Virial Identities and Energy–Momentum Relation for Solitary Waves of Nonlinear Dirac Equations

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Abstract—Solitary waves of nonlinear Dirac, Maxwell–Dirac and Klein–Gordon–Dirac equations are considered. We deduce some virial identities and check that the energy-momentum relation for solitary waves coincides with the Einstein energy-momentum relation for point particles.

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1. INTRODUCTION

The paper concerns the old problem of mathematically describing elementary particles in field theory. Einstein and Grommer [13] suggested that particles could be described as singularities of solutions to the field equations. The generalization of this result to interacting systems of particles was given by Einstein, Infeld and Hoffmann [14]. Rosen [28] was the first who proposed a description of particles for the coupled Klein–Gordon–Maxwell equations, which are invariant with respect to the Lorentz group. Namely, the particle at rest is described by a finite energy solution that has “Schrödinger’s” form $\phi(x)e^{-i\omega t}$ (“nonlinear eigenfunctions” or “solitary waves”). The particle with the nonzero velocity $v$, $|v| < 1$, is obtained by the corresponding Lorentz (or Poincare) transformation. The existence of solitary waves has been analyzed by many authors for diverse Lagrangian field theories [15, 22, 27, 28, 33, 35], such that nonlinear Dirac fields, the Maxwell–Dirac (MD) and Klein–Gordon–Dirac (KGD) equations. We describe briefly some results.

Nonlinear Dirac equations occur in the attempt to construct relativistic models of extended particles by means of nonlinear Dirac fields. The review of such models can be found in [16]. The stationary solutions of nonlinear Dirac equation were extensively studied in the literature used variational methods [17], a dynamical systems approach [2, 6, 23] and a perturbation method [24]. For details, see the survey papers [16, 18, 26] and the references therein.

The (MD) equations (see, e.g., [5, 32]) describing the interaction of an electron with its own electromagnetic field have been widely studied by many authors. The first results on the local existence and uniqueness of solutions was obtained by Gross [19], Chadam [7], Chadam and Glassey [8]. The stationary (localized) solutions of the classical (MD) system were studied numerically by Wakano [35] and Lisi [22]. Using variational methods, Esteban, Georgiev and Seré [15] have proved the existence of stationary solutions with $\omega \in (-m, 0)$. These results were extended by Abenda [1] for $\omega \in (-m, m)$.

For the (KGD) equations, the local existence and uniqueness of solutions were proved by Chadam and Glassey [8]. Numerical results on the stationary states were obtained by Ranada and Vázquez [27]. The rigorous proof of the existence for the stationary solutions was given by Esteban et al. [15]. For some Lorentz invariant complex scalar fields theories, the particle-like solutions was studied by Rosen [29, 30].

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Note that it would be of importance to develop a particle-like dynamics for moving solitons. We make a step in this direction for relativistic-invariant nonlinear Dirac, (MD) and (KGD) equations. Namely, we prove that the energy-momentum relation coincides with that of a relativistic particle.

Now we outline the main result in the case of nonlinear Dirac equations. We consider the Dirac equations of the form

$$i\dot{\psi} = -i\alpha \cdot \nabla \psi + m\beta \psi - g(\bar{\psi}\psi)\beta \psi, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}. \quad (1)$$

We use natural units, in which we have rescaled length and time so that $\hbar = c = 1$. Here unknown function $\psi = \psi(t, x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$ is a four-component Dirac spinor field, $m > 0$, $\dot{\psi} = \partial_t \psi$, $x = (x_1, x_2, x_3)$, $\nabla = (\partial_1, \partial_2, \partial_3)$, $\partial_0 = \partial / (\partial x_0)$, $k = 1, 2, 3$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. $\alpha_k, \beta$ are the $4 \times 4$ complex Pauli–Dirac matrices (in the standard $2 \times 2$ blocks representation)

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad (k = 1, 2, 3), \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where $I$ denotes the $2 \times 2$ unit matrix, and $\sigma_k$ are Pauli matrices defined as

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One verifies that $\sigma_k \sigma_l + \sigma_l \sigma_k = 2\delta_{kl}I, \sigma_k^2 = \sigma_k, k = 1, 2, 3$. Then

$$\beta^* = \beta, \quad \alpha_k^* = \alpha_k, \quad \alpha_k^2 = \beta^2 = I, \quad \alpha_k \alpha_j + \alpha_j \alpha_k = 0 \quad \text{for} \quad j \neq k, \quad \alpha_k \beta + \beta \alpha_k = 0. \quad (2)$$

Let us fix the following notations. Given two vectors of $\mathbb{C}^4$, $\psi \phi := \psi \cdot \phi$ is the inner product in $\mathbb{C}^4$, $\ast$ denotes the complex conjugate. By definition, the “adjoint spinor” is $\psi = \psi^* \beta$.

The particular nonlinearity $g(s) = \lambda s$ corresponds to the so-called Soler model of extended fermions [2, 33]. In the general case of $g(s)$, $\text{Eqn} \ (1)$ is often called the generalized Soler model (see [5, 17]). The review of models of extended particles by means of nonlinear Dirac fields can be found in [26].

The stationary solutions of nonlinear Dirac equation are considered as particle-like solutions. They are the solutions of the form $\psi_0(t, x) = e^{-i\omega t} \varphi(x)$, where $\varphi$ is a nonzero localized solution of the stationary nonlinear Dirac equation (5), see Definition 1 below.

Denote by $\psi_\nu(t, x)$ the moving solitary waves with velocity $\nu \in \mathbb{R}^3, |\nu| < 1$,

$$\psi_\nu(t, x) = S(\Lambda_\nu) \psi_0 (\Lambda_\nu^{-1}(t, x)), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R},$$

where $\Lambda_\nu$ is a Lorentz transformation (see formula (25) below), $S(\Lambda_\nu)$ is a matrix defined in (26). Put $G(s) = \int_0^s g(p)dp$. The energy functional is given by

$$\mathcal{E}(\psi) = \int_{\mathbb{R}^3} \left( -i \psi^* \alpha \cdot \nabla \psi + m \bar{\psi} \psi - G(\bar{\psi} \psi) \right) dx. \quad (3)$$

Using equalities (2), it is easy to check that $\mathcal{E}(\psi(t, \cdot)) = \text{const}$. Our main objective is to check that the energy–momentum relation coincides with one of relativistic point particle, namely,

$$\mathcal{E}(\psi_\nu) = \gamma \mathcal{E}(\psi_0), \quad \gamma = (1 - |\nu|^2)^{-1/2}. \quad (4)$$

The paper is organized as follows. In Sections 2 and 3, we check the relation (4) for nonlinear Dirac equations (1). Section 4 concerns the Dirac equations in $\mathbb{R}^1$. For (MD) and (KGD) equations, the result is obtained in Sections 5 and 6, respectively.
2. STANDING SOLITARY WAVES FOR DIRAC EQUATIONS

Denote by $H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$, the Sobolev space, i.e., the Hilbert space of distributions $\varphi \in S'(\mathbb{R}^3)$ endowed with the norm $||\varphi||_{H^s} = ||A^s\varphi||_{L^2}$, where $A^s\varphi := F^{-1}_{\xi \rightarrow x}((1 + |\xi|^2)^{s/2}\hat{\varphi}(\xi))$, and $\hat{\varphi} := F\varphi$ denotes Fourier transform.

Let $W^{1,q}(\mathbb{R}^3)$, $q \geq 2$, denote the space of distributions $\varphi \in S'(\mathbb{R}^3)$ endowed with the norm $||\varphi||_{W^{1,q}} = ||\nabla \varphi||_{L^q} + ||\varphi||_{L^q}$. In particular, $W^{1,2}(\mathbb{R}^3) = H^1(\mathbb{R}^3)$.

**Definition 1.** The stationary states or localized solutions of Eqn (1) are the solutions of the form $\psi_0(t, x) = e^{-i\omega t}\varphi_\omega(x)$, $\omega \in \mathbb{R}$, such that $\varphi_\omega \in H^1(\mathbb{R}^3; \mathbb{C}^4)$, and $\varphi \equiv \varphi_\omega$ is a nonzero localized solution of the following stationary nonlinear Dirac equation

$$i\alpha \cdot \nabla \varphi + \omega \varphi - m\beta \varphi + g(\varphi)\beta \varphi = 0, \quad x \in \mathbb{R}^3.$$  

The existence of solutions of Eqn (5) has been proved in [2, 3, 6, 17, 23] for $\omega \in (0, m)$ under some restrictions on $G$. In particular, the following restrictions were imposed in [17].

**G1** $G \in C^2(\mathbb{R}; \mathbb{R})$

**G2** For any $s \in \mathbb{R}$, $g(s)s \geq \theta G(s)$ with some $\theta > 1 (g(s) = G'(s))$

**G3** $G(0) = G'(0) = 0$

**G4** $G(s) \geq 0$ for any $s \in \mathbb{R}$, and $G(A_0) > 0$ for some $A_0 > 0$

**Theorem 1.** (see [17, Theorem 1]). Let conditions **G1**–**G4** hold and $\omega \in (0, m)$. Then there is an infinity of solutions of Eqn (5) in $\bigcap_{2 \leq q < \infty} W^{1,q}(\mathbb{R}^3; \mathbb{C}^4)$. Each of them are critical points of the functional $I^G_D$,

$$I^G_D(\varphi) = \frac{1}{2} \int_{\mathbb{R}^3} \left( i\varphi^* \cdot \nabla \varphi - m\beta \varphi + \omega|\varphi|^2 + G(\varphi) \right) dx.$$

These solutions $\varphi \equiv \varphi_\omega$ are of the form (in the spherical coordinates $(r, \phi, \theta)$ of $x \in \mathbb{R}^3$)

$$\varphi_\omega(x) = \begin{pmatrix} u(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ iu(r) \begin{pmatrix} \cos \theta \\ \sin \theta e^{i\phi} \end{pmatrix} \end{pmatrix}, \quad (x_1 = r \cos \phi \sin \theta, \quad x_2 = r \sin \phi \sin \theta, \quad x_3 = r \cos \theta, \quad r = |x|).$$

Thus they correspond to classical solutions of the O.D.E. system

$$\begin{cases} u' + \frac{2m}{r} = v[g(v^2 - u^2) - (m - \omega)], \\ v' = u[g(v^2 - u^2) - (m + \omega)]. \end{cases}$$

Finally, the solutions decrease exponentially at infinity, together with their first derivatives.

**Remarks.** (i) Denote by $L_D$ the Lagrangian density for considered Dirac fields,

$$L_D(\psi) = \bar{\psi}(ig^\mu \partial_\mu - m)\psi + G(\bar{\psi}\psi),$$

where $g^\mu \partial_\mu = \gamma^0 \partial_t + \gamma \cdot \nabla$ with Dirac matrices $\gamma^\mu$ ($\gamma^0 = \beta$, $\gamma^k = \beta \alpha_k$, $k = 1, 2, 3$), $\partial_\mu = (\partial_t, \nabla)$. It is easy to check that the Euler–Lagrange equations applied to (7) give Eqn (1). In particular, for stationary solutions $\psi_0(t, x)$ we have

$$L_D(\psi_0) = \psi^*(\omega + i\alpha \cdot \nabla - m\beta)\psi + G(\bar{\psi}\psi).$$
Note that $I_D^0(\varphi) = -(1/2) \int \mathcal{L}_D(\psi_0) dx$. Here and below, for simplicity, we omit the symbol $\mathbb{R}^3$ in the notation of the integral $\int_{\mathbb{R}^3} \ldots dx$.

(ii) In \cite{3}, the existence of solutions of the form (6) have been proved for singular self-interactions $g(s) \sim s^{-\alpha}$ with some $\alpha \in (0,1)$.

(iii) The stationary nonlinear Dirac equations of the form

\begin{equation}
\text{i} \alpha \cdot \nabla \varphi + \omega \varphi - m \beta \varphi + \nabla F(\varphi) = 0, \quad x \in \mathbb{R}^3,
\end{equation}

has been studied by Esteban and Séré \cite{17}. If $F(\varphi) = G(\bar{\varphi} \varphi)$, then Eqn (8) coincides with (5). For a more general class of nonlinearities $F$, which do not satisfy condition $F(\varphi) = G(\bar{\varphi} \varphi)$, the ansatz (6) is no more valid. In this case, the existence of solutions to Eqn (8) has been proved in \cite{17, Theorems 2, 3} with nonlinearities $F$ of the form $F(\varphi) = \lambda(\varphi|\varphi|^\kappa + b |\varphi|^3 \varphi^2)$ with $1 < \kappa_1, \kappa_2 < 3/2$, $\gamma^5 = -i\alpha_1 \alpha_2 \alpha_3$, $\lambda, b > 0$; and for a more general class of nonlinearities $F$ satisfying the following conditions:

\begin{equation}
\begin{aligned}
0 \leq F(\varphi) &\leq a_1(|\varphi|^{\kappa_3} + |\varphi|^{\kappa_4}) & \text{ with } & a > 0, \quad 2 < \kappa_3 \leq \kappa_4 < 3; \\
F &\in C^2(C^4; \mathbb{R}); & F(0) = F''(0) = 0; & |F''(\varphi)| \leq a_2 |\varphi|^{\kappa_1 - 2}, \quad a_2 > 0, \quad \forall |\varphi| \text{ large}; \\
\nabla F(\varphi) \cdot \varphi &\geq b F(\varphi), & b > 2, & \forall \varphi \in C^4; \\
\exists \kappa > 3, & \forall \delta > 0, & \exists C_\delta > 0: & |\nabla F(\varphi)| \leq (\delta + C_\delta F(\varphi)^{1/\kappa}) |\varphi|, \quad \forall \varphi \in C^4; \\
F(\varphi) &\geq a_3 |\varphi|^{\kappa_4} - a_4, & \nu > 1, & a_3, a_4 > 0, \quad \forall \varphi \in C^4.
\end{aligned}
\end{equation}

The following virial identity (or so-called Pokhozhaev identity \cite{25}) was proved in \cite{17, Proposition 3.1}.

**Lemma 1.** Let $\varphi \in H^1(\mathbb{R}^3; C^1)$ be a solution to Eqn (5). Then $\varphi(x)$ satisfies

\begin{equation}
\text{i} \int \varphi^* \alpha \cdot \nabla \varphi dx = \frac{3}{2} \int (m \bar{\varphi} \varphi - \omega \bar{\varphi} \varphi - G(\bar{\varphi} \varphi)) dx.
\end{equation}

Introduce the following notations

\begin{equation}
I_k \equiv I_k(\varphi) = -i \int \varphi^* \alpha_k \partial_k \varphi dx, \quad k = 1, 2, 3,
\end{equation}

\begin{equation}
Q \equiv Q(\varphi) = \int \varphi^* \varphi dx, \quad V \equiv V(\varphi) = \int \left( m \bar{\varphi} \varphi - G(\bar{\varphi} \varphi) \right) dx.
\end{equation}

Then the equality (9) is rewritten as

\begin{equation}
\omega Q = V + \frac{2}{3} (I_1 + I_2 + I_3).
\end{equation}

**Remark.** Formally, the identity (9) can be proved used Derrick’s technique \cite[p. 1253]{10}. Indeed, introducing $\varphi_\lambda(x) = \varphi(x/\lambda)$ gives

\begin{equation}
0 = \frac{d}{d\lambda} \bigg|_{\lambda=1} I_D^0(\varphi_\lambda) = \frac{1}{2} \frac{d}{d\lambda} \bigg|_{\lambda=1} \left[ I_1(\varphi_\lambda) + I_2(\varphi_\lambda) + I_3(\varphi_\lambda) + V(\varphi_\lambda) - \omega Q(\varphi_\lambda) \right]
= \frac{1}{2} \frac{d}{d\lambda} \bigg|_{\lambda=1} \left[ \lambda^2 I_1(\varphi) + \lambda^2 I_2(\varphi) + \lambda^2 I_3(\varphi) + \lambda^3 V(\varphi) - \lambda^3 \omega Q(\varphi) \right]
= I_1(\varphi) + I_2(\varphi) + I_3(\varphi) + \frac{3}{2} (V - \omega Q).
\end{equation}

This gives the identity (9). The similar Derrick’s technique has been used in \cite{11} for relativistic-invariant nonlinear wave equations. Using the similar reasonsing with $\varphi_\lambda(x) = \varphi(x_1/\lambda, x_2/\lambda, x_3)$, $\varphi_\lambda(x) = \varphi(x_1, x_2, x_3/\lambda)$, it is easy to check that

\begin{equation}
I_1 = I_2 = I_3 = \frac{1}{3} (I_1 + I_2 + I_3) = \frac{1}{2} (\omega Q - V).
\end{equation}
Corollary 1. Let \( \varphi \) be a solution of Eqn (5). Then the following relations hold.

\[
I_1 + I_2 + I_3 = \omega Q + \int \left( g(\bar{\varphi}) - m \right) \bar{\varphi} \varphi dx. \tag{13}
\]

\[
I_1 + I_2 + I_3 = 3 \int \left( g(s) s - G(s) \right) \bigg|_{s=\bar{\varphi} \varphi} dx > 0. \tag{14}
\]

\[\mathcal{E}_0 \equiv I_1 + I_2 + I_3 + V > 0.\]

Proof. By (5), we have

\[
\int \varphi^* i\alpha \cdot \nabla \varphi dx = \int \varphi^* \left( -\omega \varphi + m\beta \varphi - g(\bar{\varphi})\beta \varphi \right) dx.
\]

This implies the identity (13). Then, by (11) and (13), we obtain

\[
\omega Q = V + \frac{2}{3}(I_1 + I_2 + I_3) = \int \left( m - g(\bar{\varphi}) \right) \bar{\varphi} \varphi dx + I_1 + I_2 + I_3.
\]

Hence,

\[
\frac{1}{3}(I_1 + I_2 + I_3) = \int \left( g(s)s - G(s) \right) \bigg|_{s=\bar{\varphi} \varphi} dx. \tag{15}
\]

Therefore, relation (14) follows from (15) and conditions G2 and G4, since for \( \theta > 1 \),

\[g(s)s - G(s) > g(s)s - \theta G(s) \geq 0 \quad \text{ for all } s \in \mathbb{R}.\]

Using (3) and (13), the energy \( \mathcal{E}_0 := \mathcal{E}(\psi_0(t, \cdot)) \) associated with particle-like solutions \( \psi_0 \) is expressed by

\[
\mathcal{E}_0 \equiv \mathcal{E}_0(\varphi) = I_1 + I_2 + I_3 + V = \omega \int \left| \varphi(x) \right|^2 dx + \int \left( g(s)s - G(s) \right) \bigg|_{s=\bar{\varphi} \varphi} dx > 0,
\]

due to conditions G2 and G4.

Denote by \((\cdot, \cdot)\) the inner scalar product in \( L^2 \).

Lemma 2. Let \( \varphi \) be a solution to Eqn (5), \( \varphi \in H^1(\mathbb{R}^3; \mathbb{C}^4) \). Then

\[
\omega(\varphi^*, \alpha_k \varphi) = -i(\varphi^*, \partial_k \varphi), \quad k = 1, 2, 3. \tag{16}
\]

Proof. Multiply (5) on the left by \( \alpha_1 \) and obtain

\[
i\partial_1 \varphi + i\alpha_1 \alpha_2 \partial_2 \varphi + i\alpha_1 \alpha_3 \partial_3 \varphi + \omega \alpha_1 \varphi - m\alpha_1 \beta \varphi + g(\bar{\varphi})\alpha_1 \beta \varphi = 0.
\]

Hence,

\[
i(\varphi^*, \partial_1 \varphi) + i(\varphi^*, \alpha_1 \alpha_2 \partial_2 \varphi) + i(\varphi^*, \alpha_1 \alpha_3 \partial_3 \varphi) + \omega(\varphi^*, \alpha_1 \varphi) - m(\varphi^*, \alpha_1 \beta \varphi) + (\varphi^*, g(\bar{\varphi})\alpha_1 \beta \varphi) = 0. \tag{17}
\]

On the other hand, taking the adjoint of Eqn (5) and multiplying on the right by \( \alpha_1 \), one obtains

\[
-i\partial_1 \varphi^* - i\partial_2 \varphi^* \alpha_2 \alpha_1 - i\partial_3 \varphi^* \alpha_3 \alpha_1 + \omega \varphi^* \alpha_1 - m\varphi^* \beta \alpha_1 + g(\bar{\varphi})\varphi^* \beta \alpha_1 = 0.
\]

Hence,

\[
-i(\partial_1 \varphi^*, \varphi) - i(\partial_2 \varphi^* \alpha_2 \alpha_1, \varphi) - i(\partial_3 \varphi^* \alpha_3 \alpha_1, \varphi) + \omega(\varphi^* \alpha_1, \varphi) - m(\varphi^* \beta \alpha_1, \varphi) + (\varphi^* \beta \alpha_1, g(\bar{\varphi})\varphi) = 0. \tag{18}
\]

By (2), summing Eqns (17) and (18) gives (16) for \( k = 1 \). For \( k \neq 1 \), the proof is similar. \[\square\]
2.1. A Particular Ansatz for the Solutions of Dirac Equations

As in [22], we choose to orient the angular momentum along the \(x_3\)-axis and consider four families of solutions to Eqn (5) which in spherical coordinates \((r, \phi, \theta)\) (i.e. \(x_1 = r \cos \phi \sin \theta, x_2 = r \sin \phi \sin \theta, x_3 = r \cos \theta\)) are of the form

\[
\varphi^1(x) = \begin{pmatrix} v_+(r) & 0 \\ iu_+(r) & \cos \theta \end{pmatrix}, \quad \varphi^2(x) = \begin{pmatrix} v_-(r) & -\sin \theta e^{i\phi} \\ iu_-(r) & \cos \theta \end{pmatrix},
\]

\[
\varphi^3(x) = \begin{pmatrix} v_+(r) & 0 \\ iu_+(r) & -\cos \theta \end{pmatrix}, \quad \varphi^4(x) = \begin{pmatrix} v_-(r) & \sin \theta e^{-i\phi} \\ iu_-(r) & -1 \end{pmatrix}.
\]

If \(\varphi^1, \ldots, \varphi^4\) are substituted into Eqn (5), then this equation reduces to the following O.D.E. system for radial functions \(u_\pm\) and \(v_\pm\):

\[
\begin{cases}
u_+' + \frac{2m}{r} = v[g(v_+^2 - u_+^2) - (m + \omega)], \\
v_-' = u_+[g(v_+^2 - u_+^2) - (m - \omega)].
\end{cases}
\]

The existence of the solutions \(u_\pm\) and \(v_\pm\) follows from results of the works [2, 6, 17, 23].

The total angular momentum operator is \(\mathbf{M} = \mathbf{L} + \mathbf{S}\), where \(\mathbf{L} = x \times (-i\nabla)\) is the orbital angular momentum, \(\mathbf{S} = \Sigma/2\) is the spin angular momentum, \(\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}\). Here \(\mathbf{M} = (M_1, M_2, M_3), \mathbf{L} = (L_1, L_2, L_3), \sigma = (\sigma_1, \sigma_2, \sigma_3)\). In particular, in the spherical coordinates, the third component of \(\mathbf{L}\) is \(L_3 = -i\partial_\theta\).

It is easy to check the following properties of \(\varphi^a(x), a = 1, 2, 3, 4\).

(i) \(\varphi^a\) are eigenfunctions of the third component of \(\mathbf{M}\) with eigenvalue \(m_3 = \pm 1/2\). More exactly, \(M_3\varphi^a = (1/2)\varphi^a\) for \(a = 1, 2, M_3\varphi^a = -(1/2)\varphi^a\) for \(a = 3, 4\). Since \(\mathbf{M}\varphi^1 = (1/2)(\varphi^3, i\varphi^3, \varphi^1), \mathbf{M}\varphi^2 = (1/2)(-\varphi^4, -i\varphi^4, \varphi^2), \mathbf{M}\varphi^3 = (1/2)(\varphi^1, i\varphi^1, -\varphi^3), \mathbf{M}\varphi^4 = (1/2)(-\varphi^2, i\varphi^2, -\varphi^4)\), then \(M_3^k\varphi^a = (1/4)\varphi^a\) for all \(k = 1, 2, 3, M_2^2\varphi^a = (3/4)\varphi^a = j(j + 1/2)\varphi^a\) for all \(a\), where \(M^2 = M_1^2 + M_2^2 + M_3^2\). Hence, the quantum number \(j = 1/2\) for all \(\varphi^a\).

(ii) For the “spin–orbit” operator \(\mathbf{K} = \beta\Sigma \cdot \mathbf{M} - 1/2\beta = \beta(\Sigma \cdot \mathbf{L} + 1)\) (see [12, p. 19]), we have \(\mathbf{K}^2 = \mathbf{M}^2 + 1/4\). Then the eigenvalues of \(\mathbf{K}\) are \(\kappa = \pm (j + 1/2)\). Hence, for all \(a\), \(\varphi^a\) are eigenfunctions of \(\mathbf{K}\) with eigenvalues \(\kappa = \pm 1\), where the quantum number \(\kappa = 1\) for \(a = 1, 3\) and \(\kappa = -1\) for \(a = 2, 4\).

(iii) For any solution \(\varphi\) from the four families \(\varphi^1, \ldots, \varphi^4\), the following equalities hold. At first, \(\varphi^*(x)\alpha_3 \varphi(x) \equiv 0\). Secondly,

\[
\int \varphi^*(x) \nabla \varphi(x) dx = 0, \quad (19)
\]

\[
\int \varphi^*(x) \alpha_k \partial_\lambda \varphi(x) dx = 0 \quad \text{for any} \quad k \neq l. \quad (20)
\]

(iv) For stationary states \(\psi_0(t, x) = e^{-i\omega t} \varphi(x)\) with \(\varphi\) from these particular families of solutions, we have

\[
Q(\psi_0) = 4\pi \int_0^\infty \left( u_+^2 + u_-^2 \right) r^2 dr,
\]

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\[ \mathcal{E}_0 \equiv \mathcal{E}(\psi_0) = 4\pi \omega \int_0^{+\infty} \left( v_+^2 + u_+^2 \right) r^2 dr + 4\pi \int_0^{+\infty} \left. \left( g(s) s - G(s) \right) \right|_{s = v_+^2 - u_+^2} r^2 dr, \]

where \( v_\pm = v_\pm(r), u_\pm = u_\pm(r) \). Moreover, the current \( J(x) := \psi_0^*(t,x) \alpha \psi_0(t,x) \) equals
\[ J(x) = 4\kappa_m m_3 u_+ v_+ (-\sin \phi, \cos \phi, 0), \]
where the quantum numbers are \( m_3 = \pm 1/2, \kappa_\pm = \pm 1 \).

3. MOVING SOLITARY WAVES FOR NONLINEAR DIRAC EQUATIONS

As shown, e.g., in [5, 12, 34], the Dirac equation (1) with \( g \equiv 0 \) is Lorentz invariant. Namely, let \( \Lambda = (\Lambda_{\mu\nu})_{\mu,\nu=0}^3 \) be a Lorentz transformation and \( \psi(t,x) \) be a solution of Eqn (1) with \( g \equiv 0 \). Then, there exists a matrix \( S(\Lambda) \) such that \( \psi^\prime(t^\prime, x^\prime) = S(\Lambda) \psi(t,x) \) satisfies the same equation in the terms of the new variables \( (t^\prime, x^\prime) = \Lambda(t,x) \). It requires the following conditions on \( S \equiv S(\Lambda) \):
\[ \alpha_\mu = \sum_{\nu=0}^3 \beta S \beta \Lambda_{\mu\nu} \alpha_\nu S^{-1} \text{ with } \alpha_0 = I, \tag{21} \]
or \( S^{-1} \gamma^\nu S = \sum_{\mu=0}^3 \Lambda_{\mu\nu} \gamma^\mu, \nu = 0, 1, 2, 3 \), where \( \gamma^0 := \beta, \gamma^k := \beta \alpha_k, k = 1, 2, 3 \) (see, e.g., [5]). Here and below by \( I \) we denote the unit \( 4 \times 4 \) (or \( 2 \times 2 \)) matrix. The nonlinear equation (1) is Lorentz invariant, if condition (21) holds and
\[ S^\ast \beta S = \beta. \tag{22} \]
The conditions (21) and (22) can be rewritten in the form (cf formulas (23) and (27) from [12])
\[ S^\ast \beta S = \beta, \quad S^\ast \alpha_\mu S = \sum_{\nu=0}^3 \Lambda_{\mu\nu} \alpha_\nu \text{ with } \alpha_0 = I. \tag{23} \]
The existence of the matrix \( S \) satisfying conditions (23) follows from Pauli’s Fundamental Theorem (see, e.g., [31, § 4], [4, § 17]).

Let \( \Lambda_\nu : \mathbb{R}^4 \to \mathbb{R}^4 \) be a Lorentz transformation (boost) with velocity \( \nu \in \mathbb{R}^3, |\nu| < 1 \): \( \Lambda_\nu(t,x) = \left( \gamma(t + \nu \cdot x), \gamma(x^\| + \nu t) + x^\perp \right) \), where \( x^\| + x^\perp = x, x^\| || \nu, x^\perp \perp \nu, \gamma = (1 - \nu^2)^{-1/2} \). Hence (see, e.g., [34, formula (2.14)]),
\[ \Lambda_\nu = \begin{pmatrix} \gamma & \gamma \nu^T \\ \gamma \nu & I + \frac{\gamma - 1}{|\nu|^2} \nu \nu^T \end{pmatrix}, \quad \text{where } \nu \nu^T = (\nu_i \nu_j)_{i,j=1}^3, \tag{24} \]
i.e.,
\[ \Lambda_\nu(t,x) = \left( \gamma(t + \nu \cdot x), x + (\gamma - 1) \frac{\nu \cdot x}{|\nu|^2} + \gamma \nu t \right), \quad (t,x) \in \mathbb{R}^4. \tag{25} \]
Note that \( \det \Lambda_\nu = 1 \) and \( \Lambda_\nu^{-1} = \Lambda_{-\nu} \). The matrix \( S_\nu \equiv S(\Lambda_\nu) \) can be chosen as
\[ S_\nu = \sqrt{\frac{\gamma + 1}{2}} \left( I + \alpha \cdot \nu \frac{\gamma}{\gamma + 1} \right) = \exp \left( \frac{\xi}{2} \frac{\alpha \cdot \nu}{|\nu|} \right), \tag{26} \]
where number \( \xi \) such that \( \cosh(\xi/2) = \sqrt{(\gamma + 1)/2} \) (or \( \tanh(\xi) = |\nu| \)). It is easy to verify that
\[ S_0 = I, \quad S_\nu^* = S_\nu, \quad S_{-\nu} = S_\nu^{-1}, \quad S_\nu^2 = \gamma(\alpha \cdot \nu + I), \]
\[ S_\nu^* \beta S_\nu = \beta, \quad S_\nu^* \alpha_j S_\nu = \alpha_j + \gamma v_j I + v_j \frac{\gamma - 1}{|\nu|^2} \alpha \cdot \nu, \quad j = 1, 2, 3. \tag{27} \]
and conditions (23) hold. In particular,
\[
\gamma (I - \alpha \cdot v) S_v = S_v^{-1},
\]
\[
\alpha \cdot S_v \left( \nabla \varphi + v \frac{\gamma - 1}{|v|^2} \nabla \varphi \cdot v \right) - \gamma S_v \nabla \varphi \cdot v = S_v^{-1} \alpha \cdot \nabla \varphi.
\] (28)

Let \( \omega \in (0, m) \), and \( \psi_0(t, x) = e^{-i \omega t} \varphi(x) \), be a standing solitary wave. By \( \psi_v(t, x) \) we denote a (moving) solitary wave with velocity \( v \in \mathbb{R}^3, |v| < 1 \):
\[
\psi_v(t, x) = S_v \psi_0(\Lambda_v^{-1}(t, x)).
\]

In other words,
\[
\psi_v(t, x) = e^{-i \omega \gamma (t - v \cdot x)} S_v \varphi \left( x + (\gamma - 1) \frac{x \cdot v}{|v|^2} - \gamma vt \right).
\] (29)

Then, the following result is true.

**Lemma 3.** (i) Let \( \varphi(x) \) be a non-zero solution to Eqn (5). Then solitary waves \( \psi_v(t, x) \) satisfy Eqn (1). This follows from (28) and (29). Indeed, substituting \( \psi_v(t, x) \) in Eqn (1) and using (27) and (28) we obtain
\[
\frac{i \partial_t + i \alpha \cdot \nabla - m \beta + g(\varphi \varphi \beta)}{\omega \gamma (I - \alpha \cdot v) S_v \varphi} = e^{i \omega \gamma (t - v \cdot x)} \left[ \omega \gamma (I - \alpha \cdot v) S_v \varphi \right]
\]
\[
+ i \alpha \cdot S_v \left( \nabla \varphi + v \frac{\gamma - 1}{|v|^2} \nabla \varphi (y) \cdot v \right) - \gamma S_v \nabla \varphi (y) \cdot v - m \beta S_v \varphi (y) + g(\varphi \varphi \beta) S_v \varphi \right]
\]
\[
e^{-i \omega \gamma (t - v \cdot x)} S_v^{-1} \left[ \omega \gamma (I - \alpha \cdot v) S_v \varphi \right] = 0,
\]
with \( y = x + v(\gamma - 1)x \cdot v/|v|^2 - \gamma vt \).

(ii) Let \( \psi'(t', x') = S(\Lambda) \psi(t, x) \), where \( \Lambda \) is a Lorentz transformation. Denote by \( J(t', x) \) the 4-current, \( J^\mu(t, x) = \psi^*(t, x) \alpha_\mu \psi(t, x) \) (with \( \alpha_0 = 1 \)) and let \( J^\mu(t, x) = \psi^*(t, x) \alpha_\mu \psi(t, x) \). Then \( J(t', x') = \Lambda J(t, x) \), where \( (t', x') = \Lambda (t, x) \). In particular, if \( \Lambda \) is a boost, i.e., \( \Lambda = \Lambda_v \) with \( v \in \mathbb{R}^3 \), and \( \psi = \psi_v \) with \( \psi_v \) from (29), then
\[
J_v(t, x) = \Lambda_v J_0(t) = \Lambda_v \left( \varphi_v^*(y) \varphi_v(y) \right),
\]
where \( J_v = (J^\mu)_\mu = 0 \), \( J^\varphi_v = \psi_v^* \alpha_\mu \psi_v, J^\mu_v = \psi_v^* \alpha_\mu \psi_v, y = x + v(\gamma - 1)x \cdot v/|v|^2 - \gamma vt \).

For simplicity, put \( v = (0, 0, v) \in \mathbb{R}^3 \) and denote by \( \Lambda_v \) the Lorentz transformation (boost) \( \Lambda_v \) in this case:
\[
\Lambda_v : (t, x) \rightarrow (\gamma (t + vx_3), x_1, x_2, \gamma (x_3 + vt)), \quad |v| < 1.
\]

Then, the solitary waves \( \psi_v(t, x) : = \psi_v(t, x) |_{v=(0,0,v)} \) are
\[
\psi_v(t, x) = S_v \psi_0(\Lambda_v^{-1}(t, x)) = e^{-i \omega \gamma (t - vx_3)} S_v \varphi(x_1, x_2, \gamma (x_3 - vt));
\] (30)

the matrix \( S_v \) (cf (27), (28)) is defined as
\[
S_v = \sqrt{\frac{\gamma + 1}{2}} \left( I + \alpha_3 \frac{v\gamma}{\gamma + 1} \right) = \sqrt{\frac{\gamma + 1}{2}} \left( \begin{array}{cc}
I & \frac{v\gamma}{\gamma + 1} \\
\frac{v\gamma}{\gamma + 1} & I
\end{array} \right), \quad v \in \mathbb{R}^1.
\] (31)

Using the explicit formula (31), we obtain the following properties of \( S_v \):
\[
S_0 = I, \quad S_v^{*} = S_v, \quad S_{-v} = S_v^{-1}, \quad S_v^{*} \beta S_v = \beta,
\]
\[
S_v^{*} \alpha_3 S_v = \gamma (vI + \alpha_3), \quad S_v^{*} S_v = \gamma (v\alpha_3 + I), \quad S_v^{*} \alpha_k S_v = \alpha_k, \quad k = 1, 2.
\] (32)

In particular, \( \gamma S_v^{*} (\alpha_3 - vI) S_v^{*} = \alpha_3, \quad \gamma S_v^{*} (I - \alpha_3 v) S_v = I \).

Given \( v = (v_1, v_2, v_3) \in \mathbb{R}^3 \), we impose the following conditions on \( \varphi(x) \).
\( \mathbf{C1} \int \varphi^* \nabla \varphi dx \cdot \mathbf{v} = 0. \)

\( \mathbf{C2} \sum_{k,j=1,2,3,k \neq j} v_k v_j \int \varphi^* \alpha_k \partial_j \varphi dx = 0. \)

**Theorem 2.** Let \( \mathbf{v} \in \mathbb{R}^3 \) with \(|\mathbf{v}| < 1\), \( \psi(t, x) \) be a solitary wave of the form (29), and \( \varphi \) satisfy conditions \( \mathbf{C1} \) and \( \mathbf{C2} \). Then

\[
\mathcal{E}_\psi := \mathcal{E}(\psi_\psi) = \gamma \mathcal{E}_0. \tag{33}
\]

**Proof.** We first consider the particular case \( \mathbf{v} = (0, 0, v) \in \mathbb{R}^3 \). Then \( \psi_\psi(t, x) \) is defined in (30). Substitute the function \( \psi_\psi \) into (3) and apply equalities (32):

\[
\mathcal{E}_\psi := \int \left( -i \sum_{k=1}^2 \varphi^* S^k_\alpha k S^3_\alpha \partial_k \varphi \varphi^* S^3_\alpha k \partial_k \varphi - i \varphi^* S^3_\alpha k \partial_k \varphi \varphi^* S^3_\alpha k \partial_k \varphi - m \varphi^* S^3_\alpha k \beta S^3_\alpha k \varphi - G(\varphi) \right) dx
=
\int \left( \gamma^2 \varphi^* (v + \alpha_3) (\omega \varphi - i \partial_3 \varphi) - i \sum_{k=1}^2 \varphi^* \alpha_k \partial_k \varphi \varphi^* \alpha_k \partial_k \varphi - m \varphi^* \varphi - G(\varphi) \right) dx,
\]

where \( \varphi \equiv \varphi(x_1, x_2, \gamma(x_3 - vt)) \). Changing variables \( x = (x_1, x_2, x_3) \to y := (x_1, x_2, \gamma(x_3 - vt)) \), we obtain

\[
\mathcal{E}_\psi = \omega \gamma v(\varphi^*, (v + \alpha_3) \varphi^*) - i \gamma v(\varphi^*, \partial_3 \varphi) + \gamma I_3 + \frac{1}{\gamma} (I_1 + I_2) + \frac{1}{\gamma} V.
\]

In particular,

\[
\mathcal{E}_0 \equiv \mathcal{E}(\psi_0) = I_1 + I_2 + I_3 + V = 3I_3 + V,
\]

since \( I_1 = I_2 = I_3 \) by the identity (12). Applying equalities (11) and (16), one obtains

\[
\omega(\varphi^*, (v + \alpha_3) \varphi) = v_1 Q + \omega(\varphi^*, \alpha_3 \varphi) = v(V + 2I_3) - i(\varphi^*, \partial_3 \varphi).
\]

Therefore,

\[
\mathcal{E}_\psi = \gamma v(vV + 2vI_3 - i(\varphi^*, \partial_3 \varphi)) - i(\varphi^*, \partial_3 \varphi) + \gamma I_3 + \frac{2}{\gamma} I_3 + \frac{1}{\gamma} V = \gamma \mathcal{E}_0 - 2 \gamma v i(\varphi^*, \partial_3 \varphi).
\]

Hence, identity (33) holds iff \( (\varphi^*, \partial_3 \varphi) = 0 \), what follows from condition \( \mathbf{C1} \).

In the general case of \( \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 \), we substitute \( \psi_\psi \) from (29) in (3), apply equalities (27), change variables \( x \to y := x + v(\gamma - 1)x \cdot \mathbf{v}/|\mathbf{v}|^2 - \gamma vt \), use formulas (11) and (16) and obtain \( \mathcal{E}_\psi = \gamma \mathcal{E}_0 + \eta_\psi \), where, by definition,

\[
\eta_\psi := -2i \gamma (\varphi^*, \nabla \varphi) \cdot \mathbf{v} - i \gamma \sum_{j,k,j \neq k} (\varphi^*, \alpha_k \partial_j \varphi) v_k v_j. \tag{35}
\]

By conditions \( \mathbf{C1} \) and \( \mathbf{C2} \), \( \eta_\psi = 0 \). Hence, identity (33) is valid. \( \square \)

Let \( \mathbf{v} \in \mathbb{R}^3 \) and \( \psi_\psi \) be of the form (29). Write \( P_\psi := P(\psi_\psi) \), where \( P(\psi) \) stands for the momentum operator,

\[
P(\psi) := -i \int \psi^*(t, x) \nabla \psi(t, x) dx.
\]

To prove the next result for \( P_\psi \), we impose conditions \( \mathbf{C1}' \) and \( \mathbf{C2}' \) which are stronger than conditions \( \mathbf{C1} \) and \( \mathbf{C2} \).

\( \mathbf{C1}' \int \varphi^* \nabla \varphi dx = 0. \)

\( \mathbf{C2}' \) For \( \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 \) and every \( j = 1, 2, 3 \), \( \sum_{k,k \neq j} v_k \int \varphi^* \alpha_k \partial_j \varphi dx = 0. \)
Lemma 4. Let \( \varphi \) be a solution to Eqn (5) and conditions \( C1' \) and \( C2' \) hold. Then
\[
P_v = \gamma v \mathcal{E}_0,
\]
where \( \mathcal{E}_0 \) is defined in (34).

Proof. By (27) and (29), we have
\[
P_v = -i \int \varphi^* (\alpha \cdot v + I) \left( i \omega \gamma \varphi + \nabla \varphi + v \kappa \nabla \varphi \cdot v \right) dx \quad \text{with} \quad \kappa := \frac{\gamma - 1}{|v|^2},
\]
where \( \varphi \equiv \varphi(y) \) with \( y := x + v \kappa (x \cdot v) - \gamma vt. \) Since \( dy = \gamma dx, \) changing variables \( x \to y \) gives
\[
P_v = -i \int \varphi^* (\alpha \cdot v + I) \left( i \omega \gamma \varphi + \nabla \varphi + v \kappa \nabla \varphi \cdot v \right) dy.
\]
Using (9) and (16), we obtain \( P_v = \gamma v \mathcal{E}_0 - i \xi_v, \) where, by definition,
\[
\xi_v = v (\gamma + \kappa) \int \varphi^* \nabla \varphi \cdot v dy + \int \varphi^* \nabla \varphi dy + v \kappa \sum_{k,j} \int \varphi^* \alpha_k \partial_j \varphi dy v_k v_j
\]
\[
+ \left( \sum_{k \neq 1} v_k \int \varphi^* \alpha_k \partial_1 \varphi dy, \sum_{k \neq 2} v_k \int \varphi^* \alpha_k \partial_2 \varphi dy, \sum_{k \neq 3} v_k \int \varphi^* \alpha_k \partial_3 \varphi dy \right).
\]
In particular, if \( v = (0,0,v) \in \mathbb{R}^3, \)
\[
\xi_v = \left( \int \varphi^* (v \alpha_3 + 1) \partial_1 \varphi dy, \int \varphi^* (v \alpha_3 + 1) \partial_2 \varphi dy, \int \varphi^* (v \alpha_3 + 1) \partial_3 \varphi dy \right).
\]
By conditions \( C1' \) and \( C2', \) \( \xi_v = 0. \) Hence, identity (36) holds.

Remarks. (i) Conditions \( C1' \) and \( C2' \) (and also \( C1 \) and \( C2 \)) are fulfilled with any \( v \in \mathbb{R}^3 \) for four families of solutions considered in Section 2.1, see formulas (19) and (20).

(ii) Let \( \psi(t,x) \) be of the form (29) and condition \( C1' \) hold. Then, by (27), the charge functional is
\[
Q(\psi_v) = \int \psi_v^*(t,x) \psi_v(t,x) dx = \int \varphi^*(y)(\alpha \cdot v + I) \varphi(y) dy
\]
\[
= (\varphi^*, \varphi) + (\varphi^*, \alpha \varphi) \cdot v = \int |\varphi(y)|^2 dy.
\]
The last equality follows from (16) and condition \( C1'. \) Hence, \( Q(\psi_v) = Q(\psi_0) \) for any \( v \in \mathbb{R}^3. \)

(iii) Let \( v = (0,0,v) \in \mathbb{R}^3 \) and \( \varphi = \varphi_\omega \) be a solution to Eqn (5) from one of four families of solutions \( \varphi^1, \ldots, \varphi^4 \) considered in Section 2.1. Then applying the total angular momentum operator \( M_3 \) to \( \psi_v, \) we have
\[
M_3 \psi_v = e^{-i \omega \gamma (t-x_3)} S_v M_3 \varphi(x_1, x_2, \gamma(x_3 - vt)).
\]
Hence, if \( \varphi \in \{ \varphi^1, \varphi^2 \}, \) then \( M_3 \psi_v = (1/2) \psi_v. \) For \( \varphi \in \{ \varphi^3, \varphi^4 \}, M_3 \psi_v = -(1/2) \psi_v. \)

4. SOLITARY WAVES IN 1 + 1 DIMENSIONS

We consider the nonlinear Dirac equation in \( \mathbb{R}, \)
\[
i \dot{\psi} = -i \alpha \psi' + m \beta \psi - g(\bar{\psi}) \beta \psi, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}.
\]
Here \( \psi \equiv \psi(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2, \) \( \psi' := \partial_x \psi, \alpha = -\sigma_2, \beta = \sigma_3. \) In the case when \( g(s) = s, \) Eqn (38) is called the massive Gross–Neveu model or the 1D Soler model. The stationary states or localized solutions to Eqn (38) are the solutions of the form \( \psi(t,x) = e^{-i \omega t} \varphi_\omega, \omega \in (0,m), \) such that \( \varphi_\omega \in H^1(\mathbb{R}), \) and \( \varphi \equiv \varphi_\omega \) is a nonzero localized solution of the following stationary nonlinear Dirac equation
\[
i \alpha \varphi' + \omega \varphi - m \beta \varphi + g(\bar{\varphi}) \beta \varphi = 0, \quad x \in \mathbb{R}.
\]

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The solitary wave solutions have been studied, e.g., in [20, 21]. Write

\[ I = -i \int_{-\infty}^{+\infty} \varphi'^* \varphi' dy, \quad Q = \int_{-\infty}^{+\infty} \varphi'^* \varphi dx, \quad V = \int_{-\infty}^{+\infty} (m \varphi' - G(\varphi')) dy. \]

Note that

\[ \omega Q = V. \]  

(39)

This equality can be proved used Derrick’s technique similarly to (12). For \( v \in \mathbb{R}, |v| < 1, \) introduce the “moving solitary waves” as

\[ \psi_v(t, x) = e^{-i \omega \gamma (t - vx)} S_v \varphi(y = x - vt), \quad S_v = \sqrt{\gamma + 1} I + \frac{v \gamma}{\gamma + 1}, \quad x \in \mathbb{R}. \]

Note that \( \alpha^* = \alpha, \beta^* = \beta, \alpha^2 = \beta^2 = I, \alpha \beta + \beta \alpha = 0. \) Hence, \( S_v^* \beta S_v = \beta, \) \( S_v^* S_v = \gamma (vI + \alpha), \) \( S_v^* \alpha S_v = \gamma (vI + \alpha). \) Consider

\[ \mathcal{E}_v := \mathcal{E}(\psi_v) = \int_{-\infty}^{+\infty} (-i \varphi'^* \varphi' + m \psi_v \psi_v - G(\psi_v \psi_v)) \ dx. \]

Using the properties \( S_v, \) we obtain

\[ \mathcal{E}_v = \int_{-\infty}^{+\infty} \left(-i \varphi'^* S_v^* \alpha S_v (i \omega v \varphi + \gamma \varphi') + m \varphi' - G(\varphi')\right)_{\varphi \equiv \varphi(y = x - vt)} \ dx \]

\[ = -i \gamma \int_{-\infty}^{+\infty} \varphi'^* (y) (vI + \alpha)(v \omega \varphi(y) + \varphi'(y)) \ dy + \frac{1}{\gamma} V. \]

In the last integral we changed variable \( x \rightarrow y = \gamma (x - vt). \) Hence,

\[ \mathcal{E}_v = \gamma v^2 \omega Q + \gamma v \omega (\varphi^*, \alpha \varphi) - i \gamma v (\varphi^*, v \varphi') + \gamma I + \frac{1}{\gamma} V. \]

In particular,

\[ \mathcal{E}_0 = \int_{-\infty}^{+\infty} (-i \varphi'^* \varphi + m \varphi' - G(\varphi')) dx = I + V. \]

We apply equalities (39) and \( \omega(\varphi^*, \alpha \varphi) = -i(\varphi^*, \varphi') \) (cf (16)) and obtain

\[ \mathcal{E}_v = \gamma \mathcal{E}_0 - 2i \gamma v (\varphi^*, \varphi'). \]

Assuming that \( \varphi \) satisfies the property \( (\varphi^*, \varphi') = 0 \) (cf condition \( C1 \) or \( C1' \)), we have \( \mathcal{E}_v = \gamma \mathcal{E}_0. \) Moreover, under the same condition on \( \varphi, \) one obtains

\[ P_v = -i \int_{-\infty}^{+\infty} \psi_v'^* (t, x) \psi_v' (t, x) dx = \gamma v \mathcal{E}_0 - i (v^2 + 1) \int_{-\infty}^{+\infty} \varphi'^* (x) \varphi'(x) dx = \gamma v \mathcal{E}_0. \]

5. MAXWELL–DIRAC EQUATIONS

We use natural units, in which we have rescaled length and time so that \( h = c = e = 1. \) Then, in the Lorentz gauge, the (MD) system reads

\[ i \dot{\psi} = -i \alpha \cdot \nabla \psi + \Phi \psi - \alpha \cdot A \psi + m \beta \psi, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}, \]

(40)
\[
\begin{align*}
\ddot{\Phi} - \Delta \Phi &= 4\pi\rho \\
\ddot{A} - \Delta A &= 4\pi J \\
\nabla \cdot A + \dot{\Phi} &= 0.
\end{align*}
\] (41)

Here \(\psi\) describes the charged Dirac spinor, \(\psi \equiv \psi(t, x) \in \mathbb{C}^4\), \(A \equiv A(t, x) = (A^1, A^2, A^3)\) and \(\Phi \equiv \Phi(t, x)\) are the classical electromagnetic potentials, \(m > 0\), \(\beta\) and \(\alpha = (\alpha_1, \alpha_2, \alpha_3)\) are Pauli–Dirac matrices, \(\rho \equiv \rho(t, x)\) is charge density, \(J \equiv J(t, x)\) is electric current. By definition,

\[\rho = \psi^* \psi, \quad J = \psi^* \alpha \psi.\]

We also introduce notation \(J \equiv (J^\mu) = (\rho, J)\) for the 4-electromagnetic current \(J^\mu = \bar{\psi} \gamma^\mu \psi\), and \(A \equiv (A^\mu) = (\Phi, A)\) for the 4-potential of the electromagnetic field. Below we use the notations \(J_\mu\) and \(A_\mu\) with lower index \(\mu\) for vectors \((J_\mu) = (\rho, -J)\) and \((A_\mu) = (\Phi, -A)\). Eqn (42) is called the Lorentz gauge condition.

The magnetic and electric fields \(H \equiv H(t, x)\) and \(E \equiv E(t, x)\) are given by

\[H = \text{rot} A \equiv \nabla \times A, \quad E = -\dot{A} - \nabla \Phi.\] (43)

Then, by condition (42), Eqns (41) become classical Maxwell’s equations of electrodynamics

\[
\begin{align*}
\dot{H} &= -\text{rot} E, \quad \dot{E} = \text{rot} H - 4\pi J, \quad \nabla \cdot E &= 4\pi \rho, \quad \nabla \cdot H = 0.
\end{align*}
\]

As shown, e.g., in [5, 32], this model is based on the Lagrangian density \(L_Q = L_D + L_M + L_I\). Here \(L_D\) and \(L_M\) are Lagrangian densities for the free Dirac and electromagnetic fields, resp., \(L_I\) is extra term describing the interaction between \(\psi\) and the electromagnetic field,

\[L_D = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi, \quad L_M = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} = \frac{1}{8\pi} (E^2 - H^2), \quad L_I = -J_\mu A^\mu = -\rho \Phi + J \cdot A,\]

where \(F^{\mu\nu}\) stands for the electromagnetic field tensor, \(F^{\mu\nu} := \partial_\nu A_\mu - \partial_\mu A_\nu,\ \partial_\mu = \partial / \partial x^\mu,\ \mu, \nu = 0, 1, 2, 3,\ F^{\mu\nu} := \partial_\nu A^\mu - \partial_\mu A^\nu,\ \partial_\mu = \partial / \partial x^\mu.\) Other words,

\[L_Q \equiv L_Q(\psi, A) = \psi^* \left(i \partial_t + i \alpha \cdot \nabla - \Phi + \alpha \cdot A - m\beta\right) \psi + \frac{1}{8\pi} \left(|\dot{A} + \nabla \Phi|^2 - |\text{rot} A|^2\right).\] (44)

It is easy to check that the Euler–Lagrange equations applied to (44) give Eqn (40) and

\[\left(\partial_t^2 - \Delta\right) A^\mu - \partial^\mu (\partial_\nu A^\nu) = J^\mu.\]

Due to the Lorentz gauge (42), we obtain Eqns (41).

Since \(\partial L_Q / (\partial \dot{\Phi}) = 0\), the Hamiltonian density equals

\[H(\psi, A) = \frac{\partial L_Q}{\partial \dot{\psi}} \cdot \dot{\psi} + \frac{\partial L_Q}{\partial \dot{A}} \cdot \dot{A} - L_Q = i\psi^* \cdot \dot{\psi} + \frac{1}{4\pi} (\dot{A} + \nabla \Phi) \cdot \dot{A} - L_Q = \psi^* \left[\alpha \cdot (-i \nabla - A) + \dot{\Phi} + m\beta\right] \psi + \frac{1}{4\pi} E \cdot \nabla \Phi + \frac{1}{8\pi} (E^2 + H^2).\]

Hence, the energy functional of system (40)–(42) reads (cf. [5])

\[E(t) \equiv E(\psi(t, \cdot), A(t, \cdot)) = \int H(\psi(t, \cdot), A(t, \cdot)) dx\]

\[= \int \psi^* \left[\alpha \cdot (-i \nabla - A) \psi + m\beta \psi\right] dx + \frac{1}{8\pi} \int (E^2 + H^2) dx,\] (45)

where \(E\) and \(H\) are defined in (43). Here we use the fact that

\[\int E \cdot \nabla \Phi dx = -4\pi \int \rho \Phi dx.\]

Evidently, \(\dot{E}(t) = 0.\)
5.1. Standing Solitary Waves

Let \( \omega \in (-m, m) \). Consider stationary solutions \((\psi, A)\) to system (40)-(42), where \( \psi(t, x) = e^{-i\omega t}\varphi(x) \) and the field \( A = (\Phi, A) \) does not depend on \( t \). Such stationary solutions we denote by \((\psi_0, A_0)\). Substituting these solutions in system (40)-(41) we obtain

\[
\begin{align*}
(-\omega + \Phi_0 - i\alpha \cdot \nabla - \alpha \cdot A_0 + m\beta) \varphi &= 0, \quad x \in \mathbb{R}^3, \\
-\Delta \Phi_0 &= 4\pi \rho_0 = 4\pi \varphi^\ast \varphi, \\
-\Delta A_0^\mu &= 4\pi J_0^\mu = 4\pi \varphi^\ast \alpha_k \varphi, \quad k = 1, 2, 3
\end{align*}
\]

By (47), \( A_0^\mu = \varphi^\ast \alpha_\mu \varphi \ast (1/|x|) \) (with \( \alpha_0 \equiv 1 \)), \( \mu = 0, 1, 2, 3 \), i.e.,

\[
\Phi_0(x) = \int \frac{\rho_0(y)}{|x - y|} dy, \quad A_0(x) = \int \frac{J_0(y)}{|x - y|} dy \quad \text{with} \quad \rho_0 = |\varphi|^2, \quad J_0 = \varphi^\ast \alpha \varphi.
\]

Note that the Lorentz condition (42) becomes

\[
\nabla \cdot A_0 = 0,
\]

what follows from (46) and (48). Using (47) and (49), we rewrite the energy associated with stationary states \((\psi_0, A_0)\) as

\[
\mathcal{E}_0 := \mathcal{E}(\psi_0, A_0) = \int \varphi^\ast [\alpha \cdot (-i\nabla - A_0) + m\beta] \varphi dx + \frac{1}{8\pi} \int (|\nabla \Phi_0|^2 + |\text{rot} A_0|^2) dx
\]

\[
= \int \varphi^\ast [-i\alpha \cdot \nabla + m\beta] \varphi dx + \frac{1}{2} \int (\rho_0 \Phi_0 - J_0 \cdot A_0) dx.
\]

The last integral in (50) is

\[
\frac{1}{2} \int J_\mu(x) A_0^\mu(x) dx = \frac{1}{2} \int \int \frac{J_\mu(x) J_\mu^*(y)}{|x - y|} dxdy.
\]

**Definition 2.** The stationary states or standing waves \((\psi_0(t, x), A_0(x)) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4 \times \mathbb{R}^4\) are the solutions of the (MD) system of a form

\[
\psi_0(t, x) = e^{-i\omega t}\varphi_\omega(x),
\]

\[
A_0^\mu(x) = J_\mu * \frac{1}{|x|} \int \frac{J_\mu(y)}{|x - y|} dy, \quad \mu = 0, 1, 2, 3.
\]

Here \( \omega \in (-m, m) \), \( (J_\mu) = (\varphi^\ast \alpha_\mu \varphi_\omega) = (\rho_0, J_0) \), and \( \varphi \equiv \varphi_\omega \) is a solution of (46).

The stationary solutions of the (MD) system were studied numerically by Lisi [22]. Earlier, these solutions were studied numerically by Wakano [35] in the approximation \( A_0^0 \neq 0, A_0^k = 0 \) for \( k = 1, 2, 3 \), i.e., for the so-called Dirac–Poisson system. Using variational methods, Esteban, Georgiev and Séré [15] have proved the existence of stationary solutions to the (MD) system with \( \omega \in (-m, 0) \). To state this result we introduce a functional

\[
I_\omega^0(\varphi) := \frac{1}{2} \int \mathcal{L}_Q(\psi_0, A_0) dx = \frac{1}{2} \int \varphi^\ast (i\alpha \cdot \nabla - m\beta + \omega) \varphi dx - \frac{1}{4} \int \int J_\mu(x) J_\mu^*(y) \frac{1}{|x - y|} dxdy.
\]

Note that if \((\psi_0, A_0)\) is a solution to the (MD) system of the form (51), then (formally) \( \varphi_\omega \) is a critical point of \( I_\omega^0(\varphi) \).

**Theorem 3** (see [15, Theorem 1]). For any \( \omega \in (-m, 0) \), there exists a non-zero critical point \( \varphi_\omega \in H^{1/2}(\mathbb{R}^3; \mathbb{C}^4) \) of the functional \( I_\omega^0(\varphi) \). Moreover, \( \varphi_\omega \) is a smooth function of \( x \) exponentially decreasing at infinity with all its derivatives, and \( \psi(x, t) = e^{-i\omega t}\varphi_\omega(x) \), \( A_\mu(x, t) = J_\mu * (1/|x|) \) are the solutions of the (MD) system.
5.2. Virial Identities

The following virial identity was proved in [17, Proposition 3.1].

**Lemma 5.** Let \( \varphi \in H^1(\mathbb{R}^3; \mathbb{C}^4) \) be a solution to Eqn (46). Then \( \varphi(x) \) satisfies

\[
i \int \varphi^* \alpha \cdot \nabla \varphi dx = \frac{3}{2} \int \left( m \varphi^* \varphi - \omega \varphi^* \varphi + \frac{5}{6} J_\mu(x) A^\mu(x) \right) dx, \tag{52}\]

where \( J_\mu A^\mu = \rho_0 \Phi_0 - J_0 \cdot A_0, \rho_0 = |\varphi|^2, J_0 = \varphi^* \alpha \varphi. \)

Let functionals \( I_k(\varphi), k = 1, 2, 3, \) and \( Q(\varphi) \) be as in (10). Also, we put

\[
T \equiv T(\varphi) = \int \left( \rho_0(x) \Phi_0(x) - J_0(x) \cdot A_0(x) \right) dx = (2\pi)^{-3} 4\pi \int \frac{\rho_0(k)^2 - |\mathcal{J}_0(k)|^2}{k^2} dk,
\]

\[
m_0 \equiv m_0(\varphi) = m \int \varphi \phi dx. \tag{53}\]

Here \( \hat{\rho}_0(k) \) and \( \hat{\mathcal{J}}_0(k) \) denote the Fourier transform of \( \rho_0(x) \) and \( J_0(x) \), respectively.

**Remark.** Formally, the identity (52) can be proved using Derrick’s technique [10, p. 1253]. Indeed, using notations (53), we rewrite \( I_Q^0(\varphi) \) as

\[
I_Q^0(\varphi) = \frac{1}{2} \left( \omega Q(\varphi) - m_0(\varphi) - I_1(\varphi) - I_2(\varphi) - I_3(\varphi) \right) - \frac{1}{4} T(\varphi), \tag{54}\]

and introduce \( \varphi_\lambda(x) = \varphi(x/\lambda) \). Then \( T(\varphi_\lambda) = \lambda^5 T(\varphi), \ I_k(\varphi_\lambda) = \lambda^2 I_k(\varphi), \ Q(\varphi_\lambda) = \lambda^3 Q(\varphi), \ m_0(\varphi_\lambda) = \lambda^3 m_0(\varphi). \) Hence,

\[
0 = \frac{d}{d\lambda} \bigg|_{\lambda=1} I_Q^0(\varphi_\lambda) = \frac{1}{2} \frac{d}{d\lambda} \bigg|_{\lambda=1} \left[ \omega Q(\varphi_\lambda) - m_0(\varphi_\lambda) - I_1(\varphi_\lambda) - I_2(\varphi_\lambda) - I_3(\varphi_\lambda) - \frac{1}{2} T(\varphi_\lambda) \right]

= \frac{3}{2} \left( \omega Q(\varphi) - m_0(\varphi) \right) - I_1(\varphi) - I_2(\varphi) - I_3(\varphi) - \frac{5}{4} T(\varphi).

Hence,

\[
I_1 + I_2 + I_3 = \frac{3}{2} (\omega Q - m_0) - \frac{5}{4} T, \tag{55}\]

and the identity (52) holds.

**Corollary 2.** (i) Eqn (46) implies the following equality,

\[
\omega Q - m_0 - (I_1 + I_2 + I_3) = T
\]

(cf. [11, p. 238] or [15, formula (3.10)]). Hence, by (55),

\[
\omega Q - m_0 = \frac{1}{2} T. \tag{56}\]

Moreover,

\[
I_1 + I_2 + I_3 = -\frac{1}{2} T. \tag{57}\]

In particular, \( I_Q^0(\varphi) = -(I_1 + I_2 + I_3)/2 = T/4 \), by (54), (56) and (57).

(ii) Using (50) and (53), we rewrite \( \mathcal{E}_0 \) as

\[
\mathcal{E}_0 = I_1 + I_2 + I_3 + m_0 + \frac{1}{2} T. \tag{58}\]

By (57) and (58), we obtain \( \mathcal{E}_0 = m_0 \).

**Lemma 6.** The following identity holds,

\[
I_j = \frac{1}{2} (\omega Q - m_0) - \frac{3}{4} T + T_j, \quad j = 1, 2, 3, \tag{59}\]
where

\[
T_j := 4\pi(2\pi)^{-3} \int \frac{k^2(\rho_0(k)^2 - J_0(k)^2)}{k^4} \, dk = \frac{1}{4\pi} \int \left( |\partial_j \Phi_0(x)|^2 - |\partial_j A_0(x)|^2 \right) \, dx.
\]  

(60)

Proof. Introduce \( \varphi_\lambda(x) = \varphi(x_1/\lambda, x_2, x_3) \). Then \( I_1(\varphi_\lambda) = I_1(\varphi), \ I_k(\varphi_\lambda) = \lambda I_k(\varphi), \ k = 2, 3, \ Q(\varphi_\lambda) = \lambda Q(\varphi), \ m_0(\varphi_\lambda) = \lambda m_0(\varphi), \) and

\[
T(\varphi_\lambda) = (2\pi)^{-3}4\pi \lambda \int \frac{\rho_0(k)^2 - J_0(k)^2}{k^4} \, dk.
\]

Hence,

\[
0 = \frac{d}{d\lambda} \Bigg|_{\lambda=1} I_Q^\lambda(\varphi_\lambda) = \frac{1}{2} \left[ \omega Q - m_0 - (I_2 + I_3) - \frac{1}{2}(T + 2T_1) \right].
\]

Therefore, \( I_2 + I_3 = \omega Q - m_0 - (T + 2T_1)/2 \). Together with (55), the last identity implies (59) with \( j = 1 \). Similarly, introducing \( \varphi_\lambda(x) = \varphi(x_1, x_2/\lambda, x_3) \) or \( \varphi_\lambda(x) = \varphi(x_1, x_2, x_3/\lambda) \), we can verify (59) with \( j = 2, 3 \).

**Corollary 3.** Since \( T_1 + T_2 + T_3 = T \), identity (59) implies (55). Moreover, by (56), we have \( I_j = -T_j/2 + T_j, \ j = 1, 2, 3 \).

**Lemma 7.** Let \( \varphi \) be a solution to Eqn (46). Then

\[
i(\varphi^*, \nabla \varphi) = -\omega(\varphi^*, \alpha \varphi).
\]

(61)

Proof. Using Eqn (46) and relations (2), we obtain

\[
i \int \varphi^*(x) \nabla \varphi(x) \, dx + \omega \int \varphi^*(x) \alpha \varphi(x) \, dx = \int \left( J_0(x) \Phi_0(x) - \rho_0(x) A_0(x) \right) \, dx.
\]

Since the field \((\Phi_0, A_0)\) is of the form (48), then

\[
\int J_0(x) \Phi_0(x) \, dx = \int \rho_0(x) A_0(x) \, dx.
\]

(62)

Hence, if \((\varphi, \Phi_0, A_0)\) is a solution of system (46)–(47), then (61) is true (cf formula (16)).

**5.3. A Particular Ansatz of Stationary Solutions**

Abenda [1, Theorem A] extended the results of Theorem 3 to \( \omega \in (-m, m) \) and proved the existence of the particular ansatz of solutions to Eqns (46), (47) in the form

\[
\varphi_\omega(x) = \begin{pmatrix}
u_1(r, z)e^{i(m_3+1/2)\phi} \\
u_2(r, z)e^{i(m_3+1/2)\phi} \\
u_3(r, z)e^{i(m_3-1/2)\phi} \\
u_4(r, z)e^{i(m_3+1/2)\phi}
\end{pmatrix}
\text{ with } m_3 = \pm \frac{1}{2},
\]

(63)

\[
\Phi_0(x) = \Phi_\ast(r, z), \quad A_0(x) = A_\ast(r, z)(-\sin \phi, \cos \phi, 0),
\]

(64)

where \((r, z, \phi)\) are the cylindric coordinates of \( x \in \mathbb{R}^3 (x_1 = r \cos \phi, x_2 = r \sin \phi, x_3 = z) \). Moreover, \( u_1, u_2, u_3, u_4, \Phi_\ast \) and \( A_\ast \) are scalar real-valued smooth functions exponentially decreasing at infinity with all its derivatives. The system of equations for \( u_1, u_2, u_3, u_4, \Phi_\ast, A_\ast \) was derived by Lisi [22].

It is easy to check the following result.

**Lemma 8.** The solutions \((\varphi_\omega, \Phi_0, A_0)\) of the form (63) and (64) have the following properties.

(i) \( \varphi_\omega \) are eigenfunctions of the third component of the total angular momentum \( \mathbf{M} \) (see Section 2.1) with eigenvalues \( m_3 = \pm 1/2 \).

(ii) \( \int \varphi_\omega^*(x) \nabla \varphi_\omega(x) \, dx = 0 \).
(iii) $\int \varphi_\omega^2(x) \alpha_k \partial_j \varphi_\omega(x) dx = 0 \text{ for } k \neq j, k, j = 1, 2, 3.$

(iv) $\int \partial_t \Phi_0(x) \partial_j \Phi_0(x) dx = 0$ and $\int \partial_t A_0(x) \cdot \partial_j A_0(x) dx = 0 \text{ for } i \neq j, i, j = 1, 2, 3.$

(v) The charge density is $\rho_0 \equiv \rho_0(r, z) = u_1^2 + u_2^2 + u_3^2$, the current $J_0$ is

$$J_0(x) = \psi_0^*(t, x) \alpha \psi_0(t, x) = \varphi_\omega^*(x) \alpha \varphi_\omega(x) = 2(u_1 u_4 - u_2 u_3)(\sin \phi, - \cos \phi, 0).$$

Furthermore, using (64), we have

$$E_0(x) := -\nabla \Phi_0(x) = -(\cos \phi \partial_r \Phi_\omega, \sin \phi \partial_r \Phi_\omega, \partial_z \Phi_\omega),$$

$$H_0(x) := \text{rot} A_0(x) = -(\cos \phi \partial_z A_\omega, - \sin \phi \partial_z A_\omega, \partial_r A_\omega + A_\omega/r).$$

5.4. Moving Solitary Waves

Consider travelling solutions $(\psi_\omega, A_\omega)$, where $A_\omega = (\Phi_\omega, A_\omega)$, with velocity $v \in \mathbb{R}^3, |v| < 1$:

$$\psi_\omega(t, x) = S_\omega \psi_0(A_\omega^{-1}(t, x),$$

$$A_\omega(t, x) = \Lambda_\omega A_0(y) \text{ with } y = x + v \frac{(\gamma - 1)}{|v|^2} x \cdot v - \gamma vt. \quad (65)$$

Here the stationary solutions $(\psi_0, A_0)$ are introduced in Definition 2, $S_\omega$ is defined in (26), $\Lambda_\omega$ is a Lorentz transformation defined in (24). It is easily to check that $(\psi_\omega, A_\omega)$ is a solution of the (MD) system. Indeed, first, similarly to Lemma 3 (i), we obtain

$$i \dot{\psi}_\omega + i\alpha \cdot \nabla \psi_\omega - m_\beta \psi_\omega = e^{-i\omega(1-\gamma)x} S_\omega^{-1}(\alpha \cdot \nabla - m_\beta) \psi_\omega(y).$$

Here and below $y$ stands for the expression $y = x + v (\gamma - 1)x \cdot v / |v|^2 - \gamma vt$ (as in (65)). On the other hand, $S_\omega(\Phi_\omega(t, x) - \alpha \cdot A_\omega(t, x)) = S_\omega(\Phi_0(y) - \alpha \cdot A_0(y))$, hence

$$(\Phi_\omega(t, x) - \alpha \cdot A_\omega(t, x))\psi_\omega(t, x) = e^{-i\omega(1-\gamma)x} S_\omega^{-1}(\Phi_0(y) - \alpha \cdot A_0(y)) \psi_\omega(y),$$

and Eqn (40) is valid. To verify Eqn (41), we put $J_\omega^\mu = \psi_\omega \alpha^\mu \psi_\omega$. Then, by (65), (47), and Lemma 3 (ii), one obtains

$$(\partial_t^2 - \Delta) A_\omega(t, x) = \Lambda_\omega (\partial_t^2 - \Delta_x) A_0(y) = \Lambda_\omega (-\Delta_y A_0(y)) = 4\pi \Lambda_\omega J_0(y) = 4\pi J_\omega(t, x),$$

and Eqn (41) follows. Moreover, $\Phi_\omega(t, x) - \nabla x \cdot A_\omega(t, x) = \nabla y \cdot A_\omega(y) = 0$, i.e., the Lorentz gauge condition (42) is fulfilled.

Remark. Denote $E_\omega = -\nabla \Phi_0, H_\omega = \nabla \times A_0$, and $E_\omega = -A_\omega - \nabla \Phi_0, H_\omega = \nabla \times A_\omega, v \in \mathbb{R}^3$.

Then

$$E_\omega(t, x) = \gamma E_0(y) - \nu \kappa v \cdot E_0(y) - \gamma v \times H_0(y)$$

and

$$H_\omega(t, x) = \gamma H_0(y) - \nu \kappa v \cdot H_0(y) + \gamma v \times E_0(y)$$

$$\kappa := \frac{\gamma - 1}{|v|^2}. \quad (66)$$

We impose conditions $C_1$ and $C_2$ on $\varphi_\omega$ (see Section 3). Moreover, we assume the additional condition $C_0$ on $(\Phi_0, A_0)$.

$C_0$ For $v = (v_1, v_2, v_3) \in \mathbb{R}^3$, the following relation holds

$$\sum_{i,j:i \neq j} v_i v_j \int \left( \partial_t \Phi_0(x) \partial_j \Phi_0(x) - \partial_i A_0(x) \cdot \partial_j A_0(x) \right) dx = 0.$$

Remarks. (i) By Fourier transform and formulas (47), condition $C_0$ can be rewritten in the form

$$\sum_{i,j;i \neq j} v_i v_j \int \frac{k_i k_j}{k^4} \left( |\hat{\rho}_0(k)|^2 - |J_0(k)|^2 \right) dk = 0.$$
(ii) Conditions **C0**–**C2** are fulfilled, for instance, for the particular family of solutions considered in Section 5.3 (see Lemma 8 (ii)–(iv)). Obviously, conditions **C0** and **C2** are valid in the particular case when \( \mathbf{v} = (0,0,v) \).

Put \( \mathcal{E}_\mathbf{v} = \mathcal{E}(\psi_\mathbf{v}, A_\mathbf{v}), \mathbf{v} \in \mathbb{R}^3 \), where \( \mathcal{E} \) is defined in (45), i.e.,

\[
\mathcal{E}_\mathbf{v} = \int \left( \psi_\mathbf{v}^* (-i \alpha \cdot \nabla + m \beta) \psi_\mathbf{v} - J_\mathbf{v} \cdot A_\mathbf{v} \right) dx + \frac{1}{8\pi} \int (\mathbf{E}_\mathbf{v}^2 + \mathbf{H}_\mathbf{v}^2) dx.
\]

**Theorem 4.** Let \( (\psi_\mathbf{v}, A_\mathbf{v}) \) be a solitary wave of the (MD) system and conditions **C0**–**C2** hold. Then the "particle-like" energy relation holds, \( \mathcal{E}_\mathbf{v} = \gamma \mathcal{E}_0 \).

**Proof.** First we rewrite the term in \( \mathcal{E}_\mathbf{v} \) corresponding to the Dirac field,

\[
e_D := \int \left( \psi_\mathbf{v}^* (-i \alpha \cdot \nabla + m \beta) \psi_\mathbf{v} \right) dx
\]

\[
= \int \left( -i \varphi^* S \alpha \cdot S \left( i \gamma \omega \varphi + \nabla \varphi + \mathbf{v} \frac{\gamma - 1}{|\mathbf{v}|^2} \nabla \varphi \cdot \mathbf{v} \right) + m \varphi^* S \beta S \varphi \right) dx
\]

\[
= \int \left( \gamma^2 \omega \varphi^* [\alpha \cdot \mathbf{v} + \mathbf{v}^2] \varphi - i \varphi^* \left[ \gamma^2 (\alpha \cdot \mathbf{v} + 1) \mathbf{v} \cdot \nabla \varphi + \alpha \cdot \nabla \varphi \right] + m \varphi \right) dx,
\]

where \( \varphi \equiv \varphi(y) \) with \( y \) from (65). Here we apply formulas (27) and (28). We change variables \( x \to y = x + \mathbf{v}(\gamma - 1)x \cdot \mathbf{v}/|\mathbf{v}|^2 - \gamma t \), \( dt = dx/\gamma \). Using (61), we obtain

\[
e_D = \omega \gamma (\varphi^*, [\alpha \cdot \mathbf{v} + \mathbf{v}^2] \varphi) - i \gamma (\varphi^*, (\alpha \cdot \mathbf{v} + 1) \mathbf{v} \cdot \nabla \varphi) + \frac{1}{\gamma} (I_1 + I_2 + I_3) + \frac{1}{\gamma} m_0
\]

\[
= \omega Q \gamma \varphi^2 + \frac{1}{\gamma} (I_1 + I_2 + I_3) + \frac{1}{\gamma} m_0 + \gamma \sum_{j=1}^{3} v_j^2 I_j + \eta_\mathbf{v},
\]

(67)

where \( \eta_\mathbf{v} \) is defined in (35). Applying 'virial' identities (55) and (59), we obtain

\[
e_D = \gamma (I_1 + I_2 + I_3 + m_0) + \frac{1}{2} \gamma \varphi^2 T + \gamma \sum_{j=1}^{3} v_j^2 T_j + \eta_\mathbf{v}.
\]

Moreover, by (57),

\[
e_D = \gamma m_0 - \frac{1}{2\gamma} T + \gamma \sum_{j=1}^{3} v_j^2 T_j + \eta_\mathbf{v}.
\]

(68)

Second, we rewrite the "magnetic" term in \( \mathcal{E}_\mathbf{v} \), i.e., the term corresponding to the interaction. Since

\[
A_\mathbf{v}(t,x) = \gamma \mathbf{v} \Phi_0(y) + A_0(y) + \mathbf{v} \kappa A_0(y) \cdot \mathbf{v}
\]

\[
J_\mathbf{v}(t,x) \equiv \psi_\mathbf{v} \alpha \psi_\mathbf{v} = \gamma \mathbf{v} \rho_0(y) + J_0(y) + \mathbf{v} \kappa J_0(y) \cdot \mathbf{v}
\]

then, by (62), we have

\[
e_I = -\int A_\mathbf{v}(t,x) \cdot J_\mathbf{v}(t,x) dx
\]

\[
= -\int \left( \gamma \mathbf{v}^2 \Phi_0(y) \rho_0(y) + \frac{1}{\gamma} A_0(y) \cdot J_0(y) + \gamma \mathbf{v} \cdot A_0(y) \right) dy - R_\mathbf{v},
\]

(69)

where \( R_\mathbf{v} \) stands for the integral \( R_\mathbf{v} = 2\gamma \int \rho_0(y) \mathbf{v} \cdot A_0(y) dy \).

Further, using (66), we rewrite the energy corresponding to the electromagnetic field,

\[
e_M := \frac{1}{8\pi} \int (\mathbf{E}_\mathbf{v}^2 + \mathbf{H}_\mathbf{v}^2) dx = \frac{4\gamma}{8\pi} \int \mathbf{v} \cdot (\mathbf{H}_0 \times \mathbf{E}_0) dy.
\]

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Finally, substituting (68), (69) and (72) in the r.h.s. of (70) equals $R_v$. Using the formula
\begin{equation}
|a|^2 |b|^2 = |a \cdot b|^2 + |a \times b|^2 \quad \text{for} \quad a, b \in \mathbb{R}^3,
\end{equation}
the second integral in the r.h.s. of (70) can be rewritten as
\begin{equation}
\frac{\gamma(1 + v^2)}{8\pi} \int (\mathbf{E}_0^2 + \mathbf{H}_0^2) dy - \frac{\gamma}{4\pi} \int ((\mathbf{v} \cdot \mathbf{E}_0)^2 + (\mathbf{v} \cdot \mathbf{H}_0)^2) dy.
\end{equation}
Using formulas $\mathbf{E}_0 = -\nabla \Phi_0$, $\mathbf{H}_0 = \nabla \times \mathbf{A}_0$ and Eqns (47), we obtain
\begin{equation}
e_M = \frac{\gamma(1 + v^2)}{2} \int (\rho_0 \Phi_0 + \mathbf{J}_0 \cdot \mathbf{A}_0) dy - \frac{\gamma}{4\pi} \int ((\nabla \cdot \mathbf{E}_0)^2 + (\mathbf{v} \cdot (\nabla \times \mathbf{A}_0))^2) dy + R_v.
\end{equation}
Finally, substituting (68), (69) and (72) in $\mathcal{E}_v = e_D + e_I + e_M$ and using notations (53) and (60), we have
\begin{equation}
\mathcal{E}_v = \gamma m_0 - \frac{1}{2\gamma} \int (\rho_0 \Phi_0 - \mathbf{J}_0 \cdot \mathbf{A}_0) dy + \frac{\gamma}{4\pi} \sum_{j=1}^3 v_j^2 \int (|\partial_j \Phi_0|^2 - |\partial_j \mathbf{A}_0|^2) dy + \eta_v
\end{equation}
\begin{equation}\nonumber
- \int (\gamma v^2 \Phi_0 \rho_0 + \frac{1}{\gamma} \mathbf{A}_0 \cdot \mathbf{J}_0 + \gamma (\mathbf{v} \cdot \mathbf{A}_0) (\mathbf{v} \cdot \mathbf{J}_0)) dy - R_v
\end{equation}
\begin{equation}\nonumber
+ \frac{\gamma(1 + v^2)}{2} \int (\rho_0 \Phi_0 + \mathbf{J}_0 \cdot \mathbf{A}_0) dy - \frac{\gamma}{4\pi} \int ((\nabla \cdot \mathbf{E}_0)^2 + (\mathbf{v} \cdot (\nabla \times \mathbf{A}_0))^2) dy + R_v
\end{equation}
\begin{equation}\nonumber
= \gamma m_0 + \eta_v + \frac{\gamma}{4\pi} \left( \sum_{j=1}^3 v_j^2 \int |\partial_j \Phi_0|^2 dy - \int (\nabla \cdot \mathbf{E}_0)^2 dy \right)
\end{equation}
\begin{equation}
+ \frac{\gamma}{4\pi} \int \left( 4\pi v^2 \mathbf{A}_0 \cdot \mathbf{J}_0 - 4\pi (\mathbf{v} \cdot \mathbf{A}_0) (\mathbf{v} \cdot \mathbf{J}_0) - (\mathbf{v} \cdot (\nabla \times \mathbf{A}_0))^2 - \sum_{j=1}^3 v_j^2 |\partial_j \mathbf{A}_0|^2 \right) dy.
\end{equation}
Using Fourier transform, relation $\hat{\mathbf{J}}_0(k) = k^2 \hat{\mathbf{A}}_0(k)/(4\pi)$ and formula (71), we rewrite the last integral in (73) in the form
\begin{equation}
\frac{\gamma}{4\pi} (2\pi)^{-3} \int \left( |k \times (\mathbf{v} \times \hat{\mathbf{A}}_0)|^2 - \sum_{j=1}^3 v_j^2 k_j^2 |\hat{\mathbf{A}}_0|^2 \right) dk.
\end{equation}
By condition (49), the last expression is
\begin{equation}
\frac{\gamma}{4\pi} (2\pi)^{-3} \int \left( (k \cdot \mathbf{v})^2 - \sum_{j=1}^3 v_j^2 k_j^2 \right) |\hat{\mathbf{A}}_0|^2 dk = \frac{\gamma}{4\pi} \sum_{i,j: i \neq j} v_i v_j \int \partial_i \mathbf{A}_0(x) \cdot \partial_j \mathbf{A}_0(x) dx.
\end{equation}
Hence, by (73), $\mathcal{E}_v = \gamma m_0 + \eta_v + \tilde{\eta}_v$, where, by definition,
\begin{equation}
\tilde{\eta}_v := -\frac{\gamma}{4\pi} \sum_{i,j: i \neq j} v_i v_j \int \left( \partial_i \Phi_0(x) \partial_j \Phi_0(x) - \partial_i \mathbf{A}_0(x) \cdot \partial_j \mathbf{A}_0(x) \right) dx.
\end{equation}
Finally, conditions C0–C2 yield $\eta_v = \tilde{\eta}_v = 0$. Therefore, $\mathcal{E}_v = \gamma m_0 = \gamma \mathcal{E}_0$, by Corollary 2 (ii).}

Denote by $P = (P^1, P^2, P^3)$ the momentum operator for the (MD) system,
\begin{equation}
P(\psi, A) = -i \int \psi^*(t,x) \nabla \psi(t,x) dx + \frac{1}{4\pi} \int \left( \Phi(t,x) \nabla \Phi(t,x) - \sum_{k=1}^3 \hat{A}^k(t,x) \nabla A^k(t,x) \right) dx.
\end{equation}
Put $P_v := P(\psi_v, A_v), v \in \mathbb{R}^3$. We impose conditions $C1'$ and $C2'$ on $\varphi_\omega$ (see Section 3). Moreover, we impose a stronger condition $C0'$ on $A_0$ than $C0$.

**C0'** Let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. For any $k = 1, 2, 3$, the following relation holds

$$
\sum_{j:j \neq k} v_j \int \left( \partial_j \Phi_0(x) \partial_k \Phi_0(x) - \partial_j A_0(x) \cdot \partial_k A_0(x) \right) dx = 0.
$$

Note that conditions $C0'–C2'$ are fulfilled for the particular ansatz of solutions $\varphi \equiv \varphi_\omega$ considered in Section 5.3, see Lemma 8 (iv).

**Lemma 9.** Let conditions $C0'–C2'$ hold. Then $P_v = \gamma v \varepsilon_0$.

**Proof.** Using (65) and (27), we rewrite the first term in $P_v$ corresponding to $\psi_v$,

$$
P_1(\psi_v) := -i \int \psi^*(t, x) \nabla \psi_v(t, x) dx
$$

$$
= -i \int \varphi^*(y)(\alpha \cdot v + I) \left( i \omega \gamma v \varphi(y) + \nabla \varphi(y) + v \kappa \nabla \varphi(y) \cdot v \right) dy,
$$

where $\kappa := (\gamma - 1)/(|v|^2)$. Using notations (10) and formula (61), we have

$$
P_1(\psi_v) = v \gamma \omega Q + (I_1 v_1, I_2 v_2, I_3 v_3) + v \kappa \sum_{j=1}^3 v_j^2 T_j - i \xi_v,
$$

where $\xi_v$ is defined in (37). Applying (59) and (55), we obtain

$$
P_1(\psi_v) = v \gamma \varepsilon_0 + (T_1 v_1, T_2 v_2, T_3 v_3) + v \kappa \sum_{j=1}^3 v_j^2 T_j - i \xi_v. \quad (74)
$$

By conditions $C1'$ and $C2'$, one obtains $\xi_v = 0$. Further, the second term in $P_v$ corresponding to $A_v$ is

$$
P_2(A_v) := \frac{1}{4\pi} \int \left( \Phi_v(t, x) \nabla \Phi_v(t, x) - \Phi_v(t, x) \cdot \nabla A_v(t, x) \right) dx
$$

$$
= -\frac{1}{4\pi} \int \left( (v \cdot \nabla \Phi_0(y)) \nabla \Phi_0(y) + v \kappa (v \cdot \nabla \Phi_0(y))^2 \right.
$$

$$
- \sum_{n=1}^3 \left( (v \cdot \nabla A_0^n(y)) \nabla A_0^n(y) + v \kappa (v \cdot \nabla A_0^n(y))^2 \right) dy
$$

$$
= -(T_1 v_1, T_2 v_2, T_3 v_3) - v \kappa \sum_{j=1}^3 v_j^2 T_j - \tilde{\xi}_v, \quad (75)
$$

where $\tilde{\xi}_v$ stands for the following vector

$$
\tilde{\xi}_v := \left( \sum_{j \neq 1} v_j T_{1j}, \sum_{j \neq 2} v_j T_{2j}, \sum_{j \neq 3} v_j T_{3j} \right) + v \kappa \sum_{i,j \neq i,j} v_i v_j T_{ij}.
$$

Here by $T_{ij}$ we denote the integral

$$
T_{ij} := \frac{1}{4\pi} \int \left( \partial_i \Phi_0(y) \partial_j \Phi_0(y) - \partial_i A_0(y) \cdot \partial_j A_0(x) \right) dy.
$$

By condition $C0'$, $\tilde{\xi}_v = 0$. Hence, formulas (74) and (75) yield $P_v = P_1(\psi_v) + P_2(A_v) = \gamma v \varepsilon_0$. \hfill \square
We consider the Klein–Gordon–Dirac (KGD) system arising in the Yukawa model (see, for instance, [5, § 10.2], [36]) and describing the interaction between the Dirac and scalar (or pseudoscalar) fields. This system is based on the Lagrangian density

$$\mathcal{L}(\psi, \chi) = \mathcal{L}_D(\psi) + \mathcal{L}_{KG}(\chi) + \mathcal{L}_I(\psi, \chi).$$  (76)

Here $\mathcal{L}_D(\psi)$ and $\mathcal{L}_{KG}(\chi)$ are the Lagrangian densities for the nonlinear Dirac field $\psi$ and for the free Klein–Gordon field $\chi$, respectively, “extra” term $\mathcal{L}_I$ describes the Yukawa interaction between the fields. $\mathcal{L}_D(\psi)$ is defined in (7),

$$\mathcal{L}_D(\psi) = \frac{1}{2}\left(|\psi|^2 - | \nabla \psi |^2 - M^2 \chi^2 \right), \quad \mathcal{L}_I(\psi, \chi) = \eta \bar{\psi} \Gamma \psi \chi,$$

where $\chi$ is a (real) scalar field, $M > 0$, $\eta$ is a constant, and $\Gamma$ is some $4 \times 4$ matrix. This model with $G \equiv 0$ and $\Gamma = I$ has been studied by Chadam and Glassey [8] and Esteban et al. [15]. In another model presented by Ranada and Vázquez [27] the self-coupling is $G(\bar{\psi} \psi) = \lambda(\bar{\psi} \psi)^2$ (as in the Soler model) and $\Gamma = i \gamma^5$ with $\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

For simplicity, we consider the case $\Gamma = I$. Then, applying the Lagrange–Euler equations to (76), we come to the following system

$$(-i \gamma^\mu \partial_\mu + m - g(\bar{\psi} \psi))\psi = \eta \chi \chi, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R},$$  (77)

$$\left( \partial^2_t - \Delta + M^2 \right) \chi = \eta \bar{\psi} \chi,$$  (78)

where $g(s) = G'(s), n = 1, 3$ (cf [8, p. 5]). If $n = 1$, we put $\psi = (\psi_1, \psi_2),$ $\beta = \sigma_3, \alpha = -\sigma_2$. Below we consider the case $n = 3$ only. The case $n = 1$ can be studied by a similar way. By (76), the Hamiltonian density reads

$$\mathcal{E}(\psi, \chi) = \int \left( \psi^* (-i \alpha \nabla + m \beta) \psi - G(\bar{\psi} \psi) + \frac{1}{2} (|\chi|^2 + | \nabla \chi |^2 + M^2 |\chi|^2) - \eta \bar{\psi} \chi \right) dx.$$  

If $G \equiv 0$, the local existence and uniqueness of solutions to system (77)–(78) were obtained by Chadam and Glassey [8]. The existence for the stationary solutions was given by Esteban et al. [15, Theorem 2] also only in the case when $G \equiv 0$. In spite of this fact we verify below the identity (4) for (KGD) system assuming that either the self-coupling $G$ vanishes or $G$ satisfies the conditions $G1–G4$ (see Section 2).

### 6.1. Standing Waves for (KGD) Equations

**Definition 3.** Let $\omega \in (-m, m)$. The standing waves of the (KGD) system are the stationary solutions $(\psi_0, \chi_0)$ of the form

$$\psi_0(t, x) = e^{-i \omega t} \varphi(x), \quad \chi_0(x) = \frac{e^{-M|x|}}{4\pi|x|} * f_\varphi, \quad \text{with} \quad f_\varphi := \eta \bar{\varphi} \varphi,$$  (79)

where $\varphi \in H^1(\mathbb{R}^3; \mathbb{C}^4)$ satisfies the following equation

$$\left( \omega + i \alpha \nabla - m \beta + \eta \chi_0 \beta + g(\bar{\varphi} \varphi) \right) \varphi = 0.$$  (80)

Put $I^\omega(\varphi) = \frac{1}{2} \mathcal{L}(\psi_0, \chi_0) dx$. Then, by (76) and (79),

$$I^\omega(\varphi) = \frac{1}{2} \int \left( \varphi^* (i \alpha \nabla - m \beta + \omega) \varphi + G(\bar{\varphi} \varphi) \right) dx + \frac{1}{16\pi} \int \int \frac{e^{-M|x-y|}}{|x-y|} f_\varphi(x) f_\varphi(y) dx dy.$$  

Note that if $(\psi_0, \chi_0)$ is a stationary solution of the (KGD) equations, then (formally) $\varphi \equiv \varphi_\omega$ is a critical point of $I^\omega(\varphi)$.
A particular family of solutions. In the spherical coordinates \((r, \phi, \theta)\) of \(x \in \mathbb{R}^3\), the particular family of the stationary solutions \((\psi_0, \chi_0)\) is given by

\[
\psi_0(t, x) = e^{-i\omega t} \varphi(x), \quad \varphi(x) = \begin{pmatrix} v(r) \\ iu(r) \end{pmatrix},
\]

\[
\chi_0(x) = \begin{cases} \chi_*(r) \cos \theta, & \text{if } \Gamma = i\gamma^5, \\ \chi_*(r), & \text{if } \Gamma = I. 
\end{cases}
\]

In the case \(\Gamma = i\gamma^5\), this ansatz has been studied numerically in [27]. In the case when \(\Gamma = I\), the functions \(u, v\) are classical solutions of the following system:

\[
\begin{aligned}
u' + \frac{2\nu}{r} &= v[g(v^2 - u^2) - (m - \omega) + \eta \chi_*], \\
v' &= u[g(v^2 - u^2) - (m + \omega) + \eta \chi_*].
\end{aligned}
\]

The function \(\chi_*\) is a solution of the equation

\[
-\chi_*'' - \frac{2}{r} \chi_*' + M^2 \chi_* = \eta(v^2 - u^2)
\]

or

\[
\chi_*(|x|) = \eta \int \frac{e^{-M|x-y|}}{4\pi|x-y|} \left( v^2(|y|) - u^2(|y|) \right) dy.
\]

6.2. A Virial Identity

Let \(I_k \equiv I_k(\varphi)\), \(V \equiv V(\varphi)\), \(Q \equiv Q(\varphi)\) be as in (10),

\[
R \equiv R(\varphi) := \int \chi_0(x) f_\varphi(x) dx = \int \int \frac{e^{-M|x-y|}}{4\pi|x-y|} f_\varphi(x) f_\varphi(y) dy dx,
\]

\[
R_1 \equiv R_1(\varphi) := \frac{1}{4\pi} \int e^{-M|x-y|} f_\varphi(x) f_\varphi(y) dy dx. \tag{81}
\]

Note that by the Parseval identity and formulas (79),

\[
R(\varphi) = (2\pi)^{-3} \int \frac{\left| \hat{f}_\varphi(k) \right|^2}{k^2 + M^2} dk,
\]

\[
R_1(\varphi) = 2M(2\pi)^{-3} \int \frac{\left| \hat{f}_\varphi(k) \right|^2}{(k^2 + M^2)^2} dk = 2M \int |\chi_0(x)|^2 dx. \tag{82}
\]

where \(\hat{f}_\varphi\) denotes the Fourier transform of \(f_\varphi\). Using (82), we rewrite \(I^\omega(\varphi)\) as

\[
I^\omega(\varphi) = \frac{1}{2} \left( \omega Q(\varphi) - V(\varphi) - I_1(\varphi) - I_2(\varphi) - I_3(\varphi) + \frac{1}{2} R(\varphi) \right).
\]

Lemma 10. Let \(\varphi \in H^1(\mathbb{R}^3; \mathbb{C}^4)\) be a solution to Eqn (80). Then

\[
\omega Q = \frac{2}{3} (I_1 + I_2 + I_3) + V - \frac{1}{6} (5R - MR_1). \tag{83}
\]

Moreover,

\[
I_j(\varphi) = \frac{1}{2} (\omega Q(\varphi) - V(\varphi)) + \frac{3}{4} R(\varphi) - \frac{M}{4} R_1(\varphi) - P_j(\varphi), \quad j = 1, 2, 3, \tag{84}
\]

where \(P_j(\varphi)\) stands for the following functional

\[
P_j \equiv P_j(\varphi) = (2\pi)^{-3} \int \frac{k_j^2 \left| \hat{f}_\varphi(k) \right|^2}{(k^2 + M^2)^2} dk = \int \left| \partial_j \chi_0(x) \right|^2 dx. \tag{85}
\]
Proof. (i) The virial identity (83) was derived in [15] in the case when $G \equiv 0$. For any $G$, this identity can be proved (formally) used Derrick’s technique. Indeed, introduce $\varphi(x) = \varphi(x/\lambda)$. Then,

$$R(\varphi_\lambda) = \lambda^5 \int \frac{e^{-\lambda M|x-y|}}{4\pi|x-y|} f(\varphi(x)) f(\varphi(y)) dx dy,$$

$I_k(\varphi_\lambda) = \lambda^2 I_k(\varphi)$, $Q(\varphi_\lambda) = \lambda^3 Q(\varphi)$, $V(\varphi_\lambda) = \lambda^3 V(\varphi)$. Hence,

$$0 = \frac{d}{d\lambda} I^\mu(\varphi_\lambda) \bigg|_{\lambda=1} = \frac{1}{2} \frac{d}{d\lambda} \left[ \omega Q(\varphi_\lambda) - V(\varphi_\lambda) - I_1(\varphi_\lambda) - I_2(\varphi_\lambda) - I_3(\varphi_\lambda) + \frac{1}{2} R(\varphi_\lambda) \right] \bigg|_{\lambda=1}
$$

$$= \frac{3}{2} \left( \omega Q(\varphi) - V(\varphi) \right) - I_1(\varphi) - I_2(\varphi) - I_3(\varphi) + \frac{1}{4} (5R(\varphi) - MR_1(\varphi)),$$

and identity (83) holds.

(ii) Introduce $\varphi(\lambda) = \varphi(x_1/\lambda, x_2, x_3)$. Then $I_1(\varphi_\lambda) = I_1(\varphi)$, $I_k(\varphi_\lambda) = \lambda I_k(\varphi)$ for $k = 2, 3$,

$$R(\varphi_\lambda) = (2\pi)^{-3} \int \frac{e^{-\lambda M|x-y|}}{k_1^2 + k_2^2 + k_3^2 + M^2} \left| \hat{f}(k) \right|^2 dk$$

$Q(\varphi_\lambda) = \lambda Q(\varphi)$, $V(\varphi_\lambda) = \lambda V(\varphi)$. Using (85), we have

$$\frac{d}{d\lambda} R(\varphi_\lambda) \bigg|_{\lambda=1} = R + 2P_1$$. Hence

$$0 = \frac{d}{d\lambda} I^\mu(\varphi_\lambda) = \frac{1}{2} \left( \omega Q(\varphi) - V(\varphi) - I_2(\varphi) - I_3(\varphi) + \frac{1}{2} R(\varphi) + P_1(\varphi) \right).$$

Therefore,

$$I_2(\varphi) + I_3(\varphi) = \omega Q(\varphi) - V(\varphi) + \frac{1}{2} R(\varphi) + P_1(\varphi). \quad (86)$$

Therefore, identities (83) and (86) imply (84) with $j = 1$. Similarly, introducing $\varphi(\lambda) = \varphi(x_1, x_2/\lambda, x_3)$ or $\varphi(\lambda) = \varphi(x_1, x_2, x_3/\lambda)$ gives (84) with $j = 2, 3$. Note that equality (83) follows from (84), since $P_1 + P_2 + P_3 = R - MR_1/2$. \hfill \Box

Corollary 4 (cf. Corollaries 1 and 2). Let $\varphi$ be a solution to Eqn (80). Then the following relations hold. (i) By (80), we have

$$I_1 + I_2 + I_3 = \omega Q + \int (g(\varphi_\varphi) - m) \varphi_\varphi dx + R. \quad (87)$$

(ii) Using identities (83) and (87), we obtain

$$\omega Q = \frac{2}{3} (I_1 + I_2 + I_3) + V - \frac{1}{6} (5R - MR_1) = I_1 + I_2 + I_3 + \int (m - g(\varphi_\varphi)) \varphi_\varphi dx - R.$$ 

Hence,

$$I_1 + I_2 + I_3 = 3 \int (g(s)s - G(s))|_{s=\varphi_\varphi} dx + \frac{1}{2} (R + MR_1) > 0, \quad (88)$$

by (82) and condition $G2$.

(iii) The total energy associated to particle-like solutions $(\psi_0, \chi_0)$ is

$$E_0 := E(\psi_0, \chi_0) = I_1 + I_2 + I_3 + V - \frac{1}{2} R. \quad (89)$$

Using (87) and condition $G2$, we have

$$E_0 = \omega \int |\varphi(x)|^2 dx + \int (g(s)s - G(s))|_{s=\varphi_\varphi} dx + \frac{1}{2} R > 0.$$ 

(iv) Similarly to Lemma 2 it can be proved that identity (16) holds.
6.3. Moving Waves for (KGD) Equations

Consider travelling solutions \( (\psi_\nu, \chi_\nu) \) with velocity \( \nu \in \mathbb{R}^3, |\nu| < 1 \):

\[
\begin{align*}
\psi_\nu(t, x) &= S_\nu \psi_0(A_\nu^{-1}(t, x)), \\
\chi_\nu(t, x) &= \chi_0(y) \quad \text{with} \quad y = x + \nu \frac{(y - x)}{|\nu|} \cdot \nu - \gamma \nu t. 
\end{align*}
\]  

(90)

It is easy to check that \( (\psi_\nu, \chi_\nu) \) is a solution to system (77)–(78).

Denote by \( E_{\nu} := E(\psi_\nu, \chi_\nu) \) the energy of the moving solitary waves \( (\psi_\nu, \chi_\nu) \),

\[
E_{\nu} = \int \left( \psi_\nu (-i\alpha \nabla + m\beta) \psi_\nu - G(\bar{\psi}_\nu \psi_\nu) + \frac{1}{2} \left( |\chi_\nu|^2 + |\nabla \chi_\nu|^2 + M^2 |\chi_\nu|^2 \right) - \eta \bar{\psi}_\nu \bar{\psi}_\nu \psi_\nu \right) dx.
\]

Assume that conditions \( C1 \) and \( C2 \) hold (see Section 3). Moreover, we impose the additional condition \( C3 \).

\( C3 \) For given \( \nu = (v_1, v_2, v_3) \in \mathbb{R}^3, |\nu| < 1 \),

\[
\sum_{i,j: i \neq j} v_i v_j \int \partial_i \chi_0(x) \partial_j \chi_0(x) dx = 0.
\]  

(91)

The integral in (91) equals \((2\pi)^{-3} \int k_i k_j |\hat{\varphi}(k)|^2 dk \). Then condition \( C3 \) holds, for instance, if \( \bar{\varphi} \varphi(x) \) is an even function in \( x \in \mathbb{R}^3 \). In particular, conditions \( C1–C3 \) are fulfilled for the particular family of solutions considered in Section 6.1 (see also Section 2.1 and formulas (19) and (20)).

**Lemma 11.** Let conditions \( C1–C3 \) hold, \( \nu \in \mathbb{R}^3 \) with \( |\nu| < 1 \). Then \( E_{\nu} = \gamma \mathcal{E}_0 \).

**Proof.** At first, consider the term in \( E_{\nu} \) corresponding to the Dirac field (cf formula (67)),

\[
E_D := \int \left( \psi_\nu (-i\alpha \nabla + m\beta) \psi_\nu - G(\bar{\psi}_\nu \psi_\nu) \right) dx
\]

\[
= \omega Q \gamma \nu^2 + \frac{1}{\gamma} (I_1 + I_2 + I_3) + \frac{1}{\gamma} V + \gamma \sum_{j=1}^{3} v_j^2 I_j + \eta_\nu,
\]

where \( \eta_\nu \) is defined in (35). Applying formula (84) and then the identity (83), we obtain

\[
E_D = \gamma (I_1 + I_2 + I_3 + V) - \frac{1}{2} \gamma \nu^2 R - \gamma \sum_{j=1}^{3} v_j^2 P_j + \eta_\nu.
\]  

(92)

Second, we rewrite the term in \( E_{\nu} \) corresponding to the Klein–Gordon field,

\[
E_{KG} := \frac{1}{2} \int \left( |\partial_t \chi_\nu(t, x)|^2 + |\nabla \chi_\nu(t, x)|^2 + M^2 |\chi_\nu(t, x)|^2 \right) dx
\]

\[
= \frac{1}{2\gamma} \int \left( |\nabla \chi_0(y)|^2 + 2\gamma^2 |\nu \cdot \nabla \chi_0(y)|^2 + M^2 |\chi_0(y)|^2 \right) dy
\]

\[
= \frac{1}{2\gamma} \int \chi_0(y) (-\Delta + M^2) \chi_0(y) dy + \gamma \sum_{j=1}^{3} v_j^2 \int |\partial_j \chi_0(y)|^2 dy + \eta'_\nu
\]

\[
= \frac{1}{2\gamma} R + \gamma \sum_{j=1}^{3} v_j^2 P_j + \eta'_\nu,
\]  

(93)

where, by definition, \( \eta'_\nu := \gamma \sum_{i,j: i \neq j} v_i v_j \int \partial_i \chi_0(y) \partial_j \chi_0(y) dy \). Further, the term in \( E_{\nu} \) corresponding to the interaction is

\[
E_I := -\eta \int \chi_\nu(t, x) \bar{\psi}_\nu(t, x) \psi_\nu(t, x) dx = -\frac{\eta}{\gamma} \int \chi_0(y) \bar{\varphi}(y) \varphi(y) dy = -\frac{1}{\gamma} R(\varphi).
\]  

(94)
Applying (92)–(94), and (89), we obtain

\[ \mathcal{E}_v = E_D + E_{KG} + E_I = \gamma(I_1 + I_2 + I_3 + V) - \frac{1}{2} \gamma R + \eta_v + \eta'_v = \gamma E_0 + \eta_v + \eta'_v. \]

Finally, by conditions C1–C3, \( \eta_v = \eta'_v = 0 \). Therefore, \( \mathcal{E}_v = \gamma E_0 \).

**Remark.** If \( \mathbf{v} = (0, 0, v) \) with \( |v| < 1 \), then \( \eta_v = -2i \gamma v(\varphi^*, \partial_3 \varphi) \) and \( \eta'_v = 0 \). In this case, conditions C2 and C3 are fulfilled, and condition C1 is equivalent to the condition \( \varphi^*, \partial_3 \varphi = 0 \).

Denote by \( P = (P^1, P^2, P^3) \) the momentum operator, where \( P^\alpha = \int T^{\alpha \beta} dx \), and \( T^{\alpha \beta} (\alpha, \beta = 0, 1, 2, 3) \) denotes the energy–momentum tensor for the (KGD) model. Using formula (13) from [27] for the tensor \( T^{\alpha \beta} \), we have

\[ P = P(\psi, \chi) = -\int \left( i \psi^*(t, x) \nabla \psi(t, x) + \chi(t, x) \nabla \chi(t, x) \right) dx. \]

Write \( P_v := P(\psi_v, \chi_v) \), \( \mathbf{v} \in \mathbb{R}^3 \) with \( |\mathbf{v}| < 1 \). We impose conditions C1' and C2' on \( \varphi_\omega \) (see Section 3). Moreover, we assume the additional condition C3' on \( \chi_0 \).

C3' For \( \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 \) and \( k = 1, 2, 3 \), \( \sum_{j \neq k} v_j \int \partial_k \chi_0(y) \partial_j \chi_0(y) dy = 0 \).

Conditions C1'–C3' are fulfilled for the particular ansatz of solutions \( \varphi \equiv \varphi_\omega \) considered in Section 6.1.

**Lemma 12.** Let conditions C1'–C3' hold. Then \( P_v = \gamma v \mathcal{E}_0 \).

**Proof.** Using (90) and (27), we rewrite the first term in \( P_v \) corresponding to \( \psi_v \) as

\[ p_1(\psi_v) := -i \int \psi^*_v(t, x) \nabla \psi_v(t, x) dx \]

\[ = -i \int \varphi^*(y) (\alpha \cdot \mathbf{v} + I) \left( i \omega \gamma \varphi(y) \nabla \varphi(y) + \mathbf{v} \kappa \nabla \varphi(y) \cdot \mathbf{v} \right) dy, \quad \kappa = \frac{\gamma - 1}{|\mathbf{v}|^2}. \]

Applying (83) and (84), we obtain (cf (74))

\[ p_1(\psi_v) = \mathbf{v} \gamma \mathcal{E}_0 - (P_1 v_1, P_2 v_2, P_3 v_3) - \mathbf{v} \kappa \sum_{j=1}^3 v_j^2 P_j - i \xi_v, \quad (95) \]

where \( \xi_v \) is defined in (37). By conditions C1' and C2', we have \( \xi_v = 0 \). The second term in \( P_v \) corresponding to \( \chi_v \) is

\[ p_2(\chi_v) := -\int \chi_v(t, x) \nabla \chi_v(t, x) dx = \int (\nabla \chi_0(y) \cdot \mathbf{v}) \left( \nabla \chi_0(y) + \mathbf{v} \kappa \nabla \chi_0(y) \cdot \mathbf{v} \right) dy \]

\[ = (P_1 v_1, P_2 v_2, P_3 v_3) + \mathbf{v} \kappa \sum_{j=1}^3 v_j^2 P_j + \xi'_v, \quad (96) \]

where \( \xi'_v \) stands for the following vector

\[ \xi'_v := \left( \sum_{j \neq 1} v_j \int \partial_j \chi_0 \partial_1 \chi_0 dy, \sum_{j \neq 2} v_j \int \partial_j \chi_0 \partial_2 \chi_0 dy, \sum_{j \neq 3} v_j \int \partial_j \chi_0 \partial_3 \chi_0 dy \right) \]

\[ + \mathbf{v} \kappa \sum_{k, j: k \neq j} v_k v_j \int \partial_k \chi_0 \partial_j \chi_0 dy. \]

Applying condition C3' we obtain \( \xi'_v = 0 \). Therefore, formulas (95) and (96) yield

\[ P_v = -\int \left( i \psi^*_v(t, x) \nabla \psi_v(t, x) + \chi_v(t, x) \nabla \chi_v(t, x) \right) dx = p_1(\psi_v) + p_2(\chi_v) = \gamma v \mathcal{E}_0. \]

\[ \square \]
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