PURE NASH EQUILIBRIA AND BEST-RESPONSE DYNAMICS IN RANDOM GAMES

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ABSTRACT. In finite games mixed Nash equilibria always exist, but pure equilibria may fail to exist. To assess the relevance of this nonexistence, we consider games where the payoffs are drawn at random. In particular, we focus on games where a large number of players can each choose one of two possible actions, and the payoffs are i.i.d. with the possibility of ties. We provide asymptotic results about the random number of pure Nash equilibria, such as fast growth and a central limit theorem, with bounds for the approximation error. Moreover, by using a new link between percolation models and game theory, we describe in detail the geometry of Nash equilibria and show that, when the probability of ties is small, a best-response dynamics reaches a Nash equilibrium with a probability that quickly approaches one as the number of players grows. We show that a multitude of phase transitions depend only on a single parameter of the model, that is, the probability of having ties.

1. Introduction

1.1. Background and motivation. A pure equilibrium in a game is a profile of actions (one for each player) such that, given the choice of the other players, no player has an incentive to make a different choice. In other words, deviations from an equilibrium are not profitable for any player. This concept of equilibrium, although quite simple and powerful, has the drawback that not every game admits pure equilibria. John Nash’s major contribution was to introduce the more general concept of mixed equilibrium, and to show that—in a game with a finite number of players and actions—the existence of mixed equilibria is guaranteed (Nash, 1951, 1950). A mixed action of a player is a probability distribution over their action set. When mixed actions are allowed, the choice criterion is the expected payoff with respect to the product of the mixed actions. As before, a mixed equilibrium is a profile of mixed actions that does not allow profitable deviations.

Although the definition and properties of mixed actions and mixed equilibria are clear, their interpretation is far from unanimous. Section 3.2 of Osborne and Rubinstein (1994), dedicated to the interpretation of mixed equilibria, has paragraphs individually signed by each of the two authors, since they could not reach an agreement. In general, pure equilibria have a stronger epistemic foundation than mixed equilibria. As mentioned before, the main problem of pure equilibria is existence.

Some authors have tried to frame this problem in a stochastic way: given a set of players and a set of actions for each player, if payoffs are drawn at random, what is the probability that the game admits pure Nash equilibria? More precisely, what is the distribution of the number of pure Nash equilibria in a game with random payoffs? The
answer to this question clearly depends on the way random payoffs are drawn. In any case, for a fixed number of players and actions, the answer is computationally daunting. For this reason some papers have chosen to investigate the problem from an asymptotic viewpoint; that is, they have looked at the limit distribution of the number of pure Nash equilibria as the number of either actions or players goes to infinity.

The basic common assumption of much of this literature is that the distribution of the random payoffs is nonatomic and payoff profiles are independent. Under these hypotheses, the probability that two payoffs coincide is zero and, as a consequence, calculations are significantly simplified.

It is well-known that Nash equilibria are hard to compute (Daskalakis et al., 2009). One way to address the issue is to devise iterative procedures that converge to a Nash equilibrium. One very natural adaptive procedure is best-response dynamics (BRD) (see, e.g., Roughgarden (2016, chapter 16)): starting from an action profile, a single player is picked at random and allowed to choose a different action. This player will choose one of the most profitable actions among all alternative actions with a strictly higher payoff. If no such action exists, the player will not move and a different player is chosen at random. When a new action profile is reached, the process is repeated. If we reach a profile for which no player has a profitable deviation, then the process has reached a pure Nash equilibrium. The question is whether, starting from any action profile, a pure equilibrium is reached. In general the answer is negative: first, because a game may fail to have pure equilibria; second, due to the fact that even when pure equilibria exist, players may be trapped in a subset of vertices that does not contain a pure Nash equilibrium. One way to determine the severity of this failure to reach a pure Nash equilibrium via best-response dynamics is to examine games with random payoffs.

1.2. Our contribution. In the present paper we consider games where the number $N$ of players is large, each player has two actions, and payoffs are random. The main novelty of our approach is to show the strict relation between three different topics—games with random payoffs, best-response dynamics, and percolation—and to provide analytic results, rather than simulations. In games with random payoffs, a significant part of the existing literature has focused on the number of pure Nash equilibrium. Here we extend this analysis, but we also connect it to the behavior of best-response dynamics and provide results that concern not only the number of pure equilibria, but also how easily they can be reached via best-response dynamics. The main tool for this analysis is a suitable correspondence between a random directed graph that represents our random game and a suitable percolation graph.

A main feature of the present work is that we dispense with the assumption of nonatomic distribution of the payoffs and therefore allow ties to exist. We show that the probability of ties plays a crucial role in many ways. For example, it determines the asymptotic distribution of the number of pure Nash equilibria. Moreover, we establish a novel connection with percolation theory on the hypercube. We use tools from percolation theory to describe the geometry of the set of pure Nash equilibria, which also depends on the probability of ties. This description permits to analyze the performance of best-response dynamics. This has been extensively done in the literature for the class of potential games. In our paper we can show that, asymptotically in the number of players, with
high probability best-response dynamics converges to a pure Nash equilibrium, if the probability of ties is small (less than 0.55).

As mentioned before, the probability of ties in the payoffs, which we call \( \alpha \), is the fundamental parameter in this model. Different values of \( \alpha \) produce different possible behaviors in the number of pure Nash equilibria as well as in their correlation structure. We will show that, for as long as \( \alpha \) is positive, the game has many pure equilibria with very high probability, and best-response dynamics converges to one of them (Theorems 4.3 and 4.4 below). Moreover, if \( \alpha \) is strictly less than 1/2, then all pure Nash equilibria are reachable with high probability via best-response dynamics from any deterministic starting point. Conversely, some of them are unreachable when \( \alpha \) is at least 1/2. Furthermore, when \( \alpha \) is positive, Theorem 3.2 shows a concentration of the number of pure Nash equilibria around \((1 + \alpha)^N\) and establishes a central limit theorem, using the Chen-Stein method, as developed in Chatterjee (2008).

To illustrate this phenomenon, we plot in Fig. 1 the case where \( Z \) takes only the values \{-1, 1\} with equal probability (notice that \( \alpha = 0.5 \) in this context). The average number of pure Nash equilibria exactly fits the curve \((1.5)^N\), confirming our prediction. Moreover, we are able to quantify the fluctuations (see Theorem 3.2) which are of the order \((1 + \alpha)^{N/2}\). Finally, the number of pure Nash equilibria, properly rescaled, rapidly converges to a standard normal (see Fig. 2), with speed of convergence of the order \((1 + \alpha)^{-N/4}\). We emphasize that our results depend on the payoff distributions only through the parameter \( \alpha \), and remain applicable even when the payoff distributions vary among players.

1.3. Related work. As mentioned before, several papers have considered aspects related to the number of pure equilibria in games with random payoffs. In many of the papers that we consider below, the random payoffs are i.i.d. from a continuous distribution. Unless otherwise stated, this is the assumption that governs the results of the following papers.
Goldman (1957) considered zero-sum two-person games and showed that the probability of having a pure equilibrium goes to zero as the number of actions grows. He also briefly mentioned the case of payoffs with a Bernoulli distribution. Goldberg et al. (1968) considered general two-person games and showed that the probability of having at least one pure equilibrium converges to $1 - e^{-1}$ as the number of actions diverges. Dresher (1970) generalized this result to the case of an arbitrary finite number of players.

More recent papers have looked at the asymptotic distribution of the number of pure equilibria. Powers (1990) showed that, when the number of actions of at least two players goes to infinity, the distribution of the number of pure Nash equilibria goes to a Poisson(1). She then compared the case of continuous and discontinuous distributions.

Stanford (1995) derived an exact formula for the distribution of pure Nash equilibria in random games and obtained the result in Powers (1990) as a corollary. Stanford (1996) dealt with the case of two-person symmetric games and obtained Poisson convergence for the number of both symmetric and asymmetric pure Nash equilibria.

In all the above models, the expected number of pure Nash equilibria is in fact 1. Under different hypotheses, this expected number diverges. For instance, Stanford (1997, 1999) showed that this is the case for games with vector payoffs and for games of common interest, respectively. In Stanford (1999) both strictly and weakly ordinal preferences were studied.

Rinott and Scarsini (2000) weakened the hypothesis of i.i.d. payoffs, that is, they assumed that payoff vectors corresponding to different action profiles are i.i.d., but they allowed some dependence within the same payoff vector. In this setting, they proved asymptotic results when either the number of players or the number of actions diverges. More precisely, if each payoff vector has a multinormal exchangeable distribution with correlation coefficient $\rho$, then the following hold: for $\rho$ negative the number of pure Nash equilibria goes to zero in probability; for $\rho = 0$ it converges to a Poisson(1), and; for $\rho$ positive, it diverges and a central limit theorem holds.

Takahashi (2008) considered the distribution of the number of pure equilibria in a random game with two players, conditionally on the game having nondecreasing best-response functions. This assumption greatly increases the expected number of pure Nash equilibria. Daskalakis et al. (2011) extended the framework of games with random payoffs.
to graphical games. Action profiles are vertices of a graph and players’ actions are binary, like in our model. Moreover, their payoff depends only on their action and the actions of their neighbors. The authors studied how the structure of the graph affects existence of pure Nash equilibria and they examined both deterministic and random graphs.

The issue of solution concepts in games with random payoffs has been explored by various authors in different directions. For instance, Cohen (1998) studied the probability that a Nash equilibrium (pure and mixed) in a finite random game maximizes the sum of the players’ payoffs. Pei and Takahashi (2019) devoted their attention to rationalizable strategies in two-person games with random payoffs and performed an asymptotic analysis in the number of actions.

Finding a Nash equilibrium in a game is PPAD-complete (Daskalakis et al. (2009)). Therefore, given this computational difficulty, several learning procedures have been proposed to reach an equilibrium by playing the game over and over. (see, e.g., Blum and Mansour (2007), Tardos and Vazirani (2007)). Probably the simplest such procedure is best-response dynamics. The main problem that arises is that best-response dynamics is guaranteed to converge to a pure Nash equilibrium only when the game is of some specific type, such as, for instance, a potential game (Monderer and Shapley (1996)) or a weakly acyclic game (Fabrikant et al. (2013)). The performance of BRD in randomly drawn potential games has been studied in Coucheney et al. (2014), Durand et al. (2019), Durand and Gaujal (2016). To be able to deal also with games for which BRD does not converge to a pure Nash equilibrium (PNE), Goemans et al. (2005) defined the concept of sink equilibrium. A sink equilibrium is simply a fully connected set of two or more vertices with no edges leading out of the set. If players are selected at random and asked to choose a best response, the game will eventually reach a sink equilibrium and wander on its components. Christodoulou et al. (2012) considered a similar model, focusing on the rate of convergence to approximate solutions of the game. Dütting and Kesselheim (2017) considered best-response dynamics in the context of combinatorial auctions.

The idea of generating games at random to check properties of learning procedures was used by Galla and Farmer (2013) and, more recently, by Pangallo et al. (2019), who studied—mainly through simulations—the behavior of various learning procedures in games whose payoffs are drawn at random from a multinormal distribution.

Some of our results are proved by using variations of the Chen-Stein methods (see, Chen (1975), Stein (1972)). In particular, we use a result of Chatterjee (2008). Another key tool in our analysis is provided by percolation theory, introduced by Broadbent and Hammersley (1957). Since then, the theory developed very quickly and became very important both in the mathematics and the physics communities. For a general account on percolation, see Grimmett (1999). We will focus on percolation on the hypercube, and will use classical results by Erdős and Spencer (1979) and Bollobás (2001), as well as a more recent result by McDiarmid et al. (2018).

1.4. Organization of the paper. Section 2 introduces some notation and basic definitions. Section 3 deals with the number of pure Nash equilibrium in a random game. Section 4 studies the behavior of best-response dynamics in these games. The interaction between games with random payoffs and percolation is expounded in Section 5. Section 6 contains the proofs.
2. Notation

We first introduce some notation that will be adopted throughout the paper. We consider a game

$$\Gamma_N = ([N], (S_i)_{i \in [N]}, (g_i)_{i \in [N]}),$$

(2.1)

where $[N] := \{1, \ldots, N\}$ is the set of players and $S_i$ is the set of actions of each player $i \in [N]$. We set $S = \times_{i \in [N]} S_i$, and we let $g_i : S \rightarrow \mathbb{R}$ be the payoff function of player $i$. For each $s = (s_1, \ldots, s_N) \in S$, we call $s_{-i}$ the action profile of all players except $i$.

**Definition 2.1.** An action profile $s^*$ is a **pure Nash equilibrium** (PNE) of the game $\Gamma_N$ if for all $i \in [N]$ and for all $s_i \in S_i$ we have

$$g_i(s^*) \geq g_i(s_i, s_{-i}).$$

(2.2)

An action profile $s^*$ is a **strict pure Nash equilibrium** (SPNE) if for all $i \in [N]$ and for all $s_i \in S_i$ we have

$$g_i(s^*) > g_i(s_i, s_{-i}).$$

(2.3)

We will refer to the set of PNE by $\mathcal{N}_N$.

In what follows, we will assume that all players have the same binary action set, i.e., $S_i = \{0, 1\}$ for each $i \in [N]$.

Two action profiles $s, t$ are **neighbors** if one can be obtained from the other by changing the action of exactly one player, i.e.,

$$s_i \neq t_i \text{ for some } i \in [N] \text{ and } s_j = t_j \text{ for all } j \neq i.$$  

(2.4)

In this case we write $s \sim_i t$. Moreover, we write $s \sim t$ if $s \sim_i t$ for some $i \in [N]$.

We now associate to our game the graph $\mathcal{H}_N = (\mathcal{V}_N, \mathcal{E}_N)$, where the set of vertices is the set of action profiles, i.e., $\mathcal{V}_N = S$, and two vertices $s, t$ are connected by an edge in $\mathcal{E}_N$ iff they are neighbors in the sense of Eq. (2.4). We call $\mathcal{H}_N$ an $N$-cube. For each pair $s, t$ of neighbors, call $[s, t]$ the edge connecting them. This representation has been used by Candogan et al. (2011) for general finite games with an arbitrary number of actions for each player.

In the rest of the paper we will be interested in asymptotic results. For this purpose the following definition will be useful.

**Definition 2.2.** A sequence of events $(A_k)_{k}$ is said to hold **with high probability** (WHP) if

$$\lim_{k \to \infty} P(A_k) = 1.$$  

(2.5)

A sequence of events $(A_k)_{k}$ is said to hold **with very high probability** (WVHP) if

$$\sum_{k=1}^{\infty} (1 - P(A_k)) < \infty.$$  

(2.6)

Notice that ‘with very high probability’ is much stronger than ‘with high probability’. In fact, it implies, in virtue of the first Borel–Cantelli lemma, that there exists a random $N$ such that $A_k$ is true for all $k \geq N$. 

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3. Number of pure Nash equilibria and strict pure Nash equilibria

Our goal is to examine some generic properties of games with binary actions and a large number of players. To do this, we assume that the payoffs of our game are drawn at random. In particular, for each \( s \in S \), the payoff \( g_i(s) \) is the realization of a random variable \( Z^{s_i} \) and the random variables \( (Z^{s_i})_{i \in [N], s \in S} \) are i.i.d.. Denote by \( Z \) a generic independent copy of \( Z^{s_i} \). We will look at asymptotic results in the number of players.

Several results exist in the literature about the distribution of the number of PNE. When the distribution of \( Z \) has no atoms, with probability 1, the number of PNE coincides with the number of SPNE. This is not the case when atoms are present. Our first result deals with this issue.

We define
\[
\alpha := P(Z_1 = Z_2), \quad \beta := P(Z_1 < Z_2) = \frac{1 - \alpha}{2},
\]
where \( Z_1 \) and \( Z_2 \) are i.i.d. copies of \( Z \).

As we will see, all the results in the paper will depend on \( \alpha \). Most of the existing literature deals with the case \( \alpha = 0 \). Our extension to the general case has relevant implications.

**Theorem 3.1.** If \( \alpha = 0 \), then, as \( N \to \infty \), the number of SPNE converges in distribution to a Poisson\((1)\). If \( \alpha > 0 \), then the number of SPNE is 0 WVHP. 

**Remark 3.1.** As mentioned before, when \( \alpha = 0 \), the number of PNE and of SPNE are almost surely equal. In this case, convergence of the number of PNE to a Poisson distribution as the number of players increases was proved by Arratia et al. (1989), Rinott and Scarsini (2000) for any fixed number of actions.

It is interesting to note that, whenever atoms are present, the numbers of PNE and of SPNE have radically different behavior. This fact will be better described in Theorems 3.2 and 4.3 below.

We have seen that, when the law of \( Z \) has atoms, the number of SPNE vanishes asymptotically, while the number of PNE diverges. Moreover, in this case, a version of the central limit theorem holds. Call \( \Phi \) the cumulative distribution function of a standard normal random variable:
\[
\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du.
\]

**Theorem 3.2.** Assume that the law of \( Z \) has atoms, i.e., \( \alpha > 0 \). Then there exists a constant \( K_\alpha > 0 \), which depends only on \( \alpha \), such that
\[
\sup_{x} \left| P\left( \frac{|N|}{N} - \frac{(1 + \alpha)^N}{(1 + \alpha)^{N/2}} \leq x \right) - \Phi(x) \right| \leq \frac{K_\alpha}{(1 + \alpha)^{1/4} N}. \tag{3.3}
\]

The interesting feature of **Theorem 3.2** is that, besides showing convergence in distribution of the number of pure Nash equilibria to a normal distribution, it also provides a bound for the approximation error, in the spirit of Chen (1975), Stein (1972). A related result was obtained by Rinott and Scarsini (2000), who showed that the number of PNE diverges and a central limit theorem holds when the joint distribution of payoffs within the same profile has positive correlation.
Remark 3.2. Notice that Theorem 3.2 implies that the number of Nash Equilibria grows geometrically when \( \alpha > 0 \). More precisely, we can show that, for any \( \varepsilon > 0 \) small enough, Eq. (3.3) implies that
\[
|\mathcal{N}_N| > (1 + \alpha)^N - (1 + \varepsilon)^N(1 + \alpha)^{N/2}. \tag{3.4}
\]
Fix \( \varepsilon > 0 \) such that
\[
(1 + \varepsilon) < (1 + \alpha)^{1/2}.
\]
With this choice of \( \varepsilon \), the right-hand side of Eq. (3.4) is \( (1 + \alpha)^N(1 + o(1)) \). Hence, by establishing Eq. (3.4) we would prove that the number of Nash Equilibria grows like \( (1 + \alpha)^N \). Choose \( x_N = - (1 + \varepsilon)^N \). Recall the standard inequality \( \Phi(-x) \leq \phi(x)/x \), valid for \( x > 0 \), where \( \phi \) is the density of a standard normal. We then have that
\[
\Phi(x_N) \leq C(1 + \varepsilon)^{-N}
\]
for some constant \( C > 0 \).
\[
\sum_{N=1}^{\infty} P(|\mathcal{N}_N| \leq (1 + \alpha)^N - (1 + \varepsilon)^N(1 + \alpha)^{N/2})
= \sum_{N=1}^{\infty} (P(|\mathcal{N}_N| \leq (1 + \alpha)^N - (1 + \varepsilon)^N(1 + \alpha)^{N/2}) - \Phi(x_N)) + \sum_{N=1}^{\infty} \Phi(x_N)
\leq \sum_{N=1}^{\infty} \frac{K_\alpha}{(1 + \alpha)^{1/2} N} + \sum_{N=1}^{\infty} \frac{C}{(1 + \varepsilon)^N} < \infty.
\]

4. IMPROVEMENTS AND BEST-RESPONSE DYNAMICS

Once it is proved that our large binary game has many equilibria, it becomes interesting to study their geometry in the hypercube \( \mathcal{H}_N \). In particular, we will be interested in evaluating the probability of reaching a PNE starting from a generic strategy profile and moving from one profile to another via payoff improvement, as described below.

Definition 4.1. Given a strategy profile \( s \in \mathcal{H}_N \), the profile \( t \sim_i s \) is a profitable deviation for player \( i \) if
\[
g_i(t) > g_i(s). \tag{4.1}
\]
in which case the difference \( g_i(t) - g_i(s) \) is called the improvement of player \( i \).

An improvement path is a sequence of strategy profiles \( s_0, s_1, \ldots \) such that each \( s_{k+1} \) is a profitable deviation of \( s_k \) for some player \( i_k \).

If an improvement path stops, it is because it has reached a PNE. An improvement path may never stop, either because the game does not have PNE or because it gets trapped in some region.

In the case of games with two actions for each player, a BRD is just a random improvement path. In what follows, we study the probability that the BRD reaches a PNE in a random game. Again, we will see that the role of atoms in the distribution of \( Z \) is fundamental.

Starting with the \( N \)-cube \( \mathcal{H}_N = (\mathcal{V}_N, \mathcal{E}_N) \), we obtain a new partially oriented graph \( \overrightarrow{\mathcal{H}}_N = (\mathcal{V}_N, \overrightarrow{\mathcal{E}}_N) \) where some of the edges are assigned a random orientation by the following process. Let \( s \sim_i t \); then the directed edge \( [s, t] \) from \( s \) towards \( t \) is in \( \overrightarrow{\mathcal{E}}_N \) if...
and only if $Z_i^s < Z_i^t$. If the law of $Z$ is nonatomic, then the probability that two payoffs coincide is zero. Therefore, $\mathcal{H}_N$ is just a random orientation of $\mathcal{H}_N$, where each edge is independently oriented in one direction or the other with probability $1/2$. If, on the other hand, the law of $Z$ has atoms, then $\mathbf{P}(Z_i^s = Z_i^t) > 0$, so with positive probability, some edges have no orientation.

**Definition 4.2.** We say that $t$ is directly accessible from $s$ if the directed edge $[s, t] \in \mathcal{E}_N$. We say that $t$ is accessible from $s$ if there exists a finite sequence $s_0, s_1, \ldots, s_k$ such that $s = s_0$, $t = s_k$ and, for all $i \in \{0, \ldots, k - 1\}$, we have $[s_i, s_{i+1}] \in \mathcal{E}_N$.

Hence, $t$ is directly accessible from $s$ if $t$ is a profitable deviation from $s$ for some player $i$ and $t$ is accessible from $s$ if we can go from $s$ to $t$ along an improvement path. **Definition 4.2** has a natural interpretation in terms of BRD. Suppose a BRD is initiated from $s$. Then $t$ is accessible from $s$ if and only if there is a positive probability that the BRD reaches it.

An example with three players is given in *Fig. 3*. The orientation of the edges is induced by the payoffs in the table below. The absence of ties produces a complete orientation of the hypercube. The green edges are the possible paths of a BRD starting at the vertex $(1, 1, 1)$. The two vertices in red, i.e., $(0, 0, 0)$ and $(1, 1, 0)$, are the two pure Nash equilibria of the game. Either of these vertices can be reached via a BRD starting from any initial vertex. **Theorem 4.4** below proves that, with very high probability, this is always the case, when the number of players is large and $\alpha$ is small enough.

| Player 1 | Player 2 | 0       | 1       |
|----------|----------|---------|---------|
| 0        |          | (0.542, 0.709, 0.426) | (0.209, 0.659, 0.569) |
| 1        |          | (0.292, 0.684, 0.126) | (0.815, 0.774, 0.508) |

| Player 1 | Player 2 | 0       | 1       |
|----------|----------|---------|---------|
| 0        |          | (0.202, 0.549, 0.174) | (0.199, 0.097, 0.319) |
| 1        |          | (0.110, 0.567, 0.794) | (0.949, 0.530, 0.055) |

**Figure 3.** Representation of $\Gamma_3$ on $\{0, 1\}^3$.  

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Our next result shows the existence of a sharp phase transition in the accessibility of PNE. Roughly speaking, as the mass of the atoms in the distribution of $Z$ grows, so does the number of PNE, though some may not be accessible from some profile $s$. Hence, in this case, some PNE may not be reachable via BRD. Fix $s \in \mathcal{V}_N$ and partition
\[ \mathcal{V}_N = \mathcal{L}_N^s \cup \mathcal{M}_N^s \] (4.2)
in such a way that $\mathcal{L}_N^s$ is the set that contains $s$ as well as all vertices $t$ that are accessible from $s$ in the oriented graph $\overrightarrow{H}_N$.

**Theorem 4.3.** Let $\alpha$ be defined as in Eq. (3.1). Fix $s \in \mathcal{H}_N$.

(a) If $0 \leq \alpha < 1/2$, then
\[ \lim_{N \to \infty} P(\mathcal{M}_N \subset \mathcal{L}_N^s) = 1. \]

(b) If $\alpha = 1/2$, then
\[ \lim_{N \to \infty} P(|\mathcal{M}_N \cap \mathcal{M}_N^s| > 0) > 0. \]

(c) If $\alpha > 1/2$, then, for any $K > 0$,
\[ \lim_{N \to \infty} P(|\mathcal{M}_N \cap \mathcal{M}_N^s| > K) = 1. \]

The interpretation of Theorem 4.3 is that, for any fixed $s$, asymptotically,
(a) for $0 \leq \alpha < 1/2$, every PNE is potentially reachable by the BRD starting at $s$, WHP;
(b) for $\alpha = 1/2$, with positive probability there exist PNE that are not reachable by the BRD starting at $s$;
(c) for $\alpha > 1/2$, the number of PNE that are not reachable by the BRD starting at $s$ grows to infinity, WHP.

Therefore, we have an interesting phase transition at $\alpha = 1/2$.

To complete the picture, we give a result about the convergence of BRD to some PNE. The following theorem shows that BRD converges to a PNE, WVHP if $\alpha \leq 0.55$. This rules out the possibility for the BRD be trapped in a cycle, when $\alpha$ is small.

**Theorem 4.4.** If $\alpha > 0$ satisfies
\[ \left\lfloor -\frac{1}{\ln (1/2 + \alpha/2)} \right\rfloor \leq 3, \] (4.3)
i.e., if $\alpha \in (0, 0.55]$, then for any fixed starting profile $s$, the BRD converges to a PNE, WVHP.

**Remark 4.1.** To summarize our results: Theorem 3.2 proves that the number of PNE grows geometrically when $\alpha > 0$ (see Remark 3.2). Theorem 4.3 describes the set of PNE and proves that, when $\alpha$ is larger than a certain threshold, some of them are not accessible. Theorem 4.4 instead establishes that, when $\alpha$ is small enough, the BRD converges to some PNE and does not get trapped in a region not containing any PNE.

**Example 4.1.** To clarify the role of $\alpha$, assume that $Z \sim \text{Poisson}(\theta)$. Then
\[ \alpha = \sum_{k=0}^{\infty} \frac{e^{-2\theta} \theta^k}{(k!)^2} = e^{-2\theta} I_0(2\theta), \] (4.4)
where $I_0$ is the modified Bessel function of the first kind (Abramowitz and Stegun, 1965, 9.6.12, p. 375). The graph of this function is given in Fig. 4. The larger $\theta$, the smaller the probability of ties and all values of $\alpha$ can be covered. The critical value $\alpha = 1/2$ corresponds to $\theta = 0.617705$.

![Figure 4. $\alpha$ as a function of $\theta$.](image)

5. **Percolation**

Theorems 4.3 and 4.4 concerning the geometry of the set of pure Nash equilibria will be proved using a connection with percolation theory that is interesting in itself, as it creates an important bridge between disciplines. For this reason, we choose to discuss this link here, even though the percolation will be used later on, when proving the theorems of Section 4. This application shows an interesting use of tools from probability, combinatorics and graph theory in the context of game theory and opens the way for a fruitful interplay.

Fix a starting action profile $s$, i.e., a vertex of the hypercube, and consider the cluster of edges that can be traversed by a BRD starting from $s$. For example see the green cluster in Fig. 3, relative to $s = (1, 1, 1)$. In general, we will show that this cluster has the same distribution as an independent bond percolation cluster containing $s$. The parameter of the percolation is uniquely determined by $\alpha$.

5.1. **Bond percolation.** Independent bond percolation on $H_N$ is defined as follows. For each edge in $H_N$, flip a coin having probability $\beta$ of showing heads. If the toss shows heads, then declare the edge to be open; otherwise the edge is closed. The subgraph obtained from $H_N$ by deleting the closed edges is a random graph $H_N^\beta$, called percolation, that includes all vertices in $V_N$, but could be disconnected. This model allows us to give a detailed description of the geometry of the PNE.

Random subsets $\mathcal{U}_1, \mathcal{U}_2$ with values in $V_N$ are said to have the same distribution if

$$P(\mathcal{U}_1 = A) = P(\mathcal{U}_2 = A) \quad \text{for all } A \subset V_N. \quad (5.1)$$

We consider the following three random sets:

(a) $L_N^\beta$ is the largest component of the percolation $H_N^\beta$,  
(b) $L_N^{\beta, s}$ is the largest component containing the vertex $s$ in $H_N^\beta$,  
(c) $L_N^s$ is the set of vertices that are accessible from $s$ in the BRD in $H_N^\beta$.
The next two propositions show the relations between these three sets. In particular, Proposition 5.1 proves that $L_N^\beta$ and $L_N^{\beta,s}$ coincide WHP for large values of $N$. The proof of this result will use Borel-Cantelli’s lemma. Proposition 5.2 shows that, for every $N$, $L_N^s$ and $L_N^{\beta,s}$ are equal in distribution. The proof of this result will be achieved through a suitable coupling.

**Proposition 5.1.** For any fixed $s \in \mathcal{V}_N$ and $\beta > 0$, WHP we have

$$L_N^\beta = L_N^{\beta,s}. \quad (5.2)$$

**Proposition 5.2.** Let $\beta$ be defined as in Eq. (3.1). For any profile $s$ the random sets $L_N^s$ and $L_N^{\beta,s}$ have the same distribution.

The next result focuses on the nonatomic case, that is, $\beta = 1/2$, i.e., $\alpha = 0$. This corresponds to the classical bond percolation with parameter 1/2. As mentioned above, a vertex is called isolated if it has degree zero in the graph induced by the percolation.

Erdős and Spencer (1979) analyzed the asymptotic behavior of $H_N^\beta$ when $\beta = 1/2$, and showed that the random graph is connected WHP. A further inspection of their proof reveals that WHP the largest connected component of this percolation contains all the vertices in $\mathcal{V}_N$ with the exception of some ‘isolated’ vertices, i.e., vertices with degree 0 in $H_N^\beta$.

**Proposition 5.3** (Erdős and Spencer (1979, page 35)). WHP the largest connected component of $H_N^{1/2}$ contains all the vertices, with the exception of some isolated vertices, i.e., vertices with degree 0 in the induced subgraph. Let $\Xi_N$ be the set of isolated vertices. As $N \to \infty$, the distribution of $\Xi_N$ converges to a Poisson(1).

### 6. Proofs

**Proofs of Section 3.**

**Proof of Theorem 3.1.** When $\alpha = 0$, convergence of the number of PNE to a Poisson(1) was proved by Arratia et al. (1989), Rinott and Scarsini (2000). Moreover, since almost surely no two payoffs are equal, we have that each PNE is also an SPNE.

Next, we focus on the case $\alpha > 0$ and prove that the number of SPNE is zero, WVHP. Notice that $\alpha > 0$ implies that $\beta < 1/2$. For any $s \in \mathcal{V}_N$, define

$$W^s := \begin{cases} 1 \text{ if } s \text{ is an SPNE,} \\ 0 \text{ otherwise.} \end{cases} \quad (6.1)$$

Notice that $E[W^s] = \beta^N$. If we call $W_N$ the total number of SPNE in the game, we have

$$W_N = \sum_{s \in \mathcal{V}_N} W^s. \quad (6.2)$$

Therefore, $E[W_N] = (2\beta)^N$. Markov’s inequality implies

$$P(W_N \geq 1) \leq E[W_N]. \quad (6.3)$$

Since $2\beta < 1$, the upper bound goes to zero geometrically fast. \hfill \Box

Denote by $h$ the Hamming distance on $H_N$:

$$h(s, t) = \# \{i : s_i \neq t_i \}. \quad (6.4)$$
Proof of Theorem 3.2. For \( s \in \mathcal{V}_N \), define
\[
\hat{X}^s := \begin{cases} 
1 & \text{if } s \in \mathcal{A}_N, \\
0 & \text{otherwise,}
\end{cases}
\]
and
\[
X^s := \hat{X}^s - (1 - \beta)^N.
\]
Let \( \tau_N^2 := \text{Var}(|\mathcal{A}_N|) \). We first prove that
\[
\tau_N^2 = 2^N b_N + 2^N N(\alpha(1 - \beta)^{2N-2} - (1 - \beta)^{2N}),
\]
where \( b_N = (1 - \beta)^N (1 - (1 - \beta)^N) \). We note here that \( \mathbf{X} := (X^s)_{s \in \mathcal{V}_N} \) is a collection of identically distributed mean-zero random variables. Moreover, for two vertices \( s \sim t \), we know that \( X^s \) and \( X^t \) are independent, so
\[
\tau_N^2 = 2^N b_N + 2^N N \text{Cov}(X^0, X^1)
\]
where \( 0 = (0, 0, \ldots, 0) \) and \( 1 = (1, 0, \ldots, 0) \). To compute the covariance above, observe that
\[
\text{Cov}(X^0, X^1) = \text{Cov}(\hat{X}^0, \hat{X}^1) = E[\hat{X}^0 \hat{X}^1] - (1 - \beta)^{2N}.
\]
The product \( \hat{X}^0 \hat{X}^1 \) is nonzero only when both \( 0 \) and \( 1 \) are PNE, which requires their payoffs in the first dimension to be equal, giving
\[
\text{Cov}(X^0, X^1) = \alpha(1 - \beta)^{2N-2} - (1 - \beta)^{2N}.
\]
It is worthwhile noting here that \( 2^N b_N / \tau_N^2 = 1 + c_N \) where \( |c_N| \) decreases to 0 geometrically fast.

Now let
\[
V := f(\mathbf{X}) := \frac{1}{\tau_N} \sum_{s \in \mathcal{V}_N} X^s
\]
and \( \mathbf{\bar{X}} := (\bar{X}^s)_{s \in \mathcal{V}_N} \) be an independent copy of \( \mathbf{X} \). Also, for any set \( A \subset \mathcal{V}_N \), let \( \mathbf{X}_A := (X^s_A)_{s \in \mathcal{V}_N} \) be such that
\[
X^s_A := \begin{cases} 
\bar{X}^s & \text{if } s \in A, \\
X^s & \text{if } s \notin A.
\end{cases}
\]
Moreover, define
\[
\Delta_t f(\mathbf{X}) := f(\mathbf{X}) - f(\mathbf{X}_{\{t\}}) = \frac{1}{\tau_N} \left( X^t - \bar{X}^t \right).
\]
Hence, for any \( t \notin A \),
\[
\Delta_t f(\mathbf{X}_A) = f(\mathbf{X}_A) - f(\mathbf{X}_{A \cup \{t\}}) = \frac{1}{\tau_N} \left( X^t - \bar{X}^t \right) = \Delta_t f(\mathbf{X}).
\]
Furthermore, let
\[
T^s := \frac{1}{2} \sum_{A \subset (\mathcal{V}_N \setminus \{s\})} \left( 2^N \left( 2^N - 1 \right) \right)^{-1} \Delta_s f(\mathbf{X}) \Delta_s f(\mathbf{X}_A) = \frac{(X^s - \bar{X}^s)^2}{2 \tau_N^2}, \quad (6.9)
\]
and $T = \sum_{s \in V_N} T^s$. As $E[V] = 0$ and $\text{Var}(V) = 1$, using Chatterjee (2008, Theorem 2.2), we have

$$\sup_{x \in \mathbb{R}} |P(V \leq x) - \Phi(x)| \leq 2 \left( \sqrt{\text{Var}(T)} + \frac{1}{2} \sum_{t=1}^{2N} E[|\Delta_t f|^3] \right)^{1/2}.$$  \hfill (6.10)

Notice that in order to get (6.10) from Chatterjee (2008, Theorem 2.2), we have to use a simple relation between Kolmogorov’s distance $\kappa$, which we use here, and the Kantorovich-Wasserstein distance $\omega$, which is used in Chatterjee (2008, Theorem 2.2). In general, $\kappa \leq \sqrt{2\omega}$.

Notice that

$$E[T^s] = \frac{E[(X^s - \bar{X}^s)^2]}{2\tau_N^2} = \frac{b_N}{\tau_N^2}.$$

Moreover, for any pair of profiles $s, t$,

$$T^sT^t = \frac{(X^s - \bar{X}^s)^2(X^t - \bar{X}^t)^2}{4\tau_N^4}.$$

Hence, for $h(s, t) \geq 2$, we have that $E[T^sT^t] = b_N^2 \tau_N^{-4}$ by independence. If, on the other hand, $h(s, t) = 1$, then

$$E[T^sT^t] = O(2^{-4N}(1 - \beta)^{-2N}).$$

Finally, for $s = t$,

$$E[(T^s)^2] = (1 + c_N)2^{-N}\tau_N^{-2}.$$

Hence

$$\text{Var}(T^s) = O(2^{-N}\tau_N^{-2}).$$

Moreover, if $h(s, t) = 1$, then

$$\text{Cov}(T^s, T^t) = O(2^{-2N}).$$

If $h(s, t) \geq 2$, we have that $\text{Cov}(T^s, T^t) = 0$. Hence,

$$\text{Var}(T) \leq 2^N \text{Var}(T^s) + 2^N N \text{Cov}(T^0, T^1) = O(\tau_N^{-1}).$$

Similarly,

$$E[|\Delta_t f|^3] \sim \frac{(1 - \beta)^N}{\tau_N^3} = \frac{1}{2^N\tau_N^3}.$$

We obtain that there exists a constant $K_1$ such that

$$\sup_{x \in \mathbb{R}} |P(V \leq x) - \Phi(x)| \leq K_1 \tau_N^{-1/4}.$$

This implies (3.3), as $\tau_N^{-1/4} = O((1 + \alpha)^{-N/4})$. \hfill $\Box$
Proofs of Section 5.

Proof of Proposition 5.1. Fix any $\beta \in (0,1/2)$ and let $\varepsilon > 0$. We call the complement of $L_N^\beta$ the fragment of the percolation, i.e., the complement of the largest connected component. McDiarmid et al. (2018, Theorem 1 a)) states that the cardinality of the fragment is bounded WVHP by the quantity
\[
h_N := (2\beta)^N + \varepsilon\sqrt{N(2\beta)^N/2}. \tag{6.11}
\]
We suppose that this bound holds, as the probability that does not hold decays fast to 0 (it is actually summable). Hence, by symmetry, the probability that $s \notin L_N^\beta$ is bounded above by $h_N/2^N$, which is summable. We extend this to general $\beta$ by monotonicity. In fact, it is well known that there exists a coupling such that if $\beta < \beta'$, then $L_N^\beta \subset L_N^{\beta'}$, e.g. see Bollobás (2001, Theorem 2.1 page 36).

□

Proof of Proposition 5.2. We will actually prove the stronger result that there exists a coupling such that $L_{sN} = L_{\beta s}$, $sN$. To this end, we need to define the following objects.

Assume that $r,t \in V_N$ are neighbors in $H_N^-$ and define the event
\[
\{r \rightarrow t\} := \{[r,t] \in E_N\}. \tag{6.12}
\]
Since each player has only two actions, we have that $\{r \rightarrow t\}$ is independent of $\{u \rightarrow w\}$ for every $\{u \rightarrow w\} \neq \{r \rightarrow t\}$.

For any subset $U \subset V_N$, we call $\overrightarrow{\Delta U}$ the set of vertices in $U^c$ which are out-neighbors of some elements in $U$, that is,
\[
w \in \overrightarrow{\Delta U} \text{ iff } w \in U^c \text{ and } \exists u \in U \text{ such that } \{u \rightarrow w\} \text{ is true}, \tag{6.13}
\]
and we call $\Delta U$ the set of vertices in $U^c$ that are neighbors of some elements in $H_N$, that is,
\[
w \in \Delta U \text{ iff } w \in U^c \text{ and } \exists u \in U \text{ such that } u \sim w. \tag{6.14}
\]
We will prove the result by constructing a suitable coupling between the random oriented $N$-cube $H_N^-$ and the percolation graph $H_N^\beta$.

We define
\[
\mathcal{P}_1 = \{s\} \text{ and, for each } k \in \mathbb{N}, \mathcal{P}_{k+1} = \mathcal{P}_k \cup \overrightarrow{\Delta \mathcal{P}_k}. \tag{6.15}
\]
We then construct a finite sequence of random graphs such that each graph of the sequence is a bond percolation with parameter $\beta$ and the last graph in the finite sequence has the property that we want.

Start with a bond percolation with parameter $\beta$, and call the resulting graph $B_1$. Assume that this percolation is independent of $(Z^i_t : i \in [N], t \in V_N)$.

For every $k \geq 1$ we will update $B_k$ by changing the status of some edges at each stage, in such a way that $B_{k+1}$ is still a bond percolation with parameter $\beta$. More precisely, we define the sequence of random subgraphs $(B_k)_{k \in \mathbb{N}}$ of $H_N$ recursively as follows. For each edge $e \in E_N$, we define $B_k\{e\}$ the status (open or closed) of edge $e$ in $B_k$.

For every $k \in \mathbb{N}$, we obtain $B_{k+1}$ from $B_k$, by updating all and only the edges in $E_N$ that connect an element of $\mathcal{P}_k$ to an element of $\Delta \mathcal{P}_k$. More precisely: for any $u \in \mathcal{P}_k$
and any \( w \in \Delta \mathcal{P}_k \), with \( u \sim w \),

\[
\mathcal{B}_{k+1}\{[u, w]\} = \begin{cases} 
\text{open,} & \text{if } \{u \rightarrow w\}, \\
\text{closed,} & \text{otherwise;}
\end{cases}
\]

(6.16)

for all other edges \( e \in \mathcal{E}_N \), we have \( \mathcal{B}_{k+1}\{e\} = \mathcal{B}_k\{e\} \).

Since the status of edges is updated independently of the original configuration and with i.i.d. \text{Bernoulli}(\beta) random variables, we have that \( \mathcal{B}_{k+1} \) is still a bond percolation with parameter \( \beta \).

Notice that, in the worst-case scenario each of these processes explores the whole \( \mathcal{H}_N \) in \( 2^N \) iterations. That is \( \mathcal{B}_{k+1} = \mathcal{B}_k \) and \( \mathcal{P}_{k+1} = \mathcal{P}_k \) for all \( k \geq 2^N \).

By construction, \( \mathcal{P}_{2N} \) is exactly the set of vertices in the connected component that contains \( s \), in the percolation graph \( \mathcal{B}_{2N} \). In this context, \( \mathcal{L}_N^s = \mathcal{P}_{2N} \) and the set of vertices in the connected component that contains \( s \), in the percolation graph \( \mathcal{B}_{2N} \) is \( \mathcal{L}_N^{\beta, s} \). This completes the proof.

From now on, we set \( \mathcal{B}^* = \mathcal{B}_{2N} \), which is an independent bond percolation with parameter \( \beta \). In the proof of Proposition 5.2 we build a percolation \( \mathcal{B}^* \) that creates a coupling between \( \mathcal{L}_N^s \) and \( \mathcal{L}_N^{\beta, s} \). In the remaining part of the paper, we will use this percolation \( \mathcal{B}^* \) and in this context the coupling \( \mathcal{L}_N^s = \mathcal{L}_N^{\beta, s} \) will hold.

Proof of Proposition 5.3. For any graph \( \mathcal{G} \), call \( |\mathcal{G}| \) the cardinality of its vertex set. Define

\[
\gamma_k := \min \{ \# \text{ of edges on the boundary of } H \subset \mathcal{B}^*: |H| = k \},
\]

\[
\lambda_k := \# \text{ of subgraphs of } \mathcal{B}^* \text{ having } k \text{ vertices.}
\]

(6.17)

The probability of having a connected subgraph \( H \subset \mathcal{B}^* \) which is disconnected from its complement, is at most \( \lambda_k 2^{-\gamma_k} \). Hence, in order to prove that disconnected components can only be single points, we can follow the strategy in Bollobás (2001, page 385), and it is enough to prove that

\[
\sum_{k=2}^{2^N-1} \lambda_k 2^{-\gamma_k} = o(1).
\]

(6.18)

Eq. (6.18) is proved in Bollobás (2001, Lemma 14.4, page 388). Then WHP \( \mathcal{B}^* \) is connected if and only if it contains no disconnected components of size 1, i.e., isolated vertices. In particular, as Eq. (6.18) holds, then the connected component misses only the isolated vertices, WHP. In virtue of Proposition 5.3, we have that \( \mathcal{B}^* \) contains all the vertices in \( \mathcal{Y}_N \) with the exception of a \text{Poisson}(1) number of isolated vertices. This is because, with WHP the profile \( s \) is not an isolated vertex.

Proofs of Section 4. The following lemma gives a precise description of the geometry of the percolation graph when \( \beta < 1/2 \), which corresponds to the case \( \alpha > 0 \). It states that there exists a constant \( m_\beta \), depending on \( \beta \) only, and a \( \delta > 0 \), such that no ball of radius \( \delta N \) can contain more than \( m_\beta \) elements not belonging to the largest component, WVHP.

Lemma 6.1 (McDiarmid et al. (2018, Theorem 2(a))). For any \( r \in \mathbb{N} \) and any \( s \in \mathcal{Y}_N \), call

\[
B_r(s) := \{ t: h(s, t) \leq r \},
\]

(6.19)
where \( h \) is the Hamming distance defined in Eq. (6.4). Set
\[
m_\beta := \left\lfloor \frac{1}{-\ln(1-\beta)} \right\rfloor. \tag{6.20}
\]
Then, for any fixed \( s \), there exists \( \delta > 0 \) such that
\[
\sum_N \mathbb{P}(\exists t: |B_{\delta N}(t) \setminus \mathcal{L}_N^s| \geq m_\beta) < \infty.
\]

Remark 6.1. Notice that McDiarmid et al. (2018, Theorem 2(a)) is actually formulated in terms of largest connected component of \( \mathcal{B}^* \), i.e., \( \mathcal{L}_N^s \) instead of \( \mathcal{L}_N^s \). This substitution holds in virtue of Proposition 5.1 and Proposition 5.2.

Proof of Theorem 4.3. (a) The case \( \alpha = 0 \) can be inferred directly from Propositions 5.2 and 5.3. Proposition 5.2 implies that the cluster containing \( s \) has the same distribution as the cluster of profiles accessible for a BRD which starts from \( s \). Hence, in virtue of Proposition 5.3, WHP, in the percolation \( \mathcal{B}^* \), we still use the notation \( \mathcal{L}_N^s \) to denote the largest connected component which contains \( s \). Using Proposition 5.3 we infer that \( \mathcal{L}_N^s \) contains all but a Poisson number of vertices. None of these vertices can be a PNE, as the edges incident to them are oriented towards their neighbors. This implies that WHP \( \mathcal{L}_N^s \) contains all the PNE, if the game has any.

Next, we turn to the case \( 0 < \alpha \leq 1/2 \). Let \( \mathcal{T}_N \) be the number of vertices that are incident to at least \( 2N - m_\beta \) unoriented edges. Markov’s inequality yields
\[
\mathbb{P}(\mathcal{T}_N \geq 1) \leq \frac{2N}{\alpha N - m_\beta}. \tag{6.21}
\]
Since \( \alpha < 1/2 \), the right-hand side decreases geometrically to 0. Therefore, WHP no vertex in \( \mathcal{V}_N \) is incident to more than \( 2N - m_\beta \) unoriented edges. In virtue of Lemma 6.1, there exists \( \delta > 0 \) such that WHP, for any \( s \), any ball \( B_{\delta N}(s) \) contains at most \( m_\beta \) vertices of \( \mathcal{M}_N^s \), which was defined in Eq. (4.2).

We emphasize that simultaneously all the balls satisfy the property described above WHP. Under the event \( \{ \mathcal{T}_N = 0 \} \), which has a very high probability, each element of \( \mathcal{M}_N^s \cap B_{\delta N}(s) \) is incident to no less than \( N - m_\beta \) unoriented edges. Hence, for each each \( t \in \mathcal{M}_N^s \), WHP there exist at most \( m_\beta - 1 \) other vertices of \( \mathcal{M}_N^s \) whose distance from \( t \) is less than \( \delta N \). Moreover, WHP each \( t \in \mathcal{M}_N^s \) has an edge oriented towards a vertex of \( \mathcal{L}_N^s \), and this prevents \( t \) from being a PNE. Hence, WHP no element of \( \mathcal{M}_N^s \cap B_{\delta N}(s) \) can be a PNE.

(b) For this case, we introduce a different percolation \( \mathcal{H}_N \) on \( \mathcal{H}_N \) which is defined below. This percolation is related to \( \mathcal{B}^* \). For any pair of vertices \( r, t \in \mathcal{V}_N \), we declare the edge \( [r, t] \) open in \( \mathcal{H}_N \) iff
\[
\{r \to t\} \cup \{t \to r\} \tag{6.22}
\]
holds true, that is, the edge connecting the two profiles \( r \) and \( t \) is oriented in \( \mathcal{H}_N \). Otherwise the edge \( [r, t] \) is declared closed in \( \mathcal{H}_N \). Since the percolation \( \mathcal{H}_N \) has parameter \( \alpha = 1/2 \), we are in the framework studied in Erdős and Spencer (1979), and we can apply Proposition 5.3.

Call \( \mathcal{L}_N^s \) the largest connected component of \( \mathcal{H}_N \) that contains \( s \). WHP the set \( \mathcal{L}_N^s \) contains all vertices of \( \mathcal{H}_N \) except some isolated vertices. Any isolated vertex in \( \mathcal{H}_N \) is a PNE, as it is incident to non-oriented edges only, which in turns imply that each player
Moreover, \( \Theta \) then \( Z \). Fix \( I \) and define \( \Theta^t \) the indicator of the event that the vertex \( t \) is incident only to unoriented edges in \( \mathcal{H}_N \) and

\[
\Theta_N = \sum_{t \in \mathcal{V}_N} \Theta^t.
\]

Notice that \( \mathcal{H}_N \) is a bipartite graph. We have

\[
\mathcal{V}_N = \mathcal{V}_N^{\text{even}} \cup \mathcal{V}_N^{\text{odd}},
\]

where \( \mathcal{V}_N^{\text{even}} \) is the set of vertices for which the sum of coordinates is even and \( \mathcal{V}_N^{\text{odd}} \) is the set of vertices for which the sum of coordinates is odd. Edges connect only vertices from different components, so no pair of vertices in \( \mathcal{V}_N^{\text{even}} \) (or in \( \mathcal{V}_N^{\text{odd}} \)) can be neighbors. Obviously \( |\mathcal{V}_N^{\text{even}}| = |\mathcal{V}_N^{\text{odd}}| = 2^{N-1} \). Our first goal is to prove the following claim.

**Claim 6.2.** \( \{\Theta^t : t \in \mathcal{V}_N^{\text{even}}\} \) is a collection of independent events.

**Proof.** The event \( \Theta^s \) depends only on the payoffs at \( s \) and at each of its neighbors. It is enough to prove that, for every subset \( I \subset \mathcal{V}_N^{\text{even}} \), we have

\[
P\left( \bigcap_{s \in I} \Theta^s \right) = \prod_{s \in I} P(\Theta^s).
\]

Fix \( I \) and \( t \in I \) and define \( I_{-t} := I \setminus \{t\} \). We need to prove that

\[
P\left( \bigcap_{s \in I} \Theta^s \right) = P(\Theta^t) \prod_{s \in I_{-t}} P(\Theta^s).
\]

The set of profiles in \( I_{-t} \) that share a neighbor with \( t \) has cardinality at most \( N - 1 \). If this set is empty then Eq. (6.25) trivially holds. Otherwise, for \( i \in [N] \), let \( w_i, s_{i} \in I_{-t} \) be such that \( w_i \sim t \) and \( s_{i} \sim_j w_i \), with \( i \neq j \). If, for some \( i \), the event \( \Theta^{s_{i}} \) is true, then \( Z_{i}^{w_{i}} \geq Z_{i}^{w_{j}} \), and this event is independent of \( Z_{i}^{w_{j}} \). Therefore the class of events \( \{\Theta^{s_{i}}\}_{i \in [N]} \) is independent of the class of random variables \( \{Z_{i}^{w_{i}}\}_{i \in [N]} \). Since the event \( \Theta^t \) depends only on \( \{Z_{i}^{w_{i}}\}_{i \in [N]} \) and \( \Theta^t \), we have that \( \Theta^t \) is independent of \( \{\Theta^{s_{i}}\}_{i \in [N]} \). Moreover, \( \Theta^t \) is independent of \( \Theta^s \) for all \( s \in I_{-t} \). This ends the proof of Claim 6.2. \( \square \)

In turn, Claim 6.2 implies that \( \Theta_N \) is stochastically larger than a Binomial\( (2^{N-1}, \alpha^N) \). Each vertex \( t \) that is incident only to unoriented edges has the following properties:

- It is a PNE.
- It lies in \( \mathcal{M}_N^s \), unless \( t = s \).

Hence, we have that for any fixed \( K > 0 \),

\[
\lim_{N \to \infty} P(|\mathcal{M}_N \cap \mathcal{M}_N^s| > K) = 1. \quad \square
\]

**Proof of Theorem 4.4.** We reason by contradiction. BRD does not converge to PNE if either these equilibria are not accessible or if the BRD gets trapped in a subgraph where no equilibria are present. The former option is ruled out by the combination of Theorems 3.2
and 4.3(a). Notice that if BRD gets trapped in a subgraph $\mathcal{K}$ where no equilibria are present, then the number of vertices in this subgraph, which we call a trap, is at least 4, because $\mathcal{K}_N$ is bipartite. We have that each edge connecting $\mathcal{K}$ to its boundary $\Delta \mathcal{K}$ either is undirected or points towards $\mathcal{K}$.

Denote $\mathcal{B}^*$ the graph obtained by adding to $\mathcal{B}^*$ the edges which have no orientation, i.e., the ones that correspond to a tie. Call $\mathcal{C}$ the graph with vertex set $\mathcal{V}_N$ and all the edges in $\mathcal{E}_N$ which are not in $\mathcal{B}^*$. The random graph $\mathcal{C}$ is obtained through a percolation with parameter $\beta < 1/2$, as an edge is open if it is closed in $\mathcal{B}^*$, which is a percolation. Notice that the vertices of $\mathcal{C}$ are not connected, by definition of this random graph, to the largest component $\mathcal{L}_N^*$ of $\mathcal{B}^*$, i.e., using the terminology of McDiarmid et al. (2018), they are part of the so-called fragment of $\mathcal{B}^*$. By Lemma 6.1 there exists $\delta > 0$ such that, WVHP, any $\delta N$ ball contains at most $m_\beta$ vertices, where $m_\beta$ is defined as in Eq. (6.20)). Notice that, by Eq. (4.3) and Eq. (6.20) we have

$$m_\beta = \left\lfloor \frac{1}{-\ln(1-\beta)} \right\rfloor = \left\lfloor -\frac{1}{(\ln(1/2 + \alpha/2))} \right\rfloor \leq 3.$$ 

Hence, the size of $\mathcal{K}$ must be less than 4, and this yields a contradiction. □

7. Conclusion and open problems

Large random games have many PNE, as long as the probability of ties $\alpha > 0$. We identified the limiting distribution of the number of PNE and their position with respect the starting point of a BRD. The relevance of our approach is that it creates a link between different subjects. The next important question is the following. How long does it take for BRD to reach a PNE? This is equivalent to study the path-length of a non-backtracking random walk on the percolation cluster of the hypercube. Fig. 5 shows that the time seems to grow polynomially in $N$. Notice that Fig. 5 describes the behavior of a BRD where at each step a player is randomly chosen among those who are willing to deviate.

![Figure 5.](image)

**Figure 5.** Iterations needed for BRD to reach an NE for $\alpha = 0.5$, with 100 trials per $N$.

Next, it is important to study the geometry of PNE when more actions are available, and the payoffs are weakly dependent.
Acknowledgements. Marco Scarsini is a member of GNAMPA-INdAM and of the COST Action GAMENET. He gratefully acknowledges the support and hospitality of the Department of Mathematics at Monash University, where this research started. His work is partially supported by the GNAMPA-INdAM Project 2019 “Markov chains and games on networks” and by the PRIN 2017 project ALGADIMAR. Andrea Collevecchio’s work is partially supported by ARC grant DP180100613 and Australian Research Council Centre of Excellence for Mathematical and Statistical Frontiers (ACEMS) CE140100049. The authors thank Mikhail Isaev for pointing out reference McDiarmid et al. (2018). Ben Amiet’s work is supported by an Australian Government Research Training Program (RTP) Scholarship.

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8. List of Symbols

\( b_N \) \hspace{1cm} constant defined by \((1 - \beta)^N (1 - (1 - \beta)^N)\), introduced in Eq. (6.5)

\( B_r(s) \) \hspace{1cm} ball of radius \( r \) centred at \( s \), introduced in Lemma 6.1

\( \mathcal{B}^* \) \hspace{1cm} the graph obtained by adding to \( \mathcal{B}^* \) the edges which have no orientation

\( \mathcal{B}_k \) \hspace{1cm} percolation process at time \( k \), defined in Eq. (6.16)

\( \mathcal{B}^* \) \hspace{1cm} equal to \( \mathcal{B}_{2N} \)

\( \mathcal{C} \) \hspace{1cm} the graph with vertex set \( V_N \) and all the edges in \( \mathcal{E}_N \) which are not in \( \mathcal{B}^* \)

\( \mathcal{E}_N \) \hspace{1cm} edge set of \( \mathcal{H}_N \)

\( \mathcal{E}_N^* \) \hspace{1cm} edge set of \( \widetilde{\mathcal{H}}_N \)

\( f \) \hspace{1cm} defined in Eq. (6.6)

\( g_i \) \hspace{1cm} payoff function for player \( i \), introduced in Eq. (2.1)

\( h \) \hspace{1cm} Hamming distance on \( \mathcal{H}_N \)

\( h_N \) \hspace{1cm} a bounding constant, defined in Eq. (6.11)

\( \mathcal{H}_N \) \hspace{1cm} \( N \)-cube

\( \widetilde{\mathcal{H}}_N \) \hspace{1cm} partially oriented hypercube

\( \mathcal{H}_N^\beta \) \hspace{1cm} the random subgraph obtained from \( \mathcal{H}_N \) via \( \beta \)-bond percolation

\( \mathcal{H}_N \) \hspace{1cm} percolation process, defined in Eq. (6.22)

\( I \) \hspace{1cm} a subset of \( \mathcal{V}_N^{\text{even}} \), introduced in Eq. (6.25)

\( I_{-t} \) \hspace{1cm} equal to \( I \setminus \{t\} \), defined in Eq. (6.26)

\( K_\alpha \) \hspace{1cm} a constant dependent on \( \alpha \), introduced in Theorem 3.2

\( \mathcal{K} \) \hspace{1cm} subgraph of \( \mathcal{H}_N \)

\( \mathcal{L}_N^s \) \hspace{1cm} set of all vertices in \( \widetilde{\mathcal{H}}_N \) accessible from \( s \), introduced in Eq. (4.2)

\( \mathcal{L}_N^{\beta, s} \) \hspace{1cm} largest component of \( \mathcal{H}_N^\beta \)

\( \mathcal{L}_N^{\beta, 0} \) \hspace{1cm} largest component of \( \mathcal{H}_N^\beta \) containing \( s \)

\( \mathcal{L}_N^* \) \hspace{1cm} the largest connected component of \( \widetilde{\mathcal{H}}_N \) that contains \( s \)

\( m_\beta \) \hspace{1cm} constant, introduced in Lemma 6.1

\( \mathcal{M}_N^s \) \hspace{1cm} set of all vertices in \( \widetilde{\mathcal{H}}_N \) not accessible from \( s \), introduced in Eq. (4.2)

\( N \) \hspace{1cm} number of players, introduced in Eq. (2.1)

\([N]\) \hspace{1cm} set of players, introduced in Eq. (2.1)

\( \mathcal{N}_N \) \hspace{1cm} set of PNE in \( \Gamma_N \), introduced in Definition 2.1

\( \mathcal{P}_k \) \hspace{1cm} exploration process at time \( k \), defined in Eq. (6.15)

\( \{r \rightarrow t\} \) \hspace{1cm} event in which \( [s, t] \in \mathcal{E} \), introduced in Eq. (6.12)

\( s \) \hspace{1cm} an action profile, introduced after Eq. (2.1)

\( s_{-i} \) \hspace{1cm} action profile \( s \) for all players except \( i \), introduced after Eq. (2.1)

\( S_i \) \hspace{1cm} set of actions for player \( i \), introduced in Eq. (2.1)

\( S \) \hspace{1cm} set of all possible action profiles, introduced after Eq. (2.1)

\( [s, t] \) \hspace{1cm} edge connecting vertices \( s \) and \( t \)

\( [s, t] \) \hspace{1cm} edge \([s, t]\) oriented from \( s \) to \( t \)

\( t \) \hspace{1cm} an action profile, introduced in Eq. (2.4)

\( T^* \) \hspace{1cm} defined in Eq. (6.9)

\( T \) \hspace{1cm} equal to \( \sum_{s \in \mathcal{V}_N} T^s \)
\[ T_N \] number of vertices incident to at least \( 2^{N-m} \) unoriented vertices, introduced in Eq. (6.21)

\[ V \] defined in Eq. (6.6)

\[ \mathcal{V}_N \] vertex set of \( \mathcal{H}_N \)

\[ \mathcal{V}_N^{\text{even}} \] set of vertices whose sum of coordinates is even, introduced in Eq. (6.24)

\[ \mathcal{V}_N^{\text{odd}} \] set of vertices whose sum of coordinates is odd, introduced in Eq. (6.24)

\[ W^s \] Bernoulli random variable describing the event in which \( s \) is an SPNE, introduced in Eq. (6.1)

\[ W_N \] total number of SPNE in \( \Gamma_N \)

\[ \hat{X}^s \] Bernoulli random variable describing the event in which \( s \) is a PNE, introduced in proof of Theorem 3.2

\[ \hat{X} \] mean-adjusted \( \hat{X}^s \), introduced in proof of Theorem 3.2

\[ X \] collection \( (X^s)_{s \in \mathcal{V}_N} \)

\[ \tilde{X}^s \] an independent copy of \( X^s \)

\[ \tilde{X} \] collection \( (\tilde{X}^s)_{s \in \mathcal{V}_N} \)

\[ X_A^s \] defined in Eq. (6.7)

\[ Z^s_i \] random variable dictating the payoff for player \( i \) of action profile \( s \)

\[ Z \] a generic copy of \( Z^s_i \)

\[ \alpha \] probability of payoffs being equal, introduced in Eq. (3.1)

\[ \beta \] probability that one payoff is strictly less than another, introduced in Eq. (3.1)

\[ \gamma_k \] defined in Eq. (6.17)

\[ \Gamma_N \] game with \( N \) players, introduced in Eq. (2.1)

\[ \Delta_t \] a difference operator, defined in Eq. (6.8)

\[ \Delta \mathcal{U} \] set of vertices which are neighbours of vertex set \( \mathcal{U} \), introduced in Eq. (6.14)

\[ \Delta \mathcal{U}^\rightarrow \] set of vertices which are out-neighbours of vertex set \( \mathcal{U} \), introduced in Eq. (6.13)

\[ \Theta^t \] the indicator of the event that the vertex \( t \) is incident only to unoriented edges in \( \overrightarrow{\mathcal{H}}_N \), introduced in Eq. (6.23)

\[ \Theta_N \] defined in Eq. (6.23)

\[ \lambda_k \] defined in Eq. (6.17)

\[ \Xi_N \] the set of all isolated vertices in \( \mathcal{H}^{\beta}_N \), introduced in Proposition 5.3

\[ \tau_N \] standard deviation of the number of PNE in \( \Gamma_N \), introduced in Eq. (6.5)

\[ \phi \] density function of a standard normal

\[ \Phi \] cumulative distribution function of a standard normal, introduced in Eq. (3.2)

\[ 0 \] action profile \((0, 0, \ldots, 0)\)

\[ 1 \] action profile \((1, 0, \ldots, 0)\)
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