Generalized symmetries generating Noether currents
and canonical conserved quantities

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Abstract. We determine the condition for a Noether–Bessel-Hagen current, associated with
a generalized symmetry, to be variationally equivalent to a Noether current for an invariant
Lagrangian. We show that, if it exists, this Noether current is exact on-shell and generates a
canonical conserved quantity.

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1. Introduction
As it is well known, invariance properties of field dynamics are an effective tool to understand
a physical system without solving the equations themselves: the existence of conservation
laws associated with symmetries of equations strongly simplifies their study and corresponding
conserved currents along solutions (on-shell) appear to be the most significant for the description
of a system.

The Noether Theorems [15] stated that, for the description of systems the equations of which
arise from a variational problem, off-shell conserved currents are even more relevant since, for a
large class of field theories, potentials (also called superpotentials in natural and gauge-natural
theories) of such currents provide conserved quantities such as energy, momenta and charges.
Therefore, it turns out to be fundamental to understand whether conserved currents associated
with invariance of equations could be identified with Noether conserved currents for a certain
Lagrangian; in fact, a symmetry of a Lagrangian is also a symmetry of its Euler-Lagrange
form, but the converse in general is not true. We are interested to this converse problem which
belongs to aspects of inverse problems the calculus of variations. The expression for the so-
called canonical Noether current associated with a Lagrangian is the outcome of an integration
by parts, related with the precise form taken by the Lie derivative of a Lagrangian (i.e. the
Lie derivative of an horizontal n-form, n being the dimension of the base manifold). Of course,
the particular type of inverse problem we shall consider will strictly interplay with the formula
for a variational Lie derivative; analogously to what happens in the study of variationality of
equations, the solution to this problem involves a kind of variational integrating factor.

Consider then conserved currents associated with invariance properties of (locally) variational
global field equations, i.e. with so-called generalized or Bessel-Hagen symmetries [1]. Noether
currents for different local Lagrangian presentations and corresponding conserved currents
associated with each local presentation have been characterized in [3, 4, 5, 17]. There exist...
cohomological obstructions for such local currents be globalized and such obstructions are also related with the existence of global solutions for a given global field equation [6].

We refer to the geometric formulation of the calculus of variations as a subsequence of the de Rham sequence of differential forms on finite order prolongations of fibered manifolds. We assume the r-th order prolongation of a fibered manifold \( \pi : Y \to X \), with \( \dim X = n \) and \( \dim Y = n + m \), to be the configuration space; this means that fields are assumed to be (local) sections of \( \pi^r : J_r Y \to Y \). Due to the affine bundle structure of \( \pi^r : J_r Y \to Y \), we have a natural splitting \( J_r Y \times_{J_{r-1}Y} T^* J_{r-1}Y = J_r Y \times_{J_{r-1}Y} (T^* X \oplus V^* J_{r-1}Y) \), which induces natural splittings in horizontal and vertical parts of vector fields, forms and of the exterior differential on \( J_r Y \). Let \( \rho \) be a q-form on \( J_r Y \); in particular we obtain a natural decomposition of the pullback by the affine projections of a q-form \( \rho \), as \( (\pi^r)^* \rho = \sum_{i=0}^q p_i \rho \), where \( p_i \rho \) is the i-contact component of \( \rho \) (by definition a contact form is zero along any holonomic section of \( J_r Y \)). For \( q \leq n \), the representation mapping is just given by the horizontalization \( \rho \rho = h \rho \). For \( q > n \), say, \( q = n + k \), it is clear that, in this case, \( p_k \rho \) denotes the component of \( \rho \) with the lowest degree of contactness. Starting from this splitting one can define sheaves of contact forms \( \Theta_n^r \), suitably characterized by the kernel of \( p_i \) [9]; the sheaves \( \Theta_n^r \) form an exact subsequence of the de Rham sequence on \( J_r Y \) and one can define the quotient sequence

\[
0 \to \mathcal{R}_Y \to \cdots \to \mathcal{E}_{n-1,1} \left\langle \Theta_{r}^{n}/\Theta_{r}^{n+1} \right\rangle \mathcal{E}_{n+1,1}/\Theta_{r}^{n+2} \to \mathcal{E}_{n+2} \to 0
\]

the \( r \)-th order variational sequence on \( Y \to X \) which is an acyclic resolution of the constant sheaf \( \mathcal{R}_Y \); see [9].

In the following, if \( \rho \in \Lambda^n \), \( [\rho] \in \mathcal{V}_q^r \) denotes the equivalence class of \( \rho \) modulo \( \ker p_i \), \( i = 0, 1, \ldots, m, q = n + i \). The quotient sheaves \( \mathcal{V}_{k}^r \equiv \Lambda^k/\Theta^k_i \) in the variational sequence can be represented as sheaves of k-forms on jet spaces of higher order by the interior Euler operator which is uniquely intrinsically defined by the decomposition \( p_k \rho = I(\rho) + p_k dp_k \mathcal{R}(\rho) \), together with the properties

\[
(\pi^r)^* \rho - I(\rho) \in \Theta_{2r+1}^{n+1}\left\langle \Theta_{2r+1}^{n+1} \right\rangle; \quad I(p_k dp_k \mathcal{R}(\rho)) = 0; \quad \mathcal{I}^2(\rho) = (\pi^{2r+1})^* I(\rho); \quad \ker \mathcal{I} = \Theta_{2r+1}^{n+1};
\]

one can define a representation mapping \( R_q : \mathcal{V}_q^{2r+1} \to \Lambda^{2r+1}_n \); \( [\rho] \to R_q([\rho]) \), with \( R_q([\rho]) = p_0 \rho \equiv h \rho \)

for \( 0 \leq q \leq n, s = r + 1 \); \( R_q([\rho]) = I(\rho) \) for \( n + 1 \leq q \leq P, s = 2r + 1 \); \( R_q([\rho]) = \rho \) for \( P + 1 \leq q \leq N, s = r \), where \( N = \dim J_r Y \) and \( P \) is the maximal degree of non trivial contact forms on \( J_r Y \) (see e.g. [7, 8, 9, 12, 18], whereby also local coordinate expressions can be found). The representation sequence \( 0 \to R_s(\mathcal{V}_s^n), \mathcal{E}_s \), is also exact and we have \( E_q \circ R_q([\rho]) = R_{q+1} \circ E_q([\rho]) = R_{q+1}([d \rho]) \). Currents are sheaf sections \( \epsilon \) of \( \mathcal{V}_s^{n+1} \) and \( \mathcal{E}_{n+1} = d_H \) is the total divergence; Lagrangians are sections \( \lambda \) of \( \mathcal{V}_s^n \), while \( \mathcal{E}_n \) is called the Euler-Lagrange morphism; Sections \( \eta \) of \( \mathcal{V}_s^{n+1} \) are called source forms or also dynamical forms, \( \mathcal{E}_{n+1} \) is called the Helmholtz morphism.

2. Currents associated with (locally) variational dynamical forms

In the following for the sake of generality, we will consider locally variational dynamical forms. We will denote by a subscript \( i \) the fact that in general a sheaf section is defined locally. A variational Lie derivative operator \( \mathcal{L}_{j_r \Xi} \) is well defined acting on the sections of sheaves in the variational sequence: it can be characterized as a quotient Cartan formula (which in fact provides a variation formula); the basic idea is to factorize modulo contact structures [10, 11, 12, 13]. This enable us to define symmetries of classes of forms of any degree in the variational sequence and corresponding conservation theorems; see [2] and [14] for details. We shall study the interplay of such an operator with the representation by forms. Let \( j_r \Xi \) be a projectable vector field on \( J_r Y, \rho \) a q-form defined (locally) on \( J_r Y \).
We have a commutative diagram defined by \( \hat{R}_q (L_j \varepsilon [\rho]) = \hat{R}_q (L_j \varepsilon [\rho]) = L_j \varepsilon [\rho] \), where \( \hat{R}_q (L_j \varepsilon [\rho]) = L_j \varepsilon [\rho] \), \( 0 \leq q \leq n, s = r + 1 \); \( \hat{R}_q (L_j \varepsilon [\rho]) = L_j \varepsilon [\rho] \), \( n + 1 \leq q \leq P, s = 2r + 1 \); \( \hat{R}_q (L_j \varepsilon [\rho]) = L_j \varepsilon [\rho], P + 1 \leq N, s = r \). This enables us to deal with ordinary Lie derivatives of forms on \( \Lambda^n \), then apply the Cartan formula for differential forms, therefore return back to the classes of forms to obtain the following variational Cartan formulae [14].

**Proposition 1** Let \( 0 \leq q \leq n \). Let \( j_s \varepsilon \) be a projectable vector field on \( J_s \mathbf{Y} \). We have
\[
L_{j_s} \varepsilon [\rho] = \varepsilon [\rho] + \varepsilon [\rho] d_H (j_s \varepsilon V) p_{dV} [\rho] + \varepsilon [\rho] d_H \varepsilon V + \varepsilon [\rho] d_H \varepsilon V [\rho] .
\]

Note that, for \( q = n - 1 \), the formula above defines a ‘momentum’ associated with a current, we shall denote such a momentum by \( \hat{p} \); it is clear that, for \( q < n \), \( \varepsilon = d_H \).

**Proposition 2** Let \( q = n + k \), with \( k \geq 1 \). Let \( j_s \varepsilon \) be a projectable vector field on \( J_s \mathbf{Y} \). We have
\[
L_{j_s} \varepsilon [\rho] = j_s \varepsilon [\rho] + \varepsilon [\rho] d_H (j_s - 1 \varepsilon V) p_{dV} [\rho] + \varepsilon [\rho] d_H \varepsilon V [\rho] .
\]

Let for simplicity \( \eta_{\lambda_i} \) denote a global Euler–Lagrange class of forms for a (local) variational problem represented by (local) sheaf sections \( \lambda_i \).

**Proposition 1** (Noether Theorem I) reads \( L_{j_s} \varepsilon \lambda_i = \varepsilon [\rho] d_H \eta_i \), where \( \varepsilon_i = j_s \varepsilon V p_{dV} \lambda_i + \varepsilon [\rho] \lambda_i \) is the Noether current associated with it.

**Definition 1** A generalized symmetry of a (locally variational) dynamical form \( \eta_{\lambda_i} \) is a projectable vector field \( j_s \varepsilon \) on \( J_s \mathbf{Y} \) such that \( L_{j_s} \varepsilon \eta_{\lambda_i} = 0 \).

Since we assume \( \eta_{\lambda_i} \) to be closed, Proposition 2 reduces (case \( q = n + 1 \)) to \( L_{j_s} \varepsilon \eta_{\lambda_i} = \varepsilon [\rho] \lambda_i \), and if \( j_s \varepsilon \) is such that \( L_{j_s} \varepsilon \lambda_i = 0 \), then \( \varepsilon [\rho] \lambda_i = 0 \); therefore, locally we have \( \varepsilon [\rho] \lambda_i = d_H \eta_i \). Notice that, although \( \varepsilon [\rho] \lambda_i \) is global, in general it defines a non trivial cohomology class \( [3] \); it is clear that \( \eta_i \) is a (local) current which is conserved on-shell (i.e. along critical sections). On the other hand, and independently (see [15]), we get locally \( \varepsilon \lambda_i = d_H \beta_i \), thus we can write \( \varepsilon [\rho] \lambda_i + d_H (\varepsilon_i - \beta_i) = 0 \), where \( \varepsilon_i \) is the usual canonical Noether current.

**Definition 2** We call the (local) current \( \varepsilon_i - \beta_i \) a Noether–Bessel–Hagen current.

A Noether–Bessel–Hagen current \( \varepsilon_i - \beta_i \) is a current associated with a generalized symmetry (conserved along critical sections); in [5, 6] we proved that a Noether–Bessel–Hagen current is variationally equivalent to a global (conserved) current if and only if \( 0 = [\varepsilon [\rho] (\lambda_i)] \in H^n_{dr}(\mathbf{Y}) \).

### 3. Generalized symmetries generating Noether currents

In the following we investigate under which conditions a Noether–Bessel–Hagen current is variationally equivalent to a Noether conserved current for a suitable Lagrangian. We shall see that this is involved with the existence of a variationally trivial local Lagrangian \( d_H \mu_i \), and with a condition on the current associated with it.

**Proposition 3** A Noether–Bessel–Hagen current \( \varepsilon_i - \beta_i \) associated with a generalized symmetry of \( \eta_{\lambda_i} \) is a Noether conserved current if and only if of the form \( \varepsilon_i - L_{j_s} \varepsilon \mu_i \), with \( \mu_i \) a current satisfying \( L_{j_s} \varepsilon \lambda_i - d_H \mu_i = 0 \).

**Proof.** From \( L_{j_s} \varepsilon \lambda_i = 0 \), we get \( L_{j_s} \varepsilon \lambda_i = d_H \beta_i \). It is easy to see that the current \( \varepsilon_i - \beta_i \) is a Noether conserved current if and only if there exists \( \mu_i \) such that \( \beta_i = L_{j_s} \varepsilon \mu_i \), i.e. \( d_H \beta_i = d_H (j_{s-1} \varepsilon V) p_{dV} d_H \mu_i + \varepsilon [\rho] d_H \mu_i \). On the other hand \( d_H \beta_i = d_H (\varepsilon [\rho] d_H \mu_i) \). Comparing the two expression we get \( d_H (j_{s-1} \varepsilon V) p_{dV} d_H \mu_i \equiv 0 \). This identity is a consequence of the fact that \( L_{j_s} \varepsilon \lambda_i \) commutes with \( d_H \); note that it can be obtained also by the fact that \( d_H (j_{s-1} \varepsilon V) p_{dV} d_H \mu_i \.) (see [7]).
Proposition 4 The Noether current $\epsilon_{\lambda_i - d_H \mu_i}$ is exact on-shell and it is equal to $d_H (j_{s-1} \bar{\Xi}_V | \bar{p}_{dV} \mu_i + \Xi_H | \mu_i)$. 

Proof. As it is well known, along any section pulling back to zero $\Xi_V | \eta_{\lambda_i}$ we get the on-shell conservation law $d_H (\epsilon_{\lambda_i - \beta_i} = 0$ If there exists a current $\mu_i$ such that $\beta_i = \mathcal{L}_{\lambda_i} \Xi_{H} = d_H \mu_i + d_H (j_{s-1} \bar{\Xi}_V | \bar{p}_{dV} \mu_i + \Xi_H | \mu_i)$, then $j_{s-1} \bar{\Xi}_V | \bar{p}_{dV} \mu_i + \Xi_H | \mu_i$ is closed on-shell. By an uniqueness argument, we see that the latter expression must be equal to $j_{s-1} \bar{\Xi}_V | \bar{p}_{dV} \mu_i; \therefore d_H (j_{s-1} \bar{\Xi}_V | \bar{p}_{dV} \mu_i + \Xi_H | \mu_i) = \epsilon_{\lambda_i - d_H \mu_i}$ on-shell.

Remark 1 It turns out that, on-shell, a canonical potential of the Noether current $\epsilon_{\lambda_i - d_H \mu_i}$, then a corresponding conserved quantity, is defined.

Remark 2 An off-shell exact Noether current associated with the invariance of $\lambda_i - d_H \mu_i$ would be generated by a generalized symmetry $j_{s-1}$ such that $\Xi_V | \eta_{\lambda_i} = 0$; the corresponding cohomology class would be, therefore, trivial, (see the discussion in [5, 6]).

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