BIRATIONAL SMOOTH MINIMAL MODELS HAVE EQUAL HODGE NUMBERS IN ALL DIMENSIONS

TETSUSHI ITO

Abstract. This is a resume of the author's talk at the Workshop on Arithmetic, Geometry and Physics around Calabi-Yau Varieties and Mirror Symmetry (July 23-29, 2001), the Fields Institute. The aim of this note is to prove that birational smooth minimal models over \( \mathbb{C} \) have equal Hodge numbers in all dimensions by an arithmetic method. Our method is a refinement of the method of V. Batyrev and C.-L. Wang on Betti numbers who used \( p \)-adic integration and the Weil conjecture. Our ingredient is to use further arithmetic results such as the Chebotarev density theorem and \( p \)-adic Hodge theory.

1. Introduction

Smooth minimal models play an important role in birational geometry. In this paper, we study Hodge numbers of smooth minimal models by an arithmetic method. Here we recall the definition of smooth minimal models. A divisor \( D \) on a smooth projective variety \( X \) is called nef if \( D \cdot C \geq 0 \) for all curves \( C \) in \( X \). A smooth minimal model is a smooth projective variety \( X \) whose canonical bundle \( K_X = \Omega^\dim X_X \) is nef.

The purpose of this paper is to prove the following theorem.

Theorem 1.1. Let \( X \) and \( Y \) be smooth minimal models over \( \mathbb{C} \). Assume that \( X \) and \( Y \) are birational over \( \mathbb{C} \). Then, \( X \) and \( Y \) have equal Hodge numbers:

\[
\dim_{\mathbb{C}} H^i(X, \Omega^j_X) = \dim_{\mathbb{C}} H^i(Y, \Omega^j_Y) \quad \text{for all } i, j.
\]

In this paper, we firstly compute the number of rational points of reduction modulo \( p \) of \( X \) and \( Y \) by \( p \)-adic integration. Then we apply the following proposition which is proved in \([4] \) by combining the Weil conjecture, the Chebotarev density theorem and \( p \)-adic Hodge theory (for a variant in terms of zeta functions, see Corollary \([1,8]\)).

Proposition 1.2. Let \( K \) be a number field. Let \( \mathfrak{X} \) and \( \mathfrak{Y} \) be schemes of finite type over the ring of integers \( \mathcal{O}_K \) of \( K \) whose generic fibers \( X = \mathfrak{X} \otimes_{\mathcal{O}_K} K \) and \( Y = \mathfrak{Y} \otimes_{\mathcal{O}_K} K \) are proper and smooth over \( K \). If \( |\mathfrak{X}(\mathcal{O}_K/p)| = |\mathfrak{Y}(\mathcal{O}_K/p)| \) for all

Date: Sep. 20, 2002.

1991 Mathematics Subject Classification. Primary 11R42, 11S80; Secondary 14E05.

The author was supported by the Japan Society for the Promotion of Science Research Fellowships for Young Scientists.
but a finite number of maximal ideals $\mathfrak{p}$ of $\mathcal{O}_K$, then $X$ and $Y$ have equal Hodge numbers:

$$\dim_K H^i(X, \Omega^j_X) = \dim_K H^i(Y, \Omega^j_Y) \quad \text{for all } i, j.$$ 

A similar statement for Betti numbers of proper smooth varieties over a finite field is a well-known consequence of the Weil conjecture (see Proposition 1.2). We use $p$-adic Hodge theory to get information on Hodge numbers from Galois representations. It is likely that Proposition 1.2 is well-known only for specialists in arithmetic geometry. We expect that Proposition 1.2 has further interesting applications in algebraic geometry.

Here we say a few words about the history of Theorem 1.1. The case $\dim = 1$ is trivial since birational projective smooth curves are automatically isomorphic. The case $\dim = 2$ is also automatic because $X$ and $Y$ are isomorphic by uniqueness of the minimal models for surfaces. The case $\dim = 3$ is more interesting. By the minimal model program for threefolds, $X$ and $Y$ are not necessarily isomorphic, but connected by a sequence of flops. Hence we conclude, by basic properties of flops, that $X$ and $Y$ have equal Hodge structures ([KMM], [Ka1], [Kol]). However, we can’t do the same if $\dim \geq 4$ since the minimal model program in $\dim \geq 4$ is still under construction. On the other hand, Batyrev found a new way to get cohomological properties of birational varieties in all dimensions. He proved that birational smooth Calabi-Yau manifolds have equal Betti numbers in all dimensions ([Ba1]). He used $p$-adic integration and the Weil conjecture. Wang generalized Batyrev’s result for Betti numbers of smooth minimal models ([Wa1]).

We sketch the outline of this paper. In §2, we prove certain geometric properties of minimal models needed for our purpose. In §3, we recall Weil’s $p$-adic integration which computes the number of rational points of reduction modulo $p$ by integrating a gauge form on a $p$-adic manifold. In §4, we recall several facts on Galois representations and prove Proposition 1.2. Finally, in §5, we prove Theorem 1.1 by combining above results.

Remark 1.3. After this work was completed, François Loeser pointed out to the author that Theorem 1.1 can be obtained by the theory of motivic integration developed by Kontsevich [Kon] and Denef-Loeser [DL] (see also [Ba2], [Ve], [Wa1], [Wa2]). Willem Veys kindly informed the author that Theorem 1.1 is already written in his paper [Ve], Corollary of Theorem 2.7. Moreover, Chin-Lung Wang kindly informed the author that he independently obtained the same result in 2000. The author knew it in his talk at Tokyo on February 2002. The author would like to thank them for information.

Remark 1.4. Recently, the author generalized the results of this paper and obtained an application of $p$-adic Hodge theory to stringy Hodge numbers for singular varieties (for details, see [It]). Stringy Hodge numbers were introduced by Batyrev in [Ba2] where he studied them by motivic integration.
Acknowledgments. The author is grateful to Takeshi Saito and Kazuya Kato for their advice and support. He also would like to thank Takeshi Tsuji and Shinich Mochizuki for invaluable suggestion on $p$-adic Hodge theory, Keiji Oguiso, Yujiro Kawamata, and Victor Batyrev for invaluable comment and encouragement, Yasunari Nagai for discussion about the minimal model program, Yoichi Mieda for carefully reading a manuscript.

2. Geometry of minimal models

Let $X$ and $Y$ be birational projective smooth algebraic varieties over $\mathbb{C}$. Recall that $X$ and $Y$ are called $K$-equivalent if there exists a projective smooth variety $Z$ over $\mathbb{C}$ and proper birational morphisms $f : Z \to X$, $g : Z \to Y$ such that $f^*K_X = g^*K_Y$.

Recently the notion of $K$-equivalence plays an important role in birational geometry ([Ka2], [Wa2]).

Although the following property of birational smooth minimal models seems well-known for specialists in birational geometry, we write it for the reader’s convenience ([Kol], [Fu], [Wa1]).

Proposition 2.1. Birational smooth minimal models over $\mathbb{C}$ are $K$-equivalent.

Proof. Let $X$ and $Y$ be birational smooth minimal models over $\mathbb{C}$. By taking a resolution of singularities of the closure of the graph of the birational map, there exists a projective smooth variety $Z$ over $\mathbb{C}$ and proper birational morphisms $f : Z \to X$, $g : Z \to Y$.

To see $f^*K_X = g^*K_Y$, we write the canonical bundle relation:

$$K_Z = f^*K_X + F + G,$$

$$K_Z = g^*K_Y + F' + G'.$$

Here $F, G, F', G'$ are effective divisors such that $F, F'$ are exceptional for both $f$ and $g$, $G$ is exceptional for $f$ but not for $g$, and $G'$ is exceptional for $g$ but not for $f$.

By symmetry, it is enough to show $F + G \geq F' + G'$. To show this, it is enough to show $F - F' - G' \geq 0$. Write $F - F' - G' = A - B$ such that $A, B$ are effective divisors and they have no common component. It is enough to show $B = 0$. To show this, we assume $B \neq 0$. Then we have

$$g^*K_Y = f^*K_X + G + (F - F' - G') = f^*K_X + G + (A - B).$$

By taking suitable hyperplane sections and using the Hodge index theorem, we can take a curve $C$ in $Z$ such that $B \cdot C < 0$, $g(C)$ is a point and $C$ is not
contained in $G+A$ ([Fu], 1.5). Recall that $B$ is exceptional for $g$. Since $g^*K_Y\cdot C = K_Y\cdot g(C) = 0$, we have

$$0 = g^*K_Y\cdot C = f^*K_X\cdot C + G\cdot C + A\cdot C - B\cdot C.$$ 

We examine the right hand side. $f^*K_X\cdot C = K_X\cdot f(C) \geq 0$ since $K_X$ is nef. $G\cdot C \geq 0$, $A\cdot C \geq 0$ since $C$ is not contained in $G+A$. Since $B\cdot C < 0$, we conclude that the right hand side must be positive. This is contradiction. Hence $B = 0$ and the proof is completed. \hfill \Box

3. $p$-adic integration

3.1. General definitions. Let $p$ be a prime number and $\mathbb{Q}_p$ be the field of $p$-adic numbers. Let $F$ be a finite extension of $\mathbb{Q}_p$, $R \subset F$ be the ring of integers in $F$, $m \subset R$ be the maximal ideal of $R$, $\mathbb{F}_q = R/m$ be the residue field of $F$ with $q$ elements, where $q$ is a power of $p$. For an element $x \in F$, we define the $p$-adic absolute value $|x|_p$ by

$$|x|_p = \begin{cases} q^{-v(x)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

where $v : F^\times \to \mathbb{Z}$ is the normalized discrete valuation of $F$.

Let $\mathfrak{X}$ be a smooth scheme over $R$ of relative dimension $n$. We can compute the number of $\mathbb{F}_q$-rational points $|\mathfrak{X}(\mathbb{F}_q)|$ by integrating certain $p$-adic measure on the set of $R$-rational points $\mathfrak{X}(R)$. We note that $\mathfrak{X}(R)$ is a compact and totally disconnected topological space with respect to its $p$-adic topology.

Let $\omega \in \Gamma(\mathfrak{X}, \Omega^n_{\mathfrak{X}/R})$ be a regular $n$-form on $\mathfrak{X}$, where $\Omega^n_{\mathfrak{X}/R}$ is the relative canonical bundle of $\mathfrak{X}/R$. We shall define the $p$-adic integration of $\omega$ on $\mathfrak{X}(R)$ as follows. Let $s \in \mathfrak{X}(R)$ be a $R$-rational point. Let $U \subset \mathfrak{X}(R)$ be a sufficiently small $p$-adic open neighborhood of $s$ on which there exists a system of local $p$-adic coordinates $\{x_1, \ldots, x_n\}$. Then $\{x_1, \ldots, x_n\}$ defines a $p$-adic analytic map

$$x = (x_1, \ldots, x_n) : U \longrightarrow R^n$$

which is a homeomorphism between $U$ and a $p$-adic open set $V$ of $R^n$. By using the above coordinates, $\omega$ can be written as

$$\omega = f(x) \, dx_1 \wedge \cdots \wedge dx_n.$$ 

We can consider $f(x)$ as a $p$-adic analytic function on $V$. Then we define the $p$-adic integration of $\omega$ on $U$ by the equation

$$\int_U |\omega|_p = \int_V |f(x)|_p \, dx_1 \cdots dx_n,$$

where $|f(x)|_p$ is the $p$-adic absolute value of the value of $f$ at $x \in V$ and $dx_1 \cdots dx_n$ is the Haar measure on $R^n$ normalized by the condition

$$\int_{R^n} dx_1 \cdots dx_n = 1.$$
By patching the above integration, we get the $p$-adic integration of $\omega$ on $X(R)$
\[ \int_{X(R)} |\omega|_p. \]

3.2. $p$-adic integration of a gauge form. A gauge form $\omega$ on $X$ is a nowhere vanishing global section $\omega \in \Gamma(X, \Omega^n_{X/R})$. Clearly, a gauge form exists if and only if $\Omega^n_{X/R}$ is trivial. Therefore a gauge form exists at least Zariski locally. The most important property of $p$-adic integration is that the $p$-adic integration of a gauge form computes the number of $\mathbb{F}_q$-rational points.

**Proposition 3.1** ([We2], 2.2.5). Let $X$ be a smooth scheme over $R$ of relative dimension $n$ and $\omega$ be a gauge form on $X$. Then we have
\[ \int_{X(R)} |\omega|_p = \frac{|X(\mathbb{F}_q)|}{q^n}. \]

**Proof.** Let
\[ \varphi : X(R) \to X(\mathbb{F}_q) \]
be the reduction modulo $m$ map. For $\bar{x} \in X(\mathbb{F}_q)$, $\varphi^{-1}(\bar{x})$ is a $p$-adic open set of $X(R)$. Therefore, it is enough to show
\[ \int_{\varphi^{-1}(\bar{x})} |\omega|_p = \frac{1}{q^n}. \]

Let $\{x_1, \ldots, x_n\} \subset \mathcal{O}_{X, \bar{x}}$ be a regular system of parameters at $\bar{x}$. Then $\{x_1, \ldots, x_n\}$ defines a system of local $p$-adic coordinates on $\varphi^{-1}(\bar{x})$ and
\[ x = (x_1, \ldots, x_n) : \varphi^{-1}(\bar{x}) \to m^n \subset R^n \]
is a $p$-adic analytic homeomorphism. Let $\omega$ be written as $\omega = f(x) \, dx_1 \wedge \cdots \wedge dx_n$. Since $\omega$ is a gauge form, $f(x)$ is a $p$-adic unit for all $x \in \varphi^{-1}(\bar{x})$. Therefore $|f(x)|_p = 1$. Then we have
\[ \int_{\varphi^{-1}(\bar{x})} |\omega|_p = \int_{m^n} dx_1 \cdots dx_n = \frac{1}{q^n} \]
since $m^n$ is an index $q^n$ subgroup of $R^n$. \qed

3.3. A slight generalization — $p$-adic integration of local generators of a lattice. We shall consider Proposition 3.1 if $\Omega^n_{X/R}$ is not necessarily trivial. By a lattice of $\Omega^n_{X/R}$, we mean a locally free subsheaf $\mathcal{L} \subset \Omega^n_{X/R}$ of rank 1. We can define the $p$-adic integration of local generators of a lattice $\mathcal{L}$ as follows. If both $\mathcal{L}$ and $\Omega^n_{X/R}$ are free, take a generator $\omega$ of $\mathcal{L}$. Then we have the $p$-adic integration
\[ \int_{X(R)} |\omega|_p. \]

This value is independent of $\omega$ since $\omega$ is unique up to multiplication by a unit $f \in \mathcal{O}_X^*$ and such $f$ takes $p$-adic absolute value 1 as a $p$-adic analytic function.
Therefore we can patch them and get the \( p \)-adic integration of local generators of \( \mathcal{L} \) which we denote by

\[
\int_{X(R)} |\mathcal{L}|_p.
\]

Note that if \( \mathcal{L} = \Omega^n_{X/R} \) and \( \Omega^n_{X/R} \) is trivial, the above value is nothing but the \( p \)-adic integration of a gauge form in \( \S 3.2 \). Therefore, we have the following generalization of Proposition 3.1.

**Corollary 3.2.** Let \( X \) be a smooth scheme over \( R \) of relative dimension \( n \). Then we have

\[
\int_{X(R)} |\Omega^n_{X/R}|_p = \frac{|X(\mathbb{F}_q)|}{q^n}.
\]

### 3.4. An application to \( K \)-equivalent varieties.

**Proposition 3.3** ([Ba1], [Wa1]). Let \( X, Y, Z \) be smooth schemes over \( R \) of relative dimension \( n \). Assume that there exist proper birational morphisms \( \tilde{f} : Z \to X, \tilde{g} : Z \to Y \) such that \( \tilde{f}^*\Omega^n_{X/R} = \tilde{g}^*\Omega^n_{Y/R} \). Then

\[
|X(\mathbb{F}_q)| = |Y(\mathbb{F}_q)|.
\]

**Proof.** By Corollary 3.2, we have

\[
\int_{X(R)} |\Omega^n_{X/R}|_p = \frac{|X(\mathbb{F}_q)|}{q^n}, \quad \int_{Y(R)} |\Omega^n_{Y/R}|_p = \frac{|Y(\mathbb{F}_q)|}{q^n}.
\]

Since \( \tilde{f}^*\Omega^n_{X/R} = \tilde{g}^*\Omega^n_{Y/R} \), we compute

\[
\frac{|X(\mathbb{F}_q)|}{q^n} = \int_{X(R)} |\Omega^n_{X/R}|_p = \int_{Z(R)} |\tilde{f}^*\Omega^n_{X/R}|_p = \int_{Z(R)} |\tilde{g}^*\Omega^n_{Y/R}|_p = \frac{|Y(\mathbb{F}_q)|}{q^n}
\]

by using the change of variable formula for \( p \)-adic integration twice. Hence we have \( |X(\mathbb{F}_q)| = |Y(\mathbb{F}_q)| \). \( \square \)

**Remark 3.4.** By combining Proposition 2.1, Proposition 3.3, and the Weil conjecture (see \( \S 4.1 \)), we can prove the equality of Betti numbers for birational smooth minimal models over \( \mathbb{C} \) as in [Ba1], [Wa1]. However, to show the equality of Hodge numbers, we need further arithmetic results such as the Chebotarev density theorem and \( p \)-adic Hodge theory which will be explained in \( \S 4 \).
4. Review of Galois representations

4.1. The Weil conjecture. Let $X$ be a proper smooth variety over a finite field $\mathbb{F}_q$ of dimension $n$. Fix a prime number $l$ prime to $q$. Let $H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ be the $i$-th $l$-adic étale cohomology of $X_{\overline{\mathbb{F}}_q} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, where $\overline{\mathbb{F}}_q$ denotes an algebraic closure of $\mathbb{F}_q$. Let $F : X_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$ be the $q$-th power Frobenius morphism. Note that the set of fixed points of $F$ is precisely the set of $\mathbb{F}_q$-rational points $X(\mathbb{F}_q)$. Then, by the Lefschetz fixed point formula for étale cohomology, we have

\[ |X(\mathbb{F}_q)| = \sum_{i=0}^{2n} (-1)^i \text{Tr}(F^*; H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)). \]  

(4.1)

Moreover, by the Weil conjecture proved by Deligne, all eigenvalues of $F^*$ acting on $H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ are algebraic numbers and all conjugates of them have complex absolute value $q^{i/2}$. This is an analogue of the Riemann hypothesis for a proper smooth variety over a finite field.

The Hasse-Weil zeta function $Z(X, t)$ is a formal power series with coefficients in $\mathbb{Q}$ defined by

\[ Z(X, t) = \exp \left( \sum_{r=1}^{\infty} \frac{|X(\mathbb{F}_{q^r})|}{r} t^r \right). \]

Then, by (4.1), we have the following expression of $Z(X, t)$

\[ Z(X, t) = \frac{P_1(X, t) \cdots P_{2n-1}(X, t)}{P_0(X, t)P_2(X, t) \cdots P_{2n}(X, t)^n}, \]

where

\[ P_i(X, t) = \det(1 - F^*t; H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)). \]

Although the following application of the Weil conjecture is well-known and weaker than our key proposition (Proposition 1.2), we note it here for reader’s convenience.

**Proposition 4.1.** Let $X$ and $Y$ be proper smooth varieties over $\mathbb{F}_q$. If $|X(\mathbb{F}_{q^r})| = |Y(\mathbb{F}_{q^r})|$ for all $r$, then $P_i(X, t) = P_i(Y, t)$. In particular, by comparing the degrees, we have

\[ \dim_{\mathbb{Q}} H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l) = \dim_{\mathbb{Q}} H^i_{\text{ét}}(Y_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l). \]

Therefore, if such $X$ (resp. $Y$) comes from a proper smooth variety $\widetilde{X}$ (resp. $\widetilde{Y}$) over a number field $K$ by modulo $p$ reduction, then $\widetilde{X} \otimes_K \mathbb{C}$ and $\widetilde{Y} \otimes_K \mathbb{C}$ have equal Betti numbers:

\[ \dim_{\mathbb{Q}} H^i(\widetilde{X} \otimes_K \mathbb{C}, \mathbb{Q}) = \dim_{\mathbb{Q}} H^i(\widetilde{Y} \otimes_K \mathbb{C}, \mathbb{Q}) \quad \text{for all } i. \]
Proof. We have $Z(X, t) = Z(Y, t)$ by definition. Hence $P_i(X, t) = P_i(Y, t)$ for all $i$ because we can recover $P_i(X, t)$ (resp. $P_i(Y, t)$) from $Z(X, t)$ (resp. $Z(Y, t)$) by the Weil conjecture. The rest follows from basic properties of étale cohomology.

4.2. An application of the Chebotarev density theorem. In this section, we don’t need the Chebotarev density theorem itself but need its application to $l$-adic Galois representations. For details, we refer Serre’s book [F3].

**Proposition 4.2** ([F3], I.2.3). Let $K$ be a number field, $m, m' \geq 1$ be integers, and $l$ be a prime number. Let

$$\rho : \text{Gal}(\overline{K}/K) \to \text{GL}(m, \mathbb{Q}_l), \quad \rho' : \text{Gal}(\overline{K}/K) \to \text{GL}(m', \mathbb{Q}_l)$$

be continuous $l$-adic $\text{Gal}(\overline{K}/K)$-representations such that $\rho$ and $\rho'$ are unramified outside a finite set $S$ of maximal ideals of $\mathcal{O}_K$. If

$$\text{Tr}(\rho(Frob_p)) = \text{Tr}(\rho'(Frob_p)) \quad \text{for all maximal ideals} \quad p \notin S,$

then $\rho$ and $\rho'$ have the same semisimplifications as $\text{Gal}(\overline{K}/K)$-representations. Here $Frob_p$ denotes a geometric Frobenius element at $p$ which specializes to the inverse of the $|\mathcal{O}_K/p|$-th power Frobenius map on the residue field at $p$.

Proof. We only give a sketch of the proof. By the Chebotarev density theorem ([F3] I.2.2), the set of conjugates of $Frob_p$ for all $p \notin S$ is dense in $G' = \text{Gal}(\overline{K}/K)/(\text{Ker}\rho \cap \text{Ker}\rho')$. Since $\rho$ and $\rho'$ are continuous representations, $\text{Tr} \circ \rho$ and $\text{Tr} \circ \rho'$ are continuous maps from $G'$ to $\mathbb{Q}_l$ which coincide on a dense subset of $G'$. Hence they are equal. Therefore, we have $\text{Tr}(\rho(\sigma)) = \text{Tr}(\rho'(\sigma))$ for all $\sigma \in \text{Gal}(\overline{K}/K)$. Then we have the conclusion by representation theory (see, for example, Bourbaki, Algèbre, Ch. 8, §12, n° 1, Prop 3.).

4.3. $p$-adic Hodge theory. In this section, we recall $p$-adic Hodge theory. Especially, we recall Hodge-Tate decomposition which is a $p$-adic analogue of Hodge decomposition over $\mathbb{C}$.

Let $p$ be a prime number and $F$ be a finite extension of $\mathbb{Q}_p$. Let $\mathbb{C}_p$ be a $p$-adic completion of an algebraic closure $\overline{F}$ of $F$. We define the $p$-adic Tate twists as follows. We define $\mathbb{Q}_p(0) = \mathbb{Q}_p$, $\mathbb{Q}_p(1) = \left( \varprojlim_{n} \mu_{p^n} \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and, for $n \geq 1$, $\mathbb{Q}_p(n) = \mathbb{Q}_p(1)^{\otimes n}$, $\mathbb{Q}_p(-n) = \text{Hom}(\mathbb{Q}_p(n), \mathbb{Q}_p)$. Moreover, we define $\mathbb{C}_p(n) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$, on which $\text{Gal}(\overline{F}/F)$ acts diagonally. It is known that $(\mathbb{C}_p)^{\text{Gal}(\overline{F}/F)} = F$ and $(\mathbb{C}_p(n))^{\text{Gal}(\overline{F}/F)} = 0$ for $n \neq 0$ ([F3], Theorem 2).

Let $B_{HT} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}_p(n)$ be a graded $\mathbb{C}_p$-module with an action of $\text{Gal}(\overline{F}/F)$. For a finite dimensional $\text{Gal}(\overline{F}/F)$-representation $V$ over $\mathbb{Q}_p$, we define a finite dimensional graded $F$-module $D_{HT}(V)$ by $D_{HT}(V) = (V \otimes_{\mathbb{Q}_p} B_{HT})^{\text{Gal}(\overline{F}/F)}$. The graded module structure of $D_{HT}(V)$ is induced from that of $B_{HT}$. In general, it is known that

$$\dim_F D_{HT}(V) \leq \dim_{\mathbb{Q}_p} V.$$
If the equality holds, $V$ is called a Hodge-Tate representation $([Fa], [Fo])$.

**Theorem 4.3** (Hodge-Tate decomposition, [Fa], [Ts]). Let $X$ be a proper smooth variety over $F$ and $k$ be an integer. The $p$-adic étale cohomology $H^k_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)$ of $X_{\overline{F}} = X \otimes_F \overline{F}$ is a finite dimensional Gal($\overline{F}/F$)-representation over $\mathbb{Q}_p$. Then, $H^k_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)$ is a Hodge-Tate representation. Moreover, there exists a canonical and functorial isomorphism

$$
\bigoplus_{i+j=k} H^i(X, \Omega^j_X) \otimes_F \mathbb{C}_p(-j) \cong H^k_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p
$$

of Gal($\overline{F}/F$)-representations, where Gal($\overline{F}/F$) acts on $H^i(X, \Omega^j_X)$ trivially and the right hand side diagonally.

For a finite dimensional Gal($\overline{F}/F$)-representation $V$ over $\mathbb{Q}_p$, we define

$$h^n(V) = \dim_F (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(n))^{\text{Gal}(\overline{F}/F)}.$$

The following lemma seems well-known for specialists in $p$-adic Hodge theory. But we write it here for reader’s convenience.

**Lemma 4.4.** Let $W_2$ be a Hodge-Tate representation and

$$0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow W_3 \longrightarrow 0$$

be an exact sequence of finite dimensional Gal($\overline{F}/F$)-representations over $\mathbb{Q}_p$. Then, $W_1$ and $W_3$ are Hodge-Tate representations and

$$h^n(W_2) = h^n(W_1) + h^n(W_3) = h^n(W_1 \oplus W_3)$$

for all $n$.

**Proof.** Since

$$0 \longrightarrow D_{HT}(W_1) \longrightarrow D_{HT}(W_2) \longrightarrow D_{HT}(W_3)$$

is exact by definition, we have $\dim_F D_{HT}(W_2) \leq \dim_F D_{HT}(W_1) + \dim_F D_{HT}(W_3)$. On the other hand, since $W_2$ is a Hodge-Tate representation, we have

$$\dim_F D_{HT}(W_2) = \dim_{\mathbb{Q}_p} W_2 = \dim_{\mathbb{Q}_p} W_1 + \dim_{\mathbb{Q}_p} W_3 \geq \dim_F D_{HT}(W_1) + \dim_F D_{HT}(W_3).$$

Therefore, we have $\dim_F D_{HT}(W_1) + \dim_F D_{HT}(W_3) = \dim_{\mathbb{Q}_p} W_1 + \dim_{\mathbb{Q}_p} W_3$ and hence $W_1$ and $W_3$ are Hodge-Tate representations. Then,

$$0 \longrightarrow D_{HT}(W_1) \longrightarrow D_{HT}(W_2) \longrightarrow D_{HT}(W_3) \longrightarrow 0$$

is exact. If we take the dimension of each graded quotient of the above exact sequence, we have $h^n(W_2) = h^n(W_1) + h^n(W_3) = h^n(W_1 \oplus W_3)$. □

By combining above results, we can recover the Hodge numbers from the semisimplifications of the $p$-adic étale cohomology as follows.
Corollary 4.5. Let $X$ be a proper smooth variety over $F$. Then, we have
\[
\dim_F H^i(X, \Omega^j_X) = h^j(H^{i+j}(X_F, \mathbb{Q}_p)^{ss}) \quad \text{for all } i, j,
\]
where $H^{i+j}(X_F, \mathbb{Q}_p)^{ss}$ denotes the semisimplification of $H^{i+j}(X_F, \mathbb{Q}_p)$ as a $\text{Gal}(\overline{F}/F)$-representation.

Proof. By Theorem 4.3, if we take the dimension of the $\text{Gal}(\overline{F}/F)$-invariant of $H^{i+j}(X_F, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}((j))$, we have
\[
\dim_F H^i(X, \Omega^j_X) = h^j(H^{i+j}(X_F, \mathbb{Q}_p)).
\]
On the other hand, since $H^{i+j}(X_F, \mathbb{Q}_p)$ is a Hodge-Tate representation,
\[
h^j(H^{i+j}(X_F, \mathbb{Q}_p)) = h^j(H^{i+j}(X_F, \mathbb{Q}_p)^{ss})
\]
by Lemma 4.4. Hence Corollary 4.3 is proved. \qed

Remark 4.6. A proof of Theorem 4.3 was firstly given by Faltings ([Fa1], for recent developments of Faltings’ theory of almost étale extensions, see also [Fa2]). Tsuji gave another proof by using de Jong’s alteration ([Ts]). In this paper, we don’t need the full version of Theorem 4.3. For example, the theorem of Fontaine-Messing is enough for our purpose who proved Theorem 4.3 in the case $F$ is unramified over $\mathbb{Q}_p$, $\dim X < p$ and $X$ has good reduction ([FM]).

4.4. An application to Hodge numbers. Here we prove Proposition 1.2. It is a standard consequence of the above results.

Proof of Proposition 1.2. Let notation be as in Proposition 1.2. Fix a prime number $l$. Let $S$ be a sufficiently large finite set of maximal ideals of $\mathcal{O}_K$ such that $X$ and $Y$ are proper and smooth over $(\text{Spec } \mathcal{O}_K) \setminus S$, $|\mathfrak{X}(\mathcal{O}_K/p)| = |\mathfrak{Y}(\mathcal{O}_K/p)|$ for all $p \not\in S$, and $S$ contains all $p$ dividing $l$. Let $H^i_{\text{ét}}(X_K, \mathbb{Q}_l)$ (resp. $H^i_{\text{ét}}(Y_K, \mathbb{Q}_l)$) be the $i$-th $l$-adic étale cohomology of $X_K = X \otimes_K \overline{K}$ (resp. $Y_K = Y \otimes_K \overline{K}$) on which $\text{Gal}(\overline{K}/K)$ acts.

Let $p$ be a maximal ideal of $\mathcal{O}_K$ outside $S$ and $\mathbb{F}_q = \mathcal{O}_K/p$ be the residue field at $p$. Then $\mathfrak{X}_{\overline{F}_q} = \mathfrak{X} \otimes_{\mathcal{O}_K} \overline{F}_q$ is a proper smooth variety over $\overline{F}_q$. Let $F : \mathfrak{X}_{\overline{F}_q} \to \mathfrak{X}_{\overline{F}_q}$ be the $q$-th power Frobenius morphism as in §1.1. Note that $p$ doesn’t divide $l$ here. Then, by basic properties of étale cohomology, $H^i_{\text{ét}}(X_{\overline{F}_q}, \mathbb{Q}_l)$ is canonically isomorphic to $H^i_{\text{ét}}(\mathfrak{X}_{\overline{F}_q}, \mathbb{Q}_l)$ and the action of $F^*$ on $H^i_{\text{ét}}(\mathfrak{X}_{\overline{F}_q}, \mathbb{Q}_l)$ corresponds to the action of $\text{Frob}_p^*$ on $H^i_{\text{ét}}(X_{\overline{F}_q}, \mathbb{Q}_l)$, where $\text{Frob}_p$ is a geometric Frobenius element at $p$ as in Proposition 1.2. Therefore, by the Lefschetz fixed point formula for étale cohomology (§1.1, (1.1)), we have
\[
|\mathfrak{X}(\mathcal{O}_K/p)| = \sum_{i=0}^{2 \dim X} (-1)^i \text{Tr}(\text{Frob}_p^*, H^i_{\text{ét}}(X_{\overline{F}_q}, \mathbb{Q}_l)).
\]
If local zeta functions \( \zeta_p \), Then Corollary 4.8. Therefore, we don't need the
representations, where \( \text{Gal}(K/K) \). Hence, by Proposition 4.2,
and 4.3 in this paper. Therefore, we don't need the
have equal Hodge numbers. Hence, by Proposition 4.2, 
now, take a maximal ideal \( p \) of \( \mathcal{O}_K \) outside \( S \). Let \( V' \) be a simple subquotient of 
acting on \( V' \). The same is true for \( W \). Therefore, since \( H^2_{\text{et}}(X_K, \mathbb{Q}_l) \neq 0 \)
and \( H^2_{\text{et}}(Y_K, \mathbb{Q}_l) \neq 0 \) by Poincaré duality, we have \( \dim X = \dim Y \). Moreover, for each \( i \), we conclude that \( H^i_{\text{et}}(X_K, \mathbb{Q}_l) \) and \( H^i_{\text{et}}(Y_K, \mathbb{Q}_l) \) have the same semisimplifications as \( \text{Gal}(\overline{K}/K) \)-representations.

Now, take a maximal ideal \( q \) of \( \mathcal{O}_K \) dividing \( l \). Let \( F \) be the completion of 
and \( Y \) have equal Hodge numbers.

**Remark 4.7.** If we take a sufficiently large \( l \) in the above proof, we can use the
theorem of Fontaine-Messing (Remark 4.6, [FM]). Therefore, we don't need the
full version of Theorem 4.3 in this paper.

The following corollary is a variant of Proposition 4.2 in terms of zeta functions.

**Corollary 4.8.** Let \( X \) and \( Y \) be proper smooth varieties over a number field \( K \). If
local zeta functions \( \zeta_p(X, s) \) and \( \zeta_p(Y, s) \) are the same for all but a finite number of maximal ideals \( p \) of \( \mathcal{O}_K \), then \( X \) and \( Y \) have equal Hodge numbers:

\[
\dim_{\mathbb{C}} H^i(X, \Omega^j_X) = \dim_{\mathbb{C}} H^i(Y, \Omega^j_Y) \quad \text{for all } i, j.
\]
Proof. Let $Z_p(X,t)$ be the Hasse-Weil zeta function of $X$ modulo $p$ as in §4.4 for all but finitely many $p$. Then $\zeta_p(X,s) = Z_p(X,|\mathcal{O}_K/p|^s)$ by definition. The same is true for $Y$. Therefore, Corollary 4.8 is an immediate consequence of Proposition 4.2. 

Remark 4.9. Here we note a practical difference between Proposition 4.2 and Proposition 4.4. In Proposition 4.4, it is sometimes possible to compute the Hasse-Weil zeta function $Z(X,t)$ and hence Betti numbers explicitly (for example, see [We1]). However, it seems very difficult to compute the number of rational points $|\mathcal{X}(\mathcal{O}_K/p)|$ for all but finitely many $p$ for $X$. Even if they are computed, there seems no way to compute Hodge numbers explicitly. Nevertheless, we expect that Proposition 4.2 has further interesting applications in algebraic geometry.

5. Proof of the main theorem

In this section, we give a proof of Theorem 1.1. By Proposition 2.4, it is enough to show the following statement on Hodge numbers of $K$-equivalent varieties.

Proposition 5.1. Let $X$ and $Y$ be birational projective smooth algebraic varieties over $\mathbb{C}$. Assume that $X$ and $Y$ are $K$-equivalent (see §3), then $X$ and $Y$ have equal Hodge numbers:

$$\dim_\mathbb{C} H^i(X, \Omega^j_X) = \dim_\mathbb{C} H^i(Y, \Omega^j_Y) \quad \text{for all } i,j.$$ 

Proof. Since $X$ and $Y$ are $K$-equivalent, there exists a projective smooth variety $Z$ over $\mathbb{C}$ and proper birational morphisms $f : Z \to X$, $g : Z \to Y$ such that $f^*K_X = g^*K_Y$. $X, Y, Z, f, g$ are defined over a subfield $K'$ of $\mathbb{C}$ which is finitely generated over $\mathbb{Q}$. Take a number field $K$ and a variety $T$ over $K$ such that the function field $K(T)$ of $T$ is isomorphic to $K'$ over $K$. Then, we can take varieties $X', Y', Z'$ over $K$, morphisms $X' \to T$, $Y' \to T$, $Z' \to T$ over $K$, and morphisms $f' : Z' \to X'$, $g' : Z' \to Y'$ over $T$ such that the generic fibers of $X', Y', Z', f', g'$ tensored with $\mathbb{C}$ are $X, Y, Z, f, g$ respectively. Then, by shrinking $T$ if necessary, we may assume $X', Y', Z', f', g'$ are proper birational morphisms, and $f'^*\Omega^n_{X'/T} = g'^*\Omega^n_{Y'/T}$, where $n = \dim X = \dim Y$. Moreover, by replacing $K$ by its finite extension, we may assume $T$ has a $K$-rational point $s \in T(K)$. Since, for a proper smooth family of varieties in characteristic 0, all fibers have equal Hodge numbers ([Del], 5.5), we may replace $X, Y, Z, f, g$ by the fibers of $X', Y', Z', f', g'$ at $s$.

Therefore, by changing notation, we may assume $X, Y, Z, f, g$ are defined over a number field $K$. Take schemes of finite type $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ over $\mathcal{O}_K$ with generic fiber $X, Y, Z$. Let $S$ be a sufficiently large finite set of maximal ideals of $\mathcal{O}_K$ such that $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ are proper and smooth over $\mathcal{U} = (\text{Spec} \mathcal{O}_K) \setminus S$, and $f : Z \to X$, $g : Z \to Y$ extend to proper birational morphisms $\tilde{f} : \mathfrak{Z} \otimes_{\mathcal{O}_K} \mathcal{U} \to \mathfrak{X} \otimes_{\mathcal{O}_K} \mathcal{U}$, $\tilde{g} : \mathfrak{Z} \otimes_{\mathcal{O}_K} \mathcal{U} \to \mathfrak{Y} \otimes_{\mathcal{O}_K} \mathcal{U}$ over $\mathcal{U}$ satisfying $\tilde{f}^*\Omega^n_{(\mathfrak{X} \otimes_{\mathcal{O}_K} \mathcal{U})/\mathcal{U}} = \tilde{g}^*\Omega^n_{(\mathfrak{Y} \otimes_{\mathcal{O}_K} \mathcal{U})/\mathcal{U}}$. 


Then for a maximal ideal \( p \) of \( \mathcal{O}_K \) outside \( S \), the completion of \( X, Y, Z, \tilde{f}, \tilde{g} \) at \( p \) satisfy the condition of Proposition 3.3. Therefore, we have \( |X(\mathcal{O}_K/p)| = |Y(\mathcal{O}_K/p)| \) for all \( p \notin S \). By Proposition 1.2, we conclude that \( X \) and \( Y \) have equal Hodge numbers. \( \square \)

References

[Ba1] V. V. Batyrev, *Birational Calabi-Yau n-folds have equal Betti numbers*, in *New trends in algebraic geometry* (Warwick, 1996), 1–11, Cambridge Univ. Press, Cambridge, 1999.
[Ba2] V. V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, in *Integrable systems and algebraic geometry* (Kobe/Kyoto, 1997), 1–32, World Sci. Publishing, River Edge, NJ, 1998.
[De1] P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. No. 35, (1968), 259–278.
[De2] P. Deligne, *La conjecture de Weil I*, Inst. Hautes Études Sci. Publ. Math. No. 43, (1974), 273–307.
[DL] J. Denef, F. Loeser, *Germes of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), no. 1, 201–232.
[Fa1] G. Faltings, *p-adic Hodge theory*, J. Amer. Math. Soc. **1** (1988), no. 1, 255–299.
[Fa2] G. Faltings, *Almost étale extensions, Cohomologies p-adiques et applications arithmétiques*, II. Astérisque No. 279, (2002), 185–270.
[Fo] J.-M. Fontaine, *Le corps des périodes p-adiques*, With an appendix by Pierre Colmez, *Périodes p-adiques* (Bures-sur-Yvette, 1988), Astérisque No. 223, (1994), 59–111.
[FM] J.-M. Fontaine, W. Messing, *p-adic periods and p-adic étale cohomology*, in *Current trends in arithmetical algebraic geometry* (Arcata, Calif., 1985), 179–207, Contemp. Math., 67, Amer. Math. Soc., Providence, RI, 1987.
[Fu] T. Fujita, *Zariski decomposition and canonical rings of elliptic threefolds*, J. Math. Soc. Japan **38** (1986), no. 1, 19–37.
[It] T. Ito, *Stringy Hodge numbers and p-adic Hodge theory*, preprint.
[Ka1] Y. Kawamata, *Creplant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. (2) **127** (1988), no. 1, 93–163.
[Ka2] Y. Kawamata, *D-equivalence and K-equivalence*, preprint, 2002, [math.AG/0205287](http://arxiv.org/abs/math.AG/0205287).
[Kol] J. Kollár, *Flops*, Nagoya Math. J. **113** (1989), 15–36.
[Kon] M. Kontsevich, *Lecture at Orsay* (December 7, 1995).
[Se] J.-P. Serre, *Abelian l-adic representations and elliptic curves*, W. A. Benjamin, Inc., New York, 1968.
[Ta] J. T. Tate, *p-divisible groups*, in *Proc. Conf. Local Fields* (Driebergen, 1966), 158–183, Springer, Berlin, 1967.
[Ts] T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. **137** (1999), no. 2, 233–411.
[Ve] W. Veys, *Zeta functions and “Kontsevich invariants” on singular varieties*, Canad. J. Math. **53** (2001), no. 4, 834–865.
[Wa1] C.-L. Wang, *On the topology of birational minimal models*, J. Differential Geom. **50** (1998), no. 1, 129–146
[Wa2] C.-L. Wang, *K-equivalence in Birational Geometry*, preprint, [math.AG/0204160](http://arxiv.org/abs/math.AG/0204160).
[We1] A. Weil, *Numbers of solutions of equations in finite fields*, Bull. Amer. Math. Soc. **55** (1949), 497–508.
[We2] A. Weil, *Adèles and algebraic groups*, Birkhäuser, Boston, Mass., 1982.
Department of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan

E-mail address: itote2@ms.u-tokyo.ac.jp

Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany

E-mail address: tetsushi@mpim-bonn.mpg.de