Topological phase-transition characterization of the one-dimensional Ising chain in a longitudinal field

Yi Liao¹,² and Ping-Xing Chen¹,²

¹ Department of Physics, National University of Defense Technology, Changsha, 410073, China
² Interdisciplinary Center for Quantum Information, National University of Defense Technology, Changsha, 410073, China

Abstract

jiu's
I. INTRODUCTION

The Ising model firstly describing paramagnetic-ferromagnetic phases transition with an external field is of some physical interest. In the case of transverse field (TF), it corresponds to the pseudo-spin formulation of several phase transition problems such as insulating magnetic systems, order-disorder ferroelectrics, cooperative Jahn-Teller systems[1, 2]. For the one-dimensional infinite TF Ising chain, Pfeuty’s work showed an asymptotic degeneracy of the ground state leading to the appearance of order. The addition of TF eliminates this degeneracy. With the field strength varying, the non-degeneracy remains and the order disappeared. On the contrary, the ground state may also become degenerate with the state carrying the excitation[1]. This excitation spectrum is identical to the excitation spectrum of the XY model[3]. As the XY model is deeply examined, another kind of order is generalized by J. M. Kosterlitz and D. J. Thouless as a new definition called topological order corresponding to the topological phase transition[4, 5].

The Berry phase (BP), a typical topological order parameter, entered the lexicon of physics about 30 years ago[6, 7]. Since then, numerous applications and experimental confirmations of this phase have been found in various physical systems[8–11]. A classical result showed in Berry’s work was that for a closed loop the geometric phase associated with the ground state is the half of the solid angle swept out[6, 12]. The BP can be exploited as a tool to detect topological phase transition. And if there exists nonzero BP in the system, it means that there exists parity and time inverse (PT) symmetry breaking for the ground state. The relationship between BP and topological transition in quantum system has been notoriously discussed in many literatures[13, 14].

The topological transitions in the Ising models have been extensively investigated, especially in the TF[15–19]. For the one-dimensional Ising chain in TF, an analytical theory has well been developed in the form of BP. With the Jordan-Wigner transformation, it could be equivalent a two-level system. And the transfer matrix for it can be mapped to the two-dimensional Ising model. All above make it possible to well calculate BP. If someone pays his attention to the longitudinal field (LF), he will find that the Jordan-Wigner transformation brings a term related to a kind of four-fermion interaction that is hard to deal with. Due to the coherent state path-integral for this spin-1/2 particle, the state can be written in the form of a Wess-Zumino-Novikov-Witten term[17], one-dimesional Ising chains, whenever in the TF or LF, are characterized by an emergent $SU(2)$ symmetry observed precisely at criticality. Moreover a change in a BP crosses the transition[19]. Nevertheless, the quantitative dependence on LF for the discontinuous BP still poses a challenge.
up to now. A concise and explicit characterization is worth exploring.

The outline of this paper is as follows. In Sec.II, with the method of the mean-field theory, we deal with the four-fermion interaction. We map the Hamiltonian of one-dimensional LF Ising model in the momentum space to a two-level system. We solve the ground state energy. In Sec.III, we diagonalise the Hamiltonian via a Bogoliubov transformation and obtain the ground state. And we calculate the BP in the paramagnetic and diamagnetic system. In Sec.IV, we draw the conclusion and compare our results with the topological transition in the TF case.

II. ONE-DIMENSIONAL ISING MODEL IN THE LONGITUDINAL FIELD

The Hamiltonian of one-dimensional LF Ising model reads

$$H = \sum_{i=1}^{N} \epsilon S_i^z + \sum_{i=1}^{N-1} (S_i^z S_{i+1}^z).$$

(1)

Here, the notation $J$ is the exchange couple and the index $\epsilon$ represents the reduced field strength. In the paper we only consider that lattice point number $N \to +\infty$, the temperature $T \to 0$. The sign $|g>\equiv<|O>|g>$ denotes the ground state of the system. The sign $<O>\equiv<|g>O|g>$ denotes the expectation of operator $O$ in the ground state. Ones could adopt a mean-field approximation which reads

$$O^\dagger O \approx <O^\dagger O> + <O^\dagger><O>.$$  

(2)

Meanwhile ones can make Jordan-Wigner transformation which is

$$S_i^z = c_i^+ c_i - \frac{1}{2}, \{c_i, c_i^+\} = \delta_{ii}, \{c_i, c_i\} = \{c_i^+, c_i^+\} = 0.$$  

(3)

So it turns into the following form given by

$$\tilde{H} = \sum_{i=1}^{N} (\epsilon (c_i^+ c_i - \frac{1}{2})) + \sum_{i=1}^{N-1} [(c_i^+ c_i - \frac{1}{2})(c_{i+1}^+ c_{i+1} - \frac{1}{2})]$$

$$= \left(1 - 2\epsilon\right)N + \sum_{i=1}^{N-1} (\epsilon (c_i^+ c_i) + \frac{1}{2} \sum_{i=1}^{N-1} (c_{i+1}^+ c_i c_{i+1} c_i - c_{i+1}^+ c_i c_i^+ c_{i+1})$$

$$\approx \left(1 - 2\epsilon - 2(\epsilon - 1)\right) + \sum_{i=1}^{N} [(\epsilon - 1)c_i^+ c_i]$$

$$+ \frac{1}{2} \sum_{i=1}^{N-1} (c_{i+1}^+ c_i c_{i+1} c_i + c_{i+1}^+ c_i c_i^+ c_{i+1} - c_{i+1}^+ c_i c_i^+ c_{i+1})$$

$$\equiv NA_{\epsilon} + \sum_{i=1}^{N} B_{\epsilon} c_i^+ c_i$$

$$+ \frac{1}{2} \sum_{i=1}^{N} [(C_{\epsilon} c_{i+1}^+ c_i + C_{\epsilon}^* c_i^+ c_{i+1}) - (D_{\epsilon} c_i^+ c_{i+1} + D_{\epsilon}^* c_{i+1}^+ c_i)].$$
Here, \( A_\epsilon = \frac{(1-2\epsilon)}{4} \); \( B_\epsilon = (\epsilon - 1) \); \( C_\epsilon \equiv \langle c_i c_{i+1} \rangle ; D_\epsilon \equiv \langle c_i^+ c_i \rangle \). And it is noticeable that \(|C_\epsilon|^2 = |D_\epsilon|^2 = \langle S_i^z + \frac{1}{2} \rangle = - \langle (S_i^z + \frac{1}{2})(S_{i+1}^z + \frac{1}{2}) \rangle \equiv A_\epsilon \). It will be proved in Sec. \( \text{III} \) that \( C_\epsilon \) and \( D_\epsilon \) are independent of the position of the \( i \)-th lattice.

One can switch to momentum space by Fourier transformation when \( N \to +\infty \), which is

\[
c_i = \frac{1}{\sqrt{2\pi}} \int c_k e^{jik} dk; \ c_k = \frac{1}{\sqrt{2\pi}} \int c_i e^{-jik} di. \tag{5}
\]

Here, \( j \) denotes the imaginary unit satisfying \( j^2 = -1 \). The reduced Hamiltonian \( \tilde{H} \) reads

\[
\tilde{H} = N A_\epsilon + \frac{1}{2} \int_{-\pi}^{\pi} \tilde{H}(k) dk. \tag{6}
\]

Here, \( \tilde{H}(k) \) refers to the reduced Hamiltonian in the \( k \)-space. There is the particle-hole symmetry, in other words, \( \tilde{H}(-k) = \tilde{H}(k) \). So \( \tilde{H}(k) \) reads

\[
\tilde{H}(k) = B_\epsilon (c_k^+ c_k + c_{-k}^+ c_{-k}) - j \sin k (C_\epsilon c_k^+ c_{-k} - C_\epsilon^* c_{-k} c_k) - \text{Re} D_\epsilon \cos k (c_k^+ c_k + c_{-k}^+ c_{-k})
\]

\[
= [B_\epsilon - \text{Re}(D_\epsilon e^{jk})] (c_k^+ c_k + c_{-k}^+ c_{-k}) - j \sin k (C_\epsilon c_k^+ c_{-k} - C_\epsilon^* c_{-k} c_k).
\]

We choose four basic vectors \( \mid 0 >_k \mid 0 >_{-k} \mid 1 >_k \mid 1 >_{-k} \mid 0 >_k \mid 0 >_{-k} \) and \( \mid 0 >_k \mid 1 >_{-k} \) . For simplicity, we let \( h(k) \equiv \frac{B_\epsilon}{2} + \text{Re}(D_\epsilon e^{jk}) \), \( g(k) \equiv C_\epsilon \sin k \). So \( \tilde{H}(k) \) reads

\[
\tilde{H}(k) = \begin{pmatrix}
0 & jg^*(k) & 0 & 0 \\
-jg(k) & 2h(k) & 0 & 0 \\
0 & 0 & h(k) & 0 \\
0 & 0 & 0 & h(k)
\end{pmatrix} = h(k) I_4 + \begin{pmatrix}
-h(k) & jg^*(k) & 0 & 0 \\
-jg(k) & h(k) & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{8}
\]

Here, \( I_4 \) denotes the \( 4 \times 4 \) unit matrix. The four eigenvalues of the energy is

\[
E_m(k) = h(k) \pm \sqrt{h^2(k) + g(k)g^*(k)}; h(k), h(k), m = 0, 1, 2, 3. \tag{9}
\]

The corresponding eigenvector \( \mid \psi_m(k) > \) satisfies

\[
\tilde{H}(k) \mid \psi_m(k) > = E_m(k) \mid \psi_m(k) >. \tag{10}
\]

When the exchange couple \( J < 0 \), \( E_0(k) = h(k) + \sqrt{h^2(k) + g(k)g^*(k)} \) is ground state energy. When the exchange couple \( J > 0 \), \( E_0(k) = h(k) - \sqrt{h^2(k) + g(k)g^*(k)} \) is ground state energy.
We can map the Hamiltonian of one-dimensional LF Ising model in the momentum space to a two-level system with the Hamiltonian \[ H(k) = \begin{pmatrix} -h(k) & -jg^*(k) \\ jg(k) & h(k) \end{pmatrix} \sim \begin{pmatrix} -h(k) & |g(k)| \\ |g(k)| & h(k) \end{pmatrix}. \] (11)

Here, the sign \( M_1 \sim M_2 \) means that the matrix \( M_1 \) is similar to the matrix \( M_2 \). They possess same topological structure if the phase angle \( \omega_C \) of the complex number \( g(k) \) is independent of \( k \).

### III. BERRY PHASE IN THE PARAMAGNETIC AND DIAMAGNETIC SYSTEM

For calculating the BP, we can diagonalise this Hamiltonian via a Bogoliubov transformation with two real functions \( \theta_k \) and \( \phi_k \) satisfying \( \theta_{-k} = -\theta_k \) and \( \phi_{-k} = \phi_k \), yielding

\[ c_k = \cos \theta_k d_k - j e^{i\phi_k} \sin \theta_k d_{-k}^+, \quad c_{-k}^+ = -j e^{-i\phi_k} \sin \theta_k d_k + \cos \theta_k d_{-k}. \] (12)

Here we have

\[ h(k) \sin(2\theta_k) + [g e^{i\phi_k} \sin^2 \theta_k - g^* e^{i\phi_k} \cos^2 \theta_k] = 0. \] (13)

In the other words, it reads

\[ h(k) \sin(2\theta_k) = |g(k)| \cos(2\theta_k); \quad \phi_k = \omega_C. \] (14)

\( |\psi_g \rangle \) is the ground state wave function in the \( k \)-space. \( \phi_k \) is a constant. \( |\psi_g \rangle \) reads

\[ |\psi_g \rangle \equiv \prod_k |\psi_0(k) \rangle = \prod_k [\cos \theta_k |0 >_k |0 >_{-k} - j e^{i\phi_k} \sin \theta_k |1 >_k |1 >_{-k}]. \] (15)

We have

\[ <\psi_0(k)|c_{k'}^+ c_{k''}^*|\psi_0(k)> = \sin^2 \theta_k (\delta_{k',k} \delta_{k'',-k} + \delta_{k',-k} \delta_{k'',k}); \] (16)

\[ <\psi_0(k)|c_{k'} c_{k''}|\psi_0(k)> = -j e^{i\phi_k} \sin \theta_k \cos \theta_k (\delta_{k',-k} \delta_{k'',k} - \delta_{k',k} \delta_{k'',-k}). \]

And we adopt a quasi-continuous convention which is \( \Sigma_k \rightarrow \frac{1}{4\pi} \int_{-\pi}^{\pi} dk \), we get

\[ C_\epsilon = \frac{1}{4\pi^2} \int <\psi_0(k)|c_{k'}^+ c_{k''}^*|\psi_0(k)> e^{j[k'+(i+1)k'']} dk' dk'' \] (17)

\[ = \frac{e^{j\omega_C}}{4\pi^2} \int_{-\pi}^{\pi} \sin(2\theta_k) \sin kd\theta; \]

\[ D_\epsilon = \frac{1}{4\pi^2} \int <\psi_0(k)|c_{k'}^+ c_{k''}^*|\psi_0(k)> e^{j[k''-(i+1)k']} dk' dk'' \]

\[ = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \sin^2 \theta_k \cos kd\theta. \]
TABLE I: The relation between the Berry phase $\gamma_g$ of the ground state and the reduced field strength $\epsilon$ in the paramagnetic case where $J < 0$.

| $\epsilon$       | $|D_\epsilon|$| $|B_\epsilon|$| $\gamma_g$ |
|------------------|----------------|----------------|-----------|
| $(-\infty, 0)$   | $0^+$          | $\neq 0$       | $0$       |
| $0$              | $0^+$          | $1$            | not well defined |
| $(0, +\infty)$   | $0^+$          | $|\epsilon - 1|$| $-\pi \delta_{\epsilon,1}$ |

So, $|D_\epsilon| = |\text{Re}(D_\epsilon)| = \sqrt{\Lambda_\epsilon}$.

We have a close curve $\partial \Omega$ where the point $(x, y)$ satisfies

$$\left(\frac{x - B_\epsilon}{D_\epsilon}\right)^2 + \left(\frac{y}{|C_\epsilon|}\right)^2 = 1.$$ (18)

The BP of the ground state is defined by

$$\gamma_g = j \int_{-\pi}^{\pi} \psi_0(k) \left|\frac{d}{dk}\psi_0(k)\right| dk, h(k) \sin(2\theta_k) = |g(k)| \cos(2\theta_k).$$ (19)

Because $\omega_C$ is a constant which is independent of $k$, the criterion for nonzero berry phase is decided by the relation between the point $(0, 0)$ and the curve $\partial \Omega$ [7]. In other words, it depend on the size of the relationship between $|B_\epsilon|$ and $|D_\epsilon|$. The BP reads

$$\gamma_g = \frac{\text{sgn}(J)[\text{sgn}(|D_\epsilon| - |B_\epsilon|) + 1]}{2} \pi.$$ (20)

Here $\text{sgn}(\zeta > 0) = 1; \text{sgn}(\zeta = 0) = 0; \text{sgn}(\zeta < 0) = -1$.

There exists the exact solution of $<S_i^z>$ and $<S_i^z S_{i+1}^z>$. So one can know [20],

$$\Lambda_\epsilon = \frac{1}{4} - \frac{1}{4} \lim_{\beta \rightarrow +\infty} \left\{ \sinh^2\left(\frac{\beta J}{2}\right) + \left[ \cosh\left(\frac{\beta J}{2}\right) - \frac{\sqrt{\sinh^2\left(\frac{\beta J}{2}\right) + \exp(\beta J)}\cosh\left(\frac{\beta J}{2}\right)}{\cosh\left(\frac{\beta J}{2}\right) + \sqrt{\sinh^2\left(\frac{\beta J}{2}\right) + \exp(\beta J)}}\right] \exp(\beta J) \right\}$$ (21)

If the field vanishes, due to the ground-state degeneracy, it should be noticed that the BP is not well defined. Based on the Eq.(20) and Eq.(21), the BP of the ground state dependence on $J$ and $\epsilon$ is summarized in Tables I and II.

IV. RESULTS AND DISCUSSION

The present calculation for the BP of one-dimensional LF Ising model serves as the complementary step to understand topological transitions in the Ising model. The level-crossing in the
TABLE II: The relation between the Berry phase $\gamma_g$ of the ground state and the reduced field strength $\epsilon$ in the paramagnetic case where $J > 0$.

| $\epsilon$      | $|D|_\epsilon$ | $|B|_\epsilon$ | $\gamma_g$ |
|-----------------|----------------|----------------|------------|
| $(-\infty, -1)$ | $0^+$          | $\neq 0$       | 0          |
| $-1$            | $\sqrt{10-2\sqrt{5}}/20$ | 2 | 0          |
| $(-1, 0)$       | $\sqrt{2}/2$ | $(1, 2)$       | 0          |
| $0$             | $\sqrt{2}/2$ | 1              | not well defined |
| $(0, 1)$        | $\sqrt{2}/2$ | $|\epsilon - 1|\left[\text{sgn}(\epsilon - 1 + \sqrt{2}/2) + 1\right]\pi$ | $\pi$ |
| $1$             | $\sqrt{10-2\sqrt{5}}/20$ | 0 | $\pi$ |
| $(1, +\infty)$  | $0^+$          | $\neq 0$       | 0          |

model corresponds to an analytic continuation around either of the two square-root branch-point singularities. This coalescing also corresponds to the conversion of a zero-mode solution of the fermionic quasi-particle propagator for the case of a gapped spectrum into a pole at criticality. It will lead to an important consequence on the nature of the quantum phase transition. Moreover, such singularities related to nonzero BPs are referred to as exceptional points that break parity and time inverse symmetry. As shown in Table II, the nontrivial point $\epsilon_{non} = 1$. And in the case of TF, $-1 \leq \epsilon_{non} \leq 1$. On the another hand, in the case of TF, the BP is dependent on the direct comparison between the exchange couple and the field strength. In the case of LF, the field strength affect the physical quantity of the system, the BP is dependent on the kind of the stimulus-response relation. The difference brings some new features in the LF case. As shown in Tables II and III, the nonzero BP occurs when $0\epsilon = 1$ for the paramagnetic system where $J < 0$, and it only occurs when $1 - \sqrt{2}/2 \leq \epsilon \leq 1$ for the diamagnetic system where $J > 0$. As a comparison, no matter $J < 0$ or $J > 0$ in the TF case, it occurs when $0 < |\epsilon| \leq 1$. It need to be pointed out that the conclusion is strain in the LF case due to $|D|_\epsilon = |B|_\epsilon = 0$ for the diamagnetic system where $J < 0$. It means that sometimes the BP is not enough good to determine the topological transition. We have to sake for more topological orders, such as topological entanglement entropy. When $\epsilon = 1$, what role will the quantum entanglement and coherent play to determine the topological transition? It is a interesting question, which will be studied in our future works.
Acknowledgments

This work was supported by the National Basic Research Program of China under Grant No. 2016YFA0301903; the National Natural Science Foundation of China under Grants No. 11574398, No. 11174370, No. 11304387, No. 61632021, No. 11305262, No. 11574398, and No. 61205108. This work was also supported by National Natural Science Foundation of China (NSFC) under grants (Y6GJ161001).

References

[1] P. Pfeuty, Ann. Phys. 57, 79(1970).
[2] R.B. Stinchcombe, J. Phys. C: Solid State Phys. 6, 2459(1973).
[3] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. 16, 407(1961).
[4] V. L. Berezinskii, Sov. Phys.-JETP 32 493-500 (1971).
[5] J. M. Kosterlitz and D. J. Thouless J. Phys. C: Solid State Phys. 6, 1181(1973).
[6] M. V. Berry, Proc. R. Soc. Lond. A, 392, 45-57(1984).
[7] Y. Liao, Phys. Lett. A, 380, 2888-2891(2016).
[8] C. A. Mead, Rev. Mod. Phys. 64, 51-85(1992).
[9] R. Resta, Rev. Mod. Phys. 66, 899-915(1994).
[10] D. R. Yarkony, Rev. Mod. Phys. 68, 985-1013(1996).
[11] A. Garg, Am. J. Phys. 78, 661-670(2010).
[12] M. Franz and L. Molenkamp, Topological Insulators (Elsevier Press, Oxford, 2013).
[13] A. Bohm, A. Mostafazadeh, H. Koizumi, Q. Niu and J. Zwanziger, The Geometric Phase in Quantum Systems (Springer Press, New York, 2003).
[14] S. Sachdev, Quantum Phase Transitions (Cambridge University Press, New York, 2011).
[15] E. Fradkin and L. Susskind, Phys. Rev. D 17, 2637 (1978).
[16] A. C. M. Carollo and J. K. Pachos, Phys. Rev. Lett. 95, 157203 (2005).
[17] E. Fradkin, Field theories of Condensed Matter Physics (Cambridge University Press, New York, 2013).
[18] S. Suzuki, J. Inoue and B. K. Chakrabarti, *Quantum Ising Phases and Transitions in Transverse Ising Models* (Springer Press, New York, 2013).

[19] S. Jalal, R. Khare, and S. Lal, arXiv:1610.09845 (2016).

[20] J. Strečka and M. Jaščur, Acta Physica Slovaca, 65 235-367 (2015).