Affine Toda Solitons and Systems of Calogero-Moser Type

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March 28, 2022

Abstract

The solitons of affine Toda field theory are related to the spin-generalised Ruijsenaars-Schneider (or relativistic Calogero-Moser) models. This provides the sought after extension of the correspondence between the sine-Gordon solitons and the Ruijsenaars-Schneider model.
1 Introduction

The purpose of this letter is to relate the solitons of the affine Toda system to the spin extensions of the Ruijsenaars-Schneider model, the latter also being known as the relativistic Calogero-Moser model. This work generalises the known connection between the solitons of the sine-Gordon equation and the (non-spin) Ruijsenaars-Schneider model. The connection made here should be viewed as part of a larger programme that seeks to identify classes of solutions of PDEs with finite dimensional mechanical systems, whereby the evolution of the solutions to the PDE is expressed as a dynamical system on the (finite-dimensional) moduli space of solutions. Thus, for example, the pole solutions of the KP equation [1, 2] and its reductions (such as KdV [3]) are related to the non-relativistic Calogero-Moser model, while the sine-Gordon solitons are related to its relativistic counterpart, the Ruijsenaars-Schneider model (see [4, 5]). This programme also extends to include the peakon solutions appearing in fibre-optics and shallow water waves which have associated mechanical systems [6, 7]. A similar connection may well underlie the appearance of finite dimensional mechanical systems in the study of various models possessing duality [8, 9]. As the dynamics of mechanical systems are often easier to understand (or simulate) than the equations of motion for a field theory, such a programme aims at giving qualitative information about field theories by an appropriate reduction of degrees of freedom. The recent work of Babelon, Bernard and Smirnov [10] may be viewed as taking this correspondence between field theories and mechanical systems beyond the classical to include the quantum regime as well, though the ability to focus attention solely on a fixed N-particle sector of the full quantum Hilbert space appears to depend crucially on the model. Our work will reveal further new features in such correspondences, as well as provide a sought after generalisation of known results about the sine-Gordon model to the case of affine Toda solitons.

Ruijsenaars and Schneider’s seminal work [5] showed that the soliton solutions of a variety of equations were related to dynamics built from the Hamiltonians (with canonically conjugate variables $q_j, p_j$)

$$H_\pm = \sum_j e^{\pm p_j} \prod_{k \neq j} \coth \left( \frac{q_j - q_k}{2} \right),$$

and equations of motion (for either $H_\pm$)

$$\ddot{q}_j = 2 \sum_{k \neq j} \frac{\dot{q}_j \dot{q}_k}{\sinh(q_j - q_k)}, \quad (1)$$
An appropriate scaling limit of \( H = H_+ + H_- \) yields a system of Calogero-Moser type. In particular, the eigenvalues \( i e^{q_j} \) of an \( N \times N \) matrix associated with the tau function describing an \( N \)-soliton solution of the sine-Gordon equation evolve according to (1). The \( q_j \)'s and \( p_j \)'s may, at least when they are well separated, be related to the positions and rapidities of \( N \) constituent single-solitons; the dynamics of the system encodes the various soliton phase shifts. (More details of this will be given below.) Thus the system governed by \( H_+ \) describes how the space-time trajectories of the ‘constituent’ solitons interact. Of course the same system may be described via the inverse scattering transform by a free system with linearly evolving data: the point of the Ruijsenaars-Schneider description is to make greater contact with the particle description of the soliton.

Viewing the sine-Gordon model as the \( A_{1}^{(1)} \) affine Toda system (with imaginary coupling) a natural question to ask is how the above results generalise to other affine Toda systems. These systems have been extensively studied in recent years both for real and imaginary couplings. In the real coupling regime a beautiful structure was uncovered and exact S-matrices have been conjectured for the theories (see \([11, 12, 13]\) and references therein). The imaginary coupling regime has also been investigated and classically the solitons have real energy-momentum although the Lagrangian is complex (\([14]\) and references therein). Spence and Underwood \([15]\) have recently used this work to obtain the symplectic form on the space of affine Toda solitons but a dynamical description generalising the sine-Gordon/Ruijsenaars-Schneider correspondence has proved elusive. The purpose of the present letter is to give this generalisation. Just as the affine Toda systems generalise the sine-Gordon model, there are spin-generalisations of the Ruijsenaars-Schneider systems, and it is these systems which describe the dynamics of the affine Toda solitons. These models (which have been most studied in the \( A_n \) setting) are the relativistic extension of Gibbon and Hermsen’s spin generalisation of the original Calogero-Moser model \([16]\). One new feature we have found in our correspondence is the appearance of new degrees of freedom, the internal spins of the model. Although not needed to describe the solitons of the affine Toda system, these spins determine the matrix that diagonalises the Lax pair. We will comment further on this later in the letter.

An outline of the letter is as follows. First we will review the construction of affine Toda solitons, and then in section 3 consider the reduced symplectic form of the theory. We are then in a position to relate the affine Toda solitons to the spin-generalised Ruijsenaars-Schneider model in section 4. For the purposes of this letter we shall limit our discussion to the \( A_{1}^{(1)} \) case, both for simplicity and to make clear the generalisation of the sine-Gordon/Ruijsenaars-Schneider correspondence.
2 The $A_n^{(1)}$ Affine Toda Solitons

For the $A_n^{(1)}$ affine Toda theory with imaginary coupling, the equations of motion read

$$\partial_+ \partial_- \phi_j + \frac{m^2}{2i\beta}(e^{i\beta(\phi_j - \phi_{j+1})} - e^{i\beta(\phi_{j-1} - \phi_j)}) = 0,$$

(2)

$j = 0, 1, ..., n$. Here $\pm$ denotes differentiation with respect to light-cone coordinates $x_\pm = \frac{1}{\sqrt{2}}(t \pm x)$, and the indices on the components of the field $\phi$ are read modulo $(n + 1)$ where necessary. We shall be considering the solitonic sector of the theory, which means assuming $\sum_{j=0}^n \phi_j = 0$ (in other words, discarding the free field part of $\phi$).

There are various ways to construct and parametrise soliton solutions to (2). Perhaps the simplest methods to implement from a practical point of view are the application of the Bäcklund transformation derived by Fordy and Gibbons [17] or the bilinear formalism developed by Hirota [18]. There are also the powerful vertex operator techniques which make full use of the representation theory of the $A_n^{(1)}$ algebra [14]. While the latter approach is currently the most popular, we wish to make contact with the original work of Ruijsenaars and Schneider [5], which made much reference to the soliton formulae of Hirota. Hence we choose to start from the form of the N-soliton solution of (2) derived by Hollowood [19] via Hirota’s direct method. The $l$-th component of the field $\phi$ is given by

$$e^{i\beta \phi_l} = \frac{\tau_l - 1}{\tau_l},$$

(3)

where the tau function $\tau_l$ is of the form

$$\tau_l = \sum_\epsilon \exp \left( \sum_{j<k} \epsilon_j \epsilon_k B_{jk} + \sum_j \epsilon_j \zeta_{j,l}(x_+, x_-) \right).$$

(4)

In the above the $\epsilon$ indicates a summation over all possible combinations of $\epsilon_j$ taking the values 0 or 1, and the indices $j$ and $k$ take values in $\{1, ..., N\}$. We will explain shortly what the various terms in (4) mean, and how we have parametrised the $A_n^{(1)}$ affine Toda solitons. For the moment we would like to comment that expression (4) is a rather generic form of the soliton tau function for an integrable PDE, the precise nature of $B_{jk}$ and $\zeta_{j,l}$ depending on the particular PDE being considered; it may be viewed as a degeneration of the theta function solutions of the PDE given via algebraic geometry in which the $\epsilon_j$’s run over all of the integers. Ruijsenaars and Schneider
succeeded in making the connection between their relativistic Calogero-Moser systems and soliton solutions of the sine-Gordon and KdV equations, among others, by showing a direct correspondence between the coordinates of the N-particle system and the parameters of the N-soliton solution. An important part of the correspondence was that all the tau functions of form (4) being considered in [3] could be written in terms of determinants like

\[
\det (1 + M)
\]

for suitable matrices \( M \). In what follows we express all the \( N \)-soliton solutions of the \( A^{(1)}_n \) affine Toda theory in this way, and thereby obtain a relation to spin-generalised Ruijsenaars-Schneider systems.

First of all we should explain the parameters of the Toda \( N \)-soliton which appear in (4). Each soliton has a rapidity denoted by \( \eta_j \), a position parameter denoted by \( a_j \), and a discrete parameter \( \theta_j \) taking values in \( \{ \frac{2\pi k}{n+1} | k = 1, 2, ..., n \} \) (so that \( \exp(i\theta_j) \) is an \((n + 1)\)th root of unity). The rapidities are all real, while the \( a_j \) are pure imaginary for solitons (there are different reality conditions for other types of solution e.g. breathers). The different values of \( \theta_j \) give \( n \) different species of soliton in the \( A^{(1)}_n \) affine Toda theory whose masses are \( 2m \sin(\theta_j/2) \). We also need to define

\[
\mu_j^\pm = \exp(\eta_j \pm \frac{1}{2} i\theta_j).
\]

With this choice of parameters, the terms in the sum (4) are given by

\[
B_{jk} = \log \left( \frac{(\mu_j^+ - \mu_k^-)(\mu_j^- - \mu_k^+)}{(\mu_j^- - \mu_k^+)(\mu_j^+ - \mu_k^-)} \right),
\]

\[
\zeta_{jl}(x_+, x_-) = \log \left( a_j \exp \left( \sqrt{2m} (e^{-\eta_j} x_+ - e^{+\eta_j} x_-) \sin(\theta_j/2) + i\theta_j \right) \right).
\]

(To make a comparison with the vertex operator formulae, we note that in terms of the notation of reference [15], we have \( B_{jk} = \log(X_{j,k}) \), \( a_j = Q_j \). We will deal with the general formalism elsewhere.)

We are now ready to write the tau functions as determinants. In fact Olive, Turok and Liao [20] found that determinants naturally arose when they derived the \( N \)-soliton solution by the Bäcklund transformation, but the matrices involved are not of the right form for our purposes. Instead we set \( X_j = a_j (\mu_j^+ - \mu_j^-) \exp \left( \sqrt{2m} (e^{-\eta_j} x_+ - e^{\eta_j} x_-) \sin(\theta_j/2) \right) \), and define \( N \)-by-\( N \) matrices \( V, \Theta \) by

\[
V_{jk} = \frac{\sqrt{X_j X_k}}{\mu_j^+ - \mu_k^-},
\]

(5)
and

\[ \Theta = \text{diag}(\theta_1, \theta_2, \ldots, \theta_N). \]

Then we find that

\[ \tau_l = \det(1 + e^{il\Theta/2}Ve^{il\Theta/2}). \]

To verify (6) it is necessary to expand the determinant on the right-hand side in terms of the principal cofactors of \( V \), and then use Cauchy’s identity:

\[
\det \left( \frac{1}{\mu_j^+ - \mu_k^-} \right)_{j,k} = \prod_j \frac{1}{\mu_j^+ - \mu_j^-} \prod_{j<k} \left( \frac{\mu_j^+ - \mu_k^+)(\mu_j^- - \mu_k^-)}{\mu_j^- - \mu_k^+)(\mu_j^+ - \mu_k^-)} \right).
\]

Writing the right-hand side of (4) in terms of these new parameters and comparing with the cofactor expansion gives the required result. Note that in the \( A_1^{(1)} \) (sine-Gordon) case the \( \theta_j \) must all take the value \( \pi \), which means that the matrix exponentials appearing in (6) are multiples of the identity, and we reproduce the standard result

\[
e^{i\beta_0} = e^{-i\beta_1} = \frac{\det(1 - V)}{\det(1 + V)}.
\]

3 The Reduced Symplectic Form

In this section we describe the phase space of the \( N \)-soliton solution in terms of its symplectic form, before describing how spin-generalised Ruijsenaars-Schneider systems arise in the following section. The phase space of the affine Toda system has the standard symplectic form

\[
\Omega = \int_{-\infty}^{\infty} (\delta \phi_t \wedge \delta \phi) \, dx.
\]

On substitution of the \( N \)-soliton solution into (7), one obtains (after an integration) the reduced symplectic form on the \( N \)-soliton phase space. In practice it is not possible to perform the integration for anything other than the one-soliton solution [15] (except for the sine-Gordon case, where Babelon and Bernard succeeded in showing that the integrand could be written as an exact derivative for both the one- and two-soliton [4]). For the one-soliton phase space, the reduced symplectic form is (up to an irrelevant numerical factor independent of \( \theta \))

\[
\omega^{(1)} = \frac{da}{a} \wedge d\eta.
\]

The intractability of the integral (7) for the general \( N \)-soliton solution does not matter, as it is a standard result that as \( t \to \pm \infty \) (the out/in
limits) the $N$-soliton decomposes into a superposition of $N$ one solitons with a shift of the parameters. So the symplectic form may just be written

$$\omega^{(N)} = \sum_j \frac{da_j^{\text{out}}}{a_j^{\text{out}}} \wedge d\eta_j^{\text{out}} = \sum_j \frac{da_j^{\text{in}}}{a_j^{\text{in}}} \wedge d\eta_j^{\text{in}}.$$  \hspace{1cm} (8)

By direct calculation using the formula (4) for the tau functions of the $N$-soliton solution, we find the relations between the $\text{out}/\text{in}$ parameters and the standard ones:

$$\eta_j^{\text{in}} = \eta_j^{\text{out}} = \eta_j,$$

$$a_j^{\text{in}} = a_j \prod_{k>j} \frac{(\mu_j^+ - \mu_k^+)(\mu_j^- - \mu_k^-)}{(\mu_j^- - \mu_k^+)(\mu_j^+ - \mu_k^-)} a_j \prod_{k>j} \exp(B_{jk})$$

(and similarly for $a_j^{\text{out}}$ with the inequality reversed). This agrees with the formulae of Spence and Underwood \[15\] obtained via vertex operator arguments, where in their notation $a_j = Q_j$ and $\exp(B_{jk}) = X_{j,k}$. So substituting for the $\text{in}$ parameters into (8), we obtain the $N$-soliton symplectic form as

$$\omega^{(N)} = \sum_j \frac{da_j}{a_j} \wedge d\eta_j + \sum_{j<k} E_{jk}(\eta) \sinh(\eta_j - \eta_k) d\eta_j \wedge d\eta_k,$$  \hspace{1cm} (9)

where

$$E_{jk}(\eta) = \frac{1}{\cosh(\eta_j - \eta_k) - \cos((\theta_j - \theta_k)/2)} - \frac{1}{\cosh(\eta_j - \eta_k) - \cos((\theta_j + \theta_k)/2)}.$$

We observe that $\omega^{(N)}$ is clearly real if we choose the $\eta_j$ to be real and the $a_j$ to be pure imaginary (which in the $A_1^{(1)}$ case coincides with the condition on $a_j$ for sine-Gordon solitons given in \[4\]). This means that the matrix $V$ defined in (5) is anti-hermitian, which will be important in the next section when we look at the dynamics of the eigenvalues of $V$.

4 Ruijsenaars-Schneider Systems

Here we consider how the eigenvalues of the matrix $V$ evolve with respect to each of the light cone coordinates, and find that spin-generalised Ruijsenaars-Schneider equations result. Since $V$ is anti-hermitian, it may be diagonalised with a unitary matrix $U$:

$$Q := UVU^\dagger = \text{diag}(i \exp(q_1), ..., i \exp(q_N)).$$
If we let a dot denote \( \frac{d}{dx} \), then \( V \) satisfies
\[
\dot{V} = \frac{1}{2}(\Lambda V + V\Lambda),
\]
for the constant diagonal matrix \( \Lambda \) given by
\[
\Lambda = \text{diag}(\pm \sqrt{2} m \exp(\mp \eta_1) \sin(\theta_1/2), \ldots, \pm \sqrt{2} m \exp(\mp \eta_N) \sin(\theta_N/2)).
\]
Now let \( u_j \) denote the \( j \)th row of \( U \) (considered as a column vector, so that the \( u_j \) are the left eigenvectors of \( V \)). Define the Lax matrix \( L \) by
\[
L_{jk} = u_k^\dagger \Lambda u_j,
\]
so that
\[
L = U \Lambda U^\dagger.
\]
Then \( L \) satisfies the Lax equation
\[
\dot{L} = [M, L],
\]
for \( M = \dot{U} U^\dagger \). Differentiating the definition of \( Q \) gives
\[
\dot{Q} = [M, Q] + UV U^\dagger,
\]
and after substituting for \( \dot{V} \) from (10) and using the definition of \( L \) we find the identity
\[
LQ + QL = 2(\dot{Q} + [Q, M]).
\]
Upon setting \( Q_j = i e^{\theta_j} \), then (12) in components reads
\[
L_{jk}(Q_j + Q_k) = 2(\dot{Q}_j \delta_{jk} + M_{jk}(Q_j - Q_k)),
\]
which yields
\[
L_{jj} = \dot{q}_j
\]
and
\[
M_{jk} = \frac{1}{2} \left( \frac{Q_j + Q_k}{Q_j - Q_k} \right) L_{jk} = \frac{1}{2} \coth((q_j - q_k)/2)L_{jk},
\]
(for \( j \neq k \)). When \( V \) is diagonalised we may always choose the phases of the left eigenvectors \( u_j \) so that \( M_{jj} = u_j^\dagger \dot{u}_j = 0 \). Substituting these into the Lax equation (11) produces the equations of motion:
\[
\dot{L}_{jj} = \dot{\theta}_j = \sum_{k \neq j} \coth((q_j - q_k)/2)L_{jk}L_{kj},
\]
\[ \ddot{L}_{jk} = \frac{1}{2} \coth((q_j - q_k)/2)(\dot{q}_k - \dot{q}_j) L_{jk} + \sum_{l \neq j,k} \frac{1}{2} \left( \coth((q_j - q_l)/2) - \coth((q_l - q_k)/2) \right) L_{jl} L_{lk} \] (15)

(j \neq k). These are in fact the spin-generalised Ruijsenaars-Schneider equations with certain constraints, although to see this requires comparison with the formulae of Krichever and Zabrodin [21].

In [21] the generalised Ruijsenaars-Schneider model is defined in terms of \(N\) particle positions \(x_j\) and their internal degrees of freedom (spins) given by \(l\)-dimensional vectors \(a_j\) and \(l\)-dimensional covectors \(b_j^\dagger\), subject to the equations of motion

\[ \ddot{x}_j = \sum_{k \neq j} (b_j^\dagger a_k)(b_k^\dagger a_j)(\mathcal{V}(x_j - x_k) - \mathcal{V}(x_k - x_j)), \] (16)

\[ \dot{a}_j = \sum_{k \neq j} a_k(b_k^\dagger a_j)\mathcal{V}(x_j - x_k), \] (17)

\[ \dot{b}_j^\dagger = -\sum_{k \neq j} b_k^\dagger(b_k^\dagger a_k)\mathcal{V}(x_k - x_j). \] (18)

The potential \(\mathcal{V}\) is expressed in terms of the Weierstrass zeta function or its rational or hyperbolic limits. To make contact with our equations we set \(x_j = q_j\) and choose the hyperbolic potential

\[ \mathcal{V}(q_j - q_k) = \frac{1}{2} \coth((q_j - q_k)/2). \]

Then (16) generalises (1). In [21] the spin degrees of freedom were real, but here we allow them to be complex, and identify them with the eigenvectors of \(V\) by setting

\[ b_j^\dagger = u_j^\dagger, \quad a_j = \Lambda u_j. \]

So we have taken \(l = N\), and in fact our spins are expressed entirely in terms of the eigenvectors of \(V\) and the constant matrix \(\Lambda\); in particular the \(b_j^\dagger\) must form an orthonormal basis. In the notation of [21] the components of the Lax matrix are given by

\[ L_{jk} = b_k^\dagger a_j. \]

There are various other constraints that we have imposed on our system. First the equations (16), (17) and (18) have the scaling symmetry

\[ a_j \rightarrow \alpha_j a_j, \quad b_j^\dagger \rightarrow \frac{1}{\alpha_j} b_j^\dagger. \]

The corresponding integrals of motion are \(\dot{x}_j - b_j^\dagger a_j\), and setting them to zero and rewriting them in terms of our coordinates shows that this is equivalent to
equation (13). Similarly our requirement that \( M_{jj} = 0 \) is another constraint on the system. Now given these constraints we find that from the definition of \( L \) in terms of the spins we can compute \( \dot{L}_{jk} \). So for \( j = k \) (18) is equivalent to (14), while for \( j \neq k \) (17) and (18) yield (15).

To make the correspondence between the solitons and the many-body system clearer, it is worth considering the sine-Gordon case in more detail and comparing it with the general situation. The results about sine-Gordon solitons are explained in detail in [4], and we have kept our notation as similar to this reference as possible to make comparison easier. The first thing to observe is that in the \( A_1(1) \) case only knowledge of the \( q_j \) is required to specify the field components, as we have

\[
e^{i\beta_{\phi_0}} = e^{-i\beta_{\phi_1}} = \prod_{j=1}^{N} \frac{1 - i \exp(q_j)}{1 + i \exp(q_j)}.
\]

In the general case the presence of the matrix \( e^{i\Theta/2} \) in the expression for the tau functions (3) means that knowledge of both the spin vectors \( u_j \) (which make up the matrix \( U \)) and the \( q_j \) is required to evaluate these determinants. The essential difference is that for sine-Gordon there is only one soliton species, while in the \( A_n(1) \) case there are \( n \) different species corresponding to the different allowed values of \( \theta_j \). This difference is also apparent at the level of the equations of motion. In fact when we differentiate the matrix \( V \), in the case of sine-Gordon we find from (10) that

\[
\dot{V} = i(ee^\dagger)
\]

for a certain vector \( e \). But then conjugating the equation (10) with \( U \) we obtain

\[
i\tilde{e}e^\dagger = \frac{1}{2}(LQ + QL),
\]

where \( \tilde{e} = U e \). Actually \( \tilde{e} \) is a real vector, and in terms of its components \( \tilde{e}_j \) we have

\[
L_{jk} = 2 \frac{\tilde{e}_j \tilde{e}_k}{\exp(q_j) + \exp(q_k)}.
\]

Since we know the diagonal elements of \( L \) explicitly in terms of the \( q_j \) (from (13)) the above formula means that we then know all the \( \tilde{e}_j \) and hence the off-diagonal elements of \( L \) are found to be

\[
L_{jk} = \sqrt{\frac{q_j q_k}{\cosh((q_j - q_k)/2)}}.
\]

This may then be substituted into (14), (13) to give the ordinary (non-spin) Ruijsenaars-Schneider equations. In this case (14) yields (1) and (15) is a
consequence. Babelon and Bernard have shown [4] that there is a canonical transformation between the soliton parameters and the dynamical variables $q_j, \dot{q}_j$ (more precisely, they formulate this in terms of the variables $Q_j = i \exp(q_j)$). We discuss how this could possibly be extended to the $A_n^{(1)}$ case in our Conclusion.

5 Conclusion

We have shown the connection between spin-generalised Ruijsenaars-Schneider systems and $A_n^{(1)}$ affine Toda solitons. The soliton tau functions are determined by the positions $q_j$ of $n$ particles on the line as well as an orthonormal set of $n$-dimensional spin vectors $u_j$, which are together subject to the equations of a constrained spin-generalised Ruijsenaars-Schneider model. This extends the known result for the sine-Gordon equation, where the spins are no longer part of the dynamics and there is a canonical transformation between the positions and momenta of the particles and the parameters of the solitons. For the general case such a transformation is no longer apparent, although we note that the $N$-soliton phase space is still of dimension $2N$, and so it is worth exploring exactly how the extra spin degrees of freedom are absorbed in the transition from the dynamical variables to the soliton parameters. Also it would be interesting to see what rôle the spins might play in the quantum theory. Finally there remains the extension to the other affine algebras and elucidating the connections to the vertex operator constructions mentioned at various points in the text. We intend to pursue these points in the future.

6 Acknowledgements

One of us (ANWH) thanks the EPSRC for support. HWB thanks D. Bernard, D Olive and R Sasaki for stimulating discussions on related matters.

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