Risk-Averse Markov Decision Processes under Parameter Uncertainty with an Application to Slow-Onset Disaster Relief

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Abstract: In classical Markov Decision Processes (MDPs), action costs and transition probabilities are assumed to be known, although an accurate estimation of these parameters is often not possible in practice. This study addresses MDPs under cost and transition probability uncertainty and aims to provide a mathematical framework to obtain policies minimizing the risk of high long-term losses due to not knowing the true system parameters. To this end, we utilize the risk measure value-at-risk associated with the expected performance of an MDP model with respect to parameter uncertainty. We provide mixed-integer linear and nonlinear programming formulations and heuristic algorithms for such risk-averse MDPs under a finite distribution of the uncertain parameters. Our proposed models and solution methods are illustrated on an inventory management problem for humanitarian relief operations during a slow-onset disaster. The results demonstrate the potential of our risk-averse modeling approach for reducing the risk of highly undesirable outcomes in uncertain/risky environments.

Keywords: Markov decision processes, parameter uncertainty, value-at-risk, chance constraints, humanitarian supply chains, disaster relief

1. Introduction Markov Decision Processes (MDPs) are effectively used in many applications of sequential decision making in uncertain environments including inventory management, manufacturing, robotics, communication systems, and healthcare, e.g., Puterman (2014); Altman (1999); Boucherie and van Dijk (2017). In an MDP model, the decision makers take an action at specified points in time considering the current state of the system with the aim of minimizing their expected loss (resp., maximizing their expected utility), and depending on the action taken, the system transitions to another state. The evolution of the underlying process is mainly characterized by the action costs (resp., rewards) and transition probabilities between the system states, inducing two types of uncertainty. The internal uncertainty stems from the probabilistic behaviour of transitions between states, costs and actions (see, e.g., Ruszczyński, 2010; Bäuerle and Ott, 2011; Xu and Mannor, 2011; Fan and Ruszczyński, 2018 for studies addressing the risk arising from internal uncertainty). The parameter uncertainty, on the other hand, is due to the ambiguities in the parameters representing the costs and transition probabilities. In classical MDPs, these parameters are assumed to be known; they are usually estimated from historical data or learned from previous experiences. However, in practice, it is usually not possible to obtain a single estimate that fully captures the nature of the uncertainties. The actual performance of the system may significantly differ from the anticipated performance of the MDP model due to the inherent varia-
tion in the parameters (Mannor et al., 2007). Our focus in this study is on the parameter uncertainty in MDPs—decision makers are assumed to be sensitive to the risk associated with the fluctuations in parameters while being risk-neutral to the internal randomness due to state transitions, costs and randomized actions. This setting is especially suitable for applications in which the objective function is aggregated over a number of problem instances, e.g., total inventory cost over various types of supply items. In such cases, aggregation across multiple instances mitigates the variation due to internal uncertainty, and parameter uncertainty becomes the main source of variation.

A widely used approach to incorporate parameter uncertainty into MDPs is robust optimization. In the robust modeling framework, the objective is to optimize the worst-case performance over all possible realizations in a given uncertainty set. This approach is appealing in the sense that it requires no prior information on the distribution of costs or transition probabilities and it gives rise to computationally efficient solution algorithms. However, it often leads to conservative results because the focus is on the worst-case system performance, which may be rarely encountered in practice. In the initial studies on robust MDPs, uncertainty is usually described using a polyhedral set, because it leads to tractable solution algorithms (Satia and Lave Jr, 1973; White III and Eldeib, 1994; Givan et al., 2000; Tewari and Bartlett, 2007; Bagnell et al., 2001). The uncertainty sets are later extended for more general definitions with the aim of balancing the conservatism of the solutions and the tractability of the solution algorithms. Nilim and El Ghaoui (2005) model the uncertainty in transition probabilities using a set of stochastic matrices satisfying rectangularity property, i.e., when there are no correlations between transition probabilities for different states and actions. The authors devise an efficient dynamic programming algorithm for this case. Similarly, Iyengar (2005) studies robust MDPs under transition probability uncertainty with rectangularity assumption and provides robust value and policy iteration algorithms for finite-horizon nonstationary and infinite-horizon stationary problem settings. Sinha and Ghate (2016) propose a policy iteration algorithm for robust infinite-horizon nonstationary MDPs following the rectangularity assumption. Wiesemann et al. (2013) relax the rectangularity assumption of Nilim and El Ghaoui (2005) and consider a more general class of uncertainty sets in which the assumption of no correlation between transition probabilities is only made for states, not for actions. Mannor et al. (2016) define a tractable subclass of nonrectangular uncertainty sets, namely $k$-rectangular uncertainty sets, such that the number of possible conditional projections of the uncertainty set is at most $k$. Another alternative modeling approach to balance the conservatism of the solutions is to consider the distributional robustness, where the uncertain parameters are assumed to follow the worst-case distribution from a set of possible distributions described by some general properties such as expectations or moments. Unlike robust MDPs, distributionally robust MDPs incorporate the available—but incomplete—information on the a priori distribution of the uncertain parameters. For relevant studies on distributionally robust MDPs, we refer the reader to Xu and Mannor (2012); Yu and Xu (2016).
Bayesian approaches to address parameter uncertainty have been receiving increasing attention in the recent literature. This uncertainty model treats unknown parameters as random variables with corresponding probability distributions. Hence, it provides the means to incorporate complete distributional information about the unknown parameters and the attitude of decision makers towards uncertainty and risk, oftentimes at the cost of increasing problem complexity. Steimle et al. (2018b) consider a multi-model MDP, where the aim is to find a policy maximizing the weighted sum of expected total rewards over a finite horizon associated with different sets of parameters obtained by different estimation methods. This modeling framework is analogous to an expected value problem considering a finite number of scenarios in the context of stochastic programming, because each set of parameters can be treated as a scenario in which the corresponding scenario probability is set as the normalized weight value. The authors prove the existence of a deterministic optimal policy, show that the problem is NP-hard, and provide a mixed-integer linear programming (MIP) formulation and a heuristic algorithm. In a subsequent work, Steimle et al. (2018a) propose a customized branch-and-bound algorithm to solve the multi-model MDPs. Buchholz and Scheftelowitsch (2018) study multi-model MDPs, where each model corresponds to an infinite-horizon MDP. Unlike the finite-horizon case, it is demonstrated that infinite-horizon variant may not have a deterministic optimal policy. The authors propose two nonlinear formulations and a heuristic algorithm for the case where randomized policies are allowed, and an MIP model utilizing the dual linear programming formulation of MDPs for the deterministic policy case.

A potential drawback of previously mentioned modeling frameworks for multi-model MDPs is that the expected value objective ignores the risk arising from the parameter uncertainty. Considering this issue, Xu and Mannor (2009) address reward uncertainty in MDPs with respect to parametric regret. The MDP of interest is a finite-horizon, discounted model with finite state and action spaces. The authors consider two different objectives: minimax regret based on the robust approach and mean-variance trade-off of the regret based on the Bayesian approach. They propose a nonconvex quadratic program for the former objective and a convex quadratic program for the latter. Delage and Mannor (2010) consider reward and transition probability uncertainty separately and propose a chance-constrained model in the form of percentile optimization, which corresponds to the risk measure, value-at-risk (VaR). The authors give a formulation for infinite-horizon MDPs with finite state and action spaces, and stationary policies. They show that the problem is intractable in the general case but can be efficiently solved when the rewards follow a Gaussian distribution or transition probabilities are modeled using independent Dirichlet priors. Alternatively, Chen and Bowling (2012) investigate a class of percentile-based objective functions that are easy to approximate for any probability distribution and Adulyasak et al. (2015) focus on finding objective functions that are separable over realizations of uncertain parameters under a sampling framework. Our modeling approach for incorporating parameter uncertainty into MDPs is along the same line with Delage and Mannor (2010), however we allow general probability distributions with finite support.
Throughout the manuscript, we use the terms percentile, quantile and VaR interchangeably.

In this study, we adopt a Bayesian approach to address MDPs under cost and transition probability uncertainty. Our aim is to obtain a stationary policy that optimizes the quantile function value, \( \text{VaR}_\alpha \), at a certain confidence level \( \alpha \) with respect to parameter uncertainty. The \( \text{VaR}_\alpha \) assesses an estimate of the largest potential loss excluding the worst outcomes with at most \( 1 - \alpha \) probability. Conservatism of solutions in optimization problems involving \( \text{VaR}_\alpha \) can be adjusted using different confidence levels \( \alpha \), reflecting decision makers’ risk aversion—quantile optimization is equivalent to a robust optimization approach when \( \alpha = 1 \). In addition, quantile-based performance measures, such as VaR, are used in many applications in the service industry because of their clear interpretation and correspondence with the service-level requirements, e.g., minimum investment to guarantee \( 100\alpha \% \) service level (DeCandia et al., 2007; Benoit and Van den Poel, 2009; Atakan et al., 2017). Although VaR is nonconvex in general, the main challenge in our case is the combinatorial nature of policies independent from the choice of the risk measure, which makes the problem NP-hard even for the expected value objective (Steimle et al., 2018b).

Unlike Delage and Mannor (2010), who also consider the VaR objective, we assume that action costs and transition probabilities are both uncertain and follow a finite joint distribution. This approach directly represents the cases in which a finite set of possible parameter realizations can be obtained based on historical data and/or multiple estimation tools (Bertsimas and Misic, 2016; Steimle et al., 2018b). Moreover, it provides a general framework that can be used on sample approximations of a wide range of probability distributions. Finite representation of uncertainty also facilitates competitive solution methods for optimization problems incorporating VaR. Since VaR corresponds to a quantile function, optimization problems incorporating VaR can be formulated as chance-constrained programs (CCPs), which are known to be NP-hard in general and usually require multidimensional integration. When uncertain parameters follow a finite distribution, the challenges of working with multivariate distributions can be circumvented, and the CCP formulations can be stated as MIP models by employing big-M inequalities and additional binary variables for each possible realization of uncertainty (Luedtke and Ahmed, 2008) (see, e.g., Kucukyavuz, 2012; Liu et al., 2017a; Zhao et al., 2017 for further strengthenings). Majority of the studies on CCPs considers a single-state (i.e., static) decision-making framework, where the uncertainty is revealed only after all required decisions are made. Luedtke (2014) and Liu et al. (2016) extend the literature for a two-stage decision-making framework such that recourse decisions are allowed in the second-stage and provide branch-and-cut algorithms employing mixing inequalities to ensure feasibility/optimality of the second-stage problems. Along the same lines, Zhang et al. (2014) consider a multi-stage (finite-horizon) setting and propose a branch-and-cut algorithm using continuous mixing inequalities. For an overview of CCPs and related approaches, we refer the reader to Kucukyavuz and Sen (2017). Motivated by these advances in the CCP literature, Feng et al. (2015) formulate the VaR portfolio optimization problem as a single-stage CCP and provide an MIP reformulation and a branch-and-cut
algorithm utilizing the mixing inequalities of Luedtke (2014). Their results suggest superiority of the MIP approaches over branch-and-cut based algorithms and emphasize the significance of the big-M terms on computational performance of the MIP formulations for VaR optimization. Based on these conclusions, Pavlikov et al. (2018) provide a bounding scheme that produces tighter big-M values for the MIP formulation.

Optimizing VaR associated with parameter uncertainty in MDPs, on the other hand, brings out additional challenges due to the combinatorial nature of the decisions and the underlying Markovian system dynamics in an MDP. In this problem setting, the aim is to obtain a single optimal policy (selected at the beginning) minimizing the VaR associated with parameter uncertainty in an MDP, which is to be implemented over the entire planning horizon under any realization of uncertainty. Different from the previously mentioned studies on CCPs, the underlying process is assumed to be Markovian and possible actions in each state belong to a finite set. For this purpose, we provide a two-stage CCP formulation capable of modeling the dynamics of a Markov chain for any selected policy considering possible values of uncertain parameters. Note that here the second stage represents the performance of the MDP for a given policy. We additionally propose relaxations and heuristic solution algorithms that can be used for obtaining lower and upper bounds on the optimal objective function value. Although we focus on infinite-horizon MDPs, our results and algorithms can be easily extended for finite-horizon MDPs after small adjustments.

We test our modeling framework and solution algorithms on a humanitarian inventory management problem for relief items required to sustain basic needs of a population affected by a slow-onset disaster, e.g., war, political insurgency, extreme poverty, famine, or drought. Since the progress and impact of a slow-onset disaster generally depend on unpredictable political and/or natural events, the demand for the relief items is highly variable. The supply amounts are also exposed to uncertainty as they mainly rely on voluntary donations. Another critical issue in humanitarian inventory management is the perishability of many relief items such as food and medication. At the beginning of each time period, based on the current inventory level, the decision makers need to determine an additional order quantity to minimize the expected total inventory holding, stock-out and disposal costs considering the expiration dates and the uncertainty in supply and demand. This problem can be modeled as an MDP, where the current inventory level represents the state of the system, and the uncertainty in supply, demand, and inventory is captured by the transition probabilities between different states (Ferreira et al., 2018). However, the cost and transition probability parameters of the MDP model are subject to high level of uncertainty because demand and supply rates and shelf life of perishable relief items used in the estimation of these parameters may widely fluctuate. The VaR$_\alpha$ objective in this setting has a natural interpretation: to find a replenishment strategy that minimizes the budget required to cover the expected total costs considering all possible parameter realizations with at least $\alpha$ probability.
The rest of this paper is organized as follows. In Section 2, we formulate the risk-averse MDP problem under cost and transition probability uncertainty, provide mixed-integer nonlinear programming (MINLP) and MIP models and explore characteristics of the optimal policies. Section 3 presents preprocessing procedures, which can be used for initializing auxiliary parameters of the proposed mathematical models as well as reducing the problem size, and a heuristic solution algorithm. We describe a stochastic inventory management problem for slow-onset disasters in Section 4, which is later used for demonstrating effectiveness of the quantile-optimizing modeling approach and proposed solution methods in Section 5.

### 2. Problem Formulation and Structural Properties

Consider a discrete-time infinite-horizon MDP model with finite state space $\mathcal{H}$ and finite action space $\mathcal{A}$. We define $\hat{c}_i(a)$ as the immediate expected cost of taking action $a \in \mathcal{A}$ in state $i \in \mathcal{H}$ and $\hat{P}_{ij}(a)$ as the probability of transitioning from state $i \in \mathcal{H}$ to state $j \in \mathcal{H}$ under action $a \in \mathcal{A}$. The future costs are discounted by $\gamma \in [0, 1)$ and the distribution of the initial state is given as $|\mathcal{H}|$-dimensional vector $q$. A stationary policy $\pi = (\pi_1, \ldots, \pi_{|\mathcal{H}|})$ refers to a sequence of decision rules $\pi_i$ describing the action strategy for each state $i \in \mathcal{H}$. When the policy is randomized, each element of $|\mathcal{A}|$-dimensional vector $\pi_i$ denotes the probability of taking the respective action at each time state $i \in \mathcal{H}$ is encountered. For a deterministic policy $\pi$, on the other hand, $\pi_i$ refers to a unit vector in which only the element corresponding to the action selected in state $i \in \mathcal{H}$ is one. Assuming that the cost and transition probability parameters are nonnegative, stationary and bounded, the expected total discounted cost of the underlying Markov chain for a given policy $\pi$ and known system parameters $(\hat{c}, \hat{P})$ can be stated as

$$C(\pi, \hat{c}, \hat{P}) = \mathbb{E}_{x_0 \in \mathcal{H}} \left( \sum_{t=0}^{\infty} \gamma^t \hat{c}_{x_t}(\pi_{x_t}) | x_0 \propto q, \pi \right),$$

where $x_0$ and $x_t$ denote the initial state and the state of the system at decision epoch $t > 0$, respectively. Let $\Pi$ be the set of stationary Markov policies. In an MDP model, the aim is to find a policy $\pi \in \Pi$ minimizing the expected total discounted cost, i.e.,

$$\min_{\pi \in \Pi} C(\pi, \hat{c}, \hat{P}). \quad (1)$$

This problem is known to have a stationary and deterministic optimal solution over the set of all policies, and it can be solved efficiently using several well-known methods such as the value iteration algorithm, the policy iteration algorithm, and linear programming (Puterman, 2014). Using the Bellman equation (Bellman, 2013), problem (1) can be alternatively stated as

$$\min_{\pi \in \Pi, \nu \in \mathbb{R}^{|\mathcal{H}|}} \sum_{i \in \mathcal{H}} q_i \nu_i. \quad (2a)$$
\[ v_i = \hat{c}_i(\pi_i) + \gamma \sum_{j \in \mathcal{H}} \hat{P}_{ij}(\pi_i) v_j, \quad i \in \mathcal{H}, \quad (2b) \]

where \( v_i \) is the expected sum of discounted costs under the selected policy \( \pi \) when starting from state \( i \in \mathcal{H} \), which satisfies Bellman optimality condition

\[ v_i = \min_{a \in \mathcal{A}} \left\{ \hat{c}_i(a) + \gamma \sum_{j \in \mathcal{H}} \hat{P}_{ij}(a) v_j \right\}, \quad i \in \mathcal{H}. \quad (3) \]

The cost vector \( \hat{c} \) and transition probability matrix \( \hat{P} \) in MDP model (1) are assumed to be known. However, in practice, it is usually difficult to predict exact values of these parameters. For example, the prices or the weekly demand rate for relief items during a slow-onset disaster (e.g., war) are subject to a high level of uncertainty due to various factors. In addition, for cases where rare events may have a tremendous impact—as in the case of disaster management—the decision makers may prefer to incorporate their aversion towards risk into the decision support tools. Motivated by these arguments, in this study, we consider the setting where the elements of the cost vector \( \hat{c} \) and transition probability matrix \( \hat{P} \) are assumed to be random variables instead of known parameters. The decision makers need to determine a policy \( \pi \in \Pi \) in this uncertain environment with the aim of minimizing their risk of realizing a large amount of expected total discounted cost. In terms of stochastic programming terminology, policy decisions can be thought of as non-anticipative, that is, a policy minimizing the risk is selected at the beginning in the presence of uncertainty, and it will be implemented throughout the planning horizon for any realization of parameters \( \hat{c} \) and \( \hat{P} \), the values of which are not revealed. We seek a policy \( \pi \in \Pi \) minimizing the \( \alpha \)-quantile of the expected total discounted cost with respect to parameter uncertainty, which corresponds to the optimal policy of the risk-averse MDP problem

\[
\text{(RAMDP)} \quad \min_{y \in \mathbb{R}, \pi \in \Pi} y \\
\quad \text{s.t. } \mathbb{P}_{\hat{c}, \hat{P}} \left( C(\pi, \hat{c}, \hat{P}) \leq y \right) \geq \alpha.
\]

Formulation RAMDP ensures that the expected total discounted cost for the optimal policy \( \pi^* \) is less than or equal to the optimal objective function value \( y^* \) with probability at least \( \alpha \) under the distributions of \( \hat{c} \) and \( \hat{P} \). Note that the optimal value \( y^* \) corresponds to the VaR of the expected total discounted cost for the optimal policy \( \pi^* \) at confidence level \( \alpha \). Such formulations optimizing VaR (\( \alpha \)-quantile) are also referred as quantile or percentile optimization in the literature. Delage and Mannor (2010) consider special cases of the quantile optimization problem RAMDP in which cost and transition probability uncertainty are treated separately. The authors assume that the cost parameters follow a Gaussian distribution in the former case, and the transition probabilities in the latter are modeled using Dirichlet priors.
Different than their model, we consider the cost and transition probability uncertainty simultaneously without making any assumptions on the distributions of uncertain parameters other than that it can be represented/approximated with a finite discrete distribution.

Using the nominal MDP model in (2), we obtain an alternative CCP formulation for problem RAMDP,

\[
\begin{align*}
\min_{y \in \mathbb{R}, \pi \in \Pi, v \in \mathbb{R}^{\mathcal{H}}} & \quad y \\
\text{s.t.} & \quad \mathbb{P}_{\tilde{\epsilon}, \tilde{P}} \left( \sum_{i \in \mathcal{H}} q_i v_i^{(\tilde{\epsilon}, \tilde{P})} \leq y \right) \geq \alpha. \\
& \quad v_i^{(\tilde{\epsilon}, \tilde{P})} = \tilde{c}_i(\pi_i) + \gamma \sum_{j \in \mathcal{H}} \tilde{P}_{ij}(\pi_i) v_j^{(\tilde{\epsilon}, \tilde{P})} \quad i \in \mathcal{H}. \tag{5c}
\end{align*}
\]

The distribution of \( v^{(\tilde{\epsilon}, \tilde{P})} \) depends on the joint distribution of random parameters \( \tilde{\epsilon} \) and \( \tilde{P} \) and the selected policy \( \pi \). In general, such problems with chance constraints are highly challenging to solve since they require computation of the joint distribution function, which usually involves numerical integration in multidimensional spaces (Deák, 1988). On the other hand, using a discrete representation of the distribution function obtained by a sampling method significantly reduces the computational complexity and provides reliable approximations to CCPs for a sufficiently large sample size (Calafiore and Campi, 2006; Luedtke and Ahmed, 2008). In stochastic optimization, each sample of parameters is referred to as a scenario. Note that this approach yields optimal solutions for multi-model MDPs in which parameter uncertainty can be finitely discretized. For example in medical applications for designing optimal treatment and screening protocols, the system state usually represents patient health status and transition probabilities between states can be computed using multiple tools from the clinical literature, which often produce different parameters (Steimle et al., 2018b). In this context, the parameters computed using each tool can be treated as a scenario. Similarly, for the humanitarian inventory management problem, each scenario may correspond to a set of parameters computed using a different demand/supply forecasting method. In this case, the VaR objective can be interpreted as minimizing the worst-case expected total cost over \( \alpha \) fraction of the possible MDP parameters (scenarios).

Another challenge in solving problem (5) is the possibility that none of the optimal policies is deterministic. As mentioned before, for unconstrained infinite-horizon discounted MDPs with discrete state space, finite action space and known parameters, it is ensured that there always exists a deterministic stationary optimal policy. However, this is not necessarily true for MDPs under parameter uncertainty or additional constraints. Even for the expected value objective, the infinite-horizon problem may not have any deterministic optimal policy (Buchholz and Scheftelowitsch, 2018), while the existence of a deterministic optimal policy is guaranteed for the finite-horizon problem (Steimle et al., 2018b). The problem RAMDP, on the other hand, does not necessarily have a deterministic optimal policy for either infinite or finite-horizon cases. The following example demonstrates this observation for the infinite-horizon case,
and it can be easily adjusted for a finite-horizon problem with a single period and zero termination costs.

**Example 2.1** Consider a single-state infinite-horizon MDP with two actions, a and b, under two scenarios with equal probabilities. Under scenario 1, \( \tilde{c}(a) = 0 \) and \( \tilde{c}(b) = 2 \), and under scenario 2, \( \tilde{c}(a) = 2 \) and \( \tilde{c}(b) = 0 \). In case a stationary deterministic policy is applied, i.e., either action a or b is chosen, the optimal objective function value of the problem at confidence level \( \alpha = 0.9 \) is \( 2/(1-\gamma) \). However, if a randomized policy of selecting action a or b with equal probabilities is applied, then the optimal objective function value is \( 1/(1-\gamma) \), proving that no stationary deterministic policy is optimal.

For classical MDPs with known parameters, the Bellman optimality condition (3) leads to a linear programming formulation, where policy decisions are implied by the constraints that are satisfied as strict equalities. However, the quantile optimization problem RAMDP cannot utilize such implicit representation of the policies because Bellman optimality condition does not necessarily hold when the same policy should be imposed across all scenarios due to non-anticipativity. Even though the Bellman optimality condition is no longer valid, value functions in each scenario still need to obey the Bellman equation (2b) to correctly represent the dynamics of the underlying Markov chain for any given policy. Using this property, we propose an MINLP formulation for problem (5) assuming the existence of a discrete representation for parameter uncertainty as in the following statement:

A1. Joint distribution of \( \tilde{c} \) and \( \tilde{P} \) is represented as a finite set of scenarios \( \mathcal{S} = \{1, \ldots, n\} \) with corresponding probabilities \( p^1, \ldots, p^n \).

Note that our mathematical models and solution algorithms can be easily adjusted for the finite-horizon MDPs with nonstationary/stationary policies by introducing a time dimension on the decisions and/or parameters. Here, we focus on the infinite-horizon case, in which the stationarity assumption is desired for practicality and tractability, and ignore the time indices for brevity.

**Proposition 2.1** Assuming a finite representation of parameter uncertainty as described in A1, problem (5) can be restated as the following deterministic equivalent formulation,

\[
\text{(RAMDP-R)} \quad \min \ y \\
\text{s.t.} \quad \sum_{a \in A} w_{ia} = 1, \quad i \in \mathcal{H}, \quad (6a) \\
\quad \quad \sum_{s \in \mathcal{S}} z^s p^s \geq \alpha, \quad (6b) \\
\quad \quad \sum_{i \in \mathcal{H}} q_i v_i^s \leq y + (1 - z^s)M, \quad s \in \mathcal{S}, \quad (6c) \\
\quad \quad v_i^s \geq \sum_{a \in A} \tilde{c}_i^a (a) w_{ia} + \gamma \sum_{a \in A} \sum_{j \in \mathcal{H}} \tilde{P}_{ij}^s (a) v_j^s w_{ia}, \quad i \in \mathcal{H}, \ s \in \mathcal{S}, \quad (6d)
\]
where $M$ represents a large number such that constraint (6d) for $s \in S$ is redundant if $z^s = 0$. 

**Proof.** Constraint (6b) combined with the domain constraint (6g) ensures a probability distribution on the action space for each state, which is effective for all stages. Hence, $w_{ia}$ corresponds to the probability that action $a$ is taken in state $i$ under the optimal policy. Note that the variables $w$ are required for enforcing a single stationary policy across all scenarios. Constraints (6b) and (6g) also guarantee that constraint (6e) is equivalent to the Bellman equations in (5c) for the policy determined by $w$, and variable $v_i^s$ corresponds to the value of $v_i^{c,\tilde{P}}$ in (5) for parameter realizations in scenario $s \in S$. Finally, by definition of $M$, $z^s$ denotes a binary variable equal to 1 if scenario $s$ satisfies $\sum_{i \in H} q_i v_i^s \leq y$ and consequently, constraints (6c)–(6d) and (6f) are equivalent to the chance constraint (5b). □

The formulation RAMDP-R is nonlinear and nonconvex in general due to constraints (6e), which contain the bilinear terms $v_j^s w_{ia}$ for $s \in S$, $i \in H$, $a \in A$. For such nonlinear optimization models, most of the existing solution algorithms only guarantee local optimality of the resulting solutions. To obtain a lower bound on the globally optimal objective function value, we approximate each bilinear term $x_{ija} = v_j^s w_{ia}$ for $i, j \in H$, $a \in A$, $s \in S$ in (6e) by the following linear inequalities using McCormick envelopes (McCormick, 1976)

$$
\ell_j^s w_{ia} \leq x_{ija} \leq u_j^s w_{ia},
$$

$$
v_j^s - (1 - w_{ia}) u_j^s \leq x_{ija} \leq v_j^s - (1 - w_{ia}) \ell_j^s,
$$

$$
\ell_j^s \leq v_j^s \leq u_j^s,
$$

where $\ell_j^s$ and $u_j^s$ are lower and upper bounds on the value of $v_j^s$, respectively. Then, nonlinear constraint (6e) can be replaced by

$$
v_i^s \geq \sum_{a \in A} \tilde{c}_i^s(a) w_{ia} + \gamma \sum_{a \in A} \sum_{j \in H} \tilde{P}_{ij}^s(a) x_{ija}, \quad i \in H, \ s \in S,
$$

which provides a linear relaxation of RAMDP-R in a lifted space.

Despite the possibility that all optimal policies are randomized, implementation of deterministic policies may be preferred over randomized policies in certain application areas. These include the cases in which the decision makers are prone to making errors (Chen and Blankenship, 2002), and the cases that making randomized decisions raises ethical concerns as in the health care systems (Steimle et al., 2018b) and humanitarian relief operations. Considering these cases, we make an additional assumption that
A2. The policy space is restricted to the set of stationary deterministic policies, denoted as $\Pi_D$, i.e., set $\Pi$ in RAMDP is replaced by $\Pi_D$,

and propose an MINLP formulation that aims to minimize the $\alpha$-quantile value over all stationary deterministic policies in set $\Pi_D$.

**Proposition 2.2** Under assumptions A1–A2, problem (5) can be restated as the following deterministic equivalent formulation,

$$(\text{RAMDP-D}) \quad \min_y \quad s.t. \quad (6b) - (6f),$$

$$w_{ia} \in \{0, 1\}, \quad i \in \mathcal{H}, \quad a \in \mathcal{A},$$

where $M$ represents a large number such that constraint (6d) for $s \in \mathcal{S}$ is redundant if $z^* = 0$.

The proof easily follows from Proposition 2.1 and the characterization of deterministic policies.

Similar to RAMDP-R, Formulation RAMDP-D is also nonlinear and nonconvex due to the bilinear terms in constraint (6c), but in this case, binary representation of deterministic policies can be further utilized to obtain MIP reformulations. When policy vector $w$ is binary, the McCormick envelopes (7) correspond to an exact linearization of the bilinear terms $x^s_{ija} = v^s_j w_{ia}$ for $i, j \in \mathcal{H}, \ a \in \mathcal{A}, \ s \in \mathcal{S}$. Hence, the MIP obtained by replacing (6e) with (8) and adding constraints (7) is equivalent to RAMDP-D. As for RAMDP-R, this McCormick reformulation requires the addition of $\mathcal{O}(|\mathcal{S}||\mathcal{A}||\mathcal{H}|^2)$ variables and constraints. An alternative reformulation of RAMDP-D as an MIP of smaller size by a factor of $\mathcal{O}(|\mathcal{H}|)$ can be achieved by replacing (6e) in RAMDP-D with linear inequalities

$$v^s_i \geq \bar{z}^s(a) + \gamma \sum_{j \in \mathcal{H}} \hat{P}^s_{ij}(a) v^s_j - (1 - w_{ia}) M_{is}, \quad i \in \mathcal{H}, \ a \in \mathcal{A}, \ s \in \mathcal{S}. \quad (10)$$

Constraint (10) assures that the multi-scenario Bellman equations (5c) in the CCP model (5) is satisfied for the selected scenarios, i.e., $z^* = 1$, and the policy represented by the state-action pairs $(i, a)$ such that $w_{ia} = 1$. The big-$M$ term $M_{is}$ for any $i \in \mathcal{H}, \ s \in \mathcal{S}$ denotes a large number sufficient to make constraint (10) redundant when $w_{ia} = 0$ for $a \in \mathcal{A}$. Note that replacing (6e) with (10) does not provide an exact reformulation of RAMDP-R. In addition, even though the MIP reformulation of RAMDP-D with McCormick envelopes requires more variables and constraints, it is also likely to provide tighter bounds than the reformulation with constraint (10), because the big-$M$ terms are known to cause weak linear programming relaxations as also demonstrated by our computational experiments in Section 5.4.
3. Implementation Details  In this section, we propose preprocessing methods to initialize problem parameters, which may also be used for reducing the problem size. We also provide a heuristic solution algorithm. Note that the proposed methods are applicable for both RAMDP-R and RAMDP-D, but here we consider on RAMDP-R due to its generality.

3.1 Preprocessing  The formulations for problems RAMDP-R and RAMDP-D presented in the previous section require determination of big-M terms $M_i, M_i s_i, l_i^s$ and $u_i^s$ for $i \in H, s \in S$. In what follows, we provide preprocessing procedures to prespecify these big-M terms and to narrow down the solution space before solving the original problem. Existing solution algorithms for MDPs usually suffer from the size of state and action spaces, often referred to as the curse of dimensionality. In our case, the computational complexity is additionally amplified with the number of scenarios and the combinatorial nature of the quantile calculation. Hence it is worthwhile to search for methods that reduce the size of the problem.

Let $y^*$ be the optimal objective function value of problem RAMDP-R. First, we relax the requirement that the same policy should be selected over all scenarios, and use monotonicity property of the VaR function to find bounds on the value of $y^*$. We denote $\bar{b}$ as a random variable representing the maximum expected total discounted cost of the relaxed MDP model. The realization of $\bar{b}$ under scenario $s \in S$ can be computed by solving the following linear programming problem

$$\bar{b} = \min \sum_{i \in H} q_i v_i$$  

s.t. $v_i \geq \tilde{c}_i^s(a) + \gamma \sum_{j \in H} \tilde{P}_{ij}^s(a) v_j, \quad i \in H, a \in A$.  

Then we order the realizations of $\bar{b}$ under each scenario as $\bar{b}^{s_1} \leq \bar{b}^{s_2} \leq \cdots \leq \bar{b}^{s_n}$, and let $k_1 \in \{1, \ldots, n\}$ be the order of the scenario such that $\sum_{i=1}^{k_1} p_i \geq \alpha$ and $\sum_{i=1}^{k_1-1} p_i < \alpha$, and $b_u = \bar{b}^{s_{k_1}}$. Note that $b_u$ corresponds to the VaR of random variable $\bar{b}$ at confidence level $\alpha$. Hence $b_u$ provides an upper bound on $y^*$, i.e., $b_u \geq y^*$, since $\bar{b} \geq C(\pi, \tilde{c}, \tilde{P})$ for all $s \in S, \pi \in \Pi$.

Similarly, let $\bar{b}^*$ be a random variable representing the minimum expected total discounted cost of the MDP with relaxed policy selection requirements, where realizations of $\bar{b}^*$ under each scenario $s \in S$ can be computed by solving the linear programs

$$\bar{b}^* = \max \sum_{i \in H} q_i v_i$$  

s.t. $v_i \leq \tilde{c}_i^s(a) + \gamma \sum_{j \in H} \tilde{P}_{ij}^s(a) v_j, \quad i \in H, a \in A$.  

Let $b_l$ correspond to the VaR of random variable $\bar{b}^*$ at confidence level $\alpha$. The value of $b_l$ provides a lower
bound on $y^*$, i.e., $b_l \leq y^*$ as $\bar{\mu}^s \leq C(\pi, \tilde{c}^s, \tilde{P}^s)$ for all $s \in \mathcal{S}$, $\pi \in \Pi$ by definition.

Problems (11) and (12) can be efficiently solved using a policy iteration or value iteration algorithm in polynomial time. Using these bounds, we can conclude that any scenario $s$, whose lower bound is greater than the upper bound on the quantile value, i.e., $\bar{\mu}^s > b_u$, cannot satisfy the term inside chance constraint (5b) in the optimal policy, hence we can set $z^s = 0$. Similarly, any scenario $s \in \mathcal{S}$ with an upper bound value smaller than the lower bound on the quantile function value, i.e., $\bar{\mu}^s < b_l$ must satisfy the term inside chance constraint (5b) in the optimal solution. Therefore, the corresponding scenario variable can be set as $z^s = 1$. This result can be used for reducing the number of $z$ variables. Furthermore, we can add the constraints $b_l \leq y^s \leq b_u$ into our mathematical model. As previously mentioned, validity of these bounds follows from monotonicity property of the VaR function. Additionally, the inequalities

$$y \geq \bar{\mu}^s z^s, \quad s \in \mathcal{S}$$  

(13)

are valid due to constraints (6d) and the fact that $\bar{\mu}^s \leq C(\pi, \tilde{c}^s, \tilde{P}^s)$ for all $s \in \mathcal{S}$, $\pi \in \Pi$. Note that Küçükyavuz and Noyan (2016) propose similar bounding and scenario elimination ideas and demonstrate their effectiveness in the context of multivariate CVaR-constrained optimization (see also, Liu et al., 2017b; Noyan et al., 2019).

Let $\bar{v}^s_i$ and $\underline{v}^s_i$ be the optimal values of variable $v_i$ in problem (11) and (12), respectively, for state $i \in \mathcal{H}$ and scenario $s \in \mathcal{S}$. Then, we can set the lower and upper bounds on variable $v_i^s$ in (7) as $\ell^s_i = \underline{v}^s_i$ and $u^s_i = \bar{v}^s_i$, respectively. This result follows from the fixed-point and contraction properties of the value functions for classical MDPs. The information obtained by solving problems (11) and (12) can also be used to obtain tighter values of the big-M terms in constraints (6d) and (10). Clearly, we can set $M = b_u$ in (6d) because its value is bounded by $M \geq \max_{v:(6b)-(6g)} \sum_{i \in \mathcal{H}} q_i v_i^s - \min_{y:(6b)-(6g)} y$ for $s \in \mathcal{S}$ such that $z^s = 1$. Moreover, Steinle et al. (2018b) show that the big-M term in constraint (10) can be set as $M_{is} = \bar{v}^s_i - \underline{v}^s_i$ for all $i \in \mathcal{H}$, $s \in \mathcal{S}$.

### 3.2 Obtaining a Feasible Solution

Our preliminary computational experiments suggest that the upper bound $b_u$ described in the previous section can be further improved by finding a feasible solution to problem RAMDP-R. Here, we propose a polynomial time heuristic algorithm (Algorithm 1), which benefits from the connection between a substructure of our problem with the robust MDPs to attain a feasible policy effectively.

Algorithm 1 follows two sequential phases: scenario selection and policy selection. In the scenario selection phase, we decide on which scenarios will be enforced to satisfy the chance constraint, i.e., $\tilde{z}^s = 1$. As in the previous subsection, we relax the requirement that the selected policy should be the same over all scenarios, and solve problems (12) independently to obtain the optimal objective function.
Algorithm 1: \textit{initialSolution()}

1. Given distinct cost and transition matrices \( \{ \tilde{c}, \tilde{P} \}_{s \in S} \) with corresponding probabilities \( \{ p \}_{s \in S} \), set \( \hat{z} \leftarrow 0 \);

2. for each scenario \( s \in S \) do
   \begin{enumerate}
   \item Solve problem (12) to obtain its optimal objective function value \( \hat{b}^s \);
   \item Compute VaR\( _\alpha(\hat{b}) \) and find a maximal subset of scenarios \( \hat{S} \subseteq S \) such that \( \hat{b}^s \leq \text{VaR}_\alpha(\hat{b}) \) for each scenario \( s \in \hat{S} \) and \( \sum_{s \in \hat{S} \setminus \{ s' \}} p^s < \alpha \) for all \( s' \in \hat{S} \);
   \item for each scenario \( s \in \hat{S} \) do
     \begin{enumerate}
     \item Set \( \hat{z}^s \leftarrow 1 \);
     \end{enumerate}
   \end{enumerate}

3. Return \textit{robustPolicySelection}(\( \hat{z} \)).

In the policy selection phase provided in Algorithm 2, we use the scenarios selected in the first phase to obtain a feasible policy and the corresponding quantile value. In other words, we let \( S(z) = \hat{S} \) in Line 1 of Algorithm 2 and solve a relaxation of the robust MDP problem

\[
\text{rMDP}(z): \min_{\pi \in \Pi} \max_{s \in S(z)} C(\pi, \tilde{c}^s, \tilde{P}^s),
\]

where \( S(z) := \{ s \in S | z^s = 1 \} \), to find a feasible policy for RAMDP-R. For any scenario vector \( z \), the rMDP\( (z) \) can be seen as an adversarial game, where decision maker selects a stationary policy at the very beginning, and the system evolves based on the worst possible scenario for the selected policy afterwards. An important observation is that for any vector \( z \) satisfying constraints (6c) and (6f), the optimal policy obtained by solving rMDP\( (z) \) and its optimal objective function value correspond to a feasible solution of RAMDP-R for policy variables \( w \) and quantile variable \( y \), respectively. To see this, we reformulate RAMDP-R using the definition of \( C(\pi, \tilde{c}^s, \tilde{P}^s) \) as

\[
\min_{y \in \mathbb{R}, \pi \in \Pi} y \\
\text{s.t.} \quad y \geq C(\pi, \tilde{c}^s, \tilde{P}^s) - (1 - z^s)M, \quad s \in S,
\]

(6c), (6f),

or equivalently

\[
\min_{z:(6c),(6f), \pi \in \Pi} \max_{s \in S(z)} C(\pi, \tilde{c}^s, \tilde{P}^s).
\]

Note that the substructure rMDP\( (z) \) is a robust MDP, where the uncertainty in the transition matrix is coupled across the time horizon and the state space, i.e., a single realization of the cost and transition matrices is selected randomly at the beginning and it holds for all decision epochs and states of the
system. Problem rMDP(\(z\)) is known to be NP-hard even when the parameters are allowed to follow a different scenario for each state and only the scenario considered at each time a state is encountered should be consistent over time (Iyengar, 2005). Therefore, in Algorithm 2, using the scenario vector \(\hat{z}\) obtained in the scenario selection phase, we find a policy by solving a relaxed version of rMDP(\(\hat{z}\)), which we describe next.

**Algorithm 2: robustPolicySelection(\(z\))**

1. Given a small tolerance parameter \(\epsilon > 0\), set some positive \(v_1 \in \mathbb{R}^{|H|}, S(\hat{z}) \leftarrow \{s \in S\mid \hat{z}_s = 1\}\) and \(t \leftarrow 1\);
2. for each state \(i \in H\) do
   3.     for each action \(a \in A\) do
   4.         Set \(\sigma_{ia} \leftarrow \max_{s \in S(\hat{z})} \left\{ c_s^a(a) + \gamma \sum_{j \in H} \hat{P}_{ij}^s(a) \hat{v}_t(j) \right\} \);
   5.         Compute \(\hat{v}_{t+1}(i) \leftarrow \min_{a \in A} \sigma_{ia} ;\)
6.     if \(||\hat{v}_{t+1} - \hat{v}_t|| < \frac{(1-\gamma)\epsilon}{\gamma}\) then
   7.         Go to line 10;
   8.     else
   9.         Set \(t \leftarrow t + 1\) and go to line 2;
10.    Return \(\pi\) such that \(\pi_i \in \arg\max_{a \in A} \sigma_{ia} ;\)

Similar to Nilim and El Ghaoui (2005), we consider a relaxed variant of rMDP(\(\hat{z}\)) by assuming that the costs and transition probabilities are independent over different states and actions, and these parameters are allowed to be time-varying. This setting can be seen as a sequential game, where at each time step, an action is taken by the decision maker and accordingly the system generates a cost and makes a transition based on the worst possible parameter realization in scenario set \(S(\hat{z})\) for the current state and action, in an iterative fashion. Note that Nilim and El Ghaoui (2005) consider the case of transition matrix uncertainty only and prove that the optimal policy in this setting obeys a set of optimality conditions, which can be solved using a robust value iteration algorithm. Here, we further generalize their algorithm for the case of uncertainty in both cost and transition matrices to find a policy that performs well in terms of the VaR objective, but is not necessarily optimal. For the vector \(\hat{z}\) computed in the first phase, we construct the set \(S(\hat{z}) := \{s \in S\mid \hat{z}_s = 1\}\) and solve the following equations

\[
v_i = \min_{a \in A} \max_{s \in S(\hat{z})} \left\{ c_s^a(a) + \gamma \sum_{j \in H} \hat{P}_{ij}^s(a) v_j \right\}, \quad i \in H
\]

through a variant of the value iteration algorithm as presented in Algorithm 2. Note that the policy obtained by Algorithm 2 and the corresponding value functions provide upper bounds on the results of the robust problem rMDP(\(\hat{z}\)). Moreover, due to line 10, Algorithm 2 is guaranteed to terminate with a deterministic policy, hence the obtained solution is also feasible for problem RAMDP-D.
4. An Application to Inventory Management in Long-Term Humanitarian Relief Operations

Long-term humanitarian relief operations, alternatively referred to as continuous aid operations, are vital to sustain daily basic needs of a population affected by a slow-onset disaster including war (Syria, Afghanistan, Iraq), political insurrection (Syria), famines (Yemen, South Sudan, Somalia), droughts (Ethiopia) and extreme poverty (Niger, Liberia). Unlike the sudden-onset disasters (e.g., earthquakes, hurricanes, terrorist attacks), they require delivery of materials such as food, water and medical supplies to satisfy a chronic need over a long period of time. Since the progress of a slow-onset disaster usually presents irregularities in terms of its scale and location, the demand is highly uncertain. In addition, the supply levels are also exposed to uncertainty as they mainly depend on donations from multiple resources. More than 90% of the people affected by slow-onset disasters lives in developing countries, hence the required relief items are usually outsourced from resources in various locations around the world (Rottkemper et al., 2012). Another important consideration in long-term humanitarian operations is the perishability of the relief items. Many items needed during and after a disaster, e.g., food and medication, have a limited shelf life. In addition to the possible uncertainty in the initial shelf life, the remaining shelf life upon arrival is also affected by the uncertainty in the lead times due to unknown location of the donations. Hence, high level of uncertainty inherent in the supply chain makes the system prone to unwanted shortages and disposals. To prevent interruptions, the policy makers may interfere through different actions such as campaigns and advertisements that provide additional relief items.

In this section, we consider an inventory management model for a single perishable item in long-term humanitarian relief operations proposed by Ferreira et al. (2018). They formulate the problem as an infinite-horizon MDP with finite state and action spaces, where the states represent possible levels of inventory. The aim is to minimize the long-term average expected cost. Their model assumes that both demand and supply is uncertain following Poisson distributions with known demand and supply rates, respectively, and the expiration time of the supply items is deterministic. These parameters are generally obtained using various forecasting methods. However, due to multiple sources of uncertainty in the context of long-term humanitarian operations, the forecasted values may be erroneous, which may consequently affect the performance of the resulting policy. To handle the parameter uncertainty, different from Ferreira et al. (2018), we assume that demand and supply rates and the expiration time probabilistically take value from a finite set of scenarios \( \mathcal{S} \) and the objective in our model is to minimize the \( \alpha \)-quantile of the total discounted expected total cost with respect to the uncertainty in these parameters.

In what follows, we elaborate on the components and assumptions of the MDP model based on Ferreira et al. (2018).

**States:** The state of the system is represented by the number of available items in the inventory at the beginning of each decision epoch that will not expire before delivery. It is assumed that the inventory
At the beginning of each decision epoch, the decision maker takes an action \( a \) from a finite set \( \mathcal{A} \) that provides an additional \( n_a \) number of relief items. The set of actions taken for each possible state of the system that minimizes the objective function is referred to as the optimal policy.

**Actions:** At the beginning of each decision epoch, the decision maker takes an action \( a \) from a finite set \( \mathcal{A} \) that provides an additional \( n_a \) number of relief items. The set of actions taken for each possible state of the system that minimizes the objective function is referred to as the optimal policy.

**Transition Probabilities:** Under each scenario \( s \in \mathcal{S} \), the demand \( (D) \) and supply \( (U) \) for the item follow Poisson distribution with probability mass functions \( f_d^s \) and \( f_u^s \) of rates \( \mu_d^s \) and \( \mu_u^s \), respectively, and the expiration time takes value \( t_e^s \). We assume that the demand rate is at least as large as the supply rate. Let \( \Delta_{\min} \) and \( \Delta_{\max} \) be the minimum and maximum possible difference between the supply and demand at any decision epoch considering a 100(1 - \( \epsilon \))% confidence level for a small \( \epsilon > 0 \). Note that the minimum possible demand and supply amounts are assumed as zero so that \( \Delta_{\min} \) and \( \Delta_{\max} \) represent the negative of maximum possible demand and the maximum possible supply, respectively. Then, ignoring the perishability of the item and the actions taken by the decision maker, the probability of transitioning from inventory level \( i \in \mathcal{H} \) to \( j \in \mathcal{H} \) under scenario \( s \in \mathcal{S} \) is

\[
\hat{P}_{ij}^s = \begin{cases} 
\sum_{\Delta=\Delta_{\min}}^{\Delta_{\max}} \sum_{D=0}^{j-i} f_d^s(D) f_u^s(D + \Delta), & \text{if } j = 0, \\
\sum_{\Delta=\Delta_{\min}}^{\Delta_{\max}} \sum_{D=0}^{\Delta_{\max} - \Delta_{\min}} f_d^s(D) f_u^s(D + \Delta), & \text{if } j = K, \\
\sum_{D=0}^{\Delta_{\min}} f_d^s(D) f_u^s(D + j - i), & \text{otherwise.}
\end{cases}
\]

Instead of keeping track of the number of items that expire in each period, the model is simplified by assuming that the items procured in each decision epoch have the same expiration time. As a result, at the beginning of each decision epoch, the expiration probability for the whole batch of newly arriving items is computed as the probability of not being able to consume all available items before the expiration time. Note that since the demand follows Poisson distribution with rate \( \mu_d^s \), the probability of consuming \( k \) items before time \( t_e^s \) follows Erlang distribution with parameters \( \mu_d^s \) and \( k \), which can be stated as \((1 - f_d^s(t_e^s, k))\). Incorporating perishability of the items based on this simplification, the probability of a transition from inventory level \( i \in \mathcal{H} \) to inventory level \( j \in \mathcal{H} \) under scenario \( s \in \mathcal{S} \) when action \( a \in \mathcal{A} \) is taken can be computed as

\[
\hat{P}_{ij}^s(a) = \begin{cases} 
(1 - f_d^s(t_e^s, j)) \hat{P}_{i+n_a,j}^s, & \text{if } j > i + n_a, \\
\sum_{j'=i+n_a+1}^{\min(K,i+n_a+\Delta_{\max})} f_d^s(t_e^s, j') \hat{P}_{i+n_a,j'}^s + \hat{P}_{i+n_a,j}^s, & \text{if } j = i + n_a, \\
\hat{P}_{i+n_a,j}, & \text{otherwise.}
\end{cases}
\]  

\text{(15)}

Equation (15) follows from the fact that if an arriving batch of items is known to expire, the new arrivals are immediately used to fulfill the demand with priority over the existing inventory (the existing inventory
can be used later since it is guaranteed not to expire by the previous assumption), and the remaining items in the batch are disposed. Hence, the case \( j > i + n_a \) can occur only if the new items do not expire. Similarly, \( j = i + n_a \) implies either that the incoming supply is at least as much as the demand in the current period and the supply surplus is disposed, or demand is equal to the sum of supply and additional items acquired by the action taken.

**Cost function:** The expected cost of taking action \( a \in \mathcal{A} \) at inventory level \( i \in \mathcal{H} \) under scenario \( s \in \mathcal{S} \), stated as \( \tilde{c}_s^i(a) \), consists of three main components: the inventory holding cost, the expected shortage cost and the expected disposal cost. Assuming a unit disposal cost of \( \tilde{d} \), the total expected disposal cost when action \( a \in \mathcal{A} \) is taken at inventory level \( i \in \mathcal{H} \) under scenario \( s \in \mathcal{S} \) is

\[
DC(i, a, s) = \sum_{\Delta=1}^{\min(K-i-n_a, \Delta_{\text{max}})} f_c^s(t_s, i + n_a + \Delta) \hat{P}_s^{i+n_a, i+n_a+\Delta} \Delta \hat{d} + \sum_{j=K}^{K+n_a+\Delta_{\text{max}}-\Delta_{\text{min}}} \sum_{D=0}^{K} f_d^s(D) f_u^s(D + j - i - n_a) (j - K) \hat{d}.
\]

The first summation term is due to the expired items, while the second term is because of the inventory surplus. Similarly, the shortage costs are stated as

\[
SC(i, a, s) = \sum_{j=i+n_a+\Delta_{\text{min}}}^{j=K} \sum_{D=0}^{-\Delta_{\text{min}}} f_d^s(D) f_u^s(D + j - i - n_a) (-j) \hat{u}, \quad i \in \mathcal{H}, \ a \in \mathcal{A}, \ s \in \mathcal{S},
\]

where \( \hat{u} \) is the unit shortage cost. Assuming that the unit inventory holding cost is \( h \), the total expected cost of taking action \( a \in \mathcal{A} \) at inventory level \( i \in \mathcal{H} \) under scenario \( s \in \mathcal{S} \) can be computed using \( \tilde{c}_s^i(a) = hi + DC(i, a, s) + SC(i, a, s), \quad i \in \mathcal{H}, \ a \in \mathcal{A}, \ s \in \mathcal{S} \). Note that the cost structure may be different depending on the characteristics of the environment in consideration. The MDP formulation provides decision makers the flexibility to use even nonconvex cost functions. Our methodology only requires the cost terms to be bounded.

### 5. Computational Study

In this section, we conduct computational experiments on the long-term humanitarian relief operations inventory management problem described in Section 4 to examine the effects of incorporating risk aversion towards parameter uncertainty into MDP models and to compare the efficiency of different solution approaches. The problem instances are generated based on the experiments provided in Ferreira et al. (2018) considering the inventory of a blood center, which collects and distributes blood packs to support humanitarian relief operations. We suppose that the inventory replenishment decisions are made on a weekly basis, and the weekly demand and supply rates for the blood packs take values in the intervals \([30, 180]\) and \([20, 80]\), respectively. After donation, each blood pack has a shelf life of six weeks, however unknown lead times may affect the remaining shelf life at the time of arrival to
the blood center. Hence, the shelf life is assumed to be random on the interval $[1, 6]$. In case of need, the center may procure additional supply of blood packs by sending up to $V \in \{2, 3, 4\}$ blood collection vehicles to distant areas so that $\mathcal{A} = \{0, 1, \ldots, V\}$. Each vehicle collects additional 20 blood packs at the expense of incurring a certain cost. We use the cost parameters for additional procurement, inventory holding costs, disposal costs and shortage costs given in Ferreira et al. (2018). Note that state action costs are scaled by 1000 in our computations. We additionally assume that the blood center has a capacity of $K \in \{50, 100, 150\}$ units and the blood packs are in batches of 10 units so that $\mathcal{H} = \{0, 1, \ldots, \lceil K/10 \rceil\}$. Different than Ferreira et al. (2018), we generate random instances with $|\mathcal{S}| \in \{5, 10, 20\}$ equiprobable scenarios, where the parameters of shelf life and demand and supply rates for each scenario randomly take value on their respective intervals stated above. The distribution on the initial state is assumed to be uniform as well.

All experiments are performed using single thread of a Linux server with Intel Haswell E5-2680 processor at 2.5 GHz and 128 GB of RAM using Python 3.6 and Gurobi Optimizer 7.5.1. The time limit for each instance on Gurobi is set to 3600 seconds and we use the default settings for the MIP gap and feasibility tolerances. The results are obtained for the instances with $\alpha = 0.80$ and $\gamma = 0.99$. Unless otherwise stated, each reported value corresponds to the average of two replications and only the instances which can be solved to optimality within the time limit for both replications are included.

5.1 The impact of deterministic policies

We first investigate how the quantile value is affected by the choice of narrowing down the policy space to deterministic policies by comparing the solutions obtained by solving RAMDP-R, which includes randomized policies, with the solutions of RAMDP-D considering only deterministic policies. Because finding a globally optimal solution for RAMDP-R is computationally challenging, we use the open-source MINLP solver Bonmin 1.8 (Bonmin), which provides local optima to nonconvex problems, in combination with Python-based, open-source optimization modeling language Pyomo 5.5 (Hart et al., 2012). The solver is warm-started with the policy obtained by solving Algorithm 1. Since this approach only guarantees local optimality, we additionally obtain lower bounds by solving the McCormick relaxation of RAMDP-R as described in Section 2.

Table 1 compares the results for deterministic policies (RAMDP-D) reported under column “Deterministic” with the results of the case with randomized policies: the objective value of the locally optimal solution obtained by solving RAMDP-R using the MINLP solver Bonmin under column “Randomized - NL” and its lower bound provided by the McCormick relaxation under the column “Randomized - MC”. Each value under column “Obj.” refers to the objective function value of the corresponding problem. The percentage difference between the optimal objective value considering only the deterministic policies, denoted by “$o_d$”, and the objective value of each solution approach allowing for randomized policies, denoted by “$o_r$”, is reported as Diff. (%) = $100 \times \frac{o_r - o_d}{o_d}$ under the column of the corresponding solution.
Table 1: Comparison of deterministic and randomized policies

| Instance | Deterministic | Randomized - NL | Randomized - MC |
|----------|---------------|-----------------|-----------------|
|          | Obj.          | Obj. | Diff. (%) | Obj. | Diff. (%) |
| 5 6 3    | 3463.101      | 3447.642 | 1.10        | 3423.978 | 2.79 |
| 5 6 4    | 2607.511      | 2589.090 | 1.18        | 2533.563 | 3.70 |
| 5 6 5    | 2092.897      | 2076.114 | 1.15        | 1898.200 | 8.38 |
| 5 11 3   | 3145.413      | 3144.661 | 0.09        | 3141.744 | 0.43 |
| 5 11 4   | 2189.095      | 2184.847 | 0.19        | 2174.245 | 0.75 |
| 5 11 5   | 1449.466      | 1443.921 | 0.32        | 1377.446 | 3.76 |
| 5 16 3   | 3032.634      | 3031.919 | 0.11        | 3025.430 | 1.11 |
| 5 16 4   | 2049.630      | 2046.576 | 0.23        | 2039.607 | 1.03 |
| 10 6 3   | 4949.928      | 4949.893 | 0.03        | 4941.964 | 0.19 |
| 10 6 4   | 3389.317      | 3373.038 | 0.57        | 3126.111 | 8.94 |
| 10 6 5   | 2750.743      | 2711.555 | 1.39        | 2412.298 | 12.35 |
| 10 11 3  | 4888.538      | 4887.493 | 0.03        | 4886.354 | 0.05 |
| 10 11 4  | 3101.620      | 3090.454 | 0.37        | 2949.659 | 5.87 |
| 10 16 3  | 4860.883      | 4860.290 | 0.01        | 4859.601 | 0.03 |
| 20 6 3   | 6616.574      | 6616.574 | 0.00        | 6616.574 | 0.00 |
| 20 6 4   | 4698.699      | 4679.635 | 0.41        | 4650.469 | 1.03 |
| 20 6 5   | 3163.172      | 3143.856 | 0.64        | 2854.993 | 9.80 |
| 20 11 3  | 6589.076      | 6589.076 | 0.00        | 6589.076 | 0.00 |
| 20 11 4  | 4604.808      | 4602.596 | 0.05        | 4598.897 | 0.13 |
| 20 11 5  | 2721.644      | 2718.434 | 0.12        | 2655.973 | 2.43 |
| 20 16 3  | 6564.271      | 6564.271 | 0.00        | 6564.271 | 0.00 |
| 20 16 4  | 4576.195      | 4575.169 | 0.02        | 4573.356 | 0.06 |
| Average  | 0.36          | 2.86 |
| Maximum  | 1.39          | 12.35 |
| Minimum  | 0.00          | 0.00 |

Motivated by these results, in the following sections, we focus on the problem RAMDP-D, which only considers deterministic policies.
5.2 The impact of parameter uncertainty  Next we examine the effect of incorporating parameter uncertainty into the MDP model in terms of the value gained by using the available stochastic information and the potential loss due to not having perfect information on the true realization of random parameters.

In Table 2, we compare the optimal objective function value of the quantile optimization problem RAMDP-D, reported under column “OPT”, with two benchmark cases. The first benchmark assumes that the decision maker waits until observing the actual parameter realizations and makes a decision for each scenario independently. This approach does not provide a feasible solution to the original problem because it may produce distinct policies for each scenario. Clearly the quantile value in this case corresponds to a lower bound on the OPT since it can be stated as $\text{LB} = \text{VaR}_{\alpha}(\tilde{b})$, where the realization under scenario $s \in S$ is $\tilde{b}^s = \min_{\pi \in \Pi_D} C(\pi, \tilde{c}^s, \tilde{P}^s)$. Using this value, we compute the value of perfect information on MDP parameters as $\text{VPI} = \text{OPT} - \text{LB}$ and its percentage value as $\%\text{VPI} = 100 \times \frac{\text{OPT} - \text{LB}}{\text{OPT}}$. The VPI values and their respective percentages presented in Table 2 indicate that the average and maximum losses in the quantile function value due to not knowing the true parameter realizations is 2.87% and 12.35%, respectively, for our problem instances. Therefore, it is worthwhile to collect and utilize additional information on the uncertain parameters whenever possible.

Table 2: Value of perfect information and value of stochastic solution of the quantile optimization problem

| $|S|$ | $|H|$ | $|A|$ | OPT | LB | MV | VPI | VSS | %VPI | %VSS |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 5   | 6   | 3   | 3463.101 | 3423.978 | 3482.171 | 39.123 | 19.071 | 2.79 | 1.32 |
| 5   | 6   | 4   | 2607.511 | 2533.563 | 2644.759 | 73.947 | 37.248 | 3.70 | 1.80 |
| 5   | 6   | 5   | 2092.897 | 1898.200 | 2523.066 | 194.696 | 430.170 | 8.38 | 12.73 |
| 5   | 11  | 3   | 3145.413 | 3141.744 | 3157.794 | 3.669 | 12.381 | 0.43 | 1.42 |
| 5   | 11  | 4   | 2189.095 | 2174.245 | 2227.727 | 14.850 | 38.632 | 0.75 | 2.15 |
| 5   | 11  | 5   | 1449.466 | 1377.446 | 1511.640 | 72.020 | 62.174 | 3.76 | 3.73 |
| 5   | 16  | 3   | 3032.634 | 3025.430 | 3037.301 | 10.042 | 18.899 | 1.03 | 1.42 |
| 5   | 16  | 4   | 2049.630 | 2039.605 | 2068.528 | 159.250 | 145.817 | 1.03 | 2.15 |
| 10  | 6   | 3   | 4949.928 | 4941.957 | 5810.361 | 7.970 | 860.433 | 0.19 | 14.48 |
| 10  | 6   | 4   | 3389.317 | 3126.093 | 4128.468 | 263.224 | 739.151 | 8.94 | 17.29 |
| 10  | 6   | 5   | 2750.743 | 2412.291 | 3111.269 | 338.452 | 360.526 | 12.35 | 11.59 |
| 10  | 11  | 3   | 4888.538 | 4886.350 | 4904.396 | 2.188 | 15.859 | 0.05 | 0.37 |
| 10  | 11  | 4   | 3101.620 | 2942.370 | 3247.437 | 159.250 | 145.817 | 0.16 | 0.58 |
| 10  | 11  | 5   | 4860.883 | 4859.598 | 4861.658 | 1.284 | 0.775 | 0.03 | 0.01 |
| 20  | 6   | 3   | 6616.574 | 6615.574 | 6616.574 | 0.000 | 0.000 | 0.00 | 0.00 |
| 20  | 6   | 4   | 4698.699 | 4650.463 | 4749.289 | 48.237 | 50.589 | 1.03 | 1.07 |
| 20  | 6   | 5   | 3163.172 | 2854.986 | 3452.575 | 308.186 | 289.403 | 9.80 | 8.31 |
| 20  | 11  | 3   | 6589.076 | 6589.076 | 6590.956 | 0.000 | 1.880 | 0.00 | 0.03 |
| 20  | 11  | 4   | 4604.808 | 4598.894 | 4618.510 | 5.914 | 13.702 | 0.13 | 0.30 |
| 20  | 11  | 5   | 2721.644 | 2655.967 | 2896.413 | 65.678 | 174.768 | 2.43 | 6.09 |
| 20  | 16  | 3   | 6564.271 | 6564.271 | 6566.770 | 0.000 | 2.499 | 0.00 | 0.04 |
| 20  | 16  | 4   | 4576.195 | 4573.354 | 4585.490 | 2.842 | 9.294 | 0.06 | 0.20 |

The second benchmark considers the quantile function value corresponding to a policy obtained by
solving a single MDP with expected parameter values, referred as the mean value problem in stochastic programming. This provides a heuristic approach to solve RAMDP, and the corresponding quantile function value, denoted as \( MV \), can be computed by treating the expected total cost for the policy obtained by solving the mean value problem under each scenario as a realization of the corresponding random variable, and it provides an upper bound on \( OPT \). Based on \( MV \), the value of incorporating stochasticity of parameters into the MDP model can be measured as \( \text{VSS} = MV - OPT \) and \( \% \text{VSS} = 100 \times \frac{MV - OPT}{MV} \). The results show that by incorporating uncertainty in the parameters, the \( \alpha \)-quantile value can be improved by 4.09% on average with a maximum improvement of 17.29%. Hence, it is possible to achieve significant reductions in risk by incorporating parameter uncertainty into MDPs, and thereby reducing the possibility of undesirable outcomes.

### 5.3 The impact of incorporating risk aversion

Given the advantages of incorporating parameter uncertainty into MDPs, next, we analyze how the performance of a policy would be affected by decision makers’ attitude towards risk. For rarely occurring events such as disasters, it is important to ensure that the selected policy performs well even under undesirable realizations of random parameters.

Table 3 reports \( \alpha \)-quantile value for the policy obtained by solving the expected value problem given in Steimle et al. (2018b), namely “VaR-e”, and compares it to the optimal \( \alpha \)-quantile value, denoted as “VaR∗”. The table contains only the results of the instances for which both expected value and quantile minimization problems can be solved to optimality within one hour. The percentage deviation of the quantile value for the expected value policy from the optimal quantile value is reported under column “Diff. (%)” as \( 100 \times \frac{\text{VaR-e} - \text{VaR∗}}{\text{VaR-e}} \). The results in Table 3 show that the policy obtained by solving the expected value problem may perform worse than the VaR-optimizing policy by 5.87% on average (resp., 13.01% at maximum) in terms of the \( \alpha \)-quantile value. This points out that minimizing the VaR objective instead of the expected value may be beneficial for taking extreme outcomes into consideration.

We additionally analyze the structure of the optimal policies obtained for the quantile optimization problem RAMDP-D and the associated expected value problem in comparison to the optimal policies of distinct scenarios. Figure 1 illustrates the behaviour of the optimal policies under different settings for a particular instance with five scenarios, an inventory capacity of 100 units and at most three vehicles, i.e., \(|S| = 5, |H| = 11, |A| = 4\). The line marked with circles corresponds to the quantile-optimizing policy, where two vehicles are dispatched whenever the available inventory level drops to zero, one vehicle if the inventory level is in the interval \((0, 20]\) and no vehicles otherwise. Similarly, the line marked with squares depicts the optimal policy of the expected value problem and the dashed lines represent the optimal policies for each scenario independently. It can be seen from Figure 1 that the optimal policies for scenarios 2, 3 and 5, which have a cumulative probability of 0.60, are similar to each other, while scenarios 1 and 4 also have similar optimal policies and larger expected total costs. As expected, the optimal policy
Table 3: Performance of the optimal policy of the expected value problem at the quantile

| |S| |H| |A| VaR-e | VaR∗ | Diff. (%) |
|---|---|---|---|---|---|---|---|
|5|6|3|3579.981|3463.101|4.19|
|5|6|4|2863.595|2607.511|7.91|
|5|6|5|2318.715|2092.897|8.69|
|5|11|3|3257.888|3145.413|8.00|
|5|11|4|2354.455|2189.095|9.07|
|5|16|3|3121.174|3032.634|7.87|
|10|6|3|5057.267|4949.928|2.08|
|10|6|4|3624.192|3389.317|5.99|
|10|6|5|3162.039|2750.743|13.01|
|10|11|3|4919.796|4888.538|0.59|
|10|11|4|3264.104|3101.620|5.01|
|20|6|3|6659.532|6616.574|0.65|
|20|6|4|4869.421|4698.699|3.52|
|20|6|5|3560.758|3163.172|11.21|
|20|11|3|6610.031|6589.076|0.32|
|Average| | | | | | |5.87|
|Maximum| | | | | | |13.01|

for the quantile optimization problem is more aligned with the extreme scenarios 1 and 4, while the expected value problem policy presents an average behaviour. Furthermore, we observe, empirically, that the optimal policies for quantile-optimizing humanitarian inventory management problem are monotone policies, where the number of vehicles to be dispatched decreases as the inventory level increases, as illustrated in Figure 1. While this may not be optimal in general, a certain structure of the policies may be desired by the decision makers in some cases. Using such information on the characteristics of the desired policy, it is possible to add the following constraint into our model to enforce a monotone policy structure

\[ w_{ia} \leq \sum_{a' \in A: a' \leq a} w_{i'a'}, \quad a \in \mathcal{A}, \quad i, i' \in \mathcal{H} : \quad i' > i. \]  

Henceforth, we refer to the problem RAMDP-D with additional monotonicity constraint (16) as RAMDP-M. Constraint (16) ensures that if \( n_a \) vehicles are dispatched at the inventory level \( i \), then for any higher inventory level \( i' > i \), the number of vehicles dispatched can be at most \( n_a \) (assuming \( n_a > n_{a'} \) for \( a, a' \in \mathcal{A} \) such that \( a > a' \), as given in the problem statement). Incorporating more information on the characteristics of the desired policy may also provide computational advantages as it reduces the solution space.

5.4 Comparison of Solution Approaches In this section, we evaluate the computational performance of the proposed heuristic algorithms and the MIP reformulations of RAMDP-D in terms of solution times and optimality gaps achieved within the time limit of one hour. Our preliminary experiments using a branch-and-cut algorithm with mixing inequalities indicate poor computational performance because of the difficulty to balance the strength of bounds obtained from the mixing inequalities with the computational effort required to solve the subproblems in each iteration of the algorithm. Hence, these results
We first examine the accuracy of the feasible solutions generated by the heuristic methods: the mean value (MV) problem, Algorithm 1, and the problem with monotone policies (RAMDP-M), where the optimal policies are restricted to be monotone in the sense that the number of vehicles to be dispatched decreases as the inventory level increases. The RAMDP-M is formulated by adding constraint (16) into the McCormick reformulation of RAMDP-D where constraint (10) is replaced by constraints (7) and (8), and the big-M term in (6b) is set to $M = b_u$. Moreover, we add constraint (13) and apply scenario elimination using the best feasible solution as an upper bound as described in Section 3.1. Table
reports the computation time “Time” and the optimality gap “Gap (%)” for these three heuristic solution methods. The optimality gap for a policy with objective function value \(obj\) is computed as 

\[100 \times \frac{(obj - obj^*)}{obj^*},\]

where \(obj^*\) refers to the optimal objective function value of the quantile minimization problem. The results show that the MV problem produces feasible policies in at most 1.55 seconds, but the optimality gap can be as high as 13.04% and 3.52% on average. Algorithm 1, on the other hand, takes longer to terminate with a feasible policy (9.40 seconds on average and 35.69 seconds at most), but provides better optimality gaps with an average of 2.01% and a maximum of 7.96%. Note that performance of Algorithm 1 could be potentially improved by performing a local search on the scenario decision vector \(z\). Surprisingly, the heuristic imposing monotone policies, RAMDP-M, achieves the optimal solution for all instances in our computations and terminates in 6.28 seconds on average with at most 36.98 seconds.

Next we evaluate computational performance of the proposed MIP reformulations of RAMDP-D under the settings detailed below. Note that these MIP reformulations provide an exact representation of problem RAMDP-D.

- **BM**: It corresponds to formulation RAMDP-D where constraint (6e) is replaced by constraint (10) with additional big-M terms. The big-M term in (6b) is set to \(M = b_u\) and the big-M terms \(M_{is} = \bar{v}_s - \underline{v}_s, i \in H, s \in S\) in constraint (10). Moreover, constraint (13) is added and the scenario elimination procedure described in Section 3.1 is applied using the optimal monotone policy solution of RAMDP-M as an upper bound. Gurobi solver is provided with the optimal monotone policy solution as an initial feasible solution.

- **MC**: It corresponds to the McCormick reformulation of RAMDP-D with constraints (7) and (8) replacing (6e). The big-M term in (6b) is set to \(M = b_u\). The optimal monotone policy solution of RAMDP-M is embedded into the solver as an initial feasible solution. Constraint (13) is added and scenario elimination is applied using the optimal monotone policy solution as an upper bound as described in Section 3.1.

In Table 5, we report the best objective function value (“Obj.”) and optimality gap (“Gap”) achieved within the time limit, the total solution time in seconds (“Time”), and the number of nodes in the branch-and-bound tree (“Nodes”) for MIP reformulations BM and MC. The optimality gap values are computed as 

\[100 \times \frac{ub - lb}{ub},\]

where \(ub\) and \(lb\) correspond to the best upper and lower bounds on the optimal objective function value achieved at termination, respectively. The reported solution times do not include the time to obtain bounds and initial solutions. Each row corresponds to the average of two replications and each dagger (†) in column “Gap” indicates a replication with positive optimality gap obtained within the time limit. The results in Table 5 show that the MIP model with McCormick reformulation (MC) outperforms the MIP model with additional big-M terms (BM). Out of 54 instances, MC succeeds to
optimally solve 43 instances, while BM can solve 34 instances optimally within one hour. Moreover, the average solution time for model MC is approximately 870 seconds whereas model BM takes nearly twice, 1530 seconds on average. It can also be seen that McCormick reformulation provides an observable decrease in the number of explored nodes in the branch-and-bound tree.

Lastly we analyze the effectiveness of the best performing exact and heuristic solution methods MC, Algorithm 1 and RAMDP-M on problem instances with up to 1000 scenarios in Table 6. In particular, the accuracy of the heuristic methods, Algorithm 1 and RAMDP-M, is evaluated with respect to the best objective function value achieved by the exact solution method MC within the time limit of one hour.

Note that the solutions obtained by solving RAMDP-M are used as an initial feasible solution to the MIP formulation MC. Under column MC, we report the optimality gap reported by Gurobi at the time of termination ("Gap (%)"), the solution times ("Time (s)"), the number of nodes in the branch-and-bound tree ("Nodes"), and the percentage of the scenario variables $z$ fixed to zero or one by the preprocessing procedures ("Fixed (%)"). For heuristic methods, Algorithm 1 and RAMDP-M, in addition to the solution times ("Time (s)"), we also present their associated gap values with respect to the final solution of MC as $100 \times \frac{\text{obj}_H - \text{obj}_MC}{\text{obj}_MC}$, where $\text{obj}_MC$ and $\text{obj}_H$ denote the best objective function values achieved

| Instance | $|S|$ | $|H|$ | $|A|$ | BM | MC |
|--------|-----|-----|-----|-----|-----|
| | | | | Obj. | Gap (%) | Time | Nodes | Obj. | Gap (%) | Time | Nodes |
| 5 6 3 | 3463.101 | 0.00 | 0.14 | 460 | 3463.101 | 0.00 | 1.06 | 431.5 |
| 5 6 4 | 2607.511 | 0.00 | 1.36 | 4488 | 2607.511 | 0.00 | 3.81 | 5475 |
| 5 6 5 | 2189.095 | 0.01 | 7.69 | 17489 | 2189.095 | 0.01 | 16.60 | 5844 |
| 5 11 3 | 1449.466 | 2.25 | 1810.52 | 10324.5 |
| 5 11 4 | 2189.095 | 0.01 | 1186.49 | 1810.52 | 10324.5 |
| 5 11 5 | 1449.466 | 3.77 | 457900 | 10324.5 |
| 10 6 3 | 4949.928 | 0.00 | 0.13 | 281.5 | 4949.928 | 0.00 | 0.33 | 24 |
| 10 6 4 | 3389.317 | 0.00 | 2.20 | 3871.5 | 3389.317 | 0.00 | 5.23 | 998.5 |
| 10 6 5 | 2750.743 | 0.00 | 11.60 | 14044.5 | 2750.743 | 0.00 | 93.32 | 13501 |
| 10 11 3 | 4888.538 | 0.00 | 34.73 | 18520 | 4888.538 | 0.00 | 1.30 | 16.5 |
| 10 11 4 | 3101.620 | 0.00 | 1840.16 | 1212353 | 3101.620 | 4.79 | 1552.32 | 82784.5 |
| 10 11 5 | 2267.095 | 5.70 | 3600.00 | 82784.5 |
| 10 16 3 | 4860.883 | 0.03 | 1800.03 | 540690 | 4860.883 | 0.01 | 3.69 | 15.5 |
| 10 16 4 | 3071.691 | 5.40 | 3600.00 | 1085973 | 3071.691 | 5.40 | 1821.20 | 92784.5 |
| 10 16 5 | 2078.233 | 2.34 | 3600.00 | 23988.5 |
| 20 6 3 | 6616.574 | 0.00 | 0.04 | 0.00 | 6616.574 | 0.00 | 0.14 | 0 |
| 20 6 4 | 4698.699 | 0.00 | 2.59 | 2847.5 | 4698.699 | 0.00 | 3.57 | 220 |
| 20 6 5 | 3163.172 | 0.00 | 11.95 | 9932.5 | 3163.172 | 0.00 | 36.25 | 3378.5 |
| 20 11 3 | 6589.076 | 0.00 | 0.04 | 0.00 | 6589.076 | 0.00 | 0.35 | 0 |
| 20 11 4 | 4604.808 | 0.00 | 2069.50 | 481034.5 | 4604.808 | 0.01 | 8.86 | 62 |
| 20 11 5 | 2721.644 | 0.01 | 2455.02 | 40426 |
| 20 16 3 | 6564.271 | 0.00 | 0.07 | 0.00 | 6564.271 | 0.00 | 0.68 | 0 |
| 20 16 4 | 4576.227 | 0.01 | 19.49 | 10209.5 |
| 20 16 5 | 2620.774 | 0.43 | 3600.00 | 23988.5 |
| Average | 3508.655 | 0.98 | 1526.67 | 79038.70 | 3508.655 | 0.85 | 870.45 | 19599.63 |

Table 5: Comparison of MIP reformulations of RAMDP-D
Table 6: Comparison of solution methods for larger problem instances

| Instance | $|S|$ | $|H|$ | $|A|$ | MC | Algorithm 1 | RAMDP-M |
|----------|------|------|------|----------------|----------------|----------------|
|          | Gap (%) | Time (s) | Nodes | Fixed (%) | Gap (%) | Time (s) | Gap (%) | Time (s) |
| 50 6 3   | 0.00  | 17.64 | 617.0 | 55.00 | 1.34 | 13.97 | 0.00  | 6.52   |
| 50 6 4   | 0.00  | 51.89 | 1523.5 | 26.00 | 4.03 | 18.63 | 0.00  | 13.66  |
| 50 11 3  | 0.01  | 34.30 | 68.5  | 56.00 | 0.15 | 35.67 | 0.00  | 19.02  |
| 50 11 4  | 1.12  | 3345.91 | 22120.0 | 25.00 | 1.79 | 47.34 | 0.00  | 128.65 |
| 100 6 3  | 0.00  | 24.66 | 48.5  | 68.50 | 0.26 | 35.37 | 0.00  | 14.42  |
| 100 6 4  | 0.00  | 97.52 | 1233.5 | 32.00 | 6.20 | 46.23 | 0.00  | 28.38  |
| 100 11 3 | 0.00  | 42.03 | 3.5   | 68.50 | 0.04 | 85.52 | 0.00  | 40.17  |
| 100 11 4 | 0.32  | 1999.86 | 4419.5 | 31.00 | 1.33 | 114.59 | 0.00  | 142.80 |
| 200 6 3  | 0.00  | 34.37 | 14.5  | 69.75 | 5.22 | 135.29 | 0.00  | 377.62 |
| 200 6 4  | 0.01  | 1046.13 | 2429.0 | 41.25 | 0.04 | 234.93 | 0.00  | 36.69  |
| 200 11 3 | 0.00  | 7.21  | 1.0   | 69.75 | 0.00 | 247.53 | 0.00  | 247.53 |
| 200 11 4 | 0.01  | 1131.66 | 432.0  | 41.50 | 0.85 | 412.47 | 0.00  | 545.18 |
| 500 6 3  | 0.00  | 774.97 | 859.5 | 69.20 | 0.73 | 615.55 | 0.00  | 400.06 |
| 500 6 4  | 8.31  | 3600.07 | 6.0   | 36.60 | 2.28 | 600.05 | 0.00  | 3600.21 |
| 1000 6 3 | 2.38  | 3600.03 | 1   | 67.80 | 0.03 | 1566.66 | 0.40  | 3600.13 |
| 1000 6 4 | 44.40 | 3600.10 | 0.5  | 32.55 | -4.60 | 2160.85 | 0.00  | 3601.13 |
| Average  | 3.53  | 1213.02 | 2111.09 | 49.40 | 1.23 | 388.94 | 0.03  | 800.14 |
| Maximum  | 44.40 | 3600.10 | 22120.00 | 69.75 | 6.20 | 2160.85 | 0.40  | 3601.13 |

within the time limit by the model $MC$ and the heuristic approach of interest, respectively. The results show that Algorithm 1 outperforms the MIP approaches $MC$ and RAMDP-M in terms of solution times at the cost of lower solution accuracy. While average solution time of Algorithm 1 is considerably smaller (388.94 seconds) especially for larger instances, its objective value is worse than the objective function of $MC$ by 1.23% on average with up to a 6.20% difference. The MIP formulation $MC$ provides the best objective function values for all problem instances but one that is of the largest size. It is also noteworthy that the preprocessing procedures given in Section 3.1 predetermine nearly half of the scenario variables on average. RAMDP-M, on the other hand, achieves an objective value as good as $MC$ for all instances except one. It can be also seen that instances of $MC$ with 1000 scenarios terminates after exploring approximately one node after one hour. This is mainly because even solving the root relaxation in the branch-and-bound tree requires significantly longer times for larger problem instances.

6. Conclusions  In this study, we investigate the risk associated with parameter uncertainty in MDPs. We formulate the problem with the objective of minimizing the VaR of the expected total discounted cost of an MDP at a prespecified confidence level $\alpha$ and explore characteristics of the optimal policies. Assuming a discrete representation of uncertainty, we provide MINLP and MIP formulations considering randomized and deterministic policies, and propose preprocessing methods and heuristic algorithms that can be applied for both cases. The proposed modeling approach and solution algorithms are tested on an inventory management problem in the long term humanitarian relief operations context.

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Appendix A. A Branch-and-Cut Algorithm with Mixing Inequalities  Although allowing randomized policies lead to nonlinear mathematical models, the RAMDP-D that considers only deterministic policies can be formulated as an MIP using additional big-M inequalities or McCormick envelopes as presented in Section 2. The resulting MIP models can be seen as two-stage CCPs, where given a first-stage solution consisting of \(\alpha\)-quantile value, a policy and a subset of scenarios with cumulative probability of at least \(\alpha\), the second-stage subproblems enforces that the expected total discounted cost for the given policy is not larger than the predetermined \(\alpha\)-quantile value for each selected scenario of parameter realizations. Based on this perspective, here we adapt a branch-and-cut algorithm with mixing inequalities (see Luedtke et al., 2010; Liu et al., 2016) for the McCormick reformulation of RAMDP-D. Under the assumptions that there exists a finite representation for the joint distribution of \((\tilde{c}, \tilde{P})\) and only the deterministic stationary policies are of interest, the MIP reformulation of RAMDP-D is

\[
\begin{align*}
\min \quad & y \\
\text{s.t.} \quad & \sum_{a \in A} w_{ia} = 1, \quad i \in \mathcal{H}, \quad \text{(17a)} \\
\quad & \sum_{s \in S} z^s p^s \geq \alpha, \quad \text{(17b)} \\
\quad & \sum_{i \in \mathcal{H}} q_i v_i^s \leq y + (1 - z^s) M, \quad s \in \mathcal{S}, \quad \text{(17c)} \\
\quad & v_i^s \geq \sum_{a \in A} \tilde{c}_i^s (a) w_{ia} + \gamma \sum_{a \in A, j \in \mathcal{H}} \tilde{P}_{ij}^s (a) x_{ij}^s, \quad i \in \mathcal{H}, \quad s \in \mathcal{S}, \quad \text{(17d)} \\
\quad & \ell_j^s w_{ia} \leq x_{ij}^s \leq u_j^s w_{ia}, \quad i, j \in \mathcal{H}, \quad a \in \mathcal{A}, \quad s \in \mathcal{S}, \quad \text{(17e)} \\
\quad & v_j^s - (1 - w_{ia}) u_j^s \leq x_{ij}^s \leq v_j^s - (1 - w_{ia}) \ell_j^s, \quad i, j \in \mathcal{H}, \quad a \in \mathcal{A}, \quad s \in \mathcal{S}, \quad \text{(17f)} \\
\quad & z^s \in \{0, 1\}, \quad s \in \mathcal{S}, \quad \text{(17g)} \\
\quad & w_{ia} \in \{0, 1\}, \quad i \in \mathcal{H}, \quad a \in \mathcal{A}, \quad \text{(17h)}
\end{align*}
\]

where constraints (17f)-(17g) are associated with the McCormick envelope corresponding to \(x_{ij}^s = v_j^s w_{ia}\) for \(i, j \in \mathcal{H}, \quad a \in \mathcal{A}, \quad s \in \mathcal{S}\).

We relax the second-stage feasibility requirements (17d)–(17g) and consider

\[
\begin{align*}
\text{(MP)} \min \quad & y \\
\text{s.t.} \quad & \sum_{a \in A} w_{ia} = 1, \quad i \in \mathcal{H},
\end{align*}
\]
\[
\sum_{s \in S} z^s p^s \geq \alpha, \\
z^s \in \{0, 1\}, \ s \in S, \\
w_{ia} \in \{0, 1\}, \ i \in H, \ a \in A.
\]

After fixing variables \((y, w, z)\), the remaining subproblem is a linear program that can be decomposed over the set of scenarios. Recall that for any feasible solution \((\hat{y}, \hat{w}, \hat{z})\) of MP, we only need to check feasibility of the second-stage subproblems for scenarios with \(z^s = 1\) to satisfy the chance constraint. Hence, we solve the following subproblem for any scenario \(s \in S\) with \(z^s = 1\)

\[
(PS^s) \quad \min \ 0 \\
\text{s.t.} \quad \sum_{i \in H} -q_i v_i^s \geq -\hat{y}, \\
v_i^s - \gamma \sum_{a \in A} \sum_{j \in H} \tilde{P}_{ij}^s(a) x_{ija}^s \geq \sum_{a \in A} \bar{c}_i^s(a) \hat{w}_{ia}, \ i \in H, \\
x_{ija}^s \geq \ell_{ija}^s, \ i, j \in H, \ a \in A, \\
x_{ija}^s \geq -u_{ija}^s, \ i, j \in H, \ a \in A, \\
x_{ija}^s - v_j^s \geq -(1 - \hat{w}_{ia}) u_{ija}^s, \ i, j \in H, \ a \in A, \\
x_{ija}^s + v_j^s \geq (1 - \hat{w}_{ia}) \ell_{ija}^s, \ i, j \in H, \ a \in A, \\
x_{ija}^s \geq \ell_{ija}^s, \ i, j \in H, \ a \in A, \\
x_{ija}^s \geq -u_{ija}^s, \ i, j \in H, \ a \in A.
\]

The subproblem \(PS^s\) is a feasibility problem checking if the expected total discounted cost of the MDP with parameters in scenario \(s \in S\) for the policy determined by \(\hat{w}\) is no more than the value of \(\hat{y}\). The decisions in these subproblems do not correspond to recourse actions, but they are merely used for computation of the expected total discounted cost of the MDP for the given policy and scenario. Note that constraints \((19h)-(19i)\) can be removed from the formulation without affecting the optimal solution.

Let \(\rho, \eta, \lambda, \Lambda, \beta, \delta, \theta, \vartheta\) be dual variables associated with constraints \((19b)-(19i)\), respectively. Then, the dual of subproblem \(PS^s\) for scenario \(s \in S\) with \(z^s = 1\) can be written as

\[
(DS^s) \quad \max \ -\rho \hat{y} + \sum_{i \in H} \sum_{a \in A} \hat{w}_{ia} \tilde{c}_i^s(a) \eta_i + \sum_{i \in H} \sum_{j \in H} \sum_{a \in A} [\hat{w}_{ia} (\ell_{ija}^s \lambda_{ija} - u_{ija}^s \Lambda_{ija}) + (1 - \hat{w}_{ia}) (\ell_{ija}^s \beta_{ija} - u_{ija}^s \delta_{ija})] \\
+ \sum_{i \in H} (\ell_{ija}^s \theta_i - u_{ija}^s \vartheta_i) \\
\text{s.t.} \quad -q_i \rho + \eta_i + \sum_{j \in H} \sum_{a \in A} (\beta_{ija} - \delta_{ija}) + \theta_i - \vartheta_i = 0, \ i \in H, \\
-\gamma \tilde{P}_{ij}^s(a) \eta_j + (\lambda_{ija} - \Lambda_{ija}) + (\delta_{ija} - \beta_{ija}) = 0, \ i, j \in H, \ a \in A,
\]
\( \rho, \eta, \lambda, \Lambda, \beta, \delta, \theta, \vartheta \geq 0. \)

The dual subproblem \( DS^s \) is unbounded if and only if the primal subproblem \( PS^s \) is infeasible because the objective function of \( PS^s \) is bounded and \( DS^s \) is always feasible (consider a solution with all zeros). Hence, instead of \( PS^s \), we solve dual subproblem \( DS^s \) for all scenarios \( s \in S \) with \( z^s = 1 \) and obtain an extreme ray in case of unboundedness as a proof of infeasibility for problem \( PS^s \). Suppose that there exists a scenario \( \hat{s} \) whose dual subproblem \( DS^\hat{s} \) is unbounded, and the associated extreme ray is \( (\hat{\rho}, \hat{\eta}, \hat{\lambda}, \hat{\beta}, \hat{\delta}, \hat{\theta}, \hat{\vartheta}) \). Let \( \hat{\phi}_{ia} = \sum_{j \in H} [(\hat{\lambda}_{ija} - \hat{\delta}_{ija})u^j_s - (\hat{\lambda}_{ija} - \hat{\beta}_{ija})\ell^j_s] - \hat{c}^i(a)\hat{\eta}_i \) for \( i \in H, a \in A \).

Following Luedtke et al. (2010) and Liu et al. (2016), given the extreme ray of \( DS^\hat{s} \), we solve the following subproblem for all \( s \in S \)

\[
\begin{align*}
  h_s &= \min_{\hat{\rho}y + \sum_{i \in H} \sum_{a \in A} \phi_{ia} w_{ia}} \\
  &\text{s.t.} \sum_{a \in A} w_{ia} = 1, \quad i \in H, \quad (21a) \\
  &\sum_{i \in H} q_i u^s_i \leq y, \quad (21b) \\
  &v^s_i \geq \sum_{a \in A} \hat{c}^s_i(a)w_{ia} + \gamma \sum_{a \in A} \sum_{j \in H} \hat{P}^{s}_{ij}(a)x^s_{ija}, \quad i \in H, \quad (21c) \\
  &\ell^j_wia \leq x^s_{ija} \leq u^s_jw_{ia}, \quad i, j \in H, \quad a \in A, \quad (21d) \\
  &v^s_j - (1 - w_{ia})u^s_j \leq x^s_{ija} \leq v^s_j - (1 - w_{ia})\ell^s_j, \quad i, j \in H, \quad a \in A, \quad (21e) \\
  &w_{ia} \in \{0, 1\}, \quad i \in H, \quad a \in A, \quad (21f)
\end{align*}
\]

or equivalently

\[
\begin{align*}
  h_s &= \min_{\sum_{i \in H} \hat{\rho}q_i v^s_i + \sum_{i \in H} \sum_{a \in A} \phi_{ia} w_{ia}} \\
  &\text{s.t.} \sum_{a \in A} w_{ia} = 1, \quad i \in H, \quad (22a) \\
  &v^s_i = \sum_{a \in A} \hat{c}^s_i(a)w_{ia} + \gamma \sum_{a \in A} \sum_{j \in H} \hat{P}^{s}_{ij}(a)x^s_{ija}, \quad i \in H, \quad (22b) \\
  &\ell^j_wia \leq x^s_{ija} \leq u^s_jw_{ia}, \quad i, j \in H, \quad a \in A, \quad (22c) \\
  &v^s_j - (1 - w_{ia})u^s_j \leq x^s_{ija} \leq v^s_j - (1 - w_{ia})\ell^s_j, \quad i, j \in H, \quad a \in A, \quad (22d) \\
  &w_{ia} \in \{0, 1\}, \quad i \in H, \quad a \in A. \quad (22e)
\end{align*}
\]

For any scenario \( s \in S \), the value of \( h_s \) can be regarded as the optimal total expected discounted cost of an infinite horizon MDP with additional one-time state-action costs, i.e., it costs \( \phi_{ia} \) to have the option to use action \( a \in A \) in state \( i \in H \). This problem setting corresponds to finding a stationary optimal

...
policy to a non-stationary infinite horizon MDP. Note that, for the infinite-horizon MDPs, computing $h_s$ requires the solution of an MIP formulation as stated above. In case of finite-horizon MDPs, on the other hand, the one-time state-action costs $\phi_{ia}$ for $i \in \mathcal{H}$, $a \in \mathcal{A}$ can be aggregated into the first decision epoch and the problem can be solved as a regular finite-horizon MDP.

In what follows, we describe the valid inequalities used for eliminating infeasible solutions in the branch-and-cut algorithm. We present the feasibility cuts given in Luedtke (2014), which are based on the mixing inequalities with knapsack inequality $\sum_{s \in S} (1 - z^s)p^s \leq 1 - \alpha$. Let $\sigma$ be a permutation of scenarios such that $h_{\sigma_1} \geq h_{\sigma_2} \geq \ldots \geq h_{\sigma_{|S|}}$ and $k := \max\{i : \sum_{j=1}^i p^{\sigma_j} \leq 1 - \alpha\}$. Let $T = \{t_1, \ldots, t_l\} \subseteq \{\sigma_1, \ldots, \sigma_k\}$ be such that $h_{t_i} \geq h_{t_{i+1}}$ for $i = 1, \ldots, l$, where $t_1 = \sigma_1$ and $t_{l+1} = \sigma_{k+1}$. Then the inequality

\[ \hat{\rho}y - \sum_{i \in \mathcal{H}} \sum_{a \in \mathcal{A}} \phi_{ia} w_{ia} + \sum_{i=1}^l (h_{t_i} - h_{t_{i+1}})(1 - z^{t_i}) \geq h_{t_1} \quad (23) \]

is valid for problem RAMDP-D. These inequalities remain valid for any $h$ computed using a relaxation of the associated mixing subproblem, however, this may result in weaker inequalities. Moreover, similar inequalities can be derived for the MIP reformulation $BM$, but our preliminary results indicate poor computational performance due to the existence of additional big-M terms.

The branch-and-cut algorithm (B&C) starts with the relaxed problem MP. For each incumbent integer solution, it checks whether the chance constraint is violated by solving subproblems $DS_s$ for each scenario $s \in S$ with $z^s = 1$. If there exists a direction of unboundedness of $DS_s$ for some scenario $s \in S$ with $z^s = 1$, then the associated mixing inequality (23) is added to eliminate the current solution. This algorithm is guaranteed to terminate in a finite number of iterations.

We compare the computational performance of the branch-and-cut algorithm (B&C) with the best-performing MIP formulation $MC$. The B&C algorithm is implemented using the callback function of Gurobi on a single branch-and-bound tree. Instead of solving MIP formulations in each step of the B&C algorithm, we compute the $h_s$ as the optimal value of the linear programming relaxation of the associated optimization problem for $s \in S$. Table 7 reports the percentage optimality gap after one hour (“Gap(%)”), the time until termination due to optimality or time limit (“Time (s)”), and the number of nodes explored in the branch-and-bound tree (“Nodes”) for both methods. For the B&C algorithm, we additionally present the total number of times the callback function is invoked (“Callb.”) and the total time spent in the callback (“CallTime (s)”). Each dagger corresponds to an instance that can not be solved optimally within the time limit but terminated with a feasible solution, and each asterisk indicates an instance with no feasible solution obtained within the time limit. Out of the twelve instances, the B&C fails to achieve optimality for eight instances and terminates with no feasible solution for six of them. The $MC$, on the other hand, produces a feasible solution for each problem instance, while proving
optimality for all except one. Moreover, the solution times and the number of nodes are significantly larger for the B&C algorithm. Even though we solve linear programming relaxations of the additional subproblems, the B&C algorithm spends the most of the solution time in the callback for generating the mixing inequalities. Hence, the B&C algorithm is excluded from the main text.

Table 7: Comparison of the MIP formulation MC and the branch-and-cut algorithm

| Instance | MC          | B&C          |
|----------|-------------|--------------|
| | Gap (%) | Time (s) | Nodes | Gap (%) | Time (s) | Nodes | Calib. | CallTime (s) |
| | | | | | | | | |
| | | | | | | | | |
| 100 6 3 | 0.00 | 24.66 | 48.5 | 0.00 | 334.98 | 640 | 248.5 | 334.35 |
| 100 6 4 | 0.00 | 97.52 | 1233.5 | 2.62† | 3278.33 | 3210 | 1182 | 3248.48 |
| 100 11 3 | 0.00 | 42.03 | 3.5 | 0.01 | 1749.95 | 1590 | 352.5 | 1745.53 |
| 100 11 4 | 0.32† | 1999.86 | 4419.5 | ** | - | - | - | - |
| 200 6 3 | 0.00 | 34.37 | 14.5 | 0.01 | 544.54 | 409.5 | 121 | 544.24 |
| 200 6 4 | 0.01 | 1046.13 | 2429 | ** | - | - | - | - |
| 200 11 3 | 0.00 | 7.21 | 1 | 0.04† | 3343.24 | 775 | 249 | 3342.14 |
| 200 11 4 | 0.01 | 1131.66 | 432 | ** | - | - | - | - |

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