Color Skyrmions in the Quark-Gluon Plasma

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Abstract

We consider the general formulation of nonabelian fluid dynamics based on symmetry considerations. We point out that, quite generally, this admits solitonic excitations which are the color analog of skyrmions. Some general properties of the solitons are discussed.

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1 Introduction

The collision of heavy nuclei at the Relativistic Heavy Ion Collider has created an interesting new state of matter [1]. It is clear that a state with deconfinement of quarks and gluons has been achieved. There are various indications that the resulting fluid is probably best described as a color liquid. One of the surprises has been the very low shear viscosity of the fluid. The value of the viscosity, it seems, is close to what may be the theoretical lower limit possible, a value which may be understandable in terms of a gravity-dual to the theory [2]. Thus a good first approximation to the description of this color liquid may be as a ‘perfect fluid’. However, unlike the case of ordinary fluids, since the constituents carry color degrees of freedom, the transport of such degrees of freedom, in a way consistent with the nonabelian gauge symmetry, becomes an important issue for this ‘color liquid’. This can, of course, be studied via kinetic equations, starting from the basics of QCD. However, the hierarchy of kinetic equations has to be truncated for reasons of computability, very often to the level of uncorrelated single-particle distribution functions, and such an approach is then limited to dilute systems near equilibrium. Since the experimental results indicate that the color liquid is not a dilute system, the validity and fruitfulness of this approach become questionable. By contrast, experience with ordinary fluid dynamics shows that one can derive the equations of fluid dynamics from very general principles, which then shows that the equations have a range of validity significantly beyond the regime where the truncated kinetic equations apply. This latter, a priori, approach has been developed for nonabelian fluid dynamics as well [3, 4].

In the next section, we will review briefly the formulation of ordinary fluid mechanics in group theory language and its generalization to include the transport of nonabelian degrees of freedom. This subject has been reviewed recently in [5]. In section 3, we will introduce the solitons and work out their properties. We then conclude with a short discussion.

2 Group theory and fluid dynamics

We begin with the well-known observation that ordinary fluid dynamics can be described as a Poisson bracket system. With $\rho$ as the fluid density and $v_i$ as the fluid velocity, the
fundamental Poisson bracket relations are given by

\[
\begin{align*}
[\rho(\vec{x},t), \rho(\vec{y},t)] &= 0, \\
[v_i(\vec{x},t), \rho(\vec{y},t)] &= \frac{\partial}{\partial x^i} \delta^{(3)}(x - y) \\
[v_i(\vec{x},t), v_j(\vec{y},t)] &= -\frac{\omega_{ij}}{\rho} \delta^{(3)}(x - y)
\end{align*}
\] (1)

where \(\omega_{ij}\) is the vorticity defined by

\[
\omega_{ij} = \partial_i v_j - \partial_j v_i
\] (2)

Equations (1) will lead to the usual equations of fluid dynamics, where the Hamiltonian \(H\) may be taken as

\[
H = \int d^3x \left[ \frac{1}{2} \rho v^2 + V(\rho) \right]
\] (3)

The canonical structure, or the symplectic form, defined by the Lagrangian is the inverse of the fundamental Poisson bracket. In the present case, the fundamental Poisson brackets have a zero mode and so this is a complication in finding a suitable Lagrangian. This can be seen in terms of the Chern-Simons action for the velocity, namely,

\[
\mathcal{C} = \frac{1}{8\pi} \int \epsilon^{ijk} v_i \partial_j v_k
\] (4)

(The invariant defined by \(\mathcal{C}\) is the fluid helicity.) It is easily seen that \(\mathcal{C}\) Poisson commutes with any observable \(F\),

\[
[F, \mathcal{C}] \equiv \int \left[ \frac{\delta F}{\delta \rho} \frac{\partial}{\partial x^i} \left( \frac{\delta \mathcal{C}}{\delta v_i} \right) - \frac{\delta \mathcal{C}}{\delta \rho} \frac{\partial}{\partial x^i} \left( \frac{\delta F}{\delta v_i} \right) - \omega_{ij} \frac{\delta F}{\delta v_i} \frac{\delta \mathcal{C}}{\delta v_j} \right] = 0
\] (5)

This shows that \(\delta \mathcal{C}/\delta v_i\) is a zero mode for the brackets, preventing the inversion needed to define the action. The solution to this problem is also rather clear. Since \(\mathcal{C}\) commutes with all observables, we must fix its value and consider only variations in \(v_i\) which preserve this value. In particular, if we choose \(\mathcal{C} = 0\), we must consider \(v_i\) which trivializes the Chern-Simons term by making \(v_i \partial_j v_k \epsilon^{ijk}\) into a total derivative. Such \(v_i\) are given in terms of the Clebsch parametrization,

\[
v_i = \partial_i \theta + \alpha \partial_i \beta,
\] (6)

where \(\theta, \alpha, \beta\) are arbitrary functions.

A suitable action which leads to the Poisson bracket relations (1), and the continuity and Euler equations, is

\[
S = \int dt d^3x \left[ \rho \dot{\theta} + \rho \alpha \dot{\beta} - \frac{1}{2} \rho v^2 - V(\rho) \right]
\] (7)

Notice that we have the canonically conjugate pairs \((\rho, \theta), (\rho \alpha, \beta)\).
We will now rewrite this in terms of a group element

\[ g = \frac{1}{\sqrt{1 - u\bar{u}}} \begin{pmatrix} 1 & u \\ \bar{u} & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \]  
(8)

where \( u \) is a complex variable. \( g \) is an element of \( SU(1,1) \) if \( \bar{u} = \bar{u} \) and an element of \( SU(2) \) if \( \bar{u} = -\bar{u} \). (We will consider both possibilities together for a while.) Using (8), we find by direct computation that

\[ -i \text{Tr}(\sigma_3 g^{-1} dg) = d\theta + \alpha d\beta \]  
(9)

where

\[ \alpha = \frac{u\bar{u} - 1}{1 - u\bar{u}} , \quad \beta = -\frac{i}{2} \log(u/\bar{u}) \]  
(10)

Equations (9,10) show that the Clebsch parametrization can be expressed in terms of a single group element as \( v_i = -i \text{Tr}(\sigma_3 g^{-1} \partial_i g) \). Notice that \( \theta \) in (8) corresponds to the \( \sigma_3 \)-direction in \( g \); thus \( \alpha, \beta \) parametrize the space \( SU(1,1)/U(1) \) and \( SU(2)/(1) \), respectively, for \( \bar{u} = \bar{u} \) and \( \bar{u} = -\bar{u} \). Going back to (7) we notice that the action can be written as

\[ S = \int d^4x \left[ -i j^\mu \text{Tr}(\sigma_3 g^{-1} \partial_\mu g) - \left( \frac{j^i j^i}{2\rho} + V \right) \right] \]  
(11)

where \( j^0 = \rho \). Elimination of \( j^i \) using its equation of motion takes us back to (7). The form of equation (11) also shows us how to generalize to a relativistic situation,

\[ S = \int d^4x \left[ -i j^\mu \text{Tr}(\sigma_3 g^{-1} \partial_\mu g) - F(n) \right] \]  
(12)

where \( n^2 = j^\mu j^\nu g_{\mu\nu} \), \( g_{\mu\nu} \) being the metric tensor. The choice of the function \( F \) specifies the equation of state for the fluid. A four-velocity \( u^\mu \) for the fluid can be defined by \( j^\mu = nu^\mu \). The energy-momentum tensor is then

\[ T^{\mu\nu} = n F' u^\mu u^\nu - g^{\mu\nu} (n F' - F) \]  
(13)

which also identifies the pressure as \( p = n F' - F \).

The Poisson bracket relations which follow from (11) (or (12)) are

\[ [\rho(\vec{x}, t), \rho(\vec{y}, t)] = 0 \]

\[ [\rho(\vec{x}, t), g(\vec{y}, t)] = -i g(x) t_3 \delta^{(3)}(x - y) \]  
(14)

Here \( t_a = \frac{1}{2} \sigma_a \). This equation shows that \( \rho \) generates right translations on \( g \) along the \( \sigma_3 \)-direction. In the quantum theory, it is easily checked that \( U = \exp[-2\pi i \int \rho] \) acts on \( g \) as

\[ U^\dagger g U = g e^{2\pi i t_3} = -g \]  
(15)
All observables are invariant under $U$ since they have even powers of $g$, $g^\dagger$. This means that $U$ is the identity operator on observables, which, in turn, implies that $\int \rho = \text{integer}$ [7]. This is essentially the statement that the fluid is made of particles. Mathematically, this is due to the $\theta$-direction being compact. Turning this logic around, we see that it is consistent to require that the field $\theta$, which appears in the Clebsch parametrization, should be a compact direction since fluids of interest are ultimately made of particles.

From the Clebsch parametrization, $\omega_{ij} = (\partial_i \alpha \partial_j \beta - \partial_j \alpha \partial_i \beta)$. Consider now the integral of this over some compact region $V$ in $\mathbb{R}^2$. $\int_V \omega$ is the volume of $SU(1,1)/U(1)$ or $SU(2)/U(1)$ over the image of $V$ via the map $g(x)$. This will be quantized for $S^2 = SU(2)/U(1)$, but not for $SU(1,1)/U(1)$. Thus we see that the choice of the group is determined by the quantization of $\int \rho$ and by whether we want quantized vorticity or not.

We now turn to the generalization of this to other groups. A Clebsch-type parametrization for higher groups would be an obvious direction to try [3]. This is possible for the groups $SO(n)$ with $v_i^a = -i\text{Tr}(t^a R_T \partial_i R)$ where $R \in SO(2n-1)$ and $t_a$ are generators of $SO(n) \subset SO(2n-1)$. The relevant coset is $SO(2n-1)/SO(n) \times SO(n-1)$. The resulting dynamics is rather involved and does not admit the Eckart factorization $J_i^a = Q_{aj}^i$ which is what we would expect if the nonabelian charge density $Q^a$ is transported by particle motion.

A different approach is to start with the motion of particles carrying nonabelian charges. This is described by the Wong equations [8]

\[
\dot{Q}^a - f^{abc} A_i^c \dot{x}^i Q^b = 0 \\
\dot{p}_i - F_{ij}^a \dot{x}^i Q^a = 0
\]

These equations of motion can be obtained from the action [9]

\[
S = \int dt \left[ \frac{1}{2} m \dot{x}^2 + A_i^a Q^a \dot{x}^i \right] - in \int dt \text{Tr}(\sigma_3 g^{-1} \dot{g})
\]

where $Q^a = \text{Tr}(\sigma_3 g^{-1} t^a g)$ and $g \in SU(2)$. (Here we are considering $SU(2)$ color for simplicity. We will extend this to any Lie group shortly.) The first term is the usual action for a particle coupled to an external gauge field except for the color charge factor $Q^a$. The second term is what leads to the dynamics of the color charge. Notice that $Q^a$ is invariant under $g \rightarrow gh$, where $h = \exp(it_3 \dot{\varphi})$. The second term in the action changes by a surface term $\Delta S = n \int \dot{\varphi} dt = n \Delta \varphi$. Thus, the equations of motion are invariant under $g \rightarrow gh$. However, one can consider closed loops in the $U(1)$-subgroup, generated as a trajectory over time, which would have $\Delta \varphi = 2\pi$. The invariance of $e^{iS}$ then requires $n \in \mathbb{Z}$. For integer values of $n$, the theory (including the quantum theory) involves only $SU(2)/U(1) = S^2$ variables. In fact, the second term of $S$ gives the symplectic form $\Omega = in \text{Tr}(\sigma_3 g^{-1} \dot{g} \wedge g^{-1} dg)$ for the parameters in $g$. Since the phase space for this is the two-sphere, the phase volume is
finite, giving a finite-dimensional Hilbert space upon quantization. In fact, the quantization of (17) leads to one unitary irreducible representation of \( SU(2) \) with a highest weight state of \( j = n/2 \). The action (17) thus leads to the standard description of color by matrices. The term \(-in \text{Tr}(\sigma_3 g^{-1} dg)\) is often referred to as the Kostant-Kirillov-Souriau (KKS) form [10].

Focusing on the color degrees of freedom, the generalization of (the second term of) the action (17) to many particles is given by

\[
S = -i \int dt \sum_{\lambda} n_{\lambda} \text{Tr}(\sigma_3 g^{-1}_{\lambda} \dot{g}_{\lambda})
\]

where \( \lambda = 1, 2, \ldots, N \), labels the particle under consideration. We can now take the continuum limit by taking \( N \) large, with an average volume \( v \) per particle. This means that we can write \( \lambda \to \vec{x} \), \( \sum_{\lambda} \to \int d^3 x / v \) and \( n_{\lambda} / v = j_0 \). The action (18) then becomes

\[
S = -i \int d^4 x \ j^0 \ \text{Tr}(\sigma_3 g^{-1} \dot{g})
\]

where \( g = g(\vec{x}, t) \). Notice the similarity of this to the first term of the action for ordinary fluid dynamics. Taking (19) as the key term which leads to the symplectic form for the color degrees of freedom, we can write the action for fluids, where the particle carry \( SU(2) \) color degrees of freedom, as

\[
S = \int d^4 x \left[ -i j^\mu \ \text{Tr}(\sigma_3 g^{-1} D_\mu g) - F(n) - \frac{1}{4} F^a_{\mu \nu} F^{a \mu \nu} \right]
\]

where \( D_\mu g = \partial_\mu g + A_\mu g \) and \( A_\mu = -it^a A^a_\mu \) is a nonabelian background field. (By this we mean the field background in which the fluid moves; thus \( A_\mu \) represents gluon degrees of freedom which are not homogenized with the fluid, such as hard gluons.) \( j^\mu \) is a current four-vector, \( n^2 = j^\mu j^\nu g_{\mu \nu} \). \( j^i \) can be eliminated by its equation of motion to get a simpler form of \( S \), which, however, will not be manifestly Lorentz invariant. The dynamics which follows from (20) will be the \( SU(2) \) analog of magnetohydrodynamics. The color current, which couples to \( A^a_\mu \), is easily seen to be

\[
J^{a \mu} = \text{Tr}(\sigma_3 g^{-1} t^a g) \ j^\mu = Q^a j^\mu
\]

We see that the Eckart factorization is realized with \( Q^a \) as the charge density for the fluid. The equations of motion are given as

\[
D_\mu J^{a \mu} = 0 \quad F^\prime u^\mu = -i \text{Tr}(\sigma_3 g^{-1} D_\mu g)
\]

where the color flow velocity \( u^\mu \) is defined by \( j^\mu = nu^\mu \). We can also carry out a variation on \( g \) on the right, which leads to the equation \( \partial_\mu j^\mu = 0 \). This equation is not independent of equations (22). The energy-momentum tensor for the fluid is given by

\[
T^{\mu \nu} = n F^\prime u^\mu u^\nu - g^{\mu \nu} (n F^\prime - F)
\]
and obeys the expected relation

$$\partial_\mu T^{\mu\nu} = \text{Tr}(J_\mu F^{\mu\nu})$$  (24)

It is interesting at this stage to give an interpretation of the group element $g$. Let $\rho = \rho^a t^a$ be the nonabelian charge density of a distribution of particles. Under a gauge transformation $U \in SU(2)$, it transforms as

$$\rho \to \rho' = U^{-1} \rho U$$  (25)

We can diagonalize the hermitian matrix $\rho$ at each point by an $(\vec{x}, t)$-dependent transformation. We may thus write $\rho = g \rho_{\text{diag}} g^{-1}$, or

$$\rho^a = \rho_0 \text{Tr}(\sigma_3 g^{-1} t^a g) = n \text{Tr}(\sigma_3 g^{-1} t^a g)$$  (26)

where $\rho_0 = n$ is the eigenvalue of $\rho$. Evidently, $n$ is gauge invariant. Comparing with our previous expression for $\rho^a = J^a_0$, we see that the dynamical variable $g(\vec{x}, t)$ can be interpreted as the gauge transformation which diagonalizes the charge density at each point. The flow of the gauge invariant eigenvalues is given by $u^a$. Under a gauge transformation, $g \to U^{-1} g$.

The generalization to higher groups is clear from this discussion. For a group $G$, we have $\text{rank}(G)$ eigenvalues and hence $\text{rank}(G)$ $n$’s and $u^\mu$’s. The field $g$ is an element of $G$ and the action is given by

$$S = -i \sum_{s}^{\text{rank}(G)} \int d^{4}x \int j_{s}^{\mu} \text{Tr}(q_{s} g^{-1} D_{\mu} g) = -\int d^{4}x F(n_{1}, n_{2}, \ldots) + S_{\text{YM}}$$  (27)

$q_s$ are the diagonal generators of $G$, $j_{s}^{2} = n_{s}^{2}$. We have as many $n$’s as there are simultaneously diagonalizable conserved charges.

For a statistical distribution, we expect a chemical potential for each of the simultaneously diagonalizable conserved charges. The values of these chemical potentials are fixed by the values of the corresponding charges. Thus a statistical distribution of particles is specified by the values it has for the simultaneously diagonalizable conserved charges. What we find for the fluid is in conformity with this.

One can give a general justification for (27) starting from a one-particle action again. The relevant observation is the following. The general form of KKS action is [10]

$$S = -i \sum_{s} w_{s} \int dt \text{Tr}(q_{s} g^{-1} \dot{g})$$  (28)

There are quantization condition on this action. The numbers $(w_1, w_2, \ldots)$ should form the weight vector of the highest weight state of a unitary irreducible representation of the group.
Upon quantization, the action (28) then gives a Hilbert space which corresponds to the unitary irreducible representation characterized by \((w_1, w_2, \ldots)\). What we have given in (27) is indeed the appropriate generalization of this result to fluids. (The action (28) leads to the symplectic form which is the Kähler two-form on the coadjoint orbit of the element \(\sum_s w_s q_s\). So this method is also known as the coadjoint orbit method.)

The foregoing discussion also shows that ordinary hydrodynamics is a special case of this general structure where we have only one conserved charge, namely, the particle number.

For the sake of completeness, we give the equations of motion for the action (27); these are

\[
\partial_\mu j^\mu_s = 0 \\
\sum_s j^\mu_s (D_\mu Q_s)^a = 0 \\
u^\mu_s \frac{\partial F}{\partial n_s} = -i \text{Tr}(q_s g^{-1} D_\mu g)
\]

where \(Q^a_s = \text{Tr}(q_s g^{-1} t^a g)\). The basic Poisson brackets are given by

\[
[\rho^a(\vec{x}, t), \rho^b(\vec{y}, t)] = f^{abc}(\vec{x}, t) \delta^{(3)}(x - y) \\
[j^0_s(\vec{x}, t), j^0_{s'}(\vec{y}, t)] = 0 \\
[j^0_s(\vec{x}, t), g(\vec{y}, t)] = -i g(x) q_s \delta^{(3)}(x - y)
\]

Here \(\rho^a = J^{a0}\) is the color charge density and the color current is \(J^{a\mu} = \sum_s j^\mu_s Q^a_s\).

3 Color skyrmions

The description of nonabelian charge using the KKS form is a standard part of Lie group theory. In fact, if we ask whether we can find a classical theory which upon quantization gives finite-dimensional Lie algebra (color) matrices in a single irreducible representation, the answer is the KKS action (28). Since the action for fluid dynamics which we postulate is a straightforward fluid generalization of this basic result for particles, we see that, quite generally, the color degrees of freedom of the fluid are described by \(g(\vec{x}, t) \in G\) and \(j^0_s \in \mathfrak{g}\) Cartan subalgebra of \(G\). For QCD, we have \(g(\vec{x}, t) \in SU(3)\). At a fixed time, we have \(g(\vec{x}) : V \to SU(3)\), where \(V\) is a region in \(\mathbb{R}^3\) in which the fluid exists. For configurations with \(g \to 1\) on the boundary of \(V\), these functions \(g(\vec{x})\) are equivalent to \(g(\vec{x}) : S^3 \to SU(3)\). The homotopy classes, or equivalence classes up to smooth deformations, of such maps are given by \(\Pi_3[SU(3)] = \mathbb{Z}\). The topologically nontrivial configurations of \(g(\vec{x})\) are then solitons. These are mathematically the same as the usual skyrmions, although these are in the color
sector and not in the flavor sector [11, 12]. The soliton number which characterizes the homotopy classes \(\Pi_3[SU(3)]\), or the skyrmions, is

\[
Q = -\frac{1}{24\pi^2} \int d^3x \quad \varepsilon^{ijk} \quad \text{Tr}(g^{-1}\partial_ig^{-1}\partial_jg^{-1}\partial_kg)
\]  

(31)

(This is essentially the color version of helicity.)

For \(SU(3)\), there are two distinct types of maps \(g : S^3 \rightarrow SU(3)\) we can consider. One of them corresponds to the image of \(S^3\) being an \(SU(2)\) subgroup of \(SU(3)\), the other corresponds to \(S^3\) being mapped to the \(SO(3)\) subgroup defined by the generators \((\lambda_2, -\lambda_5, \lambda_7)\), \(\lambda_a\) being the usual Gell-Mann matrices of \(SU(3)\). Most of the features of the skyrmions we are considering will be clear from the first type of maps, namely, \(g : S^3 \rightarrow SU(2) \subset SU(3)\), so we shall consider only these in this paper. This means that, effectively, we can restrict attention to \(SU(2) \subset SU(3)\).

The explicit solution of the equation of motion to obtain the soliton will be very involved and will depend sensitively on the choice of Hamiltonian. But, as is usually done in the case of flavor skyrmions, we can choose an ansatz in a given topological sector, which depends on some parameters, and then variationally minimize the energy to fix the values of these parameters. In general, this will give a good qualitative (and, to some extent, even quantitative) description of the soliton. The simplest ansatz we can take for \(g(\vec{x})\) is the spherically symmetric ansatz

\[
g_s(\vec{x}) = \cos \phi(r) + i \sigma \cdot \hat{x} \quad \sin \phi(r)
\]  

(32)

which leads to

\[
Q = \frac{\phi(0) - \phi(\infty)}{\pi}
\]  

(33)

The profile of \(\phi\) as a function of \(r\) is not yet determined; for \(Q = 1\), we need one step of height \(\pi\) as we go from \(r = 0\) to \(r = \infty\). The simplest choice is the stereographic ansatz

\[
\sin \phi = \frac{2Rr}{R^2 + r^2}, \quad \cos \phi = \frac{R^2 - r^2}{R^2 + r^2}
\]  

(34)

where \(R\) sets the scale for the soliton. In what follows we will make this simple choice.

Up to this point we have not chosen a particular form for the Hamiltonian. The action (27) has the form

\[
S = -i \int d^4x \quad j^0 \quad \text{Tr}(\sigma_3g^{-1}\dot{g}) - \int dt \quad H
\]  

(35)

\(H\) is determined by the choice of \(F(n)\). We will take a simple form which is easy to work with. More specific choices, based on the equation of state of the color liquid, can be made.

\footnote{Skyrmions where the target space includes color have been considered in a different and unrelated context in [13].}
We do not expect the general features of the solitons to be changed by this. Our choice for 
\( F(n) \) is then
\[
F(n) = \frac{1}{2\mu^2} n^2 = \frac{1}{2\mu^2} \left[ j^0 j^0 - \vec{j} \cdot \vec{j} \right]
\]  
(36)

Eliminating \( \vec{j} \) from the action (27), we identify
\[
H = \int d^3 x \left[ \frac{j^0 j^0}{2\mu^2} + \frac{\mu^2}{2i} \text{Tr}(\sigma_3 g^{-1} \nabla g) \right]^2
\]  
(37)

We see that, unlike the case of the flavor skyrmions, we will need an ansatz for \( j^0 \) as well.

For this purpose, consider the collective coordinate quantization of the soliton \( g_s(\vec{x}) \). Color transformations act on the field \( g \) as \( g \rightarrow U g \). States with nonzero color charge can be generated from \( g_s(\vec{x}) \) by writing 
\[
g(\vec{x}, t) = U(t) g_s(\vec{x})
\]  
(38)

where \( U(t) \), \( R(t) \) (or its \((2 \times 2)\)-matrix version \( S(t) \)) represent the collective coordinates.

Notice that while the ansatz (32) is similar to what happens for flavor skyrmions, the collective coordinates enter into equation (38) in a very different way. Using (38) in the action, we find
\[
S = -i \int dt \left[ m \text{Tr}(\sigma_3 U^{-1} \dot{U}) + (n - m) \text{Tr}(\sigma_3 S^{-1} \dot{S}) \right] - \int dt H
\]  
(39)

where \( m, n \) are given in terms of integrals involving combinations of \( j^0 \) and \( \phi \). The action is given in terms of the KKS forms for \( U \) and \( S \) and so lead to representations of the color algebra (\( SU(2) \) in this case) and the rotation algebra. For consistent quantization, with unitary representations of the two groups, we need \( m, n \) to be integers. Our ansatz for \( j^0 \) must be consistent with this. Since there are two conditions we take an ansatz for \( j^0 \) which has two functions in it; the simplest choice is
\[
j^0 = f(r) + \left[ (v \cdot \hat{x})^2 - \frac{4\phi'^2}{3} \right] h(r)
\]  
(40)

where \( v_i = i \text{Tr}(\sigma_3 g^{-1} \partial_i g) \). Using the ansatz for \( g \) we then find
\[
\int d^3 x f(r) = n
\]
\[
\frac{4}{3} \int d^3 x \left( f - \frac{8}{15} \frac{h}{\phi'^2} \right) \sin^2 \phi = n - m
\]  
(41)

The energy functional can be worked out to be
\[
E = \frac{2\mu^2}{3} \int d^3 x \left( \phi'^2 + \frac{2\sin^2 \phi}{r^2} \right) + \frac{1}{2\mu^2} \int d^3 x \left( \frac{f^2}{45} (\phi'^2 h)^2 \right)
\]  
(42)
So far we have not used the specific form of $\phi$. We now simplify our ansatz further by taking $h(r)$ to be a constant and $f(r) = w(8\phi^2/15)$, where $w$ is another constant. Further taking $\phi$ to be given by the stereographic ansatz, the conditions (41) are evaluated as

$$n = \frac{32\pi^2 R}{15} w, \quad n - m = \frac{16\pi^2 R}{15} (w - h)$$

which lead to

$$w = \frac{15}{32\pi^2 R} n, \quad h = \frac{15}{32\pi^2 R} (2m - n)$$

The energy function (42) can also be evaluated easily as

$$E(R) = 8\pi^2 \mu^2 R + \frac{5}{16\pi^2 \mu^2 R^3} \left[ (m - s)^2 + \frac{(m + s)^2}{5} \right]$$

where $s = \frac{1}{2}(n - m)$ is the spin of the soliton and $m$ defines the color charge. The collective coordinate action, upon quantization, gives an $(m + 1)$-dimensional $SU(2)$ color multiplet and $(2s + 1)$-dimensional representation of the rotation group. The minimum of $E(R)$ occurs at

$$(\mu R)^4 = \frac{15}{128 \pi^4} \left[ (m - s)^2 + \frac{(m + s)^2}{5} \right]$$

and the variational estimate of the energy of the soliton is given as

$$E = \frac{8\mu \pi}{3} (30)^{1/4} \left[ (m - s)^2 + \frac{(m + s)^2}{5} \right]^{1/4}$$

Going back to (38), we see that the ansatz depends on $R(t) \in SO(3)$, so only the integer spins can arise. This means that the the difference $n - m$ should be an even integer. With this condition, formula (47) gives the energy as a function of the color charge and spin.

### 4 Discussion

In this paper we considered the general formulation of nonabelian fluid dynamics. Since it is based on symmetry considerations and the generalization of the mechanics of particles carrying nonabelian charges, there is a universality to this formulation. It is then a very general result that the theory admits color skyrmion configurations. We have obtained a (variational) mass formula for these as a function of spin and color charge, choosing a simple form for the Hamiltonian. Unlike the flavor skyrmions, there is no Wess-Zumino term, since there is no color anomaly. These solitons are bosonic excitations of integral spin.

Clearly there are a number of interesting questions which need further investigation. There is a hierarchy of energy scales which is important in this context. The fluid approximation works in a regime where there are few single-particle hard scattering events, the
collective effects are important, and before rehadronization sets in. The form of the Hamiltonian has to be fixed by the equation of state for the color degrees of freedom. (This is different from a general equation of state, since there are other conserved charges such as baryon number. Thus there are different partial pressures and here we need the equation of state for the color partial pressure as a function of the color charge density.) A more detailed investigation of how our results depend on the form of the Hamiltonian has to be carried out. Even after we have made the simple choice (37), the scale parameter \( \mu \) is not specified. While it can be related to the equation of state, we expect that it is closely related to the magnetic screening in the plasma. One reason for this is to notice that, if we absorb the field \( g \) as a gauge transformation of \( A \), in a way analogous to going to the unitary gauge in spontaneously broken gauge theories, our Hamiltonian has the form

\[
H = \int \frac{1}{2} \mu^2 A^3 \cdot A^3 + \ldots.
\]

Another key issue is the production, detection and destruction of these solitons. The conservation of the topological charge is vitiated only if the field \( g(\vec{x}, t) \) cannot be defined without singularities. So we expect that if such solitons are produced in nuclear collisions, it should happen at the transition to the deconfined fluid phase. Likewise, they should disappear due to rehadronization when the fluid description loses its validity. Dissipative effects can be introduced into our approach by modifying the equations of motion. While they can change many details, the qualitative features of the solitons should be unaltered.

We hope to address some of the issues raised here in a future publication.

VPN thanks I. Zahed for a useful comment. This work was supported in part by the National Science Foundation grant number PHY-0244873 and by a CUNY Collaborative Research Incentive grant.

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