Qualitative Analysis of a Reaction-Diffusion System with cubic Nonlinearity

B. Ambrosio

Normandie Univ, UNIHAVRE, LMAH, FR-CNRS-3335, ISCN, 76600 Le Havre, France

Abstract

The aim of this article is to provide some insights in the qualitative analysis of a two dimensional nonlinear reaction-diffusion system. This system can be seen, in some aspects, as a toy model of the FitzHugh-Nagumo model arising in Neuroscience context. We will exhibit some phenomena arising as a parameter is varied. In particular, we will focus on the convergence of the system toward different solutions: fixed point or periodic. The framework is the Hilbert space $L^2 \times L^2$ and its spectral decomposition.

Our analysis shows that this framework is relevant for the nonlinearity considered here.

Keywords: Hopf Bifurcation; Reaction-Diffusion; FitzHugh-Nagumo; LaSalle’s Principle

1. Introduction

In this article, we focus on the qualitative analysis of the following system:

$$\begin{align*}
    u_t &= \alpha u - u^3 - v + u_{xx}, \\
    v_t &= u
\end{align*}$$

(1)

on the domain $]0, 1[$ with Neumann Boundary conditions. System (1) can be seen as a toy example of a class of reaction-diffusion (RD) systems of FitzHugh-Nagumo (FHN) type that was studied previously by the author, see for example [1, 2, 3]. It is known that for each given initial condition (IC) in $L^2(0, 1) \times L^2(0, 1)$, system (1) admits a unique solution defined for all time, see [4, 5, 6]. Here, our goal is to prove...
different results regarding the asymptotic behavior, and particularly the co-existence of solutions, toward which the convergence depend on initial conditions (ICs). To this end, we will first give a detailed analysis of the linearized system of (4) around (0, 0), and will then move to the analysis of (4). Our article is divided as follows; after this introduction, we will analyze the linear system in section 2. In section 3, we will then consider the system (4). In section 4, we will provide some numerical simulations and conclude in section 5.

Notations
Throughout the article, we will adopt the following notations:

\[ \mathcal{H} = L^2(0, 1) \times L^2(0, 1) \]
\[ \mathcal{V} = H^1(0, 1) \times H^1(0, 1) \] where \( H^1(0, 1) \) is the classical Sobolev space.

\[ \| \cdot \| \] will denote the norm on \( \mathcal{H} \).

Subscripts t and x are used for respectively time and space derivatives.

2. The linear case

Note first that (0, 0) is a constant solution of (4). The linearized system around this point is given by:

\[
\begin{align*}
  u_t &= \alpha u - v + u_{xx} \\
  v_t &= u
\end{align*}
\]

(2)
on the domain \([0, 1]\) with Neumann Boundary conditions. Using the spectral decomposition, we can give a detailed and comprehensive analysis of the qualitative behavior of (2). Classically, we set:

\[ \varphi_0(x) = 1, \text{ and } \forall k \in \mathbb{N}^+ \varphi_k(x) = \sqrt{2}\cos(k\pi x). \]

We recall that the family \((\varphi_k)_{k \in \mathbb{N}}\) is an orthonormal basis of \( L^2 \), and that the functions \( \varphi_k \) satisfy:

\[ -(\varphi_k)_{xx} = \lambda_k \varphi_k \]
and

\((\varphi_k)_x(0) = (\varphi_k)_x(1) = 0,\)

with

\(\lambda_k = k^2 \pi^2.\)

Looking for solutions of the form,

\(u(t) = \sum_{k=0}^{\infty} u_k(t) \varphi_k, \quad v(t) = \sum_{k=0}^{\infty} v_k(t) \varphi_k\)

leads by projection on the eigenspace generated by \((\varphi_k, \varphi_k)\) to the resolution of the two dimensional ODE systems indexed by \(k\), and denoted by \(E_k:\)

\[
\begin{cases}
  u_{kt} = (\alpha_k - \lambda_k)u_k - v_k \\
  v_{kt} = u_k
\end{cases}
\]

The eigenvalues of matrix

\(A_k = \begin{pmatrix} \alpha - \lambda_k & -1 \\ 1 & 0 \end{pmatrix}\)

are given by

\[
\sigma_1^k = \frac{1}{2} \left( \alpha - \lambda_k - \sqrt{(\alpha - \lambda_k)^2 - 4} \right), \quad \sigma_2^k = \frac{1}{2} \left( \alpha - \lambda_k + \sqrt{(\alpha - \lambda_k)^2 - 4} \right).
\]

We summarize the remarkable properties of \(\sigma_1^k\) and \(\sigma_2^k\) in the following proposition.

**Proposition 1.** When \(\alpha\) crosses \(\lambda_k\) from left to right, \(\sigma_1^k\) and \(\sigma_2^k\) cross the imaginary axis from left to right. Furthermore,

\[
\lim_{k \to +\infty} \sigma_1^k = -\infty \quad \text{and} \quad \lim_{k \to +\infty} \sigma_2^k = 0^-.
\]

We now state the main results describing the behavior of (2), in the two next theorems.

**Theorem 1.** For \(\alpha < 0\), for any initial condition \((u(\cdot, 0), v(\cdot, 0))\) in \(\mathcal{H}\), we have

\[
\lim_{t \to +\infty} \|(u, v)(t)\| = 0.
\]
Proof. For any fixed $N > 0$, one can prove that there exists $\delta > 0$ such that
\[
\sum_{k=0}^{N} (|u_k(t)|^2 + |v_k(t)|^2) \leq e^{-\delta t} \sum_{k=0}^{N} (|u_k(0)|^2 + |v_k(0)|^2),
\]
where $(u_k, v_k)(t)$ is the solution of $E_k$ with $(u_k, v_k)(0) = (\int_0^1 u(x, 0) \varphi_k(x) dx, \int_0^1 v(x, 0) \varphi_k(x) dx)$. Note that since $\sigma_k^2 \to 0$ our computations do not allow to take $N = \infty$ and conserve the exponential decay. However, for any $\varepsilon > 0$ there exists $N$ large enough such that
\[
\sum_{k=N+1}^{+\infty} (|u_k(0)|^2 + |v_k(0)|^2) < \frac{\varepsilon}{2}.
\]
Since for $N$ large enough and $k > N$ we have,
\[
\frac{d}{dt} (|u_k(t)|^2 + |v_k(t)|^2) \leq 2(\alpha - \lambda_k)|u_k(t)|^2 \leq 0,
\]
the following inequality holds:
\[
\forall t > 0, \sum_{k=N+1}^{+\infty} (|u_k(t)|^2 + |v_k(t)|^2) < \frac{\varepsilon}{2}.
\]
Combining the above results, we can deduce that for any $\varepsilon > 0$ there exists $T$ such that for $t > T$,
\[
||(u, v)(t)|| < \varepsilon.
\]

\[\square\]

**Theorem 2.** Let $k \in \mathbb{N}^+$. 
For $\alpha = \lambda_k$, $(0, 0)$ is a center for system $E_k$, a source for $E_l$ if $l < k$ and a sink for $E_l$ if $l > k$. Furthermore, if: $u_l(0) = v_l(0) = 0$ for $l \in \{0, \ldots, k-1\}$ then
\[
\lim_{t \to +\infty} ||(u, v)(t) - \varphi_k(u_k(t), v_k(t))|| = 0.
\]

Otherwise,
\[
\lim_{t \to +\infty} ||(u, v)(t)|| = +\infty.
\]
For $\lambda_k < \alpha < \lambda_{k+1}$, $(0, 0)$ is a source for $E_l$ if $l \leq k$ and a sink for $E_l$ if $l > k$. Furthermore, if $u_l(0) = v_l(0) = 0$ for $l \in \{1, \ldots, k\}$ then
\[
\lim_{t \to +\infty} ||(u, v)(t)|| = 0.
\]

Otherwise
\[
\lim_{t \to +\infty} ||(u, v)(t)|| = +\infty.
\]
Remark 1. For fixed \( \alpha > 0 \), the above theorem allows to characterize eigen subspaces of IC, leading to convergence to fixed point, periodic solutions or infinity. The same idea will guide us when dealing with the nonlinear system [3].

The spectral decomposition of \( L^2 \) has allowed an exhaustive study of the asymptotic behavior. We will now move to the nonlinear case and show that the spectral decomposition remains useful regarding asymptotical qualitative analysis.

3. The Nonlinear case

We now consider the system

\[
\begin{align*}
    u_t &= \alpha u - u^3 - v + u_{xx} \\
    v_t &= u
\end{align*}
\]

on the domain \([0,1]\) with Neumann Boundary conditions.

For \( \alpha < 0 \) the fixed point \((0,0)\) is still attracting all the IC in \( \mathcal{H} \). Indeed, we have:

**Theorem 3.** For \( \alpha < 0 \),

\[
\lim_{t \to +\infty} \|(u,v)(t)\| = 0
\]

**Proof.** The proof relies on the LaSalle’s principle. First note that:

\[
\frac{d}{dt} \|(u,v)(t)\|^2 = \alpha |u|^2 - \int_0^1 u^4 dx - \int_0^1 u^2_t dx 
\leq 0
\]

which proves that \( \|(u,v)\|^2 \) is a Lyapunov function. Furthermore,

\[
\frac{d}{dt} \|(u_x,v_x)(t)\|^2 = \alpha |u_x|^2 - 3 \int_0^1 u^2 u_x^2 dx - \int_0^1 u_{xx}^2 dx 
\leq 0
\]

which proves that the trajectories are bounded in \( \mathcal{V} \). Using the LaSalle’s principle, this proves that, for all IC in \( \mathcal{H} \cap \mathcal{V} \) the trajectory tends to \((0,0)\). Now, since \( \mathcal{V} = \mathcal{H} \) and since for two solutions starting in \( \mathcal{H} \), \( \|(u_2 - u_1, v_2 - v_1)(t)\| \) is deacreasing , one can prove that the result is true for all IC in \( \mathcal{H} \).

It is well known that for \( \alpha > 0 \) the diffusion less system admits a unique limit cycle which attracts all the trajectories distinct from \((0,0)\). It follows that, for \( \alpha > 0 \),
(0, 0) becomes unstable since any solution constant in space different from (0, 0) will evolve towards the limit cycle of the ODE. However this particular system allows to construct specific solutions of interest. Before giving these solutions, we prove two lemmas which allows to identify correlations along the eigenfunctions.

**Lemma 1.** Let \( k, m, n \in \mathbb{N}, k, m, n > 0 \), then

\[
\int_0^1 \varphi_k(x) \varphi_m(x) \varphi_n(x) dx \neq 0
\]

if and only if

\[
k + m = n \text{ or } k + n = m \text{ or } m + n = k
\]

And in this case,

\[
\int_0^1 \varphi_k(x) \varphi_m(x) \varphi_n(x) dx = \frac{\sqrt{2}}{2}
\]

**Proof.** We have,

\[
\varphi_k(x) \varphi_m(x) \varphi_n(x) = \frac{\sqrt{2}}{4} (e^{ik\pi x} + e^{-ik\pi x})(e^{im\pi x} + e^{-im\pi x})(e^{in\pi x} + e^{-in\pi x})
\]

\[
= \frac{\sqrt{2}}{4} (e^{(k+m+n)\pi x} + e^{(k+m-n)\pi x} + e^{(-k-m+n)\pi x} + e^{(-k-m-n)\pi x})
\]

\[
= \frac{\sqrt{2}}{2} (\cos((k+m+n)\pi x) + \cos((k-m+n)\pi x) + \cos((k-m+n)\pi x) + \cos((-k+m+n)\pi x))
\]

which proves the result.

**Lemma 2.**

\[
\int_0^1 \varphi_k(x) \varphi_l(x) \varphi_m(x) \varphi_n(x) dx \neq 0
\]

if and only if

\[
k + l + m = n \text{ or } k + l + n = m \text{ or } k + m + n = l \text{ or } l + m + n = k
\]

Or

\[
k + l = m + n \text{ or } k + n = l + m \text{ or } k + m = n + l \text{ or } k + n = l + m.
\]

**Proof.** Similar computations show that,

\[
\varphi_k \varphi_l \varphi_m \varphi_n
\]

\[
= \frac{1}{4} \left( \cos((k+l+m+n)\pi x) + \cos((k+l+m-n)\pi x) + \cos((k+l-m+n)\pi x) + \cos((k-l+m+n)\pi x) + \cos((k+l-m-n)\pi x) + \cos((-k+l+m+n)\pi x) + \cos((-k+l-m+n)\pi x) + \cos((-k-l+m+n)\pi x) \right)
\]

which proves the result.
Proposition 2. For $0 < \alpha < \lambda_1$, if $u(x) = -u(1-x)$ and $v(x) = -v(1-x)$ then

$$\lim_{t \to +\infty} ||(u,v)(t)|| = 0$$

Proof. By symmetry, $\int_0^1 u(x,t)dx = \int_0^1 v(x,t)dx = 0$. Then, we apply the LaSalle’s Principle as in the proof of 3, within the positive invariant subspace defined by $\int_0^1 u(x,t)dx = \int_0^1 v(x,t)dx = 0$.

The following proposition, gives the description of solutions related with the spectral decomposition.

Proposition 3. If $u(x,0) = -u(1-x,0)$ and $v(x,0) = -v(1-x,0)$, then for all $t \geq 0$

$$u_{2k}(t) = v_{2k}(t) = 0.$$ 

If $u(x,0) = u(1-x,0)$ and $v(x,0) = v(1-x,0)$, then for all $t \geq 0$

$$u_{2k+1}(t) = v_{2k+1}(t) = 0.$$ 

Proof. This comes from the symmetry of the system. One can also directly check how solutions are written in each spectral subspace, see remark below.

Remark 2. As it comes from the above propositions, the subspaces $u_{2k}(t) = v_{2k}(t) = 0$ and $u_{2k+1}(t) = v_{2k+1}(t) = 0$ are positively invariant under the flow.

The analysis of the linear system provides a local stability result for arbitrary large finite approximated projection equation of (4). We write it in the following sense.

Proposition 4. For $0 < \alpha < \lambda_1$, for each positive integer $N > 1$ there exists $\mu_N > 0$ such that $\sum_{k=1}^N(u_k^2 + v_k^2) < \mu_N$ implies

$$\lim_{t \to +\infty} ||(u^N(t) - u_0(t), v^N(t) - v_0(t))|| = 0$$

where $(u^N = \sum_{k=0}^N u_k \varphi_k, v^N = \sum_{k=0}^N v_k \varphi_k)$ with $(u_k, v_k)$ solution of:

$$(E_k) \begin{cases} u_{kt} = (\alpha - \lambda_k)u_k - \sum_{l,m=0}^N u_l u_m \int_0^1 \varphi_l \varphi_m \varphi_k dx - v_k \\ v_{kt} = u_k \end{cases} \quad (5)$$
Proof. We consider the equations

\[
\begin{align*}
(E_k) \left\{ \begin{array}{l}
u_{kt} = (\alpha - \lambda_k)u_k - \sum_{i,j,m}u_iu_ju_m \int_0^1 f_i \phi_j \phi_m \phi_k dx - v_k \\
v_{kt} = u_k
\end{array} \right.
\] (6)
\]

We can write:

\[
\begin{align*}
(E_0) \left\{ \begin{array}{l}
u_{0t} = \alpha u_0 - u_0^3 - v_0 - 3u_0 \sum_{i=1}^N u_i^2 - g_0 \\
v_{0t} = u_k
\end{array} \right.
\]
\]

where

\[
g_0 = \frac{\sqrt{2}}{2} \left( \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} u_iu_ju_{i+j} + \sum_{i=1}^{N-1} \sum_{j=1}^{i-1} u_iu_ju_{i-j} \right)
\]

and for \( k \geq 1 \),

\[
(E_k) \left\{ \begin{array}{l}
u_{kt} = (\alpha - \lambda_k - 3u_0^2)u_k - v_k - 3\sqrt{2}\sum_{i=1}^N u_i(u_iu_{k+i} + \sum_{i=1}^{k-1} u_iu_{k-i} + \sum_{i=k+1}^{N-1} u_iu_{k-i} - g_k) \\
v_{kt} = u_k
\end{array} \right.
\]
\]

where

\[
g_k = \frac{1}{2} \left( \sum_{i=1}^{N} \sum_{j=1}^{N} u_iu_j(u_{i+j+k} + u_{i+j-k} + u_{i-j+k} + u_{i-j-k} + u_{-i+j+k} + u_{-i-j+k} + u_{i-j-k} + u_{-i-j-k}) \right)
\]

where the terms are only taken in account if they belong to \( \{1, \ldots, N\} \). Then, note that the two last terms in the right hand side of of the first equation of (6) are bounded by \( 3|u_0|(\sum_{i=1}^N u_i^2) + \frac{3}{2}\sqrt{N}(\sum_{i=1}^N u_i^2)^{\frac{3}{2}} \), while the two last terms of the first equation of (7) are bounded by \( 3\sqrt{2}|u_0|(\sum_{i=1}^N u_i^2) + \frac{3}{2}\sqrt{N}(\sum_{i=1}^N u_i^2)^{\frac{3}{2}} \). This implies that choosing \( \sum_{i=1}^N (u_i^2 + v_i^2) \) small enough, one can obtain a bound on \( u_0^2 + v_0^2 \) as well as an arbitrary fixed small bound for \( \sum_{i=1}^N u_i^2 \). But then, \( \sum_{i=1}^N (u_i^2 + v_i^2) \) is a lyapunov function for \( \sum_{i=1}^N u_i^2 \) small enough, which in turn implies the convergence to 0. This implies the result. \( \square \)

**Proposition 5.** For \( 0 < \alpha < \lambda_1 \), there exists a sequence \( (\mu_k)_{k \in \mathbb{N}} \) such that if

\[
(u_k(0), v_k(0)) \in B(0, \mu_k)
\]

then

\[
\lim_{t \to +\infty} ||(u^N(t) - u_0(t), v^N(t) - v_0(t))|| = 0,
\]

where \( B(0, \mu_k) \) is the ball of center \( (0, 0) \) and radius \( \mu_k \).
Proof. The non linear terms are bounded by:

\[ C \sum_{i=1}^{\infty} |u_i| \sum_{i=1}^{\infty} u_i^2, \]

where \( C \) is a constant. Thanks to the dynamics of each system \( E_k \), one can ensure that for any arbitrary small \( \varepsilon \), the following estimates are valid:

\[ |u_k| \leq \frac{\varepsilon}{|\alpha - \lambda_k|}, |v_k| \leq \varepsilon \]

But then, one can prove that

\[ \sum_{k=1}^{\infty} (u_k^2 + v_k^2) \]

is a Lyapunov function. Which by the LaSalle’s principle proves the result. \( \square \)

Remark 3. Note that the previous proposition can be adapted for \( \lambda_k < \alpha < \lambda_{k+1} \), but then, one has to take into account the dynamics of systems \( E_i \) for \( 0 \leq l \leq k \). This opens up a rich possible behavior as \( \alpha \) increases.

4. Numerical simulations

In this section, we provide some numerical simulations of system (4). Our simulations have been performed using our own C++ program with an RK4 numerical scheme. We use a time step of \( 10^{-5} \) and a space step of 0.02. We choose to illustrate simulations for two values of the parameter: \( \alpha = 1 \) and \( \alpha = 15 \). For each value of the parameter, we illustrate two pictures corresponding to odd solutions (up to a translation around \((0, 0.5)\), where only odd coefficients are taken into account and other solutions.

In figure 1, we simulate system (4) for \( 0 = \alpha = 1 < \lambda_1 \). IC satisfy the symmetric condition \( u(x) = u(1-x) \) and \( v(x) = v(1-x) \). Indeed, we choose \( u(x, 0) = v(x, 0) = 1 \) on \((0, 0.5)\), \( u(x, 0) = v(x, 0) = -1 \) on \((0.5, 1)\). This implies a symmetric solution for all time. According to theorem 3, the solution converges toward \((0, 0)\) in \( \mathcal{H} \). The observation of \( u \) illustrates this theoretical result. For \( v \) the evolution is slower.

In figure 2, we simulate system (4) for \( 0 = \alpha = 1 < \lambda_1 \). IC do not satisfy the symmetric condition \( u(x) = u(1-x) \) and \( v(x) = v(1-x) \). Indeed, we choose \( u(x, 0) = v(x, 0) = 1 \) on \((0, 0.5)\), \( u(x, 0) = v(x, 0) = -0.5 \) on \((0.5, 1)\). The simulations show that
the solution $u$ reaches a constant function in space with periodicity in time. There is also periodicity in time for $v$, but the evolution of the shape in space is slower for $v$.

In figure 3, we simulate system (4) for $\lambda_1 < \alpha = 15 < \lambda_2$. IC satisfies $u(x) = u(1-x)$ and $v(x) = v(1-x)$ which implies that only odd coefficients are taken in account, and a symmetric solution (in particular $u_0(t) = v_0(t) = 0$). We observe that the solution $u$ evolves non constantly in space with periodicity in time.

In figure 4, IC do not satisfy the symmetric condition $u(x) = u(1-x)$ and $v(x) = v(1-x)$. Indeed, we choose $u(x,0) = v(x,0) = 1$ on $(0,0.5)$, $u(x,0) = v(x,0) = -0.5$ on $(0.5,1)$. The first row illustrates $u(x,t)$ for $x \in (0,1)$ and $t = 0.1$ (left), $t = 100$ (right). We observe that the solution $u$ reaches a constant function in space with periodicity in time. We can note the difference of the amplitude of the limit cycle with the previous simulation; for these IC, $u_0(t)$ and $v_0(t)$ are no longer zero.

5. Conclusion

In this article, we have studied a RD system with a cubic nonlinearity, which may be seen as a toy model for the FHN RD system. We have provided a comprehensive analysis of the linearized system, as well as qualitative analytical results for the nonlinear system. The main tools were the spectral decomposition and the LaSalle’s principle. In a forthcoming article, we will focus on the local analysis at the bifurcation values $\alpha = \lambda_k$.

Acknowledgments

I would like to thanks my colleague A. Ducroc for short but fruiful discussions. I would like to thank Region Normandie France and the ERDF (European Regional Development Fund) project XTERM for funding.
Figure 1: Simulation of system \( 4 \) for \( \alpha = 1 < \lambda_1 \). IC satisfy the symmetric condition \( u(x) = u(1 - x) \) and \( v(x) = v(1 - x) \). Indeed, we choose \( u(x, 0) = v(x, 0) = 1 \) on \((0, 0.5)\), \( u(x, 0) = v(x, 0) = -1 \) on \((0.5, 1)\). According to theorem 3, the solution converges toward \((0, 0)\) in \( H \). The first row illustrates \( u(x, t) \) for \( x \in (0, 1) \) and \( t = 0.1 \) (left), \( t = 100 \) (right). We observe that the solution \( u \) reaches the constant function 0. The second row illustrates \( v(x, t) \) for \( x \in (0, 1) \) and \( t = 0.1 \) (left), \( t = 100 \) (right). Note that, since after some time, \( u \) is close to 0, according to the equation \( v_t = u \), the evolution is slow. The last row shows the evolution in the time interval \((0:100)\) for fixed \( x = 0.02 \) and \( x = 0.98 \).
Figure 2: Simulation of system \( \mathbf{4} \) for \( 0 = \alpha = 1 < \lambda_1 \). IC do not satisfy the symmetric condition \( u(x) = u(1-x) \) and \( v(x) = v(1-x) \). Indeed, we choose \( u(x,0) = v(x,0) = 1 \) on \( (0,0.5) \), \( u(x,0) = v(x,0) = -0.5 \) on \( (0.5,1) \). The first row illustrates \( u(x,t) \) for \( x \in (0,1) \) and \( t = 0.1 \) (left), \( t = 100 \) (right). We observe that the solution \( u \) reaches a constant function in space. Comparing with the picture in the last row indicates a periodicity in time. The second row illustrates \( v(x,t) \) for \( x \in (0,1) \) and \( t = 0.1 \) (left), \( t = 100 \) (right). In fact, complementary observation would show that the shape of \( v \) is moving slowly while its mean is moving periodically fastly. The last row shows the evolution in the time interval \( (0:100) \) for fixed \( x = 0.02 \) and \( x = 0.98 \).
Figure 3: The case $\lambda_1 < \alpha = 15 < \lambda_2$. IC satisfies $u(x) = u(1-x)$ and $v(x) = v(1-x)$ which implies that only odd coefficients are taken in account, and a symmetric solution (in particular $u_0(t) = v_0(t) = 0$). The first row illustrates $u(x,t)$ for $x \in (0,1)$ and $t = 0.1$ (left), $t = 100$ (right). We observe that the solution $u$ evolves non constantly in space with periodicity in time. The second row illustrates $v(x,t)$ for $x \in (0,1)$ and $t = 0.1$ (left), $t = 100$ (right). The last row shows the symmetric evolution in the time interval $(0,100)$ for fixed $x = 0.02$ and $x = 0.98$. 
Figure 4: The case $\lambda_1 < \alpha = 15 < \lambda_2$. IC do not satisfy the symmetric condition $u(x) = u(1-x)$ and $v(x) = v(1-x)$. Indeed, we choose $u(x,0) = v(x,0) = 1$ on $(0,0.5)$, $u(x,0) = v(x,0) = -0.5$ on $(0.5,1)$. The first row illustrates $u(x,t)$ for $x \in (0,1)$ and $t = 0.1$ (left), $t = 100$ (right). We observe that the solution $u$ reaches a constant function in space. Comparing with the picture in the last row indicates a periodicity in time. The last row shows the evolution in the time interval $(0,100)$ for fixed $x = 0.02$ and $x = 0.98$. Note the difference of the amplitude of the limit cycle with the previous simulation; for these IC, $u_0(t)$ and $v_0(t)$ are no longer zero.
References

[1] B. Ambrosio, J.-P. Francoise, Propagation of bursting oscillations, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences 367 (1908) (2009) 4863–4875. doi:10.1098/rsta.2009.0143.

[2] B. Ambrosio, M. A. Aziz-Alaoui, Basin of attraction of solutions with pattern formation in slow–fast reaction–diffusion systems Acta Biotheoretica 64 (4) (2016) 311–325. doi:10.1007/s10441-016-9294-z URL: https://doi.org/10.1007/s10441-016-9294-z

[3] B. Ambrosio, Hopf bifurcation in an oscillatory-excitable reaction–diffusion model with spatial heterogeneity, International Journal of Bifurcation and Chaos 27 (05) (2017) 1750065. doi:10.1142/s0218127417500651

[4] B. Ambrosio, Wave propagation in excitable media: numerical simulations and analytical study, Ph.D. thesis, Université Paris VI (2009).

[5] J. C. Robinson, Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors (Cambridge Texts in Applied Mathematics), Cambridge University Press, 2001. URL: https://www.amazon.com/Infinite-Dimensional-Dynamical-Systems-Introduction-Dissipati/dp/0521635640?SubscriptionId=AKIAIOBINVZXYZQZ2U3A&tag=chimbori05-20&linkCode=xm2&camp=2025&creative=165953&creativeASIN=0521635640

[6] R. Temam, Infinite Dynamical Systems in Mechanics and Physics, Springer, 1988.