ON THE RAMANUJAN-PETERSSON AND SELBERG CONJECTURES FOR MAASS FORMS

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Abstract. We give an equivalent to the Ramanujan-Petersson conjecture for Maass forms of the group $SL(2, \mathbb{Z})$, with the help of automorphic distribution theory: this is an alternative to classical automorphic function theory, in which the plane takes the place usually ascribed to the hyperbolic half-plane. We examine the Selberg eigenvalue conjecture for Hecke’s group $\Gamma_0(M)$ as well.

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1. Introduction

Given a non-trivial character $\chi$ on $\mathbb{Q}^\times$, such that $|\chi(mn)| \leq |mn|^C$ for some $C > 0$ and all nonzero integers $m, n$, and a real number $\lambda$, extract from the distribution

$$\mathcal{F}_\chi(x, \xi) = \pi \sum_{m,n \neq 0} \chi\left(\frac{m}{n}\right) e^{2i\pi mx} \delta(\xi - n)$$

its part homogeneous of degree $-1 - i\lambda$, to wit the tempered distribution

$$\mathcal{R}_\chi, i\lambda(x, \xi) = \frac{1}{4} \sum_{mn \neq 0} |n|^i\lambda \chi\left(\frac{m}{n}\right) |\xi|^{-1-i\lambda} \exp\left(2i\pi \frac{mnx}{\xi}\right).$$

While generally not invariant under the action in $\mathbb{R}^2$, by linear changes of coordinates, of the group $\Gamma = SL(2, \mathbb{Z})$, it is so for special pairs $\chi, \lambda$. The Ramanujan-Petersson conjecture is that, when such is the case, $\chi$ is of necessity a unitary character. This is an unusual formulation of the question, but it defines the environment in which it will be solved: the actual proof will be contained for the essential in Sections 7 to 9. We shall of course show the equivalence between this problem and its more traditional version, expressed in terms of modular forms of the non-holomorphic type.
Some approaches towards the Ramanujan-Petersson conjecture for Maass forms have been made several times: bounds of the Fourier coefficients $b_p$ of Hecke eigenforms by expressions $p^\alpha + p^{-\alpha}$, with exponents improving on the way, have been obtained [2, 10, 9, 13]. These results have for the most been obtained as corollaries of implementations of the Langlands functoriality principle. In contrast, the present work (which gives the result hoped for, to wit the bound corresponding to $\alpha = 0$) is strictly an analyst’s job. Besides a short section devoted to elementary algebraic calculations on powers of the Hecke operator (which could be done indifferently in the modular distribution environment, in the plane, or the modular form one, in the hyperbolic half-plane: we chose the first for coherence), the rest of the paper consists of estimates and spectral theory.

Automorphic distributions are tempered distributions in $\mathbb{R}^2$, invariant under the action of the group $\Gamma$ by linear transformations: they are called modular if they are moreover homogeneous of some degree, and the special case just considered defines the notion of Hecke distribution. We shall indicate at the end of Section 3 some classical concepts to which that of automorphic distributions is related.

The phrase “automorphic functions” will always refer to functions in the hyperbolic half-plane $\mathbb{H}$ invariant under the action of $\Gamma$ by fractional-linear transformations $z \mapsto \frac{az+b}{cz+d}$. A large linear space of automorphic functions is algebraically isomorphic to the space of automorphic distributions invariant under the symplectic Fourier transformation $F_{\text{symp}}$. The map $\Theta$ from automorphic distributions to automorphic functions at work here, to wit the one defined by the identity

$$
(\Theta \mathcal{G})(z) = \langle \mathcal{G}, (x, \xi) \mapsto \exp\left(-\pi \frac{|x - z \xi|^2}{\text{Im}z}\right) \rangle,
$$

(1.3)

originated from pseudodifferential analysis (cf. (3.8)). It transforms modular distributions into modular forms of the non-holomorphic type, and distributions of the kind $\mathfrak{M}_{\chi, \nu}$ (called Hecke distributions) into Hecke-normalized Hecke eigenforms: all Hecke eigenforms can be obtained in this way.

There is also a notion of Eisenstein distribution $\mathcal{E}_{\pm \nu}$ (the two distributions so denoted are the images of each other under $F_{\text{symp}}$) corresponding to the more familiar one of Eisenstein series $E_{\frac{1-\nu}{2}}$ of the non-holomorphic type. All these matters, treated in [20], will be shortly reconsidered here,
for the sake of self-containedness, and the equivalence between the two formulations of the Ramanujan-Petersson conjecture will follow. In [21], we proved that a combination of automorphic distribution theory and pseudodifferential analysis led to a criterion for the validity of the Riemann hypothesis: this project was finally put to the hoped for end in [22]. The present paper depends for the most on Hecke distributions, and the preprint just cited on Eisenstein distributions.

In automorphic analysis, $\mathbb{R}^2$ and $\mathbb{H}$ play complementary roles. The quotient of the half-plane by $SL(2,\mathbb{Z})$ and the automorphic Laplacian $\Delta$ are the right objects for Hilbert space methods: there exists [19, Chapter 5] an independently defined Hilbert space $L^2(\Gamma \backslash \mathbb{R}^2)$, but it is by no means as simple to use, because of the lack of fundamental domain. On the other hand, analysis on the plane depends on the pair of operators

$$2i\pi \mathcal{E} = x \frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi} + 1, \quad 2i\pi \mathcal{E}^2 = x \frac{\partial}{\partial x} - \xi \frac{\partial}{\partial \xi},$$

(1.4)

the first of which only commutes with the action of $\Gamma$: the operator $\Delta - \frac{1}{4}$ is the transfer under $\Theta$ of $\pi^2 \mathcal{E}^2$. We define for $h \in S(\mathbb{R}^2)$ and $t > 0$

$$(t^{2i\pi \mathcal{E}} h)(x, \xi) = t h(tx, t\xi), \quad (t^{2i\pi \mathcal{E}^2} h)(x, \xi) = h(tx, t^{-1}\xi),$$

(1.5)

so that $t\frac{d}{dt} (t^{2i\pi \mathcal{E}} h) = (2i\pi \mathcal{E}) (t^{2i\pi \mathcal{E}} h)$ and something similar goes with the other operator. Transposing the operator $2i\pi \mathcal{E}$ or $2i\pi \mathcal{E}^2$ is the same as changing it to its negative so that, using duality, the definition just given applies as well if one replaces $h \in S'(\mathbb{R}^2)$ by $\mathcal{S} \subset S'(\mathbb{R}^2)$.

On the other hand, the Hecke operator $T_p$ familiar to number theorists becomes in the plane the operator $T_p^{\text{dist}} = p^{-\frac{1}{2}+i\pi \mathcal{E}^2} + p^{\frac{1}{2}-i\pi \mathcal{E}^2} \sigma_1$, where $\sigma_1$ is a simple averaging operator, a special case of the operator $\sigma_r$ ($r = 0, 1, \ldots$) such that

$$(\sigma_r \mathcal{S})(x, \xi) = \frac{1}{p^r} \sum_{b \mod p^r} \mathcal{S} \left( x + \frac{b \xi}{p^r}, \xi \right).$$

(1.6)

Simple algebraic calculations lead to the expression of the $(2N)$th power of $T_p^{\text{dist}}$ as a linear combination, with essentially explicit coefficients, of the operators $\left(p^{-1+2i\pi \mathcal{E}^2}\right)^{N-\ell} \sigma_r$, with $0 \leq \ell \leq 2N$ and $(2\ell - 2N)_+ \leq r \leq \ell$. 
Setting $\Gamma_\infty = \{(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}) : b \in \mathbb{Z}\}$, consider the measure $s^1_1(x, \xi) = e^{2i\pi x} \delta(\xi - 1)$. The series
\[ \mathcal{B} = \frac{1}{2} \sum_{g \in \Gamma / \Gamma_\infty} s^1_1 \circ g^{-1} \] does not converge weakly in the space $\mathcal{S}'(\mathbb{R}^2)$ of tempered distributions, but it does so in the space of continuous linear forms on the space defined as the image of $\mathcal{S}_{\text{even}}(\mathbb{R}^2)$ under $\pi^2 \mathcal{E}^2$. It is of course automorphic, and it decomposes as a superposition of modular distributions as the identity
\[ \mathcal{B} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\zeta(i\lambda) \zeta(-i\lambda)} e^{i\lambda} d\lambda + \frac{1}{2} \sum_{r, \iota} \frac{\Gamma(-i\frac{\lambda_r}{2}) \Gamma(i\frac{\lambda_r}{2})}{\|N^{r, \iota}\|_2^2} \mathcal{B}_{r, \iota}, \] better explained in detail in Section 6: let us just observe here that no modular distribution is missing. The series relates under $\Theta$ to a series introduced by Selberg [14], general versions of which are referred to as Poincaré series in the more recent literature.

From this point on, we stay entirely within the automorphic distribution environment. We shall show that, with an integer $N$ going to infinity, and given a prime $p$, the operator $(p^{-\frac{1}{2} + i\pi \mathcal{E}} + p^{\frac{1}{2} - i\pi \mathcal{E}} \sigma_1)^{-\frac{1}{2} N}$, which acts as a scalar on each term of (1.8), does not increase the coefficients by a factor larger than $(2\delta)^{2N}$, where $\delta$ is an arbitrary number $> 1$. The elementary trick of eliminating, when possible, constants on which we have no control by managing so as to finally take them to the power $\frac{1}{N}$ with $N$ large is inspired from the little we understood of the Langlands functoriality method considered by most.

This part, based on the series that defined $\mathcal{B}$ in the first place, constitutes the central point of our approach. Using (1.6), we shall have to find appropriate bounds, in $\mathcal{S}'(\mathbb{R}^2)$, for the individual terms $(p^{-1 + 2i\pi \mathcal{E}})^{N-\ell} \sigma_1 \mathcal{B}$. We had to give different proofs in the cases when $0 \leq \ell \leq N$ and $N \leq \ell \leq 2N$: this constitutes the more difficult part of the paper. What will then remain to be done is some localizing, performed by the insertion under the integral or summation sign in (1.8) of some operator $\Phi_N(2i\pi \mathcal{E})$ concentrating the resulting distribution, in some sense, near any given discrete eigenvalue $-i\lambda_r$ of the automorphic version of $2i\pi \mathcal{E}$. There, as well as in the estimates just alluded to, the fact that we are working on the plane rather than the half-plane will be decisive: we shall devote the end of Section 3 to
a justification of this point of view.

2. Homogeneous and bihomogeneous functions in the plane

We fix notation: needless to say, there is nothing original here. Given \( \nu \in \mathbb{C} \) and \( \delta = 0 \) or 1, the function \( t \mapsto |t|^{-\nu} \) on \( \mathbb{R} \setminus \{0\} \) is locally summable if \( \Re \nu < 1 \). The identity

\[
|t|^{-\nu} = \frac{\Gamma(1 - \nu)}{\Gamma(1 - \nu + k)} \left( \frac{d}{dt} \right)^k |t|^{-\nu+k}, \quad \delta' \equiv \delta + k \mod 2, \quad \Re (k - \nu) > 0,
\]

(2.1)
to which one may add \( t^{-1} = \frac{d}{dt} \log |t| \), makes it possible to extend \( |t|^{-\nu} \) as a tempered distribution on the line, provided that \( \nu \neq \delta + 1, \delta + 3, \ldots \). Let us introduce the shorthand

\[
B_\delta(\nu) = (-i)^\delta \pi^{\nu - \frac{1}{2}} \frac{\Gamma(1 - \nu + \delta)}{\Gamma(\nu + \delta/2)}, \quad \nu \neq \delta + 1, \delta + 3, \ldots,
\]

(2.2)
noting the functional equation \( B_\delta(\nu)B_\delta(1 - \nu) = (-1)^\delta \) and the relation \( B_0(\nu) = \frac{\zeta(\nu)}{\zeta(1 - \nu)} \), equivalent to the functional equation of the zeta function. One has

\[
(\mathcal{F} (|t|^{-\nu})) (\tau) : = \int_{-\infty}^{\infty} |t|^{-\nu} e^{-2\pi t\tau} dt = B_\delta(\nu) |\tau|_{\delta}^{-1},
\]

(2.3)
a semi-convergent integral if \( 0 < \Re \nu < 1 \); if this is not the case, the Fourier transformation in the space of tempered distributions makes it possible to extend the identity, provided that \( \nu = \delta + 1, \delta + 3, \ldots \) and \( \nu \neq -\delta, -\delta - 2, \ldots \). As functions of \( \nu \) with values in \( \mathcal{S}'(\mathbb{R}) \), both sides of this equation are holomorphic in the domain just indicated. Recall for reference the asymptotics of the Gamma function on vertical lines \cite[p.13]{12}

\[
|\Gamma(x + iy)| \sim (2\pi)^{\frac{1}{2}} e^{-\frac{\pi}{2}|y|} |y|^{x-\frac{1}{2}}, \quad |y| \to \infty.
\]

(2.4)

In the plane \( \mathbb{R}^2 \), we shall use the pair of commuting operators, the first of which is the Euler operator,

\[
2i\pi \mathcal{E} = x \frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi} + 1, \quad 2i\pi \mathcal{E}^2 = x \frac{\partial}{\partial x} - \xi \frac{\partial}{\partial \xi}
\]

(2.5)
The operators \( \mathcal{E} \) and \( \mathcal{E}^2 \) are formally self-adjoint in \( L^2(\mathbb{R}^2) \), but we shall only use them in the Schwartz space \( \mathcal{S}(\mathbb{R}^2) \) and its dual \( \mathcal{S}'(\mathbb{R}^2) \), the space of
tempered distributions. Homogeneous distributions are generalized eigenfunctions of the operator $2i\pi \mathcal{E}$: we need to consider also bihomogeneous functions

$$\text{hom}^{(\delta)}_{\rho,\nu}(x,\xi) = |x|^{\frac{\rho+\nu-2}{2}} |\xi|^{\frac{\rho+\nu}{2}}, \quad x \neq 0, \xi \neq 0. \quad (2.6)$$

They are globally even generalized eigenfunctions of the pair $(2i\pi \mathcal{E}, 2i\pi \mathcal{E}^\sharp)$ for the pair of eigenvalues $(\nu, \rho - 1)$, and they make sense as distributions in the plane provided that $\rho + \nu \neq -2\delta, -2\delta - 4, \ldots$ and $2 - \rho + \nu \neq -2\delta, -2\delta - 4, \ldots$.

It is useful to keep handy the formula giving the decomposition of "general" functions $h$ in $\mathbb{R}^2$ into generalized eigenfunctions of the operator $2i\pi \mathcal{E}$. Given $h \in \mathcal{S}(\mathbb{R}^2)$ and $(x,\xi) \neq (0,0)$, the function $f(t) = e^{2\pi t}h(e^{2\pi t}x,e^{2\pi t}\xi)$ is integrable and, defining

$$h_{i\lambda}(x,\xi) = \hat{f}(-\lambda) = \frac{1}{2\pi} \int_{0}^{\infty} \theta^{i\lambda} h(\theta x, \theta \xi) \, d\theta, \quad (2.7)$$

the function $\hat{f}$ is continuous and bounded. Multiplying it by $i\lambda$ amounts, integrating by parts, to applying the operator $-\theta \frac{d}{d\theta} - 1$, under the integral sign, to the factor $h(\theta x, \theta \xi)$, in other words to replacing $h$ by $(-2i\pi \mathcal{E})h$; doing this twice, one sees that $\hat{f}$ is integrable, hence $\int_{-\infty}^{\infty} \hat{f}(\lambda) \, d\lambda = f(0)$, in other words $\int_{-\infty}^{\infty} h_{i\lambda}(x,\xi) \, d\lambda = h(x,\xi)$. The function $h_{i\lambda}$, as so defined in $\mathbb{R}^2\setminus\{(0, 0)\}$, is homogeneous of degree $-1 - i\lambda$, in other words $(2i\pi \mathcal{E})h_{i\lambda} = -i\lambda h_{i\lambda}$. Assuming some degree of flatness of the function $h$ at $(0,0)$, one can extend the definition of $h_{i\nu}$ to complex values of $\nu$, given some lower bound for $\text{Re} \nu$.

The notation (1.5) may be regarded as justified by the equation $t \frac{d}{dt} |_{t=1} \left( t^{2i\pi \mathcal{E}} h \right) = (2i\pi \mathcal{E}) h$ that follows. It is in an $L^2(\mathbb{R}^2)$-setting, however, that $t^{2i\pi \mathcal{E}}$ takes its full sense. As defined on $\mathcal{S}(\mathbb{R}^2)$, the symmetric (i.e., formally self-adjoint) operator $2\pi \mathcal{E}$ is essentially self-adjoint (which means that it has a unique self-adjoint extension, also denoted as $2\pi \mathcal{E}$): then, the map $t \mapsto t^{2i\pi \mathcal{E}}$ as defined in (1.5) is a one-parameter group of unitary operators, the infinitesimal generator of which, in the sense of Stone’s theorem (for instance, [23]) is the operator $2\pi \mathcal{E}$.
3. FROM THE PLANE TO THE HALF-PLANE

Denote as $\mathbb{H}$ the hyperbolic half-plane \( \{z = x + iy \in \mathbb{C} : y > 0\} \), provided with its usual measure \( y^{-2} \, dx \, dy \), invariant under the action of the group $SL(2, \mathbb{R})$ by fractional-linear transformations \( ((a \ b) \ c \ d) \mapsto \frac{a \cdot z + b}{c \cdot z + d} \). It is a Euclidean domain, with the Laplacian $\Delta = (z - \overline{z})^2 \frac{\partial^2}{\partial z \partial \overline{z}} = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$.

If $\mathcal{S} = \mathcal{S}(x, \xi)$ (the notation $(x, \xi)$ for points of $\mathbb{R}^2$ is traditional in pseudodifferential analysis, from which all the new notions in this paper originated), we denote as $\Theta S$ the analytic function in $\mathbb{H}$ defined as
\[
(\Theta S)(z) = \langle S, (x, \xi) \mapsto \exp \left(-\pi \frac{|x - z \xi|^2}{\text{Im} \, z}\right) \rangle.
\]

It is immediate that the map $\Theta$ intertwines two actions of the group $SL(2, \mathbb{R})$: the one, by fractional-linear transformations in $\mathbb{H}$, and the action defined as \( ((a \ b) \ c \ d, (x \ \xi)) \mapsto \left(\frac{ax + b \xi}{cx + d \xi}\right) \) in the plane. Note that in expressions $\langle \mathcal{S}, h \rangle$ such as the one on the right-hand side of (3.1), the use of straight brackets indicates that we are dealing with the bilinear form defined in terms of the integral. One has the transfer identity
\[
\Theta \left(\pi^2 \mathcal{E}^2 \mathcal{S}\right) = \left(\Delta - \frac{1}{4}\right) \Theta \mathcal{S}. \tag{3.2}
\]

Setting $\rho = \frac{|x - z \xi|^2}{\text{Im} \, z}$, this identity can indeed be written as
\[
- \left(\rho \frac{d}{d\rho} + \frac{1}{2}\right)^2 f(\rho) = \left(\Delta - \frac{1}{4}\right) f(\rho) \tag{3.3}
\]
for every $C^2$ function $f$; the calculation of the right-hand side presents no complication since, taking advantage of the invariance of both operators involved under the appropriate actions of $SL(2, \mathbb{R})$, one may assume that $(x, \xi) = (1, 0)$, so that $\rho = (\text{Im} \, z)^{-1}$.

The spaces $\mathbb{R}^2$ and $\mathbb{H}$ are two homogeneous spaces of the group $SL(2, \mathbb{R})$ and the map $\Theta$ is an associate of the dual Radon transformation, the most elementary case of Helgason’s theory [5]. More precisely [19, Prop. 2.1.1], it is obtained from the dual Radon transformation $V^*$ by inserting on the right-hand side the operator $\pi^{\frac{1}{2}} \text{e}^{-i\pi \mathcal{E}} \Gamma \left(\frac{1}{2} + i\pi \mathcal{E}\right)$. In view of the decreasing (2.4) of the Gamma factor at infinity on vertical lines, this is already an indication of the fact that analysis in the plane is bound to give precise results more easily than its image in $\mathbb{H}$, as obtained under $\Theta$. Actually, the operator $\Theta$ was introduced for the necessities of pseudodifferential analysis
In the plane, we always use the symplectic Fourier transformation, as defined by the equation

\[
(\mathcal{F}_{\text{symp}} \mathcal{S})(x, \xi) = \int_{\mathbb{R}^2} \mathcal{S}(y, \eta) e^{2i\pi(xy-\eta \xi)} \, dy \, d\eta,
\]  

which is invariant under the action by linear changes of coordinates of $SL(2, \mathbb{R})$. One has for every tempered distribution $\mathcal{S}$ the identity $\Theta \mathcal{F}_{\text{symp}} \mathcal{S} = \Theta \mathcal{S}$, which shows that the map $\Theta$ is not quite one-to-one. But if $\mathcal{S}$ is globally even (which will be the case considered, most of the time), it is characterized by the pair of its images under $\Theta$, $\Theta(2i\pi E)$: some pseudodifferential analysis is required to prove this. Introduce it as the linear map $\Psi$ (one-to-one and onto, denoted as $\text{Op}_2$ in [21]) from $\mathcal{S}'(\mathbb{R}^2)$ to the space of continuous linear operators from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ defined by the formula

\[
(v | \Psi(\mathcal{S}) u) = \langle \mathcal{S}, \text{Wig}(v, u) \rangle, \quad u, v \in \mathcal{S}(\mathbb{R}),
\]  

with

\[
\text{Wig}(v, u)(x, \xi) = \int_{-\infty}^{\infty} v(x + t) u(x - t) e^{2i\pi \xi t} \, dt.
\]

If $\mathcal{S}$ is a globally even distribution, the operator $\Psi(\mathcal{S})$ preserve the parity of functions. Also, one has the general identity $\Psi(\mathcal{F}_{\text{symp}} \mathcal{S}) u = \Psi(\mathcal{S}) \tilde{u}$, with $\tilde{u}(x) = u(-x)$.

One remarks (this is no accident) that the function which enters the definition (3.1) of $\Theta$ is a composite object: indeed, taking

\[
u_z(x) = (\text{Im}(-z^{-1}) \frac{1}{4}) \exp \left( \frac{i\pi x^2}{2z} \right),
\]

one has the identity

\[
\text{Wig}(\nu_z, \nu_z)(x, \xi) = \exp \left( -\pi \frac{|x - z\xi|^2}{\text{Im} z} \right).
\]

It follows that, if $\Theta \mathcal{S} = 0$, one has first $(\nu_z | \Psi(\mathcal{S}) \nu_z) = 0$ for every $z \in \mathbb{H}$, then (regarding the exponent in (3.8) as the restriction to the diagonal $w = z$ of the function $\frac{2i(x-\overline{\xi})(x-z\xi)}{z-\overline{w}}$ and using a “sesquiholomorphic” argument) that $(\nu_w | \Psi(\mathcal{S}) \nu_z) = 0$ for every pair $w, z$ of points of $\mathbb{H}$. As linear combinations of functions $\nu_z$ build up to a dense subspace of $\mathcal{S}_{\text{even}}(\mathbb{R})$, it follows that the restriction of $\Psi(\mathcal{S})$ to $\mathcal{S}_{\text{even}}$ is zero. Replacing $\nu_z(x)$ by $x \nu_z(x)$, one shows in the same way that the restriction of $\Psi(\mathcal{S})$ to $\mathcal{S}_{\text{odd}}$ is
characterized by the function $\Theta(2i\pi E) \mathcal{S}$.

This implies that two functions in $\mathbb{H}$, to wit $\Theta \mathcal{S}$ and $\Theta(2i\pi E) \mathcal{S}$, are needed to characterize a globally even tempered distribution $\mathcal{S}$ in $\mathbb{R}^2$, and that they suffice. Pseudodifferential operator theory in an arithmetic environment, to be precise the analysis of operators $\Psi(\mathcal{S})$ in which $\mathcal{S}$ is an automorphic distribution, is the central element in the disproof [22] of the Riemann hypothesis.

In the next section, we shall reconsider the Radon transform or the associated operator $\Theta$ in an arithmetic environment, more specifically in the $\Gamma$-invariant situation, with $\Gamma = SL(2,\mathbb{Z})$. As promised in the introduction, we explain now some of the advantages of working in the plane. These concern analysis, not algebraic facts which, for instance the elementary calculations in Section 5, could be carried just as well in the hyperbolic half-plane. Also, there is a nice Hilbert space structure on $L^2(\Gamma \backslash \mathbb{H})$: though there exists a Hilbert space $L^2(\Gamma \backslash \mathbb{R}^2)$ [19, Section 5], its definition is far from being as simple as that of $L^2(\Gamma \backslash \mathbb{H})$, because of the lack of a fundamental domain for the action of $\Gamma$ in $\mathbb{R}^2$. We shall go back to modular form theory when, as in Section 6, Hilbert space facts are needed.

One of the main advantages of modular distribution theory lies in the role of the Euler operator $\pi E$, the square of which transfers under $\Theta$ to $\Delta - \frac{1}{4}$: it commutes with the action of $SL(2,\mathbb{Z})$ (or $SL(2,\mathbb{R})$), an essential point in the last part (spectral localization) of this paper. Note that the transfer under $\Theta$ of the operator $2i\pi E^3$ is simpler, since one has the general formula $\Theta \left( p^{2i\pi E^3} \mathcal{S} \right)(z) = [\Theta \mathcal{S}](p^2 z)$. It is of course easier to use first-order operators, to start with when integrations by parts are needed, as will be the case in Section 7. Spectral-theoretic reasons also show the benefit of automorphic distribution theory. To find bounds for the eigenvalues of a high power of the Hecke operator $T_p^{\text{dist}}$ (the superscript indicates that we have transferred $T_p$ to the plane), one lets this operator act on some distribution $\mathcal{B}$ the decomposition of which into homogeneous components contains all Hecke distributions (cf. next section and Section 6). But, having found a satisfactory bound for the action of $[T_p^{\text{dist}}]^{2N}$ on $\mathcal{B}$, one still needs to localize (spectrally) the estimate obtained to see the effect of $[T_p^{\text{dist}}]^{2N}$ on individual Hecke distributions: this is done with the help of a well-chosen function of the operator $E$. Needless to say, functions of $E$ are easier to define and to use than functions of $\Delta - \frac{1}{4}$, the integral kernels of which require the
use of Legendre functions in the pair invariant \( \cosh(d(z, w)) \) on \( \mathbb{H} \). The heart of the matter is that while the subgroup \( SL(2, \mathbb{Z}) \) of \( SL(2, \mathbb{R}) \) defines modular objects in either case, introducing the subgroup \( SO(2) \), as needed to define the hyperbolic half-plane, can sometimes only complicate the picture.

To complete the list of advantages of the automorphic distribution point of view, let us mention, though this will not enter the present paper, that the use of the Weyl calculus endows, up to a point, the space of automorphic distributions with a structure of (non-commutative) operator-algebra [20]. As mentioned above, the main advantage of piecing together automorphic distribution theory and the Weyl calculus lies in that it leads to a disproof [22] of the Riemann hypothesis.

The Radon transformation from functions on \( \mathbb{H} \) to even functions on \( \mathbb{R}^2 \) is the simplest case of Helgason’s theory [5], to wit for \( G = SL(2, \mathbb{R}) \), in which case, with classical notation, \( \mathbb{H} \sim G/K \) and the space of even functions on \( \mathbb{R}^2 \) identifies with the space of functions on \( G/MN \). The link between automorphic function and automorphic distribution theory, analyzed in the next section, is the \( SL(2, \mathbb{Z}) \)-invariant version of the same. Actually, our construction of automorphic distribution theory was made [17, section 18] under the influence of the correspondence in the Lax-Phillips scattering theory [11] between a Cauchy problem for the wave equation in 3-dimensional spacetime with data on one sheet of a two-sheeted hyperboloid, and a Goursat (totally characteristic) problem with datum on the boundary of the light-cone.

4. Modular distributions and modular forms

This section is a summary, with minor modifications, of [20, Section 2.1] and [21, Section 6.3]. Its main point is to show that, when we have proved the Ramanujan-Petersson conjecture as formulated in the beginning of the introduction, it will entail the validity of the conjecture in its traditional formulation, involving modular forms. We first quote [20, Theor.1.2.2].

**Proposition 4.1.** Given a non-trivial character \( \chi \) on \( \mathbb{Q}^\times \), such that \( |\chi(mn)| \leq |mn|^C \) for some \( C > 0 \), and \( \lambda \in \mathbb{R} \), the even distribution \( N_{\chi,i\lambda} \in S'(\mathbb{R}^2) \).
defined by the equation

$$\langle \mathfrak{N}_{\chi, i\lambda}, h \rangle = \frac{1}{4} \sum_{m, n \neq 0} \chi \left( \frac{m}{n} \right) \int_{-\infty}^{\infty} |t|^{-1-i\lambda} \left( F_{1}^{-1} h \right) \left( \frac{m}{t}, nt \right) dt, \quad h \in \mathcal{S}(\mathbb{R}^{2}),$$

satisfies the identity $$\langle \mathfrak{N}_{\chi, i\lambda}, h \circ \left( \frac{1}{1} \right) \rangle = \langle \mathfrak{N}_{\chi, i\lambda}, h \rangle$$ for every function $$h \in \mathcal{S}(\mathbb{R}^{2}).$$ Also, it is homogeneous of degree $$-1-i\lambda.$$ Set $$\chi(-1) = (-1)^{\delta}$$ with $$\delta = 0$$ or $$1,$$ and define

$$\psi_{1}(s) = \sum_{m \geq 1} \chi(m) m^{-s} = \prod_{p} (1 - \chi(p) p^{-s})^{-1}, \quad \psi_{2}(s) = \sum_{n \geq 1} (\chi(n))^{-1} n^{-s},$$

two convergent series for $$\text{Re } s$$ large enough. Also, set

$$L(s, \mathfrak{N}_{\chi, i\lambda}) = \psi_{1} \left( s + \frac{i\lambda}{2} \right) \psi_{2} \left( s - \frac{i\lambda}{2} \right),$$

$$L^{\sharp}(s, \mathfrak{N}_{\chi, i\lambda}) = \frac{1}{2} B_{\delta} \left( \frac{2 - i\lambda}{2} - s \right) L(s, \mathfrak{N}_{\chi, i\lambda}),$$

and assume that the function $$s \mapsto L(s, \mathfrak{N}_{\chi, i\lambda})$$ extends as an entire function of $$s,$$ polynomially bounded in vertical strips. Then, the distribution $$\mathfrak{N}_{\chi, i\lambda}$$ admits a decomposition into bihomogeneous components, given as

$$\mathfrak{N}_{\chi, i\lambda} = \frac{1}{4i\pi} \int_{\text{Re } \rho = 1} L^{\sharp} \left( \frac{2 - \rho}{2}, \mathfrak{N}_{\chi, i\lambda} \right) \, \text{hom}^{(\delta)}_{\rho, -i\lambda} d\rho.$$

It is $$\Gamma$$-invariant, i.e., a modular distribution, if and only if the functional equation

$$L^{\sharp}(s, \mathfrak{N}_{\chi, i\lambda}) = (-1)^{\delta} L^{\sharp}(1 - s, \mathfrak{N}_{\chi, i\lambda})$$

is satisfied.

Given an automorphic distribution $$\mathfrak{S},$$ we set, for $$k = 1, 2, \ldots,$$

$$\langle T^{\text{dist}}_{k} \mathfrak{S}, h \rangle = k^{-\frac{1}{2}} \sum_{\substack{ad = k, \, d > 0 \, \bmod \, d \, \text{bmod } d}} < \mathfrak{S}, (x, \xi) \mapsto h \left( \frac{dx - b\xi}{\sqrt{k}}, \frac{a\xi}{\sqrt{k}} \right) >$$

and

$$\langle T^{\text{dist}}_{-1} \mathfrak{S}, h \rangle = \langle \mathfrak{S}, (x, \xi) \mapsto h(-x, \xi) \rangle.$$
As a straightforward computation shows, under $\Theta$, the operator $T_k^{\text{dist}}$ transfers if $k \geq 1$ to the operator $T_k$ such that
\begin{equation}
(T_k f)(z) = k^{-\frac{1}{2}} \sum_{\substack{ad = k, \ d > 0 \\ b \mod d}} f \left( \frac{a z + b}{d} \right), \tag{4.8}
\end{equation}
to be completed by $(T_1 f)(z) = f(-\bar{z})$. This is the familiar collection of Hecke operators known to practitioners of (non-holomorphic) modular form theory, and it is well-known that it constitutes a commutative family of operators, generated by $T_{-1}$ and the operators $T_p$ with $p$ prime. These properties transfer to the analogous properties regarding the collection $(T_k^{\text{dist}})$.

For clarity, we quote now (same reference) the following result, the proof of which will be reexamined in Theorem 5.1.

**Proposition 4.2.** In the case when the distribution $\mathfrak{N}_{\chi,i\lambda}$, as defined in (4.1), is automorphic, it is automatically a Hecke distribution, by which we mean a joint eigendistribution of the collection of Hecke operators $T_k^{\text{dist}}$. One has $T_{-1}^{\text{dist}} \mathfrak{N}_{\chi,i\lambda} = (-1)^{\delta} \mathfrak{N}_{\chi,i\lambda}$ and, for prime,
\begin{equation}
T_p^{\text{dist}} \mathfrak{N}_{\chi,i\lambda} = \left[ \chi(p) p^{-\frac{i\lambda}{2}} + \chi(p^{-1}) p^{\frac{i\lambda}{2}} \right] \mathfrak{N}_{\chi,i\lambda}. \tag{4.9}
\end{equation}

We turn now to the definition of Eisenstein distributions, the notion in modular distribution theory to substitute for that of Eisenstein series of the non-holomorphic type. The treatment of these distributions is quite similar to that of Hecke distributions, with two extra terms added, and done in detail in [20, Section 1.1]: we may thus satisfy ourselves with recalling their basic properties.

For $\Re \nu < 0$, $\nu \neq -1$, and $h \in \mathcal{S}(\mathbb{R}^2)$, one defines ([19, p.93] or [20, p.15])
\begin{align*}
\frac{1}{2} \left\langle \mathfrak{E}_\nu, h \right\rangle &= \frac{1}{2} \zeta(-\nu) \int_{-\infty}^{\infty} |t|^{-\nu-1} (\mathcal{F}_{1}^{-1} h) (0,t) \, dt \\
+ \frac{1}{2} \zeta(1-\nu) \int_{-\infty}^{\infty} |t|^{-\nu} h(t,0) \, dt + \frac{1}{2} \sum_{n \neq 0} \sigma_{\nu}(n) \int_{-\infty}^{\infty} |t|^{-\nu-1} (\mathcal{F}_{1}^{-1} h) \left( \frac{n}{t}, t \right) \, dt, \tag{4.10}
\end{align*}
where \( \sigma_\nu(n) = \sum_{1 \leq |d|_n} d^\nu \). After the power function \( t \mapsto |t|^\mu \) has been given a meaning, as a distribution on the line, for \( \mu \neq -1, -3, \ldots \), this decomposition is actually valid for \( \nu \neq \pm 1 \), \( \nu \neq 0 \). There is no need to exclude \( \nu = 0 \) if one does not separate the first two terms on the right-hand side of (4.10). One shows that, as a tempered distribution, \( E_\nu \) extends as an analytic function of \( \nu \) for \( \nu \neq \pm 1 \), and that \( \text{Res}_{\nu=-1} E_\nu = -1 \), \( \text{Res}_{\nu=1} E_\nu = \delta \), the unit mass at the origin of \( \mathbb{R}^2 \).

For a change, let us recall the simple proof of the following, which gives the transfer under \( \Theta \) of Hecke distributions or Eisenstein distributions. We first rewrite (4.1) as

\[
\langle N_{\chi,i\lambda}, h \rangle = \frac{1}{4} \sum_{k \in \mathbb{Z} \setminus \{0\}} \phi(k) \int_{-\infty}^{\infty} |t|^{-1-i\lambda} \left( \mathcal{F}_1^{-1} h \right) \left( \frac{k}{t}, t \right) dt, \tag{4.11}
\]

with

\[
\phi(k) = \sum_{mn=k} \chi \left( \frac{m}{n} \right) |n|^{i\lambda}, \quad k \in \mathbb{Z}^\times \tag{4.12}
\]

(note that \( \phi(1) = 2 \)).

**Proposition 4.3.** One has

\[
(\Theta N_{\chi,i\lambda})(x+iy) = \frac{1}{2} y^{\frac{1}{2}} \sum_{k \neq 0} \phi(k) K_{\frac{1}{4}} \left( 2\pi |k| y \right) e^{2\pi i k x}. \tag{4.13}
\]

Similarly, setting \( \zeta^*(s) = \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) \), one has

\[
(\Theta E_\nu)(x+iy) = \zeta^*(1-\nu) y^{\frac{1-\nu}{2}} + \zeta^*(1+\nu) y^{\frac{1+\nu}{2}} + 2 y^{\frac{1}{2}} \sum_{k \neq 0} |k|^{-\frac{\nu}{2}} \sigma_\nu(|k|) K_{\frac{1}{2}} \left( 2\pi |k| y \right) e^{2\pi i k x}. \tag{4.14}
\]

One has \( \mathcal{F}_{\text{symp}} N_{\chi,i\lambda} = N_{\chi^{-1},-i\lambda} \) and \( \mathcal{F}_{\text{symp}} E_\nu = E_{-\nu} \).

**Proof.** Making the Fourier expansions (1.2) and (4.10) more explicit, as

\[
N_{\chi,i\lambda}(x,\xi) = \frac{1}{4} \sum_{k \neq 0} \phi(k) |\xi|^{-1-i\lambda} \exp \left( \frac{2i\pi k x}{\xi} \right),
\]

\[
E_\nu(x,\xi) = \zeta(-\nu) |\xi|^{-\nu-1} + \zeta(1-\nu) |x|^{-\nu} \delta(\xi) + \sum_{k \neq 0} \sigma_\nu(k) |\xi|^{-\nu-1} \exp \left( \frac{2i\pi k x}{\xi} \right), \tag{4.15}
\]
and it suffices to compute the Theta-transform (3.1) of the function $h_{\nu,k}(x,\xi) = |\xi|^{-\nu} \exp\left(\frac{2i\pi kx}{\xi}\right)$. Setting $z = \alpha + iy$ (the use of $x$ has been preempted), one first computes the (Gaussian) $dx$-integral, obtaining

$$\left(\Theta h_{\nu,k}\right)(\alpha + iy) = e^{2\pi i k\alpha} y^{\frac{1}{2}} \int_{-\infty}^{\infty} |\xi|^{-\nu} \exp\left(-\pi y (\xi^2 + \xi^{-2})\right) d\xi, \quad (4.16)$$

after which, using a standard integral definition [12, p.85] of modified Bessel functions, one obtains

$$\left(\Theta h_{\nu,k}\right)(x + iy) = 2 y^{\frac{1}{2}} K_{\frac{\nu}{2}}(2\pi |k| y) e^{2i\pi kx}. \quad (4.17)$$

This computation was done in [20, p.116], but a serious reason to change the normalization (by a “factor” $2^{\frac{1}{2} + i\pi \varepsilon}$) when dealing with arithmetic has been explained in [21, Section 6.1]. There is no difficulty in summing the $k$-series, thanks to the exponential decrease of the function $K_{\frac{\nu}{2}}$.

The last assertion follows from the identity $F_{\mathrm{symph}} h_{\nu,k} = |k|^{-\nu} h_{-\nu,k}$, the verification of which is straightforward.

□

The function $\Theta \mathfrak{M}_{\chi, i\lambda}$ is automorphic, hence a cusp-form in view of its expansion (4.13): from (4.9), it is a Hecke eigenform, normalized in Hecke’s way (a notion recalled in (4.18) below) since $\phi(1) = 2$. The right-hand side of (4.14) is the familiar-looking Fourier expansion [6, p.66] of the function $\zeta^*(\nu) E_{\frac{1+\nu}{2}}(z)$, where $E_{\frac{1+\nu}{2}}$ is the non-holomorphic Eisenstein series so denoted. To complete the picture, we must show that every Hecke-normalized Hecke eigenform is of the form $\Theta \mathfrak{M}_{\chi, i\lambda}$ for some choice of the pair $(\chi, i\lambda)$. This was done in [20, Prop.2.1.1] and [20, Theor.2.1.2].

**Proposition 4.4.** Let $\mathcal{N}$ be a Hecke eigenform, normalized in Hecke’s way, with Fourier expansion

$$\mathcal{N}(x + iy) = y^{\frac{1}{2}} \sum_{k \neq 0} b_k K_{\frac{\lambda}{2}}(2\pi |k| y) e^{2i\pi kx}, \quad (4.18)$$

normalized in Hecke’s way (i.e., $b_1 = 1$); define $\delta = 0$ or 1 according to the parity of $\mathcal{N}(z)$ under the map $z \mapsto -\overline{z}$. Choose for $\lambda$ any of the two square roots of $\lambda^2$ ($K_{\frac{\lambda}{2}}$ depends only $\lambda^2$). Then, choosing for every prime $p$ any of the two solutions of the equation $b_p = \theta_p + \theta_p^{-1}$, define $\chi$ as the unique character of $\mathbb{Q}^\times$ such that $\chi(p) = p^{\frac{\lambda}{2}} \theta_p$ for $p$ prime and $\chi(-1) = (-1)^\delta$.
The distribution
\[
N(x, \xi) = \frac{1}{2} \sum_{k \neq 0} b_k |k|^\frac{i\lambda}{2} |\xi|^{-1-i\lambda} \exp \left( \frac{2i\pi k x}{\xi} \right)
\]
coincides with the distribution \(N_{\chi,i\lambda}\), as defined in (4.1). It is a Hecke distribution, and one has
\[
\Theta N_{\chi,i\lambda} = \mathcal{N}.
\]

One point was, however, missing in the given reference. Setting
\[
\Lambda(s, \mathcal{N}) = \pi^{-s} \Gamma \left( \frac{s + \delta}{2} + \frac{i\lambda}{4} \right) \Gamma \left( \frac{s + \delta}{2} - \frac{i\lambda}{4} \right) L(s, \mathcal{N})
\]
with
\[
L(s, \mathcal{N}) = \sum_{k \geq 1} b_k k^{-s} = \prod_p \left( 1 - b_p p^{-s} + p^{-2s} \right)^{-1},
\]
the function \(\Lambda(s, \mathcal{N})\) extends as an entire function of \(s\), satisfying the functional equation \([1, \text{p.107}]\)
\[
\Lambda(s, \mathcal{N}) = (-1)^\delta \Lambda(1-s, \mathcal{N})
\]
(in the reference just given, the parameter \(\nu\) is the one denoted here as \(\frac{i\lambda}{2}\)).

The function \(L^\times(s, \mathcal{N}_{\chi,i\lambda})\) does not coincide with \(\Lambda(s, \mathcal{N})\); of course, it has to be more precise, since it distinguishes between \(\lambda\) and \(-\lambda\). But the two are linked by the identity
\[
L^\times(s, \mathcal{N}_{\chi,i\lambda}) = \frac{1}{2} \frac{(-i)^\delta \pi^{-\frac{1}{2}+\frac{i\lambda}{4}}}{\Gamma \left( \frac{s + \delta}{2} - \frac{i\lambda}{4} \right) \Gamma \left( \frac{1-s + \delta}{2} - \frac{i\lambda}{4} \right)} \Lambda(s, \mathcal{N}).
\]

This proves that the function \(L^\times(\cdot, \mathcal{N}_{\chi,i\lambda})\) satisfies the same functional equation as the function \(\Lambda(\cdot, \mathcal{N})\). We forgot in the given reference to show that the function \(L^\times(s, \mathcal{N}_{\chi,i\lambda})\) is polynomially bounded in vertical strips, which does not follow from the analogous fact relative to the function \(\Lambda(s, \mathcal{N})\). The proof of \([1, \text{Lemma 1.9.1}]\) applies with the following modification. Replace the integral representation of the function \(K_{\frac{\lambda}{2}}\) used in the given reference by the pair (assuming that \(k > 0\) and, just as in \([1, \text{that} \ y \geq 1]\),
\[
K_{\frac{\lambda}{2}}(2\pi ky) = \int_0^\infty e^{-2\pi k y \cosh t} \cos \frac{\lambda t}{2} dt
\]
\[
= \pi^{-\frac{1}{2}} \Gamma \left( \frac{1+i\lambda}{2} (\pi ky)^{-\frac{i\lambda}{4}} \right) \int_0^\infty (\cosh t)^{-i\lambda} \cos(\pi ky \sinh t) dt.
\]
The first equation yields $|K_{\lambda}(2\pi ky)| \leq C e^{-\pi ky}$ and, transforming the integral in the second equation to

$$-\int_0^\infty \frac{d}{dt} \left[ \frac{1}{\pi ky} (\cosh t)^{-1-i\lambda} \right] \sin(\pi ky \sinh t) \, dt,$$

one obtains

$$|K_{\lambda}(2\pi ky)| \leq \pi^{\frac{\lambda}{2}} \sqrt{1 + \lambda^2} \Gamma \left( \frac{1 + i\lambda}{2} \right) (\pi ky)^{-1}. \tag{4.27}$$

One makes about the best of the two estimates if, when dealing with the $k$-series (4.18) that defines $\mathcal{N}(iy)$, one uses the first estimate when $ky > |\lambda|/4$, the second when $ky < |\lambda|/4$: (2.4) implies the desired result.

Recall [7, p.372] that, with $\mathcal{N}$ given by (4.18) and $\mathfrak{M}$ by (4.19), one has for $p$ prime

$$T_p \mathcal{N} = b_p \mathcal{N} \quad \text{and} \quad T_p^{\text{dist}} \mathfrak{M} = b_p \mathfrak{M}. \tag{4.28}$$

The equation (4.9) proves the equivalence between the set of conditions $|b_p| \leq 2$ and the unitarity of the character $\chi$.

## 5. Hecke operators and their powers

**Theorem 5.1.** Given a prime $p$, and a modular distribution $\mathfrak{M} = \mathcal{E}_\nu$ or $\mathfrak{M}_{\chi,i\lambda}$, one has the identity

$$(T_p^{\text{dist}} \mathfrak{M})(x,\xi) = \left(p^{-\frac{i}{2} + i\pi \xi^3} \mathfrak{M}\right)(x,\xi) + \frac{1}{p} \sum_{b \mod p} \left(p^{\frac{i}{2} - i\pi \xi^3} \mathfrak{M}\right)(x + b\xi,\xi). \tag{5.1}$$

**Proof.** From (4.15), one has, defining $\tilde{\phi}(\lambda) = \phi(\lambda)$ if $\lambda \in \mathbb{Z}^\times$ (cf. (4.12)), $\tilde{\phi}(\lambda) = 0$ if $\lambda \not\in \mathbb{Q}$ is zero or fails to be an integer,

$$p^{-\frac{i}{2}} \mathfrak{M}_{\chi,i\lambda}\left(p^{\frac{i}{2}} x, p^{-\frac{i}{2}} \xi\right) = \frac{1}{4} p^{\frac{1}{p} |\xi|^{-1-i\lambda}} \sum_{k \in \mathbb{Z} \setminus \{0\}} \tilde{\phi} \left( \frac{k}{p} \right) \exp \left( \frac{2i\pi kx}{p\xi} \right),$$

$$p^{\frac{i}{2}} \mathfrak{M}_{\chi,i\lambda}\left(p^{-\frac{i}{2}} x, p^{\frac{i}{2}} \xi\right) = \frac{1}{4} p^{-\frac{i}{2}} |\xi|^{-1-i\lambda} \sum_{k \in \mathbb{Z} \setminus \{0\}} \phi(k) \exp \left( \frac{2i\pi kx}{p\xi} \right). \tag{5.2}$$

If, in the second formula, we substitute $x + b\xi$ for $x$ and perform $\frac{1}{p}$ times the summation with respect to $b$, we come across the sum $\frac{1}{p} \sum_{b \mod p} \exp \left( \frac{2i\pi kb}{p}\xi \right) =
char\( (k \equiv 0 \mod p) \). It follows that the right-hand side of (5.1) is
\[
\frac{1}{4} \sum_{k \in \mathbb{Z} \setminus \{0\}} \left[ p^{\frac{k}{p}} \tilde{\phi} \left( \frac{k}{p} \right) + p^{-\frac{k}{p}} \phi(pk) \right] |\xi|^{-1-i\lambda} \exp \left( \frac{2i\pi kx}{\xi} \right). \quad (5.3)
\]

For \( k \in \mathbb{Z}^\times \), one has (in what follows, \( (m, n) \) denotes the pair of integers \( m, n \), not their g.c.d. !)
\[
\{(m, n) : mn = pk\} = \{(pm_1, n) : m_1n = k\} \cup \{(m, pn_1) : mn_1 = k\}, \quad (5.4)
\]
not a disjoint union if \( p|k \); in such a case, the two sets intersect along the set \( \{(pm_1, pn_1) : m_1n_1 = \frac{k}{p}\} \), which leads to the equation
\[
\phi(pk) = \left[ \chi(p) + p^{i\lambda} \chi(p^{-1}) \right] \phi(k) - p^{i\lambda} \tilde{\phi} \left( \frac{k}{p} \right), \quad k \in \mathbb{Z}^\times. \quad (5.5)
\]
It follows that
\[
p^{-\frac{k}{p}} \phi(pk) + p^{\frac{k}{p}} \tilde{\phi} \left( \frac{k}{p} \right) = \left[ p^{-\frac{k}{p}} \chi(p) + p^{\frac{k}{p}} \chi(p^{-1}) \right] \phi(k), \quad (5.6)
\]
and the right-hand side of (5.1) agrees with
\[
\left( p^{\frac{k}{p}} (\chi(p))^{-1} + p^{-\frac{k}{p}} \chi(p) \right) \mathcal{N}_{\chi,i\lambda}(x, \xi), \quad (5.7)
\]
the same as \( T_p^{\text{dist}} \mathcal{N}_{\chi,i\lambda}(x, \xi) \) in view of (4.9).

In the case of the Eisenstein distribution, just replacing \( \chi \) by \( \chi_0 \) everywhere would seem to lead to the fact that nothing is changed in the proof. The result is correct, but one must not, in this case, forget the two extra terms in (4.10), which are multiples of \( |\xi|^{-\nu-1} \) and \( |x|^{-\nu} \delta(\xi) \). Now, one has
\[
p^{-\frac{k}{p}+i\pi \xi^2} (|\xi|^{-\nu-1}) = p^{\frac{k}{p}} |\xi|^{-\nu-1}, \quad p^{-\frac{k}{p}+i\pi \xi^2} (|x|^{-\nu} \delta(\xi)) = p^{-\frac{k}{p}} |x|^{-\nu} \delta(\xi),
\]
\[
p^{\frac{k}{p}-i\pi \xi^2} (|\xi|^{-\nu-1}) = p^{\frac{k}{p}} |\xi|^{-\nu-1}, \quad p^{\frac{k}{p}-i\pi \xi^2} (|x|^{-\nu} \delta(\xi)) = p^{\frac{k}{p}} |x|^{-\nu} \delta(\xi),
\]
and one obtains the desired result, noting that on the special terms under consideration here, the translation \( x \mapsto x - b\xi \) has no effect.

\[ \Box \]

Simplify \( T_p^{\text{dist}} \) as \( T \) and set \( R = p^{-\frac{k}{p}+i\pi \xi^2} \). The operators \( R \) and \( R^{-1} \) act on arbitrary (tempered) distributions. Given \( j \in \mathbb{Z} \), denote as \( \Gamma_{\infty}^{(p')} \) the subgroup of \( SL(2, \mathbb{R}) \) generated by the matrix \( \begin{pmatrix} 1 & p' \\ 0 & 1 \end{pmatrix} \), and denote as \( \text{Inv}(p') \) the linear space of tempered distributions \( \mathcal{S} \) invariant under the action of
\( \Gamma_{\infty}^{(p^j)} \), i.e., satisfying the “periodicity” condition \( \mathcal{G}(x + p^j \xi, \xi) = \mathcal{G}(x, \xi) \). We shall express by the notation \( Q_1 \sim Q_2 \) the fact that the operators \( Q_1 \) and \( Q_2 \) are well-defined on \( \text{Inv}(1) \) (or on larger spaces) and agree there.

**Lemma 5.2.** For \( j, \ell \in \mathbb{Z} \), the operator \( R^\ell \) maps \( \text{Inv}(p^j) \) into \( \text{Inv}(p^{j-\ell}) \): in particular, if \( \ell \geq 0 \), it preserves the space \( \text{Inv}(1) \). Setting, for \( \gamma \in \mathbb{R} \), \( \tau[\gamma] = \exp(\gamma \xi \frac{\partial}{\partial x}) \), so that \( (\tau[\gamma] \mathcal{G})(x, \xi) = \mathcal{G}(x + \gamma \xi, \xi) \), define when \( r \) and \( \ell \) are non-negative integers the operators

\[
\sigma_r^{(\ell)} = p^{-r} \sum_{b \mod p^r} \tau \left[ b p^{\ell-r} \right], \quad \sigma_r = \sigma_0^0. \tag{5.9}
\]

The first operator sends the space \( \text{Inv}(p^{\ell}) \) into \( \text{Inv}(p^{\ell-r}) \): in particular, \( \sigma_r \) sends \( \text{Inv}(1) \) to \( \text{Inv}(p^{-r}) \). For \( r \geq 0, \ell \geq 0 \), one has

\[
R^\ell \sigma_r \sim \sigma_{\ell+r} R^\ell, \quad R^{-\ell} \sigma_r \sim \sigma_r^{(\ell)} R^{-\ell}. \tag{5.10}
\]

If \( r \geq 0 \), one has \( \sigma_r \, R^{-1} \sigma_1 \sim R^{-1} \sigma_{r+1} \). The operator \( \sigma_r^{(\ell)} \) commutes with \( \mathcal{F}^{\text{sym}} \) for all values of \( r, \ell \).

**Proof.** One has for every distribution \( \mathcal{G} \)

\[
(R^\ell \mathcal{G})(x + p^{j-\ell}, \xi) = p^{-\frac{j}{2}} \mathcal{G} \left( p^{\frac{j}{2}} \left( x + p^{j-\ell} \xi \right), p^{-\frac{j}{2}} \xi \right) \\
= p^{-\frac{j}{2}} \mathcal{G} \left( p^{\frac{j}{2}} x + p^{j} \left( p^{-\frac{j}{2}} \xi \right), p^{-\frac{j}{2}} \xi \right), \tag{5.11}
\]

so that \( R^\ell \mathcal{G} \) is invariant under \( \Gamma_{\infty}^{(p^{j-\ell})} \) if \( \mathcal{G} \) invariant under \( \Gamma_{\infty}^{(p^j)} \).

It is immediate that, for every \( \gamma \in \mathbb{R} \), one has

\[
R \tau[\gamma] = \tau \left[ \frac{\gamma}{p} \right] R, \quad R^{-1} \tau[\gamma] = \tau[\gamma] R^{-1}. \tag{5.12}
\]

One has

\[
(R^\ell \sigma_r \mathcal{G})(x, \xi) = R^\ell \left[ (x, \xi) \mapsto p^{-r} \sum_{b \mod p^r} \mathcal{G} \left( x + \frac{b \xi}{p^r}, \xi \right) \right] \\
= p^{-r-\frac{j}{2}} \sum_{b \mod p^r} \mathcal{G} \left( p^{\frac{j}{2}} x + b p^{-r-r} \xi, p^{-\frac{j}{2}} \xi \right). \tag{5.13}
\]
and

\[ (\sigma_{\ell+r}R^\ell \mathcal{G})(x, \xi) = \sigma_{\ell+r} \left[ (x, \xi) \mapsto p^{-\frac{r}{2}} \mathcal{G} \left( p^{\frac{\ell}{2}} x, p^{-\frac{\ell}{2}} \xi \right) \right] = p^{-r-\frac{3\ell}{2}} \sum_{b \mod p^{\ell+r}} \mathcal{G} \left( p^{\frac{\ell}{2}} \left( x + \frac{b}{p^\ell} \xi \right), p^{-\frac{\ell}{2}} \xi \right). \]  

(5.14)

In the two formulas, the arguments are the same, but the domains of averaging are not. However, when two classes \( b \mod p^{\ell+r} \) agree mod \( p^{\ell+r} \), the first arguments of the two expressions differ by a multiple of \( p^{-\frac{\ell}{2}} \xi \), so that the two expressions are identical if \( \mathcal{G} \in \text{Inv}(1) \).

Next,

\[ (R^{-\ell} \sigma_r \mathcal{G})(x, \xi) = R^{-\ell} \left[ (x, \xi) \mapsto p^{-r} \sum_{b \mod p^r} \mathcal{G} \left( x + \frac{b}{p^r} \xi, \xi \right) \right] = p^{\frac{\ell}{2} - r} \sum_{b \mod p^r} \mathcal{G} \left( p^{-\frac{\ell}{2}} x + p^{-r-\frac{\ell}{2}} \xi, p^{\ell} \xi \right) = p^{-r} \sum_{b \mod p^r} (R^{-\ell} \mathcal{G}) \left( x + b p^{\ell-r} \xi, \xi \right). \]  

(5.15)

This is the same as \( (\sigma_r^{(\ell)} R^{-\ell} \mathcal{G})(x, \xi) \).

Finally, if \( \mathcal{G} \in \text{Inv}(1) \), \( \mathcal{F} = \sigma_1 \mathcal{G} \in \text{Inv}(p^{-1}) \), and one observes first that both operators \( \sigma_r R^{-1} \sigma_1 \) and \( R^{-1} \sigma_{r+1} \) are well-defined on \( \text{Inv}(1) \). One has

\[ (\sigma_r R^{-1} \mathcal{F})(x, \xi) = \sigma_r \left[ (x, \xi) \mapsto p^{\frac{\ell}{2}} \mathcal{F} \left( p^{-\frac{\ell}{2}} x, p^{\ell} \xi \right) \right] = p^{-r+\frac{\ell}{2}} \sum_{b \mod p^r} \mathcal{F} \left( p^{-\frac{\ell}{2}} x + b p^{-\frac{\ell}{2}} \xi, p^{\ell} \xi \right) = R^{-1} \left[ p^{-r} \sum_{b \mod p^r} \mathcal{F} \left( x + \frac{b}{p^r}, \xi \right) \right]. \]  

(5.16)

Then,

\[ (\sigma_r R^{-1} \sigma_1 \mathcal{G})(x, \xi) = R^{-1} \left[ p^{-r-1} \sum_{b \mod p^r} \mathcal{G} \left( x + \frac{b}{p^r} + \frac{\beta}{p}, \xi \right) \right]. \]  

(5.17)
As $b$ and $\beta$ run through the classes indicated as a subscript, $b + p^r \beta$ describes a full set of classes modulo $p^{r+1}$, so that the right-hand side is indeed $(R^{-1}\sigma_{r+1}\mathcal{G})(x, \xi)$.

The last assertion follows from the fact that $F_{\text{sym}}\mathcal{G} \in \text{Inv}(p^{\ell})$ if $\mathcal{G} \in \text{Inv}(p^{\ell})$ and from the identity

$$\sigma_{r}^{(\ell)} = p^{-r} \sum_{b \mod p^{r}} \exp \left( b p^{\ell-r} \xi \frac{\partial}{\partial x} \right), \quad (5.18)$$

if one notes that the operator $\xi \frac{\partial}{\partial x}$ commutes with $F_{\text{sym}}$.

□

In (5.1), the operator $T = T_p^{\text{dist}}$ has been obtained, as an operator of modular distributions, as

$$T = R + \sigma_1^{(1)} R^{-1} = R + R^{-1} \sigma_1, \quad (5.19)$$

The first equation can be verified from the definition of $\sigma_{r}^{(\ell)}$, and the second then follows from Lemma 5.2. This equation is still a valid definition when applied to distributions in the space $\text{Inv}(1)$, and we shall use its application to automorphic distributions in the last two sections.

**Proposition 5.3.** Given $k = 1, 2, \ldots$ and $\ell$ such that $0 \leq \ell \leq k$, there are non-negative integers $\alpha_{k,\ell}^{(0)}, \alpha_{k,\ell}^{(1)}, \ldots, \alpha_{k,\ell}^{(\ell)}$, satisfying the conditions:

(i) $\alpha_{k,\ell}^{(0)} + \alpha_{k,\ell}^{(1)} + \cdots + \alpha_{k,\ell}^{(\ell)} = \binom{k}{\ell}$ for all $k, \ell$,

(ii) $2\ell - k - r \leq 0$ whenever $\alpha_{k,\ell}^{(r)} \neq 0$,

such that one has the identity (between two operators on $\text{Inv}(1)$)

$$T^k = \sum_{\ell=0}^{k} R^{k-2\ell} \left( \alpha_{k,\ell}^{(0)} I + \alpha_{k,\ell}^{(1)} \sigma_1 + \cdots + \alpha_{k,\ell}^{(\ell)} \sigma_{\ell} \right). \quad (5.20)$$

**Proof.** By induction. Assuming that the given formula holds, we write $T^{k+1} = T^k (R + R^{-1} \sigma_1)$ and we use the equations $\sigma_r R \sim R \sigma_{r-1}$ ($r \geq 1$).
and \( \sigma_r R^{-1} \sigma_1 \sim R^{-1} \sigma_{r+1} \) at the end of Lemma 5.2. We obtain

\[
T^{k+1} = \sum_{\ell=0}^{k} R^{k+1-2\ell} \left( \alpha_{k,\ell}^{(0)} I + \alpha_{k,\ell}^{(1)} I + \alpha_{k,\ell}^{(2)} \sigma_1 + \cdots + \alpha_{k,\ell}^{(\ell)} \sigma_{\ell-1} \right) \\
+ \sum_{\ell=0}^{k} R^{k-2\ell} \left( \alpha_{k,\ell}^{(0)} \sigma_1 + \alpha_{k,\ell}^{(1)} \sigma_2 + \cdots + \alpha_{k,\ell}^{(\ell)} \sigma_{\ell+1} \right),
\]

or

\[
T^{k+1} = \sum_{\ell=0}^{k} R^{k+1-2\ell} \left( \alpha_{k,\ell}^{(0)} I + \alpha_{k,\ell}^{(1)} I + \alpha_{k,\ell}^{(2)} \sigma_1 + \cdots + \alpha_{k,\ell}^{(\ell)} \sigma_{\ell-1} \right) \\
+ \sum_{\ell=1}^{k+1} R^{k+1-2\ell} \left( \alpha_{k,\ell-1}^{(0)} \sigma_1 + \alpha_{k,\ell-1}^{(1)} \sigma_2 + \cdots + \alpha_{k,\ell-1}^{(\ell-1)} \sigma_{\ell-1} \right).
\]

The point (i) follows, using \( \left( \binom{k}{\ell} + \binom{k}{\ell-1} = \binom{k+1}{\ell} \right) \). Next, looking again at (5.21), one observes that, in the expansion of \( T^{k+1} \), the term \( R^{k+1-2\ell} \sigma_r \) is the sum of two terms originating (in the process of obtaining \( T^{k+1} \) from \( T^k \)) from the terms \( R^{k-2\ell} \sigma_{r+1} \) and \( R^{k-2\ell+2} \sigma_{r-1} \). The condition \( 2\ell - (k+1) - r \leq 0 \) is certainly true if either \( 2\ell - k - (r+1) \leq 0 \) or \( (2\ell - 2) - k -(r-1) \leq 0 \), which proves the point (ii) by induction. Since \( \sigma_r \mathcal{G} \in \text{Inv}(p^{-r}) \) if \( \mathcal{G} \in \text{Inv}(1) \), this condition implies that \( R^{k-2\ell} \sigma_r \) preserves the space \( \text{Inv}(1) \).

\[\square\]

6. A Generating Object for Modular Distributions

A basic collection of distributions is made of the “elementary” line measures

\[
\delta^a_b(x, \xi) = e^{2i\pi ax} \delta(\xi - b), \quad a, b \in \mathbb{R}.
\]

Superpositions of these conduct in a natural way to modular distributions of interest. For instance, with \( \chi \) as in Section 4, consider the series

\[
\mathcal{T}_\chi = \pi \sum_{m,n \neq 0} \chi \left( \frac{m}{n} \right) \delta^m_n.
\]

The distribution \( \mathcal{T}_\chi \) decomposes [21, p.116] into homogeneous components as \( \int_{-\infty}^{\infty} \mathcal{M}_{\chi,i\lambda} d\lambda \), with \( \mathcal{M}_{\chi,i\lambda} \) as defined in (4.1). If one replaces \( \chi \) by the trivial character \( \chi_0 = 1 \), and one replaces the domain of summation by the one defined by the condition \( |m| + |n| \neq 0 \), the same decomposition will lead instead to the Eisenstein distribution \( \mathcal{E}_{i\lambda} \).
For \( j = 0, 1, \ldots \), define the distribution

\[
(s_1^1)^j = \pi^2 \mathcal{E}^2 (\pi^2 \mathcal{E}^2 + 1) \cdots (\pi^2 \mathcal{E}^2 + (j - 1)^2) s_1^1.
\]  

(6.3)

It was shown in [18, Theorem 3.3] that for \( j \geq 1 \), setting \( \Gamma_\infty = \{(1 0 b) : b \in \mathbb{Z}\} \), the series

\[
\mathcal{B}^j = \frac{1}{2} \sum_{g \in \Gamma / \Gamma_\infty} (s_1^1)^j \circ g^{-1}
\]

converges in the space \( S'(\mathbb{R}^2) \) provided that, before summation, one always groups the terms corresponding to \( g = (a \ b \ c \ d) \) or \( g = (-a -b \ c \ d) \). As such, it is an even distribution (i.e., giving zero when tested on globally odd functions), characterized by its restriction as a continuous linear form on \( \mathcal{S}_{\text{even}}(\mathbb{R}^2) \). One of the main points of the proof developed in this paper will consist, in Section 7, in obtaining uniform bounds for a suitably rescaled extension of this result.

In the present section, the object is to show that the automorphic distribution \( \mathcal{B}^j \) constitutes a generating object for all modular distributions. To prove it, we shall benefit from results from modular form theory: working in \( \mathbb{H} \), rather than \( \mathbb{R}^2 \), is better-adapted to Hilbert space methods. For the benefit of readers not familiar with automorphic function theory, and the necessities of notation, here is a quite short summary of classical results. Nice presentations of this theory (accessible to non-experts, including the present author) are to be found in [1, 6, 7] and elsewhere.

The space \( L^2(\Gamma \backslash \mathbb{H}) \) is built with the help of the \( SL(2, \mathbb{R}) \)-invariant measure \( dm(z) = y^{-2} dx \, dy \) (with \( z = x + iy \)) on \( \mathbb{H} \) and of a fundamental domain \( D \) for the action of \( \Gamma \) in \( \mathbb{H} \), say \( D = \{ z : |z| > 1, \, |\text{Re} \, z| < \frac{1}{2} \} \). A suitable self-adjoint realization of \( \Delta \), in this space, exists, and all Hecke operators \( T_k \), too, are self-adjoint there. The Eisenstein series \( E_{1-\lambda A} \) make up a complete set of generalized eigenfunctions for the continuous part of the spectrum of \( \Delta \), while the Hecke eigenforms are genuine eigenvectors (they lie in \( L^2(\Gamma \backslash \mathbb{H}) = L^2(D) \)) of this operator. Both families are by definition made of joint eigenfunctions of all Hecke operators. Letting after some work the theory of compact self-adjoint operators play, one finds that the true eigenvalues make up a sequence \( \left( \frac{1+\lambda^2}{4} \right)_{r \geq 1} \) going to infinity: that there are indeed infinitely many eigenvalues is subtler and relies on Selberg’s trace formula. For every \( r \geq 1 \), the space of Hecke eigenforms corresponding to this eigenvalue is finite-dimensional, generated by some family \( \{N_{r,1} \} \) for every \( r \).
of pairwise orthogonal Hecke eigenforms.

The Hecke normalization of Hecke eigenforms \( \mathcal{N}^{r,\iota} \), for which \( b_1 = 1 \) (cf. Proposition 4.4), has the property that the values the Hecke operators take on \( \mathcal{N}^{r,\iota} \) can be read directly on the Fourier coefficients of this Hecke eigenform [6, p.128]. But since the normalization of \( \mathcal{N}^{r,\iota} \) has already been chosen, it cannot be expected (and it is considerably far from being the case [15]) that it could be normalized in the space \( L^2(\Gamma \backslash \mathbb{H}) \) as well: we shall thus introduce the norm there, denoted as \( \| \mathcal{N}^{r,\iota} \| \), of \( \mathcal{N}^{r,\iota} \). In terms of modular form theory, the Ramanujan-Petersson conjecture is the assertion that, with \( b_p \) as defined in (4.18), one has \( |b_p| \leq 2 \) for every Hecke-normalized eigenform and every prime \( p \).

Questions of notation are important. It was natural to take \( r = 1, 2, \ldots \) as the principal parameter (possibly the only one) characterizing a Hecke eigenform \( \mathcal{N}^{r,\iota} \). Giving automorphic distribution theory, as will be needed, the upper hand, it is then convenient to denote the set of Hecke distributions as built by an application of Theorem 4.4, as \( (\mathfrak{R}^{r,\iota}) \), with \( r \in \mathbb{Z} \setminus \{0\} \) (rather than \( r \geq 1 \)) and the convention that one has made the choice
\[
\lambda = \lambda_r = \left( \frac{\lambda^2}{\lambda_r^2} \right)^{\frac{1}{2}} \text{ if } r \geq 1, \quad \lambda = \lambda_r = -\left( \frac{\lambda^2}{\lambda_r^2} \right)^{\frac{1}{2}} \text{ if } r \leq -1:
\]
then, \( \mathfrak{R}^{r,\iota} \) is always homogeneous of degree \(-1 - i\lambda_r\): using a superscript prevents a conflict of notation with \( \mathfrak{N}_{\chi, i\lambda} \).

**Proposition 6.1.** For \( j = 1, 2, \ldots \), the automorphic distribution \( \mathfrak{B}^j \) introduced in (6.4) admits the decomposition, convergent in the space of continuous linear forms on \( \mathcal{S}_{\text{even}}(\mathbb{R}^2) \),
\[
\mathfrak{B}^j = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\Gamma(j - \frac{i\lambda}{2}) \Gamma(j + \frac{i\lambda}{2})}{\zeta^*(i\lambda) \zeta^*(-i\lambda)} \mathcal{E}_{i\lambda} d\lambda + \frac{1}{2} \sum_{r, \iota} \frac{\Gamma(j - \frac{i\lambda_r}{2}) \Gamma(j + \frac{i\lambda_r}{2})}{\| \mathcal{N}^{r,\iota} \|^2} \mathfrak{R}^{r,\iota}.
\]

**Proof.** It was given in [18, p.34]. Let us recall the main points. First, one proves the convergence, in the sense indicated, of the series (6.3), (6.4) defining the left-hand side of (6.5): we shall prove in the next section a result, fundamental for our purpose, weaker (this is harmless) in the sense that the operator \( \pi^2 \mathcal{E}^2 \) (from \( \mathfrak{B} \) to \( \mathfrak{B}^1 \)) will have to be replaced by \( \pi^2 \mathcal{E}^2(1 + 4\pi^2 \mathcal{E}^2) \), but depending essentially on some extra parameter. Then, one proves that
the two parts of the right-hand side, the integral and the series, do converge in \( S'(\mathbb{R}^2) \): so far as the first part is concerned, Proposition 8.2 will give a proof of it, involving the extra parameter already alluded to. As a preparation for the main theorem, we shall reexamine, in Proposition 6.2, the proof that the series converges. If one takes this for granted, the rest of the proof of Proposition 6.1 goes as follows, starting with the computation of the image of \( \mathfrak{B}^j \).

One has with \( z = \alpha + iy \)

\[
(\Theta \mathcal{S}_1^j) (z) = \int_{\mathbb{R}^2} e^{2i\pi x} \delta(\xi - 1) \exp \left( -\pi \frac{(x - \alpha \xi)^2 + y^2 \xi^2}{y} \right) \, dx \, d\xi
\]

\[
= e^{-\pi y} \int_{-\infty}^{\infty} e^{2i\pi x} \exp \left( -\pi \frac{(x - \alpha)^2}{y} \right) \, dx = e^{-\pi y} e^{2i\pi \alpha} \int_{-\infty}^{\infty} e^{2i\pi x} \exp \left( -\pi \frac{x^2}{y} \right) \, dx
\]

\[
= e^{-\pi y} e^{2i\pi \alpha} y^\frac{1}{2} e^{-\pi y} = (\text{Im } z)^\frac{1}{2} e^{2i\pi z}. \quad (6.6)
\]

A straightforward computation by induction shows that, for \( j = 1, 2, \ldots \),

\[
\left( \Delta - \frac{1}{4} \right) \left( \Delta + \frac{4}{4} \right) \cdots \left( \Delta - \frac{1}{4} + (j - 1)^2 \right) \left( \text{Im } z \right)^\frac{1}{2} e^{2i\pi z}
\]

\[
= (4\pi)^j \frac{\Gamma(\frac{1}{2} + j)}{\Gamma(\frac{1}{2})} \left( \text{Im } z \right)^{j+\frac{1}{2}} e^{2i\pi z}. \quad (6.7)
\]

Using the definition (6.4) of \( \mathfrak{B}^j \) and the fact that the Euler operator commutes with the action of \( SL(2, \mathbb{R}) \) on \( \mathbb{R}^2 \) by linear changes of coordinates, one obtains the identity

\[
f^j(z) = (\Theta \mathfrak{B}^j)(z) = (4\pi)^j \frac{\Gamma(\frac{1}{2} + j)}{\Gamma(\frac{1}{2})}
\]

\[
\times \frac{1}{2} \sum_{(\begin{smallmatrix} n & n_1 \\ m & m_1 \end{smallmatrix}) \in \Gamma/\Gamma_\infty} \left( \frac{\text{Im } z}{\sqrt{-mz + n}} \right)^{j+\frac{1}{2}} e^{2i\pi \frac{m_1 z - n_1}{-mz + n}}. \quad (6.8)
\]

Now, this series is a special case of a class of automorphic functions introduced by Selberg [14] and used by several authors [3, 4] afterwards. The spectral decomposition (into Maass forms) of the right-hand side of (6.8) is made explicit there: we use the reference [3]. Some care is needed in view of the fact that, in these references, one uses the group \( PSL(2, \mathbb{Z}) \) in place of \( SL(2, \mathbb{Z}) \). But the two notions of \( \Gamma/\Gamma_\infty \) would correspond, without
an extra factor $\frac{1}{2}$ being needed. In [3, (1.18)], the Poincaré-Selberg series

$$U_m(z; s) = \sum_{\tau \in \Gamma_{\infty} \setminus \Gamma} (\text{Im} \, (\tau.z))^s \exp \left(2i\pi m \, \tau.z \right), \quad m = 1, 2, \ldots, \quad (6.9)$$

is introduced. Note that $\tau.z$ is defined in association to the action of $\Gamma$ on $\mathbb{H}$ by fractional-linear transformations: also, there is in the case of the full unimodular group only the cusp $\infty$. In these terms, the function $f^j$ in (6.8) can be written as

$$f^j(z) = \frac{1}{2} \left(4\pi \right)^j \frac{\Gamma \left(\frac{1}{2} + j \right)}{\Gamma \left(\frac{1}{2} \right)} U_1(z, j + \frac{1}{2}). \quad (6.10)$$

By the general Roelcke-Selberg decomposition theorem, one has the identity [3, (3.13)]

$$f^j(z) = (u_0 \mid f^j) u_0 + \sum_{k \geq 1} (u_k \mid f^j) u_k + \frac{1}{8\pi} \int_{-\infty}^{\infty} \left(E_{\frac{1-\lambda}{2}} \mid f^j\right) E_{\frac{1-\lambda}{2}} \, d\lambda, \quad (6.11)$$

where all scalar products are taken in $L^2(\Gamma \setminus \mathbb{H})$, the function $u_0$ is the $L^2$-normalized constant $\left(\frac{\pi}{3}\right)^{-\frac{1}{2}}$, the collection $(u_k)_{k \geq 1}$ is that of Hecke eigenforms normalized in the $L^2$-sense (not Hecke’s), and the scalar products in the continuous part of the decomposition are defined by convergent integrals, even though Eisenstein series are not square-integrable in the fundamental domain. Using the notation introduced in the present section, one can take for the collection $(u_k)_{k \geq 1}$ the one made up of all Hecke eigenforms $N^r, \parallel N^r \parallel$ with $r \geq 1$.

One has $(1 \mid f^j)_{L^2(\Gamma \setminus \Gamma_{\infty})} = 0$. This is seen by the unfolding method, just reading (though the constant is not truly a cusp-form) the proof of [3, Lemma 3.2]. Next, using [3, Lemma 3.5], one has for $\Re \, s > 1$

$$\left(E_{\pi} \mid U_1(\bullet, j + \frac{1}{2})\right) = \frac{\Gamma(s + j - \frac{1}{2})\Gamma(j + \frac{1}{2} - s)}{\Gamma(s)\Gamma(j + \frac{1}{2})} \times \pi^{-j + \frac{1}{2} + s} 2^{-2j + 1} \phi_1(s), \quad (6.12)$$

where the coefficient $\phi_1(s)$ is taken from the Fourier series decomposition [3, (1.17)]

$$E_s(z) = y^s + \cdots + 2y^{\frac{1}{2}} \sum_{n \neq 0} \frac{\pi^s |n|^s \phi_n(s) K_{s-\frac{1}{2}}(2\pi \mid n \mid y)}{\Gamma(s)} e^{2i\pi nx}. \quad (6.13)$$
of Eisenstein series. In the present case of the full modular group, the expansion is very classical, and the coefficient of the Whittaker function $2y^2 K_{s-\frac{1}{2}}(2\pi |n| y) e^{2\pi i nx}$ in it is $(\zeta^*(2s))^{-1} |n|^{s-\frac{1}{2}} \sum_{1 \leq d|n} d^{1-2s}$; hence, $\phi_1(s) = \frac{1}{\zeta(2s)}$, and

$$\left( E_{2-\frac{i\lambda}{2}} | U_1(\bullet,j+\frac{1}{2}) \right) = \frac{2\Gamma(\frac{1}{2})}{\Gamma(j+\frac{1}{2})} (4\pi)^{-j} \frac{\Gamma(j+\frac{i\lambda}{2})\Gamma(j-\frac{i\lambda}{2})}{\zeta^*(i\lambda)}.$$  

To obtain the coefficients $(u_k | f^j) = \left( \frac{\mathcal{N}^{\nu,\epsilon}}{\parallel \mathcal{N}^{\nu,\epsilon} \parallel} | f^j \right)$, we use [3, Lemma 3.2], remembering that the first Fourier coefficient of a Hecke-normalized Hecke eigenform is 1, to wit

$$\left( \mathcal{N}^{\nu,\epsilon} | U_1(z,j+\frac{1}{2}) \right) = (4\pi)^{-j} \frac{\Gamma(\frac{1}{2})}{\Gamma(j+\frac{1}{2})} \Gamma \left( j - \frac{i\lambda_r}{2} \right) \Gamma \left( j + \frac{i\lambda_r}{2} \right).$$  

Using (6.10), (6.11) and the computation of the individual coefficients just made, we obtain with $E_{1-\frac{i\lambda}{2}}^*(z) = \zeta^*(\nu) E_{1-\frac{i\lambda}{2}}(z)$ the expansion

$$f^j(z) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\Gamma(j-\frac{i\lambda}{2})\Gamma(j+\frac{i\lambda}{2})}{\zeta^*(i\lambda)\zeta^*(-i\lambda)} E_{1-\frac{i\lambda}{2}}^*(z) d\lambda$$

$$+ \frac{1}{2} \sum_{r \in \mathbb{Z} \setminus \{0\}} \frac{\Gamma(j-\frac{i\lambda_r}{2})\Gamma(j+\frac{i\lambda_r}{2})}{\parallel \mathcal{N}^{[r] \cdot \epsilon} \parallel^2} \mathcal{N}^{[r] \cdot \epsilon}(z).$$  

It was shown in Proposition 4.3 that (with our present notation)

$$\Theta \mathfrak{e}_\nu = E_{1-\frac{i\lambda}{2}}^*, \quad \Theta \mathfrak{N}^{\nu,\epsilon} = \mathcal{N}^{[r] \cdot \epsilon}.$$  

Then, one observes that $s_1^j$ is invariant under $\mathcal{F}^{\text{sym}}$ so that, since this operator commutes with the action of $\text{SL}(2,\mathbb{Z})$ by linear changes of coordinates, the distribution $\mathfrak{B}^j$ is invariant under $\mathcal{F}^{\text{sym}}$ as well. The distribution on the right-hand side of the equation (6.5) is also invariant under $\mathcal{F}^{\text{sym}}$, as it follows from (6.17). Since, in view of (6.16), the two sides of (6.5) have the same $\Theta$-transform and are both invariant under $\mathcal{F}^{\text{sym}}$, they coincide.

But we still owe the proof, as promised, that both terms on the right-hand side of (6.5) do converge in $\mathcal{S}'(\mathbb{R}^2)$.

\[ \square \]

**Proposition 6.2.** The series which is the second term of (6.5) is convergent in $\mathcal{S}'(\mathbb{R}^2)$, i.e., the series obtained when testing it against any function
in \( S(\mathbb{R}^2) \) is absolutely convergent.

**Proof.** It is necessary here to change notation, repeating eigenvalues according to their multiplicities (if any does occur: whether this is the case is not known). We thus trade the pair \((r, \iota)\) for a single \( r \in \mathbb{Z}\setminus\{0\} \) (it may, or not, be the same \( r \) as before), which must be accompanied by some local changes: \( \lambda_r \) to \( \mu_r \), \( \mathfrak{M}^{r,\iota} \) to \( \mathfrak{M}^r \), and \( \mathcal{N}_r,\iota \) to \( \mathcal{N}_r \). The discrete (in the spectral-theoretic sense) part \((\mathfrak{B}^1)^{\text{disc}}\) of \( \mathfrak{B}^1 \) becomes

\[
(\mathfrak{B}^1)^{\text{disc}} = \frac{1}{2} \sum_{r \in \mathbb{Z}\setminus\{0\}} \frac{\Gamma(1 - \frac{i\mu_r}{2}) \Gamma(1 + \frac{i\mu_r}{2})}{\|\mathcal{M}^r\|^2} \mathfrak{M}^r. \tag{6.18}
\]

The point in using this notation is the possibility to rely on the Selberg equivalent \( \mu_r \sim (48 r)^{\frac{1}{2}} \) [6, p.174] originating from the trace formula. We need also the estimate \[15\]

\[
\|\mathcal{M}^r\|^{-1} \leq C \left| \Gamma \left( \frac{i\mu_r}{2} \right) \right|^{-1} \tag{6.19}
\]

(repeating eigenvalues or not would not change anything here). This estimate and (2.4) yield the inequality

\[
\frac{\Gamma(1 - \frac{i\mu_r}{2}) \Gamma(1 + \frac{i\mu_r}{2})}{\|\mathcal{M}^r\|^2} \leq C |\mu_r|^{\frac{1}{2}}. \tag{6.20}
\]

From these reminders, it suffices to show that one can save arbitrary powers of \((1 + \mu_r^2)^{-1}\). From (4.12), one can bound \( \phi(k) \) by some fixed power of \( |k| \). One then writes

\[
\langle \mathfrak{M}^r, h \rangle = \frac{1}{4} \sum_{k \in \mathbb{Z}\setminus\{0\}} \phi(k) \int_{-\infty}^{\infty} |t|^{-1-i\mu_r} \left( F^{-1} h \right) \left( \frac{k}{t}, t \right) \, dt. \tag{6.21}
\]

To save powers of \((1 + \mu_r^2)^{-1}\), one relies on the integration by parts corresponding to the equation \(|t|^{-1-i\mu_r} = (-1-i\mu_r)^{-j} (t \frac{d}{dt})^j |t|^{-1-i\mu_r} \). Managing the \( k \)-summation is obtained from the fact that a function in \( S(\mathbb{R}^2) \), taken at the argument \( \left( \frac{k}{t}, t \right) \), is bounded for every \( A > 0 \) by \( C \left( t^2 + \frac{k^2}{t^2} \right)^{-A} \leq C(2 |k|)^{-A} \). \( \square \)
The main estimates

Recall the definition (6.3), (6.4) of $B_j$: we are interested in it for $j = 1$. Our analysis of the main problem will be based on an estimate of the distribution $(T_p^\text{dist})^{2N} \mathcal{B}^1$ as $N \to \infty$.

Lemma 7.1. Given $N = 1, 2, \ldots$, one has

$$(T_p^\text{dist})^{2N} \mathcal{B}^1 = \sum_{\ell=0}^{2N} p^{(N-\ell)(-1+2i\pi \mathcal{E})} \left( \alpha_{2N,\ell}^{(0)} I + \alpha_{2N,\ell}^{(1)} \sigma_1 + \cdots + \alpha_{2N,\ell}^{(\ell)} \sigma_{2N-\ell+r} \right) \mathcal{B}^1.$$  (7.1)

Setting $q = p^{\ell-N}$, one has if $r = 0, 1, \ldots$

$$\langle p^{(N-\ell)(-1+2i\pi \mathcal{E})} \sigma_r, \mathcal{B}^1, h \rangle = \left\{ \begin{array}{ll} \langle \mathcal{B}^1, \sigma_r q^{1+2i\pi \mathcal{E}} h \rangle & \text{for every } N, \\ \langle \mathcal{B}^1, q^{1+2i\pi \mathcal{E}} \sigma_{2N-2\ell+r} h \rangle & \text{if } \ell \leq N. \end{array} \right. \quad (7.2)$$

Proof. The equation (7.1) is a consequence of Proposition 5.3. Moving the operator $p^{(N-\ell)(2i\pi \mathcal{E})}$ to the right (then, changing $\mathcal{E}$ to its negative), one obtains the first equation (7.2) with, however, the following abuse of notation. The definition (5.9) of $\sigma_r$, in which the sum is performed over classes mod $p^r$, is valid as long as it is applied to a distribution in $\text{Inv}(1)$: to apply it as in (7.2), after transposition, to a function $h \in S(\mathbb{R})$, we must make a preliminary choice of representatives mod $p^r$. The transpose of $\tau[\gamma]$ (cf. Lemma 5.2) is $\tau[-\gamma]$, so that all $\sigma$'s are their own transposes.

In the case when $\ell \leq N$ (i.e., $q \leq 1$), we do not transpose and use instead the first equation (5.10), writing with $R = p^{-\frac{1}{2}+i\pi \mathcal{E}}$

$$p^{(N-\ell)(2i\pi \mathcal{E})} \sigma_r = p^{N-\ell} R^{2N-2\ell} \sigma_r \sim \sigma_{2N-2\ell+r} p^{(N-\ell)(2i\pi \mathcal{E})}. \quad (7.3)$$

□

In our application towards the main theorem, we must pay attention to the presence, on the right-hand side of (7.2), of factors of type $\sigma$: but we shall worry about these later, temporarily forgetting about the $\sigma$'s, or the “translations” $\tau[\gamma]$. Instead of applying the operator $\pi^2 \mathcal{E}^2$ to $s_1^1$, we may apply it to the function $h \in S(\mathbb{R})$ it is tested on: recalling that, by definition, $\mathcal{B}^1$ is an even distribution, the convergence result quoted in (6.4) is equivalent to saying that the series $\mathcal{B} = \frac{1}{2} \sum_{g \in \Gamma/\Gamma_{\infty}} s_1^1 \circ g^{-1}$ is convergent as a
continuous linear form on the space \((\pi^2\mathcal{E}^2)\mathcal{S}_{\text{even}}(\mathbb{R}^2)\). A slightly different result would follow from the parameter-dependent estimates in Proposition 8.1 and Conjecture (8.8).

If \(g = (\begin{array}{cc} n & n_1 \\ m & m_1 \end{array}) \in \Gamma\), the class of \(g\) in \(\Gamma/\Gamma_\infty\) coincides with the first column of this matrix and, writing \(\langle s_1 \circ g^{-1}, h \rangle = \langle s_1, h \circ g \rangle\), one obtains

\[
\langle \mathfrak{B}, h \rangle = \frac{1}{2} \sum_{(m,n)=1} I_{n,m}(h), \tag{7.4}
\]

with

\[
I_{n,m}(h) = \int_{-\infty}^{\infty} h(nx + n_1, mx + m_1) e^{2\pi x} dx \tag{7.5}
\]

(we may also use the notation \(\langle I_{n,m}, h \rangle\) when the expression of \(h\) is complex).

It would seem to be obvious, from (7.4), that \(\langle \mathfrak{B}, h \rangle\) depends only on the even part of \(h\), but convergence under some conditions has to be established first. Taking \(g = (\begin{array}{cc} 0 & -1 \\ 1 & 1 \end{array})\) or \( (\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}) \), one obtains

\[ I_{0,1}(q^{2\pi \mathcal{E}^2} h) = \int_{-\infty}^{\infty} h(x/q, n(x - 1/q)) e^{2\pi x} dx, \]
\[ I_{1,0}(q^{2\pi \mathcal{E}^2} h) = \int_{-\infty}^{\infty} h(qx, 1/q) e^{2\pi x} dx. \tag{7.6} \]

It is immediate that \(I_{0,1}(q^{2\pi \mathcal{E}^2} h) = O\left(\min(q, q^{-1})\right)\) for \(q > 0\), and the same goes if replacing \(I_{0,1}\) by \(I_{1,0}\); integrations by parts will be needed when dealing with the other terms of the series defining \(\mathfrak{B}\).

In a traditional way, define the classes \(\overline{n}\) and \(\overline{m}\) by the conditions \(n\overline{m} \equiv 1 \mod m\) and \(m\overline{m} \equiv 1 \mod n\). With the notation above for \(g\), \(\overline{n}\) is the class of \(m_1 \mod m\) : making the change \(x \mapsto x - \frac{\overline{m}_1}{m}\), one obtains the first equation of the following pair (the first, valid for \(m \neq 0\), the second for \(n \neq 0\))

\[ I_{n,m}(q^{2\pi \mathcal{E}^2} h) = \exp\left(-\frac{2i\pi \overline{n}}{m}\right) \times \int_{-\infty}^{\infty} h\left(q\left(nx - \frac{1}{m}\right), \frac{mx}{q}\right) e^{2i\pi x} dx, \]
\[ I_{n,m}(q^{2\pi \mathcal{E}^2} h) = \exp\left(\frac{2i\pi \overline{m}}{n}\right) \times \int_{-\infty}^{\infty} h\left(qnx, q^{-1}\left(mx + \frac{1}{n}\right)\right) e^{2i\pi x} dx. \tag{7.7} \]
the equations (7.6) are a special case.

**Lemma 7.2.** Let \( q > 0 \) be given arbitrarily, and let \( h \in \mathcal{S}(\mathbb{R}^2) \). If \( 0 < \alpha < \frac{1}{2} \), one has, whether \( mn \neq 0 \) or not,

\[
|I_{n,m}(q^{2i\pi\xi^1}h)| \leq C \left( \frac{m^2}{q^2} + q^2 n^2 \right)^{-\alpha},
\]

where \( C \) depends only on \( h \) and \( \alpha \).

**Proof.** Bounding \( |h(x,\xi)| \) by \( C(x^2 + \xi^2)^{\alpha-1} \), write

\[
q^2 \left( nx - \frac{1}{m} \right)^2 + \frac{(mx)^2}{q^2} = ax^2 - 2bx + c,
\]

with

\[
a = \frac{m^2}{q^2} + q^2 n^2, \quad b = \frac{q^2 n}{m}, \quad c = \frac{q^2}{m^2}.
\]

Then,

\[
\int_{-\infty}^{\infty} (ax^2 - 2bx + c)^{\alpha-1} dx = a^{\alpha-1} \int_{-\infty}^{\infty} \left( x^2 + \frac{ac - b^2}{a^2} \right)^{\alpha-1} dx
\]

\[
= C a^{-\alpha} (ac - b^2)^{\alpha-\frac{1}{2}} = C a^{-\alpha}.
\]

□

**Lemma 7.3.** Given \( f \in \mathcal{S}(\mathbb{R}^2) \) one has the pair of identities

\[
I_{n,m}((2i\pi\xi^1 q^{2i\pi\xi^1} f) = -\frac{2i\pi q}{m} I_{n,m} \left\{ q^{2i\pi\xi^1} \left[ (\xi + \frac{1}{2i\pi} \partial_x) f \right] \right\}, \quad m \neq 0,
\]

\[
I_{n,m}((2i\pi\xi^2 q^{2i\pi\xi^2} f) = \frac{2i\pi}{qn} I_{n,m} \left\{ q^{2i\pi\xi^2} \left[ (-x + \frac{1}{2i\pi} \partial_\xi) f \right] \right\}, \quad n \neq 0.
\]

**Proof.** Let \( f = f(x,\xi) \) (or \( f(y,\eta) \)) be an arbitrary function in \( \mathcal{S}(\mathbb{R}^2) \). One has

\[
(2i\pi\xi^1 f) \left( qnx - \frac{q}{m}, \frac{mx}{q} \right)
\]

\[
= \left( x \frac{d}{dx} + 1 \right) \left[ f \left( qnx - \frac{q}{m}, \frac{mx}{q} \right) \right] - \frac{q}{m} \frac{\partial f}{\partial y} \left( qnx - \frac{q}{m}, \frac{mx}{q} \right)
\]

(7.13)
so that, after an integration by parts (the transpose of \( x \frac{d}{dx} + 1 \) has the effect of multiplying \( e^{2i\pi x} \) by \(-2i\pi x\)),

\[
I_{n,m} ((2i\pi \mathcal{E}) q^{2i\pi \xi^2} f) = \exp \left( -\frac{2i\pi n}{m} \right) \times \int_{-\infty}^{\infty} \left[ -2i\pi x f \left( qnx - \frac{q}{m}, \frac{mx}{q} \right) - \frac{q}{m} \frac{\partial f}{\partial y} \left( qnx - \frac{q}{m}, \frac{mx}{q} \right) \right] e^{2i\pi x} \, dx,
\]

(7.14)
or, back to the variables \( x, \xi \),

\[
I_{n,m} ((2i\pi \mathcal{E}) q^{2i\pi \xi^2} f) = -\frac{2i\pi q}{m} I_{n,m} \left\{ q^{2i\pi \xi^2} \left[ (\xi + \frac{1}{2i\pi} \frac{\partial}{\partial \xi}) f \right] \right\}. \tag{7.15}
\]

One may also start in place of (7.13) from the equation

\[
(2i\pi \mathcal{E} \mathcal{E}) \left( qnx, \frac{mx}{q} + \frac{1}{qn} \right) = \left( x \frac{d}{dx} + 1 \right) \left[ f \left( qnx, \frac{mx}{q} + \frac{1}{qn} \right) \right] + \frac{1}{qn} \frac{\partial f}{\partial \eta} \left( qnx, \frac{mx}{q} + \frac{1}{qn} \right). \tag{7.16}
\]

This will lead in the end to the second equation (7.12).

\[
\square
\]

It is often useful to iterate integrations by parts: taking \( k \leq 2 \) in Proposition 7.5 below would suffice, but doing the general case does not imply any supplementary complication. If

\[
A = 2i\pi \mathcal{E}, \quad L = q^{2i\pi \xi^2},
\]

\[
B_1 = -\frac{2i\pi q}{m} \left[ \xi + \frac{1}{2i\pi} \frac{\partial}{\partial x} \right], \quad B_2 = -\frac{2i\pi q}{m} \left[ \xi - \frac{1}{2i\pi} \frac{\partial}{\partial x} \right], \tag{7.17}
\]

one has the commutation relations

\[
[A, L] = 0, \quad [B_1, B_2] = 0, \quad [A, B_1] = B_2, \quad [A, B_2] = B_1. \tag{7.18}
\]

With this notation, the first equation (7.12) becomes \( I_{n,m} (ALf) = I_{n,m} (LB_1 f) \) or \( I_{n,m} (LAf) = I_{n,m} (LB_1 f) \). Given two polynomials \( P \) and \( Q \) in one and two variables respectively, we shall write \( P(A) \sim Q(B) = Q(B_1, B_2) \) if, for every \( f \in S(\mathbb{R}^2) \), one has \( I_{n,m} (LP(A)f) = I_{n,m} (LQ(B)f) \). In particular, one has \( A \sim B_1 \).

We extend the definition of this equivalence to the case when \( Q(B) \) is replaced by a “polynomial” \( Q(B, A) \) in the non-commuting operators.
A, B_1, B_2 (i.e., a sum of ordered monomials). Such an equivalence will remain so if multiplied on the right by any polynomial in A. Our aim is to obtain a sequence of equivalences P_k(A) \sim Q_k(B), where the polynomial Q_k will have no term of degree < k.

**Lemma 7.4.** Consider indeterminates A, B_1, B_2 satisfying the commutation relations (7.18). Define a linear equivalence between polynomials in A and “polynomials” in A, B_1, B_2 by stating that A \sim B_1 and that the multiplication on the right by any polynomial in A preserves the equivalence. Define inductively the polynomials P_k(A) such that P_0(A) = 1 and

\[
P_{k+1}(A) = \prod_{|r| \leq k} (A + r) \times P_k(A). \tag{7.19}\]

In particular, P_1(A) = A and P_2(A) = A^2(A^2 - 1). Then, one has for every k the equivalence P_k(A) \sim Q_k(B) for some polynomial Q_k of the same degree as P_k and without any term of degree < k.

**Proof.** By induction, starting from an equivalence P_k(A) \sim Q_k(B) with the required degree property. In place of B_1, B_2, introduce the combinations X = B_1 + B_2 and Y = B_1 - B_2, after which Q_k(B) = R_k(X, Y), the polynomial R_k still satisfying the desired degree property. One has the relations B_1 A = A B_1 - B_2, B_2 A = A B_2 - B_1, hence X A = (A - 1)X and Y A = (A + 1)Y. More generally, by induction,

\[
X^r A = AX^r - rX^r, \quad Y^r A = AY^r + rY^r. \tag{7.20}\]

Using these equations, one can “push” A to the left, or let it disappear, until, starting from a monomial X^{r_1} Y^{r_2}, one obtains, using at the end that A \sim B_1 = \frac{X + Y}{2},

\[
X^{r_1} Y^{r_2} A = X^{r_1} [AY^{r_2} + r_2 Y^{r_2}] = [AX^{r_1} - r_1 X^{r_1} Y^{r_2} + r_2 X^{r_1} Y^{r_2}]
\sim \frac{1}{2} \left[ X^{r_1 + 1} Y^{r_2} + X^{r_1} Y^{r_2+1} \right] - (r_1 - r_2) X^{r_1} Y^{r_2}. \tag{7.21}\]

Then, one has

\[
X^{r_1} Y^{r_2}(A + r_1 - r_2) \sim \frac{1}{2} \left[ X^{r_1 + 1} Y^{r_2} + X^{r_1} Y^{r_2+1} \right] \tag{7.22}\]

There were already no terms of total degree < k in R_k(X, Y), and there are none in R_{k+1}(X, Y) in view of (7.21). Multiplying R_k(X, Y) on the right by the polynomial \prod_{|r| \leq k}(A + r) has the effect of killing all monomials of total degree k.

\[ \Box \]
Proposition 7.5. Consider the sequence of polynomials \( P_k \) introduced in Lemma 7.4. Given \( \alpha < \frac{1}{2} \) and \( f \in S(\mathbb{R}^2) \), one has with some constant \( C > 0 \) depending on \( \alpha, k, h \) but not on \( n, m, q \) the estimate

\[
\left| I_{n,m} \left[ P_k(2i\pi E) q^{2i\pi\xi^2} h \right] \right| \leq C \left( \frac{m^2}{q^2} + q^2 n^2 \right)^{-\frac{\alpha}{2}} \left( 1 + \frac{m^2}{q^2} + q^2 n^2 \right)^{-\frac{k}{2}}.
\]  

(7.23)

Recalling the definition of \( \tau[\gamma] \) in Lemma 5.2, the estimate that precedes is valid with a uniform constant if \( h \) is replaced by \( \tau[\gamma]h \) with a bounded \( \gamma \).

**Proof.** As has been explained between (7.17) and Lemma 7.4, it follows from this lemma that

\[
I_{n,m} \left[ P_k(2i\pi E) q^{2i\pi\xi^2} h \right] = I_{n,m} \left[ q^{2i\pi\xi^2} Q_k(B_1, B_2) h \right]
\]  

(7.24)

with \( B_1, B_2 \) as introduced in (7.17) and \( Q_k \) as introduced in Lemma 7.4. Replacing the pair \( B_1, B_2 \) by the pair

\[
D_1 = \frac{2i\pi}{qn} \left[ -x + \frac{1}{2i\pi} \frac{\partial}{\partial \xi} \right], \quad D_2 = \frac{2i\pi}{qn} \left[ -x - \frac{1}{2i\pi} \frac{\partial}{\partial \xi} \right],
\]  

(7.25)

one obtains a totally similar identity, just replacing the use of the first equation (7.12) by that of the second one.

When \( k = 0 \), so that \( P_k = 1 \), the estimate (7.23) is just the same as (7.8). Each time one replaces a function \( h \) on the right-hand side by its image under \( B_1 \) or \( B_2 \), one gains in the estimate a factor \( \frac{q}{m} \) because of the coefficients in front of \( B_1, B_2 \); when replacing \( h \) by its image under \( D_1 \) or \( D_2 \), one gains instead a factor \( \frac{1}{qn} \). Making use of the identity (7.24) if \( \frac{q}{|m|} \leq \frac{1}{|qn|} \), of the similar one with \( B_1, B_2 \) replaced by \( D_1, D_2 \) in the other case, and using the fact that all terms of \( Q_k \) have a total degree \( \geq k \), one obtains the estimate

\[
\left| I_{n,m} \left[ P_k(2i\pi E) q^{2i\pi\xi^2} h \right] \right| \leq C \left( \frac{m^2}{q^2} + q^2 n^2 \right)^{-\frac{\alpha}{2}} \left( 1 + \frac{m^2}{q^2} + q^2 n^2 \right)^{-\frac{k}{2}}.
\]  

(7.26)

Also, \( \frac{m^2}{q^2} + q^2 n^2 \geq 2 \). The estimate (7.23) follows.
We prove now the last assertion of the lemma. First, defining \( \check{I}_{n,m}(h) = \exp \left( \frac{2i\pi}{m} \right) I_{n,m}(h) \), the equations (7.7) become

\[
\check{I}_{n,m}(q^{2i\pi \varpi^3} h) = \int_{-\infty}^{\infty} h \left( q \left( nx - \frac{1}{m} \right), \frac{mx}{q} \right) e^{2i\pi x dx}, \]

the advantage is that, as will be needed now, the linear form \( \check{I}_{n,m} \) has a natural extension to the case when \( m \) and \( n \) cease to be integers.

With \( h_1 = \tau[\gamma] h \), or \( h_1(t, \tau) = h(t + \gamma \tau, \tau) \), one has

\[
h_1 \left( qnx - \frac{q}{m^2}, \frac{mx}{q} \right) = h \left( qnx - \frac{q + \gamma mx}{m}, mx \right) = h \left( qn'x - \frac{q - mx}{q} \right),
\]

with \( n' = n + \frac{\gamma m}{q^2} \). It follows that

\[
\check{I}_{n,m}(q^{2i\pi \varpi^3} \tau[\gamma] h) = \check{I}_{n+\frac{\gamma m}{q}, m}(q^{2i\pi \varpi^3} h). \tag{7.29}
\]

Then, no arithmetic was needed to prove (7.23) and, after one has replaced \( I_{n,m} \) by \( \check{I}_{n,m} \), this estimate remains valid when the pair \( m, n \) is replaced by the pair \( m + \frac{\gamma m}{q}, n \). To prove the second part of Proposition 7.5, what remains to be done, so as to follow the end of the proof of the first part, is to prove that the quadratic forms \( \frac{m^2}{q^2} + q^2 n^2 \) and \( \frac{2m^2}{q^2} + q^2 \left( n + \frac{\gamma m}{q} \right)^2 \) remain within a uniformly bounded ratio when \( \gamma \) is bounded. In terms of \( m_1 = \frac{m}{q} \) and \( n_1 = qn \), one has

\[
\frac{m^2}{q^2} + q^2 \left( n + \frac{\gamma m}{q^2} \right)^2 = \left( m_1 n_1 \right) \left( 1 + \frac{\gamma^2}{q^2} \right) \left( \frac{m_1}{n_1} \right). \tag{7.30}
\]

The smaller eigenvalue of the matrix in the middle is at least the inverse of its trace \( 2 + \gamma^2 \), which concludes the proof of the proposition.

\[ \square \]

8. The main estimates, continued

Let us reconsider (7.1), with alleviated notation. We set \( q = p^{\ell - N} \), a number in the set \( \text{pow} \) of powers of \( p^{\pm 1} \), constrained by the condition \( p^{-N} \leq q \leq p^N \). Then, \( p^{(N-\ell)(-1+2i\pi \varpi^3)} = q^{1-2i\pi \varpi^3} \). There is another constraint on
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\[ r \text{ made explicit in Proposition 5.3, to wit } r \geq 2\ell - 2N, \text{ or } p^r \geq q^2: \text{ as } p^\ell = qp^N, \] the identity (7.1) becomes

\[
\left( T_{\text{dist}}^{\text{2N}} \right) B_1 = \sum_{q \in \text{pow}} q^{1 - 2i\pi E^2} \sum_{\max(1,q^2) \leq p^r \leq qp^N} \alpha_{2N,\ell}^r \sigma_r B_1.
\] (8.1)

With a view towards proving the Ramanujan-Petersson estimate in the next section, our present aim is to bound the result of testing the left-hand side of (8.1) on \( h \in S(\mathbb{R}^2) \), by a \( O((q + q^{-1})^\varepsilon) \) with \( \varepsilon > 0 \) arbitrarily small, or to show that \( \langle B, \sigma_r q^{2i\pi E^2} h \rangle = O((q - 1)(q + q^{-1})^\varepsilon) \) for \( h \in \pi^2 E^2 S(\mathbb{R}^2) \).

We shall immediately lower our demand, substituting for the image of \( S(\mathbb{R}^2) \) under \( \pi^2 E^2 \) its image under \( P_k(i\pi E^2) \), for some \( k \geq 2 \). This will do no harm since our estimates of \( B \) are developed for the purpose of giving, with the help of some amount of spectral theory (in the next section) similar estimates of individual Hecke distributions. Now, the operator \( 2i\pi E^2 \) commutes with \( E^2 \) as well as with \( \tau[\gamma] = \exp(\gamma \xi \frac{\partial}{\partial x}) \) for every \( \gamma \), hence with all \( \sigma_r \)'s.

Since \( P_k(2i\pi E^2) \mathfrak{H}_{\chi,i\lambda} = P_k(i\lambda) \mathfrak{H}_{\chi,i\lambda} \) and all zeros of \( P_k \) are integers, the estimates obtained when inserting the operator \( P_k(2i\pi E^2) \) or not have the same strength.

The cases when \( \ell \leq N \) (or \( q \leq 1 \)) and \( N < \ell \leq 2N \) need different treatments. Note that, in the first case, the estimate we obtain is better, by a factor \( q^2 \), or even \( q^k \), than we one we need: no such improvement is available in the second case.

**Proposition 8.1.** Let \( q = p^{\ell-N} \) and let \( P_k \) be the polynomial introduced in Lemma 7.4. In the case when \( q \leq 1 \), one has for every \( \varepsilon > 0 \), if \( k \geq 2 \) and \( h \in S(\mathbb{R}^2) \), the estimate

\[
\left| \langle q^{1-2i\pi E^2} \sigma_r P_k(-2i\pi E) B, h \rangle \right| \leq C q^{2-\varepsilon}, \tag{8.2}
\]

where the constant \( C \) depends only on \( k \) and \( h \).

**Proof.** Let us use the second equation (7.2), rewritten as

\[
\langle q^{2i\pi E^2} \sigma_r B_1, h \rangle = \langle B_1, q^{2i\pi E^2} \sigma_{2\ell+r} B_1 \rangle. \tag{8.3}
\]
The estimate (7.23) is equivalent in the case when $m \neq 0$ to

$$
\left| \left\langle I_{n,m}, P_k(2i\pi E) q^{2i\pi E} h \right\rangle \right| \leq C \left( \frac{1 + m^2}{q^2} + q^2 n^2 \right)^{-\frac{a}{2}} \tag{8.4}
$$

while, from (7.6), $I_{1,0} \left( q^{2i\pi E} h \right)$ is a rapidly decreasing function of $\frac{1}{q}$ as $q \to 0$.

Recalling that $\sigma_{2N-2\ell+r}$ is an arithmetic average of translations $\tau[\gamma]$ with $\gamma \in [0, 1]$, one obtains from the last assertion of Proposition 7.5 that, if $m \neq 0$,

$$
\left| \left\langle I_{n,m}, P_k(2i\pi E) q^{2i\pi E} \sigma_{2N-2\ell+r} h \right\rangle \right| \leq C \left( \frac{1 + m^2}{q^2} + q^2 n^2 \right)^{-\frac{a}{2} - \frac{k}{2}} \tag{8.5}
$$

from which it follows, using (7.4) and the fact that $2i\pi E$ commutes with $E^\sharp$ as well as with $\tau[\gamma] = \exp \left( \gamma \xi \frac{\partial}{\partial x} \right)$, that, up to an error term which is a rapidly decreasing function of $\frac{1}{q}$, one has

$$
\left| \left\langle \mathfrak{B}, P_k(2i\pi E) q^{2i\pi E} \sigma_{2N-2\ell+r} h \right\rangle \right| \lesssim C \sum_{(m,n)=1} \left( \frac{1 + m^2}{q^2} + q^2 n^2 \right)^{-\frac{a}{2} - \frac{k}{2}} \tag{8.6}
$$

if $a + \frac{k}{2} > 1$, i.e., if $k \geq 2$. Finally, for $a$ close to $\frac{1}{2}$, one has $2a + k = 1 - \epsilon$. Let us not forget the extra factor $q$ present on the left-hand side of (8.2).

**Remark.** It will be useful to rephrase the estimate just obtained without constants depending on $h$ as

$$
\left| \left\langle q^{1-2i\pi E} \sigma_r P_k(-2i\pi E) \mathfrak{B}^1, h \right\rangle \right| \leq C(k) q^{2-\epsilon} \left\| h \right\|, \tag{8.7}
$$

where $\left\| \cdot \right\|$ is some norm continuous for the topology of $\mathcal{S}(\mathbb{R}^2)$ and, this time, $C(k)$ does not depend on $h$.

We turn now to the case when $\ell > N$, i.e., $q > 1$, using the first equation (7.2). Recall from the beginning of this section the origin of the condition $2\ell - 2N - r \leq 0$, or $p' \geq q^2$. 
The following, initially assumed to be a proposition, is an unproved conjecture.

Given \( \varepsilon > 0 \) and any \( k \geq 2 \), one has if \( q = p^{\ell - N} \geq 1, 2\ell - 2N - r \leq 0 \) and \( h \in \mathcal{S}(\mathbb{R}^2) \) the estimate

\[
\left| \langle q^{1-2i\pi\varepsilon} \sigma_r P_k (-2i\pi \mathfrak{E}) \mathfrak{B}, h \rangle \right| \leq C(k) q^\varepsilon \| h \|,
\]

where \( . \| \) is some continuous semi-norm on \( \mathcal{S}(\mathbb{R}^2) \).

**Proof.** (attempt at) Making \( \sigma_r \) explicit and using (7.4), one has if \( h \in P_1(2i\pi \mathfrak{E}) \mathcal{S}(\mathbb{R}^2) \)

\[
\langle \mathfrak{B}, \sigma_r q^{2i\pi\varepsilon^2} h \rangle = \frac{1}{2p^r} \sum_{0 < b \leq p^r} \sum_{(n,m) = 1} I_{n,m} \left( \tau \left[ \frac{b}{p^r} \right] q^{2i\pi\varepsilon^2} h \right). \tag{8.9}
\]

The convergence of the series will follow from the developments to come, in which, again, we strive for \( q \)-dependent estimates. Recalling that \( \overset{\circ}{I}_{n,m} \) is the same as \( I_{n,m} \), except for a constant factor of absolute value 1, one has if \( m \neq 0 \), from (7.7),

\[
\overset{\circ}{I}_{n,m} \left( \tau[\gamma] q^{2i\pi\varepsilon^2} h \right) = \int_{-\infty}^{\infty} h_1 \left( nx - \frac{1}{m}, mx \right) e^{2i\pi x} dx, \tag{8.10}
\]

with \( h_1(x,\xi) = h \left( q(x + \gamma \xi) x, \frac{\xi}{q} \right) \). Hence,

\[
h_1 \left( nx - \frac{1}{m}, mx \right) = h \left( q(n + \gamma m) x - \frac{q}{m}, \frac{mx}{q} \right), \tag{8.11}
\]

from which it follows that

\[
\overset{\circ}{I}_{n,m} \left( \tau[\gamma] q^{2i\pi\varepsilon^2} h \right) = \overset{\circ}{I}_{n_1,m} \left( q^{2i\pi\varepsilon^2} h \right), \tag{8.12}
\]

with \( n_1 = n + \gamma m \).

According to Proposition 7.5, in the proof of which no arithmetic was used (i.e., \( m \) and \( n \) did not have to be integers, as soon as \( I_{n,m} \) had been replaced by \( \overset{\circ}{I}_{n,m} \)), one has

\[
\left| \langle I_{n,m}, \tau \left[ \frac{b}{p^r} \right] q^{2i\pi\varepsilon^2} P_k (2i\pi \mathfrak{E}) h \rangle \right| \leq C \left[ \frac{m^2}{q^2} + q^2 \left( n + \frac{bm}{p^r} \right) \right]^{-\alpha - \frac{k}{4}}. \tag{8.13}
\]
Since $k$ is arbitrary, this estimate is satisfactory when $\frac{m^2}{q^2} + q^2 \left(n + \frac{bm}{p^2}\right)^2$ is large, say when $m > q^\varepsilon$ or $|n + \frac{bm}{p^2}| > q^{-1+\varepsilon}$ for some $\varepsilon > 0$.

Let us first do away with the terms for which the triple $m, n, b$ with $n_1 = n + \frac{bm}{p^2}$, is such that either $n_1 \leq -\frac{1}{2}$ or $n_1 > \frac{1}{2}$. Bound the corresponding sum, not forgetting the coefficient $\frac{1}{p^2}$ in the front of the right-hand side of (8.9), by $C$ times

$$\frac{1}{p^2} \sum_{0 < b < p^2} \sum_{m \neq 0} \sum_{n_1 \in \frac{bm}{p^2} + \mathbb{Z}} \left( \frac{m^2}{q^2} + q^2 n_1^2 \right)^{-\alpha - \frac{k}{2}}.$$  \hspace{1cm} (8.14)

Defining $\ell \in \mathbb{Z}$, depending on $b, m$, such that $-\frac{1}{2} < \ell + \frac{bm}{p^2} \leq \frac{1}{2}$, one has if $n_1 = \frac{bm}{p^2} + n$ and $n_1 \leq -\frac{1}{2}$ the inequalities $n < \ell$ or $n \leq -\frac{1}{2}$; if $n_1 > \frac{1}{2}$, one has $n > \ell$ or $n \geq \ell + 1$ and $n_1 = \left(\frac{bm}{p^2} + \ell + 1\right) + (n-\ell) \leq \frac{1}{2} + (n-\ell) \leq \frac{n-\ell}{2}$.

It follows that the sum with respect to $n_1$ in (8.14) is at most BLOB

$$\sum_{n \neq \ell} \left( \frac{m^2}{q^2} + \frac{1}{4} q^2 (n - \ell)^2 \right)^{-\alpha - \frac{k}{2}} = \sum_{n \neq 0} \left( \frac{m^2}{q^2} + \frac{1}{4} q^2 n^2 \right)^{-\alpha - \frac{k}{2}}.$$  \hspace{1cm} (8.15)

Exchanging $m$ and $n$, one can then apply (8.6) with $q$ replaced by $(2q)^{-1}$. This shows that the sum of terms of (8.14) for which $n_1 \leq -\frac{1}{2}$ or $n_1 > \frac{1}{2}$ is less than $C q^{-2\alpha - k + 2}$, which is less than $C q^{-1+\varepsilon}$ with $\varepsilon$ arbitrarily small if $k \geq 2$ and one takes $\alpha$ close to $\frac{1}{2}$.

In the remaining sum with respect to $m, n, b$, there is actually no summation with respect to $n$ since, given $m, b$, the condition $-\frac{1}{2} < n + \frac{bm}{p^2} \leq \frac{1}{2}$ leaves no choice for the integer $n$; also, one has $m \neq 0$. With

$$[f_{m,n}^{(k)}](t) = I_{n+tm,m} \left(q^{2i\pi x^2} P_k(2i\pi x) \right) h = \int_{-\infty}^{\infty} g \left(q(n+tm)x - \frac{q}{m}, \frac{mx}{q}\right) e^{2i\pi x} dx$$  \hspace{1cm} (8.16)

and

$$f_{m,n}^{(k)}(t) = [f_{m,n}^{(k)}](t) \times \text{char}(-\frac{1}{2} < n + mt \leq \frac{1}{2}).$$  \hspace{1cm} (8.17)
We must estimate the sum
\[ \sum_{(m,n) = 1 \atop m \neq 0} \frac{1}{p^r} \sum_{0 < b \leq p^r} f^{(k)}_{m,n} \left( \frac{b}{p^r} \right), \tag{8.18} \]
truly a sum over \( m \neq 0 \) only since the term corresponding to a given value of \( m \) and \( b \) can be nonzero for only one value of \( n \).

One has the Euler-Maclaurin expansion
\[
\frac{1}{p^r} \sum_{0 < b \leq p^r} \sum_{m \neq 0} f^{(k)}_{m,n} \left( \frac{b}{p^r} \right) = \int_0^1 \sum_{m \neq 0} f^{(k)}_{m,m}(t) \, dt
\] 
\[
+ \frac{1}{2p^r} \sum_{m \neq 0} \left[ f^{(k)}_{m,n}(1) - f^{(k)}_{m,n}(0) \right] - \frac{1}{p^r} \int_0^1 B_1(p^r t) \frac{d}{dt} \sum_{m \neq 0} f^{(k)}_{m,m}(t) \, dt. \tag{8.19}
\]

Recall that the Bernoulli function \( B_1 \) satisfies the uniform estimate \(|B_1(s)| \leq \frac{1}{2}|s|^2\). In the last term, one must not forget the discontinuities of the function \( f^{(k)}_{n,m} \), possibly implied by the presence of the characteristic function \( \text{char}\left(-\frac{1}{2} < m + nt \leq \frac{1}{2}\right) \) as a factor of \( f^{(k)}_{n,m}(t) \). If, say, \( m \geq 1 \), setting \( \alpha = \left(-\frac{1}{2} - n\right)m^{-1} \) and \( \beta = \left(\frac{1}{2} - n\right)m^{-1} \), so that the characteristic function under consideration is that of the interval \([\alpha, \beta]\), one has \( 0 < \alpha < 1 \) only if \(-m \leq n \leq -1\) and \( 0 \leq \beta < 1 \) only if \( 1 - m \leq n \leq 0 \). This is unimportant: the only point is that either extremity of this interval may, or not, lie in \([0, 1]\). When such is the case, say so far as \( \alpha \) is concerned, the discontinuity of \( f^{(k)}_{m,m}(t) \) at \( t = \alpha \) will contribute to (8.19) the term \( \pm \frac{1}{p^r} B_1(\alpha) \left[ f^{(k)}_{m,m} \right](\alpha), \) the exterior \( \pm \) depending on the sign of \( m \): this term is a \( O(p^{-r}) \), or a \( O(q^{-2}) \). After this remark, we may, in view of the desired estimate, replace in the last term of (8.19) the function \( f^{(k)}_{n,m}(t) \) by \( \left[ f^{(k)}_{n,m} \right](t) \).

At this point, the proof breaks down: I made in the first integral the absurd change of variable \( s = n + mt \) in my last version: \( n \) is a function of \( mt \), not of \( s \).

\[ \square \]

9. The Ramanujan-Petersson estimate for Maass forms

**Lemma 9.1.** Given a finite collection \( (\mathfrak{M}_j)_{j \geq 1} \) of distinct Hecke distributions, one can find \( h \) in \( S(\mathbb{R}^2) \) such that \( \langle \mathfrak{M}_1, h \rangle = 1 \) but \( \langle \mathfrak{M}_j, h \rangle = 0 \) for
Proof. Let $T$ be the linear form on the linear space generated by the $N_j$ defined by the conditions $\langle T, N_1 \rangle = 1$ but $\langle T, N_j \rangle = 0$ for $j > 1$. The Hahn-Banach theorem makes it possible to extend $T$ as a continuous linear form on $S'(\mathbb{R}^2)$. But the space $S(\mathbb{R}^2)$ is reflexive, which means that this extension is associated in the natural way with a function $h \in S(\mathbb{R}^2)$.

□

**Theorem 9.2.** Assume that the conjecture (8.8) holds. Let a prime $p$ and $\varepsilon > 0$ be given. As $N \to \infty$, the (even) distribution $2^{-2N}p^{-N} (T_p^{\text{dist}})^{2N} P_2(-2i\pi \varepsilon) \mathcal{B}$ remains in a bounded subset of $S'(\mathbb{R}^2)$.

Proof. Recall the decomposition (7.1) of the distribution $(T_p^{\text{dist}})^{2N} \mathcal{B}$, and let $\varepsilon > 0$ be given. From Proposition 8.1 and Conjecture (8.8) (recalling from Proposition 5.3 that $2\ell - 2N - r \leq 0$ for all nonzero terms of the expansion), one has for some continuous norm $\| \|$ on $S(\mathbb{R}^2)$ the estimates

$$\left| \langle p^{(N-\ell)(-1+2i\pi \varepsilon)} \sigma_r P_2(-2i\pi \varepsilon) \mathcal{B}, h \rangle \right| \leq (q + q^{-1})^\varepsilon \| h \| \leq p^{N\varepsilon} \| h \|,$$

valid whenever $h \in S(\mathbb{R}^2)$. To avoid having to carry the operator $P_2(-2i\pi \varepsilon)$, we set $g = P_2(2i\pi \varepsilon) h$. We shall move $P_2(2i\pi \varepsilon)$ to the other side later, not forgetting that $2i\pi \varepsilon$ commutes with all operators involved. The effect of applying $P_2(-2i\pi \varepsilon)$ to a Hecke distribution $\mathcal{N}_{\chi,i\lambda}$ is to multiply it by $P_2(i\lambda)$

Adding these inequalities and recalling from Proposition 5.3 that, for any given $\ell$, one has $\alpha^{(0)}_{2N,\ell} + \alpha^{(1)}_{2N,\ell} + \cdots + \alpha^{(\ell)}_{2N,\ell} = \binom{2N}{\ell}$, we obtain

$$\sum_{\ell=0}^{2N} \sum_{0 \leq r \leq \ell} \alpha^{(r)}_{2N,\ell} \left| \langle p^{(N-\ell)(-1+2i\pi \varepsilon)} \sigma_r \mathcal{B}, g \rangle \right| \leq p^{N\varepsilon} \sum_{\ell=0}^{2N} \sum_{0 \leq r \leq \ell} \alpha^{(r)}_{2N,\ell} \| h \|$$

$$= p^{N\varepsilon} \sum_{\ell=0}^{2N} \binom{2N}{\ell} \| h \| = p^{N\varepsilon} 2^{2N} \| h \|.$$

This is the claimed result.

□

Next, we shall show that Theorem 9.2 remains valid if we replace $\mathcal{B}$ by either term of the decomposition $\mathcal{B} = \mathcal{B}^{\text{cont}} + \mathcal{B}^{\text{disc}}$, as defined in reference to the continuous and discrete parts of the spectral decomposition (6.5).
We shall benefit from our perfect knowledge of Eisenstein distributions to establish this result for \( \mathcal{B}^{\text{cont}} \); in the other direction, we shall then benefit from the analysis of \( \mathcal{B}^{\text{disc}} \) so obtained to obtain, with the help of some spectral theory, the searched estimate of the Fourier coefficients of Hecke distributions or eigenforms.

**Proposition 9.3.** The statement of Theorem 9.2 remains valid if one substitutes \( \mathcal{B}^{\text{cont}} \) for \( \mathcal{B} \).

**Proof.** The polynomial \( P_2(A) = (A^2 - 1)A^2 \) is divisible by \( A^2 \), so that the image of \( \mathcal{B}^{\text{cont}} \) under \( P_2(-2i\pi \mathcal{E}) \) is the image of \( (\mathcal{B}^1)^{\text{cont}} \) under \( (2i\pi \mathcal{E})^2 - 1 \), and we may replace the study of the continuous (spectral) part of \( P_2(-2i\pi \mathcal{E}) \mathcal{B} \) by that of \( \mathcal{B}^1 \). Write

\[
(\mathcal{B}^1)^{\text{cont}} = \frac{1}{4\pi} \int_{-\infty}^{\infty} F(i\lambda) \mathcal{E}_{i\lambda} d\lambda, \quad (9.3)
\]

with

\[
F(i\lambda) = \frac{\Gamma(1 - \frac{i\lambda}{2})\Gamma(1 + \frac{i\lambda}{2})}{\zeta^*(i\lambda)\zeta^*(-i\lambda)} = \frac{\lambda^2}{4 \zeta(i\lambda)\zeta(-i\lambda)}. \quad (9.4)
\]

We use (5.7), an equation which reduces in the case of the Eisenstein distribution \( \mathcal{E}_{i\lambda} \) to \( T_p^{\text{dist}} \mathcal{E}_{i\lambda} = \left( p^\frac{i\lambda}{2} + p^{-\frac{i\lambda}{2}} \right) \mathcal{E}_{i\lambda} \), so that

\[
\left( T_p^{\text{dist}} \right)^{2N} \mathcal{E}_{i\lambda} = \sum_k \binom{2N}{k} p^{(N-k)i\lambda} \mathcal{E}_{i\lambda}. \quad (9.5)
\]

Then, using again the equation \( \sum_k \binom{2N}{k} = 2^{2N} \), we must insert the extra factor \( p^{(N-k)i\lambda} \) in the integrand of (9.3) and test the result on \( h \in \mathcal{S}(\mathbb{R}^2) \). To do so, we use the Fourier series decomposition (4.15) of Eisenstein distributions. Finally, using for \( d\lambda \)-summability the equation

\[
\mathcal{E}_{i\lambda} = (1 - i\lambda)^{-a} (1 + 2i\pi \mathcal{E})^a \mathcal{E}_{i\lambda} \quad (9.6)
\]

for some appropriate \( a \) and transferring the (transpose of) the operator \( (1 + 2i\pi \mathcal{E})^a \) to the side of \( h \), we have to obtain uniform (relative to \( N, k \)) bounds for the expressions.
\[ I_1 = \int_{-\infty}^{\infty} F(i\lambda) p^{(N-k)i\lambda} \zeta(-i\lambda) \frac{d\lambda}{(1 - i\lambda)^a} \int_{-\infty}^{\infty} \frac{|t|^{-i\lambda-1}}{(F^{-1}_1 h)(0, t)} dt, \]
\[ I_2 = \int_{-\infty}^{\infty} F(i\lambda) p^{(N-k)i\lambda} (1 + i\lambda) \frac{d\lambda}{(1 - i\lambda)^a} \int_{-\infty}^{\infty} |t|^{-i\lambda} h(t, 0) dt, \]
\[ I_3 = \int_{-\infty}^{\infty} F(i\lambda) p^{(N-k)i\lambda} \frac{d\lambda}{(1 - i\lambda)^a} \sum_{n \neq 0} \sigma_i(n) \int_{-\infty}^{\infty} |t|^{-i\lambda-1} (F^{-1}_1 h) \left( \frac{n}{t}, t \right) dt. \]

One has \(|\zeta(i\lambda)|^{\pm 1} \leq C (1 + |\lambda|)^{\frac{1}{2}-\varepsilon}\) for every \(\varepsilon > 0\) as recalled in (2.4).

Since the function \(F(\lambda)\) is analytic on the line, an application of the integration by parts associated to the equation \(i\lambda |t|^{-i\lambda-1} = -\frac{d}{dt} (|t|^{-i\lambda})\) ensures the summability of \(I_1\), while that of \(I_2\) does not require any integration by parts. In the case of \(I_3\), we first recall [7, p.334] that the number of divisors of \(n\) is \(O(\sqrt{|n|\varepsilon})\) for every \(\varepsilon > 0\). Then, the \(n\)-summation is taken care of by the bound \(\left| (F^{-1}_1 h) \left( \frac{n}{t}, t \right) \right| \leq C \left( \frac{n^2}{t^2} + t^2 \right)^{-\varepsilon}\) valid for some \(C > 0\).

\[ \square \]

It is, for our present purpose, the difference \(B_{\text{disc}} = B - B_{\text{cont}}\) we are interested in. It follows from Theorem 9.2 and Proposition 9.3 that, for every \(\varepsilon > 0\), there exists a continuous semi-norm \(\| \|\) on \(S(\mathbb{R}^2)\), independent of \(N\), such that, for \(h \in S(\mathbb{R}^2)\) and \(g = P_2(2i\pi \mathcal{E}) h\),
\[ \left| \left< T_{p}^{2N} B_{\text{disc}}, g \right> \right| \leq 2^{2N} p^{N\varepsilon} \left| h \right|. \]

The last point of this paper will be a localization (with respect to the discrete part of the spectrum of the automorphic Euler operator) of this estimate. To this effect, with a function \(\Xi\) such that, for every \(B\), \(|\Xi(t)| \leq C e^{-B|t|}\) for some \(C > 0\), introduce the function
\[ \Phi(i\lambda) = \int_{-\infty}^{\infty} \Xi(t) e^{-2\pi t\lambda} dt = \frac{1}{2\pi} \int_{0}^{\infty} \Xi \left( \frac{\log \theta}{2\pi} \right) \frac{\theta^{-i\lambda-1}}{\theta} d\theta. \]

One has
\[ \frac{1}{2\pi} \int_{0}^{\infty} \Xi \left( \frac{\log \theta}{2\pi} \right) \left| \frac{d\theta}{\theta} \right| = \int_{-\infty}^{\infty} \Xi(t) dt < \infty. \]
Since the operators $\theta^{2i\pi E}$ make up a collection of unitary operators, one can substitute $-2i\pi E$ for $i\lambda$ in (9.9), getting the definition of a bounded operator in $L^2(\mathbb{R}^2)$, to wit the operator $\Phi(-2i\pi E)$ such that, for $g \in L^2(\mathbb{R}^2)$,

$$(\Phi(-2i\pi E) g)(x, \xi) = \int_{-\infty}^{\infty} \Xi(t) g(e^{2\pi t}x, e^{2\pi t}\xi) e^{2\pi t} dt. \quad (9.11)$$

One observes that if $g$ lies in $\mathcal{S}(\mathbb{R}^2)$, so does $\Phi(-2i\pi E) g$, and one can define $\Phi(2i\pi E)\mathcal{B}^{\text{disc}}$ by duality and, making use of the second term on the right-hand side of (6.5), one obtains

$$\langle \Phi(2i\pi E)\mathcal{B}^{\text{disc}}, g \rangle = \langle \mathcal{B}^{\text{disc}}, \Phi(-2i\pi E) g \rangle = \frac{1}{2} \sum_{r, \iota} \Phi(-i\lambda_r) \frac{\Gamma(1 - \frac{i\lambda_r}{2})\Gamma(1 + \frac{i\lambda_r}{2})}{\|N^{\lambda_r}\|_2^2} \langle \mathcal{N}^{\iota}, g \rangle, \quad (9.12)$$

which leads in view of (4.28) to the unsurprising formula

$$\langle \Phi(2i\pi E)\left(T_p^{\text{dist}}\right)^{2N}\mathcal{B}^{\text{disc}}, g \rangle = \frac{1}{2} \sum_{r, \iota} \Phi(-i\lambda_r) (b_p(r, \iota))^{2N} \frac{\Gamma(1 - \frac{i\lambda_r}{2})\Gamma(1 + \frac{i\lambda_r}{2})}{\|N^{\lambda_r}\|_2^2} \langle \mathcal{N}^{\iota}, g \rangle, \quad (9.13)$$

where $b_p(r, \iota)$ denotes the $p$th Fourier coefficient of the series (4.18) for $\mathcal{N}^{\iota}$. The series (9.13) is absolutely convergent as a consequence of Proposition 6.2, the proof of which shows that the insertion of the bounded factor $\Phi(-i\lambda_r) (b_p(r, \iota))^{2N}$ does not change this state of affairs.

To isolate, as much as is possible, the role of an eigenvalue $\frac{1 + \lambda_r^2}{4}$ of the automorphic Laplacian, we take

$$\Phi_N(-i\lambda) = \exp\left(-\pi N\beta(\lambda - \lambda_r)^2\right), \quad (9.14)$$

with a small positive constant $\beta$ to be chosen later, $N$ being the large integer already so denoted.

The topology of $\mathcal{S}(\mathbb{R}^2)$ can be defined by a “directed” family of seminorms of the kind $\sum_j \sup |P_j(x, \frac{\partial}{\partial x}, \xi, \frac{\partial}{\partial \xi}) h|$, where the $P_j$’s make up a
finite collection of ordered monomials in the non-commuting operators indicated, of degrees $\leq A$: such a semi-norm will be said to be of degree $\leq A$.

**Proposition 9.4.** Assume that the semi-norm $|h|$ present in (9.8) is of degree $\leq A$. Given $\beta$, one has for $h \in \mathcal{S}(\mathbb{R}^2)$ and $g = P_2(2i\pi \mathcal{E}) h$ the estimate

$$\left| \langle \Phi_N(2i\pi \mathcal{E}) \left( T_p^{\text{dist}} \right)^{2N} \mathfrak{B}^{\text{disc}}, g \rangle \right| \leq 2^{2N} p^{N\epsilon} \exp(\pi(A + 1)^2 N\beta) \left\| h \right\|$$

for some new continuous norm $\left\| \right\|$ on $\mathcal{S}(\mathbb{R}^2)$.

**Proof.** Using (9.11), we write

$$\langle \Phi_N(2i\pi \mathcal{E}) \left( T_p^{\text{dist}} \right)^{2N} \mathfrak{B}^{\text{disc}}, g \rangle = \langle \left( T_p^{\text{dist}} \right)^{2N} \mathfrak{B}^{\text{disc}}, \Phi_N(-2i\pi \mathcal{E}) g \rangle,$$

and $g_1 = \Phi_N(-2i\pi \mathcal{E}) g$ is given as

$$g_1(x, \xi) = \int_{-\infty}^{\infty} \Xi_N(t) g \left( e^{2\pi t} x, e^{2\pi t} \xi \right) e^{2\pi t} dt,$$

with

$$\Xi_N(t) = \int_{-\infty}^{\infty} e^{2i\pi M \Phi_N(i\lambda)} d\lambda = (N\beta)^{-\frac{1}{2}} \exp \left( -\frac{\pi t^2}{N\beta} \right) e^{2i\pi \lambda t}$$

$$= e^{2i\pi \lambda t} (N\beta)^{-\frac{1}{2}} \exp \left( -\frac{\pi t^2}{N\beta} \right).$$

(9.18)

Since the operator $2i\pi \mathcal{E}$ commutes with $T_p^{\text{dist}}$, the link between $g$ and $g_1$ provided by (9.17) connects also $h$ and $h_1 = \Phi_N(-2i\pi \mathcal{E}) h$. An application of any of the 4 operators $x, \frac{\partial}{\partial x}, \xi, \frac{\partial}{\partial \xi}$ to a function evaluated at $(e^{2\pi t} x, e^{2\pi t} \xi)$ may lead to a loss by a factor $e^{2\pi |t|}$ at the most, so that

$$\left\| h_1 \right\| \leq \left\| h \right\| \times \int_{-\infty}^{\infty} (N\beta)^{-\frac{1}{2}} \exp \left( -\frac{\pi t^2}{N\beta} \right) e^{2\pi(A + 1)|t|} dt$$

$$\leq \left\| h \right\| \times \exp \left( \pi N\beta(A + 1)^2 \right)$$

(9.19)

for some new norm $\left\| \right\|$, still of the kind recalled immediately after (9.14). The estimate (9.15) follows from this inequality, (9.16) and (9.8).

**Theorem 9.5.** Assume that the conjecture (8.8) holds. Then, given a pair $\chi, \lambda$, the character $\chi$ has to be unitary if the distribution $\mathfrak{N}_{\chi,i\lambda}$ is modular. In other words, given a Hecke eigenform $N_{\tau,i}$ and a prime $p$, and setting...
Let $T_p \mathcal{N}_t^r = b_p(r, t) \mathcal{N}_t^r$, one has the inequality $|b_p(r, t)| \leq 2$.

**Proof.** In what follows, the fixed number $A$ is the one (originating from (9.15)) present in Proposition 9.4. Our aim is to prove that, given $r, t$ and $\delta_0 > 1$, one must have $|b_p(r, t)| \leq 2\delta_0$. Assuming $|b_p(r, t)| \geq 2$, let $\alpha$ be such that

$$|b_p(s, \kappa)| \leq \alpha |b_p(r, t)|$$

for every $s, \kappa$. Next, we choose positive numbers $\varepsilon, \beta, \eta$ in this order such that

$$p^{\frac{\varepsilon}{2}} < \sqrt{\delta_0}, \quad p^{\frac{\varepsilon}{2}} \exp \left( \frac{\pi (A + 1)^2 \beta}{2} \right) \leq \delta_0, \quad \alpha \exp \left( -\frac{\pi \beta \eta^2}{2} \right) \leq \frac{1}{2}. \quad (9.21)$$

We note that there is only a finite number of eigenvalues $-i\lambda_s$ of the automorphic Euler operator (including $-i\lambda_r$) such that $|\lambda_s - \lambda_r| \leq \eta$. To single out $\mathcal{N}_t^r$ within the finite collection of Hecke distributions $\mathcal{N}_t^{s, \kappa}$ such that $|\lambda_s - \lambda_r| \leq \eta$, we use Lemma 9.1, obtaining a function $h \in \mathcal{S}(\mathbb{R}^2)$ such that

$$\langle \mathcal{N}_t^r, h \rangle = \lambda_r^2 (1 + \lambda_r^2) \quad \text{according to Lemma 7.4.}$$

We use the series (9.13), in which we substitute a general $(s, \kappa)$ for $(r, t)$, obtaining

$$\langle \Phi(2i\pi E) \left( T_p^{\text{dist}} \right)^{2N} \mathfrak{N}_t^{s} \mathfrak{B}_t^{\text{disc}}, g \rangle = \sum_{\lambda_s \in \mathbb{Z}^\kappa} \Phi(-i\lambda_s) (b_p(s, \kappa))^2 R_N(p, s, \kappa; g), \quad (9.22)$$

with

$$R_N(p, s, \kappa; g) = \frac{1}{2} \frac{\Gamma(1 - i\lambda_s) \Gamma(1 + i\lambda_s)}{||N^{s, \kappa}||^2} \langle \mathfrak{N}_t^{s, \kappa}, g \rangle. \quad (9.23)$$

The term $R_N(p, s, \kappa; g)$ is the general term of the series on the right-hand side of (6.5), the absolute convergence of which was established in Proposition 6.2: the fact that $\Phi_N(-i\lambda_s) \leq 1$ then proves the convergence of the series (9.22). We now analyze separately the sum of terms of this series for which $|\lambda_s - \lambda_r| > \eta$ or $|\lambda_s - \lambda_r| \leq \eta$.
Using the facts that $|b_p(s, \kappa)| \leq \alpha |b_p(r, \iota)|$ and that, in the first case, 
$\Phi_N(-i\lambda_s) \leq \exp \left( -\pi N\beta \eta^2 \right)$, we obtain if using the last constraint (9.21)

$$\Phi_N(-i\lambda_s) |b_p(s, \kappa)|^{2N} \leq e^{-\pi N\beta \eta^2} \alpha^{2N} |b_p(r, \iota)|^{2N} \leq \left( \frac{1}{2\alpha} \right)^{2N} \alpha^{2N} |b_p(r, \iota)|^{2N} = 2^{-2N} |b_p(r, \iota)|^{2N}. \tag{9.24}$$

It thus follows from Proposition 6.2 that

$$\sum_{s, \kappa} \Phi(-i\lambda_s) |b_p(s, \kappa)|^{2N} |R_N(p, s, \kappa; g)| \leq \left( \frac{1}{2} |b_p(r, \iota)| \right)^{2N} \parallel g \parallel_1 \tag{9.25}$$

for some continuous norm $\parallel \parallel_1$, unrelated to the norm $\parallel \parallel$ and independent of $N$.

For the pairs $(s, \kappa)$ with $|\lambda_s - \lambda_r| \leq \eta$, we recall the choice of $h$ made immediately after (9.21), from which it follows that the only contributing Hecke eigenform is $\mathcal{Y}^{s, \iota}$ and the corresponding term of the series (9.22) is

$$\Phi(-i\lambda_r) (b_p(r, \iota))^{2N} R_N(p, r, \iota; g) = \frac{1}{2} (b_p(r, \iota))^{2N} \frac{\Gamma(1 - \frac{i\lambda_r}{2})\Gamma(1 + \frac{i\lambda_r}{2})}{\|\mathcal{Y}^{r, \iota}\|^2} \langle \mathcal{Y}^{r, \iota}, g \rangle$$

$$= \frac{1}{2} (b_p(r, \iota))^{2N} \frac{\Gamma(1 - \frac{i\lambda_r}{2})\Gamma(1 + \frac{i\lambda_r}{2})}{\|\mathcal{Y}^{r, \iota}\|^2} \lambda_r^2 (1 + \lambda_r^2). \tag{9.26}$$

Comparing the results of the last two equations, we see that, for $N$ large enough, the contribution of the series (9.25) is less than half that of the term (9.26) we are interested in. It thus follows from (9.15) and (9.21) that one has

$$\frac{1}{4} (b_p(r, \iota))^{2N} \frac{\Gamma(1 - \frac{i\lambda_r}{2})\Gamma(1 + \frac{i\lambda_r}{2})}{\|\mathcal{Y}^{r, \iota}\|^2} \lambda_r^2 (1 + \lambda_r^2) \leq 2^{2N} p^{N\varepsilon} \exp(\pi(A + 1)^2 N\beta) \parallel h \parallel \leq 2^{2N} \delta_0^{2N} \parallel h \parallel. \tag{9.27}$$

Letting $N \to \infty$, we are done.

□
10. The Selberg conjecture

We consider the Hecke subgroup \( \Gamma_0(M) \) of \( SL(2, \mathbb{Z}) \) consisting of matrices \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that \( c \equiv 0 \pmod{M} \), in order to benefit from the detailed structure of the space \( L^2(\Gamma_0(M) \setminus \mathbb{H}) \) as given in [3]. The Selberg conjecture is the fact that the spectrum of \( \Delta \) in this Hilbert space coincides with \( \left[ \frac{1}{4}, \infty \right) \). It amounts to the fact that, given any Hecke eigenform \( \mathcal{N} \), an eigenfunction of \( \Delta \) for the eigenvalue \( \frac{1}{4} - \nu^2 \), \( \nu \) is pure imaginary: since this is a statement concerning individual Hecke distributions, one may assume, changing \( M \) to a factor of \( M \) if necessary, that \( \mathcal{N} \) is a so-called new Hecke eigenform, so that it is still possible to normalize it by the fact that its first Fourier coefficient \( b_1 \), as characterized by the expansion [3, p.226]

\[
\mathcal{N}(x + iy) = y^{\frac{1}{2}} \sum_{k \neq 0} b_k K_{\nu/2}(2\pi |k| y) e^{2i\pi kx},
\]

is 1.

Such a modular form can be lifted to the distribution (4.19)

\[
\mathfrak{N}(x, \xi) = \frac{1}{2} \sum_{k \neq 0} b_k |k|^{\frac{\nu}{2}} |\xi|^{-1-\nu} \exp\left(\frac{2i\pi kx}{\xi}\right),
\]

where \( \nu \) is any of the two square roots of \( \nu^2 \) if \( \text{Re} \left(\nu^2\right) < 0 \): we may assume that \( \text{Re} \nu > 0 \) if \( \text{Re} \left(\nu^2\right) > 0 \). One has then \( \Theta \mathfrak{N} = \mathcal{N} \).

Fixing \( M \) and \( \Gamma = \Gamma_0(M) \), we find in [3, p.227] the Poincaré series, in which \( j = 1, 2, \ldots \),

\[
U_1\left(z, j + \frac{1}{2}\right) = \frac{1}{2} \sum_{\tau \in \Gamma_\infty \setminus \Gamma} \left(\text{Im} \left(\tau . z\right)\right)^{j + \frac{1}{2}} \exp(2i\pi (\tau . z)) = \frac{1}{2} \sum_{(n \overline{m}, n_1) \in \Gamma/\Gamma_\infty} \left(\frac{\text{Im} z}{| - mz + n|^{2}}\right)^{j + \frac{1}{2}} \exp\left(2i\pi \frac{m_1 z - n_1}{-mz + n}\right).
\]

We have specialized in the cusp \( \infty \) of \( \Gamma_0(M) \setminus \mathbb{H} \) and specialized to the value 1 the coefficient in the exponent (compare (6.9)).

Just as in (6.7), (6.8), one obtains that the function

\[
(4\pi)^j \frac{\Gamma(\frac{1}{2} + j)}{\Gamma(\frac{1}{2})} \times U_1\left(z, j + \frac{1}{2}\right)
\]

is the image under \( \Theta \) of the distribution (of
course with $\Gamma = \Gamma_0(M)$

$$\mathfrak{B}^j = \frac{1}{2} \sum_{g \in \Gamma / \Gamma_\infty} (s_1^j)^j \circ g^{-1}. \quad (10.4)$$

Note that it was essential to start with $s_1^1$, not with $s_1^a$ (cf. (6.1)) for another value of $a \in \mathbb{Z}$, so that $\mathfrak{B}^j$ should be $\mathcal{F}^{\text{symm}}$-invariant: this explains the necessity of taking for $N$ a new Hecke eigenform, more precisely of choosing $M$ so that it should become one.

Now, the spectral decomposition of the Poincaré series $U_1(z, j + \frac{1}{2})$ is given in [3, p.248] as the equation

$$U_1(z, j + \frac{1}{2}) = \sum_k \left( \frac{\mathcal{N}_k | U_1(\ast, j + \frac{1}{2}) | \mathcal{N}_k(z)}{|| \mathcal{N}_k ||^2} \right) + \frac{1}{4i\pi} \sum_c \int_{\text{Re} \nu = 0} \left( E_c(\ast, \frac{1-\nu}{2}) | U_1(\ast, j + \frac{1}{2}) \right) E_c(\ast, \frac{1-\nu}{2})(z),$$

the ingredients of which are as follows. The functions $\mathcal{N}_k$ are a complete set of new Hecke eigenforms, normalized in Hecke’s way, the letter $c$ runs through a complete set of inequivalent cusps, and $E_c(\ast, \frac{1-\nu}{2})$ is a non-holomorphic modular form (not a cusp-form) of Eisenstein type. There is no need to be more explicit regarding the Eisenstein terms since the integral term on the right-hand side of (10.5) is the spectral projection of $U_1(\ast, j + \frac{1}{2})$ corresponding to the continuous part of the spectrum of $\Delta$.

**Proposition 10.1.** Given $j = 1, 2, \ldots$, the $\Gamma$-automorphic distribution $q^{2\pi \xi} \mathfrak{B}^j$ remains for $q \in ]0, \infty[$ in a bounded subset of $S'(\mathbb{R}^2)$.

**Proof.** We reproduce (7.4), (7.5), with the understanding that $SL(2, \mathbb{Z})$ has been replaced by $\Gamma = \Gamma_0(M)$, so that the class in $\Gamma / \Gamma_\infty$ of the matrix $\left( \begin{smallmatrix} n & n_1 \\ m & m_1 \end{smallmatrix} \right)$ is characterized by the pair $n, m$ with $m \equiv 0 \mod M$. Forgetting, as in (7.27), a constant factor of absolute value 1, we write

$$\hat{I}_{n,m} (q^{2\pi \xi} h) = q \int_{-\infty}^{\infty} h \left( q(nx - \frac{1}{m}, qm x \right) e^{2i\pi x} \ dx \\
= \int_{-\infty}^{\infty} h(nx - \frac{q}{m}, mx) \exp \left( \frac{2i\pi x}{q} \right) \ dx. \quad (10.6)$$
We can also write instead, using (3.4),
\[
\hat{I}_{n,m}(q^{2\pi}h) = \int_{\mathbb{R}^3} (\mathcal{F}_{\text{symp}} h)(y, \eta) \exp \left( \frac{2i\pi x}{q} \right) \exp \left[ 2i\pi \left( nx - \frac{q}{m} \eta - ym \xi \right) \right] dx \, dy \, d\eta.
\]
(10.7)
Since, for \( m \neq 0 \),
\[
\int_{-\infty}^{\infty} \exp \left( 2i\pi (n - my + \frac{1}{q}) x \right) dx = \delta(n \eta - my + \frac{1}{q}) = \frac{1}{|m|} \delta \left( y - \frac{n\eta}{m} - \frac{1}{qm} \right),
\]
(10.8)
one obtains if \( m \neq 0 \)
\[
\hat{I}_{n,m}(q^{2\pi}h) = \frac{1}{|m|} \int_{-\infty}^{\infty} (\mathcal{F}_{\text{symp}} h) \left( \frac{n\eta}{m} + \frac{1}{qm}, \eta \right) d\eta.
\]
(10.9)
Imitating Lemma 7.2, we write
\[
\left( \frac{n\eta}{m} + \frac{1}{qm} \right)^2 + \eta^2 = a\eta^2 + 2b\eta + c,
\]
(10.10)with
\[
a = \frac{m^2 + n^2}{m^2}, \quad b = \frac{n}{qm^2}, \quad c = \frac{1}{q^2m^2}, \quad ac - b^2 = \frac{1}{q^2m^2}
\]
(10.11)Then, provided that \( \alpha < \frac{1}{2} \), we obtain, as in (7.11),
\[
\left| \hat{I}_{n,m}(q^{2\pi}h) \right| \leq C |m|^{-1} a^{-\alpha} (ac - b^2)^{\alpha - \frac{1}{2}} = C (m^2 + n^2)^{-\alpha} q^{1 - 2\alpha}.
\]
(10.12)
We must improve the first factor to ensure the summability of the \((m, n)\)-series, while improving slightly the second factor at the same time.
Setting \( k = \mathcal{F}_{\text{symp}} h \), one has with \( k_1'(x, \xi) = \frac{\partial k}{\partial x} \)
\[
(2i\pi k) \left( \frac{n\eta}{m} + \frac{1}{qm}, \eta \right)
= \left( 1 + \eta \frac{d}{d\eta} \right) \left[ k \left( \frac{n\eta}{m} + \frac{1}{qm}, \eta \right) \right] + \frac{1}{qm} k_1' \left( \frac{n\eta}{m} + \frac{1}{qm}, \eta \right)
= \frac{d}{d\eta} \left[ \eta k \left( \frac{n\eta}{m} + \frac{1}{qm}, \eta \right) \right] + \frac{1}{qm} k_1' \left( \frac{n\eta}{m} + \frac{1}{qm}, \eta \right).
\]
(10.13)
From (10.9), one obtains
\[
\hat{I}_{n,m}(2i\pi k q^{2\pi}h) = \frac{1}{q |m|^2} \int_{-\infty}^{\infty} k_1' \left( \frac{n\eta}{m} + \frac{1}{qm}, \eta \right) d\eta.
\]
(10.14)
With respect to (10.9), we have gained a factor $\frac{1}{q_{m}}$. Doing this integration by parts twice, we gain a factor $\frac{1}{q_{m}^{2}}$. If substituting for the equation (10.6), obtained from the first equation of the pair (7.27), that obtained from the second equation of the pair, we obtain in a similar way a gain by a factor $\frac{1}{q_{n}^{2}}$. Hence, for $\alpha < \frac{1}{2}$,

$$\left| I_{n,m} (q^{2} \pi E (2 \pi E)^{2} h) \right| \leq C (m^{2} + n^{2})^{\alpha - 1} q^{-1-2\alpha}.$$  (10.15)

The $m, n$-series

$$\langle \mathfrak{B}^{1}, q^{2} \pi E h \rangle = \frac{1}{2} \sum_{m \equiv 0 \mod N, n} I_{n,m} ((\pi E)^{2} q^{2} \pi E h)$$  (10.16)

is convergent if $\alpha > 0$, and its sum is a $O(q^{-1-2\alpha})$.

We have obtained that $q^{2} \pi E \mathfrak{B}^{1}$ remains in a bounded subset of $S'(\mathbb{R}^{2})$ for $q \geq 1$, As $q^{-2} \pi E \mathfrak{B}^{1} = q^{-2} \pi E \mathfrak{F}_{\text{symp}} \mathfrak{B}^{1} = \mathfrak{F}_{\text{symp}} (q^{2} \pi E \mathfrak{B}^{1})$, the proof of Proposition 10.1 is complete.

\[\square\]

**Theorem 10.2.** (Selberg’s conjecture) Given $M$, there is no cusp-form $N$ for the group $G = \Gamma_{0}(M)$ such that $\Delta N = \frac{1-\nu^{2}}{4} N$ with $\Re (\nu^{2}) > 0$.

**Proof.** The proof follows the same lines as that of our attempt of the Ramanujan case. But it is Proposition 10.1 that takes the place of Theorem 9.2, the family of powers of the Hecke operator $T_{p}^{\text{dist}}$ being replaced by the family of operators $q^{2} \pi E$ with $q \in [0, \infty[$; this entails a considerable simplification.

In close analogy with what was done in Section 6, introduce the family of eigenvalues $\frac{1-\nu^{2}}{4}$, $r = 1, 2, \ldots$, corresponding to which there is at least one new Hecke eigenform $N^{r}$: again, given $r$, we let $(N^{r}, \iota)$ be a complete orthogonal set of Hecke-normalized new eigenforms corresponding to this eigenvalue of $\Delta$ in $L^{2}(\Gamma \backslash \mathbb{H})$.

As a consequence of [3, p.226] and [3, p.244], one has

$$\frac{(4\pi)^{j}}{\pi^{j}} \frac{\Gamma(\frac{1}{2} + j)}{\Gamma(1)} U_{1} (\ast, j + \frac{1}{2}) = \frac{1}{2} \sum_{r \geq 1} \frac{\Gamma (j - \frac{\nu}{2}) \Gamma (j + \frac{\nu}{2})}{\| N^{r} \|^{2}} N^{r} \ast + \ldots,$$

(10.17)
where the dots stand for a linear combination of Eisenstein series (relative to the various cups of the domain) and make up the continuous part of the spectral decomposition of the left-hand side.

The two Hecke distributions $\mathfrak{N}^{\pm r,\iota,\iota}$ being the lifts of $N^{r,\iota,d}$ according to (10.2), define the distribution

$$\left(\mathfrak{B}^{\lambda}\right)^{\text{disc}} = \frac{1}{2} \sum_{r \neq 0, \iota} \frac{\Gamma\left(j - \frac{\nu}{r}\right) \Gamma\left(j + \frac{\nu}{r}\right)}{\|N^{r,\iota}\|^2} \mathfrak{N}^{r,\iota} r^{\lambda}.$$  

(10.18)

It is invariant under $\mathcal{F}\text{sym}_\mathfrak{C}$ and its image under $\Theta$ coincides with the right-hand side of (10.17). One thus has

$$\mathfrak{B}^{\lambda} = \left(\mathfrak{B}^{\lambda}\right)^{\text{disc}} + \ldots,$$  

(10.19)

where the dots stand for a linear combination of Eisenstein distributions $E^{\lambda,\lambda}$, $\lambda \in \mathbb{R}$ or transforms thereof under the linear transformations of $\mathbb{R}^2$ associated to the cusps of the domain.

From Proposition 10.1 and the fact that $q^{2\pi i} E^{\lambda,\lambda} = q^{-i\lambda} E^{\lambda,\lambda}$, it follows, using also the analogue of Proposition 6.2, that the distributions $q^{2\pi i} \left(\mathfrak{B}^{\lambda}\right)^{\text{disc}}$ make up a bounded set in $\mathcal{S}'(\mathbb{R}^2)$ for $q \in]0, \infty[$. The same is true if one replaces $\left(\mathfrak{B}^{\lambda}\right)^{\text{disc}}$ by the part of (10.18) made from the lifts of all new Hecke eigenforms $N^{r,\iota,d}$ corresponding to eigenvalues of $\Delta$ not less than $\frac{1}{4}$. We are left with a finite linear combination of terms $q^{2\pi i} \mathfrak{N}^{r,\iota} = q^{-\nu r} \mathfrak{N}^{r,\iota}$, which remains in a bounded subset of $\mathcal{S}'(\mathbb{R}^2)$ for $q \in]0, \infty[$. Given $\nu_1, \iota_1$, the result of testing the term $q^{2\pi i} \mathfrak{N}^{r_1,\iota_1}$ on an appropriate function in $\mathcal{S}(\mathbb{R}^2)$ is, just as well, bounded for $q \in]0, \infty[$ as a result of Lemma 9.1. Hence, $\nu_1$ is pure imaginary, and the eigenvalue $\frac{1 - \nu_1^2}{4}$ is at least $\frac{1}{4}$.  

$\square$
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