A HALFWISP TYPE FORMULA FOR THE R-MATRIX OF A SYMMETRIZABLE KAC-MOODY ALGEBRA

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Abstract. Kirillov-Reshetikhin and Levendorskii-Soibelman developed a formula for the universal R-matrix of \( U_q(\mathfrak{g}) \) of the form \( R = (X^{-1} \otimes X^{-1}) \Delta(X) \). The action of \( X \) on a representation \( V \) permutes weight spaces according to the longest element in the Weyl group, so is only defined when \( \mathfrak{g} \) is of finite type. We give a similar formula which is valid for any symmetrizable Kac-Moody algebra. This is done by replacing the action of \( X \) on \( V \) with an endomorphism that preserves weight spaces, but which is bar-linear instead of linear.

1. Introduction

Let \( \mathfrak{g} \) be a finite type complex simple Lie algebra, and let \( U_q(\mathfrak{g}) \) be the corresponding quantized universal enveloping algebra. In [KR] and [LS], Kirillov-Reshetikhin and Levendorskii-Soibelman developed a formula for the universal \( R \)-matrix

\[
R = (X^{-1} \otimes X^{-1}) \Delta(X),
\]

where \( X \) belongs to a completion of \( U_q(\mathfrak{g}) \). The element \( X \) is constructed using the braid group element \( T_{w_0} \) corresponding to the longest word of the Weyl group, so only makes sense when \( \mathfrak{g} \) is of finite type.

The element \( X \) defines a vector space endomorphism \( X_V \) on each representation \( V \), and in fact \( X \) is defined by this system \( \{ X_V \} \) of endomorphisms. With this point of view, Equation (1) is equivalent to the claim that, for any finite dimensional representations \( V \) and \( W \) and \( u \in V \otimes W \),

\[
R(u) = (X_V^{-1} \otimes X_W^{-1}) X_V \otimes W(u).
\]

In the present work we replace \( X_V \) with an endomorphism \( \Theta_V \) which preserves weight spaces. We show that, for any symmetrizable Kac-Moody algebra \( \mathfrak{g} \), and any integrable highest weight representations \( V \) and \( W \) of \( U_q(\mathfrak{g}) \), the action of the universal \( R \)-matrix on \( u \in V \otimes W \) is given by

\[
R(u) = (\Theta_V^{-1} \otimes \Theta_W^{-1}) \Theta_V \otimes W(u).
\]

There is a technical difficulty because \( \Theta_V \) is not linear over the base field \( \mathbb{Q}(q) \), but instead is compatible with the automorphism of \( \mathbb{Q}(q) \) which inverts \( q \). For this reason \( \Theta_V \) depends on a choice of a “bar involution” on \( V \). To make Equation (3) precise we define a bar involution on \( V \otimes W \) in terms of chosen involutions of \( V \) and \( W \), and then show that the composition \( (\Theta_V^{-1} \otimes \Theta_W^{-1}) \Theta_V \otimes W \) does not depend on any choices.

The system of endomorphisms \( \Theta \) was previously studied in [T], where it was used to construct the universal \( R \)-matrix when \( \mathfrak{g} \) is of finite type. Essentially we have extended this previous work to include all symmetrizable Kac-Moody algebras. However, the action of \( \Theta \) on a tensor product is defined differently here than in [T], so the constructions of \( R \) are a-priori not identical, and we have not in fact proven that the construction in [T] gives the universal \( R \)-matrix in all cases.

This note is organized as follows. In Section 2 we establish notation and review some background material. In Section 3 we construct the system of endomorphisms \( \Theta \). In Section 4 we prove our main
2.1. Conventions. We first fix some notation. For the most part we follow conventions from [CP].

- \( g \) is a complex simple Lie algebra with Cartan algebra \( \mathfrak{h} \) and Cartan matrix \( A = (a_{ij})_{i,j \in I} \).
- \( \langle \cdot , \cdot \rangle \) denotes the paring between \( \mathfrak{h} \) and \( \mathfrak{h}^* \) and \( (\cdot, \cdot) \) denotes the usual symmetric bilinear form on either \( \mathfrak{h} \) or \( \mathfrak{h}^* \). Fix the usual bases \( a_i \) for \( \mathfrak{h}^* \) and \( H_i \) for \( \mathfrak{h} \), and recall that \( \langle H_i, a_j \rangle = a_{ij} \).
- \( d_i = (\alpha_i, a_i)/2 \), so that \( (H_i, H_j) = d_j^{-1}a_{ij} \).
- \( \rho \) is the weight satisfying \( \langle \alpha_i, \rho \rangle = d_i \) for all \( i \).
- \( U_q(\mathfrak{g}) \) is the quantized universal enveloping algebra associated to \( \mathfrak{g} \), generated over \( \mathbb{Q}(q) \) by \( E_i \) and \( F_i \) for all \( i \in I \), and \( K_w \) for \( w \) in the co-weight lattice of \( \mathfrak{g} \). As usual, let \( K_i = K_{H_i} \). We use conventions as in [CP].

For convenience, we recall the exact formula for the coproduct:

\[
\begin{align*}
\Delta E_i &= E_i \otimes K_i + 1 \otimes E_i \\
\Delta F_i &= F_i \otimes 1 + K_i^{-1} \otimes F_i \\
\Delta K_i &= K_i \otimes K_i
\end{align*}
\]

2.2. The R-matrix. We briefly recall the definition of a universal \( R \)-matrix, and the related notion of a braiding.

**Definition 2.1.** A braided monoidal category is a monoidal category \( \mathcal{C} \), along with a natural system of isomorphisms \( \sigma_{V,W}^{br} : V \otimes W \to W \otimes V \) for each pair \( V, W \in \mathcal{C} \), such that, for any \( U, V, W \in \mathcal{C} \), the following two equalities hold:

\[
\begin{align*}
\sigma_{U,V,W}^{br} \otimes \text{Id} \circ \text{Id} \otimes \sigma_{V,W}^{br} &= \sigma_{U \otimes V,W}^{br} \\
\text{Id} \otimes \sigma_{U,V}^{br} \circ \sigma_{U,V,W}^{br} \otimes \text{Id} &= \sigma_{U \otimes V,W}^{br}.
\end{align*}
\]

The system \( \sigma^{br} := \{ \sigma_{V,W}^{br} \} \) is called a braiding on \( \mathcal{C} \).

Let \( U_q(\mathfrak{g}) \) be the completion of \( U_q(\mathfrak{g}) \) in the weak topology defined by all matrix elements of \( V_\lambda \otimes V_\mu \), for all ordered pairs of dominant integral weights \( (\lambda, \mu) \).

**Definition 2.2.** A universal \( R \)-matrix is an element \( R \) of \( U_q(\mathfrak{g}) \) such that \( \sigma_{V,W}^{br} := \text{Flip} \circ R \) is a braiding on the category of \( U_q(\mathfrak{g}) \) representations. Equivalently, an element \( R \) is a universal \( R \)-matrix if it satisfies the following three conditions

(i) For all \( u \in U_q(\mathfrak{g}) \), \( R\Delta(u) = \Delta^{op}(u)R \).

(ii) \( (\Delta \otimes 1)R = R_{13}R_{23} \), where \( R_{ij} \) mean \( R \) placed in the \( i \) and \( j \)th tensor factors.

(iii) \( (1 \otimes \Delta)R = R_{13}R_{12} \).
A FORMULA FOR THE R-MATRIX

The following theorem is central to the theory of quantized universal enveloping algebra. See [CP] for a discussion when \( g \) is of finite type, and [LL] for the general case. Unfortunately the conventions in [LL] are quite different from those used here. An explicit proof that our statement follows from [L, Chapter 4] can be found at http://www.ms.unimelb.edu.au/~ptingley/lecturenotes/RandquasiR.pdf.

Proposition 2.3. Let \( g \) be a symmetrizable Kac-Moody algebra. Then \( U_q(g) \) has a unique universal \( R \)-matrix of the form

\[
R = A \left( 1 \otimes 1 + \sum_{\text{positive integral weights } \beta \text{ (with multiplicity)}} X_\beta \otimes Y_\beta \right),
\]

where \( X_\beta \) has weight \( \beta \), \( Y_\beta \) has weight \(-\beta\), and for all \( v \in V \) and \( w \in W \), \( A(v \otimes w) = q^{(\text{wt}(v), \text{wt}(w))} \).

2.3. Constructing isomorphisms using systems of endomorphisms. In this section we review a method for constructing natural systems of isomorphisms \( \sigma_{V,W} : V \otimes W \rightarrow W \otimes V \) for representations \( V \) and \( W \) of \( U_q(g) \). This idea was used by Henriques and Kamnitzer in [HK], and was further developed in [KT2]. The data needed is:

(i) An algebra automorphism \( C_\xi \) of \( U_q(g) \) which is also a coalgebra anti-automorphism.
(ii) A natural system of invertible (vector space) endomorphisms \( \xi_V \) of each representation \( V \) of \( U_q(g) \) such that the following diagram commutes for all \( V \):

\[
\begin{array}{ccc}
V \otimes V & \xrightarrow{\xi_V} & V \\
\downarrow c_\xi \downarrow & & \downarrow c_\xi \\
U_q(g) & \xrightarrow{C_\xi} & U_q(g).
\end{array}
\]

It follows immediately from the definition of coalgebra anti-automorphism that

\[
\sigma^\xi := \text{Flip} \circ (\xi_V^{-1} \otimes \xi_W^{-1}) \circ \xi_{V \otimes W}
\]

is an isomorphism of \( U_q(g) \) representations from \( V \otimes W \) to \( W \otimes V \).

In the current work we require a little more freedom: we will sometimes use automorphisms \( C_\xi \) of \( U_q(g) \) which are not linear over \( \mathbb{C}(q) \), but instead are bar-linear (i.e. invert \( q \)). This causes some technical difficulties, which we deal with in Section 3.

Comment 2.4. To describe the data \( (C_\xi, \xi) \), it is sufficient to describe \( C_\xi \), and the action of \( \xi_{V_\lambda} \) on any one vector \( v \) in each irreducible representation \( V_\lambda \). This is usually more convenient then describing \( \xi_{V_\lambda} \) explicitly. Of course, the choice of \( C_\xi \) imposes a restriction on the possibilities for \( \xi_{V_\lambda}(v) \), so when we give a description of \( \xi \) in this way we are always claiming that the action on our chosen vector in each \( V_\lambda \) is compatible with \( C_\xi \).

2.4. A useful lemma. Let \( (V_\lambda, v_\lambda) \) and \( (V_\mu, v_\mu) \) be irreducible representations with chosen highest weight vectors. Every vector \( u \in V_\lambda \otimes V_\mu \) can be written as

\[
u = v_\lambda \otimes c_0 + b_1 \otimes c_1 + \ldots + b_k \otimes c_{k-1} + b_0 \otimes v_\mu,
\]

where, for \( 0 \leq j \leq k-1 \), \( b_j \) is a weight vector of \( V_\lambda \) of weight strictly less then \( \lambda \), and \( c_j \) a weight vector of \( V_\mu \) of weight strictly less then \( \mu \). Furthermore, the vectors \( b_0 \in V_\lambda \) and \( c_0 \in V_\mu \) are uniquely determined by \( u \). Thus we can define projections from \( V_\lambda \otimes V_\mu \) to \( V_\lambda \) and \( V_\mu \) as follows:
Definition 2.5. The projections \( p^1_{\lambda,\mu} : V_\lambda \otimes V_\mu \to V_\lambda \) and \( p^2_{\lambda,\mu} : V_\lambda \otimes V_\mu \to V_\mu \) are given by, for all \( u \in V_\lambda \otimes V_\mu \),

\[
\begin{align*}
(11) & \quad p^1_{\lambda,\mu}(u) := b_0 \\
(12) & \quad p^2_{\lambda,\mu}(u) := c_0.
\end{align*}
\]

Lemma 2.6. Let \( S_{\lambda,\mu} \) be the space of singular vectors in \( V_\lambda \otimes V_\mu \). The restrictions of the maps \( p^1_{\lambda,\mu} \) and \( p^2_{\lambda,\mu} \) from Definition 2.5 to \( S_{\lambda,\mu} \) are injective.

Proof. We prove the Lemma only for \( p^2_{\lambda,\mu} \), since the proof for \( p^1_{\lambda,\mu} \) is completely analogous. Let \( c_1, \ldots, c_m \) be a weight basis for \( V_\mu \). Let \( u \) be a singular vector of \( V_\lambda \otimes V_\mu \) of weight \( \nu \). Then \( u \) can be written uniquely as

\[
(13) \quad u = \sum_{j=1}^{m} v_j \otimes c_j,
\]

where each \( v_j \) is a weight vector in \( V_\lambda \). Let \( \gamma \) be a maximal weight such that there is some \( j \) with \( \text{wt}(v_j) = \gamma \) and \( v_j \neq 0 \). It suffices to show that \( \gamma = \lambda \), so assume for a contradiction that it does not. Then \( v_j \) is not a highest weight vector, so \( E_i(v_j) \neq 0 \) for some \( i \). But then

\[
(14) \quad E_i(u) = \sum_{\text{wt}(v_{j_s}) = \gamma} E_i(v_{j_s}) \otimes c_{j_s} + \text{terms whose first factors have weight strictly less then } \gamma + \alpha_i.
\]

Since the \( c_j \) are linearly independent and \( E_i(v_j) \neq 0 \) for some \( j \) with \( \text{wt}(v_j) = \gamma \), this implies that \( E_i(u) \neq 0 \), contradicting the fact that \( v \) is a singular vector. \( \square \)

3. Constructing the system of endomorphisms \( \Theta \)

Constructing and studying \( \Theta = \{ \Theta_V \} \) is the technical heart of this work. As we mentioned in the introduction, \( \Theta_V \) is bar linear instead of linear, which makes it more difficult to choose a normalization. To get around this, we introduce the notion of a bar involution \( \text{bar}_V \) on \( V \), and actually define \( \Theta \) on the category of representations with a chosen bar involution. We then define a tensor product on this new category, and show that \( (\Theta_{V, \text{bar}_V} \otimes \Theta_{W, \text{bar}_W}) \circ \Theta_{(V, \text{bar}_V) \otimes (W, \text{bar}_W)} \) does not depend on the choices of \( \text{bar}_V \) and \( \text{bar}_W \). The real work is in defining this tensor product, which essentially amounts to defining a bar involution on \( V \otimes W \) in terms of bar involutions \( \text{bar}_V \) and \( \text{bar}_W \).

3.1. Bar involution. The following \( \mathbb{Q} \) algebra involution of \( U_q(\mathfrak{g}) \) has been studied in several places, for example \[K\] Section 1.3, and is usually called bar involution. We use the notation \( C_{\text{bar}} \) because we will also work with bar involutions \( \text{bar}_V \) on representations \( V \), which are compatible with \( C_{\text{bar}} \) in the sense of Equation \[K\].

Definition 3.1. \( C_{\text{bar}} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \) is the \( \mathbb{Q} \)-algebra involution defined by

\[
\begin{align*}
C_{\text{bar}}q &= q^{-1} \\
C_{\text{bar}}K_i &= K_i^{-1} \\
C_{\text{bar}}E_i &= E_i \\
C_{\text{bar}}F_i &= F_i.
\end{align*}
\]

It is perhaps useful to imagine that \( q \) is specialized to a complex number on the unit circle (although not a root of unity), so that \( C_{\text{bar}} \) is conjugate linear.

Definition 3.2. Let \( V \) be a representation of \( U_q(\mathfrak{g}) \). A bar involution on \( V \) is a \( \mathbb{Q} \)-linear involution \( \text{bar}_V \) such that
(i) $\text{bar}_V$ is compatible with $C_{\text{bar}}$ in the sense that the following diagram commutes:

\[
\begin{array}{ccc}
V & \xrightarrow{\text{bar}_V} & V \\
\downarrow & & \downarrow \\
U_q(\mathfrak{g}) & \xrightarrow{C_{\text{bar}}} & U_q(\mathfrak{g}).
\end{array}
\]

(ii) Let $V^{\text{inv}} = \{ v \in V \text{ such that } \text{bar}_V(v) = v \}$. Then $V = \mathbb{Q}(q) \otimes_{\mathbb{Q}} V^{\text{inv}}$.

Comment 3.3. It is straightforward to check that $C_{\text{bar}}^2$ is the identity. Along with condition (ii), this implies that $\text{bar}_V$ is the identity, so the term “involution” is justified.

Comment 3.4. When it does not cause confusion we will denote $\text{bar}_V(v)$ by $\bar{v}$.

Proposition 3.5. Fix $\lambda$ and a highest weight vector $v_\lambda \in V_\lambda$. There is a unique bar involution $\text{bar}_{(V_\lambda,v_\lambda)}$ on $V_\lambda$ such that $\text{bar}_{(V_\lambda,v_\lambda)}(v_\lambda) = v_\lambda$.

Proof. Recall that $V_\lambda$ has a basis consisting of various $F_i \cdots F_i v_\lambda$. All of these vectors must be fixed by any bar involution preserving $v_\lambda$, so there is at most one possibility. On the other hand, it is clear that the unique $\mathbb{Q}$-linear map sending $f(q)F_i \cdots F_i v_\lambda$ to $f(q^{-1})F_i \cdots F_i v_\lambda$ for each of these basis vectors is a bar involution. □

Corollary 3.6. Every representation $V$ has a (non-unique) bar involution $\text{bar}_V$.

Proof. Choose a decomposition of $V$ into irreducible components, and a highest weight vector in each irreducible component, then use Proposition 3.5. □

Definition 3.7. Fix $(V, \text{bar}_V)$ and $(W, \text{bar}_W)$, where $\text{bar}_V$ and $\text{bar}_W$ are involutions of $V$ and $W$ compatible with $C_{\text{bar}}$. Let $\text{bar}_{V \otimes W}$ be the vector space involution on $V \otimes W$ defined by $f(q)v \otimes w \rightarrow f(q^{-1})\bar{v} \otimes \bar{w}$ for all $f(q) \in \mathbb{Q}(q)$ and $v \in V, w \in W$.

Comment 3.8. It is straightforward to check that the action of $(\text{bar}_V \otimes \text{bar}_W)$ on a vector in $V \otimes W$ does not depend on its expression as a sum of elements of the form $f(q)v \otimes w$. The resulting map is a $\mathbb{Q}$-linear involution.

Definition 3.9. Fix $u \in V_\lambda \otimes V_\mu$ a weight vector of weight $\nu$. Define $u^\beta$ for each weight $\beta$ as the unique element of $V_\lambda(\nu - \beta) \otimes V_\mu(\beta)$ such that

\[
u = \sum_{\text{weights } \beta} u^\beta.
\]

Lemma 3.10. Fix $(V_\lambda, \text{bar}_{V_\lambda})$ and $(V_\mu, \text{bar}_{V_\mu})$. Let $v_\nu$ be a singular weight vector in $V_\lambda \otimes V_\mu$, and write

\[
u = \sum_{j=1}^{N} b_j \otimes c_j,
\]

where each $b_j$ is a weight vector of $V_\lambda$, and each $c_j$ is a weight vector of $V_\mu$. Then

\[
\text{bar}(v_\nu) := \sum_{j=0}^{N} q^{\langle \mu,\nu \rangle - (\nu, c_j) + 2(\mu, -c_j, \rho)} \bar{b}_j \otimes \bar{c}_j
\]

is also singular.
Proof. Fix \( i \in I \). The vector \( \nu_i \) is singular, so \( E_i \nu_i = 0 \) and hence \((E_i \nu_i)^\beta = 0\) for all \( \beta \). Then:

\[
0 = (E_i \nu_i)^\beta = \sum_{\text{wt}(c_j) = \beta} q^{(\beta,\alpha_i)} E_i b_j \otimes c_j + \sum_{\text{wt}(c_j) = \beta - \alpha_i} b_j \otimes E_i c_j.
\]

Using Equation (18):

\[
(E_i \bar{\nu}_i)^\beta = \sum_{\text{wt}(c_j) = \beta} q^{(\beta,\beta)+2(\mu-\beta,\rho)} q^{(\beta,\alpha_i)} E_i \bar{b}_j \otimes \bar{c}_j
\]

\[
+ \sum_{\text{wt}(c_j) = \beta - \alpha_i} q^{(\beta-\alpha_i,\beta-\alpha_i)+2(\mu-\beta+\alpha_i,\rho)} \bar{b}_j \otimes E_i \bar{c}_j
\]

\[
= q^{(\mu-\beta,\alpha_i)+2(\mu-\beta+\alpha_i,\rho)} \times
\]

\[
\sum_{\text{wt}(c_j) = \beta} q^{(\beta,\alpha_i)} E_i \bar{b}_j \otimes \bar{c}_j + \sum_{\text{wt}(c_j) = \beta - \alpha_i} \bar{b}_j \otimes E_i \bar{c}_j
\]

\[
= q^{(\mu-\beta,\alpha_i)+2(\mu-\beta+\alpha_i,\rho)} (\bar{\nu}_\lambda \otimes \bar{\nu}_\mu)(E_i \nu_i)^\beta,
\]

where \((\bar{\nu}_\lambda \otimes \bar{\nu}_\mu)\) is the involution from Definition 3.7. But \( E_i (\nu_i)^\beta = 0 \), so we see that \( E_i (\nu_i)^\beta = 0 \). Since this holds for all \( i \) and all \( \beta \), \( \bar{\nu}_i \) is singular. \( \square \)

Definition 3.11. Let \( \bar{\nu}_\lambda \otimes \bar{\nu}_\mu \) be the unique involution on \( V_\lambda \otimes V_\mu \) which agrees with the involution \( \bar{\nu} \) from Lemma 3.10 on singular vectors, and is compatible with \( C_{\bar{\nu}} \).

Lemma 3.12. \( \bar{\nu}_{(V_\lambda v_\lambda) \otimes (V_\mu v_\mu)} \) is a bar involution.

Proof. Definition 3.2 part (i) follows immediately from the definition of \( \bar{\nu}_{(V_\lambda v_\lambda) \otimes (V_\mu v_\mu)} \). To establish Definition 3.2 part (ii), it suffices to show that there is a basis for the space \( S_{\lambda,\mu} \) of singular vectors of \( V_\lambda \otimes V_\mu \) which is fixed by \( \bar{\nu}_{(V_\lambda v_\lambda) \otimes (V_\mu v_\mu)} \). Since \( V_\lambda = \mathbb{Q}(q) \otimes V_\lambda^{\text{inv}} \), there is a basis for \( S_{\lambda,\mu} \) consisting of elements of \( V_\lambda^{\text{inv}} \otimes V_\mu \). Using Lemma 2.6 we see that there is a basis for \( S_{\lambda,\mu} \) consisting of vectors of the form

\[
v_\lambda \otimes c_0 + \cdots + b_0 \otimes v_\mu,
\]

where \( b_0 = 0 \) and the missing terms are all of the form \( b \otimes c \) with \( \text{wt}(c) < \mu \). By Definition 3.11 and Lemma 2.6, this vector is invariant under \( \bar{\nu}_{(V_\lambda v_\lambda) \otimes (V_\mu v_\mu)} \). \( \square \)

In light of Definition 3.2 part (ii), we can extend Definition 3.11 by naturality to construct a bar-involution on \( (V, \bar{\nu}_V) \otimes (W, \bar{\nu}_W) \) in terms of any bar-involutions \( \bar{\nu}_V \) and \( \bar{\nu}_W \).

3.2. The system of endomorphisms \( \Theta \). Consider the \( \mathbb{Q} \)-algebra automorphism \( C_{\Theta} \) of \( U_\Phi(q) \):

\[
\begin{cases}
C_{\Theta}(E_i) = E_i K_i^{-1} \\
C_{\Theta}(F_i) = K_i F_i \\
C_{\Theta}(K_i) = K_i^{-1} \\
C_{\Theta}(q) = q^{-1}.
\end{cases}
\]

Notice that \( C_{\Theta} \) is not linear over \( \mathbb{Q}(q) \), but instead inverts \( q \). One can easily check that \( C_{\Theta} \) is a \( \mathbb{Q} \)-algebra involution, and that it is also a coalgebra anti-involution.

Definition 3.13. Fix a representation \( V \) with a bar involution \( \bar{\nu}_V \). Then \( \Theta_{V, \bar{\nu}_V} \) is the \( \mathbb{Q} \) linear endomorphism of \( V \) defined by

\[
\Theta_{V, \bar{\nu}_V}(v) = q^{-(\text{wt}(v),\text{wt}(v))/2+(\text{wt}(v),\rho)} \bar{\nu}_V(v).
\]
Comment 3.14. Using Definitions 3.11 one can see that, for any irreducible \( V_\lambda \subset V \), \( \Theta_{V,\text{bar}_V} \) restricts to an endomorphism of \( V_\lambda \).

Comment 3.15. There are sometimes weights \( \lambda \) for which \(- (\lambda, \lambda)/2 + (\lambda, \rho)\) is not an integer. However, it is always a multiple of \( 1/k \) where \( k \) is twice the size of the weight lattice mod the root lattice. It is for this reason that we adjoin \( q^{1/k} \) to the base field.

**Lemma 3.16.** the following diagram commutes

\[
\begin{array}{ccc}
V & \xrightarrow{\Theta_V} & V \\
\downarrow{U_q(g)} & & \downarrow{U_q(g)} \\
C_\Theta & \xrightarrow{c_\Theta} & C_\Theta
\end{array}
\]

**Proof.** It is sufficient to check that \( C_\Theta(X)\Theta_V(v) = \Theta_V(Xv) \), where \( X = E_i \) or \( F_i \). We do the case of \( F_i \) and leave \( E_i \) as an exercise. Fix \( v \in V \).

\[
\begin{align*}
\Theta_V(F_i v) &= q^{-((\text{wt}(F_i v), \text{wt}(F_i)))/2 + (\text{wt}(F_i v), \rho) \text{bar}_V(F_i v)} \\
&= q^{-((\text{wt}(v) - \alpha_i, \text{wt}(v) - \alpha_i))/2 + (\text{wt}(v) - \alpha_i, \rho) F_i \text{bar}_V(v)} \\
&= q^{(\alpha_i, \text{wt}(v) - \alpha_i) - (\text{wt}(v), \rho) F_i \text{bar}_V(v)} \\
&= K_i F_i q^{-(\text{wt}(v), \text{wt}(v))/2 + (\text{wt}(v), \rho) \text{bar}_V(v)} \\
&= C_\Theta(F_i)\Theta_V(v).
\end{align*}
\]

where for Equation (29) we have used the fact that \((\alpha_i, \alpha_i)/2 = (\alpha_i, \rho) = d_i \). \( \square \)

**Definition 3.17.** Fix two representations with bar involutions \((V, \text{bar}_V)\) and \((W, \text{bar}_W)\). We set \( \Theta_{(V, \text{bar}_V) \otimes (W, \text{bar}_W)} \) to be the \( \mathbb{Q} \) linear endomorphism of \( V \otimes W \) defined by, for all \( u \in V \otimes W \),

\[
(\Theta_{(V, \text{bar}_V) \otimes (W, \text{bar}_W)})(u) = q^{-(\text{wt}(u), \text{wt}(u))/2 + (\text{wt}(u), \rho) \text{bar}(V \otimes W)}.
\]

**Comment 3.18.** By Lemma 3.16, \( \Theta_{(V, \text{bar}_V) \otimes (W, \text{bar}_W)} \) is a bar involution on \( V \otimes W \), so by Lemma 3.16 \( \Theta_{(V, \text{bar}_V) \otimes (W, \text{bar}_W)} \) is compatible with \( C_\Theta \).

## 4. Main Theorem

**Theorem 4.1.** \( (\Theta_{V, \text{bar}_V}^{-1} \otimes \Theta_{W, \text{bar}_W}^{-1})\Theta_{(V \otimes W, \text{bar}_{V \otimes W})} \) acts on \( V \otimes W \) as the standard \( R \)-matrix. This holds independent of the choice of bar involutions \( \text{bar}_V \) and \( \text{bar}_W \).

**Proof.** We will actually prove the equivalent statement that

\[
\sigma^\Theta := \text{Flip} \circ (\Theta_{V, \text{bar}_V}^{-1} \otimes \Theta_{W, \text{bar}_W}^{-1})\Theta_{(V \otimes W, \text{bar}_{V \otimes W})}
\]

acts on \( V \otimes W \) as the standard braiding \( \text{Flip} \circ R \). By Lemma 3.16 and the fact that \( C_\Theta \) is a \( \mathbb{Q} \) coalgebra anti-automorphism, the following diagram commutes:

\[
\begin{array}{ccc}
V \otimes W & \xrightarrow{\Theta_{(V \otimes W, \text{bar}_{V \otimes W})}} & V \otimes W \\
\downarrow{U_q(g)} & & \downarrow{U_q(g)} \\
C_\Theta & \xrightarrow{c_\Theta} & C_\Theta
\end{array}
\]

In particular, \( \sigma^\Theta : V \otimes W \rightarrow W \otimes V \) is an isomorphism. Thus it suffices to show that \( \sigma^\Theta(v_{\nu}) = \text{Flip} \circ R(v_{\nu}) \) for every singular weight vector \( v_{\nu} \in V \otimes W \). By naturality it is enough to consider the case when \( V \) and \( W \) are irreducible, so let \( v_{\nu} \) be a singular vector in \( V_\lambda \otimes V_\mu \). Write

\[
v_{\nu} = b_\lambda \otimes c_0 + b_{k-1} \otimes c_1 + \ldots + b_1 \otimes c_{k-1} + b_0 \otimes b_{\mu},
\]
where for $0 \leq j \leq k-1$, $b_j$ is a weight vector of $V_\mu$ of weight strictly less than $\mu$. By Definitions 3.11 and 3.13,

\begin{align}
\sigma^\Theta(v_\nu) &= \text{Flip} \circ (\Theta_{\nu_\lambda}^{-1} \otimes \Theta_{\nu_\mu}^{-1}) \Theta_{V_\lambda,\nu_\lambda} \otimes (V_\mu,\nu_\mu) (\cdots + b_0 \otimes b_\mu) \\
&= \text{Flip} \circ (\Theta_{\nu_\lambda}^{-1} \otimes \Theta_{\nu_\mu}^{-1}) (q^{-(\mu+\text{wt}(b_0),\mu+\text{wt}(b_0))/2+(\mu+\text{wt}(b_0),\rho)} (\cdots + b_0 \otimes b_\mu)) \\
&= q^{-(\text{wt}(b_0),\text{wt}(b_0))/2-(\mu,\mu)/2+(\mu+\text{wt}(b_0),\mu+\text{wt}(b_0))/2} b_\mu \otimes b_0 + \cdots \\
&= q^{\text{wt}(b_0),\mu} b_\mu \otimes b_0 + \cdots,
\end{align}

where $\cdots$ always represents terms where the factor coming from $V_\mu$ has weight strictly less than $\mu$.

It follows immediately from Proposition 2.3 that

\begin{align}
\text{Flip} \circ R(v_\nu) &= q^{\text{wt}(b_0),\mu} b_\mu \otimes b_0 + \cdots,
\end{align}

where again $\cdots$ represents terms of the form $c \otimes b$ where $\text{wt}(c) < \mu$. Both $\sigma^\Theta(v_\nu)$ and $\text{Flip} \circ R(v_\nu)$ are singular vectors in $V_\mu \otimes V_\lambda$, so by Lemma 2.6 they are equal.

\begin{comment}
\textbf{Comment 4.2.} The above proof works independent of the choice of $\text{bar}_V$ and $\text{bar}_W$. One can also see directly that $\sigma^\Theta$ does not depend on these choices. Restrict to the irreducible case, and notice that by Lemma 3.5, $\sigma^\Theta$ depends only on the choice of highest weight vectors $v_\lambda$ and $v_\mu$. It is straightforward to check that rescaling these vectors has no effect on $\sigma^\Theta$.
\end{comment}

\begin{comment}
\textbf{Comment 4.3.} One can check that $\Theta_V$ is an involution of $\mathbb{Q}$ vector spaces, so the inverses in the statement of Theorem 4 are in some sense unnecessary. We include them because $\Theta_V$ should really be thought of as an isomorphism between $V$ and the module which is $V$ as a $\mathbb{Q}$ vector space, but with the action of $U_q(\mathfrak{g})$ twisted by $C_\Theta$. We have not specified the action of $\Theta$ on this new module. The way the formula is written, $\Theta$ is always acting on $V$, $W$ or $V \otimes W$ with the usual action, where it has been defined.
\end{comment}

5. Future directions

We have two main motivations for developing our formula for the $R$-matrix.

\textbf{Motivation 1.} In work with Joel Kamnitzer [KT2], we showed that Drinfeld’s unitarized $R$-matrix $\bar{R}$ (see [D]) respects crystal basis (up to some signs). Composing with Flip, we see that $\bar{R}$ descends to a crystal map from $B \otimes C$ to $C \otimes B$, which is fact agrees with the crystal commutator defined in [HK]. We make extensive use of Equation (11), so our methods are only valid in the finite type case. However Drinfeld’s unitarized $R$-matrix is defined in the symmetrizable Kac-Moody case, as is the crystal commutator (see [KT1] and [S]). We hope that the formula given in Theorem 4.1 will help us to extend some of the results in [KT2] to the symmetrizable Kac-Moody case.

\textbf{Motivation 2.} Recall that the action of the braiding $\text{Flip} \circ R$ on $V \otimes W$ can be drawn diagrammatically as passing a string labeled $V$ over a string labeled $W$. If we use flat ribbons in place of strings, as it is often convenient to do, one can consider the following isotopy:
Roughly, if one interprets twisting a ribbon by 180 degrees as $X$, and twisting two ribbon together as at the bottom on the right side as $\text{Flip} \circ \Delta(X)$, the two sides of this isotopy correspond to the two sides of Equation (1), written as

$$\text{Flip} \circ R = \text{Flip} \circ (X^{-1} \otimes X^{-1}) \Delta(X) = (X^{-1} \otimes X^{-1}) \circ \text{Flip} \circ \Delta(X).$$

(41)

In work with Noah Snyder [ST], we make this precise. One should be able to use our new formula to give a precise interpretation of “twisting a ribbon by 180 degrees” in the symmetrizable Kac-Moody case. It is for this reason that we use the term “half twist type formula” in our title.

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