On a conjecture of Demailly and new bounds on Waldschmidt constants in $\mathbb{P}^N$

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Abstract

In the present note we prove a conjecture of Demailly for finite sets of sufficiently many very general points in projective spaces. This gives a lower bound on Waldschmidt constants of such sets. Waldschmidt constants are asymptotic invariants of subschemes receiving recently considerable attention [1], [2], [6], [11], [14].

Keywords point configurations, Waldschmidt constants, symbolic powers, projective space

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1 Introduction

In 1980 Jean-Charles Moreau proved the following version of the Schwarz Lemma in several complex variables, [15, Theorem 1.1].

Theorem 1.1 (Moreau). Let $Z \subset \mathbb{C}^N$ be a finite set of points. For every positive $m \in \mathbb{Z}$, there exists a real number $r_m(Z)$ such that for all $R \geq r \geq r_m(Z)$ and for all holomorphic functions $f$ vanishing to order at least $m$ at every point of $Z$ there is

$$|f|_r \leq \left(\frac{2e^{N/2}r}{R}\right)^{\alpha(mZ)} |f|_R,$$

where $|f|_s = \sup_{|z| \leq s} |f(z)|$ and $\alpha(kW)$ is the least degree of a polynomial vanishing at all points of a finite set $W$ to order at least $k$.

The number $\alpha(mZ)$ in the Theorem is optimal, i.e., the statement fails with any larger number. Several authors, in particular Chudnovsky, were interested in obtaining an exponent in [11] independent of $m$. To this end one defines the following quantity [16].

Definition 1.2 (Waldschmidt constant). Let $Z \subset \mathbb{C}^N$ be a finite set of points. The Waldschmidt constant of $Z$ is the real number

$$\hat{\alpha}(Z) = \lim_{m \to \infty} \frac{\alpha(mZ)}{m}.$$

The existence of the limit has been showed by Chudnovsky [3, Lemma 1]. It is well known that $\hat{\alpha}(Z) = \inf_{m \geq 1} \frac{\alpha(mZ)}{m}$. Chudnovsky established also the following fundamental fact, see [3, Theorem 1].
Theorem 1.3. Let $Z \subset \mathbb{C}^N$ be a finite set of points. Then
\[
\hat{\alpha}(Z) \geq \frac{\alpha(Z)}{N}.
\]
(2)

The bound in (2) can now be easily derived from the seminal results of Ein, Lazarsfeld and Smith [9]. We discuss it briefly below in Section 2. Chudnovsky suspected that the bound in (2) is not optimal and raised the following Conjecture, see [3, Problem 1].

Conjecture 1.4 (Chudnovsky). Let $Z \subset \mathbb{C}^N$ be a finite set of points. Then
\[
\hat{\alpha}(Z) \geq \frac{\alpha(Z) + N - 1}{N}.
\]
(3)

This has been subsequently generalized by Demailly, see [4, p. 101].

Conjecture 1.5 (Demailly). Let $Z \subset \mathbb{C}^N$ be a finite set of points. Then for all $m \geq 1$
\[
\hat{\alpha}(Z) \geq \frac{\alpha(mZ) + N - 1}{m + N - 1}.
\]
(4)

Of course, for $m = 1$ Demailly’s Conjecture reduces to that of Chudnovsky.

There has been recently considerable progress on the Chudnovsky Conjecture for general points obtained independently by Dumnicki and Tutaj-Gasińska in [8] and Fouli, Mantero and Xie in [12].

Our main result here is the following.

Main Theorem. The Demailly’s Conjecture (4) holds for $s \geq (m + 1)^N$ very general points in $\mathbb{P}^N$.

Remark 1.6. For $m = 1$ we recover the aforementioned result [8] that the Chudnovsky Conjecture holds for $s \geq 2^N$ very general points in $\mathbb{P}^N$.

Throughout the paper we work over the field $\mathbb{C}$ of complex numbers.

2 Around the Chudnovsky Conjecture

Esnault and Viehweg using methods of complex projective geometry have proved the following useful result, see [10, Inégalité A].

Theorem 2.1 (Esnault – Viehweg). Let $I$ be a radical ideal of a finite set of points in $\mathbb{P}^N$ with $N \geq 2$. Let $m \leq k$ be two integers. Then
\[
\frac{\alpha(I^{(m)}) + 1}{m + N - 1} \leq \frac{\alpha(I^{(k)})}{k},
\]

in particular
\[
\frac{\alpha(I^{(m)}) + 1}{m + N - 1} \leq \hat{\alpha}(I).
\]
(5)

For $N = 2$ the inequality in (5) establishes Demailly’s Conjecture in $\mathbb{P}^2$.

Corollary 2.2. Conjecture (4) holds for arbitrary finite sets of points in $\mathbb{P}^2$. 
Around 2000 Ein, Lazarsfeld and Smith established a uniform containment result for symbolic and ordinary powers of homogeneous ideals. For the purpose of this paper we recall here a somewhat simplified version of their general result.

**Definition 2.3 (Symbolic power).** Let $Z = \{P_1, \ldots, P_s\}$ be a finite set of points in $\mathbb{P}^N$. For an algebraic set $W \subset \mathbb{P}^N$, let $I(W)$ be its homogeneous defining ideal. Then

$$I(Z) = I(P_1) \cap \ldots \cap I(P_s)$$

and for a positive integer $m$

$$I^{(m)}(Z) = I(P_1)^m \cap \ldots \cap I(P_s)^m$$

is the $m$th symbolic power of $I(Z)$.

**Theorem 2.4 (Ein – Lazarsfeld – Smith).** Let $Z$ be a finite set of points in $\mathbb{P}^N$ and let $I = I(Z)$ be its defining ideal. Then the containment

$$I^{(m)} \subset I^r$$

holds for all $m \geq Nr$.

Theorem 1.3 is an immediate consequence of Theorem 2.4. Indeed, let $Z \subset \mathbb{C}^N \subset \mathbb{P}^N$ be a finite set of points with the defining ideal $I = I(Z)$. Then

$$\alpha(I^{(Nr)}) \geq \alpha(I^r) = r\alpha(I)$$

follows from the containment in (6). Hence

$$\frac{\alpha(I^{(Nr)})}{Nr} \geq \frac{\alpha(I)}{N}$$

for all $r \geq 1$. Passing with $r$ to infinity we obtain

$$\hat{\alpha}(I) \geq \frac{\alpha(I)}{N}.$$  

### 3 A combinatorial inequality

In this section we prove the following auxiliary fact.

**Lemma 3.1.** For all $N \geq 3$, $m \geq 1$ and $k \geq m + 1$ there is

$$\binom{k(m + N - 1) + 1}{N} \geq \binom{m + N - 1}{N}(k + 1)^N.$$  

**Proof.** It is convenient to abbreviate $q := m + N - 1$. The claim in the Lemma is equivalent to the following inequality

$$(kq + 1) \cdot (kq) \cdot \ldots \cdot (kq + 2 - N) \geq q \cdot (q - 1) \cdot \ldots \cdot m \cdot (k + 1)^N. \quad (7)$$

We will group factors in (7) and show that

$$(kq + 1 - i)(kq + 2 - N + i) \geq (q - i)(m + i)(k + 1)^2. \quad (8)$$
holds for all \( i = 0, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor \). To this end we define
\[
u(N, m, k, i) := (kq + 1 - i)(kq + 2 - N + i) - (q - i)(m + i)(k + 1)^2 \tag{9}
\]
and show that this function is non-negative.

**Reduction 1.** In the first step, we will show that the difference function
\[
dk(N, m, k, i) = u(N, m, k + 1, i) - u(N, m, k, i)
\]
is non-negative. Taking this for granted, in order to show that the function in (9) is non-negative, it suffices to check it for the least allowed value of \( k \), i.e., for \( k = m + 1 \). In other words the claim in (9) reduces to the claim that the function
\[

\nu(N, m, i) := u(N, m, m + 1, i) \tag{10}
\]
is non-negative for all \( N, m, i \) in the given range.

Turning to the proof of the Reduction 1 claim, since the difference function is linear in \( k \), it suffices to show
\begin{enumerate}
  \item the leading coefficient of \( dk(N, m, k, i) \) treated as a polynomial in \( k \) is positive
  \item the function is non-negative for \( k = m + 1 \).
\end{enumerate}
The leading coefficient in a) can be written as
\[
(i^2 - Ni + i + \frac{1}{3}N^2) + (\frac{2}{3}N^2 + mN - m - 2N + 1).
\]
It is elementary to check that the terms in brackets are non-negative.

Evaluating \( dk(N, m, m + 1, i) \) we obtain the following expression
\[
2Nm^2 - 4m^2 + 2N^2 - 4mN - 2imN + 2i^2m + 2im + 4m + 2N^2 - 2N + 5i - 5iN + 5i^2 \tag{11}
\]
The term \((2N - 4)m^2\) is positive. The remaining summands in (11) can be rearrange in the following way
\[
(2m + 2) \cdot N^2 - (4m + 2 + 5i + 2im)N + (2im + 2i^2m + 5i + 4m + 5i^2). \tag{12}
\]
This is a quadratic function in \( N \) whose discriminant
\[
\Delta = 4 - 16m^2 - 12i^2m^2 - 16m - 8im - 36i^2m - 20i - 15i^2
\]
is negative for all \( m \) and \( i \) in the allowed range. Thus the expression in (12) is positive. This concludes the proof of Reduction 1.

We study now the function \( \nu(N, m, i) \) defined in (10). Our approach is similar. We show in

**Reduction 2.** That the difference function
\[
dN(N, m, i) = \nu(N + 1, m, i) - \nu(N, m, i)
\]
is non-negative. This follows immediately from the following presentation of this function
\[
dN(N, m, i) = m^3 + (\frac{N - 1}{2} - i)m^2 + (2N - 4i - 2)m + (3\frac{N - 1}{2}m^2 - 3i + 1) \tag{13}
\]
Indeed, all terms in brackets in (13) are non-negative.

Hence, it is enough to check that the function in (9) is non-negative for \( N = 3 \) (and \( k = m + 1 \)). But this is immediate since
\[

\nu(3, m, i) = (m + 3)(m + 1)(i - 1)^2.
\]
This ends the proof of the Lemma. \( \square \)
4 A proof of the Main Theorem

In this section we prove the Main Theorem. First we recall from [8, Theorem 3] the following crucial observation.

**Theorem 4.1** (Lower bound on Waldschmidt constants). Let $Z$ be a set of $s$ very general points in $\mathbb{P}^N$. Then

$$\hat{\alpha}(Z) \geq \lfloor \sqrt[N]{s} \rfloor.$$  

Turning to the proof of the Main Theorem, let $Z$ be a set of $s \geq (m+1)^N$ very general points in $\mathbb{P}^N$. Since the result holds in $\mathbb{P}^2$ by Corollary 2.2, we may assume here $N \geq 3$. There exists a unique integer $k \geq m+1$ such that

$$k^N \leq s < (k+1)^N.$$  

By Theorem 4.1 we have $\hat{\alpha}(Z) \geq k$.

We claim that there exists a form of degree $k(m+N-1) - N + 1$ vanishing to order at least $m$ at every point of $Z$. This follows from the dimension count. Indeed, we need to show that

$$\binom{(k(m+N-1)+1)}{N} \geq \binom{(N+m-1)}{N} \cdot s$$  

holds. Since $s \leq (k+1)^N-1$, it is enough to show that

$$\binom{k(m+N-1)+1}{N} \geq \binom{N+m-1}{N} \cdot (k+1)^N$$  

holds. This is exactly the statement of Lemma 3.1.

It follows that

$$\alpha(mZ) \leq k(m+N-1) - N + 1.$$  

But then

$$\frac{\alpha(mZ) + N - 1}{m + N - 1} \leq k \leq \hat{\alpha}(Z)$$  

and we are done. \qed

We conclude this note with examples showing that the inequality in Conjecture 1.5 cannot be improved in general. To this end we recall first the notion of star configurations, see [13].

**Definition 4.2** (Star configuration of points). We say that $Z \subset \mathbb{P}^N$ is a star configuration of degree $d$ if $Z$ consists of all intersection points of $d \geq N$ general hyperplanes in $\mathbb{P}^N$. By intersection points we mean the points which belong to exactly $N$ of given $d$ hyperplanes.

The assumption general in Definition means that any $N$ of $d$ given hyperplanes meet in a single point and there is no point belonging to $N+1$ or more hyperplanes. In particular a star configuration of degree $d$ consists of exactly $\binom{d}{N}$ points.

**Example 4.3.** Let $Z \subset \mathbb{P}^N$ be a star configuration of degree $d$. Then it is easy to check that for any $k \geq 1$

$$\alpha((1 + kN)Z) = (k + 1)d - N + 1$$  

and hence

$$\frac{d}{N} = \hat{\alpha}(Z) = \frac{\alpha((1 + kN)Z) + N - 1}{1 + kN + N - 1},$$  

so that there is equality in (4) for infinitely many values of $m = 1 + kN$.  

The second example is in a sense more exotic.

**Example 4.4.** Let $Z$ be the set of points in $\mathbb{P}^2$ defined by the ideal

\[ I = \langle x(y^3 - z^3), y(z^3 - x^3), z(x^3 - y^3) \rangle. \]

The $Z$ is the union of points

- $P_1 = (1 : 0 : 0)$,
- $P_2 = (0 : 1 : 0)$,
- $P_3 = (0 : 0 : 1)$,
- $P_4 = (1 : 1 : 1)$,
- $P_5 = (1 : \varepsilon : \varepsilon^2)$,
- $P_6 = (1 : \varepsilon^2 : \varepsilon)$,
- $P_7 = (\varepsilon : 1 : 1)$,
- $P_8 = (1 : \varepsilon : 1)$,
- $P_9 = (1 : 1 : \varepsilon)$,
- $P_{10} = (\varepsilon^2 : 1 : 1)$,
- $P_{11} = (1 : \varepsilon^2 : 1)$,
- $P_{12} = (1 : 1 : \varepsilon^2)$.

which together with lines

- $L_1 : x - y = 0$,
- $L_2 : y - z = 0$,
- $L_3 : z - x = 0$,
- $L_4 : x - \varepsilon y = 0$,
- $L_5 : y - \varepsilon z = 0$,
- $L_6 : z - \varepsilon x = 0$,
- $L_7 : x - \varepsilon^2 y = 0$,
- $L_8 : y - \varepsilon^2 z = 0$,
- $L_9 : z - \varepsilon^2 x = 0$.

form a 123,94 configuration, see [7].

The Waldschmidt constant $\hat{\alpha}(Z) = 3$ has been computed in passing in the proof of Theorem 2.1 in [5]. In fact the proof shows that

\[ \alpha(3kZ) = 9k \quad (14) \]

for all $k \geq 1$. We claim now that

\[ \alpha((3k + 2)Z) = 9k + 8 \quad (15) \]

for all $k \geq 0$. For $k = 0$ this can be checked computing $I^{(2)}$ explicitly. Clearly (14) implies

\[ \alpha((3k + 2)Z) \leq 9k + 8. \]

Indeed, any partial derivative of a polynomial computing $\alpha(3(k + 1)Z)$ has degree $9k + 8$ and the right order of vanishing at $Z$.

Assume that there is a $k \geq 2$ such that

\[ \alpha((3k + 2)Z) \leq 9k + 7. \]

Then there is a divisor $D$ of degree $9k + 7$ vanishing to order at least $3k + 2$ at every point $P_i$ of $Z$. Intersecting $D$ with any of the lines $L_j$ for $j = 1, \ldots, 9$, we conclude by Bezout Theorem that $L_j$ is a component of $D$. Hence there exists a divisor $D' = D - \sum_{j=1}^{9} L_j$ of degree $9(k - 1) + 7$ vanishing to order at least $3(k - 1) + 2$ at every point of $Z$. Repeating this argument $k$ times we get a contradiction with $\alpha(2Z) = 8$.

Now, for $m = 3k + 2$ with $k \geq 1$ we obtain the equality in (4).

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