F}(x(0)) \circ (dc_s(x(0)) \circ \Phi_x(0))$$

$$= \Phi_x(st)^{-1} \circ d(c_{1/s} \circ \varphi^t_{F} \circ c_s)(x(0)) \circ \Phi_x(0)$$

$$= \Phi_x(st)^{-1} \circ d\varphi^{st}_F(x(0)) \circ \Phi_x(0)$$

$$= \Gamma_x(st).$$
The last assertion follows from the first identity:

$$\mu_{RS}(x) = \mu_{RS}(\Gamma_x) = \mu_{RS}(\Gamma_{x_s}) = \mu_{RS}(x_s).$$

Proposition 5.2 tells us that viewing 1-periodic orbits $y$ of $F$ on $\Sigma$ does not affect the Robbin–Salamon index, in other words, the $p$-shift does not affect the index.

Note that $z(t) := x(t + s)$ is also a $T$-periodic Hamiltonian orbit of $F$ if $x$ is. Similar to the $p$-shift index invariance, we claim that the time-shift does not affect the Robbin–Salamon index. This is the content of the Proposition 5.3 below).

**Proposition 5.3.** For $x(t), z(t) = x(t + s)$, and $s \in [0,T]$ as above, we have

$$\mu_{RS}(\Gamma_z^1) = \mu_{RS}(\Gamma_z^2).$$

In particular

$$\mu_{RS}(x) = \mu_{RS}(z).$$

**Proof.** The idea is to use Proposition 5.1 with the loop

$$\Theta(t) = \Gamma_z^2(t) \cdot \Gamma_z^1(t)^{-1}, \quad t \in \mathbb{R}/T\mathbb{Z}.$$ Observing that $\Theta$ is a loop based at the identity. Moreover

$$\Theta(t) = \left[ \Psi_z^2(t)^{-1} \circ d\varphi_t^F(z(0)) \circ \Psi_z^2(0) \right] \circ \left( \Psi_z^1(0)^{-1} \circ d\varphi_t^F(z(0))^{-1} \circ \Psi_z^1(t)^{-1} \right)$$

$$= \left[ \Phi_x(t)^{-1} \circ d\varphi_t^F(x(t))^{-1} \circ d\varphi_s^F(x(s)) \circ \Phi_x(0) \right] \circ \left[ \Phi_x(s)^{-1} \circ d\varphi_t^F(x(s))^{-1} \circ \Phi_x(t + s) \right]$$

$$= \left[ \Phi_x(t)^{-1} \circ d\varphi_t^F(x(t + s)) \circ \Phi_x(0) \right] \circ \left[ \Phi_x(s)^{-1} \circ d\varphi_t^F(x(s))^{-1} \circ \Phi_x(t + s) \right]$$

$$= \left[ \Phi_x(t)^{-1} \circ d\varphi_t^F(x(0)) \circ \Phi_x(0) \right] \circ \left[ \Phi_x(s)^{-1} \circ d\varphi_t^F(x(s))^{-1} \circ \Phi_x(t + s) \right].$$

Setting $s = 0$ in the expression above gives

$$\Phi_x(t)^{-1} \circ \Phi_x(t) = \text{id},$$

which then readily proves that $\Theta$ is homotopic to the constant loop $\text{id}$. In particular, using Proposition 5.1 we get:

$$\mu_{RS}(\Gamma_z^2) = \mu_{RS}(\Theta \cdot \Gamma_z^1) = \mu_{RS}(\Gamma_z^1).$$

At the same time, the above computation (see square brackets!) also proves $\Gamma_z^2(t) = \Gamma_x(t)$. In particular

$$\mu_{RS}(x) = \mu_{RS}(\Gamma_z^1) = \mu_{RS}(\Gamma_z^2) = \mu_{RS}(\Gamma_x) = \mu_{RS}(x).$$

$\square$
Instead of using the Robbin–Salamon index to extend $\mu_{CZ}$ to the whole $S(2n)$, we could have used upper/lower semicontinuous extensions $\mu_{CZ}^+$, $\mu_{CZ}^-$. For paths $\Gamma \in S^*(2n)$ all three notions agree. On the whole $S(2n)$ the following relations are well known

$$\mu_{CZ}^+(x) + \mu_{CZ}^-(x) = 2 \cdot \mu_{RS}(x)$$
$$\mu_{CZ}^+(x) = \mu_{CZ}^-(x) + \nu(x),$$

where $\nu(x)$ is the nullity i.e. the geometric multiplicity of the eigenvalue 1 of the matrix $\Gamma_x(1)$:

$$\nu(x) = \dim \ker(\Gamma_x(1) - \text{id}) = \dim \ker(d\varphi^1_F(x(0)) - \text{id}),$$

see [AM17, Section 3.1] for more details. Combining these two gives

$$\mu_{RS}(x) = \mu_{CZ}^+(x) - \frac{\nu(x)}{2} = \mu_{CZ}^-(x) + \frac{\nu(x)}{2}.$$

With this we can bound the Robbin–Salamon index of the relevant carrier from above and below:

**Proposition 5.4.** Let $\alpha \in H_i(LQ) \setminus \{0\}$ and $y$ a carrier of $c_\alpha(K)$. Then

$$i - \frac{\nu(y)}{2} \leq \mu_{RS}(y) \leq i + \frac{\nu(y)}{2}.$$

**Proof.** Indeed, we have seen in the proof of Proposition 3.8 that $y$ is the limit of genuine non-degenerate orbits of Conley–Zehnder (and thus Robbin–Salamon) index $i$. Hence, by definition of $\mu_{CZ}^\pm$:

$$\mu_{CZ}^+(y) \geq i, \quad \mu_{CZ}^-(y) \leq i.$$

This together with the identities above the proposition grants the desired bound. \qed

For $T$-periodic Reeb orbits $y$ of $\Sigma$ one defines the nullity of $y$ as follows:

$$\nu(y) = \dim \ker(d\varphi^T_R(y(0)) - \text{id}).$$

**Notation.** We adopt the following notation: if $x \in P(F)$, then $\bar{x}$ denotes the corresponding $p$-shifted periodic Hamiltonian orbit of $F$ on $\Sigma$. The nullity $\nu(\bar{x})$ is understood to be the Reeb nullity as defined above — this is not really abusive due to the next result.

**Proposition 5.5.** Let $x \in P(F)$. Then

$$\mu_{RS}(\bar{x}) = \mu_{RS}(x) \text{ and } \nu(\bar{x}) = \nu(x).$$

In particular, if $x$ is a carrier of $c_\alpha(K)$ as in Proposition 5.4, then

$$i - \frac{\nu(\bar{x})}{2} \leq \mu_{RS}(\bar{x}) \leq i + \frac{\nu(\bar{x})}{2}.$$

Moreover, the Reeb flow action on $\bar{x}$ does not affect the Robbin–Salamon index.
Proof. The first equation holds true because of Proposition 5.2, whereas the nullity equality follows from \( F \) being 2-homogeneous and a standard computation of \( d\varphi_t^F \), see [BO09, Lemma 3.3] for more details. The inequality now follows from Proposition 5.3. The Reeb flow action on \( \bar{y} \) is just a time-shift, which again does not affect the index; cf. Proposition 5.3. \( \square \)

Remark 5.6. The Conley–Zehnder index defined in [GGP08] agrees with the Robbin–Salamon index above; cf. [Gos09, Proposition 9] and [Gos09, formula (39)].

6 Main Theorem

6.1 Preliminaries for the Main Theorem and Examples

Let \( Q \) be a \( n \)-dimensional closed and connected manifold whose first Betti number does not vanish. Then there exists a homotopy class \( \eta \in \pi_1(Q) \) whose image in \( H_1(Q; \mathbb{Z}) \) is of infinite order. In particular, \( \eta \) itself is of infinite order and the conjugacy classes of each iterate \( \eta^m \) are distinct. Denote by \( L_mQ \) the connected component corresponding to the conjugacy class of \( \eta^m \).

Remark 6.1. The choice of \( \eta \) above implies that \( L_kQ \neq L_lQ \) for integers \( k \neq l \). It is not clear whether an arbitrary element \( \beta \in \pi_1(Q) \) of infinite order has the same implication. A similar issue regarding conjugacy classes is addressed in [FS04, page 4]. The same Betti number assumption is made by Allais in [All20]. Note that Allais proves, among other things, a version of Theorem 6.5 in the case of geodesic chords.

We fix once and for all a homotopy class \( \eta \in \pi_1(Q) \) as above.

Definition 6.2. A continuous \( S^1 \)-action \( \phi: S^1 \times Q \to Q \) is called non-trivial with respect to \( \beta \in \pi_1(Q,q_0) \), if \([\phi(\cdot, q_0)] = \beta\).

Note that any two orbits of \( \phi \) have the same free homotopy class since \( Q \) is assumed to be connected.

Convention. Whenever \( Q \) admits a homotopy class \( \eta \) as described above, we (abusively) call an \( S^1 \)-action \( \phi: S^1 \times Q \to Q \) non-trivial if there exists a non-zero integer \( k \) such that \( \phi \) is non-trivial with respect to \( \eta^k \) in the sense of Definition 6.2. Note that up to switching \( \eta \) and \( \eta^{-1} \), we may assume \( k > 0 \).

Let \( \Sigma \) be a starshaped hypersurface as in the previous sections. We define what it means for \( \Sigma \) to be non-degenerate:

Definition 6.3. Let \( y \) be a \( T \)-periodic Reeb orbit of \( \Sigma \) with \( T > 0 \). Then \( y \) is said to be non-degenerate if

\[
\det \left( d(\varphi_T^R)_{\xi_y(0)} - \text{id}_{\xi_y(0)} \right) \neq 0,
\]

where the contact structure \( \xi \) is defined as in Section 2. A starshaped hypersurface \( \Sigma \) is said to be non-degenerate if every non-constant Reeb orbit of \( \Sigma \) is non-degenerate.
**Remark 6.4.** Non-degeneracy of $\Sigma$ ensures that the nullity of every Reeb orbit $y$ is precisely 1. In particular
\[ \nu(x) = \nu(\bar{x}) = 1, \quad \forall x \in \mathcal{P}(F), \]

cf. Proposition 5.5. Controlling the nullity will be crucial in the proof of Proposition 6.9 and thus Theorem 6.5.

**Theorem 6.5.** Let $Q$ be a $n$-dimensional closed connected manifold with $n \geq 2$ whose first Betti number is non-trivial. Assume that $Q$ admits a non-trivial $S^1$-action. Then for any starshaped hypersurface $\Sigma \subseteq T^*Q$, whose closed Reeb orbits have nullity less or equal to $n-1$, it holds:
\[ N_{\Sigma}(T) \geq \frac{1}{\log(2n)} \cdot \log(aT - 1), \quad \forall T > 0 \]
for some $a > 0$.

**Remark 6.6.** The non-trivial action assumption can be weakened to the existence of a section
\[ s: Q \twoheadrightarrow \mathcal{L}_kQ, \]
for some $k \neq 0$. Note that the formula $\phi(t, q) := s_k(q)(t)$ does not necessarily define an $S^1$-action.

In view of Remark 6.4 we obtain the following corollary:

**Corollary 6.7.** Let $Q$ be as in Theorem 6.5. Then for any non-degenerate starshaped hypersurface $\Sigma \subseteq T^*Q$ we have
\[ \liminf_{T \to \infty} \frac{N_{\Sigma}(T)}{\log(T)} > 0. \]

Here is a list of examples that satisfy the hypotheses of Theorem 6.5:

i) any product $S^1 \times M$ with $M$ a closed manifold of dimension at least 1,

ii) any principal $S^1$-bundle $Q$ over a closed connected base manifold $B$ with $\pi_2(B) = 0$ and $\dim B \geq 1$,

iii) any $H$-space $Q$ with $\dim Q \geq 2$ and infinite fundamental group.

The last two examples require some explanation. Ad ii): From the homotopy long exact sequence we deduce that the fiber action $\varphi: S^1 \to Q$ induces an injection on the fundamental groups and therefore defines a non-trivial $S^1$-action — see also [Mac04, Proposition 3.1]. Since principal $S^1$-bundles over $B$ are parametrized by $H^2(B; \mathbb{Z})$ [Kob56], this gives a whole family of examples for any fixed $B$ meeting the above conditions.

Ad iii): The fundamental group $\pi_1(Q, e)$ is abelian since $(Q, \cdot)$ is an $H$-space. Thus $\pi_1(Q, e) \cong H_1(Q; \mathbb{Z})$ by Hurewicz, but the latter is a finitely generated $\mathbb{Z}$-module since $Q$ is a closed manifold, therefore $\pi_1(Q, e)$ contains a $\mathbb{Z}$-copy generated by some homotopy.
class \( \eta \). Let \( \gamma \) be a representative of \( \eta \) and denote by \( \mathcal{L}_1Q \) its loop space component. Associated to \( \gamma \) we have a section

\[ s : Q \rightarrow \mathcal{L}_1Q, \quad s(q) := q \cdot \gamma(\cdot) \]

While this section is not necessarily induced from an \( S^1 \)-action on \( Q \), we will shortly see that its existence is still sufficient to run the proof of Theorem 1 and Corollary 6.7, see Remark 6.6.

Let \( Q \) be as Theorem 6.5. We proceed as in the introduction and define

\[ \alpha_m := (\mathcal{I}_m) \ast \alpha_1 \in H_n(\mathcal{L}_{mk}Q) \setminus \{0\} \]

Recall that we deduced \( \alpha_m \neq 0 \) using the non-trivial section \( s : Q \rightarrow \mathcal{L}_kQ \). Finally define

\[ \tau_m := \sqrt{c_{\alpha_m}(K)}, \quad \tau_m^\pm := \sqrt{c_{\alpha_m}(G^\pm)} \]

The following lemma will be key for the growth control and is based on the pinching machinery developed in Section 4.

**Lemma 6.8.** We have

\[ \tau_m \leq m \cdot \tau_1^- = m \cdot \sqrt{c_{\alpha_1}(L)}, \quad \forall m \in \mathbb{N}. \]

**Proof.** First of all

\[ c_{\alpha_m}(K) \leq c_{\alpha_m}(G^-) = c_{\alpha_m}(L), \]

by Corollary 4.11. Let \( (\sigma^i)_{i \in \mathbb{N}} \) be a sequence of representatives of the singular homology class \( \alpha_1 \in H_n(\mathcal{L}_kQ) \) such that

\[ c_{\alpha_1}(L) = \lim_{i \to \infty} \sup_{v \in \Delta^n} \mathcal{E}(\sigma^i(v)) = \mathcal{E}(q_1) \]

with \( q_1 \) a critical point of \( \mathcal{E} \) inside \( \mathcal{L}_kQ \) — such a sequence and critical point exist by classical LS theory. By definition we know that

\[ \mathcal{I}_m \circ \sigma^i : \Delta^n \rightarrow \mathcal{L}_{mk}Q \]

defines a representative of \( \alpha_m \). In particular

\[ c_{\alpha_m}(L) \leq \lim_{i \to \infty} \sup_{v \in \Delta^n} \mathcal{E}(\mathcal{I}_m \circ \sigma^i(v)). \]

However, for every \( v \in \Delta^n \) we have

\[ \mathcal{E}(\mathcal{I}_m \circ \sigma^i(v)) = \mathcal{E}((\sigma^i(v) \ast \cdots \ast \sigma^i(v))_{m\text{-times}}) = m^2 \cdot \mathcal{E}(\sigma^i(v)), \]

Note that each loop \( s(q) \) is freely homotopic to \( s(e) = \gamma \) since \( Q \) is path connected.

We are suppressing \( d \) from the notation of \( \tau_m \) once again.

Here we are implicitly assuming that \( \sigma^i \) is not a sum of several \( n \)-simplices — otherwise we would need to take the supremum of \( \mathcal{E} \) over the union of the images of all summands. This assumption is purely for notational convenience.
therefore by choice of $\sigma^i$ we obtain

$$c_{\alpha_m}(L) \leq m^2 \cdot \lim \sup_{i \to \infty} \sup_{v \in \Delta^a} \mathcal{E}(\sigma^i(v)) = m^2 \cdot c_{\alpha_1}(L).$$

Taking square roots and putting everything together we see that

$$\tau_m = \sqrt{c_{\alpha_m}(K)} \leq \sqrt{c_{\alpha_m}(L)} \leq m \sqrt{c_{\alpha_1}(L)} = m \cdot \tau_{1},$$

which concludes the proof. \(\square\)

Now denote by $x_m \in \mathcal{L}_{mk}T^*Q$ the carrier of $\tau_m$ and recall the notation $\bar{x}_m$ from Section 5 to denote the corresponding Reeb orbit on $\Sigma$. Since $\alpha_m$ lives in degree $n$, Proposition 5.5 implies

$$n - \frac{\nu(\bar{x}_m)}{2} \leq \mu_{RS}(\bar{x}_m) \leq n + \frac{\nu(\bar{x}_m)}{2}.$$

**Notation.** If $\tilde{x} : [0, T] \to T^*Q$ is $T$-periodic, we define $\tilde{x}^r = \tilde{x} \circ \cdots \circ \tilde{x} : [0, rT] \to T^*Q$ to be the $r$-th iterate of $\tilde{x}$ as in [GGP08].

The homotopy classes of $\bar{x}_m$ are not enough to distinguish the carriers from each other — it could very well happen that the images of $\bar{x}_m^r$ and $\bar{x}_m^{rm}$ are the same for infinitely many $r \in \mathbb{N}$. The next result shows that this scenario does not occur under an additional assumption on the nullity of the Reeb orbits.

**Proposition 6.9.** Assume that all the closed Reeb orbits $y$ on $\Sigma$ satisfy

$$\nu(y) \leq n - 1.$$

Let $r \in \mathbb{N}$, and $x_m, x_{rm}$ corresponding to carriers $x_m, x_{rm}$ of $c_{\alpha_m}(K), c_{\alpha_{rm}}(K)$, respectively. Then the following implication holds true:

$$r \geq 2n \implies \text{im}(\bar{x}_m^r) \neq \text{im}(\bar{x}_{rm}).$$

**Proof.** We show the contrapositive: assume that $\bar{x}_m^r$, $\bar{x}_{rm}$ have the same image. In particular, the moreover part of Proposition 5.5 implies

$$\mu_{RS}(\bar{x}_m^r) = \mu_{RS}(\bar{x}_{rm}).$$

If $\Phi_{\bar{x}_m}$ is a vertically preserving symplectic trivialization of $\bar{x}_m : [0, T] \to T^*Q$, then its periodic extension to $[0, rT]$ defines a vertically preserving symplectic trivialization of $\bar{x}_m^r$. This together with Remark 5.6 tells us that we can apply [GGP08, Corollary 4.4] to obtain

$$\left|\mu_{RS}(\bar{x}_m^r) - r \cdot \mu_{RS}(\bar{x}_m)\right| \leq \frac{n}{2} \cdot (r - 1).$$

Combining this with the previous index inequality and the nullity assumption gives
\[ n + \frac{n-1}{2} \geq \mu_{RS}(\bar{x}_{rm}) = \mu_{RS}(x^r_m) \geq r \cdot \mu_{RS}(\bar{x}_m) - \frac{n}{2}(r-1) \]
\[ \geq r \cdot \left( n - \frac{\nu(\bar{x}_m)}{2} \right) - \frac{n}{2} \]
\[ \geq r \cdot \left( \frac{n}{2} - \frac{n-1}{2} \right) + \frac{n}{2} \]
\[ = \frac{r}{2} + \frac{n}{2}. \]

Therefore \( n - \frac{1}{2} \geq \frac{r}{2} \) and thus \( r < 2n \). \( \square \)

### 6.2 Proof of the Main Theorem

We can finally present the proof of Theorem 6.5.

**Proof of Theorem 6.5.** Let \( T > 0 \) be arbitrary and \( t = T/2 \). Pick \( m' \in \mathbb{N} \) the maximal integer such that
\[ m'\sqrt{c_{\alpha_1}(L)} \leq t. \]

Let \( d > 0 \) be so big that \( \tau_m = \tau_{m,d} \) is defined for all \( m = 1, \ldots, m' \) and such that the corresponding carriers by \( x_m \) are elements in \( \mathcal{P}(F) \) — this is possible thanks to Proposition 4.3 and the fact that the \( x_m \) are not constant because their homotopy class is non-trivial. Note that by Lemma 6.8 we have
\[ \tau_m \leq t, \quad \forall m = 1, \ldots, m'. \]

Fix \( r \in \mathbb{N} \) such that \( r \geq 2n \). We may assume that \( m' >> r \) as \( r \) is fixed throughout the proof. Observe that for any \( s \in \mathbb{N} \), \( x^s_m \) has the same homotopy class as \( x_{sm} \). The corresponding Reeb orbits \( \bar{x}^s_m \) and \( \bar{x}_{sm} \) however, are geometrically distinct if \( s \geq r \), due to Proposition 6.9 and choice of \( r \).

If \( N \in \mathbb{N} \) denotes the unique integer such that
\[ r^{(N-1)} \leq m' \leq r^N, \]
then we have \( N \) pairwise geometrically distinct Reeb orbits, namely
\[ \bar{x}_1, \bar{x}_r, \bar{x}_{r^2}, \ldots, \bar{x}_{r^{N-1}}, \]
again due to choice of \( r \) and Proposition 6.9. By construction
\[ \sqrt{A_F(x_m)} = \tau_m \leq t, \]
in particular the period of the Reeb orbits \( \bar{x}_m \) is less than \( T = 2t \), see Section 2 and therefore
\[ \mathcal{N}_\Sigma(T) \geq N. \]
The only thing left to do is finding a lower bound for $N$. To this end observe that by maximality of $m'$ we have

$$(m' + 1)\sqrt{c_{a_1}(L)} \geq t,$$

thus

$$m' \geq \frac{t}{\sqrt{c_{a_1}(L)}} - 1 = \frac{T}{2\sqrt{c_{a_1}(L)}} - 1.$$ 

Moreover,

$$N \geq \log(m') - \log(r),$$

and therefore

$$N_\Sigma(T) \geq N \geq \frac{1}{\log(r)} \cdot \log \left( \frac{T}{2\sqrt{c_{a_1}(L)}} - 1 \right).$$

Since any $r \geq 2n$ is allowed we can set $r = 2n$ and obtain the desired result.

\[\square\]

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