COHEN-MACAULAY, SHELLABLE AND UNMIXED CLUTTERS WITH A PERFECT MATCHING OF KÖNIG TYPE

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Abstract. Let \( C \) be a clutter with a perfect matching \( e_1, \ldots, e_g \) of König type and let \( \Delta_c \) be the Stanley-Reisner complex of the edge ideal of \( C \). If all c-minors of \( C \) have a free vertex and \( C \) is unmixed, we show that \( \Delta_c \) is pure shellable. We are able to describe, in combinatorial and algebraic terms, when \( \Delta_c \) is pure.

If \( C \) has no cycles of length 3 or 4, then it is shown that \( \Delta_c \) is pure if and only if \( \Delta_c \) is pure shellable (in this case \( e_i \) has a free vertex for all \( i \)), and that \( \Delta_c \) is pure if and only if for any two edges \( f_1, f_2 \) of \( C \) and for any \( e_i \), one has that \( f_1 \cap e_i \subset f_2 \cap e_i \) or \( f_2 \cap e_i \subset f_1 \cap e_i \). It is also shown that this ordering condition implies that \( \Delta_c \) is pure shellable, without any assumption on the cycles of \( C \).

Then we prove that complete admissible uniform clutters and their Alexander duals are unmixed. In addition, the edge ideals of complete admissible uniform clutters are facet ideals of shellable simplicial complexes, they are Cohen-Macaulay, and they have linear resolutions. Furthermore if \( C \) is admissible and complete, then \( C \) is unmixed. We characterize certain conditions that occur in a Cohen-Macaulay criterion for bipartite graphs of Herzog and Hibi, and extend some results of Faridi—on the structure of unmixed simplicial trees—to clutters with the König property without 3-cycles or 4-cycles.

1. Introduction

A clutter \( C \) with finite vertex set \( X \) is a family of subsets of \( X \), called edges, none of which is included in another. The set of vertices and edges of \( C \) are denoted by \( V(C) \) and \( E(C) \) respectively. Clutters are special types of hypergraphs. The set of edges of a clutter can be viewed as the set of facets of a simplicial complex. A basic example of a clutter is a graph. For a thorough study of clutters and hypergraphs from the point of view of combinatorial optimization see [8, 23].

Let \( C \) be a clutter with finite vertex set \( X = \{x_1, \ldots, x_n\} \). We shall always assume that \( C \) has no isolated vertices, i.e., each vertex occurs in at least one edge. Let \( R = K[x_1, \ldots, x_n] \) be a polynomial ring over a field \( K \). The edge ideal of \( C \), denoted by \( I(C) \), is the ideal of \( R \) generated by all monomials \( \prod_{x_i \in e} x_i = x_e \) such that \( e \in E(C) \). The assignment \( C \mapsto I(C) \) establishes a natural one to one correspondence between the family of clutters and the family of square-free monomial ideals. Edge ideals of clutters are also called facet ideals [12]. A subset \( F \) of \( X \) is called independent or stable if \( e \not\subset F \) for any \( e \in E(C) \). The dual concept of an independent vertex set is a vertex cover, i.e., a subset \( C \) of \( X \) is a vertex cover of \( C \) if and only if \( X \setminus C \) is an independent vertex set. The number of vertices in a minimum vertex cover of \( C \) is called the covering number.

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of $C$, and this number coincides with $\text{ht} \, I(C)$, the height of $I(C)$. The Stanley-Reisner complex of $I(C)$, denoted by $\Delta_C$, is the simplicial complex whose faces are the independent vertex sets of $C$. Recall that $\Delta_C$ is called pure if all maximal independent vertex sets of $C$, with respect to inclusion, have the same number of elements. If $\Delta_C$ is pure (resp. Cohen-Macaulay, Shellable), we say that $C$ is unmixed (resp. Cohen-Macaulay, Shellable). A clutter has the König property if the maximum number of pairwise disjoint edges equals the covering number. A perfect matching of $C$ of König type is a collection $e_1, \ldots, e_g$ of pairwise disjoint edges whose union is $X$ and such that $g$ is the height of $I(C)$. Any unmixed clutter with the König property and without isolated vertices has a perfect matching of König type (Lemma 2.8).

We are interested in determining what families of clutters have the property that $\Delta_C$ is pure, Cohen-Macaulay, or Shellable in the non-pure sense of Björner-Wachs [3]. The last two properties have been extensively studied, see [5, 24, 26, 29] and the references there, but to the best of our knowledge the first property has not been studied much except for the case of graphs [20, 21, 22, 30]. The aim of this paper is to examine these three properties when $C$ has a perfect matching of König type or when $C$ has the König property.

The contents of this paper are as follows. Let $C$ be a clutter with a perfect matching $e_1, \ldots, e_g$ of König type and let $I(C)$ be its edge ideal. The main theorem in Section 2 is a combinatorial description of the unmixed property of $C$, along with an equivalent algebraic formulation. Before stating the theorem, recall that the support of $x^a = x_1^{a_1} \cdots x_n^{a_n}$, denoted by $\text{supp}(x^a)$, is the set of $x_i$ such that $a_i > 0$. The colon ideal $(x^a : x^b)$ is the set of $f$ in $R$ such that $fx^b$ is in $(x^a)$. The colon ideal $(I(C)^2 : x_{e_i})$ is defined similarly.

**Theorem 2.9.** The following conditions are equivalent:

1. $C$ is unmixed.
2. For any two edges $e \neq e'$ and for any two distinct vertices $x \in e$, $y \in e'$ contained in some $e_i$, one has that $(e \setminus \{x\}) \cup (e' \setminus \{y\})$ contains an edge.
3. For any two edges $e \neq e'$ and for any $T \subset e_i$ such that $xT$ divides $x_ex_{e'}$, one has that $\text{supp}(x_ex_{e'}/x_T)$ contains an edge.
4. For any two edges $e \neq e'$ and for any $e_i$, $(x_ex_{e'} : x_{e_i}) \subset I(C)$.
5. $I(C) = (I(C)^2 : x_{e_1}) + \cdots + (I(C)^2 : x_{e_g}).$

This generalizes to balanced clutters (see Definition 2.10) and beyond an unmixedness criterion of [30] valid only for bipartite graphs (Corollary 2.11).

The notions of minor and c-minor play a prominent role in combinatorial optimization [8]. The precise definitions of these notions can be found in Section 2. Roughly speaking a minor (c-minor) is obtained from $I(C)$ by making any sequence of variables equal to 1 or 0 (resp. equal to 1 only). From the algebraic point of view, a c-minor corresponds to a colon operation or localization of $I(C)$. In Theorem 2.8 we show that for a clutter with a perfect matching of König type, if all c-minors of $C$ have a free vertex, i.e., a vertex that occurs in one edge only, and $C$ is unmixed, then $\Delta_C$ is pure shellable. This complements a result of [27] showing that if all minors of an arbitrary clutter $C$ have a free vertex, then $\Delta_C$ is shellable. Using this free vertex property, we show in Theorem 2.16 that if for any two edges $f_1, f_2$ of $C$ and for any $e_i$, one has that $f_1 \cap e_i \subset f_2 \cap e_i$ or
$f_2 \cap e_i \subset f_1 \cap e_i$, then $\Delta_C$ is pure shellable. Note that this ordering property on the edges implies $C$ is unmixed, as is seen in Theorem 2.13.

An additional property is needed to guarantee that an unmixed clutter will have the above ordering property. Let $A = (a_{ij})$ be the incidence matrix of $C$. Recall that $a_{ij} = 1$ if $x_i \in g_j$ and $a_{ij} = 0$ otherwise, where $g_1, \ldots, g_q$ are the edges of $C$. In Theorem 2.12 we assume that $C$ has no cycles of length 3 or 4, i.e., $A$ has no square submatrix of order 3 or 4 with exactly two 1’s in each row and column, and then show that if $C$ is unmixed, then for any two edges $f_1, f_2$ of $C$ and for any $e_i$, one has that $f_1 \cap e_i \subset f_2 \cap e_i$ or $f_2 \cap e_i \subset f_1 \cap e_i$. This ordering property was shown to hold for the clutter of facets of any unmixed simplicial tree [13, Remark 7.2, Corollary 7.8]. Thus our result is a wide generalization of this fact because simplicial trees are acyclic clutters [17]. In addition, when $C$ is unmixed and has no cycles of length three or four, we show that $\Delta_C$ is pure shellable (Theorem 2.15) and that $e_i$ has a free vertex for all $i$ (Proposition 2.14). Then we give a far reaching generalization of Faridi’s characterization of unmixed simplicial trees [13] (see Corollary 2.19) and show some applications of these results to totally balanced clutters (Corollary 2.20).

In Section 3 we introduce the notion of an admissible clutter. The notion of an admissible clutter was inspired by a certain ordering condition that occurs in a Cohen-Macaulay criterion for bipartite graphs of Herzog and Hibi [16] (see condition (h1) below). We show that any complete admissible clutter is unmixed (Proposition 3.6) and that the edge ideal of any complete admissible uniform clutter is the facet ideal of a shellable complex (Theorem 3.7). A clutter is called uniform if all its edges have the same size. It is shown in Lemma 3.10 that complete admissible uniform clutters are closed under taking Alexander duals. This allows us to prove Theorem 3.12: If $C$ is a complete admissible uniform clutter, then $R/I(C)$ is Cohen-Macaulay and has a linear resolution.

An interesting problem that remains unsolved is whether an unmixed admissible clutter is Cohen-Macaulay (Conjecture 3.5). For bipartite graphs this problem has a positive answer (Theorem 4.11 [16]).

Section 4 is devoted to bipartite graphs with a perfect matching of König type. An unmixed bipartite graph without isolated vertices will always have this type of matching by König’s theorem [23]. Bipartite Cohen-Macaulay graphs have been studied in [6, 11, 16, 29]. In [11] it is shown that $G$ is a Cohen-Macaulay graph if and only if $\Delta_G$ is pure shellable. In [27] a classification of all sequentially Cohen-Macaulay bipartite graphs is given. In particular, it is shown that $\Delta_G$ is shellable if and only if $R/I(G)$ is sequentially Cohen-Macaulay.

Let $G$ be a bipartite graph and let $V_1 = \{x_1, \ldots, x_g\}$ and $V_2 = \{y_1, \ldots, y_g\}$ be a bipartition of $G$ such that $\{x_i, y_k\} \in E(G)$ for all $i$. We examine the conditions (h1): “if $\{x_i, y_j\} \in E(G)$, then $i \leq j$”, and (h2): “if $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are in $E(G)$ and $i < j < k$, then $\{x_i, y_k\} \in E(G)$” that occur in the Herzog and Hibi criterion for Cohen-Macaulay bipartite graphs [16]. See Theorem 4.11 for a precise statement of this criterion. Some characterizations of these conditions have been shown by Yassemi (personal communication), and by Carrà Ferro and Ferrarello [6]. These conditions have also been examined in [27] from the point of view of digraphs following ideas introduced in [6]. Our main result of Section 4 shows that condition (h1) holds if and only if the subcomplex generated by the facets of
maximum dimension of $\Delta_C$ is shellable (Theorem 4.3). We recover a result of [30] describing all unmixed bipartite graphs in combinatorial terms (Corollary 4.2). In particular it follows that in the Herzog and Hibi criterion (Theorem 4.1) we can replace condition (h2) by condition (h2): “$G$ is unmixed”. In Corollary 4.5 we give a variation of this criterion.

The natural generalization of a bipartite graph is a balanced clutter, i.e., a clutter without odd cycles. It turns out that the ordering criterion that Herzog and Hibi used to classify Cohen-Macaulay bipartite graphs does not extend to Cohen-Macaulay balanced clutters (Example 4.6).

2. Shellable clutters with a perfect matching

Let $C$ be a clutter on the vertex set $X = \{x_1, \ldots, x_n\}$ and let $I = I(C)$ be its edge ideal. A contraction (resp. deletion) of $I$ is an ideal of the form $(I: x_i)$ (resp. $J = I \cap K[x_1, \ldots, \hat{x_i}, \ldots, x_n]$) for some $x_i$, where $(I: x_i) := \{f \in R \mid fx_i \in I\}$ is the standard colon operation in ideal theory. The ideal $I$ is regarded as a contraction. The clutter associated to the square-free monomial ideal $(I: x_i)$ (resp. $J$) is denoted by $C/x_i$ (resp. $C \setminus x_i$). A c-minor (resp. d-minor) of $I$ is an ideal obtained from $I$ by a sequence of contractions (resp. deletions). If a c-minor $I'$ contains a variable $x_i$ and we remove this variable from $I'$, we still consider the new ideal a c-minor of $I$. A minor of $I$ is an ideal obtained from $I$ by a sequence of deletions and contractions in any order. A minor (resp. c-minor) of $C$ is any clutter that correspond to a minor (resp. c-minor) of $I$. This terminology is consistent with that of [8, p. 23]. A vertex $x$ of $C$ is called isolated if $x$ does not occur in any edge of $C$. A subset $C \subset X$ is a minimal vertex cover of the clutter $C$: (c1) every edge of $C$ contains at least one vertex of $C$, and (c2) there is no proper subset of $C$ with the first property. If $C$ only satisfies condition (c1), then $C$ is called a vertex cover of $C$. Recall that $p$ is a minimal prime of $I = I(C)$ if and only if $p = (C)$ for some minimal vertex cover $C$ of $C$ [29, Proposition 6.1.16]. Thus the primary decomposition of the edge ideal of $C$ is given by

$$I(C) = (C_1) \cap (C_2) \cap \cdots \cap (C_p),$$

where $C_1, \ldots, C_p$ are the minimal vertex covers of $C$. In particular observe that the height of $I(C)$ equals the number of vertices in a minimum vertex cover of $C$. Note that the facets of $\Delta_C$ are $X \setminus C_1, \ldots, X \setminus C_p$. Thus $C$ is unmixed, equivalently $\Delta_C$ is pure, if and only if all minimal vertex covers of $C$ have the same size.

**Definition 2.1.** A perfect matching of König type of $C$ is a collection $e_1, \ldots, e_g$ of pairwise disjoint edges whose union is $X$ and such that $g$ is the height of $I(C)$.

A set of pairwise disjoint edges is called independent and a set of independent edges of $C$ whose union is $X$ is called a perfect matching. A clutter $C$ satisfies the König property if the maximum number of independent edges of $C$ equals the height of $I(C)$. It is rapidly seen that a clutter with a perfect matching of König type has the König property. In Lemma 2.3 we show the converse to be true for unmixed clutters. For uniform clutters, it is easy to check that if $C$ has the König property and a perfect matching, then the perfect matching is of König type. However the next example shows that this converse fails in general.
Example 2.2. Consider the clutter $\mathcal{C}$ with vertex set $X = \{x_1, \ldots, x_9\}$ whose edges are

$$
e_1 = \{x_1, x_2\}, \ e_2 = \{x_3, x_4, x_5, x_6\}, \ e_3 = \{x_7, x_8, x_9\}, \ f_4 = \{x_1, x_3\}, \ f_5 = \{x_2, x_4\}, \ f_6 = \{x_5, x_7\}, \ f_7 = \{x_6, x_8\}.
$$

The edges $e_1, e_2, e_3$ form a perfect matching, $f_4, f_5, f_6, f_7$ are independent edges, and $\text{ht} I(\mathcal{C}) = 4$. Thus $\mathcal{C}$ has the König property, but $\mathcal{C}$ has no perfect matching of König type.

Lemma 2.3. If $\mathcal{C}$ is an unmixed clutter with the König property and without isolated vertices, then $\mathcal{C}$ has a perfect matching of König type.

Proof. Let $X$ be the vertex set of $\mathcal{C}$. There are $e_1, \ldots, e_g$ independent edges of $\mathcal{C}$, where $g$ is the height of $I(\mathcal{C})$. If $e_1 \cup \cdots \cup e_g \subseteq X$, pick $x_r \in X \setminus (e_1 \cup \cdots \cup e_g)$. Since the vertex $x_r$ occurs in some edge of $\mathcal{C}$, there is a minimal vertex cover $C$ containing $x_r$. Thus using that $e_1, \ldots, e_g$ are mutually disjoint we conclude that $C$ contains at least $g + 1$ vertices, a contradiction.

Notation. As usual, we will use $x^a$ as an abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. The support of a monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$ is given by $\text{supp}(x^a) = \{x_i \mid a_i > 0\}$.

Proposition 2.4. Let $\mathcal{C}$ be an unmixed clutter with a perfect matching $e_1, \ldots, e_g$ of König type and let $C_1, \ldots, C_r$ be any collection of minimal vertex covers of $\mathcal{C}$. If $\mathcal{C}'$ is the clutter associated to $I' = \cap_{i=1}^r (C_i)$, then $\mathcal{C}'$ has a perfect matching $e_1', \ldots, e_g'$ of König type such that: (a) $e_i' \subseteq e_i$ for all $i$, and (b) every vertex of $e_i \setminus e_i'$ is isolated in $\mathcal{C}'$.

Proof. We denote the minimal set of generators of the ideal $I = I(\mathcal{C})$ by $G(I)$. There are monomials $x^{e_1}, \ldots, x^{e_g}$ in $G(I)$ so that $\text{supp}(x^{e_i}) = e_i$ for $i = 1, \ldots, g$. Since $x^{e_i}$ is in $I$ and $I \subseteq I'$, there is $e_i' \subseteq e_i$ such that $e_i'$ is an edge of $\mathcal{C}'$. Let $x$ be any vertex in $e_i \setminus e_i'$. If $x$ is not isolated in $\mathcal{C}'$, there would a minimal vertex cover $C_k$ of $\mathcal{C}'$ containing $x$. As $C_k$ contains a vertex of $e_j'$ for each $1 \leq j \leq g$ and since $e_1', \ldots, e_g'$ are pairwise disjoint, we get that $C_k$ contains at least $g + 1$ vertices, a contradiction. Thus (a) and (b) are satisfied. Clearly $g$ is the height of $I'$ by construction of $I'$. Let $X'$ be the vertex set of $\mathcal{C}'$. To finish the proof we need only show that $X' = e_1' \cup \cdots \cup e_g'$. Let $x \in X'$, then $x \in e_i$ for some $i$ and $x$ belongs to at least one edge of $\mathcal{C}'$. By part (b) we get that $x \in e_i'$, as required.

Remark 2.5. Let $C_1, \ldots, C_p$ be the minimal vertex covers of $\mathcal{C}$. Since $I(\mathcal{C})$ is equal to $\cap_{i=1}^p (C_i)$, one has $(I(\mathcal{C}) : x_j) = \cap_{x_j \notin C_i} (C_i)$ for any vertex $x_j \notin I(\mathcal{C})$. Under the assumptions of Proposition 2.4 we get that $\mathcal{C}/x_j$ has a perfect matching $e_1', \ldots, e_g'$ satisfying (a) and (b).

Lemma 2.6. Let $\mathcal{C}$ be an unmixed clutter with a perfect matching $e_1, \ldots, e_g$ of König type and let $I = I(\mathcal{C})$. If $e_1 = \{x_1, \ldots, x_r\}$, then

$$\bigcap_{x_1 \in C_i} ((I : x_2) : x_3) : x_4) : \cdots : x_{r-1}) : x_r),$$

where $C_1, \ldots, C_p$ are the minimal vertex covers of $\mathcal{C}$.
Proof. Let $I'$ denote the ideal on the right hand side of the equality. Then $I'$ is obtained from $I$ by making $x_i = 1$ for $i = 2,\ldots, r$, i.e., if $x^n_1,\ldots, x^n_r$ generate $I$ and we make $x_i = 1$ for $i = 2,\ldots, r$ in $x^n_1,\ldots, x^n_r$, we obtain a generating set of $I'$. Notice that $I' = (I; x_2\cdots x_r)$ by the definition of the colon operation. Take a monomial $x^a = x^{a_1}_1 x^{a_2}_{r+1} \cdots x^{a_n}_n$ in $I'$. We may assume $a_1 = 0$, otherwise $x^a$ is already in the left hand side. Then $x_2\cdots x_r x^{a_{r+1}}_r \cdots x^{a_n}_n$ is in $I$. Let $C_i$ be any minimal vertex cover of $C$ containing $x_1$. Observe that $C_i$ cannot contain $x_j$ for $2 \leq j \leq r$. Indeed if $x_j \in C_i$ for some $2 \leq j \leq r$, then $C_i$ would contain $\{x_1, x_j\}$ plus at least one vertex of each edge in the collection $e_2,\ldots, e_g$, a contradiction because $C_i$ has exactly $g$ vertices. Hence, using that $x_2\cdots x_r x^{a_{r+1}}_r \cdots x^{a_n}_n$ is in $I$, we get that $x^{a_{r+1}}_r \cdots x^{a_n}_n$ is in $(C_i)$. Consequently $x^a$ is in the left hand side of the equality. Conversely let $x^a$ be a minimal generator in the left hand side of the equality. Then $x^a \in (C_i)$ whenever $x_1 \in C_i$. If $x_1 \notin C_i$, then $x_2\cdots x_r \in (C_i)$ since $C_i$ covers $e_1$. Thus $x^a x_2\cdots x_r \in (C_i)$ for all $i$, and so $x^a x_2\cdots x_r \in \cap_{i=1}^g (C_i) = I$. Thus $x^a$ is in the right hand side of the equality. \hfill $\Box$

Definition 2.7. A simplicial complex $\Delta$ is shellable if the facets (maximal faces) of $\Delta$ can be ordered $F_1,\ldots, F_s$ such that for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $\ell \in \{1,\ldots, j-1\}$ with $F_j \setminus F_\ell = \{v\}$. We call $F_1,\ldots, F_s$ a shelling of $\Delta$.

The above definition of shellable is due to Björner and Wachs [3]. Originally, the definition of shellable also required that the simplicial complex be pure, that is, all the facets have same dimension. We will say $\Delta$ is pure shellable if it also satisfies this hypothesis. Because $I = I(C)$ is a square-free monomial ideal, it also corresponds to a simplicial complex via the Stanley-Reisner correspondence [26]. We let $\Delta_C$ represent this simplicial complex. Note that $F$ is a facet of $\Delta_C$ if and only if $X \setminus F$ is a minimal vertex cover of $C$. For use below we say $x_i$ is a free variable (resp. free vertex) of $I$ (resp. $C$) if $x_i$ only appears in one of the monomials of $G(I)$ (resp. in one of the edges of $C$), where $G(I)$ denotes the minimal set of generators of the monomial ideal $I = I(C)$.

If $C$ has the free vertex property, i.e., all minors of $C$ have a free vertex, then $\Delta_C$ is shellable [27]. We complement this result by showing that if all c-minors have a free vertex and $C$ is unmixed, then $\Delta_C$ is shellable.

Theorem 2.8. Let $C$ be a clutter with a perfect matching $e_1,\ldots, e_g$ of König type. If all c-minors of $C$ have a free vertex and $C$ is unmixed, then $\Delta_C$ is pure shellable.

Proof. The proof is by induction on the number of vertices. We may assume that $C$ is a non-discrete clutter, i.e., it contains an edge with at least two vertices. Let $z$ be a free vertex of $C$ and let $C_1,\ldots, C_p$ be the minimal vertex covers of $C$. We may also assume that $z \in e_m$ for some $e_m = \{z_1,\ldots, z_r\}$, with $r \geq 2$. For simplicity of notation assume that $z = z_1$ and $m = g$. Consider the clutters $C_1$ and $C_2$ associated with

\begin{equation}
I_1 = \bigcap_{z_1 \notin C_i} (C_i) \text{ and } I_2 = \bigcap_{z_1 \in C_i} (C_i)
\end{equation}

respectively. By Proposition 2.4 the clutter $C_2$ has a perfect matching $e'_1,\ldots, e'_g$ of König type such that: (a) $e'_i \subset e_i$ for all $i$, and (b) every vertex $x$ of $e_i \setminus e'_i$ is
isolated in $C_2$, i.e., $x$ does not occur in any edge of $C_2$. In particular all vertices of $e_g \setminus \{z_1\}$ are isolated vertices of $C_2$. Similar statements hold for $C_1$ because of Proposition 2.4. By Lemma 2.6 and Remark 2.3 we get

$$I_1 = (I : z_1) \text{ and } I_2 = \left( ((I : z_2) : z_3) : z_4 \cdot \cdot \cdot : z_{r-1}) : z_r \right),$$

that is, $C_1 = C/z_1$ and $C_2 = C/\{z_2, \ldots , z_r\}$. Hence the ideals $I_1$ and $I_2$ are $c$-minors of $I$. The number of vertices of $C_i$ is less than that of $C$ for $i = 1, 2$. Thus $\Delta_{C_1}$ and $\Delta_{C_2}$ are shellable by the induction hypothesis. Consider the clutter $C'_i$ whose edges are the edges of $C_i$ and whose vertex set is $X$. The minimal vertex covers of $C'_i$ are exactly the minimal vertex covers of $C_i$. Thus it follows that $\Delta_{C'_i}$ is shellable for $i = 1, 2$. Let $F_1, \ldots , F_s$ be the facets of $\Delta_C$ that contain $z_1$ and let $G_1, \ldots , G_t$ be the facets of $\Delta_C$ that do not contain $z_1$. Notice that the edge ideals of $C_i$ and $C'_i$ coincide, the vertex set of $C'_i$ is equal to the vertex set of $C_i$, and $I = I_1 \cap I_2$. Hence from Eq. (2.1) we get that $F_1, \ldots , F_s$ are the facets of $\Delta_{C'_1}$ and $G_1, \ldots , G_t$ are the facets of $\Delta_{C'_2}$. By the induction hypothesis we may assume $F_1, \ldots , F_s$ is a shelling of $\Delta_{C'_1}$ and $G_1, \ldots , G_t$ is a shelling of $\Delta_{C'_2}$. We now prove that

$$F_1, \ldots , F_s, G_1, \ldots , G_t$$

is a shelling of $\Delta_C$. We need only show that given $G_j$ and $F_i$ there is $v \in G_j \setminus F_i$ and $F_\ell$ such that $G_j \setminus F_\ell = \{v\}$. We can write

$$G_j = X \setminus C_j \text{ and } F_i = X \setminus C_i,$$

where $C_j$ (resp. $C_i$) is a minimal vertex cover of $C$ containing $z_1$ (resp. not containing $z_1$). Notice that $z_2, \ldots , z_r$ are not in $C_j$ because $e_1, \ldots , e_g$ is a perfect matching and $|C_j| = g$. Thus $z_2, \ldots , z_r$ are in $G_j$. Since $z_1 \in F_i$ and $F_i$ cannot contain the edge $e_g$, there is a $z_k$ so that $z_k \notin F_i$ and $k \neq 1$. Set $v = z_k$ and $F_\ell = (G_j \setminus \{z_k\}) \cup \{z_1\}$. Clearly $F_\ell$ is an independent vertex set because $z_1$ is a free vertex in $e_g$ and $G_j$ is an independent vertex set. Thus $F_\ell$ is a facet because $C$ is unmixed. To complete the proof observe that $G_j \setminus F_\ell = \{z_k\}$. 

For use below we set $x_e = \prod_{i \in e} x_i$ for any $e \subset X$. Next we give a characterization of the unmixed property of $C$. This characterization can be formulated combinatorially or algebraically.

**Theorem 2.9.** Let $C$ be a clutter with a perfect matching $e_1, \ldots , e_g$ of König type and let $I = I(C)$ be its edge ideal. Then the following are equivalent:

(a) $C$ is unmixed.

(b) For any two edges $e \neq e'$ and for any two distinct vertices $x \in e, y \in e'$ contained in some $e_i$, one has that $(e \setminus \{x\}) \cup (e' \setminus \{y\})$ contains an edge.

(c) For any two edges $e \neq e'$ and for any $T \subset e_i$ such that $x_T$ divides $x_e x_{e'}$, one has that supp$(x_e x_{e'} / x_T)$ contains an edge.

(d) For any two edges $e \neq e'$ and for any $e_i, (x_e x_{e'} : x_{e_i}) \subset I$.

(e) $I = (I^2 : x_{e_1}) + \cdot \cdot \cdot + (I^2 : x_{e_g})$.

**Proof.** (a) $\Rightarrow$ (c): We may assume $i = 1$. Let $T$ be a subset of $e_1$ such that $x_T$ divides $x_e x_{e'}$. If $T \subset e$, then $e'$ is an edge contained in $S = \text{supp}(x_e x_{e'} / x_T)$ and there is nothing to show. The proof is similar if $T \subset e'$. So we can define $T_1 = e \cap T$ and $T_2 = T \setminus T_1$ and we may assume neither $T_1$ nor $T_2$ is empty. Note that $T_1 \subset e$ and $T_2 \subset e'$. In fact, $T_2 \subset T \cap e'$, but equality does not necessarily
hold. Notice that \( S = (e \setminus T_1) \cup (e' \setminus T_2) \). If \( S \) does not contain an edge, its complement contains a minimal vertex cover \( C \). We use \( c \) to denote complement. Then

\[
C \subset X \setminus S = S^c = (e \setminus T_1)^c \cap (e' \setminus T_2)^c = (e^c \cup T_1) \cap (e'^c \cup T_2).
\]

Now \( C \cap e \neq \emptyset \), so there is an \( x \in C \cap e \). Then \( x \in e^c \cup T_1 \). This forces \( x \in T_1 \). Similarly there is a \( y \in C \cap e' \), and so \( y \in e'^c \cap T_2 \). Thus \( y \in T_2 \). By the definition of \( T_2 \), \( x \neq y \). To derive a contradiction pick \( z_k \in e_k \cap C \) for \( k \geq 2 \) and notice that \( x, y, z_2, \ldots, z_g \) is a set of \( g + 1 \) distinct vertices in \( C \), which is impossible because \( C \) is unmixed.

\( (e) \Rightarrow (b) \): Let \( x \in e \) and \( y \in e' \) be two distinct vertices contained in some \( e_i \). Let \( T = \{ x, y \} \). Then \( x_T \) divides \( x_e x_{e'} \) and

\[
S = \text{supp}(x_e x_{e'}/x_T) \subset (e \setminus \{ x \}) \cup (e' \setminus \{ y \})).
\]

By (c), \( S \) contains an edge. Thus \((e \setminus \{ x \}) \cup (e' \setminus \{ y \}) \) contains an edge.

\( (b) \Rightarrow (a) \): Let \( C \) be a minimal vertex cover of \( C \). Since the matching is perfect, there is a partition:

\[
C = (C \cap e_1) \cup \cdots \cup (C \cap e_g).
\]

Hence it suffices to prove that \( |C \cap e_i| = 1 \) for all \( i \). We proceed by contradiction. For simplicity of notation assume \( i = 1 \) and \( |C \cap e_1| \geq 2 \). Pick \( x \neq y \) in \( C \cap e_1 \). Since \( C \) is minimal, there are edges \( e, e' \) such that

\[
eq \cap (C \setminus \{ x \}) = \emptyset \text{ and } e' \cap (C \setminus \{ y \}) = \emptyset.
\]

Clearly \( x \in e \), \( y \in e' \), and \( e \neq e' \) because \( y \notin e \). Then by hypothesis the set

\[
S = (e \setminus \{ x \}) \cup (e' \setminus \{ y \}) \text{ contains an edge } e''.
\]

Take \( z \in e'' \cap C \), then \( z \in e \setminus \{ x \} \) or \( z \in e' \setminus \{ y \} \), which is impossible by Eq. (2.2).

\( (c) \Rightarrow (d) \): Let \( x^a \in (x_e x_{e'} : x_{e_i}) \) be a monomial generator of the colon ideal. Then \( x^a x_{e_i} = m x_e x_{e'} \) for some monomial \( m \). Let \( T \subset e_i \) be maximal such that \( x_T \) divides \( x_e x_{e'} \). Then \( x_{e_i \setminus T} \) divides \( m \), and \( x^a = (m/x_{e_i \setminus T})(x_e x_{e'}/x_T) \). Since \( \text{supp}(x_e x_{e'}/x_T) \) contains an edge, we have \( x_e x_{e'}/x_T \in I \). Thus \( x^a \in I \) as desired.

\( (d) \Rightarrow (c) \): Suppose \( T \subset e_i \) is such that \( x_T \) divides \( x_e x_{e'} \). Then

\[
(x_e x_{e'}/x_T)x_{e_i} = x_e x_{e'} x_{e_i \setminus T},
\]

and so \((x_e x_{e'}/x_T) \subset (x_e x_{e'} : x_{e_i}) \subset I \). Thus \((x_e x_{e'}/x_T) \) is a multiple of a monomial generator of \( I \). Hence \( \text{supp}(x_e x_{e'}/x_T) \) contains an edge.

\( (e) \Rightarrow (d) \): If equality in (e) holds, then \((I^2 : x_{e_i}) = I \) for all \( i \). Hence from the inclusion \((I^2 : x_{e_i}) \subset I \) we rapidly obtain that condition (d) holds.

\( (d) \Rightarrow (e) \): It suffices to verify that \((I^2 : x_{e_i}) = I \) for all \( i \). Since \( I \) is clearly contained in \((I^2 : x_{e_i}) \), we need only show the inclusion \((I^2 : x_{e_i}) \subset I \). Take \( x^a \in (I^2 : x_{e_i}) \), then \( x^a x_{e_i} = m x_e x_{e'} \) for some edges \( e, e' \) of \( C \) and some monomial \( m \). If \( e \neq e' \), then by hypothesis \( x^a \in (x_e x_{e'} : x_{e_i}) \subset I \), i.e., \( x^a \in I \). If \( e = e' \), then \( x^a x_{e_i} = m x_e^2 \) Thus \( x_e \) divides \( x^a \) because \( x_{e_i} \) is a square-free monomial, but this means that \( x^a \in I \), as required.

**Definition 2.10.** Let \( A \) be the incidence matrix of a clutter \( C \). A clutter \( C \) has a cycle of length \( r \) if there is a square sub-matrix of \( A \) of order \( r \geq 3 \) with exactly two 1’s in each row and column. A clutter without odd cycles is called balanced and an acyclic clutter is called totally balanced.
This definition of cycle is equivalent to the usual definition of cycle in the sense of hypergraph theory \cite{2, 17}. All minors of a balanced clutter have the König property \cite{23}. If \( G \) is a graph, then \( G \) is balanced if and only if \( G \) is bipartite and \( G \) is totally balanced if and only if \( G \) is a forest.

The following result extends—to clutters with the König property—an unmixedness criterion of \cite{30} valid for bipartite graphs. As a byproduct we obtain a full description of all unmixed balanced clutters.

**Corollary 2.11.** Let \( C \) be a clutter with the König property. Then \( C \) is unmixed if and only if there is a perfect matching \( e_1, \ldots, e_g \) of König type such that for any two edges \( e \neq e' \) and for any two distinct vertices \( x \in e, y \in e' \) contained in some \( e_i \), one has that \((e \setminus \{x\}) \cup (e' \setminus \{y\}) \) contains an edge.

**Proof.** \( \Rightarrow \) Assume that \( C \) is unmixed. By Theorem \ref{2.9} it suffices to observe that any unmixed clutter with the König property and without isolated vertices has a perfect matching of König type, see Lemma \ref{2.3}.

\[ \Leftarrow \] This implication follows at once from Theorem \ref{2.9}. \( \square \)

The following ordering property was shown to hold for the clutter of facets of any unmixed simplicial tree \cite[Remark 7.2, Corollary 7.8]{13}. The next result is a wide generalization of this fact because unmixed simplicial trees are acyclic \cite[being balanced they have the König property \cite[Theorem 83.1]{23}, and by Lemma \ref{2.3} they have a perfect matching.

**Theorem 2.12.** Let \( C \) be a clutter with a perfect matching \( e_1, \ldots, e_g \) of König type. If \( C \) has no cycles of length 3 or 4 and \( C \) is unmixed, then for any two edges \( f_1, f_2 \) of \( C \) and for any \( e_i \), one has that \( f_1 \cap e_i \subset f_2 \cap e_i \) or \( f_2 \cap e_i \subset f_1 \cap e_i \).

**Proof.** For simplicity assume \( i = 1 \). We proceed by contradiction. Assume there are \( x_1 \in f_1 \cap e_1 \setminus f_2 \cap e_1 \) and \( x_2 \in f_2 \cap e_1 \setminus f_1 \cap e_1 \). As \( C \) is unmixed, by Theorem \ref{2.9} \( \text{b) there is an edge } e \text{ of } C \text{ such that} \)

\[ e \subset (f_1 \setminus \{x_1\}) \cup (f_2 \setminus \{x_2\}) = (f_1 \cup f_2) \setminus \{x_1, x_2\}. \]

Since \( e \not\subset e_1 \), there is \( x_3 \in e \setminus e_1 \). Then either \( x_3 \in f_1 \) or \( x_3 \in f_2 \). Without loss of generality we may assume \( x_3 \in f_1 \setminus e_1 \). For use below we denote the incidence matrix of \( C \) by \( A \).

Case(I): \( x_3 \in f_2 \). Then the matrix

\[
\begin{pmatrix}
 f_1 & f_2 & e_1 \\
 x_1 & 1 & 0 & 1 \\
x_2 & 0 & 1 & 1 \\
x_3 & 1 & 1 & 0 \\
\end{pmatrix}
\]

is a submatrix of \( A \), a contradiction.

Case(II): \( x_3 \notin f_2 \). Notice that \( e \not\subset f_1 \), otherwise \( e = f_1 \) which is impossible because \( x_1 \in f_1 \setminus e \). Thus there is \( x_4 \in e \setminus f_1 \) and \( x_4 \in (e \cap f_2) \setminus f_1 \).
Subcase (II.a): $x_4 \in e_1$. Then the matrix

\[
\begin{pmatrix}
  f_1 & e & e_1 \\
  x_1 & 1 & 0 & 1 \\
  x_2 & 1 & 1 & 0 \\
  x_4 & 0 & 1 & 1
\end{pmatrix}
\]

is a submatrix of $A$, a contradiction.

Subcase (II.b): $x_4 \notin e_1$. Then the matrix

\[
\begin{pmatrix}
  f_1 & e & f_2 & e_1 \\
  x_1 & 1 & 0 & 0 & 1 \\
  x_2 & 0 & 0 & 1 & 1 \\
  x_3 & 1 & 1 & 0 & 0 \\
  x_4 & 0 & 1 & 1 & 0
\end{pmatrix}
\]

is a submatrix of $A$, a contradiction.

Conversely, the above ordering property implies unmixedness. Note that the assumption on the incidence matrix is not needed for this implication.

**Theorem 2.13.** Let $C$ be a clutter with a perfect matching $e_1, \ldots, e_g$ of König type. If for any two edges $f_1, f_2$ of $C$ and for any $e_i$, one has that $f_1 \cap e_i \subset f_2 \cap e_i$ or $f_2 \cap e_i \subset f_1 \cap e_i$, then $C$ is unmixed.

**Proof.** To show that $C$ is unmixed it suffices to verify condition (b) of Theorem 2.9. Let $f_1 \neq f_2$ be two edges and let $x \in f_1, y \in f_2$ be two distinct vertices contained in some $e_i$. For simplicity we assume $i = 1$. Set $B = (f_1 \setminus \{x\}) \cup (f_2 \setminus \{y\})$. Then $f_2 \cap e_1 \subset f_1 \cap e_1$ or $f_1 \cap e_1 \subset f_2 \cap e_1$. In the first case we have that $f_2 \subset B$. Indeed let $z \in f_2$. If $z \neq y$, then $z \in f_2 \setminus \{y\} \subset B$, and if $z = y$, then $z \in f_2 \cap e_1 \subset f_1 \cap e_1$ and $z \neq x$, i.e., $z \in f_1 \setminus \{x\} \subset B$. In the second case $f_1 \subset B$. \qed

**Proposition 2.14.** Let $C$ be an unmixed clutter without cycles of length 3 or 4. If $e_1, \ldots, e_g$ is a perfect matching of $C$ of König type, then $e_i$ has a free vertex for all $i$.

**Proof.** Fix an integer $i$ in $[1, g]$. We may assume that $e_i$ has at least one non-free vertex. Consider the set of edges:

\[ F = \{ f \in E(C) | e_i \cap f \neq \emptyset; f \neq e_i \}. \]

By Theorem 2.12, the edges of $F$ can be listed as $f_1, \ldots, f_r$ so that they satisfy the inclusions

\[ f_1 \cap e_i \subset f_2 \cap e_i \subset \cdots \subset f_r \cap e_i \nsubseteq e_i. \]

Thus any vertex of $e_i \setminus (f_r \cap e_i)$ is a free vertex of $e_i$. \qed

**Theorem 2.15.** Let $C$ be an unmixed clutter with a perfect matching $e_1, \ldots, e_g$ of König type. If $C$ has no cycles of length 3 or 4, then $\Delta_C$ is pure shellable.

**Proof.** All hypothesis are preserved under contractions, i.e., under c-minors. This follows from Remark 2.5 and the fact that the incidence matrix of a contraction of $C$ is a submatrix of the incidence matrix of $C$. Thus by Proposition 2.14 any c-minor has a free vertex and the result follows from Theorem 2.8. \qed
**Theorem 2.16.** Let $\mathcal{C}$ be a clutter with a perfect matching $e_1, \ldots, e_g$ of König type. If for any two edges $f_1, f_2$ of $\mathcal{C}$ and for any edge $e_i$ of the perfect matching, one has that $f_1 \cap e_i \subset f_2 \cap e_i$ or $f_2 \cap e_i \subset f_1 \cap e_i$, then $\Delta_{\mathcal{C}}$ is pure shellable.

**Proof.** Notice the following two assertions: (i) $\mathcal{C}$ is an unmixed clutter, which follows from Theorem 2.13, and (ii) $e_i$ has a free vertex for all $i$, which follows from the proof of Proposition 2.14. Thus by Theorem 2.8 we need only show that any $c$-minor has a free vertex. By (ii) it suffices to show that our hypotheses are closed under contractions. Let $x$ be a vertex of $\mathcal{C}$ and let $\mathcal{C}' = \mathcal{C}/x$. By Remark 2.5, we get that $\mathcal{C}/x$ has a perfect matching $e'_1, \ldots, e'_g$ satisfying: (a) $e'_i \subset e_i$ for all $i$, and (b) every vertex of $e_i \setminus e'_i$ is isolated in $\mathcal{C}'$. Let $e, e'$ be two edges of $\mathcal{C}'$ and let $e'_i$ be an edge of the perfect matching of $\mathcal{C}'$. There are edges $f, f'$ of $\mathcal{C}$ such that one of the following is satisfied: $e = f$ and $e' = f' \setminus \{x\}$, $e = f \setminus \{x\}$ and $e' = f' \setminus \{x\}$, and $e = f$ and $e' = f'$. We may assume $f \cap e_i \subset f' \cap e_i$. To finish the proof we now show that $e \cap e_i \subset e' \cap e'_i$. Take $z \in e \cap e_i$. Then $z \in f \cap e_i$ and consequently $z \in f' \cap e'_i$. Since $x \notin e'_i$, one has $z \neq x$. It follow that $z \in e' \cap e'_i$.

Let $G$ be a graph and let $V$ be its vertex set. For use below consider the graph $G \cup W(V)$ obtained from $G$ by adding new vertices $\{y_i \mid x_i \in V\}$ and new edges $\{\{x_i, y_i\} \mid x_i \in V\}$. The edges $\{x_i, y_i\}$ are called whiskers. The notion of a whisker was introduced in [24, p. 392].

**Corollary 2.17.** If $G$ is a graph and $G' = G \cup W(V)$, then $\Delta_{G'}$ is pure shellable.

**Proof.** It follows at once from Theorem 2.16. Indeed if $V = \{x_1, \ldots, x_n\}$, then $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$ is a perfect matching of $G'$ satisfying the ordering condition in Theorem 2.16.

Recall that a clutter $\mathcal{C}$ is called **totally balanced** if $\mathcal{C}$ is acyclic and that a graph $G$ is totally balanced if and only if $G$ is a forest. Faridi [12] introduced the notion of a leaf for a simplicial complex $\Delta$. Precisely, a facet $F$ of $\Delta$ is a leaf if $F$ is the only facet of $\Delta$, or there exists a facet $G \neq F$ in $\Delta$ such that $F \cap F' \subset F \cap G$ for all facets $F' \neq F$ in $\Delta$. A simplicial complex $\Delta$ is a simplicial forest if every nonempty subcollection, i.e., a subcomplex whose facets are also facets of $\Delta$, of $\Delta$ contains a leaf. Recently Herzog, Hibi, Trung and Zheng [17, Theorem 3.2] showed that $\mathcal{C}$ is the clutter of the facets of a simplicial forest if and only if $\mathcal{C}$ is a totally balanced clutter. Soleyman Jahan and X. Zheng [25, Corollary 3.1] showed that $\mathcal{C}$ is a totally balanced clutter if and only if $\mathcal{C}$ satisfies the free vertex property. Altogether one has:

**Proposition 2.18.** [17, 25] Let $\mathcal{C}$ be a clutter. Then the following conditions are equivalent

(a) $\mathcal{C}$ is the clutter of the facets of a simplicial forest.
(b) $\mathcal{C}$ has the vertex free property.
(c) $\mathcal{C}$ is totally balanced.

Thus some of the results in [13] can be examined using the combinatorial structure of totally balanced clutters [23, Chapter 83, p. 1439–1451]. Since totally balanced clutters are acyclic and satisfy the König property [23], the next result
generalizes the Cohen-Macaulay criterion for trees given in [28, Theorem 2.4] and is a far reaching generalization of Faridi’s characterization of unmixed simplicial trees [13, Remark 7.2, Corollary 7.8].

Corollary 2.19. Let $C$ be a clutter with the König property and without cycles of length 3 or 4. Then any of the following conditions are equivalent:

(a) $C$ is unmixed.
(b) There is a perfect matching $e_1, \ldots, e_g$, $g = \text{ht } I(C)$, such that $e_i$ has a free vertex for all $i$, and for any two edges $f_1, f_2$ of $C$ and for any edge $e_i$ of the perfect matching, one has that $f_1 \cap e_i \subset f_2 \cap e_i$ or $f_2 \cap e_i \subset f_1 \cap e_i$.
(c) $R/I(C)$ is Cohen-Macaulay.
(d) $\Delta_C$ is a pure shellable simplicial complex.

Proof. Using Lemma 2.3, Theorems 2.12 and 2.13, and Proposition 2.14 it follows readily that conditions (a) and (b) are equivalent. Since (a) is equivalent to (b), from Theorem 2.15 we get that (b) implies (d). That (d) implies (c) and (c) implies (a) are well known properties, see for instance [26, 29].

Next we give some applications to totally balanced clutters. We begin by recalling some notions. Let $A$ be the incidence matrix of a clutter $C$. The matrix $A$ is called perfect if the polytope defined by the system $x \geq 0; xA \leq 1$ is integral, i.e., it has only integral vertices. Here $1$ denotes the vector with all its entries equal to 1. A clique of a graph $G$ is a subset of the set of vertices that induces a complete subgraph. We will also call a complete subgraph of $G$ a clique. The vertex-clique matrix of a graph $G$ is the $\{0, 1\}$-matrix whose rows are indexed by the vertices of $G$ and whose columns are the incidence vectors of the maximal cliques of $G$. Let $G$ be a graph. A colouring of the vertices of $G$ is an assignment of colours to the vertices of $G$ in such a way that adjacent vertices have distinct colours. The chromatic number of $G$ is the minimal number of colours in a colouring of $G$. A graph is perfect if for every induced subgraph $H$, the chromatic number of $H$ equals the size of the largest complete subgraph of $H$. A clutter is called uniform if all its edges have the same size.

Corollary 2.20. Let $C$ be an unmixed totally balanced clutter with vertex set $X$. If $C$ has no isolated vertices and $g$ is the height of $I(C)$, then

(a) [13, Theorem 6.8] $C$ has a perfect matching $e_1, \ldots, e_g$ of König type such that $e_i$ has a free vertex for all $i$.
(b) [27, Corollary 5.4] $\Delta_C$ is a pure shellable simplicial complex.
(c) $C$ is the clutter of maximal cliques of a perfect graph $G$.
(d) The set of non-free vertices of $e_i$ is contained in a maximal clique of $G$.
(e) [14, Proposition 5.8] If $C$ is uniform, there is a partition $X^1, \ldots, X^d$ of $X$ such that any edge of $C$ intersects any $X^i$ in exactly one vertex.

Proof. (a) and (b) follow at once from Corollary 2.19. (c) Let $A$ be the incidence matrix of $C$. According to [11, 23, Corollary 83.1a(vii), 1441] $C$ is balanced if and only if every submatrix of $A$ is perfect. By [7] there is a perfect graph $G$ such that $A$ is the vertex-clique matrix of $G$, i.e., $C$ is the clutter of maximal cliques of $G$. (d) Consider the set $G = \{e_i \cap e| e \in E(C); e \neq e_i\}$.
By Theorem 2.12 the sets in $\mathcal{G}$ can be listed in increasing order
\[ f_1 \cap e_i \subset f_2 \cap e_i \subset \cdots \subset f_r \cap e_i \subset e_i, \]
for some edges $f_1, \ldots, f_r$. Thus $e_i \cap f_r$ is exactly the set of non-free vertices of $e_i$, and $f_r$ is the required maximal clique.

We have included part (d) as one of the properties of totally balanced uniform clutters because it serves as an introduction to the notion of admissible clutter to be defined in the next section.

3. Admissible clutters with a perfect matching

Let $X^1, \ldots, X^d$ and $e_1, \ldots, e_g$ be two partitions of a finite set $X$ such that $|e_i \cap X^j| \leq 1$ for all $i, j$. The variables of the polynomial ring $K[X]$ are linearly ordered by: $x < y$ if $(x \in X^i, y \in X^j, i < j)$ or $(x, y \in X^i, x \in e_k, y \in e_k, k < \ell)$.

Let $e$ be a subset of $X$ of size $k$ such that $|e \cap X^i| \leq 1$ for all $i$. There are unique integers $1 \leq i_1 < \cdots < i_k \leq d$ and integers $j_1, \ldots, j_k \in [1, g]$ such that
\[ \emptyset \neq e \cap X^{i_1} = \{x_1\}, \emptyset \neq e \cap X^{i_2} = \{x_2\}, \ldots, \emptyset \neq e \cap X^{i_k} = \{x_k\} \]
and $x_1 \in e_{j_1}, \ldots, x_k \in e_{j_k}$. We say that $e$ is admissible if $i_1 = 1, i_2 = 2, \ldots, i_k = k$ and $j_1 \leq \cdots \leq j_k$. We can represent an admissible set $e = \{x_1, \ldots, x_k\}$ as $e = x_{j_1}^1 \cdots x_{j_k}^k$, i.e., $x_i = x_{j_i}^i$ and $x_i^j \in X^i \cap e_j$ for all $i$. A monomial $x^a$ is admissible if $\text{supp}(x^a)$ is admissible. A clutter $\mathcal{C}$ is called admissible if $e_1, \ldots, e_g$ are edges of $\mathcal{C}$, $e_i$ is admissible for all $i$, and all other edges are admissible sets not contained in any of the $e_i$’s. We can think of $X^1, \ldots, X^d$ as color classes that color the edges.

**Lemma 3.1.** If $\mathcal{C}$ is an admissible clutter, then $e_1, \ldots, e_g$ is a perfect matching of Kőnig type.

**Proof.** It suffices to prove that $g = \text{ht} I(\mathcal{C})$. Clearly $\text{ht} I(\mathcal{C}) \geq g$ because any minimal vertex cover of $\mathcal{C}$ must contain at least one vertex of each $e_i$ and the $e_i$’s form a partition of $X$. For each $1 \leq i \leq g$ there is $y_i = x_i^1$ so that $e_i \cap X^1 = \{y_i\}$. Since the $e_i$’s form a partition we have the equality
\[ (e_1 \cap X^1) \cup \cdots \cup (e_g \cap X^1) = X^1. \]
Thus $|X^1| = g$. To complete the proof notice that $X^1$ is a vertex cover of $\mathcal{C}$ because all edges of $\mathcal{C}$ are admissible. This shows $\text{ht} I(\mathcal{C}) \leq g$, as required.

Admissible clutters with two color classes $X^1, X^2$ are special types of bipartite graphs. They will be examined in Section 4.

**Example 3.2.** Consider the following balanced admissible clutter with color classes $X^1, X^2, X^3$ and edges $e_1, e_2, e_3, f_1, f_2, f_3$.

\[
\begin{array}{ccc}
X^1 & X^2 & X^3 \\
e_1 & x_1 & y_1 & f_1 & x_1 & y_2 & z_2 \\
e_2 & x_2 & y_2 & z_2 & f_2 & x_1 & y_3 \\
e_3 & x_3 & y_3 & f_3 & x_2 & y_3 \\
\end{array}
\]

This clutter is Cohen-Macaulay, and $e_1, e_2, e_3$ is a perfect matching of Kőnig type.
Example 3.3. The uniform admissible clutters with three color classes

\[ X^1 = \{x_1, \ldots, x_g\}, \quad X^2 = \{y_1, \ldots, y_g\}, \quad X^3 = \{z_1, \ldots, z_g\} \]

are, up to permutation of variables, exactly the clutters with a perfect matching \( e_i = \{x_i, y_i, z_i\} \) for \( i = 1, \ldots, g \) such that all edges of \( \mathcal{C} \) have the form \( \{x_i, y_j, z_k\} \), with \( 1 \leq i \leq j \leq k \leq g \).

Example 3.4. Consider the following admissible uniform clutter with edges \( e_1, e_2, f_1 \), perfect matching \( e_1, e_2 \), and color classes \( X^1, X^2, X^3 \):

\[
\begin{align*}
e_1 & = x_1 y_1 z_1 \\
e_2 & = x_2 y_2 z_2 \\
f_1 & = x_1 y_1 z_2
\end{align*}
\]

This clutter is Cohen-Macaulay.

An examination of the Cohen-Macaulay and unmixed criteria for bipartite graphs (see Theorem 4.1 and Corollary 4.2) suggests the following conjecture.

Conjecture 3.5. If \( \mathcal{C} \) is an admissible clutter and \( \mathcal{C} \) is unmixed, then \( I(\mathcal{C}) \) is Cohen-Macaulay.

This conjecture is true for admissible clutters with two color classes \( X^1, X^2 \) (see Theorem 4.1) and has been verified in a large number of examples.

Let \( e_1, \ldots, e_g \) and \( X^1, \ldots, X^d \) be as in the beginning of Section 3. Suppose \( e_1, \ldots, e_g \) are admissible subsets of \( X \). The clutter \( \mathcal{C} \) on \( X \) whose set of edges is:

\[
E(\mathcal{C}) = \left\{ e \in X \mid \begin{array}{l} e_i \not\subseteq e \text{ for } i = 1, \ldots, g, e \text{ is admissible,} \\
e \not\subseteq e' \text{ for any admissible set } e' \neq e \end{array} \right\} \cup \{e_1, \ldots, e_g\}
\]

is called a complete admissible clutter. This clutter consists of the maximal admissible sets with respect to inclusion. By Lemma 3.1 we get that \( e_1, \ldots, e_g \) is a perfect matching of König type.

Proposition 3.6. If \( \mathcal{C} \) is a complete admissible clutter, then \( \mathcal{C} \) is unmixed.

Proof. To show that \( \mathcal{C} \) is unmixed it suffices to verify condition (b) of Theorem 2.9. Let \( e \neq e' \) be two edges of \( \mathcal{C} \) and let \( x \neq y \) be two vertices such that \( \{x, y\} \subset e_i \) for some \( e_i, x \in e, \) and \( y \in e' \). Since \( e, e', e_i \) are admissible we can write

\[
e = \{x_1, \ldots, x_k\}, \quad e' = \{y_1, \ldots, y_{k'}\}, \quad e_i = \{z_1, \ldots, z_{k''}\},
\]

where \( x_i \in X^i, y_i \in X^i, z_i \in X^i \). Then there are \( i_1, i_2 \) such that \( x = x_{i_1}, y = y_{i_2}, x = z_{i_1}, \) and \( y = z_{i_2} \). Without loss of generality we may assume \( i_1 < i_2 \). One has \( i_1 < k \), because if \( k = i_1 \), then \( e \subset e \cup \{z_{i_1+1}, \ldots, z_{i_2}\} \) and the right hand side is admissible, a contradiction. Set \( f = \{y_1, \ldots, y_{i_1}, x_{i_1+1}, \ldots, x_k\} \). Then

\[
f \subset e \setminus \{x\} \cup e' \setminus \{y\}.
\]

Thus to finish the proof we need only show that \( f \) is an edge of \( \mathcal{C} \). Since \( y_{i_2} \in e_i \) and \( z_{i_2} \in e_i \), then \( y_{i_2} \in e_\ell \) for some \( \ell \leq i_1 \) and \( x_{i_1+1} \in e_\ell \) for some \( i \leq \ell \). Hence \( f \) is admissible. Next we show that \( f \) is maximal. Assume that \( f \) is not maximal. Then there exists an admissible subset \( f' \) that properly contains \( f \). Then there is \( z \in f' \cap X^{k+1} \) and since \( f \cup \{z\} \subset f' \), we get that \( e \cup \{z\} = \{x_1, \ldots, x_k, z\} \) is admissible, but \( e \not\subset e \cup \{z\} \), a contradiction. Hence \( f \) is maximal. \( \square \)
Suppose $C$ is a clutter on the vertex set $X$ with a perfect matching $e_1, \ldots, e_g$ where $g$ is the height of $I(C)$, and let $X^1, \ldots, X^d$ be a partition of $X$ such that every edge of $C$ intersects each $X^i$ exactly once. If every maximal admissible subset of $X$ is an edge of $C$ and these are the only edges of $C$, then we call $C$ a complete admissible uniform clutter. Note that a complete admissible uniform clutter is in fact uniform with every edge having $d$ vertices. Also, Proposition 3.6 holds and $C$ is unmixed.

**Theorem 3.7.** If $C$ is a complete admissible uniform clutter, then the simplicial complex generated by the edges of $C$ is pure shellable.

**Proof.** Order the variables of $K[X]$ as in the beginning of Section 3. Since every monomial intersects each $X^i$ exactly once, we can represent the edges of $C$ as $F_i = x_{i_1}^1 x_{i_2}^2 \cdots x_{i_d}^d$ where $x_{i_j}^j \in X^i \cap e_j$ (example: $x_3^3 \in X^3 \cap e_2$). Since $X^i \cap e_j$ has precisely one element for each $i, j$, this notation is well-defined. Then we order the edges of $C$ lexicographically, that is $F_i = x_{i_1}^1 x_{i_2}^2 \cdots x_{i_d}^d < F_j = x_{j_1}^1 x_{j_2}^2 \cdots x_{j_d}^d$ if the first nonzero entry of $(j_1, j_2, \ldots, j_d) - (i_1, i_2, \ldots, i_d) = j - i$ is positive. Under this order, we show that $C$ is shellable.

Suppose $F_i$ and $F_j$ are two edges of $C$ with $F_i < F_j$. Suppose the first non-zero entry of $j - i$ is $j_t - i_t$. Then $1 \leq i_t < j_t$. Let $F_k = F_j \setminus \{x_{j_t}^t\} \cup \{x_{i_t}^t\}$ and let $v = x_{j_t}^t$. Since $j_1 = i_1 \leq \cdots \leq j_{t-1} = i_{t-1} \leq i_t < j_t \leq j_{t+1} \leq \cdots \leq j_d$ then $F_k$ is maximal admissible, $v \in F_j \setminus F_i$, $F_k < F_j$ and $F_j \setminus F_k = \{v\}$ as required.

**Example 3.8.** The complete admissible uniform clutter with three color classes

$$X^1 = \{x_1, \ldots, x_g\}, \quad X^2 = \{y_1, \ldots, y_g\}, \quad X^3 = \{z_1, \ldots, z_g\}$$

is the clutter $C$ whose edge set is $E(C) = \{\{x_i, y_j, z_k\} | 1 \leq i \leq j \leq k \leq g\}$. Note that $e_1 = \{x_1, y_1, z_1\}, \ldots, e_g = \{x_g, y_g, z_g\}$ is the perfect matching of $C$.

The next example illustrates the construction of the lexicographical shelling used in the proof of Theorem 3.7.

**Example 3.9.** Let $C$ be the complete admissible uniform clutter with color classes

$$X^1 = \{x_1, x_2, x_3\}, \quad X^2 = \{y_1, y_2, y_3\}, \quad X^3 = \{z_1, z_2, z_3\}.$$ 

Then the shelling of the simplicial complex generated by the edges of $C$ is:

$$F_1 = \{x_1, y_1, z_1\} < F_2 = \{x_1, y_1, z_2\} < F_3 = \{x_1, y_1, z_3\} < F_4 = \{x_1, y_2, z_2\} < F_5 = \{x_1, y_2, z_3\} < F_6 = \{x_1, y_3, z_3\} < F_7 = \{x_2, y_2, z_2\} < F_8 = \{x_2, y_2, z_3\} < F_9 = \{x_2, y_3, z_3\} < F_{10} = \{x_3, y_3, z_3\}.$$ 

Let $C$ be a clutter. The Alexander dual of $C$, denoted by $\Upsilon(C)$, is the clutter whose edges are the minimal vertex covers of $C$. The edge ideal of $\Upsilon(C)$ is called the Alexander dual of $I(C)$. In combinatorial optimization the Alexander dual of a clutter is referred to as the blocker of the clutter.

**Lemma 3.10.** If $C$ is a complete admissible uniform clutter, then the Alexander dual $\Upsilon(C)$ of $C$ is also a complete admissible uniform clutter.

**Proof.** Since $C$ is unmixed with covering number $g = \text{ht } I(C)$, then the Alexander dual is uniform with edges of size $g$. Note that $e_1, \ldots, e_g$ form a partition of the vertices of the Alexander dual. Every minimal vertex cover of $C$ must by definition
intersect each $e_i$ at least once, and since $C$ is unmixed all minimal vertex covers have exactly $g$ elements, thus every edge of $\Upsilon(C)$ intersects each $e_i$ exactly once. Also, $X^1, \ldots, X^d$ is a perfect matching of $\Upsilon(C)$ since the $X^i$ partition the vertices and since each edge of $C$ intersects each $X^i$ exactly once, $X^i$ is a minimal vertex cover of $C$, and thus an edge of the Alexander dual.

Now since every minimal vertex cover of $C$ has $g$ elements and intersects $e_i$ exactly once for each $i$, all edges of the Alexander dual have the form $M = x_1^{i_1}x_2^{i_2} \cdots x_g^{i_g}$ where $1 \leq i_t \leq d$ for all $1 \leq t \leq g$. To show that the edges of $\Upsilon(C)$ are precisely the maximal admissible subsets (with the $e_i$’s being the partition and the $X^i$’s the perfect matching), we must show that $M$ is an edge of $\Upsilon(C)$ if and only if $i_1 \leq i_2 \leq \cdots \leq i_g$.

Suppose $M$ is as above and $1 \leq i_1 \leq \cdots \leq i_g \leq d$. Suppose $F_j = x_1^{j_1} \cdots x_d^{j_d}$ is an edge of $C$. Then $F_j$ is admissible, so $1 \leq j_1 \leq \cdots \leq j_d \leq g$. We must show $M \cap F_j \neq \emptyset$. If $x_{j_1}^{i_1} \in M$ the intersection is not empty. Else, since $j_1 \in \{ 1, \ldots, g \}$, then $x_{j_1}^{i_1} \in M$ for some $i_1 > 1$. Thus $i_t \geq 2$ for $t \geq j_1$. Consider $x_2^{j_2}$. If $x_2^{j_2} \in M$, done. Else $i_t \geq 3$ for $t \geq j_2$. Since $i_4 \leq d$, this process must stop with an element in the intersection of $M$ and $F_j$, or $i_t = d$ for all $t \geq j_s$ for some $s$. If $i_t = d$ for $t \geq j_s$, then since $j_s \leq g$ and $j_d \geq j_s$, we have $x_d^{j_d} \in F_j \cap M$ and thus the intersection is not empty and so $M$ is a minimal vertex cover of $C$ and so an edge of the Alexander dual.

Now suppose $M$ is as above, but $i_t > i_s$ for some $t < s$. Choose $t$ and $s$ so that $i_j < i_t$ for $j < t$ and $i_t \geq i_s$ for $t < \ell < s$. Define $F = x_1^{i_1} \cdots x_{t-1}^{i_{t-1}}x_t^{i_t}x_{t+1}^{s} \cdots x_s^{i_s}$. Then since $t < s$, $F$ is maximal admissible and so an edge of $C$. But $M \cap e_s = \{ x_s^{i_s} \}$ and $M \cap e_t = \{ x_t^{i_t} \}$ and $F \cap e_t = \{ x_1^{i_1} \cdots x_{t-1}^{i_{t-1}} \}$ and since $i_s < i_t$, $x_t^{i_t} \notin F \cap e_s$. Thus $F \cap M = \emptyset$. Thus $M$ is not a vertex cover of $C$ and so is not an edge of the Alexander dual. 

\textbf{Lemma 3.11.} If $C$ is a complete admissible uniform clutter, then the simplicial complex $\Delta_{\Upsilon(C)}$ generated by \{ $X \setminus F | F \in E(C)$ \} is pure shellable.

\textbf{Proof.} Let $F_1, \ldots, F_r$ be the shelling of the edges of $C$ defined in Theorem 3.7. Let $G_1 = X \setminus F_1, \ldots, G_r = X \setminus F_r$ be the facets of $\Delta_{\Upsilon(C)}$. We claim that $G_1, \ldots, G_r$ is the desired shelling. Suppose $G_i < G_j$. Then $F_i < F_j$. Using the notation defined in Theorem 3.7 let $v = x_1^{i_1}$ and define $u = x_i^{j_t}$. Then $u \in G_j \setminus G_i$ and $G_j \setminus G_k = \{ u \}$ as required. \hfill \Box

\textbf{Theorem 3.12.} If $C$ is a complete admissible uniform clutter, then $R/I(C)$ is a Cohen-Macaulay ring with a $d$-linear resolution and $|E(C)| = \binom{d+g-1}{d}$. 

\textbf{Proof.} Consider the clutter $\Upsilon(C)$ of minimal vertex covers of $C$. By Lemma 3.11 and Lemma 3.10 we have that $\Delta_{\Upsilon(C)}$ is pure shellable. Now recall that the Stanley-Reisner ideal of $\Delta_{\Upsilon(C)}$ is $I(\Upsilon(C))$ and that $I(\Upsilon(C))$ is the Alexander dual of $I(C)$. Thus $R/I(\Upsilon(C))$ is Cohen-Macaulay, and by \[10] the ideal $I(C)$ has a linear resolution. Since the Alexander dual of a complete admissible uniform clutter is also a complete admissible uniform clutter and since $\Upsilon(\Upsilon(C)) = C$ it follows that $R/I(C)$ is Cohen-Macaulay. The formula for the number of edges of $C$ follows from the explicit formula given in \[19] for the Betti numbers of a Cohen-Macaulay ideal with a linear resolution. \hfill \Box
Let $C$ be a complete admissible uniform clutter. For each edge $e = x_{j_1} x_{j_2} \cdots x_{j_d}$ of $C$ consider all pairs $(x_{j_i}, x_{j_k})$ with $i < k$ and consider the union of all these pairs with $e = x_{j_1} x_{j_2} \cdots x_{j_d}$ running through all edges of $C$. This defines a poset $(P, \prec)$ on $X$ whose comparability graph $G$ is defined by all the unordered pairs $\{x_{j_i}, x_{j_k}\}$. The graph $G$ is perfect [23, Corollary 66.2a] and any $d$-minor of the clutter of maximal cliques of $G$ satisfies the König property. This follows from a variant of Dilworth’s decomposition theorem [23, Theorem 14.18]. In the terminology of [4] $G$ is clique-perfect.

**Corollary 3.13.** If $G'$ is the complement of the comparability graph $G$ defined above, then $R/I(G')$ is Cohen-Macaulay.

**Proof.** Notice that $\Delta_{G'} = \{K_v | K_v$ is a clique of $G\} = \mathcal{O}(P)$, where $\mathcal{O}(P)$ is the order complex of $P$. Since the maximal faces of $\mathcal{O}(P)$ are precisely the edges of $C$, by Theorem [5,5.7] we obtain that $\mathcal{O}(P)$ is a pure shellable complex whose Stanley-Reisner ring is $R/I(G')$. Hence $R/I(G')$ is Cohen-Macaulay. $\Box$

Let $C$ be a clutter and let $x^{v_1}, \ldots, x^{v_q}$ be the minimal set of generators of $I(C)$. Consider the ideal $I^* = (x^{v_1}, \ldots, x^{v_q})$, where $v_i + w_i = (1, \ldots, 1)$. Following the terminology of matroid theory we call $I^*$ the dual of $I$. Recall that $I^*$ has linear quotients if there is an ordering of the generators $x^{v_1}, \ldots, x^{v_q}$ such that

$$(x^{v_1}, \ldots, x^{v_{i-1}}); (x^{v_i}) = (x_{i_1}, \ldots, x_{i_t})$$

for $i = 2, \ldots, q$, i.e., all colon ideals are generated by subsets of the set of variables. If $I^*$ has linear quotients and all $x^{v_i}$ have the same degree, then $I^*$ has a linear resolution (see [12, Lemma 5.2], [31]).

**Corollary 3.14.** If $C$ is a complete admissible uniform clutter, then $I(C)^*$ has linear quotients.

**Proof.** Let $x^{v_1}, \ldots, x^{v_q}$ be the minimal set of generators of $I = I(C)$ and let $F_i = \text{supp}(x^{v_i})$ for $i = 1, \ldots, q$. By Theorem [5,5.7] we may assume that $F_1, \ldots, F_q$ is a shelling for the simplicial complex $\langle F_1, \ldots, F_q \rangle$ generated by the $F_i$’s. Thus according to [18, Theorem 1.4(c)] the ideal $I^* = (x_{F_1}^1, \ldots, x_{F_q}^q)$ has linear quotients, where $F_i^c = X \setminus F_i$ and $x_{F_i}^c = \prod_{x_i \in F_i^c} x_i$. $\Box$

We may also redefine the notion of admissible monomial to allow “gaps”. This can be done as follows. Let $S = \{x_1, \ldots, x_s\}$ be a subset of $X$ of size $s$ such that $|S \cap X^i| \leq 1$ for all $i$. There are $k_1, \ldots, k_s$ and $j_1, \ldots, j_r$ such that $x_{j_i} \in X^{k_i}$ and $x_i \in e_{j_i}$ for all $i, \ell$. The set $S$ is called admissible if $j_1 \leq \cdots \leq j_r \leq g$ and $k_1 < \cdots < k_s$. A monomial $x^a$ is admissible if $\text{supp}(x^a)$ is admissible.

**Example 3.15.** Consider the following clutter with edges $e_1, e_2, f_1, f_2$ and color classes $X^1, X^2, X^3$

\[
\begin{array}{ccc}
  e_1 & = & x_1 & y_1 & X^1 \\
  e_2 & = & y_1 & z_2 & X^2 \\
  f_1 & = & y_1 & z_2 & X^3 \\
  f_2 & = & x_1 & y_2 & X^3
\end{array}
\]

This clutter is unmixed, non-Cohen-Macaulay, has a perfect matching $e_1, e_2$ of König type, and the height of $I(C)$ is two. Thus this example shows that allowing gaps gives a negative answer to Conjecture 3.5.
4. Cohen-Macaulay bipartite graphs and shellability

Throughout this section we assume that $G$ is a bipartite graph with bipartition $V_1 = \{x_1, \ldots, x_g\}$ and $V_2 = \{y_1, \ldots, y_{g_2}\}$ and without isolated vertices.

The following nice criterion of Herzog and Hibi classifies all Cohen-Macaulay bipartite graphs.

**Theorem 4.1** ([16]). $G$ is Cohen-Macaulay bipartite graph if and only if $g = |V_1| = |V_2|$ and we can order the vertices such that: $(h_0)$ $\{x_i, y_i\} \in E(G)$ for $i = 1, \ldots, g$, $(h_1)$ if $\{x_i, y_j\} \in E(G)$, then $i \leq j$, and $(h_2)$ if $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are in $E(G)$ and $i < j < k$, then $\{x_i, y_k\} \in E(G)$.

The results of this section are inspired by this criterion. Below we study condition $(h_1)$ and a variation of condition $(h_2)$. Observe that the uniform admissible clutters with two color classes $X^1$, $X^2$ (see Section 3) are exactly the bipartite graphs that satisfy $(h_0)$ and $(h_1)$.

Next we give a combinatorial characterization—suggested by condition $(h_2)$—of all unmixed bipartite graphs.

**Corollary 4.2** ([30]). Let $G$ be a bipartite graph. Then $G$ is unmixed if and only there is a perfect matching $e_1, \ldots, e_g$ such that for any two edges $e \neq e'$ and for any two distinct vertices $x \in e$, $y \in e'$ contained in some $e_i$, one has that $(e \setminus \{x\}) \cup (e' \setminus \{y\})$ is an edge.

**Proof.** It follows at once from Corollary 2.11 because bipartite graphs satisfy the König property [23].

This corollary shows that condition $(h_2)$ is in essence an expression for the unmixed property of $G$, i.e., in Theorem 4.1 we may assume that $G$ is unmixed instead of assuming condition $(h_2)$.

Let $\Delta_G$ be the Stanley-Reisner complex of $I(G)$. Its facets are the maximal independent (stable) sets of vertices of $G$. Following [9] we define the $k$th pure skeleton of $\Delta_G$ as:

$$\Delta_G^{[k]} = \langle\{F \in \Delta_G | k = |F|\} \rangle; \quad 0 \leq k \leq \dim(\Delta_G) + 1,$$

where $\langle F \rangle$ denotes the subcomplex generated by $F$. Note that this simplicial complex is always pure. By an interesting result of Duval [9] Theorem 3.3 a simplicial complex $\Delta$ is sequentially Cohen-Macaulay if and only if $\Delta^{[k]}$ is Cohen-Macaulay for $0 \leq k \leq \dim(\Delta) + 1$. In particular $R/I(G)$ is Cohen-Macaulay if and only if $R/I(G)$ is sequentially Cohen-Macaulay and $G$ is unmixed. Here we shall be interested only in the pure skeleton of $\Delta_G$ of maximum dimension.

The following result characterizes all bipartite graphs with a perfect matching that satisfy condition $(h_1)$. It gives a combinatorial description of the admissible uniform clutters with two color classes.

**Theorem 4.3.** If $G$ is a bipartite graph with a perfect matching $e_1, \ldots, e_g$ such that $e_i = \{x_i, y_i\}$ for all $i$, then $\Gamma = \Delta_G^{[g]}$ is pure shellable if and only if we can order $e_1, \ldots, e_g$ such that $\{x_i, y_j\} \in E(\tilde{G})$ implies $i \leq j$. 


Proof. \( \Leftarrow \) It suffices to show that \( \Gamma = \Delta^{[g]}_{\Delta} \) is shellable because this simplicial complex is always pure. We proceed by induction on \( g \). Each facet of \( \Gamma \) contains exactly one vertex of each edge of the perfect matching. We set

\[
A = \{y_i | x_i \in N(y_g)\}; \quad B = A \cup N(y_g) = \bigcup_{x_i \in N(y_g)} \{x_i, y_i\},
\]

where \( N(y_g) \) is the set of vertices of \( G \) that are adjacent to \( y_g \). Consider the graph \( G' = G \setminus B \), obtained from \( G \) by removing all vertices of \( B \) and all edges incident with some vertex of \( B \).

Let \( F' = \emptyset \) if \( |A| = g \), in which case \( G' = \emptyset \). Else let \( F'_1, \ldots, F'_r \) be the facets of \( \Gamma' = \Delta^{[\ell]}_{\Delta} \) that do not intersect \( N(A) \), where \( \ell = g - |A| \). Here \( N(A) \) denotes the neighbor set of \( A \), i.e., the set of vertices of \( G \) that are adjacent to some vertex of \( A \). We claim that \( F_1 = F'_1 \cup A, \ldots, F_r = F'_r \cup A \) is the set of facets of \( \Gamma \) that contain \( y_g \). First we show that \( F_k \) is a facet of \( \Gamma \) for all \( k \). If \( F_k \) contains an edge \( e = \{x_i, y_j\} \), then \( y_j \in A \) and \( x_i \in F_k' \) because \( A \) and \( F_k' \) are independent. Then \( x_i \in N(A) \), a contradiction because \( N(A) \cap F_k' = \emptyset \). Hence \( F_k \) is independent and it is a facet of \( \Gamma \) because \( |F_k| = g \). Conversely, let \( F \) be a facet of \( \Gamma \) containing \( y_g \). Then \( F \cap N(y_g) = \emptyset \), \( A \subset F \), and \( F \cap N(A) = \emptyset \). Thus we can write \( F = F' \cup A \), where \( F' = F' \cap A \) is a facet of \( \Gamma' \) with \( F' \cap N(A) = \emptyset \), as required. By the induction hypothesis \( \Gamma' \) is shellable. Next we prove that \( F'_1, \ldots, F'_r \) is a shelling for the simplicial complex they generate. It is rapidly seen that \( F'_1, \ldots, F'_r \) is also a shelling for the simplicial complex they generate.

Next we consider the graph \( G'' = G \setminus \{x_g, y_g\} \) and the complex \( \Gamma'' = \Delta^{[g-1]}_{\Delta} \). Let \( F''_1, \ldots, F''_m \) be the facets of \( \Gamma'' \). By the induction hypothesis \( \Gamma'' \) is shellable. Thus we may assume that \( F''_1, \ldots, F''_m \) is a shelling for \( \Gamma'' \). It is not hard to see that

\[
H_1 = F''_1 \cup \{x_g\}, \ldots, H_m = F''_m \cup \{x_g\}
\]

is the set of facets of \( \Gamma \) containing \( x_g \), and that \( H_1, \ldots, H_m \) is a shelling for the simplicial complex generated by them. To finish the proof notice that

\[
H_1, H_2, \ldots, H_m, F_1, F_2, \ldots, F_r,
\]

is clearly the complete list of facets of \( \Gamma \) and they form a shelling of \( \Gamma \). Indeed for any \( F_j \) one has that \( H_k = (F_j \setminus \{y_g\}) \cup \{x_g\} \) is a facet of \( \Gamma \) with \( F_i \setminus H_k = \{y_g\} \) and \( H_k \prec F_j \).

\( \Rightarrow \) The proof is by induction on \( g \). We claim that \( G \) has a vertex of degree 1. Let \( F_1, \ldots, F_s \) be a shelling of \( \Gamma \). As \( \{y_1, \ldots, y_g\} \) and \( \{x_1, \ldots, x_g\} \) are facets of \( \Gamma \), we may assume that \( F_i = \{y_1, \ldots, y_g\}, F_j = \{x_1, \ldots, x_g\} \) and \( i < j \). Then there is \( x_k \in F_j \setminus F_i \) and \( F_j \) with \( \ell \leq j - 1 \) such that \( F_j \setminus F_\ell = \{x_k\} \). Then

\[
\{x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_g\} \subset F_\ell
\]
and there is $y_t$ in $F_t$ for some $1 \leq t \leq g$. Since

$$F_t = \{x_1, \ldots, x_{k-1}, y_t, x_{k+1}, \ldots, x_g\}$$

is an independent set of $G$, we get that $y_t$ can only be adjacent to $x_t$. Thus $\deg(y_t) = 1$ because $G$ has no isolated vertices. Thus we may order $e_1, \ldots, e_g$ so that $\deg(x_g) = 1$. Consider the graph $G' = G \setminus \{x_g, y_g\}$. Using [27, Theorem 2.9] we obtain that $\Delta^{[g-1]}_{G'}$ is a shellable complex. Hence by induction hypothesis we can order $e_1, \ldots, e_{g-1}$ so that if $\{x_i, y_j\} \in E(G')$, then $1 \leq i \leq j \leq g - 1$. To finish the proof note that any edge of $G$ is either an edge of $G'$ or an edge of $G$ containing $y_g$. $\square$

Some characterizations of condition (h1) have been shown by Yassemi (personal communication), and by Carrà Ferro and Ferrarello [6]. In [27] it is shown that if $G$ has a perfect matching and $R/I(G)$ is sequentially Cohen-Macaulay, then condition (h1) holds.

**Example 4.4 ([27]).** Let $G$ be the following bipartite graph. The ring $R/I(G)$ is not sequentially Cohen-Macaulay [27] but the complex $\Delta^{[5]}_G$ is shellable.

![Diagram of a bipartite graph]

A shelling of the facets of $\Delta^{[5]}_G$ is:

$$\{x_1, x_2, x_3, x_4, x_5\} < \{x_2, x_3, x_4, x_5, y_1\} < \{x_3, x_4, x_5, y_1, y_2\} < \{x_4, x_5, y_1, y_2, y_3\} < \{x_5, y_1, y_2, y_3, y_4\} < \{y_1, y_2, y_3, y_4, y_5\}.$$ 

**Corollary 4.5.** $G$ is a Cohen-Macaulay bipartite graph if and only if: (h1′) $\Delta^{[g]}_G$ is shellable, $g = \text{ht} I(G)$, and (h2′) $G$ is unmixed.

**Proof.** It follows using Corollary 4.2 together with Theorems 4.1 and 4.3. $\square$

This corollary shows that $G$ is Cohen-Macaulay if and only if $\Delta_G$ is pure shellable [11, 27].

The natural generalization of a bipartite graph is a balanced clutter. The next example shows that Theorems 4.3 and 4.1 do not extend to balanced clutters.

**Example 4.6.** Consider the clutter $\mathcal{C}$ whose edge ideal is generated by:

$$a_1b_1c_1d_1g_1h_1k_1, \quad a_2b_2c_2d_2g_2h_2k_2, \quad a_3b_3c_3d_3g_3h_3k_3,$$

$$a_4b_4c_4d_4g_4h_4k_4, \quad a_1b_1c_1d_1g_1h_1k_1, \quad a_1b_2c_3d_4g_2h_3k_4.$$

This clutter is balanced. Indeed its incidence matrix $A$ is totally unimodular, i.e., each $i \times i$ minor of $A$ is 0 or $\pm 1$ for all $i \geq 1$. Furthermore $\mathcal{C}$ satisfies condition (b) of Corollary 2.19. Hence $I(\mathcal{C})$ is Cohen-Macaulay. However we cannot order its vertices so that it becomes an admissible uniform clutter.
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