ON GLOBAL SOLUTIONS AND BLOW-UP FOR KURAMOTO–SIVASHINSKY-TYPE MODELS, AND WELL-POSED BURNETT EQUATIONS

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Abstract. The initial boundary-value problem (IBVP) and the Cauchy problem for the Kuramoto–Sivashinsky equation

\[ v_t + v_{xxxx} + v_{xx} = \frac{1}{2} (v^2)_x \]

and other related 2\(m\)th-order semilinear parabolic partial differential equations in one dimension and in \(\mathbb{R}^N\) are considered. Global existence and blow-up as well as \(L^\infty\)-bounds are reviewed by using:

(i) classic tools of interpolation theory and Galerkin methods,
(ii) eigenfunction and nonlinear capacity methods,
(iii) Henry’s version of weighted Gronwall’s inequalities,
(vi) two types of scaling (blow-up) arguments.

For the IBVPs, existence of global solutions is proved for both Dirichlet and “Navier” boundary conditions. For some related 2\(m\)th-order PDEs in \(\mathbb{R}^N \times \mathbb{R}_+\), uniform boundedness of global solutions of the Cauchy problem is established.

As another related application, the well-posed Burnett-type equations

\[ v_t + (v \cdot \nabla)v = -\nabla p - (-\Delta)^m v, \quad \text{div} v = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad m \geq 1, \]

are studied. For \(m = 1\), these are the classic Navier-Stokes equations. As a simple illustration, it is shown that a uniform \(L^p(\mathbb{R}^N)\)-bound on locally sufficiently smooth \(v(x,t)\) for \(p > \frac{N}{2m-1}\) implies a uniform \(L^\infty(\mathbb{R}^N)\)-bound, hence the solutions do not blow-up. For \(m = 1, N = 3\), this gives \(p > 3\) reflecting the famous Leray–Prodi–Serrin–Ladyzhenskaya regularity results \((L^p,q\) criteria\), and re-derives Kato’s class of unique mild solutions in \(\mathbb{R}^N\). Bounded classical \(L^2\)-solutions are shown to exist for \(N < 2(2m-1)\).

1. Introduction and motivation

The role of the Kuramoto–Sivashinsky equation

\[ v_t + v_{xxxx} + v_{xx} = \frac{1}{2} (v^2)_x \]
is well known in contemporary nonlinear mechanics and physics. This equation arises as a model in hydrodynamics (a thin film flow down an inclined plane in the presence of an electric field), in combustion theory (propagation of flame fronts), phase turbulence and plasma physics, as well as one model for spatio-temporal chaos and in many other physical phenomena; see [74] for a nice short review of applications with key original references.

First results on global existence of classical solutions of (1.1) go back to the 1970s (cf. [55]) and the 1980s; we refer to [5, 6, 7, 11, 19, 33, 36, 43, 45, 47, 51, 65, 68, 72, 74, 80], where further important references can be found. A large part of previous study of (1.1) was devoted to periodic problems. See [2, 47] for important references from the 1980s and also to some classes of particular solutions and the local behavior of solutions, where very interesting results (for instance, chaotic behaviour, estimates of dimension and structure of attractors, bifurcation theorems, etc.) have been obtained. Other papers with deep results are devoted to existence of periodic solutions and traveling wave solutions with complicated dynamics. Numerous contributions are dealing with dynamical analysis of the one dimensional Kuramoto–Sivashinsky equation.

The present paper is devoted to a review of approaches that lead to global existence and blow-up for the Kuramoto–Sivashinsky equation and modified versions of it in one dimension and on $\mathbb{R}^N$, the latter being much less developed in the literature.

One of the key mathematical features of the KS-type PDEs is that an a priori $L^2$-bound of solutions $v(x, t)$ of the form

$$(1.2) \quad \|v(t)\|_2 = \left( \int |v(x, t)|^2 \, dx \right)^{\frac{1}{2}} \leq \|v_0\|_2 e^{\frac{t}{4}} \quad \text{for} \quad t \geq 0,$$

is straightforward, while the main question is how to use this information to get stronger estimates in Sobolev spaces and eventually in $L^\infty$.

To this end, we shall use some known methods and compare their strength with the new techniques to be reviewed and developed throughout this paper:

(I) In Section 2 we employ classical interpolation and Galerkin methods for study global existence of the initial-boundary value problems (IBVP) in one dimension with Dirichlet boundary conditions and with “Navier” ones;

(II) Eigenfunction technique and nonlinear capacity method [61] are used in Section 3 to prove blow-up in finite time of solutions of these IBVPs with non-standard boundary conditions;

(III) In Section 4 Henry’s version of weighted Gronwall’s inequalities are used prove global existence for the Cauchy problem in $\mathbb{R}^N \times \mathbb{R}_+$ for modified Kuramoto-Sivashinsky equations;

(IV) In Section 5 we use two types of scaling blow-up techniques for global existence for the Cauchy problem and IBVPs.

Our analysis embraces a number of $2m$ th-order $N$-dimensional Kuramoto-Sivashinsky type models posed in $\mathbb{R}^N \times \mathbb{R}_+$. 2
In general, as customary in PDE theory, the following approach is classical:

\[(1.3)\quad \text{local existence } + \text{ a global a priori bound } \implies \text{ global existence.}\]

More precisely, in order to get (1.3), we use the following intermediate blow-up step:

\[(1.4)\quad \text{local existence } + \text{ a global a priori bound } \implies \text{ no finite-time blow-up.}\]

Here scaling arguments to prevent blow-up are crucial.

In this context, the problem of global solvability takes the following obvious negation form:

\[(1.5)\quad \text{global solvability } \neq \exists \text{ finite-time blow-up of local smooth solutions.}\]

Of course, here we understand the a priori estimates in the corresponding function spaces. This proclamation is also relevant to the Navier–Stokes equations in \(\mathbb{R}^3\), which have a long mysterious history of uniqueness/non-uniqueness and blow-up singularity open problems.

As rather unexpected but related application, in Section 6, we consider the Navier–Stokes equations

\[(1.6)\quad v_t + (v \cdot \nabla)v = -\nabla p + \Delta v, \quad \text{div } v = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+,\]

with bounded integrable divergence-free data \(v_0\). It is worth noting that the convective term in the first equation in (1.6) for the velocity field has indeed a nonlinear dispersion mathematical nature as in the Kuramoto-Sivashinsky equation (1.1). Therefore, it is rather natural to include the model (1.6) in the present context of semilinear KS-type equations.

**Blow-up self-similar singularities with finite energy do not exist.** The idea that the classical fundamental problem of unique solvability of (1.6) in \(\mathbb{R}^3\) is associated with existence or nonexistence of certain blow-up singularities as \(t \to T^-\), goes back to Th. von Kármán; see [44]. Later on, in 1933-34, J. Leray [52, 54] proposed a mathematical question to look for blow-up in (1.6) for \(N = 3\) driven by the self-similar solutions of the standard dimensional type, with blow-up time \(T = 1\),

\[(1.7)\quad v(x,t) = \frac{1}{\sqrt{1-t}} w(y), \quad p(x,t) = \frac{1}{1-t} P(y), \quad \text{where } y = \frac{x}{\sqrt{1-t}}.\]

Substituting (1.7) into (1.6) yields for functions \(w\) and \(P\) a “stationary” system,

\[(1.8)\quad \frac{1}{\tau} w + \frac{1}{\tau} (y \cdot \nabla)w + (w \cdot \nabla)w = -\nabla P + \Delta w, \quad \text{div } w = 0 \quad \text{in } \mathbb{R}^N.\]

During last fifteen years, a number of negative answers concerning existence of such non-trivial similarity patterns (1.7), (1.8) in \(\mathbb{R}^3\) were obtained; see [13, 62, 60], and the most advanced and justifying negative answer in [41]. Let us note an existence result in [16] for \(N = 4\).

Nonexistence of Leray’s similarity solutions (1.7) and other local types of self-similar blow-up is a definite step towards better understanding of the singularity nature for the Navier–Stokes equations. Of course, this does not settle the problem of singularity...
formation (or nonexistence of finite energy singularities which is more plausible), since there might be other ways for (1.6) to create singularities as $t \to 1^-$ rather than the purely self-similar scenario (1.7). This multiplicity question is discussed below.

On a countable set of blow-up patterns with infinite energy. For infinite $L^2$-energy, the blow-up (also called the enstrophy blow-up of vorticity) in the Navier–Stokes equations (1.6) can occur even for $N = 2$. Such global blow-up described by von Kármán solutions is explained in [32, Ch. 8], where a rigorous theory of such singularities was developed. Earlier history of such solutions can be found in [1]. This blow-up creates a plane jet. It is worth noting that the blow-up behaviour is also not of self-similar form as $t \to T^-$, and it is given by a similarity solutions of a non-local first-order Hamilton–Jacobi equation associated with such a flow. Moreover, and this is also crucial, that [32, pp. 232–235]

(1.9) there exists a countable set of such different blow-up patterns ($N = 2$).

Similar solutions can be constructed for axisymmetric flows in cylindrical coordinates for the Navier–Stokes equations (1.6) in $\mathbb{R}^3$; see an example in [1, Ch. 7, §3] and [64]. However, the mathematics of such blow-up patterns becomes more involved, and (1.9) for $N = 3$ demands difficult proofs [29].

Singular (blow-up) set has zero measure. There exists another classic direction of the singularity theory for the Navier–Stokes equations that was originated by Leray himself [54] (details are available in [20]) and in Caffarelli, Kohn, and Nirenberg [9]. It was shown that the one-dimensional Hausdorff measure of the singular (blow-up) points in a time-space cylinder is equal to zero. We refer to [63, 69] for further development and references. In particular, among other results including Leray’s one in [54], a refined criterion is obtained in [69], saying that, if $t = 1$ is the first singular (blow-up) moment for a solution $v(x, t)$ of (1.6), then

\[
\lim_{t \to 1^-} \frac{1}{1-t} \int_1^t \int_{\mathbb{R}^3} |v(x, t)|^3 \, dx \, dt = +\infty.
\]

Performing the scaling as in (1.7),

\[
v(x, t) = \frac{1}{\sqrt{1-t}} w(y, \tau), \quad y = \frac{x}{\sqrt{1-t}}, \quad \tau = -\ln(1-t) \to +\infty \text{ as } t \to 1^-,
\]

(1.10) takes the same form

\[
\lim_{\tau \to +\infty} e^\tau \int_\tau^{+\infty} e^{-s} \left( \int_{\mathbb{R}^3} |w(x, s)|^3 \, dy \right) ds = +\infty.
\]

This means that, for the existence of a singular point $t = 1$, the solution of the rescaled equations,

\[
w_\tau + \frac{1}{2} w + \frac{1}{2} (y \cdot \nabla)w + (w \cdot \nabla)w = -\nabla P + \Delta w, \quad \text{div } w = 0 \quad \text{in } \mathbb{R}^N,
\]

must diverge (blow-up) as $\tau \to +\infty$ in $L^3(\mathbb{R}^3)$. 

\[ \text{(1.13)} \]
Thus, according to the criterion (1.12), $t = 1$ is not a singular (and hence regular) point, if the corresponding locally smooth solution of (1.13) does not blow-up as $\tau \to \infty$ in a suitable functional setting. Hence, the problem of global existence and uniqueness of a smooth solutions of the Navier–Stokes equations in $\mathbb{R}^3$ reduces to nonexistence of blow-up in infinite time for the rescaled system (1.13). In such a framework, this problem falls into the scope of standard blow-up/non-blow-up theory for nonlinear evolution PDEs.

**Countable sets of blow-up patterns in combustion problems.** Actually, there are many examples of countable sets of blow-up patterns for of much simpler reaction-diffusion equations. Amongst them is the classic *Frank-Kamenetskii equation* (1938) [23] developed in combustion theory of solid fuels (also called *solid fuel model*),

\begin{equation}
(1.14) 
  u_t = \Delta u + e^u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+ \quad (N = 1, 2),
\end{equation}

for which there exists a countable set of different blow-up patterns. Rigorous mathematical theory of such blow-up patterns was known since the beginning of the 1990s (see e.g., [75, 76]) and was developed by linearization in the inner blow-up region and nonlinear matching. A similar strategy to construct a countable set of blow-up patterns is applicable to the higher-order reaction-diffusion PDEs [26]

\begin{equation}
(1.15) 
  u_t = -(-\Delta)^m u + |u|^{p-1} u \quad (m \geq 2, \ p > 1),
\end{equation}

where the analysis uses polynomial eigenfunctions and discrete spectrum of some related linear non self-adjoint differential operators [17].

On the other hand, the quasilinear 1D counterpart of (1.14) with the $p$-Laplacian,

\begin{equation}
(1.16) 
  u_t = (|u_x|^\sigma u_x)_x + e^u \quad (\sigma > 0),
\end{equation}

is also known to admit a countable set of blow-up patterns [8] (see also a general discussion in [28, pp. 30-34]), but now, depending on $\sigma > 0$, first few are self-similar, i.e., represent the case of *nonlinear eigenfunctions* (not linearized as above for (1.14)).

**Evolution completeness as a necessary ingredient.** It is a principal open problem to describe the whole set of all possible blow-up patterns (if any) for the Navier–Stokes equations (1.6). Evidently, proving nonexistence of all the blowing up patterns, i.e., nonexistence of blow-up at all, will settle the fundamental problem of global smooth (and hence unique) continuation of sufficiently arbitrary solutions. Another natural possibility is to establish that, for given class of data, the orbits do not approach any of blow-up pattern scenario, so remain regular for all times.

In this context, the problem of **evolution completeness** of the given countable set of patterns occur, meaning that these patterns exhaust all possible types of approaching the singularity for a fixed class of data. For linear problems, the evolution completeness follows from standard completeness and closure of eigenfunction subset of a linear operator or pencil in a fixed functional framework, so the evolution completeness does not have a separate meaning. For nonlinear problems, where “linear” notions of completeness and closure make no sense in general, the evolution completeness becomes key.

There are some examples of evolution completeness of countable sets of linearized or nonlinear patterns (eigenfunctions) in parabolic asymptotic theory. For instance, the full...
classification of blow-up sets for the Frank-Kamenetskii equation \((1.14)\) was performed by Velázquez \([75]\) by actual proving the evolution completeness of the countable set of linearized blow-up patterns. Concerning nonlinear patterns, it seems that there exists a unique example of a proof of evolution completeness of such a countable set for the porous medium and \(p\)-Laplacian equations in 1D or in radial geometry in \(\mathbb{R}^N\),

\[
  u_t = \Delta(|u|^{m-1}u) \quad \text{and} \quad u_t = \nabla \cdot (|\nabla u|^{m-1}\nabla u), \quad \text{with} \quad m > 1;
\]

see \([27]\), where the notion of evolution completeness was introduced.

On the well-posed Burnett-type equations. Concerning our conclusion, as a straightforward application of the scaling technique in \((IV)\), we present a simple proof of the fact that a local smooth solution \(v(x,t)\) of \((1.6)\), which is uniformly bounded in \(L^p(\mathbb{R}^N)\), with \((1.17)\)

\[
  p > N, \quad \text{i.e.,} \quad p > 3 \quad \text{for} \quad N = 3,
\]

is uniformly bounded in \(L^\infty(\mathbb{R}^N)\), so that such \(L^p\)-solutions do not blow-up. The non-blow-up also is proved in the well-known critical case \(p = N = 2\). The condition \((1.17)\) is consistent with Leray–Prodi–Serrin–Ladyzhenskaya regularity \(L^{p,q}\) criteria and other more recent results; see key references, history, details, and results in recent papers \([65, 20, 69, 25]\). In addition, \((1.17)\) re-derives Kato’s class of unique mild solutions in \(\mathbb{R}^N\), \([42]\); see details and key references in \([25, 77]\).

This approach also extends to the related 2mth-order well-posed Burnett equations,

\[
  (1.18) \quad v_t + (v \cdot \nabla)v = -\nabla p - (-\Delta)^m v, \quad \text{div} v = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+,
\]

containing the higher-order diffusion operator \(-(-\Delta)^m v\), with arbitrary \(m \geq 1\). Then the regularity criterion similar to that in \((1.17)\) for \(m = 1\) reads

\[
  p > \frac{N}{2m-1} \quad \text{for} \quad m = 1, 2, 3, \ldots.
\]

On the other hand, we prove that for smooth fast decaying divergence-free \(L^2\)-data \(v_0\), finite-time blow-up is impossible in dimensions

\[
  N < 2(2m - 1) \quad (N = 2 \quad \text{for} \quad m = 1 \quad \text{is included}, \quad [53, 50]),
\]

so there exists a unique global classic bounded solution.

2. Method of interpolation: global existence for the KSE

2.1. A priori estimates. To demonstrate these classic approaches, we consider the one dimensional KSE in the following IBVP setting (for convenience, here \(D = \frac{\partial}{\partial x}\)):

\[
  (2.1) \quad v_t + D^4v + D^2v = \frac{1}{2} Dv^2, \quad t > 0, \quad x \in (-L, L),
\]

\[
  (2.2) \quad v(x, 0) = v_0(x), \quad x \in (-L, L),
\]

with either:

\[
  (2.3) \quad v = Dv = 0, \quad x = -L, \quad x = L, \quad t > 0, \quad \text{or}
\]

\[
  (2.4) \quad v = D^2v = 0, \quad x = -L, \quad x = L, \quad t > 0.
\]
A priori estimates. Multiplying (2.1) by \(v\) and integrating by parts over \(\Omega = (-L,L)\) with regard to (2.3) or (2.4), we find

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega v^2(x,t) \, dx + \int_\Omega |D^2v|^2 \, dx - \int_\Omega |Dv|^2 \, dx = 0.
\]

Due to (2.3) or (2.4), we have

\[
\int_\Omega |Dv|^2 \, dx \leq \left( \int_\Omega v^2 \, dx \right)^{\frac{1}{2}} \left( \int_\Omega |D^2v|^2 \, dx \right)^{\frac{1}{2}}.
\]

Denoting by

\[
E(t) = \int_\Omega v^2(x,t) \, dx,
\]

from (2.5) we get

\[
\frac{1}{2} \frac{dE}{dt} \leq \left( \int_\Omega |D^2v|^2 \, dx \right)^{\frac{1}{2}} E^{\frac{1}{2}}(t) - \int_\Omega |D^2v|^2 \, dx \leq \frac{1}{4} E(t).
\]

So

\[
E(t) \leq \hat{E}(t) := E(0) e^{\frac{t}{2}}.
\]

Next, integrating (2.5) in \(t > 0\), we see that

\[
\int_0^t \int_\Omega |D^2v|^2 \, dx \, dt = \int_0^t \int_\Omega |Dv|^2 \, dx \, dt - \frac{1}{2} E(t) + \frac{1}{2} E(0).
\]

Due to (2.3) or (2.4) we have

\[
0 = \int_0^t \int_\Omega D(vDv) \, dx \, dt = \int_0^t \int_\Omega vD^2v \, dx \, dt + \int_0^t \int_\Omega |Dv|^2 \, dx \, dt.
\]

From this, it follows that

\[
\int_0^t \int_\Omega |Dv|^2 \, dx \, dt \leq \frac{1}{2} \int_0^t \int_\Omega v^2 \, dx \, dt + \frac{1}{2} \int_0^t \int_\Omega |D^2v|^2 \, dx \, dt.
\]

Thus, (2.3) implies that

\[
\frac{1}{2} \int_0^t \int_\Omega |D^2v|^2 \, dx \, dt \leq \frac{1}{2} \int_0^t E(t) \, dt - \frac{1}{2} E(t) + \frac{1}{2} E(0).
\]

Hence, by (2.7) one has

\[
\int_0^t \int_\Omega |D^2v|^2 \, dx \, dt \leq 2E(0)e^{\frac{t}{2}} - E(0).
\]
Then from (2.8) it follows that

\[(2.11)\quad \int_0^t \int_\Omega |Dv|^2 \, dx \, dt \leq (2e^\frac{t}{2} - \frac{3}{2}) E(0).\]

**Estimate of** \(\int_0^t \int_\Omega v^2 |Dv|^2 \, dx \, dt\). Thanks to embedding inequality for \(\Omega \subset \mathbb{R}\) we have

\[\|v(t)\|_\infty^2 \leq c_\infty \int_\Omega |Dv|^2(x, t) \, dx\]

since \(v|_{\partial\Omega} = 0\). Then

\[\int_\Omega v^2 |Dv|^2 \, dx \leq c_\infty \left( \int_\Omega |Dv|^2 \, dx \right)^2.\]

Next

\[\left( \int_\Omega |Dv|^2 \, dx \right)^2 \leq \int_\Omega v^2 \, dx \cdot \int_\Omega |D^2v|^2 \, dx = E(t) \int_\Omega |D^2v|^2 \, dx.\]

Applying (2.10), we find

\[\int_0^t \int_\Omega v^2 |Dv|^2 \, dx \, dt \leq c_\infty \int_0^t \hat{E}(\tau) \int_\Omega |D^2v|^2 \, dx \, d\tau\]

\[(2.12)\quad \leq c_\infty \hat{E}(t) \int_0^t |D^2v|^2 \, dx \, d\tau \leq c_\infty \hat{E}(t)(2e^\frac{t}{2} - 1) E(0)\]

\[= c_\infty e^\frac{t}{2} (2e^\frac{t}{2} - 1) E^2(0).\]

Next, by using the above estimates and the linear theory of the parabolic equations of the 4th-order we get (here \(Q_T = \Omega \times (0, T)\))

\[\|v(T)\|_{L^2(\Omega)}^2 + \|v_t\|_{L^2(Q_T)}^2 + \|D^4v\|_{L^2(Q_T)}^2 \]

\[+ \|D^2v\|_{L^2(Q_T)}^2 + \|Dv\|_{L^2(Q_T)}^2 + \|v\|_{L^2(Q_T)}^2 \leq C(\|vDv\|_{L^2(Q_T)}^2 + \|v_0\|_{L^2(\Omega)}^2)\]

\[\leq C_1(e^T E^2(0) + e^\frac{T}{2} E(0) + \|v_0\|_{L^2(\Omega)}^2)\]

with a positive constant \(C_1\) independent of \(v\) and \(T > 0\).
2.2. Global existence.

Theorem 2.1. For any \( v_0 \in L^2(\Omega) \), both the IBVPs \((2.1)-(2.3)\) and \((2.1), (2.2), (2.4)\) have solutions for any \( t > 0 \), satisfying inequality \((2.13)\).

Proof. The ground a priori estimates \((2.7), (2.9), (2.10)\) and \((2.12)\) are obtained by multiplication of equation \((2.1)\) by \( v \). Thanks to this we use the Galerkin approach and these a priori estimates are valid for the Galerkin approximations \( \{v_m\} \).

The passage to the limit \( v_m \to v \) as \( m \to \infty \) in the nonlinear term \( v(x,t)Dv(x,t) \) \( x \in \Omega \subset \mathbb{R}, t > 0 \) is proved by standard arguments (see [5] [56]). As a result we get a weak solution. Then from \((2.12)\) due to the linear theory of parabolic equations of the 4th order we obtain the final estimate \((2.13)\).

\[ \square \]

3. Method of eigenfunctions: blow-up of solutions

We now show how, using a similar interpolation-eigenfunction technique, to derive sufficient conditions of blow-up for the KSE with special “boundary conditions”.

3.1. Basic computations. Multiplying equation \((2.1)\) by a function \( \psi(x) \) belonging to \( C^4(\mathbb{R}) \), we get

\[
\frac{d}{dt} \int_{\Omega} v \psi \, dx + \int_{\Omega} v(D^4\psi + D^2\psi) \, dx = -\frac{1}{2} \int_{\Omega} v^2 D\psi \, dx + B(v, \psi),
\]

where \( \Omega = (0, L) \subset \mathbb{R} \) and

\[
B(v, \psi) = \left[ vD^3\psi - Dv \cdot D^2\psi + (D^2v + v)D\psi - (D^3v + Dv)\psi + \frac{1}{2} v^2 \psi \right]_0^L.
\]

For the function

\[
\psi(x) = \psi_\lambda(x) := |x - L|^\lambda
\]

with \( \lambda > 6 \), we have from \((3.1)\)

\[
\frac{d}{dt} \int_{\Omega} v\psi_\lambda \, dx = \frac{\lambda}{2} \int_{\Omega} v^2 \cdot |x - L|^{\lambda-1} \, dx - \int_{\Omega} v\psi_1 \, dx + B_0(v), \quad \text{where}
\]

\[
\psi_1 = \lambda(\lambda - 1)|x - L|^{\lambda-4}[(\lambda - 2)(\lambda - 3) + |x - L|^2] \quad \text{and}
\]

\[
B_0(v) = -\frac{1}{2} L^\lambda v^2(0, t) + \lambda L^{\lambda-1} \cdot [(\lambda - 1)(\lambda - 2)L^{-2}]v(0, t)
\]

\[
+ L^\lambda[1 + \lambda(\lambda - 1)L^{-2}]v(0, t) + \lambda L^{\lambda-1}D^2v(0, t) + L^\lambda D^3v(0, t).
\]

Denote by

\[
J(t) = \int_{\Omega} v(x,t)|x - L|^\lambda \, dx, \quad \Omega = (0, L).
\]
\[
\int_\Omega v\psi_1 \, dx \leq \left( \int_\Omega v^2 |x - L|^{\lambda-1} \, dx \right)^{\frac{1}{2}} \left( \int_\Omega \psi_1^2(x) \, dx \right)^{\frac{1}{2}} \\
\leq \frac{\lambda}{4} \int_\Omega v^2 |x - L|^{\lambda-1} \, dx + C_\lambda(L), \quad \text{with}
\]
\[
C_\lambda(L) = \lambda(\lambda - 1)L^{\lambda-6} \left[ \frac{\lambda-2}{\lambda-6} + \frac{2(\lambda-2)(\lambda-3)}{\lambda-4}L^2 + \frac{1}{\lambda-2}L^4 \right],
\]

from (3.3) it follows that
\[
\frac{dJ}{dt} \leq \frac{\lambda}{4} \int_\Omega v^2 |x - L|^{\lambda-1} \, dx + B_0(v) - C_\lambda(L).
\]

By using the Hölder inequality
\[
J^2(t) \leq \frac{L^{\lambda+2}}{\lambda+2} \int_\Omega v^2 |x - L|^{\lambda-1} \, dx,
\]
we get
\[
\frac{dJ}{dt} \geq \frac{\lambda(\lambda+2)}{4} \frac{1}{L^{\lambda+2}} J^2(t) + B_0(v) - C_\lambda(L).
\]

3.2. **General initial boundary value problem.** We consider a non-standard IBVP for the KSE (2.1) that includes initial data
\[
(3.6) \quad v(x, 0) = v_0(x)
\]
and some general boundary conditions to be specified,
\[
(3.7) \quad B_0(v) = h(t) \quad \text{at} \quad x = 0 \quad \text{for} \quad t > 0.
\]

We understand a solution \(v\) of (2.1), (3.6), (3.7) as a weak solution in the sense of identity (3.7) with respect to any test function \(\psi \in C^4(\mathbb{R})\) with an initial data in the sense
\[
\int_\Omega v(x, t)\psi(x) \, dx \rightarrow \int_\Omega v_0(x)\psi(x) \, dx \quad \text{as} \quad t \rightarrow 0.
\]

Of course, the function \(v\) is assumed to belong to the function space such that identity (3.7) makes sense.

3.3. **The blow-up results.** As follows from (3.5) the blow-up phenomenon for IBVP (2.1), (3.6), (3.7) depends on
\[
(3.8) \quad H_\lambda(v, L) := B_0(v) - C_\lambda(L).
\]

It means it depends on the relationship between initial data \(v_0(x)\) and boundary conditions.
Theorem 3.1. Let $H_\lambda(v, L) \geq a^2 > 0$ for some $\lambda > 6$ and $L > 0$. Then there is no global solution of (2.1), (3.6), (3.7).

Moreover, there is no global solution of (2.1), (3.6), (3.7) for $x \in (0, L)$, and

$$J(t) \geq \frac{2}{\kappa} \tan(\alpha x t + c_0),$$

where

$$\kappa = \left(\frac{\lambda(\lambda+2)}{4} L^{-(2+\lambda)}\right)^{\frac{1}{2}}, \quad a = \sqrt{a^2} > 0, \quad c_0 = \arctan \frac{\kappa J_0}{a}, \quad J_0 = \int_{\Omega} v_0(x)|x - L|^\lambda \, dx,$$

and the blow-up time $T_\infty$ is estimated as:

$$T_\infty \leq \frac{\pi}{2\alpha}.$$

The proof follows immediately from (3.5).

Corollary 3.1. Suppose that $v(x,t)$ satisfies,

$$v(0,t) = 0,$$

$$Dv(0,t) + D^3v(0,t) \geq \text{const} > 0 \quad \forall t > 0, \quad \text{and}$$

$$Dv(0,t), \quad D^2v(0,t) \geq 0.$$

Then there is $L_0 > 0$ such that all the assumptions of Theorem 2.1 are fulfilled.

Theorem 3.2. Let $H_\lambda(v, L) \geq 0$ for some $\lambda > 6$ and $L > 0$. Let

$$J_0 := \int_{\Omega} v_0(x)|x - L|\, dx > 0.$$

Then there is no global solution of (2.1), (3.6), (3.7).

Moreover, there is no global solution of (2.1), (3.6), (3.7) for $x \in (0, L)$. In particular $J$ satisfies

$$J(t) \geq \frac{J_0}{1 - \kappa J_0^2 t},$$

so that for the blow-up time $T_\infty$ we have

$$T_\infty < \frac{1}{\kappa^2 J_0}.$$

The proof follows from (3.5).

Theorem 3.3. Let $H_\lambda(v, L) \geq -a^2$ for some $\lambda > 6$ and $L > 0$. Let

$$J_0 := \int_{\Omega} v_0(x)|x - L|\, dx > \frac{|a|}{\kappa}.$$

Then there is no global solution of (2.1), (3.6), (3.7).

Moreover, there is no global solution of (2.1), (3.6), (3.7) for $x \in (0, L)$. In particular

$$J(t) \geq \frac{1 + c_0 e^{2\alpha x^2}}{1 - c_0 e^{2\alpha x^2} \kappa}, \quad \text{with} \quad c_0 = \frac{\kappa J_0 - a}{\kappa J_0 + a}.$$
Hence, the blow-up time $T_\infty$ satisfies:

$$T_\infty < \frac{1}{2\alpha} \ln \frac{\kappa \beta a}{\kappa j_0 - a}, \quad \text{where} \quad \kappa = \left( \frac{\lambda (\lambda + 2)}{4} L^{-(2+\lambda)} \right)^{\frac{1}{2}} \quad \text{and} \quad a = |a|.$$ 

Again, the proof follows directly from (3.5).

**Corollary 3.2.** Let $v_0(x) \geq c|x|^\mu$ with some $\mu > 0$ and $c > 0$. Suppose that the boundary values $v, Dv, D^2v,$ and $D^3v$ at $x = 0$ do not depend on $t > 0$. Then there exists $\lambda > 0$ such that the assumptions of Theorem 3.3 are fulfilled.

### 4. Global existence and $L^\infty$-bounds by weighted Gronwall’s inequalities

#### 4.1. A general KS-type model.

In this section, we extend global existence approaches to more general $2m$th-order parabolic equations of the KS-type. As it has been seen, the zero boundary conditions of the IBVPs under consideration reinforced the energy control and helped the global solvability of the KSE. It is then reasonable to consider the KS-type equations in unbounded domains suggesting “infinite propagation” with no spatial “obstacles” and bounds, which the Cauchy problem is most natural for.

Thus, as a basic and typical model, we consider the Cauchy problem (CP) for the equation, which includes both stable and unstable linear diffusion terms as well as the “convection”:

$$v_t = -(-\Delta)^l v + (-\Delta)^l v + B_1(|v|^p) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \quad (m = 2l = 2, 4, 6, ...),$$

where $p > 1$ and $B_1$ is the linear first-order differential operators in divergence form

$$B_1 u = \frac{1}{p} \sum_k d_k D_{x_k} u.$$ 

For simplicity, the coefficients $\{d_k\}$ of the operator are assumed to be constant, though some $(x, t)$-dependence can be easily allowed. The classic KSE (2.1) corresponds to

$$p = 2, \quad l = 1, \quad \text{and} \quad N = 1 \quad (d_1 = 1).$$

We refer to (4.1) as to the modified Kuramoto–Sivashinsky equation (the mKSE).

Thus, we consider for (4.1) the CP with bounded sufficiently smooth data

$$v(x, 0) = v_0(x) \quad \text{in} \quad \mathbb{R}^N, \quad v_0 \in L^\infty(\mathbb{R}^N) \cap H^{2m}(\mathbb{R}^N).$$

For instance, for the main demonstrations of the techniques involved, we may always assume that $v_0(x)$ is smooth and has exponential decay at infinity.

One of the key mathematical features of the mKSE (4.1) is that it admits multiplication by $v$ in $L^2$ to get the a priori $L^2$-bound. Namely, similar to related straightforward manipulations in Section 2, we have by the Hölder inequality:

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_2^2 = -\|\Delta^l v(t)\|_2^2 + \int v(-\Delta)^l v \, dx \leq -\|\Delta^l v(t)\|_2^2 + \|v(t)\|_2 \|\Delta^l v(t)\|_2$$

$$\leq \frac{1}{4} \|v(t)\|_2^2 \quad \Rightarrow \quad \|v(t)\|_2^2 \leq \|v_0\|_2^2 e^{\frac{t}{12}} \quad \text{for all} \quad t > 0.$$
Another obvious higher-order model with a similar $L^2$-control (4.5) is
\begin{equation}
(4.6) \quad v_t = -(-\Delta)^m v + \frac{1}{4} v + B_1(|v|^p) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \quad (m = 2, 3, 4, \ldots),
\end{equation}
with the same convection-nonlinear dispersion and the linear unstable zero-order term $+\frac{1}{4} v$. On the other hand, $\frac{1}{4} v$ can be replaced by $-\Delta v$ as in (1.1). However, for some $2m$th-order KS-type models, deriving $L^2$-bounds can represent a hard problem, so we avoid using those in future application of the scaling and other techniques.

Thus, as a principal issue of our analysis, the intention is to use this \textit{a priori} estimate (4.5) to derive stronger uniform bounds on solutions $v(x,t)$ in the sup$_x$-norm for all $t \geq 0$.

The developed scaling technique is rather general and applies to other higher-order KS-type models with \textit{a priori} known $L^2$, $L^p$, Sobolev, or other types of weaker or stronger integral bounds on solutions.

4.2. Fundamental solution. We will need the fundamental solution of the corresponding linear parabolic (poly-harmonic) equation
\begin{equation}
(4.7) \quad u_t = -(-\Delta)^m u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \quad (m = 2l),
\end{equation}
which has the self-similar form
\begin{equation}
(4.8) \quad b(x,t) = t^{-\frac{N}{2m}} F(y), \quad y = x/t^{\frac{1}{2m}}.
\end{equation}
The rescaled kernel $F$ is the unique radial solution of the elliptic equation
\begin{equation}
(4.9) \quad BF \equiv (-\Delta)^m F + \frac{1}{2m} y \cdot \nabla F + \frac{N}{2m} F = 0 \quad \text{in} \quad \mathbb{R}^N, \quad \int F = 1.
\end{equation}
$F$ is oscillatory for $m > 1$, has exponential decay as $|y| \to \infty$, and satisfies, for some positive constants $D$ and $d$ depending on $m$ and $N$ [18],
\begin{equation}
(4.10) \quad |F(y)| < D e^{-d|y|^\alpha} \quad \text{in} \quad \mathbb{R}^N, \quad \text{where} \quad \alpha = \frac{2m}{2m-1} \in (1, 2).
\end{equation}

4.3. Local existence and uniqueness of smooth solutions. As a standard practice (see [18] [24] and [73, Ch. 15]), for sufficiently regular initial data, \textit{local} in time existence and uniqueness of the classical solution of the CP (4.1), (4.4) is studied (via Duhamel’s principle) by using the integral equation, where we have integrated by parts once,
\begin{equation}
(4.11) \quad v(t) = b(t) * v_0 + \int_0^t (-\Delta)^l b(t-s) * v(s) \, ds + \int_0^t B_1^* b(t-s) * |v|^p(s) \, ds,
\end{equation}
where $b(t)$ is the fundamental solution (4.8) and $B_1^*$ is the adjoint operator,
\begin{equation}
(4.12) \quad B_1^*(\cdot) = -\frac{1}{p} \sum_{(k)} D_{x_k} (d_k(\cdot)).
\end{equation}
For more singular data $v_0 \in L^q$, with $q \in (1, \infty)$, the solution may not be classical and is then understood as a proper continuous curve $u : [0, T] \to L^q$ satisfying equation (4.11). Such questions of local existence and uniqueness were first systematically studied by Weissler in the 1970s and 80s, [78, 79]. See the results in [79] pp. 87-90, which actually apply to $2m$-th order equations like (4.1). More recent results on local and global existence for higher-order parabolic equations such as (4.11) can be found in [2, 15, 17, 31].
In what follows, we are interested in global existence and behaviour of solutions, so we have assumed that data \( v_0 \) are sufficiently regular, and then we will use the equivalent integral equation (4.11) for such purposes.

4.4. Global existence by Henry’s version of weighted Gronwall’s inequality.

**Theorem 4.1.** The Cauchy problem (4.1), (4.4) has a global unique classical solution if

\[
1 < p \leq 3 \quad \text{and} \quad p < p_0 = 1 + \frac{2(2m-1)}{N}.
\]

For the original KSE (2.1) with parameters (4.3), (4.13) is valid in dimension

\[
N < 6.
\]

**Proof.** Let us write (4.11) in greater detail,

\[
v(x,t) = t^{-\frac{N}{2m}} \int_{\mathbb{R}^N} F(\cdot)v_0(y) \, dy + \int_0^t (t-s)^{-\frac{N+2l}{2m}} ds \int_{\mathbb{R}^N} (-\Delta)^l F(\cdot)v(y,s) \, dy
\]

\[
+ \int_0^t (t-s)^{-\frac{N+1}{2m}} ds \int_{\mathbb{R}^N} B_1^* F\left(\frac{x-y}{(t-s)^{1/2m}}\right) |v|^p(y,s) \, dy, \quad (\cdot) = \left(\frac{x-y}{(t-s)^{1/2m}}\right),
\]

where we mean in \((-\Delta)^l F(z)\) and \(B_1^* F(z)\) that the operators act in the \(z\)-variable. Denote

\[
V(t) = \sup_{x \in \mathbb{R}^N} |v(x,t)|.
\]

Writing in (4.15) \( |v|^p = |v| \cdot |v|^{p-1} \) for \( p \leq 3 \) \((p - 1 \leq 2)\), using Hölder’s inequality yields

\[
|v(t)| \leq \sup |v_0| \int |b(t)| + \int_0^t |\Delta^l b(t-s) * v(s)| \, ds
\]

\[
+ \int_0^t |B_1^* b(t-s) * (|v|^{p-1}|v|)(s)| \, ds \leq \sup |v_0| \|F\|_1
\]

\[
+ C\|\Delta^l F\|_1 \int_0^t V(s)(t-s)^{-\frac{p-1}{m}} \, ds
\]

\[
+ C\|B_1^* F\|_2 \int_0^t V(s)\|v(s)\|_2^{p-1}(t-s)^{\beta-1} \, ds,
\]

where \( \beta = \frac{4m - 2 - N(p-1)}{4m} \).
Thus, by \((4.5)\), we obtain weighted Gronwall’s inequality (here \(\frac{1}{m} = \frac{1}{2}\))

\[
V(t) \leq C + C \int_0^t (t-s)^{-\frac{1}{2}} V(s) \, ds + C \int_0^t e^{\frac{(p-1)s}{4}} (t-s)^{\beta-1} V(s) \, ds,
\]

where, obviously, the last term on the right-hand side is key for boundedness of \(V(s)\). By Henry’s estimates for such weighted inequalities \([39, \text{p. 188}]\), it follows that \(V(s)\) is bounded for any \(t > 0\), if \(\beta > 0\), which is equivalent to the last inequality in \((4.13)\). □

4.5. **On a double exponential \(L^\infty\)-growth by weighted Gronwall’s inequality.**

As we have mentioned, for the KSE, it is principal also to establish the best estimate on the growth of global solutions for \(t \gg 1\), to be compared with the exponential one \((1.2)\).

Let us see what kind of \(L^\infty\)-bound is guaranteed by the approach applied above. Consider the principal integral operator in \((4.19)\), where we skip all the constants \(C\),

\[
V(t) \leq 1 + \int_0^t e^{\frac{(p-1)s}{4}} (t-s)^{\beta-1} V(s) \, ds \quad (\beta > 0).
\]

For \(p = 1\), Henry’s “discrete” proof in \([39, \text{p. 188}]\) also gives the fact that solutions of such Gronwall’s inequalities do not grow as \(t \to \infty\) not faster than exponentially, which is fine according to \((1.2)\).

With the presence of the multiplier \(e^{\frac{(p-1)s}{4}}\) in the kernel in \((4.20)\), this is not the case. Using the idea of Henry’s proof, a certain estimate of the behaviour of \(V(t)\) for \(t \gg 1\) can be indeed obtained. Nevertheless, rather surprisingly, this will not be an exponential growth, which may naturally look most plausible. Namely, it is easy to see that an upper bound on the growth of solutions of \((4.20)\) is *doubly exponential* in the sense that their solutions cannot grow as \(t \to +\infty\) faster than the “supersolution” (corresponding to “=” in \((4.20)\))

\[
\hat{V}(t) = \exp\left\{\frac{4}{4(p-1)} + \varepsilon\right\} t^\beta e^{\frac{(p-1)t}{4}},
\]

with arbitrarily small \(\varepsilon > 0\).

In other words, the approach based on weighted Gronwall’s inequalities, though proving global existence, supplies us with a rather non-realistic (in comparison with \((1.2)\) and \((4.5)\)) doubly exponential \(L^\infty\)-bound \((4.21)\) on solutions \(v(x,t)\). We will then need in Section 5 to improve this bound via a different scaling approach.

4.6. **Application to a non-divergent equation.** A similar technique being applied to the non-divergent diffusion-absorption equation

\[
v_t = -(-\Delta)^m v - |v|^{p-1} v \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+,
\]

yields global boundedness of solutions in the subcritical Sobolev range \([33, \text{§ 2}]\),

\[
1 < p < p_S = \frac{N+2m}{N-2m}.
\]

Further refined applications of the scaling technique for \((4.22)\) are given in \([12]\) (note that Theorem 4.1 proved for \(p > p_S\) therein in § 4 applies to sufficiently small solutions only).
5. Global existence and exponential $L^\infty$-bounds by scaling techniques

5.1. Global existence in subcritical range: $C_k$-scaling. Cf. Theorem \[4.1\]

**Theorem 5.1.** The Cauchy problem \((4.1)\), \((4.4)\) has global unique classical solution if
\[
1 < p < p_0 = 1 + \frac{2(2m-1)}{N} \quad (m = 2l).
\]

Thus, for the standard KSE \((2.1)\) with parameters \((4.3)\), in comparison with \((5.1)\), we manage to skip the first assumption \(p \leq 3\).

**Proof.** Assume that the local classical solution \(v(x, t)\) of the CP blows up first time at some finite \(t = T\). Then \(v(x, T^-)\) is unbounded in \(L^\infty(\mathbb{R}^N)\), otherwise it can be extended as a bounded solution on some interval \([T, T + \varepsilon]\), with a sufficiently small \(\varepsilon > 0\), by using the integral equation \((4.15)\) with a contractive operator in \(C(\mathbb{R}^N \times [0, \delta])\), \(\delta > 0\) small, equipped with the standard sup-norm.

Thus, we argue by contradiction and assume that there exist sequences \(\{t_k\} \to T^-\), \(\{x_k\} \subset \mathbb{R}^N\), and \(\{C_k\}\) such that
\[
\sup_{\mathbb{R}^N \times [0, t_k]} |v(x, t_k)| = |v(x_k, t_k)| = C_k \to +\infty.
\]

Using a modification of the rescaling technique in \[33\], we perform the change
\[
v_k(x, t) \equiv v(x_k + x, t_k + t) = C_k w_k(y, s), \quad \text{where} \quad x = a_k y, \quad t = a_k^2 s,
\]
where \(\{a_k\}\) is such that the \(L^2\)-norm is preserved after rescaling, i.e.,
\[
\|v_k(0)\|_2 = \|w_k(0)\|_2 \implies a_k = C_k^{-\frac{2}{p}} \to 0.
\]

Therefore, by \((4.15)\), for all \(s\) for which \(w_k(s)\) is defined,
\[
\|w_k(s)\|_2 = \frac{1}{a_k^{2m} C_k} \int v_k^2(x, t) \, dx \leq \|v_0\|_2^2 e^{\frac{s}{2}} \quad \text{for all} \quad s \in \left[-\frac{t_k}{a_k^{2m}}, \frac{T-t_k}{a_k^{2m}}\right).
\]

As usual, such a rescaling near blow-up time, in the limit \(k \to \infty\) leads to the so-called ancient solutions (i.e., defined for all \(s < 0\)) in Hamilton’s notation \[38\], which has been a typical technique of reaction-diffusion theory; see various form of its application in \[67\] \[32\].

Let, according to \((5.1)\), \(p < p_0\). Then, substituting \((5.3)\) into equation \((4.1)\) yields that \(w_k\) satisfies (as usual, \(m = 2l\))
\[
(w_k)_s = -(-\Delta)^2 w_k + \mu_k (-\Delta)^l w_k + \nu_k B_1 |w_k|^p \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}, \quad \text{where}
\]
\[
\mu_k = C_k^{\frac{2m}{2m-1}} \to 0, \quad \nu_k = a_k^{2m-1} C_k^{p-1} = C_k^{p-1 - \frac{2(2m-1)}{N}} \to 0, \quad k \to \infty.
\]

We next perform backward shifting in time by fixing \(s_0 > 0\) large enough (this is possible in the time-interval in \((5.5)\) since \(a_k \to 0\) in \((5.1)\)), and setting \(\bar{w}_k(s) = w_k(s - s_0)\). Then by construction, we have that
\[
|\bar{w}_k(s)| \leq 1 \quad \text{and} \quad \|\bar{w}_k\|_2 \leq C \quad \text{on} \quad (0, s_0)
\]
are uniformly bounded classical solutions of the uniformly parabolic equation \((4.7)\). By classic parabolic regularity theory \[18\] \[24\], we have that the sequence \(\{\bar{w}_k\}\) is uniformly
bounded and equicontinuous on any compact subset from $\mathbb{R}^N \times (0,s_0)$. Indeed, the necessary uniform gradient bound can be obtained from the integral equation for $(5.6)$, or by other usual regularity methods for uniformly parabolic equations. Note that compactness in any suitable weaker topologies (see e.g., [78, 79]) is also acceptable, since passing to the limit we arrive at a weak, and hence, classical solution of the limit (simpler) parabolic equation.

Therefore, by the Ascoli-Arzelà theorem, along a certain subsequence, $\bar{w}_k(s) \rightarrow \bar{w}(s)$ uniformly on compact subsets from $\mathbb{R}^N \times (0,s_0)$. Passing to the limit in equation $(5.6)$ and using that both scaling parameters satisfy $\mu_k, \nu_k \rightarrow 0$, yields that $\bar{w}(s)$ is a bounded weak solution and, hence, a classical solution of the Cauchy problem for the linear parabolic equation $(4.7)$,

\[
\bar{w}_s = -(-\Delta)^m \bar{w}, \quad \text{with data } |\bar{w}_0| \leq 1 \quad \text{and} \quad \|\bar{w}_0\|_2 \leq C.
\]

Using the Hölder inequality in the convolution (see the first integral in $(4.15)$) yields

\[
\bar{w}(s_0) = b(s_0) * \bar{w}_0 \quad \Rightarrow \quad |\bar{w}(y,s_0)| \leq (s_0)^{-\frac{N}{2m}} \|F\|_2 \|\bar{w}_0\|_2 \ll 1,
\]

for all $s_0 \gg 1$. Hence, the same holds for $\sup_y |\bar{w}_k(y,s_0)|$ for $k \gg 1$, from whence comes the contradiction with the assumption $\sup_y |v_k(y,s_0)| = 1$.

Thus, $v(x,t)$ does not blow-up and remains bounded for all $t > 0$. $\Box$

The proof is entirely local, so the result holds for the Cauchy problem as well as for any other homogeneous basic IBVPs for $(4.1)$, where the boundary conditions cannot generate blow-up on the boundary themselves.

5.2. Exponential $L^\infty$-bound. We next improve the double exponential $L^\infty$-bound in $(4.21)$:

**Theorem 5.2.** The global solution of the Cauchy problem $(4.7)$, $(4.4)$ in the subcritical parameter range $(5.1)$, $p < p_0$, satisfies

\[
\sup_{x \in \mathbb{R}^N} |v(x,t)| \leq C_0 e^{\gamma_0 t} \quad \text{for all } t > 0, \quad \text{where} \quad \gamma_0 = \frac{2m-1}{2N(p_0 - p)} \left( > \frac{1}{4} \right).
\]

**Proof.** We assume that $(5.2)$ holds for a monotone sequence $\{t_k\} \rightarrow +\infty$. By $(4.5)$, i.e.,

\[
\|v(t_k)\|_2 \leq \|v_0\|_2 e^{\frac{t_k}{4}},
\]

we perform scaling $(5.3)$ by taking into account the exponential factor in $(5.12)$ targeting a uniformly bounded rescaled solution in the sense of $(5.8)$. This yields (cf. $(5.4)$)

\[
a_k = C_k^{\frac{1}{2N}} e^{\frac{t_k}{8}},
\]

and eventually we arrive at the rescaled equation $(5.6)$, where

\[
\mu_k = a_k^{\frac{2}{m}} = C_k^{\frac{1}{2N}} e^{\frac{mt_k}{8N}} \quad \text{and} \quad \nu_k = C_k^{\frac{p_0}{p}} e^{\frac{(2m-1)t_k}{2N}}.
\]

Assume now that $(5.11)$ does not valid so that, along the sequence $\{t_k\}$,

\[
C_k = \kappa_k e^{\gamma_0 t_k}, \quad \text{where} \quad \kappa_k \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty.
\]
One can see that then the rescaled equation (5.6) contains the parameters (recall, $\gamma_0 > \frac{1}{4}$)

\begin{equation}
(5.16) \quad a_k = \kappa_k^{-\frac{2}{d}} e^{-\frac{4m-1}{2m} t_k} \to 0, \quad \mu_k = a_k^m \to 0, \quad \text{and} \quad \nu_k = \kappa_k^{p_0} \to 0.
\end{equation}

Hence, repeating the arguments of the proof of Theorem 5.1, we obtain the limit problem (5.9), which does not support the assumed growing behaviour. \ensuremath{\square}

5.3. Global existence for the critical exponent $p = p_0$: $(T-t)$-scaling.

**Theorem 5.3.** The Cauchy problem \((4.1), (4.4)\) has also global unique classical solution in the critical case

\begin{equation}
(5.17) \quad p = p_0 = 1 + \frac{2(2m-1)}{N} \quad (m = 2l).
\end{equation}

**Remark: on application of the $C_k$-scaling.** For $p = p_0$ in (5.6), $\mu_k \to 0$, but $\nu_k \equiv 1$, so that passing to the limit $k \to \infty$, for the limit function $w_k(s) \to w(s)$, we obtain the KSE in $\mathbb{R}^N \times (-\infty, 0)$ without the unstable diffusion-like term:

\begin{equation}
(5.18) \quad w_s = -(-\Delta)^m w + B_1 |w|^p, \quad |w(s)| \leq 1, \quad \|w(s)\|_2 \leq C, \quad \sup_y |w(y, 0)| = 1.
\end{equation}

Thus, we need to prove that such a solution defined for all $s \leq 0$ is nonexistent.

On the one hand, this looks rather reasonable, since in the class of uniformly bounded $L^2 \cap H^{2m}$-solutions, the PDE (5.18) in $\mathbb{R}^N \times \mathbb{R}_+$ is a smooth gradient dynamical system admitting positive definite Lyapunov function as in (4.5),

\begin{equation}
(5.19) \quad \frac{1}{2} \frac{d}{ds} \|w(s)\|_2^2 = -\int w(s)(-\Delta)^m w(s) \equiv -\|\tilde{D}^m w(s)\|_2^2 \leq 0.
\end{equation}

Hence, the only equilibrium is 0 that is globally asymptotically stable in $L^2(\mathbb{R}^N)$, so that, in view of the interior parabolic regularity for bounded solutions, for any suitable initial data $w_0 \in L^\infty \cap L^2$,

\begin{equation}
(5.20) \quad w(y, s) \to 0 \quad \text{as} \quad s \to +\infty \quad \text{uniformly}.
\end{equation}

On the other hand, this is not enough to complete the proof, since we need the convergence (5.20) to be uniform with respect to data satisfying conditions (5.8). This will prove the actual nonexistence of a solution to (5.18), but is not that straightforward. Therefore, for convenience, we choose another, but indeed related, scaling technique to prove non-blow-up for the critical exponent, which also emphasize other important aspects of this evolution PDE.

**Proof of Theorem 5.3.** Thus, assuming first $L^\infty$-blow-up at $t = T^-$, we perform the time-dependent $(T-t)$-scaling of the orbit $\{v(\cdot, t), t \in (0, T)\}$ of (1.1):

\begin{equation}
(5.21) \quad v(x, t) = (T-t)^{-\alpha} w(y, \tau), \quad \alpha = \frac{2m-1}{2(m+1)}, \quad y = \frac{x}{(T-t)^{1/2m}}, \quad \tau = -\ln(T-t).
\end{equation}

Without loss of generality, we assume that $x = 0$ is a blow-up point. Then $w$ solves the following exponentially perturbed equation:

\begin{equation}
(5.22) \quad w_\tau = A(w) + e^{-\tilde{x}}(-\Delta)^lw, \quad w_0 \in H^{2m} \cap L^\infty, \quad A(w) = -(-\Delta)^m w - \frac{1}{18} \tau y \cdot \nabla w - \alpha w + B_1 |w|^p.
\end{equation}
Note that the $L^2$-invariance of the scaling \((5.16)\):
\[(5.23) \quad \|v(\cdot, t)\|_2 \equiv \|w(\cdot, \tau)\|_2.\]

We next need some auxiliary results.

**Proposition 5.1.** Assume that, along a subsequence \(\{\tau_k\} \to +\infty\),
\[(5.24) \quad w(y, \tau_k) \to 0 \text{ uniformly.}\]
Then \(t = T\) is not a blow-up time for \(v(x, t)\) (so the singularity is removable).

**Proof.** Consider the sequence of solutions \(\{w_k(y, s) = w(y, \tau_k + s)\}\) with vanishing initial data in \(L^\infty\) according to \((5.24)\). We now use the good and well-developed spectral properties [17] of the linearized operator \(B^*\) in \((5.22)\) defined in the weighted space \(L^2_{\rho^*}(\mathbb{R}^N)\), where \(\rho^*(y) = e^{-\alpha|y|^\beta}, \quad \beta = \frac{2m^2}{2m - 1}, \quad \alpha > 0\) is small enough:
\[(5.25) \quad B^*_\alpha = -(-\Delta)^m - \frac{1}{2m} y \cdot \nabla - \alpha I, \quad \sigma(B^*_\alpha) = \{\lambda_l = -\alpha - \frac{l}{2m}, \quad l = 0, 1, 2, \ldots\},\]
with eigenfunctions (generalized Hermite polynomials) that compose a complete and closed set. Therefore, according to classic asymptotic parabolic theory (see e.g., [57]), we conclude that for any sufficiently large \(k\),
\[(5.26) \quad w_k(y, s) \sim O(e^{-\alpha s}), \quad s \gg 1 \implies \frac{w(y, \tau)}{e^{-\alpha \tau}}, \quad \tau \gg 1.\]
Overall, taking into account scaling \((5.21)\), this yields:
\[(5.27) \quad v(x, t) \sim (T - t)^{-\alpha}O(e^{-\alpha \tau}) = O(1) \quad \text{as} \quad t \to T^-\]
so that \(v(x, t)\) is uniformly bounded at \(t = T\). \qed

**Proposition 5.2.** There must exist a subsequence \(\{\tau_k\} \to +\infty\) such that
\[(5.28) \quad C_k \equiv \|w(\cdot, \tau_k)\|_{L^\infty} \to \infty \quad \text{as} \quad k \to +\infty.\]

**Proof.** Assume for contradiction that
\[(5.29) \quad w(y, \tau) \quad \text{is uniformly bounded in} \quad \mathbb{R}^N \times \mathbb{R}_+.\]
Then, similar to \((5.19)\), we get for uniformly bounded and smooth solution \(w(y, \tau)\) that
\[(5.30) \quad \frac{1}{2} \frac{d}{d\tau} \|w(\tau\|_2^2 = -\|D^m w(\tau)\|_2^2 + e^{-\frac{\alpha}{2}} \|D^l w(\tau)\|_2^2.\]
Therefore, in order to avoid the \(L^2\)-vanishing:
\[(5.31) \quad \|w(\tau)\|_2^2 \to 0 \quad \text{as} \quad \tau \to +\infty,\]
which by the interior parabolic regularity would imply \((5.24)\) as \(\tau \to +\infty\) uniformly and hence no blow-up, one has to have that
\[(5.32) \quad \int^\infty \|D^m w(\tau)\|_2^2 d\tau < \infty, \quad \text{i.e.,} \quad \|D^m w(\tau)\|_2^2 \in L^1((1, \infty)).\]
Obviously, the convergence of the integral in \((5.32)\) implies that
\[(5.33) \quad D^m w(\cdot, \tau + s) \to 0 \quad \text{as} \quad \tau \to \infty \quad \text{weakly in} \quad L^2_{loc}(\mathbb{R}_+; L^2).\]
Indeed, one can see that, by the Hölder inequality, for any \( \chi \in C_0^{\infty}(\mathbb{R}^N \times \mathbb{R}_+) \),
\[
| - \int (-\Delta)^m w(\tau + s) \chi(s) | \equiv \left| \int (\tilde{D}^m)^2 w(\tau + s) \chi(s) \right| \leq \left( \int (\tilde{D}^m w(\tau + s))^2 \right)^{\frac{1}{2}} \left( \int (\tilde{D}^m \chi(s))^2 \right)^{\frac{1}{2}} \to 0
\]
as \( \tau \to \infty \). Fixing a sequence \( \{\tau_k\} \to \infty \) and passing to the limit as \( \tau_k + s \to \infty \), we conclude that, in view of the remaining interior parabolic regularity (recall that, regardless the “artificial degeneracy” \( (5.33) \)), equation \( (5.22) \) is uniformly parabolic, some equicontinuous subsequence \( \{w(\tau_k + s)\} \to \hat{w}(s) \) uniformly, where \( \hat{w}(y, s) \) is a smooth solution of the first-order Hamilton–Jacobi equation
\[
\hat{w}_s = -\frac{1}{2m} y \cdot \nabla \hat{w} - \alpha \hat{w} + B_1|\hat{w}|^p, \quad \hat{w}_0 \in L^2 \cap L^\infty.
\]
However, by classic theory of conservation laws [73] (and, actually, by the standard comparison), all such solutions of \( (5.35) \) are exponentially decaying:
\[
\hat{w}(y, s) = O(e^{-\alpha s}) \quad \text{as} \quad s \to \infty.
\]
This implies that there exists always a moment \( \tau_0 = \tau_k + s_0 \), with large enough \( k, \tau_k \), and \( s_0 \) such that \( w(y, \tau_0) \) is arbitrarily uniformly. As in the proof of Proposition 5.1 overall, this implies no blow-up for \( v(x, t) \) at \( t = T \). □

Finally, the result on global existence is completed by the following:

**Proposition 5.3.** The Type I\(^2\) blow-up solutions satisfying \( (5.28) \) do not exist.

**Proof.** We now apply in \( (5.22) \) the \( C_k \)-scaling with \( \{C_k\} \) given in \( (5.28) \), i.e., as in \( (5.3) \). For \( p = p_0 \), this gives the following equation this gives the following equation for the rescaled sequence:
\[
w_k(y, \tau) \equiv w(y_k + a_k z, \tau_k + a_k^2 m s) = C_k \hat{w}_k(z, s), \quad \text{where} \quad \hat{w}_k(z, s) \text{ solve}
\]
\[
\hat{w}_s = -(-\Delta)^m \hat{w} + B_1|\hat{w}|^p - a_k^{2m} \left( \frac{1}{2m} z \cdot \nabla \hat{w} + \alpha \hat{w} \right) + a_k^m e^{-\frac{r_k + a_k^2 m s}{2} (-\Delta)^l \hat{w}}.
\]
Here, at \( s = 0 \), \( \hat{w}_0 \in L^2 \cap L^\infty \) and \( a_k = C_k^{-2/N} \). This is a uniformly parabolic equation with asymptotically small perturbations, so that on passage to the limit \( k \to \infty \) for this set \( \{\hat{w}_k\} \) of smooth solutions, one has to have in the limit that, along a subsequence,
\[
\begin{align*}
\hat{w}_k &= w(y_k + a_k z, \tau_k + a_k^2 m s) \to W(z, s), \quad \text{where} \quad W(z, s) \equiv W(z, s) \\
W_s &= -(-\Delta)^m W + B_1|W|^p \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}, \quad W \in L^2 \cap L^\infty, \quad \|W(0, 0)\|_\infty = 1.
\end{align*}
\]
Note that \( W(z, s) \neq 0 \) is an ancient solution, which is defined for all \( s \leq 0 \). At the same time, by construction, it is also a future solution, which must defined for all \( s > 0 \). Indeed, one can see that if \( W(s) \) blows up at some finite \( s = S^- > 0 \), this would contradict the

\(^2\) The terms “Type I, II” were borrowed from Hamilton [38], where Type II is also called slow blow-up. In reaction-diffusion theory [67] [32], Type I blow-up is usually called of self-similar rate, while Type II is referred to as fast and non-self-similar.
Type II solution $w(y, \tau)$ is globally defined for all $\tau > 0$. Thus, by scaling of the Type II blow-up orbit (5.28), we arrive at the problem (5.39), which defines:

$$\{ \text{heteroclinic solution } W(z, s) \neq 0 \} = \{ \text{ancient for } s < 0 \} \cup \{ \text{future for } s > 0 \}. \quad (5.40)$$

Let us more clearly specify the necessary properties of this mysterious and hypothetical heteroclinic solution $W$. Bearing in mind the possibility of multiplying (5.39) by $W$ and integrating by parts, which are again guaranteed by the convergence demand as in (5.32), we have the standard identity:

$$\frac{1}{2} \frac{d}{ds} \| W(s) \|_2^2 = - \| \bar{D}^m W(s) \|_2^2 \leq 0, \quad s \in \mathbb{R} \implies \int_{-\infty}^{+\infty} \| \bar{D}^m W(s) \|_2^2 ds < \infty. \quad (5.41)$$

Hence, in the given class $L^2 \cap L^\infty$ classical solutions, (5.39) is a sufficiently smooth gradient system, and the only equilibrium that can be approached by such bounded orbits is $W = 0$. By passing to the limit $s \to \pm \infty$, we then obtain that uniformly

$$W(s) \to 0 \quad \text{as} \quad s \to \pm \infty. \quad (5.42)$$

This actually means that

$$W(z, s) \neq 0 \text{ is a homoclinic of 0 orbit.} \quad (5.43)$$

We next easily prove that

**Proposition 5.4.** A nontrivial solution $W \neq 0$ of (5.39) does not exist.

**Proof.** It suffices to use the “ancient” part of the definition (5.40). In view of the integral convergence in (5.41) at $s = -\infty$, similar to (5.24) we have that along any monotone sequence $s_k \to -\infty$, we have that, for $k \gg 1$, $W(s_k + s) \approx V(s)$ uniformly on compact subsets in $\mathbb{R}^N \times \mathbb{R}$, where $V$ solves the Hamilton-Jacobi equation (a conservation law)

$$V_s = B_1|V|^p \quad \text{for } s > 0, \quad V_0 = W(s_k) \in L^2 \cap L^\infty. \quad (5.44)$$

Since in the class of smooth solutions, $V(s)$ decays fast (see e.g., [70, 73]), we have that

$$\| V(0) \|_\infty \gg \| V(s) \|_\infty \quad \text{for } s \gg 1, \quad (5.45)$$

so that the same is true for $W(s_k + s)$ provided that $k \gg 1$. In fact, (5.45) means that using proper theory of conservation laws (5.44) implies that a nontrivial $L^2 \cap L^\infty$ ancient solution does not exists for such gradient system, since by the convergence in (5.41), equation (5.39) does not have a sufficient mechanism for growth of solutions from $\| W(-\infty) \|_\infty = 0$ to $\| W(0) \|_\infty = 1$. To justify this more clearly, given an ancient solution $W(z, s)$ of (5.39), we perform the standard scaling:

$$W_\lambda(z, s) = \lambda^{-\alpha} W(\frac{z}{\lambda^{1/2m}}, \frac{s}{\lambda}), \quad \lambda > 0, \quad (5.46)$$

where $W_\lambda$ is also an ancient solution of (5.39) for any $\lambda > 0$. We now pass to the limit $\lambda \to 0^+$ by using these simple properties:

$$\| W_\lambda(0) \|_\infty = \lambda^{-\alpha} \to +\infty, \quad \| W_\lambda(s) \|_2^2 = \| W(\frac{s}{\lambda}) \|_2^2 \to c_0 > 0,$$

$$\text{and} \quad \frac{1}{2} \frac{d}{ds} \| W_\lambda(s) \|_2^2 = -\lambda \| \bar{D}^m W(\frac{s}{\lambda}) \|_2 \to 0. \quad (5.47)$$
Therefore, in the limit \( \lambda \to 0 \), we have to have a nontrivial \( L^2 \)-solution \( V(z,s) \) of (5.44), which blows up as \( s \to 0^- \). Obviously, this contradicts the Maximum Principle for this first-order conservation law, [73, Ch. 16].

In other words, we have that, for the dynamical system in (5.39),
\[
\text{the unstable manifold of the origin 0 is empty.} \quad \square
\]

This also completes the proof of Theorem 5.3. \( \square \)

5.4. On uniform bounds in other KS-type models. For the non-divergent equation (4.22), a similar scaling techniques yields that [33], in the Sobolev range (4.23), solutions are uniformly bounded, i.e.,
\[
|v(x,t)| \leq C \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+.
\]

For the divergent mKS-type equations without the unstable backward diffusion term,
\[
v_t = -(-\Delta)^m v + B_1(|v|^p) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+ \quad (m \geq 2),
\]
we easily extend the result as follows:

**Theorem 5.4.** The Cauchy problem (5.50) (4.4) in the parameter range (5.1) has a global unique classical solution, which is uniformly bounded, i.e., (5.49) holds.

**Proof.** Again, it suffices to consider the case when (5.2) holds for some sequence \( \{t_k\} \to +\infty \). Then the same proof leads to the contradiction with the hypothesis that \( \{C_k\} \to +\infty \), and hence \( v(x,t) \) is global and uniformly bounded. \( \square \)

5.5. On a generalization: higher-order nonlinear dispersion. In order to extend application of our final approach, we next briefly discuss mKS-type equations with a third-order (also odd) nonlinear perturbation (dispersion) of the form
\[
v_t = -(-\Delta)^m v - \Delta B_1(|v|^p) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+ \quad (m \geq 2).
\]

Writing this PDE in a pseudo-parabolic form,
\[
Pv_t = (-\Delta)^{m-1} v + B_1(|v|^p), \quad \text{where} \quad P = (-\Delta)^{-1} > 0,
\]
and multiplying by \( v \) in \( L^2(\mathbb{R}^N) \), we observe that, instead of a uniform \( L^2 \)-bound, we are given an \emph{a priori} \( H^{-1} \)-bound: for uniformly bounded data
\[
v_0 \in L^\infty(\mathbb{R}^N) \cap H^{2m}(\mathbb{R}^N),
\]
the following holds:
\[
\|v(t)\|_{-1} \leq \|v_0\|_{-1} \quad \text{for} \quad t > 0.
\]

Here, for simplicity, we assume that \( v_0(x) \) has also exponential decay at infinity, so \( v(x,t) \) does for \( t > 0 \). In (5.52) and later on, we define \( (-\Delta)^{-1} w = g \) in a standard manner:
\[
\Delta g = -w \quad \text{in} \quad \mathbb{R}^N, \quad g(x) \to 0 \quad \text{as} \quad x \to \infty.
\]

For the solvability of this problem we shall always assumes that
Clearly this property holds for the divergent equation (5.51) with exponentially decaying solutions.

**Theorem 5.5.** A unique solution \( v(\cdot, t) \in H^{-1}(\mathbb{R}^N) \) for \( t > 0 \) of the Cauchy problem (5.51), (4.4), (5.59) is uniformly bounded, i.e., (5.49) holds, in the parameter range

\[
1 < p < p_0 = 1 + \frac{2(2m-3)}{N+2}.
\]

**Proof.** Local existence of classic solutions for (5.51) also follows from the equivalent integral equation such as (4.11), with no second unstable term by replacing

\[
B_1^* \mapsto -B_1^* \Delta.
\]

One can see that we still obtain a locally integrable in \( t > 0 \) kernel that defines the operator being a contraction in \( C(\mathbb{R}^N \times [0, \delta]) \), with sup-norm.

To prove global and uniform boundedness, we use the same scaling as in the proof of Theorem 5.1 where, instead of (5.4), keeping the \( H^{-1} \)-norm of \( w_k \) yields

\[
\|v_k(t_k)\|_{-1} = \|w_k(0)\|_{-1} \implies a_k = C_k^{-\frac{2}{N+2}}.
\]

Then, for \( p < p_0 \), we arrive at the corresponding equation such as (5.6), where

\[
\nu_k = a_k^{2m-3} C_k^{p-1} = C_k^{p-1-\frac{2(2m-3)}{N+2}} \to 0 \quad (\mu_k = 0)
\]

as \( k \to \infty \). By passing to the limit \( k \to \infty \), we again arrive at the purely poly-harmonic flow as in (5.9) that cannot support the necessary properties of the sequence. As usual, this analysis applies twice: (i) \( \{t_k\} \to T^- < \infty \), to prove that no blow-up occurs, and (ii) \( \{t_k\} \to +\infty \), to prove that the solution is uniformly bounded. \( \square \)

The critical case \( p = p_0 \) can be also studied in similar lines as above. Note in addition that 0 is the unique globally asymptotically stable equilibrium of the smooth gradient system (5.51) in the corresponding class of regular solutions that are uniformly bounded in \( L^\infty \cap H^{-1} \).

Further generalizations including odd higher-order nonlinear dispersion operators are straightforward.

**5.6. On blow-up in divergent models.** Phenomena of finite-time blow-up in such semilinear models with divergent operators is most well-known for unstable limit Cahn–Hilliard equation

\[
(5.60) \quad u_t = -\Delta^2 u - \Delta(|u|^{p-1}u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \quad (p > 1).
\]

Blow-up solutions have the standard self-similar form

\[
(5.61) \quad u(x,t) = (T-t)^{-\frac{1}{2(p-1)}} f\left(\frac{y}{\sqrt{T-t}}\right), \quad y = x/(T-t)^{\frac{1}{2}},
\]
where $f$ solves the elliptic equation

$$
- \Delta^2 f - \Delta(|f|^{p-1}f) - \frac{1}{4} y \cdot \nabla f - \frac{1}{2(p-1)} f = 0 \quad \text{in} \quad \mathbb{R}^N.
$$

This equation admits a complicated set of solutions. For instance, for $N = 1$ and $p = 3$, it has a countable set of different profiles $f$ that describe various types blow-up patterns 

Concerning Leray’s scenario of similarity blow-up in (5.60) in both limits $t \to T^\pm$, see [29] and references therein.

We are not aware of reliable traces of standard blow-up for KS-type PDEs such as (4.1) or (5.51) with odd-order nonlinear dispersion terms in the supercritical range $p > p_0$. Therefore, we do not know whether conditions (5.1) and (5.57) of global solvability reflect the actual evolution properties of the PDEs under consideration, or are sometimes purely technical. In other words, one then faces another fundamental problem on construction of blow-up patterns for $L^2$-bounded solutions. Regardless a sufficiently strong progress in understanding of formation of blow-up singularities in various nonlinear PDEs achieved in the last twenty five years (see references and results in the monographs [28, 32, 61, 67, 71]), for higher-order equations, there are still several fundamental open problems in identifying admitted structures of blow-up patterns.

6. On $L^\infty$-bounds for the Navier–Stokes equations in $\mathbb{R}^N$ and well-posed Burnett equations

6.1. A classical fluid model in $\mathbb{R}^N$. Consider the Navier–Stokes equations (1.6) with given bounded $L^p$-data $v_0$. In order to apply our scaling argument, we use the fact that a classical bounded solution $v(x, t)$ can be locally extended by using the integral equation that is similar to (4.15). Existence of such a local semigroup of smooth bounded solutions for the NSEs is well known; see Majda–Bertozzi [58].

Let us present some comments that will be useful for the Burnett equations (1.18) with $m \geq 2$. Taking the fundamental solution (1.8) for $m = 1$ with the rescaled Gaussian

$$
F(y) = (4\pi)^{-\frac{N}{2}} e^{-\frac{|y|^2}{4}},
$$

we consider (1.6) with $h = -\nabla p$ as a system

$$
\begin{align*}
\mathbf{v}(t) &= b(t) \ast \mathbf{v}_0 - \int_0^t b(t-s) \ast [(\mathbf{v} \cdot \nabla)\mathbf{v}](s) \, ds + \int_0^t b(t-s) \ast \mathbf{h}(s) \, ds,
\end{align*}
$$

$$
\begin{align*}
\text{div} \, b(t) \ast \mathbf{v}_0 - \int_0^t \text{div} \, b(t-s) \ast [(\mathbf{v} \cdot \nabla)\mathbf{v}](s) \, ds + \int_0^t \text{div} \, b(t-s) \ast \mathbf{h}(s) \, ds &= 0.
\end{align*}
$$

As usual, the second equation in (6.2) is the one for the pressure corresponding to the solenoidal vector field $\mathbf{v}$. Observe that, due to the exponential decay of the Gaussian (6.1), the first equation contains the operator in $\mathbf{v}$ that contractive in a bounded closed
subset \( M_\delta \) of \( C([0, \delta], C^1(\mathbb{R}^N)) \), where \( \delta > 0 \) sufficiently small, with the sup-norm. Indeed, assuming that in \( M_\delta \),

\[
|v| \leq C \quad \text{and} \quad |Dv| \leq C,
\]

we can use the possibility of differentiating in \( x \) the equation once to control \( Dv \). For fixed vectors \( v_0 \) and \( h(s) \), the contractivity of the principal "convective" operator

\[
N(v) = \int_0^t b(t-s) \ast [(v \cdot \nabla)v](s) \, ds
\]

in \( M_\delta \) for small \( \delta > 0 \) is then straightforward. Note that the standard Picard iteration scheme for (6.2) can be put into the probability framework by using the linear semigroup associated with 3D Brownian motion; see a survey [77] for these and other details related to the Navier–Stokes equations. We stop at this moment discussing the local interior regularity theory for the Navier–Stokes equations and refer to [10, 37, 58] as a guide for detailed developments in this direction.

As a standard classic alternative way for local applications, the pressure is excluded from the NSEs

\[
v_t = H(v) \equiv \Delta v - P(v \cdot \nabla)v, \quad P = u - \nabla \Delta^{-1}(\nabla \cdot u)
\]
is the Leray–Hopf projector onto the solenoidal vector field. Using the fundamental solution of \( \Delta \) in \( \mathbb{R}^N \), \( N \geq 3 \) (\( \sigma_N \) is the surface area of the unit ball \( B_1 \subset \mathbb{R}^N \))

\[
b_N(y) = -\frac{1}{(N-2)\sigma_N} \frac{1}{|y|^{N-2}}, \quad \text{where} \quad \sigma_N = \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})},
\]
the operator in (6.4) is written in the form of Leray’s formulation [58, p. 32]

\[
H(v) \equiv \Delta v - (v \cdot \nabla)v + C_3 \int_{\mathbb{R}^3} \frac{v - z}{|y - z|^3} \, \text{tr}(\nabla v(z, \tau))^2 \, dz,
\]

where \( \text{tr}(\nabla v(z, \tau))^2 = \sum_{(i,j)} v_{ij} v_{ij} \) and \( C_3 = \frac{1}{\sigma_N} > 0 \).

Equivalently, the nonlocal parabolic equation (6.4) is known to induce a local semigroup of smooth solutions.

As usual, our intention is to show that local sufficiently smooth solutions cannot blow-up under a certain \( L^p \)-type constraint. We next clarify the conditions that prevent finite-time blow-up of solutions in \( L^\infty \) and exclude Leray’ similarity (1.7) or any other.

It is a classic matter that the \( L^2 \)-norm is natural for (1.6), since after multiplication by \( v \), the convective and pressure terms vanish on sufficiently smooth functions \( v(x, t) \) with fast decay at infinity,

\[
\langle (v \cdot \nabla)v, v \rangle = 0 \quad \text{and} \quad - \langle \nabla p, v \rangle = \langle p, \nabla \cdot v \rangle = 0.
\]

Therefore, on smooth solutions,

\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_2^2 = -\|Dv(t)\|_2^2 \quad \Rightarrow \quad \|v(t)\|_2^2 + 2 \int_0^t \|Dv(t)\|_2^2 \, dt \leq \|v_0\|_2, \quad t \geq 0.
\]
Actually, the estimate in (6.8) is the energy inequality for Leray–Hopf weak solutions of (1.6); see e.g., [65] and references therein.

Nevertheless, an $L^2$-bound is not sufficient to control non-blowing up property of solutions, and this is the origin of extensive mathematical research in the last fifty years. Recall that global weak solutions of (1.6) satisfying

$$u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; H^1(\mathbb{R}^3))$$

were constructed by Leray [54], and Hopf [40] in 1951.

6.2. Application of blow-up $C_k$-scaling in the supercritical case $p > N$. Namely, as a first simple constraint, which is inherited from our previous study of the KS-type equations, we assume that

$$\|v(t)\|_p \leq C \quad \text{for} \quad t \geq 0 \quad (p > 2).$$

Then, seeking $L^\infty$-bound and hence assuming (5.2), we perform the scaling (5.3), where we impose the preservation of the $L^p$-norm of the rescaled function, i.e.,

$$\|v_k\|_p = \|w_k\|_p \quad \implies \quad a_k = C_k^{1-\frac{p}{N}} \to 0, \quad \text{and} \quad b_k = a_k^2.$$

As usual, we next perform passage to the limit in the NSEs. This can be done in the framework of the original model (1.6), as well as of the nonlocal parabolic representation (6.4), which we actually do. Note that passage to the limit in the integral term causes no difficulty for our sequences of uniformly bounded smooth solutions $\{w_k\}$, where Ascoli-Arzelà classic theorem [46, Ch. 2] applies.

We then obtain the following rescaled equations for $w = w_k(y,s)$:

$$w_s + \nu_k \mathbb{P}(w \cdot \nabla)w = \Delta w, \quad \text{where} \quad \nu_k = C_k^{1-\frac{p}{N}} \to 0 \quad \text{for} \quad p > N.$$

Next, after time shifting, $s \mapsto s - s_0$, the solutions and data satisfy (cf. (5.8))

$$|\bar{w}_k| \leq 1 \quad \text{and} \quad \|\bar{w}_k\|_p \leq C.$$

By regularity for uniformly bounded sequence $\{\bar{w}_k(s)\}$, we conclude that there exists its partial limit $\bar{w}(s)$ satisfying the solenoidal heat equation

$$\bar{w}_s = \Delta \bar{w}.$$

Of course, this is an equivalent form of writing the nonstationary linear “convectionless” Stokes system,

$$\bar{w}_s = -\nabla \bar{q} + \Delta \bar{w}, \quad \text{div} \bar{w} = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+,$$

with data as in (6.12). It is an exercise to check, by using the integral equation with the kernel as in (4.15), that this problem for (6.13) or (6.14) admits a solution that exhibits a power decay for $s \gg 1$. Namely, an estimate similar to (5.10) can be obtained with a different exponent of the decay rate.

In other words, under the assumption (6.9), the Navier–Stokes system does not have a mechanism to create any $L^\infty$ blow-up singularities. We thus arrive at the following:
Proposition 6.1. Under the given assumptions, $L^p$-solutions, with $p > N > 2$ in (6.9) of the Navier–Stokes equations (1.6) do not blow-up, and, moreover, are uniformly bounded:

\[
|v(x,t)| \leq C \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+.
\]

The last bound (6.15) is proved as in Theorem 5.2 by assuming that \( \{t_k\} \to +\infty \).

One can change the functional space in estimate (6.9) to get $L^\infty$-bound.

6.3. Critical case $p = N = 2$: an example of application of the ($T - t$)-scaling.

In the critical case $p = N$, we have that $\nu_k = 1$ in (6.11), so that in the limit $k \to +\infty$ we arrive at the same Navier–Stokes equations, but now with uniformly bounded data and solutions in both $L^3(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$. Moreover, by passing to the limit $s_0 = s_{0j} \to +\infty$, we actually deal with the following class of solutions:

\[
\begin{align*}
\bar{w}(s) &\in L^\infty \cap L^N \quad \text{for all} \quad s \geq 0: \quad |\bar{w}(s)| \leq 1, \quad \|\bar{w}(s)\|_N \leq C.
\end{align*}
\]

In other words, using the scaling in the critical case $p = N$, we eventually get into the special class of solutions (6.16), so a key restriction of the Navier–Stokes equations is achieved. Obviously, no blow-up and other singularities are available in the class (6.16). Thus, this admits the unique globally asymptotically stable trivial equilibrium $0$.

However, for $p = N$, proving non blow-up of solutions leads to a hard problem of nonexistence of suitable ancient solutions. Similar to the previous KS problem, we demonstrate an example of application of the ($T - t$)-scaling to get the result in the critical case $p = N = 2$ of the obvious particular interest.

We begin with Leray’s blow-up scaling [54] for (6.4) by setting

\[
v(x,t) = \frac{1}{\sqrt{T-t}} w(y,\tau), \quad y = \frac{x}{\sqrt{T-t}}, \quad \tau = -\ln(T-t),
\]

to get the following rescaled equation:

\[
\bar{w}_\tau + \mathbb{P}(\bar{w} \cdot \nabla)\bar{w} = \mathcal{B}^*\bar{w}, \quad \text{where} \quad \mathcal{B}^* = \Delta - \frac{1}{2} y \cdot \nabla - \frac{1}{2} I.
\]

Here, $\mathcal{B}^*$ is the adjoint Hermite operator with the discrete spectrum

\[
\sigma(\mathcal{B}^*) = \{\lambda_k = -\frac{1}{2} - \frac{k}{2}, \quad k = 0,1,2,...\}
\]

and a complete and closed set of eigenfunctions being finite solenoidal Hermite polynomials; see details in [34] Append. A and [30] § 2, where all the aspects of the functional settings of $\mathcal{B}^*$ in the weighted space $L^2_{\rho^*}, \rho^*(y) = e^{-|y|^2/4}$, can be found. Note that scaling (6.17) implies the following behaviour of the rescaled solution:

\[
\|\bar{w}(\tau)\|_2^2 \sim e^{\frac{N-2}{2}\tau} |_{N=2} = O(1) \quad \text{as} \quad \tau \to +\infty.
\]

We next follow the same scheme as in Section 5

(i) First, the $L^2$-conservation such as (5.23) holds for $w(y,\tau)$; see (6.20).

(ii) Second, assuming (5.24) yields by using the spectral properties in (6.19) and scaling (6.17) (cf. (5.27)) the non-blow-up in $L^\infty$ at $t = T$ of $v(x,t)$. 


(iii) Thirdly, we arrive at (5.28) for $w(y, \tau)$, since its uniform boundedness will lead to $L^\infty$-decay by the identity

\begin{equation}
\frac{1}{2} \frac{d}{d\tau} \|w(\tau)\|_2^2 = - \int |\nabla w|^2 < 0.
\end{equation}

This $\nabla$-property of the dynamical system means that, in the class of bounded smooth solutions, the only equilibrium is trivial $w = 0$, so that the uniform stabilization to it again guarantees non-blow-up.

(iv) Fourth, assuming (5.28), we $C_k$-scale such an orbit to get a convergence to a nontrivial 0-homoclinic solution as in (5.39):

\begin{equation}
W_s + \mathbb{P}(W \cdot \nabla)W = \Delta W \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}, \quad \int_{-\infty}^{\infty} \|\nabla W(s)\|^2_2 ds < \infty,
\end{equation}

where, thus, $W(s) \in L^2 \cap L^\infty$, and, by (6.21), $W(s) \in H^1$ for all $s \in \mathbb{R}$. Otherwise, if $W \notin H^1$, then (6.21) will guarantee fast decay of the $L^2$-norm of $w(\cdot, \tau)$ and hence $v(\cdot, t)$ as $t \to T^-$, which is contradictory. Finally, using the converging integral in (6.22) and passing to the limit as $s_k + s \to -\infty$ leads to smooth small solutions of the Euler equations (EEs):

\begin{equation}
V_s + \mathbb{P}(V \cdot \nabla)V = 0, \quad V(s) \in L^2 \cap L^\infty.
\end{equation}

Then the void conclusion similar to (5.48) remains valid, provided that smooth $L^2$-solutions of the EEs (6.23) decay in $L^\infty$ sufficiently fast, which is true since this is a smooth gradient system (recall that we are obliged to treat the simpler critical case $p = N = 2$ only); see [73, Ch. 17] and surveys [3, 14] for further details. This shows a potential correct way to treat the relations between singularities in the NSEs and EEs, where the absence of some of those for the latter ones implies the same for the former.

Of course, our analysis of global existence of classical bounded solutions in the critical case $p = N = 2$ just reflects the classic Leray’s (1933, the CP) [53] and Ladyzhenskaya (1958, IBVPs) [48, 49] (see also [50]) existence-uniqueness results for $N = 2$. In this connection, it is worth mentioning another new proof for $N = 2$ in Mattingly–Sinai [59], which is based on using advanced Fourier Transform techniques.

6.4. 2$\text{mth}$-order well-posed Burnett equations. As a natural extension, similar to KS-type problems in Sections 4 and 5 we consider the 2$m$th-order model (1.18). Actually, for $m \geq 2$, such models are not that formal and have been known to appear as Burnett’s equations on the basis of Grad’s method in Chapman–Enskog expansions for hydrodynamics. Unlike (1.18), the original Burnett equations are ill-posed, as backward higher-order parabolic equations having a wrong sign at the diffusivity operators. Namely, Grad’s method applied to kinetic equations yields, in addition to the classic operators of the Euler equations, other viscosity parts, as follows:

\[
v_t + (v \cdot \nabla)v = \sum_{n=0}^{\infty} \varepsilon^{2n+1} \Delta^n(\mu_n \Delta v) + \ldots = \varepsilon (\mu_0 \Delta v + \varepsilon^2 \mu_1 \Delta^2 v + \ldots) + \ldots,
\]

\[3\]Actually, the proof therein looks not that “elementary” as the title of [59] suggests; we pretend that our approach is more elementary, though this is not a proper point for arguing.
where $\varepsilon > 0$ is essentially the Knudsen number $Kn$; see details in Rosenau’s regularization approach, [66]. In a full model, truncating such series at $n = 0$ leads to the Navier–Stokes equations (1.6) (with $\mu_0 > 0$), while $n = 1$ is associated with the Burnett equations. These are ill-posed since, by expansion, $\mu_1 > 0$, so a backward bi-harmonic flow occurs, etc. We will refer to (1.18) for $m \geq 2$ as to the well-posed Burnett equations.

Note also that Burnett-type equations with a small parameter appeared as higher-order viscosity approximations of the Navier–Stokes equations is an effective tool for proving existence of their weak (“turbulent” in Leray’s sense) solutions; see Lions’ classic monograph [56, § 6, Ch. 1].

For the system (1.18), we use the following parameters of scaling:

\begin{equation}
(6.24) \quad a_k = C_k^{-\frac{k}{N}}, \quad b_k = a_k^{2m}, \quad \text{and} \quad D_k = C_k^{1+\frac{p(2m-1)}{N}}.
\end{equation}

This gives the parameter of the convection term in the analogy of (6.11) as:

\begin{equation}
(6.25) \quad \nu_k = C_k^{1-\frac{p(2m-1)}{N}}.
\end{equation}

Hence $\nu_k \to 0$ as $k \to \infty$ under the following hypothesis:

**Proposition 6.2.** Under the given assumptions, $L^p$-solutions of the well-posed Burnett equations (1.18), with

\begin{equation}
(6.26) \quad p > p_0 = \frac{N}{2m-1}
\end{equation}

in (6.9), do not blow-up, and are uniformly bounded, i.e., (6.15) holds.

The scaling analysis can be applies also directly to the locally smooth integral nonlocal parabolic flow similar to (6.4). The critical case $p = p_0$ is then treated by an additional use of the $(T - t)$-scaling, where some technical difficulties may occur.

6.5. **Well-posed Burnett equations: no blow-up for $N \leq 2(2m - 1)$**. This is a simple consequence of the previous scaling analysis. We recall that, for (1.18), $L^2$-norm of $v(t)$ does not increase with time, so, for smooth solutions, the analogy of (6.8) holds.

**Proposition 6.3.** Let $v_0 \in L^2(\mathbb{R}^N) \cap H^{2m}(\mathbb{R}^N)$ be divergence free. Then there exists the unique global bounded smooth solution of the well-posed Burnett equations (1.18) if

\begin{equation}
(6.27) \quad N < 2(2m - 1).
\end{equation}

Indeed, substituting $p = 2$ into (6.25) yields that

(i) $\nu_k \to 0$ in the subcritical case $N < 2(2m - 1)$, where the proof is completed in similar lines and causes no extra difficulties.

(ii) In addition, $\nu_k \equiv 1$ in the critical case

\begin{equation}
(6.28) \quad N = 2(2m - 1),
\end{equation}

in (6.9), do not blow-up, and are uniformly bounded, i.e., (6.15) holds.
where the dynamical system is a gradient one in the class of smooth bounded solutions. However, the proof of global existence according to the corresponding scaling quite similar to that in (6.17),

\[(6.29)\quad v(x,t) = (T - t)^{-\frac{2m-1}{2m}} w(y,\tau), \quad y = \frac{x}{(T-t)^{1/2m}}, \quad \tau = -\ln(T-t),\]

will lead to the following rescaled equation:

\[(6.30)\quad w_\tau + P(w \cdot \nabla)w = B^*w, \quad B^* = -(\Delta)^m - \frac{1}{2m} y \cdot \nabla - \frac{2m-1}{2m} I.\]

Therefore, a proper spectral theory of generalized solenoidal Hermite polynomials as eigenfunctions of $B^*$, with the spectrum

\[(6.31)\quad \sigma(B^*) = \{\lambda_k = -\frac{2m-1}{2m} - \frac{k}{2m}, \quad k = 0, 1, 2, \ldots\},\]

is necessary. This is developed along the lines presented in [17], where the scalar $2m$th-order case was under scrutiny. Eventually, this will allow to treat (we do not guarantee that all steps are going to be simple as in Section 6.3) the global existence of bounded classical solution in the critical case (6.28). As we have seen, for $m = 1$, (6.28) reads $N = 2$ and reflects classic Leray’s [53] and Ladyzhenskaya’s [48, 49, 50] results.

As is well-known, existence or nonexistence of $L^2$-bounded $L^\infty$-blow-up patterns in the complement to (6.27), (6.28) range

\[(6.32)\quad N > 2(2m-1), \quad \text{or} \quad N \geq 3 \text{ for } m = 1, \text{ including the crucial } 3D \quad N = 3,\]

comprises the core of the Millennium Prize Problem for the Clay Institute; see Fefferman [22]. Concerning possible structures of blow-up patterns in the NSEs, see discussion and review of a large amount of existing literature in a [30] that seems to be a most recent survey in the area.

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