The automorphism group of the \( s \)-stable Kneser graphs\(^*\)

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Abstract

For \( k, s \geq 2 \), the \( s \)-stable Kneser graphs are the graphs with vertex set the \( k \)-subsets \( S \) of \( \{1, \ldots, n\} \) such that the circular distance between any two elements in \( S \) is at least \( s \) and two vertices are adjacent if and only if the corresponding \( k \)-subset are disjoint. Braun showed that for \( n \geq 2k + 1 \) the automorphism group of the 2-stable Kneser graphs (Schrijver graphs) is isomorphic to the dihedral group of order \( 2n \). In this paper we generalize this result by proving that for \( s \geq 2 \) and \( n \geq sk + 1 \) the automorphism group of the \( s \)-stable Kneser graphs also is isomorphic to the dihedral group of order \( 2n \).

Keywords: Stable Kneser graph, Automorphism group.

1 Introduction

Given a graph \( G \), \( V(G) \), \( E(G) \) and \( \text{Aut}(G) \) denote its vertex set, edge set and automorphism group, respectively. Let \( [n] := \{1, 2, 3, \ldots, n\} \). For positive integers \( n \) and \( k \) such that \( n \geq 2k \), the Kneser graph \( KG(n,k) \) has as vertices the \( k \)-subsets of \( [n] \) with edges defined by disjoint \( k \)-subsets. A subset \( S \subseteq [n] \) is \( s \)-stable if any two of its elements are at least "at distance \( s \) apart" on the \( n \)-cycle, i.e. \( s \leq |i - j| \leq n - s \) for distinct \( i, j \in S \). For \( s, k \geq 2 \), we denote \( [n]_s^k \) the family of \( s \)-stable \( k \)-subsets of \( [n] \). The \( s \)-stable Kneser graph \( KG(n,k)_s^{\text{stab}} \) is the subgraph of \( KG(n,k) \) induced by \( [n]_s^k \).

In a celebrated result, Lovász \([5]\) proved that the chromatic number of \( KG(n,k) \), denoted \( \chi(KG(n,k)) \), is equal to \( n - 2k + 2 \), verifying a conjecture due to M. Kneser \([3]\). After this result, Schrijver \([7]\) proved that the chromatic number remains the same for \( KG(n,k)_2^{\text{stab}} \). Moreover, this author showed that \( KG(n,k)_2^{\text{stab}} \) is \( \chi \)-critical. Due to these facts, the 2-stable Kneser graphs have been named Schrijver graphs. These results were the base for several papers devoted to Kneser graphs and stable Kneser graphs (see e.g. \([1,4,6,8,9,10]\)). In addition, it is well known that for \( n \geq 2k + 1 \) the automorphism group of the Kneser graph \( KG(n,k) \) is isomorphic to \( S_n \), the symmetric group of order \( n \) (see \([2]\) for a textbook account).

More recently, in 2010 Braun \([1]\) proved that the automorphism group of the Schrijver graphs \( KG(n,k)_2^{\text{stab}} \) is isomorphic to the dihedral group of order \( 2n \), denoted \( D_{2n} \). In this paper we generalize this result by proving that the automorphism group of the \( s \)-stable Kneser graphs is isomorphic to \( D_{2n} \) for \( n \geq sk + 1 \).

Firstly, notice that if \( n = sk \), the \( s \)-stable Kneser graph \( KG(n,k)_s^{\text{stab}} \) is isomorphic to the complete graph on \( s \) vertices and the automorphism group of \( KG(n,k)_s^{\text{stab}} \) is isomorphic to \( S_s \).

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From the definitions we have that $D_{2n}$ injects into $\text{Aut}(\overline{KG}(n,k)_{s-stab})$, as $D_{2n}$ acts on $\overline{KG}(n,k)_{s-stab}$ by acting on $[n]$. Then, we have the following fact.

**Remark 1.** $D_{2n} \subseteq \text{Aut}(\overline{KG}(n,k)_{s-stab})$.

In the sequel, the arithmetic operations are taken modulo $n$ on the set $[n]$ where $n$ represents the 0. Let us recall an important result due to Talbot.

**Theorem 1** (Theorem 3 in [8]). Let $n,s,k$ be positive integers such that $n \geq sk$ and $s \geq 3$. Then, every maximum independent set in $\overline{KG}(n,k)_{s-stab}$ is of the form $I_i = \{I \in [n]^k_s : i \in I\}$ for a fixed $i \in [n]$.

For $n \geq sk + 1$ we observe that $\{i, i+s, i+2s, \ldots, i+(k-1)s\}$ and $\{i, i+s+1, i+2s+1, \ldots, i+(k-1)s+1\}$ belong to $[n]^k_s$ for all $i \in [n]$. Then, we can easily obtain the following fact.

**Remark 2.** Let $n \geq sk + 1$ and $i,j \in [n]$. If $i \neq j$, then $I_i \neq I_j$.

### 2 Automorphism group of $\overline{KG}(n,k)_{s-stab}$

This section is devoted to obtain the automorphism group of $\overline{KG}(n,k)_{s-stab}$. To this end, let us introduce the following graph family. Let $n,s,k$ be positive integers such that $n \geq sk + 1$. We define the graph $G(n,k,s)$ with vertex set $[n]$ and two vertices $i,j \in [n]$ are adjacent if and only if it does not exist $S \in [n]^k_s$ such that $\{i,j\} \subseteq S$. See examples in Figure 1.

![Figure 1: Examples of graphs $G(n,k,s)$](image)

Two vertices $i, j$ of $G(n,k,s)$ are consecutive if $i = j + 1$. Let us see a direct result about consecutive vertices and dihedral groups, which we will use in the following theorem.

**Remark 3.** An injective function $f : [n] \rightarrow [n]$ sends consecutive vertices of $G(n,k,s)$ to consecutive vertices of $G(n,k,s)$ if and only if $f$ belongs to the dihedral group $D_{2n}$.

Next, we obtain the main result of this section that states the link between the automorphism groups of $\overline{KG}(n,k)_{s-stab}$ and $G(n,k,s)$.

**Theorem 2.** Let $n,s,k$ be positive integers such that $n \geq sk + 1$ and $s \geq 3$. Then, the automorphism group of $\overline{KG}(n,k)_{s-stab}$ is isomorphic to the automorphism group of $G(n,k,s)$.
Proof. As we have mentioned, given \( i \in [n] \), Theorem 1 guarantees that the sets \( \mathcal{I}_i \) are the maximum independent sets in \( KG(n,k)_{s-\text{stab}} \). Besides, any automorphism of \( KG(n,k)_{s-\text{stab}} \) send maximum independent sets into maximum independent sets, i.e. for each \( \alpha \in \text{Aut}(KG(n,k)_{s-\text{stab}}) \) and \( i \in [n] \), \( \alpha(I_i) = I_j \) for some \( j \in [n] \). From Remark 2, if \( i \neq j \) then \( \alpha(I_i) \neq \alpha(I_j) \) and so \( \alpha \) permutes these independent sets. Hence we define the homomorphism \( \phi \) from \( \text{Aut}(KG(n,k)_{s-\text{stab}}) \) to \( S_n \) such that

\[
\phi(\alpha)(i) = j \iff \alpha(I_i) = I_j.
\]

We will show that \( \phi \) is injective and its image is \( \text{Aut}(G(n,k,s)) \).

Given a non-trivial element \( \alpha \in \text{Aut}(KG(n,k)_{s-\text{stab}}) \), there exists \( S \in [n]^k \) such that \( \alpha(S) \neq S \), i.e. there exists \( j \in S \) such that \( j \notin \alpha(S) \). It follows that \( \alpha(S) \in \mathcal{I}_{\phi(\alpha)(j)} \), but \( \alpha(S) \notin \mathcal{I}_j \), hence \( \phi(\alpha)(j) \neq j \) and \( \phi(\alpha) \) is non-trivial. Then, \( \phi \) is injective.

Now, we first prove that \( \text{Aut}(G(n,k,s)) \subseteq \phi(\text{Aut}(KG(n,k)_{s-\text{stab}})) \). For each \( \beta \in \text{Aut}(G(n,k,s)) \) we define the function \( \gamma : V(KG(n,k)_{s-\text{stab}}) \to V(KG(n,k)_{s-\text{stab}}) \) such that for each \( S = \{s_1,\ldots,s_k\} \in V(KG(n,k)_{s-\text{stab}}) \), \( \gamma(S) = \{\beta(s_1),\ldots,\beta(s_k)\} \).

Since \( S \) is a stable set of \( G(n,k,s) \), \( \gamma(S) \) is also a stable set of \( G(n,k,s) \) and \( \gamma \) is well defined. It is not hard to see that \( \gamma \) is bijective. Furthermore, \( S \) and \( S' \) are adjacent in \( KG(n,k)_{s-\text{stab}} \) if and only if \( \gamma(S) \) and \( \gamma(S') \) are adjacent in \( KG(n,k)_{s-\text{stab}} \). Therefore \( \gamma \in \text{Aut}(KG(n,k)_{s-\text{stab}}) \) and from definition \( \phi(\gamma) = \beta \).

Let us prove that \( \phi(\text{Aut}(KG(n,k)_{s-\text{stab}})) \subseteq \text{Aut}(G(n,k,s)) \), i.e. \( \phi(\alpha) \) is an automorphism of \( G(n,k,s) \) for each \( \alpha \in \text{Aut}(KG(n,k)_{s-\text{stab}}) \). Let \( i, j \in [n] \), \( i' = \phi(\alpha)(i) \) and \( j' = \phi(\alpha)(j) \). If \( ij \in E(G(n,k,s)) \), since \( I_i \cap I_j = \emptyset \) and \( \alpha \) is injective, \( I_{i'} \cap I_{j'} = \emptyset \), i.e. \( \emptyset \neq S \in V(KG(n,k)_{s-\text{stab}}) \) such that \( \{i',j'\} \subseteq S \). Thus \( i'j' \in E(G(n,k,s)) \). Since \( \phi(\alpha) \) is bijective, we conclude that \( \phi(\alpha) \in \text{Aut}(G(n,k,s)) \).

Therefore the image of \( \phi \) is \( \text{Aut}(G(n,k,s)) \) and the proof is complete.

This result allow us to obtain \( \text{Aut}(KG(n,k)_{s-\text{stab}}) \) from \( \text{Aut}(G(n,k,s)) \). Next section is devoted to analyze the structure and the automorphism group of the graphs \( G(n,k,s) \).

### 2.1 The automorphism group of \( G(n,k,s) \).

Let \( G \) be a simple graph. For a vertex \( v \in V(G) \), the open neighborhood of \( v \) in \( G \) is the set \( N(v) = \{u \in V(G) : uv \in E(G)\} \). Then, the closed neighborhood of \( v \) in \( G \) is \( N[v] = N(v) \cup \{v\} \). The degree of a vertex \( v \in V(G) \) is \( \text{deg}(v) = |N(v)| \). For any positive integer \( d \), we denote by \( G^d \) the \( d \)-th power of \( G \), i.e. the graph with the same vertex set \( V(G) \) and such that two vertices \( u \) and \( v \) are adjacent if and only if \( \text{dist}_G(u,v) \leq d \), where \( \text{dist}_G(u,v) \) is the distance between \( u \) and \( v \) in \( G \), i.e. the length of the shortest path in \( G \) from \( u \) to \( v \). We denote by \( C_n \) the \( n \)-cycle graph with vertex set \([n]\) and edge set \( \{ij : i,j \in [n], j = i+1\} \).

**Theorem 3.** Let \( n,s,k \) be positive integers such that \( n \geq sk+1 \) and \( s \geq 3 \). Then,

1. if \( s(k+1) - 1 \leq n \), then \( G(n,k,s) \) is isomorphic to \( C_n^{s-1} \), and

2. if \( sk+1 \leq n \leq s(k+1) - 2 \). Then \( G(n,k,s) \) is the graph on \([n]\) and edges defined as follows:

\[
i j \in E(G(n,k,s)) \iff i \neq j, |j-i| \notin \bigcup_{d=1}^{k-1} \{ds,ds+1,\ldots,ds+r\},
\]
where \( r = n - sk \).

Proof. From the symmetry of \( G(n, k, s) \) (see Remark 1), to prove this result it is enough to obtain the open/closed neighborhood of vertex 1 in \( G(n, k, s) \) for each case.

1. **Case** \( s(k+1) - 1 \leq n \): We have to prove that \( [n] \setminus N[1] = \{ s + 1, \ldots, n - s + 1 \} \). By definitions, \( i \in N(1) \) for every \( i \in \{ 2, \ldots, n \} \cup \{ n - s + 2, \ldots, n \} \). We only need to prove that for all \( i \in \{ s + 1, \ldots, n - s + 1 \} \) there exists \( S_i \in [n]^k_s \) such that \( \{ 1, i \} \subseteq S_i \). So, let \( i \in \{ s + 1, \ldots, n - s + 1 \} \) and \( t = \left\lfloor \frac{i-1}{s} \right\rfloor \).

   If \( t \geq k - 1 \), let \( S_i = \{ 1, 1 + s, \ldots, 1 + (k-2)s, i \} \). Then \( S_i \in [n]^k_s \), since \( s + 1 \leq i \leq n - s + 1 \) and \( i - 1 = (1 + (t-1)s) \geq i - 1 = (t - 1) \). Hence, if \( t \leq k - 2 \), let \( S_i = \{ 1, 1 + s, \ldots, 1 + (t-1)s, i, i + s, \ldots, i + (k - t - 1)s \} \). To prove that \( S_i \in [n]^k_s \) it is enough to show that \( i - 1 + (t - 1)s \geq s \) and \( n - (i + (k - t - 1)s) \geq s - 1 \). The first inequality trivially holds. To see the second inequality, notice that

\[
i - (1 + (t-1)s) = i - 1 + s - s \left\lfloor \frac{i-1}{s} \right\rfloor < i - 1 + s - s \left( \frac{i-1}{s} - 1 \right) = 2s.
\]

Then, \( i - (1 + (t-1)s) \leq 2s - 1 \). Therefore \( n - (i + (k - t - 1)s) = n - (i - 1 - (t - 1)s + (k - 2)s + 1) = n - (i - 1 + (t - 1)s) - ((k - 2)s + 1) \geq n - (2s - 1) - ((k - 2)s + 1) = n - 2s + 1 - (k - 2)s - 1 = n - sk \geq s - 1 \).

2. **Case** \( sk + 1 \leq n \leq s(k + 1) - 2 \): Let \( F_d = \{ 1 + ds, 1 + ds + 1, \ldots, 1 + ds + r \} \) for \( d \in [k - 1] \) and \( F = \bigcup_{d=1}^{k-1} F_d \). We will prove that \( N[1] = [n] - F \), which implies that \( 1 \) and \( j \) are adjacent in \( G(n, k, s) \) if and only if \( j - 1 \not\in \bigcup_{d=1}^{k-1} \{ ds, ds + 1, \ldots, ds + r \} \), as required.

   Firstly, since \( n - (r + 1 + (k - 1)s) = s - 1 \), the set \( S_p = \{ 1, p + s, p + 2s, \ldots, p + (k - 1)s \} \in [n]^k_s \) for all \( p \in [r + 1] \). Furthermore, \( \{ 1 \} \cup F = \bigcup_{p=1}^{r+1} S_p \) and then \( N[1] \subseteq [n] - F \).

To see the converse inclusion, observe first that if \( h \in [s] \cup \{ n - s + 2, \ldots, n \} \) then \( h \in N[1] \) from definition of \( G(n, k, s) \).

Hence, if \( k = 2 \) we have finished.

Now, let \( k \geq 3 \) (see Figure 2). We have that

\[
[n] \setminus (F \cup [s] \cup \{ n - s + 2, \ldots, n \}) = \bigcup_{m=1}^{k-2} \{ ms + 2 + r, \ldots, (m + 1)s \}.
\]

Let \( h \in \bigcup_{m=1}^{k-2} \{ ms + 2 + r, \ldots, (m + 1)s \} \). We will prove that it does not exist \( S \in [n]^k_s \) such that \( \{ 1, h \} \subseteq S \). Let \( W \) be an \( s \)-stable set of \( [n] \) such that \( \{ 1, h \} \subseteq W \). Notice that \( |W \cap [h-1]| \leq \left\lfloor \frac{h-1}{s} \right\rfloor \) and \( |W \cap \{ h, \ldots, n \}| \leq \left\lfloor \frac{n - h + 1}{s} \right\rfloor \).

Consider \( m' \in [k - 2] \) such that \( h \in \{ m's + 2 + r, \ldots, (m' + 1)s \} \). Then,

- \( \left\lfloor \frac{h-1}{s} \right\rfloor \leq \left\lfloor \frac{(m'+1)s-1}{s} \right\rfloor = m' \).
• \[ \left\lfloor \frac{n-h+1}{s} \right\rfloor \leq \left\lfloor \frac{n-(m's+2+r)+1}{s} \right\rfloor \leq \left\lfloor \frac{n-r-1}{s} \right\rfloor - m' = \left\lfloor \frac{n-n+sk-1}{s} \right\rfloor - m' = k - 1 - m'. \]

Thus, \( |W| \leq \left\lfloor \frac{h-1}{s} \right\rfloor + \left\lfloor \frac{n-h+1}{s} \right\rfloor \leq k - 1. \) Therefore, any \( s \)-stable set of \( [n] \) containing the set \( \{1, h\} \) has cardinality at most \( k - 1 \), i.e. it does not exist \( S \in [n]^k_s \) such that \( \{1, h\} \subseteq S \). Hence \( h \in N[1] \) and the result follows.

\[ \square \]

Figure 2: Neighborhood of vertex 1 in \( G(n, k, s) \).

In order to obtain \( \text{Aut}(G(n, k, s)) \), let us recall a well known result on automorphism group (see, e.g. [9]).

**Remark 4.** Let \( m \) and \( q \) be positive integers such that \( m \geq 2q + 3 \). Then, the automorphism group of \( C^q_m \) is the dihedral group \( D_{2m} \).

Let \( x \) be the degree of the vertices in \( G(n, k, s) \) (which is a regular graph). Then, we have the following result.

**Theorem 4.** Let \( n, s, k \) be positive integers such that \( n \geq sk + 1 \) and \( s \geq 3 \). Then, the automorphism group of \( G(n, k, s) \) is the dihedral group \( D_{2n} \).

**Proof.** Firstly, observe that if \( s(k+1) - 1 \leq n \) the result immediately follows from Case 1 of Theorem 3 and Remark 4. Let us consider \( sk + 1 \leq n \leq s(k+1) - 2 \). From Remark 3 we only need to prove that any \( \alpha \in \text{Aut}(G(n, k, s)) \) sends consecutive vertices to consecutive vertices. Moreover, by Remark 1 and Theorem 2 it is enough to show that \( \alpha(1) \) and \( \alpha(2) \) are consecutive vertices. Without loss of generality we consider \( \alpha(1) = 1 \).

Let \( r = n - sk \). From Theorem 3

\[ N[2] \cap \{1 + ds, \ldots, 1 + ds + r\} = \{1 + ds\} \]
for \( d = 1, \ldots, k - 1, \) and

\[
[n] \setminus N[1] = \bigcup_{d=1}^{k-1} \{1 + ds, 1 + ds + 1, \ldots, 1 + ds + r\}.
\]

So,

\[
|N[1] \cap N[2]| = x + 1 - (k - 1) = x - k + 2.
\]

Analogously, \( |N[1] \cap N[n]| = x - k + 2. \)

Let \( i \in N(1). \) Recall that

\[
N(1) = \{2, \ldots, s\} \cup \{n - s + 2, \ldots, n\} \cup \left( \bigcup_{m=1}^{k-2} \{ms + 2 + r, \ldots, (m + 1)s\} \right).
\]

If \( i \in \{3, \ldots, s\} \) we have that \( \{1 + s, 2 + s\} \subseteq N[i]. \) Besides, if \( k \geq 3 \) observe that \( \{i + 1 + (d - 1)s + r, \ldots, i + ds - 1\} \subseteq N[i] \) for \( d = 2, \ldots, k - 1. \) Hence, since \( 3 \leq i \leq s, 1 + ds \leq i + ds - 1 \) and \( i + 1 + (d - 1)s + r \leq 1 + ds + r. \) Therefore, \( \{i + 1 + (d - 1)s + r, \ldots, i + ds - 1\} \cap \{1 + ds, \ldots, 1 + ds + r\} \neq \emptyset \) for \( d = 2, \ldots, k - 1. \) Then, \( |N[i] \cap \{1 + s, \ldots, 1 + s + r\}| \geq 2 \) and \( |N[i] \cap \{1 + ds, \ldots, 1 + ds + r\}| \geq 1 \) for \( d = 2, \ldots, k - 1. \) So, if \( k \geq 2, \) \( |N[1] \cap N[i]| \leq x + 1 - k \leq x - 1 \) and thus \( \alpha(2) \neq i. \) Similarly if \( i \in \{n - s + 2, \ldots, n - 1\}, \) we have \( \alpha(2) \neq i. \) So, if \( k = 2 \) the result follows.

Now, let \( k \geq 3. \) Consider \( i \in \bigcup_{m=1}^{k-2} \{ms + 2 + r, \ldots, (m + 1)s\} \) and let \( m_i \in \{1, \ldots, k - 2\} \) such that \( i \in \{m_is + 2 + r, \ldots, (m_i + 1)s\}. \)

Notice that

\[
\{1 + m_is, \ldots, 1 + m_is + r\} \cup \{1 + (m_i + 1)s, \ldots, 1 + (m_i + 1)s + r\} \subseteq \{i - (s - 1), \ldots, i + s - 1\} \subseteq N[i]. \quad (1)
\]

Therefore,

\[
\{1 + m_is, \ldots, 1 + m_is + r\} \cup \{1 + (m_i + 1)s, \ldots, 1 + (m_i + 1)s + r\} \subseteq N[i] \setminus N[1].
\]

Now, let \( m \in [k - 2]. \) If \( m < m_i \) then \( 1 + (m_i - m)s + r \leq i - (1 + ms) \leq (m_i - m)s - 1. \) From Theorem 3, we have that \( 1 + ms \in N[i] \) if \( m < m_i. \) By a similar reasoning we have that \( 1 + ms + r \in N[i] \) if \( m > m_i + 1. \) Then,

\[
\{1 + ms : m < m_i, \ m \in [k]\} \cup \{1 + ms + r : m > m_i + 1, \ m \in [k]\} \subseteq N[i] \setminus N[1].
\]

These facts together with (1) imply that

\[
|N[1] \cap N[i]| = x + 1 - (N[i] \setminus N[1]) \leq x + 1 - (2(r + 1) + (k - 4)) \leq x - k + 1.
\]

Thus \( \alpha(2) \neq i. \) Therefore \( \alpha(2) \in \{2, n\} \) and the thesis holds.

Finally, we have the main result of this work.

**Theorem 5.** Let \( n, s, k \) be positive integers such that \( n \geq sk + 1 \) and \( s \geq 2. \) Then, the automorphism group of \( KG(n, k)_{s-stab} \) is isomorphic to the dihedral group \( D_{2n}. \)

**Proof.** The result for the case \( s = 2 \) follows from (1) and for the remaining cases can be obtained from Theorems 2 and 4. \( \square \)
3 Further results

In this section we will obtain some properties of $s$-stable Kneser graphs as a consequence of the results in the previous sections. Firstly, as a consequence of Theorem 5, we have the following result.

**Theorem 6.** Let $n, k, s \geq 2$ with $n \geq sk + 1$. Then, $KG(n, k)_{s-stab}$ is vertex transitive if and only if $n = sk + 1$.

**Proof.** Without loss of generality, we assume that any vertex $S = \{s_1, s_2, \ldots, s_k\}$ of the $s$-stable Kneser graph $KG(n, k)_{s-stab}$ verifies that $s_1 < s_2 < \ldots < s_k$. Then, $S$ is described unequivocally by $s_1$ and the gaps $l_1(S), \ldots, l_k(S)$ such that for $i \in [k - 1]$, $l_i(S) = s_{i+1} - s_i$ and $l_k(S) = s_1 + n - s_k$. Observe that any automorphism of $KG(n, k)_{s-stab}$ “preserves” the gaps $l_i$, i.e. if $\phi \in \text{Aut}(KG(n, k)_{s-stab})$ there exist $\alpha \in D_{2k}$ such that $l_i(\phi(S)) = l_{\alpha(i)}(S)$ for all $i \in [k]$.

If $n \geq sk + 2$, then $S_1 = \{1, 1 + s, 1 + 2s, \ldots, 1 + (k - 1)s\} \in [n]_s^k$ and $S_2 = \{1, 2 + s, 2 + 2s, \ldots, 2 + (k - 1)s\} \in [n]_s^k$. Therefore, from Theorem 5 we have that no automorphism of $KG(n, k)_{s-stab}$ maps $S_1$ to $S_2$, since $l_1(S_2) = s + 1$ but $l_i(S_1) = s$ for $i \in [k - 1]$ and $l_k(S_1) \geq s + 2$.

Besides, in [9] it is proved that if $S \in [sk + 1]_s^k$ then exactly one gap $l_m(S)$ is equal to $s + 1$ and the remaining gaps are equal to $s$. From this fact we have that $KG(sk + 1, k)_{s-stab}$ is vertex transitive.

Next, we will analyze some aspects related to colourings of $s$-stable Kneser graphs. Let $\alpha(G)$ and $\chi^*(G)$ the independence number and fractional chromatic number of a graph $G$, respectively.

**Proposition 1.** Let $n, k, s \geq 2$ with $n \geq sk + 1$. Then, $\chi^*(KG(n, k)_{s-stab}) = \frac{n}{k}$.

**Proof.** It is immediate to observe that $\chi^*(KG(n, k)_{s-stab}) \leq \frac{n}{k}$ (see, e.g Theorem 7.4.5 in [2]). To see the converse inequality, we use the fact that for any graph $G$, $\chi^*(G) \geq \frac{|V(G)|}{\alpha(G)}$.

So, let us compute $|[n]_s^k| = |V(KG(n, k)_{s-stab})|$. From [8], since the sets $\mathcal{I}_i$ are maximum independent sets for $i \in [n]$, $\alpha(KG(n, k)_{s-stab}) = |\mathcal{I}_i| = \left( \begin{array}{c} n -(s-1)k-1 \\ k-1 \end{array} \right)$.

Then, to compute $|[n]_s^k|$, let us observe that $\bigcup_{i=1}^{n} \mathcal{I}_i = [n]_s^k$ and $\sum_{i=1}^{n} |\mathcal{I}_i| = n \left( \begin{array}{c} n -(s-1)k-1 \\ k-1 \end{array} \right)$, where each vertex of $KG(n, k)_{s-stab}$ is computed $k$ times. Then,

$|[n]_s^k| = n \left( \begin{array}{c} n -(s-1)k-1 \\ k-1 \end{array} \right)$

and the result follows.

Hence $|[n]_s^k| = \frac{n}{k} \left( \begin{array}{c} n -(s-1)k-1 \\ k-1 \end{array} \right)$ and the result follows.

As we have mentioned before, Schrijver [7] proved that the graphs $KG(n, k)_{2-stab}$ are $\chi$-critical subgraphs of $KG(n, k)$ but it is an open problem to compute the chromatic number of $s$-stable Kneser graphs. From the last result and Proposition 2 in [6] we have

$\frac{n}{k} \leq \chi(KG(n, k)_{s-stab}) \leq n - (k-1)s$.

In particular, if $n = ks + 1$ we obtain that $\chi(KG(ks + 1, k)_{s-stab}) = s + 1$, which is an alternative proof to compute the exact value of $\chi(KG(ks + 1, k)_{s-stab})$ already studied in [8] and [9].
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