The cone of $Z$-transformations on the second order cone

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Abstract

In this paper, we describe the structural properties of the cone of $Z$-transformations on the second order cone in terms of the semidefinite cone and copositive/completely positive cones induced by the second order cone and its boundary. In particular, we describe its dual as a slice of the semidefinite cone as well as a slice of the completely positive cone of the second order cone. This provides an example of an instance where a conic linear program on a completely positive cone is reduced to a problem on the semidefinite cone.

Key Words: $Z$-transformation, dual cone, second order cone, semidefinite cone, completely positive cone.

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1 Introduction

Given a proper cone $K$ in a finite dimensional real Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, a linear transformation $A : \mathcal{H} \to \mathcal{H}$ is said to be a $Z$-transformation on $K$ if

\[ x \in K, y \in K^*, \text{ and } \langle x, y \rangle = 0 \Rightarrow \langle Ax, y \rangle \leq 0, \]

where $K^*$ denotes the dual of $K$ in $\mathcal{H}$. Such transformations appear in various areas including economics, dynamical systems, optimization, see e.g., [1–4] and the references therein. When $\mathcal{H}$ is $\mathbb{R}^n$ and $K$ is the nonnegative orthant, $Z$-transformations become $Z$-matrices, which are square matrices with nonpositive off-diagonal entries.

The set $Z(K)$ of all $Z$-transformations on $K$ is a closed convex cone in the space of all (bounded) linear transformations on $\mathcal{H}$. Given their appearance and importance in various areas, describing/characterizing elements of $Z(K)$ and its interior, boundary, dual, etc., is of interest. An early result of Schneider and Vidyasagar [5] asserts that $A$ is a $Z$-transformation on $K$ if and only if $e^{-tA}(K) \subseteq K$ for all $t \geq 0$; consequently,

\[ Z(K) = \overline{\mathbb{R}I - \pi(K)}, \quad (1) \]

where $\pi(K)$ denotes the set of all linear transformations that leave $K$ invariant, $I$ denotes the identity transformation, and overline denotes the closure. To see another description of $Z(K)$, let $LL(K) := Z(K) \cap -Z(K)$ denote the lineality space of $Z(K)$, the elements of which are called Lyapunov-like transformations. Then the inclusions

\[ \mathbb{R}I - \pi(K) \subseteq LL(K) - \pi(K) \subseteq Z(K) = \overline{\mathbb{R}I - \pi(K)} \]

imply that

\[ Z(K) = \overline{LL(K) - \pi(K)}. \]

As the cones $Z(K)$, $\pi(K)$, and $LL(K)$ are generally difficult to describe for an arbitrary proper cone $K$, we consider special cases. When $K$ is the nonnegative orthant, $Z(K)$ consists of square matrices with nonpositive off-diagonal entries, $\pi(K)$ consists of nonnegative matrices, and $LL(K)$ consists of diagonal matrices. Consequently, proper polyhedral cones can be handled via isomorphism arguments. Moving away from proper polyhedral cones, in this paper, we focus on the second order cone (also called the Lorentz cone or the ice-cream cone) in the Hilbert space $\mathbb{R}^n$, $n > 1$, defined by:

\[ \mathcal{L} := \{(t, u)^T : t \in \mathbb{R}, u \in \mathbb{R}^{n-1}, t \geq ||u||\}. \quad (2) \]

This cone, being an example of a symmetric cone, appears prominently in conic optimization [6]. For this cone, Stern and Wolkowicz [7] have shown that $A \in Z(\mathcal{L})$ if and only if for some real number $\gamma$, the matrix $\gamma J - (JA + A^T J)$ is positive semidefinite, where $J$ is the diagonal matrix.
Theorem 4.2) asserts that

$$Z(L) = LL(L) - \pi(L).$$

(3)

(Going in the reverse direction, in a recent paper, Kuzma et al., [9] have shown that for an irreducible symmetric cone $K$, the equality $Z(K) = LL(K) - \pi(K)$ holds only when $K$ is isomorphic to $L$.)

Characterizations of $\pi(L)$ and $LL(L)$ appear, respectively, in [10] and [11].

In this paper, we describe $Z(L)$ and its interior, boundary, and dual in terms of the semidefinite cone and the so-called copositive and completely positive cones induced by $L$ (or its boundary $\partial(L)$) see below for the definitions. In particular, we describe the dual of $Z(L)$ as a slice of the semidefinite cone and also of the completely positive cone of $L$. This provides an example of an instance where a conic linear optimization problem over a completely positive cone is reduced to a semidefinite problem. To elaborate, consider $\mathbb{R}^n$, the Euclidean $n$-space of (column) vectors with the usual inner product, $\mathbb{R}^{n\times n}$, the space of all real $n \times n$ matrices with the inner product $\langle X, Y \rangle = \text{tr}(X^T Y)$, and $S^n$, the subspace of all real $n \times n$ symmetric matrices in $\mathbb{R}^{n\times n}$. Corresponding to a closed cone $C$ (which is not necessarily convex) in $\mathbb{R}^n$, let

$$\mathcal{E}_C := \text{copos}(C) := \left\{ A \in S^n : x^T A x \geq 0, \forall x \in C \right\}$$

denote the copositive cone of $C$ and

$$\mathcal{K}_C := \text{compos}(C) := \left\{ \sum uu^T : u \in C \right\}$$

denote the completely positive cone of $C$, where the sum is a finite sum of objects. When $C = \mathbb{R}^n$, these two cones coincide with the semidefinite cone $S^n_+$; when $C = \mathbb{R}^+_{n\times n}$, these reduce, respectively, to the (standard) copositive cone and completely positive cone. All these cones appear prominently in conic optimization. A result of Burer [12] (see also, [13, 14]) says that any nonconvex quadratic programming problem over a closed cone with additional linear and binary constraints can be reformulated as a linear program over a suitable completely positive cone. For this and other reasons, there is a strong interest in understanding copositive and completely positive cones. For the closed convex cones $\mathcal{E}_C$ and $\mathcal{K}_C$, various structural properties (such as the interior, boundary) as well as duality, irreducibility, and homogeneity properties, have been investigated in the literature, see for example, [15–18]. Taking $C$ to be one of $\mathbb{R}^n$, $L$, or $\partial(L)$, we show that

$$Z(L)^* = \left\{ B \in \mathbb{R}^{n\times n} : \langle B, I \rangle = 0, -JB \in \mathcal{K}_C \right\}$$

(4)

and deduce the equality of slices

$$\left\{ X \in \mathbb{R}^{n\times n} : \langle J, X \rangle = 0, X \in S^n_+ \right\} = \left\{ X \in \mathbb{R}^{n\times n} : \langle J, X \rangle = 0, X \in \mathcal{K}_C \right\}.$$  

(5)
2 Preliminaries

In a (finite dimensional real) Hilbert space \((H, \langle \cdot, \cdot \rangle)\), a nonempty set \(\mathcal{K}\) is said to be a closed convex cone if it is closed and \(tx + sy \in \mathcal{K}\) whenever \(x, y \in \mathcal{K}\) and \(t, s \geq 0\) in \(\mathbb{R}\). Such a cone is said to be proper if \(\mathcal{K} \cap -\mathcal{K} = \{0\}\) and has nonempty interior. Corresponding to a closed convex cone \(\mathcal{K}\), we define its dual in \(H\) as the set
\[
\mathcal{K}^* = \{x \in H : \langle x, y \rangle \geq 0, \forall y \in \mathcal{K}\}
\]
and the complementarity set of \(\mathcal{K}\) as the set \(\{(x, y) : x \in \mathcal{K}, y \in \mathcal{K}^*, \langle x, y \rangle = 0\}\). We say that a linear transformation \(A : H \to H\) is copositive on \(\mathcal{K}\) if \(\langle Ax, x \rangle \geq 0\) for all \(x \in \mathcal{K}\). We also let \(\pi(\mathcal{K}) = \{A : A(\mathcal{K}) \subseteq \mathcal{K}\}\), where \(A\) denotes a linear transformation on \(H\). For a set \(S\) in \(H\), we denote the closure, interior, and the boundary by \(\overline{S}, S^\circ,\) and \(\partial(S)\) respectively. Throughout this paper, we use the summation sign \(\sum\) to describe a finite sum of objects.

We will be considering closed convex cones in the space \(H = \mathbb{R}^n\) which carries the usual inner product and in the space \(\mathbb{R}^{n \times n}\) which carries the inner product \(\langle X, Y \rangle := \text{tr}(X^\top Y)\), where the trace of a square matrix is the sum of its diagonal entries. In \(\mathbb{R}^{n \times n}\), \(\mathcal{S}^n\) denotes the subspace of all symmetric matrices and \(\mathcal{A}^n\) denotes the subspace of all skew-symmetric matrices. We note that \(\mathbb{R}^{n \times n}\) is the orthogonal direct sum of \(\mathcal{S}^n\) and \(\mathcal{A}^n\).

We recall some (easily verifiable) properties of the second order cone \(\mathcal{L}\) given by (2). \(\mathcal{L}\) is a self-dual cone in \(\mathbb{R}^n\), that is, \(\mathcal{L}^* = \mathcal{L}\); its interior and boundary are given, respectively, by
\[
\mathcal{L}^\circ = \{(t, u)^\top : t > ||u||\},
\]
\[
\partial(\mathcal{L}) = \{(t, u)^\top : t = ||u||\} = \{\alpha (1, u)^\top : \alpha \geq 0, ||u|| = 1\}.
\]
We also have
\[
[0 \neq x, y \in \mathcal{L}, \langle x, y \rangle = 0] \Rightarrow x = \alpha (1, u)^\top \text{ and } y = \beta (1, -u)^\top, \text{ for some } \alpha, \beta > 0 \text{ and } ||u|| = 1.
\]
(6)

For a closed cone \(\mathcal{C}\) in \(\mathbb{R}^n\), we consider the copositive cone \(\mathcal{E}_\mathcal{C}\) and the completely positive cone \(\mathcal{K}_\mathcal{C}\) (defined in the Introduction). Note that these are cones of symmetric matrices.

In the Hilbert space \(\mathcal{S}^n\) (which carries the inner product from \(\mathbb{R}^{n \times n}\)), the following hold.

1. \(\mathcal{K}_\mathcal{C}\) is the dual cone of \(\mathcal{E}_\mathcal{C}\).[17]

2. When \(\mathcal{C} = \mathbb{R}^n\), both \(\mathcal{E}_\mathcal{C}\) and \(\mathcal{K}_\mathcal{C}\) are proper cones (\[12\], Proposition 2.2). In particular, this holds when \(\mathcal{C}\) is one of \(\mathbb{R}^n, \mathcal{L},\) or \(\partial(\mathcal{L})\).

3. We have \(\mathcal{E}_{\mathbb{R}^n} = \mathcal{S}_+^n \subset \mathcal{E}_\mathcal{L} \subset \mathcal{E}_{\partial(\mathcal{L})}\), or equivalently, \(\mathcal{K}_{\partial(\mathcal{L})} \subset \mathcal{K}_\mathcal{L} \subset \mathcal{K}_{\mathbb{R}^n} = \mathcal{S}_+^n\).
3 Main results

In this section, we provide a closure-free description of \( Z(L) \) and, additionally, describe the dual, interior, and the boundary of \( Z(L) \). We recall that \( J = \text{diag}(1, -1, -1, \ldots, -1) \) and \( \mathcal{A}^n \) denotes the set of all skew-symmetric matrices in \( \mathbb{R}^{n \times n} \).

**Theorem 3.1** Let \( \mathcal{C} \) denote one of \( \mathbb{R}^n \), \( L \), or \( \partial(L) \). Then,

\[
Z(L) = \mathbb{R}I - J(\mathcal{E}_L + \mathcal{A}^n).
\]

**Proof.** Let \( A \in Z(L) \). From the result of Stern and Wolkowicz [7] mentioned in the Introduction, we have

\[
2\gamma J - (JA + A^\top J) = 2P
\]

for some \( \gamma \in \mathbb{R} \) and \( P \in \mathcal{S}^n_+ \). Hence, \( JA + (JA)^\top = 2(\gamma J - P) \), which implies

\[
2JA = JA + (JA)^\top - [(JA)^\top - JA] = 2(\gamma J - P) - 2Q,
\]

where \( 2Q = (JA)^\top - JA \) is skew-symmetric. Since \( J^2 = I \), this leads to

\[
A = \gamma I - J(P + Q),
\]

where \( P \in \mathcal{S}^n_+ \) and \( Q \in \mathcal{A}^n \). As \( \mathcal{S}^n_+ \subset \mathcal{E}_L \subset \mathcal{E}_{\partial(L)} \), this proves that

\[
Z(L) \subseteq \mathbb{R}I - J(\mathcal{S}_+^n + \mathcal{A}^n) \subseteq \mathbb{R}I - J(\mathcal{E}_L + \mathcal{A}^n) \subseteq \mathbb{R}I - J(\mathcal{E}_{\partial(L)} + \mathcal{A}^n).
\]

Now, to see the reverse inclusions, suppose \( A = \gamma I - J(P + Q) \) for some \( \gamma \in \mathbb{R} \), \( P \in \mathcal{E}_{\partial(L)} \), and \( Q \) skew-symmetric. Let \( 0 \neq x, y \in L \) with \( \langle x, y \rangle = 0 \). By [10], \( x \) and \( y \) are in \( \partial(L) \), and \( Jy \) is a positive multiple of \( x \). Hence, \( \langle Px, Jy \rangle \geq 0 \) as \( P \in \mathcal{E}_{\partial(L)} \) and \( \langle Qx, Jy \rangle = 0 \) as \( Q \) is skew-symmetric. Thus,

\[
\langle Ax, y \rangle = \gamma \langle x, y \rangle - \langle JPx, y \rangle + \langle JQx, y \rangle = -\langle Px, Jy \rangle + \langle Qx, Jy \rangle \leq 0.
\]

This shows that \( A \in Z(L) \) and so, inclusions in (9) turn into equalities. Thus we have (7). \( \square \)

**Remarks.** From the above theorem, we have

\[
\mathbb{R}I - J(\mathcal{S}_+^n + \mathcal{A}^n) = \mathbb{R}I - J(\mathcal{E}_L + \mathcal{A}^n) = \mathbb{R}I - J(\mathcal{E}_{\partial(L)} + \mathcal{A}^n).
\]

Multiplying throughout by \( J \) and noting \(-\mathcal{A}^n = \mathcal{A}^n\), we get the equality of sets

\[
(\mathbb{R}J - \mathcal{S}_+^n) + \mathcal{A}^n = (\mathbb{R}J - \mathcal{E}_L) + \mathcal{A}^n = (\mathbb{R}J - \mathcal{E}_{\partial(L)}) + \mathcal{A}^n,
\]

where each set is a sum of \( \mathcal{A}^n \) and a subset of \( \mathcal{S}^n \). Since \( \mathbb{R}^{n \times n} = \mathcal{S}^n + \mathcal{A}^n \) is an (orthogonal) direct sum decomposition, we see that

\[
\mathbb{R}J - \mathcal{S}_+^n = \mathbb{R}J - \mathcal{E}_L = \mathbb{R}J - \mathcal{E}_{\partial(L)}.
\]
These equalities can also be established via different arguments. A result of Loewy and Schneider [10] asserts that a symmetric matrix $X$ is copositive on $L$ if and only if there exists $\mu \geq 0$ such that $X - \mu J \in S^n$. (This is essentially a consequence of the so-called S-Lemma [20]: If $A$ and $B$ are two symmetric matrices with $\langle Ax_0, x_0 \rangle > 0$ for some $x_0$ and $\langle Ax, x \rangle \geq 0 \Rightarrow \langle Bx, x \rangle \geq 0$, then there exists $\mu \geq 0$ such that $B - \mu A$ is positive semidefinite.) This result gives the equality 

$$E_L = S^n + \mathbb{R}J$$

and consequently $\mathbb{R}J - S^n = \mathbb{R}J - E_L$. The equality 

$$\mathcal{E}_{\partial(L)} = S^n + \mathbb{R}J$$

can be seen via an application of Finsler’ theorem [20] that says that if $A$ and $B$ are two symmetric matrices with $[x \neq 0, \langle Ax, x \rangle = 0] \Rightarrow \langle Bx, x \rangle > 0$, then there exists $\mu \in \mathbb{R}$ such that $B + \mu A$ is positive semidefinite. (For $M \in \mathcal{E}_{\partial(L)}$ and vectors $u, v \in L^n$, one has $\langle Jx, x \rangle = 0 \Rightarrow \langle M_kx, x \rangle > 0$, where $k$ is a natural number and $M_k := M + \frac{1}{k}uv^\top$. When $M_k + \mu_k J$ is positive semidefinite, it follows that the sequence $\mu_k$ is bounded.) From this equality, one gets $\mathbb{R}J - S^n = \mathbb{R}J - \mathcal{E}_{\partial(L)}$.

Our next result deals with the dual of $Z(L)$.

**Theorem 3.2** Let $C$ denote one of $\mathbb{R}^n$, $L$, or $\partial(L)$. Then, 

$$Z(L)^* = \{B \in \mathbb{R}^{n \times n} : \langle B, I \rangle = 0, -JB \in K_C \}.$$

In particular, (5) holds.

**Proof.** We fix $C$. From (7), we see that $B \in Z(L)^*$ if and only if 

$$0 \leq \langle B, \gamma I - J(P + Q) \rangle$$

for all $\gamma$ real, $P$ in $\mathcal{E}_C$, and $Q$ in $A^n$. Clearly, this holds if and only if 

$$\langle B, I \rangle = 0, \langle -JB, P \rangle \geq 0, \text{ and } \langle -JB, Q \rangle = 0$$

for all $\gamma$, $P$, and $Q$ specified above. Now, with the observation that a (real) matrix is orthogonal to all skew-symmetric matrices in $\mathbb{R}^{n \times n}$ if and only if it is symmetric, this further simplifies to 

$$\langle B, I \rangle = 0 \text{ and } -JB \in \mathcal{E}_C^*,$$

where $\mathcal{E}_C^*$ is the dual of $\mathcal{E}_C$ computed in $S^n$. Since $K_C = \mathcal{E}_C^*$ in $S^n$, we see that $B \in Z(L)^*$ if and only if $\langle B, I \rangle = 0$ and $-JB \in K_C$. This completes the proof.

We remark that (5) can be deduced directly from (10) by taking the duals in $S^n$.

In our final result, we describe the interior and boundary of $Z(L)^\circ$. First, we recall some definitions.
from [4]. Let

$$\Omega := \{ (x, y) \in L \times L : ||x|| = 1 = ||y|| \text{ and } \langle x, y \rangle = 0 \}.$$ 

It is easy to see that \(\Omega\) is compact and, from (6),

$$\Omega = \{ (x, Jx) : x \in \partial(L), ||x|| = 1 \}.$$  \hfill (11)

For any \(A \in \mathbb{R}^{n \times n}\), let

$$\gamma(A) := \max \{ \langle Ax, y \rangle : (x, y) \in \Omega \}.$$ 

Note that \(A \in \mathcal{Z}(L)\) if and only if \(\gamma(A) \leq 0\). We say that \(A \in \mathbb{R}^{n \times n}\) is a \textit{strict-\(\mathcal{Z}\)-transformation} on \(L\) if

$$[0 \neq x, y \in L, \langle x, y \rangle = 0] \Rightarrow \langle Ax, y \rangle < 0.$$ 

The set of all such transformations is denoted by \(\text{str}(\mathcal{Z}(L))\). For \(A \in \mathbb{R}^{n \times n}\), the following statements are shown in [4], Theorem 3.1:

$$\gamma(A) < 0 \iff A \in \mathcal{Z}(L)^{\circ} \iff A \in \text{str}(\mathcal{Z}(L))$$

and

$$\gamma(A) = 0 \iff A \in \partial(\mathcal{Z}(L)).$$

Recall that \(\mathcal{E}_L\) consists of all symmetric matrices that are copositive on \(L\). We say that a symmetric matrix \(P\) is \textit{strictly copositive} on \(L\) if \(0 \neq x \in L \Rightarrow \langle Px, x \rangle > 0\); the set of all such matrices is denoted by \(\text{str}(\mathcal{E}_L)\). Similarly, one defines \(\text{str}(\mathcal{E}_{\partial(L)})\).

\textbf{Corollary 3.3} The following statements hold:

$$\mathcal{Z}(L)^{\circ} = \text{str}(\mathcal{Z}(L)) = \mathbb{R} I - J \left( \text{str}(\mathcal{E}_{\partial(L)}) + A^n \right)$$

and

$$\partial(\mathcal{Z}(L)) = \mathbb{R} I - J \left( \partial(\mathcal{E}_{\partial(L)}) + A^n \right),$$

where \(\partial(\mathcal{E}_{\partial(L)})\) denotes the boundary of \(\mathcal{E}_{\partial(L)}\) in \(S^n\).

\textbf{Proof.} We first deal with the interior of \(\mathcal{Z}(L)\). The equality

$$\{ A \in \mathbb{R}^{n \times n} : \gamma(A) < 0 \} = \mathcal{Z}(L)^{\circ} = \text{str}(\mathcal{Z}(L))$$

has already been observed in [4], Theorem 3.1. To see the first assertion, we show that \(\gamma(A) < 0\) if and only if \(A = \theta I - J(P + Q)\) for some \(\theta \in \mathbb{R}\), \(P\) (symmetric) strictly copositive on \(\partial(L)\), and \(Q\) skew-symmetric. Suppose \(\gamma(A) < 0\). Then, for any \(\theta \in \mathbb{R}\),

$$\max \{ \langle (A - \theta I)x, y \rangle : (x, y) \in \Omega \} < 0,$$
which, from (11) becomes
\[ \min \left\{ \langle J(\theta I - A)x, x \rangle : x \in \partial(L), ||x|| = 1 \right\} > 0. \]

Now, fix \( \theta \) and let \( J(\theta I - A) = P + Q, \) where \( P \in S^n \) and \( Q \in A^n. \) As \( \langle Qx, x \rangle = 0 \) for any \( x, \) the above inequality implies that \( \min \left\{ \langle Px, x \rangle : x \in \partial(L), ||x|| = 1 \right\} > 0. \) This proves that \( P \) is strictly copositive on \( \partial(L). \) Rewriting \( J(\theta I - A) = P + Q, \) we see that \( A = \theta I - J(P + Q) \) which is of the required form.

To see the converse, suppose \( A = \theta I - J(P + Q), \) where \( \theta \in \mathbb{R}, \) \( P \) (symmetric) strictly copositive on \( \partial(L), \) and \( Q \) skew-symmetric. Using (11), we can easily verify that \( \gamma(A) < 0. \) Thus, \( A \in \text{str}(Z(L)). \)

An argument similar to the above will show that \( \gamma(A) = 0 \) if and only if \( A = \theta I - J(P + Q) \) for some \( \theta \in \mathbb{R}, \) \( P \in \partial_*(E_\partial(L)), \) and \( Q \) skew-symmetric. This gives the statement regarding the boundary of \( Z(L). \)

We end the paper with a remark dealing with conic linear programs. Motivated by the result of Burer (mentioned in the Introduction), we consider a conic linear program on a completely positive cone \( K_C \) (where \( C \) is a closed cone):
\[ \min \left\{ \langle c, x \rangle : Ax = b, x \in K_C \right\}. \]

While such a problem is generally hard to solve, we ask: (When) can we replace \( K_C \) by \( S^n \) and thus reduce the above problem to the semidefinite programming problem \( \min \left\{ \langle c, x \rangle : Ax = b, x \in S^n \right\}? \)

Just replacing \( K_C \) by \( S^n \) without handling the constraint \( Ax = b \) is not viable as \( K_C = S^n \) if and only if \( C \cup -C = \mathbb{R}^n \) (which fails to hold when \( n > 1 \) and \( C \) is pointed), see [18]. While we do not answer this broad question, we point out, as a consequence of (5) that for any \( C \in S^n, \)
\[ \min \left\{ \langle C, X \rangle : \langle X, J \rangle = 0, X \in K_C \right\} = \min \left\{ \langle C, X \rangle : \langle X, J \rangle = 0, X \in S^n_+ \right\}. \]

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