Linear Configurations of Complete Graphs $K_4$ and $K_5$

in $\mathbb{R}^3$, and Higher Dimensional Analogs

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Abstract

We show that the space of images of linear embeddings of a tetrahedral graph in $\mathbb{R}^3$ has the homotopy type of the double mapping cylinder $SO(3)/T \leftarrow SO(3)/\mathbb{Z}_3 \rightarrow SO(3)/D_3$, where $T$ is the symmetry group of a tetrahedron and $D_3$ is the symmetry group of a triangle. The fundamental group is presented here as an analog of braid groups and related to a subgroup of $\text{Aut}(F_3)$, generated by elements which square to inner automorphisms. The space of images of linear embeddings of the $K_5$ graph in $\mathbb{R}^3$ is shown to be homotopically equivalent to the same double mapping cylinder. This result is generalized to give a homotopy equivalence between linear configuration spaces of the $(n-2)$-skeleton of an $n$ simplex in $\mathbb{R}^n$ and linear configurations of the $(n-2)$-skeleton of an $(n+1)$-simplex in $\mathbb{R}^n$.

1 Introduction

The braid group arises as the fundamental group of the configuration space $C_n(\mathbb{R}^2)$ of $n$ points embedded in the plane, and is realized, from this picture, as a subgroup of the automorphisms of the free group. Specifically loops in $\pi_1(C_n(\mathbb{R}^2))$ act on the fundamental group $F_n$ of the complement of a configuration. This relationship was generalized to configurations of unlinked, unknotted $C^\infty$ circles in $\mathbb{R}^3$ in the thesis of Dahm [D62], and the homotopy type of the space of $C^\infty$-embedded, unlinked circles was described in [HB02]. A tetrahedral graph linearly embedded in $\mathbb{R}^3$ also has the property that its complement has a free fundamental group $F_3$, so that the fundamental group of the configuration space of unknotted, unlinked tetrahedral graphs in $\mathbb{R}^3$ maps to a subgroup of $\text{Aut}(F_3)$. The present work uses basic geometric arguments to calculate the homotopy type of the space of images of a tetrahedral graph, linearly embedded in $\mathbb{R}^3$. 

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In sections 2 through 4 we investigate the homotopy type of the space $C(K_4) = \text{Emb}(K_4, \mathbb{R}^3)/\Sigma_4$, where $K_4$ is the tetrahedral graph (which on occasion will be referred to as $(\Delta^3)^1$, the 1-skeleton of the 3-simplex), and $C, \text{Emb}$ denote linear configurations, linear embeddings, respectively. Here, $\Sigma_4$ is the symmetric group on 4 elements, a configuration is the image of an embedding, and an embedding is linear if it is affine linear when restricted to each edge. We will sometimes refer to a tetrahedral graph as a tetrahedron, even when it is planar, by a convenient abuse of terminology.

Section 5 gives a presentation of $\pi_1(C(K_4))$ in terms of a set of generators on which the automorphism group $\text{Aut}(\pi_1(C(K_4)))$ acts transitively, and this presentation is related to that given by the amalgamated free product obtained from the Van Kampen decomposition of the space.

Section 6 is a brief look at the action on the fundamental group $F_3$ of the complement of a configuration.

Section 7 introduces a higher dimensional analog where now we consider the $(n-2)$-skeleton of the $n$-simplex linearly embedded in $\mathbb{R}^n$. The result of this section is that this space has the homotopy type of the double mapping cylinder

$$SO(n)/A_{n+1} \leftarrow SO(n)/A_n \to SO(n)/\Sigma_n,$$

where $A_n$ and $\Sigma_n$ are alternating and symmetric groups, respectively.

Section 8 considers the case from section 7, but where we've increased the number of vertices by 1, while keeping the same dimension skeleton and ambient space. The main result of this section is that this configuration space is homotopy equivalent to the previous one.

In all cases the strategy is the same, to deformation retract the configuration space to a lower dimensional model which parametrizes a special subspace of symmetrical configurations, at which point the Van Kampen theorem makes evident the fundamental group. Since automorphisms of the simplex are finite, the space of embeddings (i.e., labeled) is a covering space, which is easy to understand and informs the higher homotopy groups of the configuration space.

Finally, we conclude the introduction with a few remarks. The homotopy equivalence of section 8 does not hold when we increase the simplex dimension again. For example, configurations of $K_6 = (\Delta^5)^1$ linearly embedded in $\mathbb{R}^3$ can contain 1 or 3 Hopf links depending on if they contain a trefoil or not [HJ07], so this configuration space is not connected. Nor does the homotopy equivalence hold for the case $n = 2$, where the respective spaces are $C_3(\mathbb{R}^2)$.
and $C_4(\mathbb{R}^2)$ (configurations of 3 and 4 points in the plane, respectively). Here, the skeleton of the simplex is not connected so the generalization given of the non-planarity of $K_5$ does not apply (as evidenced by the fact that $(\Delta^3)^0$ embeds in a 1-dimensional space) and the symmetry is different since for a square configuration of points, the diagonals are interchangable (i.e., the Radon point doesn’t distinguish a particular edge, as it does when $n > 2$). This case is well understood, as it gives Eilenberg-MacLane spaces for the braid group on 3 and 4 strands, respectively.

There are many possible directions to generalize the results contained here. One obvious direction is to consider the other Platonic graphs. When a platonic graph is embedded in $\mathbb{R}^2$, it has either 3, 4 or 5 extremal vertices, depending on the number of sides of one of its faces. I conjecture that the fundamental groups are amalgamated products of the binary symmetry groups of the regular solid with the dicyclic symmetry of the planar graph, amalgamated over the binary cyclic symmetries of a face, in all 5 cases.

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2 Space of Embeddings

The space $\widetilde{C(K_4)} = \text{Emb}(K_4, \mathbb{R}^3) \subset \mathbb{R}^{12}$ is precisely those ordered 4-tuples of points in $\mathbb{R}^3$ such that if all four points are coplanar, then one is in the interior of the triangle formed by the other three (the case of 2 being coincidental or 3 being collinear being special cases of this). The action of $\Sigma_4$ on $\widetilde{C(K_4)}$ gives a covering map to our space of interest, $C(K_4)$. The homotopy type of the covering space is given by our first theorem.

Theorem 2.1. The space $\widetilde{C(K_4)}$ of linear embeddings of the tetrahedral graph into $\mathbb{R}^3$ is homotopy equivalent to $(\vee^3 S^1) \times \text{SO}(3)$.

Proof. The space $\widetilde{C(K_4)}$ can be seen as a fiber bundle of possible positions for the fourth point over the space $\text{Emb}(K_3, \mathbb{R}^3)$ of labeled triangles in $\mathbb{R}^3$, where $\text{Emb}(K_3, \mathbb{R}^3)$ parametrizes the locations of the first three points. Fixing some
triangle $b \in \text{Emb}(K_3, \mathbb{R}^3)$ we see that the fourth point can be either outside of the plane spanned by $b$ or within that plane, either interior to $b$ or interior to one of the three cones extending the edges of $b$ (see figure 1).

![Diagram](image1.png)

**Figure 1:** The four regions a fourth point can occupy in the plane containing the other three, while still forming an embedded $K_4$.

The fiber over the fixed base point $b$ is thus seen to deformation retract to a wedge of three circles (see figure 2).

![Diagram](image2.png)

**Figure 2:** We remove a thickening of the planar subset of $\mathbb{R}^3$ where the fourth point would force an intersection of edges, leaving a space homotopy equivalent to $\bigvee^3 S^1$.

Furthermore, for an arbitrary fiber $F_b$, there is a unique orientation preserving affine linear transformation $T$ taking the triangle $b = \text{conv span}(b_0, b_1, b_2)$
to the standard labeled triangle \( c \), defined by \( c_0 = (0, 0, 0) \), \( c_1 = (1, 0, 0) \), \( c_2 = (0, 1, 0) \), such that \( T(b_i) = c_i \) and \( T(b_0 + (b_1 - b_0) \times (b_2 - b_0)) = (0, 0, 1) \) (see figure 3). The map \( T \), when restricted to a fiber, is a homeomorphism to the “typical” fiber \( F_c \) and varies continuously as the base point varies, thus the fiber bundle is trivial.

![Diagram](image)

Figure 3: The map \( T \) is defined by its action on \( b \), which it sends to \( c \), along with the specific extension sending \( b_0 + (b_1 - b_0) \times (b_2 - b_0) \) to \( (0, 0, 1) \).

Finally, the base space of labeled triangles in \( \mathbb{R}^3 \) is homotopy equivalent to \( SO(3) \), as is easily seen by regarding a labeled triangle \( b = \text{convex span}(b_0, b_1, b_2) \) as a basis \( ((b_1 - b_0), (b_2 - b_0), (b_1 - b_0) \times (b_2 - b_0)) \) and applying the Gram-Schmidt deformation retraction while translating to the origin. \( \square \)

### 3 Space of pyramids

A reasonable (i.e., compact, low-dimensional) model of \( C(K_4) \) is the subset of tetrahedral graphs having 3-fold symmetry. We call these pyramids, and denote the space by \( \mathcal{P} \). Specifically, \( \mathcal{P} \) will be those tetrahedra with 3 unit length edges and 3 edges of length \( \ell \in [3^{-1/2}, 1] \), with barycenter at the origin. Those with \( \ell = 1 \) are denoted \( \mathcal{R} \), for regular; those with \( \ell = 3^{-1/2} \) are denoted \( \mathcal{F} \), for flat.

The unlabeled pyramid space \( \mathcal{P} \) is covered by the space \( \tilde{\mathcal{P}} \) of labeled pyramids. From \( \tilde{\mathcal{P}} \), we have a map to \( SO(3) \) which forgets the 4th point \( v_4 \) and regards the isosceles triangle of \( v_1, v_2, v_3 \) as an equilateral labeled triangle centered at the origin, by the obvious rescaling, the space of which is parametrized by \( SO(3) \). Fixing a labeled triangle \( b \) in the image of this map, we have that \( b \) is mapped to by a set of tetrahedra parametrized by
the graph which is the suspension of 4 points, homotopically equivalent to a wedge of 3 circles. Specifically, the fourth point can make up a pyramid with \( b \) as its base, for which there is a line segment’s worth of choices, or the given base \( b \) can be the image of a pyramid with the isosceles face \( v_1, v_2, v_3 \) parallel to the plane containing \( b \). For each of \( i = 1, 2, 3 \) there is an arc of pyramids with apex \( v_i \) of the isosceles face \( v_1, v_2, v_3 \), so the preimage of a point is the graph with 4 edges, 2 vertices and no edges which are loops. There are local trivializations, which makes this map a fiber bundle, but in fact there is a global trivialization as the next theorem asserts.

**Theorem 3.1.** The space \( \tilde{\mathcal{P}} \) of labeled pyramids in \( \mathbb{R}^3 \) is homotopy equivalent to \((\bigvee^3 S^1) \times \text{SO}(3)\).

**Proof.** The map from \( \tilde{\mathcal{P}} \) to \( \text{SO}(3) \) is given above. The map to the suspension of 4 points is given by mapping the pyramid to the edge with the same label as the apex vertex of the pyramid. This edge is parametrized by the height of the apex. The edges are oriented, since in the labeled case, a triangle in \( \mathbb{R}^3 \) has a well-defined positive normal direction, so that heights vary in \([-3^{-1/2}, 3^{-1/2}]\). The vertices of the graph correspond to the two components of regular labeled tetrahedra. (See figure 4).

![Figure 4: The map from \( \tilde{\mathcal{P}} \) to the suspension of 4 points. Each labeled edge corresponds to the various heights for the corresponding apex vertex.](image-url)
4 Explicit deformation retraction from $C(K_4)$ to $\mathcal{P}$

In this section we give an explicit deformation retraction from $C(K_4)$ to $\mathcal{P}$. We begin with the regularization of non-degenerate tetrahedra.

We will require a signed-permutation-equivariant version of the Gram Schmidt process. For the sake of an explicit deformation retraction we use the method called Löwdin orthogonalization which is as follows [L50]. To each basis of $\mathbb{R}^3$ given as columns of the matrix $B$ define a path $p_B(t) = (1-t)B + tB(B^T B)^{-1/2}$. The terminal point is $B$ times the inverse of the positive square root of the matrix $B^T B$, which is a positive definite symmetric matrix. (Polar decomposition gives $B = OS$, where $O \in O(3)$ and $S = S^T$. Here $O$ is the terminal point and $S$ is $(B^T B)^{1/2}$) Multiplying this terminal point by its transpose, we have $(B(B^T B)^{-1/2}) \cdot (B(B^T B)^{-1/2})^T

= B(B^T B)^{-1/2}(B^T B)^{-1/2}B^T = BB^{-1}(B^T)^{-1}B^T = I,

so that $B(B^T B)^{-1/2}$ is orthogonal.

If we left act on the space by the linear transformation $B^{-1}$ we get $B^{-1}p_B(t) = (1-t)I + t(B^T B)^{-1/2}$ which is a convex combination of two positive definite symmetric matrices for each $t \in [0,1]$, so in particular it is invertible. Hence $p$ is a path in $GL(3)$ which terminates in $O(3)$. As we change $B$ continuously, $p_B$ changes continuously, so in fact we have a deformation retraction from $GL(3)$ to $O(3)$. Finally, let $P \in \mathbb{Z}_2 \wr \Sigma_3$ be a signed permutation matrix (i.e., a matrix with exactly one non-zero element valued in $\{\pm 1\}$ in each row and column). Then from $BP$ the defined path is

$p_{BP}(t) = (1-t)BP + tBP((BP)^T BP)^{-1/2}

= (1-t)BP + tBPBP^T(B^T B)^{-1/2}P

= ((1-t)B + tB(B^T B)^{-1/2}) \cdot P = p_B(t)P

so that the deformation retraction is equivariant with respect to the right $\mathbb{Z}_2 \wr \Sigma_3$ action. (Note, in fact, the same calculation shows equivariance for $O(3)$).

Theorem 4.1. The space of non-degenerate tetrahedra in $\mathbb{R}^3$ deformation retracts to $\mathcal{R}$, the space of regular tetrahedra.

Proof. To each tetrahedron we assign what we will call its bimedian basis which is the (unordered) collection of 3 line segments joining midpoints of
opposite (necessarily skew) edges (see figure 5). These line segments intersect at the barycenter, which bisects each line segment. It is easy to verify that $E$ the standard bimedian basis—i.e., the basis formed by the standard basis vectors and their negations—has exactly 2 tetrahedra which have $E$ for a bimedian basis, which differ by the reflection $-I$. Any other bimedian basis is then the image of this one under an invertible linear map (modulo translation, which we are not concerned with), so that the space of tetrahedra is a double cover of the space of bimedian bases. Any deformation retraction of $GL(3)$ to $O(3)$ which is $\mathbb{Z}_2 \wr \Sigma_3$-equivariant descends to a deformation retraction of bimedian bases to the orthonormal bimedian bases, which then lifts to the double cover, resulting in regular tetrahedra. Thus the Löwdin process gives a regularization of tetrahedra in $\mathbb{R}^3$. □

![Figure 5: The bimedian basis of a tetrahedron is shown in gray.](image)

As a side remark, the set of determinant 0 matrices in $\mathbb{R}^{n \times n}$ is in fact the cut locus of $O(n)$ under the Frobenius (i.e., Euclidean) norm, meaning the points in $\mathbb{R}^{n \times n}$ which have a unique nearest $O(n)$ element are precisely $GL(n)$. For general $n$, the Löwdin orthogonalization achieves the minimizing geodesic from $GL(n)$ to $O(n)$. (We omit the easy calculations showing this.) However, the method of picking a bimedian basis and thus lifting orthogonalization of bases to regularization of a simplex does not generalize to arbitrary $n$. We require a homomorphism from $\Sigma_{n+1}$, the symmetries of the $n$-simplex, to $\mathbb{Z}_2 \wr \Sigma_n$, the symmetries of the bimedian basis. The image of this homomorphism must at least generate $\Sigma_n < \mathbb{Z}_2 \wr \Sigma_n$, and so must be injective for $n > 3$, since $\Sigma_{n+1}$'s only normal subgroup is $A_{n+1}$. The respective orders are $(n+1)!$ and $2^n n!$, thus such a method can only exist when $n = 2^k - 1$ for some $k$. 
In section 7 and 8 we provide two more deformation retractions, which will generalize to similar cases in higher dimensions.

We now use the Löwdin regularization of non-degenerate tetrahedra to deformation retract all of \( C(K_4) \) to \( \mathcal{P} \).

**Theorem 4.2.** There is a deformation retraction from \( C(K_4) \) to \( \mathcal{P} \), the space of pyramids.

**Proof.** In \( C(K_4) \) we have degenerate tetrahedra with one solid angle of \( 2\pi \), but none with all solid angles 0, though we can decrease the maximum solid angle to any small \( \epsilon > 0 \). We thus have a surjective function which gives the maximum solid angle

\[ \alpha : C(K_4) \to (0, 2\pi]. \]

The sum of solid angles is in fact bounded by \( 2\pi \) [see section 7] so that any tetrahedron \( x \in \alpha^{-1}(\pi, 2\pi] \) has a vertex of uniquely greatest solid angle. For these tetrahedra, call the face opposite this vertex the *wide face* \( W_x \), and call the line perpendicular to the wide face at the barycenter of the wide face \( W_x^\perp \). For tetrahedra in \( \alpha^{-1}(\pi, 2\pi] \) any deformation retraction \( \Phi_t \) defines a map \( \phi_x : I \to \text{Aff}(\mathbb{R}^3) \), from the interval to affine transformations of \( \mathbb{R}^3 \), which sends a time \( t \) to the transformation which takes \( W_x \) to \( \Phi_t(W_x) \) and which takes \( W_x^\perp \) to \( \Phi_t(W_x^\perp) \), preserving distance and orientation. Let \( \eta \) be defined on tetrahedra in \( \alpha^{-1}(\pi, 2\pi] \) as the reparametrization

\[ \eta(x) = \frac{\alpha(x)}{\pi} - 1, \]

so that going from regular to degenerate, \( 1 - \eta(x) \) ranges from 1 to 0 (see fig 6). Note, for ease of understanding where this argument is going, that \( (\phi_x(t)^{-1} \circ \Phi_t)(x) \) is a path from \( x \), which keeps \( W_x \) fixed and ends in being height \( \sqrt{2/3} \) above \( W_x \), directly over its barycenter. We will use the solid angle of a vertex which is near its opposite face \( W_x \) as a parameter to slow down the motion of \( W_x \), so that in the limiting case of the degenerate tetrahedron, the extremal triangle will be fixed.

Let \( \psi_t \) be defined on \( \alpha^{-1}(\pi, 2\pi] \) as scaling by a factor of \( 1 - \eta(x) \) along \( W_{\Phi_t(x)}^\perp \) followed by \( (\phi_x(\eta(x)t))^{-1} \), and extended to \( \alpha^{-1}(0, \pi] \) with \( \psi_t|_{\alpha^{-1}(0, \pi]} \equiv \text{id} \). Consider the deformation retraction

\[ \Psi_t = \psi_t \circ \Phi_t \]

where \( \Phi_t \) is the deformation retraction given by the Löwdin orthogonalization above. The picture is this: as our tetrahedron approaches being degenerate, \( 1 - \eta \) approaches 0 and this parameter is used to scale the result of the regularization by \( \Phi_t \), to be a pyramid which becomes flatter as the greatest
solid angle becomes greater. The same parameter is used to undo some portion of how $\Phi_x$ moves $W_x$. In the limit $\eta \to 1$ the wide face is fixed under $\Psi_t$ (see figure 7).

Note that because $\Phi_t$ sends vertices along constant velocity paths, then so does $\Psi_t$. Then we can extend $\Phi_t$ to $\alpha^{-1}(2\pi)$ by moving the interior vertex $v$ along the straight-line path to the barycenter of $W_x$, while keeping $W_x$ fixed. This path is the limit of the straight-line paths taken by vertices of large solid angle vertex $v_t$ which approach $v$, by construction of the deformation retraction. Finally, we equilateralize the wide face while the vertex opposite is kept over the barycenter of the wide face by–extending isometrically in the perpendicular direction again–some choice of deformation retraction which equilateralizes (and normalizes the length of) triangles. The resulting space is $\mathcal{P}$.

This technique will be revisited in section 7. We give it the name damping, as $\alpha$ is used to impede the motion of the wide face under $\Phi_t$. 

Figure 6: The parameter $\eta$ varies from 0 (left) to 1 (right).

Figure 7: The solid angle opposite its wide face is used both to scale down the resulting regular tetrahedron and impede the motion of the wide face.
5 Presentations

Section 4 gave $\mathcal{P}$ as a low-dimensional, compact model of $C(K_4)$. This model makes clear the fundamental group, by an application of van Kampen’s theorem. Specifically, divide $\mathcal{P}$ into closed subsets $U$, of those with altitudes $a \geq \frac{1}{2} \sqrt{\frac{2}{3}}$, and $V$, with altitudes $a \leq \frac{1}{2} \sqrt{\frac{2}{3}}$. The first clearly deformation retracts to $\mathcal{R}$, the regular tetrahedra, and the second to $\mathcal{F}$, the planar pyramids. The intersection $U \cap V$ of half high pyramids is a neighborhood retract of $\mathcal{P}$, and has as fundamental group $2A_4 \cong \mathbb{Z}_6$, coming from lifting $A_3 < SO(3)$ to the double cover $S^3$. The symmetries of the regular tetrahedron sitting in $\mathbb{R}^3$ form a copy of $A_4$ inside $SO(3)$, so that $\pi_1(U) \cong \pi_1(\mathcal{R}) \cong 2A_4$, the double cover of $A_4$ by a subgroup of $S^3$. The symmetries of the equilateral triangle form a copy of $Dih_3 < SO(3)$, so that $\pi_1(V) \cong \pi_1(\mathcal{F}) \cong Dic_3$, the dicyclic group of order 12. Thus, from van Kampen’s theorem, we have

**Theorem 5.1.** The fundamental group of $C(K_4)$ is

$$2A_4 * 2A_3 Dic_3 \cong \langle X, R, S \mid X^2 = R^3 = S^3 = (SR)^3, XR = R^{-1}X \rangle$$

Here $X$ is the rotation of order 4 in $\pi_1(\mathcal{P})$ which reflects the planar tetrahedron (by a rotation of $\pi$ in a given direction), and $R, S$ are two face rotations as given in figure 8.

**Proof.** Given the above decomposition, this theorem follows directly from van Kampen’s theorem. □

Another presentation of $\pi_1(C(K_4))$ is given in terms of loops from a base point in $\mathcal{F}$ which transpose the center vertex and an extremal vertex by passing the center vertex up and over while passing the extremal vertex down and under. (figure 9 shows such a generator). This presentation has two advantages. First, it is particularly simple and is symmetric, in the sense that $Aut(\pi_1(C(K_4)))$ acts transitively on it. Second, it makes transparent the action of $\pi_1(C(K_4))$ on the free group on three generators $F_3$, the fundamental group of the complement of a given configuration, as section 6 explains.
Figure 8: The generators $X, R, S$. We consider $R, S$ rigid motions of the regular tetrahedron, while considering $X$ a rotation of $\pi$ of the planar tetrahedron.

**Theorem 5.2.** The fundamental group of $C(K_4)$ is generated by three elements $\{y_1, y_2, y_3\}$, subject to the following relations.

\begin{align*}
(y_j y_i^{-1})^3 &= (y_k y_l^{-1})^3 \text{ for neither side trivial,} & (i) \\
y_i y_j^{-1} y_i &= y_j^{-1} y_j y_i^{-1} \text{ for } i \neq j, & (ii) \\
y_k y_j^{-1} y_i y_j^{-1} y_k &= y_j^{-1} y_j y_i^{-1} \text{ for } i, j, k \text{ distinct.} & (iii)
\end{align*}

The isomorphism can be seen by observing one presentation in terms of the geometry of the other, the fine details of which we omit. The map is given by the identities

\[
\begin{align*}
S &= y_3^{-1} y_2 \\
R &= y_2^{-1} y_3 y_1^{-1} y_2 \\
X &= y_3^{-1} y_1 y_3^{-1}
\end{align*}
\quad \begin{align*}
y_3 &= X^{-1} S R^{-1} S^{-1} \\
y_2 &= R X^{-1} S R^{-1} S^{-1} R^{-1} \\
y_1 &= R^{-1} X^{-1} S R^{-1} S^{-1} R
\end{align*}
\]

In terms of these generators, the kernel $F_3 \times \mathbb{Z}_2$ of the map $\pi_1(C(K_4)) \rightarrow S_3$ is generated by each of the three $y_i^2$, for the left factor, and $\tau = (y_i y_j^{-1})^3$ (any two distinct $i, j$), for the right factor. Geometrically, this can be seen by viewing $y_i y_j^{-1}$ as a rotation of the tetrahedron by $2\pi/3$ so that it cubes to a rotation of $2\pi$, which explains the first set of relations (i). The second set of relations (ii) can be rewritten, by multiplying both sides by the left side, to state that $y_i y_j^{-1} y_i$ squares to $\tau$. Geometrically this is so, because $y_i y_j^{-1} y_i$...
Figure 9: The generator $y_1$ which transposes the center vertex with the one in position 1, by passing the center up and over while passing the extremal vertex down and under.

is effectively a rotation of $\pi$ about the edge $e_k$ which would get reversed by $y_k$ (see figure 10). The third set of relations (iii) can then be rewritten to state that conjugation of $y_k$ by this particular square root of $\tau$ inverts $y_k$. This is easily seen from the fact that the circle along which the end points of $e_k$ travel under the action of $y_k$ gets reversed in orientation by $y_i y_j^{-1} y_i$. It should be remarked that $\tau$ is thus central, as it commutes with $y_k$, for any $k$. Also, we note from (i) that $(y_i y_j^{-1})^3 = (y_j y_i^{-1})^3$ so that $(y_i y_j^{-1})^6 = \tau^2 = 1$.

It is worth noting that the families (i),(ii) and (iii) of relations above are independent in the sense that no two families generate the third. Without relation (iii) the quotient by the subgroup generated by $\{y_i^2\}$, $i = 1, 2, 3$, and $(y_i y_j)^3$ has the Cayley graph of figure 11. In particular it is not finite and so is not $\Sigma_4$, thus (iii) is independent. Restricting to a subgroup generated by two generators $y_i, y_j$ renders (iii) inconsequential, and gives $(y_i y_j^{-1})^3 = (y_j y_i^{-1})^3$ as the only consequence of (i), so that it’s easy to see (by a change of basis $h = y_i, g = y_i y_j^{-1}$, say) that (ii) is independent. In fact, by abelianizing this subgroup (i.e., by counting the exponents in a relator) we have relations $(6, -6) = 0$ from (i) and $(1, 1) = 0$ from (ii), in $\mathbb{Z}^2$, thus (i) is also independent.

6 Action of $\pi_1(C(K_4))$ on $F_3$

Our space $C(K_4)$ is a specific case of the more general space of images of the $(n - 2)$-skeleton of the $n$-simplex linearly embedded in $\mathbb{R}^n$. For the simplest
such case, $n = 2$, we get the configuration spaces of 3 points $(\Delta^2)^0$ in the
plane, both labeled and unlabeled, giving fundamental groups known classically as the pure braid group on 3 strands $PB_3$, and the braid group on 3 strands $B_3$, respectively. Both $PB_n$ and $B_n$ are realized as subgroups of $Aut(F_n)$ by considering the action of loops in the configuration space of $n$ points in the plane on the fundamental group of the complement of a configuration (i.e., the $n$-punctured disc), which is $F_n$. The situation for $\pi_1(C(K_4))$ is similar, since the complement of a linearly embedded tetrahedral graph in $\mathbb{R}^3$ has fundamental group $F_3$, but here a rotation of the tetrahedron by $2\pi$ effects the trivial action on $F_3$. That is, this loop, $\tau$, is in the kernel of the induced map $\psi : \pi_1(C(K_4)) \to Aut(F_3)$. By labeling the generators of $F_3$ in
correspondence with the $y_i$’s of $\pi_1(C(K_4))$, (see figure 12), we have that

$$\psi(y_i^2)(a) = a_i a_{a_i}^{-1},$$

for $a \in F_3$ and $a_i$ the generator of $F_3$ corresponding to $y_i$. Thus $\psi|_{\pi_1(Emb(K_4,\mathbb{R}^3))}$ is quotienting by the $\mathbb{Z}_2$ factor followed by the natural identification $F_3 \cong Inn(F_3)$. The action of $\pi_1(C(K_4))$ on $F_3$ is given by the identities

$$y_i \cdot a_j = a_i a_{a_j}^{-1}$$

if $i \neq j$ and otherwise

$$y_i \cdot a_i = a_i,$$

as seen in figures 12, 13. The generators of $\pi_1(C(K_4))$ are thus sent to square roots of conjugation in $Aut(F_3)$.
7 Higher dimensional analog

This result can be generalized to a higher dimensional analog involving \( \text{Emb}((\Delta^n)^{n-2}, \mathbb{R}^n) \), where \((\Delta^n)^{n-2}\) is the \((n-2)\)-skeleton of the \(n\)-simplex, for \(n > 3\). (the case \(n = 2\) gives rise to an Eilenberg-Maclane space for the braid group on 3 strands. See [P07], [H02]). That we consider the \((n - 2)\)-skeleton insures that no \(n\) points lie in an \((n - 2)\)-hyperplane, since the induced \((n - 2)\)-complex on these vertices is a simplicial \((n - 2)\)-sphere, which cannot topologically embed in \(\mathbb{R}^{n-2} \). Then any \(n\) vertices span a codimension-1 hyperplane, and the \((n+1)\)th vertex can occupy any point outside of the hyperplane or any point of \((n + 1)\) open, simply connected disjoint regions in that hyperplane, as the next proposition makes explicit.

**Proposition 7.1.** Let \((\Delta^n)^{n-2}\) be embedded in \(\mathbb{R}^{n-1}\) (identify \((\Delta^n)^{n-2}\) with the image of its embedding). Then exactly one vertex \(c\) of \((\Delta^n)^0\) is in the interior of the convex hull of \((\Delta^n)^0\). Furthermore, let \(v\) be some extremal vertex, and \(0 \leq i \leq n - 2\) enumerate the vertices \(v_i \in (\Delta^n)^0 \setminus \{c, v\}\) and let \(n_i\) be the outward normal vector to the \((n - 2)\)-face \((\Delta^n)^0 \setminus \{v, v_i\}\). Then for an embedding with \(c\) the interior point, \(v\) can occupy precisely any of the points in the open conical region

\[
\bigcup_{0 \leq i \leq n - 3} \{w \in \mathbb{R}^{n-1} \mid \langle w, n_i \rangle > 0\}.
\]

(See figure [14].)

**Proof.** That there are no more than one interior vertices is noted above: the complex induced on any \(n\) vertices is a copy of \((\Delta^{n-1})^{n-2}\), which is a topological sphere and doesn’t embed in \(\mathbb{R}^{n-2}\). Nor can the \((n + 1)\)th vertex lie on the boundary of the convex hull of the other vertices, since this boundary is in the image of the \((n - 2)\)-skeleton of the \((n-1)\)-simplex. That there is one vertex interior to the convex hull of the others is a corollary to Radon’s theorem:

**Theorem 7.2.** Given \(n+1\) vertices \(V\) in \(\mathbb{R}^{n-1}\), \(V\) can be partitioned into two disjoint non-empty sets such that the intersection of their respective convex hulls is non-empty (such a point is called a Radon point).

For a proof see [Z95]. In our case, suppose both sets contain more than 1 vertex. Then the set with more vertices contains at most \(n - 1\) vertices and so each set is contained in an \((n - 2)\)-face which implies the \((n - 2)\)-skeleton.
Figure 14: When $v \in c + \{w \in \mathbb{R}^2 | \langle w, n_i \rangle > 0; i = 1, 2\}$, $c$ is in the interior of the triangle spanned by the $v$'s.

of the $n$-simplex intersects itself away from the vertices. Then it must be the case that one set contains one vertex, and it is interior to the convex hull of the others. Finally, if we fix a vertex $c$ to be in the interior of the convex hull of the others, and allow a vertex $v$ to vary, we must have that $c$ is interior to the image of the convex hull of $(\Delta^n)^0$ under a rank 1 linear map to $\mathbb{R}$, so that for each $i$ we have $\langle v - c, n_i \rangle > 0$ and so $v$ is indeed in the open conical region defined above. (see figure 15)

It is worth noting that such embeddings do exist. To see this note that the $(n - 2)$-skeleton of $\Delta^n$ is the $(n - 2)$-skeleton of the $\Delta^{n-1}$ we get by removing a vertex $c$, together with those $(n - 2)$-faces of $\Delta^n$ that contain $c$. The former is a topological sphere, while the latter is the cone $(\Delta^{n-1})^{n-3} \times I / (\Delta^{n-1})^{n-3} \times \{0\}$ thought of as a cone to the point $c$. When $c$ is the origin of $\mathbb{R}^{n-1}$ and is inside the embedded $\partial \Delta^{n-1}$, this cone does not contain faces which intersect, since for $t \in (0, 1]$ the embedded $(\Delta^{n-1})^{n-3} \times \{t\}$ is sitting in the embedded sphere $t(\Delta^{n-1})^{n-2}$, i.e., scaled from $c$ by $t$, and these spheres are disjoint for distinct values $t_0 \neq t_1$.

Theorem 7.3. The space $\text{Emb}(\Delta^n)^{n-2}, \mathbb{R}^3)$ of labeled linear embeddings of $(\Delta^n)^{n-2}$ into $\mathbb{R}^n$, for $n > 2$, is homotopy equivalent to $(\mathbb{V}^n S^1) \times SO(n)$. 

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Figure 15: The mapping $v \mapsto \langle v - c, n_i \rangle$. Note that $c$ is mapped into the interior of the image so that $\langle v - c, n_i \rangle > 0$, for each $i$.

Proof. The proof is directly analogous to theorem 2.1. From the proposition, when the $(n+1)$th vertex is moved into the codimension-1 plane spanned by the others it must be in one of $n+1$ $(n-1)$-balls, which connect the upper half space to the lower half space. The space is thus a fiber product with fiber homotopy equivalent to a wedge of $n$ circles. The base space is the space of embeddings of the labeled $(n-1)$-simplex in $\mathbb{R}^n$ which can be viewed as the space of ordered, positively oriented frames in the tangent bundle of $\mathbb{R}^n$, which has the homotopy type of $SO(n)$ by the Gram-Schmidt process. The bundle is again trivial by defining the analogous trivializing affine linear map to a typical fiber. The space is thus homotopically equivalent to $(\bigwedge^n S^1) \times SO(n)$.

We next define a compact low dimensional model of $C((\Delta^n)^{n-2}, \mathbb{R}^n)$, analogously to that in section 4. Let $\mathcal{P}_n$ be those simplices in $\mathbb{R}^n$ with $A(n)$ symmetry (i.e., one face is regular, and the other vertex is equidistant to each vertex of that face), with barycenter at the origin, and such that the height is in $[0, \sqrt{\frac{n+1}{2n}}]$, and call these pyramids. We define a deformation retraction from $C((\Delta^n)^{n-2}, \mathbb{R}^n)$ to $\mathcal{P}_n$, first by regularizing the non-degenerate simplices. The idea is to increase the volume of the insphere while fixing the volume of the simplex. By symmetry such a flow is stationary on the regular simplices. We show that every trajectory results in a regular simplex.
Lemma 7.4. For $r_x$ the inradius of an $n$-simplex $x$ in $\mathbb{R}^n$ we have
\[ r_x = n \cdot Vol(x) / Vol(\partial x) \]

Proof. Realize $x$ as a cone over $\partial x$ to the incenter. Partition this cone into the cones over each face $f_i$. The volume of the cone over $f_i$ is \( \frac{1}{n} \cdot r_x \cdot Vol(f_i) \). Summing over the faces gives the result. (See figure 16). \qed

![Figure 16: The volume of the simplex is disassembled into simplices with height $r_x$ above base face $f_i$.](image)

By the above, flowing along the gradient of $Vol(\text{insphere}(x))$ constrained to a fixed volume is the same as flowing to minimize the surface volume, with the same constraint. We consider the component of this flow in the direction which fixes a base face $f_v$ and moves its opposite vertex $v$ at height $H$ above $f_v$, to minimize $Vol(\partial x_i)$ to prove the following.

Lemma 7.5. The flow which minimizes the surface volume of a simplex $x$, subject to maintaining a fixed volume, results in a simplex where each vertex is directly over the incenter of its opposite face.

Proof. Let the $n$ $(n-2)$-dimensional faces of $f_v$ be indexed as $g_i$, and denote the $(n-1)$-dimensional face containing $g_i$ and $v$ with $\bar{g}_i$. Let $x_i$ be the signed distance from the projection of $v$ on the hyperplane containing $f_v$ to $g_i$, signed so that $x_i$ is positive whenever the projection of $v$ is in $f_v$ (see figure 17). We have $Vol(\bar{g}_i) = \frac{1}{(n-1)} \cdot Vol(g_i) \cdot \sqrt{H^2 + x_i^2}$ so that
\[ Vol(\partial x) = Vol(f_v) + \frac{1}{(n-1)} \sum Vol(g_i) \sqrt{H^2 + x_i^2}. \]

Any $x_i$ depends affine-linearly on the others, since removing any one gives a coordinate system, so that
\begin{equation}
1 = \sum C_i x_i \tag{1}
\end{equation}
for some constants $C_i$. The value of $x_i$ for $v$ over the $i$th vertex, for which all other $x_j$’s are 0, is the altitude $A_i$ of that vertex in $f_v$, giving $C_i = \frac{1}{A_i}$ and

$$A_i Vol(g_i) = (n - 1) Vol(f_v). \tag{2}$$

Then (1) gives the constraint $\sum \frac{x_i}{A_i} = 1$ and using the method of Lagrange multipliers we get the system

$$\sum \frac{x_i}{A_i} = 1$$

and

$$\frac{1}{(n - 1)} \frac{Vol(g_i) \cdot x_i}{\sqrt{H^2 + x_i^2}} = \lambda,$$

which using (2) simplifies to

$$Vol(f_v) \cdot \frac{x_i}{\sqrt{H^2 + x_i^2}} = \lambda$$

which gives

$$\frac{x_i}{\sqrt{H^2 + x_i^2}} = \frac{x_j}{\sqrt{H^2 + x_j^2}}$$

implying

$$x_i^2 (H^2 + x_j^2) = x_j^2 (H^2 + x_i^2)$$

which necessitates $x_i = x_j$ since both are positive where a minimum is achieved.

It is therefore the case that volume of boundary is minimized when the vertices are directly over the incenters of their respective opposite faces.

It remains to argue that such a trajectory actual terminates in a simplex with the property that the vertices are directly over the incenters of their
opposite faces, as opposed to escaping to “infinity” or limiting to more than a single point.

Our flow is for simplices of fixed volume and will increase the volume bounded by the insphere. Note first that if we have vertices of arbitrary distance \( d \) from the incenter, then the cone formed by the vertex and the insphere (i.e., truncate it where its boundary intersects the insphere) is contained in the simplex \( x_t \), and has volume with \( \lim \inf \) equal to that of \( d \cdot c \cdot (1/n) \) where \( c \) is the volume of the \((n-1)\)-ball spanned by a great sphere of the insphere. Then that the volume of \( x_t \) is fixed and is an upper bound for this cone, necessitates that the inradius vanishes, contradicting the construction of the flow. Also note that if \( \ell \) is the altitude of \( v \) and \( w \) is the closest vertex of \( x \) to \( v \) with edge length \( |(v, w)| \), then for \( 2r \) the indiameter, we have \( 2r \leq \ell \leq |(v, w)| \), so that again \( r \) must vanish, contradicting the construction of the flow. (Figure 18 illustrates these two arguments). Translation to infinity is clearly not a concern. For example we can further stipulate that the incenter is fixed at the origin.

Finally, getting arbitrarily close to the critical set gives that each vertex gets arbitrarily close to being directly over the incenter of the opposite face, so that the flow results in a single limiting simplex.

![Figure 18: The trajectory does not escape to infinity.](image)

**Lemma 7.6.** A simplex for which each vertex orthogonally projects to the incenter of the opposite face is a regular simplex.

**Proof.** Let \( v, w \) be vertices of the simplex \( x \), \( c_v \) be the incenter of the face \( f_v \) opposite \( v \) and \( r_w \) be the outward pointing radial vector from \( c_v \) to the codimension 2 face excluding \( w \) and \( v \) (Figure 19 is helpful). Note that the \( r_w \) form congruent right triangles with \( v - c_v \), and that on the \( i \)th face the
gradient of the distance to $f_v$, (at $c_v + r_w$ in $f_w$) is the hypotenuse of the right triangle containing $r_w$. Thus the point on $v - c_v$ which is equidistant to some face and to $c_v$ is actually the incenter $c_x$.

![Figure 19: The condition that each vertex is over its opposite incenter implies regularity.](image)

It is therefore the case that $c_x$ projects orthogonally to $c_v$ and all other faces have equal pitch relative to $f_v$. That is, for $c_w$ the outward pointing vector from $c_x$ to $f_w$ realizing the inradius, we have that $c_w \cdot c_u = c_w \cdot c_y$ for all distinct $u, w, y$. It follows that the simplex they define has full symmetry.

The previous two lemmas piece together to give the following theorem.

**Theorem 7.7.** The space of linearly embedded $n$-simplices in $\mathbb{R}^n$ deformation retracts to that of the regular ones. In particular, this is achieved by the flow which increases the inradius while fixing the entire volume.

For those close to being degenerate, we will make use of a greatest solid angle function $\alpha$. We require the following lemma.

**Lemma 7.8.** Let $V_{n-1}$ be the $(n - 1)$-volume of the unit $(n - 1)$-sphere, for $n > 2$. The sum $S$ of the solid angles of (the $n - 2$-skeleton of) an $n$-simplex in $\mathbb{R}^n$ is tightly bounded by $(0, V_{n-1}/2]$.
Proof. Consider first only non-degenerate simplices. The idea is to translate, for each of $n + 1$ vertices, a copy of the $n$-simplex, to have its $i$th vertex at 0. The $n+1$ cones $C_i$ which are formed by extending each of these outward, along with their reflections $-C_i$ through 0, give $2n+2$ regions whose interiors are pairwise disjoint. Intersection with a unit sphere $S^{n-1}$ then gives that

$$V_{n-1} \geq \sum_i Vol(C_i \cap S^{n-1}) + \sum_i Vol(-C_i \cap S^{n-1}) = 2S.$$ 

Specifically, put the 0th vertex at the origin and let $x_i$ be the $i$th vertex, for $1 \leq i \leq n$ ranging over the other $n$ vertices. Then

$$C_0 = \text{convex span}\{x_i\} = \sum a_i x_i$$

for $a_i \geq 0$, and

$$C_i = \text{convex span}\{(x_j - x_i)_{j \neq i} \cup \{-x_i\}\}
= (\sum_j a_j (x_j - x_i)) - a_i x_i.$$ 

Putting these into coordinates $x_i$ gives $C_i$ as

$$(a_1, a_2, \ldots, a_{i-1}, -\sum_j a_j, a_{i+1}, \ldots, a_n),$$

from which it is clear that for any $i \neq j$ we have

$$\text{int}(C_i) \cap \text{int}(C_j) = \text{int}(C_i) \cap \text{int}(-C_j) = \emptyset.$$ 

It remains to show the bounds are tight. The sum being 0 could only happen for a degenerate simplex, but Proposition 7.1 says that for such a simplex one vertex is interior to the convex hull of the other $n$ points, so the solid angle here is a hemisphere, proving the bounds are tight. 

The procedure to handle the degenerate cases in the general dimension $n$ case will differ from the 3 dimensional case, because in general we use a non-linear deformation retraction to regularize the simplices, as opposed to the linear method in dimension 3. Consequently, the limiting path of the vertex with successively larger solid angles is not defined without considerable further work. Instead, we will glue the regularization deformation retraction
Figure 20: The schematic for gluing together the regularization deformation retraction and the preferred point deformation retraction.

to a second deformation retraction which is defined only in a neighborhood of the degenerate simplices.

As above, let \( V_{n-1} \) be the volume of the unit \((n-1)\)-sphere. Define some cut off point \( \ell \in (V_{n-1}/4, V_{n-1}/2) \) and let \( \beta(x) \) be the affine linear reparametrization of \( \alpha(x) \) which has \( \beta(\alpha^{-1}(V_{n-1}/4)) = 1 \) and \( \beta(\alpha^{-1}(\ell)) = 0 \). Let \( B = \beta^{-1}[0, 1] \), \( A = \beta^{-1}[1, \infty) \), \( C = \beta^{-1}(-\infty, 0] \), form a closed cover of \( C((\Delta^n)^{n-2}, \mathbb{R}^n) \) (follow along with figure 20). We define a deformation retraction \( \mathcal{P}_t \) on \( B \cup C \) to \( \mathcal{P}_n \), called the preferred point deformation retraction, which moves the vertex of large solid angle \( v \) in the hyperplane parallel to \( W_x \), the face opposite \( v \), in a constant velocity path towards the point in this hyperplane directly over the barycenter \( c_x \) of \( W_x \), followed by regularizing \( W_x \) in its own hyperplane using the geometric flow introduced earlier in this section, and extended to be an isometry in the orthogonal complement. Finally a normalization of edge lengths, and a translation, has \( \mathcal{P}_t \) arrive in \( \mathcal{P}_n \). Next we glue.

Let \( \mathcal{R}_t \) be the volume-fixing insphere-maximizing flow defined above, followed by normalization of edge lengths to arrive in \( \mathcal{P}_n \), and define \( \mathcal{P}_{(2t-1)} \circ \mathcal{R}_{\beta(x)} \) to be \( \mathcal{R}_{\beta(x)} \) followed by the preferred point deformation retraction of \( x \) by the parameter \( 2t-1 \) (i.e., if \( \mathcal{R}_{\beta(x)}(x) \) reduces the greatest solid angle to less than \( V_{n-1}/4 \) then strictly speaking \( \mathcal{P}_{(2t-1)} \) is not defined here). The maps agree on the lines separating regions of the schematic in figure 20, so
the deformation retraction glues. The picture is this: flow along $\mathcal{R}$ toward regularization. The parameter $\beta$, where it is used, is used to proportionally go less of the entire way toward the regular simplices. At the point $\mathcal{R}_{\beta(x)}$ we let the preferred point flow take over. As $\beta$ decreases to 0 there is proportionately less regularization that has been done, until $\beta = 0$ and the only the preferred point flow is used.

The theorem of Van Kampen is easily applicable to $\mathcal{P}_n$. The half-high pyramids have as fundamental group the subgroup of $\text{Spin}(n)$ which double covers the copy of $A_{n-1} < SO(n-1) < SO(n)$, the alternating group, which comes from orientation preserving symmetries of the symmetrical face. The regular pyramids have as fundamental group the subgroup of $\text{Spin}(n)$ which double covers the copy of $A_n < SO(n)$, and the degenerate pyramids have as fundamental group the subgroup of $\text{Spin}(n)$ which double covers the copy of $\Sigma_{n-1} < SO(n)$, the symmetric group which permutes the extremal vertices. Hence

**Theorem 7.9.** The space of $(n-2)$-skeleta of $n$-simplices in $\mathbb{R}^n$ is homotopy equivalent to the double mapping cylinder

$$SO(n)/A_n \leftarrow SO(n)/A_{n-1} \rightarrow SO(n)/\Sigma_{n-1}$$

where the maps are the obvious ones given by the decomposition of pyramids into those half high, those which are regular and those which are degenerate.

**Proof.** The symmetries of the spaces are straightforward, as are the inclusions. The proof follows from Van Kampen’s theorem. $\square$

Finally, let us observe by analogy to section 5 that the fundamental group of the complement of a configuration of $(\Delta^n)^{n-2}$ in $\mathbb{R}^n$ is $F_n$, the free group on $n$ generators, as can be seen from looking at a degenerate configuration, which separates a hyperplane $\mathbb{R}^{n-1} \cong P \subset \mathbb{R}^n$ into $n$ $(n-1)$-balls and one thickened $(n-2)$-sphere. A loop in this space, based in the $(n - 2)$-sphere, say, is generated by paths which pass into the northern half-space in $\mathbb{R}^n$, pass through one of the $(n-1)$-balls, and return via the southern half-space. Thus the configuration space $C((\Delta^n)^{n-2}, \mathbb{R}^n)$ gives rise to an action of $2A_n * 2A_{n-1} 2\Sigma_{n-1}$ on $F_n$. We omit further details.
8 The case of $C((\Delta^{n+1})^{n-2}, \mathbb{R}^n)$

In this section we increase the number of vertices by increasing the dimension of the simplex, while keeping both the skeleton’s dimension and the ambient dimension the same. We seek to compare the homotopy type of $C((\Delta^{n+1})^{n-2}, \mathbb{R}^n)$ with that of $C((\Delta^n)^{n-2}, \mathbb{R}^n)$, using the methods developed in section 7.

Lemma 8.1. For $n > 2$ the codimension 3 skeleton of $\Delta^{n+1}$ embedded in $\mathbb{R}^n$, has either one vertex interior to the convex hull of the others, or has exactly one edge which intersects the $n−1$ face spanning the other $n$ vertices.

Proof. Let $x \in C((\Delta^{n+1})^{n-2}, \mathbb{R}^n)$. We may apply Radon’s theorem. Let $p$ be a Radon point (see theorem 7.2) in the intersection of two faces $F_1, F_2$. Then one of these two faces is of dimension at least $n − 1$, since otherwise the $(n − 2)$-skeleton is not embedded, so that the other face is at most of dimension 1. In the case that the lower dimensional face, say $F_1$, is dimension one, then $(F_2)^{n-2}$ is a simplicial sphere in the dimension $n − 1$ hyperplane $H$ it spans. It is possible that one vertex $v_1$ of $F_1$ is in $H$. To show that the other vertex $v_2$ in $F_1$ cannot be in $H$, we generalize the argument that $K_5$ is non-planar (see figure 21). Let $W$ be the vertices of $F_2$ union $\{v_1\}$. By remarks in section 7, here we have that some vertex $w \in W$ is in the interior (in $H$) of the convex hull of the other vertices in $W$. Then for $w \neq u \in W$, $u$ is exterior to the $(n − 2)$-sphere formed by the $(n − 2)$-skeleton of $W \setminus \{u\}$. Suppose that $v_2$ is also in $H$. It is connected by edges to each such $u$, so cannot be contained in any of the spheres thus formed, so it must be exterior to $F_2$, but then cannot connect to $w$ without intersecting $F_2$. This shows that when the lower dimensional face is dimension one the Radon point is unique. In the case the lower dimensional face is dimension 0 this is immediate. The Radon point $p$ thus distinguishes either two vertices whose edge contains $p$, or one interior vertex.

On the other hand, given any non-degenerate configuration of the $(n − 2)$-skeleton of an $n$-simplex in $\mathbb{R}^n$, with some marked vertex $v$, we can introduce an $(n + 2)$th vertex $w$ and the induced $(n − 2)$-faces, by placing $w$ anywhere in the interior of the inward pointing cone at $v$, formed by its edges. To show that the $(n − 2)$-skeleton doesn’t intersect itself, let the first $n$ vertices be the standard basis $e_i$ of $\mathbb{R}^n$, the $(n + 1)$th vertex be the origin, and the $(n + 2)$th vertex be in the interior of the first orthant (see figure 22). This is
Figure 21: The graph $K_5$ is non-planar since a fifth vertex put in one of the four regions above is necessarily separated from one of the four vertices. Similarly $(\Delta^{n+1})^{n-2}$ does not embed in $\mathbb{R}^{n-1}$ for $n \geq 2$.

sufficiently general, since any non-degenerate $n$-simplex in $\mathbb{R}^n$ is sent here by an invertible affine linear map, but we will further put the $(n+2)$th vertex at $(k, \ldots, k)$, for some $k > 0$, for simplicity, since self-intersection will only change when we change the strict inequalities which partition the vertices into interiors of half-spaces formed by the others. Thus, we need only check the specific case $k = \frac{1}{n}$ and the cases to either side. Let $a$ be in the convex span of some $n - 1$ vertices amongst $\{e_i\} \cup \{0\} \cup \{(1, \ldots, 1)\}$. We have

$$a = (a_1, \ldots, a_n) + a_{n+2}(k, \ldots, k)$$

where $a_i = 0$ for at least 3 values of $i$ (possibly including the coefficient $a_{n+1}$ of the vertex at the origin). Let $b$ be the same point but with coefficients coming from another face, i.e.,

$$b = (b_1, \ldots, b_n) + b_{n+2}(k, \ldots, k) = a.$$  

Then for $a_i = 0$, $i \leq n$, we have that this coordinate of $a$ which is $a_{n+2}k$ is less than or equal to the other coordinates, so that the same holds for $b$ and thus $b_i = 0$. That values $1 \leq i \leq n$ for which $a_i = 0$ are the same as those for which $b_i = 0$ contradicts that the $(n-2)$-faces are different. Therefore, the faces do not intersect.

The above lemma say that the picture for general $n$ is much as it is for $n = 3$: there is either an edge intersecting the interior of an $(n-1)$-cell (which
Figure 22: The \((n + 2)\)th vertex \(w\) is in the cone formed by the edges emanating from \(v\), which is in this case the first orthant.

Figure 23: Generically, either some vertex is interior to the convex hull of the others or some specific edge intersects its opposite face.

is not a part of our skeleton) or there is a vertex interior to the convex hull of the others (see figure 23).

A space analogous to pyramids exists for \(C((\Delta^{n+1})^{n-2}, \mathbb{R}^n)\). Let this space consist of those configurations such that \(n + 1\) of the vertices form a regular, unit edge length image \(S\) of \((\Delta^n)^{n-2}\), and where the \((n + 2)\)th vertex is on a line connecting the centroid of one face \(F\) of \(S\) to the centroid of \(S\), at a height between the centroid and \(\sqrt{\frac{n+1}{2n}}\) above \(F\) (\(\sqrt{\frac{n+1}{2n}}\) being the height of a regular unit edge length \(n\)-simplex). Denote this space with \(Q_n\).

Let \(\mathcal{I} \subset Q_n\) be those with Radon point a vertex interior to the convex hull of the others, let \(\mathcal{E}\) be those with Radon point in the interior of an edge, and \(\mathcal{B}\) be the others, i.e., those with Radon point a vertex in the boundary
of the convex hull of the others (see figure 24). We will use a deformation retraction of \( C(\Delta^n, \mathbb{R}^n) \) to the space of regular simplices to define a deformation retraction from \( C((\Delta^{n+1})^{n-2}, \mathbb{R}^n) \) to \( Q_n \). For the subspace \( E \), we will apply slightly different deformation retractions to both halves of its elements by gluing along their shared face. For this we require a linear deformation retraction following what was done in Section 3.

![Figure 24: The three types of configurations in \( Q_n \)](image)

**Theorem 8.2.** The deformation retraction of theorem 7.7 is achieved with a linear flow.

**Proof.** Let \( A \) be the \( n \times n \) symmetric matrix whose columns form a unit edge length simplex. Explicitly \( A \) has \( \mu \) for each entry on its diagonal and \( \nu \) for each entry off the diagonal where \( \mu^2 + (n-1)\nu^2 = 1 \) and \( 2(\mu - \nu)^2 = 1 \), so that

\[
\mu = \frac{n + \sqrt{n+1} - 1}{\sqrt{2n}} \quad \text{and} \quad \nu = \frac{\sqrt{n+1} - 1}{\sqrt{2n}}.
\]

Let \( B_i \) be the \( n \times n \) identity matrix \( I \) with the \( i \)th row replaced by \([-1, \ldots, -1]\). The matrix \( B_i \) acts on the right as a column operator to change bases between vertices of an \( n \)-simplex. I.e., given a matrix \( V \) whose columns form a basis, the columns of \( VB_i \) are those emanating from \( Ve_i \) to 0 and to each of the other \( Ve_j \)'s (see figure 25). Let \( B \in \{B_i\} \). Note that

\[
AB = QA
\]

for some \( Q \in O(n) \) (this is obvious, geometrically). We have \( B^2 = I \) and \( Q^{-1} = ABA^{-1} \) so \( Q^{-1} = Q^T = Q \) and

\[
B^T = (A^{-1}QA)^T = AQ^{-1}A^{-1} = A^2BA^{-2}.
\]
Figure 25: $B_i$ swaps $V$ for the basis at $v_i$ which spans the same simplex as $V$.

Let

$$\Omega_t(x) = \left[ (1 - t)xA^{-1} + txA^{-1}((xA^{-1})^T(xA^{-1}))^{-1/2} \right] A.$$ 

Then $\Omega_t(x) \cdot A^{-1}$ is the Löwdin deformation retraction of $xA^{-1}$ to $O(n)$, and so $\Omega_t(x)$ gives a linear path from $x$ to $O(n) \cdot A$ (see figure 26). We compute $\Omega_t(xB)$ to show equivariance: $\Omega_t(xB) = \Omega_t(x)B$.

$$\begin{align*}
\Omega_t(xB) &= \left[ (1 - t)xBA^{-1} + txBA^{-1}((xBA^{-1})^T(xBA^{-1}))^{-1/2} \right] A \\
&= \left[ (1 - t)xA^{-1}Q + txA^{-1}Q((xA^{-1}Q)^TxA^{-1}Q)^{-1/2} \right] A \\
&= \left[ (1 - t)xA^{-1} + txA^{-1}((xA^{-1})^T(xA^{-1}))^{-1/2} \right] QA \\
&= \left[ (1 - t)xA^{-1} + txA^{-1}((xA^{-1})^T(xA^{-1}))^{-1/2} \right] AB \\
&= \Omega_t(x)B,
\end{align*}$$

so $\Omega_t$ is equivariant under $B$. The set $\{B_i\}$ generates a copy $\Gamma$ of $\Sigma_{n+1}$ ($B_i$ acting on the vertices, in cycle notation, is the transposition $(0, i)$) so up to translation and scaling (see figure 27), $(O(n) \cdot A)/\Gamma$ is the space of regular simplices and $\Omega_t$ descends to the quotient $(GL(n) \cdot A)/\Gamma$, to give a linear (i.e., vertices move along linear paths) $\Sigma_{n+1}$-equivariant regularization of simplices in $\mathbb{R}^n$.

**Theorem 8.3.** The space $C((\Delta^{n+1})^{n-2}, \mathbb{R}^n)$ deformation retracts to $Q_n$, the pyramid model.

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Figure 26: The $\Sigma_n$ equivariant orthogonalization is conjugated to give a $\Sigma_{n+1}$ equivariant regularization. The line segments are the deformation retractions in $GL(n)$.

Figure 27: $B_i$ is effectively the transposition $(0, i)$.

Proof. We achieve the deformation retract in three steps, the first two of which are divided into 3 cases each.

Step 1a. Let $x \in \mathcal{I}$ with interior vertex $v$. Let $\{v_i\}, 1 \leq i \leq n$, be $n$ of
Figure 28: Realize $v$ as a convex combination of $c$ and the closest vertices $v_i$ to $v$.

the closest vertices of $x$ to $v$, and let $c$ be the centroid of $x$, and $d$ to be the centroid of the face $W_x$ spanned by $\{v_i\}$ (see figure 28). We will move $v$ to lie along the line segment connecting $c$ to $d$. This can be done explicitly by putting $v$ in barycentric coordinates

$$v = qc + \sum a_i v_i.$$  (with $q + \sum a_i = 1$, and $q, a_i \geq 0$)

Set $m = \min\{a_i\}$. Note that $m = q = 0$ cannot happen since this would put $v$ in the $(n-2)$-skeleton of $x$ (see figure 29). We have $3m \leq 1 - q$ and require a parameter $s(m, q)$ so as to send $v$ to $(1 - s)d + sc$ which is continuous on $0 \leq 3m \leq 1 - q \leq 0$ minus the origin, and for which $s(0, q) = 1$, $s(m, 0) = 0$ and $s(\frac{1}{3}(1-q), q) = q$ (so that if $v$ is equidistant to two extremal vertices it gets sent to $c$, if it is in $W_x$ it gets sent to $d$, and if it is on the line connecting $c$ to $d$ it is fixed). This is accomplished with

$$s(m, q) = (1-q)\left(1-\frac{3m}{1-q}\right)^{1/q} + q,$$

which we extend continuously by $s \equiv 0$ on $q = 0$ (see figure 30). Sending $v$ to $(1 - s)d + sc$ along the straight line path $v_t = (1 - t)v + t(1 - s)d + sc$ gives a retraction of $I$ to the subspace of $I$ with internal vertex along a radial segment connecting the barycenter to the center of a face.
Step 1b. Let \( x \in \mathcal{E} \). We want to parallel transport the edge \( e \) containing the Radon point \( p \) so that the intersection of this edge with its opposite face \( W_x \) is at the barycenter \( d \) of that face. When one vertex \( v_1 \) of \( e \) is close to \( W_x \) we need the other vertex \( v_2 \) to move only a small distance so that step 1b can be continuously glued to step 1a. To do this, we follow the parallel transport with a shear back in the direction that \( v_2 \) has moved, in the plane containing \( d \) and \( e \), with origin at \( d \), in proportion to \( 1 - \frac{v_1 - p}{v_2 - p} \). Explicitly, put \( \ell_i = |v_i - p| \) (see figure 31), and send \( v_2 \) to \( v_2 + \frac{\ell_1}{\ell_2} (d - p) \), send \( p \) to \( d \), and send \( v_1 \) to \( v_1 + \frac{t_2 - t_1}{t_1} \ell_1 (d - p) \). Thus as we approach \( B \), \( \ell_1/\ell_2 \) approaches 0 and \( V_2 \) moves less and less.

Step 1c. These two deformation retractions agree on their respective extensions to \( B \). In both cases the extension is to send the Radon point vertex to the centroid of the \((n-1)\)-face it is in, in a straight line path while fixing everything else.

At the end of step 1 the Radon point of each \( x \in C((\Delta^{n+1})^{n-2}, \mathbb{R}^n) \) is along a ray extending from the centroid of \( x \) to the centroid of a face. For \( 0 \leq t \leq \frac{1}{3} \), let \( \Lambda_t \) be all three parts of step 1, simultaneously performed in the variable \( 3t \).
Figure 30: Graphs of $s$ for smaller $q$ (left), for larger $q$ (right), and in the $q - m$ plane (bottom). Note the origin is excluded.

$$\frac{1}{3}(1 - q)$$

Figure 31: Parallel transport followed by a shear, with $v_2$ going back in the direction parallel transported.

Step 2a. For $x \in \Lambda_{1/3}(\mathcal{I})$, using the same barycentric coordinates and function $s$ as in 1a. we transfix by the parameter $1 - s$ the regularization of the
extremal $n$-simplex via $\Omega$. Specifically, let $\omega : I \to \text{Aff}(\mathbb{R}^n)$ be defined as the induced map from $\Omega$ as was done in section 3 (i.e., $\omega_x(t)$ is the transformation which takes $W_x$ to $\Omega_t(W_x)$ and which takes $W_x^\perp$ to $(\Omega_t(W_x))^\perp$, preserving distance and orientation.)

For $x \in \Lambda_{1/3}(I)$ with $s \in (0, 1)$, let

$$\Lambda_t(x) = \left( \omega_x((1 - s(x))(3t - 1)) \right)^{-1} \circ \Omega_{3t-1}.$$

Then $\Lambda_{3t-1}$ extends to where $s \in \{0, 1\}$ continuously (i.e., $\Lambda|_{s-1(0)}$ keeps $W_x$ fixed, $\Lambda|_{s-1(1)} = \Omega|_{s-1(1)}$).

Step 2b. For $x \in \Lambda_{1/3}(E)$ let $\ell_1, \ell_2$ be as in 1b. We will keep the shared face fixed and apply $\Omega$ to the half-space $H_2$ containing $v_2$. To the other half-space we deformation retract not to $\Omega_1(x)$ but to $L \cdot \Omega_1(x)$ where $L$ scales in the direction of $W_x^\perp$ by $\frac{\ell_1}{\ell_2}$. Specifically, we have

$$x \mapsto \omega_x(t)^{-1} \cdot \Omega_t(x)$$
on $H_2$ and

$$x \mapsto \omega_x(t)^{-1} \cdot \bar{\Omega}_t(x)$$
on $H_1$, where $\bar{\Omega}_t(x) = (1 - t)x + L\Omega_t(x)$.

Step 2c. For $x \in \Lambda_{1/3}(B)$ the the non-degenerate half of $x$ is regularized by the process of 2a. This agrees with the limit of the process in 2b. as $x$ approaches $B$ from $E$.

For $\frac{1}{3} \leq t \leq \frac{2}{3}$, let $\Lambda_t$ be all three parts of step 2, simultaneously performed in the variable $3t - 1$.

At the end of step 2 all that remains is to regularize $W_x$ in the hyperplane it spans, and extend to an isometry on $W_x^\perp$. This is achieved by theorem 8.2. This gives the final third of $\Lambda$.

**Theorem 8.4.** $C((\Delta^{n+1})^{n-2}, \mathbb{R}^n)$ is homotopy equivalent to $C((\Delta^n)^{n-2}, \mathbb{R}^n)$, moreover $Q_n$ is homeomorphic to $P_n$.

**Proof.** The deformation retraction above gives $C((\Delta^{n+1})^{n-2}, \mathbb{R}^n) \simeq Q_n$. Both $Q_n$ and $Q_n$ have subsets representing objects with the symmetry of an $n$-simplex in $\mathbb{R}^n$, and subsets representing objects with the symmetry of an $(n - 1)$-simplex in $\mathbb{R}^n$. An obvious homeomorphism between $Q_n$ and $P_n$ is given by linearly corresponding the parameters which range between these respective subsets. (See figure 32)
Figure 32: A comparison of the symmetries found in the $K_4$ and $K_5$ pyramidal cases. Note the equality of the symmetries between horizontally adjacent figures.

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