Trans-Planckian Scale 
and 
Krein Space Quantization 

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February 23, 2012

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Abstract 

In this work, Krein space quantization method is applied to eliminate the Ultraviolet divergence of Green functions. This paper shows that the power spectrum of scalar field fluctuations can be calculated in the limit of short distance physics, trans-Planckian physics, without using the usual re-normalization process.

Keywords: Krein space quantization, power spectrum, trans-Planckian scale.

\section{Introduction}

It is proven that for the minimally coupled scalar field in de Sitter space a naturally re-normalized quantum field theory can be constructed by Krein space quantization method [1]. It is shown that in this quantization method, the Hamiltonian operator has a finite vacuum expectation value. This result is achieved in different way, without using the normal ordering [1, 2]. It is one of the most noticeable results of this quantization method. Another of the noticeable results of this method is the elimination of the Ultraviolet divergence of Green functions automatically [1, 2]. Moreover, since the minimally coupled scalar field in de Sitter space plays an important role in the inflationary universe as well as in the linear quantum gravity [3], therefore we refer to previous work in which the power spectrum of scalar field fluctuations is calculated by Krein quantization method [4]. Let us make this remark that there is a similarity between the Krein quantization method and usual re-normalization. In the curved space-time, the standard re-normalization of the Ultraviolet divergence of the vacuum energy is accomplished by subtracting the local divergence of Minkowski space [4, 5]

\begin{equation}
\langle \Omega | : T_{\mu\nu} : | \Omega \rangle = \langle \Omega | T_{\mu\nu} | \Omega \rangle - \langle 0 | T_{\mu\nu} | 0 \rangle. \tag{1}
\end{equation}
In which $|\Omega\rangle$ is the vacuum state in curved space and $|0\rangle$ is the vacuum state in Minkowski space. The minus sign in equation (1) can be interpreted as the negative norm states which is added to the positive norm states [2]. This interpretation resembles Krein space quantization method, where the negative norm states are considered. Indeed in this method field operator is built by considering two possible solutions of field equations, positive and negative norm states [1, 2]. These negative norm states are defined in Minkowski space and they are not the solutions of the wave equation in the curved space-time. The auxiliary negative norm states in our method are similar to the ghost states in the standard gauge QFT [6] and they play the role of an automatic re-normalization tool [7, 8, 9]. Therefore the auxiliary negative norm states (ghost states) can neither propagate in the physical world nor interact with physical states.

Since the effects of short distance physics, such as trans-Planckian scale, non commutative space-time and so on, might be observable in cosmological scales on the power spectrum of Cosmic Microwave Background Radiation (CMBR)[10, 11, 12, 13, 14, 15], therefore it is reasonable to present the another ability for Krein space quantization method, which is the calculation of the power spectrum for scalar field fluctuations in the early universe. To reach this goal we review the trans-Planckian scale at first and then we explain how Krein quantization method can solve our problem.

2 Trans-Planckian Scale

Let us consider a background as a de Sitter universe with metric [4, 14]

$$ds^2 = dt^2 - e^{2Ht} \, d\vec{x}^2.$$  \hspace{1cm} (2)

The equation of motion for scalar field in this background is given by [4, 5, 14]

$$\ddot{\phi} + 3H \dot{\phi} - \nabla^2 \phi = 0.$$  \hspace{1cm} (3)

Where field operator can be represented in the form [5, 6]

$$\phi(\eta, \vec{x}) = (2\pi)^{-3/2} \int d^3\vec{k} [a_k u_k(\eta)e^{i\vec{k}.\vec{x}} + a_k^\dagger u_k^\dagger(\eta)e^{-i\vec{k}.\vec{x}}].$$  \hspace{1cm} (4)

And Fourier modes, satisfies the equation

$$u_k'' + 2aH u_k - k^2 u_k = 0.$$  \hspace{1cm} (5)

The exact solutions of (5) are

$$u_k^\pm = \frac{1}{a \sqrt{2k}} (1 \mp \frac{i}{k\eta})e^{\mp ik\eta}.$$  \hspace{1cm} (6)

Continuing in the Heisenberg picture leads to Bogoliubov transformations for $a$ and $a^\dagger$ between fixed time $\eta_0$ and $\eta$ [5, 16]

$$a_k(\eta) = \alpha a_k(\eta_0) + \beta a_k^\dagger(\eta_0), \quad a_k^\dagger(\eta) = \alpha^* a_k^\dagger(\eta_0) + \beta^* a_k(\eta_0).$$  \hspace{1cm} (7)

It is convenient to choose a boundary initial condition by stating that the modes in the limit $\eta_0 \rightarrow \infty$ are positive norms of the Bunch-Davis vacuum [5, 17]. But if the boundary initial
condition chosen differently, by the trans-planckian considerations, the vacuum state would not being a Bunch-Davis one [10, 11, 12, 13, 14, 15]. Therefore one can choose the vacuum by considering \( \eta_0 = -\frac{\Lambda}{H} \) has a finite value and \( \Lambda \) is the energy scale, e.g., Planck scale [14]. If one assumes \( \frac{\Lambda}{H} \gg 1 \), the power spectrum of scalar field fluctuations is given by [14]
\[
P_\phi = \left( \frac{H}{2\pi} \right)^2 (1 - \frac{H}{\Lambda} \sin(\frac{2\Lambda}{H})).
\] (8)
Which is a scale-dependent power spectrum.

3 Power Spectrum in Krein Space Quantization

According to the first perspective of [4] one can choose the negative norms with respect to the Minkowski background. The equation of the negative norm states and it’s solution is:
\[
u''_k + \frac{k^2}{a^2} \nu_k = 0, \quad \nu_{k,n} = \frac{e^{i k \eta}}{a \sqrt{2k}}.
\] (9)
The scalar field operator in this view is given by [2, 4]
\[
\phi(\eta, \vec{x}) = (2\pi)^{-3/2} \int d^3 \vec{k} \left\{ [\nu_k(\eta) a_k e^{-i \vec{k}, \vec{x}} + \nu_k^*(\eta) a_k^\dagger e^{i \vec{k}, \vec{x}}] \\
+ [\nu_{k,n}(\eta) b_k e^{-i \vec{k}, \vec{x}} + \nu_{k,n}^*(\eta) b_k^\dagger e^{i \vec{k}, \vec{x}}] \right\}.
\] (10)
Where \( u_k \) and \( u_{k,n} \) are defined respectively in (6) and (9). Also \( a_k \) and \( b_k \) are two independent operators. Creation and annihilation operators are constrained to obey the following commutation rules
\[
[a_k, a_k^\dagger] = \delta_{kk'}, \quad [b_k, b_k^\dagger] = -\delta_{kk'}.
\] (11)
Since the nature of fluctuations is statistically Gaussian, then the variance of it’s distribution is given by [5, 17, 18, 19]
\[
\langle \phi^2 \rangle = \frac{1}{(2\pi)^3} \int |\nu_k|^2 d^3 \vec{k} - \frac{1}{(2\pi)^3} \int |\nu_{k,n}|^2 d^3 \vec{k}.
\] (12)
In which the negative norm states satisfy the equation (9) and positive ones can be written from (6) generally
\[
u_k = \frac{A}{a \sqrt{2k}} (1 - \frac{i}{k \eta}) e^{-i \vec{k}, \vec{x}} + \frac{B}{a \sqrt{2k}} (1 + \frac{i}{k \eta}) e^{i \vec{k}, \vec{x}}.
\] (13)
which by considering initial condition with finite \( \eta_0 \) and normalization condition for \( A \) and \( B \), e.g., \( |A|^2 - |B|^2 = 1 \), we have [14]
\[
B = A \frac{ie^{-2i k \eta_0}}{2k \eta_0 + i}, \quad |A|^2 = \frac{1}{1 - |\gamma|^2}, \quad \gamma = \frac{i}{2k \eta_0 + i}.
\] (14)
The next step is calculation of variance of distribution with using Krein quantization method. Using (9) and (13), we may write (12) as
\[
\langle \phi^2 \rangle = \frac{1}{(2\pi)^3} \int d^3 \vec{k} \left\{ [|A|^2 + |B|^2] (1 + \frac{1}{(k \eta)^2}) + A^* B e^{2i k \eta} (1 + \frac{i}{k \eta})^2 \right\}
\]
\[ + B^\ast A e^{-2i k \eta} (1 - \frac{i}{k \eta})^2 ] \right] \frac{1}{(2 \pi)^3} \int \frac{d^3 \vec{k}}{2 k a^2}. \] (15)

With substituting (14) in (15) one can find

\[ \langle \phi^2 \rangle = \frac{1}{(2 \pi)^3} \int \frac{d^3 \vec{k}}{2 k a^2} \left\{ \left| A \right|^2 \left[ (1 + \frac{1}{1 + 4 k^2 \eta_0^2}) (1 + \frac{1}{(k \eta)^2}) + \gamma e^{2i k (\eta - \eta_0)} (1 + \frac{i}{k \eta})^2 \right. \right. \\
+ \frac{1}{(2 \pi)^3} \int \frac{d^3 \vec{k}}{2 k a^2}. \] (16)

Calculating (16) in the limit \( \eta \to 0 \), \( \frac{A}{H} \gg 1 \), and substitute \( \eta = -\frac{1}{a H} \) leads to

\[ \langle \phi^2 \rangle = \frac{1}{(2 \pi)^3} \int \frac{d^3 \vec{k}}{2 k a^2} \left[ \left( \frac{H^2}{2 k^3} + \frac{1}{2 k a^2} \right) - \frac{H^3}{2 \Lambda k^3} \sin \left( \frac{2 \Lambda}{H} \right) \right] - \frac{1}{(2 \pi)^3} \int \frac{d^3 \vec{k}}{2 k a^2}. \] (17)

We can see explicitly vanishing Ultraviolet divergence from (17).

## 4 Conclusions

In this work we shows that Krein space quantization method can be convenient in constructing naturally re-normalized QFT. We investigated that it is possible to remove Ultraviolet divergence of Wightman two-point function which is applicable in inflationary cosmology, by considering negative norm states.

**Acknowledgements:** We would like to thank M. V. Takook for his valuable help.

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