CONVERGENCE OF DENSITY APPROXIMATIONS FOR STOCHASTIC HEAT EQUATION

CHUCHU CHEN, JIANBO CUI, JIALIN HONG, AND DERUI SHENG

ABSTRACT. This paper investigates the convergence of density approximations for stochastic heat equation in both uniform convergence topology and total variation distance. The convergence order of the densities in uniform convergence topology is shown to be exactly $1/2$ in the nonlinear case and nearly $1$ in the linear case. This result implies that the distributions of the approximations always converge to the distribution of the origin equation in total variation distance. As far as we know, this is the first result on the convergence of density approximations to the stochastic partial differential equation.

1. Introduction

In this paper, we consider the stochastic heat equation driven by space-time white noise:

$$
\partial_t u(t, x) = \partial_{xx} u(t, x) + b(u(t, x)) + \sigma \dot{W}(t, x), \quad (t, x) \in (0, T] \times [0, 1]
$$

with initial value $u(0, x) = u_0(x)$, $x \in [0, 1]$ and Neumann boundary condition $\partial_x u(t, 0) = \partial_x u(t, 1) = 0$, $t \in [0, T]$. Here, $T > 0$ is a fixed number and $\sigma \neq 0$ is a constant. Eq. (1.1) arising in many physical problems, characterizes the evolution of a scalar field in a space-time-dependent random medium. The choice of the white noise as random potential corresponds to considering those regimes with very rapid variations, the type of turbulent flows (see [3]). The density function of the solution characterizes all relevant probabilistic information. Concerning the density of $u(t, x)$, its existence, regularity and strictly positivity under suitable assumptions have been well studied (e.g. [1, 12, 15]). If the coefficient $b$ in Eq. (1.1) is infinitely differentiable with bounded derivatives, then as a direct consequence of [12], for any $0 \leq x \leq 1$, $t > 0$, $u(t, x)$ admits a smooth density, and as is shown in [1], for any $0 \leq x_1 < \cdots < x_d \leq 1$, $t > 0$, the law of $(u(t, x_1), \cdots, u(t, x_d))$ admits a strictly positive smooth density, which can be seen as a regularity result for the marginal distribution of $C([0, 1])$-valued random variable $u(t, \cdot)$. Moreover, if $b$ is continuously differentiable with bounded derivative, [15] gives the lower and upper Gaussian bound for the density of $u(t, x)$.

It is a challenge topic to obtain the density exactly or even approximately. However, for stochastic heat equations, even for stochastic partial differential equations, to the best of our knowledge, there are few results concerning the approximation of the density of the origin equation. The purpose of this paper is to develop a strategy to investigate the convergence of the density approximations of the exact solution to Eq. (1.1) in a suitable topology, via a sequence of perturbed equations

$$
\partial_t u^\delta(t, x) = \partial_{xx} u^\delta(t, x) + b(u^\delta(\delta[t/\delta], x)) + \sigma \dot{W}(t, x), \quad (t, x) \in (0, T] \times [0, 1],
$$

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where $\delta = T/N$, $N \in \mathbb{N}^+$ and $[\cdot]$ denotes the greatest-integer function. Different from Eq. (1.1), the drift term $(t, x) \to b(u^\delta(\delta[t/\delta], x))$ in Eq. (1.2) being a piecewise constant in the variable $t$ is a discontinuous function and converges to the drift term of Eq. (1.1) as $\delta$ tends to 0 formally. Hence it is natural and important to study the existence and convergence of the density of Eq. (1.2). Our main results are the convergence of density in total variation distance and convergence order of density in uniform convergence topology.

**Theorem 1.1.** Assume that $b \in C_b^\infty$, $\delta \in \left(0, \frac{T}{N} \wedge \frac{\log \frac{2}{\delta}}{4|b|_T}\right)$. Then there exists some constant $C = C(T, b, \sigma, \|u_0\|_E)$ such that for any $x \in (0, 1)$,

$$
\|q_{T,x}^\delta - q_{T,x}\|_{L^\infty(\mathbb{R})} \leq C\delta^{\frac{1}{2}},
$$

where $q_{T,x}^\delta$ and $q_{T,x}$ are the densities of $u^\delta(T, x)$ and $u(T, x)$, respectively.

In the particular case that $b$ is affine, the above convergence order $1/2$ of density can be improved to $1 - \epsilon$ with some sufficient small $\epsilon > 0$, which coincides with the strong convergence order $1 - \epsilon$ in [10]. As far as we know, this is the first result on the convergence of density approximations to the stochastic partial differential equation. Combining the uniformly boundedness of $q_{T,x}^\delta$ in $L^1(\mathbb{R})$ and Theorem 1.1, it is concluded that $q_{T,x}^\delta$ converges to $q_{T,x}$ in $L^1(\mathbb{R})$. This implies that the distribution of $u^\delta(T, x)$ converges to the distribution of $u(T, x)$ in total variation distance.

Our strategy to prove Theorem 1.1 is based on the weak convergence analysis in the following sense

$$
\|\mathbb{E}[f(u^\delta(T, x)) - f(u(T, x))]\| \leq C\delta^\mu,
$$

where $C$ is independent of $f$ and $\mu > 0$. One key ingredient for the test function-independent weak convergence analysis is the application of the Malliavin integration by parts formula, whose prerequisite is the above uniform non-degeneracy of $u^\delta(T, x)$. It is known that this non-degeneracy condition (see Definition 2.3) is exactly the condition of applying Bouleau–Hirsch criterion (see e.g. [14, Theorem 2.1.4]) to establish the existence and smoothness of the corresponding density. The major obstacle of this non-degeneracy lies in establishing the negative moments of the determinant of the corresponding Malliavin covariance matrix, which is overcome by proving a discrete version of comparison principle. Another difficulty is that the moments of the Gateaux derivatives, as well as the Malliavin derivatives, of both $u(T, x)$ and $u^\delta(T, x)$ are dominated by the multiples of the corresponding Green function associated to Neumann boundary condition, instead of being bounded by a constant. Based on the technical estimates on the Green function, we remove the infinitesimal factor in the weak convergence order of the numerical scheme in the literature (see [8]) and prove that the weak convergence order in (1.4) is $1/2$. In the particular case that $b$ in Eq. (1.1) is affine, the convergence order is improved to $1 - \epsilon$ for some sufficient small $\epsilon > 0$.

The paper is structured as follows. In Section 2, Malliavin calculus associated to a white noise and the properties of the Green function are introduced briefly. Several auxiliary results concerning the regularity estimates of densities and derivatives of solutions are analyzed in Section 3. Then we present the weak convergence analysis via Malliavin calculus in Section 4. Finally, Section 5 is devoted to the convergence of density approximations of Eq. (1.2).

2. Preliminaries

Denote by $E$ the Banach space $C([0, 1])$ endowed with the norm $\|h\|_E = \sup_{x \in [0, 1]} |h(x)|$ and by $C^\infty_\mathbb{P}$ the set of all infinitely differentiable functions with polynomial growth from $\mathbb{R}$ to $\mathbb{R}$.
Let $C^k_b$ be the set of all $k$ times continuous differentiable functions with bounded derivatives from $\mathbb{R}$ to $\mathbb{R}$ and $C^\infty_b := \bigcap_{k \geq 1} C^k_b$. For $b \in C^k_b$, denote $|b| := \sup_{x \in \mathbb{R}} |b^{(i)}(x)|$, $\forall i \in \{1, \ldots, k\}$. We denote by $\delta_z$ the Dirac delta function concentrated at $z \in \mathbb{R}$ and by $x \wedge y = \min\{x, y\}$, $\forall x, y \in \mathbb{R}$. Throughout this article, we use $C$ to denote a generic constant that may change from one place to another and depend on several parameters but never on the perturbation parameter $\delta$. When required, we will explicitly write $C(T, \sigma, \ldots)$ to emphasize the dependence of the constant $C$ upon the parameters $T, \sigma, \ldots$. In what follows, we adopt the conventions that a sum over an empty set is zero and that $\frac{4}{\delta} = \infty$ provided $\delta > 0$ is a constant.

In this section, we present some preliminaries, including some basic elements from Malliavin calculus associated to a white noise and several basic properties of the Green function associated to Eq. (1.1). Let $\{W(t, x)\}_{(t, x) \in [0, T] \times [0, 1]}$ be a Brownian sheet on $[0, T] \times [0, 1]$, defined in a complete probability space $(\Omega, \mathcal{F}, P)$. For $0 \leq t \leq T$, let $\mathcal{F}_t$ be the $\sigma$-field generated by the random variables $\{W(s, x)\}_{(s, x) \in [0, t] \times [0, 1]}$ and the $P$-null sets. In the context of Malliavin calculus, the isonormal Gaussian family $\{W(h), h \in \mathbb{H}\}$ corresponding to $\mathbb{H} := L^2([0, T] \times [0, 1])$ is given by the Wiener integral

$$W(h) = \int_0^T \int_0^1 h(s, y)W(ds, dy).$$

We are interested in Eq. (1.1) and always assume that $u_0 \in E$ is deterministic. If the coefficient $b : \mathbb{R} \to \mathbb{R}$ satisfies the global Lipschitz condition, the rigorous meaning of Eq. (1.1) is given by means of (see e.g. [19]):

$$u(t, x) = \int_0^1 G_t(x, y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x, y)b(u(s, y))dyds + \int_0^t \int_0^1 G_{t-s}(x, y)\sigma W(ds, dy), \quad (2.1)$$

where $G_t(x, y), (t, x, y) \in \mathbb{R}_+ \times (0, 1)^2$, is the Green function associated to the stochastic heat equation on $[0, 1]$ with Neumann boundary condition. Similarly, the mild solution of Eq. (1.2) is given by

$$u^\delta(t, x) = \int_0^1 G_t(x, y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x, y)b(u^\delta(s/\delta), y))dyds$$

$$+ \int_0^t \int_0^1 G_{t-s}(x, y)\sigma W(ds, dy). \quad (2.2)$$

We would like to mention that the mild solution given by (2.2) to Eq. (1.2) corresponds to the accelerate exponential Euler scheme in numerical analysis (see e.g. [10]). By denoting $t_i = i\delta$, we have

$$u^\delta(t_{i+1}, x) = \int_0^1 G_{\delta}(x, y)u^\delta(t_i, y)dy + \int_{t_i}^{t_{i+1}} \int_0^1 G_{t_{i+1}-s}(x, y)b\left(u^\delta(t_i, y)\right)dyds$$

$$+ \int_{t_i}^{t_{i+1}} \int_0^1 G_{t_{i+1}-s}(x, y)\sigma W(ds, dy).$$

2.1. Green function. The explicit formula of the Green function $G$ in (2.1) is well-known,

$$G_t(x, y) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{+\infty} e^{-\frac{(x-y-2n)^2}{4t}} + e^{-\frac{(x+y-2n)^2}{4t}}. \quad (2.3)$$
Denote by \( P_t(x, y) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{(x-y)^2}{4t} \right) \) the heat kernel on \( \mathbb{R} \). Hereafter, the following facts will be used frequently ([1, Appendix]):

1. For any \((t, x, y) \in \mathbb{R}_+ \times (0, 1)^2\), \( G_t(x, y) > 0 \) and \( \int_0^1 G_t(x, y) \, dy = 1 \).
2. Semigroup property: \( \int_0^1 G_t(x, y) G_s(y, z) \, dy = G_{s+t}(x, z) \), \( \forall s, t \in \mathbb{R}_+, x, z \in (0, 1) \).
3. There exists a constant \( K \) depending on \( T \) such that for \((t, x, y) \in (0, T) \times (0, 1)^2\), \[ \frac{1}{K} P_t(x, y) \leq G_t(x, y) \leq K P_t(x, y). \] (2.4)

For any \( t > 0 \) and \( x, y \in \mathbb{R} \), it is fairly understood that \( \rho_t^2(x, y) = \sqrt{1/8\pi t}P_{t/2}(x, y) \) and \( P_s(x, y) \leq \sqrt{t/\pi} \rho_s(x, y) \) provided \( 0 < s < t \leq T \). The explicit formula for \( G_t(x, y) \) is complicated, whose estimation will be converted into the estimation of \( P_t(x, y) \) in view of (2.4). For instance, there exists \( C = C(T) \) such that for \( (t, x, y) \in (0, 1), 0 < s < t \leq T \),

\[ G_t^2(x, y) \leq C \sqrt{t} G_{t/2}(x, y) \] (2.5)

and

\[ G_s(x, y) \leq C \sqrt{t/s} G_t(x, y). \] (2.6)

**Lemma 2.1.** For any \( \nu \in \left( \frac{1}{3}, 1 \right) \), there is \( C = C(T, \nu) \) such that for any \( 0 < s < t \leq T \),

\[ \max \left( \int_0^1 |G_t(x, y) - G_s(x, y)| \, dx, \int_0^1 |G_t(x, y) - G_s(x, y)| \, dy \right) \leq C s^{-\nu} (t - s)^\nu. \]

**Proof.** Similar to [19, Corollary 3.4], the series expansion in (2.3) shows that

\[ G_t(x, y) = P_t(x, y) + H_t(x, y) \] (2.7)

with \( H_t(x, y) \in C^\infty([0, T] \times [0, 1]^2) \). From [11, Corollary 2.2], we have for any \( \nu \in \left( \frac{1}{3}, 1 \right) \),

\[ \max \left( \int_\mathbb{R} |P_t(x, y) - P_s(x, y)| \, dx, \int_\mathbb{R} |P_t(x, y) - P_s(x, y)| \, dy \right) \leq C s^{-\nu} (t - s)^\nu. \]

Finally, the proof is completed by the facts that \( H_t(x, y) \in C^\infty([0, T] \times [0, 1]^2) \) and

\[ |G_t(x, y) - G_s(x, y)| \leq |P_t(x, y) - P_s(x, y)| + |H_t(x, y) - H_s(x, y)|. \]

\( \square \)

When considering Dirichlet boundary condition, all the results in the paper hold as well with minor modification because the Green function \( G \) corresponding to the Dirichlet boundary condition on \([0, 1]\) is

\[ G_t(x, y) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{+\infty} \left( e^{-\frac{(x-y+2n)^2}{4t}} - e^{-\frac{(x+y-2n)^2}{4t}} \right), \] (2.8)

whose property is very similar to \( G \). For more information on the properties of \( G \), the reader is referred to [18, Lemma 7]. The following two-parameter Gronwall lemma is essential in the moment estimates in section 3, whose proof is given in Appendix.

**Lemma 2.2.** Let \( g_{s,y}(t, x) \geq 0 \) satisfy

\[ g_{s,y}(t, x) \leq CG_{t-s}(x, y) + C \int_0^t \int_0^1 G_{t-r_1}(x, z_1) g_{s,y}(r_1, z_1) \, dz_1 \, dr_1, \]

for some constant \( C > 0 \) and all \( 0 < s < t \leq T \) and \( x, y \in (0, 1) \). Then for some \( C = C(T) \),

\[ g_{s,y}(t, x) \leq CG_{t-s}(x, y). \]
2.2. Malliavin calculus associated to white noise. We denote by \( S \) the class of smooth \( \mathbb{R} \)-valued random variables such that \( F \in S \) has the form \( F = f(W(h_1), \ldots, W(h_n)) \), where \( f \) belongs to \( C_p^\infty(\mathbb{R}^n), h_i \in \mathbb{H}, i = 1, \ldots, n, n \geq 1 \). Here, \( C_p^\infty(\mathbb{R}^n) \) is the set of all infinitely continuously differentiable functions \( f : \mathbb{R}^n \to \mathbb{R} \) such that \( f \) and all of its partial derivatives have polynomial growth. Then for any \( p \geq 1 \) and integer \( k \geq 1 \), we denote by \( \mathbb{D}^{k,p} \) the completion of \( S \) with respect to the norm

\[
\|F\|_{k,p} = \left( \mathbb{E} \left[ |F|^p + \sum_{j=1}^k \|D^j F\|^p_{\mathbb{H}^\otimes j} \right] \right)^{\frac{1}{p}},
\]

where \( D \) is the Malliavin derivative operator. In particular, for \( p = 1 \), we simply write \( \|F\|_p \) as an abbreviation for \( \|F\|_{0,p} \). Define

\[
L^\infty(-\Omega) := \bigcap_{p \geq 1} L^p(\Omega), \quad \mathbb{D}^{k,\infty} := \bigcap_{p \geq 1} \mathbb{D}^{k,p}, \quad \mathbb{D}^{\infty} := \bigcap_{k \geq 1} \mathbb{D}^{k,\infty}
\]

to be topological projective limits. As in the Schwartz theory of distributions, \( \mathbb{D}^{-k,p} \) is the topological dual of the Banach space \( \mathbb{D}^{k,p'} \) with \( 1/p + 1/p' = 1 \) and \( \mathbb{D}^{-\infty} = \bigcup_{p \geq 1} \bigcup_{k \geq 1} \mathbb{D}^{-k,p} \) is the space of generalized Wiener functionals. The natural coupling of \( F \in \mathbb{D}^{k,p} \) and \( \Phi \in \mathbb{D}^{-k,q} \) with \( 1/p + 1/q = 1 \) or that of \( F \in \mathbb{D}^{\infty} \) and \( \Phi \in \mathbb{D}^{-\infty} \) is denoted by \( \mathbb{E}[F \cdot \Phi] \).

**Definition 2.3.** A random vector \( F = (F_1, F_2, \ldots, F_m) \) whose components are in \( \mathbb{D}^{\infty} \) is non-degenerate if the Malliavin covariance matrix \( \Gamma_F := (\langle DF_i, DF_j \rangle_\mathbb{H})_{1 \leq i,j \leq m} \) is invertible a.s. and \( (\det \Gamma_F)^{-1} \in L^\infty(-\Omega) \).

In the special case \( m = 1 \), we still call \( \Gamma_F := \|DF\|^2_\mathbb{H} \) the Malliavin covariance matrix of \( F \), although \( \Gamma_F \) is actually a scalar variable.

3. Technical estimates

The classical weak convergence analysis of stochastic partial differential equations has been researched during the past two decades (see e.g. \([4, 5, 8, 9] \) and references therein), where the test function \( \phi \) requires to have boundedness derivatives up to some degree, and the weak convergence order relies on the regularity of \( \phi \). However, this kind of weak convergence for approximations is equivalent to the weak convergence of the associated distributions and is not sufficient to derive the convergence of densities. By \([14, \text{Lemma 2.1.7}] \), the probability of the law of \( F \) at \( z \in \mathbb{R} \) can be determined by the generalized expectation \( \mathbb{E}[\delta_z(F)] \), provided \( F \) is a non-degenerate random variable. For any fixed \( z \in \mathbb{R} \), \( \zeta > 0 \), we define the following mapping

\[
y \mapsto g_\zeta(y - z) = \frac{1}{\sqrt{2\pi \zeta}} e^{-|y-z|^2 / 2\zeta}. \tag{3.1}
\]

It is well known that \( g_{n^{-1}}(\cdot - z) \to \delta_z(\cdot) \) as \( n \) tends to \( \infty \) in the distribution sense and is natural to consider the error between \( \mathbb{E} \left[ g_{n^{-1}}(u^\delta(T, x) - z) \right] \) and \( \mathbb{E} \left[ g_{n^{-1}}(u(T, x) - z) \right] \). Therefore, an alternative space that test function \( f \) lives in to derive the convergence of density of Eq. (1.2) is

\[
\Psi := \{ f : \mathbb{R} \to \mathbb{R} | f \in C_p^\infty, \exists F : \mathbb{R} \to \mathbb{R} \text{ such that } 0 \leq F \leq 1 \text{ and } F' = f \},
\]

since \( \{g_{n^{-1}}(\cdot - z)\}_{n \geq 1, z \in \mathbb{R}} \) is an element of \( \Psi \). In this section, we prove some technical results in preparation for the following test-function independent weak convergence analysis result.
Theorem 3.1. Let \( b \in C^\infty_b, \delta \in \left(0, \frac{T}{12} \wedge \frac{\log \frac{3}{\delta}}{4|\beta|}\right). \) Then there exists some positive constant \( C = C(T, b, \sigma, \|u_0\|_E) \) such that for any \( x \in (0,1) \) and \( f \in \Psi, \) it holds that
\[
\left| E[f(u^\delta(T, x))] - E[f(u(T, x))] \right| \leq C\delta^\frac{1}{2}. \tag{3.2}
\]

3.1. Error decomposition. In order to prove Theorem (3.1), the following notations are introduced for simplicity. For \( 0 \leq s < t \leq T, x \in (0,1) \) and \( v : \Omega \to E \) being \( \mathcal{F}_s \)-measurable, we denote by \( \varphi^\delta_t(s, v) \) (resp. \( \Phi^\delta_t(s, v) \)) the exact flow of Eq. (1.1) (resp. Eq. (1.2)). More precisely,
\[
\varphi^\delta_t(s, v) = \int_0^t G_{t-s}(x, z) v(z) dz + \int_s^t \int_0^1 G_{t-r}(x, z)b(\varphi^\delta_r(s, v)) dr dz \\
+ \int_s^t \int_0^1 G_{t-r}(x, z) \sigma W(dr, dz),
\]
\[
\Phi^\delta_t(s, v) = \int_0^t G_{t-s}(x, z) v(z) dz + \int_s^t \int_0^1 G_{t-r}(x, z)b\left(\Phi^\delta_{r|\delta}(s, v)\right) dr dz \\
+ \int_s^t \int_0^1 G_{t-r}(x, z)W(dr, dz).
\]
The Gateaux derivative of \( \varphi^\delta_t(s, \cdot) \) at \( v \in E \) in the direction \( h \in E \) is defined formally by
\[
\langle \mathcal{D}f(\varphi^\delta_t(s, v), h) \rangle = \frac{d}{d\epsilon}f(\varphi^\delta_t(s, v + \epsilon h))|_{\epsilon = 0} = \frac{d}{d\epsilon}f(\varphi^\delta_t(s, v)) \langle \mathcal{D} \varphi^\delta_t(s, v), h \rangle.
\]
For \( i \in \{1, \cdots, N\} \) and \( \beta, \tau \in [0,1], \) we denote
\[
Y^\tau_i(t, r, y) := \tau \Phi^\delta_i(t_{i-1}, \Phi^\delta_{t_{i-1}}(0, u_0)) + (1 - \tau)\varphi^\delta_{t_i}(t_{i-1}, \Phi^\delta_{t_{i-1}}(0, u_0)), \tag{3.3}
\]
\[
Z^\delta_i(r, y) := \beta \Phi^\delta_i(t_{i-1}, \Phi^\delta_{t_{i-1}}(0, u_0)) + (1 - \beta)\Phi^\delta_{t_{i-1}}(0, u_0), r \in (t_{i-1}, t_i], y \in [0,1]. \tag{3.4}
\]
Since \( Y^\tau_i \in E, \) a.s, for \( y \in [0,1], \) we write
\[
Y^\tau_i(y) := \tau \Phi^\delta_i(t_{i-1}, \Phi^\delta_{t_{i-1}}(0, u_0)) + (1 - \tau)\varphi^\delta_{t_i}(t_{i-1}, \Phi^\delta_{t_{i-1}}(0, u_0)).
\]
Using the above notations, the one-step error between Eq. (1.1) and Eq. (1.2) is divided into
\[
\varphi_{t_i}(t_{i-1}, \Phi^\delta_{t_{i-1}}(0, u_0)) - \Phi_{t_i}(t_{i-1}, \Phi^\delta_{t_{i-1}}(0, u_0)) \\
= \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-r}(\cdot, y) \left( b(\varphi^\delta_r(t_{i-1}, \Phi^\delta_{t_{i-1}}(0, u_0))) - b(\Phi^\delta_{t_{i-1}}(0, u_0)) \right) dy dr \\
= \int_0^1 \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-r}(\cdot, y) b'(Z^\delta_i(\cdot, y)) \left( \varphi^\delta_{t_i}(t_{i-1}, \Phi^\delta_{t_{i-1}}(0, u_0)) - \Phi^\delta_{t_{i-1}}(0, u_0) \right) dy dr d\beta \\
= \int_0^1 \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-r}(\cdot, y) b'(Z^\delta_i(\cdot, y)) \left\{ G_r(y, \xi) - G_{t_{i-1}}(y, \xi) \right\} u_0(\xi) dy d\xi dr d\beta \\
+ \int_0^1 \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-r}(\cdot, y) b'(Z^\delta_i(\cdot, y)) \left\{ G_{r-\theta}(y, \xi) - G_{t_{i-1}-\theta}(y, \xi) \right\} d\xi d\theta dy dr d\beta \\
+ \int_0^1 \int_{t_{i-1}}^{t_i} \int_0^1 G_{t-r}(\cdot, y) b'(Z^\delta_i(\cdot, y)) \left\{ G_{r-\theta}(y, \xi) - G_{t_{i-1}-\theta}(y, \xi) \right\} \sigma W(d\theta, d\xi) dy dr d\beta.
\]
By chain rule, we have 

\[ \text{solutions } u \text{ which indicates the existence and smoothness of its density } \].

It is noteworthy that both the \( E \) in \( C \) holds for some constant \( \frac{1}{2} \).

Supposing that \( f \in \Psi \), we consider the telescoping sum

\[
E[f(\varphi^\tau_T(0,u_0))] - E[f(\Phi^\tau_T(0,u_0))]
\]

\[
= \sum_{i=1}^{N} \left[ f(\varphi^\tau_T(t_i, \varphi(t_{i-1}, \Phi(t_{i-1}(0,u_0)))) - f(\varphi^\tau_T(t_i, \Phi(t_{i-1}(0,u_0)))) \right]
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{5} E \left[ \int_{0}^{1} \left< Df(\varphi^\tau_T(t_i, Y_{i\tau}^t)), R_{ij} \right> d\tau \right].
\]

By chain rule, we have

\[
E[I_i] := E \left[ \int_{0}^{1} \left< Df(\varphi^\tau_T(t_i, Y_{i\tau}^t)), R_{ij} \right> d\tau \right] = E \left[ \int_{0}^{1} f'(\varphi^\tau_T(t_i, Y_{i\tau}^t)) \langle D\varphi^\tau_T(t_i, Y_{i\tau}^t), R_{ij} \rangle d\tau \right].
\]

The above error decomposition (3.5) is standard, however, the appearance of \( f'(\varphi^\tau_T(t_i, Y_{i\tau}^t) \) in \( E[I_i] \) and the requirement that \( C \) is independent of \( f \in \Psi \) imply that the classical estimates that \( f'(\varphi^\tau_T(t_i, Y_{i\tau}^t) \) is bounded by \( |f| \) is not apply to our case. The Malliavin integration by parts formula (see e.g. [14, Proposition 2.1.4]) has been used to remove the dependence of \( C \) upon \( f \) in the weak convergence of approximation for stochastic ordinary differential equation (see e.g. [2]). We applying this idea to the stochastic heat equation and obtain Lemma 3.2, whose proof is based on the non-degeneracy of \( \varphi^\tau_T(t_i, Y_{i\tau}^t) \) and is given in Subsection 3.2.

To avoid ambiguity, we point out that \( f^{(\alpha)}(\varphi^\tau_T(t_i, Y_{i\tau}^t)) \) denotes the composition of the \( \alpha \)-th derivative \( f^{(\alpha)} \) of \( f \) and the random variable \( \varphi^\tau_T(t_i, Y_{i\tau}^t) \).

**Lemma 3.2.** Let \( \alpha \in \mathbb{N} \), \( b \in \mathbb{C}^\infty_0 \) and \( \delta \in \left( 0, \frac{\tau}{12} \wedge \frac{\log \frac{3}{4\|b\|_1}}{\|b\|_1} \right) \). If \( G_1 \in \mathbb{D}^\infty \) and \( f \in \Psi \), then for any \( i \in \{1, \ldots, N\} \), \( x \in (0,1) \) and \( \tau \in [0,1] \),

\[
E\left[ f^{(\alpha)}(\varphi^\tau_T(t_i, Y_{i\tau}^t)) G_1 \right] \leq C||G_1||_{k,p}
\]

holds for some constant \( C = C(\alpha, k, p, T, \sigma, b, ||u_0||_E) \).

### 3.2. Regularity of densities.

In this part, we study the non-degeneracy property of

\[
\{\varphi^\tau_T(t_i, Y_{i\tau}^t)\}_{x \in (0,1), i \in \{1, \ldots, N\}, \tau \in [0,1]},
\]

which indicates the existence and smoothness of its density. It is noteworthy that both the solutions \( u(T, x) \) to Eq. (1.1) and \( u^\delta(T, x) \) to Eq. (1.2) are special cases of \( \varphi^\tau_T(t_i, Y_{i\tau}^t) \) since \( u(T, x) = \varphi^\tau_T(t_1, Y_0^t) \) and \( u^\delta(T, x) = \varphi^\tau_T(t_N, Y^\delta t_1) \). For more general SPDEs, as well as those driven by multiplicative or more rough noises, we refer to [6, 16] and references therein for a fruitful results of research on densities of their exact solutions. In particular, as a direct consequence of [12], for any \( 0 < x < 1 \), \( u(T, x) \) is non-degenerate and thereby admits a smooth density.
3.2.1. **Negative moments.** To start with, we give a uniform positive lower bound, independent of the sample $\omega$ and the perturbation parameter $\delta$, of the Malliavin covariance matrix $\Gamma_{\varphi^\gamma_t}(t, Y_t^\tau)$ by proving a discrete version comparison principle.

**Proposition 3.3.** Let $x \in (0, 1)$, $i \in \{1, \cdots, N\}$ and $\tau \in [0, 1]$, and assume that $b \in C_b^1$. Then for any $\delta \in \left(0, \frac{T}{12} \wedge \frac{\log \frac{4}{|b|_1}}{4|b|_1}\right)$, the Malliavin covariance matrix $\Gamma_{\varphi^\gamma_t}(t, Y_t^\tau)$ satisfies

$$\Gamma_{\varphi^\gamma_t}(t, Y_t^\tau) \geq c,$$

for some $c = c(T, |b|_1, \sigma) > 0$.

**Proof.** Without loss of generality, assume that $\sigma > 0$. By the Cauchy-Schwarz inequality, we infer that for $i \geq 1$,

$$\Gamma_{\varphi^\gamma(t, Y_t^\tau)} = \int_0^T \int_0^1 D_{\theta, \xi} \varphi^\gamma_T(t, Y_t^\tau)^2 d\xi d\theta \geq \int_0^T \left( \int_0^1 D_{\theta, \xi} \varphi^\gamma_T(t, Y_t^\tau) d\xi \right)^2 d\theta. \quad (3.6)$$

Denote $X(t, x; \theta) := \int_0^1 D_{\theta, \xi} \varphi^\gamma_T(t, Y_t^\tau) d\xi$. Recalling the definition of $Y_t^\tau$ in (3.3), $X(t, x; \theta)$ depends on $\tau$ and we drop its explicit dependence for simplicity. Observing that by the definition of $\varphi^\gamma_T(t, Y_t^\tau)$ and the chain rule,

$$X(T, x; \theta) = \int_0^1 G_{T-t_i}(x, y) \int_0^1 D_{\theta, \xi} Y_t^\tau(x) d\xi dy + \int_0^T \int_0^1 G_{T-\tau}(x, y) b'(\varphi_T(t, Y_t^\tau)) X(r, y; \theta) dy dr + G_{T-\theta}(x, \xi) \sigma 1_{\{\theta \in [t_i, T]\}}. \quad (3.7)$$

In order to dominate $X(T, x; \theta)$ from below, we require to estimate $X(T, x; \theta)$ in two cases $\theta > t_i$ and $\theta < t_i$. In the first case, $\theta > t_i$ implies $D_{\theta, \xi} Y_t^\tau(y) = 0$ because $Y_t^\tau$ is $\mathcal{F}_{t_i}$-measurable and in the second case, $\theta < t_i$ implies $G_{T-\theta}(x, \xi) \sigma 1_{\{\theta \in [t_i, T]\}} = 0$, but the estimation of $D_{\theta, \xi} Y_t^\tau(y)$ requires a more sophisticated treatment than the first case.

**Case 1:** Let $\theta \in (t_i, T]$. Then it follows from (3.7) that

$$\partial_t X(t, x; \theta) = \partial_{xx} X(t, x; \theta) + b'(\varphi^\gamma_T(t, Y_t^\tau)) X(t, x; \theta), \quad \theta < t \leq T$$

with initial condition $X(\theta, x; \theta) = \sigma, \forall x \in (0, 1)$ and the Neumann boundary condition. By comparison principle ([12, Lemma 4]) and the assumption $b'(\varphi^\gamma_T(t, Y_t^\tau)) \geq -|b|_1 > -\infty$, we obtain that for any $\tau \in [0, 1]$,

$$X(T, x; \theta) \geq e^{-|b|_1(T-\theta)} \sigma. \quad (3.8)$$

**Case 2:** Let $\theta \in (0, t_i)$. By (3.7), we begin with estimating $\int_0^1 D_{\theta, \xi} Y_t^\tau(y) dy$, which is equivalent to estimating $\int_0^1 D_{\theta, \xi} \varphi^\gamma_T(t_{i-1}, \Phi_{t_{i-1}}(0, u_0) d\xi$ and $\int_0^1 D_{\theta, \xi} \Phi^b_{t_i}(0, u_0) d\xi$. Therefore, we are in a position to estimate $\int_0^1 D_{\theta, \xi} \Phi_{t_i}(0, u_0) d\xi$. We denote for brevity $M_i(\theta, y) := \int_0^1 D_{\theta, \xi} \Phi^b_{t_i}(0, u_0) d\xi$. Then for any $\theta \in (t_k, t_{k+1})$ with $0 \leq k \leq i - 1$, $M_i(\theta, y)$ satisfies the following recursive relation

$$M_i(\theta, y) = \sum_{j=k+1}^{i-1} \int_{t_{j-1}}^{t_j} \int_0^1 G_{t_{j-1}}(y, z) b'(\Phi^b_{t_j}(0, u_0)) M_j(\theta, z) dz dr + \sigma.$$ 

To get a lower bound, we prove a discrete version of comparison principle. Define a two-parameter sequence $\{A^k_i\}_{1 \leq k \leq i \leq N}$ by for any $i \in \{1, \cdots, N\}$, $A^i_i = 0$, $A^{i-1}_i = \sigma$ and

$$A^k_i = \sum_{j=k+1}^{i-1} \int_{t_{j-1}}^{t_j} \int_0^1 G_{t_{j-1}}(y, z) |b|_1 A^k_z dz dr + \sigma.$$ 

\[
\sum_{j=k+1}^{i-1} |b|_1 \delta A^k_j + \sigma, \ \forall 1 \leq i \leq i - 2. \tag{3.9}
\]

By an induction argument and the construction of \(A^k_i\), we see that for any \(\theta \in (t_k, t_{k+1})\) and \(y \in (0, 1)\),
\[
|M_i(\theta, y)| \leq A^k_i.
\]

By definition, if \(i_1 - k_1 = i_2 - k_2\), then \(A^k_{i_1} = A^k_{i_2} =: A_{i_1 - k_1}\). Rearranging (3.9), we derive
\[
A_{i-k} = \sum_{j=k+1}^{i-1} |b|_1 \delta A_{j-k} + \sigma = |b|_1 \delta A_{i-1-k} + \sum_{j=k+1}^{i-2} |b|_1 \delta A_{j-k} + \sigma
\]
\[
= (1 + |b|_1 \delta) A_{i-1-k} = (1 + |b|_1 \delta)^{i-k-1} \sigma.
\]

Therefore, if \(|b|_1 > 0\), we have
\[
M_i(\theta, y) \geq \sigma - |b|_1 \delta (A^k_{i-1} + A^k_{i-2} + \cdots + A^k_{i-k+1})
\]
\[
= \left\{ 2 - (1 + |b|_1 \delta)^{i-k-1} \right\} \sigma \geq \frac{1}{2} \sigma,
\]

provided \(1 \leq i - k - 1 \leq \frac{\log 2}{\log (1+|b|_1 \delta)}\). Notice that \(0 < \log (1+x) \leq x\), \(\forall x > 0\). To summarize, for any \(y \in (0, 1)\) and \(\theta \in (t_k, t_{k+1})\) with \(\max \left\{ 0, i-1 - \frac{\log 2}{|b|_1 \delta} \right\} \leq k \leq i - 2\), it holds that
\[
M_i(\theta, y) \geq \frac{1}{2} \sigma. \tag{3.10}
\]

Obviously, if \(|b|_1 = 0\), i.e. \(b' \equiv 0\), the desired positive lower bound (3.10) for \(M_i(\theta, y)\) is valid as well.

Noticing that
\[
\int_{0}^{1} D_{\theta, \xi} \varphi_{t_i}(t_{i-1}, \Phi_{t_{i-1}}(0, u_0)) d\xi = \int_{0}^{1} G_{\delta}(y, z) \int_{0}^{1} D_{\theta, \xi} \Phi_{t_{i-1}}(0, u_0) d\xi dz
\]
\[
+ \int_{t_{i-1}}^{t_i} \int_{0}^{1} G_{t_i - r}(y, z)b'(\varphi_{t_i}(t_{i-1}, \Phi_{t_{i-1}}(0, u_0))) \int_{0}^{1} D_{\theta, \xi} \varphi_{t_i}(t_{i-1}, \Phi_{t_{i-1}}(0, u_0)) d\xi dz dr,
\]

we aim to derive a low bound for \(\int_{0}^{1} D_{\theta, \xi} \varphi_{t_i}(t_{i-1}, \Phi_{t_{i-1}}(0, u_0)) d\xi\) by the above low bound of \(\int_{0}^{1} D_{\theta, \xi} \Phi_{t_{i-1}}(0, u_0) d\xi = M_{i-1}(\theta, z)\) and the comparison principle ([12, Lemma 4]). To be precise, supposing that \(\theta \in (t_k, t_{k+1})\) and \(y \in (0, 1)\) with \(\max \left\{ 0, i-1 - \frac{\log 2}{|b|_1 \delta} \right\} \leq k \leq i - 3\) are arbitrarily fixed, then for any \(z \in (0, 1)\), it holds that \(\int_{0}^{1} D_{\theta, \xi} \Phi_{t_{i-1}}(0, u_0) d\xi \geq \frac{1}{2} \sigma\), which, together with the comparison principle indicates that
\[
\int_{0}^{1} D_{\theta, \xi} \varphi_{t_i}(t_{i-1}, \Phi_{t_{i-1}}(0, u_0)) d\xi \geq \frac{1}{2} e^{-|b|_1 \delta} \sigma. \tag{3.11}
\]

Thus by (3.10) and (3.11), we have that for any \(\tau \in [0, 1]\),
\[
\int_{0}^{1} D_{\theta, \xi} \varphi_{t_i}(t_{i-1}, \Phi_{t_{i-1}}(0, u_0)) d\xi = \tau \int_{0}^{1} D_{\theta, \xi} \varphi_{t_i}(t_{i-1}, \Phi_{t_{i-1}}(0, u_0)) d\xi + (1 - \tau) \int_{0}^{1} D_{\theta, \xi} \Phi_{t_i}(0, u_0) d\xi
\]
\[
\geq \frac{\tau}{2} e^{-|b|_1 \delta} \sigma + (1 - \tau) \frac{1}{2} \sigma \geq \frac{1}{2} e^{-|b|_1 \delta} \sigma. \tag{3.12}
\]
Now we turn to (3.7) and estimate $X(T, x; \theta)$. Taking account of (3.12) and applying the comparison principle yield that

$$X(T, x; \theta) \geq e^{-|b_1| (T-t_1)} \int_0^1 \theta \xi \gamma_i^T(y) d\xi \geq \frac{1}{2} e^{-|b_1| (T-t_{i-1})} \theta,$$

(3.13)

for any $\tau \in [0, 1]$ and $\theta \in (t_k, t_{k+1})$ with $\max\left\{0, i - 1 - \frac{\log \frac{b_1}{|b|}}{|b|} \right\} \leq k \leq i - 3$.

So far, we have dominated $X(T, x; \theta)$ from below when $\theta > t_i$ in Case 1 and when $\theta \in (t_k, t_{k+1})$ with $\max\left\{0, i - 1 - \frac{\log \frac{b_1}{|b|}}{|b|} \right\} \leq k \leq i - 3$ in Case 2, based on which, we are going to give a lower bound estimate of $\Gamma_{\varphi_i^T(t_i, Y_i^T)}$ as follows. By (3.6) and (3.8),

$$\Gamma_{\varphi_i^T(t_i, Y_i^T)} \geq \int_0^T |X(T, x; \theta)|^2 d\theta$$

$$\geq \sum_{k=0}^{i-3} \int_{t_k}^{t_{k+1}} |X(T, x; \theta)|^2 d\theta + \int_{t_i}^T e^{-2|b_1| (T-\theta)} \sigma^2 d\theta.$$

For $0 \leq i \leq \frac{N}{2} + 3$,

$$\Gamma_{\varphi_i^T(t_i, Y_i^T)} \geq \int_0^T e^{-2|b_1| (T-\theta)} \sigma^2 d\theta = \frac{1}{2} e^{-2|b_1| T} \sigma^2 =: c_1,$$

in view of $\delta \leq \frac{T}{T}$. For $\frac{N}{2} + 3 \leq i \leq N$, we have $T - t_{i-1} \leq \frac{T}{2}$ and thus

$$\Gamma_{\varphi_i^T(t_i, Y_i^T)} \geq \frac{1}{4} e^{-2|b_1| T} \sigma^2 \delta \min\left\{ \frac{N}{2}, \frac{\log \frac{2}{|b|}}{|b|} - 2 \right\} \geq \frac{1}{8} e^{-2|b_1| T} \sigma^2 \min\left\{ T, \frac{\log \frac{2}{|b|}}{|b|} \right\} =: c_2,$$

thanks to (3.13) and $\delta \leq \frac{\log \frac{3}{4|b|}}{4|b|}$. Finally, we finish the proof by choosing $c = \min\{c_1, c_2\}$.

3.2.2. Integrability of Malliavin derivatives. Next lemma states that $\varphi_i^T(t_i, Y_i^T)$ and its Malliavin derivatives of any order have bounded moments. We would like to mention that this property is still valid for stochastic heat equation driven by multiplicative noise with further assumptions (see e.g. [1]).

Lemma 3.4. Assume that $b \in C_{0}^{\infty}$. Then for any integers $k \geq 0$, $p \geq 1$, there exists $C = C(k, p, T, b, \sigma, \|u_0\|_{L})$ such that for any $\tau \in [0, 1]$,

$$\sup_{i=1, \cdots, N} \sup_{y \in (0, 1)} \|\Phi_i^y(t_i, u_0)\|_{k,p} + \sup_{i=1, \cdots, N} \sup_{y \in (0, 1)} \|\varphi_i^y(t_{i-1}, \Phi_{t_{i-1}}(t_i, Y_i^T))\|_{k,p} \leq C,$$

(3.14)

$$\sup_{i=1, \cdots, N} \sup_{t \in [t_i, t_{i+1}], x \in (0, 1)} \|\varphi_i^x(t_i, Y_i^T)\|_{k,p} \leq C.$$

(3.15)

Proof. Notice that for any $F \in \mathbb{D}_{k,p}$, it holds that

$$\|F\|_{k,p}^p = \|F\|_{k-1,p}^p + \|D^k F\|_{L_p(\Omega, \mathbb{H}^{\otimes k})}^p.$$

(3.16)
To begin with, let \( i \in \{1, \ldots, N\} \) be arbitrarily fixed. By definition,
\[
\Phi^y_{t_i}(0, u_0) = \int_0^1 G_{t_i}(y, z)u_0(z)dz + \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \int_0^1 G_{t_i-r}(y, z)b(\Phi^y_{t_j}(0, u_0))dzdr \\
+ \int_0^{t_i} \int_0^1 G_{t_i-r}(y, z)\sigma W(dr, dz).
\]
By \( u_0 \in E \) and the linear growth of \( b \), we have
\[
\sup_{y \in (0,1)} \|\Phi^y_{t_i}(0, u_0)\|_p \leq C(T, \|u_0\|_E) + \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \sup_{y \in (0,1)} \int_0^1 G_{t_i-r}(y, z) \sup_{z \in (0,1)} \|\Phi^z_{t_j}(0, u_0)\|_p dzdr \\
+ \sup_{y \in (0,1)} \left\| \int_0^{t_i} \int_0^1 G_{t_i-r}(y, z)\sigma W(dr, dz) \right\|_p.
\]
Therefore, the Burkholder’s inequality and the discrete Gronwall lemma produce
\[
\sup_{y \in (0,1)} \|\Phi^y_{t_i}(0, u_0)\|_p \leq C, \forall i = 1, \ldots, N. \quad (3.17)
\]
Similarly, by the definition of \( \varphi^y_{t_i}(t_{i-1}, \Phi_{t_{i-1}}(0, u_0)) \), the linear growth of \( b \), Burkholder’s and Minkowskii’s inequalities, we have
\[
\|\varphi^y_{t_i}(t_{i-1}, \Phi_{t_{i-1}}(0, u_0))\|_p \leq \int_0^1 G_{t_i}(y, z)\|\Phi^z_{t_{i-1}}(0, u_0)\|_p dz + C \left( \int_{t_{i-1}}^{t_i} \int_0^1 G_{t_i-r}(y, z)dzdr \right)^{\frac{1}{2}} \\
+ C \int_{t_{i-1}}^{t_i} \int_0^1 G_{t_i-r}(y, z)(1 + \|\varphi^z_{t_{i-1}}(0, u_0)\|_p)dzdr.
\]
Taking the supremum over \( y \in (0,1) \) and taking account of (3.17), we obtain that
\[
\sup_{y \in (0,1)} \|\varphi^y_{t_i}(t_{i-1}, \Phi_{t_{i-1}}(0, u_0))\|_p \leq C, \forall i = 1, \ldots, N, \quad (3.18)
\]
which together with (3.17) implies that (3.14) holds for \( k = 0 \). Similar to the process of the proof of (3.18), it can be shown that (3.15) holds for \( k = 0 \) as well.

By induction, we assume that (3.14) and (3.15) hold up to the index \( k - 1, k \geq 1 \). By utilizing Leibnitz’s rule, it holds that
\[
\|D^k \varphi^y(t_i, Y^\tau_i)\|_{L^p(\Omega, \mathbb{H}^\otimes k)} \leq \int_0^1 G_{t_i}(x, y)\|D^k Y^\tau_i(y)\|_{L^p(\Omega, \mathbb{H}^\otimes k)}dy \\
+ \sigma \left( \int_{t_i}^t \int_0^1 G_{t-i-s}(x, y)dyds \right)^{\frac{1}{2}} \mathbf{1}_{\{k=1\}} + |b| \int_{t_i}^t \int_0^1 G_{t-i-s}(x, y)\|D^k \varphi^y_{t-s}(t_i, Y^\tau_i)\|_{L^p(\Omega, \mathbb{H}^\otimes k)}dyds \\
+ \int_{t_i}^t \int_0^1 G_{t-i-s}(x, y) \sum_{j=1}^{k-1} \binom{k-1}{j} \|D^j b'(\varphi^y_{t-s}(t_i, Y^\tau_i))\|_{L^{2p}(\Omega, \mathbb{H}^\otimes j)} \|D^{k-j} \varphi^y_{s}(t_i, Y^\tau_i)\|_{L^{2p}(\Omega, \mathbb{H}^\otimes k-j)}dyds.
\]
The Faà di Bruno’s formula (see e.g. [17]) gives that
\[
D^j b'(\varphi^y_{s}(t_i, Y^\tau_i)) = \sum_{l_1! \cdots l_j!} \frac{j!}{l_1! \cdots l_j!} b^{(l+1)}(\varphi^y_{s}(t_i, Y^\tau_i)) \left( \frac{D \varphi^y_{s}(t_i, Y^\tau_i)}{l_1!} \right)^{l_1} \cdots \left( \frac{D \varphi^y_{s}(t_i, Y^\tau_i)}{l_j!} \right)^{l_j},
\]
where \( l = l_1 + \cdots + l_j \) and the sum is taken over all partitions of \( j \), i.e., values of \( l_1, \ldots, l_j \) such that \( l_1 + 2l_2 + \cdots + jl_j = j \). Using Hölder’s inequality, for \( 1/p_1 + \cdots + 1/p_j = 1/p \), we have

\[
\| D^j y'(\varphi^y(t, Y_T)) \|_{L^p(\Omega, \mathbb{H}^k)} \\
\leq C \sum_{i=l_1}^{j} \frac{j!}{l_1! \cdots l_j!} \| D^j y'(t, Y_T) \|_{L^{t_1}(\Omega, \mathbb{H})} \cdots \| D^j y'(t, Y_T) \|_{L^{t_j}(\Omega, \mathbb{H}^k)}.
\]

Therefore, by the assumption that (3.15) holds up to the index \( k-1 \), we arrive at

\[
\| D^k y'(t, Y_T) \|_{L^p(\Omega, \mathbb{H}^{\otimes k})} \leq C + \int_0^t \int_0^1 G_{t-t_i}(x, y) \| D^k y'(t, Y_T) \|_{L^p(\Omega, \mathbb{H}^{\otimes k})} dy ds.
\]

By Leibnitz’s rule,

\[
D^k \Phi^y_{t_i}(0, u_0) = \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \int_0^1 G_{t_i-r}(y, z) b'(\Phi^y_{t_j}(0, u_0)) D^k \Phi^y_{t_j}(0, u_0) dz dr + \int_0^{t_{j+1}} \int_0^1 G_{t_i-r}(y, z) \sum_{m=1}^{k-1} \frac{(k-1)!}{m!} D^m b'(\Phi^y_{t_j}(0, u_0)) D^{k-m} \Phi^y_{t_j}(0, u_0) dz dr.
\]

Similar to the proof of (3.19), the Faà di Bruno’s formula and the assumption that (3.14) holds up to \( k-1 \) imply

\[
\| D^k \Phi^y_{t_i}(0, u_0) \|_{L^p(\Omega, \mathbb{H}^{\otimes k})} \leq C + |b|_1 \int_0^{t_{j+1}} \int_0^1 G_{t_i-r}(y, z) \| D^k \Phi^y_{t_j}(0, u_0) \|_{L^p(\Omega, \mathbb{H}^{\otimes k})} dz dr
\]

and

\[
\| D^k \varphi^y_{t_i}(t_{i-1}, \Phi_{t_i-1}(0, u_0)) \|_{L^p(\Omega, \mathbb{H}^{\otimes k})} \leq C + \int_0^1 G_\delta(y, z) \| \Phi^y_{t_i-1}(0, u_0) \|_{L^p(\Omega, \mathbb{H}^{\otimes k})} dz + C \int_0^{t_i} \int_0^1 G_{t_i-r}(y, z) \| D^k \varphi^y_{t_i}(t_{i-1}, \Phi_{t_i-1}(0, u_0)) \|_{L^p(\Omega, \mathbb{H}^{\otimes k})} dz dr.
\]

Taking supremum over \( y \in (0, 1) \) on both sides of (3.20) and (3.21), then applying the Gronwall lemma, we arrive at

\[
\sup_{y \in (0, 1)} \| D^k \Phi^y_{t_i}(0, u_0) \|_{L^p(\Omega, \mathbb{H}^{\otimes k})} \leq C, \forall i = 1, \cdots, N
\]

and

\[
\sup_{y \in (0, 1)} \| D^k \varphi^y_{t_i}(t_{i-1}, \Phi_{t_i-1}(0, u_0)) \|_{L^p(\Omega, \mathbb{H}^{\otimes k})} \leq C, \forall i = 1, \cdots, N,
\]

which, together with (3.16), completes the proof of (3.14). Finally, it follows from (3.14) and (3.19) that (3.15) holds for \( k \) and the proof is completed. \( \Box \)
The proof of Lemma 3.4 is naturally extended to the following cases, whose proof is skipped.

**Corollary 3.5.** Assume that \( b \in \mathcal{C}_b^\infty \). Then for any integers \( k \geq 0, p \geq 1 \), there exists 
\[ C = C(k, p, T, b, \sigma, \|u_0\|_F) \] such that for any \( \tau, \beta \in [0, 1] \), we have 
\[
\sup_{i=1,\ldots,N} \sup_{t \in (t_i, t_{i+1}], \xi \in (0, 1)} \|b'((\varphi^\beta_{\theta_i} (t, Y^\tau_{i}))\|_{k,p} \leq C,
\]
\[
\sup_{i=1,\ldots,N} \sup_{r \in (t_{i-1}, t_i), y \in (0,1)} \|b'((Z^\beta_r (r, y)))\|_{k,p} \leq C,
\]
\[
\sup_{i=1,\ldots,N} \sup_{\theta \in [0,t_{i-1}], \xi \in (0,1)} \|b'((\Phi^\xi_{\theta} (0, u_0)))\|_{k,p} \leq C.
\]

Based on Proposition 3.3 and Lemma 3.4, we are in a position to show the regularity of the density of \( u^\delta(T, x) \) and to give the proof of Lemma 3.2.

**Theorem 3.6.** Assume that \( b \in \mathcal{C}_b^\infty \) and \( \delta \in \left(0, \frac{T}{12} \wedge \frac{\log \frac{2}{\delta}}{4|\delta|} \right) \). Then for every \( x \in (0,1) \), \( u^\delta(T, x) \) admits an infinitely differentiable density.

**Proof.** In view of Proposition 3.3 and (3.15), for every \( x \in (0,1), i \in \{1, \ldots, N\} \) and \( \tau \in [0,1], \varphi^\tau_{\theta_i}(t, Y^\tau_i) \) is non-degenerate and so is \( u^\delta(T, x) \). Consequently, a direct application of the Bouleau–Hirsch’s criterion (see e.g. [14, Theorem 2.1.4]) yields that for every \( x \in (0,1) \), \( u^\delta(T, x) \) admits an infinitely differentiable density. \[\square\]

We emphasize that Lemma 3.2 will be used repeatedly to ensure that the generic constant \( C \) appeared in Theorem 3.1 is independent of the test function \( f \).

**Proof of Lemma 3.2:** Invoking Proposition 3.3 and Lemma 3.4, it follows from [14, Proposition 2.1.4] that for any \( \alpha \in \mathbb{N}, k \geq 1 \), there exists an element \( H_{\alpha+1}(\varphi^\tau_{\theta_i}(t, Y^\tau_i), G_1) \in \mathcal{D}^\infty \) such that 
\[
\mathbb{E}\left[ f^{(\alpha)}(\varphi^\tau_{\theta_i}(t, Y^\tau_i))G_1 \right] = \mathbb{E}\left[ F(\varphi^\tau_{\theta_i}(t, Y^\tau_i))H_{\alpha+1}(\varphi^\tau_{\theta_i}(t, Y^\tau_i), G_1) \right].
\] (3.22)

Furthermore, for \( p_1 \geq 1 \), there exist constants \( C(p_1, \alpha), a, q, k', w, k, p \) such that 
\[
\|H_{\alpha+1}(\varphi^\tau_{\theta_i}(t, Y^\tau_i), G_1)\|_{p_1} \leq C(p_1, \alpha)\|\Gamma^{-1}_{\varphi^\tau_{\theta_i}(t, Y^\tau_i)}\|_{q}^w\|\varphi^\tau_{\theta_i}(t, Y^\tau_i)\|_{k,p}^w\|G_1\|_{k,p}.
\]

Hence, by \( 0 \leq F \leq 1 \), Proposition 3.3 and Lemma 3.4, we complete the proof. \[\square\]

### 3.3. Regularity of derivatives.

In this part, we present Lemma 3.7 on the moments of the Gateaux derivative and Lemma 3.8 on the moments of the Malliavin derivative of \( \varphi^\tau_{\theta_i}(t, Y^\tau_i) \), which will be used in the proof of Theorem 3.1. As we will see, the \( p \)-th moment of these derivatives are dominated by the corresponding Green function, instead of being bounded by a constant. This is the main difference in the weak convergence analysis between stochastic partial differential equations and stochastic ordinary differential equations.

**Lemma 3.7.** Assume that \( b \in \mathcal{C}_b^\infty \). Then for any integers \( k \geq 0, p \geq 1 \), there exists \( C = C(k, p, T, b, \sigma) \) such that 
\[
\|\langle D\varphi^\tau_{i}(t, Y^\tau_i), G_{t-r}(\cdot, y) \rangle\|_{k,p} \leq CG_{t-r}(x,y)
\] (3.23)
holds for every \( r \in [t_{i-1}, t_i), t_i \leq t \leq T, i \in \{1, \ldots, N\} \) and \( \tau, x, y \in (0,1) \).

**Proof.** The proof is completed by induction on \( k \). From (4.2), the Minkowskii’s inequality and the boundedness of \( b' \) give that 
\[
\|\langle D\varphi^\tau_{i}(t, Y^\tau_i), G_{t-r}(\cdot, y) \rangle\|_{p} \leq G_{t-r}(x,y)
\]
A direct application of Lemma 3.8. completes the proof of (3.23) when \(k = 0\).

Assuming that (3.23) holds up to the index \(k - 1, k \geq 1\). Hence, by applying Leibnitz’s rule, Hölder’s inequality and Corollary 3.5, it holds for \(t_i < \theta_1 < t\) that

\[
\|D^k \{b^i(t_i, Y_{i-1}^T) \langle D\varphi_{\theta_1}^i(t_i, Y_{i}^T), G_{t_i-\theta}^r(\cdot, y) \rangle \} \|_{L^p(\Omega, \mathbb{R}^k)} \\
\leq |b_i| \|D^k \{b^i(t_i, Y_{i-1}^T) \langle D\varphi_{\theta_1}^i(t_i, Y_{i}^T), G_{t_i-\theta}^r(\cdot, y) \rangle \} \|_{L^p(\Omega, \mathbb{R}^k)} \\
+ C(k, p, T) \|b^i(t_i, Y_{i}^T)\|_{k, 2p} \|D\varphi_{\theta_1}^i(t_i, Y_{i}^T), G_{t_i-\theta}^r(\cdot, y)\|_{k-1, 2p} \\
\leq |b_i| \|D^k \{b^i(t_i, Y_{i}^T), G_{t_i-\theta}^r(\cdot, y)\} \|_{L^p(\Omega, \mathbb{R}^k)} + C(k, p, T)G_{t_i-\theta}^r(y, z),
\]

which, together with the semigroup property of \(G\), indicates that

\[
\|D^k \{b^i(t_i, Y_{i}^T), G_{t_i-\theta}^r(\cdot, y)\} \|_{L^p(\Omega, \mathbb{R}^k)} \leq C(T, k, p)G_{t_i-\theta}^r(y, z)
\]

Consequently, (3.23) is valid for \(k\) thanks to Lemma 2.2 and (3.16) and the proof is completed. \(\square\)

**Lemma 3.8.** Assume that \(b \in C_0^\infty\). Then for any integers \(k \geq 0, p \geq 1\), there exists \(C = C(k, p, T, b, \sigma)\) such that for every \(i \in \{1, \cdots, N\}\), \(\theta \in (0, t_{i-1})\), \(\tau, x, y, \xi \in (0, 1)\) and \(t \in (t_i, T]\),

\[
\|D_{\theta, \xi} \Phi_{t_i}^i(0, u_0)\|_{k, p} + \|D_{\theta, \xi} \Phi_{t_i}^i(t_{i-1}, \Phi_{t_i-1}(0, u_0))\|_{k, p} \leq CG_{t_i-\theta}(y, \xi),
\]

\[
|D_{\theta, \xi} \Phi_{t_i}^i(t_i, \Phi_{t_i}^i(0, u_0))|_{k, p} \leq CG_{t_i-\theta}(x, \xi).
\]

**Proof.** We proceed by induction on \(k\), which is analogous to the proof of Lemma 3.4 and Lemma 3.7. Thus, we only give the details of the proof of the case \(k = 0\), and the induction argument for \(k \geq 1\) is omitted.

Let \(y \in (0, 1)\) and \(i \in \{1, \cdots, N\}\) be arbitrarily fixed. First, we claim that

\[
\|D_{\theta, \xi} \Phi_{t_i}^i(0, u_0)\|_{p} \leq CG_{t_i-\theta}(y, \xi), \quad \forall \theta \in (0, t_{i-1}), \quad \xi \in (0, 1).
\]

In fact, if \(i = 2\), then for any \(\theta \in (0, t_1)\), \(\xi \in (0, 1)\), \(D_{\theta, \xi} \Phi_{t_i}^i(0, u_0) = \sigma G_{t_i-\theta}(y, \xi)\) and

\[
D_{\theta, \xi} \Phi_{t_i}^i(0, u_0) = \sigma G_{t_i-\theta}(y, \xi) + \int_{t_1}^{t_2} \int_{t_1}^{1} G_{t_i-\theta}(y, z) b^i(\Phi_{t_1}^i(0, u_0)) D_{\theta, \xi} \Phi_{t_i}^i(0, u_0) dz \, dr.
\]

Taking the norm \(\| \cdot \|_p\) on both sides and by (2.6),

\[
\int_{t_1}^{t_2} \int_{t_1}^{1} G_{t_i-\theta + 1}(y, z) G_{t_i-\theta}(z, \xi) dz \, dr \\
= \int_{t_1}^{t_2} \int_{t_1}^{1} G_{t_i-\theta}(y, \xi) dz \, dr \leq C(T) \int_{t_1}^{t_2} \sqrt{\frac{t_2 - r + t_1 - \theta}{t_2 - \theta}} G_{t_i-\theta}(y, \xi) dz \, dr \leq C(T) \delta G_{t_i-\theta}(y, \xi)
\]

because of \(\theta \in (0, t_1)\) and \(t_2 = 2\delta\), then (3.26) holds true when \(i = 2\). To show (3.26) for general \(3 \leq i \leq N\), by induction, we assume that (3.26) holds up to \(i - 1\). Now assume that \(\theta \in (0, t_{i-1})\) and \(\xi \in (0, 1)\). Then there exists \(k \in \{1, \cdots, i - 1\}\) such that \(\theta \in (t_k, t_k)\) and

\[
\|D_{\theta, \xi} \Phi_{t_i}^i(0, u_0)\|_{p} \leq |b_i| \sum_{j=k}^{i-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^{1} G_{t_i-\theta}(y, z) |D_{\theta, \xi} \Phi_{t_j}^j(0, u_0)|_{p} dz \, dr + G_{t_i-\theta}(y, \xi) \sigma,
\]
where
\[ \sum_{j=k}^{i-1} \int_{t_j}^{t_{j+1}} G_{t_i-r-t_j} \left( y, \xi \right) \, dr \leq C(T) \delta \sum_{j=k}^{i-1} \int_{t_j}^{t_{j+1}} \sqrt{t_i - \theta \over t_i - r + t_j - \theta} \, dr G_{t_i-r} \left( y, \xi \right) \]
\[ = 2C(T) \delta \sum_{j=k-1}^{i-1} \sqrt{t_i - \theta \over t_i - \theta + \sqrt{t_i - \theta}} G_{t_i-r} \left( y, \xi \right) \leq C(T) G_{t_i-r} \left( y, \xi \right), \]
in view of (2.6). This completes the proof of (3.26). Notice that for \( \theta \in (t_{i-2}, t_{i-1}), \)
\[ \left\| D_{\theta, \xi} \Phi_{t_{i-1}}^y (0, u_0) \right\|_p = \left| \sigma \right| G_{t_{i-1}-r} \left( y, \xi \right) \leq C G_{t_i-r} \left( y, \xi \right), \]
and by (3.26), for \( \theta \in (0, t_{i-2}), \)
\[ \left\| D_{\theta, \xi} \Phi_{t_{i-1}}^y (0, u_0) \right\|_p \leq C G_{t_i-r} \left( y, \xi \right). \]
Hence, by the semigroup property of \( G, \) we have
\[ \left\| D_{\theta, \xi} \varphi_{t_{i-1}}^y (t_i-r, \Phi_{t_{i-1}} (0, u_0)) \right\|_p \]
\[ \leq C G_{t_i-r} \left( y, \xi \right) + C \int_{t_{i-1}}^{t_i} \int_0^1 G_{t_i-r} (y, z) \left\| D_{\theta, \xi} \varphi_{t_i}^z (t_{i-1}, \Phi_{t_{i-1}} (0, u_0)) \right\|_p \, dz \, dr. \]

By Lemma 2.2 and (3.26), we complete the proof of (3.24) when \( k = 0. \) Finally, the definition
of \( Y_i^\tau \) implies that for any \( \theta \in (0, t_{i-1}), \xi \in (0, 1), \) \( \left\| D_{\theta, \xi} Y_i^\tau (y) \right\| \leq C G_{t_i-r} \left( y, \xi \right) \) and thereby
(3.25) follows from an analogue argument by using Lemma 2.2. \( \square \)

**Corollary 3.9.** Assume that \( b \in C_b^\infty. \) Then for any integers \( k \geq 0 \) and \( p \geq 1, \) there exists
some constant \( C = C(k, p, T, b, \sigma) \) such that for every \( i \in \{1, \ldots, N\}, \) \( t_{i-1} < r \leq t_i, \theta \in (0, r) \)
and \( \beta \in [0, 1), \) it holds that
\[ \left\| D_{\theta, \xi} Z_{i-1}^\beta (r, y) \right\|_{k,p} \leq C G_{t_i-r} \left( y, \xi \right). \]

### 4. Weak convergence analysis

#### 4.1. Test function-independent analysis

In this part, we give the proof of Theorem 3.1, which is essential to obtain the convergence order of density approximations in the uniform convergence topology.

**Proof of Theorem 3.1:** Observing that
\[ \mathbb{E} [f(u^T(x))] - \mathbb{E} [f(u(T, x))] = \mathbb{E} [f(\varphi_T^x (0, u_0))] - \mathbb{E} [f(\Phi_T^x (0, u_0))], \]
we proceed to estimate the summation \( \sum_{i=1}^N \mathbb{E} [Z_i^j], \) \( j \in \{1, \ldots, 5\}, \) defined in (3.5).
4.1.1. Estimate of $I^1_t$. For fixed $0 \leq r < t_i \leq T$ and $y \in (0, 1)$, we have

$G_{t_i-r}(\cdot, y) \in E$.

Invoking [14, Proposition 1.5.6], Lemma 3.7 and Corollary 3.5, for any $k, p$,

$$
\| (D\varphi^\infty_T(t_i, Y^\infty_i), G_{t_i-r}(\cdot, y)) b'(Z_i^\beta(r, y)) \|_{k,p} \leq C(k, p)\| (D\varphi^\infty_T(t_i, Y^\infty_i), G_{t_i-r}(\cdot, y))\|_{k,2p} \leq C(k, p, T)G_{T-r}(x, y),
$$

which, combined with Lemma 3.2 and Lemma 2.1 implies that

$$
\sum_{i=2}^{N} \mathbb{E}[I^1_t] \leq \sum_{i=2}^{N} \int_0^1 \int_{t_{i-1}}^{t_i} \int_0^1 \int_0^1 \mathbb{E} \left[ f'(\varphi^\infty_T(t_i, Y^\infty_i)) \langle D\varphi^\infty_T(t_i, Y^\infty_i), G_{t_i-r}(\cdot, y) \rangle b'(Z_i^\beta(r, y)) \right] \\
\int_0^1 \left| G_r(y, \xi) - G_{t_{i-1}}(y, \xi) \right| u_0(\xi) \frac{d\xi}{d\tau} \leq C(k, p, T)\|u_0\|_E \mathbb{E} \sum_{i=2}^{N} \int_{t_{i-1}}^{t_i} (r - t_{i-1})^{-\nu}(t_{i-1})^{-\nu} dr \leq C \delta^\nu \int_0^T \frac{1}{r^\nu} dr \leq C \delta^\nu
$$

with $\nu \in \left(\frac{1}{3}, 1\right)$. In addition, for $i = 1$,

$$
\mathbb{E}[I^1_1] \leq C \int_0^1 \int_0^1 G_{T-r}(x, y) \int_0^1 G_r(y, \xi)u_0(\xi) \frac{d\xi}{d\tau} \leq 2C\|u_0\|_E \delta.
$$

4.1.2. Estimate of $I^2_t$. Similarly, we again apply Lemma 2.1 and Lemma 3.2, Lemma 3.7 and Corollary 3.5 to obtain

$$
\sum_{i=1}^{N} \mathbb{E}[I^2_t] \leq \sum_{i=1}^{N} \int_0^1 \int_{t_{i-1}}^{t_i} \int_0^1 \int_0^1 \int_0^1 \mathbb{E} \left[ f'(\varphi^\infty_T(t_i, Y^\infty_i)) \langle D\varphi^\infty_T(t_i, Y^\infty_i), G_{t_i-r}(\cdot, y) \rangle b'(Z_i^\beta(r, y)) \Phi^\xi_{[\theta]}(0, u_0) \right] \\
\left| G_r(y, \xi) - G_{t_{i-1}-\theta}(y, \xi) \right| d\xi d\theta dy dr d\tau \leq C \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathbb{E} \left[ f''(\varphi^\infty_T(t_i, Y^\infty_i)) \langle D\varphi^\infty_T(t_i, Y^\infty_i), G_{t_i-r}(\cdot, y) \rangle \right] \\
\left| G_r(y, \xi) - G_{t_{i-1}-\theta}(y, \xi) \right| d\xi d\theta dy dr d\tau \leq C \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathbb{E} \left[ f'(\varphi^\infty_T(t_i, Y^\infty_i)) \langle D\varphi^\infty_T(t_i, Y^\infty_i), G_{t_i-r}(\cdot, y) \rangle \right] \\
\left| G_r(y, \xi) - G_{t_{i-1}-\theta}(y, \xi) \right| d\xi d\theta dy dr d\tau \leq C \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathbb{E} \left[ f'(\varphi^\infty_T(t_i, Y^\infty_i)) \langle D\varphi^\infty_T(t_i, Y^\infty_i), G_{t_i-r}(\cdot, y) \rangle \right] \\
\left| G_r(y, \xi) - G_{t_{i-1}-\theta}(y, \xi) \right| d\xi d\theta dy dr d\tau \leq C \delta^\nu
$$

with $\nu \in \left(\frac{1}{3}, 1\right)$.

4.1.3. Estimate of $I^3_t$. In view of the Malliavin integration by parts formula and chain rule, $\mathbb{E}[I^3_t]$ is further decomposed into

$$
\mathbb{E}[I^3_t] = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathbb{E} \left[ f''(\varphi^\infty_T(t_i, Y^\infty_i)) \langle D\varphi^\infty_T(t_i, Y^\infty_i), G_{t_i-r}(\cdot, y) \rangle \right] \\
b'(Z_i^\beta(r, y)) \left\{ G_{t_{i-1}-\theta}(y, \xi) - G_{t_{i-1}-\theta}(y, \xi) \right\} d\xi d\theta dy dr d\tau
$$

$$
+ \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathbb{E} \left[ f'(\varphi^\infty_T(t_i, Y^\infty_i)) \langle D\varphi^\infty_T(t_i, Y^\infty_i), G_{t_i-r}(\cdot, y) \rangle \right] \\
b'(Z_i^\beta(r, y)) \left\{ G_{t_{i-1}-\theta}(y, \xi) - G_{t_{i-1}-\theta}(y, \xi) \right\} d\xi d\theta dy dr d\tau
$$

with $\nu \in \left(\frac{1}{3}, 1\right)$.
Now we proceed to show that for any \( k, p \),

First, by chain rule and the semigroup property of \( G \),

Taking the Malliavin derivative \( \frac{\partial}{\partial \theta} \) with

Estimate of \( J_1^i \): By Lemma 3.2, it holds that

for some positive constants \( k, p \). Hence, applying \([14, \text{Proposition 1.5.6}]\), the semigroup property of \( G \), (2.5) and Lemma 2.1, we have

with \( \nu \in \left( \frac{1}{3}, 1 \right) \).

Estimate of \( J_2^i \): To treat \( J_2^i \), notice that by Lemma 3.2, there exist some constants \( k, p \), such that

Now we proceed to show that for any \( \nu \in \left( \frac{1}{3}, 1 \right) \), there exists \( C = C(T, k, p, \nu) \) such that for any \( x \in (0, 1) \), \( t \in (t_i, T] \) and \( \tau \in (0, 1) \),

First, by chain rule and the semigroup property of \( G \), we obtain

Taking the Malliavin derivative \( D_{\theta, \xi} \) on both sides of (4.2) gives

\[
D_{\theta, \xi} \left( D_{\varphi_1^Z(t_i, Y_i^r)}, G_{t_i-r}(\cdot, y) \right) = \int_{t_i}^t \int_{0}^{1} G_{t-r, \theta_i}(x, z) \left( \varphi_1^Z(t_i, Y_i^r) \right) D_{\varphi_1^Z(t_i, Y_i^r), G_{t_i-r}(\cdot, y)} dz d\theta_i.
\]
Then it follows from the definition of $A_i(t, x; 0, p, \nu, \tau)$ and Hölder’s inequality that

$$A_i(t, x; 0, p, \nu, \tau) \leq \int_{t_i}^t \int_0^1 \int_{t_i-1}^{t_i} \int_0^1 \int_{t_i}^t G_{l-\theta_i}(x, z) |D_{\theta, \xi} b'(\varphi^\tau_{\theta_i}(t, Y_i^\tau))| \, dz \, d\theta_1 \, d\xi \, d\theta \, d\tau$$

$$\leq \int_{t_i}^t \int_0^1 \int_{t_i-1}^{t_i} \int_0^1 \int_{t_i}^t G_{l-\theta_i}(x, z) |G_{l-\theta_i}(x, z)| G_{l-\theta_i}(y, (y, \xi) - G_{t_i-1-\theta_i}(y, \xi)| \, dz \, d\theta_1 \, d\xi \, d\theta \, d\tau$$

$$\leq \int_{t_i}^t \int_0^1 \int_{t_i-1}^{t_i} \int_0^1 \int_{t_i}^t G_{l-\theta_i}(x, z) \, dz \, d\theta_1 \, + C_{\delta^\nu} \int_{t_i-1}^{t_i} \int_{t_i}^t \int_0^1 \int_0^1 \int_0^1 \frac{1}{\sqrt{t - t_i}} \frac{1}{\sqrt{t_i - \theta}} G_{l-\theta_i}(r, \xi) \, dz \, d\theta_1 \, d\xi \, d\theta \, d\tau$$

Taking supremum on both sides of the above inequality over $x \in (0, 1)$, we complete the proof of (4.1) when $k = 0$ by applying the Gronwall lemma. Assume that (4.1) holds up to the index $k - 1$. Then the induction argument for general $k$ is similar to the proof of Lemma 3.4 and thereby is omitted.

Estimate of $\mathcal{J}_3^i$: By Lemmas 3.2, 3.7 and Corollaries 3.5, 3.9, it holds for some constants $k, p$ that

$$\sum_{i=1}^N |\mathcal{J}_3^i| \leq \sum_{i=1}^N \int_0^1 \int_{t_i-1}^{t_i} \int_0^1 \frac{1}{\sqrt{t - t_i}} \frac{1}{\sqrt{t_i - \theta}} \frac{1}{\sqrt{t_i - \theta}} \frac{1}{\sqrt{t_i - \theta}} |G_{l-\theta_i}(r, \xi) - G_{t_i-1-\theta_i}(y, \xi)| \, dz \, d\theta_1 \, d\xi \, d\theta \, d\tau$$

$$\leq C \sum_{i=1}^N \int_{t_i-1}^{t_i} \int_0^1 \int_0^1 \int_0^1 \frac{1}{\sqrt{t_i - \theta}} \frac{1}{\sqrt{t_i - \theta}} \frac{1}{\sqrt{t_i - \theta}} \frac{1}{\sqrt{t_i - \theta}} G_{l-\theta_i}(x, z) \, dz \, d\theta_1 \, d\xi \, d\theta \, d\tau$$

$$+ C \sum_{i=1}^N \int_{t_i-1}^{t_i} \int_0^1 \int_0^1 \int_0^1 \frac{1}{\sqrt{t_i - \theta}} \frac{1}{\sqrt{t_i - \theta}} \frac{1}{\sqrt{t_i - \theta}} \frac{1}{\sqrt{t_i - \theta}} G_{l-\theta_i}(x, z) \, dz \, d\theta_1 \, d\xi \, d\theta \, d\tau$$

$$=: \mathcal{J}_{31} + \mathcal{J}_{32}. \quad (4.3)$$
Then Lemma 2.1 with $\nu \in (\frac{1}{2}, 1)$ leads to

$$J_{31} \leq C \sum_{i=1}^{N} \int_{\tau_{i-1}}^{t_{i}} \int_{0}^{1} \int_{0}^{r} G_{r-\theta}(x, y)(r - \theta)^{-\frac{1}{2}} \int_{0}^{1} |G_{r-\theta}(y, \xi) - G_{r-\theta}(y, \xi)| d\xi d\theta dy dr \leq C \delta^{\nu+1} \sum_{i=1}^{N} \int_{0}^{t_{i-2}} \frac{(t_{i-1} - \theta)^{-\nu}}{\sqrt{t_{i} - \theta + \sqrt{t_{i-1} - \theta}}} d\theta$$

Besides, the positivity of $G$ leads to

$$J_{32} \leq C \sum_{i=1}^{N} \int_{\tau_{i-1}}^{t_{i}} \int_{0}^{1} \int_{0}^{r} G_{r-\theta}(x, y)(r - \theta)^{-\frac{1}{2}} \int_{0}^{1} |G_{r-\theta}(y, \xi) - G_{r-\theta}(y, \xi)| d\xi d\theta dy dr \leq C \delta^{\nu+1} \sum_{i=1}^{N} \int_{0}^{t_{i-2}} (t_{i-1} - \theta)^{-\nu} d\theta \leq C \delta^{\frac{1}{2}}.$$
\[
G_{r-\theta}(y, \xi) d\xi d\theta dy dr d\tau \\
+ \int_0^1 \int_0^1 \int_{t_{i-1}}^1 \int_{t_{i-1}}^r \int_0^1 E \left[ f'(\varphi_T^r(t_i, Y_i^T))(D\varphi_T^r(t_i, Y_i^T), G_{t_i-r}(\cdot, y))D\theta \xi \right] d\theta d\xi dy dr d\tau \\
G_{r-\theta}(y, \xi) d\xi d\theta dy dr d\tau =: \sum_{j=1}^3 K_j^3.
\]

**Estimate of \( K_1^3 \):** By applying Lemma 3.2, Lemma 3.7, Corollary 3.5 and the semigroup property of \( G \), we obtain that for some positive constants \( k, p \),

\[
\sum_{i=1}^N |K_1^3| \leq C \sum_{i=1}^N \int_0^1 \int_{t_{i-1}}^1 \int_{t_{i-1}}^r \int_0^1 ||D\theta \xi \varphi_T^r(t_i, Y_i^T)(D\varphi_T^r(t_i, Y_i^T), G_{t_i-r}(\cdot, y)) \\|_{k, p} G_{r-\theta}(y, \xi) d\xi d\theta dy dr d\tau
\]

\[
\leq C \delta \sum_{i=1}^N \int_0^1 \int_{t_{i-1}}^1 \int_{t_{i-1}}^r \int_0^1 G_{T-\theta}(x, \xi) G_{T-r}(x, y) G_{r-\theta}(y, \xi) d\xi d\theta dy dr
\]

\[
\leq C \delta \sum_{i=1}^N \int_0^1 \int_{t_{i-1}}^1 \int_{t_{i-1}}^r \int_0^1 G_{T-\theta}(x, \xi) d\xi d\theta dy d\tau \leq C \delta.
\]

**Estimate of \( K_2^3 \):** Similar to the estimation of \( J_2^i \), we denote

\[
|K_2^3| \leq C \int_0^1 \int_{t_{i-1}}^1 \int_{t_{i-1}}^r \int_0^1 \|D\theta \xi (D\varphi_T^r(t_i, Y_i^T), G_{t_i-r}(\cdot, y))\|_{k, p} G_{r-\theta}(y, \xi) d\xi d\theta dy dr d\tau
\]

\[
= C \int_0^1 B_i(T, x; k, p, \tau) d\tau,
\]

and then prove that there exists \( C = C(T, k, p, \nu) \) such that for any \( \tau \in (0, 1) \), \( x \in (0, 1) \) and \( t \in (t_i, T) \),

\[
B_i(t, x; k, p, \nu, \tau) \leq C \delta^2,
\]

whose proof is similar to that of (4.1). Indeed, taking the Malliavin derivative \( D_{\theta, \xi} \) on both sides of (4.2), it follows from

\[
\int_{t_i}^t \int_0^1 \int_{t_{i-1}}^1 \int_{t_{i-1}}^r \int_0^1 G_{t-\theta_1}(x, z)G_{t_1-\theta}(z, \xi)G_{t_1-r}(z, y)G_{r-\theta}(y, \xi) d\xi d\theta dy dr d\xi d\theta_1 d\xi_1
\]

\[
= \int_{t_i}^t \int_0^1 \int_{t_{i-1}}^1 \int_{t_{i-1}}^r \int_0^1 G_{t-\theta_1}(x, z)G_{t_1-\theta}(z, \xi) d\xi d\theta dr d\xi d\theta_1 d\xi_1
\]

\[
\leq C \int_{t_i}^t \int_{t_{i-1}}^1 \int_{t_{i-1}}^r \frac{1}{\sqrt{\theta_1 - \theta}} d\theta dr d\xi d\theta_1 d\xi_1 \leq C \int_{t_i}^t \frac{1}{\sqrt{\theta_1 - t_i}} dt_i \int_{t_{i-1}}^1 \int_{t_{i-1}}^r d\xi d\theta_1 \leq C \delta^2,
\]

and Lemmas 3.4 and 3.7 that

\[
B_i(t, x; 0, p, \nu, \tau) \leq \int_{t_i}^t \int_0^1 G_{t-\theta_1}(x, z) B_i(\theta_1, \xi; 0, p, \tau) d\xi d\theta_1 + C \delta^2.
\]

**Estimate of \( K_3^3 \):** By Corollary 3.9, for any \( t_{i-1} < \theta < r \leq t_i \) and \( \beta \in (0, 1) \),

\[
||D_{\theta, \xi} Z_i^\beta(r, y)||_{k, p} \leq CG_{r-\theta}(y, \xi).
\]
By applying Lemma 3.2, Lemma 3.7, there exist some positive constants \( k, p \) such that
\[
\sum_{i=1}^{N} |K_{3}^{i}| \leq C \sum_{i=1}^{N} \int_{0}^{1} \int_{0}^{1} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \parallel(D_{\theta}x_{T}(t, y), G_{t_{i-1}}(\cdot, y))b''(Z_{3}^{i}(r, y)) \parallel_{k, p} \delta d\theta dy d\rho d\sigma d\tau
\]
\[
\leq C \sum_{i=1}^{N} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}} \int_{0}^{r} \int_{0}^{r} G_{t_{i-1}}(x, y)G_{t_{i-1}}^{2}(y, x) \parallel \delta d\theta dy d\rho d\sigma d\tau \leq C \delta^{\frac{1}{2}}.
\] (4.5)

Gathering all above estimates, we complete the proof. \( \square \)

**Remarks 4.1.**

1. If \( b(u) = b_{1}u + c \) is an affine function, then \( b'(Z_{3}^{i}(r, y)) \equiv 0 \) and thereby \( J_{3}^{i} = K_{3}^{i} = 0, i = 1, \cdots, N. \) In this case, by gathering the estimates on \( \{I_{i}^{j}\}_{i=1,\cdots,N, j=1,\cdots,5} \) in the proof of Theorem 3.1, we have, instead of (3.2), that
\[
\left| \mathbb{E}[f(u^{\delta}(T, x))] - \mathbb{E}[f(u(T, x))] \right| \leq C(T, b, \sigma, ||u_{0}\parallel_{E, \nu}) \delta^{\nu},
\]
for every \( \nu \in (\frac{1}{2}, 1). \)

2. With the same idea of Taylor expansion and error decomposition technique as the proof of the above theorem, we may have the following result on weak convergence as well:

Let \( f : \mathbb{R} \to \mathbb{R} \) be smooth with bounded derivatives and \( 0 < \delta \leq 1. \) Assume that \( b \in C_{b}^{2}. \)

Then there exists some positive constant \( C = C(T, b, \sigma, ||u_{0}\parallel_{E, f}) \) such that
\[
\left| \mathbb{E}[f(u^{\delta}(T, x))] - \mathbb{E}[f(u(T, x))] \right| \leq C \delta^{\frac{1}{2}}, \forall x \in (0, 1).
\] (4.6)

Note that we don’t need Lemma 3.2 since the generic constant \( C \) may depend on \( f \) here. As a result, the requirements of the perturbation parameter \( \delta \) and the regularity of \( b \) are not as strict as those of Theorem 3.1.

### 4.2. Analysis with small drift.

It is obvious that if \( b = 0 \), the solution \( u^{\delta}(T, x) \) of Eq. (1.2) is exactly the exact solution \( u(T, x) \) of Eq. (1.1). In this part, we consider the weak convergence with small drift \( b \), that is \( b(u) = \tilde{b}b(u) \) for small \( 0 < \varepsilon < 1 \) and \( \tilde{b} \in C_{b}^{3} \) is not affine. In this case, we observe that \( J_{3}^{i} \) and \( K_{3}^{i} \) are bounded by \( C(T, \tilde{b}, b_{\parallel}u_{0}\parallel_{E, f}) \delta^{\frac{1}{2}}. \) By borrowing the notation from the proof of Theorem 3.1, for any \( \nu \in (\frac{1}{2}, 1) \), there exists \( C = C(\nu, T, b, \tilde{b}, ||u_{0}\parallel_{E, f}) \) such that
\[
\sum_{j=1}^{5} \sum_{i=1}^{N} \mathbb{E}[T_{i}^{j}] - \sum_{i=1}^{N} (J_{3}^{i} + K_{3}^{i}) \leq C \varepsilon^{\nu},
\] (4.7)
and thereby for \( 0 < \delta \leq 1, \)
\[
\left| \mathbb{E}[f(u^{\delta}(T, x))] - \mathbb{E}[f(u(T, x))] \right| \leq C \varepsilon^{\nu} + \sum_{i=1}^{N} |J_{3}^{i}| + \sum_{i=1}^{N} |K_{3}^{i}| \leq C \varepsilon^{\frac{1}{2}} \delta^{\frac{1}{2}}.
\] (4.8)

The main result of this part is the following proposition.

**Proposition 4.2.** Let \( b(u) = \tilde{b}b(u) \) for small \( 0 < \varepsilon < 1 \) and \( \tilde{b} \in C_{b}^{3}, f : \mathbb{R} \to \mathbb{R} \) be smooth with bounded derivatives and \( 0 < \delta \leq 1. \) Then for any \( \nu \in (\frac{3}{4}, 1) \), there exists some positive constant \( C = C(T, \tilde{b}, \sigma, ||u_{0}\parallel_{E, f, \nu}) \) such that
\[
\left| \mathbb{E}[f(u^{\delta}(T, x))] - \mathbb{E}[f(u(T, x))] \right| \leq C \varepsilon^{\nu - \frac{1}{4}} + C \varepsilon^{2} \delta^{\frac{1}{2}}, \forall x \in (0, 1).
\] (4.9)
Proof. For the sake of simplicity, denote
\[
I_i(\beta, r, y) := \int_0^{t_i-1} \int_0^1 D_{\theta, \xi} \Phi_{t_i}^y(0, u_0) \{G_{r-\theta}(y, \xi) - G_{t_{i-1}-\theta}(y, \xi)\} \sigma d\xi d\theta,
\]
\[
+ \int_0^r \int_0^1 D_{\theta, \xi} \Phi_{t_i}^y(0, u_0) \{G_{r-\theta}(y, \xi) + G_{t_{i-1}-\theta}(y, \xi)\} \sigma d\xi d\theta.
\]
We recall that \(J_3^i + K_3^i\) is equal to
\[
\int_0^1 \int_0^1 \int_0^{t_i} \int_0^1 \mathbb{E} \left[ f'(\varphi_T(t_i, Y_i^T)) \langle D \varphi_T(t_i, Y_i^T), G_{t_{i-1}}(\cdot, y) \rangle \sigma' \left( Z_i^\beta(r, y) \right) I_i(\beta, r, y) \right] d\beta d\tau d\tau d\tau
\]
and rewrite \(J_3^i + K_3^i = L_1^i + L_2^i\) with
\[
L_1^i := \int_0^1 \int_0^1 \int_0^{t_i} \int_0^1 \mathbb{E} \left[ f'(\varphi_T(t_i, Y_i^T)) \langle D \varphi_T(t_i, Y_i^T), G_{t_{i-1}}(\cdot, y) \rangle \sigma' \left( \Phi_{t_i}^y(0, u_0) \right) \right] d\beta d\tau d\tau d\tau
\]
and
\[
L_2^i := \int_0^1 \int_0^1 \int_0^{t_i} \int_0^1 \mathbb{E} \left[ f'(\varphi_T(t_i, Y_i^T)) \langle D \varphi_T(t_i, Y_i^T), G_{t_{i-1}}(\cdot, y) \rangle \Delta_i(\beta, r, y) \right] d\beta d\tau d\tau d\tau
\]
where
\[
\Delta_i(\beta, r, y) := \sigma' \left( Z_i^\beta(r, y) \right) - \sigma' \left( \Phi_{t_i}^y(0, u_0) \right)
\]
and
\[
= \int_0^1 \sigma' \left( (1 - \zeta) \Phi_{t_i}^y(0, u_0) + \zeta Z_i^\beta(r, y) \right) d\zeta \left( Z_i^\beta(r, y) - \Phi_{t_i}^y(0, u_0) \right).
\]
First, we proceed to estimate \(I_i(\beta, r, y)\) as follows. Notice that
\[
\int_0^1 \int_0^1 \int_0^{t_i-1} D_{\theta, \xi} \Phi_{t_i}^y(0, u_0) \{G_{r-\theta}(y, \xi) - G_{t_{i-1}-\theta}(y, \xi)\} \sigma d\theta d\xi
\]
\[
= \sum_{j=1}^{i-2} \int_0^1 \int_0^{t_{j+1}} \int_{t_j}^{t_{i-1}} G_{t_{i-1}-s}(y, z) b'(\Phi_{t_j}^z(0, u_0)) D_{\theta, \xi} \Phi_{t_j}^z(0, u_0) \{G_{r-\theta}(y, \xi) - G_{t_{i-1}-\theta}(y, \xi)\} \sigma dz d\sigma d\theta d\xi
\]
\[
= \sum_{j=1}^{i-2} B_{i,1}^{j} + \int_0^1 \int_0^{t_i-1} G_{t_{i-1}-s}(y, z) \sigma^2 \{G_{r-\theta}(y, \xi) - G_{t_{i-1}-\theta}(y, \xi)\} d\theta d\xi. \tag{4.10}
\]
For any \(j\) with \(1 \leq j \leq i - 2\), we denote \(A_k^j := \sum_{j=1}^{i-2} B_{i,1}^{j}\) and decompose \(B_{i,2}^{j} = B_{i,2}^{j,1} + B_{i,2}^{j,2}\) with
\[
B_{i,1}^{j,1} := \int_0^1 \int_0^{t_{j+1}} \int_{t_j}^{t_{i-1}} G_{t_{i-1}-s}(y, z) b'(\Phi_{t_j}^z(0, u_0)) D_{\theta, \xi} \Phi_{t_j}^z(0, u_0) \{G_{r-\theta}(y, \xi) - G_{t_{i-1}-\theta}(y, \xi)\} dz d\sigma d\theta d\xi,
\]
\[
B_{i,2}^{j,2} := \int_0^1 \int_0^{t_{j+1}} \int_{t_j}^{t_{i-1}} G_{t_{i-1}-s}(y, z) b'(\Phi_{t_j}^z(0, u_0)) D_{\theta, \xi} \Phi_{t_j}^z(0, u_0) \{G_{r-\theta}(y, \xi) - G_{t_{i-1}-\theta}(y, \xi)\} dz d\sigma d\theta d\xi.
\]
because of \( D_{\theta, \xi} \Phi_{ij}^z (0, u_0) = 0 \), if \( \theta > t_j \). By choosing \( \nu \in (\frac{1}{2}, 1) \), it leads to

\[
\| B_{i, 1}^j \|_2 \leq C|b_1| \int_0^1 \int_0^{t_j} \int_0^{t_j+1} G_{t_{i-1} - s}(y, z)(t_j - \theta)^{-\frac{1}{2}} |G_r - \theta(y, \xi) - G_{t_{i-1} - \theta}(y, \xi)| \, dz \, ds \, d\xi \\
\leq C\delta^\nu |b_1| \int_0^{t_j} \int_0^{t_j+1} \int_0^{t_j} G_{t_{i-1} - s}(y, z)(t_j - \theta)^{-\frac{1}{2}} (t_{i-1} - \theta)^{-\nu} \, dz \, ds \, d\xi \\
\leq C|b_1| \delta^{1+\nu} \int_0^{t_j} (t_j - \theta)^{-\frac{1}{2}} (t_{i-1} - \theta)^{-\nu} \, d\theta \leq C|b_1| \delta^{1+\nu} \int_0^{t_j} (t_j - \theta)^{-\frac{1}{2} - \nu} \, d\theta \leq C|b_1| \delta^{\frac{3}{2}}
\]

and

\[
\| B_{i, 2}^j \|_2 \leq C \int_0^{t_j} \int_0^{t_j+1} G_{t_{i-1} - s}(y, z)|b_1(t_j - \theta)^{-\frac{1}{2}} \int_0^{t_j} |G_r - \theta(y, \xi) - G_{t_{i-1} - \theta}(y, \xi)| \, d\xi \, d\theta \\
\leq C \int_0^{t_j} \int_0^{t_j+1} G_{t_{i-1} - s}(y, z)|b_1(t_j - \theta)^{-\frac{1}{2}} \, d\xi \, d\theta \\
\leq C|b_1| \delta \int_0^{t_j} (t_j - \theta)^{-\frac{1}{2}} \, d\theta \leq C|b_1| \delta^{\frac{3}{2}}
\]

for some \( C = C(T, \sigma, \tilde{b}, \nu) \). Hence, it follows that

\[
\| \sum_{j=1}^{i-2} B_i^j \|_2 = \| \sum_{j=1}^{i-2} (B_{i, 1}^j + B_{i, 2}^j) \|_2 \leq C(T, \sigma, \tilde{b})|b_1| \delta^{\frac{3}{2}}.
\]

(4.11)

In the same way, by noticing that for \( 0 < \theta < t_{i-1} < r < t_i \),

\[
D_{\theta, \xi} \varphi_{ij}^y(t_{i-1}, \Phi_{t_{i-1}}(0, u_0)) = G_{r - \theta}(y, z)\sigma + \sum_{j=1}^{i-2} \int_{t_j}^{t_{j+1}} \int_0^1 G_{r - s}(y, z)b'(\Phi_{ij}^z(0, u_0))D_{\theta, \xi} \Phi_{ij}^z(0, u_0) \, dz \, ds \\
+ \int_0^r \int_0^1 G_{r - s}(y, z)b'(\varphi_{ij}^s(t_{i-1}, t_{i-1} - \varphi_{ij}^s)(0, u_0))D_{\theta, \xi} \varphi_{ij}^s(t_{i-1}, t_{i-1} - \varphi_{ij}^s)(0, u_0)) \, dz \, ds
\]

and the estimate

\[
\left| \int_0^1 \int_0^{t_{i-1}} \int_0^r \int_0^{t_{i-1}} G_{r - s}(y, z)b'(\varphi_{ij}^s(t_{i-1}, \Phi_{t_{i-1}}(0, u_0))D_{\theta, \xi} \varphi_{ij}^s(t_{i-1}, \Phi_{t_{i-1}}(0, u_0)) \\
\{G_{r - \theta}(y, \xi) - G_{t_{i-1} - \theta}(y, \xi)\} \, dz \, ds \, d\xi \right| \\
\leq C(T, \sigma, \tilde{b})|b_1| \int_0^1 \int_0^{t_{i-1}} \int_0^r \int_0^{t_{i-1}} G_{r - s}(y, z)G_{r - \theta}(z, \xi)|G_{r - \theta}(y, \xi) - G_{t_{i-1} - \theta}(y, \xi)| \, dz \, ds \, d\xi \, d\theta \\
\leq C(T, \sigma, \tilde{b})|b_1| \int_0^{t_{i-1}} \int_0^r (r - \theta)^{-\frac{1}{2}} \int_0^{t_{i-1}} G_{r - \theta}(y, \xi) + G_{t_{i-1} - \theta}(y, \xi) \, d\xi \, d\theta \\
\leq C(T, \sigma, \tilde{b})|b_1| \delta^{\frac{3}{2}},
\]

we can derive that for some \( A_{i_1}^2 \),

\[
\int_0^1 \int_0^{t_{i-1}} D_{\theta, \xi} \varphi_{ij}^y(t_{i-1}, \Phi_{t_{i-1}}(0, u_0)) \{G_{r - \theta}(y, \xi) - G_{t_{i-1} - \theta}(y, \xi)\} \, dz \, ds \, d\xi
\]
\[
= \int_0^1 \int_0^{t_{i-1}} G_{r-\theta}(y, \xi) \sigma^2 \{G_{r-\theta}(y, \xi) - G_{t_{i-1}-\theta}(y, \xi)\} d\theta d\xi + A_i^2
\]

with
\[
\|A_i^2\|_2 \leq C(\tilde{b}, T, \sigma)|b_1|^{\frac{1}{\sigma}}.
\]

By the semigroup property of \( G \) and (2.3), for any \( r_1 \in (t_{i-1}, t_i) \),
\[
\int_0^1 \int_0^{t_{i-1}} G_{r_1-\theta}(y, \xi) \sigma^2 \{G_{r_1-\theta}(y, \xi) - G_{t_{i-1}-\theta}(y, \xi)\} d\theta d\xi
= \int_0^{t_{i-1}} \sigma^2 \{G_{r_1-\theta+r-\theta}(y, y) - G_{r_1-\theta+t_{i-1}-\theta}(y, y)\} d\theta
= 2\sigma^2 \sum_{k=0}^{\infty} \int_0^{t_{i-1}} e^{-k^2\pi^2(r_1-\theta+r-\theta)} - e^{-k^2\pi^2(r_1-\theta+t_{i-1}-\theta)} d\theta \cos^2(k\pi y).
\]

Then (4.10) and (4.12), and the definition of \( Z_i^\beta(r, y) \) give that
\[
= 2\beta \sigma^2 \sum_{k=1}^{\infty} \int_0^{t_{i-1}} e^{-k^2\pi^2(r_1-\theta+r-\theta)} - e^{-k^2\pi^2(r_1-\theta+t_{i-1}-\theta)} d\theta \cos^2(k\pi y) + \beta A_i^2
+ 2(1 - \beta) \sigma^2 \sum_{k=1}^{\infty} \int_0^{t_{i-1}} e^{-k^2\pi^2(t_{i-1}-\theta+r-\theta)} - e^{-k^2\pi^2(t_{i-1}-\theta+t_{i-1}-\theta)} d\theta \cos^2(k\pi y) + (1 - \beta) A_i^1.
\]

Since for any \( \theta \in (t_{i-1}, r) \), \( D_{\theta, \xi}Z_i^\beta(r, y) = \beta \sigma G_{r-\theta}(y, \xi) \), we have that for \( \beta \in (0, 1) \),
\[
= \beta \int_0^{t_{i-1}} \sigma^2 G_{2(r-\theta)}(y, y) d\theta = 2\beta \sigma^2 \sum_{k=0}^{\infty} \int_0^{r} e^{-k^2\pi^2(r_1-\theta+r-\theta)} d\theta \cos^2(k\pi y)
= \beta \sigma^2 \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \left( 1 - e^{-2k^2\pi^2(r-t_{i-1})} \right) \cos^2(k\pi y) + 2\beta \sigma^2 (r - t_{i-1}).
\]

Summarizing the above calculations, we obtain
\[
\int_0^1 I_i(\beta, r, y) d\beta = 2\sigma^2 \sum_{k=1}^{\infty} \int_0^{t_{i-1}} e^{-k^2\pi^2(r_1-\theta+r-\theta)} - e^{-k^2\pi^2(t_{i-1}-\theta+t_{i-1}-\theta)} d\theta \cos^2(k\pi y)
+ \sigma^2 \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \left( 1 - e^{-2k^2\pi^2(r-t_{i-1})} \right) \cos^2(k\pi y) + \sigma^2 (r - t_{i-1}) + \frac{1}{2} (A_i^2 + A_i^1)
= \sigma^2 \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \left( e^{-2k^2\pi^2t_{i-1}} - e^{-2k^2\pi^2r} \right) \cos^2(k\pi y) + \sigma^2 (r - t_{i-1}) + \frac{1}{2} (A_i^2 + A_i^1)
= \sigma^2 \sum_{k=1}^{\infty} \int_{t_{i-1}}^{r} e^{-2k^2\pi^2s} ds \cos^2(k\pi y) + \sigma^2 (r - t_{i-1}) + \frac{1}{2} (A_i^2 + A_i^1).
\]
From (4.11), (4.13) and the inequality \( \sum_{k=1}^{\infty} e^{-2k^2\pi^2 t} \leq (8\pi t)^{-\frac{1}{2}} \), we have that for some \( C = C(|f|_1, T, \sigma, \tilde{b}) \),
\[
\left| \sum_{i=1}^{N} L_i^1 \right| \leq C|b_2| \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_{0}^{1} G_{T-r}(x, y) \left\| \int_{0}^{1} I_i(\beta, r, y) d\beta \right\|_2 dydr \\
\leq C|b_2| \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{r} s^{-\frac{1}{2}} dsdr + C|b_2| \left( |b_1|\delta^\frac{1}{2} + \delta \right) \\
\leq C|b_2| \left( |b_1|\delta^\frac{1}{2} + \delta \right). \tag{4.15}
\]

Now we estimate \( L_i^2 \). By (3.4) and the fact that the exact flow \( \varphi \) associated to Eq. (1.1) is almost 1/4-Hölder continuous in time (see e.g. [14, Proposition 2.4.3]), for \( r \in (t_{i-1}, t_i) \) and \( \nu \in (\frac{3}{4}, 1) \), we have
\[
\| \Delta_i(\beta, r, y) \|_2 \leq |b_3| \left\| \varphi^n_r(t_{i-1}, \Phi_{t_{i-1}}(0, u_0)) - \Phi_{t_{i-1}}^n(0, u_0) \right\|_2 \leq C(\nu, T, \tilde{b}, \sigma, \|u_0\|_E)|b_3|\delta^{\nu-\frac{1}{4}}.
\]

By replacing \( b''(Z_{i}^\delta(r, y)) \) by \( \Delta_i(\beta, r, y) \) in (4.3) and (4.5), it follows that for any \( \nu \in (\frac{3}{4}, 1) \),
\[
\sum_{i=1}^{N} L_i^2 \leq C(|f|_1, T, \sigma, \nu, \tilde{b})|b_3|\delta^{\nu-\frac{1}{4}}. \tag{4.16}
\]

Finally, by noticing that \( |b_i| = \epsilon |b|_i, i = 1, 2, 3 \), and combining (4.7), (4.15) and (4.16), we complete the proof. \( \square \)

5. Convergence of Density Approximations

In this section, we focus on the convergence of density approximations and the logarithmic asymptotic behavior of the densities.

5.1. Convergence of Densities. This part investigates the convergence of density approximations for Eq. (1.1) in both uniform convergence topology and total variation distance. We would like to mention that there already exist some convergence results of density approximations for stochastic ordinary differential equations (see e.g. [2, 7] and references therein), but few results on stochastic partial differential equations.

From Theorem 3.6 and [14, Lemma 2.1.7], it follows that
\[
\mathbb{E}[\delta_x(u^\delta(T, x))] = q_{T,x}^\delta(z) \quad \text{(resp. } \mathbb{E}[\delta_x(u(T, x))] = q_{T,x}(z) \text{)}
\]
is the density of \( u^\delta(T, x) \) (resp. \( u(T, x) \)) at \( z \in \mathbb{R} \). On the basis of Theorem 3.1, we show that the density \( q_{T,x}^\delta \) converges to the density \( q_{T,x} \) in the uniformly convergence topology, and the convergence order coincides with the weak convergence order. For this purpose, we begin with recalling the fact: If a random variable \( F \) has a smooth density \( q \), then
\[
q(z) = \lim_{n \to \infty} \int_{\mathbb{R}} g_{n^{-1}}(z - \xi)q(\xi) d\xi = \lim_{n \to \infty} \mathbb{E}[g_{n^{-1}}(z - F)], \tag{5.1}
\]
where \( g_{n^{-1}} \) is defined by (3.1). Now we prove Theorem 1.1.

Proof of Theorem 1.1: Let \( z \in \mathbb{R} \) and integer \( n \geq 1 \) be arbitrarily fixed. We take \( f(y) = g_{n^{-1}}(y - z) \) in Theorem 3.1. Then \( F(y) = \int_{-\infty}^{y} g_{n^{-1}}(y_1 - z) dy_1 \) satisfies \( 0 \leq F \leq 1 \), hence there exists \( C = C(T, b, \sigma, \|u_0\|_E, \nu) \) independent of \( z, n \) and \( x \) such that
\[
\left| \mathbb{E}[g_{n^{-1}}(u^\delta(T, x) - z)] - \mathbb{E}[g_{n^{-1}}(u(T, x) - z)] \right| \leq C\delta^\frac{1}{2}.
\]
Putting \( n \to \infty \) in the above inequality, then the desired result (1.3) follows from Theorem
3.6, the non-degeneracy of \( u(T, x) \) (see e.g. \([12, \text{Section 4}]\)) and (5.1).

When \( \nu \) is affine, it can be seen from Remarks 4.1 that the convergence order of density approximations can be improved to be nearly 1, which can also be proved based on the strong convergence order; see the following example.

**Example 5.1.** We discuss the affine case: \( b(u) = b_1 u + c \). On the one hand,
\[
\begin{align*}
  u(T, x) &= \int_0^T e^{b_1 T} G_T(x, y) u_0(y)dy + \int_0^T \int_0^T G_{T-s}(x, y)e^{b_1(T-s)}dyds \\
  &\quad + \int_0^T \int_0^T G_{T-s}(x, y)e^{b_1(T-s)}\sigma W(ds, dy)
\end{align*}
\]
indicates that \( u(T, x) \) is a Gaussian random variable with mean
\[
  m_1 := \int_0^1 e^{b_1 T} G_T(x, y) u_0(y)dy + \frac{c(1-e^{b_1 T})}{b_1}
\]
and variance
\[
  \sigma_1 := \int_0^T \int_0^T G_{T-s}(x, y)e^{2b_1(T-s)}\sigma^2dyds.
\]
On the other hand, we use a version of Clark-Ocone formula for two parameter processes to obtain
\[
  u^\delta(T, x) = \mathbb{E}[u^\delta(T, x)] + \int_0^T \int_0^T \mathbb{E}[D_{r,y}u^\delta(T, x)\mid \mathcal{F}_t]W(dr, dy),
\]
where
\[
  D_{r,y}u^\delta(T, x) = \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 G_{T-s}(x, y)b_1 D_{r,y}u^\delta(t_j, y)dyds + G_{T-r}(x, y)\sigma
\]
is independent of \( \omega \). Therefore, for any fixed \( x \in (0, 1) \), \( u^\delta(T, x) \) is a Gaussian random variable with mean
\[
  m_2 := \mathbb{E}[u^\delta(T, x)]
\]
and variance
\[
  \sigma_2 := \mathbb{E}[u^\delta(T, x)^2] - (\mathbb{E}[u^\delta(T, x)])^2.
\]
Although it is not easy to give the explicit expressions of \( m_2 \) and \( \sigma_2 \), the convergence order of density approximations of Eq. (1.1) with \( b(u) = b_1 u + c \) can be obtained by the strong convergence order. In fact, we observe that
\[
  |m_1 - m_2| \leq \|u^\delta(T, x) - u(T, x)\|_1,
\]
\[
  |\sigma_1 - \sigma_2| \leq 2\|u^\delta(T, x) - u(T, x)\|_2(\|u^\delta(T, x)\|_2 + \|u(T, x)\|_2),
\]
and that by the mean value theorem
\[
  \left| \frac{\partial}{\partial m} g_\sigma(z - m_1) - g_\sigma(z - m_2) \right| \leq C(\sigma_1, \sigma_2)\|u^\delta(T, x) - u(T, x)\|_2 \left( \|u^\delta(T, x)\|_2 + \|u(T, x)\|_2 \right)
\]
where \( g \) is defined by (3.1) and \( C \) is independent of \( x \in (0, 1) \), \( z \in \mathbb{R} \) and \( \theta \in (0, 1) \). Further, it can be shown that for any \( \nu \in (0, 1) \), there exists \( C = C(T, \nu, b, \sigma) \) such that (see e.g. \([10]\)),
\[
  \sup_{x \in (0, 1)} \|u^\delta(T, x) - u(T, x)\|_2 \leq C\delta^\nu.
\]
As a result, for any \( \nu \in (0, 1) \) and \( x \in (0, 1) \), there exists \( C = C(T, \nu, b_1, c, \sigma) \) such that
\[
  \sup_{z \in \mathbb{R}} |q^\delta_{T,x}(z) - q_{T,x}(z)| \leq C\delta^\nu.
\]
Recall that the total variation distance of probability measures $\mu$ and $\nu$ on a $\sigma$-algebra $\Sigma$ is defined by
\[
d_{TV}(\mu, \nu) = 2 \sup\{|\mu(A) - \nu(A)| : A \in \Sigma\}.
\] (5.2)

Let $\{B_t\}_{t \geq 0}$ be a standard 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. It is shown in [2, Theorem 2.6] that the numerical approximation $Y_N$ of the Euler-Maruyama scheme of elliptic stochastic differential equations $dY(t) = f(Y(t))dt + dB_t$, $t \in [0, T]$, has weak convergence order 1 even for bounded continuous test functions. This implies that
\[
\lim_{\delta \to 0} d_{TV}(Y(T) \circ \mathbb{P}^{-1}, Y_N \circ \mathbb{P}^{-1}) = 0.
\]

For infinite dimensional case, we can derive a similar result that for Eqs. (1.1) and (1.2) and for any fixed $x \in (0, 1)$,
\[
\lim_{\delta \to 0} d_{TV}(u(T, x) \circ \mathbb{P}^{-1}, u^\delta(T, x) \circ \mathbb{P}^{-1}) = 0.
\]

In fact, because $u(T, x)$ and $u^\delta(T, x)$ have smooth densities $q_{T,x}$ and $q_{T,x}^\delta$, respectively, it is readily to verify that the set $A = \{z : q_{T,x}(z) > q_{T,x}^\delta(z)\}$ attains the supremum of $\sup\{|\mathbb{P}(u(T, x) \in A) - \mathbb{P}(u^\delta(T, x) \in A)| : A \in \mathcal{B}(\mathbb{R})\}$, which leads to
\[
d_{TV}(u(T, x) \circ \mathbb{P}^{-1}, u^\delta(T, x) \circ \mathbb{P}^{-1}) = \int_{\mathbb{R}} |q_{T,x}^\delta(z) - q_{T,x}(z)|dz.
\]

For any $\eta \in (0, \frac{1}{2})$ and $\delta > 0$, it follows from (1.3) that
\[
\int_{-\delta^{-\eta}}^{\delta^{-\eta}} |q_{T,x}^\delta(z) - q_{T,x}(z)|dz \leq 2C\delta^{-\eta}\delta^{1/2} \leq 2C\delta^{1/2-\eta}.
\]

Accordingly, we obtain
\[
\int_{\mathbb{R}} |q_{T,x}^\delta(z) - q_{T,x}(z)|dz \leq 2C\delta^{1/2-\eta} + \int_{-\infty}^{-\delta^{-\eta}} |q_{T,x}^\delta(z)|dz + \int_{\delta^{-\eta}}^{\infty} |q_{T,x}^\delta(z)|dz
\]
\[
+ \int_{-\infty}^{-\delta^{-\eta}} |q_{T,x}(z)|dz + \int_{\delta^{-\eta}}^{\infty} |q_{T,x}(z)|dz \to 0, \text{ as } \delta \to 0,
\]

since the last four integrals tend to 0 as $\delta \to 0$ thanks to $\int_{\mathbb{R}} |q_{T,x}^\delta(y)|dy = \int_{\mathbb{R}} |q_{T,x}(y)|dy = 1$.

5.2. Logarithmic of asymptotic property. In this part, we present the logarithmic asymptotic property of the density of the exact solution of Eq. (1.1), which turn out to be preserved by Eq. (1.2) exactly. For this end, we begin with briefly recalling the Nourdin and Viens’s result (5.3) on dominating the density of a general centered random variable $Z$ from above and below by means of Malliavin calculus.

For $Z \in \mathbb{D}^{1,2}$ with mean zero, define the function $h$ by
\[
h(z) := \mathbb{E}[|DZ, -DL^{-1}Z|_Y |Z = z], \forall z \in \mathbb{R},
\]
where $L^{-1}$ is the inverse of infinitesimal generator $L$ of Ornstein-Uhlenbeck semigroup. If there exist $\sigma_{\min}, \sigma_{\max} > 0$ such that
\[
\sigma_{\min}^2 \leq h(Z) \leq \sigma_{\max}^2, \text{ a.s.,}
\]
then, by [13, Corollary 3.5], $Z$ has a density $\rho$ satisfying, for almost every $z \in \mathbb{R}$,
\[
\frac{\mathbb{E}|Z|}{2\sigma_{\min}^2} \exp\left(-\frac{z^2}{2\sigma_{\min}^2}\right) \leq \rho(z) \leq \frac{\mathbb{E}|Z|}{2\sigma_{\max}^2} \exp\left(-\frac{z^2}{2\sigma_{\max}^2}\right).
\] (5.3)
Suppose that the process $W' = \{W'(h), h \in \mathbb{H}\}$ is an independent copy of $W$. If there is no confusion caused, $W : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}^\mathbb{H}$ and $W' : (\Omega', \mathcal{F}', \mathbb{P}') \to \mathbb{R}^\mathbb{H}$ can be seen as the canonical mappings associated with the processes $W = \{W(h), h \in \mathbb{H}\}$ and $W' = \{W'(h), h \in \mathbb{H}\}$, respectively. If $Z \in \mathbb{D}^{1,2}$, we write $DZ = \Psi_Z \circ W$, where $\Psi_Z$ is a measurable mapping from $\mathbb{R}^\mathbb{H} \to \mathbb{H}$, determined $\mathbb{P} \circ W^{-1}$-almost surely ([14, Section 1.4.1]). Further, by [13, Proposition 3.5], $h(Z)$ can be rewritten as

$$h(Z) = \int_0^\infty e^{-\theta}E \left[ (\Psi_Z \circ W, \Psi_Z \circ (e^{-\theta}W + \sqrt{1 - e^{-2\theta}}W'))_{\mathbb{H}} \right] \, d\theta,$$

where $E$ denotes the expectation with respect to $\mathbb{P} \times \mathbb{P}'$. By denoting $\omega := (\omega, \omega')$ and

$$\tilde{D}Z(\omega) := \Psi_Z \circ \left( e^{-\theta}W(\omega) + \sqrt{1 - e^{-2\theta}}W'(\omega') \right),$$

we have

$$h(Z) = \int_0^\infty e^{-\theta}E' \left[ (DZ, \tilde{D}Z)_{\mathbb{H}} \right] \, d\theta,$$

where $E'$ denotes the expectation with respect to $\mathbb{P}'$ and the explicit dependence of $\tilde{D}Z$ upon $\theta$ is dropped for simplicity of notation.

Based on the above techniques, we show the following theorem.

**Theorem 5.2.** Let $b \in C_b^1$. Then for any $x \in [0, 1]$, $u(t, x)$ admits a density $q_{t, x}$ satisfying that for almost every $z \in \mathbb{R}$,

$$\lim_{t \to 0} \frac{1}{t^2} \log q_{t, x}(z) = -\frac{\sqrt{2\pi}}{4\sigma^2} (1 + \text{sgn}(x(1 - x)))(z - u_0(x))^2.$$

**Proof.** Without loss of generality, we assume that $\sigma > 0$. From [14, Proposition 2.4.4], we have $u(t, x) \in \mathbb{D}^{1,2}$. For any fixed $(r, z) \in (0, T) \times [0, 1]$, the Malliavin derivative $D_{r, z}u(t, x)$ satisfies

$$D_{r, z}u(t, x) = \sigma G_{t-r}(x, z) + \int_r^t \int_0^1 b'(u(s, y))D_{r, z}u(s, y) \, dy \, ds.$$

Noticing that $-|b_1| \leq b'(u(s, y)) \leq |b_1|$, and by the comparison principle ([12, Lemma 4]), we obtain that, except on a $\mathbb{P}$-null set, for all $(t, x) \in (r, T) \times [0, 1],

$$e^{-|b_1|(t-r)} \sigma G_{t-r}(x, z) \leq D_{r, z}u(t, x) \leq e^{|b_1|(t-r)} \sigma G_{t-r}(x, z).$$

Due to $u(t, x) \in \mathbb{D}^{1,2}$, $Du(t, x)(\omega) = \Psi_{u(t,x)}(W(\omega))$ for some measurable mapping $\Psi_{u(t,x)}$ from $\mathbb{R}^\mathbb{H}$ to $\mathbb{H}$, $\mathbb{P} \circ W^{-1}$-a.s. For any $(r, z) \in (0, T) \times (0, 1)$, we write

$$\Psi_{u(t,x)}^r_z(W) := D_{r, z}u(t, x),$$

and then conclude that

$$e^{-|b_1|(t-r)} \sigma G_{t-r}(x, z) \leq \Psi_{u(t,x)}^r_z(W) \leq e^{|b_1|(t-r)} \sigma G_{t-r}(x, z), \quad \mathbb{P} \text{-a.s.}$$

Substituting $Z$ by $u(t, x)$ in (5.4), we denote $Du(t, x) = \Psi_{u(t,x)}(e^{-\theta}W + \sqrt{1 - e^{-2\theta}}W')$. Since the process $W = \{W(h), h \in \mathbb{H}\}$ defined by

$$W(h) = e^{-\theta}W(h) + \sqrt{1 - e^{-2\theta}}W'(h), \quad h \in \mathbb{H},$$

is Gaussian on the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \times \mathbb{P}')$, with mean zero and with the same covariance function as $W$ (see [14, Section 1.4.1]), $Du(t, x) = \{D_{r, z}u(t, x), (r, z) \in \mathbb{D}^{1,2}\}$. Thus, $\Psi_{u(t,x)}^r_z(W)$ is a Gaussian random variable with density $q_{t, x}(z)$. The proof is completed.
Therefore, from (5.3) we have
\[
\sigma_G(x, t) \leq \sigma_G(x, t) \leq e^{\frac{1}{2}|b_1(t-r)|} \sigma_G(x, t).
\]
Putting the above arguments together, for any \( t > 0 \) and \( x \in [0, 1] \), we obtain
\[
\int_0^t \int_0^1 e^{-2|b_1(t-r)|} \sigma_G^2(x, r) dr = \int_0^t \int_0^1 e^{-2|b_1(t-r)|} \sigma_G^2(x, r) dr = \sigma_{\min}^2
\]
and
\[
\int_0^t \int_0^1 e^{2|b_1|} \sigma_G^2(x, r) dr = \sigma_{\max}^2.
\]
Therefore, from (5.3) we deduce immediately that \( u(t, x) \) has a density \( q_{t,x} \) satisfying, for almost all \( z \in \mathbb{R} \),
\[
\frac{E[u(t, x) - E[u(t, x)]]}{2\sigma_{\min}^2} \exp \left( -\frac{(z - E[u(t, x)])^2}{2\sigma_{\max}^2} \right) \\
\leq q_{t,x}(z) \leq \frac{E[u(t, x) - E[u(t, x)]]}{2\sigma_{\max}^2} \exp \left( -\frac{(z - E[u(t, x)])^2}{2\sigma_{\min}^2} \right). \tag{5.6}
\]
Notice that there exist some constants \( C \) and \( \tilde{C} \) independent of \( t, b \) and \( \sigma \) such that
\[
Ct^\frac{1}{4}e^{-2|b_1|} \leq \sigma_{\min}^2 \leq \sigma_{\max}^2 \leq \tilde{C}t^\frac{1}{4}e^{2|b_1|}, \quad \forall t \in (0, 1],
\]
in view of (2.4). We claim that
\[
\lim_{t \to 0} t^{\frac{1}{2}} \int_0^1 G^2_t(x, y) dy = \frac{1}{(1 + \text{sgn}(x(1-x))\sqrt{2\pi})}, \quad \forall x \in [0, 1]. \tag{5.8}
\]
In fact, the spectral decomposition
\[
G_t(x, y) = 2 \sum_{k=0}^{\infty} e^{-|k|^2 \pi^2 t} \cos(k \pi x) \cos(k \pi y)
\]
and the identity \( 2 \cos(j \pi y) \cos(k \pi y) = \cos((j + k) \pi y) + \cos((j - k) \pi y) \) allow us to calculate
\[
\int_0^1 G^2_t(x, y) dy = 2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} e^{-|k-j|^2 \pi^2 t} \cos(k \pi x) \cos(j \pi x) \int_0^1 2 \cos(j \pi y) \cos(k \pi y) dy
\]
\[
= 2 + 2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} e^{-|k+j|^2 \pi^2 t} \cos(k \pi x) \cos(j \pi x) \int_0^1 \cos((j - k) \pi y) dy
\]
\[
= 2 + 2 \sum_{k=0}^{\infty} e^{-2k^2 \pi^2 t} \cos^2(k \pi x) = 2 + G_{2t}(x, x).
\]
Accordingly, for any \( x \in [0, 1] \), we have
\[
t^{\frac{1}{2}} \int_0^1 G^2_t(x, y) dy = 2t^{\frac{1}{2}} + t^{\frac{1}{2}} G_{2t}(x, x) = 2t^{\frac{1}{2}} + \frac{1}{\sqrt{8\pi}} \sum_{n=-\infty}^{+\infty} \left( e^{-\frac{n^2}{8t}} + e^{-\frac{(x-n)^2}{2t}} \right)
\]
\[
5.9 \quad \frac{1}{\sqrt{8\pi}} \left\{ 1 + e^{-\frac{(x-1)^2}{2t}} + 2 \sum_{n=1}^{\infty} e^{-\frac{n^2}{2t}} + \sum_{n=1}^{\infty} e^{-\frac{(x-n)^2}{2t}} + \sum_{n=1}^{\infty} e^{-\frac{(x+n)^2}{2t}} \right\}.
\]

Using the Euler-Poisson integral: \( \int e^{-ux^2} \, dx = 1 \), we have

\[
0 < \sum_{n=1}^{\infty} e^{-\frac{n^2}{2t}} \leq \int_{0}^{\infty} e^{-\frac{y^2}{2t}} \, dy = \sqrt{\frac{t}{2}}.
\]

Combining (5.9) and the fact

\[
0 < \sum_{n=2}^{\infty} e^{-\frac{(x-n)^2}{2t}} + \sum_{n=1}^{\infty} e^{-\frac{(x+n)^2}{2t}} \leq 2 \sum_{n=1}^{\infty} e^{-\frac{n^2}{2t}}, \forall x \in [0, 1],
\]

we arrive at

\[
\lim_{t \to 0} t^{\frac{1}{2}} \int_{0}^{1} G_{t}^{2}(x,y) \, dy = \begin{cases} 
\frac{\sqrt{2\pi}}{2}, & \text{if } x \in \{0, 1\}, \\
\frac{1}{2\sqrt{2\pi}}, & \text{if } x \in (0, 1),
\end{cases}
\]

which is equivalent to (5.8).

**Upper bound:** By (5.7), (5.8), the right hand of (5.6) and L'Hôpital's rule, we have

\[
\lim_{t \to 0} t^{\frac{1}{2}} \log q_{t,x}(z) \leq \lim_{t \to 0} t^{\frac{1}{2}} \log \frac{E[u(t,x)]}{2\sigma_{\text{max}}^{2}} + \lim_{t \to 0} t^{\frac{1}{2}} \left( -(z - \frac{E[u(t,x)]}{2\sigma_{\text{min}}^{2}})^{2} \right)
\]

\[
= -(z - u_{0}(x))^{2} \lim_{t \to 0} \frac{2\sigma_{\text{min}}^{2}}{t^{\frac{3}{2}}} = -(z - u_{0}(x))^{2} \frac{1}{4} \lim_{t \to 0} e^{-2|b_{1}t\sigma_{\text{min}}^{2}} \frac{1}{\sqrt{2\pi}} \int_{0}^{1} G_{t}^{2}(x,z) \, dz
\]

where in the first step, we have used the facts that \( E[|u(t,x)|] \) is uniformly bounded with respect to \( t \in (0, 1) \) and \( x \in [0, 1] \) when dealing with the first limit and that \( \lim_{t \to 0} E[u(t,x)] = u(0,x) = u_{0}(x) \) when dealing with the second limit.

**Lower bound:** Similarly, from the left hand of (5.6) and L'Hôpital's rule, it follows that

\[
\lim_{t \to 0} t^{\frac{1}{2}} \log q_{t,x}(z) \geq -(z - u_{0}(x))^{2} \frac{1}{4\sigma^{2}} (1 + \text{sgn}(x(1-x)))\sqrt{2\pi}.
\]

The proof is completed. \( \square \)

From the above theorem, even if the drift term \( b \) is nonlinear, the behavior of \( q_{t,x} \) looks like a Gaussian density with mean \( u_{0}(x) \) and covariance nearly proportional to \( t^{\frac{1}{2}} \) when \( t \) is sufficiently small, since

\[
q_{t,x}(z) \approx \exp\left( -\frac{\sqrt{2\pi}(1 + \text{sgn}(x(1-x))) (z - u_{0}(x))^{2}}{4\sigma^{2}t^{\frac{1}{2}}} \right), \quad 0 < t \ll 1.
\]

Roughly speaking, for fixed \( x \in [0, 1] \), the distribution of \( u(t,x) \) decays to the distribution \( \delta_{u_{0}(x)} \) of \( u(0,x) \) exponentially as \( t \) tends to 0.

Under Dirichlet boundary condition, we denote by \( u(t,x) \) the corresponding solution to Eq. (1.1). By a slight modification, a similar result can be proved.
Corollary 5.3. Under the same condition of Theorem 5.2, except replacing the Neumann boundary condition by the Dirichlet boundary condition, for any \( x \in (0, 1) \), \( u(t, x) \) admits a density \( q_{t, x} \) satisfying that for almost every \( z \in \mathbb{R} \),

\[
\lim_{t \to 0} \frac{1}{t^2} \log q_{t, x}(z) = -\frac{\sqrt{2\pi}}{2\sigma^2}(z - u_0(x))^2.
\]

We also investigate the logarithmic asymptotic property of the density of the approximation \( \{ u^{\delta}(\delta, x) \}_{\delta > 0} \) associated to Eq. (1.2), as the perturbation parameter \( \delta \) tends to 0. It is observed that the limit \( \lim_{\delta \to 0} \delta^2 \log q_{\delta, x}(z) \) is exactly the limit \( \lim_{t \to 0} t^2 \log q_{t, x}(z) \).

Proposition 5.4. Assume that \( b \in C^1_b \). Then for any \( x \in [0, 1] \), the solution \( u^{\delta}(\delta, x) \) given by Eq. (1.2) admits a density \( q_{\delta, x} \) satisfying that for almost every \( z \in \mathbb{R} \),

\[
\lim_{\delta \to 0} \delta^2 \log q_{\delta, x}(z) = -\frac{\sqrt{2\pi}}{4\sigma^2}(1 + \text{sgn}(x(1 - x))) (z - u_0(x))^2.
\]

The proofs of Corollary 5.3 and Proposition 5.4 are similar to that of Theorem 5.2 and are postponed to Appendix.

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APPENDIX

In the Appendix, we give the proofs of some technique results for reader’s convenience.

Proof of Lemma 2.2.

Proof. Taking the supremum over \( x \in (0, 1) \), then for any \( y \in (0, 1) \),

\[
\sup_{x \in (0, 1)} g_{s,y}(t,x) \leq C \frac{1}{\sqrt{t - s}} + C \int_{s}^{t} \sup_{z \in (0, 1)} g_{s,y}(r_1, z_1) dr_1,
\]

which, together with Gronwall’s inequality, implies that for some \( C = C(T) \),

\[
\sup_{x \in (0, 1)} g_{s,y}(t,x) \leq C \frac{1}{\sqrt{t - s}} + C, \forall y \in (0, 1).
\]

By an iteration process and the semigroup property of \( G \), we have

\[
g_{s,y}(t,x) \leq CG_{t-s}(x,y) + C \int_{s}^{t} \int_{s}^{r_1} \int_{s}^{r_2} \cdots \int_{s}^{r_{n-1}} \int_{s}^{r_n} G_{t-r_1}(x, z_1) G_{r_1-s}(z_1, y) dz_1 dr_1 + \cdots
\]

\[
+ C^n \int_{s}^{t} \int_{s}^{r_1} \int_{s}^{r_2} \cdots \int_{s}^{r_{n-1}} \int_{s}^{r_n} G_{t-r_1}(x, z_1) G_{r_1-r_2}(z_1, z_2)
\]

\[
\cdots G_{r_{n-1}-r_n}(z_{n-1}, z_n) G_{r_{n-2}-s}(z_n, y) dz_n \cdots dz_2 dz_1 dr_1
\]

\[
+ C^n \int_{s}^{t} \int_{s}^{r_1} \int_{s}^{r_2} \cdots \int_{s}^{r_{n-1}} \int_{s}^{r_n} G_{t-r_1}(x, z_1) G_{r_1-r_2}(z_1, z_2)
\]

\[
\cdots G_{r_{n-1}-r_n}(z_{n-1}, z_n) G_{r_{n-2}-r_{n-1}}(z_{n-1}, z_{n+1}) g_{s,y}(r_{n+1}, z_{n+1}) dz_{n+1} \cdots dz_2 dz_1 dr_1
\]

\[
\leq \left( C + C(t-s) + \cdots + C^n \frac{(t-s)^n}{n!} \right) G_{t-s}(x,y)
\]

\[
+ C^n \frac{(t-s)^n}{n!} \int_{s}^{t} \int_{s}^{r_1} \int_{s}^{r_2} \cdots \int_{s}^{r_{n-1}} \int_{s}^{r_n} G_{t-r_{n+1}}(x, z_{n+1}) g_{s,y}(r_{n+1}, z_{n+1}) dz_{n+1} dr_{n+1},
\]
where the first term on the right-hand side is bounded by $e^{CTG_{t-s}(x,y)}$ and the second term is dominated by

$$C_n\frac{(t-s)^n}{n!}C(T) \left( \int_s^t \frac{1}{\sqrt{t-r}} \frac{1}{\sqrt{s-r}} dr_{n+1} + 1 \right),$$

which tends to 0 as $n \to \infty$, thanks to (5.10). The proof is completed. \hfill \Box

**Proof of Corollary 5.3.**

**Proof.** In contrast to (5.8), it suffices to show that for every $x \in (0,1)$,

$$\lim_{t \to 0} t^{\frac{1}{2}} \int_0^1 G_t^2(x,y)dy = \frac{1}{2\sqrt{2\pi}}.$$  \hspace{1cm} (5.11)

Fix $x \in (0,1)$. Now we proceed to verify (5.11). Indeed, the expression

$$G_t(x,y) = 2 \sum_{k=1}^\infty e^{-k^2 \pi^2 t} \sin(k \pi x) \sin(k \pi y)$$

and the identity $2 \sin(j \pi y) \sin(k \pi y) = -\cos((j + k) \pi y) + \cos((j - k) \pi y)$ yield

$$\int_0^1 G_t^2(x,y)dy = G_t(x,x).$$

Furthermore, by applying (2.8), we have

$$t^{\frac{1}{2}} \int_0^1 G_t^2(x,y)dy$$

$$= \frac{1}{\sqrt{8\pi}} \left\{ 1 - e^{-\frac{(x-1)^2}{2t}} - e^{-\frac{x^2}{2t}} + \left( \sum_{n=1}^\infty e^{-\frac{n^2}{2t}} - \sum_{n=2}^\infty e^{-\frac{(n-1)^2}{2t}} \right) + \left( \sum_{n=1}^\infty e^{-\frac{n^2}{2t}} - \sum_{n=1}^\infty e^{-\frac{(x+n)^2}{2t}} \right) \right\}$$

with

$$0 \leq \sum_{n=1}^\infty e^{-\frac{n^2}{2t}} - \sum_{n=2}^\infty e^{-\frac{(n-1)^2}{2t}} \leq \sqrt{\frac{\pi t}{2}}$$

and

$$0 \leq \sum_{n=1}^\infty e^{-\frac{n^2}{2t}} - \sum_{n=1}^\infty e^{-\frac{(x+n)^2}{2t}} \leq \sqrt{\frac{\pi t}{2}}$$

Putting $t \to 0$, the desired identity (5.11) follows and the proof is finished. \hfill \Box

**Proof of Corollary 5.4.**

**Proof.** First, we fix $x \in (0,1)$. By (2.2), $u^\delta(t,x)_{t=\delta}$ is computed by

$$u^\delta(\delta, x) = \int_0^1 G_\delta(x,y)u_0(y)dy + \int_0^\delta \int_0^1 G_{\delta-s}(x,y)b(0(y))dyds$$

$$+ \int_0^\delta \int_0^1 G_{\delta-s}(x,y)dW(ds,dy),$$

which implies that the distribution of $u^\delta(t,x)_{t=\delta}$ is Gaussian and hence is denoted by $\mathcal{N}(\mu_\delta, \nu_\delta)$. By applying the isometry formula, we have

$$\mu_\delta = \int_0^1 G_\delta(x,y)u_0(y)dy + \int_0^\delta \int_0^1 G_{\delta-s}(x,y)b(0(y))dyds,$$
\[ \nu_\delta = \int_0^\delta \int_0^1 G_{\delta-s}^2(x, y) \sigma^2 dy ds = \sigma^2 \int_0^\delta \int_0^1 G_x^2(x, y) dy ds. \]

Therefore,
\[ \lim_{\delta \to 0} \frac{1}{\delta} \log q_{\delta,x}^\beta(z) = \lim_{\delta \to 0} \frac{1}{\delta} \log \frac{1}{\sqrt{2\pi \nu_\delta}} e^{-\frac{(z-\mu_\delta)^2}{2\nu_\delta}} = \lim_{\delta \to 0} \frac{1}{\delta} \log \frac{1}{\sqrt{2\pi \nu_\delta}} + \lim_{\delta \to 0} -\frac{1}{2\nu_\delta} (z - \mu_\delta)^2. \]

Taking \( |b|_1 = 0 \) in (5.7) yields \( C_\delta^\beta \leq \int_0^\delta \int_0^1 G_x^2(x, y) dy ds \leq \tilde{C}_\delta^\beta, \forall \delta \in (0, 1) \), which indicates the first limit in the right hand of (5.12) is zero. Observing that \( \lim_{\delta \to 0} \mu_\delta = u_0(x) \) and using (5.8), as well as L’Hôpital’s rule, we finally derive that
\[ \lim_{\delta \to 0} \frac{1}{\delta} \log q_{\delta,x}^\beta(z) = \lim_{\delta \to 0} -\frac{1}{2\nu_\delta} (z - u_0(x))^2 = -\frac{\sqrt{2\pi}}{4\sigma^2} (1 + \text{sgn}(x(1-x)))(z - u_0(x))^2. \]

\[ \square \]

**Proof of Proposition 3.9.**

*Proof.* Similar to the proof of (3.24), for \( \theta \in (0, t_{i-1}) \),
\[ \| D_{\theta,\xi} Z_1^\beta (r, y) \|_{k, p} \leq C G_{r-\theta}(y, \xi). \]

Similar to the proof of Lemma 3.4, for \( \theta \in (t_{i-1}, r) \), we only give the details of the case \( k = 0 \), and the induction argument for \( k \geq 1 \) is omitted. By the definition of \( Z_1^\beta (r, y) \), for \( \theta \in (t_{i-1}, r) \),
\[ D_{\theta,\xi} Z_1^\beta (r, y) = \beta D_{\theta,\xi} \varphi_r^y(t_{i-1}, \Phi_{t_{i-1}}(0, u_0)) = \beta \sigma G_{r-\theta}(y, \xi) \]
\[ + \beta \int_0^r \int_0^1 G_{r-s}(y, z) b'(\varphi_s^y(t_{i-1}, \Phi_{t_{i-1}}(0, u_0))) D_{\theta,\xi} \varphi_s^z(t_{i-1}, \Phi_{t_{i-1}}(0, u_0)) dz ds, \]
which together with Lemma 2.2 gives
\[ \| D_{\theta,\xi} Z_1^\beta (r, y) \|_p \leq C G_{r-\theta}(y, \xi). \]

\[ \square \]

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1. LSEC, ICMSEC., Academy of Mathematics and Systems Science., Chinese Academy of Sciences., Beijing, 100190, China, 2. School of Mathematical Science., University of Chinese Academy of Sciences., Beijing, 100049, China

E-mail address: chenchuchu@lsec.cc.ac.cn

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

E-mail address: jianbocui@lsec.cc.ac.cn

1. LSEC, ICMSEC., Academy of Mathematics and Systems Science., Chinese Academy of Sciences., Beijing, 100190, China, 2. School of Mathematical Science., University of Chinese Academy of Sciences., Beijing, 100049, China

E-mail address: hjl@lsec.cc.ac.cn

1. LSEC, ICMSEC., Academy of Mathematics and Systems Science., Chinese Academy of Sciences., Beijing, 100190, China, 2. School of Mathematical Science., University of Chinese Academy of Sciences., Beijing, 100049, China

E-mail address: sdr@lsec.cc.ac.cn (Corresponding author)