BREAKDOWN OF CLUSTER DECOMPOSITION IN
INSTANTON CALCULATIONS OF THE GLUINO CONDENSATE

TIMOTHY J. HOLLOWOOD\textsuperscript{1,3}, VALENTIN V. KHOZE\textsuperscript{2}, WEONJONG LEE\textsuperscript{1}
AND MICHAEL P. MATTIS\textsuperscript{1}

\textsuperscript{1}Theoretical Division T-8, Los Alamos National Laboratory,
Los Alamos, NM 87545, USA
\textsuperscript{2}Department of Physics, University of Durham,
Durham, DH1 3LE, UK
\textsuperscript{3}Department of Physics, University of Wales Swansea,
Swansea, SA2 8PP, UK

ABSTRACT

A longstanding puzzle concerns the calculation of the gluino condensate $\langle \frac{tr\lambda^2}{16\pi^2} \rangle = c\Lambda^3$ in $\mathcal{N} = 1$ supersymmetric $SU(N)$ gauge theory: so-called weak-coupling instanton (WCI) calculations give $c = 1$, whereas strong-coupling instanton (SCI) calculations give, instead, $c = 2((N - 1)!(3N - 1))^{-1/N}$. By examining correlators of this condensate in arbitrary multistanton sectors, we cast serious doubt on the SCI calculation of $\langle \frac{tr\lambda^2}{16\pi^2} \rangle$ by showing that an essential step—namely cluster decomposition—is invalid. We also show that the addition of a so-called Kovner-Shifman vacuum (in which $\langle \frac{tr\lambda^2}{16\pi^2} \rangle = 0$) cannot straightforwardly resolve this mismatch.
I Introduction

Almost dating back to the development of QCD itself, supersymmetric versions of QCD have been closely studied, as tractable laboratories for extracting exact analytic information about both perturbative and non-perturbative phenomena in nonabelian gauge theories. One outstanding puzzle, unresolved since the mid-1980’s, concerns the calculation of the gluino condensate $\langle \frac{\text{tr} \lambda^2}{16\pi^2} \rangle$ in these models. This is an interesting quantity, as it is a measure of chiral symmetry breakdown. In pure supersymmetric Yang-Mills (SYM) theory, by dimensional analysis, one expects

$$\langle \frac{\text{tr} \lambda^2}{16\pi^2} \rangle = c \Lambda^3,$$

where $\Lambda$ is the dynamical scale in the theory (developed by dimensional transmutation as in QCD), while $c$ is a numerical constant. Remarkably, there are two approaches in the literature for calculating $\langle \frac{\text{tr} \lambda^2}{16\pi^2} \rangle$, each purporting to be exact (i.e., nonrenormalized), but which differ in their predictions of the constant $c$. This disagreement is especially vexing in light of the fact that both involve the use of supersymmetric instantons. The first approach, generally known as “strong-coupling instanton” (SCI) calculations, was developed in Refs. [1–5], while the second approach, generally known as “weak-coupling instanton” (WCI) calculations, was developed in Refs. [5–8]; for self-containedness, both will be reviewed below.

In this paper, we re-examine this old controversy, using our recently developed methods for studying supersymmetric multi-instantons [9–12]. In particular, by looking at $n$-point correlators $\langle \frac{\text{tr} \lambda^2(x_1)}{16\pi^2} \cdots \frac{\text{tr} \lambda^2(x_n)}{16\pi^2} \rangle$ of the gluino condensate, we will be able to probe arbitrary topological numbers $k$. In a nutshell, our results cast serious doubt on the validity of the SCI calculations of the condensate. Specifically, we will demonstrate that an essential technical step in the SCI approach, namely the use of cluster decomposition, is invalid. The important implications of this observation are as follows. Since cluster decomposition is an essential requirement of quantum field theories (with very mild assumptions that are certainly met by SYM theory), the exact quantum correlators must have this property. That cluster is violated by the instanton-saturated SCI correlators then means that (contrary to claims in the literature) the SCI approximation is only giving part of the full answer. Since the SCI correlators obey supersymmetric perturbative nonrenormalization theorems [7], it necessarily follows that additional non-perturbative objects must be contributing to the correlators. A fuller discussion of this point is given in Sec. VII below; however, categorizing the nature of these non-perturbative configurations is beyond the scope of the present paper.\footnote{See however Ref. [13] where, in a compactified version of the present theory, the important role played by monopoles is emphasized.}
should add that we believe that, in contrast, the WCI correlators are consistent with cluster decomposition.

In addition, we will address an ingenious, if controversial, hypothesis of Shifman’s, in which the numerical disagreement between the SCI and WCI results is taken as circumstantial evidence for the existence of an extra disconnected vacuum in SYM theory in which chiral symmetry is unbroken \[13,17\]. While this so-called “Kovner-Shifman (KS) vacuum” can indeed potentially resolve the disagreement at the 1-instanton sector \((k = 1)\), we will show that it fails to do so for the topological sectors with \(k > 1\). In other words, positing a KS vacuum cannot by itself restore the cluster property to the SCI correlators. This discouraging finding might be viewed as removing some of the impetus for positing such a vacuum in the first place.

Finally we will present a novel calculation of \(\langle \text{tr} \lambda^2 \rangle\) which relates the \(\mathcal{N} = 1\) supersymmetric models discussed herein to the exactly soluble Seiberg-Witten models with \(\mathcal{N} = 2\) supersymmetry. This calculation is of potential pedagogical interest because it bypasses the explicit use of instantons, and instead relies on functional methods. Not surprisingly, it recaptures the WCI answer.

Let us sketch in broad strokes the main differences between the SCI and WCI calculations (a more detailed review will follow). For \(\mathcal{N} = 1\) supersymmetric \(SU(N)\) gauge theory with no matter, the leading coefficient of the \(\beta\)-function is \(b_0 = 3N\), so that \(\Lambda^3\) goes like an \(“N^\text{th}”\) root of an instanton: \(\Lambda^3 \propto \exp(-8\pi^2/g^2N)\). This means that a naïve 1-instanton calculation of \(\langle \text{tr} \lambda^2 \rangle\)—in which \(\lambda\) is simply replaced by its “classical value” as an adjoint fermion zero mode in the instanton background, and all the instanton collective coordinates, both bosonic and fermionic, are integrated over—fails; specifically it gives a zero answer, due to unsaturated Grassmann integrations. In order to perform a sensible 1-instanton calculation of \(\langle \text{tr} \lambda^2 \rangle\), two alternative, and necessarily more elaborate, approaches suggest themselves. In the SCI approach, one calculates the \(N\)-point correlator of this condensate, which scales like \(\exp(-8\pi^2/g^2)\), and is indeed nonzero at the 1-instanton level. Furthermore, by a Ward identity reviewed in the Appendix, it is independent of the \(N\) space-time insertion points \(x_i\). After performing the requisite collective coordinate integration, one finds:

\[
\langle \frac{\text{tr} \lambda^2(x_1)}{16\pi^2} \cdots \frac{\text{tr} \lambda^2(x_N)}{16\pi^2} \rangle = \frac{2^N}{(N-1)!(3N-1)} \Lambda^{3N} \tag{1.2}
\]

In order to extract \(\langle \text{tr} \lambda^2 \rangle\) from the correlator (1.2), one then invokes cluster decomposition: taking \(|x_i - x_j| \gg \mu^{-1}\) where \(\mu\) is the mass gap in this theory, and remembering the constancy

---

\(^2\)See Ref. [14] and the rebuttal Ref. [15].
of the correlator, one replaces the left-hand side of Eq. (1.2) simply by $\langle \frac{\text{tr} \lambda^2}{16\pi^2} \rangle^N$. The net result thus reads:

$$
\langle \frac{\text{tr} \lambda^2}{16\pi^2} \rangle = \frac{2}{((N-1)! (3N-1))^{1/N}} \Lambda^3 e^{2\pi i u/N} \quad \text{SCI result},
$$

(1.3)

where $u = 0, \ldots, N - 1$ indexes the $N$ vacua $|u\rangle$ of the $SU(N)$ theory, and reflects the ambiguity in taking the $N^{th}$ root of unity. In retrospect (as argued in Refs. 3,4), the reason why the naïve calculation of $\langle \frac{\text{tr} \lambda^2}{16\pi^2} \rangle$ gives zero is that these $N$ vacua are being averaged over and the phases cancel.

In contrast, in the WCI approach, one modifies the pure gauge theory by adding matter superfields in such a way that $\langle \frac{\text{tr} \lambda^2}{16\pi^2} \rangle$ itself (rather than a higher-point function thereof) receives a nonzero contribution at the 1-instanton level. Next, one decouples these extraneous matter fields by giving them a mass $M$, and taking the joint limit $M \to \infty$ and $\Lambda \to 0$ in the manner dictated by renormalization group (RG) decoupling. Matching onto the effective low-energy theory without matter gives:

$$
\langle \frac{\text{tr} \lambda^2}{16\pi^2} \rangle = \Lambda^3 \quad \text{WCI result}.
$$

(1.4)

Note that the RG decoupling procedure forces the low-energy theory into one of the $N$ degenerate vacua $|u\rangle$, which by convention we take to be the one with real phase. The nomenclature “strong coupling” versus “weak coupling” used to designate these differing approaches refers to the fact that, in the former, as in QCD, the only scale in the problem is the dynamical scale $\Lambda$, whereas in the latter, the existence of VEVs $v_i$ of the matter superfields permit a standard semiclassical expansion when the dimensionless ratios $\Lambda/v_i$ are all small. (The holomorphic properties of SYM theory then permit the analytic continuation of the answer beyond this regime.)

As mentioned above, it is possible to reconcile the two calculations (1.3) and (1.4) by positing the existence of an extra vacuum $|S\rangle$ in which the condensate vanishes (16). Specifically, if $p$ and $1 - p$ represent the probability weights in the vacuum sector of the theory for the standard vacua $\{|u\rangle\}$, and for $|S\rangle$, respectively, and if one takes

$$
p = \frac{2^N}{(N-1)! (3N-1)},
$$

(1.5)

then both the 1-instanton results can be understood. Unfortunately, the multi-instanton calculations presented below show that the mismatch between the SCI and the WCI calculations becomes more severe for higher topological number $k$, and apparently cannot be reconciled in this way for $k > 1$. 

4
This paper is organized as follows. In Secs. II and III, respectively, we review the SCI and WCI calculations of \( \langle \text{tr} \lambda^2 \rangle \) for general gauge group \( SU(N) \). Also in Sec. III we present an alternate, non-instanton-based derivation of this condensate, specific to the gauge group \( SU(2) \), which starts from the Seiberg-Witten solution of the \( \mathcal{N} = 2 \) model \[18\] and flows to the \( \mathcal{N} = 1 \) model, recapturing the WCI result. In Sec. IV we discuss cluster decomposition in more depth, and motivate Shifman’s proposal for reconciling the SCI and WCI calculations by postulating an extra vacuum state. Our principal results are described in Secs. V and VI, in which (extending the SCI approach) we calculate higher-point functions of the condensate, in the topological sectors \( k > 1 \). In Sec. V we calculate, analytically, the \( (kN) \)-point functions \( \langle \text{tr} \lambda^2(x_1) \cdots \text{tr} \lambda^2(x_N) \rangle \) in \( SU(N) \) gauge theory for arbitrary instanton number \( k \), but to leading order in \( 1/N \), while in Sec. VI we calculate, numerically, the 4-point function \( \langle \text{tr} \lambda^2(x_1) \text{tr} \lambda^2(x_2) \text{tr} \lambda^2(x_3) \text{tr} \lambda^2(x_4) \rangle \) at the 2-instanton level for gauge group \( SU(2) \). In either case our SCI calculations explicitly contradict the hypothesis of cluster decomposition—both with and without an extra KS vacuum. Concluding comments are made in Sec. VII.

II Review of the Strong-Coupling Instanton Calculation

Let us review the SCI result for \( \langle \text{tr} \lambda^2(x_1) \cdots \text{tr} \lambda^2(x_N) \rangle \), for pure \( \mathcal{N} = 1 \) \( SU(N) \) gauge theory. The calculation for done originally for the \( SU(2) \) theory in \[1\] and then extended to the \( SU(N) \) theories in \[3\] (see also the very comprehensive review articles \[4, 17\]).

The correlator in question is saturated at the 1-instanton level. The gauge-invariant collective coordinate integration measure is a suitable generalization of the Bernard measure \[19\] to an \( \mathcal{N} = 1 \) theory, and reads:

\[
-\frac{2^{3N+2} \pi^{N-2} \Lambda^{3N}}{(N-1)!(N-2)!} \int d^4a' \, d\rho^2 \, (\rho^2)^{2N-4} \, d^2M' \, d^2\zeta \, d^{N-2}\nu \, d^{N-2}\bar{\nu} \ .
\] (2.1)

Here \( a'_\alpha \) is (minus) the 4-position of the instanton and \( \rho \) is its scale size, the Grassmann spinors \( M'_\alpha \) and \( \zeta_\alpha \) parametrize the supersymmetric and superconformal modes, respectively, of the

\[3\] The numerical calculation is based on a Monte Carlo integration, which (with our present statistics) is incompatible with the clustering result at the \( 5\sigma \) level, and incompatible with the modified clustering result due to the incorporation of a KS vacuum (tuned to reconcile the WCI and SCI 1-instanton results), at the \( 11\sigma \) level.

\[4\] Our choice of notation is dictated by the \( k \)-instanton generalization of this measure, Eq. (5.18) below. Following Ref. \[20\], we correct a factor of two mistake in the normalization of adjoint fermion zero modes that pervades much of the literature (e.g., Refs. \[4, 5\]). Hence our final result for the \( N \)-point function, Eq. (1.2), differs by \( 2^N \) from these references.
gluino, and the Grassmann parameters \( \nu_{\nu'} \) and \( \bar{\nu}_{\nu'} \), \( u' = 1, \ldots, N-2 \), are the superpartners to the iso-orientation modes which sweep the instanton through \( SU(2) \) subgroups of the \( SU(N) \) gauge group (note that each \( \nu_{\nu'} \) and \( \bar{\nu}_{\nu'} \) is a Grassmann number rather than a Grassmann spinor). The measure includes the Lambda parameter of the Pauli-Villars (PV) scheme which at the two-loop level is

\[
\Lambda = g(\mu)^{-2/3} e^{-8\pi^2/(3Ng(\mu)^2)}\mu. \quad (2.2)
\]

Into this measure one inserts \( \prod_{i=1}^{N} \text{tr} \lambda^2(x_i) \) where \( \lambda^\alpha(x) \) is the most general classical adjoint fermion zero mode in the 1-instanton background. In terms of these bosonic and fermionic collective coordinates, one derives (see Eq. (5.21) below):

\[
\text{tr} \lambda^2(x) = -\frac{1}{4} \square \left( \frac{2}{\rho^2 + y^2} \sum_{\nu'=1}^{N-2} \bar{\nu}_{\nu'\nu'} + \zeta_\alpha \zeta_\dot{\alpha} \frac{y^4}{(\rho^2 + y^2)^2} \right. \\
\left. + \mathcal{M}^\alpha \mathcal{M}'_\alpha \frac{2\rho^2 + y^2}{(\rho^2 + y^2)^2} - \mathcal{M}^\alpha y_{\alpha\dot{\alpha}} \zeta_\dot{\alpha} \frac{2\rho^2}{(\rho^2 + y^2)^2} \right)
\]

where

\[
y_{\alpha\dot{\alpha}} = x_{\alpha\dot{\alpha}} + a'_{\alpha\dot{\alpha}} = (x_n + a'_n){\sigma}_n^\alpha \sigma_n^{\dot{\alpha}}. \quad (2.4)
\]

Now let us carry out the Grassmann integrations in Eq. (2.1). Obviously the \( \zeta \) and \( \mathcal{M}' \) Grassmann integrations will be saturated from the condensates inserted at two points \( \{x_i, x_j\} \) chosen from among the \( N \) insertions \( x_1, \ldots, x_N \). For each such pair there are three contributions to these integrals:

\[
\frac{1}{16} \square_i \square_j \frac{y_i^4(2\rho^2 + y_j^2)}{(\rho^2 + y_i^2)(\rho^2 + y_j^2)^2} \quad (\zeta^2 \text{ at } x_i, \mathcal{M}'^2 \text{ at } x_j), \quad (2.5a)
\]

\[
\frac{1}{16} \square_i \square_j \frac{(2\rho^2 + y_j^2) y_j^4}{(\rho^2 + y_i^2)^2(\rho^2 + y_j^2)^2} \quad (\mathcal{M}'^2 \text{ at } x_i, \zeta^2 \text{ at } x_j), \quad (2.5b)
\]

\[
\frac{1}{16} \square_i \square_j \frac{2\rho^4 y_i \cdot y_j}{(\rho^2 + y_i^2)^2(\rho^2 + y_j^2)^2} \quad (\zeta \times \mathcal{M}' \text{ at } x_i, \zeta \times \mathcal{M}' \text{ at } x_j). \quad (2.5c)
\]

Adding these three contributions gives the simpler expression

\[
-36\rho^8 (x_i - x_j)^2 \\
(\rho^2 + y_i^2)^2(\rho^2 + y_j^2)^4. \quad (2.6)
\]

Now we take advantage of the fact that this \( N \)-point function is independent of the \( x_i \) (see the Appendix), to choose these insertion points for maximum simplicity of the algebra. The simplest conceivable such choice, \( x_i = 0 \) for all \( i \), turns out to give an ill-defined answer of the form “0 \( \times \infty \)” (the zero coming from the Grassmann integrations as follows from Eq. (2.4),
and the infinity from divergences in the $\rho^2$ integration due to coincident poles). In order to sidestep this ambiguity, one chooses instead:

$$x_1 = \cdots = x_{N-1} = 0, \quad x_N = x.$$  \hspace{1cm} (2.7)

This choice is the simplest one which gives a well-defined answer with no "0 $\times \infty$" ambiguity. More ambitiously, one can still perform the calculation even if all the insertion points are taken to be arbitrary \[3, 4\]; however, we find it convenient for later to take the minimal resolution provided by (2.7). From the \((x_i - x_j)^2\) dependence in Eq. (2.6), it follows that the pair of insertions \(\{x_i, x_j\}\) responsible for the \(\{\zeta, \mathcal{M}'\}\) integrations must include the point \(x_N = x\); there are \(N - 1\) possible such pairs, giving

$$\frac{-36(N - 1)\rho^8 x^2}{(\rho^2 + (x + a')^2)^4 (\rho^2 + a'^2)^4}$$ \hspace{1cm} (2.8)

for these contributions. The remaining Grassmann integrations over \(\{\nu, \bar{\nu}\}\) are saturated at \(x_i = 0\), and give

$$(N - 2)! \left(\frac{4\rho^2}{(\rho^2 + a'^2)^3}\right)^{N-2}.$$ \hspace{1cm} (2.9)

Combining the denominators in Eqs. (2.8)-(2.9) with a Feynman parameter \(\alpha\),

$$\frac{1}{(\rho^2 + a'^2)^{2N-2}} \frac{1}{(\rho^2 + (x + a')^2)^4} = \frac{(3N + 1)!}{3! (3N - 3)!} \int_0^1 d\alpha \frac{\alpha^3 (1 - \alpha)^{3N-3}}{(\rho^2 + (a' + \alpha x)^2 + \alpha(1 - \alpha)x^2)^{3N+2}}$$ \hspace{1cm} (2.10)

and performing the \(d^4a'\) integration then yields:

$$\langle \text{tr}\lambda^2(x_1) \cdots \text{tr}\lambda^2(x_N) \rangle = \frac{2^{3N+2} \pi^{2N-2} \Lambda^{3N}}{(N - 1)! (N - 2)!} \int_0^1 d\alpha \int_0^\infty d\rho^2 \frac{(\rho^2)^{2N-4}}{(3N - 1)! (3N - 3)!}$$

$$\times \frac{(36(N - 1)\rho^8 x^2)(N - 2)! (4\rho^2)^{N-2}}{2^{5N} \pi^{2N} \Lambda^{3N}} \int_0^1 d\alpha \alpha^2 (1 - \alpha)^{3N-4} \pi^2$$

$$\times \frac{(3N - 2)!}{(3N + 1)! \Lambda^{2N-2}} (\rho^2 + \alpha(1 - \alpha)x^2)^{3N}$$ \hspace{1cm} (2.11)

$$= \frac{3(3N - 2) 2^{5N-1} \pi^{2N} \Lambda^{3N}}{(N - 1)! (3N - 1)} \int_0^1 d\alpha \alpha^2 (1 - \alpha)^{3N-4}$$

in agreement with Eqs. (1.2)-(1.3).
III Review of the Weak-Coupling Instanton Calculation

Next, let us review the WCI calculation of the gluino condensate. As mentioned above, the general WCI strategy is to extend the pure gauge theory to include matter content, in such a way that \( \langle \frac{\bar{\lambda}^2 \lambda^2}{16\pi^2} \rangle \) receives a nonzero contribution at the 1-instanton level. Decoupling the extraneous matter and matching to the low-energy pure gauge theory is then accomplished using standard RG prescriptions. Since the precise nature of this extraneous matter is rather arbitrary, the WCI calculation really stands for a family of related calculations sharing this basic approach, all of which give the same result (1.4). Calculations of this type were done in [5–8] and reviewed in [17].

We will find it efficient to exploit the functional identity (see for example [21]):

\[
\langle \text{tr}\lambda^2 \rangle = -8\pi i \left\langle \frac{\partial}{\partial \tau} W_{\text{eff}} \right\rangle = \frac{16\pi^2}{b_0} \left\langle \Lambda \frac{\partial}{\partial \Lambda} W_{\text{eff}} \right\rangle .
\] (3.1)

Here \( W_{\text{eff}} \) is the effective superpotential,

\[
\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}
\] (3.2)

is the usual complexified coupling, and

\[
\Lambda = \mu e^{2\pi i r(\mu)/b_0}
\] (3.3)

is the RG-invariant 1-loop dynamical scale of the theory. This result comes from writing the microscopic gauge theory as

\[
\mathcal{L} = \frac{1}{4\pi} \text{Im} \left( \tau \int d^2 \theta \text{tr} W^\alpha W_\alpha \right) ,
\] (3.4)

where \( W^\alpha \) is the gauge field-strength chiral superfield, and promoting \( \tau \) to a “spurion superfield”,

\[
\tau \rightarrow T(y, \theta) = \tau(y) + \sqrt{2} \theta^\alpha \chi^\tau_\alpha(y) + \theta^2 F^\tau(y) .
\] (3.5)

From Eqs. (3.4)-(3.5) it trivially follows that

\[
\langle \text{tr}\lambda^2 \rangle = \frac{8\pi}{Z} \frac{\delta}{\delta F^\tau(x)} Z \bigg|_{T(y, \theta) = \tau} ,
\] (3.6)

where

\[
Z = \int \mathcal{D}W e^{i \int d^4 x \mathcal{L}}
\] (3.7)
is the partition function of the microscopic theory, in the generalized background field (3.3).

In order to derive Eq. (3.1) from Eq. (3.6), one assumes that the functional differentiation indicated in Eq. (3.6) formally commutes with the integrating-out of the microscopic degrees of freedom. In other words, $Z$ can be re-expressed in terms of the relevant effective chiral superfields $\Phi_i$ (whatever these may be):

$$
\langle \text{tr} \lambda^2 \rangle = \frac{8\pi}{Z_{\text{eff}}} \frac{\delta}{\delta F^\tau(x)} Z_{\text{eff}} \bigg|_{T(y,\theta)=\tau}, \tag{3.8}
$$

where

$$
Z_{\text{eff}} = \int D\Phi_i e^{-i \int d^4x \int d^2\theta W_{\text{eff}}(\Phi_i, T)}. \tag{3.9}
$$

Equation (3.1) then follows from the observation that $\partial W_{\text{eff}} / \partial F^\tau = \theta^2 \partial W_{\text{eff}} / \partial \tau$.

We now need an explicit expression for the effective superpotential. Following Affleck, Dine and Seiberg (ADS) [6], it is convenient to start from $SU(N)$ gauge theory where the number of flavors $N_F$ is fixed to $N_F = N - 1$. A 1-instanton calculation of the superpotential then gives:

$$
W_{\text{eff}}^{N_F,N} \equiv W_{\text{eff}}^{N-1,N} = C_{\text{ADS}} \frac{\Lambda_{b_0}^{b_0}}{\det_{N_F} (Q_f \tilde{Q}_{f'})}, \tag{3.10}
$$

where the flavor indices $f, f' = 1, \ldots, N_F$ run over the quark superfields. The coefficient of the $\beta$-function is, for general $N$ and $N_F$,

$$
b_0 = 3N - N_F. \tag{3.11}
$$

The normalization constant for the specific case $N_F = N - 1$ was fixed by an explicit 1-instanton calculation, and is simply [20, 22] $C_{\text{ADS}} = 1$. By decoupling the quark flavors one at a time, this 1-instanton expression flows into models with $N_F < N - 1$ for which the superpotential is no longer a 1-instanton phenomenon. In this way one generalizes Eq. (3.10) to (see e.g., Refs. [21, 23]):

$$
W_{\text{eff}}^{N_F,N} = C_{\text{ADS}}^{N_F,N} \left( \frac{\Lambda_{b_0}^{b_0}}{\det_{N_F} (Q_f \tilde{Q}_{f'})} \right)^{1 \over N - N_F} (N_F \leq N - 1), \tag{3.12}
$$

where [20]

$$
C_{\text{ADS}}^{N_F,N} = N - N_F. \tag{3.13}
$$

---

5In the example culminating in Eq. (3.16) below, we will find that there are in fact no residual chiral superfields ‘$\Phi_i$’, so that simply $W_{\text{eff}} = W_{\text{eff}}(T)$, whereas in the Seiberg-Witten example (3.17) below, the $\Phi_i$ are the monopole superfields $M, \tilde{M}$ as well as the dual Higgs $A_D$. 

9
Starting from this more general superpotential, let us decouple the remaining quarks, by giving them a common VEV \( v \). Viewing \( Q \) as an \( N_F \times N \) matrix, one assumes:

\[
\langle Q \rangle = \begin{pmatrix} v & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & v & 0 & \ldots \end{pmatrix}, \quad \langle \tilde{Q} \rangle = \begin{pmatrix} \tilde{v} & 0 \\ \vdots & \ddots \\ 0 & \ldots \end{pmatrix}.
\]

(3.14)

The \( D \)-flatness condition together with a global gauge rotation gives \( \tilde{v} = \bar{v} \). Taking \( |v| \rightarrow \infty \) then decouples the quarks as well as a subset of the gauge fields, leaving a pure \( SU(N') \) gauge theory with \( N' = N - N_F \) and \( b_0 = 3N' \). The 1-loop RG matching prescription reads [20]:

\[
\left( \frac{\Lambda_{N_F,N}}{|v|} \right)^{3N - N_F} = \left( \frac{\Lambda_{0,N'}}{|\tilde{v}|} \right)^{3N'}, \quad N' = N - N_F.
\]

(3.15)

Inputting Eqs. (3.13)-(3.15) into Eq. (3.12) gives:

\[
W_{\text{eff}} = (N - N_F) \left( \frac{\Lambda_{N_F,N}}{|v|^{2N_F}} \right)^{\frac{1}{N_F}} = N' (\Lambda_{0,N'})^3.
\]

(3.16)

The desired result (1.4) then follows from Eq. (3.1).

Note that the starting-point for this WCI calculation, Eq. (3.10), is a \textit{bona fide} 1-instanton calculation. The remaining steps towards the answer involve well-studied path-integral and renormalization group manipulations (principally Eq. (3.1), and Eqs. (3.12)-(3.15), respectively). Alternatively, starting again from the functional identity (3.1), we can rederive the WCI result (1.4) without any reference to an instanton calculation. Instead, one starts from the Seiberg-Witten solution of the \( \mathcal{N} = 2 \) model\(^6\) in the presence of a mass deformation which breaks the supersymmetry down to \( \mathcal{N} = 1 \). In the strong-coupling domain, in the vicinity of the monopole singularity, the superpotential looks like:

\[
W_{sw} = \sqrt{2} \tilde{M} A_D M + m U(A_D).
\]

(3.17)

Here the chiral superfields \( \{M, \tilde{M}\} \) describe the monopole multiplet, \( A_D \) is the dual Higgs, \( U \) is the quantum modulus of the theory (here, in strong coupling, expressed in terms of \( A_D \) rather than \( A \)), and \( m \) is the mass parameter. The \( F \)-flatness condition for the vacuum reads

\[
0 = \frac{\partial W_{sw}}{\partial M} = \frac{\partial W_{sw}}{\partial \tilde{M}} = \frac{\partial W_{sw}}{\partial A_D},
\]

(3.18)

\(^6\)For the remainder of the section, we focus on \( SU(2) \) gauge theory, and quote well-known formulae from Seiberg and Witten [18].
which is solved by

$$a_D \equiv \langle A_D \rangle = 0 , \quad \langle M \rangle = \langle \tilde{M} \rangle = \left( -\frac{m}{\sqrt{2}} U'(0) \right)^{1/2} .$$

(3.19)

In the vicinity of this solution, the relationship between $a_D$ and $u = \langle U \rangle$ is given by

$$a_D = \sqrt{\frac{2}{\pi}} \int_1^u dx \frac{\sqrt{x-u}}{\sqrt{x^2 - \Lambda_{SW}^4}}$$

(3.20)

from which it follows that

$$u = \Lambda_{SW}^2 - 2i\Lambda_{sw} a_D + \mathcal{O}(a_D^2) .$$

(3.21)

Here the Seiberg-Witten dynamical scale $\Lambda_{sw}$ is related to the conventional PV/DR scale $\Lambda_{N=2}$ via [20]

$$\Lambda_{SW} = \sqrt{2} \Lambda_{N=2} .$$

(3.22)

Note that the series (3.21) is not an instanton expansion (i.e., an expansion in $\Lambda_{SW}^4$); instantons emerge only in the weak-coupling regime, where $u$ is expanded in terms of $a = \langle A \rangle$ rather than $a_D$.

Applying the identity (3.1) to $W_{SW}$ using Eqs. (3.21)-(3.22) gives the gluino condensate in the vacuum (3.19):

$$\langle \text{tr} \lambda^2 \rangle = 16\pi^2 m \Lambda_{N=2}^2 .$$

(3.23)

Next we decouple the adjoint Higgs superfield, by taking $m \to \infty$. In this way we flow to the pure $\mathcal{N} = 1$ supersymmetric $SU(2)$ gauge theory. The RG matching condition between the scale $\Lambda$ of the $\mathcal{N} = 1$ theory and the scale $\Lambda_{N=2}$ of the mass-deformed $\mathcal{N} = 2$ theory reads [20]:

$$m^2 \Lambda_{N=2}^4 = \Lambda^6 .$$

(3.24)

Substituting Eq. (3.24) into Eq. (3.23) once again gives the WCI answer (1.4).

IV Comments on Cluster Decomposition

In this section, we examine the issue of cluster decomposition in the context of the gluino condensate. This issue of cluster decomposition is fundamental to a quantum field theory.
The clustering property requires that for sufficiently large separations \(|x_i - x_j|\), compared with the inverse mass gap,\(^7\)

\[
\langle \varphi_1(x_1) \cdots \varphi_n(x_n) \rangle \to \langle \varphi_1 \rangle \times \cdots \times \langle \varphi_n \rangle .
\]

(4.1)

Generally, this property breaks down when, in a statistical mechanical sense, the theory is in a mixed phase. In field theory language, this means there is more than one possible vacuum state. The clustering property is then restored by restricting the theory to the Hilbert space built on one of the vacua. In this sense, clustering is violated in a mild way, and to distinguish this from some other, potentially more serious, violations uncovered below, we will say that the theory satisfies a “generalized notion of clustering”.

Let us consider the calculation of the \(G_n\), the \(n\)-point function of the composite operator \(\text{tr}_N \lambda^2\):

\[
G_n(x_1, \ldots , x_n) = \langle \text{tr}_N \lambda^2(x_1) \cdots \text{tr}_N \lambda^2(x_n) \rangle .
\]

(4.2)

For present purposes we restrict our attention to pure \(\mathcal{N} = 1\) \(SU(N)\) gauge theory. Since \(\text{tr}_N \lambda^2\) is the lowest component of a gauge-invariant chiral superfield (namely \(\text{tr}_N W^2\) where \(W^\alpha\) is the field-strength superfield), a well-known identity—reviewed in the Appendix—says that

\[
G_n(x_1, \ldots , x_n) = \text{const. ,}
\]

(4.3)

independent of the \(x_i\). Next let us consider this constant correlator in the instanton approximation. This means that, at topological level \(k\), \(\lambda(x)\) is simply to be replaced by a general superposition of adjoint fermion zero modes in the general ADHM \(k\)-instanton background, weighted by Grassmann-valued parameters (i.e., fermionic collective coordinates). All bosonic and fermionic collective coordinates are then integrated over, in the appropriate supersymmetric way reviewed below. It can also be shown that \(G_n\) should still be a constant. (The field theory proof of the constancy of the correlation functions and its extension to the instanton approximation is discussed in the Appendix.) Now, in \(SU(N)\) gauge theory, at the topological level \(k\), a multi-instanton has precisely \(2kN\) adjoint fermion zero modes which need to be integrated over. Let us summarize the rules for Grassmann integration: if \(\xi\) is a Grassmann parameter, then

\[
\int d\xi \xi = 1 , \quad \int d\xi 1 = 0 .
\]

(4.4)

\(^7\)For a discussion of clustering and other references, see Bogolubov et al. [24].
Since \( \text{tr}_N \lambda^2 \) is a Grassmann bilinear, it follows that \( \mathcal{G}_n \) is only non-vanishing for \( n = kN \). In particular, the one-point function \( \mathcal{G}_1 \) always vanishes. In summary, in the instanton approximation, at topological level \( k \), we have the following selection rule:

\[
\langle \text{tr}_N \lambda^2(x) \rangle_{k-\text{inst}} \equiv \mathcal{G}_1 \bigg|_{k-\text{inst}} = 0 \quad \text{for all } k ; \\
\langle \text{tr}_N \lambda^2(x_1) \cdots \text{tr}_N \lambda^2(x_n) \rangle_{k-\text{inst}} \equiv \mathcal{G}_n \bigg|_{k-\text{inst}} \neq 0 \quad \text{if and only if } n = kN .
\] (4.5a)

Notice that these results already indicate a breakdown of clustering for the correlation functions (4.2), although, as we shall explain below the breakdown is of the ‘mild’ variety and can be traced to the fact that in instanton approximation the theory is in a mixed phase, i.e. the instanton approximation samples the theory in a number of distinct vacua as opposed to a single vacuum.

A general field-theoretic understanding of the selection rule (4.5a)-(4.5b) was suggested in Refs. [3, 4]. The suggestion relies on the fact that, in \( \mathcal{N} = 1 \) \( SU(N) \) gauge theory, the vacua of the theory come in an \( \mathcal{N} \)-tuplet [23]. The vacua spontaneously break the discrete \( \mathbb{Z}_{2N} \) anomaly-free remnant of the classical \( U(1)_R \) symmetry to the \( \mathbb{Z}_2 \) subgroup: \( \lambda_\alpha \to -\lambda_\alpha \).

The vacuum sector therefore consists of:

\[
\{ |u \rangle : 0 \leq u \leq N - 1 \} .
\] (4.6)

If we define the condensate \( \mathcal{J} \) via

\[
\mathcal{J} = \langle u = 0 | \text{tr}_N \lambda^2 | u = 0 \rangle ,
\] (4.7)

then the \( N \)-tuple of vacua are related by phase factors, namely the \( N \)th roots of unity:

\[
\langle u | \text{tr}_N \lambda^2 | u \rangle = \mathcal{J} e^{2\pi i u/N}.
\] (4.8)

Now let us see how the selection rule (4.5a) comes about. We define the density matrix

\[
\varrho = \frac{1}{N} \sum_{u=0}^{N-1} |u \rangle \langle u | .
\] (4.9)

Since the instanton calculation is \( \mathbb{Z}_{2N} \) symmetric, it must average over all the vacua. This means

\[
\langle \text{tr}_N \lambda^2 \rangle = \text{Tr}(\varrho \text{tr}_N \lambda^2) = \frac{1}{N} \sum_{u=0}^{N-1} \langle u | \text{tr}_N \lambda^2 | u \rangle = \frac{\mathcal{J}}{N} \sum_{u=0}^{N-1} e^{2\pi i u/N} = 0 .
\] (4.10)

Here the capitalized ‘Tr’ means a trace over the Hilbert space. In order to check the selection rule (4.5b), we need the additional assumption of a well-defined clustering limit. We have:

\[
\langle \text{tr}_N \lambda^2(x_1) \cdots \text{tr}_N \lambda^2(x_n) \rangle = \text{Tr}(\varrho \text{tr}_N \lambda^2(x_1) \cdots \text{tr}_N \lambda^2(x_n)) \\
= \frac{1}{N} \sum_{u=0}^{N-1} \langle u | \text{tr}_N \lambda^2(x_1) \prod \text{tr}_N \lambda^2(x_2) \prod \cdots \prod \text{tr}_N \lambda^2(x_n) | u \rangle ,
\] (4.11)
where $\mathbb{P}$ denotes the sum over a complete set of states. At this point the generalized notion of the clustering assumption enters. We assume there exists a mass gap $\mu$ that is dynamically generated in the theory, and we consider the $n$ insertion points are sufficiently far separated in Euclidean space compared to this scale: $|x_i - x_j| \gg \mu^{-1}$. (Since $G_n$ is a constant even in leading semiclassical order, moving to this regime does not entail any additional approximations.) In this regime, the generalized cluster decomposition (in our present usage) is equivalent to the statement that $\mathbb{P}$ collapses to $\mathbb{P}_0$ where $\mathbb{P}_0$ is the projection operator onto vacuum states only:

$$\mathbb{P} \to \mathbb{P}_0, \quad \mathbb{P}_0 = \sum_{u=0}^{N-1} |u \rangle \langle u|.$$ (4.12)

Using the fact that the operator $\text{tr}_N \lambda^2$ is diagonal in the $u$ index, it follows that with the replacement (4.12), the correlator (4.11) collapses to

$$\frac{1}{N} \sum_{u=0}^{N-1} (\mathcal{J} e^{2\pi i u/N})^n = \begin{cases} \mathcal{J}^{kN} & n = kN \\ 0 & \text{otherwise} \end{cases}.$$ (4.13)

Next we consider how this elementary analysis is modified if the $N$-tuplet of vacua $\{|u\rangle\}$ is supplemented by an extra vacuum state, the so-called Kovner-Shifman vacuum [16], which we denote $|S\rangle$. A single such vacuum is permissible under the discrete symmetry only if

$$\langle S| \text{tr}_N \lambda^2 |S\rangle = 0.$$ (4.14)

The analysis proceeds just as before, with the obvious modification that the density matrix $\rho$ should be replaced by $\rho'$, defined by

$$\rho' = (1-p) |S\rangle \langle S| + \frac{p}{N} \sum_{u=0}^{N-1} |u \rangle \langle u|,$$ (4.15)

where the probability $p$ is a real number between 0 and 1. Proceeding as before, we find that with the generalized clustering assumption

$$\mathbb{P} \to \mathbb{P}'_0, \quad \mathbb{P}'_0 = |S\rangle \langle S| + \sum_{u=0}^{N-1} |u \rangle \langle u|,$$ (4.16)

one derives

$$\langle \text{tr}_N \lambda^2(x_1) \cdots \text{tr}_N \lambda^2(x_n) \rangle = \begin{cases} p\mathcal{J}^{kN} & n = kN \\ 0 & \text{otherwise} \end{cases}.$$ (4.17)

---

8This follows from the fact that there should be no mixing between the sectors of Hilbert space built on each vacuum: they are super-selection sectors.
Obviously this modified expression also applies if there are several distinct KS vacua $|S_i\rangle$ in which $\text{tr}\lambda^2 = 0$, that is

$$g' = \sum_{i=1}^{l} q_i |S_i\rangle \langle S_i| + \frac{p}{N} \sum_{u=0}^{N-1} |u\rangle \langle u| ,$$  \hspace{1cm} (4.18)

where $p = 1 - \sum q_i$.

In the following we will calculate these $(kN)$-point correlators, first analytically for large $N$, then numerically for $N = 2$ and $k = 2$, and will find a behavior quite different from either (4.13) or (4.17).

## V Large-$N$ Calculation of Gluino Condensate Correlation Functions

We now present an explicit evaluation of $G_n$, $n = kN$, in the limit $N \to \infty$ with $k$ held fixed. Our answer turns out to be incompatible with both Eqs. (4.13) and (4.17). The cleanest way to quantify this disagreement is to consider the $(kN)^{\text{th}}$ root, $(G_{kN})^{1/kN}$. In the large-$N$ limit, from Eq. (4.13), i.e. clustering without the KS vacuum, one obtains:

$$\lim_{N \to \infty} (G_{kN})^{1/kN} = J(N) .$$  \hspace{1cm} (5.1)

We have written $J$ as $J(N)$ to allow for an unknown $N$ dependence. Using the one instanton expression (1.3) in the large-$N$ limit one expects

$$\frac{1}{16\pi^2} J(N) = \frac{2e}{N} \Lambda^3 ,$$  \hspace{1cm} (5.2)

where $e = 2.718 \ldots$. The key point is that the right-hand side of (5.1) is independent of the topological number $k$ (as well as of the space-time insertion points $x_i$). Note that Eq. (5.1) follows, not only from Eq. (4.13), but also from Eq. (4.17), so long as the constant $p$ either has a nonzero large-$N$ limit, or else vanishes at large $N$ more slowly than exponentially. Alternatively, with the “Shifman assumption” (1.3) for $p$, which vanishes faster than exponentially at large $N$, one obtains instead from Eq. (4.17):

$$\lim_{N \to \infty} (G_{kN})^{1/kN} = \left(\frac{2e}{N}\right)^{1/k} J(N) .$$  \hspace{1cm} (5.3)

Combining this with the large-$N$ limit of the 1-instanton expression (1.3), one extracts instead the expression

$$\frac{1}{16\pi^2} J(N) = \Lambda^3$$  \hspace{1cm} (5.4)
which now agrees (by construction) with the 1-instanton WCI calculation.

Below we will calculate $G_{kN}$, to leading order in $1/N$, but for all instanton number $k$, and will obtain a markedly different behavior. Explicitly we will find:

$$\frac{1}{16\pi^2} \lim_{N \to \infty} (G_{kN})^{1/kN} = \frac{2e}{N} k \Lambda^3 + \mathcal{O}(N^{-2}) \, .$$

Notice that for $k = 1$ we obviously recover the results (5.1)-(5.2) (or (5.3)-(5.4)); however the linear $k$ dependence is in sharp disagreement with the $k$ dependence of either Eq. (5.1) or Eq. (5.3). This disagreement means that the generalized clustering assumption (4.12) is invalid when combined with the instanton approximation. It also means that the extension (4.16) of this clustering assumption, in the presence of an extra KS vacuum state, is likewise invalid.

The large-$N$ calculation proceeds as follows. In supersymmetric theories, at topological level $k$, the bosonic and fermionic collective coordinates live, respectively, in complex-valued matrices $a$ and $\mathcal{M}$, with elements:

$$a = \begin{pmatrix} w_{u\dot{\alpha}j} \\ (\alpha_{\beta\dot{\alpha}})_{ij} \end{pmatrix} \, , \quad \mathcal{M} = \begin{pmatrix} \mu_{uj} \\ (\mathcal{M}_{\beta})_{ij} \end{pmatrix} \, .$$

The indices run over $u = 1, \ldots, N \, , \quad i, j = 1, \ldots, k \, , \quad \dot{\alpha}, \beta = 1, 2 \, ;$

traces over these indices are denoted ‘$\text{tr}_N$’, ‘$\text{tr}_k$’, and ‘$\text{tr}_2$’, respectively. The elements of $\mathcal{M}$ are Grassmann (i.e., anticommuting) quantities. The $k \times k$ submatrices $a'_{\beta\dot{\alpha}} \equiv a_n' \sigma^n_{\beta\dot{\alpha}}$ and $\mathcal{M}'_{\beta}$ are subject to the Hermiticity conditions

$$\bar{a}'_n = a'_n \, , \quad \bar{\mathcal{M}}'_{\dot{\alpha}} = \mathcal{M}'_{\alpha} \, .$$

In the instanton approximation, the Feynman path integral is replaced by a finite-dimensional integration over the degrees of freedom in $a$ and $\mathcal{M}$. These $k$-instanton collective coordinates are weighted according to the integration measure

$$\int d\mu_{\text{phys}}^k = \frac{2^{k^2/2} (C_1)^k}{\text{Vol}\, U(k)} \int d^{4k^2} a' d^{2kN} \bar{w} d^{2kN} w d^{2k^2} \mathcal{M}' d^{kN} \bar{\mu} d^{kN} \mu \times \prod_{r=1, \ldots, k} \prod_{\alpha=1,2,3} \delta \left( 2 \text{tr}_k T^r (\tau^c a) \right) \prod_{\dot{\alpha}=1,2} \delta \left( \text{tr}_k T^r (\bar{\text{M}} a_{\dot{\alpha}} + \bar{a}_{\dot{\alpha}} \mathcal{M}) \right) \, .$$

Our conventions are taken from [11, 12] which also provide self-contained reviews of the ADHM formalism for the $SU(N)$ gauge group.

The reason we have $2^{k^2/2}$ rather than $2^{-k^2/2}$, as in [12], is that we restore Wess and Bagger integration conventions for the $\mathcal{M}'$ integration: $\int d^2 \xi \xi^2 = 1$ rather than 2 where $\xi^2 = \bar{\xi}^2 \xi$ is the square of a Grassmann Weyl spinor.
where the two $\delta$-functions enforce the bosonic and fermionic ADHM constraint conditions, respectively. The integrals over the $k \times k$ matrices $a'_n$ and $\mathcal{M}'$ are defined as the integral over the components with respect to a Hermitian basis of $k \times k$ matrices $T^r$ normalized so that $\text{tr}_k T^r T^s = \delta^{rs}$. These matrices also provide explicit definitions of the $\delta$-function factors in the way indicated.

The form of the measure given in Eq. (5.9) is known as the “flat measure”, since the bosonic and fermionic ADHM collective coordinates are integrated over as Cartesian variables, subject to the nonlinear $\delta$-function constraints. It was uniquely constructed in Ref. [26] to obey several important consistency requirements—including cluster decomposition—so that the failure of cluster uncovered below cannot be attributed to the collective coordinate measure. In practical applications, however, the flat measure is not the most useful form available. When

$$N \geq 2k ,$$

(5.10)

it is convenient to switch to the so-called “gauge-invariant measure,” involving a new set of variables in terms of which the arguments of the $\delta$-functions are linear (and hence trivially implemented) [12]. This is the form of the measure which we will utilize in the present section. The restriction (5.10) is obviously well suited to the large-$N$ limit. As the name implies, the gauge-invariant measure can only be used to integrate gauge-invariant quantities, such as our present focus on correlators formed from $\text{tr}_N \lambda^2$. Alternatively, for the special cases $k \leq 2$, it is easy to solve the nonlinear constraints explicitly without such a change of variables [9, 28].

In order to switch from the flat measure to the gauge-invariant measure, one trades the collective coordinates $w$ and $\bar{w}$ (which transform in the $N$ of $SU(N)$) for the gauge-invariant bosonic bilinear quantity $W$, defined by [12]

\begin{equation}
(W_{\dot{\alpha} \beta})_{ij} = \bar{w}_{i \dot{\alpha}} w_{uj \beta} , \quad W^0 = \text{tr}_2 W , \quad W^c = \text{tr}_2 \tau^c W , \quad c = 1, 2, 3 .
\end{equation}

(5.11)

The appropriate Jacobian for this change of variables reads:

\begin{equation}
d^{2kN} \bar{w} d^{2kN} w \longrightarrow c_{k,N} (\det_{2k}W)^{N-2k} d^{2k} W^0 \prod_{c=1,2,3} d^{2k} W^c ,
\end{equation}

(5.12)

where

\begin{equation}
c_{k,N} = \frac{2^{2kN-4k^2+k} \pi^{2kN-2k^2+k}}{\prod_{i=1}^{2k} (N - i)!} .
\end{equation}

(5.13)

Note that the bosonic $\delta$-function in (5.9) can be rewritten in a gauge-invariant way as the condition

\begin{equation}
0 = W^c + \left[ a'_n , a'_m \right] \text{tr}_2 \tau^c \bar{\sigma}^{nm} = W^c - i \left[ a'_n , a'_m \right] \bar{\eta}^{nm}
\end{equation}

(5.14)
in terms of the gauge-invariant coordinates (here $\bar{\eta}^{c}_{nm}$ is an 't Hooft tensor \[29\]). As advertised, these constraints are linear in the new variables $W^{c}$; consequently the $W^{c}$ integrals simply remove the bosonic ADHM $\delta$-functions in Eq. (5.9) (giving rise to the numerical factor of $2^{3k^{2}}$ from the $\frac{1}{2}$’s in the arguments of the $\delta$-functions).

Next we perform a similar change of variables for the fermions, letting \[12\]
$$
\mu_{ui} = w_{uj\dot{a}}(\zeta^{\dot{a}})_{ji} + \nu_{ui}, \quad \bar{\mu}_{iu} = (\bar{\zeta}_{\dot{a}i})_{j\bar{u}}w^{\dot{a}}_{j\bar{u}} + \bar{\nu}_{iu},
$$
(5.15)
where $\nu$ lies in the orthogonal subspace to $w$:
$$
\bar{w}^{\dot{a}}_{iu} \nu_{uj} = 0, \quad \bar{\nu}_{iu} w_{uj\dot{a}} = 0.
$$
(5.16)

One finds:
$$
\int d^{kN} \mu d^{kN} \bar{\mu} \prod_{r=1,\ldots,k^{2}} \prod_{\dot{a}=1,2} \delta \left( Tr_{k} T^{r}(\bar{\cal{M}}a_{\dot{a}} + \bar{a}_{\dot{a}}\cal{M}) \right) \longrightarrow 2^{k^{2}} \int d^{2k^{2}} \zeta d^{kN-2k^{2}} \nu d^{kN-2k^{2}} \bar{\nu},
$$
(5.17)
where the $\delta$-functions have been used to eliminate the $\bar{\zeta}$ variables from the problem. In summary, the gauge-invariant measure is:
$$
\frac{2^{9k^{2}/2} (C_{1})^{k} c_{k,N}}{\text{Vol } U(k)} \int d^{4k^{2}} a' d^{k^{2}} W^{0} d^{2k^{2}} \cal{M}' d^{2k^{2}} \zeta d^{kN-2k^{2}} \nu d^{kN-2k^{2}} \bar{\nu} (\det_{2k} W)^{N-2k}
$$
(5.18)
and the constraint $\delta$-functions have been eliminated for all $k$ satisfying Eq. (5.10). For $k = 1$, one recovers the expression (2.1), a comparison which fixes the normalization constant $C_{1}$:
$$
C_{1} = -2^{N+1/2} \Lambda^{3N}.
$$
(5.19)

For $k = 2$, we recapture the Osborn measure discussed in Refs. \[9,26,28\], which we utilize in Sec. VI below.

Into this measure we now insert
$$
\text{tr}_{N} \lambda^{2}(x_{1}) \times \cdots \times \text{tr}_{N} \lambda^{2}(x_{kN}),
$$
(5.20)
where the gluino $\lambda^{\dot{a}}(x)$ is replaced in the instanton approximation by a general superposition of adjoint fermion zero modes. In terms of the previously introduced collective coordinates $a$ and $\cal{M}$, a useful identity states \[12\]:
$$
\text{tr}_{N} \lambda^{2}(x) = -\frac{1}{4} \Box \text{tr}_{k} \bar{\cal{M}}(\mathcal{P} + 1)\cal{M} f,
$$
(5.21)
where the ADHM quantities $\mathcal{P}$ and $f$ are defined as:
$$
\mathcal{P} = 1 - \Delta f \bar{\Delta}, \quad f = (\bar{\Delta}\Delta)^{-1}, \quad \Delta = a + bx,
$$
(5.22)
and $b$ is the $(N + 2k) \times (2k)$ matrix whose lower $2k \times 2k$ part is the identity $\delta_3^a \delta_{il}$ and whose upper $N \times 2k$ part is zero (quaternionic multiplication is implied in the product $bx$). As discussed earlier, $G_{kN}(x_1, \ldots, x_{kN})$ is actually a constant, independent of the $x_i$. The $x_i$ can therefore be chosen for maximum simplicity of the algebra. However, the simplest conceivable choice, $x_i = 0$ for all $i$, results in an ill-defined answer of the form “$0 \times \infty$” (the zero coming from unsaturated Grassmann integrations, and the infinity from divergences in the bosonic integrations due to coincident poles); we have already noted this fact in the 1-instanton sector in Sec. II above. The simplest choice of the $x_i$ that avoids this problem turns out to be:

$$x_1 = \cdots = x_{kN-k^2} = 0,$$
$$x_{kN-k^2+1} = \cdots = x_{kN} = x$$

which we adopt for the remainder of this section.\[11\]

In the large-$N$ limit the large preponderance of the insertions (5.23) are at $x_i = 0$, and the resulting factor of $(\text{tr}_N \lambda^2(0))^{kN-k^2}$, taken together with the Jacobian factor $(\det_{2k}W)^{N-2k}$ from the measure (5.18), dominate the integral and can be treated in saddle-point approximation. Below we will carry out this saddle-point evaluation in full detail, but we can already quite easily understand the source of the linear dependence on $k$ in the final result (5.5). The chain of argument goes as follows:

1. Let us imagine carrying out all the Grassmann integrations in the problem. The remaining large-$N$ integrand will then have the form $\exp (-N \Gamma + O(\log N))$ where $\Gamma$ might be termed the “effective large-$N$ bosonic instanton action.” The large-$N$ saddle-points are then the stationary points of $\Gamma$ with respect to the bosonic collective coordinates. By Lorentz symmetry, $\Gamma$ can only depend on the four $k \times k$ matrices $a'_n$ through even powers of $a'_n$. (This is because the bulk of the insertions have been chosen to be at $x_i = 0$; otherwise one could form the Lorentz scalar $x_n a'_n$ and so have odd powers of $a'_n$.) It follows that the ansatz

$$a'_n = 0, \quad n = 1, 2, 3, 4,$$  (5.24a)
$$W^c = 0, \quad c = 1, 2, 3$$  (5.24b)

is automatically a stationary point of $\Gamma$ with respect to these collective coordinates. (Note that (5.24b) follows automatically from (5.24a) by virtue of the ADHM constraints (5.14).) It will actually turn out that, once one assumes these saddle-point values, $\Gamma$ is independent of the remaining collective coordinate matrix $W^0$; furthermore we will verify that this saddle-point is actually a minimum of the Euclidean action.

\[11\] As a nontrivial check on our algebra, we have also numerically integrated the large-$N$ correlator for insertions other than Eq. (5.23), and verified the constancy of the answer presented below.
2. Having anticipated the saddle-point (5.24a)-(5.24b) using these elementary symmetry considerations, let us back up to a stage in the analysis prior to the Grassmann integration, and proceed a little more carefully. Evaluating the insertions $\text{tr}_N \lambda^2(x_i)$ on this saddle-point, one easily verifies that the $\zeta$ modes vanish when $x_i = 0$; consequently the $\zeta$ integrations must be saturated entirely from the $k^2$ insertions at $x_i = x$. This leaves the $\mathcal{M}'$, $\nu$ and $\bar{\nu}$ integrations to be saturated purely from the insertions at $x_i = 0$. Moreover, because $\mathcal{M}'$ carries a Weyl spinor index $\alpha$ whereas $\nu$ and $\bar{\nu}$ do not, the $\text{tr}_N \lambda^2(0)$ insertions depend on these Grassmann coordinates only through bilinears of the form $\bar{\nu} \times \nu$ or $\mathcal{M}' \times \mathcal{M}'$; there are no cross terms.

3. Performing all the Grassmann integrations then automatically generates a combinatoric factor
\[
(k^2)! (k^2)! (kN - 2k^2)! \left( \frac{kN - k^2}{k^2} \right).
\] (5.25)
Here the first three factors account for the indistinguishable bilinear insertions of the $\zeta$, $\mathcal{M}'$, and $\{\nu, \bar{\nu}\}$ modes, respectively, while the final factor counts the ways of selecting the $k^2$ bilinears in $\mathcal{M}'$ from the $kN - k^2$ insertions at $x_i = 0$. Multiplying these combinatoric factors together, as well as the normalization constants $c_{k,N}(C_1)^k$ from Eq. (5.18), and taking the $(kN)^{th}$ root yields, in the large-$N$ limit:
\[
\lim_{N \to \infty} \left[ c_{k,N}(C_1)^k (k^2)! (kN - k^2)! \right]^{1/kN} = 2^3 \pi^2 eN^{-1}k\Lambda^3 + \mathcal{O}(N^{-2}) .
\] (5.26)
Remarkably, apart from a factor of four, this back-of-the-envelope analysis precisely accounts for the previously announced final answer, Eq. (5.5). Note that most of the remaining contributions to the saddle-point analysis, which involve a specific convergent bosonic integral derived below, as well as the factor $2^{9k^2/2}/\text{Vol} U(k)$ from Eq. (5.18), reduce to unity when the $(kN)^{th}$ root is taken in the large-$N$ limit; the missing factor of four will simply come from the leading saddle-point evaluation of the bosonic integrand.

Here are the details of the large-$N$ calculation of $\mathcal{G}_{k,N}$. Since the problem has an obvious $U(k)$ symmetry [12], we will find it convenient to work in a basis where $W^0$ (which transforms in the adjoint of the $U(k)$) is diagonal:
\[
W^0 = \begin{pmatrix}
2\rho_1^2 & 0 \\
0 & 2\rho_k^2
\end{pmatrix} .
\] (5.27)
As the notation implies, in the dilute instanton gas limit $\rho_i$ can be identified with the scale size of the $i^{th}$ instanton in the $k$-instanton sector (see Sec. II.4 of [12]). The appropriate
change of variables reads:

\[
\frac{1}{\text{Vol } U(k)} \int d^k W^0 \rightarrow \frac{2^{\delta k(1)/2\pi - k}}{k!} \int_0^\infty d\rho_1^2 \cdots d\rho_k^2 \prod_{1 \leq i < j \leq k} (\rho_i^2 - \rho_j^2)^2. \tag{5.28}
\]

For \( k = 1 \) one has, of course, \( \int dW^0 \rightarrow 2 \int_0^\infty d\rho^2 \).

Now let us consider the Grassmann integrations, beginning with the \( \zeta \) modes. We assume the saddle-point conditions (5.24a)-(5.24b), in which case

\[
\Delta = \left( w \cdot 1_{[k] \times [k]} \right), \quad f = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \tag{5.29}
\]

and from Eq. (5.21),

\[
\text{tr}_N \chi^2(x) = - \sum_{i,j=1}^k (\zeta^\alpha)_{ij} (\zeta^\alpha)_{ji} F_{ij}(x) + \cdots, \tag{5.30}
\]

where

\[
F_{ij}(x) = \frac{1}{4} \square \frac{x^4}{(x^2 + \rho_i^2)(x^2 + \rho_j^2)} \tag{5.31}
\]

and the omitted terms in Eq. (5.30) represent dependence on the other Grassmann modes \( \{ \mathcal{M}', \nu, \bar{\nu} \} \). It is obvious from Eq. (5.31) that \( F_{ij}(0) = 0 \), so that the \( \zeta \) modes must be entirely saturated from the \( k^2 \) insertions at \( x_i = x \) as claimed above. Performing the \( \zeta \) integrations then yields

\[
(-1)^{k^2} (k^2)! \prod_{i,j=1}^k F_{ij}(x). \tag{5.32}
\]

Next we consider the insertions at \( x_i = 0 \). Focusing on the \( \mathcal{M}' \) modes first, one finds from Eq. (5.21):

\[
\text{tr}_N \chi^2(0) = 2 \sum_{i,j=1}^k (\mathcal{M}'^\alpha)_{ij} (\mathcal{M}'^\alpha)_{ji} (\rho_i^{-4} + \rho_j^{-4} + \rho_i^{-2} \rho_j^{-2}) + \cdots, \tag{5.33}
\]

omitting the \( \nu \times \bar{\nu} \) terms. Hence the \( \mathcal{M}' \) integrations yield

\[
\left( \frac{kN - k^2}{k^2} \right) (k^2)! 2^{k^2} \prod_{i,j=1}^k (\rho_i^{-4} + \rho_j^{-4} + \rho_i^{-2} \rho_j^{-2}), \tag{5.34}
\]
where the combinatoric factors in (5.34) (as well as in (5.32)) have been explained previously.\footnote{One can easily check that these large-$N$ formulae are consistent with the explicit 1-instanton calculation presented in Sec. II which is exact in $N$. In particular, if one takes $x_i = x$ while $x_j = 0$, then Eqs. (2.5b)-(2.5c) are suppressed vis-à-vis Eq. (2.5a) by factors of $a_n$, and in turn, $a_n \sim N^{-1/2}$ as follows from Eq. (5.36) below.}

Finally we turn to the $\{\nu, \bar{\nu}\}$ integrations. Since (unlike the $\zeta$ and $M'$ modes) the number of $\nu$ and $\bar{\nu}$ modes grows with $N$ as $kN - 2k^2$, it does not suffice merely to plug in the saddle-point values (5.24a)-(5.24b) and (5.29). One must also calculate the Gaussian determinant about the saddle-point, which provides an $O(N^0)$ multiplicative contribution to the answer. Accordingly we expand about (5.24a)-(5.24b) to quadratic order in the $a_n'$. The $\nu \times \bar{\nu}$ contribution to $\text{tr}_N \lambda^2(0)$ has the form

$$
-\frac{1}{2}p_{ju}\nu_{ui}\square f_{ij}\bigg|_{x=0} = 2\nu_{ju}\nu_{ui}\left(f \cdot \text{tr}_2 \bar{b} \mathcal{P} b \cdot f\right)_{ij}\bigg|_{x=0}
$$

as follows from Eqs. (5.21)-(5.22), and Eq. (2.63) of [12]. Performing the $\{\nu, \bar{\nu}\}$ integrations therefore gives

$$(kN - 2k^2)! \exp\left((N - 2k)\text{tr}_k \log \left(2f \cdot \text{tr}_2 \bar{b} \mathcal{P} b \cdot f\right)\right)_{x=0} = (kN - 2k^2)! \exp\left((N - 2k)\left(\log \det_k 16(W^0)^{-2} - \frac{3}{2} \sum_{i,j=1}^{k} \sum_{n=1}^{4} a_{nij}' a_{nji}' \left(\rho_i^{-2} + \rho_j^{-2}\right) + O(a_n'^4)\right)\right).$$

The negative sign in front of the quadratic term in $a_n'$ confirms that our saddle-point (5.24a)-(5.24b) is in fact a minimum of the action. Combining this expression with the measure factor in Eq. (5.18), namely

$$(\det_{2k} W)^{N-2k} = \exp\left((N - 2k) \log \det_{2k} W\right) = \exp\left((N - 2k)\left(\log \det_k (\frac{1}{2}W^0)^2 + O(a_n'^4)\right)\right),$$

and performing the Gaussian integrations over $a_n'$, yields:

$$2^{2k(N-2k)} (kN - 2k^2)! \prod_{i,j=1}^{k} \left(\frac{2\pi}{3N(\rho_i^{-2} + \rho_j^{-2})}\right)^2 + \cdots,$$

where the omitted terms are suppressed by powers of $N$.

Finally one combines Eqs. (5.18), (5.28), (5.32), (5.34) and (5.38) to obtain the leading-order result for the correlator:

$$\lim_{N \to \infty} G_{kN} = \frac{2^{5kN+k^2-k+1/2} \pi^{2kN-k+1/2} e^{kN} (k^2)! k^{kN-k^2+1/2} \mathcal{I}_k \Lambda^{3kN}}{3^{2k^2} N^{kN+k^2+1/2} k!}, \quad (5.39)$$
where $I_k$ is the convergent integral

$$I_k = \int_{0}^{\infty} d\rho_1^2 \cdots d\rho_k^2 \prod_{1 \leq i < j \leq k} (\rho_i^2 - \rho_j^2)^2 \cdot \prod_{i,j=1}^{k} F_{ij}(x) \left(1 - \frac{\rho_j}{\rho_i} + \rho_i/\rho_j\right)^{-2}.$$  \hspace{1cm} (5.40)

Note that $I_k$ is independent of $x$ as a simple rescaling argument confirms. For the simple case $k = 1$, the $(\rho_i^2 - \rho_j^2)^2$ terms in this integral are absent; one finds $I_1 = \frac{3}{2}$ and the expression (5.39) agrees—as it must—with the large-$N$ limit of the 1-instanton SCI result (2.11).

**VI The 4-Point Function of the Gluino Condensate in $SU(2)$ Gauge Theory**

We have seen that cluster decomposition fails (both with and without a KS vacuum) in the SCI calculation of the gluino condensate, for gauge group $SU(N)$ in the large-$N$ limit. In this section we focus instead on the gauge group $SU(2)$. In this case, at the 1-instanton level, the 2-point function (1.2) works out to:

$$\left\langle \frac{\text{tr} \lambda^2(x_1)}{16\pi^2} \frac{\text{tr} \lambda^2(x_2)}{16\pi^2} \right\rangle = \frac{4}{5} \Lambda^6.$$  \hspace{1cm} (6.1)

Here we will calculate the 4-point function, which receives a nonzero contribution at the 2-instanton level:

$$\left\langle \frac{\text{tr} \lambda^2(x_1)}{16\pi^2} \frac{\text{tr} \lambda^2(x_2)}{16\pi^2} \frac{\text{tr} \lambda^2(x_3)}{16\pi^2} \frac{\text{tr} \lambda^2(x_4)}{16\pi^2} \right\rangle = c \Lambda^{12}.$$  \hspace{1cm} (6.2)

In the absence of a KS vacuum, generalized cluster decomposition together with Eq. (6.1) predicts $c = (4/5)^2 = .64$. Alternatively, in the presence of a KS vacuum, weighted according to Eq. (1.3) in order to reconcile the SCI and WCI 1-instanton calculations, one expects $c = 4/5 = .8$. Instead, we have calculated $c$ numerically, and find:

$$c \approx .500 \pm .026.$$  \hspace{1cm} (6.3)

Here are the details of the calculation.

As mentioned above, for $k = 2$, one can eliminate the $\delta$-function constraints in Eq. (5.9) without changing variables. Another simplification for the particular gauge group $SU(2)$ is that one can adopt a concise quaternionic representation for the ADHM bosonic collective coordinates, taking advantage of the fact that $SU(2) \cong Sp(1)$. Specifically, the 16 gauge and
8 gaugino collective coordinates live, respectively, in the following matrices:

\[
\begin{pmatrix}
  a_1 & w_1 \\
  a_2 & w_2 \\
  a_1' & a_2' \\
  a_2' & a_2'
\end{pmatrix}, \quad
\begin{pmatrix}
  \mu_1 \gamma & \mu_2 \gamma \\
  \mathcal{M}_{11} & \mathcal{M}_{12} \\
  \mathcal{M}_{12} & \mathcal{M}_{22}
\end{pmatrix},
\]

(6.4)

where \( a = a_{\alpha \bar{\alpha}} = a_{\alpha} \sigma^\alpha_{\alpha \bar{\alpha}} \) and the matrices \( a_{n} \) as well as \( \mathcal{M}_{\gamma} \) are real-valued (unlike the complex-valued collective coordinates of the same name introduced in Eq. (5.6) which are needed for general \( SU(N) \)). The resulting 2-instanton “Osborn measure” on these collective coordinates is detailed in Refs. [9, 26, 28], and reads:

\[
\int d\mu_{\text{phys}}^2 = 2^{14} \Lambda^6 \int d^4 w_1 d^4 w_2 d^4 a_1' d^4 a_2' d^2 \mu_1 d^2 \mu_2 d^2 \mathcal{M}_{11} d^2 \mathcal{M}_{22} \left| \frac{a_3'}{a_3'} - \left| a_{12}' \right|^2 \right|. \quad (6.5)
\]

Here the \( \delta \)-function constraints from the flat measure have been used to eliminate \( a_{12}' \) and \( \mathcal{M}_{12}' \) in terms of the other collective coordinates, via:

\[
a_{12}' = \frac{1}{4|a_3'|^2} a_3' (\bar{w}_2 w_1 - \bar{w}_1 w_2),
\]

(6.6)

and

\[
\mathcal{M}_{12}' = \frac{1}{2|a_3'|^2} a_3' \left( 2\bar{a}_{12}' \mathcal{M}_3' + \bar{w}_2 \mu_1 - \bar{w}_1 \mu_2 \right),
\]

(6.7)

where we have defined

\[
a_3' = \frac{1}{2} (a_{11}' - a_{22}'), \quad \mathcal{M}_3' = \frac{1}{2} (\mathcal{M}_{11}' - \mathcal{M}_{22}').
\]

(6.8)

Into this measure one inserts the 4-point function of the classical condensate, expressed as a function of the 2-instanton collective coordinates (6.4). The 8-dimensional Grassmann integrations over \( \{ \mu_1, \mu_2, \mathcal{M}_{11}', \mathcal{M}_{22}' \} \) are then accomplished in two steps. The first step is to expand the integrand in terms of Grassmann variables using a modified version of the program “Dill”, written for Mathematica.\(^{14}\) The second step involves the explicit Grassmann integration, accomplished using an “awk-script” implemented on a UNIX system and made to perform the symbolic algebra of Grassmann integration.

The resulting 16-dimensional bosonic integration over \( \{ w_1, w_2, a_{11}', a_{22}' \} \), the remaining quaternionic variables, is carried out using a standard Monte Carlo integration procedure. The integrable singularities are handled using the standard procedure: firstly, dropping a tiny region around the integrable singularities and then making sure that the contribution from

\(^{13}\)See Ref. [6] for details of notation and conventions pertinent to Sec. VI.

\(^{14}\)“Dill” is a Mathematica package originally written by Vladan Lucic in 1994 in order to simplify SUSY algebraic expressions. This program can be modified so that it can handle the large number of Grassmann variables that we need.
this dropped region is negligibly smaller than the precision required. After 450 million points have been sampled, we have obtained the numerical value \(6.3\) given above. As a check on our numerics, we have also verified the constancy of the answer by comparing different choices for the four space-time insertion points.

VII Discussion

The mismatch between the strong coupling and weak coupling calculations is a fascinating puzzle. Previously, only the mismatch at the one instanton level was known; now we see a mismatch established at large \(N\) for all instanton numbers, and for \(N = 2\) at the 2-instanton level. Certainly we do not mean to imply that, because of this mismatch, SCI calculations are all necessarily suspect; indeed an \(\mathcal{N} = 4\) supersymmetric version of an SCI calculation performed by some of us \[12\] has recently provided a dramatic quantitative and qualitative verification of Maldacena’s conjecture. However in this case the coupling does not run and the calculation can be performed at weak coupling, actually small \(g^2N\), where the instanton approximation is fully justified. The continuation to strong coupling, large \(g^2N\), is then accomplished by means of a non-renormalization theorem. Rather, our objections to the SCI computation are more narrow and technical in scope: specifically, our calculations imply a fundamental breakdown of clustering in the instanton approximation to the gluino condensate at strong coupling.

One may wonder what the origin for this breakdown is? The usual justification for the strong coupling calculation is that one can take \(|x_i - x_j|\) much smaller than the scale of strong coupling effects \(\Lambda^{-1}\) and so the theory would be weakly coupled, due to asymptotic freedom, and the instanton calculation would be justified. Then, since the correlation functions (4.2) are independent of the positions, the result would be valid at all distances. This point-of-view has simultaneously been used and criticized by various authors \[2, 7, 8\]. The asymptotic freedom argument means that the first-order, second-order, etc., perturbative corrections to the SCI calculations are small—indeed, for the gluino condensate correlators discussed herein, these perturbative corrections are entirely absent due to a non-renormalization theorem \[7\]. However, the asymptotic freedom argument does not guarantee that the zeroth order instanton calculation is itself complete; there may be other non-perturbative configurations contributing to the correlators. Indeed, the breakdown of cluster suggests that such additional nonperturbative configurations (with size of order \(\Lambda^{-1}\)) must be present, and that they must account for the mismatch between the SCI and WCI calculations.
In contrast, it seems that the WCI calculation uses a method that has amassed a considerable pedigree. These kinds of calculations appear to be consistent in all applications and agree with other non-instanton methods \[20\]; for example, the two-instanton check of the Seiberg-Witten approach to $\mathcal{N} = 2$ theories \[9, 10\] and the latter calculation in Sec. III. Moreover, in the WCI set-up, large-scale nonperturbative configurations as just discussed, such as instanton-antiinstanton pairs of size $\Lambda^{-1}$, would be exponentially suppressed in the path integral so long as $\Lambda \ll v$. We should further note that, as the separation between insertions tends to zero, the WCI calculation does not smoothly go over to the SCI calculation as one would naively expect; there are additional important contributions which will be discussed in a separate publication (work in progress).

It is unfortunate that, based on our results, the highly original and intriguing (both theoretically and phenomenologically) proposal of Shifman, namely the existence of a chirally symmetric vacuum state, loses much of its reason d’être. Then again, we have not actually ruled out the existence of such a state. After the completion of this work, it has been suggested that the mixing parameter $p$ of the KS vacuum, defined in Eq. (4.15) above, may actually be instanton number dependent \[31\]. Prima facie, this appears to be incompatible with invariance under large gauge transformations ($|k\rangle \rightarrow |k+1\rangle$); however, if such a counterintuitive flexibility is permissible in the definition of the instanton vacuum, clustering in the presence of the KS vacuum can be saved.

Conceptual difficulties with the instanton approximation and cluster decomposition were pointed out in the context of pure (non-supersymmetric) QCD some time ago. Since this may have some bearing on the present discussion, we review some comments of Lüscher regarding this issue \[32\]. The pure instanton (i.e. no anti-instantons) approximation to QCD obviously violates parity since

$$\langle \text{tr} F_{nm} * F_{nm} \rangle_{\text{inst.}} = \rho \neq 0 ,$$

where $\rho$ here is the instanton density. Parity is then recovered by summing over instantons (I) and anti-instantons ($\bar{I}$); however, in this approximation the cluster property would not hold. To see this note that

$$\langle \text{tr} F_{nm} * F_{nm}(x) \text{tr} F_{pq} * F_{pq}(0) \rangle \rightarrow_{|x| \rightarrow \infty} \rho^2 \neq 0 ,$$

whereas

$$\langle \text{tr} F_{nm} * F_{nm} \rangle = \frac{1}{2} (\langle \text{tr} F_{nm} * F_{nm} \rangle_{\text{inst.}} + \langle \text{tr} F_{nm} * F_{nm} \rangle_{\text{anti-inst.}} ) = 0 .$$

In order to resolve this clustering conundrum, it is apparent that additional configurations, which may, in the dilute gas limit, be thought of as mixtures of instantons and anti-instantons,
would need to be incorporated in the approximation. In this case the two-point function (7.2) would indeed be zero, the result of summing the II, I, I, and I contributions which on average all contribute equally. Away from the dilute instanton gas limit, the identification and physical interpretation of these additional cluster-restoring nonperturbative configurations is necessarily more subtle.

In summary, the results of this paper imply something analogous in the \( \mathcal{N} = 1 \) theory: additional configurations must contribute to the correlators at strong coupling and resolve the breakdown of clustering (as well as repairing the mismatch between the SCI and WCI calculations). In fact, it was suspected some time ago (see Refs. [28,33] and references therein) that in strongly coupled theories, it may be more appropriate to think of instantons as composite configurations of some more basic objects: so-called “instanton partons”. The dominant contributions to the path integral at strong coupling would then arise from the partons themselves. In Ref. [13], we make this piece of folklore more precise by identifying instanton partons with the monopole configurations of the supersymmetric Yang-Mills compactified on the cylinder \( \mathbb{R}^3 \times S^1 \), with the circle having circumference \( \beta \). Each monopole has precisely two gluino zero modes, rather than four for the instanton. The instanton itself is then identified with a specific two-monopole configuration. We calculate the monopole contribution to the gluino condensate and then, at the end of the day, take the decompactification limit \( \beta \to \infty \). The value of the gluino condensate obtained in this way, is precisely the WCI result (1.4).

Acknowledgments

We thank Philippe Pouliot, Larry Yaffe, Alex Kovner, Misha Shifman and Nick Dorey for valuable discussions. VVK and MPM acknowledge a NATO Collaborative Research Grant. TJH and VVK acknowledge the TMR network grant FMRX-CT96-0012.

Appendix A: Constancy of the Correlators

Since the constancy of the correlation function (1.2) plays such an important role in our analysis, in this appendix, we review the arguments leading to this result. More importantly, we explain how the field theory proof remains valid in the instanton approximation.
First the field theory proof [1, 2]. The argument is completely general and applies to the correlation functions of any lowest component \( A \) of a gauge invariant chiral superfield \( \Phi \):

\[
\Phi = A(x) + \sqrt{2}\theta \psi(x) + \cdots .
\]  

(A.1)

In the present discussion, the operator \( \text{tr} \lambda^2 \) is the lowest component of the chiral superfield \( \text{tr} W^\alpha W_\alpha \), where \( W_\alpha \) is supersymmetric field strength. Consider the correlation function

\[
\langle A_1(x_1) \cdots A_p(x_p) \rangle .
\]  

(A.2)

We will show that this is independent of the \( x_i \)'s. To this end, one has

\[
\frac{\partial}{\partial x_i} A_i(x) = i \frac{\bar{\sigma}^{\dot{\alpha} \alpha}}{4} \{ \bar{Q}_{\dot{\alpha}}, \psi_{\alpha}(x) \} .
\]  

(A.3)

Hence

\[
\frac{\partial}{\partial x_i^n} \langle A_1(x_1) \cdots A_p(x_p) \rangle
\]

\[
= i \frac{\bar{\sigma}^{\dot{\alpha} \alpha}}{4} \langle 0 | A_1(x_1) \cdots A_{i-1}(x_{i-1}) \{ \bar{Q}_{\dot{\alpha}}, \psi_{\alpha}(x_i) \} A_{i+1}(x_{i+1}) \cdots A_p(x_p) | 0 \rangle
\]

\[
- i \frac{\bar{\sigma}^{\dot{\alpha} \alpha}}{4} \sum_{j=1}^{i-1} \langle 0 | A_1(x_1) \cdots [ \bar{Q}_{\dot{\alpha}}, A_j(x_j) ] \cdots A_{i-1}(x_{i-1}) \psi_{\alpha}(x_i) A_{i+1}(x_{i+1}) \cdots A_p(x_p) | 0 \rangle
\]

\[
+ i \frac{\bar{\sigma}^{\dot{\alpha} \alpha}}{4} \sum_{j=i+1}^{p} \langle 0 | A_1(x_1) \cdots A_{i-1}(x_{i-1}) \psi_{\alpha}(x_i) A_{i+1}(x_{i+1}) \cdots [ \bar{Q}_{\dot{\alpha}}, A_j(x_j) ] \cdots A_p(x_p) | 0 \rangle ,
\]  

(A.4)

where the last line follows by commuting the \( \bar{Q}_{\dot{\alpha}} \) through the other insertions, to the left and right, respectively, until it hits the vacuum which it annihilates. But \( \{ \bar{Q}_{\dot{\alpha}}, A_j(x) \} = 0 \) and therefore the right-hand side of (A.4) vanishes and consequently the correlation function is, indeed, independent of the insertion points. QED.

If the multiple correlator of \( \text{tr} \lambda^2 \) is constant in the full field theory, it then becomes an issue as to whether this constancy is retained in the instanton approximation. That it is, rests upon two facts. Firstly, the supersymmetry transformations of the fields can be traded for supersymmetry transformations of the collective coordinates [1, 10, 11, 26, 27]. In other words, the supersymmetry algebra is represented on the collective coordinates. Specifically, under an infinitesimal supersymmetry transformation \( \xi Q + \bar{\xi} \bar{Q} \):

\[
\delta a_{\dot{\alpha}} = \bar{\xi}_{\dot{\alpha}} \mathcal{M}, \quad \delta a^{\dot{\alpha}} = -\mathcal{\bar{M}} \bar{\xi}^{\dot{\alpha}}
\]  

(A.5a)

\[
\delta \mathcal{M} = -4i\delta^\alpha \xi_{\alpha}, \quad \delta \mathcal{\bar{M}} = 4i\xi^\alpha \bar{b}_{\alpha} .
\]  

(A.5b)

In particular, (A.3) will hold, with the fields replaced by their expression in the instanton background and with the right-hand side involving the appropriate transformation of the
collective coordinates. *Ipso facto*, the argument leading to (A.4) will hold with the transformations acting on the collective coordinates; moreover $[\bar{Q}_\alpha, A] = 0$, understood as a transformation of the collective coordinates. The remaining piece of the proof is the analogue of the fact that $\bar{Q}_\alpha$ annihilates the vacuum state. In the instanton approximation, where the functional integral is approximated by the integral over the collective coordinates, the analogue of the statement that the vacuum is a supersymmetry invariant, is the statement that the measure on the space of collective coordinates is invariant under the supersymmetry transformations (A.5a)-(A.5b). This invariance was proved in \[26, 27\].

References

[1] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, “Instanton Effects In Supersymmetric Theories,” Nucl. Phys. B229, 407 (1983).

[2] G.C. Rossi and G. Veneziano, “Nonperturbative Breakdown Of The Nonrenormalization Theorem In Supersymmetric QCD,” Phys. Lett. 138B, 195 (1984).

[3] D. Amati, G.C. Rossi and G. Veneziano, “Instanton Effects In Supersymmetric Gauge Theories,” Nucl. Phys. B249, 1 (1985).

[4] D. Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano, “Nonperturbative Aspects In Supersymmetric Gauge Theories,” Phys. Rept. 162, 169 (1988).

[5] J. Fuchs and M.G. Schmidt, “Instanton Induced Green Functions In The Superfield Formalism,” Z. Phys. C30, 161 (1986).

[6] I. Affleck, M. Dine and N. Seiberg, “Supersymmetry Breaking By Instantons,” Phys. Rev. Lett. 51, 1026 (1983); “Dynamical Supersymmetry Breaking In Supersymmetric QCD,” Nucl. Phys. B241, 493 (1984).

[7] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, “Supersymmetric Instanton Calculus (Gauge Theories With Matter),” Nucl. Phys. B260, 157 (1985).

[8] M.A. Shifman and A.I. Vainshtein, “On Gluino Condensation In Supersymmetric Gauge Theories. SU(N) And O(N) Groups,” Nucl. Phys. B296, 445 (1988).

[9] N. Dorey, V.V. Khoze and M.P. Mattis, “Multi-instanton calculus in N=2 supersymmetric gauge theory,” Phys. Rev. D54, 2921 (1996) [hep-th/9603136].

[10] N. Dorey, V.V. Khoze and M.P. Mattis, “Multi-instanton calculus in N=2 supersymmetric gauge theory. 2. Coupling to matter,” Phys. Rev. D54, 7832 (1996) [hep-th/9607202].
[11] V.V. Khoze, M.P. Mattis and M.J. Slater, “The Instanton Hunter’s guide to supersymmetric SU(N) gauge theories,” Nucl. Phys. B536, 69 (1998) hep-th/9804009.

[12] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, “Multi-instanton calculus and the AdS / CFT correspondence in N=4 superconformal field theory,” Nucl. Phys. B552, 88 (1999) hep-th/9901128. “Multiinstantons and Maldacena’s conjecture,” JHEP 06, 023 (1999) hep-th/9810243.

[13] N.M. Davies, T.J. Hollowood, V.V. Khoze and M.P. Mattis, “Gluino Condensate and Magnetic Monopoles in SUSY Gluodynamics,” hep-th/9905015. Nucl. Phys. B (to appear).

[14] C. Csaki and H. Murayama, “Discrete anomaly matching,” Nucl. Phys. B515, 114 (1998) hep-th/9710105.

[15] I.I. Kogan, A. Kovner and M. Shifman, “More on supersymmetric domain walls, N counting and glued potentials,” Phys. Rev. D57, 5195 (1998) hep-th/9712046.

[16] A. Kovner and M. Shifman, “Chirally symmetric phase of supersymmetric gluodynamics,” Phys. Rev. D56, 2396 (1997) hep-th/9702174.

[17] M. Shifman and A. Vainshtein, “Instantons versus supersymmetry: Fifteen years later,” hep-th/9902018.

[18] N. Seiberg and E. Witten, “Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory,” Nucl. Phys. B426, 19 (1994) hep-th/9407087.

[19] C. Bernard, “Gauge Zero Modes, Instanton Determinants, And QCD Calculations,” Phys. Rev. D19, 3013 (1979).

[20] D. Finnell and P. Pouliot, “Instanton calculations versus exact results in four-dimensional SUSY gauge theories,” Nucl. Phys. B453, 225 (1995) hep-th/9503115.

[21] M.E. Peskin, “Duality in supersymmetric Yang-Mills theory,” hep-th/9702094.

[22] S.F. Cordes, “The Instanton Induced Superpotential In Supersymmetric QCD,” Nucl. Phys. B273, 629 (1986).

[23] A.C. Davis, M. Dine and N. Seiberg, “The Massless Limit Of Supersymmetric QCD,” Phys. Lett. 125B, 487 (1983).

K. Intriligator, R.G. Leigh and N. Seiberg, “Exact superpotentials in four-dimensions,” Phys. Rev. D50, 1092 (1994) hep-th/9403198.
K. Intriligator and N. Seiberg, “Lectures on supersymmetric gauge theories and electric-magnetic duality,” Nucl. Phys. Proc. Suppl. **45BC**, 1 (1996) [hep-th/9509066].

[24] N.N. Bogolubov, A.A. Logunov and I.T. Todorov, “Introduction to Axiomatic Quantum Field Theory,” Benjamin 1975.

[25] E. Witten, “Constraints On Supersymmetry Breaking,” Nucl. Phys. **B202**, 253 (1982).

[26] N. Dorey, V.V. Khoze and M.P. Mattis, “Supersymmetry and the multi-instanton measure,” Nucl. Phys. **B513**, 681 (1998) [hep-th/9708036].

[27] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, “Supersymmetry and the multi-instanton measure II: From $N = 4$ to $N = 0$,” Nucl. Phys. **B519**, 470 (1998) [hep-th/9709072].

[28] H. Osborn, “Semiclassical Functional Integrals For Selfdual Gauge Fields,” Ann. Phys. **135**, 373 (1981).

[29] G. ’t Hooft, “Computation Of The Quantum Effects Due To A Four-Dimensional Pseudoparticle,” Phys. Rev. **D14**, 3432 (1976); *ibid.* **D18**, 2199 (1978).

[30] V. Lucic, “DILL: An Algorithm and a symbolic software package for doing classical supersymmetry calculations,” Comp. Phys. Commun. **92**, 90 (1995), [hep-ph/9412298].

[31] M.A. Shifman, *private communication*.

[32] M. Luscher, “Exact Instanton Gases,” *In: Cargese 1979, Proceedings, Recent Developments In Gauge Theories*, 205-216.

[33] A.A. Belavin, V.A. Fateev, A.S. Schwarz and Y.S.Tyupkin, “Quantum Fluctuations Of Multi - Instanton Solutions,” Phys. Lett. **B83B**, 317 (1979).