HQC-RMRS, an instantiation of the HQC encryption framework with a more efficient auxiliary error-correcting code

Nicolas Aragon∗ Philippe Gaborit† Gilles Zémor‡

Abstract

The HQC encryption framework is a general code-based encryption scheme for which decryption returns a noisy version of the plaintext. Any instantiation of the scheme will therefore use an error-correcting procedure relying on a fixed auxiliary code. Unlike the McEliece encryption framework whose security is directly related to how well one can hide the structure of an error-correcting code, the security reduction of the HQC encryption framework is independent of the nature of the auxiliary decoding procedure which is publicly available. What is expected from it is that the decoding algorithm is both efficient and has a decoding failure rate which can be easily modelized and analyzed. The original error-correction procedure proposed for the HQC framework was to use tensor products of BCH codes and repetition codes. In this paper we consider another code family for removing the error vector deriving from the general framework: the concatenation of Reed-Muller and Reed-Solomon codes. We denote this instantiation of the HQC framework by HQC-RMRS. These codes yield better decoding results than the BCH and repetition codes: overall we gain roughly 17% in the size of the key and the ciphertext, while keeping a simple modelization of the decoding error rate. The paper also presents a simplified and more precise analysis of the distribution of the error vector output by the HQC protocol.

1 Introduction

The first code-based cryptosystem was proposed in 1978 by McEliece. The proposed framework can be instantiated with any family of error-correcting codes having an efficient decoding algorithm. However, the security of the cryptosystem is highly dependent on the choice of the family. Even though the original instantiation, based on Goppa codes, remains secure, many others (for example using Reed-Muller or Reed-Solomon codes) have

∗XLIM, Université de Limoges
†XLIM, Université de Limoges
‡IMB, Université de Bordeaux
been broken by recovering the hidden structure of the code from the public key. The very nature of this framework makes it difficult to reduce the security of the scheme to a general decoding problem for random codes.

In [1], the authors propose a framework that can be instantiated in different metrics to build cryptosystems whose security is reduced to decoding random quasi-cyclic codes. The instantiation of this framework in the Hamming metric is HQC (Hamming Quasi Cyclic), which was submitted to the second round of the NIST Post-Quantum Standardization workshop. Thanks to the quasi-cyclic structure, the scheme features compact key sizes (about 20,000 bits for a security of 128 bits) as well as fast keygen, encryption and decryption operations.

A public structured error-correcting code $C$ is needed to remove noise inherent to the decryption process. In [1], the authors proposed tensor products of BCH and repetition codes, because encoding is fast and they allow precise DFR analysis. The analysis consists of two steps: first, the weight distribution of the error vector is studied, and then one analyzes how well the chosen codes decode errors of a given weight.

Our contribution in this paper is twofold: we provide a better analysis of the distribution of the weight of the error vector, which allows for a better DFR analysis regardless of which public code is used to decode it, and we propose using a concatenation of Reed-Muller and Reed-Solomon codes to decode the error: this code family allows one to reach a low DFR (for example $< 2^{-128}$) with shorter codes, hence leading to shorter public keys and ciphertexts.

2 Preliminaries

In this section we introduce necessary notation and the description of the HQC scheme. For more details about the protocol and the security proof, we refer the reader to [1].

**Notation:** Throughout this document, $\mathbb{Z}$ denotes the ring of integers and $\mathbb{F}_2$ the binary field. Additionally, we denote by $\omega(\cdot)$ the Hamming weight of a vector $i.e.$ the number of its non-zero coordinates, and by $S_{w}^n(\mathbb{F}_2)$ the set of words in $\mathbb{F}_2^n$ of weight $w$. Formally:

$$S_{w}^n(\mathbb{F}_2) = \{v \in \mathbb{F}_2^n, \text{ such that } \omega(v) = w\}.$$ 

$\mathcal{V}$ denotes the vector space $\mathbb{F}_2^n$ of dimension $n$ over $\mathbb{F}_2$ for some positive $n \in \mathbb{Z}$. Elements of $\mathcal{V}$ can be interchangeably considered as row vectors or polynomials in $\mathcal{R} = \mathbb{F}_2[X]/(X^n - 1)$. Vectors/Polynomials (resp. matrices) will be represented by lower-case (resp. upper-case) bold letters. A prime integer $n$ is said primitive if the polynomial $X^n - 1/(X - 1)$ is irreducible in $\mathbb{F}_2[X]$.

For $u, v \in \mathcal{V}$, we define their product similarly as in $\mathcal{R}$, i.e. $uv = w \in \mathcal{V}$ with

$$w_k = \sum_{i+j \equiv k \mod n} u_i v_j, \text{ for } k \in \{0, 1, \ldots, n - 1\}. \tag{1}$$
Our new protocol takes great advantage of the cyclic structure of matrices. Following [1], \(\text{rot}(h)\) for \(h \in V\) denotes the circulant matrix whose \(i^{th}\) column is the vector corresponding to \(hX^i\). This is captured by the following definition.

**Definition 2.0.1 (Circulant Matrix).** Let \(v = (v_0, \ldots, v_{n-1}) \in \mathbb{F}_2^n\). The circulant matrix induced by \(v\) is defined and denoted as follows:

\[
\text{rot}(v) = \begin{pmatrix}
v_0 & v_{n-1} & \cdots & v_1 \\
v_1 & v_0 & \cdots & v_2 \\
\vdots & \vdots & \ddots & \vdots \\
v_{n-1} & v_{n-2} & \cdots & v_0
\end{pmatrix} \in \mathbb{F}_2^{n \times n}
\]

As a consequence, it is easy to see that the product of any two elements \(u, v \in R\) can be expressed as a usual vector-matrix (or matrix-vector) product using the \(\text{rot}(\cdot)\) operator as

\[
u \cdot v = u \times \text{rot}(v)^\top = (\text{rot}(u) \times v^\top)^\top = v \times \text{rot}(u)^\top = v \cdot u.
\]

We now recall the HQC scheme in figure 1. In [1], the code \(C\) used for decoding is a tensor product of BCH and repetition codes. But since this code is public, its structure has no incidence on security, and one can choose any code family, influencing only the Decryption Failure Rate and the parameter sizes.

- **Setup(1^\lambda)\): generates and outputs the global parameters \(\text{param} = (n, k, \delta, w, w_r, w_e)\).
- **KeyGen(\text{param})\): samples \(h \xleftarrow{} R\), the generator matrix \(G \in \mathbb{F}_2^{k \times n}\) of \(C\), \(sk = (x, y) \xleftarrow{} R^2\) such that \(\omega(x) = \omega(y) = w\), sets \(pk = (h, s = x + h \cdot y)\), and returns \((pk, sk)\).
- **Encrypt(pk, m)\): generates \(e \xleftarrow{} R\), \(r = (r_1, r_2) \xleftarrow{} R^2\) such that \(\omega(e) = w_e\) and \(\omega(r_1) = \omega(r_2) = w_r\), sets \(u = r_1 + h \cdot r_2\) and \(v = mG + s \cdot r_2 + e\), returns \(c = (u, v)\).
- **Decrypt(sk, c)\): returns \(C.\text{Decode}(v - u \cdot y)\).

**Figure 1:** Description of HQC.

### 3 Analysis of the error vector distribution for Hamming distance

From the description of the HQC framework, decryption corresponds to decoding the received vector: \(v - u \cdot y = mG + e'\) for the error vector \(e' = x \cdot r_2 - r_1 \cdot y + e\). In this section we provide a more precise analysis of the error distribution approximation compared...
to [1]. We first compute exactly the probability distribution of each fixed coordinate $e'_k$ of the error vector

$$e' = x \cdot r_2 - r_1 \cdot y + e = (e'_0, \ldots, e'_{n-1}).$$

We obtain that every coordinate $e'_k$ is Bernoulli distributed with parameter $p^* = P[e'_k = 1]$ given by Proposition 3.2.1.

To compute decoding error probabilities, we will then need the probability distribution of the weight of the error vector $e'$ restricted to given sets of coordinates that correspond to codeword supports. We will make the simplifying assumption that the coordinates $e'_k$ of $e'$ are independent variables, which will let us work with the binomial distribution of parameter $p^*$ for the weight distributions of $e'$. This working assumption is justified by remarking that, in the high weight regime relevant to us, since the component vectors $x, y, e$ have fixed weights, the probability that a given coordinate $e'_k$ takes the value 1 conditioned on abnormally many others equalling 1 can realistically only be $\leq p^*$. We support this modeling of the otherwise intractable weight distribution of $e'$ by extensive simulations; these back up our assumption that our computations of decoding error probabilities and DFRs can only be upper bounds on their real values.

3.1 Analysis of the distribution of the product of two vectors

The vectors $x, y, r_1, r_2, e$ have been taken uniformly random and independently chosen among vectors of weight $w$, $w_r$ and $w_e$. We first evaluate the distributions of the products $x \cdot r_2$ and $r_1 \cdot y$.

**Proposition 3.1.1.** Let $x = (x_0, \ldots, x_{n-1})$ be a random vector chosen uniformly among all binary vectors of weight $w$ and let $r = (r_0, \ldots, r_{n-1})$ be a random vector chosen uniformly among all vectors of weight $w_r$ and independently of $x$. Then, denoting $z = x \cdot r$, we have that for every $k \in \{0, \ldots, n-1\}$, the $k$-th coordinate $z_k$ of $z$ is Bernoulli distributed with parameter $\tilde{p} = P(z_k = 1)$ equal to:

$$\tilde{p} = \frac{1}{\binom{n}{w}\binom{n}{n-w_r}} \sum_{\ell \text{ odd} 1 \leq \ell \leq \min(w, w_r)} C_\ell,$$

where $C_\ell = \binom{n}{\ell}\binom{n-\ell}{w_r-\ell}\binom{n-w}{w_r-\ell}$.

**Proof.** The total number of ordered pairs $(x, r)$ is $\binom{n}{w}\binom{n}{w_r}$. Among those, we need to count how many are such that $z_k = 1$. We note that

$$z_k = \sum_{i+j = k \text{ mod } n} x_i r_j.$$

We need therefore to count the number of couples $(x, r)$ such that we have $x_i r_{k-i} = 1$ an odd number of times when $i$ ranges over $\{0, \ldots, n-1\}$ (and $k - i$ is understood modulo
n). Let us count the number $C_{\ell}$ of couples $(x, r)$ such that $x_i r_{k-i} = 1$ exactly $\ell$ times. For $\ell > \min(w, w_r)$ we clearly have $C_{\ell} = 0$. For $\ell \leq \min(w, w_r)$ we have $\binom{n}{\ell}$ choices for the set of coordinates $i$ such that $x_i = r_{k-i} = 1$, then $\binom{n-\ell}{w}$ remaining choices for the set of coordinates $i$ such that $x_i = 1$ and $r_{k-i} = 0$, and finally $\binom{w_r-\ell}{w-w}$ remaining choices for the set of coordinates $i$ such that $x_i = 0$ and $r_{k-i} = 1$. Hence $C_{\ell} = \binom{n}{\ell} \binom{n-\ell}{w} \binom{w_r-\ell}{w-w}$. The formula for $\bar{p}$ follows.

\[ \sum_{\ell=0}^{w} C_{\ell} = \sum_{\ell=0}^{w_r} \binom{n}{\ell} \binom{n-\ell}{w} \binom{w_r-\ell}{w-w} \]

3.2 Analysis of $e'$

Let $x, y$ (resp. $r_1, r_2$) be independent random vectors chosen uniformly among all binary vectors of weight $w$ (resp. $w_r$).

By independence of $(x, r_2)$ with $(y, r_1)$, the $k$-th coordinates of $x \cdot r_2$ and of $r_1 \cdot y$ are independent, and they are Bernoulli distributed with parameter $\bar{p}$ by Proposition 3.1.1. Therefore their modulo 2 sum $t = x \cdot r_2 - r_1 \cdot y$ is Bernoulli distributed with

\[
\begin{align*}
\Pr[t_k = 1] &= 2\bar{p}(1 - \bar{p}), \\
\Pr[t_k = 0] &= (1 - \bar{p})^2 + \bar{p}^2.
\end{align*}
\]

Finally, by adding modulo 2 coordinatewise the two independent vectors $e$ and $t$, we obtain the distribution of the coordinates of the error vector $e' = x \cdot r_2 - r_1 \cdot y + e$ given by the following proposition:

**Proposition 3.2.1.** Let $x, y$ be uniformly chosen among vectors of weight $w$, let $r_1, r_2$ be uniformly chosen among vectors of weight $w_r$, and let $e$ be uniformly chosen among vectors of weight $w_e$. We suppose furthermore that the random vectors $x_v, y, r_1, r_2, e$ are independent. Let $e' = x \cdot r_2 - r_1 \cdot y + e = (e'_0, \ldots, e'_{n-1})$. Then, for every $k \in \{0, \ldots, n-1\}$ we have:

\[
\begin{align*}
\Pr[e'_k = 1] &= 2\bar{p}(1 - \bar{p})(1 - \frac{w_e}{n}) + ((1 - \bar{p})^2 + \bar{p}^2) \frac{w_e}{n}, \\
\Pr[e'_k = 0] &= ((1 - \bar{p})^2 + \bar{p}^2)(1 - \frac{w_e}{n}) + 2\bar{p}(1 - \bar{p}) \frac{w_e}{n}.
\end{align*}
\]

**Proof.** The vectors $x \cdot r_2, r_1 \cdot y$ and $e$ are clearly independent. The $k$-th coordinate of $e$ is Bernoulli distributed with parameter $w_e/n$. The random Bernoulli variable $e'_k$ is therefore the sum modulo 2 of three independent Bernoulli variables of parameters $\bar{p}$ for the first two and of parameter $w_e/n$ for the third one. The formula therefore follows standardly. \qed

Proposition 3.2.1 gives us the probability that a coordinate of the error vector $e'$ is 1. In our simulations, which occur in the regime $w = \alpha \sqrt{n}$ with constant $\alpha$, we make the simplifying assumption that the coordinates of $e'$ are independent, meaning that the weight of $e'$ follows a binomial distribution of parameter $p^*$, where $p^*$ is defined as in Eq. (5): $p^* = 2\bar{p}(1 - \bar{p})(1 - \frac{w_e}{n}) + ((1 - \bar{p})^2 + \bar{p}^2) \frac{w_e}{n}$. This approximation will give us, for $0 \leq d \leq \min(2 \times w \times w_r + w_e, n)$,

\[
\Pr[\omega(e') = d] = \binom{n}{d} (p^*)^d (1 - p^*)^{(n-d)}.
\]
3.3 Supporting elements for our modelization

we give in Fig. 2 and 3 simulations of the distribution of the weight of the error vector together with the distribution of the associated binomial law of parameters \( p^\star \). These simulations show that error vectors are more likely to have a weight close to the mean than predicted by the binomial distribution, and that on the contrary the error is less likely to be of large weight than if it were binomially distributed. This is for instance illustrated on parameters sets I and II corresponding to real parameters used for 128 bits security. For cryptographic purposes we are mainly interested by very small DFR and large weight occurrences which are more likely to induce decoding errors. These tables show that the probability of obtaining a large weight is close but smaller for the error weight distribution of \( e' \) rather than for the binomial approximation. This supports our modelization and the fact that computing the decoding failure probability with this binomial approximation permits to obtain an upper bound on the real DFR. This will be confirmed in the next sections by simulations with real weight parameters (but smaller lengths).

Examples of simulations. We consider two examples of parameters in Table 1: Parameter sets I and II which correspond to cryptographic parameters and for which we simulate the error distribution versus the binomial approximation together with the probability of obtaining large error weights. In order to follow [1] we computed vectors of length \( n \) (the blocksize of the double circulant code) and then, for \( n_1n_2 \) the length of the auxiliary error correcting code \( C \), we truncated the last \( l = n - n_1n_2 \) bits before measuring the Hamming weight of the vectors.

| Parameter set | \( w \) | \( w_r = w_e \) | \( n \) | \( n_1n_2 \) | \( p^\star \) |
|---------------|-------|--------------|-------|------------|----------|
| I             | 67    | 77           | 23,869| 23746      | 0.2918   |
| II            | 67    | 77           | 20,533| 20480      | 0.3196   |

Table 1: Parameters sets I and II used for simulations.

Simulations for Parameter set I. Simulation results are shown figure 2. We computed the weights such that 0.1%, 0.01% and 0.001% of the vectors are of weight greater than this value, to study how often extreme weight values occur in Table 2.

|                      | 0.1% | 0.01% | 0.001% | 0.0001% |
|----------------------|------|-------|--------|---------|
| Error vectors        | 7101 | 7134  | 7163   | 7190    |
| Binomial approximation| 7147 | 7191  | 7228   | 7267    |

Table 2: Simulated probabilities of large weights for Parameter Set I for the distributions of the error vector and the binomial approximation.
Figure 2: Comparison between error $e'$ generated using parameter set I and its binomial approximation.

**Simulations for Parameter set 2.** Simulation results are shown on figure [3]. We perform the same analysis as for the parameter set I about extreme weight values in Table 3.

Simulation results are shown on figure [3]. We perform the same analysis as for the parameter set I about extreme weight values in Table 2.

|                | 0.1% | 0.01% | 0.001% | 0.0001% |
|----------------|------|-------|--------|---------|
| Error vectors  | 6715 | 6749  | 6779   | 6808    |
| Binomial approx | 6753 | 6796  | 6834   | 6859    |

Table 3: Simulated probabilities of large weights for Parameter Set II for the distributions of the error vector and the binomial approximation.

As we can see from these, extreme weight values seem to happen more often in the case of the binomial approximation. Since these cases are the most likely to lead to decoding failure, this approximation should lead to conservative decryption failure rate estimations.

**Comparison with the previous analysis in [1]:** the present analysis is better than the previous one, in practice in the case of decoding with BCH and repetition codes for security parameter 128 bits, the present analysis leads to a DFR in $2^{-154}$ when the previous one lead to $2^{-128}$. In practice this allows to reduce by 3% the key size in the case of the BCH-repetition code decoder of [1].
Figure 3: Comparaison between error $e'$ generated using parameter set II and its binomial approximation.

4 Proposition of new auxiliary error-correcting codes: Reed-Muller and Reed-Solomon concatenated codes

In this section we study the impact of using a new family of auxiliary error-correcting codes: instead of the tensor product codes used in the HQC cryptosystem framework we propose to consider the concatenation of Reed-Muller and Reed-Solomon codes. We denote this instanciation of the HQC framework by HQC-RMRS.

4.1 Construction

Definition 4.1.1. [Concatenated codes]

A concatenated code consists of an external code $[n_e, k_e, d_e]$ over $\mathbb{F}_q$ and an internal code $[n_i, k_i, d_i]$ over $\mathbb{F}_2$, with $q = 2^{k_i}$. We use a bijection between elements of $\mathbb{F}_q$ and the words of the internal code, this way we obtain a transformation:

$$\mathbb{F}_q^{n_e} \rightarrow \mathbb{F}_2^N$$

where $N = n_e n_i$. The external code is thus transformed into a binary code of parameters $[N = n_e n_i, K = k_e k_i, D \geq d_e d_i]$.

For the external code, we chose a Reed-Solomon code of dimension 32 over $\mathbb{F}_{256}$ and, for the internal code, we chose the Reed-Muller code $[128, 8, 64]$ that we are going to
duplicate between 2 and 6 times (i.e. duplicating each bit to obtain codes of parameters [256, 8, 128], [512, 8, 256], [786, 8, 384]).

**Decoding:** We perform maximum likelihood decoding on the internal code. This yields a vector of $\mathbb{F}_q^n$ that we then decode using an algebraic decoder for the Reed-Solomon code.

**Decoding the internal Reed-Muller code:** The Reed-Muller code of order 1 can be decoded using a fast Hadamard transform (see chapter 14 of [2]). The algorithm needs to be slightly adapted when decoding duplicated codes. For example, if the Reed-Muller of length $128 = 2^7$ is duplicated three times, we create the function $F : \mathbb{F}_2^7 \to \{3, 1, -1, -3\}$ (which can be thought of as a 128-tuple of symbols from $\{3, 1, -1, -3\}$) by transforming every block of three bits $y_1y_2y_3$ of the received vector of length 384 to

$$(-1)^{y_1} + (-1)^{y_2} + (-1)^{y_3}.$$  

We then apply the Hadamard transform to the output of the function $F$. We take the maximum value in $\hat{F}$ and $x \in \mathbb{F}_2^{128}$ that maximizes the value of $|\hat{F}|$. If $\hat{F}(x)$ is positive, then the closest codeword is $xG$ where $G$ is the generator matrix of the Hadamard code (without the all-one-vector). If $\hat{F}(x)$ is negative, then we need to add the all-one-vector to it.

### 4.2 Decryption failure rate analysis

We now consider the decoding failure rate of the concatenated code which also corresponds to the decryption failure rate of the encryption scheme. We first provide two bounds on the maximum likelihood decoding error probability of the duplicated Reed-Muller code: a first simple union bound and a second more accurate one. These bounds can then be plugged into the decoding error probability for the bounded distance decoder of the Reed-Solomon code.

**Proposition 4.2.1. [Simple Upper Bound for the DFR of the internal code]**

Let $p$ be the transition probability of the binary symmetric channel. Then the DFR of a duplicated Reed-Muller code of dimension 8 and minimal distance $d_i$ can be upper bounded by:

$$p_i = 255 \sum_{j=d_i/2}^{d_i} \binom{d_i}{j} p^j (1-p)^{d_i-j}$$

**Proof.** For any linear code $C$ of length $n$, when transmitting a codeword $c$, the probability that the channel makes the received word $y$ at least as close to a word $c' = c + x$ as $c$ (for $x$ a non-zero word of $C$ and $|x|$ the weight of $x$) is:

$$\sum_{j \geq |x|/2} \binom{|x|}{j} p^j (1-p)^{n-j}.$$
By the union bound applied on the different non-zero codewords $x$ of $C$, we obtain that the probability of a decryption failure can thus be upper bounded by:

$$
\sum_{x \in C} \sum_{j \geq |x|/2} \binom{|x|}{j} p^j (1 - p)^{n-j}
$$

There are 255 non-zero words in a $[128,8,64]$ Reed-Muller code, 254 of weight 64 and one of weight 128. The contribution of the weight 128 vector is smaller than the weight 64 vectors, hence by applying the previous bound to duplicated Reed-Muller codes we obtain the result.

**Better upper bound on the decoding error probability for the internal code.**

The previous simple bound pessimistically assumes that decoding fails when more than one codeword minimizes the distance to the received vector. The following bound improves the previous one by taking into account the fact that decoding can still succeed with probability $1/2$ when exactly two codewords minimize the distance to the received vector.

**Proposition 4.2.2. [Improved Upper Bound for the DFR of the internal code]**

Let $p$ be the transition probability of the binary symmetric channel. Then the DFR of a Reed-Muller code of dimension 8 and minimal distance $d_i$ can be upper bounded by:

$$
p_i = \frac{1}{2} 255 \left( \frac{d_i}{d_i/2} \right) p^{d_i/2} (1 - p)^{d_i/2}
+ 255 \sum_{j=d_i/2+1}^{d_i} \left( \frac{d_i}{j} \right) p^j (1 - p)^{d-j}
+ \frac{1}{2} \left( \frac{255}{2} \right)^{d_i/2} \sum_{j=0}^{d_i/2} \left( \frac{d_i/2}{j} \right) p^{d_i-j} (1 - p)^{d_i/2+j}
$$

**Proof.** Let $E$ be the decoding error event. Let $e$ be the error vector.

- Let $A$ be the event where the closest non-zero codeword $c$ to the error is such that $d(e, c) = d(e, 0) = |e|$.
- Let $B$ be the event where the closest non-zero codeword $c$ to the error vector is such that $d(e, c) < |e|$.
- Let $A' \subset A$ be the event where the closest non-zero codeword $c$ to the error vector is such that $d(e, c) = |e|$ and such a vector is unique, meaning that for every $c' \in C, c' \neq c, c' \neq 0$, we have $d(e, c') > |e|$.
- Finally, let $A''$ be the event that is the complement of $A'$ in $A$, meaning the event where the closest nonzero codeword $c$ to the error is at distance $|e|$ from $e$, and there exists at least one codeword $c', c' \neq c, c' \neq 0$, such that $d(e, c') = d(e, c) = |e|$.
The probability space is partitioned as $\Omega = A \cup B \cup C = A' \cup A'' \cup B \cup C$, where $C$ is the complement of $A \cup B$. When $C$ occurs, the decoder always decodes correctly, i.e. $P(E|C) = 0$. We therefore write:

$$P(E) = P(E|A')P(A') + P(E|A'')P(A'') + P(E|B)P(B)$$

When the event $A'$ occurs, the decoder chooses at random between the two closest codewords and is correct with probability $1/2$, i.e. $P(E|A') = 1/2$. We have $P(E|B) = 1$ and writing $P(E|A'') \leq 1$, we have:

$$P(E) \leq \frac{1}{2}P(A') + \frac{1}{2}P(A'') + P(B)$$

Now we have the straightforward union bounds:

$$P(B) \leq 255 \sum_{j=d_i/2+1}^{d_i} \binom{d_i}{j} p^j (1-p)^{d_i-j}$$

$$P(A) \leq 255 \binom{d_i}{d_i/2} p^{d_i/2} (1-p)^{d_i/2}$$

and it remains to find an upper bound on $P(A'')$.

We have:

$$P(A'') \leq \sum_{c,c'} P(A_{c,c'})$$

where the sum is over pairs of distinct nonzero codewords and where:

$$A_{c,c'} = \{d(e, c) = d(e, c') = |e|\}$$

This event is equivalent to the error meeting the supports of $c$ and $c'$ on exactly half their coordinates. All codewords except the all-one vector have weight $d_i$, and any two codewords of weight $d_i$ either have non-intersecting supports or intersect in exactly $d/2$ positions. $P(A_{c,c'})$ is largest when $c$ and $c'$ have weight $d$ and non-zero intersection. In this case we have:

$$P(A_{c,c'}) = \sum_{j=0}^{d_i/2} \binom{d_i/2}{j}^3 p^{d_i-j} (1-p)^{d_i/2+j}$$

Hence
\[ P(A') \leq \sum_{c,c'} P(A_{c,c'}) \leq \left( \frac{255}{2} \right) \sum_{j=0}^{d_i/2} \left( \frac{d_i}{2} \right)^3 p^{d_i-j}(1-p)^{d_i/2+j} \]  

(10)

Plugging 9, 8 and 10 into 7 we obtain the result.

Remark 4.1. Propositions 4.2.1 and 4.2.2 give upper bounds on the Decryption Failure Rate for the internal code. The smaller the DFR, the closer the bounds become to the real value. We give a comparison of the bounds from 4.2.1 and 4.2.2 and the actual DFR for [256, 8, 128], [512, 8, 256] and [768, 8, 384] duplicated Reed-Muller using \( p^* \) values from actual parameters. Simulation results are presented in Table 4.

| Security level | \( p^* \) | Reed-Muller code | DFR from 4.2.1 | DFR from 4.2.2 | Observed DFR |
|----------------|-----------|------------------|---------------|---------------|--------------|
| 128            | 0.3196    | [256, 8, 128]    | -7.84         | -8.03         | -8.72        |
| 192            | 0.3535    | [512, 8, 256]    | -11.81        | -12.12        | -12.22       |
| 256            | 0.3728    | [768, 8, 384]    | -13.90        | -14.20        | -14.25       |

Table 4: Comparison between the observed Decryption Failure Rate and the formula from proposition 4.2.1. Results are presented as \( \log_2(DFR) \).

Remark 4.2. Propositions 4.2.1 and 4.2.2 have been derived with a binary symmetric channel model for the distribution of the HQC error vector restricted to the support of a (duplicated) Reed-Muller code. Figure 4 compares the actual weight distribution of the error vector to the binomial distribution when restricted to this relatively small number of bits. We observe that they are virtually identical, meaning that a small proportion of HQC bits do behave as i.i.d Bernoulli variables.

Theorem 4.3. [Decryption Failure Rate of the concatenated code]

Using a Reed-Solomon code \([n_e, k_e, d_e]_{256}\) as the external code, the DFR of the concatenated code can be upper bounded by:

\[ \sum_{l=\delta_e+1}^{n_e} \binom{n_e}{l} p_i^l (1-p_i)^{n_e-l} \]

Where \( d_e = 2\delta_e + 1 \) and \( p_i \) is defined as in Proposition 4.2.1

4.3 Simulation results

We tested the Decryption Failure rate of the concatenated codes against both binary symmetric channels and HQC vectors. For Reed-Muller codes, rather than considering the upper bound approximation we effectively decoded the code, which means than in practice the upper bound that we use for our theoretical DFR, is greater than what is obtained in the simulations. Simulation results are presented on Figure 5. These results show that
the DFR of the encryption scheme is smaller than the simulated error with a binomial distribution which is itself smaller than the DFR derived from the bound on the internal duplicated Reed-Muller code.

4.4 Proposed parameters

From the DFR analysis we derive new parameters for the HQC-RMRS cryptosystem. These are described on Figure 6.

5 Conclusion

In Section 3 we presented a better analysis of the error weight distribution for HQC, which leads to a better DFR estimation. This can be used to reduce the size of the parameters, no matter what family of codes is used for decoding. In Section 4 we propose using a concatenation of Reed-Muller and Reed-Solomon codes and we provide an upper bound on the DFR in this setting. This family allows us to reduce the public key and ciphertext sizes by about 17% when compared to the tensor product of BCH and repetition codes (when considering the same error weight distribution).
Figure 5: Comparison of the Decryption Failure Rate of concatenated codes against approximation by a binary symmetric channel and against HQC error vectors. Parameters simulated are derived from those of HQC-RMRS for 128 security bits: $w = 67$, $w_r = w_e = 77$, a $[256, 8, 128]$ duplicated Reed-Muller code for internal code and a $[NRS, 32]$ Reed-Solomon code for external code.

| Instance         | security | $w$  | $w_e = w_r$ | Reed-Muller | Reed Solomon | $n$    | $DFR$     | Gain over |
|------------------|----------|------|-------------|--------------|--------------|--------|-----------|-----------|
| HQC-RMRS-128     | 128      | 67   | 77          | $[256, 8, 128]$ | $[80, 32, 49]$ | 20,533 | $< 2^{-128}$ | 16.8%     |
| HQC-RMRS-192     | 192      | 101  | 117         | $[512, 8, 256]$ | $[76, 32, 45]$ | 38,923 | $< 2^{-192}$ | 16.7%     |
| HQC-RMRS-256     | 256      | 133  | 153         | $[768, 8, 384]$ | $[78, 32, 47]$ | 59,957 | $< 2^{-256}$ | 15.4%     |

Figure 6: New proposed parameters for the HQC-RMRS cryptosystem (security is in bits).

References

[1] Carlos Aguilar-Melchor, Olivier Blazy, Jean-Christophe Deneuville, Philippe Gaborit, and Gilles Zémor. Efficient encryption from random quasi-cyclic codes. *IEEE Transactions on Information Theory*, 64(5):3927–3943, 2018. [2] F.J. MacWilliams and N.J.A. Sloane. *The theory of error-correcting codes*. North-Holland, 1977. [9]