TIME-INCONSISTENT STOCHASTIC OPTIMAL CONTROL PROBLEMS: A BACKWARD STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS APPROACH

ISHAK ALIA
Department of Mathematics
University of Bordj Bou Arreridj, 34000 Algeria

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Abstract. In this paper, we investigate a class of time-inconsistent stochastic control problems for stochastic differential equations with deterministic coefficients. We study these problems within the game theoretic framework, and look for open-loop Nash equilibrium controls. Under suitable conditions, we derive a verification theorem for equilibrium controls via a flow of forward-backward stochastic partial differential equations. To illustrate our results, we discuss a mean-variance problem with a state-dependent trade-off between the mean and the variance.

1. Introduction. In recent years, there is a growing literature investigating time-inconsistent stochastic control problems, where the objective functional includes non-standard terms such that Bellman’s principle of optimality does not hold (i.e., an optimal control selected at a given moment might not remain optimal at later time moments). A typical example is the continuous-time mean-variance (MV) portfolio selection model [47], where the time-inconsistency is caused by a non-linear term of conditional expectation in the MV objectives which causes the failure of the iterated-expectations property. Some other examples of time-inconsistent models do exist as well, to cite a few, include the hyperbolic discounting related to people preferences [37], a state-dependent utility function in economics [14] and probability distortion as in behavioral finance models [23]; we refer the readers, for example, to Björk and Murgoci [8] and Delong [12] for a detailed discussion about the possible sources of time-inconsistency in typical stochastic control models.

The study of time-inconsistency goes back to Robert H. Strotz in his work [37] on a deterministic hyperbolic discounting Ramsay problem in 1952. In light of non-applicability of the classical calculus of variation approach, Strotz was the first to suggest two possible ways to deal with dynamic time-inconsistent problems: (i) The strategy of pre-commitment; (ii) and the consistent planning (game theoretic) approach. In the pre-commitment approach, the controller pre-commits to follow the initial optimal control \( \bar{u}_0 (\cdot) \) (i.e. the strategy which optimizes the performance functional at time 0), despite the fact that at future dates he will no longer be optimal according to his preferences. For example, Li and Ng [28] and Zhou and Li [47]

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introduced an embedding technique to solve for a pre-commitment MV portfolio in both discrete and continuous-time models. Anderson and Djehiche [3] dealt with the stochastic maximum principle approach for a pre-commitment MV portfolio problem. More interestingly, in the consistent planning approach, a time-inconsistent problem is considered as a non-cooperative game, in which decisions at every instant of time are selected as if various players (which represent the incarnations of the controller at each instant of time) are anticipated to optimize their own objective functions. Nash equilibriums are therefore adopted as concepts of solutions to the problem; for a detailed introduction see e.g. [37], [16], [17], [8], [43], [19], [1], [2], [42] and references therein.

Although the game theoretic formulation is easy to understand when the time-setting is discrete, in a continuous-time setting the formulation is considerably more delicate and the concept of the equilibrium solution can be adopted in quit different frameworks. In order to investigate the investment-consumption problem under hyperbolic discounting in both deterministic and stochastic models, Ekeland and Lazrak [16] and Ekeland and Pirvu [17] were the first to provide a precise definition of the equilibrium concept in continuous-time within the class of feedback controls. In addition, they derived among other things an extended Hamilton-Jacobi-Bellman (HJB) equation along with a verification theorem that characterizes feedback equilibriums. Yong [43] performed a multi-person differential game approach for a general discounting time-inconsistent stochastic optimal control problem and he characterized the Nash equilibrium controls via the so-called equilibrium HJB equation. Further work devoted to Yong’s approach can be found in [44], [38], [41], [40] and [32]. In a series of papers, Basak and Chabakauri [7], Zeng et al. [46], Böjrk et al. [9] derived feedback Nash equilibrium solutions to various dynamic optimization problems under the mean–variance criterion by solving an extended HJB system. In Hu et al. ([19], [20]), the authors considered a time-inconsistent stochastic linear-quadratic problem with random coefficients. Different from all the literature above-listed, they adopted the concept of the Nash equilibrium solution within the class of open-loop controls\(^1\) and performed a duality approach to characterize the existence as well as the uniqueness of the equilibrium solution via a flow of forward-backward stochastic differential equations (FBSDEs). They also undertook a deep study of the mean–variance portfolio selection problem within a random coefficients diffusion model. The work of Djehiche and Huang [13] extended [19] to a class of time inconsistent decision problems for dynamics that are driven by diffusion processes of mean field type. More recently, Björk et al. [10] generalized the extended HJB equations method for feedback equilibriums to a quite general class of Markov diffusion processes and a fairly general objective functional. Karnam et al. [24] introduced the idea of “dynamic utility” under which the original time-inconsistent problem is transferred to a time-consistent one. Hu et al. [21] extended the work [20] by incorporating control constraints. Wang [39] discussed open-loop equilibrium controls and their particular closed-loop representations for a class of time inconsistent stochastic linear-quadratic optimal control problems of mean-field stochastic differential equations. The work of Alia [1] discussed open-loop equilibriums for a class of non-exponential discounting time-inconsistent stochastic control problems under jump-diffusion models; we refer the readers to the survey paper [42]

\(^1\)Note that the class of feedback controls is a subset of that of open-loop ones. In classical (time-consistent) stochastic control theory, an optimal control is usually defined in the whole class of open-loops [46]; we refer the readers to [19] for more detailed discussion.
for a detailed discussion about the existing literature on time-inconsistent stochastic control problems.

In this paper we formulate a fairly general stochastic optimal control problem, where the objective functional includes simultaneously a general (non-exponential) discount term, a state-dependent term and a nonlinear term of the expected state. These non-standard terms introduce time-inconsistency into the problem in somewhat different ways. A similar problem was considered by Björk et al. [10] in which the equilibrium is, however, defined within the class of feedback controls. Here we consider equilibriums in the whole class of open-loop controls. By using some techniques of FBSDEs with random coefficients found in Ma and Yong [29], we derive a verification theorem for open-loop equilibriums through a system of coupled forward-backward stochastic partial differential equations (FBSPDEs). An intriguing feature of these FBSPDEs is that they are consisting of a single forward SDE and a flow of backward stochastic partial differential equations (BSPDEs), which are defined on different time-intervals and connected via an equilibrium condition. We point out that, the basic idea of the approach in the present paper was first introduced by Peng [35], in the framework of classical time-consistent stochastic control problems. However, to the best of our knowledge, the above-described FBSPDEs appear for the first time in the literature. To illustrate our results, we discuss a continuous-time mean-variance portfolio selection problem, where the objective of the problem is to maximize the expected terminal return and minimize the variance of the terminal wealth at the same time. We apply the verification theorem (Theorem 3.3) to derive the equilibrium solution by solving a system of ordinary differential equations. The solution we obtain coincides with that obtained in [19] by solving a flow of FBSDEs having finite-dimensional states processes.

The plan of the paper is as follows, in the second section, we formulate the problem and give necessary notations and preliminaries. Section 3 is devoted to the presentation of the verification theorem. In Section 4, we discuss the relationship between the BSPDEs approach of the present paper and the duality approach of Hu et al. ([19], [20]). Finally, in Section 5, we discuss a mean-variance portfolio selection model.

2. Formulation of the problem. Throughout this paper $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ is a filtered probability space such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets, $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed finite time horizon $T > 0$, and $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions. $\mathcal{F}_t$ stands for the information available up to time $t$ and any decision made at time $t$ is based on this information. We also assume that all processes and random variables are well defined and adapted in this filtered probability space. Let $W(\cdot)$ be a $d$-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. For simplicity, it is assumed that the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ coincides with the one generated by the Brownian motion; that is $\mathcal{F}_t = \sigma(W(r); 0 \leq r \leq t)$.

We use $C^T$ to denote the transpose of any vector or matrix $C$ and for a function $f$, we denote by $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ (resp. $\nabla^2 f = (\frac{\partial^2 f}{\partial x_i \partial x_j}; 1 \leq i \leq n, 1 \leq j \leq n)$) the gradient or Jacobian (resp. the Hessian) of $f$ with respect to the variable $x \in \mathbb{R}^n$. We denote by $\chi_A$ the indicator function of the set $A$. In addition, we use the following notations for several sets and spaces of processes on the filtered probability space, which will be used later:

- $D[0,T] = \{(t,s) \in [0,T] \times [0,T], \text{ such that } s \geq t\}$
- $S^n$ : the set of $(n \times n)$ symmetric matrices.
Here $R$ represents the control process; $X$ measurable functions, where $n$ is regarded as the initial state.

Finally, for any $p ≥ 1$ and any infinite-dimensional Banach space $E$ with a norm $||\cdot||_E$, we denote:

- $C^p_F (t, T; \mathbb{R}^l)$: the set of $\mathbb{R}^l$-valued, $(\mathcal{F}_s)_{s \in [t, T]}$-adapted continuous processes $X(\cdot)$, with
  \[
  \|X(\cdot)\|_{C^p_F (t, T; \mathbb{R}^l)} := \left( \mathbb{E} \left[ \sup_{s \in [t, T]} |X(s)|^p \right] \right)^{\frac{1}{p}} < \infty.
  \]

- $L^p_F (t, T; \mathbb{R}^l)$: the space of $\mathbb{R}^l$-valued, $(\mathcal{F}_s)_{s \in [t, T]}$-adapted processes $Y(\cdot)$, with
  \[
  \|Y(\cdot)\|_{L^p_F (t, T; \mathbb{R}^l)} := \left( \mathbb{E} \left[ \int_t^T |Y(s)|^p \, ds \right] \right)^{\frac{1}{p}} < \infty.
  \]

- $L^p (\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^l)$: the set of $\mathbb{R}^l$-valued, $\mathcal{F}_t$-measurable random variables $\zeta$, with
  \[
  ||\zeta||_{L^p (\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^l)} = (\mathbb{E} [||\zeta||^p])^{\frac{1}{p}} < \infty.
  \]

- $C^k (\mathbb{R}^n; \mathbb{R}^l)$: the set of functions from $\mathbb{R}^n$ to $\mathbb{R}^l$ that are continuously differentiable up to order $k ≥ 1$.

- $C^k_b (\mathbb{R}^n; \mathbb{R}^l)$: the set of those functions in $C^k (\mathbb{R}^n; \mathbb{R}^l)$ whose partial derivatives up to order $k$ are uniformly bounded.

- $H^k (\mathbb{R}^n)$: the usual Sobolev space.

- $C^k_F ([t, T] \times \mathbb{R}^n; \mathbb{R}^l)$: the set of all random fields $\phi(s, x)$ from $[t, T] \times \mathbb{R}^n \times \Omega$ to $\mathbb{R}^l$ such that:
  (i) for each fixed $x ∈ \mathbb{R}^n$, $\phi(\cdot, x)$ is an $(\mathcal{F}_s)_{s \in [t, T]}$-adapted continuous process;
  (ii) for each $s ∈ [t, T]$, $\phi(s, \cdot)$ is continuous almost surely.

- $C^0_F^k ([t, T] \times \mathbb{R}^n; \mathbb{R}^l)$: the set of all random fields $\phi(s, x)$ from $[t, T] \times \mathbb{R}^n \times \Omega$ to $\mathbb{R}^l$ such that:
  (i) for each fixed $x ∈ \mathbb{R}^n$, $\phi(\cdot, x)$ is an $(\mathcal{F}_s)_{s \in [t, T]}$-adapted continuous process;
  (ii) for each $s ∈ [t, T]$, $\phi(s, \cdot)$ is continuously differentiable up to order $k ≥ 1$, almost surely.

Finally, for any $p ≥ 1$ and any infinite-dimensional Banach space $E$ with a norm $||\cdot||_E$, we denote:

- $C^p_F (t, T; E)$: the set of all $(\mathcal{F}_s)_{s \in [t, T]}$-predictable $E$-valued continuous processes $\theta(\cdot)$, with
  \[
  \|\theta(\cdot)\|_{C^p_F (t, T; E)} := \sup_{s \in [t, T]} (\mathbb{E} [||\theta(s)||_E^p])^{\frac{1}{p}} < \infty.
  \]

- $L^p_F (t, T; E)$: the set of all $(\mathcal{F}_s)_{s \in [t, T]}$-predictable $E$-valued processes $\psi(\cdot)$, with
  \[
  \|\psi(\cdot)\|_{L^p_F (t, T; E)} := \left( \mathbb{E} \left[ \int_t^T ||\psi(s)||_E^p \, ds \right] \right)^{\frac{1}{p}} < \infty.
  \]

We consider a continuous-time, n-dimensional, controlled system

\[
\left\{ \begin{array}{l}
  dX^{x_0, u(\cdot)}(s) = b(s, X^{x_0, u(\cdot)}(s), u(s)) \, ds \\
  \quad + \sigma(s, X^{x_0, u(\cdot)}(s), u(s)) \, dW(s), \ s ∈ [0, T], \\
  X^{x_0, u(\cdot)}(0) = x_0.
\end{array} \right.
\]  

(1)

Here $b : [0, T] × \mathbb{R}^n × U → \mathbb{R}^n$ and $\sigma : [0, T] × \mathbb{R}^n × U → \mathbb{R}^n × \mathbb{R}^d$ are two deterministic measurable functions, where $U ⊂ \mathbb{R}^m$ is a given closed subset; $u(\cdot) : [0, T] × \Omega → U$ represents the control process; $X^{x_0, u(\cdot)}(\cdot)$ is the controlled state process; and $x_0 ∈ \mathbb{R}^n$ is regarded as the initial state.
When time evolves, we consider the following controlled system starting from the situation \((t, \xi) \in [0, T] \times \mathbb{R}^n\) and satisfied by \(X(\cdot) = X^t,\xi,u(\cdot)(\cdot)\),
\[
\begin{aligned}
\left\{ 
\begin{array}{ll}
\frac{dX(s)}{ds} = b(s, X(s), u(s)) \, ds + \sigma(s, X(s), u(s)) \, dW(s), & s \in [t, T], \\
X(t) = \xi.
\end{array}
\right.
\end{aligned}
\] (2)

At any time \(t\) with the system state \(X(t) = \xi\), in order to evaluate the cost-performance of a control process \(u(\cdot)\), we introduce the functional
\[
J(t, \xi; u(\cdot)) := \mathbb{E}_t \left[ \int_t^T f(t, \xi, s, X(s), u(s)) \, ds + F(t, \xi, X(T)) \right] + G(t, \xi, \mathbb{E}_t[X(T)]) ,
\] (3)
where \(\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]\); \(f : [0, T] \times \mathbb{R}^n \times [0, T] \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}\), \(F : [0, T] \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}\) and \(G : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}\) are three deterministic measurable functions.

This functional is not suitable for dynamic programming. Specifically, the first term \(f(t, \xi, s, X(s), u(s))\), which explicitly depends on the initial time \(t\) is motivated by the non-exponential discounting in economics, see e.g. [37], [17], [8], [43], [41]; the second term \(F(t, \xi, X(T))\) which depends on the state \(\xi\) at time \(t\), stems from a state-dependent utility function in economics see e.g. [14], [12]; and in the last term \(G(t, \xi, \mathbb{E}_t[X(T)])\), we have a non-linear function \(G\) acting on the expected value \(\mathbb{E}_t[X(T)]\), which can be motivated by the variance term in the mean–variance portfolio problem [9]. Each of these three terms leads to time-changing preferences of the Decision-Maker, thus decisions being made at different time points can be different with each other leading the problem to time-inconsistency; please refer to [8] or [42] for a more detailed discussion.

We introduce the following assumptions.

**(H1)** The maps \(b, \sigma\) are continuous and there exists a constant \(K > 0\) such that for all \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n\) and \((s, u) \in [0, T] \times U\),
\[
|b(s, x, u) - b(s, y, u)| + |\sigma(s, x, u) - \sigma(s, y, u)| \leq K|x - y|
\]
and
\[
|b(s, 0, u)| + |\sigma(s, 0, u)| \leq K(1 + |u|).
\]

**(H2)** (i) The maps \(f, F\) are continuous and quadratic growth on \(\xi, x\) and \(u\) uniformly in time, i.e. there exists a constant \(K > 0\) such that for all \((\xi, x) \in \mathbb{R}^n \times \mathbb{R}^n\) and \((s, u) \in [0, T] \times [0, T] \times U\),
\[
|f(t, \xi, s, x, u)| \leq K \left(1 + |\xi|^2 + |x|^2 + |u|^2\right),
\]
\[
|F(t, \xi, x)| \leq K \left(1 + |\xi|^2 + |x|^2\right).
\]

(ii) The map \(G\) is continuously differentiable with respect to \(\bar{x}\) and there exists a constant \(K > 0\) such that for all \((t, \xi, \bar{x}) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n\),
\[
|G(t, \xi, \bar{x})| \leq K \left(1 + |\xi|^2 + |\bar{x}|^2\right),
\]
\[
|G_{\bar{x}}(t, \xi, \bar{x})| \leq K \left(1 + |\xi| + |\bar{x}|\right).
\]

Under Assumptions (H1)-(H2), for any initial pair \((t, \xi) \in [0, T] \times \mathbb{R}^n\) and a control \(u(\cdot) \in \mathcal{L}^p_x(t, T; \mathbb{R}^m)\), with \(p \geq 2\), the state equation (2) admits a unique
solution $X(\cdot) = X^{t,\xi,u(\cdot)}(\cdot) \in C^p_F(t,T;\mathbb{R}^n)$ (see e.g. [45]) and the cost functional (3) is well-defined. Moreover, there exists a constant $K > 0$ such that

$$
\mathbb{E}\left[\sup_{t \leq s \leq T} |X(s)|^p\right] \leq K (1 + |\xi|^p).
$$

We now introduce the class of admissible controls.

**Definition 2.1** (Admissible control). An admissible control $u(\cdot)$ over $[t, T)$ is a $U$-valued $(\mathcal{F}_s)_{s \in [t,T]}$-adapted right-continuous with finite left hand limits (càdlàg) process such that

$$
\mathbb{E}\left[\sup_{s \in [t,T]} |u(s)|^4\right] < \infty.
$$

The class of admissible controls over $[t, T)$ is denoted by $U[t, T)$.

Our stochastic optimal control problem can be stated as follows.

**Problem (N).** For any given initial pair $(t, \xi) \in [0, T] \times \mathbb{R}^n$, find a $\bar{u}^{t,\xi}(\cdot) \in U[t, T)$ such that

$$
\mathbf{J} \left(t, \xi; \bar{u}^{t,\xi}(\cdot)\right) = \inf_{u(\cdot) \in U[t, T)} \mathbf{J} \left(t, \xi; u(\cdot)\right).
$$

3. **Equilibrium controls.** Each term in the cost functional (3) introduces time-inconsistency of Problem (N) in somewhat different ways. As already mentioned in the introduction section, one way to get around the time-inconsistency issue is to consider only pre-committed strategies: We fix one initial point, like for example $(0, x_0)$, and then try to find the control process $\bar{u}^{x_0}(\cdot)$ which optimizes $\mathbf{J}(0, x_0; \cdot)$. We then simply disregard the fact that at a later state such as $(t, X^{t,\bar{u}^{x_0}(\cdot)}(t))$ the control $\bar{u}^{x_0}(\cdot)$ will not be optimal for the functional $\mathbf{J}(t, X^{t,\bar{u}^{x_0}(\cdot)}(t); \cdot)$. However, as claimed by Björk and Murgoci [8], to deal with time-inconsistency in a more sophisticated way, one needs to study the problem within a game theoretic framework and look for the Nash equilibrium strategy instead of the pre-commitment strategy. This is in fact the approach of the present paper. Loosely speaking we view the game as follows (see e.g. [10]):

- Consider a non-cooperative game with one player at each point $t$ in time. This player represents the incarnation of the controller at time $t$ and is referred to as “Player $t$”.
- For each fixed $t \in [0, T)$, Player $t$ can control the system only at time $t$. He/she does that by taking his/her strategy $u(\cdot) : (\Omega, \mathcal{F}_t) \to U$.
- An admissible control process $u(\cdot) \in U[0, T)$ is then viewed as a complete description of the chosen strategies of all players in the game.
- The objective function to player $t$ is given by the functional:

$$
\mathbf{J}^t_{\{u(s)\}_{s \in (t,T)}}(u(t)) := \mathbf{J} \left(t, X^{t,x_0,u(\cdot)}(t); u(\cdot)\right),
$$

where $\{u(s)\}_{s \in (t,T)}$ represent the chosen strategies of all Players $s > t$.

With the above game perspective, the concept of the *Nash equilibrium control* $\hat{u}(\cdot)$ can be intuitively described as follows:

- $\hat{u}(\cdot) \in U[0, T)$.
- Suppose that for every $s > t$, the optimal strategy for Player $s$ is $\hat{u}(s)$. Then the optimal choice for player $t$ is that he/she also uses the strategy $\hat{u}(t)$. 

Nevertheless, the problem with this implicit definition of the equilibrium concept, is that the individual player \( t \) does not really influence the outcome of the game at all. He/she only chooses the control at the single point \( t \), and since this is a time set of Lebesgue measure zero, the control dynamics will not be influenced. We thus need to a more explicit definition of the equilibrium concept, and we will in fact follow Hu et al. [20] who suggested the following formal definition of the so-called open-loop Nash equilibrium control that uses a “local” spike variation in a natural way.

Given an admissible control \( \hat{u}(\cdot) \in \mathcal{U}[0, T) \). For any \( t \in [0, T) \), \( v(\cdot) \in \mathcal{U}[0, T) \) and for any \( \varepsilon \in [0, T - t) \), define
\[
\begin{align*}
\hat{u}_{t,\varepsilon,v}(s) = \begin{cases} 
v(s), & \text{for } s \in [t, t + \varepsilon), \\
\hat{u}(s), & \text{for } s \in [t + \varepsilon, T).
\end{cases}
\end{align*}
\]

**Definition 3.1 (Equilibrium Control [20]).** Let \( \hat{u}(\cdot) \in \mathcal{U}[0, T) \) be a given control and \( \hat{X}_{x_0}(\cdot) = X_{x_0,\hat{u}}(\cdot) \) be the state process corresponding to \( \hat{u}(\cdot) \). The control \( \hat{u}(\cdot) \) is called an open-loop Nash equilibrium control for Problem (N) if
\[
\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ J(t, \hat{X}_{x_0}(t); \hat{u}_{t,\varepsilon,v}(\cdot)) - J(t, \hat{X}_{x_0}(t); \hat{u}(\cdot)) \right\} \geq 0,
\]
where \( u_{t,\varepsilon,v}(\cdot) \) is defined by (4), for any \( t \in [0, T) \) and \( v(\cdot) \in \mathcal{U}[0, T) \).

In the rest of this paper, sometimes we simply call \( \hat{u}(\cdot) \) an equilibrium control instead of open-loop Nash equilibrium control when there is no ambiguity.

**Remark 1.** It is important to emphasize that the perturbation of the control \( \hat{u}(\cdot) \) in \([t, t + \varepsilon)\) will not change \( \hat{u}(\cdot) \) in \([t + \varepsilon, T)\), this is not the case with the closed-loop equilibrium concepts adopted by [17], [10] and [43].

### 3.1. BSPDEs and verification Theorem.

The idea of defining the equilibrium control within the whole class of open-loop controls goes back to Hu et al. ([19], [20]), where the authors undertook a deep study of a time-inconsistent stochastic LQ model with random coefficients. Inspired by the classical stochastic maximum principle approach ([34], [4]), they developed a duality approach to characterize the existence as well as the uniqueness of the open-loop equilibrium solution via a new type of stochastic systems, namely, flows of forward-backward stochastic differential equations. Loosely speaking, they are systems consisting of a single forward SDE and a flow of backward stochastic differential equations (BSDEs) plus a minimum condition called the equilibrium law. The unique (global) solvability of this type of systems remains a challenging open problem except for some special cases; we refer the readers to Hamaguchi [18] for a detailed discussion.

Our main aim in this paper is to suggest an alternative approach to characterize the open-loop equilibrium controls. Specifically, inspired by the idea of Four Step Scheme for FBSDEs with random coefficients (see e.g. Ma and Yong [29], Section 6), we prove a verification theorem for open-loop equilibriums via an intriguing class of coupled forward-backward stochastic equations. These FBSDEs differ from the ones of Hu et al. ([19], [20]) since they are consisting of a single forward SDE and a flow of BSPDEs, which are defined on different time-intervals and coupled via an equilibrium condition. We point out that the approach of the present paper is much inspired by the extended HJB-method of Björk et al. [10]. We are also deeply inspired by Yong [43], which constructed sophisticated cost functionals via some techniques of FBSDEs with deterministic coefficients. However, it is important to
mention that the basic idea of the method in our paper was introduced for the first time by Peng [35] in the framework of classical time-consistent stochastic control problems.

3.1.1. **BSPDEs, a heuristic derivation.** In this paragraph we will, in an heuristic way, derive the BSPDEs which play a fundamental role in this paper. Let \( \left( \hat{X}^{x_0}(\cdot), \hat{u}(\cdot) \right) \) be an admissible state-control pair and consider the perturbed strategy \( u^{t,\varepsilon,v(\cdot)}(\cdot) \) defined by the spike variation (4) for some fixed arbitrarily \( v(\cdot) \in \mathcal{U}[0,T], t \in [0,T] \) and \( \varepsilon \in [0,T-t] \). For simplicity of the presentation, suppose for the moment that \( n = d = 1 \).

Our goal is to determine a suitable expression to the difference
\[
\Delta \hat{J}(t) := J\left(t, \hat{X}^{x_0}(t); u^{t,\varepsilon,v(\cdot)}(\cdot)\right) - J\left(t, \hat{X}^{x_0}(t); \hat{u}(\cdot)\right),
\]
in order to be able to evaluate the limit in (5). To this end, let \( X^{t,\varepsilon,v(\cdot)}(\cdot) \) be the unique solution of the following SDE,
\[
\begin{aligned}
dX^{t,\varepsilon,v(\cdot)}(s) &= b\left(s, X^{t,\varepsilon,v(\cdot)}(s), u^{t,\varepsilon,v(\cdot)}(s)\right) ds \\
&+ \sigma\left(s, X^{t,\varepsilon,v(\cdot)}(s), u^{t,\varepsilon,v(\cdot)}(s)\right) dW(s), \quad s \in [t,T], \\
X^{t,\varepsilon,v(\cdot)}(t) &= \hat{X}^{x_0}(t).
\end{aligned}
\]
and for each fixed \( (s,x) \in [t,T] \times \mathbb{R}^n \), denote by \( \hat{X}^{s,x}(\cdot) \) the solution of the SDE
\[
\begin{aligned}
d\hat{X}^{s,x}(\tau) &= b\left(\tau, \hat{X}^{s,x}(\tau), \hat{u}(\tau)\right) d\tau + \sigma\left(\tau, \hat{X}^{s,x}(\tau), \hat{u}(\tau)\right) dW(\tau), \quad \tau \in [s,T], \\
\hat{X}^{s,x}(s) &= x.
\end{aligned}
\]

By the definition of the cost functional, we have
\[
J\left(t, \hat{X}^{x_0}(t); \hat{u}(\cdot)\right)
= \mathbb{E}_t \left[ \int_t^T f\left(t, \hat{X}^{x_0}(t), s, \hat{X}^{x_0}(s), \hat{u}(s)\right) ds + F\left(t, \hat{X}^{x_0}(t), \hat{X}^{x_0}(T)\right) \right] \\
+ G\left(t, \hat{X}^{x_0}(t), \mathbb{E}_t \left[ \hat{X}^{x_0}(T) \right] \right).
\]

Moreover, since \( u^{t,\varepsilon,v(\cdot)}(s) = v(s) \chi_{[t,t+\varepsilon]}(s) + \hat{u}(s) \chi_{[t+\varepsilon,T]}(s) \), the objective value of \( u^{t,\varepsilon,v(\cdot)}(\cdot) \) can be written as
\[
J\left(t, \hat{X}^{x_0}(t); u^{t,\varepsilon,v(\cdot)}(\cdot)\right)
= \mathbb{E}_t \left[ \int_t^{t+\varepsilon} f\left(t, \hat{X}^{x_0}(t), s, X^{t,\varepsilon,v(\cdot)}(s), v(s)\right) ds \\
+ \int_{t+\varepsilon}^T f\left(t, \hat{X}^{x_0}(t), s, \hat{X}^{t+\varepsilon,X^{t,\varepsilon,v(\cdot)}(t+\varepsilon)(s)}, \hat{u}(s)\right) ds \\
+ F\left(t, \hat{X}^{x_0}(t), \hat{X}^{t+\varepsilon,X^{t,\varepsilon,v(\cdot)}(t+\varepsilon)}(T)\right) \right] \\
+ G\left(t, \hat{X}^{x_0}(t), \mathbb{E}_t \left[ \hat{X}^{t+\varepsilon,X^{t,\varepsilon,v(\cdot)}(t+\varepsilon)}(T) \right] \right).
\]

Note that the control \( \hat{u}(\cdot) \) in \( [t+\varepsilon, T] \) is not influenced by the initial pair \( (t + \varepsilon, X^{t,\varepsilon,v(\cdot)}(t + \varepsilon)) \); this is quite different from the closed-loop control concept.
adopted by Björk et al. [10]. Now, if we define
\[
h^t(s,x) := \mathbb{E}_s \left[ \int_t^T f \left( t, X^{x_0}(t), \tau, X^{s,x}(\tau), \hat{u}(\tau) \right) d\tau + F \left( t, X^{x_0}(t), \hat{X}^{s,x}(T) \right) \right]
\]
and
\[
\hat{\zeta}(s,x) := \mathbb{E}_s \left[ \hat{X}^{s,x}(T) \right],
\]
then the objective values of \( \hat{u}(\cdot) \) and \( u^{t,\varepsilon,v(\cdot)}(\cdot) \) can be rewritten as
\[
J \left( t, X^{x_0}(t) ; \hat{u}(\cdot) \right) = h^t \left( t, X^{x_0}(t) \right) + G \left( t, X^{x_0}(t), \hat{\zeta} \left( t, X^{x_0}(t) \right) \right)
\]
and
\[
J \left( t, X^{x_0}(t) ; u^{t,\varepsilon,v(\cdot)}(\cdot) \right) = \mathbb{E}_t \left[ \int_t^{t+\varepsilon} f \left( t, X^{x_0}(t), s, X^{t,\varepsilon,v(\cdot)}(s), v(s) \right) ds + h^t \left( t + \varepsilon, X^{t,\varepsilon,v(\cdot)}(t + \varepsilon) \right) \right] \]
\[
+ G \left( t, X^{x_0}(t), \mathbb{E}_t \left[ \hat{\zeta} \left( t + \varepsilon, X^{t,\varepsilon,v(\cdot)}(t + \varepsilon) \right) \right] \right),
\]
respectively. Thus, \( \Delta \mathcal{J}^\varepsilon(t) \) takes the form
\[
\Delta \mathcal{J}^\varepsilon(t) = \mathbb{E}_t \left[ \mathbf{I}^{t,\varepsilon,v(\cdot)} + \hat{\Delta}_1^\varepsilon + \hat{\Delta}_2^\varepsilon \right],
\]
where
\[
\mathbf{I}^{t,\varepsilon,v(\cdot)} := \int_t^{t+\varepsilon} f \left( t, X^{x_0}(t), s, X^{t,\varepsilon,v(\cdot)}(s), v(s) \right) ds,
\]
\[
\hat{\Delta}_1^\varepsilon := h^t \left( t + \varepsilon, X^{t,\varepsilon,v(\cdot)}(t + \varepsilon) \right) - h^t \left( t, X^{x_0}(t) \right)
\]
and
\[
\hat{\Delta}_2^\varepsilon := G \left( t, X^{x_0}(t), \mathbb{E}_t \left[ \hat{\zeta} \left( t + \varepsilon, X^{t,\varepsilon,v(\cdot)}(t + \varepsilon) \right) \right] \right) \]
\[
- G \left( t, X^{x_0}(t), \hat{\zeta} \left( t, X^{x_0}(t) \right) \right).
\]

One may easily see that \( \hat{u}(\cdot) \) is an equilibrium control, if and only if, for any \( t \in [0,T) \) and \( v(\cdot) \in \mathcal{U}[0,T) \)
\[
\lim_{\varepsilon \downarrow 0} \inf_{\varepsilon} \mathbb{E}_t \left[ \mathbf{I}^{t,\varepsilon,v(\cdot)} + \hat{\Delta}_1^\varepsilon + \hat{\Delta}_2^\varepsilon \right] \geq 0.
\]

However, since the random fields \( \hat{h}^t(\cdot, \cdot) \) and \( \hat{\zeta}(\cdot, \cdot) \) seem to be a little too implicit, it is difficult for us to evaluate the limit in (7) under the above forms of \( \hat{\Delta}_1^\varepsilon \) and \( \hat{\Delta}_2^\varepsilon \). We would like to make them more explicit in some sense. Our situation is comparable to that of Yong [43], particularly when he constructed the sophisticated cost functional for Player (N-1) in his multi-person differential game approach. Inspired by [43] and by the idea of Four Step Scheme introduced in ([29], [31]) for FBSDEs with random coefficients, we proceed as follows. For each fixed \( (s,x) \in \)
[t, T] × \mathbb{R}^n$, we introduce the following system of FBSDEs:

\[
\begin{cases}
    d\hat{X}^{s,x}(\tau) = b\left(\tau, \hat{X}^{s,x}(\tau), \hat{u}(\tau)\right) d\tau \\
    \quad + \sigma\left(\tau, \hat{X}^{s,x}(\tau), \hat{u}(\tau)\right) dW(\tau), \text{ for } \tau [s, T], \\
    d\hat{Y}^{s,x}(\tau; t) = -f\left(t, \hat{X}^{s,x}(\tau), \hat{X}^{s,x}(\tau), \hat{u}(\tau)\right) d\tau \\
    \quad + \hat{Z}^{s,x}(\tau; t) dW(\tau), \text{ for } \tau [s, T], \\
    d\hat{K}^{s,x}(\tau) = \tilde{H}^{s,x}(\tau) dW(\tau), \text{ for } \tau [s, T], \\
    \hat{X}^{s,x}(s) = x, \hat{K}^{s,x}(T) = \hat{X}^{s,x}(T), \\
    \hat{Y}^{s,x}(T; t) = F\left(t, \hat{X}^{s,x}(t), \hat{X}^{s,x}(T)\right).
\end{cases}
\]

Under Assumptions (H1)-(H2), the above system of FBSDEs admits a unique adapted solution \(\hat{X}^{s,x}(\cdot), \hat{Y}^{s,x}(\cdot; \cdot), \hat{K}^{s,x}(\cdot)\) \(\in \mathcal{C}^2_f(s, T; \mathbb{R})^3\), \(\tilde{Z}^{s,x}(\cdot; \cdot), \hat{H}^{s,x}(\cdot)\) \(\in \mathcal{L}_2^f(s, T; \mathbb{R})^2\), uniquely depending on \((s, x)\) (see e.g. [45] or [36]). Further, one has

\[
\hat{Y}^{s,x}(s; t) = \mathbb{E}_s \left[ \int_t^T f\left(t, \hat{X}^{s,x}(t), \tau, \hat{X}^{s,x}(\tau), \hat{u}(\tau)\right) d\tau + F\left(t, \hat{X}^{s,x}(t), \hat{X}^{s,x}(T)\right) \right] = \hat{h}^t(s, x)
\]

and

\[
\hat{K}^{s,x}(s) = \mathbb{E}_s \left[ \hat{X}^{s,x}(T) \right] = \hat{\xi}(s, x).
\]

Note that, in (8), \(\hat{u}(\cdot)\) can be purely regarded as a random term incorporated into the coefficients \(b(\cdot, \cdot, \hat{u}(\cdot)), \sigma(\cdot, \cdot, \hat{u}(\cdot))\) and \(f(t, \hat{X}^{s,x}(t), \cdot, \cdot, \hat{u}(\cdot))\). Therefor, the stochastic system (8) can be considered as an FBSDEs with random coefficients. By [29] (see also [31]), we have the following representation for \(\hat{Y}^{s,x}(\cdot; \cdot)\)

\[
\hat{Y}^{s,x}(\tau; t) = \theta^t\left(\tau, \hat{X}^{s,x}(\tau)\right), \text{ for } \tau [s, T],
\]

as long as \((\theta^t(\cdot, \cdot), \psi^t(\cdot, \cdot))\) is an adapted solution of the following linear BSPDE

\[
\begin{cases}
    d\theta^t(s, x) = -\left\{ \theta^t_x(s, x) b(s, x, \hat{u}(s)) + \frac{1}{2} \sigma(s, x, \hat{u}(s))^2 \theta^t_{xx}(s, x) \\
    \quad + \psi^t_x(s, x) \sigma(s, x, \hat{u}(s)) + f\left(t, \hat{X}^{s,x}(t), s, x, \hat{u}(s)\right) \right\} ds \\
    \quad + \psi^t(s, x) dW(s), \text{ for } (s, x) \in [t, T] \times \mathbb{R}, \\
    \theta^t(T, x) = F\left(t, \hat{X}^{s,x}(t), x\right), \text{ for } x \in \mathbb{R}.
\end{cases}
\]

and we have the following representation for \(\hat{K}^{s,x}(\cdot)\)

\[
\hat{K}^{s,x}(\tau) = g\left(\tau, \hat{X}^{s,x}(\tau)\right), \text{ for } \tau [s, T],
\]

as long as \((g(\cdot, \cdot), \eta(\cdot, \cdot))\) satisfies the following BSPDE

\[
\begin{cases}
    dg(s, x) = -\left\{ g_x(s, x) b(s, x, \hat{u}(s)) + \frac{1}{2} \sigma(s, x, \hat{u}(s))^2 g_{xx}(s, x) \\
    \quad + \eta_x(s, x) \sigma(s, x, \hat{u}(s)) \right\} ds + \eta(s, x) dW(s), \\
    g(T, x) = x, \text{ for } x \in \mathbb{R},
\end{cases}
\]

for \((s, x) \in [0, T] \times \mathbb{R}\).}

Particularly, we have

\[
\hat{h}^t(s, x) = \hat{Y}^{s,x}(s; t) = \theta^t(s, x)
\]
and
\[ \hat{\zeta}(s, x) = \hat{K}^{x, \epsilon}(s) = g(s, x). \]

With the above representations of \( \hat{h}^t(\cdot, \cdot) \) and \( \hat{\zeta}(\cdot, \cdot) \), we can rewrite \( \Delta_1^\epsilon \) and \( \Delta_2^\epsilon \) as follows:
\[
\Delta_1^\epsilon := \theta^t \left( t + \epsilon, X^{t, \epsilon, v(t)}(t + \epsilon) \right) - \theta^t \left( t, \hat{X}^{x_0}(t) \right)
\]
and
\[
\Delta_2^\epsilon := G \left( t, \hat{X}^{x_0}(t), \mathbb{E}_t \left[ g \left( t + \epsilon, X^{t, \epsilon, v(t)}(t + \epsilon) \right) \right] \right) 
- G \left( t, \hat{X}^{x_0}(t), g \left( t, \hat{X}^{x_0}(t) \right) \right).
\]

Accordingly, we obtain that
\[
\Delta \hat{j}^\epsilon(t) = \mathbb{E}_t \left[ \int_0^t f \left( \hat{X}^{x_0}(t), s, X^{t, \epsilon, v(t)}(s), v(s) \right) ds \right] 
+ \mathbb{E}_t \left[ \theta^t \left( t + \epsilon, X^{t, \epsilon, v(t)}(t + \epsilon) \right) - \theta^t \left( t, \hat{X}^{x_0}(t) \right) \right] 
+ G \left( t, \hat{X}^{x_0}(t), \mathbb{E}_t \left[ g \left( t + \epsilon, X^{t, \epsilon, v(t)}(t + \epsilon) \right) \right] \right) 
- G \left( t, \hat{X}^{x_0}(t), g \left( t, \hat{X}^{x_0}(t) \right) \right) \] \tag{11}

Clearly, the above expression of \( \Delta \hat{j}^\epsilon(t) \) is more explicit than the one in (6).

Returning to the higher-dimensional case. Motivated by the heuristic derivations above, we now introduce the backward stochastic partial differential equations (BSDEs) involved in our verification theorem for open-loop equilibrium controls. For any fixed admissible control \( v(\cdot) \in \mathcal{U}[0, T] \) and \( t \in [0, T] \), we define the random fields \( b^{v(t)} : [0, T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n \), \( \sigma^{v(t)} : [0, T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^{n \times d} \), \( a^{v(t)} : [0, T] \times \mathbb{R}^n \times \Omega \to \mathbb{S}^n \), \( f^{t, \hat{X}^{x_0}(t), v(t)} : [t, T] \times \mathbb{R}^n \times \Omega \to \mathbb{R} \), \( F^{t, \hat{X}^{x_0}(t)} : \mathbb{R}^n \times \Omega \to \mathbb{R} \) and \( G^{t, \hat{X}^{x_0}(t)} : \mathbb{R}^n \times \Omega \to \mathbb{R} \) by
\[
b^{v(t)}(s, x) := b(s, x, v(s)), \\
\sigma^{v(t)}(s, x) := \sigma(s, x, v(s)), \\
a^{v(t)}(s, x) := \frac{1}{2} \sigma(s, x, v(s)) \sigma(s, x, v(s))^\top, \\
f^{t, \hat{X}^{x_0}(t), v(t)}(s, x) := f(t, \hat{X}^{x_0}(t), s, x, v(s)), \\
F^{t, \hat{X}^{x_0}(t)}(x) := F(t, \hat{X}^{x_0}(t), x), \\
G^{t, \hat{X}^{x_0}(t)}(\bar{x}) := G(t, \hat{X}^{x_0}(t), \bar{x}).
\]

For each \( t \in [0, T] \), we define in \( (s, x) \in [t, T] \times \mathbb{R}^n \), the following linear degenerate backward stochastic partial differential equation in the unknown random fields \( \theta^t(s, x) \in \mathbb{R}, \psi^t(s, x) = (\psi_1^t(s, x), ..., \psi_d^t(s, x))^\top \in \mathbb{R}^d \),
\[
\begin{cases}
\int d \theta^t(s, x) = - \left\{ \langle \theta_x^t(s, x), b^{\epsilon(t)}(s, x) \rangle + \text{tr} \left[ a^{\epsilon(t)}(s, x) \theta^t_{xx}(s, x) \right] \\
+ \text{tr} \left[ a^{\epsilon(t)}(s, x) \psi_x^t(s, x) \right] + f^{t, \hat{X}^{x_0}(t), \epsilon(t)}(s, x) \right\} ds \\
+ \psi^t(s, x)^\top dW(s), (s, x) \in [t, T] \times \mathbb{R}^n,
\end{cases}
\]
(12)

\[
\theta^t(T, x) = F^{t, \hat{X}^{x_0}(t)}(x), \quad \text{for } x \in \mathbb{R}^n,
\]
and we define in \((s, x) \in [0, T] \times \mathbb{R}^n\) the following system of linear degenerate BSPDEs in the unknown random fields \(g(s, x) = (g^1(s, x), ..., g^n(s, x))^T \in \mathbb{R}^n\), 
\(\eta(s, x) = (\eta^1(s, x), ..., \eta^n(s, x)) \in \mathbb{R}^{d \times n}\). For \(1 \leq i \leq n\),
\[
\begin{cases}
\, dg^i(s, x) = - \left\{ \left( g^i_x(s, x), b^i(\cdot)(s, x) \right) + \text{tr} \left[ a^i(\cdot)(s, x) g^i_{xx}(s, x) \right] \right. \\
\quad + \text{tr} \left[ \sigma^i(\cdot)(s, x) \eta^i_x(s, x) \right] \right\} \, ds + \eta^i(s, x)^T \, dW(s),
\end{cases}
\]
for \((s, x) \in [0, T] \times \mathbb{R}^n\),
(13)
where \(\eta^i(s, x)\) is the \(i\)-th column of \(\eta(s, x)\) and \(x_i\) denotes the \(i\)-th coordinate of \(x \in \mathbb{R}^n\).

**Remark 2.** Note that, since \(b, \sigma, f\) and \(F\) are deterministic functions, the (unknown) random fields \(\psi^i(s, x)\) and \(\eta(s, x)\) hedge the uncertainty arising from the randomness of the control process \(\hat{u}(\cdot)\). Thus it is not difficult to see that \(\psi^i(s, x) \equiv 0\) and \(\eta(s, x) \equiv 0\), if and only if, \(\hat{u}(\cdot)\) is a deterministic function of time.

In what follows, we denote \(\bar{B}_R = \{ x \in \mathbb{R}^n \text{ such that } |x| \leq R \}\), for any \(R > 0\).

**Definition 3.2 (Classical Solution [30]).** Let \((\theta^i(\cdot, \cdot), \psi^i(\cdot, \cdot))\) and \((g(\cdot, \cdot), \eta(\cdot, \cdot))\) be two pairs of random fields.

(i) \((\theta^i(\cdot, \cdot), \psi^i(\cdot, \cdot))\) is called a classical solution of (12) if
\[
\begin{cases}
\theta^i(\cdot, \cdot) \in C^2_{\mathbb{F}}(t, T; C^2(\bar{B}_R; \mathbb{R})) \, , \\
\psi^i(\cdot, \cdot) \in L^2_{\mathbb{F}}(t, T; C^1(\bar{B}_R; \mathbb{R}^d)) \, , \forall R > 0,
\end{cases}
\]
such that the following holds for all \((s, x) \in [t, T] \times \mathbb{R}^n\), almost surely:
\[
\theta^i(s, x) = F_{t, \bar{X}^x_0}(x) + \int_s^T \left\langle \theta^i_x(\tau, x), b^i(\cdot)(\tau, x) \right\rangle d\tau + \text{tr} \left[ a^i(\cdot)(\tau, x) \theta^i_{xx}(\tau, x) \right] + \text{tr} \left[ \sigma^i(\cdot)(\tau, x) \psi^i_x(\tau, x) \right] + \int_s^T f^i(\bar{X}^x_0(\cdot), \bar{\hat{u}}(\cdot))(\tau, x) d\tau - \int_s^T \psi^i(\tau, x)^T dW(\tau).
\]

(ii) \((g(\cdot, \cdot), \eta(\cdot, \cdot))\) is called a classical solution of (13) if for each \(1 \leq i \leq n\)
\[
\begin{cases}
g^i(\cdot, \cdot) \in C^2_{\mathbb{F}}(0, T; C^2(\bar{B}_R; \mathbb{R})) \, , \\
\eta^i(\cdot, \cdot) \in L^2_{\mathbb{F}}(0, T; C^1(\bar{B}_R; \mathbb{R}^d)) \, , \forall R > 0,
\end{cases}
\]
such that
\[
\begin{aligned}
g^i(s, x) &= x_i + \int_s^T \left\{ \left( g^i_x(\tau, x), b^i(\cdot)(\tau, x) \right) + \text{tr} \left[ a^i(\cdot)(\tau, x) g^i_{xx}(\tau, x) \right] \right. \\
&\quad + \text{tr} \left[ \sigma^i(\cdot)(\tau, x) \eta^i_x(\tau, x) \right] \right\} d\tau - \int_s^T \eta^i(\tau, x)^T dW(\tau),
\end{aligned}
\]
for all \((s, x) \in [0, T] \times \mathbb{R}^n\), almost surely.

Let us mention that BSPDEs of the above forms have been studied e.g. in Ma and Yong ([29], [30]); see also Hu et al. [22], Du and Zhang [15] and Ma et al. [31].

For example, if the following conditions are satisfied for some fixed \(k > 2 + \frac{d}{2}\):

1. For each fixed \((s, u) \in [0, T] \times U\), \(b(s, \cdot, u) \in C^{k}_b(\mathbb{R}^n; \mathbb{R}^n)\) and \(\sigma(s, \cdot, u) \in C^{k+1}_b(\mathbb{R}^n; \mathbb{R}^{n \times d})\) such that the maps
\[
(s, u) \rightarrow b(s, \cdot, u) \in C^{k}_b(\mathbb{R}^n; \mathbb{R}^n)
\]
and

\[(s, u) \rightarrow \sigma(s, \cdot, u) \in C_{b}^{k+1}(\mathbb{R}^{n}; \mathbb{R}^{n \times d})\]

are continuous with respect to \((s, u)\) and bounded. Moreover, \(\sigma\) satisfies the "symmetry condition"

\[\left[\sigma(s, x, v)(\sigma_{x_{i}}(s, x, v))^{\top}\right]^{\top} = \sigma(s, x, v)(\sigma_{x_{i}}(s, x, v))^{\top},\]

\[\forall v \in U, \text{ a.e. } (s, x) \in [0, T] \times \mathbb{R}^{n}, \text{ a.s.,}\]

for each \(1 \leq i \leq n\).

2. For each fixed \((t, \xi, s, u) \in [0, T] \times \mathbb{R}^{n} \times [0, T] \times U, f(t, \xi, s, \cdot, u) \in H^{k}(\mathbb{R}^{n})\)

and \(F(t, \xi, \cdot) \in H^{k}(\mathbb{R}^{n})\) such that the maps

\[(t, \xi, s, u) \rightarrow f(t, \xi, s, \cdot, u) \in H^{k}(\mathbb{R}^{n})\]

and

\[(t, \xi) \rightarrow F(t, \xi, \cdot) \in H^{k}(\mathbb{R}^{n})\]

are continuous with respect to \((t, \xi, s, u)\). Moreover, there exists a constant \(K > 0\) such that for all \((\xi, x, u) \in \mathbb{R}^{n} \times \mathbb{R}^{n} \times U, (t, s) \in [0, T] \times [0, T],\)

\[\|f(t, \xi, s, \cdot, u)\|_{H^{k}(\mathbb{R}^{n})} \leq K \left(1 + |\xi|^2 + |u|^2\right)\]

and

\[\|F(t, \xi, \cdot)\|_{H^{k}(\mathbb{R}^{n})} \leq K \left(1 + |\xi|^2\right).\]

Then it is not difficult to see that each of the BSPDEs (12)-(13) admits a unique adapted classical solution; see e.g. Theorem 3.1 in [30].

Next, associated with the family of processes (resp. random fields) \((\hat{u}(\cdot), \hat{X}^{x_{0}}(\cdot), \{\theta^{t}(\cdot, \cdot), \psi^{t}(\cdot, \cdot)\}_{t \in [0, T]}, g(\cdot, \cdot), \eta(\cdot, \cdot))\), we define an \(\mathcal{H}\)-function as follows:

For each \((t, s) \in D[0, T], v \in U\) and for each \(X \in L^{4}(\Omega, \mathcal{F}_{s}, \mathbb{P}; \mathbb{R}^{n})\),

\[\mathcal{H}(t, s, X, v) := \left\langle \psi_{x}^{t}(s, X) + \sum_{i=1}^{n} G_{\xi_{i}} \left( t, \hat{X}^{x_{0}}(t), \mathbb{E}_{t} [g(s, X)] \right) g_{x}^{i}(s, X), b(s, X, v) \right\rangle \]

\[+ \frac{1}{2} \text{tr} \left[ \left( \theta_{x}^{t}(s, X) + \sum_{i=1}^{n} G_{\xi_{i}} \left( t, \hat{X}^{x_{0}}(t), \mathbb{E}_{t} [g(s, X)] \right) g_{xx}^{i}(s, X) \right) (\sigma \sigma^{\top}) (s, X, v) \right] \]

\[+ \text{tr} \left[ \left( \psi_{x}^{t}(s, X) + \sum_{i=1}^{n} G_{\xi_{i}} \left( t, \hat{X}^{x_{0}}(t), \mathbb{E}_{t} [g(s, X)] \right) \eta_{x}^{i}(s, X) \right) \sigma (s, X, v) \right] \]

\[+ f \left( t, \hat{X}^{x_{0}}(t), s, X, v \right), \]

where for all \((t, \xi, \bar{x}) \in [0, T] \times \mathbb{R}^{n} \times \mathbb{R}^{n}, G_{\xi_{i}}(t, \xi, \bar{x}) = \frac{\partial}{\partial \xi_{i}} G(t, \xi, \bar{x}_{1}, \bar{x}_{2}, ..., \bar{x}_{n}).\]

3.1.2. Verification Theorem. The following theorem is the main result of this work; it provides a sufficient condition for open-loop equilibrium controls of Problem (N).

**Theorem 3.3 (Verification Theorem).** Let (H1)-(H2) hold. Given an admissible control \(u(\cdot) \in \mathcal{U}[0, T]\), suppose that for each \(t \in [0, T]\), the BSPDEs (12) and (13) admit the classical solutions \((\theta^{t}(\cdot, \cdot), \psi^{t}(\cdot, \cdot))\) and \((g(\cdot, \cdot), \eta(\cdot, \cdot))\), respectively, such that the following hold:

(i) For each \(x \in \mathbb{R}^{n}, \theta^{t}(\cdot, x), \psi^{t}(\cdot, x), \theta_{x}^{t}(\cdot, x), \psi_{x}^{t}(\cdot, x), \theta_{xx}^{t}(\cdot, x)\) are continuous \((\mathcal{F}_{s})_{s \in [t, T]}\) progressively measurable processes.

(ii) For each \(x \in \mathbb{R}^{n}, g^{i}(\cdot, x), \eta^{i}(\cdot, x), g_{x}^{i}(\cdot, x), \eta_{x}^{i}(\cdot, x), g_{xx}^{i}(\cdot, x), \) for \(1 \leq i \leq n,\)
are continuous \( (\mathcal{F}_t)_{t \in [0,T]} \)-progressively measurable processes.

(iii) There exists a constant \( K > 0 \) such that for all \((t, s, x) \in D[0,T] \times \mathbb{R}^n\) we have

\[
|\theta^i (s, x)| + |\psi^i (s, x)| \leq K \left( 1 + |x|^2 + |\hat{X}^{x_0} (t)|^2 + |\hat{X}^{x_0} (s)|^2 \right),
\]

\[
|\theta^i_x (s, x)| + |\psi^i_x (s, x)| \leq K \left( 1 + |x| + |\hat{X}^{x_0} (t)| + |\hat{X}^{x_0} (s)| \right),
\]

\[
|\eta^i_x (s, x)| \leq K,
\]

and, for each \( 1 \leq i \leq n \),

\[
|g^i (s, x)| + |\eta^i (s, x)| \leq K \left( 1 + |x| + |\hat{X}^{x_0} (s)| \right),
\]

\[
|g^i_x (s, x)| + |\eta^i_x (s, x)| \leq K.
\]

(iv) For all \( t \in [0,T] \),

\[
\mathcal{H} \left( t, t, \hat{X}^{x_0} (t), \hat{u} (t) \right) = \min_{v \in U} \mathcal{H} \left( t, t, \hat{X}^{x_0} (t), v \right).
\]

Then \( \hat{u} (\cdot) \) is an equilibrium control. Furthermore, the objective value of \( \hat{u} (\cdot) \) at time \( t \in [0,T] \) is given by

\[
\mathbf{J} \left( t, \hat{X}^{x_0} (t); \hat{u} (\cdot) \right) = \theta^i \left( t, \hat{X}^{x_0} (t) \right) + G \left( t, \hat{X}^{x_0} (t), g \left( t, \hat{X}^{x_0} (t) \right) \right).
\]

Before presenting some lemma which will play a very important role in the proof of Theorem 3.3, we first recall an Itô-Wentzell formula to the composition of random fields and continuous semimartingale processes; see e.g. Kunita ([25], [26], [27]), Ocone and Pardoux [33] and Buckdahn and Ma [6].

**Theorem 3.4 (Itô-Wentzell Formula).** Let \( X (\cdot) \in C^2 \left( T, T; \mathbb{R}^n \right) \) be a process of the form

\[
dX (s) = b (s) \, ds + \sigma (s) \, dW (s),
\]

where \( b (\cdot) \in L^1 \left( t, T; \mathbb{R}^n \right) \) and \( \sigma (\cdot) \in L^2 \left( t, T; \mathbb{R}^{n \times d} \right) \). Suppose that \( V (\cdot, \cdot) \in C^2 \left( t, T; C^2 \left( B_R; \mathbb{R} \right) \right), \forall R > 0, \) is a semimartingale with spatial parameter \( x \in \mathbb{R}^n \):

\[
dV (s, x) = \Gamma (s, x) \, ds + \zeta (s, x)^T \, dW (s), \quad (s, x) \in [t, T] \times \mathbb{R}^n,
\]

with \( \Gamma (\cdot, \cdot) \in C^r \left( [t, T] \times \mathbb{R}^n; \mathbb{R} \right) \cap \mathcal{C}^{r} \left( t, T; C \left( B_R; \mathbb{R} \right) \right) \) and \( \zeta (\cdot, \cdot) \in \mathcal{C}^{\frac{1}{2}} \left( [t, T] \times \mathbb{R}^n; \mathbb{R}^{d} \right) \cap \mathcal{C}^{2} \left( t, T; \mathbb{R}^{d} \right) \), \forall R > 0, such that

\[
\begin{aligned}
\Gamma (\cdot, X (\cdot)), \langle V_x (\cdot, X (\cdot)) , b (\cdot) \rangle \in \mathcal{L}^1 \left( t, T; \mathbb{R} \right), \\
\text{tr} \left[ \zeta_x (\cdot, X (\cdot)) \sigma (\cdot) \right] \in \mathcal{L}^1 \left( t, T; \mathbb{R} \right), \\
\text{tr} \left[ \sigma (\cdot) \sigma (\cdot)^T V_x (\cdot, X (\cdot)) \right] \in \mathcal{L}^1 \left( t, T; \mathbb{R} \right), \\
\zeta (\cdot, X (\cdot)) , \sigma (\cdot)^T V_x (\cdot, X (\cdot)) \in \mathcal{L}^2 \left( t, T; \mathbb{R}^{d} \right).
\end{aligned}
\]

Then the following holds for all \( s \in [t, T] \), almost surely,

\[
V (s, X (s)) - V (t, X (t))
\]

\[
= \int_t^s \left\{ \Gamma (\tau, X (\tau)) + \langle V_x (\tau, X (\tau)) , b (\tau) \rangle \\
+ \text{tr} [\zeta_x (\tau, X (\tau)) \sigma (\tau)] + \frac{1}{2} \text{tr} \left[ \sigma (\tau) \sigma (\tau)^T V_x (\tau, X (\tau)) \right] \right\} d\tau
\]

\[
+ \int_t^s \left\{ \zeta (\tau, X (\tau)) + V_x (\tau, X (\tau)) \right\} dW (\tau).
\]
Proof. See [6], Proof of Theorem 4.2. □

Note that the expression of $\Delta \tilde{J}^x (t)$ obtained in (11) has been derived in an heuristic and informal way only. The following lemma ensures that an equivalent expression to the one in (11) holds indeed, provided that some regularity conditions are satisfied.

**Lemma 3.5.** Let (H1)-(H2) hold. Given an admissible control $\tilde{u} (\cdot) \in U [0, T)$, suppose that for each $t \in [0, T]$, the BSPDEs (12) and (13) admit the classical solutions $(\theta^t (\cdot, \cdot), \psi^t (\cdot, \cdot))$ and $(g (\cdot, \cdot), \eta (\cdot, \cdot))$, respectively, such that Condition (iii) in Theorem 3.3 is satisfied. Then for each $t \in [0, T]$, $\theta^t (t, \hat{X}^x_0 (t))$ and $g (t, \hat{X}^x_0 (t))$ have the following probabilistic representations:

$$
\begin{align*}
\theta^t (t, \hat{X}^x_0 (t)) &= \mathbb{E}_t \left[ \int_t^T f^t, \hat{X}^x_0 (t) (s, \hat{X}^x_0 (s)) \, ds + F^t, \hat{X}^x_0 (t) (\hat{X}^x_0 (T)) \right] \\
\text{and} \quad g (t, \hat{X}^x_0 (t)) &= \mathbb{E}_t \left[ \hat{X}^x_0 (T) \right].
\end{align*}
$$

Furthermore, for any $v (\cdot) \in U [0, T)$ and for any $\varepsilon \in [0, T - t)$, the following equality holds

$$
\begin{align*}
J (t, \hat{X}^x_0 (t); u^{t, \varepsilon, v (\cdot)} (\cdot)) - J (t, \hat{X}^x_0 (t); \hat{u} (\cdot)) = \int_t^{t+\varepsilon} \mathbb{E}_t \left[ \mathcal{H} (t, s, X^{t, \varepsilon, v (\cdot)} (s), v (s)) - \mathcal{H} (t, s, X^{t, \varepsilon, v (\cdot)} (s), \hat{u} (s)) \right] \, ds,
\end{align*}
$$

where $\mathcal{H}$ is as introduced in (14), $u^{t, \varepsilon, v (\cdot)} (\cdot)$ is defined by (4) and $X^{t, \varepsilon, v (\cdot)} (\cdot)$ is the state process corresponding to $u^{t, \varepsilon, v (\cdot)} (\cdot)$.

**Proof.** First, we prove the equality in (17). Let $(\theta^t (\cdot, \cdot), \psi^t (\cdot, \cdot))$ be the classical solution of the BSPDE (12). Define

$$
\begin{align*}
J := \mathbb{E}_t \left[ \int_t^T f^t, \hat{X}^x_0 (t) (s, \hat{X}^x_0 (s)) \, ds + F^t, \hat{X}^x_0 (t) (\hat{X}^x_0 (T)) \right].
\end{align*}
$$

By the terminal condition in (12), we have

$$
\begin{align*}
J &= \mathbb{E}_t \left[ \int_t^T f^t, \hat{X}^x_0 (t) (s, \hat{X}^x_0 (s)) \, ds + \theta^t (T, \hat{X}^x_0 (T)) \right] \\
&= \mathbb{E}_t \left[ \int_t^T f^t, \hat{X}^x_0 (t) (s, \hat{X}^x_0 (s)) \, ds + \theta^t (t, \hat{X}^x_0 (t)) \right] \\
&\quad + \mathbb{E}_t \left[ \theta^t (T, \hat{X}^x_0 (T)) - \theta^t (t, \hat{X}^x_0 (t)) \right].
\end{align*}
$$

Moreover, by applying Itô-Wentzell formula to $\theta^t (s, \hat{X}^x_0 (s))$ on time interval $[t, T]$, we obtain that

$$
\mathbb{E}_t \left[ \int_t^T f^t, \hat{X}^x_0 (t) (s, \hat{X}^x_0 (s)) \, ds \right] = - \mathbb{E}_t \left[ \int_t^T f^t, \hat{X}^x_0 (t) (s, \hat{X}^x_0 (s)) \, ds \right].
$$
Invoking this into (20), we obtain (17). Similarly, we prove Equality (18). Define for each $1 \leq i \leq n$,

$$
\tilde{J}_i = \mathbb{E}_t \left[ \hat{X}_i^{x_0} (T) \right] \\
= \mathbb{E}_t \left[ g^i \left( T, \hat{X}_i^{x_0} (T) \right) \right] \\
= \mathbb{E}_t \left[ g^i \left( T, \hat{X}_i^{x_0} (T) \right) - g^i \left( t, \hat{X}_i^{x_0} (t) \right) \right],
$$

where $\hat{X}_i^{x_0} (T)$ denotes the $i$-th coordinate of $\hat{X}^{x_0} (T) = (\hat{X}_1^{x_0} (T), ..., \hat{X}_n^{x_0} (T))^\top$.

Applying Itô-Wentzell formula to $g^i \left( s, \hat{X}_i^{x_0} (s) \right)$, we get

$$
\mathbb{E}_t \left[ g^i \left( T, \hat{X}_i^{x_0} (T) \right) - g^i \left( t, \hat{X}_i^{x_0} (t) \right) \right] = 0.
$$

Thus, for each $1 \leq i \leq n$,

$$
\tilde{J}_i = g^i \left( t, \hat{X}_i^{x_0} (t) \right).
$$

Hence

$$
g \left( t, \hat{X}^{x_0} (t) \right) = (\tilde{J}_1, ..., \tilde{J}_n)^\top \\
= \mathbb{E}_t \left[ \hat{X}^{x_0} (T) \right].
$$

We now go on to show that the equality in (19) holds indeed. Consider the perturbed strategy $u^{t, \varepsilon, v(\cdot)} (\cdot)$ defined by the spike variation (4) for some fixed arbitrarily $v (\cdot) \in \mathcal{U} [0, T)$, $t \in [0, T]$ and $\varepsilon \in [0, T - t)$. Consider the difference

$$
\mathcal{J} \left( t, \hat{X}^{x_0} (t) ; u^{t, \varepsilon, v(\cdot)} (\cdot) \right) - \mathcal{J} \left( t, \hat{X}^{x_0} (t) ; \hat{u} (\cdot) \right) \\
= \mathbb{E}_t \left[ \int_t^T f^t, \hat{X}^{x_0} (t) u^{t, \varepsilon, v(\cdot)} (s) \left( \hat{X}^{x_0} (s) \right) ds + F^t, \hat{X}^{x_0} (t) \left( \hat{X}^{x_0} (T) \right) \right] \\
- \mathbb{E}_t \left[ \int_t^T f^t, \hat{X}^{x_0} (t) \hat{u} (s) \left( \hat{X}^{x_0} (s) \right) ds + F^t, \hat{X}^{x_0} (t) \left( \hat{X}^{x_0} (T) \right) \right] \\
+ G^t, \hat{X}^{x_0} (t) \left( \mathbb{E}_t \left[ \hat{X}^{x_0} (T) \right] \right) - G^{t, \hat{X}^{x_0} (t)} \left( \mathbb{E}_t \left[ \hat{X}^{x_0} (T) \right] \right).
$$

By the terminal conditions in BSPDEs (12)-(13) and the equalities in (17)-(18), we have

$$
\mathcal{J} \left( t, \hat{X}^{x_0} (t) ; u^{t, \varepsilon, v(\cdot)} (\cdot) \right) - \mathcal{J} \left( t, \hat{X}^{x_0} (t) ; \hat{u} (\cdot) \right) \\
= \mathbb{E}_t \left[ \int_t^T f^t, \hat{X}^{x_0} (t) u^{t, \varepsilon, v(\cdot)} (s) \left( \hat{X}^{x_0} (s) \right) ds \right] \\
+ \mathbb{E}_t \left[ \theta^t \left( T, \hat{X}^{x_0} (T) \right) - \theta^t \left( t, \hat{X}^{x_0} (t) \right) \right] \\
+ G^{t, \hat{X}^{x_0} (t)} \left( \mathbb{E}_t \left[ g \left( T, \hat{X}^{x_0} (T) \right) \right] \right) - G^{t, \hat{X}^{x_0} (t)} \left( g \left( t, \hat{X}^{x_0} (t) \right) \right). \quad (21)
$$
Recall that \( X^{t,\varepsilon,v(\cdot)}(t) = \dot{X}^{x_0}(t) \), then applying Itô-Wentzell formula to 
\( \theta^t \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \) on time interval \([t, T]\), we obtain that

\[
\mathbb{E}_t \left[ \theta^t \left( T, X^{t,\varepsilon,v(\cdot)}(T) \right) - \theta^t \left( t, \dot{X}^{x_0}(t) \right) \right] 
= \mathbb{E}_t \left[ \int_t^T \left\{ \left\{ \theta^t \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right), b^{u^{t,\varepsilon,v(\cdot)}(s)} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) - b^{\tilde{u}} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right\}, \frac{ds}{t} \right\} \right] 
+ \text{tr} \left[ \left( a^{u^{t,\varepsilon,v(\cdot)}(s)} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) - a^{\tilde{u}} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right), \theta^t_x \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right] 
+ \text{tr} \left[ \left( \sigma^{u^{t,\varepsilon,v(\cdot)}(s)} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) - \sigma^{\tilde{u}} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right), \theta^t_x \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right] 
- f^{t,\varepsilon,v(\cdot)}(t), \tilde{u}(s, X^{t,\varepsilon,v(\cdot)}(s)) \right] ds \right] .
\]

Next, for each \( 1 \leq i \leq n \), applying Itô-Wentzell formula to 
\( g^i \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \) on time interval \([t, T]\), and taking the conditional expectation \( \mathbb{E}_t[\cdot] \), we get

\[
d\mathbb{E}_t \left[ g^i \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right] 
= \mathbb{E}_t \left[ \left\{ \left\{ g^i \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right), b^{u^{t,\varepsilon,v(\cdot)}(s)} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) - b^{\tilde{u}} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right\}, \frac{ds}{t} \right\} \right] 
+ \text{tr} \left[ \left( a^{u^{t,\varepsilon,v(\cdot)}(s)} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) - a^{\tilde{u}} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right), g^i_x \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right] 
+ \text{tr} \left[ \left( \sigma^{u^{t,\varepsilon,v(\cdot)}(s)} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) - \sigma^{\tilde{u}} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right), g^i_x \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right] 
- f^{t,\varepsilon,v(\cdot)}(t), \tilde{u}(\cdot) \right] \right] ds \right] ,
\]

By the chain rule we obtain that

\[
G^{t,\varepsilon,v(\cdot)}(t) \left( \mathbb{E}_t \left[ g \left( T, X^{t,\varepsilon,v(\cdot)}(T) \right) \right] \right) - G^{t,\varepsilon,v(\cdot)}(t) \left( g \left( t, \dot{X}^{x_0}(t) \right) \right) 
= \int_t^T \sum_{i=1}^n G^{t,\varepsilon,v(\cdot)}(t) \left( \mathbb{E}_t \left[ g \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right] \right) d\mathbb{E}_t \left[ \left\{ g^i \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right\} \right] ds \right] .
\]

Combining (21) together with (22) and (23), it follows that

\[
J \left( t, \dot{X}^{x_0}(t), u^{t,\varepsilon,v(\cdot)}(\cdot) ; \cdot \right) - J \left( t, \dot{X}^{x_0}(t), \tilde{u}(\cdot) \right) 
= \mathbb{E}_t \left[ \int_t^T \left\{ \left\{ \theta^t \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right), b^{u^{t,\varepsilon,v(\cdot)}(s)} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) - b^{\tilde{u}} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right\}, \frac{ds}{t} \right\} \right] 
+ \text{tr} \left[ \left( a^{u^{t,\varepsilon,v(\cdot)}(s)} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) - a^{\tilde{u}} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right), \theta^t_x \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right] 
+ \text{tr} \left[ \left( \sigma^{u^{t,\varepsilon,v(\cdot)}(s)} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) - \sigma^{\tilde{u}} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right), \theta^t_x \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right] 
+ f^{t,\varepsilon,v(\cdot)}(t), u^{t,\varepsilon,v(\cdot)}(s), \tilde{u}(s, X^{t,\varepsilon,v(\cdot)}(s)) \right] 
+ \sum_{i=1}^n G^{t,\varepsilon,v(\cdot)}(t) \left( \mathbb{E}_t \left[ g \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right] \right) \right] 
\times \left( \text{tr} \left[ \left\{ g^i_x \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right), \sigma^{u^{t,\varepsilon,v(\cdot)}(s)} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) - \sigma^{\tilde{u}} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right\} \right] \right] 
+ \text{tr} \left[ \left( a^{u^{t,\varepsilon,v(\cdot)}(s)} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) - a^{\tilde{u}} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right), \theta^t_x \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right] 
+ \text{tr} \left[ \left( \sigma^{u^{t,\varepsilon,v(\cdot)}(s)} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) - \sigma^{\tilde{u}} \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right), \theta^t_x \left( s, X^{t,\varepsilon,v(\cdot)}(s) \right) \right] 
+ f^{t,\varepsilon,v(\cdot)}(t), u^{t,\varepsilon,v(\cdot)}(s), \tilde{u}(s, X^{t,\varepsilon,v(\cdot)}(s)) \right] \right] ds \right] .
\]
Since \( u^{t,\epsilon,v(\cdot)}(s) = v(s) \chi_{(t,t+\epsilon)}(s) + \hat{u}(s) \chi_{(t+\epsilon,T)}(s) \), we obtain that

\[
J \left( t, \hat{X}^{x_0} (t) ; u^{t,\epsilon,v(\cdot)} (\cdot) \right) - J \left( t, \hat{X}^{x_0} (t) ; \hat{u} (\cdot) \right)
\]

\[
= \mathbb{E}_t \left[ \int_t^{t+\epsilon} \{ \theta_x \left( s, X^{t,\epsilon,v(\cdot)}(s) \right), \theta_x \left( s, X^{t,\epsilon,v(\cdot)}(s) \right) - \theta_x \left( s, X^{t,\epsilon,v(\cdot)}(s) \right) \} \right]
\]

\[
+ \mathbb{E}_t \left[ \int_t^{t+\epsilon} \left\{ \left( g_x \left( s, X^{t,\epsilon,v(\cdot)}(s) \right), g_x \left( s, X^{t,\epsilon,v(\cdot)}(s) \right) - g_x \left( s, X^{t,\epsilon,v(\cdot)}(s) \right) \right) \right\} \right]
\]

\[
+ \mathbb{E}_t \left[ \int_t^{t+\epsilon} \left\{ \left( \eta_x \left( s, X^{t,\epsilon,v(\cdot)}(s) \right), \eta_x \left( s, X^{t,\epsilon,v(\cdot)}(s) \right) - \eta_x \left( s, X^{t,\epsilon,v(\cdot)}(s) \right) \right) \right\} \right] \right] ds.
\]

This completes the proof. \( \square \)

Now, we are ready to give a proof of Theorem 3.3.

**Proof of Theorem 3.3.** Throughout this proof, \( K \) represents a generic constant, which can be different from line to line. Let \( \hat{u}(\cdot) \in \mathcal{U}[0,T] \) be an admissible control for which Assumptions (i)-(iv) in Theorem 3.3 hold. Consider the perturbed strategy \( u^{t,\epsilon,v(\cdot)}(\cdot) \) defined by the spike variation (4) for some fixed arbitrarily \( v(\cdot) \in \mathcal{U}[0,T], t \in [0,T] \) and \( \epsilon \in [0,T-t] \). Let \( X^{t,v(\cdot)}(\cdot) \) be the state process corresponding to \( u^{t,\epsilon,v(\cdot)}(\cdot) \) and \( X^{t,v(\cdot)}(\cdot) \) be the solution of the following SDE,

\[
\begin{aligned}
&dX^{t,v(\cdot)}(s) = b(v(\cdot),X^{t,v(\cdot)}(s)) \, ds + \sigma(v(\cdot),X^{t,v(\cdot)}(s)) \, dW(s), \\
&X^{t,v(\cdot)}(t) = \hat{X}^{x_0}(t).
\end{aligned}
\]

Under Assumption (H1) the above SDE admits a unique solution \( X^{t,v(\cdot)}(\cdot) \) and there exists a constant \( K > 0 \) such that

\[
\mathbb{E} \left[ \sup_{s \in [t,T]} \left| X^{t,v(\cdot)}(s) \right|^4 \right] \leq K \left( 1 + \mathbb{E} \left[ |\hat{X}^{x_0}(t)|^4 \right] \right).
\]

Moreover, since \( u^{t,\epsilon,v(\cdot)}(s) = v(s) \) on \( [t,t+\epsilon) \), then

\[
X^{t,\epsilon,v(\cdot)}(s) = X^{t,v(\cdot)}(s), \text{ a.s., for } s \in [t,t+\epsilon).
\]

(24)

We claim that

\[
\lim_{\epsilon \downarrow 0} \mathbb{E} \left[ \mathcal{H} \left( t, t, \hat{X}^{x_0}(t), v(t) \right) - \mathcal{H} \left( t, t, \hat{X}^{x_0}(t), \hat{u}(t) \right) \right] = 0.
\]

Indeed, define for \( s \in [t,T] \)

\[
\Phi(s; t) = \mathcal{H} \left( t, s, X^{t,v(\cdot)}(s), v(s) \right) - \mathcal{H} \left( t, s, X^{t,v(\cdot)}(s), \hat{u}(s) \right).
\]
It is clear that $\Phi (\cdot ; t)$ is a right-continuous $(\mathcal{F}_s)_{s \in [t, T]}$-progressively measurable process. Moreover, it follows from the polynomial growth (resp. Boundedness) conditions in Assumption (iii) together with the linear growth conditions of the coefficients $b, \sigma$ and $G_x$ that there exists a constant $K > 0$, such that

$$|\mathcal{H}(t, s, X, v)|$$

$$\leq \left| \theta_x^t (s, X) + \sum_{i=1}^n G^i \left(t, \hat{X}^x(t), \mathbb{E}_t [g(s, X)] \right) g^i_x(s, X) \right| |b(s, X, v)|$$

$$+ \frac{1}{2} \left| \theta_{xx}^t (s, X) + \sum_{i=1}^n G^i \left(t, \hat{X}^x(t), \mathbb{E}_t [g(s, X)] \right) g^i_x(s, X) \right| |\sigma(s, X, v)|^2$$

$$+ \left| \psi_x^t (s, X) + \sum_{i=1}^n G^i \left(t, \hat{X}^x(t), \mathbb{E}_t [g(s, X)] \right) \eta^i_x(s, X) \right| |\sigma(s, X, v)|$$

$$+ \left| f \left(t, \hat{X}^x(t), s, X, v \right) \right|$$

$$\leq K \left( 1 + \left| f \left(t, \hat{X}^x(t), s, X, v \right) \right|^2 + |b(s, X, v)|^4 \right.$$

$$+ \left. \left| \theta_x^t (s, X) + \sum_{i=1}^n G^i \left(t, \hat{X}^x(t), \mathbb{E}_t [g(s, X)] \right) g^i_x(s, X) \right|^4 \right.$$

$$+ \left. \left| \theta_{xx}^t (s, X) + \sum_{i=1}^n G^i \left(t, \hat{X}^x(t), \mathbb{E}_t [g(s, X)] \right) g^i_x(s, X) \right|^4 + |\sigma(s, X, v)|^4 \right.$$

$$+ \left. \left( \psi_x^t (s, X) + \sum_{i=1}^n G^i \left(t, \hat{X}^x(t), \mathbb{E}_t [g(s, X)] \right) \eta^i_x(s, X) \right)^4 + |\sigma(s, X, v)|^4 \right)$$

$$\leq K \left( 1 + \left| \hat{X}^x(t) \right|^4 + |\hat{X}^x(s)|^4 + \mathbb{E}_t \left[ \left| \hat{X}^x(s) \right|^4 + \left| \hat{X}^x_0(s) \right|^4 + |\hat{u}(s)|^4 \right] + |\sigma(s, X, v)|^4 + \left| \eta^i_x(s, X) \right|^4 \right)$$

for any $(t, s, v) \in D [0, T] \times U$, $X \in \mathbb{L}^4 (\Omega, \mathcal{F}_s, \mathbb{P}; \mathbb{R}^n)$, where we have used the inequalities $a \leq \frac{1}{2} \left( a^2 + 1 \right)$, $ab \leq \frac{1}{4} \left( 2 + a^4 + b^4 \right)$ and $ab^2 \leq \frac{1}{4} \left( 1 + a^4 + 2b^4 \right)$ for any $a, b \in \mathbb{R}$. Consequently, there exists $K > 0$ such that

$$|\Phi (s; t)| \leq K \left( 1 + \left| \hat{X}^x(t) \right|^4 + |\hat{u}(s)|^4 \right)$$

$$+ \mathbb{E}_t \left[ \left| \hat{X}^x(s) \right|^4 + \left| \hat{X}^x_0(s) \right|^4 + \left| \hat{X}^{t, u(\cdot)}(s) \right|^4 + \mathbb{E}_t \left[ \left| \hat{X}^{t, u(\cdot)}(s) \right|^4 \right] \right].$$

Accordingly we have

$$\mathbb{E} \left[ \sup_{s \in [t, T]} \left| \Phi (s; t) \right| \right]$$

$$\leq K \left( 1 + \mathbb{E} \left[ \sup_{s \in [t, T]} \left| \hat{u}(s) \right|^4 \right] + \mathbb{E} \left[ \sup_{s \in [t, T]} \left| \hat{u}(s) \right|^4 \right] \right.$$}

$$+ \mathbb{E} \left[ \sup_{s \in [t, T]} \left| \hat{X}^x_0(s) \right|^4 \right] + \mathbb{E} \left[ \sup_{s \in [t, T]} \left| \hat{X}^{t, u(\cdot)}(s) \right|^4 \right]$$

$$\leq K \left( 1 + \mathbb{E} \left[ \sup_{s \in [t, T]} \left| \hat{u}(s) \right|^4 \right] + \mathbb{E} \left[ \sup_{s \in [t, T]} \left| \hat{u}(s) \right|^4 \right] + \mathbb{E} \left[ \left| \hat{X}^x(t) \right|^4 \right] \right),$$
Remark 3. The following system of FBSPDEs,
\[
\begin{align*}
\sup_{s \in [t,T]} |\Phi (s; t)| < \infty. & \text{ Thus, by applying Dominated Convergence Theorem together with (19) and (24), we get} \\
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ J \left( t, \hat{X}^{x_0} (t); u^{\varepsilon, v (\cdot)} (\cdot) \right) - J \left( t, \hat{X}^{x_0} (t); \hat{u} (\cdot) \right) \right\} \\
= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t \left[ \mathcal{H} \left( t, s, X^{t, v (\cdot)} (s), v (s) \right) - \mathcal{H} \left( t, s, X^{t, v (\cdot)} (s), \hat{u} (s) \right) \right] ds \\
= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t \left[ \mathcal{H} \left( t, s, X^{t, v (\cdot)} (s), v (s) \right) - \mathcal{H} \left( t, s, X^{t, v (\cdot)} (s), \hat{u} (s) \right) \right] ds \\
= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t \left[ \Phi (s; t) \right] ds \\
= \lim_{s \downarrow t} \mathbb{E}_t \left[ \Phi (s; t) \right] \\
= \mathcal{H} \left( t, t, \hat{X}^{x_0} (t), v (t) \right) - \mathcal{H} \left( t, t, \hat{X}^{x_0} (t), \hat{u} (t) \right),
\end{align*}
\]
which prove our claim. It then follows from (15) that
\[
\lim \inf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ J \left( t, \hat{X}^{x_0} (t); u^{\varepsilon, v (\cdot)} (\cdot) \right) - J \left( t, \hat{X}^{x_0} (t); \hat{u} (\cdot) \right) \right\} \\
= \mathcal{H} \left( t, t, \hat{X}^{x_0} (t), v (t) \right) - \mathcal{H} \left( t, t, \hat{X}^{x_0} (t), \hat{u} (t) \right) \\
\geq 0.
\]
Hence $\hat{u} (\cdot)$ is an equilibrium control. The equality in (16) is an immediate consequence of the probabilistic representations (17)-(18). This completes the proof. \(\square\)

Let us make some remarks on Theorem 3.3.

**Remark 3.** Theorem 3.3 shows that one can obtain equilibrium controls by solving the following system of FBSPDEs,
\[
\left\{ \begin{array}{l}
\quad d\hat{X}^{x_0} (s) = b \left( s, \hat{X}^{x_0} (s), \hat{u} (s) \right) ds + \sigma \left( s, \hat{X}^{x_0} (s), \hat{u} (s) \right) dW (s), \quad s \in [0, T], \\
\quad d\theta^i (s, x) = - \left\{ \theta^i_x (s, x), b (s, x, \hat{u} (s)) \right\} + \frac{1}{2} tr \left[ \left( \sigma \sigma^T (s, x, \hat{u} (s)) \right) \theta^i_x (s, x) \right] \\
\quad \quad \quad \quad \quad + tr \left[ \psi^i (s, x) \sigma (s, x, \hat{u} (s)) \right] + C (t, \hat{X}^{x_0} (t), s, \hat{u} (s)) \right\} ds \\
\quad \quad + \psi^i (s, x)^T dW (s), \quad (t, s, x) \in D [0, T] \times \mathbb{R}^n, \\
\quad dg^i (s, x) = - \left\{ g^i_x (s, x), b (s, x, \hat{u} (s)) \right\} + \frac{1}{2} tr \left[ \left( \sigma \sigma^T (s, x, \hat{u} (s)) \right) g^i_x (s, x) \right] \\
\quad \quad \quad \quad \quad + tr \left[ \eta^i_x (s, x) \sigma (s, x, \hat{u} (s)) \right] ds + \eta^i (s, x)^T dW (s), \quad (s, x) \in [0, T] \times \mathbb{R}^n, \quad 1 \leq i \leq n, \\
\quad \hat{X}^{x_0} (0) = x_0, \quad g^i (T, x) = x_i, \quad \theta^i (T, x) = F (t, \hat{X}^{x_0} (t), x), \quad \text{for } x \in \mathbb{R}^n. \\
\quad \mathcal{H} \left( t, t, \hat{X}^{x_0} (t), \hat{u} (t) \right) = \min_{v \in U} \mathcal{H} \left( t, t, \hat{X}^{x_0} (t), v \right), \quad \text{a.s., for any } t \in [0, T].
\end{array} \right.
\]
(25)

However, the above system is not standard since a flow of backward stochastic partial differential equations is involved. Proving the general existence for this type of FBSPDEs remains an outstanding open problem, even for the case $n = d = 1$. Nonetheless, we are able to solve quite thoroughly this problem in some special cases as will be demonstrated by the mean–variance portfolio selection model in Section 5.
Remark 4. On comparing between the BSPDEs approach of the present paper and the duality approach of Hu et al. [19], we find the following facts:

(i) Note that our verification theorem (Theorem 3.3) enables us to derive simultaneously an equilibrium control \( \hat{u}(\cdot) \) as well as its objective value \( J(t, X^{x_0}(t); \hat{u}(\cdot)) \), at each \( t \in [0, T] \), while the sufficient condition for equilibriums of Hu et al. [19] permits us to obtain an equilibrium solution only; it does not provide the objective value \( J(t, X^{x_0}(t); \hat{u}(\cdot)) \).

(ii) The BSPDEs (12) and (13) permit us to derive a suitable expression to the difference

\[
\Delta J^\tau(t) = J(t, X^{x_0}(t); u^{t,\tau, \mu(\cdot)}(\cdot)) - J(t, X^{x_0}(t); \hat{u}(\cdot)),
\]

without the necessity of performing the second order expansion in the spike variation.

(iii) Our verification theorem for open-loop equilibrium controls does not require any differentiability assumptions on the involved coefficients, except for the function \( G(\cdot) \); this is quite different from the duality approach (see e.g. [13] or [42]).

(iv) The BSPDEs approach provides more information on the open-loop equilibrium pair \( (\hat{X}^{x_0}(\cdot), \hat{u}(\cdot)) \) than the duality approach. For example, the system of FBSPDEs (25) permits us to derive directly an explicit expression of the expected value \( \mathbb{E}_t \left[ \hat{X}^{x_0}(T) \right] \), at each \( t \in [0, T] \), which is not the case of the duality approach.

(v) Despite all the differences between these two approaches, they are actually connected to each other via their corresponding flows of FBSDEs; see Section 4 for more details.

3.1.3. The case of general discounting only. Now let us look at an important special case. Suppose that

\[
\begin{align*}
  f(t, \xi, x, u) & \equiv f(t, x, u), \\
  F(t, \xi, x) & \equiv F(t, x), \\
  G(t, \xi, \bar{x}) & \equiv 0.
\end{align*}
\]

In this case, for each \( t \in [0, T] \), the BSPDE (12) associated to an admissible control \( \hat{u}(\cdot) \in \mathcal{U}[0, T] \) reduces to

\[
\begin{cases}
  d\theta^t(s, x) = - \{ \langle \theta^t_x(s, x), b(s, x, \hat{u}(s)) \rangle + \frac{1}{2} \text{tr} \left[ \sigma \sigma^\top(s, x, \hat{u}(s)) \theta^{xx}_x(s, x) \right] \\
  & + \text{tr} \left[ \psi^t_x(s, x) \sigma(s, x, \hat{u}(s)) \right] + f(t, s, x, \hat{u}(s)) \} ds \\
  + \psi^t(s, x)^\top dW(s), & (s, x) \in [t, T] \times \mathbb{R}^n, \\
  \theta^t(T, x) = F(t, x), & \text{for } x \in \mathbb{R}^n.
\end{cases}
\]

(26)

Note that, since \( G(t, \xi, \bar{x}) \equiv 0 \), the system of BSPDEs (13) is not necessary in this case. We have the following corollary.

Corollary 1. Let (H1) and (i) in (H2) hold. Given an admissible control \( \hat{u}(\cdot) \in \mathcal{U}[0, T] \), suppose that for each \( t \in [0, T] \), the BSPDE (26) admits a classical solution \((\theta^t(\cdot, \cdot), \psi^t(\cdot, \cdot))\) such that the following hold:

(i) For each \( x \in \mathbb{R}^n \), \( \theta^t(\cdot, x), \psi^t(\cdot, x), \theta^t_x(\cdot, x), \psi^t_x(\cdot, x), \theta^t_{xx}(\cdot, x) \) are continuous \((\mathcal{F}_s)_{s \in [t, T]}\)-progressively measurable processes.
There exists a constant \( K > 0 \) such that for all \( (t, s, x) \in D \times \mathbb{R}^n \) we have
\[
|\theta^t (s, x)| + |\psi^t (s, x)| \leq K \left( 1 + |x|^2 + |\hat{X}^{x_0} (s)|^2 \right),
\[
|\theta^t_x (s, x)| + |\psi^t_x (s, x)| \leq K \left( 1 + |x| + |\hat{X}^{x_0} (s)| \right),
\[
|\theta^t_{xx} (s, x)| \leq K.
\]

For all \( t \in [0, T] \),
\[
\hat{u} (t) \in \arg \min \left\{ \theta^t_x \left( t, \hat{X}^{x_0} (t) \right), b \left( t, \hat{X}^{x_0} (t), \cdot \right) \right\}
+ \frac{1}{2} \text{tr} \left[ \sigma \sigma^\top \left( t, \hat{X}^{x_0} (t), \cdot \right) \theta^t_x \left( t, \hat{X}^{x_0} (t) \right) \right]
+ \text{tr} \left[ \psi^t_x \left( t, \hat{X}^{x_0} (t) \right) \sigma \left( t, \hat{X}^{x_0} (t), \cdot \right) \right] + f \left( t, t, \hat{X}^{x_0} (t), \cdot \right).
\]  
\]
Then \( \hat{u} (\cdot) \) is an equilibrium control. Furthermore, the objective value of \( \hat{u} (\cdot) \) at time \( t \in [0, T] \) is given by
\[
\mathcal{J} \left( t, \hat{X}^{x_0} (t); \hat{u} (\cdot) \right) = \theta^t \left( t, \hat{X}^{x_0} (t) \right). \]

This result is comparable with those in [43] in which the equilibrium is, however, defined within the class of closed-loop controls. Therein the closed-loop equilibrium control can be constructed by solving the so-called Equilibrium HJB equation. More specifically, the equilibrium HJB equation is a deterministic fully nonlinear PDE having the following form:

\[
\begin{cases}
0 = \rho_x (t, s, x) + \langle \rho_x (t, s, x), b (s, x, \psi (s, s, x, \rho_x (s, s, x), \rho_{xx} (s, s, x))) \rangle \\
+ \frac{1}{2} \text{tr} \left[ \sigma \sigma^\top (s, x, \psi (s, s, x, \rho_x (s, s, x), \rho_{xx} (s, s, x))) \rho_{xx} (s, s, x) \right] \\
+ \hat{f} (t, s, x, \psi (s, s, x, \rho_x (s, s, x), \rho_{xx} (s, s, x))), \\
\text{for} \ (t, s, x) \in D \times \mathbb{R}^n, \\
\rho (t, T, x) = F (t, x), \text{ for} \ (t, x) \in [0, T] \times \mathbb{R}^n,
\end{cases}
\]  
\]
where \( \psi : D \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n \) is a deterministic sufficiently regular map satisfying
\[
\psi (t, s, x, p, P) \in \arg \min \left\{ \langle p, b (s, x, \cdot) \rangle + \frac{1}{2} \text{tr} \left[ \sigma \sigma^\top (s, x, \cdot) P \right] + f (t, s, x, \cdot) \right\},
\text{for all} \ (t, s, x, p, P) \in D \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n.
\]

If the equilibrium HJB equation (28) admits a classical solution \( \rho (\cdot, \cdot, \cdot) \) then the state-control pair \( \left( \hat{X}^{x_0} (\cdot), \hat{u}^{CL} (\cdot) \right) \) defined by
\[
\hat{u}^{CL} (t) = \psi \left( t, t, x, \rho_x \left( t, t, \hat{X}^{x_0} (t) \right), \rho_{xx} \left( t, t, \hat{X}^{x_0} (t) \right) \right),
\text{for all} \ t \in [0, T], \ a.s.,
\]
is a closed-loop equilibrium pair and the objective value of \( \hat{u}^{CL} (\cdot) \) at time \( t \in [0, T] \) is given by
\[
\mathcal{J} \left( t, \hat{X}^{x_0} (t); \hat{u}^{CL} (\cdot) \right) = \rho \left( t, t, \hat{X}^{x_0} (t) \right).
\]

**Remark 5.** There is a notable difference in how these two methods permit us to derive the equilibrium controls (open-loop and closed-loop):

(i) Firstly, it is seen that (26) is a BSPDE which depends on the open-loop equilibrium control \( \hat{u} (s) \), while the equilibrium HJB equation is a deterministic PDE that depends on the closed-loop equilibrium strategy \( \psi (s, s, x, \rho_x (s, s, x), \rho_{xx} (s, s, x)) \).

(ii) Secondly, the minimum condition (27) is an open-loop specification of \( \hat{u} (\cdot), \)
because \( \hat{u}(t) \) depends not only on the state \( \hat{X}^{x_0}(t) \) but also on \( \theta^t \left( t, \hat{X}^{x_0}(t) \right) \) and \( \psi^t \left( t, \hat{X}^{x_0}(t) \right) \) (which are dependent on \( \hat{u}(\cdot) \)), while (29) is a closed-loop (feedback) specification of \( \hat{u}^{CL}(\cdot) \); indeed, assuming that we know the function \( \rho(\cdot, \cdot, \cdot) \) everywhere, \( \hat{u}^{CL}(t) \) is completely determined by the current state \( \hat{X}^{x_0}(t) \).

(iii) Thirdly, Corollary 1 permits us to derive the open-loop equilibrium control where, \( \bar{\psi} \).

Relationship with the duality approach.

4. Relationship with the duality approach. In this section, we study the relationship between the BSPDEs approach of the present paper and the duality approach of Hu et al. ([19], [20]) for the time-inconsistent stochastic optimal control problem:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{dX}{ds}(s) = b(s, X(s), u(s)) ds + \sigma(s, X(s), u(s)) dW(s), \ s \in [t, T], \\
X(t) = \xi,
\end{array} \right.
\end{aligned}
\]

along with the cost functional

\[
J(t; \xi; u(\cdot)) = \mathbb{E}_t \left[ \int_t^T f(t, \xi, s, X(s), u(s)) ds + F(t, \xi, X(T)) \right] + G(t, \xi, \mathbb{E}_t[X(T)]).
\]

The main result in this section is comparable with some of the results in [35]. Without loss of generality, throughout this section, the coefficients are assumed to be one dimensional (i.e. \( n = d = 1 \)) as in [13], to avoid heavy notations in the definition of adjoint processes. Specially, with \( n = d = 1 \), the BSPDEs (12)-(13) reduce, respectively, to

\[
\begin{aligned}
&\left\{ \begin{array}{l}
d\theta^t(s, x) = -\left\{ \theta^t_x(s, x) b\hat{u}(s, x) + \frac{1}{2} \sigma\hat{u}(s, x)^2 \theta^t_{xx}(s, x) \\
+ \psi^t_x(s, x) \sigma\hat{u}(s, x) + f^t, \hat{X}^{x_0}(t), \hat{u}(\cdot)(s, x) \right\} ds \\
+ \psi^t(s, x) dW(s), \ \text{for} \ (s, x) \in [t, T] \times \mathbb{R},
\end{array} \right.
\end{aligned}
\]

\[(30)\]

and

\[
\begin{aligned}
&\left\{ \begin{array}{l}
dg(s, x) = -\left\{ g_x(s, x) b\hat{u}(s, x) + \frac{1}{2} \sigma\hat{u}(s, x)^2 g_{xx}(s, x) \\
+ \eta_x(s, x) \sigma\hat{u}(s, x) + \xi_x(s, x) \sigma\hat{u}(s, x) \right\} ds + \eta(s, x) dW(s), \\
\text{for} \ (s, x) \in [0, T] \times \mathbb{R},
\end{array} \right.
\end{aligned}
\]

\[(31)\]

Before providing the precise statement of the main result in this section, let us first present a version of a sufficient stochastic maximum principle (SMP, for short) which characterizes open-loop equilibriums. As in Hu et al. [19] and Djehiche and Huang [13], we derive this SMP by the second order Taylor’s expansion in the spike variation, in the same spirit of proving the stochastic Pontryagin’s maximum principle ([34], [4]). Let \( \hat{u}(\cdot) \in \mathcal{U}[0, T] \) be a fixed admissible control and \( \hat{X}^{x_0}(\cdot) \) be
the state process corresponding to $\hat{u}(\cdot)$. For some fixed arbitrary $u \in U$, we put for $\varphi = b, \sigma$:

$$
\begin{align*}
\varphi(s) &= \varphi \left( s, \hat{X}^{x_0}(s), \hat{u}(s) \right), \\
\varphi_x(s) &= \varphi_x \left( s, \hat{X}^{x_0}(s), \hat{u}(s) \right), \\
\varphi_{xx}(s) &= \varphi_{xx} \left( s, \hat{X}^{x_0}(s), \hat{u}(s) \right), \\
\delta \varphi(s; u) &= \varphi \left( s, \hat{X}^{x_0}(s), u \right) - \varphi \left( s, \hat{X}^{x_0}(s), \hat{u}(s) \right), \\
f_x(t, s) &= f_x \left( t, \hat{X}^{x_0}(t), s, \hat{X}^{x_0}(s), \hat{u}(s) \right), \\
\delta f(t, s; u) &= f \left( t, \hat{X}^{x_0}(t), s, \hat{X}^{x_0}(s), u \right) \\
&\quad - f \left( t, \hat{X}^{x_0}(t), s, \hat{X}^{x_0}(s), \hat{u}(s) \right).
\end{align*}
$$

The following assumptions (imposed in [4] and [13]) will be in force throughout this section. These assumptions can be made weaker, but we do not focus on this here.

(H1*) The maps $b(s, x, u)$ and $\sigma(s, x, u)$ are twice continuously differentiable with respect to $x$. They and their derivatives in $x$ are continuous in $(s, x, u)$, and bounded.

(H2*) (i) The functions $F(t, \xi, x)$ and $f(t, \xi, x, u)$ are twice continuously differentiable with respect to $x$. They and their derivatives in $x$ are continuous in $(s, x, u)$, and bounded.

(ii) The function $G(t, \xi, \bar{x})$ is twice continuously differentiable with respect to $\bar{x}$. $G$ and its derivatives in $\bar{x}$ are continuous in $\bar{x}$, and bounded.

Define the Hamiltonian as a map from $[0, T] \times \mathbb{R} \times [0, T] \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ by

$$
\mathbb{H}(t, \xi, s, x, u, p, q) := b(s, x, u)p + \sigma(s, x, u)q + f(t, \xi, s, x, u),
$$

and let us introduce the adjoint equations involved in the stochastic maximum principle which characterizes the equilibrium controls.

For each fixed $t \in [0, T]$, the first order adjoint equation associated to the state-control pair $\left( \hat{X}^{x_0}(\cdot), \hat{u}(\cdot) \right)$ is the following linear BSDE satisfied by the pair of processes $(p^t(\cdot), q^t(\cdot))$,

$$
\begin{align}
&d p^t(s) = - \left\{ b_x(s) p^t(s) + \sigma_x(s) q^t(s) + f_x(t, s) \right\} ds + q^t(s) dW(s), \\
&\quad s \in [t, T], \\
&p^t(T) = F_t^{\hat{X}^{x_0}(t)} \left( \hat{X}^{x_0}(T) \right) + G_t^{\hat{X}^{x_0}(t)} \left( \mathbb{E}_t \left[ \hat{X}^{x_0}(T) \right] \right),
\end{align}
$$

and the second order adjoint equation associated to $\left( \hat{X}^{x_0}(\cdot), \hat{u}(\cdot) \right)$ is the following BSDE satisfied by the pair of processes $(P^t(\cdot), Q^t(\cdot))$,

$$
\begin{align}
&d P^t(s) = - \left\{ \left( 2 b_x(s) + \sigma_x(s)^2 \right) P^t(s) + 2 \sigma_x(s) Q^t(s) \right\} ds \\
&\quad + \mathbb{H}_{xx} \left( t, \hat{X}^{x_0}(t), s, \hat{X}^{x_0}(s), \hat{u}(s), p^t(s), q^t(s) \right) ds + Q^t(s) dW(s), \\
&\quad s \in [t, T], \\
&P^t(T) = F_t^{\hat{X}^{x_0}(t)} \left( \hat{X}^{x_0}(T) \right).
\end{align}
$$

Under (H1*)-(H2*) the above BSDEs are uniquely solvable in $(p^t(\cdot), q^t(\cdot)) \in \mathcal{C}_F^2(t, T; \mathbb{R}) \times \mathcal{L}_F^2(t, T; \mathbb{R})$ and $(P^t(\cdot), Q^t(\cdot)) \in \mathcal{C}_F^2(t, T; \mathbb{R}) \times \mathcal{L}_F^2(t, T; \mathbb{R})$, respectively (see e.g. [5]).
Proposition 1. Let \((H1^*)-(H2^*)\) hold. Then for any \(t \in [0, T]\), \(v(\cdot) \in \mathcal{U}[0, T]\) and for any \(\varepsilon \in [0, T-t]\), the following equality holds

\[
\mathbf{J} \left( t, \hat{X}_{x_0}(t); u^{t,\varepsilon, v(\cdot)} \right) = \mathbf{J} \left( t, \hat{X}_{x_0}(t); \hat{u}(\cdot) \right) + \int_t^{t+\varepsilon} \mathbb{E}_t \left[ \delta \mathbb{H} (t, s; v(s)) + \frac{1}{2} P^t (s) (\delta \sigma (s; v(s)))^2 \right] ds + o(\varepsilon),
\]

where

\[
\delta \mathbb{H} (t, s; v(s)) = \mathbb{H} \left( t, \hat{X}_{x_0}(t), s, \hat{X}_{x_0}(s), v(s), \hat{p}^t (s), q^t (s) \right) \]

\[
- \mathbb{H} \left( t, \hat{X}_{x_0}(t), s, \hat{X}_{x_0}(s), \hat{u}(s), \hat{p}^t (s), q^t (s) \right),
\]

\[
= \delta b (s; v(s)) p^t (s) + \delta \sigma (s; v(s)) q^t (s) + \delta f (t, s; v(s)).
\]

Proof. The proof can be adapted from [4] (Proof of Theorem 2.1). We omit it. \(\square\)

The following theorem is comparable with Theorem 1 of [42]; it provides a version of a sufficient stochastic maximum principle which characterizes open-loop equilibriums.

Theorem 4.1. Let \((H1^*)-(H2^*)\) hold. Given an admissible control \(\hat{u}(\cdot) \in \mathcal{U}[0, T]\), let for each \(t \in [0, T]\), \((p^t (\cdot), q^t (\cdot))\) and \((\hat{p}^t (\cdot), \hat{q}^t (\cdot))\) be the unique solutions to the BSDEs (32) and (33), respectively. Suppose that the following hold:

(i) For each \(t \in [0, T]\), \(s \mapsto (p^t (s), q^t (s), P^t (s))\) is continuous on \(t \leq s \leq T\).

(ii) For each \(t \in [0, T]\), there exists a constant \(K > 0\) such that,

\[
K \geq \mathbb{E} \left[ \sup_{s \in [t, T]} |p^t (s)|^2 + \sup_{s \in [t, T]} |q^t (s)|^2 \right] + \mathbb{E} \left[ \sup_{s \in [t, T]} |P^t (s)|^2 \right].
\]

(iii) For each \(t \in [0, T]\),

\[
0 \leq \mathbb{H} \left( t, \hat{X}_{x_0}(t), t, \hat{X}_{x_0}(t), u, p^t (t), q^t (t) \right) \]

\[
- \mathbb{H} \left( t, \hat{X}_{x_0}(t), t, \hat{X}_{x_0}(t), \hat{u}(t), \hat{p}^t (t), \hat{q}^t (t) \right)
\]

\[
+ \frac{1}{2} P^t (t) \left( \sigma \left( t, \hat{X}_{x_0}(t), u \right) - \sigma \left( t, \hat{X}_{x_0}(t), \hat{u}(t) \right) \right)^2, \text{ a.s., } \forall u \in \mathcal{U}.
\]

Then \(\hat{u}(\cdot)\) is an equilibrium control.

Proof. Throughout this proof, \(K\) is still a generic constant, which can be different from line to line. Let \(\hat{u}(\cdot) \in \mathcal{U}[0, T]\) be an admissible strategy for which Assumptions (i)-(iii) in Theorem 4.1 hold. We claim that for any fixed \(t \in [0, T]\) and \(v(\cdot) \in \mathcal{U}[0, T]\), the following equality holds,

\[
\lim_{\varepsilon \downarrow 0} \mathbb{E}_t \left[ \delta \mathbb{H} (t, s; v(s)) + \frac{1}{2} P^t (s) (\delta \sigma (s; v(s)))^2 \right] ds
\]

\[
= \delta \mathbb{H} (t, \hat{v}(t)) + \frac{1}{2} P^t (t) (\delta \sigma (t; v(t)))^2.
\]
Indeed, define for some fixed arbitrarily \( t \in [0, T] \) and \( v (\cdot) \in \mathcal{U} [0, T] \),

\[
\Theta (s; t) = \delta \mathbb{H} (t, s; v (s)) + \frac{1}{2} P^t (s) (\delta \sigma (s; v (s)))^2 \\
= \delta b (s; v (s)) p^t (s) + \delta \sigma (s; v (s)) q^t (s) + \delta f (t, s; v (s)) \\
+ \frac{1}{2} P^t (s) (\delta \sigma (s; v (s)))^2 , \quad \text{for} \ s \in [t, T). 
\]

Clearly, \( \Theta (\cdot; t) \) is a right-continuous \( (\mathcal{F}_s)_{s \in [t, T]} \)-progressively measurable processes. Moreover, it follows from the boundedness conditions of the coefficients \( b, \sigma \) and \( f \) that there exists a constant \( K > 0 \), such that

\[
|\Theta (s; t)| \leq K \left( 1 + |p^t (s)| + |q^t (s)| + |P^t (s)| \right). 
\]

Accordingly, by (36), we have

\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |\Theta (s; t)| \right] \\
\leq K \left( 1 + \mathbb{E} \left[ \sup_{s \in [t, T]} |p^t (s)|^2 \right] + \mathbb{E} \left[ \sup_{s \in [t, T]} |q^t (s)|^2 \right] + \mathbb{E} \left[ \sup_{s \in [t, T]} |P^t (s)|^2 \right] \right) \\
\leq K,
\]

which means that \( \mathbb{E} \left[ \sup_{s \in [t, T]} |\Theta (s; t)| \right] < \infty \). Thus, by Dominated Convergence Theorem together with (34), we obtain that

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ J \left( t, \hat{X}^{x_0} (t) : u_{t, \varepsilon} (\cdot) \right) - J \left( t, \hat{X}^{x_0} (t) : \hat{u} (\cdot) \right) \right\} \\
= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t \Theta (s; t) \, ds \\
= \Theta (t; t) \\
= \delta \mathbb{H} (t; t; v (t)) + \frac{1}{2} P^t (t) (\delta \sigma (t; v (t)))^2,
\]

which prove our claim. Therefor, it follows from (37) that

\[
\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ J \left( t, \hat{X}^{x_0} (t) : u_{t, \varepsilon} (\cdot) \right) - J \left( t, \hat{X}^{x_0} (t) : \hat{u} (\cdot) \right) \right\} \\
= \delta \mathbb{H} (t; t; v (t)) + \frac{1}{2} P^t (t) (\delta \sigma (t; v (t)))^2 \\
\geq 0.
\]

Hence \( \hat{u} (\cdot) \) is an equilibrium control. This completes the proof. \( \square \)
Theorem 4.1 shows that one can obtain equilibrium controls by solving the following flow of FBSDEs,

\[
\begin{aligned}
    d\hat{X}^x_0(s) &= b\left(s, \hat{X}^x_0(s), \hat{u}(s)\right)\,ds + \sigma\left(s, \hat{X}^x_0(s), \hat{u}(s)\right)\,dW(s), \quad s \in [0, T], \\
    dp^t(s) &= -\left\{2b_x(s) + \sigma_x(s)q^t(s) + \sigma_x(s)f_x(t, s)\right\}\,ds + \sigma^t(s)\,dW(s), \quad 0 \leq t \leq s \leq T, \\
    dP^t(s) &= -\left\{\left(2b_x(s) + \sigma_x(s)^2\right)P^t(s) + 2\sigma_x(s)Q^t(s) + \mathbb{H}_{xx}\left(t, \check{X}^{x_0}(t), s, \hat{X}^{x_0}(s), \hat{u}(s), p^t(s), q^t(s)\right)\right\}\,ds \\
    &\quad + Q^t(s)\,dW(s), \quad 0 \leq t \leq s \leq T, \\
    \hat{X}^{x_0}(0) &= x_0, \quad P^t(T) = F_{x_0}^{\check{X}^{x_0}(t)}\left(\hat{X}^{x_0}(T)\right), \\
    p^t(T) &= F_{x_0}^{\check{X}^{x_0}(t)}\left(\check{X}^{x_0}(T)\right) + G_{x_0}^{\check{X}^{x_0}(t)}\left[\mathbb{E}_t\left[\check{X}^{x_0}(T)\right]\right], \\
    \quad t \in [0, T], \\
    0 \leq \delta\mathbb{H}(t, u) + \frac{1}{2}\delta^t(u) \leq \mathbb{H}(t, u), \quad \forall t \in [0, T], \quad \forall u \in U.
\end{aligned}
\tag{38}
\]

Obviously, the above system of FBSDEs is different from the system of FBSDEs (25). However, since both systems lead to open-loop equilibrium solutions, they are necessarily connected to each other. In what follows, we present a theorem which establishes the relationship between these two stochastic systems. Specifically, we show that the adjoint processes as the unique solutions to the BSDEs (32)-(33), can be expressed in terms of the derivatives of the random fields \(\theta^t(s, x), \psi^t(s, x), g(s, x)\) and \(\eta(s, x)\), which solve the BSPDEs (30)-(31).

The following fact concerning the differentiability of stochastic integrals with parameter is important for our purpose. Let \(\phi(\cdot, \cdot) \in \mathcal{L}_{C}^2(0, T; C^0_b(\mathbb{R}; \mathbb{R}))\). Then it was shown that (see, e.g. [Kunita [26], Proposition 2.3.1, Page 56] or [Kunita [25], Exercise 3.1.5, Page 78]) the stochastic integral with parameter \(\int_0^t \phi(s, \cdot)\,dW(s)\) has a modification that belongs to \(\mathcal{L}_{C}^2(0, T; C^1(\mathbb{R}; \mathbb{R}))\) and it satisfies

\[
\frac{\partial}{\partial x} \int_0^t \phi(s, x)\,dW(s) = \int_0^t \frac{\partial}{\partial x} \phi(s, x)\,dW(s), \quad \text{for } \alpha = 1, 2,
\]

where \(\frac{\partial^1}{\partial x} \phi(s, x)\) (resp. \(\frac{\partial^2}{\partial x} \phi(s, x)\)) denote the first derivative (resp. the second derivative) of \(\phi(s, x)\) with respect to the variable \(x\).

We now present a theorem which establishes the connection between Theorem 3.3 and Theorem 4.1. For brevity, we put for \(q = \theta^t, \psi^t, g, \eta\)

\[
\begin{aligned}
    q(s) &= q\left(s, \hat{X}^{x_0}(s)\right), \quad q_x(s) = q_x\left(s, \hat{X}^{x_0}(s)\right), \\
    q_{xx}(s) &= q_{xx}\left(s, \hat{X}^{x_0}(s)\right).
\end{aligned}
\]

**Theorem 4.2.** Suppose that \((H1)-(H2)\) hold. Let \(\hat{u}(\cdot) \in \mathcal{U}[0, T]\) be a fixed admissible control and \(\hat{X}^{x_0}(\cdot)\) be the state process corresponding to \(\hat{u}(\cdot)\). Suppose that, for each \(t \in [0, T]\), the BSPDEs (30)-(31) admit the classical solutions \((\theta^t(\cdot, \cdot), \psi^t(\cdot, \cdot))\) and \((g(\cdot, \cdot), \eta(\cdot, \cdot))\), respectively, such that

\[
\begin{aligned}
    \theta^t(\cdot, \cdot) &\in C^2_{x}(t, T; C^0_b(\mathbb{R}; \mathbb{R})), \quad \psi^t(\cdot, \cdot) \in C^2_{x}(t, T; C^0_b(\mathbb{R}; \mathbb{R})), \\
    g(\cdot, \cdot) &\in C^2_{x}(0, T; C^0_b(\mathbb{R}; \mathbb{R})), \quad \eta(\cdot, \cdot) \in C^2_{x}(0, T; C^0_b(\mathbb{R}; \mathbb{R})).
\end{aligned}
\]

Define the following processes for each \(0 \leq t \leq s \leq T\),
By using the fact (39) we can differentiate (30)-(31) in Proof.

\[
\begin{align*}
\left\{ \begin{array}{l}
p^t(x) = \theta^t_x(s) + G^t_x \hat{X}^{x_0}(t) \left( E_t \left[ \hat{X}^{x_0}(T) \right] \right) g_x(s), \\
q^t(s) = \sigma(s) \left( \theta^t_{xx}(s) + G^t_x \hat{X}^{x_0}(t) \left( E_t \left[ \hat{X}^{x_0}(T) \right] \right) g_{xx}(s) \right) \\
+ \psi^t_x(s) + G^t_x \hat{X}^{x_0}(t) \left( E_t \left[ \hat{X}^{x_0}(T) \right] \right) \eta_x(s), \\
\end{array} \right.
\end{align*}
\]

(40)

Then \( (p^t(\cdot), q^t(\cdot)) \) satisfies the BSDE (32) and \( (P^t(\cdot), Q^t(\cdot)) \) satisfies the BSDE (33).

Proof. By using the fact (39) we can differentiate (30)-(31) in \( x \) to obtain that:

\[
\begin{align*}
\theta^t_x(s, x) &= F_x^{t, \hat{X}^{x_0}(t)}(x) + \int_s^T \left\{ \begin{array}{l}
\theta^t_{xx}(\tau, x) b^{\hat{u}(\cdot)}(\tau, x) \\
+ \theta^t_x(\tau, x) b^{\hat{u}(\cdot)}(\tau, x) + \frac{1}{2} \theta^t_{xxx}(\tau, x) \sigma^{\hat{u}(\cdot)}(\tau, x) \\
+ \sigma^\xi(\tau, x) \sigma^{\hat{u}(\cdot)}(\tau, x) \theta^t_{xx}(\tau, x) + \psi^t_x(\tau, x) \sigma^{\hat{u}(\cdot)}(\tau, x) \\
+ \psi^t_x(\tau, x) \sigma^{\hat{u}(\cdot)}(\tau, x) + f^t_x \hat{X}^{x_0}(t), \bar{\eta}(\cdot) (\tau, x) \right\} d\tau \\
- \int_s^T \psi^t_x(\tau, x) dW(\tau)
\end{align*}
\]

and

\[
\begin{align*}
g_x(s, x) &= 1 + \int_s^T \left\{ g_{xx}(\tau, x) b^{\hat{u}(\cdot)}(\tau, x) \\
+ g_x(\tau, x) b^{\hat{u}(\cdot)}(\tau, x) + \frac{1}{2} \sigma^{\hat{u}(\cdot)}(\tau, x) \sigma^{\hat{u}(\cdot)}(\tau, x) \right\} d\tau \\
+ \eta_x(\tau, x) \sigma^{\hat{u}(\cdot)}(\tau, x) d\tau - \int_s^T \eta_x(\tau, x) dW(\tau).
\end{align*}
\]

Or, equivalently,

\[
\begin{align*}
\left\{ \begin{array}{l}
d\theta^t_x(s, x) = - \left\{ \theta^t_{xx}(s, x) b^{\hat{u}(\cdot)}(s, x) + \theta^t_x(s, x) b^{\hat{u}(\cdot)}(s, x) \\
+ \frac{1}{2} \sigma^{\hat{u}(\cdot)}(s, x) \sigma^{\hat{u}(\cdot)}(s, x) \right\} ds + \psi^t_x(s, x) dW(s), \\
\theta^t_x(T, x) = F_x^{t, \hat{X}^{x_0}(t)}(x), \text{ for } x \in \mathbb{R}
\end{array} \right.
\]

and

\[
\begin{align*}
dg_x(s, x) = - \left\{ g_{xx}(s, x) b^{\hat{u}(\cdot)}(s, x) + g_x(s, x) b^{\hat{u}(\cdot)}(s, x) \\
+ \frac{1}{2} \sigma^{\hat{u}(\cdot)}(s, x) \sigma^{\hat{u}(\cdot)}(s, x) g_{xx}(s, x) + \sigma^{\hat{u}(\cdot)}(s, x) g_{xx}(s, x) g_{xx}(s, x) \\
+ \eta_x(\tau, x) \sigma^{\hat{u}(\cdot)}(s, x) + \eta_x(s, x) \sigma^{\hat{u}(\cdot)}(s, x) \right\} ds + \eta_x(s, x) dW(s), \text{ for } (s, x) \in [0, T] \times \mathbb{R},
\end{align*}
\]

\[
g_x(T, x) = 1, \text{ for } x \in \mathbb{R}.
\]
Accordingly, applying Itô-Wentzell formula to \( \left( \theta^t_x (s, \hat{X}^{x_0} (s)), g_x (s, \hat{X}^{x_0} (s)) \right) \), we get
\[
\begin{align*}
  d\theta^t_x (s) &= - \left\{ \theta^t_x (s) b_x (s) + \sigma_x (s) \left( \sigma (s) \theta^t_{xx} (s) + \psi^t_x (s) \right) \\
  &\quad + f_x (t, s) \right\} ds + \left\{ \psi^t_x (s) + \theta^t_{xx} (s) \sigma (s) \right\} dW (s), \\
  \theta^t_x (T) &= \mathbb{F}_x^{t, \hat{X}^{x_0} (t)} \left( \hat{X}^{x_0} (T) \right)
\end{align*}
\] (42)
and
\[
\begin{align*}
  dg_x (s) &= - \left\{ g_x (s) b_x (s) + \sigma_x (s) \left( \sigma (s) g_{xx} (s) + \eta_x (s) \right) \right\} ds \\
  &\quad + \left\{ \eta_x (s) + g_{xx} (s) \sigma (s) \right\} dW (s), \quad \text{for } s \in [0, T],
\end{align*}
\] (43)
Now define for \( 0 \leq t \leq s \leq T \),
\[
\begin{align*}
  \tilde{p}^t (s) &:= \theta^t_x (s) + \mathbb{G}_x^{t, \hat{X}^{x_0} (t)} \left( \mathbb{E}_t \left[ \hat{X}^{x_0} (T) \right] \right) g_x (s), \\
  \tilde{q}^t (s) &:= \sigma (s) \left( \theta^t_{xx} (s) + \mathbb{G}_x^{t, \hat{X}^{x_0} (t)} \left( \mathbb{E}_t \left[ \hat{X}^{x_0} (T) \right] \right) g_{xx} (s) \right) \\
  &\quad + \psi^t_x (s) + \mathbb{G}_x^{t, \hat{X}^{x_0} (t)} \left( \mathbb{E}_t \left[ \hat{X}^{x_0} (T) \right] \right) \eta_x (s).
\end{align*}
\] (44)-(45)
Combining (42)-(43) together with (44)-(45), we can easily verify that, for each fixed \( t \in [0, T] \), \((\tilde{p}^t (\cdot), \tilde{q}^t (\cdot))\) satisfies
\[
\begin{align*}
  d\tilde{p}^t (s) &= - \left\{ b_x (s) \tilde{p}^t (s) + \sigma_x (s) \tilde{q}^t (s) + f_x (t, s) \right\} ds + \tilde{q}^t (s) dW (s), \\
  \tilde{p}^t (T) &= \mathbb{F}_x^{t, \hat{X}^{x_0} (t)} \left( \hat{X}^{x_0} (T) \right) + \mathbb{G}_x^{t, \hat{X}^{x_0} (t)} \left( \mathbb{E}_t \left[ \hat{X}^{x_0} (T) \right] \right).
\end{align*}
\] Hence, by the uniqueness of the solution to (32), we obtain that
\[
(\tilde{p}^t (s), \tilde{q}^t (s)) = (\tilde{p}^t (s), \tilde{q}^t (s)), \quad \text{a.s., for all } s \in [t, T].
\] Similarly, we prove that the pair of processes defined by (41) coincides with the solution of BSDE (33). Differentiate (30) and (31) twice in \( x \), then \((\theta^t_{xx} (\cdot, \cdot), \psi^t_{xx} (\cdot, \cdot))\) and \((g_{xx} (\cdot, \cdot), \eta_{xx} (\cdot, \cdot))\) satisfy
\[
\begin{align*}
  d\theta^t_{xx} (s, x) &= - \left\{ \theta^t_{xx} (s, x) b^\xi (s, x) + 2 \theta^t_{xx} (s, x) b^\xi (s, x) \\
  &\quad + 2 \sigma^\xi (s, x) \sigma^\xi (s, x) \theta^t_{xx} (s, x) + \psi^t_{xx} (s, x) \right\} ds \\
  &\quad + \sigma^\xi (s, x) g^\xi (s, x) dW (s), \\
  \theta^t_{xx} (T, x) &= \mathbb{F}^{t, \hat{X}^{x_0} (t)} (x), \quad \text{for } x \in \mathbb{R},
\end{align*}
\] for \((s, x) \in [t, T] \times \mathbb{R} \).
and
\[\begin{align*}
&dg_{xx} (s, x) = - \left\{ g_{xxx} (s, x) b^{(c)} (s, x) + 2g_{xx} (s, x) b^{(u)} (s, x) + 2\sigma^{(c)} (s, x) \sigma^{(u)} (s, x) g_{xxx} (s, x, x) + \eta_{xx} (s, x) \right\} ds \\
&\quad + \frac{1}{2} \sigma^{(u)} (s, x)^2 g_{xxx} (s, x, x) + \eta_{xxx} (s, x) \sigma^{(u)} (s, x) ds \\
&+ \eta_{xx} (s, x) dW (s), \text{ for } (s, x) \in [0, T] \times \mathbb{R},
\end{align*}\]

respectively. Applying Itô-Wentzell formula to \( \left( \theta^t_{xx} (s, \hat{X}^{x_0} (s)), g_{xx} (s, \hat{X}^{x_0} (s)) \right) \), we get

\[\begin{align*}
&d\theta^t_{xx} (s) = - \left\{ 2\theta^t_{xx} (s) b_x (s) + 2\sigma_x (s) (\sigma (s) g_{xx} (s)) \right\} ds \\
&\quad + \psi^t_{xx} (s) + \sigma_x (s)^2 \theta^t_{xx} (s) + \theta^t_x (s) b_x (s) \\
&\quad + \sigma_{xx} (s) (\sigma (s) \theta^t_{xx} (s) + \psi^t_x (s)) + f_{xx} (t, s) \right\} ds \\
&\quad + \left\{ \psi^t_{xx} (s) + \sigma (s) \theta^t_{xx} (s) \right\} dW (s), \text{ for } s \in [t, T],
\end{align*}\]

(46)

\[\begin{align*}
&d\theta^t_{xx} (s) = - \left\{ 2g_{xx} (s) b_x (s) + 2\sigma_x (s) (\sigma (s) g_{xx} (s)) \right\} ds \\
&\quad + \eta_{xx} (s) + \sigma_x (s)^2 g_{xx} (s) + g_x (s) b_x (s) \\
&\quad + \sigma_{xx} (s) (\sigma (s) g_{xx} (s) + \eta_x (s)) \right\} ds \\
&\quad + \left\{ \eta_{xx} (s) + \sigma (s) g_{xx} (s) \right\} dW (s), \text{ for } s \in [0, T],
\end{align*}\]

(47)

Define for \(0 \leq t \leq s \leq T,\)

\[\tilde{P}^t (s) = \theta^t_{xx} (s) + G^t_{x} \tilde{X}^{x_0} (t) \left( \mathbb{E}_t \left[ \tilde{X}^{x_0} (T) \right] \right) g_{xx} (s),\]

\[\tilde{Q}^t (s) = \sigma (s) \left( \theta^t_{xx} (s) + G^t_{x} \tilde{X}^{x_0} (t) \left( \mathbb{E}_t \left[ \tilde{X}^{x_0} (T) \right] \right) g_{xx} (s) \right) + \psi^t_{xx} (s) + G^t_{x} \tilde{X}^{x_0} (t) \left( \mathbb{E}_t \left[ \tilde{X}^{x_0} (T) \right] \right) \eta_{xx} (s).\]

Using (46)-(47) together with (40), it is not difficult to verify that for each \( t \in [0, T],\)

\( \left( \tilde{P}^t (\cdot), \tilde{Q}^t (\cdot) \right) \) satisfies

\[\begin{align*}
&d\tilde{P}^t (s) = - \left\{ \left( 2b_x (s) + \sigma_x (s)^2 \right) \tilde{P}^t (s) + 2\sigma_x (s) \tilde{Q}^t (s) \\
&\quad + \mathbb{H}_{xx} \left( t, \tilde{X}^{x_0} (t), s, \tilde{X}^{x_0} (s), \tilde{u} (s), p^t (s), q^t (s) \right) \right\} ds \\
&\quad + \tilde{Q}^t (s) dW (s), \text{ for } s \in [t, T],
\end{align*}\]

\[\tilde{P}^t (T) = F^{t, \tilde{X}^{x_0} (t)}_{xx} \left( \tilde{X}^{x_0} (T) \right).\]

Thus

\( (P^t (s), Q^t (s)) = \left( \tilde{P}^t (s), \tilde{Q}^t (s) \right), \) a.s., for all \( s \in [t, T].\)

\( \square \)

5. Application to the mean-variance portfolio problem. To illustrate our results, we consider a mean-variance portfolio selection problem associated to a diffusion model with deterministic coefficients. We apply the verification argument in Theorem 3.3 to derive the equilibrium investment strategy in an explicit form. Note that, this problem was considered by Hu et al. [19] in which the equilibrium
is, however, derived by solving a flow of FBSDEs having finite-dimensional states processes.

Suppose that there is a financial market in which \( d + 1 \) securities are traded continuously. One of them is a bond, with price \( S_0(s) \) at time \( s \in [0, T] \) governed by

\[
\frac{dS_0(s)}{S_0(s)} = r_0(s) \, ds, \quad S_0(0) = p_0 > 0,
\]

where \( r_0 : [0, T] \to (0, \infty) \) is a deterministic continuous function which represents the risk-free rate. The other \( d \) assets are called risky stocks, whose price processes \( S_1(\cdot), \ldots, S_d(\cdot) \) satisfy the following stochastic differential equations:

\[
\frac{dS_i(s)}{S_i(s)} = r_i(s) \, ds + \sum_{j=1}^{d} \sigma_{ij}(s) \, dW_j(s), \quad S_i(0) = p_i > 0,
\]

where, for each \( i \in \{1, \ldots, d\} \), \( r_i(\cdot) : [0, T] \to (0, \infty) \) and \( \sigma_i(\cdot) = (\sigma_{i1}(\cdot), \ldots, \sigma_{id}(\cdot)) : [0, T] \to (0, \infty)^d \) are deterministic measurable functions. Specifically, \( r_i(\cdot) \) and \( \sigma_i(\cdot) \) represent the appreciation rate and the volatility of the \( i \)-th stock, respectively. For brevity, we write \( r(s) = (r_1(s), \ldots, r_d(s))^\top \) and \( \sigma(s) = (\sigma_{ij}(s))_{1 \leq i, j \leq d} \in \mathbb{R}^{d \times d} \).

We assume that \( r(s) \) and \( \sigma(s) \) are continuous functions of \( s \in [0, T] \). We also require a non-degeneracy condition as follows:

\[
z^\top \sigma(s) \sigma(s)^\top z \geq \epsilon |z|^2, \quad \forall z \in \mathbb{R}^d, \text{ a.e. } s \in [0, T],
\]

for some \( \epsilon > 0 \).

Starting from an initial capital \( x_0 > 0 \) at time 0, during the time horizon \([0, T]\), the investors are allowed to dynamically invest in the financial market. A trading strategy is a \( d \)-dimensional stochastic process \( \pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_d(\cdot))^\top \), where \( \pi_i(s) \) represents the amount invested in the \( i \)-th risky stock at time \( s \in [0, T] \). The amount invested in the bond at time \( s \) is \( X^{x_0, \pi(\cdot)}(s) - \sum_{i=1}^{d} \pi_i(s) \), where \( X^{x_0, \pi(\cdot)}(\cdot) \) is the wealth process associated with the strategy \( \pi(\cdot) \) and the initial state \( x_0 \). Then the evolution of \( X^{x_0, \pi(\cdot)}(\cdot) \) can be described as

\[
\begin{cases}
\frac{dX^{x_0, \pi(\cdot)}(s)}{X^{x_0, \pi(\cdot)}(s)} = \left( X^{x_0, \pi(\cdot)}(s) - \sum_{i=1}^{d} \pi_i(s) \right) \frac{dS_0(s)}{S_0(s)} + \sum_{i=1}^{d} \pi_i(s) \frac{dS_i(s)}{S_i(s)}, \quad s \in [0, T], \\
X^{x_0, \pi(\cdot)}(0) = x_0.
\end{cases}
\]

Accordingly, the wealth process solves the following SDE

\[
\begin{cases}
\frac{dX^{x_0, \pi(\cdot)}(s)}{X^{x_0, \pi(\cdot)}(s)} = \left( r_0(s) X^{x_0, \pi(\cdot)}(s) + \pi(s)^\top \kappa(s) \right) ds + \pi(s)^\top \sigma(s) dW(s), \\
X^{x_0, \pi(\cdot)}(0) = x_0,
\end{cases}
\]

for \( s \in [0, T] \),

where \( \kappa(s) = (r_1(s) - r_0(s), r_2(s) - r_0(s), \ldots, r_d(s) - r_0(s))^\top \). Denote \( \rho(s) = \sigma(s)^{-1} \kappa(s) \), \( u(s) = \sigma(s)^\top \pi(s) \), then the wealth equation is equivalent to the equation of \( (X^{x_0, u(\cdot)}(\cdot), u(\cdot)) \),

\[
\begin{cases}
\frac{dX^{x_0, u(\cdot)}(s)}{X^{x_0, u(\cdot)}(s)} = \left( r_0(s) X^{x_0, u(\cdot)}(s) + u(s)^\top \rho(s) \right) ds + u(s)^\top dW(s), \\
X^{x_0, u(\cdot)}(0) = x_0.
\end{cases}
\]

We interchangeably call \( \pi(\cdot) \) and \( u(\cdot) \) as trading strategies.
As time evolves, we consider the controlled stochastic differential equation parameterized by \((t, \xi) \in [0, T] \times \mathbb{R}\) and satisfied by \(X (\cdot) = X^{t, \xi, u(\cdot)} (\cdot)\),
\[
\begin{aligned}
    dX (s) &= \left( r_0 (s) X (s) + u (s)^\top \rho (s) \right) ds + u (s)^\top dW (s), \text{ for } s \in [t, T], \\
    X (t) &= \xi.
\end{aligned}
\]

A trading strategy \(u (\cdot)\) is said be admissible over \([t, T]\), if it is a \(\mathbb{R}^d\)-valued \((\mathcal{F}_s)_{s \in [t, T]}\) adapted càdlàg process such that:
\[
    \mathbb{E} \left[ \sup_{s \in [t, T]} |u (s)|^4 \right] < \infty.
\]

We denotes by \(\Pi [t, T]\) the set of all admissible strategies over \([t, T]\).

For any fixed initial state \((t, \xi) \in [0, T] \times \mathbb{R}\), the investor’s aim is to choose an investment strategy \(u (\cdot)\) in order to maximize the conditional expectation of the terminal wealth over the period \([t, T]\), while trying at the same time to minimize financial risk. Interpreting risk as the conditional variance of the terminal wealth (i.e. \(\text{Var}_t [X (T)] = \mathbb{E}_t \left[ X (T)^2 \right] - \mathbb{E}_t [X (T)]^2\)). Then the optimization problem is therefore to

Minimize \(J (t, \xi; u (\cdot))\)
\[
\begin{aligned}
    &:= \frac{1}{2} \text{Var}_t [X (T)] - (\mu_1 \xi + \mu_2) \mathbb{E}_t [X (T)] \\
    &= \frac{1}{2} \mathbb{E}_t \left[ X (T)^2 \right] - \left( \frac{1}{2} \mathbb{E}_t [X (T)]^2 + (\mu_1 \xi + \mu_2) \mathbb{E}_t [X (T)] \right)
\end{aligned}
\]

over \(\Pi [t, T]\), where \((\mu_1 \xi + \mu_2)\), with \(\mu_1, \mu_2 \geq 0\), denotes the weight between the conditional variance (as a risk measure) and the conditional expectation.

Obviously, the above model is a special case of Problem (N) with
\[
n = 1, \quad U = \mathbb{R}^d, \quad \sigma (s, x, u) \equiv u^\top, \\
    b (s, x, u) \equiv r_0 (s) x + u^\top \rho (s), \\
    f (t, \xi, s, x, u) \equiv 0, \quad F (t, \xi, x) \equiv \frac{1}{2} x^2 \\
    \text{and } G (t, \xi, \ddot{x}) \equiv -\frac{1}{2} \ddot{x}^2 - (\mu_1 \xi + \mu_2) \ddot{x}.
\]

Accordingly, the BSPDEs associated to an admissible state-control pair \(\left( \hat{X}^{x_0 (\cdot)}, \hat{u} (\cdot) \right)\) are defined as follows:
\[
\begin{aligned}
    d\theta (s, x) &= -\left\{ \theta_x (s, x) \left( r_0 (s) x + \hat{u} (s)^\top \rho (s) \right) \\
    &+ \frac{1}{2} \hat{u} (s)^2 \theta_{xx} (s, x) + \hat{u} (s)^\top \psi_x (s, x) \right\} ds \\
    &+ \psi (s, x)^\top dW (s), \text{ for } (s, x) \in [0, T] \times \mathbb{R}, \\
    \theta (T, x) &= \frac{1}{2} x^2, \text{ for } x \in \mathbb{R}; \\
\end{aligned}
\]
\[
\begin{aligned}
    dg (s, x) &= -\left\{ g_x (s, x) \left( r_0 (s) x + \hat{u} (s)^\top \rho (s) \right) \\
    &+ \frac{1}{2} \hat{u} (s)^2 g_{xx} (s, x) + \hat{u} (s)^\top \eta_x (s, x) \right\} ds \\
    &+ \eta (s, x)^\top dW (s), \text{ for } (s, x) \in [0, T] \times \mathbb{R}, \\
    g (T, x) &= x, \text{ for } x \in \mathbb{R};
\end{aligned}
\]
and the $\mathcal{H}$–function associated to the 6-tuple $\left(\hat{u} (\cdot), \hat{X}^{x_0} (\cdot), \theta (\cdot, \cdot), \psi (\cdot, \cdot), g (\cdot, \cdot), \eta (\cdot, \cdot)\right)$ defined in (14) takes the form,

$$
\mathcal{H} (t, s, X, v) = \frac{1}{2} \left( \theta_{xx} (s, X) - \left( \mu_1 \hat{X}^{x_0} (t) + \mu_2 + \mathbb{E}_{t} [g (s, X)] \right) \right) g_{xx} (s, X) |v|^2 \\
+ v^\top \left( \psi_x (s, X) - \left( \mu_1 \hat{X}^{x_0} (t) + \mu_2 + \mathbb{E}_{t} [g (s, X)] \right) \right) \eta_x (s, X) \\
+ \left( \theta_x (s, X) - \left( \mu_1 \hat{X}^{x_0} (t) + \mu_2 + \mathbb{E}_{t} [g (s, X)] \right) \right) g_x (s, X) \left( r_0 (s) X + v^\top \rho (s) \right).
$$

5.1. Equilibrium solution. In the next, we derive the equilibrium investment solution in an explicit form. The key point in the explicit resolution of the problem is that the random fields $\theta (s, x)$ and $g (s, x)$ may be separated into functions of time and state variables. Then, one needs to solve some system of coupled ordinary differential equations in order to completely determine the equilibrium solution.

Before providing the precise statement of the main result in this section, we begin by establishing some heuristic derivations. By virtue of Remark 3, the sufficient condition for equilibriums in Theorem 3.3 leads to the following system of forward and backward stochastic partial differential equations,

$$
\begin{align*}
\frac{d\hat{X}^{x_0} (s)}{ds} &= \left( r_0 (s) \hat{X}^{x_0} (s) + \hat{u} (s)^\top \rho (s) \right) ds + \hat{u} (s)^\top dW (s), \\
\frac{d\theta (s, x)}{ds} &= - \left\{ \theta_x (s, x) \left( r_0 (s) x + \hat{u} (s)^\top \rho (s) \right) \\
&\quad + \frac{1}{2} \hat{u} (s)^2 \theta_{xx} (s, x) + \hat{u} (s)^\top \psi_x (s, x) \right\} ds \\
&\quad + \psi (s, x)^\top dW (s), \text{ for } (s, x) \in [0, T] \times \mathbb{R}, \\
\frac{dg (s, x)}{ds} &= - \left\{ g_x (s, x) \left( r_0 (s) x + \hat{u} (s)^\top \rho (s) \right) \\
&\quad + \frac{1}{2} |\hat{u} (s)|^2 g_{xx} (s, x) + \hat{u} (s)^\top \eta_x (s, x) \right\} ds \\
&\quad + \eta (s, x)^\top dW (s), \text{ for } (s, x) \in [0, T] \times \mathbb{R},
\end{align*}
$$

$$
\hat{X}^{x_0} (0) = x_0, \quad \theta (T, x) = \frac{1}{2} x^2, \quad g (T, x) = x, \text{ for } x \in \mathbb{R},
$$

with the condition: For all $s \in [0, T],

$$
\mathcal{H} \left( s, s, \hat{X}^{x_0} (s), \hat{u} (s) \right) = \min_{v \in \mathbb{R}} \mathcal{H} \left( s, s, \hat{X}^{x_0} (s), v \right), \text{ a.s.}
$$

To solve the above system, we consider the following ansatz: For all $(s, x) \in [0, T] \times \mathbb{R},$

$$
\theta (s, x) = M_1 (s) \frac{x^2}{2} + M_2 (s) \hat{X}^{x_0} (s) + M_3 (s) X^{x_0} (s) x \\
+ M_4 (s) x + M_5 (s) \hat{X}^{x_0} (s) + M_6 (s)
$$

and

$$
g (s, x) = N_1 (s) x + N_2 (s) \hat{X}^{x_0} (s) + N_3 (s),
$$

where $M_1 (\cdot), M_2 (\cdot), M_3 (\cdot), M_4 (\cdot), M_5 (\cdot), M_6 (\cdot), N_1 (\cdot), N_2 (\cdot)$ and $N_3 (\cdot)$ are deterministic continuously differentiable functions of time such that

$$
M_1 (T) = 1, \quad M_i (T) = 0, \text{ for } i = 2, \ldots, 6, \\
N_2 (T) = N_3 (T) = 0
$$
and

\[ N_1(T) = 1. \]

In this case, the partial derivative of \( \theta(s,x) \) and \( g(s,x) \) are

\[
\begin{align*}
\theta_x(s,x) &= M_1(s)x + M_3(s) \dot{X}^{x_0}(s) + M_4(s), \\
\theta_{xx}(s,x) &= M_1(s), \quad g_x(s,x) = N_1(s) \quad \text{and} \quad g_{xx}(s,x) = 0.
\end{align*}
\]

We would like to determine the equations that \( M_1(\cdot), M_2(\cdot), M_3(\cdot), M_4(\cdot), M_5(\cdot), M_6(\cdot), N_1(\cdot), N_2(\cdot) \) and \( N_3(\cdot) \) should satisfy. To this end we differentiate (53)-(54), compared them with (51) and obtain that (suppressing \( (s) \))

\[
\begin{align*}
\frac{d\theta(s,x)}{ds} &= \left\{ \frac{dM_1}{ds} x^2 + \frac{dM_2}{ds} \left( \dot{X}^{x_0} \right)^2 \right\} \\
&\quad + \frac{dM_4}{ds} x + \frac{dM_5}{ds} \dot{X}^{x_0} + \frac{dM_6}{ds} + M_2 \left( r_0 \left( \dot{X}^{x_0} \right)^2 + \dot{X}^{x_0} \dot{u}^\top \rho + \frac{1}{2} |\dot{u}|^2 \right) \\
&\quad + M_3 \left( r_0 \dot{X}^{x_0} + \dot{u}^\top \rho \right) + M_5 \left( r_0 \dot{X}^{x_0} + \dot{u}^\top \rho \right) \right\} ds \\
&\quad + \left( M_2 \dot{X}^{x_0} + M_3 x + M_5 \right) \dot{u}^\top dW \\
&= - \left\{ \left( M_1 x + M_3 \dot{X}^{x_0} + M_4 \right) \left( r_0 x + \dot{u}^\top \rho \right) \\
&\quad + \frac{1}{2} |\dot{u}|^2 M_1 + \psi_x(s,x)^\top \dot{u} \right\} ds + \psi(s,x)^\top dW
\end{align*}
\]

and

\[
\begin{align*}
\frac{dg(s,x)}{ds} &= \left( \frac{dN_1}{ds} x + \frac{dN_2}{ds} \dot{X}^{x_0} + \frac{dN_3}{ds} + N_2 r_0 \dot{X}^{x_0} + N_2 \dot{u}^\top \rho \right) ds \\
&\quad + N_2 \dot{u}^\top dW \\
&= - \left\{ N_1 \left( r_0 x + \dot{u}^\top \rho \right) + \eta_x(s,x)^\top \dot{u} \right\} ds \\
&\quad + \eta(s,x)^\top dW.
\end{align*}
\]

Thus for all \((s,x) \in [0,T] \times \mathbb{R},\)

\[
\begin{align*}
\psi(s,x) &= \left( M_2(s) \dot{X}^{x_0}(s) + M_3(s) x + M_5(s) \right) \dot{u}(s), \\
\eta(s,x) &= N_2(s) \dot{u}(s),
\end{align*}
\]

Consequently,

\[
\begin{align*}
\psi_x(s,x) &= M_3(s) \dot{u}(s), \\
\eta_x(s,x) &= 0.
\end{align*}
\]
Moreover, the minimum condition in (52) suggests that

\[ 0 = \mathcal{H}_v(s, s, \hat{X}^{x_0}(s), \hat{u}(s)) \]

\[ = (M_1(s) + M_3(s) - (\mu_1 + N_1(s) + N_2(s)) N_1(s)) \rho(s) \hat{X}^{x_0}(s) \]
\[ + (M_4(s) - \mu_2 N_1(s) - N_3(s) N_1(s)) \rho(s) \]
\[ + (M_3(s) + M_1(s)) \hat{u}(s). \]

Accordingly, we obtain that \( \hat{u}(s) \) admits the following explicit representation:

\[ \hat{u}(s) = -\Psi(s) \hat{X}^{x_0}(s) - \varphi(s), \quad (62) \]

where, for all \( s \in [0, T] \),

\[ \Psi(s) = \frac{(M_1(s) + M_3(s) - N_1^2(s) - N_1(s) N_2(s) - N_1(s) \mu_1)}{M_1(s) + M_3(s)} \rho(s) \quad (63) \]

and

\[ \varphi(s) = \frac{M_4(s) - \mu_2 N_1(s) - N_3(s) N_1(s)}{M_1(s) + M_3(s)} \rho(s). \quad (64) \]

Next, comparing the \( ds \) terms in (56)-(57) and using the expression (62), we obtain that (Again suppressing \((s)\))

\[ 0 = \left( \frac{dM_1}{ds} + 2r_0 M_1 \right) \frac{x^2}{2} \]
\[ + \left( \frac{dM_2}{ds} + M_2 2r_0 - 2(M_2 + M_3) \Psi^\top \rho \right) x \hat{X}^{x_0} \]
\[ + \left( \frac{dM_3}{ds} + 2r_0 M_3 - (M_3 + M_1) \Psi(s)^\top \rho \right) x \hat{X}^{x_0} \]
\[ + \left( \frac{dM_4}{ds} + r_0 M_4 - (M_3 + M_1) \varphi^\top \rho \right) x \]
\[ + \left( \frac{dM_5}{ds} + M_5 r_0 - (M_2 + M_3) \varphi^\top \rho - (M_5 + M_4) \Psi^\top \rho \right) \hat{X}^{x_0} \]
\[ + \frac{M_5 r_0}{M_1 + M_2 + 2M_3} \Psi^\top \varphi \right) \hat{X}^{x_0} \]
\[ + \frac{dM_6}{ds} - (M_5 + M_4) \varphi^\top \rho + \frac{1}{2} |\varphi|^2 (M_1 + M_2 + 2M_3) \]

and

\[ 0 = \left( \frac{dN_1}{ds} + N_1 r_0 \right) x \]
\[ + \left( \frac{dN_2}{ds} + r_0 N_2 - (N_1 + N_2) \Psi^\top \rho \right) \hat{X}^{x_0} \]
\[ + \frac{dN_3}{ds} - (N_1 + N_2) \varphi^\top \rho, \]
which leads to the following systems of ODEs

\[
\begin{align*}
\frac{dM_1}{ds} &= -2r_0 M_1, \\
\frac{dN_1}{ds} &= -r_0 N_1, \\
\frac{dN_2}{ds} &= -r_0 N_2 + (N_1 + N_2) \Psi^\top \rho, \\
\frac{dM_4}{ds} &= -2r_0 M_4 + (M_3 + M_1) \Psi^\top \rho, \\
\frac{dM_6}{ds} &= -r_0 M_6 + (M_5 + M_1) \varphi^\top \rho, \\
M_1 (T) &= N_1 (T) = 1, \\
N_2 (T) &= N_3 (T) = M_4 (T) = M_3 (T) = 0.
\end{align*}
\]

and

\[
\begin{align*}
\frac{dM_2}{ds} &= -2r_0 M_2 + 2 (M_2 + M_3) \Psi^\top \rho, \\
\frac{dM_5}{ds} &= -r_0 M_5 + (M_2 + M_3) \varphi^\top \rho + (M_5 + M_4) \Psi^\top \rho - (M_1 + M_2 + 2M_3) \Psi^\top \varphi, \\
\frac{dM_6}{ds} &= (M_5 + M_4) \varphi^\top \rho - \frac{1}{2} |\varphi|^2 (M_1 + M_2 + 2M_3), \\
0 &= M_2 (T) = M_5 (T) = M_6 (T).
\end{align*}
\]

Using the system of ODEs (65), it is not difficult to verify that

\[
\begin{align*}
M_1 (s) &\equiv N_1 (s)^2, \\
M_3 (s) &\equiv N_1 (s) N_2 (s), \\
M_4 (s) &\equiv N_1 (s) N_3 (s).
\end{align*}
\]

Consequently, it follows from (63)-(64) that for all \( s \in [0, T] \),

\[
\Psi (s) = -\frac{N_1 (s) \mu_1}{M_1 (s) + M_3 (s)} \rho (s) = -\frac{\mu_1}{N_1 (s) + N_2 (s)} \rho (s)
\]

and

\[
\varphi (s) = -\frac{N_1 (s) \mu_2}{M_1 (s) + M_3 (s)} \rho (s) = -\frac{\mu_2}{N_1 (s) + N_2 (s)} \rho (s).
\]

Using the above expressions for \( \Psi (\cdot) \) and \( \varphi (\cdot) \), we can easily solve the systems of ODEs (65)-(66), whose solutions are

\[
N_1 (s) = e^{\int_s^T r_0 (\tau) d\tau},
\]

where

\[
\begin{align*}
\frac{dM_1}{ds} &= -2r_0 M_1, \\
\frac{dN_1}{ds} &= -r_0 N_1, \\
\frac{dN_2}{ds} &= -r_0 N_2 + (N_1 + N_2) \Psi^\top \rho, \\
\frac{dM_4}{ds} &= -2r_0 M_4 + (M_3 + M_1) \Psi^\top \rho, \\
\frac{dM_6}{ds} &= -r_0 M_6 + (M_5 + M_1) \varphi^\top \rho, \\
M_1 (T) &= N_1 (T) = 1, \\
N_2 (T) &= N_3 (T) = M_4 (T) = M_3 (T) = 0.
\end{align*}
\]
\[ N_2(s) = \mu_1 e^{\int_s^T r_0(\tau)d\tau} \int_s^T e^{-\int_s^\tau r_0(\tau)d\tau} |\rho(\tau)|^2 d\tau, \quad (71) \]
\[ N_3(s) = \mu_2 \int_s^T |\rho(\tau)|^2 d\tau, \quad (72) \]
\[ M_1(s) = e^{\int_s^T r_0(\tau)d\tau}, \quad (73) \]
\[ M_2(s) = \mu_1 e^{\int_s^T \lambda(\tau)d\tau} \int_s^T e^{-\int_s^\tau \lambda(\tau)d\tau} |\rho(\tau)|^2 \left\{ \frac{N_1(\tau) N_2(\tau)}{N_1(\tau) + N_2(\tau)} \right\} d\tau, \quad (74) \]
\[ M_3(s) = \mu_1 e^{\int_s^T r_0(\tau)d\tau} \int_s^T e^{-\int_s^\tau r_0(\tau)d\tau} |\rho(\tau)|^2 d\tau, \quad (75) \]
\[ M_4(s) = \mu_2 e^{\int_s^T \lambda(\tau)d\tau} \int_s^T |\rho(\tau)|^2 d\tau, \quad (76) \]
\[ M_5(s) = e^{\int_s^T \lambda(\tau)d\tau} \int_s^T e^{-\int_s^\tau \lambda(\tau)d\tau} |\rho(\tau)|^2 \left\{ \frac{N_1(\tau) \mu_2 (M_2(\tau) + M_3(\tau))}{M_1(\tau) + M_3(\tau)} \right\} d\tau, \quad (77) \]
\[ M_6(s) = \int_s^T |\rho(\tau)|^2 \left\{ \frac{\mu_2^2 (M_1(\tau) + M_2(\tau) + 2M_3(\tau))}{2N_1(\tau) + N_2(\tau)} \right\} d\tau. \quad (78) \]

Remark 6. Note that if \( \mu_1 = 0 \), it is not difficult to verify that \( 0 \equiv N_2(s) \equiv M_2(s) \equiv M_3(s) \equiv M_5(s) \) and \( M_6(s) \) reduces to
\[ M_6(s) = \int_s^T \left\{ \frac{1}{2} \mu_2^2 |\rho(\tau)|^2 + \mu_2 |\rho(\tau)|^2 N_3(\tau) \right\} d\tau. \]

The above derivation can be summarized as follows.

Theorem 5.1. Let \( N_1(\cdot) \), \( N_2(\cdot) \), \( N_3(\cdot) \) be the functions given by \( (70) \)-(\( \cdot \)), respectively, and \( M_1(\cdot) \), \( M_2(\cdot) \), \( M_3(\cdot) \), \( M_4(\cdot) \), \( M_5(\cdot) \), \( M_6(\cdot) \) be the deterministic functions given by \( (73) \)-(\( \cdot \)), respectively. Then
\[ \hat{u}(s) = \frac{\mu_1 \hat{X}_0(s) + \mu_2}{N_1(s) + N_2(s)} \rho(s), \quad s \in [0,T], \quad (79) \]
is an equilibrium investment strategy to the mean-variance problem (49)-(50) and the objective value of 
\( \hat{u} (\cdot) \) at time \( t \in [0, T] \) is given by

\[
\mathbf{J} \left( t, \hat{X}^{x_0} (t) ; \hat{u} (\cdot) \right)
\]

\[
= \left( M_2 (t) - N_2 (t)^2 - 2 \mu_1 (N_2 (t) + N_1 (t)) \right) \frac{\dot{X}^{x_0} (t)^2}{2}
+ (M_5 (t) - N_2 (t) N_3 (t) - \mu_1 N_3 (t) - \mu_2 (N_1 (t) + N_2 (t))) \dot{X}^{x_0} (t)
+ M_6 (t) - \frac{1}{2} N_3 (t)^2 - \mu_2 N_3 (t) .
\]

(80)

Proof. First, substituting the equilibrium solution (79) into the wealth process results

\[
\begin{align*}
\hat{X}^{x_0} (s) &= \left\{ \left( r_0 (s) + \frac{\mu_1 \rho_1 (s)}{N_1 (s) + N_2 (s)} | \rho (s) |^2 \right) \dot{X}^{x_0} (s) + \frac{\mu_2 \rho_1 (s)}{N_1 (s) + N_2 (s)} | \rho (s) |^2 \right\} ds \\
&+ \frac{\mu_1 \hat{X}^{x_0} (s) + \mu_2 \eta_1 (s) \rho (s)^{T} \rho (s)^{T}}{N_1 (s) + N_2 (s)} dW (s),
\end{align*}
\]

\[
\hat{X}^{x_0} (0) = x_0 .
\]

The above SDE has a unique solution \( \hat{X}^{x_0} (\cdot) \in C^2 (0, T; \mathbb{R}) \). So the control \( \hat{u} (\cdot) \) defined by (79) is admissible. Moreover, defining \( \theta (\cdot, \cdot) \), \( g (\cdot, \cdot) \), \( \psi (\cdot, \cdot) \) and \( \eta (\cdot, \cdot) \) via (53), (54), (58) and (59), respectively, it is easy to check that \( \hat{u} (\cdot) \), \( \hat{X}^{x_0} (\cdot) \), \( \theta (\cdot, \cdot) \), \( \psi (\cdot, \cdot) \), \( g (\cdot, \cdot) \) and \( \eta (\cdot, \cdot) \) satisfy the system of FBSDEs (51) and that (suppressing (s))

\[
\mathcal{H}_v (s, s, \hat{X}^{x_0}, \hat{u})
\]

\[
= \left( \theta_{xx} (s, \hat{X}^{x_0}) - \left( \mu_1 \hat{X}^{x_0} + \mu_2 + g \left( s, \hat{X}^{x_0} \right) \right) g_{xx} (s, \hat{X}^{x_0}) \right) \hat{u}
+ \left( \psi_{x} (s, \hat{X}^{x_0}) - \left( \mu_1 \hat{X}^{x_0} + \mu_2 + g \left( s, \hat{X}^{x_0} \right) \right) \eta_{x} (s, \hat{X}^{x_0}) \right)
+ \left( \theta_{x} (s, \hat{X}^{x_0}) - \left( \mu_1 \hat{X}^{x_0} + \mu_2 + g \left( s, \hat{X}^{x_0} \right) \right) \eta_{x} (s, \hat{X}^{x_0}) \right) \rho
\]

\[
= -N_1 \left( \mu_1 \hat{X}^{x_0} + \mu_2 \right) \rho + N_1 \left( \mu_1 \hat{X}^{x_0} + \mu_2 \right) \rho
= 0,
\]

\[
\mathcal{H}_{yu} (s, s, \hat{X}^{x_0}, \hat{u}) = \theta_{xx} (s, \hat{X}^{x_0})
= M_1 (s)
> 0.
\]

Thus, the minimum condition in (52) holds. Furthermore, Assumptions (i), (ii) and (iii) in Theorem 3.3 are satisfied. Hence, \( \hat{u} (\cdot) \) is an equilibrium solution. Next we prove (80). By the equality (16) in Theorem 3.3, the objective value of \( \hat{u} (\cdot) \) at time \( t \in [0, T] \) is given by

\[
\mathbf{J} \left( t, \hat{X}^{x_0} (t) ; \hat{u} (\cdot) \right)
\]

\[
= \theta^t \left( t, \hat{X}^{x_0} (t) \right) + G \left( t, \hat{X}^{x_0} (t), g \left( t, \hat{X}^{x_0} (t) \right) \right)
= \theta^t \left( t, \hat{X}^{x_0} (t) \right) - \frac{1}{2} \left( g \left( t, \hat{X}^{x_0} (t) \right) \right)^2
- \left( \mu_1 \hat{X}^{x_0} (t) + \mu_2 \right) g \left( t, \hat{X}^{x_0} (t) \right),
\]

(81)
The mean-variance problem (49)-(50) was considered by Hu et al. Remark 7. with following system of FBSDEs:

\[
\left\{ \begin{array}{l}
dX^* (s) = \left( r_0 (s) X^* (s) + u^* (s) ^\top \rho (s) \right) ds + u^* (s) ^\top dW (s) , \ s \in [0, T] , \\
p^t (s) = -r_0 (s) p^t (s) ds + q^t (s) ^\top dW (s) , \ 0 \leq t \leq s \leq T , \\
\rho ^t (T) = X^* (T) - \mathbb{E}_t \left[ X^* (T) \right] - \mu_1 X^* (t) - \mu_2 , \ \text{for} \ t \in [0, T] , \\
X^* (0) = x_0 .
\end{array} \right.
\]

Invoking (82)-(83) into (81), by simple calculations, we obtain the equality in (80).

\[\square\]

**Remark 7.** The mean-variance problem (49)-(50) was considered by Hu et al. \([19], [20] \). Therein the open-loop equilibrium solution is derived by solving the following system of FBSDEs:

\[
\left\{ \begin{array}{l}
dX^* (s) = \left( r_0 (s) X^* (s) + u^* (s) ^\top \rho (s) \right) ds + u^* (s) ^\top dW (s) , \ s \in [0, T] , \\
p^t (s) = -r_0 (s) p^t (s) ds + q^t (s) ^\top dW (s) , \ 0 \leq t \leq s \leq T , \\
\rho ^t (T) = X^* (T) - \mathbb{E}_t \left[ X^* (T) \right] - \mu_1 X^* (t) - \mu_2 , \ \text{for} \ t \in [0, T] , \\
X^* (0) = x_0 .
\end{array} \right.
\]

which leads to the following explicit solution

\[
u^* (s) = \frac{e^{\int_s^T r_0 (\tau) d\tau} \left( \mu_1 X^* (s) + \mu_2 \right)}{A (s)} \rho (s) , \tag{85}\]

with \(A (\cdot)\) given by

\[
A (s) = e^{2 \int_s^T r_0 (\tau) d\tau} \left( 1 + \mu_1 \int_s^T e^{-\int_s^{\tau} t r_0 (\kappa) d\kappa} |\rho (\tau)|^2 d\tau \right) , \ \text{for} \ s \in [0, T] . \tag{86}\]

It is easy to verify that

\[
A (s) = e^{2 \int_s^T r_0 (\tau) d\tau} \left( N_1 (s) + N_2 (s) \right) = M_1 (s) + M_3 (s) , \ \text{for all} \ s \in [0, T] .
\]

Therefore,

\[
u^* (s) = \hat{u} (s) \ \text{and} \ X^* (s) = \hat{X} x_0 (s) , \ \text{for all} \ s \in [0, T] , \ a.s.. \]

Which means that our solution coincides with the one obtained by Hu et al. \([19]\). However, this is unsurprising since Theorem 5.1 in \([20]\) ensures that there is a unique open-loop equilibrium strategy for the mean-variance problem (49)-(50), which is identical to the one generated from the feedback law (85).

By virtue of Remark 6, we have the following corollary.

**Corollary 2.** In the case of the mean-variance portfolio selection model with a constant risk aversion parameter (i.e. \(\mu_1 = 0\) and \(\mu_2 = \frac{1}{\gamma}\), with \(\gamma > 0\)), the equilibrium investment strategy is given by

\[
\hat{u} (s) = \frac{1}{\gamma} e^{\int_s^T r_0 (\tau) d\tau} \rho (s) \tag{87}
\]
and the objective value of $\hat{u}(\cdot)$ at time $t \in [0, T]$ is given by

$$J(t, \hat{X}^{x_0}(t); \hat{u}(\cdot)) = -\frac{1}{\gamma} \hat{X}^{x_0}(t) e^{\int_t^T r_0(\tau) d\tau} - \frac{1}{2\gamma^2} \int_0^T |\rho(\tau)|^2 d\tau.$$

**Remark 8.** The equilibrium strategy and the equilibrium objective value presented in Corollary 2 are essentially the same as that obtained by Basak and Chabakauri [7] and Björk and Murgoci [8] by solving an extended HJB system. This shows the existence of a link between the extended HJB approach of Björk and Murgoci [8] and the BSPDEs approach of the present paper although the definitions of equilibrium are different.

6. **Conclusion and open problems.** In this paper we have presented a fairly general class of time inconsistent stochastic control problems. Using a game theoretic perspective we have derived a stochastic system that includes a flow of backward stochastic partial differential equations for the determination of the open-loop equilibrium control as well as for the objective value of the equilibrium solution. We have proved a verification theorem, and we have studied its connection with the duality approach of Hu et al. ([19], [20]). We illustrated our main results by solving a mean-variance portfolio problem with a state-dependent trade-off between the mean and the variance and the solution we obtained coincides with that obtained in [19]. However, some obvious open research problems are the following:

- Our main result (Theorem 3.3) takes the existence of a solution to the flow of FBSDEs (25) as an assumption. An open and very difficult problem is to provide mild conditions which guarantee that the flow of FBSDEs (25) admits a unique adapted solution $\left(\hat{u}(\cdot), \hat{X}^{x_0}(\cdot), \left\{\theta^t(\cdot, \cdot), \psi^t(\cdot, \cdot)\right\}_{t \in [0, T]}, g(\cdot, \cdot), \eta(\cdot, \cdot)\right)$.
- Assumptions (i), (ii) and (iii) in Theorem 3.3 may seem restrictive conditions. We hope that in our future publications, these conditions can be made weaker or removed.

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E-mail address: izacalia@yahoo.com