PREVALENT DYNAMICS AT THE FIRST BIFURCATION
OF HÉNON-LIKE FAMILIES

HIROKI TAKAHASI *

Abstract. We study the dynamics of strongly dissipative Hénon-like maps, around the first bifurcation parameter $a^*$ at which the uniform hyperbolicity is destroyed by the formation of tangencies inside the limit set. We prove that $a^*$ is a full Lebesgue density point of the set of parameters for which Lebesgue almost every initial point diverges to infinity under positive iteration. A key ingredient is that $a^*$ corresponds to “non-recurrence of every critical point”, reminiscent of Misiurewicz parameters in one-dimensional dynamics. Adapting on the one hand Benedicks & Carleson’s parameter exclusion argument, we construct a set of “good parameters” having $a^*$ as a full density point. Adapting Benedicks & Viana’s volume control argument on the other, we analyze Lebesgue typical dynamics corresponding to these good parameters.

1. Introduction

One important problem in dynamics is to describe transitions from structurally stable to unstable regimes. Equally important is to describe how strange attractors are created. A prototypical example intimately connected to these problems is given by the Hénon family

$$H_a: (x, y) \mapsto (1 - ax^2 + \sqrt{by}, \pm \sqrt{bx}), \quad 0 < b \ll 1.$$  

For all large $a$, one gets a uniformly hyperbolic horseshoe [9], a paradigmatic example of structurally stable chaotic systems. As one decreases $a$, the horseshoe loses its stability at a bifurcation parameter, and then a nonuniformly hyperbolic strange attractor is created, with positive probability in parameter space [5]. The aim of this paper is to shed some light on the process of this sort of transition from horseshoes to strange attractors.

We work within a framework set up by Palis for studying bifurcations of diffeomorphisms: consider arcs of diffeomorphisms losing their stability through generic bifurcations, and analyze which dynamical phenomena are more frequently displayed (in the sense of Lebesgue measure in parameter space) in the sequel of the bifurcation. More precisely, let $(\varphi_a)$ be a parametrized family of diffeomorphisms which undergoes a first bifurcation at $a = a^*$, i.e., $\varphi_a$ is structurally stable for $a > a^*$ and $\varphi_{a^*}$ has a cycle. We assume $(\varphi_a)$ unfolds the bifurcation generically. A dynamical phenomenon $P$ is prevalent at $a^*$ if

$$\liminf_{\varepsilon \to +0} \varepsilon^{-1} \text{Leb}(\{a \in [a^* - \varepsilon, a^*]: \varphi_a \text{ displays } P\}) > 0.$$  

This framework originates in the work of Newhouse and Palis [13], on the frequency of bifurcation sets in the unfoldings of homoclinic tangencies. In that paper, diffeomorphisms before the first bifurcation are Morse-Smale. Palis and Takes [14, 15, 16], inspired by works of Newhouse, studied the prevalence of uniform hyperbolicity in arcs of diffeomorphisms for

* Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, JAPAN
(e-mail address: takahasi@math.kyoto-u.ac.jp).
Figure 1. Manifold organization for $a = a^*$. There exist two hyperbolic fixed saddles $P$, $Q$ near $(1/2, 0), (-1, 0)$ correspondingly. In the orientation preserving case (left), $W^u(Q)$ meets $W^s(Q)$ tangentially. In the orientation reversing case (right), $W^u(P)$ meets $W^s(Q)$ tangentially. The shaded regions represent $R_0$ (see Sect.2.1).

which the non-wandering set of the diffeomorphism at the bifurcation is a union of a non-trivial basic set of saddle type and an orbit of tangency. In opposite direction, the frequency of non-hyperbolicity was studied by Palis and Yoccoz [17, 18, 19].

For the Hénon family, the first bifurcation where the horseshoe ceases to be stable corresponds to the formation of homoclinic or heteroclinic tangencies [2]. This tangency is quadratic, and $(H_a)_a$ unfolds the tangency generically [3]. The orbit of the tangency is accumulated by transverse homoclinic points, and hence contained in the limit set. In [8], all these statements are extended to Hénon-like families, a perturbation of the Hénon family (see Section 2 for a precise definition).

This sort of bifurcation is completely different from the one treated in [14, 15, 16, 17, 18, 19]. A key aspect of models treated in these papers is that the orbit of tangency at the first bifurcation is not contained in the limit set. This implies a global control on new orbits added to the underlying basic set, and moreover allows one to use its invariant foliations to translate dynamical problems to the problem on how two Cantor sets intersect each other. This argument is not viable, if the orbit of tangency, responsible for the loss of the stability of the system, is contained in the limit set, as in the case of Hénon-like families. Let us call such a bifurcation an internal tangency bifurcation.

For an Hénon-like family $(f_a)$, we aim to describe changes in the set

$$K_a = \{z \in \mathbb{R}^2 : \{f_a^n z\}_{n \in \mathbb{Z}} \text{ is bounded} \}.$$ 

By a result of [8], there is a parameter $a^*$ such that $K_a$ is a hyperbolic set for $a > a^*$, and $(f_a)_a$ unfolds a quadratic tangency at $a = a^*$ generically. This suggests that the structure of $K_a$ depends in a very discontinuous way upon $a$. For instance, $a^*$ is accumulated from left by: $a$-intervals for which $f_a$ has sinks [11]; sets with positive Lebesgue measure for which $f_a$ has nonuniformly hyperbolic attractors [12], etc. A consequence of our theorem is that the frequency of such parameters tends to zero as $a \to a^*$. Let

$$K_a^+ = \{z \in \mathbb{R}^2 : \{f_a^n z\}_{n \geq 0} \text{ is bounded} \}.$$ 

**Theorem.** For an Hénon-like family $(f_a)$ there exists a set $\Delta$ of $a$-values such that:

(a) $\lim_{\varepsilon \to +0} \varepsilon^{-1} \text{Leb}(\Delta \cap [a^* - \varepsilon, a^*]) = 1$;

(b) if $a \in \Delta$, then $K_a^+$ has zero Lebesgue measure.

(c) if $a \in \Delta$, then $f_a$ is transitive on $K_a$. 


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To grasp the meanings of the theorem, it is worthwhile to recall Jakobson’s theorem [11] for the quadratic family $x \rightarrow 1 - ax^2$, which states that $a = 2$ is a (one-sided) full Lebesgue density point of the set of parameters corresponding to absolutely continuous invariant probability measures. These measures allow one to statistically predict the asymptotic “fate” of Lebesgue almost every initial conditions. For $a > 2$, the orbit of the critical point $x = 0$ is escaping, and thus the invariant set is uniformly hyperbolic. In other words, $a = 2$ is a first bifurcation parameter of the quadratic family. Immediately right after the bifurcation one mainly gets “observable chaos”. Our theorem asserts a sharp contrast to this sort of transition. For $a \in \Delta$, $K_a$ behaves like a basic set of saddle type, in that Lebesgue typical points escape from any neighborhood of it. This means that, physically observable complicated behaviors are chaotic transient around $K_a$, not sustained in time.

This striking difference at the first bifurcation stems from a simple fact intrinsic to two-dimension: at the parameter $a^*$, the unstable manifold of the saddle fixed point(s) is not confined in any bounded region. Indeed, one key step in the proof of the theorem is to show that, for carefully chosen parameters, the unstable manifold intersects $K_a^+$ in a set with zero Lebesgue measure on the manifold.

By the continuous dependence of invariant manifolds on parameter, one can take a parameter $a' < a^*$ such that $W^u(P)$ is unbounded for $a' < a < a^*$. Let $a^{**}$ denote the smallest with this property. Our parameter set $\Delta$ is contained in $(a^{**}, a^*]$. Benedicks and Carleson [7], Mora and Viana [12] constructed a set of $a$-values near 2, corresponding to maps for which the closure of $W^u(P)$ is a nonuniformly hyperbolic strange attractor. Their parameter sets are at the left of $a^{**}$. Figure 2 indicates a landscape in the $(a, b)$-plane (as usual, $b$ controls the closeness to the quadratic family, see [11]). In the orientation preserving case, $a^{**}$ corresponds to the tangency between $W^u(P)$ and $W^s(Q)$.

In view of the theorem, one might speculate that maps in $\{f_a: a \in \Delta\}$ would retain some weak form of hyperbolicity, as a memory of the uniform hyperbolicity before the bifurcation. For the moment, we do not know if the uniform hyperbolicity is prevalent at $a^*$. To our knowledge, the only presently known result on the prevalence of hyperbolicity in internal tangency bifurcations is due to Rios [20], on arcs of surface diffeomorphisms destroying type 3 horseshoes (horseshoes with three symbols [15]).

Figure 2. Landscape in the $(a, b)$-plane near $(2, 0)$. Parameter sets constructed in [5, 12, 24] corresponding to nonuniformly hyperbolic strange attractors are contained in the blue region. Our parameter set is contained in the red region. The dynamics are uniformly hyperbolic at the right of the $a^*$-line.
To prove the theorem, we build on and develop the machinery for the analysis of strongly dissipative Hénon maps [5, 6, 7, 12, 24]. Excluding undesirable parameters inductively, we construct the parameter set $\Delta$ having $a^*$ as a full density point. We then investigate the dynamics of $f \in \{f_a : a \in \Delta\}$.

A parameter exclusion argument in the spirit of Jakobson [11], Benedicks and Carleson [4, 5] was first brought into the study of homoclinic bifurcations by Palis and Yoccoz [18, 19]. As we mentioned in the beginning, the underlying basic set at the bifurcation is used in a crucial way there, and the same approach does not work in our context of internal tangency bifurcation. In order to prove that $K^+=a^{\ast}$ has zero Lebesgue measure, we develop the volume control argument of Benedicks and Viana [6].

The rest of this paper consists of six sections and one appendix. In Section 2 we analyze one fixed map, collecting results from [5, 6, 12, 24] and [21] as far as we need them. In Section 3 we recall the procedure in [21] for finding suitable critical approximations, used as guides for orbits falling in critical regions.

The parameter set $\Delta$ is constructed in Section 4. This part closely follows the previous construction of the parameter set in [21], modulo the assertion that $a^*$ is a full density point of $\Delta$. It is at this point where the characteristic of the first bifurcation is crucial. We show that the map $f_{a^*}$ behaves as if it is a “two-dimensional Misiurewicz map”, in the sense that every critical approximation of it is non-recurrent. Then it is possible, as in the one-dimensional case [4, 11], to arrange the induction construction in such a way that less and less proportions of parameters in $[a^* - \epsilon, a^*]$ are excluded as $\epsilon \to +0$, and the total fractions of $\Delta$ in the intervals get closer to one. Consequently, $\Delta$ must have $a^*$ as a full density point.

For the remaining three sections we consider the dynamics of one fixed map $f \in \{f_a : a \in \Delta\}$. In Section 5 we identify an well-organized geometric structure of the unstable manifold, close to the one identified by Wang and Young [24]. Using this structure, in Section 6 we analyze the dynamics on the unstable manifold. Combining a classical large deviation argument [5, 7] with a continuity argument from the first bifurcation, we prove that $K^+$ intersects the unstable manifold in a set with zero Lebesgue measure. In Section 7 we study the dynamics on $K^+$. A careful adaptation of the volume control argument [6] together with the conclusion of Section 6 shows that $K^+$ cannot have positive two-dimensional Lebesgue measure.

2. Preliminaries

In this section we analyze one fixed map $f$, collecting results from [5, 6, 12, 24] and [21] as far as we need them.

2.1. Hénon-like families. We deal with a parameterized family $(f_a)$ of diffeomorphisms on $\mathbb{R}^2$ such that $f = f_a$ has the form

$$f_a : (x, y) \mapsto (1 - ax^2, 0) + b \cdot \Phi(a, b, x, y),$$

where $(a, b)$ is close to $(2, 0)$ and $\Phi$ is bounded, continuous, $C^4$ in $(a, x, y)$.

Although $f$ is globally defined on $\mathbb{R}^2$, it is possible to localize our consideration to a compact domain defined as follows. If $f$ preserves orientation, let $W^u = W^u(Q)$. Otherwise, let $W^u = W^u(P)$. Let $R_0$ denote the compact domain bounded by $W^u$ and $W^s(Q)$, as indicated in Figure 1 in the case $a = a^*$. By a result of [8], points outside of $R_0$ escape to infinity either by positive or negative iterations. Hence $K \subset R_0$ holds. Let $D_0 = \{(x, y) \notin R_0 : x \geq \sqrt{2}\}$. It can be read out from [8] that $K^+ \subset D_0 \cup R_0$ holds. By the obvious uniform hyperbolicity on
Claim 2.1. $K^+ \cap R_0 = \bigcap_{n \geq 0} f^{-n}R_0$.  

Proof. Let $z \in K^+ \cap R_0$. Suppose that $z \notin f^{-n}R_0$ holds for some $n > 0$. Let $n_0$ denote the smallest integer with this property. Then $f^{n_0+1}z \in D_1$, where $D_1$ is the set of points $(x,y)$ which is at the left of $W_{\text{loc}}^s(Q)$ and $|y| \leq \sqrt{b}$. As $D_1 \cap K^+ = \emptyset$, $z \notin K^+$ holds, which is a contradiction. Consequently, $K^+ \cap R_0 \subset \bigcap_{n \geq 0} f^{-n}R_0$ holds. The reverse inclusion is obvious. 

To structure the dynamics inside $R_0$, we construct critical points and use them as guides. Unlike the attractor context [5, 12, 24], the construction of critical points has to take into consideration possible leaks out of $R_0$ under iteration, and unbounded derivatives at infinity is a bit problematic. To bypass this problem, we work with a new family $(f_{a,b})$ which is obtained by modifying the quadratic map $x \to 1 - ax^2$, and $\Phi$ in (1) so that the following holds:

(M1) $f = \tilde{f}$ on $R_0$ and $\tilde{f}D_1 \subset D_1$;
(M2) if $z \in R_0$ and $\tilde{f}z \notin R_0$, then for any $n \geq 1$ and a nonzero tangent vector $v$ at $\tilde{f}^nz$ with slope$(v) \leq \sqrt{b}$, slope$(\tilde{f}v) \leq \sqrt{b}$ and $||\tilde{f}v|| \geq 2||v||$;
(M3) there exists a constant $C_0 > 0$ such that $||\partial^i f|| \leq C_0$ and $|\det \tilde{f}| \leq C_0b$ on $D_1 \cup R_0 \cup fR_0$ ($1 \leq i \leq 4$), where $\partial^i f$ denotes any partial derivative in $a, x, y$ of order $i$.

2.2. Hyperbolic behavior. Constructive constants are $\alpha, M, \delta$, chosen in this order. The $\alpha, \delta$ are small, and $M$ is a large integer. Having chosen all of them, we choose sufficiently small $b$. The letter $C$ denotes any generic constants which depend only on $f_{a,b}$ restricted to $[-2,2]^2$.

From this point on, let us denote $\tilde{f}$ by $f$. We start with basic properties of $f$. For $\delta > 0$, define $I(\delta) = \{(x,y) \in R_0 : |x| < \delta\}$. The next lemma establishes a uniform hyperbolicity outside of $I(\delta)$. Not only for orbits staying inside $R_0$, the hyperbolicity estimates hold for orbits which leak out of $R_0$.

Lemma 2.1. For any $\lambda_0 \in (0, \log 2)$ and $\delta > 0$, the following holds for $(a,b)$ close to $(2,0)$. Let $z \in R_0$ be such that $z, fz, \cdots, f^{n-1}z \notin I(\delta)$, and let $v$ be a tangent vector at $z$ with slope$(v) \leq \sqrt{b}$. Then:

(a) $\text{slope}(Df^n v) \leq \sqrt{b}$ and $\|Df^n v\| \geq \delta e^{\lambda_0 n}\|v\|$;
(b) if, in addition, $f^n z \in I(\delta)$, then $\|Df^n v\| \geq e^{\lambda_0 n}\|v\|$.

Proof. If $z, fz, \cdots, f^{n-1}z \in R_0$, then (a) (b) follow from the closeness of $f$ to the top quadratic map. Otherwise, the orbit splits into the part $z, fz, \cdots, f^{k-1}z$ ($k < n$) in $R_0$, and the rest out of $R_0$. (b) is vacuous because of $f^n z \notin I(\delta)$. We have slope$(Df^k v) \leq \sqrt{b}$ and $\|Df^k v\| \geq \delta e^{\lambda_0 k}\|v\|$. Combining these with (M2) we obtain (a). 

2.3. Quadratic behavior. In the next lemma we assume $\gamma$ is a horizontal curve, that is, a $C^2$-curve such that the slopes of its tangent directions are $\leq 1/10$ and the curvature is everywhere $\leq 1/10$. For $z \in \gamma$, let $t(z)$ denote any unit vector tangent to $\gamma$ at $z$. In addition, we assume there exists $\zeta \in \gamma$ such that slope$(Df t(\zeta)) \geq C\sqrt{b}$. Let $e$ denote any unit vector
tangent to \( f\gamma \) at \( f\zeta \). Split \( Df(z) = A(z) (\frac{\partial}{\partial z}) + B(z) e \). Let us agree that \( a \approx b \) indicates that \( C^{-1} \leq a/b \leq C \) holds for some \( C \geq 1 \).

**Lemma 2.2.** ([21] Lemma 2.2.) For all \( z \in \gamma \cap I(\delta) \), \( |z - \zeta| \approx |A(z)| \) and \( |B(z)| \leq C\sqrt{b} \).

**Remark 2.1.** This lemma implies the following, that is often used in what follows. A \( C^2 \)-curve of the form

\[
\{(x(y), y) : |y| \leq \sqrt{b}, |x'(y)| \leq C\sqrt{b}, |x''(y)| \leq C\sqrt{b}\}.
\]

is called a *vertical curve*. Any vertical curve near \( f\gamma \) is tangent to \( f\gamma \) and the tangency is quadratic, or else it intersects \( f\gamma \) exactly at two points.

### 2.4. Most contracting directions

Some versions of results in this section were obtained in [5, 12]. Our presentation follows [24]. Let \( M \) be a \( 2 \times 2 \) matrix. Denote by \( e \) the unit vector (up to sign) such that \( \|Me\| \leq \|Mu\| \) holds for any unit vector \( u \). We call \( e \), when it exists, the *most contracting direction* of \( M \).

For a sequence of matrices \( M_1, M_2, \ldots \), we use \( M^{(i)} \) to denote the matrix product \( M_i \cdots M_2 M_1 \), and \( e_i \) to denote the mostly contracting direction of \( M^{(i)} \).

**Hypothesis for Sect.2.2.** The matrices \( M_i \) satisfy \( |\det M_i| \leq Cb \) and \( \|M_i\| \leq C_0 \).

**Lemma 2.3.** ([24] Lemma 2.1.) Let \( i \geq 2 \), and suppose that \( \|M^{(i)}\| \geq \kappa^i \) and \( \|M^{(i-1)}\| \geq \kappa^{i-1} \) for some \( \kappa \geq b^{1/10} \). Then \( e_i \) and \( e_{i-1} \) are well-defined, and satisfy

\[
\|e_i \times e_{i-1}\| \leq \left( \frac{Cb}{\kappa^2} \right)^{i-1}.
\]

**Corollary 2.1.** ([24] Corollary 2.1.) If \( \|M^{(i)}\| \geq \kappa^i \) for \( 1 \leq i \leq n \), then:

(a) \( \|e_n - e_1\| \leq \frac{Cb}{\kappa^2} \);

(b) \( \|M^{(i)}e_n\| \leq \left( \frac{Cb}{\kappa^2} \right)^i \) holds for \( 1 \leq i \leq n \).

Next we consider for each \( i \) a parametrized family of matrices \( M_i(s_1, s_2, s_3) \) such that \( \|\partial^j_1 \det M_i(s_1, s_2, s_3)\| \leq C^j b^i \), and \( \|\partial^j_1 M_i(s_1, s_2, s_3)\| \leq C^j_0 \) for each \( 0 \leq j \leq 3 \). Here, \( \partial^j_1 \) represents any one of the partial derivatives of order \( j \) with respect to \( s_1, s_2, \) or \( s_3 \).

**Corollary 2.2.** ([24] Corollary 2.2.) Suppose that \( \|M^{(i)}(s_1, s_2, s_3)\| \geq \kappa^i \) for \( 1 \leq i \leq n \). Then for \( j = 1, 2, 3 \) and \( 2 \leq i \leq n \),

\[
(2) \quad |\partial^j_1 (e_i \times e_{i-1})| \leq \left( \frac{Cb}{\kappa^{2+j}} \right)^{i-1},
\]

\[
(3) \quad \|\partial^j_1(M^{(i)}e_i)\| \leq \left( \frac{Cb}{\kappa^{2+j}} \right)^i.
\]

Let \( e_1(z) \) denote the most contracting direction of \( Df(z) \) when it makes sense. From the form of our map ([11]), \( e_1(z) \) is defined for all \( z \notin I(\sqrt{b}) \). In view of [12] pp. 21], we have

\[
(4) \quad \text{slope}(e_1) \geq C/\sqrt{b} \quad \text{and} \quad \|\partial e_1\| \leq C\sqrt{b}.
\]

We say \( z \) is \( \kappa \)-expanding up to time \( n \), or simply *expanding*, if there exists a tangent vector \( v \) at \( z \) and \( \kappa \geq b^{1/10} \) such that for every \( 1 \leq i \leq n \),

\[
\|Df^iv\| \geq \kappa^i\|v\|.
\]
Corollary 2.3. If $z$ is $\kappa$-expanding up to time $n$, then slope($e_n$) $\geq C/\sqrt{b}$ and $\|de_n\| \leq \frac{C_\kappa}{n}$. 

2.5. Long stable leaves. In the next lemma, a $C^2$-distance $d_{C^2}$ between two vertical curves is measured by regarding them as $C^2$-functions on $[-\sqrt{b}, \sqrt{b}]$.

Lemma 2.4. (cf. [12] Section 6.) Let $\kappa \geq \delta^{15}$. If $z$ is $\kappa$-expanding up to time $n$, then for every $1 \leq i \leq n$, the maximal integral curve of $e_i$ through $z$ contains a vertical curve, denoted by $\Gamma_i(z)$. In addition, for every $1 < i \leq n$, $d_{C^2}(\Gamma_i(z), \Gamma_{i-1}(z)) \leq \left(\frac{C_\kappa}{\delta^4}\right)^{i-1}$.

By a long stable leaf of order $i$ through $z$ we mean the curve $\Gamma_i(z)$ as in the statement.

Remark 2.2. In the construction of long stable leaves, the relation between the lengths of leaves and the value of $\kappa$ is crucial [12]. In [6], long stable leaves of length $\approx 1/5$ are used. To this end, they require $\kappa \geq e^{-20}$. For our purpose, long stable leaves of length $\approx 2\sqrt{b}$ suffices. Hence, $\kappa \geq \delta^{15}$ suffices.

Lemma 2.5. (cf. [6] Proposition 2.4.) Let $\kappa \geq \delta^{15}$. If $z$ is $\kappa$-expanding, then there exists a vertical curve $\Gamma(z)$ through $z$ such that:

\begin{enumerate}[(a)]
  \item $|f^n\xi - f^n\eta| \leq \left(\frac{C_\kappa}{n}\right)^n$ for all $\xi, \eta \in \Gamma(z)$ and $n \geq 1$;
  \item if $z_1, z_2$ are $\kappa$-expanding, then $\angle(t_i(\xi_1), t_i(\xi_2)) \leq C\sqrt{b}|\xi_1 - \xi_2|$, where $t_i(\xi_i)$ denotes any unit vector tangent to $\Gamma(z_i)$ at $\xi_i; i = 1, 2$.
\end{enumerate}

We call a long stable leaf through $z$ the curve $\Gamma(z)$ as in the statement, and a stable leaf any compact curve having some iterate contained in a long stable leaf.

Let us record one consequence of Lemma 2.5. Let $\Gamma(z_1), \Gamma(z_2)$ be two long stable leaves and $\xi_1, \eta_1 \in \Gamma(z_1)$. Let $\xi_2, \eta_2$ denote the points in $\Gamma(z_2)$ whose $y$-coordinate coincides with that of $\xi_1$ and $\eta_1$ correspondingly. Gronwall’s inequality gives

$$|\xi_1 - \xi_2| \leq e^{C\sqrt{b}}|\eta_1 - \eta_2|.$$ 

2.6. Recovering expansion. Let $\gamma$ be a horizontal curve and $n \geq M$. We say $\zeta \in \gamma$ is a critical approximation of order $n$ on $\gamma$ if:

\begin{enumerate}[(i)]
  \item $\|Df_i(f\zeta)\| \geq 1/10$ for $1 \leq i \leq n$;
  \item $e_n(f\zeta)$ is tangent to $Dft(\zeta)$, where $t(\zeta)$ is any unit vector tangent to $\gamma$ at $\zeta$.
\end{enumerate}

Notation. For $z \in I(\delta)$ and $i \geq 1$, let $w_i(z) = Df^{i-1}(fz)(\frac{1}{i}z)$.

We now introduce three conditions, which are taken as inductive assumptions in the construction of the parameter set $\Delta$. Let $\lambda := \lambda_0/2$, where $\lambda_0$ is the one in Lemma 2.1. A critical approximation $\zeta$ of order $n$ on $\gamma$ has a good critical behavior if:

\begin{enumerate}[(G1)]
  \item $\|w_i(\zeta)\| \geq e^{\lambda(i-1)}$ for $1 \leq i \leq 20n$;
  \item $\|w_j(\zeta)\| \geq e^{-20\lambda}\|w_i(\zeta)\|$ for $1 \leq i < j \leq 20n$;
  \item there exists a monotone increasing function $\chi : [M, 20n] \cap \mathbb{N} \cap \mathbb{N}$ such that for each $j \in [M, 20n]$, $(1 - \sqrt{e})j \leq \chi(j) \leq j$ and $\|w_{\chi(j)}(\zeta)\| \geq \delta\|w_i(\zeta)\|$ holds for $0 \leq i < \chi(j)$.
\end{enumerate}

Hypothesis for the rest of Sect. 2.6. $\zeta$ is a critical approximation of order $n$ on $\gamma$, with a good critical behavior.
For $M \leq k \leq 20n - 1$, let
\[
D_k(\zeta) = e^{-3ak} \min_{1 \leq i \leq k} \min_{1 \leq j \leq k+1} \frac{\|w_j(\zeta)\|^2}{\|w_i(\zeta)\|^3}.
\]
Represent the long stable leaf of order $n$ through $f\zeta$ as a graph of a function $\Gamma_n(f\zeta) = \{(x_n(y), y) : |y| \leq \sqrt{b}\}$. Let
\[
V_k = \{(x, y) : |x - x_n(y)| \leq D_k(\zeta)/2, |y| \leq \sqrt{b}\}.
\]
Take a monotone increasing function $\chi$ satisfying condition (G3). Let $v$ denote any nonzero vector tangent to $\gamma$ at $z$. If $fz \in V_k \setminus V_{k+1}$, then we say $v$ is in admissible position relative to $\zeta$. Define a bound period $p = p(\zeta, z)$ by
\[
p = \chi(k),
\]
and a fold period $q = q(\zeta, z)$ by
\[
q = \min \{i \in [1, p) : |\zeta - z|^\beta \cdot \|w_j(\zeta)\| \geq 1 \text{ for every } i \leq j < p\},
\]
where
\[
(6) \quad \beta = \frac{2 \log C_0}{\log 1/b}.
\]
It is easy to check that $q$ is well-defined, by (G1-3) and the assumption on $z$. If $fz \in V_{20n-1}$, then we say $v$ is in critical position relative to $\zeta$.

**Proposition 2.1.** (Proposition 2.2.) Let $\gamma$, $\zeta$, $z$, $v$ be as above.
(i) If $v$ is in admissible position relative to $\zeta$ and $fz \in V_k \setminus V_{k+1}$, then:
(a) $\log |\zeta - z|^\frac{1}{3\log C_0} \leq p \leq \log |\zeta - z|^{-\frac{1}{b}}$;
(b) $q \leq C\beta p$;
(c) $|f^i\zeta - f^i z| \leq e^{-2\alpha p}$ for $1 \leq i \leq p$;
(d) $|\zeta - z|^\alpha \|v\| \leq \|Df^i v\| \leq |\zeta - z|^{-1-\alpha} \|v\|$;
(e) $\|Df^i v\| \geq \|v\| \cdot |\zeta - z|^{-1+\frac{\alpha}{\log C_0}} \geq e^{\frac{\alpha}{2}} \|v\|$;
(f) $\|Df^i v\| \geq (\delta/10) \|Df^i v\|$ for $0 \leq i < p$;
(ii) If $v$ is in critical position relative to $\zeta$, then $\|Df^i v\| \leq e^{-8\alpha n} \|v\|$.

A proof of this proposition follows the line that is now well understood \[5\] \[12\] \[24\]. We split $Dfv$ into the direction of $(\frac{1}{b})$ and that of $e_n(fz)$, iterate them separately, and put them together at the expiration of the fold period.

### 3. Existence of binding points

To deal with returns to the region $I(\delta)$, we look for suitable critical approximations and use them as guides to keep further evolution in track. Such critical approximations, if exists, are called binding points. In this section we recall the procedure in \[21\] for finding binding points.

#### 3.1. Creation of new critical approximations.

By a $C^2(b)$-curve we mean a $C^2$-curve such that the slopes of all its tangent vectors are $\leq \sqrt{b}$ and the curvature is everywhere $\leq \sqrt{b}$. The next two lemmas are used to create new critical approximations around the existing ones. For corresponding versions, see: \[5\] p.113, Lemma 6.1; \[12\] Sect.7A, 7B; \[24\] Lemma 2.10, 2.11.
Lemma 3.1. Let \( \gamma \) be a \( C^2(b) \)-curve in \( I(\delta) \) parameterized by arc length and such that \( \gamma(0) \) is a critical approximation of order \( n \). Suppose that:

(i) \( \gamma(s) \) is defined for \( s \in [-b^{\frac{1}{3}}, b^{\frac{1}{3}}] \);
(ii) there exists \( m \in [n/3, 20n] \) such that \( \|Df^i(f_\gamma(0))\| \geq 1 \) for \( 1 \leq i \leq m \).

There exists \( s_0 \in [-b^{\frac{1}{3}}, b^{\frac{1}{3}}] \) such that \( \gamma(s_0) \) is a critical approximation of order \( m \) on \( \gamma \).

Next we consider two \( C^2(b) \)-curves \( \gamma_1, \gamma_2 \) in \( I(\delta) \) parametrized by arc length, in a way that the \( x \)-coordinate of \( \gamma_1(0) \) coincides with that of \( \gamma_2(0) \). Let \( t_\sigma(s) \) denote any unit vector tangent to \( \gamma_\sigma \) at \( \gamma_\sigma(s) \), \( \sigma = 1, 2 \).

Lemma 3.2. Let \( \gamma_1, \gamma_2 \) be as above and suppose that:

(i) \( \gamma_1(s), \gamma_2(s) \) are defined for \( s \in [-\varepsilon^\frac{n}{3}, \varepsilon^\frac{n}{3}] \), \( \varepsilon \leq C_0^{-5} \);
(ii) \( \gamma_1(0) \) is a critical approximation of order \( n \) on \( \gamma_1 \) and \( \|Df^i(f_\gamma(0))\| \geq 1 \) for \( 1 \leq i \leq n \);
(iii) \( |\gamma_1(0) - \gamma_2(0)| \leq \varepsilon^n \) and \( \angle(t_1(0), t_2(0)) \leq \varepsilon^n \).

There exists \( s_0 \in [-\varepsilon^\frac{n}{3}, \varepsilon^\frac{n}{3}] \) such that \( \gamma_2(s_0) \) is a critical approximation of order \( n \) on \( \gamma_2 \).

3.2. Hyperbolic times. Let

\[
\theta = \alpha^3, \quad \kappa_0 = C_0^{-10}.
\]

Let \( v \) be a tangent vector at \( z \) and let \( m \geq 1 \). We say \( v \) is \( r \)-regular up to time \( m \) if for \( 0 \leq i < m \),

\[
\|Df^m v\| \geq r\|Df^i v\|.
\]

We say \( \mu \in [0, m] \) is an \( m \)-hyperbolic time of \( v \) if \( Df^\mu v \) is \( \kappa_0^\frac{1}{3} \)-expanding up to time \( m - \mu \).

Results related to the next lemma can be found in [5 Lemma 6.6], [12 Lemma 9.1], [24 Claim 5.1].

Lemma 3.3. ([21] Lemma 2.12; Abundance of well-distributed hyperbolic times) Let \( m \geq \log(1/\delta) \) and suppose that a tangent vector \( v \) at \( z \) is 1/100-regular up to time \( m \). There exist \( s \geq 2 \) and a sequence \( \mu_1 < \mu_2 < \cdots < \mu_s \) of \( m \)-hyperbolic times of \( v \) such that:

(a) \( \|Df^{\mu_j} v\| \) is \( \kappa_0^\frac{1}{3} \)-expanding up to time \( m - \mu_j \);
(b) \( 1/16 \leq (m - \mu_{j+1})/(m - \mu_j) \leq 1/4 \) for \( 1 \leq j \leq s - 1 \);
(c) \( 0 \leq \mu_1 < m/2 \) and \( m - \log(1/\delta) \leq \mu_s \leq m - \log(1/\delta)/2 \).

3.3. Nice critical approximations. Let \( \zeta \) be a critical approximation of order \( n \) on a horizontal curve \( \gamma \). We say \( \zeta \) is nice if:

(C1) \( \|Df^i(f(\zeta))\| \geq 1 \) for \( 1 \leq i \leq n \);
(C2) \( f^{-i}\zeta \in [-2, 2] \times [-\sqrt{b}, \sqrt{b}] \) for \( 1 \leq i \leq [\theta n] \);
(C3) let \( u \) denote any unit vector at \( f^{-[\theta n]} \zeta \) such that \( Df^{[\theta n]} u \) is tangent to \( \gamma \). Then \( u \) is \( \kappa_0^\frac{1}{3} \)-expanding and 1/100-regular, both up to time \( [\theta n] \).

A nonzero vector \( v \) is in tangential position relative to \( \zeta \) if there exists a horizontal curve which is tangent to both \( v \) and \( Df^{[\theta n]} u \).

Let

\[
N = \left[ \frac{\log 1/\delta}{\theta} \right],
\]

where the square bracket denotes the integer part.
Hypothesis for the rest of Sect.3: m, n are integers with \( m \geq \log(1/\delta) \), \( n \geq N \), and:

• each nice critical approximation \( \zeta \) of order \( \leq n \) has a good critical behavior;
• a tangent vector \( v \) at \( z \) is 1/10-regular up to time \( m \), and \( f^m z \in I(\delta) \).

3.4. Binding procedure. Under the above hypothesis, we describe how to choose a binding point relative to which \( Df^m v \) is in tangential position. In view of Lemma 3.3 fix once and for all a sequence \( \mu_1 < \mu_2 < \cdots < \mu_s \) of \( m \)-hyperbolic times of \( v \) satisfying

\[
m - \mu_1 \leq \theta n, \quad \frac{1}{2} \log(1/\delta) \leq m - \mu_s \leq \log(1/\delta), \quad \frac{1}{16} \leq \frac{m - \mu_{i+1}}{m - \mu_i} \quad \text{for} \quad 1 \leq i < s.
\]

Correspondingly, fix once and for all a sequence \( n \geq n_1 > \cdots > n_s > n_{s+1} > \cdots > n_{s_0} := M \) of integers such that

\[
m - \mu_i = \lfloor \theta n_i \rfloor \quad \text{for} \quad 1 \leq i \leq s,
\]

\[
n_{i+1} = n_i - 1 \quad \text{for} \quad s \leq i < s_0.
\]

We construct a family of \( C^2 \)(b)-curves tangent to \( Df^m v \), arranged in such an organized way that Lemmas 3.1 and 3.2 may be used inductively. This produces a critical approximation on each of the \( C^2 \)(b)-curves. We choose one of them as a binding point. In this way we obtain the following statement.

Lemma 3.4. (21 Proposition 3.1.) There exist \( i \in [1, s] \) and a critical approximation \( \zeta_i \) of order \( n_i \) such that \( Df^m v \) is in tangential position relative to \( \zeta_i \).

Sketch of the proof. One way to find such \( n_i \) and \( \zeta_i \) are described as follows. Let \( l_i \) denote the straight segment of length \( \kappa^n \) centered at \( f^m z \) and tangent to \( Df^m v \). Then \( \gamma_i := f^m l_i \) is a \( C^2 \)(b)-curve extending to both sides around \( f^m z \) to length \( \geq \kappa^n \). Lemma 3.1 Lemma 3.2 and the hypothesis of \( f \) allow us to show the following: if \( Df^m v \) is in critical position relative to a critical approximation of order \( n_i \) on \( \gamma_i \), then there exists a critical approximation of order \( n_{i-1} \) on \( \gamma_{i-1} \) relative to which \( Df^m v \) is in tangential position. A recursive use of this argument yields the conclusion. \( \square \)

Definition 3.1. Let \( i_0 \in [1, s] \) denote the largest integer such that there exists a critical approximation of order \( n_{i_0} \) relative to which \( Df^m v \) is in tangential position. We call any such critical approximation a binding point for \( Df^m v \).

Let \( \zeta \) denote any binding point for \( Df^m v \). By the definitions in Sect.2.6, there are two mutually exclusive cases:

(a) \( i_0 = 1 \), and \( f^m z \) is in critical position relative to \( \zeta \);
(b) \( Df^m v \) is in admissible position relative to \( \zeta \).

In case (a), the contraction estimate in (ii) Proposition 2.1 is in place. In case (b), all the estimates in (i) Proposition 2.1 are in place: the loss of expansion and regularity suffered from the return are recovered at the end of the bound period.

In case (b), one can repeat the binding procedure in the following manner. Write \( m = m_1 \). Let \( p_1 \) denote the bound period. (e,f) Proposition 2.1 implies that \( v \) is 1/10-regular up to time \( m_1 + p_1 \). Let \( m_2 \geq m_1 + p_1 \) denote the smallest such that \( f^{m_2} z \in I(\delta) \). By Lemma 2.1 \( v \) is 1/10-regular up to time \( m_2 \). Subsequently one may repeat the binding procedure once again, replacing \( m \to m_2, f^m z \to f^{m_2} z, Df^m v \to Df^{m_2} v \).
In this way, one can (if (a) does not occur) define integers
\[ m_1 < m_1 + p_1 \leq m_2 < m_2 + p_2 \leq m_3 < \ldots \]
inductively as follows: for \( k \geq 1 \), let \( p_k \) be the bound period of \( f^{m_k} z \), and let \( m_{k+1} \) be the smallest \( j \geq m_k + p_k \) such that \( f^j z \in I(\delta) \). (Note that an orbit may return to \( I(\delta) \) during its bound periods, i.e. \( \{m_k\} \) are not the only return times to \( I(\delta) \).) This decomposes the orbit of \( z \) into segments corresponding to time intervals \( (m_k, m_k + p_k) \) and \( [m_k + p_k, m_{k+1}] \), during which we describe the orbit of \( z \) as being “bound” and “free” states respectively; \( m_k \) are called times of free returns.

**Remark 3.1.** Let us consider the case where the above hypothesis is satisfied for every \( n \geq N \). Then, the binding procedure allows us to keep in track the evolution of any complete orbit in \( W^u \), decomposing it into bound and free segments. However, this procedure is not well-adapted to our phase-space construction in later sections, because:

- the choice of binding points relies only on the individual orbit under consideration and neglects a global information on \( W^u \);
- Critical approximations eligible as binding points are not unique.

These issues will be resolved in Section 5, for parameters in \( \Delta \) constructed in the next section.

### 4. Parameter exclusion

In this section we construct the parameter set \( \Delta \) in the theorem, having \( a^* \) as a full density point. The construction is done by induction: \( \Delta = \bigcap_{n \geq 0} \Delta_n \), where \( \Delta_n \) is constructed at step \( n \), excluding from \( \Delta_{n-1} \) all those undesirable parameters for which some critical approximation may not have good critical behavior up to time \( 20n \).

### 4.1. Critical approximations of \( f_{a^*} \) are non-recurrent

The construction of \( \Delta \) and a measure estimate of it closely follow [21], in which a positive measure set of parameters was constructed corresponding to Hénon-like maps with nonuniformly hyperbolic behavior. One key difference from [21] is the assertion that \( a^* \) is a full density point of \( \Delta \). A key ingredient for this is the next proposition, which states that the orbit of every critical approximation of \( f_{a^*} \) is non-recurrent.

**Proposition 4.1.** For every critical approximation \( \zeta \) of \( f_{a^*} \) of order \( n \), \( f_{a^*}^i \zeta \in \{(x, y) \in \mathbb{R}^2 : |x| \geq 9/10\} \) holds for every \( 1 \leq i < 20n \).

We postpone a proof of this proposition to Sect.4.12.

### 4.2. Definition of parameter sets

Choose sufficiently small \( \varepsilon_0 \) and \( b \) so that for any \( f \in \{f_a : a \in [a^* - \varepsilon_0, a^*]\} \), any critical approximation \( \zeta \) of \( f \) and \( 1 \leq i < 20N \), \( f^i \zeta \in \{(x, y) \in \mathbb{R}^2 : |x| \geq 9/10\} \). This requirement is feasible by the fact that any critical approximation is contained in \( I(\sqrt{b}) \). Set \( \Delta_n = [a^* - \varepsilon_0, a^*] \) for \( 1 \leq n \leq N \).

Let \( n > N \), \( a \in \Delta_{n-1} \) and suppose that every nice critical approximation of \( f_a \) of order \( < n \) has a good critical behavior. Let \( 20(n - 1) \leq m < 20n \). We say a nice critical approximation \( \zeta \) of \( f_a \) of order \( \geq n \) satisfies \((G)_m\) if:

- (i) there is an well-defined decomposition of the orbit \( w_1(\zeta), w_2(\zeta), \ldots, w_m(\zeta) \) into bound and free segments, as described at the end of Sect.3.3.
(ii) let \( n_1 - 1 < n_2 - 1 < \cdots < n_s - 1 \leq m - 1 \) denote all the free return times of \( f \zeta \), with \( z_1, \cdots, z_s \) the corresponding binding points. They are of order \(< n \) and

\[
\sum_{i=1}^{s} \log |f^{n_i} \zeta - z_i| \geq -\alpha m. \tag{12}
\]

For \( n > N \), define \( \Delta_n \) to be the set of all \( a \in \Delta_{n-1} \) for which every nice critical approximation of order \( \geq n \) satisfies \((G)_{20n-1}\). In other words,

\[
\Delta_{n-1} \setminus \Delta_n = \{ a \in \Delta_{n-1} : (G)_m \text{ fails for some } 20(n-1) \leq m < 20n\text{ and some nice critical approximation of order } \geq n \text{ of } f_a \}. \tag{13}
\]

**Remark 4.1.** Let \( \zeta \) be a critical approximation. If \( n - 1 \) is a free return time of \( w_1(\zeta) \), then for simplicity, let us call \( n \) a free return time of \( \zeta \). This terminology is in a slight disagreement with that introduced at the end of Sect. 3.4.

The next proposition indicates that, for parameters in \( \Delta_n \), critical approximations of order \( n \) can be used as binding points, and thus allows us to proceed to the definition of \( \Delta_{n+1} \).

**Proposition 4.2.** Let \( n > N \), \( a \in \Delta_n \) and let \( \zeta \) be a nice critical approximation of order \( \geq n \) of \( f_a \). Then:

(a) \( ||w_i(\zeta)|| \geq e^{\lambda(i-1)} \) for \( 1 \leq i \leq 20n \);

(b) \( ||w_j(\zeta)|| \geq e^{-2\alpha t} ||w_i(\zeta)|| \) for \( 1 \leq i < j \leq 20n \);

(c) if \( \zeta \) is of order \( n \), then it has a good critical behavior.

**Proof.** The proof is almost identical to that of [21 Proposition 5.1]. Here we suppose \( \zeta \) is of order \( n \), and just give a sketch of how to define the function \( \chi \) in \((G3)\). For details, see [21].

Let \( j \in [M, 20n] \) and \( h_0 := j \). Define a finite sequence \( h_1 > \cdots > h_{t(j)} \) of free return times of \( \zeta \) inductively as follows. Let \( \hat{h}_{k+1} \) denote the largest free return time before \( h_k \), when it makes sense. Let \( p_{k+1} \) denote the corresponding bound period. If

\[
h_k - \hat{h}_{k+1} - p_{k+1} \leq (1/\lambda_0) \log(10\delta), \tag{13}
\]

then let \( h_{k+1} = \hat{h}_{k+1} \). In all other cases, \( h_{k+1} \) is undefined, namely \( k = t(j) \). Define \( \chi(j) = h_{t(j)} \). Obviously, \( \chi(j) \leq j \) holds. If \((1 - \sqrt{\alpha})j \leq \chi(j) \) did not hold, \((13)\) would imply that the total number of bound iterates in the interval \([1 - \sqrt{\alpha})j, j\] were bigger than a constant multiple of \( \sqrt{\alpha}j \). While by condition \((G)\), the total number of bound states in the interval is smaller than a constant multiple of \( \alpha j \). If \( \alpha \) is small, then these two estimates are not compatible.

To estimate the measure of \( \Delta_{n-1} \setminus \Delta_n \), we first decompose it into a finite number of subsets, based on certain combinatorics on itineraries of critical approximations. We then estimate the measure of each subset separately, and unify them at the end. In the next two subsections we introduce two integral components of the combinatorics.

### 4.3. Deep returns.
Let \( f \in \{ f_a : a \in \Delta_{n-1} \setminus \Delta_n \} \). Let \( \zeta \) be a nice critical approximation of \( f \) of order \( \geq n \). Let \( \nu < 20n \) be a free return time of \( \zeta \), with the binding point \( z \). If \( \nu \) is not the first return time to \( I(\delta) \), then let \( n_1 < \cdots < n_t \) denote all the free return times of \( \zeta \) before \( \nu \). For \( 1 \leq s \leq t \), let \( z_s \) denote the corresponding binding point and \( p_s \) the bound period.
Write $n_{t+1} = \nu$ and $z_{t+1} = z$. We say $\nu$ is a deep return time, if it is the first return time to $I(\delta)$, or else for $1 \leq s \leq t$,

\begin{equation}
\sum_{j=s+1}^{t+1} 2 \log |f^{n_j} \zeta - z_j| \leq \log |f^{n_s} \zeta - z_s|.
\end{equation}

For each $n_s$, let

$$\sigma_{n_s}(\zeta) = \frac{|f^{n_s} \zeta - z_s|}{\|w_{n_s}(\zeta)\|}.$$  

For each $i \in [1,\nu] \setminus \bigcup_{1 \leq s \leq t} [n_s, n_s + p_s - 1]$, let

$$\sigma_i(\zeta) = \frac{\|w_{i+1}(\zeta)\|}{\|w_i(\zeta)\|^2}.$$  

Define

$$\Theta_\nu(\zeta) = \kappa_0 \cdot \left[ \sum_{i=1}^{\nu-1} \sigma_i(\zeta) \right]^{-1}.$$  

It is understood that the sum runs over all $i$ such that $f^i \zeta$ is free.

**Lemma 4.1.** ([21] Lemma 5.2.) For the above $f, \zeta, \nu, z$, if $\nu$ is a deep return time of $\zeta$, then

$$\|w_\nu(\zeta)\|\Theta_\nu(\zeta) \geq |f^\nu \zeta - z|^\frac{1}{2}.$$  

4.4. **Position of nice critical approximations.** For each $\mu \geq \theta M > 1$, fix a subdivision of $\mathbb{R} \times \{\sqrt{b}\}$ into right-open horizontal lines of equal length $\kappa_0^\mu$. We label all of them intersecting $H := [-2,2] \times \{\sqrt{b}\}$ with $l = 1, 2, 3, \cdots$, from the left to the right. By a $\mu$-address of a point $x$ on $H$ we mean the integer $l$ which is a label of the horizontal containing $x$.

In general, let $\zeta$ be a nice critical approximation of order $n$. The long stable leaf through $f^{-[\theta n]} \zeta$ of order $[\theta n]$ intersects $H$ exactly at one point. Let $A(\zeta)$ denote the $[\theta n]$-address of the point of the intersection.

Let $\zeta$ be a nice critical approximation of order $n \geq N$ on a horizontal curve $\gamma$. By definition, there exists a tangent vector $u$ at $f^{-[\theta n]} \zeta$ for which (C3) in Sect 3.3 holds. Let $\mu$ be any $[\theta n]$-hyperbolic time of $u$. We call $\mu$ a hyperbolic time of $\zeta$. The long stable leaf through $f^{\mu-[\theta n]} \zeta$ of order $[\theta n] - \mu$ intersects $H$ exactly at one point. Let $A(\zeta, \mu)$ denote the $([\theta n] - \mu)$-address of the point of the intersection.

4.5. **Decomposition of the exclude parameter set at step $n$.** Fix positive integers $m \in [20(n-1), 20n)$, $s, t, R$. Fix the following combinatorics:

- sequences $(\mu_1, \cdots, \mu_s), (x_1, \cdots, x_s)$ of $s$ positive integers;
- sequences $(\nu_1, \cdots, \nu_t), (n_1, \cdots, n_t), (r_1, \cdots, r_t), (y_1, \cdots, y_t)$ of $t$ positive integers.

Let $E_n(\ast) = E_n(m, s, t, R, \cdots)$ denote the set of all $a \in \Delta_{n-1} \setminus \Delta_n$ for which there exists a nice critical approximation $\zeta$ of $f_a = f$ of order $n' \geq n$ such that the following holds:

- $(Z1)$ $(G)_{m-1}$ holds, and $(G)_m$ fails;
- $(Z2)$ $\nu_1 < \cdots < \nu_t = m$ are all the deep return times in the first $m$ iterates of $\zeta$, with $z_1, \cdots, z_t$ the corresponding binding points;
- $(Z3)$ for each $k \in [1, t]$, the order of $z_k$ is $n_k < n$. If $\nu_k < m$, then $|f^{\nu_k} \zeta - z_k| \in [e^{-r_k}, e^{-r_{k+1}}]$. If $\nu_k = m$, which means $k = t$ and $\nu_t = m$, then $r_t$ is defined as follows. If $|f^m \zeta - z_t| > e^{-\alpha m}$, then $r_t$ is such that $|f^m \zeta - z_t| \in [e^{-r_t}, e^{-r_{t+1}}]$ holds. Otherwise, $r_t = \alpha m$;
\( \mu_1 < \cdots < \mu_s \) is a minimal sequence of hyperbolic times of \( \zeta \) satisfying

\[
\frac{1}{2} \leq \frac{[\theta n'] - \mu_s}{\log(1/\delta)} \leq 1, \quad [\theta n'] - \mu_1 \geq \theta n, \quad \frac{1}{16} \leq \frac{[\theta n'] - \mu_{i+1}}{[\theta n'] - \mu_i} \leq \frac{1}{4} \quad \text{for } 1 \leq i < s.
\]

Lemma 3.3 ensures the existence of such a sequence;

\( \mu_1 = A(\zeta, \mu_1) \);
\( \mu_s = A(z_s) \).

If \( a \in E_n(*) \), then any nice critical approximation of \( f_a \) of order \( \geq n \) for which (Z1-6) hold is called \textit{responsible} for \( a \). The parameter set \( E_n(*) \) is called an \textit{n-class}. By definition, any parameter excluded from \( \Delta_{n-1} \) belongs to some \( n \)-class. We estimate the measure of \( \Delta_{n-1} \setminus \Delta_n \) by estimating a contribution from each \( n \)-class first, and then counting the total number of \( n \)-classes.

4.6. Digestive remarks on the combinatorics. Let us remark on the meanings of the conditions in the definition of \( E_n(*) \). (Z1,Z2,Z3) are conditions on the forward orbits of responsible critical approximations. (Z2) indicates that \textit{we do exclusions of parameters only at deep return times}. (Z4,Z5) are conditions on the backward orbits of responsible critical approximations. (Z4) indicates that only the backward orbit segments of length comparable to \( \theta n \) are taken into consideration. (Z6) is a condition on binding points at each deep return time. (Z4,Z5,Z6) are used to deal with the following two problems intrinsic to two-dimension.

- \textit{Infinitely many responsible critical approximations}. The first problem is that critical approximations responsible for a single parameter are far from unique, and even infinite. All of them have to be taken into consideration in the measure estimate of \( E_n(*) \). (Z4, Z5) are used to deal with this problem. They allow us to reduce our consideration to a finite number of parameter-dependent orbits, called \textit{deformations}, introduced in Sect 4.8.

- \textit{Infinitely many binding points}. Nice critical approximations eligible as binding points are far from unique, due to the very definition of binding points in Sect 3.4. (Z6) allows us to deal with this problem, with the help of deformations as well.

4.7. Full Lebesgue density at the first bifurcation parameter. We conclude that \( \Delta \) has \( a^* \) as a full Lebesgue density point. Let \( | \cdot | \) denote the one-dimensional Lebesgue measure. For a compact interval \( I \) centered at \( x \) and \( r > 0 \), let \( r \cdot I \) denote the interval of length \( r|I| \) centered at \( x \). The main step is a proof of the next

Proposition 4.3. (Covering by intervals) Let \( m \in [20(n-1), 20n) \), \( s, t, R \) be positive integers. For any \( n \)-class \( E_n(m, s, t, R, \cdots, *) = E_n(*) \), for any \( \varepsilon \in (0, \varepsilon_0) \), \( k \in [1, t] \), there exist a finite number of pairwise disjoint intervals \( \{J_{k,i}\}_i \) with the following properties:

(a) \( E_n(*) \cap [a^* - \varepsilon, a^*] \subset \bigcup J_{k,i} \); 
(b) if \( t > 1 \), then for each \( k \in [2, t] \) and \( J_{k,i} \) there exists \( J_{k-1,j} \) such that \( J_{k,i} \subset 2e^{-r_{k-1}/3} \cdot J_{k-1,j} \); 
(c) \( \sum |J_{1,i}| \leq 3\varepsilon \).

This sort of covering originates in the works Tsujii \[22, 23\], and has been used in \[21\] for the construction of positive measure set of parameters corresponding to maps with nonuniformly hyperbolic behavior. For our purpose we need to develop it further.
Proposition 4.3 gives $|E_n(\ast) \cap [a^* - \varepsilon, a^*]| \leq 3\varepsilon e^{-\frac{1}{3}R}$, where $R = r_1 + r_2 + \cdots + r_t$. To conclude, we need to count the number of all feasible $n$-classes. The counting argument in [21] shows \[
\#(\mu_1, \ldots, \mu_s, x_1, \ldots, x_s) \leq C^{-\theta n}
\]
and \[
\#(\nu_1, \ldots, \nu_t) \#(n_1, \ldots, n_t) \#(y_1, \ldots, y_t) \leq e^{\tau(\delta)n + C\theta\alpha^{-1}R},
\]
where $\tau(\delta) \to 0$ as $\delta \to 0$. [21] Lemma 5.3 gives $r_1 + \cdots + r_t \geq \alpha m/2$. Taking contributions from all $n$-classes into consideration, \[
|\Delta \cap [a^* - \varepsilon, a^*]| \leq \varepsilon \sum_{m,s,t} \sum_{R \geq \alpha m/2} \sum_{r_1 + \cdots + r_t = R} |E_n(\ast) \cap [a^* - \varepsilon, a^*]|
\leq \varepsilon e^{\tau(\delta)n} \sum_{R \geq \alpha m} \exp \left( -\frac{R}{6} \right) \leq \varepsilon e^{-\alpha n/8}.
\]

Let \[
n_0(\varepsilon) = \frac{1}{2 \log C_0} \log \left( \frac{2\varepsilon}{\kappa_0} \right).
\]
The next lemma indicates that no parameter is deleted from $[a^* - \varepsilon, a^*]$ at least up to step $[n_0(\varepsilon)/20]$, namely $[a^* - \varepsilon, a^*] \subset \Delta_n$ holds for every $0 \leq n \leq [n_0(\varepsilon)/20]$.

**Lemma 4.2.** Let $a_0 \in [a^* - \varepsilon, a^*]$, and let $\zeta_0$ be a nice critical approximation of $f_{a_0}$ of order $\xi$. Then $f_{a_0}^n \zeta_0 \notin I(\delta)$ holds for every $1 \leq n < \min (n_0(\varepsilon), 20\xi)$.

Therefore \[
|\Delta \cap [a^* - \varepsilon, a^*]| = |\Delta_0 \cap [a^* - \varepsilon, a^*]| - \sum_{n=1}^{\infty} |(\Delta_{n-1} \setminus \Delta_n) \cap [a^* - \varepsilon, a^*]|
\]
\[
= \varepsilon - \sum_{n>[n_0(\varepsilon)/20]} |(\Delta_{n-1} \setminus \Delta_n) \cap [a^* - \varepsilon, a^*]| \geq \varepsilon \left( 1 - \sum_{n>[n_0(\varepsilon)/20]} e^{-\alpha n} \right).
\]
Since $n_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$, we obtain $\lim_{\varepsilon \to 0} \varepsilon^{-1} |\Delta \cap [a^* - \varepsilon, a^*]| = 1$ as desired. \qed
4.8. Parameter dependence of nice critical approximations. The rest of this section is entirely devoted to the proof of Proposition 4.3 and Lemma 4.2. A key ingredient is a deformation of a quasi critical approximation, developed in [21] Section 4.5 for dealing with the parameter dependence of nice critical approximations.

We begin by relaxing the definition of nice critical approximations as follows. Let \( \zeta \) be a critical approximation of order \( n \) on a horizontal curve \( \gamma \). Let \( u \) denote any unit vector at \( f^{-[\theta n]} \zeta \) such that \( Df^{[\theta n]}u \) is tangent to \( \gamma \). We say \( \zeta \) is a quasi critical approximation of order \( n \) on \( \gamma \) if \( u \) is \( \kappa_0^{\frac{1}{2}} \)-expanding up to time \( [\theta n] \).

Hypothesis for the rest of Sect.4.8. Let \( \hat{a} \in [a^* - \varepsilon_0, a^*] \). Write \( f \) for \( f_{\hat{a}} \). Let \( \gamma \) be a \( C^2(b) \)-curve in \( I(\delta) \). Let \( \zeta \) be a quasi critical approximation of order \( n \) on \( \gamma \), with \( u \) the same meaning as above. Assume:

- \( \| Df^i(f\zeta) \| \geq 1 \) for \( 1 \leq i \leq n \);
- \( u \) is \( \kappa_0^{\frac{1}{2}} \)-expanding and \( \delta/160 \)-regular, both up to time \( [\theta n] \).

Let \( r \) denote the point of intersection between \( H \) and the long stable leaf of order \( [\theta n] \) through \( \xi \). Let \( I \subset H \) denote the horizontal of length \( 2\kappa_0^{3\theta n} \) centered at \( r \). By [21] Lemma 4.1, \( f_{\hat{a}}^{[\theta n]}l \) is a \( C^2(b) \)-curve, and there exists a quasi critical approximation of order \( n \) on it, denoted by \( \zeta(\hat{a}) \) for which \( |\zeta - \zeta(\hat{a})| \leq (Cb)^{4n} \) holds. In addition, this picture persists, for a small variation of parameters within the interval

(17) \[ I_n(\hat{a}) = [\hat{a} - \kappa_n^n, \hat{a} + \kappa_n^n]. \]

By [21] Lemma 4.2, for all \( a \in I_n(\hat{a}) \), \( f_{\hat{a}}^{[\theta n]}l \) is a \( C^2(b) \)-curve. By [21] Proposition 4.1, there exists a quasi critical approximation of order \( n \) of \( f_{\hat{a}} \) on it, which we denote by \( \zeta(a) \).

Definition 4.1. The map \( a \in I_n(\hat{a}) \rightarrow \zeta(a) \) is called a deformation of \( \zeta \).

The next lemma states that the “speed” of the deformation as \( a \) sweeps \( I_n(\hat{a}) \) is uniformly bounded. We use “‘...” to denote the differentiation with respect to \( a \).

Lemma 4.3. ([21] Proposition 4.2.) The deformation \( a \in I_n(\hat{a}) \rightarrow \zeta(a) \) of \( \zeta \) is \( C^3 \) and for all \( a \in I_n(\hat{a}) \), \( \max \left( \| \dot{\zeta}(a) \|, \| \ddot{\zeta}(a) \| \right) \leq \kappa_0^{10\log \delta} \).

4.9. Evolution of critical curves. In the next proposition we assume \( \hat{a} \in \Delta_{n-1}, 0 < \nu < 20n \) and \( \hat{\zeta} \) is a nice critical approximation of \( f_{\hat{a}} \) of order \( n \), for which \( (G)_{\nu-1} \) holds and \( f_{\hat{a}}^{\nu} \hat{\zeta} \) is free. Define

\[ J_{\nu}(\hat{a}, \hat{\zeta}) = [\hat{a} - \Theta_{\nu}(\hat{\zeta}), \hat{a} + \Theta_{\nu}(\hat{\zeta})]. \]

Proposition 4.4. ([21] Section 5] There exist an integer \( m \) and a quasi critical approximation \( \zeta \) of order \( m \) of \( f_{\hat{a}} \) such that:

(i) \( |f_{\hat{a}}^{\nu} \hat{\zeta} - f_{\hat{a}}^{\nu} \hat{\zeta}| \leq (C\nu)^{2\nu} \);
(ii) for the deformation \( a \in I_m(\hat{a}) \rightarrow \zeta(a) \) of \( \zeta \), write \( f_{\hat{a}}^{\nu} \zeta(a) = \zeta(\nu)(a) \). Then:

(a) \( J_{\nu}(\zeta, \hat{a}) \subset I_m(\hat{a}); \)
(b) the set \( \{ \zeta(\nu)(a) : a \in J_{\nu}(\hat{a}, \hat{\zeta}) \} \) is a horizontal curve;
(c) \( \| \zeta(\nu)(a) - \zeta(\nu)(b) \| \approx \| w_{\nu}(\hat{\zeta}) \| | a - b | \ll 1 \) for all \( a, b \in J_{\nu}(\hat{a}, \hat{\zeta}) \).
4.10. **Proof of Proposition 4.3.** We choose each \( J_{k,i} \) so that it has the form \( J_{k,i} = J_{\nu_k}(a_{k,i}, \zeta_{k,i}) \), where \( a_{k,i} \in E_n(*) \cap [a^* - \varepsilon, a^*] \) and \( \zeta_{k,i} \) is some responsible critical approximation of \( f_{a_{k,i}} \). In what follows we describe how to choose \((a_{k,i}, \zeta_{k,i})\).

The first level. Start with \( k = 1 \). We describe how to choose \((a_{1,i}, \zeta_{1,i})\) such that (a) holds with \( k = 1 \). First, choose arbitrary \( a_{1,1} \in E_n(*) \cap [a^* - \varepsilon, a^*] \). Let \( \zeta_{1,1} \) denote any responsible critical approximation of \( f_{a_{1,1}} \). We show

\[
E_n(*) \cap (J_{1,1} \setminus e^{-r_1/3} J_{1,1}) = \emptyset.
\]

If \( J_{1,1} \) covers \( E_n(*) \), then the desired inclusion follows. Otherwise, choose \( a_{1,2} \in E_n(*) - J_{1,1} \). We claim that

\[
J_{1,1} \cap J_{1,2} = \emptyset.
\]

If \( J_{1,1} \cap J_{1,2} \) covers \( E_n(*) \), then the desired inclusion follows. Otherwise, choose \( a_{1,3} \in E_n(*) - J_{1,1} \cup J_{1,2} \). Repeat this. As the length of these intervals are uniformly bounded from below, there must come a point when the inclusion is fulfilled.

Below we sketch the proofs of (18) and (19). To ease notation, write \( a_i := a_{1,i}, \zeta_i := \zeta_{1,i} \) and \( J_i = J_{1,i}, i = 1, 2 \).

**Sketch of the proof of** (18). Choose an integer \( m \), a quasi critical approximation \( \zeta \) of \( f_{a_1} \) of order \( m \), and its deformation \( a \in I_{n}(a_1) \to \zeta(a) \) for which the conclusions of Proposition 4.4 hold up to time \( \nu_1 \). In fact, (Z4), (Z5) allow us to choose such a deformation so that the following holds:

- \( |f_{\tilde{a_1}}^m \zeta_1 - f_{\tilde{a_1}}^m \zeta(a_1)| \leq e^{-r_1} \);
- if \( a \in J_1 \cap E_n(*) \) and \( x \) is any responsible critical approximation of \( f_a \), then \( |f_a^m x - f_a^m \zeta(a)| \ll e^{-r_1} \).

The second item states that, although responsible critical approximations for a single parameter \( a \) are not unique, all of their positions at time \( \nu_1 \) are well-approximated by that of \( f_a^m \zeta(a) \).

Now, let \( z_1 \) denote the binding point of order \( \nu_1 \) for \( f_{\tilde{a_1}}^m \zeta_1 \) and let \( a_{\nu_1}(a_1) \to z_1(a) \) denote its deformation. (Z6) allows us to show that this deformation satisfies:

- \( |z_1 - z_1(a_1)| \leq e^{-r_1} \);
- if \( a \in J_1 \cap E_n(*) \) and \( x \) is any responsible critical approximation of \( f_a \), with \( y \) a binding point for \( f_a^m x \), then \( |y - z_1(a)| \ll e^{-r_1} \).

The second item states that, although binding points are not unique, they are well approximated by \( z_1(a) \).

These four conditions altogether imply (18). To see this, suppose that this is not the case and let \( a \in J_{1,1} \setminus e^{-r_1/3} J_{1,1} \), \( a \in E_n(*) \). Let \( x \) denote any critical approximation responsible for \( a \). Let \( y \) denote any binding point for \( f_a^m x \). The triangle inequality gives

\[
|f_a^m x - y| \geq |f_a^m \zeta(a) - f_{\tilde{a_1}}^m \zeta(a_1)| - |f_a^m \zeta(a) - f_a^m x| - |f_{\tilde{a_1}}^m \zeta(a_1) - f_{\tilde{a_1}}^m \zeta_1|
\]

\[
- |f_{\tilde{a_1}}^m \zeta_1 - z_1| - |z_1 - z_1(a)| - |z_1(a) - y|,
\]

where, for the last term, \( z_1(a) \) makes sense, because of \( J_1 \subset I_{\nu_1}(a_1) \). On the first term, Proposition 4.4 and Lemma 4.1 give

\[
|f_{\tilde{a_1}}^m \zeta(a_1) - f_{\tilde{a_1}}^m \zeta(a)| \approx \|w_{\nu_1}(\zeta_1)\| |a_1 - a| \gg e^{-r_1}.
\]
The remaining four terms are \( \leq e^{-r_1} \). It follows that \( |f^*_a x - y| \geq e^{-r_1} \). This yields a contradiction to the assumption that \( x \) is responsible for \( a \). Hence \( a \notin E_n(*) \) holds and we get (18).

**Sketch of the proof of (19).** In the discussions to follow, we need to introduce critical parameters [21]. For the purpose of this we make the following assumption and observation. Let \( \hat{a} \in E_n(*) \) and let \( \hat{\zeta} \) denote any critical approximation responsible for \( a \). Let \( z \) denote any binding point for \( f^*_\hat{a} \hat{\zeta} \), and let \( a \in I_n(\hat{a}) \to z(a) \) denote its deformation. Take an integer \( m \), a quasi critical approximation \( \zeta \) of \( f^*_\hat{a} \) of order \( m \), and its deformation \( a \in I_m(\hat{a}) \to \zeta(a) \) for which the conclusions of Proposition 4.3 hold up to time \( \nu_k \). The “speed” of \( z(a) \) as \( a \) sweeps the interval \( I_n(\hat{a}) \) is bounded from above by in Lemma 4.3. On the other hand, the “speed” of \( \zeta_{\nu_k}(a) \) as \( a \) sweeps the interval \( J_{\nu_k}(\hat{a}, \hat{\zeta}) \) is much faster, by Proposition 4.3. From the proposition, \( J_{\nu_k}(\hat{a}, \hat{\zeta}) \subset I_n(\hat{a}) \) holds. Hence, the comparison of the speeds and Lemma 4.1 together imply that there exists a unique parameter \( c_0 \in e^{-r_k/3} \cdot J_{\nu_k}(\hat{a}, \hat{\zeta}) \) such that the \( x \)-coordinate of \( \zeta_{\nu_k}(c_0) \) coincides with that of \( z(c_0) \).

**Definition 4.2.** The \( c_0 \) is called a critical parameter in \( J_{\nu_k}(\hat{a}, \hat{\zeta}) \).

A proof of (19) is outlined as follows. Let \( c_0, c'_0 \) denote the critical parameters in \( J_{1,1}, J_{1,2} \) respectively. Suppose that (19) does not hold. Then, from a distortion argument, \( |J_{1,1}| \approx |J_{1,2}| \) follows. As \( a_{1,2} \notin J_{1,1} \), this implies \( c_0 \neq c'_0 \). In addition, it is possible to extend the domain of definition of the deformation of \( \zeta_{1,1} \) to the larger interval \( J_{1,1} \cup J_{1,2} \), so that all the above properties of the deformation continue to hold. As \( a_{1,2} \notin J_{1,1} \), the argument used in the proof of (18) gives \( a_{1,2} \notin E_n(*) \). This is a contradiction. Hence (19) holds.

**From level \( k-1 \) to \( k \).** Having chosen \((a_{k-1,i}, \zeta_{k-1,i})_i \) and the corresponding intervals \((J_{k-1,i})_i \), we choose \((a_{k,j}, \zeta_{k,j})_j \) as follows. For each \( J_{k-1,i} \), in the same way as the proof of (18) it is possible to choose a finite number of parameters \( a_{k,1}, a_{k,2}, \cdots \in E_n(*) \cap [a^* - \varepsilon, a^*] \) such that \( J_{k,1}, J_{k,2}, \cdots \) are pairwise disjoint and altogether cover \( E_n(*) \cap e^{-r_k/3} \cdot J_{k-1,i} \). Now the issue is to show the inclusion \( \bigcup J_{k,j} \subset 2e^{-r_k/3} \cdot J_{k-1,i} \). This is a consequence of the fact that the center \( a_{k,j} \) of each \( J_{k,j} \) belongs to \( e^{-r_k/3} \cdot J_{k-1,i} \), and any \( J_{k,j} \) does not contain the critical parameter in \( J_{k-1,i} \).

**Lemma 4.4.** For every \( i \), \( \Theta_{n}(\zeta_{1,i}) \leq 2\varepsilon \).

As the intervals \((J_{1,i})_i \) are pairwise disjoint and intersect \([a^* - \varepsilon, a^*] \), Lemma 4.4 gives \( \sum_i |J_{1,i}| \leq 3\varepsilon \). This proves (c).

It is left to prove Lemma 4.4. We use the following which can be proved by slightly extending the arguments in Sect.4.1 and using the definition of quasi critical approximations.

**Claim 4.1.** Let \( \zeta \) be a quasi critical approximation of order \( n \) of \( f^*_{a^*} \). There exists a critical point \( z \) of \( f^*_{a^*} \) such that \( |\zeta - z| \leq (Ck)^{\frac{1}{6^n}} \).

**Proof of Lemma 4.4.** Take an integer \( m \), a quasi critical approximation \( \zeta \) of \( f^*_{a_{1,i}} \) of order \( m \), and its deformation \( a \in I_m(a_{1,i}) \to \zeta(a) \) for which the conclusions of Proposition 4.3 hold up to time \( \nu_1 \). If \( |J_{1,1}| > 2\varepsilon \), then \( a^* \in J_{1,1} \) holds, because of \( a_{1,1} \in (a^* - \varepsilon, a^*) \). Then \( \zeta(a^*) \) makes sense and we have \( |f^*_{a_{1,i}} \zeta_{1,i} - \zeta_{\nu_1}(a^*)| \leq |f^*_{a_{1,i}} \zeta_{1,i} - \zeta_{\nu_1}(a_{1,i})| + |\zeta_{\nu_1}(a_{1,i}) - \zeta_{\nu_1}(a^*)| \ll 1 \). As \( \nu_1 \) is a return time, \( f^*_{a_{1,i}} \zeta_{1,i} \in I(\delta) \) holds. It follows that \( \zeta_{\nu_1}(a^*) \) is near \( I(\delta) \). On the other hand,
Proposition 4.4 and Claim 4.1 together imply \( \zeta_0(a^*) \in \{(x, y) : |x| \geq 4/5\} \). A contradiction arises.

4.11. Proof of Lemma 4.2. We argue by induction on \( n \). Let \( 1 < k \leq \min(n_0(\varepsilon), 20\xi) \) and assume \( f^k_{a^0} \zeta_0 \notin I(\delta) \) for \( 1 \leq i \leq k-1 \). Then \( f^k_{a^0} \zeta_0 \) is free. Take an integer \( m \), a quasi critical approximation \( \zeta \) of \( f^k_{a^0} \) of order \( m \), and its deformation \( a \in I_m(\hat{a}) \to \zeta(a) \) for which the conclusions of Proposition 4.4 hold up to time \( k \). The definition of \( J_k(a_0, \zeta_0) \) and (10) give

\[
|J_k(a_0, \zeta_0)| \geq \kappa_0 C_k^{-2k} \geq 2\varepsilon.
\]

As \( a_0 \in [a^* - \varepsilon, a^*] \), \( a^* \in J_k(a_0, \zeta_0) \) holds. Hence, \( \zeta(a^*) \) makes sense and we have \( |f^k_{a^0} \zeta_0 - \zeta_k(a_0)| \leq |f^k_{a^0} \zeta_0 - \zeta_k(a_0)| + |\zeta_k(a_0) - \zeta_k(a^*)| \ll 1 \). Proposition 4.4 and Claim 4.1 give \( \zeta_k(a^*) \notin \{(x, y) : |x| \leq 4/5\} \). Hence \( f^k_{a^0} \zeta_0 \notin I(\delta) \) holds, recovering the assumption of the induction. \( \square \)

4.12. Proof of Proposition 4.1. Write \( f \) for \( f_{a^*} \). Let \( r \) denote the point of the quadratic tangency near \((0,0)\). Let \( S \) denote the lenticular compact domain in \( I(\delta) \) bounded by the segment in \( W^a \) and the parabola in \( W^s(Q) \) containing \( r \) (cf. Figure 1). By (M1), all points in \( fS \) do not return to \( R_0 \) under positive iteration, and thus they are expanding. By Proposition 2.3 \( fS \) is foliated by long stable leaves. Note that the leaf through \( fr \) contains the boundary of \( R_0 \).

Temporarily, let us adopt the following definition. Let \( \gamma \) be a \( C^2(\xi) \)-curve in \( W^s(Q) \) stretching across \( I(\delta) \). We say \( \zeta \in \gamma \) is a critical point on \( \gamma \) if \( z \in S \), and the long stable leaf through \( fz \) is tangent to \( W^s(Q) \) at \( fz \). For the proof of Proposition 4.1 we approximate any critical approximation by a critical point. Since the orbit of every critical point do not return to \( R_0 \), the conclusion of the proposition follows.

Lemma 4.5. Let \( \gamma \) be a \( C^2(\xi) \)-curve in \( W^s(Q) \) stretching across \( I(\delta) \). There exists a unique critical point on \( \gamma \). In addition, for every \( n \geq M \) there exists a critical approximation of order \( n \) on \( \gamma \) within the distance \( (Cb)^{2n} \) from the critical point.

Proof. By Remark 2.1 any long stable leaf at the right of \( \Gamma(fr) \) intersects \( f\gamma \) at two points, or else it is tangent to \( f\gamma \) and the point of tangency is quadratic. There exists only one leaf for which the latter holds, for otherwise two distinct leaves intersect each other, a contradiction to the remark below Lemma 2.3. The pull-back of the point of tangency is a critical point on \( \gamma \), denoted by \( \zeta \). Hence, the first statement holds.

Take \( z \in \gamma \) with \( |z| = b^\frac{n}{2} \), and write \( fz = (x_0, y_0) \). Represent the two long stable leaves as graphs of functions on \([-\sqrt{b}, \sqrt{b}]\): \( \Gamma_n(z) = \{(x(y), y)\} \) and \( \Gamma_n(f\zeta) = \{(\tilde{x}(y), y)\} \). Since the Hausdorff distance between \( \Gamma_n(f\zeta) \) and \( \Gamma(f\zeta) \) is \( \leq (Cb)^n \), Lemma 2.2 gives \( |x(y_0) - \tilde{x}(y_0)| = |x_0 - \tilde{x}(y_0)| \approx b^\frac{n}{2} \). Since \( c_n \) is Lipschitz, it follows that \( |x(y) - \tilde{x}(y)| \approx Cb^\frac{n}{2} \) for all \( y \in [-\sqrt{b}, \sqrt{b}] \). Hence \( f^{-1}\Gamma_n(fz) \) intersects \( \gamma \) at two points within \( (Cb)^{2n} \) from \( \zeta \). This and Remark 2.1 together imply the second statement. \( \square \)

Let \( \zeta_0 \) denote the critical point which is closest to \( Q \) in the Riemannian distance in \( W^s(Q) \). Let \( G \) denote the segment in \( W^s(Q) \) with endpoints \( Q, fz_0 \). A proof of the next lemma is given in Appendix A.1.

Lemma 4.6. For every \( n \geq 0 \), any component of \( f^nG \cap I(\delta) \) is a \( C^2(b) \)-curve.

We are in position to finish the proof Proposition 4.1. If \( |f^{-|\theta n|}\zeta - fr| \leq 1/10 \), then let \( m = |\theta n| - 1 \). Otherwise, let \( m = |\theta n| \). Then \( f^{-m}\zeta \) is expanding. Let \( z \) denote the point of
intersection between the long stable leaf of order $m$ through $f^{-m}\zeta$ and $G$. It is possible to take a curve $\gamma$ in $G$ extending both sides around $z$ to length $b^{2\overline{\alpha}}$. For otherwise the contraction along the long stable leaf gives $f^mQ \in I(\delta)$, a contradiction because $Q$ is a fixed point and $Q \notin I(\delta)$. By the definition of $m$, $\gamma$ avoids the turn near $f^\zeta_0$, and hence is $C^2(\delta)$. Then $f^m\gamma$ is a $C^2\gamma$-curve extending both sides around $f^mz$ to length $\geq b^{2\overline{\alpha}}$. By Lemma 3.2 there exists a critical approximation $\bar{z}$ of order $n$ on $f^m\gamma$ such that $|\zeta - \bar{z}| \leq (Cb)^{2\overline{\alpha}}$ holds. By Lemma 4.5 and Lemma 4.6 there exist a $C^2\gamma$-curve $\gamma'$ in $W^u$ containing $f^m\gamma$ and stretching across $I(\delta)$, and a critical point $\zeta''$ on $\gamma'$ such that $|\bar{z} - \zeta''| \leq (Cb)^{2\overline{\alpha}}$. It follows that $|f^i\zeta - f^i\zeta''| \leq (Cb)^{2\overline{\alpha}}$ for $1 \leq i < 20n$. As the orbit of $\zeta''$ is out of $R_0$, the claim holds.

**Standing hypothesis for the rest of the paper:** $f \in \{f_a : a \in \Delta \cap (a**, a^*)\}$. Here, $a**$ is the one defined in Introduction.

5. **Dynamics on the unstable manifold**

In this paper we develop a one-dimensional analysis on the unstable manifold $W^u$. In Sect 5.1, we define a critical set $C$ in $W^u$, as a set of accumulation points of critical approximations, and use it as a spine to structure the dynamics. Each element of $C$ is called a critical point. In Sect 5.2, 5.3 we prove some key estimates on critical points. In Sect 5.4 we identify a geometric structure of $W^u$ near the critical set.

**Notation.** For $z \in W^u$, let $t(z)$ denote any unit vector tangent to the unstable manifold at $z$. The boundaries of $R_0$ in $W^u$ is called unstable sides, and denoted by $\partial R_0$. Let $\partial R_n := f^n(\partial R_0)$.

5.1. **The critical set.** In the case $W^u = W^u(Q)$, fix a fundamental domain $\mathcal{F}$ in $W^u_{loc}(Q)$. For $z \in \mathcal{F}$, define a sequence $n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq n_3 < \cdots$ inductively as follows: $n_1$ is the smallest such that $f^{n_1}z \in I(\delta)$ and $p_1$ is the bound period of $f^{n_1}z$; $n_k \geq n_k - 1 + p_k - 1$ is the smallest such that $f^{n_k}z \in I(\delta)$, and $p_k$ is the bound period of $f^{n_k}z$. From the fact that $Q$ is a fixed saddle, it follows that this sequence is defined indefinitely, or else there exists an integer $m$ such that $Df^m t(z)$ is in critical position relative to critical approximations of arbitrarily high order. If the latter case occurs, we let $f^m z \in C$. Since each such point is isolated in $W^u$, $C$ is a countable set. In the case $W^u = W^u(P)$, $C$ is constructed in the same way, with $Q$ replaced by $P$.

**Proposition 5.1.** For each $\zeta \in C$ we have:

(a) $\|w_n(\zeta)\| \geq e^{\lambda(n-1)}$ for $n \geq 1$;
(b) $\|w_j(\zeta)\| \geq e^{-2\alpha i} \|w_i(\zeta)\|$ for $1 \leq i < j$;
(c) there exists a monotone increasing function $\chi : \mathbb{N} \to \mathbb{N}$ such that for each $n$, $(1 - \sqrt{\alpha})n \leq \chi(n) \leq n$ and $\|w_{\chi(n)}(\zeta)\| \geq \delta \|w_k(\zeta)\|$ for $1 \leq k < \chi(n)$;
(d) the long stable leaf through $f^\zeta$ is tangent to $W^u$ at $f^\zeta$ and the tangency is quadratic.

**Proof.** By definition, for each $\zeta \in C$ there exists a strictly increasing sequence $m_1 < m_2 < \cdots$ of integers and a sequence $\zeta_{m_1}, \zeta_{m_2}, \cdots$ of critical approximations with good critical behavior, such that $\zeta_{m_\ell}$ is of order $m_\ell$, and $\zeta_{m_\ell} \to \zeta$ as $\ell \to \infty$. (a) (b) (c) are direct consequences of this convergence. By the definition of $C$ and (ii) in Proposition 2.1, $t(\zeta)$ is contracted exponentially by positive iterations. Thus $t(f^\zeta)$ is tangent to $f^\Gamma(\zeta)$. By Remark 2.1, this tangency is quadratic, and (d) holds.
and a genuine critical point, and thus the estimates are available entirely on Proposition 2.1 which is proved similarly. The difference is that \( \zeta \) and a fold period \( \gamma \) does not intersect the vertical boundary of \( V \).

For each \( \zeta \) at the right and the other at the left. For simplicity, let us denote both by \( \zeta \) and \( \gamma \).

As before, write \( \Gamma(f_\zeta) = \{(x, y) : |y| \leq \sqrt{b}\} \), and for each \( k \geq M \), let \( V_k = \{(x, y) : |x - x(y)| \leq D_k(\zeta)/2, |y| \leq \sqrt{b}\} \). If \( fz \in V_k \setminus V_{k+1} \), define a bound period \( p = p(\zeta, z) \) by
\[
p = \chi(k),
\]
and a fold period \( q = q(\zeta, z) \) by
\[
q = \min \left\{ i \in [1, p) : |\zeta - z|^\beta \cdot \|w_{j+1}(\zeta)\| \geq 1 \text{ for } i \leq j < p \right\}.
\]

**Proposition 5.2.** Let \( z \in \gamma \setminus \{\zeta\} \) and let \( t(z) \) denote any unit vector tangent to \( \gamma \) at \( z \). Then:

(a) \( p \leq \log |\zeta - z|^{-\frac{3}{\lambda}} \);
(b) \( q \leq C\beta p \);
(c) \( |f^i_\zeta - f^i_z| \leq e^{-2\alpha p} \text{ for } 1 \leq i \leq p \);
(d) \( |\zeta - z| \leq \|Df^p t(z)\| \leq |\zeta - z|^{1-\beta} \);
(e) \( \|Df^p t(z)\| \geq |\zeta - z|^{-1+\frac{1}{\log 10}} \geq e^{\frac{\lambda p}{2}} \);
(f) \( \|Df^q t(z)\| \geq (\delta/10) \|Df^q t(z)\| \text{ for } 0 \leq i < p \);
(g) \( \|Df^q t(z)\| \approx |\zeta - z|\|w_i(\zeta)\| \text{ for } q \leq i \leq p \);
(h) \( \|Df^q t(z)\| < 1 \text{ for } 1 \leq i \leq q \).

**5.3. Critical partitions.** Using the family \((V_k)_k\) of vertical strips, we construct a critical partition of \( \gamma \) as follows. By Remark 2.1, \( \gamma \cap f^{-1}(V_k \setminus V_{k+1}) \) consists of two components, one at the right \( \zeta \) and the other at the left. For simplicity, let us denote both by \( \gamma_k \). If \( f\gamma_k \) does not intersect the vertical boundary of \( V_k \), then we take \( \gamma_k \) together with the adjacent \( \gamma_{k+1} \). We cut each \( \gamma_k \) into \([e^{3\lambda k}]\)-number of curves of equal length, and denote them by \( \gamma_{k,s} \) \((s = 1, 2, \cdots)\).

A proof of the next lemma is given in Appendix A.2.

**Lemma 5.1.** For each \( \gamma_{k,s} \) we have:

(a) \( f^{\lambda k}_{\gamma_{k,s}} \) is a \( C^2 \)-curve of length \( \geq e^{-4\lambda k} \);
(b) For all \( \xi, \eta \in \gamma_{k,s} \),
\[
\log \frac{\|Df^{\lambda k}_{\gamma_{k,s}}(\xi)\|}{\|Df^{\lambda k}_{\gamma_{k,s}}(\eta)\|} \leq C |f^{\lambda k}_{\gamma_{k,s}}(\xi) - f^{\lambda k}_{\gamma_{k,s}}(\eta)|^{C\alpha}.
\]
5.4. Geometry of critical regions. We identify a geometric structure of critical regions, close the one depicted in ([24] Sect.1.2). Let $C^{(0)} = \{(x,y) \in R_0 : |x| \leq \delta \}$. 

**Proposition 5.3.** There exists a nested sequence $C^{(0)} \supset C^{(1)} \supset C^{(2)} \supset \cdots$ such that the following holds for $k = 0, 1, 2, \cdots$:

1. $C^{(k)}$ has a finite number of components called $Q^{(k)}$ each one of which is diffeomorphic to a rectangle. The boundary of $Q^{(k)}$ is made up of two $C^2(b)$-curves of $\partial R_k$ connected by two vertical lines: the horizontal boundaries are $\approx \min(2\delta, \kappa_b^k)$ in length, and the Hausdorff distance between them is $O(b^k)$;

2. On each horizontal boundary $\gamma$ of each component $Q^{(k)}$ of $C^{(k)}$, there is a critical point located within $O(b^k)$ of the midpoint of $\gamma$.

3. $C^{(k)}$ is related to $C^{(k-1)}$ as follows: $Q^{(k-1)} \cap R_k$ has at most finitely many components, each of which lies between two $C^2(b)$ subsegments of $\partial R_k$ that stretch across $Q^{(k-1)}$ as shown in FIGURE 7. Each component of $Q^{(k-1)} \cap R_k$ contains exactly one component of $C^{(k)}$.

4. Let $\Xi^{(k)}$ denote the set of critical points on the horizontal boundaries of $\bigcup_{j=0}^{k} C^{(j)}$. Then $C = \bigcup_{k \geq 0} \Xi^{(k)}$.

The rest of this section is entirely devoted to an inductive proof of (S1), (S2), (S3). (S4) is a direct consequence of this. In Section 5.5, we first describe a structure of the induction, to make clear how to proceed from one to the next step. In Section 5.6, we treat an initial step of the induction. In Section 5.7, we treat a generic step.

5.5. Structure of induction. (S1), (S2) for $k = 0$ are trivial. (S3) for $k = 0$ is an empty condition. Let us say that $\partial R_0$ is controlled up to time $0$ by $\Xi^{(0)}$. Using the critical partition in Sect.5.3, we assign to all points in $\partial R_0 \cap I(\delta)$ their binding points in $\Xi^{(0)}$ and bound periods. This makes sense to refer to points in $\partial R_1$ as being free or bound.

**Definition 5.1.** Let $j \geq 1$ and assume:

1. $(I)_{j-1}$: (S1-3) hold for $0 \leq k \leq j - 1$, and $\partial R_0$ is controlled up to time $j - 1$ by $\Xi^{(j-1)}$.

Under this assumption, we say:

- a segment in $\partial R_j$ is a free segment if all points on it are free;
- a maximal free segment in $\partial R_j$ is a free segment in $\partial R_j$ which is not contained in any other free segment in $\partial R_j$;
- a bound segment in $\partial R_j$ is any connected component of $\partial R_j \setminus \{\text{maximal free segment in } \partial R_j\}$.

In the sequel we need two curvature-related estimates.

**Lemma 5.2.** Any free segment in $\partial R_j$ is a $C^2(b)$-curve.

**Proof.** Let $\gamma$ be a free segment in $\partial R_j$. Then $1 \geq C\delta\|Df^{-n}(z)t(z)\|$ holds for all $z \in \gamma$ and $n > 0$. Hence, the curvature of $\gamma$ is $\leq \sqrt{b}$, by the curvature estimate in [21] Lemma 2.4 and the boundedness of the curvature of $W^s_{loc}$. The inequality for $n = -1$ implies that the slopes of the tangent directions of $\gamma$ are $\leq \sqrt{b}$. $\square$

**Lemma 5.3.** For any free segment $\gamma$ and $n \geq 0$, the curvature of $f^{-n}\gamma$ is everywhere $\leq 5^{3n}$.
Proof. For $z \in \gamma$, let $\kappa_{-n}(z)$ denote the curvature of $f^{-n}\gamma$ at $f^{-n}z$. If $f^{-n}z$ is free, then $\kappa_{-n}(z) \leq \sqrt{b}$, by Lemma 5.2. Otherwise, let $m < -n$ denote the largest integer such that $f^mz$ is a free return. [21] Lemma 2.4] and $\kappa_m(z) \leq \sqrt{b}$ give

$$\kappa_{-n}(z) \leq \sqrt{b}(Cb)^{-n-m} \frac{\|Df^mt(z)\|^3}{\|Df^{-nt}(z)\|^3} + \sum_{i=1}^{-n-m} (Cb)^i \frac{\|Df^{-ni}t(z)\|^3}{\|Df^{-nt}(z)\|^3}.$$  

Since $z$ is free, $\|Df^{-n-i}t(z)\| \leq 10/\delta$, and thus for $1 \leq i \leq -n - m$,

$$\frac{\|Df^{-n-i}t(z)\|}{\|Df^{-nt}(z)\|} \leq 10 \cdot 5^{n-1}. $$  

Replacing all these in the above inequality, we obtain $\kappa_{-n}(z) \leq 5^n$.\hfill \Box

Definition 5.2. Suppose that (S1-3) hold for every $0 \leq k \leq j$. We say $\partial R_0$ is controlled up to time $j$ by $\Xi^{(j)}$, if for any maximal free segment $\gamma$ in $\partial R_j$ there exist a horizontal curve $\tilde{\gamma}$ which contains $\gamma$ and a critical point $\zeta \in \Xi^{(j)}$ on $\tilde{\gamma}$.

At step $j - 1$ of the induction, we show the implication $(I)_{j-1} \implies (I)_j$. Then, for all points in $\partial R_j \cap I(\delta)$ which are free, we assign their binding points as follows. For a maximal free segment $\gamma$ in $\partial R_j$, take $(\tilde{\gamma}, \zeta)$ as in Definition 5.2. We use $\zeta$ as a common binding point for points in $\gamma \cap I(\delta)$. Their bound periods are given by considering the critical partition of $\tilde{\gamma}$. This makes sense to refer to points in $\partial R_{j+1}$ as being free or bound.

5.6. From step 0 to step $N$. Let $1 \leq j \leq N$ and suppose $(I)_{j-1}$. The bound parts of $\partial R_j$ do not come back to $C^{(0)}$, and $\partial R_j \cap I(\delta)$ consists of $C^2(\delta)$ curves, each of which admits a critical point. Define $C^{(j)} = R_j \cap C^{(0)}$. $(I)_j$ obviously holds.

5.7. From step $2^mN$ to $2^{m+1}N$. The same argument cannot be continued indefinitely, because bound segments return to $I(\delta)$. To deal with these returns, we need the help of critical points.

Lemma 5.4. For each $\zeta \in C$ there exist positive integers $n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq n_3 < \cdots$ such that, for each $n_t$, $f^{n_t}\zeta \in I(\delta)$, and there exists a critical approximation $\hat{z}_t$ relative to which $w_{n_t}(\zeta)$ is in admissible position, with $|f^{n_t}\zeta - \hat{z}_t| \geq e^{-\alpha n_t}$.

The integers $n_t, n_2, \cdots$ are called free return times of $\zeta$.

Proof. We argue by induction. First, let $n_1 = \min\{n > 0: f^n\zeta \in I(\delta)\}$. As $I(\delta)$ is open, $n_1 = \min\{n > 0: f^n\zeta_{m_t} \in I(\delta)\}$ holds for all sufficiently large $\ell$. Let $z_{m_t}$ denote the binding point for $f^{n_t}\zeta_{m_t}$, with a bound period $p_{m_t}$. Passing to subsequences, we may assume that both converge as $\ell \to \infty$. Define $\hat{z}_1, p_1$ to be the corresponding limits.

Given $(n_k, \hat{z}_k, p_k)$, define $n_{k+1} = \min\{n \geq n_k + p_k: f^n\zeta \in I(\delta)\}$. Passing to subsequences again, we may assume that $f^{n_{k+1}}\zeta_{m_{k'}}$ is a free return to $I(\delta)$, with a binding point $z_{m_{k'}}$, and a bound period $p_{m_{k'}}$, both converging as $\ell \to \infty$. Define $\hat{z}_{k+1}, p_{k+1}$ to be the corresponding limits.\hfill \Box

Definition 5.3. Let $\zeta \in C$, with $n_1, n_2, \cdots$ and $\hat{z}_1, \hat{z}_2, \cdots$ as in Lemma 5.4. We say $\zeta$ is controlled up to time $n$ by $\Xi^{(k)}$ if, for each $n_t \leq n$ there exists $z_t \in \Xi^{(k)}$ such that $|z_t - \hat{z}_t| = O(b^{\xi})$, where $\xi$ is the order of $\hat{z}_t$. Such $z_t$ is called a binding point for $\zeta$. 
Clearly, every $\zeta \in C$ is controlled up to time $2N$ by $\Xi([0,N])$. To proceed from step $2mN$ to step $2m+1N$, it suffices to show

**Lemma 5.5.** Let $m \geq 0$. Suppose that $(I)_{2mN}$ holds, and that every $\zeta \in C$ is controlled up to time $2m+1N$ by $\Xi([2^{m+1}N])$. Then:

(a) $(I)_k$ holds for $2mN < k \leq 2m+1N$;
(b) every $\zeta \in C$ is controlled up to time $2m+2N$ by $\Gamma([2^{m+1}N])$.

*Proof of (a).* Assume $(I)_{j-1}$ for some $2mN < j \leq 2m+1N$. Then $\Xi^{(j-1)}$ makes sense. We prove $(I)_j$ in three steps.

**Step 1:** Treatment of bound segments in $\partial R_j$. Let $d$ denote the Hausdorff distance.

**Lemma 5.6.** Let $B$ be a bound segment in $\partial R_j$. There exist $N < l < j$ and $\zeta \in \Xi^{(j-1)}$ such that $f^l\zeta$ is free and $d(f^l\zeta, B) \leq e^{-2\alpha l}$.

*Proof.* We define a sequence $z_0, \cdots, z_s$ in $\Xi^{(j-1)}$ and a sequence $n_0, \cdots, n_s$ of positive integers inductively as follows. By the definition of bound segments, there exists $0 < n_0 \leq k$ such that $f^{-n_0}B$ contains a critical point in $\Xi^{(j-n_0)}$, denoted by $z_0$. If $f^{-n_0}z_0$ is bound, let $n_1 < n_0$ denote the free return time of $z_0$ with bound period $p_1$, such that $n_1 < n_0 < n_1 + p_1$. Let $z_1$ denote the corresponding binding point, which is in $\Xi([\theta n_1]) \subset \Xi^{(j-1)}$ by the assumption of induction. If $f^{-n_0-n_1}z_1$ is bound, then let $n_2 < n_0 - n_1$ denote the free return time of $z_1$ with bound period $p_2$, such that $0 < n_2 < n_0 - n_1 < n_2 + p_2$. Let $z_2$ denote the binding point, which is in $\Xi([\theta n_2]) \subset \Xi^{(j-1)}$, and so on.

We must reach some $n_s$ and $z_s$ such that $f^{-n_0-n_1-\cdots-n_s}z_s$ is free. By the inductive assumption, each $z_i$ is controlled up to time $k - 1$. Hence, for each $i = 1, \cdots, s$ we have $p_i < \frac{4\alpha}{\gamma}p_{i-1}$. We have

$$d(B, f^{-n_0-n_1-\cdots-n_s}z_s) \leq d(B, f^{-n_0}z_0) + |f^{-n_0}z_0 - f^{-n_0-n_1}z_1| + \sum_{k=2}^{s} |f^{-n_0-n_1-\cdots-n_{k-1}}z_{k-1} - f^{-n_0-n_1-\cdots-n_k}z_k| \leq \sum_{k=0}^{s} 2e^{-2\alpha p_k} \leq 3e^{-2\alpha p_s} \leq 3e^{-2\alpha(n_0-n_1-\cdots-n_s)},$$

where we have used (c) Proposition [3.2] for the second inequality. As $z_{s-1}$ is bound at time $n_0 - n_1 - \cdots - n_{s-1}, n_0 - n_1 - \cdots - n_{s-1} < p_s$ holds. Hence $n_0 - n_1 - \cdots - n_s < p_s$ and the last inequality holds. Take $l = n_0 - n_1 - \cdots - n_s$ and $\zeta = z_s$. The argument shows $N < l$. \qed

**Corollary 5.1.** For any bound segment $B$ in $\partial R_j$ and $\alpha j \leq i < j$, $B \cap C^{(i)} = \emptyset$.

*Proof.* Take $l < j$ and $\zeta \in \Xi^{(j-1)}$ such that the conclusion of Lemma 5.6 holds. If $f^l\zeta \in I(\delta)$, then let $z \in \Xi([\theta \delta])$ denote the binding point. We have $d(B, z) \geq |f^l\zeta - z| - \text{diam}(B) \geq e^{-\alpha l} - 6e^{-2\alpha l} \geq e^{-2\alpha l}$. This implies $B \cap C^{(\alpha l)} = \emptyset$, and the claim holds. If $f^l\zeta \notin I(\delta)$, then let $O = (0, 0)$. If $l$ is large so that $d(B, O) \geq |f^l\zeta - O| - \text{diam}(B) \geq \delta - 2e^{-2\alpha l} \geq \delta/2$ holds, then the claim follows, because $j > 0$. If $l$ is so small that the last inequality does not hold, then $f^l\zeta$ is near $fI(\delta)$, which is away from $I(\delta)$. \qed

**Step 2:** Construction of $C^{(j)}$. Let $Q^{(j-1)}$ denote any component of $C^{(j-1)}$ which intersects $\partial R_j$. By Corollary 5.1 bound segments in $\partial R_j$ do not intersect $C^{(j-1)}$. Hence, each component of
Lemma 5.7. For any free segment $\gamma$ in $\partial R_j$ stretching across $Q^{(j-1)}$, there exists a critical point on $\gamma$ within $O(b^\frac{4}{7})$ of the midpoint of $\gamma$.

**Proof.** By the closeness and the disjointness of the boundaries of $Q^{(j-1)}$, their tangent directions are close enough, for Lemma 3.2 to yield a critical approximation $\zeta_0$ of order $m_0 := j$ on $\gamma$, within $O(b^\frac{4}{7})$ of the midpoint of $\gamma$.

We inductively construct a sequence $\zeta_0, \zeta_1, \ldots$, of nice critical approximations on $\gamma$, of order $m_0 < m_1 < \cdots$, such that: (a) $m_{i+1} \in [5m_i/4, 20m_i]$; (b) $|\zeta_i - \zeta_{i+1}| \leq (Cb)^{\frac{4}{7}}$. The limit of the sequence $(\zeta_i)_i$ is a critical point with the desired property.

Given $\zeta_i$ of order $m_i$ for some $i \geq 0$, $\zeta_{i+1}$ is constructed as follows. Let $\mu_1 < \mu_2 < \cdots$ denote an infinite sequence of positive integers such that $\mu_{j+1} \leq 16\mu_j$ for $j = 1, 2, \ldots$, and $\|D^{f_{j-k}}t(\zeta_i)\| \geq \frac{\kappa_4}{2^{j-k}}$ for $0 \leq k \leq \mu_j$. Lemma 3.3 ensures the existence of such a sequence. Given $j(i)$ such that $\mu_{j(i)} \leq 20m_i < \mu_{j(i)+1}$, define $m_{i+1}$ to be the smallest such that $[\theta m_{i+1}] = \mu_{j(i)}$. We have $\theta m_{i+1} \geq \mu_{j(i)+1}/16 \geq 5\theta m_i/4$. (a) allows us to use Lemma 3.1 to create a critical approximation of order $m_{i+1}$, denoted by $\zeta_{i+1}$. (b) is a consequence of Lemma 3.1.

Since $\gamma$ is a free segment, 

$$|f^{-[\theta m_{i+1}]}\zeta_i - f^{-[\theta m_{i+1}]}\zeta_{i+1}| \leq 10\delta(Cb)^{\frac{4}{7}}.$$  

Lemma 5.3 implies, for $1 \leq j \leq \lfloor \theta m_{i+1} \rfloor$, 

$$\|Df^{j(i)}(f^{-[\theta m_{i+1}]}\zeta_{i+1})\| \geq \frac{1}{2}\kappa^4 \geq \kappa^4.$$ 

In other words, $t(f^{-[\theta m_{i+1}]}\zeta_{i+1})$ is $\kappa_4$-expanding up to time $[\theta m_{i+1}]$. Also, it is $1/10$-regular up to time $[\theta m_{i+1}]$, because $\gamma$ is a free segment. Consequently, (C3) in Sect 3.3 holds and $\zeta_{i+1}$ is a nice critical approximation of order $m_{i+1}$ on $\gamma$. This completes the construction of $(\zeta_i)_i$ and also the proof of Lemma 5.7. 

**Step 3: Verification of (I)$_j$.** To show the assertion on the Hausdorff distance in (S1), we regard the horizontal boundaries of the component of $C^{(j-1)} \cap R_j$ containing $Q^{(j)}$ as graphs of functions $\gamma_1, \gamma_2$ defined on an interval $I$ of length $2\kappa_j^{-1}$. Let $L(x) = |\gamma_1(x) - \gamma_2(x)|$. (S1) gives $L^\frac{1}{2}(x) \leq (Cb)^{\frac{4}{7}} < \text{length}(I)$. Moreover $|\gamma_1'(x) - \gamma_2'(x)| \leq L^\frac{1}{2}(x)$ holds, for otherwise $\gamma_1$ intersects $\gamma_2$. By this and the $C^2(b)$-property, $L(y) \geq L(x) - (L^\frac{1}{2}(x) - C\sqrt{b}|x-y|)|x-y|$ holds for $x, y \in I$, which is $\geq L(x)/2$ provided $|x-y| \leq L^\frac{1}{2}(x)$. Hence, area($Q^{(j)}$) $\geq L^\frac{1}{2}(x)/2$ holds. If $L(x) \geq b^\frac{4}{7}$, then area($Q^{(j)}$) $\geq b^\frac{4}{7}/2$, which yields a contradiction to area($Q^{(j)}$) $<$ area($R_j$) $\leq (Cb)^{\frac{4}{7}}$.

We show that $\partial R_0$ is controlled up to time $j$. Let $\gamma$ denote any maximal free segment in $\partial R_j$ intersecting $I(\delta)$. We indicate how to choose the horizontal curve $\tilde{\gamma}$.

If $\gamma \cap Q^{(j-1)} \neq \emptyset$, then $\gamma$ stretches across a component $Q^{(j-1)}$, and there exists a critical point on $\gamma$, by Lemma 5.7. In this case, we take $\tilde{\gamma} = \gamma$. If $\gamma \cap Q^{(k-1)} = \emptyset$, let $k_0 < k - 1$ denote the largest such that $C^{(k_0)} \cap Q^{(j)} \neq \emptyset$. Let $Q^{(k_0)}$ denote the component intersecting $\gamma$. Let $Q^{(k_0+1)}$ denote any component of $C^{(k_0+1)}$ in $Q^{(k_0)}$. Since the bound segments are small, there exists a horizontal curve $\tilde{\gamma}$ which contains $\gamma$ and a critical point on $\tilde{\gamma}$. 
(Proof of (b)). Let $\zeta \in C$, $2^{m+1}N < n_t \leq 2^{m+2}N$ and suppose that $n_t$ is a free return time of $\zeta$. Let $\tilde{z}_i$ denote the binding point of order $\xi$, as in Lemma 5.4. If $f^{-[\theta\xi]}\tilde{z}_i \notin fI(\delta)$, then the long stable leaf of order $[\theta\xi]$ through $f^{-[\theta\xi]}\tilde{z}_i$ intersects $\partial R_0$ at one point, which we denote by $x$. Otherwise, the long stable leaf of order $[\theta\xi] - 1$ through $f^{-[\theta\xi]+1}\tilde{z}_i$ intersects $\partial R_0$ at one point, which we denote by $x$. In either of the two cases, $|f^{[\theta\xi]}x - \hat{z}_i| \leq (Cb)^{\theta\xi}$, and

$$\xi \leq C\alpha n_t < 2^{m+1}N.$$

Claim 5.1. $f^{[\theta\xi]}x$ is free.

Proof. Suppose the contrary. Let $B$ denote the bound segment containing $f^{[\theta\xi]}x$, which is in $\partial R_{[\theta\xi]}$. By Lemma 5.3, $B \subset I(\delta)$ and there exists $l < [\theta\xi]$, $z \in \Xi([\theta\xi]-1)$ such that $f^lz$ is free and $d(f^lz, B) \leq e^{-2\alpha l}$. Let $z'$ denote the binding point for $f^lz$. It follows that $\zeta$ and $z'$ lie on the same horizontal curve, a contradiction. □

Let $\gamma$ denote the maximal free segment containing $f^{[\theta\xi]}x$. Lemma 5.6 implies that $\gamma$ stretches across $Q([\theta\xi]-1)$. By the assumption of induction, there exists $z_i \in \Xi([\theta\xi]) \subset \Xi([2^{m+1}+\theta\xi])$, located within $O(b^{[\theta\xi]})$ of the midpoint of $\gamma \cap Q([\theta\xi]-1)$. By Lemma 5.1, there exists a critical approximation $z$ of order $\xi$ on $\gamma$ such that $|f^{[\theta\xi]}x - z| = O(b^{p_{\xi}})$. Lemma 3.1 implies $|z_i - z| \leq (Cb)^{\theta\xi}$. Hence $|z_i - z| \leq |z_i - f^{[\theta\xi]}x| + |f^{[\theta\xi]}x - z| + |z - z_i| = O(b^{\theta\xi})$, which means that $\zeta$ is controlled up to time $n_t$ by $\Xi([2^{m+1}+\theta\xi])$. This completes the proof of Proposition 5.3.

5.8. Binding points in $C$. The following statement was obtained from the proof of Proposition 5.3.

Corollary 5.2. For all $z \in \partial R_0 \setminus \bigcup_{n \geq 0} f^{-n}C$ there exist a sequence of integers $0 \leq n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq \cdots$ and a sequence $\zeta_1, \zeta_2, \cdots$ of critical points such that for each $n_t$ we have: $f^{n_t}z \in I(\delta)$; there exists a horizontal curve $\tilde{\gamma}$ which contains the maximal free segment containing $f^{n_t}z$, and a critical point $\zeta_i$ on $\tilde{\gamma}$; $p_i$ is the bound period from the critical partition of $\tilde{\gamma}$.

We use this corollary to resolve the problems mentioned in Remark 3.1 on the ambiguities of binding points. From this point on, we call each $\zeta_i$ a binding point for the orbit of $z$, and refer to $f^iz$ as bound if $n_i < i < n_i + p_i$ for some $n_i$. Otherwise, we refer to $f^iz$ as free.

6. The measure of $W^u \cap K^+$

Let $|\cdot|$ denote the arc length measure on $W^u$ (we will also denote by $|\cdot|$ the two-dimensional Lebesgue measure, but never for both things simultaneously). The aim of this section is to prove

Proposition 6.1. $|W^u \cap K^+| = 0$.

The main step in the proof of this proposition is to show the next

Lemma 6.1. (Growth to a fixed size) Let $\omega_0$ be an element of a critical partition constructed in Section 5.2 or a free segment not intersecting $I(\delta)$ and stretching across one of the components of $I(2\delta) \setminus I(\delta)$. If $\omega_0 \cap K^+$ has positive Lebesgue measure, there exist a collection $Q$ of pairwise interior-disjoint curves in $\omega_0$ and a stopping time function $S: Q \rightarrow \mathbb{N}$ such that:

(a) for a.e. $z \in \omega_0 \cap K^+$, there exists $\omega \in Q$ containing $z$;
(b) for each \( \omega \in Q \), \( f^S(\omega)\omega \) is a free segment not intersecting \( I(\delta) \) and stretching across one of the components of \( I(2\delta) \setminus I(\delta) \). The distortion of \( f^S(\omega)\omega \) is uniformly bounded;
(c) there exists \( c > 0 \) depending only on the length of \( \omega_0 \) such that for \( n > 0 \),
\begin{equation}
|\{ S > n \}| \leq ce^{-Cn}.
\end{equation}

Here, \( \{ S > n \} \) denotes the union of all \( \omega \in Q \) such that \( S(\omega) > n \).

A large part of this section is devoted to the proof of Lemma \( 6.1 \). In Section \( 6.1 \) we define and describe the combinatorics of the partition \( Q \) and the stopping time \( S \). In Section \( 6.2 \) we estimate the size of a curve with a given combinatorics, and combine it with a counting argument, and prove Lemma \( 6.1 \). In Section \( 6.3 \) we show that stable manifolds with "good shapes" are more or less dense. Combining this topological result with Lemma \( 6.1 \) we complete the proof of Proposition \( 6.1 \).

### 6.1. Combinatorial structure.

Let \( \omega_0 \) be a free segment in \( W^u \) as in Lemma \( 6.1 \). For each \( n \geq 0 \), considering \( n \)-iterates we construct a partition \( P_n \) of \( \omega_0 \) and its subset \( Q_n \). Each element of \( P_n \) is a countable union of elements of \( P_{n+1} \). Each element of \( Q \) is an element of some \( Q_n \). If \( \omega \in Q \cap Q_n \), then \( S(\omega) = n \) holds.

If \( \omega_0 \) is an element of a critical partition, \( \omega_0 \in \mathcal{P}_n \), and \( \omega_0 \) is either cutting or non-cutting time of \( \omega \). If \( n \) is a cutting time of \( \omega \), \( f^n\omega \) is cut into pieces. A pull-back of this partition defines \( \mathcal{P}_n|\omega \). If \( n \) is a non-cutting time of \( \omega \), let \( \mathcal{P}_n|\omega = \{ \omega \} \).

We precisely describe when \( n \) is a cutting or non-cutting time of \( \omega \). If \( f^n\omega \cap I(\delta) = \emptyset \), or \( f^n\omega \) is bound, then \( n \) is a non-cutting time of \( \omega \). If \( S(\omega) \cap I(\delta) = \emptyset \), or \( f^n\omega \) is free, Let \( \gamma \) denote the horizontal curve given by Corollary \( 5.2 \). Namely, \( \gamma \) contains \( f^n\omega \), and there exists a critical point on \( \gamma \). There are two mutually exclusive cases:

- \( \omega_0 \) contains at least one element of the critical partition \( \{ \gamma_{k,s} \} \) of \( \gamma \). In this case, \( n \) is a cutting time of \( \omega_0 \). We cut \( \omega \cap I(\delta) \) into pieces, by intersecting it with the elements of \( \{ \gamma_{k,s} \} \). The partition elements containing the boundary of \( \omega \cap I(\delta) \) are taken together with the adjacent ones, so that all the resultant elements contains exactly one element of \( \{ \gamma_{k,s} \} \). If the component of \( \omega \cap I(\delta) \) is \( \geq \delta \) in length, then we treat it as an element of our partition of \( \omega \). Otherwise, we take it together with the adjacent \( \gamma_{k,s} \). Lemma \( 5.1 \) goes through for each partition element, because it is a union of at most \( \log 2/(3\alpha) \)-number of elements of the critical partition of \( \gamma \). This follows from \( \sum \).
  - \( \omega_0 \) contains no element of \( \{ \gamma_{k,s} \} \). In this case, \( n \) is a non-cutting time of \( \omega \).

Let \( \mathcal{P}'_n \) denote the collection of all elements of \( \mathcal{P}_n \) intersecting \( K^+ \).

**Lemma 6.2.** If \( \omega \in \mathcal{P}'_{n-1} \) and \( f^n\omega \) is free, then for all \( \xi, \eta \in \omega \),
\[ \log \frac{\|Df^n\omega(t)(\xi)\|}{\|Df^n\omega(t)(\eta)\|} \leq \frac{C}{\delta} |f^n\xi - f^n\eta|^{C\alpha}. \]

If \( f^n\xi, f^n\eta \in I(2\delta) \), then the factor \( \delta \) can be dropped.

**Proof:** Let \( k < n \) and suppose that \( f^k\omega \) is free. The time interval \([k, n]\) is decomposed into bound and free segments. Applying Proposition \( 5.2 \) to each bound segment and Lemma \( 2.1 \)
to each free segment, we have \( \|Df^{n-k}(z)t(z)\| \geq \delta e^{\frac{1}{2}(n-k)} \) for all \( z \in f^k(\omega) \). Since \( f^k\omega \) and \( f^n\omega \) are \( C^2(b) \), it then follows that

\[
|f^k\xi - f^k\eta| \leq \delta^{-1}e^{-\frac{1}{2}(n-k)}|f^n\xi - f^n\eta|.
\]

Let \( n_1 < \cdots < n_s < n_{s+1} := n \) denote all the free returns in the first \( n \)-iterates of \( \omega \), with \( p_j \) the corresponding bound period. By Lemma 5.1 and (21),

\[
\sum_{j=0}^{s} \log \frac{\|Df^{p_j}(f^{n_j}\xi)\|}{\|Df^{p_j}\xi\|} \leq \sum_{j=0}^{s} |f^{n_j+p_j}\xi - f^{n_j+p_j}\eta|^{C_\alpha} \leq \frac{C}{\delta^{C_\alpha}}|f^n\xi - f^n\eta|^{C_\alpha}.
\]

By Lemma 2.1, \( f^i\omega \) is a \( C^2(b) \)-curve outside of \( I(\delta) \), for \( n_j + p_j \leq i < n_{j+1} \). Hence

\[
\sum_{j=0}^{s} \sum_{i=n_j+p_j}^{n_{j+1}-1} \log \frac{\|Df^i(f^i\xi)\|}{\|Df^i\xi\|} \leq \frac{C}{\delta} \sum_{j=0}^{s} \sum_{i=n_j+p_j}^{n_{j+1}-1} |f^i\xi - f^i\eta| \leq \frac{C}{\delta}|f^n\xi - f^n\eta|.
\]

These two inequalities yield the desired one. The last assertion follows from (b) Lemma 2.1. \( \square \)

6.2. Large deviation. Let us say that \( \omega \in \mathcal{P}'_n \) is an escaping element if (i) \( \omega \cap K^+ \neq \emptyset \); (ii) \( n \) is a cutting time of the element of \( \mathcal{P}'_{n-1} \) containing \( \omega \), and \( f^n\omega \cap I(\delta) = \emptyset \). By construction, if \( \omega \in \mathcal{P}'_n \) is an escaping element, then \( f^n\omega \) is a free segment, not intersecting \( I(\delta) \) and stretching across one of the components of \( I(2\delta) \) \( \setminus \) \( I(\delta) \).

Let \( \mathcal{Q}_n \) denote the collection of all escaping elements of \( \mathcal{P}'_n \) which are not contained in escaping elements in \( \bigcup_{0 \leq k \leq n} \mathcal{P}'_k \). Define \( \mathcal{Q} = \bigcup_n \mathcal{Q}_n \). Define a stopping time function \( S: \mathcal{Q} \rightarrow \mathbb{N} \) by \( S(\omega) = n \) for each \( \omega \in \mathcal{Q}_n \). Clearly, \( \mathcal{P}'_n \setminus \bigcup_{0 \leq k \leq n} \mathcal{Q}_k = \{ \omega \in \mathcal{Q} : S(\omega) > n \} \) holds. We show (20). By construction, this would imply that the elements of \( \mathcal{Q} \) altogether cover \( \omega_0 \cap K^+ \) up to a set of zero Lebesgue measure, as desired in (a).

Let \( n_1 > 0 \) denote the cutting time of \( \omega_0 \). It is finite, and depends only on the length of \( \omega_0 \). This implies that, for \( n \geq n_1 \), any \( \omega \in \mathcal{P}'_n \setminus \bigcup_{0 \leq k \leq n} \mathcal{Q}_k \) has an well-defined itinerary that is described as follows. There exist a sequence of integers \( 0 < n_1 < \cdots < n_s \leq n \) called essential free returns, and an associated sequence \( \omega_1 \supset \cdots \supset \omega_s \supset \omega \) such that \( \omega_i \) is the element of \( \mathcal{P}'_{n_i} \) containing \( \omega \), and \( n_i \) is a cutting time of \( \omega_{i-1} \), with \( f^{n_i}\omega_i \subset I(2\delta) \). Let \( \zeta_i \in \mathcal{C} \) denote the binding point for \( f^{n_i}\omega_{i-1} \). Let \( p_i \) denote the bound period. By an itinerary of \( \omega \) we mean the sequence \( (n_1, \pm p_1, \zeta_1), (n_2, \pm p_2, \zeta_2), \ldots, (n_s, \pm p_s, \zeta_s) \), where \( +, - \) indicates whether \( f^{n_i}\omega_i \) is at the right or left of \( \zeta_i \).

From this point on we assume

\[
(22) \quad n \geq 2n_1.
\]

By construction, \( f^{n_1}\omega_i \) and \( f^{n_1+p_i}\omega_i \) are free segments. The following estimates are used in the proof:

\[
|f^{n_1}\omega_i| \leq e^{-\lambda p_i} \quad \text{and} \quad |f^{n_1+p_i}\omega_i| \geq e^{-4\alpha p_i}.
\]

The first one follows from the definition of the critical partition. The second one is from Lemma 5.1. Let \( n_{s+1} > n \) denote the cutting time of \( \omega_s \).

Claim 6.1. \( n_{i+1} - n_i - p_i \leq \frac{20p_i}{\lambda} \) for every \( 1 \leq i \leq s \).
Proof. Since $f^{n+1} \omega_i$ is also a free segment, in view of Proposition 5.2 and Lemma 6.1, we have

$$3 \geq |f_{n+1} \omega_i| \geq \delta e^{\frac{3}{2}(n+1-n_i-p_i)} e^{-4\alpha p_i}.$$ 

Rearranging gives $n_{i+1} - n_i - p_i \leq \frac{20}{\lambda} p_i$, where the last inequality follows from $p_i \geq \frac{\log 1/\delta}{2 \log 2}$.

Summing the above inequality over all $1 \leq i \leq s$ and then using (22), we have

$$n \leq \frac{40}{\lambda} \sum_{i=1}^{s} p_i.$$ 

Write $\omega = \omega_{s+1}$. Since $f^{n+1} \omega_{i+1} \subset I(2\delta)$, the factor $\delta$ in Lemma 6.2 can be dropped and

$$|\omega| \leq |\omega_s| = |\omega_1| \frac{|\omega_2|}{|\omega_1|} \cdots \frac{|\omega_s|}{|\omega_{s-1}|} \leq 2^{s-1} \prod_{i=1}^{s-1} \frac{|f^{n+1} \omega_{i+1}|}{|f^{n+1} \omega_i|} \leq e^{-(\lambda - 3\alpha)(p_1 + \cdots + p_{s-1})} e^{-\lambda p_s} \leq e^{-(\lambda - 3\alpha)R},$$

where $R = \sum_{i=1}^{s} p_i$, which is $\geq \frac{\lambda m}{40}$ by (23). Hence

$$\sum_{\omega \in \mathcal{P}_n \setminus \bigcup_{0 \leq k \leq n} \mathcal{Q}_k} |\omega| = \sum_{R} \sum_{p_1 + \cdots + p_s - R} |\omega| \leq \sum_{R} \sum_{s} 2^s \left(\frac{R}{s}\right) e^{-(\lambda - 3\alpha)R} \leq \sum_{R \geq \lambda n/40} e^{-(\lambda - 4\alpha)R}.$$

For the last inequality we have used $s/R \leq C/\log(1/\delta)$ and $(\frac{R}{s}) \leq e^{\beta(\delta)R}$, where $\beta(\delta) \to 0$ as $\delta \to 0$, which follows from Stirling’s formula for factorials. This completes the proof of (20) and that of Lemma 6.1.

As a corollary we obtain

Corollary 6.1. (Abundance of stopping times) Let $\omega$ be an element of some critical partition. If $\omega \cap K^+$ has positive Lebesgue measure, there exist a sequence $\mathcal{Q}^{(1)}, \mathcal{Q}^{(2)}, \cdots$ of collections of pairwise interior-disjoint curves in $\omega$, and a sequence of stopping time functions $S_1, S_2, \cdots, S_k: \mathcal{Q}^{(k)} \to \mathbb{N}$ such that:

(a) for a.e. $z \in \omega \cap K^+$ there exists a sequence $\omega^{(1)} \supset \omega^{(2)} \supset \cdots$ of curves such that $\omega^{(k)} \in \mathcal{Q}^{(k)}$ for each $k \geq 1$ and \{z\} = $\bigcap_{k \geq 1} \omega^{(k)}$;

(b) $0 < S_1(\omega^{(1)}) < S_2(\omega^{(2)}) < \cdots$, and $\log \frac{\|Df^{S_k(\omega^{(k)})}t_\xi\|}{\|Df^{S_k(\omega^{(k)})}t_\eta\|} \leq C\delta^{-1}$ for all $\xi, \eta \in \omega^{(k)}$;

(c) $f^{S_k(\omega^{(k)})}$ is a $C^2(b)$-curve, stretching across one of the components of $I(2\delta) \setminus I(\delta)$.

Proof. Let $\mathcal{Q}, S$ be as in Lemma 6.1 replacing $\omega_0$ by $\omega$, and define $\mathcal{Q}_1 = \mathcal{Q}, S_1 = S$. Given $\mathcal{Q}_k$ and $S_k$, for each $\omega \in \mathcal{Q}_k$ define a partition $\mathcal{Q}'$ of $f^{S_k} \omega$ and a stopping time function $S': \mathcal{Q}' \to \mathbb{N}$, replacing $\omega_0$ in Lemma 6.1 by $f^{S_k} \omega$. This defines $\mathcal{Q}_{k+1}$ in the obvious way. For $\omega' \in \mathcal{Q}_{k+1}$, define $S_{k+1}(\omega') = S_k(\omega) + S'(f^{S_k} \omega')$, and so on. The bounded distortion follows from Lemma 6.2.

6.3. Proof of Proposition 6.1. The next lemma relies on a continuity argument within a small parameter range containing the first bifurcation parameter $a^*$, and is not valid for the parameter ranges treated in [5, 12, 24].

Lemma 6.3. There exist $\varepsilon_1 \in (0, a^* - a^{**})$ and $\sigma \in (0, 1)$ such that for any $a \in [a^* - \varepsilon_1, a^*]$ and any $C^2(b)$-curve $\gamma$ in $W^u$ stretching across one component of $I(2\delta) \setminus I(\delta)$, $|\gamma \cap K^+| \leq \sigma|\gamma|$.

We finish the proof of Proposition 6.1 assuming the conclusion of the lemma. Assume $|W^u \cap K^+| > 0$. Then one can choose an element $\omega$ of some critical partition for which $|\omega \cap K^+| > 0$ holds. By Corollary 6.1 and Lemma 6.3 for a.e. $z \in \omega \cap K^+$ there exists an
arbitrarily small neighborhood of $z$ in $W^u$ in which the set of points which eventually escape from $R_0$ has a definite proportion. It follows that $z$ is not a Lebesgue density point of $\omega \cap K^+$. This yields a contradiction to the Lebesgue density theorem.

It is left to prove Lemma 6.3. The following elementary observation is used, on the quadratic map $g_2: [-1, 1] \ni \omega, g_2(x) = 1 - 2x^2$: 1/2 is a repelling fixed point, and the set of preimages $\bigcup_{n \geq 0} g_2^n(1/2)$ is dense in $[-1, 1]$, not containing 0.

By a vertical curve we mean a curve such that the slopes of its tangent directions are $\gg 1$. Let $l_0 \subset W^s(Q)$ denote the segment in $W^s(P)$ which contains $P$ and stretches across $R_0$. Clearly, $l_0$ is a vertical curve. Iterating $l_0$ backward, it is possible to choose an integer $N_0$ independent of $b$, and to define a sequence $l_0, l_1, \cdots, l_{N_0}$ of vertical curves in $W^s(P)$ which stretch across $R_0$, and with the property that any $C^2(b)$-curve as in the statement of the lemma intersects one of them in its middle third. This picture persists for all $a \in (a^{**}, a^*)$ sufficiently close to $a^*$. By the definition of $a^{**}$, $W^u(P)$ is not contained in $[-2, 2]^2$. By Inclination lemma, the conclusion holds. □

7. Dynamics of Lebesgue Typical points

In this last section we show $\bigcap_{n \geq 0} f^{-n}R_0$ has zero Lebesgue measure, and completes the proof of the theorem. The main step is a statistical argument, which enables us to show that the occurrence of infinitely many close returns is improbable. This sort of argument has been successfully undertaken by Benedicks and Viana [6] in the attractor context. We adapt it to our non-attracting context, with the help of the geometric structure of critical regions in Proposition 5.3. In addition, we dispense with any assumption on the Jacobian, which was assumed in [6, 24].

As a preliminary step, in Sect.7.1 we construct a family long stable leaves near each critical point. In Sect.7.2 using these leaves we define a certain region, and introduce close return time, as a kind of a first return time to this region. In Sect.7.3 we show that the theorem follows from Proposition 7.2, which states that the occurrence of infinitely many close return times is improbable.

For the proof of Proposition 7.2, based on preliminary geometric constructions in Sect.7.4 we construct in Sect.7.6 an infinite nested sequence $\Omega_0 \supset \Omega_1 \supset \cdots$. Each $\Omega_k$ is decomposed into rectangles, bordered by stable leaves and pieces of $W^u$ and denoted by $R_{i_0 \cdots i_k}$. The sequence $(i_0, \cdots, i_k)$ records the recurrent behavior of the iterates of the rectangle to the critical set. Combining these geometric ingredients with key analytic estimates in Sect.7.7, 7.8 we complete the proof of Proposition 7.2 in Sect.7.9.

7.1. Construction of long stable leaves. For the purpose of stating the next proposition, we introduce a truncated distance $d_C(\cdot)$ to $C$ as follows. Let $z \in W^u \setminus \bigcup_{n \geq 0} f^nC$ and suppose that $z$ is free. If $z \notin I(\delta)$, then let $d_C(z) = 1$. Otherwise, let $\zeta \in C$ denote the binding point for $z$ and let $d_C(z) = |z - \zeta|$. If $z$ is bound, then $d_C(z)$ is undefined. For a free segment $\omega$, let $d_C(\omega) = \min_{z \in \omega} d_C(z)$.

The next proposition indicates the existence of a family of long stable leaves near each critical value. In addition, these leaves have a slow recurrence property to $C$.

Proposition 7.1. (Long stable leaves through slowly recurrent points) Let $\zeta$ be a critical point on a free segment $\gamma$. For each element $\omega_0$ of the critical partition of $\gamma$ there exists $z \in \omega_0$
such that \( d_C(f^n z) \geq e^{-5\alpha n} \) holds for every \( n > 0 \) such that \( f^n z \) is free. In addition, the long stable leaf through \( fz \) exists.

**Proof.** We divide the proof into three steps. First, we prove the existence of \( z \in \omega_0 \) with the property as in the first statement. Next, we give angle estimates. Finally, we show the existence of long stable leaves through \( fz \).

**Step 1. Construction of slowly recurrent points.** Let \( n_0 = 0 \). Let \( p_0 \) denote the bound period of \( \omega_0 \). Let \( P_0, P_1, P_2, \ldots \) denote the sequence of partitions of \( \omega_0 \) constructed in the same way as in Sect 6.1. We construct a (possibly finite) sequence \( p_0 \leq n_1 < n_2 < \cdots \) and a nested sequence \( \omega_0 \supset \omega_1 \supset \omega_2 \supset \cdots \) of curves for which the following holds for every \( k \geq 0 \). Obviously, any point in the intersection \( \bigcap_{k \geq 0} \omega_k \) satisfies the desired property:

- \( \omega_k \in P_{n_k} \), and for \( 0 \leq n \leq n_k \) such that \( f^n \omega_k \) exists.
- \( n_{k+1} \) is a cutting time of \( \omega_k \). If there exists no cutting time of \( \omega_k \), then \( n_{k+1} \) is undefined.

The construction of the sequence is by induction that is described as follows. Given \( n_k \), \( \omega_k \in P_{n_k} \) such that \( f^{n_k} \omega_k \in I(\delta^2) \), with a bound period \( p_k \), define \( n_{k+1} = n_k + p_k \) to be the cutting time of \( \omega_k \). Then \( f^{n_{k+1}} \omega_k \) is a free segment of length \( \geq e^{-5\alpha n_k} \). Indeed, by Lemma 5.1, \( f^{n_k+p_k} \omega_k \) is a free segment of length \( \geq e^{-4\alpha p_k} \). Using Lemma 2.1, from time \( n_k + p_k \) to \( n_{k+1} \), \( |f^{n_k+1} \omega_k| \geq |f^{n_k+p_k} \omega_k| \geq e^{-4\alpha n_k}. \) Hence, it is possible to take an element \( \omega_{k+1} \in P_{n_{k+1}} \) such that \( \omega_{k+1} \subset \omega_k \), \( f^{n_{k+1}} \omega_{k+1} \subset I(\delta^2) \) and \( d_C(f^{n_{k+1}} \omega_{k+1}) \geq e^{-5\alpha n_{k+1}} \). To recover the assumption of the induction, it suffices to show \( d_C(f^n \omega_k) \geq e^{-5\alpha n} \) for \( n_k + p_k \leq n < n_{k+1} \) such that \( f^n \omega_k \) is free. If \( f^n \omega_k \cap I(\delta) = \emptyset \), then \( d_C(f^n \omega_k) = 1 \geq e^{-5\alpha n}. \) To treat the case where \( n \) is a free return time, we need

**Sublemma 7.1.** Let \( \tilde{n}_1 < \cdots < \tilde{n}_s \) denote all the free return times of \( \omega_k \) in \( [n_k + p_k, n_{k+1}) \), with \( \tilde{p}_1, \ldots, \tilde{p}_s \) the corresponding bound periods. Then

\[
\tilde{p}_1 + \cdots + \tilde{p}_s \leq \frac{13\alpha p_k}{\lambda}.
\]

**Proof.** Splitting the time interval \( [n_k + p_k, n_{k+1}) \) into bound and free segments, for all \( z \in f^{n_k+p_k} \omega_k \) we have \( \|D^t f^{n_k+1-n_k-p_k} t(z)\| \geq e^{\frac{5}{3}(\tilde{p}_1 + \cdots + \tilde{p}_s)}. \) Combining this with \( |f^{n_k+p_k} \omega_k| \geq e^{-4\alpha p_k} \) from Lemma 5.1, we get \( 3 > |f^{n_k+1} \omega_k| \geq e^{\frac{5}{3}(\tilde{p}_1 + \cdots + \tilde{p}_s) - 4\alpha p_k}. \) The first inequality is due to the elementary fact that the forward iterates of \( \omega_k \) cannot grow to a free segment of length \( > 3 \) without intersecting \( I(\delta) \). Taking logs we obtain the desired inequality.

For each \( \tilde{n}_i \) we have \( d_C(f^{\tilde{n}_i} \omega) \geq e^{\frac{\log C_0}{\lambda} \tilde{p}_i} \geq e^{-5\alpha \log C_0 \alpha p_k} \geq e^{-5\alpha n}. \) The last inequality follows from \( p_k \leq \frac{3\alpha n_k}{\lambda} \lambda \) as in (a) Proposition 5.2.

**Step 2. Angle estimates.** We introduce a useful language along the way.

**Definition 7.1.** Let \( z \in I(\delta) \setminus C \). A tangent vector \( v \) at \( z \) is in tangential position relative to \( \zeta \in C \) if there exists a horizontal curve \( \gamma \) which is tangent to both \( v \) and \( t(\zeta) \).

Let \( z \in \omega_0 \) have the property in Lemma 7.1. Let \( \theta_n = \angle(D^t f^n t(z), w_n(z)) \). Let \( 0 =: n_0 < n_1 < n_2 < \cdots \) denote all the free return times of \( z \), with \( \zeta_0, \zeta_1, \zeta_2, \cdots \) the corresponding binding points. The next lemma allows us to use \( \zeta_k \) as a binding point for \( w_{n_k}(\zeta) \).

**Lemma 7.1.** For every free return time \( n_k > 0 \) of \( z \), \( \theta_{n_k} \leq (Cb)^{\frac{n_k}{14}} \) holds. In addition, \( w_{n_k}(z) \) is in tangential position relative to \( \zeta_k \).
Proof. Let $p_k$ denote the binding period for $n_k$. The next three angle estimates follow from [21 Sublemma 3.2.]:

\begin{equation}
\theta_{p_0} \leq \theta_1(Cb)\left(p_0^{-1}/2\right) \frac{\|Dft(z)\|}{\|Df^{p_0 t}(z)\|} \cdot \frac{\|w_1(z)\|}{\|w_{p_0}(z)\|} \leq (Cb)^{p_0/3}; \tag{24}
\end{equation}

\begin{equation}
\theta_{n_k+1} \leq \theta_{n_k+p_k}(Cb)^{(n_k+1-n_k-p_k)/2} \frac{\|Df^{n_k+p_k t}(z)\|}{\|Df^{n_k+1 t}(z)\|} \cdot \frac{\|w_{n_k+p_k}(z)\|}{\|w_{n_k+1}(z)\|} \quad \text{for } k \geq 0; \tag{25}
\end{equation}

\begin{equation}
\theta_{n_k+p_k} \leq \theta_{n_k}(Cb)^{p_k/2} \frac{\|Df^{n_k t}(z)\|}{\|Df^{n_k+p_k t}(z)\|} \cdot \frac{\|w_{n_k}(z)\|}{\|w_{n_k+p_k}(z)\|} \quad \text{for } k \geq 1. \tag{26}
\end{equation}

Using these, we prove the statement by induction on $k$. Take $k = 0$ in (25). By (24) and Lemma 2.1, the two fractions of the right-hand side are $\leq 1/\delta$ and $\theta_{n_1} \leq (Cb)^{n_1/3}$ holds. This estimate and the distance bound in Lemma 7.1 implies that $w_{n_1}(z)$ is in tangential position relative to $\zeta_1$. Then, taking $k = 1$ in (26) we get $\theta_{n_1+p_1} \leq (Cb)^{(n_1+p_1)/3}$. Taking $k = 2$ in (25) we get $\theta_{n_2} \leq (Cb)^{n_2/3}$, and that $w_{n_2}(z)$ is in tangential position relative to $\zeta_2$, and so on. \hfill \Box

Step 3. The existence of long stable leaves. In view of Lemma 2.5 it suffices to show that $f z$ is expanding. In the next lemma, we assume $1 < \sigma < e^{\frac{4}{3}}$.

Lemma 7.2. For every $n \geq 1$, $\|w_n(z)\| \geq \sigma^{n-1}$.

Proof. The inequality for $1 \leq n \leq n(\lambda)$, where $n(\lambda) < p_0$ only depends on $\lambda$, follows from the closeness of our map $f$ to $(x, 0) \rightarrow (1 - 2x^2, 0)$. For $n(\lambda) \leq n \leq p_0$, it follows from the exponential growth and the bounded distortion along the orbit of $\zeta$. Let $n > p_0$. Suppose $f^nz$ is free. Applying Lemma 2.3 to each free segment and Proposition 2.1 to each bound segment, we have $\|w_n(z)\| \geq \delta e^{\frac{4}{3}(n-1)}$. If $n \leq n_1$, then the factor $\delta$ can be dropped by Lemma 2.4. If $n > n_1$, then using $\delta > de(f^{n_1}z) \geq e^{-5a_{n_1}}$ we have $\|w_n(z)\| \geq e^{\frac{4}{3}(n-1)} \geq de(f^{n_1}z)\|w_n(z)\| \geq e^{\frac{4}{3}(n-1)} \geq \sigma^{n-1}$. If $f^{n}z$ is bound, namely $n_k < n < n_k + p_k$ for some $n_k$, then $\|w_n(z)\| \geq 5^{-(n_k+p_k-n)} \|w_{n_k+p_k}(z)\| \geq 5^{-p_k}e^{\frac{4}{3}(n_k+p_k-1)} \geq 5^{\frac{3n_k}{2}}e^{-\frac{4}{3}n} \geq \sigma^{n-1}$. For the third inequality we have used $p_k \leq \frac{3n_k}{2}$ as in (a) Proposition 5.2. This completes the proof of Lemma 7.2 and hence that of Proposition 7.1. \hfill \Box

7.2. Close return time. Let $Q(k)$ denote any component of $C(k)$. Let $\zeta_0, \zeta_1$ denote the critical points on the horizontal boundaries of $Q(k)$. Take curves $\gamma_0, \gamma_1$ of length $\delta M$ in the horizontal boundaries of $Q(k)$ so that: (i) $\gamma_0$ (resp. $\gamma_1$) contains $\zeta_0$ (resp. $\zeta_1$) within $O(b^k)$ of the midpoint of it; (ii) $\gamma_0, \gamma_1$ are connected by two vertical lines. Let $B(k) \subset Q(k)$ denote the region bordered by $\gamma_0$ is connected to $\gamma_1$ by the two vertical lines through their endpoints.

We construct a region $B_0(k) \subset B(k)$ as follows. Assume $\Gamma(f\zeta_0)$ is at the right of $\Gamma(f\zeta_1)$. Choose a point $z \in \gamma_1$ for which $\delta^k \leq |z - \zeta_1| \leq \delta^2$, and $d_e(f^nz) \geq e^{-5a_n}$ holds for every $n \geq 1$. Proposition 7.1 ensures the existence of such a point. By Remark 2.1, $\Gamma(z)$ intersects $f\gamma_1$ exactly at two points.

By (13), the Hausdorff distance between $\Gamma(z)$ and $\Gamma(f\zeta_0)$ is $\leq C|fz - f\zeta_1| + C|f\zeta_1 - f\zeta_0| \leq C\delta^\frac{4}{3}$. Hence, $\Gamma(z)$ intersects $f\gamma_0$ at one point. By Remark 2.1, $\Gamma(z)$ intersects $f\gamma_0$ exactly at two points. Define $B_0(k)$ to be the region bordered by $\gamma_0, \gamma_1$ and the parabola $f^{-1}\Gamma(z)$. By construction, the horizontal boundaries of $B_0(k)$ extend both sides around $\zeta_0, \zeta_1$ to length from $\approx \delta^k$ to $\approx \delta^2$. Let $A(k)$ denote the collection of all $B_0(k)$. 

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HIROKI TAKAHASI *
Definition 7.2. We say $z \in I(\delta)$ is controlled up to time $\nu > 0$ if $f^n z \notin \mathcal{A}^{(n)}$ holds for every $1 \leq n < \nu$. In addition, if $f^\nu z \in \mathcal{A}^{(\nu)}$, then we say $z$ makes a close return at time $\nu$, and call $\nu$ a close return time of $z$.

7.3. Infinitely many close returns are improbable. Let $z \in I(\delta)$. Let $\nu_1, \nu_2, \ldots$ be defined inductively as follows: $\nu_1$ is a close return time of $z$; given $\nu_1, \ldots, \nu_k$, let $\nu_{k+1}$ be the close return time of $f^{\nu_1+\nu_2+\cdots+\nu_k} z \in I(\delta)$. If $\nu_1, \ldots, \nu_k$ are defined in this way, we say $z$ has $k$ close return times. If the sequence is defined is indefinitely, we say $z$ has infinitely many close return times. Otherwise, we say $z$ only finitely many close return times. We say $z$ is controlled if there is no close return time of $z$.

Let $k_0$ be a large integer, to be specified later. Let $\Omega_\infty$ denote the set of all $z \in \mathcal{A}^{(k_0)}$ which has infinitely many close return times. We have $\Omega_\infty = \bigcap_{k \geq 1} \Omega_k$, where $\Omega_k$ denotes the set of all $z \in \mathcal{A}^{(k_0)}$ which has $k$ close return times. Obviously, $\Omega_k \subset \Omega_{k-1}$ holds.

Proposition 7.2. $|\Omega_k|/|\Omega_{k-1}| \to 0$ exponentially fast, as $k \to \infty$. In particular, $\Omega_\infty$ has zero Lebesgue measure.

Let $\Lambda = \bigcap_{n \geq 0} f^{-n} \partial R_0$. We show how $|\Lambda| = 0$ follows from this proposition. We argue contradiction assuming $|\Lambda| > 0$. Lemma 7.2 indicates that $\Lambda$ intersects $\bigcup_{n \geq 0} f^{-n} I(\delta)$ in a set with positive Lebesgue measure. For almost every $z \in \Lambda \cap \bigcup_{n \geq 0} f^{-n} I(\delta)$, define $m(z) \geq 0$ to be the smallest such that $f^m(z) z$ is controlled. Let us see $m(z)$ is well-defined. This is clear in the case $z \notin \bigcup_{n \geq 0} f^{-n} \mathcal{A}^{(k_0)}$. Otherwise, take $i_0(z) \geq 0$ such that $f^{i_0(z)} z \in \mathcal{A}^{(k_0)}$. By Proposition 7.2, one of the following holds: either (i) $f^{i_0(z)} z$ is controlled, or else (ii) $f^{i_0(z)} z$ has only finitely many close return times, denoted by $\nu_1, \ldots, \nu_k$. By definition, $f^{i_0(z)+\nu_1+\cdots+\nu_k} z$ is controlled.

Let $V_j = \{z \in \Lambda \cap \bigcup_{n \geq 0} f^{-n} I(\delta): m(z) = j\}$. Take $j$ such that $|V_j| > 0$. By definition, any point in $f^j V_j$ is controlled. The next lemma indicates that $f^{j+1} V_j$ is foliated by long stable leaves.

Lemma 7.3. If $z \in I(\delta)$ is controlled up to time $\nu$, then $\|w_n(z)\| \geq \delta^{12n \log 2} / \lambda$ holds for $1 \leq n < \nu$.

Proof. We inductively define a sequence $0 < n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq \cdots \leq n_s < n_s + p_s \leq \nu$ of integers and critical points $\zeta_1, \zeta_2, \ldots, \zeta_s$ such that: (i) $f^{n_i} z \in I(\delta)$ for each $n_i$, and $w_{n_i}(z)$ is in tangential position relative to $\zeta_i$, with $p_i$ the bound period and $|f^{n_i} z - \zeta_i| \geq \delta^{2n_i}$; (ii) $n_{i+1}$ is the next time of returns to $I(\delta)$ after $n_i + p_i$.

Given $n_i, \zeta_i$ and $p_i$, let $n_{i+1} \geq n_i + p_i$ denote the smallest such that $f^{n_{i+1}} z \in I(\delta)$. By the assumption, $f^{n_{i+1}} z \notin \mathcal{A}^{(n_{i+1})}$ holds. Let $k$ denote the largest integer such that $f^{n_{i+1}} z \in \mathcal{C}^{(k)}$, and let $\mathcal{Q}^{(k)}$ denote the component of $\mathcal{C}^{(k)}$ containing $f^{n_{i+1}} z$. By (S3), $f^{n_{i+1}} z$ is in tangential position relative to critical points on the horizontal boundaries of the component of $\mathcal{C}^{(k-1)}$ containing $\mathcal{Q}^{(k)}$. Choose one of them as $\zeta_{i+1}$.

Suppose that $n_1 < n < n_1 + p_1$ holds. In the same way as in the proof of Lemma 7.2, we have $\|w_n(z)\| \geq 4^{-p_1}$. Substituting $p_1 \leq \frac{6n_i}{\lambda} \log(1/\delta)$ into the exponent yields the desired inequality. For all other $n$ it is immediate to show the desired inequality, in the same way as in the proof of Lemma 7.2. \hfill \Box

Consider the projection $\pi: f^{j+1} V_j \to \partial R_0$ along the long stable leaves. \[\Box\] says that $\pi$ is Lipschitz continuous. In particular, $\pi(f^{j+1} V_j)$ has positive one-dimensional Lebesgue measure.
in \( W^u \). By the contraction along the leaves, \( \pi(f^{j+1}V_j) \subset K^+ \) holds. This yields a contradiction to Proposition 6.1.

The rest of this paper is devoted to the proof of Proposition 7.2. Before proceeding, let us give some estimates on close return times which will be used sometimes in the sequel. Let \( z \in \Omega_k \), and let \( \nu_1, \ldots, \nu_k \) denote the corresponding sequence of \( k \) close return times of \( z \). By definition, for \( 1 \leq l \leq k-1 \), \( f^{\nu_l}z \in A^{(\nu_l)} \) holds. Let \( \zeta \) denote any critical point on the horizontal boundary of the component of \( A^{(\nu_l)} \) containing \( f^{\nu_l}z \). By definition, \( |f^{\nu_l}z-\zeta| \leq C\delta^{\nu_l} \) holds. Then

\[
|f^{\nu_l+i}z - f^i\zeta| \leq C\delta^{\nu_l}4^i \ll e^{-ai} \quad \text{for } 1 \leq i \leq 4\nu_l.
\]

This implies

\[
\nu_{l+1} \geq 4\nu_l \quad \text{for } 1 \leq l < k.
\]

The same reasoning gives \( \nu_1 \geq 4k_0 \), and thus

\[
\nu_l \geq 4^l k_0 \quad \text{for } 1 \leq l \leq k.
\]

7.4. **Partitions of rectangles.** By a rectangle \( R \) we mean a compact region bounded by two disjoint curves in \( W^u \) and two disjoint stable leaves. The boundaries of \( R \) in \( W^u \) are called unstable sides. The boundaries in the stable leaves are called stable sides.

We define partitions of rectangles, using the families of long stable leaves constructed in Section 7.1. To this end, let us fix once and for all an enumeration \( C = \{\zeta_m\}_{m=1}^\infty \) of all the critical points and let \( \gamma_m \) denote the maximal free segment containing \( \zeta_m \). We deal with a rectangle \( R \) in \( I(\delta) \) such that:

- (R1) the unstable sides of \( R \) are made up of two free segments, each contained in \( \gamma_{m_0} \) and \( \gamma_{m_1} \). In addition, \( |\zeta_{m_0} - \zeta_{m_1}| \leq (Cb)^\frac{k}{2} \) holds for some \( k \geq 1 \);
- (R2) the unstable sides of \( R \) extend to both sides around \( \zeta_{m_0}, \zeta_{m_1} \) to length \( \approx \delta^k \);
- (R3) \( \Gamma(f\zeta_{m_0}) \) is at the right of \( \Gamma(f\zeta_{m_1}) \);
- (R4) there exists a long stable leaf \( \Gamma_\infty \) such that \( f^{-1}\Gamma_\infty \) contains the stable sides of \( R \).

One typical situation we have in mind is that two maximal free segments in \( \partial R_v \) stretch across \( B^{(k)}_0 \), where \( k < \nu \). If this happens, then the region bounded by the two maximal free segments and the stable sides of \( B^{(k)}_0 \) is a rectangle satisfying all the requirements.

By Lemma 7.1 in each element of the critical partition of \( \gamma_{m_1} \) there exists a point \( z \) such that the long stable leaf through \( fz \) exists. Take just one such point from each element of the partition and denote the associated countable number of long stable leaves by \( \Gamma_\Delta \), \( \Delta = -1, -2, -3, \ldots \) from the left to the right. We repeat essentially the same construction for \( \gamma_{m_0} \). The difference is that, only those of the elements of the critical partition of \( \gamma_{m_0} \) come into play whose \( f \)-image is at the right of \( \Gamma(f\zeta_{m_1}) \). We denote by \( \Gamma_\Delta \) the associated countable number of long stable leaves at the right of \( \Gamma(f\zeta_{m_1}) \), where \( \Delta = 1, 2, 3, \ldots \) from the left to the right.

By Remark 7.1 if \( \Delta > 0 \), then \( f^{-1}\Gamma_\Delta \) intersects the unstable side of \( R \) containing \( \zeta_{m_0} \) exactly at two points, one on the right of \( \zeta_{m_0} \) and the other on the left. If \( \Delta < 0 \), then \( f^{-1}\Gamma_\Delta \) intersects the stable side of \( R \) containing \( \zeta_{m_0} \). By Remark 7.1 again, \( f^{-1}\Gamma_\Delta \) intersects each of the unstable sides of \( R \) exactly at two points. These observations and the Lipschitz continuity of the tangent directions of the leaves as in (5) altogether indicate that, the family of the long stable leaves induces a partition of \( R \). Each element of the partition is a rectangle, bounded by the unstable sides of \( R \) and two neighboring parabolas, which are preimages of \( \Gamma_\Delta, \Gamma_{\Delta+1} \).
7.5. **Symbolic coding.** Each rectangle in the partition of $R$ constructed in Section 7.4 is denoted by $R(\rho, \epsilon, \Delta, p)$. Here, the meanings of $\rho, \epsilon, \Delta, p$ are as follows:

- if the unstable sides of $R(\rho, \epsilon, \Delta, p)$ intersect both $\gamma_{m_0}$ and $\gamma_{m_1}$, then $\rho = m_1$. Otherwise, $\rho = m_0$;
- if $\rho = m_0$, then $\epsilon = 0$. If $\rho = m_1$, then $\epsilon = +$ or $-$, depending on whether the unstable sides of $R(\rho, \epsilon, \Delta, p)$ is at the “right” or the “left” of $\zeta_{m_0}$ and $\zeta_{m_1}$;
- the stable sides of $f(R(\rho, \epsilon, \Delta, p))$ are contained in $\Gamma_{\Delta} \cup \Gamma_{\Delta+1}$.
- $p = \max\{p(\zeta_{\rho}, z) : z \in \gamma_{\rho} \cap R(\rho, \epsilon, \Delta, p)\}$.

The integer $p$ is called a **bound period** of $R(\rho, \epsilon, \Delta, p)$. By the monotonicity of the function $z \mapsto p(\zeta_{\rho}, z)$, the maximum is attained at one of the edges of the rectangle $R(\rho, \epsilon, \Delta, p)$. It is immediate to see:

(i) all points in $f(R(\rho, \epsilon, \Delta, p))$ are expanding up to time $p_{i_0} - 1$;
(ii) for all $\xi, \eta \in R(\rho, \epsilon, \Delta, p)$ and $1 \leq i \leq p$, $\|w_i(\xi)\|/\|w_i(\eta)\| \leq 2$.

**Lemma 7.4.** (Geometry of rectangles at the end of bound periods) For all $z$ in the unstable sides of $R(\rho, \epsilon, \Delta, p)$, $\|Df^p t(z)\| \geq C\delta \|Df^t(z)\|$ holds for every $0 \leq i < p$. In particular, the unstable sides of $f^p(R(\rho, \epsilon, \Delta, p))$ are made up of two $C^2(b)$-curves.

**Proof.** Let $\zeta$ denote the critical point on the unstable side of $R$ which contains $z$. Let $p(\zeta, z), q(\zeta, z)$ denote the bound and fold periods of $z$ with respect to $\zeta$, as defined in Sect. 5.2. In view of (ii) as above and (g) Proposition 5.2

\begin{equation}
\|Df^i t(z)\| \approx |\zeta - z| \cdot \|w_i(\zeta)\| \quad \text{for} \quad q(\zeta, z) \leq i \leq \max(p(\zeta, z), p).
\end{equation}

Let $\xi_1, \xi_2, \xi_3, \xi_4$ denote the **edges** of the rectangle, namely, the points which belong to both the stable and the unstable sides of $R(\rho, \epsilon, \Delta, p)$. In the discussion to follow, we assume that
\( \xi_1, \xi_2 \) are on the same unstable side of \( R \), and \( f(\xi_i, f(\xi_{i+2}) \) (\( i = 1, 2 \)) are connected by the long stable leaf which defines the stable side of \( f(R(\rho, \epsilon, \Delta, p)) \).

**Case 1:** \( \epsilon = 0 \). In this case, \( \xi_1, \xi_2, \xi_3, \xi_4 \) are on the same unstable side of \( R \). We suppose that \( \xi_1 \) is closest to \( \zeta \). Then \( p = \max(p(\zeta, \xi_1), p(\zeta, \xi_3)) \) holds. \( [5] \) and Lemma \( [22] \) give \( |\zeta - \xi_1| \geq |\zeta - \xi_3| \). Hence, \( (a,b) \) Proposition \( [52] \) gives

\[
q(\zeta, z) \leq C\beta \max(\log |\zeta - \xi_1|^{-1}, \log |\zeta - \xi_3|^{-1}) < p.
\]

This means that \( (29) \) holds for \( q(\zeta, z) \leq i \leq p \) and therefore

\[
\frac{\|Df^pt(z)\|}{\|Df^t(z)\|} \geq C\alpha \|w_p(\zeta(z))\| \geq C\delta.
\]

For \( 1 \leq i \leq q(\zeta, z) \),

\[
\frac{\|Df^pt(z)\|}{\|Df^t(z)\|} \geq C\alpha |z - z'| \|w_p(z, z')(\zeta(z))\| \geq C\delta \log \eta > \delta.
\]

The first inequality follows from \( (h) \) Proposition \( [52] \). The second inequality follows from \( C\delta \|w_p(z, z')(\zeta(z))\| \geq C\delta \|w_p(z, z')(\zeta(z))\| \geq C\delta |z - z'| \|w_p(z, z')(\zeta(z))\| \geq C\delta \log \eta > \delta. \)

**Case 2:** \( \epsilon = + \) or \( - \). In this case, \( \xi_1 \) and \( \xi_3 \) (resp. \( \xi_2 \) and \( \xi_4 \)) are on different unstable sides of \( R \). We suppose that \( \Gamma(f\xi_1) \) is at the right of \( \Gamma(f\xi_2) \), and that \( \xi_1 \) and \( \zeta \) belong to the same unstable side of \( R \). Let \( \zeta' \) denote the other critical point of \( R \) on the unstable side of \( R \). Then \( p = \max\{p(\zeta, \xi_1), p(\zeta', \xi_3)\} \) holds. By \( [15] \) and Lemma \( [22] \) again, \( |\zeta - z| \geq C|\zeta - \xi_1| \) and \( |\zeta - z| \geq C|\zeta' - \xi_3| \). Hence, \( q(\zeta, z) \leq C\beta \log |\zeta - z|^{-1} < p \). This means that \( (29) \) holds for \( q(\zeta, z) \leq i \leq p \). The rest of the argument is analogous to that in Case 1.

### 7.6. Construction of partitions

Putting the results in Sections 7.3, 7.5 together, for each \( k \geq 0 \) we inductively construct a partition of each \( \Omega_k \) into a countable number of rectangles. Each element of the partition of \( \Omega_k \) will be denoted by \( R_{i_0 \cdots i_k} \), where \( (i_0, \cdots, i_k) \) are itineraries which record the behavior of the rectangle under iteration, up to time \( \nu_1 + \cdots + \nu_k \).

**Initial step.** Let \( \Omega_0 = A^{(0)} \). Take a component of \( \Omega_0 \) and denote it by \( R \). Following the steps in Sect. 7.3, define a partition of \( R \) with the family of long stable leaves. To each element of the partition, assign the set of symbols according to the rule described in Sect. 7.5. Each
element is denoted by $R_{i_0}$, where $i_0 = (\rho_0, \epsilon_0, \Delta_0, p_0)$ and $R_{i_0} = R(\rho_0, \epsilon_0, \Delta_0, p_0)$. We repeat
the same construction for any component of $\Omega_0$.

**General step.** Given the partition $\{R_{i_0\cdots i_{k-1}}\}_{i_0\cdots i_{k-1}}$ of $\Omega_{k-1}$ for some $k \geq 1$, define $R_{i_0\cdots i_{k-1}}(\nu_k) = \{z \in R_{i_0\cdots i_{k-1}} : \nu_k$ is a close return time of $f^{\nu_0+\cdots+\nu_{k-1}}z\}$. Here and for the rest of this section, we adopt the next

**Convention.** $\nu_0 = 0$.

By definition,

\[ \Omega_k = \bigcup_{(i_0, \cdots, i_{k-1})} \bigcup R_{i_0\cdots i_{k-1}}(\nu_k). \]

**Proposition 7.3.** (Geometry of rectangles at close return times) Let $z \in f^{\nu_0+\cdots+\nu_k}R_{i_0\cdots i_{k-1}}$ and suppose $f^\nu z \in B^{(\nu_k)}_0 \subset B^{(\nu_k)}_0$. Then the unstable sides of $f^{\nu_1+\cdots+\nu_k}R_{i_0\cdots i_{k-1}} \cap B^{(\nu_k)}_0$ are $C^2(b)$-curves stretching across $B^{(\nu_k)}_0$.

We finish the construction of the partition of $\Omega_k$ assuming the conclusion of the proposition. Take a component of $f^{\nu_0+\cdots+\nu_k}R_{i_0\cdots i_{k-1}}(\nu_k)$ and denote it by $R$. By the proposition and the geometric structure of critical regions in Proposition 5.3, on each unstable side of $R$ there exists a critical point, within $O(b^{(\nu_k)})$ of its midpoint. In particular, $R$ meets all the requirements (R1-4) in Sect. 7.4. Following the steps in Sect. 7.4, 7.5, define a partition of $R$ with the family of long stable leaves and assign to each element the set of symbols. Let $R_{i_0\cdots i_{k-1}} = f^{-(\nu_0+\cdots+\nu_k)}R(\rho_k, \epsilon_k, \Delta_k, p_k)$, where $i_k = (\rho_k, \epsilon_k, \Delta_k, p, \nu_k)$. We repeat the same construction for any component of $f^{\nu_0+\cdots+\nu_k}R_{i_0\cdots i_{k-1}}(\nu_k)$.

**Proof of Proposition 7.3.** Let $\Gamma_{\nu_k-1}(z) = \{(x, y) : |y| \leq \sqrt{\theta}\}$. Consider the vertical strip $V = \{(x, y) : |x - x(y)| \leq \delta^{\nu_k}, |y| \leq \sqrt{\theta}\}$.

**Lemma 7.5.** $V$ does not intersect the stable sides of $f^{\nu_0+\cdots+\nu_k+1}R_{i_0\cdots i_{k-1}}$.

**Proof.** Let $\sigma$ denote any stable side of $f^{\nu_0+\cdots+\nu_k+1}R_{i_0\cdots i_{k-1}}$. By construction, there exists $y \in W^s \cap \sigma$ such that $d_{c}(f^n y) \geq e^{-5a_n}$ holds whenever $f^n y$ is free, and $\sigma \subset \Gamma(y)$. Suppose $V \cap \sigma = \emptyset$, and let $\xi \in V \cap \sigma$. Let $\eta$ denote the point of intersection between $\Gamma$ and the horizontal through $\xi$. The definition of $V$ gives $|\xi - \eta| \leq \delta^{\frac{\nu_k}{2}}$, and thus $|f^{\nu_k-1} - f^{\nu_k-1}| \leq \delta^{\frac{\nu_k}{2}}$. Since $\eta \in \Gamma$, $|f^{\nu_k-1} - f^{\nu_k-1}| \leq (C\varphi^{\nu_k-1})$. Hence $|f^{\nu_k-1} - f^{\nu_k-1}| \leq (C\varphi^{\nu_k-1})$ follows. Meanwhile $|f^{\nu_k-1} - f^{\nu_k-1}| \leq (C\varphi^{\nu_k-1})$ holds, and the assumption on $z$ gives $|\xi - f^{\nu_k} z| \leq C\delta^{\frac{\nu_k}{2}}$, where $\xi$ is any critical point on the unstable sides of $B^{(\nu_k)}_0$. Therefore

\[ |\xi - f^{\nu_k-1} z| \leq |\xi - f^{\nu_k} z| + |f^{\nu_k} z - f^{\nu_k-1} z| + |f^{\nu_k-1} - f^{\nu_k-1}| \leq \delta^{\frac{\nu_k}{2}}. \]

This estimate and the proof of Corollary 5.1 together indicate that $f^{\nu_k-1} y$ is free. Hence, Proposition 7.1 gives a critical point $\xi'$ such that $|\xi' - f^{\nu_k-1} y| \geq e^{-5a_{\nu_k}}$. Then it is possible to choose a horizontal curve $\gamma$ such that both $\xi$ and $\xi'$ are on $\gamma$. This is a contradiction. □

By Lemma 7.5 $V$ cuts a segment in each unstable side of $f^{\nu_0+\cdots+\nu_k+1}R_{i_0\cdots i_{k-1}}$, denoted by $\gamma$. Let $\xi'$ denote the critical point on the same unstable side of $f^{\nu_0+\cdots+\nu_k+1}R_{i_0\cdots i_{k-1}}$ as that of $f^{-1} \gamma$. Let $z'$ be an arbitrary point in $\gamma$. Let $p_{k-1}$ denote the bound period of $f^{\nu_0+\cdots+\nu_k+1}R_{i_0\cdots i_{k-1}}$. The bounded distortion gives $|D f^j(z')| \approx \|D f^j(z')\| \approx |w_j(z)|$ for $1 \leq j < \nu_k$, and thus for $p_{k-1} - 1 \leq j < \nu_k$,

\[ \|D f^j(z')\| \approx |\xi' - f^{-1} z'| \cdot \|D f^j(z')\| \approx |\xi' - f^{-1} z'| \cdot |w_j(z)|. \]
By Lemma 7.3, \( f^{p_k-1} \gamma \) is \( C^2(b) \). Then, by [21] Lemma 2.3. and (30), the curvature of \( f^{\nu_k-1} \gamma \) is everywhere bounded from above by

\[
(Cb)^{v_k-p_k-1} \frac{\|w_{p_k-1}(z)\|^3}{\|w_{p_k}(z)\|^3} \sqrt{b} + \sum_{j=p_k-1}^{\nu_k-1} (Cb)^{v_k-j-1} \frac{\|w_{j+1}(z)\|^3}{\|w_{v_k}(z)\|^3}.
\]

Since \( \|w_{p_k}(z)\| \geq C\delta \|w_{j+1}(z)\| \) for \( p_k-1 \leq j < \nu_k \), it follows that the curvature is everywhere \( \leq \sqrt{b} \). (30) also implies that the slopes of the tangent directions of \( f^{\nu_k-1} \gamma \) are \( \leq \sqrt{b} \). Hence, \( f^{\nu_k-1} \gamma \) is a \( C^2(b) \)-curve.

Parametrize \( \gamma \) by arc length \( s \). Using \( |\zeta - f^{-1} \gamma(s)| \geq C^{-p_k-1} \) for all \( s \) and the fact that the width of the strip \( V \) is \( \delta_{\frac{w_k}{2}} \),

\[
\int \|Df^{\nu_k-1} t(\gamma(s))\| \, ds \geq C\|w_{v_k}(z)\| \int |\zeta - f^{-1}(\gamma(s))| \, ds \geq C^{-\nu_k} \delta_{\frac{w_k}{2}} \gg \delta_{\frac{w_k}{2}}.
\]

This implies that \( f^{\nu_k-1} \gamma \) stretches across \( B^{(v_k)} \). \( \square \)

7.7. Unstable sides are roughly parallel. A main step in the proof of Proposition 7.2 is an estimate of the measure of the set

\[
R_{i_0 \cdots i_{k-1}}(\nu_k) = \{ z \in R_{i_0 \cdots i_{k-1}} : \nu_k \text{ is a close return time of } f^{i_0 \cdots i_k+1}z \}.
\]

This subsection and the next are devoted to obtaining this estimate. For the purpose of stating the next proposition we need some definitions.

- (New constants) Choose \( C_1, C_2 \) as follows: \( |\det Df| \geq C_1 \) on \( R_0 \); for all \( \xi, \eta \) in the unstable sides of any component of \( \Omega_0 \), \( \angle(u(\xi), u(\eta)) \leq C_2 |\xi - \eta| \). Let \( C_3 = C_0 e^{\log C_0} \).

- (Attachment of collars) For each \( R_{i_0} \subset \Omega_0 \), let \( Q(R_{i_0}) \) denote the component of \( \Omega_0 \) containing \( R_{i_0} \). Let \( k \geq 1 \). For each \( R_{i_0 \cdots i_k} \subset \Omega_k \), By Proposition 7.3 there exists exactly one component \( B^{(v_k)} \) of \( A^{(v_k)} \) containing \( f^{i_0 \cdots i_k+1} R_{i_0 \cdots i_k} \). Let \( Q(R_{i_0 \cdots i_k}) \) denote the component of \( f^{-(i_0 \cdots i_k+1)} B^{(v_k)} \cap R_{i_0 \cdots i_{k-1}} \) containing \( R_{i_0 \cdots i_k} \).

- For any \( z \) in a free segment of \( W^u \), let \( u(z) \) denote the unit vector tangent to \( W^u \) at \( z \) such that the sign of the first component is positive.

**Proposition 7.4.** For every \( j \geq 0 \) and any \( \xi, \eta \) in the unstable side of \( f^{i_0 \cdots i_j+1} Q(R_{i_0 \cdots i_j}) \),

\[
\angle(u(\xi), u(\eta)) \leq C_2 C_3^{v_j} |\xi - \eta|.
\]

**Proof of Proposition 7.4.** We argue by induction on \( j \). The choice of \( C_2 \) and the convention \( \nu_0 = 0 \) give (31) for \( j = 0 \). Let \( k \geq 1 \) and assume (31) for \( j = k - 1 \).

**Lemma 7.6.** For any \( \xi, \eta \) in the unstable sides of \( f^{i_0 \cdots i_{k-1}+1} Q(R_{i_0 \cdots i_{k-1}}) \),

\[
\angle(Dfu(\xi), Dfu(\eta)) \leq C_2 C_3^{v_k} |f\xi - f\eta|.
\]

**Proof.** Let \( \theta_i = \angle(Df^i u(\xi), Df^i u(\eta)), i = 0, 1 \). A simple computation gives

\[
\theta_1 \leq \frac{Cb\theta_0 + C|\xi - \eta|}{\|Dfu(\xi)\| \|Dfu(\eta)\|}.
\]

Hence \( \theta_1 \ll 1 \), provided \( k_0 \) is sufficiently large. We have

\[
\theta_1 \leq C C_1^{-1} (|\xi - \eta| + \angle(u(\xi), u(\eta))).
\]
The inequality follows from the following elementary fact: for any nonzero vectors $u$, $v$ such that $\angle(u, v) \ll 1$, $\angle(u, v) \leq 2|u - v|/ \min\{\|u\|, \|v\|\}$. \begin{equation} \tag{51} \label{51} |\xi - \eta| + \angle(u(\xi), u(\eta)) \leq 2C_1^{-1}C_2C_3^{3\nu_k-1}|f\xi - f\eta|. \end{equation} Replacing this in the previous inequality,
\[ \theta_1 \leq CC_1^{-2}C_2C_3^{3\nu_k-1}|f\xi - f\eta| \leq C_2C_3^{\nu_k}|f\xi - f\eta|. \]
The last inequality holds for sufficiently large $k_0$, because of $C_3^{\nu_k} \leq C_3^{\nu_k}$ from \eqref{27}. \hfill \Box

For any $\xi$ on the unstable sides of $f^{\nu_0+\cdots+\nu_k-1}Q(R_{i_0\cdots i_k})$, let
\begin{equation} \label{32} v(\xi) = \rho \cdot Df \nu(f^{-1}\xi), \end{equation}
where $\rho > 0$ is the normalizing constant. If $k_0$ is sufficiently large, then $v(\xi)$ has a large slope. By the definition of $u(\cdot)$, the sign of the second component of $v(\xi)$ is constant for all $\xi$.

By Proposition \ref{7.3} and the distortion control, the contractive field $e_{\nu_k-1}$ is well-defined on $f^{\nu_0+\cdots+\nu_k-1}Q(R_{i_0\cdots i_k})$. Fix once and for all the orientation of $e_{\nu_k-1}$ so that the second component of $e_{\nu_k-1}$ and that of $v(\xi)$ have the same sign. Let $f_{\nu_k-1}$ denote the unit vector field orthogonal to $e_{\nu_k-1}$. Split $v(\xi) = A(\xi)e_{\nu_k-1}(\xi) + B(\xi)f_{\nu_k-1}(\xi)$.

\textbf{Lemma 7.7.} For any $\xi_1, \xi_2$ on the unstable sides of $f^{\nu_0+\cdots+\nu_k-1}Q(R_{i_0\cdots i_k})$,
\[ \max\{|A(\xi_1) - A(\xi_2)|, |B(\xi_1) - B(\xi_2)|\} \leq 2C_2C_3^{\nu_k}|\xi_1 - \xi_2|. \]

\textbf{Proof.} The following elementary fact is used. For $u_i = (\cos\theta_i, \sin\theta_i)$, $0 \leq \theta_i \leq \pi$, $i = 1, 2, 3, 4$,
\[ |\angle(u_1, u_2) - \angle(u_3, u_4)| \leq \angle(u_1, u_3) + \angle(u_2, u_4). \]
This can be checked using $\angle(u_i, u_j) = |\theta_i - \theta_j|$ and the triangle inequality.

We have $A(\xi_1) = \langle e_{\nu_k-1}(\xi_1), v(\xi_1) \rangle = \cos\angle(e_{\nu_k-1}(\xi_1), v(\xi_1))$, where the bracket denotes the scalar product. Then $\angle(e_{\nu_k-1}(\xi_1), v(\xi_1)) \in [0, \pi]$ holds, which is close to 0. Considering $\cos^{-1} : [-1, 1] \to [0, \pi]$ and $|\cos^{-1}'| \geq 1$ we have $|A(\xi_1) - A(\xi_2)| \leq |\cos^{-1}(A(\xi_1)) - \cos^{-1}(A(\xi_2))|$, and
\[ |\cos^{-1}(A(\xi_1)) - \cos^{-1}(A(\xi_2))| = |\angle(e_{\nu_k-1}(\xi_1), v(\xi_1)) - \angle(e_{\nu_k-1}(\xi_2), v(\xi_2))| \leq \angle(v(\xi_1), v(\xi_2)) + \angle(e_{\nu_k-1}(\xi_1), e_{\nu_k-1}(\xi_2)) \leq 2C_2C_3^{\nu_k}|\xi_1 - \xi_2|. \]
The first factor in the second line is bounded by Lemma \ref{7.6}. The second factor is bounded by Lemma \ref{2.3}. In the same way, we have $B_i = \langle f_{\nu_k-1}(\xi_i), v(\xi_i) \rangle = \cos\angle(f_{\nu_k-1}(\xi_i), v(\xi_i))$ and $\angle(f_{\nu_k-1}(\xi_i), v(\xi_i)) \in [0, \pi]$, which is close to $\pi/2$. Then
\[ |\cos^{-1}(B(\xi_1)) - \cos^{-1}(B(\xi_2))| = |\angle(f_{\nu_k-1}(\xi_1), v(\xi_1)) - \angle(f_{\nu_k-1}(\xi_2), v(\xi_2))| \leq \angle(v(\xi_1), v(\xi_2)) + \angle(f_{\nu_k-1}(\xi_1), f_{\nu_k-1}(\xi_2)) \leq 2C_2C_3^{\nu_k}|\xi_1 - \xi_2|. \]
For the last inequality we have used the orthogonality of $f_{\nu_k-1}$ to $e_{\nu_k-1}$. \hfill \Box

\textbf{Lemma 7.8.} There is a $C^1$ vector field $\phi_0$ on $f^{\nu_0+\cdots+\nu_k-1}Q(R_{i_0\cdots i_k})$ which is tangent to the unstable sides of it, with $\|\phi_0\| \leq 2$ and $\|D\phi_0\| \leq 4C_2C_3^{\nu_k}$. 

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Proof. Let \( \zeta^{(1)}, \zeta^{(2)} \) denote the critical points on the unstable sides of \( f_{\nu_0 + \cdots + \nu_{k-1}} Q(R_{i_0 \cdots i_{k-1}}) \).
We introduce a nearly orthogonal coordinate on the rectangle which trivialize \( \Gamma(f \zeta) \) and \( \Gamma(f \zeta') \). Namely, it is a \( C^1 \) coordinate \((\check{x}, \check{y})\) on \( f_{\nu_0 + \cdots + \nu_{k-1} + 1} Q(R_{i_0 \cdots i_{k-1}}) \) such that:

(i) \( 9/10 \leq \|\partial_{\check{x}}\| \leq 10/9, \|\partial_{\check{y}}\| = 1, \langle \partial_{\check{x}}, \partial_{\check{y}} \rangle = 0, \langle \partial_{\check{y}}, t(f \zeta_{1}) \rangle = 1; \)

(ii) \( \Gamma(f \zeta^{(1)}) = \{ \check{x} = 0 \}, \Gamma(f \zeta^{(2)}) = \{ \check{x} = c \}, \) where \( c \) is a constant.

It is possible to choose such a coordinate, by the properties of long stable leaves and (b) Lemma \ref{lem_unstable}. Let \( T : (x, y) \rightarrow (\check{x}, \check{y}) \) denote the coordinate transformation.

With respect to \((\check{x}, \check{y})\)-coordinate, we represent the unstable sides of \( f_{\nu_0 + \cdots + \nu_{k-1} + 1} Q(R_{i_0 \cdots i_{k}}) \) by graphs of functions \( \gamma_1, \gamma_2, \gamma_1(\check{x}) < \gamma_2(\check{x}) \). For all \( \xi \) in the unstable sides the rectangle, let

\[
(\gamma_2(\check{x}) - \gamma_1(\check{x})) \cdot v(\xi) = \tilde{A}(\xi)e_{\nu_{k-1}}(\xi) + \tilde{B}(\xi)f_{\nu_{k-1}}(\xi),
\]

where \( v(\xi) \) is the one in \([32]\) and \( T(\xi) = (\check{x}, \check{y}) \). In what follows, we shall extend \( \tilde{A}, \tilde{B} \) to \( C^1 \) functions on the the entire \( f_{\nu_0 + \cdots + \nu_{k-1} + 1} Q(R_{i_0 \cdots i_{k}}) \), in such a way that \( \max(\|\tilde{A}\|, \|\tilde{B}\|) \leq 1 \) and \( \max(\|D\tilde{A}\|, \|D\tilde{B}\|) \leq 3C_2C_3^{2\nu_k} \). For all \( z \) in the rectangle, define

\[
\phi_0(z) = \tilde{A}(z)e_{\nu_{k-1}}(z) + \tilde{B}(z)f_{\nu_{k-1}}(z).
\]

Since \( \|D\phi_{\nu_{k-1}}\|, \|Df_{\nu_{k-1}}\| \) are bounded by Lemma \ref{lem_unstable_2}, this yields the desired inequality.

To simplify notation, write \( A \) for \( A \circ T^{-1} \), and the same for \( B, \tilde{A}, \tilde{B} \). On the assumption that both \( \gamma_1(\check{x}) \) and \( \gamma_2(\check{x}) \) make sense, we extend \( \tilde{A} \) affinely along the \( \check{y} \)-direction. In other words, for \( \check{y} \in [\gamma_1(\check{x}), \gamma_2(\check{x})] \), define

\[
\tilde{A}(\check{x}, \check{y}) = \tilde{A}(\check{x}, \gamma_1(\check{x}))(\check{y} - \gamma_1(\check{x}))(A(\check{x}, \gamma_2(\check{x})) - A(\check{x}, \gamma_1(\check{x}))).
\]

In the same way, we extend \( \tilde{B} \) affinely along the \( \check{y} \)-direction. If, for instance, \( \gamma_1(\check{x}) \) makes sense and \( \gamma_2(\check{x}) \) does not, we enlarge the domain of definition of \( \gamma_2 \) so that \( \gamma_2(\check{x}) \) makes sense. It is possible to show, using the long stable leaf of order \( \nu_k - 1 \) through \( \gamma_1(\check{x}) \), that \( \gamma_2(\check{x}) \) is sufficiently close to the unstable sides of the rectangle, so that all the preceding arguments go through.

The definition gives \( \max(\|\tilde{A}\|, \|\tilde{B}\|) \leq \gamma_2(\check{x}) - \gamma_1(\check{x}) \ll 1 \). Lemma \ref{lem_unstable_2} and the choice of \((\check{x}, \check{y})\)-coordinate give \( \max(\|\partial_{\check{y}}\tilde{A}\|, \|\partial_{\check{y}}\tilde{B}\|) \leq 3C_2C_3^{2\nu_k} \). To evaluate the norms of \( \check{x} \)-derivatives, we assume that \( \zeta^{(\sigma)} \) and \( f^{-1}\gamma_{\sigma}(\check{x}) \) belong to the same unstable side, \( \sigma = 1, 2 \). Recall the symbolic coding \( i_{k-1} = (\rho_{k-1}, \epsilon_{k-1}, \Delta_{k-1}, p_{k-1}, \nu_{k-1}) \). In the case \( \epsilon_{k-1} = + \) or \(-\),

\[
\left| \frac{d\gamma_{\sigma}}{d\check{x}}(\check{x}) \right| \leq \frac{C\sqrt{b}}{|f^{-1}(\check{x}, \gamma_{\sigma}(\check{x})) - \zeta^{(\sigma)}|} \leq \frac{3\nu_k}{e^{\log C_0}} \leq e^{3\nu_k e^{\log C_0}}.
\]

In the case \( \epsilon_{k-1} = 0 \),

\[
\left| \frac{d\gamma_{\sigma}}{d\check{x}}(\check{x}) \right| \leq \frac{C\sqrt{b}}{|f^{-1}(\check{x}, \gamma_{\sigma}(\check{x})) - \zeta_{\rho_{k-1}}|} \leq e^{3
u_k e^{\log C_0}} \leq e^{3\nu_k e^{\log C_0}}.
\]

In either of the two cases, Sublemma \ref{lem_unstable_2} gives

\[
\left| \frac{d\tilde{A}}{d\check{x}}(\check{x}, \gamma_{\sigma}(\check{x})) \right| \leq 3C_2C_3^{\nu_k} \left| \frac{d\gamma_{\sigma}}{d\check{x}}(\check{x}) \right| \leq 3C_2C_3^{\nu_k} e^{3\nu_k e^{\log C_0}}.
\]

As \( \tilde{A}(\check{x}, \gamma_{\sigma}(\check{x})) = (\gamma_2(\check{x}) - \gamma_1(\check{x}))A(\check{x}, \gamma_{\sigma}(\check{x})) \),

\[
\left| \frac{d\tilde{A}}{d\check{x}}(\check{x}, \gamma_{\sigma}(\check{x})) \right| \leq C_2C_3^{\nu_k} e^{3\nu_k e^{\log C_0}}.
\]
Differentiating (35) with \( \hat{x} \) and then using (36) (37) (38) (39), we obtain \( \| \partial_\xi \tilde{A} \| \leq C_2 C_3^{2\nu k} \). In the same way we obtain the desired upper estimate of \( \| \partial_\xi \tilde{B} \| \). Transforming all these derivative estimates back to the original \((x, y)\)-coordinate, we obtain the desired estimates. \( \square \)

We now introduce the projectivization \( f_* \) of \( Df \), given by \( f_*(\xi, v) = Df(\xi)v/\| Df(\xi)v \| \), and define vector fields \( \phi_j \) on \( f^{\nu_0+\cdots+\nu_k-1+j+1}Q(R_{i_0\cdots i_k}) \) for \( 1 \leq j < \nu_k \), by push-forward under \( f_* \):

\[
\phi_j(z) = f_*(f^{-1}z, \phi_{j-1}(f^{-1}z)).
\]

If \( \epsilon_{k-1} = + \) or \( - \), then for all \( \xi, \eta \) in the unstable side of \( f^{\nu_0+\cdots+\nu_k}Q(R_{i_0\cdots i_k}) \), \( \angle(u(\xi), u(\eta)) = \angle(\phi_{\nu_k-1}(\xi), \phi_{\nu_k-1}(\eta)) \) holds. If \( \epsilon_{k-1} = 0 \), then \( \angle(u(\xi), u(\eta)) < \angle(\phi_{\nu_k-1}(\xi), \phi_{\nu_k-1}(\eta)) \) holds. Hence, (31) for \( j = k \) is a direct consequence of the next

**Lemma 7.9.** For all \( z \in f^{\nu_0+\cdots+\nu_k}Q(R_{i_0\cdots i_k}) \), \( \| D\phi_{\nu_k-1}(z) \| \leq C_2 C_3^{2\nu k} \).

**Proof.** The following estimates, proved in Appendix A.3, are used:

(40) \[
|\partial_\nu f_*(\xi, v)| \leq 2^\nu \frac{|\det Df(\xi)|}{\| Df(\xi)v \|^2}.
\]

(41) \[
|\partial_\xi f_*(\xi, v)| \leq \frac{\| D^2f(\xi)\|}{\| Df(\xi)v \|}.
\]

Differentiating the formula of \( \phi_j \) and using the result recursively we get

\[
D\phi_{\nu_k-1}(z) = \sum_{i=1}^{\nu_k-1} \partial_\nu f_*^{i+1}(f^{-i+1}z, \phi_{\nu_k-1}) \partial_\xi f_*(f^{-i}z, \phi_{\nu_k-1}) Df^{-i}(z)
\]

\[
+ \partial_\nu f_*^{\nu_k-1}(f^{-\nu_k+1}z, \phi_0) D\phi_0(f^{-\nu_k+1}z) Df^{-\nu_k+1}(z),
\]

where \( \phi_{\nu_k-1-i} \) means \( \phi_{\nu_k-1-i}(f^{-i}z) \). By (40), for every \( 1 \leq i < \nu_k \),

\[
\| \partial_\nu f_*^{i+1}(f^{-i+1}z, \phi_{\nu_k-1}) \| \leq 2^{i+1} \frac{|\det Df^{-i+1}(f^{-i+1}z)|}{\| Df^{-i+1}(f^{-i+1}z) \|},
\]

\[
= \frac{2^{i-1}}{\| Df^{-i+1}(f^{-i+1}z) \|} \| Df^{-i+1}(f^{-i+1}z) \| \| Df^{-i+1}(f^{-i+1}z) \|.
\]

(41) gives

\[
\| \partial_\xi f_*(f^{-i}z, \phi_{\nu_k-1-i}) Df^{-i}(z) \| \leq \| \partial_\xi f_*(f^{-i}z, \phi_{\nu_k-1-i}) Df^{-i+1}(z) \| \| Df^{-i+1}(z) \|
\]

\[
\leq C \frac{\| Df^{-i+1}(z) \|}{\| Df^{-i}(z) \|} \leq CC_1^{-2} \| Df^{-i+1}(z) \|.
\]

Replacing all these in the above equality,

\[
\| D\phi_{\nu_k-1}(z) \| \leq \sum_{i=1}^{\nu_k-1} 2^{i-1} CC_1^{-2} \| Df^{-i+1}(z) \| \ni Df^{-i+1}(z) \| + \| Df^{-\nu_k+1}(f^{-\nu_k+1}z) \| \| Df^{-\nu_k+1}(z) \|.\]

To evaluate the denominators of the fractions, we need

**Lemma 7.10.** For all \( \xi \) in the unstable sides of \( f^{\nu_0+\cdots+\nu_k-1+j+1}Q(R_{i_0\cdots i_k}) \) and \( 0 \leq j < \nu_k \), \( \| \phi_{\nu_k-1}(\xi) \| \geq C\delta \| \phi_j(\xi) \|.\)
Proof. Let \( \zeta \) denote the critical point on the same unstable side of \( f^{\nu_0 + \cdots + \nu_k - 1} Q(R_{i_0 \cdots i_{k-1}}) \) as that of \( f^{-1} \xi \). Let \( q \) denote the fold period for \( f^{-1} \xi \). In view of Proposition 7.2 and the bounded distortion, for \( q \leq j < \nu_k \) we have \( \| \phi_{j-1}(\xi) \| \geq C \| w_{j-1}(f^{-1} \xi) \| \geq C \delta_j \), and for \( 0 \leq j < q \), \( \| \phi_j \| > \| \phi_j(\xi) \| \).

Lemma 7.8 and Lemma 7.10 give
\[
\| D \phi_{\nu_k-1}(z) \| \leq \sum_{i=1}^{\nu_k-1} C \delta^{-2} 10^i + C \delta^{-2} 5^{\nu_k} C^2 \leq C \delta^{-2} 2^{3\nu_k},
\]
The last inequality holds for sufficiently large \( k_0 \).

7.8. Area distortion bounds. Proposition 7.4 and the next area distortion bounds together allow us to estimate the Lebesgue measure of the set in question.

Proposition 7.5. For every \( k \geq 1 \) and all \( \xi_1, \xi_2 \in f^{\nu_0 + \cdots + \nu_k - 1} Q(R_{i_0 \cdots i_k}) \),
\[
\frac{| \det D f^{\nu_k}(\xi_1) |}{| \det D f^{\nu_k}(\xi_2) |} \leq e^{C_1^{-1}}.
\]

Proof. Since \( \| D \log \det D f \| \leq C b C_1^{-1} \), it suffices to show
\[
| f^i \xi_1 - f^i \xi_2 | \leq C.
\]

Let \( \gamma \) denote one of the unstable sides of \( f^{\nu_0 + \cdots + \nu_k - 1} Q(R_{i_0 \cdots i_{k-1}}) \). Let \( \eta_\sigma \) denote the point of intersection between \( \Gamma_{\nu_k}(\xi_\sigma) \) and \( \gamma \). Let \( 0 \leq i < \nu_k \). If \( \eta_1 \) and \( \eta_2 \) are on the unstable sides of \( f^{\nu_0 + \cdots + \nu_k - 1} Q(R_{i_0 \cdots i_k}) \), Lemma 7.10 implies \( | f^i \eta_1 - f^i \eta_2 | \leq C \delta^{-1} | f^{\nu_k-1} \eta_1 - f^{\nu_k-1} \eta_2 | \leq C \delta^{-1} b \). On the other hand, the contraction along the stable leaves gives \( | f^i(\xi_{\sigma}) - f^i \eta_{\sigma} | \leq (Cb)^2 | f^i(\xi_{\sigma}) - \eta_{\sigma} | \leq (Cb)^2 \). It follows that \( | f^i \xi_1 - f^i \xi_2 | \leq C \delta^{-1} b \). Summing this over all \( 0 \leq i < \nu_k \) yields (42).

7.9. Proof of Proposition 7.2. In what follows, we assume \( k \geq k_0 \) is large so that \( C_2 C_3^{3\nu_k} \leq C_3^{4\nu_k} \). Denote by \( \gamma_1 \) and \( \gamma_2 \) the two unstable sides of \( f^{\nu_0 + \cdots + \nu_k} Q(R_{i_0 \cdots i_k}) \), and consider their graph representations \( \gamma_1 = \{(x, \gamma_1(x))\} \) and \( \gamma_2 = \{(x, \gamma_2(x))\} \). Let \( L(x) = | \gamma_1(x) - \gamma_2(x) | \). Proposition 7.3 and the Gronwall inequality give \( L(x)/L(y) \leq e^{C_3^{4\nu_k}|x-y|} \) for all \( x, y \). As \( |x-y| \leq C \delta^{-1} b \), \( L(x)/L(y) \leq 2 \) holds.

Let \( S_{\nu_{k-1}}, S_{\nu_{k-2}}, \cdots \) denote the components of \( R_{i_0 \cdots i_{k-1}}(\nu_k) \), the total number of which is clearly \( \leq 2^{\nu_k} \). For each \( S_{\nu_{k,m}} \), the above estimate and Proposition 7.3 give
\[
\frac{| B^{(\nu_k)}(S_{\nu_{k,m}}) \cap f^{\nu_0 + \cdots + \nu_k} Q(R_{i_0 \cdots i_k}) |}{| f^{\nu_0 + \cdots + \nu_k} Q(R_{i_0 \cdots i_k}) |} \leq 2 \delta^{\nu_k}.
\]

Proposition 7.3 gives \( \frac{| \det D f^{\nu_0 + \cdots + \nu_k}(\xi_1) |}{| \det D f^{\nu_0 + \cdots + \nu_k}(\xi_2) |} \leq e^{C_1^{-1} k} \) for all \( \xi_1, \xi_2 \in Q(R_{i_0 \cdots i_k}) \). Hence
\[
\frac{| f^{-(\nu_0 + \cdots + \nu_k)}(B^{(\nu_k)}(S_{\nu_{k,m}})) \cap R_{i_0 \cdots i_{k-1}} |}{| R_{i_0 \cdots i_{k-1}} |} \leq \frac{| f^{-(\nu_0 + \cdots + \nu_k)}(B^{(\nu_k)}(S_{\nu_{k,m}})) \cap Q(R_{i_0 \cdots i_k}) |}{| Q(R_{i_0 \cdots i_k}) |} \leq 2 e^{C_1^{-1} k} \delta^{\nu_k}.
\]
The first inequality follows from the obvious inclusion \( Q(R_{i_0 \ldots i_k}) \subset R_{i_0 \ldots i_{k-1}} \). Summing this over all components, and then for all feasible \( \nu_k \),

\[
\sum_{\nu_k} \sum_m |S_{\nu_k,m}| \leq 2 \sum_{\nu_k \geq 1} 2^{\nu_k} e^{C_1^{-1} k \delta \frac{d_{k_0}}{\nu_k}} \leq e^{C_1^{-1} k \delta \frac{d_{k_0}}{\nu}}.
\]

Therefore

\[
|\Omega_k| = \sum_{(i_0, \ldots, i_k)} |R_{i_0 \ldots i_k}| = \sum_{(i_0, \ldots, i_{k-1})} |R_{i_0 \ldots i_{k-1}}| \sum_{\nu_k,m} |S_{\nu_k,m}| \leq e^{C_1^{-1} k \delta \frac{d_{k_0}}{\nu}} |\Omega_{k-1}|.
\]

This completes the proof of Proposition 7.2.

7.10. **Transitivity.** We show \( f \) is transitive on \( K \). Let \( H(Q) \) denote the closure of transverse homoclinic points of \( Q \). Then \( H(Q) \subset K \) holds. It suffices to show the reverse inclusion. Let \( z \in K \), and let \( U \) be an open set containing \( z \). Since the Lebesgue measure of \( U \cap K^+ \) is zero, \( U \) intersects \( W^s(Q) \). It follows that \( W^s(Q) \) is dense in \( K \). By Inclination Lemma, \( z \) is accumulated by transverse homoclinic points of \( Q \). Hence \( K \subset H(Q) \) holds.

**APPENDIX**

A.1. **Proof of Lemma 4.6.** From the next sublemma, it follows that \( f^n G \cap I(\delta) \) is made up of \( C^2(b) \)-curves. This yields the conclusion of Lemma 4.6. For a proof of it, the correct order for the reader is to go over Sects 5.2, 5.3 first. For \( z \in W^u(Q) \), let \( t(z) \) denote any unit vector tangent to \( W^u(Q) \) at \( z \).

**Sublemma 7.2.** Let \( n \geq 0 \) and \( z \in G \). If \( f^iz \notin I(\delta) \) for \( 0 \leq i \leq n \), then there exists a sequence \( 0 \leq n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq n_3 < \cdots \leq n \) of integers such that:

(a) \( f^{n_i}z \in I(\delta) \);
(b) \( f^{j}z \in \{(x, y) \in \mathbb{R}^2 : |x| \geq 9/10\} \) for \( n_i + 1 \leq j \leq n_i + p_i \);
(c) \( \|Df^{n_i}t(z)\| \geq (\delta/10)\|Df^i t(z)\| \) for \( 0 \leq j < n_i \).

**Proof of Sublemma 7.2.** The argument is an induction on \( n \). For \( n = 0 \), the assertions are direct consequences of the definition of \( G \). Suppose that they hold for \( n = k \). From the fact that the orbits of all critical points on \( W^u(Q) \) are out of \( R_0 \), all the estimates in Proposition 5.2 remain to hold for them. This allows us to decompose the orbit of \( z \) into bound and free segments as follows: \( n_i \leq k \) is a return time to \( I(\delta) \). By the assumption of the induction, there exists a \( C^2(b) \)-curve in \( W^u(Q) \) tangent to \( Df^{n_i}t(z) \) stretching across \( I(\delta) \). Let \( p_i \) denote the bound period, given by the critical point on the \( C^2(b) \)-curve and an associated critical partition in Sect 5.3. Let \( n_{i+1} \) denote the next return time to \( I(\delta) \). By (c) in Proposition 5.2 bound parts of \( f^{k+1}G \) do not return to \( I(\delta) \). This recovers all the assertions for \( n = k + 1 \).

A.2. **Proof of Lemma 5.1.** First, for \( M \leq k < 20n - 1 \), we show

\[
e^{-3\alpha k} D_k(\zeta) \leq D_{k+1}(\zeta) \leq e^{-3\alpha} D_k(\zeta).
\]

To this end, let \( d_c(i) = \min_{j \in [i, i+1]} \|w_j(\zeta)\|/\|w_i(\zeta)\|^{-3} \). Then

\[
\frac{D_{k+1}(\zeta)}{D_k(\zeta)} = e^{-3\alpha} \min_{j \in [1, k+1]} d_{k+1}(i) d_k(i) \leq e^{-3\alpha} \min_{j \in [1, k]} d_{k+1}(i) d_k(i) \leq e^{-3\alpha},
\]

and the second inequality holds.
(G2) gives \( \|w_{k+2}(\zeta)\| \geq e^{-2\alpha(k+1)}\|w_{k+1}(\zeta)\| \), and thus for \( 1 \leq i \leq k \),
\[
d_{k+1}(i) = \min \left\{ d_k(i), \|w_{k+2}(\zeta)\|^2\|w_i(\zeta)\|^{-3} \right\} \geq e^{-4\alpha k}d_k(i) \geq e^{-4\alpha(k+1)}D_k(\zeta).
\]
Using \( \|w_{k+1}(\zeta)\| \leq C_0\|w_k(\zeta)\| \) and \( \|w_j(\zeta)\| \geq e^{-2\alpha k}\|w_{j-1}(\zeta)\| \) from (G2),
\[
d_{k+1}(k+1) \geq C_0^{-3}e^{-4\alpha(k+1)}d_k(k).
\]

These two inequalities yield the first inequality in (43).

We now show (a) (b). From (f) Proposition 5.2, [21] Lemma 2.3 and the fact that \( \gamma \) is \( C^2(b) \), \( f_{\chi(k)}\gamma_{k,s} \) is \( C^2(b) \). Using (43),
\[
\text{length}(f_{\chi(k)}\gamma_{k,s}) \geq C e^{-3\alpha k}\|w_{\chi(k)}(\zeta)\|(D_k(\zeta) - D_{k+1}(\zeta)) \geq C e^{-3\alpha k}\|w_{\chi(k)}(\zeta)\|D_k(\zeta)(1 - e^{-3\alpha})
\]
\[
\geq C e^{-3\alpha k}\|w_k(\zeta)\|D_k(\zeta)C_0^{-\sqrt{\alpha}}(1 - e^{-3\alpha}) \geq e^{-4\alpha k}.
\]

The third inequality follows from \( k - \chi(k) \leq \sqrt{\alpha}k \) in (G2). Using \( D_{k+1}(\zeta) \geq C_0^{-3k} \) and length(\( \gamma_k \)) \( \leq C e^{2\alpha k}D_{k+1}^\frac{1}{2}(\zeta) \) which follows from (43),
\[
\text{length}(\gamma_{k,s}) \leq e^{-3\alpha k} \cdot \text{length}(\gamma_k) \leq D_{k+1}^{\frac{1}{2} + \frac{\alpha}{\sqrt{\alpha}k}}(\zeta) \leq d(\gamma_{k,s}, \zeta)^{1+C\alpha}.
\]
Here, \( d(\gamma_{k,s}, \zeta) \) denotes the distance between \( \gamma_{k,s} \) and \( \zeta \). Now (b) follows from [21] Lemma 5.12. \( \square \)

**A.3. Derivative estimates of projectivization.** We prove (10) (11). Let \( v^\perp \) denotes any unit vector orthogonal to \( v \). Then
\[
|\partial_v f_*(\xi, v)| = \lim_{\Delta \theta \to 0} \left| \frac{1}{\Delta \theta} \left( \frac{Df(\xi)(v + \Delta \theta v^\perp)}{\|Df(\xi)(v + \Delta \theta v^\perp)\|} - \frac{Df(\xi)v}{\|Df(\xi)v\|} \right) \right|
\]
\[
\leq \frac{\|Df(\xi)v^\perp\|}{\|Df(\xi)v\|} + \lim_{\Delta \theta \to 0} \left| \frac{1}{\Delta \theta} \left( \frac{Df(\xi)v - \|Df(\xi)(v + \Delta \theta v^\perp)\|}{\|Df(\xi)v\|} \right) \right|
\]
\[
\leq 2 \frac{\|Df(\xi)v^\perp\|}{\|Df(\xi)v\|} = 2 \frac{\det Df(\xi)}{\|Df(\xi)v\|^2}.
\]

Let \( \xi = (x, y) \). Writing \( \xi_x = \xi + (\Delta x, 0) \) we have
\[
|\partial_x f_*(\xi, v)| = \lim_{\Delta x \to 0} \left| \frac{1}{\Delta x} \left( \frac{Df(\xi_x)v}{\|Df(\xi_x)v\|} - \frac{Df(\xi)v}{\|Df(\xi)v\|} \right) \right|
\]
\[
= \lim_{\Delta x \to 0} \left| \frac{1}{\Delta x} \left( \frac{Df(\xi_x)v - Df(\xi)v}{\|Df(\xi)v\|} \right) \right|
\]
\[
\leq 2 \lim_{\Delta x \to 0} \left| \frac{1}{\Delta x} \left( \frac{Df(\xi_x)v - Df(\xi)v}{\|Df(\xi)v\|} \right) \right| = 2 \frac{\|Df(\xi)v\|}{\|Df(\xi)v\|} \left( \frac{\partial}{\partial x} Df(\xi) \right)v.
\]
In the same way we get
\[
|\partial_y F(\xi, v)| \leq \frac{2}{\|Df(\xi)v\|} \left( \frac{\partial}{\partial y} Df(\xi) \right)v.
\]
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$$f^\nu_0 + \ldots + \nu_k (R_{i_0 \ldots i_{k-1}})$$

$\mathcal{B}^{(\nu_k)}$

$\mathcal{B}_0^{(\nu_k)}$
$\mathcal{Q}^{(k-1)}$
$2e^{-r_{k-1}/3} \cdot J_{k-1,j}$

$J_{k,i}$

$J_{k-1,j}$

$a_{(k-1,j)}$

$J_{(k-1,j)}$

$J_{(k,1)}$
