Fast generalized Nash equilibrium seeking under partial-decision information

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Abstract

We address the generalized Nash equilibrium seeking problem in a partial-decision information scenario, where each agent can only exchange information with some neighbors, although its cost function possibly depends on the strategies of all agents. The few existing methods build on projected pseudo-gradient dynamics, and require either double-layer iterations or conservative conditions on the step sizes. To overcome both these flaws and improve efficiency, we design the first fully-distributed single-layer algorithms based on proximal best-response. Our schemes are fixed-step and allow for inexact updates, which is crucial for reducing the computational complexity. Under standard assumptions on the game primitives, we establish convergence to a variational equilibrium (with linear rate for games without coupling constraints) by recasting our algorithms as proximal-point methods, opportunely preconditioned to distribute the computation among the agents. Since our analysis hinges on a restricted monotonicity property, we also provide new general results that significantly extend the domain of applicability of proximal-point methods. Besides, our operator-theoretic approach favors the implementation of provably correct acceleration schemes that can further improve the convergence speed. Finally, the potential of our algorithms is demonstrated numerically, revealing much faster convergence with respect to projected pseudo-gradient methods and validating our theoretical findings.

Key words: Nash equilibrium seeking; Proximal-point method; Distributed algorithms; Multi-agent systems

1 Introduction

Generalized games model the interaction between self-interested decision makers, or agents, that aim at optimizing their individual, yet inter-dependent, objective functions, subject to shared constraints. This competitive scenario has received increasing attention with the spreading of networked systems, due to the numerous engineering applications, including demand-side management in the smart grid [31], charging/discharging of electric vehicles [20], demand response in competitive markets [26], and radio communication [15]. From a game-theoretic perspective, the challenge is to assign the agents behavioral rules that eventually ensure the attainment of a satisfactory equilibrium.

A recent part of the literature focuses in fact on designing distributed algorithms to seek a generalized Nash equilibrium (GNE), a decision set from which no agent has interest to unilaterally deviate [13], [40], [4], [39], [10]. In these works, the computational effort is partitioned among the agents, but assuming that each of them has access to the decision of all the competitors (or to an aggregation value, in the case of aggregative games). Such an hypothesis, referred to as full-decision information, generally requires the presence of a central coordinator that communicates with all the agents, which is impractical in some cases [35], [16]. One example is the Nash–Cournot competition model described in [23], where the profit of each of a group of firms depends not only on its own production, but also on the total supply, a quantity not directly accessible by any of the firms. Instead, in this paper we consider the so-called partial-decision information scenario, where each agent estimates the actions of all the competitors by relying only on the information exchanged with some neighbors over a communication network. Thus, the goal is to design fully-distributed (namely, center-free) algorithms, based exclusively on peer-to-peer communication.

The partial-decision information setup has only been introduced very recently. Most results consider non-generalized games (i.e., games without shared con-
straints) [23], [32], [36], [33]. Even fewer algorithms can cope with the presence of coupling constraints [30], [5], [18], despite this extension arises naturally in most resource allocation problems [13, §2], e.g., due to shared capacity limitations. All the cited formulations resort to (projected) gradient and consensus-type dynamics, and are single-layer (i.e., they require a fixed finite number of communications per iteration). The main drawback is that, due to the partial-decision information assumption, theoretical guarantees are obtained only for small (or vanishing) step sizes, which significantly affect the speed of convergence. The only alternative available in literature consists of double-layer algorithms, [25], [29], where the agents must communicate multiple (virtually infinite) times to reach consensus, before each update. An extensive communication requirement is however a performance bottleneck, as the communication time can overwhelm the time spent on local useful processing – in fact, this is a common problem in parallel computing [22], Let alone the time lost in the transmission, sending large volumes of data on wireless networks results in a dramatically increased energetic cost.

**Contributions:** To improve speed and efficiency, we design the first fully-distributed single-layer GNE seeking algorithms based on proximal best-response. For the sake of generality and mathematical elegance, we take here an operator-theoretic approach [3], [39], and re-formulate the GNE problem as that of finding a zero of a monotone operator. The advantage is that several fixed-point iterations are known to solve monotone inclusions [1, §26], thus providing a unifying framework to design algorithms and study their convergence. For instance, the methods in [30], [5], [18], were developed based on the (preconditioned) forward-backward (FB) splitting [1, §26.5]. To enhance the convergence speed, we instead employ a proximal-point algorithm (PPA) [1, Th. 28.1], which typically can tolerate much larger step sizes. Nonetheless, the design of distributed GNE seeking PPA was elusive until now, because a direct implementation results in double-layer algorithms [34], [38].

The novelties of this work are summarized as follows:

- We propose the first PPA to compute a zero of a restricted monotone operator, which significantly generalizes classical results for maximally monotone operators. Differently from other recent extensions [12], [27], we also allow for set-valued resolvents and inexact updates, and we do not assume pseudomonotonicity or hypomonotonicity. This is a fundamental result of independent interest, which we exploit to prove convergence of our algorithms (§4.2).

- We introduce a novel primal-dual proximal best-response GNE seeking algorithm, which is the first non-gradient-based scheme for the partial-decision information setup. We derive our method as a PPA, where we design a novel preconditioning matrix to distribute the computation and obtain a single-layer iteration. Under strong monotonicity and Lipschitz continuity of the game mapping, we prove global convergence with fixed step sizes, by exploiting restricted monotonicity properties. Convergence is retained even if the proximal best-response is computed inexactly (with summable errors), which is crucial for practical implementation. Differently from [30, Alg. 1], the step sizes can be chosen independently of a certain restricted strong monotonicity constant. In turn, not only we allow for much larger steps, but parametric dependence is also improved: for instance, the bounds do not vanish when the number of agents grows, and the resulting convergence rate for non-generalized games is superior. Moreover our scheme requires only one communication per iteration, instead of two (§4.3, §5.1);

- We apply some acceleration schemes [21] to our preconditioned PPA (PPPA) and provide new theoretical convergence guarantees. We observe numerically that the iterations needed to converge can be halved (§5);

- We tailor our method to efficiently solve aggregative games, by letting each agent keep and exchange an estimate of the aggregate value only, instead of an estimate of all the other agents’ actions (§6);

- Via numerical simulations, we show that our PPPAs significantly outperform the pseudo-gradient methods in [30], [17] (the only other known fully-distributed, single-layer, fixed-step GNE seeking schemes), not only in terms of number of iterations needed to converge (hence with a considerable reduction of the communication burden), but also in terms of total computational cost (despite each agent must locally solve a strongly convex optimization problem, rather than a projection, at each step) (§7).

Some preliminary results of this paper appeared in [6], where we study only one special case for games without coupling constraints and with exact computation of the resolvent (and where we do not consider aggregative games or acceleration schemes), see §5.1.

**Basic notation:** \( \mathbb{N} \) is the set of natural numbers, including \( \mathbb{R} (\mathbb{R}_{\geq 0}) \) is the set of (nonnegative) real numbers. \( 0_q \in \mathbb{R}^q (\mathbf{1}_q \in \mathbb{R}^q) \) is a vector with all elements equal to \( 0 \) (1); \( I_q \in \mathbb{R}^{q \times q} \) is an identity matrix; the subscripts may be omitted when there is no ambiguity. For a matrix \( A \in \mathbb{R}^{p \times q} \), \([A]_{i,j} \) is the element on row \( i \) and column \( j \); \( \text{null}(A) := \{ x \in \mathbb{R}^q \mid Ax = 0_q \} \) and \( \text{range}(A) := \{ v \in \mathbb{R}^p \mid v = Ax, x \in \mathbb{R}^q \} \); \( \| A \|_\infty \) is the maximum of the absolute row sums of \( A \). If \( A = A^T \in \mathbb{R}^{q \times q} \), \( \lambda_{\min}(A) := \lambda_1(A) \leq \cdots \leq \lambda_q(A) := \lambda_{\max}(A) \) denote its eigenvalues. \( \text{diag}(A_1, \ldots, A_N) \) is the block diagonal matrix with \( A_1, \ldots, A_N \) on its diagonal. Given \( N \) vectors \( x_1, \ldots, x_N \), \( \text{col}(x_1, \ldots, x_N) := [x_1^\top \cdots x_N^\top]_\otimes \) denotes the Kronecker product. \( \ell_1 \) is the set of absolutely summable sequences.

**Euclidean spaces:** Given a positive definite matrix \( \mathbb{R}^{p \times q} \ni P \succ 0 \), \( \mathcal{H}_P := (\mathbb{R}^q, (\cdot | \cdot)_P) \) is the Euclidean space obtained by endowing \( \mathbb{R}^q \) with the \( P \)-weighted inner product \( (x | y)_P = x^\top P y \), and \( \| \cdot \|_P \) is the asso-
associated norm; we omit the subscripts if $P = I$. Unless otherwise stated, we always assume to work in $\mathcal{H} = \mathcal{H}_I$.

**Operator-theoretic background:** A set-valued operator $\mathcal{F} : \mathbb{R}^q \rightrightarrows \mathbb{R}^q$ is characterized by its graph $\text{gra}(\mathcal{F}) := \{(x, u) \mid u \in \mathcal{F}(x)\}$, dom$(\mathcal{F}) := \{x \in \mathbb{R}^q \mid \mathcal{F}(x) \neq \emptyset\}$, fix$(\mathcal{F}) := \{x \in \mathbb{R}^q \mid x \in \mathcal{F}(x)\}$ and $\text{zer}(\mathcal{F}) := \{x \in \mathbb{R}^q \mid 0 \in \mathcal{F}(x)\}$ are the domain, set of fixed points and set of zeros, respectively. $\mathcal{F}^{-1}$ denotes the inverse operator of the defined operator $\mathcal{F}$, as $\text{gra}(\mathcal{F}^{-1}) = \{(u, x) \mid (x, u) \in \text{gra}(\mathcal{F})\}$. $\mathcal{F}$ is ($\mu$-strongly) monotone in $\mathcal{H}_P$ if $\langle u - v, x - y \rangle_P \geq 0$ ($\geq \mu\|x - y\|_P^2$) for all $(x, u), (y, v) \in \text{gra}(\mathcal{F})$; we omit the indication “in $\mathcal{H}_P$” whenever $P = I$. $\mathcal{I}$ is the identity operator. $\mathcal{J}_x := (\mathcal{I} + \mathcal{F})^{-1}$ denotes the resolvent operator of $\mathcal{F}$. For a function $\psi_0 : \mathbb{R}^q \to \mathbb{R} \cup \{\infty\}$, $\text{dom}(\psi_0) := \{x \in \mathbb{R}^q \mid \psi_0(x) < \infty\}$, its subdifferential operator is $\partial \psi_0 : \text{dom}(\psi_0) \rightrightarrows \mathbb{R}^q : x \mapsto \{v \in \mathbb{R}^q \mid \psi_0(x) + \langle v, z - x \rangle \leq 0\}$ is the normal cone operator of $\psi_0$. If $S$ is closed and convex, then $\partial \psi_0 = N_S$ and $(\mathcal{I} + N_S)^{-1} = \text{proj}_S$ is the Euclidean projection onto $S$. Given $\mathcal{F} : S \rightrightarrows \mathbb{R}^q$, the variational inequality $\mathcal{V}(\mathcal{F}, S)$ of the problem of finding $x^* \in S$ such that $\langle \mathcal{F}(x^*) - x, x - x^* \rangle \geq 0$, for all $x \in S$ (or, equivalently, $x^*$ such that $0 \in \mathcal{F}(x^*) + N_S(x^*)$). We denote the solution set of $\mathcal{V}(\mathcal{F}, S)$ by $\text{SOL}(\mathcal{F}, S)$.

### 2 Mathematical setup

We consider a set of agents, $\mathcal{I} := \{1, \ldots, N\}$, where each agent $i \in \mathcal{I}$ shall choose its decision variable (i.e., strategy) $x_i$ from its local decision set $\Omega_i \subseteq \mathbb{R}^{n_i}$. Let $x := \text{col}(x_i)_{i \in \mathcal{I}} \in \Omega$ denote the stacked vector of all the agents’ decisions, $\Omega := \Omega_1 \times \cdots \times \Omega_N \subseteq \mathbb{R}^n$ the overall action space and $n := \sum_{i=1}^N n_i$. The goal of each agent $i \in \mathcal{I}$ is to minimize its objective function $J_i(x_i, x_{-i})$, which depends on both the local variable $x_i$ and on the decision variables of the other agents $x_{-i} := \text{col}(x_j)_{j \in \mathcal{I} \setminus \{i\}}$. Furthermore, the feasible decisions of each agent depend on the action of the other agents via coupling constraints, which we assume affine: most of the literature focuses on the case $\mathcal{I} = \{1\}$ [29, 5], which in fact accounts for the vast majority of practical applications [13, §3.2]. Specifically, the overall feasible set $\mathcal{X} := \Omega \cap \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A := [A_1, \ldots, A_N]$ and $b := \sum_{i=1}^N b_i$, $A_i \in \mathbb{R}^{m \times n_i}$ and $b_i \in \mathbb{R}^m$ being local data. The game is then represented by the interdependent optimization problems:

\[
\forall i \in \mathcal{I} : \text{minimize } J_i(y_i, x_{-i}) \text{ s.t. } (y_i, x_{-i}) \in \mathcal{X}. \tag{1}
\]

The technical problem we consider here is the computation of a GNE, namely a set of decisions that simultaneously solve all the optimization problems in (1).

**Definition 1.** A collective strategy $x^* = \text{col}(x_i^*)_{i \in \mathcal{I}}$ is a generalized Nash equilibrium if, for all $i \in \mathcal{I}$,

\[
J_i(x_i^*, x_{-i}) \leq \inf \{J_i(y_i, x_{-i}) \mid (y_i, x_{-i}) \in \mathcal{X}\}. \tag{2}
\]

Next, we postulate some common regularity and convexity assumptions for the constraint sets and cost functions, as in, e.g., [23, Asm. 1], [30, Asm. 1].

**Standing Assumption 1.** For each $i \in \mathcal{I}$, the set $\Omega_i$ is closed and convex; $\mathcal{X}$ is non-empty and satisfies Slater’s constraint qualification; $J_i$ is continuous and $J_i(\cdot, x_{-i})$ is convex and continuously differentiable for every $x_{-i}$.

As per standard practice [29], [39], among all the possible GNEs, we focus on the subclass of variational GNEs (v-GNEs) [13, Def. 3.11], which are more economically justifiable, as well as computationally tractable [24]. The v-GNEs are so called because they coincide with the solutions to the variational inequality VI($F, \mathcal{X}$), where $F$ is the pseudo-gradient mapping of the game:

\[
F(x) := \text{col}(\langle \nabla_x J_i(x_i, x_{-i}), i \in \mathcal{I} \rangle). \tag{2}
\]

Under Standing Assumption 1, $x^*$ is a v-GNE of the game in (1) if and only if there exists a dual variable $\lambda^* \in \mathbb{R}^m$ such that the following Karush–Kuhn–Tucker (KKT) conditions are satisfied [13, Th. 4.8]:

\[
0_n \in F(x^*) + A^T \lambda^* + N_\mathcal{X}(x^*) - \langle Ax^* - b, 0_n \rangle_{\mathbb{R}^m} = 0. \tag{3}
\]

**Standing Assumption 2.** The pseudo-gradient mapping $F$ in (2) is $\mu$-strongly monotone and $\theta_0$-Lipschitz continuous, for some $\mu, \theta_0 > 0$.

The strong monotonicity of $F$ is sufficient to ensure existence and uniqueness of a v-GNE [14, Th. 2.3.3]; it was always assumed for GNE seeking under partial-decision information with fixed step sizes [36, Asm. 2], [30, Asm. 3] (while it is sometimes replaced by strict monotonicity or cocoercivity, under vanishing steps and compactness of $\mathcal{X}$ [23, Asm. 2], [28, Asm. 3] [5, Asm. 5]).

### 3 Fully-distributed equilibrium seeking

In this section, we present our baseline algorithm to seek a v-GNE of the game in (1) in a fully-distributed way. Specifically, each agent $i$ only knows its own cost function $J_i$ and feasible set $\Omega_i$, and the portion of the coupling constraints $(A_i, b_i)$. Moreover, agent $i$ does not have full knowledge of $x_{-i}$, and only relies on the information exchanged locally with some neighbors over an undirected communication network $\mathcal{G}(\mathcal{I}, \mathcal{E})$. The unordered pair $(i, j)$ belongs to the set of edges $\mathcal{E}$ if and only if agent $i$ and $j$ can mutually exchange information. We denote: $W = [w_{i,j}]_{i,j \in \mathcal{I}} \in \mathbb{R}^{N \times N}$ the symmetric weight matrix of $\mathcal{G}$, with $w_{i,j} > 0$ if $(i, j) \in \mathcal{E}$, $w_{i,j} = 0$ otherwise, and the convention $w_{i,i} = 0$ for all $i \in \mathcal{I}$; $L := D - W$ the Laplacian matrix of $\mathcal{G}$, with
Algorithm 1 Fully-distributed v-GNE seeking via PPPA

Initialization: \( \cdot \) For all \( i \in \mathcal{I} \), set \( x_i^0 \in \Omega_i \), \( x_{i,-i}^0 \in \mathbb{R}^{n_i-n_{i-1}} \), \( s_i^0 = 0_m \), \( \lambda_i^0 \in \mathbb{R}_{+0}^m \).

For all \( k \in \mathbb{N} \): \( \cdot \) Communication: The agents exchange the variables \( \{x^k_i, x^k_{i,-i}, \lambda^k_i\} \) with their neighbors.

\( \cdot \) Local variables update: each agent \( i \in \mathcal{I} \) computes

\[
x_i^{k+1} = \frac{1}{\tau + d_{ii}} (x_i^k - \tau_i \sum_{j \in \mathcal{N}_i} w_{i,j} x_j^k)
\]

\[
x_i^{k+1} = \arg\min_{y \in \Omega_i} (J(y, x_{i,-i}^{k+1}) + \frac{1}{2\tau_i} \|y - x_i^k\|^2 + \frac{d}{\tau_i} \sum_{j \in \mathcal{N}_i} w_{i,j} \|x_j^k\|^2 + \frac{1}{2} (A_i^T \lambda_i^k)\top y)
\]

\[
z_i^{k+1} = z_i^k + \sum_{j \in \mathcal{N}_i} \nu_{i,j} w_{i,j} (\lambda_j^k - \lambda_i^k)
\]

\[
\lambda_i^{k+1} = \text{proj}_{\mathbb{R}_{+0}^m} (\lambda_i^k + \delta_i (A_i(2x_i^{k+1} - x_i^k) - b_i - (2z_i^{k+1} - z_i^k))).
\]

Remark 1. The functions \( J_i(\cdot, x_{i,-i}) \) are strongly convex, for all \( x_{i,-i}, i \in \mathcal{I} \), as a consequence of the Standing Assumption 2. Hence, the argmin operator in Algorithm 1 is single-valued, and the algorithm is well defined.

Remark 2. In Algorithm 1, each agent has to locally solve an optimization problem, at every iteration. Not only these subproblems are fully-decentralized (i.e., they do not require extra communication), but they are also of low dimension \( (n_i) \). This is a major departure from the procedure proposed in the PPAs [34, Alg. 2], [38, Alg. 2], where the agents have to collaboratively solve a subgame (of dimension \( n \)) before each update.

4 Convergence analysis

4.1 Definitions and preliminary results

We denote \( \mathbf{x} := \text{col}(\{x_i\}_{i \in \mathcal{I}}) \in \mathbb{R}^{Nn} \). Besides, let us define, as in [30, Eq. 13, 14], for all \( i \in \mathcal{I} \),

\[
\mathcal{R}_i := \begin{bmatrix} 0_{n_i \times n_i} & I_{n_i} & 0_{n_i \times n_{i+1}} \end{bmatrix} \in \mathbb{R}^{n_i \times n}, \quad (4a)
\]

\[
\mathcal{S}_i := \begin{bmatrix} I_{n_{i-1}} & 0_{n_{i-1} \times n_i} & 0_{n_{i-1} \times n_{i+1}} \\ 0_{n_{i+1} \times n_i} & 0_{n_{i+1} \times n_{i+1}} & I_{n_{i+1}} \end{bmatrix} \in \mathbb{R}^{n_{i-1} \times n} \quad (4b)
\]

where \( n_{<i} := \sum_{j<i,j \in \mathcal{I}} n_j, n_{>i} := \sum_{j>i,j \in \mathcal{I}} n_j \) and \( n_{-i} := n - n_i \). In simple terms, \( \mathcal{R}_i \) selects the \( i \)-th \( n_i \)-dimensional component from an \( n \)-dimensional vector, while \( \mathcal{S}_i \) removes it. Thus, \( \mathcal{R}_i \mathbf{x} = \mathbf{x}_{i,--} = x_i \) and \( \mathcal{S}_i \mathbf{x} = \mathbf{x}_{i,-} \). Let \( \mathcal{R} := \text{diag} (\{\mathcal{R}_i\}_{i \in \mathcal{I}}) \), \( \mathcal{S} := \text{diag} (\{\mathcal{S}_i\}_{i \in \mathcal{I}}) \). It follows that \( \mathbf{x} = \mathcal{R} \mathbf{x} \) and \( \text{col}(\mathbf{x}_{i,-} \in \mathcal{I}) = \mathcal{S} \mathbf{x} \in \mathbb{R}^{(N-1)n} \). Moreover, \( \mathbf{x} = \mathbf{x}^\top + S^\top S \mathbf{x} \). We define the extended pseudo-gradient mapping \( \mathbf{F} : \mathbb{R}^{Nn} \to \mathbb{R}^n \) as

\[
\mathbf{F} (\mathbf{x}) := \text{col}(\{ \nabla_{x_i} J_i(x_i, x_{i,-}) \}_{i \in \mathcal{I}}).
\]
and the operators
\[
F_n(x) := \alpha R^T F(x) + (D_n - W_n)x, \\
\mathcal{A}(\omega) := \begin{bmatrix} F_n(x) \\ 0_{Em} \end{bmatrix} + \begin{bmatrix} R^T A^\lambda \\ -V_m\lambda \end{bmatrix} + \begin{bmatrix} N_{\Omega}(x) \\ 0_{Em} \end{bmatrix} \begin{bmatrix} N_{R^\Omega}^{\geq m}(\lambda) \end{bmatrix}
\]
where \( \alpha > 0 \) is a design constant, \( \omega := \text{col}(x, v, \lambda) \), \( \nu := \text{col}(v_t)_{t \in \{1, \ldots, E\}} \in \mathbb{R}^{Em} \), \( \lambda := \text{col}(\lambda_i)_{i \in I} \in \mathbb{R}^{M_n} \), \( A := \text{diag}(A_i)_{i \in I} \), \( W_n := W \otimes I_n \), \( D_n := D \otimes I_n \), \( V_m := V \otimes I_m \), and \( \Omega := \{ x \in \mathbb{R}^{nN} \mid R(x) \in \Omega \} \).

The following lemma relates the unique v-GNE of the game in (1) to the zeros of the operator \( \mathcal{A} \). The proof is analogous to [30, Th. 1] or Lemma 10 in Appendix B, and hence it is omitted.

**Lemma 1.** Let \( \mathcal{A} \) be as in (7). It holds that \( \text{zer}(\mathcal{A}) \neq \emptyset \). Moreover, let \( x^* \in \mathbb{R}^{nN/}, \lambda^* \in \mathbb{R}^{M_n} \), then the following statements are equivalent:
1. There exists \( v^* \) such that \( \text{col}(x^*, v^*, \lambda^*) \in \text{zer}(\mathcal{A}) \).
2. \( x^* = 1_N \otimes x^* \) and \( \lambda^* = 1_N \otimes \lambda^* \), where the pair \((x^*, \lambda^*)\) satisfies the KKT conditions in (3), hence \( x^* \) is the v-GNE of the game in (1).

Effectively, Lemma 1 provides an extension of the KKT conditions in (3) and allows us to recast the GNE problem as that of computing a zero of the operator \( \mathcal{A} \), for which a number of iterative algorithms are available [1, §26-28]. In fact, in §4.3, we show that Algorithm 1 can be recast as a PPA [1, Th. 23.41].

Nonetheless, technical difficulties arise in the analysis because of the partial-decision information. Specifically, in (5), each partial gradient \( \nabla_{x_i} J_i(x_i, x_{i-1}) \) is evaluated on the local estimate \( x_{i-1} \), and not on the actual value \( x_i \). Only when the estimates \( x_i \) are at consensus, i.e., \( x_i = 1_N \otimes x \) (namely, the estimate of each agent coincides with the actual value of \( x \)), we have that \( \bar{F}(x) = F(x) \). As a result, the operator \( R^T \bar{F} \) and consequently the operator \( \mathcal{A} \) is not monotone in general, not even under the strong monotonicity of the game mapping \( F \) in Standing Assumption 2. Instead, analogously to the approaches in [33], [30], [17], our analysis is based on a restricted monotonicity property.

**Definition 2.** An operator \( F : \mathbb{R}^q \rightarrow \mathbb{R}^q \) is restricted (\( \mu \)-strongly) monotone in \( \mathcal{H}_P \) if \( \text{zer}(F) \neq \emptyset \) and \( \langle \omega - \omega^* | u \rangle \geq 0 \) for all \( (\omega, u) \in \text{gra}(F) \), \( \omega^* \in \text{zer}(F) \) (we omit the characterization “in \( \mathcal{H}_P \)” whenever \( P = I \)).

Definition 2 differs from that in [30, Lem. 3], as we only consider properties with respect to the zero set and we need to include set-valued operators. The definition comprises the nonemptiness of the zero set and it does not exclude an operator that is multi-valued on its zeros. The next lemmas show that restricted monotonicity of \( \mathcal{A} \) can be guaranteed for any game satisfying Standing Assumptions 1–3, without additional hypotheses.

**Lemma 2 ([7, Lemma 3]).** The mapping \( F \) in (5) is \( \theta \)-Lipschitz continuous, for some \( \theta \in [\mu, \theta_0] \).

**Lemma 3.** Let \( \alpha_{\text{max}} := \frac{\gamma \mu L(\lambda)}{(\theta_0 + \theta)^2 + 4 \mu \theta} \),
\[
M := \alpha \begin{bmatrix} \frac{\mu}{\theta} & \frac{\theta_0 + \theta}{2 \sqrt{\lambda \theta}} \\
-\frac{\theta_0 + \theta}{2 \sqrt{\lambda \theta}} & \frac{\theta_0 + \theta}{\alpha} - \theta \end{bmatrix}, \quad \mu_{F_+} := \lambda_{\min}(M).
\]
If \( \alpha \in (0, \alpha_{\text{max}}] \), then \( \mu_{F_+} \geq 0 \) and \( \mathcal{A} \) in (7) is restricted monotone.

**Proof.** The operator \( \mathcal{A} \) in (7) is the sum of three operators. The third is monotone by properties of normal cones [1, Th. 20.25]; the second is a linear skew-symmetric operator, hence monotone [1, Ex. 20.35]. Let \( \omega^* = \text{col}(x^*, v^*, \lambda^*) \in \text{zer}(A) \), where \( \text{zer}(A) \neq \emptyset \) by Lemma 1. By Lemma 1, \( x^* = 1_N \otimes x^* \), with \( x^* \) the v-GNE of the game in (1); hence by [30, Lemma 3], for any \( \alpha \in (0, \alpha_{\text{max}}] \), it holds that \( M \geq 0 \) and that, for all \( x \in \mathbb{R}^{nN} \),
\[
\langle x - x^* | F_n(x) - F_n(x^*) \rangle \geq \mu_{F_+} \| x - x^* \|^2.
\]
Therefore, for all \( (\omega, u) \in \text{gra}(A) \), with \( \omega = \text{col}(x, v, \lambda), (\omega - \omega^* | u - 0) \geq \mu_{F_+} \| x - x^* \|^2 \geq 0 \).

### 4.2 PPA for restricted monotone operators

In the remainder of this section, we show that Algorithm 1 is an instance of the PPA, applied to seek a zero of the (suitably preconditioned) operator \( \mathcal{A} \) in (7). Then, we show its convergence based on the restricted monotonicity result in Lemma 3.

Informally speaking, in proximal-point methods, a problem is decomposed into a sequence of regularized subproblems, which are possibly better conditioned and easier to solve. Let \( B : \mathbb{R}^q \rightarrow \mathbb{R}^q \) be maximally monotone [1, Def. 20.20] in a space \( H_P \), and \( J_B = (\text{Id} + B)^{-1} \) its resolvent. Then, \( \text{dom}(J_B) = \mathbb{R}^q \) and \( J_B \) is single-valued; moreover, if \( \text{zer}(B) \neq \emptyset \), then the sequence \( (\omega^k)_{k \in \mathbb{N}} \) generated by the PPA,
\[
(\forall k \in \mathbb{N}) \quad \omega^{k+1} = J_B(\omega^k), \quad \omega^0 \in \mathbb{R}^q
\]
converges to a point in \( \text{zer}(B) = \text{fix}(J_B) \) [1, Th. 23.41]. Note that performing the update in (10) is equivalent to solving for \( \omega^{k+1} \) the (regularized) inclusion
\[
0 \in B(\omega^{k+1}) + \omega^{k+1} - \omega^k.
\]
Unfortunately, many operator-theoretic properties are
not guaranteed if $B$ is only restricted monotone. In fact, $J_B$ might not be defined everywhere or single-valued.

**Example 1.** Let $B : \mathbb{R} \to \mathbb{R}$, with $B(\omega) = 9 - 2\omega$ if $\omega \in [3, 4)$, $B(\omega) = \omega$ otherwise. Then, $\text{zer}(B) = \{0\}$ and $B$ is restricted strongly monotone. However, $J_B(\omega) = \{\omega/2, 9 - \omega/2\}$ if $\omega \in (5, 6]$. ■

Nonetheless, some important properties carry on to the restricted monotone case, as we prove next.

**Lemma 4.** Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be restricted monotone in $H_P$. Then, $J_B$ is firmly quasinonexpansive in $H_P$: for any $(\omega, u) \in \text{gra}(J_B)$, $w^* \in \text{zer}(B) = \text{fix}(J_B)$, it holds that

$$
(\omega - u \mid w - w^*)_P - \|w - w^*\|^2_P = (\omega - u \mid u - w^*)_P \geq 0.
$$

Moreover, $J_B(w^*) = \{w^*\}$.

**Proof.** By definition of resolvent, $w^* \in J_B(w^*) \Leftrightarrow w^* + \partial B(\omega) w^* = 0 \in B(w^*)$; also, for any $(\omega, u) \in \text{gra}(J_B)$, $w^* \in \text{zer}(B) \Leftrightarrow B(u) = 0$. Hence, the inequality in (12) is the restricted monotonicity of $B$; the elementary equality follows by expanding the terms. Finally, by taking $w = w^*$ in (12), we infer that $J_B$ is single-valued on $\text{fix}(J_B)$. ■

Next, by leveraging Lemma 4, we extend classical results for the PPA [11, Th. 5.6] to the case of a restricted monotone operator (possibly with multi-valued resolvent).

**Theorem 1.** Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be restricted monotone in $H_P$, and $C := \text{zer}(B) \neq \emptyset$. Let $(\gamma_k)_{k \in \mathbb{N}}$ be a sequence in $[0, 2]$, and $(e^k)_{k \in \mathbb{N}}$ a sequence in $\mathbb{R}^n$ such that $(\gamma_k\|e^k\|_P)_{k \in \mathbb{N}} \in \ell^1$. Let $w_0 \in \mathbb{R}^n$ and let $(w^k)_{k \in \mathbb{N}}$ be any sequence such that:

$$
(\forall k \in \mathbb{N}) \quad w^{k+1} = w^k + \gamma_k(u^k - w^k + e^k), \quad u^k \in J_B(w^k).
$$

Then, the following holds:

(i) $(\forall w^* \in C)(\forall k \in \mathbb{N}) \quad \|w^{k+1} - w^*\|_P \leq \|w^k - w^*\|_P + \gamma_k\|e^k\|_P$.

(ii) $(\gamma_k(2 - \gamma_k))\|u^k - w^k\|^2_P \in \ell^1$.

(iii) Assume that every cluster point of $(w^k)_{k \in \mathbb{N}}$ belongs to $C$. Then, $(w^k)_{k \in \mathbb{N}}$ converges to a point in $C$.

(iv) Assume that $B$ is $\mu_B$-strongly restricted monotone in $H_P$. Then, $C = \{w^*\}$ and, for all $k \in \mathbb{N}$,

$$
\|w^{k+1} - w^*\|_P \leq \rho_k\|w^k - w^*\|_P + \gamma_k\|e^k\|_P, \quad \text{where} \quad \rho_k = \max(1 - \frac{2\mu_B}{1 + \mu_B}, \gamma_k - 1).
$$

**Proof.** See Appendix A. ■

**Remark 3.** The condition $\text{dom}(J_B) = \mathbb{R}^n$ is sufficient (but not necessary) for the existence of a sequence $(w^k)_{k \in \mathbb{N}}$ that satisfies (13), which can be constructed choosing arbitrarily $u_k \in J_B(w^k)$, for all $k \in \mathbb{N}$. ■

**Example 2.** Consider the VI$(\Psi, S)$, where $S \subset \mathbb{R}^n$ is compact and convex, and $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and pseudomonotone in the sense of Karamardian (i.e., for all $\omega, \omega' \in \mathbb{R}^n$, the implication $(\Psi(\omega), \omega' - \omega) \geq 0 \Rightarrow (\Psi'(\omega'), \omega' - \omega) \geq 0$ holds). It holds that $\text{SOL}(\Psi, S) = \text{zer}(B) \neq \emptyset$, where $B = \Psi + N_S$ [14, Prop. 2.2.3]. Moreover, $B$ is restricted monotone. To show this, consider any $w^* \in \text{zer}(B)$ and $(\omega, u) \in \text{gra}(B)$, so $u = \Psi(\omega) + u'$, for some $u'$ such that $(\omega, u') \in \text{gra}(N_S)$. Then, $(u \mid \omega - w^*) = (\Psi(\omega) \mid \omega - w^*) + (u' \mid u - w^*) \geq 0$, where we used that $\langle \Psi(\omega) \mid \omega - w^* \rangle \geq 0$, $\langle u' \mid u - w^\ast \rangle \geq 0$, by pseudomonotonicity and because $\langle \Psi(u^\ast) \mid \omega - w^\ast \rangle \geq 0$ by definition of VI, and $\langle u' \mid u - w^\ast \rangle \geq 0$ because $(\omega^\ast, 0) \in \text{gra}(N_S)$ and monotonicity of the normal cone.

We note that $\text{dom}(J_B) = \mathbb{R}^n$ by [14, Prop. 2.2.3]. Let us consider any sequence $(\omega^k)_{k \in \mathbb{N}}$ such that, for all $k \in \mathbb{N}$, $\omega^{k+1} = u^k, u^k \in J_B(\omega^k)$, (or equivalently (11) or $\omega^{k+1} \in \text{SOL}(\Psi + \text{Id} - w^k, S)$). By Theorem 1 (with $\gamma_k = 1, e^k = 0$, $(\omega^k)_{k \in \mathbb{N}}$ is bounded, hence it admits at least one cluster point, say $\omega$; by Theorem 1(ii) $\|u^k - w^\ast\| \to 0$. However, by definition of VI, for any $\omega \in S, (\Psi(u^k) + u^k - w^\ast) \mid w^k - u^k \rangle \geq 0$. By passing to the limit (on a subsequence) and by continuity, we obtain $\langle \Psi(\omega) \mid \omega - w^\ast \rangle \geq 0$, which shows that $\omega \in \text{SOL}(\Psi, S)$. Therefore $(\omega^k)_{k \in \mathbb{N}}$ converges to a solution to VI$(\Psi, S)$ by Theorem 1(iii). This extends the results in [12, §4.2], where hypomonotonicity of $\Psi$ is assumed and where a small-enough step size is chosen to ensure that $J_B$ is single-valued (besides, pseudomonotonicity of $\Psi$ is sufficient, but not necessary, for the restricted monotonicity of $B$, and Theorem 1 would also allow to take into account iterations with errors, cf. [12, §4.2]). ■

### 4.3 Derivation and convergence

Next, we show how that Algorithm 1 is obtained by applying the iteration in (13) to the operator $\Phi^{-1}A$, where

$$
\Phi := \begin{bmatrix}
\bar{\tau}^{-1} + W_n & 0 & -R^T A^T \\
0 & \bar{\nu}^{-1} & V_m \\
-A R & V_m^T & \delta^{-1}
\end{bmatrix}
$$

is called *preconditioning matrix*. The step sizes $\bar{\tau} := \text{diag}(\tau_i), \bar{\nu} := \text{diag}(\nu_{i,i}), \delta := \text{diag}(\delta_i)$, have to be chosen such that $\Phi > 0$. In this case, it also holds that $\text{zer}((\Phi^{-1}A)) = \text{zer}(A)$. Sufficient conditions that ensure $\Phi > 0$ are given in the next lemma, which follows by the Gerschgorin’s circle theorem.

**Lemma 5.** The matrix $\Phi$ in (14) is positive definite if $\nu_{i,j} > 2\sqrt{|w_i,j|}$ for all $(i, j) \in \mathcal{E}$ and $\bar{\tau}_i > d_i > |A^i|_\infty, \delta_i > |A_i|_\infty + \sum_{j=1}^N \sqrt{|w_{i,j}|}$ for all $i \in \mathcal{I}$. ■

In the following, we always assume that the step sizes in Algorithm 1 are chosen such that $\Phi > 0$. Then, we are able to formulate the following result.

**Lemma 6.** Algorithm 1 is equivalent to the iteration

$$
(\forall k \in \mathbb{N}) \quad w^{k+1} \in J_{\Phi^{-1}A}(w^k),
$$

with $A$ as in (7), $\Phi$ as in (14): for any initial condition $w_0 = \text{col}(x^0, v^0) = 0_{e \times m}, x^0)$, the sequence $(x^k, V_m x^k, \lambda^k)_{k \in \mathbb{N}}$ generated by (15) coincides with the
sequence \((a^k, z^k, \lambda^k)_{k \in \mathbb{N}}\) generated by Algorithm 1 with initial conditions \((a^0, z^0) = 0_{n \times m}, \lambda^0\).

**Proof.** By definition of inverse operator, we have that
\[
\omega^{k+1} \in (I + \Phi^{-1} A)^{-1}(\omega^k)
\]
\[\iff 0 \in \Phi^{-1}A(\omega^{k+1}) - \omega^k + \omega^{k+1}
\]
\[\iff 0 \in \Phi(\omega^{k+1} - \omega^k) + A(\omega^{k+1})
\]
(16)
\[\iff \begin{cases}
0 \in -\tilde{\tau}^T(x^{k+1} - x^k) + W_a x^{k+1} - W_a x^k + D_a x^{k+1} - R_a^{-1} A^T \lambda^{k+1} + \alpha R_a^{-1} F(x^{k+1})
\quad - W_a \bar{x}^{k+1} + R_a^{-1} A^T \lambda^{k+1} + \text{N}^\rho(\omega^{k+1})
0 \in -\tilde{\nu}^{-1}(\nu^{k+1} - \nu^k) + V_m \lambda^{k+1} + V_m \lambda^k - V_m \lambda^{k+1}
0 \in -\tilde{\delta}^{-1}(\lambda^{k+1} - \lambda^k) + \text{N}^\rho(\lambda^{k+1}) + b
- A R(2x^{k+1} - x^k) + V_m(2v^{k+1} - v^k)
\end{cases}
\]
(17)

In turn, the first inclusion in (17) can be split in two by left-multiplying both sides with \(R_a^{-1} A^T\) and \(S\). By \(SN = 0_{(N-1)n}, R_a S = I_n\) and \(SR_a = 0_{(N-1)n}\), we get
\[0 \in S((I + \tilde{\tau} D_a)x^{k+1} - x^k - \tilde{\tau} W_a x^k)
0 \in R((I + \tilde{\tau} D_a)x^{k+1} - x^k - \tilde{\tau} W_a x^k)
+ \text{N}^\rho(\omega^{k+1}) + \tilde{\tau} F(x^{k+1}, S x^{k+1}) + \tilde{\tau} A^T \lambda^{k+1}
\]
\[\iff \begin{cases}
\forall i \in I : \begin{cases}
x^{k+1}_i = \frac{1}{\tilde{\tau} + \alpha} (x^{k+1}_i - \tau_i \sum_{j=1}^N w_{ij} x^{k+1}_j) + \alpha R_a^{-1} F(x^{k+1}, S x^{k+1}) + A^T \lambda^{k+1}_i
\end{cases}
\end{cases}
\]
\[\forall i \in I : \begin{cases}
x^{k+1}_i = \frac{1}{\tilde{\tau} + \alpha} (x^{k+1}_i - \tau_i \sum_{j=1}^N w_{ij} x^{k+1}_j) + \alpha R_a^{-1} F(x^{k+1}, S x^{k+1}) + A^T \lambda^{k+1}_i
\end{cases}
\]
\[\forall i \in I : \begin{cases}
v^{k+1} = \tilde{\nu} V_m \lambda^{k+1}
\lambda^{k+1} = -A R(2x^{k+1} - x^k) + V_m(2v^{k+1} - v^k)
\end{cases}
\]
(18)

The conclusion follows by defining \(z^k \coloneqq V_m^T v^k\), where \(z^k = \text{col}(z_i)_{i \in I} \in \mathbb{R}^{Nn}\) and \(z^k \in \mathbb{R}^m\) are local variables kept by each agent, provided that \(z^0 = V_m^T v^0\). The latter is ensured by \(z^0 = 0_{n \times m}\), as in Algorithm 1.

**Remark 4.** The preconditioning matrix \(\Phi\) is designed to make the system in (17) block triangular, i.e., to remove the term \(W_a x^{k+1}\) and \(R_a A^T \lambda^{k+1}\) from the first inclusion, and the terms \(V_m \lambda^{k+1}\) from the second one: in this way, \(x^{k+1}_i\) and \(\lambda^{k+1}\) do not depend on \(x^{k+1}_j\), for \(i \neq j\) or \(\lambda^{k+1}\). This ensures that the resulting iteration can be computed by the agents in a fully-distributed fashion (differently from the non-preconditioned resolvent \(J_\lambda\)). Furthermore, the change of variable \(z = V_m^T v\) reduces the number of auxiliary variables and decouples the dual update in (18) from the graph structure.

**Remark 5.** By Lemma 6, Remark 1 and by \(J_{\Phi^{-1} A}\) in (18), we conclude that \(\text{dom}(J_{\Phi^{-1} A}) = \mathbb{R}^{N+n+Em+Nm}\) and that \(J_{\Phi^{-1} A}\) is single-valued.

In order to apply Theorem 1 to the iteration in (15), we still need the following lemma.

**Lemma 7.** Let \(\alpha \in (0, \alpha_{\max}]\), \(\alpha_{\max}\) as in Lemma 3. Then \(\Phi^{-1} A\) is restricted monotone in \(\mathcal{H}_\alpha\).

**Proof.** Let \((\omega, u) \in \gamma(\Phi^{-1} A), \omega^* \in \text{zer}(\Phi^{-1} A)\). Then, \((\omega, \Phi u) \in \gamma(A)\) and \(\omega^* \in \text{zer}(A)\). By Lemma 3 we conclude that \(\langle u | \omega - \omega^* \rangle = \langle \Phi u | \omega - \omega^* \rangle \geq 0\).

**Theorem 2.** Let \(\alpha \in (0, \alpha_{\max}]\), with \(\alpha_{\max}\) as in Lemma 3, and let the step sizes \(\tau, \tilde{\tau}, \tilde{\nu}, \tilde{\delta}\) be as in Remark 5. Then, the sequence \((x^k, z^k, \lambda^k)_{k \in \mathbb{N}}\) generated by Algorithm 1 converges to some equilibrium \((x^*, z^*, \lambda^*)\), where \(x^* = 1 \otimes x^*\) and \(z^*\) is the v-GNE of the game in (1).

**Proof.** By Lemma 6, we can equivalently study the convergence of the iteration in (15). In turn, (15) can be rewritten as (13) with \(\gamma^k = 1, \alpha^k = 0\), for all \(k \in \mathbb{N}\). For later reference, let us define \(u^k = J_{\Phi^{-1} A}(\omega^k)\) (here \(u^k = \omega^{k+1}\)). \(\Phi^{-1} A\) is restricted monotone in \(\mathcal{H}_\alpha\) by Lemma 7. By Theorem 1(i), the sequence \((\omega^k)_{k \in \mathbb{N}}\) is bounded, hence it admits at least one cluster point, say \(\omega\). By (16) and (7), it holds, for any \(\omega \in \mathbb{E}^{n \times m} \otimes \mathbb{R}^{Nm}\), that \(\langle A_1(u^k) + \Phi(u^k - \omega) | \omega - u^k \rangle \geq 0\), with \(A_1\) as in (7). By Theorem 1(ii), \(u^k - \omega^k \to 0\). Therefore, by continuity of \(A_1\), taking the limit on a diverging subsequence \((k_l)_{l \in \mathbb{N}}\) such that \((\omega^{k_l})_{l \in \mathbb{N}} \to \omega\), we have that for all \(\omega \in \mathbb{E}^{n \times m} \otimes \mathbb{R}^{Nm}\), \(\langle A_1(\omega) | \omega - \omega^* \rangle \geq 0\), which shows that \(\omega^* \in \text{zer}(A) = \text{fix}(J_{\Phi^{-1} A})\). Hence \((\omega^k)_{k \in \mathbb{N}}\) converges to an equilibrium of (15) by Theorem 1(iii).

The conclusion follows by Lemma 1.

**Remark 6.** While the choice of step sizes in Lemma 5 is decentralized, computing the bound \(\alpha_{\max}\) for the common parameter \(\alpha\) in Algorithm 1 requires some global information on the graph \(\mathcal{G}\) (i.e., the algebraic connectivity) and on the game mapping (the strong monotonicity and Lipschitz constants).

**Remark 7.** If \(x^0 \in \mathbb{E}^N\), then \(x^k \in \Omega^N\) for all \(k \in \mathbb{N}\) (by convexity and the updates in Algorithm 1), and Assumption 2 can be relaxed to hold only on \(\Omega\).

**Remark 8 (Inexact updates).** The local optimization problems in Algorithm 1 are strongly convex, hence they can be efficiently solved by several iterative algorithms (with linear rate). While computing the exact solutions \(z^k_i\) would require an infinite number of iterations, the
converges to an equilibrium (\(x^\ast\)) of the gradient descent method, warm-started at \(x_k^0\), with (small enough) fixed step. Then, each agent can independently ensure that 
\[\|e_k\| \leq \varepsilon_k,\]
for some \((\varepsilon_k)_{k \in \mathbb{N}} \in \ell^1\), by simply choosing
\[\varepsilon_k = \frac{1}{\sqrt{\log (\varepsilon_k) / (\|x_k - x_{k-1}\|) / \log (\rho_i),}}\]  
(19)
where \(x_k^0\) is the approximation obtained after one gradient step and \(\rho_i \in (0, 1)\) is the contractivity parameter of the gradient descent. We finally remark that \(x_k^0\) must be calculated with increasing accuracy. In practice, however, when \(x_k^0\) is converging, \(\|x_k^0 - x_{k-1}\| \to 0\). Hence \(x_k^0\) is a good initial guess for \(x_k^1\), and the computation of \(x_k^1\) often requires few gradient steps, see also §7.  

5 Accelerations

Lemma 6 shows that Algorithm 1 can be recast (modulo the change of variables \(z = V_n v\)) as
\[\omega^{k+1} = T(\omega^k),\]  
(20)
where \(T := \mathcal{J}_{\Phi^{-1}, \mathcal{A}}\). This compact operator representation allows for some modifications of Algorithm 1, that can increase its convergence speed. In particular, we consider three popular acceleration schemes [21], which have been extensively studied for the case of firmly nonexpansive operators [1, Def. 4.1], and also found application in games under full-decision information [2], [34]. Here we provide convergence guarantees for the partial-decision information setup, where \(T\) is only firmly quasi-nonexpansive. Our fully distributed accelerated algorithms are illustrated in Algorithm 2. In the following, we assume that \(\alpha \in (0, \alpha_{\text{max}}), \alpha_{\text{max}}\) as in Lemma 3, and that the step sizes \(\tau, \nu, \delta\) are chosen as in Lemma 5.

Proposition 1 (Overrelaxation). Let \(\gamma \in [1, 2].\) Then, for any \(\omega^0\), the sequence \((\omega^k)_{k \in \mathbb{N}}\) generated by
\[(\forall k \in \mathbb{N}), \quad \omega^{k+1} = \omega^k + \gamma (T(\omega^k) - \omega^k),\]  
(21)
converges to an equilibrium \((\mathbf{x}^\ast, \mathbf{v}^\ast, \lambda^\ast) \in \text{zer}(\mathcal{A}),\) where \(\mathbf{x}^\ast = 1_N \otimes x^\ast\) and \(x^\ast\) is the v-GNE of the game in (1).  

Proof. The iteration in (21) is in the form (13), with \(\gamma^k = \gamma, \epsilon^k = 0,\) for all \(k \in \mathbb{N}.\) Then, the conclusion follows analogously to Theorem 2.  

Proposition 2 (Inertia). Let \(\zeta \in [0, \frac{1}{4})\). Then, for any \(\omega^{-1} := \omega^0,\) the sequence \((\omega^k)_{k \in \mathbb{N}}\) generated by
\[(\forall k \in \mathbb{N}), \quad \omega^{k+1} = T(\omega^k + \zeta (\omega_k - \omega^{k-1})),\]  
(22)
converges to an equilibrium \((\mathbf{x}^\ast, \mathbf{v}^\ast, \lambda^\ast) \in \text{zer}(\mathcal{A}),\) where \(\mathbf{x}^\ast = 1_N \otimes x^\ast\) and \(x^\ast\) is the v-GNE of the game in (1).  

Proof (sketch). By following all the steps in the proof of [9, Th. 5] (which can be done by recalling that an operator \(T\) is firmly (quasi)nonexpansive if and only if the operator \(2T - \mathcal{I}\) is (quasi)nonexpansive [1, Prop. 4.2, 4.4]), it can be shown that, if \(\zeta \in [0, \frac{1}{3})\), then \((\omega^k)_{k \in \mathbb{N}}\) is bounded and \(\omega^{k+1} - \omega^k \to 0.\) Then, the proof follows analogously to Theorem 2.  

Proposition 3 (Alternated inertia). Let \(\eta \in [0, 1].\) Then, for any \(\omega^0\), the sequence \((\omega^k)_{k \in \mathbb{N}}\) generated by
\[\left\{\begin{array}{ll}
\omega^{k+1} = T(\omega^k) & \text{if } k \text{ is even}, \\
\omega^{k+1} = T(\omega^k + \eta (\omega^k - \omega^{k-1})) & \text{if } k \text{ is odd},
\end{array}\right.\]  
(23)
converges to an equilibrium \((\mathbf{x}^\ast, \mathbf{v}^\ast, \lambda^\ast) \in \text{zer}(\mathcal{A}),\) where \(\mathbf{x}^\ast = 1_N \otimes x^\ast\) and \(x^\ast\) is the v-GNE of the game in (1).  

Proof. For all \(k \in \mathbb{N},\) \(\omega^{2k+2} = T(T(\omega^{2k}) + \eta (T(\omega^{2k}) - \omega^{2k})),\) which is the same two-steps update obtained in (13) with \(\gamma^{2k} = 1 + \eta, \gamma^{2k+1} = 1\) (and \(\mathcal{B} := \Phi^{-1}, \mathcal{A}, \epsilon^k = 0\)). Therefore the convergence of the sequence \((\omega^k)_{k \in \mathbb{N}}\) to an equilibrium \((\mathbf{x}^\ast, \mathbf{v}^\ast, \lambda^\ast) \in \text{zer}(\mathcal{A})\) follows analogously to Theorem 2 (with a minor modification for the case \(\eta = 1\)). The convergence of the sequence \((\omega^{2k+1})_{k \in \mathbb{N}}\) then follows by Theorem 1(i).  

We note that, by Theorem 1, the convergence results in Propositions 1 and 3 hold also in the case of summable errors on the updates, as in Remark 8. Analogously to our analysis, provably convergent acceleration schemes could also be obtained for the FB algorithm in [30]; however, an advantage of our PPA is that the bounds on the inertial/relaxation parameters are fixed and independent on (unknown) problem parameters.

5.1 On the convergence rate

We conclude this section with a discussion on the convergence rate of Algorithms 1 and 2. First, even under Standing Assumption 2, the KKT operator on the right-hand side of (3) is generally not strongly monotone. Similarly, the operator \(\mathcal{A}\) in (7) is not strongly monotone and Algorithm 1 can have multiple fixed points. Therefore, one should not expect linear convergence. By Lemma 6 and the proof of Theorem 1, we can derive the following ergodic rate for the fixed-point residual in Algorithm 1:
\[\frac{1}{k} \sum_{i=0}^{k} \|\omega^{k+1} - \omega^k\|^2 \leq O(1/k).\]
This rate also holds for the iterations in (21), (22), (23); for the case of general operator splittings (and differently
Initialization: Choose acceleration: set γ > 0, ζ = 0, η = 0;
Inertia: set γ = 0, ζ > 0, η = 0;
Alternated inertia: set γ = 0, ζ = 0, η > 0;

For all i ∈ I, set x_i^{-1} = x_i^0 ∈ Ω_i, x_{i,-i}^{-1} = x_{i,-i}^0 ∈ R^{n-n_i}, z_i^{-1} = 0, λ_i^{-1} = λ_i^0 ∈ R_{≥0}^n.

For all k > 0: (Alternated) inertial step: set ˜η_k = 0 if k is even, ˜η_k = η otherwise; each agent i ∈ I computes

\[ \begin{align*}
    \tilde{x}_{i,-i}^{k+1} & = \frac{1}{1+\tau_i} (\tilde{x}_{i,-i}^k + \tau_i \sum_{j=1}^{N} w_{i,j} \tilde{x}_{j,-i}^k) \\
    \tilde{z}_{i}^{k+1} & = \tilde{z}_{i}^k + (\gamma + \tilde{\eta}_k) (z_i^k - z_i^{k-1}) \\
    \tilde{\lambda}_{i}^{k+1} & = \tilde{\lambda}_{i}^k + (\gamma + \tilde{\eta}_k) (\lambda_i^k - \lambda_i^{k-1})
\end{align*} \]

Communication: The agents exchange the variables \{x_{i,-i}^k, \tilde{x}_{i,-i}^k, \lambda_i^k\} with their neighbors.

Resolvent computation: each agent i ∈ I computes

\[ \begin{align*}
    x_{i,-i}^{k+1} & = \arg\min_{y ∈ Ω_i} \left( J_i(y, x_{i,-i}^k) + \frac{1}{\tau_i \gamma} \| y - x_{i,-i}^k \|_2^2 + \frac{1}{\tau_i \gamma} \| d_i y - \sum_{j=1}^{N} w_{i,j} x_{j,-i}^k \|_2^2 \right) \\
    z_{i}^{k+1} & = z_{i}^k + \gamma (z_i^k - z_i^{k-1}) \\
    \lambda_{i}^{k+1} & = \lambda_{i}^k + \gamma (\lambda_i^k - \lambda_i^{k-1})
\end{align*} \]

Relaxation step: each agent i ∈ I computes

\[ \begin{align*}
    x_{i,-i}^{k+1} & = \gamma \tilde{x}_{i,-i}^{k+1} + (1 - \gamma) x_{i,-i}^k \\
    z_{i}^{k+1} & = \gamma \tilde{z}_{i}^{k+1} + (1 - \gamma) z_i^k \\
    \lambda_{i}^{k+1} & = \gamma \tilde{\lambda}_{i}^{k+1} + (1 - \gamma) \lambda_i^k
\end{align*} \]

from optimization algorithms), tighter rates for accelerated schemes are only known for particular cases, and most works focus on mere convergence [21], [9]. Yet, the practice shows that relaxation and inertia often result in improved speed, see [2] or §7.

The same residual rate \(O(1/k)\) can also be shown for the pseudo-gradient method in [30, Alg. 1]. However, a major difference from Lemma 5 is that the upper bounds for the step sizes in [30, Th. 2] are proportional to the constant \(\mu_F^*\) in (8), which is typically very small (up to scaling of the whole operator \(F_n\), [8] (see also §7.1), and, most importantly, it vanishes as the number of agents increases (fixed the other parameters). In contrast, our algorithms allows for much larger steps, which can be chosen independently of the number of agents. This is a structural advantage of the PPA, whose convergence does not depend on the cocoercivity constant of the operators involved. Indeed, step sizes must be taken into account if convergence is evaluated in terms of residuals.

We finally note that linear convergence can be achieved via PPA for games without coupling constraints. For instance, Algorithm 3 corresponds to the overrelaxed method in Algorithm 2, and can be derived, as in Lemma 6, by taking \(B = \Phi_{NE}^* A_{NE}(x)\) in (13), where \(\mathcal{A}_{NE}(x) = F_n(x) + N_{Ω}(x)\) and \(\Phi_{NE} := \tilde{\tau}^{-1} + W_{\alpha}\) are obtained by removing the dual variables from \(\mathcal{A}, \Phi\).

Algorithm 2 Fully-distributed v-GNE seeking via accelerated PPA

Algorithm 3 Fully-distributed NE seeking via PPA

\[ \begin{align*}
    x_{i,-i}^{k+1} & = \frac{1}{1+\tau_i} (x_{i,-i}^k + \tau_i \sum_{j=1}^{N} w_{i,j} x_{j,-i}^k) \\
    x_{i}^{k+1} & = \arg\min_{y ∈ Ω_i} \left( J_i(y, x_{i,-i}^k) + \frac{1}{\tau_i \gamma} \| y - x_{i,-i}^k \|_2^2 + \frac{1}{\tau_i \gamma} \| d_i y - \sum_{j=1}^{N} w_{i,j} x_{j,-i}^k \|_2^2 \right) \\
    x_{i}^{k+1} & = x_{i}^k + \gamma (x_{i}^{k+1} - x_i^k)
\end{align*} \]
observed in [6, §5] that this rate compares favorably with that of the state-of-the-art algorithms – please refer to [6], also for numerical results. For instance, in the absence of coupling constraints, the FB algorithm in [30, Alg. 1] reduces to [36, Alg. 1], whose optimal linear rate $O((1-\kappa_{\mathcal{F}})^{\frac{2}{3}})$ depends quadratically on the quantity $\kappa_{\mathcal{F}_i} := \mu_{\mathcal{F}_i}/\theta_{\mathcal{F}_i} < 1$ [36, Th. 7], where $\theta_{\mathcal{F}_i} := 2 \max((d_i)_{i \in \mathcal{I}}) + \alpha \theta$. Instead, $\rho_i \leq 1 - \kappa_{\mathcal{F}_i}$, for large enough $\tau_i$’s (since $|\Phi_{\text{NE}}| + 2 \mu_{\mathcal{F}_i} \leq \max((d_i + \tau_i^{-1})_{i \in \mathcal{I}}) + 2 \alpha \theta$), as shown in Table 1.

### 6 Aggregative games

In this section we focus on the particularly relevant class of (average) aggregative games, which arises in a variety of engineering applications, e.g., network congestion control and demand-side management [20]. In aggregative games, $n_i = \bar{n} > 0$ for all $i \in \mathcal{I}$ (hence $n = \bar{n} \bar{N}$) and the cost function of each agent depends only on its local decision and on the value of the average strategy

$$\text{avg}(x) := \frac{1}{\bar{N}} \sum_{i \in \mathcal{I}} x_i.$$  

Therefore, for each $i \in \mathcal{I}$, there is a function $f_i : \mathbb{R}^\bar{n} \times \mathbb{R}^\bar{n} \to \mathbb{R}$ such that the original cost function $J_i$ in (1) can be written as

$$J_i(x_i, x_{-i}) := f_i(x_i, \text{avg}(x)).$$  

Since an aggregative game is only a particular instance of the game in (1), all the considerations on the existence and uniqueness of a v-GNE and the equivalence with the KKT conditions in (3) are still valid.

Moreover, Algorithms 1 could still be used to compute a v-GNE. This would require each agent to keep (and exchange) an estimate of all other agents’ action, i.e., a vector of $(N-1)\bar{n}$ components. In practice, however, the cost of each agent is only a function of the aggregative value $\text{avg}(x)$, whose dimension $\bar{n}$ is independent of the number $N$ of agents. To reduce communication and computation burden, in this section we introduce a PPPA specifically tailored to seek a v-GNE in aggregative games, that is scalable with the number of agents. The proposed iteration is shown in Algorithm 4, where the parameters $\alpha$, $\beta$, and $\tau_i$, $\delta_i$ for all $i \in \mathcal{I}$, $v_{i,j}$, for all $(i, j) \in \mathcal{E}$ have to be chosen appropriately, and we denote

$$\tilde{F}_i(x_i, x_j) := \nabla f_i(x_i, x_j) + \frac{1}{N} \nabla \text{avg}(x) - \delta_{i,j} \nabla f_i(x_i, x_j).$$  

We note that $\tilde{F}_i(x_i, \text{avg}(x)) = \nabla J_i(x_i, x_{-i}) = \nabla f_i(x_i, \text{avg}(x)).$

Because of the partial-decision information assumption, no agent has access to the actual value of the average strategy. Instead, we equip each agent with an auxiliary error variable $s_i \in \mathbb{R}^\bar{n}$, which is an estimate of the quantity $\text{avg}(x) - x_i$. Each agent aims at reconstructing the true aggregate value, based on the information received from its neighbors. In particular, it should hold that $s^k \to \mathbb{1}_N \otimes \text{avg}(x^k) - x^k$ asymptotically, where $s := \text{col}((s_i)_{i \in \mathcal{I}})$. For brevity of notation, we also denote

$$\sigma_i := x_i + s_i, \quad \sigma := \text{col}((\sigma_i)_{i \in \mathcal{I}}).$$  

**Remark 9.** By the updates in Algorithm 4, we can infer an important invariance property, namely that $\text{avg}(s^k) = 0_{\bar{n}}$, or equivalently $\text{avg}(x^k) = \text{avg}(s^k)$, for any $k \in \mathbb{N}$, provided that the algorithm is initialized appropriately, i.e., $s^0 = 0_{\bar{n}}$, for all $i \in \mathcal{I}$. In fact, the update of $\sigma$, as it follows from Algorithm 4, is

$$\sigma^{k+1} = \sigma^k - \beta L_{\bar{n}} \sigma^k + (x^{k+1} - x^k),$$  

where $L_{\bar{n}} := L \otimes I_{\bar{n}}$. This update is a dynamic tracking for the time-varying quantity $\text{avg}(x)$, similar to those considered for aggregative games in [23], [5], [18]. Differently from [18], here we introduce the error variables $s_i$, which allow us to directly recast the iteration in (28) in an operator-theoretic framework.

Similarly to §4, we study the convergence of Algorithm 4 by relating it to the iteration in (13). First, let us define the extended pseudo-gradient mapping

$$\tilde{F}(x, \xi) := \text{col}((\tilde{F}_i(x_i, x_j))_{i \in \mathcal{I}}),$$  

with $\xi := \text{col}((\xi_i)_{i \in \mathcal{I}}) \in \mathbb{R}^{\bar{n}}$, and the operators

$$\tilde{F}_i(x, s) := \text{col}(\alpha \tilde{F}(x, \sigma) + L_{\bar{n}} \sigma, L_{\bar{n}} \sigma,$$

$$\tilde{A}(\omega) := \left[ \begin{array}{c} \alpha \tilde{F}(x, \sigma) + L_{\bar{n}} \sigma \\ L_{\bar{n}} \sigma \\ 0_{\bar{N}_m} \\ -V_m \lambda \\ V_m \omega - Ax \end{array} \right] + \left[ \begin{array}{c} N_{\bar{O}}(x) \\ 0_{\bar{n}} \\ 0_{\bar{O}_m} \end{array} \right],$$  

where $\omega := \text{col}(x, s, v, \lambda) \in \mathbb{R}^{2\bar{n}+\bar{E}_m+\bar{N}_m}$, and we recall that $\sigma = x + s$ is just a shorthand notation.

**Lemma 8.** The mapping $\tilde{F}$ in (29) is $\tilde{\theta}$-Lipschitz continuous, for some $\tilde{\theta} > 0$. \hfill \qed

**Proof.** It follows from Lemma 2, by noticing that $\tilde{F}(x, \xi) = F((I_{\bar{N}} \otimes 1 - I_{\bar{n}}) \otimes (\bar{N}^{-1} - \bar{n}^{-1} - 1)).$ \hfill \qed

Finally, we will assume that the step sizes $\tau := \text{diag}((\tau_i I_{\bar{n}})_{i \in \mathcal{I}})$, $\beta := \beta I_{\bar{N}_m}$, $\tilde{\nu} := \text{diag}((\nu_{i,j})_{(i,j) \in \mathcal{E}}), \tilde{\nu}$.
Algorithm 4 Fully-distributed v-GNE seeking in aggregative games via PPPA

**Initialization:** For all $i \in I$, set $x_i^0 \in \Omega_i$, $s_i^0 = 0_m$, $z_i^0 = 0_m$, $\lambda_i^0 \in \mathbb{R}_{\geq 0}^m$.

For all $k > 0$: Communication: The agents exchange the variables $\{\sigma_i^k = x_i^k + s_i^k, \lambda_i^k\}$ with their neighbors.

Local variables update: each agent $i \in I$ computes

$$
\begin{align*}
\sigma_i^{k+1} &= \sigma_i^k - \beta \sum_{j=1}^{m} w_{i,j} (\sigma_i^k - \sigma_j^k) \\
\tau_i^{k+1} &\leftarrow y \text{ s.t. } \eta_i \in \alpha \mathcal{F}_i(y, y + \tau_i^{k+1}) + \frac{1}{n_i} (y - x_i^k) + A_i^T \lambda_i^k + \sum_{j=1}^{m} w_{i,j} (\sigma_i^k - \sigma_j^k) + N_{\Omega_i}(y) \\
\lambda_i^{k+1} &= \text{proj}_{\mathcal{E}} \left( \lambda_i^k + \delta_i (A_i (2x_i^{k+1} - x_i^k) - b_i - (2z_i^{k+1} - z_i^k)) \right).
\end{align*}
$$

Lemma 9. The matrix $\Phi$ in (31) is positive definite if $\beta^{-1} > 4 \max((\delta_{i}(\mathcal{E}) \nu_{(i,j)}^{-1} > 2 \sqrt{\nu_{i,j}})$ for all $(i, j) \in \mathcal{E}$, and $\tau_i^{-1} > 4 d_i + \|A_i^T\|_\infty$, $\delta_i^{-1} > \|A_i\|_\infty + \sum_{j=1}^{m} \sqrt{\nu_{i,j}}$, for all $i \in I$. \hfill $\square$

Theorem 4. Let $d_{\min} := \min((d_i)_{i \in I})$ and

$$
\alpha_{\max} := \min \left( \frac{\mu_2(L)}{\bar{\beta}^2}, \frac{2 \sqrt{\mathcal{Z}}(d_{\min})}{\bar{\beta}} \right).
$$

Let $\alpha \in (0, \alpha_{\max})$ and let the step sizes $\bar{\beta}, \bar{\nu}, \bar{\delta}$ be as in Lemma 9. Then, for all $k \in \mathbb{N}$, the inclusion in (24) has a unique solution. Moreover, the sequence $(x_i^k, s_i^k, z_i^k, \lambda_i^k)_{k \in \mathbb{N}}$ generated by Algorithm 4 converges to an equilibrium $(x^*, 1 \otimes \text{avg}(x^*) - x^*, z^*, \lambda^*)$, where $x^*$ is the v-GNE of the game in (1). \hfill $\square$

Proof. Similarly to Lemma 6, we first show that Algorithm 4 can be recast as a PPPA, applied to find a zero of the operator $\Phi^{-1} \mathcal{A}$. Then, we restrict our analysis to the invariant subspace

$$
\Sigma := \{(x, s, v, \lambda) \in \mathbb{R}_{\geq 0}^{2n + E_m + N_m} | \text{avg}(s) = 0_m \}.
$$

A detailed proof is in Appendix B. \hfill $\blacksquare$

Remark 10. The update in (24) is implicitly defined by a strongly monotone inclusion, or, equivalently, variational inequality (see Appendix B). We emphasize that there are several iterative methods to find the unique solution (with linear rate) [1, §26] and that, as in Remark 8, convergence is guaranteed even if the solution is approximated at each step (with summable errors). \hfill $\square$

Remark 11. If, for some $i \in I$, there exists a function $\varphi_i$ such that $\nabla_y \varphi_i(y, s_i^{k+1}) = \mathcal{F}_i(y, y + s_i^{k+1})$, then the update of $x_i^k$ in Algorithm 4 can be simplified as

$$
x_i^{k+1} = \arg\min_{y \in \Omega_i} \left( \varphi_i(y, s_i^{k+1}) + \frac{1}{2 \alpha_i} \|y - x_i^k\|^2 \\
+ \frac{1}{\alpha_i} (A_i^T \lambda_i^k)^\top y + \frac{1}{\alpha_i} \sum_{j=1}^{n_i} \omega_{i,j} (\sigma_i^k - \sigma_j^k)^\top y \right),
$$

as in Lemma 6. For scalar games (i.e., $\bar{n} = 1$) this condition holds for all $i \in I$. Another noteworthy example is that of a cost $f_i(s_i, \text{avg}(x)) = f_i(x_i) + (Q_i \text{avg}(x))^\top x_i$, for some function $f_i$ and symmetric matrix $Q_i$, which models applications as the Nash–Cournot game described in [23] and the resource allocation problem considered in [3]. In this case, $\varphi_i(x_i, s_i) = f_i(x_i) + (Q_i(s_i + x_i))^\top x_i - \frac{1}{2 \alpha_i} x_i^\top Q_i x_i$. \hfill $\square$

Remark 12. Inertial/relaxed versions of Algorithm 4 can be studied as in §5; further, linear convergence can be established for aggregative games without coupling constraints, based on the restricted strong monotonicity of $\mathcal{F}_i$ (see the proof of Lemma 11 in Appendix B), as in Theorem 3. \hfill $\square$

7 Numerical simulations

7.1 Nash–Cournot game

We consider a Nash–Cournot game [30, §6], where $N$ firms produce a commodity that is sold to $m$ markets. Each firm $i \in I = \{1, \ldots, N\}$ participates in $n_i \leq m$ of the markets, and decides on the quantities $x_i \in \mathbb{R}^{n_i}$ of commodity to be delivered to these $n_i$ markets. The quantity of product that each firm can deliver is bounded by the local constraints $0_{n_i} \leq x_i \leq X_i$. Moreover, each market $k = 1, \ldots, m$ has a maximal capacity $r_k$. This results in the shared affine constraint $Ax \leq r$, with $r = \text{col}((r_k)_{k=1, \ldots, m})$ and $A = [A_1 \ldots A_N]$, where $A_i \in \mathbb{R}^{m \times n_i}$ is the matrix that expresses which markets firm $i$ participates in. Specifically, $[A_i]_{k,j} = 1$ if $x_i[j]$ is the amount of product sent to the $k$-th market by agent $i$, $[A_i]_{k,j} = 0$ otherwise, for all $j = 1, \ldots, n_i, k = 1, \ldots, m$. Hence, $Ax = \sum_{i=1}^{N} A_i x_i \in \mathbb{R}^m$ is the vector of the quantities of total product delivered to the markets. Each firm $i$ aims at maximizing its profit, i.e., minimizing the cost function $J_i(x_i, x_{-i}) = 10^{-3} \ast (c_i(x_i) - p(Ax)^\top A_i x_i)$. 

\[11\]
Here, $c_i(x_i) = x_i^T Q_i x_i + q_i^T x_i$ is firm $i$'s production cost, with $Q_i \in \mathbb{R}^{n_i \times n_i}$, $Q_i \succ 0$, $q_i \in \mathbb{R}^{n_i}$. Instead, $p : \mathbb{R}^m \to \mathbb{R}^m$ associate to each market a price that depends on the amount of product delivered to that market. Specifically, the price for the market $k$, for $k = 1, \ldots, m$, is $[p(x)]_k = P_k - \chi_k [A x]_k$, where $P_k, \chi_k \succ 0$.

We set $N = 20, m = 7$. The market structure (i.e., which firms are allowed to participate in which of the $m$ markets) is defined as in [30, Fig. 1]; thus $x = \text{col}((x_i))_{i \in \mathcal{I}} \in \mathbb{R}^n$ and $n = 32$. The firms cannot access the production of all the competitors, but they are allowed to communicate with their neighbors on a randomly generated connected graph. We select randomly with uniform distribution $\pi_k$ in $[1, 2]$, $Q_k$ diagonal with diagonal elements in $[1, 8]$, $q_k$ in $[1, 2]$, $P_k$ in $[10, 20]$, $\chi_k$ in $[1, 3]$, $X_i$ in $[5, 10]$, for all $i \in \mathcal{I}$, $k = 1, \ldots, m$.

The resulting setup satisfies all our theoretical assumptions [30, §VI]. We set $\alpha = \alpha_{\text{max}} \approx 0.7$ as in Lemma 3 and we choose the step sizes as in Lemma 5 to satisfy all the conditions of Theorem 2.

We compare the performance of Algorithm 1 versus that of the pseudo-gradient method in [30, Alg. 1], which is to the best of our knowledge the only other available single-layer fixed-step scheme to solve GNE problems under partial-decision information. In [30, Alg. 1], we choose the parameter $c$ that maximize the step sizes $\tau, \nu, \sigma$, provided that the conditions in [30, Th. 2] are satisfied. This results in very small step sizes, e.g., $\tau^* \approx 10^{-5}$.

The results are illustrated in Figure 1, where the two Algorithms are initialized with the same random initial conditions. [30, Alg. 1] is extremely slow, due to the small step sizes; and our PPPA method shows a much faster convergence. According to our numerical experience, the impact is remarkable, up to halving the number of iterations needed for convergence, as shown in Figure 3.

7.2 Charging of plug-in electric vehicles

We consider the charging scheduling problem for a group of plug-in electric vehicles, modeled by an aggregative game [20]. Each user $i \in \mathcal{I} = \{1, \ldots, N\}$ plans the charging of its vehicle for an horizon of 24 hours, discretized into $\bar{n}$ intervals; the goal is to choose the energy injections $x_i \in \mathbb{R}^{\bar{n}}$ of each time interval to minimize its cost $J_i(x_i, \text{avg}(x)) = g_i(x_i) + p(\text{avg}(x))^T x_i$, where $g_i(x_i) = x_i^T Q_i x_i + c_i^T x_i^2$ is the battery degradation cost, and $p(\xi) = a(\xi + d) + b 1_{\bar{n}}$ is the cost of energy, with $b$ a baseline price, $a$ the inverse of the price elasticity and $d \in \mathbb{R}$ the inelastic demand (not related to vehicle charging) along the horizon. We assume a maximum injection per interval and a desired final charge for all $i \in \mathcal{I}$. We set $\Omega_i = \{ y \in [0, \bar{n}] \mid 1_{\bar{n}}^T y = \gamma_i \}$.

We set $N = 1000$, $\bar{n} = 12$. For all $i \in \mathcal{I}$, we select with uniform distribution $c_i$ in $[0.55, 0.95]$, $Q_i \succ 0$ with diagonal and off-diagonal elements in $[0.2, 0.8]$ and $[0.05, 0.05]$, respectively, $\gamma_i$ in $[0.6, 1]$; $[\bar{x}_i]_j = 0.25$ with probability $20\%$, $[\bar{x}_i]_j = 0$ otherwise. We set $[c_j]$ as $0.04$ if $j \in \{1, 2, 3, 11, 12\}$, as $0.01$ otherwise (corresponding to more restrictive limitations in the daytime); $a = 0.38$.
Standing Assumptions 1, 2 hold, and let the agents communicate over a randomly generated connected graph.

Differently from our PPPA, the upper bounds for the step sizes in [18] are proportional to the quantity $\mu \lambda$ in [18, Lem. 4], hence they depend on $\lambda_2(L)$, $\theta_0$, $\mu$, $\theta$ (but not on $N$, cf. § 7.1, 5.1): in turn, we expect these parameters to affect to a larger extent the convergence speed for the FB method. In Figure 6 we compare the two algorithms, with $N = 10$, for different values of the communication graph connectivity: in the considered range, the number of iterations to converge varies by a factor 2 for Algorithm 4, by a factor $10^3$ for [18, Alg. 1].

8 Conclusion

Inexact preconditioned proximal-point methods are extremely efficient to design fully-distributed single-layer generalized Nash equilibrium seeking algorithms. The advantage is that convergence can be guaranteed for much larger step sizes compared to pseudo-gradient-based algorithms. In fact, in our numerical experience, our algorithms proved much faster than the existing methods, resulting in a considerable reduction of communication and computation requirements. Besides, our operator-theoretic approach facilitates the design of acceleration schemes, also in the partial-decision information setup. As future work, it would be highly valuable to relax our monotonicity and connectivity assumptions, namely to allow for merely monotone game mappings and jointly connected networks, and to address the case of nonlinear coupling constraints.

A Proof of Theorem 1

For all $k \in \mathbb{N}$, let $z^k := \omega^k + \gamma^k (u^k - \omega^k)$, so that $\omega^{k+1} = z^k + \gamma^k e^k$. Consider any $\omega^* \in C$. We have, for all $k \in \mathbb{N}$,

$$\| z^k - \omega^* \|^2_P = \| \omega^k - \omega^* \|^2_P - 2 \gamma^k \langle \omega^k - u^k | \omega^k - \omega^* \rangle_P + (\gamma^k)^2 \| u^k - \omega^k \|^2_P$$

$$\leq \| \omega^k - \omega^* \|^2_P - \gamma^k (2 - \gamma^k) \| u^k - \omega^k \|^2_P,$$

(A.1)

where the inequality follows by Lemma 4.

(i) By (A.1), $\| z^k - \omega^* \|^2_P \leq \| \omega^k - \omega^* \|^2_P$, and the conclusion follows by the Cauchy–Schwartz inequality.

(ii) By $(\gamma^k \| e^k \|_P)_{k \in \mathbb{N}} \in \ell^1$ and point (i), $(\omega^k)_{k \in \mathbb{N}}$ is bounded. Let $c := \sup_{k \in \mathbb{N}} \| \omega^k - \omega^* \|^2_P < \infty$ and $e^k := 2 \gamma^k (\| e^k \|_P + (\gamma^k \| e^k \|_P)^2)$. For all $k \in \mathbb{N}$.

$$\| u^k - \omega^* \|^2_P \leq (\| z^k - \omega^* \|^2_P + \gamma^k (\| e^k \|_P)^2) \leq \| u^k - \omega^* \|^2_P - \gamma^k (2 - \gamma^k) \| u^k - \omega^k \|^2_P + e^k,$$

(A.2)

and the thesis follows by recursion.

(iii) By (A.2), [11, Prop. 3.2(i)] and [11, Th. 3.8].

(iv) By definition of resolvent, $\omega^k - u^k \in B(u^k)$; hence

$$\langle u^k - \omega^* | \omega^k - u^k \rangle_P \geq \mu_B \| u^k - \omega^* \|^2_P.$$

(A.3)
By the Cauchy–Schwarz inequality, \( \| \omega^k - u^k \|_p \geq \mu_B \| u^k - \omega^* \|_p \). Thus, (A.3) yields
\[
\| \omega^k - \omega^* \|^2_p = \| u^k - \omega^* \|^2_p + 2(\omega^k - u^k \cdot \omega^k - u^k)_p + \| \omega^k - u^k \|^2_p \\
\geq (1 + \mu_B)^2 \| u^k - \omega^* \|^2_p.
\] (A.4)

If \( \gamma^k \leq 1 \), by the Cauchy–Schwarz inequality and (A.4), we have \( \| z^k - \omega^* \|_p \leq (1 - \gamma^k) \| \omega^k - \omega^* \|_p + \gamma^k \| u^k - \omega^* \|_p \leq (1 - \gamma^k) \| \omega^k - \omega^* \|_p + \gamma^k \| u^k - \omega^* \|_p \). For \( \gamma^k > 1 \), we can write
\[
\| z^k - \omega^* \|^2_p = (1 - \gamma^k)^2 \| \omega^k - \omega^* \|^2_p + \gamma^k (2 - \gamma^k) \| u^k - \omega^* \|^2_p \\
+ 2\gamma^k (1 - \gamma^k) (\| u^k - \omega^* \|_p + \| \omega^k - u^k \|_p) \\
\leq (1 - \gamma^k)^2 \| \omega^k - \omega^* \|^2_p \\
+ \gamma^k (2(1 + \mu_B) - \gamma^k (1 + 2\mu_B)) \| u^k - \omega^* \|^2_p \] (A.5)
\[
\leq \max(1 - \gamma^k, 1 - \gamma^k, 1 - \gamma^k) \| \omega^k - \omega^* \|^2_p. \] (A.6)

where the first equality follows by rearranging the terms in (A.1); in the first inequality we used (A.3); the last inequality follows by taking into account that the second term in (A.5) is nonpositive if \( \gamma^k \in \{1 + \frac{1}{2\mu_B}, 1 \} \) and can be upper bounded via (A.4) if \( \gamma^k \in \{1 + \frac{1}{2\mu_B}, 2 \} \).

Finally, assume that \( \omega^k \in C \), and choose \( u^k = \omega^k \). Then (A.6) implies \( \omega^k = \omega^* \), hence \( C \) must be a singleton. \( \blacksquare \)

B Proof of Theorem 4

Analogously to Lemma 6, it can be shown that Algorithm 4 is equivalent to the iteration
\[
\omega^{k+1} \in J_{\tilde{\Delta}_{-1, \Delta}}(\omega^k), \quad \omega^0 = \omega^0, \] (B.1)

where \( \omega^0 = (x^0, 0, 0, 0, \lambda^0) \), for some \( x^0 \in \Omega, \lambda^0 \in \mathbb{R}^m \), modulo the transformation \( z^k = V^m v^k \).

First, we show that the iteration in (B.1) is uniquely defined. For all \( i \in I \), let \( F_i(y, \omega^k) := \alpha \tilde{F}_i(y, x^i + s^k + \frac{1}{\tau}(y - x^i) + A^i \lambda^k + \sum_{j=1}^N w_{ij}(\sigma^k - \sigma^j) + N_{i\ell}(y) \), where \( \omega^k = (x^k, s^k, \lambda^k) \). We note that \( \tilde{F}_i \) is \( \theta \)-Lipschitz, because \( \tilde{F} \) is \( \theta \)-Lipschitz by Lemma 8. Then, by monotonicity of the normal cone, we have \( \langle y - y' \mid F_i(y, \omega^k) - F_i(y', \omega^k) \rangle \geq (\tau^{-1} - \alpha \sqrt{2\theta}) \| y - y' \|^2_p \), for any \( y, y' \in \mathbb{R}^n \), and for any \( \omega^k \). By the assumption on \( \alpha, F_i \) is strongly monotone in \( y \) for any \( \omega^k \), and the inclusion in (24) has a unique solution, for any \( \omega^k \) [1, Cor. 23.37]. Therefore, it also holds that \( \text{dom}(J_{\tilde{\Delta}_{-1, \Delta}}) = \mathbb{R}^m + E + \mathbb{N} \) and that \( J_{\tilde{\Delta}_{-1, \Delta}} \) is single-valued.

We turn our attention to the set \( \Sigma \) in (33). As in Remark 9, for any \( \in \Sigma, J_{\tilde{\Delta}_{-1, \Delta}}(\in) \in \Sigma \); hence \( \Sigma \) is invariant for (B.1). Moreover, \( \omega^0 \in \Sigma \). Hence, in (B.1), it is enough to consider the operator \( J_{\tilde{\Delta}_{-1, \Delta}} \Sigma \); \( \Sigma \to \Sigma \), where \( B|\Sigma \) is the restriction of the operator \( B \) to \( \Sigma \), i.e., \( B|\Sigma(\omega) = B(\omega) \) if \( \omega \in \Sigma, B|\Sigma(\omega) = \emptyset \) otherwise. By invariance and (16), it also follows that \( J_{\tilde{\Delta}_{-1, \Delta}} \Sigma = J_{\tilde{\Delta}_{-1, \Delta}} \Sigma \). Thus, the iteration in (B.1) is rewritten as
\[
\omega^{k+1} = J_{\tilde{\Delta}_{-1, \Delta}}(\omega^k), \quad \omega^0 = \omega^0. \] (B.2)

We show the convergence of (B.2) by studying the properties of \( \tilde{\Delta}_{-1, \Delta} \). We start by characterizing the zero set.

Lemma 10. The following statements holds:
(i) If \( \text{col}(x^* s^*, v^*, \lambda^*) \in \text{zer}(\tilde{\Delta}_{-1, \Delta}) \), then \( s^* = 1_N \otimes \text{avg}(v^*) - x^* \) and \( v^* \) is the v-GNE of the game in (1).
(ii) \( \text{zer}(\tilde{\Delta}_{-1, \Delta}) \neq \emptyset \). \( \square \)

Proof. Let \( V_q := V \otimes I_q, L_q := L \otimes I_q = V_q^T V_q \) for any \( q > 0 \); hence, under Standing Assumption 3, we have
\[
\text{null}(L_q) = \text{null}(V_q) = \text{range}(1_N \otimes I_q) \] (B.3)
\[
\text{range}(V_q^T) \supseteq \text{range}(L_q) = \text{null}(1_N \otimes I_q). \] (B.4)

(i) Let us consider any \( \omega^* = \text{col}(x^*, s^*, v^*, \lambda^*) \in \text{zer}(\tilde{\Delta}_{-1, \Delta}) \), and let \( \sigma^* = x^* + s^* \); then we have
\[
0_n \in \alpha \tilde{F}(x^*, \sigma^*) + L_0 \sigma^* + N_0(x^*) + A^T \lambda^* \] (B.5a)
\[
0_n = L_0 \sigma^* \] (B.5b)
\[
0_{Em} = -V_m \lambda^* \] (B.5c)
\[
0_{Nm} = b + N_{R_{\geq m}^n}(\lambda^*) - Ax^* + V_m v^* \] (B.5d)

By (B.5c) and by (B.3), we have \( \lambda^* = 1_N \otimes \lambda^* \), for some \( \lambda^* \in \mathbb{R}^m \); by (B.5b) and since \( \omega^* \in \Sigma \), it must hold \( \sigma^* = x^* + s^* = 1_N \otimes \text{avg}(v^*) \). It is then enough to prove that the pair \( (x^*, \lambda^*) \) satisfies the KKT conditions in (3). By (B.5a), by recalling that \( A^T (1_N \otimes \lambda^*) = \lambda^* \) and \( \tilde{F}(x^*, 1_N \otimes x^*) = F(x^*) \), we retrieve the first KKT condition in (3). We obtain the second KKT condition by left-multiplying both sides of (B.5d) with \( (1_N \otimes I_m) \) and using that \( (1_N \otimes I_m)b = b, (1_N \otimes I_m)L_m = 0 \) by (B.3) and symmetry of \( L, (1_N \otimes I_m)A = A \) and \( (1_N \otimes I_m)N_{R_{\geq m}^n} = NN_{R_{\geq m}^n}(\lambda^*) \). Clearly, \( \omega^* \in \Sigma \). Besides, \( \omega^* \) satisfies the conditions (B.5a)-(B.5c), as in point (i). By (3), there exists \( u^* \in N_{R_{\geq m}^n}(\lambda^*) \) such that \( Ax^* - b - u^* = 0_n \).

Also, \( N_{R_{\geq m}^n}(1_N \otimes \lambda^*) = \prod_{i \in I} N_{R_{\geq m}^n}(\lambda^*) \), and it follows by properties of cones that \( \text{col}(u_1^*, \ldots, u_N^*) \in N_{R_{\geq m}^n}(1_N \otimes \lambda^*) \), with \( u_i^* = \cdots = u_N^* = \frac{1}{\lambda^*} u_i^* \).

Hence \( (1_N \otimes I_m)(-Ax^* + b + \text{col}(u_1^*, \ldots, u_N^*)) = b - Ax^* + u^* = 0_m, \) or \( -Ax^* + b + \text{col}(u_1^*, \ldots, u_N^*) \in \text{null}(1_N \otimes I_m) \subseteq \text{range}(V_m^T) \), by (B.4). Therefore there
exists $v^*$ such that also the condition (B.5d) is satisfied, for which $\omega^* \in \text{zer}(A)$.

Next, similar to Lemma 3, we show restricted monotonicity of the operator $A|_{\Sigma}$.

**Lemma 11.** Let $\alpha \in (0, \tilde{\alpha}_{\text{max}})$, with $\tilde{\alpha}_{\text{max}}$ as in (32). Then $A|_{\Sigma}$ is restricted monotone.

**Proof.** The operator $A|_{\Sigma}$ is the sum of three components, as in (30). The third is monotone by properties of the normal cones [1, Th. 20.25], the second because it is a linear skew-symmetric operator [1, Ex. 20.35] (and restriction does not cause loss of monotonicity, by definition). For the first term, let $(\omega, u) \in \text{gra}(A|_{\Sigma})$, $\omega := \text{col}(x, s, v, \lambda)$, $\omega^* := \text{col}(x^*, s^*, v^*, \lambda^*) \in \text{zer}(A|_{\Sigma})$, $\sigma = x+s, \sigma^* = s^*+x^*$. By Lemma 10, $s^* = 1_N \otimes \text{avg}(x^*)-x^*$. Then, by [18, Lemma 4], there is a $\tilde{\mu} > 0$ such that $\langle \text{col}(x-x^*, s-s^*) | F_a(x, s) - F_a(x^*, s^*) \rangle = \langle x-x^* | \alpha F_a(x, \sigma) - \alpha F_a(x^*, \sigma^*) + (\sigma-\sigma^*) | L_n(\sigma-\sigma^*) \rangle \geq \tilde{\mu} \|\text{col}(x-x^*, s-s^*)\|^2 \geq \mu F_a(3-s^*/2)$, and the last inequality follows by definition of $\sigma$ and bounds on quadratic forms.

Finally, the preconditioning matrix $\hat{\Phi}$ is positive definite by Lemma 9. As in Lemma 7, by Lemma 11, it holds that $\hat{\Phi}^{-1}A|_{\Sigma}$ is restricted monotone in $H_{\Sigma}$. In view of (B.2), the conclusion follows analogously to Theorem 2.

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