HIGHER NEWTON POLYGONS AND INTEGRAL BASES
JORDI GUÀRDIA, JESÚS MONTES, AND ENRIC NART

Abstract. Let \( p \) be a prime number. We present an algorithm that computes \( p \)-integral bases in number fields. The algorithm is obtained as a by-product of a \( p \)-adic factorization method based on Newton polygons of higher order. The running-time of the algorithm appears to be very good: it computes the 2-integral basis of a number field of degree 1152 in a few seconds.

1. Introduction

The theory of higher order Newton polygons developed in [Mon99] and revised in [HN] (HN standing for “higher Newton”) has revealed itself as a powerful tool for the analysis of the decomposition of a prime \( p \) in a number field. Higher order Newton polygons are \( p \)-adic objects, and their computation involves no extension of the ground field, but only extensions of the residue field; thus, they constitute an excellent computational tool. Based on this technique, in [GMN08] we presented an algorithm, due to the second author, that computes the prime ideal decomposition of \( p \) in a number field, and the \( p \)-valuation of the discriminant. In this paper we show that a slight modification of this algorithm provides a \( p \)-integral basis too, as a by-product.

Let \( f(x) \in \mathbb{Z}[x] \) be a monic irreducible polynomial of degree \( n \). Let \( \theta \in \overline{\mathbb{Q}} \) be a root of \( f(x) \), \( K = \mathbb{Q}(\theta) \) the number field determined by \( f(x) \), and \( \mathbb{Z}_K \) the ring of integers of \( K \). Along the flow of Montes’ algorithm [GMN08 Sects.2,3], some special polynomials \( \phi(x) \) are constructed such that the \( \phi \)-adic developments of \( f(x) \) provide the combinatorial data that are necessary to build the higher order Newton polygons of \( f(x) \). Just by keeping trace of the quotients that are obtained along the computation of these \( \phi \)-adic developments, we can construct \( n \) integral elements of the form \( q(\theta)/p^{\nu} \), where \( q(x) \) is a polynomial that is a product of some of these quotients, and the exponent \( \nu \) of the denominator is expressed in terms of combinatorial data attached to the polygons (Proposition 3.6). We conjecture that these elements are always a \( p \)-integral basis. The conjecture is proven in two extreme (and opposite) cases: when the algorithm uses only polygons of order one, regardless of the number of \( p \)-adic irreducible factors of \( f(x) \) [EN09], and when \( f(x) \) is irreducible over \( \mathbb{Z}_p \), regardless of the orders of the Newton polygons that are considered by the algorithm (Theorem 4.4). Thus, the remaining task is to prove the conjecture in all in-the-middle (or mixed) cases.

In spite of the apparent uncompleteness of the paper, the algorithm is quite useful in practice because it does compute a \( p \)-integral basis. In fact, the \( p \)-valuation \( \text{ind}(f) := v(\mathbb{Z}_K: \mathbb{Z}[\theta]) \) of the index of \( f(x) \) is one of the outputs of the algorithm;

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therefore, it is trivial to check numerically that the output family of integral elements is a $p$-integral basis indeed. More specifically, if we denote by $\mathcal{O}$ the $\mathbb{Z}$-module generated by this output family, the algorithm computes the $p$-valuation of $(\mathbb{Z}_K : \mathcal{O})$ and prints this value together with a message of “maximal order” or “non-maximal order”. Thus, the algorithm either provides a $p$-integral basis or it warns the user that this were not the case. On the other hand, in an exhaustive numerical exploration that we carried out, we always obtained the output message of “maximal order”.

This method is very efficient and it makes possible to compute $p$-integral bases in number fields of high degree. In fact, the running-time of the algorithm appears to be very good. Even in some bad case, chosen to test the limit of its capabilities, it computes the $p$-integral basis of a number field of degree 1920 and $p$-index 958560 in a few seconds, in a personal computer. However, the $p$-integral bases produced by the algorithm are not triangular; if we add at the end of the algorithm a triangulation process, the running-time may increase in a significant way, because the matrix of the coefficients of the polynomials $q(x)$ tends to be far from sparse. One can try to avoid this by considering only generators of $\mathbb{Z}_{K,p} := \mathbb{Z}_K \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ as a $\mathbb{Z}_{(p)}[\theta]$-module and by using a Groebner Basis algorithm over $\mathbb{Z}[x]$ to find a $p$-stem of the integral basis (a set of generators of $\mathbb{Z}_{K,p}$ as a $\mathbb{Z}_{(p)}[\theta]$-module, with minimal degrees). This procedure leads to a much faster triangulation process, but at the cost of an increase of the memory requirements.

The outline of the paper is as follows. In section 2 we review Montes’ algorithm. In section 3 we study the partial quotients $q(x)$ obtained along the computation of $\phi$-adic developments of $f(x)$, and we determine the highest exponent $\nu$ such that $q(\theta)/p^\nu$ are integral (Proposition 3.6). In section 4 we construct our candidates to be a $p$-integral basis and we describe how to adapt Montes’ algorithm to compute them as a by-product. In Theorem 4.4 we prove that these integral elements are a $p$-integral basis when $f(x)$ is irreducible over $\mathbb{Z}_p[x]$. In section 5 we present some numerical explorations with times of execution of the algorithm in comparison with the standard procedures to get a $p$-integral basis of PARI, MAGMA and SAGE.

Conventions and general notations. We fix a number field $K = \mathbb{Q}(\theta)$, generated by a monic irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of degree $n$, such that $f(\theta) = 0$. We denote by $\mathbb{Z}_K$ the ring of integers of $K$.

We fix also a prime number $p \in \mathbb{Z}$. We denote by $v: \overline{\mathbb{Q}}_p^* \longrightarrow \mathbb{Q}$, the $p$-adic valuation normalized by $v(p) = 1$; we extend $v$ to a discrete valuation of $\mathbb{Q}_p(x)$ by letting it act as

$$v \left( \sum_{0 \leq i} a_i x^i \right) := \min_{0 \leq i} \{ v(a_i) \},$$

on polynomials. We shall introduce $p$-adic valuations $v_r$ of higher order of the same field $\mathbb{Q}_p(x)$, and we convene that $v_1 = v$.

We denote by

$$\text{ind}(f) := v \left( (\mathbb{Z}_K : \mathbb{Z}[\theta]) \right),$$

the $p$-adic value of the index of the polynomial $f(x)$. Recall the well-known relationship, $v(\text{disc}(f)) = 2 \text{ind}(f) + v(\text{disc}(K))$, between $\text{ind}(f)$, the discriminant of $f(x)$ and the discriminant of $K$. 


If $\mathbb{F}$ is a finite field and $\varphi(y), \psi(y) \in \mathbb{F}[y]$, we write $\varphi \sim \psi$ to indicate that the two polynomials coincide up to multiplication by a nonzero constant in $\mathbb{F}$.

We shall freely use the notations and results of [HN]. A short review can be found in [GMN08 Sect.2].

2. Review of Montes’ algorithm

The flow of Montes’ algorithm can be represented by a tree. The starting node is labelled by the input polynomial $f(x)$, and the first branches that sprout from this node are labelled by the different irreducible factors of $f(x)$ modulo the input prime $p$. For each one of these branches, we lift the corresponding irreducible factor modulo $p$ to some monic polynomial $\phi_1(x) \in \mathbb{Z}[x]$; then, we compute the $\phi_1$-adic development of $f(x)$:

$$f(x) = a_0(x) + a_1(x)\phi_1(x) + \cdots + a_s(x)\phi_1(x)^s,$$

with $a_i(x) \in \mathbb{Z}[x]$, deg $a_i(x) < \deg \phi_1(x)$. This development leads to a Newton polygon of the first order, $N_1(f)$, and we compute its principal part, $N_1^-(f)$, that gathers only the sides of negative slope. The length of this polygon $N_1^-(f)$ (that is, the length of its projection to the horizontal axis) is the highest exponent $n_1$ such that $\phi_1(x)^{n_1}$ divides $f(x)$ modulo $p$.

Caution: there is a different Newton polygon of the first order for each branch, but we use the notation $N_1^-(f)$ whenever it is implicitly understood that we are working on a concrete branch.

To each side of $N_1^-(f)$, with slope $\lambda$, we attach a residual polynomial, $R_\lambda(f)(y) \in \mathbb{F}_1[y]$, with coefficients in the finite field $\mathbb{F}_1 := \mathbb{Z}[x]/(p, \phi_1(x))$. The branch we are analyzing ramifies into as many branches (of first order) as pairs $(\lambda, \psi(y))$, where $\lambda$ is one of the slopes of $N_1^-(f)$ and $\psi(y)$ is a monic irreducible factor of $R_\lambda(f)(y)$ in $\mathbb{F}_1[y]$.

Let us fix now one of these first order branches, and let $(\phi_2(x); \lambda_1, \psi_1(y))$ be the data that characterize the branch. Once these data are fixed, for any polynomial $P(x) \in \mathbb{Z}[x]$ we shall denote simply by $R_1(P)(y)$ the residual polynomial attached to the side of slope $\lambda_1$ of the Newton polygon $N_1(P)$ constructed from the $\phi_1$-development of $P(x)$. We construct now a polynomial $\phi_2(x)$ such that:

1. $\phi_2(x)$ is congruent to a power of $\phi_1(x)$ modulo $p$,
2. $N_1(\phi_2)$ is one-sided, with slope $\lambda_1$,
3. $R_1(\phi_2) \sim \psi_1(y)$ in $\mathbb{F}_1[y]$.

The data $(\phi_1(x); \lambda_1, \psi_1(y))$ determine also a $p$-adic valuation $v_2$ of the field $\mathbb{Q}_p(x)$. We can use this valuation of second order to construct Newton polygons of second order of any polynomial $P(x) \in \mathbb{Z}_p[x]$, with respect to its $\phi_2$-development; if $P(x) = \sum a_i(x)\phi_2(x)^i$, then $N_2(P)$ is the lower convex envelope of the set of points $(i, v_2(a_i(x)\phi_2(x)^i))$ of the Euclidean plane. The algorithm computes the principal part $N_2^-(f)$; to each side of this polygon, with slope (say) $\lambda$, we attach a residual polynomial of second order, $R_\lambda(f)(y) \in \mathbb{F}_2[y]$, with coefficients in the finite field $\mathbb{F}_2 := \mathbb{F}_1[y]/(\psi_1(y))$. The length of $N_2^-(f)$ is equal to the highest exponent $n_2$ such that $\psi_1(y)^{n_2}$ divides $R_1(f)(y)$. The branch we are analyzing ramifies into as many branches of second order as pairs $(\lambda, \psi(y))$, where $\lambda$ is one of the slopes of $N_2^-(f)$ and $\psi(y)$ is a monic irreducible factor of $R_\lambda(f)(y)$ in $\mathbb{F}_2[y]$.
Going on with this procedure, a branch of order \( r \) of the algorithm is labelled by a type of order \( r \):
\[
\mathbf{t} = (\phi_1(x) ; \lambda_1 , \phi_2(x) ; \ldots ; \lambda_{r-1} , \phi_r(x) ; \lambda_r , \psi_r(y)),
\]
where \( \phi_1(x), \ldots, \phi_r(y) \in \mathbb{Z}[x] \) are monic polynomials, \( \lambda_1, \ldots, \lambda_r \in \mathbb{Q}^- \) are negative rational numbers, and \( \psi_r(y) \in \mathbb{F}_r[y] \) is a monic irreducible polynomial. With these data one is able to construct a \( p \)-adic valuation \( v_{r+1} \) and a special monic polynomial \( \phi_{r+1}(x) \), called the representative of the type, and the pair \( v_{r+1}, \phi_{r+1}(x) \) allows us to build \((r+1)\)-th order Newton polygons.

For the precise definition of a type see [HN, Sect.2.1]. For our purposes, it is sufficient to recall that for all \( 1 \leq i \leq r \), these objects satisfy the following properties:

1. The Newton polygon of \( i \)-th order of \( \phi_{i+1}(x) \) with respect to \( \mathbf{t} \), \( N_i(\phi_{i+1}) \), is one-sided, with slope \( \lambda_i \).
2. The residual polynomial of \( i \)-th order of \( \phi_{i+1}(x) \) with respect to \( \mathbf{t} \) satisfies:
   \[
   R_i(\phi_{i+1})(y) \sim \psi_i(y) \text{ in } \mathbb{F}_i[y],
   \]
   for certain monic irreducible factor \( \psi_i(y) \) of \( R_i(f)(y) \) in \( \mathbb{F}_i[y] \).
3. The length of \( N_{i+1}(f) \) is the highest exponent \( n_{i+1} \) such that \( \psi_i(y)^{n_{i+1}} \) divides \( R_i(f)(y) \) in \( \mathbb{F}_i[y] \).

Every type carries implicitly a certain amount of extra data, whose notation we fix now. For all \( 1 \leq i \leq r \):

- \( \mathbb{F}_{i+1} := \mathbb{F}_i[y]/(\psi_i(y)) \).
- \( h_i, e_i \) are a pair of positive coprime integers such that \( \lambda_i = -h_i/e_i \).
- \( f_i := \deg \psi_i(y) \), and \( f_0 := m_1 := \deg \phi_1(x) \).
- \( m_{i+1} := \deg \phi_{i+1}(x) \). Note that \( m_{i+1} = e_1 f_1 m_i = f_0 (e_1 f_1) \cdots (e_i f_i) \).

Also, for all \( 1 \leq i \leq r + 1 \), the type carries certain \( p \)-adic discrete valuations \( v_i : \mathbb{Q}_p(x)^* \to \mathbb{Z} \) and semigroup homomorphisms,

\[
\omega_i : \mathbb{Z}_p[x] \setminus \{0\} \to \mathbb{Z}_{\geq 0}, \quad P(x) \mapsto \text{ord}_{\phi_{i-1}} (R_{i-1}(P)),
\]

where \( R_0(P)(y) \in \mathbb{F}_0[y] \) is by definition the reduction modulo \( p \) of \( P(y)/p^{\nu(P)} \). For instance, the integers \( \omega_1(f), \ldots, \omega_{r+1}(f) \) are the above mentioned exponents \( n_1, \ldots, n_{r+1} \).

To avoid confusion, in case of working simultaneously with different types, we add a superscript with the type to every component or datum: \( \phi_1^t(x), \lambda_1^t, e_1^t, \) etc.

**Definition 2.1.** A type \( \mathbf{t} \) of order \( r \) is said to be complete if \( \omega_{r+1}(f) = 1 \).

**Theorem 2.2** ([HN Sect.3]). Let \( \mathbf{t} \) be a type of order \( r \) such that \( n_{r+1} := \omega_{r+1}(f) > 0 \). Then, \( \mathbf{t} \) singles out a monic \( p \)-adic factor \( f_k(x) \in \mathbb{Z}_p[x] \) of \( f(x) \), uniquely determined by

\[
n_k := \deg f_k = m_{r+1} n_{r+1}, \quad \omega_{r+1}(f_k) = n_{r+1}.
\]

If \( \mathbf{t} \) is complete then \( f_k(x) \) is irreducible in \( \mathbb{Z}_p[x] \), so that \( \mathbf{t} \) actually singles out a prime ideal \( \mathfrak{p} \) dividing \( p \mathbb{Z}_K \). The ramification index and residual degree of \( \mathfrak{p} \) are given by:

\[
e(\mathfrak{p}/p) = e_1 \cdots e_r, \quad f(\mathfrak{p}/p) = f_0 f_1 \cdots f_r.
\]

Therefore, the branches whose type \( \mathbf{t} \) is complete are end branches that bear no further ramification. Whenever the algorithm constructs a type, it is sent to a list \texttt{stack}, if it is non-complete, or to a list \texttt{completetypes} if it is complete.
The algorithm runs till the list stack is empty, or equivalently, till all types are complete.
Along the analysis of the branch labelled by \( t \), the algorithm computes the positive integer
\[
\text{ind}_t(f) := f_0 \cdots f_{r-1} \text{ind}(N_{r-1}^-(f)),
\]
where \( \text{ind}(N_{r-1}^-(f)) \) counts the number of points with integral coordinates that lie below or on the polygon, strictly above the horizontal line that passes through the last point of \( N_{r-1}^-(f) \) and strictly beyond the vertical axis.

**Definition 2.3.** For any natural number \( r \geq 1 \), we define
\[
\text{ind}_r(f) := \sum_{t \in t_{r-1}(f)} \text{ind}_t(f),
\]
where \( t_{r-1}(f) \) is the set of all types of order \( r-1 \) that are computed by the algorithm.

**Theorem 2.4** ([HN Sect.4]). There is some \( r \geq 0 \) such that all types of \( t_r(f) \) are complete. In this case,
\[
(2.1) \quad \text{ind}(f) = \text{ind}_1(f) + \cdots + \text{ind}_r(f).
\]

In particular, by a simple accumulation of all \( \text{ind}_t(f) \) to a variable totalindex the algorithm computes \( \text{ind}(f) \) as a by-product.

Finally, the algorithm incorporates a crucial optimization. A \textit{refinement process} looks for an optimal choice of the representative \( \phi_{r+1}(x) \) of the type \( t \), for each of the ramified branches of order \( r+1 \) that sprout from the branch represented by \( t \).
This process requires the introduction of a variable \( H = H^t \) called the \textit{cutting slope}, attached to each type. Initially one takes \( H = 0 \), and, if \( \phi_{r+1}(x) \) is not optimal for certain subbranch, then for the type \( t' \) that will be responsible for the analysis of the subbranch, \( H^{t'} \) is given certain positive value. This refinement causes a slight modification in the way the higher order indices \( \text{ind}_t(f) \) are accumulated to totalindex.

The main loop analyzes a type from stack and determines its ramification.

**Main loop of Montes’ algorithm.** At the input of a type \( t \) of order \( r-1 \), for which \( \omega_r(f) > 0 \), with representative \( \phi_r(x) \) and cutting slope \( H \):

1. Compute the partial polygon, \( N_r^H(f) \), that gathers all sides of slope less than \( -H \) of the Newton polygon of \( r \)-th order \( N_r(f) \), and accumulate to totalindex the value
   \[
   \text{ind}_t^H(f) := f_0 \cdots f_{r-1} \left( \text{ind}(N_r^H(f)) - \frac{1}{2}H(\ell - 1) \right),
   \]
   where \( \ell \) is the length of \( N_r^H(f) \).
2. FOR every side of \( N_r^H(f) \), let \( \lambda < -H \) be its slope, and do
   \[
   \begin{align*}
   \text{3. } & \quad \text{Compute the residual polynomial of } r \text{-th order, } R_\lambda(f)(y) \in \mathbb{F}_r[y], \text{ and factorize this polynomial in } \mathbb{F}_r[y]. \\
   \text{4. } & \quad \text{FOR every irreducible factor } \psi(y) \text{ do} \\
   \text{5. } & \quad \text{Compute a representative } \phi_{r+1}(x) \text{ of the type } t' := (\ldots, \phi_r(x); \lambda, \psi(y)). \\
   \text{6. } & \quad \text{If } t' \text{ is complete, then add } t' \text{ to completetypes, and continue to the next factor of } R_\lambda(f)(y).
   \end{align*}
   \]
7. If $\deg \psi = 1$ and $\lambda \in \mathbb{Z}$ (the type must be refined), set $\phi^t(x) \leftarrow \phi_{r+1}(x)$, $H^t \leftarrow \lambda$, add $t$ to stack and continue to the next factor of $R_\lambda(f)(y)$.

8. (Build a higher order type) Set $H^t \leftarrow 0$, add $t'$ to stack and continue to the next factor of $R_\lambda(f)(y)$.

After each iteration of the main loop, the set $T = \text{stack} \cup \text{completetypes}$, of all types produced by the algorithm determines a factorization of $f(x)$ in $\mathbb{Z}_p[x]$, and the corresponding partition of the set $P$ of prime ideals of $K$ lying above $p$:

$$f(x) = \prod_{t \in T} f_t(x), \quad P = \prod_{t \in T} P_t,$$

where $P_t$ denotes the set of prime ideals of $P$ such that the corresponding $p$-adic irreducible factor of $f(x)$ divides $f_t(x)$ in $\mathbb{Z}_p[x]$. Clearly,

$$n_t := \deg f_t = \sum_{p \in P_t} e(p/p)f(p/p), \quad n = \sum_{t \in T} n_t.$$

3. Quotients of $\phi$-adic developments

**Definition 3.1.** Let $\phi(x) \in \mathbb{Z}[x]$ be a monic polynomial, and let

$$f(x) = a_0(x) + a_1(x)\phi(x) + \cdots + a_s(x)\phi(x)^s,$$

with $a_i(x) \in \mathbb{Z}[x]$, $\deg a_i(x) < \deg \phi(x)$, be the $\phi$-adic development of $f(x)$.

The quotients attached to this $\phi$-development are, by definition, the different quotients $q_1(x), \ldots, q_s(x)$ that are obtained along the computation of the coefficients of the development:

$$
\begin{align*}
    f(x) &= \phi(x)q_1(x) + a_0(x), \\
    q_1(x) &= \phi(x)q_2(x) + a_1(x), \\
    \cdots &= \cdots \\
    q_{s-1}(x) &= \phi(x)q_s(x) + a_{s-1}(x), \\
    q_s(x) &= \phi(x) \cdot 0 + a_s(x) = a_s(x).
\end{align*}
$$

Equivalently, $q_i(x)$ is the quotient of the division of $f(x)$ by $\phi(x)^i$; we denote by $r_i(x)$ the residue of this division. Thus, for all $1 \leq i \leq s$ we have,

$$f(x) = r_1(x) + q_1(x)\phi(x)^i, \quad r_i(x) = a_0(x) + a_1(x)\phi(x) + \cdots + a_{i-1}(x)\phi(x)^{i-1}.$$

The aim of this section is to compute, for certain quotients $q(x)$ attached to certain $\phi$-adic developments of $f(x)$, the highest power $p^m$ of $p$ such that $q(\theta)/p^m$ is integral. To this end we shall need three lemmas. The first one is an easy generalization of [HN Prop.2.7], in a spirit similar to that of [HN Lem.4.21].

**Lemma 3.2.** Let $t = (t_1; \cdots, t_r; \lambda, \psi_r(y))$ be a type of order $r$. Let

$$J = \{(j_1, \ldots, j_r) \in \mathbb{N}^r \mid 0 \leq j_i < e_i f_i, \ 1 \leq i \leq r\},$$

and for any $j \in J$ denote $\Phi(x)^j = \phi_1(x)^j_1 \cdots \phi_r(x)^j_r$. Let $P(x) \in \mathbb{Z}[x]$ be a polynomial of degree less than $m_{r+1}$, and consider its $\Phi$-multiadic development:

$$P(x) = \sum_{j \in J} a_j(x)\Phi(x)^j, \quad \deg a_j < m_1, \ \forall j \in J.$$

Then, $v_{r+1}(P) = \min_{j \in J} \{v_{r+1}(a_j(x)\Phi(x)^j)\}$.
Proof. Since $v_{r+1}$ is a valuation, it is sufficient to check that $v_{r+1}(a_j(x)\Phi(x)^j) \geq v_{r+1}(P)$, for all $j \in J$. Let us prove this inequality by induction on $r \geq 1$. For $r = 1$ this is proven in [HN] Prop. 2.7. Suppose that $r \geq 2$ and the lemma is true in order $1, \ldots, r - 1$. Consider the $\phi_r$-adic development of $P(x)$:

$$P(x) = \sum_{0 \leq j < e_r f_r} P_j(x)\phi_r(x)^j.$$ 

By [HN] Prop. 2.7,

$$v_{r+1}(P) \leq v_{r+1}(P_j(x)\phi_r(x)^j) = e_r v_r(P_j) + j v_{r+1}(\phi_r),$$

for all $0 \leq j < e_r f_r$. Consider now the set

$$J^0 = \{(j_1, \ldots, j_{r-1}) \in \mathbb{N}^{r-1} \mid 0 \leq j_i < e_i f_i, 1 \leq i < r\},$$

and for any $0 \leq j < e_r f_r$ consider the embedding

$$J^0 \hookrightarrow J, \quad j^0 = (j_1, \ldots, j_{r-1}) \mapsto (j^0, j) := (j_1, \ldots, j_{r-1}, j).$$

The $(\phi_1, \ldots, \phi_{r-1})$-multiadic development of $P_j(x)$ is:

$$P_j(x) = \sum_{j^0 \in J^0} a_{(j^0, j)}(x)\phi_1(x)^{j_1} \cdots \phi_{r-1}(x)^{j_{r-1}},$$

and by induction hypothesis:

$$v_r(P_j) \leq v_r(a_{(j^0, j)}(x)\phi_1(x)^{j_1} \cdots \phi_{r-1}(x)^{j_{r-1}})$$

$$= v_r(a_{(j^0, j)}) + j_1 v_r(\phi_1) + \cdots + j_{r-1} v_r(\phi_{r-1}),$$

for all $j^0 \in J^0$. By [HN] Prop. 2.7,(1) and [HN] Lem. 2.2,(2) we have

$$e_r v_r(a_{(j^0, j)}) = v_r(a_{(j^0, j)}); \quad e_r v_r(\phi_i) = v_{r+1}(\phi_i), \forall 1 \leq i < r,$$

so that the last inequality, together with (3.2) gives the desired inequality:

$$v_{r+1}(P) \leq v_{r+1}(a_{(j^0, j)}) + j_1 v_r(\phi_1) + \cdots + j_{r-1} v_{r+1}(\phi_{r-1}) + j v_{r+1}(\phi_r),$$

for all $j^0 \in J^0$ and all $0 \leq j < e_r f_r$. \hfill \Box

We shall also use the following elementary combinatorial result.

Lemma 3.3. Let $M_1, \ldots, M_r$ be positive integers. Then, the set

$$\{j_1 + j_2 M_1 + \cdots + j_r M_1 M_2 \cdots M_{r-1} \mid j_i \in \mathbb{Z} \cap [0, M_i), 1 \leq i \leq r\},$$

coincides with the set $\mathbb{Z} \cap [0, M_1 M_2 \cdots M_r).$ \hfill \Box

For any pair of indices, $s, t$, such that $s \leq t \leq r$, let us denote

$$A_{s, t} := e_{s+1} \cdots e_{r-1}(e_s f_s) \cdots (e_{t-1} f_{t-1}).$$

With this notation, the closed formulas for $v_r(\phi_t)$ and $v_s(\phi_s)$ from [HN] Prop. 2.15 can be written as:

$$v_r(\phi_t) = \sum_{k=1}^t A_{k, t} h_k, \quad t < r,$$

$$v_r(\phi_r) = \sum_{k=1}^{r-1} A_{k, r} h_k,$$

$$e_s A_{s, t} v_s(\phi_s) = \sum_{k=1}^{t-1} A_{k, t} h_k, \quad s \leq t \leq r.$$
Lemma 3.4. Suppose that \( s < r \), and let \((j_1, \ldots, j_{r-1}) \in \mathbb{N}^{r-s} \) be such that \( 0 \leq j_t < e_t f_t \), for all \( s \leq t < r \). Then,

\[
\sum_{s \leq t < r} j_t A_{s,t} < A_{s,r}.
\]

Proof. The inequality is equivalent to

\[
\sum_{s \leq t < r} j_t (e_s f_s) \cdots (e_{t-1} f_{t-1}) < (e_s f_s) \cdots (e_{r-1} f_{r-1}),
\]

and this is an immediate consequence of the previous lemma.

\[ \square \]

Corollary 3.5. Let \( r \geq 2 \), and let \( t = (\phi_1(x); \cdots, \phi_{r-1}(x); \lambda_{r-1}, \varphi_{r-1}(y)) \), be a type of order \( r - 1 \), with representative \( \phi_r(x) \). Let \((j_1, \ldots, j_{r-1}) \in \mathbb{N}^{r-1} \) be such that \( 0 \leq j_t < e_t f_t \), for all \( 1 \leq t < r \). Then,

\[
j_1 v_r(\phi_1) + \cdots + j_{r-1} v_r(\phi_{r-1}) < v_r(\phi_r).
\]

Proof. By (3.3), we want to show that

\[
\sum_{1 \leq t < r} j_t \sum_{k=1}^t A_{k,t} h_k < \sum_{k=1}^{r-1} A_{k,r} h_k.
\]

For each value of \( 1 \leq k < r \), the terms involving \( h_k \) in both sides satisfy this inequality by the previous lemma:

\[
\sum_{k \leq t < r} j_t A_{k,t} h_k < A_{k,r} h_k.
\]

\[ \square \]

Proposition 3.6. Let \( r \geq 1 \) and let \( t = (\phi_1(x); \cdots, \phi_{r-1}(x); \lambda_{r-1}, \varphi_{r-1}(y)) \), be a type of order \( r - 1 \), with representative \( \phi_r(x) \), such that \( n_r := \omega_r(f) > 0 \). Let \( 1 \leq i \leq n_r \) be an integral abscissa, and let \( Y_i \in \mathbb{Q} \) be the ordinate of the point of \( N_r^{-}(f) \) with abscissa \( i \). Let \( q_i(x) \) be the \( i \)-th quotient attached to the \( \phi_r \)-development of \( f(x) \). Then, for every prime ideal \( \mathfrak{p} \) of \( K \) lying above \( p \) we have:

\[
\frac{v_{\mathfrak{p}}(q_i(\theta))}{e(\mathfrak{p}/p)} \geq \frac{Y_i - iv_r(\phi_r)}{e_0 e_1 \cdots e_{r-1}}.
\]

where \( e_0 = 1 \) by convention, and \( v_{\mathfrak{p}} \) is the discrete valuation of \( K \) determined by \( \mathfrak{p} \).

In particular, if \( H_1 := [(Y_i - iv_r(\phi_r))/(e_0 e_1 \cdots e_{r-1})] \), then \( q_i(\theta)/p^{H_1} \) belongs to \( \mathbb{Z}_K \).

Proof. Let \( \mathfrak{p} \in \mathcal{P} \) be a prime ideal of \( K \) lying above \( p \), and let \( F_\mathfrak{p}(x) \in \mathbb{Z}_p[x] \) be the monic irreducible factor of \( f(x) \) corresponding to \( \mathfrak{p} \). Let us choose a root \( \theta^\mathfrak{p} \) of \( F_\mathfrak{p}(x) \); consider the embedding \( \iota_\mathfrak{p} : K \hookrightarrow \overline{\mathbb{Q}}_p \) determined by sending \( \theta \) to \( \theta^\mathfrak{p} \). If we denote, \( w_\mathfrak{p} := e(\mathfrak{p}/p)^{-1}v_\mathfrak{p} \), then we have: \( w_\mathfrak{p}(\alpha) = v(\iota_\mathfrak{p}(\alpha)) \), for all \( \alpha \in K \). In particular, for any polynomial \( P(x) \in \mathbb{Z}[x] \),

\[
w_\mathfrak{p}(P(\theta)) = v(P(\theta^\mathfrak{p})).
\]

Let \( \lambda_0 \) be the slope of the side \( S \) of \( N_r^{-}(f) \), whose projection to the horizontal axis contains the abscissa \( i \). If \( i \) is the abscissa of a vertex of \( N_r^{-}(f) \), then we let \( S \) be one of the two adjacent sides. The prime ideal \( \mathfrak{p} \) satisfies one and only one of the following conditions:

(1) \( \mathfrak{p} \in \mathcal{P}_\mathfrak{r} \), for \( \mathfrak{r} = (\phi_1(x); \cdots, \phi_r(x); \lambda, \psi(y)) \).
(2) \( p \in \mathcal{P}_t, \) for \( t' = (\phi_1(x); \cdots, \phi_s(x); \lambda, \psi(y)) \), for some \( 1 \leq s < r \), with \( \psi(y) \neq \psi_s(y) \) if \( \lambda = \lambda_s \).

(3) \( p \nmid \phi_1(\theta) \) in \( \mathbb{Z}_K \).

We shall prove the inequality (3.4) by an independent argument in each case. In the two first cases, since \( F_p(x) \) is a factor of \( f_{t'}(x) \), the element \( \theta_p \) will be a root of \( f_{t'}(x) \) too. This allows us to apply the results of [HN, Sect.3] to the pair \( f_{t'}(x), \theta_p \). We shall denote throughout the proof: \( e = e_0 e_1 \cdots e_{r-1} \) and \( v_t = v_t^t \), for all \( 1 \leq t \leq r \).

Case 1: \( p \in \mathcal{P}_t', \) for \( t' = (\phi_1(x); \cdots, \phi_r(x); \lambda, \psi(y)) \)

By (3.1), the Newton polygon \( N_{-r}^-(f) \) splits essentially in two parts: \( N_{-r}(r_i) \) and \( N_{-r}(q_i \phi_i^r) \).

This is not always true because, depending on the values of \( v_r(a_{i-1} \phi_i^{i-1}) \) and \( v_r(a_i \phi_i^i) \), the two parts of the side \( S \) in the polygons \( N_{-r}(r_i) \) and \( N_{-r}(q_i \phi_i^r) \) might change. The next figure shows different possibilities for these parts of \( S \). However, the line \( L_{\lambda_0} \), of slope \( \lambda_0 \) that first touches both polygons from below is still the line determined by \( S \).
If $|\lambda| \geq |\lambda_0|$, we apply [HN] Prop.3.5,(3)] directly to the polynomial $q_i(x)\phi_r(x)^i$ to get:

$$w_p(q_i(\theta)\phi_r(\theta)^i) \geq \frac{Y_i + i|\lambda|}{e}.$$  \hfill (3.5)

If $|\lambda| < |\lambda_0|$, we apply [HN] Prop.3.5,(3)] to the polynomial $r_i(x)$ and we get the same result, thanks to (3.4):

$$w_p(q_i(\theta)\phi_r(\theta)^i) \geq \frac{Y_i + i|\lambda|}{e}. $$  \hfill (3.6)

Recall that $r_i(x) = \sum_{0 \leq j < i} a_j(x)\phi_r(x)^j$. Our aim is to show that $w_p(a_j(\theta))$ is sufficiently large, for all $0 \leq j < i$. More precisely, we want to show the following two inequalities:

$$w_p(\phi_r(\theta)) = v(\phi_r(\theta^p)) = (v_r(\phi_r) + |\lambda|)/e,$$

and we get the desired inequality:

$$w_p(q_i(\theta)) \geq \frac{Y_i + i|\lambda|}{e} - i \cdot \frac{v_r(\phi_r) + |\lambda|}{e} = \frac{Y_i - iv_r(\phi_r)}{e}. $$  \hfill (3.7)

**Case 2:** $p \in P_r$, for $\theta' = (\phi_1(x); \cdots, \phi_s(x); \lambda, \psi(y))$, for some $1 \leq s < r$, with $\psi(y) \neq \psi_s(y)$ if $\lambda = \lambda_s$.

Recall that $r_i(x) = \sum_{0 \leq j < i} a_j(x)\phi_r(x)^j$. Our aim is to show that $w_p(a_j(\theta))$ is sufficiently large, for all $0 \leq j < i$. More precisely, we want to show the following two inequalities:

$$w_p(\phi_r(\theta)) \leq \frac{v_r(\phi_r)}{e}, $$  \hfill (3.6)

$$w_p(a_j(\theta)) \geq \frac{v_r(a_j)}{e} - \left(\frac{v_r(\phi_r)}{e} - w_p(\phi_r(\theta))\right), \quad \forall 0 \leq j < i. $$  \hfill (3.7)

Let us show first that these two inequalities imply the desired inequality (3.4). In fact, from (3.6) and (3.7) we get:

$$w_p(a_j(\theta)) \geq \frac{v_r(a_j)}{e} - (i - j) \left(\frac{v_r(\phi_r)}{e} - w_p(\phi_r(\theta))\right), \quad \forall 0 \leq j < i, $$  \hfill (3.8)
and we can argue as follows:

\[ w_p(q_i(\theta)) = w_p(q_i(\theta)\phi_r(\theta)^i) - iw_p(\phi_r(\theta)) \]
\[ \geq \min_{0 \leq j < 1} \{ w_p(a_j(\theta)\phi_r(\theta)^j) \} - iw_p(\phi_r(\theta)) \]
\[ = w_p(a_{j_0}(\theta)\phi_r(\theta)^{j_0}) - iw_p(\phi_r(\theta)) = w_p(a_{j_0}(\theta)) - (i - j_0)w_p(\phi_r(\theta)) \]

Let us compute now the exact value of \( w_p(\phi_t(\theta)) = v(\phi_t(\theta^p)) \), for all \( s \leq t \leq r \).

For \( t = s \), the theorem of the polygon \( \text{[HN Thm.3.1]} \) shows that:

\[ w_p(\phi_s(\theta)) = v(\phi_s(\theta^p)) = \frac{v_s(\phi_s)}{e_0e_1\cdots e_{s-1}}. \]

For \( s < t \leq r \), \( v(\phi_t(\theta^p)) \) can be computed by applying \( \text{[HN Prop.3.5,(3)]} \) to the polynomial \( \phi_t(x) \). We know that \( N_s(\phi_t) \) is one-sided with slope \( \lambda_s \); thus, there are two different situations according to \( |\lambda| \geq |\lambda_s| \), or \( |\lambda| < |\lambda_s| \), reflected in the following figures:

If \( \lambda \neq \lambda_s \), the \( \lambda \)-component of \( N_s(\phi_t) \) reduces to a point, and the residual polynomial of \( s \)-th order \( R_s(\phi_t)(y) \) is a constant \( \text{[HN Defs.1.5,2.21]} \). If \( \lambda = \lambda_s \), then \( R_s(\phi_t)(y) \) is a power of \( v_s(y) \); since \( v(y) \neq v_s(y) \), we see that \( R_s(\phi_t)(y) \) is not divisible by \( \psi(y) \) either. Thus, \( \text{[HN Prop.3.5,(5)]} \) shows that \( v(\phi_t(\theta^p)) \) is always equal to \( H/(e_0e_1\cdots e_{s-1}) \), where \( H \) is the ordinate of the point of intersection of the vertical axis with the line of slope \( \lambda \) that first touches \( N_s(\phi_t) \) from below. By taking a glance at the figures above we see that:

\[ w_p(\phi_t(\theta)) = v(\phi_t(\theta^p)) = \frac{v_s(\phi_t) + (e_s f_s)\cdots (e_{t-1} f_{t-1}) \mu}{e_0 e_1 \cdots e_{s-1}}, \quad s < t \leq r, \]

where \( \mu := \min\{|\lambda|, |\lambda_s|\} \). Since \( N_s(\phi_t) \) is one-sided with last point \( (m_t/m_s, v_s(\phi_t)) \) \( \text{[HN Lem.2.17.(1)]} \), we have

\[ v_s(\phi_t) = v_s(\phi_{t/m_s}) = (m_t/m_s)v_s(\phi_s) = (e_s f_s)\cdots (e_{t-1} f_{t-1})v_s(\phi_s). \]

so that \( \text{(3.10)} \) can be rewritten as

\[ v(\phi_t(\theta^p)) = (e_s f_s)\cdots (e_{t-1} f_{t-1}) \frac{v_s(\phi_s) + \mu}{e_0 e_1 \cdots e_{s-1}}, \quad s < t \leq r. \]
Since $|\lambda| \geq \mu$, we deduce from (3.9), (3.11) and (3.3):

\[ \text{ev}(\phi_r(\theta^p)) \geq e_s A_{s,t}(v_s(\phi_s) + \mu) = v_r(\phi_t) - \sum_{k=s}^{\min(t,r-1)} A_{k,t}h_k + e_s A_{s,t}\mu, \]

for all $s \leq t \leq r$.

Let $J_s := \{(j_1, \ldots, j_{r-1}) \in \mathbb{N}^{r-s} \mid 0 \leq j_t < e_t f_t, s \leq t < r\}$, and consider the following multiadic development of $a_j(x)$:

\[ a_j(x) = \sum_{j \in J_s} b_j(x)\phi_s(x)^{j_s} \cdots \phi_{r-1}(x)^{j_{r-1}}, \quad \deg b_j < m_s. \]

Fix a multiindex $j \in J_s$ such that

\[ v(a_j(\theta^p)) \geq v \left( b_j(\theta^p)\phi_s(\theta^p)^{j_s} \cdots \phi_{r-1}(\theta^p)^{j_{r-1}} \right) \]

= $v(b_j(\theta^p)) + j_s v(\phi_s(\theta^p)) + \cdots + j_{r-1} v(\phi_{r-1}(\theta^p))$.

By [HN, Prop.2.9], $v(b_j(\theta^p)) = v_r(b_j)/e$. Therefore, by (3.12) we get

\[ e_{w_p}(a_j(\theta)) = ev(a_j(\theta^p)) \]

\[ \geq v_r(b_j) + \sum_{s \leq t < r} j_t \left( v_r(\phi_t) + A_{s,t}(e_s\mu - h_s) - \sum_{k=s+1}^{t} A_{k,t}h_k \right) \]

\[ \geq v_r(a_j) + \sum_{s \leq t < r} j_t \left( A_{s,t}(e_s\mu - h_s) - \sum_{k=s+1}^{t} A_{k,t}h_k \right). \]

the last inequality by Lemma 3.2. On the other hand, for $t = r$ we have by (3.12):

\[ v_r(\phi_r) - e_{w_p}(\phi_r(\theta)) = -e_s A_{s,r}\mu + \sum_{k=s}^{r-1} A_{k,r}h_k = A_{s,r}(h_s - e_s\mu) + \sum_{k=s+1}^{r} A_{k,r}h_k \geq 0, \]

because $h_s \geq e_s\mu$. This proves (3.6). Also, (3.7) is reduced now to:

\[ \sum_{s \leq t < r} j_t \left( A_{s,t}(h_s - e_s\mu) + \sum_{k=s+1}^{t} A_{k,t}h_k \right) \leq \sum_{k=s}^{r-1} A_{k,r}h_k - e_s A_{s,r}\mu. \]

If we consider only the summands involving $h_s$ and $\mu$, we get the desired inequality by Lemma 3.4

\[ (h_s - e_s\mu) \sum_{s \leq t < r} j_t A_{s,t} < (h_s - e_s\mu) A_{s,r}. \]

Finally, if we consider the summands involving $h_k$, for any fixed $s < k < r$, we get

\[ h_k \sum_{k \leq t < r} j_t A_{k,t} < h_k A_{k,r}. \]

**Case 3: $p \nmid \phi_1(\theta)$ in $\mathbb{Z}_K$**

We have now $w_p(\phi_1(\theta)) = 0$, for all $1 \leq t \leq r$. Thus, (3.6) is obvious, and arguing as in Case 2, it is sufficient to check the inequality (3.7). Let us consider the multiadic development of $a_j(x)$:

\[ a_j(x) = \sum_{j \in J} b_j(x)\phi_1(x)^{j_1} \cdots \phi_{r-1}(x)^{j_{r-1}}, \quad \deg b_j < m_1. \]
We fix $j \in J$ such that $v(a_j(\theta^p)) \geq v(b_j(\theta^p))$. By [HN] Prop.2.7(1)], $v_r(b_j) = ev_1(b_j)$, so that
\[
e w_r(a_j(\theta)) = ev(a_j(\theta^p)) \geq ev(b_j(\theta^p)) \geq ev_1(b_j) = v_r(b_j) \geq v_r(a_j) - j_1 v_r(\phi_1) - \cdots - j_{r-1} v_r(\phi_{r-1}) > v_r(a_j) - v_r(\phi_r).
\]
the last inequality by Corollary 3.5.

\[\square\]

4. Computation of a $p$-integral basis

4.1. Computation of $n$ integral elements. Let $t = (\phi_1(x); \cdots; \phi_r(x); \lambda_r, \psi_r(y))$ be a type of order $r$. We recall that the reduced type determined by $t$ is by definition
\[t^0 := (\phi_1(x); \cdots; \phi_r(x); \lambda_r).
\]
Note that $t^0$ is not actually a type.

**Definition 4.1.** Let $t^0 = (\phi_1(x); \cdots; \phi_r(x); \lambda_r)$ be a reduced type of order $r$. We say that $t^0$ is regular (or more properly: $f$-regular), if the $r$-th order residual polynomial $R_{\lambda_r}(f)(y) \in \mathbb{F}_r[y]$ is separable.

Imagine that in one iteration of the main loop of Montes’ algorithm, we analyze a type $t$ of order $r - 1$, with representative $\phi_r(x)$. We consider one of the sides $S$ of $N_{\lambda_r}^-(f)$, with slope (say) $\lambda_r$, and we factorize the $r$-th order residual polynomial in $\mathbb{F}_r[y]$:
\[R_{\lambda_r}(f)(y) = \psi_{r,1}(y)^{u_1} \cdots \psi_{r,g}(y)^{u_g}.
\]
Then, we are about to ramify $t$ into $g$ types of order $r$
\[t_i := (\phi_1(x); \cdots; \phi_r(x); \lambda_r, \psi_{r,i}(y)), \quad 1 \leq i \leq g,
\]
extcept for the branches that might be refined (if $\lambda_r \in \mathbb{Z}$, deg $\psi_{r,i} = 1$ and $t_i$ is not complete).

**Remark 4.2.** If the reduced type $(\phi_1(x); \cdots; \phi_r(x); \lambda_r)$ is regular, then $t_1, \ldots, t_g$ are all complete. Moreover, in this case
\[n_{t_1} + \cdots + n_{t_g} = m_1(e_1 f_1) \cdots (e_{r-1} f_{r-1}) \ell(S),
\]
where $\ell(S)$ is the length of the projection of $S$ to the horizontal axis.

Suppose that $CT := \text{completetypes}$ is the output list of complete types provided by Montes’ algorithm, and let $CT^{\text{reg}}$ be the set of all regular reduced types $t^0$, for $t \in CT$. Let us show how these two sets can be used to construct a $p$-integral basis.

We introduce some notation attached to any $t = (\phi_1(x); \cdots; \phi_r(x); \lambda_r, \psi_r(y)) \in CT$ of order (say) $r$. For all $1 \leq t \leq r$, let $[a_t, b_t]$ be the projection to the horizontal axis of the side of slope $\lambda_t$ of $N_{\lambda}^-(f)$. Now, for all integer abscissas, $a_t < j \leq b_t$, we denote by $Y_{t,j}$ the ordinate of the point of $N_{\lambda_t}^-(f)$ of abscissa $j$, and
\[q_{t,j}(x) = \begin{cases} 1, & \text{if } t < r \text{ and } j = b_t, \\ j\text{-th quotient of the } \phi_t\text{-development of } f(x), & \text{otherwise.} \end{cases}
\]
\[H_{t,j}(x) = \begin{cases} 0, & \text{if } t < r \text{ and } j = b_t, \\ \frac{Y_{t,j} - j v_t(\phi_t)}{\prod_{i=1}^{r-1} e_i}, & \text{otherwise.} \end{cases}
\]
By Proposition 3.6, \( q_{i,j}(\theta)/p^{[H_{i,j}]} \) is integral, for all \( t, j \). We shall construct a set \( B \) of \( n \) elements of \( \mathbb{Z}_K \); the basic idea is that each \( t \in \text{CT} \) contributes with \( n_t \) elements to the basis \( B \). However, when for some \( t \in \text{CT} \), the reduced type \( t^0 \) is regular, then we compute the contribution of the complete types \( t_1, \ldots, t_g \) of Remark 4.2 altogether.

**Case \( t^0 \) regular.** We take all complete types \( t_1, \ldots, t_g \in \text{CT} \), that have a common reduced type, \( t^0 = (\phi_1(x); \cdots, \phi_r(x); \lambda_r) \), which is regular. Let \( J \) be the set:

\[
\{(j_0, \ldots, j_r) \in \mathbb{N}^{r+1} \mid 0 \leq j_0 < m_1; \ b_t - e_t f_t < j_t \leq b_t, \ 1 \leq t < r; \ a_t < j_r \leq b_r \}.
\]

In this case, we add to \( B \) the set:

\[
Q_{t^0} := \left\{ \frac{q_{r,j_0}(\theta) \cdots q_{1,j_1}(\theta) \theta^{b_0}}{p^{[H_{r,j_r} + \cdots + H_{1,j_1}]}} \mid (j_0, \ldots, j_r) \in J \right\}.
\]

Note that \(|Q_{t^0}| = m_1(e_1 f_1) \cdots (e_{r-1} f_{r-1})(b_r - a_r) = n_{t_1} + \cdots + n_{t_g} \), the last equality by Remark 4.2.

**Case \( t^0 \) not regular.** We take a complete type \( t \in \text{CT} \), such that \( t^0 \) is not regular, and we compute a representative \( \phi_{r+1}(x) \) of \( t \). The polygon \( N_{r+1}^{-1}(f) \) has length one, with integer slope \( \lambda_{r+1} \), and its last point is \((1, v_{r+1}(f))\). We compute the first quotient of the \( \phi_{r+1} \)-adic development of \( f(x) \):

\[
f(x) = \phi_{r+1}(x)q(x) + a(x), \quad \deg a(x) < m_{r+1}.
\]

By Proposition 3.5, if \( H := (v_{r+1}(f) - v_{r+1}(\phi_{r+1}))/e_0 e_1 \cdots e_r \), then \( q(\theta)/p[H] \) belongs to \( \mathbb{Z}_K \). We add then to \( B \) the set:

\[
Q_t := \left\{ \frac{q(\theta)q_{r,j_0}(\theta) \cdots q_{1,j_1}(\theta) \theta^{b_0}}{p^{[H + H_{r,j_r} + \cdots + H_{1,j_1}]}} \mid j_t \in (b_t - e_t f_t, b_t], \ 1 \leq t \leq r; \ j_0 \in [0, m_1] \right\}.
\]

Caution: \( q_{r,j}(x) \) are now considered as quotients of \( \phi_r \)-developments with respect to a type of order \( r + 1 \); thus, following the criterion of (4.2) and (4.3), we take \( q_{r,b_t}(x) = 1, H_{r,b_t} = 0 \).

Note that \(|Q_t| = m_1(e_1 f_1) \cdots (e_r f_r) = n_t \).

**Conjecture 4.3.** The following family is a \( p \)-integral basis of \( K \):

\[
B = \left( \bigcup_{t^0 \in \text{CTreg}} Q_{t^0} \right) \cup \left( \bigcup_{t \in \text{CT}, v^0 \in \text{CTreg}} Q_t \right).
\]

4.2. Montes’ algorithm suitably modified to compute a \( p \)-integral basis.

Montes’ algorithm computes already the quotients of the different \( \phi_r \)-adic developments of \( f(x) \) along the process of determining the coefficients of the development. Thus, the algorithm can be easily adapted to compute a \( p \)-integral basis, just by computing the exponents \( H_{r,j} \) of the denominators and the adequate products of quotients to build the families \( Q_{t^0} \) and/or \( Q_t \).

A part of these products is shared by the subbranches of a type, when it ramifies; therefore it is more efficient to compute the partial products before a type ramifies, so that all subbranches can share these partial products without repeating their computation. To this end, we attach to each type \( t \) two lists \( \text{ProductOfQuotients} \) and \( \text{ProductOfQuotientsValuations} \), denoted \( \text{PQ} = \text{PQ}^t \), \( \text{PQVals} = \text{PQVals}^t \) in the sequel for simplicity. These lists are initialized by

\[
\text{PQ}^t = \{1, x, \ldots, x^{\deg \phi_{r+1} - 1}\}, \quad \text{PQVals}^t = \{0, \cdots, 0\},
\]
and they are enlarged in the following way. Suppose that we analyze a type of
order \( r - 1 \) from the list stack: \( t = (\phi_1(x); \cdots, \phi_{r-1}(x); \lambda_{r-1}, \psi_{r-1}(y)) \), with
representative \( \phi_r(x) \): we compute the Newton polygon \( N_r^+(f) \), we consider one of
its sides \( S \) with slope (say) \( \lambda_r = -h_r/e_r \), and we factorize the residual polynomial
as in (4.1). If the reduced type \( (\phi_1(x); \cdots, \phi_{r-1}(x); \lambda_{r-1}, \phi_r(x); \lambda_r) \) is not regular,
we compute \( f_{\text{max}} := \max_{1 \leq i \leq 9} \{\deg \psi_{r,i}\} \) and we construct the following two lists:

(4.4) \[ \text{EnlargedPQ } \leftarrow \text{PQ}^t \cup q_{r,b_r-1} \cdot \text{PQ}^t \text{ (mod f(x)) } \cup \cdots \]
\[ \cup q_{r,b_r-e_{r,\text{max}}} \cdot \text{PQ}^t \text{ (mod f(x))}, \]

(4.5) \[ \text{EnlargedPQVals } \leftarrow \text{PQVals}^t \cup H_{r,b_r-1} + \text{PQVals}^t \cup \cdots \]
\[ \cup H_{r,b_r-e_{r,\text{max}}} + \text{PQVals}^t. \]

For each branch \( t' = ((\phi_1(x); \cdots, \phi_{r-1}(x); \lambda_{r-1}, \phi_r(x); \lambda_r, \psi_{r,i}(y)), \) that sprouts
from \( t \), the lists \( \text{PQ}^{t'}, \text{PQVals}^{t'} \) are the sublists of the first \( e_{r,f_r} \) \( \text{PQ}^t \) elements,
respectively of EnlargedPQ, EnlargedPQVals.

Finally, we introduce two global lists numators, denominators. When the
algorithms ends, the \( p \)-integral basis is made up by all elements \( q_i(\theta)/p^\nu, 1 \leq i \leq n, \)
with \( q_i(x) \in \text{numators} \) and \( \nu_i \in \text{denomators} \). The main loop of Montes’
algorithm is then modified in the following way:

**Main loop of Montes’ algorithm, modified to compute a \( p \)-integral basis.**

At the input of a type \( t \) of order \( r - 1 \), for which \( \omega_r(f) > 0 \), with representative
\( \phi_r(x) \) and cutting slope \( H \):

1. Compute the partial polygon, \( N_r^H(f) \), that gathers all sides of slope less than
\( -H \) of the Newton polygon of \( r \)-th order \( N_r(f) \), and accumulate to totalindex
the value
\[
\text{ind}_r^H(f) := f_0 \cdots f_{r-1} \left( \text{ind}(N_r^H(f)) - \frac{1}{2} H \ell (\ell - 1) \right),
\]
where \( \ell \) is the length of \( N_r^H(f) \).

2. FOR every side of \( N_r^H(f) \), let \( \lambda_r < -H \) be its slope, and do

3. Compute the residual polynomial of \( r \)-th order, \( R_{\lambda_r}(f)(y) \in \mathbb{F}_r[y], \) and
factorize this polynomial in \( \mathbb{F}_r[y] \).

3.1 If \( R_{\lambda_r}(f)(y) \) is separable, then

3.1.1 Add \( \bigcup_{\lambda_r < j \leq b_n} q_{r,j} \cdot \text{PQ} \text{ (mod f(x))} \) to numators.
Add \( \bigcup_{\lambda_r < j \leq b_n} |H_{r,j} + \text{PQVals}| \) to denominators.

3.1.2 Add the types \( t' = (\phi_1(x); \cdots, \phi_r(x); \lambda_r, \psi_r(y)) \) to completetypes,
where \( \psi_r(y) \) runs on all irreducible factors of \( R_{\lambda_r}(f)(y) \). Continue to the next side
of \( N_r^H(f) \).

3.2 Build the lists EnlargedPQ, EnlargedPQVals, as indicated in (4.4) and (4.5).

4. FOR every irreducible factor \( \psi_r(y) \) do

5. Compute a representative \( \phi_{r+1}(x) \) of the type \( t' := (\ldots, \phi_r(x); \lambda_r, \psi_r(y)) \).

6. If \( t' \) is complete, then add \( t' \) to completetypes, and

6.1 \( \text{PQ } \leftarrow \{\text{EnlargedPQ}(i) \mid 1 \leq i \leq e_{r,f_r}\}, \)
\( \text{PQVals } \leftarrow \{\text{EnlargedPQVals}(i) \mid 1 \leq i \leq e_{r,f_r}\}. \)
6.2 Compute the first quotient $q(x)$ of the $\phi_{r+1}$-adic development of $f(x)$, and the exponent $H$ given by Proposition 3.6. Add $q(x) \text{PQ} \pmod{f \{x\}^r}$ to numerators and $[H + \text{PQVals}]$ to denominators, and continue to the next factor of $R_{\lambda}(f)(y)$.

7. If $\deg \psi = 1$ and $\lambda \in \mathbb{Z}$ (the type must be refined), set $\phi_+^t(x) \leftarrow \phi_+(x)$, $H^t \leftarrow \lambda$, add $t$ to stack and continue to the next factor of $R_{\lambda}(f)(y)$.

8. (Build a higher order type) Set $H^t \leftarrow 0$.

8.1 $\text{PQ} \leftarrow \{\text{EnlargedPQ}(i) \mid 1 \leq i \leq e_r f_r\}$, $\text{PQVals} \leftarrow \{\text{EnlargedPQVals}(i) \mid 1 \leq i \leq e_r f_r\}$.

8.2 Add $t'$ to stack, and continue to the next factor of $R_{\lambda}(f)(y)$.

When the algorithm ends, the list numerators contains $n$ polynomials $q_i(x)$ with integer coefficients, and the list denominators contains $n$ non-negative integers $\nu_i$. We explain now how can be checked that the elements $q_i(\theta)/p^{\nu_i}$ form a $p$-integral basis indeed.

Consider the $\mathbb{Z}$-modules

$$O^\text{num} := \langle q_i(\theta) \mid q_i(x) \in \text{numerators} \rangle \subseteq \mathbb{Z}[\theta],$$

$$O := \langle \frac{q_i(\theta)}{p^{\nu_i}} \mid q_i(x) \in \text{numerators}, \nu_i(x) \in \text{denominators} \rangle \subseteq \mathbb{Z}_K,$$

We compute $\text{ind}^\text{num} := v(\mathbb{Z}[\theta] : O^\text{num})$ as the $p$-adic value of the determinant of the matrix made up with the coefficients of all $q_i(x)$. Also, we can compute $v(O : O^\text{num}) = \sum_{i=1}^n \nu_i$.

On the other hand, since $\text{ind}(f) = v(\mathbb{Z}_K : \mathbb{Z}[\theta])$ is computed by the algorithm, we know the value of $v(\mathbb{Z}_K : O^\text{num}) = \text{ind}^\text{num} + \text{ind}(f)$. Thus, $B$ is a $p$-integral basis if and only if

$$(4.6) \quad \sum_{i=1}^n \nu_i = \text{ind}^\text{num} + \text{ind}(f).$$

4.3. Triangular basis. We say that a $p$-integral basis is triangular if it consists of $n$ integral elements of the type $q_i(\theta)/p^{\nu_i}$, with $q_i(x) \in \mathbb{Z}[x]$ monic of degree $i$.

It is well-known that the chinese remainder theorem can be used to construct an integral basis of $\mathbb{Z}_K$ from triangular $p$-integral bases for the different prime divisors $p$ of $\text{disc}(f)$.

Unfortunately, the $p$-integral basis $B = \{q_i(\theta)/p^{\nu_i} \mid 1 \leq i \leq n\}$, produced by the above modification of Montes’ algorithm, is not triangular. On the other hand, for $n$ large the computation of the determinant of the matrix of the coefficients of the numerators $q_i(x)$ is inefficient. We can save these two handicaps by computing a $p$-stem of a $p$-integral basis from the output list $B$. We recall that a $p$-stem is a family of generators of $\mathbb{Z}_K \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ as a $\mathbb{Z}_{(p)}[\theta]$-module, of the form $q_i(\theta)/p^{\nu_i}$, with monic polynomials $q_i(x) \in \mathbb{Z}[x]$ having minimal degrees.

The $p$-stem is obtained by a single call to the MAGMA routine GroebnerBasis in order to compute Groebner generators of the ideal of $\mathbb{Z}[x]$ generated by $\{p^r\} \cup \{p^{r-\nu_i} q_i(x) \mid 1 \leq i \leq n\}$, where $\nu = \max_i \{\nu_i\}$. The output of this routine is a list of polynomials $p^r$, $p^{r-\nu_1} g_1(x), \ldots, p^{r-\nu_k} g_k(x)$, with $g_i(x) \in \mathbb{Z}[x]$ monic polynomials
having strictly increasing degrees; the $p$-stem of $\mathbb{Z}_K$ is given by

$$1, \frac{g_1(\theta)}{p^{\mu_1}}, \ldots, \frac{g_k(\theta)}{p^{\mu_k}},$$

and this is the final output of the algorithm. If $d_i := \deg g_i$ for $1 \leq i \leq k$, the triangular $p$-integral basis determined by the $p$-stem is:

$$1, \theta, \ldots, \theta^{d_1-1}, \frac{g_1(\theta)}{p^{\mu_1}}, \ldots, \frac{g_k(\theta)\theta^{d_2-d_1-1}}{p^{\mu_1}}, \ldots, \frac{g_k(\theta)\theta^{n-d_k-1}}{p^{\mu_k}}.$$

Once the $p$-stem is computed, in order to show that we obtained a $p$-integral basis it is sufficient to check that

$$(d_2 - d_1)\mu_1 + \cdots + (n - d_k)\mu_k = \text{ind}(f).$$

If this equality holds, the algorithm prints a message of maximal order, and otherwise a message of non-maximal order.

This results into a more efficient computation, but the routine $\text{GroebnerBasis}$ uses much more memory.

4.4. An example. Let us show how the algorithm works with an example. Take $p = 2$ and $f(x) = x^{12} + 4x^6 + 16x^3 + 64$.

Since $f(x) \equiv x^{12} \pmod{2}$, the set $t_0(f)$ contains only one type $t_0$ of order zero, represented by the irreducible polynomial $\psi_0(y) = y$; we take $\phi_1(x) = x$ as a representative of this type. The Newton polygon of first order of $f(x)$ has two sides, with slopes $-2/3$ and $-1/3$, and $\text{ind}_1(f) = \text{ind}(N_1(f)) = 23$.

![Newton polygon of $f(x)$](image)

The residual polynomials are:

$$R_{-2/3}(f)(y) = y^2 + y + 1, \quad R_{-1/3}(f)(y) = (y + 1)^2.$$

Thus, the type $t_0$ ramifies into two types of order one:

$$t = (x; -2/3, y^2 + y + 1), \quad t' = (x; -1/3, y + 1),$$

and $t_1(f) = \{t, t'\}$. The type $t$ is complete, and it singles out a prime ideal $p_1$ with $e(p_1/2) = 3, f(p_1/2) = 2$. Since its reduced type $t^0$ is regular, the six first quotients of the $\phi_1$-adic development of $f(x)$ determine six elements from the final
list $B$: 

\begin{align*}
q_1(x) &= x^{11} + 4x^5 + 16x^2, & H_1 &= 16/3, \quad \rightsquigarrow \quad (\theta) / 2^5 \\
q_2(x) &= x^{10} + 4x^4 + 16x, & H_2 &= 14/3, \quad \rightsquigarrow \quad (\theta) / 2^4 \\
q_3(x) &= x^9 + 4x^3 + 16, & H_3 &= 4, \quad \rightsquigarrow \quad (\theta) / 2^3 \\
q_4(x) &= x^8 + 4x^2, & H_4 &= 10/3, \quad \rightsquigarrow \quad (\theta) / 2^3 \\
q_5(x) &= x^7 + 4x, & H_5 &= 8/3, \quad \rightsquigarrow \quad (\theta) / 2^2 \\
q_6(x) &= x^6 + 4, & H_6 &= 2 \quad \rightsquigarrow \quad (\theta) / 2^6.
\end{align*}

The type $t'$ is not complete, and it requires to do some more work in order two. Before analyzing the future ramifications of $t'$ we enlarge the list $\text{PQ}^t$ with the quotients $q_{10}(x) = x^2$ and $q_{11}(x) = x$ of the $\phi_1$-adic development of $f(x)$:

$$\text{PQ}^t \leftarrow \{1, x, x^2\}, \quad \text{PQVals}^t \leftarrow \{0, 1/3, 2/3\}.$$ 

Let us choose $\phi_2(x) = x^3 + 6$ as a representative of $t'$. The $\phi_2$-adic development of $f(x)$ is:

$$f(x) = \phi_2(x)^4 - 24\phi_2(x)^3 + 220\phi_2(x)^2 - 896\phi_2(x) + 1408.$$ 

The principal part of the Newton polygon of second order has length $2 = \omega_2^t(f)$; hence we have to draw only the first three points to determine $N_2^{-2}(f)$. Since $v_2(\phi_2) = v_2(2) = 3$, we have $v_2(1408) = 21$, $v_2(896\phi_2) = 24$, $v_2(220\phi_2^3) = 12$, so that $N_2^{-2}(f)$ has only one side of slope $-9/2$, and $\text{ind}_2(f) = \text{ind}(N_2^{-2}(f)) = 4$:

![Newton polygon](image)

The residual polynomial of second order is already separable: $R_{-9/2}(f)(y) = y + 1$. Thus, $t'$ is extended to a unique type of order two:

$$t'' = (x; -1/3, x^3 + 6; -9/2, y + 1),$$

which is already complete. It singles out a prime ideal $p_2$ with $e(p_2/2) = 6$, $f(p_2/2) = 1$. Since the reduced type $(t'')^0$ is regular, the two first quotients of the $\phi_2$-adic development of $f(x)$ determine six elements of the final list $B$. The two quotients and the exponents of their denominators, given by Proposition 3.6, are:

$$Q_1(x) = \phi_2(x)^3 - 24\phi_2(x)^2 + 220\phi_2(x) - 896 = x^9 - 6x^6 + 40x^3 - 224,$$

$$Q_1(x) = (16.5 - 3)/3 = 9/2,$$

$$Q_2(x) = \phi_2(x)^2 - 24\phi_2(x) + 220 = x^6 - 12x^3 + 112,$$

$$Q_2(x) = (12 - 6)/3 = 2.$$
By multiplying them by $PQ^t$, each one determines three elements of $B$:

$$\frac{Q_1(\theta)}{2^4}, \frac{Q_1(\theta)\theta}{2^4}, \frac{Q_1(\theta)\theta^2}{2^5}, \frac{Q_2(\theta)}{2^4}, \frac{Q_2(\theta)\theta}{2^4},$$

and our list $B$ is complete, with 12 integral elements.

The output list of complete types is $CT = \{t, t''\}$, parameterizing the prime ideals $p_1, p_2$. The index $\text{ind}(f) = v(\mathbb{Z}_K : \mathbb{Z}[\theta])$ is equal to: $\text{ind}(f) = \text{ind}_1(f) + \text{ind}_2(f) = 23 + 4 = 27$. Since $v(\text{disc}(f)) = 69$, we get $v(\text{disc}(K)) = 15$.

The reader can check that the sum of the exponents of the denominators of the elements in $B$ is 39, and $\text{ind}^{\text{sum}} = 12$; thus, $B$ is a $p$-integral basis indeed, by criterion (4.6).

However, the algorithm does not check this. It outputs directly the $p$-stems:

$$1, \frac{1}{2}, \frac{\theta^3}{2}, \frac{\theta^5 + 2\theta^2}{2^2}, \frac{\theta^6 + 2\theta^3}{2^3}, \frac{\theta^8 + 2\theta^5 + 8\theta^2}{2^4}, \frac{\theta^{11} + 4\theta^5 + 16\theta^2}{2^5},$$

by a call to the GroebnerBasis routine, as explained above. Then, it checks that criterion (4.6) is satisfied:

$$2 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 1 \cdot 5 = 27;$$

thus, it outputs a message of maximal order.

4.5. **Proof of the conjecture when $f(x)$ is irreducible over $\mathbb{Z}_p[x]$.** In [EN09] it is proved that $B$ is a $p$-integral basis if all complete types of the output list $CT$ have order less than or equal to one. We deal now with the case $f(x)$ irreducible over $\mathbb{Z}_p[x]$, or equivalently, the output list $CT$ consists of only one type.

**Theorem 4.4.** Suppose that $f(x)$ is irreducible over $\mathbb{Z}_p[x]$. Let

$$t = (\phi_1(x); \ldots, \phi_r(x); \lambda_r, \psi_r(y));$$

be a complete type of order $r$, and denote $n_i := \omega_i(f)$, for $1 \leq i \leq r + 1$. Let $e_0 = 1,$ $e_i = m_i$ and $J := \{ (j_0, \ldots, j_r) \in \mathbb{N}^{r+1} \mid 0 \leq j_i < e_i f_i, \text{ for all } 0 \leq i \leq r \}$. Then, with the notation of (4.2) and (4.3), the family of all

$$q_{r,n_r-j_r}(\theta) \cdots q_{1,n_1-j_1}(\theta)\theta^{j_0} \in \mathbb{Z}_p \cdot \text{disc}(f), \quad (j_0, \ldots, j_r) \in J,$$

is a $p$-integral basis of $K$.

**Proof.** By the Theorem of the polygon and the Theorem of the residual polynomial [HN Thms.3.1,3.7], for each $1 \leq i \leq r$:

1. $N_i(f)$ is one-sided, with slope $\lambda_i$ and length $n_i = (e_i f_i) \cdots (e_r f_r)$,
2. $\text{ind}_i(f) = f_0 \cdots f_{i-1} \text{ind}(N_i(f))$,
3. $R_i(f)(y) = \psi_i(y)^{n_i+1}$ in $\mathbb{F}_r[y]$.

Since $t$ is complete, we have $n_{i+1} = 1$, and

$$n = m_i n_i = m_t = f_0 (e_1 f_1) \cdots (e_r f_r),$$

for all $1 \leq i \leq r + 1$. Thus, the family of (4.8) consists of $n$ algebraic numbers, all of them integral by Proposition (3.6). Let $\mathcal{O} \subseteq \mathbb{Z}_K$ be the sub-$\mathbb{Z}$-module generated by these elements.

By (3.11), the degree of each $q_{r,n_r-j_r}(x)$ is equal to $n = (n_i - j_i) m_i = j_i m_i$. By Lemma (3.3), the degrees of the numerators of the $n$ integral elements of (4.8) take the values $0, 1, \ldots, n - 1$; therefore, the elements $q_{r,n_r-j_r}(\theta) \cdots q_{1,n_1-j_1}(\theta)\theta^{j_0}$, for
\((j_0, \ldots, j_r) \in J\), are a \(Z\)-basis of the order \(Z[\theta]\). In particular, \(O\) is a free \(Z\)-module of rank \(n\) that contains \(Z[\theta]\), and
\[
v(O: Z[\theta]) = \sum_{(j_0, \ldots, j_r) \in J} [H_{r,n-r-j_r} + \cdots + H_{1,n_1-j_1}]
\]
\[
= f_0 \sum_{(0,j_1,\ldots,j_r) \in J} [H_{r,n-r-j_r} + \cdots + H_{1,n_1-j_1}].
\]

Therefore, in order to show that \(v(Z_K: O) = 0\) it is sufficient to check that \(v(O: Z[\theta]) = v(Z_K: Z[\theta]) = \operatorname{ind}(f)\). By \ref{2.1}, this is equivalent to
\[
(4.9) \quad f_0 \sum_{(0,j_1,\ldots,j_r) \in J} [H_{r,n-r-j_r} + \cdots + H_{1,n_1-j_1}] = \operatorname{ind}_1(f) + \cdots + \operatorname{ind}_r(f).
\]

By a simple glance at the Newton polygon \(N_i(f)\):

\[
\begin{array}{c}
0 \\
\hline
j \\
\hline
n_i \\
\hline
\end{array}
\]

we see that \(v_i(f) = v_i(\phi^{n_i})\) and \(Y_{i,j} = v_i(f) + (n_i - j)|\lambda_i|\); hence,
\[
H_{i,n_i-j_i} = \frac{Y_{i,n_i-j_i} - (n_i - j_i)v_i(\phi_i)}{\epsilon_0 \cdots \epsilon_{i-1}} = \frac{v_i(f) + j_i|\lambda_i| - (n_i - j_i)v_i(\phi_i)}{\epsilon_0 \cdots \epsilon_{i-1}} = j_i \frac{|\lambda_i| + v_i(\phi_i)}{\epsilon_0 \cdots \epsilon_{i-1}} = j_i v(\phi_i(\theta)),
\]
the last equality by the Theorem of the polygon \cite[Thm.3.1]{HN}.

Therefore, the equality \ref{4.9} can be rewritten as
\[
f_0 \sum_{(0,j_1,\ldots,j_r) \in J} \left[ \sum_{i=1}^r j_i v(\phi_i(\theta)) \right] = \operatorname{ind}_1(f) + \cdots + \operatorname{ind}_r(f),
\]
and this equality is proved in \cite[p.55]{HN}, as one of the final steps of the proof of the Theorem of the index. \(\square\)

5. Some numerical examples

We dedicate this section to illustrate the performance of (our implementation of) Montes’ algorithm with several polynomials chosen to force its capabilities at maximum in three directions: polynomials with a unique associate type of large order, polynomials which require a lot of refinements, and polynomials with many different types.

All the tests have been done in a Linux server, with two Intel Quad Core processors, running at 3.0 Ghz, with 32Gb of RAM memory.
Example 1: Take \( p = 2 \). Consider the irreducible polynomials

\[
\begin{align*}
\phi_1 &= x^2 + 4x + 16; \\
\phi_2 &= \phi_1^3 + 16x\phi_1 + 1024; \\
\phi_3 &= \phi_2^3 + 2^{11}u\phi_2 + 2^{18}x\phi_1; \\
\phi_4 &= \phi_3^3 + 2^{25}x\phi_3 + 2^{35}\phi_1\phi_2; \\
\phi_5 &= \phi_4^3 + 2^{29}\phi_3\phi_1^2 + 2^{139}\phi_3 + 2^{153}\phi_2; \\
\phi_6 &= \phi_5^3 + 2^{141}\phi_3\phi_5 + 2^{279}\phi_4; \\
\phi_7 &= \phi_6^3 + 2^{998}\phi_1 + 2^{1003}; \\
\phi_8 &= \phi_7^3 + 2^{1505}(\phi_5 + 2^{167})\phi_6; \\
\phi_9 &= \phi_8^3 + ((2^{683}(xv\phi_2 + 2^{13}w)\phi_3 + 2^{710}(w\phi_2 + 2^{11}xv))\phi_4^2 + 2^{743}(x(\phi_2 + 2^{7})\phi_3 + 2^{25}(w\phi_2 + 2^{7}(u\phi_1 + 64)))\phi_4 + 2^{793}(u\phi_1\phi_2 + 2^{13}w)\phi_3 + 2^{26}(xv\phi_2 + 2^{13}\phi_1))\phi_5 + 2^{867}(u\phi_3 + 2^{10}(w\phi_2 + 2uv))\phi_2^2 + 2^{906}((u\phi_1 + 64)\phi_2 + 2^{12}u\phi_1)\phi_3 + 2^{31}u(\phi_2 + 2^{7}\phi_1))\phi_4 + 2^{960}((x\phi_1 + 64)\phi_2 + 2^{12}xv)\phi_3 + 2^{986}((u\phi_1 + 32x)\phi_2 + 2^{13}(\phi_1 + 16u)))\phi_7\phi_8 + ((2^{3364}(u\phi_2 + 2^{12})\phi_3 + 2^{22}(v\phi_2 + 2^{11}u\phi_1))\phi_3^2 + 2^{3420}(u\phi_1 + 64uv)\phi_3 + 2^{209}(u\phi_1 + 32x)\phi_2 + 2^{233}w)\phi_4 + 2^{3469}(x\phi_1\phi_2 + 2^{13}w)\phi_3 + 2^{4095}((x\phi_1 + 32u)\phi_2 + 2^{12}(w\phi_1 + 32x))\phi_5 + 2^{3531}(u\phi_2 + 2^{7}(x\phi_1 + 32u))\phi_3 + 2^{22}(\phi_1 + 16u)\phi_2 + 2^{12})\phi_4^2 + 2^{3582}(v\phi_2 + 2^{16}x)\phi_3 + 2^{23}u\phi_2\phi_4 + 2^{3641}(u\phi_2 + 2^{13})(\phi_3 + 2^{21}(xv\phi_2 + 2^{13}(\phi_1 + 16u)))\phi_6);
\end{align*}
\]

where \( u = x + 2, v = \phi_1 + 32, w = \phi_1 + 16x \).

For each \( j \), the corresponding polynomial \( \phi_j \) has a unique associate complete type of order \( j \), so that in the corresponding number field \( K_j \) the ideal \( 2\mathbb{Z}_K \) is the power of a unique prime ideal \( \mathfrak{p}_j \). The following table contains the running time of the computation of a 2-integral basis in these fields with our MAGMA implementation of the algorithm, compared to the routines incorporated in PARI, MAGMA and SAGE.

In the column “2-basis”, we give the time spent in the computation of the prime ideal decomposition of 2 and a 2-integral basis. The column “2-stem”, shows the total time of the algorithm, that adds to the previous data the computation of a 2-stem. In the columns PARI, MAGMA and SAGE we display the time spent in the computation of a 2-stem (by defect these programs compute always a triangular 2-integral basis).

Times are expressed in seconds. An “M” in a column means that the program exhausted the computer memory giving no result. Those executions lasting more than three days were interrupted.

| \( \phi_j \) | \( \deg \phi_j \) | \( \text{ind}(\phi_j) \) | 2-basis | 2-stem | PARI 2.3.4 | MAGMA 2.11 | SAGE 3.2.3 |
|---|---|---|---|---|---|---|---|
| \( \phi_2 \) | 4 | 12 | 0.00 | 0.01 | 0.00 | 0.01 | 0.01 |
| \( \phi_3 \) | 8 | 72 | 0.00 | 0.01 | 0.004 | 0.02 | 0.01 |
| \( \phi_4 \) | 16 | 352 | 0.00 | 0.02 | 0.016 | 4.67 | 0.05 |
| \( \phi_5 \) | 48 | 3996 | 0.03 | 0.6 | 2.4 | 42474 | 4.06 |
| \( \phi_6 \) | 96 | 15408 | 0.08 | 0.38 | 101 | > 72h | 196 |
| \( \phi_7 \) | 288 | 142416 | 0.97 | 16 | 47157 | > 72h | 119047 |
| \( \phi_8 \) | 576 | 573696 | 6.8 | M | > 72h | > 72h | > 72h |
| \( \phi_9 \) | 1152 | 2303520 | 34.5 | M | > 72h | > 72h | > 72h |
Example 2: Let \( f_k(x) = (x^2 + x + 1)^2 - p^{2k+1} \), with \( p \equiv 1 \mod 3 \) a prime number. When we apply Montes' algorithm to factor the ideal \( p\mathbb{Z}_K \), we obtain two types of order zero with liftings \( \phi(x) \in \mathbb{Z}[x] \) of degree one. For both of them the Newton polygon has only one side, with slope \(-1\) and end points \((2,0)\) and \((0,2)\), and the residual polynomial is the square of a linear factor. After approximately \( 2k \) total refinements, both types become \( f_k \)-complete. The ideal \( p\mathbb{Z}_K \) splits as the product of two prime ideals with ramification index 2 and residual degree 1, and the \( p \)-index of \( f_k(x) \) is \( 2k \).

This is almost the ill-conditioned quartic polynomial for the algorithm, since the index of every type is increased a unit per refinement in general, and the total \( p \)-index of \( f_k(x) \) is \( 2k \). Thus, the program has to make about \( 2k \) iterations of the main loop. Numerical experimentation shows that even in this worst case the running time of the algorithm is very low. In the following table we show the running time of the program for different values of \( k \) and \( p \), compared to those of PARI, MAGMA and SAGE. In this case the computation of the triangular basis with our algorithm is almost costless, so that we have included a single column for the running-times.

| \( p \) | \( \text{ind}(f_k) \) | \( p\)-stem | PARI 2.3.4 | MAGMA 2.11 | SAGE 3.2.3 |
|-------|-----------------|----------|----------|----------|----------|
| 7     | 1000            | 0.41     | 2.14     | 0.89     | 2.4      |
| 7     | 2000            | 1.14     | 15.03    | 3.35     | 16.4     |
| 7     | 4000            | 4.02     | 111.7    | 15.6     | 121      |
| 7     | 8000            | 18.9     | 747      | 84.6     | 841      |
| 7     | 16000           | 105      | 5573     | 486      | 6374     |
| 7     | 20000           | 187      | 11520    | 859      | 12242    |
| 13    | 1000            | 0.5      | 3.8      | 1.37     | 4.4      |
| 13    | 2000            | 1.5      | 27.4     | 5.16     | 30.7     |
| 13    | 10000           | 53.7     | 2585     | 231      | 3071     |
| 19    | 10000           | 65.7     | 3444     | 284      | 4213     |
| 31    | 10000           | 86.5     | 4741     | 364      | 6000     |
| 37    | 10000           | 93.7     | 5238     | 395      | 6715     |
| 43    | 10000           | 100.6    | 5089     | 422      | 7370     |
| 1003  | 10000           | 140      | 9120     | 596      | 11913    |
| 1009  | 10000           | 0.99     | 27.9     | 3.65     | 37       |
| 1009  | 20000           | 4.49     | 189      | 19.6     | 266.2    |
| 1009  | 40000           | 24.5     | 1380     | 112      | 2032     |
| \( 10^9 + 9 \) | 10000 | 3.94 | 188 | 23.2 | 519 |
| \( 10^9 + 9 \) | 20000 | 22.9 | 1409 | 133 | 4085 |
| \( 10^9 + 9 \) | 40000 | 139 | 10098 | 763 | 42790 |
| \( 10^{69} + 9 \) | 100 | 1.59 | 12.4 | 5.61 | 165 |
| \( 10^{69} + 9 \) | 200 | 4.14 | 88.5 | 30.1 | 1322 |
| \( 10^{69} + 9 \) | 400 | 14.3 | 688 | 167 | 10802 |

Example 3: Take \( p = 3 \) and let \( \psi_1(y), \ldots, \psi_{36}(y) \in \mathbb{F}_9[y] \) be all the irreducible quadratic polynomials over the finite field of nine elements. Consider the order two types
\[
t^k = (x; -3, x^2 + 3^6; -2/3, \psi_k(y)), \quad k = 1, \ldots, 36,
\]
and take representatives $\phi_1(x), \ldots, \phi_{36}(x)$ of these types. Define finally the polynomial
\[
f(x) := 3^{2000} + \prod_{k=1}^{36} \phi_k(x)^2.
\]
The degree of $f(x)$ is 864. Let $K$ the number field defined by $f(x)$. The 3-valuation of the index of $f(x)$ is 757296. Our implementation of the algorithm computes this index and a 3-basis for the ring of integers $\mathbb{Z}_K$ in 21 seconds. The triangulation of the basis, obtained after a Groebner basis computation, lasts about 3553 seconds. The total time for computing the 3-stem of $\mathbb{Z}_K$ is 3576 seconds.

6. Some statistics

We have tested our program with different sets of random polynomials, both to check the validity of conjecture [EN09] and to estimate its performance. Results are gathered in the following tables. In the first table, we have chosen random polynomials of types of orders 3, 4 and 5 at $p = 2$, while in the second table we have worked with random polynomials of types of order 1, 2 and 3 at different random primes less than 1024.

$p = 2$

| Order | Tests | Mean Degree | Mean Index | Mean Time |
|-------|-------|-------------|------------|-----------|
| 3     | 1800  | 65          | 6735       | 1.065     |
| 4     | 5054  | 117         | 25774      | 3.936     |
| 5     | 300   | 172         | 67411      | 19.605    |

$1 < p < 1024$

| Order | Tests | Mean Degree | Mean Index | Mean Time |
|-------|-------|-------------|------------|-----------|
| 1     | 20000 | 7           | 33         | 0.002     |
| 2     | 10000 | 25          | 777        | 0.151     |
| 3     | 6000  | 65          | 6605       | 4.09      |

References

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[Mon99] Montes, J., *Polígonos de Newton de orden superior y aplicaciones aritméticas*, Tesi Doctoral, Universitat de Barcelona 1999.
