Quantum Properties of Topological Black Holes

Dietmar Klemm

Dipartimento di Fisica, Università di Trento, Italia

Luciano Vanzo

Dipartimento di Fisica, Università di Trento
and Istituto Nazionale di Fisica Nucleare
Gruppo Collegato di Trento, Italia

Abstract

We examine quantum properties of topological black holes which are asymptotically anti-de Sitter. First, massless scalar fields and Weyl spinors which propagate in the background of an anti-de Sitter black hole are considered in an exactly soluble two-dimensional toy model. The Boulware-, Unruh-, and Hartle-Hawking vacua are defined. The latter results to coincide with the Unruh vacuum due to the boundary conditions necessary in asymptotically adS spacetimes. We show that the Hartle-Hawking vacuum represents a thermal equilibrium state with the temperature found in the Euclidean formulation. The renormalized stress tensor for this quantum state is well-defined everywhere, for any genus and for all solutions which do not have an inner Cauchy horizon, whereas in this last case it diverges on the inner horizon. The four-dimensional case is finally considered, the equilibrium states are discussed and a luminosity formula for the black hole of any genus is obtained. Since spacelike infinity in anti-de Sitter space acts like a mirror, it is pointed out how this would imply information loss in gravitational collapse. The black hole’s mass spectrum according to Bekenstein’s view is discussed and compared to that provided by string theory.

04.20.-q, 04.20.Gz, 04.70.Bw
I. INTRODUCTION

Since the discovery of black holes whose event horizons have nontrivial topology \[1,4\] there has been much research activity in this area. Charged versions of these black holes were presented in \[3,4\], and also rotating generalizations are known \[3,4\]. Moreover, they also exist in dilaton gravity \[7\]. Mann \[8\] and Lemos \[9\] showed that topological black holes can form by gravitational collapse. Finally, from a thermodynamical point of view, they are well–behaved objects obeying the entropy–area law \[3,10\].

In this paper we want to investigate questions such as particle production by the black holes, and their equilibrium state, the Israel–Hartle–Hawking quantum state familiar in the Schwarzschild case \[11–14\]. As the local temperature of the black holes is zero at infinity due to a diverging lapse function (infinite redshift), it is possible that there is no net flux of radiated particles at infinity, and that therefore no Unruh–like states exist.

We will show that this is indeed the case, and that one can define equilibrium thermal states. Part of the interest in doing so is one important feature that distinguishes anti–de Sitter from asymptotically flat black holes, namely the asymptotic behaviour at infinity. The timelike character of the boundary of the space manifests itself in the black hole spacetime by making the exterior static region non–globally hyperbolic. This is remedied by imposing boundary conditions at spacelike infinity \[15,16\] to prevent radiation from escaping out. As a result, the particles emitted by the black hole will be ultimately recaptured back and the black hole evolution will not be complete evaporation. Hence one suspects that the final state should be a thermal equilibrium state, implying a maximum of information loss.

In fact, one can imagine a pure state collapsing in anti–de Sitter space to end into a mixture with the largest entropy available. Of course, this is due to infinity in anti–de Sitter acting like a boundary, which seems a rather artificial situation. However, it is an important matter of principle showing that gravity can imply information loss, since now there is no point for the information to return, apart from the existence of white holes \[17\].

Another point of interest is the discrete quantization of the one–particle energies in anti–de Sitter space. When a black hole is present, this is no longer true and a continuous spectrum appears. The density of states is then blue shifted to infinity near the horizon, so one cannot explain the black hole entropy as resulting from entangled states, without invoking a short distance cut–off \[18\] (see \[19,20\] for recent reviews).

Finally, there is the question of the black hole mass spectrum in the spirit of Bekenstein \[21,22\]. To match with the Euclidean partition function, we obtain a mass spectrum \[M_n = \sigma n^{3/2}\], which seems to be difficult to conciliate with string theory. However, recently the entropy of three– and five–dimensional anti–de Sitter black holes has been explained by string theory \[23\]. Even more interesting is the agreement between entropy and the degeneracy of states in a conformal field theory \[24,25\] realizing the asymptotic symmetries of three–dimensional anti–de Sitter gravity \[26\]. These important new developments may have an impact on the more difficult four–dimensional case.

The rest of the paper is structured as follows:
We begin in section (II) by presenting the geometry of the problem.

In section (III) we consider massless scalar fields and Weyl spinors propagating in a two–
dimensional black hole background. The Boulware–, Unruh–, and Hartle–Hawking vacua are
defined. We then calculate the renormalized stress tensor for the three states and show that
in the Hartle–Hawking case it indeed describes a thermal equilibrium state. Furthermore we will see that, unlike the Schwarzschild black hole, the Unruh– and the Hartle–Hawking states coincide, and that this can be traced back to the boundary conditions which have to be imposed at infinity, as our spacetime fails to be globally hyperbolic. Though the absence of a true Unruh state in its original meaning, we show that a black hole luminosity can be defined in a certain sense, and calculate it using the Bogoljubov coefficients relating the Unruh– and the Boulware vacuum. 

In section (IV) we consider the more complicated four–dimensional case, in which backscattering is present. We argue that the most likely final state is a thermal Israel–Hartle–Hawking state. This implies information loss.

In section (V) we address the question of the mass spectrum from the simple perspective of Bekenstein–Mukhanov arguments. We discuss this spectrum from the Susskind–Horowitz–Polchinski point of view and we point out the difficulty to explain the black hole entropy with it, at least naively. At last, our results will be summarized and discussed in section (VI).

In this paper we shall use the curvature conventions of Hawking–Ellis’ book [28] and employ Planck’s dimensionless units.

II. SPACETIME GEOMETRY

All black hole metrics we shall consider are of the form

\[ ds^2 = -V(r)dt^2 + V^{-1}(r)dr^2 + r^2d\sigma^2, \]

where \( d\sigma^2 \) is the metric of a constant curvature Riemann surface with genus \( g \), denoted \( S_g \). For \( g = 1 \) we shall take this metric to be

\[ d\sigma^2 = dx^2 + |\tau|^2dy^2 + 2\text{Re}\tau dx dy \quad \text{Im} \tau > 0, \]

where \((x, y)\) ranges over the closed unit square \([0, 1] \times [0, 1]\). The element \( \tau \) is known as the Teichmüller parameter of the torus. The lapse \( V(r) \) is given by

\[ V(r) = -1 + \delta_{g,1} - \frac{2\eta}{r} + \frac{r^2}{\ell^2}, \]

where \( \Lambda = -3\ell^{-2} \) is the cosmological constant. Denoting with \( r_+ \) the position of the horizon (the outermost zero of the lapse), the surface gravity of the black holes is

\[ \kappa = \frac{3r_+^2 - \ell^2}{2r_+\ell^2}, \quad \kappa = \frac{3r_+^2}{2\ell^2}, \]

for \( g > 1 \) and \( g = 1 \) respectively. As will be seen, the quantity \( T = \kappa/2\pi \) is the Hawking temperature of the black hole.

For \( g > 1 \), the parameter \( \eta \) is related to the mass of the black hole by \( M = (\eta + \ell/3\sqrt{3})(g - 1) \geq 0 \), and for \( g = 1 \) by \( M = |\text{Im} \tau|/4\pi \geq 0 \) [3]. The positivity condition for the mass means that the solutions have a black hole interpretation for \( \eta > -\ell/3\sqrt{3}(1 - \delta_{g,1}) \), the other
values giving naked singularities or extreme black holes, in general.
The reason for this connection between the ADM mass and the parameter \( \eta \), in the higher genus case, can be understood by finding the zero temperature state. For a toroidal black hole, the temperature vanishes precisely when \( \eta = 0 \), but in the \( g > 1 \) case, the zero temperature state has \( \eta = -\ell/3\sqrt{3} \). This represents an extremal black hole, where the Cauchy inner horizon has merged with the event horizon, and is to be considered as the ground state.

The Euclidean section of the extreme state can be identified to any period, without introducing conical singularities at the origin. We may then compute the Euclidean action of a black hole relative to the extreme state, and define the mass as the thermal energy in the canonical ensemble corresponding to the Hawking temperature. As has been shown in [3], this yields

\[
M = \frac{(g-1)4\pi^3\ell^4 T^3}{27} \left( 1 + \sqrt{1 + 3/4\pi^2\ell^2 T^2} \right) \left( 2 - \frac{3}{2\pi^2\ell^2 T^2} + 2\sqrt{1 + 3/4\pi^2\ell^2 T^2} \right) + \left( \frac{\ell}{3\sqrt{3}} \right) (g-1).
\]

(5)

This mass is an increasing function of \( T \) in the full range \( 0 \leq T \leq \infty \), with a large-\( T \) behaviour \( M \sim T^3 \). Any other choice for the background gives a negative contribution to the entropy of the black hole, because one cannot identify the solution and the background with the same temperature without introducing conical singularities, except for the extreme state. In other words, one cannot shift the mass and leave unaffected the entropy at the same time.

By expressing the mass in (5) in terms of \( \eta \), one finds the relation we started with, \( M = (\eta + \ell/3\sqrt{3})(g-1) \). One can also show that this is precisely the on-shell value of the Hamiltonian, relative to the zero temperature state.

Let us restrict our considerations to the case \( g = 1 \), i.e. to the torus. The generalization to \( g > 1 \) can be done in a straightforward way, if the parameter \( \eta > 0 \) is positive. The fact that \( \eta \) can become negative in the higher genus case, and an inner horizon forms for \( \eta < 0 \), has many implications for the results we will obtain, but will not affect the discussion of the Hawking radiation perceived by an external observer. Instead, global classical and quantum properties, such as global hyperbolicity and the existence of the Hartle-Hawking state, will be affected heavily. Throughout this paper, we will indicate the modifications of our results in the \( g > 1 \) case at the appropriate places. For our subsequent discussion it will be convenient to introduce several other coordinate systems. First of all, define the so-called tortoise coordinate \( r_* \) by

\[
r_* = \int \frac{dr}{V(r)} = \frac{\ell^2}{r_+} \left[ \frac{1}{6} \ln \frac{(r - r_+)^2}{r^2 + rr_+ + r_+^2} + \frac{1}{\sqrt{3}} \arctan \frac{2r + r_+}{r_+\sqrt{3}} - \frac{\pi}{2\sqrt{3}} \right],
\]

(6)

where \( r_+ = (2\eta\ell^2)^{1/3} \) is the location of the event horizon and the integration constant is chosen so that \( r_+ \rightarrow 0 \) for \( r \rightarrow \infty \). Then introduce retarded/advanced null coordinates \( u, v \) according to

\[\text{...}
\]
\[ u = t - r_*, \quad v = t + r*. \] (7)

In these coordinates the line element reads
\[ ds^2 = -V(r)du dv + r^2 d\sigma^2. \] (8)

Finally, Kruskal coordinates \( U, V \) are defined by
\[ U = -\exp(-\kappa u), \quad V = \exp(\kappa v), \] (9)

with \( \kappa \) given by (4). In Kruskal coordinates we obtain for the metric
\[ ds^2 = \frac{V(r)}{\kappa^2 U V} dU dV + r^2 d\sigma^2, \] (10)

which is regular on the horizon. Figure 1 shows the conformal diagram of the spacetime.

It is an important fact that the solutions so far discussed may appear as a result of gravitational collapse of a configuration of pressureless dust [8]. The exterior metric matches with an interior Robertson–Walker spacetime, and complete collapse occurs in a finite amount of co–moving time. The fact that the exterior metric is given by (1) at all times results from a generalization of Birkhoff’s theorem [27].

III. THE TWO–DIMENSIONAL CASE

In our two–dimensional toy model we forget the part \( r^2 d\sigma^2 \) in (1), i. e. we limit the calculations to the toroidal analogue of the spherical s–wave sector. This restriction “does not throw out the baby with the water”, as expressed A. Strominger [29]. Indeed, we shall see that most of the essential features are present in this model. In two dimensions, there is no backscattering, the metric is conformally flat, and the Klein–Gordon or the Weyl equation can be solved exactly.

A. Propagation of Massless Scalar Particles

Since the metric is conformally flat, the Klein–Gordon equation for a massless field \( \phi \) is just the same as in Minkowski space, namely
\[ \frac{\partial^2 \phi}{\partial u \partial v} = 0, \quad \text{or} \quad \frac{\partial^2 \phi}{\partial U \partial V} = 0, \] (11)

which has the general solution
\[ \phi(u, v) = f(u) + g(v) \]
\[ \text{or} \quad \phi(U, V) = F(U) + G(V). \] (12)

\( f, g, F, G \) are arbitrary functions. Now in the Unruh model [30] of an evaporating Schwarzschild black hole, one defines e. g. a complete set of Boulware modes by \( \varphi^B_m \propto \)
exp(−iωv) and \( \varphi^B_{\text{out}} \propto \exp(−iωu) \), which means that we have neither particles incoming from past infinity nor particles outgoing to future infinity in the corresponding vacuum state. Analogously, Hartle–Hawking (HH) modes are defined by \( \varphi^{HH}_{\text{in}} \propto \exp(−iωU) \) and \( \varphi^{HH}_{\text{out}} \propto \exp(−iωV) \), i.e. the Hartle–Hawking vacuum does not contain Kruskal particles emerging from the white hole horizon \( H^- \) or crossing the future horizon \( H^+ \). Finally, Unruh modes are given by \( \varphi^{U}_{\text{in}} \propto \exp(−iωU) \) and \( \hat{\varphi}^{U}_{\text{in}} \propto \exp(−iωv) \). Now this procedure is no more possible in the case of topological black holes. The Penrose diagram in figure [4] shows that the spacetime is not globally hyperbolic; in order to obtain a well-posed Cauchy problem we have to impose boundary conditions at infinity. There are three natural choices of boundary conditions at \( r = \infty \) (which is equivalent to \( u = v \) or \( UV = −1 \)):

\[
\begin{align*}
\phi|_{r=\infty} &= 0 \quad \text{(Dirichlet)} \\
\nabla_n \phi|_{r=\infty} &= 0 \quad \text{(Neumann)} \\
[K(r,t)\phi + \nabla_n \phi]|_{r=\infty} &= 0 \quad \text{(Robin)}
\end{align*}
\] (13)

Here \( n \) denotes the unit normal to the surface \( r = \infty \), and \( K \) is a function which will be given below. Obviously the above-defined functions for the Schwarzschild case do not fulfill any of the boundary conditions (13). Therefore we define the Boulware modes to be

\[
\varphi^B_\omega(u, v) = \frac{1}{\sqrt{4\pi\omega}}(\exp(-iωu) \pm \exp(-iωv)),
\] (14)

where the minus (plus) sign corresponds to Dirichlet (Neumann) boundary conditions. For Robin boundary conditions we take

\[
\varphi^B_\omega R = \frac{1}{1 + \omega^2 k - 2} \left( \varphi^B_\omega D + i\omega k^{-1} \varphi^B_\omega N \right)
\] (15)

with \( k \in \mathbb{R} \). Here the function \( K \) is simply a constant which equals \(-k\). The limit cases \( k = 0 \) (\( k \to \infty \)) represent Neumann (Dirichlet) boundary conditions. The modes (14), (15) form a complete orthonormal set with respect to the Klein–Gordon scalar product

\[
(\alpha, \beta)_{KG} = i \int_{\Sigma} (\bar{\alpha} \nabla a \beta - \beta \nabla a \bar{\alpha}) n^a dx,
\] (16)

where \( \Sigma \) is a spacelike hypersurface with unit normal \( n^a \), and \( dx \) is the induced volume element on \( \Sigma \). Thus we have

\[
(\varphi^B_\omega, \varphi^B_{\omega'}) = \delta(\omega - \omega')
\] (17)

We can then expand the field in terms of the Boulware modes

\[
\phi(u, v) = \int_0^\infty d\omega (b_\omega \varphi^B_\omega + b^\dagger_\omega \bar{\varphi}^B_\omega).
\] (18)

The Boulware vacuum \(|B\rangle\) is now determined by

\[
b_\omega |B\rangle = 0.
\] (19)
We define Unruh modes according to

\[
\varphi^U_\omega(U, V) = \frac{1}{\sqrt{4\pi\omega}}(\exp(-i\omega U) \pm \exp(i\omega/V)),
\]

(20)

the +/- again denoting Neumann/Dirichlet boundary conditions respectively. For Robin boundary conditions we have

\[
\varphi^U_\omega R \left( \frac{1}{1 + \omega^2k^{-2}} \left( \varphi^U_\omega D + i\omega k^{-1} \varphi^U_\omega N \right) \right)
\]

(21)

with \( k \in \mathbb{R} \) and \( K(U) = kU \). The modes (20), (21) resemble the solutions of the moving mirror problem (see e. g. [31,32]). Indeed, the infinity \( UV = -1 \) can be viewed as a moving mirror \( V = V(U) = -1/U \), and a positive frequency wave \( e^{-i\omega U} \) outgoing from the past horizon is reflected by the mirror and becomes a wave \( e^{i\omega/V} \) travelling towards \( H^+ \).

On the past horizon \( V = 0 \) only the first summand of (21) survives, since the second oscillates infinitely fast, and hence does not give any contribution if one constructs wave packets. The functions (20), (21) represent again a complete orthogonal system, i. e. they satisfy

\[
(\varphi^U_\omega, \varphi^U_{\omega'}) = \delta(\omega - \omega').
\]

(22)

One can also expand the field in terms of Unruh modes

\[
\phi(U, V) = \int_0^\infty d\omega (a_\omega \varphi^U_\omega + a^*_\omega \varphi^U_{\omega'}).
\]

(23)

Finally we define the Unruh vacuum \(|U\rangle\) by

\[
a_\omega |U\rangle = 0.
\]

(24)

Obviously \(|U\rangle\) does not contain particles emerging from the past horizon. Last, Hartle–Hawking modes are given by

\[
\varphi^{HH}_\omega(U, V) = \frac{1}{\sqrt{4\pi\omega}}(\exp(-i\omega V) \pm \exp(i\omega/U)).
\]

(25)

For Robin boundary conditions we take an expression corresponding to (21). If one constructs wave packets, only the first term of (25) gives a contribution on the future horizon \( U = 0 \). The Hartle–Hawking vacuum \(|H\rangle\) is defined by the usual procedure, analogous to the two preceding cases. \(|H\rangle\) does not contain particles crossing the future horizon, but there are also no particles emerging from the white hole, as we shall see later by investigating the renormalized stress tensor.

Note that in contrast to two–dimensional adS space, where the spectrum is discrete [16,33], in our case we have a continuous frequency spectrum. This will also hold true in four dimensions (see section [17]). It should not be surprising, because our spacetime is only asymptotically locally adS, and one boundary condition at \( UV = -1 \) clearly does not cause a discrete spectrum.
B. Weyl Fermions

Let us now devote attention to Weyl fermions propagating in the black hole background. This case presents no particular difficulties, but there are also significant differences. The Weyl equation in a two–dimensional curved spacetime enjoys the conformal invariance on the same footing of the massless Klein-Gordon equation. We introduce a zweibein field \( e_i^a(x) \), deserving the latin indices \( i, j, k, \ldots \) to denote local Lorentz tensors and \( a, b, c, \ldots \) to denote coordinate tensor components, and the two–dimensional Dirac matrices \( \rho^i \). Then given a pair \((g_{ab}, \psi)\) consisting of a metric tensor and a Weyl spinor satisfying the Weyl equation

\[
\rho^i e_i^a \nabla_a \psi = 0,
\]

the transformed pair \( \tilde{\psi} = \lambda^{-1/2} \psi, \tilde{g}_{ab} = \lambda^2 g_{ab} \), will also satisfy the Weyl equation in the metric \( \tilde{g}_{ab} \). In both coordinate systems we have employed so far, the retarded/advanced system \((u, v)\) and the Kruskal system \((U, V)\), the metric is conformal to a flat one of the form \( du \, dv \) or \( dU \, dV \). The Weyl equation for such flat metrics splits into a pair of decoupled equations for the positive/negative chirality spinors, \( \psi_\pm \), which read

\[
\partial_u \psi_+ = \partial_v \psi_- = 0 \quad (u, v) \text{– system},
\]

with identical equations in the \((U, V)\)-system. In string terminology, \( \psi_+ \) is left moving and \( \psi_- \) is right moving. The spinor in spacetime can now be obtained multiplying the flat \( \psi_\pm \) with the respective conformal factors.

Normalizable, positive frequency solutions of (27) in the interval \( r^* \in (-\infty, 0) \) are given by

\[
\psi_+ = e^{-i\omega_v \sqrt{2\pi}}, \quad \psi_- = e^{-i\omega_u \sqrt{2\pi}}.
\]

There is no question to impose Dirichlet or other boundary conditions here, because the Weyl equations fix a unique form to the positive frequency modes. Indeed, that Dirichlet boundary conditions on the Dirac equation lead to inconsistencies has long been known, as a consequence of the first order character of the equation. Note however that for the given solutions, the component of the conserved Dirac current along the normal to the boundary at infinity, namely \( j = \bar{\psi} \rho^1 \psi \) evaluated at \( r^*_\ast = 0 \), vanishes identically. This is a much more weaker boundary condition than Dirichlet or Neumann, allowing to find non–trivial solutions. In the Kruskal frame, we take advantage from the fact that both horizons are Cauchy surfaces for Weyl spinors in anti-de Sitter space. The normalization measure is \( (\kappa U)^{-1} dU \) along the past sheet of the horizon, and \( (\kappa V^{-1}) dV \) along the future sheet, so normalizable solutions are

\[
\psi_+ = \frac{(\kappa |V|)^{1/2} e^{-i\omega V}}{\sqrt{2\pi}}, \quad \psi_- = \frac{(\kappa |U|)^{1/2} e^{-i\omega U}}{\sqrt{2\pi}}.
\]

We shall use these to compute the relevant Bogoljubov coefficients in the next but one section.
C. Stress Tensor

We now wish to determine the expectation value of the stress tensor

\[ T_{ab} = \phi_a \phi_b - \frac{1}{2} g_{ab} \phi_c \phi^c \]  

(30)

for the Unruh-, the Hartle–Hawking–, and the Boulware vacuum state (see [34] for a readable account). Now, it is well known [35] that (30) is mathematically ill-defined, involving products of two distributions on spacetime. Therefore some kind of regularization procedure is needed, e.g. a point-splitting method. We follow here the lines of Davies and Fulling [37], who presented in detail the renormalization theory of the stress tensor of a two-dimensional massless scalar field, including boundary conditions. They consider the line element

\[ ds^2 = c(u, v) dudv, \]  

(31)

where \( c(u, v) \) is the conformal factor. For mode functions of the form

\[ \phi_\omega = (4\pi\omega)^{-\frac{1}{2}} (e^{-i\omega u} \pm e^{-i\omega v}) \]  

(32)

the renormalized vacuum expectation value of the energy–momentum tensor is given by

\[ \langle T_{ab} \rangle = \theta_{ab} - (48\pi)^{-1} R g_{ab}, \]  

(33)

where \( R \) is the scalar curvature of the manifold, which gives rise to the conformal anomaly, and \( \theta_{ab} \) depends on the conformal factor \( c \) according to

\[ \theta_{aa} = - (12\pi)^{-1} c^{-2} \left[ \frac{3}{4} (\partial_a c)^2 - \frac{1}{2} c \partial_a^2 c \right], \quad (a = u, v) \]

\[ \theta_{uv} = \theta_{vu} = 0. \]  

(34)

In the following we omit the brackets for the expectation value of \( T_{ab} \). Using (33), one obtains for the renormalized stress tensor in the Boulware state

\[ T^B_{uu} = \frac{RV(r)}{96\pi}, \]

\[ T^B_{vv} = \frac{\eta}{16\pi r} \left( \frac{\eta}{r^3} - \frac{2}{\ell^2} \right), \]

\[ T^B_{uv} = \frac{-RV(r)}{96\pi UV}. \]  

(35)

where \( R = -d^2 V/dr^2 \).

In order to examine whether \( T_{ab} \) is well-defined everywhere or not, we have to express it in the global nonsingular coordinate system \((U, V)\). The components then read

\[ T^B_{UU} = \frac{1}{\kappa^2 U^2} T^B_{uu}, \]

\[ T^B_{VV} = \frac{1}{\kappa^2 V^2} T^B_{vv}, \]

\[ T^B_{UV} = -\frac{RV(r)}{96\pi UV}. \]  

(36)
Thus $T_{UU}^B$ diverges on the future horizon, whereas $T_{VV}^B$ diverges on the past horizon.

In coordinates $r, t$ we have

$$
T_t^{Bt} = -\frac{2}{V(r)}(T_{uu}^B + T_{uw}^B),
$$
$$
T_r^{Br} = \frac{2}{V(r)}(T_{uu}^B - T_{uw}^B),
$$
$$
T_r^{Bt} = T_t^{Br} = 0,
$$

which behaves on the horizon as

$$
T_a^{Bb}|_{r \approx r_+} = -\frac{\eta}{16\pi r_+^2(r - r_+)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
$$

This behaviour is similar to that of the stress tensor in the Boulware state for the Schwarzschild spacetime [38]. There it diverges like $(r - r_+)^{-2}$ in four dimensions, our exponent $-1$ instead of $-2$ stems from the reduction to two dimensions.

The energy density $\rho$ in the Boulware state, measured by a Killing observer with four-velocity $v^a = V(r)^{-1/2}\partial_t$ (i. e. at constant $r$), is given for large $r$ by

$$
\rho = T_{ab} v^a v^b = V(r)^{-1} T_{tt}^B = -\frac{r^2}{24\pi V(r)\ell^4} \approx -\frac{1}{24\pi \ell^2}.
$$

This represents the Casimir energy of the state, resulting from the boundary conditions at infinity. The ”would be” divergence for $r \to \infty$ (i. e. on the boundary) is a well-known behaviour of the Casimir energy, but here it is cancelled by the growing of the lapse at infinity. Later we shall see that in the HH state there is an additional term in the expression for $\rho$, coming from black hole radiation.

Let us now focus our attention on the Hartle–Hawking state. First of all, note that the formulas given in [37] for the renormalized stress tensor are only valid for time-independent boundary conditions with respect to $T := \frac{1}{2}(V + U)$. This is not the case here. Therefore we must introduce new coordinates $\bar{V} = V$, $\bar{U} = -1/U$. In these coordinates the boundary $UU = -1$ is located at $\bar{U} = \bar{V}$, i. e. at $\bar{X} := \frac{1}{2}(\bar{V} - \bar{U}) = 0$. Reexpressing $T_{ab}^{HH}$ in Kruskal coordinates after the calculation, one gets

$$
T_{VV}^{HH} = -\frac{1}{48\pi V^2} w(r),
$$
$$
T_{UU}^{HH} = -\frac{1}{48\pi U^2} w(r),
$$
$$
T_{UV}^{HH} = -\frac{R g_{UV}}{48\pi},
$$

where we have defined

$$
w(r) := -1 + \frac{V'(r)^2}{4\kappa^2} - \frac{V(r)V''(r)}{2\kappa^2}.
$$

Now near the horizon we have $w(r) \approx -6e^{\pi/\sqrt{3}\ell^2} V^2$, cancelling the divergence of the prefactor, therefore $T_{HH}^{HH}$ is well-defined everywhere.
Here a new feature arises in the higher genus case for \( \eta < 0 \), i.e. when an inner horizon forms. Using the \( g > 1 \) expression for \( V(r) \) in (41), and transforming the stress tensor to coordinates in which the metric is regular at the inner horizon \( r_- \), one finds a divergence of the stress tensor at \( r = r_- \). Having adjusted the quantum state of the scalar field so that the stress tensor is finite at the outer horizon, it diverges at the inner horizon. This behaviour is identical to what has been found in two dimensions for a Reissner–Nordström black hole [36], and is related to an infinite frequency blue-shift occurring on the Cauchy horizon. This suggests that there is no Hartle–Hawking equilibrium state, and that strong back reaction effects take place near the inner horizon. In two dimensions, however, a semiclassical calculation shows that this last sentence is not true, in the sense that the singularity appearing at the horizon is very mild [36].

Let us now return to the toroidal black hole. We observe that the \( V \)-component of the energy current four–vector \( J^a \), measured by an "observer" with four–velocity \( v^a = \partial_U \) on the past–horizon, namely

\[
J^V = -T_b^{HH} V v^b = -g^{VV} T_b^{HH}
\]

vanishes on \( H_- \). Therefore \( |H \rangle \) does not contain Kruskal particles emerging from the white hole, as already pointed out in the previous section.

For large \( r \), the energy density \( \rho \), measured by a Killing observer with four–velocity \( v^a = V(r)^{-1/2}\partial_t \), is now given by

\[
\rho = V(r)^{-1} T_t^{HH} = \frac{\kappa^2}{24\pi V(r)} - \frac{r^2}{24\pi V(r)\ell^4}.
\]

Again we meet the Casimir energy \(-1/24\pi\ell^2\), but now there is an additional term. It is connected to the thermal radiance of the black hole, as can be readily seen by calculating the radiated energy

\[
E = \frac{1}{2\pi} \cdot 2 \cdot \int_{0}^{\infty} \omega d\omega \frac{\omega d\omega}{e^{2\pi\omega/\kappa} - 1},
\]

which yields exactly \( \kappa^2/24\pi \). (\( 1/2\pi \) is the density of states in one spatial dimension, and the additional factor 2 has to be included, because we have left– and right–moving waves). The factor \( V(r)^{-1} \) is the usual Tolman red–shift of the local temperature, making the energy density vanishingly small at infinity.

We now want to show that \( |H \rangle \) describes a thermal equilibrium state. To this aim, we calculate the net null flux through a surface \( r = \text{const.} \). The energy current four–vector \( J^a \), measured by an observer at constant \( r \) with four–velocity \( v^a \), is represented by

\[
J_a = -T_a^{HH} v^b.
\]

Now the net energy flux through the surface \( r = \text{const.} \), i.e. \( UV = \text{const.} < 0 \), is given by the integral

\[
\int_{UV=\text{const.}} J_a n^a dS,
\]

11
where \( n^a \) denotes the unit normal to the surface, and \( dS \) is the induced "volume" element. One easily verifies that for \( UV = \text{const.} \) the integrand \( J_a n^a \) is zero, hence the incoming null flux through the constant \( r \) surface equals the outgoing, and \( |H\rangle \) represents indeed a thermal equilibrium state with the Hawking temperature.

For the Unruh vacuum we find a stress tensor identical to that of the Hartle–Hawking state. This suggests that in our case these two states coincide. Indeed, the Unruh modes (24) can be obtained from the HH modes (25) by interchanging \( U \) and \( V \). This is an isometry of the metric (1), because it leaves \( r \) invariant and maps \( t \) to \( -t \). Due to the reflective boundary conditions we were compelled to impose (for all three choices in (13) the component of the Klein–Gordon current normal to the boundary at infinity vanishes), no Unruh state like in the Schwarzschild case can be defined, as all the Hawking radiation emitted by the black hole is reflected at infinity and travels back to the future horizon. Therefore necessarily a thermal equilibrium state results, and \( |U\rangle \) coincides with \( |H\rangle \).

D. Black Hole Temperature and Luminosity

First of all, let us remark that in principle the definition of a luminosity makes only sense if there exists an Unruh state. As in the case under consideration the Unruh state also describes thermal equilibrium, i.e. the black hole absorbs the same amount of radiation as it emits, the net luminosity is zero. Nevertheless we can calculate the emitted radiation (which, of course, is reflected at the boundary), and call this the luminosity of the black hole. To this end, we have to find the Bogoljubov transformation relating Unruh modes to Boulware modes, i.e.

\[
\varphi^B_{\omega}(u,v) = \int_0^\infty (\alpha_{\omega\omega'} \varphi^V_{\omega'}(U,V) + \beta_{\omega\omega'} \varphi^V_{\omega'}(U,V)) d\omega'.
\]

A nice calculation yields

\[
\alpha_{\omega\omega'} = -\frac{i}{2\pi} \frac{\omega' - i\omega/\kappa}{\omega \omega'} \Gamma \left( 1 + i \frac{\omega}{\kappa} \right) \exp \left( \frac{\pi\omega}{2\kappa} \right),
\]

\[
\beta_{\omega\omega'} = \frac{i}{2\pi} \frac{\omega' - i\omega/\kappa}{\omega \omega'} \Gamma \left( 1 + i \frac{\omega}{\kappa} \right) \exp \left(-\frac{\pi\omega}{2\kappa} \right).
\]

Inserting (47) into equation (18), we obtain the relation between Boulware and Unruh operators

\[
b_{\omega'} = \int_0^\infty d\omega [\bar{\alpha}_{\omega\omega'} a_{\omega} - \bar{\beta}_{\omega\omega'} a_{\omega}].
\]

The matrix element \( \langle U | b_{\omega}^\dagger b_{\omega'} | U \rangle \) can now be calculated, and we obtain

\[
\langle U | b_{\omega}^\dagger b_{\omega'} | U \rangle = \int_0^\infty \beta_{\omega k} \bar{\beta}_{\omega' k} dk = \frac{\delta(\omega - \omega')}{\exp \left( \frac{2\pi\omega}{\kappa} \right) - 1}.
\]
\[
d_{n_{\omega}} = \frac{1}{2\pi} \frac{d\omega}{\exp\left(\frac{2\pi \omega}{\kappa}\right) - 1}
\]

zero rest mass particles flowing near infinity per unit time in the frequency range between \(\omega\) and \(\omega + d\omega\). As already pointed out, the radiation is reflected at infinity, so the net particle flux is zero. From (52) we also infer that the radiation temperature of the black hole is \(T = \kappa/2\pi\), in accordance with the temperature found in the Euclidean formulation [3,10].

Note that, in order to obtain the "luminosity", it would have been sufficient to calculate the Bogoljubov coefficients relating an outgoing mode \(\exp(-i\omega U)\) to the modes \(\exp(-i\omega u)\). This yields the same Bogoljubov coefficients as above, which can be understood as follows: The monochromatic components \(\exp(\pm i\omega u)\) in the expansion of the outgoing mode \(\exp(-i\omega U)\) are reflected at the boundary, becoming ingoing modes \(\exp(\pm i\omega v)\). These give the expansion of the reflected wave in Kruskal coordinates, namely of \(\exp(i\omega/V)\).

At this point one may ask how it is possible to assign the outgoing flux only to the modes \(\exp(-i\omega U)\), and the reflected one only to \(\exp(i\omega/V)\). This is a legitimate question, in view of the stress tensor being quadratic in \(\phi\). However, the energy current four–vector \(J^u\) measured by an observer with four–velocity \(V(r)^{-1/2}\partial_t\), is given by

\[
J^u = -g^{uv}V(r)^{-1/2}(T_{uv} + T_{vu}),
\]

and one can show that only the modes \(\exp(-i\omega U)\) contribute to \(T_{uu}\), and only the \(\exp(i\omega/V)\) contribute to \(T_{vv}\) [37]. As \(T_{uv}\) is completely fixed by the manifold via the conformal anomaly, it is independent of the modes. Therefore the outgoing null flux \(J^v\) is determined by the modes \(\exp(-i\omega U)\) only, whereas the reflected (ingoing) flux is determined exclusively by the \(\exp(-i\omega/V)\).

For the Weyl fermions the relevant Bogoljubov coefficient is

\[
\beta_{\omega\omega'} = e^{i\pi/4} \frac{1}{2\pi \sqrt{\kappa \omega'}} e^{-\pi\omega/\kappa} e^{i\omega/\kappa} \Gamma\left(\frac{1}{2} + \frac{i\omega}{\kappa}\right),
\]

from which one obtains the Fermi–Dirac distribution

\[
\int_0^\infty \beta_{\omega\sigma} \beta_{\omega'\sigma} d\sigma = \frac{\delta(\omega - \omega')}{e^{2\pi\omega/\kappa} + 1}.
\]

**IV. GENERALIZATION TO FOUR DIMENSIONS**

We now consider the full metric (1) and recall some results about the family of black holes it describes. We will not specify a particular horizon metric for \(g > 1\) (which depends on the \(6g - 6\) moduli of a Riemann surface), since its precise form is not important in what follows. For \(g = 1\) the moduli space of the torus is \(H^+\text{/}\text{SL}(2, \mathbb{Z})\) (\(H^+\) denotes the upper complex half plane), and the torus flat metric is given by Eq. (2) in terms of its Teichmüller parameter \(\tau \in H^+\). Any two such parameters related by \(\text{SL}(2, \mathbb{Z})\) fractional linear transformations describe conformally equivalent tori.
The metric (1) can be continued to imaginary values of the Killing time \((T = it)\) as a Riemannian non–singular metric everywhere. This metric takes the form

\[
ds^2 = V(r)dT^2 + V^{-1}(r)dr^2 + r^2d\sigma^2,
\]

where we recall that \(\Lambda = -3/\ell^2\) and

\[
V(r) = -1 + \delta_{g,1} - \frac{2\eta}{r} + \frac{r^2}{\ell^2},
\]

and \(r > r_+\) is required for this to be of positive signature. Close to the horizon \(V(r) = 2\kappa(r - r_+)\), where \(\kappa\) is the surface gravity, and the near horizon geometry is described in terms of proper distance, \(s^2 = 2(r - r_+)/\kappa\), by the metric

\[
ds^2 = \kappa^2 s^2 dT^2 + ds^2 + r_+^2 d\sigma^2
\]

Regularity of the metric then requires the period of \(T\) to be \(\beta_+ = 2\pi\kappa^{-1}\), so this fixes the black hole’s temperature. The zero temperature state has \(\kappa = 0\) and is a naked singularity with parameter \(\eta = 0\) for \(g = 1\), and an extremal black hole with parameter \(\eta = -\ell/3\sqrt{3}\) for \(g > 1\), while the positive temperature states above it have positive mass. The member of the family with mass \(M = \ell(g - 1)/\sqrt{27}\) has parameter \(\eta = 0\) and is the quotient of anti–de Sitter space by a discrete subgroup of its isometry group \([1]\), in particular it is a space of constant curvature.

Along with the metric, one can analytically continue the wave equation. This then gives an elliptic operator with non–singular coefficients and positive spectrum. The Schwinger function is the symmetric two–point function which decays to zero at infinity, it is regular at the origin and solves the Euclidean wave equation. As for the metric, regularity at the origin (the horizon in the Lorentzian sector) demands that the Schwinger function be periodic in \(T\) with period \(\beta_+\). The analytically continued function in real time will then be periodic in imaginary time and regular all over the event horizon, but for \(\eta < 0\), the function can be extended only up to the inner Cauchy horizon. The quantum state to which it corresponds is the equilibrium Israel–Hartle–Hawking state, and describes a topological black hole in thermal equilibrium with black body radiation at the Hawking temperature. Later we shall discuss this state from a proper quantum field theory approach.

The contribution of the black hole to the partition function is \(\ln Z = -I_E\), the on–shell value of the Euclidean action of the black hole \([39]\). The Euclidean action can also be evaluated off–shell \((\beta \neq \beta_+)\), relative to the zero temperature ground state, and is \([3]\)

\[
I_E = \beta M - \frac{A}{4},
\]

where \(A\) is the area of the event horizon. Since \(M\) and \(\beta\) are here independent variables, this quickly leads to an entropy \(S = A/4\) which, when expressed as a function of the mass, has the large mass behaviour \(S \simeq CM^{2/3}\). This means that the density of states grows as \(\exp(CM^{2/3})\), so the partition function will converge. This is not a special feature of topological black holes, but also holds for the genus-0 anti–de Sitter black hole \([10]\) and is related to a negative cosmological constant rather than to topology.
A. The Israel–Hartle–Hawking State

To discuss black hole emission, we shall consider a scalar field obeying the conformally invariant Klein-Gordon equation

\[ \frac{1}{\sqrt{-g}} \partial_a (g^{ab} \sqrt{-g} \partial_b) \phi - \frac{1}{6} R \phi = 0. \] (60)

This equation can be separated into the following eigenvalue equations: Setting \( \phi = r^{-1} F_j(t, r) u_j(x) \) and \( \partial_* = \partial_{r_*} \), we have a two-dimensional wave equation for \( F_j \), with a potential barrier \( P_\lambda(r) \),

\[ \partial_t^2 F_j - \partial_*^2 F + P_\lambda(r) F_j = 0, \] (61)

together with the eigenvalue equation for the Laplacian on \( S_g \)

\[ \Delta u_j = -\lambda_j^2 u_j, \] (62)

where the potential barrier is given by

\[ P_\lambda(r) = V(r) \left( \frac{\lambda^2}{r^2} + \frac{V'(r)}{r} + \frac{R}{6} \right) = V(r) \left( \frac{\lambda^2}{r^2} + \frac{2\eta}{r^3} \right), \] (63)

and the scalar curvature is \( R = -12\ell^{-2} \). This term precisely cancels the divergent (as \( \propto r^2 \) at infinity) anti–de Sitter gravitational potential, which is why conformal scalar emission will be greater than that of minimally coupled scalars. For these the barrier is parabolic at infinity, with behaviour \( P_\lambda(r) \simeq 2\ell^{-4} r^2 \), and the modes behave like Bessel functions at infinity.

In all cases the potential vanishes at the horizon and approaches the asymptotic value \( \lambda^2/\ell^2 \) for \( r \to \infty \). For \( g = 1 \) or \( g \geq 1 \) and \( \eta > 0 \) we have also a local maximum outside the horizon. The behaviour of \( P_\lambda(r) \) is shown in figure 2 for the torus or a \( g > 1 \) black hole with \( \eta > 0 \), and in figure 3 for \( g > 1 \) and \( \eta < 0 \). Again, new features arise when \( \eta < 0 \), i. e. when an inner horizon forms. For sufficiently large eigenvalues \( \lambda^2 \), there is a potential well in between the two horizons, causing amplification for waves entering from the outer horizon (c. f. Chandrasekhar’s monography [41]). For \( g > 1 \) and \( \eta = 0 \), the potential is zero at the horizon, and then monotonically increases to reach the asymptotic value \( \lambda^2/\ell^2 \).

Let us consider now the eigenvalue equation (62). On a general Riemann surface there are comparatively little informations on the eigenvalues \( \lambda_j \), except that they are finitely degenerate and form an unbounded increasing sequence. For the torus we have instead an exact formula for all the eigenvalues and, at least for \( \tau = i \) (symmetric torus), for the respective degeneracies. Indeed, solutions of (62) must be automorphic functions under the identification group

\[ x \simeq x + n \quad n \in \mathbb{Z} \] (64)
\[ y \simeq y + m \quad m \in \mathbb{Z}. \] (65)

This fixes the normalized eigenfunctions to be (the torus area element is \( dS = \text{Im} \tau dx \, dy \))
\[ u_{nm}(x, y) = \sqrt{\text{Im} \tau}^{-1} \exp(2\pi i(nx + my)), \]  

(66)

and therefore the eigenvalues are

\[ \lambda_{nm}^2 = (2\pi)^2 n^2 + \left(\frac{2\pi}{\text{Im} \tau}\right)^2 (m - n \text{Re} \tau)^2. \]  

(67)

For arbitrary values of the Teichmüller parameter \( \tau \) the degeneracy \( g_\lambda \) is difficult (if not impossible) to calculate. For \( \tau = i \) however, \( g_\lambda \) equals the number of representations of \( \lambda^2/4\pi^2 \) in the form

\[ \frac{\lambda_{nm}^2}{4\pi^2} = n^2 + m^2, \]  

(68)

which is given by \[ \text{[42]\text{]}\]  

\[ g_\lambda = 4 \sum d | \frac{\lambda_{nm}^2}{4\pi^2} \chi(d), \]  

(69)

where one has to sum over all divisors \( d \) of \( \lambda^2/4\pi^2 \), and \( \chi(d) \) is defined by

\[ \chi(d) = \begin{cases} 0 & 2 \mid d \\ (-1)^{\frac{1}{2}(d-1)} & 2 \not| d. \end{cases} \]  

(70)

Knowing the eigenvalues, we can write explicitly the potential barrier felt by a mode in a toroidal black hole, which is

\[ P_\lambda(r) = \left( -\frac{2\eta}{r} \right) \left[ \frac{\lambda_{nm}^2}{r^2} + \frac{2\eta}{r^3} \right]. \]  

(71)

The potential is zero at the horizon. (It falls off exponentially in the tortoise coordinate \( r_* \) for \( r_* \to -\infty \), i.e. on the horizon). There is a maximum of \( P_\lambda(r) \) at \( r = r_{\text{max}} \), where

\[ r_{\text{max}} = \left\{ \begin{array}{ll} 2\lambda_{mn}\ell \cosh \frac{\varphi}{3}, & \lambda_{mn}^3 < \frac{4\eta}{\ell} \quad (\cosh \varphi := \frac{4\eta}{\ell \lambda_{mn}^3}) \\ 2\lambda_{mn}\ell \cos \frac{\varphi}{3}, & \lambda_{mn}^3 \geq \frac{4\eta}{\ell} \quad (\cos \varphi := \frac{4\eta}{\ell \lambda_{mn}^3}). \end{array} \right. \]  

(72)

For \( \lambda = 0 \), \( r_{\text{max}} = 4^{1/3}r_+ \) is just outside the black hole, and it is increasing to infinity as \( \lambda \to \infty \). For \( \lambda_{mn}^3 < 4\eta/\ell \) the potential maximum \( P_\lambda(r_{\text{max}}) \) is given by

\[ P_\lambda(r_{\text{max}}) = \frac{3 \left( \frac{\varphi}{3} + \lambda_{mn}^3 \cosh \frac{\varphi}{3} \right)^2}{4\lambda_{mn}^3 \ell^2 \cosh^4 \frac{\varphi}{3}}. \]  

(73)

For \( \lambda_{mn}^3 \geq 4\eta/\ell \) the \( \cosh \) has to be replaced by a \( \cos \). At infinity the potential equals the constant \( \lambda_{nm}^2 \ell^{-2} \). The potential curve is shown in figure \[ \text{[43]}\text{].}

For any genus, a set of Boulware modes, normalized to \( \delta(\omega) \), can be defined by the asymptotic conditions
\[ B_{\omega \lambda} \simeq (4\pi \omega)^{-1/2} e^{-i\omega t} u_\lambda(x, y) r^{-1} \left\{ \begin{array}{ll}
\frac{e^{i\omega r_*} + R_\lambda(\omega)e^{-i\omega r_*}}{T_\lambda(\omega) \sin(\Omega_\lambda r_*)} \quad & r_* \to -\infty \\
T_\lambda(\omega) \sin(\Omega_\lambda r_*) \quad & r_* \to 0,
\end{array} \right. \] (74)

where \( u_\lambda(x, y) \) are the eigenfunction of the scalar Laplacian on a Riemann surface of genus \( g \geq 1 \), \( \Omega_\lambda = \sqrt{\omega^2 - \lambda^2 m \ell^2} \), and \( R_\lambda(\omega) \) and \( T_\lambda(\omega) \) are the reflection– and the transmission coefficients of the potential barrier, respectively.

For \( \Omega_\lambda \) imaginary, the mode at infinity acquires an additional phase \( \pm i \). This is a Dirichlet set, but a Neumann set can also be defined by replacing the sine function with a cosine.

The modes appear to emerge from the past horizon in the eternal black hole spacetime.

The phase of \( R_\lambda \) is then twice the phase of \( T_\lambda \), as a consequence of the boundary condition.

These phase shifts have no singularities in the lower half complex \( \omega \)-plane, because the potential admits no bound states. Note that for \( \eta < 0 \), the potential for modes with sufficiently small eigenvalues has a well outside the event horizon (see figure [3]). This occurs for \( 0 < \lambda^2 < 2/3 \), and such small eigenvalues exist in general on any Riemann surface.

The classical counterpart is that there are no closed null geodesics around the black hole within the potential well. Resonant diffusion is not excluded, but we have not analyzed this any further (at the large frequencies which are relevant to the Hawking radiation, there is certainly no problem with resonances).

The Dirichlet coefficients \( R_\lambda(\omega) \) and \( T_\lambda(\omega) \) are not related by current conservation, due to the boundary conditions. However, they can be related to the coefficients describing scattering off the barrier without the boundary conditions at infinity, i.e. by replacing \( \sin \Omega_\lambda r_* \) by \( \exp(\pm i \Omega_\lambda r_*) \) in (74). We shall denote these outgoing reflection/transmission coefficients by right pointing arrows, \( \vec{R}_\lambda(\omega) \) and \( \vec{T}_\lambda(\omega) \) respectively, and the ingoing coefficients with left pointing arrows. Current conservation then gives the unitarity conditions

\[ \omega[1 - |\vec{R}_\lambda(\omega)|^2] = \sqrt{\omega^2 - \lambda^2 m \ell^2} |\vec{T}_\lambda(\omega)|^2 \] (75)

\[ \sqrt{\omega^2 - \lambda^2 m \ell^2} \vec{T}_\lambda(\omega) = \omega \vec{T}_\lambda(\omega). \] (76)

The original coefficients are then given in terms of \( \vec{R}_\lambda(\omega) \) and \( \vec{T}_\lambda(\omega) \) by the equations

\[ R_\lambda(\omega) = -\frac{Z}{\vec{R}_\lambda(\omega)}, \quad T_\lambda(\omega) = -\frac{2i|\vec{T}_\lambda(\omega)|^2}{Z}, \] (77)

where \( Z = \vec{R}_\lambda(\omega) \vec{T}_\lambda(\omega) - \vec{T}_\lambda(\omega) \).

Clearly, with either boundary conditions \( R_\lambda(\omega) = \exp(i \delta_\lambda(\omega)) \), so all the emitted radiation is ultimately reflected back into the black hole. In fact an eternal black hole can only exist in a thermal equilibrium state.

To introduce this equilibrium state, we will find the solutions of the wave equation that are positive frequency along one sheet of the event horizon, with respect to its canonical affine parameter.

We define the Hartle–Hawking modes to be solutions of the wave equation which obey Dirichlet boundary conditions at infinity and are positive frequency on the past horizon.
$H^-$, with respect to the canonical affine parameter $U$, i.e. $\partial_U H_{\lambda\omega} = -i\omega H_{\lambda\omega}$. Outside the horizon they will be superpositions of Boulware modes, which we write in the form

$$H_{\lambda\omega}(p) = \int_0^\infty \frac{d\omega'}{\sqrt{4\pi\omega'}} [\bar{\gamma}_{\omega'} B_{\omega'\lambda}(p) - \epsilon_{\omega'\omega} \bar{B}_{\omega'\lambda}(p)],$$

where $p = (u, v, x, y)$ belongs to the outer region. The boundary conditions at infinity are then automatically satisfied. By definition, on the past horizon $H_{\lambda\omega}(p)$ converges to the function $(4\pi\omega)^{-1/2} \exp(-i\omega U) u_\lambda$. Using the Fourier transform

$$e^{-i\omega U} \theta(-U) = \int_{-\infty}^\infty \frac{d\omega'}{2\pi\kappa} e^{\pi\omega'/2\kappa} \omega^{i\omega'/\kappa} \Gamma(-i\omega'/\kappa) e^{-i\omega'u}$$

and the asymptotic condition (74), we find that the phase $\delta_\lambda(\omega)$ of $R_\lambda(\omega)$ disappears along $H^-$ and we get the $\gamma$- and $\epsilon$-coefficients in the form

$$\gamma_{\omega'} = -i \frac{\omega^{-i\omega'/\kappa}}{2\pi \sqrt{\omega'}} \Gamma(1 + i\kappa^{-1}\omega') e^{\pi\omega'/2\kappa},$$

$$\epsilon_{\omega'} = \frac{i}{2\pi} \frac{\omega^{-i\omega'/\kappa}}{\sqrt{\omega'}} \Gamma(1 + i\kappa^{-1}\omega') e^{-\pi\omega'/2\kappa}.$$ (80)

On the future horizon $H^+$, we now claim that $H_{\omega\lambda}(p)$ is a superposition of positive frequency solutions with respect to the canonical affine parameter $V$ of the future horizon. In fact, using (74) and going on $H^+$ we obtain

$$H_{\omega\lambda}(p) = \int_0^\infty \frac{d\omega'}{\sqrt{4\pi\omega'}} [\bar{\gamma}_{\omega'} e^{-i\omega'u + i\delta_\lambda(\omega')} - \epsilon_{\omega'} \omega e^{i\omega'u - i\delta_\lambda(\omega')}],$$

where now the phase shifts do give a contribution to the mode.

From Eq. (74) and the uniqueness of Dirichlet solution we deduce $B_{-\omega\lambda} = i\bar{B}_{\omega\lambda}$, up to a phase, so $R_\lambda(-\omega) = R_\lambda(\omega)$, which gives the antisymmetry condition $\delta_\lambda(-\omega) = -\delta_\lambda(\omega)$.

We are now in position to prove our claim. One may recast (81) in the form of a single integral over the real line, as the pole in $\omega'$ is 0 in the two integrals cancel. From Eq. (80), the functions $\gamma_{\omega'}$ and $\epsilon_{\omega'}$ have infinitely many simple poles in the lower half complex $\omega'$-plane, at $\omega'_n = -i\kappa n$, for $n \geq 0$ and integer. As the potential barrier has no bound states, the phase shifts $\delta_\lambda(\omega')$ have no poles for $\Im \omega' < 0$. By analytic continuation arguments, the Boulware modes along the imaginary axis are real functions, so (77) gives $R_\lambda(-i\omega) = -1$, or $\delta_\lambda(-i\omega) = \pm \pi$. We may then compute $H_{\lambda\omega}$ along $H^+$ by summing over the residues (omitting the pole in 0) and putting $\delta_\lambda(-in\kappa) = \pm \pi$, after which we obtain the result

$$H_{\lambda\omega}(p) = (4\pi\omega)^{-1/2} \left[ 1 - \exp \left( \frac{i\omega}{V} \right) \theta(V) \right], \quad p \in H^+.$$ (82)

The "1" above is the relict of $\exp(-i\omega U)$ along $H^+$ (which is the set $U = 0$) and, apart from it, the function $H_{\lambda\omega}(p)$ is analytic for $\Im V < 0$, so its Fourier expansion must contain only positive frequencies (it is actually a superposition of Bessel functions of order zero).

We have obtained an interesting result. We started with a function like $\exp(-i\omega U)$ along $H^-$, as in the definition of the Unruh vacuum, and we ended with a function like $\exp(i\omega V^{-1})$. 
along $H^+$, which is a superposition of positive frequency $V$-modes. This means that the state defined by the modes $H_{\lambda\omega}$ is a true vacuum for particles defined in the Kruskal time, $T = (U + V)/2$, and therefore in particular it is an equilibrium state. This is the Israel–Hartle–Hawking state $|H\rangle$: writing the quantum field as

$$\phi(p) = \int_0^\infty [A_{\lambda\omega} H_{\lambda\omega}(p) + A^\dagger_{\lambda\omega} \bar{H}_{\lambda\omega}(p)] d\omega,$$

it is defined by $A_{\lambda\omega} |H\rangle = 0$. When analyzed in terms of Boulware modes, however, we will find it to contain a thermal distribution of particles with the black hole’s temperature.

From the above it also follows that we could have defined the Hartle–Hawking modes to be positive frequency along the future horizon with respect to $V$ (which would be the usual definition for asymptotically flat black holes). In this case we would have ended with modes which are positive frequency along the past horizon in the time $U$, and therefore we would not have changed the definition of the state.

We see then that there is no Unruh state, since modes which are positive frequency relative to $U$ along $H^-$ and obey Dirichlet boundary conditions at infinity, are also positive frequency along $H^+$ in the time $V$.

For $\eta > 0$, the Hartle–Hawking modes are defined everywhere. For $\eta < 0$, the Hartle–Hawking modes are defined in the region contained within the inner Cauchy horizon, where they stay bounded. However, any flux of energy coming from outside the event horizon diverges relative to a local frame crossing the Cauchy horizon [11], due to an infinite blue shift. Hence we suspect that there will be divergences in the quantum expectation value of the stress tensor in the Hartle–Hawking state, near the Cauchy horizon (in two dimensions it diverges, in fact). If this is the case, then one cannot ignore the back reaction of the thermal energy on the spacetime, as it is done implicitly in defining the Hartle-Hawking state. In the analogous situation of a Reissner–Nordström black hole, this question is yet unsettled to the authors’ knowledge (c.f. [43] for this case), since the Killing approximation devised by Frolov–Zel’nikov [44] fails near the horizons, as well as the analytic approximation devised by Hiscock et al. [43]. However, this is a global question that will not affect our subsequent results.

From Eq. (80) and the expansion (78), we can easily determine the mean occupation number near infinity, for Boulware particles with energy $\omega$ (i.e. for particles defined by the static time parameter) in the Israel–Hartle–Hawking state: it is a Planck distribution with the black hole’s temperature

$$dN_{\lambda\omega} = \frac{g_\lambda d\omega}{e^{2\pi\omega/\kappa} - 1},$$

where $g_\lambda$ is the degeneracy of $\lambda$. To find the energy density from this is slightly non–trivial, as one would use the density of states and then sum over the degeneracy $g_\lambda$. We will present a calculation of this kind when discussing particle production by the black hole, using Weyl’s asymptotic formula.
B. Particle Production

In a thermal equilibrium state there will be no net flux of particles from the black hole, of course. However, for a black hole that formed from gravitational collapse \[8\], the thermal equilibrium state will settle down only asymptotically at large times, for the black hole will start to radiate only near and after the formation of the event horizon. If the universe is large enough, there will be a long time before infinity scatters the radiation back, and during this time there will be a net outgoing flux. With this in mind, we now want to calculate the black hole luminosity, i.e. that part of the total flux which is outgoing to infinity. To this end let us consider an outgoing Unruh–like mode, which near the horizon takes the form

\[ \phi_{\lambda \omega} = (4\pi \omega)^{-1/2} e^{-i\omega U} u_{\lambda}(x, y). \] (85)

The reason for considering this is that to an external stationary observer, the collapse approach of the dust surface to the event horizon is exponentially fast in retarded time (this easily follows from the fact that the exterior metric is static all the time and equals the eternal black hole metric), i.e. we have for the radial coordinate \( R \) of the dust surface

\[ R(u) - r_+ \simeq C e^{-\kappa u}. \] (86)

Thus the waves emitted from the surface of the dust appear enormously red–shifted with a continuously varying frequency of the form \( \omega \exp(-\kappa u) \), which is just the phase of the Unruh mode (85). Using (79) and (80), we can express (85) in the form

\[
\phi_{\lambda \omega} = \int_0^\infty \frac{d\omega'}{4\pi \omega'} r_+^{-1} \bar{\gamma}_{\omega'} e^{-i\omega' U} u_{\lambda}(x, y) \\
- \int_0^\infty \frac{d\omega'}{4\pi \omega'} r_+^{-1} \epsilon_{\omega'} e^{i\omega' U} u_{\lambda}(x, y),
\] (87)

Every component \( \exp(-i\omega' U) \) in the wave packet (87) arrives at infinity as a mode \( \mathbf{T}_\lambda (\omega') \exp(-i\omega't + i\Omega') \), with \( \Omega' = \sqrt{\omega'^2 - \lambda^2/\ell^2} \). Note that only waves with \( \omega' > \lambda \ell^{-1} \) have oscillatory character when they arrive at infinity, the others are damped exponentially. (The fact that in adS space the ratio of angular momentum and energy is limited above, is well–known, see e.g. [15].) Having noted this, it is an easy matter to find the Bogoljubov coefficients relating the \(|\text{in}\rangle\) to \(|\text{out}\rangle\) vacuum. They are

\[ \alpha_{\lambda,\omega',\omega} \equiv \sqrt{\frac{\Omega'}{\omega'}} \mathbf{T}_\lambda (\omega') \gamma_{\omega',\omega}, \]
\[ \beta_{\lambda,\omega',\omega} \equiv \sqrt{\frac{\Omega'}{\omega'}} \mathbf{T}_\lambda (\omega') \epsilon_{\omega',\omega}. \] (88)

The relevant \( \beta \)–coefficients satisfy the relations

\[
\int_0^\infty \beta_{\lambda,\omega',\omega} \beta_{\lambda,\omega'',\omega'} d\omega = \frac{\sqrt{\omega'^2 - \lambda^2\ell^2}}{\omega'} \frac{\mathbf{T}_\lambda (\omega')^2}{e^{\omega'/T} - 1} \delta(\omega' - \omega''),
\] (89)
where $T = \kappa/2\pi$. As $\delta(0) = T/2\pi$ for large time $T$, from this we conclude that the luminosity of the black hole is

$$L = \frac{1}{2\pi} \sum_{\lambda} g_\lambda \int_0^\infty \omega^{-1} \sqrt{\omega^2 - \lambda^2 \ell^{-2}} \left| \mathcal{T}_\lambda(\omega) \right|^2 \frac{\omega d\omega}{e^{\omega/T} - 1}. \quad (90)$$

As usual, the spectrum is not precisely planckian due to the presence of the grey body factor; however, it is only for large $\omega$ that it approaches a form similar to that for the asymptotically flat Schwarzschild black hole. We obtained the grey body factor

$$\Gamma_\lambda(\omega) = \omega^{-1} \sqrt{\omega^2 - \lambda^2 \ell^{-2}} \left| \mathcal{T}_\lambda(\omega) \right|^2, \quad (91)$$

which means that a fraction $\lambda \rightarrow R_\lambda(\omega) = 1 - \Gamma_\lambda(\omega)$ of the emitted particles can not reach infinity and is recaptured by the black hole (this fraction is also equal to $\lambda \rightarrow R_\lambda(\omega)$, the reflection coefficient for ingoing waves).

We may estimate $L$ in the geometrical optics approximation, i.e. in the high frequency limit. In this limit there is no reflection of emitted particles down the black hole if the inequality $\omega^2 > P_\lambda(r_{max})$ holds. In this case the transmission coefficient is proportional to a step function, and it follows from Eq. (75) that

$$\left| \mathcal{T}_\lambda(\omega) \right|^2 = \omega(\omega^2 - \lambda^2 \ell^{-2})^{-1/2} \theta(\omega^2 - P_\lambda(r_{max})). \quad (92)$$

From Eq. (73), there is a $\lambda_{max}$ for which the inequality is true, which can be estimated to be $\lambda_{max}(\omega) \sim \omega \ell$ for large enough $\omega$. Also $\omega > \omega_0 = 3^{1/2} 2^{-1} \ell^{-4/3} \eta^{1/3}$, for otherwise the inequality is violated for small $\lambda$. In the $g > 1$ case, we also have $\lambda_{max}(\omega) \sim \omega \ell$, and $\omega > \omega_0$, with $\omega_0$ being identical to the value for $g = 1$ given above, provided $\eta > 0$ and $\eta \gg \ell$. For $\eta > 0$ and $\eta \ll \ell$, or for $\eta \leq 0$, one has $\omega_0 = 0$. If the event horizon is spherical ($g = 0$), however, the situation changes. One now obtains for $\lambda_{max}(\omega)$

$$\lambda_{max}(\omega)^2 \sim \frac{2\ell^2 \omega^2}{\ell^2 + 27\eta^2}. \quad (93)$$

In order to get the luminosity in the various cases, we finally have to sum over $\lambda$ in (90), at first sight a difficult task to perform since the degeneracy $g_\lambda$, where known, is a rather complicated expression. However, for compact manifolds there is the general Weyl’s asymptotic formula [45], which in our case reads

$$\sum_{\lambda=0}^{\lambda_{max}} g_\lambda = \frac{A_g}{4\pi} \lambda_{max}^2 + \mathcal{O}(\lambda_{max}), \quad (94)$$

where $A_g$ is the area of a Riemann surface with genus $g$ and Gaussian curvature $K = -1$, $K = 0$ or $K = 1$ for $g > 1$, $g = 1$ or $g = 0$, respectively. So for a genus $g > 1$ black hole, $A_g = 4\pi(g - 1)$, for a torus $A_1 = \text{Im} \tau$, and for a sphere $A_0 = 4\pi$.

Using this, we get for large $\lambda_{max}$ (which is fulfilled for $\omega$ sufficiently large, i.e. in the geometrical optics approximation) the estimate

$$L = \frac{A_g}{8\pi^2} \int_{\omega_0}^{\infty} \lambda_{max}(\omega)^2 \frac{\omega d\omega}{e^{\omega/T} - 1}. \quad (95)$$
This yields for $g \geq 1$

$$L = C \pi^2 A_g \ell^2 T^4,$$

(96)

with

$$C \equiv \frac{4^{1/3}}{18} \int_1^\infty \frac{x^3 \, dx}{e^{4^{1/3} \pi x/\sqrt{3}} - 1} \approx 0.0052,$$

(97)

for $g = 1$ or for $g > 1$, $\eta > 0$ and $\eta \gg \ell$. For $g > 1$ and $0 < \eta \ll \ell$, or for $g > 1$ and $\eta \leq 0$, we get $C = 1/120$.

In the spherical case, the luminosity is

$$L = C \pi^2 A_0 \frac{27 \eta^2 \ell^2}{\ell^2 + 27 \eta^2} T^4,$$

(98)

with some numerical constant $C$, which we have not calculatedexplicitly here.

Writing $L = -dM/dt$, and inserting the $M$–dependence $T \propto M^{1/3}$, one derives an infinite lifetime for the toroidal black hole, in contrast to the Schwarzschild case. (Of course, this is valid only in the semiclassical limit. When the black hole mass approaches the Planck mass, quantum gravity effects will occur). Therefore, if the universe is not too large to allow the black hole reaching the Planck mass before the radiation is reflected back, then, sooner or later, the black hole must start to grow until it reaches the temperature of the reflected radiation again. At this point the black hole should settle down to an equilibrium thermal state with a large entropy. In fact we have seen that a wave like \(85\) in a stationary anti-de Sitter black hole, will propagate so as to become a positive frequency, ingoing wave on the future horizon relative to its canonical affine parameter. This is the behaviour that marks the appearance of the thermal equilibrium state and it means that an ingoing flux of energy enters the black hole and balances the emitted, outgoing flux.

Inspecting (96) we observe that, if we consider the black hole as a black body radiating with the Hawking temperature, the area $A$ entering Stefan’s law $L \propto AT^4$ is not the area of the event horizon, but an area determined by $\ell^2$, i. e. by the cosmological constant! This is another intriguing feature of topological black holes, different from asymptotically flat cases. Note that the luminosity of a black hole with spherical event horizon differs from that of the topological ones. Indeed, in the prefactor of (98) also the parameter $\eta$ (which is equal to the mass $M$ for $g = 0$) enters, and in the limit $\ell \gg \eta$ (i. e. small cosmological constant) we recover the known Schwarzschild result.

The found luminosity behaviour can be understood already at a classical level, by examining null geodesics in the black hole spacetime. We shall do this in the following. Using the fact that to every Killing vector there is an associated constant of motion, for the radial coordinate $r$ one gets the equation

$$\frac{1}{2} \dot{r}^2 + \frac{1}{2} V(r) \frac{L^2}{r^2} = \frac{1}{2} E^2,$$

(99)

where the dot denotes the derivative with respect to an affine parameter. $V(r)$ is the square of the lapse function, $E$ is the constant of motion associated to $\partial_t$, and $L^2 = L_x^2 + L_y^2$, where
$L_x$ and $L_y$ belong to the Killing vectors on the torus, namely $\partial_x$ and $\partial_y$ respectively. (For the Schwarzschild (AdS) black hole, or for $g > 1$, we limit ourselves to a fixed value of $\theta$, e. g. $\theta = \pi/2$ for $g = 0$, so $L$ belongs to $\partial_\phi$. (Note that for a Riemann surface of genus $g > 1$, $\partial_\phi$ is a Killing vector only locally)). Now the potential in (99) is given by

$$P(r) = V(r) \frac{L^2}{2r^2} = \begin{cases} \frac{L^2}{2r^2} - \frac{ML^2}{r^2} & , \text{Schwarzschild} \\ \frac{L^2}{2r^2}(\delta_{g,1} - 1) + \frac{L^2}{r^2} - \frac{\eta L^2}{r^2} & , \text{ } g \geq 1 \end{cases}$$ (100)

As is well–known, in Schwarzschild spacetime, this potential is zero at the horizon, has a maximum $P_{\text{max}} = L^2/54M^2$ at $r = 3M$, and then falls off to zero at infinity. Hence a particle coming from infinity is captured by the black hole, if its "energy" $E^2/2$ exceeds the potential maximum. This means that the apparent impact parameter $b \equiv L/E$ must be smaller then $\sqrt{27}M$ [46], and leads to the capture cross section

$$\sigma = \pi b^2 = 27\pi M^2$$ (101)

for the Schwarzschild geometry. Thus a Schwarzschild black hole absorbs like a black body with area $27\pi M^2$, a number directly proportional to the horizon area. For the $g \geq 1$ black hole, however, the situation is different. The potential $P(r)$ is also zero at the horizon, but then increases monotonically to reach the constant value $L^2/2\ell^2$ at infinity. Therefore every massless particle at infinity with $E^2/2 > L^2/2\ell^2$, i. e. $b < \ell$, travelling towards the black hole, is captured. This gives an absorption cross section $\sigma = \pi \ell^2$, i. e. a $g \geq 1$ topological black hole absorbs like a black body with area $\pi \ell^2$, not like a black body with the event horizon area. Of course, quantum mechanically there arises a local maximum in the potential (see figure 2), but this does not alter the situation essentially.

For the $g = 0$ Schwarzschild–AdS black hole, $P(r)$ has a local maximum at $r = 3M$, which leads to the capture cross section

$$\sigma = \frac{27\pi M^2 \ell^2}{\ell^2 + 27M^2},$$ (102)

encountered already (if we identify $\eta = M$) in (99). In this case, $\sigma$ is determined both by the cosmological constant and the mass parameter, whereas for the $g \geq 1$ topological black holes only the cosmological constant enters the capture cross section, and thus the prefactor in the luminosity formula.

V. BLACK HOLE SPECTRUM AND STRING STATES

We have seen that an isolated black hole in anti–de Sitter space will ultimately settle down to a thermal equilibrium state with the Hawking temperature $T = \kappa/2\pi$ and some mass $M$. Such a black hole contributes to the total entropy its own entropy, $S_{bh} = A/4$. For a large mass black hole, the entropy and temperature depend on the mass as

$$S = aM^{2/3}, \quad T = bM^{1/3},$$ (103)

with $a$ and $b$ computable constants (the two formulas are exact for the toroidal black hole, but only asymptotically correct for higher genus black holes and for $g = 0$, which is the
spherical anti–de Sitter black hole studied by Hawking–Page \[40\]). Hence the degeneracy of black hole states decreases in anti–de Sitter space, and the level density grows like

\[ \rho_{bh}(M) \simeq \exp(aM^{2/3}). \]

(104)

According to \[49\], the same phenomenon occurs for strings in anti–de Sitter space where the level density at very large masses is (\(\ell^{-2}\) is proportional to the cosmological constant \(\Lambda\))

\[ \rho_s(M) \simeq \exp(\sqrt{M\ell}). \]

(105)

We now want to understand the black hole result by assuming a certain discrete spectrum for the black hole mass, with a certain degeneracy, and computing the corresponding partition function. The adiabatic invariant argument of Bekenstein would work in this case also, and suggests an area spectrum \(A_n = \sigma n\), with \(\sigma\) a number of order one \[21,22\] (this result has been obtained also in loop quantum gravity \[17\] and the membrane approach \[18\], for large quantum numbers). Then a mass spectrum arises of the form

\[ M_n = \alpha n^{3/2}, \]

(106)

in sharp contrast with either the Schwarzschild or the string spectrum, \(M_n \simeq \alpha \sqrt{n}\), or with the spectrum for strings in anti-de Sitter space, \(M_n \simeq \ell^{-1}n\) for large \(n\) \[19\]. The degeneracy will be assumed to be an increasing function \(d(n)\), and the partition function takes the form

\[ Z = \sum_{n=0}^{\infty} e^{\ln d(n)} e^{-\beta \alpha n^{3/2}}. \]

(107)

We shall evaluate this quantity for large masses, i.e. small \(\beta\), by using the steepest descent method. Replacing the sum with an integral we get

\[ Z = \int_0^\infty dt e^{\ln d(t)} e^{-\beta \alpha t^{3/2}}. \]

(108)

With \(f(t) = \ln d(t) - \alpha \beta t^{3/2}\), the stationary point \(t_0\), occurs at \(f'(t_0) = 0\), and assuming also \(f''(t_0) > 0\) we obtain the partition function

\[ \ln Z = \ln d_0 - \left( \frac{d_0'}{d_0} \right)^3 \frac{8}{27 \alpha^2 \beta^2} + \frac{1}{2} \ln \left( \frac{2\pi}{f''(t_0)} \right), \]

(109)

where \(d_0 = d(t_0)\). The partition function can also be computed in Euclidean quantum gravity for the asymptotically anti–de Sitter black holes \[14,15\], with the result

\[ \ln Z = \frac{4\pi^2 \ell^4}{27 \beta^2}. \]

(110)

This is exact for the toroidal black hole and valid approximatively for higher genus or spherical black holes. Comparing the two partition functions requires \(d'/d = \alpha\), \(G\) being a constant, for a wide range of masses. In other words, \(d(n) = G^n\) asymptotically for large \(n\), with \(G\) of order \(\exp((\pi \alpha \ell^2)^{2/3})\) to match with the (robust) Euclidean result.
The obtained mass spectrum seems to be difficult to reconcile with string theory, even in anti–de Sitter space where \( M \simeq \ell^{-1} n \), asymptotically at mass level \( n \) (in flat space this is \( M \simeq \alpha' \sqrt{n} \)). On the other hand, the degeneracy of string states grows as \( \exp \sqrt{n} \) (as in flat space), and therefore it is not obvious how the Susskind–Horowitz–Polchinski argument \cite{50,51} should work. According to this argument, the black hole description breaks down when the horizon is of order the string scale, and the black hole becomes a highly excited string state. The mass of the black hole is \( M_{bh} \simeq r_+^2 G^{-1} \ell^{-2} \), and the mass of a string state at level \( n \) is \( M_s \simeq \ell^{-1} n \), for \( n \gg \ell_s / \ell_s \), where \( \ell_s = \sqrt{\alpha'} \) is the string scale. The Newton constant is \( G = g^2 \alpha' \), where \( g \) is the string coupling constant. Requiring the two masses to coincide (within a factor of order unity) when \( r_+ \simeq \ell_s \) gives

\[
g^{-2} \simeq \ell \ell_s^{-1} n, \tag{111}
\]

and the entropy is

\[
S = \frac{r_+^2}{4G} \simeq \frac{\ell_s^2}{4g^2 \ell_s^2} = \ell \ell_s^{-1} n, \tag{112}
\]

which disagrees with the string entropy \( S \simeq \sqrt{n} \). This argument should be regarded as a very naive one. Strings in adS are not as well understood as in flat space and the mass formula is very complicated. For example, there is a regime where the mass–to–level relation is as in flat space if \( \ell / \ell_s \gg 1 \). On the other hand, the cosmological constant is not a completely free parameter if string theory is to be anomaly free \cite{52}. In view of these facts, the correspondence principle of Horowitz and Polchinski can not be rejected on the above basis, but it remains to see how exactly it works.

### VI. SUMMARY AND DISCUSSION

We have discussed quantum aspects of fields in the background of anti–de Sitter black holes. All the properties of them which are expected from the classical laws to the Euclidean approach are confirmed. However, also new features emerge, which are related to the special asymptotic behaviour of anti–de Sitter space. Most surprising is the area dependence of the radiation formula \cite{16}, which is not determined by the area of the horizon. We also hope to have made clear that no Unruh–like states exist for eternal black holes. In contrast, a black hole formed by collapse will radiate away its mass for a while after formation, until infinity will reflect it back. The black hole’s temperature will then rise again up to the radiation temperature. At this point it should settle down to an equilibrium state at a certain Hawking temperature. Although we did not made efforts to compute the actual black hole evolution, this is a very reasonable picture, because anti–de Sitter space does not permit radiation to disperse to infinity. But then we have another version of the information loss paradox, because if the black hole does not completely evaporate there is no point for information to return.

The thermodynamical properties of anti–de Sitter black holes also lead to a peculiar mass spectrum, according to Bekenstein’s view of the quantum structure of a black hole. This we
have briefly discussed in relation to string theory, too. We think there is no simple way to understand the string–black hole correspondence principle in adS space, but we regard the question as unsettled for the time being, the point being that strings in adS behave very differently than in flat space.

ACKNOWLEDGEMENT

The part of this work due to D.K. has been supported by a research grant within the scope of the Common Special Academic Program III of the Federal Republic of Germany and its Federal States, mediated by the DAAD.
REFERENCES

[1] S. Åminneborg, I. Bengtsson, S. Holst and P. Peldán, Class. Quantum Grav. 13, 2707 (1996).
[2] R. B. Mann, Class. and Quantum Grav. 14, L109 (1997); R. B. Mann, gr-qc/9709039 (1997).
[3] L. Vanzo, Phys. Rev. D 56, 6475 (1997).
[4] R. G. Cai and Y. Z. Zhang, Phys. Rev. D 54, 4891 (1996).
[5] J. P. S. Lemos and V. T. Zanchin, Phys. Rev. D 54, 3840 (1996).
[6] D. Klemm, V. Moretti and L. Vanzo, Phys. Rev. D 57, 6127 (1998).
[7] R. G. Cai, J. Y. Ji and K. S. Soh, gr-qc/9708063.
[8] R. B. Mann and W. L. Smith, Phys. Rev. D 56, 4942 (1997).
[9] J. P. S. Lemos, gr-qc/9709013 (1997) (to be published in Phys. Rev. D).
[10] D. R. Brill, J. Louko and P. Peldán, Phys. Rev. D 56, 3600 (1997).
[11] W. Israel, Phys. Lett. 57 A, 107 (1976).
[12] J. B. Hartle and S. W. Hawking, Phys. Rev. D 13, 2188 (1976).
[13] S. W. Hawking, Comm. Math. Phys. 43, 199 (1975).
[14] R. M. Wald, Comm. Math. Phys. 45, 9 (1975).
[15] S. J. Avis, C. J. Isham and D. Storey, Phys. Rev. D 18, 3565 (1978).
[16] P. Breitenlohner and D. Z. Friedmann, Ann. Phys. 177, 76 (1982).
[17] S. W. Hawking, Phys. Rev. D 13, 191 (1976).
[18] G. ’t Hooft, Nucl. Phys. B 256, 727 (1985).
[19] J. D. Bekenstein, 7th Marcel Grossman Meeting on General Relativity, p. 39 (1994), gr-qc/9409013.
[20] V. P. Frolov and D. V. Fursaev, hep-th/9802010 (1998).
[21] V. Mukhanov, JETP Lett. 44, 63 (1986);
    Yu. I. Kogan, JETP Lett. 44, 267 (1986);
    J. G. Bellido, hep-th/9302127 (unpublished).
[22] J. D. Bekenstein and V. Mukhanov, Phys. Lett. B 360, 7 (1995);
    Yu. I. Kogan, hep-th/9412232 (1994);
    H. A. Kastrup, Phys. Lett. B 413, 267 (1997).
[23] D. Birmingham, I. Sachs and S. Sen, hep-th/9801019 (1998);
    D. Birmingham, hep-th/9801143 (1998).
[24] A. Strominger, hep-th/9712251 (1997).
[25] M. Bañados, T. Brotz and M. Ortiz, hep-th/9802076 (1998).
[26] J. D. Brown and M. Henneaux, Comm. Math. Phys. 104, 207 (1986).
[27] K. A. Bronnikov and M. A. Kovalchuk, J. Phys. A13, 187 (1980).
[28] S. W. Hawking and G. F. Ellis, The Large Scale Structure of Spacetime, Cambridge University Press, Cambridge, England (1973).
[29] A. Strominger, Les Houches Lectures on Black Holes, hep-th/9501071 (1995).
[30] W. G. Unruh, Phys. Rev. D 14, 870 (1976).
[31] S. A. Fulling and P. C. W. Davies, Proc. R. Soc. Lond. A 348, 393 (1976).
[32] L. H. Ford, gr-qc/9707062 (1997) (to be published in the proceedings of the IX Jorge André Swieca Summer School, Campos dos Jordão, SP, Brazil, February 1997).
[33] N. Sakai and Y. Tanii, Nucl. Phys. B 255, 401 (1985).
[34] A. Wipf, hep-th/9801025 (1998).
[35] R. M. Wald, *Quantum Field Theory and Black Hole Thermodynamics in Curved Spacetime*, The University of Chicago Press, Chicago (1984).
[36] S. P. Trivedi, Phys. Rev. D 47, 4233 (1993).
[37] P. C. W. Davies and S. A. Fulling, Proc. R. Soc. Lond. A 354, 59 (1977).
[38] P. Candelas, Phys. Rev. D 21, 2185 (1980).
[39] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15, 2738 (1977).
[40] S. W. Hawking and D. N. Page, Comm. Math. Phys. 87, 577 (1983).
[41] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Press Inc., NY (1983).
[42] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, Oxford (1979).
[43] P. R. Anderson, W. A. Hiscock and D. Samuel, Phys. Rev. Lett. 70, 1739 (1993).
[44] V. P. Frolov and A. I. Zel’nikov, Phys. Rev. D 35, 3031 (1987).
[45] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, Orlando, USA (1984).
[46] R. M. Wald, *General Relativity*, The University of Chicago Press, Chicago (1984).
[47] C. Rovelli and L. Smolin, Nucl. Phys. B 442, 593 (1995).
[48] M. Maggiore, Nucl. Phys. B 429, 205 (1994).
[49] A. L. Larsen and N. Sánchez, Phys. Rev. D 52, 1051 (1995).
[50] L. Susskind, hep-th/9309145 (unpublished);
  E. Halyo, A. Rajaraman and L. Susskind, Phys. Lett. B 392, 319 (1996);
  E. Halyo, B. Kol, A. Rajaraman and L. Susskind, Phys. Lett. B 401, 15 (1996).
[51] G. T. Horowitz and J. Polchinski, Phys. Rev. D 55, 6189 (1997);
  H. Yang, hep-th/9801073 (1998).
[52] E. S. Fradkin and V. Ya. Linetsky, Phys. Lett. B 261, 26 (1991).
FIG. 1. Penrose–Carter diagram for the toroidal black hole with $\eta > 0$. 
FIG. 2. Potential barrier for a massless scalar particle in the toroidal black hole spacetime, or in the $g > 1$ spacetime with $\eta > 0$. 
FIG. 3. Potential barrier for a massless scalar particle in the $g > 1$ spacetime with $\eta < 0$. $r_1$ is the zero of the expression $(\lambda^2/r^2 + 2\eta/r^3)$, appearing in the potential (63). In the figure the case $\lambda^2 > 2|\eta|/r_-$ is shown. For $2|\eta|/r_+ < \lambda^2 < 2|\eta|/r_-$ one has $r_- < r_1 < r_+$, and for $\lambda^2 < 2|\eta|/r_+$ one has $r_1 > r_+$. (The course of the potential is the same in all three cases).