CONVERSION OF SECOND-CLASS CONSTRAINTS AND RESOLVING THE ZERO-CURVATURE CONDITIONS IN THE GEOMETRIC QUANTIZATION THEORY

I. A. Batalin* and P. M. Lavrov†

In the approach to geometric quantization based on the conversion of second-class constraints, we resolve the corresponding nonlinear zero-curvature conditions for the extended symplectic potential. From the zero-curvature conditions, we deduce new linear equations for the extended symplectic potential. We show that solutions of the new linear equations also satisfy the zero-curvature condition. We present a functional solution of these new linear equations and obtain the corresponding path integral representation. We investigate the general case of a phase superspace where boson and fermion coordinates are present on an equal basis.

Keywords: symplectic potential, second-class constraint, conversion method

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1. Introduction

We are happy for the opportunity to contribute an article in honor of the 75th birthday of our friend and colleague Professor Igor Viktorovich Tyutin. We have worked together with Igor Tyutin for many years, and we have valued his extraordinary scientific potential in full measure, as well as his brilliant personal qualities. We cordially wish Igor Tyutin many new scientific achievements together with further successes in his personal life.

Berezin’s fundamental concept of quantization [1] has a nontrivial projection on the geometric quantization [2]–[5], the Batalin–Fradkin–Vilkovisky (BFV) formalism [6], [7], and the deformation quantization [8]–[10]. It is well known that the conversion of second-class constraints serves as one of the most powerful modern approaches to geometric quantization. The standard scenario of the conversion method [11]–[16] (also see [17] and further developments in [18]–[20]) is formulated as follows. The starting point is some phase manifold with a complicated nonlinear Poisson bracket. New momenta are then introduced as canonically conjugate to the original phase variables, which are now regarded as mutually commuting. Second-class constraints are simultaneously introduced equating the new momenta to the components of the symplectic potential. Additional degrees of freedom (conversion variables) [14], [21] are then introduced to convert the
second-class constraints into first-class constraints. The Poisson brackets for the conversion variables are defined to be constant. The first-class constraints obtained after the conversion equate the momenta to the components of the extended symplectic potential, which now also depend on the conversion variables. It is usually assumed that these first-class constraints mutually commute (Abelian conversion). Their commutation relations then have the form of a zero-curvature condition. These zero-curvature conditions are in fact nonlinear equations that are technically difficult to solve for the extended symplectic potential.

Here, we derive new linear equations for the extended symplectic potential by multiplying the zero-curvature conditions times the original phase variables. We then show that solutions of these linear equations also satisfy nonlinear zero-curvature conditions. The situation here is very similar to the case of the Maurer–Cartan equation in standard group theory. Finally, we present a functional solution of the new linear equations and then derive the corresponding path integral representation.

2. Classical mechanics in the general setting

Let \( Z^A, \varepsilon(Z^A) = \varepsilon_A \), be coordinates of the original phase space in the Hamiltonian formalism. Let \( V_A(Z), \varepsilon(V_A) = \varepsilon_A \), be a symplectic potential and \( S \) be an action with an original Hamiltonian \( H = H(Z) \),

\[
S = \int dt \mathcal{L}, \quad \mathcal{L} = V_A \partial_t Z^A - H. \tag{2.1}
\]

Making an arbitrary variation \( \delta Z^A \), we obtain the equations of motion

\[
\omega_{AB} \partial_t Z^B - \partial_A H = 0, \tag{2.2}
\]

where \( \omega_{AB} \) is a symplectic metric,

\[
\omega_{AB} = \partial_A V_B + \partial_B V_A (-1)^{\varepsilon_A+1}(\varepsilon_B+1). \tag{2.3}
\]

It follows from (2.3) that

\[
\partial_C \omega_{AB} (-1)^{(\varepsilon_C+1)\varepsilon_B} + \text{cyclic perm.}(A, B, C) = 0. \tag{2.4}
\]

Hereafter, we assume that metric (2.3) is invertible and let

\[
\omega^{AB} = -\omega^{BA} (-1)^{\varepsilon_A\varepsilon_B} \tag{2.5}
\]

denote its inverse. From (2.4), we hence obtain

\[
\omega^{AD} \partial_D \omega^{BC} (-1)^{\varepsilon_A\varepsilon_C} + \text{cyclic perm.}(A, B, C) = 0. \tag{2.6}
\]

Multiplying (2.3) by \( Z^A \) from the left, we obtain

\[
(Z^A \partial_A + 1)V_B - \mathcal{F}_0 \overrightarrow{\partial}_B = Z^A \omega_{AB} \tag{2.7}
\]

for \( V_B \) with an arbitrary function \( \mathcal{F}_0 = Z^A V_A (-1)^{\varepsilon_A} \). In turn, it follows from (2.7) that the Lagrangian in (2.1) can be rewritten in the form

\[
\mathcal{L} = Z^A \omega_{AB} \partial_t Z^B - H + \partial_t \chi. \tag{2.8}
\]

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