A Riccati type PDE for
light-front higher helicity vertices

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ABSTRACT: This paper is based on a curious observation about an equation related to the tracelessness constraints of higher spin gauge fields. The equation also occurs in the theory of continuous spin representations of the Poincaré group. Expressed in an oscillator basis for the higher spin fields, the equation becomes a non-linear partial differential operator of the Riccati type acting on the vertex functions. The consequences of the equation for the cubic vertex is investigated in the light-front formulation of higher spin theory. The classical vertex is completely fixed but there is room for off-shell quantum corrections.

KEYWORDS: Higher spin field theory, light-front field theory, Cubic interactions, Cubic counterterms.

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1 Introduction

The structure of light-front higher spin cubic vertices has been investigated by Metsaev in a series of papers (see [1] for references). These results are for arbitrary dimensions and mixed symmetry fields. The general cubic vertex in four dimensions was considered in [2]. That paper reviews and extends the original results of [3] and [4]. It was found that the restrictions on the allowed vertices from the dynamical part of the Poincaré algebra are very weak. There is a countable infinite sequence of vertex operators in powers of oscillators and transverse momenta. This fact has not been noted – as far as I know – in the literature before.

Furthermore, in the first paper [5] on the covariant BRST-approach to interacting higher spin gauge fields in Minkowski spacetime, it was conjectured that the cubic vertex, apart from being gauge (BRST) invariant, also satisfies a special partial differential equation. This was shown for the pieces of the vertex then known and to the order that could be checked. Although the evidence is weak, the idea itself is quite intriguing, in particular as it can be connected to physical arguments. These arguments were set out in [6] and will be repeated below. Since the light-front cubic vertex is known in its entirety, it is interesting to check if it satisfies a similar equation. This is the aim of the present paper.

It is found that the cubic vertex indeed satisfies a partial differential equation of the Riccati type. The coefficients of the vertex operator terms are related in such a way as to fix the classical vertex while leaving room for independent quantum corrections in the form of possible off-shell counterterms.

In section 2 the results from [2] that we need are summarized. Section 3 sets up and motivates the PDE. In section 4 the consequences of the equation are calculated. Section 5 contains conclusions and outlook. Conventions used are the same as in [2] and the basic ones are listed in section 6.

2 The general light-front cubic vertex in four dimensions

The framework we are working in is spelled out in full in [2]. Here we will only repeat the bare minimum needed for the present calculation. To begin with, fields of all helicities are collected in a Fock-space field

$$|\Phi(p)\rangle = \sum_{\lambda=0}^{\infty} \frac{1}{\sqrt{\lambda!}} \left( \phi_{\lambda}(p)(\bar{\alpha}^\dagger)^{\lambda} + \bar{\phi}_{\lambda}(p)(\alpha^\dagger)^{\lambda} \right) |0\rangle. \quad (2.1)$$

We are using a complex notation where $\phi_{\lambda}(p)$ is a field of helicity $+\lambda$ and its complex conjugate $\bar{\phi}_{\lambda}(p)$ has helicity $-\lambda$. Complex transverse momenta are denoted by $p$ and $\bar{p}$. The Fock field satisfies the constraint $\bar{\alpha}\alpha|\Phi(p)\rangle = 0$ which is a remnant of the covariant theory tracelessness constraint. The full spectrum of Fock component fields is detailed in [2].

The free theory Hamiltonian is

$$H_{(0)} = \frac{1}{2} \int \gamma d\gamma dpd\bar{p} \langle \Phi | h | \Phi \rangle. \quad (2.2)$$
where $\gamma$ denotes the $p^+$ component of momentum, and $h$ is $\frac{p\bar{p}}{\gamma}$. The cubic interaction can be written as

$$H_{(1)} = \frac{1}{3} \int \prod_{r=1}^{3} \gamma_r d\gamma_r dp_r d\bar{p}_r \langle \Phi_r | V_{123} \rangle.$$ (2.3)

where the product over $r$ is over field enumeration 1, 2, 3. The cubic vertex operator $|V_{123}\rangle$ as calculated in [4] can be written

$$|V_{123}\rangle = \frac{g}{\kappa} \exp(\Delta_{st} + \Delta_{hs}) |0_{123}\rangle \Gamma^{-1}(\sum_r \gamma_r) \delta(\sum_r p_r) \delta(\sum_r \bar{p}_r)$$

(2.4)

with

$$\Delta_{st} = \sum_{r,s} Y_{rs} \alpha^\dagger_r \alpha^\dagger_s + \kappa \sum_r Y^r (\alpha^\dagger_1 \bar{p} + \bar{\alpha}^\dagger_1 P),$$ (2.5)

$$\Delta_{hs} = \kappa \sum_{r,s,t} Y_{rst} (\alpha^\dagger_r \alpha^\dagger_1 \alpha^\dagger_s \bar{p} + \bar{\alpha}^\dagger_r \bar{\alpha}^\dagger_1 \bar{\alpha}^\dagger_s P).$$ (2.6)

We use $|\mathcal{O}_{123}\rangle$ as a shorthand for the Fock space vacua and the momentum conservation delta-functions and normalizing factors. The transverse momentum combination $\mathbb{P}$ is defined by

$$\mathbb{P} = - \frac{1}{3} \sum_{r=1}^{3} \bar{\gamma}_r p_r \quad \text{with} \quad \bar{\gamma}_r = \gamma_{r+1} - \gamma_{r+2}$$ (2.7)

and correspondingly for $\bar{\mathbb{P}}$.

In the formula (2.4), $g$ is a dimensionless coupling constant and $\kappa$ is of dimension $-1$. For the cubic interactions $g$ becomes the spin-1 coupling and $g\kappa$ becomes the spin-2 coupling. The higher spin coupling constants $g_\lambda$ come out as $g_\lambda \lambda^{-1}$. $\Gamma$ is $\gamma_1 \gamma_2 \gamma_3$ and it compensates the measure factor in (2.3).

The $Y^r$, $Y^{rs}$ and $Y^{rst}$ in formulas (2.5) and (2.6) are rational functions of $\gamma$ that are determined by the Poincaré algebra. In (2.5) we have

$$Y^{rs} = \delta_{rs}, \quad Y^r = \frac{1}{\gamma_r},$$ (2.8)

and in (2.6)

$$Y^{rst} = \frac{\gamma^r}{\gamma_r \gamma_s}.$$ (2.9)

The form of the string-like functions (2.8) are such that they cannot produce any self-interactions among the fields. For that the operator in (2.6) with $Y$-functions of the form (2.9) (as discovered in reference [4]) is needed.

The general vertices as derived in [2] can be systematically listed using a shorthand notation for products of oscillators

$$\alpha^\dagger_{r_{k}} = \alpha^\dagger_{r_{1}} \ldots \alpha^\dagger_{r_{k}}, \quad \text{and} \quad \bar{\alpha}^\dagger_{s_{l}} = \bar{\alpha}^\dagger_{s_{1}} \ldots \bar{\alpha}^\dagger_{s_{l}},$$ (2.10)

as

$$\Delta = \kappa^{(n+m)} Y^{r_{1} \ldots r_{k}} Y^{s_{1} \ldots s_{l}} (\alpha^\dagger_{r_{k}} \bar{\alpha}^\dagger_{s_{l}} \mathbb{P}^{m_{n}} \bar{\mathbb{P}}^{m_{n}} + \bar{\alpha}^\dagger_{r_{k}} \alpha^\dagger_{s_{l}} \mathbb{P}^{m_{n}} \bar{\mathbb{P}}^{m_{n}}).$$ (2.11)
The functions are enumerated with \( k = 1, 2, 3, \ldots \) distinguishing two cases: \( l < k \) and \( l = k \). For the first case \( l < k \) we get

\[
Y^{r_1 \ldots r_k s_1 \ldots s_l} = (\gamma_{s_1} \cdots \gamma_{s_l} / \gamma_{r_1} \cdots \gamma_{r_k}) (n + m) / \gamma_{r_1} \cdots \gamma_{r_k}, \tag{2.12}
\]

with \( n - m = k - l \). Taking \( m = 0 \) yields terms with either \( \mathbb{P} \) or \( \bar{\mathbb{P}} \). With non-zero \( m \) we get terms with products of \( \mathbb{P} \) or \( \bar{\mathbb{P}} \). Such terms correspond to on-shell field redefinition terms.

Hence, the case \( m = 0 \) is the most interesting.

For the second case \( l = k \) we get

\[
Y^{r_1 \ldots r_k s_1 \ldots s_k} = \delta_{r_1 s_1} \delta_{r_2 s_2} \ldots \delta_{r_k s_k}. \tag{2.13}
\]

Excluding field redefinition terms we thus have the vertex functions

\[
Y^{r_1 \ldots r_k s_1 \ldots s_l} = \frac{\gamma_{s_1} \cdots \gamma_{s_l}}{\gamma_{r_1} \cdots \gamma_{r_k}} \quad \text{for} \quad l < k, \tag{2.14}
\]

\[
Y^{r_1 \ldots r_k s_1 \ldots s_k} = \delta_{r_1 s_1} \delta_{r_2 s_2} \ldots \delta_{r_k s_k} \quad \text{for} \quad l = k, \tag{2.15}
\]

listed by enumerating \( k = 0, 1, 2, \ldots \) and \( 0 \leq l \leq k \). These are the functions we will work with in this paper. To be explicit, the cubic vertex operator we are considering is

\[
\Delta = \sum_{k=1}^{\infty} \kappa^{(k-l)} y_{kl} Y^{r_1 \ldots r_k s_1 \ldots s_l} (\alpha^\dagger_{r_k} \bar{\alpha}^\dagger_{s_1} \bar{\alpha}^\dagger_{s_k} + \bar{\alpha}^\dagger_{r_k} \alpha^\dagger_{s_1} \alpha^\dagger_{s_k}) + \sum_{k=1}^{\infty} \kappa^{(k-l)} y_{kl} \frac{\gamma_{s_1} \cdots \gamma_{s_l}}{\gamma_{r_1} \cdots \gamma_{r_k}} (\alpha^\dagger_{r_k} \bar{\alpha}^\dagger_{s_1} \bar{\alpha}^\dagger_{s_k} + \bar{\alpha}^\dagger_{r_k} \alpha^\dagger_{s_1} \alpha^\dagger_{s_k}) \tag{2.16}
\]

Since the relative coefficients for the terms are not fixed by the Poincaré algebra we have inserted numerical coefficients \( y_{kl} \).

### 3 The PDE

The explicit form of the differential equation is as follows

\[
\sum_{r=1}^{3} \left( \alpha_r \alpha_r - \alpha^\dagger_r \alpha_r - \alpha^\dagger_r \alpha_r \right) \Delta + (\bar{\alpha}_r \Delta)(\alpha_r \Delta) = - \sum_{r=1}^{3} \alpha^\dagger_r \alpha^\dagger_r + \rho^2 \sum_{r=1}^{3} \frac{\bar{\mathbb{P}} \mathbb{P}}{\gamma^2_r} + 3\eta. \tag{3.1}
\]

where the sum is over the three transverse Fock-spaces connected by the cubic vertex, suggesting a generalization to higher order vertices. All the terms (in particular the oscillators) in the equation are acting on an implicit vacuum \( |\Omega_{123}\rangle \). Momentum conservation is therefore also encoded. For practical calculation we can think of the annihilators \( \alpha \) and \( \bar{\alpha} \) as derivatives with respect to the variables \( \alpha^\dagger \) and \( \bar{\alpha}^\dagger \)

\[
\alpha = \frac{\partial}{\partial \alpha^\dagger} \quad \text{and} \quad \bar{\alpha} = \frac{\partial}{\partial \bar{\alpha}^\dagger}. \tag{3.2}
\]

The equation can be seen as a multidimensional Riccati-type differential equation (see section 3.3). The parameters \( \rho^2 \) and \( \eta \) will be explained below.
3.1 Rationale

The rationale for the equation was explained in [6] based on observations made in [5]. The argument can be briefly stated as follows. Consider a two-particle mechanical system with coordinates \( t^\mu \) and \( b^\mu \) and corresponding canonical momenta \( u_{\mu} \), \( d_{\mu} \). Call these variables "end-point" variables. The center of motion \( x^\mu \) and relative coordinates \( \xi^\mu \) are defined by

\[
x^\mu = \frac{1}{2}(t^\mu + b^\mu), \quad \xi^\mu = \frac{1}{2}(t^\mu - b^\mu).
\]

(3.3)

The corresponding canonical conjugate momenta \( p_{\mu} \), \( \pi_{\mu} \) are

\[
p^\mu = u^\mu + d^\mu, \quad \pi^\mu = u^\mu - d^\mu.
\]

(3.4)

In terms of these variables, the mechanical first class constraints of higher spin gauge field theory can be expressed as \( p^2 \approx 0 \), \( \xi \cdot p \approx 0 \) and \( \pi \cdot p \approx 0 \). Using equations (3.3) and (3.4) the constraints become

\[
(t - b) \cdot (u - d) \approx 0, \quad u^2 + u \cdot d \approx 0, \quad d^2 + u \cdot d \approx 0.
\]

(3.5)

Then requiring the endpoints to move with the velocity of light forces the further constraint \( u \cdot d \approx 0 \). This equation is naturally interpreted as one of the Wigner equations defining the continuous spin representations of the Poincaré group [7]. It can also be related to the tracelessness constraints of higher spin theory. The reader is referred to [6] for fuller a discussion of these questions. Using equations (3.4) we have \( u \cdot d = (p^2 - \pi^2)/4 \) so that the constraint can also be written \( p^2 - \pi^2 \approx 0 \).

The constraint is then applied to the cubic vertex

\[
\sum_{r=1}^{3} u_r \cdot d_r |V_{123}\rangle = 0.
\]

(3.6)

In order to arrive at the PDE of equation (3.1) we must fix the light-front gauge. We will first do it for free fields and then discuss modifications in the interacting theory.

3.2 light-front gauge fixing

The simplest way is to start with the form \( p^2 - \pi^2 \approx 0 \). Since in the light-front gauge we are on the free field mass shell we have \( p^2 \approx 0 \). For \( \pi^2 \) we have \( \pi^2 = 2(\pi^+ - \pi^+ \pi^-) \). The light-front gauge for is reached by putting \( \pi^+ = 0 \) so what remains is simply \( \pi^\pi \approx 0 \).

The result can be checked by working from the \( u \cdot d \approx 0 \) form of the constraint. In light-front coordinates we have \( u \cdot d = \bar{u}d + u\bar{d} - u^-d^+ - u^+d^- \). The gauge is reached by putting \( \alpha^+ = \alpha^{++} = 0 \). This gives for the + components of \( u \) and \( d \)

\[
u^+ = (p^+ + \pi^+)/2 = p^+/2 - i(\alpha^+ - \alpha^{++})/2\sqrt{2} = p^+/2,
\]

\[
d^+ = (p^+ + \pi^+)/2 = p^+/2 - i(\alpha^+ - \alpha^{++})/2\sqrt{2} = p^+/2.
\]

(3.7)

(3.8)
For the $-$ components we have
\[ u^- = (p^- - \pi^-)/2 = p^-/2 - i(\alpha^- - \alpha^{\dagger-})/2\sqrt{2}, \] (3.9)
\[ d^- = (p^- - \pi^-)/2 = p^-/2 - i(\alpha^- - \alpha^{\dagger-})/2\sqrt{2}, \] (3.10)
\[ (3.11) \]

The $-$ components of $\alpha$, $\alpha^{\dagger}$ and $p$ are solved for from the higher spin constraints $\alpha \cdot p \approx 0$, $\alpha^{\dagger} \cdot p \approx 0$ and $p^2 \approx 0$ with the result
\[ \alpha^- = \frac{\bar{\alpha}p + \alpha \bar{p}}{p^+}, \] (3.12)
\[ \alpha^{\dagger-} = \frac{\bar{\alpha}^{\dagger} p + \alpha^{\dagger} \bar{p}}{p^+}, \] (3.13)
\[ p^- = \frac{p \bar{p}}{p^+}. \] (3.14)

We can now calculate $u^-d^+ - u^+d^- = p\bar{p}/2$. On the other hand a short calculation shows that $\bar{u}d + u\bar{d} = (p\bar{p} - \pi \bar{\pi})/2$. In this way we reproduce the result $u \cdot d = -\pi \bar{\pi}/2$.

In any way, when the light-front gauge fixed operator $-\pi^2 = -2\pi \bar{\pi}$ is expressed in terms of transverse oscillators it becomes
\[ -\pi^2 = -2\pi \bar{\pi} = -2(i/\sqrt{2})^2(\alpha^{\dagger} - \alpha)(\bar{\alpha}^{\dagger} - \bar{\alpha}) \]
\[ = \alpha \bar{\alpha} + \alpha^{\dagger} \bar{\alpha}^{\dagger} - \alpha^{\dagger} \bar{\alpha} - \alpha \bar{\alpha}^{\dagger}. \] (3.15)

There is a normal ordering issue that we have to resolve. We can choose to normal order either before or after fixing the light-front gauge. Suppose we want to normal order $\alpha \cdot \alpha^{\dagger}$ in $D$ space-time dimensions. Normal ordering (N.O) first and then light-front gauge fixing (L.F) would yield $\alpha^{\dagger}_1 \alpha_1 + D$. On the other hand, first gauge fixing and then normal ordering would yield $\alpha^{\dagger}_1 \alpha_1 + D - 2$. In four dimensions we parametrize this choice by writing
\[ -\pi^2 = \alpha \bar{\alpha} + \alpha^{\dagger} \bar{\alpha}^{\dagger} - \alpha^{\dagger} \bar{\alpha} - \alpha \bar{\alpha}^{\dagger} - \eta, \] (3.16)

with $\eta = D/2$ in the first case (first N.O then L.F) and $\eta = (D - 2)/2$ in the second (first L.F then N.O). The choice will have consequences.

### 3.3 Why Riccati?

We can now see why the equation can be designated as being of Riccati type. A one-dimensional analogue to the $\pi^2$ operator would be
\[ \left( \frac{d}{dx} - x \right) \left( \frac{d}{dx} - x \right). \]

If we let this operator act on a function $e^f$ and equate to zero, we get
\[ f'' - 2xf' + (f')^2 = 1 - x^2. \]

Then substituting $y = f'$ we get a Riccati-type equation
\[ y' - 2xy + y^2 = 1 - x^2. \]

Equation (3.1) can be seen as a multidimensional generalization of this simple differential equation.
3.4 Interacting theory

Generically we write the vertex as $e^{\Delta} |\varnothing\rangle$ as in (2.4). Now, in determining the equations that govern the form of the $\Delta$-operators we act with light-front Poincaré generators $g$ on the vertex. Since they are linear operators we then get sums of terms of the form $(g\Delta)e^{\Delta} |\varnothing\rangle$ equal to zero. Thus, for the cubic vertex it does not really matter if we work with a vertex of the form $e^{\Delta} |\varnothing\rangle$ or simply $\Delta |\varnothing\rangle$. The equations for $\Delta$ become exactly the same.

This cubic ambiguity is reflected in an observation made in [2] that the restrictions on $\Delta$ from the Poincaré algebra are very weak, allowing an countable infinite set of vertex operators as listed in (2.11).

The KLT-relations [8], in the field theory limit, relates gravity amplitudes to Yang-Mills amplitudes. In a certain sense gravity tree amplitudes can be considered as squares of Yang-Mills tree amplitudes. In the light-front approach this is explicit for cubic vertices and it can be generalized to arbitrary integer spin. The momentum structure of a pure helicity $\lambda$ cubic vertex being simply

$$\left(\frac{\gamma_1}{\gamma_2\gamma_3}\right)^{\lambda}. \tag{3.17}$$

This was observed already in [4] (see formula (22) of that paper). This structure has also been observed by Ananth in [9] using MHV-notation. Indeed, the $\mathbb{P}$ and $\bar{\mathbb{P}}$ and are essentially the same thing as the spinor products $\langle k l \rangle$ and $[k l]$ respectively (see also comments in [2]).

Anyway, working with a vertex of the form $e^{\Delta} |\varnothing\rangle$, all pure higher helicity cubic interactions can be generated from the operators $\Delta_{hs}$ of formula (2.6) by expanding out the powers in $e^{\Delta}$. Working with a vertex of the form $\Delta |\varnothing\rangle$ the higher helicity interactions come instead from the operators

$$\Delta = \kappa^{(k-l)} Y^{r_1\ldots r_k s_1\ldots s_l} \left( \bar{\alpha}^\dagger_{r_1} \bar{\alpha}^\dagger_{r_2} \alpha^\dagger_{s_1} \alpha^\dagger_{s_2} \bar{\mathbb{P}}^{(k-l)} + \bar{\alpha}^\dagger_{s_1} \alpha^\dagger_{s_2} \bar{\mathbb{P}}^{(k-l)} \right), \tag{3.18}$$

with

$$Y^{r_1\ldots r_k s_1\ldots s_l} = \frac{\gamma_{s_1} \ldots \gamma_{s_l}}{\gamma_{r_1} \ldots \gamma_{r_k}}. \tag{3.19}$$

Here, the helicity is $\lambda = k - l$. Since for a cubic pure helicity $\lambda$ vertex we must also have the total number of oscillators $k + l = 3\lambda$ we get $k = 2\lambda$ and $l = \lambda$. For instance for spin 2 we get the operator

$$\frac{\gamma_1 \gamma_2}{\gamma_1 \gamma_2 \gamma_3 \gamma_4} \left( \bar{\alpha}^\dagger_{r_1} \bar{\alpha}^\dagger_{r_2} \alpha^\dagger_{r_3} \alpha^\dagger_{r_4} \bar{\alpha}^\dagger_{s_1} \bar{\alpha}^\dagger_{s_2} \bar{\mathbb{P}}^{2} + \bar{\alpha}^\dagger_{s_1} \alpha^\dagger_{s_2} \bar{\mathbb{P}}^{2} \right). \tag{3.20}$$

This generalizes to higher helicity in an obvious way.

An argument for choosing a vertex of the form $e^{\Delta} |\varnothing\rangle$ rather than $\Delta |\varnothing\rangle$ can therefore hardly be construed from the cubic theory alone. However, once quartic vertices are considered it is probably essential to work with $e^{\Delta} |\varnothing\rangle$ since the Poincaré algebra then will contain combinations of the form

$$e^{\Delta_{1234}} |\varnothing_{1234}\rangle e^{\Delta_{134}} |\varnothing_{134}\rangle,$$
with a contraction over one of the Fock spaces (indexed by $j$ in the qualitative formula above). Then the properties of the exponential function are likely to be crucial. As many authors have commented, the fate of the light-front approach to higher spin is likely to be settled by a calculation of the full quartic vertex – if it exists.

Now applying the light-front gauge fixed operator $-\pi^2$ from equation (3.16) to the $e^\Delta|\emptyset\rangle$ vertex yields

$$\left(\sum_{r=1}^{3}(\bar{\alpha}_r\alpha_r - \bar{\alpha}^\dagger_r\alpha_r - \alpha^\dagger_r\bar{\alpha}_r)\Delta + (\bar{\alpha}_r\Delta)(\alpha_r\Delta) + \bar{\alpha}^\dagger_r\alpha^\dagger_r - \eta\right)e^\Delta|\emptyset\rangle_{123}. \quad (3.21)$$

Then there is one more issue to deal with. We started with the operator $p^2 - \pi^2$ but concluded that $p^2 = 0$ on the light-front. But that is for free fields. In the cubic interacting theory we should expect to have instead a term with a factor $\bar{P}\bar{P}$. The simplest such term that is dimensionally correct (without any coupling constants) and symmetric in the field labels is

$$\sum_{r=1}^{3}\frac{\bar{P}\bar{P}}{\gamma^2_r}$$

Adding this term with a parameter $\rho^2$ to (3.21) and equating to zero and dropping $e^\Delta|\emptyset\rangle$ finally yields the PDE recorded above in formula (3.1).

### 3.5 Classical interaction terms and counterterms

It must be realized that the vertex operators in (2.16) generate not just the pure helicity $\lambda-\lambda-\lambda$ classical interaction terms but also a large set of interactions between fields of different helicities. It also generates higher-derivative counterterms. Let us first consider pure helicity $\lambda$ interactions.

Suppose we want cubic interaction terms for helicity $\lambda$. The terms in the action are of the form $\phi_1\phi_2\phi_3, \bar{\phi}_1\phi_2\phi_3, \bar{\phi}_1\bar{\phi}_2\phi_3$ or $\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3$ where the last two can be gotten from the first two using complex conjugation and cyclic symmetry in field labels. To extract an interaction of the form $\bar{\phi}_1\bar{\phi}_2\phi_3$, we chose a bra state excited by $\bar{\phi}_1\bar{\phi}_2\phi_3\alpha^\lambda_1\bar{\alpha}^\lambda_2\bar{\alpha}^\lambda_3$ and insert it into (2.3) using (2.16). The matrix element to compute is

$$\bar{\phi}_1\bar{\phi}_2\phi_3\langle0|\alpha^\lambda_1\bar{\alpha}^\lambda_2\bar{\alpha}^\lambda_3e^\Delta|\emptyset\rangle_{123}. \quad (3.22)$$

The annihilators will saturate any combination of creators $(\bar{\alpha}^\dagger_1)^\lambda(\alpha^\dagger_2)^\lambda(\alpha^\dagger_3)^\lambda$ that appear in the expansion of $\exp(\Delta)$. This will give the familiar basic higher spin interactions with minimal number of transverse derivatives equal to the helicity $\lambda$. But it will also pick out higher derivative terms. These can be interpreted as counterterms. For instance, in the case of helicity 2 we will get various terms with transverse momentum factors of $\bar{P}\bar{P}$ and $\bar{P}\bar{P}$ corresponding to one loop and two loops respectively.

On the other hand, to extract an interaction of the form $\phi_1\phi_2\phi_3$, we chose a bra state excited by $\phi_1\phi_2\phi_3\bar{\alpha}^\lambda_1\alpha^\lambda_2\bar{\alpha}^\lambda_3$ and insert it into (2.3) using (2.16). The matrix element to compute is

$$\phi_1\phi_2\phi_3\langle0|\bar{\alpha}^\lambda_1\alpha^\lambda_2\bar{\alpha}^\lambda_3e^\Delta|\emptyset\rangle_{123}. \quad (3.23)$$
For helicity 2 this will produce an off-shell two loop counterterm with transverse derivative structure $\mathbb{P}^0$ \cite{10}. I will comment further on the possible counterterms in sections 4.3 and 4.4.

**General analysis of cubic interactions** The overall structure of the possible interaction terms can be analyzed in the following way. First again consider an interaction of the form $\tilde{\phi}_1 \phi_2 \phi_3$, but now with fields of three, possibly different, helicities. Chose a bra state excited by $\tilde{\phi}_\lambda \phi_{\lambda_2} \phi_{\lambda_3} \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \alpha_3^{\lambda_3}$. This saturates an operator combination $\mathbb{P}^m \mathbb{P}^n (\tilde{\alpha}_1^\dagger)^{\lambda_1} (\alpha_2^\dagger)^{\lambda_2} (\alpha_3^\dagger)^{\lambda_3}$ from the exponential $\exp \Delta$. We then get the condition $n - m = \lambda_2 + \lambda_3 - \lambda_1$. Taking $m = 0$ we get interactions with pure powers of $\mathbb{P}^n$ where $n = \lambda_2 + \lambda_3 - 1$. That is with $\lambda_2 + \lambda_3 \geq 1$. This corresponds to the terms classified as (ii) in \cite{4}. On the other hand, taking $n = 0$ we get interactions with pure powers of $\mathbb{P}^m$ where $m = \lambda_1 - \lambda_2 - \lambda_3 \geq 0$, that is with $\lambda_1 \geq \lambda_2 + \lambda_3$. This corresponds to the terms classified as (iii) in \cite{4} (see formula A1.5). The case (i) is included with $\lambda_1 = \lambda_2 + \lambda_3$.

On-shell pure helicity $\lambda$ counterterms are produced if we take $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. Then $n - m = \lambda$. The maximum value of $m$ is $\lambda$ (there are just $\lambda$ oscillators $\alpha$ in the state), we get a sequence of interaction terms with transverse momentum structure $\mathbb{P}^\lambda, \mathbb{P}^{\lambda+1} \mathbb{P}, \mathbb{P}^{\lambda+2} \mathbb{P}^2, \ldots, \mathbb{P}^{2\lambda} \mathbb{P}^\lambda$. We recognize the helicity 2 sequence $\mathbb{P}^2, \mathbb{P}^3 \mathbb{P}, \mathbb{P}^4 \mathbb{P}^2$ corresponding to zero, one and two loops.

Next consider interactions excited by $\phi_{\lambda_1} \phi_{\lambda_2} \phi_{\lambda_3} \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \alpha_3^{\lambda_3}$. In this case we simply get interactions with transverse momentum structure $\mathbb{P}^n$ where $m = \lambda_1 + \lambda_2 + \lambda_3$. This corresponds to the terms classified as (iv) in \cite{4} (see formula A1.6).

The interactions with field combinations $\tilde{\phi}_1 \tilde{\phi}_2 \phi_3$ and $\tilde{\phi}_1 \phi_2 \tilde{\phi}_3$ should of course be added for hermiticity, but they are gotten by complex conjugation and cyclic symmetry in field labels.

**4 Consequences of the PDE**

Returning now to the vertex operators of (2.16) everything is fixed by the Poincaré algebra except the relative numerical coefficients of these operators. We will use the notation $y_{kl}$ (see formula (2.16)) for the as yet undetermined coefficient of the $Y_{r_1 \ldots r_k s_1 \ldots s_l}$ term in $\Delta$. For instance the coefficient for the term $Y_{rst}(\alpha_1^s, \alpha_2^s, \alpha_3^s)$ is $y_{21}$.

**4.1 Fixing the coefficients**

The procedure is straightforward to check the consequences of the PDE. We will first record the lowest order in oscillators and momenta as parameterized by $k$ and $l$.

$k = 0, l = 0$: Here we get two equations

\begin{align}
3y_{11} &= 3\eta \\
y_{10}^2 &= \rho^2
\end{align}

(4.1) (4.2)
$k = 1, l = 0$: Here we get the equation

$$y_{10}(y_{11} - 1) + 6y_{21} = 0 \quad (4.3)$$

We see at once that we can have neither $\eta = 1$ nor $\rho = 0$ because then $y_{21} = 0$. Then we wouldn’t have any cubic interactions since this is the coefficient in front of the crucial cubic vertex operator of (2.6). As shown above, $\eta = 1$ corresponds to normal ordering after light-front gauge fixing. Instead $\eta$ could be treated as a parameter $\neq 1$ but I will conveniently choose its value to be 2 corresponding to normal ordering before light-front gauge fixing.

$k = 1, l = 1$: Here we get the equation

$$y_{11}(y_{11} - 2) + 12y_{22} = -1 \quad (4.4)$$

where the $-1$ comes from the term $-\sum_{r=1}^{3} \bar{\alpha}_r \alpha_r$.

The solution to equations (4.1), (4.2), (4.3) and (4.4) is

$$y_{10} = \rho, \quad y_{11} = 2, \quad y_{21} = -\frac{\rho}{6}, \quad y_{22} = -\frac{1}{12}. \quad (4.5)$$

On the next $k$ level up we get three equations

$$k = 2, l = 0 : \quad 2y_{20}(y_{11} - 1) + 9y_{31} + y_{21}y_{10} = 0, \quad (4.6)$$
$$k = 2, l = 1 : \quad 3y_{21}(y_{11} - 1) + 18y_{32} + 2y_{22}y_{10} = 0, \quad (4.7)$$
$$k = 2, l = 2 : \quad 4y_{22}(y_{11} - 1) + 27y_{33} = 0. \quad (4.8)$$

Here we have four new coefficients: $y_{20}$ and $y_{31}, y_{32}, y_{33}$. The equations can be solved in terms of $y_{20}$ with the result

$$y_{31} = \frac{1}{54} \left( \rho^2 - 12y_{20} \right), \quad y_{32} = \frac{\rho}{27}, \quad y_{33} = \frac{1}{81}. \quad (4.9)$$

This pattern continues as can be seen on the next $k$ level up, where we get four equations

$$k = 3, l = 0 : \quad 3y_{30}(y_{11} - 1) + 12y_{41} + y_{31}y_{10} + 2y_{21}y_{20} = 0, \quad (4.10)$$
$$k = 3, l = 1 : \quad 4y_{31}(y_{11} - 1) + 24y_{42} + 2y_{32}y_{10} + 4y_{22}y_{20} + 2y_{21}^2 = 0, \quad (4.11)$$
$$k = 3, l = 2 : \quad 5y_{32}(y_{11} - 1) + 36y_{43} + 3y_{33}y_{10} + 6y_{21}y_{22} = 0, \quad (4.12)$$
$$k = 3, l = 3 : \quad 6y_{33}(y_{11} - 1) + 48y_{44} + 4y_{22}y_{22} = 0. \quad (4.13)$$

There are now five new coefficients: $y_{30}$ and $y_{41}, y_{42}, y_{43}, y_{44}$ that can be solved for in terms of $y_{30}$ and $y_{20}$. This pattern continues. On level $k$ we get $k + 1$ equations for the $k + 2$ coefficients $y_{k0}$ and $y_{k+1,1}, y_{k+1,2}, \ldots, y_{k+1,k+1}$ where the $y_{k+1,j}$ coefficients can be solved for in terms of $y_{k,0}, y_{k-1,0}, \ldots, y_{20}$.

For the record, here is the solution to the level $k = 3$ equations.

$$y_{41} = \frac{1}{648} \left( -\rho^3 + 30\rho y_{20} - 162y_{30} \right), \quad y_{42} = -\frac{11}{1296} \left( \rho^2 - 6y_{20} \right),$$
$$y_{43} = -\frac{11\rho}{1296}, \quad y_{44} = -\frac{11}{5184}. \quad (4.9)$$
4.2 General structure of the equations

We can outline the general structure of the equations. Consider level \( k \). There are \( k + 1 \) equations parameterized by \( l = 0, 1, \ldots, k \). Each equation has two linear terms. One is \( 3(k + 1)(l + 1)y_{k+1,l+1} \) coming from the sum over \( r \) of \( \bar{\alpha}_r \alpha_r \Delta \) part of the PDE. The coefficients in these terms are determined at this level. The other linear terms are \(- (k + l)y_{kl}\) coming from the sum over \( r \) of \(- \bar{\alpha}_r^\dagger \alpha_r - \bar{\alpha}_r \alpha_r^\dagger \) part of the PDE. The coefficients in these latter terms are already known from the level \( k - 1 \) equations. The rest of the terms are bilinear in already determined coefficients, coming from the sum over \( r \) of \( (\bar{\alpha}_r \Delta)(\alpha_r \Delta) \) part of the PDE. The equations take the general form

\[
y_{k+1,l+1} = -\frac{1}{3(k + 1)(l + 1)}((k + l)y_{kl}(y_{11} - 1) + \text{bilinears})
\]  

for \( k \geq 2 \) and \( 0 \leq l \leq k \).

The first equation of this set, the one for \( y_{k+1,1} \), contains the new undetermined parameter \( y_{k0} \). The rest of the coefficients \( y_{k+1,l+1} \) with \( 1 \leq l \leq k \) are solved for in terms of the \( y_{kl} \).

The final result is a set of recursive equations that can be solved in terms of the parameters \( y_{10}, y_{20}, y_{30}, \ldots \). Putting all the \( y_{k0} \) with \( k > 1 \) to zero (which could be seen as a choice of boundary-values for the PDE), all coefficients can be expressed in terms of the parameter \( \rho = y_{10} \). The equations for the coefficients \( y_{k+1,1} \) then simplify to \( 3(k+1)y_{k+1,1} = \rho y_{k,1} \) (for \( k > 1 \)) with solution

\[
y_{k+1,1} = \left(-\frac{\rho}{3}\right)^k \frac{1}{(k + 1)!}.
\]

I haven’t been able to derive closed formulas for the rest of the coefficients. However, a numerical study shows that they too drop off rapidly with increasing \( k \). Furthermore \( y_{kl} \sim \rho^{k-l} \).

4.3 Off-shell counterterms

The theory considered here is a multi-helicity theory. Therefore even with just cubic vertices, loop diagrams will have vertices connecting fields with different helicities and internal lines will propagate various helicities.

Vertex operator terms with coefficients \( y_{k0} \) with \( k \leq 3\lambda \) will produce pure helicity \( \lambda \) counterterms with the transverse derivative structure \( \mathbb{P}^{3\lambda} \) and \( \bar{\mathbb{P}}^{3\lambda} \). It is tempting to put all the \( y_{k0} \) coefficients to zero. Then all the other coefficients in the vertex are fixed in terms of \( y_{10} = \rho \). Even if we do this, we will still generate this kind of off-shell counterterm through the vertex operators with coefficient \( y_{10} \) which is non-zero. The coefficients of these terms, then proportional to \( (y_{10})^{3\lambda} \), are however fixed. Switching on the \( y_{k0} \) terms again give us off-shell counterterms with adjustable coefficients. So it seems that the PDE fixes the cubic vertex completely at the classical level while still allowing for quantum corrections to absorbed.
4.4 A note on gravity cubic counterterms

In [10] we studied cubic counterterms for light-front gravity. Apart from the two-loop off-shell term with transverse derivative structure $\bar{p}^6$ we found an infinite set of on-shell one-loop terms with transverse derivative structure $\bar{p}^3p$ differing in their $p^+$ (that is $\gamma$) structure. This is somewhat strange because in a pure helicity 2 theory we would expect just one such term corresponding a $\bar{2} - 2 - 2$ vertex (where helicity $-2$ is denoted by $\bar{2}$). We explained this as a result of higher helicity cubic interactions leaking into the helicity 2 calculation. This is indeed so because in a formulation that from the outset contains fields of all helicities (as the present one does) there will be vertices connecting fields of helicities $\bar{4} - 3 - 3, \bar{6} - 4 - 4, \bar{8} - 5 - 5$, et cetera, all with transverse structure $\bar{p}^3p$ but with differing $p^+$ structure.

4.5 A note on a $\Delta|0\rangle$ vertex as opposed to a $\exp \Delta|\emptyset\rangle$

As argued above in section 3.4 a cubic vertex of the type $\exp \Delta|\emptyset\rangle$ is probably implied by higher orders in the interaction. Still it is interesting to see what would be the consequences or requiring the PDE to hold for a vertex of the type $\Delta|\emptyset\rangle$. In that case all the bilinear terms disappear and the equations simplify to

\begin{align}
  y_{11} &= \eta \\
  y_{10}^2 &= \rho^2 \\
  6y_{21} &= y_{10} \\
  12y_{22} &= 2y_{11} - 1 \\
  y_{k+1,l+1} &= \frac{k + l}{3(k + 1)(l + 1)}y_{kl} \quad \text{for } k \geq 2 \text{ and } 0 \leq l \leq k
\end{align}

Here too, each level $k \geq 2$ introduces one new undetermined coefficient $y_{k0}$. Putting all these to zero leads to two non-zero sequences of coefficients

\begin{align}
  y_{10} &\rightarrow y_{21} \rightarrow y_{32} \rightarrow y_{43} \rightarrow \ldots \\
  y_{11} &\rightarrow y_{22} \rightarrow y_{33} \rightarrow y_{44} \rightarrow \ldots
\end{align}

Such a vertex however does not suffice to produce higher helicity than 1 self-interactions unless the $y_{k0}$-terms are kept. If the $y_{k0}$ with $k > 1$ are kept, they will start new sequences of coefficients

\begin{align}
  y_{20} &\rightarrow y_{31} \rightarrow y_{42} \rightarrow y_{53} \rightarrow \ldots \\
  y_{30} &\rightarrow y_{41} \rightarrow y_{52} \rightarrow y_{63} \rightarrow \ldots \\
  y_{40} &\rightarrow y_{51} \rightarrow y_{62} \rightarrow y_{73} \rightarrow \ldots
\end{align}

in this way filling out the full spectrum of coefficients.
5 Conclusion and outlook

The analysis performed here underlines the urgent need to proceed to an attempt to calculate contributions to the quartic higher spin light-front vertex. It could very well be that the existence of a quartic vertex puts strong restrictions on the cubic vertices. It is interesting to see if such restrictions are consistent with the restrictions on the cubic vertex imposed by the PDE considered in the present paper, and if the PDE holds for a quartic vertex as well.

6 Conventions

Coordinates and momenta are given by

$$x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3), \quad x^- = \frac{1}{\sqrt{2}}(x^0 - x^3),$$

(6.1)

$$x = \frac{1}{\sqrt{2}}(x^1 + ix^2), \quad \bar{x} = \frac{1}{\sqrt{2}}(x^1 - ix^2),$$

(6.2)

$$p^+ = \frac{1}{\sqrt{2}}(p_0 + p_3) = -p, \quad p^- = \frac{1}{\sqrt{2}}(p_0 - p_3) = -p^+, \quad p = \frac{1}{\sqrt{2}}(p_1 + ip_2), \quad \bar{p} = \frac{1}{\sqrt{2}}(p_1 - ip_2).$$

(6.3)

With a mostly-plus Minkowski metric $-+++$, the light-front scalar product becomes

$$A_\mu B^\mu = AB + \bar{A}B + A_+B^- + A_-B^+$$

$$= AB + \bar{A}B - A^+B^--A^-B^+$$

(6.5)

The transverse oscillators are

$$\alpha = \frac{1}{\sqrt{2}}(\alpha_1 + i\alpha_2), \quad \bar{\alpha} = \frac{1}{\sqrt{2}}(\alpha_1 - i\alpha_2),$$

(6.6)

$$\alpha^\dagger = \frac{1}{\sqrt{2}}(\alpha^\dagger_1 + i\alpha^\dagger_2), \quad \bar{\alpha}^\dagger = \frac{1}{\sqrt{2}}(\alpha^\dagger_1 - i\alpha^\dagger_2).$$

They obey commutators

$$[\alpha, \alpha^\dagger] = 1, \quad [\bar{\alpha}, \alpha^\dagger] = 1.$$ 

(6.7)

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