CONSECUTIVE PRIMES AND BEATTY SEQUENCES

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Abstract. Fix irrational numbers \(\alpha, \hat{\alpha} > 1\) of finite type and real numbers \(\beta, \hat{\beta} \geq 0\), and let \(B\) and \(\hat{B}\) be the Beatty sequences
\[
B := (\lfloor \alpha m + \beta \rfloor)_{m \in \mathbb{N}} \quad \text{and} \quad \hat{B} := (\lfloor \hat{\alpha} m + \hat{\beta} \rfloor)_{m \in \mathbb{N}}.
\]
In this note, we study the distribution of pairs \((p, p^\#)\) of consecutive primes for which \(p \in B\) and \(p^\# \in \hat{B}\). Under a strong (but widely accepted) form of the Hardy-Littlewood conjectures, we show that
\[
\left| \left\{ p \leq x : p \in B \text{ and } p^\# \in \hat{B} \right\} \right| = (\alpha \hat{\alpha})^{-1} \pi(x) + O(x(\log x)^{-3/2 + \varepsilon}).
\]

1. Introduction

For any given real numbers \(\alpha > 0\) and \(\beta \geq 0\), the associated (generalized) Beatty sequence is defined by
\[
B_{\alpha, \beta} := \left( \lfloor \alpha m + \beta \rfloor \right)_{m \in \mathbb{N}},
\]
where \([t]\) is the largest integer not exceeding \(t\). If \(\alpha\) is irrational, it follows from a classical exponential sum estimate of Vinogradov [7] that \(B_{\alpha, \beta}\) contains infinitely many prime numbers; in fact, one has
\[
\# \left\{ \text{prime } p \leq x : p \in B_{\alpha, \beta} \right\} \sim \alpha^{-1} \pi(x) \quad (x \to \infty),
\]
where \(\pi(x)\) is the prime counting function.

Throughout this paper, we fix two (not necessarily distinct) irrational numbers \(\alpha, \hat{\alpha} > 1\) and two (not necessarily distinct) real numbers \(\beta, \hat{\beta} \geq 0\), and we denote
\[
B := B_{\alpha, \beta} \quad \text{and} \quad \hat{B} := B_{\hat{\alpha}, \hat{\beta}}. \quad (1.1)
\]
Our aim is to study the set of primes \(p \in B\) for which the next larger prime \(p^\#\) lies in \(\hat{B}\). The results we obtain are conditional, relying only on the Hardy-Littlewood conjectures in the following strong form. Let \(\mathcal{H}\) be a finite subset of \(\mathbb{Z}\), and let \(1_p\) denote the indicator function of the primes. The Hardy-Littlewood conjecture for \(\mathcal{H}\) asserts that the estimate
\[
\sum_{n \leq x} \prod_{h \in \mathcal{H}} 1_p(n + h) = \mathfrak{S}(\mathcal{H}) \int_2^x \frac{du}{(\log u)^{|\mathcal{H}|}} + O(x^{1/2 + \varepsilon}) \quad (1.2)
\]
holds for any fixed \(\varepsilon > 0\), where \(\mathfrak{S}(\mathcal{H})\) is the singular series given by
\[
\mathfrak{S}(\mathcal{H}) := \prod_p \left( 1 - |(\mathcal{H} \mod p)| \right) \left( 1 - \frac{1}{p} \right)^{-|\mathcal{H}|}.
\]
Our main result is the following.

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Theorem 1.1. Fix irrational numbers $\alpha, \hat{\alpha} > 1$ of finite type and real numbers $\beta, \hat{\beta} \geq 0$, and let $\mathcal{B}$ and $\hat{\mathcal{B}}$ be the Beatty sequences given by (1.1). For every prime $p$, let $p^\sharp$ denote the next larger prime. Suppose that the Hardy-Littlewood conjecture (1.2) holds for every finite subset $H$ of $\mathbb{Z}$. Then, for any fixed $\varepsilon > 0$, the counting function

$$
\pi(x; \mathcal{B}, \hat{\mathcal{B}}) := \left| \{ p \leq x : p \in \mathcal{B} \text{ and } p^\sharp \in \hat{\mathcal{B}} \} \right|
$$

satisfies the estimate

$$
\pi(x; \mathcal{B}, \hat{\mathcal{B}}) = (\alpha \hat{\alpha})^{-1} \pi(x) + O(x (\log x)^{-3/2 + \varepsilon}),
$$

where the implied constant depends only on $\alpha$, $\hat{\alpha}$ and $\varepsilon$.

Our results are largely inspired by the recent breakthrough paper of Lemke Oliver and Soundararajan [3], which studies the surprisingly erratic distribution of pairs of consecutive primes amongst the $\phi(q)^2$ permissible reduced residue classes modulo $q$. In [3] a conjectural explanation for this phenomenon is given which is based on the strong form of the Hardy-Littlewood conjectures considered in this note, that is, under the hypothesis that the estimate (1.2) holds for every finite subset $H$ of $\mathbb{Z}$.

2. Preliminaries

2.1. Notation. The notation $\lfloor t \rfloor$ is used to denote the distance from the real number $t$ to the nearest integer; that is,

$$
\lfloor t \rfloor := \min_{n \in \mathbb{Z}} |t - n| \quad (t \in \mathbb{R}).
$$

We denote by $|t|$ and $\{t\}$ the greatest integer $\leq t$ and the fractional part of $t$, respectively. We also write $e(t) := e^{2\pi i t}$ for all $t \in \mathbb{R}$, as usual.

Let $\mathbb{P}$ denote the set of primes in $\mathbb{N}$. In what follows, the letter $p$ always denotes a prime number, and $p^\sharp$ is used to denote the smallest prime greater than $p$. In other words, $p$ and $p^\sharp$ are consecutive primes with $p^\sharp > p$. We also put

$$
\delta_p := p^\sharp - p \quad (p \in \mathbb{P}).
$$

For an arbitrary set $S$, we use $1_S$ to denote its indicator function:

$$
1_S(n) := \begin{cases} 
1 & \text{if } n \in S, \\
0 & \text{if } n \notin S.
\end{cases}
$$

Throughout the paper, implied constants in symbols $O$, $\ll$ and $\gg$ may depend (where obvious) on the parameters $\alpha, \hat{\alpha}, \varepsilon$ but are absolute otherwise. For given functions $F$ and $G$, the notations $F \ll G$, $G \gg F$ and $F = O(G)$ are all equivalent to the statement that the inequality $|F| \leq c|G|$ holds with some constant $c > 0$.

2.2. Discrepancy. We recall that the discrepancy $D(M)$ of a sequence of (not necessarily distinct) real numbers $x_1, x_2, \ldots, x_M \in [0, 1)$ is defined by

$$
D(M) := \sup_{\mathcal{I} \subseteq [0,1]} \left| \frac{V(\mathcal{I}, M)}{M} - |\mathcal{I}| \right|, \quad (2.1)
$$
where the supremum is taken over all intervals $\mathcal{I} = (b, c)$ contained in $[0, 1)$, the quantity $V(\mathcal{I}, M)$ is the number of positive integers $m \leq M$ such that $x_m \in \mathcal{I}$, and $|\mathcal{I}| = c - b$ is the length of $\mathcal{I}$.

For any irrational number $a$ we define its type $\tau = \tau(a)$ by the relation

$$\tau := \sup \{ t \in \mathbb{R} : \liminf_{n \to \infty} n^t \lfloor an \rfloor = 0 \}.$$ 

Using Dirichlet’s approximation theorem, one sees that $\tau \geq 1$ for every irrational number $a$. Thanks to the work of Khinchin [1] and Roth [5, 6] it is known that $\tau = 1$ for almost all real numbers (in the sense of the Lebesgue measure) and for all irrational algebraic numbers, respectively.

For a given irrational number $a$, it is well known that the sequence of fractional parts $\{a\}, \{2a\}, \{3a\}, \ldots$, is uniformly distributed modulo one (see, for example, [2, Example 2.1, Chapter 1]). When $a$ is of finite type, this statement can be made more precise. By [2, Theorem 3.2, Chapter 2] we have the following result.

**Lemma 2.1.** Let $a$ be a fixed irrational number of finite type $\tau$. For every $b \in \mathbb{R}$ the discrepancy $D_{a,b}(M)$ of the sequence of fractional parts $\{am+b\}_{m=1}^M$ satisfies the bound

$$D_{a,b}(M) \leq M^{-1/\tau+o(1)} \quad (M \to \infty),$$

where the function implied by $o(\cdot)$ depends only on $a$.

2.3. **Indicator function of a Beatty sequence.** As in §1 we fix (possibly equal) irrational numbers $\alpha, \hat{\alpha} > 1$ and (possibly equal) real numbers $\beta, \hat{\beta} \geq 0$, and we set

$$\mathcal{B} := \mathcal{B}_{\alpha,\beta} \quad \text{and} \quad \hat{\mathcal{B}} := \mathcal{B}_{\hat{\alpha},\hat{\beta}}.$$ 

In what follows we denote

$$a := \alpha^{-1}, \quad \hat{a} := \hat{\alpha}^{-1}, \quad b := \alpha^{-1}(1 - \beta) \quad \text{and} \quad \hat{b} := \hat{\alpha}^{-1}(1 - \hat{\beta}).$$ 

It is straightforward to show that

$$1_{\mathcal{B}}(m) = \psi_a(am+b) \quad \text{and} \quad 1_{\hat{\mathcal{B}}}(m) = \psi_{\hat{a}}(\hat{a}m+\hat{b}) \quad (m \in \mathbb{N}), \quad (2.2)$$

where for any $t \in (0, 1)$ we use $\psi_t$ to denote the periodic function of period one defined by

$$\psi_t(x) := \begin{cases} 
1 & \text{if } 0 < \{x\} \leq t, \\
0 & \text{if } t < \{x\} < 1 \text{ or } \{x\} = 0.
\end{cases}$$

2.4. **Modified Hardy-Littlewood conjecture.** For their work on primes in short intervals, Montgomery and Soundararajan [4] have introduced the modified singular series

$$\mathfrak{S}_0(\mathcal{H}) := \sum_{\mathcal{T} \subseteq \mathcal{H}} (-1)^{|\mathcal{H}\setminus\mathcal{T}|} \mathfrak{S}(\mathcal{T}),$$

for which one has the relation

$$\mathfrak{S}(\mathcal{H}) = \sum_{\mathcal{T} \subseteq \mathcal{H}} \mathfrak{S}_0(\mathcal{T}).$$
Note that \( \mathcal{G}(\emptyset) = \mathcal{G}_0(\emptyset) = 1 \). The Hardy-Littlewood conjecture (1.2) can be reformulated in terms of the modified singular series as follows:

\[
\sum_{n \leq x} \prod_{h \in \mathcal{H}} \left( 1 - \frac{1}{\log n} \right) = \mathcal{G}_0(\mathcal{H}) \int_2^x \frac{du}{(\log u)^{\mathcal{H}}} + O(x^{1/2+\varepsilon}). \tag{2.3}
\]

**Lemma 2.2.** We have

\[
\sum_{1 \leq t \leq h-1} \mathcal{G}_0(\{0, t\}) \ll h^{1/2+\varepsilon},
\]

\[
\sum_{1 \leq t \leq h-1} \mathcal{G}_0(\{t, h\}) \ll h^{1/2+\varepsilon},
\]

\[
\sum_{1 \leq t_1 < t_2 \leq h-1} \mathcal{G}_0(\{t_1, t_2\}) = -\frac{1}{2} h \log h + \frac{1}{2} Ah + O(h^{1/2+\varepsilon}),
\]

where \( A := 2 - C_0 - \log 2\pi \) and \( C_0 \) denotes the Euler-Mascheroni constant.

**Proof.** Let us denote

\[
B := \sum_{1 \leq t \leq h-1} \mathcal{G}_0(\{0, t\}), \quad C := \sum_{1 \leq t \leq h-1} \mathcal{G}_0(\{t, h\}),
\]

and

\[
D_{\pm} := \sum_{1 \leq t_1 < t_2 \leq h-1} \mathcal{G}_0(\{t_1, t_2\})
\]

for either choice of the sign \( \pm \). Clearly,

\[
\mathcal{G}_0(\{0, h\}) + B + C + D_+ = D_+ \quad \text{and} \quad B = \sum_{1 \leq t \leq h-1} \mathcal{G}_0(\{0, h-t\}) = C.
\]

From [4, Equation (16)] we derive the estimates

\[
D_{\pm} = -\frac{1}{2} h \log h + \frac{1}{2} Ah + O(h^{1/2+\varepsilon}).
\]

Using the trivial bound \( \mathcal{G}_0(\{0, h\}) \ll \log \log h \) and putting everything together, we finish the proof. \(\square\)

2.5. Technical lemmas. Let \( \nu(u) := 1 - 1/\log u \). Note that \( \nu(u) \approx 1 \) for \( u \geq 3 \).

**Lemma 2.3.** Let \( c > 0 \) be a constant, and suppose that \( f \) is a function such that \( |f(h)| \leq h^c \) for all \( h \geq 1 \). Then, uniformly for \( 3 \leq u \leq x \) and \( \lambda \in \mathbb{R} \) we have

\[
\sum_{h \leq \lfloor \log x \rfloor^3} f(h) \nu(u)^h e(\lambda h) = \sum_{h \geq \lfloor \log x \rfloor^3} f(h) \nu(u)^h e(\lambda h) + O_c(x^{-1}).
\]

**Proof.** Write \( \nu(u)^h = e^{-h/H} \) with \( H := -(\log \nu(u))^{-1} \). Since \( H \leq \log u \) for \( u \geq 3 \), for any \( h > (\log x)^3 \) we have \( h/H \geq h^{2/3} \) as \( u \leq x \); therefore,

\[
\left| \sum_{h > (\log x)^3} f(h) \nu(u)^h e(\lambda h) \right| \leq \sum_{h > (\log x)^3} h^c e^{-h^{2/3}} \leq x^{-1} \sum_{h > (\log x)^3} h^c e^{h^{1/3}-h^{2/3}} \ll x^{-1},
\]

and the result follows. \(\square\)
The next statement is an analogue of [3, Proposition 2.1] and is proved using similar methods.

**Lemma 2.4.** Fix $\theta \in [0, 1]$ and $\vartheta = 0$ or $1$. For all $\lambda \in \mathbb{R}$ and $u \geq 3$, let

$$R_{\theta, \vartheta; \lambda}(u) := \sum_{h \geq 1 \atop 2 \nmid h} h^\theta (\log h)^\vartheta \nu(u)^h e(\lambda h),$$

$$S_{\lambda}(u) := \sum_{h \geq 1 \atop 2 \nmid h} \mathcal{S}_0(\{0, h\}) \nu(u)^h e(\lambda h).$$

When $\lambda = 0$ we have the estimates

$$R_{\theta,0;0}(u) = \frac{1}{2} \Gamma(1 + \theta)(\log u)^{1+\theta} + O(1),$$

$$R_{\theta,1;0}(u) = \frac{1}{2} (\log 2) \Gamma(1 + \theta)(\log u)^{1+\theta} + O(1),$$

$$S_{0}(u) = \frac{1}{2} \log u - \frac{1}{2} \log \log u + O(1).$$

On the other hand, if $\lambda$ is such that $|\lambda| \geq (\log u)^{-1}$, then

$$\max \{|R_{\theta,0;\lambda}(u)|, |S_{\lambda}(u)|\} \ll \lambda^{-4}.$$

**Proof.** We adapt the proof of [3, Proposition 2.1]. As in Lemma 2.3 we write

$$\nu(u)^h = e^{-h/H}$$

with $H := -(\log \nu(u))^{-1}$. We simplify the expressions $R_{\theta,0;\lambda}(u)$, $S_{\lambda}(u)$ and $T_{\lambda}(u)$ by writing

$$\nu(u)^h e(\lambda h) = e^{-h/H_{\lambda}}$$

with $H_{\lambda} := \frac{H}{1 - 2\pi i \lambda H}$.

Since $\Re(h/H_{\lambda}) = h/H > 0$ for any positive integer $h$, using the Cahen-Mellin integral we have

$$R_{\theta,0;\lambda}(u) = \sum_{h \geq 1 \atop 2 \nmid h} h^\theta (\log h)^\vartheta e^{-h/H_{\lambda}} = \frac{1}{2\pi i} \int_{4-i\infty}^{4+i\infty} \left( \sum_{h \geq 1 \atop 2 \nmid h} \frac{h^\theta (\log h)^\vartheta}{h^s} \right) \Gamma(s) H_{\lambda}^s \, ds.$$  

In particular,

$$R_{\theta,0;\lambda}(u) = \frac{2^\theta}{2\pi i} \int_{4-i\infty}^{4+i\infty} 2^{-s} \zeta(s-\theta) \Gamma(s) H_{\lambda}^s \, ds$$  

and

$$R_{\theta,1;\lambda}(u) = R_{\theta,0;\lambda}(u) \log 2 - \frac{2^\theta}{2\pi i} \int_{4-i\infty}^{4+i\infty} 2^{-s} \zeta'(s-\theta) \Gamma(s) H_{\lambda}^s \, ds.$$  

When $\lambda \neq 0$ we have

$$|R_{\theta,0;\lambda}(u)| \leq \frac{2^{\theta-4}|H_{\lambda}|^4}{2\pi} \int_{-\infty}^\infty |\zeta(4-\theta + it)\Gamma(4 + it)| \, dt$$

$$\ll |H_{\lambda}|^4 = \left( \frac{H^2}{1 + 4\pi^2 \lambda^2 H^2} \right)^2,$$

hence the bound $R_{\theta,0;\lambda}(u) \ll \lambda^{-4}$ holds if $|\lambda| \geq (\log u)^{-1}$ since $H \asymp \log u$ for $u \geq 3$. In the case that $\lambda = 0$, the stated estimate for $R_{\theta,0;0}(u)$ is obtained by shifting the line of integration in (2.4) to the line $\{\Re(s) = -\frac{1}{3}\}$ (say), taking into account the residues of the poles of the integrand at $s = 1 + \theta$ and $s = 0$. 


Our estimates for $R_{\theta,1,\lambda}(u)$ are proved similarly, using (2.5) instead of (2.4) and taking into account that $\zeta'(s-\theta) = (s-1-\theta)^{-1} + O(1)$ for $s$ near $1+\theta$.

Next, for all $\lambda \in \mathbb{R}$ and $u \geq 3$, let
\[ T_\lambda(u) := \sum_{h \geq 1} \mathcal{G} \{0, h\} e^{-h/H_\lambda}. \]

Since $\mathcal{G}_0 \{0, h\} = \mathcal{G} \{0, h\} - 1$ for all integers $h$, and $\mathcal{G} \{0, h\} = 0$ if $h$ is odd, it follows that
\[ S_\lambda(u) = T_\lambda(u) - R_{0,0,\lambda}(u) = T_\lambda(u) - \frac{1}{2} \log u + O(1). \]

Hence, to complete the proof of the lemma, it suffices to show that
\[ T_0(u) = \log u - \frac{1}{2} \log \log u + O(1) \quad \text{and} \quad T_\lambda(u) \ll \lambda^{-4} \text{ if } |\lambda| \geq (\log u)^{-1}. \]

As in the proof of [3, Proposition 2.1], we consider the Dirichlet series
\[ F(s) := \sum_{h \geq 1} \frac{\mathcal{G} \{0, h\}}{h^s}, \]
which can be expressed in the form
\[ F(s) = \frac{\zeta(s)\zeta(s+1)}{\zeta(2s+2)} \prod_p \left( 1 - \frac{1}{(p-1)^2} + \frac{2p}{(p-1)^2(p^{s+1}+1)} \right), \]
and the final product is analytic for $\Re(s) > -1$. Using the Cahen-Mellin integral we have
\[ T_\lambda(u) = \frac{1}{2\pi i} \int_{4-i\infty}^{4+i\infty} F(s) \Gamma(s) H_\lambda^s ds. \] (2.6)

For $\lambda \neq 0$ we have
\[ |T_\lambda(u)| \leq \frac{|H_\lambda|^4}{2\pi} \int_{-\infty}^{\infty} |F(4+it)\Gamma(4+it)| \, dt \ll |H_\lambda|^4 = \left( \frac{H^2}{1 + 4\pi^2\lambda^2H^2} \right)^2 \]
hence $T_\lambda(u) \ll \lambda^{-4}$ holds provided that $|\lambda| \geq (\log u)^{-1}$. For $\lambda = 0$, we shift the line of integration in (2.6) to the line $\{\Re(s) = -\frac{1}{3}\}$ (say), taking into account the double pole at $s = 0$ and the simple pole at $s = 1$. This leads to the stated estimate for $T_0(u)$. \qed

We also need the following integral estimate (proof omitted).

**Lemma 2.5.** For all $\lambda \in \mathbb{R}$ and $x \geq 3$, let
\[ I_\lambda(x) := \int_{3}^{x} \frac{e(\lambda u)}{\nu(u) \log u} \, du. \]

When $\lambda = 0$ we have the estimate
\[ I_0(x) = \frac{x}{\log x} + O\left( \frac{x}{(\log x)^2} \right), \]
whereas for any $\lambda \neq 0$ we have
\[ I_\lambda(x) \ll |\lambda|^{-1}. \]
3. Proof of Theorem 1.1

For every even integer $h \geq 2$ we denote
$$\pi_h(x; B, \hat{B}) := \left| \{ p \leq x : p \in B, \ p^r \in \hat{B} \text{ and } \delta_p = h \} \right| = \sum_{n \leq x} 1_B(n) 1_{\hat{B}}(n+h)f_h(n),$$
where
$$f_h(n) := 1_{\mathbb{P}}(n) 1_{\mathbb{P}}(n+h) \prod_{0 < t < h} (1 - 1_{\mathbb{P}}(n+t)) = \begin{cases} 1 & \text{if } n = p \in \mathbb{P} \text{ and } \delta_p = h, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,
$$\pi(x; B, \hat{B}) = \sum_{h \leq (\log x)^3} \pi_h(x; B, \hat{B}) + O\left( \frac{x}{(\log x)^3} \right). \quad (3.1)$$

Fixing an even integer $h \in [1, (\log x)^3]$ for the moment, our initial goal is to express $\pi_h(x; B, \hat{B})$ in terms of the function $S_h(x)$ recently introduced by Lemke Oliver and Soundararajan [3, Equation (2.5)]. In view of (2.2) we can write
$$\pi_h(x; B, \hat{B}) = \sum_{n \leq x} \psi_a(an + b)\psi_\hat{a}(\hat{a}(n + h) + \hat{b})f_h(n). \quad (3.2)$$

According to a classical result of Vinogradov (see [8, Chapter I, Lemma 12]), for any $\Delta$ such that
$$0 < \Delta < \frac{1}{8} \quad \text{and} \quad \Delta \leq \frac{1}{2} \min\{a, 1 - a\}$$
there is a real-valued function $\Psi_a$ with the following properties:

(i) $\Psi_a$ is periodic with period one;
(ii) $0 \leq \Psi_a(t) \leq 1$ for all $t \in \mathbb{R}$;
(iii) $\Psi_a(t) = \psi_a(t)$ if $\Delta \leq \{t\} \leq a - \Delta$ or if $a + \Delta \leq \{t\} \leq 1 - \Delta$;
(iv) $\Psi_a$ is represented by a Fourier series
$$\Psi_a(t) = \sum_{k \in \mathbb{Z}} g_a(k)e(kt),$$
where $g_a(0) = a$, and the Fourier coefficients satisfy the uniform bound
$$|g_a(k)| \ll \min\{|k|^{-1}, |k|^{-2}\Delta^{-1}\} \quad (k \neq 0). \quad (3.3)$$

For convenience, we denote
$$\mathcal{I}_a := [0, \Delta) \cup (a - \Delta, a + \Delta) \cup (1 - \Delta, 1),$$
so that $\Psi_a(t) = \psi_a(t)$ whenever $\{t\} \notin \mathcal{I}_a$. Defining $\Psi_\hat{a}$ and $\mathcal{I}_\hat{a}$ similarly with $\hat{a}$ in place of $a$, and taking into account the properties (i)–(iii), from (3.2) we deduce that
$$\pi_h(x; B, \hat{B}) = \sum_{n \leq x} \Psi_a(an + b)\Psi_\hat{a}(\hat{a}(n + h) + \hat{b})f_h(n) + O(V(x)), \quad (3.4)$$
where $V(x)$ is the number of positive integers $n \leq x$ for which
\[
\{an + b\} \in \mathcal{I}_a \quad \text{or} \quad \{\hat{a}(n + h) + \hat{b}\} \in \hat{\mathcal{I}}_a.
\]
Since $\mathcal{I}_a$ and $\hat{\mathcal{I}}_a$ are unions of intervals with overall measure $4\Delta$, it follows from the definition (2.1) and Lemma 2.1 that
\[
V(x) \ll \Delta x + x^{1-1/\tau + o(1)} \quad (x \to \infty).
\] (3.5)

Now let $K \geq \Delta^{-1}$ be a large real number, and let $\Psi_{a,K}$ be the trigonometric polynomial given by
\[
\Psi_{a,K}(t) := \sum_{|k| \leq K} g_a(k) e^{ikt}.
\]
Using (3.3) it is clear that the estimate
\[
\Psi_{a}(t) = \Psi_{a,K}(t) + O(K^{-1}\Delta^{-1})
\] (3.6)
holds uniformly for all $t \in \mathbb{R}$. Defining $\hat{\Psi}_{a,K}$ in a similar way, combining (3.6) with (3.4), and taking into account (3.5), we derive the estimate
\[
\pi_h(x; \mathcal{B}, \hat{\mathcal{B}}) = \Sigma_h + O(\Delta x + x^{1-1/\tau + \varepsilon} + K^{-1}\Delta^{-1}x),
\]
where
\[
\Sigma_h := \sum_{n \leq x} \Psi_{a,K}(an + b)\Psi_{\hat{a},\hat{K}}(\hat{a}(n + h) + \hat{b})f_h(n)
\] \[= \sum_{n \leq x} \sum_{|k|,|\ell| \leq K} g_a(k)e^{k(an + b)}g_a(\ell)e^{\ell(\hat{a}(n + h) + \hat{b})}f_h(n)
\] \[= \sum_{|k|,|\ell| \leq K} g_a(k)e^{k(b)}g_a(\ell)e^{\ell(\hat{b})}e^{(\ell\hat{a})h} \sum_{n \leq x} e((ka + \ell\hat{a})n)f_h(n).
\]
Therefore
\[
\pi_h(x; \mathcal{B}, \hat{\mathcal{B}}) = \sum_{|k|,|\ell| \leq K} g_a(k)e^{k(b)}g_a(\ell)e^{\ell(\hat{b})}e^{(\ell\hat{a})h} \int_{3^{-}}^{x} e((ka + \ell\hat{a})u) d(S_h(u))
\] \[+ O(\Delta x + x^{1-1/\tau + \varepsilon} + K^{-1}\Delta^{-1}x),
\] (3.7)
which completes our initial goal of expressing $\pi_h(x; \mathcal{B}, \hat{\mathcal{B}})$ in terms of the function $S_h$. To proceed further, it is useful to recall certain aspects of the analysis of $S_h$ that is carried out in [3]. First, writing $\bar{1}_p(n) := 1_p(n) - 1/\log n$, up to an error term of size $O(x^{1/2+\varepsilon})$ the quantity $S_h(x)$ is equal to
\[
\sum_{n \leq x} \left(\bar{1}_p(n) + \frac{1}{\log n}\right) \left(\bar{1}_p(n + h) + \frac{1}{\log n}\right) \prod_{0 < t < h} \left(1 - \frac{1}{\log n} - \bar{1}_p(n + t)\right)
\] \[= \sum_{\mathcal{A} \subseteq \{0,h\}} \sum_{\mathcal{T} \subseteq [1,h-1]} (-1)^{|\mathcal{T}|} \sum_{n \leq x} \left(\frac{1}{\log n}\right)^{2-|\mathcal{A}|} \left(1 - \frac{1}{\log n}\right)^{h-1-|\mathcal{T}|} \prod_{t \in \mathcal{A} \cup \mathcal{T}} \bar{1}_p(n + t);
By the modified Hardy-Littlewood conjecture (2.3) the estimate
\[
\sum_{n \leq x} (\log n)^{-c} \prod_{t \in \mathcal{H}} \tilde{I}_P(n + t) = \int_3^x (\log u)^{-c} \left( \sum_{n \leq u, t \in \mathcal{H}} \tilde{I}_P(n + t) \right) \, du = \mathcal{S}_0(\mathcal{H}) \int_3^x (\log u)^{-c-|\mathcal{H}|} \, du + O(x^{1/2+\varepsilon})
\]
holds uniformly for any constant \(c > 0\); consequently, up to an error term of size \(O(x^{1/2+\varepsilon})\) the quantity \(S_h(x)\) is equal to
\[
\sum_{A \subseteq \{0,h\}} \sum_{T \subseteq [1,h-1]} (-1)^{|T|} \mathcal{S}_0(A \cup T)(\nu(u) \log u)^{-|T|} \nu(u)^h,
\]
where
\[
\nu(u) := 1 - \frac{1}{\log u} \quad (u > 1)
\]
(note that \(\nu(u)\) is the same as \(\alpha(u)\) in the notation of [3]). For every integer \(L \geq 0\) we denote
\[
D_{h,L}(u) := \sum_{A \subseteq \{0,h\}} \sum_{T \subseteq [1,h-1]} (-1)^{|T|} \mathcal{S}_0(A \cup T)(\nu(u) \log u)^{-|T|} \nu(u)^h,
\]
so that
\[
S_h(x) = \sum_{L=0}^{h+1} \int_3^x \nu(u)^{-1}(\log u)^{-2} D_{h,L}(u) \, du.
\]
We now combine this relation with (3.7), sum over the even natural numbers \(h \leq (\log x)^3\), and apply (3.1) to deduce that the quantity \(\pi(x;\mathcal{B},\hat{\mathcal{B}})\) is equal to
\[
\sum_{h \leq (\log x)^3} \sum_{L=0}^{h+1} \sum_{|k|,|\ell| \leq K} g_a(k) e(kb) g_a(\ell) e(\ell\hat{a}) \cdot e(\ell\hat{a}h) \int_3^x \frac{e((ka + \ell\hat{a})u)}{\nu(u)(\log u)^2} D_{h,L}(u) \, du
\]
up to an error term of size
\[
\ll \frac{x}{(\log x)^3} + (\Delta x + x^{1-1/\tau+\varepsilon} + K^{-1} \Delta^{-1} x)(\log x)^3.
\]
Choosing \(\Delta := (\log x)^{-6}\) and \(K := (\log x)^{12}\) the combined error is \(O(x/(\log x)^3)\), which is acceptable.

Next, arguing as in [3] and noting that
\[
\sum_{|k|,|\ell| \leq K} |g_a(k) g_a(\ell)| \ll (\log x)^2,
\]
one sees that the contribution to \(\pi(x;\mathcal{B},\hat{\mathcal{B}})\) coming from terms with \(L \geq 3\) does not exceed \(O(x/(\log x)^{5/2})\). Since \(D_{h,1}\) is identically zero (as \(\mathcal{S}_0\) vanishes on singleton sets), this leaves only the terms with \(L = 0\) or \(L = 2\). The function \(D_{h,2}\)
splits naturally into four pieces according to whether \( \mathcal{A} = \emptyset, \{0\}, \{h\} \) or \( \{0, h\} \). Consequently, up to \( O(x/(\log x)^{5/2}) \) we can express the quantity \( \pi(x; \mathcal{B}, \hat{\mathcal{B}}) \) as

\[
\sum_{j=1}^{5} \sum_{|k|, |\ell| \leq K} g_a(k) e(kb) g_a(\ell) e(\ell b) \int_{3}^{x} \frac{e((ka + \ell a)u)}{\nu(u)(\log u)^2} F_{j, \ell}(u) \, du, \tag{3.8}
\]

where (taking into account Lemma 2.3) we have written

\[
\sum_{h < (\log x)^{3/2}} e(\ell h) D_{h, L}(u) = \sum_{j=1}^{5} F_{j, \ell}(u) + O(x^{-1})
\]

with

\[
F_{1, \ell}(u) := \sum_{h \geq 1 \atop 2 | h} \nu(u)^{h} e(\ell h),
\]

\[
F_{2, \ell}(u) := \sum_{h \geq 1 \atop 2 | h} \mathcal{S}_{0}(\{0, h\}) \nu(u)^{h} e(\ell h),
\]

\[
F_{3, \ell}(u) := \frac{(-1)}{\nu(u) \log u} \sum_{h \geq 1 \atop 2 | h} \sum_{1 \leq t \leq h-1} \mathcal{S}_{0}(\{0, t\}) \nu(u)^{h} e(\ell h),
\]

\[
F_{4, \ell}(u) := \frac{(-1)}{\nu(u) \log u} \sum_{h \geq 1 \atop 2 | h} \sum_{1 \leq t \leq h-1} \mathcal{S}_{0}(\{t, h\}) \nu(u)^{h} e(\ell h),
\]

\[
F_{5, \ell}(u) := \frac{1}{(\nu(u) \log u)^{2}} \sum_{h \geq 1 \atop 2 | h} \sum_{1 \leq t_{1} < t_{2} \leq h-1} \mathcal{S}_{0}(\{t_{1}, t_{2}\}) \nu(u)^{h} e(\ell h).
\]

First, we show that certain terms in (3.8) make a negligible contribution that does not exceed \( O(x/(\log x)^{3/2-\varepsilon}) \).

For any \( \ell \neq 0 \), using Lemma 2.4 with \( \lambda = \ell \hat{a} \) we have

\[
F_{1, \ell}(u) = R_{0,0,\ell \hat{a}}(u) \ll \ell^{-4}
\]

provided that \( |\ell \hat{a}| \geq (\log u)^{-1} \), and for this it suffices that \( u \geq \exp(\hat{a}) \). Thus,

\[
\int_{3}^{x} \frac{e((ka + \ell \hat{a})u)}{\nu(u)(\log u)^2} F_{1, \ell}(u) \, du \ll 1 + \ell^{-4} \frac{x}{(\log x)^{2}}.
\]

In view of (3.3), the contribution to (3.8) from terms with \( j = 1 \) and \( \ell \neq 0 \) is

\[
\ll \sum_{|k|, |\ell| \leq K \atop \ell \neq 0} |g_{a}(k)| \cdot |\ell|^{-1} \left( 1 + \ell^{-4} \frac{x}{(\log x)^{2}} \right) \ll \frac{x \log x}{(\log x)^{2}} \ll \frac{x}{(\log x)^{3/2-\varepsilon}}.
\]

Similarly, for \( \ell \neq 0 \) and \( u \geq \exp(\hat{a}) \) we have \( F_{2, \ell}(u) = S_{\ell \hat{a}}(u) \ll \ell^{-4} \) by Lemma 2.4, so the contribution to (3.8) from terms with \( j = 2 \) and \( \ell \neq 0 \) is also \( O(x/(\log x)^{3/2-\varepsilon}) \).
For any \( \ell \in \mathbb{Z} \), by Lemma 2.2 and Lemma 2.4 we have
\[
\max \{ |F_{3,\ell}(u)|, |F_{4,\ell}(u)| \} \ll \frac{1}{\log u} \sum_{\substack{h \geq 1 \atop 2|h}} h^{1/2+\varepsilon/2} \nu(u)^h \ll (\log u)^{1/2+\varepsilon/2},
\]
hence for \( j = 3, 4 \) we see that
\[
\int_3^x \frac{e((ka + \ell \hat{a})u)}{\nu(u)(\log u)^2} F_{j,\ell}(u) \, du \ll \frac{x}{(\log x)^{3/2-\varepsilon/2}}.
\]
By (3.3), it follows that the contribution to (3.8) from terms with \( j = 3, 4 \) is
\[
\ll \frac{x}{(\log x)^{3/2-\varepsilon/2}} \sum_{|k|, |\ell| \in K} |g_a(k)g_\hat{a}(\ell)| \ll \frac{x(\log \log x)^2}{(\log x)^{3/2-\varepsilon/2}} \ll \frac{x}{(\log x)^{3/2-\varepsilon}}.
\]
Finally, for any \( \ell \in \mathbb{Z} \) and \( u \geq \exp(\hat{a}) \), by Lemma 2.2 and Lemma 2.4 we have
\[
F_{5,\ell}(u) = \frac{1}{(\nu(u) \log u)^2} \sum_{\substack{h \geq 1 \atop 2|h}} (-\frac{1}{2} h \log h + \frac{1}{2} Ah + O(h^{1/2+\varepsilon/2})) \nu(u)^h e(\ell \hat{a} h) = -\frac{1}{2} R_{1,1;\ell \hat{a}}(u) + \frac{1}{2} AR_{1,0;\ell \hat{a}}(u) + O(R_{1,2+\varepsilon/2,0,0}(u))
\]
\[
\ll \frac{\lambda^{-4} + (\log u)^{3/2+\varepsilon/2}}{(\log u)^2},
\]
and arguing as before we see that the contribution to (3.8) coming from terms with \( j = 5 \) does not exceed \( O(x/(\log x)^{3/2-\varepsilon}) \).

Applying the preceding bounds to (3.8) we see that, up to \( O(x/(\log x)^{3/2-\varepsilon}) \), the quantity \( \pi(x; \mathcal{B}, \hat{\mathcal{B}}) \) is equal to
\[
\hat{a} \sum_{j=1,2} \sum_{|k| \in K} g_a(k)e(kb) \int_3^x \frac{e(kau)}{\nu(u)(\log u)^2} F_{j,0}(u) \, du,
\]
where we have used the fact that \( g_\hat{a}(0) = \hat{a} \). By Lemma 2.4 we have
\[
F_{1,0}(u) = \sum_{\substack{h \geq 1 \atop 2|h}} \nu(u)^h = R_{0,0,0}(u) = \frac{1}{2} \log u + O(1)
\]
and
\[
F_{2,0}(u) = \sum_{\substack{h \geq 1 \atop 2|h}} \mathcal{S}_0(\{0, h\}) \nu(u)^h = S_0(u) = \frac{1}{2} \log u - \frac{1}{2} \log \log u + O(1);
\]
therefore,
\[
\int_3^x \frac{e(kau)}{\nu(u)(\log u)^2} F_{j,0}(u) \, du = \frac{1}{2} \int_3^x \frac{e(kau)}{\nu(u) \log u} \, du + O\left( \frac{x \log \log x}{(\log x)^2} \right) \quad (j = 1, 2).
\]
Consequently, up to \( O(x/(\log x)^{3/2-\varepsilon}) \) we can express the quantity \( \pi(x; \mathcal{B}, \hat{\mathcal{B}}) \) as
\[
\hat{a} \sum_{|k| \in K} g_a(k)e(kb) \int_3^x \frac{e(kau)}{\nu(u) \log u} \, du. \tag{3.9}
\]
To complete the proof of Theorem 1.1, we apply Lemma 2.5, which shows that the term $k = 0$ in (3.9) contributes
\[
\frac{a\hat{a}}{\log x} + O\left(\frac{x}{(\log x)^2}\right) = (a\hat{a})^{-1}\pi(x) + O\left(\frac{x}{(\log x)^2}\right)
\]
to the quantity $\pi(x; \mathcal{B}, \mathcal{B})$ (and thus accounts for the main term), whereas the terms in (3.9) with $k \neq 0$ contribute altogether only a bounded amount.

References

[1] A. Khinchin, Zur metrischen Theorie der diophantischen Approximationen. Math. Z. 24 (1926), no. 4, 706–714.
[2] L. Kuipers and H. Niederreiter, Uniform distribution of sequences. Pure and Applied Mathematics. Wiley-Interscience, New York-London-Sydney, 1974.
[3] R. J. Lemke Oliver and K. Soundararajan, Unexpected biases in the distribution of consecutive primes. Proc. Natl. Acad. Sci. USA, to appear. arXiv:1603.03720
[4] H. L. Montgomery and K. Soundararajan, Primes in short intervals. Comm. Math. Phys. 252 (2004), no. 1-3, 589–617.
[5] K. Roth, Rational approximations to algebraic numbers. Mathematika 2 (1955), 1–20.
[6] K. Roth, Corrigendum to “Rational approximations to algebraic numbers.” Mathematika 2 (1955), 168.
[7] I. M. Vinogradov, A new estimate of a certain sum containing primes (Russian). Rec. Math. Moscou, n. Ser. 2(44) (1937), no. 5, 783–792. English translation: New estimations of trigonometrical sums containing primes. C. R. (Dokl.) Acad. Sci. URSS, n. Ser. 17 (1937), 165–166.
[8] I. M. Vinogradov, The method of trigonometrical sums in the theory of numbers. Dover Publications, Inc., Mineola, NY, 2004.