Elliptic Stable Envelopes and Finite-dimensional Representations of Elliptic Quantum Group

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Abstract

We construct a finite dimensional representation of the face type, i.e dynamical, elliptic quantum group associated with $\hat{sl}_N$ on the Gelfand-Tsetlin basis of the tensor product of the $n$-vector representations. The result is described in a combinatorial way by using the partitions of $[1,n]$. We find that the change of basis matrix from the standard to the Gelfand-Tsetlin basis is given by a specialization of the elliptic weight function obtained in the previous paper [33]. Identifying the elliptic weight functions with the elliptic stable envelopes obtained by Aganagic and Okounkov, we show a correspondence of the Gelfand-Tsetlin bases (resp. the standard bases) to the fixed point classes (resp. the stable classes) in the equivariant elliptic cohomology $E_T(X)$ of the cotangent bundle $X$ of the partial flag variety. As a result we obtain a geometric representation of the elliptic quantum group on $E_T(X)$.

1 Introduction

It has long been conjectured that there is a parallelism between the infinite dimensional (quantum) algebras and (equivariant) cohomology, K-theory, and elliptic cohomology [17,19]. In [16,39,40], finite-dimensional representations of symmetrizable Kac-Moody algebras $\mathfrak{g}$ were constructed in terms of homology groups of quiver varieties. Their extension to the quantized universal enveloping algebras $U_q(\mathfrak{g})$ and to their affinization were constructed on equivariant K-theory of quiver varieties [18,20,39,41,53,55]. Note that Yangian $Y(\mathfrak{g})$ is obtained by replacing equivariant K-theory by equivariant homology [41,54]. The basic tool in these works are convolution operation and correspondences in homology and equivariant K-theory. See for example [41]. However the elliptic case still remains conjecture.

Stable envelopes introduced by Maulik and Okourov [35] are new tools to tackle this problem. For a quiver variety $X$, stable envelope is a map from the equivariant cohomology of the torus $A$-fixed point set $X^A$ to the equivariant cohomology of $X$. It was extended to equivariant K-theory [43] and equivariant elliptic cohomology [1]. In terms of stable envelopes Maulik and
Okoukov constructed rational $R$ matrices geometrically and obtained a geometric realization of the Yangian $Y_Q$ associated with a quiver $Q$ \cite{35}. Such geometric construction of $R$ matrices was extended to the trigonometric \cite{43} and the elliptic \cite{1} cases. The stable envelopes also proved to be useful in solving integrable systems \cite{2,35,49}.

This new approach was enhanced by a discovery of a connection to the weight functions appearing in the hypergeometric integral solutions to the difference KZ equations. Gorbounov, Rimányi, Tarasov and Varchenko found an identification of rational weight functions with stable envelopes for torus-equivariant cohomology of the partial flag variety $T^*F_\lambda$ \cite{21} and extended this to the trigonometric ones for the equivariant K-theory \cite{14}. Furthermore they succeeded to construct a geometric representation of the Yangian $Y(\mathfrak{gl}_N)$ \cite{21} and the quantum affine algebra $U_q(\hat{\mathfrak{gl}}_N)$ \cite{44} on the equivariant cohomology and the equivariant K-theory, respectively. In these works, a correspondence between finite-dimensional representations of quantum groups on the Gelfand-Tsetlin basis of the tensor product of the vector representations and geometric representations is a key to construction. Furthermore Felder, Rimányi and Varchenko \cite{14} proposed a geometric representation of the dynamical elliptic quantum group $E_{\tau,y}(\mathfrak{gl}_2)$ by using the $\hat{\mathfrak{sl}}_2$ type elliptic weight function obtained in \cite{13,50}.

The elliptic weight functions of type $\hat{\mathfrak{sl}}_N$ were derived in the previous paper \cite{33} by using representation theory of the elliptic quantum group $U_{q,p}(\mathfrak{sl}_N)$ \cite{7,24,28,29}. The $U_{q,p}(\mathfrak{sl}_N)$ is a Drinfeld realization of the dynamical elliptic quantum group and is isomorphic to the central extension of Felder’s elliptic quantum group $E_{q,p}(\mathfrak{sl}_N)$ \cite{32}. Furthermore, in \cite{33} their properties such as triangularity, transition property, orthogonality, quasi-periodicity and shuffle algebra structure were investigated. Comparing these properties with those of the elliptic stable envelopes in \cite{11}, we conjectured that the elliptic weight functions can be identified with the elliptic stable envelopes. Some of similar but slightly different results were presented in \cite{46}.

The purpose of this paper is to formulate a geometric representation of the higher rank dynamical elliptic quantum group associated with $\hat{\mathfrak{sl}}_N$. Constructing the Gelfand-Tsetlin basis of the tensor product of the $n$-vector representations explicitly (Theorem \ref{thm:gtb}), we obtain finite-dimensional representations of $E_{q,p}(\mathfrak{sl}_N)$ on it. In particular, we obtain an action of the half-currents of $E_{q,p}(\mathfrak{sl}_N)$ and of the associated elliptic currents of $U_{q,p}(\mathfrak{sl}_N)$ on the Gelfand-Tsetlin basis (Theorem \ref{thm:cur} and Corollary \ref{cor:cur}). The resultant representations are described in a combinatorial way by using the partitions of $[1,n]$. It turns out that in the trigonometric and non-dynamical limit their combinatorial structures coincide with those of $U_q(\mathfrak{sl}_N)$ on the equivariant K-theory obtained by Ginzburg and Vasserot \cite{20,55} and by Nakajima \cite{41}.

We then lift these representations to the geometric ones by identifying the elliptic weight
functions with the elliptic stable envelopes. We make a direct comparison of the elliptic weight functions with the abelianization formula of the elliptic stable envelopes, which was obtained by Shenfeld [48] in the rational case and extended to the elliptic case in [1]. We also obtain an identification of certain specializations of the elliptic weight functions with the elliptic stable envelopes restricted to the torus fixed points. In this restriction, the stable envelopes play a role of the change of basis matrix elements from the stable classes to the fixed point classes in $E_T(T^*F_\lambda)$. This allows us to define the fixed point classes in $E_T(T^*F_\lambda)$ as transformations from the stable classes.

We then find that this defining relation of the fixed point classes (5.15) is identical to the change of basis relation from the standard basis to the Gelfand-Tsetlin basis $[4,2]$. Then a correspondence between the Gelfand-Tsetlin bases (resp. the standard bases) and the fixed point classes (resp. the stable classes) in $E_T(T^*F_\lambda)$ yields a definition of the actions of the half-currents of $E_{q,p}(\hat{\mathfrak{g}}_N)$ and of the elliptic currents of $U_{q,p}(\hat{\mathfrak{g}}_N)$ on the fixed point classes in $E_T(T^*F_\lambda)$, and provides a geometric representation of the elliptic quantum group on $E_T(T^*F_\lambda)$ (Theorem 5.1 and Corollary 5.2).

In [46], a similar formula for elliptic weight functions of type $\hat{\mathfrak{g}}_N$ and their triangularity and the orthogonality properties are presented without derivation. There the triangular property agrees with ours but the orthogonality property seems wrong due to a lack of the dynamical shift. There are also no formulas for the shuffle algebra in [46]. In addition, it seems that in [46] a different formulation of elliptic stable envelopes from the one in [1] is presented. The relation between them is not clear for us. However, we would like to stress that in [1,46] there are neither statements on the definition of the fixed point classes in $E_T(T^*F_\lambda)$ nor the correspondence between the Gelfand-Tsetlin bases and the fixed point classes, which are the keys to our results.

A part of the results has been presented at the workshops “Elliptic Hypergeometric Functions in Combinatorics, Integrable Systems and Physics”, March 20-24, 2017, ESI, Vienna, “Topological Field Theories, String Theory and Matrix Models”, August 25-31, 2017, ITEP, Moscow, Infinite Analysis 17 “Algebraic and Combinatorial Aspects in Integrable Systems”, December 4-7, 2017, Osaka City University, “Geometric $R$-Matrices: From Geometry to Probability”, December 18-23, 2017, University of Melbourne, Creswick and at the MSJ Autumn meeting, September 12, 2017, Yamagata University.

This paper is organized as follows. In Section 2 we prepare some notations including the elliptic dynamical $R$ matrices. We also provide defining relations of the elliptic quantum groups $E_{q,p}(\hat{\mathfrak{g}}_N)$ and $U_{q,p}(\hat{\mathfrak{g}}_N)$, and their basic properties. Definition of the half-currents of $E_{q,p}(\hat{\mathfrak{g}}_N)$
and their relationship to the elliptic currents of $U_{q,p}(\mathfrak{gl}_N)$ are also exposed. Section 3 is devoted to a summary of the properties of the elliptic weight functions obtained in [33], such as the triangular property, transition property, orthogonality, quasi-periodicity and the shuffle algebra structures. In Section 4, we discuss a construction of finite-dimensional representations of $E_{q,p}(\mathfrak{gl}_N)$ and $U_{q,p}(\mathfrak{sl}_N)$ on the Gelfand-Tsetlin basis of the tensor product of the vector representations. In particular we show that the change of basis matrix from the standard to the Gelfand-Tsetlin basis is given by a specialization of the elliptic weight functions. In Section 5, we discuss an identification between the elliptic weight functions and the elliptic stable envelopes and give a geometric representation of $E_{q,p}(\mathfrak{gl}_N)$ and $U_{q,p}(\mathfrak{sl}_N)$. In Appendix A we summarize a co-algebra structure of $E_{q,p}(\mathfrak{gl}_N)$ and $U_{q,p}(\mathfrak{sl}_N)$. In Appendix B we present a proof of our main Theorem 4.7. In Appendix C we present a direct check of Corollary 4.8 for the relation (2.33).

2 Preliminaries

Through this paper we follow the notations in [33]. We here list the basic ones.

2.1 The commutative algebra $H$

- $A = (a_{ij})$ ($i, j \in \{0, 1, \ldots, N - 1\}$) : the generalized Cartan matrix of $\mathfrak{sl}_N = \mathfrak{sl}(N, \mathbb{C})$
- $\mathfrak{h} = \bar{\mathfrak{h}} \oplus \mathbb{C}d$, $\mathfrak{h} = \bar{\mathfrak{h}} \oplus \mathbb{C}c$, $\bar{\mathfrak{h}} = \oplus_{i=1}^{N-1} \mathbb{C}h_i$ : the Cartan subalgebra of $\mathfrak{sl}_N$
- $\mathfrak{h}^* = \bar{\mathfrak{h}}^* \oplus \mathbb{C}\delta$, $\bar{\mathfrak{h}}^* = \bar{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0$, $\bar{\mathfrak{h}}^* = \oplus_{i=1}^{N-1} \mathbb{C}\Lambda_i$ : the dual space of $\mathfrak{h}$
- $\alpha_i \in \bar{\mathfrak{h}}^*$ ($1 \leq i \leq N - 1$) : simple roots such that $<\alpha_i, h_j> = a_{ji}$
- $Q = \oplus_{i=1}^{N-1} \mathbb{Z}\alpha_i$ : root lattice, $\mathcal{P} = \oplus_{i=1}^{N-1} \mathbb{Z}\Lambda_i$ : weight lattice

Let $\{\epsilon_j \ (1 \leq j \leq N)\}$ be the orthonormal basis in $\mathbb{R}^N$ with the inner product $(\epsilon_j, \epsilon_k) = \delta_{j,k}$. We set $\bar{\epsilon}_j = \epsilon_j - \sum_{k=1}^{N} \epsilon_k/N$, $1 \leq j \leq N$. Then we have a realization $\alpha_i = \bar{\epsilon}_i - \bar{\epsilon}_{i+1}$ and $\Lambda_i = \epsilon_1 + \cdots + \epsilon_i$, $1 \leq i \leq N - 1$. For $\alpha \in \bar{\mathfrak{h}}^*$ we define $h_\alpha \in \bar{\mathfrak{h}}$ by $<\beta, h_\alpha> = (\beta, \alpha) \ \forall \beta \in \bar{\mathfrak{h}}^*$. We regard $\bar{\mathfrak{h}} \oplus \bar{\mathfrak{h}}^*$ as the Heisenberg algebra by

$$[h_\alpha, \beta] = (\alpha, \beta), \quad [h_\alpha, h_\beta] = 0 = [\alpha, \beta] \quad \alpha, \beta \in \bar{\mathfrak{h}}^*. \quad (2.1)$$

Similarly,

- $\{P_\alpha, Q_\beta\}$ ($\alpha, \beta \in \bar{\mathfrak{h}}^*$) : the Heisenberg algebra defined by

$$[P_\alpha, Q_\beta] = (\alpha, \beta), \quad [P_\alpha, P_\beta] = 0 = [Q_\alpha, Q_\beta], \quad (2.2)$$
Then we define

- \( H = \sum_{j=1}^{N} \mathbb{C}(P + h)_{\epsilon_j} + \sum_{j=1}^{N} \mathbb{C}P_{\epsilon_j} + \mathbb{C}c \), where \( (P + h)_{\epsilon_j} \) is an abbreviation of \( P_{\epsilon_j} + h_{\epsilon_j} \).

- \( H^* = \tilde{\mathbb{H}}^* \oplus \sum_{j=1}^{N} \mathbb{C}Q_{\epsilon_j} \): the dual space of \( H \) with paring \( < \Lambda_0, c >= 1, < \hat{\Lambda}_i, h_j >= \delta_{i,j}, < Q_\alpha, P_\beta >= (\alpha, \beta) \) and the others vanish.

- \( \mathbb{F} = \mathcal{M}_{H^*} \): the field of meromorphic functions on \( H^* \).

2.2 \( q \)-integers, infinite products and theta functions

Let \( q \) be generic complex numbers satisfying \(|q| < 1\).

\[
[u]_q = \frac{q^n - q^{-n}}{q - q^{-1}},
\]

\[
(x; q)_\infty = \prod_{n=0}^{\infty} (1 - xq^n), \quad (x; q, t)_\infty = \prod_{n,m=0}^{\infty} (1 - xq^n t^m).
\]

Let \( r \) be a generic positive real number and set \( p = q^{2r} \). We use the following Jacobi’s odd theta functions.

\[
[u] = q^{\frac{u^2}{2}} \Theta_p(z), \quad \Theta_p(z) = (z; p)_\infty (p/z; p)_\infty (p; p)_\infty
\]

\[
[u + r] = -[u], \quad [u + r\tau] = -e^{-\pi i \tau} e^{-\pi i u/r} [u],
\]

where \( z = q^{2u}, p = e^{-2\pi i \tau} \). For \( k \in \mathbb{R} \), we also need the theta function \([u]^*\) whose elliptic nome is given by \( p^* = q^{2r^*}, r^* = r - k \). We assume \( r^* > 0 \).

\[
[u]^* = p^{\frac{u^2}{2}} \Theta_{p^*}(z).
\]

2.3 The elliptic dynamical \( R \)-matrix of the \( \tilde{\mathfrak{sl}}_N \) type

Let \( \hat{V} = \bigoplus_{\mu=1}^{N} \mathbb{F}v_\mu \) be the \( N \)-dimensional vector space over \( \mathbb{F} \). The elliptic dynamical \( R \)-matrix \( R^\pm(z_1/z_2, \Pi) \in \text{End}_\mathbb{C}(\hat{V} \otimes \hat{V}) \) of type \( \tilde{\mathfrak{sl}}_N \) is given by

\[
R^\pm(z, \Pi) = \rho^\pm(z) \tilde{R}(z, \Pi),
\]

\[
\tilde{R}(z, \Pi) = \sum_{j=1}^{N} E_{j,j} \otimes E_{j,j} + \sum_{1 \leq j_1 < j_2 \leq N} \left( b(u, (P + h)_{j_1,j_2}) E_{j_1,j_1} \otimes E_{j_2,j_2} + \tilde{b}(u) E_{j_2,j_2} \otimes E_{j_1,j_1} \right.
\]

\[
+ \left. c(u, (P + h)_{j_1,j_2}) E_{j_1,j_2} \otimes E_{j_2,j_1} + \tilde{c}(u, (P + h)_{j_1,j_2}) E_{j_2,j_1} \otimes E_{j_1,j_2} \right),
\]
where $E_{i,j}v_\mu = \delta_{j,\mu}v_i$, $z = q^{2u}$, $\Pi_{j,k} = q^{2(P+h)_{j,k}}$, $(P+h)_{j,k} := (P+h)\epsilon_j - (P+h)\epsilon_k$,

$$\rho^+(z) = q^{-\frac{N-1}{2}} z^{-\frac{N-1}{2}} \{ q^{2N}q^{-2z} \} \{ q^{2z} \} \frac{pqq^{2N}/z \{ p/z \} \{ pqq^{2N}/q^{-2}z \} \{ pqq^{2}/z \}}{\{ pqq^{2N}/z \} \{ z \} \{ pqq^{2}/z \}},$$

(2.7)

$$\rho^-(z) = \rho^+(pz),$$

(2.8)

$$b(u, s) = \frac{(s + 1)(s - 1)[u]}{[s]^2[u + 1]}, \quad \tilde{b}(u) = \frac{[u]}{[u + 1]},$$

(2.9)

$$c(u, s) = \frac{[1][s + u]}{[s][u + 1]}, \quad \tilde{c}(u, s) = \frac{[1][s - u]}{[s][u + 1]}$$

and $\{ z \} = (z; p, q^{2N})_\infty$. This $R$ matrix is gauge equivalent to Jimbo-Miwa-Okado’s $A^{(1)}_{N-1}$ face type Boltzmann weight [22] and can be obtained [30] by taking the vector representation of the universal elliptic dynamical $R$ matrix [23].

The $R^\pm(z, q^{2s})$ satisfies the dynamical Yang-Baxter equation

$$R^{(12)}(z_1/z_2, q^{2(s+h^{(3)})})R^{(13)}(z_1/z_3, q^{2s})R^{(23)}(z_2/z_3, q^{2(s+h^{(1)})}) = R^{(23)}(z_2/z_3, q^{2s})R^{(13)}(z_1/z_3, q^{2(s+h^{(2)})})R^{(12)}(z_1/z_2, q^{2s}),$$

(2.10)

where $q^{2h^{(l)}_{j,k}}$ acts on the $l$-th tensor space $\hat{V}$ by $q^{2h^{(l)}_{j,k}}v_\mu = q^{2<\xi_{j,k}v_\mu, v_\mu>}$, and the unitarity

$$R(z, q^{2s})R^{(21)}(z^{-1}, q^{2s}) = \text{id}_{\hat{V} \otimes \hat{V}}.$$

(2.11)

### 2.4 The elliptic quantum groups $E_{q,p}(\hat{gl}_N)$ and $U_{q,p}(\hat{gl}_N)$

We consider the dynamical elliptic quantum group realized in the two ways $E_{q,p}(\hat{gl}_N)$ and $U_{q,p}(\hat{gl}_N)$. The elliptic algebra $E_{q,p}(\hat{gl}_N)$ is a central extension of Felder’s elliptic quantum group [11,32], whereas $U_{q,p}(\hat{gl}_N)$ is an elliptic and dynamical analogue [29,32] of Drinfeld’s new realization of the quantum affine algebra $U_q(\hat{gl}_N)$ [34]. For the details of the definitions we refer the reader to Sec.3 and Appendix D.1 in [32] and Appendix A in [33].

#### 2.4.1 The elliptic algebra $E_{q,p}(\hat{gl}_N)$

For simplicity of presentation, we treat the elliptic algebra $E_{q,p}(\hat{gl}_N)$ as a unital associative algebra over $\mathbb{F}$ generated by (the Laurent coefficients of) $L_{ij}^+(z)$ ($1 \leq i, j \leq N$) and the central element $q^{2c/2}$. Let $L^+(z) = \sum_{1 \leq i, j \leq N} E_{ij}L_{ij}^+(z)$. In the level $k \in \mathbb{R}$ representation, where $c = k$, the defining relations are given as follows.

$$g(P)\hat{L}_{ij}(z) = \hat{L}_{ij}(z) g(P - <Q_{\epsilon_j}, P>),$$

(2.12)

$$g(P + h)\hat{L}_{ij}(z) = \hat{L}_{ij}(z) g(P + h - <Q_{\epsilon_j}, P + h>),$$

(2.13)

$$R^{+ (12)}(z_1/z_2, \Pi)\hat{L}^{+(1)}(z_1)\hat{L}^{+(2)}(z_2) = \hat{L}^{+(2)}(z_2)\hat{L}^{+(1)}(z_1)R^{+(12)}(z_1/z_2, \Pi^*),$$

(2.14)
where \(g(P + h), g(P) \in \mathbb{F}\), and

\[
R^\pm(z, \Pi^*) = R^\pm(z, \Pi)|_{p \to p^*, r \to r^*, [u] \to [u]^*, P + h \to P}
\]

with \(\Pi^*_{jl} = q^{2P_{jl}}, p^* = pq^{-2k} = q^{2r^*}\).

Setting \(\hat{L}^-(z) = \hat{L}^+(pq^{-k}z)\), we have from Proposition D.2 in [32]

\[
R^{-(12)}(z_1/z_2, \Pi)\hat{L}^{-(1)}(z_1)\hat{L}^{-(2)}(z_2) = \hat{L}^{-(2)}(z_2)\hat{L}^{-(1)}(z_1)R^{-*-(12)}(z_1/z_2, \Pi^*),
\]

\[
R^{\pm(12)}(q^{\pm k}z_1/z_2, \Pi)\hat{L}^{\pm(1)}(z_1)\hat{L}^{\pm(2)}(z_2) = \hat{L}^{\mp(2)}(z_2)\hat{L}^{\pm(1)}(z_1)R^{\pm*-(12)}(q^{\mp k}z_1/z_2, \Pi^*)
\]

where

\[
R^-(z, \Pi) = \rho^-(z)\hat{R}(z, \Pi), \quad \rho^-(z) = z^{2N-1}N \rho^+(pz),
\]

\[
R^*-(z, \Pi^*) = R^-(z, \Pi)|_{p \to p^*, r \to r^*, [u] \to [u]^*, P + h \to P}.
\]

**Definition 2.1.** We define the half-currents of \(E_{q,p}(\hat{n}(N))\), \(F_{j,l}^\pm(z)\), \(E_{l,j}^\pm(z)\), \((1 \leq j < l \leq N)\) and \(\hat{R}_l^\pm(z)\) \((1 \leq l \leq N)\), as the following Gauss components of \(\hat{L}^\pm(z)\).

\[
\hat{L}^\pm(z) = \begin{pmatrix}
1 & F_{1,2}^\pm(z) & F_{1,3}^\pm(z) & \cdots & F_{1,N}^\pm(z) \\
0 & 1 & F_{2,3}^\pm(z) & \cdots & F_{2,N}^\pm(z) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & F_{N-1,N}^\pm(z) \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}
\times
\begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
E_{2,1}^\pm(z) & 1 & \ddots & \vdots \\
E_{3,1}^\pm(z) & E_{3,2}^\pm(z) & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & 0 \\
E_{N,1}^\pm(z) & E_{N,2}^\pm(z) & \cdots & E_{N,N-1}^\pm(z) & 1
\end{pmatrix}
\]

One can express the half-currents in terms of the quantum minor determinant of the \(L\) operators [32]. For \(1 \leq a, b \leq N\), let us define \(\hat{L}^+(z)_{a,a} = (\hat{L}^+_{i,j}(z))_{a \leq i,j \leq N}\) and

\[
\hat{L}^+(z)_{a,b} = \begin{pmatrix}
\hat{L}^+_{ab}(z) & \hat{L}^+_{a+1}(z) & \cdots & \hat{L}^+_{aN}(z) \\
\hat{L}^+_{a+1b}(z) & \hat{L}^+_{a+1a+1}(z) & \cdots & \hat{L}^+_{a+1N}(z) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{L}^+_{Nb}(z) & \hat{L}^+_{N+1}(z) & \cdots & \hat{L}^+_{NN}(z)
\end{pmatrix}
\text{ for } a > b
\]
Then we obtain

\[
\begin{pmatrix}
\hat{\mathcal{L}}_{ab}(z) & \hat{\mathcal{L}}_{ab+1}(z) & \cdots & \hat{\mathcal{L}}_{aN}(z) \\
\hat{\mathcal{L}}_{b+1b}(z) & \hat{\mathcal{L}}_{b+1b+1}(z) & \cdots & \hat{\mathcal{L}}_{b+1N}(z) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\mathcal{L}}_{Nb}(z) & \hat{\mathcal{L}}_{Nb+1}(z) & \cdots & \hat{\mathcal{L}}_{NN}(z)
\end{pmatrix}
\]

for \( a < b \). (2.19)

Then we obtain

\[ \hat{K}_j(z) = \mathcal{N}_{N-j+1}^{-1} q \cdot \det \hat{\mathcal{L}}^\pm(z)_{j,j} \left( q \cdot \det \hat{\mathcal{L}}^\pm(zq^{-2})_{j+1,j+1} \right)^{-1}, \]

\[ E_{k,j}^\pm(z) = \left( q \cdot \det \hat{\mathcal{L}}^\pm(z)_{k,k} \right)^{-1} q \cdot \det \hat{\mathcal{L}}^\pm(z)_{k,j}, \]

\[ F_{j,k}^\pm(z) = q \cdot \det \hat{\mathcal{L}}^\pm(z)_{j,k} \left( q \cdot \det \hat{\mathcal{L}}^\pm(z)_{k,k} \right)^{-1} \quad (1 \leq j < k \leq N), \]

where

\[
\mathcal{N}_k = \frac{\mathcal{N}_{k-1}}{\mathcal{N}_k}, \quad \mathcal{N}_k = \prod_{1 \leq a < b \leq k} \frac{\rho_0^*[a][1]}{\rho_0[a][1]^*},
\]

\[
\rho_0 = - \lim_{z \to q^{-1}} z \cdot \rho^+(z) \left[ \frac{1}{u+1} \right] = q^{-N+2} \frac{z^{2N}; q^{2N}\infty}{(p^q \infty \{pq^2\})^2 \{pq^4\}^{-1} q^{2N}},
\]

\[
\rho_0^* = \rho_0|_{p \to q^*, r \to r^*}
\]

**Corollary 2.3.** Let us set

\[ \hat{K}(z) = \hat{K}_1^+(z) \hat{K}_2^+(zq^{-2}) \cdots \hat{K}_N^+(zq^{-2(N-1)}). \]

Then the \( q \)-determinant of \( \hat{\mathcal{L}}^+(z) \) is given by

\[ q \cdot \det \hat{\mathcal{L}}^+(z) = \mathcal{N}_N \hat{K}(z) \]

and belongs to the center of \( E_{q,p}(\widehat{\mathfrak{gl}}_N) \).

Moreover from Proposition 6.4 in [32], we have

**Corollary 2.4.** For \( l = 1, \cdots, N \), the \( q \)-principal minor determinant \( q \cdot \det \hat{\mathcal{L}}^+(z)_{ll} \) is given by

\[ q \cdot \det \hat{\mathcal{L}}^+(z)_{ll} = \mathcal{N}_{N-l+1} \hat{K}_1^+(z) \hat{K}_{l+1}^+(zq^{-2}) \cdots \hat{K}_N^+(zq^{-2(N-l)}) \]

and belongs to the center of the subalgebra \( E_{q,p}(\widehat{\mathfrak{gl}}_{N-l+1}) \) of \( E_{q,p}(\widehat{\mathfrak{gl}}_N) \).

We define the elliptic algebra \( E_{q,p}(\widehat{\mathfrak{sl}}_N) \) as the quotient algebra \( E_{q,p}(\widehat{\mathfrak{gl}}_N)/<\hat{K}(z) - 1> \).
2.4.2 The elliptic algebra $U_{q,p}(\mathfrak{sl}_N)$

For simplicity of presentation, we treat the elliptic algebra $U_{q,p}(\mathfrak{g}(\mathfrak{l}_N))$ as a unital associative algebra over $\mathbb{F}$ generated by (the Laurent coefficients of) the elliptic currents $E_j(z), F_j(z), K^+_i(z) \ (1 \leq j \leq N - 1, 1 \leq l \leq N)$. In the level-$k$ ($k \in \mathbb{R}$) representation, the defining relations are given in the sense of analytic continuation as follows. For $g(P), g(P + h) \in \mathbb{F}$,

\begin{align*}
g(P + h)E_j(z) &= E_j(z)g(P + h), \quad g(P)E_j(z) = E_j(z)g(P - <Q_{\alpha_j}, P>), \quad (2.20) \\
g(P + h)F_j(z) &= F_j(z)g(P + h - <\alpha_j, P + h>), \quad g(P)F_j(z) = F_j(z)g(P), \quad (2.21) \\
g(P)K^+_i(z) &= K^+_i(z)g(P - <Q_{\epsilon_i}, P>), \quad g(P + h)K^+_i(z) = K^+_i(z)g(P + h - <Q_{\epsilon_i}, P>), \quad (2.22) \\
K^+_i(z_1)K^+_i(z_2) &= \frac{\rho^{+*}(z_1/z_2)}{\rho^+(z_1/z_2)} K^+_i(z_2)K^+_i(z_1), \quad (2.23) \\
K^+_j(z_1)K^+_l(z_2) &= \frac{\rho^{+*}(z_1/z_2)}{\rho^+(z_1/z_2)} [u_1 - u_2 - 1] [u_1 - u_2 - 1] K^+_l(z_2)K^+_j(z_1) \quad (1 \leq j < l \leq N), \quad (2.24)
\end{align*}

\begin{align*}
K^+_j(z_1)E_j(z_2) &= \left[ u_1 - u_2 + \frac{j-N-k+1}{2} \right]^* E_j(z_2)K^+_j(z_1), \quad (2.25) \\
K^+_{j+1}(z_1)E_j(z_2) &= \left[ u_1 - u_2 + \frac{j-N-k+1}{2} \right]^* E_j(z_2)K^+_{j+1}(z_1), \quad (2.26) \\
K^+_i(z_1)E_j(z_2) &= E_j(z_2)K^+_i(z_1) \quad (l \neq j, j + 1), \quad (2.27) \\
K^+_j(z_1)F_j(z_2) &= \left[ u_1 - u_2 + \frac{j+N+1}{2} - 1 \right] F_j(z_2)K^+_j(z_1), \quad (2.28) \\
K^+_{j+1}(z_1)F_j(z_2) &= \left[ u_1 - u_2 + \frac{j+N+1}{2} + 1 \right] F_j(z_2)K^+_{j+1}(z_1), \quad (2.29) \\
K^+_i(z_1)F_j(z_2) &= F_j(z_2)K^+_i(z_1) \quad (l \neq j, j + 1), \quad (2.30)
\end{align*}

\begin{align*}
[u - v - \frac{a_{ij}}{2}]^* E_i(z)E_j(w) &= [u - v + \frac{a_{ij}}{2}]^* E_j(w)E_i(z), \quad (2.31) \\
[u - v + \frac{a_{ij}}{2}] F_i(z)F_j(v) &= [u - v - \frac{a_{ij}}{2}] F_j(v)F_i(z), \quad (2.32) \\
[E_i(z), F_j(w)] &= \frac{\delta_{i,j}}{q - q^{-1}} \left( \delta(q^{-k-1} z/w)H_i^-(q^{k/2} w) - \delta(q^{-1} z/w)H_i^+(q^{-k/2} w) \right), \quad (2.33) \\
K^+_i(z) &= K^+_i(pq^{-k}z), \quad (2.34) \\
H^+_j(z) &= qK^+_j(q^{-j-1} q^{k/2} z)K^+_{j+1}(q^{N-j-1} q^{k/2} z)^{-1}. \quad (2.35)
\end{align*}
\[ z_1 \frac{1}{2} \left( \frac{p^* q^{-1} z_2 / z_1 ; p^*}{(p^* q^{-2} z_2 / z_1 ; p^* )} \right) \left( \frac{p^* q^{-1} z / z_1 ; p^*}{(p^* q z / z_1 ; p^* )} \right) E_i(z_1) E_i(z_2) E_j(z) \]

\[ - \left[ q \right] \frac{p^* q^{-1} z / z_1 ; p^*}{(p^* q z / z_1 ; p^* )} E_i(z_1) E_j(z_2) \]

\[ + \left( z_1 / z \right) \frac{1}{2} \left( \frac{p^* q^{-1} z_1 / z ; p^*}{(p^* q z_1 / z ; p^* )} \right) E_j(z_1) E_i(z_2) \] + (z_1 \leftrightarrow z_2) = 0, \quad (i \neq j = 1). \tag{2.36} \]

\[ z_1 \frac{1}{2} \left( \frac{p q^{-2} z_2 / z_1 ; p}{(p q^2 / z_2 z_1 ; p) } \right) \left( \frac{(p q z / z_1 ; p)}{(p q^{-1} z_2 / z_1 )} \right) F_i(z_1) F_i(z_2) F_j(z) \]

\[ - \left[ q \right] \frac{(p q z / z_1 ; p)}{(p q^{-1} z_2 / z_1 ; p)} F_i(z_1) F_j(z_2) \]

\[ + \left( z_1 / z \right) \frac{1}{2} \left( \frac{(p q z_1 / z ; p)}{(p q^{-1} z_2 / z ; p)} \right) F_j(z_1) F_i(z_2) \] + (z_1 \leftrightarrow z_2) = 0, \quad (i \neq j = 1). \tag{2.37} \]

where

\[ \varrho = \frac{(p ; p) \infty (p^* q^2 ; p^* ) \infty}{(p^* ; p^* ) \infty (p q^2 ; p ) \infty}. \]

Proposition 2.5. \[ K(z) = K_1^+(z) K_2^+(z q^{-2}) \cdots K_N^+(z q^{-2(N-1)}). \]

We define the elliptic algebra \( U_{q,p}(\widehat{\mathfrak{gl}}_N) \) as the quotient algebra \( U_{q,p}(\widehat{\mathfrak{gl}}_N) / < K(z) - 1 >. \)

2.4.3 Isomorphism between \( E_{q,p}(\widehat{\mathfrak{gl}}_N) \) and \( U_{q,p}(\widehat{\mathfrak{gl}}_N) \)

Proposition 2.6. \[ E_j(z q^{i-1} - c/2) := \mu^* \left( E_{j+1,j}^+(z q^{c/2}) - E_{j+1,j}^-(z q^{-c/2}) \right), \]

\[ F_j(z q^{i-1} - c/2) := \mu \left( F_{j+1,j}^+(z q^{-c/2}) - F_{j+1,j}^-(z q^{c/2}) \right), \]

where \( \mu \) and \( \mu^* \) satisfy

\[ \mu \mu^* = - \frac{\varrho}{q - q^{-1}} \frac{[0]}{[1]}. \tag{2.38} \]

Then identifying \( \widehat{K}_i(z) \) with \( K_i^+(z), \) \( \widehat{K}_i(z), \) \( E_j(z), \) \( F_j(z) \) satisfy the defining relations of \( U_{q,p}(\widehat{\mathfrak{gl}}_N) \) in Sec. 2.4.2.

Furthermore let us set

\[ H_{j}^\pm(z) := g \widehat{K}_j^\pm (q^{N-j-1} q^{c/2} z) \widehat{K}_j^\pm (q^{N-j-1} q^{-c/2} z). \]
Corollary 2.7. Under the constraint \( \hat{K}(z) = 1 \), the generating functions \( H_j^\pm(z), E_j(z), F_j(z) \) \((1 \leq j \leq N - 1)\) satisfy the defining relations of \( U_{q,p}(\hat{\mathfrak{g}}_N) \).

Theorem 2.8. \([32]\)

\[ U_{q,p}(\hat{\mathfrak{g}}_N) \cong E_{q,p}(\hat{\mathfrak{g}}_N). \]

Definition 2.9. The Gelfand-Tsetlin subalgebra \( \mathfrak{G} \) of \( E_{q,p}(\hat{\mathfrak{g}}_N) \) as well as of \( U_{q,p}(\hat{\mathfrak{g}}_N) \) is defined to be a unital subalgebra generated by (the Laurent coefficients of) \( \hat{K}_l^+(z) \) \((1 \leq l \leq N)\).

From (2.23)-(2.24) we obtain

Proposition 2.10. The Gelfand-Tsetlin subalgebra \( \mathfrak{G} \) becomes a commutative subalgebra at the level \( 0 \) \((c = 0)\).

2.4.4 Dynamical L operators and half-currents

For later convenience (see Sec.4) we introduce the dynamical \( L \) operators \([24, 28]\) by

\[ L^\pm(z, P) = \hat{L}^\pm(z)e^{\sum_{j=1}^N \pi(h_{\epsilon_j})Q_{\epsilon_j}}, \]

where \( \pi(h_{\epsilon_j}) = E_{jj} \). Then \( L^\pm(z, P) \) commutes with the elements in \( \mathbb{F} \) and satisfy the full dynamical RLL relations \([11]\)

\[ R^{\pm(12)}(z_1/z_2, \Pi)L^{\pm(1)}(z_1, P)L^{\pm(2)}(z_2, P + h^{(1)}) = L^{\pm(2)}(z_2, P)L^{\pm(1)}(z_1, P + h^{(2)})R^{\pm(12)}(z_1/z_2, \Pi^*), \]

(2.39)

\[ R^{\pm(12)}(q^{\pm c}z_1/z_2, \Pi)L^{\pm(1)}(z_1, P)L^{\mp(2)}(z_2, P + h^{(1)}) = L^{\mp(2)}(z_2, P)L^{\mp(1)}(z_1, P + h^{(2)})R^{\pm(12)}(q^{\mp c}z_1/z_2, \Pi^*), \]

(2.40)

Accordingly we define the dynamical half-currents \( \hat{K}_l^+(z), E_{j+1,j}^+(z, P), F_{j,j+1}^+(z, P) \) \((1 \leq l \leq N, 1 \leq j \leq N - 1)\) as the corresponding Gauss coordinates of \( L^\pm(z, P) \). Then the relation between the half-currents from \( \hat{L}^\pm(z) \) and those from \( L^\pm(z, P) \) is given as follows.

Proposition 2.11. \([32]\)

\[ \hat{K}_l^\pm(z) = K_l^\pm(z)e^{-Q_{\epsilon_j}}, \]

\[ \hat{E}_{j+1,j}^\pm(z) = e^{Q_{\epsilon_j+1}}E_{j+1,j}^\pm(z, P)e^{-Q_{\epsilon_j}}, \]

\[ \hat{F}_{j,j+1}^\pm(z) = F_{j,j+1}^\pm(z, P). \]
3 Elliptic Weight Functions

In this section we summarize some basic properties of the elliptic weight functions obtained in [33].

3.1 Combinatorial notations

Let \( \widetilde{V} = \bigoplus_{\mu=1}^{N} Fv_{\mu} \) be the same as in Sec. 2.3 and consider its tensor product \( \widetilde{V} \otimes^{n} \). The standard basis of \( \widetilde{V} \otimes^{n} \) is given by \( \{ v_{\mu_{1}} \otimes \cdots \otimes v_{\mu_{n}} \mid \mu_{1}, \cdots, \mu_{n} \in \{1, \cdots, N\} \} \), where \( \otimes \) is defined in Appendix A.

Let \([1,n]=\{1,\cdots,n\}\). For a vector \( v_{\mu_{1}} \otimes \cdots \otimes v_{\mu_{n}} \), we define the index set \( I_{l}:=\{ i \in [1,n] \mid \mu_{i}=l \} \) \((l=1,\cdots,N)\) and set \( \lambda_{l}:|I_{l}|, \lambda:=(\lambda_{1},\cdots,\lambda_{N}) \). Then \( I=I_{1} \cup \cdots \cup I_{N} \) is a partition of \([1,n]\), i.e.

\[ I_{1} \cup \cdots \cup I_{N} = [1,n], \quad I_{k} \cap I_{l} = \emptyset \quad (k \neq l). \]

We often denote thus obtained partition \( I \) by \( I_{\mu_{1}, \cdots, \mu_{n}} \). We also write \( v_{I}=v_{\mu_{1}} \otimes \cdots \otimes v_{\mu_{n}} \). Let \( \mathbb{N} = \{ m \in \mathbb{Z} \mid m \geq 0 \} \). For \( \lambda=(\lambda_{1}, \cdots, \lambda_{N}) \in \mathbb{N}^{N} \) satisfying \( |\lambda|=\lambda_{1}+\cdots+\lambda_{N}=n \), let \( \mathcal{I}_{\lambda} \) be the set of all partitions \( I=(I_{1}, \cdots, I_{N}) \) of \([1,n]\) satisfying \( |I_{l}|=\lambda_{l} \) \((l=1,\cdots,N)\). We also set \( \lambda^{(l)}:=\lambda_{1}+\cdots+\lambda_{l}, I^{(l)}:=I_{1} \cup \cdots \cup I_{l} \) and let \( I^{(l)}:=\{ i^{(l)}_{1}<\cdots<i^{(l)}_{\lambda(l)} \} \). For \( I \in \mathcal{I}_{\lambda} \), all vectors \( v_{I} \) have the same weight \( \sum_{j=1}^{n} \bar{\epsilon}_{\mu_{j}} \), which we call the weight associated with \( \lambda \). For each \( i^{(l)}_{a} \) \((l=1,\cdots,N, a=1,\cdots,\lambda^{(l)})\), we consider the variables \( i^{(l)}_{a}=t(i^{(l)}_{a}) \) with \( i_{b}^{(N)}=z_{a} \) \((a=1,\cdots,n)\), and set \( t=(i^{(l)}_{a}) \) \((l=1,\cdots,N, a=1,\cdots,\lambda^{(l)})\).

For \( \lambda=(\lambda_{1}, \cdots, \lambda_{N}) \in \mathbb{N}^{N}, |\lambda|=n \), we consider in Sec.5 the partial flag variety \( \mathcal{F}_{\lambda}=\mathcal{F}(\lambda^{(1)}, \cdots, \lambda^{(N-1)}, n) \) consisting of \( 0=\mathcal{V}_{0} \subset \mathcal{V}_{1} \subset \cdots \subset \mathcal{V}_{N}=\mathbb{C}^{n} \) with \( \dim_{\mathbb{C}} \mathcal{V}_{k}=\lambda^{(k)} \). A representation theoretical meaning to this parametrization is given in [33].

3.2 The elliptic weight functions of type \( \mathfrak{sl}_{N} \)

We consider the following elliptic weight functions [33].

\[
\widetilde{W}_{I}(t,z,\Pi) = \text{Sym}_{\lambda^{(1)}} \cdots \text{Sym}_{\lambda^{(N-1)}} \widetilde{U}_{I}(t,z,\Pi), \tag{3.1}
\]

\[
\widetilde{U}_{I}(t,z,\Pi) = \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda^{(l)}} \left( \frac{v_{b}^{(l+1)} - v_{a}^{(l)} + (P+h)\mu_{s,l+1} - C_{\mu_{s},l+1}(s)}{v_{b}^{(l+1)} - v_{a}^{(l)} + 1} \right) \left( P+h \right)^{\mu_{s,l+1} - C_{\mu_{s},l+1}(s)} \left. v_{b}^{(l+1)} \right|_{v_{b}^{(l+1)}=v_{a}^{(l)}} = s, \tag{3.2}
\]

\[
\times \prod_{b=1}^{\lambda^{(l+1)}} \frac{v_{b}^{(l+1)} - v_{a}^{(l)}}{v_{b}^{(l+1)} - v_{a}^{(l)} + 1} \prod_{b=a+1}^{\lambda^{(l)}} \frac{v_{b}^{(l)} - v_{a}^{(l)} - 1}{v_{b}^{(l)} - v_{a}^{(l)}} \right),
\]

\[ 12 \]
where we set \( t_{a}^{(l)} = q^{2a_{l}} \) \((l = 1, \ldots, N - 1, a = 1, \ldots, \lambda^{(l)})\), \( z_{k} = q^{2u_{k}} \) \((k = 1, \ldots, n)\), \( v_{s}^{(N)} = u_{s} \) \((s = 1, \ldots, n)\) and \( C_{\mu_{s}, l+1}(s) := \sum_{j=s+1}^{n} \epsilon_{\mu_{j}, h_{\mu_{k}, l+1}} > (\mu_{s} \leq l)\). The symbol \( \text{Sym}_{l}^{(i)} \) denotes the symmetrization over the variables \( t_{1}^{(l)}, \ldots, t_{\lambda^{(l)}}^{(l)}\).

For \( I = (I_{1}, \ldots, I_{N}) \in \mathcal{I}_{X}\), let \( I_{k} = \{i_{k,1} < \cdots < i_{k,\lambda_{k}}\} \) \((k = 1, \ldots, N)\). Then \( C_{\lambda_{s}, l+1}\) has the following combinatorial expression.

**Proposition 3.1.**

\[
C_{\lambda_{s}, l+1}(s) = \begin{cases} 
\lambda_{s} - \lambda_{l+1} - \bar{s} + m_{\mu_{s}, l+1}(s) - 1 & \text{if } s \leq i_{l+1, \lambda_{l+1}} \\
\lambda_{s} - \bar{s} & \text{if } s > i_{l+1, \lambda_{l+1}}
\end{cases}
\]

where for \( s \in [1, n] \) we define \( \bar{s} \) by \( i_{\mu_{s}, \bar{s}} = s \) and \( m_{\mu_{s}, l+1}(s) \) by

\[
m_{\mu_{s}, l+1}(s) = \min\{1 \leq j \leq \lambda_{l+1} \mid s < i_{l+1, j}\} \quad \text{for } s \leq i_{l+1, \lambda_{l+1}}.
\]

### 3.3 Entire function version

Let us set

\[
H_{\lambda}(t, z) := \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda^{(l)}} \prod_{b=1}^{\lambda^{(l)}} \left[ v_{b}^{(l+1)} - v_{a}^{(l)} + 1 \right]. \tag{3.3}
\]

The following gives an entire function version of the elliptic weight function.

\[
W_{I}(t, z, \Pi) = H_{\lambda}(t, z) \widetilde{W}_{I}(t, z, \Pi) = \text{Sym}_{l}^{(1)} \cdots \text{Sym}_{l}^{(N-1)} U_{I}(t, z, \Pi), \tag{3.4}
\]

where

\[
U_{I}(t, z, \Pi) = \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda^{(l)}} \left[ v_{b}^{(l+1)} - v_{a}^{(l)} + (P + h)_{\mu_{s}, l+1} - C_{\mu_{s}, l+1}(s) \right] \left[ 1 \right] \left| \begin{array}{c}
\mid v_{b}^{(l+1)} - v_{a}^{(l)} + (P + h)_{\mu_{s}, l+1} - C_{\mu_{s}, l+1}(s) \\
\mid v_{b}^{(l+1)} = \bar{s} \end{array} \right| \times \prod_{l=1}^{\lambda^{(l+1)}} \left[ v_{b}^{(l+1)} - v_{a}^{(l)} \right] \prod_{l=1}^{\lambda^{(l+1)}} \left[ v_{b}^{(l+1)} - v_{a}^{(l)} + 1 \right] \prod_{b=\alpha+1}^{\lambda^{(l+1)}} \left[ v_{b}^{(l+1)} - v_{a}^{(l)} + 1 \right] \prod_{b=\alpha+1}^{\lambda^{(l+1)}} \left[ v_{b}^{(l+1)} - v_{a}^{(l)} + 1 \right]. \tag{3.5}
\]

Furthermore, in order to compare with the stable envelopes, it is convenient to consider the following expression. See Sec. 5.3

\[
W_{I}(t, z, \Pi) = \frac{W_{I}(t, z, \Pi)}{E_{\lambda}(t)} = \text{Sym}_{l}^{(1)} \cdots \text{Sym}_{l}^{(N-1)} U_{I}(t, z, \Pi), \tag{3.6}
\]

\[
U_{I}(t, z, \Pi) = \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda^{(l)}} u_{I}^{(l)} \left( t_{l}^{(l+1)}, \Pi_{l}^{(1)}, l+1 q_{l}^{2} C_{\mu_{s}, l+1, \lambda_{l+1}}^{(i)} \right) \prod_{1 \leq a < b \leq \lambda^{(l)}} \left[ v_{a}^{(l)} - v_{b}^{(l)} \right] \left[ v_{b}^{(l)} - v_{a}^{(l)} - 1 \right]. \tag{3.7}
\]

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where \( t^{(l+1)} = (t_1^{(l+1)}, \ldots, t_{\lambda^{(l)}}^{(l+1)}) \), and

\[
E_{\lambda}(t) = \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda^{(l)}} \prod_{b=1}^{\lambda^{(l)}} [v_b^{(1)} - v_a^{(1)} + 1],
\]

(3.8)

\[
u^{(l)}_I(i^{(l)}_a, t^{(l+1)}, \Pi_{j,k}) = \left[ v_b^{(l+1)} - v_a^{(l)} + (P + h)_{j,k} \right] \left[ (P + h)_{j,k} \right]_{i^{(l+1)}_b = i^{(l)}_a} \times \prod_{b=1}^{l^{(l+1)} > i^{(l)}_b \in \lambda^{(l)}} \left[ v_b^{(l+1)} - v_a^{(l)} \right] \prod_{b=1}^{l^{(l+1)} < i^{(l)}_b \in \lambda^{(l)}} \left[ v_b^{(l+1)} - v_a^{(l)} + 1 \right]
\]

(3.9)

for \( 1 \leq j < k \leq N \).

**Remark.** In the trigonometric \((p \to 0)\) and non-dynamical (neglecting the factors depending on \(P + h\)) limit \(W_I\) and \(\hat{W}_I\) coincide with \(W_I\) and \(\hat{W}_I\) discussed in [44], respectively. See also [36,37].

### 3.4 Properties of the elliptic weight functions

#### 3.4.1 Triangular property

For \(I, J \in \mathcal{I}_\lambda\), let \(I^{(l)} = \{i_1^{(l)} < \cdots < i_{\lambda^{(l)}}^{(l)}\}\) and \(J^{(l)} = \{j_1^{(l)} < \cdots < j_{\lambda^{(l)}}^{(l)}\}\) \((l = 1, \ldots, N)\). Define a partial ordering \(\leq\) by

\[
I \leq J \iff i_a^{(l)} \leq j_a^{(l)} \quad \forall I, a.
\]

Let us denote by \(t = z_I\) the specialization \(t_a^{(l)} = z_a^{(l)}\) \((l = 1, \ldots, N-1, a = 1, \ldots, \lambda^{(l)})\) [44]. The weight function has the following triangular property.

**Proposition 3.2.** For \(I, J \in \mathcal{I}_\lambda\),

1. \(W_I(z_I, z, \Pi) = 0\) unless \(I \leq J\).

2. 

\[
W_I(z_I, z, \Pi) = \prod_{1 \leq k < l \leq N} \prod_{a \in I_k} \prod_{b \in I_l} \left[ u_b - u_a \right] \prod_{b \in I_l} \left[ u_b - u_a + 1 \right].
\]

For \(\sigma \in \mathcal{S}_n\), let us denote \(\sigma^{-1}(I) = I_{\mu_\sigma(1) \cdots \mu_\sigma(n)}\) and \(\sigma(z) = (z_{\sigma(1)}, \ldots, z_{\sigma(n)})\). Following [44], let us set \(W_{\sigma,I}(t, z, \Pi) = W_{\sigma,I}(t, \sigma(z), \Pi)\) and \(W_{id,I}(t, z, \Pi) = W_I(t, z, \Pi)\). Let us consider the matrix \(\hat{W}_\sigma(z, \Pi)\), whose \((I, J)\)th element is given by \(W_{\sigma,I}(z_I, z, \Pi)\) \((I, J \in \mathcal{I}_\lambda)\). We put the matrix elements in the decreasing order with respect to \(\leq\). Then Proposition [32] yields that the matrix \(\hat{W}_{id}(z, \Pi)\) is lower triangular, whereas \(\hat{W}_\sigma(z, \Pi)\) with \(\sigma_0\) being the longest element in \(\mathcal{S}_n\) is upper triangular. In particular, for generic \(u_a\) \((a = 1, \ldots, n)\), \(\hat{W}_\sigma(z, \Pi)\) is invertible.
3.4.2 Transition property

Proposition 3.3. Let $I = I_{\mu_1 \cdots \mu_i+1 \cdots \mu_n} \in \mathcal{I}_\lambda$.

\[
W_{I_{\mu_1 \cdots \mu_i+1 \cdots \mu_n}}(t, \cdots, z_{i+1}, z_i, \cdots, \Pi) = \sum_{\mu_i', \mu_{i+1}'} R(z_i/z_{i+1}, \Pi q^{-2} \sum_{j=1}^{\mu_i'} \langle \epsilon_{\mu_{j}}, h \rangle) W_{I_{\mu_i' \cdots \mu_{i+1}'}}(t, \cdots, z_i, z_{i+1}, \cdots, \Pi). \tag{3.10}
\]

Note that since $H_\lambda(t, z)$ is a symmetric function in $z_1, \cdots, z_n$, $\tilde{W}_I(t, z, \Pi)$ has the same property.

3.4.3 Orthogonality

Noting (3.6) and the remark in Sec.5.3 in [33], where $E_\lambda(t, z)$ is the same as $E_\lambda(t)$ in (3.8), we have the following property.

Proposition 3.4. For $J, K \in \mathcal{I}_\lambda$,

\[
\sum_{I \in \mathcal{I}_\lambda} W_J(z_I, z, \Pi q^{-2} \sum_{j=1}^{\mu_j} \langle \epsilon_{\mu_{j}}, h \rangle) W_{\sigma_0(K)}(z_I, \sigma_0(z), \Pi) Q(z_I) R(z_I) = \delta_{J,K},
\]

where $\sum_{j=1}^{\mu_j} \tilde{\epsilon}_{\mu_j}$ is the weight associated with $\lambda$ (Sec.3.1), and

\[
Q(z_I) = \prod_{1 \leq k < l \leq N} \prod_{a \in I_k} \prod_{b \in I_l} [u_b - u_a + 1],
\]

\[
R(z_I) = \prod_{1 \leq k < l \leq N} \prod_{a \in I_k} \prod_{b \in I_l} [u_b - u_a].
\]

In Sec.5.4, a consistency between this property and the formula in Theorem 3.3 becomes a key to obtain a geometric representation of the elliptic quantum group.

3.4.4 Quasi-periodicity

Remember that we set $t_a^{(l)} = q^{2u_a^l}$, $z_k = q^{2u_k}$ and $\Pi_{j,k} = q^{2(P+h)_{j,k}}$. Note that $t_a^{(l)} \mapsto pt_a^{(l)} \iff v_a^{(l)} \mapsto v_a^{(l)} + r$ and $t_a^{(l)} \mapsto e^{-2\pi i t_a^{(l)}} \iff v_a^{(l)} \mapsto v_a^{(l)} + r \tau$. From (2.4) and Proposition 3.1 we obtain the following statement.

---

1In [14, 46], the dynamical shift in the orthogonality relation is missing.
Proposition 3.5. For $I \in \mathcal{I}_\lambda$, the weight functions $W_I(t, z, \Pi)$ has the following quasi-periodicity.

$$W_I(\cdots, pt_a^{(l)}, \cdots, z, \Pi) = (-1)^{\lambda_{l+1} - \lambda_l + 2}W_I(\cdots, t_a^{(l)}, \cdots, z, \Pi),$$

$$W_I(\cdots, e^{2\pi i}t_a^{(l)}, \cdots, z, \Pi) = (-e^{-2\pi i})^{\lambda_{l+1} - \lambda_l + 2} \times \exp \left\{ -\frac{2\pi i}{r} \left( (\lambda_{l+1} - \lambda_l)v_a^{(l)} - \sum_{b=1}^{\lambda_{l+1}} v_b^{(l+1)} + 2 \sum_{b=1}^{\lambda_{l+1}} v_b^{(l)} - \sum_{b=1}^{\lambda_{l+1}} v_b^{(l-1)} - (P + h)t_{l+1} - \lambda_{l+1} \right) \right\} \times W_I(\cdots, t_a^{(l)}, \cdots, z, \Pi) \quad (1 \leq a \leq \lambda^{(l)}, 1 \leq l \leq N - 1).$$

Remark. For $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{N}^N$, let $x = t(x_1^{(1)}, \cdots, x_N^{(1)}, \cdots, x_1^{(N-1)}, \cdots, x_N^{(N-1)}) \in \mathbb{C}^M$, where $M = \sum_{l=1}^{N-1} \lambda^{(l)} = \sum_{l=1}^{N-1} (N - l)\lambda_l$. From Proposition 3.5 one can deduce a symmetric integral $M \times M$ matrix $N$ and a vector $\xi \in (\mathbb{C}/r\mathbb{Z})^M$, which imply the following quadratic form $N(x) = t^*Nx$ and the linear form $\langle x, \Pi \rangle = \lambda^{(l)}x_l$.

Then by Appel-Humbert theorem [34], a pair $(N, \xi)$ characterizes a line bundle $L(N, \xi) : (\mathbb{C}^M \times \mathbb{C})/\Lambda^M \to \mathbb{C}^M$, where $\Lambda = r\mathbb{Z} + r\mathbb{Z}\tau$, with action

$$\omega \cdot (x, \eta) = (x + \omega, e_\omega(x)\eta), \quad \omega \in \Lambda^M, \ x \in \mathbb{C}^M, \ \eta \in \mathbb{C},$$

and cocycle

$$e_{nr+mr\tau}(x) = (-1)^n n N(-e^{2\pi i})^m N e^{2\pi i} n (N x + \xi), \quad n, m \in \mathbb{Z}^M.$$ 

Hence $\text{Span}_\mathbb{C}\{ W_I(t, z, \Pi) \ (I \in \mathcal{I}_\lambda) \}$ is a space of meromorphic sections of $L(N, \xi)$.

3.4.5 Shuffle algebra structure

For $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{N}^N$, $|\lambda| = n$, let $z^{(n)} = (z_1, \cdots, z_n) \in (\mathbb{C}^*)^n$. For $I = I_{\mu_1 \cdots \mu_n} \in \mathcal{I}_\lambda$, we denote by $\Pi_I$ a set of dynamical parameters $\{\Pi_{\mu_1^1 \cdots \mu_n^j} = q^{2(P + h)_{\mu_1^1 \cdots \mu_n^j}} \ (k = 1, \cdots, n, j = \mu_k + 1, \cdots, N)\}$, where $(P + h)_{j,k} \in \mathbb{C}/r\mathbb{Z}$ $(1 \leq j < k \leq N)$, and set $\Pi_\lambda = \cup_{I \in \mathcal{I}_\lambda} \Pi_I$.

Definition 3.6. For $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{N}^N$, $|\lambda| = n$, we define $M^{(n)}_{\lambda}(z^{(n)}, \Pi_\lambda)$ to be the space of meromorphic functions $F(t; z, \Pi)$ of $M$ variables $t = (t_1^{(1)}, \cdots, t_1^{(\lambda_1)}, \cdots, t_1^{(N-1)}, \cdots, t_1^{(N-1)})$ such that
(1) $F(t; z, \Pi)$ is symmetric in $t^{(l)}_1, \ldots, t^{(l)}_l$ for each $l \in \{1, \ldots, N-1\}$.

(2) $F(t; z, \Pi)$ has the quasi-periodicity

$$F(\cdots, pt^{(l)}_a, \cdots; z, \Pi) = F(t; z, \Pi),$$

$$F(\cdots, e^{-2\pi i t^{(l)}_a}; \cdots, z, \Pi) = \exp \left\{ \frac{2\pi i}{r} \left( (P + h)_{l,l+1} - \lambda_l \right) \right\} F(t; z, \Pi)$$

$(l = 1, \ldots, N-1, a = 1, \ldots, \lambda(l))$.

Let us consider the subspace space $\mathcal{M}^{(n)}_\lambda(z(n), \Pi_\lambda) := \text{Span}_\mathbb{C}\{ \widehat{W}_I(t, z, \Pi) \ (I \in \mathcal{I}_\lambda) \}$ of $\mathcal{M}^{(n)}_\lambda(z, \Pi_\lambda)$. From Proposition 3.2, we obtain

**Proposition 3.7.** $\dim_\mathbb{C} \mathcal{M}^{(n)}_\lambda(z(n), \Pi_\lambda) = \frac{n!}{\lambda_1! \cdots \lambda_N!}$.

Consider a graded $\mathbb{C}$-vector space

$$\mathcal{M}(z, \Pi) = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\lambda \in \mathbb{N}^N} \mathcal{M}^{(n)}_\lambda(z(n), \Pi_\lambda)$$

with $\mathcal{M}^{(0),\ldots,0}_\lambda(\zeta, \Pi) = \mathbb{C}1$.

**Definition 3.8.** For $F(t; z^{(m)}, \Pi_I) \in \mathcal{M}^{(m)}(z^{(m)}, \Pi_\lambda)$, $G(t'; z^{(n)}, \Pi_{I'}) \in \mathcal{M}^{(n)}(z^{(n)}, \Pi_{\lambda'})$, we define the bilinear product $\star$ on $\mathcal{M}(z, \Pi)$ by

$$\begin{align*}
(F \star G)(t^{(1)}_1, \ldots, t^{(1)}_{\lambda(1)}; \ldots; t^{(N-1)}_1, \ldots, t^{(N-1)}_{\lambda(N-1)}; z_1, \ldots, z_{m+n}, \Pi_{I+I'}) := \\
\frac{1}{\prod_{l=1}^{N-1} \lambda(l)! \lambda'(l)!} \text{Sym}^{(1)} \cdots \text{Sym}^{(N-1)} \left[ F(t, z, \Pi_I q^{-2\sum \lambda_l <\xi_{\mu'_l}, h>}) G(t', z', \Pi_{I'}) \right] \Xi(t, t', z, z'),
\end{align*}$$

(3.11)

where $I' = I'_{\mu'_1, \ldots, \mu'_{n}}$ and

$$\Xi(t, t', z, z') = \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda(l)} \left( \frac{\lambda'(l+1)}{\lambda(l)} \prod_{b=1}^{\lambda'(l+1)} \frac{v_b^{(l+1)} - v_a^{(l)}}{v_b^{(l+1)} - v_a^{(l)} + 1} \prod_{c=1}^{\lambda(l)} [v_c^{(l)} - v_a^{(l)} + 1] \right).$$

In the LHS of (3.11), we set $t^{(l)}_{\lambda(l)+a} := t^{(l)}_a$ $(a = 1, \ldots, \lambda(l))$, $z_{m+k} := z'_k$ $(k = 1, \ldots, n)$ and $\Pi_{I+I'} = \{ \Pi_{\mu_{k}, j} \ (k = 1, \ldots, m + n, j = \mu_k + 1, \ldots, N) \}$, where $\Pi_{\mu_{m+k}, j} := \Pi'_{\mu'_k, j}$ $(k = 1, \ldots, n, j = \mu'_k + 1, \ldots, N)$.

This endows $\mathcal{M}(z, \Pi)$ with a structure of an associative unital algebra with the unit 1. In [14], a $\mathfrak{sl}_2$ version of the $\star$-product is given.
Let us consider the subspace of $\mathcal{M}(z, \Pi)$.

$$\mathcal{M}^{+}(z, \Pi) = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\lambda \in \mathbb{N}^{N}} \mathcal{M}_{\lambda}^{+(n)}(z^{(n)}, \Pi_{\lambda}).$$

All the elements in $\mathcal{M}^{+}(z, \Pi)$ satisfy the following pole and wheel conditions. For $F(t; z, \Pi) \in \mathcal{M}_{\lambda}^{+(n)}(z^{(n)}, \Pi_{\lambda})$,

1) there exists an entire function $f(t; z, \Pi) \in \Theta_{\lambda}^{+}(z^{(n)}, \Pi_{\lambda}) = \text{Span}_{\mathbb{C}} \{ W_{I}(t, z^{(n)}, \Pi) \, (I \in \mathcal{I}_{\lambda}) \}$ such that

$$F(t; z, \Pi) = \frac{f(t; z, \Pi)}{H_{\lambda}(t, z)}.$$

2) $f(t; z, \Pi) = 0$ once $t_{a}^{(l)}/t_{c}^{(l+\varepsilon)} = q^{2\varepsilon}$ and $t_{c}^{(l+\varepsilon)}/t_{b}^{(l)} = 1$ for some $l, \varepsilon, a, b, c$, where $\varepsilon \in \{\pm 1\}$, $l = 1, \ldots, N$, $a, b = 1, \ldots, \lambda^{(l)}$, $c = 1, \ldots, \lambda^{(l+\varepsilon)}$ and $t_{a}^{(N)} = z_{a}$.

Proposition 3.9. The subspace $\mathcal{M}^{+}(z, \Pi) \subset \mathcal{M}(z, \Pi)$ is $*$-closed.

4 Finite Dimensional Representations

In this section we construct finite dimensional tensor product representations of the elliptic quantum group $E_{q,p}(\hat{\mathfrak{gl}}_{N})$ and $U_{q,p}(\hat{\mathfrak{gl}}_{N})$ on the Gelfand-Tsetlin basis.

4.1 Finite dimensional tensor product representations

Let $(\pi_{z}, \hat{V}_{z})$ denote the $N$-dimensional dynamical evaluation representation of $E_{q,p}(\hat{\mathfrak{gl}}_{N})$: $\hat{V}_{z} = \hat{V}[z, z^{-1}]$ with $\hat{V} = \oplus_{\mu=1}^{N} \mathbb{F}v_{\mu}$. The level-0 action of the $L$-operator $\hat{L}^{\pm}(z)$ or the dynamical $L$-operator $L^{\pm}(z, P)$ introduced in Sec.2.4.4 is given by

$$\pi_{z}(\hat{L}^{\pm}_{I\bar{J}}(1/w))v_{\nu} = \pi_{z}(L^{\pm}_{I\bar{J}}(1/w, P) e^{-Q_{I\bar{J}}})v_{\nu} = \sum_{\mu=1}^{N} \tilde{R}(z/w, \Pi^{*})^{\mu\nu}_{I\bar{J}} v_{\mu}$$

with $e^{Q_{\alpha}v_{\mu}} = v_{\mu}$ ($\alpha \in \hat{\mathfrak{h}}^{*}$), where $\Pi^{*}_{j,l} = q^{2P_{j,l}}$ as before.

The action on the tensor product space is obtained by the co-algebra structure presented in Appendix A.

Proposition 4.1. $\hat{L}^{\pm}(1/w)$ acts on $\hat{V}_{w} \otimes \hat{V}_{z_{1}} \otimes \cdots \otimes \hat{V}_{z_{n}}$ by

$$(\pi_{z_{1}} \otimes \cdots \otimes \pi_{z_{n}})\Delta^{(n-1)}(\hat{L}^{\pm}(1/w))$$

$$= \tilde{R}^{(0n)}(z_{n}/w, \Pi^{*}q^{2\sum_{j=1}^{n-1} h^{(j)}})\tilde{R}^{(0n-1)}(z_{n-1}/w, \Pi^{*}q^{2\sum_{j=1}^{n-2} h^{(j)}}) \cdots \tilde{R}^{(01)}(z_{1}/w, \Pi^{*}).$$

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Proof. It is enough to show the $n = 2$ case.

$$(\pi z_1 \otimes \pi z_2) \Delta'(\hat{L}^\pm_{ij}(1/w)) v_\mu \otimes v_\nu = \sum_k \pi z_1(\hat{L}^\pm_{kj}(1/w)) v_\mu \otimes \pi z_2(\hat{L}^\pm_{ik}(1/w)) v_\nu$$
$$= \sum_{k, \mu', \nu'} \Delta' \left(\pi z_1 \otimes \pi z_2\right)(\hat{L}^\pm_{ij} \otimes \hat{L}^\pm_{ij}) (z_1/w, \Pi^*)^{\mu \nu}_{\mu' \nu'} v_\mu \otimes v_\nu$$
$$= \sum_{\mu', \nu'} \sum_k \Delta' \left(\pi z_2 \otimes \pi z_2\right)(\hat{L}^\pm_{ij} \otimes \hat{L}^\pm_{ij}) (z_2/w, \Pi^*)^{\mu \nu}_{\mu' \nu'} v_\mu \otimes v_\nu$$
$$= \sum_{\mu', \nu'} \left(\Delta' \left(\pi z_1 \otimes \pi z_1\right)(\hat{L}^\pm_{ij} \otimes \hat{L}^\pm_{ij}) (z_1/w, \Pi^*)\right)^{\mu \nu}_{\mu' \nu'} v_\mu \otimes v_\nu.$$

To obtain the third equality we used (A.2).

It is also useful to write down the comultiplication formula of the dynamical $L$-operator, which is equivalent to Proposition 4.1.

Proposition 4.2. The dynamical $L$-operator $L^+(1/w, P)$ acts on $\hat{V}_w \otimes \hat{V}_{z_1} \otimes \cdots \otimes \hat{V}_{z_n}$, where $\otimes$ denotes the usual tensor product, by

$$(\pi z_1 \otimes \cdots \otimes \pi z_n) \Delta^{(n-1)}(L^+_{ij}(1/w, P)) = \sum_{k_1, \cdots, k_{n-1}=1}^N L^+_{k_1 j_1}(z_1/w, P) \otimes L^+_{k_2 j_2}(z_1/w, P + h^{(1)}) \otimes \cdots \otimes L^+_{i_{n-1} k_{n-1}}(z_n/w, P + \sum_{j=1}^{n-1} h^{(j)}).$$

4.2 The Gelfand-Tsetlin basis

Definition 4.3. The Gelfand-Tsetlin basis is a basis of the level-$0$ representation of $E_{q,p}(\hat{g}(N))$ or $U_{q,p}(\hat{g}(N))$ consisting of the simultaneous eigenvectors of the Gelfand-Tsetlin subalgebra $\mathcal{G}$ in Definition 2.9.

We consider the the Gelfand-Tsetlin (GT) basis in $\hat{V}_{z_1} \otimes \cdots \otimes \hat{V}_{z_n}$. Following [4] we construct it as follows. Firstly we realize $\mathcal{G}_n$ in terms of the elliptic dynamical $R$ matrix in (2.6). Let define $\tilde{S}_i(P)$ by

$$\tilde{S}_i(P) := \mathcal{P}(g+1) \tilde{R}^{(g+1)}(z_i/z_{i+1}, \Pi^* q^{2\sum_{j=1}^{i-1} h^{(j)}}) s_{i}^z,$$

where

$$\mathcal{P} : v \otimes w \mapsto w \otimes v, \quad s_{i}^z f(\cdots, z_i, z_{i+1}, \cdots) = f(\cdots, z_{i+1}, z_i, \cdots)$$

Then by using the dynamical Yang-Baxter equation (2.10) and the unitarity relation (2.11) one can show the following.
Proposition 4.4.

\[ \widetilde{S}_i(P)\widetilde{S}_{i+1}(P)\widetilde{S}_i(P) = \widetilde{S}_i(P)\widetilde{S}_{i+1}(P)\widetilde{S}_i(P), \]
\[ \widetilde{S}_i(P)\widetilde{S}_j(P) = \widetilde{S}_j(P)\widetilde{S}_i(P) \quad (|i - j| > 1) \]
\[ \widetilde{S}_i(P)^2 = 1. \]

For \( \lambda \in \mathbb{N}^N, |\lambda| = n, I = I_{\mu_1}\ldots\mu_n \in \mathcal{I}_\lambda \), we set
\[ v_I = v_{\mu_1}\ldots\mu_n := v_{\mu_1} \otimes \cdots \otimes v_{\mu_n}. \]

We define the Gelfand-Tsetlin basis \( \{ \xi_I \}_{I \in \mathcal{I}_\lambda} \) by
\[ \xi_{I_{\max}} := v_{I_{\max}}, \quad \xi_{s_i(I)} := \widetilde{S}_i(P)\xi_I, \quad (4.1) \]
where
\[ I_{\max} = I_{\lambda_N\ldots\lambda_1}. \]

Let us consider the change of basis matrix \( \hat{X} = (X_{IJ}(z,P))_{I,J \in \mathcal{I}_\lambda} \):
\[ \xi_I = \sum_{J \in \mathcal{I}_\lambda} X_{IJ}(z,P)v_J. \quad (4.2) \]

Here we put the matrix elements in the decreasing order \( I_{\max} \geq \cdots \geq I_{\min} \). Then by construction, \( \hat{X} \) is a lower triangular matrix. Furthermore the following remarkable relationship between \( \hat{X} \) and the specialized elliptic weight functions becomes a key to obtain a geometric interpretation of the results in the next subsection. See Sec 5.4.

Theorem 4.5.

\[ X_{IJ}(z,P) = \hat{W}_J(z^{-1},z^{-1},\Pi^{s_i}q^{2\sum_{j=1}^{n-1}<\epsilon_{\mu_j},h>}). \quad (4.3) \]

Proof. Let \( J = I_{\mu_1}\ldots\mu_i\mu_{i+1}\ldots\mu_n \in \mathcal{I}_\lambda \). By definition,
\[ \xi_{s_i(I)} = \sum_{J} X_{s_i(I),J}(z,P)v_J \]
\[ = \widetilde{S}_i(P)\xi_I = \sum_{J} X_{IJ}(s_i(z),P)\widetilde{S}_i(P)v_J \]
\[ = \sum_{J,\mu_{i}',\mu_{i+1}'} X_{IJ}(s_i(z),P)R(z_{i}/z_{i+1},\Pi^{s_i}q^{2\sum_{j=1}^{i-1}<\epsilon_{\mu_j},h>})_{\mu_i\mu_{i+1}'} v_{\mu_1} \otimes \cdots \otimes v_{\mu_i'} \otimes v_{\mu_{i+1}'} \otimes \cdots \otimes v_{\mu_n}. \]

Hence we obtain
\[ X_{s_i(I),J}(z,P) = X_{IJ}(s_i(z),P) \quad (4.4) \]
for $\mu_i = \mu_{i+1}$, and
\[
(X_{s_i(I)}(z, P) X_{s_i(J)}(z, P)) = (X_{s_i(I)}(z, P) X_{s_i(J)}(z, P)) P_2^t \mathcal{R}(z_i/z_{i+1}, \Pi^* q^2 \sum_{j=1}^{i-1} \langle \epsilon_{\mu_j}^1, h \rangle)_{\mu_i, \mu_{i+1}}
\] (4.5)
for $\mu_i > \mu_{i+1}$. Here we set
\[
P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{R}(z, \Pi^*)_{\mu_i, \mu_{i+1}} = \begin{pmatrix} \mathcal{R}(z, \Pi^*)_{\mu_i+1, \mu_i} & \mathcal{R}(z, \Pi^*)_{\mu_i+1, \mu_i} \\ \mathcal{R}(z, \Pi^*)_{\mu_i+1, \mu_i} & \mathcal{R}(z, \Pi^*)_{\mu_i+1, \mu_i} \end{pmatrix}. \]
(4.6)

Note that (4.4) and (4.5) determine the whole matrix elements in $\tilde{X}$ recursively starting from $X_{I_{\text{max}}, I_{\text{max}}}(z, P) = 1$.

On the other hand, from Proposition 3.3 with replacing $\Pi$ by $\Pi^*$ we have
\[
\tilde{W}_J(t, s_i(z), \Pi^*) = \tilde{W}_J(t, z, \Pi^*)
\] (4.7)
if $\mu_i = \mu_{i+1}$, and
\[
(\tilde{W}_J(t, s_i(z), \Pi^*) \tilde{W}_{s_i(J)}(t, s_i(z), \Pi^*)) = (\tilde{W}_J(t, z, \Pi^*) \tilde{W}_{s_i(J)}(t, z, \Pi^*)) P_2^t \mathcal{R}(z_i/z_{i+1}, \Pi^* q^2 \sum_{j=1}^{i-1} \langle \epsilon_{\mu_j}^1, h \rangle)_{\mu_i, \mu_{i+1}}
\]
if $\mu_i \neq \mu_{i+1}$. Using
\[
(P_2^t \mathcal{R}(z, \Pi^*)_{\mu_i, \mu_{i+1}})^{-1} = P_2^t \mathcal{R}(z^{-1}, \Pi^*)_{\mu_i, \mu_{i+1}},
\]
we obtain in particular for $\mu_i > \mu_{i+1}$
\[
(\tilde{W}_J(t, z, \Pi^*) \tilde{W}_{s_i(J)}(t, z, \Pi^*)) = (\tilde{W}_J(t, s_i(z), \Pi^*) \tilde{W}_{s_i(J)}(t, s_i(z), \Pi^*)) P_2^t \mathcal{R}(z_i/z_{i+1})^{-1}, \Pi^* q^2 \sum_{j=1}^{i-1} \langle \epsilon_{\mu_j}^1, h \rangle)_{\mu_i, \mu_{i+1}}
\] (4.8)

Specializing $t = s_i(z)_J$ and noting
\[
\tilde{W}_J(s_i(z)_J, z, \Pi^*) = \tilde{W}_J(z_{s_i(J)}, z, \Pi^*)
\]
etc., we obtain from (4.7) and (4.8)
\[
\tilde{W}_J(s_i(z)_J, s_i(z), \Pi^*) = \tilde{W}_J(z_{s_i(J)}, z, \Pi^*)
\] (4.9)
if $\mu_i = \mu_{i+1}$, and
\[
(\tilde{W}_J(z_{s_i(J)}, z, \Pi^*) \tilde{W}_{s_i(J)}(z_{s_i(J)}, z, \Pi^*)) = (\tilde{W}_J(s_i(z)_J, s_i(z), \Pi^*) \tilde{W}_{s_i(J)}(s_i(z)_J, s_i(z), \Pi^*)) P_2^t \mathcal{R}(z_i/z_{i+1})^{-1}, \Pi^* q^2 \sum_{j=1}^{i-1} \langle \epsilon_{\mu_j}^1, h \rangle)_{\mu_i, \mu_{i+1}}
\] (4.10)
if $\mu_i > \mu_{i+1}$. Therefore one finds that $\tilde{W}_I(z^{-1}, z^{-1}, \Pi^* q^2 \sum_{j=1}^n <\epsilon_{\nu_j}, h>)$ satisfy the same recursion relations as (4.4) and (4.5) for $X_I(z, P)$. In addition their initial conditions coincide: $\tilde{W}_{I_{\text{max}}}(z^{-1}, z^{-1}, \Pi^* q^2 \sum_{j=1}^n <\epsilon_{\nu_j}, h>) = 1 = X_{I_{\text{max}}}(z, P)$. \hfill \Box

**Example.** The case $N = 2, n = 3, \lambda = (2, 1)$. We have $I_\lambda = \{ I_{211} \geq I_{121} \geq I_{112} \}$ and

$$
\begin{pmatrix}
\xi_{211} \\
\xi_{121} \\
\xi_{112}
\end{pmatrix} = 
\begin{pmatrix}
1 & 0 & 0 \\
c(u_{1,2}, P_{1,2}) & b(u_{1,2}) & 0 \\
c(u_{1,3}, P_{1,2}) & b(u_{1,3})c(u_{2,3}, P_{1,2} + 1) & b(u_{1,3})b(u_{2,3})
\end{pmatrix}
\begin{pmatrix}
v_{211} \\
v_{121} \\
v_{112}
\end{pmatrix},
$$

where $u_{i,j} = u_i - u_j$. On the other hand we have

$$
\tilde{W}_{I_{\text{id}}}(z, \Pi^*) = 
\begin{pmatrix}
\tilde{W}_{I_{211}}(z_{I_{211}}, z, \Pi^*) & 0 & 0 \\
\tilde{W}_{I_{121}}(z_{I_{121}}, z, \Pi^*) & \tilde{W}_{I_{121}}(z_{I_{121}}, z, \Pi^*) & 0 \\
\tilde{W}_{I_{112}}(z_{I_{112}}, z, \Pi^*) & \tilde{W}_{I_{112}}(z_{I_{112}}, z, \Pi^*) & \tilde{W}_{I_{112}}(z_{I_{112}}, z, \Pi^*)
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
[u_{1,2}^{-1} + u_{2,1}][P_{1,2}^{-1}] & [u_{2,1}] & 0 \\
[u_{3,1}^{-1} + u_{3,2}][P_{1,2}^{-1}] & [u_{3,1}] & [u_{3,2}]
\end{pmatrix}.
$$

### 4.3 Action of the elliptic currents

In order to derive an action of the elliptic quantum group on the GT basis $\{ \xi_I \}$, the following property of the symmetrization operators $\tilde{S}_I(P)$ is useful. \[\tilde{S}_I(P)\Delta^{(n-1)}(\hat{L}^\pm(w)) = \Delta^{(n-1)}(\hat{L}^\pm(w))\tilde{S}_I(P + h^{(0)})\]

**Proposition 4.6.**

**Proof.** Use the dynamical Yang-Baxter equation. \hfill \Box

Thanks to this proposition it suffices to construct an action of $\Delta^{(n-1)}(\hat{L}^\pm(w))$ on $\xi_{I_{\text{max}}}$. From Theorem 2.2 and Proposition 4.4 we obtain the following level-0 action of the half-currents of $E_{q,p}(\hat{g}_N)$ on the GT basis. Note that at the level 0 we have $L^-(w, P) = L^+(pw, P)$, hence $K^{-}_j(w) = K^+_j(pw), E^{-}_{j+1,j}(w, P) = E^+_{j+1,j}(pw, P), F^{-}_{j,j+1}(w, P) = F^+_{j,j+1}(pw, P)$. \[\mathcal{K}^{-}_j(w) = (\pi_{z_1} \otimes \cdots \otimes \pi_{z_n}) \Delta^{(n-1)}(E^+_{j+1,j}(1/w, P)) \text{ and } F^+_{j,j+1}(1/w, P) = (\pi_{z_1} \otimes \cdots \otimes \pi_{z_n}) \Delta^{(n-1)}(F^+_{j,j+1}(1/w, P)), \]

**Theorem 4.7.** Let $\{ \xi_I \mid I \in I_\lambda, \lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{N}^N, |\lambda| = n \}$ be the GT basis of $\tilde{V}_{\lambda_1} \otimes \cdots \otimes \tilde{V}_{\lambda_N}$. Under the abbreviation $\mathcal{K}^\pm_j(1/w) = (\pi_{z_1} \otimes \cdots \otimes \pi_{z_n}) \Delta^{(n-1)}(\mathcal{K}^\pm_j(1/w)), E^\pm_{j+1,j}(1/w, P) = (\pi_{z_1} \otimes \cdots \otimes \pi_{z_n}) \Delta^{(n-1)}(E^\pm_{j+1,j}(1/w, P))$ and $F^\pm_{j,j+1}(1/w, P) = (\pi_{z_1} \otimes \cdots \otimes \pi_{z_n}) \Delta^{(n-1)}(F^\pm_{j,j+1}(1/w, P))$. \[\left. \begin{array}{c}
\end{array} \right\} \text{ for } 1 \leq j < n. \]
we have

\[
K_j^\pm (1/w) \xi_I = \prod_{k=1}^{j-1} \prod_{a \in I_k} \frac{[u_a - v]}{[u_a - v + 1]} \prod_{i=-j+1}^{N} \prod_{b \in I_i} \frac{[u_b - v - 1]}{[u_b - v]} \xi_I, \tag{4.11}
\]

\[
E_{j+1,j}^\pm (1/w, P) \xi_I = \sum_{i \in I_{j+1}} \frac{[P_{j+1} - u_i + v][1]}{[P_{j+1}][u_i - v]} \prod_{k \in \delta_{j+1}^i} \frac{[u_i - u_k + 1]}{[u_i - u_k]} \xi_{I'}, \tag{4.12}
\]

\[
F_{j,j+1}^\pm (1/w, P) \xi_I = \sum_{i \in I_j} \frac{[P_{j+1} + \lambda_j - \lambda_{j+1} + u_i - v - 1][1]}{[P_{j+1} + \lambda_j - \lambda_{j+1} - 1][u_i - v]} \prod_{k \in I_j} \frac{[u_k - u_i + 1]}{[u_k - u_i]} \xi_{I'}, \tag{4.13}
\]

where \( w = q^{2 \nu}, z_i = q^{2u_i} (i = 1, \cdots, n), I = (I_1, \cdots, I_N), \) and \( I' \in \mathcal{I}(\lambda_1, \cdots, \lambda_j, 1, \lambda_{j+1} - 1, \cdots, \lambda_N) \) and \( I' \in \mathcal{I}(\lambda_1, \cdots, \lambda_j - 1, \lambda_{j+1} + 1, \cdots, \lambda_N) \) are defined by

\[
(I')_j = I_j \cup \{i\}, \quad (I')_{j+1} = I_{j+1} - \{i\}, \quad (I'_k) = I_k \quad (k \neq j, j + 1),
\]

\[
(I')_j = I_j - \{i\}, \quad (I')_{j+1} = I_{j+1} \cup \{i\}, \quad (I'_k) = I_k \quad (k \neq j, j + 1).
\]

The symbols \( |_\pm \) distinguish contributions from the plus- and the minus-half-currents and specify their expansion directions as

\[
\begin{align*}
\frac{[s + v]}{[s][v]} & = w^\pm \frac{\Theta_p(q^{2s}w)}{\Theta_p(q^{2s})} \text{ expand in } w, \\
\frac{[s + v]}{[s][v]} & = (pw)^\pm \frac{\Theta_p(pq^{2s}w)}{\Theta_p(q^{2s})} \text{ expand in } 1/w.
\end{align*}
\]

A proof is given in Appendix B.

Example. Let us consider the case \( N = 3, \quad n = 5, \quad \lambda = (2, 2, 1) \). We use the abbreviation \( \xi_{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5} = \xi_{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5} \), as well as \( \hat{L}^\pm_{ij} (1/w) = (\pi_{z_1} \otimes \cdots \otimes \pi_{z_5}) \Delta^{(4)}(\hat{L}^\pm_{ij} (1/w)) \) on \( \hat{V}_{z_1} \otimes \cdots \otimes \hat{V}_{z_4} \). Noting \( I_{\text{max}} = I_{32211} \in \mathcal{I}_\lambda \), let us consider the action on \( \xi_{32211} \).

- \( K_3^\pm (1/w) \): from Theorem 2.22 and Proposition 4.1 we obtain
  \[
  K_3^+(1/w) \xi_{32211} = \hat{L}^+_{33}(1/w) \xi_{32211} = \hat{b}(u_2 - v) \hat{b}(u_3 - v) \hat{b}(u_4 - v) \hat{b}(u_5 - v) \xi_{32211}.
  \]

- \( E_{3,2}^+(1/w, P) \): similarly we have
  \[
  \hat{L}^+_{32}(1/w) \xi_{32211} = \hat{c}(u_1 - v, P_{2,3}) \hat{b}(u_2 - v) \hat{b}(u_3 - v) \hat{b}(u_4 - v) \hat{b}(u_5 - v) \xi_{22111}.
  \]

Note that \( I_{22111} \) is the maximal partition in \( \mathcal{I}_{(2,3,0)} \). We also obtain

\[
\hat{L}^+_{33}(1/w) \xi_{22111} = \hat{b}(u_1 - v) \hat{b}(u_2 - v) \hat{b}(u_3 - v) \hat{b}(u_4 - v) \hat{b}(u_5 - v) \xi_{22111}.
\]
Hence
\[ E^+_3(1/w, P)\xi_{32211} = \hat{L}^+_3(1/w)^{-1}\hat{L}^+_3(1/w)\xi_{32211} = \frac{c(u_1 - v, P_{2,3})}{b(u_1 - v)} \xi_{2211}. \]

- \( F^+_2(1/w, P) \): similarly we obtain
\[ \hat{L}^+_2(1/w)\xi_{32211} = c(u_2 - v, P_{2,3} - 1)\hat{b}(u_4 - v)\hat{b}(u_5 - v)\xi_{32211} + \hat{b}(u_2 - v)c(u_3 - v, P_{2,3})\hat{b}(u_4 - v)\hat{b}(u_5 - v)v_{32311}. \]

Again note that \( I_{3211} \) is the maximal partition in \( J_{(2,1,2)} \). From (4.1) we have
\[ v_{32311} = \frac{1}{b(u_{23})} \xi_{32311} - \frac{c(u_{23}, P_{2,3} - 1)}{b(u_{23})} \xi_{32211}. \]

Substituting this and using the identity
\[ c(u_2 - v, P_{2,3} - 1) - \hat{b}(u_2 - v)c(u_{23}, P_{2,3} - 1)b(u_3 - v, P_{2,3}) = \frac{c(u_2 - v, P_{2,3})}{b(u_{32})} \xi_{32211}, \]
we obtain
\[ \hat{L}^+_2(1/w)\xi_{32211} = \frac{c(u_2 - v, P_{2,3})}{b(u_{32})} \hat{b}(u_3 - v)\hat{b}(u_4 - v)\hat{b}(u_5 - v)\xi_{32211} + \hat{b}(u_2 - v)\frac{c(u_3 - v, P_{2,3})}{b(u_{23})} \hat{b}(u_4 - v)\hat{b}(u_5 - v)v_{32311}, \]
where \( u_{ij} = u_i - u_j \). Hence
\[ F^+_2(1/w, P)\xi_{32211} = \hat{L}^+_2(1/w)\hat{L}^+_2(1/w)^{-1}\xi_{32211} = \frac{c(u_2 - v, P_{2,3})}{b(u_{23})} \xi_{32211} + \frac{c(u_2 - v, P_{2,3})}{b(u_{32})} \xi_{32211}. \]

- \( K^+_2(1/w) \): let us consider the action
\[ \hat{L}^+_2(1/w)\xi_{32211} = b(u_1 - v, P_{2,3})\hat{b}(u_4 - v)\hat{b}(u_5 - v)\xi_{32211} + \hat{b}(u_2 - v)c(u_3 - v, P_{2,3} + 1)\hat{b}(u_4 - v)\hat{b}(u_5 - v)v_{23211} + \hat{b}(u_2 - v)c(u_3 - v, P_{2,3} + 2)\hat{b}(u_4 - v)\hat{b}(u_5 - v)v_{22311}. \]

On the other hand we have
\[ F^+_2(1/w, P)K^+_2(1/w)E^+_3(1/w, P)\xi_{32211} = c(u_1 - v, P_{2,3})\hat{b}(u_4 - v)\hat{b}(u_5 - v)\xi_{32211} + \hat{b}(u_2 - v)c(u_3 - v, P_{2,3} + 1)\hat{b}(u_4 - v)\hat{b}(u_5 - v)v_{23211} + \hat{b}(u_2 - v)c(u_3 - v, P_{2,3} + 2)\hat{b}(u_4 - v)\hat{b}(u_5 - v)v_{22311}. \]
Therefore
\[
\mathcal{K}_2^+(1/w)\xi_{3211} = \left(\hat{L}_{22}^+(1/w) - F_{2,3}^+(1/w, P)\mathcal{K}_2^+(1/w)E_{3,2}^+(1/w, P)\right)\xi_{3211}
\]
\[
= \frac{\bar{b}(u_4 - v)\bar{b}(u_5 - v)}{b(-u_1 + v)}\xi_{3211}.
\]
The last equality follows from the identity
\[
b(u_1 - v, P_3) = \bar{c}(u_1 - v, P_3) = 1.
\]
Furthermore noting the formula
\[
\left|\frac{s + u}{|s|u}\right|_+ - \left|\frac{s + u}{|s|u}\right|_- = \frac{1}{[0]}\delta(w)
\]
and using Propositions 2.6 and 2.11 we obtain the following level-0 action of the elliptic currents
of \(U_{q,p}(\hat{\mathfrak{sl}}_N)\) on the GT basis.

**Corollary 4.8.**

\[
\begin{align*}
H_j^{\pm}(q^{j-N-1}/w)\xi_I &= \varrho \prod_{a \in I_j} \frac{[u_a - v + 1]}{[u_a - v]} \prod_{b \in I_{j+1}} \frac{[u_b - v - 1]}{[u_b - v]} e^{-Q_{ij}} \xi_I, \\
E_j(q^{j-N+1}/w)\xi_I &= \frac{\mu^*[1]}{[0]} \sum_{i \in I_{j+1}} \delta(z_i/w) \prod_{k \in I_{j+1} \setminus \{i\}} \frac{[u_i - u_k + 1]}{[u_i - u_k]} e^{-Q_{ij}} \xi_{I'} \\
F_j(q^{j-N+1}/w)\xi_I &= \frac{\mu^*[1]}{[0]} \sum_{i \in I_j} \delta(z_i/w) \prod_{k \in I_j \setminus \{i\}} \frac{[u_k - u_i + 1]}{[u_k - u_i]} \xi_{I'}.
\end{align*}
\]

In the trigonometric and non-dynamical limit, the combinatorial structures of the formulas in this Corollary are the same as those in the geometric representation of \(U_q(\mathfrak{sl}_N)\) on the equivariant
\(K\)-theory of the quiver variety of type \(A_{N-1}\) obtained by Ginzburg and Vasserot [20, 55], and
by Nakajima [41]. One can directly check that these actions of the elliptic currents satisfy the
defining relations of the level-0 \(U_{q,p}(\mathfrak{sl}_N)\) in the same way as in [41, 55]. We give a check of the
most non-trivial relation (2.33) in Appendix C.

**Proposition 4.9.** The finite dimensional representation given in Corollary 4.8 is an irreducible
highest weight representation with the highest weight vector \(\xi_{11 \ldots 1}\). The elliptic analogue of the
Drinfeld polynomials of this representation are given by

\[
P_1(w) = \prod_{a=1}^{n} [u_a - v + 1], \quad P_l(w) = 1 \quad (l = 2, \ldots, N - 1).
\]
Proof. The statement follows from a similar argument to Theorem 4.11 in [31] and

\[ E_j(1/w)\xi_{11\ldots j} = 0 \quad (j = 1, \cdots, N - 1), \]

\[ H^+_1(1/w)\xi_{11\ldots j} = \varrho \prod_{a=1}^{\varrho} \frac{[u_a - v + 1]}{[u_a - v]} \xi_{11\ldots j}, \]

\[ H^+_1(1/w)\xi_{11\ldots j} = \xi_{11\ldots j} \quad (l = 2, \cdots, N - 1). \]

5 Geometric Representation

5.1 Equivariant elliptic cohomology $\text{Ell}_T(X)$

For $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{N}^N$, $|\lambda| = n$, let $\mathcal{F}_\lambda$ denote the partial flag variety as before and consider the cotangent bundle $X = T^*\mathcal{F}_\lambda$. Let us set $X = A \times \mathbb{C}^*$, $A = (\mathbb{C}^*)^n$. The torus $A$ has a natural action on $\mathcal{F}_\lambda$ and the extra $\mathbb{C}^*$ acts on the fibers of $T^*\mathcal{F}_\lambda \to \mathcal{F}_\lambda$ by multiplication with weight $h$. Let $E = \mathbb{C}^*/p^\mathbb{Z}$ ($|p| < 1$). We regard the elliptic curve $E$ as a group scheme over $\mathbb{C}$. We follow [14, 15, 17, 19, 46] for the definition of the $T$-equivariant elliptic cohomology $\text{Ell}_T(X)$. The basic facts on $\text{Ell}_T(X)$ are summarized as follows.

1. The $T$-equivariant elliptic cohomology, $\text{Ell}_T(X)$, is a functor from finite $T$-spaces $X$ to superschemes, covariant in both $T$ and $X$, satisfying a set of axioms (1.4.1 in [15] and 2.1.2 in [1]). In particular, $\text{Ell}_T(X)$ is a scheme over $\text{Ell}_T(pt) \cong E^n \times E$. Moreover associated with a construction of $X$ as a hyper-Kähler quotient

\[ T^*\mathbb{P}\left( \bigoplus_{l=1}^{N-1} \text{Hom}(\mathbb{C}^{\lambda(l)}, \mathbb{C}^{\lambda(l+1)}) \right) \big/G \]

with $G = \prod_{l=1}^{N-1} \text{GL}(\lambda(l), \mathbb{C})$, we have a collection of tautological vector bundles $\{\mathbb{C}^{\lambda(l)}\}$ of $\text{rk} = \lambda(l)$ $(l = 1, \cdots, N - 1)$ over $X$ and a map

\[ \text{Ell}_T(X) \to \text{Ell}_T(pt) \times E^{(\lambda(1))} \times \cdots \times E^{(\lambda(N-1))}, \tag{5.1} \]

where $E^{(m)} = E^m/\mathfrak{S}_m$ denotes the symmetric product of $E$. This map is expected to be an embedding near the origin of $\text{Ell}_T(pt)$ (2.2 in [1]).

2. The Thom class map $\Theta : K_T(X) \to \text{Pic}(\text{Ell}_T(X))$ is a map of a $T$-equivariant complex vector bundle $\xi$ to a line bundle $\mathbb{L}_T^\xi$ over $\text{Ell}_T(X)$. The line bundle $\mathbb{L}_T^\xi$ is called the Thom sheaf of $\xi$. See 2.3.2 in [1] and Definition 6.1 in [15].

3. Let $f : X \to Y$ be a holomorphic map of $T$-spaces. Pull-back in the elliptic cohomology is the contravariant functoriality map $\text{Ell}(f) : \text{Ell}_T(X) \to \text{Ell}_T(Y)$ (1.7.4) in [17] and 2.3.1
If \( f \) is proper, pushforward is a morphism
\[
f_* : \text{Ell}(f)_* \Theta(-N_f) \to \mathcal{O}_{\text{Ell}_T(Y)}
\]
of sheaves on \( \text{Ell}_T(Y) \), where \( N_f = f^*T_Y - TX \in K_T(X) \). See (2.3.2) in \([17]\) and (11) in \([1]\).

(4) The dynamical parameter dependence is introduced by extending \( \text{Ell}_T(X) \) to
\[
\text{E}_T(X) := \text{Ell}_T(X) \times \mathcal{E}_{\text{Pic}_T(X)}
\]
where
\[
\mathcal{E}_{\text{Pic}_T(X)} = \text{Pic}_T(X) \otimes \mathbb{Z} E,
\]
as a scheme over \( \mathcal{B}_{T,X} = \text{Ell}_T(\text{pt}) \times \mathcal{E}_{\text{Pic}_T(X)} \). The variables in the two factors of \( \mathcal{B}_{T,X} \), \( z_1, \cdots, z_n \), in \( \text{Ell}_T(\text{pt}) \cong E^n \times E \) and \( \Pi^*_{j,j+1} \) (1 \( \leq j \leq N - 1 \)) in \( \mathcal{E}_{\text{Pic}_T(X)} \), are called the equivariant and the Kähler parameters, respectively. See 2.4.3 in \([1]\). In Sec 5.3 we identify \( \Pi^*_{j,j+1} \) with the dynamical parameters.

(5) \( \mathcal{E}_{\text{Pic}_T(X)} \) and \( \mathcal{E}^\vee_{\text{Pic}_T(X)} := \text{Hom}((\text{Pic}_T(X), E) \) are dual abelian varieties each other. Hence there exists a universal line bundle \( \mathcal{U}_{\text{Poincaré}} \) over \( \mathcal{E}^\vee_{\text{Pic}_T(X)} \times \mathcal{E}_{\text{Pic}_T(X)} \). By using a map
\[
\tilde{c} : \text{Ell}_T(X) \to \mathcal{E}^\vee_{\text{Pic}_T(X)},
\]
which is obtained from the Chern class of line bundles over \( X \) (2.4.1 in \([1]\)), one obtains a line bundle \( \mathcal{U} \) on \( \text{E}_T(X) \) as
\[
\mathcal{U} = (\tilde{c} \times 1)^* \mathcal{U}_{\text{Poincaré}}.
\]

5.2 Elliptic stable envelopes

5.2.1 Chamber structure

Let \( \text{Hom}_{\text{grp}}(\mathbb{C}^*, A) \) be the space of one parameter subgroups \( \rho \) in \( A \) and \( \text{Hom}_{\text{grp}}(\mathbb{C}^*, A) \otimes \mathbb{Z} \mathbb{R} \subset \text{Lie} A \) be its real form. The latter space can be decomposed into finitely many chambers \( \mathcal{C} \) defined as a connected component of the compliment of the union of hyperplanes given by \( \rho \) such that \( X^{\rho(\mathbb{C}^*)} \neq X^A \) \([42]\).

The \( A \)-fixed points on \( X \) are described by the partitions in \( \mathcal{I}_\lambda \). Let \( X^A \) be the \( A \)-fixed point locus in \( X \) and \( X^A = \bigsqcup_{t \in \mathcal{I}_\lambda} F_t \) a decomposition to connected components. Let \( \rho \in \mathcal{C} \). For every \( \mathcal{S} \subset X^A \) we define its attracting set
\[
\text{Attr} (\mathcal{S}) = \{ (x,s), s \in \mathcal{S}, \lim_{t \to 0} \rho(t)x = s \} \subset X \times X^A,
\]

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and denote by $\text{Attr}^f(S)$ the full attracting set, which is the minimal closed subset of $X$ that contains the diagonal $S \times S$ and is closed under taking $\text{Attr}(\cdot)$. We then define a partial ordering on $\{F_I\}$ by

$$F_J \leq F_I \iff \text{Attr}^f(F_I) \cap F_J \neq \emptyset.$$ 

### 5.2.2 Definition

For a pair $(\mu, \nu), \mu \in \text{char}(T) = \text{Hom}(\text{Ell}_T(pt), E), \nu \in \text{Pic}_T(X) = \text{Hom}(E, E_{\text{Pic}_T(X)}),$ let $\varsigma$ denote the automorphism of $B_{T,X}$

$$\varsigma(\mu \nu)(z, \Pi^*) \mapsto (z, \nu(\mu(z))\Pi^*).$$

Let $\iota : X^A \to X$ be the inclusion map, which is proper. For each chamber $C$ of Lie $A$, one can consider the polarization $T^{1/2}X \in K_T(X)$ of $X$ and its restriction $T^{1/2}X|_{X^A}$ to $X^A$. Let us denote by $\text{ind} := T^{1/2}X|_{X^A,>0}$ the attracting part of $T^{1/2}X|_{X^A}$. We have

$$\det \text{ind} \in \text{Pic}_T(X^A)$$

and a translation

$$\varsigma(-h \det \text{ind}) : B_{T,X^A} \to B_{T,X^A}.$$ 

For the line bundle $U_{\text{Ell}_T(X^A)}$ on $\text{Ell}_T(X^A) \times E_{\text{Pic}_T(X)}$ we set

$$U' = (1 \times \iota^*)^*\varsigma(-h \det \text{ind})^*U_{\text{Ell}_T(X^A)},$$

where $\iota^*$ is the pull-back of line bundles from $X$ to $X^A$.

The elliptic stable envelop $\text{Stab}_C$ is defined to be a map of $\mathcal{O}_{B_{T,X}}$-modules

$$\Theta(T^{1/2}X^A) \otimes U' \to \Theta(T^{1/2}X) \otimes U \otimes \cdots,$$

where $\Theta(T^{1/2}X)$ denotes the Thom sheaf of a polarization, and $\cdots$ stands for a certain line bundle pulled back from

$$B' = B_{T,X}/\text{Ell}_A(pt).$$

$\text{Stab}_C$ is subjected to the following two conditions (3.3.4 in [1]).

1. (triangularity) Let $s_K$ be an elliptic cohomology class supported on $F_K$ locally over $B_{T,X}$.

Then $\text{Stab}_C(s_K)$ is supported on $\text{Attr}^f(F_K)$. In particular if $F_K < F_I$ we have

$$\text{Stab}_C(s_K)|_{F_I} = 0.$$
(ii) (normalization) Near the diagonal in $X \times F_K$, we have

$$\text{Stab}_\xi = (-1)^{\text{rk ind } j} \pi^*,$$

where

$$F_K \xleftarrow{\pi} \text{Attr}(F_K) \xrightarrow{j} X$$

are the natural projection and inclusion maps.

### 5.3 Direct comparison with the elliptic weight functions

In this and the following subsections, we consider the elliptic weight functions $W_I(t, z, \Pi^*)$ obtained from $W_I(t, z, \Pi)$ in (3.6) by replacing $\Pi_{j,l} = q^{2(P+j)(l)}$ by $\Pi_{j,l}^* = q^{2P(l)}$, which also satisfy all the properties in Sec. 3 under the same replacement.

The symmetry structure in the target of (5.1) coincides with the one of the elliptic weight function $W_I(t, z, \Pi^*)$ with respect to the variables $t_{a}^{(l)} (l = 1, \cdots, N - 1, a = 1, \cdots, \lambda^{(l)})$. This suggests that $\{t_{a}^{(l)}\}$ can be identified with the Chern roots of the tautological vector bundles over $X$. This structure as well as the quasi periodicity in Proposition 3.5 allow us to identify the elliptic weight functions $W_I(t, z, \Pi^*)$ with meromorphic sections of line bundles over $E_T(X)$ near the origin of $B_{T,X}$.

More precisely one can compare the elliptic weight functions $W_{\sigma_0(I)}(\tilde{t}, \sigma_0(z^{-1}), \Pi^{*-1})$, where $\tilde{t}$ denotes a set of $t$-variables associated with the partition $\sigma_0(I)$, with the abelianization formula of the elliptic stable envelopes [1, 48].

Let $G$ be a reductive group acting on a vector space $M$. It induces the Hamiltonian action on $T^*M$. Let $\mu_G$ be the corresponding moment map. Let $S \subset G$ be the maximal torus and let $\pi_S : (\text{Lie } G)^* \rightarrow (\text{Lie } S)^*$ be a projection, and set $\mu_S = \pi_S \circ \mu_G$. For the hyper-Kähler quotient

$$X = \mu_G^{-1}(0)/G,$$

the associated abelian quotient

$$X_S = \mu_S^{-1}(0)/S$$

is a hypertoric variety called the abelianization of $X$.

Let $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{N}^N$, $I \in \mathcal{I}_\lambda$ and $\lambda^{(l)} = \lambda_1 + \cdots + \lambda_l$, $I^{(l)} = \{i_1^{(l)} < \cdots < i_{\lambda^{(l)}}^{(l)}\}$ ($l = 1, \cdots, N$) as in Sec 3.1.
5.3.1 The case $X = T^* \mathbb{P}(C^{(\lambda^{(l+1)})})$

We follow a construction of Stab for $X$ given in 3.4 from [I]. For $a \in [1, \lambda^{(l)})$, let us fix a variable $t_a^{(l)}$ and regard it as the coordinate on $GL(1)/p^Z = E = E^{\lambda}$. The variable $t_a^{(l)}$ gives the Chern root of the line bundle $\mathcal{O}(1)$ over $X$. We denote by $\Pi^*_{t_a^{(l+1)}} = q^{2P_{t_a^{(l+1)}}}$ the Kähler parameter dual to $t_a^{(l)}$.

Let us take a basis $\{e_{ib}^{(l+1)} \mid b = 1, \ldots , \lambda^{(l+1)}\}$ of $C^{(\lambda^{(l+1)})}$, on which the torus $A^{(l+1)}$ acts as $\text{diag}(t_1^{(l+1)}, \ldots , t_{\lambda^{(l+1)}})$. Hence $F_{ib}^{(l+1)} = C e_{ib}^{(l+1)} \mid b = 1, \ldots , \lambda^{(l+1)}$ give the $A^{(l+1)}$-fixed points. We chose the chamber $C^{(l+1)}$ such that

$$F_{i_1^{(l+1)}} > F_{i_2^{(l+1)}} > \cdots > F_{i_{\lambda^{(l+1)}}^{(l+1)}}.$$  

Let $\phi^{(l)} : [1, \lambda^{(l)}) \to [1, \lambda^{(l+1)}]$ be a map defined by $\phi^{(l)}_{(l)(a)} = \phi^{(l)}(t_a)$. For $a \in [1, \lambda^{(l)})$, the elliptic stable envelopes of $\text{Stab}_{(l)}(F_{\phi^{(l)}(a)})$ for $X = T^* \mathbb{P}(C^{(\lambda^{(l+1)})})$ is then given by

$$f^{(l)}(t_a^{(l)}, t^{(l+1)}, \Pi_{t_a^{(l+1)}}^{*} q^{2(\lambda^{(l+1)} - \phi^{(l)}(a))}) = \frac{[-v^{(l+1)}_{\phi^{(l)}(a)} + v^{(l)}_{a} + P_{l,t^{(l+1)} + \lambda^{(l+1)} - \phi^{(l)}(a)}]}{\prod_{b < \phi^{(l)}(a)} [v_{b}^{(l+1)} - v_{a}^{(l)}] \prod_{b > \phi^{(l)}(a)} [v_{b}^{(l+1)} - v_{a}^{(l)} + 1]},$$

where we set $t_a^{(l)} = q^{2t_a^{(l)}}$, $t_a^{(l+1)} = q^{2t_a^{(l+1)}}$, $t^{(l+1)} = (t_1^{(l+1)}, \ldots , t_{\lambda^{(l+1)}})$, and made an identification $h = q^{-2}$.

Let us compare this with [3.3]. Let $\tilde{I} = \sigma_0(I)$ with $\tilde{I}^{(l)} = \{t_1^{(l)} < \cdots < t_{\lambda^{(l)}}^{(l)}\} \mid l = 1, \cdots , N\}$ and consider the variables $t_a^{(l)} \equiv t^{(l)}$. Then one finds

$$\tilde{t}_a^{(l)} = \sigma_0(t_a^{(l)}),$$

where $\sigma_0^{(l)} \in \mathfrak{S}_{\lambda^{(l)}}$ denotes the longest element. Furthermore for a map $\tilde{\phi}^{(l)} : [1, \lambda^{(l)}) \to [1, \lambda^{(l+1)}]$ such that $\tilde{\phi}^{(l)}_{(l)(a)} = \tilde{t}_a^{(l)}$, we have

$$\tilde{\phi}^{(l)}(\sigma_0^{(l)}(a)) = \sigma_0^{(l+1)}(\phi^{(l)}(a)).$$

We then find that (5.2) coincides with

$$u_{\sigma_0(I)}^{(l)}(\tilde{t}_a^{(l)}, \tilde{t}^{(l+1)}, \Pi_{\tilde{t}}^{*} q^{2(\tilde{\phi}^{(l)}(\tilde{a}) - 1)})$$

from [3.3] under the identification $\tilde{a} = \sigma_0^{(l)}(a)$, $\tilde{b} = \sigma_0^{(l+1)}(b)$, $t_a^{(l)} = t^{(l)}(t_a^{(l)})$, $t_b^{(l)} = t^{(l)}(t_b^{(l)})$, $t_a^{(l)} = t_a^{(l)}$, $t_b^{(l+1)} = t_b^{(l+1)}$ and in particular $t_a^{(N)} = z_{\sigma_0(a)}^{-1}$. This is due to the identity

$$\tilde{\phi}^{(l)}(\tilde{a}) - 1 = \lambda^{(l+1)} - \phi^{(l)}(a),$$
and the equality such as

$$
\prod_{i_b^{(l+1)} < i_a^{(l)}} [v_{b}^{(l+1)} - v_{a}^{(l)}] + 1 = \prod_{b > \phi^{(l)}(a)} [v_{b}^{(l+1)} - v_{a}^{(l)}] + 1,
$$

which follows from

$$
\frac{i_b^{(l+1)}}{i_a^{(l)}} < \frac{i_a^{(l)}}{i_a^{(l)}} \iff \tilde{b} < \tilde{a} \iff b > \phi^{(l)}(a).
$$

### 5.3.2 The case \( X = T^*\text{Gr}(\lambda^{(l)}, \lambda^{(l+1)}) \)

We follow 4.4 in [1]. For each \( l (l = 1, \ldots, N - 1) \), let \( S^{(l)} = \text{diag}(t_1^{(l)}, \ldots, t_{\lambda^{(l)}}^{(l)}) \subset GL(\lambda^{(l)}) \) be

the maximal torus. The abelianization \( X_{S^{(l)}} \) of \( X \) is given by

$$
X_{S^{(l)}} = (T^*\mathbb{P}(C^{(l+1)}))^{\lambda^{(l)}}.
$$

Correspondingly the product

$$
\text{Stab}_{T^{(l+1)}}(F_{T^{(l)}}) = \prod_{a=1}^{\lambda^{(l)}} f^{(l)}(t_a^{(l)}, t^{(l+1)}, \Pi_{l,l+1}^* 2C_{l,l+1}(i_a^{(l)}))
$$

gives the elliptic stable envelopes for \( X_{S^{(l)}} \). Here \( F_{T^{(l)}} = \text{Span}_\mathbb{C}\{ e_{i_a^{(l+1)}} | i_a^{(l+1)} = i_a^{(l)} (a = 1, \ldots, \lambda^{(l)}) \} = \text{Span}_\mathbb{C}\{ e_{i_a^{(l)}} (a = 1, \ldots, \lambda^{(l)}) \} \) are the \( A^{(l+1)} \)-fixed points, and \( i_a^{(l)} \)'s are identified with the Chern roots of the tautological bundle over \( X \). The dynamical shift \( C_{l,l+1}(i_a^{(l)}) \) is given by

$$
C_{l,l+1}(i_a^{(l)}) = 2(\lambda^{(l)} - a) - \lambda^{(l+1)} + \phi^{(l)}(a).
$$

Note that this is identical to the one in Proposition 5.3.1 for \( i_a^{(l)} = s, \mu_s = l \).

Then the abelianization formula [1,43] gives the stable envelopes \( \text{Stab}_{T^{(l+1)}}(F_{T^{(l)}}) \) for \( X \) in terms of \( \text{Stab}_{T^{(l+1)}}(F_{T^{(l)}}) \) as follows.

$$
\text{Stab}_{T^{(l+1)}}(F_{T^{(l)}}) = \text{Sym}_{\lambda^{(l)}} \frac{\text{Stab}_{T^{(l+1)}}(F_{T^{(l)}})}{\prod_{1 \leq a < b \leq \lambda^{(l)}} [v_a^{(l)} - v_b^{(l)}] [v_b^{(l)} - v_a^{(l)} - 1]},
$$

In particular, the case \( N = 2 \), where \( \lambda^{(2)} = n \) and \( i_b^{(2)} = z_b^{-1} \), we have for \( a \in [1, \lambda^{(1)}] \)

$$
f^{(1)}(t_a^{(1)}, z^{-1}, \Pi_{1,2}^* q^{2C_{1,2}(i_a^{(1)})})
= \frac{[u_s + v_a^{(1)} + P_{1,2} + C_{1,2}(i_a^{(1)})]}{[P_{1,2} + C_{1,2}(i_a^{(1))}]}
\prod_{1 \leq b < \phi^{(1)}(a)} [u_b + v_a^{(1)}] \prod_{\phi^{(1)}(a) < b \leq n} [u_b + v_a^{(1)} - 1].
$$
The resultant $\text{Stab}_{c(2)}(F_{l(t)})$ is the elliptic stable envelope for $T^*\text{Gr}(\lambda^{(1)}, n)$ given in (60) from [1]. Combining this with the identification between $f^{(l)}$ and $u^{(l)}_{\sigma_0(l)}$ for $l = 1$ in the last subsection, we find

$$\text{Stab}_{c(2)}(F_{l(t)}) = \mathcal{W}_{\sigma_0(l)}(\tilde{t}, \sigma_0(z^{(1)}), \Pi^{*-1}),$$

where $\tilde{t} = (\tilde{t}_a(l))$ ($l = 1, 2$, $a = 1, \cdots, \lambda^{(1)}$).

5.3.3 The case $X = T^*F_\lambda$

Let $N > 2$. The partial flag variety $F_\lambda = F(\lambda^{(1)}, \cdots, \lambda^{(N-1)}, n)$ is given by a hyper-Kähler quotient

$$F_\lambda = \mathbb{P} \left( \bigoplus_{l=1}^{N-1} \text{Hom}(\mathbb{C}^{\lambda^{(l)}}, \mathbb{C}^{\lambda^{(l+1)}}) \bigg) / G$$

by the action of $G = \prod_{l=1}^{N-1} GL(\lambda^{(l)}, \mathbb{C})$, and has a description as a tower of Grassmannian bundles [3]

$$\text{Gr}(\lambda^{(l)}, \lambda^{(l+1)}) \rightarrow F(\lambda^{(l)}, \lambda^{(l+1)}, \cdots, \lambda^{(N-1)}, n) = \text{Gr}(\lambda^{(l)}, S_{l+1})$$

$$\downarrow$$

$$F(\lambda^{(l+1)}, \cdots, \lambda^{(N-1)}, n),$$

where $S_1 \subset S_2 \subset \cdots \subset S_{N-1} \subset \mathbb{C}^n \otimes \mathcal{O}_F$ denotes the universal bundle. Correspondingly, the abelianization $(F_\lambda)_S$ of $F_\lambda$ by $S = \prod_{l=1}^{N-1} S^{(l)} \subset G$ is a tower of product-of-projective-space bundles:

$$(\mathbb{P}^{\lambda^{(l+1)}-1})^{\lambda^{(l)}} \rightarrow \mathcal{F}_l = \mathbb{P}(V_{l+1}) \times_{\mathcal{F}_{l+1}} \cdots \times_{\mathcal{F}_{l+1}} \mathbb{P}(V_{l+1})$$

$$\downarrow$$

$$\mathcal{F}_{l+1},$$

where $\mathcal{F}_l$ is the abelianization of $F(\lambda^{(l)}, \lambda^{(l+1)}, \cdots, \lambda^{(N-1)}, n)$, and

$$V_{l+1} = \bigoplus_{j=1}^{\lambda^{(l+1)}} \mathcal{O}(0, \cdots, 0, j, 0, \cdots, 0).$$

is the vector bundle on $\mathcal{F}_{l+1}$ corresponding to $S_{l+1}$. The abelianization $X_S$ of $X = T^*F_\lambda$ contains $T^*(F_\lambda)_S$ as a dense open subset, and is obtained by gluing together the flopped versions of $T^*(F_\lambda)_S$ (5.1.2 in [3]).
For \( I = \{ \mu_1, \ldots, \mu_n \} \in \mathcal{I}_\lambda \), let \( F_I \) denote the \( A = \text{diag}(z_1, \ldots, z_n) \)-fixed point \( 0 \subset F_{I^{(1)}} \subset \cdots \subset F_{I^{(N-1)}} \subset \mathbb{C}^n \) and \( \{ t_a^{(l)} \} \) be the Chern roots of the tautological bundle over \( X \). We choose the chamber \( \mathcal{C} \) consistently to the choice of \( \mathcal{C}^{(l+1)} \) for \( T^* \text{Gr}(\lambda^{(l)}, \lambda^{(l+1)}) \) \( (l = 1, \ldots, N - 1) \). Namely

\[
zb / za > 0 \iff a < b.
\]

Then the elliptic stable envelop for \( X_S \) is given by

\[
\text{Stab}_s^S(F_I) = \prod_{l=1}^{N-1} \prod_{a=1}^{l} f(l) \left( t_a^{(l)}, l^{(l+1)}, \Pi_{\mu, (l)}^{*}, l+1q^{2C_{\mu, (l)}, l+1(t_a^{(l)})} \right).
\] (5.8)

Here we defined for \( s \in I^{(l)} \), \( \mu_s \leq l \),

\[
\Pi_{\mu, (l)}^{*, l+1} := \prod_{k=\mu_s}^{l} \Pi_{k, k+1}, \quad q^{2C_{\mu, l+1}(s)} := \prod_{k=\mu_s}^{l} q^{2C_{k, k+1}(s)}
\] (5.9)

with \( C_{k, k+1}(s) \) given in \( (5.7) \) for \( s = t_a^{(l)} \). The factor \( \prod_{a=1}^{l} f(l) \left( t_a^{(l)}, l^{(l+1)}, \Pi_{\mu, (l)}^{*}, l+1q^{2C_{\mu, (l)}, l+1(t_a^{(l)})} \right) \)

is a contribution from \( T^* \text{Gr}(\lambda^{(l)}, \lambda^{(l+1)}) \) with the Kähler parameters \( \Pi_{\mu, (l)}^{*, l+1} \) dual to \( t_a^{(l)} \). Note that for \( N > 2 \), \( \mu, (l) \) in general takes a value in the range \([1, l]\) corresponding to the embedding structure \( F_{I^{(1)}} \subset \cdots \subset F_{I^{(\mu, (l))}} \subset \cdots \subset F_{I^{(l+1)}} \). The formulas in \( (5.9) \) are then natural in the sense that the Kähler parameters as well as their dynamical shift \( q^{2C_{\mu, (l)}, l+1(t_a^{(l)})} \)

are given as built-up contributions from a sequence of the Grassmannians filling a gap between \( F_{I^{(\mu, (l))}} \) and \( F_{I^{(l+1)}} \). Note that the resultant \( C_{\mu, l+1}(s) \) coincides with the one in Proposition 3.1.

Then again the abelianization formula \( \text{[48]} \) yields the following expression of \( \text{Stab}_s^S(F_I) \) for \( X = T^*F_\lambda \).

\[
\text{Stab}_s^S(F_I) = \text{Sym}_{t^{(1)}} \cdots \text{Sym}_{t^{(N-1)}} \prod_{l=1}^{N-1} \prod_{1 \leq a < b \leq \lambda(l)} [v_a^{(l)} - v_b^{(l)}][v_a^{(l)} - v_a^{(l)} - 1] 
\]

\[
= \text{Sym}_{t^{(1)}} \cdots \text{Sym}_{t^{(N-1)}} \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda(l)} f(l) \left( t_a^{(l)}, l^{(l+1)}, \Pi_{\mu, (l)}^{*, l+1}q^{2C_{\mu, (l)}, l+1(t_a^{(l)})} \right) 
\]

\[
= \text{Sym}_{t^{(1)}} \cdots \text{Sym}_{t^{(N-1)}} \prod_{l=1}^{N-1} \prod_{1 \leq a < b \leq \lambda(l)} [v_a^{(l)} - v_b^{(l)}][v_a^{(l)} - v_a^{(l)} - 1].
\] (5.10)

Under the same identification as in the previous subsections, we thus obtain

\[
\text{Stab}_s^S(F_I) = W_{s_0(t)}(\tilde{t}, \sigma_0(z^{-1}), \Pi^{*-1}).
\] (5.11)
5.3.4 Restriction to the fixed points

For \( J \in \mathcal{I}_\lambda \), let us consider the restriction to the fixed point \( F_J \) i.e. the specialization \( t = z_J^{-1} \) given by

\[
t_a^{(l)} = z_J^{-1} \quad (l = 1, \ldots, N - 1, a = 1, \ldots, \lambda^{(l)}).
\]

From (5.10) we obtain

\[
\text{Stab}_\xi(F_I)|_{F_J} = \text{Sym}_{\lambda^{(1)}} \cdots \text{Sym}_{(N-1)} \prod_{l=1}^{N-1} \frac{\lambda^{(l)}(t_a^{(l)}, t^{(l+1)}, \Pi^{(l)}_{\mu_t^{(l)+1}})}{\prod_{1 \leq a < b \leq \lambda^{(l)}} [u_{j_a^{(l)}} - u_{j_b^{(l)}}] [u_{j_a^{(l)}} - u_{j_b^{(l)}} + 1]},
\]

where

\[
f^{(l)}(t_a^{(l)}, t^{(l+1)}, \Pi^{(l)}_{\mu_t^{(l)+1}}) = \frac{[u_{j_a^{(l)+1}} - u_{j_a^{(l)}} + P_{\mu_t^{(l)+1}} + C_{\mu_t^{(l)+1}}(s)]}{[P_{\mu_t^{(l)+1}} + C_{\mu_t^{(l)+1}}(s)]} \prod_{j_b^{(l+1)} < s} [u_{j_b^{(l+1)}} - u_{j_a^{(l)}}] \prod_{j_b^{(l+1)} > s} [u_{j_b^{(l+1)}} - u_{j_a^{(l)}} - 1].
\]

This is an elliptic and dynamical analogue of the formula in Theorem 5.2.1 in [48].

By using the identification (5.11) and the equivalence of the specializations

\[
t = z_J^{-1} \iff \tilde{t} = \sigma_0(z^{-1}) \sigma_0(J), \quad \text{i.e.} \quad t_a^{(l)} = z_J^{-1} \iff \tilde{t}_a^{(l)} = z_0(J_a^{(l)}),
\]

where \( \tilde{J} = \sigma_0(J) \), we obtain the following identification.

\[
\text{Stab}_\xi(F_I)|_{F_J} = \mathcal{W}_{\sigma_0(I)}(\sigma_0(z^{-1}) \sigma_0(J), \sigma_0(z^{-1}), \Pi^{-1}) = \mathcal{W}_{\sigma_0(I)}(z_J^{-1}, \sigma_0(z^{-1}), \Pi^{-1}),
\]

where in the second equality we used the identity

\[
\mathcal{W}_I(\sigma(z) \sigma(J), z, \Pi) = \mathcal{W}_I(z_J, z, \Pi^*) \quad \forall I, J \in \mathcal{I}_\lambda, \forall \sigma \in \mathcal{S}_n.
\]

5.4 Geometric representation

Let \( X = T^* \mathcal{F}_\lambda \) and fix a chamber \( \mathcal{C} \) as above. By definition, the stable classes \( \text{Stab}_\xi(F_K) \) \((K \in \mathcal{I}_\lambda)\) are triangular with respect to the fixed point classes \( \{[I]\}_{I \in \mathcal{I}_\lambda} \) in \( \mathcal{E}_T(X) \). See 3.3.4 in [11]. Namely we have the following expansion formula

\[
\text{Stab}_\xi(F_K) = \sum_{I \in \mathcal{I}_\lambda} \frac{\text{Stab}_\xi(F_K)|_{F_I}}{R(z_I^{-1})} [I]. \quad (5.13)
\]
Here we chose a normalization by $R(z)$ given in Proposition 3.4 for later convenience. We regard this as the definition of the fixed point classes.

From (5.12), we have

$$\text{Stab}_\xi(F_K)|_{F_I} = W_{\sigma_0(K)}(z_I^{-1}, \sigma_0(z^{-1}), \Pi^{-1}).$$

(5.14)

Note also that by the replacement $z \mapsto z^{-1}$ and $\Pi \mapsto \Pi^{*-1}$ one can rewrite Proposition 3.4 as

$$\sum_{I \in \mathcal{I}_\lambda} W_J(z_I^{-1}, z_I^{-1}, \Pi^{*}q^{2\sum_{s=1}^{n}<\ell_{ij}, h>}) W_{\sigma_0(K)}(z_I^{-1}, \sigma_0(z^{-1}), \Pi^{*-1}) = \delta_{J,K}. \tag{5.14}$$

Then using this and (5.14) one can invert (5.13) and obtain

$$[I] = \sum_{J \in \mathcal{I}_\lambda} \tilde{W}_J(z_I^{-1}, z_I^{-1}, \Pi^{*}q^{2\sum_{s=1}^{n}<\ell_{ij}, h>}) \text{Stab}_\xi(F_J). \tag{5.15}$$

Comparing this with Theorem 4.5 we find that (5.15) is identical to the relation (4.12) under the correspondence

the Gelfand-Tsetlin base $\xi_I \iff \text{the fixed point class } [I]$,

the standard base $v_J \iff \text{the stable class } \text{Stab}_\xi(F_J)$.

On the basis of this correspondence as well as Theorem 4.7 and Corollary 4.8 we obtain the following statement on the level-0 action of $E_{q,p}(\mathfrak{g}_N)$ and $U_{q,p}(\mathfrak{g}_N)$ on $E_T(X)$.

**Theorem 5.1.** Under the same notation as Theorem 4.7 let us define the action of the half-currents $K_j^{\pm}(1/w), E_{j+1,j}^{\pm}(1/w, P), F_{j,j+1}^{\pm}(1/w, P)$ on the fixed point classes by

$$K_j^{\pm}(1/w)[I] = \prod_{k=1}^{j-1} \prod_{a \in l_k} \frac{[u_a - v]}{[u_a - v + 1]} \prod_{i = j+1}^{N} \prod_{b \in l_i} \frac{[u_b - v - 1]}{[u_b - v]} [I], \tag{5.16}$$

$$E_{j+1,j}^{\pm}(1/w, P)[I] = \sum_{i \in \mathcal{I}_{j+1}} \frac{[P_{j,j+1} - u_i + v][1]}{[P_{j,j+1}][u_i - v]} \prod_{k \in l_{j+1}} \frac{[u_i - k u + 1]}{[u_i - u_k]} [I'], \tag{5.17}$$

$$F_{j,j+1}^{\pm}(1/w, P)[I] = \sum_{i \in \mathcal{I}_j} \frac{[P_{j+1,j} + \lambda_j - \lambda_{j+1} + u_i - v - 1][1]}{[P_{j+1,j} + \lambda_j - \lambda_{j+1} - 1][u_i - v]} \prod_{k \in l_j} \frac{[u_k - u_i + 1]}{[u_k - u_i]} [I'][I], \tag{5.18}$$

Then this gives an irreducible finite-dimensional representation of $E_{q,p}(\mathfrak{g}_N)$ on $E_T(X)$.  

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Corollary 5.2. The level-0 action of $U_{q,p}(\widehat{\mathfrak{sl}_N})$ on $E_T(X)$ is given by

\[
H_j^\pm(q^{j-N+1}/w)[I] = q \prod_{a \in I_j} \frac{[u_a - v + 1]}{[u_a - v]} \prod_{b \in I_{j+1}} \frac{[u_b - v - 1]}{[u_b - v]} e^{-Q_\alpha_j[I]},
\]

\[
E_j(q^{j-N+1}/w)[I] = \mu^*[1] \sum_{i \in I_j} \delta(z_i/w) \prod_{k \in I_{j+1}, k \neq i} \frac{[u_i - u_k + 1]}{[u_i - u_k]} e^{-Q_\alpha_j[I']},
\]

\[
F_j(q^{j-N+1}/w)[I] = \mu^*[1] \sum_{i \in I_j} \delta(z_i/w) \prod_{k \in I_{j+1}, k \neq i} \frac{[u_k - u_i + 1]}{[u_k - u_i]} [I'].
\]

Remark. Similar correspondences between the Gelfand-Tsetlin basis and the fixed point classes were studied in [8,9,26,38,51]. In [38] for the level-(0,1) representation of the quantum toroidal algebra of type $A$, the Gelfand-Tsetlin basis on the $q$-Fock space [52] was identified with the fixed point basis of the equivariant K-theory of corresponding cyclic quiver variety [53]. Affine Yangian analogue of this result was obtained in [26]. In [8,9,51], certain geometric actions of the universal enveloping algebra $U(\mathfrak{gl}_N)$ on the Laumon spaces, of the affine Yangian of type $A^{(1)}_{N-1}$ on the affine Laumon spaces and of the quantum toroidal algebra $\tilde{U}_q(\mathfrak{sl}_N)$ on the K-theory of the affine Laumon spaces were constructed, respectively.

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A The $H$-Hopf algebroid structure of $E_{q,p}(\widehat{\mathfrak{gl}_N})$ and $U_{q,p}(\widehat{\mathfrak{gl}_N})$

We summarize an $H$-Hopf algebroid structure based on the opposite coproduct to the one used in the previous papers [31,32]. This opposite $H$-Hopf algebroid structure is used in Sec[4] to construct the finite dimensional representations of $E_{q,p}(\widehat{\mathfrak{gl}_N})$ and $U_{q,p}(\widehat{\mathfrak{gl}_N})$. 
Let $A$ denote $E_{q,p}(\mathfrak{gl}_N)$ or $U_{q,p}(\mathfrak{gl}_N)$. The $A$ is bi-graded over $H^*$ by

$$A = \bigoplus_{\alpha, \beta \in H^*} A_{\alpha, \beta}$$

$$A_{\alpha, \beta} = \left\{ x \in A \mid q^{P+h} x q^{-(P+h)} = q^{<\alpha, P+h>} x, \quad q^P x q^{-P} = q^{<\beta, P>} x \quad \forall P + h, \, P \in H \right\},$$

and possesses two moment maps $\mu_l, \mu_r : F \to A_{0,0}$ defined by

$$\mu_l(f) = f(P + h, p) \in F[[p]], \quad \mu_r(f) = f(P, p^*) \in F[[p]].$$

The $\mu_l$ and $\mu_r$ satisfy

$$\mu_l(f) a = a \mu_l(T_\alpha f), \quad \mu_r(f) a = a \mu_r(T_\beta f), \quad a \in A_{\alpha, \beta}, \quad \tilde{f} \in F,$$

where $T_\alpha = e^\alpha \in \mathbb{C}[R_Q]$ denotes the automorphism of $F$

$$T_\alpha \tilde{f} = e^\alpha f(P, P + h) e^{-\alpha} = f(P + <\alpha, P>, P + h + <\alpha, P>).$$

Hence both $E_{q,p}(\mathfrak{gl}_N)$ and $U_{q,p}(\mathfrak{gl}_N)$ are the $H$-algebras [6, 27, 32].

Let $A$ and $B$ be two $H$-algebras. The tensor product $A \otimes B$ is the $H^*$-bigraded vector space with

$$(A \otimes B)_{\alpha\beta} = \bigoplus_{\gamma \in H^*} (B_{\gamma\beta} \otimes_F A_{\alpha\gamma}),$$

where $\otimes_F$ denotes the usual tensor product modulo the following relation.

$$\mu_l^A(\tilde{f}) b \otimes a = b \otimes \mu_r^A(\tilde{f}) a, \quad a \in A, \quad b \in B, \quad \tilde{f} \in F.$$  \hspace{1cm} (A.1)

The tensor product $A \otimes B$ is again an $H$-algebra with the multiplication $(b \otimes a)(d \otimes c) = bd \otimes ac$ ($a, c \in A, \, b, d \in B$) and the moment maps

$$\mu_l^A \otimes B = 1 \otimes \mu_l^A, \quad \mu_r^A \otimes B = \mu_r^B \otimes 1.$$  \hspace{1cm} (A.2)

We also consider the $H$-algebra of the shift operators [6]

$$\mathcal{D} = \left\{ \sum_{\alpha} \hat{f}_\alpha T_\alpha \mid \hat{f}_\alpha \in M_{H^*}, \, \alpha \in R_Q \right\},$$

$$\mathcal{D}_{\alpha, \alpha} = \left\{ \hat{f} T_{-\alpha} \right\}, \quad \mathcal{D}_{\alpha, \beta} = 0 \ (\alpha \neq \beta),$$

$$\mu_l^D(\tilde{f}) = \mu_r^D(\tilde{f}) = \hat{f} T_0 \quad \hat{f} \in M_{H^*}.$$  \hspace{1cm} (A.3)

Then we have the $H$-algebra isomorphism

$$A \cong A \otimes \mathcal{D} \cong \mathcal{D} \otimes A.$$  \hspace{1cm} (A.4)
The two $H$-algebras $E_{q,p}(\mathfrak{gl}_N)$ and $U_{q,p}(\mathfrak{gl}_N)$ are equipped with the common $H$-Hopf algobroid structure [32] defined by the two $H$-algebra homomorphisms, the co-unit $\varepsilon : A \to D$ and the (oposite) co-multiplication $\Delta' : A \to A \otimes A$

$$\varepsilon(\tilde{L}_{i,j}^+(z)) = \delta_{i,j}T_{Q_{i,j}} \quad (n \in \mathbb{Z}), \quad \varepsilon(e^Q) = e^Q, \quad (A.5)$$

$$\varepsilon(\mu_l(\tilde{f})) = \varepsilon(\mu_r(\tilde{f})) = \tilde{f}T_0, \quad (A.6)$$

$$\Delta'(\tilde{L}_{i,j}^+(z)) = \sum_k \tilde{L}_{k,j}^+(z) \otimes \tilde{L}_{i,k}^+(z), \quad (A.7)$$

$$\Delta'(e^Q) = e^Q \otimes e^Q, \quad (A.8)$$

$$\Delta'(\mu_l(\tilde{f})) = 1 \otimes \mu_l(\tilde{f}), \quad \Delta'(\mu_r(\tilde{f})) = \mu_r(\tilde{f}) \otimes 1, \quad (A.9)$$

and the algebra antihomomorphism $S : A \to A$

$$S(\tilde{L}_{i,j}^+(z)) = (\tilde{L}^+(z)^{-1})_{ij},$$

$$S(e^Q) = e^{-Q}, \quad S(\mu_r(\tilde{f})) = \mu_l(\tilde{f}), \quad S(\mu_l(\tilde{f})) = \mu_r(\tilde{f}).$$

B Proof of Theorem 4.7

In this section we prove that the action of the half-currents (1.11)-(1.13) satisfies the defining relations of the elliptic algebra $E_{q,p}(\mathfrak{gl}_N)$ at $k = 0$. From Lemma 6.10 in [32] it is enough to show the following relations, which are the dynamical counterpart of those listed in Sec.C.1 in [32] through Proposition 2.11. Or one can directly derive them from (2.39) with corresponding Gauss decomposition of the dynamical $L$-operator.

$$\mathcal{K}_{j+1}(1/w_1)^{-1}E_{j+1,j}^+(1/w_2, P)\mathcal{K}_{j+1}^+(1/w_1)$$

$$= E_{j+1,j}^+(1/w_2, P + \varphi_{j+1}, h) \frac{1}{b^*(v_{-12})} - E_{j+1,j}^+(1/w_1, P + \varphi_{j+1}, h) \frac{c^*(v_{12}, P_{j+1})}{b^*(v_{-12})}, \quad (B.1)$$

$$\mathcal{K}_{j+1}(1/w_1)F_{j,j+1}^+(1/w_2, P + \varphi_{j+1}, h) \mathcal{K}_{j+1}^+(1/w_1)^{-1}$$

$$= \frac{1}{b(-v_{12})} E_{j,j+1}^+(1/w_2, P) - \frac{\varphi(v_{12}, (P + h)_{j,j+1})}{b(v_{12})} E_{j,j+1}^+(1/w_1, P), \quad (B.2)$$

$$\frac{1}{b^*(v_{12})} E_{j+1,j}^+(1/w_1, P + \varphi_{j+1}, h) E_{j+1,j}^+(1/w_2, P + \varphi_{j}, h)$$

$$- E_{j+1,j}^+(1/w_2, P + \varphi_{j+1}, h) E_{j+1,j}^+(1/w_2, P + \varphi_{j}, h) \frac{c^*(v_{12}, P_{j+1})}{b^*(v_{12})}, \quad (B.3)$$

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\[
\frac{1}{b(-v_{12})} F_{j,j+1}^+(1/w_1, P) F_{j,j+1}^+(1/w_2, P) - F_{j,j}^+(1/w_1, P)^2 \frac{\bar{c}(-v_{12}, (P + h)_{j,j+1} - 2)}{b(-v_{12})} \\
= \frac{1}{b(v_{12})} F_{j,j+1}^+(1/w_2, P) F_{j,j+1}^+(1/w_1, P) - F_{j,j+1}^+(1/w_2, P)^2 \frac{\bar{c}(v_{12}, (P + h)_{j,j+1} - 2)}{b(v_{12})},
\]

(B.4)

\[
E_{j+1j}^+(1/w_1, P) F_{j,j+1}^+(1/w_2, P + \epsilon_j, h) - F_{j,j+1}^+(1/w_2, P + \epsilon_{j+1}, h) E_{j+1j}^+(1/w_1, P) \\
= K_j^+(1/w_2) K_j^+(1/w_1) - \frac{1}{b(-v_{12})} \frac{\bar{c}(-v_{12}, (P + h)_{j,j+1})}{b(-v_{12})}.
\]

(B.5)

where \( w_i = q^{2i} (i = 1, 2) \) and we set \( v_{12} = v_1 - v_2 \). Note that at \( k = 0, p = p^*, r = r^* \), hence \( b(u, P) = b^*(u, P), b(u) = b^*(u), c(u, P) = c^*(u, P), \bar{c}(u, P) = \bar{c}^*(u, P) \) etc.

(B.1): Noting \( \epsilon_{j+1}, h_{j,j+1} = -1, \) from (4.11) and (4.12) we have

\[
\text{LHS} = \sum_{k \in I_{j+1}} \frac{\bar{c}(u_k - v_2, P_{j,j+1})}{b(u_k - v_2)} \frac{1}{b(-v_{12})} \prod_{\substack{i \in I_{j+1} \setminus k \neq k}} b(u_{kl})^{-1} \xi_{I^{l'}}.
\]

RHS = \sum_{k \in I_{j+1}} \left( \frac{\bar{c}(u_k - v_2, P_{j,j+1} - 1)}{b(u_k - v_2)} - \frac{\bar{c}(u_k - v_1, P_{j,j+1} - 1)}{b(u_k - v_1)} \right) \frac{1}{b(-v_{12})} \prod_{\substack{i \in I_{j+1} \setminus k \neq k}} b(u_{kl})^{-1} \xi_{I^{l'}}.

Then the equality follows from the identity

\[
\frac{\bar{c}(-v_2, s)}{b(-v_2)} \frac{1}{b(-v_{12})} = \frac{\bar{c}(-v_2, s - 1)}{b(-v_2)} - \frac{\bar{c}(-v_1, s - 1)}{b(-v_1)} \frac{c(-v_{12}, s)}{b(-v_{12})}.
\]

(B.6)

One can prove (B.2) similarly.

(B.3): The action of the LHS on \( \xi_I \) yields

\[
\sum_{a,b \in I_{j+1}} \left( \frac{[v_{12} + 1][P - 1 - u_b + v_1][P + 1 - u_a + v_2]}{[v_{12}][u_b - v_1][u_a - v_2]} - \frac{[P + v_{12}][P - 1 - u_b + v_2][P + 1 - u_a + v_2]}{[P][v_{12}][u_b - v_2][u_a - v_2]} \right) \frac{1}{u_{ab}} \prod_{\substack{k \in I_{j+1} \setminus a \neq b}} b(u_{ak})^{-1} b(u_{bk})^{-1} \xi_{I^{(l')}^{l'}}
\]

\[
= \frac{[P - 1]}{[P]} \sum_{a,b \in I_{j+1}} \left( \frac{[P + 1 - u_a + v_2][v_1 - u_b + P][v_2 - u_b - 1][u_{ab} + 1]}{[u_a - v_2][u_b - v_1][u_b - v_2]} \right) \frac{1}{u_{ab}} \prod_{\substack{k \in I_{j+1} \setminus a \neq b}} b(u_{ak})^{-1} b(u_{bk})^{-1} \xi_{I^{(l')}^{l'}}
\]

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where we set $P = P_{j,j+1}$ and $u_{ak} = u_a - u_k$ etc. The second equality follows from the identity

$$[v_{12} + 1][P - 1 - u_b + v_1][P][u_b - v_2] - [P + v_{12}][1][P - 1 - u_b + v_2][u_b - v_1] = -[P - 1][v_{12}][v_1 - u_b + P][v_2 - u_b - 1].$$

Similarly the RHS of (B.3) yields

$$\sum_{a,b \in I_{j+1} \atop a \neq b} \left( \frac{[v_{12} - 1][P - 1 - u_b + v_2][P + 1 - u_a + v_1]}{v_{12}][u_b - v_2][u_a - v_1] \right) \frac{u_{ab} + 1}{u_{ab}} \prod_{k \in I_{j+1} \atop \neq a,b} \bar{b}(u_{ak})^{-1} b(u_{bk})^{-1} \xi_{(I')^j}^{(v)},$$

$$= -\frac{[P - 1]}{[P]} \sum_{a,b \in I_{j+1} \atop a \neq b} \left( \frac{[P + 1 - u_a + v_2][v_1 - u_b + P][v_2 - u_b - 1]}{u_a - v_2}[u_b - v_1][u_b - v_2] \right) \frac{u_{ab} + 1}{u_{ab}} \prod_{k \in I_{j+1} \atop \neq a,b} \bar{b}(u_{ak})^{-1} b(u_{bk})^{-1} \xi_{(I')^j}^{(v)}.$$

Taking the difference between the LHS and the RHS, we obtain

$$-\frac{[P - 1]}{[P]} \sum_{a,b \in I_{j+1} \atop a \neq b} \left( \frac{[P + 1 - u_a + v_2][v_1 - u_b + P][v_2 - u_b - 1]}{u_a - v_2}[u_b - v_1][u_b - v_2] \right) \frac{u_{ab} - 1}{u_{ab}} \frac{u_{ab} + 1}{u_{ab}} f(u_a, u_b) \xi_{(I')^j}^{(v)}$$

$$= \frac{[P - 1][P + 1][v_{12}]}{[P]} \sum_{a,b \in I_{j+1} \atop a \neq b} \left( u_{ab} - 1 \right) \left( u_{ab} + 1 \right) f(u_a, u_b) \xi_{(I')^j}^{(v)}$$

where the equality follows from the identity

$$-[P + 1 - u_a + v_2][v_1 - u_b + P][v_2 - u_b - 1][u_a - v_1] + [P + 1 - u_a + v_1][v_2 - u_b + P][v_1 - u_b - 1][u_a - v_2]$$

$$= [P + 1][v_{12}][u_{ab} - 1][P + v_1 + v_2 - u_a - u_b]$$

and we set

$$f(u_a, u_b) = \frac{[P + v_1 + v_2 - u_a - u_b]}{[u_a - v_1][u_a - v_2][u_b - v_1][u_b - v_2]} \prod_{k \in I_{j+1} \atop \neq a,b} \bar{b}(u_{ak})^{-1} b(u_{bk})^{-1}.$$

Since $f(u_a, u_b) = f(u_b, u_a)$ and $(I')^j = (I')^{j'}$ ($a, b \in I_{j+1}, a \neq b$), the summation in (B.7) vanishes. □
One can prove (B.4) similarly.

(B.5) In the LHS, noting $< \bar{\epsilon}_j, h_{j,j+1} > = 1$ we obtain

$$
E^+_{j,j+1}(1/w_1, P)E^+_{j,j+1}(1/w_2, P+< \bar{\epsilon}_j, h >)\xi_I
$$

$$
= \sum_{a \in I_j} \frac{c(u_a - v_2, P_{j,j+1} + \lambda_j - \lambda_{j+1}) \bar{c}(u_a - v_1, P_{j,j+1})}{b(u_a - v_2)} \prod_{k \in I_j} \bar{b}(u_{ka})^{-1} \prod_{l \in I_{j+1}} \bar{b}(u_{al})^{-1} \bar{\xi}_{(I^{'a})'}
$$

$$
F^+_{j,j+1}(1/w_2, P+< \bar{\epsilon}_{j+1}, h >)E^+_{j+1,j}(1/w_1, P)\xi_I
$$

$$
= \sum_{b \in I_{j+1}} \frac{c(u_b - v_2, P_{j,j+1} + \lambda_j - \lambda_{j+1}) \bar{c}(u_b - v_1, P_{j,j+1})}{b(u_b - v_2)} \prod_{k \in I_j} \bar{b}(u_{kb})^{-1} \prod_{l \in I_{j+1}} \bar{b}(u_{lb})^{-1} \bar{\xi}_{I_{j+1}, j}
$$

where we set $u_{ka} = u_k - u_a$ etc. Since $(I^{'a})' = (I^{'a})' \forall a \in I_j, \forall b \in I_{j+1}$, the second terms in (B.8) and (B.9) coincide each other. Hence we have

$$
\left( E^+_{j,j+1}(1/w_1, P)E^+_{j,j+1}(1/w_2, P+< \bar{\epsilon}_j, h >) - F^+_{j+1,j}(1/w_2, P+< \bar{\epsilon}_{j+1}, h >)E^+_{j+1,j}(1/w_1, P) \right) \xi_I
$$

$$
= \left( \sum_{a \in I_j} \frac{c(u_a - v_2, P_{j,j+1} + \lambda_j - \lambda_{j+1}) \bar{c}(u_a - v_1, P_{j,j+1})}{b(u_a - v_2)} \prod_{k \in I_j} \bar{b}(u_{ka})^{-1} \prod_{l \in I_{j+1}} \bar{b}(u_{al})^{-1} \right) \xi_I
$$

In the RHS of (B.5), noting $\Delta^{(n-1)}(h_{j,j+1})\xi_I = (\lambda_j - \lambda_{j+1})\xi_I$ we obtain

$$
K^+_j(1/w_2)K^+_j(1/w_2)^{-1} \bar{c}(v_{12}, P_{j,j+1}) b(-v_{12}) \xi_I
$$

$$
= \frac{\bar{c}(v_{12}, P_{j,j+1})}{b(-v_{12})} \prod_{a \in I_j} \bar{b}(u_a - v_2)^{-1} \prod_{b \in I_{j+1}} \bar{b}(v_2 - u_b)^{-1} \xi_I,
$$

(B.11)

$$
K^+_{j+1}(1/w_1)^{-1}K^+_j(1/w_1) \bar{c}(v_{12}, (P + \Delta^{(n-1)}(h))_{j,j+1}) b(-v_{12}) \xi_I
$$

$$
= \frac{\bar{c}(v_{12}, P_{j,j+1} + \lambda_j - \lambda_{j+1})}{b(-v_{12})} \prod_{a \in I_j} \bar{b}(u_a - v_1)^{-1} \prod_{b \in I_{j+1}} \bar{b}(v_1 - u_b)^{-1} \xi_I.
$$

(B.12)
Hence

$$\text{LHS} - \text{RHS} = \left( \sum_{a \in I_j} \frac{c(u_{a,2}, P_{j,j+1} + \lambda_j - \lambda_{j+1})}{b(u_{a,2})} \frac{c(u_{a,1}, P_{j,j+1})}{b(u_{a,1})} \prod_{k \in I_j} b(u_{ka})^{-1} \prod_{l \in I_{j+1}} b(u_{al})^{-1} \right. \left. - \sum_{b \in I_{j+1}} \frac{c(u_{b,2}, P_{j,j+1} + \lambda_j - \lambda_{j+1})}{b(u_{b,2})} \frac{c(u_{b,1}, P_{j,j+1})}{b(u_{b,1})} \prod_{k \in I_j} b(u_{kb})^{-1} \prod_{l \in I_{j+1}} b(u_{bl})^{-1} \right) \xi_I,$$ \hspace{1cm} (B.13)

where we set \( u_{c,i} = u_c - v_i \) (\( c = a, b, i = 1, 2 \)). For \( a \in I_j \) let us regard \((\cdots)\) in the RHS as a function of \( u_a \) and denote it by \( F(u_a) \). It is not so hard to find

$$F(u_a + r) = F(u_a), \quad F(u_a + r \tau) = e^{-2\pi i r} F(u_a)$$

and all the residues at the poles \( u_a = v_i, v_j, u_k, u_l \) \((k \in I_j, k \neq a, l \in I_{j+1})\) vanish. Hence \( F(u_a) \) should be identically zero, because \( F(u_a)[u_a]/[u_a + 1] \) becomes a order 1 elliptic function unless \( F(-1) = 0 \).

\( \Box \)

\section{A Direct Check of Corollary 4.8 for (2.33) at \( k = 0 \)}

In this section we give a direct check that the action of the elliptic currents in Corollary 4.8 satisfies (2.33). We start from the following partial fraction expansion formula, which can be obtained from (4.2) in [47] by changing the multiplicative notation to the additive one and setting \( t = v, b_k = u_k - 1 \) \((k = 1, \cdots, m), b_l = a_l + 1 \) \((l = m + 1, \cdots, n)\) and \( b_{n+1} = v + 2m - n \).

\begin{align*}
\prod_{k=1}^{m} \frac{[v - u_k + 1]}{[v - u_k]} \prod_{l=m+1}^{n} \frac{[v - u_l - 1]}{[v - u_l]} \\
= \sum_{a=1}^{n} \frac{[v - u_a + 2m - n]}{[2m - n][v - u_a]} \prod_{k=1}^{m} \frac{[u_a - u_k + 1]}{[u_a - u_k]} \prod_{l=m+1}^{n} \frac{[u_a - u_l - 1]}{[u_a - u_l]}.
\end{align*}

\hspace{1cm} (C.1)

Let us denote the LHS of (C.1) by \( F(v) \). Then

\[ \text{Res}_{v=u_a} F(v)dv = \frac{1}{[0]'\prod_{k=1}^{m} \frac{[u_a - u_k + 1]}{[u_a - u_k]} \prod_{l=m+1}^{n} \frac{[u_a - u_l - 1]}{[u_a - u_l]}}. \]

Then from (4.14), we obtain
Lemma C.2.

\[ F(v)|_+ - F(v)|_- = \sum_{a=1}^{n} \delta(z_a/w) \text{Res}_{v=u_a} F(v) dv, \]

where \( z_a = q^{2a}, w = q^2 \).

Now let us check the relation [2.33].

1) the cases \(|i - j| > 1 \) and \( i = j + 1 \): It is obvious that \((I^a)' = (I^b)'a, \forall a \in I_j, \forall b \in I_{i+1} \). Hence \([E_i(1/w_1), F_j(1/w_2)]\xi_I = 0 \).

2) the case \( i = j - 1 \): It is easy to show \((I^a)' = (I^b)'a, \forall a, b \in I_j, (a \neq b) \). Then

\[ E_{j-1}(q^{j-N}/w_1)F_j(q^{j-N+1}/w_2)\xi_I = C e^{Q_{a_j-1}} \sum_{a\neq b} \delta(z_a/w_2)\delta(z_b/w_1) \prod_{k \in I_j, k \neq a} \bar{b}(u_{ka})^{-1} \prod_{l \in (I^a)'j, l \neq b} \bar{b}(u_{bl})^{-1} \xi_{(I^a)'a}, \]

whereas

\[ F_j(q^{j-N+1}/w_2)E_{j-1}(q^{j-N}/w_1)\xi_I = C e^{Q_{a_j-1}} \sum_{a\neq b} \delta(z_a/w_2)\delta(z_b/w_1) \prod_{k \in (I^b)'j, k \neq a} \bar{b}(u_{ka})^{-1} \prod_{l \in I_j, l \neq b} \bar{b}(u_{bl})^{-1} \xi_{(I^b)'a}. \]

Here we set

\[ C = \mu \mu^* \left( \frac{[1]}{[0]} \right)^2 = -\frac{\rho}{q - q^{-1}} \frac{[1]}{[0]}, \]

(C.2)

The last equality follows from [2.38]. Noting

\[ \prod_{l \in I_j, l \neq b} \bar{b}(u_{bl})^{-1} = \bar{b}(u_{ba})^{-1} \prod_{l \in (I^a)'j, l \neq b} \bar{b}(u_{bl})^{-1}, \]

we obtain \([E_{j-1}(q^{j-N}/w), F_j(q^{j-N+1}/w)]\xi_I = 0 \).

3) the case \( i = j \): We have

\[ E_j(q^{j-N+1}/w_1)F_j(q^{j-N+1}/w_2)\xi_I \]

\[ = C e^{-Q_{a_j}} \left( \sum_{a \in I_j} \delta(z_a/w_2) \delta(z_a/w_1) \prod_{k \in I_j, k \neq a} \bar{b}(u_{ka})^{-1} \prod_{l \in I_{j+1}, l \neq b} \bar{b}(u_{bl})^{-1} \times \bar{b}(u_{ba})^{-1} \xi_{(I^a)'a} \right. \]

\[ + \sum_{a \in I_j} \delta(z_a/w_2) \delta(z_a/w_1) \prod_{k \in I_j, k \neq a} \bar{b}(u_{ka})^{-1} \prod_{l \in I_{j+1}} \bar{b}(u_{bl})^{-1} \xi_I \), \quad (C.3) \]
Let us set the eigenvalue of $H$

Hence we obtain

Then from Lemma C.2 and (C.2), we obtain the desired formula.

Then we have for $a \in I_j$ and $b \in I_{j+1}$,

Therefore one can write (C.5) as

Then from Lemma C2 and C2, we obtain the desired formula.

Since $(I^a)^b = (I^b)^a \forall a \in I_j, \forall b \in I_{j+1}$, the first terms in (C.3) and (C.4) coincide each other. Hence we obtain

Let us set the eigenvalue of $H_j^+(q^{-N+1}/w)$ on $\xi_I$ by $g h_j(v)$ with $g$ in (2.38) i.e.

Then we have for $a \in I_j$ and $b \in I_{j+1}$,

Therefore one can write (C.5) as

Then from Lemma C2 and C2, we obtain the desired formula.

$$[E_j(q^j/w_1), F_j(q^j/w_2)]\xi_I = -\frac{\theta}{q - q^{-1}} \delta(w_1/w_2) \sum_{c \in I_j \cup I_{j+1}} \delta(z_c/w_1) \text{Res} h_j(v) d\nu_1 \xi_I.$$ (C.6)
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