Nonlinear model identification and spectral submanifolds for multi-degree-of-freedom mechanical vibrations

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Abstract

In a nonlinear oscillatory system, spectral submanifolds (SSMs) are the smoothest invariant manifolds tangent to linear modal subspaces of an equilibrium. Amplitude-frequency plots of the dynamics on SSMs provide the classic backbone curves sought in experimental nonlinear model identification. We develop here a methodology to compute analytically both the shape of SSMs and their corresponding backbone curves from a data-assimilating model fitted to experimental vibration signals. Using examples of both synthetic and real experimental data, we demonstrate that this approach reproduces backbone curves with high accuracy.

1 Introduction

Modal decomposition into normal modes is a powerful tool in linear system identification (see., e.g., Ewins [6]), but remains inapplicable to nonlinear systems due to the lack of a superposition principle. Various nonlinear normal mode (NNM) concepts nevertheless offer a conceptual simplification in the description of small-amplitude nonlinear vibrations.

For conservative oscillatory systems with no resonance, the Lyapunov subcenter-manifold theorem (Kelley [14]) guarantees the existence of a unique, analytic surface of periodic orbits that is tangent to any selected two-dimensional modal subspace (or eigenspace) of the linearised system at the origin. Each periodic orbit in such a subcenter manifold is a NNM by the classic definition of Rosenberg [23]. In contrast, Shaw and Pierre [24] call the subcenter manifold itself a NNM.

Shaw and Pierre [24] also extend the latter view to dissipative systems, envisioning NNMs as invariant manifolds tangent to modal subspaces of an equilibrium point (see the reviews of Vakakis [29], Kerschen et al. [15], Peeters et al. [21], and Avramov and Mikhlin [17, 1]). As observed

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Fig. 1: The three main types of NNMs (trivial, periodic and quasiperiodic) and their corresponding SSMs (autonomous, periodic and quasiperiodic). The NNMs are, or are born out of perturbations of, a fixed point. The SSMs are, in contrast, the smoothest invariant manifolds tangent to a subbundle along the NNM whose fibers are close to a specific modal subspace $E_{j_1,\ldots,j_q}$ of the linearised system. Here the indices $j_1,\ldots,j_q$ refer to an arbitrary selection of $q$ two-dimensional modal subspaces of the linearised system (cf. [10] for more detail).

recently, however, by multiple authors (Neild et al. [18], Cirillo et al. [4], and Haller and Ponsioen [10]), such invariant manifolds are non-unique even for linear systems, let alone for nonlinear ones. Formal Taylor expansions and operational numerical procedures do nevertheless yield approximate invariant surfaces in most problems. This effectiveness of the Shaw–Pierre approach has inspired its formal extension to invariant manifolds modelled over multiple modes (Pescheck et al. [22]), as well as to time-dependent invariant manifolds under external harmonic forcing (see, e.g., Jiang, Pierre and Shaw [13], Gabale and Sinha [9]).

In a recent mathematical treatment, Haller and Ponsioen [10] unites the Rosenberg and Shaw–Pierre NNM concepts for dissipative systems under possible time-dependent forcing. In this setting, a *nonlinear normal mode* (NNM) is a near-equilibrium oscillation with finitely many frequencies. This NNM concept includes the trivial case of an equilibrium with no (i.e. zero) oscillation frequencies; Rosenberg’s case of a periodic orbit; and the case of a quasiperiodic oscillation with finitely many rationally independent frequencies. Haller and Ponsioen [10] then defines a *spectral submanifold* (SSM) as the smoothest invariant manifold tangent to a spectral subbundle along a NNM. For a trivial NNM (equilibrium), a spectral subbundle is a modal subspace of the linearised system at the equilibrium, and hence an SSM is the smoothest Shaw–Pierre-type invariant manifold tangent to this modal subspace. Similarly, for periodic or quasiperiodic NNMs, an SSM is the smoothest invariant manifold among those sought formally in time-dependent extensions of the Shaw-Pierre surfaces (see Fig. [1] for illustration).

Here we adopt the above distinction between NNMs and SSMs and restrict our attention to SSMs of trivial NNMs (fixed points). Even in this simplest setting, it is not immediate that a single smoothest invariant manifold tangent to a modal subspace of the fixed point actually exists. This question, however, is positively answered under certain conditions by the abstract invariant manifold results of Cabré et al. [3], as explained by Haller and Ponsioen [10]. These results also provide a computationally efficient way of computing SSMs using the parametrisation method (cf.
Haro et al. [11] for a general introduction.

The reduced dynamics on a single-mode SSM gives an exact nonlinear extension of the linear dynamics of the modal subspace to which the SSM is tangent. This extension is characterised by a backbone curve, i.e., a graph expressing the instantaneous vibration amplitude as a function of the instantaneous vibration frequency along the SSM.

Without specific concern for SSMs, backbone curves have been approximated operationally in a number of numerical studies. One approach assumes that the mechanical system is conservative apart from a weak damping term that is a linear (or at least odd), position-independent function of the velocities. In such a system, a periodic forcing producing a $90^\circ$ out-of-phase response preserves exactly a periodic orbit (i.e., Rosenberg’s NNM) of the conservative limit (Peeters et al. [19, 20]). The systematic construction of external forcing that yields the required $90^\circ$ phase lag for various frequencies is usually referred to as the force appropriation method. In practice, force appropriation involves a tedious tuning process that also suffers from unintended interactions between a shaker and the nonlinear system.

To expedite the backbone curve construction, one may locate a single high-amplitude periodic NNM from force appropriation, then turn off the forcing, and identify – by signal processing – the instantaneous amplitude-frequency relation of the decaying vibration as the backbone curve. Usually referred to as the resonance decay method, this process tacitly assumes that the decaying vibrations closely follow a Lyapunov subcenter manifold of the conservative limit. In our terminology, the assumption is that the analytic subcenter manifold of the conservative limit perturbs smoothly to a unique SSM under small enough damping. While this statement seems exceedingly difficult to establish mathematically, it appears to hold true for small enough viscous damping (see Kerschen et al. [16] and Peeters et al. [19]). Therefore, under weak viscous damping, the resonance decay approach gives consistent results for SSMs, provided that the decaying vibrations are close to the (yet unknown) SSM. Small errors in this initialisation are expected to be persistent for fast SSMs, i.e., SSMs tangent to the modal subspaces with higher damping. This is because the off-SSM components (errors) in the initial conditions decay much slower than the in-SSM components (useful signal), which adds substantial inaccuracy to the backbone curve construction.

A third approach to backbone-curve construction uses time-dependent normal forms to construct approximate reduced-order nonlinear models of the system near each natural frequency. The backbone curve is then obtained approximately by the method of harmonic balance applied to the reduced model under resonant parametric forcing (see, e.g., [18]). An advantage of this method is its ability to deal with internal resonances, producing intricate multi-dimensional backbone surfaces. The underlying assumption for all this is that the higher-order normal form terms coupling the reduced model to the remaining degrees of freedom are small, and that the oscillations have small enough amplitudes for the harmonic balance method to be reasonably accurate. A further important assumption is that an exact nonlinear model for the system is available for the purposes of computing a normal form. This tends to limit the practical use of this approach to simple geometries and materials.

In summary, several methods for numerical or experimental backbone-curve construction are available, but all make assumptions limiting their range of applicability. These assumptions include small, position-independent, and linear viscous damping; small enough oscillation amplitudes; an accurate initial knowledge of the SSM; and yet unproven results on the smooth persistence of Lyapunov subcenter manifolds as SSMs under nonzero damping.

Here we develop a backbone-curve identification method that addresses most of the above challenges. We infer backbone curves directly from the dynamics on the SSM, without making any assumptions on the damping or any reference to the Lyapunov subcenter manifold of the con-
servative limit. As input, we assume that tracks of decaying vibration data are available in the vicinity of \( N \) natural frequencies. We simultaneously assimilate all this data into a nonlinear discrete mapping model of the near-equilibrium dynamics of the system. We then construct backbone curves analytically from the nonlinear dynamics on the SSMs of this discrete mapping.

We illustrate the generality and accuracy of this approach on two examples. Our first example is a two-degree-of-freedom nonlinear mechanical system, for which we perform both an analytic and a data-assimilating construction of the backbone curves for comparison. Our second example is a clamped-clamped beam experiment [5], in which we determine the first three SSMs simultaneously from measurements of decaying vibration signals.

2 Set-up

We start with an \( n \)-degree of freedom, autonomous mechanical system of the general form

\[
M(q)\ddot{q} - f(q, \dot{q}) = 0, \quad f(0, 0) = 0, \tag{1}
\]

where the mass matrix \( M(q) \in \mathbb{R}^{n \times n} \) and its inverse \( M^{-1}(q) \) are of class \( C^r \), with \( r \geq 1 \), in the generalised coordinate vector \( q \in \mathbb{R}^n \). The forcing vector \( f(q) \in \mathbb{R}^n \) is also \( C^r \) in its arguments, containing all conservative and non-conservative autonomous forces, both linear and nonlinear.

Beyond taking nonnegative integer values, the smoothness parameter \( r \) is also allowed to be \( r = \infty \) (arbitrarily many times differentiable functions) or \( r = a \) (analytic functions, i.e., \( C^\infty \) functions with a convergent Taylor expansion in a complex neighbourhood of \((q, \dot{q}) = (0, 0)\)). The degree of freedom \( n \geq 1 \) is allowed to be arbitrarily high and may also be in principle infinity (continuum vibrations), although some of our assertions about properties of the solutions would need to be verified on a case-by-case basis in the infinite-dimensional setting. By the formulation in (1), \( q \equiv 0 \) is an equilibrium point for the system.

The equivalent first-order form of the differential equation (1) is obtained by letting \( x = (q, \dot{q}) \in \mathbb{R}^{2n} \), which leads to

\[
\dot{x} = \mathcal{F}(x), \quad \mathcal{F}(0) = 0, \quad \mathcal{F}(x) = \left( \begin{array}{c} \dot{q} \\ M^{-1}(q)f(q, \dot{q}) \end{array} \right), \tag{2}
\]

where \( \mathcal{F}(x) \in \mathbb{R}^{2n} \) is \( C^r \) in its arguments. The solutions \( x(t) \) of (2) give rise to the flow map

\[
\Xi_t: x_0 \mapsto x(t),
\]

where \( x_0 = x(0) \).

The linearisation of (2) at the equilibrium point \( x = 0 \) is given by

\[
\dot{x} = \mathcal{A}x, \quad \mathcal{A} = D\mathcal{F}(0). \tag{3}
\]

We assume that \( \mathcal{A} \) has \( n \) pairs of complex conjugate eigenvalues \( \lambda_1, \bar{\lambda}_1, \ldots, \lambda_n, \bar{\lambda}_n \), satisfying

\[
\text{Re}\lambda_n \leq \ldots \leq \text{Re}\lambda_1 < 0, \tag{4}
\]

and hence the equilibrium point is linearly asymptotically stable. This context is relevant for underdamped structural vibrations, in which the nonlinear system (1) is known to have a stable equilibrium, but the exact nature of its nonlinearities is unknown.
3 Sampled nonlinear vibrations

To reduce the complexity of the flow map $\Xi_t$ in our analysis, we will focus on temporally sampled approximations to $\Xi_t$. Iterating such discrete approximations, one can still reproduce the main features of the nonlinear dynamics at regular time intervals. Constructing the sampled dynamics via a stroboscopic (or Poincaré) map is, however, only feasible when the full dynamical system (1) is precisely known, and hence trajectories from arbitrary initial conditions can be generated. In practice, this is generally not the case.

Instead, we seek to reconstruct a sampled representation of $\Xi_t$ from a limited number of observations of trajectories. The scalar observable along trajectories can be, for instance, a position or a velocity coordinate of a certain material point of the mechanical system (1). We denote this observable by $\varphi(x) : \mathbb{R}^{2n} \to \mathbb{R}$, i.e., as a scalar function of the state variable $x$ alone. We then build a new state vector $\xi \in \mathbb{R}^{2\nu}$ out of $2\nu$ subsequent observations along trajectories of (1) by letting

$$\xi = \Phi(x), \quad \Phi(x) := (\varphi(x), \varphi(\Xi_T(x)), \ldots, \varphi(\Xi_T^{2\nu-1}(x))) \in \mathbb{R}^{2\nu}, \quad \nu \geq 1. \quad (5)$$

We have selected the dimension of $\xi$ to be even (i.e., $2\nu$) to ensure basic spectral compatibility between the dynamics of $\xi$ and the dynamics of $x$, as discussed in more detail below.

A sampling map $F$ can be defined as the discrete mapping advancing the current $2\nu$ observations by one, i.e., from the observation vector $\Phi(x)$ to the observation vector $\Phi(\Xi_T(x))$. Specifically, we define the mapping $F : \mathbb{R}^{2\nu} \to \mathbb{R}^{2\nu}$ via the relation

$$\xi_{k+1} := F(\xi_k) = F^k(\xi_0), \quad k \in \mathbb{N}, \quad \xi_0 = \Phi(x_0), \quad (6)$$

or, equivalently, as

$$\Phi \circ \Xi_T = F \circ \Phi. \quad (7)$$

By construction, the $x = 0$ equilibrium point of system (1) is mapped into a fixed point $\xi^0 = \Phi(0)$ of the sampling map $F$ under $\Phi$. If necessary, we shift the $\xi$ coordinates as $\xi \to \xi - \xi^0$ to achieve $\xi^0 = 0$. Therefore, without loss of generality, we may assume

$$F(0) = 0. \quad (8)$$

As a consequence, whenever $\varphi \in C^r$ holds, a Taylor expansion of $F$ at the origin must be of the form

$$F(x) = \sum_{|m|=1}^{r} a_m \xi_1^{m_1} \cdots \xi_{2\nu}^{m_{2\nu}} + o(|x|^r) = \sum_{|m|=1}^{r} a_m \xi^m + o(|x|^r) \quad (9)$$

for appropriate coefficient vectors $a_m \in \mathbb{R}^{2\nu}$ and integer index vector $m = (m_1, \ldots, m_{2\nu}) \in \mathbb{N}^{2\nu}$, whose norm we measure as $|m| = \sum_{i=1}^{2\nu} m_i$. We have used here the short-hand notation $\xi^m = \xi_1^{m_1} \cdots \xi_{2\nu}^{m_{2\nu}}$.

4 Delay embedding

The definition (6) does not immediately clarify the relation between the dynamics of the flow map $\Xi_T$ and the dynamics of the sampling map $F$. The Takens Embedding Theorem [28], however, guarantees that such a relationship exists on invariant manifolds of generic flow maps $\Xi_T$, at least for generic observables $\varphi$, as long as the sample length $2\nu$ is long enough.
Specifically, if $W$ is a compact, $d$-dimensional inflowing-invariant manifold $\mathbf{S}$ of system (1) and $2\nu \geq 2d + 1$ holds, then the set of function pairs $(\Xi_T, \varphi)$ for which $\Phi(W)$ is diffeomorphic to $W$ is open and dense in the product space $\mathcal{D}^r(W) \times C^r(W, \mathbb{R})$. Here $\mathcal{D}^r(W)$ denotes the space of $C^r$ diffeomorphisms of $W$, and $C^r(W, \mathbb{R})$ denotes the space of $C^r$ scalar functions defined on $W$, with both spaces endowed with the $C^r$ topology.

Takens’s theorem can further be strengthened (Huke [12], Stark [20]) when $\Xi_T$ has only a finite number of periodic orbits of periods less than $2\nu$, with all periodic orbit admitting distinct Floquet multipliers. In this case, for any $\Xi_T$, there is an open and dense set of observables $\varphi \in C^r(W, \mathbb{R})$ such that $\Phi$ is an embedding of $W$ into $\mathbb{R}^{2\nu}$. This version of the theorem is particularly helpful in our setting, as close enough to its asymptotically stable equilibrium at $x = 0$, the flow map $\Xi_T$ will have no periodic orbits. Therefore, it is enough for us to require the observable $\varphi$ to be generic, without having to assume anything further for $\Xi_T$. This simplification holds true on any extended neighbourhood of the origin that has the required low number of nondegenerate periodic orbits discussed above.

For such generic observables, $\Phi(W) \subset \mathbb{R}^{2\nu}$ is a diffeomorphic copy of the invariant manifold $W \subset \mathbb{R}^{2n}$. Importantly, $\Phi(W)$ is then an invariant manifold for the discrete dynamical system (6) by definition. On this invariant manifold, the map $F$ is conjugate to the flow map $\Xi_T$ by formula (7), which can now be re-written as

$$F = \Phi \circ \Xi_T \circ \Phi^{-1} : \Phi(W) \to \Phi(W)$$

(10)

given that $\Phi$ is a diffeomorphism onto its image.

Consequently, any coordinate-independent dynamical feature of $\Xi_T$ will be shared by the mapping $F$. This will be a crucial point in our strategy to build a faithful reduced-order model for system (1). Specifically, we will use an experimentally observed scalar $\varphi$ to approximate the Taylor expansion (7) of the mapping $F$.

Our focus here is the reconstruction of the dynamics of $F$ on two-dimensional invariant manifolds $W$ tangent to two-dimensional modal subspaces of the linearised flow map $D\Xi_T(0)$ at the equilibrium point. We thus have $d = 2$, and hence the minimal dimension for the embedding space $\mathbb{R}^{2\nu}$ required by Takens’s theorem is $2\nu \geq 5$, implying $\nu \geq 3$ (For our first example of a two-degree-of-freedom model in section 9.1, a comparison with exact analytic computation shows that a reconstruction with $\nu = 2$ already suffices, but this cannot be generally guaranteed.)

The tangent space $T_0W$ of $W$ at the origin is a two-dimensional invariant subspace for $D\Xi_T(0)$. Specifically, $T_0W$ is the modal subspace corresponding to a pair of complex conjugate eigenvalues $(\mu_\ell, \bar{\mu}_\ell) = (e^{\lambda_\ell T}, e^{\bar{\lambda}_\ell T})$, where $(\lambda_\ell, \bar{\lambda}_\ell)$ are eigenvalues of $\mathcal{A}$, ordered as in (4). The conjugacy relationship (10) and formula (8) then implies that $(\mu_\ell, \bar{\mu}_\ell)$ are also eigenvalues of the linearised sampling map $DF(0)$ at $\xi = 0$, i.e., we have

$$\{\mu_\ell, \bar{\mu}_\ell\} = \{e^{\lambda_\ell T}, e^{\bar{\lambda}_\ell T}\} \subset \text{Spect}\{DF(0)\},$$

(11)

where $\text{Spect}\{DF(0)\}$ denotes the spectrum (i.e., set of eigenvalues) of the Jacobian matrix $DF(0)$.

5 Spectral submanifolds of the sampling map

The linearised sampled dynamics near the fixed point $\xi = 0$ is governed by the Jacobian $A = DF(0)$ of the sampling map $F$. We assume that this Jacobian is diagonalisable and collect its
complex eigenvectors in a matrix $V \in \mathbb{C}^{2\nu \times 2\nu}$. Introducing the new coordinate $y \in \mathbb{C}^{2\nu}$ via the relation

$$\xi =Vy,$$  \hspace{1cm} (12)

we obtain the transformed form of (6) as

$$y_{k+1} = \Lambda y_k + G(y_k), \quad \Lambda = \text{diag}(\mu_1, \mu_2, \ldots, \mu_{2\nu}) = V^{-1}AV, \quad \mu_{2l} = \bar{\mu}_{2l-1}, \quad l = 1, \ldots, \nu.$$  \hspace{1cm} (13)

where $G(y)$ are nonlinear coupling terms with $D G(0) = 0$. If, specifically, the $l^{th}$ linear mode of system (1) is brought to the standard form

$$\ddot{\eta}_l + 2\zeta_l \omega_l \dot{\eta}_l + \omega_l^2 \eta_l = 0,$$

with the damping ratio $\zeta_l < 1$ and undamped natural frequency $\omega_l \in \mathbb{R}$, then we obtain

$$\lambda_l, \lambda_l^* = -\zeta_l \omega_l \pm i \sqrt{1 - \zeta_l^2} \omega_l,$$

and hence the eigenvalues of $A$ and $\Lambda$ can be expressed as

$$\mu_l, \bar{\mu}_l = e^{-\zeta_l \omega_l T} e^{\pm i \sqrt{1 - \zeta_l^2} \omega_l T}.$$  \hspace{1cm} (14)

We recall that if the sampling map $F$ was constructed from observables along a two-dimensional invariant manifold $W$ of system (2), then $\Lambda$ has a complex conjugate pair of eigenvalues related to a pair of eigenvalues of $A$ through the relationship (11). In particular, $W$ is tangent to an underdamped modal subspace $E$ of the equilibrium $y = 0$ corresponding to the eigenvalue pair $\mu_\ell, \bar{\mu}_\ell$ for some $\ell \in [1, 2n - 1]$, as implied by assumption (4).

The existence of a two-dimensional invariant manifold $W$ tangent to a two-dimensional spectral subspace $E$ of the linearised system (3) was first envisioned in the seminal work of Shaw and Pierre [24], then extended to more general settings by the same authors and collaborators (see, e.g., Vakakis [29], Kerschen et al. [15], Peeters et al. [21], and Avramov and Mikhlin [17, 1] for reviews). On closer inspection, one finds that such invariant manifolds indeed exist under certain nonresonance conditions, but are non-unique and may have a low order of differentiability (Neild et al. [18], Cirillo et al. [4], Haller and Ponsioen [10]).

Following Haller and Ponsioen [10], we address this uniqueness and smoothness issue with the help of the following definition:

**Definition 1.** A spectral submanifold (SSM) $W(E)$ corresponding to a spectral subspace $E$ of the operator $A$ is an invariant manifold of $F$ with the following properties:

(i) $W(E)$ is tangent to $E$ at $y = 0$ and has the same dimension as $E$;

(ii) $W(E)$ is strictly smoother than any other invariant manifold satisfying (i).

If exists, an SSM serves as the unique nonlinear continuation of the modal subspace $E$ to the nonlinear system (2). By definition, all other invariant manifolds tangent to the same modal subspace have only finitely many derivatives, and hence any high-enough order Taylor expansion is only valid for the SSM.

Haller and Ponsioen [10] has pointed out that the more general and abstract results of Cabré et al. [3] imply the existence of spectral submanifolds for $F$ under appropriate conditions on the spectrum of $A$. Below we recall these results stated specifically in the context of the sampling
Spectral submanifolds of the sampling map

We note that by the conjugacy relation (10), the existence of a two-dimensional SSM for the sampling map $F$ is equivalent to the existence of a two-dimensional SSM for the mechanical system (2), as long as the observable $\varphi$ is generically chosen.

We start by considering a two-dimensional eigenspace $E \subset \mathbb{C}^2_\nu$ of the linearised sampling map $A$, corresponding to the eigenvalue pair $\mu_\ell, \overline{\mu}_\ell$. We define the relative spectral quotient $\sigma(E)$ as the positive integer

$$\sigma(E) = \text{Int} \left[ \frac{\min_{j \neq \ell, \ell + 1} \log |\mu_j|}{\log |\mu_\ell|} \right] \in \mathbb{N}^+. \quad (15)$$

For the linearised sampling map $A (3)$, the constant $\sigma(E)$ is the integer part of the ratio of the strongest decay rate towards the spectral subspace $E$ to the decay rate along $E$. This integer ratio turns out to control the smoothness of the spectral submanifold $W(E)$, as we shall see shortly.

We assume now that $\sigma(E) \leq r$, (16) i.e., the degree of smoothness of the sampling map $F$ is at least as high as the relative spectral quotient of the modal subspace $E$. Finally, we assume that no resonance relationships between order 2 and order $\sigma(E)$ hold between the eigenvalues $\mu_\ell, \overline{\mu}_\ell$ and the rest of the spectrum of $A$:

$$\mu_s^{s_1} \overline{\mu}_s^{s_2} \neq \mu_j, \quad \forall j \neq \ell, \ell + 1, \quad 2 \leq s_1 + s_2 \leq \sigma(E). \quad (17)$$

We then have the following existence and uniqueness result for spectral submanifolds of the sampling map.

**Theorem 1.** Assume that conditions (16)-(17) are satisfied. Then the following statements hold:

(i) There exists an SSM, $W(E)$, for the nonlinear sampling map $F$, that is tangent to the invariant subspace $E$ at the $\xi = 0$ fixed point.

(ii) The invariant manifold $W(E)$ is class $C^r$ smooth and unique among all two-dimensional, class $C^{\sigma(E)+1}$ invariant manifolds of $F$ that are tangent to $E$ at $\xi = 0$.

(iii) The SSM $W(E)$ can be viewed as a $C^r$ immersion of an open set $U \subset \mathbb{C}^2$ into the phase space $\mathbb{C}^{2\nu}$ of $F$ via a map

$$W : U \subset \mathbb{C}^2 \rightarrow \mathbb{C}^{2\nu}, \quad W(U) = W(E). \quad (18)$$

(iv) There exists a $C^r$ polynomial map $R : U \rightarrow U$ such that

$$F \circ W = W \circ R, \quad (19)$$

i.e., the dynamics on the SSM, expressed in the coordinates $z = (z_\ell, \overline{z}_\ell) \in U$, is given by the polynomial mapping $R$. This polynomial mapping only has terms up to order $O \left( |z|^{\sigma(E)} \right)$.

(v) If, for some integer $j_0 \geq 2$, all internal non-resonance conditions

$$\mu_s^{s_1} \overline{\mu}_s^{s_2} \neq \mu_\ell, \mu_{\ell+1}, \quad j_0 \leq s_1 + s_2 \leq \sigma(E) \quad (20)$$

hold within $E$, then the polynomial $R$ in (19) can be selected to contain only terms up to order $j_0 - 1$. 
(vi) If the observable $\varphi$ used in the construction of the sampling map $F$ is generic, then a two-dimensional SSM, $W(E)$, tangent to the subspace $E$ at $x = 0$ exists for the original system \([2]\). The invariant manifold $W(E)$ shares the properties (i)-(v) of $W(E)$ due to the conjugacy relationship \([10]\).

Proof. As explained in detail by Haller and Ponsioen \([10]\), the proofs of statements (i)-(v) follow from a direct application of the more general Theorem 1.1 of Cabré et al. \([3]\) on invariant manifolds tangent to spectral subspaces of arbitrary dimension, for mappings defined on Banach spaces. Statement (vi) can be concluded by invoking the strengthened version of Taken’s theorem (Huke \([12]\), Stark \([26]\)) that we recalled in section 4, then interpreting the resulting structures of the Poincaré map $\Xi_T$ for the flow map of \([2]\).

Remark 1. If the linearisation of the original mechanical system \([1]\) satisfies the classic proportional damping hypothesis, we can use \([14]\) to rewrite the relative spectral quotient defined in \([15]\) as

$$\sigma(E) = \text{Int} \left[ \max_{j \neq \ell, \ell+1} \frac{\zeta_j \omega_j}{\zeta_{\ell} \omega_{\ell}} \right]. \quad (21)$$

In this case, the expressions in the external nonresonance conditions \([17]\) for $E$ take the specific form

$$\left( (s_1 - s_2) \omega_\ell \sqrt{1 - \zeta_\ell^2} \mod \frac{2\pi}{T} \right) \neq \left( \frac{\zeta_j \omega_j}{\omega_{nj} \sqrt{1 - \zeta_j^2}} \mod \frac{2\pi}{T} \right), \quad 2 \leq s_1 + s_2 \leq \sigma(E),$$

where $\mod$ denotes the modulo operation that takes sampling into account. In the limit of zero damping, an external resonance for $E$ means that a frequency $\omega_j$ outside $E$ is an integer multiple of the frequency $\omega_\ell$ inside $E$.

Statements (iii)-(v) of Theorem 1 imply that, unlike in the Shaw-Pierre \([24]\) construction, the SSM inferred from the results of Cabré et al. \([3]\) is not assumed to be a graph over the subspace $E$ in the phase space of $F$. This allows $W(E)$ to be constructed on larger domains on which it can produce folds over $E$. This parametrisation approach to SSM construction was also re-discovered recently by Cirillo et al. \([4]\) under the assumption that the flow is analytically linearisable near the fixed point $x = 0$. Analytic linearisation does not allow for any resonance in the spectrum of $A$ and, in return, transforms the full dynamics of the mapping $F$ into that of $\Lambda y$. In the case of a near-resonance – which arises for all weakly underdamped modes, as we shall see below – analytic linearisation can therefore only be constructed on a very small domain near the fixed point. This disallows the type of direct identification of nonlinear terms that we discuss next.

6 Dynamics on SSMs: Backbone curves

Since $|\mu_\ell| < 1$ holds by assumption \([4]\), we find that, strictly speaking, the internal non-resonance condition \([20]\) is always satisfied for nonzero damping.

As seen in the construct of Cabré et al. \([3]\), however, even an approximate resonance $\mu_\ell^{s_1} \bar{\mu}_\ell^{s_2} \approx \mu_j$ causes the near-identity transformation $(\xi_\ell, \tilde{\xi}_\ell) \mapsto (z_\ell, \tilde{z}_\ell)$ to have small denominators, limiting the existence of this transformation to a tiny neighbourhood of the $\xi = 0$ fixed point. Since our interest here is to obtain an approximation of the dynamics of $F$ on a sizeable neighbourhood of the fixed point within the SSM, we do not insist on the removal of approximately resonant terms in
the \((z_\ell, \bar{z}_\ell)\) coordinate system. Rather, we observe that for small damping ratios (i.e., for \(|\mu_\ell| \approx 1\)), the low-order near-resonance relationships

\[
\mu_\ell^2 \mu_\ell \approx \mu_\ell, \quad \mu_\ell \bar{\mu}_\ell \approx \bar{\mu}_\ell
\]  

are always satisfied, and hence the minimal possible integer \(j_0\) satisfying \((20)\) (with \(\neq\) replaced with \(\approx\)) is \(j_0 = 1\). Accordingly, the approximately failing resonance conditions in \((20)\) prompt us to seek \(R\) (cf. statement (iv) of Theorem \(1\)) as a cubic polynomial of the form

\[
R(z) = \left( \begin{array}{c}
\mu_\ell z_\ell + \beta_\ell \bar{\mu}_\ell \bar{z}_\ell + \ldots \\
\bar{\mu}_\ell z_\ell + \beta_\ell \mu_\ell \bar{z}_\ell + \ldots
\end{array} \right).
\]  

Introducing polar coordinates \(z = \rho \text{e}^{i\theta}\), we can further transform \((23)\) to the real amplitude-phase components

\[
\rho_\ell \mapsto \rho_\ell |\mu_\ell + \beta_\ell \rho_\ell^2|, \quad \theta_\ell \mapsto \theta + \text{arg}(\mu_\ell + \beta_\ell \rho_\ell^2).
\]

Equation \((25)\) then provides an instantaneous frequency of nonlinear oscillations, with the instantaneous oscillation amplitude governed by equation \((24)\). Given that the sampling period we have used is \(T\), the leading-order approximation of the instantaneous oscillation frequency in the original nonlinear system \((2)\) is

\[
\omega(\rho_\ell) = \frac{\text{arg}(\mu_\ell + \beta_\ell \rho_\ell^2)}{T}.
\]

We take the instantaneous leading-order amplitude of the corresponding trajectories of \((2)\) to be the norm of \(z(\rho_\ell, \theta_\ell) = (\rho \text{e}^{i\theta_\ell}, \rho \text{e}^{-i\theta_\ell})\) in the original \(\xi\) coordinates. A nominal instantaneous amplitude \(\text{Amp}(\rho)\) of the vibration can then be calculated from \((12)\) as the \(L_2\) norm of the norm of \(z(\rho_\ell, \theta_\ell)\) in the original \(\xi\) coordinates:

\[
\text{Amp}(\rho_\ell) = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} |V \ W(z(\rho_\ell, \theta_\ell))|^2 \ d\theta}.
\]

Here, the linear map \(V\) is the one appearing in \((12)\), and the mapping \(W\) in the one appearing in \((18)\).

**Definition 2.** We call the parametrised curve

\[
B_\ell = \{ \omega(\rho_\ell), \text{Amp}(\rho_\ell) \}_{\rho_\ell \in \mathbb{R}^+} \subset \mathbb{R}^2
\]

the backbone curve associated with the nonlinear dynamics on the SSM, \(W(\mathcal{E})\).

The key to the computation of the backbone curve \((28)\) is, therefore, the computation of the single complex coefficient \(\beta_\ell\) and of the mapping \(W(z)\). This is because both the eigenvalue \(\mu_\ell\) and the sampling time \(T\) are already assumed to be known.

**Remark 2.** It is often desirable to translate the \(\varphi\)-based backbone curve \(B_\ell\) defined in \((28)\) to a backbone curve observed directly for a given mechanical coordinate \(q_j\). When the observable is an invertible function of such a \(q_j\), that is, \(\varphi(q, \dot{q}) = \varphi(q_j)\), we can use the inverse, defined by \(q_j = P(\xi) = P(\varphi(q_j))\). Also notice that by the definition of the observable space, coordinates of \(\xi\)
are just sampled values of the same observed quantity. Therefore, when calculating an amplitude, it is reasonable to consider just a single component of $\xi$, for example, $\xi_1$. With this in mind, we consider $P$ functions in the particular form $P(\xi) = P(\xi_1)$. As a result, the observed amplitude in the $q_j$ mechanical coordinate can be computed as

$$\text{Amp}(\rho_\ell) = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} |P(VW(z(\rho_\ell, \theta_\ell)))|^2 \, d\theta}.$$ 

To compute the complex parameter $\beta$ in equation (26), we need to solve (19). To this end, we seek the Taylor series coefficients of the $j$th coordinate functions, $W_j(z) \in \mathbb{C}$, $j = 1, \ldots, 2\nu$, of the mapping $W(z)$ up to third order. Similarly, we seek the third-order Taylor coefficient $\beta_\ell \in \mathbb{C}$ of the polynomial mapping $R(z)$ defined in (23). All these unknowns should be expressed in the end as functions of the $j$th coordinate functions $G_j(y) \in \mathbb{C}$, $j = 1, \ldots, 2\nu$, of the nonlinear part $G(y)$ of the transformed sampling map $F$. The relevant Taylor expansions are in the general form

$$G_j(y) = \sum_{|m| \geq 2} g_j^m y^m, \quad m \in \mathbb{N}^{2\nu}, \quad g_j^m \in \mathbb{C}, \quad j = 1, \ldots, 2\nu,$$

$$W_j(z) = \sum_{|s| \geq 1} w_j^s z^s, \quad s \in \mathbb{N}^2, \quad w_j^s \in \mathbb{C}, \quad j = 1, \ldots, 2\nu.$$

In expressing the solutions of (19) in terms of these coefficients, we will use the short-hand notation $(p_{@j})$ for an integer multi-index whose elements are zero, except for the one at the $j$th position, which is equal to $p$:

$$(p_{@j}) := \left(0, \ldots, 0_{j-1}, p, 0_{j+1}, \ldots, 0\right) \in \mathbb{N}^{2\nu}.$$ 

We will also concatenate this notation to refer to multi-indices whose entries are zero except at prescribed locations:

$$(p_{@j_1}, q_{@j_2}) := \left(0, \ldots, 0_{j_1-1}, p_{j_1}, 0_{j_1+1}, \ldots, 0_{j_2-1}, q_{j_2}, 0_{j_2+1}, \ldots, 0\right) \in \mathbb{N}^{2\nu}.$$ 

For $j_1 \equiv j_2 = j$, we let

$$(p_{@j}, q_{@j}) := ((p + q)_{@j}) = \left(\ldots, 0_{j-1}, p + q, 0_{j+1}, \ldots\right) \in \mathbb{N}^{2\nu}.$$ 

With all this notation, we obtain the following result:

**Theorem 2.** Suppose that the assumptions of Theorem 1 hold but with the strengthened version

$$\mu_1^{s_1} \mu_2^{s_2} \not\sim \mu_j, \quad \forall j \neq \ell, \ell + 1, \quad 1 \leq s_1 + s_2 \leq \sigma(E)$$

of the external non-resonance condition (17). Then, for any $j \in [1, 2\nu]$, the $j$th coordinate function $W_j$ of the mapping $W$ and the cubic Taylor coefficient $\beta_\ell$ of the conjugate map $R$ are given by the following formulas:

$$w_j^{(1,0)} = \delta_{j\ell}, \quad w_j^{(0,1)} = \delta_{j(\ell+1)},$$
\[ w_j^{(2,0)} = \frac{g_j^{(2\ell)}}{\mu_\ell^2 - \mu_j}, \quad w_j^{(1,1)} = \frac{g_j^{(1\ell,1\ell)}}{\mu_\ell^2 \mu_\ell^2 - \mu_j}, \quad w_j^{(0,2)} = \frac{g_j^{(2\ell,1\ell)}}{\mu_\ell^2 - \mu_j}, \]

\[ w_j^{(3,0)} = \sum_{q=1}^{2\nu} \frac{(1 + \delta_{\ell q}) g_j^{(1\ell,1\ell,q)}}{\mu_\ell^2 - \mu_j} w_q^{(2,0)} + \frac{g_j^{(3\ell)}}{\mu_\ell^2 - \mu_j}, \]

\[ w_j^{(2,1)} = (1 - \delta_{\ell j}) \sum_{q=1}^{2\nu} \frac{(1 + \delta_{\ell q}) g_j^{(1\ell,1\ell,q)}}{\mu_\ell^2 \mu_\ell^2 - \mu_j} w_q^{(1,1)} + \frac{g_j^{(2\ell,1\ell)}}{\mu_\ell^2 - \mu_j}, \]

\[ w_j^{(1,2)} = (1 - \delta_{\ell (\ell + 1)}) \sum_{q=1}^{2\nu} \frac{(1 + \delta_{\ell q}) g_j^{(1\ell,1\ell,q)}}{\mu_\ell^2 \mu_\ell^2 - \mu_j} w_q^{(2,0)} + \frac{g_j^{(2\ell,1\ell)}}{\mu_\ell^2 - \mu_j}, \]

\[ \beta_\ell = \sum_{q=0}^{2\nu} \frac{(1 + \delta_{\ell q}) g_j^{(1\ell,1\ell,q)}}{\mu_\ell^2 \mu_\ell^2 - \mu_j} w_q^{(1,1)} + \frac{g_j^{(2\ell,1\ell)}}{\mu_\ell^2 - \mu_j}. \]

**Proof.** See Appendix A. \(\square\)

**Remark 3.** Theorem 2 only provides the solution of the homological equation (19) up to cubic order. This equation, however, can be solved by symbolic computations up to any order for the Taylor coefficients of the functions \( W \) and \( R \). For instance, up to quintic order, the near-resonance conditions (22) imply the general form

\[ R(z) = \begin{pmatrix} \mu_\ell z_\ell + \beta_\ell \bar{z}_\ell^2 z_\ell + \gamma_\ell z_\ell z_\ell \bar{z}_\ell^2 + \ldots \\ \bar{\mu}_\ell \bar{z}_\ell + \beta_\ell \bar{z}_\ell^2 \bar{z}_\ell + \gamma_\ell \bar{z}_\ell \bar{z}_\ell \bar{z}_\ell^2 + \ldots \end{pmatrix} \]

for the polynomial conjugate dynamics on the SSM \( E \). The coefficient \( \gamma_\ell \) as well as the quartic and quintic terms of \( W \) can be found recursively from equation (19), following the procedure outlined in Appendix A. The sampling map restricted to the SSM \( W(E) \) can be written in polar coordinates up to quintic order as

\[ \rho_\ell \rightarrow \rho_\ell |\mu_\ell + \beta_\ell \rho_\ell^2 + \gamma_\ell \rho_\ell^4|, \]

\[ \theta_\ell \rightarrow \theta + \text{arg} \left( \mu_\ell + \beta_\ell \rho_\ell^2 + \gamma_\ell \rho_\ell^4 \right), \]

yielding the instantaneous oscillation frequency in the original nonlinear system (2) as

\[ \omega(\rho_\ell) = \frac{\text{arg} (\mu_\ell + \beta_\ell \rho_\ell^2 + \gamma_\ell \rho_\ell^4)}{T}. \]
7 Reconstruction of the sampling map from data

In an experimental setting, backbone-curve identification via Theorem 2 requires the fitting of a model of $F$ to observations using an appropriate set of basis functions. Due to the polynomial form of $F$, the required basis functions are precisely vector-valued monomials of the variables $\xi_1, \ldots, \xi_{2\nu}$, not including constant terms. The lack of constant terms follows from the assumption that can always be satisfied by an appropriate shift of coordinates, if necessary.

For the polynomial-based model-identification for $F$, we employ a nonlinear autoregressive model (NAR) (Billings [2]). We order all integer vectors $(8)$, which can always be satisfied by an appropriate shift of coordinates, if necessary.

We can then write the yet unknown, $r\textsuperscript{th}$-order Taylor expansion of $F$ in the compact form

$$F(\xi) = K\psi(\xi) + r(\xi), \quad \psi_1(\xi) = \xi^{m_1},$$

where $K \in \mathbb{R}^{2\nu \times N}$ is a rectangular matrix, to be determined by minimising the residual term $r(\xi) \in \mathbb{R}^{2\nu}$ on assimilated data in the $\ell^2$ norm.

The input data to be assimilated into the NAR model consists of $P$ sequences of $M_p > 2\nu$-long observations, $\{\xi_{k}^{p}\}_{k=0}^{M_p-2\nu}$, $p = 1, \ldots, P$, with each observation sequence $\{\xi_{k}^{p}\}_{k=0}^{M_p-2\nu}$ defined as in [5]. The $\ell^2$ norm of $r(\xi)$ on $\{\xi_{k}^{p}\}_{k=0}^{M_p-2\nu}$ over all $P$ observation sequences is then given by

$$Err = \sum_{p=1}^{P} \sum_{k=0}^{M_p-2\nu} |r(\xi_{k}^{p})|^{\nu} = \sum_{p=1}^{P} \sum_{k=0}^{M_p-2\nu} |K\psi(\xi_{k}^{p}) - \xi_{k+1}^{p}|^{\nu}.$$

The matrix $K$ that minimises this norm is obtained by solving the equation $dErr/dK = 0$ for $K$. This classic computation yields $K = QP^{-1}$, where

$$P = \sum_{p=1}^{P} M_p^{-1} \sum_{k=0}^{M_p-2\nu} \psi(\xi_{k}^{p})\psi^{*}(\xi_{k}^{p}),$$

$$Q = \sum_{p=1}^{P} M_p^{-2\nu} \sum_{k=0}^{M_p-2\nu} \xi_{k+1}^{p}\psi^{*}(\xi_{k}^{p}),$$

with $*$ denoting the transposition. With this notation, the reconstructed nonlinear sampling map is

$$\tilde{F}(\xi) = QP^{-1}\psi(\xi),$$

which we will use instead of the exact sampling map $F$ in our analysis.

Assimilating multiple measurement sequences (i.e., using $P > 1$) generally reduces the effect of zero-mean additive noise on the model reconstruction. More importantly, using measurements from vibrations decaying near $P$ natural frequencies of interest allows us to build a single reduced-order discrete model map $\tilde{F}$ that simultaneously captures nonlinear behaviour near all these natural frequencies. The choice of the $\ell^2$ optimisation above was mostly dictated by convenience; in some situations, minimisation of $r(\xi)$ in the $\ell^1$ or $\ell^\infty$ norms might be more beneficial.

Since we do not know the invariant manifold $W(\mathcal{E})$ exactly, we will construct (36) from observed nonlinear vibration decay measurements initiated along two-dimensional modal subspaces of $D\Xi T(0)$. In practice, these subspaces can be approximated from linear modal analysis.
8 Summary of SMM-based backbone-curve identification algorithm

We now briefly summarise the steps in the approach we have developed in the preceding sections:

1. Fix a generic scalar observable \( \varphi(q, \dot{q}) \) and a sampling time \( T > 0 \) for the mechanical system \([1]\). Also fix an integer \( \nu \geq 3 \) as the number of SSMs to be identified for system \([1]\). Finally, select an integer \( r = \max |m| \) for the maximum degree of the polynomials used in the construction of the NAR model \([35]\) for the sampling map \( \tilde{F}(\xi) \) with \( \xi \in \mathbb{R}^{2\nu} \).

2. Collect \( P \) sequences of \( M_p \)-long observations, \( \{\xi^p_k\}_{k=0}^{M_p-2\nu} \), by letting
   \[
   \xi^p_k = (\varphi(q(kT), \dot{q}(kT)), \ldots, \varphi(q((k+2\nu-1)T), \dot{q}((k+2\nu-1)T))),
   \]
   \( p = 1, \ldots, P, \quad k = 0, \ldots, M_p-2\nu. \)

3. Compute the approximate \( 2\nu \)-dimensional sampling map \( \tilde{F}(\xi) \) from formula \([36]\).

4. Transform \( \tilde{F}(\xi) \) to its complex diagonal form \([13]\).

5. Using Theorem 1, compute the leading order Taylor coefficients of the mapping \( W(z_\ell) \) and the leading order polynomial coefficient \( \beta_\ell \) for each SSM, \( W(\mathcal{E}) \), provided that the nonresonance condition \([17]\) holds.

6. Calculate the backbone curve \( \mathcal{B}_\ell \) defined in \([28]\) for \( W(\mathcal{E}) \). Higher-order approximations to \( \mathcal{B}_\ell \) can be computed similarly, as summarised briefly in Remark \([3]\).

This algorithm provides the simplest possible first-order approach to SSM-based backbone curve reconstruction. This simplest approach does not fully exploit the uniqueness class \( C^{\sigma(\mathcal{E})+1} \) of \( W(\mathcal{E}) \), as guaranteed by Theorem 1. To obtain higher precision approximations to \( \mathcal{B}_\ell \), one must derive higher-order Taylor coefficients of \( W(z_\ell) \) and \( \beta_\ell \) from the invariance condition \([19]\), which we do not pursue here.

9 Examples

We now demonstrate the application of SSM-based model reduction and backbone-curve reconstruction in two examples. First, we consider a two-degree-of-freedom damped, nonlinear oscillator model to benchmark data-based SSM reconstruction in a case where analytic, model-based computations are also possible. Second, we use vibration decay data from an oscillating beam experiment to illustrate the direct computation of backbone curves \( \mathcal{B}_\ell \) from an experimentally reconstructed sampling map \( \tilde{F} \).

9.1 Modified Shaw–Pierre example

We slightly modify here the two-degree-of-freedom oscillator studied by Shaw and Pierre \([24]\) by making the damping matrix proportional to the stiffness matrix in the linearised problem. The first-order equations of motion we study are

\[
\begin{align*}
\dot{x}_1 &= v_1, \\
\dot{x}_2 &= v_2, \\
\dot{v}_1 &= -cv_1 - k_0 x_1 - k_0^3 x_1^3 - k_0 (x_1 - x_2) - c(v_1 - v_2), \\
\dot{v}_2 &= -cv_2 - k_0 x_2 - k_0 (x_2 - x_1) - c(v_2 - v_1).
\end{align*}
\]

(37)
We first calculate SSMs and backbone curves for this system using a formulation for continuous dynamical systems, as described in Appendix A. We then emulate an experimental sampling of the vibrations of system (37) and reconstruct SSMs and backbone curves from the sampled data using the discrete methodology described in Sections 3-7.

System (37) is analytic, hence we have \( r = a \) in our notation. The natural frequencies and damping ratios are

\[
\omega_1 = \sqrt{k_0}, \quad \omega_2 = \sqrt{3k_0}, \quad \zeta_1 = \frac{c}{2\sqrt{k_0}}, \quad \zeta_2 = \frac{\sqrt{3}c}{2\sqrt{k_0}},
\]

yielding the complex eigenvalues

\[
\lambda_{1,2} = -\frac{c}{2} \pm i \sqrt{k_0 \left(1 - \frac{c^2}{4k_0}\right)}, \quad \lambda_{3,4} = -\frac{3c}{2} \pm i \sqrt{3k_0 \left(1 - \frac{3c^2}{4k_0}\right)},
\]

where we have assumed that both modes are underdamped, i.e., \( c < 2\sqrt{k_0/3} \).

For the corresponding two-dimensional modal subspaces \( E_1 \) and \( E_2 \), Remark 1 gives

\[
\sigma(E_1) = \text{Int} \left[ \frac{\text{Re} \lambda_3}{\text{Re} \lambda_1} \right] = \text{Int} \left[ \frac{\sqrt{3}c}{2\sqrt{k_0}} \sqrt{3k_0} \right] \approx 3, \quad \sigma(E_2) = \text{Int} \left[ \frac{\text{Re} \lambda_1}{\text{Re} \lambda_3} \right] = \text{Int} \left[ \frac{\frac{c}{2\sqrt{k_0}} \sqrt{k_0}}{\sqrt{3}c} \sqrt{3k_0} \right] = 0.
\]

Therefore, by Theorem 3 of Appendix B, there exist two-dimensional, analytic SSMs, \( W(E_1) \) and \( W(E_2) \) that are unique among \( C^4 \) and \( C^1 \) invariant manifolds tangent to \( E_1 \) and \( E_2 \), respectively, at the origin.

By the analytic calculations detailed in Appendix C, we obtain the corresponding backbone curve parametrisations

\[
\omega(\rho_1) = \frac{1}{2} \left( \sqrt{4k_0 - c^2} + \frac{3\kappa}{\sqrt{4k_0 - c^2}} \rho_1^2 \right), \quad \text{Amp}(\rho_1) \approx 2\rho_1,
\]

\[
\omega(\rho_2) = \frac{1}{2} \left( \sqrt{3(4k_0 - 3c^2)} + \frac{\sqrt{3}\kappa}{\sqrt{4k_0 - 3c^2}} \rho_2^2 \right), \quad \text{Amp}(\rho_2) \approx 2\rho_2.
\]

To determine these backbone curves for the damping and stiffness values \( c = 0.003, k_0 = 1, \) and \( \kappa = 0.5 \), we emulate a hammer experiment that gives an initial condition in the modal subspaces \( E_1 \) and \( E_2 \) to the full nonlinear system. The precise initial conditions of the two decaying signals are

\[
\mathbf{x}^{(1)}(0) = \frac{1}{\sqrt{3}} (2, 2, 0, 0)^T \in E_1, \quad \mathbf{x}^{(2)}(0) = \frac{1}{3} (-2, 2, 0, 0)^T \in E_2.
\]

We sample the solutions starting from these points 8000 times with the sampling interval \( T = 0.8 \). In terms of our notation, we therefore have \( P = 2, M_1 = M_2 = 8000 \). As observable, we choose the velocity of the first mass was i.e., let \( \varphi(\mathbf{x}) = v_1 \), to emulate an experimental procedure that renders only velocities (as in our second example below). As minimal embedding dimension for the sampling map \( \hat{\mathbf{F}}(\mathbf{\xi}) \), Step 1 of the algorithm in Section 8 gives \( 2\nu = 6 \). In the present example, however, we know that \( E_1 \) and \( E_2 \) are properly embedded already in the four-dimensional system (37), and hence we select \( 2\nu = 4 \) instead.

The red curve in Figure 2 shows a closed-form quintic computation (cf. Remark 3) of the backbone curves \( B_1 \) and \( B_2 \) from the data-assimilating discrete algorithm described in Section 8.
The two trajectories used as inputs for this algorithm were launched from the initial conditions (38).

For comparison, the green dashed line in the same figure shows a cubic analytic computation of the backbone curves based on the continuous-time (vector-field) formulation we have given in Theorem 4 of Appendix B. Finally, we have used numerical continuation [27] at various amplitudes of forcing to find periodic orbits for low damping with \( c = 0.0005 \). The resulting periodic response amplitudes are shown in Figure 2 in blue as functions of the forcing frequency. The \( O(5) \) backbone curve fits remarkably well with the peaks of the blue curves, especially considering that these backbone curves were computed from just two sampled trajectories. The robustness of the backbone curves is also noteworthy, given that the blue curves were obtained for substantially lower damping values.

Fig. 2: Backbone curves and forced response of the mechanical system (37) Blue curves show forced responses of the lightly damped system \( c = 0.0005 \). Red continuous lines show the fifth-order backbone curves recovered from our algorithm by sampling two freely decaying trajectories with initial conditions (38). Green dashed lines show the \( O(3) \) analytic calculation of the same backbone curves using Theorem 4 of Appendix B.

9.2 Clamped-clamped beam

We now test the trajectory-data-assimilating backbone-curve reconstruction algorithm of Section 8 on experimental data obtained from the vibration tests described in [5]. We show the experimental device, a beam clamped at both ends, in Figure 3.

Fig. 3: The experimental set-up for constructing backbone curves for a clamped-clamped beam. Reproduced from [5].
The data comprises freely decaying velocity signals measured at the midpoint of the beam with initial conditions selected near three assumed SSMs. These initial conditions were obtained experimentally by force appropriation (cf. the Introduction). The decaying signals were initialised at maximal response amplitudes obtained from single-frequency force appropriation. Three signals were assimilated, corresponding to each natural frequency, which gives \( P = 3 \) in our notation. Each signal was re-sampled with time period \( T = 0.97656 \) ms. The length of the three signals were \( M_1 = 3892 \), \( M_2 = 2458 \) and \( M_3 = 1055 \) samples.

The second mode was not analysed in [5], because the node of this mode is precisely at the midpoint of the beam, which can significantly deteriorate measurement accuracy. We list the natural frequencies identified from the NAR model in Table 1. In the last row of the same table, we also show the spectral quotients obtained from formula (21) for the three modes.

| Mode | \( l = 1 \) | \( l = 2 \) | \( l = 3 \) |
|------|-------------|-------------|-------------|
| \( \omega_l \) [Hz] | 47.4921 | 167.1512 | 368.4577 |
| \( \zeta_l \) | 0.1833 | 0.0183 | 0.0019 |
| \( \sigma(\mathcal{E}_l) \) | 0 | 2 | 12 |

Tab. 1: Natural frequencies and damping ratios for the first three modes of the clamped-clamped beam as determined by our algorithm. Also shown are the spectral quotients \( \sigma(\mathcal{E}_l) \). The \( \omega_l \) values are close to those linearly identified in [5], but the \( \zeta_l \) values are markedly different.

Based on Table 1, Theorem 1 gives a unique SSM \( W(\mathcal{E}_1) \) within the class of \( C^1 \) manifolds. This is because the first mode represents the fastest decaying linear subspace of oscillations, admitting a unique nonlinear continuation in the form of the fast SSM \( W(\mathcal{E}_1) \). The second (slow) SSM \( W(\mathcal{E}_2) \) and the third (intermediate) SSM, \( W(\mathcal{E}_3) \), are only unique among \( C^3 \) and \( C^{13} \) invariant manifolds tangent to the spectral subspaces \( \mathcal{E}_2 \) and \( \mathcal{E}_3 \), respectively. This suggests that backbone reconstruction techniques that do not consider the smoothness of the underlying SSM are expected to show greater uncertainty for the second and the third mode.

We seek to obtain an NAR model for the delay embedding of all three modes in Table 1. This means we have \( \nu = 3 \), and hence the required minimal dimension of the reconstructed nonlinear sampling map \( \tilde{F}(\xi) \) is \( 2\nu = 6 \). We employ a third-order polynomial model \( (r = 3) \) in the NAR model of Section 7. Accordingly, we construct the dynamics on the three SSMs up to cubic order (cf. formula (23)), with the Taylor coefficients of \( W \) and \( R \) computed from the formulas given in Theorem 2.

Figure 4 shows the results of our computations. To be consistent with Ehrhardt and Allen [5], we compute the response amplitudes by dividing the available instantaneous velocity amplitudes with their corresponding instantaneous frequencies. This simple division, therefore, represents the function \( \mathcal{P} \) from the observable space to the relevant coordinate space (cf. Remark 2). The resulting backbone curve of the first SSM matches well previous results. This is expected, because this SSM is the most robust among the three SSMs considered here (unique already among \( C^1 \) invariant manifolds tangent to the spectral subspace \( \mathcal{E}_1 \)). The kink at about 90 Hz appears to be an artefact of \( O(3) \) model fitting. Higher amplitude results for this SSM (not shown) are even less reliable because of the relative sparsity of the data there.

There is no comparison available from Ehrhardt and Allen [5] for the second backbone curve, but the backbone curve we compute for this case is consistent with the instantaneous amplitude-frequency data (green) inferred from decaying vibrations.

For the third SSM, there is a noticeable offset between the force appropriation result and the
rest of the curves. Our calculations, however, match closely the instantaneous amplitude-frequency data, with the backbone curve obtained from resonance decay. Capturing the SSM corresponding to this mode uniquely would theoretically require a high-order, $\mathcal{O}(13)$ approximation. This, however, would be unfeasible given the limited amount of data available.

Fig. 4: The first three backbone curves of a clamped-clamped beam. Red curves: backbone curves computed from a data-assimilating cubic-order SSM reduction, as summarised in Section 8. Blue dashed lines: backbone curves obtained from individual decaying signals using a Hilbert transform approach [7]. Blue circles: force-appropriation results using stepped sine forcing. Green diamonds: Instantaneous amplitude-frequency curves inferred from decaying vibration data by calculating zero crossings of the signal to estimate vibration period. Apart from the red curves, all data was obtained directly from the experiments of Ehrhardt and Allen [5].

10 Discussion

We have developed a method to extract two-dimensional spectral submanifolds (SSMs) and their associated backbone curves for multi-degree-of-freedom nonlinear mechanical vibrations. We computed the SSMs explicitly as two-dimensional invariant manifolds of a low-order, discrete model system fitted to sampled trajectory data. Restricted to the SSMs, this model is guaranteed to be conjugate to the full mechanical system by the classic Takens embedding theorem, as long as the data assimilated into the model is from a generic observable.

We have illustrated the power of this approach by calculating backbone curves of the reconstructed dynamics on the SSMs in two examples. In our first example, a two-degree-of-freedom analytic model, we verified the trajectory-data-based backbone-curve computation via an analytic calculation of the same curve for the full, continuous-time system, as well as by numerical continuation. In our second example, we compared the data-assimilated construction of the backbone curves with various experimentally inferred curves and found close agreement.

To obtain SSMs and their reduced dynamics analytically, we use the parameterisation method of [3], which is generally not limited to a small neighbourhood of a fixed point. In addition, the parametrisation method allows for the presence of resonances or near-resonances that unavoidably arise in underdamped oscillations (cf. eq. (22)). This is in contrast with parametrised SSM constructions based on Sternberg’s analytic linearisation theorem (cf. Cirillo et al. [4]) that exclude...
any resonance in the linearised spectrum. When applied in the near-resonant case, the domain of validity of the analytic linearisation and the manifolds construction is, therefore, exceedingly small. In addition, reliance on analytic linearisation excludes the possibility of extracting backbone curves, which arise from the nonlinear dynamics on the reconstructed SSM.

The parametrisation method enables us to identify SSMs with high accuracy on larger domains, even from relatively low-amplitude trajectory samples, as long as we use high-enough order in the approximations for the SSMs and its reduced dynamics. This high-enough order ensures the accurate interrogation of nonlinearities even from low-amplitude signals. In our examples, a fifth-order computation yielded remarkably accurate results even for higher-amplitude ranges of the backbone curve, while a third-order computation was effective for lower-amplitude backbone-curve ranges.

Our algorithm is devised in a way so that an arbitrary number of decaying vibrations can be assimilated into the underlying reduced-order discrete NAR model. Unlike normal forms derived specifically for given modes of interest, our model incorporates the dynamics of all modes of interest simultaneously. This should make the reconstructed sampling map $\tilde{F}$ an ideal tool for use in model-based control.

We also envisage a closed loop identification of SSMs and backbone curves, similar to control-based continuation techniques [25]. In this case, a measure of invariance derived from equation (39) would serve as a test functional.

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**11 Appendix A: Proof of Theorem 2**

By the relationship (19), we need to solve the algebraic equation

$$\Lambda W + G \circ W = W \circ R$$  \hspace{1cm} (39)

for the unknown Taylor coefficients $w_j^{s}$ and $r_j^{s}$. We carry this out step by step for increasing powers of $z$:

$O(|z|)$: Since the Taylor series of $G$ starts with second-order terms, the first-order monomials of $z$ arising from substitution into (39) satisfy $\Lambda W = W \circ R$, which simplifies to $\Lambda W = W \Lambda$, because the linear part of $R$ is equal to the diag $\{\mu_\ell, \bar{\mu}_\ell\}$ block of the diagonal matrix $\Lambda$ (cf. (23)). Consequently, (39) can be written at leading order as

$$\mu_j w_j^{(1,0)} = w_j^{(1,0)} \mu_\ell, \quad \bar{\mu}_j w_{j+1}^{(0,1)} = w_{j+1}^{(0,1)} \bar{\mu}_\ell,$$

whose simplest solution is

$$w_j^{(1,0)} = \delta_{j\ell}, \quad w_j^{(0,1)} = \delta_{j(\ell+1)},$$

with $\delta_{j\ell}$ denoting the Kronecker delta. This proves the formulas for the first-order coefficients in Theorem 2. We note that $w_\ell^{(1,0)}$ and $w_{\ell+1}^{(0,1)}$ are only determined up to a constant, which we have chosen to be equal to 1.
$O(|z|^2)$: Since $r_j^s = 0$ for $|s| = 2$ by (23), the quadratic terms in $|z|$ on the right-hand side of (39) only arise from the substitution of linear terms of $R(z)$ into the quadratic terms of $W(z)$. As a consequence, equating the coefficients of $O(|z|^2)$ terms on both sides of (39) gives the equation

$$
\mu_j w_j^{(s_1, s_2)} + g_j^{(s_1 @ \ell, s_2 @ (\ell+1))} = \mu_{\ell} \mu_{\ell}^2 w_j^{(s_1, s_2)}, \quad |s| = 2,
$$

(42)

whose solution for $w_j^{(s_1, s_2)}$ is

$$
w_j^{(s_1, s_2)} = \frac{g_j^{(s_1 @ \ell, s_2 @ (\ell+1))}}{\mu_{\ell}^2 \mu_{\ell}^2 - \mu_j}, \quad |s| = 2,
$$

(43)

proving the formulas for the second-order coefficients $w_j^{(2,0)}$, $w_j^{(1,1)}$ and $w_j^{(0,2)}$ in the statement of Theorem 2. Note that the denominator in (43) is guaranteed to be nonzero by the nonresonance condition (17).

$O(|z|^3)$: We write out the $j^{th}$ coordinates in the three terms of eq. (30) in detail to obtain the following cubic terms:

$$
(\Lambda W)^{(3)}_j = \mu_j \left( w_j^{(3,0)} z_{\ell}^3 + w_j^{(2,1)} z_{\ell}^2 \bar{z}_{\ell} + w_j^{(1,2)} z_{\ell} \bar{z}_{\ell}^2 + w_j^{(0,3)} \bar{z}_{\ell}^3 \right),
$$

(44)

$$
(G \circ W)^{(3)}_j = \sum_{|m|=2} g_j^m y^m + \sum_{|m|=3} g_j^m y^m,
$$

(45)

$$
(W \circ R)^{(3)}_j = \sum_{|s|=1,2,3} \sum_{r^{(s)}} w_j^{(s_1, s_2)} r_{\ell}^{s_1} \bar{r}_{\ell}^{s_2}
$$

$$
= \sum_{|s|=1,2,3} \sum_{r^{(s)}} w_j^{(s_1, s_2)} \left( \mu_{\ell} z_{\ell} + \beta_{\ell} \bar{z}_{\ell}^2 \bar{z}_{\ell} \right)^{s_1} \left( \bar{\mu}_{\ell} \bar{z}_{\ell} + \bar{\beta}_{\ell} z_{\ell} \bar{z}_{\ell}^2 \right)^{s_2}
$$

(46)

We now write out the individual terms in eq. (45). For $y = W(z)$, we have

$$
y_p y_q = \left( \delta_{pq} z_{\ell} + \delta_{p(\ell+1)} \bar{z}_{\ell} + w_{p}^{(2,0)} z_{\ell}^2 + w_{p}^{(1,1)} z_{\ell} \bar{z}_{\ell} + w_{p}^{(0,2)} \bar{z}_{\ell}^2 \right) \times \left( \delta_{pq} z_{\ell} + \delta_{q(\ell+1)} \bar{z}_{\ell} + w_{q}^{(2,0)} z_{\ell}^2 + w_{q}^{(1,1)} z_{\ell} \bar{z}_{\ell} + w_{q}^{(0,2)} \bar{z}_{\ell}^2 \right)
$$

$$
= \left( \delta_{pq} w_{q}^{(2,0)} + \delta_{q(\ell+1)} \bar{w}_{q}^{(2,0)} \right) z_{\ell}^3 + \left( \delta_{pq} w_{p}^{(2,0)} + \delta_{q(\ell+1)} \bar{w}_{p}^{(2,0)} \right) \bar{z}_{\ell}^3
$$

$$
+ \left( \delta_{pq} w_{q}^{(0,2)} + \delta_{q(\ell+1)} \bar{w}_{q}^{(0,2)} + \delta_{p(\ell+1)} \bar{w}_{q}^{(1,1)} + \delta_{q(\ell+1)} \bar{w}_{q}^{(0,2)} \right) z_{\ell} \bar{z}_{\ell}^2
$$

$$
+ \left( \delta_{pq} w_{p}^{(0,2)} + \delta_{q(\ell+1)} \bar{w}_{q}^{(0,2)} + \delta_{p(\ell+1)} \bar{w}_{q}^{(1,1)} + \delta_{q(\ell+1)} \bar{w}_{q}^{(0,2)} \right) \bar{z}_{\ell} z_{\ell}^2
$$

$$
+ O(|z|^2, |z|^4),
$$
thus, for the first sum in (45), we obtain

\[
\sum_{|m|=2} g_j^m y^m = \sum_{p,q=1}^{2\nu} g_j^{(1@p,1@q)} \left[ \left( \delta_{pq} w_q^{(2,0)} + \delta_{qe} w_p^{(2,0)} \right) z_\ell^3 + \left( \delta_{p(\ell+1)} w_q^{(0,2)} + \delta_{q(\ell+1)} w_p^{(0,2)} \right) z_\ell^3 \right]
\]

\[
+ \sum_{p,q=1}^{2\nu} g_j^{(1@p,1@q)} \left[ \delta_{pq} w_q^{(1,1)} + \delta_{qe} w_p^{(1,1)} + \delta_{p(\ell+1)} w_q^{(0,2)} + \delta_{q(\ell+1)} w_p^{(0,2)} \right] z_\ell^2 z_\ell
\]

\[
+ \sum_{p,q=1}^{2\nu} g_j^{(1@p,1@q)} \left[ \delta_{pq} w_q^{(0,2)} + \delta_{qe} w_p^{(0,2)} + \delta_{p(\ell+1)} w_q^{(1,1)} + \delta_{q(\ell+1)} w_p^{(1,1)} \right] z_\ell^2 z_\ell
\]

\[
= 2 \sum_{q=1}^{2\nu} g_j^{(1@q,1@q)} \left[ \left( \delta_{qe} + \delta_{p(\ell+1)} \right) w_q^{(2,0)} z_\ell^3 + \left( \delta_{pq} + \delta_{q(\ell+1)} \right) w_q^{(0,2)} z_\ell^3 \right]
\]

\[
+ 2 \sum_{q=1}^{2\nu} \left[ \left( \delta_{qe} + \delta_{p(\ell+1)} \right) w_q^{(1,1)} + \left( \delta_{pq} + \delta_{q(\ell+1)} \right) w_q^{(0,2)} \right] z_\ell^2 z_\ell
\]

\[
+ 2 \sum_{q=1}^{2\nu} \left[ \left( \delta_{qe} + \delta_{p(\ell+1)} \right) w_q^{(0,2)} + \left( \delta_{pq} + \delta_{q(\ell+1)} \right) w_q^{(1,1)} \right] z_\ell^2 z_\ell
\]

\[
= 2 \sum_{q=1}^{2\nu} g_j^{(1@q,1@q)} \left[ \left( \delta_{qe} + \delta_{p(\ell+1)} \right) w_q^{(2,0)} z_\ell^3 + \left( \delta_{pq} + \delta_{q(\ell+1)} \right) w_q^{(0,2)} z_\ell^3 \right]
\]

\[
+ 2 \sum_{q=1}^{2\nu} \left[ \left( \delta_{qe} + \delta_{p(\ell+1)} \right) w_q^{(1,1)} + \left( \delta_{pq} + \delta_{q(\ell+1)} \right) w_q^{(0,2)} \right] z_\ell^2 z_\ell
\]

\[
+ 2 \sum_{q=1}^{2\nu} \left[ \left( \delta_{qe} + \delta_{p(\ell+1)} \right) w_q^{(0,2)} + \left( \delta_{pq} + \delta_{q(\ell+1)} \right) w_q^{(1,1)} \right] z_\ell^2 z_\ell
\]

\[
(47)
\]

For the second sum in (45), we have

\[
\sum_{|m|=3} g_j^m y^m = \sum_{p=1}^{2\nu} g_j^{(3@p)} y_p^3 + \sum_{p,q=1}^{2\nu} g_j^{(2@q,1@q)} y_p y_q y_p y_q
\]

\[
\sum_{p,q=1}^{2\nu} g_j^{(1@p,1@q,1@q)} y_p y_q y_u
\]

\[
\sum_{p,q, u=1}^{2\nu} \left[ g_j^{(1@p,1@q,1@u)} y_p y_q y_u \right]
\]

\[
(48)
\]
Working out these expressions in detail, we find that

\[
A = \sum_{\ell=1}^{2^{\nu}} g_j^{(3@\ell)} (3\delta_{\ell\ell}^2 z_\ell^3 + 3\delta_{\ell\ell}^2 \delta_{p(\ell+1)} z_\ell^2 \bar{z}_\ell + 3\delta_{\ell\ell}^2 \delta_{p(\ell+1)} z_\ell \bar{z}_\ell^2 + \delta_{p(\ell+1)}^3 \bar{z}_\ell^3)
\]
\[
= g_j^{(3@\ell)} z_\ell^3 + g_j^{(3@\ell+1)} \bar{z}_\ell^3,
\]

\[
B = \sum_{\ell=1}^{2^{\nu}} g_j^{(2@\ell,0@q)} (2\delta_{\ell\ell}^2 \delta_{q(\ell+1)} z_\ell^2 \bar{z}_\ell + \delta_{p(\ell+1)}^3 \bar{z}_\ell^2) (\delta_{q\ell} z_\ell + \delta_{q(\ell+1)} \bar{z}_\ell)
\]
\[
= g_j^{(2@\ell,1@q)} z_\ell^2 \bar{z}_\ell + g_j^{(2@\ell,1@q+1@\ell)} z_\ell \bar{z}_\ell^2,
\]

\[
C = \sum_{\ell=1}^{2^{\nu}} g_j^{(0@\ell,0@q,0@u)} (\delta_{p\ell} z_\ell + \delta_{p(\ell+1)} \bar{z}_\ell) (\delta_{q\ell} z_\ell + \delta_{q(\ell+1)} \bar{z}_\ell) (\delta_{u\ell} z_\ell + \delta_{u(\ell+1)} \bar{z}_\ell)
\]
\[
= 0.
\]

Substituting the expressions for \(A, B\) and \(C\) into (48), then substituting (48) and (47) into the invariance condition (39), we equate equal powers of \(z\) to obtain the following linear equations for the cubic coefficients of the mapping \(W\) and of the mapping \(R\):

\[
\mathcal{O}_j (z^3) : \quad \mu_j w_j^{(3,0)} + 2 \sum_{q=1}^{2^{\nu}} g_j^{(1@\ell,1@q)} w_q^{(2,0)} + g_j^{(3@\ell)}
\]
\[
= w_j^{(3,0)} \mu_j^3,
\]

\[
\mathcal{O}_j (z^2 \bar{z}) : \quad \mu_j w_j^{(2,1)} + 2 \sum_{q=1}^{2^{\nu}} g_j^{(1@\ell,1@q)} w_q^{(1,1)} + g_j^{(1@\ell+1,0@q)} w_q^{(2,0)}
\]
\[
= w_j^{(2,1)} \mu_j^2 \mu_\ell + w_j^{(1,0)} \beta_\ell,
\]

\[
\mathcal{O}_j (z \bar{z}^2) : \quad \mu_j w_j^{(1,2)} + 2 \sum_{q=1}^{2^{\nu}} g_j^{(1@\ell+1,1@q)} w_q^{(2,0)} + g_j^{(2@\ell+1,1@\ell)}
\]
\[
= w_j^{(1,2)} \mu_j \mu_\ell^2 + w_j^{(0,1)} \bar{z}_\ell,
\]

\[
\mathcal{O}_j (\bar{z}^3) : \quad \mu_j w_j^{(0,3)} + 2 \sum_{q=1}^{2^{\nu}} g_j^{(1@\ell+1,1@q)} w_q^{(0,2)} + g_j^{(3@\ell+1)}
\]
\[
= w_j^{(0,3)} \mu_\ell^3.
\]
From the first and last equation in (50), we obtain

\[ w^{(3,0)}_j = \sum_{q=1}^{2\nu} \frac{(1 + \delta_{(\ell+1)q}) g^{(10\ell,10q)}_j w^{(2,0)}_q + g^{(30\ell)}_j}{\mu^2_\ell - \mu_j}, \]

\[ w^{(3,0)}_j = \sum_{q=1}^{2\nu} \frac{(1 + \delta_{(\ell+1)q}) g^{(10\ell,10q)}_j w^{(2,0)}_q + g^{(30\ell)}_j}{\mu^2_\ell - \mu_j}. \]

(51)

We select \( j \neq \ell \) and assume that there is no first-order near-resonance (or exact resonance) involving the eigenvalues \( \mu_\ell \) and \( \mu_j \) (stated as \( \mu_j \neq \mu_\ell \) under the assumptions of the theorem). Recalling \( w^{(1,0)}_j = \delta_{j\ell} \), we then obtain from the second equation of (50) that

\[ w^{(2,1)}_j = \sum_{q=1}^{2\nu} \frac{(1 + \delta_{q}) g^{(10\ell,10q)}_j w^{(1,1)}_q + (1 + \delta_{(\ell+1)q}) g^{(10\ell,10q)}_j w^{(2,0)}_q + g^{(20\ell+1)}_j}{\mu_\ell^2 - \mu_j}, \]

whenever \( \delta_{j\ell} = 0 \). Similarly, selecting \( j \neq \ell + 1 \), assuming no first-order near-resonance (or exact resonance) involving the eigenvalues \( \mu_{\ell+1} \) and \( \mu_j \) (i.e., \( \mu_j \neq \mu_{\ell+1} \)), and recalling \( w^{(0,1)}_j = \delta_{j(\ell+1)} \), we obtain from the third equation of (50) that

\[ w^{(1,2)}_j = \sum_{q=1}^{2\nu} \frac{(1 + \delta_{q}) g^{(10\ell,10q)}_j w^{(0,2)}_q + (1 + \delta_{(\ell+1)q}) g^{(10\ell,10q)}_j w^{(1,1)}_q + g^{(20\ell+1,10)}_j}{\mu_\ell^2 - \mu_j}, \]

(53)

whenever \( \delta_{j(\ell+1)} = 0 \). Next we select \( j = \ell \) in the second equation of (50), and select \( j = \ell + 1 \) in the third equation of (50). These choices force us to select

\[ w^{(2,1)}_{\ell} = 0, \quad \delta_{j\ell} = 1, \quad w^{(1,2)}_{\ell+1} = 0, \quad \delta_{j(\ell+1)} = 1, \]

(54)

in these equations to avoid small denominators arising from the near-resonances. Then the second equation of (50) with \( j = \ell \) gives the solution

\[ \beta_\ell = \sum_{q=1}^{2\nu} \left[ (1 + \delta_{eq}) g^{(10\ell,10q)}_\ell w^{(1,1)}_q + (1 + \delta_{(\ell+1)q}) g^{(10\ell,10q)}_\ell w^{(2,0)}_q + g^{(20\ell+1,10)}_\ell \right]. \]

(55)

But equations (51)-(55) prove the formulas for the cubic coefficients of \( W_j \) and \( \beta_\ell \) in the statement of Theorem 2.

12 Appendix B: Analogous results for continuous dynamical systems

Here we discuss spectral submanifolds, backbone curves and their leading-order computation for continuous dynamical systems. The formulas we derive are useful for benchmarking our database SSM and backbone-curve approach on exactly known mechanical models. The concepts and formulas derived here, however, are also of independent interest in computing the dynamics on SSMs in analytically defined mechanical models.

We start with the continuous analogue of the the complex mapping (13), which is a complex differential equation of the form
\[
\dot{y} = \Lambda y + G(y), \quad y \in \mathbb{C}^{2\nu}, \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{2\nu}), \quad \lambda_{2l-1} = \bar{\lambda}_{2l}, \quad l = 1, \ldots, \nu \quad (56)
\]

with a fixed point at \( y = 0 \), and with a class \( C^r \) function \( G \). The eigenvalues of \( \Lambda \) are ordered so that
\[
\text{Re} \lambda_{2\nu} \leq \ldots \leq \text{Re} \lambda_1 < 0, \quad (57)
\]
and hence \( y = 0 \) is asymptotically stable. If (56) is the equivalent first-order complexified form of a mechanical system of the form (1), then we specifically have \( \nu = n \). If, furthermore, the mechanical system has linear and weak proportional damping, then we can write
\[
\text{Re} \lambda_l = -\zeta_l \omega_l, \quad \text{Im} \lambda_l = \sqrt{1 - \zeta_l^2} \omega_l, \quad (58)
\]
with \( \zeta_l \) and \( \omega_l \) denoting Lehr’s damping ratio and undamped natural frequency, respectively, for the \( l^{th} \) mode of the linearised system at the \( q = 0 \) equilibrium.

Finally, we assume that \( \mathcal{E} \) is a two-dimensional spectral subspace (eigenspace) of the operator \( \Lambda \), corresponding to the complex pair of simple eigenvalues \( \lambda_\ell = \bar{\lambda}_{\ell+1} \) for some \( \ell \in [1, 2\nu - 1] \).

### 12.1 Existence and uniqueness of SSMs

Following [10], we address this issue via the following definition:

**Definition 3.** A spectral submanifold (SSM) \( W(\mathcal{E}) \) corresponding to a spectral subspace \( \mathcal{E} \) of \( \Lambda \) is

(i) an invariant manifold of the dynamical system (56) that is tangent to \( \mathcal{E} \) at \( y = 0 \) and has the same dimension as \( \mathcal{E} \);

(ii) strictly smoother than any other invariant manifold of (56) satisfying (i).

We now recall from Haller and Ponsioen [10] the specific existence and uniqueness result pertaining to two-dimensional SSMs, deducible from the more general results of Cabré et al. [3]. The relative spectral quotient of \( \mathcal{E} \) is now defined as the positive integer
\[
\sigma(\mathcal{E}) = \text{Int} \left[ \min_{l \neq \ell, \ell+1} \frac{\text{Re} \lambda_l}{\text{Re} \lambda_\ell} \right] \in \mathbb{N}^+, \quad (59)
\]
whose meaning is the same as pointed out after formula (59) for mappings. In case of a proportionally damped mechanical system, one may use the formulas (58) and conclude that Remark 1 continues to provide the correct specific form of \( \sigma(\mathcal{E}) \) in this case.

We again assume that
\[
\sigma(\mathcal{E}) \leq r, \quad (60)
\]
and that no resonance relationships up to order \( \sigma(\mathcal{E}) \) hold between the eigenvalues \( \lambda_\ell, \bar{\lambda}_{\ell+1} \) and the rest of the spectrum of \( \Lambda \), i.e.,
\[
s_1 \lambda_\ell + s_2 \bar{\lambda}_{\ell} \neq \lambda_j, \quad \forall j \neq \ell, \ell + 1, \quad 2 \leq s_1 + s_2 \leq \sigma(\mathcal{E}). \quad (61)
\]
The alternative form of this nonresonance condition given in Remark 1 again applies whenever formulas (59) hold.
Appendix B: Analogous results for continuous dynamical systems

Theorem 3. Assume that conditions \((60)-(61)\) are satisfied. Then following statements hold:

(i) There exists an SSM, \(W(E)\), for the dynamical system \((56)\) that is tangent to the invariant subspace \(E\) at the \(y = 0\) fixed point.

(ii) The invariant manifold \(W(E)\) is class \(C^r\) smooth and unique among all two-dimensional, class \(C^{σ(E)+1}\) invariant manifolds of \((56)\) that are tangent to \(E\) at \(y = 0\).

(iii) The SSM \(W(E)\) can be viewed as a \(C^r\) immersion of an open set \(U \subset \mathbb{C}^2\) into the phase space \(\mathbb{C}^{2ν}\) of system \((56)\) via a map

\[
W : U \subset \mathbb{C}^2 \rightarrow \mathbb{C}^{2ν}, \quad W(U) = W(E).
\]

(iv) There exists \(C^r\) polynomial function \(R: U \rightarrow U\) such that

\[
ΛW + G \circ W = DW\circ R,
\]

i.e., the dynamics on the SSM, expressed in coordinates \(z = (z_ℓ, \bar{z}_ℓ) \in U\), is governed by the polynomial ODE

\[
\dot{z} = R(z), \quad DR(0) = \text{diag}(λ_ℓ, \bar{λ}_ℓ),
\]

whose right-hand side has only terms up to order \(O(\lvert z\rvert^{σ(E)})\).

(v) Under the further internal non-resonance assumption

\[
s_1λ_ℓ + s_2\bar{λ}_ℓ \neq λ_j, \quad j = ℓ, ℓ + 1, \quad 2 \leq s_1 + s_2 \leq σ(E_ℓ),
\]

within \(E\), the mapping \(W\) in \((62)\) can be selected such that the \(j^{th}\) coordinate component \(R_j\) of \(R\) does not contain the term \((z_ℓ^{s_1}, \bar{z}_ℓ^{s_2})\).

Proof. This is merely the re-statement of the main theorem of Haller and Ponsioen [10] (deduced from Cabré et al. [3]) for the case of a two-dimensional SSM corresponding to a simple pair of complex eigenvalues with negative real parts.

12.2 Backbone curves and their computation

When the spectral subspace \(E\) of \((56)\) is lightly damped (\(|\text{Re}λ_ℓ| \ll 1\)), the low-order near-resonance relationships

\[
2λ_ℓ + \bar{λ}_ℓ \approx λ_ℓ, \quad λ_ℓ + 2\bar{λ}_ℓ \approx \bar{λ}_ℓ
\]

always hold. As in the case of mappings, this prompts us to seek the polynomial dynamics on the SSM (cf. statement (iv) of Theorem 3) in the form

\[
\dot{z} = R(z) = \left(\begin{array}{c}
λ_ℓz_ℓ + β_ℓz_ℓ^2\bar{z}_ℓ \\
\bar{λ}_ℓ\bar{z}_ℓ + β_ℓ\bar{z}_ℓ^2\bar{z}_ℓ^2
\end{array}\right).
\]

Introducing polar coordinates \(z = re^{iθ}\), we can further transform \((65)\) to the real amplitude-phase equations

\[
\dot{ρ} = ρ \left(\text{Re}λ_ℓ + \text{Re} β_ℓρ^2\right), \quad (66)
\]

\[
\dot{θ} = \text{Im}λ_ℓ + \text{Im} β_ℓρ^2.
\]

(67)
Equation \( \text{[67]} \) gives instantaneous frequency of nonlinear oscillations as
\[
\omega(\rho) = \text{Im}\lambda_\ell + \text{Im}\beta_\ell \rho^2,
\]
whereas as instantaneous amplitude \( \text{Amp}(\rho) \) of the vibration can be calculated as
\[
\text{Amp}(\rho) = \frac{1}{2\pi} \int_0^{2\pi} |W(W(z(\rho)))| \, d\theta,
\]
where \( W \) is the mapping featured in statement (iii) of Theorem \( 3 \) and \( V \) is the linear mapping that transform the original, first-order dynamical system to its standard complex form \( \text{[56]} \). With the quantities defined in \( \text{[68]} \) and \( \text{[69]} \), the definition of a backbone curve \( B_\ell \) given in Definition \( 28 \), we need to find expressions for the complex coefficient \( \beta_\ell \) and the mapping \( W(z) \), as the eigenvalue \( \lambda_\ell \) is assumed to be known.

To this end, we seek the Taylor series coefficients of the \( j^{th} \) coordinate functions, \( W_j(z) \in \mathbb{C}, \) \( j = 1, \ldots, 2\nu, \) of the mapping \( W(z) \), and the third-order Taylor coefficient \( \beta_\ell \in \mathbb{C} \) of the polynomial function \( R(z) \) defined in \( \text{[65]} \). These unknowns will again be expressed as functions of the \( j^{th} \) coordinate functions \( G_j(y) \in \mathbb{C}, \) \( j = 1, \ldots, 2\nu, \) of the nonlinear part \( G(y) \) of the right-hand side of the dynamical system \( \text{[56]} \). Using the same notation as in Theorem \( 4 \), we obtain the following expressions for the required Taylor coefficients.

**Theorem 4.** Suppose that the assumptions of Theorem \( 3 \) hold but with the strengthened version
\[
s_1 \lambda_\ell + s_2 \bar{\lambda}_\ell \neq \lambda_j, \quad \forall j \neq \ell, \ell + 1, \quad 1 \leq s_1 + s_2 \leq \sigma(E)
\]
of the external non-resonance condition \( \text{[61]} \). Then, for any \( j \in \{1, 2\nu\} \), the \( j^{th} \) coordinate function \( W_j \) of the mapping \( W \) in \( \text{[62]} \) and the cubic Taylor coefficient \( \beta_\ell \) of the conjugate map \( R \) in \( \text{[63]} \) are given by the following formulas:

\[
\begin{align*}
w_j^{(1,0)} &= \delta_j \ell, & w_j^{(0,1)} &= \delta_j(\ell+1), \\
w_j^{(2,0)} &= \frac{g_j^{(2@\ell)}}{2\lambda_\ell - \lambda_j}, & w_j^{(1,1)} &= \frac{g_j^{(1@\ell,1@\ell+1)}}{\lambda_\ell + \lambda_\ell - \lambda_j}, & w_j^{(0,2)} &= \frac{g_j^{(1@\ell+1)}}{2\lambda_\ell - \lambda_j}, \\
w_j^{(3,0)} &= \frac{\sum_{q=1}^{2\nu} (1 + \delta_\ell q) g_j^{(1@\ell,1@\ell+1,q)} w_j^{(2,q)} + g_j^{(3@\ell)}}{3\lambda_\ell - \lambda_j}, \\
w_j^{(2,1)} &= \frac{\sum_{q=1}^{2\nu} \left[(1 + \delta_\ell q) g_j^{(1@\ell,1@\ell+1,q)} w_q^{(1,1)} + \left(1 + \delta_\ell(\ell+1) q\right) g_j^{(1@\ell+1,1@\ell+1,q)} w_q^{(2,0)}\right] + g_j^{(2@\ell,1@\ell+1)}}{2\lambda_\ell + \lambda_\ell - \lambda_j}, \\
w_j^{(1,2)} &= \frac{\sum_{q=1}^{2\nu} \left[(1 + \delta_\ell q) g_j^{(1@\ell,1@\ell+1,q)} w_q^{(0,2)} + \left(1 + \delta_\ell(\ell+1) q\right) g_j^{(1@\ell+1,1@\ell+1,q)} w_q^{(1,1)}\right] + g_j^{(2@\ell,1@\ell+1)}}{\lambda_\ell + 2\lambda_\ell - \lambda_j}, \\
\beta_\ell &= \sum_{q=1}^{2\nu} \left[(1 + \delta_\ell q) g_\ell^{(1@\ell,1@\ell+1,q)} w_\ell^{(1,1)} + \left(1 + \delta_\ell(\ell+1) q\right) g_\ell^{(1@\ell+1,1@\ell+1,q)} w_\ell^{(2,0)}\right] + g_\ell^{(2@\ell,1@\ell+1)}.
\end{align*}
\]
Proof. The algebraic equation (63) is similar to the equation (19), which we have solved in detail up to cubic order in the proof of Theorem (2). The first difference between the two equations is that the term $\lambda W$ in (19) has the $j$th component

$$(\lambda W)_j = \lambda_j \sum_{|s| \geq 1} w_j^s z^s, \ s \in \mathbb{N}^2, \ w_j^s \in \mathbb{C}. \quad (71)$$

The second difference is that instead of $(W \circ R)_j$, the $j$th coordinate component of the right-hand side of (19) is given by

$$(DW \circ R)_j = \partial_{\bar{z}} W_j(z) (\lambda_2 \bar{z}_2 + \beta \bar{z}_2^2 \bar{z}_2) + \partial_{\bar{z}} W_j(z) (\lambda_2 \bar{z}_2 + \beta \bar{z}_2^2 \bar{z}_2) + 3 \lambda w_j^{(3,0)} z_2^3$$

Substituting formulas (71)-(72) into (19), and using the expression for (56). This involves the coordinate change $eigenvectors and its inverse are $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1, \lambda_4 = 1, V^{-1}$

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \lambda_1 & \lambda_1 & \lambda_3 & \lambda_3 \\ \lambda_1 & \lambda_1 & -\lambda_3 & -\lambda_3 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} -\frac{\lambda_1}{2 (\lambda_1 - \lambda_1)} & \frac{\lambda_1}{2 (\lambda_1 - \lambda_1)} & \frac{1}{2 (\lambda_1 - \lambda_1)} & \frac{1}{2 (\lambda_1 - \lambda_1)} \\ \frac{\lambda_2}{2 (\lambda_2 - \lambda_2)} & \frac{\lambda_2}{2 (\lambda_2 - \lambda_2)} & \frac{1}{2 (\lambda_2 - \lambda_2)} & \frac{1}{2 (\lambda_2 - \lambda_2)} \\ -\frac{\lambda_3}{2 (\lambda_3 - \lambda_3)} & \frac{\lambda_3}{2 (\lambda_3 - \lambda_3)} & \frac{1}{2 (\lambda_3 - \lambda_3)} & \frac{1}{2 (\lambda_3 - \lambda_3)} \\ \frac{\lambda_4}{2 (\lambda_4 - \lambda_4)} & \frac{\lambda_4}{2 (\lambda_4 - \lambda_4)} & \frac{1}{2 (\lambda_4 - \lambda_4)} & \frac{1}{2 (\lambda_4 - \lambda_4)} \end{pmatrix}.$$

The transformed system (37) then takes the form

$$\dot{y} = \Lambda y + G(y), \quad y \in \mathbb{C}^4, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \quad G(y) = O(|y|^2), \quad G(y) = V^{-1} f(V y) = -\kappa \gamma(y) = \frac{i \kappa}{2} \gamma(y) \begin{pmatrix} \lambda_1 + \lambda_1 \\ \lambda_2 + \lambda_2 \\ \lambda_3 + \lambda_3 \\ \lambda_4 + \lambda_4 \end{pmatrix} = \frac{i \kappa}{4} \gamma(y) \begin{pmatrix} \lambda_1 \lambda_1 \\ \lambda_2 \lambda_2 \\ \lambda_3 \lambda_3 \\ \lambda_4 \lambda_4 \end{pmatrix},$$

where

$$\gamma(y) = y_1^3 + y_2^3 + y_3^3 + y_4^3 + 3 (y_1^2 y_2 + y_1^2 y_3 + y_1^2 y_4 + y_2^2 y_1 + y_2^2 y_3 + y_2^2 y_4 + y_3^2 y_1 + y_3^2 y_2 + y_3^2 y_4 + y_4^2 y_1 + y_4^2 y_2 + y_4^2 y_3) + 6 (y_1 y_2 y_3 + y_1 y_2 y_4 + y_1 y_3 y_4 + y_2 y_3 y_4)$$

13 Appendix C: Analytic SSM and backbone calculations for Example 1

To compute the SSMs $W(E_1)$ and $W(E_3)$ in Example 1, we transform (37) to its complex standard form (56). This involves the coordinate change $x = (x_1, x_2, v_1, v_2)^T = V y$, where the matrix $V$ of eigenvectors and its inverse are

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \lambda_1 & \lambda_1 & \lambda_3 & \lambda_3 \\ \lambda_1 & \lambda_1 & -\lambda_3 & -\lambda_3 \end{pmatrix}.$$
Theorem 4 of Appendix B then gives the following coefficients for \( \ell = 1 \):

\[
  w_1^{(1,0)} = 1, \quad w_2^{(1,0)} = w_3^{(1,0)} = w_4^{(1,0)} = 0, \quad w_2^{(0,1)} = 1, \quad w_1^{(0,1)} = w_3^{(0,1)} = w_4^{(0,1)} = 0,
\]

\[
  w_j^{(2,0)} = w_j^{(1,1)} = w_j^{(0,2)} = 0, \quad j = 1, 2, 3, 4,
\]

\[
  w_1^{(3,0)} = \frac{i \kappa}{8 \lambda_1 \text{Im} \lambda_1}, \quad w_2^{(3,0)} = -\frac{i \kappa}{4 (3 \lambda_1 - \lambda_1) \text{Im} \lambda_1}, \quad w_3^{(3,0)} = \frac{i \kappa}{4 (3 \lambda_1 - \lambda_3) \text{Im} \lambda_3}, \quad w_4^{(3,0)} = -\frac{i \kappa}{4 (3 \lambda_1 - \lambda_3) \text{Im} \lambda_3},
\]

\[
  w_1^{(0,3)} = \frac{i \kappa}{4 (3 \lambda_1 - \lambda_1) \text{Im} \lambda_1}, \quad w_2^{(0,3)} = -\frac{i \kappa}{8 \lambda_1 \text{Im} \lambda_1}, \quad w_3^{(0,3)} = \frac{i \kappa}{4 (3 \lambda_1 - \lambda_3) \text{Im} \lambda_3}, \quad w_4^{(0,3)} = -\frac{i \kappa}{4 (3 \lambda_1 - \lambda_3) \text{Im} \lambda_3},
\]

The coefficients for \( \ell = 2 \) are the complex conjugates of the above. The transformation for the SSM \( W(\mathcal{E}_1) \) up to cubic order is therefore of the form

\[
  W(z_1, \bar{z}_1) = \begin{pmatrix}
    z_1 + \frac{\kappa z_1^3}{8 \lambda_1 \text{Im} \lambda_1} + \frac{i \kappa z_1^3}{4 (3 \lambda_1 - \lambda_1) \text{Im} \lambda_1} + \frac{i \kappa z_1 \bar{z}_1^2}{8 \lambda_1 \text{Im} \lambda_1} & + \frac{i \kappa z_1 \bar{z}_1^2}{4 \lambda_1 + 2 \lambda_1 - \lambda_3) \text{Im} \lambda_3} \\
    
    \bar{z}_1 - \frac{\kappa z_1^3}{4 (3 \lambda_1 - \lambda_1) \text{Im} \lambda_1} + \frac{i \kappa z_1^3}{8 \lambda_1 \text{Im} \lambda_1} + \frac{i \kappa z_1 \bar{z}_1^2}{4 \lambda_1 + 2 \lambda_1 - \lambda_3) \text{Im} \lambda_3} & - \frac{i \kappa z_1 \bar{z}_1^2}{4 \lambda_1 + 2 \lambda_1 - \lambda_3) \text{Im} \lambda_3}
  \end{pmatrix}.
\]

(73)

Passing to polar coordinates via the substitution \( z_1 = \rho_1 e^{i \theta_1} \), the corresponding leading-order dynamics \([66]-[67]\) on \( W(\mathcal{E}_1) \) is given by the equations

\[
  \dot{\rho}_1 = -\frac{c}{2} \rho_1, \quad \dot{\theta}_1 = \frac{1}{2} \sqrt{4k_0 - c^2} + \frac{3 \kappa}{2 \sqrt{4k_0 - c^2}} \rho_1^2.
\]

As obtained in formula \([68]\), the instantaneous frequency of the oscillations on the SSM \( \mathcal{E}_1 \) is then

\[
  \omega(\rho_1) = \frac{1}{2} \left( \sqrt{4k_0 - c^2} + \frac{3 \kappa}{2 \sqrt{4k_0 - c^2}} \rho_1^2 \right).
\]
The squared $L^2$ norm of the amplitude in the original $x$ coordinates

$$[\text{Amp}(\rho_1)]^2 = \frac{1}{2\pi} \int_0^{2\pi} |V_x W(\rho_1 e^{i\theta}, \rho_1 e^{-i\theta})|^2 d\theta =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |W_1 + \bar{W}_1 + W_3 + \bar{W}_3|^2 d\theta + \frac{1}{2\pi} \int_0^{2\pi} |W_1 + \bar{W}_1 - W_3 - \bar{W}_3|^2 d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |z_1 + \bar{z}_1 + \mathcal{O}(|z|^2)|^2 d\theta + \frac{1}{2\pi} \int_0^{2\pi} |z_1 + \bar{z}_1 + \mathcal{O}(|z|^3)|^2 d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} [4z_1 \bar{z}_1 + \mathcal{O}(|z|^4)] d\theta = 4\rho_1^2 + \mathcal{O}(\rho_1^4),$$

which gives $\text{Amp}(\rho_1) \approx 2\rho_1$ for small $\rho_1$. For higher values of $\rho_1$, the exact dependence form of $\text{Amp}(\rho_1)$ can be found by evaluating the above integral numerically.

Similarly, for $\ell = 3$, Theorem 1 gives the coefficients

$$w_3^{(1,0)} = 1, \quad w_1^{(1,0)} = w_2^{(1,0)} = w_4^{(1,0)} = 0, \quad w_4^{(0,1)} = 1, \quad w_1^{(0,1)} = w_2^{(0,1)} = w_3^{(0,1)} = 0,$$

$$w_j^{(2,0)} = w_j^{(1,1)} = w_j^{(0,2)} = 0, \quad j = 1, 2, 3, 4,$$

$$w_1^{(3,0)} = \frac{i\kappa}{4(3\lambda_3 - \lambda_1)} \text{Im} \lambda_1, \quad w_2^{(3,0)} = -\frac{i\kappa}{4(3\lambda_3 - \lambda_1)} \text{Im} \lambda_1,$$

$$w_3^{(3,0)} = -\frac{i\kappa}{8\lambda_3 \text{Im} \lambda_1}, \quad w_4^{(3,0)} = -\frac{i\kappa}{4(3\lambda_3 - \lambda_3)} \text{Im} \lambda_3,$$

$$w_1^{(0,3)} = \frac{i\kappa}{4(3\lambda_3 - \lambda_1)} \text{Im} \lambda_1, \quad w_2^{(0,3)} = -\frac{i\kappa}{4(3\lambda_3 - \lambda_1)} \text{Im} \lambda_1,$$

$$w_3^{(0,3)} = \frac{i\kappa}{4(3\lambda_3 - \lambda_3)} \text{Im} \lambda_3, \quad w_4^{(0,3)} = -\frac{i\kappa}{8\lambda_3 \text{Im} \lambda_3},$$

$$w_1^{(2,1)} = \frac{i3\kappa}{4(2\lambda_3 + \lambda_3 - \lambda_1)} \text{Im} \lambda_1, \quad w_2^{(2,1)} = -\frac{i3\kappa}{4(2\lambda_3 + \lambda_3 - \lambda_1)} \text{Im} \lambda_1, \quad w_3^{(2,1)} = 0, \quad w_4^{(2,1)} = -\frac{i3\kappa}{8\lambda_3 \text{Im} \lambda_3},$$

$$w_1^{(1,2)} = \frac{i3\kappa}{4(\lambda_3 + 2\lambda_3 - \lambda_1)} \text{Im} \lambda_1, \quad w_2^{(1,2)} = -\frac{i3\kappa}{4(\lambda_3 + 2\lambda_3 - \lambda_1)} \text{Im} \lambda_1, \quad w_3^{(1,2)} = \frac{i3\kappa}{8\lambda_3 \text{Im} \lambda_3}, \quad w_4^{(1,2)} = 0,$$

$$\beta_3 = \frac{i3\kappa}{4 \text{Im} \lambda_3},$$

whose complex conjugates are the corresponding coefficients for $\ell = 4$.

Similarly, the transformation for the SSM $W(\mathcal{E}_2)$ up to cubic order is of the form

$$W(z_3, \bar{z}_3) = \begin{pmatrix}
\frac{i\kappa z_3^2}{4(3\lambda_3 - \lambda_1) \text{Im} \lambda_1} + \frac{i\kappa \bar{z}_3^2}{4(3\lambda_3 - \lambda_1) \text{Im} \lambda_1} + \frac{i3\kappa z_3^2 \bar{z}_3}{4(2\lambda_3 + \lambda_3 - \lambda_1) \text{Im} \lambda_1} + \frac{i3\kappa z_3 \bar{z}_3^2}{4(3\lambda_3 - \lambda_1) \text{Im} \lambda_1} \\
\frac{\bar{z}_3^2}{4(\lambda_3 + 2\lambda_3 - \lambda_1) \text{Im} \lambda_1} - \frac{\bar{z}_3^2}{8\lambda_3 \text{Im} \lambda_3} + \frac{i\kappa \bar{z}_3^2}{4(3\lambda_3 - \lambda_1) \text{Im} \lambda_1} + \frac{i\kappa \bar{z}_3}{4(3\lambda_3 - \lambda_1) \text{Im} \lambda_1} \\
\frac{z_3^2}{8\lambda_3 \text{Im} \lambda_3} + \frac{z_3^2}{4(3\lambda_3 - \lambda_1) \text{Im} \lambda_1} + \frac{i\kappa z_3^2 \bar{z}_3}{4(2\lambda_3 + \lambda_3 - \lambda_1) \text{Im} \lambda_1} + \frac{i\kappa z_3 \bar{z}_3^2}{4(3\lambda_3 - \lambda_1) \text{Im} \lambda_1} \\
\frac{\bar{z}_3}{4(3\lambda_3 - \lambda_1) \text{Im} \lambda_1} - \frac{\bar{z}_3}{8\lambda_3 \text{Im} \lambda_3} - \frac{i\kappa \bar{z}_3^2}{4(3\lambda_3 - \lambda_1) \text{Im} \lambda_1} - \frac{i\kappa \bar{z}_3}{4(3\lambda_3 - \lambda_1) \text{Im} \lambda_1}
\end{pmatrix}, \quad (76)
with the corresponding dynamics on the first SSM is described by (cf. (66)-(67))

\[
\begin{align*}
\dot{\rho}_3 &= -\frac{3c}{2}\rho_3, \\
\dot{\theta}_3 &= \frac{1}{2}\sqrt{3} (4k_0 - 3c^2) + \frac{\sqrt{3}\kappa}{2\sqrt{4k_0 - 3c^2}}\rho_3^2.
\end{align*}
\] (77) (78)

Following the same calculation as for \(\ell = 1\), we obtain

\[
\begin{align*}
\omega(\rho_3) &= \frac{1}{2} \left( \sqrt{3} (4k_0 - 3c^2) + \frac{\sqrt{3}\kappa}{\sqrt{4k_0 - 3c^2}}\rho_3^2 \right), \\
\text{Amp}(\rho_3) &\approx 2\rho_3.
\end{align*}
\]

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