ROTATIONAL SURFACES WITH SECOND FUNDAMENTAL FORM OF CONSTANT LENGTH

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ABSTRACT. We obtain an infinite family of complete non embedded rotational surfaces in $\mathbb{R}^3$ whose second fundamental forms have length equal to one at any point. Also we prove that a complete rotational surface with second fundamental form of constant length is either a round sphere, a circular cylinder or, up to a homothety and a rigid motion, a member of that family. In particular, the round sphere and the circular cylinder are the only complete embedded rotational surfaces in $\mathbb{R}^3$ with second fundamental form of constant length.

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1. INTRODUCTION

A surface $S$ in the 3-dimensional Euclidean space is called a Weingarten surface if there exists some relation

$$W(\lambda_1, \lambda_2) = 0,$$  \hspace{1cm} (1.1)

among its principal curvatures $\lambda_1$ and $\lambda_2$. Since the principal curvatures of a surface can always be determined from its mean curvature $H$ and its Gaussian curvature $K$, and vice-versa, the relation (1.1) can always be rewritten as a relation $U(H, K) = 0$.

Date: December 21, 2018.

2010 Mathematics Subject Classification. Primary 53A05, 53C42 ; Secondary 53C40, 14Q10.

Key words and phrases. Rotational surface; length of the second fundamental form; Weingarten surface.

All authors are partially supported by CNPq(Brasil). 

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Weingarten surfaces is a classical topic in Differential Geometry that began with the works of Weingarten in the middle of the 19th century \([20, 21]\) and that has been a subject of interest for many authors since then (see Chern \([2]\), Hartman and Winter \([7]\), Hopf \([8]\), Voss \([19]\), Rosenberg and Sá Earp \([18]\), Kühnel and Steller \([11]\), López \([12, 13, 14, 15]\), to name just a few).

Minimal surfaces, surfaces with constant mean curvature and surfaces with constant Gaussian curvature are classical examples of Weingarten surfaces. Another well known class (generalizing the previous ones) is that of the linear Weingarten surfaces, i.e., Weingarten surfaces verifying either the relation

\[ W(\lambda_1, \lambda_2) = a\lambda_1 + b\lambda_2 = c \]  

or the relation

\[ U(H, K) = aH + bK = c, \]  

where \(a, b, c \in \mathbb{R}\) are constants such that \(a\) and \(b\) do not vanish simultaneously.

The complete classification of Weingarten surfaces is far from being achieved. The existent results deal mostly with the linear case, sometimes making use of additional topological/geometric hypothesis and/or working with important subclasses of surfaces such as revolution surfaces \([8, 14, 11, 18]\), tubes along curves and cyclic surfaces \([13, 15]\), ruled surfaces and helicoidal surfaces \([10]\), translation surfaces \([3, 16]\), etc. In general, the approaches used to treat the linear case do not apply to the non-linear case. Therefore, results concerning non-linear Weingarten surfaces are more rare \([18, 11]\).

In this paper we study rotational surfaces in the 3-dimensional Euclidean space whose second fundamental forms have constant length (recall that the squared length \(|A|^2\) of the second fundamental form of a surface in \(\mathbb{R}^3\) is defined as the trace of \(A^2\), where \(A\) is its shape operator). In other words, we study rotational Weingarten surfaces that satisfy the non-linear relation

\[ W(\lambda_1, \lambda_2) = (\lambda_1)^2 + (\lambda_2)^2 = c, \]  

or equivalently

\[ U(H, K) = 4H^2 - 2K = c, \]  

for some \(c > 0\).

In this case we prove the following result (notice that since the property of having constant \(|A|\) is invariant by homotheties in \(\mathbb{R}^3\), we can assume without loss of generality that \(c = 1\)):
Theorem 1.1. There are two infinite families $F_1$ and $F_2$ of complete non embedded rotational surfaces in $\mathbb{R}^3$ with $|A| = 1$. The family $F_1$ is one parameter and its members are periodic $C^\infty$ surfaces, while the members of $F_2$ are $C^3$ surfaces. Moreover, any complete rotational $C^2$ surface with $|A| = 1$ is either a round sphere of radius $\sqrt{2}$, a circular cylinder of radius 1 or, up to a rigid motion in $\mathbb{R}^3$, a member of one of the two families.

Corollary 1.2. The only complete embedded rotational $C^2$-surfaces in $\mathbb{R}^3$ with second fundamental form of constant length are the round sphere and the circular cylinder.

Non trivial examples of compact surfaces (embedded or immersed) with second fundamental form of constant length are unknown by the authors. In view of Theorem 1.1 and Corollary 1.2, one can then formulate the following question:

Question 1.3. - Is there a compact surface, other than the round sphere, embedded/immersed in the Euclidean 3-space whose second fundamental form has constant length?

It is worth to point out that the above question has a negative answer in the class of the compact surfaces with positive Gaussian curvature [1, Theorem 5 on p. 347 and Section 4] (see also [9] or [5, Theorem 2.3]).

The importance of the class of hypersurfaces whose second fundamental forms have constant length goes beyond the context of Weingarten surfaces. We mention [4] (see also [6]), where it is proved that the generalized cylinders are the only complete embedded self-shrinkers in $\mathbb{R}^3$ with polynomial volume growth whose second fundamental forms have constant length.

The paper is organized as follows. In Section 2, we prove that the profile curve of any rotational surface in $\mathbb{R}^3$ with $|A| \equiv 1$ is convex, i.e., its signed curvature does not change signal. This fact enable us to reduce the study of rotational surfaces with $|A| \equiv 1$ to the study of the trajectories of a certain vector field in the plane. The study of this vector field is made in Section 3. Finally, we prove Theorem 1.1 in Section 4.

Acknowledgments. The authors would like to thank Thiago de Melo (IGCE-UNESP) for helpful conversations during the preparation of this work and for his help with the figures.
2. Convexity of the profile curves

Our goal in this section is to prove that the profile curve $C$ of any rotational $C^2$-surface $M \subset \mathbb{R}^3$, whose shape operator $A$ has length $|A| \equiv 1$, is convex. By applying a rigid motion of $\mathbb{R}^3$ if necessary, we can assume that $C$ is contained in the $xz$-plane and that the axis of revolution is the $x$-axis.

Let $\alpha(t) = (x(t), 0, z(t))$, $t \in (a, b)$, be a parametrization of $C$ such that $||\alpha'(t)|| = 1$ and $z(t) > 0$ for all $t$, and let $\theta : (a, b) \rightarrow \mathbb{R}$ be a continuous (and, hence, of class $C^1$) function satisfying

$$\alpha'(t) = (x'(t), 0, z'(t)) = (\cos \theta(t), 0, \sin \theta(t)), \quad t \in (a, b). \quad (2.1)$$

It is easy to see that the function $\theta$ satisfying Eq. (2.1) is unique up to an integer multiple of $2\pi$. The principal curvatures of $M$ are given by (see e.g., [14])

$$\lambda_1(t) = \theta'(t), \quad \lambda_2(t) = -\frac{\cos \theta(t)}{z(t)}. \quad (2.2)$$

Since $|A|^2 = \lambda_1^2 + \lambda_2^2 \equiv 1$ by hypothesis, one then has

$$\theta'(t)^2 + \frac{\cos^2 \theta(t)}{z^2(t)} = 1, \quad t \in (a, b). \quad (2.3)$$

As we observed in the introduction, the proof of Theorem 1.1 will be based on a careful study of the trajectories of a suitable vector field in the plane. The fundamental property of the profile curves that makes this approach possible is provided by the following proposition (recall that the signed curvature of $\alpha$ is $\theta'$):

**Proposition 2.1.** The function $\theta : (a, b) \rightarrow \mathbb{R}$ is monotone.

Before proceeding to the proof of Proposition 2.1, let us explain how to relate profile curves with the trajectories of a specific vector field.

Let $\alpha$ and $\theta$ be as above. Assuming that $\theta$ is monotone, reparametrizing $\alpha$ we can assume that $\theta' \geq 0$. Then, by Eq. (2.1) and (2.3),

$$\begin{cases} 
\theta'(t) = \sqrt{1 - \frac{\cos^2 \theta(t)}{z^2(t)}}, \\
\zeta'(t) = \sin \theta(t).
\end{cases} \quad (2.4)$$

Let $X : \Omega \rightarrow \mathbb{R}^2$ be the (smooth) vector field defined by

$$X(\theta, z) = \left(\sqrt{1 - \frac{\cos^2 \theta}{z^2}}, \sin \theta\right), \quad (2.5)$$
where \( \Omega = \{ (\theta, z) \in \mathbb{R}^2 : z > |\cos \theta| \} \). As long as \( z(t) > |\cos \theta(t)| \), the system in Eq. (2.4) can be rewritten as

\[
(\theta'(t), z'(t)) = X(\theta(t), z(t)),
\]

and so the curve \( t \mapsto (\theta(t), z(t)) \) is a trajectory of \( X \). A representation of \( \Omega \) and the vector field \( X \) can be seen in Figure 1.

![Figure 1](image.png)

**Figure 1.** Graphic representation of \( \Omega \) and the vector field \( X \) for \( \theta \in [0, 2\pi] \).

Conversely, given a trajectory \( \varphi(t) = (\theta(t), z(t)) \), \( t \in (a, b) \), of \( X \) and \( t_0 \in (a, b) \), consider the curve \( \alpha(t) = (x(t), 0, z(t)) \), \( t \in (a, b) \), where

\[
x(t) = \int_{t_0}^{t} \cos \theta(s) ds.
\]

Using Eq. (2.2), (2.5) and (2.6) one easily proves that the surface in \( \mathbb{R}^3 \) obtained by the rotation of the image of \( \alpha \) around the x-axis satisfies \( \|A\| \equiv 1 \).

In the proof of Proposition 2.1, as well as in the proofs of later results, we will use the following technical lemma. In its statement, \( \alpha(t) = (x(t), 0, z(t)) \) and \( \theta(t) \) are as in the beginning of this section.

**Lemma 2.2.** For any \( t \in (a, b) \), the following assertions hold:

(i) \( \theta'(t) = 0 \) if, and only if, \( z(t) = 1 \) and \( \sin \theta(t) = 0 \).

(ii) If \( z(t) < 1 \) then \( |z'(t)| \geq |\theta'(t)| \).

(iii) If \( \theta'(t) = 0 \), then there exists \( \delta > 0 \) such that \( z(s) \geq 1 \), \( s \in (t - \delta, t + \delta) \).
Proof. (i) If $\theta'(t) = 0$ then, by Eq. (2.3), the function $\cos^2 \theta/z^2$ attains a maximum at $t$. Hence,

$$0 = \left[ \frac{\cos^2 \theta}{z^2} \right]'(t) = \left[ \frac{-2 \theta' \cos \theta \sin z^2 - 2 \cos^2 \theta z z'}{z^4} \right](t) = -2 \cos^2 \theta(t) z'(t) / z^3(t).$$

Since $\cos \theta(t) \neq 0$, one obtains from the above equality and Eq. (2.1) that

$$\sin \theta(t) = z'(t) = 0.$$ 

Using this information in Eq. (2.3), one concludes that $z(t) = 1$. The converse is an immediate consequence of Eq. (2.3).

(ii) From Eq. (2.3) and $z(t) < 1$ one obtains

$$1 = \theta'(t)^2 + \frac{\cos^2 \theta(t)}{z^2(t)} \geq \theta'(t)^2 + \cos^2 \theta(t),$$

and so

$$\theta'(t)^2 \leq 1 - \cos^2 \theta(t) = \sin^2 \theta(t) = z'(t)^2.$$ 

The conclusion now follows by taking square roots in the above inequality.

(iii) Supposing, by contradiction, that the conclusion does not hold, we have $z(t_n) < 1$ for some sequence $(t_n)$ that converges to $t$. Since, by (i), $t_n \neq t$ for all $n$, passing to a subsequence and reparametrizing $\alpha$ if necessary, one can assume that $t_n < t$, for all $n$.

We claim that

$$z(s) < 1, \quad s \in (a, t). \quad (2.7)$$

Indeed, if $z(c) \geq 1$ for some $c \in (a, t)$ then, since $t_n \to t$ and $z(t_n) < 1$ for all $n$, there is $d \in (c, t)$ such that

$$z(d) = \inf \{ z(s) : c \leq s \leq t \} < 1. \quad (2.8)$$

Hence, $z'(d) = 0$. On the other hand, from (i), (ii) and Eq. (2.8) one obtains $|z'(d)| \geq |\theta'(d)| > 0$. This contradiction proves Eq. (2.7).

By (i), (ii) and Eq. (2.7),

$$\theta'(s) \neq 0 \quad \text{and} \quad z'(s) \neq 0, \quad s \in (a, t). \quad (2.9)$$

Then, by Eq. (2.7) and $z(t) = 1$,

$$\sin \theta(s) = z'(s) > 0, \quad s \in (a, t). \quad (2.10)$$

Since $\sin \theta(t) = 0$ by (i), we have two possibilities:

a) $\theta(t) = 2k\pi$, for some $k \in \mathbb{Z}$.

b) $\theta(t) = (2k + 1)\pi$, for some $k \in \mathbb{Z}$. 

Assuming a), from Eq. (2.10) one obtains
\[ \theta(t) + \pi > \theta(s) > \theta(t), \quad s \in (a, t). \]
Then, by the first inequality of Eq. (2.9),
\[ \theta'(s) < 0, \quad s \in (a, t). \quad (2.11) \]
From Eq. (2.7), (2.10), (2.11) and (ii), we obtain
\[ z'(s) \geq -\theta'(s), \quad s \in (a, t). \]
Hence, for fixed \( s_1 \in (a, t) \), we have
\[ 1 = z(t) = z(s_1) + \int_{s_1}^{t} z'(s)ds \geq z(s_1) - \int_{s_1}^{t} \theta'(s)ds = z(s_1) + \theta(s_1) - \theta(t), \]
and so
\[ 1 - z(s_1) \geq \theta(s_1) - \theta(t). \]
It now follows from Eq. (2.11) and the fact that the cosine function is decreasing on \([0, \pi]\), that
\[ \cos(1 - z(s_1)) \leq \cos(\theta(s_1) - \theta(t)) \]
\[ = \cos \theta(s_1) \cos \theta(t) + \sin \theta(s_1) \sin \theta(t) \]
\[ = \cos \theta(s_1). \quad (2.12) \]
Since, by Eq. (2.3) and (2.11), \( z(s_1) > |\cos \theta(s_1)| \), inequalities Eq. (2.7) and (2.12) imply
\[ \cos(1 - z(s_1)) < z(s_1) < 1, \quad (2.13) \]
contradicting the fact, easily verified, that \( \cos(1 - x) > x \), for all \( x \in [0, 1] \).

A reasoning entirely similar to the above shows that b) cannot occur either. Hence, \( z(s) \geq 1 \) on a neighbourhood of \( t \). \( \square \)

**Proof of Proposition 2.1:** Suppose, by contradiction, that \( \theta \) is not monotone. Then there exists \( t_1 < t_2 < t_3 \in (a, b) \) such that either i) or ii) below holds:

i) \( \theta(t_1) < \theta(t_2) \) and \( \theta(t_2) > \theta(t_3) \).

ii) \( \theta(t_1) > \theta(t_2) \) and \( \theta(t_2) < \theta(t_3) \).

Assuming i), we have
\[ \lambda := \sup \{\theta(t) : t \in [t_1, t_3]\} > \max \{\theta(t_1), \theta(t_3)\}. \]
Define
\[ \xi := \inf \{t \geq t_1 : \theta(t) = \lambda\}, \quad \eta := \sup \{t \leq t_3 : \theta(t) = \lambda\}. \]
Since \( \theta \) attains a local maximum at \( \xi \) and at \( \eta \), we have \( \theta'(\xi) = \theta'(\eta) = 0 \). Then, by Lemma 2.2 (i), \( z(\xi) = z(\eta) = 1 \) and \( \sin \theta(\xi) = \sin \theta(\eta) = 0 \). The latter implies that either \( \theta(\xi) = \theta(\eta) = 2k\pi \) or
\[ \theta(\xi) = \theta(\eta) = (2k + 1)\pi, \text{ for some } k \in \mathbb{Z}. \] If \( \theta(\xi) = \theta(\eta) = 2k\pi, \) one has \( \mu \in (\eta, t_3], \) such that

\[ z'(t) = \sin \theta(t) < 0, \quad t \in (\eta, \mu). \]

Then, \( z(t) < z(\eta) = 1, \) for all \( t \in (\eta, \mu), \) contradicting Lemma 2.2 (iii).

If \( \theta(\xi) = \theta(\eta) = (2k + 1)\pi, \) there exists \( \nu \in [t_1, \xi) \) such that

\[ z'(t) = \sin \theta(t) > 0, \quad t \in (\nu, \xi). \]

Then, \( z(t) < z(\xi) = 1, \) for all \( t \in (\nu, \xi), \) which also contradicts Lemma 2.2 (iii).

A reasoning entirely similar to the above shows that ii) can not occur either. Hence, the function \( \theta \) is monotone. \( \square \)

3. Phase portrait of the fundamental vector field

With the aim to prove Theorem 1.1, we study in this section the trajectories of the vector field \( X \) defined by Eq. (2.5). This study will be carried out through a series of technical lemmas.

Since the trajectories of \( X \) are invariant by horizontal translations by multiples of \( 2\pi \) (that is, if \( \varphi(t) = (\theta(t), z(t)) \) is a trajectory of \( X \) then so is the curve \( \psi(t) = (\theta(t) + 2n\pi, z(t)) \) for any \( n \in \mathbb{Z} \)), it is sufficient to consider the trajectories that pass through some point \( (\theta_0, z_0) \in \Omega \) such that \( 0 \leq \theta_0 \leq 2\pi. \)

**Lemma 3.1.** Let \( (\theta_0, z_0) \) be a point in \( \Omega \) such that \( 0 < \theta_0 < \pi \) and \( z_0 \leq 1. \) If \( \varphi(t) = (\theta(t), z(t)), \ t \in (a, b), \) is the maximal integral curve of \( X \) satisfying \( \varphi(0) = (\theta_0, z_0), \) then there exists \( c \in (0, b) \) such that \( z(c) > 1. \)

**Proof.** We can assume that \( z_0 < 1, \) for otherwise the conclusion follows immediately from \( z'(0) = \sin \theta(0) > 0. \) Suppose, by contradiction, that the conclusion does not hold. Then, by the definition of \( \Omega, \)

\[ \theta_0 < \theta(t) < \pi, \quad t \in (0, b), \quad (3.1) \]

and so

\[ z'(t) = \sin \theta(t) > 0 \text{ and } z(t) < 1, \quad t \in (0, b). \quad (3.2) \]

Let

\[ \theta_+ = \lim_{t \to b^-} \theta(t) \quad \text{and} \quad z_+ = \lim_{t \to b^-} z(t). \quad (3.3) \]

By Eq. (3.1) and (3.2), \( \frac{\pi}{2} < \theta_+ \leq \pi \) and \( z_+ \leq 1. \) From the maximality of \( \varphi \) and the fact that \( X \) has no singularities in \( \Omega, \) one obtains \( (\theta_+, z_+) \in \partial \Omega, \) and so

\[ z_+ = |\cos \theta_+| = -\cos \theta_. \quad (3.4) \]
We have two cases to consider:

i) $\theta_+ < \pi$ (and so $z_+ < 1$).

ii) $\theta_+ = \pi$ (and so $z_+ = 1$).

Since the vectors of $X$ on the boundary of $\Omega$ points inward, we can use transversality to conclude that case i) cannot occur. However, we will discard this case by a direct argument. Consider the (positive) function $\xi : (0, b) \to \mathbb{R}$ defined by $\xi(t) = z(t) + \cos \theta(t)$.

By Eq. (3.3) and (3.4),

$$\lim_{t \to b} \xi(t) = z_+ + \cos \theta_+ = 0.$$  (3.5)

Using now Eq. (2.4), (3.3) and (3.4), one obtains

$$\lim_{t \to b} \xi'(t) = \lim_{t \to b} \left( \sin \theta(t) - \theta'(t) \sin \theta(t) \right)$$

$$= \sin \theta_+ - \sin \theta_+ \sqrt{1 - \frac{\cos^2 \theta_+}{z_+^2}}$$

$$= \sin \theta_+ > 0.$$  

Then, there is $t_0 \in (0, b)$ such that $\xi'(t) > \sin \theta_+ / 2$, $t \in [t_0, b)$, and so

$$\xi(t) - \xi(t_0) = \int_{t_0}^{t} \xi'(s)ds > \frac{\sin \theta_+}{2}(t - t_0), \quad t > t_0.$$

Letting $t \to b$ in the above inequality, and using Eq. (3.5), one obtains

$$-\xi(t_0) \geq \frac{\sin \theta_+}{2}(b - t_0) > 0,$$

contradicting the fact that $\xi(t) > 0$, for all $t$.

Suppose now ii). From Eq. (3.2) and Lemma 2.2 (ii), we obtain $z'(t) \geq \theta'(t)$ for all $t \in (0, b)$, and so

$$1 > z(t) = z(0) + \int_{0}^{t} z'(s)ds \geq z(0) + \int_{0}^{t} \theta'(s)ds = z_0 + \theta(t) - \theta_0,$$

for every $t \in (0, b)$. Taking the limit when $t \to b$ in the above inequality, and using Eq. (3.3) and (b), we obtain

$$1 \geq z_0 + \theta_+ - \theta_0 = z_0 + \pi - \theta_0 > \pi - \theta_0,$$  (3.6)

and thus $0 < \pi - \theta_0 < 1$. Choosing $k \in \mathbb{N}$ such that

$$\frac{1}{k + 1} \leq \pi - \theta_0 < \frac{1}{k},$$  (3.7)

one has, since the cosine function is decreasing on $(0, \pi)$,

$$0 < \cos \left( \frac{1}{k} \right) < \cos(\pi - \theta_0) = -\cos \theta_0 \leq |\cos \theta_0|.$$
Then, by Eq. (2.3) and the above inequality,
\[
\frac{\cos^2(1/k)}{z_0^2} < \frac{\cos^2 \theta(0)}{z^2(0)} \leq 1,
\]
and so \( z_0 > \cos(1/k) \). Using now that \( \cos x > 1/(1 + x) \) for every \( x \in [0, 1] \), one concludes that \( z_0 > k/(k + 1) \). Hence, by Eq. (3.6) and (3.7),
\[
1 \geq z_0 + \pi - \theta_0 > \frac{k}{k+1} + \frac{1}{k+1} = 1,
\]
which is obviously false. This contradiction finishes the proof of the lemma.

**Lemma 3.2.** Let \( (\theta_0, z_0) \in \Omega \) such that \( 0 \leq \theta_0 < \pi \). If \( \varphi(t) = (\theta(t), z(t)), \ t \in (a, b), \) is the maximal integral curve of \( X \) satisfying \( \varphi(0) = (\theta_0, z_0) \), then there exists \( t_0 \in (0, b) \) such that \( \theta(t_0) = \pi \).

**Proof.** Assuming, by contradiction, that the conclusion does not hold, one has \( \theta < \theta(t) < \pi, \ t \in (0, b) \), and so
\[
\varphi'(t) = \sin \theta(t) > 0, \quad t \in (0, b).
\]
Let \( c \in [0, b) \) such that \( z(c) > 1 \) (such a number \( c \) exists by Lemma 3.1). Since \( \theta \) is bounded above and, by Eq. (2.4) and (3.8),
\[
\theta'(t) \geq \sqrt{1 - \frac{1}{z^2(t)}} > \sqrt{1 - \frac{1}{z^2(c)}} > 0, \quad t \in (c, b),
\]
one concludes that \( b < \infty \). Then, since \( \varphi'(t) \leq 1 \), one also has that \( z \) is bounded. Hence, \( \varphi(t) = (\theta(t), z(t)) \) converges to a point in \( \Omega \) when \( t \to b \), but this can not occur because \( b < \infty \) (see, for instance, [17, p. 91]).

**Lemma 3.3.** Given \( z_0 > 1 \) and \( n \in \mathbb{Z} \), let \( \varphi(t) = (\theta(t), z(t)), \ t \in (a, b) \), be the maximal integral curve of \( X \) satisfying \( \varphi(0) = (n\pi, z_0) \). Then, \( a = -b \) and \( \varphi(-t) = R(\varphi(t)) \) for every \( t \in (-b, b) \), where \( R \) denotes the reflection in \( \mathbb{R}^2 \) with respect to the line \( \theta = n\pi \). In short, \( \varphi \) is symmetric with respect to the line \( \theta = n\pi \).

**Proof.** Consider the curve \( \sigma : (-b, -a) \to \Omega \) defined by
\[
\sigma(t) = R(\varphi(-t)) = (2n\pi - \theta(-t), z(-t)).
\]
It is easy to see that \( \sigma \) is an integral curve of \( X \). Since \( \sigma(0) = (n\pi, z_0) = \varphi(0) \), it follows from the maximality of \( \varphi \) that \( a = -b \) and
\[
\varphi(t) = \sigma(t) = R(\varphi(-t)), \quad t \in (-b, b).
\]
Lemmas 3.2 and 3.3 tell us that to obtain a picture of the phase portrait of X it is sufficient to consider the family of trajectories \( \{ \varphi_{\lambda} \}_{\lambda > 1} \), where \( \varphi_{\lambda} : (-b_{\lambda}, b_{\lambda}) \to \Omega \) is the maximal integral curve of X such that \( \varphi_{\lambda}(0) = (\pi, \lambda) \).

From Lemma 3.2 and the fact that \( z'(t) = \sin \theta(t) \) is positive on \( \theta^{-1}((0, \pi)) \), one concludes, for each \( \lambda > 1 \), that the trajectory \( \varphi_{\lambda} : (-b_{\lambda}, b_{\lambda}) \to \Omega \) either crosses the ray \( \{ (\theta, z) \in \mathbb{R}^2 : \theta = 0 \text{ and } z > 1 \} \) or converges to a point \( p_{\lambda} \in \partial \Omega \) when \( t \to -b_{\lambda} \). Moreover, each point \( (\theta, z) \in \partial \Omega \) such that \( 0 \leq \theta < \pi \) is the limit point \( p_{\lambda} \) of some \( \varphi_{\lambda} \). The later is clear when \( \theta \neq 0 \) and \( \theta \neq \frac{\pi}{2} \), as the vector field X can be continuously extended, without singularities, to a neighbourhood of \( (\theta, z) \), and follows easily for the other two values of \( \theta \) by a continuity argument.

The following lemma shows that two distinct trajectories of the family \( \{ \varphi_{\lambda} \}_{\lambda > 1} \) can not converge to the same point in \( \partial \Omega \). Note that this fact does not follow from the standard theory of ordinary differential equations, because the vector field X does not admit a differentiable extension to a neighbourhood of any given point in \( \partial \Omega \).

**Lemma 3.4.** With the same notation as above, assume for some \( (\theta_0, z_0) \in \partial \Omega \) that
\[
\lim_{t \to -b_{\lambda_1}} \varphi_{\lambda_1}(t) = (\theta_0, z_0) = \lim_{t \to -b_{\lambda_2}} \varphi_{\lambda_2}(t). \tag{3.10}
\]
Then \( \lambda_1 = \lambda_2 \) (and hence \( \varphi_{\lambda_1} = \varphi_{\lambda_2} \)).

**Proof.** Assume, by contradiction, that \( \lambda_1 \neq \lambda_2 \), say \( \lambda_1 < \lambda_2 \). Setting \( \varphi_{\lambda_1} = (\varphi_1, \varphi_2) \) and \( \varphi_{\lambda_2} = (\varphi_1, \varphi_2) \), from Eq. (3.10) one obtains that \( \varphi_1 = \varphi_1_{\lambda_1}$\( \mid_{(-b_{\lambda_1}, 0)} \) (respectively, \( \varphi_2 = \varphi_2_{\lambda_2}$\( \mid_{(-b_{\lambda_2}, 0)} \)) is a diffeomorphism from \( (-b_{\lambda_1}, 0) \) (respectively, \( (-b_{\lambda_2}, 0) \)) to \( (\theta_0, \pi) \). Let \( \psi = \varphi_2^{-1} \circ \varphi_1 : (-b_{\lambda_1}, 0) \to (-b_{\lambda_2}, 0) \). By the Chain Rule and the Inverse Function Theorem,
\[
\psi'(t) = (\varphi_2^{-1})'(\varphi_1(t))\varphi_1'(t) = \frac{\partial_1'(t)}{\partial_2'(\psi(t))} > 0, \quad t \in (-b_{\lambda_1}, 0). \tag{3.11}
\]
Since \( \lambda_1 < \lambda_2 \) and \( \varphi_2(\psi(t)) = \varphi_1(t) \) for \( t \in (-b_{\lambda_1}, 0) \), we have \( z_2(\psi(t)) > z_1(t) \) and so
\[
\varphi_1'(t) = \sqrt{1 - \frac{\cos^2 \varphi_2(\psi(t))}{z_2^2(\psi(t))}} \geq \sqrt{1 - \frac{\cos^2 \varphi_1(t)}{z_1^2(t)}} = \varphi_1'(t),
\]
for all \( t \in (-b_{\lambda_1}, 0] \). Using this inequality in Eq. (3.11), we obtain
\[
\psi'(t) \leq 1, \quad t \in (-b_{\lambda_1}, 0]. \tag{3.12}
\]
Using again the equality $\theta_2(\psi(t)) = \theta_1(t)$, it follows from the Change of Variables Formula that

$$z_2(0) - z_2(\psi(t)) = \int_{\psi(t)}^0 z_2'(s)\,ds = \int_t^0 z_2'(\psi(u))\psi'(u)\,du$$

$$= \int_t^0 \sin \theta_2(\psi(u))\psi'(u)\,du = \int_t^0 \sin \theta_1(u)\psi'(u)\,du,$$

for every $t \in (-b,0]$. Hence, by Eq. (3.12),

$$z_2(0) - z_2(\psi(t)) = \int_t^0 z_1'(u)\psi'(u)\,du$$

$$\leq \int_t^0 z_1'(u)\,du = z_1(0) - z_1(t), \quad t \in (-b,0].$$

Taking the limit when $t \to -b$, and using Eq. (3.10), one obtains $\lambda_2 = z_2(0) \leq z_1(0) = \lambda_1$, contradicting our assumption $\lambda_1 < \lambda_2$. Hence $\lambda_1 = \lambda_2$. 

\[\Box\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Graphic representation of the phase portrait of the vector field $X$ for $\theta \in [0,2\pi]$.}
\end{figure}

4. Proof of Theorem 1.1

As before, for each $\lambda > 1$ denote by $\varphi_\lambda(t) = (\theta(t),z(t))$, $t \in (-b, b)$, the maximal integral curve of $X$ such that $\varphi_\lambda(0) = (\pi, \lambda)$. From Lemma 3.4 and the discussion that precedes its statement one concludes that there is a unique $\lambda_0 > 1$ such that $\varphi_{\lambda_0}(t) \to (0,1)$
when $t \to -b_\lambda$. Moreover, when $\lambda \neq \lambda_0$, the trajectory $\varphi_\lambda$ either crosses the ray $\{(\theta, z) \in \mathbb{R}^2 : \theta = 0 \text{ and } z > 1\}$ or converges to a point in $\partial \Omega$ depending on whether $\lambda > \lambda_0$ or $1 < \lambda < \lambda_0$ (see Figure 2).

For each $\lambda > 1$, consider the curve $\alpha_\lambda : (-b_\lambda, b_\lambda) \to \mathbb{R}^3$ defined by

$$\alpha_\lambda(t) = (x(t), 0, z(t)),$$

and the surface $M_\lambda$ of $\mathbb{R}^3$ obtained by the rotation of the image of $\alpha_\lambda$ around the $x$-axis. As we have seen in Section 2, the length of the shape operator of $M_\lambda$ equals 1 at every point. The detailed classification of the surfaces $M_\lambda$ reads:

**Theorem 4.1.** Let $M_\lambda$ be as above.

(i) If $\lambda > \lambda_0$ then $M_\lambda$ is a complete $C^\infty$-surface. Moreover, $M_\lambda$ is periodic and has self-intersections.

(ii) $M_{\lambda_0}$ is incomplete, but it can be extended in infinite many ways to a complete $C^3$-surface satisfying $|A| \equiv 1$. Any such extension has self-intersections.

(iii) $M_{\sqrt{2}}$ is the sphere with center at $(-\sqrt{2}, 0, 0)$ and radius $\sqrt{2}$ (minus two points).

(iv) If $\sqrt{2} < \lambda < \lambda_0$ or $1 < \lambda < \sqrt{2}$ then $M_\lambda$ is incomplete and cannot be extended to a surface with $|A| \equiv 1$.

Concerning the Gaussian curvature of the surfaces obtained in the above theorem, we observe that the only surfaces with positive Gaussian curvature are the surfaces $M_\lambda$ with $1 < \lambda \leq \sqrt{2}$. For all the others, the Gaussian curvature changes the signal.

Theorem 4.1 is a direct consequence of the following result:

**Theorem 4.2.** Let $\alpha_\lambda : (-b_\lambda, b_\lambda) \to \mathbb{R}^3$ be as above.

(i) If $\lambda > \lambda_0$ then $b_\lambda = +\infty$ and $\alpha_\lambda$ is of class $C^\infty$. Moreover, $\alpha_\lambda$ is periodic and has self-intersections.

(ii) $b_{\lambda_0} < +\infty$ and $\alpha_{\lambda_0}$ can be extended in infinite many ways to a profile curve of class $C^3$ defined on $\mathbb{R}$. Any such extension has self-intersections.

(iii) $\alpha_{\sqrt{2}}$ is a parametrization by arc length of the semicircle in the $xz$-plane with center at $(-\sqrt{2}, 0, 0)$ and radius $\sqrt{2}$.

(iv) If $\sqrt{2} < \lambda < \lambda_0$ or $1 < \lambda < \sqrt{2}$ then $b_\lambda < +\infty$ and $\alpha_\lambda$ cannot be extended to a profile curve defined on an open interval containing $(-b_\lambda, b_\lambda)$ properly.
Figure 3. Graphic representation of the curves $\alpha_{\lambda}$
Proof. (i) From $\lambda > \lambda_0$ and the discussion in the beginning of this section one infers that there exists $t_0 > 0$ such that $\theta(-t_0) = 0$. Applying Lemma 3.3 with $n = 0$ and $n = 1$ one concludes that $b_{t_0} = +\infty$. Being the trajectory of a vector field of class $C^\infty$, $\varphi_{t_0}$, and hence $\alpha_{t_0}$, is of class $C^\infty$.

We will now prove that $\alpha_{t_0}$ is periodic. Since, by Eq. (2.5) and Lemma 3.3, the maps $t \in \mathbb{R} \mapsto \varphi_{t_0}(t + 2t_0)$ and $t \in \mathbb{R} \mapsto (\theta(t) + 2\pi, z(t))$ are both trajectories of $X$ passing through $(3\pi, \lambda)$, one has

$$\varphi_{t_0}(t + 2t_0) = (\theta(t) + 2\pi, z(t)), \quad t \in \mathbb{R}.$$  \hfill (4.2)

On the other hand, by Eq. (4.1) one has

$$x(t + 2t_0) = \int_0^{t+2t_0} \cos \theta(s)ds$$

$$= \int_0^t \cos \theta(s)ds + \int_t^{t+2t_0} \cos \theta(s)ds$$

$$= x(t) + \int_t^{2t_0} \cos \theta(s)ds + \int_{2t_0}^{t+2t_0} \cos \theta(s)ds$$

$$= x(t) + \int_t^{2t_0} \cos \theta(s)ds + \int_0^t \cos \theta(s + 2t_0)ds.$$  \hfill (4.3)

Since $\theta(s + 2t_0) = \theta(s) + 2\pi$ by Eq. (4.2), it follows that

$$x(t + 2t_0) = x(t) + \int_t^{2t_0} \cos \theta(s)ds + \int_0^t \cos \theta(s)ds$$

$$= x(t) + \int_t^{2t_0} \cos \theta(s)ds = x(t) + x(2t_0), \quad t \in \mathbb{R}.$$  \hfill (4.4)

Since $z(t + 2t_0) = z(t)$ for all $t \in \mathbb{R}$ by Eq. (4.2), the curve $\alpha_{t_0}$ is periodic.

To complete the proof of (i), it remains to show that $\alpha_{t_0}$ is non-embedded. In fact, we will show that the restriction of $\alpha_{t_0}$ to the interval $(-t_0, t_0)$ has already self-intersections. For that observe first that, since $\theta(-t_0) = 0$, $\theta(0) = \pi$ and $\theta' > 0$, the function $\theta = \theta|_{[-t_0, 0]}$ is a diffeomorphism from $[-t_0, 0]$ to $[0, \pi]$. In particular, there exists a unique $t_1 \in (0, t_0)$ such that $\theta(-t_1) = \pi/2$. Let $\xi : [-t_0, -t_1] \to [-t_1, 0]$ be defined by

$$\xi(t) = \theta^{-1}(\pi - \theta(t)).$$  \hfill (4.5)

Clearly, $\xi$ is a diffeomorphism, $\xi(-t_0) = 0$ and $\xi(-t_1) = -t_1$. Moreover,

$$\xi'(t) = -\frac{\theta'(t)}{\theta'(\xi(t))}, \quad t \in [-t_0, -t_1],$$  \hfill (4.6)
and
\[ \cos \theta(x(t)) = \cos(\pi - \theta(t)) = -\cos \theta(t), \quad t \in [-t_0, -t_1]. \]  
(4.7)
By Eq. (2.4) and (4.7),
\[ \theta'(x(t)) = \sqrt{1 - \frac{\cos^2 \theta(x(t))}{z^2(x(t))}} = \sqrt{1 - \frac{\cos^2 \theta(t)}{z^2(x(t))}}. \]  
(4.8)
Since, by Eq. (2.4),
\[ z(x(t)) - z(t) = \int_t^{x(t)} z'(s) \, ds = \int_t^{x(t)} \sin \theta(s) \, ds > 0, \quad t \in [-t_0, -t_1], \]
one then has
\[ \theta'(x(t)) > \sqrt{1 - \frac{\cos^2 \theta(t)}{z^2(t)}} = \theta'(t), \quad t \in [-t_0, -t_1]. \]  
(4.9)
Using the informations collected above, we will now compare the values of \( x(t) \) for \( t = -t_0, t = -t_1 \) and \( t = 0 \). Since \( \pi/2 \leq \theta(t) \leq \pi \) for \( t \in [-t_1, 0] \), from Eq. (4.1) we obtain
\[ x(-t_1) = \int_0^{-t_1} \cos \theta(s) \, ds = -\int_{-t_1}^{0} \cos \theta(s) \, ds > 0 = x(0). \]  
(4.10)
On the other hand, by Eq. (4.6), (4.7) and (4.9), one has
\[ \int_{-t_1}^{0} \cos \theta(t) \, dt = \int_{-t_1}^{-t_0} \cos \theta(x(s)) x'(s) \, ds = \int_{-t_0}^{-t_1} \cos \theta(x(s)) \theta'(s) \, ds > \int_{-t_0}^{-t_1} \cos \theta(x(s)) \, ds = -\int_{-t_0}^{-t_1} \cos \theta(s) \, ds. \]  
(4.11)
Hence, by Eq. (4.1) and inequality above,
\[ x(-t_0) = -\int_{-t_0}^{0} \cos \theta(t) \, dt = -\int_{-t_0}^{-t_1} \cos \theta(t) \, dt - \int_{-t_1}^{0} \cos \theta(t) \, dt < 0. \]  
(4.12)
The curve \( \alpha_\lambda \) is symmetric with respect to the line \( x = x(0) = 0 \). Indeed, by Lemma 3.3 one has
\[ \theta(t) = 2\pi - \theta(-t), \quad z(t) = z(-t), \quad t > 0, \]  
(4.13)
and so
\[ x(t) = \int_0^t \cos \theta(s) \, ds = \int_0^t \cos \theta(-s) \, ds \]  
(4.14)
\[ = -\int_0^{-t} \cos \theta(s) \, ds = -x(-t), \]
for every \( t > 0 \).
Since \( x' > 0 \) on \((-t_0, t_1)\) and, by Eq. (4.10) and (4.12), \( x(-t_0) < 0 < x(-t_1)\), there exists a unique \( t_2 \in (t_1, t_0) \) such that \( x(-t_2) = 0 = x(0)\). Then, by Eq. (4.14),

\[
x(t_2) = -x(-t_2) = 0 = x(-t_2).
\] (4.15)

Since \( z(t_2) = z(-t_2) \) by Eq. (4.13), it follows that \( \alpha_\lambda(t_2) = \alpha_\lambda(-t_2) \). Hence, the restriction of \( \alpha_\lambda \) to the interval \((-t_0, t_0)\) has a self-intersection.

(ii) We begin by showing that \( b_{\lambda_0} < +\infty \). Let \( t_1 \in (-b_{\lambda_0}, 0) \) such that \( \theta(t_1) = \pi/2 \). For every \( t \in (-b_{\lambda_0}, t_1) \) we have

\[
t_1 - t = \int_t^{t_1} 1 \, ds = \int_{\theta(t)}^{\pi/2} (\theta^{-1})'(u) \, du = \int_{\theta(t)}^{\pi/2} \frac{1}{\theta'(\theta^{-1}(u))} \, du
\]

\[
= \int_{\theta(t)}^{\pi/2} \frac{z}{\sqrt{z^2 - \cos^2 \theta}} (\theta^{-1}(u)) \, du
\]

\[
\leq \lambda_0 \int_{\theta(t)}^{\pi/2} \frac{1}{\sqrt{z - \cos \theta}} (\theta^{-1}(u)) \, du,
\] (4.16)

where in the last inequality we used the fact that \( 1 < z \circ \theta^{-1} < \lambda_0 \) and the cosine function is nonnegative on \([0, \pi/2]\).

Claim. There is \( C_1 > 0 \) such that

\[
\frac{1}{\sqrt{z(t) - \cos \theta(t)}} \leq \frac{C_1}{\sin^2 \theta(t)}, \quad t \in (-b_{\lambda_0}, t_1].
\] (4.17)

Indeed, since \( \theta(t) \to 0 \) and \( z(t) \to 1 \) when \( t \to -b_{\lambda_0} \), from Eq. (2.4) we obtain

\[
\lim_{t \to -b_{\lambda_0}} \theta'(t) = 0.
\] (4.18)

Then, again by Eq. (2.4),

\[
\lim_{t \to -b_{\lambda_0}} \frac{\sin^2 \theta(t)}{\theta'(t)^2} = \lim_{t \to -b_{\lambda_0}} \frac{z^2(t)}{z(t) + \cos \theta(t)} \frac{\sin^2 \theta(t)}{z(t) - \cos \theta(t)}
\]

\[
= \frac{1}{2} \lim_{t \to -b_{\lambda_0}} \frac{\sin^2 \theta(t)}{z(t) - \cos \theta(t)}
\]

\[
= \frac{1}{2} \lim_{t \to -b_{\lambda_0}} \frac{2\theta'(t) \sin \theta(t) \cos \theta(t)}{\sin \theta(t)(1 + \theta'(t))}
\]

\[
= \frac{1}{2} \lim_{t \to -b_{\lambda_0}} \frac{2\theta'(t) \cos \theta(t)}{1 + \theta'(t)} = 0,
\] (4.19)
and so
\[ \lim_{t \to -b_{\lambda_0}} \frac{z(t) - \cos \theta(t)}{\sin \theta(t)} = \lim_{t \to -b_{\lambda_0}} \frac{\sin \theta(t)(1 + \theta'(t))}{\theta'(t) \cos \theta(t)} = \lim_{t \to -b_{\lambda_0}} \frac{\sin \theta(t)}{\theta'(t)} = 0. \] (4.20)

From Eq. (2.4) and the above equality one obtains
\[ \lim_{t \to -b_{\lambda_0}} \frac{\theta'(t)^2}{\sin \theta(t)} = \lim_{t \to -b_{\lambda_0}} \frac{z(t) + \cos \theta(t)}{z'(t)} \frac{z(t) - \cos \theta(t)}{\sin \theta(t)} = 0. \]

Therefore,
\[ \lim_{t \to -b_{\lambda_0}} \frac{\sin^3 \theta(t)}{z(t) - \cos \theta(t)} = \lim_{t \to -b_{\lambda_0}} \frac{3\theta'(t) \sin \frac{\theta(t)}{2} \theta(t) \cos \theta(t)}{\sin \theta(t)(1 + \theta'(t))} = \frac{3}{2} \lim_{t \to -b_{\lambda_0}} \frac{\theta'(t)}{\sin^2 \theta(t)} = 0, \] (4.21)

and the claim follows.

From Eq. (4.16) and (4.17), we obtain
\[ t_1 - t \leq \lambda_0 C_1 \int_{\theta(t)}^{\pi/2} \sin^{-\frac{3}{2}} u \, du, \quad -b_{\lambda_0} < t < t_1. \] (4.22)

Setting \( C_2 = \inf \{ \sin u/u : u \in (0, \frac{\pi}{2}] \}, \) one has
\[ \sin^{-\frac{3}{2}} u \leq C_2^{-\frac{3}{2}} u^{-\frac{3}{2}}, \quad u \in (0, \frac{\pi}{2}). \]

Using this information in Eq. (4.22), we obtain
\[ t_1 - t \leq \lambda_0 C_1 C_2^{-\frac{3}{2}} \int_{\theta(t)}^{\pi/2} u^{-\frac{3}{2}} \, du = 4\lambda_0 C_1 C_2^{-\frac{3}{2}} u^{-\frac{1}{2}} \bigg|_{\theta(t)}^{\pi/2} = 4\lambda_0 C_1 C_2^{-\frac{3}{2}} \left[ (\pi/2)^{\frac{1}{2}} - \theta(t)^{\frac{1}{2}} \right] \] (4.23)
\[ < 4\lambda_0 C_1 C_2^{-\frac{3}{2}} (\pi/2)^{\frac{1}{2}}, \]

for every \( t \in (-b_{\lambda_0}, t_1). \) Therefore, \( b_{\lambda_0} < +\infty. \)

In order to prove that \( \alpha_{\lambda_0} \) can be extended to a profile curve of class \( C^3 \) defined on \( \mathbb{R}, \) we need to evaluate the limits of \( \theta'' \) and \( \theta''' \) when \( t \to -b_{\lambda_0}. \) From Eq. (2.4) one obtains, after some work,
\[ \theta''(t) = \frac{\sin \theta(t)}{z(t) \theta'(t)} + \frac{\sin \theta(t) \cos \theta(t)}{z'(t)} - \frac{\theta'(t) \sin \theta(t)}{z(t)}. \] (4.24)

Since \( \theta(t) \to 0, z(t) \to 1 \) and \( \theta'(t) \to 0 \) when \( t \to -b_{\lambda_0}, \) it follows from Eq. (4.20) and (4.24) that
\[ \lim_{t \to -b_{\lambda_0}} \theta''(t) = \lim_{t \to -b_{\lambda_0}} \frac{\sin \theta(t)}{\theta'(t)} = 0. \] (4.25)
As to \( \lim_{t \to -b} \), observe that, since \( z(t) \to 1 \), \( \theta(t) \to 0 \), \( \theta'(t) \to 0 \) and \( \theta''(t) \to 0 \), the derivatives of either of the last two terms on the right hand side of Eq. \((4.24)\) goes to zero when \( t \to -b \lambda_0 \). Therefore,

\[
\lim_{t \to -b} \theta''(t) = \lim_{t \to -b} \left( \frac{\sin \theta}{z \theta'} \right)'(t)
\]

\[
= \lim_{t \to -b} \left\{ -\frac{\sin \theta \sin \theta}{z^2 \theta'} + \frac{1}{z} \left( \frac{\sin \theta}{\theta'} \right)' \right\}(t), \quad (4.26)
\]

where in the last equality we used Eq. \((4.19)\). Since, by Eq. \((2.4)\),

\[
\sin \frac{\theta}{\theta'} = \frac{z \sin \theta}{\sqrt{z + \cos \theta}} \frac{\sin \theta}{\sqrt{z - \cos \theta}}, \quad (4.27)
\]

one has

\[
\left( \frac{\sin \theta}{\theta'} \right)' = \left( \frac{\sin \theta}{\sqrt{z + \cos \theta}} - \frac{z \sin \theta(1 - \theta')}{2(z + \cos \theta)^{3/2}} \frac{\sin \theta}{\sqrt{z - \cos \theta}} \right)
\]

\[
+ \cos \theta - \frac{z(1 + \theta')}{2\sqrt{z + \cos \theta}} \frac{\sin^2 \theta}{(z - \cos \theta)^{3/2}}. \quad (4.28)
\]

The first term on the right hand side goes to zero by Eq. \((4.19)\) and \((4.27)\). Hence,

\[
\lim_{t \to -b} \left( \frac{\sin \theta}{\theta'} \right)' = 1 - \frac{1}{2\sqrt{2}} \lim_{t \to -b} \frac{\sin^2 \theta}{(z - \cos \theta)^{3/2}}. \quad (4.29)
\]

Using Eq. \((2.4)\) again, one obtains

\[
\lim_{t \to -b} \frac{\sin^2 \theta}{(z - \cos \theta)^{3/2}} = \lim_{t \to -b} \frac{4 \cos \theta \sqrt{z + \cos \theta}}{3z(1 + \theta')} = \frac{4\sqrt{2}}{3}. \quad (4.30)
\]

It now follows from Eq. \((4.26)\), Eq. \((4.29)\) and the above equality that

\[
\lim_{t \to -b} \theta''(t) = 1 - \frac{1}{2\sqrt{2}} \frac{4\sqrt{2}}{3} = \frac{1}{3}. \quad (4.30)
\]

It is possible to extend \( \alpha_{\lambda_0} \) gluing together copies of \( \alpha_{\lambda_0} \). By Eq. \((4.18)\), \((4.25)\) and \((4.30)\), this extension is (at least) \( C^4 \) (recall that if a profile curve is of class \( C^s \) then its corresponding angle function is of class \( C^{s-1} \)). Another way to extend \( \alpha_{\lambda_0} \) is gluing together copies of \( \alpha_{\lambda_0} \) and horizontal segments with any length and with height equal to \( 1 \). By the same equations, these extensions are \( C^3 \) but not \( C^4 \) (see Figure 3, items (B), (C) and (D), for a sample of these extensions).

To complete the proof of (ii), it remains to show that any extension of \( \alpha_{\lambda_0} \) is non-embedded. But clearly this follows from the fact that
\(\alpha_{\lambda_0}\) has a self-intersection, which in turn can be proved as in (i) (with \(b_{\lambda_0}\) playing the role of \(t_0\)).

(iii) As can be easily seen, the curve
\[
\psi(t) = (\pi + t/\sqrt{2}, \sqrt{2}\cos(t/\sqrt{2})), \quad t \in (-\sqrt{2}\pi/2, \sqrt{2}\pi/2),
\]
is a trajectory of \(X\). Since \(\psi(0) = (\pi, \sqrt{2})\), one has \(\varphi_{\sqrt{2}} = \psi\). Then, by Eq. (4.1),
\[
\alpha_{\sqrt{2}}(t) = (-\sqrt{2}\sin(t/\sqrt{2}) - \sqrt{2}, \sqrt{2}\cos(t/\sqrt{2})), \quad t \in \left(-\frac{\sqrt{2}\pi}{2}, \frac{\sqrt{2}\pi}{2}\right),
\]
which is a parametrization by arc length of the portion of the circle in the \(xz\)-plane with center \((-\sqrt{2}, 0, 0)\) and radius \(\sqrt{2}\) that is above the \(x\)-axis.

(iv) Let \((\theta_0, z_0) = \lim_{t \to -b_\lambda} \varphi_\lambda(t)\). Since \(\lambda \neq \sqrt{2}\) and, by item (iii), \(\varphi_{\sqrt{2}}(t) \to (\pi/2, 0)\) when \(t \to -b_{\sqrt{2}} = -\sqrt{2}\pi/2\), it follows from Lemma 3.4 that \(0 < z_0 < 1\) and either \(0 < \theta_0 < \pi/2\) or \(\pi/2 < \theta_0 < \pi\).

Suppose, by contradiction, that \(b_\lambda = +\infty\). Since \(\theta(t) \to \theta_0\) when \(t \to -b_\lambda\) and either \(0 < \theta_0 < \pi/2\) or \(\pi/2 < \theta_0 < \pi\), there exist \(\varepsilon > 0\) and \(t_1 \in \mathbb{R}\) such that
\[
\sin \theta(t) > \varepsilon, \quad t \leq t_1.
\]
Then, by Eq. (2.1),
\[
z(t_1) - z_0 > z(t_1) - z(t) = \int_t^{t_1} \sin \theta(s) ds > \varepsilon(t_1 - t),
\]
for every \(t < t_1\), a contradiction. Hence, \(b_\lambda < \infty\).

Suppose, by contradiction, that \(\alpha_\lambda\) can be extended to a profile curve \(\widetilde{\alpha}_\lambda(t) = (\widetilde{x}(t), 0, \widetilde{z}(t))\), \(t \in (a, b)\), where \(a < -b_\lambda\), say. Let \(\widetilde{\theta} : (a, b) \to \mathbb{R}\) be a function satisfying
\[
\widetilde{\alpha}'_\lambda(t) = (\cos \widetilde{\theta}(t), 0, \sin \widetilde{\theta}(t)), \quad t \in (a, b).
\]
Since \(\widetilde{\alpha}'_\lambda(t) = \alpha'_\lambda(t)\) for all \(t \in (-b_\lambda, b_\lambda)\), and, by Eq. (4.1),
\[
\alpha'_\lambda(t) = (x'(t), 0, z'(t)) = (\cos \theta(t), 0, \sin \theta(t)), \quad t \in (-b_\lambda, b_\lambda),
\]
one concludes that \(\theta\) and the restriction of \(\widetilde{\theta}\) to the interval \((-b_\lambda, b_\lambda)\) differ by an integer multiple of \(2\pi\). Assuming without loss of generality that \(\theta = \theta\big|_{(-b_\lambda, b_\lambda)}\), one has
\[
\widetilde{\theta}'(-b_\lambda) = \lim_{t \to -b_\lambda} \widetilde{\theta}'(t) = \lim_{t \to -b_\lambda} \theta'(t) = 0
\]
and
\[
\widetilde{z}(-b_\lambda) = \lim_{t \to -b_\lambda} \widetilde{z}(t) = \lim_{t \to -b_\lambda} z(t) = z_0,
\]
which contradicts Lemma 2.2 (i) as \(0 < z_0 < 1\). \qed
Now we finish the prove of our main theorem.

**Completion of proof of Theorem 1.1:** Let $\alpha_\lambda$ and $M_\lambda$, $\lambda > 1$, be as in the beginning of this section, and let $\mathcal{F}_1 = \{M_\lambda : \lambda > \lambda_0\}$. Let $A$ be the set of extensions of $\alpha_\lambda$ constructed in the proof of Theorem 4.2 (ii), and $\mathcal{F}_2$ the (infinite) family of surfaces obtained by the rotation of the images of the curves in $A$ around the $x$-axis. By Theorem 4.1 (i), the surfaces in $\mathcal{F}_1$ have the properties stipulated in the statement of the theorem. That the surfaces in $\mathcal{F}_2$ also meet the stipulated conditions is a consequence of the proof of Theorem 4.2 (ii). This establishes the first part of the theorem.

Let $M$ be a complete rotational surface of class $C^2$ satisfying $|A| \equiv 1$. Applying a rigid motion if necessary, we can assume that its axis of revolution is the $x$-axis and that its profile curve $C$ is contained in the $xz$-plane. Let $\alpha(t) = (x(t), 0, z(t))$, $t \in (a, b)$, be a parametrization of $C$ such that $z(t) > 0$ and $||\alpha'(t)|| = 1$ for all $t \in (a, b)$. Let $\theta : (a, b) \to \mathbb{R}$ be a $C^1$-function such that

$$(x'(t), 0, z'(t)) = (\cos \theta(t), 0, \sin \theta(t)), \quad t \in (a, b).$$

As we have seen in Section 2, the function $\theta$ satisfies

$$\theta''(t) + \frac{\cos^2 \theta(t)}{z^2(t)} = 1, \quad t \in (a, b).$$

If $\theta$ is constant, by the above equality one has that $z$ is constant. Then $\sin \theta = z' = 0$ and so $z = |\cos \theta| \equiv 1$. Being complete, $M$ is then a right circular cylinder of radius 1.

Suppose now that $\theta$ is not constant. Since $\theta$ is monotone by Proposition 2.1, reparametrizing $\alpha$ if necessary we can assume that $\theta'(t) \geq 0$ for all $t \in (a, b)$. Let $t_0 \in (a, b)$ such that $\theta'(t_0) > 0$. Without loss of generality, we can assume that $0 \leq \theta(t_0) < 2\pi$. Let $(a_1, b_1) \subset (a, b)$ be the maximal interval containing $t_0$ on which $\theta' > 0$. As we have seen in Section 2, the map $t \in (a_1, b_1) \mapsto \varphi(t) := (\theta(t), z(t))$ is an integral curve of the vector field $X$ defined by Eq. (2.5). By the results in Section 3, there is a unique trajectory $\varphi_\lambda : (-b_\lambda, b_\lambda) \to \Omega$ in the family $\{\varphi_\lambda \}_{\lambda > 1}$ that passes through $(\theta(t_0), z(t_0))$. Let $s_0 \in (-b_\lambda, b_\lambda)$ such that $\varphi_\lambda(s_0) = (\theta(t_0), z(t_0)) = \varphi(t_0)$. Assuming without loss of generality that $t_0 = s_0$, it follows from the maximality of $\varphi_\lambda$ that $\varphi = \varphi_\lambda|_{(a_1, b_1)}$. Write $\varphi_\lambda = (\theta_\lambda, z_\lambda)$ and consider its associated profile curve $t \in (-b_\lambda, b_\lambda) \mapsto \alpha_\lambda(t) = (x_\lambda(t), 0, z_\lambda(t))$ (cf. the beginning of this section). Since

$$x_\lambda'(t) = \cos \theta_\lambda(t) = \cos \theta(t) = x'(t), \quad t \in (a_1, b_1),$$

it holds that

$$\alpha(t) = \alpha_\lambda(t) + (d, 0, 0), \quad t \in (a_1, b_1), \quad (4.31)$$
for some $d \in \mathbb{R}$.

**Claim.** $a \leq -b_\lambda$ and $b \geq b_\lambda$.

Assuming, by contradiction, that $-b_\lambda < a$, one has $-b_\lambda < a_1$. Then, since $\theta'_{\alpha}(t) > 0$ on $(-b_\lambda, b_\lambda)$ and $\theta|_{(a_1, b_1)} = \theta|_{(a_1, b_1)}$, there exists $t \to a_1$ such that

$$
\lim_{t \to a_1} \theta'(t) = \lim_{t \to a_1} \theta'_{\alpha}(t) = \theta'_{\alpha}(a_1) > 0.
$$

From the above inequality and definition of $(a_1, b_1)$ one obtains $a = a_1$. It now follows from Eq. (4.31) that $\alpha$ can be extended to an interval containing $(a, b)$ properly, contradicting the completeness of $M$. This contradiction proves that $a \leq -b_\lambda$. In the same way, one proves that $b \geq b_\lambda$.

It follows from the Claim that $a_1 = -b_\lambda$. Indeed, if we had $-b_\lambda < a_1$, reasoning as above one would obtain $\lim_{t \to a_1} \theta'(t) > 0$. On the other hand, from $a < a_1$ one would obtain $\lim_{t \to a_1} \theta'(t) = 0$, a contradiction. In the same manner, one proves that $b_1 = b_\lambda$. Hence $(a_1, b_1) = (-b_\lambda, b_\lambda)$ and, by Eq. (4.31),

$$
\alpha(t) = \alpha_\lambda(t) + (d, 0, 0), \quad t \in (-b_\lambda, b_\lambda),
$$

for some $d \in \mathbb{R}$.

From Eq. (4.32) one obtains that either $\lambda \geq \lambda_0$ or $\lambda = \sqrt{2}$. In fact, if we had $1 < \lambda < \sqrt{2}$ or $\sqrt{2} < \lambda < \lambda_0$, from Theorem 4.2 (iv) we would obtain that $M$ is a translation of $M_{\lambda'}$, and so $M$ would be incomplete, contradicting the hypothesis.

In the case $\lambda > \lambda_0$, it follows from Eq. (4.32) and Theorem 4.2 (i) that $\alpha(t) = \alpha_\lambda(t) + (d, 0, 0)$ for all $t \in \mathbb{R}$, and therefore $M = M_\lambda$ (up to translation).

In the case $\lambda = \sqrt{2}$, it follows from Eq. (4.32) and Theorem 4.2 (iii) that $\alpha(t) = \alpha_{\sqrt{2}}(t) + (d, 0, 0)$, $t \in (-\sqrt{2}, b_{\sqrt{2}})$. Since $M$ is complete, one concludes that $M$ is, up to translation, the sphere with center at $(-\sqrt{2}, 0, 0)$ and radius $\sqrt{2}$.

Finally, consider the case $\lambda = \lambda_0$. By Eq. (4.32), $M$ is an extension of $M_{\lambda_0}$. Since $M$ is complete and, by Theorem 4.1 (ii), $M_{\lambda_0}$ is incomplete, one has $a < -b_{\lambda_0}$ and $b > b_{\lambda_0}$. We will conclude that, up to congruence, $\alpha$ belongs to the family $A$ and so $M \in F_2$. For that, we can assume that $\theta'$ is not identically zero on $(a, -b_{\lambda_0}) \cup (b_{\lambda_0}, b)$, for otherwise $z = 1$ outside $(-b_{\lambda_0}, b_{\lambda_0})$ and the conclusion holds trivially. We claim that for any $s \in (a, -b_{\lambda_0}) \cup (b_{\lambda_0}, b)$ at which $\theta'(s) > 0$, there is an open interval $I$ of length $2b_{\lambda_0}$ containing $s$ such that $\alpha(I)$ differs from the image of $\alpha_{\lambda_0}$ by a horizontal vector. We will prove the claim in the case $b_{\lambda_0} < s < b$ (the proof in the case $a < s < -b_{\lambda_0}$ is analogous). Denote by $(a_2, b_2) \subset (a, b)$ the maximal interval containing $s$ on which $\theta' > 0$. By what we have
already proved (cf. Eq. (4.32)), \( \alpha'((a_2, b_2)) \) coincides with a horizontal translation of the image of \( \alpha_{\lambda_0} \), for some \( \lambda > 1 \). Since \( a_2 > -\infty \) and \( z(a_2) = 1 \) (by Lemma 2.2(i), since \( \theta'(a_2) = 0 \)), we have \( \lambda = \lambda_0 \) and the claim is proved. Let \( I, J \) be subintervals of \((a, b)\) such that \( \alpha(I) \) and \( \alpha(J) \) are both horizontal translations of the image of \( \alpha_{\lambda_0} \). If the distance between \( I \) and \( J \) is positive but less than \( 2b_{\lambda_0} \), then, by the previous claim, one has \( \theta' = 0 \), and hence \( z = 1 \), in the interval between \( I \) and \( J \). It is now clear that the image of \( \alpha \) is made up of curves congruent to \( \alpha_{\lambda_0} \) and eventually of horizontal segments of height equal to 1. Therefore, \( \alpha \in \mathcal{A} \) and so \( M \in \mathcal{F}_2 \).

\[ \square \]

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