Implicit Functions from Topological Vector Spaces to Fréchet Spaces in the Presence of Metric Estimates

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Abstract
We prove an implicit function theorem for Keller $C^k_c$-maps from arbitrary real or complex topological vector spaces to Fréchet spaces, imposing only a certain metric estimate on the partial differentials. As a tool, we show the $C^k$-dependence of fixed points on parameters for suitable families of contractions of a Fréchet space. The investigations were stimulated by a recent metric approach to differentiability in Fréchet spaces by Olaf Müller. Our results also subsume generalizations of Müller’s Inverse Function Theorem for mappings between Fréchet spaces. As an application, we study existence, uniqueness and parameter-dependence of solutions to suitable ordinary differential equations in Fréchet spaces.

AMS Subject Classification. Primary 58C15; Secondary 26E15, 26E20, 35A07, 46A04, 46A13, 46A61, 46G20, 47H10, 58C20

Keywords. Fréchet space, implicit function theorem, inverse function theorem, global inverse function theorem, dependence on parameters, existence and uniqueness, ordinary differential equation, ODE, non-locally convex space, metric differential calculus, locally convex vector group, continuous inverse algebra, Nash-Moser Theorem, analytic map, holomorphic map

Introduction
One of the most famous and useful results of infinite-dimensional differential calculus beyond Banach spaces is the Nash-Moser Inverse Function Theorem (see [19], [26]; cf. [31], [29]), which provides a smooth local inverse under restrictive conditions in terms of a given fundamental sequence of seminorms (a “grading”) on the space. A variant of the Nash-Moser Theorem for implicit functions is also available [35]. These theorems are difficult to prove, and also their hypotheses are usually difficult to check in applications. Besides these results (and some variants), inverse and implicit function theorems are available for mappings between bornological spaces in the framework of “bounded differential calculus” by Colombeau (see [23] Chapter 13 for a survey). Implicit functions from topological vector spaces to Banach spaces have been studied in various settings of infinite-dimensional calculus and in varying generality (see [20], [36], [15], [17]). Furthermore, [21] provides results concerning the solutions $\phi$ to equations $f(x, \phi(x)) = 0$, where $F = \lim_{\leftarrow} F_j$ is a projective limit of Banach spaces and $f: E \times F \to F$ of the form $f = \lim_{\leftarrow} f_j$ for suitable maps $f_j: E \times F_j \to F_j$. 

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Recently, Olaf Müller formulated a metric approach to differential calculus for mappings between Fréchet spaces and provided an Inverse Function Theorem for certain “bounded differentiable” maps \[30\] (which we call “MC\(^1\)-maps” to avoid confusion with Colombeau’s venerable “bounded differential calculus”). Müller does not need to introduce gradings on \(F\) and work with “tame” smooth maps as in the case of the Nash-Moser Theorem. Rather, he equips \(F\) with a translation invariant metric \(d\) defining its topology (in which case \((F,d)\) is called a “metric Fréchet space”), and then introduces metric concepts which strongly depend on the choice of \(d\). Using \(d\) systematically, he succeeds in adapting many familiar arguments and results from the Banach case to the Fréchet case, and obtains a simple and natural proof of his inverse function theorem.

Müller asserts (see \[30, \S\]) that the metric approach is general enough to cover some of the standard applications of the Nash-Moser Theorem (like those given by Hamilton \[19\]). But this claim is too optimistic, as shown by Hiltunen \[22\] (cf. also Section 2 below). Rather, the metric approach (and its variants described in the current article) should be seen as a method which yields very strong conclusions, but only in quite restrictive situations. Despite its natural limitations, the method produces valuable new results (and we shall even encounter novel aspects of differential calculus in Banach spaces in this article).

One of the essential ideas of Müller is to replace the (unwieldy) space \(\mathcal{L}(E,F)\) of all continuous linear operators between metric Fréchet spaces \((E,d)\) and \((F,d')\) by the space\(^2\)

\[
\mathcal{L}_{d,d'}(E,F)
\]

of all linear maps from \(E\) to \(F\) which are (globally) Lipschitz continuous as mappings between the metric spaces \((E,d)\) and \((F,d')\). Then \(\mathcal{L}_{d,d'}(E,F)\) is a vector space, and also a topological group under addition with respect to the topology defined by the complete metric \((A,B) \mapsto \|A - B\|_{d,d'}\), where

\[
\|A\|_{d,d'} := \sup_{x \in E \setminus \{0\}} \frac{d'(A(x),0)}{d(x,0)}
\]

is the (minimal) Lipschitz constant \(\text{Lip}(A)\) of \(A \in \mathcal{L}_{d,d'}(E,F)\). The spaces \(\mathcal{L}_{d,d'}(E,F)\) have good properties which would be impossible for \(\mathcal{L}(E,F)\) (cf. \[27\]): For example, the evaluation map \(\mathcal{L}_{d,d'}(E,F) \times E \to F\) is continuous, and \(\mathcal{L}_d(E) := \mathcal{L}_{d,d}(E,E)\) is a topological ring with open unit group \(\mathcal{L}_d(E)^\times\) and continuous inversion \(\mathcal{L}_d(E)^\times \to \mathcal{L}_d(E)^\times, A \mapsto A^{-1}\).

In the present article, we combine Müller’s ideas with the approach to implicit functions from topological vector spaces to Banach spaces developed in \[17\].

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\(^1\)See \[22\] for a class of examples to which the method does not (and was not intended to) apply. In the case of bounded metrics, this is also clear from our Propositions \[24\] and \[25\].

\(^2\)We use the notational conventions of the present article here, rather than those from \[30\]. Also, we tacitly assume that \(d\) and \(d'\) have absolutely convex balls.
contrast to Müller, we formulate all of our results in a standard setting of differential calculus: Our $C^k_K$-maps are $C^k$-maps over $K \in \{\mathbb{R}, \mathbb{C}\}$ in the sense of Michal and Bastiani (also known as Keller’s $C^k_c$-maps)\(^3\). These are the maps widely used as the basis of infinite-dimensional Lie theory (except for the literature based on the convenient differential calculus as in [26]). By contrast, Müller uses “bounded differentiable” maps ($MC^1$-maps): these are $C^1$-maps $f : U \to F$ from an open subset $U \subseteq E$ of a metric Fréchet space $(E,d)$ to a metric Fréchet space $(F,d')$ such that $f'(U) \subseteq L_{d,d'}(E,F)$ and $f' : U \to L_{d,d'}(E,F)$ is continuous (where $f'(x) : E \to F$ is the differential of $f$ at $x$). For our results, this continuity property is not required, and this is a real advantage because the class of $MC^1$-maps can be quite small in some cases (see Remark 2.16).

Among our main results is the following Implicit Function Theorem for Keller $C^k_c$-maps from arbitrary topological vector spaces to Fréchet spaces.

**Theorem A (Generalized Implicit Function Theorem).** Let $K \in \{\mathbb{R}, \mathbb{C}\}$, $E$ be a topological $K$-vector space, $F$ be a Fréchet space over $K$, and $f : U \times V \to F$ be a $C^k_K$-map, where $U \subseteq E$ and $V \subseteq F$ are open sets. Given $x \in U$, abbreviate $f_x := f(x, \cdot) : V \to F$. Assume that $f(x_0, y_0) = 0$ for some $(x_0, y_0) \in U \times V$ and that $f'_x(y_0) : F \to F$ is invertible. Furthermore, assume that there exists a translation-invariant metric $d$ on $F$ defining its topology such that all $d$-balls are absolutely convex and

$$ \sup_{(x,y) \in U \times V} \| \id_F - f'_x(y_0)^{-1} f'_x(y) \|_{d,d} < 1 \, . $$

Then there exist open neighborhoods $U_0 \subseteq U$ of $x_0$ and $V_0 \subseteq V$ of $y_0$ such that

$$ \{(x,y) \in U_0 \times V_0 : f(x,y) = 0\} = \text{graph } \lambda $$

for a $C^k_K$-map $\lambda : U_0 \to V_0$.

Note that Theorem A also covers the case of complex analytic maps (in the usual sense, as in [4]) because a map from an open subset of a complex topological vector space to a complex locally convex space is $C^\infty$ if and only if it is complex analytic (see [3, Propositions 7.4 and 7.7]). We can also deal with local inverses.

**Theorem B (Local inverses for $C^k$-maps between Fréchet spaces).** Let $F$ be a Fréchet space over $K \in \{\mathbb{R}, \mathbb{C}\}$ and $f : U \to F$ be a $C^k_K$-map on an open subset $U \subseteq F$, where $k \in \mathbb{N} \cup \{\infty\}$. Let $x_0 \in U$. If $f'(x_0) : F \to F$ is invertible and there exists a translation-invariant metric $d$ on $F$ defining its topology such that all $d$-balls are absolutely convex and

$$ \sup_{x \in U} \| \id_F - f'(x_0)^{-1} f'(x) \|_{d,d} < 1 \, , $$

\(^3\)See [2], [9], [18], [19], [28] for discussions of such maps, in varying generality.
then there exists an open neighborhood $U_0 \subseteq U$ of $x_0$ such that $f(U_0)$ is open in $F$ and $f|_{U_0} : U_0 \to f(U_0)$ is a $C^k_F$-diffeomorphism.

We remark that, slightly more generally, (2) can be replaced by the following condition: There exist isomorphisms $S, A, T : F \to F$ of topological vector spaces such that $S \circ A \circ T \in \mathcal{L}_d(F)^\times$ and

$$\sup_{x \in U} \|S \circ (A - f'(x)) \circ T\|_{d,d} < \frac{1}{\|(S \circ A \circ T)^{-1}\|_{d,d}}.$$  

Likewise, (1) can be replaced by the condition: There exist isomorphisms of topological vector spaces $S, A, T : F \to F$ such that $S \circ A \circ T \in \mathcal{L}_d(F)^\times$ and

$$\sup_{(x,y) \in U \times V} \|S \circ (A - f'_x(y)) \circ T\|_{d,d} < \frac{1}{\|(S \circ A \circ T)^{-1}\|_{d,d}}.$$  

Both Theorem A and B will be deduced from a suitable “Inverse Function Theorem with Parameters” (Theorem 5.1), dealing with families of local diffeomorphisms. This theorem is our main result (whence we should count it as Theorem C, although we shall not restate it here in the introduction). As a technical tool, in Section 3 we prove $C^k$-dependence of fixed points on parameters, for certain “uniform families of special contractions” (as in Definition 3.6 below):

**Theorem D (Dependence of Fixed Points on Parameters).** Let $(F,d)$ be a metric Fréchet space over $K \in \{\mathbb{R}, \mathbb{C}\}$ with absolutely convex balls, and $E$ be a topological $K$-vector space. Let $P \subseteq E$ and $U \subseteq F$ be open sets, and $f : P \times U \to F$ be a continuous map such that $f_p := f(p, \cdot) : U \to F$ defines a uniform family $(f_p)_{p \in P}$ of contractions. Then the following holds:

(a) The set $Q$ of all $p \in P$ such that $f_p$ has a fixed point $x_p$ is open in $P$. Furthermore, the map $\phi : Q \to U, \phi(p) := x_p$ is continuous.

(b) If $f$ is $C^k_F$ for some $k \in \mathbb{N} \cup \{\infty\}$ and $(f_p)_{p \in P}$ is a uniform family of special contractions, then also $\phi$ is $C^k_K$.

**Variants for non-open domains.** We mention that, if $E$ is locally convex, then Theorem A and Theorem D hold just as well if $U \subseteq E$ (resp., $P \subseteq E$) is a locally convex subset with dense interior. Our proofs also cover these variants.

**The case of mappings into Banach spaces.** In the case of mappings into Banach spaces, we recover the inverse function theorem with parameters and the theorem on implicit functions from topological vector spaces to real or complex Banach spaces from [17]. The proofs of Theorem A and Theorem D are direct adaptations of the proofs in [17].

**Applications to ODEs in Fréchet spaces.** In Section 10, we prove existence, uniqueness and $C^k$-dependence on parameters for $C^k$-solutions to suitable
ordinary differential equations in Fréchet spaces. Our results (recorded as Theorem 10.3) are slightly more general than the following.

**Theorem E (Existence and Uniqueness Theorem for ODEs in Fréchet Spaces).** Let \((F,d)\) be a metric Fréchet space over \(\mathbb{R}\), with absolutely convex balls. Let \(k \in \mathbb{N}_0 \cup \{\infty\}\), \(J \subseteq \mathbb{R}\) be an interval, \(E\) be a locally convex space and \(P \subseteq E\) as well as \(U \subseteq F\) be open subsets. Let \(f : J \times U \times P \to F\) be a \(C^k\)-map which satisfies a local special contraction condition (SCC) in its second argument (as in Definition 9.5). Let \(t_0 \in J\), \(x_0 \in U\) and \(p_0 \in P\). Then there exist open neighborhoods \(U_1 \subseteq U\) of \(x_0\), \(P_1 \subseteq P\) of \(p_0\) and \(r > 0\) such that for all \((x_1,p_1) \in U_1 \times P_1\) and \(t_1 \in J_1 := [t_0 - r,t_0 + r] \cap J\), the initial value problem

\[
x'(t) = f(t,x(t),p_1), \quad x'(t_1) = x_1
\]

has a unique \(C^k\)-solution \(\phi_{t_1,x_1,p_1} : J_1 \to U\) and also the following map is \(C^k\):

\[
\Psi : J_1 \times J_1 \times U_1 \times P_1 \to U, \quad \Psi(t_1,t,x_1,p_1) := \phi_{t_1,x_1,p_1}(t).
\]

To prove Theorem E, we use Theorem A and a Lipschitz version thereof (Corollary 4.5), combined with some preparatory results concerning differentiability properties of pushforwards depending on parameters provided in Section 9. We remark that, if \((F,\|\cdot\|)\) is a Banach space and \(d(x,y) = \|x - y\|\), then the local SCC can be replaced with the ordinary local Lipschitz condition (in the middle argument) for \(f\). If also the space of parameters \(E\) is a Banach space, then an analogue of Theorem E for \(k\) times continuously Fréchet differentiable maps \((FC^k\text{-maps})\) is known (see the classical literature or also [6, §3.1, Theorem 1.1], where \(F\) is assumed finite-dimensional and \(k \geq 1\)). But for Keller \(C^k_c\text{-maps}, the result is new even in the Banach case.

**Global Inverse Function Theorems for Fréchet Spaces.** Beyond the standard theorems on local inverses, there is Hadamard’s Global Inverse Function Theorem for continuously Fréchet-differentiable self-maps of a Banach space (see [5] Chapter II.C, §4, Theorem 1], [6] Chapter 2, Theorem 3.9], or [24] Theorem 6.2.4] for a more restricted version). In Section 8 we prove analogous global inverse function theorems for self-maps of a Fréchet space, both for \(C^k\text{-maps (Theorem 8.1)} and \(MC^k\text{-maps (Theorem 8.3)}\).

**Further variations.** In [30], one also finds a discussion of left and right inverses. Along the lines of Theorem C and its proof, one could use Theorem D also to prove parameter-dependent versions of these one-sided inverse function theorems, providing left (resp., right) inverses depending on a parameter in a general topological vector space. However, we refrain from doing so here and prefer to concentrate on the central results.

**Prospects.** As the next stage, it would be interesting to study examples and to explore the scope of the approach. For example, it might go along well with certain topologically nilpotent Fréchet Lie algebras and corresponding Lie groups.
Complications of metric differential calculus. Let us mention in closing that a problem has been overlooked in [30]: Contrary to claims made there (after [30, Definition 3.13]), $L_{d,d'}(E,F)$ is not always a Fréchet space. In fact, examples show that $L_{d,d'}(E,F)$ is, in general, not a topological vector space because balls around 0 are not absorbing (see Proposition 2.2). It merely is a locally convex vector group in the sense of Raikov (as in [33], also [1]). Fortunately, this does not endanger the use of higher order differentiability properties in [30] (as clarified in Remark 2.17). It implies, however, that the class of $MC^1$-maps is quite small in many typical situations (see Remark 2.16).

Another comment concerns the type of metrics used by Müller: These are somewhat problematic, because they need not have convex balls (see Remark 1.12). By contrast, we prefer to use metrics with absolutely convex balls.

1 Preliminaries and basic facts

In this section, we set up our notation and terminology concerning differential calculus in infinite-dimensional spaces and mappings between Fréchet spaces.

Throughout the article, $K \in \{\mathbb{R}, \mathbb{C}\}$. All topological vector spaces and all topological groups are assumed Hausdorff. Our basic terminology concerning locally convex spaces follows [34]. We write $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Prerequisites concerning $C^k$-maps

Naturally, we are mainly interested in results concerning mappings from open subsets of real or complex locally convex spaces to Fréchet spaces. However, most of the results (and their proofs) apply just as well to mappings on open subsets of non-locally convex spaces, and also to mappings on suitable subsets with dense interior. Since mappings on non-open sets are useful and frequently encountered in infinite-dimensional analysis and Lie theory, we present our results in full generality. This is also vital for our main application: The approach to ODEs in Fréchet spaces in Section 10 hinges on the consideration of $C^k$-maps on sets of the form $[0,1] \times U$, with $U$ an open subset of a locally convex space.

The exact framework of differential calculus will be described now.

Definition 1.1 Given $K \in \{\mathbb{R}, \mathbb{C}\}$, let $E$ be a topological $K$-vector space and $F$ be a locally convex topological $K$-vector space. If $E$ is not locally convex, let $U \subseteq E$ be an open set. If $E$ is a locally convex space, then more generally let $U \subseteq E$ be a subset with dense interior which is locally convex in the sense that each $x \in U$ has a convex neighborhood $V \subseteq U$ (and hence arbitrarily small convex neighborhoods). Let $f : U \to F$ be a map. The map $f$ is called $C^0_K$ if it is continuous. The map $f$ is called $C^1_K$ if it is continuous and there exists a
(necessarily unique) continuous map $df : U \times E \to F$ such that
\[
df(x, y) = \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}
\]
for all $x$ in the interior $U^0$ of $U$ and all $y \in E$ (with $0 \neq t \in K$ sufficiently small).
Since $U \times E$ is open in $E \times E$ (resp., a locally convex subset with dense interior),
we can proceed by induction: Given $k \in \mathbb{N}$, we say that $f$ is $C^{k+1}_K$ if $f$ is $C^k_K$ and
$df : U \times E \to F$ is $C^k_K$. We say that $f$ is $C^\infty_K$ if $f$ is $C^k_K$ for each $k \in \mathbb{N}_0$. If $K$ is
understood, we simply write $C^k$ instead of $C^k_K$, for $k \in \mathbb{N}_0 \cup \{\infty\}$.

If $f : E \supseteq U \to F$ is $C^1_K$, then $f'(x) := df(x, \cdot) : E \to F$ is a continuous $\mathbb{K}$-linear
map (cf. [18, Chapter 1] and [9, Lemma 1.9]).

If $U \subseteq K$, we shall occasionally write $f'(x)$ also for $f'(x)(1) = \frac{df}{dx} f(x)$, in particular when dealing with solutions to differential equations. It will always be clear
from the context which meaning of $f'(x)$ is intended.

At some places, we use an alternative approach to $C^1_K$-maps based on continuous
extensions $f^{[1]}$ of directional difference quotients, which even remains meaningful
for mappings into non-locally convex spaces. This alternative approach is not
an unnecessary ballast, but invaluable for our purposes, because the proof of our
main technical result (Lemma 3.8) is best formulated in terms of the maps $f^{[1]}$.

**Definition 1.2** Let $E$ and $F$ be topological $\mathbb{K}$-vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and
$U \subseteq E$ be a subset with dense interior. Given a map $f : U \to F$, its directional
difference quotients
\[
f^{[1]}(x, y, t) := \frac{f(x + ty) - f(x)}{t}
\]
make sense for all $(x, y, t) \in U \times E \times \mathbb{K}$ such that $x + ty \in U$. Allowing now
also the value $t = 0$, we define
\[
U^{[1]} := \{(x, y, t) \in U \times E \times \mathbb{K} : x + ty \in U\}
\]
and say that $f : U \to F$ is $C^1_K$ if $f$ is continuous and there exists a (necessarily
unique) continuous map
\[
f^{[1]} : U^{[1]} \to F
\]
which extends the difference quotient map, i.e., $f^{[1]}(x, y, t) = f^{[1]}(x, y, t)$ for all
$(x, y, t) \in U^{[1]}$ such that $t \neq 0$.

**Remark 1.3** If $f$ is $C^1_K$ in the sense of Definition 1.2 then the mapping
$df : U \times E \to F$, $df(x, y) := f^{[1]}(x, y, t)$ is continuous and the differential $f'(x) :=
\frac{df}{dx}(x, \cdot) : E \to F$ is continuous linear, for each $x \in U$ (cf. [3, Proposition 2.2]).

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As introduced in [3] for maps on open sets and in [17] for maps on sets with dense interior.
We mention that no ambiguity occurs because if $E, F$ and $U \subseteq E$ happen to satisfy the hypotheses of both Definition 1.1 and Definition 1.2 then a map \( f: E \supseteq U \to F \) is $C^1_K$ in the sense of Definition 1.1 if and only if it is $C^1_K$ in the sense of Definition 1.2 (cf. [3, Proposition 7.4] or [18, Chapter 1]).

1.4 We need two versions of the Chain Rule (cf. [9, Proposition 1.15], [18, Chapter 1] and [3, Proposition 3.1]):

(a) If $E$, $F$ and $H$ are topological $\mathbb{K}$-vector spaces, $U \subseteq E$ and $V \subseteq F$ are subsets with dense interior, and \( f: U \to V \subseteq F \), \( g: V \to H \) are $C^1_K$-maps, then also the composition $g \circ f: U \to H$ is $C^1_K$, and \( (g \circ f)'(x) = g'(f(x)) \circ f'(x) \) for all $x \in U$.

(b) Let $E$ be a topological $\mathbb{K}$-vector space and $F$ as well as $H$ be locally convex topological $\mathbb{K}$-vector spaces. Let $U \subseteq E$ be open (if $E$ is not locally convex) or a locally convex subset with dense interior (if $E$ is locally convex). Let $V \subseteq F$ be a locally convex subset with dense interior. If $k \in \mathbb{N}_0 \cup \{\infty\}$ and both $f: U \to V \subseteq F$ and $g: V \to H$ are $C^1_K$-maps, then also their composition $g \circ f: U \to H$ is $C^1_K$.

Given a linear map $A: E \to F$ between vector spaces, we shall frequently write $Ax$ instead of $A(x)$.

Metric Fréchet spaces and linear, Lipschitz maps

Given a metric space $(X, d)$, we write $B^d_r(x) := \{y \in X: d(x, y) \leq r\}$ for $x \in X$ and $r \in [0, \infty]$ and $B^d_r(x) := \{y \in X: d(x, y) < r\}$ if $r > 0$. If $d$ or $X$ is understood, we also write $B^X_r(x)$ for $B^d_r(x)$, or simply $B_r(x)$. Likewise for $\overline{B}^d_r(x)$.

1.5 A metric Fréchet space is a Fréchet space $F$, equipped with a metric $d: F \times F \to [0, \infty]$ defining its topology which is translation invariant, i.e., $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in F$. In this case, we define $\|x\|_d := d(x, 0)$ for $x \in F$ and note that $d$ can be recovered from $\|\|_d: F \to [0, \infty]$ via $d(x, y) = \|x - y\|_d$. Recall that a 0-neighborhood $U \subseteq F$ is absolutely convex if it is convex and balanced, i.e., $\overline{B}^X_1(0)U \subseteq U$. We say that $d$ has symmetric (resp., balanced, resp., convex, resp., absolutely convex) balls if $\overline{B}^d_r(0) = -\overline{B}^d_r(0)$ (resp., $\overline{B}^d_r(0)$ is balanced, resp., it is convex, resp., absolutely convex) for each $r \geq 0$. Then $\overline{B}^d_r(0)$ has analogous properties, for each $r > 0$.

Example 1.6 Every Fréchet space $F$ admits a translation invariant metric $d$ which has absolutely convex balls and defines the topology of $F$. In fact, pick any sequence $w = (w_n)_{n \in \mathbb{N}}$ of real numbers $w_n > 0$ such that $\lim_{n \to \infty} w_n = 0$, and any sequence $p = (p_n)_{n \in \mathbb{N}}$ of continuous seminorms $p_n: F \to [0, \infty]$ which...
define the topology of $F$ in the sense that finite intersections of sets of the form $p_n^{-1}([0, \varepsilon[)$ (with $n \in \mathbb{N}$, $\varepsilon > 0$) form a basis of 0-neighborhoods in $F$. Then
\[
d_{w,p}: F \times F \to [0, \infty[, \quad d_{w,p}(x, y) := \sup_{n \in \mathbb{N}} \frac{p_n(x - y)}{1 + p_n(x - y)}
\]
is a metric with the desired properties.

Metrics of the form $d_{w,p}$ (as just defined) will occasionally be called standard metrics in the following.

**Lemma 1.7** Let $(F, d)$ be a metric Fréchet space with balanced balls, $t \in \mathbb{K}$ and $x \in F$. Then $\|tx\|_d \leq \|x\|_d$ if $|t| \leq 1$; $\|tx\|_d = \|x\|_d$ if $|t| = 1$; and $\|tx\|_d \leq 2|t| \cdot \|x\|_d$ if $|t| \geq 1$. In any case,
\[
\|tx\|_d \leq \max\{1, 2|t|\} \|x\|_d. \tag{7}
\]

**Proof.** The first assertion is clear since $\overline{B}_{\|x\|_d}(0)$ is a balanced 0-neighborhood. The second assertion follows from the first and the observation that $\|tx\|_d = \|t^{-1}(tx)\|_d \leq \|tx\|_d$ by the first assertion, if $|t| = 1$. If $|t| \geq 1$, set $n := \lfloor |t| \rfloor + 1 \geq |t|$, using the Gauß bracket (integer part). Then $\|tx\|_d \leq \|nx\|_d \leq n \|x\|_d \leq 2|t| \cdot \|x\|_d$. \qed

**Definition 1.8** Given metric Fréchet spaces $(E, d)$ and $(F, d')$, we let $L_{d,d'}(E, F)$ be the set of all linear maps $A: E \to F$ such that
\[
\|A\|_{d,d'} := \sup_{x \in E \setminus \{0\}} \frac{\|Ax\|_{d'}}{\|x\|_d} < \infty. \tag{8}
\]

We abbreviate $L_d(E) := L_{d,d}(E, E)$: occasionally, we write $\|A\|_d := \|A\|_{d,d}$ for $A \in L_d(E)$ (as there is little risk of confusion with $\|x\|_d := d(x, 0)$ for $x \in E$).

Condition (8) means that $A$ is Lipschitz continuous as a map $(E, d) \to (F, d')$.

To prevent misunderstandings, let us mention that although $f'(x)$ denotes the differential of $f$, we shall frequently use the symbol $d'$ in a different meaning (it denotes a metric on the range space of a map).

**Remark 1.9** The following simple properties of $L_{d,d'}(E, F)$ and the functions $\|\cdot\|_{d,d'}$ will be used later.

(a) In the situation of Definition 1.8,
\[
\|Ax\|_{d'} \leq \|A\|_{d,d'} \|x\|_d \quad \text{for all } x \in E, \tag{9}
\]
as is clear from the definition of $\|\cdot\|_{d,d'}$. Furthermore, $0 \in L_{d,d'}(E, F)$ with $\|0\|_{d,d'} = 0$ and
\[
\|A\|_{d,d'} > 0 \quad \text{if } A \in L_{d,d'}(E, F) \setminus \{0\}, \tag{10}
\]
because there is a $x \in E$ with $A.x \neq 0$ and thus $\|A\|_{d,d'} \geq \frac{\|Ax\|_{d'}}{\|x\|_d} > 0$.  

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(b) If also \((G, d'')\) is a metric Fréchet space, then
\[
\|B \circ A\|_{d,d''} \leq \|B\|_{d',d''} \|A\|_{d,d'} \quad \text{for } A \in \mathcal{L}_{d,d'}(E, F), \ B \in \mathcal{L}_{d',d''}(F, G),
\]
as an immediate consequence of (10).

(c) If \(A, B \in \mathcal{L}_{d,d'}(E, F)\), then also \(A + B \in \mathcal{L}_{d,d'}(E, F)\) and
\[
\|A + B\|_{d,d'} \leq \|A\|_{d,d'} + \|B\|_{d,d'} < \infty,
\]
because \(\|(A+B)\cdot x\|_d \leq \|A\cdot x\|_d + \|B\cdot x\|_d \leq \|A\|_{d,d'} + \|B\|_{d,d'}\) for all \(x \in E \setminus \{0\}\). Thus \(\mathcal{L}_{d,d'}(E, F)\) is a monoid under addition.

(d) If \(d'\) or \(d\) has symmetric balls, then \(\|\cdot - A\|_d = \|\cdot - A\|_{d,d'}\) \(\text{resp.}, \  \|\cdot - A\|_d = \|\cdot - A\|_{d,d'}\)
entailing that \(-A \in \mathcal{L}_{d,d'}(E, F)\) and \(\|\cdot - A\|_{d,d'} = \|A\|_{d,d'}\).

Hence \(\mathcal{L}_{d,d'}(E, F)\) is a subgroup of \(F^E\) in this case, and it follows from (a) and (c) that
\[
D_{d,d'} : \mathcal{L}_{d,d'}(E, F) \times \mathcal{L}_{d,d'}(E, F) \to [0, \infty], \ (A, B) \mapsto \|A - B\|_{d,d'}
\]
is a translation invariant metric on the abelian group \(\mathcal{L}_{d,d'}(E, F)\) which defines a topology on \(\mathcal{L}_{d,d'}(E, F)\) turning the latter into a topological group.

We shall always equip \(\mathcal{L}_{d,d'}(E, F)\) with the metric \(D_{d,d'}\) and the corresponding topology.

It is essential to have estimates on the size of integrals.

**Lemma 1.10** Let \((F, d)\) be a metric Fréchet space with convex balls, and \(\gamma : [0, 1] \to F\) a continuous curve. Then
\[
\left\| \int_0^1 \gamma(t) \, dt \right\|_d \leq \max_{t \in [0, 1]} \|\gamma(t)\|_d.
\]

**Proof.** Set \(r := \max_{t \in [0, 1]} \|\gamma(t)\|_d\). The ball \(B := B^d_r(0)\) is convex and contains \(\gamma(t)\) for each \(t \in [0, 1]\). Each Riemann sum of \(\gamma\) is a convex combination of values of \(\gamma\), whence it lies in \(B\). Since \(B\) is closed, it follows that also the limit \(\int_0^1 \gamma(t) \, dt\) of the Riemann sums lies in \(B\). \(\square\)

We record a variant of [30, Proposition 3.18]:

**Lemma 1.11** Let \((E, d_E)\) and \((F, d_F)\) be metric Fréchet spaces such that \(d_F\) has absolutely convex balls. Let \(U \subseteq E\) be a convex subset with non-empty interior and \(f : U \to F\) a \(C^1\)-map. Then
\[
\|f(y) - f(x)\|_{d_F} \leq \|y - x\|_{d_E} \sup_{t \in [0, 1]} \|f'(x + t(y - x))\|_{d_{d_E}, d_F} \quad \text{for all } x, y \in U.
\]
Proof. Apply (15) to \( \gamma: [0, 1] \to F, \gamma(t) = f'(x + t(y - x)).(y - x) \) with \( \int_0^1 \gamma'(t) \, dt = f(y) - f(x) \) and use (9) to estimate \( \|f'(x + t(y - x)).(y - x)\|_{d,F} \). \( \square \)

Remark 1.12 We warn the reader that, in the situation of Example 1.6, the metric \( D \) on \( F \) given by \( D(x,y) := \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x,y)}{1 + p_n(x,y)} \) does not have convex balls in general. For instance, \( \mathbb{R}^N \) with \( D(x,y) := \sum_{n=1}^{\infty} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} \) does not have convex balls. To see this, let \( e_n := (0, \ldots, 0, 1, 0, \ldots) \in \mathbb{R}^N \) with 1 only in the \( n \)-th slot. Then \( v_1 := 10e_1 \) and \( v_2 := 10e_2 \) are elements of the ball \( B_D^1(0) \), but \( \frac{1}{2}v_1 + \frac{1}{2}v_2 \notin B_D^1(0) \) because \( \frac{1}{2}v_1 + \frac{1}{2}v_2 \|_D = \frac{5}{8} > \frac{1}{2} \).

Since not all of the \( D \)-balls are convex, it is not clear whether Formula (16) also holds if the metric \( D \) is used. The contrary is claimed in [30, Proposition 3.18], but the author cannot make sense of its proof. This may be a serious problem for [30], because sums of metrics are used in the main results of that paper.

For many of our main results (outside Section 2), the crucial point is the validity of Lemma 1.10 rather than the absolute convexity of balls. As long as balls are balanced, the validity of the lemma should suffice to carry out the proofs.

2 The space \( \mathcal{L}_{d,d'}(E,F) \) and linear contractions

In our studies, linear contractions \( A: F \to F \) of a metric Fréchet space \((F,d)\) will play an important role, i.e., mappings \( A \in \mathcal{L}_d(F) \) such that \( \|A\|_{d,d} < 1 \). It is therefore useful to know how linear contractions look like, and moreover elements in \( \mathcal{L}_d(F) \) close to 0. With this motivation, in the current section we discuss the groups \( \mathcal{L}_{d,d'}(E,F) \) and the associated group norms \( \|\cdot\|_{d,d'} \) in more detail. In particular, we shall see that although \( \mathcal{L}_{d,d'}(E,F) \) is a vector space if \( d' \) has absolutely convex balls, it frequently happens that \( \mathcal{L}_{d,d'}(E,F) \) is not a topological vector space. We also discuss various examples which illustrate the concept of a linear contraction, and hint towards the limitations of the theory.

Although neither the vector space structure on \( \mathcal{L}_{d,d'}(E,F) \) nor other results of this section will be used later, they seem indispensable for a deeper understanding.

The following proposition slightly expands [30, Theorem 4.2]. We relegate its simple proof to Appendix A.

Proposition 2.1 Let \((E,d)\) and \((F,d')\) be metric Fréchet spaces such that all \( d' \)-balls are absolutely convex. Then the following holds:

(a) \( \mathcal{L}_{d,d'}(E,F) \) is a vector subspace of \( F^E \).
(b) The evaluation map \( \mathcal{L}_{d,d'}(E,F) \times E \rightarrow F, (A,x) \mapsto A.x \) is continuous bilinear.

c) If also \((G,d'')\) is a metric Fréchet space with absolutely convex balls, then the composition map
\[
\mathcal{L}_{d',d''}(F,G) \times \mathcal{L}_{d,d'}(E,F) \rightarrow \mathcal{L}_{d,d''}(E,G), \quad (A,B) \mapsto A \circ B
\]
is continuous bilinear.

d) \(D_{d,d'}\) from (14) is a complete metric, and has absolutely convex balls.

e) \(\mathcal{L}_d(E)\) is a unital associative \(\mathbb{K}\)-algebra, and the topology defined by \(D_{d,d}\) turns \(\mathcal{L}_d(E)\) into a topological ring.

(f) The group of units \(\{A \in \mathcal{L}_d(E) : (\exists B \in \mathcal{L}_d(E)) B \circ A = A \circ B = \text{id}_E\}\)
\(=: \mathcal{L}_d(E)^\times\) is open in \(\mathcal{L}_d(E)\) and the inversion map \(i: \mathcal{L}_d(E)^\times \rightarrow \mathcal{L}_d(E)^\times, A \mapsto A^{-1}\) is continuous.

\[\Box\]

We recall that a locally convex vector group is \(\mathbb{K}\)-vector space \(E\), equipped with a topology making \((E,+)\) a topological group and such that 0 has a basis of absolutely convex neighborhoods (see [33], also [1] §9). Unlike the case of a topological vector space, 0-neighborhoods in \(E\) need not be absorbing. Quite surprisingly, we have:

**Proposition 2.2** In the situation of Proposition 2.1, \(\mathcal{L}_{d,d'}(E,F)\) is a locally convex vector group. In some cases, \(\mathcal{L}_{d,d'}(E,F)\) is not a topological vector space. It can even happen that \(\mathcal{L}_{d,d'}(E,F)\) is discrete (but \(\neq \{0\}\)).

**Proof.** We already know that \(\mathcal{L}_{d,d'}(E,F)\) is a topological group, a vector space and that all \(\|\cdot\|_{d,d'}\)-balls around 0 are absolutely convex. Hence \(\mathcal{L}_{d,d'}(E,F)\) is a locally convex vector group.

To get an example which is not a topological vector space, equip \(\mathbb{R}\) with the unusual metric \(d: \mathbb{R} \times \mathbb{R} \rightarrow [0,\infty[, d(s,t) := \frac{|s-t|}{1+|s-t|}\). Then \(\lambda \text{id}_\mathbb{R} \in \mathcal{L}_d(\mathbb{R})\) for each \(\lambda \in \mathbb{R}\), and
\[
\|\lambda \text{id}_\mathbb{R}\|_{d,d} \geq 1 \quad \text{for all } \lambda \in \mathbb{R} \setminus \{0\}. \tag{17}
\]

In fact, \(\|\lambda \text{id}_\mathbb{R}\|_{d,d} \leq \max\{1,2|\lambda|\} < \infty\) by (7) and thus \(\lambda \text{id}_\mathbb{R} \in \mathcal{L}_d(\mathbb{R})\). If \(\lambda \neq 0\), we have
\[
\lim_{t \to \infty} \frac{d(\lambda t,0)}{d(t,0)} = \lim_{t \to \infty} \frac{|\lambda|}{1+|\lambda||t|} = 1
\]
and thus \(\|\lambda \text{id}_\mathbb{R}\|_{d,d} \geq 1\). Hence 0 is an isolated point in \(\mathcal{L}_d(\mathbb{R})\) and hence \(\mathcal{L}_d(\mathbb{R})\) is discrete (being also a topological group). \(\Box\)
Remark 2.3 In [30] p.11, it is claimed that \( \mathcal{L}_{d,d}(E,F) \) always is a Fréchet space (and hence a topological vector space), contrary to Proposition 2.2. If \( \mathcal{L}_{d,d}(E,F) \) is not a Fréchet space, then the map \( f'' : U \to \mathcal{L}_{d,d}(E,F) \) used in [30] Theorem 4.7 requires interpretation, as well as the use of higher order differentiability properties in [30] Theorem 4.6 and its proof. However, a suitable interpretation is possible (see Definition 2.15).

Our simple counterexample can be generalized further.

Proposition 2.4 Let \((F,d)\) be a metric Fréchet space with absolutely convex balls, such that \( \|Rx\|_d \subseteq \mathbb{R} \) is bounded for some non-zero vector \( x \in F \) (such \( x \) exists, e.g., if \( F \neq \{0\} \) and \( d \) has bounded image). Then \( \mathcal{L}_d(F) \) is not a topological vector space.

Proof. It is clear from the definition that \( \|\text{id}_F\|_{d,d} = 1 \), whence \( \text{id}_F \in \mathcal{L}_d(F) \). We claim that \( \|t\text{id}_F\|_d \geq 1 \) for all real numbers \( t > 0 \). If this is so, then \( \|t\text{id}_F\|_d \neq 0 \) as \( t \to 0 \), whence \( t\text{id}_F \neq 0 \) in \( \mathcal{L}_d(F) \). Thus scalar multiplication \( \mathbb{K} \times \mathcal{L}_d(F) \to \mathcal{L}_d(F) \) is discontinuous. To prove the claim, let \( x \) be as described in the proposition. Since \( d \) has convex balls, the map \( h : [0,\infty[ \to [0,\infty[ \), \( h(s) := \|sx\|_d \) is monotonically increasing. Because \( h \) is bounded by hypothesis and \( h(1) = \|x\|_d > 0 \), the limit \( \lim_{s \to \infty} h(s) \) exists and coincides with \( \sigma := \sup h([0,\infty[) > 0 \). For each \( s > 0 \), we have

\[
\|t\text{id}_F\|_{d,d} \geq \frac{\|t\text{id}_F(sx)\|_d}{\|sx\|_d} = \frac{h(ts)}{h(s)}.
\]  

(18)

For \( s \to \infty \), the right hand side of (18) tends to \( \frac{t \sigma}{\sigma} = 1 \). Thus \( \|t\text{id}_F\|_{d,d} \geq 1 \). \( \square \)

We record a crucial property of the metrics \( d_{w,p} \) from Example 1.6

Lemma 2.5 Assume that \( w = (w_n)_{n \in \mathbb{N}} \) is monotonically decreasing in the situation of Example 1.6. Given non-zero vectors \( x,y \in F \), there exist minimal numbers \( n,m \in \mathbb{N} \) such that \( p_n(x) > 0 \) and \( p_m(y) > 0 \), respectively. Then

\[
\sup_{t \in \mathbb{K} \times} \frac{d_{w,p}(ty,0)}{d_{w,p}(tx,0)} \geq \frac{w_m}{w_n}.
\]  

(19)

Proof. Abbreviate \( d := d_{w,p} \). For each \( t \in \mathbb{K} \times \), we have \( p_k(tx) = 0 \) for \( k < n \) and thus \( \|tx\|_d \leq w_n \), entailing that \( \frac{\|ty\|_d}{\|tx\|_d} \geq \frac{\|ty\|_d}{w_n} \geq \frac{w_m}{w_n} \frac{p_m(ty)}{p_m(ty)} \). Since the right hand side tends to \( \frac{w_m}{w_n} \) as \( |t| \to \infty \), the assertion follows. \( \square \)

The following example shows that \( \mathcal{L}_d(F) \) can be quite large.

Example 2.6 Let \((F_n,\|\cdot\|_n)_{n \in \mathbb{N}} \) be a sequence of Banach spaces and \( w = (w_n)_{n \in \mathbb{N}} \) be a sequence of real numbers \( w_n > 0 \) such that \( \lim_{n \to \infty} w_n = 0 \). We turn the
each
A
shall presently see. Each linear contraction
More generally, repeating the argument used to prove Proposition 2.8, we see:
F
σ
with absolutely convex balls. Then
A
x
σ
y
n
Example 2.7
General contractions of \((R^n, d)\) share a property of the shift.
Proposition 2.8
Moreover, (20) holds for each \(\ell \in \mathbb{N}\) such that \(\|A\|_{d,d} < a^{\ell-1}\).
Proof.
If the first assertion is false, there exists \(0 \neq x \in F_k\) for some \(k\) such that
y := A.x \notin F_{k+1}. Let \(n\) and \(m\) be as in Lemma 2.6. Then \(n \geq k\) and \(m \leq k\). Hence \(\|A||_{d,d} \geq \frac{w_n}{w_m} \geq \frac{w_k}{w_m} = 1\), contradicting the hypothesis that \(\|A\|_{d,d} < 1\). If the final assertion is false, instead we find \(x\) with \(y = A.x \notin F_{k+\ell}\). Then \(m \leq k + \ell - 1\) and we conclude as before that \(\|A||_{d,d} \geq \frac{w_m}{w_n} \geq \frac{w_{k+\ell-1}}{w_k} = a^{\ell-1}\), contradicting the choice of \(\ell\).
Each standard metric \(d_{w,p}\) (with \(w\) monotonically decreasing) goes along with a filtration \(F = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots\) of closed vector subspaces of \(F\), as we shall presently see. Each linear contraction \(A : F \to F\) satisfies \(A.F_k \subseteq F_{k+1}\) for each \(k\) and hence behaves, essentially, like the contractions of \(R^n\) just discussed. More generally, repeating the argument used to prove Proposition 2.8, we see:
Proposition 2.9 Let $E$ and $F$ be Fréchet spaces. Let $d := d_{w,p}$ $d' := d_{v,q}$ be metrics on $E$, resp., $F$ of the form described in Example 1.5 such that the sequences $(w_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ are monotonically decreasing. Set $E_0 := E$ and $E_k := \bigcap_{j=1}^k p_{E_0}(0)$ for $k \in \mathbb{N}$. Then $E = E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$ is a descending sequence of closed vector subspaces of $E$ such that $\bigcap_{k \in \mathbb{N}} E_k = \{0\}$. Likewise, set $F_0 := F$ and $F_k := \bigcap_{j=1}^k q_{F_0}(0)$. If $A \in \mathcal{L}_{d,d'}(E,F)$ and $\|A\|_{d,d'} < 1$, then there exists $\ell \in \mathbb{N}$ such that

$$A.E_k \subseteq F_{k+\ell} \quad \text{for each } k \in \mathbb{N}. \tag{21}$$

Moreover, $(21)$ holds for each $\ell \in \mathbb{N}$ such that $\|A\|_{d,d'} < \inf \left\{ \frac{w_{k+\ell+1}}{w_k} : k \in \mathbb{N} \right\}$. \hfill \Box

Let us sum up our observations and discuss their relevance concerning linear contractions. We have seen that, if $0 \neq A \in \mathcal{L}_d(F)$, then $tA$ need not be a contraction for any $t \neq 0$ (no matter how small). This naturally leads to the question which elements $A \in \mathcal{L}_d(F)$ have the property that $\lim_{t \to 0} tA = 0$. Since $|tS|_d = a$ for each $t \in \mathbb{R}$ such that $|t| \leq 1$ in the situation of Example 2.7 (as a consequence of Lemma 2.5), we see that $\lim_{t \to 0} tA = 0$ need not even hold if $A$ is a contraction.

Because $\mathcal{L}_{d,d'}(E,F)$ is a locally convex vector group, the following proposition provides in particular a characterization of those $A \in \mathcal{L}_{d,d'}(E,F)$ such that $tA$ can be made arbitrarily small.

Proposition 2.10 Let $E$ be a locally convex vector group over $\mathbb{K}$ and $E_0$ be its connected component of $0$. The following conditions are equivalent for $x \in E$:

(a) $tx \to 0$ in $E$ as $t \to 0$ in $\mathbb{K}$;

(b) The map $\mathbb{K} \to E$, $t \mapsto tx$ is continuous on $\mathbb{K}$ with the usual topology.

(c) $x \in E_0$.

Furthermore, $E_0$ coincides with the path component of $0$ and $E_0$ is the largest vector subspace of $E$ which is a topological vector space in the induced topology. Also, $E_0 = \bigcap_U \mathbb{K} U$, where $U$ ranges through the set of all absolutely convex $0$-neighborhoods in $E$.

Proof. For each absolutely convex $0$-neighborhood $U$ in $E$, the set $\mathbb{K} U = \bigcup_{n \in \mathbb{N}} n U = : V$ is an open vector subspace of $E$ and hence also closed. It follows that $E_0 \subseteq \bigcap_U \mathbb{K} U$. Since $U \cap V$ is absorbing in $V$ for each absolutely convex $0$-neighborhood $U \subseteq E$ (by definition of $V$) and furthermore $U \cap V$ is absolutely convex, we deduce that the topology induced by $E$ on $V$ is a vector topology. Hence $V$ is connected and thus $V \subseteq E_0$. Hence $V = E_0$.

(a) $\Rightarrow$ (b): If $tx \to 0$ as $t \to 0$, then the homomorphism of additive groups $\mathbb{K} \to E$, $t \mapsto tx$ is continuous at $0$ and hence continuous.
Example 2.12 Consider a Fréchet space $F$, equipped with a standard metrics $d = d_{w, p}$, where $w = \{w_n\}_{n \in \mathbb{N}}$ is of the form $w_n = a^n$ for some $a \in [0, 1]$. Set $F_0 := F$ and $F_k := \bigcap_{j=1}^{k} p_j^{-1}(0)$. Then $\mathcal{L}_d(F)_0 = \{0\}$. To see this, let $A \in \mathcal{L}_d(F)_0$. Given $\ell \in \mathbb{N}$, we find $t \in \mathbb{K} \setminus \{0\}$ such that $\|t A\|_{d,w} < a^{\ell-1}$. Since $a^{\ell-1} = \inf \left\{ \frac{w_k}{w_k} : k \in \mathbb{N} \right\}$, we deduce with Proposition 2.9 that $A.F =$.

(b)\(\Rightarrow\)(c): If the map $f : \mathbb{K} \to E$, $f(t) := tx$ is continuous, then $f(\mathbb{K})$ is path connected and $0 \in f(\mathbb{K})$, whence $f(\mathbb{K}) \subseteq E_0$. But $x = f(1) \in E_0$.

(c)\(\Rightarrow\)(a): If $x \in E_0$, then $\lim_{t \to 0} tx = 0$ because $E_0$ is a topological vector space, as observed at the beginning. □

The identity component $\mathcal{L}_d(F)_0$ of $\mathcal{L}_d(F)$ is a two-sided ideal in $\mathcal{L}_d(F)$ and hence a (not necessarily unital) subalgebra. It has beautiful properties.

**Proposition 2.11** Let $(F, d)$ be a metric Fréchet space over $\mathbb{K}$ with absolutely convex balls. Then $\mathcal{L}_d(F)_0$ is a Fréchet space and a so-called continuous quasi-inverse algebra, i.e., $\mathcal{L}_d(F)_0$ is a (not necessarily unital) locally convex, associative topological $\mathbb{K}$-algebra whose group $Q(\mathcal{L}_d(F)_0)$ of quasi-invertible elements is open in $\mathcal{L}_d(F)_0$ and whose quasi-inversion map $Q(\mathcal{L}_d(F)_0) \to Q(\mathcal{L}_d(F)_0)$ is continuous (hence $C^\infty_\mathbb{K}$, and even $\mathbb{K}$-analytic). In some cases, $\text{id}_F \notin \mathcal{L}_d(F)_0$.

**Proof.** Since $\mathcal{L}_d(F)_0$ is a topological vector space and closed in the complete metric abelian group $\mathcal{L}_d(F)$, it is a Fréchet space. Since $\mathcal{L}_d(F)_0$ is a topological ring with bilinear multiplication, $\mathcal{L}_d(F)_0$ is a topological algebra. Let $A \in \mathcal{L}_d(F)_0$ with $\|A\| < 1$. Then $\text{id}_F - A \in \mathcal{L}_d(F)\times$ and $(\text{id}_F - A)^{-1} = \sum_{n=0}^{\infty} A^n = \text{id}_F - B$ with $B := \sum_{n=1}^{\infty} A^n \in \mathcal{L}_d(F)_0$ (see [30] Theorem 4.1). Here $B$ is the quasi-inverse of $A$ in $\mathcal{L}_d(F)$ and hence also the quasi-inverse of $A$ in $\mathcal{L}_d(F)_0$, since $B \in \mathcal{L}_d(F)_0$ (see [11] Lemmas 2.3 and 2.5). Thus $Q(\mathcal{L}_d(F)_0)$ is a 0-neighborhood in $\mathcal{L}_d(F)_0$ and hence open, by [11] Lemma 2.6]. The inversion map $\mathcal{L}_d(F)^\times \to \mathcal{L}_d(F)^\times$ is continuous by [30] Theorem 4.1, whence also the quasi-inversion $q : Q(\mathcal{L}_d(F)) \to Q(\mathcal{L}_d(F))$ is continuous. As a consequence, the quasi-inversion map $q_0$ of $\mathcal{L}_d(F)_0$ is continuous on some 0-neighborhood (because we have seen above that it coincides with $q$ on some 0-neighborhood). By [11] Lemma 2.8], this implies continuity of $q_0$ on all of $Q(\mathcal{L}_d(F)_0)$. Now $q_0$ is $C^\infty_\mathbb{K}$ and even $\mathbb{K}$-analytic automatically (cf. Lemma 3.1, Proposition 3.2 and Proposition 3.4 in [11]). Here, $\mathbb{K}$-analyticity is understood as in [4], or as in [28] and [9]. To complete the proof, we recall that $t \text{id}_\mathbb{K} \rightarrow 0$ in $\mathcal{L}_d(\mathbb{K})$ as $t \to 0$ for $d$ as in the proof of Proposition 2.7. Hence $\text{id}_F \notin \mathcal{L}_d(\mathbb{R})_0$.

Unfortunately, it frequently happens that $\mathcal{L}_d(F)_0 = \{0\}$, as the next example shows. This can occur even if the set of contractions is large and $\mathcal{L}_d(F)$ is non-discrete (in which case $\mathcal{L}_d(F)_0$ is not open in $\mathcal{L}_d(F)$), for instance in the situation of Example 2.7.

Example 2.12 Consider a Fréchet space $F$, equipped with a standard metrics $d = d_{w, p}$, where $w = \{w_n\}_{n \in \mathbb{N}}$ is of the form $w_n = a^n$ for some $a \in [0, 1]$. Set $F_0 := F$ and $F_k := \bigcap_{j=1}^{k} p_j^{-1}(0)$. Then $\mathcal{L}_d(F)_0 = \{0\}$. To see this, let $A \in \mathcal{L}_d(F)_0$. Given $\ell \in \mathbb{N}$, we find $t \in \mathbb{K} \setminus \{0\}$ such that $\|t A\|_{d,w} < a^{\ell-1}$. Since $a^{\ell-1} = \inf \left\{ \frac{w_k}{w_k} : k \in \mathbb{N} \right\}$, we deduce with Proposition 2.9 that $A.F =$.

\(^7\)Since $\mathcal{L}_d(F)_0$ is a Fréchet space, real analyticity as in [4] coincides with real analyticity as in [28] and [9] (cf. [1] Theorem 7.1).
Proposition 2.13. The preceding example extends to much more general situations, due to the following proposition.

Proposition 2.13 Let \((F, d)\) be a metric Fréchet space with absolutely convex balls, and such that \(d\) is bounded, say \(d(F \times F) \subseteq [0, M]\) with some \(M \in ]0, \infty[\). Then there exists a sequence \(p = (p_n)_{n \in \mathbb{N}}\) of continuous seminorms \(p_1 \leq p_2 \leq \cdots\) on \(F\) such that the identity map \(\text{id}_F : (F, d) \to (F, D)\) is a quasi-isometry for the standard metric \(D = d_{w, p}\) with \(w = (2^{-n})_{n \in \mathbb{N}}\). More precisely,

\[
\frac{1}{2} \|x\|_D \leq \|x\|_d \leq \max\{4, 4M\} \|x\|_D \quad \text{for all } x \in F.
\] (22)

Proof. We define \(C_n := \overline{B}_{2^{-n}}(0)\) for \(n \in \mathbb{N}\) and let \(p_n := \mu_{C_n}\) be the Minkowski functional of \(C_n\) (as in [34, §1.33]). Then \(p_1 \leq p_2 \leq \cdots\) is an ascending sequence of continuous seminorms on \(F\), with unit balls \(\overline{B}_{1}^{p_n}(0) = C_n\). Let \(0 \neq x \in F\). We first verify the second inequality in (22).

If \(\|x\|_d > \frac{1}{2}\), then \(x \not\in C_1\) and hence \(p_1(x) > 1\), whence \(\|x\|_D \geq \frac{1}{2} p_1(x) > \frac{1}{4} \geq 4 \|x\|_d\), as required.

If \(\|x\|_d \leq \frac{1}{2}\), there exists a minimal \(n \in \mathbb{N}\) such that \(2^{-n} < \|x\|_d\). Then \(p_n(x) > 1\) and thus \(\|x\|_D \geq 2^{-n} \frac{p_n(x)}{1 + p_n(x)} \geq 2^{-n-1} \geq \frac{1}{4} \|x\|_d\), using in the last step that \(2^{-n+1} \geq \|x\|_d\) by minimality of \(n\).

To check the first inequality, pick \(n \in \mathbb{N}\) minimal such that \(2^{-n} < \|x\|_D\). Then \(n \geq 2\) (since \(\|x\|_D < \frac{1}{2}\) by definition of \(D\)), and \(2^{1-n} \geq \|x\|_D\). The definition of \(\|\cdot\|_D\) as a supremum now entails that there exists \(m \in \mathbb{N}\) such that \(2^{-m} \frac{p_m(x)}{1 + p_m(x)} > 2^{-n}\). Then \(m < n\) and \(p_m(x) \geq \frac{p_m(x)}{1 + p_m(x)} > \frac{1}{2^{n-m}}\). Thus \(p_n(2^{-m}x) > 1\) and hence \(2^{n-m}x \not\in \overline{B}_{1}^{p_n}(0) = C_m = \overline{B}_{2^{-m}}(0)\). Therefore \(2^{-m} < \|2^{-m}x\|_d \leq 2^{1-m} \|x\|_d\) (using the triangle inequality) and hence \(\|x\|_d \geq 2^{-n} \geq \frac{1}{2} \|x\|_D\).

Note that if \(\text{id}_F : (F, d) \to (F, D)\) is a quasi-isometry, then \(L_d(F) = L_D(F)\) and the identity map \(L_d(F) \to L_D(F)\) is a quasi-isometry for the metrics on operators determined by \(\|\cdot\|_d\) and \(\|\cdot\|_D\). Combination of Example 2.12 with Proposition 2.13 now shows:

Corollary 2.14 If \((F, d)\) is a metric Fréchet space with a bounded metric and absolutely convex balls, then \(L_d(F)_0 = \{0\}\). □

Varying [30], we define maps with certain metric differentiability properties.

Definition 2.15 Let \((E, d)\) and \((F, d')\) be metric Fréchet spaces over \(\mathbb{K}\), with absolutely convex balls. Let \(U \subseteq E\) be a locally convex subset with dense interior
and \( f: U \to F \) be a map. We say that \( f \) is \( MC^0_R \) if it is continuous. If \( f \) is \( C^1_K \), \( f''(U) \subseteq L_{d,d'}(E,F) \) and the map \( f': U \to L_{d,d'}(E,F) \) is continuous, then \( f \) will be called an \( MC^1_k \)-map. We also write \( f^{(0)} := f \) and \( f^{(1)} := f' \). If \( f \) is \( MC^1_K \), then \( x_0 \in U \) and \( V \subseteq U \) a connected open neighborhood of \( x_0 \) (e.g., an open convex neighborhood), then \( f'(V) \) is connected and hence contained in the connected component \( f'(x_0) + L_{d,d'}(E,F)_{x_0} \) of \( f'(x_0) \) in \( L_{d,d'}(E,F) \) (cf. Proposition 2.10). Thus \( f'|_V - f'(x_0): V \to L_{d,d'}(E,F)_{x_0} \) is again a map between subsets of Fréchet spaces. This enables a recursive definition:

If \( f \) is \( MC^1_R \) and \( V \) (as before) can be chosen for each \( x_0 \in U \) such that \( f'|_V - f'(x_0): V \to L_{d,d'}(E,F)_{x_0} \) is \( MC^{k-1}_K \), then \( f \) is called an \( MC^k_K \)-map, and we make a piecewise definition of \( f^{(k)} \) via \( f^{(k)}|_V := (f'|_V - f'(x_0))^{(k-1)} \) for \( x_0 \) and \( V \) as above. The map \( f \) is \( MC^\infty \) if it is \( MC^k \) for each \( k \in \mathbb{N}_0 \).

We mention that a suitable version of the Chain Rule holds: Compositions of composable \( MC^K \)-maps are \( MC^K \) (see Lemma B.1(f) in Appendix B).

Remark 2.16 In the setting of Corollary 2.14, we have \( L_d(F)_0 = \{0\} \), whence \( f' \) has to be locally constant for any map \( f: F \supseteq U \to F \) which is \( MC^1 \). Therefore, locally around a given point \( x_0 \), \( f \) merely is an affine linear map (with linear part in \( L_d(F) \)), since the derivative of \( f - f'(x_0) \) vanishes close to \( x_0 \). This observation shows that the class of \( MC^1 \)-maps (used as the basis of \([30 \text{ Theorem 4.7}]\) can be quite small in some cases. By contrast, our Theorem B does not require continuity of \( x \mapsto f'(x_0)^{-1}f'(x) \) as a map into \( L_d(F) \).

Remark 2.17 We mention that the topology on \( L_{d,d'}(E,F) \) arising from the metric is not the only useful one: Occasionally, it might be convenient to equip \( L_{d,d'}(E,F) \) with the translation-invariant manifold structure which makes \( L_{d,d'}(E,F)_0 \) an open \( MC^\infty \)-submanifold (and the corresponding finer topology).

Besides the preceding definition of \( MC^K \)-maps between metric Fréchet spaces, it might be interesting to explore the possibility of a metric differential calculus of \( MC^K \)-maps in arbitrary metric locally convex vector groups.

Presumably, to obtain a meaningful differential calculus for mappings between metric locally convex vector groups \((E,d)\) and \((F,d')\), one should differentiate a map \( f: E \supseteq U \to F \) only along directions in \( E_0 \); thus \( df: U \times E_0 \to F_0 \).

Motivated by the fact that inverses in \( L_d(E)^\times \) close to \( id_E \) can be expressed in terms of the Neumann series, it would also be natural to consider a certain (restrictive) class of analytic functions between metric locally convex vector groups, which are locally given by series of (metrically) Lipschitz continuous, homogeneous polynomials, with sufficiently strong convergence.

---

8These are Müller’s “bounded differentiable” maps. We avoid his terminology because of the risk of confusion with Colombeau’s venerable “bounded differential calculus,” and also because not boundedness is the main feature of the approach, but Lipschitz continuity with respect to a given choice of metrics.
3 \( C^k \)-dependence of fixed points on parameters

We now study the dependence of fixed points of contractions on parameters. In particular, we shall establish \( C^k \)-dependence under natural hypotheses. These results form the technical backbone of our generalizations of the inverse- and implicit function theorems.

**Definition 3.1** A mapping \( f: X \rightarrow Y \) between metric spaces \((X,d_X)\) and \((Y,d_Y)\) is called a contraction if there exists \( \theta \in [0,1] \) (a “contraction constant”) such that \( d_Y(f(x),f(y)) \leq \theta d_X(x,y) \) for all \( x,y \in X \).

Banach’s Contraction Theorem is a paradigmatic fixed point theorem for contractions. We recall it as a model for the slight generalizations which we actually need for our purposes:

**Lemma 3.2** Let \((X,d)\) be a (non-empty) complete metric space and \( f: X \rightarrow X \) be a contraction, with contraction constant \( \theta \in [0,1] \). Then \( f(p) = p \) for a unique point \( p \in X \). Given any \( x_0 \in X \), we have \( \lim_{n \rightarrow \infty} f^n(x_0) \) exists. Furthermore, the following a priori estimate holds, for each \( n \in \mathbb{N}_0 \):

\[
d(f^n(x_0),p) \leq \frac{\theta^n}{1-\theta} d(f(x_0),x_0).
\]

Unfortunately, we are not always in the situation of this lemma. But the simple variants compiled in the next proposition are flexible enough for our purposes.

**Proposition 3.3** Let \((X,d)\) be a metric space, \( U \subseteq X \) be a subset and \( f: U \rightarrow X \) be a contraction, with contraction constant \( \theta \). Then the following holds:

(a) \( f \) has at most one fixed point.

(b) If \( x_0 \in U \) is a point and \( n \in \mathbb{N}_0 \) such that \( f^{n+1}(x_0) \) is defined, then

\[
d(f^{k+1}(x_0),f^k(x_0)) \leq \theta^k d(f(x_0),x_0)
\]

for all \( k \in \{0,\ldots,n\} \), and \( d(f^{n+1}(x_0),x_0) \leq \frac{1-\theta^{n+1}}{1-\theta} d(f(x_0),x_0) \).

(c) If \( x_0 \in U \) is a point such that \( f^n(x_0) \) is defined for all \( n \in \mathbb{N} \), then \( (f^n(x_0))_{n \in \mathbb{N}} \) is a Cauchy sequence in \( U \), and

\[
d(f^{n+k}(x_0),f^n(x_0)) \leq \frac{\theta^n(1-\theta^k)}{1-\theta} d(f(x_0),x_0) \quad \text{for all } n,k \in \mathbb{N}_0.
\]

If \( (f^n(x_0))_{n \in \mathbb{N}} \) converges to some \( x \in U \), then \( x \) is a fixed point of \( f \), and

\[
d(x,f^n(x_0)) \leq \frac{\theta^n}{1-\theta} d(f(x_0),x_0) \quad \text{for all } n \in \mathbb{N}_0.
\]

If \( f^n(x_0) \) is defined for all \( n \in \mathbb{N} \) and \( f \) has a fixed point \( x \), then \( f^n(x_0) \to x \) as \( n \to \infty \).
(d) Assume that $U = \overline{B}_r(x_0)$ is a closed ball of radius $r$ around a point $x_0 \in X$, and $d(f(x_0), x_0) \leq (1 - \theta)r$. Then $f^n(x_0)$ is defined for all $n \in \mathbb{N}_0$. Hence $f$ has a fixed point inside $\overline{B}_r(x_0)$, provided $X$ is complete. Likewise, $f$ has a fixed point in the open ball $B_r(x_0)$ if $X$ is complete, $U = B_r(x_0)$, and $d(f(x_0), x_0) < (1 - \theta)r$.

**Proof.** (a) If $x, y \in U$ are fixed points of $f$, then $d(x, y) = d(f(x), f(y)) \leq \theta d(x, y)$, entailing that $d(x, y) = 0$ and thus $x = y$.

(b) For $k = 0$, the formula \(23\) is trivial. If $k < n$ and $d(f^{k+1}(x_0), f^k(x_0)) \leq \theta^k d(f(x_0), x_0)$, then $d(f^{k+2}(x_0), f^{k+1}(x_0)) = d(f(f^{k+1}(x_0)), f(f^k(x_0))) \leq \theta d(f^{k+1}(x_0), f^k(x_0)) \leq \theta^{k+1} d(f(x_0), x_0)$. Thus \(23\) holds in general.

Using the triangle inequality and the summation formula for the geometric series, we obtain the estimates $d(f^{n+1}(x_0), x_0) \leq \sum_{k=0}^{n} d(f^{k+1}(x_0), f^k(x_0)) \leq \sum_{k=0}^{n} \theta^k d(f(x_0), x_0) = \frac{1 - \theta^{n+1}}{1 - \theta} d(f(x_0), x_0)$, as asserted.

(c) Using both of the estimates from (b), we obtain

$$d(f^{n+k}(x_0), f^n(x_0)) \leq \frac{1 - \theta^k}{1 - \theta} d(f^{n+1}(x_0), f^n(x_0)) \leq \frac{1 - \theta^n}{1 - \theta} \theta^n d(f(x_0), x_0).$$

Thus \(24\) holds, and thus $(f^n(x_0))_{n \in \mathbb{N}}$ is a Cauchy sequence. If $f^n(x_0) \to x$ for some $x \in U$, then $x = \lim_{n \to \infty} f^{n+1}(x_0) = f(\lim_{n \to \infty} f^n(x_0)) = f(x)$ by continuity of $f$, whence indeed $x$ is a fixed point of $f$. Letting now $k \to \infty$ in \(24\), we obtain \(25\).

To prove the final assertion, assume that $f$ has a fixed point $x$ and that $f^n(x_0)$ is defined for all $n$. We choose a completion $\overline{X}$ of $X$ (with $X \subseteq \overline{X}$) and let $\overline{U}$ be the closure of $U$ in $\overline{X}$. Then $f$ extends to a contraction $\overline{U} \to \overline{X}$, which we also denote by $f$. Since $(f^n(x_0))_{n \in \mathbb{N}}$ is a Cauchy sequence in $\overline{U}$ and $\overline{U}$ is complete, we deduce that $f^n(x_0) \to y$ for some $y \in \overline{U}$. Then both $y$ and $x$ are fixed points of $f$ and hence $x = y$, by (a).

(d) We show by induction that $f^n(x_0)$ is defined for all $n \in \mathbb{N}$. For $n = 1$, this is trivial. If $f^n(x_0)$ is defined, then

$$d(f^n(x_0), x_0) \leq \frac{1 - \theta^n}{1 - \theta} d(f(x_0), x_0) \leq \frac{1 - \theta^n}{1 - \theta} (1 - \theta)r \leq r$$

by (b) and thus $f^n(x_0) \in \overline{B}_r(x_0)$, whence also $f^{n+1}(x_0) = f(f^n(x_0))$ is defined. Then $f^n(x_0) \in \overline{B}_r(x_0)$ for each $n \in \mathbb{N}$. By (c), $(f^n(x_0))_{n \in \mathbb{N}}$ is a Cauchy sequence. If $X$ is complete, then so is $\overline{B}_r(x_0)$ and thus $(f^n(x_0))_{n \in \mathbb{N}}$ converges to some point $x \in \overline{B}_r(x_0)$, which is a fixed point of $f$ by (c). Finally, if $U = B_r(x_0)$ and $d(f(x_0), x_0) < (1 - \theta)r$, there exists $s \in [0, r]$ such that $d(f(x_0), x_0) \leq (1 - \theta)s$. By the preceding, $f^n(x_0) \in \overline{B}_s(x_0) \subseteq B_r(x_0)$ for all $n \in \mathbb{N}$, and $f$ has a fixed point in $\overline{B}_s(x_0) \subseteq B_r(x_0)$.

A certain class of contractions of Fréchet spaces is of utmost importance for our purposes.
Definition 3.4. Let \((F,d)\) be a metric Fréchet space over \(\mathbb{K}\) and \(U \subseteq F\). A map \(f: U \rightarrow F\) is called a special contraction if there exists \(\theta \in [0,1]\) such that

\[
(\forall s \in \mathbb{K}) (\forall x,y \in U) \quad d(sf(x),sf(y)) \leq \theta d(sx, sy).
\]

We then call \(\theta\) a special contraction constant for \(f\).

Lemma 3.5. Let \((F,d)\) be a metric Fréchet space with absolutely convex balls, \(U \subseteq F\) be a locally convex subset with dense interior and \(f: U \rightarrow F\) be a \(C^1\)-map. Consider the following conditions:

(a) \(f\) is a special contraction;

(b) \(f'(U) \subseteq \mathcal{L}_d(F)\) and \(\sup_{x \in U} \|f'(x)\|_{d,d} < 1\).

Then (a) implies (b). If \(U\) is convex, then (a) and (b) are equivalent.

Proof. Assume that \(f\) is a special contraction with special contraction constant \(\theta \in [0,1]\). Let \(x \in U^0\), \(y \in F\) and \(t \in \mathbb{R}^+\) with \(x+ty \in U\). Since \(f\) is a special contraction, we have \(\|\frac{1}{t}(f(x+ty)-f(x))\|_d \leq \theta \|\frac{1}{t}((x+ty)-x)\|_d = \theta \|y\|_d\). Letting \(t \to 0\), we deduce that \(\|f'(x).y\|_d \leq \theta \|y\|_d\). Since \(U^0\) is dense in \(U\), it follows by continuity that \(\|f'(x).y\|_d \leq \theta \|y\|_d\) for all \(x \in U\) and \(y \in F\). For each \(x \in U\), this gives \(\|f'(x)\|_{d,d} \leq \theta\), since \(y\) was arbitrary. Thus \(\sup \|f'(U)\|_{d,d} \leq \theta\).

Conversely, suppose that \(U\) is convex and \(\theta := \sup \|f'(U)\|_{d,d} < 1\). Given \(x, y \in U\) and \(s \in \mathbb{K}\), the map \(\gamma: [0,1] \rightarrow F\), \(\gamma(t) := sf(x+t(y-x))\) is \(C^1\) and

\[
sf(y) - sf(x) = \gamma(1) - \gamma(0) = \int_0^1 \gamma'(t) dt,
\]

where \(\gamma'(t) = sf'(x+t(y-x)).(y-x) = f'(x+t(y-x)).(sy-sx)\) with \(\|\gamma'(t)\|_d \leq \|f'(x+t(y-x))\|_{d,d} \|sy-sx\|_d \leq \|sy-sx\|_d\). Hence \(d(sf(y), sf(x)) \leq \theta d(sy, sx)\), by Lemma 1.1 and so \(f\) is a special contraction. \(\square\)

We are interested in uniform families of contractions.

Definition 3.6. Let \((F,d)\) be a metric Fréchet space over \(\mathbb{K}\), \(U \subseteq F\) and \(P\) be a set. A family \(\{f_p\}_{p \in P}\) of mappings \(f_p: U \rightarrow F\) is called a uniform family of contractions if there exists \(\theta \in [0,1]\) (a "uniform contraction constant") such that

\[
\|f_p(x) - f_p(y)\|_d \leq \theta \|x - y\|_d \quad \text{for all } x,y \in U \text{ and } p \in P.
\]

If \(\|s(f_p(x) - f_p(y))\|_d \leq \theta \|s(x - y)\|_d\) for all \(s \in \mathbb{K}\), \(x,y \in U\) and \(p \in P\), we call \(\{f_p\}_{p \in P}\) a uniform family of special contractions and \(\theta\) a uniform special contraction constant.

If \(U\) is closed and each \(f_p\) is a self-map of \(U\) here, then Banach’s Contraction Theorem ensures that, for each \(p \in P\), the map \(f_p\) has a unique fixed point \(x_p\). Our goal is to understand the dependence of \(x_p\) on the parameter \(p\). In particular,
for \( P \) a subset of a topological \( \mathbb{K} \)-vector space, we want to find conditions ensuring that the map \( P \to F, \ p \to x_p \) is continuously differentiable. Let us discuss continuous dependence of fixed points on parameters first.

**Lemma 3.7** Let \( P \) be a topological space and \((F,d)\) be a metric Fréchet space. Let \( U \subseteq F \) and \( f : P \times U \to F \) be a continuous map such that \( (f_p)_{p \in P} \) is a uniform family of contractions, where \( f_p := f(p, \bullet) : U \to F \). We assume that \( f_p \) has a fixed point \( x_p \), for each \( p \in P \). Furthermore, we assume that \( U \) is open or \( f(P \times U) \subseteq U \) (whence every \( f_p \) is a self-map of \( U \)). Then the map \( \phi : P \to F, \ \phi(p) := x_p \) is continuous.

**Proof.** Let \( \theta \in [0,1[ \) be a uniform contraction constant for \( (f_p)_{p \in P} \). Given \( p \in P \) and \( \varepsilon > 0 \), we find a neighborhood \( Q \subseteq P \) of \( p \) such that \( \|f_p - f_q\| \leq \|f_p - f_q(x_p)\|d \leq (1 - \theta)d \) for all \( q \in Q \). If \( f(P \times U) \subseteq U \), then \( \|x_p - x_q\|d \leq \frac{1}{1-\theta} \|x_p - f_q(x_p)\|d \leq \varepsilon \), by (23) in Proposition 3.3 (c). If \( U \) is open, we may assume that \( \overline{B}^d(x_p) \subseteq U \) after shrinking \( \varepsilon \) and \( Q \). Then Proposition 3.3 (d) applies to \( f_q \) as a map \( \overline{B}^d(x_p) \to F \) for each \( q \in Q \), showing that \( f'_q(x_p) \) is defined for each \( n \in \mathbb{N} \) and \( x_q = \lim_{n \to \infty} f^n_q(x_p) \in \overline{B}^d(x_p) \), that is, \( \|x_p - x_q\|d \leq \varepsilon \). \( \square \)

**Lemma 3.8** Let \( E \) be a topological \( \mathbb{K} \)-vector space and \( P \subseteq E \) be a subset with dense interior. Let \((F,d)\) be a metric Fréchet space over \( \mathbb{K} \) and \( U \subseteq F \) be a subset with dense interior. Furthermore, let \( f : P \times U \to F \) be a \( C^1_K \)-map such that \((f_p)_{p \in P} \) is a uniform family of special contractions, where \( f_p := f(p, \bullet) : U \to F \). We assume that \( f_p \) has a fixed point \( x_p \), for each \( p \in P \). Finally, we assume that \( U \) is open or \( f(P \times U) \subseteq U \) (whence every \( f_p \) is a self-map of \( U \)). Then the map \( \phi : P \to F, \ \phi(p) := x_p \) is \( C^1_K \).

**Proof.** Since \( f \) is continuous, Lemma 3.7 shows that \( \phi \) is continuous. Thus \( \phi^{[1]} \) (as in \( \phi \) in Definition 3.2) is continuous. To see that \( \phi \) is \( C^1 \), it only remains to show that, for all \( p_0 \in P \) and \( q_0 \in E \), there exists an open neighborhood \( W \subseteq P^{[1]} \) of \( (p_0,q_0,0) \) and a continuous map \( g : W \to F \) which extends the difference quotient map \( \phi^{[1]}|_{W \cap P^{[1]}}, W \cap P^{[1]} \to F \). Then \( \phi^{[1]} \) has a continuous extension \( \phi^{[1]} \) to all of \( P^{[1]} \) (cf. 3.2 Exercise 3.2 A (b))] and thus \( \phi \) will be \( C^1 \). Our strategy is the following: We write

\[
(f_{p+tq}^{n+1}(x_p) - x_p)/t = \sum_{k=0}^{n} (f_{p+tq}^{k+1}(x_p) - f_{p+tq}^{k}(x_p))/t
\]

for \( (p,q,t) \) in a suitable neighborhood \( W \) of \((p_0,q_0,0)\), with \( t \neq 0 \). For \( W \) sufficiently small, the left hand side converges to \( \frac{x_{p+tq} - x_p}{t} = \frac{\phi(q)-\phi(p)}{t} \) as \( n \to \infty \). Furthermore, we can achieve that each term on the right hand side extends continuously to all of \( W \), and that the series converges uniformly to a continuous function on \( W \). This will be our desired continuous extension \( g \).
For all $k$ and note that $h_{k}$ is a continuous map such that $B\bar{z}_{2\varepsilon}(x_{p_{0}}) \subseteq U$. Since $f$ and $\phi$ are continuous and $f_{p_{0}}(x_{p_{0}}) = x_{p_{0}}$, we find an open neighborhood $Q \subseteq P$ of $p_{0}$ such that $\|x_{p} - x_{p_{0}}\|_{d} \leq \varepsilon$ and $\|x_{p} - f_{q}(x_{p})\|_{d} \leq (1 - \theta)\varepsilon$ for all $p, q \in Q$. Then $f_{q}^{k}(x_{p})$ is defined for all $k \in \mathbb{N}$, $f_{q}^{k}(x_{p}) \in B_{\bar{z}_{\varepsilon}}(x_{p})$, and $x_{q} = \lim_{k \to \infty} f_{q}^{k}(x_{p}) \in B_{\bar{z}}(x_{p})$, by Proposition 3.3(d). We now set $W_{0} := Q^{[1]}$ and note that, if $(p, q, t) \in Q^{[1]}$, then $p, p + tq \in Q$, whence $f_{p+tq}^{k}(x_{p})$ is defined for all $k \in \mathbb{N}$ and $\lim_{k \to \infty} f_{p+tq}^{k}(x_{p}) = x_{p+tq}$ (by the preceding considerations).

In either case, we define

$$h_{0}: W_{0} \to F, \quad h_{0}(p, q, t) = f^{[1]}(p, x_{p}, q, 0, t)$$

and note that $h_{0}$ is a continuous map such that

$$h_{0}(p, q, t) = (f_{p+tq}(x_{p}) - f_{p}(x_{p}))/t = (f_{p+tq}(x_{p}) - x_{p})/t \quad \text{if } t \neq 0. \quad (27)$$

For all $k \in \mathbb{N}$ and $(p, q, t) \in W_{0}$ with $t \neq 0$, we have

$$
\frac{f_{p+tq}^{k+1}(x_{p}) - f_{p+tq}^{k}(x_{p})}{t}
= f \left( p + tq, f_{p+tq}^{k-1}(x_{p}) + t \frac{f_{p+tq}^{k}(x_{p}) - f_{p+tq}^{k-1}(x_{p})}{t}, f_{p+tq}^{k-1}(x_{p}), 0, f_{p+tq}^{k-2}(x_{p}), f_{p+tq}^{k-1}(x_{p}), t \right).
\quad (28)
$$

Recursively, we define

$$h_{k}: W_{0} \to F, \quad h_{k}(p, q, t) := f^{[1]}(p + tq, f_{p+tq}^{k-1}(x_{p}), 0, h_{k-1}(p, q, t), t)$$

for $k \in \mathbb{N}$. A simple induction based on (27) and (28) shows that the definition of $h_{k}$ makes sense for each $k \in \mathbb{N}$, and that

$$h_{k}(p, q, t) = \frac{f_{p+tq}^{k+1}(x_{p}) - f_{p+tq}^{k}(x_{p})}{t} \quad \text{for all } (p, q, t) \in W_{0} \text{ with } t \neq 0. \quad (29)$$

The function $h_{0}: W_{0} \to F$, $(p, q, t) \mapsto f^{[1]}(p, x_{p}, q, 0, t)$ being continuous, we find an open neighborhood $W \subseteq W_{0}$ of $(p_{0}, q_{0}, 0)$ and $C \in [0, \infty)$ such that

$$\|f^{[1]}(p, x_{p}, q, 0, t)\|_{d} \leq C \quad \text{for all } (p, q, t) \in W.$$

For all $(p, q, t) \in W$ such that $t \neq 0$, we have

$$\|t^{-1} f_{p+tq}^{k+1}(x_{p}) - t^{-1} f_{p+tq}^{k}(x_{p})\|_{d} \leq \theta^{k} \|t^{-1} f_{p+tq}^{k}(x_{p}) - t^{-1} x_{p}\|_{d} = \theta^{k} \|f^{[1]}(p, x_{p}, q, 0, t)\|_{d} \leq \theta^{k} C;$$

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to obtain the inequality, we used repeatedly that \( f_{p+tq} \) is a special contraction with special contraction constant \( \theta \). Combining the preceding estimates with \( \|h_k(p, q, t)\|_d \leq \theta^k C \) for all \((p, q, t) \in W\) such that \( t \neq 0\), and thus

\[
\|h_k(p, q, t)\|_d \leq \theta^k C \quad \text{for all } (p, q, t) \in W,
\]

because \( h_k \) is continuous and \( W \cap P^{[1]} \) is dense in \( W \). As a consequence, \( \sum_{k=0}^{\infty} \sup_{W} \|h_k(W)\|_d \leq \sum_{k=0}^{\infty} \theta^k C \leq \frac{1}{1-\theta} C < \infty \), whence the series \( \sum_{k=0}^{\infty} h_k|W| \) of continuous functions into \((F, d)\) converges uniformly. Thus

\[
g(p, q, t) := \sum_{k=0}^{\infty} h_k(p, q, t)
\]

exists for all \((p, q, t) \in W\), and \( g: W \to F \) is continuous. It only remains to observe that

\[
\frac{f_{p+tq}(x_p) - x_p}{t} = \sum_{k=0}^{\infty} \frac{f_{p+tq}^k(x_p) - f_{p+tq}^k(x_p)}{t} = \sum_{k=0}^{\infty} h_k(p, q, t)
\]

for all \((p, q, t) \in W\) such that \( t \neq 0\). Since the left hand side converges to \( \frac{x_{p+tq} - x_p}{t} \) as \( n \to \infty\) and the right hand side converges to \( g(p, q, t) \), we obtain

\[
\frac{\phi(p + tq) - \phi(p)}{t} = \sum_{k=0}^{\infty} h_k(p, q, t) = g(p, q, t).
\]

Thus \( g: W \to F \) is a continuous map which extends \( \phi^{[1]}|_{W \cap P^{[1]}} \), as desired. This completes the proof.

We are now in the position to prove Theorem D from the introduction (the main result of this section).

**Proof of Theorem D.** (a) For the proof of (a), we assume only that \( P \) is a topological space (the hypotheses that \( P \subseteq E \) for a topological vector space \( E \) is not required). Let \( \theta \in [0, 1] \) be a uniform contraction constant for \((f_p)_{p \in P}\). If \( p \in Q \), there is \( r > 0 \) such that \( \overline{B}_r(x_p) \subseteq U \). There is a neighborhood \( S \subseteq Q \) of \( p \) such that \( \|f_q(x_p) - x_p\|_d = \|f_q(x_p) - f_p(x_p)\|_d \leq (1-\theta)r \) for all \( q \in S \). Now Proposition 3.3(d) shows that \( f_q \) has a fixed point \( x_q \) in \( \overline{B}_r(x_p) \), for each \( q \in S \). Thus \( S \subseteq Q \) and we deduce that \( Q \) is open. By Lemma 3.7, the map \( \phi: Q \to U \), \( \phi(p) := x_p \) is continuous.

(b) Let \( \theta \in [0, 1] \) be a uniform special contraction constant for \((f_p)_{p \in P}\). We may assume that \( k < \infty \). The proof is by induction on \( k \in \mathbb{N} \). Since \( \phi \) is continuous by (a), and since we only need to check that \( \phi \) is \( C^k \) on an open neighborhood of a given point \( p_0 \in Q \), after replacing \( U \) by some open ball around \( \phi(p_0) \) and
replacing \( Q \) by a smaller open neighborhood of \( p_0 \) in \( Q \) we may assume henceforth that \( U \) is convex.

The case \( k = 1 \) is covered by Lemma 3.8. Since \( \phi(p) = f(p, \phi(p)) \) for all \( p \in Q \), the Chain Rule shows that

\[
\phi'(p).q = f'(p, \phi(p)).(q, \phi'(p).q) \quad \text{for all } p \in Q \text{ and } q \in E.
\]  

(31)

If \( k \geq 2 \) and \( \phi \) is \( C^{k-1} \) by induction, then the map

\[
g: (Q \times E) \times F \to F, \quad g(p, q, y) := f'(p, \phi(p)).(q, y)
\]

is \( C^{k-1} \). By linearity in \((q, y)\), for the partial differential with respect to \( y \) we obtain \( g'_{(p,q)}(y).z = f'(p, \phi(p)).(0, z) = f'_p(\phi(p)).z \), whence \( g'_{(p,q)}(y) = f'_p(\phi(p)) \) and thus \( \|g'_{(p,q)}(y)\|_d = \|f'_p(\phi(p))\|_d \leq \theta \), using Lemma 3.5. Hence, by Lemma 3.5, \((g_{(p,q)})(p,q)\in\mathbb{Q}\times\mathbb{E}\) is a uniform family of special contractions. Since \( d\phi_{(p,q)} = \phi'(p).q \) is the fixed point of \( g_{(p,q)} \) by (31), applying the inductive hypothesis we see that \( d\phi: Q \times E \to F \) is \( C^{k-1} \). Hence \( \phi \) is \( C^k \), which completes the inductive proof. \( \square \)

4 Preparatory results concerning local inverses

We now prove an Inverse Function Theorem for self-maps of a Fréchet space, which provides local inverses which are Lipschitz continuous with respect to a given metric. We also provide a variant dealing with families of local inverses, and use it to construct continuous implicit functions.

Our first theorem, and its proof, is the direct analogue of [17, Theorem 5.3] (dealing with Banach spaces) for Fréchet spaces. It is a variant of [30, Theorem 4.5]. Parts of the proof will be re-used later.

**Theorem 4.1 (Lipschitz Inverse Function Theorem)** Let \((E, d)\) be a metric Fréchet space over \( \mathbb{K} \), with absolutely convex balls. Let \( r > 0 \), \( x \in E \), and \( f: B_d^r(x) \to E \) be a map. We suppose that there exists \( A \in \mathcal{L}_d(E) \) such that

\[
\sigma := \sup \left\{ \frac{\|f(z) - f(y) - A.(z - y)\|_d}{\|z - y\|_d} : y, z \in B_r(x), y \neq z \right\} < \frac{1}{\|A^{-1}\|_d}. \quad (32)
\]

Then the following holds:

(a) \( f \) has open image and is a homeomorphism onto its image.

(b) The inverse map \( f^{-1}: f(B_r(x)) \to B_r(x) \) is Lipschitz continuous with respect to the metric \( d \), with

\[
\text{Lip}(f^{-1}) \leq \frac{1}{\|A^{-1}\|_d - \sigma}. \quad (33)
\]
\[ (c) \] Abbreviating \( a := \| A^{-1} \|^{-1}_d - \sigma > 0 \) and \( b := \| A \|_d + \sigma, \) we have
\[
\| z - y \|_d \leq \| f(z) - f(y) \|_d \leq b \| z - y \|_d \quad \text{for all } y, z \in B_r(x). \tag{34}
\]

(d) The following estimates for the size of images of balls are available: For every \( y \in B_r(x) \) and \( s \in [0, r - \| y - x \|_d], \)
\[
B_{as}(f(y)) \subseteq f(B_s(y)) \subseteq B_{bs}(f(y)) \tag{35}
\]
holds. In particular, \( B_{ar}(f(x)) \subseteq f(B_r(x)) \subseteq B_{br}(f(x)). \)

Remark 4.2 Note that the condition (32) means that the remainder term
\[
\hat{f} : B_r(x) \to E, \quad \hat{f}(y) := f(y) - f(x) - A.(y - x)
\]
in the affine-linear approximation \( f(y) = f(x) + A.(y - x) + \hat{f}(y) \) is Lipschitz continuous with respect to the metric \( d, \) with \( \text{Lip}(\hat{f}) = \sigma < \| A^{-1} \|^{-1}_d. \)

Remark 4.3 To understand the constants in Theorem 4.1 better, we recall that \( \| A^{-1} \|^{-1}_d \) can be interpreted as a minimal distortion factor, in the following sense: For each \( u \in E, \) we have \( \| u \|_d = \| A^{-1}.(A.u) \|_d \leq \| A^{-1} \|_d \cdot \| A.u \|_d \) and thus
\[
\| A.u \|_d \geq \| A^{-1} \|^{-1}_d \| u \|_d \quad \text{for all } u \in E. \tag{36}
\]

Thus \( A \) increases the distance of each given vector from 0 by a factor of at least \( \| A^{-1} \|^{-1}_d. \) Furthermore, \( \| A^{-1} \|^{-1}_d \) is maximal among such factors, as one verifies by going backwards through the preceding lines. Similarly, since \( A^{-1}B_s(0) \subseteq B_{\| A^{-1} \|^{-1}_d.s}(0) \) and thus \( B_s(0) \subseteq A.B_{\| A^{-1} \|^{-1}_d.s}(0) \) for each \( s > 0, \) we find that
\[
A.B_s(0) \supseteq B_{\| A^{-1} \|^{-1}_d.s}(0) \quad \text{for each } s > 0. \tag{37}
\]

Proof of Theorem 4.1 (c) Given \( y, z \in B_r(x), \) we have
\[
\| f(z) - f(y) \|_d = \| f(z) - f(y) - A.(z - y) \|_d \\
\leq \| f(z) - f(y) - A.(z - y) \|_d + \| A.(z - y) \|_d \\
\leq (\sigma + \| A \|_d) \| z - y \|_d = b \| z - y \|_d
\]
and
\[
\| z - y \|_d = \| A^{-1}.(f(z) - f(y) - A.(z - y)) - (A^{-1}.f(z) - A^{-1}.f(y)) \|_d \\
\leq \| A^{-1} \|_d \cdot \| f(z) - f(y) - A.(z - y) \|_d + \| A^{-1}.f(z) - A^{-1}.f(y) \|_d \\
\leq \sigma \| A^{-1} \|_d \cdot \| z - y \|_d + \| A^{-1} \|_d \cdot \| f(z) - f(y) \|_d,
\]
whence \( \| f(z) - f(y) \|_d \geq (\| A^{-1} \|^{-1}_d - \sigma) \| z - y \|_d = a \| z - y \|_d. \) Thus (34) holds.

(b) As a consequence of (33), \( f \) is injective, a homeomorphism onto its image, and \( \text{Lip}(f^{-1}) \leq a^{-1} = (\| A^{-1} \|^{-1}_d - \sigma)^{-1}. \)
Given a Fréchet space with absolutely convex balls, and let the corresponding results for the theorem with parameters. The result, and its proof, can be re-used later to prove we are now in the position to formulate the first version of an inverse function consequence of (d), the image of $f$ hence $v$ such that $c \in f(y) + A_B(0)$. Then the following holds:

$$
\|g(v) - y\|_d \leq \frac{\|v_y - A^{-1}.f(v) + A^{-1}.f(y)\|_d + \|A^{-1}.c - A^{-1}.f(y)\|_d}{\lambda} \leq \lambda c
$$

Thus $g(\overline{B}_d(y)) \subseteq \overline{B}_d(y)$. The map $g: \overline{B}_d(y) \to \overline{B}_d(y)$ is a contraction, since

$$
\|g(v) - g(w)\|_d = \|v - w - A^{-1}.(f(v) - f(w))\|_d \\
\leq \|A^{-1}\|_d \cdot \|f(v) - f(w) - A.(v - w)\|_d \\
\leq \sigma \cdot \|A^{-1}\|_d \cdot \|v - w\|_d
$$

for all $v, w \in \overline{B}_d(y)$, where $\sigma \|A^{-1}\|_d < 1$. By Banach’s Contraction Theorem (Lemma 32), there exists a unique element $v_0 \in \overline{B}_d(y)$ such that $g(v_0) = v_0$ and hence $f(v_0) = c$.

(a) We have already seen that $f$ is a homeomorphism onto its image. As a consequence of (d), the image of $f$ is open. \hfill \Box

We are now in the position to formulate the first version of an inverse function theorem with parameters. The result, and its proof, can be re-used later to prove the corresponding results for $C^k$-maps.

**Theorem 4.4 (Continuous families of local inverses)** Let $(F, d)$ be a metric Fréchet space with absolutely convex balls, and $P$ be a topological space. Let $r > 0$, $x \in F$, and $f: P \times B \to F$ be a continuous mapping, where $B := B^d_r(x)$. Given $p \in P$, we abbreviate $f_p := f(p, \cdot): B \to F$. We suppose that there exists $A \in \mathcal{L}_d(F)$ such that

$$
\sigma := \sup_{p \in P, y, z \in B, y \neq z} \left\{ \frac{\|f_p(z) - f_p(y) - A.(z - y)\|_d}{\|z - y\|_d} : \right\} < \frac{1}{\|A^{-1}\|_d}.
$$

Then the following holds:

(a) $f_p(B)$ is open in $F$ and $f_p|_B$ is a homeomorphism onto its image, for each $p \in P$. 27
Corollary 4.5 (Continuous Implicit Functions)
As a direct consequence, we obtain an implicit function theorem. Theorem 4.1 (d), all \((q,z)\) with dense interior if \(h\) is an absolutely convex ball. Let \(W\) be given. There is an open neighborhood \(Q\) of \((p,z)\) such that \(f_p(y) = z\). Let \(\varepsilon \in ]0, r - \|y - x\|_d]\) be given. There is an open neighborhood \(Q\) of \(p\) in \(P\) such that \(d(f_q(y), f_p(y)) < \frac{\varepsilon}{2}\) for all \(q \in Q\), by continuity of \(f\). Then, by (35) in Theorem 4.1(d),
\[
f_q(B_z(y)) \supseteq B_{\varepsilon z}(f_q(y)) \supseteq B_{\varepsilon p}(f_p(y)) = B_{\varepsilon p}(z).
\]
By the preceding, \(Q \times B_{\varepsilon p}(z) \subseteq W\), whence \(W\) is a neighborhood of \((p,z)\). Furthermore, \(\psi(q, z') = (f_q)^{-1}(z') \in B_z(y) \supseteq B_{\varepsilon z}(f_q(y)) \supseteq B_{\varepsilon p}(f_p(y)) = B_{\varepsilon p}(z)\) for all \((q, z')\) in the neighborhood \(Q \times B_{\varepsilon p}(z)\) of \((p,z)\). Thus \(W\) is open and \(\psi\) is continuous. The assertions concerning \(\xi\) follow immediately. \(\square\)

As a direct consequence, we obtain an implicit function theorem.

Corollary 4.5 (Continuous Implicit Functions) In the situation of Theorem 4.1 let \((p_0, y_0) \in P \times X\). Then there exists an open neighborhood \(Q \subseteq P\) of \(p_0\) such that \(z_0 := f(p_0, y_0) \in f_p(B)\) for all \(p \in Q\). The mapping \(\lambda : Q \rightarrow B\), \(\lambda(p) := \psi(p, z_0)\) is continuous, satisfies \(\lambda(p_0) = y_0\), and
\[
\{(p, y) \in Q \times B : f(p, y) = z_0\} = \text{graph}(\lambda).
\]
Proof. Because \(W\) is an open neighborhood of \((p_0, z_0)\) in \(P \times F\), there exists an open neighborhood \(Q\) of \(p_0\) in \(P\) such that \(Q \times \{z_0\} \subseteq W\). Then \(\lambda(p) := \psi(p, z_0)\) makes sense for all \(p \in Q\). The rest is obvious from Theorem 4.4. \(\square\)

5 Inverse Function Theorem with Parameters

We are now in the position to formulate and prove our main result, an Inverse Function Theorem with Parameters for Keller \(C^k_c\)-maps in the presence of metric estimates on partial differentials.

Theorem 5.1 (Inverse Function Theorem with Parameters) Let \(E\) be a topological \(\mathbb{K}\)-vector space, and \((F, d)\) be a metric Fréchet space over \(\mathbb{K}\), with absolutely convex balls. Let \(P_0 \subseteq E\) be an open subset, or a locally convex subset with dense interior if \(E\) is locally convex. Let \(U \subseteq F\) be open, \(k \in \mathbb{N} \cup \{\infty\}\) and \(f : P_0 \times U \rightarrow F\) be a \(C^k_{\mathbb{K}}\)-map. Abbreviate \(f_p := f(p, \cdot) : U \rightarrow F\) for \(p \in P_0\).
Assume that \((p_0, x_0) \in P_0 \times U\) and \(f'_{p_0}(x_0): F \to F\) is invertible. Furthermore, assume that
\[
\sup_{(p, x) \in P_0 \times U} \| \id_F - f'_{p_0}(x_0)^{-1} f'_p(x) \|_{d, d} < 1; \tag{41}
\]
or, more generally, assume that there exist isomorphisms of topological vector spaces \(S, A, T: F \to F\) such that \(S \circ A \circ T \in \mathcal{L}_d(F)^\times\) and
\[
\sup_{(p, x) \in P_0 \times U} \| (S \circ (A - f'_p(x)) \circ T) \|_{d, d} < \frac{1}{\| (S \circ A \circ T)^{-1} \|_{d, d}}. \tag{42}
\]
Then there exists an open neighborhood \(P \subseteq P_0\) of \(p_0\) and \(r > 0\) such that \(B := B_r(x_0) \subseteq U\) and the following holds:

(a) \(f_p(B)\) is open in \(F\), for each \(p \in P\), and \(\phi_p: B \to f_p(B), \phi_p(x) := f_p(x) = f(p, x)\) is a \(C^k\)-diffeomorphism.

(b) \(W := \bigcup_{p \in P} \{p\} \times f_p(B)\) is open in \(P_0 \times F\), and the map
\[
\psi: W \to B, \quad \psi(p, z) := \phi_p^{-1}(z)
\]
is \(C^k\). Furthermore, the map
\[
\xi: P \times B \to W, \quad \xi(p, x) := (p, f(p, x))
\]
is a \(C^k\)-diffeomorphism with inverse \(\xi^{-1}(p, z) = (p, \psi(p, z))\).

(c) \(P \times B_{\delta}(f_{p_0}(x_0)) \subseteq W\) for some \(\delta > 0\).

In particular, for each \(p \in P\) there is a unique element \(\lambda(p) \in B\) such that \(f(p, \lambda(p)) = f(p_0, x_0)\), and the map \(\lambda: P \to B\) so obtained is \(C^k\).

**Remark 5.2** Typical choices of \(S, A, T\) are as follows:

(a) With \(S = f_{p_0}(x_0)^{-1}, A = f_{p_0}(x_0)\) and \(T = \id_F\), we recover \([41]\).

(b) If \(f'_{p_0}(x_0) \in \mathcal{L}_d(F)^\times\), then a typical choice is \(A := f'_{p_0}(x_0), S := T := \id_F\).

In this case, we make a requirement concerning \(\sup_{x, p} \| f'_{p_0}(x_0) - f'_p(x) \|_{d, d}\).

**Proof of Theorem 5.1** Let \(h := T^{-1} \circ A^{-1} \circ f \circ (\id_{P_0} \times T): P_0 \times T^{-1}(U) \to F\). Then \(h\) satisfies
\[
\sup_{p, x} \| \id_F - h'_p(x) \|_d = \sup_{p, x} \| (SAT)^{-1} S(A - f'_p(x)) T \|_d
\]
\[
\leq \| (SAT)^{-1} \|_d \sup_{p, x} \| S(A - f'_p(x)) T \|_d < 1,
\]
by \([42]\). After replacing \(f\) with \(h\), we may assume henceforth that \(S = A = T = \id_F\) and
\[
\theta := \sup_{(p, x) \in P_0 \times U} \| \id_F - f'_p(x) \|_d < 1. \tag{43}
\]
Let \( r > 0 \) such that \( B := B_{r}(x_{0}) \subseteq U \). Given \( y, z \in B \) and \( p \in P_{0} \), we have
\[
\| f_{p}(z) - f_{p}(y) \| = \int_{0}^{1} \left| f'_{p}(y + t(z - y)) \right| \cdot |(z - y)| \, dt \quad \text{and} \quad z - y = \int_{0}^{1} (z - y) \, dt.
\]
Then
\[
\| f_{p}(z) - f_{p}(y) - (z - y) \|_{d} \leq \sup_{t \in [0,1]} \| (f'_{p}(y + t(z - y)) - \text{id}_{F})(z - y) \|_{d}
\]
\[
\leq \sup_{t \in [0,1]} \| f'_{p}(y + t(z - y)) - \text{id}_{F} \|_{d} \| z - y \|_{d} \leq \theta \| z - y \|_{d},
\]
using Corollary 9.1 and (10). Hence
\[
\kappa := \sup \left\{ \frac{\| f_{p}(z) - f_{p}(y) - (z - y) \|_{d}}{\| z - y \|_{d}} : p \in P_{0}, z \neq y \in B \right\} \leq \theta < 1. \tag{44}
\]
Thus Theorem 4.4 applies to \( f|_{P_{0} \times B} \) with \( A = \text{id}_{F} \), whence \( f_{p}(B) \) is open in \( F \) and \( \phi_{p} := f_{p}|_{B} \) a homeomorphism onto its image, for each \( p \in P_{0} \); the set \( W := \bigcup_{p \in P_{0}} (p) \times f_{p}(B) \) is open in \( P_{0} \times F \); the map \( \psi : W \rightarrow B, \psi(p, z) := \phi_{p}^{-1}(z) \) is continuous; and the mapping \( \xi : P_{0} \times B \rightarrow W, \xi(p, y) := (p, f(p, y)) \) is a homeomorphism, with inverse given by \( \xi^{-1}(p, z) = (p, \psi(p, z)) \). Set \( \alpha := 1 - \kappa \) and \( \beta := 1 + \kappa \). In view of (44), Theorem 4.1 applies to \( f_{p}|_{B} \), for each \( p \in P_{0} \). Hence
\[
f_{p}(x) + B_{\alpha s}(0) \subseteq f_{p}(B_{s}(x)) \subseteq f_{p}(x) + B_{\beta s}(0) \tag{45}
\]
holds for all \( p \in P_{0}, x \in B \) and \( s \in [0, r - \| x - x_{0} \|_{d}] \).

Also (c) is easily established: we set \( \delta := \frac{\alpha \kappa}{2} \). There is an open neighborhood \( P \subseteq P_{0} \) of \( p \) such that \( \| f(p, x_{0}) - f(p_{0}, x_{0}) \|_{d} < \delta \) for all \( p \in P \). Then, using (45) with \( x := x_{0} \) and \( s := r \), we get \( f_{p}(B) \supseteq B_{\alpha r}(f_{p}(x_{0})) = B_{2\alpha}(f_{p}(x_{0})) \supseteq B_{2}(f_{p}(x_{0})) \), for all \( p \in P \). Thus (c) holds.

(a) and (b): If we can show that \( \psi \) is \( C_{K}^{k} \), then clearly all of the maps \( \psi, \xi, \lambda \) and \( \phi_{q} \) will have the desired properties. It suffices to show that \( \psi \) is \( C_{K}^{k} \) on an open neighborhood of each given element \((p, z) \in W \). Given \((p, z) \in W \), there exists \( y \in B \) such that \( f_{p}(y) = z \). Let \( \varepsilon \in [0, r - \| y - x_{0} \|_{d}] \); then \( B_{\varepsilon}(y) \subseteq B \). There is an open neighborhood \( Q \) of \( p \) in \( P \) such that \( \| f(q, y) - f(p, y) \|_{d} < \frac{\alpha \kappa}{2} \) for all \( q \in Q \), by continuity of \( f \). Then, using (45),
\[
f_{p}(B_{\varepsilon}(y)) \supseteq B_{\alpha r}(f_{q}(y)) \supseteq B_{\alpha r}(f_{p}(y)) = B_{2\alpha}(z).
\]
By the preceding, \( Q \times B_{2\alpha}(z) \subseteq W \) and \( \psi(Q \times B_{2\alpha}(z)) \subseteq B_{\varepsilon}(y) \). Now consider the \( C_{K}^{k} \)-map
\[
g: Q \times B_{2\alpha}(z) \times B_{\varepsilon}(y) \rightarrow F, \quad g(q, c, v) := v - (f_{q}(v) - c).
\]
For all \((q, c) \in Q \times B_{2\alpha}(z) \), the map \( g_{(q, c)} := g(q, c, *) : B_{\varepsilon}(y) \rightarrow F \) satisfies \( g'_{(q, c)}(v) = \text{id}_{F} - f'_{q}(v) \). Thus \( \sup_{(q, c)} \| g'_{(q, c)}(v) \|_{d} \leq \theta < 1 \) using (43), and so
$(g_{(q,c)})_{(q,c)}$ is a uniform family of special contractions, with uniform special contraction constant $\theta$ (see Lemma 3.5). Note that $g_{(q,c)}(v) = v$ if and only if $f_q(v) = c$, i.e., if and only if $v = \psi(q,c)$. Thus $\psi(q,c)$ is a fixed point of $g_{(q,c)}$. Since $g$ is $C^k_K$, Theorem D shows that $\psi$ is $C^k_K$ on $Q \times B_{2\epsilon}(z)$.

**Remark 5.3** Theorem 5.1 remains valid in the $C^1_K$-case if $P_0 \subseteq E$ is any subset with non-empty interior, no matter whether $E$ and $P_0$ are locally convex. To achieve this, use Lemma 3.8 instead of Theorem D at the end of the proof of Theorem 5.1.

**Remark 5.4** Note that Theorem 5.1 subsumes Theorem A from the introduction as its final assertion. Using a singleton set of parameters, we also obtain Theorem B as a special case.

**Remark 5.5** Let $E$, $(F,d)$, $P_0$ and $U$ be as in Theorem 5.1. If a $C^k_K$-map $f: P_0 \times U \to F$ satisfies $f'_p(x) \in \mathcal{L}_d(F)$ for all $(p,x) \in P_0 \times U$ and also

$$P_0 \times U \to \mathcal{L}_d(F), \quad (p,x) \mapsto f'_p(x) \tag{46}$$

is continuous at $(p_0, x_0)$ and $f'_{p_0}(x_0) \in \mathcal{L}_d(F)^\times$, then (46) is satisfied after shrinking $P_0$ and $U$ if necessary, by continuity of composition in $\mathcal{L}_d(F)$. However, only (46) (or (47)) is needed as an hypothesis for the theorem, not any continuity property concerning the map in (46) because this would be too restrictive (at least continuity on an open set), as we have seen in Remark 2.10.

**Remark 5.6** We mention that $f'_p(x) \in \mathcal{L}_d(F)^\times$ for all $p \in P_0$ and $x \in U$ if (13) holds, because $A \in \mathcal{L}_d(F)^\times$ with $A^{-1} = \sum_{n=0}^{\infty} (\text{id}_F - A)^n$ for all $A \in \mathcal{L}_d(F)$ such that $\|\text{id}_F - A\|_{d,d} < 1$ (see [30, Theorem 4.1]).

### 6 Application to families of linear operators

This section describes a simple application of the inverse function theorem with parameters (Theorem 5.1) concerning the inversion of linear operators (which cannot be deduced from the results in [30]).

For the purposes of infinite-dimensional Lie theory, it is useful to be able to speak of smooth mappings from a smooth manifold $M$ to certain groups $G$ which are not manifolds. For example, $G$ might be the diffeomorphism group of an infinite-dimensional manifold, or the group $\mathcal{L}(F)^\times$ of automorphisms of a (non-Banach) topological vector space (see [18], [32]). We now discuss the following concept.

**Definition 6.1** Let $F$ be a locally convex topological $K$-vector space, and $k \in \mathbb{N}_0 \cup \{\infty\}$. If $k = 0$, let $P$ be a topological space. If $k \geq 1$, let $P$ be a locally convex subset with dense interior of a locally convex topological $K$-vector space $E$. Let

$$\iota: \mathcal{L}(F)^\times \to \mathcal{L}(F)^\times, \quad \iota(A) := A^{-1}$$

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be the inversion map. We say that a map \( g: P \to \mathcal{L}(F)^\times \) is \( k \) times pseudo-differentiable (or \( PC_k^k \), for short), if both of the maps
\[
g^\wedge : P \times F \to F, \quad g^\wedge(x,y) := g(x) \cdot y
\]
and \((\iota \circ g)^\wedge : P \times F \to F\) are \( C^k_k\).

Our application concerns a case where the condition on \((\iota \circ g)^\wedge\) is superfluous.

**Proposition 6.2** Let \( F \) be a Fréchet space over \( \mathbb{K} \) and \( k \in \mathbb{N}_0 \cup \{\infty\} \). If \( k = 0 \), let \( P \) be a topological space. If \( k \geq 1 \), let \( E \) be a locally convex space over \( \mathbb{K} \) and \( P \subseteq E \) be a locally convex subset with dense interior. Let \( g: P \to \mathcal{L}(F)^\times \) be a map. Assume that there exists a translation invariant metric \( d \) on \( F \) defining its topology and having absolutely convex balls, and isomorphisms \( S,A,T: F \to F \) of topological vector spaces with
\[
\sup_{p \in P} \|S(A - g(p))T\|_{d,d} < \frac{1}{\|SAT\|_{d,d}^{-1}}.
\]
Then \( g \) is \( PC_k^k \) if and only if \( g^\wedge : P \times F \to F \) is \( C^k_k \).

**Proof.** If \( k \geq 1 \), then the function \( f := g^\wedge \) satisfies the hypotheses of Theorem 5.1 with \( x_0 := 0 \), \( P_0 := P \) and any \( p_0 \in P_0 \). Noting that only the proof of Theorem 5.1(c) required to shrink \( P_0 \) (which is inessential for us here), Part (a) and (b) of the theorem show that there is an open neighborhood \( W \) of \( P \times \{0\} \) in \( P \times F \) such that the map
\[
(\iota \circ g)^\wedge|_W : W \to F, \quad (p,x) \mapsto g(p)^{-1} \cdot x = (f_p)^{-1}(x)
\]
is \( C^k_k \). For \( n \in \mathbb{N} \), set \( W_n := \{(p,nx) : (p,x) \in W\} \). Then \( W_n \) is open in \( P \times F \) and \( \bigcup_{n \in \mathbb{N}} W_n = P \times F \). Since \( (\iota \circ g)^\wedge(p,x) = n(\iota \circ g)^\wedge(p,\frac{1}{n}x) \) for each \((p,x) \in W_n\) by linearity in the second argument, we see that \((\iota \circ g)^\wedge|_{W_n} \) is \( C^k_k \) for each \( n \in \mathbb{N} \). Hence \((\iota \circ g)^\wedge \) is \( C^k_k \). If \( k = 0 \), we use Theorem 4.4 instead Theorem 5.1 to reach the desired conclusion. \( \square \)

Note that, if also \( E \) happens to be a Fréchet space, we need not assume that \( g^\wedge : P \times F \to F \) is \( MC^1 \); we only need the \( C^1 \)-property. This is essential, as the following example shows.

**Example 6.3** In the situation of Example 2.7, set \( F := \mathbb{R}^N \) and consider the curve
\[
g: [0,1] \to \mathcal{L}_d(F), \quad g(t) := \text{id}_F - tS.
\]
Then \( \|\text{id}_F - g(t)\|_{d,d} = \|tS\|_{d,d} \leq \|S\|_{d,d} = a < 1 \) for each \( t \in [0,1] \), and furthermore the map
\[
g^\wedge : [0,1] \times F \to F, \quad (t,x) \mapsto g(t)(y) = y - tS.y
\]

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is $C^\infty$. By Proposition 6.2, also the mapping $h := (t \circ g)^\wedge \colon [0,1] \times F \to F$, $(t,x) \mapsto (\id_F - tS)^{-1}(x)$ is a $C^\infty$-map.\footnote{This implies that also the map $[0,1] \to \mathcal{L}(F)_b, t \mapsto dh((t,0),(0,\bullet)) = (\id_F - tS)^{-1}$ is $C^\infty$, by general facts of infinite-dimensional calculus (see \cite{IEEEexample}). Here $\mathcal{L}(F)_b$ denotes $\mathcal{L}(F)$, equipped with the topology of uniform convergence on bounded sets.}

Since $g$ is discontinuous as a map into $\mathcal{L}_d(F)$ (because $g(0) - \id_F = 0$ but $\|g(t) - \id_F\|_{d,d} = a > 0$ for each $t > 0$), it follows that $g^\wedge$ is not $MC^1$ (using the maximum metric on $\mathbb{R} \times F$). Therefore, we cannot get smoothness of $h$, say, by trying to apply an inverse function theorem for $MC^1$-maps (or $MC^\infty$-maps) to $\mathbb{R} \times F \to \mathbb{R} \times F$, $(t,x) \mapsto (t,g(t)x)$.

## 7 Inverse and implicit $MC^k$-maps

For later use, we record a variant of Müller’s Inverse Function Theorem \footnote{Theorem 4.7 in \cite{IEEEexample} and the inductive proof of his Theorem 4.6 are our models here.} which can do with slightly weaker hypotheses.

**Proposition 7.1 (Müller’s Inverse Function Theorem for $MC^k$-maps)**

Let $(F,d)$ be a metric Fréchet space over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, with absolutely convex balls. Let $k \in \mathbb{N} \cup \{\infty\}$ and $f : U \to F$ be an $MC^k_{\mathbb{K}}$-map on an open subset $U \subseteq F$. Let $x_0 \in U$. If $f'(x_0) \in \mathcal{L}(F)^\times$, then there exists an open neighborhood $V \subseteq U$ of $x$ such that $f(V)$ is open in $F$ and $f|_V : V \to f(V)$ is an $MC^k_{\mathbb{K}}$-diffeomorphism.

Before we prove the proposition, let us record a standard consequence.

**Corollary 7.2 (Implicit Function Theorem for $MC^k$-maps.)** Let $(E,d_E)$ and $(F,d_F)$ be metric Fréchet spaces over $\mathbb{K}$, with absolutely convex balls. Equip $E \times F$ with the metric $d$ given by

$$d((x_1,y_1),(x_2,y_2)) := \max\{d_E(x_1,x_2),d_F(y_1,y_2)\}.$$  

Let $f : U \times V \to F$ be an $MC^k_{\mathbb{K}}$-map, where $U \subseteq E$ and $V \subseteq F$ are open sets. Given $x \in U$, abbreviate $f_x := f(x,\bullet) : V \to F$. If $f(x_0,y_0) = 0$ for some $(x_0,y_0) \in U \times V$ and $f'_x(y_0) \in \mathcal{L}(F)^\times$, then there exist open neighborhoods $U_0 \subseteq U$ of $x_0$ and $V_0 \subseteq V$ of $y_0$ such that

$$\{(x,y) \in U_0 \times V_0 : f(x,y) = 0\} = \text{graph}\lambda$$

for an $MC^k_{\mathbb{K}}$-map $\lambda : U_0 \to V_0$. \hfill $\Box$

**Proof.** Apply Proposition 7.1 to the $MC^k$-map $U \times V \to E \times F$, $g(x,y) := (x,f(x,y))$. \hfill $\Box$

The following lemma will help us to deduce Proposition 7.1 from Theorem B. Its proof (recorded in Appendix B) is simple but requires longish preparations, because one has to struggle with the fact that $\mathcal{L}_d(F)$ only is a locally convex vector group.
Lemma 7.3 Let \((F,d)\) be a metric Fréchet space with absolutely convex balls, and \(A \in \mathcal{L}_d(F)^\times\). Let \(\Omega\) be the connected component of 0 of the set
\[((A + \mathcal{L}_d(F)_0) \cap \mathcal{L}_d(F)^\times) - A.\]
Then \((A + B)^{-1} - A^{-1} \in \mathcal{L}_d(F)_0\) for each \(B \in \Omega\), and the map \(\iota_A : \Omega \to \mathcal{L}_d(F)_0\), \(\iota_A(B) := (A + B)^{-1} - A^{-1}\) is \(MC_\infty\).

Proof of Proposition 7.1 By continuity of \(f' : U \to \mathcal{L}_d(F)\), the point \(x_0\) has an open connected neighborhood \(V \subseteq U\) with \(\sup_{x \in V} \|\id_F - f'(x_0)^{-1}f'(x)\|_{d,d} < 1\). Set \(y_0 := f(x_0)\). Then \(f(V)\) is an open neighborhood of \(y_0\) and \(f|_V\) is a \(C^k\)-diffeomorphism, by Theorem B. Set \(g := (f|_V)^{-1} : f(V) \to V\). Then \(g'(y) = f'(g(y))^{-1} \in \mathcal{L}_d(F)^\times\) for each \(y \in f(V)\), and the formula shows that \(g'\) is continuous. Hence \(g\) is \(MC^1\). By connectedness of \(V\), \(g'(f(V)) \subseteq g'(y_0) + \mathcal{L}_d(F)\). Hence, setting \(A := g'(y_0)\), we have
\[g' = \iota_A \circ (\tau_{-A} \circ f') \circ g\]
with \(\iota_A\) as in Lemma 7.3 and the translation map \(\tau_{-A} : A + \mathcal{L}_d(F)_0 \to \mathcal{L}_d(F)_0\), \(B \mapsto B - A\). Now assume that \(g\) is \(MC^{k-1}\), by induction. Since \(\iota_A\) is \(MC^\infty\), \(\tau_{-A} \circ f' : U \to \mathcal{L}_d(F)_0\) is \(MC^{k-1}\), it follows that \(g'\) is \(MC^{k-1}\). Hence \(g\) is \(MC^k\), which completes the inductive proof.

8 Global Inverse Function Theorems

In this section, we generalize Hadamard’s global inverse function theorem from the classical Banach case to the case of Fréchet spaces.

Theorem 8.1 (Global Inverse Function Theorem for \(C^k\)-Maps) Let \((E,d)\) and \((F,d')\) be metric Fréchet spaces over \(K\) with absolutely convex balls and \(f : E \to F\) be a \(C^k_K\)-map which is a local \(C^k_K\)-diffeomorphism around each point. We assume that there exist isomorphisms of topological vector spaces \(S : F \to F\) and \(T : E \to E\) such that \(Sf'(x)T : E \to F\) is invertible with inverse \((Sf'(x)T)^{-1} \in \mathcal{L}_{d',d}(F,E)\) for each \(x \in E\), and
\[M := \sup_{x \in E} \|(Sf'(x)T)^{-1}\|_{d',d} < \infty.\]
Then \(f\) is a \(C^k_K\)-diffeomorphism from \(E\) onto \(F\).

Remark 8.2 Conditions ensuring that \(f\) is a local diffeomorphism can be deduced from Theorem B. If \(E = F\) and \(d = d'\), \(f : F \to F\) is \(C^k_K\) and
\[\sup_{x \in F} \|\id_F - f'(x)\|_{d,d} < 1,\]
then all hypotheses of Theorem 8.1 are satisfied with \(S = T = \id_F\) (noting that the estimate (43) is ensured by [30] Theorem 4.1)). This case is useful for the construction of Lie groups of diffeomorphisms (see [10, 11, 12]). The injectivity of \(f\) is quite obvious in the special case when (49) holds (cf. [14] Lemma 5.1)).
We also have a version for $MC^k$-maps.

**Theorem 8.3 (Global Inverse Function Theorem for $MC^k$-Maps)** Let $(E,d)$ and $(F,d')$ be metric Frechet spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, with absolutely convex balls. Let $k \in \mathbb{N} \cup \{\infty\}$ and $f: E \to F$ be an $MC^k_{\mathbb{K}}$-map such that $f'(x): E \to F$ is invertible for each $x \in E$ and $\sup_{x \in E} \|f'(x)^{-1}\|_{d',d} < \infty$. Then $f$ is an $MC^k_{\mathbb{K}}$-diffeomorphism from $E$ onto $F$.

**Proof of Theorems 8.1 and 8.3**

Step 1. In the situation of Theorem 8.1, after replacing $f$ with $S \circ f \circ T$, we may assume that $S = \text{id}_F$ and $T = \text{id}_E$. In the situation of Theorem 8.3, $f$ is a local $MC^k$-diffeomorphism, by Theorem 7.1. We therefore only need to show that $f$ is a bijection.

Step 2. For each continuous map $\gamma: Z \to F$ on a connected topological space $Z$ and elements $z_0 \in Z$ and $x_0 \in f^{-1}(\gamma(z_0))$, there exists at most one continuous map ("lift") $\eta: Z \to E$ such that $\eta(z_0) = x_0$, because $E$ and $F$ are Hausdorff spaces and $f$ is a local homeomorphism (see [8, Theorem 4.8]).

Step 3. We now show that for each $C^1$-curve $\gamma: \mathbb{R} \to F$ and each $x_0 \in f^{-1}(\gamma(0))$, there exists a $C^1$-curve $\eta: \mathbb{R} \to E$ such that $f \circ \eta = \gamma$, and $\eta(0) = x_0$.

Lifts being unique by Step 2, there exists a largest interval $I \subseteq \mathbb{R}$ such that $\gamma|_I$ admits a lift $\eta$ with $\eta(0) = x_0$. Since $\{0\} \to E$, $0 \mapsto x_0$ is a lift, $I$ is non-empty. Because $f$ is a local homeomorphism, $I$ is an open interval. Furthermore, $\eta$ is $C^1$, because for each $t_0 \in I$, there exists an open neighborhood $U \subseteq E$ of $\eta(t_0)$ on which $f$ is injective, and thus

$$\eta|_J = (f|_U)^{-1} \circ \gamma|_J$$

by uniqueness of lifts, where $J \subseteq I$ is a connected open neighborhood of $t_0$ such that $\gamma(J) \subseteq f(U)$.

From (50), we also deduce that

$$\|\eta'(t)\|_d \leq M \|\gamma'(t)\|_{d'}$$

holds for $\eta'(t) \in E$ and $\gamma'(t) \in F$. Write $I = ]a,b]$ with $-\infty \leq a < 0 < b \leq +\infty$. If not, then $J := [0,b]$ has compact closure $\overline{J}$ and hence $L := \max\{\|\gamma'(t)\|_{d'} : t \in \overline{J}\} < \infty$. Using (51), we deduce that

$$\|\eta(t) - \eta(s)\|_d = \sup_{\tau \in J} \|\eta'(\tau)\|_{d'} \cdot |t - s| \leq ML \cdot |t - s|$$

for all $t, s \in J$.

By (52), $(\eta(t))_{t \in J}$ is a Cauchy net indexed by $J$, which is a directed set with respect to the order on $J$ induced by $\mathbb{R}$. Since $E$ is complete, the limit $\eta(b) := \lim_{t \uparrow b} \eta(t)$ exists and provides a continuous extension of $\eta$ to a map on the interval $I \cup \{b\}$. By continuity of $f$, the extended function is a lift. As $I$ is a proper subset
Step 4. $f$ is surjective. To see this, let $z_0 \in F$ be given. Pick any $x_0 \in E$ and set $y_0 := f(x_0)$. Then $\gamma: \mathbb{R} \to F$, $\gamma(t) := y_0 + t(z_0 - y_0)$ is a $C^1$-curve such that $\gamma(0) = y_0$ and $\gamma(1) = z_0$. By Step 3, there exists a $C^1$-curve $\eta: \mathbb{R} \to E$ such that $\eta(0) = x_0$ and $f \circ \eta = \gamma$. Thus $z_0 = \gamma(1) = f(\eta(1))$ in particular.

Step 5. To see that $f$ is injective, let $x_0, y_0 \in E$ such that $f(x_0) = f(y_0)$. Then $\eta: \mathbb{R} \to E$, $\eta(t) := x_0 + t(y_0 - x_0)$ is a $C^1$-curve in $E$, and $\gamma := f \circ \eta: \mathbb{R} \to F$ is a $C^1$-curve such that $\gamma(0) = \gamma(1)$. Now consider the $C^1$-map $\Gamma: \mathbb{R} \times \mathbb{R} \to F$, $\Gamma(t, s) := z_0 + s(\gamma(t) - z_0)$.

Then $\Gamma(0, s) = \Gamma(1, s) = z_0$ for each $s \in \mathbb{R}$ and $\Gamma(\ast, 0) \equiv z_0$, $\Gamma(\ast, 1) = \gamma$. By Step 3, for each $s \in \mathbb{R}$ the curve $\Gamma(\ast, s): \mathbb{R} \to F$ can be lifted to a curve $\zeta_s$ such that $\zeta_s(0) = x_0$. Then $\eta = \zeta_1$, since $f \circ \eta = \gamma = \Gamma(\ast, 1) = f \circ \zeta_1$ and lifts are unique. Likewise, $\zeta_0 \equiv x_0$ since $\Gamma(\ast, 0) \equiv z_0$. Now [8, Theorem 4.10] shows that $x_0 = \zeta_0(1) = \zeta_1(1) = \eta(1) = y_0$.

\[ \boxed{\text{9 Function spaces and mappings between them}} \]

As a preliminary for our studies of ODEs in Fréchet spaces, we now study differentiability properties of certain types of mappings between spaces of continuous vector-valued functions on compact topological spaces.

9.1 If $E$ is a locally convex topological $\mathbb{K}$-vector space and $K$ a compact topological space, we equip the space $C(K, E)$ of continuous $E$-valued maps in $K$ with the topology of uniform convergence. This topology makes $C(K, E)$ a locally convex topological $\mathbb{K}$-vector space; the sets $C(K, U)$ with $U \subseteq E$ open 0-neighborhood form a basis of open 0-neighborhoods in $C(K, E)$. If $(E, d)$ is a metric Fréchet space, we set

$$
\| \gamma \|_{d, \infty} := \max\{ \| \gamma(x) \|_d : x \in K \}
$$

for $\gamma \in C(K, E)$ and note that $(\gamma, \eta) \mapsto \| \gamma - \eta \|_{d, \infty}$ is a metric on $C(K, E)$ which defines the given topology and has absolutely convex balls if this is the case of $d$.

We shall always equip $C(K, E)$ with the latter metric, and shall refer to it as the “maximum metric with respect to $d$.”

9.2 If $U \subseteq E$ is an open subset, then $C(K, U)$ is open in $C(K, E)$. In fact, given $\gamma \in C(K, U)$, the image $\gamma(K) \subseteq U$ is compact and hence has a uniform neighborhood of the form $\gamma(K) + V \subseteq U$ for some open 0-neighborhood $V \subseteq E$. Then $C(K, V)$ is an open 0-neighborhood in $C(K, E)$ and $\gamma + C(K, V) \subseteq C(K, U)$. 

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Proposition 9.3 Let $E$ and $F$ be locally convex topological $\mathbb{K}$-vector spaces, $U \subseteq E$ be open, $P$ be a topological space and $K$ a compact topological space. Let

$$f : K \times U \times P \to F$$

be a continuous map. Given $p \in P$, abbreviate $f^p := f(\cdot, p) : K \times U \to F$. Define $(f^p)_* (\gamma) \in C(K, F)$ for $\gamma \in C(K, U)$ via

$$(f^p)_* (\gamma)(x) := f^p(x, \gamma(x)) = f(x, \gamma(x), p) \quad \text{for } x \in K.$$ 

Then the map $\phi : C(K, U) \times P \to C(K, F)$, $\phi(\gamma, p) := (f^p)_* (\gamma)$ is continuous.

**Proof.** Let $\gamma \in C(K, U)$, $p \in P$, and $V \subseteq F$ be an open 0-neighborhood. Let $W \subseteq F$ be an open 0-neighborhood such that $W - W \subseteq V$. For each $x \in K$, we find an open neighborhood $A_x \subseteq K$ of $x$, an open neighborhood $C_x \subseteq P$ of $p$ and an open 0-neighborhood $B_x \subseteq E$ such that $\gamma(A_x) + B_x \subseteq U$ and

$$f(y, u, q) - f(x, \gamma(x), p) \in W$$

for all $y \in A_x$, $u \in \gamma(A_x) + B_x$, and $q \in C_x$. By compactness, $K \subseteq \bigcup_{x \in I} A_x$ for some finite subset $I \subseteq K$. Then $B := \bigcap_{x \in I} B_x \subseteq E$ is an open 0-neighborhood and $C := \bigcap_{x \in I} C_x \subseteq P$ an open neighborhood of $p$. Let $\eta \in \gamma + C(K, B)$ and $q \in C$. Given $y \in K$, there is $x \in I$ with $y \in A_x$. Then $f(y, \eta(y), q) - f(y, \gamma(y), p) = f(y, \eta(y), q) - f(x, \gamma(x), p) - f(x, \gamma(x), p) - f(x, \gamma(x), p) \in W - W \subseteq V$. We have shown that $\phi(\eta, q) - \phi(\gamma, p) \in C(K, V)$ for all $(\eta, q)$ in the open neighborhood $(\gamma + C(K, B)) \times C$ of $(\gamma, p)$. Thus $\phi$ is continuous. \qed

If $k \in \mathbb{N}$ and $f : E \supseteq U \to F$ is a $C^k$-map, we can associate iterated differentials with $f$ via $d^0 f := f : U \to F$, $d^1 f := df : U \times E \to F$,

$$d^2 f := d(df) : (U \times E) \times (E \times E) \to F$$

(if $k \geq 2$), and recursively $d^k f := d^{k-1}(df) : U \times E^{2k-1} \to F$.

**Proposition 9.4** Let $K$ be a compact topological space, $E$ and $F$ be locally convex topological $\mathbb{K}$-vector spaces, $Z$ be a topological $\mathbb{K}$-vector space, and $k \in \mathbb{N}_0 \cup \{\infty\}$. Let $U \subseteq E$ be an open subset and $P \subseteq Z$ be a subset with dense interior. If $k \geq 2$, we assume that $P$ is open or that both $Z$ and $P$ are locally convex. Let $f : K \times (U \times P) \to F$ be a map such that

(a) $f_x := f(x, \cdot) : U \times P \to F$ is $C^k_x$ for each $x \in K$; and

(b) For each $j \in \mathbb{N}_0$ such that $j \leq k$, the map $K \times (U \times P) \times (E \times Z)^{2j-1} \to F$,

$$(x, u, p, y) \mapsto (d^j f)(x, u, p, y) := (d^j f_x)(u, p; y)$$

for $x \in K$, $u \in U$, $p \in P$, $y \in (E \times Z)^{2j-1}$ is continuous.
Proposition 9.3, we may assume that $k$.

Proof. We may assume that $k$.

The case $k$ where $g$.

Then $f : C(K, U) \times P \to C(K, F)$, $\phi(\gamma, p) := (f^p)_*(\gamma)$ is a $C^1_p$-map, where $f^p := f(\cdot, p) : K \times U \to F$ for $p \in P$ and $(f^p)_*(\gamma)(x) := f(x, \gamma(x), p)$ for $x \in K$. Furthermore, the differentials of $\phi$ are given by

$$
\phi'(\gamma, p).(\eta, q) = (g^{(p,q)})_*(\gamma, \eta),
$$

where $g^{(p,q)} := g(\cdot, (p, q))$ with

$g : K \times (U \times E) \times (P \times Z) \to F$,

$g(x, (u, v), (p, q)) := d_2f(x, u, p, q, v)$.

Proof. We may assume that $k < \infty$. The case $k = 0$ having been settled in Proposition 9.3, we may assume that $k \geq 1$. The proof is by induction.

The case $k = 1$. Let $\gamma \in C(K, U)$, $\eta \in C(K, E)$, $q \in Z$ and $p \in P^0$, the interior of $P$. Since $C(K, U)$ and $P^0$ are open, there exists $r > 0$ such that $\gamma + B^p_\varepsilon(0)\eta \subseteq C(K, U)$ and $p + B^p_\varepsilon(0)q \subseteq P^0$. For each $x \in K$ and $t \in B^p_\varepsilon(0)$ such that $t \neq 0$, we have

$$
\Delta_t(x) := \frac{\phi(\gamma + t\eta, p + tq) - \phi(\gamma, p)}{t}(x)
$$

$$
= \frac{f(x, \gamma(x) + t\eta(x), p + tq) - f(x, \gamma(x), p)}{t}
$$

$$
= \int_0^1 d_2f(x, (\gamma(x), p) + s(t\eta(x, q); (q, \eta(x)))) ds.
$$

The map $h : B^p_\varepsilon(0) \times K \times [0, 1] \to F$,

$h(t, x, s) := d_2f(x, (\gamma(x), p) + s(t\eta(x, q); (q, \eta(x))))$

is continuous. By (55), the weak integral $H(t, x) := \int_0^1 h(t, x, s) ds$ exists in $F$ for all $x \in K$ and $0 \neq t \in B^p_\varepsilon(0)$. But it also exists for $t = 0$ because the integrand is constant in this case. Now the continuity of $h$ implies continuity of the parameter-dependent weak integral $H : B^p_\varepsilon(0) \times K \to F$ (see, e.g., [18, Chapter 1]). By the first half of the exponential law ([7, Theorem 3.4.1]), continuity of $H$ implies continuity of

$$
H^\gamma : B^p_\varepsilon(0) \to C(K, F), \quad H^\gamma(t) := H(t, \gamma).
$$

Since $H^\gamma(t) = \Delta_t$ for $t \neq 0$ by (55) and $H^\gamma$ is continuous, we deduce that $\Delta_t \to H^\gamma(0)$ as $t \to 0$, where $(H^\gamma(0))(x) = H(0, x) = d_2f(x, (\gamma(x), p), (q, \eta(x)))$ for all $x \in K$. Hence $d\phi((\gamma, p), (\eta, q))$ exists for $(\gamma, p, \eta, q)$ as before, and is given by (53). Since $g$ from (54) is continuous by hypothesis (b), Proposition 9.3 shows that the map described in (53) is continuous. As the map in (53) extends $d\phi$ (defined so far only on $C(K, U) \times P^0$), we see that $\phi$ is $C^k_p$ with $d\phi$ given by (53).

Induction step. Let $k \geq 2$ and assume that the proposition holds when $k$ is
replaced with $k - 1$. We already know that $\phi$ is $C^1_K$ and that $d\phi$ is given by $\lbrack 53 \rbrack$. Since, by hypothesis (b), $g$ satisfies a condition analogous to hypothesis (b) with $k - 1$ in place of $k$, the parameter-dependent pushforward in $\lbrack 53 \rbrack$ is $C^{k-1}_K$ by induction. Thus $\phi$ is $C^1_K$ with $d\phi$ a $C^{k-1}_K$-map and hence $\phi$ is $C^K_K$. □

**Definition 9.5** Let $(E, d)$ and $(F, d')$ be metric Fréchet spaces over $\mathbb{K}$, $U \subseteq E$ and $X$ be a topological space. We say that a function $f: X \times U \to F$ satisfies the local contraction condition (or “local CC”) in its second argument, if for $x_0 \in X$ and $y_0 \in U$, there exist neighborhoods $X' \subseteq X$ of $x_0$ and $U' \subseteq U$ of $y_0$ such that $(f_x|_{U'})_{x \in X'}$ is a family of special contractions, where $f_x|_{U'}: U' \to F$, $y \mapsto f(x, y)$. If we can always find $X'$ and $U'$ as before such that $(f_x|_{U'})_{x \in X'}$ is a uniform family of special contractions, we say that $f$ satisfies the local special contraction condition (or “local SCC”) in its second argument. Likewise, we speak of a local SCC (resp., a local CC) in the second argument if $f: X \times U \times Z \to F$ with topological spaces $X$ and $Z$ and $(x_0, y_0, z_0)$ always has a box neighborhood $X' \times U' \times Z'$ such that the maps $U' \to F$, $y \mapsto f(x, y, z)$ form a uniform family of special contractions for $(x, z) \in X' \times Z'$ (resp., a uniform family of contractions).

The following simple observation is useful:

**Lemma 9.6** Let $K$ be a compact topological space, $(X, d)$ and $(Y, d')$ be metric spaces and $f: K \times X \to Y$ be a map such that $f_x := f(x, \cdot): X \to Y$ is Lipschitz continuous for each $x \in K$ and $\theta := \sup_{x \in K} \text{Lip}(f_x) < \infty$. Equip $C(K, X)$ and $C(K, Y)$ with the maximum metrics. Then also the map

$$f_*: C(K, X) \to C(K, Y), \quad f_*(\gamma)(x) := f(x, \gamma(x)) \quad \text{for} \quad \gamma \in C(K, X), \quad x \in K$$

is Lipschitz continuous, with minimal Lipschitz constant $\text{Lip}(f_*) \leq \theta$. In particular, if $g: X \to Y$ is a Lipschitz continuous map, then also

$$C(K, g): C(K, X) \to C(K, Y), \quad \gamma \mapsto g \circ \gamma$$

is Lipschitz continuous, with $\text{Lip}(C(K, g)) \leq \text{Lip}(g)$.

**Proof.** Let $\gamma, \eta \in C(K, X)$. Then

$$d'(f_*(\gamma))(x), f_*(\eta)(x)) = d'(f(x, \gamma(x)), f(x, \eta(x))) \leq \text{Lip}(f_x)d(\gamma(x), \eta(x)) \leq \theta \max_{y \in K} d(\gamma(y), \eta(y))$$

for each $x \in K$ and thus $\max_{x \in K} d'(f_*(\gamma))(x), f_*(\eta)(x)) \leq \theta \max_{y \in K} d(\gamma(y), \eta(y))$, from which the assertions follow. □

**Remark 9.7** Proposition $[9, \text{Proposition 3.3}]$ is a variant of $[\text{[13], Proposition 3.3}]$; corresponding results without parameters are well-known (see, e.g., $[\text{[10], Proposition 3.10}]$). Certain pushforwards (without parameters) between certain spaces of sections in finite-dimensional fibre bundles (with a different type of metric) have also been discussed in $[\text{[30, Theorem 3.31}]$. 

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10 ODEs in Fréchet spaces

This section is devoted to applications. We use our preceding results to discuss existence and uniqueness for solutions to ordinary differential equations in Fréchet spaces, as well as their dependence on parameters and initial conditions. To this end, we adapt a classical idea by Chow and Hale concerning ordinary differential equations in Banach spaces (see [9, Chapter 3, proof of Theorem 1.1]), who reduced the problems in contention to the implicit function theorem in Banach spaces.

Besides the real case spelled out in Theorem E, also local solutions to complex differential equations are of interest (which are suitable complex differentiable vector-valued maps on a connected, locally convex subset of \( \mathbb{C} \) with dense interior), but also mixed cases where we look for ordinary solutions (on intervals in \( \mathbb{R} \)) with values in a complex Fréchet space and would like to establish complex differentiable dependence on initial values and parameters. Such mixed situations are of interest for infinite-dimensional Lie theory, where they can simplify the proof of regularity for a given Lie group (cf. [15, Theorem 8.1]).

We begin with a simple uniqueness result.

**Proposition 10.1** Let \((F,d)\) be a metric Fréchet space with absolutely convex balls, \(U \subseteq F\) be a subset, \(J \subseteq \mathbb{K}\) be a locally convex, connected subset with dense interior, \(f : J \times U \rightarrow F\) be a continuous function and \(\gamma, \eta : J \rightarrow U\) be \(C^1_{\mathbb{R}}\)-solutions to the differential equation \(x'(t) = f(t, x(t))\) such that \(\gamma(t_0) = \eta(t_0)\) for some \(t_0 \in J\). If \(f\) satisfies a local contraction condition in its second argument, then \(\gamma = \eta\).

**Proof.** Local uniqueness: We show first that \(\gamma\) and \(\eta\) coincide on some neighborhood of \(t_0\). To this end, after shrinking \(J\) and \(U\), we may assume that \(J\) is convex, of diameter \(\leq 1\), and that \(f(t, \bullet) : U \rightarrow F\) is a uniform family of contractions for \(t \in J\), with some uniform contraction constant \(\theta \in ]0,1[\). We may also assume that \(M := \sup_{t \in J} \|f(J \times U)\|_d < \infty\). For each \(t \in J\), we have
\[
\|\gamma'(t) - \eta'(t)\|_d = \|f(t, \gamma(t)) - f(t, \eta(t))\|_d \leq \min\{2M, \theta \|\gamma(t) - \eta(t)\|_d\}. 
\]
Hence
\[
\|\gamma(t) - \eta(t)\|_d = \left\| \int_0^1 (t - t_0) \cdot (\gamma' - \eta')(t_0 + t - t_0) \, ds \right\|_d
\leq \sup_{s \in [0,1]} \|\gamma' - \eta'\|(t_0 + t - t_0)\|_d
\leq \theta \sup_{s \in [0,1]} \|[(\gamma - \eta)(t_0 + t - t_0)]_d\|, 
\]
where (56) is also \(\leq 2M\). Hence \(\Delta := \sup_{t \in J} \|\gamma(t) - \eta(t)\|_d < \infty\). If \(\Delta > 0\), we pick \(t \in J\) such that \(\Delta < \theta^{-1}\|\gamma(t) - \eta(t)\|_d\). Since the right hand side of (57) is \(\leq \Delta\), we obtain the contradiction \(\|\gamma(t) - \eta(t)\|_d < \|\gamma(t) - \eta(t)\|_d\).
The set $E := \{ t \in J : \gamma(t) = \eta(t) \}$ is closed in $J$ by continuity of $\gamma$ and $\eta$. By (a), $E$ is also a neighborhood in $J$ of any of its points and hence open in $J$. Since $E \neq \emptyset$ (as $t_0 \in E$) and $J$ is connected, it follows that $E = J$. \hfill \Box

10.2 Our general setting is as follows. We let $(F,d)$ be a metric Fréchet space over $\mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \}$, with absolutely convex balls. Also, we let $k \in \mathbb{N}_0 \cup \{ \infty \}$, $J \subseteq \mathbb{K}$ be a locally convex subset with dense interior, and $L \in \{ \mathbb{R}, \mathbb{K} \}$. If $k = 0$, we let $P$ be a topological space and assume that $\mathbb{K} = L = \mathbb{R}$. If $k \geq 1$, we let $E$ be a topological $\mathbb{K}$-vector space and $P \subseteq E$ be a subset with dense interior. If $k \geq 2$, we assume that both $P \subseteq E$ and $J \subseteq \mathbb{K}$ are open or that $E$ and $P$ are locally convex. We let $f : J \times U \times P \rightarrow F$ be a $C^k_P$-map, $t_0 \in J \cap L$, $x_0 \in U$ and $p_0 \in P$.

Theorem 10.3 (Solutions to ODEs in Fréchet Spaces) Let $f$ be as in 10.2. If $k = 0$, assume that $f$ satisfies a local contraction condition in its second argument. If $k \geq 1$, assume that $f$ satisfies a special local contraction condition in its second argument. Then there exists a convex open neighborhood $J_1 \subseteq J$ of $t_0$ and open neighborhoods $U_1 \subseteq U$ of $x_0$ and $P_1 \subseteq P$ of $p_0$ such that for all $(t_1,x_1,p_1) \in (J_1 \cap L) \times U_1 \times P_1$, the initial value problem

$$x'(t) = f(t,x(t),p_1), \quad x(t_1) = x_1$$

has a $C^k_L$-solution $\phi_{t_1,x_1,p_1} : J_1 \cap L \rightarrow U$ with the following properties:

(a) The map $\Psi : (J_1 \cap L) \times (J_1 \cap L) \times U_1 \times P_1 \rightarrow U$, $\Psi(t_1,t,x_1,p_1) := \phi_{t_1,x_1,p_1}(t)$ is $C^k_L$.

(b) For fixed $(t_1,t) \in (J_1 \cap L) \times (J_1 \cap L)$, the map $\Psi(t_1,t,\cdot) : U_1 \times P_1 \rightarrow F$ is $C^k_P$.

(c) If $(t_1,x_1,p_1) \in (J_1 \cap L) \times U_1 \times P_1$ and $\psi : W \rightarrow F$ is a $C^1_W$-solution to (59) on a convex neighborhood $W \subseteq J_1 \cap L$ of $t_1$, then $\psi = \phi_{t_1,x_1,p_1}|_{J}$.

Proof. Once $\Psi$ exists, (c) is a special case of Proposition 10.1. To construct solutions, we assume first that $\mathbb{K} = L$. By the local CC (resp., SCC), we may assume that $f(t,\cdot,\cdot) : U \rightarrow F$, for $(t,p) \in J \times P$, is a uniform family of contractions (resp., special contractions) with uniform (special) contraction constant $\theta \in [0,1]$, after replacing $J$, $U$ and $P$ with smaller neighborhoods of $t_0$, $x_0$ and $p_0$, respectively (with properties as described in the hypotheses). We may also assume that $J$ is convex.

Let $V \subseteq U$ be an open neighborhood of $x_0$ and $W \subseteq F$ be an open 0-neighborhood such that $V + W \subseteq U$. Define $g : [0,1] \times W \times J \times U \times P \rightarrow F$,

$$g(\tau,w,t_1,t_2,x_1,p_1) := (t_2 - t_1) f(t_1 + \tau(t_2 - t_1), w + x_1,p_1).$$

Given $(t_1,t_2,x_1,p_1) \in J \times J \times V \times P$, a continuous map $\eta : [0,1] \rightarrow W$ is $C^0_W$ and satisfies

$$\eta'(\tau) = g(\tau,\eta(\tau),t_1,t_2,x_1,p_1) \quad \text{for all } \tau \in [0,1], \text{ and } \eta(0) = 0$$
if and only if

\[ (\forall \tau \in [0, 1]) \quad \eta(\tau) = \int_0^\tau g(\sigma, \eta(\sigma), t_1, t_2, x_1, p_1) \, d\sigma, \]

if and only if

\[ (\forall \tau \in [0, 1]) \quad \eta(\tau) = \int_\tau^1 \tau g(\sigma, \eta(\sigma), t_1, t_2, x_1, p_1) \, d\sigma. \quad (61) \]

Using notation as in Proposition 9.4, the preceding equation can be rewritten as

\[ (\forall \tau \in [0, 1]) \quad \eta(\tau) = \tau \cdot \int_0^1 (g^{t_1, t_2, x_1, p_1})(\eta(\sigma)) \, d\sigma. \quad (62) \]

To obtain a more transparent formula, we introduce the continuous mapping

\[ m : [0, 1] \times [0, 1] \to [0, 1], \quad m(\tau, \sigma) := \sigma \tau \]

and the pullback

\[ C(m, F) : C([0, 1], F) \to C([0, 1] \times [0, 1], F), \quad \zeta \mapsto \zeta \circ m \]

which is \( K \)-linear and Lipschitz continuous with \( \text{Lip}(C(m, F)) \leq 1 \), as we are using maximum metrics on the function spaces. Given \( \zeta \in C([0, 1] \times [0, 1], F) \), the map

\[ \zeta^\vee : [0, 1] \to C([0, 1], F), \quad \zeta^\vee(\sigma) := \zeta(\tau, \sigma) \]

is continuous and the \( K \)-linear map

\[ \Phi : C([0, 1] \times [0, 1], F) \to C([0, 1], C([0, 1], F)), \quad \Phi(\zeta) := \zeta^\vee \]

is continuous (see \[7, Theorem 3.4.7\]). Since maximum metrics are used on the function spaces, it is obvious that \( \Phi \) is isometric and hence Lipschitz continuous with \( \text{Lip}(\Phi) \leq 1 \). We also need the integration operator

\[ I : C([0, 1], F) \to F, \quad \zeta \mapsto \int_0^1 \zeta(\sigma) \, d\sigma \]

which is \( K \)-linear and Lipschitz continuous with \( \text{Lip}(I) \leq 1 \) (see Lemma 1.10). Finally, we need the map

\[ C([0, 1], I) : C([0, 1], C([0, 1], F)) \to C([0, 1], F), \quad \zeta \mapsto I \circ \zeta \]

which is \( K \)-linear (as is clear) and Lipschitz continuous with \( \text{Lip}(C([0, 1], I)) \leq 1 \) (by Lemma 9.6); and the multiplication operator

\[ \mu : C([0, 1], F) \to C([0, 1], F), \quad \mu(\zeta)(\tau) := \tau \zeta(\tau) \]

which is \( K \)-linear, and Lipschitz continuous with \( \text{Lip}(\mu) \leq 1 \) (again by Lemma 9.6).

We can now rewrite (62) as

\[ h(t_1, t_2, x_1, p_1, \eta) = 0 \]
Furthermore, Theorem A can be applied (if \( k \) in (4), because the supremum on the left hand side of (4) is \( \leq \)). Corollary 4.5 can be applied with \( A \) and a map which is a uniform family of contractions (resp., of special contractions if \( h \leq \)). Now the corollary or theorem provides open neighborhoods for \( (t, x) \). Hence the case \( \phi \) is \( \gamma \) as in (59). Hence

\[ h(t_1, t_2, x_1, p_1, \eta) = \eta - \tilde{h}(t_1, t_2, x_1, p_1, \eta) \]

with \( \tilde{h}: J \times J \times V \times P \times C([0, 1], W) \to C([0, 1], F) \) defined via

\[ \tilde{h}(t_1, t_2, x_1, p_1, \eta) := (\mu \circ C([0, 1], I) \circ \Phi \circ C(m, F) \circ (g^{t_1, t_2, x_1, p_1}_{\sim})))(\eta). \]  

(63)

By Lemma 9.6 \( \text{Lip}((g^{t_1, t_2, x_1, p_1}_{\sim})) \leq \theta \) for all \( (t_1, t_2, x_1, p_1) \in J \times J \times V \times P \). If \( k \geq 1 \), for each \( s \in \mathbb{K}^x \) we can apply Lemma 9.6 also with the metric given by \( d_s(x, y) := d(sx, sy) \) (instead of \( d \)), from which we conclude that \( (g^{t_1, t_2, x_1, p_1}_{\sim}) \), for \( (t_1, t_2, x_1, p_1) \in J \times J \times V \times P \), is a uniform family of special contractions with constant \( \theta \). Since all other maps involved in (63) are \( \mathbb{K} \)-linear and Lipschitz continuous with constant \( \leq 1 \) (as explained before), we deduce that

\[ h(t_1, t_2, x_1, p_1, \cdot): C([0, 1], W) \to C([0, 1], F), \]

for \( (t_1, t_2, x_1, p_1) \in J \times J \times V \times P \), is a uniform family of contractions (resp., of special contractions if \( k \geq 1 \)), with constant \( \theta \). Furthermore, \( h \) is \( C^k_\mathbb{K} \) as a composition of continuous \( \mathbb{K} \)-linear maps and a map which is \( C^k_\mathbb{K} \) by Proposition 9.4. Also, \( h(t_0, t_0, x_0, p_0, 0) = 0 \). Hence Corollary 1.5 can be applied with \( A := \text{id}: C([0, 1], F) \to C([0, 1], F) \), if \( k = 0 \). Furthermore, Theorem A can be applied (if \( k \geq 1 \)) with \( A = S = T = \text{id} \) in [11], because the supremum on the left hand side of [11] is \( \leq \theta \). Then \( D \), the maximum metric on \( C([0, 1], F) \). Now the corollary or theorem provides open neighborhoods \( J_1 \subseteq J, V_1 \subseteq V \) and \( P_1 \subseteq P \) of \( t_0, x_0, \) resp. \( p_0 \), and a \( C^k_\mathbb{K} \)-map \( \lambda: J_1 \times J_1 \times V_1 \times P_1 \to C([0, 1], F) \) such that

\[ h(t_1, t_2, x_1, p_1, \lambda(t_1, t_2, x_1, p_1)) = 0 \]

for all \( (t_1, t_2, x_1, p_1) \in J_1 \times J_1 \times V_1 \times P_1 \).

The case \( \mathbb{K} = \mathbb{L} = \mathbb{R} \). Given \( (t_1, t_2, x_1, p_1) \in J_1 \times J_1 \times V_1 \times P_1 \), consider the map \( \eta := \lambda(t_1, t_2, x_1, p_1): [0, 1] \to W \). Since \( \eta \) is continuous and satisfies (61), it is \( C^k_\mathbb{R} \) and satisfies (60). If \( t_2 \neq t_1 \), we define \( \gamma: [t_1, t_2] \to F, t \mapsto x_1 + \eta(\frac{t - t_1}{t_2 - t_1}) \) on the line segment \( [t_1, t_2] \) joining \( t_1 \) and \( t_2 \). Then \( \gamma \) is \( C^1_\mathbb{R} \), \( \gamma(t_1) = x_1 \), and \( \gamma'(t) = \frac{1}{t_2 - t_1} \eta(\frac{t - t_1}{t_2 - t_1}) \) for \( t \in [t_1, t_2] \). Hence \( \gamma \) is a solution to (55) on \( [t_1, t_2] \), and its value at \( t_2 \) is \( x_1 + \eta(1) = x_1 + \lambda(t_1, t_2, x_1, p_1)(1) \). By uniqueness of solutions (Proposition 10.1), the former solutions on smaller intervals combine to a solution \( \phi_{t_1, t_1, p_1}: J_1 \to W \), given by

\[ \phi_{t_1, t_1, p_1}(t) = x_1 + \lambda(t_1, t, x_1, p_1)(1). \]  

(64)

Since \( \lambda \) is \( C^k_\mathbb{R} \) and the evaluation map

\[ \text{ev}_1: C([0, 1], F) \to F, \quad \zeta \mapsto \zeta(1) \]  

is continuous and linear, we deduce that \( \Psi \) (and hence also the map in (b)) is \( C^k_\mathbb{R} \).

The case \( \mathbb{K} = \mathbb{L} = \mathbb{C} \). Then \( k \geq 1 \). In this case, we simply use (64) to define
and a condition on the diameter of the image of $f$ solves (58). Then $U$ balls. For example, if $\eta$ just as well for higher order equations under appropriate analogous conditions, for $s$ a global such condition, for $\eta$ solves (58), it only remains to show that $\phi'_{x_1,t_1,p_1}(t_2) = f(t_2, \phi_{x_1,t_1,p_1}(t_2), p_1)$ holds for each $t_2 \in J_1$. By continuity, it suffices to check this for $t_2 \neq t_1$. But then we can define $\eta$ and a $C^1_\mathbb{C}$-map $\gamma: [t_1, t_2] \to U \subseteq F$ by the same formulas as in the proof of the case $K = L = \mathbb{R}$, considering now the line segment $[t_1, t_2]$ joining $t_1, t_2$ as a 1-dimensional real manifold with boundary immersed into $\mathbb{C}$. Again, $\gamma$ solves $\xi_t$ (considered now as an ODE on the manifold $[t_1, t_2]$), and we deduce as above that $\gamma(t) = x_1 + \lambda(t_1, \tau, x_1, p_1)(1) = \phi_{t_1, x_1, p_1}(\tau)$ for each $\tau \in [t_1, t_2]$. Calculating the complex derivative as a suitable real directional derivative, we find that $\phi'_{t_1, x_1, p_1}(t_2) = \gamma'(t_2) = f(t_2, \phi_{t_1, x_1, p_1}(t_2), p_1)$, as desired.

The case $K = \mathbb{C}, L = \mathbb{R}$. Then $k \geq 1$, and the case $K = L = \mathbb{C}$ provides $J_1, U_1$ and $P_1$ as described in the theorem such that $\xi_t$ admits a $C^k$-solution $\xi_{t_1, x_1, p_1}: J_1 \to U$ for all $t_1 \in J_1$, $x_1 \in U_1$ and $p_1 \in P_1$, and such that $\Theta: J_1 \times J_1 \times U_1 \times P_1 \to U, \Theta(t_1, t_2, x_1, p_1) := \xi_{t_1, x_1, p_1}(t)$ is $C^k_\mathbb{C}$. Then $\phi_{t_1, x_1, p_1} := \xi_{t_1, x_1, p_1}|_{J_1 \cap \mathbb{R}}: J_1 \cap \mathbb{R} \to U$ is a $C^k_\mathbb{R}$-solution to $\xi_t$ whenever $t_1 \in J_1 \cap \mathbb{R}$, and the map $\Psi$ (defined in (a)) is $C^k_\mathbb{R}$, being the restriction of the $C^k_\mathbb{C}$-map $\Theta$ to $(J_1 \cap \mathbb{R}) \times (J_1 \times \mathbb{R}) \times U_1 \times P_1$. Since $\Psi(t_1, t, \cdot) = \Theta(t_1, t, \cdot)$ is $C^k_\mathbb{C}$, also (b) is verified.

Remark 10.4 If $F$ is a Banach space and $f: J \times U \times P \to F$ satisfies a Lipschitz condition in its second argument, then $sf$ satisfies a local SCC in its second argument (even a global such condition), for $s \in \mathbb{R}^+$ sufficiently small. If $\gamma$ solves $\xi_t$, then $\eta: s^{-1}J_1 \to U, \eta(t) := \gamma(st)$ solves $\eta'(t) = sf(st, \eta(t), p_1)$, $\eta(0) = x_1$. Similarly, we can pass from $\eta$ back to $\gamma$. As a consequence, all conclusions of Theorem 10.3 remain valid if no local CC or local SCC is assumed, but $F$ is a Banach space and $f$ satisfies a local Lipschitz condition in its second argument.

Remark 10.5 Of course, we can prove existence, uniqueness and $C^k$-dependence just as well for higher order equations under appropriate analogous conditions, by rewriting them as first-order systems.

Remark 10.6 It is possible to extract quantitative information from the proof of Theorem 10.3 because Theorem A and Corollary 4.3 can be traced back to Theorem 4.4 which provides quantitative information on the size of the images of balls. For example, if $U$ is a ball and $U_1$ a ball with same center of half the radius of $U$, it is possible to describe explicit conditions on the size of the differentials and a condition on the diameter of the image of $f$ which ensure that $J_1 = J$ and $P_1 = P$ can be chosen in Theorem 10.3.

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Remark 10.7 If $E$ is a metric Fréchet space and $k \geq 1$, it is possible to prove an analogue of Theorem A.11 for $f$ an $MC^k_F$-map, in which case $\Psi$ will be $MC^k_F$ and the map in (b) will be $MC^k_F$. For the proof, note that Proposition 9.2 has an analogue for $MC^k_F$-maps, and use Corollary 7.2 instead of Theorem A.

10.8 Prospect: A new class of infinite-dimensional Lie groups.

Using results from this article, it is possible to construct certain Lie groups of rapidly decreasing diffeomorphisms of Fréchet spaces.

Let $(F,d)$ be a metric Fréchet space with absolutely convex balls, and $W$ be a closed set of functions $w: F \to \mathbb{R} \cup \{\infty\}$ containing the constant function 1. Let $C_W^\infty(F,F)$ be the “weighted function space” of all $MC^\infty$-maps $\gamma: F \to F$ such that $\sup_{x \in F} |w(x)| \cdot \|\gamma(x)\|_d < \infty$ and $\sup_{x \in F} |w(x)| \cdot \|\gamma^{(k)}(x)\|_{d,d} < \infty$ for all $w \in W$ and $k \in \mathbb{N}$. For example, take $F = \mathbb{R}$ and let $W$ be the set of all polynomial functions $\mathbb{R} \to \mathbb{R}$; then $C_W^\infty(\mathbb{R},\mathbb{R}) = S(\mathbb{R})$ is the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$. Returning to the general case, let $Diff_W(F)$ be the set of all diffeomorphisms $\gamma: F \to F$ such that $\gamma - id_F$, $\gamma^{-1} - id_F \in C_W^\infty(F,F)$. It was shown recently that $Diff_W(F)$ can be made a Lie group modelled on $C_W^\infty(F,F)$, for each Banach space $F$ (see [37]); this Lie group has a smooth exponential map and is regular (in Milnor’s sense, as in [28]). Using results provided in this article (notably, Theorem A.3) instead of the standard facts of Banach differential calculus used in [37], it is possible to turn $Diff_W(F)$ into a Lie group along the lines of [37]. Using the results on ODEs in Fréchet spaces sketched in Remark 10.7 one also sees similarly as in the Banach case that $Diff_W(F)$ is regular.

A Proof of Proposition 2.1

(a) We know from Remark 1.9 that $\mathcal{L}_{d,d'}(E,F)$ is an additive subgroup of $F^E$. To see closedness under scalar multiplication, let $A \in \mathcal{L}_{d,d'}(E,F)$ and $t \in \mathbb{K}$. Then $\|tA.x\|_{d'} \leq \max\{1,2|t|\} \cdot \|A.x\|_{d'} \leq \max\{1,2|t|\} \cdot \|A\|_{d,d'} \cdot \|x\|_d$ for all $x \in E$ (see Lemma 1.7 and [9]). Hence $\|tA\|_{d,d'} \leq \max\{1,2|t|\} \cdot \|A\|_{d,d'} < \infty$ and thus $tA \in \mathcal{L}_{d,d'}(E,F)$.

(b) The evaluation map $\varepsilon$ is continuous at $(0,0)$ by (9) in Remark 1.9(a), and furthermore $\varepsilon(A,\star)$ and $\varepsilon(\star,x)$ are continuous at 0 for all $A \in \mathcal{L}_{d,d'}(E,F)$ and $x \in E$, by (9). Hence $\varepsilon$ is continuous, being bilinear (see Lemma A.11 below).

(c) The composition mapping $\Gamma: \mathcal{L}_{d,d'}(F,G) \times \mathcal{L}_{d,d'}(E,F) \to \mathcal{L}_{d,d'}(E,G)$, $(A,B) \mapsto A \circ B$ is continuous at $(0,0)$ by (11) in Remark 1.9(b), and the maps $\Gamma(A,\star)$ and $\Gamma(\star,x)$ are continuous at 0, as a consequence of (11). Since $\Gamma$ is bilinear, this implies continuity of $\Gamma$.

(d) We already know from Remark 1.9(d) that $D := D_{d,d'}$ is a metric. Now $\|tA\|_{d,d'} \leq \|A\|_{d,d'}$ for all $A \in \mathcal{L}_{d,d'}(E,F)$ and $t \in \mathbb{K}$ such that $|t| \leq 1$, by the case $|t| \leq 1$ of Lemma 1.7. Hence $D$ has absolutely convex balls.
To see that \((\mathcal{L}_{d,d'}(E,F), D)\) is complete, let \((A_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in \(\mathcal{L}_{d,d'}(E,F)\). Given \(x \in E\), the point evaluation \(\mathcal{L}_{d,d'}(E,F) \rightarrow F, B \mapsto B x\) is continuous linear. Hence \((A_n x)_{n \in \mathbb{N}}\) is a Cauchy sequence in \(F\) and hence convergent, to \(A x\) say. It is clear that the map \(A : E \rightarrow F\) so obtained is linear.

Given \(\varepsilon > 0\), there is \(N \in \mathbb{N}\) such that \(\|A_n - A_m\|_{d,d'} \leq \varepsilon\) for all \(n, m \geq N\). Given \(x \in E\), this implies that \(\|A_n.x\|_{d'} \leq \|(A_n - A_N).x\|_{d'} + \|A_N.x\|_{d'} \leq \|A_n - A_N\|_{d,d'} \|x\|_d + \|A_N\|_{d,d'} \|x\|_d \leq (\varepsilon + \|A_N\|_{d,d'}) \|x\|_d\) for all \(n \geq N\) and hence also \(\|A.n\|_{d'} \leq (\varepsilon + \|A_N\|_{d,d'}) \|x\|_d\), letting \(n \rightarrow \infty\). Thus \(\|A\|_{d,d'} \leq (\varepsilon + \|A_N\|_{d,d'}) \|x\|_d < \infty\) and hence \(A \in \mathcal{L}_{d,d'}(E,F)\). Let \(\varepsilon\) and \(N\) be as before.

Given \(m \geq N\) and \(x \in E\), we have \(\|(A_n - A_m).x\|_{d'} \leq \|A_n - A_m\|_{d,d'} \|x\|_d \leq \varepsilon \|x\|_d\) for all \(n \geq N\) and hence \(\|(A - A_m).x\|_{d'} \leq \varepsilon \|x\|_d\), letting \(m \rightarrow \infty\). Since \(x\) was arbitrary, we deduce that \(\|A - A_m\|_{d,d'} \leq \varepsilon\) for all \(n \geq N\). Thus \(A = \lim_{n \rightarrow \infty} A_n\) in \(\mathcal{L}_{d,d'}(E,F)\).

(e) This is [30, Theorem 4.2].

We used the following simple fact.

**Lemma A.1** Let \(A, B, C\) be abelian topological groups and \(\beta : A \times B \rightarrow C\) be a bi-additive map (viz., a \(\mathbb{Z}\)-bilinear map). If \(\beta\) is continuous at \((0,0)\) and all of the maps \(\beta(a, \cdot) : B \rightarrow C\) for \(a \in A\) and \(\beta(\cdot, b) : A \rightarrow C\) for \(b \in B\) are continuous at \(0\), then \(\beta\) is continuous.

**Proof.** Let \((a_j, b_j)_{j \in J}\) be a convergent net in \(A \times B\), with limit \((a, b)\). Since

\[
\beta(a_j, b_j) - \beta(a, b) = \beta(a_j - a, b_j - b) + \beta(a, b_j - b) + \beta(a_j - a, b) \rightarrow 0
\]

by the hypotheses, we see that \(\beta(a_j, b_j) \rightarrow \beta(a, b)\).

\[\square\]

## B Basic facts concerning \(MC^k\)-maps

In this appendix, we prove compile various basic facts concerning \(MC^k\)-maps, and deduce Lemma B.3 from them.

On a product \(E \times F\) of metric Fréchet spaces \((E, d)\) and \((F, d')\), we shall always use the maximum metric

\[
(E \times F)^2 \rightarrow [0, \infty[, \quad ((x_1, y_1), (x_2, y_2)) \mapsto \max\{d(x_1, x_2), d'(y_1, y_2)\}. \tag{66}
\]

**Lemma B.1** For all metric Fréchet spaces \((E, d)\), \((F, d')\) and \((G, d'')\) with absolutely convex balls, the following holds:

\begin{itemize}
  \item[(a)] Each \(A \in \mathcal{L}_{d,d'}(E,F)\) is an \(MC^\infty\)-map \(E \rightarrow F\). Furthermore, the translation \(\tau_x : E \rightarrow E, y \mapsto x + y\) is \(MC^\infty\) for each \(x \in E\).
\end{itemize}
(b) For each $A \in \mathcal{L}_d(E)$, both the left multiplication map

$$\lambda_A : \mathcal{L}_d(E)_0 \to \mathcal{L}_d(E)_0, \quad B \mapsto AB$$

and the right multiplication map $\rho_A : \mathcal{L}_d(E)_0 \to \mathcal{L}_d(E)_0, \ B \mapsto BA$ are $MC^K_\infty$. More generally, for each $A \in \mathcal{L}_{d',d''}(F,G)$ and $C \in \mathcal{L}_{d,d'}(E,F)$, the maps

$$\lambda_A : \mathcal{L}_{d,d'}(E,F)_0 \to \mathcal{L}_{d,d''}(E,G)_0, \quad B \mapsto AB$$

and $\rho_A : \mathcal{L}_{d',d''}(F,G)_0 \to \mathcal{L}_{d,d'}(E,G)_0, \ B \mapsto BA$ are $MC^K_\infty$.

(c) The map $L : \mathcal{L}_d(E)_0 \to \mathcal{L}_D(\mathcal{L}_d(E)_0)_0, \ L(A) := \lambda_A$ is $MC^K_\infty$ and also the map $R : \mathcal{L}_d(E)_0 \to \mathcal{L}_D(\mathcal{L}_d(E)_0)_0, \ R(A) := \rho_A$ is $MC^K_\infty$, where $D := D_{d,d}$ is the natural metric on $\mathcal{L}_d(E)$.

(d) Let $\beta : E \times F \to G$ be a bilinear map such that $\|\beta(x,y)\|_{d''} \leq \|x\|_d \|y\|_{d'}$ for all $x \in E$, $y \in F$. Then $\beta$ is $MC^K_\infty$. In particular, the composition map $\Gamma$ is $MC^K_\infty$, where

$$\Gamma : \mathcal{L}_d(E)_0 \times \mathcal{L}_d(E)_0 \to \mathcal{L}_d(E)_0, \quad (A,B) \mapsto A \circ B.$$  

(e) If $U \subseteq E$ is a locally convex subset with dense interior and both $f : U \to F$ and $g : U \to G$ are $MC^K_\infty$, then also $(f,g) : U \to F \times G$ is $MC^K_\infty$.

(f) If $U \subseteq E$ and $V \subseteq F$ are locally convex subsets with dense interior and $f : U \to V \subseteq F$, $g : V \to G$ are $MC^K_\infty$, then also $g \circ f : U \to G$ is $MC^K_\infty$.

(g) The quasi-inversion map $q : Q(\mathcal{L}_d(E)_0) \to \mathcal{L}_d(E)_0$ is $MC^K_\infty$.

**Proof.** (a) Being continuous, $A$ is $MC^0$. Furthermore, being continuous linear, $A$ is $C^1$ with $A' : E \to \mathcal{L}(E,F)$, $x \mapsto A$ a constant (and hence continuous) map into $\mathcal{L}_d(E,F)$. Hence $A$ is $MC^1$ and it follows by a trivial induction that $A$ is $MC^k$ for each $k \geq 2$ with $A^{(k)} = 0$.

The translation $\tau_x$ is $C^1$ with $(\tau_x)'(y) = id_E$ for each $y \in E$. Thus $(\tau_x)'$ is a constant map to $\mathcal{L}_d(E)$ and hence continuous. As before, we deduce that $\tau_x$ is $MC^K_\infty$ with $(\tau_x)^{(k)}(0) = 0$ for each $k \geq 2$.

(b) By Remark 4.4(b), the map $\lambda_A$ is Lipschitz continuous. Since $\lambda_A$ is a linear map, it follows with (a) that $\lambda_A$ is $MC^K_\infty$. The maps $\rho_A$ (and $\rho_C$) can be discussed analogously.

(c) Since $\|AB\|_d \leq \|A\|_d \|B\|_d$ for all $B \in \mathcal{L}_d(E)$, it follows that $\|\lambda_A\|_D \leq \|A\|_d$ and $\lambda_A \in \mathcal{L}_D(\mathcal{L}_d(E)_0)$. If $A \in \mathcal{L}_d(E)_0$, then $tA \to 0$ as $t \to 0$, whence $\|tA\|_d \to 0$ and thus $\|t\lambda_A\|_D = \|\lambda_A\|_D \leq \|tA\|_d \to 0$. Hence $\lambda_A \in \mathcal{L}_D(\mathcal{L}_d(E)_0)_0$. Summing up, $L : \mathcal{L}_d(E)_0 \to \mathcal{L}_D(\mathcal{L}_d(E)_0)_0$ is a Lipschitz continuous linear map and thus $MC^K_\infty$, by (a). The map $R$ can be discussed along the same lines.

(d) Being continuous bilinear, $\beta$ is $C^1$ with $\beta'(x,y).(u,v) = \beta(x,v) + \beta(u,y)$. Since $\|\beta(x,v) + \beta(u,y)\|_{d''} \leq 2 \max \{\|x\|_d,\|y\|_{d'}\} \cdot \max \{\|u\|_d,\|v\|_{d'}\}$, we deduce
that \( \beta'(x, y) \in \mathcal{L}_{D, d'}(E \times F, G) \), where \( D \) is the maximum metric (as in (66)). Furthermore, \( \|\beta'(x, y)\|_{D, d'} \leq 2\|(x, y)\|_D \). Thus \( \beta': E \times F \to \mathcal{L}_{D, d'}(E \times F, G) \) is Lipschitz continuous and linear. The space \( E \times F \) and hence also its image being connected, we deduce that \( \beta'(E \times F) \subseteq \mathcal{L}_{D, d'}(E \times F, G)_0 \). Now \( \beta' \) is \( MC^\infty \) by (a). Hence also \( \beta \) is \( MC^\infty \).

(e) and (f): We may assume that \( k \in \mathbb{N}_0 \). We now prove (e) and (f) in parallel for \( k \in \mathbb{N}_0 \), by induction. In both cases, the case \( k = 0 \) is trivial. Thus, let \( k \) be an integer \( \geq 1 \) now and assume that (e) and (f) hold with \( k - 1 \) in place of \( k \).

**Induction step for (f):** After shrinking \( U \) and \( V \), we may assume that both sets are connected. Pick \( x_0 \in U \) and set \( y_0 := f(x_0) \).

We know that \( g \circ f \) is \( C^1 \), with

\[
(g \circ f)'(x) = \frac{g'(f(x)) \circ f'(x)}{\rho_{f(x)} \circ (g' - g'(y_0)) \circ f \circ f'} + \lambda_{g'(y_0)} \circ (f' - f'(x_0)),
\]

using suitable left and right translations (which are \( MC^\infty \)) and the composition map \( \Gamma: \mathcal{L}_{d', d''}(F, G)_0 \times \mathcal{L}_{d, d'}(E, F)_0 \to \mathcal{L}_{d, d''}(E, G)_0 \), which is \( MC^\infty \) by (d). All maps involved being continuous, we infer from (67) that \( (g \circ f)'(x) \) is continuous. Assume now that compositions of \( MC^{k-1} \)-maps are \( MC^{k-1} \). Using the \( MC^{k-1} \)-case of (e), we then deduce from (67) that the mapping \( (g \circ f)' - (g \circ f)(x_0) : U \to \mathcal{L}_{d, d''}(E, G)_0 \) is \( MC^{k-1} \), whence \( g \circ f \) is \( MC^k \).

**Induction step for (e):** We may assume that \( U \) is connected and pick \( x_0 \in U \). We let \( pr_1: F \times G \to F \) and \( pr_2: F \times G \to G \) be the projections onto the first and second component, respectively. These maps are Lipschitz continuous and linear. Also, we let \( \alpha: \mathcal{L}_{d, d'}(E, F)_0 \times \mathcal{L}_{d, d''}(E, G)_0 \to \mathcal{L}_{d, D}(E, F \times G)_0 \), \( (A, B) \mapsto (x \to (Ax, Bx)) \) be the natural isomorphism of vector spaces, which is a linear contraction. Then

\[
(f, g)' - (f, g)(x_0) = \alpha \circ (pr_1 \times pr_2) \circ (f' - f'(x_0), g' - g'(x_0)).
\]

Using the inductive hypotheses (both for (e) and (f)) and (b), the preceding formula shows that \( (f, g)' - (f, g)(x_0) \) is \( MC^{k-1} \) and thus \( (f, g) \) is \( MC^k \).

(g) We already know from Proposition 2.11 that \( Q(\mathcal{L}_d(E)_0) \) is open in \( \mathcal{L}_d(E)_0 \) and that \( q \) is \( C^\infty \) (and hence continuous). Since \( q(A) = id_E - (id_E - A)^{-1} \), the well-known formula \( b^{-1} - a^{-1} = b^{-1} (a - b) a^{-1} \) for invertible elements in a unital...
algebra implies that
\[
q(B) - q(A) = (\text{id}_E - A)^{-1} - (\text{id}_E - B)^{-1} = (\text{id}_E - A)^{-1}(A - B)(\text{id}_E - B)^{-1}
\]
\[
= (q(A) - \text{id}_E)(A - B)(q(B) - \text{id}_E)
\]
\[
= q(A)(A - B)q(B) - (A - B)q(B) - q(A)(A - B) + (A - B) \quad (68)
\]
for all \(A, B \in Q(L_d(E))\). Let \(A \in Q(L_d(E)_0)\) and \(B \in L_d(E)_0\) now. For \(0 \neq t \in \mathbb{K}\) sufficiently small, using (68) we see that
\[
\frac{q(A + tB) - q(A)}{t} = -q(A)Bq(A + tB) + Bq(A + tB) + q(A)B - B;
\]
which tends to \(dq(A, B) = -q(A)Bq(A) + Bq(A) + q(A)B - B\) as \(t \to 0\). Thus, writing \(1 := \text{id}_{L_d(E)_0}\), we have \(q'(A) + 1 = -\lambda_q(A) \circ \rho_q(A) + \rho_q(A) + \lambda_q(A) \in L_D(L_d(E)_0)\) by (b) and (c), and
\[
q' + 1 = \Gamma \circ (L, R) \circ q + R \circ q + L \circ q \quad (69)
\]
where \(L\) and \(R\) are the \(MC^\infty\)-maps from (c) and the composition map
\[
\Gamma : L_D(L_d(E)_0) \times L_D(L_d(E)_0) \to L_D(L_d(E)_0)
\]
is \(MC^\infty\) by (d). Since \(q\) is continuous, (69) shows that also the mapping \(q' + 1 : Q(L_d(E)_0) \to L_D(L_d(E)_0)\) is continuous, whence \(q\) is \(MC^1\). If \(q\) is \(MC^K\) by induction, then \(69\) shows that also \(q' + 1\) is \(MC^K\) and so \(q\) is \(MC^{K+1}\). 

**Proof of Lemma 7.3.** Since \(L_d(F)^\infty\) is open, \(M := ((A + L_d(F)_0) \cap L_d(F)^\infty)\) is open in the affine space \(A + L_d(F)_0\) which is homeomorphic to \(L_d(F)_0\) and hence locally connected. Thus, the connected component of \(M\) containing \(A\) is open in \(A + L_d(F)_0\). Since \(L_d(F)_0 \to L_d(F)_0\), \(C \mapsto C - A\) is a homeomorphism, openness of \(\Omega\) follows.

Since \(\iota_A(B) = (\text{id}_F + A^{-1}B)^{-1}A^{-1} - A^{-1} = ((\text{id}_F + A^{-1}B)^{-1} - \text{id}_F)A^{-1} = -q(-A^{-1}B)A^{-1}\) for \(B \in \Omega\) using the quasi-inversion map \(q\) of \(L_d(F)_0\), we see that \(\iota_A = -\rho_{A^{-1}} \circ q \circ (-\lambda_{A^{-1}})|\Omega\). Hence \(\iota_A\) is an \(MC^{\infty}\)-map, by Part (b) and (g) of Lemma 15.1

**References**

[1] Außenhofer, L., “Contributions to the Duality Theory of Abelian Topological Groups and to the Theory of Nuclear Groups,” Diss. Math. **384**, 1999.

[2] Bastiani, A., *Applications différentiables et variétés différentiables de dimension infinie*, J. Analyse Math. **13** (1964), 1–114.

[3] Bertram, W., H. Glöckner and K.-H. Neeb, *Differential calculus over general base fields and rings*, Expo. Math. **22** (2004), 213–282.
[4] Bochnak, J. and J. Siciak, Analytic functions in topological vector spaces, Studia Math. 39 (1971), 77–112.

[5] Choquet-Bruhat, Y., C. DeWitt-Morette and M. Dillard-Bleik, “Analysis, Manifolds and Physics,” North-Holland, Amsterdam, 1977.

[6] Chow, S. N. and J. K. Hale, “Methods of Bifurcation Theory,” Springer-Verlag, New York, 1982.

[7] Engelking, R., “General Topology,” Heldermann Verlag, 1989.

[8] Forster, O., “Lectures on Riemann Surfaces,” Springer, New York, 1999.

[9] Glöckner, H., Infinite-dimensional Lie groups without completeness restrictions, pp. 43–59 in: Strasburger, A. et al. (Eds.), “Geometry and Analysis on Finite- and Infinite-dimensional Lie Groups,” Banach Center Publ. 55, Warsaw, 2002.

[10] Glöckner, H., Lie group structures on quotient groups and universal complexifications for infinite-dimensional Lie groups, J. Funct. Anal. 194 (2002), 347–409.

[11] Glöckner, H., Algebras whose groups of units are Lie groups, Studia Math. 153 (2002), 147–177.

[12] Glöckner, H., Bundles of locally convex spaces, group actions, and hypocontinuous bilinear mappings, manuscript, November 2002.

[13] Glöckner, H., Lie groups over non-discrete topological fields, preprint, [arXiv:math/0408008](http://arxiv.org/abs/math/0408008).

[14] Glöckner, H., Diff(\(\mathbb{R}^n\)) as a Milnor-Lie group, Math. Nachr. 278 (2005), 1025–1032.

[15] Glöckner, H., Fundamentals of direct limit Lie theory, Compos. Math. 141 (2005), 1551–1577.

[16] Glöckner, H., Implicit functions from topological vector spaces to Banach spaces, Israel J. Math. 155 (2006), 205–252.

[17] Glöckner, H., Finite order differentiability properties, fixed points and implicit functions over valued fields, preprint, [arXiv:math/0511218](http://arxiv.org/abs/math/0511218).

[18] Glöckner, H. and K.-H. Neeb, “Infinite-Dimensional Lie Groups,” Vol. I, book in preparation.

[19] Hamilton, R. S., The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. 7 (1982), 65–222.
[20] Hiltunen, S., *Implicit functions from locally convex spaces to Banach spaces*, Studia Math. **134** (1999), 235–250.

[21] Hiltunen, S., *Differentiation, implicit functions, and applications to generalized well-posedness*, preprint, arXiv:math/0504268.

[22] Hiltunen, S., *On an assertion about Nash-Moser applications*, preprint, arXiv:math/0702063.

[23] Hogbe-Nlend, H., “Théorie des Bornologies et Applications,” Springer LNM **213**, Springer, Berlin, 1971.

[24] Keller, H. H., “Differential Calculus in Locally Convex Spaces,” Springer, 1974.

[25] Krantz, S. G. and H. R. Parks, “The Implicit Function Theorem,” Birkhäuser, Boston, 2002.

[26] Kriegl, A. and P. W. Michor, “The Convenient Setting of Global Analysis,” Amer. Math. Soc., Providence, 1997.

[27] Maissen, B., *Über Topologien im Endomorphismenraum eines topologischen Vektorraumes*, Math. Ann. **151** (1963), 283–285.

[28] Milnor, J., *Remarks on infinite-dimensional Lie groups*, pp. 1008–1057 in: DeWitt, B., and R. Stora (Eds.), “Relativity, Groups and Topology II,” North Holland, 1984.

[29] Moser, J., *A new technique for the construction of solutions of nonlinear differential equations*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 1824–1831.

[30] Müller, O., *Bounded Fréchet geometry*, preprint, arXiv:math/0612379v3, December 20, 2006.

[31] Nash, J., *Real algebraic manifolds*, Ann. of Math. (2) **56** (1952), 405–421.

[32] Neeb, K.-H. and C. Vizman, *Flux homomorphisms and principal bundles over infinite-dimensional manifolds*, Monatsh. Math. **139** (2003), 309–333.

[33] Raǐkov, D. A., *On B-complete topological vector groups*, Studia Math. **31** (1968), 295–306.

[34] Rudin, W., “Functional Analysis,” McGraw-Hill, New York, 1991.

[35] Sergeraert, F., *Un théorème de fonctions implicites. Applications*, Ann. Inst. Fourier (Grenoble) **23** (1973), 151–157.

[36] Teichmann, J., *A Frobenius theorem on convenient manifolds*, Monatsh. Math. **134** (2001), 159–167.
[37] Walter, B., “Liegruppen von Diffeomorphismen von Banach-Räumen,” Diplomarbeit, Darmstadt University of Technology, 2006 (advisor: H. Glöckner).

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