\textbf{L log log L VERSIONS OF STEIN’S AND ZYGMUND’S THEOREMS FOR THE HARDY SPACE $H^{\log}(\mathbb{R}^d)$}

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Abstract. We obtain versions of some classical results of Zygmund and Stein for functions belonging to the Hardy space $H^{\log}(\mathbb{R}^d)$ introduced by Bonami, Grellier, and Ky. We present further applications in the context of more general Orlicz spaces. This yields slight extensions of results previously obtained by Bonami-Madan, Iwaniec-Verde, and others.

1. Introduction

The Hardy-Littlewood maximal function is a fundamental object in harmonic analysis, defined for a locally integrable function $f: \mathbb{R}^d \to \mathbb{C}$ by setting

$$M(f)(x) := \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)|dy \quad x \in \mathbb{R}^d,$$

where $B(x, r)$ denotes the open ball in $\mathbb{R}^d$ centered at $x$ with radius $r > 0$ and $|A|$ denotes the Lebesgue measure of $A \subseteq \mathbb{R}^d$. It is a basic fact that the mapping $f \mapsto M(f)$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$. The maximal operator is also bounded from $L^1(\mathbb{R}^d)$ to weak-$L^1$, but does not map $L^1(\mathbb{R}^d)$ to itself (see, for instance, [14] for an in-depth discussion).

However, $M(f)$ is locally integrable provided $f$ is compactly supported and satisfies the $L \log L$ condition

$$\int_{\mathbb{R}^d} |f(x)| \log^+ |f(x)| dx < \infty,$$

where, as usual, $\log^+ |x| = \max\{\log |x|, 0\}$. In a 1969 paper, E.M. Stein [12] proved that this $L \log L$ condition is both sufficient and necessary for integrability of the Hardy-Littlewood maximal function, in the following sense: if $f$ is supported in some finite ball $B = B(r)$ of radius $0 < r < \infty$, then

$$\int_{B} M(f)dx < \infty \quad \text{if, and only if,} \quad \int_{B} |f(x)| \log^+ |f(x)| dx < \infty.$$

Another classical result that involves the space $L \log L$ is due to Zygmund, and asserts that the periodic Hilbert transform maps $L \log L(\mathbb{T})$ to $L^1(\mathbb{T})$; see e.g. Theorem 2.8 in Chapter VII of [16]. This implies that $L \log L(\mathbb{T})$ is contained in the real Hardy space $H^1(\mathbb{T})$ consisting of integrable functions on the torus whose Hilbert transforms are integrable. Moreover, as shown by Stein in [12], Zygmund’s theorem has a partial converse, namely if $f \in H^1(\mathbb{T})$ and $f$ is non-negative, then $f$ necessarily belongs to $L \log L(\mathbb{T})$. Therefore, in view of the aforementioned results of

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Zygmund and Stein, the Hardy space $H^1(\mathbb{T})$ is, in terms of magnitude, associated with the Orlicz space $L\log L(\mathbb{T})$.

In this note, we obtain versions of these results for the Musielak-Orlicz Hardy space $H^{\log}(\mathbb{R}^d)$ that was recently introduced by A. Bonami, S. Grellier, and L.D. Ky in [3] and further studied by Ky in [9]. See also [2] and [15]. To do this, we identify the correct analog of $K_y$ in [3] and further studied by Ky in [9]. See also [2] and [15]. To do this, we identify the correct analog of $K_y$ in [3] and further studied by Ky in [9].

We shall also fix a non-negative function $\phi \in C^\infty(\mathbb{R}^d)$, which is supported in the unit ball of $\mathbb{R}^d$ and has $\int_{\mathbb{R}^d} \phi(y)dy = 1$ and $\phi(x) = c_0$ for all $|x| \leq 1/2$, where $c_0$ is a constant. Given an $\epsilon > 0$, we use the standard notation $\phi_\epsilon(x) := \epsilon^{-d}\phi(\epsilon^{-1}x)$, $x \in \mathbb{R}^d$.

**Definition ($H^{\log}$, see [2, 13]).** If $\phi$ is as above, consider the maximal function

$$M_\phi(f)(x) := \sup_{\epsilon > 0} |(f * \phi_\epsilon)(x)|, \quad x \in \mathbb{R}^d.$$

The Hardy space $H^{\log}(\mathbb{R}^d)$ is defined to be the space of tempered distributions $f$ on $\mathbb{R}^d$ such that $M_\phi(f) \in L_{\Psi}(\mathbb{R}^d)$, that is, $M_\phi(f)$ satisfies

$$\int_{\mathbb{R}^d} \Psi(x, M_\phi(f)(x)) dx < \infty.$$

The motivation for defining the space $H^{\log}$ comes from the study of products of functions in the real Hardy space $H^1(\mathbb{R}^d)$ and functions in $\text{BMO}(\mathbb{R}^d)$, the class of functions of bounded mean oscillation. Following earlier work by Bonami, T. Iwaniec, P. Jones, and M. Zinsmeister in [2], it was shown by Bonami, Grellier, and Ky [3] that the product $fg$, in the sense of distributions, of a function $f \in H^1(\mathbb{R}^d)$ and a function $q \in \text{BMO}(\mathbb{R}^d)$ can be represented as a sum of a continuous bilinear mapping into $L^1(\mathbb{R}^d)$ and a continuous bilinear operator into $H^{\log}(\mathbb{R}^d)$.

Here is our version of Stein’s lemma for $L_{\Psi}$.

**Theorem 1.** Let $f$ be a measurable function supported in a closed ball $B \subseteq \mathbb{R}^d$.

Then $M(f) \in L_{\Psi}(B)$ if, and only if, $f \in L \log \log L(B)$.

Our proof in fact leads to a more general version of Theorem 1. We discuss this, and give a proof of Theorem 1 in Section 2.

Next is the analog of Zygmund’s result for $H^{\log}(\mathbb{R}^d)$.

**Theorem 2.** Let $B$ denote the closed unit ball in $\mathbb{R}^d$.

If $f$ is a measurable function satisfying $f \in L \log \log L(B)$ and $\int_B f(y)dy = 0$, then $f \in H^{\log}(\mathbb{R}^d)$.
We remark that the mean-zero condition in the hypothesis is in fact necessary in order to place a compactly supported function in $H^\log$. The proof of Theorem 2 is presented in Section 3.

In Section 4, we discuss further extensions to the periodic setting.

Remark 3. After posting a first version of this note, the authors were informed that our main results can be derived from results previously obtained in the setting of Orlicz spaces; see for instance [4, 8]. We are grateful for having been directed to the appropriate sources. In this note, we give a self-contained account, including a discussion of sharpness, and indicate some minor modifications that need to be made to obtain results in the Musielak-Orlicz setting.

2. Proof of the Stein-type Theorem for $L_\Psi$ and further extensions

We begin with an elementary observation that will be implicitly used several times in the sequel: if $\Phi : [0, \infty) \to [0, \infty)$ is an increasing function, then for every positive constant $\alpha_0$ one has

$$\int_B \Phi(|g(x)|)dx \leq \Phi(\alpha_0)|B| + \int_{\{|g(x)| > \alpha_0\}} \Phi(|g(x)|)dx$$

for each measurable set $B$ in $\mathbb{R}^d$ with finite measure.

We now turn to the proof of our first theorem.

Proof of Theorem. Assume first that $f \in L \log \log L(B)$. The main observation is that locally the space $L_\Psi$ essentially coincides with the Orlicz space defined in terms of the function $\Psi(t) := t \cdot [\log(e + t)]^{-1}$, $t \geq 0$ and so, one can employ the arguments of Stein [12]. In view of this observation, we remark that the fact that $f \in L \log \log L(B)$ implies $M(f) \in L_{\Psi_0}(B)$ is well-known; see for instance [4, p.242], [8, Sections 4 and 7]. We shall also include the proof of this implication here for the convenience of the reader.

To be more precise, we note that for $x \in B$ one has

$$\log(e + M(f)(x)) \leq \log((e + |x|)(e + M(f)(x))) \leq c \log(e + M(f)(x)),$$

for a constant $c$ that only depends on $B$. Next, an integration by parts yields

$$\int_e^y \frac{1}{\log \alpha} d\alpha = \frac{y}{\log y} - e + \int_e^y \frac{1}{\log^2 \alpha} d\alpha,$$

so that

$$\frac{y}{\log y} \leq e + \int_e^y \frac{1}{\log \alpha} d\alpha, \quad \text{for } y > e.$$

Together, these two observations imply that

$$\int_B \Psi(x, M(f)(x))dx \lesssim_B 1 + \int_{B \setminus \{M(f) > e\}} \left( \int_{\{M(f) > e\}} \frac{1}{\log \alpha} d\alpha \right) dx + \int_e^\infty \frac{1}{\log \alpha} \cdot |\{x \in B : M(f)(x) > \alpha\}| d\alpha.$$

To estimate the last integral, note that there exists an absolute constant $C_d > 0$ such that

$$|\{x \in \mathbb{R}^d : M(f)(x) > \alpha\}| \leq \frac{C_d}{\alpha} \int_{\{f > \alpha/2\}} |f(x)| dx$$
for all $\alpha > 0$; see e.g. [12, (5)] or Section 5.2 (a) in Chapter I in [13]. We thus deduce from (2.2) that

$$\int_B \Psi(x, M(f)(x)) \, dx \lesssim B \left[ 1 + \int_B |f(x)| \cdot \left( \int_{e^\alpha}^{2|f(x)|} \frac{1}{\alpha \log \alpha} \, d\alpha \right) \right] dx$$

$$\lesssim 1 + \int_B |f(x)| \log \log^+ |f(x)| \, dx,$$

which implies that $M(f) \in L_\Psi(B)$.

To prove the reverse implication, assume that for some $f$ supported in $B$ with $f \in L^1(B)$ we have $M(f) \in L_\Psi(B)$. Our task is to show that $f \in L_{\log \log L}(B)$. In order to accomplish this, we shall make use of the fact that there exists a $\rho > 2$, depending only on $\|f\|_{L^1(B)}$ and $B$, such that we also have $M(f) \in L_{\Psi(\rho B)}$ and moreover, for every $\alpha \geq e^e$,

$$\{|x \in \rho B : M(f)(x) > c_1 \cdot \alpha\} \geq \frac{c_2}{\alpha} \int_{|f| > \alpha} |f(x)| \, dx,$$

(2.3)

where $c_1, c_2$ are positive constants that can be taken to be independent of $\rho$ and $\alpha$. Indeed, arguing as in the proof of [12, Lemma 1], note that for every $r > 2$ one has

$$M(f)(x) \lesssim \frac{1}{(r-1)^d |B|} \cdot \|f\|_{L^1(B)} \quad \text{for all } x \in \mathbb{R}^d \setminus rB.$$

Hence, if we choose $\rho > 2$ to be large enough, then $M(f)(x) < e^e \leq \alpha$ for all $x \in \mathbb{R}^d \setminus \rho B$ and so, (2.3) follows from [12, Inequality (6)].

Furthermore, one can check that $M(f) \in L_\Psi(\rho B)$. Indeed, if we write $B = B(x_0, r_0)$ then, as in [12], it follows from the definition of $M$ and the fact that $\supp(f) \subseteq B$ that there exists a constant $c_0 > 0$, depending only on the dimension, such that for every $x \in 2B \setminus B$ one has

$$M(f)(x) \leq c_0 \cdot M(f) \left( x_0 + r_0^2 \cdot \frac{x - x_0}{|x - x_0|^2} \right)$$

(2.5)

and so, $M(f) \in L_\Psi(2B)$. To show that (2.5) implies that $M(f) \in L_\Psi(B)$, observe first that the function $\Psi_0(s) = s/\log(e + s)$ is increasing on $[0, +\infty)$, and for all $t \geq 1$ and all $s > 0$,

$$1 \geq \frac{\log(e + s)}{\log(e + ts)} = \frac{\log(e + s)}{\log(e + s) + \log t} \geq \frac{\log(e + s)}{\log(e + s) + \log t} \geq \frac{1}{1 + \log t},$$

so $\Psi_0$ satisfies

$$t(1 + \log t)^{-1} \Psi_0(s) \leq \Psi_0(ts) \leq t \Psi_0(s),$$

(2.6)

which implies that for all $c > 0$ and all $s > 0$

$$\Psi_0(cs) \sim_c \Psi_0(s).$$
Observe that a change to polar coordinates, followed by another a change of variables and elementary estimates yield
\[
\int_{2B \setminus B} \Psi_0(M f(x)) \, dx \lesssim \int_{r_0}^{2r_0} s^{d-1} \int_{S^{d-1}} \Psi_0(M f(x_0 + r_0 \theta/s)) \, d\sigma(\theta) \, ds \\
\sim r_0^{-1} \int_{1/2}^1 t^{-1-d} \int_{S^{d-1}} \Psi_0(M f(x_0 + r_0 t \theta)) \, d\sigma(\theta) \, dt \\
\sim \int_{1/2}^1 t^{-d-1} \int_{S^{d-1}} \Psi_0(M f(x_0 + r_0 t \theta)) \, d\sigma(\theta) \, dt \\
\lesssim \int_{B} \Psi(x, M f(x)) \, dx.
\]

Moreover, we deduce from (2.4) that \( M(f) \) belongs to \( L_\Phi(\rho B \setminus 2B) \) and it thus follows that \( M(f) \in L_\Phi(\rho B) \), as desired.

Next, note that by the same reasoning as in the proof of sufficiency and by Fubini’s theorem,
\[
\int_{\rho B} \Psi(x, M(f)(x)) \, dx \gtrsim \int_{\rho B \setminus \{ M(f) > \max\{ e^c, |x_0| + r_0 \} \}} \frac{M(f)(x)}{\log(M(f)(x))} \, dx \\
\gtrsim \int_{\rho B \setminus \{ M(f) > \max\{ e^c, |x_0| + r_0 \} \}} \left( \int_{e^c}^{M(f)(x)} \frac{1}{\log \alpha} \, d\alpha \right) \, dx \\
\gtrsim \int_{\max\{ e^c, |x_0| + r_0 \}}^\infty \frac{1}{\log \alpha} \cdot |\{ x \in \rho B : M(f)(x) > c_2 \cdot \alpha \}| \, d\alpha.
\]

By using (2.3), we now get
\[
\infty > \int_{\rho B} \Psi(x, M(f)(x)) \, dx \gtrsim \int_{B} |f(x)| \cdot \left( \int_{\max\{ e^c, |x_0| + r_0 \}}^{\log M(f)(x)} \frac{1}{\alpha \log \alpha} \, d\alpha \right) \, dx \\
\gtrsim 1 + \int_{B} |f(x)| \log^+ \log^+ |f(x)| \, dx
\]
and this completes the proof of Theorem 4. □

Remark 4. Let \( B_0 \) denote the closed unit ball in \( \mathbb{R}^d \). Given a small \( \delta \in (0, e^{-c}) \), if, as on pp. 58–59 in [2], one considers \( f := \delta^{-d} \chi_{\{|x| < \delta\}} \) then \( M(f)(x) \sim |x|^{-d} \) for all \( |x| > 2\delta \) and so,
\[
(2.7) \quad \int_{B_0} |f(x)| \log^+ \log^+ |f(x)| \, dx \sim \log(\log(\delta^{-1})) \sim \int_{B_0} \Psi(x, M(f)(x)) \, dx.
\]

This shows that given \( L_\Phi(B_0) \), the space \( L \log \log L(B_0) \) in the statement of Theorem 4 is best possible in general, in terms of size.

Indeed, the left-hand side of (2.7) follows by direct calculation. On the other hand, using (2.1), (2.6), a change to polar coordinates, and further change of variables yield
\[
\int_{B_0} \Psi(x, M(f)(x)) \, dx \sim 1 + \int_{2\delta}^{1} \frac{1}{\log(e + s^{-d})} \, ds \\
\sim 1 + \int_{2}^{(28)^{-1}} \frac{1}{\log(e + u^{-d})} \, du \sim 1 + \int_{e}^{(28)^{-1}} \frac{1}{\log(u)} \, du,
\]
from where the right-hand side of (2.7) follows.
2.1. Further generalizations. Assume that $\Psi : \mathbb{R}^d \times [0, \infty)$ is a non-negative function satisfying the following properties:

(1) For every $x \in \mathbb{R}^d$ fixed, $\Psi(x, t) = \Psi_x(t)$ is Orlicz in $t \in [0, \infty)$, namely $\Psi_x(0) = 0$, $\Psi_x$ is increasing on $[0, \infty)$ with $\Psi_x(t) > 0$ for all $t > 0$ and $\Psi_x(t) \to \infty$ as $t \to \infty$.

Moreover, assume that there exists an absolute constant $C_0 > 0$ such that $\Psi_x(2t) \leq C_0 \Psi_x(t)$ for all $x \in \mathbb{R}^d$ and every $t \in [0, \infty)$.

(2) If $K$ is a compact set in $\mathbb{R}^d$, then there exist $x_1, x_2 \in K$ and a constant $C_K > 0$ such that

$$C_K^{-1} < \Psi(x_1, t) \leq \Psi(x, t) \leq \Psi(x_2, t) < C_K$$

for every $x \in K$ and for all $t > 0$.

(3) If we write $\Psi(x, t) = \Psi_x(t) = \int_0^t \psi_x(s) ds$, then for every $\alpha_0, \beta_0$ with $0 < \alpha_0 < \beta_0$ one has

$$\int_{\alpha_0}^{\beta_0} \frac{\psi_x(s)}{s} ds < \infty$$

for every $x \in \mathbb{R}^d$.

By carefully examining the proof of Theorem 1, one obtains the following result.

**Theorem 5.** Let $\Psi(x, t) = \int_0^t \psi_x(s) ds$, $(x, t) \in \mathbb{R}^d \times [0, \infty)$, be as above.

Fix a closed ball $B$ with $B \subseteq \mathbb{R}^d$ and let $f$ be such that $\text{supp}(f) \subseteq B$. Then, $M(f) \in L_\Psi(B)$ if, and only if,

$$\int_{\{|f| > \alpha_0\}} |f(x)| \cdot \left( \int_{\alpha_0}^{\alpha_0} \frac{|\psi_x(s)|}{s} ds \right) dx < \infty$$

for every $\alpha_0 > 0$.

Theorem 5 applies to certain Orlicz spaces considered in connection with convergence of Fourier series, see e.g. [1, 11], and the recent paper by V. Lie [10]; we give some sample applications in Subsection 4.1.

3. Proof of the Zygmund-type Theorem for $H^{\log}(\mathbb{R}^d)$

We begin with the following elementary lemmas.

**Lemma 6.** Consider the function $g : [0, \infty)^2 \to [0, \infty)$ given by

$$g(s, t) := \frac{1}{\log(e + t) + \log(e + s)}, \quad (s, t) \in [0, \infty)^2.$$

Then one has

$$\Psi(x, t) \leq \int_0^t g(|x|, \tau) d\tau \leq 2\Psi(x, t)$$

for all $(x, t) \in \mathbb{R}^d \times [0, \infty)$.

**Proof.** The function $t \mapsto g(s, t) = \frac{1}{\log((e + t)(e + s))}$ is decreasing, so clearly

$$\int_0^t g(|x|, s) ds \geq tg(|x|, t) = \Psi(x, t).$$

We now address the upper bound. A calculation yields that

$$\partial_t (t^\epsilon g(|x|, t)) = \frac{t^\epsilon}{\log(e + t) + \log(1 + |x|)} \left( \frac{\epsilon}{t} - \frac{1}{(e + t)(\log(e + t) + \log(e + |x|))} \right),$$

and we observe that the term within the parenthesis is positive if, and only if,

$$\frac{\epsilon}{t} - \frac{1}{(e + t)(\log(e + t) + \log(e + |x|))} > 0,$$
which for \( \epsilon = \frac{1}{2} \) is equivalent to the inequality
\[
(e + t)(\log(e + t) + \log(e + |x|)) > 2t.
\]
But clearly
\[
(e + t)(\log(e + t) + \log(e + |x|)) \geq 2(e + t) > 2t.
\]
Thus \( s \mapsto s^\epsilon g(|x|, s) \) is increasing for \( \epsilon = 1/2 \), which implies that
\[
\int_0^t g(|x|, s) ds = \int_0^t s^{-\epsilon} s^\epsilon g(|x|, s) ds \leq \frac{1}{1 - \epsilon} \Psi(x, t) = 2\Psi(x, t)
\]
and this completes the proof of the lemma. \( \square \)

**Lemma 7.** Let \( x_0 \in \mathbb{R}^d \) be fixed and for \( u \in S(\mathbb{R}^d) \) define \( \langle \tau_{x_0} f, u \rangle := \langle f, \tau_{-x_0} u \rangle \), where \( \tau_{-x_0} u(x) := u(x - x_0), \ x \in \mathbb{R}^d \).

Then \( f \in H^{\log}(\mathbb{R}^d) \) if, and only if, \( \tau_{x_0} f \in H^{\log}(\mathbb{R}^d) \).

**Proof.** Note that it suffices to prove that for any \( x_0 \in \mathbb{R}^d \) and \( f \in H^{\log}(\mathbb{R}^d) \) one also has that \( \tau_{x_0} f \in H^{\log}(\mathbb{R}^d) \).

Towards this aim, fix an \( x_0 \in \mathbb{R}^d \) and an \( f \in H^{\log}(\mathbb{R}^d) \). Observe that, by using a change of variables and the translation invariance of \( M_\phi \), we may write
\[
I := \int_{\mathbb{R}^d} \frac{M_\phi(\tau_{x_0} f)(x)}{\log(e + |x|) + \log(e + M_\phi(\tau_{x_0} f)(x))} dx
\]
as
\[
I = \int_{\mathbb{R}^d} \frac{M_\phi(f)(x)}{\log(e + |x - x_0|) + \log(e + M_\phi(f)(x))} dx.
\]
To prove that \( I < \infty \), we split
\[
I = I_1 + I_2,
\]
where
\[
I_1 := \int_{|x| > 4|x_0|} \frac{M_\phi(f)(x)}{\log(e + |x - x_0|) + \log(e + M_\phi(f)(x))} dx
\]
and
\[
I_2 := \int_{|x| \leq 4|x_0|} \frac{M_\phi(f)(x)}{\log(e + |x - x_0|) + \log(e + M_\phi(f)(x))} dx.
\]
To show that \( I_1 < \infty \), observe that for \( |x| > 4|x_0| \) one has
\[
\frac{4|x - x_0|}{5} < |x| < \frac{4|x - x_0|}{3}
\]
and so,
\[
I_1 \leq \int_{|x| > 4|x_0|} \frac{M_\phi(f)(x)}{\log(e + |x|) + \log(e + M_\phi(f)(x))} dx
\]
\[
\leq \int_{\mathbb{R}^d} \frac{M_\phi(f)(x)}{\log(e + |x|) + \log(e + M_\phi(f)(x))} dx.
\]
Since \( f \in H^{\log}(\mathbb{R}^d) \), the last integral is finite and we thus deduce that \( I_1 < \infty \).

Next, to show that \( I_2 < \infty \), we have
\[
I_2 \leq \int_{|x| \leq 4|x_0|} \frac{M_\phi(f)(x)}{1 + \log(e + M_\phi(f)(x))} dx
\]
\[
\leq \int_{|x| \leq 4|x_0|} \frac{M_\phi(f)(x)}{\log(e + |x|) + \log(e + M_\phi(f)(x))} dx
\]
\[
\leq \int_{\mathbb{R}^d} \frac{M_\phi(f)(x)}{\log(e + |x|) + \log(e + M_\phi(f)(x))} dx
\]
and so, \( I_2 < \infty \), as \( f \in H^{\log}(\mathbb{R}^d) \). Therefore, \( I < \infty \) and it thus follows that \( \tau_{x_0} f \in H^{\log}(\mathbb{R}^d) \). \( \square \)
To obtain the desired variant of Zygmund’s theorem, we shall use the fact that functions in $H^\log(\mathbb{R}^d)$ have mean zero; see Lemma 1.4 in [12]. For the convenience of the reader, we present a detailed proof of this fact below.

**Lemma 8 ([2]).** If $f \in H^\log(\mathbb{R}^d)$ is a compactly supported integrable function, then $\int_{\mathbb{R}^d} f(y)dy = 0$.

**Proof.** Let $f$ be a given function in $H^\log(\mathbb{R}^d)$ with compact support. In light of Lemma 6 we may assume, without loss of generality, that $f$ is supported in a closed ball $B_r$ centered at 0 with radius $r > 0$, i.e. $\text{supp}(f) \subseteq B_r := \{x \in \mathbb{R}^d : |x| \leq r\}$.

To prove the lemma, take an $x \in \mathbb{R}^d$ with $|x| > 2r$ and observe that, by the definition of $\phi_\epsilon$, we can take $\epsilon = 4|x|$ to get

$$|f * \phi_\epsilon(x)| = \frac{1}{e^d} \left| \int_{B_r} f(y) \phi \left( \frac{x-y}{\epsilon} \right) dy \right| \gtrsim \frac{1}{|x|^d} \left| \int_{B_r} f(y)dy \right|$$

as we then have $\phi(\epsilon^{-1}(x-y)) = c_0$ for $y \in B_r$. Therefore, for all $|x| > 2r$ and $\epsilon = 4|x|$, we have

$$M_\phi(f)(x) \gtrsim \frac{1}{|x|^d} \left| \int_{B_r} f(y)dy \right|$$

and so, we deduce from Lemma 3 that

$$\Psi(x, M_\phi(f)(x)) \gtrsim \frac{1}{|x|^d \log(e + |x|)} \cdot \left| \int_{B_r} f(y)dy \right|$$

for $|x|$ large enough.

Hence, if $\int f(y)dy \neq 0$, then the function $\Psi(x, M_\phi(f)(x))$ does not belong to $L^1(\mathbb{R}^d)$, which is a contradiction. \(\square\)

We are now ready to prove Theorem 2.

**Proof of Theorem 2.** Let $B$ denote the unit closed ball in $\mathbb{R}^d$. Fix a function $f$ with $\text{supp}(f) \subseteq B$, $\int_B f(y)dy = 0$ and $f \in L \log \log L(B)$. First of all, observe that

$$M_\phi(f)(x) \lesssim M(f)(x) \quad \text{for all } x \in \mathbb{R}^d,$$

where $M(f)$ denotes the Hardy-Littlewood maximal function of $f$; see e.g. Theorem 2 on pp. 62–63 in [13]. We thus deduce from Lemma 3 that

$$\Psi(x, M_\phi(f)(x)) \lesssim \Psi(x, M(f)(x)) \quad \text{for all } x \in \mathbb{R}^d$$

and hence, by using Theorem 1 we obtain

$$(3.1) \quad \int_{2B} \Psi(x, M_\phi(f)(x))dx \lesssim 1 + \int_{2B} |f(x)| \log^+ \log^+ |f(x)|dx,$$

where $2B := \{x \in \mathbb{R}^d : |x| \leq 2\}$.

To estimate the integral of $\Psi(x, M_\phi(f)(x))$ for $x \in \mathbb{R}^d \setminus 2B$, we shall make use of the cancellation of $f$. To be more specific, observe that if $|x| > 2$ then for every $\epsilon < |x|/2$, one has that

$$f * \phi_\epsilon(x) = \frac{1}{e^d} \int_{B} f(y) \phi \left( \frac{x-y}{\epsilon} \right) dy = 0$$

since $|x-y|/\epsilon > 1$ whenever $y \in B$. Therefore, we may restrict ourselves to $\epsilon \geq |x|/2$ when $|x| > 2$. Hence, for $\epsilon \geq |x|/2$, by exploiting the cancellation of $f$ and using
Lipschitz estimate on $\phi$, we obtain
\[
|f \ast \phi(x)| = \frac{1}{e^\epsilon} \left| \int_B f(y) \phi \left( \frac{x - y}{\epsilon} \right) dy \right| = \frac{1}{e^\epsilon} \left| \int_B f(y) \left[ \phi \left( \frac{x - y}{\epsilon} \right) - \phi \left( \frac{x}{\epsilon} \right) \right] dy \right|
\]
\[
\lesssim \phi \frac{1}{e^{\epsilon d + 1}} \int_B |y \cdot f(y)| dy \lesssim \frac{1}{|x|^{d + 1}} \left[ 1 + \int_B |f(y)| \log^+ \log^+ |f(y)| dy \right].
\]
We thus deduce that, for every $x \in \mathbb{R}^d \setminus 2B$,
\[
|M_\phi(f)(x)| \lesssim \frac{1}{|x|^{d + 1}} \left[ 1 + \int_B |f(y)| \log^+ \log^+ |f(y)| dy \right]
\]
and so,
\[
\int_{\mathbb{R}^d \setminus 2B} \Psi(x, M_\phi(x)) dx
\]
\[
\lesssim \left[ 1 + \int_B |f(y)| \log^+ \log^+ |f(y)| dy \right] \cdot \int_{\mathbb{R}^d \setminus 2B} \frac{1}{|x|^{d + 1} \log(e + |x|)} dx
\]
\[
\lesssim 1 + \int_B |f(y)| \log^+ \log^+ |f(y)| dy,
\]
as desired. Therefore, Theorem 2 is now established by using the last estimate combined with (3.1).

3.1. A partial converse. As in the classical setting of the real Hardy space $H^1$, see [12], Theorem 2 has a partial converse. To be more precise, if a function $f$ is positive on an open set $U$ and $f$ belongs to $H^{log}(\mathbb{R}^d)$, then the function $f \in L \log \log L(K)$ for every compact set $K \subset U$.

Indeed, to see this, note that if $f$ is as above then
\[
M_\phi(f)(x) \gtrsim M(f \cdot \eta_K)(x) \quad \text{for all } x \in K,
\]
where $\eta_K$ is an appropriate Schwartz function with $\eta_K \sim 1$ on $K$; see e.g. Section 5.3 in Chapter III in [14]. Hence, by using Lemma 3 and Theorem 1 we get
\[
\int_{\mathbb{R}^d} \Psi(x, M_\phi(f)(x)) dx \geq \int_K \Psi(x, M_\phi(f)(x)) dx \gtrsim \int_K \Psi(x, M(\eta_K \cdot f)(x)) dx
\]
\[
\gtrsim 1 + \int_K |f(x)| \log^+ \log^+ |f(x)| dx.
\]

4. Variants in the periodic setting

Following [2], define $H^{log}(\mathbb{T})$ to be the space of all holomorphic functions $F$ on the unit disk $\mathbb{D}$ of $\mathbb{C}$ such that
\[
\sup_{0 < r < 1} \int_0^{2\pi} \frac{|F(re^{i\theta})|}{\log(e + |F(re^{i\theta})|)} d\theta < \infty.
\]
For $0 < q \leq \infty$, let $H^q(\mathbb{T})$ denote the classical Hardy space on $\mathbb{T}$ consisting of analytic functions $F$ having
\[
\sup_{0 < r < 1} \int_0^{2\pi} |F(re^{i\theta})|^p d\theta < \infty;
\]
see for instance [8]. Then
\[
H^1(\mathbb{T}) \subsetneq H^{log}(\mathbb{T}) \subsetneq H^p(\mathbb{T}) \quad \text{for all } 0 < p < 1.
\]
Hence, if $F \in H^{log}(\mathbb{T})$ then $F$ has a non-tangential limit $F^*$ at almost every point of $T = \partial \mathbb{D}$, and this non-tangential limit lies in $L^p(\mathbb{T})$ for $0 < p < 1$. See [8].
Theorem 2.2] for details. Moreover, by using [2, Proposition 8.2], one may identify $H^{log}(D)$ with the space of all measurable functions $f$ on the torus such that

$$\int_0^{2\pi} \Psi_0 \left( \sup_{0<r<1} |P_r \ast f(\theta)| \right) d\theta < \infty,$$

where $\Psi_0(t) := t \cdot [\log(e + t)]^{-1}$ ($t \geq 0$) and for $0 < r < 1$, $\theta \in [0, 2\pi)$,

$$P_r(\theta) := \frac{1 - r^2}{1 - 2r \cos(\theta) + r^2}$$

denotes the Poisson kernel in the unit disk.

There is a periodic version of Theorem [10, as well as Proposition [9] and Lemma [6] one can show that if $f \in L \log L(T)$.

**Proposition 9.** One has the inclusion

$$L \log L(T) \subseteq H^{log}(T).$$

Moreover, arguing as in the previous section and using the necessity in Theorem [11] as well as Proposition [9] and Lemma [6] one can show that if $f \in H^{log}(T)$ and $f$ is non-negative, then $f \in L \log L(T)$.

**Proposition 10.** One has

$$\{ f \in L \log L(T) : f \geq 0 \text{ a.e. on } T \} = \{ f \in H^{log}(T) : f \geq 0 \text{ a.e. on } T \}.$$  

**Proof.** Note that Proposition [9] implies that

$$\{ f \in L \log L(T) : f \geq 0 \text{ a.e. on } T \} \subseteq \{ f \in H^{log}(T) : f \geq 0 \text{ a.e. on } T \}.$$  

To prove the reverse inclusion, take a non-negative function $f \in H^{log}(T)$ and notice that it follows from the work of Stein [12] that

$$|\{ \theta \in T : M(f)(\theta) > c_1 \alpha \}| \geq \frac{c_2}{\alpha} \int_{\{|f|>\alpha\}} |f(\theta)| d\theta,$$

where $c_1, c_2 > 0$ are absolute constants. Hence, by arguing as in the proof of Theorem [11] it follows from (4.2) (noting that the periodic case is easier as one does not need to consider the contribution away from the support of $f$) that

$$\int_T \Psi_0(M(f)(\theta)) d\theta \geq 1 + \int_T |f(x)| \log^+ \log^+ |f(\theta)| d\theta.$$  

Since $f \geq 0$ a.e. on $T$, as in the Euclidean case, one has

$$\sup_{0<r<1} |P_r \ast f(\theta)| \gtrsim M(f)(\theta) \quad \text{for a.e. } \theta \in T.$$  

Hence, by using (4.3), (4.4) and Lemma [6] we deduce that $f \in L \log L(T)$ and so,

$$\{ f \in H^{log}(T) : f \geq 0 \text{ a.e. on } T \} \subseteq \{ f \in L \log L(T) : f \geq 0 \text{ a.e. on } T \}.$$  

The desired fact is a consequence of (4.1) and (4.5).  

**4.1. Some further applications.** We conclude with some applications of Theorem [5] in the periodic setting. The function

$$\Psi(x, t) = \Psi(t) = t \log^+ t \log^+ \log^+ t$$

appearing in [11] satisfies the hypotheses of Theorem [5] and we now determine which space maps into $L_q$ via the maximal function. With the associated $\psi$ defined as before, an integration by parts yields

$$\int \frac{\psi(s)}{s} ds = \frac{1}{2} \log^+ s \log^+ s + \log^+ s \log^+ \log^+ s - \frac{1}{4} (\log^+ s)^2.$$
This allows us to conclude that, for this choice of $\Psi$,\
$$M(f) \in L_\Psi(\mathbb{T})$$ if, and only if, $f \in L \log^2 L \log \log L(\mathbb{T})$.

Turning to the space $L \log \log L \log \log \log L \log \log \log \log L(\mathbb{T})$ appearing in Lie’s paper [10], we can check where the maximal operator maps this space. Performing the appropriate computations, we obtain that\
$$\int_\mathbb{T} \frac{M(f)}{\log(M(f) + \varepsilon)} \log^+ \log^+ \log^+ \log^+ M(f) dx < \infty$$ if, and only if, $f \in L \log \log L \log \log \log L \log \log \log \log L(\mathbb{T})$.

Roughly speaking, the contents of Theorem 5 and the computations presented above can be summarized as follows. Let $\Phi_0$ be a given Orlicz function, namely $\Phi_0 : [0, \infty) \to [0, \infty)$ is an increasing function with $\Phi_0(0) = 0$ and $\Phi_0(t) \to \infty$ as $t \to \infty$. Suppose that one can find non-negative, increasing functions $M, S$ with\
$$\Phi_0(t) = M(t) \cdot S(t) \quad (t > 0)$$ and such that, for $0 < \alpha < t$, one can easily compute\
$$F_{\alpha}(t) := \int_\alpha^t \frac{M'(s)}{s} ds$$ in closed form and, moreover, that there exists an $\alpha_0 > 0$ with the property that for every $\alpha \geq \alpha_0$ one has\
$$F_{\alpha}(t) \cdot S(t) \geq \int_\alpha^t \left( \frac{M(s)}{s} + F(s) \right) \cdot S'(s) ds \quad \text{for all } t \geq \alpha.$$ Then, by arguing as in Section 2 one deduces the “concrete” relation\
$$f \in L_{\Phi_0}(\mathbb{T})$$ if, and only if, $M(f) \in L_{F_{\alpha}, S}(\mathbb{T})$, for any $\alpha \geq \alpha_0$.

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