Note on Morita Inequality for Planar Noncommutative Inverted Oscillator

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A recent conjecture of Morita predicts a lower bound in temperature $T$ of a chaotic system, $T \geq (\hbar/2\pi)\Lambda$, $\Lambda$ being the Lyapunov exponent, which was demonstrated for a one dimensional inverse harmonic oscillator. In the present work we discuss the robustness of this demonstration in an extended version of the above model, where the inverse harmonic oscillator lives in a two dimensional noncommutative space. We show that, without noncommutativity, Morita’s conjecture survives in an essentially unchanged way in two dimensions. However, if noncommutativity is switched on, the noncommutativity induced correction terms conspire to produce, in classical framework, a purely oscillating non-chaotic system without any exponential growth so that Lyapunov exponent is not defined. On the other hand, following Morita’s analysis, we show that quantum mechanically an effective temperature with noncommutative corrections is generated. Thus Morita’s conjecture is not applicable in the noncommutative plane. A dimensionless parameter $\sigma = m\alpha\theta^2$, (where $m, \alpha, \theta$ are the particle mass, coupling strength with inverse oscillator and the noncommutative parameter respectively) plays a crucial role in our analysis.

I. INTRODUCTION

In a seminal paper Maldacena, Shenker, and Stanford \cite{1} argued that a thermodynamic quantum system possesses a sharp upper bound on the rate of growth of chaos. In particular the authors of \cite{1} conjectured that $\Lambda \leq 2\pi T/\hbar$, $\Lambda$ being the Lyapunov exponent and $T$ the temperature. In a recent interesting work Morita \cite{2} turned the inequality on its head and came up with an alternative lower bound on temperature:

$$T \geq \frac{\hbar}{2\pi} \Lambda, \quad (1)$$

with a far reaching consequence that in semi-classical regime a chaotic system (with non-zero Lyapunov exponent) has to have a non-vanishing temperature. An important achievement of \cite{2} was to demonstrate viability of the above bound in a simple toy model, a single particle in an inverted oscillator potential. Although, strictly speaking \cite{3} the classical model is not chaotic but nevertheless it has a (formally defined) non-zero Lyapunov exponent. In the quantum mechanical framework Morita \cite{2} showed that the tunneling phenomena induces an effective non-zero temperature resulting in a net (analogue) Hawking flux. Interestingly enough, the effective temperature generated in the quantum framework saturates the bound (1). This demonstration was possible primarily because the classical Lyapunov exponent is exactly and analytically computable for the simple system considered in \cite{2, 3}.

The question that naturally comes to mind is how robust is this demonstration of validity of the inequality (1) that is whether it will survive in a more complicated (but still toy!) model. Towards this end we consider a (spatially) Non-Commutative (NC) extension of this model which also brings in another extension in dimension: a two dimensional non-relativistic system is essential to have a non-trivial non-commutativity. Subsequently we rederive results analogous to that of Morita \cite{2, 3}. In recent times, extending models, both non-relativistic as well as relativistic, to NC space and spacetime, has paid rich dividends (for reviews see for example \cite{4}) since it introduces a dimensional scale via the NC parameter $\theta_{ij}$. Different structures of NC generalization brings in new interactions in the resulting theories and has been effective in ushering in possible quantum gravity corrections in a phenomenological perspective.

The paper is organized as follows. In Section II the NC inverted oscillator in two dimensional space is introduced, along with the Darboux-type of canonical phase space that we subsequently exploit. In Section III we discuss the above model in quantum mechanical framework where NC corrections in the effective temperature

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are computed. Section IV is devoted to a classical analysis of the NC inverted oscillator. In Section V we study
commutative version of the above model. Section VI contains a summary of the work presented together with
future directions. In Appendices 1 and 2 some detailed results are appended.

II. INVERTED OSCILLATOR IN NONCOMMUTATIVE PLANE

The planar Hamiltonian of the particle in inverted oscillator potential is given by
\[ H = \frac{m}{2} \dot{x}_i \dot{x}_i - \frac{\alpha}{2} x_i x_i , \quad i = 1, 2 \]  
(2)
along with the NC-extended classical Poisson bracket structure,
\[ \{ x_i, p_j \} = \delta_{ij}, \quad \{ p_i, p_j \} = 0, \quad \{ x_i, x_j \} = \theta \epsilon_{ij}; \quad \epsilon_{12} = 1. \]  
(3)
It is straightforward to obtain the NC equation of motion (in terms of non-canonical variables)
\[ \dot{x}_i = \{ x_i, H \} = \frac{1}{m} p_i - \alpha \theta \epsilon_{ij} x_j, \quad \dot{p}_i = \{ p_i, H \} = \alpha x_i \]  
(4)
the last equation \[ \{ \] being a modified version of Newton’s law. In a classical scenario non-canonical (instead of
non-commutative) degrees of freedom would be a better nomenclature but we stick to the conventional one. We
use \textit{time}, \textit{mass}, \textit{distance} as the fundamental dimensional units, in terms which \[ [x_i] = \text{length} \] and the constant
parameters have \[ [\alpha] = (\text{mass})/(\text{time})^2, \quad [\theta] = (\text{time})/(\text{mass}). \] It is conventional and more convenient to use the
canonical variables \[ X_i, P_i \]
obeying \[ \{ X_i, X_j \} = \{ P_i, P_j \} = 0; \quad \{ X_i, P_j \} = \delta_{ij} \] such that \[ x_i, p_i \]
\begin{align*}
can be mapped to \[ X_i, P_i \] (\sim \text{Darboux map}):
\begin{align*}
x_1 &= X_1 - \frac{\theta}{2} P_2, \quad x_2 = X_2 + \frac{\theta}{2} P_2, \quad p_1 = P_1, \quad p_2 = P_2.
\end{align*}
(5)
The Hamiltonian is rewritten as
\[ H = \bar{a} P_i P_i - \frac{\alpha}{2} X_i X_i + \frac{\theta \alpha}{2} \epsilon_{ij} X_i P_j; \quad \bar{a} = \frac{1}{m} - \frac{\theta^2 \alpha}{4}, \]  
(6)
where the NC modification has induced a constant magnetic field like effect \[ [5]. \]

III. NC EFFECT ON QUANTUM DYNAMICS

In the quantum version we use the polar coordinates \[ X_1 = \rho \cos \phi, X_2 = \rho \sin \phi \] in order to exploit the rotational
symmetry of the problem where \[ \phi \] becomes cyclic and the problem reduces to a one dimensional one with the
Hamiltonian
\[ H = \{- \frac{\hbar^2 \bar{a}}{2} (\partial^2_\rho + \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} \partial^2_\phi) - \frac{\alpha}{2} \rho^2 - \frac{i \hbar \theta \alpha \partial_\phi}{2} \} \]  
(7)
to be used in time independent Schrödinger equation \[ H \psi = E \psi. \] For \[ \psi \] of the form \[ \psi(\rho, \phi) = F(\rho) \text{exp}(i L \phi) \] the equation for \[ F(\rho) \]
\[ \partial_\rho^2 F + \frac{1}{\rho} \partial_\rho F - \frac{L^2}{\rho^2} F + \frac{\alpha}{h^2 \bar{a}} \rho^2 F + \frac{2}{h^2 \bar{a}} \left( \frac{\hbar \theta \alpha L}{2} + E \right) F = 0 \]  
(8)
re-expressed in terms of dimensionless variables \[ z_{NC}, E_{NC} \]
\[ z_{NC} = \left( \frac{4 \alpha}{h^2 \bar{a}} \right)^{1/4} \rho, \quad E_{NC} = \sqrt{\frac{1}{h^2 \bar{a}} \left( E + \frac{\hbar \theta \alpha L}{2} \right)}, \]  
(9)
FIG. 1: Real part of solution of (10) (hypergeometric function) for $\sigma = 0$, green curve; $\sigma = 4$, brown curve and $\sigma = 12$, red curve and parabolic cylinder function of $[2, 3]$, blue curve, are plotted for $L = 0$ in each case.

FIG. 2: Imaginary part of solution of (10) (hypergeometric function) for $\sigma = 0$, green curve; $\sigma = 4$, brown curve and $\sigma = 12$, red curve and parabolic cylinder function of $[2, 3]$, blue curve, are plotted for $L = 0$ in each case.

turns out to be,

$$\partial_{z_{\text{NC}}}^2 F + \frac{1}{z_{\text{NC}}} \partial_{z_{\text{NC}}} F - \frac{l^2}{z_{\text{NC}}^2} F + \left(\frac{z_{\text{NC}}^2}{4} - E_{\text{NC}}\right) F = 0. \quad (10)$$

Comparing with the analogous equation A3 of $[3]$ we find that there is a qualitative change due to additional $1/z_{\text{NC}}$-term and the $1/z_{\text{NC}}^2$ centrifugal term apart from NC-corrections of the constant parameters.

We are interested in the large $z_{\text{NC}}$ behavior $[2, 3]$ in which case the simplest and crudest option is to drop these new terms from (10) thus yielding

$$\partial_{z_{\text{NC}}}^2 F + \left(\frac{z_{\text{NC}}^2}{4} - E_{\text{NC}}\right) F = 0. \quad (11)$$

Now (11) is structurally identical to the equation considered in $[3]$ and all we need to do is to replace $E$ by $E_{\text{NC}}$. 
containing the NC and non-zero $L$ corrections. Hence for large $z_{NC}$

$$F(z_{NC}) \approx C(1)\Gamma_1(L)e^{i\frac{\sqrt{2}z_{NC}}{4} - E_{NC} \log z_{NC}}$$

$$+C(2)\Gamma_2(L)e^{-i\frac{\sqrt{2}z_{NC}}{4} - E_{NC} \log z_{NC}}$$  \hspace{1cm} (12)$$

where $\Gamma_i(L)$ are $L$-dependent numerical constants and $z_{NC}, E_{NC}$ are explicitly given by

$$z_{NC} = \left(\frac{4\alpha}{\hbar^2 a}\right)^{1/4}\left(1 - \frac{\sigma}{4}\right)^{-\frac{1}{4}} \approx z_M(1 + \frac{\sigma}{16}) + O(\sigma^2),$$

$$E_{NC} \approx -E_M(1 + \frac{\sigma}{8}) - \sqrt{\sigma L} + O(\sigma^{3/2}).$$  \hspace{1cm} (13)$$

In the above the subscript $M$ refers to Morita’s parameterization [2, 3] and finally we have kept up to $O(\theta^2)$ terms. Once again, following [3] we easily obtain the tunneling probability

$$P_T(E) = \frac{1}{e - \frac{2\pi\hbar}{\alpha^2} \sqrt{\frac{\alpha}{m} (1 - \frac{\sigma^2}{8})} e^{\pi L\sqrt{\sigma}} + 1} = \frac{1}{e - \frac{2\pi\hbar}{\alpha^2} \sqrt{\frac{\alpha}{m} (1 - \frac{\sigma^2}{8})} e^{\pi L\sqrt{\sigma}} + 1}$$  \hspace{1cm} (14)$$

which for $L = 0$ provides the temperature

$$T_{NC} = \frac{\hbar}{2\pi} \sqrt{\frac{\alpha}{m} (1 - \frac{\sigma^2}{8})^{-1}} \approx T_M(1 + \frac{\sigma}{8}).$$  \hspace{1cm} (15)$$

Thus an NC correction appears through $\sigma$ in the effective temperature. The effect of non-zero $L$ can be read off from (15).

In Appendix 1, in (a7), we have provided full solution of (10) which comprises of a combination of two linearly independent solutions given by hypergeometric and Laguerre functions. The solutions of the one dimensional commutative model [3] appear as parabolic cylinder functions. We have picked the Hypergeometric function in our case since in the commutative limit it is closer to the parabolic cylinder solutions. In Figure 1 and Figure 2 we have plotted respectively real and imaginary parts of the Hypergeometric function for $L = 0$ and different values of $\sigma = 0, 4, 12$. We have also included profiles of the real and imaginary parts of the parabolic cylinder function for the sake of comparison.

In Figure 3 the profiles of real parts of the solution of (10) for fixed $\sigma = 2$ and different values of $L = 0, \pm 1$ are drawn. A common feature in all the NC and $L \neq 0$ graphs is that the peak at $z = 0$ is much higher than the one dimensional commutative model of [2, 3] but the solutions die out for large $z$ in a similar fashion, which is not surprising.
Using (6) the equations of motion

\[ \dot{X}_i = \{ X_i, H \} = \left( 1 - \frac{\alpha \theta^2}{4} \right) P_i - \frac{\alpha \theta}{2} \epsilon_{ij} X_j, \quad \dot{P}_i = \{ P_i, H \} = \alpha X_i - \frac{\alpha \theta}{2} \epsilon_{ij} P_j \]  

(16)
can be decoupled for \( X_i \) to yield a fourth order equation of motion,

\[ X_i^{(4)} + (B^2 - 2A) X_i^{(2)} + A^2 X_i = 0 \]  

(17)

where \( X^{(n)} = (d^n X)/(dt^n) \) and \( A = \ddot{\alpha} + \frac{\theta^2}{4} = 1/m, \quad B = -\theta \alpha^2 \). Incidentally this is reminiscent of the Pais-Uhlenbeck oscillator dynamics \([6, 7]\) although we will not go into that topic in the present work.

The four linearly independent solutions are

\[ X_i = C^{(\mu)} \exp(\pm \sqrt{A - \frac{B^2}{2}} \pm \sqrt{B^4 - 4AB^2}), \]  

(18)

where each independent solution has integration constant \( C^{(\mu)}, \mu = 1, 2, 3, 4 \) and the \( \pm \) signatures operate independently. Defining a dimensionless parameter \( \sigma = m \alpha \theta^2 \) the exponents of the solutions appear as

\[ \omega_{NC}^{(\mu)} = \pm \sqrt{\frac{\alpha}{m} \left( 1 - \frac{\sigma}{2} \pm \frac{1}{2} \sqrt{\sigma^2 - 4\sigma} \right)} = \pm \omega_M \sqrt{(1 - \frac{\sigma}{2} \pm \frac{1}{2} \sqrt{\sigma^2 - 4\sigma})}. \]  

(19)

In \([10]\) subscripts \( NC \) and \( M \) stand for noncommutative and Morita respectively. For \( \sigma = 0 \) the result of Morita \([2]\) is recovered with Lyapunov exponent \( \omega_M(\sigma = 0) = \sqrt{\alpha/m} \). Note that although \( \theta \) is expected to be small \( \sigma \) can be large with proper tuning of \( \alpha, m \).

The general solutions can be expressed as,

\[ X_1 = A_1 e^{\omega_1 t} + A_2 e^{-(\omega_1 t)} + A_3 e^{\omega_3 t} + A_4 e^{-(\omega_3 t)} \]
\[ X_2 = B_1 e^{\omega_1 t} + B_2 e^{-(\omega_1 t)} + B_3 e^{\omega_3 t} + B_4 e^{-(\omega_3 t)} \]  

(20)

where

\[ \omega_{1(2)} = +(-) \omega_M \sqrt{(1 - \frac{\sigma}{2} \pm \frac{1}{2} \sqrt{\sigma^2 - 4\sigma})}; \quad \omega_{3(4)} = +(-) \omega_M \sqrt{(1 - \frac{\sigma}{2} \pm \frac{1}{2} \sqrt{\sigma^2 - 4\sigma})} \]

are the four possible values of \( \omega_{NC} \) in \([19]\). Substituting the solutions in \((16)\) the integration constants \( A_i, B_i \) are restricted to the following relations:

\[ \frac{A_1}{B_1} = -\frac{A_2}{B_2} = \frac{\omega^2 - \alpha/m}{\alpha \theta \omega_1} \equiv \Lambda_1; \quad \frac{A_3}{B_3} = -\frac{A_4}{B_4} = \frac{\omega^2 - \alpha/m}{\alpha \theta \omega_3} \equiv \Lambda_2. \]  

(21)

Hence, the solutions appear as

\[ X_1 = \Lambda_1 A_1 e^{\omega_1 t} - B e^{-\omega_1 t} + \Lambda_2 C_2 e^{\omega_3 t} - D e^{-\omega_3 t}, \]
\[ X_2 = A(e^{\omega_1 t} + B e^{-\omega_1 t}) + C(e^{\omega_3 t} + D e^{-\omega_3 t}). \]  

(22)

Using \(5\) in \(6\) the Hamiltonian is written completely in terms of \( X_i, \dot{X}_i \)

\[ H = \frac{1}{2\alpha} (\dot{X}_1^2 + \dot{X}_2^2) - \frac{\alpha}{2} X^2 - \frac{\alpha \theta}{4a} (X_1 \dot{X}_2 - X_2 \dot{X}_1). \]  

(23)

Finally using the above explicit forms of \( X_i \) given in \((22)\) in the above equation, we find expression for the energy,

\[ H = A^2 B [\alpha^2 - 1] (\omega_1^2 a + \alpha) + \frac{\alpha \theta}{a} \Lambda_1 \omega_1] + C^2 D [\alpha^2 - 1] (\omega_3^2 a + \alpha) + \frac{\alpha \theta}{a} \Lambda_2 \omega_3] \]
\[ + 2\alpha AC [e^{(\omega_1 + \omega_3) t} + B De^{-(\omega_1 + \omega_3) t}]. \]  

(24)
FIG. 4: Plot of $\omega^2$ against $\sigma$ for the two roots, $\omega_1^2$, blue line, and $\omega_3^2$, red line, showing that $\omega^2$ is always negative for the allowed range of $\sigma$.

Notice that because of the mixing of $\omega_1$ and $\omega_3$ in $X_i$ in (22) there appears an unwanted time dependence in $H$ from the last two terms of (24). To avoid that we choose $X_i$ to be of the form,

$$X_1 = \Lambda_1 A(e^{\omega_1 t} - Be^{-\omega_1 t}), \quad X_2 = A(e^{\omega_1 t} + Be^{-\omega_1 t})$$ (25)

with energy

$$H = -A^2 B[\frac{1}{a}(\omega_1^2 + \frac{\alpha}{m}) + 2\alpha] = -2\alpha A^2 B[3 + \frac{2\sqrt{\sigma^2 - 4\sigma}}{(4 - \sigma)}].$$ (26)

or

$$X_1 = \Lambda_2 C(e^{\omega_3 t} - De^{-\omega_3 t}), \quad X_2 = C(e^{\omega_3 t} + De^{-\omega_3 t})$$ (27)

with energy

$$H = -C^2 D[\frac{1}{a}(\omega_3^2 + \frac{\alpha}{m}) + 2\alpha] = -2\alpha C^2 D[3 - \frac{2\sqrt{\sigma^2 - 4\sigma}}{(4 - \sigma)}].$$ (28)

Furthermore, requiring $H$ to be real one has to impose the condition $\sigma^2 - 4\sigma \geq 0$ so that $\sigma = 0$ (commutative case) or $\sigma \geq 4$. Now comes the surprising result. For $\sigma \geq 4$ both $(\omega_1)^2 = (\Lambda_1^1)^2$ and $(\omega_3)^2 = (\Lambda_3^3)^2$ turn out to be always negative where $\Lambda_{NC}$ notation is used to identify them with NC corrected Lyapunov exponents.

This is shown in Figure 4 where we have plotted $\omega^2$ against $\sigma$. The two colored lines correspond to the two roots, $\omega_1^2$ for blue line and $\omega_3^2$ for red line. This in turn means that the $\Lambda_{NC}^1$ and $\Lambda_{NC}^3$ are purely imaginary thereby restricting the solutions to be purely oscillatory without any exponentially divergent chaotic behavior.

Since there are no Lyapunov exponents it is not possible to verify Morita’s conjecture. Thus, (even though in (15) $\sigma$-correction is qualitatively similar to (19) actually the latter is purely imaginary), we conclude that the planar noncommutative inverted oscillator model is not appropriate to discuss the Morita conjecture [2, 3].

V. PLANAR COMMUTATIVE INVERTED OSCILLATOR

In the classical planar commutative case with $\theta = 0 \rightarrow \sigma = 0$ (considered by us) the frequencies are same as the case considered by Morita, i.e. $\omega_{NC} = \omega_M$ with same Lyapunov exponent. On the other hand, as we have discussed earlier in Section 3, for large $z$ the equation (11) for $\sigma = 0$ reduces to the one considered in [2, 3], and hence the Morita conjecture will be verified in the same way. In Appendix 1 The solutions (computed with Mathematica package) of both (10) and (11), (the latter for the commutative case $\sigma = 0$), are given. Note
that, as observed in [2, 3], for large $\rho \sim z_{NC}$, both the solutions oscillate as $G \exp(i\chi)$ with identical form of $\chi \approx z_{NC}^2/4 - E_{NC} \log z_{NC}$ but differing in $G$ which varies as $1/\sqrt{z_{NC}}$ and $1/z_{NC}$ in (11) and (10) respectively. Since the large distance behavior is responsible for the transition and reflection rates we can safely assume that we will reach similar conclusion as [14, 15].

It is worthwhile to mention one point in our classical analysis that is discussed in detail in Appendix 2. Recall that we have used polar coordinates in quantum analysis and Cartesian coordinates in the classical analysis and so it seems natural to relate the quantum and classical results for $L = 0$ case only. It would have been more convenient to study the classical case in polar coordinates as well but recasting the noncommutative phase space algebra in polar form (and the subsequent Darboux map) is quite difficult and we have not been able to achieve a satisfactory solution (see Appendix 2 for details). This has forced us to do the classical analysis in cartesian coordinates.

VI. DISCUSSION

The present work is the first attempt to generalize the recent important conjecture of Morita [2, 3] that proposes a lower bound in the temperature of a chaotic semi-classical system. The demonstration in [2, 3] exploits the simplest possible toy model - one dimensional system consisting of a particle in inverted oscillator potential. In this case the classically computed Lyapunov exponent yields the bound (via Morita’s relation) that saturates the effective temperature derived from quantum analysis of the model. We have extended the model in two directions by considering the same model in two dimensional noncommutative space.

In the two dimensional version in canonical space we find that Morita’s conjecture is satisfied, that is, even though in the quantum analysis, two dimensional configuration space generates additional non-trivial terms in the Hamiltonian, in the large distance limit these can be neglected thus leading to no qualitative change in the results and conclusion. On the other hand, quite surprisingly (and somewhat disappointingly!) we find that the noncommutative extension does not allow the application of Morita’s inequality because, in the classical analysis, the noncommutative correction terms conspire to render the system purely oscillatory, hence non-chaotic. Although, as we have shown, the quantum analysis does yield an effective temperature with noncommutative corrections, there is no way to relate it to the Lyapunov exponent in its classical version simply because the latter does not exist.

Furthermore, as our analysis has revealed, setting up a classical noncommutative coordinate algebra in polar form and deriving the resulting canonical solution (by way of a local Darboux map) is quite involved and unsolved.

As future extension of the present work an immediate task is to compare the above mentioned results more rigorously, in particular by formulating the noncommutative geometry in polar coordinates.

One option is to incorporate other, more involved forms of noncommutativity (such as the algebra compatible to Generalized Uncertainty Principle or the Snyder form of noncommutativity).

Another interesting area of research would be the fluid model. In Morita’s paper [2, 3] it has been shown that in Fermi fluid, consisting of non-relativistic fermions obeying the canonical fluid equations (continuity and Euler equations with characteristic pressure term), acoustic Hawking radiation appears above a critical fluid velocity. We have constructed noncommutative extensions of ideal fluid model in a series of earlier works [11, 12] and it would indeed be worthwhile to test Morita’s conjecture in such system.

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Appendix 1: The solution for (10) using Mathematica is given by

$$F(z_{NC}) = 2^{(1+L)/2} e^{-iz_{NC}^2/4} (z_{NC})^L \left\{ C(1) \frac{1}{2} \left[ F[(1 + L - iE_{NC})^{1/2} \frac{1}{2} \left( 1 + L, \frac{i z_{NC}^2}{2} \right) ] \right. \right. + \left. \left. C(2) L G \left[ (-1 - L + iE_{NC}) \frac{1}{2} \left( 1, \frac{i z_{NC}^2}{2} \right) \right] \right\} ,$$

(29)
where $F$ and $L_G$ are respectively the hypergeometric and Laguerre functions. For large $z$ this reduces to

$$F(z_{NC}) \approx C(1)e^{-iz^2z_{NC}/4} \frac{1}{z_{NC}^{L+iE_{NC}/2}} \left\{ \frac{(1)}{\zeta_{NC}^{L+iE_{NC}/2}} + O(1/z_{NC}^2) \right\}$$

$$+ C(2)e^{-iz^2z_{NC}/4} \frac{1}{z_{NC}^{L+iE_{NC}/2}} \left\{ \frac{(1)}{\zeta_{NC}^{L+iE_{NC}/2}} + O(1/z_{NC}^2) \right\}$$

$$+ C(2)e^{-iz^2z_{NC}/4} \frac{1}{z_{NC}^{L+iE_{NC}/2}} \left\{ \frac{(1)}{\zeta_{NC}^{L+iE_{NC}/2}} + O(1/z_{NC}^2) \right\}.$$  \hspace{1cm} (30)

On the other hand the result of Morita [3], solution of (11) with $Q$ system in polar coordinates for comparing. It is pertinent to ask why we did not use polar coordinates in the classical case. We now reveal former in Cartesian coordinates. Hence it seems natural to consider the $L$ between classical and semi-classical NC corrected results, treating the latter in polar coordinates and the $E$.

Hamiltonian formulation of constraint analysis, we start from the Lagrangian (in Cartesian coordinates)

$$\{A,B\} = \{A,\chi_{ij}\} \chi_{ij}^{-1} \{\chi_{ij},B\}.$$  \hspace{1cm} (35)

In the present case the Dirac brackets (with the subscript $D$ omitted) are given by

$$\{A,B\}_D = \{A,B\} - \{A,\chi_{ij}\} \chi_{ij}^{-1} \{\chi_{ij},B\}.$$  \hspace{1cm} (36)

However recall that instead of the above brackets we have used [3], i.e. $\{x_i,p_j\} = -\theta \epsilon_{ij}$, $\{x_i,p_j\} = \delta_{ij},$ $\{p_i,p_j\} = 0.$ We are allowed to do it because this deformation is not operatorial in nature and algebraically does not clash with the Jacobi identities among $x_i, p_i.$ (Dirac brackets satisfy Jacobi identity.)

Let us now come to polar coordinates $\rho, \phi$ defined by $x_1 = \rho \cos \phi, x_2 = \rho \sin \phi.$ Indeed, deriving the NC brackets for polar variables, in a purely algebraic way, starting from Cartesian one, is quite involved (see eg. [9]). However generating them dynamically using Dirac constraint framework is very easy. The Lagrangian (33) in polar coordinates reduces to

$$L = \frac{1}{2\theta} \rho^2 \dot{\phi}.$$  \hspace{1cm} (37)
that has a pair of Second Class constraints

\[ \psi_1 \equiv p_\rho, \ \psi_2 \equiv p_\phi - \frac{1}{2\theta} \rho^2. \]  
(38)

The Dirac brackets are given by

\[ \{\rho, \phi\} = -\frac{\theta}{\rho}, \ \{\phi, p_\phi\} = 1, \]

\[ \{\rho, p_\phi\} = 0, \ {\phi, p_\rho\} = {p_\phi, p_\rho} = {\rho, p_\rho} = 0. \]  
(39)

Notice that the NC brackets are operatorial so that one is no longer allowed to take, for instance, a structure

\[ \{\rho, \phi\} = -\frac{\theta}{\rho}, \ {\rho, p_\rho\} = {\phi, p_\phi} = 1 \]

with rest of the brackets vanishing since the Jacobi identity

\[ \{\{\phi, \rho\}, p_\rho\} + \{\{\rho, p_\rho\}, \phi\} + \{\{p_\rho, \phi\}, \rho\} = -\frac{\theta}{\rho^2} \neq 0 \]

is non-zero and the Dirac bracket structure with \{\rho, p_\rho\} = 0 is demanded. However the Darboux map that we have exploited is difficult to construct (in closed form) for this set operatorial Dirac brackets.

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