EXISTENCE OF NON-TRIVIAL SOLUTIONS FOR NONLINEAR FRACTIONAL SCHRÖDINGER-POISSON EQUATIONS

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ABSTRACT. We study the nonlinear fractional Schrödinger-Poisson equations

\[
\begin{aligned}
(-\Delta)^s u + u + \phi u &= f(x, u), \quad \text{in } \mathbb{R}^3, \\
(-\Delta)^t \phi &= u^2, \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]

where \( s, t \in (0, 1] \), \( 2t + 4s > 3 \). Under some assumptions on \( f \), we obtain the existence of non-trivial solutions. The proof is based on the perturbation method and the mountain pass theorem.

1. Introduction

In this paper, we are concerned with the existence of non-trivial solutions for the following fractional Schrödinger-Poisson equation

\[
\begin{aligned}
(-\Delta)^s u + u + \phi u &= f(x, u), \quad \text{in } \mathbb{R}^3, \\
(-\Delta)^t \phi &= u^2, \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]

where \( s, t \in (0, 1] \), \( 2t + 4s > 3 \). \((-\Delta)^s\) denotes the fractional Laplacian.

When \( s = t = 1 \), the equation (1.1) reduces to Schrödinger-Poisson equation, which describes system of identical charged particles interacting each other in the case where magnetic effects can be neglected [3, 6]. When \( \phi = 0 \), (1.1) reduces to a fractional Schrödinger equation, which is a fundamental equation in fractional quantum mechanics [7, 8].

Recently, some authors proposed a new approach called perturbation method to study the quasi-linear elliptic equations, see [10]. The idea is to get the existence of critical points of the perturbed energy functional \( I_\lambda \) for \( \lambda > 0 \) small and then taking \( \lambda \to 0 \) to obtain solutions of original problems. Very recently, Feng [1] used the perturbation method to study the Schrödinger-Poisson equation

\[
\begin{aligned}
-\Delta u + u + \phi u &= f(x, u), \quad \text{in } \mathbb{R}^3, \\
(-\Delta)^{\alpha/2} \phi &= u^2, \quad \lim_{|x|\to\infty} \phi(x) = 0, \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]

where \( \alpha \in (1, 2] \). Under some conditions, the problem (1.2) possesses at least a nontrivial solution.

We point out that when \( s = 1 \) and \( t \in (\frac{1}{2}, 1] \), the problem (1.1) boils down to (1.2). The main result of this paper is described as follows.

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Theorem 1.1. Suppose $f$ satisfies the following conditions:

(A1) For every $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$, there exist constants $C_1 > 0$ and $p \in [2, 2^*_s)$ such that

$$|f(x, u)| \leq C_1(|u| + |u|^{p-1}),$$

where $2^*_s = \frac{6}{3 - 2s}$ is the fractional critical Sobolev exponent;

(A2) $f(x, u) = o(|u|)$, $|u| \to 0$, uniformly on $\mathbb{R}^3$;

(A3) there exists $\mu > 4$ such that

$$0 < \mu F(x, u) \leq uf(x, u)$$

holds for every $x \in \mathbb{R}^3$ and $u \in \mathbb{R} \setminus \{0\}$, where $F(x, u) = \int_0^u f(x, s) ds$;

Then problem (1.1) has at least a nontrivial solution.

The paper is organized as follows. In Section 2, we will present some preliminaries results. In Section 3, we will prove Theorem 1.1.

2. Preliminaries

For $p \in [1, \infty)$, we denote by $L^p(\mathbb{R}^3)$ the usual Lebesgue space with the norm $\|u\|_p = \left(\int_{\mathbb{R}^3} |u|^p dx\right)^{\frac{1}{p}}$. For any $p \in [1, \infty)$ and $s \in (0, 1)$, we recall some definitions of fractional Sobolev spaces and the fractional Laplacian $(-\Delta)^s$, for more details, we refer to [5]. $H^s(\mathbb{R}^3)$ is defined as follows

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi < \infty \right\}$$

with the norm

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi\right)^{\frac{1}{2}},$$

where $\mathcal{F}u$ denotes the Fourier transform of $u$. By $\mathcal{S}(\mathbb{R}^n)$, we denote the Schwartz space of rapidly decaying $C^\infty$ functions in $\mathbb{R}^n$. For $u \in \mathcal{S}(\mathbb{R}^n)$ and $s \in (0, 1)$, $(-\Delta)^s$ is defined by

$$(-\Delta)^s f = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} f), \quad \forall \xi \in \mathbb{R}^n.$$ 

By Plancherel’s theorem, we have $\|\mathcal{F}u\|_2 = \|u\|_{2}$, $\|\xi^s \mathcal{F} u\|_2 = \|(-\Delta)^{\frac{s}{2}} u\|$. Then by (2.1), we get the equivalent norm

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$ 

For $s \in (0, 1)$, the fractional Sobolev space $D^{s, 2}(\mathbb{R}^3)$ is defined as follows

$$D^{s, 2}(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : |\xi|^s \mathcal{F} u(\xi) \in L^2(\mathbb{R}^3) \right\},$$

which is the completion of $C^\infty_0(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{s, 2}} = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^3} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$ 

Lemma 2.1. (Theorem 2.1 in [12]). For any $s \in (0, \frac{3}{2})$, $D^{s, 2}(\mathbb{R}^3)$ is continuously embedded in $L^{2s'}(\mathbb{R}^3)$, i.e., there exists $c_s > 0$ such that

$$\left(\int_{\mathbb{R}^3} |u|^{2s'} dx\right)^{2/2s'} \leq c_s \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx, \quad u \in D^{s, 2}(\mathbb{R}^3).$$
We consider the variational setting of (1.1). From Theorem 6.5 and Corollary 7.2 in [5], it is known that the space $H^s(\mathbb{R}^3)$ is continuously embedded in $L^q(\mathbb{R}^3)$ for any $q \in [2, 2^*_s)$ and the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is locally compact for $q \in [1, 2^*_s)$. If $2t + 4s > 3$, then $H^s(\mathbb{R}^3) \hookrightarrow L^\frac{12}{3+2t}(\mathbb{R}^3)$. For $u \in H^s(\mathbb{R}^3)$, the linear operator $T_u : D^{s,2}(\mathbb{R}^3) \rightarrow \mathbb{R}$ defined as

$$T_u(v) = \int_{\mathbb{R}^3} u^2vdx,$$

By Hölder inequality and Lemma 2.1

$$|T_u(v)| \leq \|u\|_{12/(3+2t)}^2\|v\|_{2^*_t} \leq C\|u\|_{H^s}^2\|v\|_{D^{s,2}}.$$

Set

$$\eta(u, v) = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u \cdot (-\Delta)^{\frac{s}{2}}vdx, \quad u, v \in D^{s,2}(\mathbb{R}^3).$$

It is clear that $\eta(u, v)$ is bilinear, bounded and coercive. The Lax-Milgram theorem implies that for every $u \in H^s(\mathbb{R}^3)$, there exists a unique $\phi^t_u \in D^{s,2}(\mathbb{R}^3)$ such that $T_u(v) = \eta(\phi^t_u, v)$ for any $v \in D^{s,2}(\mathbb{R}^3)$, that is

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}\phi^t_u(-\Delta)^{\frac{s}{2}}vdx = \int_{\mathbb{R}^3} u^2vdx.$$

Therefore, $(-\Delta)^{\frac{s}{2}}\phi^t_u = u^2$ in a weak sense. Moreover,

$$\|\phi^t_u\|_{D^{s,2}} = \|T_u\| \leq C\|u\|_{H^s}^2.$$

Since $t \in (0, 1]$ and $2t + 4s > 3$, then $\frac{12}{3+2t} \in (2, 2^*_s)$. From Lemma 2.1, (2.2) and (2.3), it follows that

$$\|\phi^t_u\|_{D^{s,2}} \leq C\|u\|_{12/(3+2t)}^2.$$

For $x \in \mathbb{R}^3$, we have

$$\phi^t_u(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}}dy,$$

which is the Riesz potential [9], where

$$c_t = \frac{\Gamma\left(\frac{3-2t}{2}\right)}{\pi^{3/2}2^{2t}\Gamma(t)}.$$

Substituting $\phi^t_u$ in (1.1), we have the fractional Schrödinger equation

$$(-\Delta)^{s}u + u + \phi^t_u u = f(x, u), \quad x \in \mathbb{R}^3,$$

The energy functional $I : H^s(\mathbb{R}^3) \rightarrow \mathbb{R}$ corresponding to problem (2.8) is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 + u^2dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_u u^2dx - \int_{\mathbb{R}^3} F(x, u)dx.$$
It is easy to see that $I$ is well defined in $H^s(\mathbb{R}^3)$ and $I \in C^1(H^s(\mathbb{R}^3), \mathbb{R})$, and
\begin{equation}
(2.9)
(I'(u), v) = \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + \phi'_u uv - f(x, u)v)\, dx, \quad v \in H^s(\mathbb{R}^3).
\end{equation}

**Definition 2.2.** (1) We call $(u, \phi) \in H^s(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a weak solution of (1.1) if $u$ is a weak solution of (2.8).
(2) We call $u$ is a weak solution of (2.8) if
\begin{equation}
(2.10)
\int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + \phi'_u uv - f(x, u)v)\, dx = 0,
\end{equation}
for any $v \in H^s(\mathbb{R}^3)$.

Assume that the potential $V(x)$ satisfies the condition
(V) $V \in C(\mathbb{R}^3)$, $\inf_{x \in \mathbb{R}^3} V(x) \geq V_0 > 0$, where $V_0$ is a constant. For every $M > 0$, $\text{meas} \{ x \in \mathbb{R}^3 : V(x) \leq M \} < \infty$, where $\text{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^3$.

Let
\begin{equation}
E = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)u^2)\, dx < \infty \right\}.
\end{equation}
Then $E$ is a Hilbert space with the inner product
\begin{equation}
(2.11)
\langle u, v \rangle_E = \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + V(x)uv)\, dx
\end{equation}
and the norm $\|u\|_E = \langle u, u \rangle_E^{\frac{1}{2}}$. By Lemma 2.3 in [4], it is known that $E$ is compactly embedded in $L^p(\mathbb{R}^3)$ for $2 \leq p \leq 2^*_s$.

For fixed $\lambda \in (0, 1]$, we introduce the following inner product
\begin{equation}
(2.12)
\langle u, v \rangle_{H^s_\lambda} = \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + \lambda V(x)uv)\, dx
\end{equation}
and the norm $\|u\|_{H^s_\lambda} = \langle u, u \rangle_{H^s_\lambda}^{\frac{1}{2}}$. Denote $E_\lambda = (E, \| \cdot \|_{H^s_\lambda})$.

Define the perturbed functional $I_\lambda : E \to \mathbb{R}$:
\begin{equation}
(2.13)
I_\lambda(u) = I(u) + \frac{\lambda}{2} \int_{\mathbb{R}^3} V(x)u^2\, dx, \quad \lambda \in (0, 1].
\end{equation}

**Lemma 2.3.** Suppose that $V(x) \geq 0$ and (A1), (A2) hold. Then there exist $\rho > 0$, $\eta > 0$ such that for fixed $\lambda \in (0, 1]$,
\begin{equation}
\inf_{u \in E, \|u\|_E = \rho} I_\lambda(u) > \eta,
\end{equation}
where $\rho$ and $\eta$ are independent of $\lambda$.

**Proof.** By (A1) and (A2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
\[ |f(x, u)| \leq \varepsilon |u| + C_\varepsilon |u|^{p-1}, \quad x \in \mathbb{R}^3, \quad u \in \mathbb{R}. \]
Then
\[ |F(x, u)| \leq \frac{\varepsilon}{2} |u|^2 + \frac{C_\varepsilon}{p} |u|^p. \]
For $\rho > 0$, set
\[ \Sigma_\rho = \{ u \in E : \| u \|_E \leq \rho \}. \]
It is known that $E$ is continuously embedded into $L^q(R^3)$ for $q \in [2, 2^*]$ ($2^* = \frac{6}{3-2\mu}$), then $\|u\|_q \leq C_0 \|u\|_E$. Since $p \in (2, 2^*)$, for $u \in \partial \Sigma_p$,
\[
I_\lambda(u) = \frac{1}{2} \|u\|_2^2 + \frac{\lambda}{2} \int_{R^3} V(x)u^2 dx + \frac{1}{4} \int_{R^3} \phi'_u(x)u^2 dx - \int_{R^3} F(x, u)dx \\
\geq (1 - \varepsilon) p^2 - C_1 C_0 \rho p^p.
\]

For $\varepsilon \in (0, 1)$ and sufficiently small $\rho$, the conclusion holds.

**Lemma 2.4.** Assume that (A3) and (A4) hold. Then there exists $e \in E$ with $\|e\|_E > \rho$ such that $I_\lambda(e) < 0$ for fixed $\lambda \in (0, 1)$. where $\rho$ is the same as in Lemma 2.3.

**Proof.** By (A3), there exists a constant $C > 0$ such that
\[
F(x, u) \geq C|u|^\mu, \quad u \in \mathbb{R}.
\]

By (2.11),
\[
\int_{R^3} \phi'_u u^2 dx = \|\phi'_u\|^2_{H^\mu} \leq C\|u\|^4_{H^\mu}.
\]

For $\xi > 0$ and $v \in C_0^\infty(R^3)$, by (2.10), (2.11) and (2.12), we have
\[
I_\lambda(\xi v) = \frac{\xi^2}{2} \|v\|^2_{H^\mu} + \frac{\xi^2}{2} \|v\|^2_{L^2} + \frac{\xi^4}{4} \int_{R^3} \phi'_u v^2 dx - \int_{R^3} F(x, \xi v)dx \\
\leq \frac{\xi^2}{2} \|v\|^2_{H^\mu} + \frac{\xi^2}{2} \|v\|^2_{L^2} + \frac{\xi^4}{4} \|v\|^4_{H^\mu} - C\xi^\mu \|v\|^\mu \rightarrow -\infty
\]
as $\xi \rightarrow +\infty$. Define a path $h : [0, 1] \rightarrow E$ by $h(\eta) = \eta \xi v$. For $\xi$ large enough, we get
\[
\|h(1)\|_E = \left(\int_{R^3} ((-\Delta)^{\frac{\mu}{2}} h(1))^2 + V(x) h^2(1)) dx \right)^{\frac{1}{2}} > \rho \quad \text{and} \quad I_\lambda(h(1)) < 0.
\]

Choose $e = h(1)$, we obtain the conclusion.

3. PROOF OF THEOREM 1.1

**Lemma 3.1.** Assume that (V), (A1), (A3) hold. Then $I_\lambda$ satisfies the Palais-Smale condition on $E$ for fixed $\lambda \in (0, 1)$.

**Proof.** Let $\{u_n\}$ be a Palais-Smale sequence in $E$, i.e., $I_\lambda(u_n)$ is bounded and $I'_\lambda(u_n) \rightarrow 0$. We will show that $\{u_n\}$ has a convergent subsequence in $E$. Then
\[
C + \|u_n\|_E \geq I_\lambda(u_n) - \frac{1}{\mu} \langle I'_\lambda(u_n), u_n \rangle \\
= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2_{H^\mu} + \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2_{L^2} + \left(\frac{1}{4} - \frac{1}{\mu}\right) \int_{R^3} \phi'_u u_n^2 dx \\
+ \int_{R^3} \left( \frac{u_n f(x, u_n)}{\mu} - F(x, u_n) \right) dx \\
\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \lambda \|u_n\|^2_{E}.
\]

This implies that $\{u_n\}$ is bounded in $E$.

Up to a subsequence, we assume that $u_n \rightarrow u$ in $E$. Since $E$ is compactly embedded
in $L^p(\mathbb{R}^3)$ for $2 \leq p < 2^*_s$, then $u_n \to u$ in $L^p(\mathbb{R}^3)$, $2 \leq p < 2^*_s$. By (2.9), (2.10), we get

$$
\|u_n - u\|^2_\Lambda = (I'_\lambda(u_n) - I'_\lambda(u), u_n - u) - \|u_n - u\|^2_2 - \int_{\mathbb{R}^3} (\phi'_{u_n} u_n - \phi'_u u)(u_n - u)dx
$$

(3.2)

$$
+ \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u)dx.
$$

Clearly, we have

(3.3) 

$$
(I'_\lambda(u_n) - I'_\lambda(u), u_n - u) \to 0 \quad \text{and} \quad \|u_n - u\|^2_2 \to 0 \quad \text{as} \quad n \to \infty.
$$

By the generalization of Hölder inequality, Lemma 2.1 and (2.6), it follows that

$$
\left| \int_{\mathbb{R}^3} \phi'_{u_n} u_n(u_n - u)dx \right| \leq \|\phi'_{u_n}\|_{L^\infty} \|u_n\|^2_2 \|u_n - u\|^{12}_{\frac{2}{3 + 2n}}
\leq C\|\phi'_{u_n}\|_{L^2} \|u_n\|^2_2 \|u_n - u\|^{12}_{\frac{2}{3 + 2n}}
\leq C\|u_n\|^3_3 \|u_n - u\|^{12}_{\frac{2}{3 + 2n}}
\leq C\|u_n\|^3_E \|u_n - u\|^{12}_{\frac{2}{3 + 2n}}.
$$

Similarly,

$$
\left| \int_{\mathbb{R}^3} \phi'_u u(u_n - u)dx \right| \leq C\|u\|^2_E \|u_n - u\|^{12}_{\frac{2}{3 + 2n}}.
$$

We have

(3.4)

$$
\left| \int_{\mathbb{R}^3} (\phi'_{u_n} u_n - \phi'_u u)(u_n - u)dx \right| \leq \left| \int_{\mathbb{R}^3} \phi'_{u_n} u_n(u_n - u)dx \right| + \left| \int_{\mathbb{R}^3} \phi'_u u(u_n - u)dx \right| \to 0 \quad \text{as} \quad n \to \infty.
$$

By (A1), Hölder inequality and Minkowski inequality,

$$
\left| \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u)dx \right|
\leq C_1 \int_{\mathbb{R}^3} (|u_n| + |u|)|u_n - u|dx + C_1 \int_{\mathbb{R}^3} (|u_n|^{p-1} + |u|^{p-1})|u_n - u|dx
\leq C_1 \|u_n\|_p + \|u\|_p \|u_n - u\|_2 + C_1\|u_n\|^{p-1} + \|u|^{p-1}\|u_n - u\|_p
\leq C_1 (\|u_n\|_2 + \|u\|_2)\|u_n - u\|_2 + C_1 (\|u_n\|_p^{p-1} + \|u\|_p^{p-1})\|u_n - u\|_p
$$

(3.5)

$$
\leq C(\|u_n\|_E + \|u\|_E)\|u_n - u\|_2 + C(\|u_n\|_E^{p-1} + \|u\|_E^{p-1})\|u_n - u\|_p \to 0 \quad \text{as} \quad n \to \infty.
$$

By (3.2), (3.3), (3.4) and (3.5), we see that $\{u_n\}$ converges strongly in $E$ for fixed $\lambda \in (0, 1)$, therefore $I_\lambda$ satisfies the Palais-Smale condition on $E$ for fixed $\lambda \in (0, 1)$.

\begin{theorem}
Assume that (A3) hold. Let $\lambda_n \to 0$ and let $\{u_n\} \subset E$ be a sequence of critical points of $I_{\lambda_n}$ satisfying $I'_{\lambda_n}(u_n) = 0$ and $I_{\lambda_n}(u_n) \leq C$ for some $C$ independent of $n$. Then up to a subsequence as $n \to \infty$, $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, $u$ is a critical point of $I$.
\end{theorem}
Proof. By $I'_{\lambda_n}(u_n) = 0$ and $I_{\lambda_n}(u_n) \leq C$, we have

\[ C \geq I_{\lambda_n}(u_n) - \frac{1}{\mu} \langle I'_{\lambda_n}(u_n), u_n \rangle \]

\[ = \left( \frac{1}{2} - \frac{1}{\mu} \right) \| u_n \|_{H^s_\lambda}^2 + \left( \frac{1}{2} - \frac{1}{\mu} \right) \| u_n \|_2^2 + \int_{\mathbb{R}^3} \phi_{u_n}^* u_n^3 \, dx \]

\[ + \int_{\mathbb{R}^3} \left( \frac{u_n f(x, u_n)}{\mu} - F(x, u_n) \right) \, dx \]

(3.6)

\[ \geq \left( \frac{1}{2} - \frac{1}{\mu} \right) \| u_n \|_{H^s_\lambda}^2 + \left( \frac{1}{2} - \frac{1}{\mu} \right) \int_{\mathbb{R}^3} \lambda_n V(x) u_n^2 \, dx. \]

Then up to a subsequence, we have $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$. By Lemma 2.3 in [11], $\phi_{u_n}^* \rightarrow \phi_u^*$ in $D^{1,2}(\mathbb{R}^3)$. Taking $v \in C_0^\infty(\mathbb{R}^3)$, then

\[ \int_{\mathbb{R}^3} \phi_{u_n}^* u v \, dx \rightarrow \int_{\mathbb{R}^3} \phi_u u v \, dx, \text{ as } n \to \infty. \]

From the generalization of Hölder inequality, it follows that

\[ \left| \int_{\mathbb{R}^3} \phi_{u_n}^* (u_n - u) v \, dx \right| \leq \| \phi_{u_n}^* \|_{2^*} \| u_n - u \|_{L^{12/(3+2r)}(\Omega)} \| v \|_{L^{12/(3+2r)}(\Omega)} \to 0 \text{ as } n \to \infty, \]

where $\Omega$ is the support of $v$. Then,

\[ \left| \int_{\mathbb{R}^3} \phi_{u_n}^* u v \, dx - \int_{\mathbb{R}^3} \phi_u u v \, dx \right| \leq \int_{\mathbb{R}^3} (\phi_{u_n}^* - \phi_u^*) v \, dx + \int_{\mathbb{R}^3} \phi_{u_n}^* (u_n - u) v \, dx \to 0 \]

as $n \to \infty$, for all $v \in C_0^\infty(\mathbb{R}^3)$. By (2.14), (2.11),

\[ \langle I'_{\lambda_n}(u_n), v \rangle = \int_{\mathbb{R}^3} \left( (-\Delta) \frac{3}{2} u_n (-\Delta) \frac{3}{2} v + u_n v + \phi_{u_n}^* u_n v - f(x, u_n) v \right) \, dx + \lambda_n \int_{\mathbb{R}^3} V(x) u_n v \, dx, \]

where $v \in C_0^\infty(\mathbb{R}^3)$. By (3.6), Hölder inequality,

\[ \lambda_n \int_{\mathbb{R}^3} V(x) u_n v \, dx = \lambda_n \int_{\mathbb{R}^3} (\sqrt{V(x)} u_n) (\sqrt{V(x)} v) \, dx \leq \lambda_n^{1/2} \left( \int_{\mathbb{R}^3} \lambda_n V(x) v^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^3} V(x) v^2 \, dx \right)^{1/2} \]

\[ \to 0 \text{ as } n \to \infty. \]

Thus, we see that $I'(u)v = 0$ for all $v \in C_0^\infty(\mathbb{R}^3)$. It is known that $C_0^\infty(\mathbb{R}^3)$ is dense in $H^s(\mathbb{R}^3)$, see Theorem 2.4 in [3]. Therefore, $I'(u)v = 0$ for all $v \in H^s(\mathbb{R}^3)$, $u$ is a critical point of $I$. \hfill \square

Lemma 3.3. (Lemma 2.3 in [11]). Let $B_\sigma(x)$ be the open ball in $\mathbb{R}^3$ of radius $\sigma$ centred at $x$. If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$ and for $q \in [2, 2^*_s)$, we have

\[ (3.7) \quad \sup_{x \in \mathbb{R}^3} \int_{B_\sigma(x)} |u_n|^q \to 0 \text{ as } n \to \infty, \]

then $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for $r \in (2, 2^*_s)$. \hfill \square

Proof of Theorem 1. Choose $\phi \in C_0^\infty(\mathbb{R}^3)$ and $\xi > 0$. Define a path $h : [0, 1] \to E$ by $h(\eta) = \eta \xi \phi$. When $\xi$ is large enough, by lemma [2.4], we have $I_{\lambda}(h(1)) < 0$, $\|h(1)\|_E > \rho$ for small $\rho$ and $\sup_{\eta \in [0, 1]} I_{\lambda}(\gamma(\eta)) \leq c$ for some $c$ independent of $\eta \in [0, 1]$. Define

\[ c_{\lambda} = \inf_{\gamma \in \Gamma} \sup_{\eta \in [0, 1]} I_{\lambda}(\gamma(\eta)), \]
where $\Gamma = \{ \gamma | \gamma \in C([0, 1], E), \gamma(0) = 0, \gamma(1) = \xi \phi \}$. By Lemma 2.3, the mountain pass theorem holds and $c_3$ is a critical value of $I_\lambda$. Therefore, we can choose $\lambda_n \to 0$, and a sequence of critical points $\{u_n\} \subset E$ satisfying $I'_\lambda(u_n) = 0$ and $I_\lambda(u_n) \leq c_3$.

By Lemma 3.2, up to a subsequence $u_n \rightharpoonup u$, and $u$ is a critical point of $I$ in $H^s(\mathbb{R}^3)$. We need to show that $u \neq 0$. Note that $H^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ for any $q \in [2, 2^*_s]$, then

$$0 = I'_\lambda(u_n)u_n = \int_{\mathbb{R}^3} \left( |(-\Delta)^{\frac{s}{2}} u_n|^2 + u_n^2 \right) dx + \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n^2 dx - \int_{\mathbb{R}^3} f(x, u_n) dx$$

$$\geq \|u_n\|_{H^s}^2 - \epsilon \|u_n\|^2_2 - C \|u_n\|_p^p$$

$$\geq C \|u_n\|_{p}^2 - C \|u_n\|_p^p.$$

Then $\|u_n\|_p \geq \left( \frac{C}{C_p} \right)^{\frac{1}{p-2}}$. If $u = 0$, then given any $x \in \mathbb{R}^3$, $u_n \rightharpoonup 0$ in $H^s(B_\sigma(x))$, since the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is locally compact for $q \in [1, 2^*_s]$, then $u_n \to 0$ in $L^p(B_\sigma(x))$. By Lemma 3.3 we have $u_n \to 0$ in $L^p(\mathbb{R}^3)$, which is a contradiction with $\|u_n\|_p \geq \left( \frac{C}{C_p} \right)^{\frac{1}{p-2}}$. Therefore, $u \neq 0$. The proof is complete.

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