ON GALOIS ACTION IN RIGID DAHA MODULES

IVAN CHEREDNIK †

Abstract. Given an elliptic curve over a field $K$ of algebraic numbers, we associate with it an action of the absolute Galois group $G_K$ in the type $A_1$ rigid DAHA-modules at roots of unity $q$ and over the rings $\mathbb{Z}[q^{1/4}]/(p^m)$ for sufficiently general prime $p$. We describe rigid modules in characteristic zero and for such rings. The main examples of rigid modules are generalized nonsymmetric Verlinde algebras; their deformations for arbitrary $q$ are constructed in this paper, which is of independent interest on its own. The Galois action preserves the images of the elliptic braid group in the groups of automorphisms of rigid modules over $\mathbb{Z}[q^{1/4}]/(p^m)$. If they are finite in characteristic zero, then $G_K$ acts there and no reduction modulo $(p^m)$ is needed; we find all such cases. In the case of 3-dimensional DAHA-modules, these images are quotients of equilateral triangle groups directly related to the Livné groups. Also, this paper can be viewed as a certain extension of the DAHA theory of refined Jones polynomials of torus knots (for $A_1$) to $G_K$.

Key words: elliptic curve; braid group; Hecke algebra; Verlinde algebra; Deligne-Simpson Problem; Galois group; Tate module; triangle group

MSC (2010): 17B45, 20C08, 33D52, 12F12, 14G32, 30F10, 11G05, 20F65, 20H10, 51M10

Contents

0. Introduction 3

0.1. Tate modules via rigidity 3

0.2. The main results of the paper 5

1. DAHA and rigidity 6

1.1. Main definitions 6

1.2. Finite-dimensional modules 9

1.3. Rigidity at roots of unity 12

† February 2, 2014. Partially supported by NSF grant DMS-1101535, the Fulbright Program and the Simons Foundation.
| Section | Title                                      | Page |
|---------|--------------------------------------------|------|
| 1.4.    | Deformations of Verlinde algebras          | 16   |
| 1.5.    | Boundary cases                             | 18   |
| 1.6.    | Finite and discrete images                 | 20   |
| 1.7.    | Series of dim=$2, 4$                       | 24   |
| 2.      | The Galois action                          | 28   |
| 2.1.    | Deligne-Simpson Problem                     | 28   |
| 2.2.    | Absolute Galois group                      | 31   |
| 2.3.    | Main Theorem                               | 33   |
| 2.4.    | Special cases                              | 37   |
| 2.5.    | Triangle groups                            | 40   |
| 2.6.    | The Livné groups                           | 42   |
| 2.7.    | Some perspectives                          | 46   |
| References |                                             | 52   |
0. Introduction

Given an elliptic curve defined over a field of algebraic numbers $K$, we defined the corresponding action of the absolute Galois group $G_K = \text{Gal} (\overline{Q}/K)$ in the rigid DAHA-modules of type $A_1$. The rigidity we use is actually that in the corresponding Deligne-Simpson Problem. These modules are considered at roots of unity $q$ and over the rings $\mathbb{Z}[q^{1/4}]/(p^m)$ for odd prime $p$ such that $\gcd(p, N) = 1$ for the order $N$ of $q$ and for $m \in \mathbb{N}$. Generalized nonsymmetric Verlinde algebras are the main examples of rigid modules; we construct their deformations for any $q$, which is of interest in its own right.

The elliptic braid group, denoted by $B_1$, plays the key role in this paper; $G_K$ acts in its images in the groups of automorphisms of rigid modules considered over the rings $\mathbb{Z}[q^{1/4}]/(p^m)$. We describe all rigid DAHA-modules in characteristic 0; they are defined over $\mathbb{Z}[q^{1/4}]$ and remain rigid modulo $(p^m)$ for $p$ as above. We find all cases when $B_1$ has finite images in characteristic 0; then $G_K$ acts there without further reduction modulo $(p^m)$. Also, we post a conjectural list of the cases when the images of $B_1$ are arithmetic discrete. A link to equilateral triangle groups and the Livné lattices in $PU(2, 1)$ is established.

Given an elliptic curve $E$ with a puncture at the origin 0, the elliptic braid group controls its covers equivariant with respect to the reflection $x \mapsto -x$ in $E$. Considering such covers is related to [Bel] and Grothendieck’s program of dessins d’enfants. Let us also mention here [BL] (the construction of the polylogarithm sheaf for $E \setminus 0$).

In contrast to the Tate modules of elliptic curves based on the unramified covers of $E$, covers of $E \setminus 0$ form a huge class. A much more restrictive system of covers of $E \setminus 0$ can be obtained via the DAHA modules at roots of unity. They are similar to the unramified ones, though the functoriality with respect to the roots of unity $q$ is unclear. Even without this, such an action of $G_K$ is interesting; for instance, the DAHA-based theory of refined Jones polynomials of torus knots from [Ch2] has a counterpart for $G_K$ (in type $A_1$).

0.1. Tate modules via rigidity. Let $V_{2N}$ be the unique irreducible nonzero module of the extended Weyl algebra $W_{2N}$ generated by the elements $X^{\pm 1}, Y^{\pm 1}$ and $S$ with the relations

$$Y^{-1}X^{-1}XYq^{1/2} = 1, \quad SXSX = 1 = SYSY, \quad S^2 = 1, \quad X^{2N} = 1 = Y^{2N},$$
where $q^{1/2}$ is assumed a primitive $(2N)$th root of unity. Switching to the generators $A = XS$, $B = q^{1/4}XY$, $C = SY$, we interpret these relations as the following (very special) case of the multiplicative Deligne-Simpson problem, DSP:

$$A^2 = 1 = \tilde{B}^2 = C^2, \quad A\tilde{B}C = \tilde{S} \overset{\text{def}}{=} q^{1/4}S, \quad A, \tilde{B}, C, \tilde{S} \in \text{GL}(2N, \mathbb{C}),$$

provided that the multiplicities of the eigenvalues $\pm 1$ coincide (and equal to $N$) for $A, \tilde{B}, C$ and are $N + 1, N - 1$ for the eigenvalues $q^{1/4}, -q^{-1/4}$ of $\tilde{S}$. This DSP is rigid; it has a unique solution (up to a conjugation) due to the uniqueness of $V_{2N}$. Up to a conjugation, the matrices $A, \tilde{B}, C, \tilde{S}$ can be assumed here over $\mathbb{Q}(q^{1/4})$.

For an elliptic curve $E_C$, let $o, o_1, o_2, o_3 \in P^1_C$ be the images of $0 \in E_C$ and the remaining points $\{0_1, 0_2, 0_3\}$ of the 2nd order in $E_C$ upon the identification $E/\{s\} \xrightarrow{\sim} P^1$, where $s : z \mapsto -z$ in $E_C \ni z$.

Finite quotients of the group $\mathcal{B}_1 = < A, \tilde{B}, C, \tilde{S} >$ result in (connected) Galois covers of $P^1_C$ ramified at $\{o, o_1, o_2, o_3\}$, which can be naturally considered as covers of $E_C$ ramified only at $0$ associated with the corresponding quotients of $< \tilde{X} = q^{-1/4}X, \tilde{Y} = q^{1/4}Y, \tilde{S}^2 >$. We apply this to the image $\mathfrak{B}$ of $\mathcal{B}_1 = < A, \tilde{B}, C, \tilde{S} >$ in $\text{GL}(2N, \mathbb{Q}(q^{1/4}))$. Dividing $\mathfrak{B}$ by the center, the corresponding unramified covering $\tilde{\pi}_{2N} : E \simeq \tilde{E} \rightarrow E$ is the multiplication by $2N$; its kernel $\tilde{E}_{2N} \simeq \mathbb{Z}_{2N}^2$ is the Galois group.

Let $q^{1/2} = e^{\frac{x}{2N}}, e_1, e_2 \in \tilde{E}_{2N}$ be the generators corresponding to $X, Y$. The group $< X, Y, q^{1/2} >$ corresponds to the extension of $\mathbb{C}(E)$ by the function $f^{\mathfrak{B}}$ for $f$ with the divisor $(f) = \sum_{i=1}^{2N-1} D^i - (2N - 1)D$, where $D = \sum_{i=0}^{2N-1} e_i^i$ and $e_j^i = i e_j$. The action of $\tilde{E}_{2N}$ is pointwise.

Assume that $E$ and $0$ are defined over a field $K \subset \overline{\mathbb{Q}}$, as well as $o \in P^1$ and the set $\{o_1, o_2, o_3\}$. The rigidity of DSP above gives that

1. the group $\text{Gal}(\overline{\mathbb{Q}}/K)$ acts in the group $\mathfrak{B}$ by automorphisms;
2. $\text{Gal}(\overline{\mathbb{Q}}/K(q^{1/4}))$ acts via conjugations by matrices in $\text{GL}(V_{2N})$.

Recall that the classical Tate module is the projective limit $\varprojlim E_{\ell^m}$ as $m \rightarrow \infty$ for a prime $\ell$; it has a standard action of $G_K$. Thus our constructions for $N = \ell^m$ extends such modules.

Note that $V_{2N} \simeq H^0(\mathcal{O}(D))$ for a divisor $D$ of degree $2N$ in $E$ and $q = e^{\pi i/N}$, where $\mathfrak{B}$ acts due to $\mathcal{O}(D^\ell) \simeq \mathcal{O}(D)$ (the theory of Kummer-Weil pairing). Using $A, \tilde{B}, C, \tilde{S}$ establishes the isomorphisms $\mathcal{O}(D^\ell) \simeq$
$\mathcal{O}(D)$ via the trivialization of $\mathcal{O}(D)$ at $E \setminus 0$; here $\tilde{S}^2 = q^{1/2}$ is the monodromy of $\mathcal{O}(D)$ at 0.

Claims (I, II) are extended in this paper to any rigid DAHA-modules of type $A_1$ at roots of unity (and in finite characteristic); these are the boundary case of our construction from $(\alpha^*)$ in Section 1.3. However the compatibility for different roots of unity is missing in such a generality. Taking $N = \ell^m$ for odd prime numbers $\ell$ together with making $\mathbb{Z}[q^{1/4}]/(\ell^m)$ the ring of definition is a possibility here, but this is beyond the present paper. See also an example in (2.24) below.

We note that the approach from [Bel] (and other works) devoted to the (Regular) Inverse Galois Problem is mainly based on rigid triples, which are the triples $a, b, c \in G$ generating $G$ and satisfying $abc = 1$. They are assumed from the corresponding (given) conjugacy classes in $G$. This kind of rigidity means the uniqueness of such a triple up to a simultaneous conjugation in $G$. We need 4 points in $P^1$ and the so-called linear rigidity (in matrices) based on Katz’ theory of rigid systems. Enriching Belyi’s approach by means of the multiplicative DSP (M. Dettweiler and others) was an important development.

0.2. The main results of the paper. We begin the paper with adjusting the classification results from [Ch1], mainly Sections 2.8-9 there, to our objectives. For a primitive $N$th root of unity $q$, we need DAHA-modules over the rings $\mathbb{Z}[q^{1/4}]/(p^m)$ for prime numbers $p$ such that $\gcd(p, 2N) = 1$ and any $m \in \mathbb{N}$. The gcd-constraint makes such a modification of [Ch1] relatively straightforward. We also need to address the case when $q^{1/2}$ is not primitive of order $2N$ for odd $N$.

(i) Theorem 1.4 describes all rigid DAHA-modules $V$ in characteristic 0 for primitive $q^{1/2}$ of order $2N$. The dim-rigidity, when (by definition) only finitely many irreducible DAHA-modules exist of dimension equal to $\dim V$, is sufficient for $A_1$. Some of them are unirigid, when the dimension alone entirely determines their isomorphism classes; they automatically have the projective action of $\text{PSL}_2(\mathbb{Z})$.

(ii) Proposition 1.6 generalizes Section 2.10.5 of [Ch1] and provides $q$–deformations of the generalized nonsymmetric Verlinde algebras (in nonsymmetric polynomials and for arbitrary admissible $t$). The classical Verlinde algebra of $\hat{\mathfrak{sl}}_2$ is for $t = q$ and upon the symmetrization.

(iii) Theorem 1.8 describes all dim-rigid modules with finite images of $B_1$. We analyze the positivity of the hermitian inner product at (all)
primitive roots of unity \( q^{1/2} \). There are two series of dimension 2 and 4 for all \( N \) and two exceptional cases \( N = 6, 10 \) of \( \text{dim} = 8 \). We also provide a conjectural list of arithmetic discrete \( \text{Image}(\mathcal{B}_1) \).

(iv) The finiteness/discreteness of this kind is a very classical topic (cf. Schwarz’ table for hypergeometric functions). If the image of \( \mathcal{B}_1 \) is finite, then \textit{Riemann Existence Theorem} provides an action of \( G_K \) in \( \text{Image}(\mathcal{B}_1) \) without modular considerations. Using DAHA in finite characteristic is needed when \( \text{Image}(\mathcal{B}_1) \) is not finite.

(v) Theorem 2.2 defines the action of \( G_K \) in the images of \( \mathcal{B}_1 \) in rigid modules over \( \mathbb{Z}[q^{1/4}]/(q^n) \). The basic examples are (a) the Tate modules (see (I, II) above), (b) subgroups of the Livn´e groups considered modulo \( (q^n) \) and (c) the Verlinde algebra \( (t = q) \) over \( \mathbb{Z}[q^{1/4}]/(q^n) \).

(vi) Importantly, we can define the action from (c) above only via the nonsymmetric theory. Using the group \( \mathcal{B}_1 \) and \( \text{DSP} \) is possible only \textit{before} the symmetrization of the modules \( V \). However after the action of \( G_K \) is introduced in the whole \( V \), it can be normalized by the condition \( T \mapsto T \). Then the symmetric part of \( V \), defined as \( \{ v \in V \mid Tv = t^{1/2}v \} \), becomes invariant under this action.

(vii) The rigidity in our paper is closely related to that in [ObS], where \( \text{DSP} \) was used for the classification of rigid irreducible DAHA modules for \( C\vee C_1 \) (apart from the roots of unity), which system is a natural setting here. Theorem 2.2 generally can be extended to \( C\vee C_1 \), though there is no complete theory at roots of unity for \( C\vee C_1 \) so far.

(viii) The case of \( A_1 \) is of special interest due to its connection with covers of elliptic curves ramified at one point. Also, the images of the elliptic braid group \( \mathcal{B}_1 \) in rigid DAHA-modules at roots of unity for \( A_1 \) are directly related to the \textit{triangle groups} of type \( (n, n, n; n) \); see Proposition 2.4. We establish a connection with the Livn´e groups \( (\text{dim}V = 3) \); see [Par1]. Links to [LLM] and other papers on finite quotients of triangle groups are expected.

1. DAHA AND RIGIDITY

1.1. Main definitions. We will consider only the case of \( A_1 \) in this paper. Let \( \alpha = \alpha_1, s = s_1, \omega = \omega_1 \) be the fundamental weight; then \( \alpha = 2\omega \) and \( \rho = \omega \). The extended affine Weyl group \( \hat{W} = \langle s, \omega \rangle \) is a free group generated by the involutions \( s \) and \( \pi \overset{\text{def}}{=} \omega s \).

The generators of double affine Hecke algebra \( \mathcal{H} = \mathcal{H}_{q^{1/2}, t^{1/2}} \) are

\[
Y = Y_{\omega_1} = \pi T, \quad T = T_1, \quad X = X_{\omega_1}
\]
subject to the quadratic relation \((T - t^{1/2})(T + t^{-1/2}) = 0\) and the cross-relations:

\begin{equation}
T XT = X^{-1}, \quad T^{-1}YT^{-1} = Y^{-1}, \quad Y^{-1}X^{-1}YT^{2}q^{1/2} = 1.
\end{equation}

Using \(\pi = YT^{-1}\), the second relation becomes \(\pi^{2} = 1\). This algebra is defined over

\[ \mathbb{Z}_{q,t} \overset{\text{def}}{=} \mathbb{Z}[q^{\pm 1/2}, t^{\pm 1/2}] \]

It is important that \(\mathcal{H}_{q^{1/2},1}\) is the extended Weyl algebra generated by \(X^{\pm 1}, Y^{\pm 1}, s\) subject to the relation \(Y^{-1}X^{-1}YXq^{1/2} = 1\) for the involution \(s = T(t^{1/2} = 1)\) such that \(sXs = X^{-1}\) and \(sYs = Y^{-1}\).

**Automorphisms.** The following maps can be extended to automorphisms of \(\mathcal{H}\):

\begin{equation}
\begin{aligned}
\tau_{+}(X) &= X, \quad \tau_{+}(T) = T, \quad \tau_{+}(Y) = q^{-1/4}XY, \quad \tau_{+}(\pi) = q^{-1/4}X\pi, \\
\tau_{-}(Y) &= Y, \quad \tau_{-}(T) = T, \quad \tau_{-}(X) = q^{1/4}YX, \quad \tau_{-}(\pi) = \pi.
\end{aligned}
\end{equation}

They require adding \(q^{\pm 1/4}\) to the ring of definition of \(\mathcal{H}\).

The generalized Fourier transform corresponds to the following automorphism of \(\mathcal{H}\) (it is not an involution):

\[ \sigma(X) = Y^{-1}, \quad \sigma(T) = T, \quad \sigma(Y) = q^{-1/2}Y^{-1}XY = XT^{2}, \quad \sigma(\pi) = XT, \]

\begin{equation}
\sigma = \tau_{+}\tau_{-}^{-1}\tau_{+} = \tau_{-}^{-1}\tau_{+}\tau_{-}^{-1}.
\end{equation}

The last relation is the defining one of the projective \(PSL_{2}(\mathbb{Z})\). Therefor the latter group acts in \(\mathcal{H}\) by outer automorphisms. Check that \(\sigma\tau_{+} = \tau_{-}^{-1}\sigma, \quad \sigma\tau_{+}^{-1} = \tau_{-}\sigma\).

Due to the group nature of the definition of \(\mathcal{H}\), we have the inversion anti-involution \(\mathcal{H} \ni H \mapsto H^{*}\):

\[ X^{*} = X^{-1}, \quad Y^{*} = Y^{-1}, \quad T^{*} = T^{-1}, \quad (q^{1/4})^{*} = q^{-1/4}, \quad (t^{1/2})^{*} = t^{-1/2}. \]

It commutes with all automorphisms of \(\mathcal{H}\).

**Polynomial representation.** It is defined as \(\mathcal{A} = \mathbb{Z}_{q,t}[X^{\pm 1}]\) over the ring of definition of \(\mathcal{H}\); recall that \(\mathbb{Z}_{q,t} = \mathbb{Z}[q^{\pm 1/2}, t^{\pm 1/2}]\). The operator naturally \(X\) acts by the multiplication. The formulas for the other generators are

\[ T = t^{1/2}s + \frac{t^{1/2} - t^{-1/2}}{X^{2} - 1} \circ (s - 1), \quad Y = \pi T \]

in terms of the (multiplicative) reflection \(s(X^{n}) = X^{-n}\) and \(\pi(X^{n}) = q^{n/2}X^{-n}\) for \(n \in \mathbb{Z}\); here \(\circ\) stays for the composition. This module is
induced from the one-dimensional representation \( \{ T = t^{1/2} = Y \} \) of the affine Hecke subalgebra \( \mathcal{H}_Y = \langle Y, T \rangle \).

Let \( \langle f \rangle \) be the constant term of \( f \in \mathbb{Q}(q^{1/2}, t^{1/2})[X, X^{-1}] \),

\begin{equation}
\mu(X; q, t) \overset{\text{def}}{=} \prod_{j=0}^{\infty} \frac{(1 - q^j X^2)(1 - q^j + 1 X^{-2})}{(1 - t q^j X^2)(1 - t q^j + 1 X^{-2})},
\end{equation}

\begin{equation}
\mu_o \overset{\text{def}}{=} \frac{\mu}{\langle \mu \rangle} = 1 + \frac{t - 1}{1 - qt} (X^2 + qX^{-2}) + \ldots,
\end{equation}

where \( \langle \mu \rangle = \prod_{j=1}^{\infty} \frac{(1 - t q^j)^2}{(1 - t^2 q^j)(1 - q^j)} \).

We define the (symmetric) inner product in \( \mathscr{B} \):

\begin{equation}
\langle f, g \rangle \overset{\text{def}}{=} \langle fg^* \mu_o \rangle, \text{ satisfying } \langle f, H(g) \rangle = \langle H^*(f), g \rangle,
\end{equation}

where \( * \) in \( \mathscr{B} \) is the restriction of * from \( \mathcal{H} \) and we add the denominators of the coefficients of \( \mu_o \) to the ring of definition of \( \mathscr{B} \). We note that the automorphism \( \tau_- \) naturally acts in \( \mathscr{B} \).

The e-polynomials. Their coefficients are from the ring \( \mathbb{Z}_{a,b}^{\text{loc}} \), which we define as the localization of \( \mathbb{Z}_{a,b} \) by the expressions in the form \( 1 - t^a q^b \) for integral \( a, b \) assuming that \( t^a q^b \neq 1 \). For generic \( q, t \), the following relations fix \( e_n (n \in \mathbb{Z}) \) uniquely up to proportionality:

\begin{equation}
Y e_n = (t^{1/2} q^{\frac{|n|}{2}})^{-\text{sign}(n)} e_n \text{ for } n \in \mathbb{Z}, \text{ where }
\text{sign}(0) \overset{\text{def}}{=} -1, \text{ i.e. } 0 \text{ is treated as negative.}
\end{equation}

We normalize \( \{ e_n \} \) as follows: \( e_n = X^n + \text{ "lower terms"} \), where by “lower terms”, we mean polynomials in terms of \( X^{\pm m} \) as \( |m| < n \) and, additionally, \( X^{|n|} \) for negative \( n \).

The \( e_n (n \in \mathbb{Z}) \) are called the nonsymmetric Macdonald polynomials or simply e-polynomials; their coefficients actually are given only in terms of \( q, t \) (i.e. no \( q^{1/2}, t^{1/2} \) are needed). One has:

\begin{equation}
\langle e_m e_n^* \mu_o \rangle = \delta_{mn} \prod_{0<j<|n|} \frac{(1 - q^j)(1 - q^j t^2)}{(1 - q^j t)(1 - q^j t)},
\end{equation}

\begin{equation}
|\bar{n}| = |n| + 1 \text{ if } n \leq 0 \text{ and } |\bar{n}| = |n| \text{ if } n > 0,
\end{equation}

\begin{equation}
e_n(t^{-1/2}) = t^{-|n|/2} \prod_{0<j<|n|} \frac{1 - q^j t^2}{1 - q^j t} \text{ for } n \in \mathbb{Z}.
\end{equation}
1.2. **Finite-dimensional modules.** We first assume that \( q \) is generic. The classification of finite-dimensional irreducible modules for such \( q \) is as follows.

Let us introduce the following automorphisms of \( \mathcal{H} \):

\[
\begin{align*}
\iota & : T \mapsto -T, \ X \mapsto X, \ Y \mapsto Y, \ q^{1/2} \mapsto q^{1/2}, \ t^{1/2} \mapsto t^{-1/2}, \\
\varsigma_x & : T \mapsto T, \ X \mapsto -X, \ Y \mapsto Y, \ q^{1/2} \mapsto q^{1/2}, \ t^{1/2} \mapsto t^{1/2}, \\
\varsigma_y & : T \mapsto T, \ X \mapsto X, \ Y \mapsto -Y, \ q^{1/2} \mapsto q^{1/2}, \ t^{1/2} \mapsto t^{1/2}.
\end{align*}
\]

The automorphism \( \varsigma_x \) obviously preserves \( \mathcal{F} \); the polynomial \( e_m \) is multiplied by \((-1)^m\). Note that \( \iota \) changes \( \mathcal{H}_{q^{1/2}, t^{1/2}} \) to \( \mathcal{H}_{q^{1/2}, t^{-1/2}} \simeq \mathcal{H}_{q^{1/2}, t^{-1/2}} \) and commutes with the projective action of \( PSL_2(\mathbb{Z}) \), i.e. with the \( \tau_{\pm} \) defined above. The latter act in an obvious way on the group generated by \( \varsigma_x, \varsigma_y \), isomorphic to \( \mathbb{Z}_2^2 \):

\[
\tau_+(\varsigma_x) = \varsigma_x, \ \tau_+(\varsigma_y) = \varsigma_x \varsigma_y, \ \tau_-(\varsigma_y) = \varsigma_y, \ \tau_-(\varsigma_x) = \varsigma_x \varsigma_y.
\]

The following theorem is from [CHL], Theorem 2.8.1 and Theorem 2.8.2 (see also the references therein). We set \( t = q^k \) for \( k \in \mathbb{C} \), assuming that \( q \) is generic in the following theorem. By (ir)reducibility and \( Y \)-semisimplicity we mean those over the field of fractions of \( \mathbb{Z}_{q^{1/2}} \), though all \( \mathcal{H} \)-modules discussed below are over \( \mathbb{Z}_{q^{1/2}} \).

**Theorem 1.1.** (i) For generic \( q \), irreducible finite-dimensional \( \mathcal{H} \)-modules over the field of fractions \( \mathbb{Q}(q^{1/2}, t^{1/2}) \) of \( \mathbb{Z}_{q^{1/2}} \) are either \( Y \)-spherical, i.e. quotients of \( \mathcal{F} \), or their images under the automorphism \( \varsigma_y \), or the images of \( Y \)-spherical modules defined for \( \mathcal{H}_{q^{1/2}, t^{-1/2}} \) with the parameters \( q^{1/2}, t^{-1/2} \) upon the application of \( \iota \) or \( \varsigma_y \).

The polynomial representation \( \mathcal{F} \) considered over \( \mathbb{Q}(q^{1/2}, t^{1/2}) \) is not \( Y \)-semisimple if and only if \( t = q^{-n} \) for \( n \in \mathbb{N} \). It is reducible if and only if \( t = \zeta q^{-n/2} \) for \( \zeta = \pm 1 \) and \( n \in \mathbb{N} \), where \( \zeta = -1 \) when \( n \in 2\mathbb{Z}_+ \). Therefore if \( \mathcal{F} \) is reducible, then this module is \( Y \)-semisimple; the series \( \mu_\circ \), the pairing \( \langle f, g \rangle \) on \( \mathcal{F} \) and all polynomials \( e_m \ (m \in \mathbb{Z}) \) are well defined in this case.

(ii) For \( n \in \mathbb{Z}_+ \), let us fix \( q^{1/2} \). We set \( k = -1/2 - n \) and \( t = (q^{1/2})^{-1-2n} \). The scalar squares \( \langle e_m, e_m \rangle \) are nonzero exactly at \( \{ m \mid m \in \mathbb{Z}, -2n \leq m \leq 2n + 1 \} \). The radical \( \text{Rad}_0 \) of the pairing \( \langle f, g \rangle \) is \( \oplus_{m \in \mathbb{Z}, e_m} \mathbb{C}e_m \). Furthermore \( \text{Rad}_0 = (e_{-2n-1}) \) as an ideal in \( \mathcal{F} \), and the \( \mathcal{H} \)-module \( V_{4n+2} \triangleq \mathcal{F}/\text{Rad}_0 \) considered over \( \mathbb{Z}_{q^{1/2}}^{\text{loc}} \) is the greatest \( \mathbb{Z}_{q^{1/2}}^{\text{loc}} \)-free finite-dimensional quotient of \( \mathcal{F} \).
The coefficients of the polynomial $e_{-2n-1}$ belong to $\mathbb{Z}[q^{\pm 1/2}]$ and the module $W = W_{4n+2} = \mathcal{X}/(e_{-2n-1})$ is well defined over $\mathbb{Z}_{qt} = \mathbb{Z}[q^{\pm 1/4}]$ (i.e. without the localization with respect to $(1 - q^{j/2})$ for $j \in \mathbb{N}$).

It is a direct sum of the following two $\mathbb{Z}_{qt}$–free non-isomorphic $\mathcal{H}$–submodules of dimension $2n + 1 = 2|k|$:

\begin{align}
V_{2n+1}^\pm &\cong \mathcal{X}/(e_{n+1} \mp t^{-1/2}e_{-n}); \\
\text{see Theorem 2.8.1 from [Ch1].}
\end{align}

These modules are orthogonal to each other with respect to $\langle, \rangle$ restricted to $W$; also, $V_{2n+1}^\pm \cong \mathfrak{s}_x(V_{2n+1}^\pm)$.

The module $V_{2n+1}^+$ is $\tau_+$–invariant. The modules $\mathfrak{s}_x \mathfrak{s}_y \mathfrak{s}_\delta(V_{2n+1}^+)$ are pairwise non-isomorphic for different choices of $\epsilon, \delta = 0, 1$; they are transformed by $\tau_+$, $\tau_-$ according to formulas \([1.4]\).

(iii) The remaining case of reducibility of $\mathcal{X}$ is when $t = -(q^{1/2})^{-n}$ for $n \in \mathbb{N}$ (i.e. for $k' \equiv -n/2$ and $\zeta = -1$), where we set $t = \zeta(q^{1/2})^2k'$. Then the maximal $\mathbb{Z}_{qt}$–free finite-dimensional quotient of $\mathcal{X}$ is irreducible of even dimension $2n = 4|k'|$. It is isomorphic to $V'_{2n} = \mathcal{X}/(e_{-n})$, where $e_{-n} \in \mathcal{X}$ and the ideal $(e_{-n})$ is the radical $\text{Rad}_0$ of the pairing $\langle f, g \rangle$. This module is linearly generated over $\mathbb{Z}_{qt}$ by $e_m$ for $1 - n \leq m \leq n$. The squares $\langle e_m, e_m \rangle$ are nonzero exactly at $m = 0, 1, -1, 2, \ldots, 1 - n, n$. The automorphisms $\mathfrak{s}_x, \mathfrak{s}_y$ and $\tau_\pm$ do not change the isomorphism class of $V'_{2n}$.

The following is the explicit description of $V_{2n+1}^\pm$ over $\mathbb{Z}_{qt}^{\text{loc}}$ from [Ch1]:

\begin{align}
V_{2n+1}^\pm &= \mathbb{Z}_{qt}^{\text{loc}}(e_m \pm e_{-2n-1+m}) \mod \text{Rad}_0, \\
e_{n+1} \pm t^{-1/2}e_{-n} &= e_{n+1}(t^{-1/2}(e_{n+1} \mp e_{-n}) = X^{-n} \prod_{m=-n}^n (X \pm q^{1/4+m/2}).
\end{align}

Here $e_m(t^{-1/2})$ is from \([1.10]\). Note that $e_m$ and $e_{-2n-1+m}$ have coinciding $Y$–eigenvalues, namely $t^{1/2}q^{m/2} = t^{-1/2}q^{m/2-2n-1}$; see \([1.8]\).

Roots of unity. We assume now that $q$ is a primitive root of unity of degree $N \geq 1$ and also always pick $q^{1/2}$ in primitive $(2N)$th roots of unity. We will actually allow taking $q^{1/2}$ in $N$th roots of unity for odd $N$, but will incorporate such a case using squaring $q, t$ or via Little DAHA to be considered below.
We set \( t = (q^{1/2})^{2k} \) for \( k \in \mathbb{Z}/2 \). Then \( \mathbb{Z}_{q,t} \) becomes \( \mathbb{Z}_{q}^{\text{def}} = \mathbb{Z}[\bar{q}] \), where \( \bar{q} = q^{1/2} \) for even \( 2k \) or \( \bar{q} = q^{1/4} \) for odd \( 2k \). Accordingly, \( \mathbb{Z}^{\text{loc}}_{q,t} = \mathbb{Z}^{(2N)}_{\bar{q}} \), where the latter is the localization of \( \mathbb{Z}_{\bar{q}} \) with respect to \( 2N \). Thus coefficients of \( e \)-polynomials, if well defined, are from \( \mathbb{Z}[q]^{(N)} \) or \( \mathbb{Z}[q^{1/2}]^{(2N)} \) respectively for even and odd \( 2k \).

As above, we call a \( \mathbb{Z}_{\bar{q}} \)-free \( \mathcal{H} \)-module \( V \) irreducible if it is irreducible over the field of fractions of \( \mathbb{Z}_{\bar{q}} \). In all examples below such irreducibility implies \( \text{loc-irreducibility} \), which means that \( V \) remains irreducible over \( \mathbb{Z}_{\bar{q}}/p \) for any prime ideal \( p \) in \( \mathbb{Z}_{\bar{q}} \) over \((p) \subset \mathbb{Z} \) for prime \( p \) such that \( \gcd(p, 2N) = 1 \). The latter conditions gives that any \( \mathcal{H} \)-submodule of \( V_{p}^{m} \overset{\text{def}}{=} V \otimes_{\mathbb{Z}_{\bar{q}}} (\mathbb{Z}_{\bar{q}}/p^{m}) \) for \( m \in \mathbb{N} \) has the form \( aV_{p}^{m} \) for proper \( a \in \mathbb{Z}_{\bar{q}}/p^{m} \) (assuming \( \gcd(p, 2N) = 1 \)). Accordingly, \( V \) is called \( \text{loc-semisimple} \) if \( Y \) is semisimple over all such \( \mathbb{Z}_{\bar{q}}/p^{m} \).

Following mainly Theorems 2.9.3, 2.9.8 from [Ch1], we will begin with the \( \mathcal{H} \)-modules \( V_{2N}^{(2)} \overset{\text{def}}{=} \mathcal{X}/(X^{N} + X^{-N}) \),

\[
V_{4N}^{(-2)} \overset{\text{def}}{=} \mathcal{X}/(X^{2N} + X^{-2N} - 2), \quad V_{4N}^{(2)} \overset{\text{def}}{=} \mathcal{X}/(X^{2N} + X^{-2N} + 2).
\]

The module \( V_{4N}^{(-2)} \) will be used in the case of \( k \in \mathbb{Z} \) in the next theorem. The module \( V_{2N}^{(2)} \), which is a natural quotient of \( V_{4N}^{(2)} \), will be needed below for \( k \in 1/2 + \mathbb{Z} \). Respectively, \( t^{N} = q^{kN} = \pm 1 \) in these two cases.

The next theorem is part of the general description of all irreducible \( \mathcal{H} \)-modules at roots of unity from [Ch1]; we aim at the \( \dim \)-rigidity to be considered below.

**Theorem 1.1.** (i) Let \( 2k \in \mathbb{Z} \) and \( 0 < 2k < N \). Then \( V_{4N}^{(-2)} \) for \( k \in \mathbb{Z} \) or, correspondingly, \( V_{2N}^{(2)} \) for \( k \in 1/2 + \mathbb{Z} \) has a unique \( \mathbb{Z}_{\bar{q}} \)-free irreducible nonzero quotient, which is \( V_{2N-4k} = \mathcal{X}/(e_{-n}) \) of dimension \( 2N - 4k \) for \( n \overset{\text{def}}{=} N - 2k \). It is also \( \text{loc-irreducible} \). Its \( Y \)-spectrum is simple: \( V_{2N-4k} = \oplus_{m=2k-N+1}^{N-2k} \mathbb{Z}_{\bar{q}} e'_{m} \), where the eigenvectors \( e'_{m} \) are the images of \( e_{m} \) for \( -n + 1 \leq m \leq n \), which are all well defined. The spectrum remains simple over the rings \( \mathbb{Z}_{\bar{q}}/p^{m} \) (for \( \gcd(p, 2N) = 1 \) and any \( m \in \mathbb{N} \)). Also,

\[
(1.18) \quad T(e'_{-n}) = -t^{-1/2}e'_{-n} = Y(e'_{-n}).
\]

The isomorphism class of this module is invariant under the action of \( \varsigma_{x} \) and \( \varsigma_{y} \).

These statements can be extended to \( k = 0 \), i.e. for \( t^{1/2} = 1 \), when \( X^{N} - X^{-N} \) is taken instead of \( e_{-n} \).
(ii) Let \( k \equiv -1/2 - n \) for integral \( 0 \leq n < (N - 1)/2 \). Then \( V^{(2)}_{2N} = \mathcal{H}(X^N + X^{-N}) \) has two non-isomorphic \( \mathbb{Z}_q \)-free and \( Y \)-semisimple quotients, namely those from (1.15), which are
\[
V^{\pm}_{2n+1} \cong \mathcal{H}/(e_{n+1} \mp t^{-1/2}e_{-n}).
\]
Here \( e_{n+1}, e_{-n} \) are well defined over \( \mathbb{Z}_q \). These modules are also loc-irreducible and loc-semisimple.

The binomials \( (iX)^{n+1} \pm (iX)^{-n} \), where \( i = \sqrt{-1} \), must be taken instead of \( e_{n+1} \pm t^{-1/2}e_{-n} \) to extend these statements to the case when \( N = 2n + 1, t^{1/2} = i \).

The kernel \( (e_{-2n-1}) \) of the homomorphism from \( V^{(2)}_{2N} \) to the direct sum of \( V^+_{2n+1} \) and \( V^-_{2n+1} \) is a free \( \mathbb{Z}_q \)-module of dimension \( 2N - 4|k| \) and is isomorphic to \( V^{(2)}_{2N-4k} \) from (i) with \( t^{1/2} \) replaced by \( t^{-1/2} \) under the action of involution \( \iota \) (sending \( T \mapsto -T \); see above). The vector \( e = e_{-2n-1} \) satisfies the relations \( T(e) = -t^{-1/2}e \) and \( Y(e) = t^{-1/2}e \).

In Part (ii), the polynomials \( \epsilon_m \) are well defined for \(-2n \leq m \leq 2n + 1\) over \( \mathbb{Z}_q^{(N)} \) and the linear decompositions of \( V^\pm_{2n+1} \) in terms of \( \epsilon'_m \pm \epsilon'_{m-2n-1} \) from (1.17) holds over this ring. Also \( \langle \epsilon_m, \epsilon_m \rangle = \langle \epsilon_{m-2n+1}, \epsilon_{m-2n+1} \rangle \) in this range.

Both families of modules, \( V = V_{2N-4k} \) and \( V = V^{\pm}_{2n+1} \), have the invariant hermitian inner products \( \langle f, g \rangle \), which can be defined directly by formulas (1.9) for the images of \( \epsilon_m \) in the proper range of \( m \). Here we extend \( \mathbb{Z}_q \) to \( \mathbb{Z}_q^{(N)} \). One has \( \langle Hf, g \rangle = \langle f, H^{-1}g \rangle \) for \( H = X, Y, T \) and \( f, g \in V \); see (1.7). To justify the existence of such a form, it suffices to observe that \( \langle e, e \rangle = 0 \) for \( e = e_{-n} \) in (i) and correspondingly for \( e = e_{-2n-1} \) in (ii) (and then use the intertwiners).

1.3. Rigidity at roots of unity. We will begin with the following exact sequences from [Ch1], which clarify Theorem 1.2.

**Proposition 1.3.** Let \( k \in \mathbb{Z}/2, |k| < N/2 \). There exist two exact sequences over \( \mathbb{Z}_q \) for \( k > 0 \):

\[
(1.20) \quad 0 \to \iota_{s_q}(V_{2N+4k}) \to V^{(-2)}_{4N}\to V_{2N-4k} \to 0 \quad \text{for} \quad k \in \mathbb{Z}_+,
\]
\[
(1.21) \quad 0 \to \iota(V_{2k}^+ \oplus V_{2k}^-) \to V^{(2)}_{2N}\to V_{2N-4k} \to 0 \quad \text{for} \quad k \in 1/2 + \mathbb{Z}_+.
\]
The arrows must be reversed here for $k < 0$:

\begin{align}
(1.22) & \quad 0 \to \iota \varsigma_y(\mathcal{V}_{2N-4|k|}) \to \mathcal{V}_{4N}^{(-2)} \to \mathcal{V}_{2N+4|k|}^{(-2)} \to 0, \quad k \in -1 - \mathbb{Z}_+, \\
(1.23) & \quad 0 \to \iota \mathcal{V}_{2N-4|k|} \to \mathcal{V}_{2N}^{(2)} \to \mathcal{V}_{2|k|}^{(+2)} \oplus \mathcal{V}_{2|k|}^{(-2)} \to 0, \quad k \in -\frac{1}{2} - \mathbb{Z}_+.
\end{align}

If $k \notin \mathbb{Z}/2$, equivalently $t^2 \notin q^{\mathbb{Z}}$, then $\mathcal{V}_{4N}^{(-2)}$ and $\mathcal{V}_{2N}^{(2)}$ are both loc-irreducible.

Here $\varsigma_y$ can be omitted, since it does not change the classes of isomorphism of the modules $\mathcal{V}_{2N\pm|k|}$; though the composition map $\iota \varsigma_y$ naturally appears in the corresponding homomorphism.

Given arbitrary $q^{1/2}$ and $t^{1/2}$, a $\mathbb{Z}_{qt}$–free irreducible finite-dimensional module $\mathcal{V}$ of $\mathcal{H}$ is called *dim-rigid* if all isomorphism classes of irreducible modules of dimension equal to $\dim \mathcal{V}$ over the field of fractions of $\mathbb{Z}_{qt}$ and its arbitrary algebraic extensions $\mathbb{A}$ constitute a finite family. Such family is obviously projective $\text{PSL}_2(\mathbb{Z})$–invariant (where $q^{1/4}$ must be added to the definition field). Accordingly, for $q,t$ as above, we call $\mathcal{V}$ a *loc-dim-rigid* module if there are only finitely many isomorphism classes of irreducible free modules of dimension $\dim \mathcal{V}$ over the rings $\mathbb{A}/\mathfrak{p}^m$ for prime ideals $\mathfrak{p} \subset \mathbb{A}$ over $(p)$ for prime $p$ such that $\gcd(p,2N) = 1$ and any $m \in \mathbb{N}$.

If there is only one module in such family, then $\tau_\pm$ naturally act in this module over $\mathbb{Q}(q^{1/4})$ or, correspondingly, over $\mathbb{A}/\mathfrak{p}^m$ (for $p,\mathfrak{p},m$ as above). We will call these modules *unirigid* or, correspondingly, *loc-unirigid*. Recall that $t^{1/2} = \pm(q^{1/2})^k$ and $q^{1/2}$ is assumed primitive $(2N)$th root of unity.

The classification of *dim-rigid* modules is similar to that for spherical projective $\text{PSL}_2(\mathbb{Z})$–invariant modules, (we will begin with in the next theorem), but there are some deviations for the family (γ) below.

**Theorem 1.4.** (i) Given $N > 1$, projective $\text{PSL}_2(\mathbb{Z})$–invariant irreducible modules, i.e. where the action of $\tau_\pm$ can be defined, exist only if $t = \pm q^{\mathbb{Z}/2}$. They are isomorphic to one of the following spherical quotients of $\mathcal{X}$ or their $\iota$–images:

\begin{align}
(1.24) & \quad \begin{array}{ll}
(a) & \quad \mathcal{V}_{2N-4k} \quad \text{for} \quad 2k \in \mathbb{Z}_+, \quad N/2 > k > 0, \\
(\beta) & \quad \mathcal{V}_{2N+4|k|} \quad \text{for} \quad k \in -\mathbb{Z}_+, \quad -N/2 < k < 0, \\
(\gamma) & \quad \mathcal{V}_{2|k|}^+ \quad (k = -1/2 - m > -N/2, \quad m \in \mathbb{Z}_+),
\end{array}
\end{align}

where $t = (q^{1/2})^{2k}$ and $\iota : k \mapsto -k, \quad T \mapsto -T, \quad \mathcal{H}_{q^{1/2},t^{1/2}} \mapsto \mathcal{H}_{q^{1/2},t^{-1/2}}.$
(ii) The modules from \((\alpha, \beta)\) are \(\varsigma\)-invariant, namely, their isomorphism classes are invariant under the action of \(\varsigma_x\) and \(\varsigma_y\). The modules \((\varphi, \delta)\) for \(\epsilon, \delta = 0, 1\), incl. \(V^+_{2|k|}\), \(V^-_{2|k|} = \varsigma_x(V^+_{2|k|})\), are non-isomorphic for different \(\epsilon, \delta\); \(\tau_+\) and \(\tau_-\) transform them as in (1.14). The modules from \((\alpha, \gamma)\) are \(Y\)-semisimple and loc-semisimple, \(V_{2N+4|k|}\) from \((\beta)\) is not \(Y\)-semisimple; these hold upon applying \(\iota\).

(iii) The modules from \((\alpha, \beta)\), the \(\varsigma\)-orbit of \(V^+_{2|k|}\) from \((\varphi)\) and the \(\iota\)-images of all these modules constitute all dim-rigid irreducible modules of \(\mathcal{H}\) (for any \(t\)); they are also loc-dim-rigid. Moreover, for a fixed primitive \((2N)\)th root \(q^{1/2}\), the dimensions of modules in \((\alpha, \beta, \gamma)\) are all pairwise distinct. In particular, the modules of type \((\alpha, \beta)\) are unirigid, which provides a conceptual (without explicit formulas) proof of their projective \(\text{PSL}_2(\mathbb{Z})\)-invariance (especially valuable for \((\beta)\)).

**Proof.** This theorem is essentially from [Ch1]; for instance, Part (iii) follows from the classification of Theorem 2.9.2 there. Given \(N\) and a primitive \((2N)\)th root \(q^{1/2}\), the dimensions of modules \((\alpha, \beta, \gamma)\) determine uniquely the corresponding \(k\) (and therefore \(t\)). Indeed, the dimensions are all different inside the union \((\alpha)\cup(\beta)\) and inside \((\gamma)\); also, the dimension is even in the first group and odd in the second.

Note that the rigidity over \(\mathbb{Z}_q/p^n\) in Part (iii) requires knowing the dimensions of all free irreducible \(\mathcal{H}\)-modules over this ring (and its algebraic extensions). The classification from [Ch1], Theorem 2.9.2 is based on the intertwining operators, which are generally well defined only over \(\mathbb{Z}^\text{loc}_{qd}\). This approach can be extended to the rings \(\mathbb{Z}[q]/p^n\) for \(p \supset (p)\) under the condition \(\gcd(2N, p) = 1\); the latter condition is sufficient here, though not always necessary.

**Squaring parameters for odd** \(N\). Following Sections 2.10.1-2.10.2 from [Ch1], let us address the case when \(q^{1/2}\) is not primitive \((2N)\)th root of unity; it may occur only if \(N\) is odd. Instead of changing \(q^{1/2}\) and the previous considerations, we will incorporate such a case by performing the substitution

\[
q \mapsto q^\epsilon = q^2, \quad q^{1/2} \mapsto (q^\epsilon)^{1/2} = q, \quad t \mapsto t^\epsilon = t^2,
\]

assuming that \(N\) is odd. I.e. we define the action of \(\mathcal{H}^\epsilon \overset{\text{def}}{=} \mathcal{H}_{q,t}^{\epsilon} \) in representations \(V\) of \(\mathcal{H} = \mathcal{H}_{q^{1/2}, t^{1/2}}\) considered above (in their space, to be exact), by squaring the parameters \(q, t\) in the \(\epsilon\)-polynomials and the corresponding matrices of the generators; this is closely related to
Section 1.4 below. In type (β), the first exact sequence from (1.22) is used. We will denote the resulting HH-modules by $V^\vee$.

The HH-modules $V^\vee$ for $V$ in (α) for integral $k$ and in (β) are reducible, namely they are direct sums of their two HH-submodules of half-dimension, which is equal to $N-2k$ and $N+2k$ correspondingly. The $Y$-spectra in these two submodules coincide and are simple for (α). In the case of (β), the Jordan form of $Y$ is the same in both irreducible components.

The justification of next proposition follows case (γ) from the theorem. See also Proposition 1.7 below. We use the exact sequences from (1.20) and (1.22) and the evaluation pairings:

$$ (1.27) \quad \{f, g\}_\pm \overset{\text{def}}{=} \left( f((Y^\vee)^{-1})(g) \right)(X \mapsto \pm (t^\vee)^{1/2} = \pm t) \quad \text{in} \quad \mathcal{R}^\vee. $$

**Proposition 1.5.** For odd $N = 2n + 1 > 1$, let $q, t = q^k$ be from Theorem 1.4, including the inequality $0 < |k| < N/2$ there. We denote the polynomials $\epsilon_m$ upon the substitution $\sqrt{\cdot}$ by $\epsilon_m^\vee$.

(i) Upon the substitution (1.26) for integral $k$ in the cases of (α, β),

$$ (1.28) \quad V_{2N-4k}^\vee = V_{N-2k}^\vee \oplus V_{N-2k}^\perp, \quad V_{2N+4k}^\vee = V_{N+2k}^+ \oplus V_{N+2k}^-; $$

where the HH-modules $V_{N-2k}^\pm, V_{N+2k}^\pm$ are the quotients of $V_{2N-4k}^\vee$ and $V_{2N+4k}^\vee$ by the radicals of the corresponding evaluation pairings $\{f, g\}_\pm$. One has

$$ V_{N-2k}^\pm = \oplus_{m=1}^{N-2k} \mathbb{Z}_{qt}^{\max} (\epsilon_m^\vee \pm \epsilon_{m-N+2k}^\vee) = \mathcal{R}^\vee/(\epsilon_{n-k+1}^\vee \pm \epsilon_{k-n}^\vee). $$

(ii) Continuing (i), the modules $V_{N-2k}^\pm, V_{N+2k}^\pm$ are $\tau_\pm$-invariant. Their images under the action of $\zeta_{x,y}^\epsilon \delta$ for $\epsilon, \delta = 0, 1$ are transformed by $\tau_\pm$ as for (γ) from the theorem and are pairwise non-isomorphic. The corresponding families

$$ (1.29) \quad {\alpha'} : V_{N-2k}^{\epsilon, \delta, \vee} = \{\epsilon_{x,y}^\epsilon \delta(V_{N-2k}^+)\}, \quad {\beta'} : V_{N+2k}^{\epsilon, \delta, \vee} = \{\epsilon_{x,y}^\epsilon \delta(V_{N+2k}^-)\} $$

constitute all HH-modules of dimensions $N-2k$ and $N+2k$.

(iii) The modules $V_{2N-4k}^\vee$ for $k = 1/2+n$ and $V_{2N+4k}^\vee$ for $k = -1/2-n$ in (γ) are (remain) irreducible for $n \in \mathbb{Z}_+$ upon the substitution (1.26). The modules $V_{2N-4k}^\vee$ here remain $\tau_\pm$-invariant and become direct sums of Jordan 2-blocks under the action of $Y$ or $X$. One has

$$ V_{2N-4k}^\vee = \oplus_{m=1}^{2|k|} \mathbb{Z}_{qt}^{\max} (\epsilon_m^\vee - \epsilon_{m-2|k|}^\vee). $$

Thus (γ') : $V_{2N-4k}^{\epsilon, \delta, \vee} = \{\epsilon_{x,y}^\epsilon \delta(V_{|k|}^+)\}$ consists of all pairwise non-isomorphic HH-modules of dimension $2n+1$. The action of $\tau_\pm$ in this family remains the same as that for (γ).
Proof. Let us focus here on the series \((\alpha)\) for integral \(k\). The key is the following fact. The \(e\)–polynomials (when exist) and the inner products \(\langle, \rangle\pm\) depend only on \(q, t\), though \(T, Y\) and the evaluation pairings from (1.27) do involve \(t^{1/2}\). Thus changing \(t^{1/2}\) to \(-t^{1/2}\), which sends \(T \mapsto -T\) and \(Y \mapsto -Y\), does not influence the \(e\)–polynomials.

Generally, if the quotient of \(X\) by the radical of \(\langle, \rangle\pm\) is already irreducible, then using the evaluation pairing on top of it will produce no further reduction. However upon changing \(q, t\) by \(q^2, t^2\), the spectrum of \(Y\) becomes of multiplicity 2, the radicals of \(\{, \}\pm\) in \(V_{2N-4k}\) become nonzero and they produce the split from (1.28). Note that for odd \(N\) and integral \(k\), the transform \(q, t \mapsto q, t^2\) is a Galois automorphism sending the coefficients of \(e_m\) to those of \(e_m'\); so these polynomials exist simultaneously.

One can also use here the exact sequence (1.20). The \(\mathcal{H}\)–module \(V_{4N}^{(-2)} = \mathcal{H}/(X^{2N} + X^{-2N} - 2)\) becomes the following direct sum of its submodules: \(V_{4N}^{(-2)} = \mathcal{H}/(X^N + X^{-N} - 2) \oplus \mathcal{H}/(X^N + X^{-N} + 2)\). Note that these submodules are transposed by \(\varsigma\), which sends \(X \mapsto -X\). Accordingly, (1.20) will be reduced for \(k \in \mathbb{Z}_+\) to

\[
(1.30) \quad 0 \to \mathfrak{g}_y(V_{N+2k}^{\pm}) \to \mathcal{H}/(X^N + X^{-N} \mp 2) \to V_{N-2k}^{\pm} \to 0.
\]

Concerning (iii), let us briefly consider the series \((\alpha)\) for half-integral \(k\). We determine when the \(e\)–polynomials from \(V_{2N-4k}\) become nonexistent upon applying \(\sqrt{\text{−}}\), which results in the loss of \(Y\)–semisimplicity in this case. This is essentially sufficient for the remaining claims of the proposition for \((\alpha')\). The case of \((\tilde{\alpha}')\) is straightforward. \(\square\)

1.4. Deformations of Verlinde algebras. The subalgebra \(V_{2N-4k}^{\text{sym}}\) of \(V_{2n-4}^{\text{sym}}\) for \(k = 1\) of symmetric elements \(v \in V_{2N-4}\) is isomorphic to the usual Verlinde algebra for \(\tilde{\mathfrak{sl}}_2\) of level \(N\) (with the central charge \(N-2\)). Its dimension is \(N - 1\); generally, the dimensions of the symmetric subalgebras, defined as \(\{v \in V : Tv = t^{1/2}v\}\), of the modules under consideration are as follows:

\[
\dim V_{2N-4k}^{\text{sym}} = N - 2k + 1, \quad \dim V_{2N+4|k}^{\text{sym}} = N + 2|k| + 1, \quad (1.31) \quad \dim (V_{2|k}^{\pm})^{\text{sym}} = m + 1 \quad \text{for} \quad k = -1/2 - m, \quad m \in \mathbb{Z}_+.
\]

It matches the exact sequences from (1.20) and (1.22), since the symmetric parts of the middle modules \(V_{4N}^{(-2)}\), \(V_{2N}^{(2)}\) are of dimension \(2N\).
and \( N \) correspondingly; \( \iota \) transposes the eigenvalues \( t^{1/2} \) and \( -t^{-1/2} \) of \( T \).

Here and further by Verlinde algebras, we mean generalized (any admissible \( k \)) and nonsymmetric ones (without taking the symmetrization). We note that the projective action of \( \text{PSL}_2(\mathbb{Z}) \) in \( V_{2N-4k}^{\text{sym}} \) for integral \( 0 < k < N/2 \) was defined in [K1] using the interpretation of Macdonald polynomials via quantum \( GL \) (Etingof, Kirillov).

Now let us address the absence of modules \( V'_{4|k'|} \) for \( -2k' \leq N \), defined in (iii) of Theorem 1.1, in the classification of projective \( \text{PSL}_2(\mathbb{Z}) \)–invariant and dim-rigid modules from Theorem 1.4.

The next proposition connects the series (\( \alpha \)) with the modules \( V'_{2N-4k} \) from (\( \alpha \)) of Theorem 1.1. Cf. Section 2.10.5, the remark after Theorem 2.9.9 from [Ch1] and (1.33) below. This explains why (\( \alpha, \beta, \gamma \)) are sufficient in Theorem 1.1, and we do not need to add there the specializations of modules \( V'_{4|k'|} \) at roots of unity.

**Proposition 1.6.** Continuing to assume that \( q^{1/2} \) is a primitive \( (2N) \)th root of unity and considering the modules \( V'_{2N-4k} \) from (\( \alpha \)), let \( k' = k - N/2 \). Then

\[
 t = (q^{1/2})^{2k} = (q^{-1/2})^{N+2k'} = -(q^{1/2})^{2k'} \quad \text{and} \quad V_{2N-4k} \cong V'_{4|k'|}, \quad \text{setting} \ k' = k - N/2,
\]

where the latter module is well defined for primitive \( q^{1/2} \) of order \( 2N \). Since \( V'_{4|k'|} \) are well defined for any \( q \neq 0 \) and are irreducible for sufficiently general \( q \), they are flat \( q \)-deformations of modules \( V_{2N-4k} \) with all their structures. Namely, the dimension remains the same and the action of projective \( \text{PSL}_2(\mathbb{Z}) \) and the multiplication in \( V'_{4|k'|} \) (it is \( Y \)-spherical) deform those in \( V_{2N-4k} \). Then, \( V'_{4|k'|} \) is \( X \)- and \( Y \)-semisimple for generic \( q \) and the standard Hermitian inner product in \( V_{2N-4k} \) is the specialization of that in \( V'_{4|k'|} \), which is direct from (1.33).

Moreover, the inner product in \( V'_{4|k'|} \) remain positive definite if \( |q| = 1 \) provided \( |\arg(q)| \leq 2\pi/N \). \( \square \)

For instance, the \( q \)-deformation of the standard \( \hat{\mathfrak{sl}}_2 \)-Verlinde algebra \( V_{2N-4}^{\text{sym}} \) of (for \( k = 1 \) and of dimension \( N - 1 \)) is \( (V'_{4|k'|})^{\text{sym}} \) for \( k' = 1 - N/2 \). We note that Verlinde algebras of type (\( \alpha \)) and their deformations can be defined for arbitrary (reduced) root systems.
The deformations of nonsymmetric and symmetric generalized Verlinde algebras (of type \( (\alpha) \)) were constructed in \([\text{Ch1}]\) only for odd \( N \) and integral \( 0 < k < N/2 \). Also, it was done there upon the restriction to the nonsymmetric Little Verlinde algebra, defined as the image \( V_{N-2k}^{\text{even}} \) of even polynomials from \( \mathcal{X} \) in \( V_{2N-4k} \). It is a module over Little \( \text{DAHA} \) \( \mathcal{H}^{\text{ev}} \), defined as the span of \( X_{\pm}^{2}, Y_{\pm}^{2}, T \) in \( \mathcal{H} \).

The polynomials \( e_m \) are of the same parity as \( m \), so the subspace (and a subalgebra) \( V_{N-2k}^{\text{even}} \) is linearly generated by \( e'_m \) over \( \mathbb{Z}
\left(\frac{N}{}\right) q \) for even \( m \) taken from the set \( \{2k - N + 1, \ldots, N - 2k\} \) (note that the endpoints here have different parity). It is irreducible for odd \( N \) and has two irreducible components of dimension \( N/2 - k \) for even \( N \).

Accordingly, \( V_{N-2k}^{\text{odd}} \) is the \( \mathcal{H}^{\text{ev}} \)-module generated by the images of odd polynomials from \( \mathcal{X} \); it has the same dimension \( N - 2k \).

The claim from \([\text{Ch1}]\) is as follows:

\[
V_{N-2k}^{\text{even}} \simeq V_{2|k'|}, \text{ where } k' = k - N/2 \text{ for odd } N, 
\]

which is upon the restriction to \( \mathcal{H}^{\text{ev}} \) and when \( q^{1/2} = -e^{\nu \pi i/N} \) for \( \gcd(\nu, 2N) = 1 \). Therefore \( V_{2|k'|} \) for generic \( q \) are flat deformations of \( V_{N-2k}^{\text{even}} \) (with all their structures). Note that the choice of the sign of \( q^{1/2} \) influences \( t' = q^{k'} \) (\( k' \) is a half-integer) and makes the inner product in \( V_{2|k'|} \) coinciding with that in \( V_{N-2k}^{\text{even}} \). The inner product in the latter module involves only \( q \); see \((1.9)\). Picking \( q = e^{2\pi i/N} \), the “smallest” primitive \( N \)th root of unity, makes the inner product in \( V_{N-2k}^{\text{even}} \) positive definite for any \( 2k \in \mathbb{N} \) here and above.

The following proposition is related to \((1.33)\), however it is not its reformulation.

**Proposition 1.7.** For odd \( N \) and \( \mathbb{Z} \ni k < N/2 \), let \( q = e^{2\nu \pi i/N} \), where \( \gcd(\nu, N) = 1 \). Then the module \( V_{N-2k}^{\alpha'} \) of type \( (\alpha') \) from Proposition 1.5 is isomorphic to \( V_{2|k'|}^{\alpha'} \) for \( k' = k - N/2 \).

Note the operator \( X^{\alpha'} \) coincides with \( X^2 \) in \( V_{2N-4k} \) for the basis of \( X \)-eigenfunctions. However the corresponding \( Y^{\alpha'} \) (in the same basis) is not connected with \( Y \) in any direct way.

1.5. **Boundary cases.** With minor reservations, the statements of Theorem 1.4 can be extended to the boundary cases, namely when
$t = \pm 1$ and, respectively, $k = 0, -N/2$. The list of irreducible $\tau_{\pm}$-invariant quotients of $\mathcal{X}$ from (ii) then becomes:

$$
(1.34) \quad \begin{align*}
(a^*) \ t = 1, k = 0 : & \ V_{2N} = \mathcal{X}/(X^N - X^{-N}), \\
(b^*) \ t = -1, k = -\frac{N}{2}, \text{ even } N : & \ V_{4N} = \mathcal{X}/(X^{2N} + X^{-2N} - 2), \\
(c^*) \ t = -1, k = -\frac{N}{2}, \text{ odd } N : & \ V_N^\pm = \mathcal{X}/((\sqrt{-1}X)^N - 1).
\end{align*}
$$

When $k = 0$ and $t = 1$, the $\mathcal{H}$-module $V_{2N}$ is irreducible $\tau_{\pm}$-invariant and dim-rigid. It is not unirigid, but is unique of dimension $2N$ up to applying $\iota : T \mapsto -T$. We note that applying $\iota$ does not change the quadratic equation for $T$ when $t = 1$, however $\iota(V_{2N})$ is not a $Y$-spherical module.

In this case, we can set $T = s$, where $s^2 = 1$, $sXs = X^{-1}$ and $sYs = Y^{-1}$. The restriction of $V_{2N}$ to $X, Y$ is the classical (unique) finite-dimensional module of the Weyl subalgebra $W \subset \mathcal{H}$ which is generated by $X^{\pm 1}, Y^{\pm 1}$ subject to the relation $Y^{-1}X^{-1}YXq^{1/2} = 1$. The action of $s$ in this module is determined uniquely up to the multiplication by $\pm 1$, which is exactly applying $\iota$. The isomorphism class of this $\mathcal{H}$-module is invariant under $s_x$ and $s_y$; the latter automorphism corresponds to the conjugation by the polynomial $X^N + X^{-N}$.

Now let $t = -1$. Then the eigenvalues of $T$ coincide and $T$ becomes non-semisimple in $\mathcal{X}$ and its quotients $V$ from $(b^*, c^*)$ unless $N = 1$. Indeed, the semisimplicity of $T$ would result in $T = t^{1/2}$ in such $V$ (but it is not a scalar).

For even $N$, the module $V_{4N} = V^{-2}$ is irreducible; it is a direct sum of Jordan 2-blocks for $X, Y$. This module is $\tau_{\pm}$-invariant and invariant with respect to $s_x$ and $s_y$. The invariance with respect to $s_y$ can be seen by extending the formulas for $t \neq -1$ to this special case. Here and for the description of $V_N^+$ for odd $N$ to be discussed next, see (iii, iv) in Theorem 2.9.3 from [Ch1].

Let us (re)establish the projective $PSL_2(\mathbb{Z})$-invariance of $V_{4N}$ using the rigidity approach. The problem is that the dimension $4N$ of $V_{4N}$ is maximal among irreducible representations for such $q, t$, so it is not dim-rigid. However the Jordan decomposition of $T$ in $V_{4N}$ is unique among all modules $V^C = \mathcal{X}/(X^{2N} + X^{-2N} + C)$. One has

$$
T(P\Delta_-) = tP\Delta_- - 2tP\Delta_+ \quad \text{for } \Delta_\pm = X\pm X^{-1} \text{ and any symmetric } P.
$$
Let \( P = (X^{2N} - X^{-2N})^2/(X + X^{-1}) \), which is a symmetric Laurent polynomials since \( N \) is even. Then

\[
(1.35) \quad T(P \Delta_) = iP \Delta_ - 2t(X^{2N} - X^{-2N})^2 = iP \Delta_ \quad \text{in} \quad V^{-2}.
\]

For \( C \neq -2 \), there will be an extra term \(-(C + 2)\) in the right-hand side. Also, there can be no nonsymmetric polynomials of degree smaller than \( P \) satisfying \( T(P) = iP \) in any \( V^C \).

Thus \( V^{-2} \) is a direct sum of \((2N - 1)\) Jordan 2-blocks with respect to \( T \) and 2 one-dimensional \( T \)-eigenspaces. The modules \( V^C \) for \( C \neq -2 \) are direct sums of \( 2N \) Jordan 2-blocks under the action of \( T \). Therefore \( V_{4N} = V^{-2} \) for \( t \) from \( (\beta^*) \) is, indeed, projective \( PSL_2(\mathbb{Z}) \)-invariant.

The \( \bullet \)-module \( V_N = V_N^+ \) for odd \( N \) is \textit{dim-rigid} and \( Y \)-semisimple; its \( Y \)-spectrum is simple. It is \( \tau_\pm \)-invariant, but not invariant with respect to \( \varsigma_X \) and \( \varsigma_y \). Let \( (\gamma^*) \) denote the natural boundary of the family \( (\gamma) \) from (1.25) for such \( t = -1, k = N/2 \).

The simplest (but instructional) example of odd \( N \) is the module \( V_1^+ \), i.e. when \( N = 1 \) and \( k = -1/2 \):

\[
t = q^{-1/2} = 1 \quad \text{and} \quad t^{1/2} = q^{-1/4} = i = \sqrt{-1},
\]

\[
V_1 = \mathcal{X}/(\iota X - 1), \quad \text{where} \quad T = Y = \iota X = X^{-1}.
\]

Note that \( T(\iota X - 1) = X^{-1} - \iota, \quad \pi(\iota X - 1) = -\iota X^{-1} - 1 \), so the ideal \( (\iota X - 1) \) is obviously \( \mathcal{H} \)-invariant in this case.

The existence of the action of \( \tau_+ \) in \( V_1^+ \) is equivalent to the relation

\[
\tau_+(Y) = Y, \quad \text{which obviously holds since} \quad \tau_+(Y) \overset{\text{def}}{=} q^{-1/4}XY = iXY = \iota(-i) = i = Y \quad \text{in} \quad V_1^+. \quad \text{Similarly,} \quad \tau_- \quad \text{acts in} \quad V_1^+ \quad \text{too and} \quad \tau_+ \tau_- = id \quad \text{in} \quad \text{this module. Note that} \quad \tau_+(Y) = -Y \quad \text{in} \quad \varsigma_y(V_1^+) \quad \text{and therefore} \quad \tau_+ \quad \text{does not act in} \quad \varsigma_y(V_1^+) \quad (\text{it is not a conjugation by a matrix}).
\]

1.6. \textbf{Finite and discrete images.} Let us define the elliptic \( q \)-braid group as follows:

\[
(1.36) \quad \mathcal{B}_q \overset{\text{def}}{=} \langle X, T, Y, q^{1/2} \rangle \quad \text{for a primitive \( (2N) \)th root} \ q,
\]

subject to \( TXT = X^{-1}, \ T^{-1}YT^{-1} = Y^{-1}, \ Y^{-1}X^{-1}YXT^{q^{1/2}} = X^{-1} \).

We will also denote by \( \mathcal{B}_q^{ev} \) its subgroups generated by \( X^2, T, Y^2 \) associated with \( \mathcal{H}^{ev} \). Recall that the latter algebra acts in \( V_{even}^{N-2k} \) and \( V_{odd}^{N-2k} \) (of dimension \( N - 2k \) over \( \mathbb{Z}_q \)) generated by the images of even and odd polynomials in \( V_{2N-4k} \) for the modules from \( (\alpha) \) in (1.24).
The elliptic braid group \( B_1 \) is obtained by omitting \( q^{1/2} \) in \( B_q \). It is the following renormalization of \( B_q \) (which will be used later):

\[
(1.37) \quad \text{setting } \tilde{T} = q^{1/4}T, \tilde{Y} = q^{1/4}Y, \tilde{X} = q^{-1/4}X, \quad \tilde{T}\tilde{X}\tilde{T} = \tilde{X}^{-1}, \quad \tilde{T}^{-1}\tilde{Y}\tilde{T}^{-1} = \tilde{Y}^{-1}, \quad \tilde{Y}^{-1}\tilde{X}^{-1}\tilde{Y}\tilde{X}\tilde{T}^2 = 1.
\]

We will describe in this section all cases of \textit{dim-rigid} modules such that the image \( \text{Image}(B_q) \) of \( B_q \) in the corresponding matrix groups is finite. Without loss of generality, we can restrict ourselves with \((\alpha, \beta, \gamma)\) from Theorem 1.4 and \((\alpha', \beta', \gamma')\) from Proposition 1.5; in these cases, accordingly, the field \( \mathbb{Q} \) (embedded into \( \mathbb{C} \)) does not change the finiteness.

We analyze the positivity of the inner product \( \langle \cdot, \cdot \rangle \). The same approach provides examples when \( \text{Image}(B_q) \) is \textit{discrete} to be discussed next (though not a classification of all such cases).

**Theorem 1.8.** We omit one-dimensional modules, those from \((\alpha^*)\) in Section 1.3 and \((\gamma^*)\) for \( t^r = 1 \) from Proposition 1.5 in these cases the images are finite. Then among the modules \((\alpha, \beta, \gamma)\) from Theorem 1.4 and those from Proposition 1.5, the group \( \text{Image}(B_q) \) can be finite only for \((\alpha)\) and \((\alpha^*)\) for the following \( k \):

- \((\alpha^{-1}_{N}) : k = (N - 1)/2, \ dimV = 2, \ (\alpha^{2}_{N}) : k = (N - 2)/2, \ dimV = 4.\)
- Apart from these series, there are only two exceptional cases:
  - \((\alpha^{1}_{6}) : N = 6, \ k = 1, \ dimV = 8, \ (\alpha^{3}_{10}) : N = 10, \ k = 3, \ dimV = 8.\)
  - Also, upon \( q \mapsto q^t = q^2, \ t \mapsto t^r = t^2 \) as in Proposition 1.5, there is the following additional series of such modules \( V_{N-2k}^{+\gamma} \) for even \( N > 2 \):
    - \((\alpha^{2\gamma}_{N}) : k = (N - 2)/2, \ dimV^{\gamma} = 2.\)

**Proof.** The operators \( T, X, Y \) are represented by matrices with the entries in \( \mathbb{Z}[q^{1/4}] \) (over \( \mathbb{Q}(q^{1/4}) \) in the basis of \( e \)-polynomials) for the modules from \((\alpha, \gamma)\). Actually \( \mathbb{Z}[q^{1/2}] \) or \( \mathbb{Q}(q^{1/2}) \) is sufficient here if we switch from \( T \) to \( t^{-1/2}T \) and \( Y \) to \( t^{-1/2}Y \), which holds for \((\beta)\) as well. Accordingly, the field \( \mathbb{Q}(q) \) is sufficient for modules \( V^{\gamma} \).

Now assume that \( \text{Image}(B_q) \) in \( \text{GL}(V) \) is finite for all \( (2N) \)th primitive root of unity \( q^{1/2} \) and the corresponding \( t^{1/2} \). Then for every such \( q \) (embedded into \( \mathbb{C} \)), there exists a positive definite hermitian inner
product $(\cdot, \cdot)$ in $V$ such that $(Hu, v) = (u, H^{-1}v)$ for $H = X, T, Y$ and any vectors $u, v \in V$. This results in the semisimplicity of $X, Y$ in $V$; therefore the cases $(\beta), (\beta^\vee)$ and $(\alpha^\vee)$ for odd $2k$ can be excluded from the consideration. Moreover, such an inner product must be proportional to the standard one $(\cdot, \cdot)$ in the cases $(\alpha, \gamma)$ due to the irreducibility of $V$.

We conclude that the finiteness of $\text{Image}(B_q)$ may occur only for $(\alpha, \gamma)$ and for $(\alpha^\vee)$ when $k$ is integral. This is equivalent to the positivity of $(\cdot, \cdot)$ for any primitive $q^{1/2}$, i.e. for any embedding $\mathbb{Z}[q^{1/2}] \rightarrow \mathbb{C}$. It is straightforward to show that only $(\alpha_N^{-1}), (\alpha_N^{-2}), (\alpha_0^2), (\alpha_0^3), (\alpha_N^{-2\alpha^\vee})$ posses such positivity, though we use computers to find that there only 2 exceptional cases (apart from the infinite series).

**Comment.** One can examine the total positivity of the form $(\cdot, \cdot)$ upon the restriction to $V^{\text{sym}}$. The only new case of total positivity (for all $\nu$) is the Verlinde series $k = 1$; in this case, $\langle p_n, p_n \rangle = 1$ for any Rogers-Macdonald polynomials $p_n$. Furthermore, there are no new cases of total positivity upon the restriction to the even part $V^{\text{sym}+\text{even}}$ of $V^{\text{sym}}$ from $(\alpha, \gamma)$. However we have the following additional cases of total positivity for $\tilde{V} = V^{\text{sym}+\text{odd}}_{N-2k} = V^{\text{sym}}_{N-2k} \cap V^{\text{odd}}_{N-2k}$:

$$k = (N-4)/2, \quad \dim \tilde{V} = 4 \text{ for all } N > 4, \quad N = 12, \quad k = 2, \quad \dim \tilde{V} = 8.$$ 

The groups $B_q, B^e_q$ do not preserve $V^{\text{sym}}_{N-2k}$, though the projective $\text{PSL}_2(\mathbb{Z})$ and the absolute Galois group (see below) act there. Note that $B^e_q$ preserves $V^{\text{odd}}$ and $V^{\text{even}}$, but the (total) positivity property remains the same in these modules as for $V$. Indeed, $(e_n, e_n) = \langle e_{1-n}, e_{1-n} \rangle$ for any $n$ (provided the existence of $e_n$).

**Proposition 1.9.** The center of the group $\text{Image}(B_q)$ for $V$ of types $(\alpha, \beta, \gamma)$ belongs to the group generated by the scalar matrix $\dot{q}$, where $\dot{q} = q^{1/2}$ for $2k \in \mathbb{Z}$ and $\dot{q} = q^{1/4}$ for $2k \in 1/2 + \mathbb{Z}$.

**Proof.** Due to the irreducibility of $V$, the center elements are scalars $z \in \mathbb{Z}[\dot{q}]$. Following the proof of Theorem 1.8, $z$ is unimodular for types $(\alpha, \gamma)$ for any embeddings $\mathbb{Z}[\dot{q}] \rightarrow \mathbb{C}$. The corresponding hermitian forms can be non-positive, but this is not a problem since we consider only central elements. Thus $z$ is a root of unity, which can be only a power of $\dot{q}$. Modules of type $(\beta)$ posses hermitian invariant (non-positive) forms too, the leading terms of those in the polynomial representations covering $V$. \qed
Discrete braid-images. It is of interest to apply the method used in the proof of the theorem, to construct examples of discrete groups \( \text{Image}(B_q) \). We can use that upon the switch from \( T \) to \( t^{1/2}T \) and \( Y \) to \( t^{1/2}Y \), the images of \( T, Y, X \) can be made with entries in \( \mathbb{Z}[q^{1/2}] \); in the basis of \( e_m \) for \( m = 2k - N + 1, \ldots, N - 2k \), they are matrices over \( \mathbb{Q}(q^{1/2}) \), which field is sufficient here.

The discreteness of non-finite \( \text{Image}(B_q) \) holds if there is exactly one complex place (valuation) of \( \mathbb{Q}(q^{1/2}) \) where the form \( \langle \cdot, \cdot \rangle \) is non-positive; in this case the groups \( \text{Image}(B_q) \) are arithmetic discrete. Respectively, \( \mathbb{Q}(q) \) must be considered for the \( \sqrt{\cdot} \)-series. Omitting \( (\beta') \) and the series \( (\alpha') \) for odd \( 2k \) from Proposition 1.5, our considerations show that there are only finitely many such cases.

Let us list what was found. We set \( q^{1/2} = e^{\pm \pi i \nu / N} \) when \( 1 \leq \nu \leq 2N \), \( (\nu, 2N) = 1 \). Only the modules of type \((\alpha', \gamma')\) will be considered below. The results of our analysis and computer calculations are as follows.

There exists exactly one pair \( q^{\pm 1/2} = e^{\pm \pi i \nu / N} \) of primitive \( (2N) \)th roots of unity such that the inner product \( \langle \cdot, \cdot \rangle \) is non-positive only (presumably) for the following modules of type \((\alpha)\):

\[
\begin{align*}
\{N = 4, k = 1/2, \dim V = 6, \nu = 3\}, & \quad \{N = 5, k = 1/2, \dim V = 8, \nu = 3\}, \\
\{N = 5, k = 1, \dim V = 6, \nu = 3\}, & \quad \{N = 6, k = 1/2, \dim V = 10, \nu = 5\}, \\
\{N = 6, k = 3/2, \dim V = 6, \nu = 5\}, & \quad \{N = 7, k = 3/2, \dim V = 8, \nu = 3\}, \\
\{N = 7, k = 2, \dim V = 6, \nu = 5\}, & \quad \{N = 9, k = 2, \dim V = 10, \nu = 7\}, \\
\{N = 9, k = 5/2, \dim V = 8, \nu = 5\}, & \quad \{N = 15, k = 4, \dim V = 14, \nu = 13\}, \\
\{N = 15, k = 11/2, \dim V = 8, \nu = 7\}. & \quad (1.38)
\end{align*}
\]

Let us now discuss the type \((\alpha')\) for integral \( k \) and \((\gamma')\). In type \((\alpha')\), we have the following 4 additional cases (all with \( \nu = 1 \)):

\[
\begin{align*}
\{N = 5, k = 1, N = 7, k = 2 \dim V = 3\}, & \quad \{N = 9, k = 2, \dim V = 5\}, \quad \{N = 15, k = 4, \dim V = 7\}. \\
\end{align*}
\]

These additional cases exactly correspond to the ones in (1.38) for odd \( N \) and integral \( k \), but they are of half-dimension. They also coincide with those obtained in (1.41) below from \((\gamma')\). The coincidence is due to the isomorphism from Proposition 1.7.
In type $(\gamma)$, the inner product is always indefinite for $\nu = \pm 1$; it is positive definite at all other places in the following cases:

$$(1.40) \quad \{ N = 3, 4, 5, 6, 9, k = -3/2, \dim V = 3 \},$$
$$(1.40) \quad \{ N = 6, k = -5/2, \dim V = 5 \}.$$

For $(\gamma')$, we also have the following 4 additional cases:

$$(1.41) \quad \{ N = 5, 7, k = -3/2, \dim V = 3 \},$$
$$(1.41) \quad \{ N = 9, k = -5/2, \dim V = 5 \}, \{ N = 15, k = -7/2, \dim V = 7 \}.$$

This list is actually that in (1.39).

Summarizing, the groups $\text{Image}(B_q)$ from (1.38-1.40), are discrete and non-finite in $\text{GL}_C(V)$ with respect to the embedding $\mathbb{Q}(q^{1/2}) \hookrightarrow \mathbb{C}$ such that $q^{1/2} \mapsto e^{\pi i \nu N}$, where $\nu$ is from these lists.

The sequences (1.40), (1.41) for $\dim V = 3$ are connected in Proposition 2.5 below with the arithmetic Livné discrete groups. Moreover, the theory of the latter provides an interesting example, when $B_q$ is discrete but not arithmetic. It occurs for $\dim V = 3, N = 9$; this case is missing in (1.41) because there are two places (and four $\nu$) such that the inner product $\langle \cdot, \cdot \rangle$ is non-positive. Nevertheless the corresponding group is discrete! We mention that the Livné groups are examples of Mostow’s groups [Mos]. It is interesting to compare the cases with $\dim V > 3$ listed above with the Mostow list.

**Conjecture 1.10.** Up to isomorphisms and changing the signs of the images of $T, Y, X$, arithmetic discrete non-finite groups $\text{Image}(B_q)$ for irreducible $\mathcal{H}$–modules and irreducible $\mathcal{H}'$–modules with semisimple $X$ and $Y$ can be only those from the lists (1.38-1.41).

According to R. Schwartz, discrete images cannot appear if $T$ is elliptic and $X$ or $Y$ is parabolic when $\dim V = 3$; see Theorem 1.1 from [Par1]. Thus the semisimplicity of $X, Y$ does not seem really necessary.

**1.7. Series of $\dim = 2, 4$.** Let us give a complete description of the series $(\alpha_N^{-1})$: $k = (N - 1)/2, \dim V = 2$. Using Section 1.4 it suffices to consider $V'_q$ defined for $t = -q^{-1/2}$ for any $q$; then we will make $t = q^{N-1}$ for primitive $(2N)$th roots of unity $q^{1/2}$. 
One has $V'_2 = \mathcal{X}/(e_{-1}) = \mathbb{Z}_{q^t} e'_0 \oplus \mathbb{Z}_{q^t} e'_1$, where here and below $e'_i$ is the image of $e_i$ in $V'$. Explicitly,

$$e_0 = 1, \ e_1 = X, \ e_{-1} = X^{-1} + \frac{1 - t}{1 - tq} X = X^{-1} - tX,$$

$$Y(e_0) = t^{1/2}, \ Y(e_1) = -t^{1/2} e_1, \ T(e'_i) = t^{1/2} e'_i \ (i = 0, 1)$$

Thus this representation is as follows:

$$X = \begin{pmatrix} 0 & t^{-1} \\ 1 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} t^{1/2} & 0 \\ 0 & -t^{1/2} \end{pmatrix}, \ T = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{1/2} \end{pmatrix}. \tag{1.42}$$

The relations $Y^{-1} X^{-1} Y X = -1, \ Y^2 = X^{-2} = t = -q^{-1/2}$ and $T = t^{1/2}$ determine $\text{Image}(\mathcal{B}_q)$ up to isomorphisms. Equivalently,

$$\tilde{Y}^{-1} \tilde{X}^{-1} \tilde{Y} \tilde{X} = -1 = \tilde{T}^2 = \tilde{Y}^2 = \tilde{X}^2,$$

in the normalization of (1.37) from $\mathcal{B}_q$ to $\mathcal{B}_1$.

If $q^{1/2}$ is a primitive $(2N)$th root of unity, the group $\text{Image}(\mathcal{B}_q)$ is finite of order $16N$ unless $(t^{1/2})^N = 1$ for odd $N$ and the order is $8N$. Accordingly, $\text{Image}(\mathcal{B}_1)$ divided by its center, generated by $i$, is $\mathbb{Z}_2$. 

**Dimension 4.** The series $(\alpha_N^{-2})$ of dimension 4 is as follows. We will begin with the image of $\mathcal{B}_q$ in (the automorphisms of) $V'_4$. Recall that the latter module is defined for any $q$ and $t = -q^{-1}$. The module in $(\alpha_N^{-2})$ is the specialization of $V'_4$ for $t = q^{N/2}$ and a primitive $(2N)$th root of unity $q^{1/2}$. We set $t^{1/2} = q^{1/2}$ for $i = \sqrt{-1}$.

One has $V'_4 = \mathcal{X}/(e_{-2}) = \oplus_{i=0}^3 \mathbb{Z}_{q^{it}} e'_i$, where $e_{-2} = (X^2 + 1)(X^{-2} - t)$,

$$e_0 = 1, \ e_1 = X, \ e_{-1} = X^{-1} + \frac{1 - t}{2} X, \ e_2 = X^2 + \frac{1 - t^{-1}}{2},$$

$$Y(e_0) = t^{1/2}, \ Y(e_1) = -e_1, \ Y(e_{-1}) = i e_1, \ Y(e_2) = -t^{1/2} e_2,$$

$$T(e_0) = t^{1/2} e_0, \ T(e_1) = \frac{t^{1/2} - t^{-1/2}}{2} e_1 + t^{-1/2} e_{-1}, \tag{1.43}$$

$$T(e_{-1}) = \frac{t + t^{-1} + 2}{4 t^{-1/2}} e_1 + \frac{t^{1/2} - t^{-1/2}}{2} e_{-1}, \ T(e'_2) = t^{1/2} e'_2,$$

$$T(e'_{-2}) = -t^{-1/2} e'_{-2}, \ X e_0 = e_1, \ X e_1 = -\frac{1 - t^{-1}}{2} e_0 + e_2,$$

$$X e_{-1} = \frac{t + t^{-1} + 2}{4} e_0 + \frac{1 - t}{2} e_2, \ X e'_2 = t^{-1} e'_{-1} \text{ (in } V'_4).$$
We see that $Y^2$ is scalar in the even and odd subspaces of $V'_4$, which are $\mathbb{Z}_{q,t} e'_0 \oplus \mathbb{Z}_{q,t} e'_2$ and $\mathbb{Z}_{q,t} e'_1 \oplus \mathbb{Z}_{q,t} e'_{-1}$. Since $T$ and $X^2$ preserve these subspaces, we obtain that $TY^2 = Y^2T$ and $X^2Y^2 = Y^2X^2$. Applying $\sigma^{-1}$ from $(1.4)$ in $V'_4$ (this module is projective $PSL_2(\mathbb{Z})$-invariant), we conclude that $TX^2 = X^2T$. Therefore $\text{Image}(B_q)$ contains the commutative subgroup $N \overset{\text{def}}{=} \langle T, X^2, Y^2, q^{1/2} \rangle$. It is normal due to the following relations:

\begin{equation}
(1.44) \quad XTX^{-1} = T^{-1}X^{-2}, \quad YT^{-1}Y^{-1} = TY^{-2}, \quad Y^{-1}X^2Y = qX^{-2}, \quad X^{-1}Y^2X = q^{-1}Y^{-2}.
\end{equation}

The latter two relations result from $Y^{-1}XY = q^{1/2}XT^2$ and commutativity of $X^2, Y^2$ with $T$. For instance,

\[
Y^{-1}X^2Y = (q^{1/2}XT^2)^2 = qX(T^2X)T^2 = qX(TXX^{-2}T^{-1})T^2 = qX(X^{-3}T^{-2})T^2 = qX^{-2}.
\]

Due to $X^{-1}Y^{-1}XY = q^{1/2}T^2$ and relations from $(1.44)$, the commutator subgroup of $\text{Image}(B_q)$ (the span of all group commutators) equals $C = < q^{1/2}T^2, q^{1/2}Y^2, q^{-1/2}X^2 > = < \tilde{T}^2, \tilde{Y}^2, \tilde{X}^2 >$ in notations from $(1.37)$.

Note that the switch from $\text{Image}(B_q)$ to $\text{Image}(B_1)$ and correspondingly from $X, Y, T$ to $\tilde{X}, \tilde{Y}, \tilde{T}$ will make the relations from $(1.44)$ without $q$. For instance, one has $\tilde{Y}^{-1}\tilde{X}^2\tilde{Y} = \tilde{X}^{-2}, \tilde{X}^{-1}\tilde{Y}^2\tilde{X} = \tilde{Y}^{-2}$.

We conclude that $\text{Image}(B_q)$ is an extension of $N$ by $\mathbb{Z}_2^2$. Accordingly, $\text{Image}(B_1)$ is an extension of $C$ by $\mathbb{Z}_2^3$. Also, the center of $\text{Image}(B_q)$ belongs to $N$ (actually, it is trivial in $\text{Image}(B_1)$; see below). Indeed, any product of $X, Y, T$ can be written as $X^\epsilon Y^\delta Z$ for $Z \in N$, $\epsilon \in \{0, 1\} \ni \delta$; thus, $\epsilon = 0 = \delta$ for central elements, since $X^\epsilon Y^\delta$ commutes with $N$ only when $\epsilon = 0 = \delta$.

Let $V_{\text{even}} = \mathbb{Z}_{q,t} e'_0 \oplus \mathbb{Z}_{q,t} e'_2$, $V_{\text{odd}} = \mathbb{Z}_{q,t} e'_1 \oplus \mathbb{Z}_{q,t} e'_{-1}$. One can check that in this basis

\[
X_{\text{even}}^2 = \left( \begin{array}{cc} t^{-1} & t^{-1} + 4 \frac{t^{-1} + 1}{2} \\ 2 & t^{-1} \end{array} \right), \quad X_{\text{odd}}^2 = \left( \begin{array}{cc} t^{-1} & t^{-1} + 4 \frac{t^{-1} + 1}{2} \\ t^{-1} & 2 \end{array} \right).
\]

Therefore $(TX^2)_{\text{odd}} = t^{-1/2} = (X^2T)_{\text{odd}}$.

Since $T = t^{1/2}, Y^2 = t$ in $V_{\text{even}}$ and $X^2$ has eigenvalues $-1, t^{-1}$ there, $X^{2M}$ in $V_{\text{even}}$ can be a product of powers of $T$ and $Y^2$ there if and only
if \((-t)^M = 1 = q^M\). This results in \(X^{2M} = (-1)^M\). Similarly, \(T\) in \(V^{\text{odd}}\) has eigenvalues \(t^{1/2}, -t^{-1/2}\) and \(T^M\) in \(V^{\text{odd}}\) can be a product of powers of \(TX^2 = t^{-1/2}\) and \(Y^2 = -1\) there if and only if \((-t)^M = 1 = q^M\). Thus \(T^M = t^{-M/2}\). Finally, the relation \(X^{2a}Y^{2b}T^c = 1\), where at least one of \(a, b, c\) is nonzero, implies \((-t)^N = 1 = q^N\) for some \(N \in \mathbb{N}\).

Continuing, let \(q\) be a primitive \(N\)th root of unity. We conclude that \(N \mid a\) and \(N \mid c\), which makes \(X, T\) scalars in \(V\). Since the eigenvalues of \(Y^2\) in \(V\) are \(-1, t\), we obtain that \(N \mid b\) and, finally, the group \(\text{Image}(\mathcal{N})\) modulo scalars is isomorphic to \(\mathbb{Z}_N^3\). Thus the order of \(\text{Image}(\mathcal{B}_q)\) modulo scalars is \(4N^3\) in this case. If there is no such \(N > 0\), then \(\mathcal{N}\) is isomorphic to \(\mathbb{Z}^3\).

Finally, the scalars form the center of \(\text{Image}(\mathcal{B}_q)\) due to the irreducibility of \(V\). The center is always \(\{q^{Z/2}\}\) (either for \(N > 0\) or if \(q\) is not a root of unity); it is trivial for the group \(\text{Image}(\mathcal{B}_1)\). This is checked as above.

**Compatibility for different \(N\).** Let \(d \mid N\) and \(\bar{N} = N/d > 2\) for odd \(d\). Then \(\mathcal{B}_q\) is irreducible for \(q^{1/2} \overset{\text{def}}{=} q^{d/2}\) and \(\hat{\mathcal{H}} = \mathcal{H}_{q^{d/2}, t^{d/2}}\). The quotient of \(\text{Image}(\mathcal{B}_q)\) by the center for a primitive \((2N)\)th root of unity \(q^{1/2}\) and that for \(q^{1/2}\) will be denoted by \(\mathfrak{B}_N^d\) and \(\mathfrak{B}_N^1\).

Then \(\mathfrak{B}_N^1\) is the extension of the commutative subgroup

\[
\mathcal{N}_N^1 = \langle T, X^2, Y^2 \rangle / \langle T^N, X^{2N}, Y^{2N} \rangle \simeq \mathbb{Z}_N^3
\]

by \(\mathbb{Z}_2^2 = \langle X, Y \rangle / \langle X^2, Y^2, XYX^{-1}Y^{-1} \rangle\) subject to the relations

\[
Y^{-1}XY = XT^2 = YXY^{-1}, \quad XTX^{-1} = T^{-1}X^{-2} = X^{-1}TX, \\
YT^{-1}Y^{-1} = TY^{-2} = Y^{-1}T^{-1}Y, \quad Y^{-1}X^2Y = X^{-2} = YX^2Y^{-1},
\]

\[
X^{-1}Y^2X = Y^{-2} = XY^2X^{-1}.
\]

Furthermore, the following two equivalent relations hold in \(\mathfrak{B}_q^1\) for any \(q \in \mathbb{C}^*\) and odd \(m = 2l + 1\):

\[
(Y^mX^{-m})^2 = Z \overset{\text{def}}{=} Y^2T^{-2}X^{-2}, \quad Y^{-m}X^{-m}Y^mX^m = ZY^{-2m}X^{2m}.
\]

Indeed, using (1.46) and the commutativity of \(\mathcal{N}\),

\[
Y^{-m}X^{-m}Y^mX^m = (Y^{-m}X^{-2m}Y^m)(Y^{-m}X^{-1}Y^m)X^m = X^{2l}(T^{-2}Y^{-4}X^{-1})X^{2l+1} = X^{4l}T^{-2}Y^{-4l}, \quad \text{where}
\]

\[
Y^{-m}X^{-1}Y^m = Y^{-2l-1}X^{-1}Y^{2l+1} = Y^{-1}(Y^{-4l}X^{-1})Y = Y^{-4l}T^{-2}X^{-1}.
\]
We note that $XZX^{-1} = Z^{-1} = YZY^{-1}$, $TZ = ZT$ in $B_q^\dagger$. Also, $Z = q^{-1/2} \tau_-(X^2)$; $Z = X^{-2}$ in $V^{even}$ and $Z = -t^{1/2} T^{-1}$ in $V^{odd}$. The eigenvalues of $Z$ in each of these two spaces coincide and are $\{ -1, t \} = \{ -1, -q^{-1} \}$; recall that $t = -q^{-1}$. Actually, the eigenvalues of $Z$ are important here only up to a common factor.

Next, $Y^m T^{-m} Y^{-m} = Y T^m Y^{-1} = T^m Y^{-2m}$ and $Y^{-m} X^2m Y^{-m} = Y^{-1} X^2 m Y^{-1} X^{-2m}$ for odd $m$, which follows from (1.46) and the commutativity of $\mathcal{N}$. The same holds for $X^{-1}$ instead of $Y$. Finally, setting $Z = Y^{2d} T^{-2d} X^{-2}$ (see (1.47)), we obtain the homomorphism

$$
(1.48) \quad \mathfrak{B}^\dagger_N / <Z> \to \mathfrak{B}^\dagger_N / <\tilde{Z}>, \quad X \mapsto X^d, Y \mapsto Y^d, T \mapsto T^d.
$$

2. The Galois action

2.1. Deligne-Simpson Problem. Following our approach to the Tate modules from the Introduction and Section 2.7.3 from [Ch1], let us switch to the $A, B, C$-generators of $B_q$.

Setting $A = XT$, $B = XTY$, $C = T^{-1}Y$, the relations of $B_q$ and the action of $\tau_\pm$ become

$$
A^2 = 1 = C^2 = q^{1/2} B^2, \quad \text{where} \quad ABC = A^2 YT^{-1} Y = YY^{-1} T = T,
$$

(2.1)

$$
\tau_+ : A \mapsto A, \quad B \mapsto q^{-1/4} C, \quad C \mapsto q^{1/4} C^{-1} B C,
$$

$$
\tau_- : A \mapsto q^{1/4} A B A^{-1}, \quad B \mapsto q^{-1/4} A, \quad C \mapsto C.
$$

Also, the automorphisms $\varsigma_x, \varsigma_y$ from (1.12), (1.13) act as follows:

$$
(2.2) \quad \varsigma_x : A \mapsto -A, B \mapsto -B, C \mapsto C, \quad \varsigma_y : A \mapsto A, B \mapsto -B, C \mapsto -C.
$$

Let us now fix a primitive root of unity $q^{1/2}$ of order $2N$. We take an $\mathcal{H}^\vee$-modules $V$ that is dim-rigid and loc-dim-rigid from Theorem 1.3 and pick prime $p$ such that $\gcd(p, 2N) = 1$. Then we choose a prime ideal $\mathfrak{p}$ in $\mathbb{Z}_q$ over $(p)$ for $\bar{q} = q^{1/4} = (q^{1/2})^{1/2}$ for even $2k$ and $\bar{q} = q^{1/2}$ if $k \in 1/2 + \mathbb{Z}$. Also, all boundary modules from (1.34) will be allowed (though not all of them are dim-rigid).

Similarly, we can take a dim-rigid $\mathcal{H}^\vee$-module $V^\vee$ for odd $N$ from Proposition 1.5. In this case the condition $\gcd(p, N) = 1$ is sufficient and $\bar{q}$ must be modified as follows:

$$
\bar{q}^\vee = q^{1/2} \quad \text{for even} \ 2k \quad \text{and} \quad \bar{q}^\vee = q \quad \text{if} \ k \in 1/2 + \mathbb{Z}.
$$

Let $\mathfrak{B}_{p,m} = \mathfrak{B}(g, t, p, m)$ be $\text{Image}(\mathfrak{B}_q)$ and $\tilde{\mathfrak{B}}_{p,m} = \text{Image}(\mathfrak{B}_1)$ from (1.37). The groups $\mathfrak{B}$ and $\tilde{\mathfrak{B}}$ are the images of $\mathfrak{B}_q$ and $\mathfrak{B}_1$ without the
reduction modulo $p^m$. The group $\mathcal{B}_1$ is obtained from $\mathcal{B}_q$ by sending $q^{1/4} \mapsto 1$ in all relations. It is generated by

\[
\tilde{T} = q^{1/4}T, \quad \tilde{Y} = q^{-1/4}Y, \quad \tilde{X} = q^{-1/4}X \quad \text{or}
\]

\[
\tilde{A} = XT, \quad \tilde{B} = q^{1/4}XTY, \quad \tilde{C} = T^{-1}Y.
\]

(2.3)

For instance, relations (1.42) for $V_2$ and $t^{1/2}q^{1/4} = \pm t$ read

\[
\tilde{X} = \begin{pmatrix} 0 & q^{1/4} \\ q^{-1/4} & 0 \end{pmatrix}, \quad \tilde{Y} = \begin{pmatrix} \pm t & 0 \\ 0 & \mp tq^{1/2} \end{pmatrix}, \quad \tilde{T} = \pm t,
\]

\[
\tilde{A} = XT = \begin{pmatrix} 0 & t^{-1/2} \\ t^{1/2} & 0 \end{pmatrix}, \quad \tilde{C} = T^{-1}Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
\tilde{B} = q^{1/4}XTY = q^{1/4} \begin{pmatrix} 0 & -1 \\ t & 0 \end{pmatrix} = \begin{pmatrix} 0 & \pm t^{-1/2} \\ \pm t^{1/2} & 0 \end{pmatrix}.
\]

Thus the matrices $\tilde{A}, \tilde{B}, \tilde{C}$ for $V_2$ are conjugated to the classical Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ for $\pm = +$ (plug in here $t^{1/2} = 1$).

Thus $\mathfrak{B}_{p,m}$ is represented by invertible matrices of dimension dim$V$ or dim$V'$ with entries in $\mathbb{Z}_q = \mathbb{Z}[\tilde{q}]/p^m$ for $p > 0$ and with entries in $\mathbb{Q}(\tilde{q})$ if $p = 0 = m$ (by definition). Here $\tilde{q}'$ must be used for $V'$, which is the square of $\tilde{q}$.

Accordingly, $\mathfrak{B}_{(p),m}$ and $\tilde{\mathfrak{B}}_{(p),m}$ are defined for the image of $\mathcal{B}_q, \mathcal{B}_1$ in

\[
\text{GL}(V \otimes_{\mathbb{Z}_q} \mathbb{Z}_q/(p^m)) = \prod_{p \supset (p)} \text{GL}(V \otimes_{\mathbb{Z}_q} \mathbb{Z}_q/(p^m)), \quad q' = \tilde{q}, \tilde{q}' \quad \text{for} \quad \mathfrak{B}, \tilde{\mathfrak{B}};
\]

where $p$ are prime ideals in $\mathbb{Z}_q'$ over $(p)$, $\gcd(p, 2N) = 1$. This is unless for $p = 0$, where the definition field is $\mathbb{Q}(q')$. One can take here $V'$ for odd $N$ provided $\gcd(p, N) = 1$; then $q'$ must be replaced by $(q')^2$.

From now on, we will mainly switch from $V$ to $\tilde{V}$ and from $V'$ to $\tilde{V}'$; the tilde-modules are (by definition) the same vector spaces supplied with the action of $\mathcal{B}_1$ via the generators $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{T}$. The matrix entries of these generators can be assumed to belong to $\mathbb{Z}_q$ or $\mathbb{Z}_{\tilde{q}'}$.

Also, unless stated otherwise, we will always perform the reduction modulo $p^m$ or $(p^m)$ except for the finite $\text{Image} (\mathcal{B}_q)$ (see Theorem 1.8).

In the latter case, we set $p = 0 = m$ and consider $\tilde{V}$ and $\tilde{V}'$ over $\mathbb{Q}(\tilde{q})$ and $\mathbb{Q}(\tilde{q}')$. Thus $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{T}$ will be considered below over $\mathbb{Z}_q/p^m$ for $m \in \mathbb{N}$ or over $\mathbb{Z}_q/(p^m)$ (and with $\tilde{q} \mapsto \tilde{q}'$ for $V'$).
Deligne-Simpson problem. Using (2.1), let us reformulate Theorem 1.4 as the multiplicative (irreducible) Deligne-Simpson problem, abbreviated as DSP. See [Kos] for a survey on DSP.

We will mainly focus in this section on the case without ✓ (when \(q^{1/2}\) is primitive of order 2N). Though see Part (ii) of Theorem 2.1 below on the modules \(V^\gamma\) for odd \(N\); recall that one must change \(q, t\) to \(q^2, t^2\) and square \(\tilde{q}\) in this case.

We will use \(\sim\) for the equivalence of matrices over the field \(\mathbb{Q}(\tilde{q})\) for \(\tilde{q} = q^{1/4}\) (even 2k) and \(\tilde{q} = q^{1/2}\) (odd 2k) or, correspondingly, over \(\mathbb{Z}_{\tilde{q}}/p^m\) for prime \(p \supset (p)\) when \(p > 0\) and \(\gcd(p, 2N) = 1\). We will denote by \(D^{(l)}_{\lambda, \mu}\), the diagonal matrix with \(l\) diagonal entries \(\lambda\) and the diagonal entries \(\mu\) otherwise. We take \(q^{1/2}, t^{1/2} = \pm (q^{1/2})^k\) as in Theorem 1.4 for the series \((\alpha, \beta, \gamma)\).

**Theorem 2.1.** (i) Let \(M = \dim V = 2N \pm 4|k|, 2|k|\) for \(V\) from \((\alpha, \beta, \gamma)\) of Theorem 1.4 (without applying ✓ for the sake of definiteness); we set \(n = [(M + 1)/2], n' = [M/2] + 1\). We will assume that \(t \neq -1\) to ensure the semisimplicity of \(T\). The following irreducible DSP in invertible \(M \times M\) matrices over \(\mathbb{Q}(\tilde{q})\) or \(\mathbb{Z}_{\tilde{q}}/p^m\) (as above) has a unique solution up conjugation by a (common) invertible matrix:

\[
\tilde{A} \sim D^{(l_a)}_{1, -1}, \quad \tilde{B} \sim D^{(l_b)}_{1, -1}, \quad \tilde{C} \sim D^{(l_c)}_{1, -1}, \quad \tilde{T} = \tilde{A} \tilde{B} \tilde{C} \sim D^{(n')}_{q^{1/4} t^{1/2}, -q^{1/4} t^{1/2}},
\]

(2.5) where \(l_a, l_b, l_c = n\) for the modules from \((\alpha), (\beta)\) and \(V^+_2\) or \(V^-_2\), respectively for the modules \(\varsigma_x(V^+_2), \varsigma_y(V^+_2), \varsigma_x \varsigma_y(V^+_2)\).

Furthermore, there are no irreducible solutions of (2.5) with such \(M\) and \(k\) for any other combinations of \(l_{a,b,c}, n, n'\).

(ii) In the cases \((\alpha', \beta', \gamma')\) for odd \(N\), the dimension \(M\) is correspondingly \(N \pm 2|k|\) and \(2|k|\), except for \((\alpha')\) with odd 2k, when \(\dim = 2N - 4k\) and the action of \(X, Y\) in \(V^\gamma\) becomes non-semisimple. The eigenvalues of \(\tilde{T}^\gamma\) are obtained by squaring \(t\) (\(t^2 \neq -1\) since \(N\) is odd). In parallel with (i), \(l_a = l_b = l_c = n\) for the modules \(V^\gamma_M\). The transformations of \(l\)–multiplicities under applying \(\varsigma_x, \varsigma_y\) to \(V^\gamma_{2N-4k}\) for even 2k and to \(V^\gamma_{2N+4k}\) are described exactly as for \(\varsigma_x \varsigma_y(V^-_2)\) in (2.5). Accordingly, the ✓–variant of the DSP from (2.5) has solutions only for such \(l_{a,b,c}, n, n'\). \(\square\)
Let us briefly comment on $V^+_2$ from (γ); its dimension is $M = 2n - 1$ and $l_a, l_b, l_c = n$. The fact that the outer automorphisms $\zeta_x, \zeta_y$, and $\zeta_x\zeta_y$ (when applying to $V^+_2$) correspondingly diminish $l_a = l_b = l_c$ and $l_a = l_c$ by 1 follows directly from the definition of $\zeta_x, \zeta_y$. The same holds for the $\check{\phi}$-modules except for $(\alpha^\ell)$ and odd $2k$.

Note that the DAHA-parameters $q^{1/2}, t^{1/2}$ are not uniquely recoverable from this DSP. By altering the sign of $q^{1/4}$, one can change the sign of $t^{1/2}$, which exactly corresponds to applying $\iota$. Recall that $\iota$ can be interpreted as the substitution $k \mapsto -k$, sending $q^{1/4}t^{1/2} \mapsto q^{1/4}t^{-1/2}$. This transformation changes $n'$ to $M - n'$ and does not influence $l_{a,b,c}$.

Also, the irreducibility of DSP in (2.2.3) can be omitted for $q, t = q^k$ and $M$ as in $(\alpha, \gamma)$. The minimality of dimensions of modules $V_{2n-4k}$ and $V^+_2$ gives that $\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{\mathcal{C}}$ have no nontrivial invariant subspaces.

We see that, generally, irreducible $\mathcal{H}$-modules provide solutions of DSP for the quiver of type $D^{(1)}_4$. The other way around, one can use this DSP for finding irreducible DAHA-modules. See Theorem 1 from [ObS] devoted to the $C_1$-case (apart from the roots of unity), which is directly related to Theorem 1.5 from [CrB]. This paper is based on the roots of unity. Any rigid modules for generic $q$ remain rigid at sufficiently general roots of unity, but we need more than these.

The cases of non-semisimple $T$ for $t = -1$, including the boundary case $(\beta^\star)$, can be readily managed as well; this requires using Jordan normal forms in DSP as in [CrB]. Note that $t' = t^2 = q^{2k}$ cannot be $-1$, since $N$ is odd; so $T'$ is always semisimple. Generally, there is no connection with non-semisimplicity of $T$ and that for $X$ and $Y$, which occurs for $(\beta), (\beta^\ell)$ and for $(\alpha^\ell)$ when $2k$ is odd.

2.2. Absolute Galois group. The passage to $\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{\mathcal{C}}$ identifies $\mathcal{B}_1$, considered as the orbifold fundamental group of $(E_\mathcal{C} \setminus \{0\})/\mathbb{S}_2$ for an elliptic curve $E_\mathcal{C}$ (over $\mathbb{C}$) punctured at 0, with the standard fundamental group $\pi_1(P^1_\mathbb{C} \setminus \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}, b_0)$. See [Ch1], Proposition 2.7.6. The base point $b_0$ here and below is sufficiently general; $s \in \mathbb{S}_2$ is the reflection $x \mapsto -x$ in $E_\mathcal{C}$. The normal subgroup $<X, Y, T^2> \subset \mathcal{B}_1$ of index 2 is associated then with the corresponding covers of $E_\mathcal{C}$.

In the main theorem, $E \ni 0$ will be a (smooth projective) elliptic curve and its zero defined over the field $K \subset \overline{\mathbb{Q}} \subset \mathbb{C}$. The points $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ will be the images of the points $\{0, 0, 0, 0\}$ of the 2nd order in $E$ under the isomorphism $E/\mathbb{S}_2 \to P^1$ over $K$. 

The Riemann Existence Theorem, RET, provides the existence of a (connected) Riemann surface $F_{\mathbb{C}}$ such that it is a Galois cover of $P^1_{\mathbb{C}}$ ramified at points $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and with the Galois group $\text{Aut}(F_{\mathbb{C}}/P^1_{\mathbb{C}})$ isomorphic to $\tilde{\mathfrak{B}}_{p,m}$. Furthermore, $\tilde{T} = \tilde{A}\tilde{B}\tilde{C}$, $\tilde{A}, \tilde{B}, \tilde{C}$ can be identified with the generators of the (cyclic) stabilizers of certain ramification points $P_0, P_1, P_2, P_3 \in F_{\mathbb{C}}$ over $\alpha_0, \alpha_1, \alpha_2, \alpha_3$. Galois covers are those with transitive action of $\text{Aut}(F_{\mathbb{C}}/P^1_{\mathbb{C}})$ in the fibers. See e.g. Theorem 2.13 from [Vol] and also Proposition 4.23 there.

The isomorphism classes of such covers together with the identification of $\text{Aut}(F_{\mathbb{C}}/P^1_{\mathbb{C}})$ with the group $\tilde{\mathfrak{B}}_{p,m}$ are in one-to-one correspondence with the triples of generators $\{A', B', C'\}$ of $\tilde{\mathfrak{B}}_{p,m}$ considered up to a conjugation and satisfying the following. They must be taken from the corresponding conjugacy classes of $\tilde{A}, \tilde{B}, \tilde{C}$ and $T' = A'B'C'$ must belong to the conjugacy class of $\tilde{T}$ in $\tilde{\mathfrak{B}}_{p,m}$.

Comment. The set of such isomorphism classes is denoted by $E^{im}_4$ in [FrV]; see also Section 3.1 in [Det]. Actually $E^{ab}_4$ is sufficient in this paper, where the triples $\{A', B', C'\}$ are considered modulo the action of the group of all automorphisms of $\tilde{\mathfrak{B}}_{p,m}$, not only inner. □

The key here is the following. Let us choose an ordered standard homotopy basis in $\pi_1(P^1_{\mathbb{C}} \setminus \{\alpha_i\}, b_o)$, where $\gamma_i$ is represented by a loop at $b_o$ that winds once around $\alpha_i$ counterclockwise for $i = 1, 2, 3$ and clockwise for $i = 0$ and winds around no other $\alpha_j$. It is of course not unique; see e.g. [CoH]. Then RET provides that there is a canonical Galois cover of $P^1_{\mathbb{C}}$ together with the identification of its Galois group with $\tilde{\mathfrak{B}}_{p,m}$ such that the generators $A', B', C'$ in the latter group correspond to $\gamma_1, \gamma_2, \gamma_3$ and $T' = A'B'C'$ corresponds to $\gamma_0$.

Switching to the fields of rational functions, $\mathbb{C}(F)/\mathbb{C}(P^1)$ is the Galois extension of the fields of rational functions on $F$ and $P^1$. Therefore $\text{Gal}(\mathbb{C}(F)/\mathbb{C}(x)) \cong \tilde{\mathfrak{B}}_{p,m}$, where we set $\mathbb{C}(P^1) = \mathbb{C}(x)$.

Since $\mathbb{C}(F)/\mathbb{C}(P^1)$ is Galois, there exists the minimal field of definition of this extension of fields, which contains the field of definition of the set of points $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$. See Proposition 2.5 from [CoH], Section 1.5 from [FrV] (and references therein) and Proposition 7.12 from [Vol]. Let us assume that the definition field is a field of algebraic numbers $K \subset \overline{\mathbb{Q}}$. We will also assume that $\alpha_0$ and the base point $b_o$ are defined over $K$, though $\alpha_1, \alpha_2, \alpha_3$ may be permuted by $\text{Gal}(\overline{\mathbb{Q}}/K)$. 


2.3. Main Theorem. We fix a primitive root of unity $q^{1/2}$ of order $2N$ and a $\text{dim-rigid}$ module $V$ from Theorem 1.4 which is also $\text{loc-dim-rigid}$. Recall that $\tilde{V}$ is $V$ supplied with the action of $B_1$ via $\tilde{A}, \tilde{B}, \tilde{C}$, which can be assumed with the entries in $\mathbb{Z}_q$ for $\bar{q} = q^{1/4} = (q^{1/2})^{1/2}$ for even $2k$ and $\bar{q} = q^{1/2}$ if $k \in 1/2 + \mathbb{Z}$. We will consider $\tilde{V}$ over $\mathbb{Z}_q/p^m$ for $m \in \mathbb{N}$ unless the image $\text{Image}(B_q)$ of $B_q$ in $\text{GL}(V)$ is finite (see Theorem 1.8). In the latter case, we set $p = 0 = m$ and consider $\tilde{V}$ over $\mathbb{Q}(\bar{q})$. Also, see Part (v) of Theorem 2.2 concerning the case of modules $V^{\vee}$ for odd $N$.

Let $G_K = \text{Gal}(\overline{\mathbb{Q}}/K)$ be the absolute Galois group of $K$; we also set $G_K = \text{Gal}(\overline{\mathbb{Q}}/K^o)$ for $K^o = K(\bar{q})$.

The main application of the rigidity of the DAHA-modules $V$ under consideration in this paper is the action of $G_K$ and $G_K^o$ in the groups $\tilde{B}_{(\bar{p},m)}$ and $\tilde{B}_{\bar{p},m}$, which are the images of $B_1$ in the groups of automorphisms of $\tilde{V}$ over the rings $\mathbb{Z}_{\bar{q}}/(p^m)$ and, correspondingly, over $\mathbb{Z}_{\bar{q}}/p^m$. The ring of definition is the field $\mathbb{Q}(\bar{q})$ if $p = 0$; this is the case of finite $\tilde{B}$, which will denote the image of $B_1$ for $V$ in characteristic 0.

The justification of the following theorem is actually similar to the proof of projective $\text{PSL}_2(\mathbb{Z})$–invariance of $\text{unirigid}$ modules. Recall that $\tilde{V}_{\text{sym}} = \{v \in V : T(v) = t^{1/2}v\} = \{\bar{v} \in \tilde{V} : \tilde{T}(v) = q^{1/4}t^{1/2}\bar{v}\}$. However the $G_K$–invariance holds even for the modules $V_{2[k]}^{\vee}, V_{2[k]}^{\vee}$ and for $V_{N=2[2k]}^{\vee}$, which are not $\text{dim-rigid}$. It even works in the case of $V_{4N}$ from (1.34), which is not dim-rigid. In the non-unirigid cases we need to use the DSP at greater potential, which always appears sufficient. For instance, the Jordan form of $\tilde{T}$ is used for $V_{4N}$.

**Theorem 2.2.** Let us consider any module $\tilde{V}$ over $\mathbb{Z}_q$ of type $(\alpha, \beta, \gamma)$ from Theorem 1.4 (including applying i there) and pick prime $p$ such that $\text{gcd}(p, 2N) = 1$ and a prime ideal $\bar{p} \subset \mathbb{Z}_{\bar{q}}$ over $(p)$. For $m \in \mathbb{N}$ (we set $m = 0$ for $p = 0$), let $f : F_{\bar{C}} \to P_{\bar{C}}^1$ be the Galois cover such that $\text{Gal}(\mathbb{C}(F)/\mathbb{C}(P^1)) \to \tilde{B}_{\bar{p},m}$, where $\tilde{T} = \tilde{A}\tilde{B}\tilde{C}$, $\tilde{A}, \tilde{B}, \tilde{C}$ generate the ramification (cyclic) subgroups at certain $P_0, P_1, P_2, P_3$ in the fibers of $f$ over $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in P^1$.

(i) The cover $F_{\bar{C}} \to P_{\bar{C}}^1$ and the field extension $\mathbb{C}(F)/\mathbb{C}(P^1)$ can be defined over $K^o = K(\bar{q})$. Switching to the group $\tilde{B}_{(\bar{p},m)}$ which has natural projections onto all groups $\tilde{B}_{\bar{p},m}$ for $(p) \subset \bar{p}$, the cover $f^o : F_{\bar{C}}^o \to P_{\bar{C}}^1$ introduced for this group in the same way as above
and the corresponding field extension $\mathbb{C}(F^\circ)/\mathbb{C}(P^1)$ can be defined over $K$. Accordingly, one has homomorphisms $G_{K}^\circ \to Aut(\mathfrak{B}_{P,m})$ for any $p \supset (p)$ and a homomorphism $G_{K} \to Aut(\mathfrak{B}_{(p),m})$.

(ii) For each prime ideal $p \subset \mathbb{Z}_q$ over $(p)$ as above, the action of $G_{K}^\circ$ in $\mathfrak{B}_{P,m}$ is by inner automorphisms of $\mathfrak{V}$ considered over $\mathbb{Z}_q/p^m$. The action of any $g \in G_{K}$ in $\mathfrak{B}_{(p),m}$ is by inner automorphisms of $\mathfrak{V}$ considered over $\mathbb{Z}_q/(p^m)$ (or $\mathbb{Q}(\tilde{q})$ for $p = 0$) composed with the Galois conjugation $\sigma_g \in Gal(\mathbb{Q}(\tilde{q})/\mathbb{Q})$ acting naturally in $GL(\mathfrak{V})$ considered over $\mathbb{Z}_q/(p^m)$. Then the map $g \mapsto \sigma_g$ coincides with the natural restriction homomorphism $G_K \to Gal(\mathbb{Q}(\tilde{q})/\mathbb{Q})$.

(iii) Continuing (ii), there exists a group homomorphism $\tilde{g} \mapsto \phi_{\tilde{g}} = h_{\tilde{g}}\sigma_{\tilde{g}}$ for $\tilde{g}$ from a proper central extension $\varrho: \tilde{G}_K \to G_K$ and $g = \varrho(\tilde{g})$ such that the action of $g$ in $\mathfrak{B}_{(p),m}$ is the action of $\sigma_g$ followed by the conjugation by $h_{\tilde{g}} \in GL(\mathfrak{V})$ for $\mathfrak{V}$ over $\mathbb{Z}_q/(p^m)$ (over $\mathbb{Q}(\tilde{q})$ when $p = 0$). Moreover, applying a proper automorphism from $Aut(F^\circ \to P^1)$, one can assume that $g(T) = T^M$, where $\sigma_g(\tilde{q}) = \tilde{q}^M$, which is equivalent to the relations $\sigma_g(T) = T^M$, $h_{\tilde{g}} T h_{\tilde{g}}^{-1} = T$. Then $G_K$ preserves $\mathfrak{V}^{sym}$ over $\mathbb{Z}_q/(p^m)$; accordingly, $G_{K}^\circ$ preserves $\mathfrak{V}^{sym}$ over $\mathbb{Z}_q/p^m$.

(iv) Furthermore, $h_{\tilde{g}} (\tilde{g} \in \tilde{G}_K)$ are unitary with respect to the inner product $\langle \cdot, \cdot \rangle$ considered over $\mathbb{Z}_q/(p^m)$ for types $(\alpha, \gamma)$. In the case $(\gamma)$, this claim and those from (i, ii, iii) hold for the modules $\mathfrak{V} = \xi_x^\delta \mathfrak{V}_{(2n+1)}$ for $\{\epsilon, \delta\} \in \{\{1, 0\}, \{0, 1\}, \{1, 1\}\}$ associated correspondingly with the pairs $\{\alpha, \alpha_i\} \in \{\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_3\}, \{\alpha_1, \alpha_3\}\}$ subject to the following adjustment. The automorphism $\sigma_g$ must be multiplied by $\xi_x^{\gamma - \epsilon}\xi_y^{\delta - \delta}$ for the pair of indices $\{\epsilon, \delta\}$ corresponding to the pair $g\{\alpha_i, \alpha_j\}$ of ramification points in $P^1$.

(v) All claims above hold for the boundary cases from (1.34) and for the $H^\epsilon$ -modules $V^\epsilon$ for odd $N$ from Proposition 1.3, namely for $(\alpha^\epsilon), (\beta^\epsilon), (\gamma^\epsilon)$. The component $\sigma_g$ in $\phi_{\tilde{g}}$ must be adjusted here by $\xi_x^{\epsilon - \epsilon}\xi_y^{\delta - \delta}$ in the same way as it was done in Part (iv) for $(\gamma)$ in the following cases: $(\alpha^\epsilon)$ for even $2k$, $(\beta^\epsilon)$ and $(\gamma^\epsilon)$. This is not needed for $(\alpha^\epsilon)$ for odd $2k$, since $\xi_x^{\epsilon}\xi_y^{\delta}$ (and $\tau_\pm$) do not change the isomorphism class of $V_{2N-4k}$, which remains irreducible.

Proof. We will mainly consider here the cases $(\alpha, \beta, \gamma)$. The adjustments to the boundary cases from (1.34) and the cases $(\tilde{\alpha}^\epsilon, \tilde{\beta}^\epsilon, \tilde{\gamma}^\epsilon)$
from Proposition 1.35 are straightforward with one reservation. The boundary module $V_{1N}$ of type $(\beta^*)$ is not dim-rigid. However one can use here formula (1.35) instead of dim-rigidity. Note that when $\tilde{T}$ becomes non-semisimple, the rigidity argument continues to work.

Let us take a standard homotopy basis in $B_1 = \pi_1(P_1 \setminus \{\alpha_i, b_0\})$. Then $F_\mathcal{C}$ corresponds to a normal subgroup $U \subset B_1$ such that $B_1/U = \text{Aut}(F_\mathcal{C} \to P_1^\alpha) \sim \widetilde{B}_{p,m}$, where the standard generators $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ map to $\tilde{T}, \tilde{A}, \tilde{B}, \tilde{C}$. The latter four elements are generators of the (cyclic) stabilizers at certain ramification points $P_0, P_1, P_2, P_3 \in F_\mathcal{C}$ in the fibers of $f'$ over the points $\alpha_0, \alpha_1, \alpha_2, \alpha_3$.

The covering $F/P^1$ can be defined over $\tilde{K} \subset \overline{K}$ for a finite Galois extension $\tilde{K}/K^\circ$ (since there are only finitely many covers with the same Galois group and given ramification points). For any element $g \in \text{Gal}(\tilde{K}/K^\circ)$, let $f': F' \to P^1$ be the covering obtained by applying $g$ to $F/P^1$. It has the same ramification points in $P^1$ and corresponds to the following homomorphism $B_1/U' = \text{Aut}(F_\mathcal{C} \to P_1^\alpha) \sim \widetilde{B}_{p,m}$.

It sends the standard generators to another systems of generators of $\widetilde{B}_{p,m}$, elements $\tilde{T}', \tilde{A}', \tilde{B}', \tilde{C}'$ generating the stabilizers at $P_0', P_1', P_2', P_3'$, which are the $g$–images of $P_0, P_1, P_2, P_3$. These points belong to the fibers of $f'$ over $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and correspond to the permutation $\{123\} \mapsto \{1\, 2\, 3\}$ induced by $g$. Recall that the base point $b_0, \alpha_0$ and the set $\{\alpha_1, \alpha_2, \alpha_3\}$ are defined over $K$; so one can set $g(\alpha_i) = \alpha_i^g$ for $i \geq 1$.

Furthermore, there exists $M \in \mathbb{N}$ such that $\tilde{T}', \tilde{A}', \tilde{B}', \tilde{C}'$ belong to the same conjugacy classes as to $\tilde{T}^M, \tilde{A}^M, \tilde{B}^M, \tilde{C}^M$ under the permutation of $\tilde{A}, \tilde{B}, \tilde{C}$ induced by action of $g$ on $\alpha_1, \alpha_2, \alpha_3$.

Namely, $M$ is determined from the relation $g(\zeta) = \zeta^M$, where $\zeta$ is a primitive root of unity of order $\tilde{N}$ that is $\gcd$ of the orders of $\tilde{T}, \tilde{A}, \tilde{B}, \tilde{C}$. Thus $M$ is relatively prime to $\tilde{N}$ and $\tilde{T}^M, \tilde{A}^M, \tilde{B}^M, \tilde{C}^M$ are generators of the stabilizers at $P_0, P_1, P_2, P_3$.

Note that applying proper Galois automorphisms in the fibers gives that $\tilde{T}'$ is conjugated to a generator of the stabilizer at $P_0$ and so on for $P_1, P_2, P_3$. See e.g. Section 2.1 from [FTV]. Which gives the existence of individual $M$ above, for each of the ramification points. We need the branch cycle argument to obtain that a single $M$ serves all 4 ramification points and that it can be uniquely determined modulo $\tilde{N}$ from the relation $g(\zeta) = \zeta^M$. This argument uses a well-known presentation of ramified covers in the form $z = x^m$ for suitable local
parameters. See Lemma 2.8 from [Vol], [Bel] and also Section 3.1 from [Det] and references therein.

Since $M$ is odd, we conclude that $\tilde{A}', \tilde{B}', \tilde{C}'$ are conjugated to $\tilde{A}, \tilde{B}, \tilde{C}$ upon a permutation of $o_1, o_2, o_3$ induced by $g$. For $\tilde{T'}$, we know that it is conjugated to $\tilde{T}^M$, where $\gcd(M, \ord(\tilde{T})) = 1$.

Using (2.5), $\ord(\tilde{T} \mod \mathbb{C}^*) = \ord(-t) = 2N \gcd(2N, N - 2|k|)$. Therefore $\ord(\tilde{q}^{1/2}) = 2N$ divides $\ord(\tilde{T})$ and $\ord(\tilde{T}) = 2Nl$ for $l \in \mathbb{N}$. Moreover, $\tilde{q}^{N/2lN}$ must be 1, which gives that $l = 2$ for $k \in \mathbb{Z}$ and $l = 1$ for $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. Finally, $\tilde{q} \mapsto \tilde{q}^M$ can be extended to $\sigma \in \Gal(\mathbb{Q}(\tilde{q})/\mathbb{Q})$ for this $\tilde{M}$. Using the branch cycle argument we conclude that $\sigma = \sigma_g$ is the restriction of $g$ to $\mathbb{Q}(\tilde{q})$, which is claimed in (ii).

Now, on the DAHA end, applying $\sigma$ to the entries of the matrices representing $\tilde{T}, \tilde{A}, \tilde{B}, \tilde{C}$, one obtains an irreducible $\mathcal{H}$-module $\tilde{V}^\sigma$ of the same dimension $\dim V$, for the same $k$ and with $q^M$ instead of $q$. If $\tilde{V}$ is considered over $\mathbb{Z}_q/p^m$, then the module $\tilde{V}^\sigma$ is naturally defined over $\mathbb{Z}_q/(\sigma(p)^m)$.

Using the unirigidity in types $(\alpha, \beta)$, we see that the collection of matrix images of $\tilde{T}', \tilde{A}', \tilde{B}', \tilde{C}'$ in $\widetilde{\mathfrak{B}}_{p,m}$ can be extended to an action of $\mathcal{H}$ and the resulting module must be isomorphic to $\tilde{V}^\sigma$. Therefore the kernel $U'$ of the homomorphism

$$B_1 \to B_1/ U' = \Aut(F_C \to P_C^1) \simeq \widetilde{\mathfrak{B}}_{p,m},$$

where

$$\gamma_0 \mapsto \tilde{T}', \gamma_1 \mapsto \tilde{A}', \gamma_2 \mapsto \tilde{B}', \gamma_3 \mapsto \tilde{C},$$

coincides with the initial $U$.

Thus the cover $f : F_C \to P_C^1$ and the field $K^\circ(F)$ has a natural action of the group $G^\circ_K$. This gives the claims involving the ideal $\mathfrak{p} \supset (p)$ from Parts (i),(ii) for types $(\alpha, \beta)$.

In the case of type $(\tilde{\gamma})$ (where we have dim-rigidity but no unirigidity), these claims hold as well. Indeed, $\tilde{A}', \tilde{B}', \tilde{C}'$ must be conjugated in $\widetilde{\mathfrak{B}}_{p,m}$ to $\tilde{A}, \tilde{B}, \tilde{C}$ up to a permutation in this triple. This excludes applying $\varsigma_x, \varsigma_y$ or their product for $\tilde{V}$ of type $(\tilde{\gamma})$; see (2.5) and (2.2).

Actually applying $\varsigma_{x,y}$ to $\tilde{V}$ of type $(\tilde{\gamma})$ does not change the kernel of the homomorphism from $\Aut(f' : F' \to P^1)$ to $\Aut(f : F \to P^1)$, so the $G^\circ_K$ invariance of $K^\circ(F)$ from Part (i) follows directly from the dim-rigidity for $(\tilde{\gamma})$. However this does not prove that $G^\circ_K$ acts via conjugations for such modules, as stated in Part (ii).
Thus we established Parts (i, ii) for \( \tilde{\mathfrak{B}}_{p,m} \). This formally results in the corresponding claims there concerning the action of complete \( G_K \) in the cover \( F^\circ \to P^1 \) corresponding to the group \( \mathfrak{B}_{(p),m} \) from (i, ii).

The justification of (iii, iv) is similar. We use the uniqueness of \( \langle \cdot, \cdot \rangle \) in \( \tilde{V} \) up to proportionality in Part (iv). Note that \( (\tilde{g}(v), \hat{g}(v)) \) for \( \hat{g} \in \hat{G}_K \) and \( v \in \tilde{V} \) is defined over \( \mathbb{Z}_{\tilde{q}}/(p^m) \) (not over \( \mathbb{Z}_{\tilde{q}}/(p^m) \)) and remains unchanged if \( \hat{g} \) is multiplied by any root of unity.

Generally, it can happen that \( (\tilde{g}(v), \hat{g}(v)) = \zeta_g(\tilde{g}(v), \hat{g}(v)) \) for some \( \hat{g} \in \hat{G}_K \) and a root of unity \( \zeta_g \) (serving all \( v \in V \)). When \( p = 0 \), then \( \zeta_g \) can be only 1 in the case of \( (\alpha) \) due to the positivity of \( \langle \cdot, \cdot \rangle \) at \( q = e^{2\pi i/N} \). This is parallel to the proof of unitarity of the projective action of \( PSL_g(\mathbb{Z}) \) in \( V \) for types \( (\alpha, \beta) \); in the case of \( (\beta) \), one uses here that \( \dim V \) is odd (when \( p = 0 \)).

When \( p > 0 \), i.e., for \( \tilde{V} \) considered over \( \mathbb{Z}_{\tilde{q}}/(p^m) \), this arguments can be readily replaced by considering the exact lists of squares of \( \langle \cdot, \cdot \rangle \) upon the diagonalization of this form in \( \tilde{V} \) (in the basis of \( \{e_n'\} \)).

The extension from Part (iv) of the previous claims to the modules \( \zeta_t^q(V_{2n+1}^{\pm}) \) is straightforward; use (2.5) and (2.2).

To adjust these considerations to the cases of \( \sqrt{\cdot} \)-modules from Part (v), one needs to replace \( \bar{q} \) by its square and use that \( \gcd(M, \text{ord}(\bar{q})) = 1 \) for \( g(\zeta) = \zeta^M \) (see above). Recall that here \( q, t, \bar{q} \) must be replaced by \( q' = q^2, t' = t^2, \bar{q}' = \bar{q}^2 \). The order of \( \tilde{T}' \) divides \( 2N \) (\( \tilde{T}' \) is always semisimple). This gives the claims from (i - iii) with \( \sqrt{\cdot} \).

The same branch cycle argument gives that \( G_K^0 \) and \( G_K \) transpose the \( \zeta_t^q \)-orbits of the modules from (\( \tilde{\alpha}' \)) for even \( 2k \), (\( \tilde{\beta}' \)) and those from (\( \tilde{\gamma}' \)) via their action from (iv) for the family (\( \tilde{\gamma} \)). This action is trivial for (\( \tilde{\alpha}' \)) for odd \( 2k \).

\[ \Box \]

2.4. Special cases. Following essentially [CoH], Example 2.4, let us calculate the automorphism \( \eta \) of \( \mathfrak{B}_{p,m} \) corresponding to the complex conjugation in \( K(\bar{q}) \), sending \( \mathbb{C} \ni z \mapsto \bar{z} \). Here we can consider the whole \( \mathfrak{B}_1 \), since the complex conjugation is continuous (the other automorphisms from \( \text{Gal}(\bar{Q}/Q) \) are discontinuous).

We will pick the standard homotopy basis. Namely \( \gamma_i \) will be represented by a loop at \( b_i \) that winds once around \( \alpha_i \) counterclockwise for \( i = 1, 2, 3 \) and clockwise for \( i = 0 \) and winds around no other \( \alpha_j \). Let us assume that \( \alpha_0 < b_0 < \alpha_1 < \alpha_2 < \alpha_3 \) are all real. Then \( \gamma_0 = \gamma_3 \gamma_2 \gamma_1 \),
\( \gamma_1 \) and \( \gamma_2 \gamma_1 \) can be chosen to have support invariant under complex conjugation. The composition of the paths \( \gamma_i \) and then \( \gamma_j \) is written \( \gamma_j \gamma_i \). One has

\[
\begin{align*}
\gamma_0 &= \gamma_0^{-1}, & \gamma_1 &= \gamma_1^{-1}, & \gamma_2 \gamma_1 &= \gamma_1^{-1} \gamma_2^{-1} ,
\gamma_2 &= \gamma_1^{-1} \gamma_2^{-1} \gamma_1, & \gamma_3 &= \gamma_1^{-1} \gamma_2^{-1} \gamma_3^{-1} \gamma_2 \gamma_1 .
\end{align*}
\]

Switching to \( T', A', B', C' \) from \( \mathcal{B}_1 \) corresponding to \( \gamma_0, \gamma_1, \gamma_2, \gamma_3 \) and using that \( T' = A'B'C' = (A')^{-1} (B')^{-1} (C')^{-1} \) \( (A', B', C' \) are involutions in \( \mathcal{B}_1 \),

\[
\text{(2.7)} \quad \overline{T}' = (T')^{-1}, \quad \overline{A}' = A', \quad \overline{B}' = A'B'A', \quad \overline{C}' = T'B'A' = T'C'(T')^{-1}. 
\]

Now let us substitute \( \tilde{A} = XT, \tilde{B} = q^{1/4}XYT, \tilde{C} = T^{-1}Y, \tilde{T} = q^{1/4}T \) for \( T', A', B', C' \), see \( \text{(2.3)} \). To avoid confusion with the complex conjugation acting in matrices, we will denote the resulting automorphism of \( \mathcal{B}_q \) by \( \eta' \); then \( \eta'(q^{1/2}) = q^{-1/2} \).

Recalculating now \( \text{(2.7)} \) in terms of \( X, T, Y \) we obtain that \( \eta' \) acts as follows:

\[
\text{(2.8)} \quad \eta'(T) = T^{-1}, \quad \eta'(X) = T^{-1} X^{-1} T = XT^2, \quad \eta'(Y) = Y^{-1}.
\]

Indeed, \( \eta'(T^{-1}Y) = TCT^{-1} = T\eta'(Y) = YT^{-1} \), which gives the last formula. This is the DAHA-Kazhdan-Lusztig involution \( \eta \) from Section 2.5.5 of [Ch1] defined in terms of \( X \) instead of \( Y \).

**Quotient by the center.** Aiming at obtaining the action of \( G_K \) instead of smaller \( G^*_K \), one can try to consider its quotient of \( \mathfrak{B}_{p,m} \) by its center instead of enlarging \( \mathfrak{B}_{p,m} \) to \( \mathfrak{B}_{(p),m} \). This is actually what we are supposed to do for obtaining the *Tate module* in the case \( t = 1 \), which was discussed in the Introduction.

**Corollary 2.3.** In the notations from (i) of the theorem, let \( \overline{\mathfrak{B}}^+_1 \) be the quotient of \( \mathfrak{B}_{p,m} \) by its center. We assume that either \( \mathfrak{B} \) is finite (we set \( p = 0 = m \) in this case) or \( p \) is a primitive element modulo \( N \) for \( N = \ell^n \) and odd prime \( \ell \). Then \( \overline{\mathfrak{B}}^+_1 \) does not depend on the choice of the prime ideals \( p \supset (p) \) up to isomorphisms and the action of \( G^*_K \) in \( \mathfrak{B}_{p,m} \) and \( \overline{\mathfrak{B}}^+_1 \) provides a homomorphism \( G_K \to \operatorname{Aut}(\overline{\mathfrak{B}}^+_1) \) for any \( p \supset (p) \) and \( m \in \mathbb{N} \) and its counterpart for \( \overline{\mathfrak{B}}^+_1 \). If \( \operatorname{Image}(\mathcal{B}_1) \) is finite (and \( p = 0 \)), then \( G_K \) acts directly in \( \overline{\mathfrak{B}}^+_1 \overset{\text{def}}{=} \mathfrak{B}/\operatorname{Center}(\mathfrak{B}) \).
Proof. The consideration is essentially the same as in the proof of Theorem 2.2. We pick \( g \in G_k \) and consider two homomorphisms
\[
\mathcal{B}_1/U^1 = \text{Aut}(F^1_c \to P^1_c) \cong \tilde{\mathcal{B}}_{1,p,m},
\]
\[
\mathcal{B}_1/U^{1'} = \text{Aut}(F_1^{1'} \to P_1^1) \cong \tilde{\mathcal{B}}_{1,p,m},
\]
where the (normal) subgroups \( U^1, U^{1'} \) correspond to the cover \( F_1^{1'} \) associated with \( \tilde{\mathcal{B}}_{1,p,m} \) and its \( g \)-image \( F_1^{1'} \).

Then we conclude that the conjugation by \( \phi_g = h_\sigma g \) in GL(\( \tilde{V} \)) will not change \( U^1 \). Indeed, the relations in \( \mathcal{B}_{p,m} \) will be transformed by the automorphisms from Gal(\( \mathbb{Q}(q^{1/2})/\mathbb{Q} \)) and therefore do not depend on the particular choice of \( q^{1/2} \) modulo the center of \( \mathcal{B}_{p,m} \). Thus the whole \( G_K \), not only \( G^*_K = \text{Gal}(\tilde{\mathbb{Q}}/\mathbb{K}^\circ) \) as in the theorem, will act in the cover \( F_1^{1'} \to P_1^1 \). This gives the required for \( \tilde{\mathcal{B}}_{1,p,m} \); the passage to \( \tilde{\mathcal{B}}_{1'}^{1'} \) is straightforward.

Dimension 4. Let us consider the module \( V = V'_4 \) from (1.43) as an example. It is irreducible for a primitive \( (2N) \)th root of unity \( q^{1/2} \) and remains irreducible when \( q^{1/2} \to \tilde{q}^{1/2} \) for a primitive root \( \tilde{q}^{1/2} \) of order \( 2\tilde{N} \).

Following Section 1.7 we denote by \( \mathcal{B}_{N}^{1} \) and \( \mathcal{B}_{N}^{1'} \), the quotients of \( \text{Image}(\mathcal{B}_q) \) or \( \text{Image}(\mathcal{B}_1) \) by the center. Let us also consider the commutator subgroup
\[
\mathcal{C}_{N}^{1} \overset{\text{def}}{=} <T^2, X^2, Y^2>/ <T^{2M}, X^{2N}, Y^{2N} > \simeq \mathbb{Z}_M \times \mathbb{Z}_{N},
\]
where \( M = N \) for odd \( N \) and \( M = N/2 \) otherwise.

The quotient of \( \mathcal{B}_{N}^{1}/\mathcal{C}_{N}^{1} \) is isomorphic to \( \mathbb{Z}_{2}^{2} \) and corresponds to the cover of \( P^1 \) that is the composition \( E \simeq E' \to E'/E'_2 \simeq E \to P^1; \) the first map is the multiplication by \( 2 \) \( (E'_2 \) is the group of points of the 2nd order). The action of \( \mathcal{B}_{N}^{1} \) in \( K^\circ(E') \) is via translations by \( E'_2; \) \( T \) is the reflection \( s : z \mapsto -z \) there. Note that \( E'/\{s\} \simeq P^1 \) covers \( P^1 \) with the Galois group \( \mathbb{Z}_{2}^{2} \) and 6 ramification points of index 2 over \( \alpha_1, \alpha_2, \alpha_3 \).

Let us briefly describe the cover \( F \) of \( E \) corresponding to the group \( \mathcal{B}_{N}^{1} \) and the cover of \( E' \) corresponding to the group \( \mathcal{C}_{N}^{1}. \) One needs to extend the field \( K^\circ(E') \) by the functions \( \phi_i = f_i^{1/N} \) for \( f_i \in K^\circ(E') \) with the divisors \( (f_i) = 2(0'_i - 20') \), where \( 0', 0_i' (i = 1, 2, 3) \) are the points of the second order in \( E' \). Note that \( f_1f_2f_3 \) is a perfect square in \( K^\circ(E') \). Recall that \( K^\circ = K(q^{1/4}) \) (actually \( q^{1/2} \) is sufficient below).
We associate the translations $e_1, e_2$ by the points $0_1', 0_2' \in E'_1$ respectively with $X$ and $Y$. Then the automorphisms in $Gal(K^\circ(F)/K^\circ(P^1))$ corresponding to the generators $X, Y, T \in \mathfrak{B}_N^1$ (modulo the center) can be presented as follows. For a primitive $(2N)$th root of unity $\omega$, $\alpha, \beta \in \mathbb{C}^*$ and $\alpha/\beta = \omega$, one can pick $\phi_{1,2,3}$ to ensure the relations

\[
X \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \alpha \omega \phi_1^{-1} \\ \beta \omega \phi_2^{-1} \\ \phi_3 \end{pmatrix}, \quad Y \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \alpha \omega \phi_1^{-1} \\ \beta \omega \phi_2^{-1} \\ \phi_3 \end{pmatrix}, \quad T \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \omega^2 \phi_2 \\ \phi_3 \end{pmatrix},
\]

\[
X^2 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \omega^{-2} \phi_2 \\ \omega^{-2} \phi_3 \end{pmatrix}, \quad Y^2 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \omega^{-2} \phi_1 \\ \phi_2 \\ \omega^{-2} \phi_3 \end{pmatrix}, \quad T^2 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \omega^4 \phi_2 \\ \phi_3 \end{pmatrix}.
\]

Check the relations $(TX)^2 = 1 = (TY)^2 = Y^{-1}X^{-1}YXT^2$ using

\[
X^{-1} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \alpha \omega \phi_1^{-1} \\ \alpha \omega \phi_3 \phi_1^{-1} \end{pmatrix}, \quad Y^{-1} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \beta \omega \phi_2^{-1} \\ \beta \omega \phi_2^{-1} \end{pmatrix}.
\]

We obtain the required action. Note that $K^\circ(E')$ extended by the $\phi_1 \phi_2/\phi_3$, which is invariant with respect to $X^2$ and $Y^2$, corresponds to the homomorphism $C^\circ_N \rightarrow C^\circ_N/ \langle X^2, Y^2 \rangle \cong \mathbb{Z}_M$.

In terms of $P^1$ and the images $\alpha', \alpha'_1, \alpha'_2, \alpha'_3$ of $0', 0'_1, 0'_2, 0'_3$, one can take $f_i = \frac{\alpha'_i - z}{2}$, where $z$ is the coordinate of $P^1$. Then for $j = 1, 2,$

\[
f_i^{\epsilon_j} = u_{ij} \frac{f_k}{f_j}, \quad u_{ij} = \frac{\alpha'_i - \alpha'_j}{\alpha'_i}, \quad \text{where} \quad 1 \leq i, k \leq 3, \quad i \neq j \neq k, \quad k \neq i,
\]

and $f_j^{\epsilon_j} = u_{ij}u_{kj}f_j^{-1}$. Setting $\alpha'_1 = \alpha'_3 - \alpha'_2$, we arrive at the relations above for the action of $X, Y$ with $\omega = -1$ and $\alpha = -\beta = \frac{\alpha'_1 - \alpha'_2}{\alpha'_3}$.

2.5. **Triangle groups.** As it has been already mentioned, the root system $C^0C_1$ is a natural setting for Theorem [2.2] and its proof (based on the Riemann Existence Theorem for $\mathbb{CP}^1$ with 4 ramification points). However, $B_1$ of type $A_1$ and its quotients can be associated with the covers of elliptic curves, which is of obvious importance. Moreover, the case of $A_1$ has the following interesting relation to the triangle groups.
Given $l, m, n \in \{2, 3, 4, \ldots \}$, the corresponding triangular group is

$$\Delta(l, m, n) \overset{\text{def}}{=} \langle a, b, c \mid a^2 = b^2 = c^2 = 1 = (ab)^l = (bc)^m = (ca)^n \rangle.$$  

It contains a normal subgroup of index 2, the von Dyck group,

$$D(l, m, n) = \langle x, y, z \mid x^l = y^m = z^n = 1 = xyz \rangle,$$

where $x = ab, y = bc, z = ca, axa = x^{-1}, aya = xz,aza = z^{-1}.$

It is commonly called a triangle group, as well as $\Delta(l, m, n)$, especially in the papers devoted to simple quotients of triangle groups.

The groups $\Delta(l, m, n; r), D(l, m, n; r)$ are defined by imposing the additional relation $\langle (abac)^r = 1 = (xz^{-1})^r \rangle$. This definition is related to Conjecture 5.3 from [Sch] concerning the discreteness of the images of $\Delta(l, m, n)$ in $PU(2, 1)$.

**Proposition 2.4.** (i) For the modules $V = V_{2N-4k}, V_{2l[k]}^\alpha, V_{2l[k]}^\gamma$ over $\mathbb{Z}_q$ of types $(\alpha, \gamma)$ from Theorem 1.4 and those obtained upon applying $\iota, \iota'$, the group $\mathfrak{B}_l$, the image $\mathfrak{B}$ of $B_1$ in automorphisms of $V$ divided by its center, is a natural quotient of $\Delta(2N, 2N; 2N)$:

$$a \mapsto \tilde{A} = XT, b \mapsto \tilde{B} = q^{1/4}XTY, c \mapsto \tilde{C} = T^{-1}Y, x \mapsto q^{1/4}Y,$$

$$y \mapsto q^{1/4}T^{-1}X^{-1}T = q^{1/4}XT^2, z \mapsto Y^{-1}X^{-1} = q^{1/4}Y^{-1}T_-(X^{-1})Y.$$  

Namely, the orders of the images of $x, y, z$ are $2N$, as well as the order of $\text{Image}(abac) = YXY = \tau_+ \tau_-(\tilde{Y}).$

Assuming the semisimplicity of $T$ (i.e. for $t \neq -1$), the order of $\text{Image}(abc) = \tilde{T} = q^{1/4}T = \tilde{A}\tilde{B}\tilde{C}$ modulo scalar matrices is $\text{ord}(-t) = 2N/\text{gcd}(2N, N - 2|k|)$. Also, the order of $\text{Image}(xz^{-1}y) = \tilde{T}^2$ in $\text{GL}(V)/C^*$ equals $\text{ord}(t^2) = N/\text{gcd}(N, 2|k|)$.

(ii) In the case of odd $N$ for $(\tilde{\alpha}^\vee)$ when $2k$ is even and for $(\tilde{\gamma}^\vee)$, we substitute $q \mapsto q^\prime = q^2, t \mapsto t^\prime = t^2$ in the formulas from (2.12):

$$a \mapsto XT^\vee, b \mapsto q^{1/2}XT^\vee Y^\vee, c \mapsto (T^\vee)^{-1}Y^\vee,$$

$$x \mapsto q^{1/2}Y^\vee, y \mapsto q^{1/2}(T^\vee)^{-1}X^{-1}T^\vee, z \mapsto Y^{-1}X^{-1}.$$  

Then the orders of the images of $x, y, z$ modulo scalars are $N$, as well as the order of $\text{Image}(abc) = \tau_+^\vee \tau_-(\tilde{Y}^\vee)$. Accordingly, the images of $a, b, c, x, y, z$ satisfy the relations from $\Delta(N, N, N; N)$. The dimensions of the corresponding spaces $V^\vee$ are $N - 2k$ for $(\tilde{\alpha}^\vee)$ and $2|k|$ for $(\tilde{\gamma}^\vee)$.  

Since $t^\vee \neq -1$ ( $N$ must be odd when $\check{\vee}$ is used), $\text{Image}(abc) = q^{1/2}T^\vee$ modulo scalar matrices is of order $\text{ord}(-t^2) = 2N/gcd(N,2|k|)$ and the order of $(T^\vee)^2$ is $N/gcd(N,2|k|)$.

Proof. Calculating the orders of $\tilde{T}$ and $\tilde{T}^2$ is straightforward. Since we need them modulo scalar matrices, the coefficients of proportionality do not matter here. Concerning $\text{Image}(x) = q^{1/4}Y$, its spectrum in modules of types $\alpha, \gamma$ is described in (1.8); see also Theorem 1.1 (ii) and Theorem 1.2 (i). This gives $\text{ord}(Y \bmod C^*)$. Applying conjugations and the automorphisms $\tau_{\pm}$, which act in $V_{2N-4k}$ and in $V_{2k}^\pm$, we see that the images of $y, z, abac = xz^{-1}$ will have the same orders modulo $C^*$ as $Y$. So the corresponding triangular groups are equilateral.

The cases with $\check{\vee}$ are straightforward. □

Note that we excluded in the theorem the modules of type $(\beta), (\beta^\vee)$ and $(\alpha^\vee)$ for odd $2k$ due to the presence of the Jordan blocks in $X, Y$. See Theorem 2.9.3 from Section 2.9.2 of [Ch1]. The images of the elements $x, y, z, xz$ become of infinite order in these cases. Their orders in $\mathfrak{B}'_{p,m}$ and $\mathfrak{B}_{p,m}'$ are $2Np^n$ for $(\beta)$ and $Np^n$ for $(\alpha^\vee, \beta^\vee)$.

2.6. The Livné groups. Interestingly, the DAHA-modules of dimension 3 already provide rich theory, matching the so-called Livné groups.

The images of the triangular group in $PU(2,1)$ are extensively studied algebraically and using the methods of complex hyperbolic geometry. Generally, the images of the product $abc$ (our $\tilde{T}$) do not satisfy quadratic equations. For instance the element $abc$ in the well-studied $\Delta(4,4,4;5)$ has 3 distinct eigenvalues in $PU(2,1)$; see [Der].

Our discrete groups for $\dim V = 3$ appeared exactly the canonical subgroups of the Livné lattices (examples of the Mostow groups). Conjecture [L10] provides examples of arithmetic discrete groups $\text{Image}(\mathcal{B}_1)$ in higher dimensions.

Let us demonstrate what our construction gives for $\dim V = 3$, when it results in subgroups of $PU(2,1)$. It occurs only for the series $(\check{\gamma})$ and $(\check{\gamma}^\vee)$, so the group $\text{Image}(\mathcal{B}_1)$ is never finite when $\dim V = 3$, except for the boundary case $(\gamma^*)$ from [L34].

Let us consider $(\gamma)$, including $(\gamma^*)$. Then $k = -3/2$, $t = q^{-3/2}$, $N = 3, 4, \ldots$. Using (1.17),

$V_3^+ = \mathfrak{X}/(e_2 - e_{-1}) = \mathbb{Q}[q^{1/4}][X^{\pm 1}]/((X - q^{1/4})(X - q^{-1/4})(X - q^{3/4})).$
Also, \( V^+_3 = \oplus_{i=1}^3 \mathbb{Q}[q^{1/4}](\epsilon^i_i + \epsilon^i_{i-3}) \subset W_6 = \mathcal{X}/(\epsilon_3) \) as DAHA-modules. Here \( \epsilon'_m \) are the images in \( V^+_3 \) of \( \epsilon_0 = 1, \epsilon_1 = t^{1/2}X, \)

\[
\begin{align*}
\epsilon_{-1} &= t^{1/2}(\frac{1 - tq}{1 - t^2q} X^{-1} + \frac{1 - t}{1 - t^2q} X), \\
\epsilon_{-2} &= \frac{t(t - tq)(1 - tq^2)}{(1 - t^2q)(1 - t^2q^2)} \left( X^{-2} + \frac{1 - t}{1 - t^2q} X^2 + \frac{(q + 1)(1 - t)}{1 - t^2q} \right), \\
\epsilon_3 &= \frac{t^{3/2}(1 - tq)(1 - tq^2)}{(1 - t^2q)(1 - t^2q^2)} \left( X^3 + \frac{q^2}{1 - t^2q} X^{-1} + \frac{(q + 1)(1 - t)}{1 - t^2q} X \right).
\end{align*}
\]

Since we calculate modulo \( \langle \epsilon_2, \epsilon_1 \rangle \), it suffices to take here \( \epsilon'_i \) for \( i = 1, 2, 0 \); the exact sums \( \epsilon'_i + \epsilon'_{i-3} \) are needed if we realize \( V^+_3 \) inside \( W_6 \), which is \( \mathcal{X}/(\epsilon_3) \). Moreover, one can easily recalculate the formulas to the basis \( \{\epsilon'_0, \epsilon'_2, \epsilon'_3\} \) in \( V^+_3 \); \( Y \) will remain unchanged and the matrix for \( T \) from (2.16) will be conjugated by \( \text{diag}(e_m(t^{-1/2}), m = 1, 2, 0) \).

For \( m = 1, 2, 3 \), one has:

\[
(2.14) \quad Y(\epsilon_m - \epsilon_{m-3})/(\epsilon_m - \epsilon_{m-3}) = Y(\epsilon_m)/\epsilon_m = t^{-1/2} q^{-m/2} = t^{1/2} q^{3-m/2}.
\]

The spectrum of \( X \) in \( V_3 \) is the same as that of \( Y^{-1} \). The spectrum of \( \tilde{Y} = q^{1/4}Y \) is \( \{1, q^{1/2}, q^{-1/2}\} \).

The action of \( T \) on the vectors \( \{\epsilon_m + \epsilon_{m-3}, \epsilon_{3-m} + \epsilon_{-m}\} \) for \( 0 < m < 3 \) is the same as its action on \( \{\epsilon_m, \epsilon_{-m}\} \). It is given by the matrix

\[
(2.15) \quad T_{\{\epsilon_m, \epsilon_{-m}\}} = \begin{pmatrix}
\frac{1}{t^{1/2} - t^{-1/2}} & t^{1/2} 1 - q^{m/2} \\
\frac{1}{1 - t} - q^{m/2} & \frac{1}{t^{1/2} - t^{-1/2}}
\end{pmatrix}.
\]

This presentation formally works even for \( m = 0, 3 \); then \( T \) acts as \( t^{1/2} \), which is the sum of the entries in either column (the basic vectors then become proportional for such \( m \)).

Thus in the basis \( \{\epsilon_m + \epsilon_{3-m}, m = 1, 2, 3\} \),

\[
(2.16) \quad T = \begin{pmatrix}
\frac{1}{t^{1/2} - t^{-1/2}} & t^{1/2} 1 - q^{m/2} & 0 \\
\frac{1}{t^{1/2} - t^{-1/2}} & \frac{1}{t^{1/2} - t^{-1/2}} & 0 \\
0 & 0 & t^{1/2}
\end{pmatrix}.
\]

Its eigenvalues are \( \{t^{1/2}, t^{1/2}, -t^{-1/2}\} \); the eigenvalues of \( \tilde{T} = q^{1/4}T \) are \( \{q^{-1/2}, q^{-1/2}, -q\} \). The matrix \( T \) has coinciding eigenvalues and a Jordan 2–block when \( N = 3 \) at the eigenvalue \( t^{1/2} = q^{-3/4} = i \).
Since \( Y^{-1} = \text{diag} \left( t^{1/2} q^{1/2} = q^{-1/4}, t^{1/2} q = q^{1/4}, t^{1/2} q^{3/2} = q^{3/4} \right) \) in the basis above, the matrix for \( \widetilde{C} = T^{-1} Y = Y^{-1} T \) equals

\[
(2.17) \quad \widetilde{C} = \begin{pmatrix}
q^{1/2} t^{-1} & q^{1/2} t^{-1} & 0 \\
q^{1/2} t^{-1} & q^{1/2} t^{-1} & 0 \\
0 & 0 & q^{1/2} t
\end{pmatrix} = \begin{pmatrix}
q^{1/2} t^{2 - 1} & q^{1/2} t^{2 - 1} & 0 \\
q^{1/2} t^{2 - 1} & q^{1/2} t^{2 - 1} & 0 \\
0 & 0 & q^{1/2} t
\end{pmatrix}.
\]

Following [Par1] and the notations from this paper (see formula (2.4) and Theorem 1.2 there), let

\[
(2.18) \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad I_1 = \begin{pmatrix} 1 & \tau & \overline{\tau} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

We set \( I_2 = J I_1 J, I_2 = J^2 I_1 J, I_{123} = I_1 I_2 I_3 = (J I_1)^3 \); use that \( J^3 = 1 \). Here \( \tau \in \mathbb{C}^* \). If \( |\tau| = 2 \cos(\pi/l) = 2 \cos(\phi/2) \) for \( \phi \overset{\text{def}}{=} 2\pi/l \) and \( |\tau^2 - \overline{\tau}| = 2 \cos(\pi/m) \), then \( \Delta[l; m] \overset{\text{def}}{=} \langle I_1, I_2, I_3 \rangle \) is a quotient of \( \Delta(l, l, l; m) \) for \( 3 \leq l, m \in \mathbb{N} \). We send \( a \mapsto I_1, b \mapsto I_2, c \mapsto I_3 \).

The groups \( \Delta[l, l, l] \) for

\[
\tau = \tau_l \overset{\text{def}}{=} e^{2i\phi/3} + e^{-i\phi/3} = 2e^{i\phi/3} \cos(\phi/2) \quad \text{for} \quad \phi \overset{\text{def}}{=} 2\pi/l \ (l \geq 3)
\]

are natural subgroups of the Livné groups, which are \(< J, I_1 >\). In this case, \( |\tau^2 - \overline{\tau}| = |\tau| \) and \( m = l \), so they are of the type \( \Delta[l; l] \). See (2.22) and around for further discussion. The eigenvalues of \( I_1, I_{12} = I_1 I_2 \) (coinciding with those for \( I_{23} = I_2 I_3, I_{31} = I_3 I_1 \)) and the eigenvalues of \( I_{123} \) can be found following entry (ii) in the table after Theorem 3.1 in [Par1]. Namely, they are

\[
(2.19) \quad \{-1, -1, 1\}, \ \{1, e^{i\phi}, e^{-i\phi}\}_1, \ \{e^{2i\phi}, -e^{-i\phi}, -e^{-i\phi}\}_{123}.
\]

Here \( I_{123} \) is parabolic at \( l = 6 \) and semisimple otherwise.

**Proposition 2.5.** Let \( q^{1/2} = e^{i\phi} \) for \( \phi = 2\pi/(2N) = \pi/N \).

(i) The image of \( B_1 \) in \( GL(V_{3}^+) \) divided by the center is conjugate to \( \Delta[2N; 2N] \). More exactly, conjugation by a proper matrix sends

\[
(2.20) \quad \widetilde{A} \mapsto -I_1, \quad \widetilde{B} \mapsto -I_2, \quad \widetilde{C} \mapsto -I_3, \quad \widetilde{T} = q^{1/4} T \mapsto -I_{123},
\]

\[
\widetilde{Y} = q^{1/4} Y \mapsto I_{12}, \quad \widetilde{X} T^2 = q^{1/4} X T^2 \mapsto I_{23}, \quad Y^{-1} X^{-1} \mapsto I_{31}.
\]

Confirming Conjecture (1.10) above for \( V_{2[k]}^+ \) of type \( (\gamma) \), the sequence \( 2N = 6, 8, 10, 12, 18 \) from (1.40) for \( \text{dim} \mathcal{C} = 3 \) coincides with the even
part of the list of Proposition 4.8 from [Par1]. These are the only values of $2N$ when the image of $B_1$ in $GL(V^+_3)$ is discrete non-finite; all images are arithmetic for such $2N$.

(ii) Additionally, let us substitute $q \mapsto q^\varphi = q^2, q^{1/2} \mapsto (q^{1/2})^\varphi = q$ for odd $N \geq 3$. Then the action of $\mathcal{H}_I^\varphi = \mathcal{H}_{q, q^{1/2}}$ in $\hat{V}^+_3$ is given by the same formulas (2.14) and (2.16) upon the substitution above and this module is (remains) irreducible due to Proposition 1.5. Then the conjugation equivalence from (2.20) holds and the image of $B_1$ in $GL(V^+_3)$ modulo the center is conjugate to $\Delta[N; N]$, which corresponds to $\phi^\varphi = 2\pi/N$.

Accordingly, the odd cases from [Par1], Proposition 4.8 give that the image of $B_1$ in $GL(V^+_3)$ is discrete and non-finite if and only if $N = 5, 7, 9$. The images for $N = 5, 7$ are arithmetic, which matches Conjecture 1.10; the image of $B_1$ for $N = 9$ is discrete and non-arithmetic.

**Proof.** We only need to establish the equivalence of the theory of images of triangular groups in $PU(2, 1)$ and that concerning the images of $B_1$ in 3-dimensional irreducible modules of DAHA of type $A_1$. It suffices to observe that $I_{123}$ satisfies the quadratic equation in the first theory, which uses (2.19) and semisimplicity of $I_{123}$ for generic real $\phi$. Then we use the classification of 3-dimensional modules of $\mathcal{H}_I$ and its variant for $\mathcal{H}_I^\varphi$ from the second theory.

We note that our analysis of the discreteness of the images of $B_1$ for the lists in (1.40), including the case $\dim V = 3$, is similar to the approach from [Par1] and other papers devoted to triangle groups in $PU(2, 1)$. We analyze the signature of the DAHA inner product, which is proportional to that in [Par1 Par2] in the Livné case. Though our Conjecture 1.10 is *in any dimensions*; it is for the images of $\Delta(2N, 2N; 2N)$ and $\Delta(N, N, N; N)$ subject to the quadratic relation for $\mathcal{T} \sim abc$ in irreducible rigid modules of $\mathcal{H}_I$ and $\mathcal{H}_I^\varphi$.

The *parabolic case* is interesting, which is for $V_{2[k]}^+$ with $2N = 6, k = -3/2$ and $q^{1/2} = e^{\pi i/3}$. Recall that here $t^{1/2} = q^{-3/4} = e$ and $T$ has one Jordan 2–blocks at the eigenvalue $t^{1/2}$; this is the boundary case $(\gamma^*)$ in (1.32). Then the Livné group $< J, I_1 >$ is conjugate to the Eisenstein-Picard modular group, which is $PU(2, 1; \mathbb{Z}[q^{1/2}])$ supplied with the natural inner product. See [Par2] and references therein.
Here $\tilde{T} \sim I_{123} = (I_1J)^3$. Assuming that $p \neq 2, 3$, we can calculate in this case the image of $B_{p,m}^\dagger$ of $B_q$ in automorphisms of $V_3^+$ (modulo scalars) considered over $\mathbb{Z}[q^{1/2}]/p^m$. Multiplying $T, X, Y$ by $t^{1/2}$, we make $\mathbb{Z}[q^{1/2}]$ the ring of definition of $V_3^+$ (instead of $\mathbb{Z}[q^{1/4}]$). Let $p \supset (p)$ be a prime ideal in $\mathbb{Z}[q^{1/2}]$. We obtain that for $q^{1/2} = e^{\pi i/3}$,

\[(2.21) \quad B_{p,m}^\dagger = PU(2, 1; \mathbb{Z}[q^{1/2}]/p^n).\]

Thus Theorem 2.2 and Corollary 2.3 supply $B_{p,m}^\dagger$ and $B_{(p),m}^\dagger$ with an action of $G_K^0$ and $G_K$, where $K$ is the field of definition of $\alpha_0 \in P^1 \supset \{\alpha_1, \alpha_2, \alpha_3\}$ and the corresponding elliptic curve $E \ni 0$ in this theorem. Recall that $\{\alpha_1, \alpha_2, \alpha_3\}$ are not assumed to be individually defined over $K$ (only as a set). The group $G_K$ permutes the corresponding generators $\tilde{A}, \tilde{B}, \tilde{C}$ accordingly; see Part (iv) of Theorem 2.2.

We would like to mention here that Livné in his thesis considered his groups in connection with a certain branched cover of degree 2 of the universal elliptic curve.

2.7. Some perspectives. One can try to obtain more general groups $\Delta[l, m, n]$, the images of triangle groups $\Delta(l, m, n)$ in $PU(2,1)$ with the DAHA-type quadratic relation for $I_{123}$. Let us recall the basic construction of the group $\Delta[l, m, n]$ (see e.g. [Sch]). For $u, v, w \in \mathbb{C}^*$, we begin with unitary complex reflection

\[
I_1 = \begin{pmatrix} 1 & u & \bar{v} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} -1 & 0 & 0 \\ \bar{v} & 1 & w \\ 0 & 0 & -1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ v & \bar{w} & 1 \end{pmatrix}.
\]

with respect to the Hermitian form

\[
(e_1, e_1) = (e_2, e_2) = (e_3, e_3) = 2, \quad (e_1, e_2) = \bar{u}, \quad (e_2, e_3) = \bar{v}, \quad (e_3, e_1) = \bar{w}
\]

of determinant $det = 2(\Re(uvw) - |u|^2 - |v|^2 - |w|^2 + 4)$ for the basic vectors $\{e_i\}$. Namely, $I_i(z) = -z + (z, e_i)e_i$ for $z \in \mathbb{C}^3$.

The eigenfunctions of matrices $I_{12}, I_{23}$ and $I_{31}$ are respectively

\[
\{1, e^{\pm i\alpha} \}, \quad \{1, e^{\pm i\beta} \}, \quad \{1, e^{\pm i\gamma} \} \quad \text{for} \quad 0 \leq \alpha, \beta, \gamma \leq \pi
\]

if and only if $|u| = 2 \cos(\alpha/2), \quad |v| = 2 \cos(\beta/2), \quad |w| = 2 \cos(\gamma/2)$. Accordingly, the matrices $I_{12}, I_{23}, I_{31}$ are of orders $l, m, n$ exactly when $\alpha = 2\nu_1 \pi/l, \quad \beta = 2\nu_2 \pi/m, \quad \gamma = 2\nu_3 \pi/n$, where $\nu_i \in \mathbb{N}, \quad \nu_1 < l, \nu_2 < m, \nu_3 < n, \quad \gcd(\nu_1, l) = 1$ and so on for $\nu_{2,3}$. 


Setting \( q = e^{2i\phi} \) for \( 0 \leq \phi \leq \pi \), the condition
\[
3 + uuv - (|u|^2 + |v|^2 + |w|^2) = e^{2i\phi} - 2e^{-i\phi}
\]
is necessary and sufficient for \( I_{123} \) to have the following eigenvalues:
\[
\{-q^{1/4}t^{1/2}, -q^{1/4}t^{1/2}, q^{1/4}t^{-1/2}\} = \{-q^{-1/2}, -q^{-1/2}, q\} \text{ for } t^{1/2} = q^{-3/4}.
\]
This condition is a direct calculation of \( Tr(I_{123}) \); the corresponding relation for \( Tr(I_{123}^2) \) follows from (2.22) automatically due to \( \phi \in \mathbb{R} \).

Note that \( \det < 0 \) and the Hermitian form is of signature \((2,1)\) for \( 0 < \phi < \pi/2 \).

Equation (2.22) reduces to that with \( u, v \in \mathbb{R}^+ \) via the substitutions
\[
uw = (\sin(2\phi) + 2 \sin(\phi))/(uw) \text{ and one arrives at a quadratic equation for } \Re w, \text{ which always has solutions in } \Re \text{ for sufficiently large } u, v; \text{ the corresponding } I_{123} \text{ will be generally non-semisimple.}
\]

Let us impose (2.22) for generic real \( \phi \) and assume furthermore that \( I_{123} \) is semisimple. Then \( T \overset{\text{def}}{=} -q^{-1/4}I_{123} \) will satisfy the DAHA quadratic relation for \( t^{1/2} = q^{-3/4} \) and we can employ Part (ii) of Theorem 1.1. We conclude that the set of matrices \( \{-I_1, -I_2, -I_3, -I_{123}\} \) is conjugated to \( \{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{T}\} \) in \( V_{2[k]}^+ \) for \( k = -3/2 \).

In terms of (2.22), the semisimplicity of \( I_{123} \) for sufficiently general real \( \phi \) implies that \( u = \epsilon_1 \tau, v = \epsilon_2 \tau, w = \epsilon_3 \tau \text{ for } \tau = e^{2i\phi/3} + e^{-i\phi/3} \) and unimodular \( \epsilon_i \) such that \( \epsilon_1 \epsilon_2 \epsilon_3 = 1 \). We do not see how to derive this fact directly from (2.22) and the semisimplicity of \( I_{123} \). Recall that the coincidence of \( \{u, v, w\} \) with \( \{\tau, \tau, \tau\} \) up to \( \epsilon_i \) reflects the projective \( PSL_2(\mathbb{Z}) \)-invariance of \( V_{2[k]}^+ \).

Now let \( \phi = \nu\pi/N \) for \( N \geq 3 \) and \( (\nu, N) = 1 \). Using Theorem 2.9.2 of [Ch1], we obtain that the irreducible \( \mathcal{H} \)-modules of dimension 3 for \( q = e^{2i\phi} \) can be only \( V_3^+ \) or \( V_3^{+\vee} \) up to a possible change of signs of \( T, X, Y \). This results in \( l = m = n \) and the group \( \Delta[2N, 2N, 2N; 2N] \) or that with \( 2N \mapsto N \) for \( \check{\nu} \). We conclude that 3-dimensional representations of DAHA of type \( A_1 \) for \( q = e^{2i\nu\pi/N} \) with \( N \geq 3 \) and \( \nu \) as above generate only (subgroups of) the Livné groups; see Proposition 2.5.

Non-rigid theory. The connection with triangle groups from Proposition 2.4 links our work to [LLM] and quite a few papers devoted to finite/simple quotients of triangular groups. For instance, Corollary 1.2 from [LLM] establishes that there are infinitely many prime numbers \( p \) (their density is nonzero) such that linear and orthogonal
groups over \( \mathbb{Z}/(p^n) \) can be presented as quotients of \( D(2N, 2N; 2N) \), or \( D(N, N, N; N) \), which we can reach via rigid DAHA-modules for \( A_1 \).

Given \( V \) of types \((\alpha, \tilde{\gamma})\) with the standard DAHA inner product, finding \( p \) such that \( \tilde{\mathcal{B}}_{p,m} = PU(V) \) for \( V \) considered over \( \mathbb{Z}_q/p^n \) is of obvious interest; cf. (2.21). Note that we need unitary groups here (not considered in [LLM]); also our \( \tilde{\mathcal{B}}_{p,m} \) are of special kind due to the quadratic relation for \( \tilde{T} \) (which is the key in DAHA theory).

Using all, not only rigid, DAHA-modules from Theorem 2.9.2 of [Ch1] is quite natural for establishing a connection with [LLM], which is essentially based on the deformation theory of the character varieties for triangle groups, i.e. non-rigid representations. Even in type \( A_1 \), one can obtain here interesting examples of quotients of various triangular groups. The action of \( G_K \) can be generally defined for non-rigid representations; it will then transform the parameters of DAHA-modules.

Finding the transformation of these parameters under the projective action of \( \text{PSL}_2(\mathbb{Z}) \) (a similar and simpler problem) is important in the theory of the Lamé and Heun equations. The latter equation corresponds to the root system \( C^\vee C_1 \) and is directly related to the Painlevé VI equation. These equations require DAHA at the critical center charge \( q = 1 \) (the \( t \)-parameters can be arbitrary). The unitarity of the corresponding monodromy representations (for proper \( t \) and the spectral parameter), which are DAHA modules of dimension 2 at \( q = 1 \), and the existence of the action of the Fourier transform \( \sigma \) there are among the core questions in the theory of these equations.

Let us mention here the \( 3 \times 3 \) Fuchsian representation of the Painlevé VI equation from [Boa] and related topics considered there. This representation is connected with our using 3-dimensional DAHA modules, but this is beyond the present paper (we study only rigid modules). Also, the monodromy nature of DAHA modules combined with Proposition 2.5 provide an interpretation of the Livné groups in terms of the monodromy of the KZB and similar local systems.

**Compatibility at different roots of unity.** A challenging problem concerning Theorem 2.2 is in establishing compatibility of groups \( \tilde{\mathcal{B}}_{p,m} \) for different roots of unity \( q \). By contrast, given \( q \) and prime \( p \), the compatibility with respect to \( m \) is straightforward. One can readily define the action of \( G_K \) and \( G_K \) in

\[
\tilde{\mathcal{B}}_{p,\infty} = \lim_{m \to \infty} \tilde{\mathcal{B}}_{p,m}, \quad \tilde{\mathcal{B}}_{p,\infty} = \lim_{m \to \infty} \tilde{\mathcal{B}}_{p,m}, \quad \tilde{\mathcal{B}}_{(p),\infty} = \lim_{m \to \infty} \tilde{\mathcal{B}}_{(p),m}.
\]
Accordingly, the conjugation matrices $h_{q}$ from Theorem 2.2 will have entries in the $p$-adic rings $\lim_{\longleftarrow m} \mathbb{Z}_{q}/p^{m}$ and $\lim_{\longleftarrow m} \mathbb{Z}_{q}/(p^{m})$. Recall that the $p$-adic limit is not needed if $B$ is already finite.

The dependence of $\mathfrak{B}_{p,m}^{\dagger}$ on $N$ is very far from straightforward even in case of the triangle groups. The deformation construction from Section 1.4 suggests the following (general) approach, which seems important in its own right. Let us focus on $V'_{2N-4k}$ of type $(\alpha)$ from Proposition 1.6; the module $V'_{4|k'|}$ for $k' = k - N/2$ is its deformation, well defined for any nonzero $q$. Upon the substitution $q \mapsto q^{d}$ for $d|N$, one obtains the action of $\mathfrak{H} \overset{\text{def}}{=} \mathfrak{H}_{q^{d/2},t,d/2}$ in $V'_{4|k'|}$. The decomposition of this module is interesting in its own right.

This construction is connected with the passage $q \mapsto q^{\ell}$ for odd $N$, which is used to address the case of odd $N$ when $q^{1/2}$ is not a primitive root of unity. However, $q \mapsto q^{\ell}$ cannot be associated with any Galois automorphisms due to $d|N$ (unless $N = \ell^{a}$ and we take $p = \ell$). Potentially there can be links here with Lusztig’s Frobenius map in quantum groups, but this is beyond this paper.

Let $k = k \bmod N$ for $N = N/d$, $k, \tilde{k} \in \mathbb{Z}_{+}/2$. Assuming that $\tilde{k} < N/2$, the irreducible $\mathfrak{H}$–module $V_{2N-4k}$ is a quotient of $V_{2N-4k}$. This links $N$ and $\tilde{N}$, but does not result in any immediate connections at level of the corresponding images of $\mathcal{B}_{1}$.

For instance, one can assume here that $k < \tilde{N}/2$ or that $d$ is odd and $|k'| < \tilde{N}/2$; then we have that $\tilde{k} = k$ and, respectively, $\tilde{k}' = k'$. In the latter case, the module $V'_{2|k'|}$ remains irreducible for $q \mapsto \tilde{q}$.

**Dimension 4.** Under the inequality $|k'| < \tilde{N}/2$ for odd $d$, let us consider the module $V = V'_{4}$ from (1.43) as an example. It is irreducible for a primitive $(2N)$th root of unity $q^{1/2}$ and remains irreducible when $q^{1/2} \mapsto \tilde{q}^{1/2}$ for a primitive root $\tilde{q}^{1/2}$ of order $2\tilde{N}$. Following Section 1.7 we denote by $\mathfrak{B}_{N}^{\dagger}$ and $\mathfrak{B}_{\tilde{N}}^{\dagger}$, the quotients of $\text{Image} (\mathcal{B}_{q})$ or $\text{Image} (\mathcal{B}_{1})$ by the center. Then one has the homomorphisms from (1.48)

$$\mathfrak{B}_{N}^{\dagger}/<Z> \to \mathfrak{B}_{\tilde{N}}^{\dagger}/<\tilde{Z}> , \ X \mapsto X^{d}, \ Y \mapsto Y^{d}, \ T \mapsto T^{d}, \ (2.23)$$

for $Z = Y^{2} T^{-2} X^{-2}$ and $\tilde{Z} = Y^{2d} T^{-2d} X^{-2d}$; there are no such homomorphisms without the division by $<\tilde{Z}>$ unless $d = 1, N$. Obviously such a division (which means imposing the relation $T^{2} = Y^{2} X^{-2}$ in $\mathfrak{B}_{N}^{\dagger}$) results in loss of information. However the real problem here is
in establishing compatibility of the maps from (2.23) with the action of the Galois group $G_K$. We see no reasons for the group $\langle Z \rangle$ to be fixed under $G_K$ (upon a proper conjugation in $\mathfrak{B}_N^+$). This would hold if $\langle Z \rangle$ were conjugated in $\mathfrak{B}_N^+$ to the ramification subgroups $\langle T \rangle$, $\langle A \rangle$, $\langle B \rangle$ or $\langle C \rangle$, but this is not the case.

Instead of division by $\langle Z \rangle$ and trying the maps from (2.23), switching to the commutator subgroups seems the best option here. Let

$$C_N^1 \overset{\text{def}}{=} \langle T^2, X^2, Y^2 \rangle / \langle T^{2M}, X^{2N}, Y^{2N} \rangle \simeq \mathbb{Z}_M \times \mathbb{Z}_N^2,$$

where $M = N$ for odd $N$ and $M = N/2$ otherwise. We follow (2.9) and the corresponding part of Section 2.4.

Recall that $\mathfrak{B}_N^+/C_N^1 \simeq \mathbb{Z}_N^2$ corresponds to the cover of $P^1$ by $E' \simeq E$ for the composition $E' \to E'/E_2' \simeq E \to P^1$, where the fist map is the multiplication by 2 and $E_2'$ is the group of points of order 2. The cover of $E'$ associated with $C_N^1$ corresponds to $K^\circ(E')$ extended by the functions $f_i^{1/N}$ for $f_i \in K^\circ(E')$ with the divisors $(f_i) = 20_i - 20$, where $0_i (i = 1, 2, 3)$ are nonzero points of the second order of $E'$.

We obtain the following system of homomorphisms:

$$C_N^1 \to C_N^1, \quad \text{where } X \mapsto X^d, \; Y \mapsto Y^d, \; T^2 \mapsto T^{2d} \text{ for odd } d,$$

which is compatible with the projective action of $PSL_2(\mathbb{Z})$ and the action of $G_K$. Thus the latter action results in the following “ramified” analogue of the Tate module (though not that interesting). For an odd prime $\ell$ and $N = \ell^u$, the absolute Galois group $G_K$ acts in $T_\ell \overset{\text{def}}{=} \varprojlim_{\ell^n} C_N^1$ by $3 \times 3$-matrices with $\ell$-adic entries.

The action of the absolute Galois group is actually quite similar to that of the projective $PSL_2(\mathbb{Z})$ (more generally, the corresponding braid groups). Technically, we use the same rigidity argument for both. In the case of $t = 1(k = 0)$, the connection is the most direct; namely, the absolute Galois group acts via the projective $PSL_2(\mathbb{Z})$. Both groups are important ingredients in the theory of the corresponding Hurwitz spaces; see e.g. Section 3.1 in [Det].

The projective action of $PSL_2(\mathbb{Z})$ in $C_N^1$ can be readily calculated. Since we divide by the center, we omit the powers of $q$ in the formulas below. It is determined by the relations

$$\tau_-(X^2) = Z = \tau_+(Y^{-2}) \quad \text{for } Z = Y^2T^{-2}X^{-2} \text{ from (1.47)}.$$
For instance, let us check the first formula:

$$\tau_-(X^2) = YXYX = Y(YXT^2)X = Y^2(T^{-1}X^{-1})(X^{-1}T^{-1}) = Z.$$  

Recall that $\tau_-(Y) = Y, \tau_+(X) = X, \tau_\pm(T) = T$. We see that the action of $PSL_2(\mathbb{Z})$ in $C_N^+\mathbf{f}$ factors through its projection onto $PSL_2(\mathbb{Z}_2)$.

**Refined Jones polynomials.** Formula (2.25) with $q$ (i.e. considered in the whole $\mathcal{B}_q$) and upon its extension to all powers of $X$ and $Y$ can be used to calculate the images of the refined Jones polynomials of torus knots from [Ch2] upon the substitution $t = -q^{-1}$ (in the case of $A_1$). The torus knots are encoded by the (first columns) of elements of $\gamma \in PSL_2(\mathbb{Z})$ and their refined Jones polynomials are the evaluations in $\mathcal{H}$ of the elements $\gamma(p_n)$, considered as elements of $\mathcal{H}$, for arbitrary symmetric Macdonald polynomials $p_n$, which add colors to the theory.

For $V$ of types $(\alpha, \beta, \gamma)$ from Theorem 1.4 (and their generalizations for any root systems), the evaluations of $\gamma(p_n) \in \mathcal{H}$ coincide with the values of $\gamma(p'_n)$ at $t^{-1/2}$ for the corresponding $q, t$, where $p'_n$ is the image of $p_n$ in $V$ assuming that it is well defined (i.e. for sufficiently small $n$). We employ the fact that $\gamma$ act in $V$ (the rigidity of $V$).

The nonsymmetric polynomials $e_n$ can be used instead of $p_n$ in this construction. For instance, $\gamma(X)$ is needed for $n = 1$, which requires a calculation entirely within the group $\text{Image}(\mathcal{B}_q)$. In this case, one can disregard powers of $q$ and calculate modulo the center; the formulas from [Ch2] are anyway up to powers of $q, t$. The colored case ($n > 1$) is similar, though we need to track now the powers of $q$. For example, formulas (2.25) with proper $q$-corrections provide simple expressions for $\gamma(p_n)$ for even $n$ (under the substitution $t = -q^{-1}$). Note that $\gamma(p'_n)(t^{-1/2})$ becomes a pure power of $q$, i.e. trivial, for $t = 1(k = 0)$ and any $n$ (the same holds for any root systems), since $e_n$ are monomials in this case.

Due to Theorem 2.2, the Galois group $G_\mathcal{H}^\circ$ acts in $V$ for the modules $V$ considered there. Hence the evaluation of $g(p'_n)$, where $p'_n$ is the image of $p_n$ in $V$ (assuming that $p_n$ is well defined) can be considered as certain arithmetic refined Jones polynomial of $g \in G_\mathcal{H}^\circ$. It is a collection of elements from $\mathbb{Z}[\hat{q}]/(p^m)$ for $p, m$ as above (trivial for $t = 1$). They are actually from $\mathbb{Q}(\hat{q})$ for finite $\text{Image}(\mathcal{B}_q)$; can this be true for infinite $\text{Image}(\mathcal{B}_q)$?
Concluding remarks. (a) A generalization of Theorem 2.2 to rigid modules for any root systems, at least to \textit{perfect representations} from [Ch1], is a natural challenge. Such modules describe the monodromy of certain local systems connected with the KZB or \textit{elliptic QMBP}, which can be hopefully defined over algebraic numbers. This would give an alternative approach to the action of the absolute Galois group in the rigid DAHA modules, without using the Riemann Existence Theorem. The natural setting for the approach based on \textit{RET} is the root system $C^\vee C_1$, where quite a ramified theory at roots of unity is expected.

(b) On the other hand, even for $A_1$ and for the simplest nontrivial 3-dimensional modules of DAHA, the corresponding images of the elliptic braid group exactly match the \textit{Livné groups}. Furthermore, the simplest nontrivial case $q = e^{2m/3}$, $t = -1$ of Proposition 2.5 results in the Livné lattice for $\phi = \pi/3$, directly related to the Eisenstein-Picard modular group, which has interesting algebraic and analytic theory. Starting with an elliptic curve over $\overline{K} \subset \overline{\mathbb{Q}}$, we obtain an action of $\text{Gal}(\overline{\mathbb{Q}}/K)$ in $p$-adic completions of this modular group for $p \neq 2, 3$ and generalize this construction to arbitrary rigid DAHA-modules of type $A_1$.

(c) One of the major facts of the DAHA theory is the projective action of $PSL_2(\mathbb{Z})$ in the algebra itself and its rigid irreducible representations. This property is entirely conceptual and is directly related to the topology of the elliptic configuration space. In the present paper, we use such an approach for the absolute Galois group $G_K$ instead of the projective $PSL_2(\mathbb{Z})$. As an application, certain \textit{refined Jones polynomials} of $g \in G_K$ can be defined in type $A_1$, which are systems of elements from $\mathbb{Z}[\tilde{q}]/(p^m)$, instead of those in [Ch2] for torus knots encoded by the corresponding matrices $\gamma \in PSL_2(\mathbb{Z})$.

Acknowledgements. The author thanks David Kazhdan, Maxim Kontsevich, Nikita Nekrasov and Yan Soibelman for useful discussions and Hebrew University, IHES and SCGP (where this work was reported) for invitations and hospitality.

References

[BL] A. Beilinson, and A. Levin, \textit{The Elliptic Polylogarithm}, In: Motives (Seattle, WA, 1991), Proc. Symp. in Pure Math. 55 (1994), 123–192.

[Bel] G.V. Belyi, \textit{On extensions of the maximal cyclotomic field having a given classical Galois group}, J. Reine Angew. Math. 341 (1983), 147–156.
ON GALOIS ACTION IN RIGID DAHA MODULES

1. \footnotesize{[Boa]} P. Boalch, \textit{From Klein to Painlevé via Fourier, Laplace and Jimbo}, Proc. London Math. Soc. (3) 90 (2005), 167–208.

2. \footnotesize{[Ch1]} I. Cherednik, \textit{Double affine Hecke algebras}, London Mathematical Society Lecture Note Series, 319, Cambridge University Press, Cambridge, 2006.

3. \footnotesize{[Ch2]} \textit{Jones polynomials of torus knots via DAHA}, Int. Math. Res. Notices, 2013:23 (2013), 5366–5425.

4. \footnotesize{[CoH]} K. Coombes, and D. Harbater, \textit{Hurwitz families and arithmetic Galois group}, Duke Math. J. 52:4 (1985), 821–839.

5. \footnotesize{[CrB]} W. Crawley-Boevey, \textit{Indecomposable parabolic bundles and the existence of matrices in prescribed conjugacy class closures with product equal to the identity}, Publ. Math. Inst. Hautes Etudes Sci. 100 (2004), 171–207.

6. \footnotesize{[Der]} M. Deraux, \textit{Deforming the R-Fuchsian (4,4,4)-triangle group into a lattice}, Topology, 45 (2006), 989-1020.

7. \footnotesize{[Det]} M. Dettweiler, \textit{Plane curve complements and curves on Hurwitz spaces}, J. Reine Angew. Math. 573 (2004), 19–43.

8. \footnotesize{[FrV]} M.D. Fried, and H. Völklein, \textit{The inverse Galois problem and rational points on moduli spaces}, Math. Ann. 290 (1991), 771–800.

9. \footnotesize{[Ki]} A. Kirillov, Jr., \textit{On inner product in modular tensor categories. I}, Jour. of AMS 9 (1996), 1135–1170.

10. \footnotesize{[Kos]} V.P. Kostov, \textit{The Deligne-Simpson problem a survey}, Journal of Algebra 281:1 (2004), 83-108.

11. \footnotesize{[LLM]} M. Larsen, and A. Lubotzky, and C. Marion, \textit{Deformation theory and finite simple quotients of triangle groups II}, Preprint [arXiv:1301.2955] (math.GR) (2013).

12. \footnotesize{[Mos]} G.D. Mostow, \textit{On a remarkable class of polyhedra in complex hyperbolic space}, Pacific J. Math. 86:1 (1980), 171-276.

13. \footnotesize{[ObS]} A. Oblomkov, and E. Stoica, \textit{Finite dimensional representations of the double affine Hecke algebra of rank 1}, Journal of Pure and Applied Algebra, 213:5 (2009), 766-771.

14. \footnotesize{[Par1]} J.R. Parker, \textit{Unfaithful complex hyperbolic triangle groups I}, Pacific Journal of Mathematics 238 (2008), 145–169.

15. \footnotesize{[Par2]} — , \textit{Cone metrics on the sphere and Livnés lattices}, Acta Math. 196:1 (2006), 1-64.

16. \footnotesize{[Sch]} R.E. Schwartz, \textit{Complex hyperbolic triangle groups}, Proceedings of the International Congress of Mathematicians (Beijing, 2002), vol. II, edited by T. Li, Higher Education Press, Beijing, 2002, 339-349.

17. \footnotesize{[Vol]} H. Völklein, \textit{Groups as Galois Groups}, New York: Cambridge Studies in Advanced Mathematics 53 (1996).

\footnotesize{(I. Cherednik) DEPARTMENT OF MATHEMATICS, UNC CHAPEL HILL, NORTH CAROLINA 27599, USA, chered@email.unc.edu)