A CHARACTERIZATION OF ZONOIDS

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1. Introduction

A function defined on the $n$-dimensional sphere $\mathbb{S}^{n-1}$ will be called zonal if its value at a point $x \in \mathbb{S}^{n-1}$ depends only on the angle between $x$ and a fixed axis. Thus if $u$ is a unit vector in the direction of the fixed axis, then a function is zonal with respect to $u$ if its value at $x$ depends only on the standard inner product $\langle x, y \rangle$. A natural way to generate zonal functions is as follows. Let $f \in C(\mathbb{S}^{n-1})$, and fix a direction $u \in \mathbb{S}^{n-1}$. Let $(O(u), m)$ denote the subgroup of orthogonal transformations which keep the point $u$ fixed, equipped with the normalized Haar measure, $m$. Define

$$(S_u f)(x) = \int_{O(u)} f(Tx) \, dm(T)$$

Clearly, the function $S_u f$ is zonal. Applying this procedure to support functions leads to the following definition.

Definition 1.1. Suppose $u \in \mathbb{S}^{n-1}$ and $K$ is a centrally symmetric convex body, with support function $h_K$. The $u$-spin of $K$ is the convex body $K_u$ whose support function is $S_u h_K$.

For example, computing the $e_n$-spin of the unit cube in $\mathbb{R}^n$, gives:

$$(S_{e_n} h_{B_\infty^n})(x) = \int_{O(e_n)} \|Tx\|_1 \, dm(T) = c_n \left( \sum_{i=1}^n x_i^2 \right)^{1/2} + |x_n|,$$

($c_n$ is a positive number depending on $n$), which is the support function of a cylinder. This illustrates the choice of the word 'spin'.

Thus, with each body there is associated a system of rotation-bodies, one for each possible direction. Since $S_u f(u) = f(u)$ for each $u \in \mathbb{S}^{n-1}$ and every function $f$, a body is seen to be uniquely defined by its spins. The main result of this note is the following.

Theorem 1.2. A centrally symmetric convex body is a zonoid if and only if all its spins are zonoids.

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2. Preliminaries

$C^\infty(S^{n-1})$ is the Frechet space of functions on $S^{n-1}$ that have derivatives of every order. Elements of this space are called test functions. Its dual space, denoted $D(S^{n-1})$, is the space of distributions on $S^{n-1}$. If a subscript $e$ is added to any of the above spaces, it is to designate the subspace of even objects.

The cosine transform is the operator $C : C^\infty_e(S^{n-1}) \to C^\infty_e(S^{n-1})$ defined by:

$$(Cf)(x) = \int_{S^{n-1}} |\langle x, y \rangle| f(y) \, d\sigma_{n-1}(y),$$

where $\sigma_{n-1}$ is the normalized rotation-invariant measure on the sphere. It is well known that $C$ is a continuous bijection of $C^\infty_e(S^{n-1})$ onto itself, and that it can be extended by duality to a bi-continuous bijection of the dual space $D_e(S^{n-1})$. Hence, if $\rho$ is a distribution and $f \in C^\infty_e(S^{n-1})$ is a test function, then

$$\langle C\rho, f \rangle = \langle \rho, Cf \rangle$$

Since $C^\infty_e(S^{n-1})$ and its dual space, the even measures, are both naturally embedded in $D_e(S^{n-1})$, it makes sense to speak about the cosine transform of a measure, or of a continuous function. A fundamental connection between distributions and centrally symmetric convex bodies was discovered by Weil, in [1]. Weil proved that for every centrally symmetric convex body $K \subset \mathbb{R}^n$ there corresponds a unique distribution $\rho_K$ (called the generating distribution of $K$) such that $C\rho_K = h_K$, where $h_K$ is the support function of $K$. Suppose $f \in C^\infty(S^{n-1})$. Then for every direction $u \in S^{n-1}$, the function $S_u f$ also belongs to $C^\infty(S^{n-1})$. Therefore, $S_u$ can be defined to act on distributions by duality:

$$\langle S_u \rho, f \rangle = \langle \rho, S_u f \rangle, \quad \rho \in D(S^{n-1}), \quad f \in C^\infty(S^{n-1})$$

A routine verification shows that the transforms $S_u$ and $C$ commute on test functions, and therefore also as transforms of distributions.

3. Proof of Theorem 1.1

If $K$ is a zonoid, then $h_K = C\mu$ for some positive measure $\mu$, and for every $u \in S^{n-1}$,

$$S_u h_K = S_u (C(\mu)) = C(S_u \mu)$$

Since $S_u \mu$ is a positive measure for every $\mu$, every spin of $K$ is a zonoid. This proves the easy part of the theorem.

Suppose $K$ is a centrally symmetric convex body every spin of which is a zonoid. That is, for each direction $u \in S^{n-1}$ there exists a positive measure $\mu_u$ such that $S_u h_K = C\mu_u$. There exists a distribution $\rho$ such that $h_K = C\rho$, and so the assumption reduces to $C(S_u \rho) = C\mu_u$, where the commuting of $S_u$ and $C$ was used. Since $C$ is one-to-one, the distribution $\rho$ is seen to satisfy $S_u \rho = \mu_u$ for every $u \in S^{n-1}$. It therefore remains to prove:

**Lemma 3.1.** A distribution is positive if and only if $S_u \rho$ is positive for every $u \in S^{n-1}$. 

Proof. The “only if” part is obvious. Suppose \( \rho \in \mathcal{D}(\mathbb{S}^{n-1}) \) is a distribution such that \( S_u \rho \) is positive for every \( u \in \mathbb{S}^{n-1} \). Then \( \langle \rho, f \rangle \geq 0 \) for every positive zonal test function \( f \). Therefore if \( g = \sum_{i=1}^{m} a_i f_i \), where \( a_i \geq 0 \) and \( f_i \) are positive zonal test functions, then also \( \langle \rho, g \rangle \geq 0 \).

Choose a positive test function \( f \in C^\infty(\mathbb{S}^{n-1}) \) and write its Poisson integral:

\[
P_r f(x) = \int_{\mathbb{S}^{n-1}} \frac{1 - r^2}{\|x - ry\|^n} f(y) \, d\sigma_{n-1}(y)
\]

It is well known that \( \lim_{r \to 0^+} P_r f = f \) in the topology of \( C^\infty(\mathbb{S}^{n-1}) \). Therefore, if \( \langle \rho, P_r f \rangle \geq 0 \) for every \( 0 < r < 1 \) and every positive test function \( f \), then \( \rho \) is positive, by continuity.

Fix a sequence of convex combinations of Dirac measures of the form

\[
(1) \quad \nu_N = \sum_{i \geq 1} \lambda_i N \delta_{y_i}, \quad y_i \in \mathbb{S}^{n-1}
\]

such that \( \nu_N \to \sigma_{n-1} \) in the \( w^* \) topology of measures. If \( \varphi \) is a test function, then

\[
(2) \quad \int_{\mathbb{S}^{n-1}} \frac{1 - r^2}{\|x - ry\|^n} \varphi(y) \, d\nu_N(y) \to \int_{\mathbb{S}^{n-1}} \frac{1 - r^2}{\|x - ry\|^n} \varphi(y) \, d\sigma_{n-1}(y) = P_r \varphi(x)
\]

If \( F(x, y) \) is a continuous function on \( \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \) then for every test function \( \varphi \), the integrals \( \langle F(x, y) \varphi(y), \nu_N \rangle \) converge to the integral \( \langle F(x, y) \varphi(y), \sigma_{n-1} \rangle \) uniformly in \( x \). Therefore, the sequence of functions of the variable \( x \) in the l.h.s of (2) converge to the r.h.s in the topology of \( C^\infty(\mathbb{S}^{n-1}) \), because the function \( \|x - ry\|^{-n} \) for fixed \( y \) and \( 0 < r < 1 \) is \( C^\infty \) with respect to \( x \). Moreover, the distribution \( \rho \) is positive on every term of the l.h.s of (2), because for each \( y_i \) appearing in (1) one has

\[
\|x - ry_i\|^{-n} = \int_{O(y_i)} \|Tx - ry_i\|^{-n} \, dm(T), \quad \forall x
\]

so

\[
\langle \rho, \|x - ry_i\|^{-n} \rangle = \langle \rho, S_{y_i}(\|x - ry_i\|^{-n}) \rangle = \langle S_{y_i} \rho, \|x - ry_i\|^{-n} \rangle \geq 0
\]

Hence by continuity \( \rho \) is also positive of the r.h.s of (2) as well. It follows that \( \rho \) is a positive distribution.

It is well known that a positive distribution is in fact a positive measure, that is, it satisfies \( |\rho(f)| \leq \rho(1) \|f\|_{\infty} \) for every test function, where \( 1 \) is the constant function \( 1 \). Consequently, there is a unique extension of \( \rho \) to a bounded linear functional on \( C_c(\mathbb{S}^{n-1}) \), and so in this sense it represents a measure. This completes the proof of the theorem.

References

[1] Weil, W., Centrally symmetric convex bodies and distributions, Israel J. Math., 24, 352–367 (1976)