Black body radiation in a model universe with large extra dimensions and quantum gravity effects

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Abstract. We analyze the problem of black body radiation in a model universe with large extra dimensions where quantum gravity effects are taken into account through modified dispersion relations. In this context, modified form of Planck distribution, Jeans number, equipartition theorem, spectral energy density, Stefan-Boltzmann law and Wien’s law are found and the corresponding results are interpreted. As a generic feature, the correction terms are temperature dependent. Then, the entropy and specific heat of black body radiation are obtained in this setup. Finally, modified form of Debye law and Dulong-Petit law are obtained and departure from standard results are explained.

Key Words: Modified Dispersion Relation, Black Body Radiation, Large Extra Dimensions

PACS numbers: 04.60.-m, 05.70.Ce, 51.30.+i

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1. Introduction

String theory as a possible candidate for quantum gravity proposal implies existence of extra spacetime dimensions. In fact, existence of possible extra spatial dimensions has been a principal key to address the long-lived Hierarchy problem [1]-[2]. The first attempt to adopt extra dimensions in theoretical physics was made by Kaluza and Klein to unify gravity with electromagnetism in a 5-dimensional spacetime model. Currently the idea of having extra dimensions has been widely accepted by theoretical physics community and there is also hope to detect some footprints of these extra dimensions in experiments such as the LHC. After revolutionary report on current positively accelerating phase of the universe expansion [3], possible modification of General Relativity has been in the center of some important research programs. One may modify the geometric part of the Einstein’s equations by allowing to have extra spatial dimensions in the framework of braneworld scenarios [4]. These attempts have been applied successfully to explain the issue of dark energy and interpretation of new astrophysical data. These are just parts of possible motivation to introduce extra dimensional scenarios in theoretical physics. Nevertheless, the presence of extra dimensions would remain concealed somehow, may be because the extra dimensions are compactified to small radius with the size around Planck scale. Extra dimensional scenarios open also new perspectives, at least on phenomenological ground, to quantum gravity proposal. A common feature of all approaches to quantum gravity, such as string theory, noncommutative geometry, loop quantum gravity and Doubly Special Relativity, is modification of the standard dispersion relation (see for instance [5], [6], [7] and references therein). In fact, in the study of loop quantum gravity, Doubly Special Relativity and models based on non-commutative geometry there has been strong indication to modify the standard energy-momentum dispersion relation. In this framework issues such as possible violation of the Lorentz invariance in quantum gravity regime are studied too (see [8] and references therein). Recently modified dispersion relations are formulated in model universes with extra dimensions by focusing on black hole thermodynamics in these models [9]. Here we focus on another important issue: the black body radiation in a model universe with large extra dimensions and by incorporating possible quantum gravity effects encoded in modified dispersion relations. A black body is an ideal body which allows the whole of the incident radiation to pass into itself and absorbs within itself this whole incident radiation. One cannot ignore the significance of study of black body spectrum and its possible modification since it could be as a reliable source to give some information about the remnant of early universe radiation which could be detectable today as cosmic background microwaves (CMB). In fact, the cosmic microwave background radiation observed today is the most perfect black body radiation ever observed in nature, with a temperature of about 2.725 K. It is a snapshot of the radiation at the time of decoupling between matter and radiation in the early stages of the universe evolution. Therefore, any modification of black body spectrum due to quantum gravity effects can open new
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windows to know more about quantum gravitational features of the early universe through study of CMB spectrum. This sort of investigation has been considered by some authors recently [10]. Also formulation of black body radiation in model universes with extra dimensions, but in the absence of quantum gravity effects, has been considered by some authors [11]. The crucial feature of our work is the combination of MDR (as a phenomenological outcome of quantum gravity proposal) with possible existence of extra dimensions in problem of black body radiation. This study is an attempt to take a small step toward a deeper understanding, at least phenomenologically, of quantum gravity in a model universe with large extra dimensions. In this framework we investigate the impact of modified dispersion relations (MDR) on black body spectrum in a model universe with large extra dimensions. We establish modified form of the Planck distribution, equipartition theorem, Stefan-Boltzmann law and Wien’s law in this setup. Also modified Debye law and modified Dulong-Petit law are obtained and the corresponding results are interpreted. Then we study entropy and specific heat of black body radiation in this framework. The important ingredient and generic feature of these modified quantities is that these modifications are themselves temperature-dependent. So, it seems that quantum gravity effects encoded in modified dispersion relations are temperature-dependent for black body radiation and also related issues. This feature may be related to the foamy structure of space-time at the quantum gravity level, and taking the quantum structure of spacetime to be as a granular media which has some interaction (through quantum fluctuation of the background metric) with the events which are happened in it. So, these effects can be attributed to some yet not well-understood features of quantum spacetime at the Planck scale. We note also that the assumption of black body in a solid box, to some extent, helps us to know more about the modified terms within MDR in vibrating solids. Therefore the modified heat capacity of solids would be achieved through Dulong-petit model in high temperature.

In which follows, we set $\hbar = k_B = c = 1$ for simplicity.

2. Black body radiation in a model universe with extra dimensions

In order to study a black body radiation and it’s features in a D-dimensional space (D spatial dimensions), we consider a conducting cavity with the shape of a D-dimensional cube of side $L$. This cavity is filled with electromagnetic radiation which is in thermal equilibrium with walls at temperature $T$ and is linked with the outside by a small hole. The electromagnetic radiation inside the cavity can be treated as standing waves. We are going to seek for the functional form of energy density for one-dimensional cavity that is placed in a D-dimensional universe (see Ref. [11] for a discussion about validity of this assumption). The most proper method for investigating the black body radiation is deriving the total number of modes in order to obtain the energy density. Let us write the components of the electric field in a cavity by choosing a system of orthogonal coordinates with origin at one of the cavity’s vertices. To satisfy boundary conditions,
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at the walls of the cavity the parallel components of the electric field must vanish. Taking these considerations into account, we have the following relation in terms of wave number

\[ n_i = \frac{k_i}{\pi} L \]  

(1)

where \( k_i \) is the component of wave vector and \( n_i = 1, 2, \ldots \) represents the possible modes of vibration. The total number of modes in one-dimensional space is given by

\[ N(k_i)dk_i = \frac{L}{\pi} dk_i \]  

(2)

which can be converted to the following form in a spacetime with \( D \) spatial dimensions

\[ N(k)dk = (D - 1) \frac{V}{2^D \pi^D} dV_k \]  

(3)

where \( V = L^D \) is the volume of the cavity and \( dV_k \) is the infinitesimal element of volume in the \( k \)-space [11]. Coefficient \((D - 1)\) is related to polarization states and the presence of \( 2^D \) in the denominator of equation (3) is due to the fact that we should consider only the positive part of the volume in \( k \)-space. The \( D \)-dimensional element of volume in terms of \( \nu \) is given by

\[ dV_\nu = 2 \frac{\pi^D}{\Gamma(D/2)} (2\pi)^D \nu^{(D-1)} d\nu \]  

(4)

By using equation (4), now equation (3) in \( \nu \)-space can be rewritten as follows

\[ N(\nu)d\nu = 2(D - 1)V \frac{\pi^D}{\Gamma(D/2)} \nu^{(D-1)} d\nu . \]  

(5)

This relation expresses the total number of modes with frequencies between \( \nu \) and \( \nu + d\nu \).

In the next section, by using the average energy per mode, we will construct the spectral energy density to explain black body radiation in the presence of quantum gravity effects in a spacetime with \( D \) spatial dimensions.

3. MDR-modified black body radiation in a model universe with \( D \) spatial dimensions

The modified dispersion relation in a model universe with \( D \) spatial dimensions can be written in the following form (see [5]-[9] for more details)

\[ p^2 = f(E, m; L_P) \simeq E^2 - \mu^2 + \alpha L_P^2 E^4 + \alpha' L_P^4 E^6 + O(L_P^6 E^8) , \]  

(6)

where \( L_P \), the Planck length in spacetime with \( D \) spatial dimensions, depends on spacetime dimensionality (see [9] for instance), and \( f \) is a function that gives the exact dispersion relation [5]. On the right hand side of equation(6) we have adopted a Taylor-series expansion for \( E \ll \frac{1}{L_P} \). The parameter \( \alpha_i \) is dimensionless and takes different
values depending on the details of quantum gravity candidates. The rest energy of the
particle is \( m \) which is different from the mass parameter \( \mu \); we expect \( \mu \neq m \) if the
coefficients \( \alpha_i \) do not all vanish [5].

Based on the analysis of the previous section, the number of modes in the momentum
interval between \( p \) and \( p + dp \), is given by

\[
N(p)dp = (D - 1)V \frac{2\pi p}{\Gamma(D/2)} p^{D-1} dp.
\]

(7)

To find the modified form of the total number of electromagnetic modes in MDR
framework, we start with differentiating and applying a Taylor expansion to equation
(6) to obtain

\[
dp = dE \left[ 1 + \frac{3}{2} \alpha L_p^2 E^2 + \left( -\frac{5}{8} \alpha^2 + \frac{5}{2} \alpha' \right) L_p^4 E^4 \right],
\]

(8)

where we have saved only terms up to the forth order of the Planck length and the rest
mass has been neglected. Substituting the expressions for \( p \) and \( dp \), we find

\[
N(E)dE = (D - 1)V \frac{2\pi E}{\Gamma(D/2)} \left( 1 + \alpha L_p^2 E^2 + \alpha' L_p^4 E^4 \right)^{(D-1)/2} \times
\]

\[
\left[ 1 + \frac{3}{2} \alpha L_p^2 E^2 + \left( -\frac{5}{8} \alpha^2 + \frac{5}{2} \alpha' \right) L_p^4 E^4 \right] dE
\]

(9)

This is the modified number of electromagnetic modes in a cavity and in a model universe
with \( D \) spatial dimensions. Note that the average energy per mode is independent on
the dimensionality of space and it can be given by (see for instance [11])

\[
\bar{E} = \frac{\nu}{e^{\nu T} - 1}.
\]

(10)

Then, the modified spectral energy density is

\[
\rho(\nu)d\nu = (D - 1) \frac{2\pi}{\Gamma(D/2)} \nu^D \left( \frac{1}{e^{\nu T} - 1} \right) \left\{ 1 + \left( \frac{3}{2} + \frac{(D-1)}{2} \right) \alpha L_p^2 \nu^2 + \right. 
\]

\[
\left. \left[ \alpha^2 \left( \frac{5}{8} + \frac{3}{2} \frac{(D-1)}{2} \right) + \alpha' \left( \frac{5}{2} + \frac{(D-1)}{2} \right) \right] L_p^4 \nu^4 \right\} d\nu
\]

(11)

This is the modified Planck distribution. The influence of modified dispersion relation
on spectral energy density in a model universe with large extra dimensions is obviously
considerable. Figure 1 shows the variation of the spectral energy density versus \( \nu \) in
Planck temperature for a model universe with \( D \) spatial dimensions but without MDR
effect. This figure shows that when the number of spatial dimensions increases, the
frequency at which the distribution maximizes (the location of the distribution peak)
will attain a noticeable shift toward higher frequencies. Also the distribution becomes
wider and its height decreases by increasing \( D \). This shows that by increasing \( D \), the
number of modes located around the most probable frequency increases considerably.
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Figure 2 depicts the variation of the \textit{modified} spectral energy density versus $\nu$ in $D$-spatial dimensions when quantum gravity effects are taken into account through MDR and temperature is fixed at the Planck temperature. While the location of peaks in this case are $D$-dependent in the same way as in Fig. 1, the height of peaks \textit{increases} by increasing $D$ and the distribution width increases also with $D$. So, there is a sharp difference between the results obtained in the presence of quantum gravity effects and their absence. This feature may provide in principle a direct clue to test the underlying theory in the lab.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Spectral energy density as a function of frequency in a model universe with $D$ spatial dimensions and without MDR effect.}
\end{figure}

Comparison between these two figures shows that when quantum gravitational effects are taken into account, the frequency at which the distribution maximizes will attain a noticeable shift toward high frequencies. This is because of the impact of quantum gravitational terms on spectral energy density. Now we explain an important achievement of our analysis through figures 3 and 4. These two figures show the spectral energy density as a function of frequency with and without incorporation of quantum gravity effects for $D = 3$ and 8 (just for example) respectively. By incorporation of quantum gravity effects via MDR, in each figure the frequency at which the distribution maximizes has significant shift toward the higher frequencies and the height of the distribution increases too. Comparison among these two figures shows that as the number of spatial dimensions increases, the difference between two peaks (frequency shift) increases. Based on these figures, the effects of quantum gravity corrections get enhances as the number of spatial dimension’s of the universe increases.
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Figure 2. Spectral energy density as a function of frequency in a model universe with $D$ spatial dimensions and in the presence of quantum gravity effect encoded in MDR.

Figure 3. Spectral energy density as a function of frequency for $D=3$ in two different regimes.

In 4-dimensional space-time, classical equipartition theorem states that average energy per electromagnetic modes of black-body radiation is $T$ (with $k_B = 1$). In ordinary quantum mechanics, this average is given by $\nu e^{-\frac{\nu}{\Theta}}$. Now in a model universe with large extra dimensions and in the presence of quantum gravitational effects encoded in MDR, this average generalizes to

$$\bar{E} = \left( \frac{\nu}{e^{\frac{\nu}{\Theta}} - 1} \right) \left[ 1 + \left( \frac{3}{2} + \frac{(D-1)}{2} \right) \alpha L_p^2 \nu^2 \right]$$
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Figure 4. Spectral energy density as a function of frequency for D=8 in two different regimes.

\[ \rho(\nu, T) = \left( (D - 1) \frac{2\pi^{D/2}}{\Gamma(D/2)} \nu^{D-1} \right) \left\{ 1 + \left( \frac{3}{2} + \frac{(D - 1)}{2} \right) \alpha L_p^4 \nu^4 \right\} T. \]

(12)

3.1. Modified Rayleigh-Jeans law

The classical Rayleigh-Jeans law describes the spectral energy density of electromagnetic waves in black body radiation, which fits with experimental results at low frequencies, while in high frequencies it tends to infinity known historically as the Ultraviolet Catastrophe. The modified form of the Rayleigh-Jeans law in our framework is written as

\[ \rho(\nu, T) = \left( (D - 1) \frac{2\pi^{D/2}}{\Gamma(D/2)} \nu^{D-1} \right) \left\{ 1 + \left( \frac{3}{2} + \frac{(D - 1)}{2} \right) \alpha L_p^4 \nu^4 \right\} T. \]

(13)

Thus, the modified Jean’s number can be written as

\[ \mathcal{N}_J = \left( (D - 1) \frac{2\pi^{D/2}}{\Gamma(D/2)} \nu^{D-1} \right) \left\{ 1 + \left( \frac{3}{2} + \frac{(D - 1)}{2} \right) \alpha L_p^4 \nu^4 \right\} T. \]

(14)

Assuming \( \alpha_i \) are positive quantities, the Jeans’s number as number density of modes in a given frequency, increases in the presence of quantum gravity effects. We note that since the Rayleigh-Jeans theory has a classical basis, we have used \( k_B T \) (with \( k_B = 1 \))
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as mean energy per mode in the above calculations. Nevertheless, in a general framework we should take into account the modification of equipartition theorem and use equation (11) as our starting point. The result is the same as equation (14).

3.2. Modified Stefan-Boltzmann law

In ordinary quantum mechanics with $D = 3$, the total energy density of black body radiation is proportional to the forth power of temperature, that is, $\rho(T) = a T^4$. In a higher dimensional world, when ordinary quantum mechanics is taken into account, the total energy density is $\rho(T) = a_D T^{D+1}$. Now, in a model universe with large extra dimensions and in the presence of quantum gravity effects encoded in the MDR, we integrate equation (11) over frequency to find

$$\rho(T) = \frac{(D - 1)2\pi^{\frac{D}{2}} T^{D+1}}{\Gamma\left(\frac{D}{2}\right)} \left\{ \zeta(D+1)\Gamma(D+1) + \left[\frac{3}{2} + \frac{(D-1)}{2}\right] \alpha L_p^2 T^2 \zeta(D+3)\Gamma(D+3) \right\}$$

In comparison with $\rho(T) = a_D T^{D+1}$, the coefficient $a_D$ in the presence of quantum gravity effects is given by

$$a_D = \frac{(D - 1)2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \left\{ \zeta(D+1)\Gamma(D+1) + \left[\frac{3}{2} + \frac{(D-1)}{2}\right] \alpha L_p^2 T^2 \zeta(D+3)\Gamma(D+3) \right\}$$

where $\zeta$ and $\Gamma$ are the Riemann Zeta function and gamma function respectively. The Radiacy is defined by the total energy emitted by a black body per unit area per unit time. The energy density $\rho(T)$ is proportional to the Radiancy $R(T)$ by a pure geometric factor

$$R(T) = \frac{\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D+1}{2}\right)} \frac{1}{2\sqrt{\pi}} \rho(T).$$

Therefore, the modified Stefan-Boltzmann law may be expressed as

$$R(T) = \pi^{\frac{D-1}{2}} \frac{(D - 1)}{\Gamma\left(\frac{D+1}{2}\right)} T^{D+1} \left\{ \zeta(D+1)\Gamma(D+1) + \left[\frac{3}{2} + \frac{(D-1)}{2}\right] \alpha L_p^2 T^2 \zeta(D+3)\Gamma(D+3) \right\}$$
Now, $\sigma_M$ as the modified Stefan-Boltzmann factor in a model universe with $D$ spatial dimensions and in the presence of quantum gravity effects encoded in MDR can be written as

$$\sigma_M = \frac{\pi^{\frac{D+1}{2}}}{\Gamma\left(\frac{D+1}{2}\right)} \left\{ \zeta(D+1)\Gamma(D+1) + \left(\frac{3}{2} + \frac{D-1}{2}\right) \alpha L_p^2 T^2 \zeta(3D)\Gamma(D+3) \right. $$

$$+ \left. \left[ \alpha^2 \left( -\frac{5}{2} + \frac{3}{2}(D-1) \right) + \alpha' \left( \frac{5}{2} + \frac{D-1}{2} \right) \right] L_p^4 T^4 \zeta(3D)\Gamma(D+5) \right\}$$. (19)

We note that the modified Stefan-Boltzmann factor in this framework is not constant and varies with temperature. Also the correction terms contain only the even power of temperature. This temperature dependence of modifications is a generic feature and probably has its origin in the quantum fluctuation of background metric in quantum gravity regime.

### 3.3. Modified Wien’s law

In ordinary quantum mechanics and in three space dimensions, there is an inverse relationship between the wavelength of the peak of the emission distribution of a black body and it’s temperature, which is called Wien’s displacement law. This law is expressed as $\lambda_{\text{max}} = \frac{b}{T}$, where $b$ is Wien’s constant and $T$ is the absolute temperature of the black body when $\lambda_{\text{max}}$ is the distribution peak wavelength. In a higher dimensional universe and by incorporating the quantum gravity effects, we should rewrite the modified energy density, equation (11), as a function of wavelength. Then we find

$$\rho_\lambda(T) = (D-1) \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \frac{1}{\lambda^{D+2}} \left(1 + \frac{3}{2} + \frac{D-1}{2}\right) \alpha L_p^2 \frac{1}{\lambda^2} + \left[ \alpha^2 \left( -\frac{5}{2} + \frac{3}{2}(D-1) \right) \right. $$

$$+ \left. \left[ \alpha' \left( \frac{5}{2} + \frac{D-1}{2} \right) \right] L_p^4 \frac{1}{\lambda^4} \right\}$. (20)

In order to find the wavelength of maximum emission, we differentiate $\rho_\lambda(T)$ (expressed up to the second order of $L_p$) with respect to $\lambda$, and then by using the approximation $e^{\frac{1}{\lambda T}} \approx 1 + \frac{1}{\lambda T}$ and putting the derivative equal to zero, we find

$$\lambda_{\text{max}} = -\frac{1}{3A} \left( \frac{36BA - 108FA^2 - 8 + 12\sqrt{3}\sqrt{4B^3A - B^2 - 18BAF + 27F^2A^2 + 4FA}}{6A} \right) $$

$$\frac{3}{3} A \left( 36BA - 108FA^2 - 8 + 12\sqrt{3}\sqrt{4B^3A - B^2 - 18BAF + 27F^2A^2 + 4FA} \right)^\frac{1}{3}$. (21)

where $A$, $B$ and $F$ are defined as
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\[ A = -T(D + 1), \quad B = -T(D + 3)\left(\frac{3}{2} + \frac{(D - 1)}{2}\right)\alpha L_p^2, \]
\[ F = \left(\frac{3}{2} + \frac{(D - 1)}{2}\right)\alpha L_p^2. \]

Note that the first term in (21) gives the standard Wien’s law in the absence of quantum gravity effects. The extra terms in (21) are quantum gravity corrections and as usual these corrections are temperature dependent.

4. Entropy and specific heat

Now we calculate entropy and specific heat of black body radiation in a model universe with \( D \) spatial dimensions and in the presence of quantum gravity effects encoded in MDR. We start with total energy in cavity as

\[ U(T) = V \rho(T) = V \frac{(D - 1)2\pi^\frac{D}{2}T^{D+1}}{\Gamma\left(\frac{D}{2}\right)} \left\{ \zeta(D + 1)\Gamma(D + 1) \right. \]
\[ \left. + \left(\frac{3}{2} + \frac{(D - 1)}{2}\right)\alpha L_p^2T^2\zeta(D + 3)\Gamma(D + 3) + \right. \]
\[ \left[ \alpha^2\left(-\frac{5}{8} + \frac{3}{2}\frac{(D - 1)}{2}\right) + \alpha'\left(\frac{5}{2} + \frac{D - 1}{2}\right)\right] L_p^4T^4\zeta(D + 5)\Gamma(D + 5) \right\}. \quad (22) \]

Then the specific heat and entropy of black body radiation are expressed respectively as

\[ C_V = \frac{\partial U}{\partial T} \bigg|_{V=cte} = V \frac{(D - 1)2\pi^\frac{D}{2}}{\Gamma\left(\frac{D}{2}\right)} \left\{ \frac{(D + 1)T^D\zeta(D + 1)\Gamma(D + 1)}{D} \right. \]
\[ \left. + \left(\frac{3}{2} + \frac{(D - 1)}{2}\right)\alpha L_p^2\zeta(D + 3)\Gamma(D + 3)\frac{T^{D+2}}{D + 2} + \right. \]
\[ \left[ \alpha^2\left(-\frac{5}{8} + \frac{3}{2}\frac{(D - 1)}{2}\right) + \alpha'\left(\frac{5}{2} + \frac{D - 1}{2}\right)\right] L_p^4\zeta(D + 5)\Gamma(D + 5)\frac{T^{D+4}}{(D + 4)} \right\}. \quad (23) \]

\[ S = \int_0^T \left(\frac{C_V}{T}\right) dT = V \frac{(D - 1)2\pi^\frac{D}{2}}{\Gamma\left(\frac{D}{2}\right)} \left\{ \frac{(D + 1)T^D\zeta(D + 1)\Gamma(D + 1)}{D} \right. \]
\[ \left. + \left(\frac{3}{2} + \frac{(D - 1)}{2}\right)\alpha L_p^2\zeta(D + 3)\Gamma(D + 3)\frac{(D + 3)T^{D+2}}{(D + 2)} + \right. \]
\[ \left[ \alpha^2\left(-\frac{5}{8} + \frac{3}{2}\frac{(D - 1)}{2}\right) + \alpha'\left(\frac{5}{2} + \frac{D - 1}{2}\right)\right] L_p^4\zeta(D + 5)\Gamma(D + 5)\frac{(D + 5)T^{D+4}}{(D + 4)} \right\}. \quad (24) \]
Figures 5 and 6 demonstrate the variation of specific heat versus temperature without and with MDR respectively. As figure 5 shows, the heat capacity decreases by increasing the number of spatial dimensions. Comparison between figures 5 and 6 shows that for a given temperature, the value of specific heat with quantum gravity effects is larger than the corresponding value in the absence of these effects. In other words, incorporation of quantum gravity effects through MDR increases specific heat of black body radiation for a given temperature.

![Figure 5](image1.png)
**Figure 5.** Black body radiation specific heat in a model universe with $D$ spatial dimensions without MDR effect.

![Figure 6](image2.png)
**Figure 6.** Black body radiation specific heat in a model universe with $D$ spatial dimensions in the presence of MDR effect.
Figures 7 and 8 demonstrate the behavior of black body entropy versus temperature for different numbers of spatial dimensions without and with MDR effects respectively. Figure 7 shows that the entropy decreases as the number of spatial dimensions increases. Comparison between figures 7 and 8 shows that quantum gravity effects increase the entropy content of black body radiation in a given temperature.

**Figure 7.** Black body entropy in a model universe with $D$ spatial dimensions without MDR effect.

**Figure 8.** Black body entropy in a model universe with $D$ spatial dimensions in the presence of MDR effect.
5. Modified Debye’s law

Imagine a black body as a solid box, which is compound of vibrating atoms. These vibrating atoms are recognized as the so-called phonons in this box. The method adopted in this section resembles more or less calculation of number of electromagnetic modes in black body radiation. Nevertheless, there are some discrepancies between these two cases since in a vibrating solid there are three types of waves (one longitudinal and two transverse) in three spatial dimensions while there are only two polarization states for electromagnetic modes. So, if we generalize this situation to a model universe with $D$ spatial dimensions, we would have $D$ possible polarization states for phonons. Unlike photon, phonon frequency is bounded by upper limit. In fact, the total vibrational states for phonons is $D \times N$ where $N$ is the number of atoms in vibrating solid. Since there is no limitation on the number of phonons in each energy levels, phonon could be considered as bosons. Therefore, in a model universe with $D$ spatial dimensions, we have

$$U(\nu) = D L^D \frac{2\pi \nu^D}{\Gamma\left(\frac{D}{2}\right)} \int_0^{\nu_{\text{max}}} \frac{\nu^D}{(e^{\frac{\nu}{T}} - 1)} d\nu$$

which determines the total energy of vibrating solid in a model universe with $D$ spatial dimensions. By setting $x = \frac{\nu}{T}$, we obtain

$$U(x) = D^2 T N \left(\frac{T}{T_D}\right)^D \int_0^{x_{\text{max}}} \frac{x^D}{e^x - 1} dx$$

where $T_D$ is introduced as Debye temperature which is equal to

$$\left(\frac{T_D}{T}\right)^D = \frac{N \Gamma\left(\frac{D}{2}\right) D}{2 V \pi^{\frac{D}{2}}}.$$

Now we incorporate quantum gravity effects through MDR. Based on our previous analysis, we have

$$U(x) = D^2 T N \left(\frac{T}{T_D}\right)^D \int_0^{x_{\text{max}}} \frac{x^D}{e^x - 1} \left\{1 + \left(\frac{3}{2} + \frac{(D-1)}{2}\right) \alpha L_p^2 x^2 T^2 + \left[\alpha^2 \left(\frac{-5}{8} + \frac{3}{2} \frac{(D-1)}{2}\right) + \alpha' \left(\frac{5}{2} + \frac{(D-1)}{2}\right)\right] L_p^4 x^4 T^4\right\}$$

This relation leads us to the modified Debye law straightforwardly. Since the integral cannot be calculated analytically, we consider some limiting cases. The temperature of a Debye solid is said to be low if $T \ll T_D$. In this limit we find

$$U(T) = D^2 T N \left(\frac{T}{T_D}\right)^D \left\{\zeta(D+1) \Gamma(D+1) + \left(\frac{3}{2} + \frac{(D-1)}{2}\right) \alpha L_p^2 T^2 \zeta(D+3) \Gamma(D+3) + \left[\alpha^2 \left(\frac{-5}{8} + \frac{3}{2} \frac{(D-1)}{2}\right) + \alpha' \left(\frac{5}{2} + \frac{(D-1)}{2}\right)\right] L_p^4 T^4 \zeta(D+5) \Gamma(D+5)\right\}.$$
By differentiating this relation with respect to $T$, we find

$$
C_V = D^2(D + 1)N \left( \frac{T}{T_D} \right)^D \zeta(D + 1)\Gamma(D + 1)
$$

$$
+ \left( \frac{3}{2} + \frac{(D - 1)}{2} \right) \alpha L_p^2 D^2 N \frac{T^{D+2}}{(T_D)^D} \zeta(D + 3)\Gamma(D + 3)
$$

$$
+ \left[ \alpha^2 \left( -\frac{5}{8} + \frac{3(D - 1)}{2} \right) + \alpha' \left( \frac{5}{2} + \frac{D - 1}{2} \right) \right] L_p^4 D^2 N(D+5) \frac{T^{D+4}}{(T_D)^D} \zeta(D+5)\Gamma(D+5)
$$

This is the modified Debye law at very low temperature in the MDR framework. Note that the first term in the right hand side gives the standard Debye law at very low temperature in the absence of quantum gravity effects. It’s worth mentioning to note that modified specific heat changes with temperature. So, quantum gravity corrections in this case are temperature dependent as usual. Figure 9 shows the heat capacity of solid based on Debye model at low temperature as a function of temperature. Based on

![Figure 9](image)

**Figure 9.** Debye specific heat of solid at very low temperature in two different regime for $D=3$ and at a fixed $T_D$.

Debye model, the value of specific heat for solid at low temperature with MDR is larger than the corresponding value in the absence of MDR at a given temperature.
5.1. Modified Dulong-petit law

Dulong and Petit found that the heat capacity of a mole of many solid substances is about $3R$, where $R$ is the universal gas constant. An equivalent statement of the Dulong-Petit law in modern terms is that, regardless of the nature of the substance or crystal, the specific heat capacity $C$ of a solid substance (measured in Joule per Kelvin per Kilogram) is equal to $\frac{3R}{M}$ where $M$ is the molar mass (measured in Kilogram per Mole). Thus, the heat capacity per mole of many solids is $3R$. Actually this statement is valid only at high temperature and in this case the specific heat is assumed to be temperature-independent in classical viewpoint. Now we reconsider this model in the MDR framework in order to see how quantum gravity effects modify the classical Dulong-Petit law. At high temperature when $T \gg T_D$, we can use the approximation $e^x - 1 \approx x$. Therefore, based on Eq. (27), the total energy of vibrating solid is given by

$$U(T) = DTN + \frac{D^2}{(D+1)}NT_D \left(\frac{3}{2} + \frac{(D-1)}{2}\right) \alpha L_p^2 T^2 + \left(\alpha^2 \left(\frac{-5}{8} + \frac{3(D-1)}{2}\right) + \alpha' \left(\frac{5}{2} + \frac{(D-1)}{2}\right) \right) \frac{D^2}{(D+3)^2} NT_D^2 T^3. \quad (30)$$

By differentiating with respect to $T$, we find

$$C_V = DN + \left\{ \frac{2D^2}{(D+1)}NT_D \left(\frac{3}{2} + \frac{(D-1)}{2}\right) \alpha L_p^2 + 2 \left[ \alpha^2 \left(\frac{-5}{8} + \frac{3(D-1)}{2}\right) + \alpha' \left(\frac{5}{2} + \frac{(D-1)}{2}\right) \right] \frac{D^2}{(D+3)^2} NT_D^2 \right\} T \quad (31)$$

This is the modified Dulong-petit law. In the absence of quantum gravity effects, ordinary quantum mechanics result will be achieved, which states that the heat capacity is independent of temperature. However, in quantum gravity domain one cannot ignore temperature dependence of heat capacity as is evident in correction terms. These temperature-dependent corrections reveal some phenomenological aspects of quantum gravity proposal. These features may provide also a direct basis to test quantum gravity in the lab. Figure 10 shows the heat capacity of solid based on Dulong-petit model at high temperature as a function of temperature.

Specific heat of solid in Dulong-petit law within MDR has linear dependence on temperature (at least up to our adopted approximations), while without MDR it is temperature-independent.
6. Conclusion

It is now a well-established fact that the Cosmic Background Microwave Radiation has a perfect black body spectrum. Modification of this spectrum due to tiny quantum gravity effects and possible detection of these footprints in CMB spectrum via observation opens new windows to know more about quantum gravitational features of the early universe and some phenomenological aspects of quantum gravity itself. Also it is interesting in essence to know the black body spectrum if our observable universe could be braneworld in nature. Based on these motivations in this paper we studied the problem of black body radiation in a model universe with extra dimensions by incorporating quantum gravity effects encoded in modified dispersion relations. We calculated with details some important physical and thermodynamical quantities attributed to black body radiation. As a generic result we have shown that quantum gravity corrections are generally temperature dependent. We have shown that in the absence of quantum gravity effects, when the number of spatial dimensions increases, the frequency at which the modified Planck distribution maximizes (the location of the distribution peak) will attain a noticeable shift toward higher frequencies. This distribution becomes wider and its height decreases by increasing the number of spatial dimensions. This means that by increasing $D$, the number of modes located around the most probable frequency increases considerably. When quantum gravity effects are taken into account through MDR the location of peaks are $D$-dependent in the same way as in the absence of quantum gravity effects, but the height of picks and the distribution width get increase by increasing the number of spatial dimensions. This feature shows that there is crucial difference between the results obtained in the presence of quantum gravity effects and
their absence and this may provide, in principle, a direct clue to test the underlying quantum gravity candidate. We have shown also that the effects of quantum gravity corrections get enhances as the number of spatial dimension’s of the universe increases. As another important result, we have shown that the Jeans’s number as number density of modes in a given frequency increases in the presence of quantum gravity effects. The modified Stefan-Boltzmann factor in this framework is no longer a constant and varies with temperature. The correction terms to Stefan-Boltzmann law in this framework contain only the even power of temperature. We obtained the modified Wien’s law which shows a small, temperature-dependent shift in $\lambda_{\text{max}}$. In the next step, we have shown that incorporation of quantum gravity effects through MDR increases the specific heat of black body radiation for a given temperature. The entropy of the system decreases as the number of spatial dimensions increases. However, quantum gravity effects generally increase the entropy content of black body radiation in a given temperature. Based on the modified Debye model presented here, the value of specific heat for solid at low temperature with MDR is larger than the corresponding value in the absence of MDR at a given temperature. We obtained also the modified Dulong-Petit law. While in ordinary quantum mechanics picture of this law the heat capacity is independent of temperature, in quantum gravity domain one cannot ignore temperature dependence of heat capacity as is evident in correction terms. Once again this temperature dependent feature can be tested essentially in future experiments. Finally we have shown that up to adopted approximations, the specific heat of solid in Dulong-petit law within MDR framework has linear dependence on temperature but without MDR it is temperature-independent.

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