Abstract: In this letter, I solve a model for the dynamics of vortices in a decaying two-dimensional turbulent fluid. The model describes their effective diffusion, and the merging of pairs of vortices of same vorticity sign, when they get too close. The merging process is characterized by the conservation of energy and of the quantity $N r^n$, where $r$ is the mean vortex radius, and $N$ their number. $n = 4$ corresponds to a constant peak vorticity, and $n = 2$ to a constant kurtosis. I found the scaling laws for various physical quantities ($r$, enstrophy, kurtosis...), and for instance, it is shown that $N \sim (t/\ln(t))^{-\frac{2n}{3n-4}}$ for $n > 2$, and $N \sim t^{-2}$ for $n = 2$, in good agreement with extensive numerical simulations. I also discuss some recent experiments in view of these results.
Recent experimental [1-2] and theoretical [3-7] works have emphasized the importance of coherent vortex dynamics during the decay of turbulence in $d = 2$. This process consists mainly in three stages: during an initial transient period, the fluid self-organized and a network of coherent vortices appears. Once the coherent vortices have emerged, vortices disappear essentially through merging of same sign vortices, such that their number decreases and their average size increases, in a process somewhat reminiscent of a coarsening stage [8][9]. Finally, when only one (or very few) dipole is left, it decays diffusively, due to the finite viscosity.

In order to describe this second stage of the dynamics of decaying turbulence, the authors of [6][7] introduced the following simple model. The starting point lies in the well known fact that the evolution dynamics of a set of far apart vortices is a conservative process, such that vortex centers positions $\{X_i\}$ evolve according to Kirchoff laws [10],

$$\Gamma_i \frac{dX_i}{dt} = \nabla_i \times \mathcal{H}, \quad \mathcal{H} = - \sum_{i,j} \Gamma_i \Gamma_j \ln |X_i - X_j| \quad (1)$$

where the circulation $\Gamma_i = \pi r_i^2 \omega_i$ is the area of the $i$-th vortex multiplied by its average core vorticity. In addition to the dynamics, one must describe the dissipation process when some vortices get too close. Experiments [1][2] or numerical simulations [5-7] have consistently shown that this happens mainly through a complicated merging process between vortices of same vorticity sign, while dipoles form a very stable state [2]. In [6-7], this was modelized by assuming that when two vortices with radii $r_1$ and $r_2$ are at a distance less than a $d_c(r_1, r_2)$, they instantly merge. The authors of [6] found that a consistent form for this minimal approach distance is $d_c(r_1, r_2) \approx 2.592r_2 + 0.609r_1^2/r_2$, for $r_1 \leq r_2$. In the present paper, the claim is made that the precise form of $d_c$ is not important as far as the critical properties of the model are concerned, provided $d_c$ is of order of the mean vortex radius $r$. When two vortices of same sign merge, we still need to define the properties of the resulting vortex. Motivated by their numerical results for the full Navier-Stokes equation, the authors of [5-6] were lead to impose local conservation of energy and a time independent typical peak vorticity $\omega$. This was also found to be
consistent with the experiment by Tabeling et al. [1]. Since the energy of a vortex is of order \( \Gamma^2 \sim \omega^2 r^4 \), the new vortex has a radius \( r_3 \) satisfying, \( r_3^4 = r_1^4 + r_2^4 \). On the other hand, following the observation of the experiment described in [2] (see below), we could alternatively impose that the energy and the total area occupied by vortices remains constant, which is essentially equivalent to have a constant kurtosis \( K \). The new merging rule is then, \( r_3^2 = r_1^2 + r_2^2 \), and \( \omega \) must decay in order to keep the energy constant. More generally, we will treat the general case of a dynamics with the conservation of the local “charge” \( q_i = r_i^n \).

We now derive the scaling relations between the number of vortices \( N \), their mean radius \( r \), their typical distance \( l \), the ratio \( Z/E \) (\( Z \) is the enstrophy), the kurtosis \( K \), and \( \omega/\sqrt{E} \). For large time, we expect that \( r \sim t^\nu \) [1-2][5-7][11][12]. The assumed conservation law, and the fact that \( E \sim N\omega^2 r^4 \), \( Z \sim N\omega^2 r^2 \), \( K \sim Nr^2 \), leads to,

\[
\begin{align*}
r &\sim t^\nu, \quad l \sim t^{\nu\frac{n}{2}}, \quad N \sim t^{-n\nu}, \\
\frac{Z}{E} &\sim t^{-2\nu}, \quad K \sim t^{(2-n)\nu}, \quad \frac{\omega}{\sqrt{E}} \sim t^{-(2-\frac{n}{2})\nu}
\end{align*}
\]

(2)

which was obtained in [5-6] for the case \( n = 4 \). We need \( n \geq 2 \) to keep \( l \geq r \), while for \( n < 2 \), one can show that \( N \sim N_0 \exp(-t/t_0) \) [9]. We see that the problem now amounts to determine the unknown exponent \( \nu \) which should depend explicitly on the conservation law considered in the dynamics. Ultimately, one should be able to infer on physical grounds (role of initial conditions [7], viscosity, or \( d = 3 \) effects [2]) which effective conservation law, or \( n \), is relevant in experiments or numerical simulations of (quasi-) two-dimensional decaying turbulence. I now focus on the relation between \( \nu \) and \( n \), and sketch a rather simple and physical derivation for it, while a more elaborate solution is possible (see conclusion). This solution is based on the mapping of the original problem on a solvable dynamics for a self-organized system [13-14]. This, maybe, gives a more solid justification for the description of the coherent vortex structure in terms of a self-organized system [15], as mentioned by some authors [6].

The first step of the solution uses the claim that the motion of vortices in the absence
of merging is effectively diffusive, which should result from the chaotic nature of their motion. From Kirchoff equations, and using the fact that the total circulation is zero, the mean square velocity of vortex centers is, 

\[ v^2 = \left\langle \sum_j \frac{\Gamma_j^2}{R_{ij}^2} \right\rangle \sim N \omega^2 r^4 \int_r^L R \frac{dR}{R^2} \sim E \ln(L/r), \]

where the upper cut-off is the linear size of the system \( L \), and the lower one is the mean vortex radius \( r \). Note that \( v^2 \) is strictly constant with the correct definition of energy \( E \sim N \omega^2 r^4 \ln(L/r) \), but decays slowly with the definition \( E \sim N \omega^2 r^4 \) considered in the simulations of [6] and also here, for the moment. The diffusion constant of a vortex is then of order \( D \sim v^2/\omega \), which varies in time if \( \omega \) is not constant. We can now consider separately the two gas of vortices with positive and negative vorticity, since the effect of one on the other is taken into account in the diffusion constant, and since mergings only involve same sign vortices [2][5-7]. It now remains to treat correctly the apparently complicated merging condition. This can be done by mapping the problem to an effective dynamics on a lattice of suitable constant \( a \), and by choosing consistently a time step \( \tau = a^2/D \). If one takes \( a = r \), the merging process is correctly described by saying that two charges add together when they hop on the same lattice site. Indeed, in this case their mutual distance is then less than the mean vortex radius, which is in accordance with the kind of merging conditions introduced above. We have thus separated the dynamics in two processes: (i) free diffusion of far apart charges (vortices), with a diffusion constant \( D \sim v^2/\omega \), and (ii) addition (merging) of charges hopping on the same site (vortices at a distance less that \( a = r \)). What makes the problem solvable is the fact that the dynamical equations for the time evolution of the \( q_i \)'s are now linear:

\[ q_i(t + \tau) = \sum_{|i-j|=a} w_{j \rightarrow i} q_j(t), \quad w_{j \rightarrow i} = 0 \text{ or } 1, \quad \sum_i w_{j \rightarrow i} = 1 \quad (3) \]

where the \( w_{j \rightarrow i} \)'s are random variables, and the last condition expresses that the vortex and its associated charge on site \( j \) hops on one of its neighboring sites (with equal probabilities). This problem can be solved exactly [13-14], even in the case when one adds a random charge \( I_i(t) \) to the left hand side of Eq. (3), which could simulate the driving noise in fully developed turbulence [9]. For the purpose of this letter, we do not need the
full solution of Eq. (3), but only need to know that the upper critical dimension beyond which mean-field theory is exact is \( d = 2 \) [16][14]. Assuming no correlation (mean-field), the number of collisions at time \( t \) is proportional to \( N^2 \). Then, \( \frac{dN}{dt} \sim -(a/L)^d N^2/\tau \), which immediately leads to \( N \sim (L/a)^d(t/\tau)^{-1} \), where \( L \) is the linear size of the system and \( d > 2 \) is the space dimension. For \( d = 2 \) and for \( n > 2 \) (\( l/r \) then diverges when \( t \to \infty \) and vortices are point-like objects), one can show exactly [17] the existence of logarithmic corrections in this kind of problems, such that \( N(t) \sim L^2 \ln(t/\tau)/(Dt) \).

In order to illustrate the quantitative effect of such a correction on the numerical determination of the decay exponent, I show in fig. 1 numerical simulations of Eq. (3) with a constant \( D = a^2/\tau \) on a 1500 \( \times \) 1500 square lattice with all sites initially occupied, and up to \( t = 10^5 \tau \). The best fit by the law \( N(t) = N_0(1 + t/t_0)^\alpha \) (as used in [6]) gives a remarkably linear fit with \( \alpha = 0.90 \pm 0.01 \) even after five decades in time (see fig. 1-a)! Of course, the plot of \( tN(t) \) vs \( \ln(t) \) immediately reveals the logarithmic correction (fig. 1-b). This remark motivates my next comment: there have been some claims that the vortex dynamics in decaying turbulence presents some similarities with the coarsening dynamics of the \( XY \) model [18], for which the number of spin vortices was apparently seen numerically to decay as \( N \sim t^{-0.75} \) [18-19], as in the Kirchoff vortex model [6]. Recent more extensive simulations [20], have shown that this apparent behavior is due to an important logarithmic correction, and that \( N \sim \ln(t)/t \), as expected from the theory of coarsening systems with non conserved order parameter [8][20]. Since the anomalous exponent \( \xi \) seen in the \( XY \) model seems to be an artifact of unanticipated logarithmic corrections, it is natural to challenge the observation of such exponents in the field of decaying turbulence. Notice however that the vortices in the \( XY \) model behave in a qualitatively quite different fashion as in a turbulent fluid [9]. First of all, they are quantized, and after a short transient time, only vortices with winding number \( \pm 1 \) remain [20][8]. Contrary to the merging dynamics of fluid vortices already described, that of the \( XY \) model only involves the annihilation of opposite (quantized) vortices.
After these preliminary remarks, I end the derivation of the scaling exponent $\nu$, and confirm the role of logarithmic corrections. Using the scaling equations Eq. (2), and the above result for $N(t)$, one easily finds, $r \sim (v^2 t / \ln(t))^\nu$, with $\nu = \frac{2}{3n-4}$, since $v$ does not behaves faster than a logarithm in time.

For the first case of a dynamics with conserved energy and constant $\omega$ ($n = 4$) studied in [5-6][7], the result is $r \sim (v^2 t / \ln(t))^{1/4}$, with $v = \ln(t_{\text{end}}/t)$, and where $t_{\text{end}}$ is of the order of the time for which only one (pair of positive and negative) vortex is left in the sample. We thus find $\nu = 1/4$, or in the language of [5-6], $\xi = 4\nu = 1$. The authors in [6] found $\xi = 0.72 \pm 0.02$. I will now show that the discrepancy with the present analytical result is certainly due to the two logarithmic corrections, the first one in $v$, and the second one coming from the critical $d = 2$ corrections, which both tend to lower the effective exponent $\xi$. In fact, it would have been more correct to consider the definition $E \sim N \omega^2 r^4 \ln(L/r)$ [6], taking into account the long range interactions between vortices, leading to the local conservation of $r^4 \ln(L/r)$. This implies that the typical vortex square velocity $v^2$ is now constant. Still, there remains the most important logarithmic correction (due to $d = 2$), which in principle, should be present in the numerical solution of the full Navier-Stokes equation, for which the authors of [6] also found $\xi \approx 0.70 \sim 0.75$. In fig. 1, we show the result of numerical simulations of the problem of diffusing vortices merging when their distance is less than $d_c(r_1, r_2) = r_1 + r_2$, and with local conservation of $r^4$, but with a constant $v^2$ (and $D$ since $\omega$ is constant for $n = 4$), in order to avoid the introduction of extra logarithmic corrections (present in [6]).

One finds $\xi = 4\nu = 0.84 \pm 0.01$ (fig. 1-a), from a fit to the form $N(t) \sim N_0 (1 + t/t_0)^{-\xi}$ as used in [5-6], even when using much bigger samples than in [6-7] ($N_0 = 20000$ instead of $N_0 \sim 400$ in [6] and $N_0 = 300$ in [7]), which is allowed by the simpler nature of the model. However, for this model, the exact solution presented here predicts $N(t) \sim \ln(t)/t$, which is illustrateded on fig. 1-b by plotting $tN(t)$ $vs$ $\ln(t)$. I now present numerical simulations of the full model introduced in [6-7], where the vortices evolve according to...
Kirchoff dynamics (Eq. (1)), and for which the same merging criterion as in [6] as been used. The calculations are much heavier than for the simple diffusing vortex model, which drastically limits the number of vortices and integration time that one can simulate. In order to obtain the properties of the scaling regime, the authors of [6] introduced a clever renormalization scheme: they have simulated the decay of the system from 400 to 100 vortices, and used the final configuration to generate a new initial condition with 400 vortices, by copying it four times. They repeated this renormalization scheme until they obtained an apparent scaling regime and enough statistics. The drawback of this method is that it only allows the precise measurement of the decay of the system during less than a decade in time, during which the number of vortices is divided by a factor 4 (in fact less [6][9]; see also fig. 1). They did not find a pure power law decay, but found a good fit to $N(t) \sim N_0(1 + t/t_0)^{-\xi}$, with $\xi \approx 0.72$. This form simulates the effective (sharp) increase with time of the exponent $\xi$, as they found quite large values for $t_0 \sim t_{end}/3$ ($t_{end}$ is the final time of the simulation). This scaling was shown in [11] to be consistent with the assumption that the population of vortices is at all time in local Boltzmann equilibrium. Note however that this assumption would be wrong in a coarsening dynamics (for instance for the $XY$ model), for which the correlation functions during the dynamics are unrelated to the equilibrium ones [8][9][20]. Such a scaling with roughly the same effective $n = 4$ and $\xi \approx 0.7$ was also observed experimentally in [1].

As far as the direct numerical simulation of the Navier-Stokes equation is concerned, the authors of [6] claim to see the same behavior as described above, while some other groups discuss the existence of such growth exponents [21], or their values [22]. The simulations presented in fig. 1 use the same method as in [6], but with an initial and final number of vortices respectively equal to 1800 and 200 (instead of 400 and 100). The details will be given in [9]. The apparent exponent as extracted by the method of [6] is $\xi = 0.76 \pm 0.02$ (fig. 1-a), systematically larger than in [6]. In [7], where smaller samples ($N = 300$) and shorter times than in [6] where considered, the authors found an even lower $\xi \sim 0.6$. In
fig. 1-b, I show the plot of $t N(t)$ vs $\ln(t)$, which is found to be reasonably linear. Note that logarithmic corrections as found here due to the merging (coarsening) dynamics of point-like objects ($r \ll l$) in $d = 2$ could also be generated by taking into account [9] the long range interaction between vortices as in the $XY$ model [20]. Finally, the mean-field vortex radii distribution can be computed [9] using Eq. (3) and the results of [14]. Once radii are expressed in unit of the average radius $r$, the result is $P(x) = 4 \lambda x^3 \exp(-\lambda x^4)$, where $\lambda = \Gamma(5/4)$. The distribution is shown on fig. 2. The agreement with numerical simulations is satisfactory, the slight discrepancy being probably due to the difference between the merging distance used in the simulations ($d_c = r_1 + r_2$) and in the exact calculation ($d_c = r$), and the fact that some small correlation (non mean-field) effects should be present in $d = 2$. The distribution for the Kirchoff dynamics is slightly wider [6], but $d_c$ is probably too different from the merging condition in the solvable model.

We now study the case of the coherent vortex dynamics with conserved kurtosis. In other words, the area covered by the vortex cores remains constant. This phenomenologically reproduces the observed feature of the experiment presented in [2]. If we also impose conservation of energy, the maximal vorticity must then decay as $\omega \sim r^{-1}$ according to the relation $E \sim N r^4 \omega^2 \sim r^2 \omega^2$, for constant $K$. Noting that $D \sim v^2 / \omega \sim r \sim l$ increases with time, one finds $r \sim v^2 t$, such that $\nu = 1$, with no logarithmic corrections (for constant $v$) since $r \sim l$, and thus vortices are no more point-like objects [9]. This exactly reproduces Batchelor phenomenology [12], in the framework of a microscopic theory. I have simulated Kirchoff’s dynamics for $n = 2$, with initially $N = 1600$ vortices. After one renormalization step, $N$ is left decaying. In this case the dynamics is much faster than for $n = 4$, which permits to observe the decay up to $N \sim O(1)$ (fig. 3). The fit to the form $N(t) = N_0 (1 + t/t_0)^{-2}$ is excellent (fig. 3-a). Similar results are obtained (on more decades) for the diffusing vortex model with $D \sim r$, confirming $\nu = 1$ (fig. 3). In the experiment described in [2], the obtained scaling relations are exactly of the form given in Eq. (2) with $n = 2$, but the experimental estimate for $\nu$ (in the
first time decade) lies in the range $\nu \approx 0.18 \sim 0.26$. In fact, a naive determination of $\nu$ in a Kirchoff vortex simulation with initially $N_0 = 100$ vortices, in the time range for which $0.2N_0 \leq N \leq 0.8N_0$ (as in [2]), leads to $2\nu = 0.54 \pm 0.04$ in surprisingly good agreement with experiment (see fig. 3 with $N_0 = 1600$). Note that three-dimensional effects, leading to the diffusion of vorticity in the third dimension, could be at the origin of the constant area occupied by vortices in this experiment [2].

In conclusion, I have solved a simple model of diffusing vortices, with the effective diffusion constant derived from the Kirchoff dynamics and conservation laws. I have found long transient regimes (due to logarithmic corrections for $n > 2$) which, I argue, are at the origin of the smaller exponents observed in the literature. For a constant peak vorticity $\omega (n = 4)$, vortices are effectively diffusive ($D \sim v^2/\omega$ is then constant), while for a constant kurtosis $K (n = 2)$, their motion is effectively ballistic ($D \sim t$). In the presence of a finite viscosity (diffusive dissipation), one could expect that the first dynamics is more stable than the Batchelor regime, the motion of ballistic vortices being more easily damped than that of diffusing ones. Note finally that the problem of diffusing vortices can be solved in many other situations: finite driving noise, initial long range distribution of vortex radii (as numerically studied in [7]),... This, and more details on the numerics, as well as a new field theory for $d = 2$ decaying turbulence in the spirit of the vortex model will be presented in [9].

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FIGURE CAPTIONS

Fig. 1: Number of remaining vortices $N(t)$ after a time $t$ for the different models introduced in the text ($n = 4$). The time and number scales are arbitrary and have been chosen appropriately in order to compare the effective simulation time ranges. The initial numbers of particles are respectively $N = 2.25 \times 10^6$, $N = 20000$ and $N \sim 1500 < 1800$ due to mergings right after a renormalization step [6][9] (respectively 4, 10 and 10 samples). (a) fits to $N(t) = N_0(1 + t/t_0)^{-\xi}$ and the effective values for $\xi$. (b) $tN(t)$ is now plotted vs log$(1+t)$. For clarity, the curve for the Kirchoff dynamics has been enlarged by a factor 3, due to the much smaller time range, and the different curves have been shifted.

Fig. 2: Normalized vortex radii distributions obtained from the diffusing vortex simulation for different times. This is compared to the exact mean-field one (see text).

Fig. 3: For the Batchelor case ($n = 2$), $N(t)$ is shown for the diffusing vortex model (40 samples) and Kirchoff dynamics (10 samples). (a) In the same units, $N(t)$ is plotted vs $(1 + t)$. 
Fig. 1 (C. Sire)
Figure 2 (C. Sire)

\[ <P(r)> \text{ (t/1000=0.5,1,2,4,8,16)} \]

- Solid line: Mean-field theory
- Dashed line: \( t=500 \)
- Dash-dotted line: \( t=2000 \)
- Dashed-dotted line: \( t=16000 \)
Fig. 3 (C. Sire)