Abstract

There are several reformulations of the Viète’s formula for pi that have been reported in the modern literature. In this paper we show another analog to the Viète’s formula for pi by Chebyshev polynomials of the first kind.

Keywords: Chebyshev polynomials, sinc function, cosine infinite product, Viète’s formula, constant pi

1 Introduction

The sinc function, also known as the cardinal sine function, is defined as

\[
sinc(t) = \begin{cases} 
\frac{\sin(t)}{t}, & t \neq 0 \\
1, & t = 0.
\end{cases}
\]

The sinc function finds many applications in sampling, spectral methods, differential equations and numerical integration [1, 2, 3, 4, 5, 6, 7, 8].

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More than four centuries ago the French lawyer and amateur mathematician François Viète found a fabulous relation showing how the sinc function can be represented elegantly as an infinite product of the cosines \[ \prod_{m=1}^{\infty} \cos \left( \frac{t}{2^m} \right). \] (1)

Since \[ \text{sinc} \left( \frac{\pi}{2} \right) = \frac{2}{\pi} \] we may attempt to substitute the argument \( t = \frac{\pi}{2} \) into right side of equation (1). Thus, using repeatedly for each \( m \) the following cosine identity for double angle \[ \cos (2\theta_m) = 2\cos^2 (\theta_m) - 1 \] or \[ \cos (\theta_m) = 2\cos^2 (\theta_{m+1}) - 1 \iff \cos (\theta_{m+1}) = \sqrt{\frac{\cos (\theta_m) + 1}{2}}, \]
where \( \theta_m = \frac{\pi}{2^{2^m}}, \, \theta_{m+1} = \frac{\theta_m}{2} \)
and taking into account that \[ \cos (\theta_1) = \cos \left( \frac{\pi}{2^{2^1}} \right) = \frac{\sqrt{2}}{2}, \]
we can find the following sequence

\[ \cos \left( \frac{\pi}{2^{2^2}} \right) = \frac{\sqrt{2} + \sqrt{2}}{2}, \]
\[ \cos \left( \frac{\pi}{2^{2^3}} \right) = \frac{\sqrt{2 + \sqrt{2}}}{2}, \]
\[ \vdots \]
\[ \cos \left( \frac{\pi}{2^{2^m}} \right) = \frac{\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}{2}. \] (2)
Therefore, from the equations (1) and (2) we obtain the Vi`ete’s infinite product formula for the constant pi (in radicals consisting of square roots and twos only) \[9, 10, 11, 12, 13\]

\[
sinc\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2^{2^1}}\right) \cos\left(\frac{\pi}{2^{2^2}}\right) \cos\left(\frac{\pi}{2^{2^3}}\right) \cdots
\]

\[
= \frac{\sqrt{2}}{2} \frac{\sqrt{2 + \sqrt{2}}}{2} \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots = \frac{2}{\pi}
\]

that can be conveniently rewritten as

\[
\frac{2}{\pi} = \lim_{M \to \infty} M \prod_{m=1}^{M} \frac{a_m}{2},
\]

where \(a_m = \sqrt{2 + a_{m-1}}\) and \(a_1 = \sqrt{2}\).

Several reformulations of the Vi`ete’s formula (3) for pi have been reported in the modern literature \[9, 10, 11, 12, 13\]. Notably, Osler has shown by “double product” generalization a direct relationship between the classical Vi`ete’s and Wallis’s infinite products for pi (see equation (3) in [9]). In this work we derive another equivalent to the Vi`ete’s formula for pi expressed in terms of the Chebyshev polynomials of the first kind.

### 2 Derivation

The Chebyshev polynomials \(T_m(x)\) of the first kind can be defined by the following recurrence relation [14, 15, 16]

\[
T_m(x) = 2xT_{m-1}(x) - T_{m-2}(x),
\]

where \(T_1(x) = x\) and \(T_0(x) = 1\). It should be noted that the recurrence procedure is not required in computation since these polynomials can also be determined directly by using, for example, a simple identity

\[
T_m(x) = x^{\frac{m}{2}} \sum_{n=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \binom{m}{2n} (1 - x^{-2})^n.
\]

Due to a remarkable property of the Chebyshev polynomials

\[
\cos(m \alpha) = T_m(\cos(\alpha)),
\]
making change of the variable in form

$$\alpha = \cos \left( \frac{t}{2M} \right)$$

results in

$$\cos \left( \frac{2m - 1}{2M} t \right) = T_{2m-1} \left( \cos \left( \frac{t}{2M} \right) \right). \quad (4)$$

Consequently, substituting equation (4) into the following product-to-sum identity [6, 7, 8]

$$\prod_{m=1}^{M} \cos \left( \frac{t}{2m} \right) = \frac{1}{2^{M-1}} \sum_{m=1}^{2M-1} \cos \left( \frac{2m - 1}{2M} t \right) \quad (5)$$

yields

$$\prod_{m=1}^{M} \cos \left( \frac{t}{2m} \right) = \frac{1}{2^{M-1}} \sum_{m=1}^{2M-1} T_{2m-1} \left( \cos \left( \frac{t}{2M} \right) \right). \quad (6)$$

It can be shown that the right side of the equation (6) can be further simplified and represented by a single Chebyshev polynomial of the second kind (see Appendix A).

Comparing equations (1) and (5) we can see that the infinite product of cosines for the sinc function can be transformed into infinite sum of cosines [7]

$$\text{sinc} \left( t \right) = \lim_{M \to \infty} \frac{1}{2^{M-1}} \sum_{m=1}^{2M-1} \cos \left( \frac{2m - 1}{2M} t \right). \quad (7)$$

Since the right side of equation (5) represents a truncation of the limit (7) by a finite value of upper integer $2^{M-1}$ in summation, it is simply the incomplete cosine expansion of the sinc function. Indeed, if the condition $2^{M-1} >> 1$ is satisfied, then the incomplete cosine expansion of the sinc function quite accurately approximates the original sinc function as given by [7, 8]

$$\frac{1}{2^{M-1}} \sum_{m=1}^{2M-1} \cos \left( \frac{2m - 1}{2M} t \right) \approx \text{sinc} \left( t \right).$$

It is interesting to note that comparing equations (1) and (6) we can also write now

$$\text{sinc} \left( t \right) = \lim_{M \to \infty} \frac{1}{2^{M-1}} \sum_{m=1}^{2M-1} T_{2m-1} \cos \left( \frac{t}{2M} \right) = \lim_{M \to \infty} \frac{1}{2^{M-1}} \sum_{m=1}^{2M-1} T_{2m-1} \cos \left( \frac{t/2}{2M-1} \right).$$
or

\[
sinc (t) = \lim_{L \to \infty} \frac{1}{L} \sum_{\ell=1}^{L} T_{2\ell-1} \cos \left( \frac{t}{2L} \right),
\]

since we can imply that \( L = 2^{M-1} \).

At \( t = \pi/2 \) the equation (6) provides

\[
\prod_{m=1}^{M} \cos \left( \frac{\pi/2}{2^m} \right) = \frac{1}{2^{M-1}} \sum_{m=1}^{2^{M-1}} T_{2m-1} \left( \cos \left( \frac{\pi/2}{2^m} \right) \right).
\]  

(8)

Applying equation (2) again for each \( m \) repeatedly, the product-to-sum identity (8) can be rearranged in form

\[
\sqrt{2} \sqrt{2 + \sqrt{2}} \sqrt{2 + \sqrt{2 + \sqrt{2}}} \cdots \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} = 1
\]

or

\[
\prod_{m=1}^{M} \frac{\alpha_m}{2} = \frac{1}{2^{M-1}} \sum_{m=1}^{2^{M-1}} T_{2m-1} \left( \frac{\alpha_M}{2} \right).
\]  

(9)

Increase of the integer \( M \) approximates the product of cosines on the left side of equation (9) closer to the value \( 2/\pi \). This signifies that the right side of the equation (9) also tends to \( 2/\pi \) as the integer \( M \) increases. Consequently, this leads to

\[
\frac{2}{\pi} = \lim_{M \to \infty} \frac{1}{2^{M-1}} \sum_{m=1}^{2^{M-1}} T_{2m-1} \left( \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}} \right)}
\]
or
\[
\frac{2}{\pi} = \lim_{M \to \infty} \frac{1}{2^{M-1}} \sum_{m=1}^{2M-1} T_{2m-1}\left(\frac{a_M}{2}\right). \tag{10}
\]

The equation (10) is completely identical to the Viète infinite product for the constant pi. Since the relation (8) remains valid for any integer $M$, the equation (10) can be regarded as a product-to-sum transformation of the Viète’s formula for pi.

It should be noted that the equation (10) can be readily rearranged as a single Chebyshev polynomial of the second kind (see Appendix B).

3 Conclusion

We show a new analog to the Viète’s formula for pi represented in terms of the Chebyshev polynomials of the first kind. This approach is based on a product-to-sum transformation of the Viète’s formula.

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Appendix A

The Chebyshev polynomials $U_m(x)$ of the second kind can also be defined by the recurrence relation. Specifically, we can write [16]

\[
U_m(x) = 2x U_{m-1}(x) - U_{m-2}(x),
\]

where $U_1(x) = 2x$ and $U_0(x) = 1$.

There is a simple relation for sum of the odd Chebyshev polynomials of the first kind

\[
U_K(x) = 2 \sum_{k \text{ odd}}^{K} T_k(x), \tag{A.1}
\]
where $K$ is an odd integer. Consequently, substituting equation (6) into relation (A.1) provides:

$$
\prod_{m=1}^{M} \cos \left( \frac{t}{2^m} \right) = \frac{1}{2^M} U_{2M-1} \left( \cos \left( \frac{t}{2^M} \right) \right).
$$

According to equation (1) tending $M$ to infinity leads to the limit

$$
sinc (t) = \lim_{M \to \infty} \frac{1}{2^M} U_{2M-1} \left( \cos \left( \frac{t}{2^M} \right) \right)
$$
or

$$
sinc (t) = \lim_{N \to \infty} \frac{1}{N} U_{N-1} \left( \cos \left( \frac{t}{N} \right) \right), \quad (A.2)
$$
since we can imply that $2^M = N$. Obviously at $N >> 1$, one can truncate equation (A.2) to approximate the sinc function by a single Chebyshev polynomial of the second kind as

$$
sinc (t) = \frac{1}{N} U_{N-1} \left( \cos \left( \frac{t}{N} \right) \right) + \epsilon (t),
$$

where $\epsilon (t)$ is the error term. For example, taking $N = 16$ results in

$$
sinc (t) = 2048 \cos^{15} \left( \frac{t}{16} \right) - 7168 \cos^{13} \left( \frac{t}{16} \right) + 9984 \cos^{11} \left( \frac{t}{16} \right)
- 7040 \cos^9 \left( \frac{t}{16} \right) + 2640 \cos^7 \left( \frac{t}{16} \right) - 504 \cos^5 \left( \frac{t}{16} \right)
+ 42 \cos^3 \left( \frac{t}{16} \right) - \cos \left( \frac{t}{16} \right) + \epsilon (t),
$$

where within the range $-10 \leq t \leq 10$ the error term satisfies $|\epsilon (t)| < 0.006$. As we can see, this approach quite accurately approximates the sinc function even if the integer $N$ in the limit (A.2) is not very large.

\footnote{The subscript $2^M - 1$ should not be confused with notation $2^{M-1}$ that has been used in some equations earlier.}
Appendix B

Substituting equation (10) into relation (A.1) we can express the Viéte’s formula for \( \pi \) by a single Chebyshev polynomial of the second kind as given by

\[
\frac{2}{\pi} = \lim_{M \to \infty} \frac{1}{2^M} U_{2M-1} \left( \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}/2 \right)
\]

or

\[
\frac{2}{\pi} = \lim_{M \to \infty} \frac{1}{2^M} U_{2M-1} \left( \frac{a_M}{2} \right).
\]  \hspace{1cm} (B.1)

Although the equation (B.1) is more simple, the equation (10) reflects explicitly the product-to-sum transformation of the Viéte’s formula (3) for the constant \( \pi \).

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