Entanglement entropy as a witness of the Aharonov–Bohm effect in QFT

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Abstract
We study the dependence of the entanglement entropy with a magnetic flux and show that the former quantity witnesses an Aharonov–Bohm-like effect. In particular, we consider free charged scalar and Dirac fields living on a two-dimensional cylinder and study how the entanglement entropy for a strip-like region on the surface of the cylinder is affected by a magnetic field enclosed by it.

Keywords: entanglement entropy, Aharonov–Bohm effect, quantum field theory

(Some figures may appear in colour only in the online journal)

1. Introduction

The Aharonov–Bohm (AB) effect is a fundamental quantum phenomenon in which an electrically charged particle is affected by an electromagnetic potential $A_\mu$, even if the magnetic and electric components of this field vanish in the region where the particle is confined. The AB effect emerges as a consequence of the fact that the circulation of $A_\mu$ around a curve $C$ ($\Phi := \oint_C A_\mu \, dx^\mu$) can be sensed in the wave function of a charged particle $\psi(x)$, which acquires an additional phase factor $e^{i\Phi} \psi(x)$, regardless of the precise values $A_\mu$ takes on the region where the particle is confined. This phase factor can therefore be seen in interference experiments of particles traveling in different paths.

This effect has first been noted by Ehrenberg and Siday in [1] and Aharonov and Bohm in [2], and has been observed in the laboratory [3]. The response of the expectation value of certain QFT operators (on cylindrical geometries) under a magnetic flux were recently analyzed using holography on [4]. In the condensed matter literature, the effect of the magnetic flux on unconventional superconductors with cylindrical geometry has also been studied (see
for example [5]). In this work, we analyze the AB effect on the vacuum fluctuations using entanglement entropy.

The entanglement entropy refers to the von Neumann entropy $S(V)$ of the vacuum state reduced to a region $V$ of the space

$$S(V) = -\text{tr}\left(\rho_V \log \rho_V\right);$$

with $\rho_V$ the reduced density matrix. It essentially measures the entanglement between $V$ and its complement.

Though it originated in an attempt to explain the entropy of black holes, the entanglement entropy has nowadays become an exceptional theoretical tool that provides new insights into a variety of topics in physics (see [6] for some general reviews on entanglement entropy in extended systems). It can be used to distinguish new topological phases or different critical points in condensed matter systems [7–9]. It has also been proposed as a useful probe of phase transitions in gauge quantum field theories [10] and has brought a new perception on the structure of renormalization group flows [11–13], being essential to prove the c-theorem in three dimensions [14].

In this paper we show that the entanglement entropy exhibits a dependence on the AB phase $\Phi$, thus becoming an attractive tool to explore related topological phenomena\(^1\). Specifically, we compute the entanglement entropy for free charged scalar and Dirac fields in the presence of an electromagnetic potential in a simple two-dimensional example.

### 2. The AB effect on entanglement entropy

We are going to analyze the case of a free scalar field, $\phi$, charged with respect to an external gauge field, $A_\mu$, which is pure gauge in the region of interest. Hence we start with the Lagrangian

$$\mathcal{L} = -\left(\partial_\mu + ieA_\mu\right)\phi^\ast \left(\partial^\mu - ieA^\mu\right)\phi - m^2\phi^\ast \phi$$

for a charged free scalar field with mass $m$ in $d$ spatial dimensions. To keep the calculation as simple as possible we consider the case of a space compactified in a circle of size $D$ in the $x^1$ direction (see figure 1) with periodic boundary conditions for the field, $\phi(x^0, 0, x^2, \ldots, x^d) = \phi(x^0, D, x^2, \ldots, x^d)$. We choose a constant gauge field in the $x^1$ direction. When the pure gauge field $A_\mu = \partial_\mu \alpha(x)$ is turned on we can eliminate it by a gauge transformation

$$\phi(x) \rightarrow e^{-ie\int_1^x d^{d-1}x^\mu \phi(x)},$$

where the base point $\bar{x}$ of the integral is arbitrary. This has the consequence that the scalar field has now the following boundary condition

$$\phi(x^0, 0, \ldots, x^d) = e^{-ie\oint_{A_1} dx^1}\phi(x^0, D, \ldots, x^d).$$

The integral

$$e\oint A_1 dx^1 = \varphi = e\Phi$$

\(^1\) The entanglement entropy has already been used to study quantum Hall states, a phenomenon intimately linked to the AB effect [15].
can be thought as proportional to the flux $\Phi$ of a magnetic field through the circle $S^1$. This magnetic field is fully outside of the space, and its effect on the scalar field is only through the AB effect. It gives a phase $\phi$ on the boundary condition of the field $\phi$ which is now decoupled from any external sources.

We can use Lagrangian (2) with $eA_1 = \phi/D$ constant, and decompose it into Fourier modes in the $x^1$ direction, $\phi = \sum_n e^{i2\pi n x^0/D} \phi^{(n)}$

$$\mathcal{L} = \sum_{n=-\infty}^{\infty} \left( -\partial_\mu \phi^{(n)} * \partial^\mu \phi^{(n)} - \left( m^2 + \frac{(2\pi n + \phi)^2}{D^2} \right) \phi^{(n)} * \phi^{(n)} \right),$$

where now the fields $\phi^{(n)}$ depend on time and $d - 1$ spatial dimensions with coordinates $x^2, \ldots, x^d$.

The entropy of a strip of width $L$ around the cylinder (see figure 1) will be given by the sum over the different modes of the entropies of these massive fields living in dimension $d - 1$.

We will focus in the case $d = 2$ (two spatial dimensions) from now on. As long as we are interested in evaluating the entropies of annulus around the direction $x^1$, the calculation reduces to computing the entropy of an interval of size $L$ in one dimension, for an infinite tower of massive fields with masses given by

$$M(n, \phi) = \sqrt{m^2 + \frac{(2\pi n + \phi)^2}{D^2}}. \quad (7)$$

The same dimensional reduction holds for Dirac fields with Lagrangian

$$\mathcal{L} = i\bar{\Psi} \gamma^\mu \left( \partial_\mu - ieA_\mu \right) \Psi - m\bar{\Psi} \Psi,$$

where the effective masses for the one-dimensional fields are given again by (7). Hence, the entanglement entropy of the annulus is

$$S(L, m, \phi) = \sum_n S_1(L, M(n, \phi)),$$

where $S_1(L, M)$ is the vacuum entropy for the massive $d = 1$ Dirac field of mass $M$ for an interval of length $L$ (we are using the letter $M$ to refer to the function $M(n, \phi)$).

The effect of boundary conditions on entanglement entropy has been studied for some models in $1 + 1$ dimensions, see for example [9, 16].
These one-dimensional entropies have been computed in [17]. We have

$$S_L(L, M) = -\int_{\epsilon}^{\infty} dy \frac{C(y)}{y} - C(0) \log (Me),$$  \hspace{1cm} (10)

where $\epsilon$ is a short distance ultraviolet cutoff, $M$ stands for the effective mass of the field, and

$$C(ML) = \frac{dS_L(L, M)}{dL}$$  \hspace{1cm} (11)

is the entropic $C$-function [11]. This is positive and monotonically decreasing. For zero $ML$ it takes the value $C(0)$ given by one third of the conformal central charge in the limit $M \to 0$. This is $C(0) = 1/3$ for Dirac fermions and $C(0) = 2/3$ for a complex scalar. For large mass $C(ML)$ is exponentially decreasing. More precisely, the limits of small and large argument for this function are [17]

$$C(y) \approx \frac{2}{3} + \frac{1}{\log (y)} + \ldots \text{ for } y \ll 1,$$

$$C(y) \approx \frac{1}{2} y K_1(2y) \text{ for } y \gg 1,$$

for a complex scalar, and

$$C(y) \approx \frac{1}{3} - \frac{1}{3} y^2 \log^2(y) + \ldots \text{ for } y \ll 1,$$

$$C(y) \approx \frac{1}{2} y K_1(2y) \text{ for } y \gg 1,$$

for a Dirac field. The expressions for short distances are the leading logarithmic terms. The complete $C$-function can be calculated numerically with high precision by integrating the solutions of an ordinary differential equation [17].

The first term in (10) gives the shape of the one-dimensional entropy as a function of $L$. We have to include the second term in (10) which is an integration constant (note the definition of $C(y)$, equation (11)) that only depends on the mass. The one-dimensional entropy saturates to a constant given by this term for large $L$ [9]. It will be affected by changes on the mass due to the magnetic flux, equation (7). The cutoff dependence in (10) does not play a role because we want evaluate how the entropy changes with the magnetic flux. It gives a constant overall ambiguity which is independent on the mass and $L$. For a scalar field the entropy includes an additional $L$ independent term that depends on the mass

$$\log (\log (-Me)).$$  \hspace{1cm} (14)

This is due to infrared divergences for massless scalars in two dimensions [17]. However, this mass dependent term has to be thought as giving an overall infrared constant term because its derivatives with respect to mass vanish for the limit of small cutoff. Then, we are neglecting this term in the following.

We can focus on the universal (cutoff independent) part of the change of entropy with magnetic flux by computing the quantity

$$S(\varphi) = \int_0^\varphi d\varphi' \frac{d}{d\varphi'} S(L, m, \varphi') = S(L, m, \varphi) - S(L, m, \varphi = 0).$$  \hspace{1cm} (15)
The contribution to $S(\varphi)$ of the second term in (10) is given by

$$\sum_{n=-\infty}^{\infty} \int_0^\varphi \frac{d\varphi'}{d\varphi} \left( -C(0) \log(M(n, \varphi')\epsilon) \right)$$

$$= - \int_0^\varphi \frac{d\varphi'}{d\varphi} \sum_{n=-\infty}^{\infty} C(0) \frac{2m + \varphi'}{m^2D^2 + (2m + \varphi')^2}$$

$$= - \int_0^\varphi \frac{d\varphi'}{d\varphi} \int_0^\varphi \frac{d\varphi''}{d\varphi'} C(0) \sum_{n=-\infty}^{\infty} \frac{(mD)^2 - (2m + \varphi'')^2}{(mD)^2 + (2m + \varphi'')^2}$$

$$= - \int_0^\varphi \frac{d\varphi'}{d\varphi} \int_0^\varphi \frac{d\varphi''}{d\varphi'} C(0) \cos (mD) \cos (\varphi'') - 1$$

$$= - \int_0^\varphi \frac{d\varphi'}{d\varphi} \frac{C(0) \sin \varphi'}{2 \left( \cosh (mD) - \cos (\varphi') \right)}$$

$$= - \frac{C(0)}{2} \log \left( \frac{\cosh (mD) - \cos (\varphi)}{\cosh (mD) - 1} \right).$$

(16)

Fortunately, the sum in the second line is done by Mathematica. The contribution in (16) is independent of the width of the strip $L$, and is always negative.

Hence, we have from equations (9), (10) and (15)

$$S(\varphi) = - \sum_{n=-\infty}^{\infty} \int_{LM(n, \varphi)}^{\infty} \frac{dy}{y} \frac{C(y)}{y} + \sum_{n=-\infty}^{\infty} \int_{LM(n, 0)}^{\infty} \frac{dy}{y} \frac{C(y)}{y} - \frac{C(0)}{2} \log \left( \frac{\cosh (mD) - \cos (\varphi)}{\cosh (mD) - 1} \right).$$

(17)

This expression is finite, showing the $\varphi$ dependence of the entropy is regularization independent. Some general features of $S(\varphi)$ follow directly from (17) without further calculation. Evidently, from (17) the entropy $S(\varphi)$ will be a periodic function of the phase $\varphi$ with period $2\pi$. When an integer number of quantum flux $\Phi = e/2\pi$ runs through the cylinder we have $S(\varphi) = 0$ and there is no net effect on the vacuum entropy. From (7) we can also see the effect is symmetrical under $\varphi \to -\varphi$ and $\varphi \to \pi - \varphi$. We can compute $S(\varphi)$ numerically from the knowledge of the $C$-function. The result shows $S(\varphi)$ is always negative; the maximum of $S(\varphi)$ is achieved for $\varphi = \pi$. This means the AB effect always decreases the entanglement with respect to the vacuum without magnetic field.

3. Various limits

In order to study the massless case we begin by considering equation (17) for $mD \ll 1$ and $mL \ll 1$. Up to first order in $mD$ the third term in (17) gives

$$C(0) \log(mD) - \frac{C(0)}{2} \log(2 - 2\cos(\varphi)).$$

(18)

We can set $m = 0$ in the first infinite summation of (17) and no divergences will arise (unless $\varphi$ is an integer multiple of $2\pi$). The second sum carries a divergence for the mode $n = 0$ when we take $m = 0$, but we can easily verify that it cancels out with the logarithmic term given by (18). We isolate the term with $n = 0$ in this second summation and extract the logarithmic term.
where
\[
\gamma = \lim_{y_0 \to 0} \left( \int_0^\infty \frac{C(y)}{y} \, dy + C(0) \log(y_0) \right).
\] (20)

We can evaluate (20) numerically using the results for \(C(y)\) given in [11] and for a Dirac field we have \(\gamma \approx -0.528\). On the other hand, for a scalar field, \(\gamma\) is controlled by infrared physics and can be large. If the infrared cutoff for the zero mode is set by a small mass we have \(\gamma \sim -mL\log(\log(mL))\). This is due to the first subleading term in the small \(mL\) expansion of the \(C(mL)\) function, equation (12). If some other mechanism set the infrared cutoff this can greatly change. For example, imposing an antiperiodic boundary condition in the \(x^2\) direction we would have \(\gamma \sim -\log(R/L)\), with \(R\) the compact size of the \(x^2\) direction.

If we write the complete expression for \(S(\phi)\) the logarithmic terms involving \(m\) in (18) and (19) cancel out and we get the expression for the entropy of the massless field
\[
S(\phi) = \sum_{n \neq 0} \int_{|2\pi n|}^{\pi} \frac{C(y)}{y} \, dy - \int_{|2\pi|}^{\pi} \frac{C(y)}{y} \, dy - \frac{C(0)}{2} \log(2 - 2\cos(\phi))
+ \gamma - C(0) \log\left(\frac{L}{D}\right).
\] (21)

 Naturally this is a function of \(L/D\). Figures (2) and (3) show \(S(\phi)\) for some values of \(L/D\) for the scalar and Dirac fields respectively.

When \(L/D \gg 1\) and \(|L/D \phi| \gg 1\) the first two terms are exponentially small (taking \(\phi \in (-\pi, \pi)\)), and the shape of the oscillations is given by
Except for a factor of two and an overall additive constant, this is the same for fermions and scalars. The maximal size $|S(x)|$ of the oscillations is in this case

$$S(x) = \frac{C(0)}{2} \log(4) - \gamma + C(0) \log \left( \frac{L}{D} \right).$$

(22)

This can be as large as we want for large $L$ and fixed $D$. The reason for this large variations is that the one-dimensional massless $n = 0$ mode has an entropy increasing logarithmically with $L$, and this is cut off by the effective mass provided by the magnetic field. However, it is interesting to note that the dependence on $x$ in (22) given by the first term is produced by a coherent contribution of all modes through the saturation term $-C(0) \log(M)$ in the entropy.

The inessential infrared divergence (14) makes its comeback in $S(x)$ for the massless scalar field through the infrared divergent constant $\gamma$, and the large variations of the entropy near $x = 0$. The change of entropy for a massless scalar field with and without magnetic field is infrared divergent. This large susceptibility is not present for the fermion field because it is due to the classical zero mode of the scalar field. However the large value of $\gamma$ does not affect finite variations of $S(x)$ between different values of $x \neq 0$. It does not change the shape of the curves away from $x = 0$ but just displaces them to large negative values (see figure 2). On the opposite limit, for small $L/D \ll 1$, the different modes add incoherently and this diminishes the size of the oscillations. In this limit of small width of the annulus it is better to study directly the derivative $S'(x)$. According to (7) and (9) in the massless limit this is

$$S'(x) = \frac{L}{D} \sum_{n=-\infty}^{\infty} \int \left( \frac{L}{D} (2\pi n + x) \right).$$

(24)

where

$$f(x) = \frac{C(|x|) - C(0)}{x}.$$  

(25)

Figure 3. $S(x)$ for the massless Dirac field and various ratios of $L/D$. From top to bottom: $L/D = 1/10$, $1/2$, $1$, $2$. For large $L/D$ the curves are similar in shape, but half the overall size, as the ones for the scalar (compare with figure 2 and formula (22)). For small $L/D$, in contrast to the scalar case, the function $S(x)$ decays to zero.
The function $f(x)$ is antisymmetric and falls to zero exponentially fast at infinity. If $f(x)$ was analytic we could use Euler MacLaurin formula in (24) to conclude that $\phi'(S)$, and hence $\phi(S)$, vanish exponentially fast with $LD$ for small $LD$. However, this is not the case since $f(x)$ is non-analytic at the origin, going as $f(x) \sim -1/3x \log(|x|)^2$ for fermions and $f(x) \sim (x \log(|x|))^{-1}$ for scalars (see (12) and (13)). As a consequence, the amplitude of the oscillations falls as $(L/D)^3 \log(L/D)$ for the fermions in the limit of small $L/D$, while the derivative $\phi'(S)$ falls only logarithmically, as $(\log(L/D))^{-1}$, for the scalar field (where $\phi$ is held fixed as $L/D \to 0$). This difference can be appreciated in the figures 2 and 3.

For massive fields the effect of the magnetic field on the entropy is reduced with respect to the massless case. For large $mL \gg 1$ the first term of (17) gives an exponentially small number $\sim 1/2mL K_1(2mL) \sim (mL)^{1/2}e^{-2mL}$. In this situation the third term of (17) gives the leading part of the entropy

$$S(\phi) \sim -C(0) \log \left( \frac{\cosh(mD) - \cos(\phi)}{\cosh(mD) - 1} \right). \quad (26)$$

When we also have $mD \gg 1$, this last term gives again an exponentially small number

$$S(\phi) \sim -C(0)e^{-mD}(1 - \cos(\phi)), \quad (27)$$

this time with a pure sinusoidal form. In figure (4) we show the lowest value of the entropy $S(\pi)$ for a particular value $L/D = 1$ as a function of mass.

4. AB effect on mutual information

Instead of considering the entanglement entropy of an annulus of size $L$ we could also think in the mutual information

$$I(A, B) = S(A) + S(B) - S(A \cup B) \quad (28)$$

between two semi-infinite half-cylinders $A$ and $B$ separated by a distance $L$ in the $x^2$ direction. $A$ and $B$ are here the two disjoint regions forming the complement of the strip in figure (1). This quantity measures the amount of information shared by the two subsystems and has the
advantage of being regularization independent from the beginning, thus becoming a useful tool to study a variety of problems in QFT. There have been recent achievements in the study of the mutual information in one and higher dimensional conformal field theory [18].

The calculation proceeds by dimensional reduction in the same way as for the entropy but we have to use the one-dimensional mutual information \( I_1(ML) \) for a field of mass \( M \) between two half-lines separated by \( L \) in one dimension. This gives

\[
I(L, m, \phi) = \sum_n I_n \left( L \sqrt{m^2 + (2n\pi + \phi)^2/D^2} \right). \tag{29}
\]

In (28) only \( S(A \cup B) \) changes with \( L \). Hence \( dI_n(ML)/dL = -dS_1(A \cup B)/dL \). We also have that \( I_1(ML) \) vanish for large \( L \) because the states in \( A \) and \( B \) decouple in this limit, and that \( S_1(A \cup B) \) is equal to the entropy of its complement \( S_1(L) \), because of the global state is pure. Using all this we obtain

\[
I_n(ML) = \int_{LM} dy \frac{C(y)}{y}. \tag{30}
\]

This is just the opposite of the entropy (10), but it does not contain the boundary term \( \log(M) \) which is independent of \( L \) and cancel out in the mutual information (28). Then we get for the variation of the mutual information a formula similar to (17) but without the last term,

\[
I(\phi) = I(L, m, \phi) - I(L, m, 0) = \sum_{n=-\infty}^{\infty} \int_{LM(n,\phi)}^{\infty} dy \frac{C(y)}{y} - \sum_{n=-\infty}^{\infty} \int_{LM(n,0)}^{\infty} dy \frac{C(y)}{y}. \tag{31}
\]

Concavity of the one-dimensional entropy gives \( I^\prime_1(ML) > 0 \), and this in turn implies that the sum in (31) is decreasing for \( \phi \in (0, \pi) \). Consequently \( I(\phi) \) is always negative, achieving its minimum for \( \phi = \pi \). This shows that the AB effect of the magnetic field always decreases the mutual information.

Notice however that the span of the fluctuations of mutual information \( I(\phi) \) diverges in the massless limit \( m \to 0 \). This is because the mutual information of the mode \( n = 0 \) diverges in one dimension for semi-infinite regions, and this is not the case for non-zero \( \phi \).

5. Summary and outlook

The Bohm–Aharonov effect produces changes in vacuum entanglement entropy periodic in the flux. We studied a simple example in two dimensions where it always decreases the entanglement in vacuum for non-zero holonomies. This can be interpreted as a consequence of the AB interference for the modes, where the holonomy induces an effective mass for the fields, reducing correlations. We found that the precise form of the effect is model dependent and, for particular cases, the AB oscillation of the entropy can achieve very large values.

Other scenarios where the AB effect can be computed are higher dimensional analogs of our calculation for free fields, amenable to dimensional reduction, or the case of a magnetic flux vortex in two dimensions using the numerical technique of Srednicki [19]. In higher dimensions one has to use mutual information in order to eliminate spurious divergences of the change on the entropy with the magnetic flux due to the change in the mass induced area terms [20]. For two regions on both sides of an annulus on a plane the variations with magnetic flux of the mutual information is not expected to diverge in the massless limit, unlike in the example discussed in this paper. This is because this quantity is finite for the zero magnetic field case. It would also be interesting to explore this effect in the context of holography.
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