THE EQUALITY CASE IN CHEEGER’S AND BUSER’S INEQUALITIES ON RCD SPACES

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Abstract. We prove that the sharp Buser’s inequality obtained in the framework of RCD(1,∞) spaces by the first two authors [26] is rigid, i.e. equality is obtained if and only if the space splits isomorphically a Gaussian. The result is new even in the smooth setting.

We also show that the equality in Cheeger’s inequality is never attained in the setting of RCD(K,∞) spaces with finite diameter or positive curvature, and we provide several examples of spaces with Ricci curvature bounded below where these assumptions are not satisfied and the equality is attained.

As a consequence of the two main results, we obtain improved versions of Buser’s and Cheeger’s inequalities for RCD(K,N) spaces which are new even for smooth Riemannian manifolds of dimension N with Ricci curvature bounded below by K ∈ R.

1. Introduction

In the paper we consider a complete and separable metric space (X,d) endowed with a Borel measure m, finite on bounded sets. The triple (X,d,m) is called metric measure space, m.m.s. for short. The space of real-valued Lipschitz (resp. bounded Lipschitz, Lipschitz with bounded support, Lipschitz on bounded sets) functions over X will be denoted by Lip(X) (resp. Lip_0(X), Lip_{bs}(X), Lip_{loc}(X)). The slope of a function f : X → R at x ∈ X is defined by

\[ \text{lip}(f)(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y,x)}, \quad (1) \]

with the convention lip(f)(x) = 0 if x is an isolated point.

We introduce the following relevant definitions: in case m(X) < ∞ we set

\[ \lambda_1(X) = \inf \left\{ \frac{\int_X \text{lip}(f)^2 \, dm}{\int_X |f|^2 \, dm} : 0 \neq f \in \text{Lip}_{bs}(X), \int_X f \, dm = 0 \right\}, \quad (2) \]

If m(X) = ∞ instead we set

\[ \lambda_0(X) = \inf \left\{ \frac{\int_X \text{lip}(f)^2 \, dm}{\int_X |f|^2 \, dm} : 0 \neq f \in \text{Lip}_{bs}(X) \right\}. \quad (3) \]

When there is no risk of confusion we will drop the dependence on the ambient space writing λ₀ and λ₁. Under quite general assumptions on the m.m.s. (X,d,m), the quantities λ₀ and λ₁ correspond to the first two eigenvalues of the Laplace operator (see Theorem 2.16).

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Let \( A \subset X \) be a Borel set, the *perimeter* \( \text{Per}(A) \) is defined as:

\[
\text{Per}(A) := \inf \left\{ \liminf_{n \to \infty} \int_X \text{lip}(f_n) \, dm : f_n \in \text{Lip}_\text{loc}(X), f_n \to \chi_A \text{ in } L^1(X, m) \right\},
\]

where we denote by \( \chi_A : X \to \{0,1\} \) the indicator function of the set \( A \subset X \).

The *Cheeger constant* of the metric measure space \( (X,d,m) \) is defined as follows:

\[
h(X) := \begin{cases} 
\inf \left\{ \frac{\text{Per}(A)}{m(A)} : A \subset X \text{ Borel with } 0 < m(A) \leq m(X)/2 \right\} & \text{if } m(X) < \infty, \\
\inf \left\{ \frac{\text{Per}(A)}{m(A)} : A \subset X \text{ Borel with } 0 < m(A) < \infty \right\} & \text{if } m(X) = \infty.
\end{cases}
\]

(4)

In [23] Cheeger obtained the following celebrated inequality, now known as Cheeger’s inequality:

\[
\lambda_1 \geq \frac{1}{4} h(X)^2.
\]

(5)

The original result of Cheeger was in the framework of smooth and compact Riemannian manifolds, but the argument of the proof can be extended to general metric measure spaces (see [26, Appendix A]).

When \( X \) is a compact Riemannian manifold of dimension \( n \) and Ricci curvature that satisfies \( \text{Ric} \geq K, \ K \leq 0 \), Buser [20] proved that also the following *upper bound* for \( \lambda_1 \) in terms of \( h(X) \) holds:

\[
\lambda_1(X) \leq 2\sqrt{-(n-1)K}h(X) + 10h(X)^2.
\]

(6)

Thanks to a result of Ledoux [38], we also know that the constants in Buser’s inequality (6) can be chosen to be dimension-independent. More precisely, Ledoux proved the following inequality for all smooth connected Riemannian manifolds of finite volume:

\[
\lambda_1(X) \leq \max\{6\sqrt{-K}h(X), 36h(X)^2\}.
\]

(7)

Recently, De Ponti and Mondino [26] sharpened the aforementioned theorems of Buser and Ledoux by improving the constants in both the Buser-type inequalities (6)-(7) and by extending the results to (possibly non-smooth) \( \text{RCD}(K,\infty) \) spaces. Recall that \( \text{RCD}(K,\infty) \) spaces are (possibly non-smooth) metric measure spaces having Ricci curvature bounded below by \( K \in \mathbb{R} \) and no upper bound on the dimension, in a synthetic sense. More precisely, \( \text{RCD}(K,\infty) \) spaces are the sub-class of \( \text{CD}(K,\infty) \) spaces introduced in the seminal works of Sturm [48] and Lott-Villani [39] having the canonical energy functional (called "Cheeger energy") satisfying the parallelogram identity. The reader is referred to Section 2 for the precise definitions, and to [2] for a survey. The class of \( \text{RCD}(K,\infty) \) spaces was singled out by Ambrosio-Gigli-Savaré [8] (see also [6]) who developed a powerful calculus in this setting.

The subclass of \( \text{RCD}(K,\infty) \) spaces having an upper bound on the dimension by \( N \in [1,\infty) \) in a synthetic sense is denoted by \( \text{RCD}(K,N) \), see [30, 27, 11, 21]. Remarkable examples of \( \text{RCD}(K,\infty) \) spaces are pmgH-limits of Riemannian manifolds with Ricci curvature bounded below (the so-called Ricci limits) [32], finite dimensional Alexandrov spaces [44], weighted Riemannian manifolds with \( \infty \)-Bakry-Émery Ricci curvature bounded below by \( K \) [48], stratified spaces [16], (possibly singular) quotients of Riemannian manifolds with Ricci bounded below [29].
In order to state the outcomes of [26], we firstly set

\[
J_K(t) := \begin{cases} \frac{\pi^2}{2K} \arctan \left( \sqrt{e^{2Kt}} - 1 \right) & \text{if } K > 0, \\ \frac{2}{\pi^2} \sqrt{t} & \text{if } K = 0, \\ \frac{2}{\pi^2 K} \arctanh \left( \sqrt{1 - e^{2Kt}} \right) & \text{if } K < 0. 
\end{cases}
\] (8)

**Theorem 1.1** (Theorem 1.1 [26]). Let \((X, d, m)\) be an \(\text{RCD}(K, \infty)\) metric measure space for some \(K \in \mathbb{R}\), with \(m(X) < \infty\). Then

\[
h(X) \geq \sup_{t > 0} \frac{1 - e^{-\lambda_1 t}}{J_K(t)}. \tag{9}
\]

We refer again to [26] for a discussion on how to obtain more explicit bounds of \(\lambda_1\) in terms of \(h(X)\) starting from the inequality (9) (improving the constants in both (6)-(7)), and for an analogous result that can be applied to spaces with \(m(X) = \infty\) (in this case \(\lambda_1\) is replaced by \(\lambda_0\)).

As noticed in [26], another important consequence of Theorem 1.1 is that the inequality is sharp in the case \(K > 0\), as equality is achieved in the Gaussian space.

A first goal of the present work is to show that the inequality (9) is also rigid:

**Theorem 1.2.** Let \((X, d, m)\) be an \(\text{RCD}(K, \infty)\) metric measure space with \(K > 0\). Let us suppose that

\[
h(X) = \sup_{t > 0} \frac{1 - e^{-\lambda_1 t}}{J_K(t)}. \tag{10}
\]

Then

\[(X, d, m) \cong (Y, d_Y, m_Y) \times (\mathbb{R}, |\cdot|, \sqrt{K/(2\pi)} e^{-Kt^2/2} dt)
\]

for some \(\text{RCD}(K, \infty)\) space \((Y, d_Y, m_Y)\), where \(\cong\) denotes isomorphism as metric measure spaces.

Let us stress that the rigidity result of Theorem 1.2 is new even in the smooth setting of (possibly weighted) Riemannian manifolds.

Using the compactness of the class of \(\text{RCD}(K, N)\) spaces under measured Gromov-Hausdorff convergence [49, 32], the stability properties of \(\lambda_1\) and \(h(X)\) under such convergence [32, 9], and the fact that no \(\text{RCD}(K, N)\) space can split isomorphically a Gaussian (since the former is measure-doubling while the latter is not), we obtain the next dimensional improvement of (9) by a straightforward argument by contradiction.

**Corollary 1.3** (Dimensional improvement of Buser’s inequality). For every \(K > 0\) and \(N \in [1, \infty)\) there exists \(\varepsilon = \varepsilon(K, N)\) with the following property. For every \(\text{RCD}(K, N)\) space \((X, d, m)\), the following improved Buser’s inequality holds:

\[
h(X) \geq \sup_{t > 0} \frac{1 - e^{-\lambda_1 t}}{J_K(t)} + \varepsilon. \tag{11}
\]

Let us stress that Corollary 1.3 is new even for smooth Riemannian manifolds with dimension \(\leq N\) and Ricci curvature \(\geq K > 0\).

A second goal of the paper is to study the equality case in Cheeger’s inequality (5). In a series of now classical papers [18, 19, 20], Buser proved that equality in Cheeger’s inequality is never attained for compact Riemannian manifolds and gave compact examples where the equality is almost attained (up to an error \(\varepsilon > 0\) arbitrarily small), showing the sharpness
of (5) among smooth manifolds. Since in Buser’s examples the diameters of the spaces grow as the error $\varepsilon > 0$ decreases, it is natural to ask if Cheeger’s inequality can be improved once an upper bound on the diameter is assumed. Linked to this question, it is also natural to ask if the equality in (5) can be attained either in the non-compact or in the non-smooth compact setting. We prove that the answer is positive for the former and is negative for the latter, even under more general assumptions. More precisely, the second main result of the paper is the following:

**Theorem 1.4.** Let $(X, d, m)$ be an RCD$(K, \infty)$ metric measure space with $m(X) < \infty$ and $K \in \mathbb{R}$. Assume that $(X, d, m)$ admits an isoperimetric profile (this is always satisfied if $\text{diam}(X) < \infty$ or $K > 0$).

Then the equality in Cheeger’s inequality is never attained, i.e.

$$\lambda_1 > \frac{1}{4} h(X)^2.$$  

We refer to the preliminaries given below for the Definition 2.11 of isoperimetric profile, and to Theorem 4.1 for the case $m(X) = \infty$.

Along the same lines of the arguments for Corollary 1.3, one can obtain the next improvement of Cheeger’s inequality which is new even for smooth Riemannian manifolds with dimension $\leq N$, Ricci curvature $\geq K$ and diameter $\leq D$:

**Corollary 1.5 (Improved Cheeger’s inequality).** For every $K \in \mathbb{R}$, $N \in [1, \infty)$ and $D \in (0, \infty)$ there exists $\varepsilon = \varepsilon(K, N, D) > 0$ with the following property. For every RCD$(K, N)$ space $(X, d, m)$ with $\text{diam}(X) \leq D$, the following improved Cheeger’s inequality holds:

$$\lambda_1 \geq \frac{1}{4} h(X)^2 + \varepsilon.$$  

In the last part of the paper, we provide several examples of spaces with Ricci curvature bounded below where the equality in Cheeger’s inequality is attained.

We conclude the introduction by mentioning that rigidity results involving the spectrum of RCD spaces received a lot of attention in the recent literature, a non-exhaustive list follows: Ketterer [36] extended the validity of Obata’s rigidity theorem to the non-smooth setting, Cavalletti-Mondino [22] proved rigidity results involving the first eigenvalue of the $p$-Laplace operator with Neumann boundary conditions and Mondino-Semola [42] for Dirichlet boundary conditions, Gigli-Ketterer-Kuwada-Ohta [31] established the rigidity in the RCD$(K, \infty)$ spectral gap, Ambrosio-Brué-Semola [3] proved rigidity in the 1-Bakry-Émery inequality of RCD$(0, N)$ spaces, later extended to RCD$(K, \infty)$ spaces with $K > 0$ by Han [34].

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2. Preliminaries

**2.1. Curvature bounds and heat flow.** Unless otherwise stated, we assume $(X, d)$ is a complete and separable metric space endowed with a $\sigma$-finite, non-negative reference measure
m over the Borel σ-algebra \( \mathcal{B} \). We also assume \( \text{supp}(m) = X \) and the existence of \( x_0 \in X \), \( M > 0 \) and \( c \geq 0 \) such that

\[
m(B_r(x_0)) \leq M \exp(cr^2) \quad \text{for every } r \geq 0.
\]

Possibly enlarging \( \mathcal{B} \) and extending the measure \( m \), we can assume that \( \mathcal{B} \) is \( m \)-complete without loss of generality. We call \((X,d,m)\) a metric measure space, m.m.s for short.

We denote by \((\mathcal{P}_2(X),W_2)\) the space of probability measures on \( X \) with finite second moment endowed with the quadratic Kantorovich-Wasserstein distance \( W_2 \).

In the sequel we use the notation:

\[
D(\text{Ent}_m) := \{\mu \in \mathcal{P}_2(X) : \text{Ent}_m(\mu) \in \mathbb{R}\}.
\]

**Definition 2.1** \((\text{CD}(K,\infty)\) condition). Given \( K \in \mathbb{R} \), a m.m.s. \((X,d,m)\) verifies the \( \text{CD}(K,\infty) \) condition if for any \( \mu^0,\mu^1 \in D(\text{Ent}_m) \) there exists a \( W_2 \)-geodesic \((\mu_t)\) connecting \( \mu^0 \) and \( \mu^1 \) and such that, for any \( t \in [0,1] \),

\[
\text{Ent}_m(\mu_t) \leq (1-t)\text{Ent}_m(\mu_0) + t\text{Ent}_m(\mu_1) - \frac{K}{2}t(1-t)W_2^2(\mu_0,\mu_1).
\]

This class of spaces was introduced independently by Sturm \[48\] and Lott-Villani \[39\].

If \((X,d,m)\) satisfies \( \text{CD}(K,\infty) \) for \( K \in \mathbb{R} \), then, for every \( \alpha,\beta > 0 \), the metric measure space \((X,\alpha d,\beta m)\) satisfies the \( \text{CD}(K/\alpha^2,\infty) \) condition. In particular, it is not restrictive to assume that a \( \text{CD}(K,\infty) \) m.m.s. with \( m(X) < \infty \) is a probability space. Moreover, \( K > 0 \) implies \( m(X) < \infty \).

The **Cheeger energy** is defined as

\[
\text{Ch}_m(f) := \inf \left\{ \liminf_{n \to \infty} \frac{1}{2} \int_X \text{lip}(f_n)^2 \, dm : f_n \in \text{Lip}_{bs}(X), f_n \to f \text{ in } L^2(X,m) \right\}.
\]

As proved in \[7\], \( \text{Ch}_m(f) \) can be represented in terms of the so called *minimal weak upper gradient* \( |\nabla f| \) as

\[
\text{Ch}_m(f) = \frac{1}{2} \int_X |\nabla f|^2 \, dm.
\]

The Cheeger energy is a 2-homogeneous, lower semicontinuous and convex functional on \( L^2(X,m) \) whose proper domain

\[
W^{1,2}(X,d,m) := \{f \in L^2(X,m) : \text{Ch}_m(f) < \infty\}
\]

is a dense linear subspace of \( L^2(X,m) \). The space \( W^{1,2}(X,d,m) \) is Banach when endowed with the norm

\[
\|f\|_{W^{1,2}}^2 := \|f\|_{L^2}^2 + 2\text{Ch}_m(f).
\]

For all \( f \in W^{1,2}(X,d,m) \), the subdifferential \( \partial \text{Ch}_m(f) \) of the Cheeger energy at \( f \) is defined as

\[
\partial \text{Ch}_m(f) := \left\{ \ell \in L^2(X,m) : \int_X \ell(g-f) \, dm \leq \text{Ch}_m(g) - \text{Ch}_m(f) \quad \forall g \in L^2(X,m) \right\}.
\]
We denote by $(H_t)_{t \geq 0}$, and we refer to it as heat flow, the $L^2(X, \mathfrak{m})$-gradient flow of the Cheeger energy, i.e. for any $f \in L^2(X, \mathfrak{m})$ the map $t \mapsto H_tf$ is a locally Lipschitz map from $(0, \infty)$ to $L^2(X, \mathfrak{m})$ such that $H_tf \to f$ in $L^2(X, \mathfrak{m})$ as $t \to 0$ and

$$
\frac{d}{dt} H_tf = -\partial\text{Ch}_\mathfrak{m}(H_tf) \quad \text{for a.e. } t \in (0, \infty).
$$

(18)

We recall now the RCD condition, a reinforcement of the CD condition introduced by Ambrosio, Gigli and Savaré [8] (in case $\mathfrak{m}(X) < \infty$; see also [6] for the current axiomatization and the extension to $\sigma$-finite measures).

**Definition 2.2** (RCD$(K, \infty)$ condition). A metric measure space $(X, d, \mathfrak{m})$ satisfies the RCD$(K, \infty)$ condition, $K \in \mathbb{R}$, if it is CD$(K, \infty)$ and the Cheeger energy $\text{Ch}_\mathfrak{m}$ is quadratic.

The quadraticity of the Cheeger energy is equivalent to the fact that $W^{1,2}(X, d, \mathfrak{m})$ is Hilbert. Such an extra requirement singles out the “Riemannian” m.m.s structures out of the “possibly Finsler” ones.

The set $D(\Delta)$ is defined as the set of $f \in L^2(X, \mathfrak{m})$ such that $\partial \text{Ch}_\mathfrak{m}(f) \neq \emptyset$. In particular, $D(\Delta) \subset W^{1,2}(X, d, \mathfrak{m})$. For $f \in D(\Delta)$ we define $-\Delta f$ as the element of minimal $L^2(X, \mathfrak{m})$ norm in $\partial \text{Ch}_\mathfrak{m}(f)$.

We recall that on an RCD$(K, \infty)$ space the heat flow can be extended to a linear semigroup of contractions in $L^p(X, \mathfrak{m})$ for every $p \in [1, \infty)$. For every $f \in L^2(X, \mathfrak{m})$ and any $t > 0$ we have $H_tf \in D(\Delta)$. The maximum principle ensures that for any $C > 0$ and any $f \in L^2(X, \mathfrak{m})$ with $0 \leq f \leq C$ $\mathfrak{m}$-a.e., it holds $0 \leq H_tf \leq C$.

The semigroup $H_t$ admits an $\mathfrak{m} \otimes \mathfrak{m}$-measurable density kernel $\rho_t(x, y)$, so that

$$
H_tf(x) = \int_X f(y) \rho_t(x, y) \, d\mathfrak{m}(y), \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X, \quad \text{for any } f \in L^2(X, \mathfrak{m}).
$$

We also know (see [8, Theorem 6.1]) that, up to a suitable choice of $\mathfrak{m}$-a.e. representative, $H_tf$ belongs to $C(X) \cap L^\infty((0, \infty) \times X)$ whenever $f \in L^\infty(X, \mathfrak{m})$, where $C(X)$ denotes the set of real valued continuous functions over $X$. Moreover, for any $f \in L^2 \cap L^\infty(X, \mathfrak{m})$ and for every $t > 0$ the regularizing property of the heat flow yields $H_tf \in \text{Lip}_b(X)$ with the bound (sharp in the case $K > 0$) [26, Proposition 3.1]

$$
\| \nabla H_tf \|_\infty \leq \sqrt{\frac{2K}{\pi(e^{2kt} - 1)}} \| f \|_\infty \quad \text{if } K \neq 0,
$$

$$
\| \nabla H_tf \|_\infty \leq \frac{1}{\sqrt{\pi t}} \| f \|_\infty \quad \text{if } K = 0.
$$

(19)

The 1-Bakry-Émery inequality, proved in the RCD setting by Savaré [46, Corollary 3.5], ensures that

$$
|\nabla H_tf| \leq e^{-Kt} H_t(|\nabla f|), \quad \mathfrak{m}\text{-a.e. } \text{ for any } f \in W^{1,2}(X, d, \mathfrak{m}).
$$

(20)

An important role in the following will be played by the notion of ultracoactivity.

**Definition 2.3.** The semigroup $H_t$ is $L^2 \to L^\infty$ ultracoactive, or simply ultracoactive, if there exists a positive function $\theta(t)$ such that for any $t > 0$ and any $f \in L^2(X, \mathfrak{m})$ we have $H_tf \in L^\infty(X, \mathfrak{m})$ with

$$
\| H_tf \|_\infty \leq \theta(t) \| f \|_2
$$

.
Under this assumption, for every \( t > 0 \) and every \( x \in X \) there exists a Lipschitz version of the density kernel \( \rho_t(x, \cdot) \). Moreover, it is easy to prove (see e.g. [33, Theorem 14.4]) that \( H_t \) is ultracontractive if and only if for every \( t > 0 \) and \( x \in X \) we have \( \rho_{2t}(x, x) \leq \theta^2(t) \).

We conclude the section by stating a result which ensures the ultracontractivity of the heat semigroup.

**Proposition 2.4.** Let \((X, d, m)\) be an \( \text{RCD}(K, \infty) \) metric measure space for some \( K \in \mathbb{R} \). Assume that there exists a positive function \( A(r) \) such that

\[
m(B_r(x)) > A(r) \quad \text{for every } x \in X, \ r \in (0, \infty).
\]  

(21)

Then \( H_t \) is ultracontractive.

**Proof.** By a recent result of Tamanini [50, Corollary 3.3], the heat kernel on an \( \text{RCD}(K, \infty) \) space satisfies the following point-wise Gaussian bounds: there exists \( C_K > 0 \) depending only on \( K \) (if \( K \geq 0 \), one can choose \( C_K = 0 \)) and for every \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that

\[
0 \leq \rho_t(x, y) \leq \frac{1}{\sqrt{m(B_{\sqrt{t}}(x)) \cdot m(B_{\sqrt{t}}(y))}} \exp \left( C_\varepsilon (1 + C_K t) - \frac{d(x, y)^2}{(4 + \varepsilon) t} \right). \tag{22}
\]

The assumption (21) combined with (22) gives that the heat kernel is uniformly bounded from above on the diagonal. This implies the ultracontractivity. \( \square \)

### 2.2. Functions of bounded variation and perimeter.

**Definition 2.5 (BV space).** A function \( f \in L^1(X, m) \) belongs to the space \( \text{BV}(X, d, m) \) of functions of bounded variation (see [41, 4]) if there exists a sequence \( (f_n)_{n \in \mathbb{N}} \in \text{Lip}_{loc}(X) \) converging to \( f \) in \( L^1(X, m) \) and such that

\[
\limsup_{n \to \infty} \int_X \text{lip}(f_n) \, dm < +\infty.
\]

If \( f \in \text{BV}(X, d, m) \) and \( A \subset X \) is open, we define

\[
|Df|(A) := \inf \left\{ \liminf_{n \to \infty} \int_A \text{lip}(f_n) \, dm : f_n \in \text{Lip}_{loc}(X), f_n \to f \text{ in } L^1(A, m) \right\}.
\]  

(23)

It is known that this function is the restriction to open sets of a finite Borel measure, called *total variation* of \( f \) and denoted by \( |Df| \).

By the very definition of \( |Df|(X) \), it is immediate to see that for all \( f, f_n \in \text{BV}(X, d, m) \)

\[
|Df|(X) \leq \liminf_n |Df_n|(X) \quad \text{whenever } f_n \to f \text{ in } L^1(X, m).
\]  

(24)

Moreover, for all \( \varphi : \mathbb{R} \to \mathbb{R} \) 1-Lipschitz with \( \varphi(0) = 0 \) we have

\[
|D(\varphi \circ f)|(X) \leq |Df|(X).
\]

When \((X, d, m)\) is an \( \text{RCD}(K, \infty) \) space and \( f \in \text{BV}(X, d, m) \), a result of Ambrosio and Honda [9, Proposition 1.6.3] ensures that \( H_t f \in \text{BV}(X, d, m) \) with the explicit inequality

\[
|D H_t f|(X) \leq e^{-Kt} |Df|(X). \tag{25}
\]

Given a Borel set \( E \) of finite measure, we say that \( E \) is a set of finite perimeter if \( \chi_E \in \text{BV}(X, d, m) \) and we set \( \text{Per}(E) := |D\chi_E|(X) \), i.e.

\[
\text{Per}(E) = \inf \left\{ \liminf_{n \to \infty} \int_X \text{lip}(f_n) \, dm : f_n \in \text{Lip}_{loc}(X), f_n \to \chi_E \text{ in } L^1(X, m) \right\}.
\]  

(26)
We remark that we can replace the set $\text{Lip}_{\text{loc}}(X)$ in definition (26) with the set $\text{Lip}_{\text{bs}}(X)$, and we can also suppose that $0 \leq f_n \leq 1$ (see [5, Remark 3.4, 3.5]).

**Proposition 2.6** (Coarea inequality and coarea formula). Let $(X, d)$ be a complete metric space and let $\mathfrak{m}$ be a non-negative Borel measure finite on bounded subsets.

Let $f \in \text{Lip}_{\text{bs}}(X)$, $f : X \to [0, \infty)$ and set $M = \sup_X f$. Then for $L^1$-a.e. $t > 0$ the set $\{f > t\}$ has finite perimeter and

$$\int_0^M \text{Per}(\{f > t\}) \, dt \leq \int_X |\text{lip}(f)| \, d\mathfrak{m}. \quad (27)$$

If in addition $(X, d)$ is separable, then the coarea formula for BV functions holds, i.e. for every $f : X \to [0, \infty)$ with $f \in \text{BV}(X, d, \mathfrak{m})$ it holds

$$\int_0^\infty \text{Per}(\{f > t\}) \, dt = |Df|(X). \quad (28)$$

**Proof.** For a proof of the first part, see for instance [26, Proposition 3.5].

The second claim was already observed in the introduction of [4] and can be proved along the lines of [41]. □

The next corollary will be useful later in the paper.

**Corollary 2.7** (Finiteness of the Cheeger constant). Let $(X, d, \mathfrak{m})$ be as in Proposition 2.6 (first part) with $\mathfrak{m}(X) \in (0, \infty)$ and $\text{diam}(X) > 0$. Then the Cheeger constant defined in (4) is finite, i.e. $h(X) \in [0, \infty)$.

**Proof.** Let $x_0 \in \text{supp} \mathfrak{m}$ and consider $f(\cdot) := \max\{1 - d(x_0, \cdot), 0\} \in \text{Lip}_{\text{bs}}(X)$. By the non triviality assumptions on $(X, d, \mathfrak{m})$ and the coarea inequality (Proposition 2.6), it follows that there exists $r \in (0, 1)$ such that the metric ball $B_r(x_0)$ satisfies $\mathfrak{m}(B_r(x_0)) \in (0, \mathfrak{m}(X)/2)$ and $\text{Per}(B_r(x_0)) \in (0, \infty)$. Thus the set of competitors with finite energy in the variational problem (4) defining $h(X)$ is non empty and the conclusion follows. □

Let $K > 0$, we denote by $I_K : [0, 1] \to [0, \sqrt{K/(2\pi)}]$ the Gaussian isoperimetric profile function defined as $I_K := \varphi_K \circ \Phi_K^{-1}$, where

$$\Phi_K(x) := \frac{\sqrt{K}}{2\pi} \int_{-\infty}^x e^{-Kt^2/2} \, dt, \quad x \in \mathbb{R},$$

and $\varphi_K := \Phi_K'$. The function $I_K$ satisfies $I_K(1/2) = \sqrt{K/(2\pi)}$ and $I_K(x) = \sqrt{K} I_1(x)$. For simplicity of notation we set $I := I_1$. We recall the following asymptotic (see [14])

$$\lim_{x \to 0} \frac{I_K(x)}{x \sqrt{2K \log \frac{1}{x}}} = 1. \quad (29)$$

As proved by Ambrosio and Mondino [10, Theorem 4.2], the celebrated Gaussian isoperimetric inequality of Bakry-Ledoux [14] extends to the class of $\text{RCD}(K, \infty)$ spaces with positive $K$:

**Proposition 2.8.** Let $(X, d, \mathfrak{m})$ be an $\text{RCD}(K, \infty)$ space with $\mathfrak{m}(X) = 1$ and $K > 0$. Then, for every Borel subset $A \subset X$ we have

$$\text{Per}(A) \geq I_K(\mathfrak{m}(A)). \quad (30)$$
Thanks to a recent result by Han [34, Corollary 4.4] building on top of [3], we also have at our disposal a rigidity statement for the Gaussian isoperimetric inequality.

**Proposition 2.9.** Let \((X, d, m)\) be an \(\text{RCD}(K, \infty)\) metric measure space with \(m(X) = 1\) and \(K > 0\). Let us suppose there exists a Borel set \(E \subset X\) with positive measure such that
\[
\text{Per}(E) = I_K(m(E)).
\]

Then
\[
(X, d, m) \cong (Y, d_Y, m_Y) \times (\mathbb{R}, |\cdot|, e^{-Kt^2/2} dt)
\]
for some \(\text{RCD}(K, \infty)\) space \((Y, d_Y, m_Y)\), where \(\cong\) denotes isomorphism as metric measure spaces.

We will take for granted the following key lemma, which can be obtained as a simpler variant of [9, Lemma 1.5.8] in the case of a fixed ambient space.

**Lemma 2.10.** Let \((X, d, m)\) be an \(\text{RCD}(K, \infty)\) metric measure space. Let \((f_k)_{k \in \mathbb{N}}\) be a sequence converging in \(L^2(X, m)\) to \(f\) and assume that
\[
\sup_{k \in \mathbb{N}} \text{Ch}_m(f_k) < \infty.
\]

Then, for any lower semicontinuous function \(g : X \to [0, \infty]\), it holds that
\[
\int_X g|\nabla f| dm \leq \liminf_{k \to \infty} \int_X g|\nabla f_k| dm.
\]

Another key property is the compactness in \(\text{BV}\). In order to state the result, we recall the next crucial definition.

**Definition 2.11.** Let \((X, d, m)\) be a m.m.s. with \(m(X) = 1\). A function \(\omega : (0, \infty) \to (0, 1/2]\) is an isoperimetric profile for \((X, d, m)\) if for all \(\varepsilon > 0\) it holds:
\[
m(E) \leq \omega(\varepsilon) \implies m(E) \leq \varepsilon \text{Per}(E)
\]
for any Borel set \(E \subset X\).

It is easy to check that the Gaussian isoperimetric inequality (Proposition 2.8) combined with the asymptotic (29) imply that if \((X, d, m)\) is an \(\text{RCD}(K, \infty)\) space for some \(K > 0\), then it admits an isoperimetric profile. We also know [9, Theorem 7.2] that any \(\text{RCD}(K, \infty)\) space with finite diameter has an isoperimetric profile.

**Proposition 2.12** (Compactness in \(\text{BV}\) and \(L^2\)). Let \((X, d, m)\) be an \(\text{RCD}(K, \infty)\) metric measure space with \(m(X) < \infty\) admitting an isoperimetric profile (this is satisfied in case \(K > 0\) or \(\text{diam}(X) < \infty\)). Then, for any sequence of functions \((f_k)_{k \in \mathbb{N}} \subset \text{BV}(X, d, m)\) such that
\[
\sup_{k \in \mathbb{N}} \left\{ \int_X |f_k| dm + |Df_k|(X) \right\} < \infty
\]
there exist a function \(f \in \text{BV}(X, d, m)\) and a subsequence \(k_j\) such that
\[
\text{sign}(f_{k_j}) \sqrt{|f_{k_j}|} \to \text{sign}(f) \sqrt{|f|} \quad \text{in } L^2(X, m).
\]
Moreover, for any sequence of functions \((f_k)_{k \in \mathbb{N}} \subset \text{W}^{1,2}(X, d, m)\) such that
\[
\sup_{k \in \mathbb{N}} \|f_k\|_{\text{W}^{1,2}} < \infty
\]
there exist a function \(f \in L^2(X, m)\) and a subsequence \(k_j\) such that
\[
f_{k_j} \to f \quad \text{in } L^2(X, m).
\]
Proof. The argument can be obtained arguing as in the proof of [9, Proposition 1.7.5].

Proposition 2.13 (Stability in BV). Let $(X, d, m)$ be an RCD$(K, \infty)$ metric measure space. Let $(f_k)_{k \in \mathbb{N}} \subset \text{BV}(X, d, m)$ be such that $|\text{sign}(f_k)| \sqrt{|f_k|}$ converge to $|\text{sign}(f)| \sqrt{|f|}$ in $L^2(X, m)$ and assume that $\sup_k |Df_k|(X) < \infty$. Then $f \in \text{BV}(X, d, m)$ and

$$|Df|(X) \leq \liminf_{k \to \infty} |Df_k|(X).$$

Proof. The proof is strongly inspired by [9, Theorem 1.6.4], we report it for reader’s convenience.

Step 1. In the first step we reduce to the case of uniformly bounded functions. To this aim we observe that, for any $N \in \mathbb{N}$ the truncated functions $f^N_k := N \land f_k \lor -N$ and $f^N := N \land f \lor -N$ verify the assumptions of the statement. Moreover, for any $g \in L^1(X, m)$ it holds that $g^N \to g$ in $L^1$ as $N \to \infty$. Then, if we are able to prove that

$$|Df^N|(X) \leq \liminf_{k \to \infty} |Df^N_k|(X),$$

the general conclusion will follow by lower semicontinuity of the variation with respect to $L^1$ convergence (see (24)) recalling that $|Dg^N|(X) \leq |Dg|(X)$ for any $N \in \mathbb{N}$ and any $g \in \text{BV}(X, d, m)$.

Step 2. Let us fix now $t > 0$ and observe that the functions $H_t f_k$ are uniformly bounded, uniformly Lipschitz, they belong to $W^{1,2}(X, d, m)$ and they still verify the assumptions of the statement. If we are able to prove that

$$|DH_t f|(X) \leq \liminf_{k \to \infty} |DH_t f_k|(X),$$

then the conclusion will follow by the 1-Bakry-Émery inequality, yielding $|DH_t f_k|(X) \leq e^{-Kt} |Df_k|(X)$, and the lower semicontinuity of the total variation w.r.t. $L^1$ convergence again, passing to the lim inf as $t \downarrow 0$.

Recalling the representation formula for the total variation [9, Proposition 1.6.3 (a)]

$$|Df|(X) = \int_X |\nabla f| \, dm$$

for any $f \in \text{Lip}_b(X) \cap L^1(X, m) \cap W^{1,2}(X, d, m)$, it remains to prove that

$$\int_X |\nabla f| \, dm \leq \liminf_{k \to \infty} \int_X |\nabla f_k| \, dm,$$

whenever $(f_k) \subset W^{1,2}(X, d, m)$ are uniformly bounded in $W^{1,2}(X, d, m)$ and converge to $f$ in $L^2(X, m)$.

Step 3. To conclude we just point out that (38) above follows from Lemma 2.10 choosing $g \equiv 1$. \hfill \Box

Let us state and prove a general existence result for optimizers of the variational problem defining the Cheeger constant (4).

Proposition 2.14. Let $(X, d, m)$ be an RCD$(K, \infty)$ metric measure space with $m(X) = 1$ and admitting an isoperimetric profile (this is satisfied in case $K > 0$ or $\text{diam}(X) < \infty$). Then there exists a Borel set $E \subset X$ with finite perimeter and $0 < m(E) \leq 1/2$ which is an optimizer for the variational problem defining the Cheeger constant (4), i.e.

$$\frac{\text{Per}(E)}{m(E)} = h(X).$$
Proof. First of all we note that \( h(X) \in [0, \infty) \) in virtue of Corollary 2.7. By the very definition of the Cheeger constant \( h(X) \) we can find a sequence of Borel sets of finite perimeter \( E_n \subset X \) such that \( \frac{\text{Per}(E_n)}{\text{m}(E_n)} \leq 1/2 \) for any \( n \in \mathbb{N} \) and \[
abla \frac{\text{Per}(E_n)}{\text{m}(E_n)} = h(X). \quad (40)
\]

Let us set \( f_n := \chi_{E_n} \), where we denote by \( \chi_E : X \to \{0,1\} \) the indicator function of an arbitrary Borel set \( E \subset X \). We claim that \( \text{m}(E_n) \) are bounded away from 0. If this is not the case, i.e. \( \liminf_{n \to \infty} \text{m}(E_n) = 0 \), from the existence of an isoperimetric profile (33) for \((X,d,m)\) we infer that \[
\liminf_{n \to \infty} \frac{\text{Per}(E_n)}{\text{m}(E_n)} = \infty,
\]
contradicting (40).

It follows from (40) that the functions \( f_n \) have uniformly bounded BV-norms, since \( |Df_n|(X) = \text{Per}(E_n) \) and \( \|f_n\|_{L^1} = \text{m}(E_n) \). Thus by Proposition 2.12 there exist a function \( f \in \text{BV}(X,d,m) \) and a subsequence, that we do not relabel, such that \( \text{sign}(f_n) \sqrt{|f_n|} \to \text{sign}(f) \sqrt{|f|} \) in \( L^2(X,m) \). In particular,

\[
\|f\|_{L^1} = \lim_{n \to \infty} \text{m}(E_n).
\]

We claim that there exists a Borel set \( E \subset X \) such that \( f = \chi_E \), m.a.e.. In order to prove the claim it is sufficient to verify that

\[
f(1-f) = 0, \quad \text{m.a.e.}\quad (42)
\]

To this aim we observe that \( f_n(1-f_n) = 0 \), m.a.e. for any \( n \in \mathbb{N} \), since \( f_n \) are indicator functions. Moreover, using that \( f_n \to f \) in \( L^1(X,m) \) and \( \|f_n\|_{L^\infty(X,m)} = 1 \), it is easily seen that \( f_n(1-f_n) \to f(1-f) \) in \( L^1(X,m) \), proving the claim (42).

Combining (41) with (42) we obtain that

\[
\text{m}(E) = \lim_{n \to \infty} \text{m}(E_n).
\]

In particular \( 0 < \text{m}(E) \leq 1/2 \). By Proposition 2.13 we also infer that \( \text{Per}(E) \leq \liminf_{n \to \infty} \text{Per}(E_n) \).

The combination of (40), (43) and (44) yields that

\[
h(X) \leq \frac{\text{Per}(E)}{\text{m}(E)} \leq \liminf_{n \to \infty} \frac{\text{Per}(E_n)}{\text{m}(E_n)} = h(X)
\]

and we obtain that \( E \) is an optimizer for the variational problem defining the Cheeger constant. \( \square \)

The next result will be useful later in the proof of Theorem 1.4.

**Lemma 2.15.** Let \((X,d,m)\) be a complete and separable metric measure space with \( m \) finite on bounded sets, and let \( f \in W^{1,2}(X,d,m) \). Then \( f^2 \in \text{BV}(X,d,m) \) and

\[
|D(f^2)| \leq 2 |f| |\nabla f| m \quad \text{as measures}.
\]

(45)
Proof. For brevity, we refer to [7, 4] for the notions of p-test plans and p-a.e. curve, $p = 1, 2$. From [7, Proposition 5.7], if $f \in W^{1,2}(X, d, m)$ then $f$ is Sobolev along 2-a.e. curve and
\[ \frac{d}{dt} f \circ \gamma \leq |\nabla f| \circ |\gamma|, \text{ a.e. in } [0, 1], \text{ 2-a.e. } \gamma \in \text{AC}((0, 1); (X, d)). \]

At this point, it is easy to check that $f^2$ is a BV function in the “weak” formulation of [4, Definition 5.5] with “weak” total variation satisfying $|D(f^2)|_w \leq 2|f| |\nabla f| m$ as measures. We conclude thanks to the identification result [4, Theorem 1.1]. \hfill \Box

2.3. Spectrum of the Laplacian. We start by recalling some classical notions of spectral theory (see for instance [13, Appendix A]). Let $(X, d, m)$ be an RCD$(K, \infty)$ metric measure space for some $K \in \mathbb{R}$ and denote by $\Delta$ the Laplacian defined in Section 2.1. We know that $-\Delta$ is a densely defined, self-adjoint operator on the Hilbert space $L^2(X, m)$. A number $\lambda \in \mathbb{C}$ is a regular value of $-\Delta$ if $(\lambda \text{Id} + \Delta)$ has a bounded inverse. The resolvent set $\rho(-\Delta)$ is the set of regular values of $-\Delta$, while the spectrum $\sigma(-\Delta)$ is defined as $\sigma(-\Delta) := \mathbb{C} \setminus \rho(-\Delta)$. Since $-\Delta$ is nonnegative, it holds $\sigma(-\Delta) \subset [0, \infty]$. If there exists a non-zero function $f \in D(\Delta)$ such that $-\Delta f = \lambda f$, we call $\lambda \in \sigma(-\Delta)$ an eigenvalue and $f$ the associated eigenfunction. The set of all eigenvalues forms the so-called point spectrum. The discrete spectrum $\sigma_d(-\Delta)$ is the set of all eigenvalues that are isolated in the point spectrum with the corresponding eigenspace which is finite dimensional. The essential spectrum is the closed set defined as $\sigma_{\text{ess}}(-\Delta) := \sigma(-\Delta) \setminus \sigma_d(-\Delta)$. We denote by $\Sigma$ the infimum of the essential spectrum of $-\Delta$, i.e.
\[ \Sigma := \inf \sigma_{\text{ess}}(-\Delta) \quad \text{and} \quad \Sigma := +\infty \text{ if } \sigma_{\text{ess}}(-\Delta) = \emptyset. \]

The space $W^{1,2}(X, d, m)$ is compactly embedded in $L^2(X, m)$ if for any sequence $\{f_k\} \subset W^{1,2}(X, d, m)$ with uniformly bounded $W^{1,2}$-norm we can extract a subsequence $f_{k_i}$ strongly converging in $L^2(X, m)$. In the next theorem, we collect some results about the spectrum of RCD$(K, \infty)$ spaces which will be useful later on. For the compact embedding $W^{1,2}(X, d, m) \subset \subset L^2(X, m)$ in RCD$(K, \infty)$ spaces under different assumptions see [32, Proposition 6.7].

**Theorem 2.16.** Let $(X, d, m)$ be an RCD$(K, \infty)$ metric measure space for some $K \in \mathbb{R}$.

**Case** $m(X) < \infty$. Assume that $(X, d, m)$ admits an isoperimetric profile and that $m(X) < \infty$. Then:

1. The embedding of $W^{1,2}(X, d, m)$ in $L^2(X, m)$ is compact.
2. The heat semigroup $H_t : L^2(X, m) \to L^2(X, m)$ is a compact operator, for any $t > 0$.
3. The spectrum of $-\Delta$ is discrete, non-negative and it diverges to $+\infty$, i.e. $\sigma(-\Delta) = \sigma_d(-\Delta) = (\lambda_k)_{k \in \mathbb{N} \cup \{0\}} \subset [0, \infty)$ with $\lambda_k \to \infty$ as $k \to \infty$. Moreover:
\[ \lambda_0 = 0 < \lambda_1 = \min \left\{ 2\text{Ch}_m(f) : f \in L^2(X, m), \|f\|_2 = 1, \int_X f \, dm = 0 \right\}. \quad (46) \]

In particular, $\lambda_1$ coincides with the value introduced in (2).
Moreover, $f \in W^{1,2}(X, d, m)$ is a $\lambda_0$-eigenfunction if and only if $f$ is a non-zero constant function, while $f \in W^{1,2}(X, d, m)$ is a (normalized) $\lambda_1$-eigenfunction if and only if
\[ \int_X f^2 \, dm = 1, \quad \int_X f \, dm = 0 \quad \text{and} \quad \int_X |\nabla f|^2 \, dm = \lambda_1. \quad (47) \]
**Case** $m(X) = \infty$. Assume $m(X) = \infty$ and $\lambda_0 := \inf \sigma_d(-\Delta) < \Sigma$. Then:

1. The eigenvalue $\lambda_0$ satisfies

   $$0 < \lambda_0 = \min \{2\text{Ch}_m(f) : f \in L^2(X, m), \|f\|_2 = 1\}.$$

   In particular, $\lambda_0$ coincides with the value introduced in (3).

   Moreover, $f \in W^{1,2}(X, d, m)$ is a (normalized) $\lambda_0$-eigenfunction if and only if

   $$\int_X f^2 \, dm = 1 \text{ and } \int_X |\nabla f|^2 \, dm = \lambda_0.$$  \hfill (48)

**Proof. Case** $m(X) < \infty$. Since $X$ admits an isoperimetric profile, the fact that the embedding of $W^{1,2}(X, d, m)$ in $L^2(X, m)$ is compact is a direct consequence of Proposition 2.12. We can thus appeal to the standard spectral theory (see e.g. [13, Theorem A.6.4]) to infer that $H_t$ is a compact operator for every $t > 0$ and that $\sigma(-\Delta)$ consists of a sequence of isolated eigenvalues $(\lambda_k)_{k \in \mathbb{N}} \subset [0, \infty)$ with $\lambda_k \to \infty$ as $k \to \infty$ and finite dimensional associated eigenspaces.

We can also appeal to the variational characterisation of the eigenvalues (see [25, Theorem 4.5.1]) to infer that

$$\lambda_k = \min_{S_{k+1}} \max_{f \in S_{k+1}, \|f\|_2 = 1} 2\text{Ch}_m(f),$$  \hfill (49)

where $S_k \subset W^{1,2}(X, d, m)$ denotes an arbitrary $k$-dimensional subspace.

Since $f \equiv 1$ is an element of $L^2(X, m)$, we infer that $\lambda_0 = 0$. Moreover, from the Sobolev-to-Lipschitz property satisfied by RCD($K, \infty$) spaces [8], it holds that $\text{Ch}_m(f) = 0$ if and only if $f$ is constant $m$-a.e.. Thus $\lambda_0 = 0$ if and only if $m(X) < \infty$ and, in this case, $f$ is a $\lambda_0$-eigenfunction if and only if $f$ is constant $m$-a.e..

Specialising (49) to $k = 1$ in case $m(X) < \infty$ gives the claimed variational formula for $\lambda_1$ which trivially coincides with (2) by the very definition of Cheeger energy.

We finally prove the implication "if $f$ satisfies (47) then $f$ is a $\lambda_1$-eigenfunction". We first show that $\lambda_1 f \in \partial \text{Ch}_m(f)$: for every $g \in L^2(X, m)$ with $\int_X g \, dm = 0$ we have

$$\text{Ch}_m(g) \geq \frac{\lambda_1}{2} \int_X g^2 \, dm \geq \lambda_1 \int_X fg \, dm - \frac{\lambda_1}{2} \int_X f^2 \, dm = \text{Ch}_m(f) + \int_X \lambda_1 f (g - f) \, dm$$

as a consequence of the variational characterization of $\lambda_1$, a trivial inequality and (47). The function $\lambda_1 f$ is also the element of minimal norm in the set $\partial \text{Ch}_m(f)$, and thus $-\Delta f = \lambda_1 f$, since

$$\|\Delta f\|_2 \geq \int_X -f \Delta f \, dm = \int_X |\nabla f|^2 \, dm = \lambda_1 = \|\lambda_1 f\|_{L^2},$$

where we have used again (47) and the Cauchy-Schwarz inequality.

**Case** $m(X) = \infty$. The assumption on the spectrum implies the existence of the first eigenvalue $\lambda_0$ of $-\Delta$ which is isolated and with non-empty finite dimension eigenspace. Using the Sobolev-to-Lipschitz property satisfied by RCD($K, \infty$) spaces [8] and observing that the constant functions are not in $L^2(X, m)$, we have $\lambda_0 > 0$. The variational characterization of $\lambda_0$ still holds (see in this case [25, Theorem 4.5.2]) and gives

$$\lambda_0 = \min_{\|f\|_2 = 1} \left\{ 2\text{Ch}_m(f) \right\},$$
so that $\lambda_0$ trivially corresponds to the definition given in (3) by the very definition of Cheeger energy. It remains to prove that a function $f$ such that
\[
\int_X f^2 \, dm = 1 \quad \text{and} \quad \int_X |\nabla f|^2 \, dm = \lambda_0
\]
is a $\lambda_0$-eigenfunction. We first show that $\lambda_0 f \in \partial \text{Ch}_m(f)$: for every $g \in L^2(X, m)$ we have
\[
\text{Ch}_m(g) \geq \frac{\lambda_0}{2} \int_X g^2 \, dm \geq \lambda_0 \int_X fg \, dm - \frac{\lambda_0}{2} \int_X f^2 \, dm = \text{Ch}_m(f) + \int_X \lambda_0 f(g - f) \, dm
\]
as a consequence of the variational characterization of $\lambda_0$, a trivial inequality and (48). The function $\lambda_0 f$ is also the element of minimal norm in the set $\partial \text{Ch}_m(f)$, and thus $-\Delta f = \lambda_0 f$, since
\[
\|\Delta f\|_2 \geq \int_X -f \Delta f \, dm = \int_X |\nabla f|^2 \, dm = \lambda_0 = \|\lambda_0 f\|_{L^2},
\]
where we have used again (47) and the Cauchy-Schwarz inequality. □

Even though we do not need this result in the paper, we mention that the assumption $m(X) < \infty$ and the ultracontractivity of the heat semigroup yield the same conclusions of the first case of Theorem 2.16 (see again [13, Appendix A]).

3. Proof of Theorem 1.2

Proof of Theorem 1.2. By the scaling property of the RCD($K, \infty$) condition, it is enough to show the result in the case $K = 1$.

Along the proof of Theorem 1.1 in [26] (see in particular [26, Eq. (49), (50), (51)]), it is shown that for all $t > 0$ and for all $A \subset X$ Borel it holds
\[
J_1(t) \text{Per}(A) \geq 2 \left( m(A) - \frac{m(A)^2}{\|H_{1/2}(\chi_A - m(A))\|_2^2} \right) \geq 2 \left( m(A) - m(A)^2 - e^{-\lambda_1 t} \|\chi_A - m(A)\|_2^2 \right) = 2m(A)(1 - m(A))(1 - e^{-\lambda_1 t}). \tag{50}
\]
The inequality (9) then follows directly from (50) by minimizing over all the Borel subsets $A$ with $m(A) \leq 1/2$.

By Proposition 2.14 and assumption (10) there exists a Borel set $E \subset X$ with $0 < m(E) \leq 1/2$ such that
\[
\frac{\text{Per}(E)}{m(E)} = \sup_{t > 0} \frac{1 - e^{-\lambda_1 t}}{J_1(t)}. \tag{51}
\]

Since $(1 - e^{-\lambda_1 t})/J_1(t)$ is a continuous function of $t \in (0, \infty)$, we have to analyse three different cases:

1. the supremum in the right hand side of (51) is achieved as $t \to 0$;
2. the supremum in the right hand side of (51) is achieved as $t \to \infty$;
3. the supremum in the right hand side of (51) is achieved for a certain $\tilde{t} \in (0, \infty)$.

Case (1): this case is easily ruled out since $(1 - e^{-\lambda_1 t})/J_1(t)$ is a positive function in $(0, \infty)$ and
\[
\lim_{t \to 0} \frac{1 - e^{-\lambda_1 t}}{J_1(t)} = 0.
\]
In particular, the supremum can never occur as $t \to 0$. 

**Case (2):** in this situation we have

\[
\sup_{t>0} \frac{1 - e^{-\lambda t}}{J_1(t)} = \frac{1}{2} \sqrt{\frac{2}{\pi}} = \sqrt{\frac{2}{\pi}},
\]

so that from (51) it follows

\[
\text{Per}(E) \leq \frac{1}{2} \sqrt{\frac{2}{\pi}} = \frac{1}{\sqrt{2\pi}} = I(1/2),
\]

where \(I\) is the Gaussian isoperimetric profile function. Since equality holds in the Buser inequality, equality holds in both the inequalities in (50). In particular \(m(E) = 1/2\). Therefore \(E\) achieves equality in the Gaussian isoperimetric inequality (30) and the conclusion is thus a consequence of Proposition 2.9.

**Case (3):** we claim that also this case can be ruled out. Indeed, if the supremum is attained for some \(0 < \bar{t} < \infty\), then by (50) we get \(m(E) = 1/2\) and

\[
\|H_{t/2}(\chi_E - m(E))\|_2^2 = e^{-\lambda \bar{t}} \|\chi_E - m(E)\|_2^2.
\]

Therefore \(f := \chi_E - m(E)\) attains the equality for \(t = \bar{t}/2\) in the inequality

\[
\|H_t f\|_2 \leq e^{-\lambda t} \|f\|_2,
\]

that has been proved in [26, eq. (46)] through an application of Gronwall’s lemma. It follows that

\[
2\lambda_1 \int_X |H_t f|^2 \text{d}m = 2 \int_X |\nabla H_t f|^2 \text{d}m = -\frac{d}{dt} \int_X |H_t f|^2 \text{d}m,
\]

for a.e. \(t \in (0, \bar{t})\) (cf. with [26, eq. (47)]). Hence, by the classical characterization of the first eigenvalue of the Laplacian, \(-\Delta H_t f = \lambda_1 H_t f\) for a.e. \(t \in (0, \bar{t})\). From this we infer by the heat equation that \(H_t f = e^{-\lambda t} f\) for any \(t \in (0, \infty)\). It follows by the regularizing properties of the heat semigroup that \(f \in D(\Delta)\), in particular \(f \in W^{1,2}(X, \text{d}m)\). We claim that this yields a contradiction with the explicit expression of \(f\). In order to do so it is sufficient to apply Lemma 3.1 below to \(u := f + 1/2\).

\[\square\]

**Lemma 3.1.** Let \((X, \text{d}, m)\) be an \(\text{RCD}(K, \infty)\) metric measure space and assume that \(u \in W^{1,2}(X, \text{d}, m)\) does not admit an \(m\)-a.e. constant representative. Then \(u\) cannot attain values in \(\{0, 1\}\) almost everywhere with respect to \(m\).

**Proof.** Let us suppose by contradiction that \(u\) takes values only in \(\{0, 1\}\). Since \(u\) does not admit an \(m\)-a.e. constant representative, we can find bounded sets \(E_0, E_1 \subset X\) such that

\[
0 < m(E_0), m(E_1) < \infty, \quad u = 0 \text{ m-a.e. on } E_0 \text{ and } u = 1 \text{ m-a.e. on } E_1.
\]

Let \(\mu_0\) and \(\mu_1\) be the probability measures obtained restricting \(m\) to \(E_0\) and \(E_1\) respectively and normalizing. By [45] there exists a unique \(W_2\) geodesic connecting \(\mu_0\) and \(\mu_1\), let us call it \((\mu_t)_{t \in [0,1]}\). Moreover we can lift \((\mu_t)\) to an optimal geodesic plan \(\Pi\) such that \((\epsilon_t)_\sharp \Pi = \mu_t\) and \(\epsilon_t\) are absolutely continuous with respect to \(m\) with uniformly bounded densities, where \(\epsilon_t\) denotes the classical evaluation map. Hence \(\Pi\) is a test plan according to [7].

By the equivalent characterization of Sobolev spaces via test plans [7] we infer that for \(\Pi\)-a.e. \(\gamma\) it holds that \(u \circ \gamma : [0,1] \to \mathbb{R}\) is a \(W^{1,2}\)-Sobolev function. Moreover, by Fubini’s theorem and the assumption that \(u\) takes values in \(\{0, 1\}\) almost everywhere we infer that for \(\Pi\)-a.e. \(\gamma\) it holds that \(u \circ \gamma\) takes values in \(\{0, 1\}\) for \(L^1\)-a.e. \(t \in (0,1)\). Hence we conclude that, for \(\Pi\)-a.e. \(\gamma\), \(u \circ \gamma\) is constant a.e. on \((0,1)\) and, by [7] again we get that for \(\Pi\)-a.e. \(\gamma\) it holds that \(u \circ \gamma\) takes values in \(\{0, 1\}\) for \(L^1\)-a.e. \(t \in (0,1)\). Hence we conclude that, for \(\Pi\)-a.e. \(\gamma\), \(u \circ \gamma\) is constant a.e. on \((0,1)\) and, by [7] again we get that for \(\Pi\)-a.e.

\[\square\]
\( \gamma \) it holds that \( u(\gamma(0)) = u(\gamma(1)) \). This gives a contradiction since \((e_i)_{2}\Pi = \mu_i \) for \( i = 0, 1 \) and by construction \( u = 0 \) almost everywhere w.r.t. \( \mu_0 \) and \( u = 1 \) almost everywhere w.r.t. \( \mu_1 \).

4. On the equality case in Cheeger’s inequality

In this section we discuss the equality case in Cheeger’s inequality. We will constantly use the notation and the results mentioned in the preliminary section.

**Proof of Theorem 1.4.** Up to replacing \( m \) by \( \frac{m}{m(X)} \), we can assume that \( m(X) = 1 \) without loss of generality. We are going to argue by contradiction.

**Step 1.** Aim for this step is to prove that, assuming equality holds in Cheeger’s inequality, we obtain the existence of a function \( f \in D(\Delta) \) such that \( -\Delta f = \lambda_1 f \) and

\[
|\nabla f| = \sqrt{\lambda_1} |f|, \text{ m-a.e.} \tag{53}
\]

First of all, Theorem 2.16 implies that \( \lambda_1 \) (and thus also \( h(X) \)) is positive and that there exists \( f \in D(\Delta) \subset W^{1,2}(X, d, m) \) with \( -\Delta f = \lambda_1 f \), \( \int_X f \, dm = 0 \) and \( \|f\|_2 = 1 \). Let \( m \) be a median for \( f \) and set \( f^\pm = \max\{f - m, 0\}, f^- = \min\{f - m, 0\}. \)

From Lemma 2.15 we know that

\[
|D(f^\pm)|^2(X) \leq 2 \int_X |f^\pm| \|\nabla f^\pm\| \, dm.
\]

Noticing that

\[
|\nabla (f^+)| \leq |\nabla f|, \quad |\nabla (f^-)| \leq |\nabla f|,
\]

we can apply the Cauchy-Schwarz inequality to infer

\[
2 \left( \int_X |\nabla f|^2 \, dm \right)^{\frac{1}{2}} \left( \int_X |f - m|^2 \, dm \right)^{\frac{1}{2}} \geq |D(f^+)|^2(X) + |D(f^-)|^2(X) \tag{54}
\]

where we have used that \( |f^+| + |f^-| = |f - m| \).

Using the coarea formula (28), we obtain

\[
2\sqrt{\lambda_1} = 2 \left( \int_X |\nabla f|^2 \, dm \right)^{\frac{1}{2}} \geq \frac{1}{\int_X |f - m|^2 \, dm} \left( |D(f^+)|^2(X) + |D(f^-)|^2(X) \right) \tag{54} \]

\[
\geq \frac{h(X)}{\int_X m \, dm} \left( \int_0^\infty \Per(\{(f^+)^2 > t\}) \, dt + \int_0^\infty \Per(\{(f^-)^2 > t\}) \, dt \right) \tag{55}
\]

\[
= \frac{h(X)}{\int_X |f - m|^2 \, dm} \left( \int_0^\infty m \{(f^+)^2 > t\} \, dt + \int_0^\infty m \{(f^-)^2 > t\} \, dt \right) \]

\[
= \frac{h(X)}{\int_X |f - m|^2 \, dm} \left( \int_X (f^+)^2 \, dm + \int_X (f^-)^2 \, dm \right) = h(X) \left( \int_X |f - m|^2 \, dm \right)^{\frac{1}{2}} \]

\[
= h(X) \left( \int_X f^2 \, dm + \int_X m^2 \, dm \right)^{\frac{1}{2}} \geq h(X) \left( \int_X f^2 \, dm \right)^{\frac{1}{2}} = h(X) = 2\sqrt{\lambda_1},
\]
where the last identity comes from the assumption that equality is achieved in Cheeger’s inequality. It follows that all the inequalities in (55) are actually equalities. In particular $m = 0$,

$$\left( \int_X |\nabla f|^2 \, dm \right)^{\frac{1}{2}} = \int_X |\nabla f| \, df$$

and

$$\int_0^\infty \text{Per}(\{(f^+)^2 > t\}) \, dt + \int_0^\infty \text{Per}(\{(f^-)^2 > t\}) \, dt$$

$$= h(X) \int_0^\infty m(\{(f^+)^2 > t\}) \, dt + h(X) \int_0^\infty m(\{(f^-)^2 > t\}) \, dt.$$  

By the equality case (56) in the Cauchy-Schwartz inequality we also know that $|\nabla f| = C|f|$ m-a.e., for some constant $C > 0$. Since

$$\lambda_1 = \int_X |\nabla f|^2 \, dm = \int_X C^2 f^2 \, dm = C^2$$

the claim (53) follows.

**Step 2.** In this step we prove that the eigenfunction $f$ is in $L^\infty(X, m)$. We argue by contradiction: if $f \notin L^\infty(X, m)$ we have

$$\max \{ \sup \text{ess}\{|(f^+)^2\}, \sup \text{ess}\{|(f^-)^2\} \} = +\infty.$$ 

We suppose $\sup \text{ess}\{|(f^+)^2\} = +\infty$ (the other case being analogous). Recalling the definition of $h(X)$ and the equality (57) we can infer that

$$\text{Per}(\{(f^+)^2 > t\}) = h(X) \, m(\{(f^+)^2 > t\}) \text{ for a.e. } t > 0.$$ 

We notice that $\{(f^+)^2 > t\} \subset \{(f^+)^2 > s\}$ if $s < t$ and by the previous step

$$\int_X m(\{(f^+)^2 > t\}) \, dt < +\infty.$$ 

In particular there exists a sequence $t_n \to \infty$ such that $0 < m(\{(f^+)^2 > t_n\}) \to 0$ and this leads to a contradiction recalling (58) and the fact that $X$ admits an isoperimetric profile.

Thus $f \in L^\infty(X, m)$ which trivially implies $\Delta f \in L^\infty(X, m)$ since $f$ is an eigenfunction. Moreover, for every $t > 0$ we have $H_t f = e^{-\lambda_1 t} f$ so that by recalling the $L^\infty - \text{ Lip}$ regularization of the heat semigroup (19) it holds $f \in \text{ Lip}_b(X)$ (up to a suitable choice of m-a.e. representative). From now on we will tacitly assume to deal with this representative of $f$.

**Step 3.** Applying [12, Theorem 9.6 (b)], for any $T > 0$ we can find a Test plan $\Pi \in \mathcal{P}(C([0, T], X))$ such that $(e_0)_2 \Pi = m$ and

$$f(\gamma(t)) - f(\gamma(s)) = \int_s^t |\nabla f|^2 \gamma(r) \, dr = \int_s^t \lambda_1 f^2 \gamma(r) \, dr,$$

for any $0 \leq s \leq t \leq T$ and for $\Pi$-almost every $\gamma$. Observe that, fixing any such curve $\gamma$ and setting $F(t) := f(\gamma(t))$, $F$ is smooth and it verifies the ODE

$$F'(t) = \lambda_1 F^2(t)$$

in the classical sense. Indeed it is absolutely continuous, it solves the ODE in the almost everywhere sense and the derivative itself is continuous, allowing to bootstrap the regularity. Observe that if $F(0) = 0$, then $F(t) = 0$ for every $t \in [0, T]$. Otherwise, if $F(0) > 0$ there is no bounded solution of (60) up to time $1/(\lambda_1 F(0))$. 

We claim that \( f \) vanishes identically, contradicting the assumption \( \int_X f^2 \, dm = 1 \). If this is not the case we can find \( \epsilon > 0 \) and a set of positive measure \( E \subset X \) such that for any \( x \in E \) it holds \( f(x) \lambda_1 > \epsilon \). Then we apply the construction above with \( T = 1/\epsilon \) finding \( \Pi \in \mathcal{P}(C([0,T],X)) \) verifying (59) and such that, for a set of positive \( \Pi \)-measure of curves \( \gamma \), it holds \( f(\gamma(0)) \lambda_1 > \epsilon \). Recalling what we pointed out above concerning bounded solutions of (60), we obtain a contradiction, since \( f \) is bounded.

\[ \square \]

In the case \( m(X) = \infty \), we have the following:

**Theorem 4.1.** Let \((X,d,m)\) be an \( RCD(K,\infty) \) metric measure space with \( m(X) = \infty \) and \( K \in \mathbb{R} \). Let us suppose that \( H_t \) is \( L^2 - L^\infty \) ultracontractive and \( \inf \sigma_d(-\Delta) < \Sigma \) (this is always the case if the spectrum is discrete).

Then the equality in Cheeger’s inequality is never attained, i.e.

\[ \lambda_0 > 1/4 h(X)^2. \] (61)

**Proof.** The argument follows the same lines of the proof of Theorem 1.4, thus we only give a sketch.

First of all, the assumption \( \inf \sigma_d(-\Delta) < \Sigma \) together with Theorem 2.16 ensures that \( \lambda_0 > 0 \) and that there exists \( f \in D(\Delta) \subset W^{1,2}(X,d,m) \) with \( -\Delta f = \lambda_0 f, \|f\|_2 = 1 \). By assuming directly \( m := 0 \) (here we do not need that \( m \) is a median, since \( h(X) \) has a different definition (4) in case \( m(X) = \infty \)) and replacing \( \lambda_1 \) with \( \lambda_0 \), one can follow verbatim the arguments of Step 1 in the proof of Theorem 1.4 in order to obtain

\[ |\nabla f| = \sqrt{\lambda_0} |f|, \text{ m-a.e.}. \]

Since for every \( t > 0 \) we have \( H_t f = e^{-\lambda_0 t} f \) and \( H_t \) is ultracontractive, it follows that \( f, \Delta f \in L^\infty(X,m) \) and also (up to a suitable choice of m-a.e. representative) \( f \in \text{Lip}_b(X) \) by the \( L^\infty - \text{Lip} \) regularization of the heat semigroup (19). We are now in position to reach a contradiction arguing as in Step 3 of the proof of Theorem 1.4. \[ \square \]

**Remark 4.2.** We notice that there exist \( RCD(K,\infty) \) metric measure spaces satisfying the assumptions of Theorem 4.1. For instance, let us consider \( \mathbb{R} \) endowed with the Euclidean distance \( d(x,y) = |x-y| \) and the measure \( m := e^{x^2/2} d\mathcal{L}^1 \). One can easily see that it satisfies the \( RCD(-1,\infty) \) condition and, by using Proposition 2.4, that the associated heat semigroup is ultracontractive. Finally, a result of Wang [51, Example 5.1] ensures that the spectrum is discrete.

We also observe that the assumption on the ultracontractivity of the heat semigroup in Theorem 4.1 can be relaxed by the following qualitative assumption:

\[ H_t \text{ maps } L^2(X,m) \text{ into } L^\infty(X,m), \text{ for some } t > 0. \]

### 4.1. Examples

We next collect a series of examples of RCD spaces (actually, smooth Riemannian manifolds) where the assumptions of the previous theorems are not satisfied and equality in Cheeger’s inequality is achieved.

**Example 4.3.** An \( RCD(0, n) \) space with infinite measure satisfying \( \lambda_0(X) = h(X) = 0 \).

A classical example of equality in Cheeger’s inequality is obtained in the Euclidean space \( (\mathbb{R}^n, |\cdot|, d\mathcal{L}^n) \). Indeed, it is well known that \( \lambda_0(\mathbb{R}^n) = 0 \) (see e.g. [33, Example 10.9]), By Cheeger’s inequality and the trivial nonnegativity of the Cheeger constant (or by direct
computation, considering balls of increasing radii as competitors in the definition), we also have $h(\mathbb{R}^n) = 0.$

**Example 4.4.** An $\text{RCD}(K, 2)$ space with finite measure satisfying $\lambda_1(X) = h(X) = 0.$

The following example is strongly inspired by [17]. For $|t| > 1,$ let us define the function $f(t) := e^{-\sqrt{|t|}}.$ We consider an extension $F : \mathbb{R} \to \mathbb{R}$ of $f$ such that $F \in C^\infty(\mathbb{R}),$ $F$ is even and $F(t) > 0$ for every $t \in \mathbb{R}.$ We denote by $S$ the surface of revolution parametrized by

$$
(t, \theta) \mapsto (F(t) \cos(\theta), F(t) \sin(\theta), t), \quad (t, \theta) \in \mathbb{R} \times [0, 2\pi].
$$

The surface $S$ is a 2-dimensional Riemannian manifold with the warped product structure $\mathbb{R} \times_F S^1,$ where the $\mathbb{R}$ factor is endowed with the arc-length metric $dx^2 = (1 + F'(t)^2)dt^2$ and the $S^1$ factor is endowed with the standard metric.

We claim that $S$ has finite volume, Gaussian curvature bounded from below and $\lambda_1(X) = 0.$

Indeed,

$$
\text{vol}(S) = 2\pi \int_{-\infty}^{\infty} F(t) \sqrt{1 + (F'(t))^2} \, dt < \infty,
$$

since the integrand is continuous and integrable at infinity as a consequence of the asymptotic

$$
F(t) \sqrt{1 + (F'(t))^2} \sim e^{-\sqrt{|t|}} \quad \text{as } |t| \to \infty.
$$

The Gaussian curvature $K$ can be computed using a classical formula for surfaces of revolution, i.e.

$$
K_S(t, \theta) = -\frac{F''(t)}{F(t) \sqrt{1 + (F'(t))^2}}.
$$

We thus observe that $K_S$ is bounded from below since $F$ is smooth, strictly positive and

$$
-\frac{F''(t)}{F(t) \sqrt{1 + (F'(t))^2}} = -\frac{\sqrt{|t| + 1}}{4|t|^{3/2} \sqrt{1 + \frac{2\sqrt{|t|}}{4|t|}}} > -\frac{1}{2}, \quad \text{for } |t| > 1.
$$

Thus $S$ is an $\text{RCD}(K, 2)$ space, for some $K \in \mathbb{R}.$ It remains to show that $\lambda_1(S) = 0$ (which will imply in turn that $h(S) = 0$ by Cheeger’s inequality). In order to prove this, it is sufficient to show (see also [38, Section 3])

$$
\mu_S := \limsup_{r \to \infty} \frac{1}{r} \log \left( \text{vol}(S) - \text{vol}(B_r) \right) = 0,
$$

where $B_r$ denotes the geodesic ball of centre $(0, 0)$ and radius $r.$

Since by elementary considerations

$$
\text{vol}(S) - \text{vol}(B_r) \geq 2\pi \int_{|t| > x} F(t) \sqrt{1 + (F'(t))^2} \, dt
$$

where $x$ is defined so that

$$
r = \int_0^x \sqrt{1 + (F'(t))^2} \, dt,
$$

we obtain

$$
\mu_S \geq \limsup_{x \to \infty} \frac{\log \left( \int_x^\infty F(t) \sqrt{1 + (F'(t))^2} \, dt \right)}{\int_0^x \sqrt{1 + (F'(t))^2} \, dt}.
$$
and the limit superior in the right hand side is actually a limit equal to 0. To see this, one can apply twice L’Hospital’s rule and then use the explicit expression of $F(x)$ for $x > 1$. Since it trivially holds that $\mu_S \leq 0$, we have $\mu_S = 0$ and the claim follows.

**Example 4.5.** An $\text{RCD}(-1,2)$ space with infinite measure satisfying $\lambda_0(X) = \frac{1}{4} h(X)^2 > 0$.

We claim that the hyperbolic plane $\mathbb{H}^2$ realizes the equality in Cheeger’s inequality with the additional property, with respect to Example 4.3, that the bottom of the spectrum and the Cheeger constant are non-trivial. Of course, the volume of the hyperbolic plane is infinite.

Let us recall that, as proved for instance in [40], on the hyperbolic plane (equipped with the canonical volume measure) it holds $\lambda_0 = 1/4$.

Let us verify that the Cheeger constant of the hyperbolic plane equals $1/4$. In order to do so we recall the isoperimetric inequality

$$\text{Per}(A)^2 \geq 4\pi \text{m}(A) + \text{m}(A)^2,$$

for any set of finite perimeter $A \subset \mathbb{H}^2$, see [15, 43, 47] dealing with sets with smooth boundary, the extension to sets of finite perimeter can be obtained with standard approximation arguments. Moreover, we recall that geodesic balls realize the equality in (62). From (62) we easily deduce that

$$\frac{\text{Per}(A)}{\text{m}(A)} \geq 1,$$

for any $A \subset \mathbb{H}^2$ with finite perimeter. This proves that $h(\mathbb{H}^2) \geq 1$. To prove that $h(\mathbb{H}^2) = 1$ we just observe that geodesic balls with radii going to infinity verify

$$\frac{\text{Per}(B_r)}{\text{m}(B_r)} \to 1,$$

as $r \to \infty$, by direct computation or by equality in (62). This proves that $h(\mathbb{H}^2) = 1$ and therefore equality holds in Cheeger’s inequality.

**Example 4.6.** An $\text{RCD}(-1,2)$ space with finite measure satisfying $\lambda_1(X) = \frac{1}{4} h(X)^2 > 0$.

We claim that an example of (actually smooth) metric measure space with finite reference measure, verifying the $\text{RCD}(-1,2)$ condition and the equality in Cheeger’s inequality is given by the symmetric three-punctured sphere with hyperbolic metric, that we shall denote by $\mathbb{D}$. Moreover in this case

$$\lambda_1(\mathbb{D}) = \frac{1}{4} h(\mathbb{D})^2 > 0.$$

The example is strongly inspired by [19] where sharpness of the Cheeger inequality was pointed out exhibiting a family of compact Riemannian manifolds almost attaining the inequality.

Let us briefly recall how a hyperbolic metric on the three-punctured sphere can be built, referring to [28, Section 10.5] for a more detailed construction and all the relevant background on hyperbolic geometry.

This hyperbolic manifold can be seen as a degenerate pair of hyperbolic pants, with cusps in place of the three boundary components. More in detail we can also obtain it considering a degenerate hexagon on the Poincaré disk model of the hyperbolic plane (i.e. we consider three points on the boundary of the disk equidistant with respect to the standard metric and connect them with hyperbolic geodesics) and gluing it with itself along the three boundary
components (i.e. we consider the double of the starting triangle $T$). Observe that the resulting Riemannian manifold, that we shall denote by $\mathbb{D}$, is a non compact, complete hyperbolic manifold. In particular it has constant sectional curvature $-1$ and therefore it is an $\text{RCD}(-1, 2)$ metric measure space when endowed with the canonical volume measure $\text{vol}$.

We claim that $\mathbb{D}$ has finite volume, in particular it holds that $\text{vol}(\mathbb{D}) = 2\pi$. We just provide a sketch of the strategy to verify this conclusion, since the result is well known. The more direct way to check this conclusion is by directly computing $\text{vol}(T) = \pi$, using the explicit formulas for the Poincaré disk model, and then to argue that $\text{vol}(D) = 2\pi$, since $D$ is the double of $T$. Alternatively one can rely on a general version of Gauss-Bonnet formula $[35]$ taking into account the fact that $D$ is homeomorphic to the sphere with three punctures and therefore it has Euler characteristic $\chi(D) = -1$. Therefore, denoting by $K_D$ the Gaussian curvature,

$$-\text{vol}(\mathbb{D}) = \int_{\mathbb{D}} K_D \, d\text{vol} = 2\pi \chi(\mathbb{D}) = -2\pi.$$ 

We divide the verification of (63) in two steps. First let us prove that $h(\mathbb{D}) = 1$. In order to do so we rely on the study of the isoperimetric problem on hyperbolic surfaces pursued in [1]. Since $\mathbb{D}$ has three cusps (corresponding to the three punctures of the sphere), by the last part of the statement of [1, Theorem 2.2] (see also the remark after its proof) we get that, for any value of the area $0 < v \leq \pi = \text{vol}(\mathbb{D})/2$, it holds that

$$\text{Per}(A) \geq \text{vol}(A) = v,$$

for any set of finite perimeter $A$ such that $\text{vol}(A) = v$. Moreover there are sets for which equality is attained in the above inequality (neighbourhoods of cusps bounded by horocycles). Therefore, by the very definition of the Cheeger constant, it holds $h(\mathbb{D}) = 1$.

We are thus left with the verification of the identity $\lambda_1(\mathbb{D}) \leq 1/4$. Observe that thanks to Cheeger’s inequality it is sufficient to prove that $\lambda_1(\mathbb{D}) \leq 1/4$, the other inequality will follow from our estimate on the Cheeger constant. In order to do so we exhibit a sequence of Lipschitz functions $f_n : \mathbb{D} \to \mathbb{R}$ such that

$$\int_{\mathbb{D}} f_n \, d\text{vol} = 0, \quad \int_{\mathbb{D}} f_n^2 \, d\text{vol} = 1,$$

for any $n \in \mathbb{N}$ and

$$\int_{\mathbb{D}} |\nabla f_n|^2 \, d\text{vol} \to 1/4, \quad \text{as } n \to \infty.$$

The conclusion $\lambda_1(\mathbb{D}) \leq 1/4$ will follow from (2).

Let us denote by $\lambda_1^D(\Omega)$ the first Dirichlet eigenvalue of the Laplacian on a smooth domain $\Omega$ contained in a Riemannian manifold. Recall that there is a variational characterization for $\lambda_1^D$ analogous to (2).

As we already observed, $\mathbb{D}$ has constant Gaussian curvature $-1$. Therefore Cheng’s inequality [24, Theorem 1.1] applies and yields that for any $x \in \mathbb{D}$ and for any $r > 0$ it holds

$$\lambda_1^D(B^\mathbb{D}_r(x)) \leq \lambda_1^D(B^\mathbb{H}^2_r(\bar{x})),$$  

where $B^\mathbb{H}^2_r(\bar{x})$ is the ball of radius $r$ and centre $\bar{x}$ in the hyperbolic plane. Moreover it is known (see for instance the top of [24, p. 294]) that

$$\lambda_1^D(B^\mathbb{H}^2_r(\bar{x})) \leq \frac{1}{4} + \left(\frac{2\pi}{r}\right)^2,$$  

(65)
for any $r > 0$. Combining (64) with (65) we infer that for any $\epsilon > 0$ there exists $r > 0$ such that for any $x \in \mathbb{D}$ it holds
\[
\lambda_1^D(B_r^\mathbb{D}(x)) \leq \frac{1}{4} + \epsilon.
\]
(66)

Next we choose points $x_1, x_2 \in \mathbb{D}$ such that $d(x_1, x_2) > 2r$, where we denoted by $d$ the Riemannian distance induced by the hyperbolic metric on $\mathbb{D}$. By (66) and the variational characterization of the first Dirichlet eigenvalue we can find non negative Lipschitz functions $f_1^\epsilon, f_2^\epsilon$ with compact support in $B_r(x_1)$ and $B_r(x_2)$ respectively and such that
\[
\int_{\mathbb{D}} (f_1^\epsilon)^2 \, d\text{vol} = \int_{\mathbb{D}} (f_2^\epsilon)^2 \, d\text{vol} = 1
\]
and
\[
\int_{\mathbb{D}} |\nabla f_1^\epsilon|^2 \, d\text{vol} \leq \frac{1}{4} + \epsilon, \quad \int_{\mathbb{D}} |\nabla f_2^\epsilon|^2 \, d\text{vol} \leq \frac{1}{4} + \epsilon.
\]
(67)

Next we observe that we can find coefficients $a_1^\epsilon, a_2^\epsilon \in \mathbb{R}$ such that, setting $f^\epsilon := a_1^\epsilon f_1^\epsilon + a_2^\epsilon f_2^\epsilon$, it holds
\[
\int_{\mathbb{D}} f^\epsilon \, d\text{vol} = 0, \quad \int_{\mathbb{D}} (f^\epsilon)^2 \, d\text{vol} = 1
\]
and
\[
\int_{\mathbb{D}} |\nabla f^\epsilon|^2 \, d\text{vol} \leq \frac{1}{4} + \epsilon.
\]

Since $f^\epsilon$ is an admissible competitor in the variational definition of $\lambda_1(\mathbb{D})$ and $\epsilon$ is arbitrary we infer that $\lambda_1(\mathbb{D}) \leq 1/4$, as desired.

In view of the previous results and examples it is tempting to raise the following:

**Open question.** Let $(X, d, \mathcal{m})$ be an RCD($K, \infty$) space for some $K \in \mathbb{R}$. Assume that $\lambda_1 < \Sigma$ (in the case $\mathcal{m}(X) < \infty$), or $\lambda_0 < \Sigma$ (in the case $\mathcal{m}(X) = \infty$). Is it true that the corresponding Cheeger’s inequality is always strict?

**References**

[1] C. Adams, F. Morgan, “Isoperimetric curves on hyperbolic surfaces”, Proc. Amer. Math. Soc. 127 (1999), no. 5, pp. 1347–1356.
[2] L. Ambrosio, “Calculus, heat flow and curvature-dimension bounds in metric measure spaces”, Proceedings of the ICM 2018, Rio de Janeiro, Vol. 1, pp. 301–340.
[3] L. Ambrosio, E. Brué, D. Semola, “Rigidity of the 1-Bakry-Émery inequality and sets of finite perimeter in RCD spaces”, Geom. Funct. Anal. 29 (2019), no. 4, pp. 949–1001.
[4] L. Ambrosio, S. Di Marino, “Equivalent definitions of BV space and of total variation on metric measure spaces”, J. Funct. Anal., 266 (2014), pp. 4150-4188.
[5] L. Ambrosio, S. Di Marino, N. Gigli, “Perimeter as relaxed Minkowski content in metric measure spaces”, Nonlinear Analysis: Theory, Methods and Applications, 153, (2017), pp. 78-88.
[6] L. Ambrosio, N. Gigli, A. Mondino, T. Rajala “Riemannian Ricci curvature lower bounds in metric measure spaces with $\sigma$-finite measure”, Trans. Amer. Math. Soc., 367, (2015), no. 7, pp. 4661–4701.
[7] L. Ambrosio, N. Gigli, G. Savaré, “Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below”, Invent. Math. 195, (2014), no. 2, pp. 289-391.
[8] L. Ambrosio, N. Gigli, G. Savaré, “Metric measure spaces with Riemannian Ricci curvature bounded from below”, Duke Math. J., 163, (2014), pp. 1405-1490.
[9] L. Ambrosio, S. Honda, “New stability results for sequences of metric measure spaces with uniform Ricci bounds from below”, Measure theory in non-smooth spaces, 1–51, Partial Differ. Equ. Meas. Theory, De Gruyter Open, Warsaw, (2017).
[10] L. Ambrosio, A. Mondino, “Gaussian-type isoperimetric inequalities in RCD($K, \infty$) probability spaces for positive $K$”. Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 27, (2016), no. 4, pp. 497–514.
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11. L. Ambrosio, A. Mondino, G. Savaré, “Nonlinear diffusion equations and curvature conditions in metric measure spaces”, Mem. Amer. Math. Soc., Volume 262, Number 1270, (2019).
12. L. Ambrosio, D. Trevisan, “Well-posedness of Lagrangian flows and continuity equations in metric measure spaces”, Anal. PDE 7 (2014), no. 5, pp. 1179–1234.
13. D. Bakry, I. Gentil, M. Ledoux, “Analysis and geometry of Markov diffusion operators”, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 348, Springer, Cham., (2014), pp. xx+552.
14. D. Bakry, M. Ledoux, “Lévy-Gromov’s isoperimetric inequality for an infinite-dimensional diffusion generator”, Invent. Math., 123, (1996), no. 2, pp. 259–281.
15. F. Bernstein, “Über die isoperimetrische Eigenschaft des Kreises auf der Kugeloberfläche und in der Ebene”, Math. Ann. 60 (1905), no. 1, pp. 117–136.
16. J. Bertrand, C. Ketterer, I. Mondello, T. Richard, “Stratified spaces and synthetic Ricci curvature bounds” to appear in Annales de l’Institut Fourier, preprint arXiv:1804.08870.
17. R. Brooks, “On the spectrum of non-compact manifolds with finite volume”, Mathematische Zeitschrift volume 187, (1984), pp. 425–432.
18. P. Buser, “Über eine Ungleichung von Cheeger” [On an inequality of Cheeger]. Math. Z. (in German). 158, (1978), no. 3, pp. 245–252.
19. “On Cheeger’s inequality $\lambda_1 \geq h^2/4$.” Geometry of the Laplace operator, (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), pp. 29–77, Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., (1980).
20. A note on the isoperimetric constant”, Ann. Sci. Ecole Norm. Sup., (4), 15, (1982), no. 2, pp. 213–230.
21. F. Cavalletti, E. Milman, “The Globalization Theorem for the Curvature Dimension Condition”, preprint arXiv:1612.07623.
22. F. Cavalletti, A. Mondino, “Sharp geometric and functional inequalities in metric measure spaces with lower Ricci curvature bounds”, Geom. & Topol., Vol. 21, (2017), no. 1, pp. 603–645.
23. J. Cheeger “A lower bound for the smallest eigenvalue of the Laplacian”, In Gunning, Robert C. Problems in analysis (Papers dedicated to Salomon Bochner, 1969). Princeton, N. J.: Princeton University Press., pp. 195–199.
24. S.-Y. Cheng, “Eigenvalue comparison theorems and its geometric applications”, Math. Z. 143 (1975), no. 3, pp. 289–297.
25. Davies, E, “Spectral Theory and Differential Operators”, (Cambridge Studies in Advanced Mathematics), Cambridge: Cambridge University Press, (1995).
26. N. De Ponti, A. Mondino, “Sharp Cheeger-Buser type inequalities in RCD(K,∞) spaces”, J. Geom. Anal. (2020). https://doi.org/10.1007/s12220-020-00358-6.
27. M. Erbar, K. Kuwada, K.T. Sturm, “On the Equivalence of the Entropic Curvature-Dimension Condition and Bocher’s Inequality on Metric Measure Space”, Invent. Math., Vol. 201, 3, (2015), pp. 993–1071.
28. B. Farb, D. Margalit, “A primer on mapping class groups”, Princeton Mathematical Series, 49, Princeton University Press, Princeton, NJ, (2012), pp. xiv+472.
29. F. Galaz-Garcia, M. Kell, A. Mondino, G. Sosa, “On quotients of spaces with ricci curvature bounded below”, J. Funct. Anal., Vol. 275, 6, (2018), pp. 1368–1446.
30. N. Gigli, “On the differential structure of metric measure spaces and applications”, Mem. Amer. Math. Soc., 236, (2015), pp. vi+91.
31. N. Gigli, C. Ketterer, K. Kuwada, S.I. Ohta, “Rigidity for the spectral gap on RCD(K,∞)-spaces”, preprint arXiv:1709.04017, to appear in American Journ. Math.
32. N. Gigli, A. Mondino, G. Savaré, “Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows”, Proc. Lond. Math. Soc., (3), 111, (2015), no. 5, pp. 1071-1129.
33. A. Grigor’yan, “Heat Kernel and Analysis on Manifolds”, AMS/IP Studies in Advanced Mathematics, 47. American Mathematical Society, Providence, RI; International Press, Boston, MA, (2009), pp. xviii+482
34. B.-X. Han, “Rigidity of some functional inequalities on RCD spaces”, preprint arXiv:2001.07930v2, to appear in Journal de Mathématiques Pures et Appliquées, (2020) https://doi.org/10.1016/j.matpur.2020.07.004.
[35] A. Huber, “On subharmonic functions and differential geometry in the large”, Comment. Math. Helv. 32 (1957), pp. 13–72.
[36] C. Ketterer, “Obata’s rigidity theorem for metric measure spaces”, Anal. Geom. Metr. Spaces, Vol. 3, (2015), pp. 278–295.
[37] M. Ledoux, “A Simple Analytic Proof of an Inequality by P. Buser”, Proceedings of the American Mathematical Society, Vol. 121, No. 3 (1994), pp. 951-959
[38] ———, “Spectral gap, logarithmic Sobolev constant, and geometric bounds”, Surveys in differential geometry. Vol. IX, 219-240, Int. Press, Somerville, MA, (2004).
[39] J. Lott, C. Villani, “Ricci curvature for metric-measure spaces via optimal transport”, Ann. of Math., 169 (2009), pp. 903-991.
[40] H. P. McKean, “An upper bound to the spectrum of $\Delta$ on a manifold of negative curvature”, J. Differential Geometry, 4 (1970), pp. 359–366.
[41] M. Miranda, Jr., “Functions of bounded variation on good metric spaces”, J. Math. Pures Appl. (9), 82, (2003), pp. 975-1004.
[42] A. Mondino, D. Semola, “Polya-Szego inequality and Dirichlet $p$-spectral gap for non-smooth spaces with Ricci curvature bounded below”, J. Math. Pures Appl., 137, (2020), pp. 238–274
[43] R. Osserman, “The isoperimetric inequality”, Bull. Amer. Math. Soc. 84 (1978), no. 6, pp. 1182–1238.
[44] A. Petrunin, “Alexandrov meets Lott-Sturm-Villani”, Münster J. Math., Vol. 4, (2011), pp. 53–64.
[45] T. Rajala, K.-T Sturm “Non-branching geodesics and optimal maps in strong $\text{CD}(K,\infty)$-spaces”, Calc.Var. Partial Differential Equations, 50, no. 3-4, (2014).
[46] G. Savaré, “Self-improvement of the Bakry-Émery condition and Wasserstein contraction of the heat flow in $RCD(K,\infty)$ metric measure spaces”, Discrete Contin. Dyn. Syst., 34, no. 4, (2014), pp. 1641–1661.
[47] E. Schmidt, “Über die isoperimetrische Aufgabe im n-dimensionalen Raum konstanter negativer Küm- mung. I. Die isoperimetrischen Ungleichungen in der hyperbolischen Ebene und für Rotationskörper im n-dimensionalen hyperbolischen Raum”, Math. Z. 46, (1940), pp. 204–230.
[48] K.T. Sturm, “On the geometry of metric measure spaces”, Acta Math. 196 (2006), pp. 65-131.
[49] K.T. Sturm, “On the geometry of metric measure spaces. II”, Acta Math., 196 (2006), pp. 133-177.
[50] L. Tamanini, “From Harnack inequality to heat kernel estimates on metric measure spaces and applications”, preprint arXiv:1907.07163.
[51] F-Y. Wang, “Functional inequalities and spectrum estimates: the infinite measure case.”, J. Funct. Anal., 194, (2002), no. 2, pp. 288–310.