REGGE'S EINSTEIN-HILBERT FUNCTIONAL ON THE DOUBLE TETRAHEDRON

DANIEL CHAMPION, DAVID GLICKENSTEIN, AND ANDREA YOUNG

Abstract. The double tetrahedron is the triangulation of the three-sphere gotten by gluing together two congruent tetrahedra along their boundaries. As a piecewise flat manifold, its geometry is determined by its six edge lengths, giving a notion of a metric on the double tetrahedron. We study notions of Einstein metrics, constant scalar curvature metrics, and the Yamabe problem on the double tetrahedron, with some reference to the possibilities on a general piecewise flat manifold. The main tool is analysis of Regge’s Einstein-Hilbert functional, a piecewise flat analogue of the Einstein-Hilbert (or total scalar curvature) functional on Riemannian manifolds. We study the Einstein-Hilbert-Regge functional on the space of metrics and on discrete conformal classes of metrics.

1. Introduction

It is well-known that Ricci-flat metrics on closed Riemannian manifolds of dimension at least three are critical points of the Einstein-Hilbert functional

\[ \mathcal{EH}(M, g) = \int_M R_g dV_g, \]

where \( R_g \) and \( dV_g \) are the scalar curvature and volume form for the Riemannian manifold \((M, g)\). Since there are topological restrictions to being Ricci-flat (e.g., the Cheeger-Gromoll splitting theorem [9]), one may restrict to the subset of Riemannian manifolds with volume equal to 1 so that critical points of the constrained problem are Einstein manifolds. Equivalently, one can consider a normalized Einstein-Hilbert functional

\[ \mathcal{NEH}(M^n, g) = \frac{\int_M R_g dV_g}{(\int_M dV_g)^{(n-2)/n}}, \]

whose critical points are Einstein manifolds. Einstein manifolds are of interest because the Einstein metric is, in some sense, a most symmetric or "best" geometry for the manifold. In trying to prove a classification theorem such as Thurston’s geometrization conjecture, one may try to find a best geometry by trying to optimize a geometric functional such as \( \mathcal{EH} \) and by studying both convergence and degenerations to try to capture all possible “best” geometries (see [2]).

Related to the study of \( \mathcal{EH} \) and Einstein manifolds is the well-known Yamabe problem, which asks whether one can find a constant scalar curvature metric within a conformal class or, equivalently, if one can find a critical point for \( \mathcal{NEH} \) restricted
to a conformal class. The solution was completed by R. Schoen, based on important contributions from Yamabe, Trudinger, and Aubin (see [19] for an overview and complete proof).

In assigning geometry to a topological manifold, an alternative to the Riemannian approach is that of piecewise flat geometry. A piecewise flat manifold is a triangulation together with edge lengths that determine a Euclidean geometry on each simplex in the triangulation. In 1961, T. Regge [22] suggested a functional defined on piecewise flat manifolds which is analogous to $\mathcal{E}_H$. We call this functional the Einstein-Hilbert-Regge functional and denote it as $\mathcal{EHR}$. Study of this functional as an action for general relativity has led to a wide array of work on Regge calculus and lattice gravity (for a survey, see [15]). It was later shown that $\mathcal{EHR}$ and $\mathcal{EH}$ are related in the sense that appropriately finer piecewise flat triangulations which converge to a Riemannian manifold lead to convergence of the functionals. In fact, it was proven that the associated curvature measures converge [10]. Thus $\mathcal{EHR}$ is a discretization of $\mathcal{EH}$, and could potentially be used to approximate $\mathcal{EH}$. Such an approach is an alternative to discretizing the Einstein equations themselves. By discretizing the functional instead of its Euler-Lagrange equation, we hope to produce an approximation of the Euler-Lagrange equation whose behavior mimics that of the smooth case. This approach has been applied in a number of contexts, such as computer graphics, computational mechanics, and computational dynamics, and it is the main focus of the fields of discrete differential geometry and discrete exterior calculus (see, e.g., [6], [11], [12], [21]).

In addition, we can use a definition of conformal class in [14] to formulate a discrete version of the Yamabe problem. However, this does not allow us to reformulate the functional in the same way as in the smooth setting, which allows $\mathcal{NEH}$ to be rewritten in a relatively simply way as function of the conformal factor. Instead, we are forced to work entirely with variation formulas for curvature.

The purpose of this paper is to consider the Einstein-Hilbert-Regge functional on the simplest possible triangulation of a three-manifold without boundary: the double tetrahedron. Even on this small triangulation, the behavior of the $\mathcal{EHR}$ functional is rich and complex. In particular, on the double tetrahedron we do not have a complete answer to the uniqueness of Einstein metrics, a complete understanding of the Yamabe problem, or a calculation of the Yamabe invariant.

In §2 we give some background on piecewise flat manifolds, and we define the $\mathcal{EHR}$ functional and two normalized versions of it. Additionally, we describe the geometry of the double tetrahedron, and we set the notation for the remainder of the paper. In §3 we consider length variations of piecewise flat metrics on the double tetrahedron. Critical points of the normalized $\mathcal{EHR}$ functionals are geometrically significant and yield definitions of Einstein metrics in the piecewise flat setting. We study the convexity of the functionals at these points. In §4 we discuss discrete conformal variations of a piecewise flat metric described in [14] (following [20], [23], [26]). The critical points of the normalized $\mathcal{EHR}$ functionals with respect to a conformal variation give rise to a notion of constant scalar curvature piecewise flat metrics. On the double tetrahedron, we are able to provide a partial classification of such metrics and are able to show existence in every conformal class. Additionally, we study the convexity of the curvature functionals at Einstein metrics. Finally, in §5 we discuss the Yamabe invariant on both the double tetrahedron and on general piecewise flat manifolds.
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2. Background and notation

In this section we will secure notation for the rest of the paper. Most of the notation follows [14]. We will also provide the necessary background on piecewise flat manifolds, and we will define the double tetrahedron.

2.1. Geometry of the tetrahedron. Consider a Euclidean tetrahedron determined by four vertices numbered 1, 2, 3, 4. The tetrahedron has six edge lengths, and we denote the length of the edge between vertices $i$ and $j$ by $\ell_{ij}$. Since edge lengths arise from a nondegenerate tetrahedron, they satisfy a particular condition.

**Definition 2.1.** Consider the matrix $A$:

$$A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & \ell_{12}^2 & \ell_{13}^2 & \ell_{14}^2 \\
1 & \ell_{12}^2 & 0 & \ell_{23}^2 & \ell_{24}^2 \\
1 & \ell_{13}^2 & \ell_{23}^2 & 0 & \ell_{34}^2 \\
1 & \ell_{14}^2 & \ell_{24}^2 & \ell_{34}^2 & 0
\end{bmatrix}.$$ 

Let $CM_3 = \det (A)$.

The quantity $CM_3$ is related to the volume of the tetrahedron:

$$Vol = \sqrt{\frac{CM_3}{288}}.$$ 

The quantity $CM_3$ is a special case of a Cayley-Menger determinant $CM_n$, which determines the volume of an $n$-simplex in a similar way. If $CM_3 \leq 0$, then there is no nondegenerate Euclidean tetrahedron with those edge lengths. Notice that $CM_3$ is a polynomial of degree six in the edge lengths.

The angles of a Euclidean triangle are determined by the edge lengths via the cosine law. The dihedral angles of a tetrahedron can then be calculated from the angles at the faces using the spherical cosine law. We use $\beta_e$ to refer to the dihedral angle of a tetrahedron at edge $e$. If we wish to emphasize that it is in tetrahedron $t$, we denote it as $\beta_{e<t}$. In the sequel, $\tau < \sigma$ or $\sigma > \tau$ will mean that $\tau$ is a sub-simplex of $\sigma$.

2.2. Piecewise flat manifolds. In this section we recall some definitions related to piecewise flat manifolds.

The double tetrahedron is a particular case of a *triangulated piecewise flat manifold*. By a triangulation, we mean a collection of simplices identified along sub-simplices. Note that the triangulation need not be a simplicial complex (for instance, in the double tetrahedron there are two tetrahedra associated to the same collection of vertices). The dimension of a triangulation is that of its highest dimensional simplex. A three-dimensional triangulation $T = (V, E, F, T)$ has a collection of vertices (denoted $V$), edges (denoted $E$), faces (denoted $F$), and tetrahedra (denoted $T$). A triangulated piecewise flat manifold is denoted as $(M, T, \ell)$, where $M$
is a manifold, \( \mathcal{T} \) is a triangulation of \( M \), and \( \ell \) is a metric according to the following definition.

**Definition 2.2.** A vector \( \ell \in \mathbb{R}^{|E|} \) such that each simplex can be realized as a Euclidean simplex with edge lengths determined by \( \ell \) is called a metric for the triangulated manifold \( (M, \mathcal{T}) \), and \( (M, \mathcal{T}, \ell) \) is called a triangulated piecewise flat manifold. The space of all metrics is denoted \( \text{met}(M, \mathcal{T}) \).

Note that the condition for a metric can be described using Cayley-Menger determinants of the type described in Section 2.1.

We will restrict to the case that \( M \) is three-dimensional. There are several quantities associated to \( (M, \mathcal{T}, \ell) \):

**Definition 2.3.** The edge curvature of an edge \( e \) is

\[
K_e = \left(2\pi - \sum_{t \in \mathcal{T}} \beta_{e<t}\right) \ell_e,
\]

where \( \beta_{e<t} \) is the dihedral angle at edge \( e \) in tetrahedron \( t \), and \( \ell_e \) is the edge length.

Now we can define some functionals on piecewise flat manifolds.

**Definition 2.4.** The total length of \( (M, \mathcal{T}, \ell) \) is

\[
\mathcal{L}(M, \mathcal{T}, \ell) = \sum_{e \in E} \ell_e.
\]

Let \( V_t \) be the volume of tetrahedron \( t \). Then the volume of \( (M, \mathcal{T}, \ell) \) is

\[
\mathcal{V}(M, \mathcal{T}, \ell) = \sum_{t \in \mathcal{T}} V_t.
\]

The Einstein-Hilbert-Regge functional is

\[
\mathcal{EHR}(M, \mathcal{T}, \ell) = \sum_{e \in E} K_e.
\]

We will also consider two normalizations of the \( \mathcal{EHR} \) functional. Volume normalization is quite natural and is the usual normalization considered in the Riemannian setting (see [5]). However, since the formula for volume of a simplex is quite complicated, one may also consider a normalization which is linear in the edge lengths.

**Definition 2.5.** The length normalized Einstein-Hilbert-Regge functional is

\[
\mathcal{L}\mathcal{EHR}(M, \mathcal{T}, \ell) = \frac{\mathcal{EHR}(M, \mathcal{T}, \ell)}{\mathcal{L}(M, \mathcal{T}, \ell)}.
\]

The volume normalized Einstein-Hilbert-Regge functional is

\[
\mathcal{V}\mathcal{EHR}(M, \mathcal{T}, \ell) = \frac{\mathcal{EHR}(M, \mathcal{T}, \ell)}{\mathcal{V}(M, \mathcal{T}, \ell)^{1/3}}.
\]

The normalizations are defined so that the functionals take the same value if all lengths are scaled by the same positive constant.

Following [14], we can now define what we consider to be Einstein metrics, which will depend on the normalization. We call these the Einstein metrics because they are the critical points of the corresponding normalized functionals.
Definition 2.6. The metric \( \ell \in \text{met}(M, T) \) is an \( \mathcal{L} \)-Einstein metric if there exists \( \lambda_\mathcal{L} \in \mathbb{R} \) such that for all \( e \in E \),
\[
K_e = \lambda_\mathcal{L} \ell_e.
\]
Here \( \lambda_\mathcal{L} = \mathcal{L} \frac{EH}{R} \). The metric \( \ell \in \text{met}(M, T) \) is a \( \mathcal{V} \)-Einstein metric if there exists \( \lambda_\mathcal{V} \in \mathbb{R} \) such that for all \( e \in E \),
\[
K_e = \lambda_\mathcal{V} V_e.
\]
Here \( \lambda_\mathcal{V} = \mathcal{V} \frac{EH}{3V} \), and \( V_e = \ell_e \frac{\partial V}{\partial \ell_e} \).

Remark 2.7. Note that \( \mathcal{L} \)-Einstein metrics satisfy
\[
\frac{\partial \mathcal{L} \mathcal{H} \mathcal{R}}{\partial \ell_e} = \lambda_\mathcal{L} \frac{\partial \mathcal{L}}{\partial \ell_e},
\]
and \( \mathcal{V} \)-Einstein metrics satisfy
\[
\frac{\partial \mathcal{L} \mathcal{H} \mathcal{R}}{\partial \ell_e} = \lambda_\mathcal{V} \frac{\partial \mathcal{V}}{\partial \ell_e}.
\]

2.3. Geometry and topology of the double tetrahedron. Most of this paper will be concerned with the double tetrahedron.

Definition 2.8. The double tetrahedron \( DT \) is the triangulation of the three-sphere obtained by identifying the corresponding boundary faces of two disjoint tetrahedra.

One should consider the double tetrahedron as \( DT = (S^3, T) \), where \( T \) is the triangulation described in Definition 2.8. The double tetrahedron is given a geometry by specifying the edge lengths (or metric), and these six edge lengths determine the two Euclidean tetrahedra which make up the manifold. Note that the two tetrahedra are necessarily congruent, which leads to the following definition.

Definition 2.9. A single tetrahedron in \( DT \) is called the generating tetrahedron.

The set of metrics on the double tetrahedron can be described succinctly as follows.

Definition 2.10. The space of metrics on the double tetrahedron is the set:
\[
\text{met}(DT) = \left\{ \ell \in \mathbb{R}^6 : \text{CM}_\mathcal{L} > 0 \right\}.
\]

Definition 2.10 gives the same set as Definition 2.2, but describe it more explicitly. We will often use the term double tetrahedron to refer to the double tetrahedron with an arbitrary metric \( \ell \). The edge curvatures of the double tetrahedron can be expressed succinctly as
\[
(2.7) \quad K_e = (2\pi - 2\beta_e) \ell_e,
\]
where \( \beta_e \) is the dihedral angle in the generating tetrahedron at edge \( e \). Note the following important property of the double tetrahedron.

Lemma 2.11. On the double tetrahedron, \( K_e > 0 \) for any metric \( \ell \).

Proof. Since in a (nondegenerate) tetrahedron, each dihedral angle is less than \( \pi \), the lemma follows from formula (2.7). \( \square \)

We will label the vertices \( \{1, 2, 3, 4\} \) and edges will be denoted as \( ij \), where \( i, j \in \{1, 2, 3, 4\} \). For instance, in regard to the edge 12 between vertices 1 and 2, we will refer to the length of the edge as \( \ell_{12} \), the dihedral angle at the edge (in the generating tetrahedron) as \( \beta_{12} \), and the edge curvature as \( K_{12} \).
3. Metric variations

In this section we will study two normalizations of the Einstein-Hilbert-Regge functional on the double tetrahedron with the primary goal of finding Einstein metrics. To do so, we need to define the following subspaces of \( \text{met}(DT) \).

**Definition 3.1.** The space of length normalized edge lengths on the double tetrahedron is the set:

\[
\text{met}_L(DT) = \left\{ \vec{\ell} \in \text{met}(DT) : \sum_{(i,j) \in E} \ell_{ij} = 1 \right\}.
\]

The space of volume normalized edge lengths on the double tetrahedron is the set:

\[
\text{met}_V(DT) = \left\{ \vec{\ell} \in \text{met}(DT) : V = 1 \right\}.
\]

The space \( \text{met} \) is defined with an open condition, and hence has the structure of an open six-dimensional manifold. Note that 1 is a regular value of the functions \( L \) and \( V \), and hence \( \text{met}_L \) and \( \text{met}_V \) have the structures of smooth five-dimensional submanifolds of \( \text{met} \).

In this section, we will analyze the variational properties of the \( \mathcal{LEHR} \) and \( \mathcal{VEHR} \) functionals. To this end, we require the following variational results which follow from the Schlafli formula (see, e.g., [14]).

**Proposition 3.2.** For the double tetrahedron \( DT \),

\[
\begin{align*}
\frac{\partial \mathcal{LEHR}(DT, \ell)}{\partial \ell_{ij}} &= 2\pi - 2\beta_{ij} = \frac{K_{ij}}{\ell_{ij}}, \\
\frac{\partial \mathcal{VEHR}(DT, \ell)}{\partial \ell_{ij}} &= V^{-\frac{1}{3}} \left( \frac{K_{ij}}{\ell_{ij}} - \frac{\mathcal{VEHR}(M, T, \ell)}{3V(M, T, \ell)} \frac{\partial V}{\partial \ell_{ij}} \right),
\end{align*}
\]

One may guess that the double tetrahedron with all lengths equal is somehow special. We call this the equal length metric and note that it is unique up to scaling. Our main results in this section are the following:

**Theorem 3.3.**

1. On the double tetrahedron, equal length metrics are Einstein metrics with respect to both \( \mathcal{LEHR} \) and \( \mathcal{VEHR} \).
2. The eigenspaces and eigenvalues for the Hessian matrices of \( \mathcal{LEHR} \) and \( \mathcal{VEHR} \) at equal length metrics (with edge lengths \( k \)) are the following:

| Eigenspace | Spanning Vectors | Eigenvalues |
|------------|-----------------|-------------|
| \( V_{\lambda_1} \) | \( (1, 0, 0, 0, -1) \) | \( \lambda_{1\mathcal{LEHR}} = \frac{2\pi^2}{3} k^{-2} \approx 0.313 \cdot k^{-2}, \) |
| | \( (0, 1, 0, -1, 0) \) | \( \lambda_{1\mathcal{VEHR}} = 2 \pi^3 - \frac{4}{3} k^{-2} \left( 2\pi + 9\pi - 9\arccos \left( \frac{1}{3} \right) \right) \approx 21.611 \cdot k^{-2}, \) |
| | \( (0, 0, 1, -1, 0) \) | |
| \( V_{\lambda_2} \) | \( (0, 1, -1, 1, 0) \) | \( \lambda_{2\mathcal{LEHR}} = -\frac{2\pi^2}{3} k^{-2} \approx -0.94 \cdot k^{-2}, \) |
| | \( (1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 1) \) | \( \lambda_{2\mathcal{VEHR}} = 2 \pi^3 + \frac{4}{3} k^{-2} \left( 7\pi - 2\pi - 7\arccos \left( \frac{4}{3} \right) \right) \approx 34.145 \cdot k^{-2}, \) |
| \( V_0 \) | \( (1, 1, 1, 1, 1) \) | 0. |
(3) Equal length metrics are the only critical points of the $\mathcal{LEHR}$ functional and hence are the only $L$-Einstein metrics. These metrics are saddle points.

(4) Equal length metrics are local minima of the $\mathcal{VEHR}$ functional.

The proof of Theorem 3.3.1 follows directly from the formulas presented in Proposition 3.2.

One can compute the eigenvalues and eigenvectors of the Hessian matrix for both the $\mathcal{LEHR}$ and $\mathcal{VEHR}$ functionals at equal length metrics to obtain the decomposition presented in Theorem 3.3.2. In fact, we prove a more general lemma which will be used later.

**Lemma 3.4.** Let $\mathcal{F}$ be either $\mathcal{LEHR}$ or $\mathcal{VEHR}$. For the length variation:

\[
\ell(t) = (\ell_{12}(t), \ell_{13}(t), \ell_{14}(t), \ell_{23}(t), \ell_{24}(t), \ell_{34}(t)),
\]

\[
= (t, 1, 1, 1, 1, t),
\]

define the following quantities

\[
a_F(t) = \left. \frac{\partial^2 \mathcal{F}}{\partial \ell_{12} \partial \ell_{13}} \right|_{\ell(t)},
\]

\[
b_F(t) = \left. \frac{\partial^2 \mathcal{F}}{\partial \ell_{12}^2} \right|_{\ell(t)},
\]

\[
c_F(t) = \left. \frac{\partial^2 \mathcal{F}}{\partial \ell_{12} \partial \ell_{14}} \right|_{\ell(t)},
\]

\[
d_F(t) = \left. \frac{\partial^2 \mathcal{F}}{\partial \ell_{13}^2} \right|_{\ell(t)},
\]

\[
e_F(t) = \left. \frac{\partial^2 \mathcal{F}}{\partial \ell_{13} \partial \ell_{14}} \right|_{\ell(t)},
\]

\[
f_F(t) = \left. \frac{\partial^2 \mathcal{F}}{\partial \ell_{14}^2} \right|_{\ell(t)},
\]

\[
a_F(t),
\]

Then the eigenspaces and eigenvalues of the Hessian of $\mathcal{F}$ are

| eigenspace | spanning vectors | eigenvalues |
|------------|-----------------|-------------|
| $V_{\lambda_1}$ | $(1, 0, 0, 0, 0, -1)$ | $b_F - c_F$ |
| $V_{\lambda_2}$ | $(0, 1, 0, 0, -1, 0)$ | $d_F - f_F$ |
| $V_{\lambda_3}$ | $(0, 1, -1, 0, 0, 0)$ | $-4c_F - 2t a_F$ |
| $V_{\lambda_4}$ | \(\left(\frac{t}{4}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}\right)\) | \((-2t + \frac{t}{4}) a_F\) |
| $V_0$ | $(t, 1, 1, 1, 1, t)$ | $0$ |

**Proof.** We see that the Hessian matrix is

\[
Hess(\mathcal{F})|_{\ell(t)} = \begin{bmatrix}
    b_F & a_F & a_F & a_F & a_F & e_F \\
    a_F & d_F & c_F & c_F & f_F & a_F \\
    a_F & c_F & d_F & f_F & c_F & a_F \\
    a_F & c_F & f_F & d_F & c_F & a_F \\
    a_F & f_F & c_F & c_F & d_F & a_F \\
    e_F & a_F & a_F & a_F & a_F & b_F
\end{bmatrix}.
\]

Note that for each choice of $\mathcal{F}$, we have $\mathcal{F}(c \ell(t)) = \mathcal{F}(\ell(t))$ for any scalar $c > 0$. Thus $\ell(t)$ is in the nullspace of the Hessian of $\mathcal{F}$. This implies the following two equalities:

\[
4a_F + 2c_F + tb_F = 0,
\]

\[
2c_F + d_F + f_F + 2ta_F = 0.
\]
One can then check the vectors in the statement of the lemma to confirm that they are eigenvectors with the corresponding eigenvalues. \(\square\)

**Proof of Theorem 3.3.2.** Let \(\ell_k\) denote the equal length metric with all lengths equal to \(k\). With the added symmetry of this length structure we get the following equalities:

\[
\begin{align*}
a_F &= \frac{\partial^2 F}{\partial t_{12} \partial t_{13}} |_{\ell_k} = \frac{\partial^2 F}{\partial t_{13} \partial t_{14}} |_{\ell_k} = c_F, \\
b_F &= \frac{\partial^2 F}{\partial t_{12}^2} |_{\ell_k} = \frac{\partial^2 F}{\partial t_{13}^2} |_{\ell_k} = d_F, \\
e_F &= \frac{\partial^2 F}{\partial t_{12} \partial t_{14}^2} |_{\ell_k} = \frac{\partial^2 F}{\partial t_{13} \partial t_{14}^2} |_{\ell_k} = f_F.
\end{align*}
\]

By Lemma 3.4, the eigenspaces and eigenvalues are:

| eigenspace \(V_{\lambda_1}\) | spanning vectors | eigenvalues |
|-----------------------------|------------------|------------|
| \(V_{\lambda_1}\)          | \((1, 0, 0, 0, 0, -1)\) | \(b_F - c_F\) |
|                             | \((0, 1, 0, 0, -1, 0)\) |            |
|                             | \((0, 0, 1, -1, 0, 0)\) |            |

| eigenspace \(V_{\lambda_2}\) | spanning vectors | eigenvalues |
|-----------------------------|------------------|------------|
| \(V_{\lambda_2}\)          | \((0, 1, -1, -1, 1, 0)\) | \(-6a_F\) |
|                             | \((1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 1)\) |           |
| \(V_0\)                    | \((1,1,1,1,1,1)\) | \(0\) |

A calculation yields the following:

\[
\begin{align*}
a_{\mathcal{LEHR}} &= \frac{\sqrt{2}}{\pi}, & a_{\mathcal{VEHR}} &= 2^{\frac{3}{2}}3^{-\frac{3}{2}}k^{-2} \left(2\sqrt{2} - 7\pi + 7 \arccos \left(\frac{1}{3}\right)\right), \\
b_{\mathcal{LEHR}} &= \frac{-\sqrt{2}}{\pi}, & b_{\mathcal{VEHR}} &= 2^{\frac{3}{2}}3^{-\frac{3}{2}}k^{-2} \left(-4\sqrt{2} + 46\pi - 46 \arccos \left(\frac{1}{3}\right)\right), \\
e_{\mathcal{LEHR}} &= \frac{-\sqrt{2}}{3\pi}, & e_{\mathcal{VEHR}} &= 2^{\frac{3}{2}}3^{-\frac{3}{2}}k^{-2} \left(-6\sqrt{2} + 5\pi - 5 \arccos \left(\frac{1}{3}\right)\right).
\end{align*}
\]

\(\square\)

That the equal length metric is a local minimum of \(\mathcal{VEHR}\) (Theorem 3.3.4) follows as a simple corollary, since the nonzero eigenvalues of \(\mathcal{VEHR}\) are both positive. Also, note that the eigenspaces of the \(\mathcal{LEHR}\) and \(\mathcal{VEHR}\) functionals are the same.

We will now show that, on the double tetrahedron, the only critical points of the \(\mathcal{LEHR}\) functional are the equal length metrics. Thus the \(\mathcal{L}\)-Einstein metrics are unique. The fact that these critical points are saddle points follows directly from the eigenvalues of the Hessian.

**Proof of Theorem 3.3.3.** By Proposition 3.2, a critical point of \(\mathcal{LEHR}\) on the double tetrahedron satisfies

\[
2\pi - 2\beta_{ij} = \lambda \text{ for all } \{i, j\} \in E,
\]

where \(\lambda = \frac{\mathcal{EH}}{\mathcal{L}}\). Thus,

\[
\beta_{ij} = \frac{2\pi - \lambda}{2} \text{ for all } \{i, j\} \in E.
\]

Equation (3.3) implies that all dihedral angles are equal. Since all dihedral angles are equal, the face angles will necessarily all be equal, since the spherical cosine law.
shows that the face angles are determined by the dihedral angles. The faces are thus all equilateral, and hence the generating tetrahedron has all edge lengths equal. Therefore, the critical points of $\mathcal{LEHR}$ occur only at equal length metrics.

Remark 3.5. An immediate consequence of Theorems 3.3.2 and 3.3.3 is that the global extrema of the $\mathcal{LEHR}$ functional occur on the boundary of $\text{met}_L$, i.e. on degenerate (zero volume) length structures.

Despite the fact that the equal length metrics are local minima of the $\mathcal{VEHR}$ functional, we cannot conclude that they are global minima since the Hessian of the $\mathcal{VEHR}$ functional is not globally positive semidefinite.

Proposition 3.6. Consider the one-parameter family of (admissible) length structures given by (3.2) for $t \in [1, \sqrt{2})$. There is a constant $t_*$ ($\approx 1.26836$) such that for $t < t_*$, the Hessian of $\mathcal{VEHR}$ is positive semidefinite with a one-dimensional nullspace, and for $t > t_*$, the Hessian has a mixed signature. For $t = t_*$, the nullspace is two-dimensional consisting of a scaling direction and an additional eigenvector $v = (0, 1, -1, -1, 1, 0)$.

Proof. Using the computation of the eigenvalues in Lemma 3.4 and Theorem 3.3.2, one sees that the eigenvalue associated to the eigenvector $v = (0, 1, -1, -1, 1, 0)$ can be continuously expressed as

$$\lambda_v^{\mathcal{VEHR}}(t) = -4c_F(t) - 2ta_F(t),$$

with the following value at $t = 1$:

$$\lambda_v^{\mathcal{VEHR}}(1) = 2\sqrt{3} \left(7\pi - 2\sqrt{2} - 7 \arccos \left(\frac{1}{3}\right)\right),$$

$$\approx 34.145.$$  

However,

$$\lambda_v^{\mathcal{VEHR}}(1.3) \approx -5.97897,$$

thus by continuity there exists a value $t_*$ such that $\lambda_v^{\mathcal{VEHR}}(t_*) = 0$. Using Newton’s method $t_*$ can be approximated as

$$t_* \approx 1.26836.$$  

In addition, one might ask if the functional $\mathcal{VEHR}$ stays bounded on $\text{met}_V$. It does not.

Proposition 3.7. The $\mathcal{VEHR}$ functional is unbounded on the double tetrahedron.  

Proof. This follows from the more general Proposition 5.2.

4. Conformal variations

In this section, we describe the behavior of the Einstein-Hilbert-Regge functional within a conformal class. We will describe the general set-up, and then we will specialize to the setting of the double tetrahedron.
4.1. Introduction to discrete conformal structures. We will consider a certain conformal structure that has been studied in [14], [20], [23]. For the following, let $V^*$ denote the real-valued functions on the vertices.

**Definition 4.1.** Let $\{L_e\}_{e \in E}$ be such that $(M, T, L)$ is a piecewise flat manifold. Let $U \subset V^*$ be an open set. A conformal structure is a map $U \to \text{met}(M, T)$ determined by

$$\ell_e(f) = \exp \left[ \frac{1}{2} (f_v + f_{v'}) \right] L_e,$$

where $e$ is the edge between $v$ and $v'$. The conformal class is the image of $U$ in $\text{met}(M, T)$, and it is entirely determined by $L$. A conformal variation $f(t)$ is a smooth curve $(-\varepsilon, \varepsilon) \to U$ for small $\varepsilon > 0$, and it induces a conformal variation of metrics $\ell(f(t))$.

**Remark 4.2.** There is a more general notion of conformal structure on piecewise flat manifolds that is described in [14]. The conformal structure described here is called the perpendicular bisector conformal structure in that paper.

A useful fact about the conformal structure is that the length cross ratio is a conformal invariant.

**Proposition 4.3 ([26]).** For a fixed conformal class and for any tetrahedron $\{1234\}$ in the triangulation, there exist constants $c_{13}$ and $c_{14}$ depending only on the conformal class such that

$$\frac{\ell_{12}\ell_{34}}{\ell_{14}\ell_{23}} = c_{13},$$

$$\frac{\ell_{12}\ell_{34}}{\ell_{13}\ell_{24}} = c_{14}.$$

In particular, $c_{13} = \frac{\ell_{12}\ell_{34}}{\ell_{14}\ell_{23}}$, and $c_{14} = \frac{\ell_{12}\ell_{34}}{\ell_{13}\ell_{24}}$.

**Proof.** If one considers the length cross ratio, $\frac{\ell_{12}\ell_{34}}{\ell_{14}\ell_{23}}$, one can easily see that the conformal factors cancel, and we are left with $\frac{\ell_{12}\ell_{34}}{\ell_{14}\ell_{23}}$, which depends only on the choice of conformal class. The same idea works for the other cross ratio. \qed

The conformal structure gives rise to certain geometric structures within each tetrahedron (or, more precisely, within the tangent space of each tetrahedron). Each face $f$ has a circumcenter $c_f$ within its tangent plane. Each tetrahedron $t$ has a circumcenter $c_t$. The segment from $c_t$ to $c_f$ (in the tangent space of $t$) is orthogonal to the tangent plane of the face $f$, and this segment has a signed height $h_f$. Within the tangent plane to each face, the segment from $c_f$ to the midpoint of one of its edges is orthogonal to that edge, and it has a signed height $h_e$. Both $h_f$ and $h_e$ can be computed explicitly (see [13]).

We can use the results from §3 to show the following theorem.

**Theorem 4.4.** At an equal length metric, both the $\mathcal{LEH}_R$ and $\mathcal{VEH}_R$ functionals are convex within a conformal class.

**Proof.** We need only show that the conformal variations at an equal length metric lie in the nonnegative eigenspaces. Note that the conformal directions are spanned by

$$\frac{\partial}{\partial f_1} = \frac{\ell_{12}}{2} \frac{\partial}{\partial \ell_{12}} + \frac{\ell_{13}}{2} \frac{\partial}{\partial \ell_{13}} + \frac{\ell_{14}}{2} \frac{\partial}{\partial \ell_{14}}.$$
etc. At equal length metrics, this corresponds to vectors (1, 1, 0, 0, 0), (1, 0, 1, 1, 0), (0, 1, 0, 1, 0, 1), and (0, 0, 1, 0, 1) in the notation of Theorem 3.3. One can verify that the conformal variations lie in the span of the nonnegative eigenspaces of the Hessian of \( \mathcal{L}_{\mathcal{EHR}} \). The argument for \( \mathcal{V}_{\mathcal{EHR}} \) is trivial since the equal length metrics are local minima in the space of all metrics. □

### 4.2. Constant Scalar Curvature Metrics

In this section, we consider the constant scalar curvature metrics (Definition 4.11) on the double tetrahedron. We show existence of constant scalar curvature metrics in each conformal class, and we also find interesting geometric examples of these metrics. Additionally, we will show that constant scalar curvature metrics in a given conformal class are not necessarily unique.

The main results in this section are the following.

**Theorem 4.5.** (1) An equihedral metric on the double tetrahedron has constant scalar curvature with respect to the \( \mathcal{L}_{\mathcal{EHR}} \) functional.

(2) An equihedral metric on the double tetrahedron has constant scalar curvature with respect to the \( \mathcal{V}_{\mathcal{EHR}} \) functional.

(3) There is a unique equihedral metric up to scaling in every conformal class on the double tetrahedron.

(4) With respect to the \( \mathcal{L}_{\mathcal{EHR}} \) functional, constant scalar curvature metrics are not necessarily unique in a conformal class on the double tetrahedron.

First, we will define a notion of vertex curvature which will lead to a definition of constant scalar curvature metrics.

**Definition 4.6.** The vertex curvature of a vertex \( v \) is

\[ K_v = \frac{1}{2} \sum_{e \uparrow v} K_e, \]

where the sum is over all edges containing vertex \( v \).

**Remark 4.7.** \( \mathcal{EHR} \) can be written in terms of vertex curvatures as

\[ \mathcal{EHR}(M, T, \ell) = \sum_{v \in V} K_v. \]

**Remark 4.8.** As described in [14], the formula for vertex curvature depends on the choice of conformal structure, which comes from the variation formula for \( \mathcal{EHR} \).

One can immediately see the connection between discrete conformal variations and those in the smooth category when one considers the following variation formulas.

**Lemma 4.9.** For a conformal variation \( f(t) \) of a three-dimensional, piecewise flat manifold \((M, T, \ell)\), we have

\[ \frac{\partial}{\partial f_v} \mathcal{EHR}(M, T, \ell(f)) = K_v, \]

\[ \frac{\partial}{\partial f_v} \mathcal{L}_{\mathcal{EHR}}(M, T, \ell(f)) = \mathcal{L}^{-1}(K_v - (\mathcal{L}_{\mathcal{EHR}})L_v), \]

\[ \frac{\partial}{\partial f_v} \mathcal{V}_{\mathcal{EHR}}(M, T, \ell(f)) = \mathcal{V}^{-\frac{1}{2}}(K_v - \frac{\mathcal{EHR}}{3\mathcal{V}} V_v), \]

where \( L_v = \frac{1}{2} \sum_{e > v} \ell_e, V_v = \frac{1}{3} \sum_{t > f > v} h_{f < t} A_f \), and \( A_f \) is the area of face \( f \).
Proof. These formulas are mostly in [14], and the rest follow easily. 

Remark 4.10. In the previous formulas, \( L_v = \frac{\partial \mathcal{L}}{\partial f_v} \) and \( V_v = \frac{\partial \mathcal{V}}{\partial f_v} \). Also note that \( \sum_{v \in V} L_v = \mathcal{L}, \) and \( \sum_{v \in V} V_v = 3\mathcal{V} \).

These variation formulas motivate the following notion of constant scalar curvature in the discrete setting.

Definition 4.11. A three-dimensional piecewise flat manifold \((M, T, \ell)\) has constant scalar curvature if it is a critical point of one of the normalized \(\mathcal{EHR}\) functionals with respect to conformal variations. We say \((M, T, \ell)\) has constant \(L\)-scalar curvature \(\lambda_L\) if

\[
K_v = \lambda_L L_v,
\]

for all \(v \in V\). Here \(\lambda_L = \mathcal{L}\mathcal{EHR}\). We say \((M, T, \ell)\) has constant \(V\)-scalar curvature \(\lambda_V\) if

\[
K_v = \lambda_V V_v,
\]

for all \(v \in V\). Here \(\lambda_V = \frac{\mathcal{EHR}}{3V}\).

The following proposition is essentially shown in [14].

Proposition 4.12. If \((M, T, \ell)\) is a three-dimensional piecewise flat manifold which is \(L\)-Einstein (\(V\)-Einstein), then it has constant \(L\)-scalar curvature (\(V\)-scalar curvature).

Now we will define the notion of an equihedral metric. We will also collect the necessary pieces to prove that the equihedral metrics have both constant \(L\)-scalar curvature and constant \(V\)-scalar curvature.

Definition 4.13. Let \(\sigma^3\) be a tetrahedron such that all faces are congruent. Then \(\sigma^3\) is called an equihedral tetrahedron.

Theorem 4.14. Let \(\sigma^3\) be a tetrahedron. One can show that the following conditions are equivalent:

1. The tetrahedron \(\sigma^3\) is equihedral.
2. The opposite edges have equal lengths.
3. The opposite edges have equal dihedral angles.
4. The inscribed sphere touches each face at its circumcenter.
5. The center of the inscribed sphere corresponds to the circumcenter of the tetrahedron.

In fact, according to [4], there are over 100 equivalent conditions that characterize equihedral tetrahedra. We note that in the literature, these tetrahedra are sometimes called “isosceles tetrahedra” or “equifacial tetrahedra.” We will say that a piecewise flat metric \(\ell\) on the double tetrahedron is equihedral if its generating tetrahedron satisfies any of the equivalent conditions in Theorem 4.14.

Lemma 4.15. In an equihedral tetrahedron, \(L_v = L_{v'}\) for all pairs of vertices \(v, v' \in V\).

Proof. Consider, for example, \(L_1 = \frac{1}{2}(\ell_{12} + \ell_{13} + \ell_{14})\). Using Theorem 4.12, one sees that this sum is equal to, say, \(\frac{1}{2}(\ell_{21} + \ell_{24} + \ell_{23}) = L_2\). A similar argument holds for the other vertices. 

□
Lemma 4.16. In an equihedral tetrahedron, the \( h_f = h_{f'} \) for all faces \( f, f' \in F \), and hence the \( V_v = V_{v'} \) for all vertices \( v, v' \in V \).

Proof. The definition of equihedral tetrahedron implies that the \( A_f \) are equal for all faces. Since the geometric center \( c_f \), corresponds to the circumcenter of \( f \), Theorem 4.14.4 and 4.14.5 combine to show that the \( h_f \) are all equal. Since, \( V_v = \frac{1}{3} \sum_{t > f > v} h_{f < t} A_f \), the \( V_v \) are equal as well. \( \square \)

We will now show that equihedral metrics have constant scalar curvature in the sense of both (4.4) and (4.5).

Proof of Theorem 4.5.1. Let \( \ell \) be an equihedral metric on the double tetrahedron. Label the generating tetrahedron 1234. We would like to show that \( \ell \) has constant \( \mathcal{L} \)-scalar curvature \( \lambda \); i.e., that \( \ell \) satisfies \( \text{(4.4)} \). By Theorem 4.14.2 and 4.14.3, an equihedral tetrahedron has opposite edge lengths and opposite dihedral angles equal. This implies that \( \ell_{12} = \ell_{34}, \ell_{13} = \ell_{24}, \ell_{14} = \ell_{23}, \beta_{12} = \beta_{34}, \beta_{13} = \beta_{24}, \) and \( \beta_{14} = \beta_{23} \). One can easily check that this implies that the \( K_v \) are equal for all \( v \in V \); hence \( 4K_v = \mathcal{EHR} \). By Lemma 4.15, \( 4L_v = \mathcal{L} \). Thus, \( \frac{K_v}{L_v} = \frac{\mathcal{EHR}}{\mathcal{L}} \) as required. \( \square \)

Proof of Theorem 4.5.2. Let \( \ell \) be an equihedral metric on the double tetrahedron. We would like to show that \( \ell \) has constant \( \mathcal{V} \)-scalar curvature \( \lambda \); i.e., that \( \ell \) satisfies (4.5). As above, all of the \( K_v \) are equal. Then for each \( v, K_v = \frac{1}{4} \mathcal{EHR} \). Since all of the \( V_v \) are equal by Lemma 4.16, we have that \( V_v = \frac{3}{4} \mathcal{V} \) for every \( v \in V \). Then the ratio \( \frac{K_v}{V_v} = \frac{\mathcal{EHR}}{3 \mathcal{V}} \) as required. \( \square \)

We now show that there is a one-parameter family of constant scalar curvature metrics in every conformal class by showing the existence of equihedral metrics in every conformal class.

Proof of Theorem 4.5.3. We begin by fixing a conformal class, \( \{ L_{ij} \} \). We would like to solve the following system of equations \( \ell_{ij} = \ell_{kl}, \ell_{ik} = \ell_{jl}, \) and \( \ell_{il} = \ell_{jk} \) in our given conformal class. Using the conformal invariants from Proposition 4.4 one obtains

\[
\begin{align*}
\ell_{ij} &= \sqrt{\frac{L_{ij} L_{kl}}{L_{ij} L_{kl}}} e^{\frac{L_{ij} L_{kl}}{L_{ij} L_{kl}}} \\
\ell_{ik} &= \sqrt{\frac{L_{ik} L_{jl}}{L_{ik} L_{jl}}} e^{\frac{L_{ik} L_{jl}}{L_{ik} L_{jl}}} \\
\ell_{il} &= \sqrt{\frac{L_{il} L_{jk}}{L_{il} L_{jk}}} e^{\frac{L_{il} L_{jk}}{L_{il} L_{jk}}} \\
\end{align*}
\]

We see that, modulo scaling, there is a unique equihedral metric in every conformal class as required. \( \square \)

A natural question is the uniqueness of constant scalar metrics in a conformal class. On the double tetrahedron, constant \( \mathcal{L} \)-scalar curvature metrics are not necessarily unique within a conformal class. Notice that the previous proof implies that such additional constant \( \mathcal{L} \)-scalar curvature metrics are not equihedral.
Proof of Theorem 4.5.4. There can be multiple constant $L$-scalar curvature metrics within a fixed conformal class. As a specific example of this, consider the conformal class given by

$$L_e = 1, \text{ for all } e \in E.$$ 

Constant scalar curvature metrics occur for the following conformal parameters:

- $F_a = (f_1, f_2, f_3, f_4) = (0, 0, 0, 0)$,
- $F_b = (f_1, f_2, f_3, f_4) \approx (-1.233, -1.233, 0, 0)$.

Notice that the equal length metric is given by $F_a$, and another non-equihedral metric is given by $F_b$. We conjecture that non-equihedral constant scalar curvature metrics exist in every conformal class. $\Box$

4.3. Convexity of $\mathcal{LEHR}$. In this section, we consider the convexity of the $\mathcal{LEHR}$ functional within a fixed conformal class. We have previously shown that at an equal length metric, the $\mathcal{LEHR}$ functional restricted to a conformal class is convex. Direct computation of the Hessian allows us to analyze the $\mathcal{LEHR}$ functional at equihedral metrics. The main results of this section are the following.

Theorem 4.17. 

1. The discrete Laplacian is negative semidefinite at equihedral metrics.
2. The $\mathcal{LEHR}$ functional is convex in a conformal class at some, but not all, equihedral metrics.

In order to analyze the $\mathcal{LEHR}$ functional in a conformal class, we will compute the Hessian at a constant scalar curvature metric. First we need the definition of dual length.

Definition 4.18. Let $(M, \mathcal{T}, \ell)$ be a three-dimensional piecewise flat manifold. The dual length $\ell^*_e$ is defined as

$$(4.6) \quad \ell^*_e = \sum_{t \succ e} \frac{1}{2}(h_{e < f} h_{f < t} + h_{e < f'} h_{f' < t}),$$

where $f$ and $f'$ are the faces of $t$ containing $e$.

Remark 4.19. Although we call $\ell^*_e$ a “dual length,” it is actually a (signed) area of a (dual) cell orthogonal to the edge $e$. For further details see [13].

Now we will compute the Hessian of the $\mathcal{LEHR}$ functional. Since we will only analyze the Hessian at constant scalar curvature metrics, we include only that formula. The more general formula follows in a straightforward fashion from the results presented in this paper and in [13].

Proposition 4.20. The Hessian of the $\mathcal{LEHR}$ functional at a constant $L$-scalar curvature metric is

$$\frac{\partial^2 \mathcal{LEHR}}{\partial f_e \partial f_{e'}} = -\frac{2}{L} \Delta_{vv'} + \frac{1}{L} N_{vv'},$$

where

$$\Delta_{vv'} = \begin{cases} 
\ell^*_e & \text{if } e = vv', \\
- \sum_{v < v'} \ell^*_e & \text{if } v = v', \\
0 & \text{otherwise}
\end{cases}$$
is the Laplacian matrix, and
\[
N_{vv'} = \begin{cases} 
\frac{1}{4}(K_e - (\mathcal{LHR})\ell_e) & \text{if } e = vv', \\
\frac{1}{2}(K_v - (\mathcal{LHR})L_v) & \text{if } v = v', \\
0 & \text{otherwise.}
\end{cases}
\]

**Corollary 4.21.** At an equihedral metric,
\[
N_{vv'} = \begin{cases} 
\frac{1}{4}(K_e - K_v L_v) & \text{if } e = vv', \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** We have shown in the proofs of Theorem 4.5.2 and Lemma 4.15 that at an equihedral metric, the vertex curvatures and the \(L_v\) are constant for all \(v \in V\). Then \(K_v = \frac{1}{4}\mathcal{LHR}\), and \(L_v = \frac{1}{4}L\). So at an equihedral metric, \((\mathcal{LHR})\ell_e = -\frac{K_v L_v}{\ell_e}\) if \(e = vv'\), and \((\mathcal{LHR})L_v = K_v\) if \(v = v'\). These formulas give the desired result. \(\square\)

We would like to analyze the convexity of the \(\mathcal{LHR}\) functional at equihedral metrics. First, we will show that the Laplacian matrix \(\Delta\) is negative semidefinite at equihedral metrics.

Recall that a matrix, \(A\), is said to be **diagonally dominant** if \(|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|\), where \(a_{ij}\) denotes the entry in the \(i^{th}\) row and the \(j^{th}\) column. The following theorem is well-known.

**Theorem 4.22.** A Hermitian, diagonally dominant matrix with real nonnegative diagonal entries is positive semidefinite.

**Proposition 4.23.** The Laplacian \(\Delta\) is negative semidefinite if the dual lengths \(\ell_e^*\) are nonnegative for a given metric.

**Proof.** Notice that if \(\ell_e^* \geq 0\), then \(\Delta\) is diagonally dominant, and so by Theorem 4.22 \(\Delta\) is negative semidefinite. \(\square\)

**Remark 4.24.** The set of metrics for which \(\Delta\) is negative semidefinite forms an open set containing the equal length metrics, as does the set of metrics for which \(\ell_e^*\) are positive. It is possible that our analogies for Riemannian geometry depend on one of these two conditions, and that other metrics correspond to some sort of more general infinitesimal geometry such as sub-Riemannian geometry.

We will now show that the Laplacian is negative semidefinite at equihedral metrics.

**Lemma 4.25.** In an equihedral tetrahedron, all faces are acute triangles. Thus \(h_{e<} f \geq 0\) for each edge \(e\) and face \(f\).

**Proof.** The proof that the faces of an equihedral tetrahedron are acute triangles follows that suggested in [18]. Let 1234 be an equihedral tetrahedron such that \(\ell_{12} = \ell_{34} = a, \ell_{13} = \ell_{24} = b,\) and \(\ell_{14} = \ell_{23} = c\). Flatten the tetrahedron by fixing \(a\) and \(c\); allow \(\ell_{24}\) to grow until faces 123 and 134 become coplanar, say when \(\ell_{24} = d\). One obtains the parallelogram 1234 with sides of lengths \(a\) and \(c\) and with diagonals of lengths \(b\) and \(d\). Since \(d > b\), the law of cosines implies that the angle at vertex 2 < 123 is acute. A similar argument can be used to show all face angles are acute.
It is well-known that the circumcenter of an acute triangle lies in the interior of the triangle, so \( h_{e<f} \geq 0 \) as required.

**Lemma 4.26.** For any equihedral metric on \( DT \), \( \ell^*_e \geq 0 \) for every edge.

*Proof.* Since \( \sum_{f \in F} h_f A_f = 3V \), both \( V \) and \( A_f \) are positive, and \( h_f \) is the same for each face by Lemma 4.16, we must have that \( h_f \geq 0 \) for all \( f \in F \). It follows from Proposition 4.25 that \( h_{e<f} \geq 0 \). Since \( \ell^*_e = h_{e<f} h_f + h_{e<f} h_f' \), the result follows.

*Proof of Theorem 4.17.1.* That the Laplacian is negative semidefinite at equihedral metrics follows directly from Proposition 4.23 and Lemma 4.26.

We will show below that many, but not all, equihedral metrics are local minima of \( LEHR \). Note that we are still unable to find which equihedral metrics are local minima of \( VEHR \) due to the influence of the normalization factor on the Hessian.

*Proof of Theorem 4.17.2.* We will show that the \( LEHR \) functional is convex at some but not all equihedral metrics. Consider the one parameter family of length vectors (3.2), which is a variation through equihedral metrics. Note that along this family, each vertex looks the same, and so we can write the Hessian of \( LEHR \) restricted to the conformal class as

\[
Hess (LEHR)|_{\ell(t)} = \begin{bmatrix}
    b & c & a & a \\
    c & b & a & a \\
    a & a & b & c \\
    a & a & c & b
\end{bmatrix}
\]

where

\[
a = \frac{\partial^2 LEHR}{\partial f_1 \partial f_3}, \quad b = \frac{\partial^2 LEHR}{\partial f_1^2}, \quad c = \frac{\partial^2 LEHR}{\partial f_1 \partial f_2},
\]

and we always calculate at \( f_1 = f_2 = f_3 = f_4 = 0 \). Furthermore, we know that \( LEHR \) is invariant under uniform scaling, so \( (1, 1, 1, 1) \) is a zero eigenvalue, which implies that \( 2a + b + c = 0 \). One can now check that the eigenspaces decompose as follows:

| eigenspace | spanning vectors | eigenvalues |
|------------|-----------------|-------------|
| \( V_{\lambda_1} \) | \((1, -1, 0, 0)\) | \(-2a - 2c\), |
| \( V_{\lambda_2} \) | \((0, 0, 1, -1)\) | \(-4a\), |
| \( V_0 \) | \((1, 1, 1, 1)\) | \(0\). |

One can show that at the equal length metric \( \ell_1 \) where all lengths equal 1, the eigenvalues of the Hessian of \( LEHR \) are \( \left( \frac{4\sqrt{2}}{9}, \frac{4\sqrt{2}}{9}, \frac{4\sqrt{2}}{9}, 0 \right) \). Since the eigenvalues are continuous and the zero eigenvalue persists, the Hessian of \( LEHR \) is positive semidefinite in a neighborhood of \( \ell_1 \). For \( t = 1.35 \), the eigenvalue \( \lambda_2 \approx -0.238 \), so by continuity of the eigenvalues, it must be zero for some \( t \). A computer algebra package allows us to approximate this value to be \( t \approx 1.31471 \). 

\[ \square \]
5. The Yamabe invariant

As in the smooth case, one can consider invariants based on the Einstein-Hilbert functional. In the smooth setting, one considers the Yamabe constant of a conformal class of Riemannian metrics. Recall that if $g$ is a Riemannian metric on a manifold $M$, the conformal class $[g]$ is defined by
\[ [g] = \{ e^f g : f \in C^\infty(M) \} . \]

One then considers the Yamabe constant $Y(M, [g])$ to be
\[ Y(M, [g]) = \inf \{ \mathcal{NEH}(M, g_0) : g_0 \in [g] \}. \]

One can show (see, e.g., [25]) that for any $M^n$ and $g$,
\[ Y(M^n, [g]) \leq Y(S^n, [g_{\text{can}}]) \]
where $g_{\text{can}}$ is any round metric, and so it makes sense to consider the Yamabe invariant for $M$ (also called the sigma constant), namely
\[ Y(M^n) = \sup Y(M^n, [g]), \]
where the sup is over all conformal classes. Note that the Yamabe invariant has been computed for a number of manifolds, including the sphere, real projective space, sphere bundles over the circle, and hyperbolic manifolds ([1], [3], [7], [17], [24]).

One may consider similar questions for the Einstein-Hilbert-Regge functional. We now have two formulations of Yamabe constants based on different normalizations:

**Definition 5.1.** Let $[\hat{\ell}]$ denote the set of all metrics which are conformal to $\ell$, i.e.,
\[ [\hat{\ell}] = \{ \hat{\ell} \in \text{met} : \hat{\ell}(e) = \exp \left( \frac{1}{2} (f(v) + f(v')) \right) \ell(e) \text{ for all } e = vv' \}. \]

The $\mathcal{L}$-Yamabe constant $Y_C(M, \mathcal{T}, [\hat{\ell}])$ is defined to be
\[ Y_C(M, \mathcal{T}, [\hat{\ell}]) = \inf \{ \mathcal{LEHR}(M, \mathcal{T}, \hat{\ell}) : \hat{\ell} \in [\hat{\ell}] \}. \]

The $V$-Yamabe constant $Y_V(M, \mathcal{T}, [\hat{\ell}])$ is defined to be
\[ Y_V(M, \mathcal{T}, [\hat{\ell}]) = \inf \{ \mathcal{VEHR}(M, \mathcal{T}, \hat{\ell}) : \hat{\ell} \in [\hat{\ell}] \}. \]

The $\mathcal{L}$-Yamabe invariant $Y_C(M, \mathcal{T})$ is defined to be
\[ Y_C(M, \mathcal{T}) = \sup \{ Y_C(M, \mathcal{T}, [\hat{\ell}]) : \hat{\ell} \in \text{met} \} . \]

The $V$-Yamabe invariant $Y_V(M, \mathcal{T})$ is defined to be
\[ Y_V(M, \mathcal{T}) = \sup \{ Y_V(M, \mathcal{T}, [\hat{\ell}]) : \hat{\ell} \in \text{met} \} . \]

The $\mathcal{L}$-Yamabe invariant seems well-motivated on the double tetrahedron, since Theorem 3.3 indicates that at the unique $\mathcal{L}$-Einstein metric, with regard to the Hessian of the $\mathcal{LEHR}$, conformal directions span the eigenspaces corresponding to nonnegative eigenvalues and the orthogonal space is spanned by eigenspaces corresponding to negative eigenvalues. The definition is less well-motivated for the $V$-Yamabe invariant since the equal length metrics on DT are local minima for $\mathcal{VEHR}$. We still define the Yamabe invariant in analogy because it is known that for sufficiently fine triangulations, $\mathcal{EH}$ and $\mathcal{V}$ converge to $\mathcal{E}$ and volume.
(see [10]), and so for certain triangulations one would expect the definition to be meaningful.

There is additional motivation for these definitions in slightly different contexts. In particular, there are the following results for the corresponding boundary value problem where the boundary geometry is fixed. In [5], Bobenko and Izmestiev show that for a convex polyhedron with one interior vertex, the Hessian of $\mathcal{E}_{HR}$ has only one positive eigenvector with a one-dimensional eigenspace. Also, in [16], Izmestiev and Schlenker show that for any convex polyhedron, the eigenspace corresponding to positive eigenvectors has dimension at least as large as the number of interior vertices. These two results indicate that $\mathcal{E}_{HR}$ tends to be convex within a conformal class.

We now consider the well-posedness of computing the Yamabe invariants. It is an easy consequence of the definition of $\mathcal{L}\mathcal{E}_{HR}$ that

$$0 \leq \mathcal{L}\mathcal{E}_{HR} \leq 2\pi,$$

and so $Y_{\mathcal{L}}(\text{DT})$ certainly exists. We are not yet able to compute its value, however. In fact, although we have found a large class of constant $\mathcal{L}$-scalar curvature metrics, we are not even able to compute the $\mathcal{L}$-Yamabe constant for the conformal class containing the equal length metrics.

**Problem 1.** Does $Y_{\mathcal{L}}(\text{DT}, [\ell_k]) = \mathcal{L}\mathcal{E}_{HR}(\text{DT}, \ell_k)$, where $\ell_k$ is an equal length metric? Does $Y_{\mathcal{L}}(\text{DT}) = \mathcal{L}\mathcal{E}_{HR}(\text{DT}, \ell_k)$?

Normalization with respect to volume is more complicated. A natural question is whether $\mathcal{V}\mathcal{E}_{HR}$ is bounded in any sense. On the double tetrahedron, it must be bounded below by 0. For the Yamabe invariant to be meaningful, we would need the supremum of Yamabe constants to be bounded. One might simply ask if $\mathcal{V}\mathcal{E}_{HR}$ is bounded on the set of constant scalar curvature metrics (which are critical points to $\mathcal{V}\mathcal{E}_{HR}$ in a conformal class). It turns out this is not true.

**Proposition 5.2.** On the double tetrahedron, the $\mathcal{V}\mathcal{E}_{HR}$ functional is not bounded above on the set of constant scalar curvature metrics.

**Proof.** Consider the one-parameter family of equihedral, hence constant scalar curvature, metrics on the double tetrahedron given by (3.2). As $t$ increases to $\sqrt{2}$, the generating tetrahedron becomes a flat square with two diagonals. It follows that the volume $V(t)$ goes to zero, the dihedral angles $\beta_{12}$ and $\beta_{23}$ go to $\pi$, and the dihedral angles at all other edges go to zero. Thus $\mathcal{E}_{HR}$ approaches $8\pi$. Therefore, we can make the $\mathcal{V}\mathcal{E}_{HR}$ functional as large as we want simply by letting $t$ approach $\sqrt{2}$ from below. \qed

Note that Proposition 5.2 does not by itself imply that the Yamabe invariant of DT is infinity, since it is possible that there are smaller constant scalar curvature metrics than the ones considered.

**Problem 2.** Is $Y_{\mathcal{V}}(\text{DT}) = \infty$?

In conclusion, we will make some comments about computation of the Yamabe constants and Yamabe invariants for general triangulated piecewise flat manifolds. In general, we will need to prove a lower bound for $\mathcal{L}\mathcal{E}_{HR}$ or $\mathcal{V}\mathcal{E}_{HR}$ within a conformal class just to compute the Yamabe constant for that class, and then upper bounds to compute the Yamabe invariant. There are fairly easy bounds for
\( \mathcal{LEHR} \), but we will need an additional “fatness” criterion for \( \mathcal{VEHR} \). Note that fatness requirements are quite common in theorems in piecewise flat geometry (e.g., [10], [27]). The most natural fatness condition for the volume normalized functional is the following:

**Definition 5.3.** A piecewise flat manifold \((M, T, \ell)\) is \( \varepsilon \)-fat if

\[
(5.3) \quad \frac{V}{(\sum \ell_e)^3} \geq \varepsilon > 0.
\]

The idea is that the volume cannot become too small without the lengths becoming small. Notice that the fatness condition is satisfied if, for instance, there is lower bound for the volume of each tetrahedron in terms of its maximum edge length, though in this case \( \varepsilon \) would depend on the number of edges in \( T \).

The following theorem gives bounds on the \( \mathcal{LEHR} \) and \( \mathcal{VEHR} \) functionals.

**Proposition 5.4.** Let \((M, T, \ell)\) be a triangulated piecewise flat three-manifold and let \( D_M \) be the maximal edge degree (tetrahedra incident on an edge). Then

- For \( \mathcal{LEHR} \) \((M, T, \ell)\) we have
  \[
  2\pi - \pi D_M \leq \mathcal{LEHR} \leq 2\pi.
  \]

- If the triangulation is \( \varepsilon \)-fat, then there exists a constant \( C(D_M, \varepsilon) \) such that
  \[
  \mathcal{VEHR} \geq C
  \]

**Proof.** For \( \mathcal{LEHR} \), the estimate is:

\[
2\pi - \pi D_M \leq \sum \frac{2\pi - \sum \beta e,t \ell_e}{\sum e \ell_e} \leq 2\pi.
\]

For \( \mathcal{VEHR} \), we see that

\[
\mathcal{VEHR} = \sum \frac{2\pi - \sum \beta e,t \ell_e}{V_{1/3}} \geq \min \{0, 2\pi - \pi D_M\} \varepsilon^{-1/3}.
\]

\( \square \)

**Remark 5.5.** The condition of fatness in [10] is slightly different, stating that there is a constant \( \delta > 0 \) such that for each \( n \)-simplex \( \sigma^n \),

\[
\max \{\ell_e : e \in E\}^{-n} |\sigma^n| \geq \delta,
\]

where \( |\sigma^n| \) is the \( n \)-dimensional volume of \( \sigma \). If one assumes the condition from [10], then

\[
\delta \leq \frac{|\sigma|^3}{\max \{\ell_e : e \in E\}^{3}} \leq \frac{|E|^V}{(\sum \ell_e)^3},
\]

and so we can take \( \varepsilon = \delta/|E| \) in (5.3), and we get a bound

\[
\mathcal{VEHR} \geq \min \{0, 2\pi - \pi D_M\} |E|^{1/3} \delta^{-1/3}.
\]

Proposition 5.4 justifies taking the infimum of \( \mathcal{LEHR} \) or \( \mathcal{VEHR} \) over appropriate subsets of a conformal class. Note that the estimates depends on edge degree, so that it would not be useful if one wished to take a sequence of triangulations converging to a smooth manifold. This motivates the following problem.

**Problem 3.** Within a discrete conformal class, is \( \mathcal{LEHR} \) or \( \mathcal{VEHR} \) bounded below by a constant independent of the triangulation?
Note that in the smooth case, one can essentially use a reformulation of $N\mathcal{E}\mathcal{H}$ and a Hölder estimate to prove that such a lower bound exists (see, e.g., [25, Chapter V]).

Recall that for DT, the known $\mathcal{V}$-Einstein metric is a local minimum in $\text{met}$. A natural question is whether there are any negative eigenvalues at all for the Hessian of $\mathcal{V}\mathcal{E}\mathcal{H}\mathcal{R}$ at a $\mathcal{V}$-Einstein metric; in other words, is there a necessity for maximizing to find a critical point. We have found the following:

**Theorem 5.6.** There is a triangulation of the three-sphere which admits a family of constant scalar curvature metrics such that the maximum of $\mathcal{L}\mathcal{E}\mathcal{H}\mathcal{R}$ over that family is a $\mathcal{L}$-Einstein metric and the maximum of $\mathcal{V}\mathcal{E}\mathcal{H}\mathcal{R}$ over that family is a $\mathcal{V}$-Einstein metric.

**Proof (sketch).** This example was first suggested to us by J. Weeks. One can consider the 600-cell, which is the boundary of a regular polytope in $\mathbb{R}^4$. It can be given a metric such that every tetrahedron is isometric to the other. The combinatorics are such that every vertex looks like every other, and it is an easy consequence that such a metric must have constant scalar curvature in either normalization. This gives a six parameter family of constant scalar curvature metrics on the 600-cell. One can then check that this family has a maximum when all lengths are the same, which gives an Einstein metric. More details will appear in [8]. □

Theorem 5.6 indicates that there are positive and negative eigenvalues for $\mathcal{L}\mathcal{E}\mathcal{H}\mathcal{R}$ and $\mathcal{V}\mathcal{E}\mathcal{H}\mathcal{R}$ at Einstein metrics. A major problem is to classify these directions.

**Problem 4.** Is there a geometric way to classify the positive, negative, and zero eigenspaces for the Hessian of $\mathcal{L}\mathcal{E}\mathcal{H}\mathcal{R}$ or $\mathcal{V}\mathcal{E}\mathcal{H}\mathcal{R}$? Do we need to restrict to certain classes of triangulations (e.g., with sufficient fatness)?

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(Daniel Champion) UNIVERSITY OF ARIZONA, TUCSON AZ, 85721 E-mail address: champion@math.arizona.edu

(David Glickenstein) UNIVERSITY OF ARIZONA, TUCSON AZ, 85721 E-mail address: glickenstein@math.arizona.edu

(Andrea Young) UNIVERSITY OF ARIZONA, TUCSON AZ, 85721 E-mail address: ayoung@math.arizona.edu