HARMONIC SPHERES IN OUTER SYMMETRIC SPACES, THEIR CANONICAL ELEMENTS AND WEIERSTRASS TYPE REPRESENTATIONS

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Abstract. Making use of Murakami’s classification of outer involutions in a Lie algebra and following the Morse-theoretic approach to harmonic two-spheres in Lie groups introduced by Burstall and Guest, we obtain a new classification of harmonic two-spheres in outer symmetric spaces and a Weierstrass-type representation for such maps. Several examples of harmonic maps into classical outer symmetric spaces are given in terms of meromorphic functions on $S^2$.

1. Introduction

The harmonicity of maps $\varphi$ from a Riemann surface $M$ into a compact Lie group $G$ with identity $e$ amounts to the flatness of one-parameter families of connections. This establishes a correspondence between such maps and certain holomorphic maps $\Phi$ into the based loop group $\Omega G$, the extended solutions [15]. Evaluating an extended solution $\Phi$ at $\lambda = -1$ we obtain a harmonic map $\varphi$ into the Lie group. If an extended solution takes values in the group of algebraic loops $\Omega_{\text{alg}} G$, the corresponding harmonic map is said to have finite uniton number. It is well known that all harmonic maps from the two-sphere into a compact Lie group have finite uniton number [15].

Burstall and Guest [1] have used a method suggested by Morse theory in order to describe harmonic maps with finite uniton number from $M$ into $G$. One of the main ingredients in that paper is the Bruhat decomposition of the group of algebraic loops $\Omega_{\text{alg}} G$. Each piece $U_\xi$ of the Bruhat decomposition corresponds to an element $\xi$ in the integer lattice $\mathfrak{h}(G) = (2\pi)^{-1} \exp^{-1}(e) \cap \mathfrak{t}$ and can be described as the unstable manifold of the energy flow on the Kähler manifold $\Omega_{\text{alg}} G$. Any extended solution $\Phi : M \to \Omega_{\text{alg}} G$ takes values, off some discrete subset $D$ of $M$, in some of these unstable manifolds $U_\xi$ and can be deformed, under the gradient flow of the energy, to an extended solution with values in some conjugacy class of a Lie group homomorphism $\gamma_\xi : S^1 \to G$. A normalization procedure allows us to choose $\xi$ among the canonical elements of $\mathfrak{h}(G)$; if $G$ has trivial centre, there are precisely $2^n$ canonical elements, where $n = \text{rank}(G)$, and consequently $2^n$ classes of harmonic maps. Burstall and Guest [1] introduced also a Weierstrass-type representation for such harmonic maps in terms of meromorphic functions on $M$.

Given an involution $\sigma$ of $G$, the compact symmetric $G$-space $N = G/G^\sigma$, where $G^\sigma$ is the subgroup of $G$ fixed by $\sigma$, can be embedded totally geodesically in $G$ via the corresponding Cartan embedding $\iota_\sigma$. Hence harmonic maps into compact symmetric spaces can be interpreted as special harmonic maps into Lie groups. For inner involutions $\sigma = \text{Ad}(s_0)$, where $s_0 \in G$ is the geodesic reflection at some base point $x_0 \in N$, the composition of the Cartan embedding with left multiplication by $s_0$ gives a totally geodesic embedding of $G/G^\sigma$ in $G$ as a connected component of $\sqrt{e}$. Reciprocally, any connected component of $\sqrt{e}$ is a compact inner symmetric $G$-space. As shown by Burstall and Guest [1], any harmonic map into a connected component of $\sqrt{e}$ admits an extended solution $\Phi$ which is invariant under the involution $I(\Phi)(\lambda) = \Phi(-\lambda)\Phi(-1)^{-1}$. Off a discrete set, $\Phi$ takes values in some unstable
manifold \( U_\xi \) and can be deformed, under the gradient flow of the energy, to an extended solution with values in some conjugacy class of a Lie group homomorphism \( \gamma_\xi : S^1 \to G^\sigma \). An appropriate normalization procedure which preserves both \( I \)-invariance and the underlying connected component of \( \sqrt{\epsilon} \) allows us to choose \( \xi \) among the canonical elements of \( \mathfrak{J}(G) \). As a matter of fact, since \( \sigma \) is inner, \( \text{rank}(G) = \text{rank}(G^\sigma) \) and we have \( \mathfrak{J}(G) = \mathfrak{J}(G^\sigma) \), that is the canonical elements of \( \mathfrak{J}(G) \) coincide with the canonical elements of \( \mathfrak{J}(G^\sigma) \). Consequently, if \( G \) has trivial center, we have \( 2^n \) classes of harmonic maps with finite uniton number into all inner symmetric \( G \)-spaces.

The theory of Burstall and Guest [1] on harmonic two-spheres in compact inner symmetric \( G \)-spaces was extended by Eschenburg, Mare and Quast [8] to outer symmetric spaces as follows: each harmonic map from a two-sphere into an outer symmetric space \( G/G^\sigma \), with outer involution \( \sigma \), corresponds to an extended solution \( \Phi \) which is invariant under a certain involution \( T_\sigma \) induced by \( \sigma \) on \( \Omega G \) (see also [11]); \( \Phi \) takes values in some unstable manifold \( U_\xi \), off some discrete set; under the gradient flow of the energy any such invariant extended solution is deformed to an extended solution with values in some conjugacy class of a Lie group homomorphism \( \gamma_\xi : S^1 \to G^\sigma \); applying the normalization procedure of extended solutions introduced by Burstall and Guest for Lie groups, \( \xi \) can be chosen among the canonical elements of \( \mathfrak{J}(G^\sigma) \subset \mathfrak{J}(G) \); if \( G \) has trivial centre, there are precisely \( 2^k \) canonical homomorphisms, where \( k = \text{rank}(G^\sigma) < \text{rank}(G) \); hence there are at most \( 2^k \) classes of harmonic two-spheres in \( G/G^\sigma \) if \( G \) has trivial centre. However, this classification does not take into account the following crucial facts concerning extended solutions associated to harmonic maps into outer symmetric spaces: although any harmonic map from a two-sphere into an outer symmetric space \( G/G^\sigma \) admits a \( T_\sigma \)-invariant extended solution, not all \( T_\sigma \)-invariant extended solutions correspond to harmonic maps into \( G/G^\sigma \); the Burstall and Guest’s normalization procedure does not necessarily preserve \( T_\sigma \)-invariance. In the present paper we will establish a more accurate classification and establish a Weierstrass formula for such harmonic maps. These will allow us to produce some explicit examples of harmonic maps from two-spheres into outer symmetric spaces from meromorphic functions on \( S^2 \).

Our strategy is the following. The existence of outer involutions of a simple Lie algebra \( g \) depends on the existence of non-trivial involutions of the Dynkin diagram of \( g^C \) [2,8,12,13]. More precisely, if \( \varphi \) is a non-trivial involution of the Dynkin diagram of \( g^C \), then it induces an outer involution \( \sigma_\varphi \) of \( g^C \), which we call the fundamental outer involution, and, as shown by Murakami [13], all the other outer involutions are, up to conjugation, of the form \( \sigma_{\varphi,i} := \text{Ad}\exp(\pi\zeta_i \circ \sigma_\varphi) \) where \( \zeta_i \) is a certain element in the integer lattice \( \mathfrak{J}(G^\sigma) \). Each connected component of \( P^\sigma = \{ g \in G | \sigma(g) = g^{-1} \} \) is a compact outer symmetric \( G \)-space associated to some involution \( \sigma_\varphi \) or \( \sigma_{\varphi,i} \); reciprocally, any outer symmetric space \( G/G^\sigma \), with \( \sigma \) equal to \( \sigma_\varphi \) or \( \sigma_{\varphi,i} \), can be totally geodesically embedded in the Lie group \( G \) as a connected component of \( P^\sigma \) (see Proposition [10]). As shown in Section 4.2, any harmonic map \( \varphi \) into a connected component \( N \) of \( P^\sigma \) admits a \( T_{\sigma_\varphi} \)-invariant extended solution \( \Phi \); off a discrete set, \( \Phi \) takes values in some unstable manifold \( U_\xi \). In Section 4.2.2 we introduce an appropriate normalization procedure in order to obtain from \( \Phi \) a normalized extended solution \( \tilde{\Phi} \) with values in some unstable manifold \( U_\zeta \) such that: \( \zeta \) is a canonical element of \( \mathfrak{J}(G^\sigma) \); \( \tilde{\Phi} \) is \( T_\tau \)-invariant, where \( \tau \) is the outer involution given by \( \tau = \text{Ad}\exp(\pi(\xi - \zeta) \circ \sigma_{\varphi}) \); \( \tilde{\Phi}(1) \) takes values in some connected component of \( P^\sigma \) which is an isometric copy of \( N \) completely determined by \( \zeta \) and \( \tau \); moreover, \( \tilde{\Phi}(1) \) coincides with \( \varphi \) up to isometry. Hence, we obtain a classification of harmonic maps of finite uniton number from \( M \) into outer symmetric \( G \)-spaces in terms of the pairs \( (\zeta, \tau) \).
Dorfmeister, Pedit and Wu [7] have introduced a general scheme for constructing harmonic maps from a Riemann surface into a compact symmetric space from holomorphic data, in which the harmonic map equation reduces to a linear ODE similar to the classical Weierstrass representation of minimal surfaces. Burstal and Guest [1] made this scheme more explicit for the case $M = S^2$ by establishing a “Weierstrass formula” for harmonic maps with finite uniton number into Lie groups and their inner symmetric spaces. In Proposition 22 we establish a version of this formula to outer symmetric spaces, which allows us to describe the corresponding $T^r$-invariant extended solutions in terms of meromorphic functions on $M$. For normalized extended solutions and “low uniton number”, such descriptions are easier to obtain. In Section 5 we give several explicit examples of harmonic maps from the two-sphere into classical outer symmetric spaces: Theorem 25 interprets old results by Calabi [3] and Eells and Wood [9] concerning harmonic spheres in real projective spaces $\mathbb{R}P^{2n-1}$ in view of our classification; harmonic two-spheres into the real Grassmannian $G_3(\mathbb{R}^6)$ are studied in detail; we show that all harmonic two spheres into the Wu manifold $SU(3)/SO(3)$ can be obtained explicitly by choosing two meromorphic functions on $S^2$ and then performing a finite number of algebraic operations.

2. Groups of algebraic loops

For completeness, in this section we recall some fundamental facts concerning the structure of the group of algebraic loops in a compact Lie group. Further details can be found in [1] [4] [13].

2.1. The Bruhat decomposition. Let $G$ be a compact matrix semisimple Lie group with Lie algebra $\mathfrak{g}$ and identity $e$. Denote the free and based loop groups of $G$ by $\Lambda G$ and $\Omega G$, respectively, whereas $\Lambda_+ G^C$ stands for the subgroup of $\Lambda G^C$ consisting of loops $\gamma : S^1 \to G^C$ which extend holomorphically to the unitary disc $|\lambda| < 1$.

Taking account the Iwasawa decomposition $\Lambda G^C \cong \Omega G \times \Lambda_+ G^C$, each $\gamma \in \Lambda G^C$ can be written uniquely in the form $\gamma = \gamma_R\gamma_+$, with $\gamma_R \in \Omega G$ and $\gamma_+ \in \Lambda_+ G^C$. Consequently, there exists a natural action of $\Lambda_+ G$ on $\Omega G$: if $g \in \Omega G$ and $h \in \Lambda_+ G$, then $h \cdot g = (hg)_R$.

Fix a maximal torus $T$ of $G$ with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Let $\Delta \subset \mathfrak{t}^*$ be the corresponding set of roots, where $i := \sqrt{-1}$, and, for each $\alpha \in \Delta$, denote by $\mathfrak{g}_\alpha$ the corresponding root space. The integer lattice $\mathfrak{I}(G) = (2\pi)^{-1} \exp^{-1}(e) \cap \mathfrak{t}$ may be identified with the group of homomorphisms $S^1 \to T$, by associating to $\xi \in \mathfrak{I}(G)$ the homomorphism $\gamma_\xi$ defined by $\gamma_\xi(\lambda) = \exp(-i\ln(\lambda)\xi)$. Let $\Omega_\xi(G)$ be the conjugacy class of homomorphisms $S^1 \to G$ which contains $\gamma_\xi$, that is $\Omega_\xi(G) = \{g \gamma_\xi g^{-1} | g \in G\}$.

Each $\xi \in \mathfrak{I}(G)$ endows $\mathfrak{g}^C$ with a structure of graded Lie algebra: for each $j \in \mathbb{Z}$, let $\mathfrak{g}_j^\xi$ be the $ji$-eigenspace of $\text{ad}\xi$, which is given by the direct sum of those root spaces $\mathfrak{g}_\alpha$ satisfying $\alpha(\xi) = ji$; then

$$\mathfrak{g}^C = \bigoplus_{j \in \{-r(\xi) \ldots r(\xi)\}} \mathfrak{g}_j^\xi, \quad [\mathfrak{g}_j^\xi : \mathfrak{g}_j^\xi] \subset \mathfrak{g}_{i+j}^\xi,$$

where $r(\xi) = \max\{j \mid \mathfrak{g}_j^\xi \neq 0\}$.

Proposition 1. [1] The conjugacy class $\Omega_\xi(G)$ of homomorphisms has a structure of complex homogeneous space. More precisely,

$$\Omega_\xi(G) \cong G^C/P_\xi, \quad \text{with} \quad P_\xi = G^C \cap \gamma_\xi \Lambda^+ G^C \gamma_\xi^{-1}.$$

The Lie algebra $\mathfrak{p}_\xi$ of the isotropy subgroup $P_\xi$ is the parabolic subalgebra induced by $\xi$, that is $\mathfrak{p}_\xi = \bigoplus_{i \leq 0} \mathfrak{g}_i^\xi$. 

Choose a fundamental Weyl chamber $\mathcal{W}$ in $\mathfrak{t}$, which corresponds to fix a positive root system $\Delta^+$. The intersection $\mathcal{Y}(G) := \mathfrak{Y}(G) \cap \mathcal{W}$ parameterizes the conjugacy classes of homomorphisms $S^1 \to G$:

$$\text{Hom}(S^1, G) = \bigsqcup_{\xi \in \mathcal{Y}(G)} \Omega_\xi(G).$$

Let $\Omega_{\text{alg}} G$ be the subgroup of algebraic based loops. The Bruhat decomposition states that $\Omega_{\text{alg}} G$ is the disjoint union of the orbits $\Lambda^+ \mathfrak{g} \cdot \gamma_\xi$, with $\xi \in \mathcal{Y}(G)$. This admits the following Morse theoretic interpretation \cite{14}. Consider the usual energy functional on paths $E : \Omega G \to \mathbb{R}$. The critical manifolds of this Morse-Bott function are precisely the conjugacy classes of homomorphisms $S^1 \to G$ and, for each $\xi \in \mathcal{Y}(G)$,

$$U_\xi(G) := \Lambda^+ \mathfrak{g} \cdot \gamma_\xi$$

is the unstable manifold of $\Omega_\xi(G)$ under the flow induced by the gradient vector field $-\nabla E$; each $\gamma \in U_\xi$ flows to some homomorphism $u_\xi(\gamma)$ in $\Omega_\xi(G)$.

**Proposition 2.** \cite{1} For each $\xi \in \mathcal{Y}(G)$, the unstable manifold $U_\xi(G)$ is a complex homogeneous space of the group $\Lambda^+ \mathfrak{g} \mathfrak{c}$, and the isotropy subgroup at $\gamma_\xi$ is the subgroup $\Lambda^+ \mathfrak{g} \mathfrak{c} \cap \gamma_\xi \Lambda^+ \mathfrak{g} \mathfrak{c} \gamma_\xi^{-1}$. Moreover, $U_\xi(G)$ carries a structure of holomorphic vector bundle over $\Omega_\xi(G)$ and the bundle map $u_\xi : U_\xi(G) \to \Omega_\xi(G)$ is precisely the natural projection

$$\Lambda^+ \mathfrak{g} \mathfrak{c} / \Lambda^+ \mathfrak{g} \mathfrak{c} \cap \gamma_\xi \Lambda^+ \mathfrak{g} \mathfrak{c} \gamma_\xi^{-1} \to \mathfrak{g} / P_\xi$$

given by $[\gamma] \mapsto [\gamma(0)]$.

Define a partial order $\preceq$ over $\mathcal{Y}(G)$ as follows: $\xi \preceq \xi'$ if $p_1^\xi \subset p_1^{\xi'}$ for all $i \geq 0$, where $p_i^\xi = \sum_{j \leq i} \Phi_j^\xi$.

**Lemma 3.** \cite{4} Take two elements $\xi, \xi' \in \mathcal{Y}(G)$ such that $\xi \preceq \xi'$. Then

$$\Lambda^+ \mathfrak{g} \mathfrak{c} \cap \gamma_\xi \Lambda^+ \mathfrak{g} \mathfrak{c} \gamma_\xi^{-1} \subset \Lambda^+ \mathfrak{g} \mathfrak{c} \cap \gamma_{\xi'} \Lambda^+ \mathfrak{g} \mathfrak{c} \gamma_{\xi'}^{-1}.$$ 

This lemma allows one to define a $\Lambda^+ \mathfrak{g} \mathfrak{c}$-invariant fibre bundle morphism $U_{\xi, \xi'} : U_\xi(G) \to U_{\xi'}(G)$ by

$$U_{\xi, \xi'}(\Psi \cdot \gamma_\xi) = \Psi \cdot \gamma_{\xi'}, \quad \Psi \in \Lambda^+ \mathfrak{g} \mathfrak{c},$$

whenever $\xi \preceq \xi'$. Since the holomorphic structures on $U_\xi(G)$ and $U_{\xi'}(G)$ are induced by the holomorphic structure on $\Lambda^+ \mathfrak{g} \mathfrak{c}$, the fibre-bundle morphism $U_{\xi, \xi'}$ is holomorphic.

### 3. Harmonic spheres in Lie groups

Harmonic maps from the two-sphere $S^2$ into the compact matrix Lie group $G$ can be classified in terms of certain pieces of the Bruhat decomposition of $\Omega_{\text{alg}} G$. Next we recall briefly this theory from \cite{1 4 5 6}.

#### 3.1. Extended solutions.

Let $M$ be a connected Riemann surface, $\varphi : M \to G$ be a smooth map and $\rho : G \to \text{End}(V)$ a finite representation of $G$. Equip $G$ with a bi-invariant metric. Define $\alpha = \varphi^{-1} d\varphi$ and let $\alpha = \alpha' + \alpha''$ be the type decomposition of $\alpha$ into $(1,0)$ and $(0,1)$-forms. As first observed by K. Uhlenbeck \cite{15}, $\varphi : M \to G$ is harmonic if and only if the loop of 1-forms given by $\alpha_\lambda = \frac{1}{2}(1 - \lambda^{-1}) \alpha' + \frac{1}{2}(1 - \lambda) \alpha''$ satisfies the Maurer-Cartan equation $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ for each $\lambda \in S^1$. Then, if $\varphi$ is harmonic and $M$ is simply connected, we can integrate to obtain a map $\Phi : M \to \Omega G$, the extended solution associated to $\varphi$, such that $\alpha_\lambda = \Phi_\lambda d\Phi_\lambda$ and $\Phi_{-1} = \varphi$. Moreover,
Φ is unique up to left multiplication by a constant loop. If \( \tilde{\Phi} = \gamma\Phi \) for some \( \gamma \in \Omega G \), we say that the extended solutions \( \tilde{\Phi} \) and \( \Phi \) are equivalent.

An extended solution \( \Phi : M \to \Omega G \) is said to have finite uniton number if \( \Phi(M) \subseteq \Omega_{\text{alg}}G \), that is \( \rho \circ \Phi = \sum_{i=r}^s \zeta_i \lambda_i \) for some \( r \leq s \in \mathbb{Z} \). The corresponding harmonic map \( \varphi = \Phi_{-1} \) is also said to have finite uniton number. The number \( s - r \) is called the uniton number of \( \Phi \) with respect to \( \rho \), and the minimal value of \( s - r \) (with respect to all extended solutions associated to \( \varphi \)) is called the uniton number of \( \varphi \) with respect to \( \rho \) and it is denoted by \( r_{\rho}(\varphi) \). K. Uhlenbeck [15] proved that all harmonic maps from the two-sphere have finite uniton number. Off a discrete subset, any such extended solution takes values in a single unstable manifold.

**Theorem 4.** [1] Let \( \Phi : M \to \Omega_{\text{alg}}G \) be an extended solution. Then there exists some \( \xi \in \mathfrak{J}(G) \), and some discrete subset \( D \) of \( M \), such that \( \Phi(M \setminus D) \subseteq U_{\xi}(G) \).

Given a smooth map \( \Phi : M \setminus D \to U_{\xi}(G) \), consider \( \Psi : M \setminus D \to \Lambda_{\text{alg}}^+ G^\mathbb{C} \) such that \( \Phi = \Psi \cdot \gamma_{\xi} \), that is \( \Psi \gamma_{\xi} = \Phi b \) for some \( b : M \setminus D \to \Lambda_{\text{alg}}^+ G^\mathbb{C} \). Write

\[
\Psi^{-1}\Psi_z = \sum_{i \geq 0} X'_i \lambda_i, \quad \Psi^{-1}\Psi_z = \sum_{i \geq 0} X''_i \lambda_i.
\]

Proposition 4.4 in [1] establishes that \( \Phi \) is an extended solution if, and only if,

\[
\text{Im}X'_i \subset p^\xi_i, \quad \text{Im}X''_i \subset p'^{\xi}_i,
\]

where \( p^\xi_i = \bigoplus_{j \leq i} g^\xi_j \). The derivative of the harmonic map \( \varphi = \Phi_{-1} \) is given by the following formula.

**Lemma 5.** [1] Let \( \Phi = \Psi \cdot \gamma_{\xi} : M \to \Omega_{\text{alg}}G \) be an extended solution and \( \varphi = \Phi_{-1} : M \to G \) the corresponding harmonic map. Then

\[
\varphi^{-1}\varphi_z = -2 \sum_{i \geq 0} b(0)X'^{i+1}_i b(0)^{-1},
\]

where \( X'^{i+1}_i \) is the component of \( X'_i \) over \( g_{i+1}^\xi \), with respect to the decomposition \( g^\mathbb{C} = \bigoplus g^\xi_j \).

Both the fiber bundle morphism \( u_{\xi,\xi'} : U_{\xi}(G) \to U_{\xi'}(G) \) and the bundle map \( u_{\xi} : U_{\xi}(G) \to \Omega_{\xi}(G) \) preserve harmonicity.

**Proposition 6.** [1] Let \( \Phi : M \setminus D \to U_{\xi}(G) \) be an extended solution. Then

a) \( u_{\xi} \circ \Phi : M \setminus D \to \Omega_{\xi} \) is an extended solution, with \( \xi \in \mathfrak{J}(G) \);

b) for each \( \xi' \in \mathfrak{J}(G) \) such that \( \xi \leq \xi' \), \( u_{\xi,\xi'}(\Phi) = U_{\xi,\xi'} \circ \Phi : M \setminus D \to U_{\xi'}(G) \) is an extended solution.

3.1.1. **Weierstrass representation.** Taking a larger discrete subset if necessary, one obtain a more explicit description for harmonic maps of finite uniton number and their extended solutions as follows.

**Proposition 7.** [1] Let \( \Phi : M \to \Omega_{\text{alg}}G \) be an extended solution. There exists a discrete set \( D' \supseteq D \) of \( M \) such that

\[
\Phi|_{M \setminus D'} = \exp C \cdot \gamma_{\xi}
\]

for some holomorphic vector-valued function \( C : M \setminus D' \to u^0_{\xi} \), where \( u^0_{\xi} \) is the finite dimensional nilpotent subalgebra of \( \Lambda_{\text{alg}}^+ g^\mathbb{C} \) defined by

\[
u^0_{\xi} = \bigoplus_{0 \leq i < r(\xi)} \lambda^i (p^\xi_i)^\perp, \quad (p^\xi_i)^\perp = \bigoplus_{i < j \leq r(\xi)} g^\xi_j.
\]
Moreover, $C$ can be extended meromorphically to $M$.

Conversely, taking account (1) and the well-known formula for the derivative of the exponential map, we see that if $C : M \to u^1_\xi$ is meromorphic then $\Phi = \exp C \cdot \gamma_\xi$ is an extended solution if and only if in the expression

$$
(\exp C)^{-1}(\exp C)_z = C_z - \frac{1}{2!}(\text{ad} C)C_z + \ldots + (-1)^{r(\xi)-1} \frac{1}{r(\xi)!}(\text{ad} C)^{r(\xi)-1}C_z,
$$

the coefficient $\lambda^i$ have zero component in each $\mathbf{g}^{\xi}_{i+2}, \ldots, \mathbf{g}^{\xi}_{r(\xi)}$.

3.1.2. $S^1$-invariant extended solutions. Extended solutions with values in some $\Omega_\xi(G)$, off a discrete subset, are said to be $S^1$-invariant. If we take a unitary representation $\rho : G \to U(n)$ for some $n$, then for any such extended solution $\Phi$ we have $\rho \circ \Phi_\lambda = \sum_{i=1}^r \lambda^i \pi_{W_i}$, where, for each $i$, $\pi_{W_i}$ is the orthogonal projection onto a complex vector subbundle $W_i$ of $\mathbb{C}^n := M \times \mathbb{C}^n$ and $\mathbb{C}^n = \bigoplus_{i=1}^r W_i$ is an orthogonal direct sum decomposition. Set $A_i = \bigoplus_{j \leq i} W_j$ so that

$$
\{0\} \subset A_r \subset \ldots \subset A_{i-1} \subset A_i \subset A_{i+1} \subset \ldots \subset A_s = \mathbb{C}^n.
$$

The harmonicity condition amounts to the following conditions on the the flag (3): for each $i$, $A_i$ is a holomorphic subbundle of $\mathbb{C}^n$; the flag of holomorphic subbundles (3) is superhorizontal, in the sense that, for each $i$, we have $\partial A_i \subseteq A_{i+1}$, that is, given any section $s$ of $A_i$ then $\frac{\partial s}{\partial z}$ is a section of $A_{i+1}$ for any local complex coordinate $z$ of $M$.

3.2. Normalization of harmonic maps. Let $\Delta_0 := \{\alpha_1, \ldots, \alpha_r\} \subset \Delta^+$ be the basis of positive simple roots, with dual basis $\{H_1, \ldots, H_r\} \subset \mathfrak{t}$, that is $\alpha_i(H_j) = i \delta_{ij}$, where $r = \text{rank}(\mathfrak{g})$. Given $\xi = \sum n_i H_i$ and $\xi' = \sum n'_i H_i$ in $\mathfrak{t}'(G)$, we have $n_i, n'_i \geq 0$ and observe that $\xi \preceq \xi'$ if and only if $n'_i \leq n_i$ for all $i$. For each $I \subseteq \{1, \ldots, r\}$, define the cone

$$
\mathfrak{C}_I = \left\{ \sum_{i=1}^r n_i H_i | n_i \geq 0, n_j = 0 \text{ iff } j \notin I \right\}.
$$

**Definition 1.** Let $\xi \in \mathfrak{t}'(G) \cap \mathfrak{C}_I$. We say that $\xi$ is a $I$-canonical element of $G$ with respect to $\mathcal{W}$ if it is a maximal element of $(\mathfrak{t}'(G) \cap \mathfrak{C}_I, \preceq)$, that is: if $\xi \preceq \xi'$ and $\xi' \in \mathfrak{t}'(G) \cap \mathfrak{C}_I$ then $\xi = \xi'$.

**Remark 1.** When $G$ has trivial centre, which is the case considered in [1], the duals $H_1, \ldots, H_r$ belong to the integer lattice. Then, for each $I$ there exists a unique $I$-canonical element, which is given by $\xi_I = \sum_{i \in I} H_i$. When $G$ has non-trivial centre, it is not so easy to describe the $I$-canonical elements of $G$ (see [5] [6]).

For simplicity of exposition, henceforth we will take $M = S^2$. However, all our results still hold for harmonic maps of finite uniton number from an arbitrary connected Riemann surface $M$.

Any harmonic map $\varphi : S^2 \to G$ admits a normalized extended solution, that is, an extended solution $\Phi$ taking values in $U_\xi$, off some discrete set, for some canonical element $\xi$. This is a consequence of the following generalization of Theorem 4.5 in [1].

**Theorem 8.** [4] Let $\Phi : S^2 \setminus D \to U_\xi(G)$ be an extended solution. Take $\xi' \in \mathfrak{t}'(G)$ such that $\xi \preceq \xi'$ and $\mathbf{g}^{\xi}_0 = \mathbf{g}^{\xi'}_0$. Then $\gamma^{-1} := U_{\xi, \xi - \xi'}(\Phi)$ is a constant loop in $\Omega_{\text{alg}} G$ and $\gamma \Phi : S^2 \setminus D \to U_{\xi'}(G)$.

The uniton number of a normalized extended solution can be computed with respect to any finite representation as follows.
Proposition 9. Let \( \rho : G \to \text{End}(V) \) be an irreducible \( n \)-dimensional representation of \( G \) with highest weight \( \omega^* \) and lowest weight \( \varpi^* \), and \( \xi \) a \( I \)-canonical element of \( \mathfrak{g} \). Then, the uniton number of \( \Phi : S^2 \setminus D \to U_\xi(G) \) is given by \( r_\rho(\xi) := \omega^*(\xi) - \varpi^*(\xi) \).

4. Harmonic spheres in outer symmetric spaces

The classification of harmonic two-spheres into outer symmetric spaces by Eschenburg, Mare and Quast \cite{8} does not take into account the following crucial facts concerning extended solutions associated to harmonic maps into outer symmetric spaces: although any harmonic map from a two-sphere into an outer symmetric space \( G/K \) admits a \( T_\sigma \)-invariant extended solution, not all \( T_\sigma \)-invariant extended solutions correspond to harmonic maps into \( G/K \); the Burstall and Guest’s normalization procedure, as described in Section 3.2, does not necessarily preserve \( T_\sigma \)-invariance. In the following sections we will establish a more accurate classification and establish a Weierstrass formula for such harmonic maps. These will allow us to produce some explicit examples of harmonic maps from two-spheres into outer symmetric spaces from meromorphic data.

4.1. Symmetric \( G \)-spaces and Cartan embeddings. Let \( N = G/K \) be a symmetric space, where \( K \) is the isotropy subgroup at the base point \( x_0 \in N \), and let \( \sigma : G \to G \) be the corresponding involution: we have \( G^\sigma_0 \subseteq K \subseteq G^\sigma \), where \( G^\sigma \) is the subgroup fixed by \( \sigma \) and \( G^\sigma_0 \) denotes its connected component of identity. We assume that \( N \) is a bottom space, i.e. \( K = G^\sigma \). Let \( \mathfrak{g} = \mathfrak{t}_\sigma \oplus \mathfrak{m}_\sigma \) be the \( \pm 1 \)-eigenspace decomposition associated to the involution \( \sigma \), \( \mathfrak{t}_\sigma \) is the Lie algebra of \( K \). Consider the (totally geodesic) Cartan embedding \( \iota_\sigma : N \to G \) defined by \( \iota_\sigma(g \cdot x_0) = g\sigma(g^{-1}) \). The image of the Cartan embedding is precisely the connected component \( P^\sigma_\xi \) of

\[
P^\sigma_\xi = \{ g \in G | \sigma(g) = g^{-1} \}
\]

containing the identity \( e \) of the group \( G \). Observe that, given \( \xi \in \mathfrak{h}(G) \cap \mathfrak{t}_\sigma \), then \( \exp(\pi \xi) \in P^\sigma_\xi \). We denote by \( P^\sigma_\xi \) the connected component of \( P^\sigma \) containing \( \exp(\pi \xi) \).

Proposition 10. Given \( \xi \in \mathfrak{h}(G) \cap \mathfrak{t}_\sigma \), we have the following.

a) \( G \) acts transitively on \( P^\sigma_\xi \) as follows: for \( g \in G \) and \( h \in P^\sigma_\xi \),

\[
g \cdot_\sigma h = gh\sigma(g^{-1}).
\]

b) \( P^\sigma_\xi \) is a bottom symmetric \( G \)-space totally geodesically embedded in \( G \) with involution

\[
\tau = \text{Ad}(\exp(\pi \xi)) \circ \sigma.
\]

c) For any other \( \xi' \in \mathfrak{h}(G) \cap \mathfrak{t}_\sigma \) we have \( \exp(\pi \xi') \in P^\tau \) and \( P^\tau_\xi = \exp(\pi \xi)P^\sigma_\xi \).

d) The \( \pm 1 \)-eigenspace decomposition \( \mathfrak{g} = \mathfrak{t}_\tau \oplus \mathfrak{m}_\tau \) associated to the symmetric \( G \)-space \( P^\sigma_\xi \) at the fixed point \( \exp(\pi \xi) \in P^\sigma_\xi \) is given by

\[
\mathfrak{t}_\tau^C = \bigoplus \mathfrak{g}_{2i+1}^\xi \cap \mathfrak{t}_\sigma^C \oplus \bigoplus \mathfrak{g}_{2i+1}^\xi \cap \mathfrak{m}_\sigma^C
\]
\[
\mathfrak{m}_\tau^C = \bigoplus \mathfrak{g}_{2i+1}^\xi \cap \mathfrak{t}_\sigma^C \oplus \bigoplus \mathfrak{g}_{2i}^\xi \cap \mathfrak{m}_\sigma^C.
\]

Proof. Take \( h \in P^\sigma \). We have

\[
\sigma(g \cdot_\sigma h) = \sigma(g h\sigma(g^{-1})) = \sigma(g) h^{-1} g^{-1} = (gh\sigma(g^{-1}))^{-1} = (g \cdot_\sigma h)^{-1}.
\]
Then $g \cdot_\sigma h \in P_\sigma^e$ and we have a continuous action of $G$ on $P_\sigma^e$. Since $G$ is connected, this action induces an action of $G$ on each connected component of $P_\sigma^e$. Since $g \cdot_\sigma e = g\sigma(g^{-1}) = \iota_\sigma(g \cdot x_0)$ and $\iota_\sigma(N) = P_\sigma^e$, the action $\cdot_\sigma$ of $G$ on $P_\sigma^e$ is transitive.

Take $\xi \in \mathcal{I}(G) \cap \mathfrak{k}_\sigma$, so that $\sigma(\xi) = \xi$ and $\exp 2\pi \xi = e$. Consider the involution $\tau$ defined by (5) if $g \in P_\sigma^e$, then

$$\tau(\exp(\pi \xi)g) = \exp(\pi \xi)\sigma(\exp(\pi \xi)g) \exp(\pi \xi) = \sigma(g) \exp(\pi \xi) = (\exp(\pi \xi)g)^{-1},$$

which means that $\exp(\pi \xi)g \in P_\tau^e$. Reciprocally, if $\exp(\pi \xi)g \in P_\tau^e$, one can check similarly that $g \in P_\sigma^e$. Hence $P_\tau^e = \exp(\pi \xi)P_\sigma^e$. In particular, by continuity, $P_\tau^e = \exp(\pi \xi)P_\tau^e - \xi$ for any other $\xi' \in \mathcal{I}(G)$ with $\sigma(\xi') = \xi'$.

Reversing the rules of $\sigma = \text{Ad}(\exp \pi \xi) \circ \tau$ and $\tau$, we also have

$$P_\xi^e = \exp(\pi \xi)P_\tau^e.$$  

Since $G$ acts transitively on $P_\tau^e$, for each $h \in P_\tau^e$ there exists $g \in G$ such that

$$h = \exp(\pi \xi)(g \cdot e) = (\exp(\pi \xi)g) \cdot_\sigma \exp(\pi \xi).$$

This shows that $G$ also acts transitively on $P_\tau^e$. The isotropy subgroup at $\exp(\pi \xi)$ consists of those elements $g$ of $G$ satisfying $g \exp(\pi \xi)\sigma(g^{-1}) = \exp(\pi \xi)$, that is those elements $g$ of $G$ which are fixed by $\tau$:

$$\exp(\pi \xi)\sigma(g) \exp(\pi \xi) = g.$$  

Hence $P_\tau^e \cong G/G^\tau$, which is a bottom symmetric $G$-space with involution $\tau$. Since $P_\tau^e \subset G$ totally geodesically and $P_\xi^e$ is the image of $P_\tau^e$ under an isometry (left multiplication by $\exp \pi \xi$), then $P_\xi^e \subset G$ totally geodesically.

Differentiating (8) at the identity we get

$$\mathfrak{k}_\tau = \{X \in \mathfrak{g} \mid X = \text{Ad}(\exp \pi \xi) \circ \sigma(X)\}.$$  

Taking account the formula $\text{Ad}(\exp(\pi \xi)) = e^{\pi \text{ad} \xi}$ and that $\sigma$ commutes with ad$\xi$, we obtain (6); and (7) follows similarly.

4.1.1. Outer symmetric spaces. The existence of outer involutions of a simple Lie algebra $\mathfrak{g}$ depends on the existence of non-trivial involutions of the Dynkin diagram of $\mathfrak{g}^\mathbb{C}$ [2, 8, 12, 13]. Fix a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a Weyl chamber $W$ in $\mathfrak{t}$, which amounts to fix a system of positive simple roots $\Delta_0 = \{\alpha_1, \ldots, \alpha_r\}$, where $r = \text{rank}(\mathfrak{g})$. Let $\varrho$ be a non-trivial involution of the Dynkin diagram and construct an involution $\sigma_\varrho$ on $\mathfrak{g}$ as follows [2, 13]. Extend $\varrho$ by linearity and duality to give an involution of $\mathfrak{t}$. This is the restriction of $\sigma_\varrho$ to $\mathfrak{t}$. For a suitable choice of root vectors $X_\alpha$ of $\mathfrak{g}_\alpha$, with $\alpha \in \Delta_0$, the restriction of $\sigma_\varrho$ to the span of these vectors is given by $\sigma_\varrho(X_\alpha) = X_{\varrho(\alpha)}$. The fundamental outer involution $\sigma_\varrho$ associated to $\varrho$ is the unique extension of this to an outer involution of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{t}_\varrho \oplus \mathfrak{m}_\varrho$ be the corresponding $\pm 1$-eigenspace decomposition of $\mathfrak{g}$. As shown in Proposition 3.20 of [2], the Lie subalgebra $\mathfrak{t}_\varrho$ is simple and the orthogonal projection of $\Delta_0$ onto $\mathfrak{t}_\varrho$, $\pi_{\mathfrak{t}_\varrho}(\Delta_0)$, is a basis of positive simple roots of $\mathfrak{t}_\varrho$ associated to the maximal abelian subalgebra $\mathfrak{t}_\varrho := \mathfrak{t} \cap \mathfrak{t}_\varrho$. We can then compute the inner products of these roots in order to identify the simple Lie algebra $\mathfrak{t}_\varrho$ via its Dynkin diagram: the (local isometry classes of) outer symmetric spaces of compact type associated to involutions of the form $\sigma_\varrho$ are precisely

$$SU(2n)/Sp(n), \, SU(2n+1)/SO(2n+1), \, E_6/F_4$$

and the real projective spaces $\mathbb{R}P^{2n-1}$. 


We call these spaces the fundamental outer symmetric spaces. The remaining conjugacy classes of outer involutions are obtained as follows.

Consider the split $t = t_\varphi \oplus t_m$ with respect to $\mathfrak{g} = \mathfrak{t}_\varphi \oplus \mathfrak{m}_\varphi$. Set $s = r - k$, where $k = \text{rank}(t_\varphi)$. We can label the basis $\Delta_0$ in order to get the following relations: $\varrho(\alpha_j) = \alpha_j$ for $1 \leq j \leq k - s$ and $\varrho(\alpha_j) = \alpha_{s+j}$ for $k - s + 1 \leq j \leq k$. Let $\pi_{t_\varphi}$ be the orthogonal projection of $t$ onto $t_\varphi$, that is $\pi_{t_\varphi}(H) = \frac{1}{2}(H + \sigma_\varphi(H))$ for all $H \in t$. Set $\pi_{t_\varphi}(\Delta_0) = \{\beta_1, \ldots, \beta_k\}$, with

$$\beta_j = \begin{cases} \alpha_j & \text{for } 1 \leq j \leq k - s \\ \frac{1}{2}(\alpha_j + \alpha_{j+s}) & \text{for } k - s + 1 \leq j \leq k \end{cases}.$$  

(9)

This is a basis of $i_{t_\varphi}^*$ with dual basis $\{\zeta_1, \ldots, \zeta_k\}$ given by

$$\zeta_j = \begin{cases} H_j & \text{for } 1 \leq j \leq k - s \\ H_j + H_{j+s} & \text{for } k - s + 1 \leq j \leq k \end{cases}.$$  

(10)

**Theorem 11.** [13] Let $\varrho$ be an involution of the Dynkin diagram of $\mathfrak{g}$. Let

$$\omega = \sum_{j=1}^{k-s} n_j \beta_j + \sum_{j=k-s+1}^{k} n'_j \beta_j$$

be the highest root of $\mathfrak{t}_\varphi$ with respect to $\pi_{t_\varphi}(\Delta_0) = \{\beta_1, \ldots, \beta_k\}$, defined as in (9). Given $i$ such that $n_i = 1$ or 2, define an involution $\sigma_{\varphi,i}$ by

$$\sigma_{\varphi,i} = \text{Ad}(\exp \pi \zeta_i) \circ \sigma_\varphi.$$  

(11)

Then any outer involution of $\mathfrak{g}$ is conjugate in $\text{Aut}(\mathfrak{g})$, the group of automorphism of $\mathfrak{g}$, to some $\sigma_\varphi$ or $\sigma_{\varphi,i}$. In particular, there are at most $k - s + 1$ conjugacy classes of outer involutions.

The list of all (local isometry classes of) irreducible outer symmetric spaces of compact type is shown in Table 1 (cf. [2] [8] [12]).

| $G/K$ | rank($G$) | rank($K$) | rank($G/K$) | dim($G/K$) |
|-------|-----------|-----------|-------------|-------------|
| $SU(2n)/SO(2n)$ | $2n-1$ | $n$ | $2n-1$ | $(2n-1)(n+1)$ |
| $SU(2n+1)/SO(2n+1)$ | $2n$ | $n$ | $2n$ | $n(2n+3)$ |
| $SU(2n)/Sp(n)$ | $2n-1$ | $n$ | $n-1$ | $(n-1)(2n+1)$ |
| $G_p(\mathbb{R}^{2n})$ ($p$ odd $\leq n$) | $n$ | $n-1$ | $p$ | $p(2n-p)$ |
| $E_6/Sp(4)$ | 6 | 4 | 6 | 42 |
| $E_6/F_4$ | 6 | 4 | 2 | 26 |

**Table 1.** Irreducible outer symmetric spaces.

Given an outer involution $\sigma$ of the form $\sigma_{\varphi,i}$ or $\sigma_\varphi$ and its $\pm 1$-eigenspace decomposition $\mathfrak{g} = \mathfrak{t}_\sigma \oplus \mathfrak{m}_\sigma$, set $t_{\varphi} = t \cap \mathfrak{t}_\sigma$, which is a maximal abelian subalgebra of $\mathfrak{t}_\sigma$. Following [8], a non-empty intersection of $t_{\varphi}$ with a Weyl chamber in $t$ is called a compartment. Each compartment lies in a Weyl chamber in $t_{\varphi}$ and the Weyl chambers in $t_{\varphi}$ can be decomposed into the same number of compartments [8].

When $\sigma$ is a fundamental outer involution $\sigma_\varphi$, the compartment $\mathcal{W} \cap t_{\varphi}$ is itself a Weyl chamber in $t_{\varphi}$. In particular, whereas the intersection of the integer lattice $\mathcal{J}(G)$ with the Weyl chamber $\mathcal{W}$
in \( t \), which we have denoted by \( \mathcal{I}'(G) \), is described in terms of the dual basis \( \{ H_1, \ldots, H_r \} \subset t \), with \( r = \text{rank}(\mathfrak{g}) \), by

\[
\mathcal{I}'(G) = \left\{ \sum_{i=1}^{r} n_i H_i \in \mathcal{I}(G) \, | \, n_i \in \mathbb{N}_0 \text{ for all } i \right\},
\]

for its part, the intersection of the integer lattice \( \mathcal{I}(G^{\sigma_\varrho}) \) with the Weyl chamber \( \mathcal{W} \cap t_{\varrho} \), is given by

\[
\mathcal{I}'(G^{\sigma_\varrho}) = \left\{ \sum_{i=1}^{k} n_i \zeta_i \in \mathcal{I}(G) \, | \, n_i \in \mathbb{N}_0 \text{ for all } i \right\} = \mathcal{I}'(G) \cap t_{\varrho}.
\]

4.1.2. Cartan embeddings of fundamental outer symmetric spaces. Next we describe those elements \( \xi \) of \( \mathcal{I}'(G^{\sigma_\varrho}) \) for which the connected component \( P^\varrho_\xi \) of \( P^\varrho \) containing \( \exp(\pi \xi) \) can be identified with the fundamental outer symmetric \( G \)-space associated to \( \varrho \). Start by considering the following \( \sigma_\varrho \)-invariant subsets of the root system \( \Delta \subset \mathfrak{t}^* \) of \( \mathfrak{g} \):

\[
\Delta(\varrho) = \{ \alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{t}^C_\varrho \}, \quad \Delta(\varpi_\varrho) = \{ \alpha \in \Delta \mid \mathfrak{m}_\alpha \subset \mathfrak{t}^C_\varrho \}, \quad \Delta_\varrho = \Delta \setminus (\Delta(\varrho) \cup \Delta(\varpi_\varrho)).
\]

Then

\[
\mathfrak{t}^C_\varrho = \mathfrak{t}^C_\varrho + \bigoplus_{\alpha \in \Delta(\varrho)} \mathfrak{g}_\alpha, \quad \mathfrak{m}^C_\varrho = \mathfrak{m}^C_\varrho + \bigoplus_{\alpha \in \Delta(\varpi_\varrho)} \mathfrak{g}_\alpha,
\]

where \( \mathfrak{t}_\varrho = \bigoplus_{\alpha \in \Delta_\varrho} \mathfrak{g}_\alpha \). Since the involution \( \varrho \) acts on \( \Delta_\varrho \) as a permutation without fixed points, we can fix some subset \( \Delta_\varrho' \) so that \( \Delta_\varrho \) is the disjoint union of \( \Delta_\varrho' \) with \( \varrho(\Delta_\varrho') \):

\[
\Delta_\varrho = \Delta_\varrho' \cup \varrho(\Delta_\varrho').
\]

For each \( \alpha \in \Delta_\varrho' \), \( \sigma_\varrho \) restricts to an involution in the subspace \( \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\varrho(\alpha)} \subset \mathfrak{t}_\varrho \). Hence we have the following.

**Lemma 12.** The orthogonal projections of \( \mathfrak{t}_\varrho \) onto \( \mathfrak{t}^C_\varrho \) and \( \mathfrak{m}^C_\varrho \) are given by

\[
\pi_{\mathfrak{t}^C_\varrho}(\mathfrak{t}_\varrho) = \bigoplus_{\alpha \in \Delta_\varrho'} \mathfrak{t}^C_\varrho \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\varrho(\alpha)}), \quad \pi_{\mathfrak{m}^C_\varrho}(\mathfrak{t}_\varrho) = \bigoplus_{\alpha \in \Delta_\varrho'} \mathfrak{m}^C_\varrho \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\varrho(\alpha)}),
\]

and, for each \( \alpha \in \Delta_\varrho' \),

\[
\mathfrak{t}^C_\varrho \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\varrho(\alpha)}) = \{ X_\alpha + \varrho_\varrho(X_\alpha) \mid X_\alpha \in \mathfrak{g}_\alpha \}, \quad \mathfrak{m}^C_\varrho \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\varrho(\alpha)}) = \{ X_\alpha - \varrho_\varrho(X_\alpha) \mid X_\alpha \in \mathfrak{g}_\alpha \}.
\]

In particular, \( \dim \mathfrak{t}_\varrho = 2 \dim \pi_{\mathfrak{t}^C_\varrho}(\mathfrak{t}_\varrho) = 2 \dim \pi_{\mathfrak{m}^C_\varrho}(\mathfrak{t}_\varrho) \).

**Proposition 13.** Consider the dual basis \( \{ \zeta_1, \ldots, \zeta_k \} \) defined by (10). Given \( \xi \in \mathcal{I}'(G^{\sigma_\varrho}) \) with \( \xi = \sum_{i=1}^{k} n_i \zeta_i \) and \( n_i \geq 0 \), then \( P_{\xi}^{\sigma_\varrho} \) is a fundamental outer symmetric space with involution (conjugated to) \( \sigma_\varrho \) if and only if \( n_i \) is even for each \( 1 \leq i \leq k - s \).

**Proof.** There is only one class of outer symmetric \( SU(2n + 1) \)-spaces and, in this case, the involution \( \varrho \) does not fix any simple root, that is \( k - s = 0 \). Hence the result trivially holds for

\[
N = SU(2n + 1)/SO(2n + 1).
\]

Next we consider the remaining fundamental outer symmetric spaces, which are precisely the symmetric spaces of rank-split type [3], those satisfying \( \Delta(\varpi_\varrho) = \emptyset \). For such symmetric spaces, the
reductive symmetric term $m_\theta$ satisfies $m_\theta = t_\theta \oplus \pi m_\theta (r_\theta)$. On the other hand, in view of (7), we have, for $\tau = \text{Ad}(\exp \pi \xi) \circ \sigma_\theta$, 

$$m_\tau^C = \bigoplus g_{2i+1}^C \cap t_\theta^C \oplus \bigoplus g_\alpha \sqcup m_\theta^C$$

$$= t_\theta^C \oplus \bigoplus \bigoplus g_\alpha \sqcup (g_\alpha \oplus g_\theta^{(\alpha)}) \oplus \bigoplus m_\theta \cap (g_\alpha \oplus g_\theta^{(\alpha)}),$$

where $\Delta_\xi^+ := \{ \alpha \in \Delta | \alpha(\xi) \text{ is even} \}$ and $\Delta_\xi^- := \{ \alpha \in \Delta | \alpha(\xi) \text{ is odd} \}$. Taking into account Lemma 12 from this we see that $\dim m_\tau = \dim m_\theta$ (which means, by Table 1, that $P_\xi^\sigma$ is a fundamental outer symmetric space with involution conjugated to $\sigma_\theta$) if and only if

$$\bigoplus \bigoplus g_\alpha = \{ 0 \},$$

which holds if and only if $\xi = \sum_{i=1}^k n_i \xi_i$ with $n_i$ even for each $1 \leq i \leq k - s$. \hfill \Box

4.2. Harmonic spheres in symmetric $G$-spaces. Given an involution $\sigma$ on $G$, define an involution $T_\sigma$ on $\Omega G$ by

$$T_\sigma(\gamma)(\lambda) = \sigma(\gamma(-\lambda)\gamma(-1)^{-1}).$$

Let $\Omega^\sigma G$ be the fixed set of $T_\sigma$.

**Lemma 14.** If $\gamma \in \Omega^\sigma G$, then $\gamma(-1) \in P^\sigma$.

**Proof.** If the based loop $\gamma$ is $T_\sigma$-invariant, then $\sigma(\gamma(-\lambda)\gamma(-1)^{-1}) = \gamma(\lambda)$, and evaluating at $\lambda = -1$ we get $\sigma(\gamma(-1)^{-1}) = \gamma(-1)$, that is $\gamma(-1) \in P^\sigma$. \hfill \Box

**Theorem 15.** [8] [11] Given $\xi \in C(\mathfrak{g}) \cap \mathfrak{t}_0$, any harmonic map $\varphi : S^2 \to P_\xi^\sigma \subset G$ admits an $T_\sigma$-invariant extended solution $\Phi : S^2 \to \Omega^\sigma G$. Conversely, given an $T_\sigma$-invariant extended solution $\Phi$, the smooth map $\varphi = \Phi_1$ from $S^2$ is harmonic and takes values in some connected component of $P^\sigma$.

**Proof.** Let $\tilde{\Phi} : S^2 \to \Omega \text{ad} G$ be an extended solution associated to $\varphi : S^2 \to P_\xi^\sigma \subset G$, that is $\tilde{\Phi}_1 = \varphi$. We assume that for a fixed point $p \in S^2$ we have $\varphi(p) = \gamma(p)$, Set

$$\gamma = \gamma_\xi \tilde{\Phi}(p)^{-1}$$

and $\Phi = \gamma \tilde{\Phi}$. Observe that $\Phi$ is the unique algebraic extended solution satisfying $\Phi_1 = \varphi$ and $\Phi(p) = \gamma(p)$. A simple computation shows that $T_\sigma(\Phi)$ is also an extended solution associated to $\varphi$ and satisfies $T_\sigma(\Phi)(p) = \gamma_\xi$. Hence, by unicity, we conclude that $\Phi = T_\sigma(\Phi)$. Conversely, if $\Phi$ is $T_\sigma$-invariant, by Lemma 14 $\Phi_1$ takes values in some connected component of $P^\sigma$. \hfill \Box

**Remark 2.** When $N = G/K$ is an inner symmetric space and $\sigma = \text{Ad}(s_0)$, with $s_0 \in G$ satisfying $s_0^2 = e$, one easily check that $s_0 P^\sigma \subseteq \sqrt{\mathfrak{e}}$ and we can identify $N$ with the connected component of $\sqrt{\mathfrak{e}} = \{ h \in G : h^2 = e \}$ containing $s_0$. Under this identification, harmonic maps into $N$ correspond to extended solutions which are invariant with respect to the involution $I : \Omega G \to \Omega G$ given by $I(\gamma)(\lambda) = \gamma(-\lambda)\gamma(-1)^{-1}$. This is the point of view used in [11].

**Proposition 16.** [8] Given $\Phi \in U_\xi^\sigma(G) := U_\xi(G) \cap \Omega^\sigma G$, with $\xi \in \mathfrak{g}(G) \cap \mathfrak{t}_0$, set $\gamma = u_\xi \circ \Phi$. Then $\gamma$ takes values in $K$. Moreover, $\Phi_1$ and $\gamma(-1)$ take values in the same connected component of $P^\sigma$. 

Proof. Since the energy $E$ is a $T_\sigma$-invariant function on $\Omega_{\text{alg}} G$, the flow $-\nabla E$ preserves $\Omega^\sigma G$. Then, if $\Phi \in U_\xi^\sigma (G)$, the loop $\gamma := u_\xi \circ \Phi \in \Omega_\xi (G)$ is also $T_\sigma$-invariant, that is $T_\sigma (\gamma) = \gamma$. A simple computation shows that $\gamma$ takes values in $K$ (see proof of Lemma 5 in [8]). Again, by continuity $\Phi_{-1}$ and $\gamma(-1)$ take values in the same connected component of $P^\sigma$. □

Hence, together with Theorems [4] and [15] this implies the following.

**Theorem 17.** Any harmonic map $\varphi$ from $S^2$ into a connected component of $P^\sigma$ admits an extended solution $\Phi : S^2 \setminus D \to U_\xi^\sigma (G) := U^\xi (G) \cap \Omega^\sigma G$, for some $\xi \in \mathcal{Y}(G) \cap \kappa_\sigma$ and some discrete subset $D$. If $\sigma = \sigma_{\null}\varphi$ is the fundamental outer involution, then $\varphi = \Phi_{-1}$ takes values in $P_{\xi}^\sigma$.

**Proof.** By Proposition [16] $\Phi$ and $\gamma := u_\xi \circ \Phi$ take values in the same connected component of $P^\sigma$ when evaluated at $\lambda = -1$. Since $\gamma : S^1 \to G^\sigma$ is a homeomorphism, then $\gamma$ is in the $G^\sigma$-conjugacy class of $\gamma_{\xi'}$ for some $\xi' \in \mathcal{Y}(G^\sigma)$, where $G^\sigma$ is the subgroup of $G$ fixed by $\sigma$. Consequently,

$$\gamma(-1) = g \gamma_{\xi'}(-1) g^{-1} = g \cdot \gamma_{\xi'}(-1),$$

for some $g \in G^\sigma$, which means that $\gamma(-1)$ takes values in the connected component $P_{\xi}^\sigma$. On the other hand, $\gamma$ is in the $G$-conjugacy class of $\gamma_{\xi}$, with $\xi \in \mathcal{Y}(G) \cap \kappa_\sigma$. If $\sigma$ is the fundamental outer involution $\sigma_{\null}\varphi$, then $\mathcal{Y}(G^\sigma) = \mathcal{Y}(G) \cap \kappa_\sigma$; and we must have $\xi = \xi'$ □

**Remark 3.** If $\sigma$ is not a fundamental outer involution, then each Weyl chamber $W_\sigma$ in $\kappa_\sigma$ can be decomposed into more than one compartments: $W_\sigma = C_1 \cup \ldots \cup C_l$, where $C_1 = W \cap \kappa_\sigma$ and the remaining compartments are conjugate to $C_1$ under $G$ [8], that is, there exists $g_i \in G$ satisfying $C_i = \text{Ad}(g_i)(C_1)$ for each $i$. Hence, if we have an extended solution $\Phi : S^2 \setminus D \to U_\xi^\sigma (G)$ with $\xi \in \mathcal{Y}(G) \cap \kappa_\sigma \subset C_1$, the corresponding harmonic map $\Phi_{-1}$ takes values in one of the connected components $P_{\null}^\sigma_{g_i,\xi g_i^{-1}}$.

4.2.1. $g$-canonical elements. Let $I$ be a subset of $\{1, \ldots, k\}$, with $k = \text{rank}(\kappa_\sigma)$, and set

$$\mathcal{C}_I^\sigma = \left\{ \sum_{i=1}^k n_i \xi_i | n_i \geq 0, n_j = 0 \text{ iff } j \notin I \right\}.$$ 

Let $\xi \in \mathcal{Y}(G^\sigma) \cap \mathcal{C}_I^\sigma$. We say that $\xi$ is a $g$-canonical element of $G$ (with respect to the choice of $W$) if $\xi$ is a maximal element of $(\mathcal{Y}(G^\sigma) \cap \mathcal{C}_I^\sigma, \preceq)$, that is: if $\xi \preceq \xi'$ and $\xi' \in \mathcal{Y}(G^\sigma) \cap \mathcal{C}_I^\sigma$ then $\xi = \xi'$.

**Remark 4.** When $G$ has trivial centre, the duals $\xi_1, \ldots, \xi_k$ belong to the integer lattice. Then, for each $I$ there exists a unique $g$-canonical element, which is given by $\xi_I = \sum_{i \in I} \xi_i$. In this case, our definition of $g$-canonical element coincides with that of $S$-canonical element in [8].

Now, consider a fundamental outer involution $\sigma_\varphi$ and let $N$ be an associated outer symmetric $G$-space, that is, $N$ corresponds to an involution of $G$ of the form $\sigma_\varphi$ or $\sigma_{\varphi,i}$, with $\xi_i$ in the conditions of Theorem [II] If $G$ has trivial centre, we certainly have $\xi_i \in \mathcal{Y}(G^\sigma)$, as a matter of fact, as we will see later, in most cases we have $\xi_i \in \mathcal{Y}(G^\sigma)$, whether $G$ has trivial centre or not, with essentially one exception: for $G = SU(2n)$ and $N = SU(2n)/SO(2n)$. So, we will treat this case separately and assume henceforth that $\xi_i \in \mathcal{Y}(G^\sigma)$. Taking into account Proposition [10] we can identify $N$ with the connected component $P_{\xi_i}^\sigma = \text{exp}(\pi \xi_i) P_{\xi_i}^\sigma$, which is a totally geodesic submanifold of $G$, via

$$g \cdot x_0 \in N \mapsto \text{exp}(\pi \xi_i) g_{\sigma_{\varphi,i}}(g^{-1}) \in P_{\xi_i}^\sigma. \quad (14)$$
By Theorem 17, each harmonic map $\varphi : S^2 \to N \cong P_{\xi}^{g_1}$ admits a $T_{\sigma_\varphi}$-invariant extended solution with values, off a discrete set, in some unstable manifold $U_{\xi}(G)$, with $\xi \in \mathcal{J}(G^{\sigma_\varphi}) \cap \mathcal{C}_f^0$. By Theorem 8, this extended solution can be multiplied on the left by a constant loop in order to get a normalized extended solution with values in some unstable manifold $U_{\xi}(G)$ for some $\rho$-canonical element $\xi$. Hence, if $G$ has trivial centre, the Bruhat decomposition of $\Omega_{\text{alg}}^G$ gives rise to $2^k$ classes of harmonic maps into $P_{\xi}^{g_1}$, that is $2^k$ classes of harmonic maps into all outer symmetric $G$-spaces.

However, the normalization procedure given by Theorem 8 does not preserve $T_{\sigma_\varphi}$-invariance, and consequently, as we will see next, normalized extended solutions with values in the same unstable manifold $U_{\xi}(G)$, for some $\rho$-canonical element $\xi$, correspond in general to harmonic maps into different outer symmetric $G$-spaces. Hence the classification of harmonic two-spheres into outer symmetric $G$-spaces in terms of $\rho$-canonical elements is manifestly unsatisfactory since it does not distinguishes the underlying symmetric space. In the following sections we overcome this weakness by establishing a classification of all such harmonic maps in terms of pairs $(\xi, \sigma)$, where $\xi$ is a $\rho$-canonical element and $\sigma$ an outer involution of $G$.

4.2.2. Normalization of $T_\sigma$-invariant extended solutions. Let $\sigma$ be an outer involution of $G$. The fibre bundle morphisms $U_{\xi,\zeta}$ preserve $T_\sigma$-invariance:

**Proposition 18.** If $\xi \preceq \xi'$ and $\xi, \xi' \in \mathcal{J}(G) \cap I_\sigma$, then $U_{\xi,\zeta'}(U_{\xi}(G)) \subset U_{\xi'}(G)$.

**Proof.** For $\Phi \in U_{\xi}(G)$, write $\Phi = \Psi \cdot \gamma_{\xi}$ for some $\Psi \in \Lambda^+_{\text{alg}} G^C$. If $\Phi$ is $T_\sigma$-invariant we have $\Psi(\lambda) \cdot \gamma_{\xi} = \sigma(\Psi(-\lambda)) \cdot \gamma_{\xi}$. Consequently, we also have $\Psi(\lambda) \cdot \gamma_{\xi'} = \sigma(\Psi(-\lambda)) \cdot \gamma_{\xi'}$, which means in turn that $U_{\xi,\zeta'}(\Phi) = \Psi \cdot \gamma_{\xi'}$ is $T_\sigma$-invariant. □

Hence, if $\Phi : S^2 \setminus D \to U_{\xi}(G)$ is an extended solution and $\xi \preceq \xi'$, with $\xi, \xi' \in \mathcal{J}(G) \cap I_\sigma$, by Theorem 8 and Proposition 18 we know that $\gamma^{-1} := U_{\xi,\zeta'}(\Phi)$ is a constant $T_\sigma$-invariant loop if $g_0 = g_\xi'$. However, in general, the product $\gamma \Phi$ is not $T_\sigma$-invariant.

**Lemma 19.** Assume that $\gamma^{-1}, \Phi \in \Omega^G$ and $\gamma(-1) \in P_{\xi}^\sigma$ for some $\xi \in \mathcal{J}(G) \cap I_\sigma$. Take $h \in G$ such that $\gamma(-1) = h_{\gamma} \cdot \exp(\pi \xi)$. Then $h_{\gamma} h^{-1} \in \Omega^G$, with $\tau = \text{Ad}(\exp(\pi \xi) \circ \sigma$.

**Proof.** Since $\gamma^{-1}, \Phi \in \Omega^G$, a simple computation shows that

$$T_{\sigma} (\gamma \Phi) = \gamma(-1)^{-1} \gamma \Phi \gamma(-1).$$

Since $\gamma(-1) \in P_{\xi}^\sigma$, there exists $h \in G$ such that $\gamma(-1) = h_{\gamma} \cdot \exp(\pi \xi) = h^{-1} \exp(\pi \xi) \sigma(h)$. One can check now that $T_\gamma (h_{\gamma} h^{-1}) = h_{\gamma} h^{-1}$. □

**Proposition 20.** Take $\xi, \xi' \in \mathcal{J}(G) \cap I_\sigma$ such that $\xi \preceq \xi'$. Let $\Phi : S^2 \setminus D \to U_{\xi}(G)$ be a $T_\sigma$-invariant extended solution. If $\gamma^{-1} := U_{\xi,\zeta'}(\Phi)$ is a constant loop, there exists $h \in G$ such that $\Phi := h_{\gamma} h^{-1}$ takes values in $U_{\xi'}(G)$, with

$$\tau = \text{Ad}(\exp(\pi(\xi - \xi')) \circ \sigma.$$

Additionally, if $\sigma$ is the fundamental outer involution $\sigma_\varphi$, the harmonic map $\Phi^{-1}$ takes values in $P_{\xi}^\sigma$ and $\Phi_{-1}$ takes values in $P_{\xi'}^\sigma$, which implies that $\Phi_{-1}$ is given, up to isometry, by

$$\exp(\pi(\xi - \xi')) \Phi_{-1} : S^2 \to P_{\xi}^\sigma.$$
Proof. Assume that \( \gamma^{-1} := U_{\xi-\xi'}(\Phi) = \Psi \cdot \gamma_{\xi-\xi'} \) is a constant loop. We can write \( \Psi \gamma_{\xi-\xi'} = \gamma^{-1}b \) for some \( b : S^2 \setminus D \to \Lambda_{\text{alg}}^+ G \). Then

\[
\Phi = \Psi \cdot \gamma \xi = \Psi \cdot \gamma_{\xi-\xi'} \gamma_{\xi'} = \gamma^{-1}b \cdot \gamma_{\xi'},
\]
which implies that \( \gamma \Phi \) takes values in \( U_{\xi'}(G) \). On the other hand, since \( \gamma^{-1} \) is \( T_\sigma \)-invariant (by Proposition 18, \( \gamma(-1) \in P_\sigma \)).

Take \( \eta \in \mathcal{J}'(G_\sigma^\gamma) \) and \( h \in G \) such that \( \gamma(-1) \in P_\sigma^\gamma \) and \( \gamma(-1) = h^{-1} \cdot \sigma \exp \pi \eta \). From Lemma 19 we see that \( \tilde{\Phi} := h \gamma \Phi h^{-1} \) is \( T_\sigma \)-invariant. Hence \( \tilde{\Phi} \) takes values in \( U_{\xi'}^\sigma(G) \). Since \( \gamma \) is constant, \( \tilde{\Phi} \) is an extended solution.

If \( \sigma = \sigma_\xi \), then \( \mathcal{J}'(G_\sigma^\gamma) = \mathcal{J}'(G) \cap G_\sigma \), which implies that \( \eta = \xi - \xi' \). The element \( h \in G \) is such that

\[
\gamma(-1) = h^{-1} \exp(\pi(\xi - \xi')) \sigma_\xi(h).
\]
On the other hand, since, by Theorem 14, \( \Phi^{-1} \) takes values in \( P_\xi^\sigma \), we also have

\[
\Phi^{-1} = g \exp(\pi(\xi)) \sigma_\xi(g) \cdot \exp(\pi \xi)
\]
for some lift \( g : S^2 \to G \). Hence

\[
\tilde{\Phi}^{-1} = h \gamma(-1) \Phi^{-1} h^{-1} = \exp(\pi(\xi - \xi')) \sigma_\xi(h) g \exp(\pi(\xi)) \sigma_\xi(g) h^{-1} = \exp(\pi(\xi - \xi')) \sigma_\xi(g) \cdot \sigma_\sigma \exp \pi \xi
\]
Hence, in view of Proposition 10, \( \tilde{\Phi}^{-1} \) takes values in \( P_\xi^\sigma = \exp(\pi(\xi - \xi')) P_\xi^\sigma \).

Under some conditions on \( \xi \preceq \xi' \), the morphism \( U_{\xi-\xi'}(\Phi) \) is always a constant loop.

**Proposition 21.** Take \( \xi, \xi' \in \mathcal{J}'(G) \cap G_\sigma \) such that \( \xi \preceq \xi' \). Assume that

\[
g^{\xi}_{2i} \cap m_{\sigma}^C \subseteq \bigoplus_{0 \leq j < 2i} g^\xi_{j-\xi'}, \quad g^\xi_{2i-1} \cap m_{\sigma}^C \subseteq \bigoplus_{0 \leq j < 2i-1} g^\xi_{j-\xi'},
\]
for all \( i > 0 \). Then, \( U_{\xi-\xi'} : U_{\xi}^\sigma(G) \to U_{\xi'}^\sigma(G) \) transforms \( T_\sigma \)-invariant extended solutions in constant loops.

**Proof.** Given an extended solution \( \Phi : S^2 \setminus D \to U_{\xi}^\sigma(G) \), choose \( \Psi : S^2 \setminus D \to \Lambda_{\text{alg}}^+ G^C \) such that \( \Phi = \Psi \cdot \gamma \xi \) and \( T_\sigma(\Psi) = \Psi \). Differentiating this we see that

\[
\text{Im} \Psi^{-1} \Psi \subseteq \bigoplus_{i \geq 0} \Lambda^{2i} \mathfrak{t}_\sigma^C \oplus \bigoplus_{i \geq 0} \Lambda^{2i+1} m_{\sigma}^C.
\]
Write \( \Psi^{-1} \Psi = \sum_{r \geq 0} \Lambda^r X_r. \) Since \( \xi \preceq \xi - \xi' \), by Proposition 5 and Proposition 13, \( U_{\xi-\xi'}(\Phi) \) is an extended solution with values in \( U_{\xi-\xi'}^\sigma \). Hence, taking into account Lemma 5, in order to prove that \( U_{\xi-\xi'}(\Phi) \) is constant we only have to check that the component of \( X_r \) over \( g^\xi_{r-\xi'} \) vanishes for all \( r \geq 0 \).

From (11) and (10) we see that, for \( r = 2i \), \( X_{2i}^\xi \) takes values in \( \bigoplus_{j \leq 2i+1} g^\xi_j \cap m_{j}^C \). But, since \( \xi \preceq \xi - \xi' \) and, by hypothesis, (15) holds, we have

\[
\bigoplus_{j \leq 2i+1} g^\xi_j \cap m_{j}^C = \left( \bigoplus_{j \leq 2i} g^\xi_j \cap m_{j}^C \right) \oplus \left( g^\xi_{2i+1} \cap m_{2i+1}^C \right) = \left( \bigoplus_{j \leq 2i} g^\xi_j \cap m_{j}^C \right) \oplus \bigoplus_{0 \leq j < 2i+1} g^\xi_{j-\xi'}.
\]
Hence the component of $X'_{2i}$ over $\mathfrak{g}_{2i-1}^{\xi-\xi'}$ vanishes for all $i \geq 0$. Similarly, for $r = 2i - 1$, $X'_{2i-1}$ takes values in $\bigoplus_{j \leq 2i} \mathfrak{g}_j^{\xi} \cap \mathfrak{m}_{2i}^{\xi}$, and we can check that the component of $X'_{2i-1}$ over $\mathfrak{g}_{2i-1}^{\xi-\xi'}$ vanishes for all $i > 0$.

Hence $\gamma^{-1} := U_{\xi,\xi-\xi'}(\Phi) = \Psi \cdot \gamma_{\xi-\xi'}$ is a constant loop. □

**Definition 2.** We say that $\zeta \in \mathcal{H}(G^\sigma) \cap \mathfrak{c}^0_I$ is a $\varphi$-semi-canonical element if $\zeta$ is of the form $\zeta = \sum_{i \in I} n_i \zeta_i$ with $1 \leq n_i \leq 2m_i$, where $m_i$ is the least positive integer which makes $m_i \zeta_i \in \mathcal{H}(G^\sigma)$.

**Corollary 1.** Take $\xi \in \mathcal{H}(G^\sigma) \cap \mathfrak{c}^0_I$, with $I \subset \{1, \ldots, k\}$. Let $\Phi : S^2 \setminus D \to U_{\xi}^{M}(G)$ be a $T_{\xi}$-invariant extended solution, and let $\varphi : S^2 \to P_{\xi}^{M}$ be the corresponding harmonic map. Then there exist $h \in G$, a constant loop $\gamma$, and a $\varphi$-semi-canonical $\zeta$ such that $\Phi := h \gamma \Phi h^{-1}$ defined on $S^2 \setminus D$ takes values in $U_{\zeta}^{M}(G)$. The harmonic map $\Phi^{-1}$ takes values in $P_{\zeta}^{M} = P_{\xi}^{M}$ and coincides with $\varphi$ up to isometry.

**Proof.** Write $\xi = \sum_{i \in I} r_i \zeta_i$, with $r_i > 0$. For each $i \in I$, let $n_i$ be the unique integer number in $\{1, \ldots, 2m_i\}$ such that $n_i = r_i \mod 2m_i$. Set $\zeta = \sum_{i \in I} n_i \zeta_i$. It is clear that $\xi \leq \zeta$ and $\zeta \in \mathcal{H}(G^\sigma) \cap \mathfrak{c}^0_I$. Observe also that conditions (15) hold automatically for any $\xi' \in \mathcal{H}(G^\sigma) \cap \mathfrak{c}^0_I$ satisfying $\xi \leq \xi'$. In particular they hold for $\xi' = \zeta$. Finally, since $\xi - \zeta = 2 \sum_{i \in I} m_i k_i \zeta_i$, any nonnegative integer numbers $k_i$, then $\pi(\xi - \zeta) = e$, and the result follows from Propositions 20 and 21. □

4.2.3. **Classification of harmonic two-spheres into outer symmetric spaces.** To sum up, in order to classify all harmonic two-spheres into outer symmetric spaces we proceed as follows:

1. Start with a fundamental outer involution $\sigma_{\varphi}$ and let $N$ be an outer symmetric $G$-space corresponding to an involution of the form $\sigma_{\varphi}$ or $\sigma_{\varphi,i}$ of $G$, according to (11), where the element $\zeta_i$ is in the conditions of Theorem 11. We assume that $\exp 2\pi \zeta_i = e$, that is, $\zeta_i \in \mathcal{H}(G^\sigma)$. Let $\varphi : S^2 \to N$ be an harmonic map and identify $N$ with $P_{\xi}^{M} = \exp(\pi \zeta_i)P_{\xi}^{M}$ via the totally geodesic embedding (14). If $N$ is the fundamental outer space with involution $\sigma_{\varphi}$ we simply identify $N$ with $P_{\xi}^{M}$ via $\nu_{\sigma_{\varphi}}$.

2. By Theorem 17, $\varphi : S^2 \to N \cong P_{\xi}^{M}$ admits a $T_{\xi}$-invariant extended solution $\Phi : S^2 \to \Omega_{\xi}^{M}G$ which takes values, off some discrete subset $D$, in some unstable manifold $U_{\xi}^{M}(G)$, with $\xi' \in \mathcal{H}(G^\sigma)$; moreover, $P_{\xi}^{M} = P_{\xi}^{M}$.

3. By Corollary 1, we can assume that $\xi'$ is a $\varphi$-semi-canonical element in $\mathcal{H}(G^\sigma) \cap \mathfrak{c}^0_I$. If $\xi$ is a $\varphi$-canonical element such that $\xi' \leq \zeta$ and $U_{\xi',\xi'-\zeta}(\Phi)$ is constant, then, taking into account Proposition 20 there exists a $T_{\tau}$-invariant extended solution $\Phi : S^2 \setminus D \to U_{\xi}^{M}(G)$, where

$$\tau = \text{Ad}(\exp \pi(\xi' - \zeta)) \circ \sigma_{\varphi},$$

such that the harmonic map $\varphi$ is given, up to isometry, by $\Phi^{-1} : S^2 \to P_{\xi}^{M}$. Here we identify $N$ with $P_{\xi}^{M} = \exp(\pi(\xi' - \zeta))P_{\xi}^{M}$ via the composition of (14) with the left multiplication by $\exp(\pi(\xi' - \zeta))$.

4. By Proposition 21 there always exists a $\varphi$-canonical element $\zeta$ in such conditions.

Hence, we classify harmonic spheres into outer symmetric $G$-spaces in terms of pairs $(\zeta, \tau)$, where $\zeta$ is a $\varphi$-canonical element and $\tau$ is an outer involution of the form (17) for some $\varphi$-semi-canonical element $\zeta'$ with $\xi' \leq \zeta$.
4.2.4. Weierstrass Representation for $T_\sigma$-invariant Extended Solutions. From (16) and Proposition 7 we obtain the following.

**Proposition 22.** Let $\Phi : M \to \Omega^\sigma_{\text{alg}} G$ be an extended solution. There exists a discrete set $D' \supset D$ of $M$ such that

$$
\Phi |_{M \setminus D'} = \exp C \cdot \gamma_\xi
$$

for some holomorphic vector-valued function $C : M \setminus D' \to (u_\xi)_\sigma$, where $(u_\xi)_\sigma$ is the finite dimensional nilpotent subalgebra of $\Lambda^+_{\text{alg}} g^C$ defined by

$$(u_\xi)_\sigma = \bigoplus_{0 \leq r < r(\xi)} \lambda^{2r} (p_{2r})^\perp \cap \mathfrak{f}_\sigma^C \oplus \bigoplus_{0 \leq r+1 < r(\xi)} \lambda^{2r+1} (p_{2r+1})^\perp \cap \mathfrak{m}_\sigma^C,$$

with $(p_i^\xi)^\perp = \bigoplus_{1 \leq j \leq r(\xi)} g^C_j$. Moreover, $C$ can be extended meromorphically to $M$.

5. Examples

Next we will describe explicit examples of harmonic spheres into classical outer symmetric spaces. For the exceptional case, we only observe the following:

Consider the Dynkin diagram of $\mathfrak{e}_6$:

```
\begin{center}
\begin{tikzpicture}
  \node[circle, fill=black, inner sep=1pt] at (0,0) (a1) {1};
  \node[circle, fill=black, inner sep=1pt] at (1,0) (a3) {3};
  \node[circle, fill=black, inner sep=1pt] at (2,0) (a4) {4};
  \node[circle, fill=black, inner sep=1pt] at (3,0) (a5) {5};
  \node[circle, fill=black, inner sep=1pt] at (4,0) (a6) {6};
  \path (a1) edge (a3)
        (a3) edge (a4)
        (a4) edge (a5)
        (a5) edge (a6);
\end{tikzpicture}
\end{center}
```

This admits a unique nontrivial involution $\varrho$. Let $\{H_1, \ldots, H_6\}$ be the dual basis of $\Delta_0 = \{\alpha_1, \ldots, \alpha_6\}$. The semi-fundamental basis $\pi_{\text{ve}}(\Delta_0) = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ is given by $\beta_1 = \alpha_2, \beta_2 = \alpha_4, \beta_3 = \alpha_3 + \alpha_5$ and $\beta_4 = \alpha_1 + \alpha_6$, whereas the dual basis is given by $\zeta_1 = H_2, \zeta_2 = H_4, \zeta_3 = H_1 + H_6$ and $\zeta_4 = H_3 + H_5$. Taking account that the elements $H_i$ are related with the duals $\eta_i$ of the fundamental weights by

$$
[H_i] = \begin{bmatrix}
4/3 & 1 & 5/3 & 2 & 4/3 & 2/3 \\
2 & 3 & 2 & 3 & 2 & 1 \\
5/3 & 2 & 10/3 & 4 & 8/3 & 4/3 \\
2 & 3 & 4 & 6 & 4 & 2 \\
4/3 & 2 & 8/3 & 4 & 10/3 & 5/3 \\
2/3 & 1 & 4/3 & 2 & 5/3 & 4/3
\end{bmatrix} [\eta_i],
$$

we see that the elements $\zeta_i$ are in the integer lattice $\mathcal{Y}(\tilde{E}_6) \subset \mathcal{Y}(E_6)$, where $\tilde{E}_6$ is the compact simply connected Lie group with Lie algebra $\mathfrak{e}_6$, which has centre $\mathbb{Z}_3$, and $E_6$ is the adjoint group $\tilde{E}_6/\mathbb{Z}_3$.

5.1. Outer symmetric $SO(2n)$-spaces. For details on the structure of $so(2n)$ see [10]. Consider on $\mathbb{R}^{2n}$ the standard inner product $\langle \cdot, \cdot \rangle$ and fix a complex basis $u = \{u_1, \ldots, u_n, \overline{u}_1, \ldots, \overline{u}_n\}$ of $\mathbb{C}^{2n} = (\mathbb{R}^{2n})^C$ satisfying

$$
\langle u_i, u_j \rangle = 0, \quad \langle u_i, \overline{u}_j \rangle = \delta_{ij}, \quad \text{for all } 1 \leq i, j \leq n.
$$

(18)

Throughout this section we will denote by $V_l$ the $l$-dimensional isotropic subspace spanned by $\overline{u}_1, \ldots, \overline{u}_l$. 

Set \( E_i = E_{i,i} - E_{n+i,n+i} \), where \( E_{i,j} \) is a square matrix, with respect to the basis \( u \), whose \((j,j)\)-entry is \( i \) and all other entries are 0. The complexification \( t^C \) of the algebra of diagonal matrices

\[
t = \{ \sum a_i E_i | a_i \in \mathbb{R}, \sum a_i = 0 \}
\]

is a Cartan subalgebra of \( \mathfrak{so}(2n)^C \). Let \( \{ L_1, \ldots, L_n \} \) be the dual basis in \( \mathfrak{t}^* \) of \( \{ E_1, \ldots, E_n \} \), that is \( L_i(E_j) = \delta_{ij} \). The roots of \( \mathfrak{so}(2n) \) are the vectors \( \pm L_i \pm L_j \), with \( i \neq j \) and \( 1 \leq i, j \leq n \).

Consider the non-trivial involution \( \varrho \) of the corresponding Dynkin diagram,

\[
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\ldots \\
\alpha_{n-3} \\
\alpha_{n-2} \\
\alpha_{n-1} \\
\alpha_n
\end{array}
\]

This involution fixes \( \alpha_i \) if \( i \leq n - 2 \) and \( \varrho(\alpha_{n-1}) = \alpha_n \). The corresponding semi-fundamental basis \( \pi_{t_\varrho}(\Delta_0) = \{ \beta_1, \ldots, \beta_{n-1} \} \) is given by

\[
\beta_i = \alpha_i = L_i - L_{i+1}, \quad \text{if} \quad i \leq n - 2, \quad \text{and} \quad \beta_{n-1} = \frac{1}{2}(\alpha_{n-1} + \alpha_n) = L_{n-1},
\]

whereas the dual basis \( \{ \zeta_1, \ldots, \zeta_{n-1} \} \) is given by

\[
\zeta_i = E_1 + \ldots + E_i,
\]

with \( i = 1, \ldots, n - 1 \). Since each \( \zeta_i \) belongs to the integer lattice \( \mathcal{H}(SO(2n)^{\sigma_0}) \), we have:

**Proposition 23.** The \( \varrho \)-semi-canonical elements of \( SO(2n) \) are precisely the elements \( \zeta = \sum_{i=1}^{n-1} m_i \zeta_i \) such that \( m_i \in \{ 0, 1, 2 \} \) for \( 1 \leq i \leq n - 1 \).

The fundamental outer symmetric \( SO(2n) \)-space is the real projective space \( \mathbb{R}P^{2n-1} \), and the associated outer symmetric \( SO(2n) \)-spaces are the real Grassmannians \( G_p(\mathbb{R}^{2n}) \) with \( p > 1 \) odd.

### 5.1.1. Harmonic maps into real projective spaces \( \mathbb{R}P^{2n-1} \)

Consider as base point the real span \( V_0 \) of \( e_n = \frac{1}{\sqrt{2}} (u_n + \pi_n) \) in \( \mathbb{R}^{2n} \), which establishes an identification of \( \mathbb{R}P^{2n-1} \) with \( SO(2n)/S(O(1)O(2n-1)) \).

Denote by \( \pi_{V_0} \) and \( \pi_{V_0}^{\perp} \) the orthogonal projections onto \( V_0 \) and \( V_0^{\perp} \), respectively. The fundamental involution is given by \( \sigma_\varrho = \text{Ad}(s_0) \), where \( s_0 = \pi_{V_0} - \pi_{V_0}^{\perp} \). Following the classification procedure established in Section 4.2.3, we start by identifying \( \mathbb{R}P^{2n-1} \) with \( P_e^{\sigma_\varrho} \).
Theorem 24. Each harmonic map $\varphi : S^2 \to \mathbb{R}P^{2n-1}$ belongs to one of the following classes: $(\zeta_l, \sigma_{0,l})$, with $1 \leq l \leq n - 1$.

Proof. Let $\zeta$ be a $g$-semi-canonical element and write

$$\zeta = \sum_{i \in I_1} \zeta_i + \sum_{i \in I_2} 2\zeta_i$$

(21)

for some disjoint subsets $I_1$ and $I_2$ of $\{1, \ldots, n-1\}$. By Proposition 13 $P^{\varphi_e}_{\zeta} \cong \mathbb{R}P^{2n-1}$ if and only if either $I_1 = \emptyset$ or $I_1 = \{n-1\}$. Suppose that $I_1 = \{n-1\}$. In this case, $\exp \pi \zeta = \exp \pi \zeta_{n-1} \in P^{\varphi_e}_{\zeta_{n-1}}$. We claim that $P^{\varphi_e}_{\zeta_{n-1}}$ is not the connected component of $P^{\varphi_e}$ containing the identity $e$. Write

$$\exp \pi \zeta_{n-1} = \pi V - \pi V'$$

where $V$ is the two-dimensional real space spanned by $e_n$ and $e_{2n}$. For each $g \in P^{\varphi_e}_{\zeta}$, since the $G$-action of $\sigma_e$ defined by (11) is transitive, we have $g = h \cdot \sigma_e e = h s_0 h^{-1} s_0$ for some $h \in G$, which means that $g s_0 = h s_0 h^{-1}$. In particular, the $+1$-eigenspaces of $g s_0$ must be 1-dimensional. However, a simple computation shows that the $+1$-eigenspace of $\exp(\pi \zeta_{n-1}) s_0$ is 3-dimensional, which establishes our claim.

Then, any harmonic map $\varphi : S^2 \to \mathbb{R}P^{2n-1} \cong P^{\varphi_e}_{\zeta}$ admits a $T_{\sigma_e}$-invariant extended solution $\Phi : S^2 \setminus D \to U_\zeta^{\varphi_e}(SO(2n))$ with $\zeta$ a $g$-semi-canonical element of the form $\zeta = \sum_{i \in I_2} 2\zeta_i$. Set $l = \max I_2$.

Next we check that $\zeta$ and $\zeta_l$ satisfy the conditions of Proposition 21 with $\xi = \zeta$ and $\xi' = \zeta_l$. It is clear that $\zeta \leq \zeta_l$. Now, according to (12) and (13), we can take $\Delta_0 = \{L_i - L_n, L_n - L_i\}$. Hence, for $i > 0$,

$$g_{2i}^\zeta \cap m_c^\zeta = \bigoplus_{\alpha \in \Delta_0 \cap \Delta_2^\zeta} (g_{\alpha} \oplus g_{\phi(\alpha)}) \cap m_c^\zeta,$$

where

$$\Delta_2^\zeta = \{\alpha \in \Delta | \alpha(\zeta) = 2\alpha\}.$$ 

Since

$$(L_j - L_n)(\zeta) = (\alpha_j + \alpha_{j+1} + \ldots + \alpha_{n-1})(\zeta) = 2|I_2 \cap \{j, \ldots, n-1\}| i,$$

we have

$$\Delta_0 \cap \Delta_2^\zeta = \{L_j - L_n | 1 \leq j \leq l, \text{ and } |I_2 \cap \{j, \ldots, l\}| = i\}.$$ 

Then, given a root $\alpha = L_j - L_n \in \Delta_0 \cap \Delta_2^\zeta$ (in particular, $j \leq l$) we have $\alpha(\zeta - \zeta_l) = (2i - 1)i$, which means that $g_\alpha \subset g_{2i-1}^{\zeta - \zeta_l}$. Consequently,

$$g_{2i}^\zeta \cap m_c^\zeta \subset \bigoplus_{0 \leq j < 2i} g_{j}^{\zeta - \zeta_l}.$$ 

Since $g_{2i-1}^{\zeta - \zeta_l} = \{0\}$ for all $i$, we conclude that (15) holds, and the statement follows from Propositions 20 and 21.

It is known [3] that there are no full harmonic maps $\varphi : S^2 \to \mathbb{R}P^{2n-1}$. The class of harmonic maps associated to $(\zeta_l, \sigma_{0,l})$ consists precisely of those $\varphi$ with $\varphi(S^2)$ contained, up to isometry, in some $\mathbb{R}P^{2l}$, as shown in the next theorem.

Theorem 25. Given $1 \leq l \leq n - 1$, any harmonic map $\varphi : S^2 \to \mathbb{R}P^{2n-1}$ in the class $(\zeta_l, \sigma_{0,l})$ is given by

$$\varphi = R \cap (A \oplus \overline{A})^\perp,$$

(22)
where $R$ is a constant $2l + 1$-dimensional subspace of $\mathbb{R}^{2n}$ and $A$ is a holomorphic isotropic subbundle of $S^2 \times R$ of rank $l$ satisfying $\partial A \subseteq \mathbb{R}^1$. The corresponding extended solutions have uniton number $2$ with respect to the standard representation of $SO(2n)$.

**Proof.** Let $\varphi : S^2 \to \mathbb{R}P^{2n-1}$ be a harmonic map in the class $(\zeta_l, \sigma_{\varphi, l})$. This means that $\varphi$ admits an extended solution $\Phi : S^2 \setminus D \to U_{\zeta_l}^{\sigma_{\varphi, l}}(SO(2n))$. Up to isometry, $\varphi$ is given by $\Phi_{-1}$, which takes values in $P_{\zeta_l}^{\sigma_{\varphi, l}} = \exp(\pi \zeta_l)P_{\varphi, l}^{\sigma_{\varphi, l}}$. This connected component is identified with $\mathbb{R}P^{2n-1}$ via

$$\quad g \cdot V_0 \mapsto \exp(\pi \zeta_l)g\sigma_{\varphi}(g^{-1}).$$

(23)

Write $\gamma_{\zeta_l}(\lambda) = \lambda^{-1}\pi_{V_1} + \pi_{V_1 \oplus V_i} + \lambda\pi_{V_i}$, where $V_i$ is the $l$-dimensional isotropic subspace spanned by $\pi_1, \ldots, \pi_i$. We have $r(\zeta_l) = 2$ if $l > 1$ and $r(\zeta_1) = 1$. Consequently, by Proposition [22]

$$(u'_{\zeta_l})_{\sigma_{\varphi, l}} = (p_0^{\zeta_l})^\perp \cap \mathfrak{t}_{\sigma_{\varphi, l}}^C + \lambda(p_1^{\zeta_l})^\perp \cap m_{\sigma_{\varphi, l}}^C.$$  

Here $(p_0^{\zeta_l})^\perp = g_2^l$, which is the null space for $l = 1$. For $l > 1$, since $\zeta_l = E_1 + \ldots + E_l$, we have $g_2^l = \{L_i + L_j | 1 \leq i < j \leq l\} \subset \Delta_{\zeta_l}$ and, from (21),

$$m_{\sigma_{\varphi, l}}^C = \bigoplus g_{2i+1}^l \cap \mathfrak{t}_{\varphi}^C \oplus \bigoplus g_{2i}^l \cap m_{\varphi}^C.$$  

Hence

$$(p_1^{\zeta_l})^\perp \cap m_{\sigma_{\varphi, l}}^C = g_2^l \subset m_{\varphi}^C = \{0\}.$$  

Then, for any $l \geq 1$, we can write $\Phi = \exp C \cdot \gamma_{\zeta_l}$ for some holomorphic function

$$C : S^2 \setminus D \to (p_0^{\zeta_l})^\perp \cap \mathfrak{t}_{\sigma_{\varphi, l}}^C = (g_1^l \oplus g_2^l) \cap \mathfrak{t}_{\sigma_{\varphi, l}}^C,$$  

which means that $\Phi$ is a $S^1$-invariant extended solution with uniton number $2$:

$$\Phi_\lambda = \lambda^{-1}\pi_W + \pi_{W \oplus \tilde{W}} + \lambda\pi_{\tilde{W}},$$

(24)

where $W$ is a holomorphic isotropic subbundle of $S^2 \times \mathbb{R}^{2n}$ of rank $l$ satisfying the superhorizontality condition $\partial W \subseteq \overline{W}^\perp$.

Set $\tilde{V}_l = V_l \oplus \tilde{V}_l$ and $\tilde{W} = W \oplus \tilde{W}$. The $T_{\sigma_{\varphi, l}}$-invariance of $\Phi$ implies that

$$[\pi_W, \pi_{V_0 \oplus \tilde{V}_l}] = 0.$$  

(25)

Now, write $\varphi = g \cdot V_0$ and consider the identification (23). We must have

$$\Phi_{-1} = \exp(\pi \zeta_l)g\sigma_{\varphi}(g^{-1}) = \exp(\pi \zeta_l)(\pi_{\varphi} - \pi_\varphi)\sigma_{\varphi}(g^{-1}).$$

(26)

From (24) and (26) we obtain

$$\pi_\varphi - \pi_{\varphi} = \text{Ad}(s_0)(\pi_{V_0 \oplus V_1} \pi_W - \pi_{V_0 \oplus V_1} \pi_{\tilde{W}} - \pi_{V_0} \pi_{\tilde{W}} - \pi_{V_0} \pi_{\tilde{W}}).$$

(27)

In view of (25), we see that $\pi_{V_0 \oplus V_1} \pi_W - \pi_{V_0 \oplus V_1} \pi_{\tilde{W}}$ is an orthogonal projection, and (27) implies that this must be an orthogonal projection onto a $1$-dimensional real subspace. Then, one of its two terms vanishes, that is either $\tilde{W} \subset V_0 \oplus \tilde{V}_l$ or $\tilde{W}^\perp \subset (V_0 \oplus \tilde{V}_l)^\perp$. For dimensional reasons, we see that the second case can not occur. Hence, we have

$$\pi_\varphi = \text{Ad}(s_0)(\pi_{V_0 \oplus V_1} \pi_W) = \pi_{V_0 \oplus V_1} \text{Ad}(s_0)(\pi_W),$$

that is (22) holds with $R = V_0 \oplus V_1 \oplus \nabla_l$ and $A = s_0(W)$. 


Remark 5. If $\varphi$ is full in $R$, then the isotropic subbundle $A$ is the $l$-osculating space of some full totally isotropic holomorphic map $f$ from $S^2$ into the complex projective space of $R$, the so called directrix curve of $\varphi$. That is, in a local system of coordinates $(U, z)$, we have $A(z) = \text{Span}\{g, g', \ldots, g^{(l-1)}\}$, where $g$ is a lift of $f$ over $U$ and $g^{(r)}$ the $r$-th derivative of $g$ with respect to $z$. Hence, formula \cite{22} agrees with the classification given in Corollary 6.11 of \cite{9}.

Example 1. Let us consider the case $n = 2$. We have only one class of harmonic maps: $(\zeta_1, \sigma_{\varphi,1})$. From Theorem \cite{25} any such harmonic map $\varphi : S^2 \to \mathbb{R}P^3$ is given by $\varphi = R \cap (A \oplus \overline{A})^\perp$, where $R$ is a constant 3-dimensional subspace of $\mathbb{R}^4$ and $A$ a holomorphic isotropic subbundle of $S^2 \times R$ of rank 1 such that $\partial A \subseteq \overline{A}$. Taking into account Proposition \cite{22} any such holomorphic subbundles $A$ can be obtain from a meromorphic function $a$ on $S^2$ as follows.

We have $\zeta_1 = E_1$ and $r(\zeta_1) = 1$. Any extended solution $\Phi : S^2 \setminus D \to U^\sigma_{\varphi,1}(SO(4))$ is given by $\Phi = \exp C \cdot \gamma_{\zeta_1}$, with $\gamma_{\zeta_1}(\lambda) = \lambda^{-1} \pi Y_1 + \pi^\perp_{Y_1 \oplus V_1} + \lambda \pi_{\overline{V}_1}$, for some holomorphic vector-valued function $C : S^2 \setminus D \to \mathbb{C}^r$, where

\[
(u^0_{\zeta_1})_{\sigma_{\varphi,1}} = (p^0_{\zeta_1})^\perp \cap \mathfrak{f}^C_{\sigma_{\varphi,1}} = \mathfrak{g}^C_1 \cap \mathfrak{f}^C_{\sigma_{\varphi,1}} = (\mathfrak{g}_{L_1-L_2} \oplus \mathfrak{g}_{L_1+L_2}) \cap \mathfrak{f}^C_{\sigma_{\varphi,1}}.
\]

Considering the root vectors $X_{1,j}, Y_{1,j}, Z_{i,j}$ as defined in \cite{19}, we have $Y_{1,2} = \sigma_{\varphi,1}(X_{1,2})$. Hence $C = a(z)(X_{1,2} + Y_{1,2})$ where $a(z)$ is a meromorphic function on $S^2$. In this case, from \cite{2}, it follows that $(\exp C)^{-1}(\exp C)_z = C_z$, and it is clear that the extended solution condition for $\Phi$ holds independently of the choice of the meromorphic function $a(z)$. Then, with respect to the complex basis $u = \{u_1, u_2, \overline{u}_1, \overline{u}_2\}$,

\[
\exp C \cdot \gamma_{\zeta_1} = \begin{bmatrix}
1 & a & -a^2 & a \\
0 & 1 & -a & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -a & 1
\end{bmatrix} \cdot \gamma_{\zeta_1}
\]

and the subbundle $A$ of $R = \text{Span}\{u_1, \overline{u}_1, u_2 + \overline{u}_2\}$ is given by $A = \exp C \cdot V_1 = \text{span}\{(a^2, a, -1, a)\}$, which satisfies $\partial A \subseteq \overline{A}$.\]

Example 2. Any harmonic two-sphere into $\mathbb{R}P^5$ in the class $(\zeta_1, \sigma_{\varphi,1})$ takes values in some $\mathbb{R}P^3$ inside $\mathbb{R}P^5$ and so it is essentially of the form \cite{28}. Next we consider the Weierstrass representation of harmonic spheres into $\mathbb{R}P^5$ in the class $(\zeta_2, \sigma_{\varphi,2})$, which are given by $\varphi = R \cap (A \oplus \overline{A})^\perp$, where $R$ is a constant 5-dimensional subspace of $\mathbb{R}^6$ and $A$ a holomorphic isotropic subbundle of $S^2 \times R$ of rank 2 such that $\partial A \subseteq \overline{A}$. We have $\zeta_2 = E_1 + E_2$, then $r(\zeta_2) = 2$. Any extended solution $\Phi : S^2 \setminus D \to U^\sigma_{\varphi,2}(SO(6))$ is given by $\Phi = \exp C \cdot \gamma_{\zeta_2}$, with

\[
\gamma_{\zeta_2}(\lambda) = \lambda^{-1} \pi_{\overline{V}_2} + \pi^\perp_{\overline{V}_2 \oplus \overline{V}_2} + \lambda \pi_{\overline{V}_2},
\]

for some holomorphic vector-valued function $C : S^2 \setminus D \to \mathbb{C}^r$, where

\[
(u^0_{\zeta_2})_{\sigma_{\varphi,2}} = \left(\mathfrak{g}_{L_1-L_3} \oplus \mathfrak{g}_{L_1+L_3}\right) \cap \mathfrak{f}^C_{\sigma_{\varphi,2}} + \left(\mathfrak{g}_{L_2-L_3} \oplus \mathfrak{g}_{L_2+L_3}\right) \cap \mathfrak{f}^C_{\sigma_{\varphi,2}} \oplus \mathfrak{g}_{L_1+L_2}.
\]

We have $Y_{1,3} = \sigma_{\varphi,2}(X_{1,3})$ and $Y_{2,3} = \sigma_{\varphi,2}(X_{2,3})$. Hence we can write

\[
C = a(z)(X_{1,3} + Y_{1,3}) + b(z)(X_{2,3} + Y_{2,3}) + c(z)Y_{1,2}
\]
where $a(z)$, $b(z)$ and $c(z)$ are meromorphic functions on $S^2$.

Now, $\Phi = \exp C \cdot \gamma_Q$ is an extended solution if and only if, in the expression $C_z - \frac{1}{2} (\text{ad} C) C_z$, which does not depend on $\lambda$, the component on $g_2^2 = g_{L_1 + L_2}$ must vanish. Since $Y_{1,2} = [Y_{2,3}, X_{1,3}] = [X_{2,3}, Y_{1,3}]$ and $[X_{1,3}, X_{2,3}] = [Y_{1,3}, Y_{2,3}] = 0$, this holds if and only if $c' = ba' - ab'$, where prime denotes $z$-derivative. Since $A = \exp C \cdot V_2$, we can compute $\exp C$ in order to conclude that the holomorphic subbundle $A$ of $R = \text{Span} \{ u_1, u_2, \overline{u}_1, \overline{u}_2, u_3 + \overline{u}_3 \}$ is given by

$$A = \text{Span} \{ (a^2, ab + c, a, -1, 0, a), (ab - c, b^2, b, 0, -1, b) \}.$$ 

5.1.2. Harmonic maps into Real Grassmannians. Let $\zeta'$ be a $\varrho$-semi-canonical element of $SO(2n)$ given by (21), for some disjoint subsets $I_1$ and $I_2$ of $\{1, \ldots, n-1\}$. By Proposition 13 we know that $P_{\zeta'}^{\varrho} \cong \mathbb{R}P^{2n-1}$ if and only if either $I_1 = \emptyset$ or $I_1 = \{ n-1 \}$. More generally we have:

**Proposition 26.** If $I_1 = \{ i_1 > i_2 > \ldots > i_r \}$ and $d = \sum_{j=1}^r (-1)^{j+1} i_j$, then $P_{\zeta'}^{\varrho} \cong G_{2d+1}(\mathbb{R}^{2n})$.

**Proof.** For $\zeta'$ of the form (21), set $\zeta'_{I_1} = \sum_{i \in I_1} \zeta_i$. Clearly, $\exp \pi \zeta' = \exp \pi \zeta'_{I_1}$, and, by Proposition 10, $P_{\zeta'}^{\varrho}$ is a symmetric space with involution

$$\tau = \text{Ad}(\exp \pi \zeta'_{I_1}) \circ \sigma_{\varrho} = \text{Ad}(s_0 \exp \pi \zeta'_{I_1}).$$

We have

$$\zeta'_{I_1} = r(E_1 + \ldots + E_n) + (r-1)(E_{i_r+1} + \ldots + E_{i_1}) + \ldots + (E_{i_2+1} + \ldots + E_{i_1}),$$

and consequently, with the convention $V_{i_0} = V_n$ and $V_{i_r+1} = \{ 0 \}$,

$$\exp \pi \zeta'_{I_1} = \sum_{j=0}^r (-1)^j \pi_{i_j - i_{j+1}} + \sum_{j=0}^r (-1)^j \pi_{i_j - i_{j+1}},$$

where $\pi_{i_j - i_{j+1}}$ is the orthogonal projection onto $V_{i_j} \cap V_{i_{j+1}}$ and $\pi_{i_j - i_{j+1}}$ the orthogonal projection onto the corresponding conjugate space. Hence, the $+1$-eigenspace of $s_0 \exp \zeta'_{I_1}$ has dimension $2d+1$, with $d = \sum_{j=1}^r (-1)^{j+1} i_j$, which means that $P_{\zeta'}^{\varrho} \cong G_{2d+1}(\mathbb{R}^{2n})$. \qed

In particular, we have $P_{\zeta'}^{\varrho} \cong G_{2d+1}(\mathbb{R}^{2n})$ for each $d \in \{ 1, \ldots, n-1 \}$.

**Theorem 27.** Each harmonic map into the real Grassmannian $G_{2d+1}(\mathbb{R}^{2n})$ belongs to one of the following classes: $(\zeta, \text{Ad} \exp \pi (\zeta - \zeta') \circ \sigma_{\varrho, l})$, where $\zeta$ and $\zeta'$ are $\varrho$-canonical elements such that $\tilde{\zeta} \preceq \zeta$ and $\zeta = \sum_{i \in I_1} \zeta_i + \zeta_l$, where

a) $I_1 = \{ i_1 > i_2 > \ldots > i_r \}$ satisfies $d = \sum_{j=1}^r (-1)^{j+1} i_j$;

b) $l \in \{ 0, 1, \ldots, n-1 \}$ and $l \notin I_1$ (if $l = 0$, we set $\zeta_0 = 0$).

**Proof.** We consider harmonic maps into $P_{\zeta'}^{\varrho} \cong G_{2d+1}(\mathbb{R}^{2n})$. Let $\zeta'$ be a $\varrho$-semi-canonical element and write $\zeta' = \sum_{i \in I_1} \zeta_i + \sum_{i \in I_2} 2 \zeta_i$ for some disjoint subsets $I_1$ and $I_2$ of $\{1, \ldots, n-1\}$. By Proposition 26 $P_{\zeta'}^{\varrho} \cong G_{2d+1}(\mathbb{R}^{2n})$ if and only if either $d = \sum_{j=1}^r (-1)^{j+1} i_j$ or $n - d - 1 = \sum_{j=1}^r (-1)^{j+1} i_j$, since $G_{2d+1}(\mathbb{R}^{2n})$ and $G_{2d'+1}(\mathbb{R}^{2n})$, with $d' = n - d - 1$, can be identified via $V \mapsto V ^{\perp}$. However, it follows from the same reasoning as in the proof of Theorem 24 that, in the second case, $P_{\zeta'}^{\varrho}$ does not coincide with the connected component $P_{\zeta'}^{\varrho}$. So we only consider the $\varrho$-semi-canonical elements $\zeta'$ with $d = \sum_{j=1}^r (-1)^{j+1} i_j$.
Set \( l = \max I_2 \). Next we check that the pair \( \zeta' \preceq \zeta = \sum_{i \in I_1} \zeta_i + \zeta \) satisfies the conditions of Proposition 21. Considering the same notations we used in the proof of Theorem 24 for each \( i > 0 \) we have

\[
\Delta' \cap \Delta_{\zeta}^{2i} = \{ \Delta_{\theta} - L_n | \| j_l \cap \{ j, \ldots, l \} + | I_l \cap \{ j, \ldots, n - 1 \} = 2i \}. \]

In particular, for \( i > 0 \) and \( \alpha = L_j - L_n \in \Delta' \cap \Delta_{\zeta}^{2i} \), it is clear that \( \alpha(\zeta' - \zeta)/i \leq 2i - 1 \), and consequently

\[
\mathfrak{g}^{\mathfrak{h}}_{2i} \cap m^C_\theta \subset \bigoplus_{0 \leq j < 2i} \mathfrak{g}^{\mathfrak{h}}_{j} \cap \mathfrak{g}^{\mathfrak{h}}_{-\zeta}.
\]

For \( i > 0 \), we have the decomposition

\[
\mathfrak{g}^{\mathfrak{h}}_{2i-1} \cap \mathfrak{h}^C_\theta = \bigoplus_{\alpha \in \Delta(\zeta) \cap \Delta^{2i-1}_{\zeta}} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Delta' \cap \Delta^{2i-1}_{\zeta}} (\mathfrak{g}_\alpha \oplus \mathfrak{g}(\alpha)) \cap \mathfrak{h}^C_\theta.
\]

Given \( \alpha \in \mathfrak{g}^{\mathfrak{h}}_{2i-1} \), since \( \alpha(\zeta')/i \) is odd, we must have \( \alpha(\zeta) \neq 0 \) for some \( j \in I_1 \). Hence \( \alpha(\zeta' - \zeta)/i < \alpha(\zeta)/i \) and we conclude that

\[
\mathfrak{g}^{\mathfrak{h}}_{2i-1} \cap \mathfrak{h}^C_\theta \subset \bigoplus_{0 \leq j < 2i-1} \mathfrak{g}^{\mathfrak{h}}_{j} \cap \mathfrak{g}^{\mathfrak{h}}_{-\zeta}.
\]

The statement of the theorem follows now from Propositions 20 and 21. \( \square \)

Next we will study in detail the case \( G_3(\mathbb{R}^6) \). Take as base point of \( G_3(\mathbb{R}^6) \) the 3-dimensional real subspace \( V_0 \oplus V_1 \oplus \overline{V}_1 \), where \( V_1 \) is the one-dimensional isotropic subspace spanned by \( \pi_1 \). This choice establishes the identification

\[
G_3(\mathbb{R}^6) \cong SO(6)/SO(3) \times O(3)
\]

and the corresponding involution is \( \sigma_{\theta,1} = \text{Ad}(\exp \pi \zeta_1) \circ \sigma_\theta \). Following our classification procedure, we also identify \( G_3(\mathbb{R}^6) \) with \( \mathbb{C}P^0_\theta \) via the totally geodesic embedding \( (14) \). From Theorem 27 we have six classes of harmonic maps into \( G_3(\mathbb{R}^6) \):

\[
(\zeta_1, \sigma_\theta), \ (\zeta_1 + \zeta_2, \sigma_\theta), \ (\zeta_2, \sigma_{\theta,1}), \ (\zeta_1, \sigma_{\theta,2}), \ (\zeta_1 + \zeta_2, \sigma_{\theta,2}), \ (\zeta_2, \text{Ad}(\exp \pi \zeta_2) \circ \sigma_{\theta,1}).
\]

**Theorem 28.** Let \( \varphi : S^2 \to G_3(\mathbb{R}^6) \) be an harmonic map.

1. If \( \varphi \) is associated to the pair \( (\zeta_1, \sigma_\theta) \) then \( \varphi \) is \( S^1 \)-invariant and, up to isometry, is given by

\[
\varphi = V_0 \oplus V \oplus \overline{V},
\]

where \( V \) is a holomorphic isotropic subbundle of \( S^2 \times V_0^\perp \) of rank 1 satisfying \( \partial V \subset \overline{V}^\perp \).

2. If \( \varphi \) is associated to the pair \( (\zeta_1 + \zeta_2, \sigma_\theta) \) and is \( S^1 \)-invariant, then, up to isometry,

\[
\varphi = V_0 \oplus (W \cap V^\perp) \oplus (\overline{W} \cap V^\perp),
\]

where \( V \subset W \) are holomorphic isotropic subbundles of \( S^2 \times V_0^\perp \) of rank 1 and 2, respectively, satisfying \( \partial V \subset W \) and \( \partial W \subset \overline{W}^\perp \).

3. If \( \varphi \) is associated to the pair \( (\zeta_2, \sigma_{\theta,1}) \) and is \( S^1 \)-invariant, then, up to isometry,

\[
\varphi = \{(L_1 \oplus \overline{L}_1)^\perp \cap (V_0 \oplus V_1 \oplus \overline{V}_1)\} \oplus (L_2 \oplus \overline{L}_2),
\]

where \( L_1 \) and \( L_2 \) are holomorphic isotropic bundle lines of \( S^2 \times (V_0 \oplus V_1 \oplus \overline{V}_1) \) and \( S^2 \times (V_0 \oplus V_1 \oplus \overline{V}_1)^\perp \), respectively.
The corresponding extended solutions have uniton number 2, 4, and 2, respectively, with respect to the standard representation of $SO(6)$. The harmonic maps in the classes $(\zeta_1, \sigma_2)$, $(\zeta_1 + \zeta_2, \sigma_2)$, and $(\zeta_2, \text{Ad}(\exp \pi \zeta_2) \circ \sigma_{e,1})$ are precisely the orthogonal complements of the harmonic maps in the classes $(\zeta_1, \sigma_2)$, $(\zeta_1 + \zeta_2, \sigma_2)$, and $(\zeta_2, \sigma_{e,1})$, respectively.

Proof. For the first two classes, and according to our classification procedure, we identify $G_3(\mathbb{R}^6)$ with $P_{\zeta_1}^{\sigma_{e,1}}$ via the totally geodesic embedding

$$g \cdot (V_0 \oplus V_1 \oplus \overline{V}_1) \mapsto \exp(\pi \zeta_1)g_{\sigma_{e,1}}(g^{-1}).$$

In these two cases, $T_{\sigma_{e,1}}$-invariant extended solutions $\Phi$ associated to harmonic maps $\varphi = g \cdot (V_0 \oplus V_1 \oplus \overline{V}_1)$ satisfy

$$\Phi_{-1} = \exp(\pi \zeta_1)g_{\sigma_{e,1}}(g^{-1}) = \exp(\pi \zeta_1)(\pi \varphi - \pi \varphi)\exp(\pi \zeta_1)s_0. \quad (31)$$

First we consider the harmonic maps associated to the pair $(\zeta_1, \sigma_2)$. We have $r(\zeta_1) = 1$ and

$$(u_{\zeta_1})_{\sigma_{e,1}} = (p_{\zeta_1})^\perp \cap \mathfrak{g}^\mathfrak{c} = g_{\zeta_1}^\perp \cap \mathfrak{g}_e^\mathfrak{c}.$$ Consequently any such harmonic map is $S^1$-invariant. Write

$$\gamma_{\zeta_1}(\lambda) = \lambda^{-1}\pi V_1 + \pi_{V_1 \oplus V_1} + \lambda \pi \overline{V}_1,$$

where $V_1$ is the one-dimensional isotropic space spanned by $\overline{V}_1$. Let $\Phi : S^2 \setminus D \to U_{\zeta_1}^{\sigma_{e,1}}$ be an extended solution associated to the harmonic map $\varphi$. Then, by $S^1$-invariance, we can write

$$\Phi_\lambda = \lambda^{-1}\pi V + \pi_{V \oplus \overline{V}} + \lambda \pi \overline{V}, \quad (32)$$

where $V$ is a holomorphic isotropic subbundle of $S^2 \times \mathbb{R}^6$ of rank 1 satisfying $\partial V \subseteq \overline{V}^\perp$. The $T_{\sigma_{e,1}}$-invariance of $\Phi$ implies that $V_0 \subset (V \oplus \overline{V})^\perp$. Equating (31) and (32), we get, up to isometry, $\varphi = V_0 \oplus V \oplus \overline{V}$.

For the case $(\zeta_1 + \zeta_2, \sigma_2)$, since

$$\gamma_{\zeta_1+\zeta_2}(\lambda) = \lambda^{-2}\pi V_1 + \lambda^{-1}\pi_{V_1 \oplus V_1} + \pi_{V_2 \oplus \overline{V}_2} + \lambda \pi_{V_2 \oplus \overline{V}_1} + \lambda^2 \pi \overline{V}_1,$$

any $S^1$-invariant harmonic map $\varphi$ in this class admits an extended solution of the form

$$\Phi_\lambda = \lambda^{-2}\pi V + \lambda^{-1}\pi_{W \oplus \overline{V}_1} + \pi_{W \oplus \overline{V}_1} + \lambda \pi_{\overline{W} \oplus \overline{V}_1} + \lambda^2 \pi \overline{V}_1,$$ \quad (33)

where $V \subset W$ are holomorphic isotropic subbundles of rank 1 and 2, respectively, satisfying $\partial V \subset W$ and $\partial W \subset \overline{W}^\perp$. By $T_{\sigma_{e,1}}$-invariance, we must have $V_0 \subset (W \oplus \overline{W})^\perp$, hence $V \subset W$ are subbundles of $S^2 \times V_0^\perp$. Equating (31) and (33), we get (29).

For the case $(\zeta_{2, \sigma_{e,1}})$, we identify $G_3(\mathbb{R}^6)$ with $P_{\zeta_2}^{\sigma_{e,1}} = \exp \pi \zeta_1 P_{\zeta_1}^{\sigma_{e,1}}$ via the totally geodesic embedding

$$g \cdot (V_0 \oplus V_1 \oplus \overline{V}_1) \mapsto g_{\sigma_{e,1}}(g^{-1}).$$

Extended solutions $\Phi$ associated to $S^1$-invariant harmonic maps in this class must be of the form

$$\Phi_\lambda = \lambda^{-1}\pi W + \pi_{W \oplus \overline{W}} + \lambda \pi W,$$ \quad (36)

where $W$ is a holomorphic isotropic subbundle of rank 2. By $T_{\sigma_{e,1}}$-invariance, we must have

$$[\pi W, \pi_{V_0 \oplus V_1 \oplus \overline{V}_1}] = 0,$$

which means that $W$ must be of the form $W = L_1 \oplus L_2$, where $L_1$ and $L_2$, respectively, are holomorphic isotropic bundle lines of $S^2 \times (V_0 \oplus V_1 \oplus \overline{V}_1)$ and $S^2 \times (V_0 \oplus V_1 \oplus \overline{V}_1)^\perp$. 


On the other hand, in view of (35), we have

$$\Phi_{-1} = (\pi \varphi - \pi_1^+) \exp(\pi \zeta_1) s_0.$$  

Equating this with (36), we conclude that (30) holds.

The remaining cases are treated similarly. □

Let us consider in detail the case ($\zeta_1 + \zeta_2, \sigma_0$). Taking into account the Weierstrass representation of Proposition 22, any extended solution $\Phi : S^2 \setminus D \to U_{\zeta}^{so}(SO(6))$, with $\zeta = \zeta_1 + \zeta_2$, can be written as $\Phi = \exp C \cdot \gamma$, for some meromorphic vector-valued function $C : S^2 \to (u_0^0)^{\sigma_0}$. We have $r(\zeta) = 3$ and

$$\Phi = \exp C \cdot \gamma = (g_1^\zeta + g_2^\zeta + g_3^\zeta) \cap \mathfrak{e}_g^\zeta + \lambda (g_2^\zeta + g_3^\zeta) \cap m_0^\zeta + \lambda^2 g_\zeta \cap \mathfrak{e}_g^\zeta.

Moreover,

$$(g_1^\zeta \cap \mathfrak{e}_g^\zeta = g_{L_1-L_2} + \{(g_{L_2-L_3} + g_{L_2+L_3}) \cap \mathfrak{e}_g^\zeta\},
\quad g_2^\zeta \cap \mathfrak{e}_g^\zeta = (g_{L_1+L_3} + g_{L_1-L_3}) \cap \mathfrak{e}_g^\zeta,
\quad g_3^\zeta \cap \mathfrak{e}_g^\zeta = g_{L_1+L_2},
\quad (g_2^\zeta + g_3^\zeta) \cap m_0^\zeta = g_2^\zeta \cap m_0^\zeta = (g_{L_1-L_3} + g_{L_1+L_1}) \cap m_0^\zeta.

Write

$$C = C_0 + \lambda C_1 + \lambda^2 C_2, \quad C_0 = c_0^0 + c_0^1 + c_0^2, \quad C_1 = c_1^0 + c_1^1, \quad C_2 = c_2^1 \quad (37)

$$

where the functions $c_0^0 : S^2 \to g_0^\zeta \cap \mathfrak{e}_g^\zeta, c_1^0 : S^2 \to g_0^\zeta \cap m_0^\zeta,$ and $c_2^0 : S^2 \to g_0^\zeta \cap \mathfrak{e}_g^\zeta$ are meromorphic functions. Clearly, $c_1^0 = 0$. Consider the root vectors defined by (19). Since $\sigma_0(X_{2,3}) = -Y_{2,3}$ and $\sigma_0(X_{1,3}) = -Y_{1,3}$, we can write

$$c_0^1 = aX_{1,2} + b(X_{2,3} - Y_{2,3}), \quad c_0^2 = c(X_{1,3} - Y_{1,3}), \quad c_0^3 = dY_{1,2}, \quad c_1^2 = e(X_{1,3} + Y_{1,3}), \quad c_2^2 = f X_{1,2}

$$

in terms of $\mathbb{C}$-valued meromorphic functions $a, b, c, d, e, f$.

Taking into account the results of Section 3.1.1, $\Phi = \exp C \cdot \gamma$ is an extended solution if and only if, in the expression

$$(\exp C)^{-1} (\exp C)z = C_z - \frac{1}{2!} (adC) C_z + \frac{1}{3!} (adC)^2 C_z,$

we have:

a) the independent coefficient should have zero component in each $g_0^\zeta$ and $g_1^\zeta$, that is

$$c_{0z}^1 - \frac{1}{2}[c_0^0, c_{0z}^1] = 0, \quad c_{0z}^2 - \frac{1}{2}[c_0^0, c_{0z}^2] - \frac{1}{2}[c_0^1, c_{0z}^1] + \frac{1}{6}[c_0^0, [c_0^0, c_{0z}^1]] = 0; \quad (38)

$$

b) the $\lambda$ coefficient should have zero component in $g_3^\zeta$, that is

$$[c_0^1, c_{1z}^1] + [c_1^1, c_{0z}^1] = 0. \quad (39)

$$

From equations (38) we get the equations (prime denotes $z$-derivative)

$$2c' = ab' - ba', \quad 3d' = 3cb' - bc'; \quad (40)

$$

on the other hand, observe that (39) always holds since

$$[c_0^1, c_{1z}^1] + [c_1^1, c_{0z}^1] \in [g_1^\zeta \cap \mathfrak{e}_g^\zeta, g_2^\zeta \cap m_0^\zeta] \subset g_3^\zeta \cap m_0^\zeta = \{0\}.

$$

Hence we conclude that, any extended solution $\Phi : S^2 \setminus D \to U_{\zeta}^{so}(SO(6))$, with $\zeta = \zeta_1 + \zeta_2$, of the form $\Phi = \exp C \cdot \gamma$, can be constructed as follows: choose arbitrary meromorphic functions $a, b, c$ and $f$; integrate equations (40) to obtain the meromorphic functions $c$ and $d$; $C$ is then given by (37).
Example 3. Choose $a(z) = b(z) = z$. From (10), we can take $c(z) = 1$ and $d(z) = z$. This data defines the matrix $C_0$ and the $S^1$-invariant extended solution $\exp C_0 \cdot \gamma_\zeta$, where the loop $\gamma_\zeta$, with $\zeta = \zeta_1 + \zeta_2$, is given by (33). The extended solutions $\Phi : S^2 \to U^{\gamma_\zeta}(SO(6))$ satisfying $\Phi^0 = u_\zeta \circ \Phi$ are of the form $\Phi = \exp C \cdot \gamma_\zeta$, where the matrix $C = C_0 + C_1 \lambda + C_2 \lambda^2$ is given by

$$C = \begin{pmatrix}
0 & z & 1 & 0 & z & -1 \\
0 & 0 & z & -z & 0 & -z \\
0 & 0 & 0 & 1 & z & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -z & 0 & 0 \\
0 & 0 & 0 & -1 & -z & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & e\lambda & 0 & f\lambda^2 & -e\lambda \\
0 & 0 & 0 & -f\lambda^2 & 0 & 0 \\
0 & 0 & 0 & e\lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -e\lambda & 0 & 0
\end{pmatrix},$$

with respect to the complex orthonormal basis $u = \{u_1, u_2, u_3, \overline{u}_1, \overline{u}_2, \overline{u}_3\}$, where $e$ and $f$ are arbitrary meromorphic functions on $S^2$. The holomorphic vector bundles $V$ and $W$ associated to the $S^1$-invariant extended solution $\exp C_0 \cdot \gamma_\zeta$ are given by $V = \exp C_0 \cdot V_1$ and $W = \exp C_0 \cdot V_2$, and we have, with respect to the basis $u$,

$$V = \text{span}\{(12 - 12z^2 - z^4, -4z^3, 12 - 6z^2, 12, -12z, -12 + 6z^2)\}$$

$$W = \text{span}\{(6z + z^3, 3z^2, 3z, 0, 3, -3z)\} \oplus V.$$

5.2. Outer symmetric $SU(2m+1)$-spaces. Let $E_j$ be the square $(m \times m)$-matrix whose $(j,j)$-entry is $i$ and all other entries are $0$. The complexification $t^C$ of the algebra of diagonal matrices

$$t = \{ \sum a_i E_i | a_i \in \mathbb{C}, \sum a_i = 0 \}$$

is a Cartan subalgebra of $\mathfrak{su}(m)^C$. Let $\{L_1, \ldots, L_m\}$ be the dual basis of $\{E_1, \ldots, E_m\}$, that is $L_i(E_j) = i \delta_{ij}$. The roots of $\mathfrak{su}(m)$ are the vectors $L_i - L_j$, with $i \neq j$ and $1 \leq i, j \leq m - 1$. The positive root system $\Delta^+ = \{L_i - L_j\}_{1 < j}$ is a set of positive simple roots $\alpha_i = L_i - L_{i+1}$, for $1 \leq i \leq m - 1$. For $i \neq j$, the matrix $X_{ij}$ whose $(i,j)$ entry is 1 and all other entries are 0 generate the root space $g_{L_i - L_j}$. The dual basis of $\Delta_0 = \{\alpha_1, \ldots, \alpha_{m-1}\}$ in $t^*$ is formed by the matrices

$$H_i = \frac{m - i}{m} (E_1 + \ldots + E_i) - \frac{i}{m} (E_{i+1} + \ldots + E_m).$$

5.2.1. Special Lagrangian spaces. Consider on $\mathbb{R}^{2m}$ the standard inner product $\langle \cdot, \cdot \rangle$ and the canonical orthonormal basis $e^{2m} = \{e_1, \ldots, e_{2m}\}$. Define the orthogonal complex structure $I$ by $I(e_i) = e_{2m+1-i}$, for $i \in \{1, \ldots, m\}$. A Lagrangian subspace of $\mathbb{R}^{2m}$ (with respect to $I$) is an $m$-dimensional subspace $L$ such that $IL \perp L$. Let $\mathcal{L}_m$ be the space of all Lagrangian subspaces of $\mathbb{R}^{2m}$ and $L_0 \in \mathcal{L}_m$ the Lagrangian subspace generated by $e^m = \{e_1, \ldots, e_m\}$. The unitary group $U(m)$ acts transitively on $\mathcal{L}_m$, with isotropy group at $L_0$ equal to $SO(m)$, and $\mathcal{L}_m$ is a reducible symmetric space that can be identified with $U(m)/SO(m)$ (see [16] for details).

The space $\mathcal{L}_m$ can also be interpreted as the set of all orthogonal linear maps $\tau : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ satisfying $\tau^2 = e$ and $I\tau = -\tau I$. Indeed, if $V_\pm$ are the $\pm 1$ eigenspaces of $\tau$, then $IV_+ = V_-$ and $IV_+ \perp V_+$, that is $V_+$ is Lagrangian. From this point of view, $U(m)$ acts on $\mathcal{L}_m$ by conjugation: $g \cdot \tau = g\tau g^{-1}$. Let $\eta_0 \in \mathcal{L}_m$ be the orthogonal linear map corresponding to $L_0$, that is, $\tau_0|_{L_0} = e$ and $\tau_0|_{L_0} = -e$. The corresponding involution on $U(m)$ is given by $\sigma(g) = \tau_0 g \tau_0$ and the Cartan embedding $\iota : \mathcal{L}_m \hookrightarrow U(m)$ is given by $\iota(\tau) = \tau \tau_0$. 
The totally geodesic submanifold $L^* := SU(m)/SO(m)$ of $U(m)/SO(m)$ is also known as the space of special Lagrangian subspaces of $\mathbb{R}^{2m}$. It is an irreducible outer symmetric $SU(m)$-space.

5.2.2. Harmonic maps into $L^*_2$. Take $m = 2n + 1$. The non-trivial involution $\varrho$ of the Dynkin diagram of $su(2n + 1)^\mathbb{C}$ is given by $\varrho(\alpha_i) = \alpha_{2n+1-i}$. In particular, $\varrho$ does not fix any root in $\Delta_0$ and there exists only one class of outer symmetric $SU(2n + 1)$-spaces. The semi-fundamental basis $\tau_{\varrho}(\Delta_0) = \{\beta_1, \ldots, \beta_n\}$ is given by $\beta_i = \frac{1}{2}(\alpha_i + \alpha_{2n+1-i})$ whereas the dual basis $\{\zeta_1, \ldots, \zeta_n\}$ is given by

$$\zeta_i = H_i + H_{2n+1-i} = E_1 + \ldots + E_i - (E_{2n+2-i} + \ldots + E_{2n+1}),$$

for $1 \leq i \leq n$. Since each $\zeta_i$ belongs to the integer lattice $3(SU(2n + 1))$, the $\varrho$-semi-canonical elements of $SU(2n + 1)$ are precisely the elements $\zeta = \sum_{i=1}^n m_i \zeta_i$ with $m_i \in \{0, 1, 2\}$.

Let $e^{2n+1} = \{e_1, \ldots, e_{2n+1}\}$ be the canonical orthonormal basis of $\mathbb{R}^{2n+1}$. Identify $\mathbb{C}^{2n+1}$ with $(\mathbb{R}^{4n+2}, I)$, where $I$ is defined as above. Set

$$v_j = \frac{1}{\sqrt{2}}(e_j + ie_{2n+2-j}),$$

for $1 \leq j \leq n$, $v_{n+1} = e_{n+1}$ and $v_{2n+2-j} = \overline{v_j}$. Take the matrices $E_j$ with respect to the complex basis $v = \{v_1, \ldots, v_{2n+1}\}$ of $\mathbb{C}^{2n+1}$. Hence $\tau_{\varrho} E_j \tau_{\varrho} = -E_{2n+2-j}$ and the fundamental involution $\sigma_\varrho$ is given by $\sigma_\varrho (g) = \tau_{\varrho} g \tau_{\varrho}$. The fundamental outer symmetric $SU(2n + 1)$-space is the space of special Lagrangian subspaces $L^*_2 = SU(2n + 1)/SO(2n + 1)$, and this is the unique outer symmetric $SU(2n + 1)$-space.

Next we consider in detail harmonic maps into $L^*_3$. In this case we have two non-zero $\varrho$-semi-canonical elements, $\zeta_1$ and $2\zeta_1$, and consequently two classes of harmonic maps, $(\zeta_1, \sigma_\varrho)$ and $(\zeta_1, \sigma_{\varrho,1})$. Since $\zeta_1 = E_1 - E_3$, we have $r(\zeta_1) = (L_1 - L_3)(\zeta_1)/i = 2$. Let $W_1$, $W_2$ and $W_3$ be the complex one-dimensional images of $E_1$, $E_2$ and $E_3$, respectively. Any extended solution $\Phi : S^2 \setminus D \rightarrow U^\varrho_{\zeta_1}(SU(2n + 1))$ is given by $\Phi = \exp C \cdot \gamma_{\zeta_1}$, with $\gamma_{\zeta_1}(\lambda) = \lambda^{-1} \pi W_3 + \pi W_2 + \lambda \pi W_1$, for some holomorphic vector-valued function $C : S^2 \setminus D \rightarrow (u^0_{\zeta_1})_{\varrho}$, where

$$(u^0_{\zeta_1})_{\varrho} = (p^0_{\varrho})^\perp \cap f^C_\varrho + \lambda(p^{1}_{\varrho})^\perp \cap m^C_\varrho$$

and

$$(p^0_{\varrho})^\perp \cap f^C_\varrho = (g_{L_1-L_2} \oplus g_{L_2-L_3} \oplus g_{L_1-L_3}) \cap f^C_\varrho, \quad (p^{1}_{\varrho})^\perp \cap m^C_\varrho = g_{L_1-L_3} \cap m^C_\varrho.$$

Let $X_{i,j}$ be the square matrix whose $(i, j)$ entry is 1 and all the other entries are 0, with respect to the basis $v$. The root space $g_{L_1-L_j}$ is spanned by $X_{i,j}$. We have $\sigma_{\varrho}(X_{1,2}) = -X_{2,3}$ and $\sigma_{\varrho}(X_{1,3}) = -X_{1,3}$ (consequently, $g_{L_1-L_3} \subset m^C_\varrho$). Hence we can write $C = C_0 + C_1 \lambda$, with

$$C_0 = a(X_{1,2} - X_{2,3}), \quad C_1 = b X_{1,3}$$

for some meromorphic functions $a, b$ on $S^2$. The harmonicity equations do not impose any condition on these meromorphic functions, hence any harmonic map

$$\varphi : S^2 \rightarrow L^*_3 = \{\tau \in SO(6) \mid \tau^2 = e, \tau I = -I \sigma \} \cong P^e_{\varrho,1}$$
admits an extended solution of the form
\[
\Phi = \exp \left( \begin{array}{ccc}
0 & a & b \lambda \\
0 & 0 & -a \\
0 & 0 & 0
\end{array} \right) \cdot \gamma_{\zeta_1} = \left( \begin{array}{ccc}
1 & a & \frac{1}{2}(-a^2 + 2b\lambda) \\
0 & 1 & -a \\
0 & 0 & 1
\end{array} \right) \cdot \gamma_{\zeta_1},
\]
and \( \varphi \) is recovered by setting \( \varphi = \Phi_{-1/2} \).

Under the natural embedding \( SU(3) \hookrightarrow SO(6) \), harmonic maps into \( SU(3)/SO(3) \) belong to the classes \( (\zeta_2, \sigma_{\theta,1}) \) and \( (\zeta_2, \text{Ad}(\exp \pi \zeta_2) \circ \sigma_{\theta,1}) \) of harmonic maps into \( G_3(\mathbb{R}^6) \). This can be seen as follows. Fix on \( \mathbb{R}^6 \) the canonical basis \( e^6 = \{ e_1, e_2, e_3, e_4, e_5, e_6 \} \) and consider the complex structure \( I \) defined as above: \( I e_1 = e_6 \), \( I e_2 = e_5 \) and \( I e_3 = e_4 \). Identify \( \mathbb{C}^3 \cong (\mathbb{R}^6, I) \). Define the complex basis \( \{ u_1, u_2, u_3, \overline{u}_1, \overline{u}_2, \overline{u}_3 \} \) of \( \mathbb{C}^6 = (\mathbb{R}^6)^{\mathbb{C}} \) by
\[
\begin{align*}
u_1 &= \frac{1}{\sqrt{2}}(e_1 + i e_3), \\
u_2 &= \frac{1}{\sqrt{2}}(I e_1 + i I e_3), \\
u_3 &= \frac{1}{\sqrt{2}}(I e_2 + i e_2),
\end{align*}
\]
which satisfies (13). With respect to these choices, the image of the \( g \)-canonical element \( \zeta_1 \) of \( \mathfrak{su}(3) \) under the natural embedding \( SU(3) \hookrightarrow SO(6) \) is precisely the \( g \)-canonical element \( \zeta_2 \) of \( \mathfrak{so}(6) \) given by (20).

5.3. Outer symmetric \( SU(2n) \)-spaces. With the same notations of Section 5.2, the non-trivial involution \( \vartheta \) of the Dynkin diagram of \( \mathfrak{su}(2n) \) is given by \( \vartheta(\alpha_i) = \alpha_{2n-i} \), and \( \vartheta \) fixes the root \( \alpha_n \). The semi-fundamental basis \( \pi_{\vartheta}(\Delta_0) = \{ \beta_1, \ldots, \beta_{n-1} \} \) is given by \( \beta_1 = \alpha_n \) and \( \beta_i = \frac{1}{2}(\alpha_i + \alpha_{2n-i}) \) if \( i \geq 2 \); whereas its dual basis \( \{ \zeta_1, \ldots, \zeta_{n-1} \} \) is given by
\[
\begin{align*}
\zeta_1 &= H_n = \frac{1}{2}(E_1 + \ldots + E_n) - \frac{1}{2}(E_{n+1} + \ldots + E_{2n}) \\
\zeta_i &= H_{n-i} + H_{2n-i+1} = E_1 + \ldots + E_{n-i - 1} - (E_{2n+2-i} + \ldots + E_{2n}), \quad \text{for } 2 \leq i \leq n - 1.
\end{align*}
\]
By Theorem [11] there exist two conjugacy classes of outer involutions: the fundamental outer involution \( \sigma_{\theta} \) and \( \sigma_{\theta,1} \). These outer involutions correspond to the symmetric spaces \( SU(2n)/Sp(n) \) and \( SU(2n)/SO(2n) \), respectively. Observe that \( \zeta_1 \) does not belong to the integer lattice \( \mathfrak{j}(SU(2n)^{\mathbb{C}}) \) since \( exp 2\pi \zeta_1 = -e \).

5.3.1. Harmonic maps into the space of special unitary quaternionic structures on \( \mathbb{C}^{2n} \). A unitary quaternionic structure on the standard hermitian space \( (\mathbb{C}^{2n}, \langle \cdot, \cdot \rangle) \) is a conjugate linear map \( J : \mathbb{C}^{2n} \to \mathbb{C}^{2n} \) satisfying \( J^2 = -Id \) and \( \langle v, w \rangle = \langle Jw, Jv \rangle \) for all \( v, w \in \mathbb{C}^{2n} \). Consider as base point the quaternionic structure \( J_o \) defined by \( J_e(e_i) = e_{2n+1-i} \) for each \( 1 \leq i \leq n \), where \( e^{2n} = \{ e_1, \ldots, e_{2n} \} \) is the canonical hermitian basis of \( \mathbb{C}^{2n} \). The unitary group \( U(2n) \) acts transitively on the space of unitary quaternionic structures on \( \mathbb{C}^{2n} \) with isotropy group at \( J_o \) equal to \( Sp(n) \), and thus \( M = U(2n)/Sp(n) \). This is a reducible symmetric space with involution \( \sigma : U(2n) \to U(2n) \) given by \( \sigma(X) = J_o X J_o^{-1} \), but the totally geodesic submanifold \( Q^s_o := SU(2n)/Sp(n) \) is an irreducible symmetric space, which we call the space of special unitary quaternionic structures on \( \mathbb{C}^{2n} \) (see [16] for details). If we consider the matrices \( E \) with respect to the complex basis \( v = \{ v_1, \ldots, v_{2n} \} \) defined by
\[
v_j = \frac{1}{\sqrt{2}}(e_j + i e_{2n+1-j}),
\]
for \( 1 \leq j \leq n \), and \( v_{2n+1-j} = \overline{v}_j \), we see that \( J_o E_j J_o^{-1} = -E_{2n+1-j} \), and consequently we have \( \sigma = \sigma_{\theta} \).

Next we consider with detail harmonic maps into \( Q^s_2 \).
Proposition 29. Each harmonic map \( \varphi : S^2 \to Q^5_2 \) belongs to one of the following classes: \((2\zeta_1, \sigma_\varphi)\), and \((\zeta_2, \sigma_{\varphi, 2})\).

Proof. We start by identifying \( Q^5_2 \) with \( P^\sigma_{\varphi} \).

The \( \sigma \)-semi-canonical elements of \( SU(4) \) are precisely the elements

\[
2\zeta_1, 4\zeta_1, \zeta_2, 2\zeta_2, 2\zeta_1 + \zeta_2, 2\zeta_1 + 2\zeta_2, 4\zeta_1 + \zeta_2, 4\zeta_1 + 2\zeta_2.
\]

By Proposition 13 all these elements correspond to the symmetric space \( Q^5_2 \).

We claim that \( \exp \pi \zeta_2 \) is not in the connected component

\[
P^\sigma_{\varphi} = \{ gJ_0g^{-1}J_0^{-1} \mid g \in SU(4) \}.
\]

In fact, \( \exp(\pi \zeta_2)J_0 = gJ_0g^{-1} \cong gSp(n) \) for the unitary transformation \( g \) defined by \( g(e_1) = e_4, g(e_1) = e_1, g(e_2) = e_3 \) and \( g(e_3) = -e_2 \). Similarly, one can check that \( \exp \pi \zeta_2 \) does not belong to \( P^\sigma_{\varphi} \).

Hence, since \( \exp 2\zeta_1 \) belongs to the centre of \( SU(4) \), any harmonic map \( \varphi : S^2 \to Q^5_2 \cong P^\sigma_{\varphi} \) belongs to one of the following classes: \((2\zeta_1, \sigma_\varphi)\), \((\zeta_2, \sigma_{\varphi, 2})\), and \((2\zeta_1 + \zeta_2, \sigma_{\varphi, 2})\). It remains to check that, in view of Proposition 21 harmonic maps in the class \((2\zeta_1 + \zeta_2, \sigma_{\varphi, 2})\) can be normalized to harmonic maps in the class \((\zeta_2, \sigma_{\varphi, 2})\).

It is clear that \( 2\zeta_1 + \zeta_2 \leq \zeta_2 \). On the other hand, for any positive root \( L_i - L_j \in \Delta^+ \), with \( i < j \), we have \((L_i - L_j)(2\zeta_1)/i \leq (L_i - L_j)(2\zeta_1 + \zeta_2)/i \), where the equality holds in just one case: \((L_2 - L_3)(2\zeta_1) = (L_2 - L_3)(2\zeta_1 + \zeta_2) = 2i \). However, \(gL_2 - L_3 \subset \mathfrak{e}_{\sigma_{\varphi, 2}} \), which means that the conditions of Proposition 21 hold for \( \zeta = 2\zeta_1 + \zeta_2 \) and \( \zeta' = \zeta_2 \), and consequently harmonic maps in the class \((2\zeta_1 + \zeta_2, \sigma_{\varphi, 2})\) can be normalized to harmonic maps in the class \((\zeta_2, \sigma_{\varphi, 2})\).

Following the same procedure as before, one can see that any harmonic map \( \varphi \to Q^5_2 \) in the class \((2\zeta_1, \sigma_\varphi)\) admits an extended solution of the form

\[
\Phi = \begin{pmatrix}
1 & 0 & c_1 + a\lambda & c_2 \\
0 & 1 & c_3 & c_1 - a\lambda \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \cdot \gamma_{2\zeta_1},
\]

where \( c_1, c_2, c_3 \in \mathbb{C} \) are constants, \( a \) is a meromorphic function on \( S^2 \). The harmonic map is recovered by setting \( \varphi = \Phi_{-1}J_0 \). Reciprocally, given arbitrary complex constants \( c_1, c_2, c_3 \) and a meromorphic function \( a : S^2 \to \mathbb{C} \), such \( \Phi \) is an extended solution associated to some harmonic map in the class \((2\zeta_1, \sigma_\varphi)\) (the harmonicity equations do not impose any restriction to \( a \)).

Similarly, any harmonic map \( \varphi \to Q^5_2 \) in the class \((\zeta_2, \sigma_{\varphi, 2})\) admits an extended solution of the form

\[
\Phi = \begin{pmatrix}
1 & b & a & c \\
0 & 1 & 0 & a \\
0 & 0 & 1 & -b \\
0 & 0 & 0 & 1
\end{pmatrix} \cdot \gamma_{\zeta_2},
\]

where \( a \), \( b \) and \( c \) are meromorphic functions satisfying \( c' = ba' - b'a \). Since \( P^\sigma_{\varphi, 2} = \exp(\pi \zeta_2)P^\sigma_{\varphi} \), the harmonic map is recovered by setting \( \varphi = \exp \pi \zeta_2 \Phi_{-1}J_0 \).
5.3.2. Harmonic maps into $\mathcal{L}^s_{2n}$. The outer symmetric $SU(2n)$-space that corresponds to the involution $\sigma_{\varrho,1}$ is the space of special Lagrangian subspaces $\mathcal{L}^s_{2n} \cong SU(2n)/SO(2n)$. Take as base point the Lagrangian space $L_0 = \text{Span}\{e_1, \ldots, e_{2n}\}$ of $\mathbb{R}^{4n}$ and let $\tau_0$ be the corresponding conjugation, so that the Cartan embedding of $\mathcal{L}^s_{2n}$ into $SU(2n)$ is given by $\tau = g\tau_0 g^{-1} \mapsto g\tau_0 g^{-1} \in P^s_{\varrho,1}$.

**Lemma 30.** For each $\zeta \in \mathcal{F}(SU(2n)^{s_{\varrho,1}})$ we have $\exp \pi \zeta \in P^s_{\varrho,1}$.

**Proof.** Each $\zeta \in \mathcal{F}(SU(2n)^{s_{\varrho,1}})$ can be written as

$$\zeta = \sum_{i=1}^{n} n_i(E_i - E_{2n+1-i}).$$

Hence, $\exp \pi \zeta = \pi V - \pi V^\perp$, where $V = \bigoplus_{n_i \text{ even}} \text{Span}\{e_i, e_{2n+1-i}\}$. Define $g \in SU(2n)$ as follows: if $n_i$ is even, then $g(e_i) = e_i$ and $g(e_{2n+1-i}) = e_{2n+1-i}$; if $n_i$ is odd, then $g(e_i) = i e_i$ and $g(e_{2n+1-i}) = -i e_{2n+1-i}$. We have $\exp \pi \zeta = g\tau_0 g^{-1} \tau_0$, that is $\exp \pi \zeta \in P^s_{\varrho,1}$. $\square$

Now, identify $\mathcal{L}^s_{2n}$ with $P^s_{\varrho,1}$ via its Cartan embedding. By Theorem [17] any harmonic map $\varphi : S^2 \to P^s_{\varrho,1}$ admits an extended solution $\Phi : S^2 \setminus D \to U^s_{\varrho,1}(SU(2n))$, for some $\zeta' \in \mathcal{F}(SU(2n)) \cap \mathfrak{r}_{\varrho,1}$ and some discrete subset $D$. We can assume that $\zeta'$ is a $g$-semi-canonical element. The corresponding $S^1$-invariant solution $u_{\zeta} \circ \Phi$ takes values in $\Omega_{\zeta}(SU(2n)^{s_{\varrho,1}})$, with $\xi \in \mathcal{F}(SU(2n)^{s_{\varrho,1}})$; and both $\Phi_{-1}$ and $(u_{\zeta} \circ \Phi)_{-1}$ take values in $P^s_{\varrho,1}$. A priori, $\xi$ can be different from $\zeta$ since $\sigma_{\varrho,1}$ is not a fundamental outer involution. However, by Lemma 30 we have $P^s_{\varrho,1} = P^s_{\varrho,1} = P^s_{\varrho,1}$.

If $\zeta$ is a $g$-canonical element such that $\zeta' \preceq \zeta$ and $U_{\zeta',\zeta'}(\Phi)$ is constant, then, taking into account Proposition [20] there exists $T_{\tau}$-invariant extended solution $\tilde{\Phi} : S^2 \setminus D \to U^s_{\zeta}(SU(2n))$, where

$$\tau = \text{Ad}(\exp \pi (\zeta' - \zeta)) \circ \sigma_{\varrho,1}. \quad (42)$$

such that $\tilde{\Phi}_{-1}$ take values in $P^s_{\zeta}$ and $\varphi$ is given up to isometry by

$$\varphi = \exp(\zeta' - \zeta) \tilde{\Phi}_{-1} \tau_0. \quad (43)$$

We conclude that, given a pair $(\zeta, \tau)$, where $\zeta \in \mathcal{F}(SU(2n)^{s_{\varrho}})$ is a $g$-canonical element and $\tau$ is an outer involution of the form (42), any extended solution $\Phi : S^2 \setminus D \to U^s_{\zeta}(SU(2n))$ gives rise via (43) to a harmonic map $\varphi$ from the two-sphere into $\mathcal{L}^s_{2n}$ and, conversely, all harmonic two-spheres into $\mathcal{L}^s_{2n}$ arise in this way.

For $\mathcal{L}^s_4$, since $\exp 2\zeta_1$ belongs to the centre of $SU(4)$, we have five classes of harmonic maps into $\mathcal{L}^s_4$:

$$(2\zeta_1, \sigma_{\varrho,1}), (\zeta_2, \sigma_{\varrho,1}), (2\zeta_1 + \zeta_2, \sigma_{\varrho,1}), (2\zeta_2, \text{Ad} \exp \pi \zeta_2 \circ \sigma_{\varrho,1}), (2\zeta_1 + \zeta_2, \text{Ad} \exp \pi \zeta_2 \circ \sigma_{\varrho,1}).$$

Let us consider in detail the class $(\zeta_2, \sigma_{\varrho,1})$. Clearly $r(\zeta_2) = 2$. Let $W_1$, $W_2$, $W_3$ and $W_4$ be the complex one-dimensional images of $E_1$, $E_2$, $E_3$ and $E_4$, respectively. That is, $W_i = \text{Span}\{v_i\}$, where $v_i$ are defined by (41). Any extended solution $\Phi : S^2 \setminus D \to U^s_{\zeta_2}(SU(2n))$ is given by $\Phi = \exp C \cdot \gamma_{\zeta_2}$, with

$$\gamma_{\zeta_2}(\lambda) = \lambda^{-1} \pi W_4 + \pi W_3 \otimes W_2 + \lambda \pi W_1,$$

for some holomorphic vector-valued function $C : S^2 \setminus D \to (u^0_{\zeta_2})_{\sigma_{\varrho,1}}$, where

$$(u^0_{\zeta_2})_{\sigma_{\varrho,1}} = (p^0_{\zeta_2})_1 \cap \mathfrak{r}^C_{\sigma_{\varrho,1}} + \lambda(p^0_{\zeta_1})_1 \cap \mathfrak{m}^C_{\sigma_{\varrho,1}}.$$
and
\[
(\mathfrak{p}_0^\perp)^{\perp} \cap \mathfrak{p}_{\sigma, 1}^C = (\mathfrak{g}_{L_1 - L_2} \oplus \mathfrak{g}_{L_3 - L_4} \oplus \mathfrak{g}_{L_1 - L_4} \oplus \mathfrak{g}_{L_2 - L_4}) \cap \mathfrak{p}_{\sigma, 1}^C,
\]
\[
(\mathfrak{p}_1^\perp)^{\perp} \cap \mathfrak{m}_{\sigma, 1}^C = \mathfrak{g}_{L_1 - L_4} \cap \mathfrak{m}_{\sigma, 1}^C = \mathfrak{g}_{L_1 - L_4}.
\]
We have \(\sigma_{\hat{g}, 1}(X_{1, 2}) = -X_{3, 4}\) and \(\sigma_{\hat{g}, 1}(X_{1, 3}) = X_{2, 4}\). Hence we can write \(C = C_0 + C_1 \lambda\), with
\[
C_0 = a(X_{1, 2} - X_{3, 4}) + b(X_{1, 3} + X_{2, 4}), \quad C_1 = cX_{1, 4}
\]
for some meromorphic functions \(a, b, c\) on \(S^2\). The harmonicity equations impose that \(ab' - ba' = 0\), which means that \(b = \alpha a\) for some constant \(\beta \in \mathbb{C}\). Hence given arbitrary meromorphic functions \(a, c\) on \(S^2\) and a complex constant \(\alpha\),
\[
\Phi = \begin{pmatrix} 1 & a & \alpha a & c\lambda \\ 0 & 1 & 0 & -\alpha a \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \gamma_{\zeta_1},
\]
is an extended solution associated to an harmonic map in the class \((\zeta_2, \sigma_{\hat{g}, 1})\). Reciprocally, any harmonic map in such class arises in this way.

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