Value Functions for Depth-Limited Solving in Zero-Sum Imperfect-Information Games

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Abstract

Depth-limited look-ahead search is an essential tool for agents playing perfect-information games. In imperfect information games, the lack of a clear notion of a value of a state makes designing theoretically sound depth-limited solving algorithms substantially more difficult. Furthermore, most results in this direction only consider the domain of poker. We consider two-player zero-sum extensive form games in general. We provide a domain-independent definitions of optimal value functions and prove that they can be used for depth-limited look-ahead game solving. We prove that the minimal set of game states necessary to define the value functions is related to common knowledge of the players. We show the value function may be defined in several structurally different ways. None of them is unique, but the set of possible outputs is convex, which enables approximating the value function by machine learning models.

1 Introduction

Depth-limited look-ahead game solving with heuristic evaluation function recently lead to beating human professionals in go [Silver et al., 2017] and poker [Moravčík et al., 2017]. While for perfect information games, it was a well established methodology since the very beginning [Shannon, 1950], creating a theoretically sound depth-limited solving in imperfect-information games is substantially more difficult. One key complication is that, in imperfect-information games, it is not possible to define a value of a single state of a game in isolation. Instead, the value depends on the other states that the players consider likely in a particular situation, on the states that they think their opponent considers likely, and so on.

Existing applications of theoretically sound depth-limited look-ahead game solving are limited to the game of poker. Depth-limited continual resolving in DeepStack [Moravčík et al., 2017] enabled human-speed expert-level game play with computing resources commonly available in a commodity laptop. Brown et al. [2018] used depth-limited solving to create a poker bot, which outperforms the strongest submissions from Annual Computer Poker Competition and can be build form scratch on a common desktop. Alternative approaches for creating top-performing poker bots usually require a supercomputer [Bowling et al., 2015; Brown and Sandholm, 2017].

This success of depth-limited game solving in an imperfect-information game has never been replicated beyond the Texas hold’em poker domain. Texas hold’em has a very particular structure of the available information. After the initial cards are dealt, all the following actions of the players, as well as chance, are perfectly observable. As a result, the amount of hidden information is constant throughout the game. Furthermore, since the players’ moves alternate, it is always clear...
We focus on two-player zero-sum extensive-form games with imperfect information. Based on [Osborne and Rubinstein, 1994], game $G$ can be described by

- $H$ – the set of **histories**, representing sequences of actions.
- $Z$ – the set of terminal histories (those $z \in H$ which are not a prefix of any other history).
- $N$ – the set of actions (possible outputs from a node).
- $\epsilon$ – the set of terminal histories.
- $\sigma$ – the set of strategies.

For a non-terminal history $h \in H \setminus Z$, $A(h) := \{a \in N | ha \in H\}$ is the set of actions available at $a$.

- $P : H \setminus Z \rightarrow N \cup \{c\}$ is the **player function** partitioning non-terminal histories into $H_p$, $p = 1, \ldots, N, c$ depending on which player acts at $h$.

- The **utility function** $u = (u_p)_{p \in N}$ assigns to each terminal history $z$ a reward $u_p(z) \in \mathbb{R}$ received by each player upon reaching $z$.

- The **information-partition** $I = (I_p)_{p \in N}$ captures the imperfect information of $G$. For each player, $I_p$ is a partition of $H_p$. If $g, h \in H_p$ belong to the same $I \in I_p$ then $p$ cannot distinguish between them. For each $I \in I_p$, the available actions $A(h)$ are the same for each $h \in I$, and we overload $A(\cdot)$ as $A(I) := A(h)$. We only consider games with **perfect recall**, where the players remember their past actions and the information sets visited so far.

We focus on two-player **zero-sum games**, where $N = \{1, 2\}$ and $u_2 = -u_1$. However, this assumption is only truly necessary for results related to optimal game values (Section 5).

A **behavioral strategy** $\sigma_p \in \Sigma_p$ of player $p$ assigns to each $I \in I_p$ a probability distribution $\sigma_p(I)$ over available actions $a \in A(I)$. A tuple $\sigma = (\sigma_p)_{p \in N}$ is called a **strategy profile**.

The **expected utility** for player $i$ of a strategy profile $\sigma$ is $u_p(\sigma) := E_{z \sim \sigma} u_p(z)$. The profile $\sigma$ is an $\epsilon$-**Nash equilibrium** if the benefit of switching to some alternative $\sigma'_p$ is limited by $\epsilon$, i.e.,
Finally, player \( p \).

We now extend this notation to \( I \).

We start by introducing the notation related to reach probabilities in EFGs. The (counterfactual) probability of reaching \( \eta \) is the average of exploitabilities \( \exp_1(\sigma, p) \), \( p = 1, 2 \), where

\[
\exp_p(\sigma) := u_p(\sigma) - \min_{\sigma_{-p} \in \Sigma_{-p}} u_p(\sigma, \sigma_{-p}).
\]

2.2 Augmented information sets and subgames

To speak about the information available to player \( i \) in histories where he doesn’t act, we will use the augmented sets \( I^\text{aug} \) introduced in [Burch et al., 2014] and note that in general EFGs, augmented information sets can be “thick” i.e. they can contain both some \( h \in H \) and its parent \( g \neq h \). Since it is straightforward to circumvent this issue by only considering the upper frontier ([Halpern and Pass, 2016; Halpern, 1997]) of each \( I \), we ignore the issue by assuming none of the discussed domains contains thick augmented information sets.

Public states are super-sets of elements of \( I^\text{aug} \) that capture the information that everybody knows:

**Definition 2.** Let \( g \sim h \iff \bar{O}_1(g) = \bar{O}_1(h) \lor \cdots \lor \bar{O}_N(g) = \bar{O}_N(h) \). A **public tree** is any partition \( S \) of \( H \) that forms a tree and is closed under \( \sim \). Its elements are called **public states**.

Using the concept of public states, [Sustr et al., 2019] define a subgame in EFGs as follows.

**Definition 3.** A **subgame** \( G(S) \) rooted at a public state \( S \) in \( S \) is the set \( \{ h \in H \mid \exists g \in S : g \sim h \} \).

3 Revisiting reach probabilities and values

In this section, we take a closer look at reach probabilities, list particularly useful versions of this concept, and use them to give several equivalent definitions of the notion of range. We also clarify the properties of and the relationship between expected utilities and counterfactual values.

3.1 Reach probabilities

We start by introducing the notation related to reach probabilities in EFGs. The **reach probability** of a history \( h \in H \) under a strategy \( \sigma \) is defined as \( \eta(h) \) the product of probabilities of taking each action between the root and \( h \). The **counterfactual reach probability** \( \eta^c(h) \) is defined analogously, except that player \( p \) now aims to reach \( h \), so the probabilities of his actions are replaced by \( 1 \). Finally, player \( p \)’s **reach probability** \( \eta^p(h) \) is the probability that they will play to reach \( h \), i.e. the quantity missing in \( \eta^c(h) \) (so we have \( \eta(h) = \eta^p(h)\eta^{c-p}(h) \)).

We now extend this notation to \( I^\text{aug} \). Note that naively defining \( \eta^c(I) := \sum_{h \in A} \eta^c(h) \) for general \( A \subset H \), such as \( \eta^c(I) \) for an information set \( I \) not belonging to \( p \), will often result in “reach probabilities” that behave nothing like actual probabilities (e.g. having values over 1).

The non-problematic part is the (counterfactual) probability of reaching \( I \) in \( I^\text{aug} \), which we simply denote as \( \eta^c(I) := \sum_{h \in I} \eta^c(h) \), resp. \( \eta^p(I) := \sum_{h \in I} \eta^p(h) \). Some care needs to be taken with \( \eta^p(I) \): rather than defining it as the sum over \( h \in I \), we set \( \eta^p(I) := \eta^p(h_1) \), where \( h_1 \in I \) is chosen arbitrarily (since \( I \in I^\text{aug}, \eta^c(h) \) is the same for all \( h \in I \)). This ensures that \( \eta^c(I) = \eta^c(h)\eta^p(h_1) \).

† However, our results would still hold if we replaced \( I^\text{aug} \) by a different “augmented information partition”, i.e. a partition \( \mathcal{J}_p \) of \( H \) which coincides with \( I_p \) on \( H_p \) and for which \( (\mathcal{J}_p, \subset) \) forms a (topological) tree w.r.t. the partial order \( I \subset J \iff \exists g \in I, h \in J : g \subset h \).
For \( g \sqsubseteq h \), \( \eta^g_h(g, h) \) denotes the probability of reaching \( h \) conditional on already being in \( g \). We compute it analogously to \( \eta^g_h(h) \), except that we only take the product of probabilities of actions between \( g \) and \( h \). For \( h \in I \sqsubseteq J \) (where \( I, J \in \mathcal{I}^{\text{aug}}_p \)) we define the conditional reach probabilities as \( \eta^g_h(I, J) := \eta^g_h(J)/\eta^g_h(I) \) and denote

\[
\eta^g(h|I) := \frac{\eta^g_h(h)}{\eta^g_h(I)} = \frac{\eta^g_{\sigma_p}(h)}{\eta^g_{\sigma_p}(I)} =: \eta^g_{\sigma_p}(h|I).
\] (1)

While the denominator in \( \eta^g_h(I, J) \) and (1) can be 0, we can extend both notions even to unreachable \( I \) (Section A.1).

3.2 Range in general imperfect information games

In poker, range can be (informally and ambiguously) defined as “the likelihood of having various private cards in a given situation”. However, a useful formalization of this concept isn’t obvious, in particular in EFGs other than poker. One option would be to set “range of \( p \) at \( S := (\eta^g_h(I)/\eta^g(S))_{I \in \mathcal{I}^{\text{aug}}_p} \)” — in other words, “the probability of being in each \( I \), conditional on being in the current public state”. However, this definition depends on the strategy of the opponent, and thus a player wouldn’t know their own range. (Indeed, just consider the difference between an opponent who plays randomly and an opponent who will always fold unless he holds two aces.)

**Proposition 4** (Equivalent definitions of range). With the knowledge of \( \sigma_1 \) and the game rules, we can calculate the following range-like variables for \( S \in \mathcal{S} ((4) and (5) are two-player games only):

\begin{align*}
(1) & \quad \eta^f(I) \text{ for } I \in \mathcal{I}^{\text{aug}}_1(S) \quad \text{(information set reach probabilities for player 1)}, \\
(2) & \quad \eta^f(h) \text{ for } h \in S \quad \text{(history reach probabilities for player 1)}, \\
(3) & \quad \eta^f(h) \text{ and } \eta^f(z) \text{ for } h \in S \quad \text{(history reach probabilities for player 1 and chance)}, \\
(4) & \quad \eta^f(J) \text{ for } J \in \mathcal{I}^{\text{aug}}_2(S) \quad \text{(the opponent’s counterfactual reach probabilities)}, \\
(5) & \quad \eta^f(h,J) \text{ for } h \in J \in \mathcal{I}^{\text{aug}}_2(S) \quad \text{(conditional probabilities for the opponent’s information sets)}.
\end{align*}

If we forget \( \sigma_1 \), any of the lines (1)-(3) can be used to recover (1)-(5) via a simple calculation.

**Proof.** The “derivations” (3) \( \rightarrow (2) \rightarrow (1) \) and (1) \( \rightarrow (2) \) are trivial, since \( \eta^f(I) \) is defined as \( \eta^f_h(h) \) for arbitrary choice of \( h \in I \) (and \( \eta^f(h) \) is the same for all \( h \in I \in \mathcal{I}^{\text{aug}}_1 \)). \( \eta^f_z(h) \) can always be recovered from the rules of the game, which gives (2) \( \rightarrow (3) \), (3) \( \rightarrow (4) \) holds because \( \eta^f(J) = \sum_{h \in J} \eta^f(h) \eta^f_z(h) \). (3) \( \rightarrow (5) \) holds because \( \eta^f(h,J) = \eta^f(h)/\eta^f(J) = \eta^f_{\sigma_2}(h)/\eta^f_{\sigma_2}(J). \)

The corollary of Proposition 4 is that among (1)-(3), we are free to use whichever option suits us the best — (3) is the most descriptive, while (1) requires the least space to store. This becomes relevant for example when using range as an input for a neural network.

**Definition 5** (Range). For \( \sigma \in \Sigma \), \( \text{rng}_\sigma(S) := (\eta^f_h(h), \eta^f_z(h), \eta^f_c(h))_S \) and \( (\pi^\sigma(h))_S \) denote the **separated** and **joint range** at \( S \). The **compact representation** of a range is the tuple \( (\eta^f_{\sigma_p}(I))_{I \in \mathcal{I}^{\text{aug}}_p(S)} \) for \( p \in \mathcal{N} \).

3.3 Values of non-optimal strategies

We now define standard and counterfactual values for a given, typically sub-optimal, strategy. We start with histories and history-action pairs and, since the players cannot observe histories directly, proceed by extending the notation to augmented information sets and action-infoset pairs. We then list the properties of these values. The use of letters \( v \) and \( q \) is inspired by and consistent with [Srinivasan et al., 2018] and (apart from using capital letters to differentiate between values of infosets and histories).

Player \( p \)'s **value of the history** \( h \in \mathcal{H} \) under \( \sigma \in \Sigma \) is the expected utility conditional on visiting \( h \): \( v^\sigma_p(h) := \mathbb{E}_{z \sim \sigma} [u_p(z) \mid h \text{ reached}] = \sum_{z \ni h} \eta^g(h, z) u_p(z). \)
Analogously, we define the value of an action \( a \in A_p(h) \) taken at \( h \in H_p \) under \( \sigma \):

\[
q_p^\sigma(h, a) := \mathbb{E}_{z \sim \sigma} [u_i(z) \mid h \text{ reached}, a \text{ taken}] = \sum_{z \supseteq h} \eta^\sigma(ha, z)u_i(z).
\]

As in MDPs, we have \( v_p^\sigma(h) := \sum_{a \in A_p(h)} \sigma_p(h, a)q_p^\sigma(h, a) \). A very useful variant of these concepts are the counterfactual values \( \text{Zinkevich et al.} [2008] \), defined as \( v_{p,c}^\sigma(h) := \eta^\sigma_p(h)v_{p,c}^\sigma(h) \) and \( q_{p,c}^\sigma(h) := \eta^\sigma_p(h)q_{p,c}^\sigma(h) \).

To extend the notation for \( I \in \mathcal{T}_p^{\text{avg}} \), we define \( V_p^\sigma(I) \) as the weighted average of \( v_p^\sigma(h) \), \( h \in I \):

\[
V_p^\sigma(I) := \mathbb{E}_{z \sim \sigma} [u_i(z) \mid I \text{ reached}] = \sum_{h \in I} \eta^\sigma(h|I)v_p^\sigma(h).
\]

We define \( V_{p,c}^\sigma(I) := \eta^\sigma_p(I)V_p^\sigma(I) \). The Q-values \( Q_p^\sigma(I, a) \) and \( Q_{p,c}^\sigma(I, a) \) are defined analogously and satisfy \( V_p^\sigma(I) = \sum_{a \in A_p(I)} \sigma_p(I, a)Q_p^\sigma(I, a) \) and \( V_{p,c}^\sigma(I) = \sum_{a \in A_p(I)} \sigma_p(I, a)Q_{p,c}^\sigma(I, a) \).

### 4 Motivation: depth-limited solving in imperfect information games

We now describe the CFR algorithm, recall two depth-limited solving algorithms that build on top of it, and discuss their connection to the present paper. For a strategy \( \sigma \in \Sigma \), \( I \in \mathcal{I} \), and \( a \in A(I) \), we define the counterfactual regret for not playing \( a \) in \( I \) under \( \sigma \) as

\[
r_p^\sigma(I, a) := Q_{p,c}^\sigma(I, a) - V_{p,c}^\sigma(I).
\]

and define the corresponding (cumulative) immediate counterfactual regret as

\[
R_{p,\text{imm}}^T(I) := \max_{a \in A(I)} R_{p,\text{imm}}^T(I, a) := \max_{a \in A(I)} \frac{1}{T} \sum_{t=1}^T r_p^\sigma(I, a).
\]

The Counterfactual Regret minimization (CFR) algorithm \( \text{Zinkevich et al.} [2008] \) works by independently minimizing \( R_{p,\text{imm}}^T(I) \) at each \( I \in \mathcal{I} \). Its average strategy \( \bar{\sigma}^T \) provably converges to a NE \( \text{Lanctot et al.} [2009, \text{Theorem 1}] \). CFR usually traverses the whole game tree in depth-first search in each iteration. On the way from the root to leaves, it computes the reach probabilities of individual players and on the way back, it computes the expected utilities and updates the regrets.

CFR-D \( \text{Burch et al.} [2014] \) is a “decomposition” variant of CFR that computes the NE strategy for only a limited number of first moves in the game, called trunk. In each iteration, it (I) computes the reach probabilities at the end of the trunk, (II) computes the Nash equilibrium in each subgame rooted at the bottom of the trunk given the fixed reach probabilities, (III) computes counterfactual best response values in the root of the subgame (w.r.t the trunk strategy extended by the subgame NE), and (IV) uses the values from (III) to update the regrets and strategy in the trunk via CFR.

Suppose that \( \mu \) is a strategy defined in a trunk \( T \subset H \). The standard variant of CFR-D solves each subgame \( G(S, \text{rng}^\mu(S)) \) at the bottom of \( T \) using CFR, this subgame-solving part of the algorithm (Subgame Values method, \( \text{Burch} [2017, \text{Section 5.3.1}] \)) can be instantiated by any mapping \( \nu_p : \mathcal{T}_p(S) \rightarrow \mathbb{R} \) for which

\[
(\exists\sigma \in \Sigma, \ \text{rng}^\sigma(S) = r) : \sigma|_{G(S)} \text{ is a NE in } G(S, r) \text{ & } \nu_p(I) = \max_{\sigma_p} V_{p,c}^\sigma |_{G(S)}(I).
\]

Eventually, one might wish to use a depth-limited solver that is more scalable than CFR-D: Continual resolving \( \text{Moravčík et al.} [2017, \text{Sustr et al.} [2019] \) utilizes the same decomposition principle as CFR-D, but it adds the option to start the search not just from the root, but rather from any public state. DeepStack \( \text{Moravčík et al.} [2017] \) is a poker-specific instance of CR. To solve subgames, it uses a neural network trained on a large number of sample subgame solutions.

### 5 Optimal Value functions in zero-sum games

While the previous section already describes the notion of value of a specific strategy, depth-limited solving requires the notion of “what would happen if we played this strategy for the next few moves and then started playing optimally”. To investigate this stronger notion of a value, we restrict our attention to two-player zero-sum EFGs.
5.1 The definition of optimal values

We first define a set of strategies that are compatible with what has happened in the game so far and "optimal" for the upcoming decisions. To get a robust notion of an optimal value, we focus on strategies that not only find the NE of the current subgame, but also play optimally even in the whole root of the game (even its unreachable parts). It suffices to use a slightly relaxed version of "subgame perfection", where the strategy

**Definition 6 (Compatible root-optimal strategy).** Let \( r \) be a range at \( S \in \mathcal{S} \). By \( \text{CROS}(S, r) \) we denote the set of strategies \( \sigma \in \Sigma \) that satisfy \( \text{rng}^*(S) = r \) and for which \( V_p^{\sigma^*}(I) = \max\{V_p^{\sigma'_p,\sigma'^-p}(I) \mid \sigma'_p \in \Sigma_p\} \) holds for each \( I \in \mathcal{T}_p^\text{aug}(S), \ p = 1, 2 \).

By \( G(S, r) \) we denote the "unsafe resolving subgame" ([Burch, 2017]) obtained by taking \( G(S) \) and turning it into a proper game by adding a chance node \( \text{root}_S \). This root connects to each (topmost, [Sustr et al., 2019]) \( h \in S \) via some action \( \sigma_h \) whose probability is proportional to \( \eta^\mu(h) \). For fully mixed ranges, there is a straightforward connection between \( \text{CROS}(S, r) \) and \( G(S, r) \). Lemma 7 (iii) uses this connection to prove the existence of \( \text{CROS}(S, r) \) strategies for general \( r \).

**Lemma 7 (Properties of SOS).** For every \( r \) and \( S \), \( \text{CROS}(S, r) \) satisfies the following:

(i) The restriction \( \sigma|_G(S) \) of any \( \sigma \in \text{CROS}(S, r) \) is a NE in \( G(S, r) \).

(ii) For fully mixed \( r \), any NE in \( G(S, r) \) extends into a \( \text{CROS}(S, r) \) strategy.

(iii) \( \text{CROS}(S, r) \) is non-empty.

This justifies the following definition of optimal value functions:

**Definition 8 (Optimal value functions).** Let \( \mu \) be a strategy defined in some trunk \( T \subset \mathcal{H} \). A function \( V_{p,\mu}^{\ast} : \mathcal{S} \times \mathcal{T}_p^\text{aug} \to \mathbb{R} \) is said to be an **optimal value function** for \( p \) and \( \mu \) if it satisfies

\[
(\forall S \in \mathcal{S} \text{ at the bottom of } T) \ (\exists \sigma \in \text{CROS}(S, \text{rng}^\mu(S))) \ (\forall I \in \mathcal{T}_p^\text{aug}) : V_{p,\mu}^{\ast}(S, I) = V_p^{\ast}(I).
\]

An optimal value function \( v_{p,\mu}^{\ast} : \mathcal{S} \times \mathcal{H} \to \mathbb{R} \) for histories is defined analogously, replacing \( V_{p,\mu}^{\ast}(S, I) = V_p^{\ast}(I) \) by \( v_{p,\mu}^{\ast}(S, h) = v_p^{\ast}(h) \). The **optimal counterfactual value functions** \( V_{c,p,\mu}^{\ast} \) and \( v_{c,p,\mu}^{\ast} \) are defined analogously, adding a \( c \) subscript into the condition above.

5.2 Properties of the optimal values

Our definition coincides with other standard definitions of value: Indeed, if \( G \) is in fact also an (acyclic) MDP (i.e. it has perfect information and no player 2 nodes), then public states coincide with states, their optimal value is uniquely defined, and hence both \( V_{p,\mu}^{\ast} \) and \( v_{p,\mu}^{\ast} \) coincide with the standard MDP state-value \( v^* \) ([Sutton and Barto, 2018] independently of \( \sigma \)). If \( G \) is a two-player zero-sum perfect-information game, then \( V_{p,\mu}^{\ast} \) and \( v_{p,\mu}^{\ast} \) again coincide and are equal to the minimax value of the state. This is true even when \( G \) admits simultaneous moves ([Bošanský et al., 2016]).

These coincidences are the main reason for considering the normalized values. On the other hand, the non-normalized values are more straightforward to compute. By Proposition 9, \( \text{Values} \) can be applied in depth-limited solving, which proves that it is indeed a useful notion of a value function:

**Proposition 9 (V_{\sigma,p}^{\ast} can be used for depth-limited solving).** For every trunk strategy \( \mu \), any optimal counterfactual value function \( V_{c,p,\mu}^{\ast} \) satisfies Equation (3).

**Proof.** By Definition 8 we have \( V_{c,p,\mu}^{\ast}(S, I) = V_{p,\mu}^{\ast}(I) \) for some \( \sigma \in \text{CROS} \). By definition of CROS, \( \sigma \) has satisfies \( \text{rng}^*(S) = r \), \( \sigma|_G(S) \in \text{NE}(G(S, r)) \) and \( V_p^{\sigma}(I) = \max_{\sigma'_p} V_p^{\sigma'_p,\sigma'^-p}(I) \) for each \( I \in \mathcal{T}_p^\text{aug}(S) \). Since \( V_{p,\mu}^{\ast}(I) = V_{p,\mu}^{\ast}(I) V_{p,\mu}^{\ast}(I) \), we get \( V_{p,\mu}^{\ast}(I) = \max_{\sigma'_p} V_{p,\mu}^{\ast}(I) V_{p,\mu}^{\ast}(I) \) as well. □

To make the computation of values feasible, we use two practical tweaks: Firstly, for a fixed range, we do not use the subgame-optimal strategies, but instead solve the subgame by CFR (w.r.t. the counterfactual values in the whole game). A solution \( \sigma \) found by CFR doesn’t necessarily have the
We now compare our definition of optimal value functions with values in other models, note the ways this proves the correctness of the function-approximation step of DeepStack [Moravčík et al., 2017]. Because of numerical instability, very similar ranges might correspond to very different (but still optimal) values. The neural network will converge to a convex combination of these values. The following result shows that both of this tweak still produce data useful for depth-limited solving.

**Proposition 10** (Approximating \(V_{c,p}^{\sigma,*}\) by a neural net). Let \(r\) be a range at \(S \in S\) and \(p \in \{1, 2\}\). If \(V\) and \(V'\) are two mappings satisfying Eq. (3), then so does their convex combination \(W = \lambda V + (1 - \lambda)V'\) for any \(\lambda \in (0, 1)\).

This proves the correctness of the function-approximation step of DeepStack [Moravčík et al., 2017].

Using infoset values over history values is often beneficial, not only because it reduces the dimension of value-vector, but also because it removes some of the ambiguity:

**Proposition 11** (Infoset aggregation reduces ambiguity). If \(\sigma, \mu \in \text{CROS}(S, r)\) satisfy \(\sigma_{-p} = \mu_{-p}\), then we have \(V_{p}^{\sigma}(I) = V_{p}^{\mu}(I)\) and \(V_{p,c}^{\sigma}(I) = V_{p,c}^{\mu}(I) = V_{p,c}^{\text{CROS}_{p}(\sigma_{-p}), \sigma_{-r}}(I)\) for each \(I \in I_{\text{aug}}(S)\).

In other words, the optimal infoset values are uniquely determined by the opponent’s strategy.

### 5.3 The rationale behind our definition

We now compare our definition of optimal value functions with values in other models, note the ways in which it is more complicated, and illustrate on examples why these complications are necessary. Consider the minimax value of a history \(h\) in a perfect-information zero-sum game. This value is uniquely defined, depends only on \(h\) (and the subgame below it), and corresponds to any-and-all NE strategies in the subgame rooted at \(h\). In contrast, we have made no claims about the uniqueness of \(v_{p*,\sigma}^{\sigma}(S, h)\), this value depends on the strategy above \(h\), and in fact above the whole \(S\), and requires not just any NE, but its particular refinement. We now explain why these these complications are necessary.

Firstly, it is possible to define a single-number value for the whole \(S\) and range as the value of the corresponding game \(G(S, r)\). This is similar to the approach taken by [Wiggers et al., 2016] in partially-observable stochastic games, where beliefs serve a similar purpose as ranges in EFGs. We choose our definition partly because knowing the value of each individual history/information set seems inherently interesting, but more importantly because unlike public-state values, these values are sufficiently informative to enable depth-limited solving (Proposition 9).

Second, why does the definition of our values use a refinement of Nash equilibrium, and not just any NE? We want our definition of value to capture the notion of “what would have happened, if the game actually reached a specific history/information set?”. However, each \(h\) in the root of \(G(S, r)\) is weighted proportionally to \(r(h)\), implying that \(G(S, r)\) effectively ignores the parts of \(G(S)\) that have zero probability under \(r\). To avoid distorting the values of such \(h\), we only consider the CROS\((S, r)\) strategies, under which even unreachable histories in \(S\) have optimal values.

For a specific example of this behavior, see the game from Figure 2(c): The bolded-out strategy \((L, B)\) is a NE, but if player 1 took this to mean that he is free to play \(R\) because the “value of \(L = \text{value of } R = 1\)”, they would be in for an unpleasant surprise – upon reaching \(X\), a rational player 2 would not play the “bad” action \(B\). Intuitively, the correct strategy is \((L, G)\), which yields

![Figure 1: A zero-sum game where different equilibria produce different expected utilities. The utility of player 1 is the sum of the obtained rewards.](image)
“value of \( L = 1 \), value of \( R = 0 \)”. This aligns with our definition of value, since the range at \( S = \{ L, R \} \) corresponding to \( L \) is \((1, 0)\), and we have \((L, G) \in \text{CROS}(\{L, R\}, (1, 0))\), but \((L, B) \notin \text{CROS}(\{L, R\}, (1, 0))\).

Moreover, a value of a history depends on the \textit{whole} range:

\textbf{Theorem 12} (Values depend on whole ranges). Let \( G \) be an EFG, \( S \in S \) a common-knowledge public state, \( h_0 \in S \), and \( r \) a joint range at \( S \). Assume that \( S \) isn’t thick and \( r \) is fully mixed. Then for each \( g \in S \), there is \( r’ \) that only differs from \( r \) at \( g \), and an EFG \( \tilde{G} \) which only differs from \( G \) below \( S \), s.t. \( \{v^*_g(h_0) \mid \sigma \in \text{CROS}(S, r)\} \neq \{v^*_g(h_0) \mid \sigma \in \text{CROS}(S, r’)\} \).

The intuition behind \( \tilde{G} \) is that until we have understood \( G \), we cannot be certain it doesn’t behave like \( \tilde{G} \). One could prove a more general version of the statement for thick \( S \) and (nearly) arbitrary \( r \) and \( r’ \). However, we believe Theorem 12 illustrates the need for full range sufficiently. Even for a fixed range, the value of a history might be different under different equilibrium strategies:

\textbf{Example 13.} \( \text{CROS}(S, r) \) strategies that have the same range can still have different history values.

This is readily witnessed by the game from Figure 1 where any player can make a trade-off between the expected utilities at \( g_1 \) and \( g_2 \) (while still playing optimally).

By Proposition 11, the ambiguity of values can be reduced by aggregating values over information sets. The game from Figure 2(a) shows that even this aggregation doesn’t fully resolve the ambiguity, since the opponent can still make tradeoffs between states that he can’t tell apart, but we can (in the depicted game, player 1 has perfect information and hence \( \{A\} \) and \( \{B\} \) are his information sets, but different NE strategies give different values for \( A \) and \( B \)):

\textbf{Example 14.} Distinct \( \text{CROS}(S, r) \) strategies can, for some \( I \in \mathcal{I}^{\text{aug}}_p(S) \), have different values of \( I \).

One could also wonder whether it perhaps doesn’t suffice to only consider ranges that correspond to Nash equilibria, and whether these might not all lead to the same value function. When running CFR-D, the trunk strategy will often be highly sub-optimal — in this sense, we truly do need values even for non-optimal ranges. The game depicted in Figure 2(b) shows that the latter property doesn’t hold either:

\textbf{Example 15.} Suppose that trunk strategies \( \sigma \) and \( \sigma’ \) can both be extended into NE in the full game and \( V^{\sigma, \ast}_p \) is an optimal value function for \( \sigma \). Then \( V^{\sigma, \ast}_p \) might not be optimal for \( \sigma’ \).

\section{5.4 Multi-valued states}

\textit{Multi-valued states} is an approach to value estimation in imperfect-information games developed by [Brown et al., 2018]. The authors propose to fix some equilibrium strategy \( \sigma_1 \) and remember, for each \( h \) whose “value” we might need, the vector of values \( v_{\sigma_1, \nu^2}(h) \) for undominated pure strategies \( \nu^2 \) of player 2 below \( h \). The part of the game below \( h \) is replaced by a single decision where player

\footnote{The common-knowledge public states, defined by [Sustr et al., 2019] as the equivalence classes of \( \sim \), are public states that cannot be further split into smaller ones.}

\footnote{That is, \( r(h) > 0 \) holds for each \( h \in S \).}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{(a) Different NE produce different infoset values. (b) Different NE ranges lead to different values. (c) Subgame-optimal strategy produces accurate values, an ordinary “subgame NE” doesn’t.}
\end{figure}
2 picks \( \nu_2 \), and receives the corresponding utility. Section 7 of \cite{Brown2018} contains a nice comparison of multi-valued states and the approach of \cite{Moravcik2017}. For our purposes, the key differences are that multi-valued states are easier to learn, but require the prior knowledge of an approximate equilibrium strategy \( \sigma_1 \) in the full game \( G \). This assumption is very unrealistic, since the whole point of solving \( G \) is to find such \( \sigma_1 \). However, the approach works well – at least for poker – even when \( \sigma_1 \) is sub-optimal \cite{Brown2018}. Ultimately, both approaches deserve further investigation, since their effectiveness might vary depending on the specific domain and setting.

6 Conclusion

We reviewed the properties of reach probabilities and gave a definition of the notion of range that can be applied not just in poker, but in any extensive-form game. We showed that our definition of range is equivalent to several other range-like variables and argued against defining range intuitively as “the probability of the real situation being \( X \) conditional on the player seeing \( Y \)”. We also listed several equivalent descriptions of a value of a strategy, and contrasted this with a more complicated notion of optimal value. We gave examples illustrating that any definition of optimal value has to take range into account, and that even with a fixed range, optimal values don’t have to be unique. However, any vector of optimal values can serve as a heuristic for depth-limited search in imperfect information EFGs.

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A Appendix

A.1 Conditional reach probabilities for unreachable $I$

We now aim to define the conditional reach probability for $I \subset J, J \in \mathcal{I}_p^\text{aug}$. To avoid undefined behavior in cases when $\eta^\sigma(I) = 0$, we define the “noisy version” of $\sigma$ as $\sigma^n(I, a) := (1 - \frac{1}{n})\sigma(I, a) + \frac{1}{n} \frac{1}{|\mathcal{A}(I)|}$ and, for $h \in I \in \mathcal{I}_p^\text{aug}$,

$$\eta^\sigma(h|I) := \lim_{n \to \infty} \mathbb{P}_{\sigma^n} [h \text{ reached | } I \text{ reached}]$$

$$= \lim_{n \to \infty} \eta^{\sigma^n}(h) = \lim_{n \to \infty} \eta_{-p}^{\sigma^n}(h).$$

Note that “$\eta_{-p}^{\sigma^n}(h|I)$” would coincide with $\eta^\sigma(h|I)$, and $\eta_{p}^{\sigma^n}(h|I)$ wouldn’t make any sense. Equipped by this tool, we define the reach probability of $J$ conditional on being in $I$ as

$$\eta^\sigma(I, J) := \sum_{g \in I} \eta^\sigma(g|I) \sum_{g \subset h \in J} \eta^\sigma(g, h).$$

The analogous definition works for $\eta_{-p}^\sigma(I, J)$ and, denoting $\eta_{p}^{\sigma^n}(I, J) := \eta_{p}^{\sigma^n}(g_p, h_J)$ for arbitrary choice of $I \ni g_p \subset h_J \in J$, leads to $\eta^\sigma(I, J) = \eta_{p}^{\sigma^n}(I, J)$ when $\eta^\sigma(I) > 0$, these definitions coincide with the intuitive definition $\eta^\sigma_n(I, J) = \eta_{p}^{\sigma^n}(I, J)$.

**Remark 16** (Properties of $\eta^\sigma(\cdot|I)$). While the conditional reach probability is defined as a limit, it is straightforward to check that it can always be rewritten as one of the following cases:

- When $\eta^\sigma(I) > 0$, $\eta^\sigma(h|I)$ is actually equal to the probability of being in $h$ conditional of being in $I$.
- When we at least have $\eta_{-p}^\sigma(I) > 0$, $\eta^\sigma(h|I) = \eta_{-p}^\sigma(h)/\eta_{-p}^\sigma(I)$ is the same conditional probability under the assumption that $p$ played to reach $I$.
- Finally, when even $\eta_{-p}^\sigma(I)$ is zero, $\eta^\sigma(h|I) = \eta^\text{ind}(h)/\eta^\text{ind}(I)$ is equal to the same conditional probability, but under the assumption that both players act (uniformly) randomly.

A.2 Properties of non-optimal value functions

The following two theorems summarize the properties of $V^\sigma_p$ and $V^\sigma_.$. By a slice of $\mathcal{I}_p^\text{aug}$ below (not necessarily directly below) $I$ we mean any collection $SL \subset \mathcal{I}_p^\text{aug}$ s.t. (i) ($\forall J \in SL$) $J \supseteq I$ (ii) $J \supseteq J'$ for no distinct $J, J' \in \mathcal{I}_p^\text{aug}$ (iii) $SL$ is a maximal set with this property. By $\text{ims}(I)$, we denote the set of immediate successors of $I$ in $\mathcal{I}_p^\text{aug}$.

**Theorem 17** (Properties of $V^\sigma_p$). For $\sigma \in \Sigma, p, \text{ and } V : \mathcal{I}_p^\text{aug} \to \mathbb{R}$, the following are equivalent:

1. ($\forall I \in \mathcal{I}_p^\text{aug} : V(I) = \sum_{h \in I} \eta^\sigma(h|I)\eta^\sigma(h) \text{ (and this is equal to } \sum_{h \in I} \eta_{-p}^\sigma(h|I)\eta_{p}^\sigma(h)).}$
2. ($\forall z \in Z : V(z) = u_p(z) \text{ and } (\forall I \in \mathcal{I}_p^\text{aug} \text{ non-terminal}) : V(I) = \sum_{J \in \text{ims}(I)} \eta^\sigma(I, J)V(J)$.
3. $V = V^\sigma_p$ is the counterfactual utility from [Zinkevich et al., 2008].

Moreover, if $V$ satisfies (1)-(3), then $V(\text{root}) = u_p(\sigma), V : \mathcal{I}_p^\text{aug} \to [\min_Z u_p(z), \max_Z u_p(z)],$ and

4. ($\forall I \in \mathcal{I}_p^\text{aug}(\forall \text{SL slice below } I) : V(I) = \sum_{J \in \text{SL}} \eta^\sigma(I, J)V(J)$.

**Proof.** We prove the theorem for fully mixed $\sigma$. The general case follows by transition to the limit. The “and this is equal to...” case follows from Remark 16.

1 $\iff$ 3: Using the formula $\eta^\sigma_p(h) = \sum_{h \subset z \in Z} \eta^\sigma(h, z)u_p(z)$, we get that $V(I)$ from 1 is equal to $\sum_{h \in I} \eta_{-p}^\sigma(h) \sum_{h \subset z \in Z} \eta^\sigma(h, z)u_p(z)$, which is the definition used in [Zinkevich et al., 2008].
\textbf{Theorem 18} (Properties of $V^\sigma_{p,c}$). For $\sigma \in \Sigma$, $p$, and $V_c : T^\text{aug}_p \to \mathbb{R}$, the following are equivalent:

\begin{enumerate}[(1')]
\item $\forall I \in T^\text{aug}_p : V_c(I) = \sum_{h \in I} \eta^\sigma_{p,c}(h)$ \hspace{1cm} (and this is equal to $\sum_{h \in I} \eta^\sigma_{p,c}(h)\eta^\sigma_{p,h}(h)$).
\item $\forall z \in Z : V_c(z) = \eta^\sigma_{p,c}(z)u_p(z)$ and $\{z\} \neq I \in T^\text{aug}_p : V_c(I) = \sum_{J \in \text{imm}(I)} \eta^\sigma_{p,c}(I, J)V_c(J)$.
\item $V_c = V^\sigma_{p,c} = \eta^\sigma_{p,c}(\cdot)V^\sigma_p$ is the counterfactual value from [Zinkevich et al., 2008].
\end{enumerate}

Moreover, (1)-(3) implies $V_c(\text{root}) = u_p(\sigma)$, $V_c(I) \in [\min_z u_p(z), \max_z u_p(z)]$, and (4') $\forall I \in T^\text{aug}_p(\forall SL \text{ slice below } I) : V_c(I) = \sum_{J \in SL} \eta^\sigma_{p,c}(I, J)V_c(J)$.

The proof of this theorem is analogous to the proof of Theorem 17.

\section{A.3 Proofs}

In this section, we give the full proofs for the theoretical results presented in the paper.

\textbf{Lemma 7 (Properties of SOS).} For every $r$ and $S$, CROS$(S, r)$ satisfies the following:

\begin{enumerate}[(i)]
\item The restriction $\sigma|_{G(S)}$ of any $\sigma \in \text{CROS}(S, r)$ is a NE in $G(S, r)$.
\item For fully mixed $r$, any NE in $G(S, r)$ extends into a CROS$(S, r)$ strategy.
\item CROS$(S, r)$ is non-empty.
\end{enumerate}

\textbf{Proof.} (i): Suppose that a restriction of some $\sigma$ is not a NE in $G(S, r)$. Since the game starts by a chance node, there must be some chance action $a_h$ at the root s.t. one of the players can change their action at $I_p(h)$ or at one of the infosets below it. This is equivalent to $V^\sigma_p(I_p(h))$ not being equal to $\max\{V^\sigma_{p,c}(I_p,h) | \sigma'_{p,c} \in \Sigma_p\}$, and thus $\sigma$ doesn’t belong to CROS$(S, r)$.

(ii): This follows from the fact that the root utility in $G(S, r)$ is equal to $\sum_{I \in T_p(S)} \eta^\sigma(\text{root}, I)V^\sigma_p(I)$, which can further be rewritten as $\sum_{I \in T_p(S)} \frac{\eta^\sigma(I)}{\eta(S)} V^\sigma_p(I)$ (where $\sigma$ the $G(S, r)$ strategy extended in a way that is compatible with $r$). If the strategy $\sigma$ is fully mixed above $S$, each of the fractions is non-zero, and each of the $V^\sigma_p(I)$-s has to be maximal.
(iii): Let μ be a strategy satisfying \( \text{rng}^\mu(S) = r \). For \( n \in \mathbb{N} \), denote by \( \mu^1 \) the fully mixed strategy defined as \( \mu^1(I, a) := (1 - \frac{1}{n})\mu(I, a) + \frac{1}{n}\frac{1}{|S|} \) and by \( r^n \) the corresponding range at \( S \). Moreover, let \( \rho^n \) be some NE of \( G(S, r^n) \) and denote by \( \sigma^n \) the strategy obtained by replacing \( \mu^n \) by \( \sigma^n \) in the subgame \( G(S) \).

Since the space of strategies is compact, we can select a sub-sequence \( (\sigma^{n_k})_{k \in \mathbb{N}} \) that converges to some \( \sigma \in \Sigma \) (to simplify the notation, we assume that \( n_k = k \), so \( \sigma^n \to \sigma \)). We shall prove that \( \sigma \in \text{CROS}(S, r) \). Firstly, we have \( \text{rng}^\sigma(S) = r \) since \( \mu^n \) coincides with \( \sigma^n \) above \( S \) and we have \( \mu^n \to \mu \) and \( \text{rng}^\sigma(S) = r \). Moreover, each \( \sigma^n \) belongs to \( \text{CROS}(S, r^n) \) by (ii), so we have \( V^\prime_p(I) = \max(V^\sigma_p, \sigma_p, \sigma^\prime_p (I_p(h)) \mid \sigma^\prime_p \in \Sigma_p) \). Since the function \( V^\prime_p(I) \) is continuous in the strategy parameter, we get \( V^\prime_p(I) = \max(V^\sigma_p, \sigma_p, \sigma^\prime_p (I_p(h)) \mid \sigma^\prime_p \in \Sigma_p) \), which concludes the proof.

**Proposition 10** (Approximating \( V^\mu, c_{CBR} \) by a neural net). Let \( r \) be a range at \( S \in S \) and \( p \in \{1, 2\} \). If \( V \) and \( V' \) are two mappings satisfying Eq. (3), then so does their convex combination \( W = \lambda V + (1 - \lambda)V' \) for any \( \lambda \in (0, 1) \).

**Proof.** Recall that a counterfactual best-response of player 1 to \( \sigma \) is a strategy satisfying \( V^\sigma_{1,c}(\sigma_2) = \max_{\sigma_1} V^\sigma_{1,c}(\sigma_1, \sigma_2) \). Note that the conclusion of the proposition isn’t trivial since a convex combination of cf. values \( V^\sigma_{1,c} + (1 - \lambda)\bar{V}^\sigma_{1,c} \) typically isn’t equal to \( V^\sigma_{1,c} + (1 - \lambda)\bar{V}^\sigma_{1,c} \).

Firstly, observe that any pair \((W_1, \sigma_2)\) satisfies \( \sigma_2 \in \Sigma_2 \) is a part of NE and \( W_1(I) = V^\sigma_{1,c}(\sigma_2) \sigma_2(I) \) for each \( I \in \Omega_1 \) if and only if \( (W_1, r_2(\sigma_2)) \) is a solution to the sequence form dual linear program (LP) (see [Shoham and Leyton-Brown, 2008, (5.10-5.13)]), where \( r_2(\sigma_2) \) denotes the realization plan corresponding to \( \sigma_2 \). (The NE part is proven in [Shoham and Leyton-Brown, 2008]. The CBR part easily follows from [Shoham and Leyton-Brown, 2008, (5.10-5.11)] by backward induction.)

Denote by \( (C) \) the constraint “a realization plan coincides with \( r_2(\sigma) \) outside of \( G(S) \)”. It follows that a vector \( W \) satisfies Eq. (3) if and only if there is some strategy \( \sigma \) for which \( (W_1, r_2(\sigma)) \) is a solution of the LP (5.10-5.14) \&(C), and analogously for \( (W_2, r_1(\sigma)) \).

Let \( \lambda \in (0, 1) \), \( V', V'' \) be as in the proposition, and let \( \sigma' \) be s.t. \( V'_p(I) = V^\sigma_{1,c}(\sigma'_2(I) \sigma'_2(I)) \) for each \( I \in \Omega_1 \). Denoting \( V''_p(I) := V^\sigma_{1,c}(\sigma'_2(I) \sigma'_2(I)) \) for all \( I \), we have that the pair \((V'_1, r_2(\sigma')) \) is a solution to (5.10 - 5.14) \&(C). The same will then hold for \( V'' \) (and the analogously defined) \( \sigma'' \). Since the space of all solutions of an LP is convex, the pair \((\lambda V'_1 + (1 - \lambda)V''_1, \lambda r_2(\sigma') + (1 - \lambda)r_2(\sigma'')) \) is also a solution to this LP. Since \( V_1 = \lambda V'_1 + (1 - \lambda)V''_1 \), the same holds for \( V_2 \). Since an analogous result is true for \( V_2 \), the observation we made earlier implies that \( V \) is a valid output for the SolveSubgame method.

**Proposition 11** (Infoset aggregation reduces ambiguity). If \( \sigma, \mu \in \text{CROS}(S, r) \) satisfy \( \sigma_{-p} = \mu_{-p} \), then we have \( V^\sigma_p(I) = V^\mu_p(I) \) and \( V^\sigma_{p,c}(I) = V^\mu_{p,c}(I) \) for each \( I \in \Omega^p_{\text{aug}}(S) \).

**Proof.** By definition of SOS, \( \sigma \) has to satisfy \( V^\sigma_p(I) = \max_{\sigma'} V^\sigma_{p,c}(\sigma', \sigma_p, \sigma^\prime_p (I_p(h)) \mid \sigma^\prime_p \in \Sigma_p) \) for each \( I \in \Omega^p_{\text{aug}}(S) \), and analogous formula holds for \( \mu \). Since \( \sigma_{-p} = \mu_{-p} \), we get the first part of the proposition. By definition of CBR, we have \( V^\sigma_{p,c}(\sigma_p, \sigma^\prime_p (I_p(h)) \mid \sigma^\prime_p \in \Sigma_p) \). Since \( \lambda V^\sigma_p(I) = V^\mu_p(I) \) and \( \lambda V^\sigma_p(I) = V^\mu_p(I) \), we get the second part as well.

**Example 13.** \( \text{CROS}(S, r) \) strategies that have the same range can still have different history values.

**Proof.** This can be witnessed on simple one-player game (Figure 2 left), where the root is a chance node with two actions, \( A, B \) and uniform strategy, where the following two nodes belong to the same \( I \in \Omega \) and have actions \( L, R \). The utilities are 0, 1 for \( AL, AR \) and 1, 0 for \( BL, BR \). Any strategy in this game is an equilibrium with counterfactual value \( V^\sigma_{1,c}(I) = 0.5 \), but any \( \sigma \neq \nu \) will have different expected utilities and counterfactual values for histories (i.e. \( v^\sigma_{1,c}(A) \neq v^\nu_{1,c}(A) \), \( v^\sigma_{1,c}(A) \neq v^\nu_{1,c}(A) \), and likewise for \( B \).}
We shall prove the theorem by constructing $\tilde{G}$ and showing that it has the desired properties.

We start by making two simplifying assumptions. First, we assume that each $h \in S$ only has one legal (dummy) action that we denote $d$. In the general case, each $h_0, h_b$ would be extended in the identical manner, complicating the notation, but not introducing any actual challenges. Since the public state cannot be further refined, there exists a sequence of histories satisfying $h_0 \sim h_1 \sim \cdots \sim h_n = g$ in $S$. We assume that $h_p \neq h_j$ for $i \neq j$. We only show the proof in the case where both $h_0, h_1$ and $h_n, h_{n-1}$ are indistinguishable by the first player (the proofs of the remaining three cases are similar). (Note that the proof actually only requires the range to be non-zero on each of the histories $h_p$, rather than on the whole $S$.)

The game $\tilde{G}$ is identical to $G$ at $(H \setminus G(S)) \cup S$, and is defined as follows at $\{hd \mid h \in S\}$ and below:

- For $h \in S \setminus \{h_p \mid i = 0, \ldots, n\}$, $hd$ is a terminal node with utility 0 (for all strategic considerations, this replaces $S$ by $\{h_p \mid i = 0, \ldots, n\}$).
- For $h_p$, $i = 0, \ldots, n-1$, $h_p d$ leads to a matching pennies game (a matrix game with actions $U, D$ for player 1, actions $L, R$ for player 2, and corresponding utilities 1 for $U, L$ and $D, R$, resp. 0 for $U, R$ and $D, L$).
- For $h_n$, $h_n d$ leads to a game where only player 1 acts, choosing between $U$ (utility 0) and $D$ (utility 1).
- The information sets below $h_p$ are defined in such a way that player 1 has to use the same strategy below $h_0$ and $h_1, h_2$ and $h_3, \ldots, h_{n-1}$ and $h_n$, and player 2 has to use the same strategy below $h_1$ and $h_2, h_3$ and $h_4, \ldots, h_{n-2}$ and $h_{n-1}$ (player 2 strategy below $h_0$ is independent of everything else).

Since $h_n$ is unreachable under $r'$, $\tilde{G}(S, r')$ is effectively a collection of (interconnected) matching pennies games. It follows that the uniform strategy of both players is a Nash equilibrium (clearly, no player can improve his overall utility). On the other hand, it is not a NE strategy in $\tilde{G}(S, r)$ (since player 1 could improve his utility by deviating to “$D$ everywhere”).

In particular, 0.5 is an expected utility of $h_0$ under some NE strategy in $\tilde{G}(S, r)$. Suppose that some NE strategy $\sigma$ in $\tilde{G}(S, r)$ has $\nu^* \sigma(h_0) = 0.5$. We will show that such $\sigma$ has to be uniformly random, and thus prove the theorem by contradiction.

Firstly, if $\sigma_1$ wasn’t uniformly random at the information set $\{hd, h_1d\}$, player 2 could increase his overall utility by changing his strategy below $h_0$ to either $L$ or $R$, and thus $\sigma$ wouldn’t be a NE.

We proceed inductively. We know that $\nu^* \sigma(h_0) = 0.5$, and that in $\{h_0d, h_1d\}$, $\sigma_1$ takes both $U$ and $D$ with non-zero probability. If $\sigma_1$ is to be a NE, player 1 has to be indifferent between playing $U$ and $D$ in $\{h_0d, h_1d\}$. Since $\nu^* \sigma(h_0) = 0.5$, this can only be achieved if $\sigma_2$ takes both $L$ and $R$ below $h_1$ with the same probability. In particular, $\nu^* \sigma(h_1) = 0.5$. Since $\tilde{G}$ forces the strategy of player 2 to be the same below $h_1$ and $h_2$, we get that $\sigma_2$ is uniformly random below $h_2$ as well.

We repeat the argument above for each $h_p$, eventually showing that if the players are to be indifferent between the actions they take with non-zero probability, $\sigma_1$ has to be uniformly random in the whole $\tilde{G}(S)$ and $\sigma_2$ has to be uniformly random below $h_1, \ldots, h_{n-1}$.

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4The common-knowledge public states, defined by [Sustr et al., 2019] as the equivalence classes of $\sim$, are public states that cannot be further split into smaller ones.

5That is, $r(h) > 0$ holds for each $h \in S$. 

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choose according to \( r(h) / \sum_s r(h') \)

Figure 3: An example of a game \( \tilde{G} \) used in the proof of Theorem 12.

Figure 4: Games where different equilibria produce different expected utilities (left) and information-set values (right). The chance strategies are uniform. (Repeated from the main text.)

\( h_0 \) (but was below \( h_1 \)), player 1 could increase his utility by deviating to either \( U \) or \( D \). This implies that the whole \( \sigma \) is uniformly random, which contradicts our earlier observation.

**Example 15.** Suppose that trunk strategies \( \sigma \) and \( \sigma' \) can both be extended into NE in the full game and \( V_{p}^{\sigma*,*} \) is an optimal value function for \( \sigma \). Then \( V_{p}^{\sigma*,*} \) might not be optimal for \( \sigma' \).

**Proof.** Indeed, consider \( G \) from the right side of Figure 4 first to act is the player 2, who decides between preparing for A (PfA) and B (PfB). A chance outcome then randomly (50:50) chooses which of the options A and B actually comes to pass. Finally, player 1 has a dummy action\( ^6 \) where he observes the chance outcome, but not the action of player 2. Player 1 then receives 0 utility if player 2 guessed the chance event correctly and 1 utility if he guessed incorrectly. The structure of the game implies that any strategy of player 2 is optimal. Moreover, we have \( V_{1}^{\sigma}(I) \neq V_{1}^{\sigma'}(I) \) whenever \( \sigma_2 \) and \( \sigma'_2 \) are distinct.

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\( ^6 \)While the actions of player 1 are irrelevant in \( G \), one can easily modify it such that the inclusion of player 1 is meaningful.