ALGEBRAICITY OF CRITICAL VALUES OF ADJOINT L-FUNCTIONS FOR GSp$_4$

SHH-YU CHEN

Abstract. We prove an algebraicity result for certain critical value of adjoint $L$-functions for GSp$_4$ over a totally real number field in terms of the Petersson norm of normalized generic cuspidal newforms on GSp$_4$. This is a generalization of our previous result [CI19].

1. Introduction

1.1. Main result. Let $f$ be a normalized elliptic newform of weight $\kappa \geq 2$ and level $\Gamma_1(N)$. Denote by $L(s,f,\text{Ad})$ the completed adjoint $L$-function of $f$. By the result of Sturm [Stu89], the algebraicity of $L(1,f,\text{Ad})$ is expressed in terms of the Petersson norm

$$\|f\| = \int_{\Gamma_0(N) \backslash \mathcal{H}} |f(\tau)|^2 \text{Im}(\tau)^{\kappa-2} \, d\tau.$$ 

More precisely, we have

$$\sigma \left( \frac{L(1,f,\text{Ad})}{\|f\|} \right) = \frac{L(1,\sigma f,\text{Ad})}{\|\sigma f\|}$$

for all $\sigma \in \text{Aut}(\mathbb{C})$, as predicted by Deligne’s conjecture [Del79]. The purpose of this paper is to prove an analogue of it for GSp$_4$.

We give a description of our main result. Let $\Pi = \bigotimes_v \Pi_v$ be an irreducible globally generic cuspidal automorphic representation of GSp$_4(\mathbb{A}_F)$ with central character $\omega_\Pi$ over a totally real number field $F$. Denote by

$$L(s,\Pi,\text{Ad}) = \prod_v L(s,\Pi_v,\text{Ad})$$

the adjoint $L$-function of $\Pi$, where $v$ runs through the places of $F$. Assume $\Pi_{\infty} = \bigotimes_v |\infty \Pi_v$ is a discrete series representation of GSp$_4(\mathbb{F}_\infty)$. Then

$$\Pi_{\infty}|_{\text{Sp}_4(\mathbb{F}_\infty)} = \bigoplus_{v\in \{\infty\}} D_{(\lambda_1,v,\lambda_2,v)} \oplus D_{(-\lambda_2,v,-\lambda_1,v)},$$

where $D_{(\lambda_1,v,\lambda_2,v)}$ is the discrete series representation of Sp$_4(\mathbb{F}_v) \simeq$ Sp$_4(\mathbb{R})$ with Blattner parameter $(\lambda_1,v,\lambda_2,v)$ such that $2 - \lambda_1,v \leq \lambda_2,v \leq -1$ for each real place $v$. Here we follow [Mor04] for the choice of the Cartan subalgebra in sp$_4(\mathbb{R})$ and the positive systems. Let $f = \bigotimes_v f_v \in \Pi$ be a non-zero cusp form satisfying the following conditions:

- $f_v$ is a paramodular newform of $\Pi_v$ for all finite places $v$;
- $f_v$ is a lowest weight vector of the minimal U(2)-type of $D_{(-\lambda_2,v,-\lambda_1,v)}$ for all real places $v$.

These conditions characterize $f \in \Pi$ up to scalars. Let $W_f$ be the Whittaker function of $f$ with respect to $\psi_U$ defined by

$$W_f(g) = \int_{U(F) \backslash U(\mathbb{A}_F)} f(ug)\overline{\psi_U(u)} \, du^{\text{Tam}}.$$ 

Here $U$ is the standard maximal unipotent subgroup of GSp$_4$, $\psi_U$ is the standard non-degenerate character of $U(F) \backslash U(\mathbb{A}_F)$ (see §1.3 for precise definition), and $du^{\text{Tam}}$ is the Tamagawa measure on $U(\mathbb{A}_F)$. We may decompose $W_f = \prod_v W_v$ as a product of local Whittaker functions of $\Pi_v$ with respect to $\psi_{U,v}$. We normalize $f$ as follows:

- for finite place $v$, we have

$$W_v(\text{diag}(\varpi_v^{-3\tau_v}, \varpi_v^{-2\tau_v},1,\varpi_v^{-\tau_v})) = 1,$$

where $\varpi_v$ is a uniformizer of $\mathbb{F}_v$ and $\tau_v$ is the valuation of the different ideal of $\mathbb{F}_v$;
Here dg \text{ in a similar way. Let } \| \Pi_1.2. F L \text{ conjecture for } \sigma \text{ for all real places}

We write \( f_L \) and call it the normalized newform of \( \Pi \). We defined the normalized newform \( f_{\Pi^\vee} \) of \( \Pi^\vee \) in a similar way. Let \( \| f_{\Pi} \| \) be the Petersson norm of \( f_{\Pi} \) defined by

\[
\| f_{\Pi} \| = \int_{GSp_4(\mathbb{F}) \backslash GSp_4(\mathbb{A}_F)} f_{\Pi}(g) f_{\Pi^\vee}(g \cdot \text{diag}(-1, -1, 1, 1) \infty) \, dg_{\text{Tam}}.
\]

Here \( dg_{\text{Tam}} \) is the Tamagawa measure on \( \mathbb{A}_F^\times \backslash GSp_4(\mathbb{A}_F) \).

For \( \sigma \in \text{Aut}(\mathbb{C}) \), let \( ^\sigma \Pi \) be the irreducible admissible representation of \( GSp_4(\mathbb{A}_F) \) defined by

\[
^\sigma \Pi = ^\sigma \Pi_\infty \otimes ^\sigma \Pi_f,
\]

where \( ^\sigma \Pi_f \) is the \( \sigma \)-conjugate of \( \Pi_f = \bigotimes_{\mathfrak{p} \nmid \infty} \Pi_{\mathfrak{p}} \) and \( ^\sigma \Pi_\infty \) is the representation of \( GSp_4(\mathbb{F}_\infty) \) so that its \( \nu \)-component is equal to \( \Pi_{\sigma^{-1} \nu} \). Assume further that \( \Pi \) is motivic, that is, there exists \( \omega \in \mathbb{Z} \) such that \( |\omega_\Pi| = |\omega|_{\mathbb{A}_F} \) and

\[
\lambda_{1, \nu} - \lambda_{2, \nu} \equiv \omega (\text{mod } 2)
\]

for all real places \( \nu \). In Lemma \cite{CI19} below, we show that \( ^\sigma \Pi \) is cuspidal automorphic and globally generic. The rationality field \( \mathbb{Q}(\Pi) \) of \( \Pi \) is the fixed field of \( \{ \sigma \in \text{Aut}(\mathbb{C}) | ^\sigma \Pi = \Pi \} \) and is a number field.

The following theorem is our main result on the algebraicity of the critical adjoint \( L \)-value \( L(1, \Pi, \text{Ad}) \) in terms of the Petersson norm of the normalized newform of \( \Pi \).

**Theorem 1.1.** Let \( \Pi \) be an irreducible motivic globally generic cuspidal automorphic representation of \( GSp_4(\mathbb{A}_F) \). For \( \sigma \in \text{Aut}(\mathbb{C}) \), we have

\[
\sigma \left( \frac{L(1, \Pi, \text{Ad})}{\zeta_{\mathbb{F}}(2) \zeta_{\mathbb{F}}(4) \cdot \| f_{\Pi} \|} \right) = \frac{L(1, ^\sigma \Pi, \text{Ad})}{\zeta_{\mathbb{F}}(2) \zeta_{\mathbb{F}}(4) \cdot \| f_{^\sigma \Pi} \|}.
\]

Here \( \zeta_{\mathbb{F}}(s) \) is the completed Dedekind zeta function of \( \mathbb{F} \). In particular, we have

\[
\frac{L(1, \Pi, \text{Ad})}{\zeta_{\mathbb{F}}(2) \zeta_{\mathbb{F}}(4) \cdot \| f_{\Pi} \|} \in \mathbb{Q}(\Pi).
\]

**Remark 1.2.** In \cite{CI19}, we compute the ratio explicitly when \( \omega_\Pi \) is trivial and the paramodular conductor of \( \Pi \) is square-free. The theorem can be regarded as a generalization of \cite{CI19}.

**Remark 1.3.** The Petersson norm \( \| f_{\Pi} \| \) can be factorized into product of periods which are obtained by comparing the rational structures via the Whittaker model and via the coherent cohomology (cf. \cite{HK92}). We expect these periods to capture the transcendental part of the critical values of certain automorphic \( L \)-functions. Moreover, the expected period relation implies that Theorem \cite{CI19} is compatible with Deligne’s conjecture for \( L(1, \Pi, \text{Ad}) \). This is an ongoing project considered by the author.

### 1.2. An outline of the proof.

The first step is to show that for any non-trivial additive character \( \psi \) of \( \mathbb{F} \backslash \mathbb{A}_F \), we have

\[
\| f_{\Pi} \| = C \cdot \frac{L(1, \Pi, \text{Ad})}{\zeta_{\mathbb{F}}(2) \zeta_{\mathbb{F}}(4)} \prod_v C_{\psi_v}(\Pi_v)
\]

for some non-zero constant \( C_{\psi_v}(\Pi_v) \) depending only on \( \Pi_v \) and \( \psi_v \), and some constant \( C \in \mathbb{Q}_\times \) depending only on \( \mathbb{F} \) and the type of \( \Pi \) (stable or endoscopic). This equality can be proved by proceeding exactly as in the proof of \cite{CI19} Proposition 5.4 (see also \cite{CI19} §1.2 for brief outline), subject to the well-definedness of \( C_{\psi_v}(\Pi_v) \). We have

\[
C_{\psi_v}(\Pi_v) = \frac{1}{L(1, \Pi_v, \text{Ad})} \frac{Z_v(1, \Pi_v, F_{\psi_v})}{Z_v(\frac{1}{2}, \Pi_v, F_{\psi_v})},
\]

where \( F_{\psi_v} \) and \( F_{\psi_v} \) are sections in degenerate principal series representations of \( GSp_4(\mathbb{F}_v) \), and \( Z_v(s, \Pi_v, F_{\psi_v}) \) and \( Z_v(s, \Pi_v, F_{\psi_v}) \) are local zeta integral for \( \Pi_v \times \Pi_v^\vee \) and doubling local zeta integral for \( \Pi_v \) defined and
studied by Jiang [Jia96] and Piatetski-Shapiro and Rallis [PSR87], respectively. We prove that the constant
is finite, let $c_{\psi}$ be the exponent of $\psi$ on $F$. Here $\Gamma(s)$ is the gamma function.

(1.3) $$\sigma \left( \prod_{\nu\mid \infty} C_{\psi, v}(\Pi_{c}) \right) = \prod_{\nu\mid \infty} C_{\psi, v}(\sigma \Pi_{c})$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. First we show that for all $\nu \mid \infty$ and $\sigma \in \text{Aut}(\mathbb{C})$, we have

(1.4) $$\sigma Z_{v}(\frac{1}{2}, \Pi_{c}, F_{\psi}) = Z_{v}(\frac{1}{2}, \sigma \Pi_{c}, F_{\psi}) \quad \sigma Z_{v}(1, \Pi_{c}, F_{\psi}) = Z_{v}(1, \sigma \Pi_{c}, F_{\psi})$$

which imply that

(1.5) $$\sigma C_{\psi, v}(\Pi_{c}) = C_{\psi, v}(\sigma \Pi_{c})$$

For $\sigma \in \text{Aut}(\mathbb{C})$, in general $\bigotimes_{\nu \mid \infty} \psi_{v}$ is not the finite part of a non-trivial additive character of $\mathbb{F}\setminus \mathbb{A}_{F}$. Nonetheless, since we have freedom to vary $\psi$ in (1.2), we show that (1.3) holds by a global argument together with (1.5). By (1.2) and (1.3), we then have

(1.6) $$_\sigma \left( \frac{L(1, \Pi, \text{Ad})}{\zeta_{F}(2)\zeta_{F}(4) \cdot \|f_{\Pi}\|_{\psi}} \right) \prod_{\nu\mid \infty} C_{\psi, v}(\Pi_{c}) = \frac{L(1, \sigma \Pi, \text{Ad})}{\zeta_{F}(2)\zeta_{F}(4) \cdot \|f_{\sigma \Pi}\|_{\psi}} \prod_{\nu\mid \infty} C_{\psi, v}(\Pi_{c})$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. Finally, we show that

(1.7) $$\prod_{\nu\mid \infty} C_{\psi, v}(\Pi_{c}) \in \mathbb{Q}^\times$$

This is a local problem, but we address it by a global argument. Based on the Rallis inner product formula [GOT14] and archimedean computations, we prove that Theorem 1.1 holds when $\Pi$ is endoscopic. Choose an endoscopic irreducible globally generic cuspidal automorphic representation $\Pi'$ of $\text{GSp}_{4}(\mathbb{A}_{F})$ such that $\Pi'_{\infty} = \Pi_{\infty}$. We thus obtain (1.7) by comparing Theorem 1.1 with (1.6). (Strictly speaking, we need only to consider endoscopic lifts for $F = \mathbb{Q}$)

This paper is organized as follows. In §2 we recall the definition of the local zeta integrals $Z_{v}$ and $Z_{v}$, and state the precise form of (1.2) in Proposition 2.4. The proposition holds subject to Lemmas 2.1 (1) and 2.3 (1) on the convergence of the local zeta integrals. In §3 we prove (1.3) in Proposition 3.7 subject to Lemmas 2.1 (2) and 2.3 (2) on the Galois equivariant property (1.4) of the local zeta integrals. In §4 we prove in Theorem 4.8 that Theorem 1.1 holds when $\Pi$ is endoscopic and $F = \mathbb{Q}$. The proposition is proved based on the Rallis inner product formula and the arithmeticity of global theta lifting in Proposition 4.5. As we sketched above, Theorem 1.1 follows from Propositions 2.4 3.7 and Theorem 4.8. The context of §5 is purely local and we prove Lemmas 2.1 and 2.3 in this section.

1.3. Notation. Fix a totally real number field $F$. Let $\mathfrak{o}$ and $\mathfrak{O}$ be the ring of integers and the absolute discriminant of $F$, respectively. Let $\mathbb{A} = \mathbb{A}_{F}$ be the ring of adeles of $F$ and $\mathbb{A}_{F}$ be its finite part. We denote by $\phi$ the closure of $\mathfrak{o}$ in $\mathbb{A}_{F}$. For a finite dimensional vector space $V$ over $F$, let $S(V(\mathbb{A}))$ be the space of Schwartz functions on $V(\mathbb{A})$. We will write $v$ for places of $F$ and $\infty$ the archimedean place of $\mathbb{Q}$.

Let $v$ be a place of $F$. If $v$ is a finite place, let $\mathfrak{o}_{v}$, $\mathfrak{w}_{v}$, and $q_{v}$ be the maximal compact subring of $F_{v}$, a generator of the maximal ideal of $\mathfrak{o}_{v}$, and the cardinality of $\mathfrak{o}_{v}/\mathfrak{w}_{v} \mathfrak{o}_{v}$, respectively. Let $|v|$ be the absolute value on $F_{v}$ normalized so that $|\mathfrak{w}_{v}|_{v} = q_{v}^{-1}$. If $v$ is a real place, let $|v| = |v|$ be the ordinary absolute value on $F_{v} \simeq \mathbb{R}$. For a character $\chi$ of $F_{v}^\times$, let $e(\chi) \in \mathbb{R}$ be the exponent of $\chi$ defined so that $|\chi| = |v|^{e(\chi)}$. When $v$ is finite, let $c(\chi) \in \mathbb{Z}_{\geq 0}$ be the smallest integer so that $\chi$ is trivial on $1 + \mathfrak{w}_{v}^{c(\chi)} \mathfrak{o}_{v}$.

Let $\zeta(s) = \zeta_{F}(s) = \prod_{\nu} \zeta_{\nu}(s)$ be the completed Dedekind zeta function of $F$, where $\nu$ ranges over the places of $F$ and

$$\zeta_{\nu}(s) = \begin{cases} (1 - q_{v}^{-s})^{-1} & \text{if } v \text{ is finite}, \\ \pi^{-s/2} \Gamma(s/2) & \text{if } v \text{ is real}. \end{cases}$$

Here $\Gamma(s)$ is the gamma function.
Let $\psi_0 = \bigotimes_v \psi_{v,0}$ be the standard additive character of $\mathbb{Q}\backslash \mathbb{A}_{\mathbb{Q}}$ defined so that
\[
\psi_{p,0}(x) = e^{-2\pi \sqrt{-1} x} \quad \text{for} \quad x \in \mathbb{Z}[p^{-1}],
\]
\[
\psi_{\mathbb{Q},0}(x) = e^{2\pi \sqrt{-1} x} \quad \text{for} \quad x \in \mathbb{R}.
\]
Let $\psi = \bigotimes_v \psi_v$ be a non-trivial additive character of $\mathbb{F}\backslash \mathbb{A}$. We say $\psi$ is standard if $\psi = \psi_0 \circ \text{tr}_{\mathbb{F}/\mathbb{Q}}$. In this case, $\psi_v$ is called the standard additive character of $\mathbb{F}_v$. For $a \in \mathbb{F}^\times$ (resp. $a \in \mathbb{F}_v^\times$), let $\psi^a$ (resp. $\psi_v^a$) be the additive character of $\mathbb{A}$ (resp. $\mathbb{A}_v$) defined by $\psi^a(x) = \psi(ax)$ (resp. $\psi_v^a(x) = \psi_v(ax)$). For each finite place $v$, let $\mathbb{O}_v^c$ be the largest fractional ideal of $\mathbb{F}_v$ on which $\psi_v$ is trivial. The absolute conductor $\text{cond}(\psi)$ of $\psi$ is the $\delta$-submodule of $\mathbb{A}_f$ defined by
\[
\text{cond}(\psi) = \prod_{v | \infty} \mathbb{O}_v^{[\mathbb{Q}(v) : \mathbb{Q}]}.
\]
We write $v | \text{cond}(\psi)$ if $c_v \neq 0$.

If $S$ is a set, then we let $\mathbb{I}_S$ be the characteristic function of $S$. Let $M_{n,m}$ be the matrix algebra of $n$ by $m$ matrices. Let $\text{GSp}_{2n}$ and $\text{Sp}_{2n}$ be the symplectic similitude group and symplectic group, respectively, defined by
\[
\text{GSp}_{2n} = \left\{ g \in \text{GL}_{2n} \left| g \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} g^t = \nu(g) \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \nu(g) \in \text{GL}_1 \right\}, \quad \text{Sp}_{2n} = \ker(\nu).
\]
Let
\[
\mathcal{B} = \left\{ \begin{pmatrix} t_1 & * & * & * \\ 0 & t_2 & * & * \\ 0 & 0 & \nu t_1^{-1} & 0 \\ 0 & 0 & * & \nu t_2^{-1} \end{pmatrix} \right\} \subset \text{GSp}_4 \quad t_1, t_2, \nu \in \text{GL}_1
\]
be the standard Borel subgroup of $\text{GSp}_4$ and $U$ be its unipotent radical. Let $T \subset \mathcal{B}$ be the standard maximal torus of $\text{GSp}_4$. For a non-trivial additive character $\psi$ of $\mathbb{F}\backslash \mathbb{A}$, let $\psi_U$ be the associated additive character of $U(\mathbb{F})\backslash U(\mathbb{A})$ defined by
\[
\psi_U \left( \begin{pmatrix} 1 & x & * & * \\ 0 & 1 & * & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \right) = \psi(-x - y).
\]
We call $\psi_U$ standard if $\psi$ is standard. Similar notation apply to additive character $\psi_v$ of $\mathbb{Q}_v$. In $\text{GL}_2$, let $B$ be the Borel subgroup consisting of upper triangular matrices, and put
\[
a(\nu) = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}, \quad d(\nu) = \begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix}, \quad m(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
for $\nu, t \in \text{GL}_1$ and $x \in \mathbb{G}_a$. Let $v$ be a finite place of $\mathbb{F}$ and $c \in \mathbb{Z}_{\geq 0}$. For $n \geq 2c$, the quasi-paramodular group $K_v((\mathbb{O}_v^c)^n; c)$ of level $(\mathbb{O}_v^c)^n$ is the open compact subgroup of $\text{GSp}_4(\mathbb{F}_v)$ consisting of $g \in \text{GSp}_4(\mathbb{F}_v)$ such that $\nu(g) \in \mathbb{O}_v^c$ and
\[
g \in \left\{ \begin{pmatrix} \mathbb{O}_v & \mathbb{O}_v & \mathbb{O}_v & \mathbb{O}_v \\ \mathbb{O}_v & \mathbb{O}_v & \mathbb{O}_v & \mathbb{O}_v \\ \mathbb{O}_v & \mathbb{O}_v & \mathbb{O}_v & \mathbb{O}_v \\ 1 + \mathbb{O}_v & \mathbb{O}_v & \mathbb{O}_v & \mathbb{O}_v \end{pmatrix} \right\}.
\]

Let $\sigma \in \text{Aut}(\mathbb{C})$. Define the $\sigma$-linear action on $\mathbb{C}(X)$, which is the field of formal Laurent series in variable $X$ over $\mathbb{C}$, as follows:
\[
^\sigma P(X) = \sum_{n \geq -\infty} \sigma(a_n)X^n
\]
for $P(X) = \sum_{n \geq -\infty} a_n X^n \in \mathbb{C}(X)$. For a complex representation $\Pi$ of a group $G$ on the space $V_\Pi$ of $\Pi$, let $^\sigma \Pi$ of $\Pi$ be the representation of $G$ defined
\[
^\sigma \Pi(g) = t \circ \Pi(g) \circ t^{-1},
\]
where $t : V_\Pi \to V_\Pi$ is a $\sigma$-linear isomorphism. Note that the isomorphism class of $^\sigma \Pi$ is independent of the choice of $t$. We call $^\sigma \Pi$ the $\sigma$-conjugate of $\Pi$. When $v$ is a finite place and $\varphi$ is a complex-valued function on $\mathbb{F}_v^n$ or $(\mathbb{F}_v^\times)^n$, we define $^\sigma \varphi(x) = \sigma(\varphi(x))$ for $x \in \mathbb{F}_v^n$ or $x \in (\mathbb{F}_v^\times)^n$. 
1.4. Measures. Let \( v \) be a place of \( \mathbb{F} \). If \( v \) is finite, we normalize the Haar measures on \( \mathbb{F}_v \) and \( \mathbb{F}_v^\times \) so that \( \text{vol}(\mathbb{O}_v) = 1 \) and \( \text{vol}(\mathbb{O}_v^\times) = 1 \), respectively. If \( v \) is real, we normalize the Haar measures on \( \mathbb{F}_v \simeq \mathbb{R} \) and \( \mathbb{F}_v^\times \simeq \mathbb{R}^\times \) so that \( \text{vol}([1, 2]) = 1 \) and \( \text{vol}([1, 2]) = \log 2 \), respectively. Let \( m \) be a positive integer. Let \( dg_v \) be the Haar measure on \( GL_m(\mathbb{F}_v) \) defined as follows: For \( \phi \in L^1(GL_m(\mathbb{F}_v)) \), we have

\[
\int_{GL_m(\mathbb{F}_v)} \phi(g_v) \, dg_v = \prod_{1 \leq i < j \leq m} \int_{\mathbb{F}_v} du_{ij} \prod_{1 \leq i \leq m} \int_{\mathbb{F}_v} d^x t_i \int_{K_v} \frac{dk}{k} \phi \begin{pmatrix} t_{11} & u_{12} & \cdots & u_{1m} \\ 0 & t_{2} & \cdots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_m \end{pmatrix} \prod_{1 \leq i \leq m} |t_i|^{-m+i},
\]

where

\[
K_v = \begin{cases} GL_m(\mathbb{O}_v) & \text{if } v \text{ is finite}, \\ O(m) & \text{if } v \text{ is real}, \end{cases}
\]

and \( \text{vol}(K_v) = 1 \). Let \( H \) be a connected reductive linear algebraic group defined and split over \( \mathbb{F}_v \). Fix a Chevalley basis of \( \text{Lie}(H) \). The basis determines a top differential form on \( G(Sp_{2n}^\wedge(\mathbb{F}_v)) \) with respect to the Haar measure on \( G(Sp_{2n}^\wedge(\mathbb{F}_v)) \). We call it the local Tamagawa measure on \( H(\mathbb{F}_v) \). For any compact group \( K \), we take the Haar measure on \( K \) such that \( \text{vol}(K) = 1 \).

1.5. Weil representation. Let \((V, (\cdot, \cdot))\) be a non-degenerate quadratic space of even dimension \( m \) over \( \mathbb{F} \). Define the orthogonal similitude group \( GO(V) \) by

\[
GO(V) = \{ h \in GL(V) \mid (hx, hy) = \nu(h)(x, y) \text{ for } x, y \in V \},
\]

here \( \nu : GO(V) \to GL_1 \) is the scale map. Let

\[
GSO(V) = \{ h \in GO(V) \mid \det(h) = \nu(h)^{m/2} \}.
\]

Let \( O(V) \) and \( SO(V) \) be the orthogonal group and special orthogonal group defined by

\[
O(V) = \{ h \in GO(V) \mid \nu(h) = 1 \},
\]

\[
SO(V) = \{ h \in GO(V) \mid \det(h) = \nu(h) = 1 \}.
\]

Let \( \psi = \bigotimes \psi_v \) be a non-trivial additive character of \( \mathbb{F}\backslash \mathbb{A} \). We denote by \( \omega_{\psi, V, n} = \bigotimes \omega_{\psi_v, V, n} \) the Weil representation of \( Sp_{2n}(\mathbb{A}) \times O(V)(\mathbb{A}) \) on \( S(V^n(\mathbb{A})) \) with respect to \( \psi \) (cf. [Kud94] § 5 and [Ich05] § 4.2 for explicit formulas). Let \( S(V^n(\mathbb{A})) \) be the subspace of \( S(V^n(\mathbb{A})) \) consisting of functions which correspond to polynomials in the Fock model at the archimedean places. Let

\[
G(Sp_{2n} \times O(V))(\mathbb{A}) = \{ (g, h) \in GSp_{2n} \times O(V) \mid \nu(g) = \nu(h) \}.
\]

We extend \( \omega_{\psi, V, n} \) to a representation of \( G(Sp_{2n} \times O(V))(\mathbb{A}) \) as follows:

\[
\omega_{\psi, V, n}(g, h) \varphi = \omega_{\psi, V, n} \left( g \begin{pmatrix} 1_n & 0 \\ 0 & \nu(g)^{-1} n \end{pmatrix}, 1 \right) L(h) \varphi
\]

for \( (g, h) \in G(Sp_{2n} \times O(V))(\mathbb{A}) \) and \( \varphi \in S(V^n(\mathbb{A})) \). Here

\[
L(h)(\varphi)(x) = |\nu(h)|_A^{-nm/4} \varphi(h^{-1} x).
\]

1.6. Adjoint L-functions. Let \( \Pi = \bigotimes \Pi_v \) be an irreducible globally generic cuspidal automorphic representation of \( GSp_4(\mathbb{A}) \). By [AS06] and [GT11a] Theorem 12.1, \( \Pi \) has a strong functorial lift \( \tilde{\psi} \) to \( GL_4(\mathbb{A}) \). By [GT11a] Theorem 12.1, either \( \tilde{\psi} \) is cuspidal or \( \tilde{\psi} = \tau_1 \boxplus \tau_2 \) for some irreducible cuspidal automorphic representations \( \tau_1 \) and \( \tau_2 \) of \( GL_2(\mathbb{A}) \) with equal central character such that \( \tau_1 \neq \tau_2 \). We say that \( \Pi \) is stable (resp. endoscopic) if \( \tilde{\psi} \) is resp. (non)-cuspidal.

Recall that the dual group of \( GSp_4 \) is \( GSp_4(\mathbb{C}) \). Let \( Ad \) denote the adjoint representation of \( GSp_4(\mathbb{C}) \) on \( pgs\!p_4(\mathbb{C}) \), and std the composition of the projection \( GSp_4(\mathbb{C}) \to PGSp_4(\mathbb{C}) \) with the standard representation of \( PGSp_4(\mathbb{C}) \simeq SO_5(\mathbb{C}) \) on \( \mathbb{C}^5 \). Let \( S \) be a finite set of places of \( \mathbb{F} \) including the archimedean places such
that, for \( v \notin S \), \( \Pi_v \) is unramified. Then the partial adjoint and standard \( L \)-functions of \( \Pi \) are defined as the Euler products

\[
L^S(s, \Pi, \text{Ad}) = \prod_{v \notin S} L(s, \Pi_v, \text{Ad}), \quad L^S(s, \Pi, \text{std}) = \prod_{v \notin S} L(s, \Pi_v, \text{std})
\]

for \( s \in \mathbb{C} \), which are absolutely convergent for \( \Re(s) \) sufficiently large. Also, we have

\[
L^S(s, \Psi, \text{Sym}^2 \otimes \omega^{-1}_{\Pi}) = L^S(s, \Pi, \text{Ad}),
\]

\[
L^S(s, \Psi, \lambda^2 \otimes \omega^{-1}_{\Pi}) = \zeta(s)L^S(s, \Pi, \text{std}).
\]

In particular, \( L^S(s, \Pi, \text{Ad}) \) and \( L^S(s, \Pi, \text{std}) \) admit meromorphic continuations to \( \mathbb{C} \). (In a more general context, the meromorphic continuation of \( L^S(s, \Pi, \text{std}) \) was established by Piatetski-Shapiro and Rallis [PSR87] much earlier.) By [GT11a, Theorem 12.1], \( L^S(s, \Psi, \lambda^2 \otimes \omega_{\Pi}^{-1}) \) has a simple (resp. double) pole at \( s = 1 \) if \( \Pi \) is stable (resp. endoscopic). Hence \( L^S(s, \Pi, \text{std}) \) is holomorphic and non-zero (resp. has a simple pole) at \( s = 1 \) if \( \Pi \) is stable (resp. endoscopic). In particular, \( L^S(s, \Pi, \text{Ad}) \) is holomorphic and non-zero at \( s = 1 \).

For any place \( v \) of \( \mathbb{F} \), we denote by \( \phi_{\Pi_v} : L_{\mathcal{P}_v} \to \text{GSp}_4(\mathbb{C}) \) the local \( L \)-parameter attached to \( \Pi_v \) by the local Langlands correspondence established by Gan and Takeda [GT11a] if \( v \) is finite and by Langlands [Lan89] if \( v \) is real. Here \( L_{\mathcal{P}_v} \) is the Weil–Deligne group of \( \mathcal{P}_v \), if \( v \) is finite but the Weil group of \( \mathcal{P}_v \), if \( v \) is real. Since \( \Psi_v \) is essentially unitary and generic (and hence “almost tempered”), the adjoint \( L \)-factor

\[
L(s, \Pi_v, \text{Ad}) = L(s, \text{Ad} \circ \phi_{\Pi_v})
\]

defined as in [Hat79, §3] is holomorphic at \( s = 1 \). In fact, the same holds for any irreducible admissible generic representation of \( \text{GSp}_4(\mathcal{P}_v) \) (see [GP92, Conjecture 2.6], [AS08], [GT11a], [GI16, Proposition B.1]). Hence the completed adjoint \( L \)-function \( L(s, \Pi, \text{Ad}) \) is holomorphic and non-zero at \( s = 1 \).

2. Formula for Petersson norms

Let \( H = \text{GSp}_8 \) and

\[
\mathbf{G} = \{ (g_1, g_2) \in \text{GSp}_4 \times \text{GSp}_4 \mid \nu(g_1) = \nu(g_2) \}.
\]

Denote by \( Z_H \) the center of \( H \). We identify \( \mathbf{G} \) with its image under the embedding

\[
\mathbf{G} \to H, \quad \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \mapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & -b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & -c_2 & 0 & d_2 \end{pmatrix}.
\]

Let \( V_{3,3} = \mathbb{F}^6 \) be the space of column vectors equipped with a non-degenerate symmetric bilinear form \((\ , \)\) given by

\[
(x, y) = \langle x, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} y \rangle
\]

for \( x, y \in V_{3,3} \). With respect to the standard basis of \( V_{3,3} \), we identify \( \text{GO}(V_{3,3}) \) with the split orthogonal similitude group

\[
\text{GO}_{3,3} = \left\{ h \in \text{GL}_6 \mid h \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} h = \nu(h) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \nu(h) \in \text{GL}_1 \right\}.
\]

For a non-trivial additive character \( \psi_v \) of \( \mathbb{F}_v \), we write \( \omega_{\psi_v} = \omega_{\psi_v, \nu_{3,3,4}} \) for the Weil representation of \( \text{Sp}_6(\mathbb{F}_v) \times \text{O}_{3,3}(\mathbb{F}_v) \) on \( S(V'_{3,3}(\mathbb{F}_v)) \) with respect to \( \psi_v \).

Let \( \Pi = \bigotimes_v \Pi_v \) be an irreducible globally generic cuspidal automorphic representation of \( \text{GSp}_4(\mathbb{A}) \) with central character \( \omega_{\Pi} \). We assume

\[
\Pi_v|_{\text{Sp}_4(\mathbb{F}_v)} = D_{(\lambda_1, v, \lambda_2, v)} \oplus D_{(-\lambda_2, v, -\lambda_1, v)}
\]

for each real place \( v \), where \( D_{(\lambda_1, v, \lambda_2, v)} \) is the discrete series representation of \( \text{Sp}_4(\mathbb{F}_v) \simeq \text{Sp}_4(\mathbb{R}) \) with Blattner parameter \((\lambda_1, v, \lambda_2, v) \in \mathbb{Z}^2 \) such that \( 2 - \lambda_1, v \leq \lambda_2, v \leq -1 \). For each finite place \( v \), by the newform theory of Robert–Schmidt [RS07, Theorem 7.5.4] and Okazaki [Oka19, Main Theorem], there exists a smallest non-negative integer \( n_v \geq 2c(\omega_{\Pi_v}) \) such that \( \Pi_v^{K_1(\pi_v^n; c(\omega_{\Pi_v}))} \neq 0 \). In this case, we have \( \dim \Pi_v^{K_1(\pi_v^n; c(\omega_{\Pi_v}))} = 1. \)
A paramodular newform of $Π_v$ is a non-zero vector in this one-dimensional space. The paramodular conductor $\text{cond}(Π)$ of $Π$ is the $δ$-submodule of $H_f$ defined by

$$\text{cond}(Π) = \prod_{v|\infty} \mathcal{O}_v^{n_v} \mathfrak{o}_v.$$ 

We write $v \mid \text{cond}(Π)$ if $n_v > 0$. For a place $v$ and a non-trivial additive character $ψ_v$ of $𝔽_v$, we denote by $\mathcal{W}(Π_v,ψ_U)$ the space of Whittaker functions of $Π_v$ with respect to $ψ_U$.

2.1. Doubling local zeta integrals. Let $P$ be the standard Siegel parabolic subgroup of $H$ defined by

$$P = \left\{ \begin{pmatrix} a & * \\ 0 & νa^{-1} \end{pmatrix} \in H \mid a \in \text{GL}_4, ν \in \text{GL}_1 \right\}.$$ 

Let $v$ be a place of $𝔽$. Denote by $I_v(s)$ the degenerate principal series representation $\text{Ind}_{P(𝔽_v)}^H(δ_P)\delta^{s/5}$ of $H(𝔽_v)$. Here $δ_P$ is the modulus character of $P(𝔽_v)$ given by

$$δ_P \left( \begin{pmatrix} a & * \\ 0 & νa^{-1} \end{pmatrix} \right) = |\det(a)|_v^{5|ν|_v^{-10}}.$$ 

Let $φ_v$ be a matrix coefficient of $Π_v$ and $F_v \in I_v(s)$ be a holomorphic section. We define the local zeta integral

$$Z_v(s,φ_v,F_v) = \int_{\text{Sp}_4(𝔽_v)} F_v(δ(g_v,1),s)φ_v(g_v) dg_v,$$

where

$$δ = \begin{pmatrix} 0 & 0 & -\frac{1}{2}1_2 & \frac{1}{2}1_2 \\ \frac{1}{2}1_2 & \frac{1}{2}1_2 & 0 & 0 \\ 1_2 & -1_2 & 0 & 0 \\ 0 & 0 & 1_2 & 1_2 \end{pmatrix}.$$ 

Here $dg_v$ is a Haar measure on $\text{Sp}_4(𝔽_v)$ normalized so that $\text{vol}(\text{Sp}_4(𝔽_v), dg_v) = 1$ if $v$ is finite and is the local Tamagawa measure if $v$ is real. Note that

$$δ(g,g)δ^{-1} ∈ P(𝔽_v)$$

for all $g ∈ \text{GSp}_4(𝔽_v)$ and all places $v$.

Let $v$ be a finite place. Let $φ_v$ be a matrix coefficient of $Π_v$ and $F_v ∈ I_v(s)$. For $σ ∈ \text{Aut}(ℂ)$, we define the matrix coefficient $σφ_v$ of $σΠ_v$ (cf. Lemma 5.9) and $σF_v$ by

$$σφ_v(g) = σ(φ_v(g)), \quad σF_v(h,s) = σ(F_v(h,s))$$

for $g ∈ \text{GSp}_4(𝔽_v)$ and $h ∈ H(𝔽_v)$. Note that $σF_v|_s = a^{-n}_v σF_v ∈ I_v(s)$ for all odd integers $n$.

Lemma 2.1. Let $φ_v$ be a matrix coefficient of $Π_v$ and $F_v ∈ I_v(s)$ be a holomorphic section.

1. The integral $Z_v(s,φ_v,F_v)$ is absolutely convergent for $\text{Re}(s) ≥ \frac{3}{2}$.

2. Assume $v$ is finite. For $σ ∈ \text{Aut}(ℂ)$, we have

$$σZ_v(\frac{1}{2},φ_v,F_v) = Z_v(\frac{1}{2},σφ_v,σF_v).$$

Proof. The assertions will be proved in Proposition 5.10 below. □

We recall the local Siegel-Weil sections. Let $ψ_v$ be a non-trivial additive character of $𝔽_v$. Define a $H(𝔽_v)$-intertwining map

$$S(V^4_{3,3}(𝔽_v)) → I_v(\frac{1}{2}), \quad ϕ ↦ F_{ψ_v}(ϕ)$$

by

$$F_{ψ_v}(ϕ)(g,\frac{1}{2}) = ω_{ψ_v}(g,h)ϕ(0),$$

where $ν(g) = ν(h)$. We extend $F_{ψ_v}(ϕ)$ to a holomorphic section $F_{ψ_v}(ϕ)$ of $I_v(s)$ such that its restriction to $K_v$ is independent of $s$, where $K_v$ is the maximal compact subgroup of $H(𝔽_v)$ defined by

$$K_v = \begin{cases} H(ℚ) & \text{if } v \text{ is finite,} \\ H(ℝ) ∩ O(2n) & \text{if } v \text{ is real.} \end{cases}$$
Lemma 2.2. Let \( \psi_v \) be a non-trivial additive character of \( \mathbb{F}_v \). There exists \( \varphi \in S(V_{3,3}^4(\mathbb{F}_v)) \) such that
\[
Z_v(\frac{1}{2}, \psi_v, F_{\psi_v}(\varphi)) \neq 0.
\]

Proof. The assertion was proved for real \( v \) in [CI19 Lemma 5.3]. We assume \( v \) is finite. Let \( H_0 \) be an irreducible component of \( \Pi_v|_{\text{Sp}_4(\mathbb{F}_v)} \). Fix a bilinear equivariant pairing \( \langle , \rangle \) on \( \Pi_0 \times \Pi_0' \). For \( f_1 \in \Pi_0 \) and \( f_2 \in \Pi_0' \), define a matrix coefficient \( \phi_{f_1 \otimes f_2} \) of \( H_0 \) by
\[
\phi_{f_1 \otimes f_2}(g) = \langle \Pi_0(g)f_1, f_2 \rangle.
\]
Then the local zeta integral \( Z_v(\frac{1}{2}, \phi_{f_1 \otimes f_2}, F) \) is absolutely convergent by Lemma 2.1(1), and defines an \( \text{Sp}_4(\mathbb{F}_v) \times \text{Sp}_4(\mathbb{F}_v) \)-intertwining map
\[
\ell : I_v(\frac{1}{2}) \rightarrow \Pi_0' \otimes \Pi_0, \quad F_v \mapsto [f_1 \otimes f_2 \mapsto Z_v(\frac{1}{2}, \phi_{f_1 \otimes f_2}, F_v)].
\]
Let \( V_0 \) be the quaternionic quadratic space of dimension 4 over \( \mathbb{F}_v \). Let \( V_1 \) (resp. \( V_2 \)) be the split quadratic space (resp. quaternionic quadratic space) of dimension 6 over \( \mathbb{F}_v \). For \( i = 1, 2 \), let \( R(V_i) \) be the image of the \( \text{Sp}_8(\mathbb{F}_v) \)-intertwining map
\[
S(V_i^4(\mathbb{F}_v)) \rightarrow I_v(\frac{1}{2}), \quad \varphi \mapsto F_{\psi_v}(\varphi),
\]
where \( F_{\psi_v}^{(i)}(\varphi)(g, \frac{1}{2}) = \omega_{\psi_v}(V_i, \varphi(g, 1, \varphi)^i) \). By [KR92, Proposition 7.2.1], the intertwining map \( \ell \) is non-zero. If \( \ell|_{R(V_1)} \) is zero, then it follows from [2.6] that \( \ell|_{R(V_2)} \) must be non-zero and defines a non-zero element in
\[
\text{Hom}_{\text{Sp}_4(\mathbb{F}_v) \times \text{Sp}_4(\mathbb{F}_v)}(R(V_0), \Pi_0' \otimes \Pi_0).
\]
Therefore the local theta lift of \( \Pi_0 \) to \( O(V_0)(\mathbb{F}_v) \) is non-zero by [HKS96, Proposition 3.1]. This contradicts the genericity of \( \Pi_0 \) (cf. [GT11a, Corollary 4.2-(i)]). Hence \( \ell|_{R(V_1)} \) is non-zero. This completes the proof. \( \square \)

2.2. Local zeta integrals for \( \text{GSp}_4 \times \text{GSp}_4 \). Let \( \mathcal{P} \) be the parabolic subgroup of \( H \) defined by
\[
\mathcal{P} = \left\{ \begin{pmatrix} a & * & * & * \\ 0 & a' & * & b' \\ 0 & 0 & \nu^{-1} & 0 \\ 0 & c' & * & d' \end{pmatrix} \in H \right\}.
\]
Let \( v \) be a place of \( \mathbb{F} \) and \( \psi_v \) be a non-trivial additive character of \( \mathbb{F}_v \). Denote by \( I_v(s) \) the degenerate principal series representation \( \text{Ind}^{H(\mathbb{F}_v)}_{\mathcal{P}(\mathbb{F}_v)}(\delta_P^s) \) of \( H(\mathbb{F}_v) \). Here \( \delta_P \) is the modulus character of \( \mathcal{P}(\mathbb{F}_v) \) given by
\[
\delta_P \begin{pmatrix} a & * & * & * \\ 0 & a' & * & b' \\ 0 & 0 & \nu^{-1} & 0 \\ 0 & c' & * & d' \end{pmatrix} = |\text{det}(a)|^{\delta_P} |\nu|^{-9}.
\]
Let \( W_{1,v} \in \mathcal{W}(\Pi_v, \psi_{U,v}) \) and \( W_{2,v} \in \mathcal{W}(\Pi_v', \psi_{U,v}^{-1}) \) be Whittaker functions of \( \Pi_v \) and \( \Pi_v' \) with respect to \( \psi_{U,v} \) and \( \psi_{U,v}^{-1} \), respectively, and \( \mathcal{F}_v \in I_v(s) \) be a holomorphic section. We define the local zeta integral
\[ (2.6) \quad Z_v(s, W_{1,v}, W_{2,v}, \mathcal{F}_v) = \int_{Z_H(\mathbb{F}_v)O(\mathbb{F}_v) \backslash G(\mathbb{F}_v)} \mathcal{F}_v(\eta_{\mathcal{F}_v}, s)(W_{1,v} \otimes W_{2,v}^*)(g_v) d\tilde{g}_v, \]
where
\[
\tilde{U} = \left\{ \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, u \right\} \quad u \in U, x \in \mathbb{G}_a.
\]
Define a \( F \) for \((2.7)\) by
\[
F_{\psi}(\varphi, g, s) = \int_{G_S(\mathbb{F}_v)} \hat{\omega}_{\psi}(g, k_v h)^{\varphi (0_{3 \times 4}, t_a v, 0_{3 \times 1})} f_{\psi, v}(k_v h, s) |\det(a_v)|^{|s| + 3} \, dk_v \, da_v.
\]
where \( \nu(h) = \nu(g) \). The Haar measure \( da_v \) on \( \text{GL}_4(\mathbb{F}_v) \) is normalized as in (1.9). Note that the integral is absolutely convergent for \( \text{Re}(s) > -1 \) and admits meromorphic continuation to \( s \in \mathbb{C} \) (cf. [GJ72]). Therefore, \( F_{\psi_v}(\varphi) \) defines a meromorphic section of \( \mathcal{I}_v(s) \) which is holomorphic for \( \text{Re}(s) > -1 \).

2.3. Petersson norms and adjoint \( L \)-values. Let \( f_{\Pi} = \bigotimes_v f_v \in \Pi \) and \( f_{\Pi}^\vee = \bigotimes_v f_v^\vee \in \Pi^\vee \) be the normalized newforms of \( \Pi \) and \( \Pi^\vee \). Choose local bilinear equivariant pairings \( \langle , \rangle_v \) on \( \Pi_v \times \Pi_v^\vee \) such that

\[
\| f_{\Pi} \| = \prod_{v \mid \infty} \langle f_v, \Pi_v^\vee(\text{diag}(-1,-1,1,1))f_v^\vee \rangle_v \prod_{v \not\mid \infty} \langle f_v, f_v^\vee \rangle_v,
\]

and define a matrix coefficient \( \phi_v \) of \( \Pi_v \) by

\[
\phi_v(g) = \begin{cases} 
\langle \Pi_v(g)f_v, f_v^\vee \rangle_v & \text{if } v \text{ is finite,} \\
\langle \Pi_v(g)f_v, \Pi_v^\vee(\text{diag}(-1,-1,1,1))f_v^\vee \rangle_v & \text{if } v \text{ is real.}
\end{cases}
\]

Let \( v \) be a place of \( \mathbb{F} \) and \( \psi_v \) a non-trivial additive character of \( \mathbb{F}_v \). Let \( W_{\psi_v} \in W(\Pi_v, \psi_{U,v}) \) be a non-zero Whittaker function satisfying the following condition:

- \( W_{\psi_v} \) is a paramodular newform if \( v \) is finite;
- \( W_{\psi_v} \) is a lowest weight vector of the minimal \( U(2) \)-type of \( D_{(-\lambda_2,v,-\lambda_1,v)} \) if \( v \) is real.

The condition characterizes \( W_{\psi_v} \) up to scalars, we normalize it so that

- \( W_{\psi_v}(\text{diag}(a_v^3, a_v^2, 1, a_v)) = 1 \) if \( v \) is finite;
- \( W_{\psi_v}(\text{diag}(a_v^3, a_v^2, 1, a_v)) \) is normalized as in (1.1) if \( v \) is real.

Here \( a_v \in \mathbb{F}_v^\times \) is chosen so that

- \( a_v \psi_v \) is the largest fractional ideal of \( \mathbb{F}_v \) on which \( \psi_v \) is trivial if \( v \) is finite;
- \( \psi_v^\vee \) is the standard additive character of \( \mathbb{F}_v \) if \( v \) is real.

Note that \( W_{\psi_v}(\text{diag}(a_v^3, a_v^2, 1, a_v)) \neq 0 \) for finite place \( v \) by the results of Robert–Schmidt [RS07] and Okazaki [Oka19], hence the normalization is valid. We call \( W_{\psi_v} \) the normalized Whittaker newform of \( \Pi_v \) with respect to \( \psi_{U,v} \). By definition, for \( a \in \mathbb{F}_v^\times \), we have

\[
W_{\psi_v}(g) = W_{\psi_v}(\text{diag}(a^3, a^2, 1, a))
\]

for \( g \in \text{GSp}_4(\mathbb{F}_v) \). We define the normalized Whittaker newform \( W_{\psi_v}^\vee \) of \( \Pi_v^\vee \) with respect to \( \psi_v \) in a similar way.

**Proposition 2.4.** Let \( \psi \) be a non-trivial additive character of \( \mathbb{F} \backslash \mathbb{A} \). We have

\[
\| f_{\Pi} \| = 2^c \cdot \mathcal{D}^{-13} \frac{L(1, \Pi, \text{Ad})}{\zeta(2)\zeta(4)} \prod_v C_{\psi_v}(\Pi_v).
\]

Here

\[
c = \begin{cases} 
1 & \text{if } \Pi \text{ is stable,} \\
2 & \text{if } \Pi \text{ is endoscopic,}
\end{cases}
\]

and \( C_{\psi_v}(\Pi_v) \) is a non-zero constant depending only on \( \Pi_v \) and \( \psi_v \) given by

\[
C_{\psi_v}(\Pi_v) = \zeta_v(1)^{-1}\zeta_v(3)^{-1}\zeta_v(4) L(1, \Pi_v, \text{Ad})^{-1}
\]

\[
\times \begin{cases} 
\zeta_v(2)^{-1}\zeta_v(4)^{-1} \cdot \frac{Z_v(1, W_{\psi_v}, W_{\psi_v}^\vee, F_{\psi_v}(\varphi_v))}{Z_v(1, W_{\psi_v}, \Pi_v^\vee(\text{diag}(-1,-1,1,1))W_{\psi_v}^\vee, F_{\psi_v}(\varphi_v))} & \text{if } v \text{ is finite,} \\
\zeta_v(1)^{-1} \cdot \frac{Z_v(1, W_{\psi_v}, \Pi_v^\vee(\text{diag}(-1,-1,1,1))W_{\psi_v}^\vee, F_{\psi_v}(\varphi_v))}{Z_v(1, W_{\psi_v}, F_{\psi_v}(\varphi_v))} & \text{if } v \text{ is real.}
\end{cases}
\]

where \( \varphi_v \in S(V^A_{1,3}(\mathbb{F}_v)) \) is any Schwartz function such that \( Z_v(1, \varphi_v, F_{\psi_v}(\varphi_v)) \neq 0 \). Moreover, if \( v \) is a finite place such that \( \Pi_v \) and \( \psi_v \) are unramified, then we have \( C_{\psi_v}(\Pi_v) = 1 \).
Proof. The assertion can be proved by proceeding exactly as in the proof of [CI19, Proposition 5.4] with Lemmas 4.2, 4.4, and 5.3 in [CI19] replaced by Lemmas 2.1(1), 2.3(1), and 2.2 respectively. Note that the constant $C_{\upsilon}(\Pi_v)$ is well-defined, since $L(s, \Pi_v, \text{Ad})$ is holomorphic and non-zero at $s = 1$ by the genericity of $\Pi_v$. Also note that the factors $\zeta(v^{-1})$ and $\zeta(4)^{-1}$ for finite places $v$ comparing with [CI19, Proposition 5.4] are due to different normalization of Haar measures used to define the local zeta integrals $Z_v$ and $Z_\infty$ for finite places $v$, and the partial Fourier transform $\hat{\phi}_v$.

3. Proof of Main Theorem

We keep the notation of §2. Assume further that $\Pi$ is motivic, that is, there exists $w \in \mathbb{Z}$ such that $|\omega_\Pi| = |w|_{\kappa}$ and

$$\lambda_{1,v} - \lambda_{2,v} \equiv w \ (\text{mod} \ 2)$$

for all real places $v$. In other words, we have $\omega_{\Pi_v} = \text{sgn}^w |\cdot|^v$ for all real places $v$. In the following lemma, we show that being globally generic is an arithmetic property of an irreducible cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A})$ of motivic discrete type at the archimedean places.

Lemma 3.1. Assume $\Pi$ is motivic. For $\sigma \in \text{Aut}(\mathbb{C})$, the representation $\sigma \Pi$ is an irreducible motivic globally generic cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A})$.

Proof. Fix $\sigma \in \text{Aut}(\mathbb{C})$. We denote by $\Psi = \bigotimes_v \Psi_v$ the strong functorial lift of $\Pi$ to $\text{GL}_4(\mathbb{A})$. By [GT11a, Theorem 12.1], the automorphic representation $\Psi \boxtimes \omega_\Pi$ is equal to the global theta lift of $\Pi$ from $\text{GSp}_4(\mathbb{A})$ to $\text{GSO}_{3,3}(\mathbb{A})$, where we identify $\text{GSO}_{3,3}$ with

$$(\text{GL}_4 \times \text{GL}_1)/\{(a I_4, a^{-2}) | a \in \text{GL}_1\}.$$  

For a real place $v$, by the assumption on $\Pi_v$, we have

$$\Psi_v \simeq \text{Ind}_{P_{2,2}(\mathbb{A})}^{\text{GL}_4(\mathbb{A})}(D(\lambda_{1,v} + \lambda_{2,v}) \boxtimes D(\lambda_{1,v} - \lambda_{2,v})) \otimes |w|^v,$$

where $P_{2,2}$ is the standard maximal parabolic subgroup of $\text{GL}_4(\mathbb{R})$ with minimal weight $\kappa \geq 2$ and central character $\text{sgn}^w$. In particular, we see that $\Psi$ is regular algebraic in the sense of Clozel [Clo90] §1.2.3 and Definition 3.12. Therefore, $\sigma \Psi = \sigma \Psi_\infty \otimes \sigma \Psi_f$ is automorphic by [Clo90, Théorème 3.13], where $\sigma \Psi_f$ is the $\sigma$-conjugate of $\Psi_f = \bigotimes_v \Psi_v$ and $\sigma \Psi_\infty$ is the representation of $\text{GL}_4(\mathbb{A})$ so that its $v$-component is equal to $\Psi_{\sigma^{-1}v}$. We claim that there exists a global irreducibly generic cuspidal automorphic representation $\Pi^\sharp = \bigotimes_v \Pi^\sharp_v$ of $\text{GSp}_4(\mathbb{A})$ with central character $\omega_\Pi = \omega_{\Pi_\infty} \cdot \omega_{\Pi_f}$ such that $\sigma \Psi$ is the strong functorial lift of $\Pi^\sharp$. Assume the claim holds. For any finite place $v$, $\sigma \Psi_v \boxtimes \omega_{\Pi_v}$ is the local theta lift of both $\sigma \Pi_v$ and $\Pi^\sharp_v$ from $\text{GSp}_4(\mathbb{A})$ to $\text{GSO}_{3,3}(\mathbb{A})$. Indeed, since $\Psi_v \boxtimes \omega_{\Pi_v}$ is the local theta lift of $\Pi_v$ from $\text{GSp}_4(\mathbb{F}_v)$ to $\text{GSO}_{3,3}(\mathbb{A}_v)$, it follows from [Rob01, Proposition 1.9] and [Mor12, Proposition 5.7] that $\sigma \Psi_v \boxtimes \omega_{\Pi_v}$ is the local theta lift of $\sigma \Pi_v$ from $\text{GSp}_4(\mathbb{F}_v)$ to $\text{GSO}_{3,3}(\mathbb{F}_v)$. Also note that $\sigma \Psi_\infty \boxtimes \omega_{\Pi_\infty}$ is the local theta lift of both $\sigma \Pi_\infty$ and $\Pi^\sharp_\infty$ from $\text{GSp}_4(\mathbb{F}_v)$ to $\text{GSO}_{3,3}(\mathbb{F}_v)$ (cf. [Pan03a]). By the Howe duality principle, we have

$$\sigma \Pi_v \simeq \Pi^\sharp_v, \quad \sigma \Pi_\infty \simeq \Pi^\sharp_\infty$$

for all finite places $v$. We conclude that $\Pi^\sharp \simeq \Pi^\sharp_\infty$ is an irreducible globally generic cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A})$.

It remains to verify the claim. Firstly we consider the case when $\Pi$ is stable. In this case, $\sigma \Psi$ is cuspidal by [Clo90, Theorem 3.13] and $L(s, \sigma, \Psi, \wedge^2 \otimes \omega_{\Pi}^{-1})$ has a pole at $s = 1$. By [GT11a, Theorem 12.1], the claim holds if and only if $L(s, \sigma, \Psi, \wedge^2 \otimes \omega_{\Pi}^{-1})$ has a pole at $s = 1$. The last assertion was proved by Gan in [GR14, Theorem 3.6.2] based on the result on functorial lifts from $\text{GSpin}_4(\mathbb{A})$ to $\text{GL}_4(\mathbb{A})$. Now we assume $\Pi$ is endoscopic. In this case, we write $\Psi = \tau_1 \boxplus \tau_2$ for some non-isomorphic irreducible cuspidal automorphic representations $\tau_1$ and $\tau_2$ of $\text{GL}_2(\mathbb{A})$ with equal central character $\omega_\Pi$ such that

$$\tau_{1,v} = D(\lambda_{1,v} + \lambda_{2,v}) \otimes |w|_{v}^{2}, \quad \tau_{2,v} = D(\lambda_{1,v} - \lambda_{2,v}) \otimes |w|_{v}^{2}$$

for all real places $v$. In particular, we see that $\tau_1$ and $\tau_2$ are regular algebraic. Therefore, $\sigma \Psi = \sigma \tau_1 \boxplus \sigma \tau_2$ is an isobaric automorphic representation of $\text{GL}_4(\mathbb{A})$. We then take $\Pi^\sharp$ be the global theta lift of $\sigma \tau_2 \boxplus \sigma \tau_1^\vee$ from $\text{GSO}_{2,2}(\mathbb{A})$ to $\text{GSp}_4(\mathbb{A})$, where we identify $\text{GSO}_{2,2}$ with

$$(\text{GL}_2 \times \text{GL}_2)/\{(a I_2, a I_2) | a \in \text{GL}_1\}.$$
as in §1.2. Then $\Pi^\sharp$ is globally generic cuspidal and $^\sigma\Psi$ is the strong functorial lift of $\Pi^\sharp$ by [GT11a, Theorem 12.1]. This completes the proof.

**Lemma 3.2.** The rationality field $\mathbb{Q}(\Pi)$ of $\Pi$ is equal to the fixed field of $\{\sigma \in \text{Aut}(\mathbb{C}) \mid ^\sigma\Pi_f = \Pi_f\}$ and is a number field.

**Proof.** Let $\Psi = \bigotimes_v \Psi_v$ be the strong functorial lift of $\Pi$ to $\text{GL}_4(\mathbb{A})$. The rationality field $\mathbb{Q}(\Psi)$ of $\Psi$ is the fixed field of $\{\sigma \in \text{Aut}(\mathbb{C}) \mid ^\sigma\Psi = \Psi\}$. By the result of Clozel [Clo90, Théorème 3.13], $\mathbb{Q}(\Psi)$ is a number field. It follows from the strong multiplicity one theorem for isobaric automorphic representations [JS81, Theorem 4.4] that $\mathbb{Q}(\Psi)$ is equal to the fixed field of $\{\sigma \in \text{Aut}(\mathbb{C}) \mid ^\sigma\Psi_f = \Psi_f\}$.

Let $\sigma \in \text{Aut}(\mathbb{C})$. For each place $v$, we have explained in the proof of Lemma 3.1 that $^\sigma\Psi_v$ is the functorial lift of $^\sigma\Pi_v$ to $\text{GL}_4(\mathbb{F}_v)$ via the local theta correspondence. It then follows from the Howe duality principle that $^\sigma\Psi_v = \Psi_v$ if and only if $^\sigma\Pi_v = \Pi_v$. Therefore, we conclude that $\mathbb{Q}(\Pi) = \mathbb{Q}(\Psi)$ and is equal to the fixed field of $\{\sigma \in \text{Aut}(\mathbb{C}) \mid ^\sigma\Pi_f = \Pi_f\}$. This completes the proof. \hfill \qed

In the following lemma, we prove the Galois equivariant property of the local adjoint $L$-functions. The argument is standard and we give a proof for convenience of the reader (cf. [Rag10, Proposition 3.17] and [Mor12, Proposition 5.4]).

**Lemma 3.3.** Let $v$ be a finite place. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$\sigma L(1, \Pi_v, \text{Ad}) = L(1, ^\sigma\Pi_v, \text{Ad}).$$

**Proof.** For $n \geq 1$, let $\text{Irr}(\text{GL}_n)$ be the set of isomorphism classes of irreducible admissible representations of $\text{GL}_n(\mathbb{F}_v)$ and $\Phi(\text{GL}_n)$ the set of equivalence classes of admissible $n$-dimensional representation of the Weil–Deligne group $L_{\mathbb{F}_v}$ of $\mathbb{F}_v$. Let

$$\text{Irr}(\text{GL}_n) \rightarrow \Phi(\text{GL}_n), \quad \Psi \mapsto \phi_{\Psi}$$

be the local Langlands correspondence established in [HT01] and [Hen99]. Let $\sigma \in \text{Aut}(\mathbb{C})$. By [Clo90, Lemme 4.6] and [Hen01, Propriété 3, §7], we have

$$^\sigma L(s + \frac{n-1}{2}, \phi_{\Psi}) = L(s + \frac{n-1}{2}, ^\sigma\phi_{\Psi}),$$

$$^\sigma\phi_{\Psi} = \phi_{^\sigma\Psi} \otimes \chi_{\sigma}^{-1}.$$

Here we regard the $L$-functions as rational functions in $q_v^{-s}$ and the $\sigma$-linear action is defined as in §1.3 and $\chi_{\sigma}$ is the quadratic character of $L_{\mathbb{F}_v}$ associated to the quadratic character $\sigma(| \cdot |_v^{-1/2}) \cdot | \cdot |_v^{-1/2}$ of $\mathbb{F}_v^\times$.

Let $\Psi_v \in \text{Irr}(\text{GL}_4)$ be the local functorial lift of $\Pi_v$ to $\text{GL}_4(\mathbb{F}_v)$. We see from the proof of Lemma 3.1 that $^\sigma\Psi_v$ is the local functorial lift of $^\sigma\Pi_v$ to $\text{GL}_4(\mathbb{F}_v)$. Let $\text{Sym}^2 : \text{GL}_4(\mathbb{C}) \rightarrow \text{GL}_{10}(\mathbb{C})$ be the symmetric square representation. It is easy to verify that

$$^\sigma(\text{Sym}^2 \circ \phi) = \text{Sym}^2 \circ ^\sigma\phi,$$

$$\text{Sym}^2 \circ (\phi \otimes \chi) = (\text{Sym}^2 \circ \phi) \otimes \chi^2$$

for any $\phi \in \Phi(\text{GL}_4)$ and character $\chi$ of $L_{\mathbb{F}_v}$. In particular, we have

$$^\sigma(\text{Sym}^2 \circ \phi_{\Psi_v}) = \text{Sym}^2 \circ ^\sigma\phi_{\Psi_v}.$$ 

Therefore, we deduce that

$$^\sigma L(s, \Pi_v, \text{Ad}) = ^\sigma L(s, \text{Sym}^2 \circ \phi_{\Psi_v})$$

$$= ^\sigma L(s + \frac{1}{2}, (\text{Sym}^2 \circ \phi_{\Psi_v}) \otimes | \cdot |_v^{-1/2})$$

$$= L(s + \frac{1}{2}, ^\sigma(\text{Sym}^2 \circ \phi_{\Psi_v}) \otimes \sigma(| \cdot |_v^{-1/2})\chi_{\sigma})$$

$$= L(s + \frac{1}{2}, (\text{Sym}^2 \circ \phi_{^\sigma\Psi_v}) \otimes | \cdot |_v^{-1/2})$$

$$= L(s, \text{Sym}^2 \circ ^\sigma\phi_{\Psi_v})$$

$$= L(s, ^\sigma\Pi_v, \text{Ad}).$$

We obtain the assertion by evaluating at $s = 1$. This completes the proof. \hfill \qed
Lemma 3.4. Let $\psi_v$ be a non-trivial additive character of $\mathbb{F}_v$ and $\varphi \in S(V^4_v(F_v))$.

(1) Let $a \in \mathbb{F}_v^\times$ so that $a \in \mathfrak{o}_v^\times$ if $v$ is finite. We have

$$F_{\psi_v,2}(\varphi) = |a|_v^{12} \cdot F_{\psi_v}(\varphi'), \quad F_{\psi_v,2}'(\varphi) = |a|_v^{-3s-9} \cdot F_{\psi_v}(\varphi'),$$

where

$$\varphi' = \omega_{\psi_v}(a^{-1} 1_4, 0, a 1_4) \cdot \varphi.$$

(2) Assume $v$ is finite. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$\sigma F_{\psi_v}(\varphi)(g, \frac{1}{2}) = F_{\psi_v}(\sigma \varphi)(g, \frac{1}{2}), \quad \sigma F_{\psi_v}(\varphi)(g, 1) = F_{\psi_v}(\sigma \varphi)(g, 1).$$

Proof. Note that we have

$$(3.1) \quad \omega_{\psi_v,2} = \omega_{\psi_v}(a 1_4, 0, a^{-1} 1_4) \cdot \omega_{\psi_v} \circ \omega_{\psi_v}(a^{-1} 1_4, 0, a 1_4).$$

Therefore, one can easily verify that

$$(3.2) \quad \omega_{\psi_v,2}(g, h) \hat{\varphi}(u, v) = \omega_{\psi_v}(g, h) \hat{\varphi}(au, av)$$

for $(g, h) \in G(\text{Sp}_8 \times O_{3,3}(F_v))$ and $u, v \in M_{3,4}(F_v)$. The first assertion for $F$ follows immediately from (5.1).

By definition, we have $f_{\psi_v}^\sigma = f_{\psi_v}^\omega, \hat{\varphi}$ if $v$ is real. The assertion $a \in \mathfrak{o}_v^\times$ if $v$ is finite implies that $f_{\psi_v}^\sigma = f_{\psi_v}^\omega, \hat{\varphi}$. The first assertion for $F$ thus follows from (3.2).

Assume $v$ is finite. Let $m$ be a positive integer and $\phi \in S(M_{m, m}(F_v))$. The integral

$$Z(s, \phi) = \int_{\text{GL}_m(F_v)} \phi(a) |\det(a)|_v^s \cdot da$$

is absolutely convergent for $\text{Re}(s) > m - 1$. Moreover, we can deduce from the proof of [GJ72, Proposition 1.1] that $Z(s, \phi)$ defines a rational function in $q_v^{-s}$ and satisfies the Galois equivariant property

$$(3.3) \quad \sigma Z(s, \phi) = Z(s, \sigma \phi).$$

By the explicit formula for the Weil representation, we have

$$\sigma \omega_{\psi_v}(g, h) \hat{\varphi} = \omega_{\psi_v}(g, h) \hat{\varphi}, \quad \sigma \omega_{\psi_v}(g, h) \hat{\varphi} = \omega_{\psi_v}(g, h) \hat{\varphi}$$

for $(g, h) \in G(\text{Sp}_8 \times O_{3,3}(F_v))$. Also note that $\sigma f_{\psi_v}^\sigma(g, 1) = f_{\psi_v}^\omega, \hat{\varphi}(g, 1)$ by definition. The second assertion then follows immediately from (3.3) and (3.4). This completes the proof.

Lemma 3.5. Let $\psi_v$ be a non-trivial additive character of $\mathbb{F}_v$ and $a \in \mathbb{F}_v^\times$. Assume either $v$ is finite and $\Pi_v$ is unramified, or $v$ is real and $a > 0$. Then we have

$$C_{\psi_v}(\Pi_v) = |a|_v^{-8} \cdot C_{\psi_v}(\Pi_v).$$

Proof. First assume $v$ is finite and $\Pi_v$ is unramified. Let $F_v^\circ \in I_v(s)$ and $F_v^\circ, \phi \in I_v(s)$ be the $H(\mathfrak{o}_v)$-invariant sections such that $F_v^\circ(1, s) = F_v^\circ(1, s) = 1$. Let $\varpi_v^{\phi} \mathfrak{o}_v$ be the largest fractional ideal of $F_v$ on which $\psi_v$ is trivial. For $a \in \mathbb{F}_v^\times$, define $\varphi = \varphi_a \in S(V^4_v(F_v))$ by

$$\varphi(x, y) = \mathbb{I}_{M_{3,4}, a^{-1} \varpi_v^{\phi}, \mathfrak{o}_v}(x) \mathbb{I}_{M_{3,4}, \mathfrak{o}_v}(y).$$

Then the partial Fourier transform (2.27) of $\varphi$ with respect to $\psi_v \circ \text{tr}$ is equal to

$$\hat{\varphi} = |a|_v^{12} q_v^{-12d_v} \cdot \mathbb{I}_{M_{3,4}, \mathfrak{o}_v}.$$
Note that \( F_{\psi_{v}'}(\varphi)(1, \frac{1}{2}) = \varphi(0) = 1 \) and
\[
\mathcal{F}_{\psi_{v}'}(\varphi)(1, s) = |a_v|^{-12} q_v^{-12d_v} \int_{\text{GL}_3(F_v)} |I_{3,3}(s,t)| \det(t)^{s+3} dt
= |a_v|^{-12} q_v^{-12d_v} \zeta_v(s+1) \zeta_v(s+2) \zeta_v(s+3).
\]
On the other hand, a change of variables \( g \mapsto (\text{diag}(a^3, a^2, 1, a), \text{diag}(a^3, a^2, 1, a))^{-1}g \) shows that
\[
Z_v(s, W_{\psi_{v}'}', W_{\psi_{v}'}, \mathcal{F}) = |a_v|^{-3s/2+11/2} Z_v(s, W_{\psi_{v}}, W_{\psi_{v}-1}', \mathcal{F}).
\]
for any holomorphic section \( \mathcal{F} \) of \( T_v(s) \). We conclude that
\[
Z_v(s, \phi_v, F_{\psi_{v}'}(\varphi)) = Z_v(s, \phi_v, F_{\psi_{v}}'),
\]
\[
Z_v(s, W_{\psi_{v}'}', W_{\psi_{v}-1}'', \mathcal{F}_{\psi_{v}'}(\varphi)) = |a_v|^{-3s/2-13/2} q_v^{-12d_v} \zeta_v(s+1) \zeta_v(s+2) \zeta_v(s+3) Z_v(s, W_{\psi_{v}}, W_{\psi_{v}-1}'', \mathcal{F}_v').
\]
The assertion then follows immediately.

Now we assume \( v \) is real and \( a > 0 \). Put \( \tilde{W}_{\xi_v} = H_{\xi_v}(\text{diag}(-1, -1, 1, 1)) W_{\psi_{v}}' \) for any additive character \( \xi_v \) of \( F_v \). Let \( \varphi \in S(V_{3,3}^3(F_v)) \) such that \( Z_v(\frac{1}{2}, \phi_v, F_{\psi_{v}}(\varphi)) \neq 0 \). By Lemma 3.4(1), we have
\[
F_{\psi_{v}}'(\varphi) = |a_v|^6 \cdot F_{\psi_{v}}(\varphi'), \quad F_{\psi_{v}}(\varphi) = |a_v|^{-3s/2-9/2} \cdot F_{\psi_{v}}(\varphi'),
\]
where
\[
\varphi' = \omega_{\psi_{v}} \left( \left( \frac{v\sqrt{a}-1}{\sqrt{a}} \frac{d}{\sqrt{a}} \right), 1 \right) \varphi.
\]
Similarly, a change of variables shows that
\[
Z_v(s, W_{\psi_{v}}, \tilde{W}_{\psi_{v}-1}', \mathcal{F}) = |a_v|^{-3s/2+11/2} Z_v(s, W_{\psi_{v}}, \tilde{W}_{\psi_{v}-1}, \mathcal{F})
\]
for any holomorphic section \( \mathcal{F} \) of \( T_v(s) \). We conclude that
\[
Z_v(s, \phi_v, F_{\psi_{v}}'(\varphi)) = |a_v|^6 Z_v(s, \phi_v, F_{\psi_{v}}(\varphi')),
\]
\[
Z_v(s, W_{\psi_{v}}, \tilde{W}_{\psi_{v}-1}', F_{\psi_{v}}(\varphi)) = |a_v|^{-3s+1} Z_v(s, W_{\psi_{v}}, \tilde{W}_{\psi_{v}-1}, F_{\psi_{v}}(\varphi')).
\]
This completes the proof. \( \square \)

**Lemma 3.6.** Let \( \psi \) be a non-trivial additive character of \( F \backslash \mathbb{A} \). Let \( S \) be any finite set of finite places of \( F \) containing the divisors of \( \text{cond}(\Pi) \). For \( a \in \bigcap_{v \in S} \mathfrak{o}_v^\times \cap \mathbb{F}_{>0}^\times \), we have
\[
\prod_{v \in S} C_{\psi_{v}}(\Pi_v) = \prod_{v \in S} C_{\psi_{v}}(\Pi_v).
\]
Here \( \mathbb{F}_{>0}^\times \) denotes the set of totally positive elements in \( F \).

**Proof.** By Proposition 2.4, we have
\[
\prod_{v \in S} C_{\psi_{v}}(\Pi_v) = \prod_{v \in S} C_{\psi_{v}}(\Pi_v).
\]
On the other hand, since \( a \) is totally positive, by Lemma 3.5, we have
\[
\prod_{v \in S} C_{\psi_{v}}(\Pi_v) = \prod_{v \in S} |a_v|^{-8} C_{\psi_{v}}(\Pi_v).
\]
Since \( a \in \bigcap_{v \in S} \mathfrak{o}_v^\times \cap \mathbb{F}^\times \), we have \( \prod_{v \in S} |a_v| = |a|_{\mathbb{A}} = 1 \). This completes the proof. \( \square \)

In the following theorem we prove the Galois equivariance property of the product of local constant \( C_{\psi_{v}}(\Pi_v) \) over finite places.

**Proposition 3.7.** Let \( \psi \) be a non-trivial additive character of \( F \backslash \mathbb{A} \). For \( \sigma \in \text{Aut}(\mathbb{C}) \), we have
\[
\sigma \left( \prod_{v \in S} C_{\psi_{v}}(\Pi_v) \right) = \prod_{v \in S} C_{\psi_{v}}(\sigma(\Pi_v)).
\]
Proof. Let \( v \) be a place of \( \mathbb{F} \) and \( \xi_v \) a non-trivial additive character of \( \mathbb{F}_v \). For \( \varphi_v \in S(V_{\mathbb{Q}_p}(\varphi_v)) \) such that 
\[ Z_v(\frac{1}{2}, \varphi_v, F_{\xi_v}(\varphi_v)) \neq 0, \]
define 
\[ C_{\xi_v}(\Pi_v, \varphi_v) = \zeta_v(1)^{-1} \zeta_v(3)^{-1} \zeta_v(4)L(1, \Pi_v, \text{Ad})^{-1} \]
\[ \times \begin{cases} \zeta_v(2)^{-1} \zeta_v(4)^{-1} \cdot \frac{Z_v(1, W_{\xi_v}, W_{\psi_v}^\vee, F_{\xi_v}(\varphi_v))}{Z_v(1, W_{\xi_v}, \Pi_v^\vee (\text{diag}(-1, -1, 1, 1)) W_{\psi_v}^\vee, F_{\xi_v}(\varphi_v))} & \text{if } v \text{ is finite,} \\
\frac{Z_v}{Z_v} \left( \frac{1}{2}, \varphi_v, F_{\xi_v}(\varphi_v) \right) & \text{if } v \text{ is real.} \end{cases} \]
When \( \xi_v \) is the local component at \( v \) of a non-trivial additive character of \( \mathbb{F}\backslash \mathbb{A} \), it follows from Proposition 2.4 that \( C_{\xi_v}(\Pi_v, \varphi_v) = C_{\xi_v}(\Pi_v) \) is the non-zero constant in (2.9) and does not depend on the choice of \( \varphi_v \).

Let \( \sigma \in \text{Aut}(\mathbb{C}) \). Let \( u \in \mathbb{Z}^\times \) be the unique element such that \( \sigma(\psi(x)) = \psi(ux) \) for \( x \in \mathbb{A}_f \) and any non-trivial additive character \( \psi \) of \( \mathbb{F}\backslash \mathbb{A} \). For each finite place \( v \) lying over a rational prime \( p \) and any non-trivial additive character \( \xi_v \) of \( \mathbb{F}_v \), define the \( \sigma \)-linear isomorphism 
\[ t_{\sigma, v} : \mathcal{W}(\Pi_v, \xi_{U,v}) \rightarrow \mathcal{W}(\sigma \Pi_v, \xi_{U,v}), \]
\[ t_{\sigma, v} W(g) = \sigma W(\text{diag}(u_p^{2s}, u_p, 1, u_p^{-1})). \]
Note that \( \sigma \xi_v = \xi_v^u \). Then \( t_{\sigma, v} W_{\xi_v} \) is the normalized Whittaker newform of \( \sigma \Pi_v \) with respect to \( \xi_{U,v} \) (cf. §2.3 ). We define \( \sigma \)-linear isomorphism \( t_{\sigma, v} : \mathcal{W}(\Pi_v, \xi_{U,v}) \rightarrow \mathcal{W}(\sigma \Pi_v, \xi_{U,v}) \) in the same way. By the Chinese remainder theorem, there exists \( a \in \prod_{p|N_{L/\mathbb{Q}}(\text{cond}(\Pi) \cdot \text{cond}(\psi))} \mathbb{Z}_p^\times \cap \mathbb{Q}^\times \) such that 
\[ au_p = s_p^2 \]
for some \( s_p \in \mathbb{Z}_p^\times \) for \( p \mid N_{L/\mathbb{Q}}(\text{cond}(\Pi) \cdot \text{cond}(\psi)) \). Let \( v \mid \text{cond}(\Pi) \cdot \text{cond}(\psi) \) lying over a prime \( p \). We have 
\[ C_{\psi_v^\sigma}(\Pi_v, \varphi_v) = \frac{\sigma C_{\psi_v^\sigma}(\Pi_v)}{C_{\psi_v^\sigma}(\sigma \Pi_v, \varphi_v)} = \frac{L(1, \sigma \Pi_v, \text{Ad})}{\sigma L(1, \Pi_v, \text{Ad})} \cdot \frac{\sigma Z_v(1, W_{\psi_v^\sigma}, W_{\psi_v}^\vee, F_{\psi_v^\sigma}(\varphi_v))}{Z_v(1, t_{\sigma, v} W_{\psi_v^\sigma}, t_{\sigma, v} W_{\psi_v}^\vee, F_{\psi_v^\sigma}(\varphi_v))} \]
\[ \times \frac{Z_v \left( \frac{1}{2}, \varphi_v, F_{\psi_v^\sigma}(\varphi_v) \right)}{\sigma Z_v \left( \frac{1}{2}, \varphi_v, F_{\psi_v^\sigma}(\varphi_v) \right)}. \]
Note that \( \sigma W_{\psi_v^\sigma} = t_{\sigma, v} W_{\psi_v^\sigma} \) by definition. It follows from Lemmas 2.1-(2), 2.3-(2), 3.3 and 3.4-(2) that the above ratio is equal to 1. Similarly, we also have 
\[ C_{\psi_v^\sigma}(\sigma \Pi_v, \varphi_v') = C_{\psi_v}(\sigma \Pi_v, \varphi_v') = C_{\psi_v}(\sigma \Pi_v), \]
where 
\[ \varphi_v' = \omega_{\psi_v} \left( \begin{pmatrix} s_p^{-1} 1_4 \\ 0 \\ s_p 1_4 \\ 0 \end{pmatrix} \right) \sigma \varphi_v. \]
Indeed, since \( au_p \in (\mathbb{Z}_p^\times)^2 \), by Lemma 3.3-(1) we have 
\[ C_{\psi_v^\sigma}(\sigma \Pi_v, \varphi_v') = \frac{Z_v(1, t_{\sigma, v} W_{\psi_v^\sigma}, t_{\sigma, v} W_{\psi_v}^\vee, F_{\psi_v}(\varphi_v'))}{Z_v(1, t_{\sigma, v} W_{\psi_v}, t_{\sigma, v} W_{\psi_v}^\vee, F_{\psi_v}(\varphi_v'))}. \]
Note that 
\[ t_{\sigma, v} W_{\psi_v^\sigma}(g) = t_{\sigma, v} W_{\psi_v}(\text{diag}(s_p^2, 1, s_p^{-4}, s_p^{-2}) g) \]
and 
\[ F_{\psi_v}(\varphi_v')(\eta(\text{diag}(s_p^2, 1, s_p^{-4}, s_p^{-2}), \text{diag}(s_p^2, 1, s_p^{-4}, s_p^{-2})), s) = F_{\psi_v}(\varphi_v'(\eta g, s)) \]
for \( g \in G(\mathbb{F}_v) \). We see that the above ratio is also equal to 1. We conclude that 
\[ \sigma \left( \prod_{v \mid \text{cond}(\Pi) \cdot \text{cond}(\psi)} C_{\psi_v}(\Pi_v) \right) = \prod_{v \mid \text{cond}(\Pi) \cdot \text{cond}(\psi)} C_{\psi_v}(\sigma \Pi_v). \]
It then follows from Theorem 3.6 that 
\[ \sigma \left( \prod_{v \mid \text{cond}(\Pi) \cdot \text{cond}(\psi)} C_{\psi_v}(\Pi_v) \right) = \prod_{v \mid \text{cond}(\Pi) \cdot \text{cond}(\psi)} C_{\psi_v}(\sigma \Pi_v). \]
Finally, note that for \( u \parallel \infty \cdot \text{cond}(\Pi) \cdot \text{cond}(\psi) \), we have \( C_{\psi_v}(\Pi_v) = C_{\psi_v}(\sigma\Pi_v) = 1 \). This completes the proof.

By Propositions 2.4 and 5.7, our main result Theorem 1.1 then follows from the following result.

**Theorem 3.8.** Let \( v \) be a real place and \( \psi_v \) the standard additive character of \( \mathbb{F}_v \). We have

\[
C_{\psi_v}(\Pi_v) \in \mathbb{Q}^\times.
\]

**Proof.** Applying [Shi12, Theorem 5.7] to \( \text{GSO}_{2,2}(\mathbb{A}_Q) \), there exist cohomological irreducible cuspidal automorphic representations \( \tau_1 \) and \( \tau_2 \) of \( \text{GL}_2(\mathbb{A}_Q) \) with central character inverse to each other such that

\[
\tau_{1,\infty} = D(\lambda_1, \nu - \lambda_2, \nu) \otimes |\nu|^{w/2}, \quad \tau_{2,\infty} = D(\lambda_1, \nu + \lambda_2, \nu) \otimes |\nu|^{-w/2}
\]

and \( \tau_1 \neq \tau_2 \). We regard \( \tau_1 \otimes \tau_2 \) as an irreducible cuspidal automorphic representation of \( \text{GSO}_{2,2}(\mathbb{A}_Q) \) and consider its global theta lift \( \Pi' = \theta(\tau_1 \otimes \tau_2) \) to \( \text{GSp}_4(\mathbb{A}_Q) \). Then \( \Pi' \) is an irreducible motivic globally generic cuspidal automorphic representation of \( \text{GSp}_4(\mathbb{A}_Q) \) such that

\[
\Pi'_\infty = \Pi_v.
\]

Let \( \psi_0 \) be the standard additive character of \( \mathbb{Q} \setminus \mathbb{A}_Q \). Thus \( \psi_{\infty,0} = \psi_v \). By Propositions 2.4 and 5.7 applied to \( \psi_0 \) and \( \Pi' \), we have

\[
\sigma \left( \frac{L(1, \Pi', \text{Ad})}{\zeta(2)\zeta(4)} \cdot C_{\psi_{\infty,0}}(\Pi'_\infty) \right) = \frac{L(1, \sigma\Pi', \text{Ad})}{\zeta(2)\zeta(4)} \cdot C_{\psi_{\infty,0}}(\Pi'_\infty)
\]

for all \( \sigma \in \text{Aut}(\mathbb{C}) \). On the other hand, we proved in Theorem 4.8 below that Theorem 1.1 holds for \( \Pi' \). It follows that

\[
\sigma C_{\psi_{\infty,0}}(\Pi'_\infty) = C_{\psi_{\infty,0}}(\Pi'_\infty)
\]

for all \( \sigma \in \text{Aut}(\mathbb{C}) \). Hence \( C_{\psi_v}(\Pi_v) = C_{\psi_{\infty,0}}(\Pi'_\infty) \in \mathbb{Q} \). This completes the proof. \( \square \)

4. Petersson norms of endoscopic lifts

The purpose of this section is to prove Theorem 1.1 for endoscopic lifts based on Rallis inner product formula. For simplicity, we assume \( \mathbb{F} = \mathbb{Q} \) in this section.

4.1. Cohomological cusp forms on \( \text{GL}_2 \). Let \( \tau = \bigotimes_v \tau_v \) be a cohomological irreducible cuspidal automorphic representation of \( \text{GL}_2(\mathbb{A}) \). There exist \( \kappa \in \mathbb{Z}_{\geq 2} \) and \( w \in \mathbb{Z} \) with \( \kappa \equiv w(\text{mod } 2) \) such that

\[
\tau_{\infty} = D(\kappa) \otimes |\nu|^{w/2}.
\]

Here \( D(\kappa) \) denotes the discrete series representation of \( \text{GL}_2(\mathbb{R}) \) with minimal weight \( \kappa \) and central character \( \text{sgn}^\kappa \). Let \( \tau_+ \) (resp. \( \tau^- \)) be the space of holomorphic (resp. anti-holomorphic) cusp forms in \( \tau \). For a non-trivial additive character \( \psi_v \) of \( \mathbb{Q}_v \), let \( W(\tau_v, \psi_v) \) be the space of Whittaker functions of \( \tau_v \) with respect to \( \psi_v \). When \( v = p \) is finite, for \( \sigma \in \text{Aut}(\mathbb{C}) \), we define the \( \sigma \)-linear isomorphism

\[
t_{\sigma,p} : W(\tau_p, \psi_p) \rightarrow W(\sigma\tau_p, \psi_p),
\]

\[
t_{\sigma,p} W(g) = W(\text{diag}(u_p^{-1}, 1)g),
\]

where \( u_p \in \mathbb{Z}_p^\times \) is the unique element such that \( \sigma(\psi_p(x)) = \psi_p(u_p x) \) for \( x \in \mathbb{Q}_p \). When \( v = \infty \) and \( \psi_\infty = \psi_\infty^0 \), let \( W(\pm\kappa; w, \psi_\infty) \in W(\tau_\infty, \psi_\infty) \) be the weight \( \pm\kappa \) Whittaker function given by

\[
W(\pm\kappa; w, \psi_\infty)(z) = \text{sgn}(\kappa)^{w/2} e^{2\pi i \kappa w/2} \psi(\kappa w/2) \cdot \Pi_{\mathbb{R}_{\geq 0}}(\pm ay) \cdot I_{\mathbb{R}_{\geq 0}}(\pm ay)
\]

for \( x \in \mathbb{R} \), \( y, z \in \mathbb{R}^\times \), and \( \kappa = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in \text{SO}(2) \). Let \( \psi \) be a non-trivial additive character of \( \mathbb{Q} \setminus \mathbb{A} \).

For \( f \in \tau \), let \( W_{f,\psi} \) be the Whittaker function of \( f \) with respect to \( \psi \) defined by

\[
W_{f,\psi}(g) = \int_{\mathbb{Q} \setminus \mathbb{A}} f(n(x)g) \overline{\psi(x)} \, dx_{\text{Tam}}.
\]

Here \( dx_{\text{Tam}} \) is the Tamagawa measure on \( \mathbb{A} \). For \( f \in \tau^\pm \), let \( W_{f,\psi}^{(\infty)} \in \bigotimes_v W(\tau_v, \psi_v) \) be the unique Whittaker function such that

\[
W_{f,\psi} = W_{(\pm\kappa; w, \psi_\infty), f, \psi_\infty}^{(\infty)} W_{f,\psi}^{(\infty)}.
\]
Then we obtain the $GL_2(A_f)$-equivariant isomorphism
$$\tau^\pm \longrightarrow \bigotimes_p W(\tau_p, \psi_p), \quad f \mapsto W_f^{(\infty)}.$$ For $\sigma \in \text{Aut}(\mathbb{C})$, it is well-known that the irreducible admissible representation
$$\sigma_\tau = \tau_\infty \otimes \sigma_{\tau_f}$$
of $GL_2(A)$ is automorphic and cuspidal, where $\tau_f = \bigotimes_p \tau_p$. Let
\begin{equation}
\tau^\pm \longrightarrow \sigma_{\tau^\pm}, \quad f \mapsto \sigma f
\end{equation}
be the $\sigma$-linear $GL_2(A_f)$-equivariant isomorphism defined such that the diagram
$$\begin{array}{ccc}
\tau^\pm & \longrightarrow & \sigma_{\tau^\pm} \\
\downarrow & & \downarrow \\
\bigotimes_p W(\tau_p, \psi_p) & \stackrel{\otimes_p f\tau_p}{\longrightarrow} & \bigotimes_p W(\sigma_{\tau_p}, \psi_p)
\end{array}$$
commutes. In other words, we have
$$W_{\sigma f, \psi}^{(\infty)} = \bigotimes_p t_{\sigma, p} W_{f, \psi}^{(\infty)}.$$  

4.2. Arithmeticity of global theta lifting. Let $(V, Q)$ be the quadratic space over $\mathbb{Q}$ defined by $V = M_{2,2}$ and $Q[x] = \det(x)$. Let $\iota$ be the main involution on $M_{2,2}$ defined by
\begin{equation}
\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}.
\end{equation}
The associated symmetric bilinear form is given by $(x, y) = \text{tr}(xy^\iota)$. We have an exact sequence
\begin{equation}
1 \longrightarrow GL_1 \xrightarrow{\Delta} (GL_2 \times GL_2) \xrightarrow{\rho} GSO(V) \longrightarrow 1,
\end{equation}
where $\Delta(a) = (a1_2, a1_2)$ and $\rho(h_1, h_2)x = h_1 x h_2^{-1}$ for $a \in GL_1, h_1, h_2 \in GL_2$, and $x \in V$. For $h_1, h_2 \in GL_2$, we write $\rho(h_1, h_2) = [h_1, h_2]$. Note that $\nu([h_1, h_2]) = \det(h_1 h_2^{-1})$. For a non-trivial additive character $\psi_v$ of $\mathbb{Q}_v$, we write $\omega_{\psi_v} = \omega_{\psi_v, V, 2}$ for the Weil representation of $Sp_4(V, \mathbb{Q}_v)$ on $S(V^2(Q_v))$ with respect to $\psi_v$. Let $\psi$ be a non-trivial additive character of $\mathbb{Q}\backslash A$. For $\varphi \in S(V^2(A))$, the theta function associated to $\varphi$ with respect to $\psi$ is defined by
$$\Theta_{\psi}(g, h; \varphi) = \sum_{x \in V^2(\mathbb{Q})} \omega_{\psi}(g, h) \varphi(x)$$
for $(g, h) \in G(\mathbb{Q}) \times GSO(V, \mathbb{A})$. Let $f$ be a cusp form on $GSO(V)(\mathbb{A})$ and let $\varphi \in S(V^2(A))$. For $g \in GSp_4(A)$, choose $h \in GSO(V)(\mathbb{A})$ such that $\nu(h) = \nu(g)$, and put
$$\theta_{\psi}(f, \varphi)(g) = \int_{SO(V) \backslash SO(V)(\mathbb{A})} f(h_1 h) \Theta_{\psi}(g, h_1 h; \varphi) \, dh_1^{\text{Tam}}.$$ Here $dh_1^{\text{Tam}}$ is the Tamagawa measure on $SO(V)(\mathbb{A})$. Then $\theta_{\psi}(f, \varphi)$ is an automorphic form on $GSp_4(A)$. 

Let $\tau_1$ and $\tau_2$ be cohomological irreducible cuspidal automorphic representations with central character inverse to each other. There exist $\kappa_1, \kappa_2 \in \mathbb{Z}_{\geq 2}$ and $w \in \mathbb{Z}$ with $\kappa_1 \equiv \kappa_2 \equiv w (\text{mod } 2)$ such that
$$\tau_{1, \infty} = D(\kappa_1) \otimes |w/2 \rangle, \quad \tau_{2, \infty} = D(\kappa_2) \otimes |w/2 \rangle.$$ We assume $\kappa_1 \geq \kappa_2$ and regard $\tau_1 \boxtimes \tau_2$ as an automorphic representation of $GSO(V)(\mathbb{A})$ via the exact sequence (4.2). We assume further that
$$\tau_1 \neq \tau_2^\vee.$$ Then the global theta lift
$$\theta(\tau_1 \boxtimes \tau_2) = \{ \theta_{\psi}(f, \varphi) \mid f \in \tau_1 \boxtimes \tau_2, \varphi \in S(V^2(\mathbb{A})) \}$$
is an irreducible motivic globally generic cuspidal automorphic representation of $GSp_4(A)$ (cf. [GTI1a Theorem 12.1]). As the notation suggests, the global theta lift does not depend on the choice of $\psi$ (cf. [Rob01 Proposition 1.9]). Write
$$\Pi = \theta(\tau_1 \boxtimes \tau_2).$$
Note that $\omega_{\Pi} = \omega_{\tau_1} = \omega_{\tau_2}^{-1}$. Moreover, $\Pi_{\infty}$ is a generic discrete series representation of $\text{GSp}_4(\mathbb{R})$ with

$$\Pi_{\infty}|_{\text{Sp}_4(\mathbb{R})} = D_{(\lambda_1, \lambda_2)} \oplus D_{(-\lambda_2, -\lambda_1)}$$

and

$$\langle \lambda_1, \lambda_2 \rangle = \langle \frac{\lambda_1 + \lambda_2}{2}, \frac{\lambda_1 - \lambda_2}{2} \rangle.$$ 

Let $\Pi_{\text{mot}}$ be the space of cusp forms in $\Pi$ such that the archimedean component is a lowest weight vector of the minimal $U(2)$-type of $D_{(-\lambda_2, -\lambda_1)}$. For $\sigma \in \text{Aut}(\mathbb{C})$ and $\psi_p$ a non-trivial additive character of $\mathbb{Q}_p$, we define the $\sigma$-linear isomorphism

$$t_{\sigma, p}: \mathcal{W}(\Pi_{p}, \psi_{U,p}) \rightarrow \mathcal{W}(\sigma \Pi_{p}, \psi_{U,p}),$$

$$t_{\sigma, p} W(g) = \sigma W(\text{diag}(u_p^{-3} \cdot u_p^{-1} \cdot 1, u_p^{-1}) g),$$

where $u_p \in \mathbb{Z}_p^\times$ is the unique element such that $\sigma(\psi_p(x)) = \psi_p(u_p x)$ for $x \in \mathbb{Q}_p$. For $v = \infty$ and $\psi_\infty$ the standard additive character of $\mathbb{R}$, let $W_{(\lambda_1, \lambda_2; w), \psi_{U, \infty}} \in \mathcal{W}(\Pi_{\infty}, \psi_{U, \infty})$ be the lowest weight Whittaker function of the minimal $U(2)$-type of $D_{(-\lambda_2, -\lambda_1)}$ with respect to $\psi_{U, \infty}$ normalized as in (1.1). For $a \in \mathbb{R}^\times$, we then define $W_{(\lambda_1, \lambda_2; w), \psi_{U, \infty}}(a)$ by

$$W_{(\lambda_1, \lambda_2; w), \psi_{U, \infty}}(g) = W_{(\lambda_1, \lambda_2; w), \psi_{U, \infty}}(\text{diag}(a^3, a^2, 1, a) g).$$

Let $\psi$ be a non-trivial additive character of $\mathbb{Q}\backslash \mathbb{A}$. For $f \in \Pi$, let $W_{f, \psi_U}$ be the Whittaker function of $f$ with respect to $\psi_U$ defined by

$$W_{f, \psi_U}(g) = \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} f(ug) \psi_U(u) \ du_{\text{Tam}}.$$ 

Here $du_{\text{Tam}}$ is the Tamagawa measure on $U(\mathbb{A})$. For $f \in \Pi_{\text{mot}}$, let $W_{f, \psi_U}^{(\infty)} \in \bigotimes_{p} \mathcal{W}(\Pi_{p}, \psi_{U,p})$ be the unique Whittaker function such that

$$W_{f, \psi_U} = W_{(\lambda_1, \lambda_2; w), \psi_{U, \infty}} \cdot W_{f, \psi_U}^{(\infty)}.$$ 

Then we obtain the GSp$_4(\mathbb{A}_f)$-equivariant isomorphism

$$\Pi_{\text{mot}} \rightarrow \bigotimes_{p} \mathcal{W}(\Pi_{p}, \psi_{U,p}), \ f \mapsto W_{f, \psi_U}^{(\infty)}.$$

For $\sigma \in \text{Aut}(\mathbb{C})$, the irreducible admissible representation

$$^\sigma \Pi = \Pi_{\infty} \otimes \sigma \Pi_f$$

of GSp$_4(\mathbb{A})$ is automorphic, globally generic, and cuspidal by Lemma 3.1 where $\Pi_f = \bigotimes_{p} \Pi_{p}$. Let (4.3)

$$\Pi_{\text{mot}} \rightarrow ^\sigma \Pi_{\text{mot}}, \ f \mapsto ^\sigma f$$

be the $\sigma$-linear GSp$_4(\mathbb{A}_f)$-equivariant isomorphism defined such that the diagram

$$\begin{array}{ccc}
\Pi_{\text{mot}} & \longrightarrow & ^\sigma \Pi_{\text{mot}} \\
\bigotimes_{p} \mathcal{W}(\Pi_{p}, \psi_{U,p}) & \longrightarrow & \bigotimes_{p} \mathcal{W}(^\sigma \Pi_{p}, \psi_{U,p})
\end{array}$$

commutes. In other words, we have

$$W_{f, \psi_U}^{(\infty)} = \bigotimes_{p} t_{\sigma, p} W_{f, \psi_U}^{(\infty)}.$$ 

The main result of this section is Proposition 4.5. Roughly speaking, we show that global theta lifting commutes with the Galois actions (1.1) and (1.3).

We begin with the following formula for Whittaker functions of global theta lifts. For each place $v$ of $\mathbb{Q}$, let $dh_{1,v}$ be the Haar measure on $\text{SO}(V(\mathbb{Q}_v))$ defined as in [CH19, §7.1].

**Lemma 4.1.** Let $\psi$ be a non-trivial additive character of $\mathbb{Q}\backslash \mathbb{A}$. Let $\varphi = \bigotimes_v \varphi_v \in S(V^2(\mathbb{A}))$ and $f = f_1 \otimes f_2 \in \tau_1 \otimes \tau_2$ with $W_{\varphi, \psi} = \prod_v W_{1,v}$ and $W_{f_1, \psi} = \prod_v W_{2,v}$. Then we have

$$W_{\theta_{\varphi}(f), \psi_U} = \zeta(2)^{-2} \prod_v \mathcal{W}_{\psi_v}(W_{1,v} \otimes W_{2,v}, \varphi_v).$$
Here

\[ \mathbb{W}_{\psi}(W_{1,v} \otimes W_{2,v}, \varphi_{v})(g_v) = \int_{\Delta N(Q_v) \backslash \text{SO}(V)(Q_v)} (W_{1,v} \otimes W_{2,v})(h_{1,v} h_{v}) \omega_{\psi}(g_v, h_{1,v} h_{v}) \varphi_v(x_0, y_0) d\mathcal{H}_{1,v} \]

for \((g_v, h_v) \in G(\text{Sp}_4 \times \text{O}(V))(Q_v)\) with \(h_v \in \text{GSO}(V)(Q_v)\),

\[ x_0 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad y_0 = a(-1), \]

\(\Delta N = \{ [n(x), n(-x)] \in \text{SO}(V) \mid x \in \mathbb{G}_a \}\), and \(d\mathcal{H}_{1,v}\) is the quotient measure defined by the Haar measures on \(\Delta N(Q_v)\) and \(\text{SO}(V)(Q_v)\).

**Proof.** This is [CT19 Lemma 7.1]. The factor \(\zeta(2)^{-2}\) is due to the comparison between quotient of Tamagawa measures on \(\Delta N(\mathbb{A})\) and \(\text{SO}(V)(\mathbb{A})\) with \(\prod_v d\mathcal{H}_{1,v}\). \(\square\)

For \(v = p\), let \(\varphi_p^0 \in S(V^2(Q_p))\) defined by

\[ \varphi_p^0 = \mathbb{I}_{V^2(Z_p)}. \]

Note that for \(a \in Q_p^\times\) and \(\psi_p\) the standard additive character of \(Q_p\), we have

\[ (4.5) \quad \omega_{\psi_p}(k, [k_1, k_2]) \varphi_p^0 = \varphi_p^0 \]

for \(k \in \text{diag}(1, 1, a, a) \text{GSp}_4(Q_p)\text{diag}(1, 1, a, a)^{-1}\) and \((k_1, k_2) \in \text{GL}_2(Z_p) \times \text{GL}_2(Z_p)\) such that \(\nu(k) = \text{det}(k_1 k_2^{-1})\). When \(\tau_{1,p}\) and \(\tau_{2,p}\) are both unramified, let \(W_{p, a}^\circ \in W(\tau_{1,p}, \psi_p^0) \otimes W(\tau_{2,p}, \psi_p^0)\) be the \(\text{GL}_2(Z_p) \times \text{GL}_2(Z_p)\)-invariant Whittaker function normalized so that

\[ W_{p, a}^\circ((a(a^{-1}), a(a^{-1}))) = 1. \]

By (4.5), the Whittaker function \(\mathbb{W}_{\psi_p}(W_{p, a}^\circ, \varphi_p^0) \in W(\Pi_p, \psi_p^0)\) is \text{diag}(1, 1, a, a) \text{GSp}_4(Q_p)\text{diag}(1, 1, a, a)^{-1}-invariant.

**Lemma 4.2.** Let \(\psi_p\) be the standard additive character of \(Q_p\) and \(a \in Q_p^\times\). Assume both \(\tau_{1,p}\) and \(\tau_{2,p}\) are unramified. We have

\[ \mathbb{W}_{\psi_p}(W_{p, a}^\circ, \varphi_p^0)(\text{diag}(a^{-3}, a^{-2}, a^{-1}, a^{-2})) = \omega_{\Pi_p}(a)^{-2} |a|^{-1}. \]

**Proof.** The computation is similar to [CT19 Lemma 7.3] and we leave the detail to the readers. \(\square\)

For \(v = \infty\), let \(\varphi_{\infty}^\pm \in S(V^2(\mathbb{R}))\) defined by

\[ \varphi_{\infty}^+(x, y) = 2^{\lambda_1 + 1}(\sqrt{-1} x_1 - x_2 - x_3 + \sqrt{-1} x_4) \lambda_1 (y_1 + \sqrt{-1} y_2 - \sqrt{-1} y_3 + y_4) - \lambda_2 e^{-\pi t r(x'y'x')}, \]
\[ \varphi_{\infty}^-(x, y) = 2^{\lambda_1 + 1}(\sqrt{-1} x_1 - x_2 + x_3 + \sqrt{-1} x_4) \lambda_1 (-y_1 + \sqrt{-1} y_2 + \sqrt{-1} y_3 + y_4) - \lambda_2 e^{-\pi t r(x'y'x')} \]

Let \(\psi_{\infty}\) be the standard additive character of \(\mathbb{R}\), then we have

\[ \omega_{\psi_{\infty}}(1, [k_{\theta_1}, k_{\theta_2}]) \varphi_{\infty}^+ = e^{-\pi t r(\kappa_1 \theta_1 + \kappa_2 \theta_2) \varphi_{\infty}^+}, \quad \omega_{\psi_{\infty}}(1, [k_{\theta_1}, k_{\theta_2}]) \varphi_{\infty}^- = e^{-\pi t r(\kappa_1 \theta_1 - \kappa_2 \theta_2) \varphi_{\infty}^-} \]

for \(k_{\theta_1}, k_{\theta_2} \in \text{SO}(2)\), and

\[ (4.6) \quad \omega_{\psi_{\infty}^+}(Z, 1) \cdot \varphi_{\infty}^+ = -\lambda_1 \cdot \varphi_{\infty}^+, \quad \omega_{\psi_{\infty}^+}(Z', 1) \cdot \varphi_{\infty}^+ = -\lambda_2 \cdot \varphi_{\infty}^+, \quad \omega_{\psi_{\infty}^-}(N, 1) \cdot \varphi_{\infty}^+ = 0. \]

Here \(Z, Z', N, \) are elements in \(\text{sp}_4(\mathbb{R}) \otimes \mathbb{R} \mathbb{C}\) defined by

\[ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & \sqrt{-1} \\ -1 & 0 & 0 & \sqrt{-1} \\ 0 & -\sqrt{-1} & 0 & 1 \\ -\sqrt{-1} & 0 & -1 & 0 \end{pmatrix}. \]

Define \(W_{\psi_{\infty}^+} \in W(\tau_{1,\infty}, \psi_{\infty}^+) \otimes W(\tau_{2,\infty}, \psi_{\infty}^+)\) by

\[ W_{\psi_{\infty}^+} = W_{(\kappa_1; w), \psi_{\infty}^+} \otimes W_{(\kappa_2; w), \psi_{\infty}^+} \quad W_{\psi_{\infty}^-} = W_{(\kappa_1; w), \psi_{\infty}^-} \otimes W_{(\kappa_2; w), \psi_{\infty}^-}. \]

By (4.6), the Whittaker function \(\mathbb{W}_{\psi_{\infty}^\pm}(W_{\psi_{\infty}^\pm}, \varphi_{\infty}^\pm) \in W(\Pi_{\infty}, \psi_{\infty}^\pm)\) is a lowest weight Whittaker function of the minimal \(U(2)\)-type of \(D(-\lambda_2, -\lambda_1)\) with respect to \(\psi_{U,\infty}^\pm\).
Lemma 4.3. Let $\psi_\infty$ be the standard additive character of $\mathbb{R}$. We have

$$\mathcal{W}_{\psi_\infty}(W^\pm_\infty, \varphi^\pm_\infty) = W_{(\lambda_1, \lambda_2; w), \psi^\pm_\infty}.$$ 

Proof. The assertion for $\mathcal{W}_{\psi_\infty}(W^+_\infty, \varphi^+_\infty)$ was proved in [C19] Lemma 7.7]. The computation for the other case is similar and we leave the detail to the readers. □

In the following lemma, we establish an explicit relation between $\mathcal{W}_{\psi_v}$ and $\mathcal{W}_{\psi_v^2}$.

Lemma 4.4. Let $\psi_v$ be a non-trivial additive character of $\mathbb{Q}_v$ and $a \in \mathbb{Q}_v^\times$. For $\varphi \in S(V^2(\mathbb{Q}_v))$ and $W \in \mathcal{W}(\tau_1, \psi_v) \otimes \mathcal{W}(\tau_2, \psi_v)$, we have

$$\mathcal{W}_{\psi_v^2}(\ell((a^2), (a^2)))W, \varphi) (\text{diag}(1, 1) \cdot \mathcal{W}_{\psi_v}(W, \varphi_v) \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, 1, 1) \varphi (g)$$

for $g \in \text{GSp}_4(\mathbb{Q}_v)$. Here $\ell((a^2), (a^2))W \in \mathcal{W}(\tau_1, \psi_v^2) \otimes \mathcal{W}(\tau_2, \psi_v^2)$ is defined by

$$\ell((a^2), (a^2))W(h) = W((a^2), (a^2))h).$$

Proof. First note that

$$\omega_{\psi_v^2} = \omega_{\psi_v} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, 1 \circ \omega_{\psi_v} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, 1$$

and

$$\omega_{\psi_v}(\text{diag}(b^{-1}, 1, b, 1), [a(b^{-1}), a(b^{-1})]) \varphi'(x_0, y_0) = |b|^{-1} \varphi'(x_0, y_0)$$

for any $\varphi' \in S(V^2(\mathbb{Q}_v))$ and $b \in \mathbb{Q}_v^\times$. Let $(g, h) \in \text{GSp}_4 \times \text{O}(V)(\mathbb{Q}_v)$ with $h \in \text{GSO}(V)(\mathbb{Q}_v)$. Thus we have

$$\mathcal{W}_{\psi_v^2}(\ell((a^2), (a^2)))W, \varphi) (\text{diag}(1, 1) \cdot \mathcal{W}_{\psi_v}(W, \varphi_v) \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, 1, 1) \varphi (g)$$

for $g \in \text{GSp}_4(\mathbb{Q}_v)$. Here $\ell((a^2), (a^2))W(h) = W((a^2), (a^2))h).$ This completes the proof. □

Let $\psi$ be a non-trivial additive character of $\mathbb{Q}\backslash A$ and $S$ an finite set of finite places of $\mathbb{Q}$. We write

$$\tau_{i,S} = \bigotimes_{p \in S} \tau_{i,p}, \quad \Pi_S = \bigotimes_{p \in S} \Pi_p, \quad Q_S = \prod_{p \in S} Q_p, \quad \psi_S = \prod_{p \in S} \psi_p, \quad \omega_{\psi_S} = \bigotimes_{p \in S} \omega_{\psi_p}$$

for $i = 1, 2$. Assume $\psi$ is standard. For $W_S \in \mathcal{W}(\tau_{1,S}, \psi_S^2) \otimes \mathcal{W}(\tau_{2,S}, \psi_S^2)$, let

$$f_{W_S} \in \tau_{1,S}^+ \otimes \tau_{2,S}^+$$

defined so that

$$W^{(\infty)}_{f_{W_S}, \psi_S} = W_S \cdot \prod_{p \notin S} W^\circ_{p, \pm a}.$$ 

For $a \in \mathbb{Q}^\times$, by the definition of global Whittaker function, we easily see that

$$W^{(\infty)}_{f_{W_S}, \psi_S} = \ell((a, a))W_S \cdot \prod_{p \notin S} W^\circ_{p, \pm a}.$$
For \( \sigma \in \text{Aut}(\mathbb{C}) \), let \( u_\sigma \in \hat{\mathbb{Z}}^\times \) be the unique element such that \( \sigma(\psi(x)) = \psi(u_\sigma x) \) for all \( x \in \mathbb{A}_f \). We have the \( \sigma \)-linear \( \text{SO}(V)(\mathbb{A}_f) \)-equivariant isomorphism

\[
\tau^+_1 \otimes \tau^+_2 \longrightarrow \sigma \tau^+_1 \otimes \sigma \tau^+_2, \quad f \mapsto \sigma f
\]

defined by the \( \sigma \)-linear isomorphisms in (4.1) for \( \tau^+_1 \) and \( \tau^+_2 \).

**Proposition 4.5.** Let \( \psi \) be the standard additive character of \( \mathbb{Q} \setminus \mathbb{A} \) and \( S \) a finite set of finite places of \( \mathbb{Q} \) containing the prime divisors of \( \text{cond}(\Pi) \). Let \( \sigma \in \text{Aut}(\mathbb{C}) \) and \( a \in \mathbb{Q}_{>0} \). Assume \( u_\sigma S = at^2 \) for some \( t \in \mathbb{Q}_0^\times \). For \( \varphi_S \in S(V^2(\mathbb{Q}_S)) \) and \( W_S \in W(\tau_1, S_1, \psi_S^\tau) \otimes W(\tau_2, S_2, \psi_S^\tau) \), we have

\[
\sigma \Pi^{S \cup \{\infty\}}(\text{diag}(1, 1, a, a)) \theta_{\psi^\tau} \left( f_{W_S}, \varphi_S \otimes \varphi_S \otimes \otimes_{p \notin S} \varphi_p^\tau \right)
\]

\[
= \frac{\zeta(2)^2}{\sigma(\zeta(2))^2} \cdot \omega_{H_\infty}(\sqrt{a})^{-1} \cdot \omega_{H_\infty}(a^2 t^3)^{-1} \times \theta_{\psi^\tau a} \left( f_{W_S}, \varphi_S \otimes \varphi_S \otimes \otimes_{p \notin S} \varphi_p^\tau \right)
\]

(4.8)

Here \( \sigma \Pi^{S \cup \{\infty\}} = \otimes_{\psi \in S \cup \{\infty\}} \sigma \Pi_\psi \) and the Galois action on the cusp form on the left-hand side (resp. right-hand side) is defined in (4.3) (resp. (4.7)).

**Proof.** To prove the assertion, it suffices to show that the Whittaker functions of both sides with respect to \( \psi^\tau_U \) are equal. By Lemmas [4.3] and [4.4], we have

\[
\mathcal{W}_{\psi^\tau_{\infty}}(\ell((ab)(ab))) W^\pm_{\infty, \psi^\tau_{\infty}} \left( \begin{pmatrix} \sqrt{b} 1_2 \\ 0 \\ \sqrt{a^{-1}} 1_2 \end{pmatrix}, 1 \right) \varphi^\pm_{\infty} \otimes \varphi^\psi_{\infty} \left( \frac{t^{-1} 1_2}{t_1 2}, 1 \right) \otimes_{p \notin S} \varphi_p^\tau \right)
\]

for all \( b > 0 \). We have

\[
W_{f_{W_S}, \psi^\tau_{\infty} \otimes \psi^\tau_S} = W^\pm_{\infty} \cdot W_S \cdot \prod_{p \notin S} W^\circ_p.
\]

By Lemma [4.1] for the additive character \( \psi^\pm, [(4.1)], \) and [(4.2)] with \( b = 1 \), we see that the Whittaker function of the global theta lift on the left-hand side of (4.8) with respect to \( \psi^\tau_U \) is equal to

\[
\sigma(\zeta(2))^{-2} \cdot W_{(\ell_1, \ell_2; w), \psi^\tau_{\infty}}(g_S) \cdot t_{\sigma, S} \mathcal{W}_{\psi^\tau_S}(W_S, \varphi_S)(g_S) \cdot \prod_{p \notin S} t_{\sigma, p} \mathcal{W}_{\psi^\tau_p}(W^\circ_p, \varphi^\circ_p)(g_p) \cdot \text{diag}(1, 1, a, a)
\]

(4.10)

for \( g \in \text{GSp}_4(\mathbb{A}) \). Let \( p \notin S \). By Lemma [4.2], we have

\[
\mathcal{W}_{\psi^\tau_p}(W^\circ_p, \varphi^\circ_p)(g_p) \cdot \text{diag}(1, 1, a, a)
\]

\[
= \omega_{H_p}(a^2) |a_p| \cdot \mathcal{W}_{\psi^\tau_p}(t_{\sigma, p} W^\circ_p, \varphi^\circ_p)(\text{diag}(a^{-3}, a^{-2}, 1, a^{-1}) g_p)
\]

for \( g_p \in \text{GSp}_4(\mathbb{Q}_p) \). Both \( t_{\sigma, p} \mathcal{W}_{\psi^\tau_p}(W^\circ_p, \varphi^\circ_p) \) and \( \mathcal{W}_{\psi^\tau_p}(t_{\sigma, p} W^\circ_p, \varphi^\circ_p) \) are \( \text{GSp}_4(\mathbb{Z}_p) \)-invariant Whittaker functions of \( \sigma \Pi_p \) with respect to \( \psi^\tau_{U,p} \). By evaluating these Whittaker functions at \( 1 \), we see that they must be equal. Thus we have

\[
t_{\sigma, p} \mathcal{W}_{\psi^\tau_p}(W^\circ_p, \varphi^\circ_p)(g_p) \cdot \text{diag}(1, 1, a, a)
\]

\[
= \omega_{H_p}(a^2) |a_p| \cdot \mathcal{W}_{\psi^\tau_p}(t_{\sigma, p} W^\circ_p, \varphi^\circ_p)(\text{diag}(a^{-3}, a^{-2}, 1, a^{-1}) g_p)
\]

(4.11)

for \( g_p \in \text{GSp}_4(\mathbb{Q}_p) \). Let \( p \in S \). By the explicit formula for the Weil representation, we have

\[
\omega_{\xi_S}(g, h) \varphi'_S = \omega_{\xi_S}(g, h) \varphi'_S
\]

for any non-trivial additive character \( \xi_S \) of \( \mathbb{Q}_S \) and \( \varphi'_S \in S(V^2(\mathbb{Q}_S)) \). It follows that

\[
\sigma \mathcal{W}_{\psi^\tau_S}(W_S, \varphi_S) = \mathcal{W}_{\psi^\tau_{\infty}}(\sigma W_S, \sigma \varphi_S).
\]

21
Therefore, by Lemma 4.4 we have

\[(4.12)\] 
\[t_{\sigma, S} \mathcal{W}_{\psi, \pm} (W_S, \varphi_S, (g_s)) = \sigma_{\omega_H} (W_S, \varphi_S) (\text{diag}(u_{\sigma, S}^{-3}, u_{\sigma, S}^{-2}, 1, u_{\sigma, S}^{-1}) g_S) \]

\[= \mathcal{W}_{\psi, \pm, \sigma} (\sigma_{\varphi_S}) (\text{diag}(u_{\sigma, S}^{-3}, u_{\sigma, S}^{-2}, 1, u_{\sigma, S}^{-1}) g_S) \]

\[= \omega_{\Pi_S} (t)^{-3} |t|_{S}^{-2} \cdot \mathcal{W}_{\psi, \pm, a} \left( \ell((a(t^{-2}), a(t^{-2}))) W_{S}, \omega_{\psi_S} (\mathcal{h}(t^{-1} 1_2 0 1_2), 1) \sigma_{\varphi_S} \right) (\text{diag}(a^{-3}, a^{-2}, 1, a^{-1}) g_S) \]

\[= \omega_{\Pi_S} (t)^{-3} |t|_{S}^{-2} \cdot \mathcal{W}_{\psi, \pm, a} \left( \ell((a(a), a(a))) t_{\sigma, S} W_{S}, \omega_{\psi_S} (\mathcal{h}(t^{-1} 1_2 0 1_2), 1) \sigma_{\varphi_S} \right) (\text{diag}(a^{-3}, a^{-2}, 1, a^{-1}) g_S) \]

for \( g_S \in \text{GSp}_4(Q_S) \). Note that

\[| t |_{S}^{-2} \prod_{p \notin S} \omega_{\Pi_p} (a)^2 | a |_p = \omega_{\Pi_{\infty}} (a)^{-2} | a |_{\infty}^{-1} \omega_{\Pi_S} (a)^{-2} \]

by the product formula and the automorphy of \( \sigma_{\omega_H} = \omega_{\Pi_{\infty}} \cdot \omega_{\Pi_S} \). By (4.11) and (4.12), we conclude that (4.10) is equal to

\[(4.13)\]

\[\sigma(\zeta(2))^{-2} \cdot \omega_{\Pi_{\infty}} (a)^{-2} | a |_{\infty}^{-1} \cdot \omega_{\Pi_S} (a^2 t^3)^{-1} \cdot W_{(\lambda_1, \lambda_2; W), \psi_{\Pi, \infty}} (g) \]

where \( W_{f, \psi_{\Pi}} (g) = W_{f, \psi_{\Pi}} (\text{diag}(a^{-3}, a^{-2}, 1, a^{-1}) g) \).

Since

\[W_{f, \psi_{\Pi}} (g) = W_{f, \psi_{\Pi}} (\text{diag}(a^{-3}, a^{-2}, 1, a^{-1}) g) \]

by Lemma 4.4 for the additive character \( \psi_{\pm} \) and (4.9) with \( b = a \), we deduce that (4.13) is equal to the Whittaker function of the global theta lift on the right-hand side of (4.8) with respect to \( \psi_{\Pi} \). This completes the proof.

4.3. Petersson norms of endoscopic lifts. Let \( \langle \cdot, \cdot \rangle_{SO(V)} : (\tau_1 \boxtimes \tau_2) \times (\tau_1^\vee \boxtimes \tau_2^\vee) \to \mathbb{C} \) and \( \langle \cdot, \cdot \rangle_{\text{GSp}_4} : \Pi \times \Pi^\vee \to \mathbb{C} \) be the Petersson bilinear pairing defined by

\[\langle f, f' \rangle_{SO(V)} = \int_{SO(V)(Q) \backslash SO(V)(A)} f(h_1) f'(h_1) \cdot dh_{1, \text{Tam}}^{\text{Tam}},\]

\[\langle f_1, f_2 \rangle_{\text{GSp}_4} = \int_{A^4 \times \text{GSp}_4(Q) \backslash \text{GSp}_4(A)} f_1(g) f_2(g) \cdot dg_{\text{Tam}}^{\text{Tam}}.\]

Here \( dh_{1, \text{Tam}}^{\text{Tam}} \) and \( dg_{\text{Tam}}^{\text{Tam}} \) are the Tamagawa measures on \( SO(V)(A) \) and \( A^\times \backslash \text{GSp}_4(A) \), respectively. For each place \( v \) of \( Q \), we fix a non-zero \( SO(V)(Q_v) \)-equivariant bilinear pairing

\[\langle \cdot, \cdot \rangle_v : (\tau_1, v) \boxtimes (\tau_2, v) \times (\tau_1^\vee, v) \boxtimes (\tau_2^\vee, v) \to \mathbb{C}.\]

We assume the pairings are chosen so that if \( f = \bigotimes_v f_v \in \tau_1 \boxtimes \tau_2 \) and \( f' = \bigotimes_v f'_v \in \tau_1^\vee \boxtimes \tau_2^\vee \), then \( (f_v, f'_v)_v = 1 \) for almost all \( v \) and

\[\langle f, f' \rangle_{SO(V)} = \prod_v \langle f_v, f'_v \rangle_v.\]

Let \( \mathcal{B}_v : \mathcal{S}(V^2(Q_v)) \times \mathcal{S}(V^2(Q_v)) \to \mathbb{C} \) be the bilinear pairing defined by

\[\mathcal{B}_v(\varphi_v, \varphi_v') = \int_{V^2(Q_v)} \varphi_v(x_v) \varphi_v'(x_v) \cdot dx_v,\]
where \( dx_v \) is defined by the Haar measure on \( Q_v \) in \([1.14]\). Note that \( B_v \) is equivariant under the Weil representation \( \omega_{\psi_v} \otimes \omega_{\psi_{v^{-1}}} \) for any non-trivial additive character \( \psi_v \) of \( Q_v \). For \( f_v \in \tau_{1,v} \otimes \tau_{2,v} \), \( f'_v \in \tau^\vee_{1,v} \otimes \tau^\vee_{2,v} \), and \( \varphi_v, \varphi'_v \in S(V^2(Q_v)) \), we define the local zeta integral

\[
Z_v(f_v, f'_v, \varphi_v, \varphi'_v) = \frac{\zeta_v(2)\zeta_v(4)}{L(1, \tau_{1,v} \otimes \tau_{2,v})} \cdot \int_{SO(V)(Q_v)} B_v(h_{1,v} \cdot \varphi_v, \varphi'_v) (\langle \tau_{1,v} \otimes \tau_{2,v} \rangle(h_{1,v}) f_v, f'_v)_v \, dh_{1,v}.
\]

Here \( L(s, \tau_{1,v} \otimes \tau_{2,v}) \) is the Rankin–Selberg \( L \)-function of \( \tau_{1,v} \otimes \tau_{2,v} \), \( dh_{1,v} \) is the Haar measure on \( SO(V)(Q_v) \) defined as in \([1.13] \S 7.1\), and \( h_{1,v} \cdot \varphi_v(x) = \varphi_v(h_{1,v}^{-1} x) \). Note that the integral converges absolutely (cf. \([GHI] \text{Lemma 7.7}\)). We recall the Rallis inner product formula in the following theorem. Let

\[
L(s, \tau_1 \times \tau_2) = \prod_v L(s, \tau_{1,v} \otimes \tau_{2,v})
\]

be the Rankin–Selberg \( L \)-function of \( \tau_1 \times \tau_2 \).

**Theorem 4.6 (Rallis inner product formula).** Let \( \psi \) be a non-trivial additive character of \( Q \setminus A \). For \( f \in \tau_1 \otimes \tau_2 \), \( f' \in \tau^\vee_1 \otimes \tau^\vee_2 \), and \( \varphi, \varphi' \in S(V^2(A)) \) with \( f = \bigotimes_v f_v, f' = \bigotimes_v f'_v \), \( \varphi = \bigotimes_v \varphi_v, \varphi' = \bigotimes_v \varphi'_v \), we have

\[
\langle \theta_v(f, \varphi), \theta_v^{-1}(f', \varphi') \rangle_{\text{GSp}_4} = 2\zeta(2)^{-2} \cdot \frac{L(1, \tau_1 \times \tau_2)}{\zeta(2)^{\frac{1}{2}}} \cdot \prod_v Z_v(f_v, f'_v, \varphi_v, \varphi'_v).
\]

**Proof.** This is a special case of the Rallis inner product formula proved in \([GQT] \text{Theorem 11.3}\) (see also \([GHI] \text{§7}\)). The factor \( 2\zeta(2)^{-2} \) is the ratio between the Tamagawa measure and the product measure \( \prod_v dh_{1,v} \) on \( SO(V)(A) \).

For each rational prime \( p \) and \( \sigma \in \text{Aut}(\mathbb{C}) \), we fix \( \sigma \)-linear isomorphisms

\[
T_{\sigma,p} : \tau_{1,p} \otimes \tau_{2,p} \rightarrow \sigma \tau_{1,p} \otimes \sigma \tau_{2,p}, \quad T_{\sigma,p}^\vee : \tau_{1,p}^\vee \otimes \tau_{2,p}^\vee \rightarrow \sigma \tau_{1,p}^\vee \otimes \sigma \tau_{2,p}^\vee.
\]

Let \( \langle , \rangle_{\sigma,p} : (\tau_{1,p} \otimes \tau_{2,p}) \times (\sigma \tau_{1,p} \otimes \sigma \tau_{2,p}) \rightarrow \mathbb{C} \) be the bilinear pairing defined by

\[
\langle T_{\sigma,p}^\vee(f_p, f'_p), T_{\sigma,p}^\vee(f''_p) \rangle_{\sigma,p} = \sigma(f_p, f''_p).
\]

for \( f_p \in \tau_{1,p} \otimes \tau_{2,p} \) and \( f''_p \in \tau_{1,p}^\vee \otimes \tau_{2,p}^\vee \). It is easy to verify that \( \langle , \rangle_{\sigma,p} \) is \( SO(V)(Q_p) \)-equivariant (cf. \( \text{Lemma 5.9} \)).

**Lemma 4.7.** Let \( f_p \in \tau_{1,p} \otimes \tau_{2,p} \), \( f'_p \in \tau^\vee_{1,p} \otimes \tau^\vee_{2,p} \), and \( \varphi_p, \varphi'_p \in S(V^2(Q_p)) \).

1. If \( p \nmid \text{cond}(II) \), \( f_p \) and \( f'_p \) are \( SO(V)(Z_p) \)-invariant, and \( \varphi_p = \varphi'_p \), then

\[
Z_p(f_p, f'_p, \varphi_p, \varphi'_p) = \langle f_p, f'_p \rangle_p.
\]

2. We have

\[
\sigma Z_p(f_p, f'_p, \varphi_p, \varphi'_p) = Z_p(T_{\sigma,p} f_p, T_{\sigma,p} f'_p, \sigma \varphi_p, \sigma \varphi'_p)
\]

for all \( \sigma \in \text{Aut}(\mathbb{C}) \).

**Proof.** The first assertion is a special case of \([PSR] \text{Proposition 6.2}\) (see also \([LR] \text{Proposition 3}\)). Let \( \sigma \in \text{Aut}(\mathbb{C}) \). An argument similar to the proof of \( \text{Lemma 5.3} \) shows that

\[
\sigma L(1, \tau_{1,p} \times \tau_{2,p}) = L(1, \sigma \tau_{1,p} \times \sigma \tau_{2,p}).
\]

Note that the integral defining \( Z_p \) is actually a doubling local zeta integral (cf. \([GHI] \S 7\)). Therefore, proceeding as in the proof of \( \text{Lemma 5.10} \) below, we have

\[
\sigma \left( \int_{SO(V)(Q_p)} B_p(h_{1,p} \cdot \varphi_p, \varphi'_p) (\tau_{1,p} \otimes \tau_{2,p})(h_{1,p}) f_p, f'_p \bigotimes dh_{1,p} \right)
\]

\[
= \int_{SO(V)(Q_p)} \sigma B_p(h_{1,p} \cdot \varphi_p, \varphi'_p) (\tau_{1,p} \otimes \tau_{2,p})(h_{1,p}) f_p, f'_p \bigotimes dh_{1,p}.
\]

Since \( \varphi_p \) and \( \varphi'_p \) have compact support, \( B_p(h_{1,p} \cdot \varphi_p, \varphi'_p) \) is a finite sum involving \( \varphi_p \) and \( \varphi'_p \). Thus we have

\[
\sigma B_p(h_{1,p} \cdot \varphi_p, \varphi'_p) = B_p(h_{1,p} \cdot \varphi_p, \varphi'_p).
\]

Also note that

\[
\sigma ((\tau_{1,p} \otimes \tau_{2,p})(h_{1,p}) f_p, f'_p) = ((\sigma \tau_{1,p} \otimes \sigma \tau_{2,p})(h_{1,p}) T_{\sigma,p} f_p, T_{\sigma,p} f'_p)_{\sigma,p}
\]

by definition. This completes the proof. \( \square \)
Theorem 4.8. Theorem 11 holds for $\Pi = \theta(\tau_1 \boxtimes \tau_2)$.

Proof. For $p \nmid \text{cond}(\Pi)$, let $f_p^s \in \tau_1,p \boxtimes \tau_2,p$ and $(f_p^s)^\vee \in \tau_1,p \boxtimes \tau_2,p$ be the SO(V)-invariant vectors defining the restricted tensor products $\boxtimes_v \tau_1,v \boxtimes \tau_2,v$ and $\boxtimes_v \tau_1,v' \boxtimes \tau_2,v'$, respectively. Let

$$f_{\tau_1,\tau_2} = \bigotimes_v f_{\tau_1,\tau_2,v} \in \tau_1^+ \otimes \tau_2^+, \quad (f_{\tau_1,\tau_2})^\vee = \bigotimes_v (f_{\tau_1,\tau_2,v}^\vee) \in (\tau_1^\vee)^+ \otimes (\tau_2^\vee)^+$$

be the normalized newforms of $\tau_1 \boxtimes \tau_2$ and $\tau_1^\vee \boxtimes \tau_2^\vee$, respectively. We assume $f_{\tau_1,\tau_2} = f_p$ and $(f_{\tau_1,\tau_2})^\vee = (f_p)^\vee$ for $p \nmid \text{cond}(\Pi)$. The Petersson norm $\|f_{\tau_1,\tau_2}\|$ of $f_{\tau_1,\tau_2}$ is defined by

$$\|f_{\tau_1,\tau_2}\| = \langle f_{\tau_1,\tau_2}, (\tau_1,\tau_2)[(a(-1),a(-1))]f_{\tau_1,\tau_2} \rangle_{\text{SO(V)}}.$$ 

Let $S$ be the set of prime divisors of $\text{cond}(\Pi)$. Fix

$$f = f_{\tau_1,\tau_2,\infty} \otimes f_S \otimes (\otimes_p f_p^s), \quad f' = (\tau_1,\tau_2)(1,a(-1))f_{\tau_1,\tau_2,\infty} \otimes f_S \otimes (\otimes_p f_p^s),$$

$$\varphi = \varphi_\infty \otimes \varphi_S \otimes (\otimes_p \varphi_p^s), \quad \varphi' = \varphi_\infty \otimes \varphi_S \otimes (\otimes_p \varphi_p^s)$$

for some $f_S \in \tau_1,S \boxtimes \tau_2,S$, $f_p^s \in \tau_1,S' \boxtimes \tau_2,S'$, and $\varphi_S, \varphi_p^s \in S(V^2(Q_S))$ such that

$$(\theta_{\psi}(f,\varphi), \Pi^\vee_\infty(\text{diag}(-1,-1,1,1))\theta_{\psi_{-1}}(f',\varphi'))_{\text{GSp}_4} \neq 0.$$ 

It is clear that $f \in \tau_1^+ \otimes \tau_2^+$ and $f' \in \tau_1^+ \otimes \tau_2^+$. Note that we have the factorization of $L$-functions:

$$L(s,\Pi,Ad) = L(s,\tau_1 \times \tau_2) \cdot L(s,\tau_1,Ad) \cdot L(s,\tau_2,Ad),$$

where $L(s,\tau_i,Ad)$ is the adjoint $L$-function of $\tau_i$ for $i = 1,2$. By Theorem 4.3 and Lemma 4.7(1), we have

$$\frac{\zeta(2)^4 \cdot \langle f_1,\Pi_\infty^\vee(\text{diag}(-1,-1,1,1))\Pi_{-1}^\vee(f',\varphi') \rangle_{\text{GSp}_4}}{\|f_1\|} = \frac{2 \cdot L(1,\Pi,Ad) \cdot \zeta(2)^2 \cdot \|f_{\tau_1,\tau_2}\| \cdot C_\infty \cdot Z_{\text{S}^\infty}(f_S,\varphi_S,\varphi_S^s)}{\|f_{\tau_1,\tau_2}\|},$$

Here

$$C_\infty = \frac{Z_{\infty}(f_{\tau_1,\tau_2,\infty} \otimes f_{\tau_1,\tau_2,\infty} \otimes f_{\tau_1,\tau_2,\infty} \otimes \omega_{\psi_{-1}}(\text{diag}(-1,-1,1,1))\varphi_\infty)}{\langle f_{\tau_1,\tau_2,\infty} \otimes f_{\tau_1,\tau_2,\infty} \rangle_{\text{GSp}_4}}.$$ 

Let $\sigma \in \text{Aut}(S)$. It is easy to see that

$$\sigma \left( \frac{\langle f_1,\Pi_\infty^\vee(\text{diag}(-1,-1,1,1))\Pi_{-1}^\vee(f',\varphi') \rangle_{\text{GSp}_4}}{\|f_1\|} \right) = \frac{\langle f_1,\Pi_\infty^\vee(\text{diag}(-1,-1,1,1))\Pi_{-1}^\vee(f',\varphi') \rangle_{\text{GSp}_4}}{\|f_1\|}$$

for all $f_1 \in \Pi_{\text{mot}}$ and $f_2 \in \Pi_{\text{mot}}$. By the result of Sturm [Stu89], we have

$$\sigma \left( \frac{L(1,\tau_1,Ad) \cdot L(1,\tau_2,Ad)}{\zeta(2)^2 \cdot \|f_{\tau_1,\tau_2}\|} \right) = \frac{\langle f_1,\Pi_\infty^\vee(\text{diag}(-1,-1,1,1))\Pi_{-1}^\vee(f',\varphi') \rangle_{\text{GSp}_4}}{\|f_1\|}.$$ 

By Lemma 4.7 we have

$$\sigma \left( \frac{Z_{\text{S}^\infty}(f_S,\varphi_S,\varphi_S^s)}{\langle f_{\tau_1,\tau_2,\infty} \otimes f_{\tau_1,\tau_2,\infty} \rangle_{\text{GSp}_4}} \right) = \frac{Z_{\text{S}^\infty}(T_{\tau,\tau_1,S}f_{\tau_1,\tau_2,S}^\vee f_{\tau_1,\tau_2,S}^\vee,\varphi_S,\varphi_S^s)}{\langle T_{\tau,\tau_1,S}f_{\tau_1,\tau_2,S}^\vee f_{\tau_1,\tau_2,S}^\vee \rangle_{\text{GSp}_4}}.$$ 

By the Chinese remainder theorem, there exists $a \in \mathbb{Q}_2^\times$ such that $u_{\sigma,s} = at^2$ for some $t \in \mathbb{Q}_2^\times$. Now we apply $\sigma$ to both sides of (1.15). It then follows from Proposition 4.3 that

$$\frac{\zeta(2)^4 \cdot \langle \varphi_{\sigma,a},\Pi_\infty^\vee(\text{diag}(-1,-1,1,1))\Pi_{-1}^\vee(f',\varphi'_{\sigma,a}) \rangle_{\text{GSp}_4}}{\|f_{\Pi}\|} = \frac{2 \cdot \langle L(1,\Pi,Ad) \cdot \zeta(2)^2 \cdot \|f_{\tau_1,\tau_2}\| \cdot L(1,\tau_1,Ad) \cdot L(1,\tau_2,Ad) \cdot \sigma C_\infty \cdot Z_{\text{S}^\infty}(T_{\tau,\tau_1,S}f_{\tau_1,\tau_2,S}^\vee f_{\tau_1,\tau_2,S}^\vee,\varphi_S,\varphi_S^s)}{\langle T_{\tau,\tau_1,S}f_{\tau_1,\tau_2,S}^\vee f_{\tau_1,\tau_2,S}^\vee \rangle_{\text{GSp}_4}}.$$ 

Here

$$\varphi_{\sigma,a} = \omega_{\psi_{-1}} \begin{pmatrix} \sqrt{a} 1_2 & 0 \\ 0 & \sqrt{a}^{-1} 1_2 \end{pmatrix}, \quad \varphi'_{\sigma,a} = \omega_{\psi_{-1}} \begin{pmatrix} \sqrt{a} 1_2 & 0 \\ 0 & \sqrt{a}^{-1} 1_2 \end{pmatrix},$$

$$\varphi_S = \omega_{\psi_{-1}} \begin{pmatrix} t^{-1} 1_2 & 0 \\ 0 & t_2 \end{pmatrix}, \quad \varphi_S^s (\otimes_p \varphi_p^s),$$

$$\varphi_S = \omega_{\psi_{-1}} \begin{pmatrix} t^{-1} 1_2 & 0 \\ 0 & t_2 \end{pmatrix}, \quad \varphi_S^s (\otimes_p \varphi_p^s).$$
By the equivariance under the Weil representation, we have
\[
B_\infty \left( h_{1,\infty} \cdot \omega_{\phi_\infty} \left( \left( \frac{\sqrt{a}}{a}, 0 \right), 1 \right) \varphi_+^{\infty}, \omega_{\phi_\infty} \left( \left( \frac{-\sqrt{a}}{a}, 0 \right), 1 \right) \varphi_\infty \right) = B_\infty \left( h_{1,\infty} \cdot \omega_{\phi_\infty} \left( \left( -1, 1, 1 \right), 1 \right) \varphi_\infty \right),
\]
(4.17)

We denote by \( \gamma_1, \gamma_2 \) the representations as 
\[
\begin{align*}
B_8 \left( h_{1,1} \cdot \omega_{\phi_8} \left( \left( t^{-1}I_2, 0 \right), 1 \right) \varphi_{\sigma, S}, \omega_{\phi_8} \left( \left( t^{-1}I_2, 0 \right), 1 \right) \varphi'_{\sigma, S} \right) = & \ B_8 \left( h_{1,1} \cdot \omega_{\phi_8} \left( \left( -1, 1, 1 \right), 1 \right) \varphi_{\sigma, S}, \omega_{\phi_8} \left( \left( -1, 1, 1 \right), 1 \right) \varphi'_{\sigma, S} \right),
\end{align*}
\]

for all \( h_{1,\infty} \in SO(V)(\mathbb{R}) \) and \( h_{1,1} \in SO(V)(\mathbb{Q}_S) \). We may assume the isomorphisms \( \gamma_1, \gamma_2 \) are normalized so that
\[
\sigma f = f_{\gamma_1, \gamma_2} \circ (\otimes_p s T_{\sigma, p} f_p^s),
\]

\[
\sigma f' = (\gamma_1, \gamma_2)((1, a(-1))) \circ (\otimes_p s T_{\sigma, p} f_p^s).
\]

Then, by Theorem 4.10, we see that the left-hand side of (4.10) is equal to
\[
2 \cdot \frac{L(1, \sigma H, \text{Ad})}{\zeta(2) \zeta(4) \cdot \|f_H\|} \cdot \frac{\zeta(2) \cdot \|f_{\gamma_1, \gamma_2}\|}{L(1, \gamma_1, \text{Ad}) \cdot L(1, \gamma_2, \text{Ad})} \cdot \frac{C_{\infty}}{C_{\infty}}.
\]

By our assumption (4.14), we have \( Z_S(f_s, f'_s, \varphi_s, \varphi'_s) \neq 0 \). We thus conclude that
\[
\sigma \left( \frac{L(1, \sigma H, \text{Ad})}{\zeta(2) \zeta(4) \cdot \|f_H\|} \cdot C_{\infty} \right) = \frac{L(1, \sigma H, \text{Ad})}{\zeta(2) \zeta(4) \cdot \|f_H\|} \cdot C_{\infty}.
\]

Finally, it was proved in [CI19, Lemma 8.10] that \( C_{\infty} \in Q^\times \). This completes the proof. \( \square \)

5. Local Zeta Integrals

In this section, we study the convergence and the Galois equivariant properties of the doubling local zeta integrals and the Rankin-Selberg local zeta integrals defined in (2.2) and (2.6), respectively. The main results are Propositions 5.10 and 5.11.

Let \( \mathbb{F} \) be a non-archimedean local field of characteristic zero. Let \( \mathfrak{o}, \varpi, \) and \( q \) be the maximal compact subring of \( \mathbb{F} \), a generator of the maximal ideal of \( \mathfrak{o} \), and the cardinality of the residue field \( \mathfrak{o}/\varpi \mathfrak{o} \), respectively. Let \( \| \cdot \| \) be the absolute value on \( \mathbb{F} \) normalized so that \( |\varpi| = q^{-1} \). Fix a non-trivial additive character \( \psi \) of \( \mathbb{F} \).

5.1. Some representation theory of \( GSp_4 \). In this section, we recall some results for \( GSp_4 \) on the unitarizability criterion of generic representations and the asymptotic behavior of Whittaker functions.

Let \( Q_1 \) and \( Q_2 \) be the standard Siegel parabolic subgroup and the standard Klingen parabolic subgroup of \( GSp_4 \), respectively, defined by
\[
Q_1 = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in GSp_4 \right\},
\]
\[
Q_2 = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in GSp_4 \right\}.
\]

We denote by \( N_{Q_1} \) and \( N_{Q_2} \) the unipotent radical of \( Q_1 \) and \( Q_2 \), respectively. The standard Levi components of \( Q_1 \) and \( Q_2 \) are given by
\[
M_{Q_1} = \left\{ m_1(A, \nu) = \begin{pmatrix} A & 0 \\ 0 & \nu^t A^{-1} \end{pmatrix} \mid A \in GL_2, \nu \in GL_1 \right\},
\]
\[
M_{Q_2} = \left\{ m_2(t, g) = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & \nu^t & 0 \\ 0 & c & d & 0 \end{pmatrix} \mid t \in GL_1, \ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in GL_2, \ \nu = \det(g) \in GL_1 \right\}.
\]
Let $M^1_\Omega = M_{Q^1} \cap \text{Sp}_4$ for $i = 1, 2$. For a character $\chi$ of $\mathbb{F}^\times$ and an irreducible admissible representation $\tau$ of $GL_2(\mathbb{F})$, let $\tau \times \chi$ be the normalized induced representation acting via the right translation $\rho$ on the space consisting of smooth functions $f : \text{GSp}_4(\mathbb{F}) \to \mathcal{V}_\tau$ such that

$$f(nm_1(A, \nu)g) = \delta_{Q_2}^{1/2}(m_1(A, \nu))\chi(\nu)\tau(A)f(g)$$

for all $n \in N_{Q_1}(\mathbb{F})$, $m_1(A, \nu) \in M_{Q_1}(\mathbb{F})$, and $g \in \text{GSp}_4(\mathbb{F})$. Here $\mathcal{V}_\tau$ is the representation space of $\tau$ and $\delta_{Q_1}$ is the modulus character of $Q_1(\mathbb{F})$ given by $\delta_{Q_1}(m_1(A, \nu)) = |\det(A)|^{|\nu|/2}$. Similarly, let $\chi \times \tau$ be the normalized induced representation acting via the right translation $\rho$ on the space consisting of smooth functions $f : \text{GSp}_4(\mathbb{F}) \to \mathcal{V}_\tau$ such that

$$f(nm_2(t, g')g) = \delta_{Q_2}^{1/2}(m_2(t, g'))\chi(t)\tau(g')f(g)$$

for all $n \in N_{Q_2}(\mathbb{F})$, $m_2(t, g') \in M_{Q_2}(\mathbb{F})$, and $g \in \text{GSp}_4(\mathbb{F})$. Here $\mathcal{V}_\tau$ is the representation space of $\tau$ and $\delta_{Q_2}$ is the modulus character of $Q_2(\mathbb{F})$ given by $\delta_{Q_2}(m_2(t, g)) = |t|^{|\nu|/2}$. Note that the central characters of $\tau \times \chi$ and $\chi \times \tau$ are equal to $\omega_\tau \chi^2$ and $\omega_\tau \chi$, respectively. By the results of Sally and Tadić [ST93], any non-supercuspidal irreducible admissible generic representation of $\text{GSp}_4(\mathbb{F})$ is the generic subrepresentation of an induced representation in one of the following types:

(I) $\text{Ind}_{B(\mathbb{F})}^{GL_2(\mathbb{F})}(\chi_1 \boxtimes \chi_2) \times \chi$ for some characters $\chi_1, \chi_2, \chi$ of $\mathbb{F}^\times$ such that $\chi_1 \neq |1|^{\pm 1}$, $\chi_2 \neq |1|^{\pm 1}$, and $\chi_1 \neq |1|^{\pm 1}, \chi_2^{\pm 1}$;

(IIa) $(\text{St} \otimes \mu) \times \chi$ for some characters $\mu, \chi$ of $\mathbb{F}^\times$ such that $\mu \neq |1|^{\pm 3/2}$ and $\mu^2 \neq |1|^{\pm 1}$;

(IIb) $\chi \times (\text{St} \otimes \mu)$ for some characters $\mu, \chi$ of $\mathbb{F}^\times$ such that $\chi \neq 1$ and $\chi \neq |1|^{\pm 2}$;

(IVa) $|\sigma|^{1/2} \times (\text{St} \otimes \mu)$ for some character $\mu$ of $\mathbb{F}^\times$;

(Va) $\chi \times (\text{St} \otimes \mu)$ for some characters $\mu, \chi$ of $\mathbb{F}^\times$ such that $\mu^2 = |1|$ and $\mu \neq |1|^{1/2}$;

(Vb) $1 \times (\text{St} \otimes \mu)$ for some character $\mu$ of $\mathbb{F}^\times$;

(VII) $\chi \times \tau$ for some character $\chi$ of $\mathbb{F}^\times$ and irreducible supercuspidal representation $\tau$ of $GL_2(\mathbb{F})$ such that $\chi \neq 1$ and either $\chi^2 \neq |1|^{\pm 2}$ or $\chi = |1|^{\pm 1}$ and $\tau \otimes \chi |^{\pm 1} \neq \tau$;

(VIIa) $1 \times \tau$ for some irreducible supercuspidal representation $\tau$ of $GL_2(\mathbb{F})$;

(VIa) $\chi \times \tau$ for some character $\chi$ of $\mathbb{F}^\times$ and irreducible supercuspidal representation $\tau$ of $GL_2(\mathbb{F})$ such that $\chi^2 = |1|^{\pm 2}$, $\chi \neq |1|$, and $\tau \otimes \chi |^{\pm 1} = \tau$;

(X) $\tau \times \chi$ for some character $\chi$ of $\mathbb{F}^\times$ and irreducible supercuspidal representation $\tau$ of $GL_2(\mathbb{F})$ such that $\omega_\tau \neq |1|^{\pm 1}$;

(XIIa) $\tau \times \chi$ for some character $\chi$ of $\mathbb{F}^\times$ and irreducible supercuspidal representation $\tau$ of $GL_2(\mathbb{F})$ such that $\omega_\tau = |1|$.

Here $\text{St}$ denotes the Steinberg representation of $GL_2(\mathbb{F})$ and we follow [RS07] for the labelling of types.

**Lemma 5.1.** Let $\Pi$ be a non-supercuspidal irreducible admissible generic representation of $\text{GSp}_4(\mathbb{F})$. Assume $\Pi$ is unitary. If $\Pi$ is of one of the types (IIa), (VIa), and (VIIa), then the inducing data are unitary. In the remaining cases, the following conditions are satisfied:

(I) $|e(\chi_1)| + |e(\chi_2)| < 1$ and $\chi_1 \chi_2 \chi^2$ is unitary;

(IIa) $|e(\mu)| < \frac{1}{2}$ and $\mu \chi$ is unitary;

(IVa) $|\mu|$ is unitary;

(Va) $|\chi|$ is unitary;

(VII) $|e(\chi)| < 1$ and $\omega_\tau \chi$ is unitary;

(XIIa) $|\omega_\tau|$ is unitary;

(XIIa) $|\omega_\tau|$ is unitary;

Moreover, if $\Pi$ is of one of the types (IVa), (Va), (IXa), and (XIIa), then $\Pi$ is a discrete series representation.

**Proof.** The conditions for unitarizability were proved in [ST93] Theorem 4.4, Propositions 4.7 and 4.9]. The assertion for discrete series representations was proved in [ST93] Theorem 4.1, Propositions 4.6 and 4.8. □

**Lemma 5.2.** Let $\Pi$ be an irreducible admissible generic representation of $\text{GSp}_4(\mathbb{F})$. There exist a finite set $\mathcal{H}$ of characters of $\mathbb{F}(\mathbb{F})$ and a positive integer $N_\Pi$ such that for any Whittaker function $W$ of $\Pi$, we have

$$W(tk) = \delta_B(t)^{1/2} \sum_{0 \leq n_1 \leq N_\Pi} \sum_{0 \leq n_2 \leq N_\Pi} \sum_{\eta \in \mathcal{H}} \eta(t)(\log_q |a|)^{n_1}(\log_q |b|)^{n_2} \varphi_{n_1, n_2, n}(a, b, k)$$

26
for some locally constant function $\varphi_{n_1,n_2,\eta}$ on $\mathbb{F} \times \mathbb{F} \times \text{GSp}_4(\mathfrak{o})$ with compact support for $0 \leq n_1, n_2 \leq N$ and $\eta \in \mathfrak{x}_n$. Here $t = \text{diag}(ab, a, b^{-1}, 1) \in \mathbf{T}(\mathbb{F})$. Moreover, the set $\mathfrak{x}_n$ is given as follows: for $\eta \in \mathfrak{x}_n$,

$$
\eta(\text{diag}(ab, a, b^{-1}, 1)) = \eta_1(a)\eta_2(b)
$$

with $\eta_1 = |e^{(\omega_2)/2}$, $\eta_2 = 1$ if $H$ is supercuspidal and

(I) $(\eta_1, \eta_2) \in \{(x, x^{-1}), (x, x^{-1}), (x, x^{-1}), (x, x^{-1}), (x, x^{-1}), (x, x^{-1}), (x, x^{-1}), (x, x^{-1}), (x, x^{-1})\};$

IIa) $(\eta_1, \eta_2) \in \{(| \eta |^{1/2} \mu^{-1}), (| \eta |^{1/2} \mu^{-1}), (| \eta |^{1/2} \mu^{-1}) | \eta \in \mathfrak{m}^\infty, \eta \in \mathfrak{m}^\infty \};$

IIa) $(\eta_1, \eta_2) \in \{(| \eta |^{1/2} \mu^{-1}), (| \eta |^{1/2} \mu^{-1}) | \eta \in \mathfrak{m}^\infty, \eta \in \mathfrak{m}^\infty \};$

IVA) $\eta_1 = |^{5/2} \mu, \eta_2 = |^{5/2} \nu ;$

IVa) $\eta_1 = |^{5/2} \mu, \eta_2 = |^{5/2} \nu ;$

VII) $\eta_1 = |^{5/2} \mu, \eta_2 = |^{5/2} \nu ;$

VII) $\eta_1 = |^{5/2} \mu, \eta_2 = |^{5/2} \nu ;$

VIIIa) $\eta_1 = |^{5/2} \mu, \eta_2 = |^{5/2} \nu ;$

IXA) $\eta_1 = |^{5/2} \mu, \eta_2 = |^{5/2} \nu ;$

X) $\eta_1 = |^{5/2} \mu, \eta_2 = |^{5/2} \nu ;$

XIa) $\eta_1 = |^{5/2} \mu, \eta_2 = |^{5/2} \nu ;$

XIa) $\eta_1 = |^{5/2} \mu, \eta_2 = |^{5/2} \nu ;$

Proof. The assertion follows from the result of Lapid and Mao \cite{LM09} Theorem 3.1 for the special case $G = \text{GSp}_4$ (see also \cite{Jum99} p. 155, Proposition 1.1.1). The formula for $\mathfrak{x}_n$ is then a consequence of the explicit formula for the semisimplification of the normalized Jacquet module of $H$ with respect to the parabolic subgroup of $\text{GSp}_4$ determined by the cuspidal support of $H$ (cf. \cite{RS07} Tables A.3 and A.4). For types (I)-(VIIIa), we consider the Borel subgroup $B$. For types (VII), (VIIIa), and (IXa) (resp. (X) and (XIa)), we consider the Klingen parabolic subgroup $Q_2$ (resp. Siegel parabolic subgroup $Q_1$).

\textbf{Remark 5.3.} If $H$ is either supercuspidal or of one of the types (VII), (VIIIa), and (IXa), then $\varphi_{n_1,n_2,\eta} = 0$ when $|a|$ is sufficiently small. Thus in this case, $\eta_1$ can be any character. We take $\eta_1 = |^{(\omega_2)/2}$ so that the estimation in the proof of Proposition \ref{prop:analogue} is more uniform.

The following lemma is on the asymptotic behavior of matrix coefficients for $\text{GL}_2$ and will be used in the proof of Proposition \ref{prop:analogue}.

\textbf{Lemma 5.4.} Let $\tau$ be an irreducible admissible generic representation of $\text{GL}_2(\mathbb{F})$ and $\phi$ a matrix coefficient of $\tau$. There exist characters $\eta_1, \eta_2$ of $\mathbb{F}^\times$ and locally constant functions $\Phi_1, \Phi_2$ on $\mathbb{F} \times \text{GL}_2(\mathfrak{o}) \times \text{GL}_2(\mathfrak{o})$ such that

$$
\phi(t, k_1, k_2) = \eta_1(t)|t|^{\omega/2} \Phi_1(t, k_1, k_2) + \eta_2(t)|t|^{\omega/2} \Phi_2(t, k_1, k_2)
$$

for $(t, k_1, k_2) \in \{(0, \chi_1, \chi_2) \times \text{GL}_2(\mathfrak{o}) \times \text{GL}_2(\mathfrak{o})$. Moreover, if $\tau = \text{Ind}_{\text{B}(\mathbb{F})}^{\text{GL}_2(\mathbb{F})}(\chi_1 \boxtimes \chi_2)$ for some characters $\chi_1, \chi_2$, then $\eta_1, \eta_2 \in \{\chi_1, \chi_2\}$. If $\tau = \text{St} \otimes \mu$ for some character $\mu$, then $\eta_1 = \eta_2 = |\mu|^{1/2}$. If $\tau$ is supercuspidal, then $\eta_1 = \eta_2 = |^{(\omega_2)/2}$.

Proof. This is well-known. Indeed, the assertion follows from the explicit computation of the normalized Jacquet module of $\tau$ with respect to $B$ (cf. \cite{GHII} § 8.12).

\textbf{Remark 5.5.} If $\tau$ is supercuspidal, then $\Phi_1 = \Phi_2 = 0$ when $|t|$ sufficiently small. Thus in this case, $\eta_1$ and $\eta_2$ can be any characters. We take $\eta_1 = \eta_2 = |^{(\omega_2)/2}$ so that the estimation in the proof of Proposition \ref{prop:analogue} is more uniform.

\textbf{5.2. Intertwining operators.} Let $B_8$ be the standard Borel subgroup of $\text{Sp}_8$ defined by

$$
B_8 = \begin{pmatrix}
* & * & * & * & * & * & * & * \\
0 & * & * & * & * & * & * & * \\
0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * \\
\end{pmatrix} \in \text{Sp}_8
$$
Denote by $N_S$ the unipotent of $B_S$ and $T_S \subset B_S$ the maximal torus of $\text{Sp}_8$ consisting of diagonal matrices. Let $W_S = N_{\text{Sp}_8}(T_S)/T_S$ be the Weyl group of $T_S$ in $\text{Sp}_8$. For $i = 1, 2, 3, 4$, let $\epsilon_i : T_S \rightarrow \text{GL}_1$ be the algebraic character defined by

$$\epsilon_i(\text{diag}(t_1, t_2, t_3, t_4, t_1^{-1}, t_2^{-1}, t_3^{-1}, t_4^{-1})) = t_i.$$  

The set of positive roots for $(\text{Sp}_8, T_S)$ is given by

$$\{\epsilon_i \pm \epsilon_j | 1 \leq i < j \leq 4, k = 1, 2, 3, 4\}.$$  

For each positive root $\epsilon$, we normalize the associated embedding $\iota_\epsilon : \text{SL}_2 \rightarrow \text{Sp}_8$ as follows:

$$\iota_{\epsilon_i + \epsilon_j}(n(x)) = \begin{pmatrix} 1 & x(E_{i,j} + E_{j,i}) \\ 0 & 1 \end{pmatrix}^t, \quad \iota_{\epsilon_i - \epsilon_j}(n(x)) = \begin{pmatrix} 1 & xE_{i,j} \\ 0 & 1 - xE_{j,i} \end{pmatrix}^t, \quad \iota_{2\epsilon_i}(n(x)) = \begin{pmatrix} 1 & xE_{i,i} \\ 0 & 1 \end{pmatrix}^t.

(5.1)

Here $E_{i,j} \in M_{4,4}$ is the matrix with $(i, j)$-entry equal to 1 and zero otherwise. Let $N_\epsilon \subset \text{Sp}_8$ be the image of the unipotent radical of $B$ under $\iota_\epsilon$ and identify $N_\epsilon$ with $G_a$ via $\iota_\epsilon$. For a character $\chi$ of $T_S(\mathbb{F})$, let $I(\chi)$ be the normalized induced representation acting via the right translation $\rho$ on the space consisting of smooth functions $f : \text{Sp}_8(\mathbb{F}) \rightarrow \mathbb{C}$ such that

$$f(n t g) = \delta_{B_S}^{1/2}(t) \chi(t) f(g)$$

for $n \in N_\epsilon(\mathbb{F})$, $t \in T_S(\mathbb{F})$, and $g \in \text{Sp}_8(\mathbb{F})$. The modulus character $\delta_{B_S}$ of $B_S(\mathbb{F})$ is given by

$$\delta_{B_S}(\text{diag}(t_1, \ldots, t_4, t_1^{-1}, \ldots, t_4^{-1})) = |t_1|^3 |t_2|^6 |t_3|^4 |t_4|^2.$$  

For $w \in W_S$, we define the intertwining operator

$$M_w : I(\chi) \rightarrow I(\chi^w),$$

$$M_w f(g) = \int_{N_w(\mathbb{F})} f(w n g) \, d n.$$  

(5.2)

Here $\chi^w(t) = \chi(w t w^{-1})$ and

$$N_w = \prod_{\epsilon \in \chi} N_\epsilon.$$  

The Haar measure $d n$ is normalized so that $\text{vol}(N_w(\mathbb{F}), d n) = 1$. The integral is absolutely convergent if $\chi$ belongs to some open subset and can be meromorphically continued to all $\chi$. If we write $w = w_1 \cdots w_\ell$ into a reduced decomposition, then we have

$$M_w = M_{w_\ell} \circ \cdots \circ M_{w_1}.$$  

(5.3)

The following lemma is on the analytic and Galois equivariant properties of the intertwining integrals in the simplest case. For characters $\chi_1, \chi_2$ of $\mathbb{F}^\times$, let $\text{ind}_{\text{B}_2(\mathbb{F})}^{\text{GL}_2(\mathbb{F})}(\chi_1 \boxtimes \chi_2)$ be the (non-normalized) induced representation on the space consisting of smooth functions $f : \text{GL}_2(\mathbb{F}) \rightarrow \mathbb{C}$ such that

$$f(n(x)a(t_1)d(t_2)g) = \chi_1(t_1)\chi_2(t_2)f(g)$$

for $x \in \mathbb{F}$, $t_1, t_2 \in \mathbb{F}^\times$, and $g \in \text{GL}_2(\mathbb{F})$.

**Lemma 5.6.** Let $\chi_1, \chi_2$ be characters of $\mathbb{F}^\times$ and $f \in \text{ind}_{\text{B}_2(\mathbb{F})}^{\text{GL}_2(\mathbb{F})}(\chi_1 \boxtimes \chi_2)$. The intertwining integral

$$\int_{\mathbb{F}} f(w n(x)) \, d x$$

is absolutely convergent if $e(\chi_1 \chi_2^{-1}) > 1$. We have

$$\sigma \left( \int_{\mathbb{F}} f(w n(x)) \, d x \right) = \int_{\mathbb{F}} \sigma(f(w n(x))) \, d x$$

for all $\sigma \in \text{Aut}(\mathbb{C})$ when both sides are absolutely convergent.
Proof. Indeed, we have
\[ \int_{\mathcal{F}} f(wn(x)) \, dx = \int_{|x| \leq q^N} f(wn(x)) \, dx + \chi_1(-1)\chi_2(-1)f(1) \int_{|x| > q^N} \chi_1^{-1}\chi_2(x) \, dx. \]
Here \( N \) is sufficiently large so that
\[ f\left( \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right) = f(1) \]
for all \( |x| < q^{-N} \). The first integral is a finite sum and the second integral converges for \( e(\chi_1\chi_2^{-1}) > 1 \). For \( \sigma \in \text{Aut}(\mathbb{C}) \), we have
\[ \sigma \left( \int_{|x| \leq q^N} f(wn(x)) \, dx \right) = \int_{|x| \leq q^N} \sigma(f(wn(x))) \, dx \]
and
\[ \sigma \left( \int_{|x| > q^N} \chi_1^{-1}\chi_2(x) \, dx \right) = \int_{|x| > q^N} \sigma\chi_1^{-1}\chi_2(x) \, dx \]
for \( e(\chi_1\chi_2^{-1}) > 1 \) and \( e(\sigma\chi_1\chi_2^{-1}) > 1 \). The assertions then follow at once. \( \square \)

Let \( \chi \) be a character of \( T_8(\mathbb{F}) \). For \( \sigma \in \text{Aut}(\mathbb{C}) \), we have the \( \sigma \)-linear isomorphism
\[ I(\chi) \rightarrow I(\sigma\chi), \quad f \mapsto \sigma f \]
with \( \sigma f(g) = \sigma(f(g)) \) for \( g \in \text{Sp}_8(\mathbb{F}) \).

Corollary 5.7. Let \( \chi \) be a character of \( T_8(\mathbb{F}) \) and \( w \in W_8 \). Then we have
\[ \sigma(M_w f(g)) = M_w \sigma f(g) \]
for all \( \sigma \in \text{Aut}(\mathbb{C}), f \in I(\chi), \) and \( g \in \text{Sp}_8(\mathbb{F}) \) when both sides are absolutely convergent.

Proof. The assertion is an immediate consequence of (5.3) and Lemma 5.6. \( \square \)

5.3. Doubling local zeta integrals. Let \( H_1 = \text{GL}_4 \) and \( H_2 = \text{GL}_2 \times \text{Sp}_4 \). We define embeddings
\[ \iota_1: H_1 \rightarrow \text{Sp}_8, \quad g \mapsto \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \]
\[ \iota_2: H_2 \rightarrow \text{Sp}_8, \quad \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \mapsto \begin{pmatrix} a & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & a_{12} & 0 & b_{11} & 0 & b_{12} \\ c & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{21} & 0 & a_{22} & 0 & b_{21} & 0 & b_{22} \\ 0 & 0 & 0 & a' & 0 & b' & 0 & 0 \\ 0 & c_{11} & 0 & c_{12} & 0 & d_{11} & 0 & d_{12} \\ 0 & 0 & 0 & c' & 0 & d' & 0 & 0 \\ 0 & c_{21} & 0 & c_{22} & 0 & d_{21} & 0 & d_{22} \end{pmatrix}. \]
Here \( \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} \), \( A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), \) and \( D = (d_{ij}) \). Recall we have identify \( \text{Sp}_4 \times \text{Sp}_4 \) as a subgroup of \( \text{Sp}_8 \) via the embedding \( (2.1) \). The image of \( M_{\tilde{Q}_i}^1 \times M_{\tilde{Q}_i}^1 \subset \text{Sp}_4 \times \text{Sp}_4 \) in \( \text{Sp}_8 \) factors through the embedding \( \iota_1 \), thus induces an embedding from \( M_{\tilde{Q}_i}^1 \times M_{\tilde{Q}_i}^1 \) into \( H_1 \). We identify \( M_{\tilde{Q}_i}^1 \times M_{\tilde{Q}_i}^1 \) as a subgroup of \( H_i \) in this way. Let \( R_i \) be the maximal parabolic subgroup of \( H_i \) defined by
\[ R_1 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} \in H_1 \right\}, \]
\[ R_2 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * \\ * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} \in H_2 \right\}. \]

29
Put

\[
\xi_i = \begin{cases} 
\left( \begin{array}{cc} 0 & 1_2 \\ 1_2 & -1_2 \end{array} \right) & \in H_1(\mathbb{F}) \\
\left( \begin{array}{ccc} 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & -1 & 0 \end{array} \right) & \in H_2(\mathbb{F}) \text{ if } i = 1,
\end{cases}
\]

Note that we have \(\xi_i(m, m)\xi_i^{-1} \in R_i(\mathbb{F})\) for all \(m \in M_{Q_i}(\mathbb{F})\).

Let \(F \in I(s)\) be a holomorphic section, where \(I(s)\) is the degenerate principal series representation defined in §[2.1]. For \(i = 1, 2\), we define the intertwining integral

\[
\Psi_i(g, s; F) = \int_{N_{Q_i}(\mathbb{F})} F(\delta(n, 1)\xi_i^{-1}(\xi_i^{-1} g), s) \, dn,
\]

where \(g \in H_i(\mathbb{F})\). By [LR05] Proposition 1, the integral \(\Psi_i(g, s; F)\) converges absolutely for \(\text{Re}(s)\) sufficiently large and admits meromorphic continuation to \(s \in \mathbb{C}\). Moreover, \(\Psi_i(\ ; F) \in \text{Ind}_{H_i(\mathbb{F})}^{H_i(\mathbb{F})}(\mu_i, s)\), where \(\mu_i, s\) is the character of the standard Levi component of \(R_i(\mathbb{F})\) given by

\[
\mu_{1, s} \left( \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right) = |\det(A)|^{s+3/2} |\det(D)|^{-s+3/2},
\]

\[
\mu_{2, s} \left( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & tA^{-1} \end{pmatrix} \right) = |a|^{s+2} |d|^{-s+2} |\det(A)|^s.
\]

In the following lemma, we explicitly determine the region of convergence and prove the Galois equivariant property of the intertwining integrals.

**Lemma 5.8.** Let \(F \in I(s)\) be a holomorphic section. For \(i = 1, 2\), the integral \(\Psi_i(g, s; F)\) converges absolutely for \(\text{Re}(s) > \frac{1}{2}\) and satisfies the Galois equivariant property

\[
\sigma \Psi_i(g, \frac{n}{2}; F) = \Psi_i(g, \frac{n}{2}; \sigma F)
\]

for all \(\sigma \in \text{Aut}(\mathbb{C}), g \in H_i(\mathbb{F})\), and odd positive integers \(n\).

**Proof.** Let \(w_1, w_2 \in W_8\) be Weyl elements defined by

\[
w_1 = \nu_{2\epsilon_4}(w)\nu_{\epsilon_3-\epsilon_4}(w)\nu_{2\epsilon_4}(w), \quad w_2 = \nu_{2\epsilon_4}(w)\nu_{\epsilon_3-\epsilon_4}(w)\nu_{\epsilon_2-\epsilon_3}(w).
\]

Here \(\nu_{\epsilon}\) is the embedding defined in [5.1] for each positive root \(\epsilon\). A direct calculation shows that

\[
\delta(N_{Q_1}(\mathbb{F}), 1)\delta^{-1} \in P(\mathbb{F}) \cdot t_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} w_1 N_{w_1}(\mathbb{F}) w_1^{-1} t_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]

\[
\delta(N_{Q_2}(\mathbb{F}), 1)\delta^{-1} \in P(\mathbb{F}) \cdot t_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} w_2 N_{w_2}(\mathbb{F}) w_2^{-1} t_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

Note that \(F \in I(s) \subset I(\chi_s)\), where \(\chi_s\) is the character of \(T_8(\mathbb{F})\) defined by

\[
\chi_s(\text{diag}(t_1, \cdots, t_4, t_1^{-1}, \cdots, t_4^{-1})) = |t_1|^{s-3/2} |t_2|^{s-1/2} |t_3|^{s+1/2} |t_4|^{s+3/2}.
\]
Therefore, we have

\[ \Psi_1(g, s; F) = M_{w_1} F \left( \begin{array}{c}
  w_1^{-1} t_1 \\
  0 0 0 1 \\
  0 0 1 0 \\
  0 1 0 0 \\
  1 0 0 0 \\
\end{array} \right) \delta_1(\xi_1^{-1} g), \]

\[ \Psi_2(g, s; F) = M_{w_2} F \left( \begin{array}{c}
  w_2^{-1} t_1 \\
  0 0 1 0 \\
  0 0 0 1 \\
  0 1 0 0 \\
  1 0 0 0 \\
\end{array} \right) \delta_2(\xi_2^{-1} g), \]

for \( g \in H_s(\mathbb{F}) \). Here \( M_{w_i} : I(\chi_s) \to I(\chi'_s) \) is the intertwining operator defined in (5.2) for \( i = 1, 2 \). The absolute convergence for \( \text{Re}(s) > -\frac{1}{2} \) then follows immediately from Lemma 5.6. Indeed, we have

\[ \chi_{2m}(w)(\text{diag}(t_1, \cdots, t_4, t_1^{-1}, \cdots, t_4^{-1})) = |t_1|^{-s-3/2}|t_2|^{s-1/2}|t_3|^{s+1/2}|t_4|^{-s-3/2}, \]

\[ \chi_{2m}(w)(t_1^{-1}) = |t_1|^{-s-3/2}|t_2|^{-s-1/2}|t_3|^{-s-3/2}|t_4|^{s+1/2}. \]

Hence \( M_{w_1} F = M_{2m-\varepsilon}(w) \circ M_{2m-\varepsilon}(w) \circ M_{2m}(w) F \) and \( M_{w_2} F = M_{2m-\varepsilon}(w) \circ M_{2m-\varepsilon}(w) \circ M_{2m}(w) F \) are absolutely convergent for \( \text{Re}(s) > \max\{ -\frac{3}{2}, -1, -\frac{1}{2} \} = -\frac{3}{2}, \)

For \( \sigma \in \text{Aut}(\mathbb{C}), \) we have \( \sigma \chi_{n/2} = \chi_{n/2} \) and \( \sigma F = \sigma F \) for all odd integers \( n \). The Galois equivariant property for \( \Psi_i \) then follows from Corollary 5.7. This completes the proof. \( \square \)

Let II be an irreducible admissible representation of \( \text{GSp}_4(\mathbb{F}) \).

**Lemma 5.9.** Let \( \phi \) be a matrix coefficient of \( \Pi \). Then \( \sigma \phi \) is a matrix coefficient of \( \sigma \Pi \) for all \( \sigma \in \text{Aut}(\mathbb{C}). \)

**Proof.** Let \( \sigma \in \text{Aut}(\mathbb{C}). \) Let \( \mathcal{V}_II \) and \( \mathcal{V}_{II'} \) be the representation spaces of \( \Pi \) and \( \Pi' \), respectively, and fix \( \sigma \)-linear isomorphisms \( t: \mathcal{V}_II \to \mathcal{V}_II \) and \( t': \mathcal{V}_{II'} \to \mathcal{V}_{II'} \). The representations \( \sigma \Pi \) and \( \sigma \Pi' \) are realized on \( \mathcal{V}_II \) and \( \mathcal{V}_{II'} \), respectively, with actions defined in (1.8). Let \( \langle , \rangle \) be a non-zero equivariant bilinear pairing for \( \Pi \times \Pi' \) realized on \( \mathcal{V}_II \times \mathcal{V}_{II'} \). Define a bilinear pairing \( \langle , \rangle' \) on \( \mathcal{V}_II \times \mathcal{V}_{II'} \) by

\[ \langle v, v' \rangle' = \sigma(t^{-1} v, (t')^{-1} v') \]

for \( v \in \mathcal{V}_II \) and \( v' \in \mathcal{V}_{II'} \). It is easy to verify that \( \langle , \rangle' \) defines an equivariant pairing for \( \sigma \Pi \times \sigma \Pi' \). Assume \( \phi(g) = \langle \Pi(g) v, v' \rangle \) for some \( v, v' \). Then \( \sigma \phi(g) = \langle t v, t v' \rangle' \) is a matrix coefficient of \( \sigma \Pi \). This completes the proof. \( \square \)

**Proposition 5.10.** Let \( \phi \) be a matrix coefficient of \( \Pi \) and \( F \in I(s) \) be a holomorphic section. The local zeta integral \( Z(s, \phi, F) \) converges absolutely for \( \text{Re}(s) \) sufficiently large, and satisfies the Galois equivariant property

\[ \sigma Z(\frac{n}{2}, \phi, F) = Z(\frac{n}{2}, \sigma \phi, \sigma F) \]

for all \( \sigma \in \text{Aut}(\mathbb{C}) \) and sufficiently large odd integers \( n \). Assume that \( \Pi \) is essentially unitary and generic, then \( Z(s, \phi, F) \) converges absolutely for \( \text{Re}(s) \geq \frac{1}{2} \).

**Proof.** If \( \Pi \) is supertcuspidal, then

\[ Z(s, \phi, F) = \sum_{i=1}^{n} F(\delta(g_i, 1), s) \phi(g_i) \]

for some \( g_1, \cdots, g_n \) depending only on the support of \( \phi \) and on a sufficiently small open compact subgroup of \( \text{GSp}_4(\mathbb{F}) \) which stabilizes \( F \). The assertions then follow at once.

Suppose that \( \Pi \) is a subrepresentation of an induced representation of the form \( \tau \times \chi \) or \( \chi \times \tau \) for some irreducible admissible generic representation \( \tau \) of \( \text{GL}_2(\mathbb{F}) \) and some character \( \chi \) of \( \mathbb{F}^\times \). Let \( \eta_1 \) and \( \eta_2 \) be the characters of \( \mathbb{F}^\times \) depending on \( \tau \) described in Lemma 5.4. Let \( Q = Q_1 \) (resp. \( Q = Q_2 \)) in if \( \Pi \subset \tau \times \chi \) (resp. \( \Pi \subset \chi \times \tau \)). We write \( \Psi = \Psi_i, \quad \iota = \iota_i, \quad \xi = \xi_i, \quad \mu_s = \mu_{i,s}, \quad H = H_i, \quad R = R_i \).
if $Q = Q_i$. Fix a non-zero equivariant bilinear pairing $\langle \cdot, \cdot \rangle$ on $\tau \times \tau'$. We may assume that

$$\phi(g) = \int_{Sp_4(\mathbb{F})} \langle f(kg), f'(k) \rangle \, dk$$

for some $f$ and $f'$. For $\text{Re}(s) > -\frac{1}{2}$, we have

$$Z(s, \phi, F) = \int_{Sp_4(\mathbb{F})} F(\delta(g, 1), s) \int_{Sp_4(\mathbb{F})} \langle f(kg), f'(k) \rangle \, dk \, dg$$

$$= \int_{Sp_4(\mathbb{F})} \int_{Sp_4(\mathbb{F})} F(\delta(g, k), s) \langle f(g), f'(k) \rangle \, dg \, dk$$

$$= \int_{Sp_4(\mathbb{F})} \int_{Sp_4(\mathbb{F})} \int_{N_Q(\mathbb{F})} \int_{M_Q^1(\mathbb{F})} dm \, \delta_Q(m) \, \langle f(mk_1), f'(m_k) \rangle$$

$$= \int_{Sp_4(\mathbb{F})} \int_{Sp_4(\mathbb{F})} \int_{M_Q^1(\mathbb{F})} dm \, \delta_Q(m) \, \langle \xi(m, 1), \rho(k_1, k_2)F \rangle \langle f(mk_1), f'(m_k) \rangle.$$

Here we use (2.3) in the second line and Lemma 5.8 in the fourth line. Also note that we have identify $M_Q^1 \times M_Q^1$ as a subgroup of $H$ via $\iota$. By the Cartan decomposition, for all $\varphi \in L^1(M_Q^1(\mathbb{F}))$, we have

$$\int_{M_Q^1(\mathbb{F})} \varphi(m) \, dm = \int_{\mathbb{F}^\times} d^\times t_1 \int_{|t_2| \leq 1} d^\times t_2 \int_{GL_2(\mathbb{O})^2} dk \langle GL_2(\mathbb{O})a(t_2) GL_2(\mathbb{O})/ GL_2(\mathbb{O}) \rangle \varphi(k_1t_1a(t_2)k_2)$$

if $Q = Q_1$; and

$$\int_{M_Q^1(\mathbb{F})} \varphi(m) \, dm = \int_{\mathbb{F}^\times} d^\times t_1 \int_{|t_2| \leq 1} d^\times t_2 \int_{SL_2(\mathbb{O})^2} dk \langle SL_2(\mathbb{O})m(t_2) SL_2(\mathbb{O})/ SL_2(\mathbb{O}) \rangle \varphi(t_1, k_1m(t_2)k_2)$$

if $Q = Q_2$. Here we identify $GL_2$ and $GL_1 \times SL_2$ with $M_Q^1$ and $M_Q^1$ via the embeddings

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & tA^{-1} \end{pmatrix}, \quad \begin{pmatrix} t \quad a \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & t^{-1} & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

Note that for any $\Psi \in \text{Ind}^{H(\mathbb{F})}_{R(\mathbb{F})}(\mu_s)$, we have

$$\Psi(\xi(t_1a(t_2), 1)g, s) = |t_1^2t_2|^{-s+5/2} \Psi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t_1t_2 & 0 & -1 & 0 \\ 0 & t_1 & 0 & -1 \end{pmatrix} g, s$$

$$= |t_1^2t_2|^{-s+1/2} \Psi \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -t_1^{-1}t_2^{-1} & 0 \\ 0 & 1 & 0 & -t_1^{-1} \end{pmatrix} g, s$$

$$= |t_1|^3|t_2|^{s+5/2} \Psi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ t_1t_2 & 0 & -1 & 0 \\ 0 & 1 & 0 & -t_1^{-1} \end{pmatrix} g, s$$
for $t_1, t_2 \in \mathbb{F}^\times, g \in H(\mathbb{F})$ if $Q = Q_1$; and

$$
\Psi(\xi((t_1, m(t_2)), 1), g, s) = |t_1|^{s+5/2}|t_2|^{s+3/2} \Psi \left( \begin{pmatrix} 1 & 0 \\ t_1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ t_2 & -1 & 0 & 0 \\ 0 & 0 & 1 & t_2 \end{pmatrix} \right) g, s
$$

(5.7)

$$
= |t_1|^{-s+3/2}|t_2|^{s+3/2} \Psi \left( \begin{pmatrix} 0 & 1 \\ 1 & -t_1^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ t_2 & -1 & 0 & 0 \\ 0 & 0 & 1 & t_2 \end{pmatrix} \right) g, s
$$

for $t_1, t_2 \in \mathbb{F}^\times, g \in H(\mathbb{F})$ if $Q = Q_2$. Assume $Q = Q_1$. By Lemma 5.4, there exist locally constant functions $\Phi_1, \Phi_2$ on $\mathbb{F} \times \text{GL}_2(\mathfrak{o}) \times \text{GL}_2(\mathfrak{o}) \times \text{Sp}_4(\mathfrak{o}) \times \text{Sp}_4(\mathfrak{o})$ such that

$$
\{ f((k_1, \mathbf{m}_1)(t_1)k_2, 1), f'((k_2')^1) \} = \eta_1(t)|t|^2 \Phi_1(t, k_1, k_2, k_1', k_2') + \eta_2(t)|t|^2 \Phi_2(t, k_1, k_2, k_1', k_2')
$$

for $(t, k_1, k_2, k_1', k_2') \in (\mathfrak{o} \setminus \{0\}) \times \text{GL}_2(\mathfrak{o}) \times \text{GL}_2(\mathfrak{o}) \times \text{Sp}_4(\mathfrak{o}) \times \text{Sp}_4(\mathfrak{o})$. Combining with (5.4) and (5.6), for $\text{Re}(s) > -\frac{3}{2}$, we see that the integral $Z(s, \phi, F)$ is a finite sum of integrals of the following forms:

$$
I_1(t, \eta, \phi_1, \phi_2, \phi_3, s) = \int_{(\mathbb{F} \times \mathfrak{o})^2} |t_1|^{2s+2}\omega_3(t) |t_2|^{s+1/2} \eta(t_1) \phi_2(t_1) \phi_3(t_2) d(t_1, t_2),
$$

$$
I_2(t, \eta, \phi_1, \phi_2, \phi_3, s) = \int_{(\mathbb{F} \times \mathfrak{o})^2} |t_1|^{2s+2}\omega_3(t) |t_2|^{s+3/2} \eta(t_2) \phi_2(t_1) \phi_3(t_2) d(t_1, t_2),
$$

$$
I_3(t, \eta, \phi_1, \phi_2, \phi_3, s) = \int_{(\mathbb{F} \times \mathfrak{o})^2} \omega_3(t_1)|t_2|^{s+1/2} \eta(t_2) \phi_2(t_1) \phi_3(t_1^{-1}t_2) d(t_1, t_2),
$$

where $\eta \in \{\eta_1, \eta_2\}$ and $\phi_i$ is a locally constant function on $\mathbb{F}$ with compact support for $i = 1, 2, 3$. The integrals $I_1, I_2, I_3$ converge absolutely for

(5.8)

$$
\text{Re}(s) > \max \{-1 + \frac{1}{2}|\epsilon(\omega_3)|, -\frac{1}{2} - \epsilon(\eta), -\frac{1}{2} + \epsilon(\omega_3) - \epsilon(\eta)\}.
$$

For an odd positive integer $n$ such that $s = \frac{n}{2}$ belongs to the above region of convergence, it is easy to verify that

$$
\sigma I_1(t, \eta, \phi_1, \phi_2, t, \frac{n}{2}) = I_1(\sigma t, \sigma \eta, \sigma \phi_1, \sigma \phi_2, \sigma t, \frac{n}{2}),
$$

(5.9)

$$
\sigma I_2(t, \eta, \phi_1, \phi_2, \phi_3, \frac{n}{2}) = I_2(\sigma t, \sigma \eta, \sigma \phi_1, \sigma \phi_2, \sigma \phi_3, \sigma \frac{n}{2}),
$$

$$
\sigma I_3(t, \eta, \phi_1, \phi_2, \phi_3, \frac{n}{2}) = I_3(\sigma t, \sigma \eta, \sigma \phi_1, \sigma \phi_2, \sigma \phi_3, \sigma \frac{n}{2})
$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. Assume $Q = Q_2$. By Lemma 5.4, there exist locally constant functions $\Phi_1', \Phi_2'$ on $\mathbb{F} \times \text{SL}_2(\mathfrak{o}) \times \text{SL}_2(\mathfrak{o}) \times \text{Sp}_4(\mathfrak{o}) \times \text{Sp}_4(\mathfrak{o})$ such that

$$
\{ f((1, k_1 \mathbf{m}_1(t_1)k_2, 1), k_1'), f''((k_2')^1) \} = \omega_2^{-1}\eta_1^2(t)|t| \Phi_1'(t, k_1, k_2, k_1', k_2') + \omega_2^{-1}\eta_2^2(t)|t| \Phi_2'(t, k_1, k_2, k_1', k_2')
$$

for $(t, k_1, k_2, k_1', k_2') \in (\mathfrak{o} \setminus \{0\}) \times \text{SL}_2(\mathfrak{o}) \times \text{SL}_2(\mathfrak{o}) \times \text{Sp}_4(\mathfrak{o}) \times \text{Sp}_4(\mathfrak{o})$. Combining with (5.5) and (5.7), for $\text{Re}(s) > -\frac{1}{2}$, we see that the integral $Z(s, \phi, F)$ is a finite sum of integrals of the following forms:

$$
I_4(\chi, \eta, \phi_1, \phi_2, s) = \int_{(\mathbb{F} \times \mathfrak{o})^2} |t_1|^{s+1/2} \chi(t_1) |t_2|^{s+1/2} \eta(t_2) \phi_1(t_1) \phi_2(t_2) d(t_1, t_2),
$$

$$
I_5(\chi, \eta, \phi_1, \phi_2, s) = \int_{(\mathbb{F} \times \mathfrak{o})^2} |t_1|^{s+1/2} \chi(t_1)^{-1} |t_2|^{s+1/2} \eta(t_2) \phi_1(t_1) \phi_2(t_2) d(t_1, t_2),
$$

where $\eta \in \{\omega_2^{-1}\eta_1^2, \omega_2^{-1}\eta_2^2\}$ and $\phi_i$ is a locally constant function on $\mathbb{F}$ with compact support for $i = 1, 2$. The integrals $I_4, I_5$ converge absolutely for

(5.10)

$$
\text{Re}(s) > \max \{-\frac{1}{2} + |\epsilon(\chi)|, -\frac{1}{2} + \epsilon(\omega_3) - 2e(\eta_1), -\frac{1}{2} + \epsilon(\omega_3) - 2e(\eta_2)\}.
$$

For an odd positive integer $n$ such that $s = \frac{n}{2}$ belongs to the above region of convergence, it is easy to verify that

$$
\sigma I_4(\chi, \eta, \phi_1, \phi_2, t, \frac{n}{2}) = I_4(\sigma \chi, \sigma \eta, \sigma \phi_1, \sigma \phi_2, \sigma t, \frac{n}{2}),
$$

(5.11)

$$
\sigma I_5(\chi, \eta, \phi_1, \phi_2, t, \frac{n}{2}) = I_5(\sigma \chi, \sigma \eta, \sigma \phi_1, \sigma \phi_2, \sigma t, \frac{n}{2})
$$
for all \( \sigma \in \text{Aut}(C) \). Let \( \sigma \in \text{Aut}(C) \) and \( n \) an odd positive integer such that \( s = \frac{n}{2} \) belongs to the region of convergence. We conclude from (5.9) and (5.11) that

\[
\sigma Z(\frac{n}{2}, \phi, F) = \int_{M_2^0(\mathbb{F})} dk \int_{M_2^0(\mathbb{F})} dm \delta_0(m)^{-1} \sigma \Psi(\xi(m,1), \frac{n}{2}; \rho((k_1,k_2))F) \sigma(f(mk_1), f'(k_2)).
\]

Note that

\[
\sigma \Psi(\xi(m,1), \frac{n}{2}; \rho((k_1,k_2))F) = \Psi(\xi(m,1), \frac{n}{2}; \rho((k_1,k_2))^sF)
\]

for \( m \in M_2^0, (k_1,k_2) \in \text{Sp}_4(\mathbb{O})^2 \) by Lemma 5.8 and

\[
\sigma \phi(g) = \int_{\text{Sp}_4(\mathbb{O})} \sigma(f(kg), f'(k)).
\]

Therefore, we have

\[
\sigma Z(\frac{n}{2}, \phi, F) = Z(\frac{n}{2}, \sigma \phi, F).
\]

Assume \( II \) is non-supercuspidal, essentially unitary, and generic. Note that

\[
(\tau \times \chi) \otimes | |^t \tau = \tau \times \chi | |^t, \quad (\chi \times \tau) \otimes | |^t \chi = \chi \times (\tau \otimes | |^t)
\]

for \( t \in \mathbb{C} \). By Lemma 5.1 the inequalities (5.8) and (5.10) are satisfied when \( \text{Re}(s) \geq \frac{1}{2} \) except when \( II \) is either of type (IVa) or (IXa) or (Xia). Indeed, if \( II \) is of one of the types (I), (II), (Va), and (X), then \( Q = Q_1 \) and (5.8) is satisfied for \( \text{Re}(s) \geq \frac{1}{2} \). If \( II \) is of one of the types (IIIa), (Vla), (VII), and (VIIIa), then \( Q = Q_2 \) and (5.10) is satisfied for \( \text{Re}(s) \geq \frac{1}{2} \). For the remaining types (IVa), (IXa), and (Xia), \( II \) is a discrete series representation. Hence the integral \( Z(s, \phi, F) \) converges absolutely for \( \text{Re}(s) \geq -\frac{1}{2} \) by [47] Lemma 9.5. This completes the proof.

\[\Box\]

5.4. Local zeta integrals for \( \text{GSp}_4 \times \text{GSp}_4 \). We write

\[
n^-(w, y, x, u, v) = \ell_{c_3+c_4} \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}, \ell_{c_3} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \ell_{c_2+c_4} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \ell_{c_1+c_4} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \ell_{c_2+c_3} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}
\]

and \( n^-(w, y, x) = n^-(w, y, x, 0, 0) \). Here we follow [6.1] for the notation.

Let \( \mathcal{I}(s) \) be the degenerate principal series representation defined in §2.2. For \( \alpha, \beta \in \mathbb{F} \) and \( F \in \mathcal{I}(s) \), define the twisted intertwining integral

\[
I(\alpha, \beta; F, \psi) = \int_{\mathbb{F}^3} F(n^-(w, y, x, 0, 0))\psi(-\alpha x + \beta y) \, dw \, dy \, dx.
\]

Lemma 5.11. Let \( u, v \in \mathbb{F} \).

1. Suppose \( u \in \mathbb{F}^\times \). We have

\[
I(\alpha, \beta; \rho(n^-(0, 0, 0, u, 0))F, \psi) = |u|^{-s-1} I \left( u\alpha, \beta; \rho \left( \ell_{c_1+c_4} \begin{pmatrix} 0 & -1 \\ u & -1 \end{pmatrix} \right) F, \psi \right).
\]

2. Suppose \( v \in \mathbb{F}^\times \). We have

\[
I(\alpha, \beta; \rho(n^-(0, 0, 0, 0, v))F, \psi) = |v|^{-2s-2} I \left( v\alpha, v^2\beta; \rho \left( \ell_{c_2+c_3} \begin{pmatrix} 0 & -1 \\ v & -1 \end{pmatrix} \right) F, \psi \right).
\]

3. Suppose \( u, v \in \mathbb{F}^\times \). We have

\[
I(\alpha, \beta; \rho(n^-(0, 0, 0, u, v))F, \psi) = |u|^{-s-1} |v|^{-s-2} I \left( u\alpha, v^2\beta; \rho \left( \ell_{c_1+c_4} \begin{pmatrix} 0 & -1 \\ u & -1 \end{pmatrix}, \ell_{c_2+c_3} \begin{pmatrix} 0 & -1 \\ v & -1 \end{pmatrix} \right) F, \psi \right).
\]

Proof. Note that

\[
\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = m(x^{-1})n(x) \begin{pmatrix} 0 & -1 \\ 1 & x^{-1} \end{pmatrix},
\]

\[
n^-(w, y, x) \ell_{c_1+c_4}(m(u)) n^-(w, y, x)^{-1} \in \mathcal{P}(\mathbb{F}) \cap \ker(\delta p),
\]

\[
n^-(w, y, x) \ell_{c_2+c_3}(n(v)) n^-(w - xxy, y, x)^{-1} \in \mathcal{P}(\mathbb{F}) \cap \ker(\delta p).
\]

One can easily verify that the first (resp. second) assertion follows from (5.12) and (5.13) (resp. 5.14). The third assertion is a direct consequence of (1) and (2). We leave the detail to the readers. \(\Box\)
Lemma 5.12. Let $F \in I(s)$ be a holomorphic section. The integral $I(\alpha, \beta; F, \psi)$ converges absolutely for $\text{Re}(s) > -1$ and satisfies the Galois equivariant property

$$\sigma(I(\alpha, \beta; F, \psi)|_{s=n}) = I(\alpha, \beta; \sigma F, \sigma \psi)|_{s=n}$$

for all $\sigma \in \text{Aut}(\mathbb{C})$ and integers $n \geq 0$. Moreover, for $\text{Re}(s) > -1$, the integral $I(\alpha, \beta; \rho(n^{-1}(0, 0, 0, u, v))F, \psi)$ as a function in $(\alpha, \beta, u, v) \in (\mathbb{P}^\times)^4$ is a finite sum of functions of the form

$$\varphi_{1,1}(\alpha)\varphi_{1,2}(\beta)\varphi_{1,3}(u)\varphi_{1,4}(v) + \varphi_{2,1}(u\alpha)\varphi_{2,2}(\beta)\varphi_{2,3}(u^{-1})\varphi_{2,4}(v),$$

$$+ \varphi_{3,1}(u\alpha)\varphi_{3,2}(u^2\beta)\varphi_{3,3}(u)\varphi_{3,4}(v^{-1}) + \varphi_{4,1}(u\alpha)\varphi_{4,2}(u^2\beta)\varphi_{4,3}(u^{-1})\varphi_{4,4}(v^{-1}),$$

where $\varphi_{i,j}$ is a locally constant function on $\mathbb{P}^\times$ so that $\varphi_{i,j}(x) = 0$ for $|x|$ sufficiently large and there exist $c_{i,j} \in \mathbb{C}$ and character $\chi_{i,j}$ of $\mathbb{P}^\times$ such that $\varphi_{i,j}(x) = c_{i,j} \chi_{i,j}(x)$ for $|x|$ sufficiently small.

Proof. We rewrite the integral $I(\alpha, \beta; F, \psi)$ into 8 terms according to whether $|x|$, $|y|$, and $|w|$ are sufficiently large or not. Note that

$$n^-(w, 0, 0)\iota_{2^2+4}(n(x))n^-(w, 0, 0) \in \mathcal{P}(\mathbb{P}) \cap \ker(\delta_F),$$

$$n^-(w, 0, 0)\iota_{2^3+4}(n(y))n^-(w, 0, 0) \in \mathcal{P}(\mathbb{P}) \cap \ker(\delta_F).$$

It follows that for all sufficiently larger integers $N_1$, $N_2$, and $N_3$ depending only on $F$, we have

$$I(\alpha, \beta; F, \psi)$$

$$= \int_{|x| \leq q^N} \frac{dx}{|y| \leq q^{N_2}} \int_{|w| \leq q^{N_3}} dw \mathcal{F}(n^-(w, y, x))\psi(-\alpha x + \beta y)$$

$$+ \int_{|x| > q^{N_1}} |x|^{-s} \psi(-\alpha x) dx \int_{|y| \leq q^{N_2}} \frac{dy}{|w| \leq q^{N_3}} \mathcal{F}(n^-(w, y, 0)\iota_{2^2+4}(w)) \psi(\beta y)$$

$$+ \int_{|y| > q^{N_2}} |y|^{-s} \psi(\beta y) dy \int_{|x| \leq q^{N_1}} dx \int_{|w| \leq q^{N_3}} \mathcal{F}(n^-(w, 0, x)\iota_{2^3}(w)) \psi(-\alpha x)$$

$$+ \int_{|x| > q^{N_1}} |x|^{-s} \psi(-\alpha x) dx \int_{|y| \leq q^{N_2}} \frac{dy}{|w| \leq q^{N_3}} \mathcal{F}(\iota_{2^2+4}(w) \iota_{2^3+4}(w) \psi(\beta y))$$

$$+ \int_{|y| > q^{N_2}} |y|^{-s} \psi(\beta y) dy \int_{|x| \leq q^{N_1}} dx \int_{|w| \leq q^{N_3}} \mathcal{F}(\iota_{2^2+4}(w) \iota_{2^3}(w)) \psi(-\alpha x)$$

$$+ \int_{|x| > q^{N_1}} |x|^{-s} \psi(-\alpha x) dx \int_{|y| > q^{N_2}} \mathcal{F}(\iota_{2^2+4}(w) \iota_{2^3+4}(w) \psi(\beta y))$$

$$+ \int_{|y| > q^{N_2}} |y|^{-s} \psi(\beta y) dy \int_{|x| > q^{N_1}} \frac{dx}{|w| \leq q^{N_3}} \mathcal{F}(\iota_{2^2+4}(w) \iota_{2^3+4}(w)) \psi(-\alpha x)$$

We see that $I(\alpha, \beta; F, \psi)$ is absolutely convergent for $\text{Re}(s) > -1$. Note that

$$\int_{|x| \leq q^N} \psi(ax) dx = q^N \cdot \mathcal{I}_{\pi^N+4}(a),$$

$$\int_{|x| \geq q^N} |x|^{-s} \psi(ax) dx = \left[\frac{1 - \alpha^{-1}}{1 - q^{-s}} \cdot (q^{-N(s+1)} - q^{(d-1)(s+1)}|a|^{s+1}) - q^{(d-1)(s+1)-1}|a|^{s+1}\right] \cdot \mathcal{I}_{\pi^{N-1}+4}(a).$$
where \( \varpi^d \) is the largest fractional ideal of \( \mathbb{F} \) on which \( \psi \) is trivial. Let \( \text{Re}(s) > -1 \). Combining with Lemma 5.11, we deduce that each term of \( I(\alpha, \beta; \rho(n(0, 0, 0, u, v))F, \psi) \) as a function in \((\mathbb{F}^\times)^4\) is equal to a finite sum of functions satisfying the conditions in the lemma. The Galois equivariant property then also follows at once. Indeed, for an integer \( n \), we have \( \sigma(n) = |n|^3 \) and \( \sigma(F(g, n)) = \sigma(F(g, n)) \) for all \( g \in \text{Sp}_4(\mathbb{F}) \) by definition. This completes the proof.

**Lemma 5.13.** Let \( \varphi_1, \ldots, \varphi_7 \) be locally constant functions on \( \mathbb{F}^\times \) so that \( \varphi_i(x) = 0 \) for \( |x| \) sufficiently large and there exist \( c_i \in \mathbb{C} \), character \( \chi_i \) of \( \mathbb{F}^\times \), and integer \( m_i \geq 0 \) such that \( \varphi_i(x) = c_i \cdot \chi_i(x)(\log_q |x|)^{m_i} \) for \( |x| \) sufficiently small. Let \( I_1(\varphi_1, \ldots, \varphi_7) \) be the integral defined by

\[
I_1(\varphi_1, \ldots, \varphi_7) = \int_{(\mathbb{F}^\times)^4} \varphi_1(a)\varphi_2(b)\varphi_3(av^{-2})\varphi_4(bu^{-1}v)\varphi_5(bu)\varphi_6(u)\varphi_7(v) \, d(a, b, u, v).
\]

Then we have

\[
\sigma I_1(\varphi_1, \ldots, \varphi_7) = I_1(\sigma \varphi_1, \ldots, \sigma \varphi_7)
\]

for all \( \sigma \in \text{Aut}(\mathbb{C}) \) when both sides are absolutely convergent. Similar assertion holds for the following integrals:

\[
I_2(\varphi_1, \ldots, \varphi_6) = \int_{(\mathbb{F}^\times)^4} \varphi_1(a)\varphi_2(b)\varphi_3(av^{-2})\varphi_4(bu)\varphi_6(v) \, d(a, b, u, v),
\]

\[
I_3(\varphi_1, \ldots, \varphi_6) = \int_{(\mathbb{F}^\times)^4} \varphi_1(a)\varphi_2(b)\varphi_3(av^2)\varphi_4(bu^{-1}v)\varphi_6(u)\varphi_7(v) \, d(a, b, u, v),
\]

\[
I_4(\varphi_1, \ldots, \varphi_7) = \int_{(\mathbb{F}^\times)^4} \varphi_1(a)\varphi_2(b)\varphi_3(av^2)\varphi_4(bu^{-1})\varphi_6(u)\varphi_7(v) \, d(a, b, u, v).
\]

**Proof.** We recall a type of local integral of the form

\[
\int_{\mathbb{F}^\times} \chi(x)(\log_q |x|)^m \cdot \varphi(x) \, d^\times x,
\]

where \( \varphi \) is a locally constant function on \( \mathbb{F} \) with compact support, \( \chi \) is a character of \( \mathbb{F}^\times \), and \( m \geq 0 \) is an integer. The integral converges absolutely when \( e(\chi) > 0 \). In this case, it is easy to verify that the integral satisfies the Galois equivariant property

\[
\sigma \left( \int_{\mathbb{F}^\times} \chi(x)(\log_q |x|)^m \cdot \varphi(x) \, d^\times x \right) = \int_{\mathbb{F}^\times} \sigma \chi(x)(\log_q |x|)^m \cdot \varphi(x) \, d^\times x
\]

for all \( \sigma \in \text{Aut}(\mathbb{C}) \) (cf. [Gro18 Proposition A]) when both sides are absolutely convergent.

We only consider the integral \( I_1(\varphi_1, \ldots, \varphi_7) \). The assertion for other three integrals \( I_2, I_3, I_4 \) can be proved in a similar way and we omit it. First we consider the case when \( \varphi_6 \) and \( \varphi_7 \) vanish unless \( |u| \) and \( |v| \) are sufficiently small. Then the integral \( I_1(\varphi_1, \ldots, \varphi_7) \) is a finite sum of integrals of the form

\[
\int_{(\mathbb{F}^\times)^2} \varphi_1'(a)\varphi_3'(av^{-2})\varphi_6'(v) \, d(a, v) \cdot \int_{(\mathbb{F}^\times)^2} \varphi_2'(b)\varphi_4'(bu^{-1})\varphi_6'(u) \, d(b, u),
\]

where \( \varphi_i' \) satisfies the same conditions of the lemma and \( \varphi_5', \varphi_6' \) vanish unless \( |u| \) and \( |v| \) are sufficiently small. Firstly we consider the integral

\[
\int_{(\mathbb{F}^\times)^2} \varphi_1'(a)\varphi_3'(av^{-2})\varphi_6'(v) \, d(a, v).
\]

When the support of \( \varphi_1' \) is contained in a bounded set away from zero, the above integral is a finite sum of integrals of the form \( \int_{\mathbb{F}^\times} \cdot \). Similar assertion holds when the support of \( \varphi_6' \) is contained in a bounded set away from zero. Therefore, we may assume

\[
\varphi_1'(a) = \chi(a)(\log_q |a|)^m I_{c, \varpi^m} \sigma(a), \quad \varphi_6'(v) = \mu(v)(\log_q |v|)^m I_{c, \varpi^m} \sigma(v)
\]

for some characters \( \chi, \mu \) of \( \mathbb{F}^\times \) and integers \( n, m, N_1, N_2 \geq 0 \). Furthermore, by the condition on \( \varphi_5' \), we may assume that either

\[
\varphi_5' = I_{c, \varpi^m} \sigma
\]

36
for some \( c \in F \) and integer \( N_3 \) with \( c \not\in \varpi^{N_3} \). or
\[
(5.18) \quad \varphi_3'(x) = \nu(x)(\log_q |x|)^r I_{\varpi^{N_3}}(x)
\]
for some character \( \nu \) of \( F^\times \) and integers \( r, N_3 \geq 0 \). Write
\[
I_{\varpi^{N_3}} = I_{\varpi^{N_4}} + (I_{\varpi^{N_2}} - I_{\varpi^{N_4}})
\]
for some sufficiently large \( N_4 \) such that \( c \in \varpi^{-2N_4+N_1} \) (resp. \( \varpi^{N_3} \in \varpi^{-2N_4+N_1} \)) if \( \varphi_3' \) is a function of the form \( (5.19) \) (resp. \( 5.18 \)). Then the integral \( (5.19) \) over \( v \in \varpi^{N_4} \) is a finite sum of integrals of the form \( (5.15) \). Suppose that \( \varphi_3' \) is a function of the form \( (5.17) \). Then the integral \( (5.16) \) over \( v \in \varpi^{N_4} \) is equal to
\[
\int_{|a| \leq q^{-N_1}} \int_{|v| \leq q^{-N_4}} \chi(a)(\log_q |a|)^n \mu(v)(\log_q |v|)^m I_{\varpi^{N_3}}(a, v) = 0
\]
Now we assume that \( \varphi_3' \) is a function of the form \( (5.15) \). Then the integral \( (5.16) \) over \( v \in \varpi^{N_4} \) is equal to
\[
\int_{|a| \leq q^{-N_1}} \int_{|v| \leq q^{-N_4}} \chi(a)(\log_q |a|)^n \mu(v)(\log_q |v|)^m I_{\varpi^{N_3}}(a, v) = 0
\]
In any case, we conclude that the integral \( (5.16) \) satisfies the Galois equivariant property, since it is a finite sum of products of integrals of the form \( (5.15) \). By a similar argument, the integral
\[
\int_{(F^\times)^2} \varphi_2'(b) \varphi_4'(bu^{-1}) \varphi_5'(u) d(b, u)
\]
also satisfies the Galois equivariant property.

Now we consider the remaining cases. If \( \varphi_6 \) vanishes when \( |u| \) is sufficiently small, then the integral \( I_1(\varphi_1, \cdots, \varphi_7) \) is a finite sum of integrals of the form
\[
(5.19) \quad \int_{(F^\times)^3} \varphi_1'(a) \varphi_2'(b) \varphi_3'(av^{-2}) \varphi_4'(bv) \varphi_5'(v) d(a, b, v),
\]
where \( \varphi_1' \) satisfies the same conditions of the lemma. If \( \varphi_7 \) vanishes when \( |v| \) is sufficiently small, then the integral \( I_1(\varphi_1, \cdots, \varphi_7) \) is a finite sum of integrals of the form
\[
(5.20) \quad \int_{(F^\times)^2} \varphi_1'(a) \varphi_2'(b) \varphi_3'(bu^{-1}) \varphi_4'(u) d(a, b, u),
\]
where \( \varphi_1' \) satisfies the same conditions of the lemma. Proceeding similarly as in the previous paragraph, one can prove that the integrals of types \( (5.19) \) and \( (5.20) \) also satisfy the Galois equivariant property. We leave the detail to the readers. This completes the proof. \( \square \)

Let \( \Pi \) be an irreducible admissible generic representation of \( \text{GSp}_4(F) \).

**Proposition 5.14.** Let \( W_1 \in \mathcal{W}(\Pi, \psi \nu) \), \( W_2 \in \mathcal{W}(\Pi^\vee, \psi_u^{-1}) \), and \( F \in \mathcal{I}(s) \) be a holomorphic section. The local zeta integral \( \mathcal{Z}(s, W_1, W_2, F) \) converges absolutely for \( Re(s) \) sufficiently large, and satisfies the Galois equivariant property
\[
\sigma \mathcal{Z}(n, W_1, W_2, F) = \mathcal{Z}(n, \sigma W_1, \sigma W_2, \sigma F)
\]
for all \( \sigma \in \text{Aut}(C) \) and sufficiently large odd integers \( n \). Assume \( \Pi \) is essentially unitary, then \( \mathcal{Z}(s, W_1, W_2, F) \) converges absolutely for \( Re(s) \geq 1 \).

**Proof.** By Lemma [5.2] we may assume
\[
W_1(utk) = \psi_\nu(u) \delta_B(t)^{1/2} \eta(t)(\log_q |a|)^{n_1}(\log_q |b|)^{n_2} \varphi_1(a, b, k),
\]
\[
W_2(utk) = \psi_u^{-1}(u) \delta_B(t)^{1/2} \eta'(t)(\log_q |a|)^{n_1}(\log_q |b|)^{n_2} \varphi_2(a, b, k)
\]

for some $\eta \in \mathcal{X}_H$ and $\eta' \in \mathcal{X}_{H'}$, some integers $0 \leq n_1, n_2, n'_1, n'_2 \leq N_H$, and locally constant functions $\varphi_1, \varphi_2$ on $\mathbb{F} \times \mathbb{F} \times \text{GSp}_4(\mathfrak{o})$ with compact support. Here $u \in U(\mathbb{F})$, $t = \text{diag}(ab, a, b^{-1}, 1) \in T(\mathbb{F})$, and $k \in \text{GSp}_4(\mathfrak{o})$.

Write

$$
\eta(\text{diag}(ab, a, b^{-1}, 1)) = \eta_1(a)\eta_2(b), \quad \eta'(\text{diag}(ab, a, b^{-1}, 1)) = \eta'_1(a)\eta'_2(b).
$$

In the notation as in the proof of Lemma 9.5, we have

$$
k_a = e(\eta_1) - e(\eta'_1) - e(\omega_H), \quad k_b = e(\eta_2), \quad k_c = 2e(\eta'_1) + e(\omega_H), \quad k_d = e(\eta'_2).
$$

Following the same argument as in the proof of that lemma, we see that the integral $Z(s, W_1, W_2, F)$ converges absolutely for

$$
\Re(s) > \max\{-e(\eta_1) + \frac{1}{2}e(\omega_H) - 1, -e(\eta_2) - 1, -2e(\eta'_1) - e(\omega_H) - 1, -e(\eta'_2) - 1, -e(\eta_1) - \frac{1}{2}e(\omega_H) + \frac{1}{2}e(\omega_H) - 1, -2e(\eta_1) - 2e(\eta'_1) - 1, -e(\eta_2) - e(\eta'_2) - 1, -2e(\eta'_1) + e(\eta'_2) - e(\omega_H) - 1\}.
$$

When $\Pi$ is essentially unitary, by Lemmas 5.1 and 5.2 one can verify case by case that the above inequality holds for $\Re(s) \geq 1$.

Now we show that the integral $Z(s, W_1, W_2, F)$ satisfies the Galois equivariant property. Write

$$
W_i(tk) = \delta_B(t)^{1/2}\Phi_i(a, b, k)
$$

for $i = 1, 2, t = \text{diag}(ab, a, b^{-1}, 1) \in T(\mathbb{F})$, and $k \in \text{GSp}_4(\mathfrak{o})$. Let

$$
(\text{GSp}_4(\mathfrak{o}) \times \text{GSp}_4(\mathfrak{o}))^c = \{(k_1, k_2) \in \text{GSp}_4(\mathfrak{o}) \times \text{GSp}_4(\mathfrak{o}) \mid \nu(k_1) = \nu(k_2)\}.
$$

We define $(B \times B)^c$ and $(T \times T)^c$ in a similar way. We have

$$
Z(s, W_1, W_2, F) = \int_{Z_H(\mathbb{F}) \setminus G(\mathbb{F})} \mathcal{F}(\eta, s)(W_1 \otimes W_2)(g) \, dg
$$

$$
= \int_{(\text{GSp}_4(\mathfrak{o}) \times \text{GSp}_4(\mathfrak{o}))^c} \int_{Z_H(\mathbb{F}) \setminus (T \times T)^c(\mathbb{F})} \delta_{(B \times B)^c}(t)^{-1}(W_1 \otimes W_2)(tk)
$$

$$
\times \int_{U(\mathbb{F}) \setminus U(\mathbb{F})} \mathcal{F}(\eta(u, 1)tk)\psi_U(u) \, du \, dt \, dk.
$$

Note that

$$
\{(\text{diag}(ab, a, b^{-1}, 1), \text{diag}(cd, c, c^{-1}d^{-1}a, c^{-1}a)) \mid a, b, c, d \in \mathbb{F}^\times\}
$$

is a set of representatives for $Z_H(\mathbb{F}) \setminus (T \times T)^c(\mathbb{F})$. Let

$$
t_1 = \text{diag}(ab, a, b^{-1}, 1), \quad t_2 = \text{diag}(cd, c, c^{-1}d^{-1}a, c^{-1}a).
$$

For

$$
u = \begin{pmatrix} 1 & x & * & w \\ 0 & 1 & * & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \in U(\mathbb{F}),
$$

a direct calculation gives that

$$
\eta(u, 1)(t_1, t_2)\eta^{-1} = p(u, t_1, t_2)n(-ac^{-2}d^{-1}w, -ac^{-2}y, ac^{-1}d^{-1}x, abc^{-1}d^{-1} - 1, ac^{-1} - 1)
$$

for some $p(u, t_1, t_2) \in \mathcal{P}(\mathbb{F})$ with $\delta_{\mathcal{P}}(p(u, t_1, t_2)) = |a^{3/2}bc^{-1}|^6$. Therefore,

$$
\int_{U(\mathbb{F}) \setminus U(\mathbb{F})} \mathcal{F}(\eta(u, 1)tk)\psi_U(u) \, du
$$

$$
= \delta_{(B \times B)^c}((t_1, t_2))^{1/2}|a^{3/2}bc^{-1}|^{s+1}
$$

$$
\times \int_{\mathbb{F}^3} \mathcal{F}(n^-(-u, y, x, abc^{-1}d^{-1} - 1, ac^{-1} - 1))\psi(-a^{-1}cdx + a^{-1}c^2y) \, dw \, dy \, dx
$$

$$
= \delta_{(B \times B)^c}((t_1, t_2))^{1/2}|a^{3/2}bc^{-1}|^{s+1} \cdot I(a^{-1}cd, a^{-1}c^2; \rho(n^-(-0, 0, 0, abc^{-1}d^{-1} - 1, ac^{-1} - 1))\mathcal{F}, \psi).
$$
We conclude that
\[
Z(s, W_1, W_2, \mathcal{F}) = \int_{(\text{GSp}_4(\mathfrak{o}) \times \text{GSp}_4(\mathfrak{o}))^s} dk \int_{(\mathbb{F}^\times)^4} d(a, b, c, d) |a^{3/2}bc^{-1}|^{s+1} \omega_I(a^{-1}c) \Phi_1(a, b, k_1) \Phi_2(a^{-1}c^2, d, k_2) \\
\times I(a^{-1}cd, a^{-1}c^2; \rho(n^- (0, 0, 0, 0, abc^{-1}d^{-1} - 1, ac^{-1} - 1)\eta k)\mathcal{F}, \psi)
\]
\[
= \int_{(\text{GSp}_4(\mathfrak{o}) \times \text{GSp}_4(\mathfrak{o}))^s} dk \int_{(\mathbb{F}^\times)^4} d(a, b, u, v) |a^{1/2}bv|^{s+1} \omega_I(v^{-1}) \Phi_1(a, b, k_1) \Phi_2(av^{-2}, bu^{-1}v, k_2) \\
\times I(bu^{-1}, av^{-2}; \rho(n^{- (0, 0, 0, 0, u - 1, v - 1)}\eta k)\mathcal{F}, \psi).
\]
Here \(d(a, b, u, v)\) is the Haar measure on \((\mathbb{F}^\times)^4\) with \(\text{vol}(\mathbb{F}^\times)^4, d(a, b, u, v) = 1\). By Lemma 5.12 for \(k = (k_1, k_2) \in (\text{GSp}_4(\mathfrak{o}) \times \text{GSp}_4(\mathfrak{o}))^s\) and \(\Re(s)\) sufficiently large, the integral
\[
\int_{(\mathbb{F}^\times)^4} d(a, b, u, v) |a^{1/2}bv|^{s+1} \omega_I(v^{-1}) \Phi_1(a, b, k_1) \Phi_2(av^{-2}, bu^{-1}v, k_2) \\
\times I(bu^{-1}, av^{-2}; \rho(n^{- (0, 0, 0, 0, u - 1, v - 1)}\eta k)\mathcal{F}, \psi)
\]
is a finite sum of integrals of the forms \(I_1, I_2, I_3, I_4\) in Lemma 5.13. Let \(\sigma \in \text{Aut}(\mathbb{C})\) and \(n\) an odd integer so that the above integrals are all absolutely convergent, we have
\[
\sigma(|a^{1/2}bv|^{n+1}) = |a^{1/2}bv|^{n+1}
\]
and
\[
\sigma(I(\alpha, \beta; \rho(n^- (0, 0, 0, 0, u - 1, v - 1)}\eta k)\mathcal{F}, \psi)|_{s = n} = I(\alpha, \beta; \rho(n^- (0, 0, 0, 0, u - 1, v - 1)}\eta k)^s\mathcal{F}, \sigma(\psi)|_{s = n}
\]
by Lemma 5.12. It then follows from the Galois equivariant property proved in Lemma 5.13 that
\[
\sigma Z(n, W_1, W_2, \mathcal{F})
\]
\[
= \int_{(\text{GSp}_4(\mathfrak{o}) \times \text{GSp}_4(\mathfrak{o}))^s} dk \int_{(\mathbb{F}^\times)^4} d(a, b, u, v) |a^{1/2}bv|^{n+1} \omega_I(v^{-1})^{\sigma} \Phi_1(a, b, k_1)^{\sigma} \Phi_2(av^{-2}, bu^{-1}v, k_2) \\
\times I(bu^{-1}, av^{-2}; \rho(n^{- (0, 0, 0, 0, u - 1, v - 1)}\eta k)^{\sigma}\mathcal{F}, \sigma(\psi)).
\]
Finally, note that
\[
^{\sigma}W_1(t_1k_1)^{\sigma}W_2(t_2k_2) = \delta_B(t_1)^{1/2} \delta_B(t_2)^{1/2} \omega_{\mathbb{F}}((a^{-1}c)^{\sigma} \Phi_1(a, b, k_1)^{\sigma} \Phi_2(a^{-1}c^2, d, k_2)
\]
for \(t_1 = \text{diag}(ab, a, b^{-1}, 1), t_2 = \text{diag}(cd, c, c^{-1}d^{-1}a, c^{-1}a) \in T(\mathbb{F})\) and \(k_1, k_2 \in \text{GSp}_4(\mathfrak{o})\). Therefore, we have
\[
\sigma Z(n, W_1, W_2, \mathcal{F}) = Z(n,^{\sigma}W_1,^{\sigma}W_2,^{\sigma}\mathcal{F}).
\]
This completes the proof. \(\square\)

REFERENCES

[AS06] M. Asgari and F. Shahidi. Generic transfer from \text{GSp}(4) to \text{GL}(4). Compos. Math., 142:541–550, 2006.
[AS08] M. Asgari and R. Schmidt. On the adjoint \(L\)-function of the \(p\)-adic \text{GSp}(4). J. Number Theory, 128:2340–2358, 2008.
[CII9] S.-Y. Chen and A. Ichino. On Petersson norms of generic cusp forms and special values of adjoint \(L\)-functions for \text{GSp}_4. 2019. Submitted. [arXiv:1002.06429]
[Clo90] L. Clozel. Motifs et Formes Automorphes: Applications du Principe de Fonctorialité. In Automorphic Forms, Shimura Varieties, and \(L\)-functions, Vol. I, Perspectives in Mathematics, pages 77–159, 1990.
[Del79] P. Deligne. Valeurs de fonctions \(L\). Springer, 1979. Submitted. [arXiv:1002.06429]
[Del79] P. Deligne. Valeurs de fonctions \(L\). In Automorphic Forms, Representations and \(L\)-Functions, volume 33, pages 313–346. Proceedings of Symposia in Pure Mathematics, 1979. Part 2.
[GH11] D. Goldberg and J. Hundley. Automorphic representations and \(L\)-functions for the general linear group, volume I. Cambridge Univ. Press, Cambridge, 2011.
[GI11] W. T. Gan and A. Ichino. On endoscopy and the refined Gross-Prasad conjecture for \((\text{SO}_5, \text{SO}_4)\). J. Inst. Math. Jussieu, 10:235–324, 2011.
[GI14] W. T. Gan and A. Ichino. Formal degrees and local theta correspondence. Invent. Math., 195:509–672, 2014.
[GI16] W. T. Gan and A. Ichino. The Gross-Prasad conjecture and local theta correspondence. Invent. Math., 206:705–799, 2016.
[GJ72] R. Godement and H. Jacquet. Zeta functions of simple algebras, volume 260 of Lecture Notes in Mathematics. Springer, 1972.
[GP92] B. H. Gross and D. Prasad. On the decomposition of a representation of SO\(_n\) when restricted to SO\(_{n-1}\). Canad. J. Math., 44:974–1002, 1992.
[GQT14] W. T. Gan, Y. Qiu, and S. Takeda. The regularized Siegel-Weil formula (the second term identity) and the Rallis inner product formula. Invent. Math., 198:739–831, 2014.
