Nonlinear Instability for Nonhomogeneous Incompressible Viscous Fluids ✤

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Abstract

We investigate the nonlinear instability of a smooth steady density profile solution of the three-dimensional nonhomogeneous incompressible Navier-Stokes equations in the presence of a uniform gravitational field, including a Rayleigh-Taylor steady-state solution with heavier density with increasing height (referred to the Rayleigh-Taylor instability). We first analyze the equations obtained from linearization around the steady density profile solution. Then we construct solutions of the linearized problem that grow in time in the Sobolev space $H^k$, thus leading to a global instability result for the linearized problem. With the help of the constructed unstable solutions and an existence theorem of classical solutions to the original nonlinear equations, we can then demonstrate the instability of the nonlinear problem in some sense. Our analysis shows that the third component of the velocity already induces the instability, this is different from the previous known results.

Keywords: Nonhomogeneous Navier-Stokes equations, steady density profile, Rayleigh-Taylor instability, incompressible viscous flows.

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1. Introduction

This paper is concerned with the nonlinear instability of a smooth steady density profile solution of the three-dimensional nonhomogeneous incompressible Navier-Stokes equations in the presence of a uniform gravitational field, including a Rayleigh-Taylor steady-state solution with heavier density with increasing height (referred to the Rayleigh-Taylor instability).

The motion of a nonhomogeneous incompressible viscous fluid in the presence of a uniform gravitational field in $\mathbb{R}^3$ is governed by the Navier–Stokes equations:

$$\begin{aligned}
\rho_t + \mathbf{v} \cdot \nabla \rho &= 0, \\
\rho \mathbf{v}_t + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p &= \mu \Delta \mathbf{v} - \rho \mathbf{g} e_3, \\
\text{div} \mathbf{v} &= 0,
\end{aligned}$$

where the unknowns $\rho$, $\mathbf{v}$ and $p$ denote the density, the velocity, and the pressure of the fluid, respectively. In the system (1.1) we have written $\mu > 0$ for the coefficient of shear viscosity,
$g > 0$ for the gravitational constant, $e_3 = (0, 0, 1)$ for the vertical unit vector, and $-ge_3$ for the gravitational force.

In this paper we consider the problem of the Rayleigh-Taylor (RT) instability, so we assume that a smooth steady density profile $\bar{\rho} := \bar{\rho}(x_3) \in C^\infty(\mathbb{R})$ exists which satisfies

$$\bar{\rho}' \in C_0^\infty(\mathbb{R}), \quad \inf_{x_3 \in \mathbb{R}} \bar{\rho} > 0, \quad (1.2)$$

$$\bar{\rho}'(x_3^0) > 0 \quad \text{for some point } x_3^0 \in \mathbb{R}, \quad (1.3)$$

where $' = d/dx_3$, see Remark 1.4 on the construction of such $\bar{\rho}$. Clearly, such $\bar{\rho}$ with $v(t, x) \equiv 0$ defines a steady state to (1.1), provided

$$\nabla p = -\bar{\rho}ge_3, \quad \text{i.e., } \frac{d\bar{\rho}}{dx_3} = -\bar{\rho}g. \quad (1.4)$$

**Remark 1.1.** We point out that by virtue of the condition (1.3), there is at least a region in which the steady density solution has larger density with increasing $x_3$ (height), thus this will lead to the classical Rayleigh-Taylor instability as shown in Theorem 1.1 below.

Let the perturbation be

$$\varrho = \rho - \bar{\rho}, \quad u = v + 0, \quad q = p - \bar{\rho},$$

then, $(\varrho, u, q)$ satisfies the perturbed equations

$$\begin{cases}
\varrho_t + u \cdot \nabla (\varrho + \bar{\rho}) = 0, \\
(\varrho + \bar{\rho})u_t + (\varrho + \bar{\rho})u \cdot \nabla u + \nabla q + g\varrho e_3 = \mu \Delta u,
\end{cases} \quad (1.5)$$

To complete the statement of the perturbed problem, we specify the initial and boundary conditions:

$$\varrho(t, x)|_{t=0} = (\varrho_0, \ u_0) \text{ in } \mathbb{R}^3 \quad (1.6)$$

and

$$\lim_{|x| \to +\infty} u(t, x) = 0 \quad \text{for any } t > 0. \quad (1.7)$$

Moreover, the initial data should satisfy $\text{div} u_0 = 0$.

If we linearize the equations (1.5) around the steady state $(\bar{\rho}, 0)$, then the resulting linearized equations read as

$$\begin{cases}
\varrho_t + \bar{\rho}'u_3 = 0, \\
\bar{\rho}u_t + \nabla q + g\varrho e_3 = \mu \Delta u, \\
\text{div} u = 0.
\end{cases} \quad (1.8)$$

It has been known for over a century that the steady states $(\bar{\rho}, 0)$ to the linearized RT problem (1.6)–(1.8) with $\mu \geq 0$ is unstable [1, 19], i.e., there exists a unstable solution to (1.6)–(1.8). Such instability to (1.6)–(1.8) is often called linear RT instability. However, there have been only few results on the mathematically rigorous justification of the RT instability for (1.5)–(1.7). In 2003, Hwang and Guo [12] proved the nonlinear RT instability to (1.5)–(1.6) with boundary condition $u \cdot n|_{\partial \Omega} = 0$ for the two-dimensional inviscid case (i.e. $\mu = 0$) where
Ω = {(x₁, x₂) ∈ ℝ² | −l < x₂ < m} and n denotes the outer normal vector to ∂Ω. To our best knowledge, it is still open mathematically whether there exists an unstable solution to the nonlinear RT problem (1.5)–(1.7) of viscous fluids with variable density. The aim of this article is to show rigorously the instability for the nonlinear RT problem (1.5)–(1.7) in some proper sense. The main result read as follows.

**Theorem 1.1.** Let the steady density profile $\bar{\rho}$ satisfy (2.3)–(2.5). Then, the steady state $(\bar{\rho}, 0)$ of (1.5)–(1.7) is unstable under the Lipschitz structure, that is, for any $s ≥ 2$, $δ > 0$, $K > 0$, and $F$ satisfying

$$F(y) ≤ Ky \quad \text{for any } y ∈ [0, ∞),$$

there exist a constant $i₀ := i₀(s) > 0$ and smooth initial data

$$(\varrho₀, u₀) ∈ (H^∞(ℝ³))^4 \text{ with } \| (\varrho₀, u₀) \|_{H^s(ℝ³)} < δ,$$

but the unique classical solution $(\varrho, u)$ of (1.5)–(1.7), emanating from the initial data $(\varrho₀, u₀)$, satisfies

$$\| u₃(tK) \|_{L²(ℝ³)} > F(\| (\varrho₀, u₀) \|_{H^s(ℝ³)}) \text{ for some } tK ∈ \left(0, \frac{2}{Λ}\frac{2K}{i₀}\right] ⊂ (0, T_{max}),$$

where the constant $Λ$ is given by (2.5), $H^∞(ℝ³) = \bigcap_{k=1}^∞ H^k(ℝ³)$, and $T_{max}$ denotes the maximal time of existence of the solution $(\varrho, u)$.

**Remark 1.2.** It should be noted that we can not obtain the same instability result for the domain $Ωₘ := \{ x ∈ ℝ³ | −l < x₃ < m \}$ in place of $ℝ³$, due to lack of an existence result of the classical solution to the RT problem (1.5) in the domain $Ωₘ$. We also mention that Theorem 1.1 still holds if we define $\| (\varrho₀, u₀) \|_{H^s(ℝ³)}² := \| \varrho₀ \|_{H^{s-1}(ℝ³)}² + \| u₀ \|_{H^s(ℝ³)}²$.

**Remark 1.3.** Our result shows that the problem (1.5)–(1.7) does not possess the following stability structure:

$$∃ \text{ a constant } C > 0, \text{ such that } \sup_{t ∈ [0, T]} \| u₃(t) \|_{L²(ℝ³)} ≤ C \| (\varrho₀, u₀) \|_{H^s(ℝ³)} \text{ for any } T > 0,$$

which should be quite general and reasonable for a global stability theory. Notice that $s ≥ 2$ in Theorem 1.1 is arbitrary. Thus, even if the initial data of the (1.5)–(1.7) are smooth and small, the failure of the stability structure (1.11) means that it is not possible to use (1.11) to control the norm of $\| u(t) \|_{L²(ℝ³)}$ for long time.

**Remark 1.4.** Here we give an example of a steady density profile solution satisfying the conditions of Theorem 1.1. Assume

$$\hat{\rho} = \begin{cases} \hat{\rho}^h & \text{for } x₃ ≥ 1, \\ (\hat{\rho}^h + \hat{\rho}^l)/2 & \text{for } x₃ ∈ (-1, 1), \\ \hat{\rho}^l & \text{for } x₃ ≤ -1, \end{cases}$$

and $0 < \hat{\rho}^l < \hat{\rho}^h < +∞$, then $\hat{\rho} := S_ε(\hat{\rho}^a)$ and $\bar{\rho} := g \int_{x₃}^{0} \hat{\rho}(s)ds$ satisfy (1.2)–(1.4), where $S_ε$ is a standard mollifier operator.
Remark 1.5. The constant \( \lambda \) (in (2.54)) is often called maximal linear growth rate. By virtue of (2.54) and (2.29), \( \Lambda < \infty \), and \( \Lambda \to 0 \) if \( g\|\tilde{\rho}'/\tilde{\rho}\|_{L^\infty(\mathbb{R})} \to 0 \) or \( \mu \to \infty \). In contrast, \( \Lambda = \infty \) in the corresponding inviscid case. This clearly shows that the viscosity plays an stabilizing role in the linear RT instability.

The proof of Theorem 1.1 inspired by [8, 10], is divided into four steps: (i) First we notice that the coefficients in the linearized equations (1.8) depend only on the vertical variable \( x_3 \in \mathbb{R}^3 \), this allows us to seek “normal mode” solutions by taking the horizontal Fourier transform of the equations and assuming the solutions grow exponentially in time by the factor \( e^{\lambda(|\xi|)} \), where \( \xi \in \mathbb{R}^2 \) is the horizontal spatial frequency and \( \lambda(|\xi|) > 0 \). This reduces the equations to a system of ordinary differential equations (ODEs) defined on \( \mathbb{R} \) with \( \lambda(|\xi|) > 0 \) for each \( \xi \). Then, solving this ODE system by a modified variational method, we can show that \( \lambda(|\xi|) > 0 \) is continuous function on \((0, \infty)\), and the normal modes with spatial frequency grow in time, providing thus a mechanism for the global linear RT instability. Consequently, we form a Fourier synthesis of the normal mode solutions constructed for each spatial frequency \( \xi \) to construct solutions of the linearized equations that grow in time, when measured in \( H^k(\mathbb{R}^3) \) for any \( k \geq 0 \). This is the content of Section 2. (ii) In Section 3, we show a uniqueness result of the linearized problem (see Theorem 3.1) in the sense of strong solutions. In spite of the uniqueness, the linearized problem is global unstable in \( H^k(\mathbb{R}^3) \) for any \( k \). (iii) Then we derive some nonlinear energy estimates of the perturbed problem with small initial data, which make it possible to take to the limit in the scaled perturbed problem to obtain the corresponding linearized equations. (iv) Finally, in Section 5, with the help of the results established in Sections 2–4 and the Lipschitz structure of \( F \), we can obtain the instability of the nonlinear problem in the sense of (1.10). In the proof, we shall see that the stability structure (1.11) would give rise to certain estimates of solutions to the linearized problem (1.6)–(1.8) that cannot hold in general because of Theorem 2.1.

We should point out that the RT instability based on the Lipschitz structure was studied by Guo and Tice in [8] for compressible inviscid fluids, where the instability was shown in the \( H^3 \)-norm of \((\varrho, \mathbf{u})\) and the flow map (see (5.12) in [8]). Our instability result Theorem 1.1 differs from that of Guo and Tice in that only \( \|u_3\|_2 \) is needed here to describe the instability. This is also different from that of Guo and Hwang [12], in which the instability for an inhomogeneous incompressible inviscid fluid is described by the norm \( \|(\varrho, \mathbf{u})\|_{L^2(\Omega)} \). Roughly speaking, our instability in terms of \( \|u_3\|_2 \) only is based on two important observations: (i) one can construct a solution \((\varrho, \mathbf{u})\) with \( \|u_3(0)\|_2 > 0 \) of the linearized problem; (ii) more regularity of the solution \((\varrho, \mathbf{u})\) to the corresponding nonlinear problem can be derived from the problem (1.5)–(1.7), we refer to Section 5 for details. We also mention that in the current paper we have to employ new techniques to construct growing in time solutions to the linearized problem. To construct such solutions, we shall first transform the linearized equations to an ODE system. In the incompressible inviscid fluid case, the ODE system can be viewed as an eigenvalue problem with eigenvalue \( -\lambda^2 \) (cf. ODE (10) in [12] for incompressible fluids, or ODE system (3.11) in [8] for compressible fluids). Unfortunately, when the viscosity is present, the linear term multiplied by \( \lambda \) breaks down the natural variational structure, such that the variational method can not be used. In order to circumvent this problem, for compressible viscous fluids, recently Guo and Tice [10] artificially removed the linear dependence on \( \lambda \) by first defining \( s := \lambda > 0 \), then solving the family of modified problems for each \( s > 0 \), and finally showing \( s = \lambda(s) \) for some \( s \). In [10] the ODE system is defined in a bounded domain, and the compact imbedding, an important step in their construction, can thus be applied. In our case, however, our ODE (see (2.4)) is defined on \( \mathbb{R} \), and consequently the compact embedding does not hold. To overcome this difficult, here we exploit the property of weak convergence and the structure of the energy functional \( E(\psi) \) corresponding
to our ODE. In particular, we develop a new analysis technique to prove \( \lambda(|\xi|) \in C^0(0, \infty) \) by first showing \( \lambda(|\xi|, s) \in C^0(0, \infty) \) for each fixed \( s \), and then exploiting the monotonicity of \( \lambda(\cdot, s) \) to further verify \( \lambda(\cdot) := \lambda(\cdot, \lambda) \in C^0(0, \infty) \), see the proof of Proposition 2.5 for details.

We end this section by briefly reviewing some of the previous results on the nonlinear RT instability for two layer incompressible fluids with a free interface, where the RT steady state solution is a denser fluid lying above a lighter one separated by a free interface. When the densities of two layer fluids are two constants, Prüss and Simonett [18] used the \( C^0 \)-semigroup theory and the Henry instability theorem to show the (local) existence of nonlinear unstable solutions in the Sobolev-Slobodeckii spaces, where the instability term is described by the sum of \( \|u\|_{W^{2-2/p}} \) and \( \|h\|_{W^{3-2/p}} \) (see [18, Theorem 1.2] for details). When densities of two layer fluids are variable, to our best knowledge, the (local) existence of solutions to the nonlinear problem is still not known unfortunately, and thus the nonlinear instability is still open. For compressible fluids there are very few results on the nonlinear RT instability. Guo and Tice proved the instability of immiscible compressible inviscid fluids in the frame of Lagrangian coordinates under the assumption of the existence of solutions [9], which is in some sense a compressible analogue to the local ill-posedness of the RT problem for incompressible fluids given in [7]. Recently, Jiang, Jiang and Wang [15] adapted Guo and Tice’s approach to investigate the nonlinear instability of two immiscible incompressible fluids with or without surface tension in Eulerian coordinates without the help of a coordinate transformation. We remark that the analogue of the RT instability arises when the fluids are electrically conducting and a magnetic field is present, and the growth of the instability will be influenced by the magnetic field due to the generated electromagnetic induction and the Lorentz force. Some authors have extended the partial results concerning the RT instability of superposed flows to the case of MHD flows by overcoming the more complicated structure due to presence of the magnetic field, see [5, 11, 14, 16, 24].

Notation: Throughout this article we shall repeatedly use the abbreviations:

\[
W^{m,p} := W^{m,p}(\mathbb{R}^3), \quad H^m := H^m(\mathbb{R}^3), \quad L^p := L^p(\mathbb{R}^3),
\]

\[
\| \cdot \|_{W^{m,p}} := \| \cdot \|_{W^{m,p}(\mathbb{R}^3)}, \quad \| \cdot \|_{H^m} := \| \cdot \|_{H^m(\mathbb{R}^3)}, \quad \| \cdot \|_{L^p} := \| \cdot \|_{L^p(\mathbb{R}^3)}, \quad \text{etc.}
\]

2. Construction of solutions to the linearized problem

We wish to construct a solution to the linearized equations (1.8) that has growing \( H^k \)-norm for any \( k \). We will construct such solutions via Fourier synthesis by first constructing a growing mode for any but fixed spatial frequency.

2.1. Linear growing modes

To begin, we make a growing mode ansatz of solutions, i.e.,

\[
\rho(x) = \tilde{\rho}(x)e^{\lambda t}, \quad u(x) = \tilde{v}(x)e^{\lambda t}, \quad q(x) = \tilde{p}(x)e^{\lambda t} \quad \text{for some } \lambda > 0.
\]

Substituting this ansatz into (1.8), and then eliminating \( \tilde{\rho} \) by using the first equation, we arrive at the time-invariant system for \( \tilde{v} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) \) and \( \tilde{p} \):

\[
\begin{cases}
\lambda^2 \tilde{\nabla} \tilde{v} + \lambda \nabla \tilde{p} = \lambda \mu \Delta \tilde{v} + g \tilde{p} \tilde{v}_3 \epsilon_3, \\
\text{div } \tilde{v} = 0
\end{cases}
\]

(2.1)

with

\[
\lim_{|x| \to +\infty} \tilde{v}(x) = 0.
\]
We fix a spatial frequency \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \) and take the horizontal Fourier transform of \( (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) \) in (2.1), which we denote with either \( \hat{\cdot} \) or \( \mathcal{F} \), i.e.,

\[
\hat{f}(\xi, x_3) = \int_{\mathbb{R}^2} f(x', x_3) e^{-ix' \cdot \xi} dx'.
\]

Define the new unknowns

\[
\varphi(x_3) = i\tilde{v}_1(\xi, x_3), \quad \theta(x_3) = i\tilde{v}_2(\xi, x_3), \quad \psi(x_3) = \tilde{v}_3(\xi, x_3), \quad \pi(x_3) = \hat{\pi}(\xi, x_3),
\]

so that

\[
\mathcal{F}(\text{div}\tilde{\mathbf{v}}) = \xi_1\varphi + \xi_2\theta + \psi',
\]

where \( ' = d/dx_3 \). Then, for \( \varphi, \theta, \psi \) and \( \lambda = \lambda(\xi) \) we arrive at the following system of ODEs.

\[
\begin{aligned}
\lambda^2 \rho \varphi' - \lambda \xi_1 \pi + \lambda \mu (|\xi|^2 \varphi - \varphi'') &= 0, \\
\lambda^2 \rho \theta' - \lambda \xi_2 \pi + \lambda \mu (|\xi|^2 \theta - \theta'') &= 0, \\
\lambda^2 \rho \psi' + \lambda \pi' + \lambda \mu (|\xi|^2 \psi - \psi'') &= g \rho \psi, \\
\lambda_1 \varphi + \lambda_2 \theta + \psi' &= 0
\end{aligned}
\]

with

\[
\varphi(-\infty) = \theta(-\infty) = \psi(-\infty) = \varphi(+\infty) = \theta(+\infty) = \psi(+\infty) = 0.
\]

Eliminating \( \pi \) from the third equation in (2.2) we obtain the following ODE for \( \psi \)

\[
- \lambda^2 [\rho |\xi|^2 \psi - (\rho \psi')'] = \lambda \mu (|\xi|^4 \psi - 2|\xi|^2 \psi'' + \psi''') - g |\xi|^2 \rho \psi'
\]

with

\[
\psi(-\infty) = \psi'(-\infty) = \psi(+\infty) = \psi'(+\infty) = 0.
\]

Similarly to [10], we can apply the variational method to construct a solutions of (2.4)–(2.5). The idea of the proof can be found in Guo and Tice’s paper for viscous compressible flows [6, 13, 14, 24], and was adapted later by other authors to investigate the instability for other fluid models [10].

Now we fix a non-zero vector \( \xi \in \mathbb{R}^2 \) and \( s > 0 \). From (2.4)–(2.5) we get a family of the modified problems

\[
- \lambda^2 [\rho |\xi|^2 \psi - (\rho \psi')'] = s \mu (|\xi|^4 \psi - 2|\xi|^2 \psi'' + \psi''') - g |\xi|^2 \rho \psi',
\]

coupled with (2.5). We define the energy functional of (2.6) by

\[
E(\psi) = \int_{\mathbb{R}} s \mu (4|\xi|^2 |\psi'|^2 + ||\xi|^2 |\psi| + \psi''^2) - g |\xi|^2 \rho \psi^2 dx_3
\]

with a associated admissible set

\[
\mathcal{A} = \left\{ \psi \in H^2(\mathbb{R}) \left| \int_{\mathbb{R}} \rho (|\xi|^2 |\psi| + |\psi'|^2) dx_3 = 1 \right. \right\}.
\]

Thus we can find a \( -\lambda^2 \) by minimizing

\[
- \lambda^2(|\xi|) = \alpha(|\xi|) := \inf_{\psi \in \mathcal{A}} E(\psi).
\]
In order to emphasize the dependence on \( s \in (0, \infty) \) we will sometimes write

\[
E(\psi, s) := E(\psi) \quad \text{and} \quad \alpha(s) := \inf_{\psi \in \mathcal{A}} E(\psi, s).
\]

Next we show that a minimizer of \((2.3)\) exists for the case of \( \inf_{\mathcal{A}} E(\psi, s) < 0 \), and that the corresponding Euler-Lagrange equations are equivalent to \((2.5), (2.6)\).

**Proposition 2.1.** For any fixed \( \xi \) with \( |\xi| \neq 0 \), \( \inf_{\psi \in \mathcal{A}} E(\psi, s) > -\infty \). In particular, if there exists a \( \tilde{\psi} \in \mathcal{A} \), such that \( E(\tilde{\psi}) < 0 \), then \( E \) achieves its infimum on \( \mathcal{A} \). In addition, let \( \psi \) be a minimizer and \( -\lambda^2 := E(\psi) \), then the pair \((\psi, \lambda^2)\) satisfies \((2.5), (2.6)\). Moreover, \( \psi \in H^k(\mathbb{R}) \) for any positive integer \( k \).

**Proof.** We first note that for any \( \psi \in \mathcal{A} \),

\[
E(\psi) \geq -g|\xi|^2 \int_{\mathbb{R}} \rho' \psi^2 \, dx_3 \geq -g \left\| \frac{\rho'}{\rho} \right\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \rho |\xi|^2 \psi^2 \, dx_3 \geq -g \left\| \frac{\rho'}{\rho} \right\|_{L^\infty(\mathbb{R})}.
\]

Hence \( E \) is bounded from below on \( \mathcal{A} \) by virtue of \((1.2)\). Let \( \psi_n \in \mathcal{A} \) be a minimizing sequence, then \( E(\psi_n) \) is bounded. This together with \((2.8)\) and \((2.7)\) again implies that \( \psi_n \) is bounded in \( H^2(\mathbb{R}) \). So, there exists a \( \psi \in H^2(\mathbb{R}) \), such that \( \psi_n \rightarrow \psi \) weakly in \( H^2(\mathbb{R}) \) and strongly in \( H^1_{loc}(\mathbb{R}) \). Moreover, by the lower semi-continuity, locally strong convergence, \((1.2)\) and the assumption that \( E(\tilde{\psi}) < 0 \) for some \( \tilde{\psi} \in \mathcal{A} \), we have

\[
E(\psi) \leq \liminf_{n \to \infty} E(\psi_n) = \inf_{\mathcal{A}} E < 0, \quad \text{and} \quad 0 < J(\psi) \leq 1.
\]

Suppose by contradiction that \( J(\psi) < 1 \). By the homogeneity of \( J \) we may find an \( \alpha > 1 \) so that \( J(\alpha \psi) = 1 \), i.e., we may scale up \( \psi \) so that \( \alpha \psi \in \mathcal{A} \). From this we deduce that

\[
E(\alpha \psi) = \alpha^2 E(\psi) \leq \inf_{\mathcal{A}} E < \inf_{\mathcal{A}} E < 0,
\]

which is a contradiction since \( \alpha \psi \in \mathcal{A} \). Hence \( J(\psi) = 1 \) so that \( \psi \in \mathcal{A} \). This shows that \( E \) achieves its infimum on \( \mathcal{A} \).

Notice that since \( E \) and \( J \) are homogeneous of degree 2, \((2.9)\) is equivalent to

\[
\alpha(s) = \inf_{\psi \in H^2(\mathbb{R})} \frac{E(\psi)}{J(\psi)}.
\]

For any \( \tau \in \mathbb{R} \) and \( \psi_0 \in H^2(\mathbb{R}) \) we take \( \psi(\tau) = \psi + \tau \psi_0 \), then \((2.11)\) implies

\[
E(\psi(\tau)) + \lambda^2 J(\psi(\tau)) \geq 0.
\]

If we set \( I(\tau) = E(\psi(\tau)) + \lambda^2 J(\psi(\tau)) \), then we see that \( I(\tau) \geq 0 \) for all \( \tau \in \mathbb{R} \) and \( I(0) = 0 \). This implies \( I'(0) = 0 \). By virtue of \((2.7)\) and \((2.8)\), a direct computation leads to

\[
\begin{align*}
& s\mu \int_{\mathbb{R}} (4|\xi|^2 \psi' \psi_0'(\psi + \psi''(\psi^2 + \psi_0'')) \, dx_3 \\
& = g|\xi|^2 \int_{\mathbb{R}} \rho \psi \psi_0 \, dx_3 - \lambda^2 \int_{\mathbb{R}} \rho (|\xi|^2 \psi \psi_0 + \psi' \psi_0') \, dx_3,
\end{align*}
\]

where we have used the upper boundedness of \( \rho \).
By further assuming that $\psi_0$ is compactly supported in $\mathbb{R}$, we find that $\psi$ satisfies the equation (2.6) in the weak sense on $\mathbb{R}$. In order to improve the regularity of $\psi$, we rewrite (2.12) as

$$
\int_\mathbb{R} \psi''\psi' dx_3 = \frac{1}{s\mu} \int_\mathbb{R} \left( g|\xi|^2\tilde{\rho}\psi - \lambda^2(|\xi|^2\tilde{\rho}\psi - (\tilde{\rho}\psi')) + s\mu(2|\xi|^2\psi'' - |\xi|^4\psi) \right) \psi_0 dx_3
$$

$$
= \int_\mathbb{R} f\psi_0 dx_3.
$$

(2.13)

For any $n \geq 1$, let $\psi_{1,n}, \psi_2 \in C_0^\infty(\mathbb{R})$ satisfy $\psi_{1,n}(x_3) \equiv 1$ for $|x_3| \leq n$. If we take $\psi_0 = \psi_{1,n} \int_{-\infty}^{x_3} \psi_2 d\tau$ in (2.13), then we have

$$
\int_\mathbb{R} (\psi_{1,n}\psi'')\psi_2 dx_3 = \int_\mathbb{R} \left( f\psi_{1,n} \int_{-\infty}^{x_3} \psi_2 d\tau - \psi_{1,n}\psi'' \int_{-\infty}^{x_3} \psi_2 d\tau - 2\psi_{1,n}'\psi''\psi_2 \right) dx_3
$$

$$
= \int_\mathbb{R} \left( \int_{x_3}^{+\infty} (f\psi_{1,n} - \psi_{1,n}'\psi'') d\tau - 2\psi_{1,n}'\psi'' \right) \psi_2 dx_3,
$$

which, recalling $\psi \in H^2(\mathbb{R})$, implies $\psi'' \in H^1_{loc}(\mathbb{R})$ and

$$
\psi''' = (\psi_{1,n}'\psi'')' = \int_{x_3}^{+\infty} (f\psi_{1,n} - \psi_{1,n}'\psi'') d\tau \quad \text{for any } x_3 \text{ with } |x_3| \leq n.
$$

Integrating by parts, we can rewrite (2.13) as

$$
- \int_\mathbb{R} \psi'''\psi_0 dx_3 = \frac{1}{s\mu} \int_\mathbb{R} \left( g|\xi|^2\tilde{\rho}\psi - \lambda^2(|\xi|^2\tilde{\rho}\psi - (\tilde{\rho}\psi')) + s\mu(2|\xi|^2\psi'' - |\xi|^4\psi) \right) \psi_0 dx_3,
$$

which, keeping in mind that $\psi \in H^2(\mathbb{R})$, yields $\psi''' \in L^2(\mathbb{R})$. Hence $\psi \in H^4_{loc}(\mathbb{R}) \cap C^{3,1/2}_{loc}(\mathbb{R})$, and $\psi'(\infty) = \psi''(\infty) = \psi'''(\infty) = 0$. Using these facts, Hölder’s inequality, and integration by parts, we deduce that

$$
\|\psi'''\|_{L^2(\mathbb{R})}^2 = \int_\mathbb{R} |\psi'''|^2 dx_3 = - \int_\mathbb{R} \psi'''\psi'' dx_3 \leq \|\psi''\|_{L^2(\mathbb{R})}\|\psi'''\|_{L^2(\mathbb{R})},
$$

(2.14)

i.e., $\psi''' \in L^2(\mathbb{R})$. Consequently, $\psi \in H^4(\mathbb{R})$ and solves (2.5)–(2.6). This immediately gives that $\psi \in H^k(\mathbb{R})$ for any positive integer $k \geq 5$.

Next, we want to show that there is a fixed point such that $\lambda = s$. To this end, we first give some properties of $\alpha(s)$ as a function of $s > 0$.

**Proposition 2.2.** The function $\alpha(s)$ defined on $(0, \infty)$ enjoys the following properties:

1. For any $a, b \in (0, \infty)$ with $a < b$, there exist constants $c_1, c_2 > 0$ depending on $\tilde{\rho}, \mu, g, a$ and $b$, such that

$$
\alpha(s) \leq -c_1 + sc_2 \quad \text{for all } |\xi| \in [a, b].
$$

(2.15)

2. $\alpha(s) \in C_{loc}^{0,1}(0, \infty)$ is nondecreasing.

**Proof.** (1) In view of (1.3), there exists a $\tilde{\psi} \in C_0^\infty(\mathbb{R})$ such that

$$
\frac{ga^2 \int_\mathbb{R} \tilde{\rho}\tilde{\psi}^2 dx_3}{\int_\mathbb{R} \tilde{\rho}(b^2|\psi|^2 + |\psi'|^2) dx_3} := c_1 > 0,
$$

(2.16)
where the constant $c_1$ depends on $a$, $b$, $g$ and $\bar{\rho}$. Now, we use (2.11) and (2.16) to find that

$$
\alpha(s) = \inf_{\psi \in H^2(\mathbb{R})} \frac{E(\psi)}{J(\psi)} \leq \frac{\int_\mathbb{R} s \mu (4|\xi|^2|\tilde{\psi}|^2 + |\xi|^2|\tilde{\psi} + \tilde{\psi}''|^2 - g |\xi|^2 \bar{\rho}^2 \psi^2 dx)}{\bar{\rho} (4|\xi|^2|\tilde{\psi}|^2 + |\tilde{\psi}'|^2) dx} \\
\leq \frac{\bar{\rho} (4|\xi|^2|\tilde{\psi}|^2 + |\tilde{\psi}'|^2) dx}{\bar{\rho} (4|\xi|^2|\tilde{\psi}|^2 + |\tilde{\psi}'|^2) dx} = s c_2 - c_1,$$

where the positive constant $c_2$ depends on $\bar{\rho}$, $\mu$, $g$, $a$ and $b$. Hence, (2.13) holds.

(2) To show the second assertion, we let $Q := [a, b] \subset \mathbb{R}$ be a bounded interval, and

$$E_1(\psi) = \int_\mathbb{R} (4|\xi|^2|\psi'|^2 + |\xi|^2 \psi + \psi''|^2) dx.$$ 

For any $s \in Q$, there exists a minimizing sequence $\{\psi_n^s\} \subset A$ of $\inf_{\psi \in A} E(\psi, s)$, such that

$$|\alpha(s) - E(\psi_n^s, s)| < 1. \quad (2.17)$$

Making use of (2.7), (2.10), (2.13) and (2.17), we infer that

$$E_1(\psi_n^s, s) = \frac{E(\psi_n^s, s)}{s} + \frac{g |\xi|^2}{s} \int_\mathbb{R} \bar{\rho} \psi^2 dx + \frac{g |\xi|^2}{s} \int_\mathbb{R} \bar{\rho}^2 \psi dx \\
\leq 1 + \max \{ |lb c_2 - c_1|, \frac{g \bar{\rho}^2}{\bar{\rho} L^\infty(\mathbb{R})} \} + \frac{g \bar{\rho}^2}{\bar{\rho} L^\infty(\mathbb{R})} := K. \quad (2.18)$$

For $s_i \in Q$ ($i = 1, 2$), we find that

$$\alpha(s_1) \leq \limsup_{n \to \infty} E(\varphi_{s_2}^n, s_1) \leq \limsup_{n \to \infty} E(\psi_{s_2}^n, s_2) + |s_1 - s_2| \limsup_{n \to \infty} E_1(\psi_{s_2}^n) \\
\leq \alpha(s_2) + K |s_1 - s_2|, \quad (2.19)$$

where $\{\psi_{s_2}^n\} \subset A$ is a minimizing sequence of $\inf_{\psi \in A} E(\psi, s_2)$ and the constant $K$ is given in (2.18). Reversing the role of the indices 1 and 2 in the derivation of the inequality (2.19), we obtain the same boundedness with the indices switched. Therefore, we deduce that

$$|\alpha(s_1) - \alpha(s_2)| \leq K |s_1 - s_2|,$$

which yields $\alpha(s) \in C_{\text{loc}}^0(0, \infty)$.

Finally, from (2.7) and (2.9) it follows that

$$\alpha(s_1) \leq \limsup_{n \to \infty} E(\psi_{s_2}^n, s_1) \leq \limsup_{n \to \infty} E(\psi_{s_2}^n, s_2) = \alpha(s_2) \quad \text{for any } 0 < s_1 < s_2 < \infty.$$ 

Hence $\alpha(s)$ is nondecreasing on $(0, \infty)$. This completes the proof of Proposition 2.2. \qed

Given $\xi \in \mathbb{R}^2$ with $|\xi| \neq 0$, by virtue of (2.13), there exists a $s_0 > 0$ depending on the quantities $\bar{\rho}$, $\mu$, $g$, $|\xi|$, such that for any $s \leq s_0$, $\alpha(s) < 0$. Let

$$\mathcal{G}_{|\xi|} := \sup \{ s \mid \alpha(\tau) < 0 \text{ for any } \tau \in (0, s) \} > 0, \quad (2.20)$$

then $\mathcal{G}_{|\xi|} > 0$. This allows us to define $\lambda(s) = \sqrt{-\alpha(s)} > 0$ for any $s \in \mathcal{S}_{|\xi|} := (0, \mathcal{G}_{|\xi|})$. Therefore, as a result of Proposition 2.1, we have the following existence for the modified problem (2.6), (2.7).
Proposition 2.3. For each $|\xi| \neq 0$ and $s \in S_{|\xi|}$ there is a solution $\psi = \psi(|\xi|, x_3) \neq 0$ with $\lambda = \lambda(|\xi|, s) > 0$ to the problem (2.4), (2.5). Moreover, $\psi \in H^k(\mathbb{R})$ for any positive integer $k$.

Now, we can use Proposition 2.2 (2.20) and (2.10) to check that $\lambda(s) \in C^{0,1}_{loc}(S_{|\xi|})$ is nonincreasing (in fact, we can further show that $\lambda(s)$ is strictly increasing, we refer to the proof of [10, Proposition 3.6]), $\lambda(s) \leq \sqrt{\frac{g}{\rho}} \frac{\sqrt{\rho}}{f_{\psi}} \|_{L^{\infty}(\mathbb{R})}$, and $\lim_{\xi \to \xi_{|\xi|}} \lambda(S_{|\xi|}) = 0$ if $S_{|\xi|} < +\infty$. Hence, we can employ a fixed-point argument to find $s \in S_{|\xi|}$ so that $s = \lambda(|\xi|, s)$, thus and obtain a solution to the original problem (2.4), (2.5).

Proposition 2.4. Let $|\xi| \neq 0$, then there exists a unique $s \in S_{|\xi|}$, such that $\lambda(|\xi|, s) = \sqrt{-\alpha(s)} > 0$ and $s = \lambda(|\xi|, s)$.

Proof. We refer to [10, Theorem 3.8] (or [24, Lemma 3.7]) for a proof. □

Consequently, in view of Propositions 2.3 and 2.4, we conclude the following existence for the problem (2.4), (2.5).

Theorem 2.1. For each $|\xi| \neq 0$, there exist $\psi = \psi(|\xi|, x_3) \neq 0$ and $\lambda(|\xi|) > 0$ satisfying (2.4), (2.5). Moreover, $\psi \in H^k(\mathbb{R})$ for any positive integer $k$.

We end this subsection by giving some properties of the solutions established in Theorem 2.1 in terms of $\lambda(|\xi|)$, which show that $\lambda$ is a bounded, continuous function of $|\xi|$.

Proposition 2.5. The function $\lambda : (0, \infty) \to (0, \infty)$ is continuous and satisfies

$$
\sup_{0 < |\xi| < \infty} \lambda(|\xi|) \leq \sqrt{g} \|\sqrt{\frac{\rho}{\sqrt{\rho}}}\|_{L^{\infty}(\mathbb{R})}.
$$

Proof. The boundedness of $\lambda$ (2.21) follows from (2.10). To show the continuity of $\lambda$, we see that for any but fixed $\xi_0 \neq 0$, there exists an interval $[a, b] \subset (0, \infty)$ so that $|\xi_0| \in (a, b)$. Assume $|\xi| \to |\xi_0|$ with $|\xi| \in (a, b)$, and denote $\kappa = |\xi|^2 - |\xi_0|^2$, then $\kappa \to 0$ as $|\xi| \to |\xi_0|$.

(i) We first show

$$
\lim_{|\xi| \to |\xi_0|} \alpha(|\xi|, s) = \alpha(|\xi_0|, s) \quad \text{for any } s \in S_{|\xi|}.
$$

By virtue of Proposition 2.4, for any $|\xi| \in (a, b)$, there exists a functions $\psi_{|\xi|} \in A$, such that

$$
\alpha(|\xi|) = \int_{\mathbb{R}} s \mu(4|\xi|^2|\psi_{|\xi|}'|^2 + ||\xi|^2 \psi_{|\xi|}' + \psi_{|\xi|}''|^2) - g\rho|\xi|^2 \psi_{|\xi|}'^2 dx_3
$$

(2.23)

Utilizing (2.8) and (2.15), we have

$$
\|\psi_{|\xi|}\|_{H^2(\mathbb{R})} \leq c_5,
$$

where $c_5$ depends on $\rho$, $\mu$, $g$, $a$, $b$ and $s$.

Substitution of $|\xi|^2 = |\xi|^2 + \kappa$ into (2.23) results in

$$
\alpha(|\xi|) = \int_{\mathbb{R}} s \mu(4|\xi_0|^2|\psi_{|\xi|}'|^2 + ||\xi_0|^2 \psi_{|\xi|}' + \psi_{|\xi|}''|^2) - g|\xi_0|^2 \rho \psi_{|\xi|}'^2 dx_3 + \kappa f(\kappa, \psi_{|\xi|})
$$

(2.25)

where

$$
f(\kappa, \psi_{|\xi|}) = \int_{\mathbb{R}} s \mu(4|\xi|^2|\psi_{|\xi|}'|^2 + 2\psi_{|\xi|}'(||\xi|^2 \psi_{|\xi|}' + \psi_{|\xi|}'') + \kappa \psi_{|\xi|}'^2 - g|\xi|^2 \rho \psi_{|\xi|}'^2 dx_3.
$$
By Hölder’s inequality and (2.24), we can bound
\[ |f(\kappa, \psi_\xi)| \leq c_6 \text{ for some constant } c_6. \]  

(2.26)

Similarly to (2.25) and (2.26), we also have
\[ \alpha(|\xi_0|) \geq \alpha(|\xi|) - \kappa f(-\kappa, \psi_{|\xi_0|}) \text{ and } |f(-\kappa, \psi_{|\xi_0|})| \leq c_6. \]  

(2.27)

Combining (2.25) with (2.27), we get
\[ \kappa f(-\kappa, \psi_{|\xi_0|}) \geq \alpha(|\xi|) - \alpha(|\xi_0|) \geq \kappa f(\kappa, \psi_{|\xi|}), \]  

which, together with (2.26) and (2.27), implies that (2.22). Hence
\[ \lim_{|\xi| \to |\xi_0|} \lambda(|\xi|, s) = \lambda(|\xi_0|, s) \text{ for any } s \in S_{|\xi|}. \]  

(2.28)

because of \( \lambda(|\xi|, s) = \sqrt{-\alpha(|\xi|, s)} \).

(ii) Exploiting (2.28) and Propositions 2.4, we know that for any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( |\lambda(|\xi|, s_{|\xi|}) - \lambda(|\xi_0|, s_{|\xi_0|})| < \varepsilon \) and \( s_{|\xi_0|} = \lambda(|\xi_0|, s_{|\xi_0|}) = \sqrt{-\alpha(|\xi_0|, s_{|\xi_0|})} \) for any \( ||\xi| - |\xi_0|| < \delta. \) On the other hand, for each \( |\xi| > 0 \), \( \lambda(s) \) is nonincreasing and continuous on \( S_{|\xi|} \), and there exists a unique \( s_{|\xi|} \in S_{|\xi|} \) satisfying \( \lambda(|\xi|, s_{|\xi|}) = s_{|\xi|} > 0 \) by Propositions 2.4. Consequently, we immediately infer that \( |\lambda(|\xi|, s_{|\xi|}) - \lambda(|\xi_0|, s_{|\xi_0|})| < \varepsilon \) with \( s_{|\xi|} = \lambda(|\xi|, s_{|\xi|}). \)

Hence \( \lambda(|\xi|) \) is continuous. This completes the proof of the proposition. \( \square \)

Remark 2.1. In addition, since \( \bar{\rho} \in C^\infty_0(\mathbb{R}) \), we can bound \( \lambda \) as follows.

(i) Applying integrating by parts and Hölder inequality,
\[
\lambda^2(|\xi|) \leq \int_{\mathbb{R}} g|\xi|^2 \bar{\rho} \psi_{|\xi|}'^2 dx_3 = 2g|\xi| \int_{\mathbb{R}} \bar{\rho}|\xi| \psi_{|\xi|}' \psi_{|\xi|}' dx_3 \\
\leq 2g|\xi| \left( \int_{\mathbb{R}} \bar{\rho}|\xi|^2 \psi_{|\xi|}'^2 dx_3 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \bar{\rho}|\psi_{|\xi|}'|^2 dx_3 \right)^{\frac{1}{2}} \leq 2g|\xi|. 
\]

Consequently we have \( \lim_{|\xi| \to 0} \lambda(|\xi|) = 0. \)

(ii) There exists a functions \( \psi_{|\xi|} \in \mathcal{A} \) such that
\[
-\lambda^2(|\xi|) = \lambda(|\xi|) \mu \int_{\mathbb{R}} (4|\xi|^2 |\psi_{|\xi|}'|^2 + ||\xi|^2 \psi_{|\xi|} + \psi_{|\xi|}''|^2) - g\bar{\rho}'|\xi|^2 \psi_{|\xi|}'^2 dx_3, 
\]
which implies that there exists a constant \( C(\bar{\rho}) \) depending on \( \bar{\rho} \) such that
\[
0 < 4\mu \lambda(|\xi|) \int_{\mathbb{R}} |\psi_{|\xi|}'|^2 dx_3 \leq gC(\bar{\rho}) \int_{\mathbb{R}} |\psi_{|\xi|}'|^2 dx_3.
\]

We immediately get that
\[
\lambda(|\xi|) \leq \frac{gC(\bar{\rho})}{4\mu}. \]  

(2.29)
2.2. Construction of a solution to the system \((2.2), (2.3)\)

A solution to \((2.4), (2.5)\) gives rise to a solution of the system \((2.2), (2.3)\) for the growing mode velocity \(u\) as well.

**Theorem 2.2.** For each \(\xi \in \mathbb{R}^2\) with \(|\xi| > 0\), there exists a solution \((\varphi, \theta, \psi, \pi) = (\varphi(\xi, x_3), \theta(\xi, x_3), \psi(|\xi|, x_3), \pi(|\xi|, x_3))\) with \(\lambda = \lambda(|\xi|) > 0\) to \((2.2), (2.3)\), and the solution belongs to \((H^k(\mathbb{R}))^4\) for any positive integer \(k\).

**Proof.** With the help of Theorem 2.1, we first construct a solution \((\varphi, \theta, \psi, \pi) = (\varphi(\xi, x_3), \theta(\xi, x_3), \psi(|\xi|, x_3), \pi(|\xi|, x_3))\) satisfying \((2.4), (2.5)\). Recalling \(\lambda > 0\) and \(\psi \in A \cap H^k(\mathbb{R})\) for any positive integer \(k\), multiplying \((2.2)_1\) and \((2.2)_2\) by \(\xi_1\) and \(\xi_2\) respectively, adding the resulting equations, and utilizing \((2.2)_4\), we find that \(\pi\) can be expressed by \(\psi\), i.e.,

\[\pi = \pi(|\xi|, x_3) = [\mu\psi''' - (\lambda\bar{\rho} + \mu|\xi|^2)\psi']|\xi|^{-2}. \quad (2.30)\]

Next, we construct the solution \((\varphi, \theta)\). To this end, we shall exploit the fact that the problem \((2.2), (2.3)\) is invariant under simultaneous rotations of \((\varphi, \theta)\) and \((\xi_1, \xi_2)\). Indeed, it is easy to see that if \(R \in SO(2)\) is a rotation operator, then \(R(\varphi, \theta), R(\xi_1, \xi_2)\) is also a solution with the same \(\psi, \pi\) and \(\lambda\). Thus, given any \(\xi\) we choose a rotation operator \(R_\xi\) so that \(R_\xi \xi = (|\xi|, 0)\). Hence,

\[ (\tilde{\varphi}, \tilde{\theta}, \psi, \pi, \lambda) = \left(-\psi'_{|\xi|}/|\xi|, 0, \psi(|\xi|), \pi \text{ (given by } (2.30)\text{), } \lambda(|\xi|)\right) \]

is a solution to \((2.2), (2.3)\) with \((\xi_1, \xi_2) = (|\xi|, 0)\). Now, if we define

\[ (\varphi, \theta, \psi, \pi, \lambda) := (R_\xi^{-1}(\tilde{\varphi}, \tilde{\theta}), \psi, \pi, \lambda) = ((\xi_1, \xi_2)\psi'_{|\xi|}/|\xi|^2, \psi, \pi, \lambda), \quad (2.31)\]

we find that \((\psi, \varphi, \theta, \pi, \lambda)\) constructed above is indeed a solution to the problem \((2.2), (2.3)\).

**Remark 2.2.** For each \(x_3\), it is easy to see that the solution \((\varphi(\xi, \cdot), \theta(\xi, \cdot), \psi(|\xi|, \cdot), \pi(|\xi|, \cdot), \lambda(|\xi|))\) constructed in Theorem 2.2 has the following properties:

1. \(\lambda(|\xi|), \psi(|\xi|, \cdot)\) and \(\pi(|\xi|, \cdot)\) are even on \(\xi_1\) or \(\xi_2\), when the another variable is fixed;
2. \(\varphi(\xi, \cdot)\) is odd on \(\xi_1\), but even on \(\xi_2\), when the another variable is fixed;
3. \(\theta(\xi, \cdot)\) is even on \(\xi_1\), but odd on \(\xi_2\), when the another variable is fixed.

The next lemma provides an estimate for the \(H^k\)-norm of the solution \((\varphi, \theta, \psi, \pi)\) with \(\xi\) varying, which will be useful in the next section when such a solution is integrated in a Fourier synthesis. To emphasize the dependence on \(\xi\), we write these solutions as \(\varphi(\xi) = \varphi(\xi, x_3), \theta(\xi) = \theta(\xi, x_3), \psi(\xi) = \psi(|\xi|, x_3), \pi(\xi) = \pi(|\xi|, x_3)\).

**Lemma 2.1.** Let \(\xi \in \mathbb{R}^2\) with \(0 < R_1 < |\xi| < R_2\), \(\varphi(\xi), \theta(\xi), \psi(\xi), \pi(\xi)\) and \(\lambda(\xi)\) be constructed as in Theorem 2.2, then for any \(k \geq 0\) there exist positive constants \(A_k, B_k, C_k\) and \(D\), which may depend on \(R_1, R_2, \bar{\rho}, \mu\) and \(g\), such that

\[\|\psi(\xi)\|_{H^k(\mathbb{R})} \leq A_k, \quad (2.32)\]
\[\|\pi(\xi)\|_{H^k(\mathbb{R})} \leq B_k, \quad (2.33)\]
\[\|\varphi(\xi)\|_{H^k(\mathbb{R})} + \|\theta(\xi)\|_{H^k(\mathbb{R})} \leq C_k. \quad (2.34)\]

Moreover,

\[\|\psi\|^2_{L^2(\mathbb{R})} > 0. \quad (2.35)\]
Proof. Throughout this proof, we denote by $\tilde{c}$ a generic positive constant which may vary from line to line, and depend on $R_1$, $R_2$, $\bar{\rho}$, $\mu$ and $g$.

(i) First, since $\psi \in A$, we see that (2.35) holds, and there exists a constant $\tilde{c}$, such that

$$\| \psi(\xi) \|_{H^1(\mathbb{R})} \leq \tilde{c}. \tag{2.36}$$

On the other hand, in view of Proposition 2.5, we have

$$\lambda(\xi) \geq \tilde{c} > 0 \quad \text{for any } |\xi| \in (R_1, R_2). \tag{2.37}$$

Similarly to (2.24), we use (2.36), (2.37) and (2.23) with $\lambda(\xi)$ in place of $\alpha(\xi)$ to deduce that

$$\| \psi(\xi) \|_{H^2(\mathbb{R})} \leq \tilde{c}. \tag{2.38}$$

We now rewrite (2.4) as

$$\psi^{(\prime\prime\prime)}(\xi) = \left[\lambda(\lambda \bar{\rho} + 2\mu |\xi|^2)\psi^{(\prime)}(\xi) + \lambda^2 \bar{\rho} \psi^{(\prime\prime)}(\xi) - |\xi|^2 (\lambda^2 \bar{\rho} + \lambda \mu |\xi|^2 - g \bar{\rho}) \psi(\xi)\right] / \lambda \mu, \tag{2.39}$$

which, together with (2.14), (2.37) and (2.38), yields

$$\| \psi(\xi) \|_{H^4(\mathbb{R})} \leq \tilde{c}. \tag{2.40}$$

Differentiating (2.39) with respect to $x_3$ and using (2.40), we find, by induction on $k$, that

$$\| \phi^{(\prime\prime)}(\xi) \|_{L^2(\mathbb{R})} + \| \theta^{(\prime\prime)}(\xi) \|_{L^2(\mathbb{R})} \leq \tilde{c}. \tag{2.41}$$

Noticing that (2.2)1 and (2.2)2 can be rewritten as

$$\phi^{(\prime)}(\xi) = \left(\frac{\lambda \bar{\rho}}{\mu} + |\xi|^2\right) \phi(\xi) - \frac{\xi_1 \pi(\xi)}{\mu}, \tag{2.42}$$

and

$$\theta^{(\prime)}(\xi) = \left(\frac{\lambda \bar{\rho}}{\mu} + |\xi|^2\right) \theta(\xi) - \frac{\xi_2 \pi(\xi)}{\mu}, \tag{2.43}$$

we apply (2.41) to (2.42) and (2.43) to obtain

$$\| \phi^{(\prime\prime)}(\xi) \|_{L^2(\mathbb{R})} + \| \theta^{(\prime\prime)}(\xi) \|_{L^2(\mathbb{R})} \leq \tilde{c}. \tag{2.44}$$

Using (2.41) and (2.44), analogously to (2.14), we infer that

$$\| \phi^{(\prime)}(\xi) \|_{L^2(\mathbb{R})} + \| \theta^{(\prime)}(\xi) \|_{L^2(\mathbb{R})} \leq \tilde{c}. \tag{2.45}$$

Putting (2.41)–(2.45) together, we immediately obtain (2.34). This completes the proof. \qed
2.3. Exponential growth rate

In this subsection we use the Fourier synthesis to build growing solutions to (1.8) out of the solutions constructed in the previous subsection (Theorem 2.2) for any fixed spatial frequency \( \xi \in \mathbb{R}^2 \) with \( |\xi| > 0 \). The constructed solutions will grow in-time in the Sobolev space of order \( k \).

**Theorem 2.3.** Let \( 0 < R_1 < R_2 < \infty \) and \( f \in C^\infty_0(R_1, R_2) \) be a real-valued function. For \( \xi \in \mathbb{R}^2 \) with \( |\xi| \in (0, \infty) \), define

\[
\begin{align*}
\varphi(\xi, x_3) & = -i\varphi(\xi, x_3)e_1 - i\theta(\xi, x_3)e_2 + \psi(\xi, x_3)e_3,
\end{align*}
\]

where \( (\varphi, \theta, \psi)(\xi, x_3) \) with \( \lambda(|\xi|) > 0 \) is the solution given by Theorem 2.2. Denote

\[
\begin{align*}
\varphi(t, x) & = -\frac{\rho'(x_3)}{4\pi^2} \int_{\mathbb{R}^2} f(|\xi|)v_3(\xi, x_3)e^{\lambda(|\xi|)t}e^{ix\cdot\xi}d\xi, \\
u(t, x) & = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \lambda(\xi)f(|\xi|)v(\xi, x_3)e^{\lambda(|\xi|)t}e^{ix\cdot\xi}d\xi, \\
q(t, x) & = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \lambda(\xi)f(|\xi|)\pi(\xi, x_3)e^{\lambda(|\xi|)t}e^{ix\cdot\xi}d\xi,
\end{align*}
\]

Then, \((\varphi, \nu, q)\) is a real-valued solution to the linearized problem (1.8) along with (1.7). For every \( k \in \mathbb{N} \), we have the estimate

\[
\|\varphi(0)\|_{H^k} + \|\nu(0)\|_{H^k} + \|q(0)\|_{H^k} \leq D_k \left( \int_{\mathbb{R}^2} (1 + |\xi|^2)^{k+2}|f(|\xi|)|^2d\xi \right)^{1/2} < \infty,
\]

where \( D_k > 0 \) is a constant depending on \( k, \bar{\rho}, R_1, R_2 \) and \( g \). Moreover, for every \( t > 0 \) we have \((\varphi(t), \nu(t), q(t)) \in H^k\), and

\[
\begin{align*}
e^{t\lambda_0(f)}\|\varphi(0)\|_{H^k} & \leq \|\varphi(t)\|_{H^k} \leq e^{t\Lambda}\|\varphi(0)\|_{H^k}, \\
e^{t\lambda_0(f)}\|\nu_i(0)\|_{H^k} & \leq \|\nu_i(t)\|_{H^k} \leq e^{t\Lambda}\|\nu_i(0)\|_{H^k}, & i = 1, 2, 3, \\
e^{t\lambda_0(f)}\|q(0)\|_{H^k} & \leq \|q(t)\|_{H^k} \leq e^{t\Lambda}\|q(0)\|_{H^k},
\end{align*}
\]

where

\[
\lambda_0(f) = \inf_{|\xi| \in \text{supp}(f)} \lambda(|\xi|) > 0
\]

and

\[
\Lambda = \sup_{0<|\xi|<+\infty} \lambda(|\xi|) < \sqrt{g} \left\| \sqrt{\rho/\bar{\rho}} \right\|_{L^\infty(\mathbb{R})}.
\]

In particular,

\[
\|u_3(0)\|_{H^k} > 0 \quad \text{if } f \neq 0,
\]

and we can further take proper constants \( R_1, R_2 \), such that

\[
\lambda_0(f) = \Lambda/2.
\]

**Proof.** Obviously, (2.53), (2.54) and (2.56) follow from Proposition 2.5. For each fixed \( \xi \in \mathbb{R}^2 \),

\[
\begin{align*}
\tilde{\varphi}(t, x) & = -\bar{\rho}f(|\xi|)v_3(\xi, x_3)e^{\lambda(|\xi|)t}e^{ix\cdot\xi}, \\
\tilde{\nu}(t, x) & = \lambda(|\xi|)f(|\xi|)v(\xi, x_3)e^{\lambda(|\xi|)t}e^{ix\cdot\xi}, \\
\tilde{q}(t, x) & = \lambda(|\xi|)f(|\xi|)\pi(\xi, x_3)e^{\lambda(|\xi|)t}e^{ix\cdot\xi},
\end{align*}
\]
Proof. Let \((\text{the linearized system (1.8) with}) \) then integrating by parts, we arrive at which will be employed to derive an energy-like estimate. In fact, recalling as a test function to get Using the standard density argument (see, for instance, [4] or [17, Section 2.1]), we could take

3. Uniqueness of the linearized equations

In this section, we will show the uniqueness of solutions to the linearized problem, which will be used in the proof of Theorem 1.1 in Section 5. We first define the function space of strong solutions.

\[
Q(T) := \{ (\bar{\varrho}, \bar{u}, \bar{q}) \mid \bar{\varrho} \in C^0([0, T], L^2), \nabla q \in L^2(0, T; H^1_{\text{loc}}),
\quad u \in C^0([0, T], (L^2)^3) \cap L^2(0, T; (H^2)^3), \partial_t u \in (L^2((0, T) \times \mathbb{R}^3))^3 \}. \]

We claim that the solution to the linearized problem is unique in the function space \(Q\).

**Theorem 3.1.** (Uniqueness) Assume that \((\bar{\varrho}, \bar{u}, \bar{q}), (\tilde{\varrho}, \tilde{u}, \tilde{q}) \in Q(T)\) are two strong solutions of (1.8) with \((\bar{\varrho}, \bar{u})(0) = (\tilde{\varrho}, \tilde{u})(0)\). Then, \((\bar{\varrho}, \bar{u}, \nabla \bar{q}) = (\tilde{\varrho}, \tilde{u}, \nabla \tilde{q})\).

**Proof.** Let \((\varrho, u, q) = (\bar{\varrho} - \tilde{\varrho}, \bar{u} - \tilde{u}, \bar{q} - \tilde{q})\). Then \((\varrho, u, q) \in Q(T)\) is still a strong solution to the linearized system (1.8) with \((\varrho, u)(0) = (0, 0)\).

Multiplying (1.8) by \(\varphi \in (C^\infty_0((0, t) \times \mathbb{R}^3))^3\) with \(\text{div}\varphi = 0\), integrating over \((0, t) \times \mathbb{R}^3\), and then integrating by parts, we arrive at

\[
\int_0^t \int_{\mathbb{R}^3} \bar{\varrho} \partial_t u \cdot \varphi \, dx \, dt + \int_0^t \int_{\mathbb{R}^3} \mu \nabla u : \nabla \varphi \, dx \, dt = -g \int_0^t \int_{\mathbb{R}^3} \rho \varphi_3 \, dx \, dt.
\]

Using the standard density argument (see, for instance, [4] or [17, Section 2.1]), we could take \(u\) as a test function to get

\[
\int_0^t \int_{\mathbb{R}^3} \bar{\varrho} \partial_t u \cdot u \, dx \, dt + \int_0^t \int_{\mathbb{R}^3} \mu \nabla u : \nabla u \, dx \, dt = -g \int_0^t \int_{\mathbb{R}^3} \rho u_3 \, dx \, dt, \quad (3.1)
\]

which will be employed to derive an energy-like estimate. In fact, recalling

\[
u \in L^2(0, T; (H^1)^3) \cap C^0([0, T], (L^2)^3) \text{ and } \partial_t u \in (L^2((0, T) \times \mathbb{R}^3))^3,\]

Finally, we can use (2.53), (2.54) and (2.57) to obtain the estimates (2.50)–(2.52). \(\square\)
and $u(0) = 0$, we easily deduce

$$\int_0^t \int_\Omega \bar{\rho} \partial_t u \cdot ud\tau = \frac{1}{2} \int_\Omega \bar{\rho} u^2(t) dx.$$  (3.2)

Since $\rho \in C^0([0, T], L^2)$ and $\rho(0) = 0$, the equation (1.8)1 gives

$$\rho(t, x) = \int_0^t \bar{\rho} u_3(s, x) ds \text{ for any } t \geq 0,$$

Consequently, with the help of the regularity of $\partial_t u_3$, the property of absolutely continuous functions and Fubini’s theorem, we conclude that

$$\int_0^t \int_{\mathbb{R}^3} \rho u_3 dx dt = \frac{1}{2} \int_{\mathbb{R}^3} \rho' \left( \int_0^t u_3(\tau, x) d\tau \right)^2 dx,$$

$$\leq \frac{t}{2} \int_0^t \int_{\mathbb{R}^3} \bar{\rho} u_3^2 dx d\tau,$$

which yields

$$\frac{1}{2} \int_{\mathbb{R}^3} \bar{\rho} u(t)^2 dx + \int_0^t \int_{\mathbb{R}^3} \mu \nabla u : \nabla u dx d\tau \leq \frac{t}{2} \left\| \frac{\rho'}{\rho} \right\|_{L^\infty(\mathbb{R})} \int_0^t \int_{\mathbb{R}^3} \bar{\rho} u^2 dx d\tau,$$

which yields

$$\left\| \sqrt{\rho} u(t) \right\|_{L^2}^2 \leq T \left\| \frac{\rho'}{\rho} \right\|_{L^\infty(\mathbb{R})} \int_0^t \left\| \sqrt{\rho} u \right\|_{L^2}^2 d\tau.$$  (3.4)

Applying Grownwall’s inequality to (3.4), we get

$$\left\| \sqrt{\rho} u(t) \right\|_{L^2}^2 = 0 \text{ for any } t \in [0, T],$$

which yields $u = 0$, i.e., $\bar{u} = \tilde{u}$, since $\bar{\rho} > 0$. This, combined with (1.8)1 and (1.8)2, proves that

$$(\tilde{\rho}, \tilde{u}, \nabla \tilde{q}) = (\bar{\rho}, \bar{u}, \nabla \bar{q}) \text{ for any } t \in (0, T].$$

Thus, the desired conclusion follows.

4. Nonlinear energy estimates of the perturbed problem

In this section, we shall derive some nonlinear energy estimates for the perturbed problem, which will also be used in the proof of Theorem 1.1 in Section 5. To this end, let $(\varrho, u, q)$ be a classical solution of the perturbed problem (1.5)–(1.7) with $\rho := \varrho + \bar{\rho} > 0$ in $[0, T] \times \mathbb{R}^3$ for some $T > 0$. Moreover, we assume that the classical solution $(\varrho, u, q)$ satisfies the initial condition

$$\sqrt{\|\varrho_0\|_{H^1} + \|u_0\|_{H^2}^2} = \delta_0 \leq 1.$$  (4.1)
The restricted relation between $T$ and $\delta_0$ will be given at the end of Subsection 4.1.

In what follows, we denote by $C_i$ ($i = 1, 2, \cdots$), $C(T)$ and $C_z$ generic positive constants depending on $\mu$, $g$ and $\bar{\rho}$. In addition, $C(T)$ also depends on $T$ and is nondecreasing with respect to $T$; the subscript $i$ of $C_i$ emphasizes that we may repeat to use the constant $C_i$ in the process of estimates.

4.1. Estimates for $\|u_t\|_{L^2}$ and $\|\nabla u\|_{H^1}$

We first observe that the continuity equation \([1.5]_1\) and the incompressibility condition \([1.5]_3\) imply immediately that for any $t \in (0, T]$,

$$\alpha := \inf_{x \in \mathbb{R}^3}\{\rho_0(x)\} \leq \rho(t) \leq \sup_{x \in \mathbb{R}^3}\{\rho_0(x)\} := \beta \quad \text{or} \quad \alpha - \bar{\rho} \leq \rho(t) \leq \beta - \bar{\rho}, \quad (4.2)$$

and

$$\frac{d}{dt}\|\rho(t)\|_{L^2}^2 = -2 \int_{\mathbb{R}^3} \rho' \rho u_3 dx \leq 2 \left\| \rho' \rho \right\|_{L^2} \|\sqrt{\rho} u\|_{L^2} \leq 2\alpha^{-1} \|\rho'\|_{L^\infty} \|\rho\|_{L^2} \|\sqrt{\rho} u\|_{L^2}. \quad (4.3)$$

Multiplying \([1.5]_2\) by $u$, using \([1.5]_1\), and then integrating (by parts) over $(0, t) \times \mathbb{R}^3$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho|u|^2(t) dx + \int_{\mathbb{R}^3} |\nabla u|^2 dx = -g \int_{\mathbb{R}^3} g u_3 dx. \quad (4.4)$$

Since the integral on the right-hand side is bounded from above by $g\alpha^{-\frac{1}{2}} \|\rho\|_{L^2} \|\sqrt{\rho} u\|_{L^2}$, we get

$$\frac{d}{dt}\|\sqrt{\rho} u(t)\|_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 \leq 2g\alpha^{-\frac{1}{2}} \|\rho\|_{L^2} \|\sqrt{\rho} u\|_{L^2}. \quad (4.5)$$

Combining \((4.3)\) with \((4.4)\) and using Cauchy-Schwarz’s inequality, we obtain

$$\frac{d}{dt}(\|\rho(t)\|_{L^2}^2 + \|\sqrt{\rho} u(t)\|_{L^2}^2) + \mu \|\nabla u\|_{L^2}^2 \leq C_1(\|\rho(t)\|_{L^2}^2 + \|\sqrt{\rho} u(t)\|_{L^2}^2), \quad (4.5)$$

which implies

$$\|\rho(t)\|_{L^2}^2 + \|\sqrt{\rho} u(t)\|_{L^2}^2 \leq \delta_0 e^{C_1 t}. \quad (4.6)$$

In particular, making use of \((4.2)\), \((4.5)\) and \((4.6)\), we arrive at

$$\|\rho(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \leq C \delta_0 e^{C_1 t}. \quad (4.7)$$

To control $u_t$, we multiply \((1.5)\) by $u_t$ in $L^2$ and apply Cauchy-Schwarz’s inequality to infer that

$$\frac{1}{2} \int_{\mathbb{R}^3} \rho u_t^2 dx + \frac{1}{\mu} \frac{d}{dt} \int_{\mathbb{R}^3} \rho u(t)^2 dx \leq C(\|\rho\|_{L^2}^2 + \|\sqrt{\rho} u \cdot \nabla u\|_{L^2}^2). \quad (4.8)$$

To bound the second term on the right-hand side of \((4.8)\), we recall that $(u, q)$ is a solution of the Stokes equations:

$$-\mu \Delta u + \nabla q = -\rho u_t - \rho (u \cdot \nabla u) - q g e_3, \quad \text{div} \ u = 0 \text{ in } \mathbb{R}^3.$$
which combined with (4.8) results in
\[
\frac{1}{4} \| \sqrt{\rho} u \|^2_{L^2} + \varepsilon \| \nabla^2 u \|^2_{L^2} + \mu \frac{d}{dt} \| \nabla u(t) \|^2_{L^2} \leq C_3 (\| \varrho \|^2_{L^2} + \| u \cdot \nabla u \|^2_{L^2}) \quad \text{with } C_3 > 1.
\] (4.9)

If we apply Hölder’s and Sobolev’s inequalities, we have
\[
\| u \cdot \nabla u \|^2_{L^2} \leq \| u \|^2_{L^6} \| \nabla u \|^2_{L^3} \leq \| u \|^2_{L^6} \| \nabla u \|^6_{L^2} \leq \frac{2C^3_1}{\varepsilon} \| \nabla u \|^6_{L^2} + \frac{\varepsilon}{8C^3_1} \| \nabla u \|^2_{H^1}.
\]

Substituting the above inequality into (4.9) and integrating over \((0, t)\), we conclude that
\[
\frac{1}{4} \| \sqrt{\rho} u \|^2_{L^2} + \frac{7}{8} \| \nabla^2 u \|^2_{L^2} + \mu \frac{d}{dt} \| \nabla u(t) \|^2_{L^2} \leq C (\| \varrho \|^2_{L^2} + \| \nabla u \|^6_{L^2}).
\] (4.10)

Letting \(\delta_0 e^{C_1 T} \leq 1\), we get from (4.5) and (4.9) that
\[
\mu \| \nabla u(t) \|^2_{L^2} \leq \int_0^t C (\| \varrho \|^2_{L^2} + \| \nabla u \|^6_{L^2}) ds + \mu \| \nabla u_0 \|^2_{L^2}
\leq C^3_0 (e^{C_1 t} - 1) + C \int_0^t \| \nabla u \|^6_{L^2} ds + \mu \| \nabla u_0 \|^2_{L^2}
\leq C_4 (\delta_0 + \int_0^t \| \nabla u \|^6_{L^2} ds),
\] (4.11)

which yields
\[
\| \nabla u(t) \|^2_{L^2} \leq \sqrt{\delta_0^2 C^2_4 \frac{1}{1 - 2\delta_0^2 C^3_4}}.
\] (4.12)

Now, we take
\[
T = \min \left\{ \frac{1}{C_1} \ln \frac{1}{\delta_0}, \frac{1}{4\delta_0^2 C^3_4} \right\}.
\]

In particular, there exists a sufficiently small constant \(\delta_1 > 0\), such that
\[
T = \frac{-\ln \delta_0}{C_1} \quad \text{for any } \delta_0 \in (0, \delta_1).
\] (4.13)

From now on, we always assume that \(\delta_0 \) and \(T\) satisfy the relation (4.13). Thus, (4.12) gives
\[
\| \nabla u(t) \|^2_{L^2} \leq \sqrt{2\delta_0^2 C^2_4} \leq C \delta_0 \quad \text{for any } t \in (0, T].
\] (4.14)

Making use of (4.2), (4.6), (4.12), and the first inequality in (4.11), we deduce from (4.10) that
\[
\| \varrho(t) \|^2_{L^2} + \| u(t) \|^2_{H^1} + \int_0^t (\| u(t) \|^2_{L^2} + \| \nabla u(s) \|^2_{L^2}) ds \leq C(T) \delta_0^2 \quad \text{for any } t \in (0, T].
\] (4.15)
4.2. Estimates for $\|\nabla u_t\|_{L^2}$ and $\|\nabla^2 u\|_{L^6}$

Using (1.5)$_1$ and keeping in mind that $p = q + \bar{p}$, we can rewrite (1.5)$_2$ as

$$\rho u_t + \rho u \cdot \nabla u + \nabla p = \mu \Delta u - \rho \varepsilon_3,$$

whence, by taking the time derivative,

$$\rho u_{tt} + \rho u \cdot \nabla u_t - \mu \Delta u_t + \nabla p_t = -\rho \varepsilon_3 (u_t + u \cdot \nabla u + \varepsilon_3) - \rho u_t \cdot \nabla u,$$

which, by using the continuity equation, can be written as

$$\rho \left( \frac{1}{2} |u_t|^2 \right)_t + \rho u \cdot \nabla \left( \frac{1}{2} |u_t|^2 \right) - \mu \Delta u_t \cdot u_t + \nabla p_t \cdot u_t$$

$$= \text{div}(\rho u)(u_t + u \cdot \nabla u + \varepsilon_3) \cdot u_t - \rho (u_t \cdot \nabla u) u_t.$$

Hence, by integrating by parts, we see that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u_t(t)|^2 \, dx + \mu \int_{\mathbb{R}^3} |\nabla u_t|^2 \, dx$$

$$\leq \int_{\mathbb{R}^3} 2\rho |u| |u_t| |\nabla u| + \rho |u| |u_t| |\nabla u|^2 + \rho |u|^2 |u_t| |\nabla u|^2$$

$$+ \rho |u|^2 |\nabla u_t| + \rho |u_t|^2 |\nabla u| + g\rho |u| |\nabla u_t| := \sum_{i=1}^{6} I_j,$$

where $I_j$ can be bounded as follows, employing straightforward calculations.

$$I_1 \leq 2 \|\rho \|_{L^{1/2}}^{1/2} \|u\|_{L^{6}} \|\nabla u_t\|_{L^{2}} \leq 2 \|\rho \|_{L^{1/2}}^{1/2} \|u\|_{L^{6}} \|\nabla u_t\|_{L^{2}} \leq C \|\rho\|_{L^\infty}^{3/4} \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^2} \leq C(\varepsilon) \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla u\|_{H^1} \leq C(\varepsilon) \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} + \varepsilon \|\nabla u_t\|_{L^2}^2,$$

$$I_2 \leq C \|\rho\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} \leq C(\varepsilon) \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} + \varepsilon \|\nabla u_t\|_{L^2}^2,$$

$$I_3 \leq C \|\rho\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla^2 u\|_{L^2} \leq C(\varepsilon) \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} + \varepsilon \|\nabla u_t\|_{L^2}^2,$$

$$I_4 \leq C \|\rho\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} \|\nabla u\|_{L^2} \leq C(\varepsilon) \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} + \varepsilon \|\nabla u_t\|_{L^2}^2,$$

$$I_5 \leq C \|\rho\|_{L^\infty}^{3/4} \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^2} \leq C(\varepsilon) \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} + \varepsilon \|\nabla u_t\|_{L^2}^2,$$

$$I_6 \leq C(\varepsilon) \|u\|_{L^2}^2 + \varepsilon \|\nabla u_t\|_{L^2}^2,$$

where $C(\varepsilon)$ is a positive constant which may depend on $\varepsilon$. Inserting all the above estimates into (4.16), we conclude

$$\frac{d}{dt} \|\sqrt{\rho} u_t(t)\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \leq C \left( \|\nabla u\|_{L^2}^2 \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 \right) + \|u_t\|_{L^2}^2,$$

which, by integrating over $(\tau, t)$ and using (4.15), leads to

$$\|\sqrt{\rho} u_t(t)\|_{L^2}^2 + \int_{\tau}^{t} \|\nabla u_t(s)\|_{L^2}^2 \, ds \leq \|\sqrt{\rho} u_t(\tau)\|_{L^2}^2 + C(T)\delta_0^2.$$  (4.17)

On the other hand, multiplying (1.5)$_2$ by $u_t$ in $L^2$ and recalling $\text{div} u_t = 0$, we find that

$$\int_{\mathbb{R}^3} \rho |u_t(t)|^2 \, dx = \int_{\mathbb{R}^3} (-\rho \varepsilon_3 - \rho u \cdot \nabla u + \mu \Delta u - \nabla q) \cdot u_t \, dx$$

$$= \int_{\mathbb{R}^3} (-\rho \varepsilon_3 - \rho u \cdot \nabla u + \mu \Delta u) \cdot u_t \, dx,$$
whence,
\[ \int_{\mathbb{R}^3} \rho |u_t(t)|^2 \, dx \leq C \int_{\mathbb{R}^3} (\rho^2 + |u|^2 |\nabla u|^2 + |\Delta u|^2)(t) \, dx. \]

Taking \( t \to 0 \) in the above inequality and using (4.1), one gets
\[ \lim_{t \to 0} \sup \int_{\mathbb{R}^3} \rho |u_t(t)|^2 \, dx \leq C \delta_0^2. \]

Therefore, letting \( \tau \to 0 \) in (4.17), we conclude that
\[ \| \sqrt{\rho} u_t(t) \|_{L^2}^2 + \int_0^t \| \nabla u_t(s) \|_{L^2}^2 \, ds \leq C(T) \delta_0^2. \tag{4.18} \]

To derive estimates of higher derivatives, we recall again that the pair \((u, q)\) solves the Stokes equations:
\[ -\mu \Delta u + \nabla q = -\rho u_t - \rho (u \cdot \nabla u) - \rho g e_3, \quad \text{div} u = 0 \text{ in } \mathbb{R}^3. \]

It follows from the classical regularity theory for Stokes equations that
\[ \| \nabla^2 u \|_{L^2}^2 + \| \nabla q \|_{L^2}^2 \leq C \| -\rho u_t - \rho (u \cdot \nabla u) - \rho g e_3 \|_{L^2}^2 \]
\[ \leq C(\| \rho u_t \|_{L^2}^2 + \| \rho (u \cdot \nabla u) \|_{L^2}^2 + \| \rho g e_3 \|_{L^2}^2) \]
\[ \leq C(\| \rho u_t \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 + \| \rho g e_3 \|_{L^2}^2 ) \tag{4.19} \]
\[ \leq C(\| \sqrt{\rho} u_t \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 + \| \rho g e_3 \|_{L^2}^2 ) + \frac{1}{2} \| \nabla u \|_{H^1}^2. \]

Putting the estimate (4.15), (4.18) and (4.19) together, we obtain
\[ \| \nabla^2 u(t) \|_{L^2}^2 + \| \nabla q(t) \|_{L^2}^2 \leq C(T) \delta_0^2 \text{ for any } t \in (0, T]. \tag{4.20} \]

Arguing analogously to (4.3), we can deduce from (1.8) \(_1\) that
\[ \frac{d}{dt} \| q(t) \|_{L^6}^2 \leq 6 \| \rho' \|_{L^\infty} \| u(t) \|_{L^6} \text{ for any } t \in (0, T]. \tag{4.21} \]

By virtue of the estimates (4.1), (4.15), (4.18), (4.20) and (4.21), and the fact that
\[ \| \nabla^2 u \|_{L^6}^2 + \| \nabla q \|_{L^6}^2 \leq C(\| \rho u_t \|_{L^6}^2 + \| \rho u \cdot \nabla u \|_{L^6}^2 + g \| \rho e_3 \|_{L^6}^2), \]
we deduce that
\[ \int_0^T \left( \| \nabla u \|_{W^{1,6}}^2 + \| \nabla q \|_{L^6}^2 + \| \nabla u \|_{L^\infty}^2 \right)(s) \, ds \leq C(T) \delta_0^2. \tag{4.22} \]

4.3. Estimates for \( \| q \|_{H^1} \)

Observing that each \( q \) satisfies
\[ (|\rho_x|^2)_{t} + \text{div}(|\rho_x|^2 u) = -2 \rho_{x_j} ((\nabla \rho \cdot u)_{x_j} + \nabla q \cdot u_{x_j}), \]
we integrate over \( \mathbb{R}^3 \), sum over \( j \) and use (1.2) to infer that
\[ \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla q(t)|^2 \, dx \leq C \int_{\mathbb{R}^3} \left( |\nabla u| |\nabla q|^2 + (|u| + |\nabla u|) |\nabla q| \right) \, dx \]
\[ \leq C(\| \nabla u \|_{L^\infty} \| \nabla q \|_{L^2} + \| u \|_{H^1} \| \nabla q \|_{L^2}. \]
Hence,
\[
\|\nabla \varrho(t)\|_{L^2} \leq e^{C \int_0^t \|\nabla \varrho(s)\|_{L^\infty} ds} \left( \|\nabla \varrho_0\|_{L^2} + C \int_0^t \|\varrho(s)\|_{H^1} ds \right). \tag{4.23}
\]

From (4.7), (4.22) and (4.23) it follows that
\[
\|\varrho(t)\|_{H^1} \leq C(T)\delta_0 \quad \text{for any } t \in (0, T].
\]

Summing up the above estimates, we arrive at the following property:

**Proposition 4.1.** There exists a \(\delta_1 \in (0, 1]\), such that if \(\sqrt{\|\varrho_0\|_{H^1}^2 + \|\varrho_0\|_{H^2}^2} = \delta_0 \in (0, \delta_1)\), then any classical solution \((\varrho, u, q)\) to (1.5), emanating from the initial data \((\varrho_0, u_0)\), satisfies
\[
\sup_{0 < s \leq T_\delta} \left( \|\varrho\|_{H^1}^2 + \|u\|_{H^2}^2 + \|u_t\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 \right)(t) + \int_{T_\delta}^{T_\delta} \left( \|\nabla u\|_{H^1}^2 + \|\nabla u_t\|_{L^2}^2 \right) ds \leq C(T)\delta_0^2,
\]

where \(T_\delta = \min\{T^{\max}, -C_1^{-1} \ln \delta_0\}\) and \(T^{\max}\) denotes the maximal time of existence of the classical solution \((\varrho, u, q)\).

**Remark 4.1.** The local existence of classical solutions and global existence of classical small solutions to the 3D nonhomogeneous incompressible Navier-Stokes equations have been established by many authors, see [22, 25] for example. In particular, by a slight modification, one can follow the proof of [22, Theorem 1.1] and use the expanding domain technique in [2, 3] to obtain a local existence result of classical solutions to the Rayleigh-Taylor instability problem (1.5)–(1.7) defined on \((0, T^{\max}) \times \mathbb{R}^3\), where \(T^{\max} \to \infty\) as the initial data \(\|\varrho_0\|_{H^3(\mathbb{R}^3)} + \|u_0\|_{H^4(\mathbb{R}^3)} \to 0\). The proof is standard by means of energy estimates, and hence we omit it here.

5. **Proof of Theorem 1.1**

In this section we start to show Theorem 1.1. To this end, we first construct a linear solution, and a family of nonlinear solutions which possess some special properties. Then, by contradiction argument, we show that there exists a nonlinearly unstable solution satisfying the properties as stated in Theorem 1.1. Suppose that \(s \geq 2, \delta > 0, K > 0,\) and \(F\) satisfying (1.9), are arbitrary but given.

5.1. **Construction of a solution to the linearized problem**

In view of Theorem 2.3, we can construct a classical solution \((\varrho, u, q)\) to (1.6)–(1.8) satisfying the properties (2.50)–(2.52), (2.55) and (2.56). Noticing that the solution \((\varrho, u)\) is independent of \(s\) and \(\|u_3(0)\|_{H^s} > 0\), we define
\[
(\tilde{\varrho}, \tilde{\varphi}, \tilde{\rho}) = \frac{\delta(\varrho, u, q)}{\|\varrho(0)\|_{H^s}}, \tag{5.1}
\]
and find that \((\tilde{\varrho}, \tilde{\varphi}, \tilde{\rho})\) is still a classical solution to (1.6)–(1.8) satisfying all the properties of \((\varrho, u, q)\). Moreover,
\[
\|\tilde{(\varrho, \varphi)}(0)\|_{H^s} = \delta.
\]

Now, we denote
\[
i_0 := i_0(s) := \frac{\|u_3(0)\|_{L^2}}{\|\varrho(0)\|_{H^s}} = \frac{\|\tilde{v}_3(0)\|_{L^2}}{\delta} \leq 1,
\]
then \(i_0 > 0\). Consequently, defining \(t_K = \frac{2}{\lambda} \ln \frac{2K}{i_0}\) and recalling that \(\tilde{v}_3\) satisfies (2.51), we obtain
\[
\|\tilde{v}_3(t_K)\|_{L^2} \geq e^{\frac{\lambda}{\Lambda} K^2/2i_0} \delta \geq 2K\delta. \tag{5.2}
\]
5.2. Construction of a solution to the corresponding nonlinear problem

Based on the initial data \((\bar{\rho}, \bar{v})(0)\) of the solution \((\tilde{\rho}, \tilde{v}, \tilde{p})\) given in \((\ref{initial})\), we proceed to construct a family of solutions to the perturbed nonlinear problem. Define

\[(\tilde{\rho}_0^\varepsilon, \tilde{u}_0^\varepsilon) := \varepsilon(\tilde{\rho}, \tilde{v})(0) \quad \text{for} \ \varepsilon \in (0, 1).\]

Noticing that

\[(\tilde{\rho}_0^\varepsilon, \tilde{u}_0^\varepsilon) \in (H^\infty)_s, \quad \text{and} \quad \|\tilde{\rho}_0^\varepsilon\|_{H^s} < \delta \varepsilon < \delta,\]  

we see by Remark \(4.1\) and Proposition \(4.1\) that there exists a constant \(\varepsilon_1\), such that for any \(\varepsilon \in (0, \varepsilon_1)\), there exists a classical solution \((\tilde{\rho}^\varepsilon, \tilde{u}^\varepsilon, \tilde{q}^\varepsilon)\) to the nonlinear RT problem \((\ref{RT})\)–\((\ref{RT1})\) on \((0, T_{\varepsilon}^{\max}) \times \mathbb{R}^3\) with \(T_{\varepsilon}^{\max} > t_K\), satisfying

\[
\sup_{0 < t \leq T_K} (\|\tilde{\rho}^\varepsilon(t)\|^2_{H^1} + \|\tilde{u}^\varepsilon(t)\|^2_{H^2} + \|\tilde{u}_t^\varepsilon(t)\|^2_{L^2} + \|\nabla \tilde{q}^\varepsilon(t)\|^2_{L^2}) + \int_0^{T_K} (\|\nabla \tilde{u}^\varepsilon(s)\|^2_{H^1} + \|\nabla \tilde{u}^\varepsilon(s)\|^2_{L^2}) ds \leq C(T_K)\delta^2 \varepsilon^2,
\]

where \(C(T_K)\) is independent of \(\varepsilon\). In addition,

\[
\sup_{0 < t \leq T_K} \|\tilde{\rho}^\varepsilon + \tilde{\rho}(t)\|_{L^\infty} \leq C(\delta) \quad \text{for some constant} \ C(\delta) \ \text{depending} \ \delta,
\]

\[
\sup_{0 < t \leq T_K} \|\tilde{q}^\varepsilon\|_{H^1(\Omega')}^2 \leq C(T_K, \Omega')\delta^2 \varepsilon^2 \quad \text{for any} \ \Omega' \subset \subset \mathbb{R}^3.
\]

Obviously, to complete the proof of Theorem \(1.1\) it suffices to show the following lemma.

**Lemma 5.1.** There exists an \(\varepsilon_0 \in (0, \varepsilon_1)\), such that the classical solution \((\tilde{\rho}^\varepsilon_0, \tilde{u}^\varepsilon_0)\) of \((\ref{RT})\)–\((\ref{RT1})\), emanating from the initial data \((\tilde{\rho}_0^\varepsilon_0, \tilde{u}_0^\varepsilon_0)\), satisfies

\[
\|\tilde{u}^\varepsilon_0(t_K)\|_{L^2} > F(\|\tilde{\rho}_0^\varepsilon_0, \tilde{u}_0^\varepsilon_0\|_{H^s}) \quad \text{for some} \ t_K \in \left(0, \frac{2}{\Lambda} \frac{2K}{\ln 2} \right] \subset (0, T_{\varepsilon_0}^{\max}),
\]

where \(T_{\varepsilon_0}^{\max}\) denotes the maximum time of existence to the solution \((\tilde{\rho}^\varepsilon_0, \tilde{u}^\varepsilon_0)\).

**Proof.** We shall show the lemma by contradiction. Suppose that for any \(\varepsilon \in (0, \varepsilon_1)\), the classical solution \((\tilde{\rho}^\varepsilon, \tilde{u}^\varepsilon, \tilde{q}^\varepsilon)\), emanating from the initial data \((\tilde{\rho}_0^\varepsilon, \tilde{u}_0^\varepsilon)\), satisfies

\[
\|\tilde{u}_t^\varepsilon(t)\|_{L^2} \leq F(\|\tilde{\rho}_0^\varepsilon, \tilde{u}_0^\varepsilon\|_{H^s}) \quad \text{for any} \ t \in (0, T_{\varepsilon}^{\max}),
\]

which, together with \((\ref{est1})\) and \((\ref{est2})\), yields

\[
\|\tilde{u}_t^\varepsilon(t)\|_{L^2} \leq K(\|\tilde{\rho}_0^\varepsilon, \tilde{v}_0^\varepsilon\|_{H^s}) \leq K\delta \varepsilon, \quad \forall \ t \in (0, T_{\varepsilon}^{\max}).
\]

We denote \((\tilde{\rho}^\varepsilon, \tilde{u}^\varepsilon, \tilde{q}^\varepsilon) := (\tilde{\rho}^\varepsilon, \tilde{u}^\varepsilon, \tilde{q}^\varepsilon)/\varepsilon\), then they satisfy

\[
\begin{aligned}
\tilde{\rho}_t^\varepsilon + \tilde{u}^\varepsilon \cdot \nabla (\varepsilon \tilde{\rho}^\varepsilon + \tilde{\rho}) &= 0, \\
(\varepsilon \tilde{\rho}^\varepsilon + \tilde{\rho}) \tilde{u}_t^\varepsilon + \varepsilon(\varepsilon \tilde{\rho}^\varepsilon + \tilde{\rho}) \tilde{u}^\varepsilon \cdot \nabla \tilde{u}^\varepsilon + \nabla \tilde{q}^\varepsilon + g \tilde{q}^\varepsilon e_3 &= \mu \Delta \tilde{u}^\varepsilon, \\
\text{div} \tilde{u}^\varepsilon &= 0.
\end{aligned}
\]

with initial data

\[(\tilde{\rho}^\varepsilon, \tilde{u}^\varepsilon)(0) = (\tilde{\rho}, \tilde{v})(0).
\]
Thus, we may chain together (5.2) and the inequality (5.13) to get, according to Theorem 3.1, 
\[ \sup_{0 < t \leq T_K} \| (\varv^e + \varrho^*) \|_{L^\infty} \leq C(\delta), \quad \sup_{0 < t \leq T_K} \| \varrho_0^e(t) \|_{L^2} \leq K\delta, \]  
(5.9)
\[ \sup_{0 < t \leq T_K} \| \varrho^* \|_{H^1(\Omega')}^2 \leq C(T_K, \Omega') \delta^2 \]  
for any \( \Omega' \subset \subset \mathbb{R}^3 \),  
(5.10)
\[ \sup_{0 < t \leq T_K} (\| \varrho^e \|_{H^1}^2 + \| \varv^e \|_{H^2}^2 + \| \varrho_0^e \|_{L^2}^2 + \| \nabla \varrho^e \|_{L^2}^2)(t) \]  
\[ + \int_0^{T_K} (\| \nabla \varv^e \|_{H^1}^2 + \| \nabla \varrho_0^e \|_{L^2}^2)(s) ds \leq C(T_K) \delta^2. \]  
(5.11)
The continuity equation (1.8) combined with (5.9)–(5.11) immediately implies  
\[ \sup_{0 < t \leq T_K} \| \varrho^e \|_{L^2}^2 \leq C(T_K) \delta^2. \]  
(5.12)
Thus, from (5.9)–(5.12) we immediately infer that there exists a subsequence (not relabeled) of \( \{(\varrho^e, \varv^e, \varrho^*)\} \), such that  
\[ (\varrho^e_t, \varv^e, \varrho^*) \rightarrow (\varrho_t, \varv_t, \varrho) \text{ weakly-star in } L^\infty(0, T_K; (L^2)^4 \times H^1_{loc}), \] 
(5.9)
\[ (\varrho^e, \varv^e) \rightarrow (\varrho, \varv) \text{ weakly-star in } L^\infty(0, T_K; H^1) \times (H^2)^3, \] 
(5.10)
\[ (\varrho^e, \varv^e) \rightarrow (\varrho, \varv) \text{ strongly in } C^0([0, T_K], (L^2)^4), \] 
and  
\[ \sup_{0 < t \leq T_K} \| \varrho_3^e(t) \|_{L^2} \leq K\delta, \quad (\varrho, \varv) \in C^0([0, T_K], (L^2)^4). \]  
(5.13)
If one takes to the limit as \( \varepsilon \to 0 \) in the equations (5.7), one gets  
\[ \begin{cases} 
\partial_t \varrho + \varv \cdot \nabla \varrho = 0, \\
\varrho \partial_t \varv + \nabla \varrho + g \varrho \varv_{c_3} = \mu \Delta \varv, \\
\text{div} \varv = 0.
\end{cases} \]
Therefore, we see that \( (\varrho, \varv) \) is just a strong solution of the linearized problem (1.6)–(1.8). Of course, \( (\varrho, \varv) \) is also a strong solution of (1.6)–(1.8). Moreover, \( (\varrho, \varv)(0) = (\varrho, \varv)(0) \). Hence, according to Theorem 5.1 
\[ \varv = \varv \text{ on } [0, T_K] \times \mathbb{R}^3. \]
Thus, we may chain together (5.2) and the inequality (5.13) to get  
\[ 2K\delta \leq \| \varv_3(t_K) \|_{L^2(\mathbb{R}^3)} \leq \| \varrho_3(t_K) \|_{L^2(\mathbb{R}^3)} \leq K\delta, \]  
which is a contraction. This completes the proof of Lemma 5.1 and hence the proof of Theorem 1.1. \( \square \)

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