FIRST-ORDER LAGRANGIANS AND PATH-INTEGRAL QUANTIZATION IN THE t-J MODEL.

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Abstract

By using the supersymmetric version of the Faddeev-Jackiw symplectic formalism, a family of first-order constrained Lagrangians for the t-J model is found. In this approach the Hubbard $\hat{X}$-operators are used as field variables. In this framework, we first study the spinless fermion model which satisfies the graded algebra spl(1,1). Later on, in order to satisfy the Hubbard $\hat{X}$-operators commutation rules satisfying the graded algebra spl(2,1), the number and kind of constraints that must be included in a classical first-order Lagrangian formalism for the t-J model are found. This model is also analyzed in the context of the path-integral formalism, and so the correlation generating functional and the effective Lagrangian are constructed.

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I. INTRODUCTION

In a recent paper [1] the classical and quantum Lagrangian dynamics for the SU(2) bosonic algebra was constructed. By means of the path-intergral techniques, the perturbative formalism was also developed. In that case the Hubbard operators [2] are boson-like one, and they are a suitable representation for the spin-1/2 Heisenberg model.

The well known t-J model is one of the most important candidate to explain the phenomenology of High-Tc superconductivity. This model contains the main physics of doped holes on an antiferromagnetic background. In the case of the t-J model, the Hubbard operator representation is quite natural to treat the electronic correlation effects [3]. In this model in which spin and charge degrees of freedom are present, the Hubbard $\hat{X}$-operators satisfies the graded algebra $spl(2,1)$ [4].

Like as it was shown in Ref.[1] for the bosonic case, it must be expected that the path-integral formalism applied to the t-J model described in terms of a first-order Lagrangian can be useful. As it is well known, these techniques are powerful in quantum field theory as well as in solid state physics. This is clearly proved when the perturbative formalism can be assumed, and consequently the Feynman rules and the diagrammatics of the model can be implemented.

In the t-J model the only three possible states on a lattice site are $|\alpha> = |0>, |+>, |-$. These states correspond respectively to an empty site, an occupied site with an electron of spin-up, or an occupied site with an electron of spin-down. Double occupancy is forbidden in the t-J model. In terms of these states the Hubbard $\hat{X}$-operators are defined as [2],

$$\hat{X}_i^{\alpha\beta} = |i\alpha><i\beta|.$$  \hspace{1cm} (1.1)

In Eq.(1.1), when one of the index is zero and the other different from zero, the corresponding $\hat{X}$-operator is fermion-like, otherwise boson-like.

The Hubbard $\hat{X}$-operators satisfy the following graded commutation relations

$$[\hat{X}_i^{\alpha\beta}, \hat{X}_j^{\gamma\delta}]_\pm = \delta_{ij}(\delta^{\beta\gamma}\hat{X}_i^{\alpha\delta} \pm \delta^{\alpha\delta}\hat{X}_i^{\gamma\beta}),$$  \hspace{1cm} (1.2)
where the + sign must be used when both operators are fermion-like, otherwise it corresponds the − sign.

Using the $\hat{X}$-operators definition it is easy to see that the following conditions hold:

a) the completeness condition

$$\sum_{\alpha} \hat{X}_{\alpha i} = \hat{I},$$

(1.3)

b) the multiplication rules in a given site

$$\hat{X}_{\alpha \beta i} \hat{X}_{\gamma \delta i} = \delta_{\beta \gamma} \hat{X}_{\alpha \delta i}.$$  

(1.4)

One of the purpose of this paper is to construct a family of first-order Lagrangians written in terms of fermion-like and boson-like Hubbard $\hat{X}$-operators, describing the dynamics of the t-J model.

The graded commutators between the field dynamical variables i.e., the graded quantum Dirac brackets of the model must verify the graded algebra $\text{spl}(2,1)$ given in (1.2) for the Hubbard $\hat{X}$-operators.

This problem will be treated by extending our results of Ref.[1] to the t-J model case. To this aim we will use the supersymmetric extension of the symplectic Faddeev-Jackiw (FJ) method.

Subsequently, once the family of Lagrangians and the constraint structure of the model were determined, by using the path-integral formalism the correlation generating functional can be found in terms of a suitable effective Lagrangian.

The paper is organized as follows. In section II, the constraint structure of the spinless fermion case is briefly analyzed in the framework of the symplectic FJ Lagrangian method. Next, by using the path-integral representation, the partition function is written in terms of an effective Lagrangian. The final expression we find agrees perfectly with the current form of the partition function for the spinless fermion model, showing the validity of our approach. In section III, a general treatment for systems containing the Hubbard $\hat{X}$-operators of the graded algebra $\text{spl}(2,1)$ as dynamical variables is constructed. A family of classical first-order Lagrangians describing these dynamical systems is found. In section IV, using these results
and by applying techniques used in quantum field theories, the path-integral quantization formalism is developed and the correlation generating functional is given in terms of an appropriate effective Lagrangian. Conclusions are written in section V.

II. THE SPINLESS FERMION CASE

The treatment of the spinless fermion model is useful for two main reasons: a) to understand how the algorithm is applied when both, boson-like and fermion-like Hubbard $\hat{X}$-operators are present; and b) to show that our approach produces the correct and well known path-integral representation for the correlation generating functional of the model.

By generalizing the results of Ref. [1], our purpose is to construct a family of classical first-order Lagrangians written in terms of the Hubbard $\hat{X}$-operators whose graded commutation relations or graded quantum Dirac brackets between field variables are given by the Eq. (1.2). This approach clearly shows how the constraint structure i.e., the kind and the number of constraints present in these models is provided by the symplectic FJ quantum method.

The notation and key equation related to the symplectic FJ method we will use are those of Refs. [6,7].

We assume that the family of classical first-order Lagrangians in terms of the Hubbard $\hat{X}$-operators can be written as follows

$$L = a_{\alpha\beta}(X)\dot{X}^{\alpha\beta} - V^{(0)}.$$ \hspace{5cm} (2.1)

In the FJ language the symplectic potential $V^{(0)}$ is defined by

$$V^{(0)} = H(X) + \lambda^a \Omega_a ,$$ \hspace{5cm} (2.2)

and so the constraints are given by

$$\frac{\partial V^{(0)}}{\partial \lambda^a} = \Omega_a .$$ \hspace{5cm} (2.3)

Looking at the Lagrangian Eq. (2.1) we see that the initial set of dynamical symplectic variables defining the extended configuration space is given by $(X^{\alpha\beta}, \lambda^a)$. 
In Eq. (2.2), $H(X)$ is a proper Hamiltonian for the model also given in terms of the Hubbard $X$-operators. It is important to remark that at this level the $X$-variables must be treated as classical fields.

The site subscript indices $i, j$, appearing in the definition of the Hubbard operators were dropped since they are irrelevant in the analysis we will develop. Without any difficulty the site indices can be opportunely included.

In Eq. (2.1) the coefficients $a_{\alpha\beta}(X)$ are a priori unknown and they are determined in such a way that the graded algebra (1.2) for the Hubbard $X$-operators must be verified. The reality condition on the Lagrangian implies that $a_{\alpha\beta}(X) = (-1)^{|a|}a_{\beta\alpha}^*(X)$, where $|a|$ indicate the Fermi grading of the coefficient. The $\lambda^a$ parameters used in Eq.(2.2) are suitable bosonic or fermionic Lagrange multipliers which allow the introduction of the constraints in the Lagrangian formalism. $\Omega_a(X)$ is the set of unknown bosonic or fermionic constraints, initially considered ad hoc in the Lagrangian. Both the constraints $\Omega_a(X)$ as well as the range of the index $a$ (i.e., kind and number of contraints) must be determined later on by consistency.

By following the steps developed in Refs.[7,8] the FJ method must be implemented on the Lagrangian (2.1), and so the symplectic supermatrix $M_{AB}(X)$ can be constructed straightforward. The matrix elements of the symplectic supermatrix are given by the generalized curl constructed with the partial derivatives involving the set of variables.

When the symplectic supermatrix $M_{AB}(X)$ is singular it is necessary to carry out the iteration procedure by enlarging the configuration space. Otherwise, when the symplectic supermatrix $M_{AB}(X)$ is non-singular, the inverse supermatrix $(M_{AB})^{-1}$ is unique and their matrix elements are the generalized FJ brackets, corresponding to the graded Dirac brackets of the theory.

As usual the relation between the graded Dirac Brackets $\{\ ,\ \}$ and the graded commutation relations (2.1) for the Hubbard $X$-operators is: $i\{X, X\}_\pm \to [\hat{X}, \hat{X}]_\pm$.

As it is well known the main feature of the symplectic formalism is that the classification of constrained or unconstrained systems is related to the singular or non singular behavior of the fundamental symplectic two-form which gives rise to the symplectic supermatrix. On
the contrary to the Dirac’s language, the classification of constraints in primary, secondary
and so on; or in first-class and second-class has no meaning.

On the other hand, when the FJ symplectic method is applied to gauge models having
constraints associated to gauge symmetries, the algorithm is unable to produce an invertible
symplectic supermatrix. Therefore, the existence of the inverse of the symplectic supermatrix
necessarily implies that the model has only constraints which in the Dirac picture correspond
to second-class one.

Now we are going to apply the symplectic formalism to the spinless fermion model. As it
is known this model is obtained from the graded algebra (1.2) when the indices \( \alpha, \beta \) can take
only two values: 0 denoting an empty site, and 1 denoting an occupied site with one fermion.
So, the four Hubbard \( \hat{X} \)- operators close the graded algebra \( \text{spl}(1,1) \) \[10\]. Two of them \( \hat{X}^{11} \)
and \( \hat{X}^{00} \) are boson-like operators, while the other two \( \hat{X}^{01} \) and \( \hat{X}^{10} \) are fermion-like one.

It is easy to show that in this case the symplectic supermatrix obtained from the La-
grangian (2.1) in which the field variables are \( (X^{\alpha\beta}, \lambda^a) \) is singular, and so it is necessary to
carry out one iteration to obtain an invertible symplectic supermatrix. As commented above,
this is done by enlarging the configuration space redefining the \( \lambda^a \) variables as: \( \lambda^a = -\dot{\xi}^a \).

Consequently, after the iteration is done the constraints are written in the symplectic
part and the first-iterated Lagrangian \( L^{(1)} \) writes as follows

\[
L^{(1)} = a_{11}(X)\dot{X}^{11} + a_{00}(X)\dot{X}^{00} + a_{01}(X)\dot{X}^{01} + a_{10}(X)\dot{X}^{10} + \dot{\xi}^a\Omega_a - \mathbf{V}^{(1)}.
\]

where \( \mathbf{V}^{(1)} = \mathbf{V}^{(0)} \mid_{\Omega_a=0} = H(X) \).

Therefore, the modified symplectic supermatrix associated to the Lagrangian (2.4) can
be formally written

\[
M_{AB} = \begin{pmatrix}
\frac{\partial a_{\alpha\beta}}{\partial X^{\gamma\delta}} & -\frac{\partial a_{\alpha\gamma}}{\partial X^{\beta\delta}} & \frac{\partial \lambda_b}{\partial X^{\alpha\beta}} \\
-\frac{\partial a_{\alpha\gamma}}{\partial X^{\beta\delta}} & \frac{\partial \lambda_b}{\partial X^{\alpha\delta}} & 0 \\
\end{pmatrix},
\]

where the compound indices \( A = \{(\alpha\beta), a\} \) and \( B = \{(\gamma\delta), b\} \) run in the different ranges of
the complete set of variables defining the extended configuration space.

Now, in order to obtain an invertible symplectic supermatrix, the problem is to determine
both, the Lagrangian coefficients \( a_{\alpha\beta}(X) \) and how many constraints \( \Omega_a \) are provided by the
symplectic FJ algorithm. Finally, by solving the equations on the constraints the functions \( \Omega_a(X) = 0 \) must be found.

Accounting that the inverse of the symplectic supermatrix \( M_{AB}(X) \) is written
\[
(M^{AB})^{-1} = \begin{pmatrix}
\{X^{\alpha\beta}, X^{\gamma\delta}\} & \{X^{\alpha\beta}, \xi^b\} \\
\{\xi^a, X^{\gamma\delta}\} & \{\xi^a, \xi^b\}
\end{pmatrix},
\] (2.6)
each matrix element of the submatrix \( \{X^{\alpha\beta}, X^{\gamma\delta}\} \) must be equalled to each one of the following Hubbard commutation relations of the spl(1,1) graded algebra
\[
\begin{align*}
\{X^{00}, X^{00}\}_- &= \{X^{00}, X^{11}\}_- = \{X^{11}, X^{11}\}_- = 0, \\
\{X^{00}, X^{01}\}_- &= -iX^{01}, \quad \{X^{00}, X^{10}\}_- = iX^{10}, \\
\{X^{11}, X^{10}\}_- &= -iX^{10}, \quad \{X^{11}, X^{01}\}_- = iX^{01}, \\
\{X^{01}, X^{10}\}_+ &= -i(X^{00} + X^{11}).
\end{align*}
\] (2.7)

Of course in the Eq.(2.6) the three remaining submatrices \( \{X^{\alpha\beta}, \xi^b\} \), \( \{\xi^a, X^{\gamma\delta}\} \) and \( \{\xi^a, \xi^b\} \) are unknown.

It is easy to show that the invertible symplectic supermatrix \( M_{AB}(X) \) given in Eq. (2.5) is a square \( 6 \times 6 \) dimensional one, and it can be written in the form
\[
M_{AB} = \begin{pmatrix}
A_{bb} & B_{bf} \\
C_{fb} & D_{ff}
\end{pmatrix},
\] (2.8)
whose Bose-Bose parts \( A_{bb} \) and Fermi-Fermi parts \( D_{ff} \) are even elements of the Grassmann algebra and whose Bose-Fermi parts \( B_{bf} \) and Fermi-Bose parts \( C_{fb} \) are odd elements. As it is well known [11] the inverse \((M^{AB})^{-1}\) exists if and only if \( A_{bb} \) and \( D_{ff} \) have an inverse.

In the present case the Bose-Bose part \( A_{bb} \) is an ordinary non-singular \( 4 \times 4 \) dimensional matrix and the Fermi-Fermi part \( D_{ff} \) is an ordinary non-singular \( 2 \times 2 \) dimensional matrix.

By using the equation \( M_{AB}(M^{BC})^{-1} = \delta^C_A \), and taking into account the equation
\[
(M^{BC})^{-1} = -i(-1)^{|\varepsilon_B|} [\hat{B}, \hat{C}]_{\pm},
\] (2.9)
where \(|\varepsilon_B|\) is the Fermi grading of the variable \( B \), differential equations on the Lagrangian coefficients \( a_{\alpha\beta}(X) \) and on the constraints \( \Omega_a \) are obtained.
In particular, the system of four homogeneous differential equations on the constraints $\Omega_a$ can be written

$$i(-1)^{|\varepsilon_{\alpha\beta}|(|a|+1)} \frac{\partial \Omega_a}{\partial X_{\alpha\beta}} \left[ X^{\alpha\beta}, X^{\gamma\delta} \right]_{\pm} = 0 ,$$

(2.10)

where $|\varepsilon_{\alpha\beta}|$ and $|a|$ are the Fermi grading of the Hubbard operators and of the constraints respectively.

By solving the partial differential equations system (2.10), two bosonic solutions are found. So, the associated constraints reads

$$\Omega_1 = X^{00} + X^{11} - 1 = 0 ,$$

(2.11a)

$$\Omega_2 = X^{11} + X^{01}X^{10} - 1 = 0 .$$

(2.11b)

We emphasize that once the invertibility of the symplectic supermatrix (2.5) is assumed, necessarily it must be understood that the constraints (2.11) are second-class one, as really occurs.

Analogously, by solving the remaining system of partial differential equations on the Lagrangian coefficients $a_{\alpha\beta}(X)$, the values we find are

$$a_{00} = \frac{1}{2}X^{11} , \ a_{11} = -\frac{1}{2}X^{00} , \ a_{10} = \frac{i}{2}X^{01} , \ a_{01} = \frac{i}{2}X^{10} .$$

(2.12)

The constraint (2.11a) provided by the symplectic FJ algorithm is precisely the completeness condition (1.3) which is necessary because the ”double occupancy” at each site is forbidden.

Therefore, from the Lagrangian (2.1) with the values of the coefficients given in (2.12), together with the bosonic constraints (2.11), it is straightforward to prove that the graded algebra $\text{spl}(1,1)$ given in Eq. (2.7) is recovered.

At this stage, we are able to write the correlation generating functional by using the path-integral approach of Faddeev-Senjanovich [12]. Thus, the spinless fermion model partition function can be initially written as follows

$$Z = \int \mathcal{D}X \ \delta(X^{01}X^{10} + X^{11} - 1) \ \delta(X^{11} + X^{00} - 1) \ \exp i \int dt \ L(X, \dot{X}) ,$$

(2.13)
where the Lagrangian $L(X, \dot{X})$ is given by

$$
L = \frac{1}{2} \left( X^{11} \dot{X}^{00} - X^{00} \dot{X}^{11} \right) + i \frac{1}{2} \left( X^{01} \dot{X}^{10} + X^{10} \dot{X}^{01} \right) - H(X). \tag{2.14}
$$

We note that in the path-integral (2.13) the superdeterminant of the supermatrix constructed from the constraints was omitted because it is field independent and can be included in the path-integral normalization factor.

By integrating out the bosonic variables $X^{11}$ and $X^{00}$, the partition function (2.13) becomes

$$
Z = \int D X^{10} D X^{01} \exp i \int dt L_{\text{eff}}(X, \dot{X}), \tag{2.15}
$$

where

$$
L_{\text{eff}}(X, \dot{X}) = \frac{i}{2} (X^{01} \dot{X}^{10} + X^{10} \dot{X}^{01}) - H(X) |_{\Omega_a = 0}. \tag{2.16}
$$

As it can be seen, this model initially has two bosonic degrees of freedom and the FJ algorithm provides two bosonic constraints, so the bosonic dynamics is lost. Therefore, Eq. (2.15) is only dependent on one complex Grassmann variable, and it is precisely the path-integral representation for the partition function of the spinless fermion model.

This example shows how our approach produces the correct effective Lagrangian for the model.

### III. FAMILY OF FIRST-ORDER LAGRANGIAN AND CONSTRAINTS IN THE $t-J$ MODEL.

In the usual approach to study the quantization problem in the $t-J$ model through the path-integral representation both, the slave-fermion and the slave-boson representations are available [3]. In these representations the real excitations are forced to be decoupled. We must note that even using the slave-particle representations, the constrained Dirac theory is needed [13].

Recently, by using the $t$-$J$ model to solve a particular problem of superconductivity, a discrepancy between the results of the slave-boson and the $X$-operator approaches was found.
Such a discrepancy is probably an artifice of the slave representation. From our point of view this situation is important and must be taken into account since we behave in such a way that the Hubbard $X$-operators are treated as indivisible objects.

Another alternative way without any decoupling assumption, is to study the system from the point of view of the coherent state phase path-integration \cite{13}.

In this section following our approach, the Lagrangian dynamics generated by the most general graded algebra spl(2,1) is constructed. In this case, the four quantities $(X^+, X^-, X^{++}, X^{--})$ are boson-like operators and the four quantities $(X^{0+}, X^{0-}, X^{+0}, X^{-0})$ are fermion-like. As we will see later on, in our approach the remaining boson-like operator $X^{00}$ is really a function of the fermion-like operators.

The Lagrangian (2.1) for the t-J model explicitly reads

$$L = a_{++}(X) \dot{X}^+ + a_{--}(X) \dot{X}^- + a_{++}(X) \dot{X}^{++} + a_{--}(X) \dot{X}^{--} + a_{0+}(X) \dot{X}^{0+} + a_{0-}(X) \dot{X}^{0-} + a_{+0}(X) \dot{X}^{+0} + a_{-0}(X) \dot{X}^{-0} - V^{(0)}.$$  \hspace{1cm} (3.1)

By defining $a_u = \frac{1}{2}(a^{++} - a^{--})$ and $a_v = \frac{1}{2}(a^{++} + a^{--})$, and by calling $u = X^{++} - X^{--}$ and $v = X^{++} + X^{--}$ the Lagrangian (3.1) can be written in the more useful form

$$L = a_{++}(X) \dot{X}^+ + a_{--}(X) \dot{X}^- + a_u(X) \dot{u} + a_v(X) \dot{v} + a_{0+}(X) \dot{X}^{0+} + a_{0-}(X) \dot{X}^{0-} + a_{+0}(X) \dot{X}^{+0} + a_{-0}(X) \dot{X}^{-0} - V^{(0)}.$$  \hspace{1cm} (3.2)

Analogously to that developed above, also in the general case of the graded algebra spl(2,1) the symplectic supermatrix $M_{AB}(X)$ can be constructed straightforward. The starting symplectic supermatrix $M_{AB}(X)$ is again singular, and one iteration is necessary to obtain an invertible supermatrix.

Taking into account that there are only four bosonic fields, only two bosonic constraints are possible (see discussion in Ref.[1]). Therefore, the antisymmetric ordinary bosonic sub-matrix $A_{bb}$ given in Eq. (2.8) must be a $6 \times 6$ dimensional square matrix.

In the t-J model case, the system of Eqs.(2.10) on the constraints is given by eight
homogeneous differential equations. Such a system has two bosonic solutions, and the corresponding constraints writes

\[ \Omega_1 = X^{++} + X^{--} + \rho - 1 = 0 , \quad (3.3a) \]

\[ \Omega_2 = X^{+-}X^{-+} + \frac{1}{4}u^2 - (1 - \frac{1}{2}v)^2 + \rho = 0 , \quad (3.3b) \]

where \( \rho \) is defined by

\[ \rho = X^{0+}X^{0+} + X^{0-}X^{-0} . \quad (3.4) \]

It is important to notice that in the present case the constraint (3.3a) solves the differential equations system (2.10), if and only if the following fermionic constraints hold

\[ \Xi_1 = X^{0+}X^{0+} - X^{0-}X^{-+} = 0 , \quad (3.5a) \]

\[ \Xi_2 = X^{0+}X^{0-} - X^{-0}X^{++} = 0 , \quad (3.5b) \]

\[ \Xi_3 = X^{0+}X^{+-} - X^{0-}X^{++} = 0 , \quad (3.5c) \]

\[ \Xi_4 = X^{0+}X^{++} - X^{-0}X^{++} = 0 . \quad (3.5d) \]

Of course, the four fermionic constraints (3.5), are also solutions of the differential equations system.

As it can be seen only two of the fermionic constraints (3.5) must be considered as independent. In fact, from Eqs. (3.5c), (3.5d) and using the Eq.(3.3b), it is easy to show that the Eqs.(3.5a) and (3.5b) can be recovered.

Therefore, in the t-J model under consideration, there are two bosonic constraints given by Eqs. (3.3) and two fermionic constraints given by Eqs. (3.5c) and (3.5d).

In such a condition the symmetric non-singular ordinary bosonic submatrix \( D_{ff} \) also results a \( 6 \times 6 \) dimensional square matrix. Thus, the symplectic supermatrix written in Eq. (2.8) has dimension \( 12 \times 12 \).
As in the spinless fermion case, in the t-J model the completeness condition (1.3) is also obtained as one of the bosonic constraints (Eq.[3.3a]). So, in Eq. (3.3a) $\rho$ must be identified with the hole density (i.e., proportional to the number of holes $X^{00}$).

We remember that such a condition has an important physical meaning, and it must be imposed to avoid at quantum level the configuration with double occupancy at each site.

Therefore, we must emphasize that by means of our approach the completeness condition appears as necessary by consistency.

The next step is to determine the Lagrangian coefficients functions $a_{\alpha\beta}(X)$ appearing in the equation (3.2). This is straightforward by using $M_{AB}(M_{BC})^{-1} = \delta^C_A$, and solving the system of partial differential equations on such coefficients. The algebraic manipulations are rather similar to that given in the pure bosonic case (see Ref.[1]) but more complicated. So, we only write here the final results.

After some algebra, it can be shown that a family of solutions of the partial differential equations system can be written as follows
a) For the bosonic coefficients:

\[ a_{+-} = i F(u, v, \rho) X^{-+}, \quad (3.6a) \]
\[ a_{-+} = a_{+-}^* = -i F(u, v, \rho) X^{+-}, \quad (3.6b) \]

and $u, v$ are respectively arbitrary functions of the $u$ and $v$ variables. For simplicity these two last coefficients can be also taken equal to zero.

b) For the fermionic coefficients:

\[ a_{-0} = \frac{i}{2} X^{0-}, \quad (3.6c) \]
\[ a_{0-} = \frac{i}{2} X^{-0}, \quad (3.6d) \]
\[ a_{+0} = \frac{i}{2} X^{0+}, \quad (3.6e) \]
\[ a_{0+} = \frac{i}{2} X^{+0}. \quad (3.6f) \]
The real function $F(u, v, \rho)$ appearing in Eqs. (3.6a,b) verifies the following system of partial differential equations

\[ 4X^{+ -}X^{- +} \frac{\partial F}{\partial u} - 2F = 1 + \rho, \]  

(3.7a)

\[ 2X^{+ -}X^{- +} \left( \frac{\partial F}{\partial X^{- -}} - \frac{\partial F}{\partial \rho} \right) + 2X^{+ +}F + X^{- +} \right] X^{0 +} = 0, \]  

(3.7b)

\[ 2X^{+ -}X^{- +} \left( \frac{\partial F}{\partial X^{+ +}} - \frac{\partial F}{\partial \rho} \right) + 2X^{- -}F - X^{- -} \right] X^{0 +} = 0, \]  

(3.7c)

\[ 2F \rho + X^{+ -}X^{- +} \rho \left[ \frac{1}{X^{+ +}} \left( \frac{\partial F}{\partial X^{- -}} - \frac{\partial F}{\partial \rho} \right) + \frac{1}{X^{- -}} \left( \frac{\partial F}{\partial X^{+ +}} - \frac{\partial F}{\partial \rho} \right) \right] = 0. \]  

(3.7d)

A family of solutions of the system (3.7) which has physical interest is given by

\[ F(u, v, \rho) = (1 + \rho)u + \alpha \frac{(2 - v)}{(2 - v)^2 - 4\rho - u^2}, \]  

(3.8)

being $\alpha$ an arbitrary and non trivial integration constant.

In the bosonic limit when $\rho = 0$ and $v = 1$ it results

\[ F(u) = \frac{u + \alpha}{1 - u^2}. \]  

(3.9)

recovering the solution for the pure bosonic case (see Refs.[1,16]).

For simplicity, we choose a particular family of solutions by taking $a_u = a_v = 0$ and $\alpha = -1$, so the Lagrangian (3.2) without accounting the constraints can be written

\[ L(X, \dot{X}) = i \sum_i \frac{(1 + \rho_i)u_i - 1}{(2 - v_i)^2 - 4\rho_i - u_i^2} \left( X_i^{+ -} \dot{X}_i^{+ -} - X_i^{+ +} \dot{X}_i^{+ +} \right) \]
\[ + \frac{i}{2} \sum_{i, \sigma} \left( X_i^{0 \sigma} \dot{X}_i^{0 \sigma} + X_i^{0 \sigma} \dot{X}_i^{0 \sigma} \right) - H_{t-j}(X), \]  

(3.10)

where $\sigma$ takes the values + and −, and the site index $i$ was added.

In Eq. (3.10) the well known Hamiltonian $H_{t-j}$ for the t-J model in terms of the Hubbard operators is given by

\[ H_{t-j} = \sum_{i,j,\sigma} t_{ij} X_i^{\sigma \sigma} X_j^{\sigma \sigma} + \frac{1}{4} \sum_{i,j,\sigma,\bar{\sigma}} J_{ij} X_i^{\sigma \sigma} X_j^{\bar{\sigma} \bar{\sigma}} - \frac{1}{4} \sum_{i,j,\sigma,\bar{\sigma}} J_{ij} X_i^{\sigma \sigma} X_j^{\bar{\sigma} \bar{\sigma}}. \]  

(3.11)

At this stage we are ready to carry out the quantization of the model by using functional techniques.
IV. PATH INTEGRAL REPRESENTATION IN TERMS OF A REAL VECTOR FIELD.

Now, we go on writing the correlation generating functional for the t-J model by using again the path-integral Faddeev-Senjanovich approach. In terms of the Hubbard $X$-operators the partition function can be formally written as follows

$$Z = \int \mathcal{D}X_i \delta(\Omega_{i1}) \delta(\Omega_{i2}) \delta(\Xi_{i3}) \delta(\Xi_{i4}) \ sdetM_{AB} \ \exp i\int dt \ L(X, \dot{X}),$$

(4.1)

where $L(X, \dot{X})$ is given in Eq. (3.10) and the constraints $\Omega_1, \Omega_2, \Xi_3$ and $\Xi_4$ are given in Eqs. (3.3a,b), (3.5c) and (3.5d) respectively.

We note that the function $sdetM_{AB}$ appearing in equation (4.1), is the superdeterminant of the symplectic supermatrix (2.8). Really, in the path-integral formalism of Faddeev-Senjanovich such superdeterminant correspond to the supermatrix constructed from the second-class constraints provided by the Dirac formalism. As it can be shown both, the symplectic supermatrix and the supermatrix constructed from the second-class constraints are equals [6].

Such superdeterminant is computed, and after some algebra we get

$$sdetM_{AB} = detA \left[det(D - CA^{-1}B)\right]^{-1} = \frac{4(1+\rho)^2}{(1-\rho+u)^2},$$

(4.2)

where $A, B, C$ and $D$ are the submatrices defined in (2.8).

At this stage, it is useful to relate the boson-like $X$-Hubbard operators with the real components $S_i$ ($i = 1, 2, 3$) of a vector field $S$, by means of a transformation. By using explicitly the constraint (3.3a), the independent bosonic degrees of freedom are reduced to three, and we can write

$$X^{++} = \frac{1}{2s}(1 - \rho)(s + S_3),$$

(4.3a)

$$X^{--} = \frac{1}{2s}(1 - \rho)(s - S_3),$$

(4.3b)

$$X^{+-} = \frac{1}{2s}(1 - \rho)(S_1 + iS_2),$$

(4.3c)
where $s$ is a constant.

Note that only when $\rho = 0$ (pure bosonic case), the real vector field $\mathbf{S}$ can be identified with the spin.

Moreover, the fermion-like $X$-Hubbard operators can be related with Grassmann variables i.e., suitable component spinors

$$X^{-0} = \Psi_+ \quad X^{0-} = \Psi_+^*,$$  

$$X^{+0} = \Psi_- \quad X^{0+} = \Psi_-^*,$$

where now $\rho = \Psi_+^*\Psi_+ + \Psi_-^*\Psi_-$ and accounting the fermionic constraints (3.5) it results $(1 - \rho)(1 + \rho) = 1$.

The remaining bosonic constraint (3.3b) as function of the real vector field variable $\mathbf{S}$ writes

$$\Omega_2 = S_1^2 + S_2^2 + S_3^2 - s^2 = 0.$$  

Analogously, the two fermionic constraints (3.5c) and (3.5d) can be written

$$\Xi_3 = \Psi_- (S_1 + iS_2) - \Psi_+^* (s + S_3) = 0,$$  

$$\Xi_4 = \Psi_- (S_1 - iS_2) - \Psi_+ (s + S_3) = 0.$$

Consequently, by neglecting a total time derivative the Lagrangian (3.10) in terms of these new fields reads

$$L = -\frac{1}{2s} \sum_i S_{i2} \frac{\dot{S}_{i1} - S_{i1} \dot{S}_{i2}}{s + S_{i3}} + i \sum_{i,\sigma} \Psi_{i,\sigma} \dot{\Psi}_{i,\sigma}^* - H_{t-J}.$$  

The Hamiltonian $H_{t-J}$ given in (3.11) written in term of the real vector variable $\mathbf{S}$ and the component spinors (4.4), takes the form

$$H_{t-J} = \sum_{i,j,\sigma} t_{ij} \psi_{i,\sigma} \psi_{j,\sigma}^* + \frac{1}{8s^2} \sum_{i,j} J_{ij} (1 - \rho_i)(1 - \rho_j) \left[ S_{i1} S_{j1} + S_{i2} S_{j2} + S_{i3} S_{j3} - s^2 \right].$$  

(4.8)
Finally, we note that the fermionic constraints (3.5) can be written in matrix notation as follows

$$\Xi = (sI + S.\sigma) \Psi = 0, \quad (4.9a)$$

$$\Xi^* = \Psi^* (sI + S.\sigma) = 0, \quad (4.9b)$$

where $\sigma$ are the Pauli matrices, and the two-component spinor $\Psi = (-\Psi_+, \Psi_-)$ is defined. The fermionic constraints (4.9) are precisely those used in Ref. [15] in the framework of the coherent state representation.

The Eq. (4.1) can be written in an alternative way by using the integral representation for the delta functions on the constraints $\Phi$

$$\delta(\Phi) = \int D\chi \exp \left( i \int dt \chi \Phi \right),$$

where the quantities $\chi$ are suitable bosonic or fermionic Lagrange multipliers.

Consequently, the correlation generating functional (4.1) takes the form

$$Z = \int DX_i D\lambda_1^i D\lambda_2 D\xi_i D\xi_i^* \ sdet M_{AB} \ exp \ i \int dt \ L_{eff}(X, \dot{X}). \quad (4.10)$$

The effective Lagrangian $L_{eff}(X, \dot{X})$ appearing in Eq. (4.10) is defined by

$$L_{eff}(X, \dot{X}) = L(X, \dot{X}) + \sum_i \lambda_1^i \left( X_i^{++} + X_i^{--} + \rho_i - 1 \right)$$

$$+ \sum_i \lambda_2^i \left[ X_i^{+-} X_i^{-+} + \frac{1}{4} u_i^2 - (1 - \frac{1}{2} v_i)^2 + \rho_i \right]$$

$$+ \left( X_0^{+} X_i^{+-} - X_0^{-} X_i^{++} \right) \xi_i + \xi_i^* \left( X_i^{+0} X_i^{--} - X_i^{-0} X_i^{++} \right). \quad (4.11)$$

where $L(X, \dot{X})$ was given in Eq. (3.10).

In Eq. (4.11) the parameters $\lambda^a$ ($a = 1, 2$) and $\xi$ are respectively bosonic and fermionic Lagrange multipliers.

The functional $sdet M_{AB}$ written in terms of the real vector field $S$ and the two-component spinor $\Psi$ results

$$sdet M_{AB} = -\frac{4s^2(1 + \rho)^2}{(1 - \rho)^2(s + S_3)^2} \quad (4.12)$$
The exponentiation of the superdeterminant is realized as usual by introducing the Faddeev-Popov ghosts in the effective Lagrangian. Therefore, we assume that the functional $sdetM_{AB}$ is written as follows

$$sdetM = \int \mathcal{D}C \exp iC^A_1 M_{AB} C^B_2,$$  \hspace{1cm} (4.13)

where $\mathcal{D}C = \prod A \mathcal{D}C_A^A \mathcal{D}C_A^1$ and $C_a^A (a = 1, 2)$ denote commuting as well as anticommuting ghosts.

It is useful to analyze the partition function (4.10) when it is written in terms of the real vector field variable $S$.

Making use of the transformation (4.3) and (4.4), Eq. (4.10) takes the form

$$Z = \int DS_1 DS_2 DS_3 D\Psi_i \ D\Psi^*_i \ D\chi \ D\chi^* \ sdetM_{AB} \frac{\partial X}{\partial S} \ exp \ (i \int dt \ L_{eff}) \ . \hspace{1cm} (4.14)$$

where the quantity $\frac{\partial X}{\partial S}$ is the super Jacobian of the transformation (4.3 - 4.4), also a field dependent functional whose value is

$$\frac{\partial X}{\partial S} = -i (1 - \rho)^3 \ . \hspace{1cm} (4.15)$$

The effective Lagrangian $L_{eff}$ given in Eq. (4.14), in terms of the new variables reads

$$L_{eff} = \frac{1}{2s} \sum_i \left[ S_{i1} \dot{S}_{i2} - S_{i2} \dot{S}_{i1} \right]_{s + S_{i3}} + i \sum_{i, \sigma} \Psi_{i\sigma} \dot{\Psi}^*_{i\sigma} - H_{i-J}$$

$$+ \sum_i \left[ \chi_i^2 (S_{i1}^2 + S_{i2}^2 + S_{i3}^2 - s^2) + \xi_i^* (\Psi_-(S_{i1} - iS_{i2}) - \Psi_+(s + S_{i3})) \right.$$  

$$\left. + \left( \Psi_-^*(S_{i1} + iS_{i2}) - \Psi_+^*(s + S_{i3}) \right) \xi_i \right] \ . \hspace{1cm} (4.16)$$

As a last comment we must say that the treatment of the path-integral (4.14) is cumbersome.

By one hand, the effective Lagrangian (4.16) depends on the hole density $\rho$. When there is a small number of holes it can be assumed that the hole density $\rho = constant$. In this situation the super Jacobian of the transformation (4.3 - 4.4) is constant and it contributes only to the normalization factor of the path-integral (4.14).

On the other hand, the non-polynomial structure of the effective Lagrangian (4.16) is due to the contribution of both, the bosonic kinetic part and the terms coming from the
functional $sdetM$. This problem is also present in the pure bosonic case (see for instance Refs. [1,16]), and is solved in the framework of the perturbative formalism. The non-polynomial character of $L_{\text{eff}}$ is due to the presence of the component $S_3$ of the real vector field $\mathbf{S}$ in the denominator. So, this problem can be treated by considering the effective Lagrangian fluctuating around the antiferromagnetic background.

In these conditions, and at least when only first-order terms of the perturbative development are retained the $sdetM$ is constant, and so it is possible to obtain results without introducing ghosts.

In a forthcoming paper under preparation, by following these prescriptions the non-polynomial effective Lagrangian is studied. On the basis of our path-integral formulation and by applying the perturbative formalism the Feynmann rules and the diagrammatics of the $t$-$J$ model will be given.

**V. CONCLUSIONS**

In this paper a discussion about the construction of a family of first-order Lagrangian describing the dynamics of the $t$-$J$ model is presented. In this approach any decoupling is used, but the field variables are directly the Hubbard $\mathbf{X}$-operators satisfying the graded algebra $\text{spl}(2,1)$. Using the supersymmetric version of the Faddeev-Jackiw symplectic formalism, we have shown that it is possible to find a family of first-order Lagrangian able to reproduce at classical level the generalized FJ brackets or graded Dirac brackets of the $t$-$J$ model. When the transition to the quantum theory is realized as usual in a canonical quantum formalism, the graded quantum Dirac brackets are precisely the graded commutators of the Hubbard $\mathbf{X}$-operators algebra. Moreover, in both cases the spinless fermion model as well as the $t$-$J$ model, the unique possible set of constraints is naturally provided by the symplectic FJ method.

From our approach applied to the simple case of the graded algebra $\text{spl}(1,1)$ (i.e, the spinless fermion model), we have shown that the partition function can be written in terms of the same effective Lagrangian obtained by means of other methods.
Also the t-J model is treated in the framework of the path-integral representation by using the Hubbard $X$-operators as field variables. In this context the correlation generating functional is constructed.

Later on, by making a transformation from the boson-like Hubbard $X$-operators to a real boson vector field $\mathbf{S}$, we have rewritten the effective Lagrangian appearing in the partition function. The real vector field $\mathbf{S}$ has the particularity that in the bosonic limit i.e., when the fermion-like operators are withdrawn, it is not other than the spin vector field. As it is also shown, in the bosonic limit the remaining bosonic part of our non-polynomial Lagrangian is equal to that given in Refs. [1,16], checking in this way the expression of the partition function for the pure bosonic case.

In summary, we can conclude that starting from a total independent scenario, the path-integral representation we found is equivalent to that obtained by means of the coherent states method.

We think that it would be interesting to compare the present formulation with those that could be obtained by using the coherent state quantization of constrained systems formalism recently developed in Ref. [17].

Finally, it must be noted (see Ref. [1]) that among the solutions we have found, also the solution given in Ref. [15] for the bosonic kinetic part is obtained. This is done by defining an appropriate vector $\mathbf{a}$ which verifies the equation $(\nabla \times \mathbf{a})\mathbf{S} = 1 + \rho$. However we believe that the bosonic solution we choose to write the Lagrangian (4.7) is more convenient for our future purposes.

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