A realization of Yangian and its applications to the bi-spin system in an external magnetic field

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Abstract

Yangian $Y(sl(2))$ is realized in the bi-spin system coupled with a time-dependent external magnetic field. It is shown that $Y(sl(2))$ generators can describe the transitions between the “spin triplet” and the “spin singlet” that evolve with time. Furthermore, new transition operators between the states with Berry phase factor and those between the states of Nuclear Magnetic Resonance (NMR) are presented.

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1. Introduction

Yangian Algebras were established by Drinfeld [1, 2] based on the investigation of Yang-Baxter equation. In recent years, many works in studying Yangian and its applications have been made, including the Yangian symmetry in quantum integrable models (such as Haldane-Shastry model [3], Calogero-Sutherland model [4], and Hubbard model [5, 6]) and the realization of Yangian in quantum mechanics [7, 8, 9]. In quantum mechanics, the Yangian associated with $sl(2)$ called $Y(sl(2))$ has been realized in angular momentum quantum mechanics systems [7], hydrogen atom [8, 9] and other systems. The sense of Yangian generators’ transition operator has also been known. This is natural because Yangian algebras belong to hopf algebras and regard Lie algebras as their subalgebras.

It was pointed out in [7] that, $Y(sl(2))$ can be constructed in a bi-spin system with spin $\hat{S}^1$ and $\hat{S}^2$ ($\hat{S}^k$ is the $k$th spin operator). It has been known that the Yangian generators $\{\hat{I}, \hat{J}\}$ can realize the transitions between the spin triplet and the spin singlet, which are the bases of quantum states of the bi-spin system (each $S^k$ has spin-1/2) that does not evolve with time. An interesting problem arises that if this system evolves with time, such as is coupled with a time-dependent external magnetic field, what will happen for the transition operators. A physical picture of the transition between the states is useful to help us understanding this time-dependent problem. In this paper, we will consider this issue. First, we will illustrate that $Y(sl(2))$ generators $\{\hat{I}(t), \hat{J}(t)\}$ still play the role of transition operators. Then, we discuss the transitions between the states with Berry phase factor of this system under the adiabatic condition. And finally, using the combination of the $Y(sl(2))$ generators $\{\hat{I}(t), \hat{J}(t)\}$, the transition operators between the Nuclear Magnetic Resonance (NMR) states are also obtained.
2. Realization of $Y(sl(2))$ and transition operators for the bi-spin system
in an external magnetic field

$Y(sl(2))$ is formed by a set of operators $\{\hat{I}, \hat{J}\}$ obeying the commutation relations [1, 7]:

\[
[\hat{I}_\alpha, \hat{I}_\beta] = i\epsilon_{\alpha\beta\gamma} \hat{I}_\gamma, \\
[\hat{I}_\alpha, \hat{J}_\beta] = i\epsilon_{\alpha\beta\gamma} \hat{J}_\gamma \\
(\alpha, \beta, \gamma = 1, 2, 3), \\
[\hat{J}_\pm, [\hat{J}_3, \hat{J}_\pm]] = \hat{I}_\pm(\hat{J}_3 \hat{I}_\mp - \hat{I}_\pm \hat{J}_3), \\
[\hat{J}_3, [\hat{J}_+, \hat{J}_-]] = \hat{I}_3(\hat{I}_+ \hat{J}_- - \hat{J}_+ \hat{I}_-) .
\]

(1)

$\hat{I}$ stands for the generators of $sl(2)$. Hereafter for any operators $\hat{A}_\pm = \hat{A}_1 \pm i \hat{A}_2$ ($\hat{A} = \hat{I}, \hat{J}$) are understood. In a bi-spin system with spin $\hat{S}^1$ and $\hat{S}^2$, the $Y(sl(2))$ generators take the form [7]:

\[
\hat{I} \equiv \hat{S} = \hat{S}^1 + \hat{S}^2, \\
\hat{J} = \mu(\hat{S}^1 - \hat{S}^2) + \frac{ih}{2} \hat{S}^1 \times \hat{S}^2
\]

(2)

where $\mu$ and $h$ are arbitrary parameters. In this paper, we consider the case that the two spins are all equal to 1/2. Then direct calculation shows that the Yangian generators $\hat{J}_\alpha$ ($\alpha = \pm, 3$) describe the transitions between the spin triplet ($S = 1$)

\[
|X_{11}\rangle = |\uparrow\uparrow\rangle, \quad |X_{10}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad |X_{1-1}\rangle = |\downarrow\downarrow\rangle
\]

(3)

and the spin singlet ($S = 0$)

\[
|X_{00}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).
\]

(4)

The transition relations are:

\[
\hat{J}_+ |X_{1-1}\rangle = \sqrt{2}(\mu - \frac{h}{4}) |X_{00}\rangle, \\
\hat{J}_3 |X_{10}\rangle = (\mu - \frac{h}{4}) |X_{00}\rangle,
\]
\[ \hat{J}_{-} | X_{11} \rangle = -\sqrt{2}(\mu - \frac{h}{4}) | X_{00} \rangle, \]
\[ \hat{J}_{\pm} | X_{00} \rangle = -\sqrt{2}(\mu + \frac{h}{4}) | X_{11} \rangle, \]
\[ \hat{J}_3 | X_{00} \rangle = (\mu + \frac{h}{4}) | X_{10} \rangle, \]
\[ \hat{J}_{-} | X_{00} \rangle = \sqrt{2}(\mu + \frac{h}{4}) | X_{1-1} \rangle. \] (5)

Obviously the Lie algebra generators \( \hat{I}_{\pm} \) can only describe the transitions between the spin triplet, but \( \hat{J}_\alpha (\alpha = \pm, 3) \) can realize the transitions between the states with different Lie-algebra weights.

The Hamiltonian of the bi-spin system coupled with a time-dependent external field \( \mathbf{B}(t) \) reads
\[ \hat{H}(t) = -\frac{1}{2} \hat{S}^1 \cdot \hat{S}^2 - \gamma \mathbf{B}(t) \cdot \hat{S} \] (6)
where \( \gamma \) is gyromagnetic ratio. To study this system, we firstly write a simple hamiltonian describing the bi-spin system in a steady magnetic field as follows:
\[ \hat{H} = -\frac{1}{2} \hat{S}^1 \cdot \hat{S}^2 - g \hat{S}_3 \] (7)
where \( g \) is a constant. It is easy to see that the eigenstates of \( \hat{H} \) are still the spin singlet and the spin triplet. So Yangian generators \( \{ \hat{I}, \hat{J} \} \) can work well in this system. An interesting picture will appear if the magnitude of the magnetic field \( | \mathbf{B}(t) | \) is chosen to be a time-independent constant \( B_0 \). Comparing Eq.(7) with Eq.(6), we find that through introducing the following unitary transformation operator
\[ \hat{U}(t) = \prod_{i=1}^{2} \hat{U}^i(t), \]
\[ \hat{U}^i(t) = \left( \frac{2}{B_0 + B_3(t)B_0} \right)^{1/2} \mathbf{G}(t) \cdot \hat{S}^i \quad (i = 1, 2) \] (8)
where
\[ \mathbf{G}(t) = (B_1(t), B_2(t), B_3(t) + B_0), \] (9)
there exists the following transformation relation:
\[ \hat{H}(t) = \hat{U}(t) \hat{H} \hat{U}^{-1}(t) \] (10)
when we choose $g = -\gamma B_0$.

From Eq.(10), it is shown that the eigenvalues of $\hat{H}(t)$ are the same as those of $\hat{H}$, and the eigenstate $|\mathcal{X}_{jm}(t)\rangle$ of $\hat{H}(t)$ can be got from the transformation of the eigenstate $|X_{jm}\rangle$ of $\hat{H}$, i.e.,

$$|\mathcal{X}_{jm}(t)\rangle = \hat{U}(t)|X_{jm}\rangle \quad (jm = 11, 10, 1 - 1, 00)$$

(11)

where $|X_{jm}\rangle$ is the spin triplet (Eqs.(3)) or the spin singlet (Eq.(4)).

Through the unitary transformation $\hat{U}(t)$, the spin realization of $Y(sl(2))$ has the time-dependent generators

$$\hat{I}(t) = \hat{U}(t)\hat{I}\hat{U}^{-1}(t),$$

$$\hat{J}(t) = \hat{U}(t)\hat{J}\hat{U}^{-1}(t),$$

(12)

which can be verified to still satisfy the definition Eqs.(1) of $Y(sl(2))$. $\hat{J}_\alpha(t) (\alpha = \pm, 3)$ are the transition operators between the “spin triplet” ($S = 1$) $\{|\mathcal{X}_{11}(t)\rangle, |\mathcal{X}_{10}(t)\rangle, |\mathcal{X}_{1-1}(t)\rangle\}$ and the “spin singlet” ($S = 0$) $|\mathcal{X}_{00}(t)\rangle$ in a time-dependent magnetic field, and the transition relations take the same form as Eqs.(5):

$$\hat{J}_+(t) |\mathcal{X}_{1-1}(t)\rangle = \sqrt{2}(\mu - \frac{h}{4}) |\mathcal{X}_{00}(t)\rangle,$$

$$\hat{J}_3(t) |\mathcal{X}_{10}(t)\rangle = (\mu - \frac{h}{4}) |\mathcal{X}_{00}(t)\rangle,$$

$$\hat{J}_-(t) |\mathcal{X}_{11}(t)\rangle = -\sqrt{2}(\mu - \frac{h}{4}) |\mathcal{X}_{00}(t)\rangle,$$

$$\hat{J}_+(t) |\mathcal{X}_{00}(t)\rangle = -\sqrt{2}(\mu + \frac{h}{4}) |\mathcal{X}_{11}(t)\rangle,$$

$$\hat{J}_3(t) |\mathcal{X}_{00}(t)\rangle = (\mu + \frac{h}{4}) |\mathcal{X}_{10}(t)\rangle,$$

$$\hat{J}_-(t) |\mathcal{X}_{00}(t)\rangle = \sqrt{2}(\mu + \frac{h}{4}) |\mathcal{X}_{1-1}(t)\rangle.$$

(13)

In fact, the generators $\hat{I}(t)$ and $\hat{J}(t)$ vary with $\hat{U}(t)$ because the selection of $\hat{U}(t)$ is not exclusive. The choice of $\hat{U}(t)$ does not affect the action of generators of Yangian $Y(sl(2))$, so
we can choose the unitary operator $\hat{U}(t)$ as simple as possible. Taking advantages of the above results, we will give two physical applications in the following.

3. Transition operators between the states with Berry phase factor

It has been shown that Berry phase [10] play a fundamental and an important role in quantum mechanics in the past two decades. Berry phase can be verified by experiments [11], and it can not be neglected in many physics lessons. Recently, Berry phase of the bi-spin system coupled with an external magnetic field has been studied [12, 13]. Now we will give the transition operators between the states with Berry phase factor.

Consider the bi-spin system in a rotating magnetic field described by

$$B_1(t) = B_0 \sin \theta \cos \omega_0 t, \quad B_2(t) = -B_0 \sin \theta \sin \omega_0 t, \quad B_3(t) = B_0 \cos \theta$$

where constant $B_0$ is the strength (as referred to before) of the magnetic field. Substituting Eqs.(14) into Eqs.(8) and Eq.(11), we can give the eigenstates of Hamiltonian Eq.(6) that is the “triplet” ($S = 1$)

$$|\chi_{11}(t)\rangle = \frac{1}{2}(1 + \cos \theta) |X_{11}\rangle + \frac{1}{\sqrt{2}} e^{-i\omega_0 t} \sin \theta |X_{10}\rangle + \frac{1}{2}(1 - \cos \theta) e^{-2i\omega_0 t} |X_{1-1}\rangle,$$

$$|\chi_{10}(t)\rangle = \frac{1}{\sqrt{2}} \sin \theta e^{i\omega_0 t} |X_{11}\rangle - \cos \theta |X_{10}\rangle - \frac{1}{\sqrt{2}} \sin \theta e^{-i\omega_0 t} |X_{1-1}\rangle,$$

$$|\chi_{1-1}(t)\rangle = \frac{1}{2}(1 - \cos \theta) e^{2i\omega_0 t} |X_{11}\rangle - \frac{1}{\sqrt{2}} e^{i\omega_0 t} \sin \theta |X_{10}\rangle + \frac{1}{2}(1 + \cos \theta) |X_{1-1}\rangle,$$

and the “singlet” ($S = 0$)

$$|\chi_{00}(t)\rangle = - |X_{00}\rangle.$$  

(15)

Under the adiabatic condition, the state with Berry phase factor has the form:

$$|\psi_{jm}(t)\rangle = \exp \left\{ -\frac{i}{\hbar} E_{jm} t \right\} \exp \{ i\gamma_{jm}(t) \} |\chi_{jm}(t)\rangle$$

(17)
where $E_{jm}$ is the eigenvalue of Hamiltonian Eq.(6) and is given by

$$E_{11} = -\frac{1}{8} - \gamma B_0, \quad E_{10} = -\frac{1}{8}, \quad E_{1-1} = -\frac{1}{8} + \gamma B_0, \quad E_{00} = -\frac{3}{8}.$$  \hspace{1cm} (18)

On the other hand, the Berry phase $\gamma_{jm}(t)$ reads

$$\gamma_{11}(t) = \omega_0 (1 - \cos \theta) t, \quad \gamma_{1-1}(t) = -\omega_0 (1 - \cos \theta) t, \quad \gamma_{10}(t) = \gamma_{00}(t) = 0.$$  \hspace{1cm} (19)

By comparing Eqs.(13) with Eq.(17), we immediately find that

$$\hat{J}_+(t) \mid \psi_{1-1}(t) > = \sqrt{2}(\mu - \frac{h}{4}) \exp \left\{ -\frac{i}{\hbar} (E_{1-1} - E_{00}) t \right\} \exp \{ i \gamma_{1-1}(t) \} \mid \psi_{00}(t) >,$$

$$\hat{J}_3(t) \mid \psi_{10}(t) > = (\mu - \frac{h}{4}) \exp \left\{ -\frac{i}{\hbar} (E_{10} - E_{00}) t \right\} \mid \psi_{00}(t) >,$$

$$\hat{J}_-(t) \mid \psi_{11}(t) > = -\sqrt{2}(\mu - \frac{h}{4}) \exp \left\{ -\frac{i}{\hbar} (E_{11} - E_{00}) t \right\} \exp \{ i \gamma_{11}(t) \} \mid \psi_{00}(t) >,$$

$$\hat{J}_+(t) \mid \psi_{00}(t) > = -\sqrt{2}(\mu - \frac{h}{4}) \exp \left\{ -\frac{i}{\hbar} (E_{11} - E_{00}) t \right\} \exp \{ -i \gamma_{1-1}(t) \} \mid \psi_{11}(t) >,$$

$$\hat{J}_3(t) \mid \psi_{00}(t) > = (\mu + \frac{h}{4}) \exp \left\{ \frac{i}{\hbar} (E_{10} - E_{00}) t \right\} \mid \psi_{10}(t) >,$$

$$\hat{J}_-(t) \mid \psi_{00}(t) > = \sqrt{2}(\mu + \frac{h}{4}) \exp \left\{ \frac{i}{\hbar} (E_{1-1} - E_{00}) t \right\} \exp \{ -i \gamma_{1-1}(t) \} \mid \psi_{1-1}(t) >.$$  \hspace{1cm} (20)

We have solved the transition problems between the states with Berry phase factor by applying $\hat{J}_\alpha(t) \ (\alpha = \pm, 3)$.

4. Transition operators between the states of NMR

NMR is a very important experiment technique based on quantum mechanics [14, 15]. It has been made rapid progress since 1945. Very recently, NMR is used to realize the Geometric Quantum Computation [16, 17]. The motivation for this section is to find the transition operators between the NMR states. We choose the same magnetic field and the Hamiltonian as that in the former sections.
By solving the Schrödinger equation and utilizing the magnetic resonance condition (MNC)

\[
\omega_0 = \gamma B_3
\]  

(21)

we can get the states of NMR by choosing different initial states. These states are the time-dependent combination of the eigenstates of the Hamiltonian Eq.(6). The “triplet” \((S = 1)\) has the forms:

\[
|\phi_{11}(t)\rangle = a_1(t) \; |\chi_{11}(t)\rangle + a_2(t) \; |\chi_{10}(t)\rangle + a_3(t) \; |\chi_{1-1}(t)\rangle,
\]

\[
|\phi_{10}(t)\rangle = b_1(t) \; |\chi_{11}(t)\rangle + b_2(t) \; |\chi_{10}(t)\rangle + b_3(t) \; |\chi_{1-1}(t)\rangle,
\]

\[
|\phi_{1-1}(t)\rangle = c_1(t) \; |\chi_{11}(t)\rangle + c_2(t) \; |\chi_{10}(t)\rangle + c_3(t) \; |\chi_{1-1}(t)\rangle.
\]

(22)

In the process of calculating, the eigenvalues of deformed wave functions under MNC Eq.(21) have the exact values:

\[
E'_{11} = -\frac{1}{8} - \gamma B_1, \quad E'_{10} = -\frac{1}{8}, \quad E'_{1-1} = -\frac{1}{8} + \gamma B_1, \quad E'_{00} = -\frac{3}{8}.
\]

(23)

The time-dependent coefficients in Eqs.(22) are

\[
a_1(t) = \frac{1}{2} \exp \{-\frac{i}{\hbar} E'_{10} t\} \exp \{i \omega_0 t\}
\]

\[
\quad (\cos \theta \cos \omega_0 t + \cos \omega_0 t \cos \omega_1 t + i \sin \omega_0 t + i \cos \theta \sin \omega_0 t \cos \omega_1 t + i \sin \theta \sin \omega_1 t),
\]

\[
a_2(t) = \frac{1}{\sqrt{2}} \exp \{-\frac{i}{\hbar} E'_{10} t\} (\sin \theta \cos \omega_0 t + i \sin \theta \sin \omega_0 t \cos \omega_1 t - i \cos \theta \sin \omega_1 t),
\]

\[
a_3(t) = -\frac{1}{2} \exp \{-\frac{i}{\hbar} E'_{10} t\} \exp \{-i \omega_0 t\}
\]

\[
\quad (\cos \theta \cos \omega_0 t - \cos \omega_0 t \cos \omega_1 t - i \sin \omega_0 t + i \cos \theta \sin \omega_0 t \cos \omega_1 t + i \sin \theta \sin \omega_1 t),
\]

\[
b_1(t) = \frac{1}{\sqrt{2}} \exp \{-\frac{i}{\hbar} E'_{10} t\} \exp \{i \omega_0 t\} (i \cos \omega_0 t \sin \omega_1 t - \cos \theta \sin \omega_0 t \sin \omega_1 t + \sin \theta \cos \omega_1 t),
\]

\[
b_2(t) = - \exp \{-\frac{i}{\hbar} E'_{10} t\} (\sin \theta \sin \omega_0 t \sin \omega_1 t + \cos \theta \cos \omega_1 t),
\]

\[
b_3(t) = \frac{1}{\sqrt{2}} \exp \{-\frac{i}{\hbar} E'_{10} t\} \exp \{-i \omega_0 t\} (i \cos \omega_0 t \sin \omega_1 t + \cos \theta \sin \omega_0 t \sin \omega_1 t - \sin \theta \cos \omega_1 t),
\]

\[
c_1(t) = \frac{1}{2} \exp \{-\frac{i}{\hbar} E'_{10} t\} \exp \{i \omega_0 t\}
\]
From Eqs.(13), Eqs.(22) and Eqs.(25), after a rather long calculation we get the following relations:

\[
c_2(t) = \frac{1}{\sqrt{2}} \exp \left\{ -\frac{i}{\hbar} E_1' t \right\} (\sin \theta \cos \omega_0 t - i \sin \theta \sin \omega_0 t \cos \omega_1 t + i \cos \theta \sin \omega_0 t),
\]

\[
c_3(t) = -\frac{1}{2} \exp \left\{ -\frac{i}{\hbar} E_1' t \right\} \exp \left\{ -i \omega_0 t \right\}
\]

\[
(- \cos \theta \cos \omega_0 t - \cos \omega_0 t \cos \omega_1 t + i \sin \omega_0 t + i \cos \theta \sin \omega_0 t \cos \omega_1 t + i \sin \theta \sin \omega_1 t)
\]

where \( \omega_1 = \gamma B_1 \). The “singlet” \((S = 0)\) is

\[
| \phi_{00}(t) >= -\exp \left\{ -i E_{00}' t \right\} | \mathcal{X}_{00}(t) >.
\]

From Eqs.(13), Eqs.(22) and Eqs.(25), after a rather long calculation we get the following relations:

\[
\hat{\mathcal{J}}_-(t) | \phi_{11}(t) >= -\sqrt{2} a_1(t)(\mu - \frac{\hbar}{4}) \exp \{ i E_{00}' t \} | \phi_{00}(t) >,
\]

\[
\{ a_3(t) \hat{\mathcal{J}}_-(t) - a_1(t) \hat{\mathcal{J}}_+(t) + \sqrt{2} a_2(t) \hat{\mathcal{J}}_3(t) \} | \phi_{00}(t) >= \sqrt{2} (\mu + \frac{\hbar}{4}) \exp \left\{ -i E_{00}' t \right\} | \phi_{11}(t) >,
\]

\[
\hat{\mathcal{J}}_3(t) | \phi_{10}(t) >= b_2(t)(\mu - \frac{\hbar}{4}) \exp \{ i E_{00}' t \} | \phi_{00}(t) >,
\]

\[
\{ b_3(t) \hat{\mathcal{J}}_-(t) - b_1(t) \hat{\mathcal{J}}_+(t) + \sqrt{2} b_2(t) \hat{\mathcal{J}}_3(t) \} | \phi_{00}(t) >= \sqrt{2} (\mu + \frac{\hbar}{4}) \exp \left\{ -i E_{00}' t \right\} | \phi_{10}(t) >,
\]

\[
\hat{\mathcal{J}}_+(t) | \phi_{-1}(t) >= \sqrt{2} c_3(t)(\mu - \frac{\hbar}{4}) \exp \{ i E_{00}' t \} | \phi_{00}(t) >,
\]

\[
\{ c_3(t) \hat{\mathcal{J}}_-(t) - c_1(t) \hat{\mathcal{J}}_+(t) + \sqrt{2} c_2(t) \hat{\mathcal{J}}_3(t) \} | \phi_{00}(t) >= \sqrt{2} (\mu + \frac{\hbar}{4}) \exp \left\{ -i E_{00}' t \right\} | \phi_{-1}(t) >.
\]

From Eqs.(26), we can draw the conclusion that \( \hat{\mathcal{J}}_\alpha(t) (\alpha = \pm, 3) \) and its combination construct the transition operators between the NMR states (for simplicity, among the above relations, we have chosen the constants which can be modified to zero or 1 if it is possible).
5. Conclusions

In summary, we get a time-dependent realization of $Y(sl(2))$ in the bi-spin system coupled with a time-dependent external magnetic field. Although we can verify that $Y(sl(2))$ does not describe the symmetry of the system which we study because $\hat{J}(t)$ does not commute with $\hat{H}(t)$, we concentrate on the transition function of Yangian. We find that the generators $\{\hat{I}(t), \hat{J}(t)\}$ of $Y(sl(2))$ can describe a new picture of transition between two quantum states at any time. For briefness, we have neglected the part of transitions that are described by the Lie algebra operators. As far as we know, many interesting investigations have relations with the bi-spin model coupled with a time-dependent external magnetic field, such as Geometric Quantum Computation [16, 17], entanglement [18]. At last, we emphasis that Yangian algebras belong to hopf algebras and take Lie algebras as their subalgebras, so the Yangian operators can connect the physical states with different Lie-algebra weights. It’s reasonable to believe that the more interesting Yangian realization and the more useful physical applications should be found.

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