ARITHMETIC OF ABELIAN VARIETIES WITH CONSTRAINED TORSION

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Abstract. Let $K$ be a number field. We present several new finiteness results for isomorphism classes of abelian varieties over $K$ whose $\ell$-power torsion fields are arithmetically constrained for some rational prime $\ell$. Such arithmetic constraints are related to an unresolved question of Ihara regarding the kernel of the canonical outer Galois representation on the pro-$\ell$ fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$.

Under GRH, we demonstrate the set of classes is finite for any fixed $K$ and any fixed dimension. Without GRH, we prove a semistable version of the result. In addition, several unconditional results are obtained when the degree of $K/\mathbb{Q}$ and the dimension of abelian varieties are not too large, through a careful analysis of the special fiber of such abelian varieties. In some cases, the results (viewed as a bound on the possible values of $\ell$) are uniform in the degree of the extension $K/\mathbb{Q}$.

1. Introduction

1.1. Introduction. Let $\ell$ be a rational prime number, let $\mu_N$ denote the $N$th roots of unity, and $\mu_{\ell^\infty} = \bigcup_{n \geq 1} \mu_{\ell^n}$. Set $\mathbb{P}_{01\infty}^1 := \mathbb{P}_{01\infty}^1 - \{0, 1, \infty\}$. For a given number field $K$, we let $\mathbb{T} = \mathbb{T}(K, \ell)$ denote the maximal pro-$\ell$ extension of $K(\mu_{\ell^\infty})$ which is unramified away from $\ell$. Let $G_K$ denote the absolute Galois group $\text{Gal}(\bar{K}/K)$, and consider the natural outer Galois representation $\Phi: G_K \to \text{Out}(\pi_{1}^f(\mathbb{P}_{01\infty}^1))$. We let $\mathbb{U} = \mathbb{U}(K, \ell)$ denote the subfield of $\bar{K}$ which is fixed by the kernel of $\Phi$. Anderson and Ihara [AI88] have shown that $\mathbb{U}$ is precisely the minimal field of definition (containing $K$) of all curves appearing in the pro-$\ell$ tower of (Galois) coverings of $\mathbb{P}^1$, branched only over $\{0, 1, \infty\}$. Moreover, they demonstrate many properties of $\mathbb{U}$, including the containment $\mathbb{U} \subseteq \mathbb{T}$.

Ihara has asked the following question ([Iha86]), which is still open: For $K = \mathbb{Q}$, does $\mathbb{U} = \mathbb{T}$? (The choice of the notation is motivated as follows: the kanji 天, read ten, means “heaven,” and the kanji 山, read san, means “mountain.” Both 天 and 山 are infinite pro-$\ell$ extensions of $K(\mu_{\ell})$. Ihara’s question is roughly as follows: “Does the mountain reach the heavens?”)

There is a natural source for subextensions of 天. Let $A$ denote an abelian variety over $K$ which possesses good reduction away from $\ell$. Then by the theory of Serre-Tate, the extension $K(A[\ell^\infty])/K(A[\ell])$ is pro-$\ell$ and unramified away from $\ell$; hence, in certain cases one finds $K(A[\ell^\infty]) \subseteq \mathbb{T}$. Reflecting on Ihara’s question, it is natural to then study whether or not $K(A[\ell^\infty]) \subseteq \mathbb{U}$. In several cases where $A$ is the Jacobian variety of a curve $C$, this is known to occur. For example, the containment holds for the following curves $C$ which appear in the pro-$\ell$ tower over $\mathbb{P}_{01\infty}^1$:

- Fermat curves and Heisenberg curves for any $\ell$ [AI88],
Conjecture 1. For any \( \ell \) (1.1)
\[
A \text{ fixed rational prime number}.
\]
We also define \( Q \) to Ihara’s question.

In addition, those elliptic curves \( E/\mathbb{Q} \) with good reduction away from \( \ell \) and which have CM by \( \mathbb{Q}(\sqrt{-\ell}) \) are also known to satisfy \( K(E[\ell^\infty]) \subseteq \mathcal{K} \), although they do not lie in the pro-\( \ell \) tower over \( \mathbb{P}_{01}^1 \).

Despite the existence of these examples, the abelian varieties \( A/K \) which satisfy \( K(A[\ell^\infty]) \subseteq \mathcal{K} \) appear to be quite rare. For an abelian variety \( A/K \), let \( [A] \) denote its \( K \)-isomorphism class. For any fixed number field \( K \), fixed integer \( g > 0 \), and fixed rational prime number \( \ell \), set
\[
\mathcal{A}(K, g, \ell) := \{ [A] : \dim A = g \text{ and } K(A[\ell^\infty]) \subseteq \mathcal{K} \}.
\]

For fixed \( \ell \), this set is necessarily finite by the Shafarevich Conjecture (Faltings’ theorem). We also define
\[
\mathcal{A}(K, g) := \{ ([A], \ell) : [A] \in \mathcal{A}(K, g, \ell) \}.
\]

**Conjecture 1.** For any \( K \) and \( g \), the set \( \mathcal{A}(K, g) \) is finite. Equivalently, the set \( \mathcal{A}(K, g, \ell) \) is non-empty for only finitely many \( \ell \).

**Remark.** If \( A \) has everywhere good reduction, it is at least possible that \( [A] \in \mathcal{A}(K, g, \ell) \) for more than one \( \ell \). Hence, the reader should not assume that the natural surjection \( \mathcal{A}(K, g, \ell) \rightarrow \bigcup \mathcal{A}(K, g, \ell), ([A], \ell) \mapsto [A] \), is a bijection.

In [RT08], the authors prove this conjecture in the case \( (K, 1) \) for \( K = \mathbb{Q} \) and for \( K \) a quadratic extension of \( \mathbb{Q} \) other than the nine imaginary quadratic extensions of class number one. Moreover, the set \( \mathcal{A}(\mathbb{Q}, 1) \) is determined explicitly. It contains 50 \( \mathbb{Q} \)-isomorphism classes, spanning 21 \( \mathbb{Q} \)-isogeny classes. The containment related to Ihara’s question, \( \mathbb{Q}(E[\ell^\infty]) \subseteq \mathcal{K}(\mathbb{Q}, \ell) \) is demonstrated for almost all classes \( ([E], \ell) \in \mathcal{A}(\mathbb{Q}, 1) \). There are 4 isomorphism classes, spanning 2 isogeny classes, which remain open. In each of these classes, \( \ell = 11 \) and the representative curve \( E \) does not have complex multiplication.

In the present article, we prove the finiteness of \( \mathcal{A}(K, g) \), for arbitrary \( K \) and \( g \), under the assumption of the Generalized Riemann Hypothesis. In addition, several new cases of the conjecture are proven unconditionally. When possible, we give proofs for uniform versions of the conjecture, meaning we demonstrate the existence of a constant \( C \), possibly dependent on \( g \) and \( [K : \mathbb{Q}] \), but not \( K \) itself, so that \( \ell > C \) implies \( \mathcal{A}(K, g, \ell) = \emptyset \).

The organization of the paper is as follows. In §2 several well-known results from analytic number theory are collected. In §3, the behavior of the Galois representation \( \rho \) on \( A[\ell] \) is studied for any \( [A] \in \mathcal{A}(K, g, \ell) \), leading to constraints on the indices of semistable reduction. This yields a proof of the conjecture when we restrict to abelian varieties with semistable reduction. In §4, we construct a character \( \chi(m_\mathbb{Q}) \) from \( \rho \), and demonstrate the remarkable property that \( \chi(m_\mathbb{Q}) \) never vanishes on the Frobenius elements of small primes. This will play a key role in the proofs of both the conditional and unconditional finiteness results.

In §5, we prove the conjecture under the assumption of the Generalized Riemann Hypothesis in various forms. Actually, two proofs are given. The first proves the finiteness of \( \mathcal{A}(K, g) \) for any choice of \( (K, g) \). The second proves a version of the conjecture which is uniform in the degree of \( K/\mathbb{Q} \). Unfortunately, the second
proof requires the assumption that \([K : Q]\) is odd. Finally, this uniform result is
generalized to the case of extensions of odd, bounded degree of a fixed but arbitrary
number field \(F\).

The remainder of the paper is dedicated to unconditional proofs of the conjecture
for certain choices of \(K\) and \(g\). In \(\S 6\), the behavior of the special fiber is used to
further constrain the numerical invariants introduced in \(\S 3, \S 4\). These results are
then used in \(\S 7\) to prove the conjecture unconditionally in several new cases:

- \(K = \mathbb{Q}\) and \(g = 2, 3\),
- \([K : \mathbb{Q}] = 2\) and \(g = 1\),
- \([K : \mathbb{Q}] = 3\) and \(g = 1\),
- \(K/\mathbb{Q}\) is a Galois extension of exponent 3 and \(g = 1\).

Moreover, in the case of cubic extensions and \(g = 1\), we are able to give a uniform
version of the result.

1.2. Notations. For any number field \(F\), we let \(\Delta_F\) denote the absolute discrimi-
nant of \(F/\mathbb{Q}\), and let \(n_F = [F : \mathbb{Q}]\). For any extension of number fields \(E/F\), if \(P\)
is a prime of \(E\) above a prime \(p\) of \(F\), we let \(e_{P/p}\) and \(f_{P/p}\) denote, respectively,
the ramification index and the degree of the residue field extension. We let \(\kappa(p)\)
denote the residue field of \(p\).

Throughout, the notation \(C_j = C_j(x, y, \ldots, z)\) indicates a constant \(C_j\) which is
dependent on \(x, y, \ldots, z\) and no other quantities.

2. Ingredients from Analytic Number Theory

In this section, we accumulate a few results from analytic number theory that
will be needed in the sequel.

2.1. Prime \(m\)-th power residues. Let \(\ell\) be a prime number. Whenever \(m \geq 1\)
is a divisor of \(\ell - 1\), it will be useful to find a small rational prime \(p\) which is an
\(m\)-th power residue modulo \(\ell\); that is, for which \(p \equiv r^{m^{\ell-1}} \pmod{\ell}\).

Without further restriction on \(m\), the best known bound for \(p\) is
\(p = O(\ell^{5/4})\), given by Heath-Brown in \(\cite{HB92}\). However, for \(m < 23\), the following result of Elliott gives a stronger
bound \(\cite{Ell71}\):

**Proposition 2.1.** Let \(m\) be a positive integer and \(\varepsilon > 0\). There exists a constant
\(C_1' = C_1'(m, \varepsilon)\) such that for any prime \(\ell\), there exists a prime \(p < C_1' \cdot \ell^{m^{-1}+\varepsilon}\)
which is an \(m\)-th power residue modulo \(\ell\).

**Remark.** In fact, Elliott assumes \(m \mid \ell - 1\). But note that in case \(m \nmid (\ell - 1)\), we
have \(\mathbb{F}_{\ell^m}^m = \mathbb{F}_{\ell^m'}^m\), where \(m' := \gcd(m, \ell - 1) < m\). So in fact the result holds as
stated.

We re-interpret Elliott’s result as follows. For any integer \(g > 0\) and any positive
\(\varepsilon < \frac{1}{4}\), set
\[C_1 = C_1(m, g, \varepsilon) := (4gC_1')^{(m-6)/g-\varepsilon}.

**Corollary 2.2.** Suppose \(1 \leq m \leq 4\) and \(0 < \varepsilon < \frac{1}{4}\). For any prime \(\ell > C_1\), there
exists a prime number \(p < \frac{\ell^m}{4^g}\) which is an \(m\)-th power residue modulo \(\ell\).

**Proof.** The quantity \(\ell^{m^{-1}+\varepsilon}\) is sub-linear in \(\ell\); hence, there must be some lower
bound for which such a \(p\) is guaranteed to exist. More precisely, one may check
directly that \(\ell > C_1\) implies \(C_1' \ell^{m^{-1}+\varepsilon} < \frac{1}{4^g}\). \(\square\)
2.2. Goldfeld’s Theorem. We recall a result of Goldfeld which will be used in the proof of finiteness over quadratic fields when $g = 1$. Let $K$ be a number field, and let $S$ be a finite set of rational primes. Consider the following two properties possibly satisfied by an integer $N$:

(Go 1) There is a quadratic extension $L/\mathbb{Q}$ such that $-N$ is the discriminant of $L/\mathbb{Q}$. (Automatically, $L = \mathbb{Q}(\sqrt{-N})$.)

(Go 2) If $p < \frac{|N|}{4}$ is a rational prime, $p \notin S$, and $p$ splits completely in $K$, then $p$ does not split in $\mathbb{Q}(\sqrt{-N})$.

The following result of Goldfeld is proved in the Appendix of [Maz78]. (It is unfortunately not effective.)

**Theorem 2.3** (Theorem A, [Maz78]). Consider the set

$$\mathcal{N}(K, S) := \{N \in \mathbb{Z} : N \text{ satisfies both (Go 1) and (Go 2)}\}.$$  

If $n_K \leq 2$, then $\mathcal{N}(K, S)$ is finite.

We will rely on the following corollary to demonstrate finiteness of $\mathcal{A}(K, 1)$ for quadratic fields $K$.

**Corollary 2.4.** Suppose $n_K = 2$. There exists an ineffective constant $C_2 = C_2(K)$ such that, for any prime $\ell > C_2$, there exists a prime number $p < \frac{\ell}{4}$ which is a square residue modulo $\ell$ and which satisfies $f_{p/p} = 1$ for any prime $p$ of $K$ above $p$.

**Proof.** For any odd prime $\ell$, let $\ell^* = (-1)^{\frac{\ell-1}{2}} \ell$, and notice that $\ell^*$ is the discriminant of $\mathbb{Q}(\sqrt{\ell^*})/\mathbb{Q}$. Further, $p$ splits in $\mathbb{Q}(\sqrt{\ell^*})$ if and only if $(\frac{\ell}{p}) = 1$. Let us set

$$\mathcal{N}'(K) := \{-\ell^* : \ell \text{ odd prime such that for every } p < \frac{\ell}{4}, \text{ if } p \text{ splits in } K \text{ then } (\frac{\ell}{p}) = -1\}.$$  

As $\mathcal{N}'(K) \subseteq \mathcal{N}(K, \emptyset)$, the result follows immediately. \[\square\]

2.3. Chebotarev Density Theorem. Let $E/F$ be a Galois extension of number fields, and let $p$ be a prime of $F$, unramified in $E/F$. The Frobenius elements of primes $\mathfrak{p}$ of $E$ above $p$ form a conjugacy class

$$\left[\frac{E/F}{p}\right] := \{\text{Frob}_p : \mathfrak{p} \subseteq \mathfrak{o}_E, \mathfrak{p} \mid p\}$$

inside $\text{Gal}(E/F)$. If the particular choice of $\mathfrak{p}$ is irrelevant, and no confusion arises, we will write $\text{Frob}_p$ to denote any one element from this class.

Let $\sigma \in \text{Gal}(E, F)$. The Chebotarev Density Theorem states that there are infinitely many primes $p$ of $F$, unramified in $E/F$, for which $\sigma \in \left[\frac{E/F}{p}\right]$. We now recall an effective version of this result, due to Lagarias and Odlyzko [LO77], conditional on the Generalized Riemann Hypothesis (GRH).

**Theorem 2.5.** There exists an absolute constant $C_3 > 0$ with the following property. Let $E/F$ be a Galois extension of number fields, and suppose the Generalized Riemann Hypothesis holds for the Dedekind zeta function of $E$. For any $\sigma \in \text{Gal}(E/F)$, there exists a prime $p$ of $F$, unramified in $E/F$, with the following properties:

- $\sigma \in \left[\frac{E/F}{p}\right]$,
- $N_{F/Q}p \leq C_3(\log \Delta_E)^2$ (provided $E \neq \mathbb{Q}$).
Remark. This statement combines both Corollary 1.2 and the discussion on pages 461–462 of \([LO77]\).

In exchange for a weakening of the bound on the norm, we may place an additional constraint on \(p\).

**Corollary 2.6.** Let \(E/F\) be a Galois extension of number fields, and let \(\tilde{E}\) denote the Galois closure of \(E\) over \(Q\). Let \(\sigma\) be a fixed element of \(\text{Gal}(E/F)\). Assume \(\text{GRH}\) holds for the Dedekind zeta function of \(\tilde{E}\). Then there exists a prime \(p\) of \(F\), unramified in \(E/F\), such that

- \(\sigma \in \left[\frac{E/F}{p}\right]\),
- \(N_{F/Q}p \leq C_3(\log \Delta_{\tilde{E}})^2\) (provided \(E \neq Q\)),
- \(f_{p/p} = 1\), where \(p\) is the rational prime below \(p\).

**Proof.** As \(\sigma\) fixes \(Q\) and \(\sigma(E) \subseteq \tilde{E}\), there exists \(\tilde{\sigma} \in \text{Gal}(\tilde{E}/Q)\) such that \(\tilde{\sigma}|_E = \sigma\). Applying Theorem 2.5 to the extension \(\tilde{E}/Q\), we know there exists a rational prime \(p < C_3(\log \Delta_{\tilde{E}})^2\) such that \(p\) is unramified in \(\tilde{E}/Q\) and \(\tilde{\sigma} \in \left[\frac{\tilde{E}/Q}{\tilde{P}}\right]\). Thus, there is a prime ideal \(\tilde{P} | p\) of \(\tilde{E}\) such that \(\tilde{\sigma} = \text{Frob}_{\tilde{P}}\). Necessarily, the decomposition group \(D_{\tilde{P}} \leq \text{Gal}(\tilde{E}/Q)\) is generated by \(\tilde{\sigma}\). As \(\tilde{\sigma}\) fixes \(F\), we in fact have \(D_{\tilde{P}} \leq \text{Gal}(\tilde{E}/F)\).

Let \(F_1\) denote the subextension of \(\tilde{E}/F\) fixed by \(D_{\tilde{P}}\), and set \(p_1 := \tilde{P} \cap \mathcal{O}_{F_1}\). Necessarily, the residue fields \(\mathcal{O}_{F_1}/p_1\) and \(\mathbb{Z}/p\mathbb{Z}\) coincide, and so \(f_{p_1/p} = 1\). Setting \(p = p_1 \cap \mathcal{O}_F\), we have \(f_{p/p} = 1\), also. Moreover, \(\sigma \in \left[\frac{E/F}{p}\right]\); since \(N_{F/Q}p = p\), the result is shown. \(\square\)

Suppose \(\ell\) is a rational prime and \(m \mid (\ell - 1)\). We let \(Q(\mu_\ell)_m\) denote the unique subfield of \(Q(\mu_\ell)\) which is a degree \(m\) extension of \(Q\).

**Proposition 2.7.** Let \(m \geq 1, g > 0,\) and \(n \geq 1\) be fixed integers. Let \(K\) be a fixed number field, with Galois closure \(\tilde{K}\) over \(Q\). There exists a constant \(C_6 = C_6(m, g, n, K)\) with the following property. Suppose \(\ell > C_6\) is a prime number, set \(L_0 = Q(\mu_\ell)_m\), and suppose the Generalized Riemann Hypothesis holds for the Dedekind zeta function of \(L_0\tilde{K}\). Then for any \(\sigma \in \text{Gal}(L_0\tilde{K}/K)\), there exists a rational prime \(p < \left(\frac{\ell}{4g}\right)^{1/n}\) and a prime \(p \mid p\) of \(K\), such that

- \(p\) is unramified in \(L_0\tilde{K}/K\),
- \(f_{p/p} = 1\),
- \(\sigma \in \left[\frac{L_0\tilde{K}/K}{p}\right]\).

Consequently, for \(\ell > C_6\), there exists a prime \(p < \left(\frac{\ell}{4g}\right)^{1/n}\) which is an \(m\)-th power residue modulo \(\ell\).

**Proof.** Since \(C_6\) may depend on \(K\), we may assume \(\ell \nmid \Delta_K\) without loss of generality. Let \(L = L_0\tilde{K}\), \(\tilde{L} = L_0\tilde{K}\). Then \(\tilde{L}\) is the Galois closure of \(L\) over \(Q\). The fields \(L_0\) and \(\tilde{K}\) are linearly disjoint over \(Q\) and have no common factor in their discriminants. So (\cite[Prop. 17]{Lan94}) we have

\[
\Delta_{\tilde{L}} = \Delta_{L_0\tilde{K}} = \Delta_{\tilde{K}}^{n_{L_0}} \cdot \Delta_{L_0}^{n_{\tilde{K}}} = \Delta_{\tilde{K}}^{n_{L_0}} \cdot (\ell^{m-1})^{n_{\tilde{K}}}.
\]
Consequently, we always have the bound
\[ \log \Delta_L = m \log \Delta_K + (m - 1)n_K \log \ell \]
\[ \leq m \log \Delta_K + (m - 1)n_K \log \ell. \]

If \( L = \mathbb{Q} \) (i.e., if \( m = 1 \) and \( K = \mathbb{Q} \)), the assertion clearly holds with any \( C_6 > 4g \cdot 2^n \), since we may then take \( p = 2 \). So we may assume \( L \neq \mathbb{Q} \). Combining with Corollary 2.6 we see there exists a rational prime \( p \) and a prime \( \mathfrak{p} \mid p \) of \( K \), such that \( \mathfrak{p} \) is unramified in \( L \), \( f_{\mathfrak{p}/p} = 1 \), and \( \sigma \in \left[ \frac{L/K}{\mathfrak{p}} \right] \). Moreover, \( p \) may be chosen so that
\[ p \leq C_3 \cdot (C_4 + C_5 \log \ell)^2, \]
where
\[ C_4 = C_4(m, K) := m \log \Delta_K, \]
\[ C_5 = C_5(m, n_K) := \max\{1, (m - 1)n_K!\}. \]

As \( C_3 \cdot (C_4 + C_5 \log \ell)^2 < \left( \frac{4g}{\ell} \right)^{1/n} \) for \( \ell \gg 0 \), this proves the first claim. For the second claim, choose \( \sigma \in \text{Gal}(\tilde{L}/K) \) such that \( \sigma|_{\Lambda_0} = \text{id} \). Then the prime \( \mathfrak{p} \) guaranteed by the first claim has an associated Frobenius element which is trivial on \( \Lambda_0 \); this implies that \( p \) is an \( m \)-th power residue modulo \( \ell \). \( \square \)

**Remark.** Notice that this result generalizes (in fact, implies, under GRH), the earlier results of the section which guarantee a small prime \( m \)-th power residue.

**Remark.** Theorem 2.5 remains valid even if \( C_3 \) is replaced by a larger constant. So we may and do assume \( C_3 \geq 1 \). Let \( \ell' \) denote the largest prime divisor of \( \Delta_K \). For the constant \( C_6 \), we may take the value (provided \((m, K) \neq (1, \mathbb{Q})\))
\[ C_6(m, g, n, K) := \max\left\{ \ell', 16g^2 C_3^2 n C_5^4 (2n)^4 \exp\left( \frac{C_4}{C_5} \right) \right\}. \]

This follows from a lengthy argument that when \( \ell > C_6 \), \( \ell \) also satisfies the inequality
\[ \left( \frac{\ell}{4g} \right)^{1/n} > C_3 \cdot (C_4 + C_5 \log \ell)^2. \]

The details of the argument are given in the appendix.

### 3. Constraints on the Indices of Semistable Reduction

Let \( K \) be a number field, and \( A/K \) an abelian variety of dimension \( g > 0 \). Let \( \ell \) be a rational prime. For any prime \( \lambda \) of \( K \) above \( \ell \), denote by \( K_\lambda \) the \( \ell \)-adic completion of \( K \). Let \( A_{K_\lambda} \) denote the base change of \( A \) over \( K_\lambda \), and let \( e_{A_{K_\lambda}} \) be the minimal ramification index at \( \lambda \) for which semistable reduction for \( A_{K_\lambda} \) is achieved. In this section, we record some constraints on \( e_{A_{K_\lambda}} \) in general, and also under the assumption that \( [A] \in \mathcal{A}(K, g, \ell) \).

#### 3.1. The Index of Semistable Ramification

Let \( K_{\text{nr}} \) denote the maximal unramified extension of \( K_\lambda \), and let \( I_\lambda = G_{K_{\text{nr}}} \subset G_{K_\lambda} \) denote the inertia group at \( \lambda \). The ramification index \( e_{\lambda/\ell} \) divides \( [K_\lambda : \mathbb{Q}_\ell] \leq n_K \). Moreover, \( e_{\lambda/\ell} > 1 \) for some \( \lambda | \ell \) if and only if \( \ell | \Delta_K \). It is known that, for any prime \( \ell' \neq \ell \), the kernel \( J_\lambda \) of the natural representation \( \rho_{A_{K_\lambda}}^{ss} : I_\lambda \to \text{GL}(V_{\ell'}(A)^{ss}) \) is an open subgroup of \( I_\lambda \) independent of the choice of \( \ell' \). Further, \( J_\lambda \) has index \( e_{A_{K_\lambda}} \) in \( I_\lambda \), so we may be
sure that $e_{A_{K'}} \mid \#GL_{2g}(F_{\ell'})$ for every $\ell' \nmid 2\ell$. (These are consequences of [SGA71, Exposé IX].) Hence, the following is useful:

**Lemma 3.1.** Fix an integer $n > 0$. For any prime $p$ and any odd prime $\ell'$, the $p$-part of $\#GL_n(F_{\ell'})$ is divisible by $p^{u_p}$, where

$$u_2 := v_2(n! + n + \left\lceil \frac{n}{2} \right\rceil),$$

$$u_p := v_p \left( \left\lceil \frac{n}{p-1} \right\rceil + \left\lfloor \frac{n}{p-1} \right\rfloor \right) \quad (p > 2).$$

Here, $v_p$ denotes the $p$-adic valuation, and $\lceil \cdot \rceil$ denotes the greatest integer function. Moreover, for any $p$, there are infinitely many $\ell'$ such that the $p$-part of $\#GL_n(F_{\ell'})$ is exactly $p^{u_p}$.

**Proof.** The result is not new. For the case of odd $p$, a proof is given in [GL06 Lemma 7]; for the even case, a similar argument can be constructed by considering primes $\ell' \equiv 3 \pmod{8}$. The formulas given in [Ser79, pg. 120] are helpful. \[ \Box \]

Note, in particular, that $u_p = 0$ for $p > n + 1$. Consequently, the product

$$M'(n) := \prod_{p \text{ prime}} p^{u_p}$$

is always finite, and gives the greatest common divisor of $\{ \#GL_n(F_{\ell'}) : \ell' \nmid 2\ell \}$. The notation $M'(n)$ is inspired by the similarity to the quantity $M(n)$, which gives the least common multiple of all the orders of finite subgroups of $GL_n(F)$. This was first computed by Minkowski [Min1887] – see [GL06] for a modern account. In any case, we obtain the following:

**Corollary 3.2.** The index $e_{A_{K'}}$ divides $M'(2g)$. Moreover, if $p$ is a prime with $p \mid e_{A_{K'}}$, then $p \leq 2g + 1$.

### 3.2. Structure of $G_K$-action on $\ell$-torsion.

For the remainder of this section, we always work under the following assumption:

(A1) \[ [A] \in \mathcal{A}(K, g, \ell) \]

Let $\chi: \mathbb{G}_m \to F_\ell^\times$ denote the cyclotomic character modulo $\ell$. Set $\delta := [F_\ell^\times : \chi(G_K)]$, and note that $\delta$ divides both $n_K$ and $\ell - 1 = \#F_\ell^\times$. We let $\rho_{A, \ell}$ denote the representation of $G_K$-action on $A[\ell]$. If $[A] \in \mathcal{A}(K, g, \ell)$, the abelian variety $A$ must have good reduction away from $\ell$. Moreover, the structure of $\rho_{A, \ell}$ is constrained as follows:

**Lemma 3.3.** Under (A1), there is a basis of $A[\ell]$ with respect to which

$$\rho_{A, \ell} = \begin{pmatrix}
\chi_{i_1}^1 & * & \cdots & * \\
\chi_{i_2}^2 & \cdots & * & \\
\vdots & \ddots & \ddots & \\
\chi_{i_{2g}}^{2g}
\end{pmatrix}.$$

Moreover, the indices $i_r$ may be chosen so that $i_r \in \mathbb{Z} \cap [0, \frac{\ell - 1}{\delta})$ for all $1 \leq r \leq 2g$.

**Proof.** This is an obvious generalization of [RT08 Lemma 3], and in fact, the proof given there may be followed almost verbatim, with $G = \text{Gal} \left( \overline{\mathbb{Q}}/K \right)$, $N = \text{Gal} \left( \overline{\mathbb{Q}}/\mu_\ell \right)$, $\Delta = \text{Gal} \left( K(\mu_\ell)/K \right)$. In [RT08], it is only claimed that $i_r < \ell - 1$. We may be sure that the stronger bounds on $i_r$ hold, by the following lemma. \[ \Box \]
Lemma 3.4. Let $G$ be a profinite group, and let $N < G$ be a normal pro-$\ell$ subgroup of $G$. Suppose that $\Delta$ is a finite cyclic group and $\ell \nmid \# \Delta$. Let $\chi: G \to \Delta$ be a group homomorphism with $\ker \chi = N$, and let $\psi: G \to \Delta$ be any other group homomorphism. Then there exists $b \in \mathbb{Z} \cap [0, \# \chi(G))$ such that $\psi = \chi^b$.

Proof. As $\ell \nmid \# \Delta$, we clearly have $N \leq \ker \psi$. Now, both $G/N$ and $\chi(G)$ are cyclic, so let $qN$ and $x$ be generators of these respective groups such that $\chi(q) = x$. By the containments $N \leq \ker \psi \leq G$, we see #($\psi(G) = [G : \ker \psi]$ divides $[G : N] = \# \chi(G)$. We must have $\psi(G) \leq \chi(G)$, since these are subgroups of the same cyclic group $\Delta$. So $\psi(g) = x^b$ for some $0 \leq b < \# \chi(G)$. Necessarily, $\psi = \chi^b$. \hfill $\square$

3.3. Tate-Oort Theory. Let $L$ be the Galois extension of $K^w_{\xi}$ of degree $e_{A_{K_L}}$ corresponding to $J_\lambda$. Note that the extension $L/K^w_{\xi}$ descends (non-canonically) to a (possibly non-Galois) extension of $K_\lambda$ of degree $e_{A_{K_{\lambda}}}$, and even descends to an extension of $K$ of degree $e_{A_{K_{\lambda}}}$ (by approximation). As $A_{L}$ is semistable over $G_{L}$, by \cite[Exposé IX, Prop. 5.6]{SGA7}, we see that each character $\chi^{(i)}: J_{\lambda} = G_{L} \to F^\times_{\ell}$ extends to a finite group scheme over $G_{L}$. Let $\psi_{\lambda}: J_{\lambda} \to F^\times_{\ell}$ be the fundamental character over $L$. Thus, $\chi = \psi_{\lambda}^{e_{\lambda}}$ on $J_{\lambda}$, where $e_{\lambda} := e_{A_{K_{\lambda}}} \cdot e_{A_{L}}$, by \cite[§1, Prop. 8]{Ser72}.

By the theory of Tate-Oort \cite{TO70}, $\chi^{(i)} = \psi_{\lambda}^{e_{\lambda} r}$ on $J_{\lambda}$, where $j_{\lambda, r} \in \mathbb{Z} \cap [0, e_{\lambda}]$. From this, we obtain:

\begin{equation}
(3.1) \quad e_{\lambda} r \equiv j_{\lambda, r} \pmod{\ell - 1}.
\end{equation}

Among all primes of $K$ which do not divide $\ell$, choose $p_0$ whose residue field $k(p_0)$ is of minimal order, and set $q_0 := \# k(p_0)$. Considering primes of $K$ above 2 and 3, we see $q_0 \leq 3^{n_\kappa} \leq 2^n$ if $\ell \neq 2$. For any integer $n > 0$, let $P_{p_0, n} \in \mathbb{Z}[T]$ denote the characteristic polynomial of $\text{Frob}^{n}_{p_0}$ acting on $V(A)$, which has degree $2g$. Fix an algebraic closure of $\mathbb{Q}$ and let $\{\alpha_{p_0, r}\}_{r=1}^{2g}$ denote the roots of $P_{p_0, 1}$ (counting multiplicity). These roots satisfy $|\alpha_{p_0, r}| = q_0^{1/2}$, and the $n$-th powers of the $\alpha_{p_0, r}$ give exactly the roots of $P_{p_0, n}$. On the other hand, modulo $\ell$, the roots of $P_{p_0,n}$ are given by $\{\chi^{(i)}(\text{Frob}^{n}_{p_0})\}_{r=1}^{2g}$. From this and the congruence $(3.1)$, we have

\begin{equation}
(3.2) \quad \prod_{r=1}^{2g} (T - \alpha^{e_{\lambda}}_{p_0, r}) = P_{p_0, e_{\lambda}}(T) \equiv \prod_{r=1}^{2g} (T - q_0^{j_{\lambda, r}}) \pmod{\ell}.
\end{equation}

Let

\[ S(T, x_1, \ldots, x_{2g}) := \prod_{i=1}^{2g} (T - x_i) \in \mathbb{Z}[x_1, \ldots, x_{2g}][T], \]

and let $S_k(x_1, \ldots, x_{2g})$ denote the coefficient of $T^{2g-k}$ in $S$. The polynomials $S_k$ are symmetric in the $x_j$, and so $S_k(\alpha^{n}_{p_0, 1}, \ldots, \alpha^{n}_{p_0, 2g}) \in \mathbb{Z}$ for any $n \geq 1$. Using \cite[(3.2)]{Ser72}, we have

\begin{equation}
(3.3) \quad S_k(\alpha^{e_{\lambda}}_{p_0, 1}, \ldots, \alpha^{e_{\lambda}}_{p_0, 2g}) \equiv S_k(q_0^{j_{\lambda, 1}}, \ldots, q_0^{j_{\lambda, 2g}}) \pmod{\ell}.
\end{equation}

As $S_k$ is a homogeneous polynomial of degree $k$ with $(2g)^k$ terms, and $j_{\lambda, r} \leq e_{\lambda}$, we certainly have by the triangle inequality:

\[ S_k(\alpha^{e_{\lambda}}_{p_0, 1}, \ldots, \alpha^{e_{\lambda}}_{p_0, 2g}) - S_k(q_0^{j_{\lambda, 1}}, \ldots, q_0^{j_{\lambda, 2g}}) \leq \binom{2g}{k} q_0^{e_{\lambda} k / 2} + \binom{2g}{k} q_0^{e_{\lambda} k}. \]
We add the following assumption:

\[(A2) \quad \ell > \max \left\{ \left(\frac{2g}{k}\right) \left(\frac{q_0^{e_\lambda k + q_0^{r_\lambda}}}{\ell}\lambda \mid \ell, 1 \leq k \leq 2g \right) \right\}.\]

For example, this is certainly satisfied if

\[(3.4) \quad \ell > C_7 = C_7(g, n_K) := 2 \cdot \left(\frac{2g}{g}\right) \cdot 3^{2g} n_K \cdot M'(2g).\]

Take \(k = 1\); as \(q_0 \geq 2\) and \(e_\lambda \geq 1\), we note that \(\ell > 2g + 1\) always under \((A2)\).

Under \((A2)\), the congruences \((3.3)\) require equality in \(\mathbb{Z}\), which means the sets (possibly with multiplicity) \(\{a_{p_0 r, r}\}_{r=1}^{2g}\) and \(\{q_0^{e_\lambda r}\}_{r=1}^{2g}\) must be equal. By the Weil conjectures, we must have \(j_{\lambda, r} = \frac{1}{2}e_\lambda\) for each \(r\). Thus, \(2 \mid e_\lambda\). Moreover, \(A_L\) has good reduction with \(\ell\)-rank 0. (Otherwise, we would have \(j_{\lambda, r} = e_\lambda\) for some \(r\).) Combining with \((3.1)\), we obtain:

\[(3.5) \quad e_\lambda i_r = e_\lambda (\text{mod } (\ell - 1)), \quad 1 \leq r \leq 2g.\]

Set \(e = \gcd\{e_\lambda : \lambda \mid \ell\}.

**Lemma 3.5.** Assume \((A1)\) and \((A2)\) hold. Then

\((a)\) \(e \mid M'(2g)n_K\),
\((b)\) \(e \mid M'(2g)\) if \(\ell \mid \Delta_K\),
\((c)\) \((\ell, e - 1) = (\ell, \ell - 1)\),
\((d)\) \(4 \mid e\),
\((e)\) For any \(1 \leq r, s \leq 2g\), \(\ell(i_r + i_s - 1) \equiv 0 \pmod{\ell - 1}\).

**Proof.** Let \(n_\lambda\) denote the local degree \([K_\lambda : \mathbb{Q}_\ell]\) at \(\lambda\). As \(\sum_{\lambda \mid \ell} n_\lambda = n_K\) and \(e_\lambda \mid n_\lambda\), we see that \(\gcd\{e_\lambda / \lambda \mid \ell\} \mid n_K\). Now, by Corollary \((3.2)\)

\[e = \gcd\{e_{A_{K_\lambda}} \cdot e_\lambda / \lambda \mid \ell\} \mid \gcd\{M'(2g) \cdot e_\lambda / \lambda \mid \ell\} = M'(2g) \cdot \gcd\{e_\lambda / \lambda \mid \ell\} = M'(2g)n_K,\]

which proves \((a)\). When \(\ell \mid \Delta_K\), all \(e_\lambda / \lambda = 1\), so that \(e = \gcd\{e_{A_{K_\lambda}} : \lambda \mid \ell\}\), and \((b)\) follows by Corollary \((3.2)\) also. Since \(2 \mid e_\lambda\) for all \(\lambda, e\) must be even. Now, from \((3.5)\), we deduce

\[(3.6) \quad e(i_r + i_s - 1) \equiv 0 \pmod{\ell - 1}.\]

As \((2i_r - 1)\) is odd, this implies \(\text{ord}_2(e) > \text{ord}_2(\ell - 1)\). Thus, \((c)\) holds. Under \((A2)\), \(\ell > 2\), so \(\text{ord}_2(e) > 1\), proving \((d)\). Finally, adding the congruence \((3.6)\) for two indices \(1 \leq r, s \leq 2g\) gives

\[e(i_r + i_s - 1) \equiv 0 \pmod{\ell - 1}.\]

This, combined with \((c)\), implies \((e)\). \(\square\)

Already, we may prove a finiteness result for everywhere semistable abelian varieties. For fixed \(K, g, \ell\), let \(\mathcal{A}_{ss}(K, g, \ell)\) denote the subset of \(\mathcal{A}(K, g, \ell)\) containing only classes of abelian varieties with everywhere semistable reduction; likewise, let \(\mathcal{A}_{ss}(K, g)\) denote the set of pairs \(([A], \ell) \in \mathcal{A}(K, g)\) for which \(A\) has everywhere semistable reduction.

**Theorem 3.6.** For any \(K\) and any \(g > 0\), the set \(\mathcal{A}_{ss}(K, g)\) is finite. Equivalently, \(\mathcal{A}_{ss}(K, g, \ell) = \emptyset\) for \(\ell \gg 0\).
Proof. For sufficiently large $\ell$, we may be sure that both $\ell \nmid \Delta_K$, and that $\{A\}$ holds. Suppose $[A] \in \mathcal{A}^\infty(K, g, \ell)$. As $\ell \nmid \Delta_K$, we know $e_{\lambda/\ell} = 1$ for every prime $\lambda \mid \ell$ in $K$. Hence, $e_\lambda = e_{A_K}\lambda$ for every $\lambda$. But as $A$ is already semistable at $\lambda$, $e_{A_K}\lambda = 1$. Thus $e_\lambda = 1$, and so $e = 1$ also. But under $\{A\}$, $e > 1$, a contradiction. Thus, $\mathcal{A}^\infty(K, g, \ell) = \emptyset$. □

In fact, a uniform version of Theorem 3.6 is available for many values of $n_K$.

Lemma 3.7. Suppose $[A] \in \mathcal{A}^\infty(K, g, \ell)$ and $\ell > C_7(g, n_K)$.

(a) If $2 \nmid e_{A_K}\lambda$ for every $\lambda \mid \ell$, then $4 \mid n_K$.
(b) If $4 \nmid e_{A_K}\lambda$ for every $\lambda \mid \ell$, then $2 \mid n_K$.

Proof. By Lemma 3.5, we know $4 \mid e_{A_K}\lambda e_{\lambda/\ell}$ for each $\lambda \mid \ell$. If $2 \nmid e_{A_K}\lambda$ for every $\lambda$, then we must have $4 \mid e_{\lambda/\ell}$. Since $n_K = \sum_{\lambda|\ell} e_{\lambda/\ell}\varphi_{\lambda/\ell}$, we obtain (a). Part (b) may be argued the same way. □

Thus, we obtain a uniform version of Theorem 3.6 for many values of $n_K$.

Corollary 3.8. Let $n$ be a positive integer, not divisible by 4. For any number field $K/Q$ with $n_K = n$, any integer $g > 0$, and any rational prime $\ell > C_7(g, n)$, $\mathcal{A}^\infty(K, g, \ell) = \emptyset$.

Proof. Were $\mathcal{A}^\infty(K, g, \ell)$ non-empty, it would contain a class $[A]$ for which $e_{A_K}\lambda = 1$ for every $\lambda \mid \ell$. However, this contradicts Lemma 3.7(a). □

Remark. Note that the proof of Theorem 3.6 actually yields a stronger result, as we only need the existence of one $\lambda \mid \ell$ for which $A$ possesses semistable reduction. Hence, we have actually proven the finiteness of the subset of pairs $([A], \ell)$ in $\mathcal{A}(K, g)$ for which $A$ possesses semistable reduction for at least one prime of $K$ dividing $\ell$. (To be clear, this improvement is not available in the uniform version of the corollary, which requires semistable reduction at every prime above $\ell$.)

4. Supersingularity at small primes

4.1. The homomorphism $\epsilon$. We keep the notations of the previous section, and assume the hypotheses $\{A\}$ and $\{A\}$ hold. Recall $\chi$ denotes the cyclotomic character modulo $\ell$. For any $r$ and $s$ with $1 \leq r, s \leq 2g$, set $\varepsilon_{r,s} := \chi^{i_r+i_s-1}$. We further define

$$
\epsilon := (\varepsilon_{r,s})_{1 \leq r,s \leq 2g}: G_Q \rightarrow (\mathbb{F}_\ell^\times)^{(2g)^2},
$$

$$
\epsilon_0 := (\varepsilon_{r,r})_{1 \leq r \leq 2g}: G_Q \rightarrow (\mathbb{F}_\ell^\times)^{2g}.
$$

Set $m_Q := \#\epsilon(G_Q)$, and $m_{0,Q} := \#\epsilon_0(G_Q)$. Then $m_Q$ is the least common multiple of the orders of the $\varepsilon_{r,s}$, and $m_{0,Q}$ is likewise the least common multiple of the orders of the $\varepsilon_{r,r}$. Hence, $m_{0,Q} \mid m_Q$. Clearly $\epsilon$ factors through $\mathbb{F}_\ell^\times$, so $m_Q \mid (\ell - 1)$ and $\epsilon(G_Q)$ is cyclic. Further, the image has exponent $\frac{\ell}{2}$, by Lemma 3.5(a). Thus, $m_Q \mid \frac{\ell}{2}$, and so $m_{0,Q} \mid m_Q \mid (\frac{\ell}{2}, \ell - 1)$.

Lemma 4.1. We have $m_{0,Q} = m_Q$. Moreover, $\operatorname{ord}_2 m_Q = \operatorname{ord}_2(\ell - 1)$. In particular, $2 \mid m_Q$.

Proof. For any $r$ and $s$, note that $\varepsilon_{r,r} \cdot \varepsilon_{s,s} = \varepsilon_{r,s}^2$. Thus, $m_Q \mid 2m_{0,Q}$. We certainly have:

$$
\operatorname{ord}_2(m_{0,Q}) \leq \operatorname{ord}_2(m_Q) \leq \operatorname{ord}_2(\frac{\ell}{2}, \ell - 1) \leq \operatorname{ord}_2(\ell - 1).
$$
Since $\chi^{(2i_r - 1) m_0, q} = \varepsilon_{r, r}^{m_0, q} = 1$, we have $(2i_r - 1) m_0, q \equiv 0 \pmod{\ell - 1}$. Since $2i_r - 1$ is odd, it follows that $ord_q m_0, q \geq ord_q (\ell - 1)$, and so all four terms in the inequality are equal. This implies $m_q | m_0, q$, and so $m_q = m_0, q$. \hfill \Box

4.2. The characters $\chi(m)$. Let $m$ be an integer dividing $\ell - 1$. We let $\chi(m) : G_{\mathbb{Q}} \to \mathbb{F}_\ell^\times / \mathbb{F}_\ell^\times m$ denote the character $\chi$ modulo $m$-th powers. Then the character $\chi(m_q)$ carries essentially the same information as the homomorphism $\epsilon$. Indeed, we have just seen that $\epsilon$ factors through $\mathbb{F}_\ell^\times$. Since the image is cyclic of order $m_q$, we have an isomorphism $\mathbb{F}_\ell^\times / \mathbb{F}_\ell^\times m_q \cong \epsilon(G_{\mathbb{Q}})$, and the following diagram commutes:

\[
\begin{array}{ccc}
G_{\mathbb{Q}} & \xrightarrow{\chi} & \mathbb{F}_\ell^\times \\
\downarrow & & \downarrow \\
\mathbb{F}_\ell^\times / \mathbb{F}_\ell^\times m_q & \cong & \epsilon(G_{\mathbb{Q}})
\end{array}
\]

Let $p \neq \ell$ be a rational prime. Since $\chi(Frob_p) \equiv p \pmod{\ell}$, we see that $\epsilon(Frob_p)$ is trivial precisely when $p$ is an $m_q$-th power residue modulo $\ell$.

4.3. A Technique of Mazur. In [Maz78 §7], Mazur deduces congruences from the existence of an isogeny of elliptic curves. Here, we follow the spirit of Mazur’s idea and use it to study the behavior of $\chi(m_q)$. Fix a prime number $\ell$, a number field $K$, and $q > 0$. Suppose $A/K$ is an abelian variety for which [A1] and [A2] hold. Let $p$ be a rational prime, and suppose $\mathfrak{p}$ is a prime of $K$ which divides $p$. We let $q = N_{K/\mathbb{Q}} \mathfrak{p} = p^{1/r_1}$. Finally, let $Frob_{\mathfrak{p}} \in G_K$ denote a Frobenius element associated to $\mathfrak{p}$.

**Proposition 4.2.** Suppose $f_{p/\mathfrak{p}}$ is odd, and $q < \frac{1}{4g}$. Then $\chi(m_q)(Frob_{\mathfrak{p}}) \neq 1$.

**Proof.** For the sake of contradiction, suppose $\chi(m_q)(Frob_{\mathfrak{p}}) = 1$; in particular, this forces $\varepsilon_{r, s}(Frob_{\mathfrak{p}}) = 1$ for all $r, s$. Let $\{\alpha_{p, r}\}_{r=1}^{2g}$ be the eigenvalues for $Frob_{\mathfrak{p}}$. Note that $\{\alpha_{p, r}^n\}$ are the eigenvalues for $Frob_{\mathfrak{p}}^n$. If we let $a_{p, n} := \sum_{r=1}^{2g} \alpha_{p, r}^n \in \mathbb{Z}$ denote the trace of $Frob_{\mathfrak{p}}$, we have:

\[
a_{p, 2} = tr(Frob_{\mathfrak{p}}^2) = \sum_{r=1}^{2g} \chi^{i'r}(Frob_{\mathfrak{p}}^2) = \sum_{r=1}^{2g} \chi^{2i'r - 1}(Frob_{\mathfrak{p}}) \cdot \chi(Frob_{\mathfrak{p}})
\]

\[
= \sum_{r=1}^{2g} \varepsilon_{r, r}(Frob_{\mathfrak{p}}) \cdot \chi(Frob_{\mathfrak{p}}) = 2g\chi(Frob_{\mathfrak{p}}) \equiv 2gq \pmod{\ell}.
\]

On the other hand, from the Weil conjectures, we know the $\alpha_{p, r}$ are $q$-Weil numbers, and so the trace $a_{p, 2}$ is a rational integer satisfying $|a_{p, 2}| \leq 2gq$. As $q < \frac{1}{4g}$, we must have $a_{p, 2} = 2gq$. This forces $\alpha_{p, r}^2 = q$ for all $r$ (any other choice of eigenvalues gives $a_{p, 2} < 2gq$). Consequently:

\[
\alpha_{p, r} = \pm q^{1/2} = \pm p^{f_{p/\mathfrak{p}}/2}.
\]

Let $s^+$ and $s^-$ denote, respectively, the number of indices $r$ for which $\alpha_{p, r}$ is $+q^{1/2}$ or $-q^{1/2}$. Then the rational integer $a_{p, 1}$ satisfies

\[
a_{p, 1} = (s^+ - s^-)p^{f_{p/\mathfrak{p}}/2},
\]
which is only possible (since \( f_{p/p} \) is odd) if \( s^+ = s^- \) and \( a_{p,1} = 0 \). We now have:

\[
0 = a_{p,1}^2 = (\text{tr Frob}_p)^2 \equiv \left( \sum_{r=1}^{2g} \chi^{ir}(\text{Frob}_p) \right)^2 = \sum_{r=1}^{2g} \sum_{s=1}^{2g} \chi^{ir+is}(\text{Frob}_p)
\]

\[
= \sum_{r,s} \varepsilon_{r,s}(\text{Frob}_p) \cdot \chi(\text{Frob}_p) \equiv 4g^2 q \pmod{\ell}.
\]

Consequently, \( \ell \mid 4g^2q \), or what is the same, \( \ell \mid 4gq \). Clearly this contradicts \( q < \frac{\ell}{4g} \), and so it must be that \( \chi(m_Q)(\text{Frob}_p) \neq 1 \), as claimed. \( \square \)

**Corollary 4.3.** In case \( K = \mathbb{Q} \) we have \( \chi(m_Q)(\text{Frob}_p) \neq 1 \) for all \( p < \frac{\ell}{4g} \).

Moreover, also in the special case \( K = \mathbb{Q} \), we have the following result. For any positive \( \varepsilon < \frac{1}{12} \), set

\[
C_8 = C_8(g, \varepsilon) := \max \left\{ C_7(g, 1), C_1(2, g, \varepsilon), C_1(4, g, \varepsilon), (4gC_1(2, \varepsilon)^{3})^{4/(1-12\varepsilon)} \right\}.
\]

**Proposition 4.4.** Suppose \( 0 < \varepsilon < \frac{1}{12} \) and \( \ell > C_8 \). If \( [A] \in \mathfrak{A}(\mathbb{Q}, g, \ell) \), then \( m_Q > 6 \).

**Proof.** As \( \ell > C_7(g, 1) \), \( \{A2\} \) holds and \( m_Q \) must be even. If \( m_Q \leq 4 \), then by Corollary 2.2 there is a prime \( p < \frac{\ell}{4g} \) which is an \( m_Q \)-th power residue modulo \( \ell \). Thus, \( \chi(m_Q)(\text{Frob}_p) = 1 \), which contradicts the previous result. It only remains to eliminate the possibility \( m_Q = 6 \). We argue by contradiction. Suppose \( m_Q = 6 \), so that \( 6 \mid (\ell - 1) \). By Proposition 2.1 we know there exists \( p < C'_1(2, \varepsilon)^{\ell/4+\varepsilon} \) such that \( \chi(2)(\text{Frob}_p) = 1 \). As \( \ell > (4gC_1(2, \varepsilon)^{3})^{4/(1-12\varepsilon)} \), we have

\[
p^3 < C'_1(2, \varepsilon)^3 \cdot \ell^{3/4+3\varepsilon} < \frac{\ell}{4g}.
\]

A priori, the characters \( \varepsilon_{r,s} \) always take values in \( \mu_6 \). However, as \( p \) is a square modulo \( \ell \), we must have \( \varepsilon_{r,s}(\text{Frob}_p) \in \mu_3 \). Hence

\[
a_{p,6} = \text{tr}(\text{Frob}_p^6) = \sum_{r=1}^{2g} \chi^{6r}(\text{Frob}_p) = \sum_{r=1}^{2g} \chi^{2ir}(\text{Frob}_p)
\]

\[
= \sum_{r=1}^{2g} \varepsilon_{r,r}(\text{Frob}_p)^3 \chi(\text{Frob}_p)^3 \equiv 2gp^3 \pmod{\ell}.
\]

From the Weil conjectures, however, we have \( |a_{p,6}| \leq 2gp^3 \), and so we must have \( a_{p,6} = 2gp^3 \). Consequently:

\[
2gp^3 = a_{p,1}^6 + \cdots + a_{p,2g}^6.
\]

As each \( \alpha_{p,r} \) has absolute value \( p^{1/2} \), we must have \( \alpha_{p,r}^6 = p^3 \), hence \( \alpha_{p,r} = \eta^{t_r}p^{1/2} \), where \( 0 \leq t_r \leq 5 \) and \( \eta \) is a primitive sixth root of unity. Thus, \( \alpha_{p,r} \in \mathbb{Q}(\eta, \sqrt[p]{p}) \).

For each \( 0 \leq t \leq 5 \), set \( \kappa_t := \# \{ r : \alpha_{p,r} = \eta^{t}p^{1/2} \} \). The group \( \text{Gal}(\mathbb{Q}(\eta, \sqrt[p]{p})/\mathbb{Q}) = \langle \sigma, \tau \rangle \), where

\[
\sigma : \left\{ \begin{array}{c} \sqrt[p]{\eta} \mapsto -\sqrt[p]{\eta} \\ \eta \mapsto \eta \end{array} \right., \quad \tau : \left\{ \begin{array}{c} \sqrt[p]{\eta} \mapsto \sqrt[p]{\eta} \\ \eta \mapsto \eta^{-1} \end{array} \right. .
\]
As a set (possibly with multiplicity), \{\alpha_{p,1}, \ldots, \alpha_{p,2g}\} is Galois stable, which yields \(\kappa_0 = \kappa_3\) and \(\kappa_1 = \kappa_2 = \kappa_4 = \kappa_5\). Moreover,
\[
a_{p,j} = p^{j/2} \left(1 + (-1)^{j}\right)\kappa_0 + (\eta^j + \eta^{3j} + \eta^{4j} + \eta^{5j})\kappa_1,
\]
which vanishes if \(j\) is odd. So \(a_{p,3} = 0\). Consequently,
\[
0 = a_{p,3}^2 = \left(\text{tr}(\text{Frob}_p^3)\right)^2 = \sum_{r=1}^{2g} \sum_{s=1}^{2g} \chi_{r}^i(\text{Frob}_p^3)\chi_{s}^i(\text{Frob}_p^3)
= \sum_{r=1}^{2g} \sum_{s=1}^{2g} \varepsilon_{r,s}(\text{Frob}_p)^3\chi(\text{Frob}_p)^3 \equiv 4g^2\chi(\text{Frob}_p)^3 \equiv 4g^2p^3 \pmod{\ell}.
\]
As \(\ell\) is prime, this implies \(\ell | 2gp\). However, since \(p < p^3 < \frac{\ell}{4g}\), this is impossible; thus, \(m_\mathbb{Q} \neq 6\).

5. Conditional Results

In this section, we provide two proofs of the finiteness conjecture (Conjecture 1) under the assumption of the Generalized Riemann Hypothesis. The first proof is completely general, in that it demonstrates the finiteness of \(\mathcal{A}(K, g)\) for any \(K/\mathbb{Q}\). The second result is weaker, because we must add the assumption that \(n_K\) is odd. However, it is a finiteness result which is uniform in the degree \(n_K\); that is, we demonstrate the existence of at least one bound \(L\), dependent only on \(g\) and \(n_K\), but not \(K\) itself, for which \(\ell > L\) implies \(\mathcal{A}(K, g, \ell) = \emptyset\).

5.1. Finiteness via Effective Chebotarev.

**Theorem 5.1.** Let \(K\) be a number field, and let \(g > 0\). For all \(\ell \gg 0\), assume the Generalized Riemann Hypothesis holds for the Dedekind zeta functions of number fields of the form \(LK\), where \(L\) is a subfield of \(\mathbb{Q}(\mu_\ell)\). Then \(\mathcal{A}(K, g)\) is finite.

**Remark.** In fact, we need only assume the Generalized Riemann Hypothesis for the Dedekind zeta functions of \(LK\), where \(L = \mathbb{Q}(\mu_\ell)'m\) and \(m | (M'(2g)n_K, \ell - 1)\).

**Proof.** We show \(\mathcal{A}(K, g, \ell)\) is non-empty for at least finitely many \(\ell\). First, let us define:
\[
C_6(m, g, K) := \max \left\{C_6(m, g, 1, K) : m \mid \frac{1}{2}M'(2g)n_K \right\}
\]
Let \(\ell\) be a prime number with
\[
\ell > \max\{C_7(g, n_K), C_9(m, g, K)\}.
\]
We claim \(\mathcal{A}(K, g, \ell) = \emptyset\). If not, then there exists an abelian variety \(A/K\) with \([A] \in \mathcal{A}(K, g, \ell)\). Then \(A(\mathbb{Q})\) holds, and we define the quantities \(e\) and \(m_\mathbb{Q}\) associated to \(A\) as in §3, §4, respectively. By Lemma 3.3 and the observation \(m_\mathbb{Q} \mid \frac{e}{\ell} (\S4.1)\), we have \(\ell > C_6(m_\mathbb{Q}, g, 1, K)\), so we may apply Proposition 2.7 (with \(L_0 = \mathbb{Q}(\mu_\ell)m_\mathbb{Q}\) and \(\sigma = 1\)). Thus, there exist a rational prime \(p < \frac{\ell}{4g}\) and a prime \(p \mid p\) in \(K\), for which \(f_{p/p} = 1\), and for which \(L_0/K(p) = \{1\}\). As \(\chi(m_\mathbb{Q})(\text{Frob}_p) = 1\). On the other hand, by Proposition 1.12 we know \(\chi(m_\mathbb{Q})(\text{Frob}_p) \neq 1\), a contradiction. \(\square\)
5.2. A Uniform Version. Let $F$ be a field and $n > 0$ an integer. Define the following collection of extensions of $F$:

$$\mathcal{F}(F, n) := \{K : F \subset K, [K : F] = n\}.$$

**Conjecture 2** (Uniform Version). Let $g > 0$ and $n > 0$. Then there exists a bound $N = N(g, n) > 0$ such that $\mathcal{A}(K, g, \ell) = \emptyset$ for any $K \in \mathcal{F}(\mathbb{Q}, n)$ and any prime $\ell > N$.

We remark that the uniform version for $n = 1$ is exactly equivalent to the original finiteness conjecture for $\mathcal{A}(\mathbb{Q}, g)$. Thus, when considering the uniform version, we may assume $n > 1$. In this section, we prove the following version of the uniform conjecture:

**Theorem 5.2.** Assume the Generalized Riemann Hypothesis. Then Conjecture 2 holds for any $g$ and any odd $n$.

In fact, we will prove a stronger result, of which Theorem 5.2 is the specific case $F = \mathbb{Q}$.

**Theorem 5.3.** Let $F$ be any number field, and assume the Generalized Riemann Hypothesis for all Dedekind zeta functions of number fields. For any $g > 0$ and any odd $n > 0$, there exists a bound $N = N(g, n, F)$ such that $\mathcal{A}(K, g, \ell) = \emptyset$ for any $K \in \mathcal{F}(F, n)$ and any prime $\ell > N$.

**Remark.** The assumption of GRH is only needed for the Dedekind zeta functions of number fields of the form $LK$, where $L \subseteq \mathbb{Q}(\mu_n)$ for some prime $\ell$, and $K/F$ is an extension of degree $n$.

**Proof.** Set $\mathcal{M}(g, n, F) := \{m \in \mathbb{Z}_{>0} : m | \frac{1}{2} M'(2g)n_Fn\}$. Define

$$N_1 = N_1(g, n, F) := \max\{C_\mu(m, g, n, F) : m \in \mathcal{M}(g, n, F)\},$$

$$N = N(g, n, F) := \max\{N_1, C_\gamma(g, n_Fn)\}.$$  

Suppose $\ell > N$ is a prime number. Let $K \in \mathcal{F}(F, n)$ (and hence $n_K = n_Fn$), and for the sake of contradiction, suppose $[A] \in \mathcal{A}(K, g, \ell)$. By the definition of $N$, $\ell > C_\gamma(g, n_K)$, and so (A1) holds. By Lemma 3.3 and §4.1, we know the quantities $m_Q$ and $e$ associated to $A$ satisfy $m_Q | \frac{1}{2} M'(2g)n_Fn$. Now, again by the definition of $N$, $\ell > C_\gamma(m_Qg, n, F)$.

Let $\tilde{F}$ denote the Galois closure of $F$ over $\mathbb{Q}$, and set $\tilde{L} := \mathbb{Q}(\mu_n, m_Q, F)$. We have assumed $\ell > C_\mu(m_Q, g, n, F)$, and so Proposition 2.7 applies. Thus, there exists a rational prime $p < \left(\frac{Q}{2}\right)^{1/n}$ and a prime $p_F$ of $F$ dividing $p$ for which $\tilde{f}_{p_F/p} = 1$; moreover, we may assume $p$ is an $m_Q$-th power modulo $\ell$. Thus, $\chi(m_Q)(\text{Frob}_{p_F}) = 1$.

However, as $n = [K : F]$ is odd, we may choose a prime $p | p_F$ of $K$ such that $f_{p_F/p} = 1$. Thus $\tilde{f}_{p_F/p} = f_{p_F/p} = 1$ is odd and at most $n$, and $N_{K/Q} p = p^{\ell/n} \leq p^n < \frac{Q}{2}$. Hence, by Proposition 1.2, $\chi(m_Q)(\text{Frob}_{p_F}) \neq 1$. But $\text{Frob}_{p_F} = \text{Frob}_{p_F/p}$, so $\chi(m_Q)(\text{Frob}_{p_F}) \neq 1$, which gives a contradiction. \qed

**Remark.** Fix an algebraic closure $\bar{Q}$ of $\mathbb{Q}$. For a number field $K \subset \bar{Q}$, $g > 0$, and a prime $\ell$, define $\mathcal{A}(K, g, \ell)$ to be the image of $\mathcal{A}(K, g, \ell)$ in the set $\mathcal{A}(\bar{Q}, g)$ of isomorphism classes of $g$-dimensional abelian varieties over $\bar{Q}$. For any $n > 0$, define

$$\mathcal{A}(n, g, \ell) := \bigcup_{K \subset \bar{Q}, K \in \mathcal{F}(\bar{Q}, n)} \mathcal{A}(K, g, \ell) \subseteq \mathcal{A}(\bar{Q}, g).$$
Conjecture 2 may be restated as follows: Given $n > 0$ and $g > 0$, $\mathcal{A}(n, g, \ell) = \emptyset$ for $\ell$ sufficiently large. One might hope that even the set $\mathcal{A}(n, g, \ell)$ is always finite, but this is not the case.

**Proposition 5.4.** $\mathcal{A}(2, 1, 2)$ is infinite.

**Proof.** For each $i \geq 0$, let $K_i \subseteq \bar{Q}$ be the splitting field for $x^2 + 2^{i+1}x - 1$, and let $\epsilon_i$ denote the root of this polynomial given by $-2^i + \sqrt{2^{2i} + 1}$. Then $[K_i : Q] = 2$ for all $i$. Moreover, as the defining polynomial is monic with unit constant term, $\epsilon_i \in \mathcal{O}_{K_i}$. On the other hand, $\epsilon_i - 1$ satisfies $x^2 + (2^{i+1} + 2)x + 2^{i+1}$, and so $\epsilon_i - 1$ lies in $\mathcal{O}_{K_i} \cap \mathcal{O}_{K_i}[\frac{1}{2}]$.

Let $E_i$ be the elliptic curve over $K_i$ defined by the equation $y^2 = x(x-1)(x-\epsilon_i)$. Immediately we see that $E_i$ has good reduction away from 2. Moreover, $E_i[2]$ is rational over $K_i$, and so $[E_i] \in \mathcal{A}(K_i, 1, 2)$. However, this family corresponds to infinitely many distinct $j$-invariants, and so the collection $\{|E_i \times K_i, \bar{Q}\}$ is an infinite subset of $\mathcal{A}(2, 1, 2)$. □

### 6. Ingredients from the Structure of the Special Fiber

**6.1. Constraints from the action of inertia, I.** The aim of this section is to state and prove a formula relating the dimension of an abelian variety to certain invariants. This will extend the results of [Tam95, §2] into a more general setting. We return to the notations of §3. In particular, the extension $L/K^{\text{nr}}$ corresponds to a subgroup $J_\lambda$ of $I_\lambda$. Let $\kappa = \kappa(\lambda)$ denote the residue field of $\mathcal{O}_{K_\lambda}$, and let $M := I_\lambda/J_\lambda$.

Let $A$ be the semistable Néron model of $A_L := A_{K_\lambda} \times_{K_\lambda} L$ over $\mathcal{O}_L$. Then the Néron property implies that the natural action of $M$ on $A_L$ (which is compatible with the natural faithful action on $L$) extends to an action of $M$ on $A$ (itself compatible with the natural faithful action on $\mathcal{O}_L$). The latter action induces a natural action of $M$ on the special fiber, here denoted $A_\kappa$. It also induces a natural action on the connected component $B := A^\vee_{K_\lambda}$ at the origin. The actions of $M$ on $A_{\bar{\kappa}}$ and $B$ are compatible with the natural action of $M$ on $\kappa$, which is trivial as $M$ is a quotient of $I_\lambda$. Equivalently, $M$ acts on $A_{\bar{\kappa}}$ and $B$ over $\bar{\kappa}$. As $A_L$ has semistable reduction, $B$ is a semi-abelian variety over $\bar{\kappa}$. Let $T$ denote the torus part of $B$, so that we have the following canonical exact sequence:

\[(6.1) \quad 0 \to T \to B \to B/\overline{T} \to 0,\]

where $\overline{T} := T/T$ is an abelian variety over $\bar{\kappa}$. As this exact sequence is canonical, the action of $M$ preserves it. In particular, $M$ acts on $T$ and $B$.

Let $A^\vee_{K_\lambda}$ be the dual abelian variety of $A_{K_\lambda}$ over $K_\lambda$, and fix a polarization $\pi: A_{K_\lambda} \to A^\vee_{K_\lambda}$ over $K_\lambda$. For each prime $\ell' \neq \ell$, $\pi$ induces an isomorphism

\[(6.2) \quad V_{\ell'}(A_{K_\lambda}) \to V_{\ell'}(A^\vee_{K_\lambda})\]

of $G_{K_\lambda}$-modules. In particular, the field $L^\vee$, defined to be the minimal Galois extension of $K^\vee$ over which $A^\vee_{K_\lambda}$ obtains semistable reduction, coincides with $L$. Hence, the analogous quantities for the dual $A^\vee_{K_\lambda}$ also coincide with those for $A_{K_\lambda}$. That is, $e_{A^\vee_{K_\lambda}} = e_{A_{K_\lambda}}$, $J^\vee_{K_\lambda} = J_\lambda$, etc. If we denote by $\mathcal{A}^\vee$ the semistable Néron model of $A^\vee_{K_\lambda} := A^\vee_{K_\lambda} \times_{K_\lambda} L$ over $\mathcal{O}_L$, we similarly obtain a canonical exact sequence

\[(6.3) \quad 0 \to T^\vee \to B^\vee \to B^\vee/\overline{T}^\vee \to 0,\]
where $B^\vee$ is the connected component of $A^\vee_d$ at the origin, etc.

In the current context, we let $(\cdot)^\ast$ denote the functor $X \mapsto \text{Hom}(X, \mathbb{Q}[\ell](1))$.

**Lemma 6.1.** Let $\ell' \neq \ell$ be a prime number. Then

$$V_{\ell'}(A_{K,\ell})^\ast \cong V_{\ell'}(B) \oplus V_{\ell'}(T^\vee)^\ast$$

as $M$-modules.

**Proof.** Set $V := V_{\ell'}(A_{K,\ell})$ and $V^\vee := V_{\ell'}(A_{K,\ell}^\vee)$. As above, we let $A^\vee$ denote the semistable Néron model of $A^\vee_d$ over $\mathcal{O}_L$. Then there is a natural perfect pairing

$$V \times V^\vee \longrightarrow \mathbb{Q}[[\ell]](1).$$

Furthermore, by [SGA7I, Exposé IX], $V$ and $V^\vee$ admit the following natural filtrations:

$$V \supseteq V^f \supseteq V^t \supseteq \{0\}, \quad V^\vee \supseteq (V^\vee)^f \supseteq (V^\vee)^t \supseteq \{0\}.$$ 

Here, we have the following definitions/equalities:

$$V^f := V^{J_\lambda} = V_{\ell'}(B),$$

$$(V^\vee)^f := (V^\vee)^{J_\lambda} = V_{\ell'}(B^\vee),$$

$$V^t := V_{\ell'}(T),$$

$$(V^\vee)^t := V_{\ell'}(T^\vee)).$$

With respect to the pairing (6.5), $V^t$ and $(V^\vee)^t$ are exact annihilators of each other. Likewise, $V^f$ and $(V^\vee)^f$ are exact annihilators of each other. Since the action of $J_\lambda$ on $V$ is unipotent of level $\leq 2$ (meaning that for every $\xi \in J_\lambda$, $(\xi - \text{id})^2 = 0$ on $V$), we have

$$V^\ast \cong V^f + (V/V^f) = V^f \oplus (V/V^f) \cong V^f \oplus ((V^\vee)^t)^\ast$$

as $M$-modules, as desired. \hfill \Box

Let $M' = \langle \gamma \rangle$ be a cyclic subgroup of $M$, and let $e'$ be the order of $M'$. We consider the group algebras $\mathbb{Z}[M']$ and $\mathbb{Q}[M']$ of $M'$ over $\mathbb{Z}$ and $\mathbb{Q}$, respectively. Note that we have:

$$\mathbb{Q}[M'] \cong \mathbb{Q}[x]/(x^{e'} - 1) \cong \prod_{d | e'} \mathbb{Q}(\zeta_d),$$

where the isomorphisms are given by identifying the generators $\gamma$, $x$ (mod $x^{e'} - 1$), and $(\zeta_d)_{d | e'}$ in each algebra. For each divisor $d | e'$, let $\Phi_d(x) \in \mathbb{Z}[x]$ denote the $d$-th cyclotomic polynomial. Then $\Phi_d(\gamma) \in \mathbb{Z}[M']$ generates $a_d$, the kernel of the natural homomorphism $\mathbb{Z}[M'] \rightarrow \mathbb{Z}[\zeta_d]$ defined by $\gamma \mapsto \zeta_d$.

There is a natural $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}[M'] \rightarrow \text{End}(B)$, and so we may consider the quotient group scheme $B_d := B/a_dB$ of $B$, on which $\mathbb{Z}[M']$ acts via $\mathbb{Z}[M'] \rightarrow \mathbb{Z}[M']/a_d \cong \mathbb{Z}[\zeta_d]$. Similarly, we have homomorphisms from $\mathbb{Z}[M']$ into the endomorphism rings of $T$, $B^\vee$ and $T^\vee$, and so we may define quotient group schemes $T_d := T/a_dT$, $B_d^\vee := B^\vee/a_dB^\vee$, $T_d^\vee := T^\vee/a_dT^\vee$. We observe by [Tam95]...
Lem. 2.1(i)] that $V_\ell(B_d)$ and $V_\ell(T_d^\vee)$ are free $\mathbb{Z}[\zeta_d] \otimes \mathbb{Q}_\ell$-modules. Define

\[
\begin{align*}
\bd &:= \text{rank}_{\mathbb{Z}[\zeta_d] \otimes \mathbb{Q}_\ell} V_\ell(B_d), \\
\td &:= \text{rank}_{\mathbb{Z}[\zeta_d] \otimes \mathbb{Q}_\ell} V_\ell(T_d^\vee), \\
\nd &:= \bd + \td, \\
gd &:= \dim B_d = \dim B_d^\vee, \\
h_d &:= \dim T_d = \dim T_d^\vee.
\end{align*}
\]

**Proposition 6.2.** Let $\varphi$ denote Euler’s totient function. There are non-negative integers $n_d$, indexed by the divisors of $e'$, such that

\[
\begin{equation}
2g = \sum_{d \mid e'} n_d \varphi(d), \quad e' = \text{lcm}\{d : n_d > 0\}.
\end{equation}
\]

Moreover, if $d \leq 2$, then $2 \mid n_d$.

**Proof.** The indices $n_d$ will be precisely as defined above. Note that $V_\ell(T_d^\vee)^* \cong$ is also a free $\mathbb{Z}[\zeta_d] \otimes \mathbb{Q}_\ell$-module of rank $\td$. Indeed, this follows from the fact that the automorphism $\gamma \mapsto \gamma^{-1}$ of the group algebra $\mathbb{Z}[M']$ induces an automorphism of $\mathbb{Z}[\zeta_d]$ (namely, $\zeta_d \mapsto \zeta_d^{-1}$). Further, the natural morphisms

\[
\begin{equation}
\begin{aligned}
B &\to \bigoplus_{d \mid e'} B_d, \\
T^\vee &\to \bigoplus_{d \mid e'} T_d^\vee,
\end{aligned}
\end{equation}
\]

induce isomorphisms

\[
\begin{equation}
\begin{aligned}
V^f &= V_\ell(B) \cong \bigoplus_{d \mid e'} V_\ell(B_d), \\
(V^\vee)^* &= V_\ell(T^\vee) \cong \bigoplus_{d \mid e'} V_\ell(T_d^\vee),
\end{aligned}
\end{equation}
\]

respectively. The latter decomposition induces $V_\ell(T_d^\vee)^* \cong \bigoplus_{d \mid e'} V_\ell(T_d^\vee)^*$. In summary:

\[
\begin{equation}
V_\ell(A_{K_\ell})^\text{ss} \cong V_\ell(B) \oplus V_\ell(T^\vee)^* \cong \bigoplus_{d \mid e'} (V_\ell(B_d) \oplus V_\ell(T_d^\vee)^*).
\end{equation}
\]

Of course, $\mathbb{Z}[\zeta_d] \otimes \mathbb{Q}_\ell$ has dimension $\varphi(d)$ as a $\mathbb{Q}_\ell$-vector space, so by counting the dimensions of $\mathbb{Q}_\ell$-vector spaces on each side of (6.13), we obtain $2g = \sum_{d \mid e'} n_d \varphi(d)$. As $M$ acts on $V_\ell(A_{K_\ell})^\text{ss}$ faithfully, we must have $\text{lcm}\{d : d \mid e', n_d > 0\} = e'$.

Next, we observe that $h_d = t_d \varphi(d)$, and also

\[
\begin{align*}
\bd \varphi(d) &= \dim_{\mathbb{Q}_\ell} V_\ell(B_d) \\
&= \dim_{\mathbb{Q}_\ell} V_\ell(T_d^\vee) + \dim_{\mathbb{Q}_\ell} V_\ell(T_d) \\
&= 2(g_d - h_d) + h_d = 2g_d - t_d \varphi(d).
\end{align*}
\]

Consequently $2g_d = n_d \varphi(d)$. Finally, for $1 \leq d \leq 2$, we have $\varphi(d) = 1$, and so $2 \mid n_d$ as claimed. \hfill \Box

This allows us to improve the result of Corollary 3.2.

**Proposition 6.3.** Let $p$ be a prime divisor of $e'$. Then

\[
p \leq 2 \cdot \max_{d \mid e'} (g_d) + 1.
\]
Proof. By the lcm property satisfied by $e'$, there exists $d | e'$, $n_d > 0$, such that $p | d$. Then

$$p = \varphi(p) + 1 \leq \varphi(d) + 1 \leq n_d \varphi(d) + 1 = 2gd + 1.$$  

\[ \square \]

**Corollary 6.4.** If $\ell > 2g + 1$, then $e_{A_{K\lambda}}$ is prime to $\ell$, and $M$ is cyclic.

**Proof.** If $e_{A_{K\lambda}} = \#M$ is divisible by $\ell$, then there exists a cyclic subgroup $M'$ of $M$ of order $e' = \ell$. By the previous proposition (or Corollary 3.2), we then have $\ell \leq 2g + 1$, which contradicts the assumption. Since $e_{A_{K\lambda}}$ is prime to $\ell$, $M$ arises as a tame inertia group, which is therefore cyclic.

Thus, if $\ell > 2g + 1$, we may apply the results in this section to $M' = M$ and $e' = e_{A_{K\lambda}}$.

### 6.2. Constraints from the action of inertia, II.

Let us continue the notations of the previous subsection. However, we now let $d$ denote a fixed divisor of $e_{A_{K\lambda}}$, and keep $d$ fixed throughout the current subsection. We will deduce further constraints under the following assumptions:

(A3) $A_L$ has good reduction.

(A4) $\ell \nmid e_{A_{K\lambda}}$.

**Remark.** If we assume both (A1) and (A2), then (A3) and (A4) hold automatically. For (A3), this follows from the discussion prior to Lemma 3.5. For (A4), this follows from Corollary 3.2 and the observation that $\ell > 2g + 1$ under (A2).

Under (A4), $M$ is cyclic. Under (A3), $B = A_{K\lambda}^d = A_{K\lambda}^s$; we obtain the following decomposition and associated formula

$$B \sim \bigoplus_{d | e_{A_{K\lambda}}} B_d, \quad 2g = \sum_{d | e_{A_{K\lambda}}} n_d \varphi(d).$$

Let $\gamma_d$ denote the $\ell$-rank of $B_d$, so that $0 \leq \gamma_d \leq gd$. (As we typically will assume (A1) and (A2), usually $\gamma_d = 0$. See the discussion prior to Lemma 3.5.) Let $f$ and $f_{\lambda}$ denote, respectively, the orders of $\ell$ (mod $d$) and $\ell / \ell_{/I}$ (mod $d$) in $(\mathbb{Z}/d\mathbb{Z})^\times$.

Let us consider the decomposition of $A_L$ in more detail. The uniqueness of $L$, together with the fact that $K_{\lambda}^w/K_{\lambda}$ is a Galois extension, implies that $L/K_{\lambda}$ is Galois. We have the following exact sequence:

$$1 \longrightarrow \text{Gal}(L/K_{\lambda}^w) \longrightarrow \text{Gal}(L/K_{\lambda}) \longrightarrow \text{Gal}(K_{\lambda}^w/K_{\lambda}) \longrightarrow 1.$$  

As $\text{Gal}(K_{\lambda}^w/K_{\lambda}) \cong G_{\kappa(\lambda)} \cong \hat{\mathbb{Z}}$ is a free profinite group, this exact sequence splits. So there is a group-theoretic section $s$: $\text{Gal}(K_{\lambda}^w/K_{\lambda}) \hookrightarrow \text{Gal}(L/K_{\lambda})$. Let $L_0/K_{\lambda}$ be the subextension of $L/K_{\lambda}$ corresponding to the subgroup $\text{Im}(s) \subseteq \text{Gal}(L/K_{\lambda})$. Then $L_0/K_{\lambda}$ is a finite, totally ramified extension of degree $e_{A_{K\lambda}}$, and $L = L_0K_{\lambda}^w$.

In case $A_{L_0}$ has good reduction, then the inertia group $I_{L_0} \leq G_{L_0}$ acts trivially on $V_{\ell'}(A_{K\lambda})$ (as before, $\ell'$ is a prime distinct from $\ell$). Hence, $A_{L_0} := A_{K\lambda} \times_{K_{\lambda}} L_0$ has good reduction. Let $A_0$ be the proper smooth Néron model of $A_{L_0}$ over $\mathcal{O}_{L_0}$, and let $B_0$ be the special fiber $(A_0)_{K(\lambda)}$ of $A_0$. As $A_{L_0}$ has good reduction, $B_0$ is an abelian variety over $K(\lambda)$. Necessarily, $B = B_0 \times_{K(\lambda)} K(\lambda)$.

Note that the abelian subvariety $a_d B$ is stable under the action of $G_{\kappa(\lambda)} = \text{Gal}(K(\lambda)/K(\lambda))$. This follows from the fact that the prime ideal $a_d \subseteq \mathbb{Z}[M]$ is
stable under the natural action of $G_{r(\lambda)}$ (induced by the natural action of $G_{r(\lambda)}$ on $M$). Thus, the abelian subvariety $a_d B \subseteq B$ and the quotient abelian variety $B \to B/a_d B = B_d$ descend to a unique abelian subvariety and a unique quotient abelian variety $B_0 \to B_{0,d}$, respectively.

Let $\text{Frob}_\lambda \in G_{r(\lambda)}$ denote the $\ell^{|\lambda/\ell|}$-th power Frobenius element of $G_{r(\lambda)}$, and let $F_\lambda \in \text{End}_{\kappa(\lambda)}(B_{0,d}) \subseteq \text{End}_{\kappa(\lambda)}(B_d)_{\mathbb{Q}} = \text{End}(B_d)_{\mathbb{Q}}$

be the $\ell^{|\lambda/\ell|}$-th power Frobenius endomorphism. It is well-known that on $V_\mathbb{F}(B_d)$ the actions of $F_\lambda$ and $\text{Frob}_\lambda$ coincide with each other. Since $\text{End}(B_d)_{\mathbb{Q}} \subseteq \text{End}(V_\mathbb{F}(B_d))$, this implies that the conjugate action of $F_\lambda$ on $V_\mathbb{F}(B_d)_{\mathbb{Q}}$ induces an action of $\mathbb{Q}(\zeta_d) \leftrightarrow \text{End}(B_d)_{\mathbb{Q}}$ (provided $n_d > 0$, of course). Necessarily, this coincides with the natural (Galois) action of $\text{Frob}_\lambda$ on $\mathbb{Q}(\zeta_d)$. Note that the latter action is induced by $\ell^{|\lambda/\ell|}$ (mod $d$) in $(\mathbb{Z}/d\mathbb{Z})^\times \cong \text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$. In particular, we get that $F := F^{\ell^{|\lambda/\ell|}}_\lambda$ acts trivially on $\mathbb{Q}(\zeta_d)$. Equivalently, $F$ and $\mathbb{Q}(\zeta_d)$ commute with each other in $\text{End}(B_d)_{\mathbb{Q}}$. So let $\mathbb{Q}(\zeta_d)'$ denote the centralizer of $\mathbb{Q}(\zeta_d)$ in $\text{End}(B_d)_{\mathbb{Q}}$. We have shown:

**Proposition 6.5.** If $[(A3)]$ and $[(A4)]$ hold, then $\mathbb{Q}(\zeta_d)[F] \subseteq \mathbb{Q}(\zeta_d)'$.

**Lemma 6.6.** If $n_d = 1$, then $\mathbb{Q}(\zeta_d)' = \mathbb{Q}(\zeta_d)$.

**Proof.** Certainly $\mathbb{Q}(\zeta_d)$ commutes with itself, and so we need only demonstrate that $\mathbb{Q}(\zeta_d)' \subseteq \mathbb{Q}(\zeta_d)$. However, $V_\mathbb{F}(B_d)$ is a free $\mathbb{Q}(\zeta_d) \otimes \mathbb{Q}_\ell$-module of rank $n_d = 1$. Hence $\mathbb{Q}(\zeta_d)' \otimes \mathbb{Q}_\ell \subseteq \mathbb{Q}(\zeta_d) \otimes \mathbb{Q}_\ell$, which is only possible if $\mathbb{Q}(\zeta_d)' \subseteq \mathbb{Q}(\zeta_d)$. \hfill $\square$

The next proposition demonstrates the implications among the following conditions:

(C1) $\gamma_d = 0$ and $g_d \leq 2$,
(C2) $-1$ (mod $d$) $\in \langle \ell \ (\text{mod } d) \rangle$ in $(\mathbb{Z}/d\mathbb{Z})^\times$,
(C3) $B_d$ is supersingular (i.e., isogenous to a product of supersingular elliptic curves),
(C4) $\gamma_d = 0$,
(C5) $\gamma_d < g_d$ or $g_d = 0$,
(C6) either $2 \mid f$ or $d \leq 2$,
(C7) $2 \mid n_df$,
(C8) $n_df\lambda/\ell \neq 1$.

**Proposition 6.7.** (i) Under assumptions $[(A3)]$ and $[(A4)]$, the following implications always hold:

(C1) \hspace{0.5cm} (C2) \hspace{0.5cm} (C3) \hspace{0.5cm} (C4) \hspace{0.5cm} (C5) \hspace{0.5cm} (C6) \hspace{0.5cm} (C7) \hspace{0.5cm} (C8)
Proof. First, let us show that (C2) implies (C6). Notice that

\[ 6.2, 2 \equiv 1 \mod n \]

that \( \text{End}(B, Q) \) remains a division algebra after tensoring with \( D \) and \( Q \), we deduce that \( 2 \nmid (\ell - 1) \). (This occurs, for example, when \( K = Q, \lambda = (\ell) \), and \( n_d = 1 \).)

Remark. Before starting the proof, we make the following observations:

- If (C2) does not hold then instead of (C6) we have \( 2 \equiv f \mod n \)
- From the definitions, we have \( f_{\lambda} = \frac{f}{f_{\lambda/\ell}} \). Note that if \( f_{\lambda/\ell} = 1 \) and \( n_d = 1 \), then (C8) is equivalent to \( d \nmid (\ell - 1) \). (This occurs, for example, when \( K = Q, \lambda = (\ell) \), and \( n_d = 1 \).)

Assume (C6). If \( 2 \nmid f \), then of course (C7) holds. If \( d \leq 2 \), then by Proposition 6.2 \( 2 \nmid n_d \) and again (C7) holds. If \( n_d = 1 \), we clearly have the reverse implication.

Assume (C7). By the remark above, note that \( f_{\lambda} f_{\lambda/\ell} = \frac{f_{\lambda/\ell}}{f_{\lambda/\ell}} \), and so is divisible by \( f \). Hence, \( 2 \nmid n_d f_{\lambda} f_{\lambda/\ell} \), and so (C8) holds.

Now, assume (C3). We may write \( B_d \sim E^{94} \) for some supersingular elliptic curve \( E \). Then \( \text{End}(B_d) \cong M_{g_\ell}(D) \), where \( D = \text{End}(E) \) is a quaternion algebra over \( Q \) whose set of ramified primes is exactly \( \{ \ell, \infty \} \). (This determines \( D \) uniquely up to isomorphism.) In particular, \( \text{End}(B_d) \) is a central simple algebra over \( Q \) and we have \( \dim Q \text{End}(B_d) = \dim_Q M_{g_\ell}(D) = 4g_\ell^2 \). As \( n_d > 0 \), we have \( Q(\zeta_d) \hookrightarrow \text{End}(B_d) \). Let \( Q(\zeta_d)' \) denote the centralizer of \( Q(\zeta_d) \) in \( \text{End}(B_d) \). Then, by [Coh77, Thm. 10.7.5], \( Q(\zeta_d)' \) is a central simple algebra over \( Q(\zeta_d) \),

\[ \dim Q(\zeta_d) Q(\zeta_d)' = \frac{(2g_\ell)^2}{\varphi(d)^2} = n_d^2, \]

and

\[ \text{End}(B_d) \otimes_Q Q(\zeta_d) \cong M_{\varphi(d)}(Q(\zeta_d)'). \]

However, \( \text{End}(B_d) \otimes_Q Q(\zeta_d) \) is also isomorphic to \( M_{g_\ell}(D \otimes_Q Q(\zeta_d)) \), and so \( D \otimes_Q Q(\zeta_d) \) and \( Q(\zeta_d)' \) are similar (i.e., the elements of \( \text{Br}(Q(\zeta_d)) \) associated to these algebras coincide).

If \( D \otimes_Q Q(\zeta_d) \) splits, i.e., \( D \cong M_2(Q(\zeta_d)) \), let \( \mu \) be any prime of \( Q(\zeta_d) \) dividing \( \ell \). Then \( f = [Q(\zeta_d)_\mu : Q] \), and by observing the local invariant at \( \ell \) of \( [D] \in \text{Br}(Q) \), we deduce that \( 2 \nmid f \). On the other hand, if \( D \otimes_Q Q(\zeta_d) \) does not split (that is, \( D \) remains a division algebra after tensoring with \( Q(\zeta_d) \)), then by the definition of similarity we must have \( D \otimes_Q Q(\zeta_d) \subseteq Q(\zeta_d)' \), and so \( 2 \nmid n_d \). Thus, (C3) \( \Rightarrow \) (C7).

Next, we show that (C5) \( \Rightarrow \) (C8). If \( n_d \neq 1 \), (C8) always holds. So let us assume that \( n_d = 1 \). By Proposition 6.3 and Lemma 6.4, we may view \( F \) as an element of \( Q(\zeta_d) \). Since the characteristic polynomial of \( F \) has coefficients in \( \mathbb{Z} \) with constant
term a power of \( \ell \), we see that

\[(6.14) \quad F \in \mathbb{Z}[\zeta_d] \cap \mathbb{Z}[\zeta_d][\frac{1}{d}]^\times.\]

Moreover, since the action of \( F \) on \( V_{\ell'}(B_d) \) is given by the scalar action of \( \mathbb{Q}(\zeta_d) \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell'} \) on the rank one free \( \mathbb{Q}(\zeta_d) \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell'} \)-module \( V_{\ell'}(B_d) \), \( F \) must be an \( \ell f_{\lambda, \ell}/f \)-Weil number (when viewed as an element of \( \mathbb{Q}(\zeta_d) \)). Let \( c \in G := \text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q}) \) denote the restriction of complex conjugation to \( \mathbb{Q}(\zeta_d) \). We must have \( F^c F = \ell f_{\lambda, \ell}/f \).

As \( \ell \nmid d \) by (A1), the ideal \((\ell)\mathbb{Z}[\zeta_d]\) splits as a product of distinct prime ideals in \( \mathbb{Z}[\zeta_d] \). Let \( \lambda_1, \ldots, \lambda_r \) be these prime ideals, where \( r = \varphi(d)/f \). By (6.14), the prime ideal factorization of the principal ideal \((F)\) must be

\[(F) = \lambda_1^{m_1} \lambda_2^{m_2} \ldots \lambda_r^{m_r}.\]

Let \( D_\ell \leq G \) be the decomposition group of \( \ell \) (or equivalently, any of the \( \lambda_i \)). Under the isomorphism \( G \cong (\mathbb{Z}/d\mathbb{Z})^\times \), \( c \) corresponds to \(-1 \mod d \). Further, \( D_\ell \) corresponds to the subgroup \( (\ell \mod d) \). Hence, the condition \( c \in D_\ell \) is equivalent to (C2). So, if \( c \notin D_\ell \), we have (C2) \( \Rightarrow \) (C6) \( \Rightarrow \) (C7) \( \Rightarrow \) (C8).

In case \( c \notin D_\ell \), then \( r \) is even, and \( \lambda_i^c \neq \lambda_i \) for every \( 1 \leq i \leq r \). Relabel the ideals so that \( \lambda_{2j-1} = \lambda_{2j} \). Now,

\[(\lambda_1 \lambda_2 \cdots \lambda_r)^f f_{\lambda, \ell} = \ell f_{\lambda, \ell} f_{\lambda, \ell} = (F^e \cdot F) = (\lambda_1 \lambda_2)^{m_1 + m_2} \cdots (\lambda_{r-1} \lambda_r)^{m_{r-1} + m_r},\]

and so

\[f_{\lambda} \cdot f_{\lambda, \ell} = m_1 + m_2 = \cdots = m_{r-1} + m_r.\]

For the sake of contradiction, suppose (C8) does not hold; i.e., \( f_{\lambda} f_{\lambda, \ell} = 1 \). Then clearly \( m_i = 0 \) for exactly half of the indices \( i \). But (under a suitable normalization) the \( \{m_i\} \) represent the slopes of the Newton polygon of \( B_d \). More specifically, the slopes are given by:

\[
\begin{array}{cccc}
\frac{m_1}{f_{\lambda} f_{\lambda, \ell}} & \frac{m_1}{f_{\lambda} f_{\lambda, \ell}} & \frac{m_2}{f_{\lambda} f_{\lambda, \ell}} & \frac{m_2}{f_{\lambda} f_{\lambda, \ell}} \\
\frac{m_3}{f_{\lambda} f_{\lambda, \ell}} & \frac{m_3}{f_{\lambda} f_{\lambda, \ell}} & \frac{m_4}{f_{\lambda} f_{\lambda, \ell}} & \frac{m_4}{f_{\lambda} f_{\lambda, \ell}} \\
\frac{m_5}{f_{\lambda} f_{\lambda, \ell}} & \frac{m_5}{f_{\lambda} f_{\lambda, \ell}} & \frac{m_6}{f_{\lambda} f_{\lambda, \ell}} & \frac{m_6}{f_{\lambda} f_{\lambda, \ell}} \\
\end{array}
\]

But this implies the Newton polygon has \( f \cdot f \) slopes of value 0, and (as \( n_d = 1 \)), this implies \( \gamma_d = g_d > 0 \), which contradicts (C5). Thus, (C5) \( \Rightarrow \) (C8).

Finally, we show, assuming \( n_d = 1 \), that (C2) \( \Rightarrow \) (C3). Assuming (C2), we have \( c \in D_\ell \), and \( \lambda_i^c = \lambda_i \) for all \( i \). So \((F^e) = (F)\). We consider the prime factorization:

\[(\lambda_1 \lambda_2 \cdots \lambda_r)^f f_{\lambda, \ell} = (F^e \cdot F) = (F)^2 = \lambda_1^{2m_1} \lambda_2^{2m_2} \cdots \lambda_r^{2m_r},\]

which gives \( f_{\lambda} f_{\lambda, \ell} = 2m_i \) for every \( i \). Again we interpret this in terms of the Newton polygon of \( B_d \), and see that every slope is \( \frac{m_i}{f_{\lambda} f_{\lambda, \ell}} = \frac{1}{2} \). This is equivalent to the condition that \( B_d \) is supersingular, i.e., (C3).

\[
\square
\]

7. UNCONDITIONAL RESULTS

In this section, we apply the results of §6 to obtain as many finiteness results as possible without the assumption of GRH.
7.1. Unconditional finiteness results over $\mathbb{Q}$. When $[A] \in \mathcal{A}(\mathbb{Q}, g, \ell)$, note that $e = e_{A\ell}$ always. Already the finiteness of $\mathcal{A}(\mathbb{Q}, 1)$ has been established in [RT08]. The information coming from the special fiber allows us to settle the conjecture over $\mathbb{Q}$ for $g \leq 3$.

Proposition 7.1. The set $\mathcal{A}(\mathbb{Q}, 2)$ is finite.

Proof. Suppose $\ell \gg 0$ and $[A] \in \mathcal{A}(\mathbb{Q}, 2, \ell)$. By Proposition 4.4, $m_\mathbb{Q} > 6$. But this is impossible under the condition $4 = \sum n_d \varphi(d)$. To see this, note that

$$\{d : \varphi(d) \leq 4\} = \{1, 2, 3, 4, 5, 6, 8, 10, 12\}.$$

Thus, the condition $4 | e$ (Lemma 3.5) implies one of $n_{12}, n_8, n_4$ must be positive. The only solution with $n_{12} > 0$ is $4 = \varphi(12)$; in this case, $e = 12$ and $m_\mathbb{Q} \mid 6$. Likewise, $n_8 > 0$ only for the solution $4 = \varphi(8)$. In this case, $e = 8$ and $m_\mathbb{Q} \mid 4$.

There are a handful of equations with $n_4 > 0$; each, however, has $e = 4$ or $e = 12$, and so $m_\mathbb{Q} \mid 6$. By contradiction, $\mathcal{A}(\mathbb{Q}, 2, \ell)$ must be empty. $\square$

Proposition 7.2. The set $\mathcal{A}(\mathbb{Q}, 3)$ is finite.

Proof. Suppose $\ell \gg 0$ and $[A] \in \mathcal{A}(\mathbb{Q}, 3, \ell)$. As in the case $g = 2$, the conditions $6 = \sum n_d \varphi(d)$ and $4 | e$ imply $m_\mathbb{Q} \leq 6$, with only four exceptions:

$$6 = \varphi(8) + \varphi(6)$$
$$6 = \varphi(8) + \varphi(3)$$
$$6 = \varphi(4) + \varphi(10)$$
$$6 = \varphi(4) + \varphi(5)$$

So we may assume $A$ has a decomposition corresponding to one of the above exceptions. For each exception, the only possible value for $m_\mathbb{Q}$ is $\frac{5}{2}$, as every other even divisor of $\frac{5}{2}$ is at most 6. However, in each case we observe a choice of $d$ such that $n_d = 1$ and $d \mid \frac{5}{2}$. Let $f$ denote the order of $\ell$ (mod $d$). We may be sure that $\gamma_d = 0$ (cf. the discussion of $\ell$-rank in §3.3). Since $K = \mathbb{Q}$, we have $f_{\lambda/d} f_{\lambda} = f$. Hence, by Proposition 6.7, $n_d f \neq 1$, and so $f \neq 1$. This forces $d \mid (\ell - 1)$, and so $d \mid m_\mathbb{Q}$. Consequently, $m_\mathbb{Q} < \frac{5}{2}$, and by contradiction, $\mathcal{A}(\mathbb{Q}, 3, \ell) = \emptyset$. $\square$

Unconditional finiteness in the case $g = 4$ is not settled, but we have the following description of possible decompositions of the special fiber.

Proposition 7.3. If $\ell \gg 0$ and $[A] \in \mathcal{A}(\mathbb{Q}, 4, \ell)$, then the decomposition of $A$ corresponds to one of the following sums, and $\ell$ must satisfy congruence conditions as follows:

| Sum | Congruence |
|-----|------------|
| $2\varphi(3) + \varphi(8)$ | $\ell \equiv 13 \pmod{24}$ |
| $2\varphi(6) + \varphi(8)$ | $\ell \equiv 13 \pmod{24}$ |
| $\varphi(16)$ | $\ell \equiv 9 \pmod{16}$ |
| $\varphi(20)$ | $\ell \equiv 11 \pmod{20}$ |
| $\varphi(24)$ | $\ell \equiv 13 \pmod{24}$ |
Proof. Take $\ell \gg 0$ and $[A] \in \mathcal{A}(\mathbb{Q}, 4, \ell)$. If the decomposition of $A$ does not correspond to one of those given in the table, then by arguments similar to the previous cases, one may show that $m_Q \leq 6$. However, the five cases above remain valid. For example, in case $m_3 = 2$ and $m_8 = 1$, we have $e = 24$, and the available results do not eliminate the possibility $m_Q = 12$. We may conclude only that $\ell \equiv 1 \pmod{12}$ and $\ell \not\equiv 1 \pmod{8}$, i.e., $\ell \equiv 13 \pmod{24}$. The congruences in the remaining cases may be deduced similarly. □

7.2. Finiteness of $\mathcal{A}(K, 1)$ when $n_K = 2$. In the authors’ earlier work, it was shown that $\mathcal{A}(K, 1)$ must be finite when $K/\mathbb{Q}$ is a quadratic extension, except possibly when $K$ is imaginary with class number one [KT08 Thm. 4]. There, the proof uses the result of Momose [Mom95] (generalizing the previous work of Mazur [Maz78]) classifying $K$-rational points on modular curves. We give a different proof, which removes the exception for imaginary fields with class number one. The new proof relies on the theorem of Goldfeld from §2.

Proposition 7.4. Suppose $n_K = 2$. Then the set $\mathcal{A}(K, 1)$ is finite.

Proof. If $\mathcal{A}(K, 1)$ is infinite, then we may choose $\ell$ and $[A]$ so that $\ell > C_2(K)$, $\ell \nmid \Delta_K$, and $[A] \in \mathcal{A}(K, 1, \ell)$. The only solution to $2 = \sum n_d \varphi(d)$ for which $4 \mid e$ is $2 = \varphi(4)$. So in fact $e = 4$ and $m_Q = 2$. By Corollary 2.3 there exists $p < \frac{d}{4}$, which is a square residue modulo $\ell$. Moreover, $f_{p/p} = 1$ for any prime $p$ of $K$ above $p$. Consequently, $\chi(2)(\text{Frob}_p) = \chi(2)(\text{Frob}_p) = 1$. However, by Proposition 1.2, $\chi(2)(\text{Frob}_p) \neq 1$. By contradiction, $\mathcal{A}(K, 1)$ is finite. □

Remark. Unfortunately, since Goldfeld’s result is not effective, Proposition 7.4 is not effective, even for a particular choice of quadratic field $K$, and cannot be made uniform at present.

7.3. Additional finiteness results when $g = 1$. We present two more unconditional finiteness results. The first establishes finiteness for cubic fields in a uniform manner; that is, there exists a bound $N$ such that $\ell > N$ implies $\mathcal{A}(K, 1, \ell) = \emptyset$ for any cubic field $K$. The second result is not uniform, but provides finiteness for $\mathcal{A}(K, 1)$ for any Galois extension $K/\mathbb{Q}$ whose Galois group has exponent 3.

We begin with the result for cubic fields. We require an extension of the result of Proposition 1.2. Let $K/\mathbb{Q}$ be a finite extension, and suppose $[A] \in \mathcal{A}(K, 1, \ell)$. Note that as (A2) holds for $\ell > C_7$, which depends only on $g$ and $n_K$, we may use the results of §5 without invalidating uniformity.

Provided $\ell > 3$, for each prime $\lambda \mid \ell$ in $K$, we have a decomposition of $V_\ell(A_{Ks})$ which yields

$$2 = 2g = \sum_{d|e_{A_{Ks}}} n_d \varphi(d), \quad e_{A_{Ks}} = \text{lcm}\{d : n_d > 0\}.$$ 

(Note that the collection $\{n_d\}$ depends on $\lambda$, even though this is suppressed in the notation.) Necessarily, $e_{A_{Ks}} \in \{1, 2, 3, 4, 6\}$.

Proposition 7.5. Conjecture 2 holds in case $(g, d) = (1, 3)$.

Proof. In this case, we take $\ell > C_7(1, 3) > 3$, so that the above relations hold, and that further $4 \mid e := \gcd\{e_{A_{Ks}}e_{\lambda/\ell} : \lambda \mid \ell\}$. Additionally, we have $m_Q \mid (\mathcal{F}, \ell - 1)$. As $K$ is a cubic extension, we have $\sum_{\lambda\mid\ell} e_{\lambda/\ell} = n_K = 3$. If there exists a prime $\lambda \mid \ell$ for which $e_{\lambda/\ell} = 1$, then as $4 \mid e_{A_{Ks}}e_{\lambda/\ell}$, we must have $e_{A_{Ks}} = 4$, and...
consequently $e = 4$. The only other possibility is that there is a unique prime $\lambda | \ell$, for which $e_{A/\ell} = 3$. In this case, we have $e_{A,K,\lambda} = 4$, $e = 12$. Consequently, we may be sure that $m_Q = 2$ or $m_Q = 6$.

Choose $0 < \varepsilon < \frac{1}{12}$. By Proposition 2.1 there exists $p = O(\ell^{\frac{4}{3} + \varepsilon})$ such that $\chi(2)(\text{Frob}_p) = 1$. As $n_K = 3$, we may always choose $p | p$ in $K$ such that $f_{p/p} \in \{1, 3\}$. Let $q = \#p(p)$. We have $q \leq p^3 = O(\ell^{\frac{4}{3} + 3\varepsilon})$. There is an absolute constant $C_{10}$, independent of $K$, for which $\ell > C_{10}$ guarantees $q < \frac{\ell}{4}$. Below, we give a mild extension of Proposition 4.2 showing that $\chi(2)(\text{Frob}_p) \neq 1$. But this is a contradiction, since $\chi(2)(\text{Frob}_p) = (\chi(2)(\text{Frob}_p))^{f_{p/p}} = 1$.

**Proposition 7.6.** Suppose $n_K = 3$, $\ell > C_{7}(1, 3)$, and $[A] \in \mathcal{A}(K, 1, \ell)$. Let $p$ be a rational prime, and $p | p$ a prime in $K$. If $f_{p/p}$ is odd and $q = \#K(p) < \frac{\ell}{4}$, then $\chi(2)(\text{Frob}_p) \neq 1$.

**Proof.** By Proposition 2.1 we already have $\chi(m_Q)(\text{Frob}_p) \neq 1$. If $m_Q = 2$, we are done. Otherwise, we have $\chi(6)(\text{Frob}_p) \neq 1$. For the sake of contradiction, suppose that $\chi(2)(\text{Frob}_p) = 1$. Then $\chi(6)(\text{Frob}_p)$ has exact order 3; so $\varepsilon_{r, r}(\text{Frob}_p)$ must be order 3 for some $r \in \{1, 2\}$. Since the determinant of $\rho_{A, \ell}$ must be $\chi$, we have $\chi^{i_1} \chi^{i_2} = \chi$; that is, $i_2 = 1 - i_1 \pmod{(\ell - 1)}$. Hence,

$$\varepsilon_{1, 1} \varepsilon_{2, 2} = \chi^{2i_1 - 1} \chi^{2i_2 - 1} = 1.$$ 

Let $\omega = \varepsilon_{1, 1}(\text{Frob}_p) \in \mathbb{F}_p^\times$. It is necessarily a primitive cube root of unity in $\mathbb{F}_p$, and so $\varepsilon_{2, 2}(\text{Frob}_p) = \omega^2$. Computing the trace of $\text{Frob}_p^2$, we have

$$a_{p, 2} = \chi^{i_1}(\text{Frob}_p^2) + \chi^{i_2}(\text{Frob}_p^2) = (\varepsilon_{1, 1}(\text{Frob}_p) + \varepsilon_{2, 2}(\text{Frob}_p)) \chi(\text{Frob}_p) \equiv -q \pmod(\ell).$$

As $|a_{p, 2}| \leq 2q$ by the Weil conjectures and $q < \frac{\ell}{4}$, we have $a_{p, 2} = -q$. Thus, the eigenvalues $\alpha_{p, j}^2$ (in $\bar{Q}$) for $\text{Frob}_p^2$ are roots of $T^2 + qT + q^2$, hence are $\{\zeta q, \zeta^2 q\}$, where $\zeta$ denotes a primitive cube root of unity in $\bar{Q}$. Consequently, the eigenvalues for $\text{Frob}_p$ over $\bar{Q}$ are contained in $\{\pm \zeta \sqrt{3}, \pm \zeta^2 \sqrt{3}\}$, and each of these possible eigenvalues has a minimal polynomial of degree 4 over $\bar{Q}$. But this is absurd; the eigenvalues should have a minimal polynomial of degree 2 over $\bar{Q}$. By contradiction, $\chi(2)(\text{Frob}_p) \neq 1$. \qed

We now turn to the result on Galois extensions of exponent 3.

**Proposition 7.7.** Suppose $K/\mathbb{Q}$ is a Galois extension and $\text{Gal}(K/\mathbb{Q})$ has exponent 3. Then the set $\mathcal{A}(K, 1)$ is finite.

**Proof.** Take $\ell \gg 0$. Suppose $[A] \in \mathcal{A}(K, 1, \ell)$. By avoiding the primes dividing $\Delta_K$, we know the decomposition $2g = \sum n_d \varphi(d)$ corresponds to $n_4 = 1, e = 4$, and $m_Q = 2$.

Let $0 < \varepsilon < \frac{1}{12}$. From Proposition 2.1 we know there exists a prime $p = O(\ell^{\frac{4}{3} + \varepsilon})$ which is a square modulo $\ell$; hence, $\chi(2)(\text{Frob}_p) = 1$. Let $p$ be a prime in $K$ above $p$ of norm $q$. Because $\text{Gal}(K/\mathbb{Q})$ is exponent 3, we must have $f_{p/p} \in \{1, 3\}$. In particular $f_{p/p}$ is odd, and $q \leq p^3 = O(\ell^{\frac{4}{3} + 3\varepsilon}) < \frac{\ell}{4}$ if we take $\ell$ sufficiently large. By Proposition 4.2 we also have $\chi(2)(\text{Frob}_p) \neq 1$. This is a contradiction, since $\text{Frob}_p = \text{Frob}_{p/p}$. By contradiction, we have that $\mathcal{A}(K, 1, \ell)$ is empty for all large $\ell$, and so $\mathcal{A}(K, 1)$ is finite. \qed


In §2, we noted the existence of a constant $C_6$, such that $\ell > C_6$ implies 

$$
(7.1) \quad \left(\frac{\ell}{4g}\right)^\frac{1}{n} > C_3 \cdot (C_4 + C_5 \log \ell)^2.
$$

We now give a brief derivation for a choice of $C_6$. The principal tool will be the Lambert $W$-function; this is a multivalued complex function defined as follows: for every $z \in \mathbb{C}$, $W(z)$ is a solution to the equation $W(z) \exp W(z) = z$. The basic properties of $W(z)$ that we will need are all thoroughly explained in [CGH+96].

If we restrict our consideration to real values, there are only two branches of $W$ of interest, which we denote $W_0(x)$ and $W_{-1}(x)$. The real function $W_0$ is an increasing function defined on $[-e^{-1}, \infty)$, and the real function $W_{-1}$ is a decreasing function defined on $[-e^{-1}, 0)$. We have $-1 = W_0(-e^{-1}) = W_{-1}(-e^{-1})$; there are no real solutions for $x < -e^{-1}$. (All of this may be observed by na"ively ‘inverting’ the real function $x(w) = we^w$.)

**Lemma 1.** Suppose $c$ and $N$ are positive constants, and $c \geq \left(\frac{e}{N}\right)^N$. Then the largest real solution to the equation $x^\frac{1}{N} = \log(c x)$ is given by

$$
x_0 = x_0(c, N) = \frac{1}{c} \exp \left( -NW_{-1}(\frac{1}{N}c^{-\frac{1}{N}}) \right).
$$

**Proof.** From $x^\frac{1}{N} = \log(c x)$, we have

$$
-\frac{1}{N}c^{-\frac{1}{N}} \cdot (cx)^\frac{1}{N} = -\frac{1}{N} \log(cx),
$$

$$
-\frac{1}{N}c^{-\frac{1}{N}} = -\frac{1}{N} \log(cx) \cdot \exp \left( -\frac{1}{N} \log(cx) \right).
$$

The assumptions guarantee $-\frac{1}{N}c^{-\frac{1}{N}} \in [-e^{-1}, 0)$, so by the definition of $W_j$, we see $-\frac{1}{N} \log(cx) = W_j(-\frac{1}{N}c^{-\frac{1}{N}})$ for $j \in \{-1, 0\}$. Solving for $x$, we find

$$
x = \frac{1}{c} \exp \left( -NW_j\left(\frac{1}{N}c^{-\frac{1}{N}}\right)\right).
$$

That the larger solution corresponds to the index $j = -1$ follows from the fact that $W_{-1} \leq W_0 < 0$ on $[-e^{-1}, 0)$. \qed

For the remainder, we write $W$ for the specific branch $W_{-1}$. At any point $x \in (-e^{-1}, 0)$, differentiating $W(x)e^{W(x)} = x$ yields

$$
W'(x) = \frac{e^{-W(x)}}{1 + W(x)} = \frac{1}{x} \cdot \frac{W(x)}{1 + W(x)}.
$$

As $W$ is a decreasing function and $y/(1 + y)$ is increasing for all $y \neq -1$, we have for any $x \in (-\frac{1}{3}, 0)$:

$$
W(x) < \frac{W\left(-\frac{1}{3}\right)}{1 + W\left(-\frac{1}{3}\right)} \approx 1.867 \cdots < 2.
$$

We define

$$
L(x) = \begin{cases} 
W\left(-\frac{1}{3}\right) & x \in [-e^{-1}, -\frac{1}{3}) \\
2\log(-x) & x \in [-\frac{1}{3}, 0) 
\end{cases}.
$$

**Lemma 2.** For all $x \in [-e^{-1}, 0)$, $-1 \geq W(x) \geq L(x)$. 

Proof. As $W$ is decreasing, the inequality is clear for $x \leq -\frac{1}{4}$. Now, set $f(x) = W(x) - L(x)$, and observe $f(-\frac{1}{4}) > 0$. For $x > -\frac{1}{4}$, we have
\[
f'(x) = \frac{1}{x} \cdot \left( \frac{W(x)}{1 + W(x)} - 2 \right) > 0,
\]
since $x < 0$ and $W(x)/(1 + W(x)) < 2$. So $W(x) \geq L(x)$, as claimed. □

Lemma 3. Suppose $c \geq \left(\frac{1}{N}\right)^N$. Then $x_0 \leq c \cdot N^{2N}$.

Proof. We have $-\frac{1}{4}c^{-\frac{1}{N}} > -\frac{1}{4}$, and so, combining the previous Lemmas,
\[
x_0 \leq \frac{1}{c} \exp \left(-N \cdot 2 \log \left(\frac{1}{N}c^{-\frac{1}{N}}\right)\right) = c \cdot N^{2N}.
\]
□

Corollary 4. If a positive integer $\ell$ satisfies (7.1), then
\[
\ell \leq 16g^2C_3^2C_5^4n^4 \exp \left(\frac{C_4}{C_5}\right).
\]

Proof. We rewrite the inequality (7.1) as
\[
x^\frac{1}{N} > \log(cx), \quad x = \frac{\ell}{4g^2C_3^2C_5^4n}, \quad N = 2n, \quad c = 4g^2C_3^2C_5^4n \exp \left(\frac{C_4}{C_5}\right).
\]
We observe that $c \geq 4 \geq \left(\frac{1}{N}\right)^N$. Thus, $x \leq x_0 \leq c \cdot N^{2N}$, and the claim follows immediately. □

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