A NOTE ON THE “LOGARITHMIC-$\mathfrak{W}_3$” OCTUPLLET ALGEBRA AND ITS NICHOLS ALGEBRA

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ABSTRACT. We describe a Nichols-algebra-motivated construction of an octuplet chiral algebra that is a “$\mathfrak{W}_3$-counterpart” of the triplet algebra of $(p,1)$ logarithmic models of two-dimensional conformal field theory.

1. INTRODUCTION

Logarithmic models of two-dimensional conformal field theory can be defined as centralizers of Nichols algebras [1, 2]. For this, the generators $F_i$ of a given Nichols algebra $\mathcal{B}(X)$ with diagonal braiding [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] are to be realized as

$$F_i = \int e^{\alpha_i \varphi}, \quad 1 \leq i \leq \text{rank} \equiv \theta,$$

where $\varphi(z)$ is a $\theta$-plet of scalar fields and $\alpha_i \in \mathbb{C}^\theta$ are chosen so as to reproduce the given braiding coefficients $q_{i,j}$ in

$$\Psi : F_i \otimes F_j \mapsto q_{i,j} F_j \otimes F_i, \quad 1 \leq i, j \leq \theta.$$

The coefficients are standardly arranged into a braiding matrix $(q_{i,j})_{1 \leq i \leq \text{rank}}$. The relation between the braiding matrix and the screening momenta is postulated [2] in the form of equations

$$q_{j,j} = e^{i \pi \alpha_j}, \quad q_{j,k} q_{k,j} = e^{2i \pi \alpha_j} \alpha_k$$

and the logical-“or” conditions

$$a_{i,j} \alpha_i \alpha_j = 2 \alpha_i \alpha_j, \quad \sqrt{(1 - a_{i,j}) \alpha_i \alpha_j = 2}$$

imposed for each pair $i \neq j$ and involving the Cartan matrix $a_{i,j}$ associated with the given braiding matrix (see, e.g., [18] and the references therein).

In this note, we describe some details related to the construction of the octuplet algebra [2] that can be considered a “logarithmic extension” of the $\mathfrak{W}_3$ algebra [19] similarly to how the triplet algebra [21, 22, 23] is a “logarithmic extension” of the Virasoro algebra. The starting point is a particular item in Heckenberger’s list of rank-2 Nichols algebras with diagonal braiding (which is item 5.7(1) in [20])—the braiding matrix

$$q_{ij} = \begin{pmatrix} q^2 & q^{-1} \\ q^{-1} & q^2 \end{pmatrix}.$$
where $q^2$ is a primitive $2p$th root of unity. We choose
\begin{equation}
q = e^{i \pi / p}
\end{equation}
with $p = 2, 3, \ldots$. This choice leads to $(p, 1)$-type logarithmic CFT models \cite{Chu2002,Kashiwara2003,Moore1998,Moore1997,Moore1994,Moore1993}, in contrast to $(p, p')$ models that follow if $q$ is chosen as $e^{i \pi p'/p}$ instead. The main expectation associated with $(p, 1)$-type models is that their representation categories are “very closely related” \cite{Kashiwara2003,Moore1997,Moore1994} to an appropriate representation category on the algebraic side, which in the braided case is some category of Yetter–Drinfeld modules (cf. \cite{Lauda2010}). In this paper, we therefore proceed along two routes: (i) describing the structure of the Yetter–Drinfeld modules, and (ii) discussing some properties of the octuplet algebra that centralizes this $\mathcal{B}(X)$. None of the two directions is pursued to the point where they actually meet (which would mean constructing a functor), but the results presented here hopefully bring us somewhat closer to that point.

2. The Nichols algebra

2.1. Presentation for $\mathcal{B}(X)$. We first recall the presentation of the relevant Nichols algebra, as a quotient of the tensor algebra. Our starting point is a two-dimensional braided vector space $X$ with the preferred basis $F_1, F_2$ and the above braiding matrix in this basis. The Nichols algebra $\mathcal{B}(X)$ is the quotient by a graded ideal $I$ \cite{Jimbo1985,Jimbo1987},
\begin{equation}
\mathcal{B}(X) = T(X)/\langle [F_1, [F_1, F_2]], [F_2, [F_2, F_1]], F_1^p, [F_2, F_1]^p, F_2^p \rangle, \quad \dim \mathcal{B}(X) = p^3,
\end{equation}
If $p = 2$, the double-bracket generators of the ideal are absent. The brackets here denote $q$-commutators determined by the braiding matrix: $[F_1, F_2] = F_1 F_2 - q^{-1} F_2 F_1$, $[F_2, F_1] = F_2 F_1 - q^{-1} F_1 F_2$, and so on by multiplicativity of the “$q$”-factor, whence the two double commutators in the ideal are explicitly given by
\begin{align*}
[F_1, [F_1, F_2]] &= F_1^2 F_2 - (q + q^{-1}) F_1 F_2 F_1 + F_2 F_1^2, \\
[F_2, [F_2, F_1]] &= F_2^2 F_1 - (q + q^{-1}) F_2 F_1 F_2 + F_1 F_2^2.
\end{align*}

A PBW basis in $\mathcal{B}(X)$ is given by $F_1^r F_3^s F_2^t$, $0 \leq r, s, t \leq p - 1$ \cite{Jimbo1987}, where
\begin{equation}
F_3 = [F_2, F_1].
\end{equation}
The double-bracket relations in the ideal can also be rewritten as $F_2 F_3 = q F_3 F_2$ and $F_3 F_1 = q F_1 F_3$.

Multiplication in $\mathcal{B}(X) = T(X)/I$ is the one induced by “concatenation” in the tensor algebra, $X^\otimes m \otimes X^\otimes n \to X^\otimes (m+n)$, $(x_1, \ldots, x_m) \otimes (y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_m, y_1, \ldots, y_n)$. It is then relatively straightforward to show that the multiplication table of the PBW basis elements is
Comultiplication is by “deshuffling,” determined by the defining property of a braided Hopf algebra and the fact that $F_1$ and $F_2$ are primitive.

2.2. $\mathfrak{B}(X)$ as a subalgebra in $T(X)$. For any Nichols algebra $\mathfrak{B}(X)$, the graded ideal $J$ such that $\mathfrak{B}(X) = T(X)/J$ is known to be the kernel of the total braided symmetrizer map in each grade, $\mathfrak{S}_n : X^{\otimes n} \to X^{\otimes n}$. Mapping by $\mathfrak{S}_n$ in each grade therefore results in an equivalent description of $\mathfrak{B}(X)$ with multiplication given by the shuffle product

$$\ast : (x_1, \ldots, x_m) \otimes (y_1, \ldots, y_n) \mapsto \mathfrak{S}_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n),$$

and comultiplication by deconcatenation (see [1] for the definition of shuffles and the braided symmetrizer; the only notational difference is that $\ast$ is not used for the shuffle product there).

We let $B(r,t,s)$ be the image of $F_1^r F_3^t F_2^s$ under the map by the braided symmetrizer, or more precisely,

$$B(r,t,s) = \frac{1}{\langle r \rangle! \langle s \rangle! \langle t \rangle! (1 - q^2)^r} \mathfrak{S}_{r+2t+s}(F_1^r F_3^t F_2^s).$$

In particular,

$$B(1,0,0) = F_1, \quad B(2,0,0) = F_1 F_1,$$

$$B(0,0,1) = F_2, \quad B(1,0,1) = F_1 F_2 + q^{-1}F_2 F_1,$$

$$B(0,0,2) = F_2 F_2,$$

$$B(0,1,0) = -q^{-2}F_2 F_1.$$

2.2.1. The shuffle product of $B(r_1,t_1,s_1)$ and $B(r_2,t_2,s_2)$ follows from (2.2):

$$B(r_1,t_1,s_1) \ast B(r_2,t_2,s_2) =$$

$$\sum_{i=0}^{\min(s_1,r_2)} \langle r_1 + r_2 - i \rangle \langle s_1 + s_2 - i \rangle \langle t_1 + t_2 + i \rangle (1 - q^2)^\langle t_1 \rangle! \langle t_2 \rangle! \langle i \rangle! q^{\langle i \rangle!(r_2-i)+t_2(s_1-i)-s_1r_2+i(i+1)/2}$$

$$\times B(r_1 + r_2 - i, t_1 + t_2 + i, s_1 + s_2 - i).$$

and the coproduct is

$$\Delta : B(r,t,s) \mapsto \sum_{j=0}^{r} \sum_{m=0}^{s} \sum_{k=0}^{t} \sum_{i=0}^{k} (-1)^i q^{-i(i+3)/2+(k-m-2i)j+m(t-i-k)}$$

$$\times \langle i+j \rangle \langle i+m \rangle \langle i \rangle! B(r-j,k-i,i+m) \otimes B(j+i,t-k,s-m),$$
where terms with the lowest grades in the first tensor factor are

\[ = 1 \otimes B(r, t, s) + F_1 \otimes B(r - 1, t, s) + q^{r-2}F_2 \otimes B(r, t, s - 1) - q^{r-2}(r + 1)F_2 \otimes B(r + 1, t - 1, s) + \ldots \]

(the dots stand for terms \( B(r', t', s') \otimes B(r'', t'', s'') \) with \( r' + 2t' + s' \geq 2 \)).

2.2.2. Remark. Although this is obvious, we note explicitly that the “Serre relations”—the double \( q \)-commutators in the ideal—are resolved in terms of the shuffle product in the sense that the relations

\[ F_1 \ast F_1 \ast F_2 - (q + q^{-1})F_1 \ast F_2 \ast F_1 + F_2 \ast F_1 \ast F_1 = 0, \]

\[ F_2 \ast F_2 \ast F_1 - (q + q^{-1})F_2 \ast F_1 \ast F_2 + F_1 \ast F_2 \ast F_2 = 0 \]

hold identically for the shuffle product defined by the braiding matrix (1.1).

2.2.3. The action of the antipode on the PBW basis elements is defined by the formulas

\[ S(B(r, 0, 0)) = (-1)^r q^{r(r-1)} B(r, 0, 0), \]

\[ S(B(0, t, 0)) = \sum_{i=0}^{t} (-1)^{i} q^{\frac{1}{2}j(i-1)-(i+3)i+2} B(i, t-i, i), \]

\[ S(B(0, 0, s)) = (-1)^s q^{s(s-1)} B(0, 0, s) \]

and by the fact that \( S \) is a braided antiautomorphism:

\[ S(B(r, t, s)) = q^{rt-rs+ts} S(B(0, 0, s)) \ast S(B(0, t, 0)) \ast S(B(r, 0, 0)). \]

2.3. Vertex operators and Yetter–Drinfeld \( \mathcal{B}(X) \) modules. Multivertex \( \mathcal{B}(X) \) module comodules, which are Yetter–Drinfeld modules, were defined in [11]. We here realize simple Yetter–Drinfeld modules of our \( \mathcal{B}(X) \) in terms of one-vertex modules.

2.3.1. The \( \mathcal{X} \) spaces. Let \( Y^{n_1, n_2} \) be a one-dimensional vector space with basis \( V^{n_1, n_2} \) and braiding \( \psi : \mathcal{B}(X) \otimes Y^{n_1, n_2} \to Y^{n_1, n_2} \otimes \mathcal{B}(X) \) and \( Y^{n_1, n_2} \otimes \mathcal{B}(X) \to \mathcal{B}(X) \otimes Y^{n_1, n_2} \) defined by

\[ \psi(F_i \otimes V^{n_1, n_2}) = q^{1-n_i} V^{n_1, n_2} \otimes F_i, \]

\[ \psi(V^{n_1, n_2} \otimes F_i) = q^{1-n_i} F_i \otimes V^{n_1, n_2}, \]

\[ i = 1, 2. \] Every space \( \mathcal{B}(X) \otimes V^{n_1, n_2} \otimes \mathcal{B}(X) \otimes V^{n_1, n_2} \otimes \ldots \otimes V^{n_1, n_2} \) is a Yetter–Drinfeld \( \mathcal{B}(X) \) module. Taking the \( a_i^j \) to be generic leads to continuum families of such modules, leaving us with no chance of a nice correspondence with any type of “reasonably rational” CFT model. The choice of the possible \( a_i^j \) values is governed by the requirement that all of them (and the braided vector space \( X \) itself) be objects of a suitable \( H^\mathcal{D} \).
category of Yetter–Drinfeld modules over a nonbraided Hopf algebra $H$. In the case of
diagonal braiding, more specifically, $H = k\Gamma$ for an Abelian group $\Gamma$, which can then be
considered the origin of the appropriate discreteness in the $a_i^j$ values. We do not pursue
this line in this paper, and simply assume that the $a_i^j$ take integer values.

We consider one-vertex modules $\mathcal{B}(X) \otimes V^{\{n_1, n_2\}}$ and for brevity write
$\mathcal{B}(r, t, s)^{\{n_1, n_2\}} = \mathcal{B}(r, t, s) \otimes V^{\{n_1, n_2\}} \in \mathcal{B}(X) \otimes Y^{\{n_1, n_2\}}$,
and, in particular,
$F_i^{\{n_1, n_2\}} = F_i \otimes V^{\{n_1, n_2\}} \in \mathcal{B}(X) \otimes Y^{\{n_1, n_2\}}$
(but $\mathcal{B}(0, 0, 0)^{\{n_1, n_2\}} = 1 \otimes V^{\{n_1, n_2\}}$ is normally written as $V^{\{n_1, n_2\}}$).

### 2.3.2. Left adjoint action. The formulas for the product, coproduct, and antipode in
[2.2.1-2.2.3] allow calculating the left adjoint action of the $\mathcal{B}(X)$ generators on one-vertex modules:
$F_1 \triangleright \mathcal{B}(r, t, s)^{\{n_1, n_2\}} = \langle r + 1 \rangle (1 - q^{2(s - t + 1 - n_1)}) \mathcal{B}(r + 1, t, s)^{\{n_1, n_2\}}$
$- q^{2r - 2s + t - 2n_1 + 3} \langle t + 1 \rangle (1 - q^2) \mathcal{B}(r, t + 1, s - 1)^{\{n_1, n_2\}}$
and
$F_2 \triangleright \mathcal{B}(r, t, s)^{\{n_1, n_2\}} = q^{1-r} \langle t + 1 \rangle (1 - q^2) \mathcal{B}(r - 1, t + 1, s)^{\{n_1, n_2\}}$
$+ q^{t-r} \langle s + 1 \rangle (1 - q^{2(s+1-n_2)}) \mathcal{B}(r, t, s + 1)^{\{n_1, n_2\}}$.

These formulas depend on $n_1$ and $n_2$ only through $(a_i \mod p)$. The $\mathcal{B}(X)$ coaction is given
by literally applying formula (2.4) to $\mathcal{B}(r, t, s) \otimes V^{\{n_1, n_2\}}$ (and is entirely independent of $a_i$).

### 2.3.3. Simple Yetter–Drinfeld modules. A simple Yetter–Drinfeld $\mathcal{B}(X)$-module $\mathcal{Y}^{\{n_1, n_2\}}$
is generated from $V^{\{n_1, n_2\}}$ under the action of $\mathcal{B}(X)$; its dimension is given by
$d(p, n_1, n_2) = \begin{cases}
\frac{d(n_1, n_2)}{\bar{p}}, & \bar{p} \leq p, \\
\frac{d(n_1, n_2) - d(p - \bar{n_1}, p - \bar{n_2})}{\bar{p} + 1}, & \bar{p} > p + 1,
\end{cases}$
where $d(n_1, n_2) = \frac{1}{2} n_1 n_2 (n_1 + n_2)$ and
$\underline{x} = \begin{cases}
p, & (x \mod p) = 0, \\
x \mod p, & \text{otherwise.}
\end{cases}$

3. The octuplet algebra centralizing $\mathcal{B}(X)$

We next discuss a CFT construction related to our $\mathcal{B}(X)$. 
3.1. Screenings and their zero-momentum centralizer. We identify the \( \mathcal{B}(X) \) generators with two screenings

\[
F_\alpha = F_1 = \int e^{\Phi_\alpha}, \quad F_\beta = F_2 = \int e^{\Phi_\beta},
\]

where \( \Phi_\alpha(z) \) and \( \Phi_\beta(z) \) are two scalar fields whose OPEs are defined in accordance with the braiding matrix as follows:

\[
\Phi_\alpha(z) \Phi_\alpha(w) = \frac{2}{p} \log(z - w), \quad \Phi_\alpha(z) \Phi_\beta(w) = -\frac{1}{p} \log(z - w),
\]

\[
\Phi_\beta(z) \Phi_\beta(w) = \frac{2}{p} \log(z - w).
\]

It follows from the formulas in [2] that the centralizer ("kernel") of screenings (3.1) contains a Virasoro algebra with the central charge

\[
c = 50 - \frac{24}{p} - 24p = -\frac{2(3p - 4)(4p - 3)}{p}.
\]

This Virasoro algebra is represented by the energy–momentum tensor

\[
T(z) = \frac{p}{3} \partial \Phi_\alpha \partial \Phi_\alpha(z) + \frac{p}{3} \partial \Phi_\alpha \partial \Phi_\beta(z) + \frac{p}{3} \partial \Phi_\beta \partial \Phi_\beta(z) - (p - 1) \partial^2 \Phi_\alpha(z) - (p - 1) \partial^2 \Phi_\beta(z).
\]

In addition to the Virasoro algebra, the kernel of the screenings contains the dimension-3 Virasoro primary field (omitting the conventional \( z \) arguments of fields)

\[
W(z) = \partial \Phi_\alpha \partial \Phi_\alpha \partial \Phi_\alpha \partial \Phi_\alpha + \frac{3}{2} \partial \Phi_\alpha \partial \Phi_\alpha \partial \Phi_\beta \partial \Phi_\beta - \frac{3}{2} \partial \Phi_\alpha \partial \Phi_\alpha \partial \Phi_\beta \partial \Phi_\beta - \partial \Phi_\beta \partial \Phi_\beta \partial \Phi_\beta \partial \Phi_\beta
\]

\[
- \frac{9(p - 1)}{2p} \partial^2 \Phi_\alpha \partial \Phi_\alpha - \frac{9(p - 1)}{4p} \partial^2 \Phi_\alpha \partial \Phi_\beta + \frac{9(p - 1)}{4p} \partial^2 \Phi_\alpha \partial \Phi_\beta + \frac{9(p - 1)}{4p} \partial^2 \Phi_\beta \partial \Phi_\beta + \frac{9(p - 1)}{4p^2} \partial^3 \Phi_\alpha - \frac{9(p - 1)^2}{4p^2} \partial^3 \Phi_\beta.
\]

The operator product of this field with itself is given by

\[
W(z) W(w) = \frac{81(3p - 5)(3p - 4)(4p - 3)(5p - 3) - 243(3p - 5)(5p - 3)T(w)}{4p^4 (z - w)^6} - \frac{243(3p - 5)(5p - 3)T(w)}{8p^4 (z - w)^3} + \frac{243(3p - 5)(5p - 3)\partial T(w)}{16p^4 (z - w)^2} + \frac{243(3p - 5)(5p - 3)\partial^2 T(w)}{8p^4 (z - w)} - \frac{243(3p - 5)(5p - 3)\partial^3 T(w)}{16p^4 (z - w)},
\]

where \( TT(w) \) is the normal-ordered product \( T(w)T(w) \) (and similarly for \( \partial T(w) \)). This OPE defines the \( \mathcal{W}_3 \) algebra [19] (also see [31]).

In an equivalent description, the \( \mathcal{W}_3 \) algebra relations for the modes introduced as

\[
T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n - 2} \quad \text{and} \quad W(z) = \sum_{n \in \mathbb{Z}} W_n z^{-n - 3}
\]

are

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(50 - \frac{24}{p} - 24p)(m - 1)m(m + 1) \delta_{m+n,0},
\]

\[
[L_m, W_n] = (2m - n)W_{m+n},
\]

\[
[W_m, W_n] = -\frac{81(3p - 5)(5p - 3)}{8p^4} \frac{(m - n)(m + n + 3)(m + n + 2)}{5} - \frac{(m + 2)(n + 2)}{2} L_{m+n}.
\]
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\[ + \frac{243}{4p^3}(m-n)\Lambda_{m+n} + \frac{27(3p-5)(3p-4)(4p-3)(5p-3)}{160p^3}m(m^2-1)(m^2-4)\delta_{m+n,0}, \]

where

\[ \Lambda_m = \sum_{n \leq -2} L_n L_{m-n} + \sum_{n \geq 1} L_{m-n} L_n - \frac{3}{10} (m+3)(m+2)L_m. \]

3.2. Long screenings. The \( \mathcal{W}_3 \) algebra is also centralized by two “long” screenings

\[ (3.4) \quad \mathcal{E}_\alpha = \oint e^{-p\phi_\alpha} \quad \text{and} \quad \mathcal{E}_\beta = \oint e^{-p\phi_\beta}. \]

Because

\[ [F_i, \mathcal{E}_j] = 0, \]

the long screenings act on the kernel of the \( F_\alpha \) and \( F_\beta \), and are therefore a useful tool in studying that kernel.

3.3. Remark. We note that, generally, given the screenings \( F_i = \oint e^{\phi_i} = \oint e^{\alpha_i \phi} \), \( i = 1, \ldots, \theta \), the Virasoro dimension of a vertex \( e^{\mu \cdot \varphi(z)} \) with \( \mu = \sum_{i=1}^\theta c_i \alpha_i \) is

\[ \Delta(c) = \sum_{i=1}^\theta c_i \left( 1 - \frac{\alpha_i}{2} \right) + \frac{1}{2} \sum_{i,j=1}^\theta c_i c_j \alpha_i \alpha_j. \]

We list the generators of the ideal in (2.1) together with the vertex operators that naively (by momentum counting) correspond to them, and with the Virasoro dimensions of these vertices:

\[ (3.5) \quad e^{2\phi_\alpha(z)+\phi_\beta(z)}, \quad e^{\phi_\alpha(z)+2\phi_\beta(z)}, \quad e^{p\phi_\alpha(z)}, \quad e^{p\phi_\alpha(z)+p\phi_\beta(z)}, \quad e^{p\phi_\beta(z)}, \quad e^{3}, \quad e^{3}, \quad 2p-1, \quad 3p-2, \quad 2p-1. \]

3.4. The octuplet algebra. The field

\[ \mathcal{W}(z) = e^{p\phi_\alpha(z)+p\phi_\beta(z)}, \]

which is the top-dimension field in (3.5), is in the kernel of \( F_\alpha \) and \( F_\beta \) and is a \( \mathcal{W}_3 \)-primary field of dimension \( \Delta = 3p-2 \) and the \( W_0 \) eigenvalue zero. To describe how it is mapped by the long screenings, we need a reminder on \( \mathcal{W}_3 \) singular vectors.

3.4.1. Singular vectors in \( \mathcal{W}_3 \) Verma modules. We recall from [32] (also see [31] and the references therein) that highest-weight vectors of the \( \mathcal{W}_3 \) algebra can be conveniently parameterized by \( (x, y) \) such that

\[ L_m |x, y\rangle = 0, \quad m \geq 1, \]

\[ W_m |x, y\rangle = 0, \quad m \geq 1, \]

\[ L_0 |x, y\rangle = \left( \frac{x^2+y^2+xy}{3} - \frac{(p-1)^2}{p} \right) |x, y\rangle, \]
and built on that state. The singular vector has the highest-weight parameters $a^c$

The two numbers $x$ and $y$ are defined not uniquely but up to a Weyl transformation; the
Weyl group orbit of $(x, y)$ also contains $(-x, x + y)$, $(x + y, -y)$, $(y, -x - y)$, $(-x - y, x)$, and $(-y, -x)$. We write $\mathcal{V}(z) \doteq |x, y\rangle$ for any field/state $\mathcal{V}(z)$ that satisfies the above conditions.

In what follows, we use the conditions for the existence of singular vectors in Verma
modules of the $\mathcal{W}_3$ algebra [33] [34] [32]. Whenever a state can be represented as $|x, y\rangle$ with $x = a\sqrt{p} - \frac{c}{\sqrt{p}}$ for integer $a$ and $c$ such that $ac > 0$, there is a singular vector on the level $ac$ built on that state. The singular vector has the highest-weight parameters $(x', y') = (x - 2a\sqrt{p}, y + a\sqrt{p})$. Similarly, if $y = b\sqrt{p} - \frac{d}{\sqrt{p}}$ with $bd > 0$, then a singular vector occurs on the level $bd$ and has the highest-weight parameters $(x'', y'') = (x + b\sqrt{p}, y - 2b\sqrt{p})$.

**3.4.2.** It follows that

$$\mathcal{W}(z) = e^{p\phi_\alpha(z) + p\phi_\beta(z)} \doteq |2\sqrt{p} - \frac{1}{\sqrt{p}}, 2\sqrt{p} - \frac{1}{\sqrt{p}}, 0\rangle,$$

and hence the corresponding Verma-module state has two singular vectors at level 2. Both
of them vanish in our free-field realization. Of the two fields $e\mathcal{W}(z)$ and $e\mathcal{W}(z)$, we
concentrate on the second; it lands in the module generated from

$$e^{p\phi_\alpha(z)} \doteq |3\sqrt{p} - \frac{1}{\sqrt{p}}, \frac{1}{\sqrt{p}}, 0\rangle.$$

The corresponding highest-weight state in the Verma module has singular vectors at levels
3 and $p - 1$. The first of these vanishes in the free-boson realization, but the second does not, yielding just the field $\mathcal{W}_\beta(z) = e\mathcal{W}(z)$, as we show in Fig. 1. We note that

$$\mathcal{W}_\beta(z) = P_{\beta}^{[p-1]}(\partial \phi(z)) e^{p\phi_\alpha(z)}$$

with a differential polynomials in $\partial \phi_\alpha(z)$, $\partial \phi_\beta(z)$ in front of the exponential; here and
hereafter, we indicate the degree $d$ of a differential polynomial as $P[d]$.

Totally similarly,

$$\mathcal{W}_\alpha(z) = e\mathcal{W}(z) = P_{\alpha}^{[p-1]}(\partial \phi(z)) e^{p\phi_\beta(z)}$$

is a descendant of

$$e^{p\phi_\beta(z)} \doteq |3\sqrt{p} - \frac{1}{\sqrt{p}}\rangle.$$

The maps of $\mathcal{W}_\alpha(z)$ by $e\mathcal{W}$ and of $\mathcal{W}_\beta(z)$ by $e\mathcal{W}$ are differential polynomials (not involving exponentials). They are not descendants of the unit operator, however. We have

$$1 \doteq |\sqrt{p} - \frac{1}{\sqrt{p}}, \sqrt{p} - \frac{1}{\sqrt{p}}\rangle,$$

which implies singular vectors at levels 1, 1, 4, $2p - 1$, and $2p - 1$. All of these vanish in the free-field realization. In each of the grades where a level-$(2p - 1)$ singular vector vanishes, another state is produced as $e\mathcal{W}(e^{p\phi_\alpha(z)})$ and
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Figure 1. Maps by the long screenings \(\varepsilon_\alpha\) and \(\varepsilon_\beta\). Crosses and downward arrows leading to them show \(W_3\) singular vectors that vanish in the free-boson realization. Bullets (and downward arrows) show nonvanishing states in the same grades; the relative levels of singular vectors are indicated at the arrows. An open circle superimposed with a cross shows a vanishing \(W_3\) singular vector and a (\(W_3\)-primary) state in the same grade, but not in the same \(W_3\)-module (and downward arrows drawn from such \(\times\) show singular vectors built on those primary states).

Two more modules—those with \(e^{p\Phi_\alpha}\) and \(e^{-p\Phi_\alpha}\) at the top—are not shown here; their structure repeats that of the "\(e^{p\Phi_\alpha}\)" and "\(e^{-p\Phi_\alpha}\)" modules with \(\alpha \leftrightarrow \beta\). Dotted arrows show the maps by \(\varepsilon_\alpha\) and \(\varepsilon_\beta\) from the missing modules.
We Weyl-reflect the highest-weight parameters to the same pair of singular vectors. These two next-generation singular vectors vanish in these singular vectors therefore has two level-

Further maps by the long screenings do not produce \( \mathbb{W}_3 \)-descendants of the corresponding exponentials either. We consider \( \mathcal{E}_\beta \mathcal{W}_\alpha (z) \) and \( \mathcal{E}_\beta \mathcal{W}_\beta (z) \). In the module associated with

\[
e^{-p\Phi_\beta(z)} = |2\sqrt{p} - \frac{1}{\sqrt{p}} - \sqrt{p} - \frac{1}{\sqrt{p}}\rangle,
\]

two singular vectors at level 2 and two at level \( 2p - 2 \) vanish; located at the grades of the last two are \( \mathcal{E}_\beta \mathcal{E}_\alpha e^{p\Phi_\alpha(z)} \) (the maps shown in Fig. 1) and \( \mathcal{E}_\beta \mathcal{E}_\beta e^{p\Phi_\beta(z)} \)\(^1\) Now, \( \mathcal{E}_\beta \mathcal{E}_\alpha e^{p\Phi_\alpha(z)} \) and \( \mathcal{E}_\beta \mathcal{E}_\beta e^{p\Phi_\beta(z)} \) have a level-(\( p - 1 \)) singular-vector descendant each. In our free-field realization, these two singular vectors evaluate the same up to a nonzero overall factor, thus producing a \( \mathbb{W}_3 \)-primary field

\[
\mathcal{W}_\beta \alpha (z) = \mathcal{E}_\beta \mathcal{W}_\alpha (z) = \mathcal{P}[{3p-2}]_\beta_\alpha (\hat{\varphi}(z)) e^{-p\Phi_\beta(z)}.
\]

Everything with the replacement \( \alpha \leftrightarrow \beta \) applies to the field

\[
\mathcal{W}_\alpha \beta (z) = \mathcal{E}_\alpha \mathcal{W}_\beta (z) = \mathcal{P}[{3p-3}]_\alpha_\beta (\hat{\varphi}(z)) e^{-p\Phi_\alpha(z)}.
\]

Finally, mapping by the long screenings once again gives a field

\[
\mathcal{W}_\alpha \beta \beta (z) = \mathcal{E}_\alpha \mathcal{W}_\beta \beta (z) = \mathcal{P}[{4p-4}]_\alpha_\beta_\beta (\hat{\varphi}(z)) e^{-p\Phi_\alpha(z)-p\Phi_\beta(z)}
\]

(which is also \( \mathcal{E}_\beta \mathcal{W}_\alpha \beta (z) \) up to a factor), which is not in the module associated with \( e^{-p\Phi_\alpha(z)-p\Phi_\beta(z)} \), however. In the Verma module associated with the highest-weight vector

\[
e^{-p\Phi_\alpha(z)-p\Phi_\beta(z)} = \left| -\frac{1}{\sqrt{p}}, -\frac{1}{\sqrt{p}} \right\rangle,
\]

there are two singular vectors at level \( p - 1 \), both of which are nonvanishing in the free-field realization and are in fact the images of \( e^{-p\Phi_\beta(z)} \) (and \( e^{-p\Phi_\alpha(z)} \); see Fig. 1). Each of these singular vectors therefore has two level-(\( 2p - 2 \)) singular vectors, which are in fact the same pair of singular vectors. These two next-generation singular vectors vanish in

\(^1\)We illustrate the use of \( \mathbb{W}_3 \). In the Verma module with the highest-weight vector \( |x,y\rangle = \left[2\sqrt{p} - \frac{1}{\sqrt{p}}, -\sqrt{p} - \frac{1}{\sqrt{p}}\right] \) associated with \( e^{-p\Phi_\beta(z)} \), one of the level-(\( 2p - 2 \)) singular vectors exists due to the representation \( y = \frac{\nu - 1}{2\sqrt{p}}, \) and therefore the singular vector has the highest-weight parameters \( (\nu',\nu'') = (-\frac{1}{\sqrt{p}},3\sqrt{p} - \frac{1}{\sqrt{p}}) \), i.e., those of \( e^{p\Phi(z)} \). The other level-(\( 2p - 2 \)) singular vector is seen immediately if we Weyl-reflect the highest-weight parameters to \( (\tilde{x},\tilde{y}) = (-x,x+y) \). We then have \( \tilde{y} = \frac{2(p-1)}{\sqrt{p}} - \sqrt{p}, \) and hence the singular vector has the parameters \( (-3\sqrt{p} + \frac{1}{\sqrt{p}},3\sqrt{p} - \frac{2}{\sqrt{p}}) \). After the same Weyl reflection, the parameters \( (3\sqrt{p} - \frac{1}{\sqrt{p}},\frac{1}{\sqrt{p}}) \) correspond to \( e^{p\Phi(z)} \).
our free-field realization, but the maps by $\mathcal{E}_\alpha$ (and by $\mathcal{E}_\beta$) land in the same grades. The two vectors in the image of the long screenings share a singular-vector descendant at the level-$(p - 1)$ and this descendant is the $W_{\alpha\alpha\beta}(z)$ field.

**3.4.3.** We summarize the octuplet structure of $\mathcal{W}_3$ primary fields generated by long screenings from $\mathcal{W}(z)$:

![Diagram of the octuplet structure of $\mathcal{W}_3$ primary fields](image)

The dashed arrows represent maps to the target field up to a nonzero overall factor. All the fields in the diagram are $\mathcal{W}_3$-primaries, with the same Virasoro dimension $3p - 2$.

We follow [2] in proposing these fields as generators of the octuplet algebra $\mathfrak{W}_{p,1}$—the extended algebra of logarithmic $\mathcal{W}_3$ models.

**3.4.4.** Calculations with particular examples show the OPE

$$W(z) W_{\alpha\alpha\beta}(w) = \frac{c_1 \cdot 1}{(z-w)^{6p-4}} + \frac{c_2 T(w)}{(z-w)^{6p-6}} + \frac{c_2/2 \partial T(w)}{(z-w)^{6p-7}} + \ldots$$

with nonzero $p$-dependent coefficients (and no dimension-3 $W(w)$ field), and

$$W_{\alpha}(z) W_{\beta\alpha\beta}(w) = \frac{(-1)^{p+1} c_1 \cdot 1}{(z-w)^{6p-4}} + \frac{(-1)^{p+1} c_2 T(w)}{(z-w)^{6p-6}} + \frac{(-1)^{p+1} c_2/2 \partial T(w)}{(z-w)^{6p-7}} + \ldots,$$

$$W_{\beta}(z) W_{\alpha\beta\alpha}(w) = \frac{(-1)^{p+1} c_1 \cdot 1}{(z-w)^{6p-4}} + \frac{(-1)^{p+1} c_2 T(w)}{(z-w)^{6p-6}} + \frac{(-1)^{p+1} c_2/2 \partial T(w)}{(z-w)^{6p-7}} + \ldots$$

with nonzero coefficients, and the OPEs $W_{\alpha}(z) W_{\beta\alpha\beta}(w)$ and $W_{\beta}(z) W_{\alpha\beta\alpha}(w)$ that start very similarly. The adjoint-$s\ell(3)$ nature of the octuplet manifests itself in the OPEs such as

$$W_{\alpha}(z) W_{\beta}(w) = \frac{c_3 W(w)}{(z-w)^{3p-2}} + \ldots,$$

$$W_{\alpha}(z) W_{\alpha\beta\alpha}(w) = O(z-w),$$

$$W_{\beta}(z) W_{\beta\alpha\beta}(w) = O(z-w),$$
\[ W_{\alpha\beta\alpha}(z) W_{\beta\alpha\beta}(w) = \frac{c_{3} W_{\alpha\beta\beta}(w)}{(z - w)^{p-2}} + \ldots \]

3.4.5. Some octuplet algebra representations. To construct CFT counterparts of the modules introduced in 2.3, we first define the “fundamental weights” \( \omega_i \) such that \( \omega_i \cdot \alpha_j = \delta_{i,j} \):

\[ \omega_1 = \frac{p}{3}(2\alpha_1 + \alpha_2), \quad \omega_2 = \frac{p}{3}(\alpha_1 + 2\alpha_2). \]

We let \( \omega_\alpha(z) \) and \( \omega_\beta(z) \) denote the corresponding fields:

\[ \omega_\alpha(z) = \frac{p}{3}(2\varphi_\alpha(z) + \varphi_\beta(z)), \quad \omega_\beta(z) = \frac{p}{3}(\varphi_\alpha(z) + 2\varphi_\beta(z)). \]

Then the field

\[ \mathcal{F}_{n_1,n_2}(z) = e^{\frac{1-n_1}{p} \omega_\alpha(z) + \frac{1-n_2}{p} \omega_\beta(z)} \]

has the same braiding with \( F_i \) as \( V^{(n_1,n_2)} \) has in 2.3.1. The dimension of \( \mathcal{F}_{n_1,n_2}(z) \) is

\[ \Delta_{n_1,n_2} = p - n_1 - n_2 + \frac{n_1^2 + n_1 n_2 + n_2^2}{3p} - \frac{(p-1)^2}{p} \]

and, in fact, \( \mathcal{F}_{n_1,n_2}(z) \cong |x,y> \) with \( (x,y) \) given by any pair from the Weyl orbit:

\[ (\sqrt{p} - \frac{n_1}{\sqrt{p}}, \sqrt{p} - \frac{n_2}{\sqrt{p}}), (\frac{n_2}{\sqrt{p}} - \sqrt{p}, \frac{n_1}{\sqrt{p}} - \sqrt{p}), \]

\[ (\sqrt{p} - \frac{n_2}{\sqrt{p}}, \frac{n_1 + n_2}{\sqrt{p}} - 2\sqrt{p}), (\frac{n_1 + n_2}{\sqrt{p}} - 2\sqrt{p}, \sqrt{p} - \frac{n_1}{\sqrt{p}}), \]

\[ (\frac{n_1}{\sqrt{p}} - \sqrt{p}, -\frac{n_1 + n_2}{\sqrt{p}} + 2\sqrt{p}), (-\frac{n_1 + n_2}{\sqrt{p}} + 2\sqrt{p}, \frac{n_2}{\sqrt{p}} - \sqrt{p}). \]

The corresponding \( \mathcal{W}_3 \) singular vectors vanish in the free-field realization. We propose the irreducible \( \mathcal{O}_{p,1} \)-modules generated from \( \mathcal{F}_{n_1,n_2}(z) \) as counterparts of the corresponding simple Yetter–Drinfeld \( \mathcal{B}(X) \) modules, as a starting point to study the relation between the two representation categories.

4. Conclusions

We have outlined some details of the construction of the octuplet extended algebra \( \mathcal{O}_{p,1} \) proposed in [2], and described the corresponding Nichols algebra \( \mathcal{B}(X) \) in rather explicit terms. Systematically comparing \( \mathcal{O}_{p,1} \) representations with Yetter–Drinfeld \( \mathcal{B}(X) \) modules is very interesting from the perspective of whether the relation existing in the \( W_{p,1} \) (triplet-algebra) case [24, 25, 28, 29] extends to the current \( \mathcal{W}_3 \)-related octuplet setting.

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