Proper connection number and graph products *

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Abstract

A path \( P \) in an edge-colored graph \( G \) is called a proper path if no two adjacent edges of \( P \) are colored the same, and \( G \) is proper connected if every two vertices of \( G \) are connected by a proper path in \( G \). The proper connection number of a connected graph \( G \), denoted by \( pc(G) \), is the minimum number of colors that are needed to make \( G \) proper connected. In this paper, we study the proper connection number on the lexicographical, strong, Cartesian, and direct product and present several upper bounds for these products of graphs.

Keywords: connectivity; vertex-coloring; proper path; proper connection number; direct product; lexicographic product; Cartesian product; strong product.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [3]. For a graph \( G \), we use \( V(G) \), \( E(G) \), \( n(G) \), \( m(G) \), \( \delta(G) \), \( \kappa(G) \), \( \kappa'(G) \), \( \delta(G) \) and \( \text{diam}(G) \) to denote the vertex set, edge set, number of vertices, number of edges, connectivity, edge-connectivity, minimum degree and diameter of \( G \), respectively. The rainbow connections of a graph which are applied to measure the safety of a network are introduced by Chartrand, Johns, McKeon and Zhang [7]. Readers can see [7, 8, 9] for details. An edge-coloring of a graph \( G \) is an assignment \( c \) of colors to the edges of \( G \), one color to each edge of \( G \). Consider an edge-coloring (not necessarily proper) of a graph \( G = (V, E) \). We say that a path of \( G \) is rainbow, if no two edges on the path have the same color. An edge-colored graph \( G \) is rainbow connected if every two vertices are connected by a rainbow path. The

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minimum number of colors required to rainbow color a graph $G$ is called the rainbow connection number, denoted by $rc(G)$. For more results on the rainbow connection, we refer to the survey paper [15] of Li, Shi and Sun and a new book [16] of Li and Sun.

If adjacent edges of $G$ are assigned different colors by $c$, then $c$ is a proper (edge-)coloring. The minimum number of colors needed in a proper coloring of $G$ is referred to as the chromatic index of $G$ and denoted by $\chi'(G)$. Recently, Andrews, Laforge, Lumduanhom and Zhang [1] introduce the concept of proper-path colorings. Let $G$ be an edge-colored graph, where adjacent edges may be colored the same. A path $P$ in $G$ is called a proper path if no two adjacent edges of $P$ are colored the same. An edge-coloring $c$ is a proper-path coloring of a connected graph $G$ if every pair of distinct vertices $u, v$ of $G$ is connected by a proper $u$-$v$ path in $G$. A graph with a proper-path coloring is said to be proper connected. If $k$ colors are used, then $c$ is referred to as a proper-path $k$-coloring. The minimum number of colors needed to produce a proper-path coloring of $G$ is called the proper connection number of $G$, denoted by $pc(G)$.

Let $G$ be a nontrivial connected graph of order $n$ and size $m$. Then the proper connection number of $G$ has the following bounds.

$$1 \leq pc(G) \leq \min\{\chi'(G), rc(G)\} \leq m.$$ 

Furthermore, $pc(G) = 1$ if and only if $G = K_n$ and $pc(G) = m$ if and only if $G = K_{1,m}$ is a star of order $m + 1$. For more details on the proper connection number, we refer to [1] [17] [21].

The standard products (Cartesian, direct, strong, and lexicographic) draw a constant attention of graph research community, see some recent papers [2] [27] [31] [34].

In this paper, we consider four standard products: the lexicographic, the strong, the Cartesian and the direct with respect to the proper connection number. Every of these four products will be treated in one of the forthcoming sections.

## 2 The Cartesian product

The Cartesian product of two graphs $G$ and $H$, written as $G \square H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices $(g, h)$ and $(g', h')$ are adjacent if and only if $g = g'$ and $(h, h') \in E(H)$, or $h = h'$ and $(g, g') \in E(G)$. Clearly, the Cartesian product is commutative, that is, $G \square H$ is isomorphic to $H \square G$.

**Lemma 1** [13] Let $gh$ and $g'h'$ be two vertices of $G \square H$. Then

$$d_{G \square H}(gh, g'h') = d_{G}(gg') + d_{H}(hh').$$
Theorem 1 Let $G$ and $H$ be connected graphs with $|V(G)| \geq 2$ and $|V(H)| \geq 2$. Then

$$pc(G \Box H) \leq \min\{pc(G), pc(H)\} + 1.$$ 

Moreover, the bound is sharp.

Proof. Without loss of generality, we assume $pc(H) \leq pc(G)$. Suppose $\{0, 1, \cdots, pc(H) - 1\}$ be a proper coloring of $H$. Clearly, Since $G$ is connected, there is a path connecting $g$ and $g'$, say $P = g_1g_2 \cdots g_{\ell-1}g'$ where $g' = g_{\ell}$. By the same reason, there is a path connecting $h$ and $h'$, say $Q = hh_1 \cdots h_{k-1}h'$ where $h' = h_k$. Now we give a coloring of $G \Box H$ using $pc(H) + 1$ colors. To show that $pc(G \Box H) \leq pc(H) + 1$, we provide a proper-coloring $c$ of $G \Box H$ with $pc(H) + 1$ colors as follows.

$$\begin{cases} 
    c(gh_s, gh_t) = c(h_sh_t), & \text{if } s \neq t, \\
    c(g_ih, gjh) = pc(H) + 1, & \text{if } i \neq j;
\end{cases}$$

It suffices to check that there is a proper-path between any two vertices $(g, h), (g', h')$ in $G \Box H$. If $g = g'$ or $h = h'$, then $P$ or $Q$, respectively, is a trivial one vertex path. We distinguish the following two cases to prove this theorem.

Case 1. $h = h'$

If $\ell$ is even, then we let $h_1$ be an arbitrary neighbor of $h$. The path induced by the edges in

$$\{(gh, g_1h), (g_1h, g_1h_1), (g_1h_1, g_2h_1), \cdots, (g_{\ell-1}h_1, g'_{h_1}), (g'h_1, g'h')\}$$

is proper $(g, h), (g', h')$-path in $G \Box H$.

If $\ell$ is odd, then we let $h_1$ be an arbitrary neighbor of $h$. The path induced by the edges in

$$\{(gh, g_1h), (g_1h, g_1h_1), (g_1h_1, g_2h_1), \cdots, (g_{\ell-1}h_1, g_{\ell-1}h), (g_{\ell-1}h, g'h')\}$$

is proper $(g, h), (g', h')$-path in $G \Box H$.

Case 2. $h \neq h'$

If $g = g'$, then $(g, h), (g', h') \in H(g)$. Clearly, there is a proper-path connecting $(g, h)$ and $(g', h')$. Now we consider $g \neq g'$. If $\ell$ is even, then we let $h_1$ be an arbitrary neighbor of $h$. The path induced by the edges in

$$\{(gh, g_1h), (g_1h, g_1h_1), (g_1h_1, g_2h_1), \cdots, (g_{\ell-1}h_1, g_{\ell-1}h), (g_{\ell-1}h, g'h_2)$$

$$, (g'h_2, g'h_3) \cdots (g'h_{k-1}, g'h')\}$$

is proper $(g, h), (g', h')$-path in $G \Box H$.
If \( \ell \) is odd, then we let \( h_1 \) be an arbitrary neighbor of \( h \). The path induced by the edges in
\[
\{(gh, g_1 h), (g_1 h, g_1 h_1), (g_1 h_1, g_2 h_1), \ldots, (g_{\ell-1} h_1, g_\ell h), (g_{\ell-1} h, g'h)
\}
\]
is a proper \((g, h), (g', h')\)-path in \( G \Box H \). \( \square \)

To show the sharpness of the above bound, we consider the following example.

**Example 1:** Let \( G = P_2 \) and \( H = K_n \). Then \( pc(G \Box H) \leq \min\{pc(G), pc(H)\} + 1 = 2 \) by Theorem 1. From Lemma 1, we have \( diam(G \Box H) = diam(G) + diam(H) = 2 \) and hence \( pc(G \Box H) \geq 2 \). Therefore, \( pc(G \Box H) = 2 = \min\{pc(G), pc(H)\} + 1 \).

### 3 The strong product

The strong product \( G \boxtimes H \) of graphs \( G \) and \( H \) has the vertex set \( V(G) \times V(H) \). Two vertices \((g, h)\) and \((g', h')\) are adjacent whenever \( gg' \in E(G) \) and \( h = h' \), or \( g = g' \) and \( hh' \in E(H) \), or \( gg' \in E(G) \) and \( hh' \in E(H) \).

**Lemma 2** [13] If \( G \) is a nontrivial connected graph and \( H \) is a connected spanning subgraph of \( G \), then \( pc(G) \leq pc(H) \).

The strong product is connected whenever both factors are and the vertex connectivity of the strong product was solved recently by Spacapan in [23].

By Lemma 2 we have \( pc(G \boxtimes H) \leq pc(G \Box H) \). By Theorem 1 the following proposition is immediate.

**Proposition 1** Let \( G \) and \( H \) be connected graphs. Then \( pc(G \boxtimes H) \leq \min\{pc(G), pc(H)\} + 1 \).

Moreover, the bound is sharp.

**Lemma 3** [13] Let \( gh \) and \( g'h' \) be two vertices of \( G \Box H \). Then
\[
d_{G \Box H}(gh, g'h') = \max\{d_G(gh'), d_H(hh')\}.
\]

To show the sharpness of the upper bound in Proposition 1 we consider the following example.

**Example 2:** Let \( G = P_n \) be a complete graph and \( H = P_2 \). From Proposition 1 we have \( pc(G \boxtimes H) \leq \min\{pc(G), pc(H)\} + 1 = 2 \). By Lemma 3 \( diam(G \boxtimes H) \geq 2 \) and hence \( pc(G \boxtimes H) \leq 2 \). Therefore, \( pc(G \boxtimes H) = 2 = \min\{pc(G), pc(H)\} + 1 \).
4 The lexicographical product

The lexicographical product $G \circ H$ of graphs $G$ and $H$ has the vertex set $V(G \circ H) = V(G) \times V(H)$. Two vertices $(g, h), (g', h')$ are adjacent if $gg' \in E(G)$, or if $g = g'$ and $h h' \in E(H)$. The lexicographical product is not commutative and is connected whenever $G$ is connected.

In this section, let $G$ and $H$ be two connected graphs with $V(G) = \{g_1, g_2, \ldots, g_n\}$ and $V(H) = \{h_1, h_2, \ldots, h_m\}$, respectively. Then $V(G \circ H) = \{(g_i, h_j) | 1 \leq i \leq n, 1 \leq j \leq m\}$. For $h \in V(H)$, we use $G(h)$ to denote the subgraph of $G \circ H$ induced by the vertex set $\{(g_i, h) | 1 \leq i \leq n\}$. Similarly, for $g \in V(G)$, we use $H(g)$ to denote the subgraph of $G \circ H$ induced by the vertex set $\{(g, h_j) | 1 \leq j \leq m\}$.

**Theorem 2** Let $G$ and $H$ be connected graphs.

(i) For $pc(G), pc(H) \geq 2$, we have

\[
\begin{align*}
pc(G \circ H) &\leq pc(H), \quad \text{if } pc(G) > pc(H); \\
\text{or } pc(G \circ H) &\leq pc(G) + 1, \quad \text{if } pc(G) < pc(H); \\
\text{or } pc(G \circ H) &\leq pc(G), \quad \text{if } pc(G) = pc(H).
\end{align*}
\]

(ii) If $pc(G) = 1, pc(H) \geq 2$, then $pc(G \circ H) = 2$;

(iii) If $pc(H) = 1, pc(G) \geq 2$, then $pc(G \circ H) = 2$;

(iv) If $pc(G) = 1, pc(H) = 1$, then $pc(G \circ H) = 1$.

Moreover, the bound is sharp.

**Proof.** (i) If $pc(G) > pc(H)$, then we give a coloring of $G \circ H$ using $pc(H)$ colors. Suppose $pc(H) = \{1, 2, \ldots, pc(H)\}$ is a proper-coloring of $H$. We color the edges $c(gh_i, gh_j)$ $(i \neq j)$ the same as $H$, and the edges $c(g_i h_s, g_j h_t) = 1$ $(i \neq j)$. It suffices to check that there is a proper-path between any two vertices $(g, h), (g', h')$ in $G \circ H$. If $g = g'$, then there is a proper path in $H(g)$ as desired. Now suppose $g \neq g'$. Since $pc(H) \geq 2$, there is an edge $h_i h_j \in E(H)$ such that $c(h_i h_j) \neq 1$. The path induced by the edges in

\[
\{(gh, gh_1), (gh_1, gh_2), \ldots, (gh_{i-1} gh_j, gh_i gh_j), (gh_i gh_j, g_1 h_j), (g_1 h_j, g_1 h_i),
\]

\[
(g_1 h_i, g_2 h_j), (g_2 h_j, g_2 h_i), \ldots, (g_{i-1} h_j, g' h')\}
\]

is a proper-path connected $gh$ and $g' h'$.

If $pc(G) < pc(H)$, then $pc(G \circ H) \leq pc(G) + 1$ by Lemma[2] and Theorem[1].

If $pc(G) = pc(H)$, then we color $G \circ H$ as follows.

\[
\begin{align*}
c(g, h, g_j h) & = c(g, g_j), \quad \text{if } i \neq j; \\
c(gh_s, gh_t) & = c(h_s h_t), \quad \text{if } s \neq t; \\
c(g, h_s, g_j h_t) & = c(g, g_j), \quad \text{if } i \neq j \text{ and } s \neq t.
\end{align*}
\]
It suffices to check that there is a proper-path between any two vertices \((g, h), (g', h')\) in \(G \circ H\). If \(h = h'\), then there is a proper-path connecting \((g, h)\) and \((g', h')\) in \(G(h)\), as desired. Suppose \(h \neq h'\). If \(g = g'\), then \((g, h), (g', h') \in H(g)\). There is a proper-path connecting \((g, h)\) and \((g', h')\). We now assume \(g \neq g'\). Since \(G\) is connected, it follows that there is a proper-path connecting \(g\) and \(g'\) in \(G\), say \(P = gg_1g_2 \cdots g_{\ell-1}g'\). Then the path induced by the edges in \(\{(gh, g_1h), (g_1h, g_2h), \cdots (g_{\ell-1}h, g'h')\}\) is a proper-path connecting \((g, h)\) and \((g', h')\). Therefore, the above coloring is a proper-path coloring of \(G \circ H\), and hence \(pc(G \circ H) = pc(G) = pc(H)\).

(ii) If \(pc(G) = 1, pc(H) \geq 2\), then \(pc(G \circ H) \leq 2\) by Lemma 1 and Theorem 2. Since \(diam(G \circ H) \geq 2\), \(pc(G \circ H) \geq 2\). So \(pc(G \circ H) = 2\).

(iii) The same as (ii).

(iv) If \(pc(G) = 1, pc(H) = 1\), then both \(G\) and \(H\) are complete. So \(pc(G \circ H) = 1\).

To show the sharpness of the upper bound in Theorem 2, we consider the following example.

**Example 3:** Let \(G = P_n\) be a path of order \(n\) \((n \geq 2)\) and \(H = P_m\) be a path of order \(m\) \((m \geq 2)\). If \(m, n \geq 3\), then \(pc(G) = pc(H) = 2\), so \(pc(G \circ H) \leq pc(G) = pc(H) = 2\) by Theorem 2. Since \(diam(G \circ H) \geq 2\), \(pc(G \circ H) \geq 2\). So \(pc(G \circ H) = 2\). If \(m = 2, n \geq 3\), then \(pc(H) = 1, pc(G) = 2\), so \(pc(G \circ H) \leq 2\) by Theorem 2. Since \(diam(G \circ H) \geq 2\), it follows that \(pc(G \circ H) \geq 2\). So \(pc(G \circ H) = 2\). If \(n = 2, m \geq 3\), then \(pc(G) = 1, pc(H) = 2\), then \(pc(G \circ H) = 2 \leq 2\) by Theorem 2. Since \(diam(G \circ H) \geq 2\), we have \(pc(G \circ H) \geq 2\). So \(pc(G \circ H) = 2\). If \(m = n = 2\), then \(pc(G) = 1, pc(H) = 1\) and \(pc(G \circ H) = 1\) by Theorem 2.

**Corollary 1** Let \(G\) and \(H\) be connected graphs, then \(pc(G \circ H) \leq \max\{pc(G), pc(H)\}\).

5 **The direct product**

The direct product \(G \times H\) of graphs \(G\) and \(H\) has the vertex set \(V(G) \times V(H)\). Two vertices \((g, h)\) and \((g', h')\) are adjacent if the projections on both coordinates are adjacent, i.e., \(gg' \in E(G)\) and \(hh' \in E(H)\). It is clearly commutative and associative also follows quickly. For more general properties we recommend [13]. The direct product is the most natural graph product in the sense of categories. But this also seems to be the reason that it is, in general, also the most elusive product of all standard products. For example, \(G \times H\) needs not to be connected even when both factors are. To gain connectedness of \(G \times H\) at least one factor must additionally be nonbipartite.
as shown by Weichsel [33]. Also, the distance formula
\[
d_{G \times H}((g,h), (g',h')) = \min\{\max\{d^e_G(g,g'), d^s_H(h,h')\}, \max\{d^o_G(g,g'), d^s_H(h,h')\}\}
\]
for the direct product is far more complicated as it is for other standard products. Here \(d^e_G(g,g')\) represents the length of a shortest even walk between \(g\) and \(g'\) in \(G\), and \(d^o_G(g,g')\) the length of a shortest odd walk between \(g\) and \(g'\) in \(G\). The formula was first shown in [25] and later in [19] in an equivalent version. There is no final solution for the connectivity of the direct product, only some partial results are known (see [4, 20]).

In this section we construct different upper bounds for the proper connection number of the direct product with respect to some invariants of the factors that are related to the rainbow vertex-connection number of the factors. A similar concept as for the distance formula is used and is due to the rainbow odd and even walks between vertices (and not only rainbow paths) and is thus, in a way, related with the formula. We say that \(G\) is \textit{odd-even proper connected} if there exists a proper colored odd path and a proper colored even path between every pair of (not necessarily different) vertices of \(G\). The \textit{odd-even proper connection number} of a graph \(G\), \(oepv(G)\), is the smallest number of colors needed for \(G\) to be odd-even proper connected and it equals infinity if no such a coloring exists. A bipartite graph has either only even or only odd paths between two fixed vertices, thus there is no odd-even proper coloring of such a graph. On the other hand, let \(G\) be a graph in which every vertex lies on some odd cycle. Then \(oepc(G)\) is finite since coloring every vertex with its own color produces an odd-even proper coloring of \(G\).

One can see that a odd cycle is an example where this coloring is optimal, and \(oepc(G) \leq |V(G)|\) for a connected graph \(G\).

It is also easy to see that \(oepc(K_3) = 3\). For \(n \geq 3\), and \(n\) is odd, \(oepc(C_n) = 3\). For \(n \geq 3\), and \(n\) is even, \(oepc(C_n) = 2\).

Let \(G\) be a graph. We split \(G\) into two spanning subgraphs \(O^G\) and \(B^G\), where the set \(E(O^G)\) consists of all edges of \(G\) that lie on some odd cycle of \(G\), and the set \(E(B^G) = E(G) \setminus E(O^G)\). Clearly, \(O^G\) and \(B^G\) are not always connected. Let \(O^G_1, O^G_2, \ldots, O^G_k\) and \(B^G_1, B^G_2, \ldots, B^G_\ell\) be components of \(O^G\) and \(B^G\), respectively, each one containing more than one vertex. Let
\[
o(G) = oepc(O^G_1) + oepc(O^G_2) + \cdots + oepc(O^G_k),
\]
and
\[
b(G) = pc(B^G_1) + pc(B^G_2) + \cdots + pc(B^G_\ell)
\]

Note that \(o(G)\) is finite since it is defined on nontrivial components \(O^G_i\), \(i \in \{1, 2, \ldots, k\}\).
Theorem 3 Let $G$ and $H$ be a nonbipartite connected graph. Then

$$pc(G \times H) \leq \min\{pc(H)((b(G) + o(G)), pc(G)(b(H) + o(H)))\}.$$ 

Proof. Without loss of generality, $pc(H)((b(G) + o(G)) \leq pc(G)(b(H) + o(H))$. Denote by $c_G^B$ an optimal proper-coloring of components of $B^G$. Let $c_G^O$ be an optimal odd-even proper-coloring of components of $O^G$.

We give a proper-coloring of $G \times H$ as follows. If $e \in E(G \times H)$ projects on $G$ to $e' \in B_G$, we set $c(e) = (c_G^B(e'), c_H(e''))$, and if $e$ projects on $G$ to $e' \in O_G$, we set $c(e) = (c_G^O(e'), c_H(e''))$. where $e'' \in E(H)$ is the projection of $e$ on $H$. By this way, we get a coloring of $V(G \times H)$ with $pc(H)(o(G) + b(G))$ colors and it remains to show that this is a rainbow coloring of $G \times H$.

Let $(g, h)$ and $(g', h')$ be arbitrary vertices from $G \times H$. Clearly, there is a proper path connecting $g$ and $g'$, say $P = gg_1, \ldots, g_{\ell-1}g'$. By the same reason, there is a proper path connecting $h$ and $h'$, say $Q = hh_1, \ldots, h_{k-1}h'$. Observe that $P$ is a shortest proper $g, g'$-path in $G$ induced by $B_G$ and $O_G$, and $Q$ is a shortest proper $h, h'$-path in $H$. If $g = g'$ or $h = h'$, then $P$ or $Q$, respectively, is a trivial one vertex path.

We distinguish the following two cases to prove this theorem.

Case 1. $\ell$ and $k$ have the same parity.

If $h = h'$, then we let $h_{k-1}$ be an arbitrary neighbor of $h$. Then the path induced by the edges in

$$\{(gh, g_1h_{k-1}), (g_1h_{k-1}, g_2h), (g_2h, g_3h_{k-1}), \ldots, (g_{\ell-1}h_{k-1}, g' h')\}$$

is a proper $(g, h), (g', h')$-path in $G \times H$.

If $g = g'$, then we let $g_{\ell-1}$ be an arbitrary neighbor of $g$. Then the path induced by the edges in

$$\{(gh, g_{\ell-1}h_1), (g_{\ell-1}h_1, gh_2), (gh_2, g_{\ell-1}h_3), \ldots, (g_{\ell-1}h_{k-1}, g' h')\}$$

is a vertex-rainbow $(g, h), (g', h')$-path in $G \times H$.

If $g \neq g'$, and $h \neq h'$, then the path induced by the edges in

$$\{(gh, g_1h_1), (g_1h_1, g_2h_2), \ldots, (g_kh', g_{k+1}h_{k-1}), (g_{k+1}h_{k-1}, g_{k+2}h') \ldots (g_{\ell-1}h_{k-1}, g' h')\}$$

is a proper $(g, h), (g', h')$-path in $G \times H$ whenever $\ell \geq k$, and the path induced by the edges in

$$\{(gh, g_1h_1), (g_1h_1, g_2h_2), \ldots, (g_{\ell-1}h_{\ell-1}, g' h'), (g' h', g_{\ell-1}h_{\ell+1}), \ldots, (g_{\ell-1}h_{k-1}, g' h')\}$$

is a proper $(g, h), (g', h')$-path in $G \times H$ whenever $\ell < k$.

Case 2. $\ell$ and $k$ have different parity.
If there exists a $g_i, g_j$-subpath of $P$ in $O_p^G$, we replace this subpath by a rainbow $g_i, g_j$-path of different parity in $O_p^G$ to obtain a proper path $P'$ between $g$ and $g'$. If this is the case, then $|E(P')|$ and $k$ have the same parity and we can use Case 1.

We now assume that all the $g_i, g_j$-subpaths of $P$ in $B_p^G$, that is, all vertices of $P$ are in $B_p^G$. To find a proper $(g, h), (g', h')$-path in $G \times H$, we find out a $g, g'$-walk in $G$. Note that $P$ is contained in one component $B_q^G$. Let $g_i \in V(P)$ be a vertex that is closest to any component $O_p^G$ of $G$ and let $v_1 \in O_p^G$ be closest to $g_i$. Let $R = g_i g_{i+1} \ldots, g_{i+r}$ ($g_{i+r} = v_1$) be a shortest $g_i, v_1$-path. From the definition of odd-even rainbow vertex-coloring, we know that there exists an odd vertex-rainbow $v_1, v_1$-cycle $C = v_1 v_2 \ldots v_p v_1$ in $O_p^G$. Now we insert a closed walk that follows $R C R$ from $g_i$ into a path $P$ to obtain a $g, g'$-walk

$$W = gg_1 \ldots g_i g_{i+1} \ldots g_{i+r} v_2 v_3 \ldots v_p v_1 g_{i+r-1} g_{i+r-2} \ldots g_{i+1} g_i g_{i+1} \ldots g'$$

of length $t = \ell + 2r + p$. Note that $t$ and $\ell$ have different parity since $p$ is an odd number, and thus $t$ and $k$ have the same parity. If $k \geq t$, then the path induced by the edges in

$$\{(u_0 h, u_1 h_1), (u_1 h_1, u_2 h_2), \ldots (u_t h_t, u_{t+1} h_{t+1}), (u_{t-1} h_{t+1}, u_t h_{t+2}), \ldots (u_{t-1} h_{k-1}, u_t h')\}$$

is a proper-coloring connected $gh$ and $g'h'$.

If $k < t$, then the path induced by the edges in

$$\{(u_0 h, u_1 h_1), (u_1 h_1, u_2 h_2), \ldots (u_k h_{k-1}, u_k h'), (u_k h', u_{k+1} h_{k-1}), \ldots (u_{t-1} h_{k-1}, u_t h')\}$$

is a proper-coloring connected $gh$ and $g'h'$. \hfill \Box

**Corollary 2** Let $G$ and $H$ be connected graphs, where $G$ is nonbipartite and $H$ is bipartite. Then

$$pc(G \times H) \leq pc(H)(b(G) + o(G))$$

A bipartite graph $G = (V_0 \cup V_1, E)$ is said to have a property $\pi$ if $G$ admits of an automorphism $\psi$ such that $x \in V_0$ if and only if $\psi(x) \in V_1$. For more details, we refer to $\text{[23]}$.

**Lemma 4** $\text{[23]}$ If $G$ and $H$ are bipartite graphs one of which has property $\pi$, then the two components of $G \times H$ are isomorphic.

**Proposition 2** Let $G$ be a nonbipartite connected graph. Then

$$pc(G \times K_2) \leq o(G) + b(G).$$
Proof. Let \( c_G^o \) be an optimal odd-even proper-coloring of \( O^G \) and let \( c_G^B \) be an optimal proper-coloring of \( B^G \) (for both cases it holds that no color appears in two different components). Observe that \( c_G^o = o(G) \) and \( c_G^B = b(G) \). We provide a coloring \( c \) of \( G \times K_2 \) with \( o(G) + b(G) \) colors as follows.

Recall that \( O^G_1, O^G_2, \cdots, O^G_k \) and \( B^G_1, B^G_2, \cdots, B^G_\ell \) are all the components of \( O^G \) and \( B^G \), respectively. By the definition, \( B^G_i \) is bipartite graph. From Lemma 3 \( B^G_i \times K_2 \) can be decomposed into two subgraphs isomorphic to \( B^G_i \). Color both components of \( B^G_i \times K_2 \) (which are isomorphic to \( B^G_i \)) optimally with \( pc(B^G_i) \) colors for every \( i \in \{1, 2, \ldots, \ell\} \). For this we use \( b(G) \) colors. Now, we assign \( o(G) \) new colors to the remaining vertices. For an edge \((gh, g'h')\) of \( G \times K_2 \), it project on \( G \) to an edge \( gg' \) of \( O^G \) receive color \( c(gh, g'h') = c_G^o(gg') \). For an edge \((gh, g'h')\) of \( G \times K_2 \), it project on \( G \) to an edge \( gg' \) of \( B^G \) receive color \( c(gh, g'h') = c_G^B(gg') \). For the introduced coloring \( o(G) + b(G) \) colors are used and we need to show that \( c \) is a proper-coloring of \( G \times K_2 \).

Set \( V(K_2) = \{k_1, k_2\} \). Let \((g, h)\) and \((g', h')\) be arbitrary vertices in \( G \times K_2 \). Let \( P = gg_1 \cdots gg_{-1}g' \) be a proper \( g, g' \)-path under the proper-coloring of \( G \) induced by \( c_G^o \) and \( c_G^B \). We distinguish two cases to show this proposition.

Case 1. Let \( \ell \) and \( d_{K_2}(h, h') \) have the same parity.

Without loss of generality we may assume that \( h = k_1 \). Consequently \( h' = k_2 \) if \( \ell \) is an even number and \( h' = k_2 \) otherwise. Thus

\[
(gk_1)(g_1k_2)(g_2k_1) \cdots (g'h')
\]

is a proper \((g, h), (g', h')\)-path in \( G \times K_2 \).

Case 2. Let \( \ell \) and \( d_{K_2}(h, h') \) have different parity.

Suppose first that \( P \) has a nonempty intersection with some \( O^G_p \) and let \( g_i \) be the first and \( g_j \) the last vertex of \( P \) in \( O^G_p \). Then we can find a proper \( g_i, g_j \)-path in \( O^G_p \) with length of different parity as is the length of the \( g_i, g_j \)-subpath of \( P \) in \( O^G_p \). Replacing the \( g_i, g_j \)-subpath of \( P \) by this proper \( g_i, g_j \)-path in \( O^G_p \) we obtain a proper \( g, g' \)-path of the same parity as \( d_{K_2}(h, h') \) and we continue as in Case 1.

Suppose now that \( P \) has an empty intersection with every \( O^G_p \), \( p \in \{1, 2, \ldots, k\} \). Then \( P \) is contained in \( B^G_q \) for some \( q \), and \((g, h)\) and \((g', h')\) are in different components \((B^G_q)_1\) and \((B^G_q)_2\) of \( B^G_q \times K_2 \), respectively. Since \( G \) is nonbipartite, there exists a vertex \( g'' \) in some component of \( O^G_p \). Set \( \{h_r, h_s\} = \{k_1, k_2\} \). Take a proper path from \((g, h)\) to \((g'', h_r)\) in \((B^G_q)_1\), a proper odd path from \((g'', h_r)\) to \((g'', h_s)\) in \( O^G_p \), and a rainbow path from \((g'', h_s)\) to \((g', h')\) in \((B^G_q)_2\). This is a proper \((g, h), (g', h')\)-path in \( G \times K_2 \) since we have used different colors for \((B^G_q)_1, (B^G_q)_2\), and \( O^G_p \).
6 Applications

In this section, we demonstrate the usefulness of the proposed constructions by applying them to some instances of Cartesian and lexicographical product networks.

The following results will be used later.

**Lemma 5** Let \((gh)\) and \((g'h')\) be two vertices of \(G \circ H\). Let \(d_G(g)\) denote the degree of vertex \(g\) in \(G\). Then

\[
d_{G \circ H}(gh, g'h') = \begin{cases} 
  d_G(gg'), & \text{if } g \neq g'; \\
  d_H(hh'), & \text{if } g = g' \text{ and } d_G(g) = 0; \\
  \min\{d_H(hh'), 2\}, & \text{if } g = g' \text{ and } d_G(g) \neq 0.
\end{cases}
\]

**6.1 Two-dimensional grid graph**

A two-dimensional grid graph is an \(m \times n\) graph \(G_{n,m}\) that is the graph Cartesian product \(P_n \square P_m\) of path graphs on \(m\) and \(n\) vertices. See Figure 1 (a) for the case \(m = 3\). For more details on grid graph, we refer to [5, 22]. The network \(P_n \circ P_m\) is the graph lexicographical product \(P_n \circ P_m\) of path graphs on \(m\) and \(n\) vertices. For more details on \(P_n \circ P_m\), we refer to [30]. See Figure 1 (b) for the case \(m = 3\).

![Figure 1](image)

Figure 1: (a) Two-dimensional grid graph \(G_{3,3}\); (b) The network \(P_3 \circ P_3\).

**Proposition 3** (i) For network \(P_n \square P_m\) \((n \geq 2, m \geq 2)\), \(2 \leq pc(P_n \square P_m) \leq 3\).

(ii) For network \(P_n \circ P_m\), \(pc(P_n \circ P_m) = 1\) when \(m = n = 2\), \(pc(P_n \circ P_m) = 2\) when \(m = 2, n > 2\) or \(n = 2, m > 2\) or \(m, n > 2\).

Proof. (i) By Theorem 1, we have \(pc(P_n \square P_m) \leq \min\{pc(P_n), pc(P_m)\} + 1 = 2 + 1 = 3\). Observe that \(diam(P_n \square P_m) \geq 2\). So \(2 \leq pc(P_n \square P_m) \leq 3\).

(ii) The same as Example 3.
6.2 $n$-dimensional mesh

An $n$-dimensional mesh is the Cartesian product of $n$ linear arrays. By this definition, two-dimensional grid graph is a 2-dimensional mesh. An $n$-dimensional hypercube is a special case of an $n$-dimensional mesh, in which the $n$ linear arrays are all of size 2; see [24].

Proposition 4 (i) For $n$-dimensional mesh $P_{L_1} \square P_{L_2} \square \cdots \square P_{L_n}$,

$$pc(P_{L_1} \square P_{L_2} \square \cdots \square P_{L_n}) = 2.$$ 

(ii) For network $P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}$, if there exists some $L_j$ such that $L_j \neq 2$ ($1 \leq j \leq n$), then $pc(P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}) = 2$; If $L_1 = L_2 = \cdots = L_n = 2$, then $pc(P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}) = 1$.

Proof. (i) By Lemma 1 we have $diam((P_{L_1} \square P_{L_2} \square \cdots \square P_{L_n}) = \sum_{i=1}^{n} diam(P_{L_i}) = \sum_{i=1}^{n} (L_i-1) = \sum_{i=1}^{n} L_i - n \geq 2$. By Theorem 2 $pc(P_{L_1} \square P_{L_2} \square \cdots \square P_{L_n}) \leq \min\{pc(P_{L_1}), pc(P_{L_2}), \cdots , pc(P_{L_n})\} + 1 = 2$. So $pc(P_{L_1} \square P_{L_2} \square \cdots \square P_{L_n}) = 2$.

(ii) If there exists some $L_j$ such that $L_j \neq 2$ ($1 \leq j \leq n$), then $pc(P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}) \leq \max\{P_{L_1}, P_{L_2}, \cdots , P_{L_n}\} = 2$ by Corollary 1. Since $diam(P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}) \geq 2$, $pc(P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}) \leq 2$. So $pc(P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}) = 2$.

If $L_1 = L_2 = \cdots = L_n = 2$, then $P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}$ is a complete graph. So $pc(P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}) = 1$.

6.3 $n$-dimensional torus

An $n$-dimensional torus is the Cartesian product of $n$ rings $R_1, R_2, \cdots, R_n$ of size at least three. (A ring is a cycle in Graph Theory.) The rings $R_i$ are not necessary to have the same size. Ku et al. [29] showed that there are $n$ edge-disjoint spanning trees in an $n$-dimensional torus. The network $R_1 \circ R_2 \circ \cdots \circ R_n$ is investigated in [30]. Here, we consider the networks constructed by $R_1 \square R_2 \square \cdots \square R_n$ and $R_1 \circ R_2 \circ \cdots \circ R_n$.

Proposition 5 (i) For network $R_1 \square R_2 \square \cdots \square R_n$,

$$2 \leq pc(R_1 \square R_2 \square \cdots \square R_n) \leq \min\{pc(R_1), pc(R_2), \cdots , pc(R_n)\} + 1 = 3$$

where $r_i$ is the order of $R_i$ and $3 \leq i \leq n$.

(ii) For network $R_1 \circ R_2 \circ \cdots \circ R_n$,

$$pc(R_1 \circ R_2 \circ \cdots \circ R_n) = 2.$$
Proof. (i) By Lemma [1] we have $diam(R_1 \square R_2 \square \cdots \square R_n) = \sum_{i=1}^n diam(R_i) = \sum_{i=1}^n [r_i/2] \geq 2$ and hence $pc(R_1 \square R_2 \square \cdots \square R_n) \geq 2$. By Theorem [4] we have

$$pc(R_1 \square R_2 \square \cdots \square R_n) \leq \min\{pc(R_1), pc(R_2), \cdots pc(R_n)\} + 1 = 3.$$ 

Therefore, $2 \leq pc(R_1 \square R_2 \square \cdots \square R_n) \leq 3$.

(ii) From Corollary [1] we have $pc(R_1 \circ R_2 \circ \cdots \circ R_n) \leq \max\{pc(R_1), pc(R_2), \cdots pc(R_n)\} = 2$. Since $diam(R_1 \circ R_2 \circ \cdots \circ R_n) \geq 2$, $pc(R_1 \circ R_2 \circ \cdots \circ R_n) \geq 2$. So $pc(R_1 \circ R_2 \circ \cdots \circ R_n) = 2$.

\[ \blacksquare \]

6.4 $n$-dimensional generalized hypercube

Let $K_m$ be a clique of $m$ vertices, $m \geq 2$. An $n$-dimensional generalized hypercube [11] [12] is the Cartesian product of $m$ cliques. We have the following:

**Proposition 6** (i) For network $K_{m_1} \square K_{m_2} \square \cdots \square K_{m_n}$ ($m_i \geq 2$, $n \geq 2$, $1 \leq i \leq n$)

$$pc(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_n}) = 2$$

(ii) For network $K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_n}$,

$$pc(K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_n}) = 1.$$ 

**Proof.** (1) Observe that $diam(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_n}) = \sum_{i=1}^n diam(K_{m_i}) = n \geq 2$. So $pc(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_n}) \geq 2$. By Theorem [4] we have $pc(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_n}) \leq \min\{pc(K_{m_1}), pc(K_{m_2}), \cdots pc(K_{m_n})\} + 1 = 2$. So $pc(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_n}) = 2$.

(2) Observe that $K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_n}$ is a complete graph. So $pc(K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_n}) = 1$. 

\[ \blacksquare \]

6.5 $n$-dimensional hyper Petersen network

An $n$-dimensional hyper Petersen network $HP_n$ is the Cartesian product of $Q_{n-3}$ and the well-known Petersen graph [10], where $n \geq 3$ and $Q_{n-3}$ denotes an $(n - 3)$-dimensional hypercube. The cases $n = 3$ and 4 of hyper Petersen networks are depicted in Figure 5. Note that $HP_3$ is just the Petersen graph (see Figure 5 (a)).

The network $HL_n$ is the lexicographical product of $Q_{n-3}$ and the Petersen graph, where $n \geq 3$ and $Q_{n-3}$ denotes an $(n - 3)$-dimensional hypercube; see [30]. Note that $HL_3$ is just the Petersen graph, and $HL_4$ is a graph obtained from two copies of the Petersen graph by add one edge between one vertex in a copy of the Petersen graph and one vertex in another copy. See Figure 5 (c) for an example (We only show the edges $v_1u_i$ ($1 \leq i \leq 10$)).
Figure 2: (a) Petersen graph; (b) The network $HP_4$; (c) The structure of $HL_4$.

**Proposition 7** (1) For network $HP_3$ and $HL_3$, $pc(HP_3) = pc(HL_3) = 2$;
(2) For network $HL_4$ and $HP_4$, $2 \leq pc(HP_4) \leq 3$ and $pc(HL_4) = 2$.

**Proof.** (1) Since $diam(HP_3) = diam(HL_3) = 2$, it follows that $pc(HP_3) = pc(HL_3) \geq 2$. One can check that there is a proper-coloring with two colors. So $pc(HP_3) = pc(HL_3) = 2$.

(2) From Theorem 1, $pc(HP_4) \leq 3$. Since $diam(HP_4) = 2$, it follows that $pc(HP_4) \geq 2$. So $pc(HP_4) = 2$. From Corollary 1 we have $pc(HL_4) \leq 2$. Since $diam(HL_4) = 2$, $pc(HL_4) \geq 2$. So $pc(HL_4) = 2$. 

**References**

[1] E. Andrews, E. Laforge, C. Lumduanhom, P. Zhang, *On proper-path colorings in graphs*, J. Combin. Math. Combin. Comput, to appear.

[2] B.S. Anand, M. Changat, S. Klavžar, I. Peterin, *Convex sets in lexicographic products of graphs*, Graphs Combin. 28(2012), 77–84.

[3] J.A. Bondy, U.S.R. Murty, *Graph Theory*, GTM 244, Springer, 2008.

[4] B. Brešar, S. Špacapan, *On the connectivity of the direct product of graphs*, Australas. J. Combin. 41(2008), 45–56.

[5] N.J. Calkin, H.S. Wilf, *The number of independent sets in a grid graph*, SIAM J. Discrete Math. 11(1)(1998), 54–60.

[6] S. Chakraborty, E. Fischer, A. Matsliah, R. Yuster, *Hardness and algorithms for rainbow connectivity*, 26th International Symposium on Theoretical Aspects of Com-
puter Science STACS (2009), 243–254. Also, see J. Combin. Optim. 21(2011), 330–347.

[7] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, *Rainbow connection in graphs*, Math. Bohem. 133(2008), 85–98.

[8] G. Chartrand, G. L. Johns, K. A. McKeon, P. Zhang, *The rainbow connectivity of a graph*, Networks 54(2009), 75–81.

[9] G. Chartrand, F. Okamoto, P. Zhang, *Rainbow trees in graphs and generalized connectivity*, Networks 55(2010), 360–367.

[10] S.K. Das, S.R. Öhring, A.K. Banerjee, *Embeddings into hyper Petersen network: Yet another hypercube-like interconnection topology*, VLSI Design, 2(4)(1995), 335–351.

[11] K. Day, A.-E. Al-Ayyoub, *The cross product of interconnection networks*, IEEE Trans. Parallel and Distributed Systems 8(2)(1997), 109–118.

[12] P. Fragopoulou, S.G. Akl, H. Meijer, *Optimal communication primitives on the generalized hypercube network*, IEEE Trans. Parallel and Distributed Computing 32(2)(1996), 173–187.

[13] R. Hammack, W. Imrich, Sandi Klavžr, *Handbook of product graphs*, Second edition, CRC Press, 2011.

[14] M. Krivelevich, R. Yuster, *The rainbow connection of a graph is (at most) reciprocal to its minimum degree three*, IWOCA 2009, LNCS 5874(2009), 432–437.

[15] X. Li, Y. Shi, Y. Sun, *Rainbow connections of graphs–A survey*, Graphs Combin. 29(1)(2013), 1–38.

[16] X. Li, Y. Sun, *Rainbow Connections of Graphs*, SpringerBriefs in Math., Springer, New York, 2012.

[17] X. Li, M. Wei, J. Yue, *Proper connection number and connected dominating sets*, arXiv 1501. 05717 v1 [math. CO] 23 Jan 2015.

[18] T. Gologranc, Gašper Mekiš, I. Peterin, *Rainbow connection and graph products*, 30(3)(2014), 591–607.

[19] A.A. Ghidewon, R. Hammack, *Centers of tensor product of graphs*, Ars Combin. 74(2005), 201–211.
[20] R. Guji, E. Vumar, A note on the connectivity of Kronecker products of graphs, Appl. Math. Lett. 22(2009), 1360–1363.

[21] F. Huang, X. Li, S. Wang, Proper connection numbers of complementary graphs, arXiv 1504. 02414 v2 [math. CO] 29 Apr 2015.

[22] A. Itai, M. Rodeh, The multi-tree approach to reliability in distributed networks, Information and Computation 79(1988), 43–59.

[23] P.K. Jha, S. Klavžar, B. Zmazek, Isomorphic components of Kronecker product of bipartite graphs, Discuss. Math. Graph Theory 17(1997), 301–309.

[24] S.L. Johnsson, C.T. Ho, Optimum broadcasting and personalized communication in hypercubes, IEEE Trans. Computers 38(9)(1989), 1249–1268.

[25] S.R. Kim, Centers of a tensor composite graph, Congr. Numer. 81 (1991) 193–203.

[26] S. Klavžar, G. Mekiš, On the rainbow connection of Cartesian products and their subgraphs, Discuss. Math. Graph Theory 32 (2012), 783–793.

[27] S. Klavžar, S. Špacapan, On the edge-connectivity of Cartesian product graphs, Asian-Eur. J. Math. 1 (2008), 93–98.

[28] M. Krivelevich, R. Yuster, The rainbow connection of a graph is (at most) reciprocal to its minimum degree, J. Graph Theory 63 (2009), 185–191.

[29] S. Ku, B. Wang, T. Hung, Constructing edge-disjoint spanning trees in product networks, Parallel and Distributed Systems, IEEE Transactions on parallel and disjoited systems 14(3) (2003), 213-221.

[30] Y. Mao, Path-connectivity of lexicographical product graphs, Int. J. Comput. Math., in press.

[31] R.J. Nowakowski, K. Seyffarth, Small cycle double covers of products. I. Lexicographic product with paths and cycles, J. Graph Theory 57 (2008), 99–123.

[32] S. Špacapan, Connectivity of strong products of graphs, Graphs Combin. 26 (2010), 457–467.

[33] P.M. Weichsel, The Kronecker product of graphs, Proc. Amer. Math. Soc. 13(1962), 47–52.

[34] X. Zhu, Game coloring the Cartesian product of graphs, J. Graph Theory 59(2008), 261–278.