Kähler Potential for Global Symmetry Breaking in Supersymmetric Theories

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Abstract

We have developed \( N = 1 \) supersymmetric nonlinear realization methods, which realize global symmetry breaking in \( N = 1 \) supersymmetric theories. The target space of nonlinear sigma models with a linear model origin is a \( G^C \)-orbit, which is a non-compact non-homogeneous Kähler manifold. We show that, if and only if the orbit is open, it includes a compact homogeneous Kähler manifold as a submanifold, and a class of strictly \( G \)-invariant Kähler potentials reduces to a Kähler potential \( G \)-invariant up to a Kähler transformation on the submanifold. Hence, in the case of an open orbit, the most general low-energy effective Kähler potential can be written as the sum of those of the compact submanifolds and an arbitrary function of strictly \( G \)-invariants.

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1 Introduction

Supersymmetric nonlinear sigma models \[1, 31\] have been used in many physical applications, such as low-energy effective Lagrangians of supersymmetric gauge theories \[4, 8\] and coset unification models \[4\].

In gauge theories, the target spaces of nonlinear sigma models are the classical (or quantum modified) moduli spaces. They are obtained by integrating out gauge fields which obtain masses by the Higgs mechanism. In supersymmetric theories, this procedure can be understood by the method of the Kähler quotient \[5\]. On the other hand, in the case of global symmetry breaking, low-energy effective Lagrangians can be obtained by integrating out massive fields, such as Higgs fields. The target manifolds can be constructed by a method involving the nonlinear realization of global symmetry \[7\]. (These two procedures just correspond to solving the D-term or F-term flatness conditions \[6\].) Much progress concerning nonlinear realization in supersymmetric theories has been made by many authors \[1, 8\]–\[25\]. A coupling to gauge fields has been discussed in Ref. \[23\], a coupling to matter fields in Ref. \[8, 19, 24\] and supersymmetric Wess-Zumino-Witten terms in Ref. \[25\].

In this paper, we consider global symmetry breaking without gauge symmetry, and develop a supersymmetric nonlinear realization method.

In general, when a global symmetry $G$ breaks down to its subgroup $H$, there appear quasi Nambu-Goldstone (QNG) bosons besides ordinary Nambu-Goldstone (NG) bosons. The low-energy effective Lagrangian is a nonlinear sigma model, whose target space is a Kähler manifold, and is parameterized by NG and QNG bosons. (Low-energy theorems of these scattering amplitudes are discussed in Refs. \[21, 22\].) The target space can be written as a complex coset space as $G^C/\hat{H}$, where $G^C$ is a complexification of $G$ and $\hat{H}$ is a complex group which includes a complexification of $H$. The Kähler potentials of a $G$-invariant effective Lagrangian are constructed by Bando, Kuramoto, Maskawa and Uehara (BKMU) \[8\]. There are three types of $G$-invariant Kähler potentials: A-, B- and C-type. The A- and C-type Kähler potential

\footnote{For a review of the supersymmetric nonlinear realization, see Refs. \[16, 17, 26, 27\]. Especially, we recommend Ref. \[27\].}
potentials are strictly $G$-invariant; on the other hand, the B-type Kähler potentials are, in general, $G$-invariant up to a Kähler transformation.

If there is any QNG boson, the target manifold becomes a non-compact non-homogeneous manifold; on the other hand, if there is no QNG boson, the target manifold is a compact homogeneous manifold. Itoh, Kugo and Kunitomo (IKK) showed that a Kähler potential of a compact homogeneous Kähler manifold can be completely written as a sum of B-type Kähler potentials. (These are called pure realizations.) [10]. All compact homogeneous Kähler manifolds have been completely classified in Ref. [11]. Moreover, coset unification models are based on these compact models [4]. Hence, many authors have studied compact models [10]–[13]. However, unfortunately, these models have no linear-model origin in the case of global symmetry breaking (without gauge symmetry) [3]. The target space of global symmetry breaking must be a non-compact non-homogeneous manifold [3, 17, 18].

Buchmuller and Ellwanger have considered models where some central charges in $\hat{H}$ are broken by hand from compact homogeneous models [14, 15, 16]. (We call these models as “broken center models”.) These models are non-compact non-homogeneous, and have strictly $G$-invariant Kähler potentials besides B-type Kähler potentials. However, it is not known whether they have linear-model origins.

We, thus, consider non-compact non-homogeneous Kähler manifolds which have linear-model origins. It is known that such models can have strictly $G$-invariant Kähler potentials [4, 17, 18, 20]. Thus, there remain a question: is there any B-type Kähler potential in a model with a linear-model origin? In this paper, we give an answer to this question. The idea is that we can consider submanifolds of the total target manifold and can add Kähler potentials of the submanifolds to the total Kähler potential. We find that, in a model with a linear-model origin, B-type Kähler potentials are strictly $G$-invariant and are not independent of A- or C-type invariants. Nevertheless, we show that if the orbit is open, the Kähler manifold, $G^C/\hat{H}$, has a compact Kähler submanifold, $G^C/\tilde{H}$, and B-type Kähler potentials on $G^C/\hat{H}$ reduce to those on $G^C/\tilde{H}$; also, if the orbit is closed, it does not have

\footnote{In the case where there is a gauge symmetry, we can sometimes construct compact models with linear origins.}
any compact Kähler submanifold, and B-type Kähler potentials are still strictly $G$-invariant. Here, $\tilde{H}$ is a complex subgroup of $G^C$ and includes $\hat{H}$ as a subgroup, $\hat{H} \subset \tilde{H}$; it is obtained by changing some broken generators in $G^C - \hat{H}$ to unbroken generators by hand. Therefore, $G^C / \hat{H}$ includes $G^C / \tilde{H}$ as a submanifold.) We can thus add B-type Kähler potential on compact submanifolds to the full Kähler potential in an open orbit. Moreover, this open-orbit model coincides with a special class of the broken-center models considered by Buchmüller and Ellwanger [14, 15, 16]. We, thus, also find a linear origin of special cases of these models.

We can conclude that pure realizations [10] are not just mathematical models, but are embedded in open orbits with linear-model origins.

This paper is organized as follows. In Sec. 2, we show that target spaces of sigma models are obtained as $G^C$-orbits of the vacuum. They can be classified by the value of $G^C$-invariants. To be precise, we treat the $O(N)$ model. In this model, there are two kinds of $G^C$-orbits: one is a closed orbit, and the other is an open orbit. We show that, although these orbits have very similar properties, an essential point is different on the both orbits: the closed orbit does not have a Borel subalgebra in the complex isotropy, $\hat{H}$; on the other hand, the open orbit has it.

In Sec. 3, strictly $G$-invariant Kähler potentials, the A- and C-types, are constructed by the method of BKMU. Although this section does not have any new result, we use the result in Sec. 4.

In Sec. 4, we discuss Kähler potentials $G$-invariant up to a Kähler transformation, the B-type Kähler potentials. We find that, although they are strictly $G$-invariant and are not independent of A- or C-type Kähler potentials, they can be reduced to B-type Kähler potentials on a compact submanifold, if and only if the orbit is open.

Sec. 5 is devoted to conclusions and discussions.

In App. A, we construct the Kähler potential of the compact homogeneous Kähler manifold $O(N)/O(N - 2) \times U(1)$ by using the method of IKK [10]. Since it is embedded to the open orbit, this appendix is used in Sec. 4.

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3 We use Latin characters as Lie algebras of the corresponding Lie groups.
2 Classification of $G^C$-orbits

$G^C$-orbits can be characterized by the values of $G^C$-invariants composed of fundamental fields. In general, depending on their value, there exist closed orbits and open orbits. As an example, let us consider the $O(N)$ model. It has both closed and open orbits. The closed orbits of this model have been discussed in Refs. [20, 22]. In this section we investigate both orbits, and find that they have essentially different properties.

2.1 $G^C$-invariants and $G^C$-orbits

We first prepare fundamental fields, $\vec{\phi} \in V$, which belong to a representation $N$ of $G = O(N)$. Here, $V$ is a representation space, and it is complexified by the supersymmetry: $V = \mathbb{C}^N$. Since a superpotential includes only chiral superfields, its $G$-invariance leads to $G^C$-invariance. Assume that there appear an effective superpotential,

$$W = g\phi_0(\vec{\phi}^2 - f^2),$$

(2.1)

where $\phi_0$ is a $G^C$-singlet Lagrange multiplier field, which has no D-term and $g$ is a constant. Here $f^2$ is also a constant and can be taken as being real by a field redefinition of $\phi_0$. By integrating out $\phi_0$ (eliminating $\phi_0$ by its equation of motion), we obtain a nonlinear field space, $M$ of $\vec{\phi}$ which is embedded in $V$ by a constraint,

$$\vec{\phi}^2 - f^2 = 0.$$ (2.2)

Since there is only one $G^C$-invariant and it is fixed, the field space $M$ is a manifold with a $G^C$-transitive action, namely a $G^C$-orbit:

$$M = \{g \cdot \vec{v}| \vec{v} \in V, \forall g \in G^C\},$$ (2.3)

where $\vec{v} \in V$ is an arbitrary vector satisfying $\vec{v}^2 = f^2$. The complex dimension of $M$ is $\dim \mathbb{C} M = \dim \mathbb{C} V - 1 = N - 1$, since one $G^C$-invariant is fixed.
The vertical axis is taken as $\text{Re} \, V^N$, the horizontal axis is taken as $\text{Im} \, V^{N-1}$ and the central point is the origin. There exist four kinds of orbits. Orbit I consists of the upper and lower hyperbola and orbit II of the left and right hyperbola. Although they look separated in this figure, they are continuous through other directions of $V$. Orbit III looks like a cone, and orbit IV is the origin.

There exist four types of $G^C$-orbits, which can be classified by the value of the $G^C$-invariant $f^2$ (see Fig. 1):

I) $f^2 > 0$: closed orbits,
II) $f^2 < 0$: closed orbits,
III) $f^2 = 0$, $\vec{\phi} \neq \vec{0}$: an open orbit.
IV) $f^2 = 0$, $\vec{\phi} = \vec{0}$: a closed orbit,

Since orbits I and II can be changed to each other by inverting the sign of $f^2$, it is sufficient to consider orbits of type I. Moreover, we do not consider the trivial orbit of type IV. We, thus, consider the closed orbits I and the open orbit III.

Since the group action is transitive, the orbit can be written as a coset space. Namely, if we define the complex isotropy group of the vacuum, $\bar{v} = < \vec{\phi} >$, as

$$\tilde{H}_v = \{ g \in G^C | g \cdot \bar{v} = \bar{v} \},$$

(2.4)
the orbit can be written as\footnote{Here, since the isotropy group at \( \vec{v}' = g_0 \vec{v} \) can be obtained by the isotropy group at \( \vec{v} \) as \( \hat{H}_{v'} = g_0 \hat{H} g_0^{-1} \), we have not written a subscript \( v \) below \( \hat{H} \).}

\[ M \simeq G^C / \hat{H}. \]

The representative of this coset manifold is generated by broken generators \( Z_i \) in \( G^C - \hat{H} \) as

\[ \xi = \exp(i \Phi \cdot Z) \in G^C / \hat{H}, \]

where \( \Phi^i \) are chiral superfields, whose scalar components are the coordinate of \( G^C / \hat{H} \).

In general, the complex isotropy group is larger than a complexification \( H^C \) of the real isotropy group, \( H_v = \{ g \in G | g \cdot \vec{v} = \vec{v} \} \).

Namely, there is a point such that \( \hat{H} \) can be written as

\[ \hat{H} = H^C \oplus B, \]

where \( B \) is a nilpotent Lie algebra, called a Borel subalgebra in \( H \). \( B \) can be written as lower- (or upper-) half triangle matrices in a suitable basis \footnote{Therefore \( \hat{H} \) can be written as a semidirect product of \( H^C \) and \( B \): \( \hat{H} = H^C \uplus B \). Here the symbol \( \uplus \) denotes a semidirect. If there are two elements of \( \hat{H} \), \( h b \) and \( h' b' \), where \( h, h' \in H^C \) and \( b, b' \in B \), their product is defined as \( (h b)(h' b') = hh'(h'^{-1} b h) b' = (hh') (b' b) \), where \( b'' = h'^{-1} b h' \).}.

The complex broken generators \( Z_i \) can be classified to two types. One is an Hermitian generator, and the other is a non-Hermitian generator. The latter has a corresponding non-Hermitian unbroken generator, \( B_i \), in the sense that two Hermitian generators can be composed of the linear combinations of \( Z_i \) and \( B_i \). (Here \( B_i \) need not to be an element of a Borel algebra. If \( B_i \) is an element of the Borel algebra, \( Z_i \) and \( B_i \) are Hermitian conjugate to each other.) The chiral superfields \( \Phi^i \) in (2.6), whose scalar components parameterize \( M \), are also classified to two
types, corresponding to broken generators. The chiral superfields, corresponding to Hermitian broken generators, are called *mixed types*; on the other hand, the chiral superfields, corresponding to non-Hermitian broken generators, are called *pure types*. These names come from the fact that the mixed type includes a QNG boson besides an NG boson as a scalar component, and the pure type includes two NG bosons without any QNG boson.

These can be illustrated as follows. As stated above, if there are no n-Hermitian broken generators, $Z_i$, there exist non-Hermitian unbroken generators, $B_i$, such that linear combinations of $Z_i$ and $B_i$ are Hermitian (broken) generators $X_i$ and $X_i'$. Hence, pure-type generators $Z_i$ in the coset representation can be transformed to $aX_i + bX_i'$, where $a$ and $b$ are some real constants, by a local $\hat{H}$ transformation from the right of the coset representative. Since $aX_i + bX_i'$ are Hermitian, these generate compact directions, corresponding NG bosons. Therefore, the scalar components of a pure-type multiplet $\Phi^i$ corresponding to $Z_i$ are both NG bosons.

On the other hand, since there is no unbroken partner of Hermitian broken generators, imaginary parts of scalar components of mixed-type multiplets generate non-compact directions, corresponding to QNG bosons.

In the next two subsections, we show how these two kinds of broken generators and the Borel algebra appear in the closed and open orbits, respectively.

### 2.2 Closed orbits

Closed orbits in the $O(N)$ model are discussed in Refs. [20, 22]. This subsection is devoted to a brief review to them. The essential feature of closed orbits is a supersymmetric vacuum alignment [28, 18, 19, 20, 22]. (Especially, a geometrical meaning of a vacuum alignment has been discussed in our previous paper [20].) By this property, the unbroken real symmetry can change from point to point.

In closed orbits, there exists a point $\vec{v}$ such that $\hat{H}$ becomes a reductive group; $\hat{H}_v = H_v^C$; hence, there is no Borel algebra [6, 7]. Actually, there exist a $G^C$-
transformation such that the vacuum can be transformed to
\[ \vec{v} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix}, \tag{2.9} \]
where \( v \) is a constant equal to \( f \): \( v^2 = f^2 \). We call this point a symmetric point.\(^8\)

The breaking pattern is \( G \to H = O(N - 1) \) and the broken generators are
\[ X_i = \begin{pmatrix} 0 & \cdots & i \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & -i \end{pmatrix} \in G - H \quad (i = 1, \ldots, N - 1). \tag{2.10} \]

A real coset space, \( G/H = O(N)/O(N - 1) \), is generated by these generators. Namely, \( U \overset{\text{def}}{=} \exp(i\pi \cdot X) \) is a representative of \( G/H \), where \( \pi^i \) \( (i = 1, \ldots, N - 1) \) are NG bosons, which parameterize \( G/H \). This manifold is a submanifold of the full target manifold \( M \), and \( M \) is just its complexification: \( M \simeq G^C/\hat{H} = O(N)^C/O(N - 1)^C \).

The symmetric point is a special point in the full target manifold \( M \). There exist other vacua, transformed by \( G^C \) from the symmetric point. To consider them, we transform the vacuum by \( G^C \) transformation as
\[ \vec{v}' = g_0 \vec{v} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -iv \sinh \tilde{\theta} \\ v \cosh \tilde{\theta} \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha \\ \beta \end{pmatrix}. \tag{2.11} \]

Here we have put \( g_0 \in G^C \) as
\[ g_0 = \exp(i\theta X_{N-1}) \]
\( K = \phi^\dagger \phi \) is a linear Kähler potential, is satisfied \( [6, 3] \). An iso-Kähler surface, which looks like a circle in Fig. 1, touches the \( G^C \)-orbit at \( \vec{v} \) (the nearest point to the origin in Fig. 1).

\(^8\) The points transformed by \( G \) from this point also satisfy the same property, and are therefore symmetric points. It is known that the \( G \)-orbit consisting of symmetric points is unique \( [3] \). Such the \( G \)-orbit is called D-orbit if we gauge \( G \).
\[
\begin{bmatrix}
1 & 0 \\
\cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta \\
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
\cosh \tilde{\theta} & -i \sinh \tilde{\theta} \\
i \sinh \tilde{\theta} & \cosh \tilde{\theta} \\
\end{bmatrix},
\] (2.12)

where \( \tilde{\theta} = i\theta \) is a pure imaginary constant. Other \( G^C \)-transformations are not independent of \( g_0 \), since any vacuum can be reached from the symmetric point by using \( g_0 \) and a \( G \)-transformation. This is because the broken generators at the symmetric point belong to a single representation, \( \mathbf{N} - \mathbf{1} \), of the unbroken symmetry \( H \) \[20\]. We call this vacuum the non-symmetric point. The breaking pattern of the global symmetry is \( G \rightarrow H' = O(N - 2) \) and broken generators are \( X_i \) and

\[
X_i' = \begin{pmatrix}
0 & i \\
\vdots & \vdots \\
-i & 0 \\
\vdots & \vdots & 0 & 0 \\
\end{pmatrix} \quad (i' = 1, \cdots, N - 2).
\] (2.13)

Although these generators, \( X_i' \), were unbroken in the symmetric point, they generate newly emerged NG bosons at the non-symmetric point. The number of NG bosons is \( \dim(G/H') = 2N - 3 \). It has increased more than at the symmetric point. Namely, some of QNG bosons at the symmetric point have changed to NG bosons, and there remains only one QNG boson \[20, 22\]. The unbroken symmetry has been changed there. This phenomenon is called “supersymmetric vacuum alignment” \[28, 18, 19, 20, 22\].

The complex broken generators are transformed as

\[
\begin{cases}
g_0 X_i g_0^{-1} = \frac{\alpha}{\nu} X_i' + \frac{\beta}{\nu} X_i \overset{\text{def}}{=} Z_i \\
g_0 X_{N-1} g_0^{-1} = X_{N-1} \overset{\text{def}}{=} Z_{N-1} \in G^C - \hat{H}'.
\end{cases}
\] (2.14)

Except for the generator \( Z_{N-1} = X_{N-1} \), most of broken generators, \( Z_i \), become non-Hermitian and, thus, are pure-type generators. Hence, there are \( N_P = N - 2 \) pure-type multiplets and the \( N_M = 1 \) mixed-type multiplet. These are consistent with the numbers of NG and QNG bosons counted above. See Table 1. (Subscriptions, P and M, under \( H \)-representation of the complex broken generators denote pure-types and
Table 1: $O(N)$ with $N$, closed orbit

| $v$  | $O(N - 1)$  | $N - 1$ | 0 | $N - 1$ | $N - 1$ | $(N - 1)_M$ |
|------|-------------|---------|---|---------|---------|-------------|
| $v' $| $O(N - 2)$  | 1       | $N - 2$ | $2N - 3$ | 1       | $(N - 2)_P \oplus 1_M$ |

mixed-types respectively.) The complex unbroken generators are also transformed as

$$
\begin{align*}
\{g_0x'_ig_0^{-1} &= \frac{2}{v}X'_i - \frac{v}{2}X_i \equiv B_i \\
g_0h'_ig_0^{-1} &= h'_i \in \mathcal{H}'
\}
\end{align*}
$$

There appear non-Hermitian generators $B_i$. $B_i$ are partners of $Z_i$ in a sense that, from linear combinations of $B_i$ and $Z_i$, we can construct Hermitian generators $X_i$ and $X'_i$. $B_i$ does not constitute a Borel subalgebra, since $[\mathcal{H}', B_i] \sim B_j$ is not satisfied. It can be understood from the fact that there is no Borel subalgebra at the symmetric point and that $G^\mathbb{C}$-transformation does not change the commutation relations.

The target spaces are same with being independent of the vacua: $G^\mathbb{C}/\mathcal{H} \simeq G^\mathbb{C}/\mathcal{H}'$.

2.3 Open orbit

In the last subsection, we discussed the closed orbit, characterized by $f^2 > 0$. There was a supersymmetric vacuum alignment, and pure-type multiplets appeared at the non-symmetric point. In this subsection, we discuss the open orbit characterized by $f^2 = 0$ and show that there is no vacuum alignment. Moreover, we find a Borel subalgebra in the complex isotropy, differently from the closed orbit.
By a $G^C$-transformation, any vacuum on the open orbit can be transformed to
\[
\vec{v} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
iv/\sqrt{2} \\
v/\sqrt{2}
\end{pmatrix},
\] (2.16)
where $v$ can be taken as an arbitrary constant. The $G$ transformation can also bring any vacuum to this form, but $v$ is not arbitrary. In both cases, the breaking patterns are $G \to H = O(N - 2)$. This coincidence of the breaking patterns means that there is no vacuum alignment. The number of NG bosons is $\dim(G/H) = 2N - 3$. Besides the Hermitian generators of $O(N-1)$, there are additional complex unbroken generators. The additional unbroken generators and their broken partners are
\[
B_i = \begin{pmatrix}
0 & i \\
\vdots & \vdots \\
-1 & \cdots
\end{pmatrix},
Z_i = \begin{pmatrix}
0 & i \\
\vdots & \vdots \\
-1 & \cdots
\end{pmatrix}.
\] (2.17)
Here, the index $i$ runs over 1 to $N - 2$. They can be written as $B_i = X_i' - iX_i$ and $Z_i = X_i' + iX_i$, where $X_i$ and $X_i'$ are given in Eqs. (2.10) and (2.13). Hence, by their linear combinations, we can obtain Hermitian generators, $X_i$ and $X_i'$. The complex broken generators are $Z_i$ and $Z_{N-1} = X_{N-1}$. Here, $Z_i$ are pure-type generators belonging to a representation $N - 2$ of $H$ and $X_{N-1}$ is a mixed-type generator belonging to a $H$-singlet. Since all mixed-types are $H$-singlets, we can make sure that there is no vacuum alignment in the open orbit by using the results given in Ref. [20]. We summarize these in Table 2.

Since $B_i$ satisfy commutation relations, $[H, B_i] \sim B_j$ and $[B_i, B_j] \sim B_k$, they are elements of a Borel subalgebra $\mathcal{B}$ in $\hat{H}$. Hence, the target space $M$ can be written as $M \simeq G^C/\hat{H} = O(N)^C/O(N - 2)^C \wedge B$, where $B$ denotes a Borel group. As stated in Subsec. 2.1, there exist a basis such that the Borel subalgebra is represented by lower (or upper) half off-diagonal matrices. We can thus change the basis to such a
Table 2: \(O(N)\) with N, open orbit

|   | \(H\) | \(N_M\) | \(N_P\) | \(N\) | \(Q\) | \(H\)-sector |
|---|---|---|---|---|---|---|
| \(\vec{v}\) | \(O(N - 2)\) | 1 | \(N - 2\) | 2\(N - 3\) | 1 | \((N - 2)_P \oplus 1_M\) |

basis by

\[ U = \begin{pmatrix}
1 & 0 \\
0 & i/\sqrt{2} \\
0 & -i/\sqrt{2}
\end{pmatrix}. \tag{2.18} \]

(In the new basis, \(X_{N-1}\) becomes a diagonal matrix.) Since \(U\) is a unitary matrix, \(UU^\dagger = U^\dagger U = 1\), the D-term \(\vec{\phi}^\dagger \vec{\phi}\) does not change. The vacuum in this basis is

\[ U\vec{v} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
v
\end{pmatrix}. \tag{2.19} \]

The complex broken generators, \(Z_i\) and \(X_{N-1}\), and the unbroken generators, \(B_i\), are represented in the basis as

\[ UZ_i U^\dagger = \begin{pmatrix}
\cdots & 1 & \cdots \\
0 & -1 & \cdots \\
\vdots & \vdots & \ddots 
\end{pmatrix}, \]

\[ UX_{N-1} U^\dagger = \begin{pmatrix}
1 \\
0 \\
\ddots \\
0 \\
\ddots \\
-1
\end{pmatrix}, \tag{2.20} \]

\[ UB_i U^\dagger = \begin{pmatrix}
\vdots \\
1 & 0 & \cdots \\
\vdots \\
\cdots & -1 & \cdots
\end{pmatrix}. \]

Here, we have rearranged an order of the blocks so that a \(O(N - 2)\) part comes to
the center. They can be summarized as

\[
G^C - \hat{H} = \begin{pmatrix} M & P \\ P & 0 \\ 0 & M \end{pmatrix}, \quad \hat{H} = \begin{pmatrix} B & H^C_{SS} \\ H^C_{SS} & B \end{pmatrix},
\]

(2.21)

where P, M, B and \(H^C_{SS}\) represent the pure-types, the mixed-type, the Borel algebra and a semisimple part of \(\hat{H}\), namely \(O(N - 2)\) part, respectively.

In the rest of paper, we argue the closed orbit in the ordinary basis, but the open orbit in the basis changed by the unitary matrix. Of course, no result depends on the basis. We do not explicitly write \(U\) in the open orbit.

Here, we give a summary of Sec. 2. In general, \(G^C\)-orbits can be characterized by the values of \(G^C\)-invariants. There are two types of \(G^C\)-orbit: one is the closed orbit (the orbit I in Fig. 1) and the other is the open orbit (orbit III in Fig. 1). (Orbit II is a mirror of orbit I and orbit IV is trivial.) Both orbits can be written as complex coset spaces as \(G^C/\hat{H} = O(N)^C/O(N - 1)^C\) and \(G^C/\hat{H} = O(N)^C/O(N - 2)^C \land B\), respectively. Although these two orbits are topologically different near to the origin, they become close to each other at infinity, as in Fig. 1. Especially, many properties in the non-symmetric point on the closed orbit and generic points on the open orbit are very similar. Namely, the number of pure-type and mixed-type multiplets are the same in both cases. (See Eqs. (2.14), (2.15) and (2.17).) Moreover, their \(H\) representations are the same in both cases. (Compare the second line of Table 1 and Table 2.) However, there is the Borel subalgebra on the open orbit, but not on the closed orbit. In the next section we show that this difference brings essentially distinct results concerning to the Kähler potentials on these orbits.

3 Strictly \(G\)-invariant Kähler potentials

In this section, we construct strictly \(G\)-invariant Kähler potentials. Although this section does not have any new feature, the results are used in the next section. Hence, we briefly discuss them.
3.1 Invariants composed of fundamental fields

In the $O(N)$-model, there exist one $G$-invariant comprising fundamental fields: $\vec{\phi}^\dagger \vec{\phi}$. This parameterizes a moduli space of global symmetry [20]. (We do not discuss this feature.) Since this is strictly $G$-invariant, a low-energy effective Kähler potential can be written as an arbitrary function of this quantity. We showed, in a previous paper [20], that this is just the A-type invariant of BKMU [9, 8].

There is a relation between the fundamental fields and the coset representative of $G^C/\hat{H}$,

$$\vec{\phi} = \xi \vec{v}|_F,$$  

where $F$ denotes the F-term constraint, $\vec{\phi}^2 = f^2$. From this equation, the Kähler potential can be written as

$$K_A = f(\vec{\phi}^\dagger \vec{\phi})|_{\phi^2 = f^2} = f(\vec{v}^\dagger \xi \xi \vec{v}).$$  

This form does not depend on whether the orbit is closed, $f^2 > 0$, or open, $f^2 = 0$.

3.2 Invariants constructed by using projections

Other strictly $G$-invariants, called as the C-type, were found by BKMU [8]. We review it here.

Consider projection matrices $\eta_i$, which project $V$ onto $\hat{H}$-invariant subspaces $\eta_i V$. These satisfy projection conditions,

$$\eta^\dagger = \eta, \; \eta \hat{H} \eta = \hat{H} \eta, \; \eta^2 = \eta.$$  

We construct quantities

$$P_i \overset{\text{def}}{=} \xi \eta_i [\xi^\dagger \xi]^{-1}_{\eta_i} \eta_i \xi^\dagger,$$  

where $[\cdots]_{\eta_i}^{-1}$ denotes the inverse matrix in the $\eta_i$ projected space. These transform under $g \in G$ as $P_i \xrightarrow{g} g P_i g^\dagger$. Thus, quantities $\text{tr}(P_i P_j)$, $\text{tr}(P_i P_j P_k)$, $\cdots$, are strictly  

\footnote{ It is not known whether these invariants can be written in fundamental fields.}
G-invariant. Hence, a Kähler potential can be written as an arbitrary function of them:

\[ K_C = f(\text{tr} (P_i P_j), \ldots) \quad (i \neq j \text{ etc.}, \; \eta_i V \not\subset \eta_j V \text{ etc.}). \] (3.5)

We next apply this to the \( O(N) \) model. Projections \( \eta \) are different between in the closed orbit and in the open orbit. We first consider the closed orbit. At the symmetric point, we can find two projections:

\[ \eta_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \] (3.6)

Hence, there is one C-type invariant on the closed orbit:

\[ K_C = f(\text{tr} (P_1 P_2)). \] (3.7)

Of course, the most general strictly \( G \)-invariant Kähler potential is an arbitrary function of both the A- and C-type invariants.

In the open orbit, \( \hat{H} \) is the form of

\[ \hat{H} = \begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix}. \] (3.8)

Therefore, we can find only one projection,

\[ \eta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \] (3.9)

\( \text{tr} (P_i) = \text{const.} \) is trivial. Since \( P_i^2 = P_i \), we require \( i \neq j \text{ etc.} \). If two projections satisfy \( \eta_i V \subset \eta_j V \) then \( P_i P_j = P_j P_i = P_i \), hence we require \( \eta_i V \not\subset \eta_j V \text{ etc.} \).

\( ^{11} \) At the non-symmetric point, as the complex isotropy \( \hat{H} \) transforms to \( \hat{H}' = g_0 \hat{H} g_0^{-1}, \eta_i \to \eta_i' = g_0 \eta_i g_0^{-1} \). Although the first condition in Eq. (3.3) is not satisfied, it is sufficient to modify \( P_i \) as \( P_i' = \xi' \eta_i' |\eta_i'|^2 \xi'^\dagger \eta_i |^2 \eta_i'^\dagger \xi'^\dagger \), where \( \xi' \in G^C / \hat{H}' \). Under \( g \in G \) transformations, \( \eta_i' \) also transforms as \( \eta_i' \to g \eta_i' g^\dagger \).
and there is no C-type invariant on the open orbit.\footnote{12}

The values of the C-type invariants are constant on each \( G \)-orbit as the A-type invariants. But we do not know whether they can be constructed by the fundamental fields and whether they have a geometric meaning, such as the moduli space of global symmetry in the case of A-types. We do not investigate these aspects in this paper, and concentrate on the B-type invariants.

4 Kähler potentials \( G \)-invariant up to a Kähler transformation

In the last section we discussed the strictly \( G \)-invariant Kähler potentials. In this section we discuss Kähler potentials \( G \)-invariant up to a Kähler transformation. BKMU showed that they can be written in B-type Kähler potentials \footnote{8}, which are generalizations of Zumino’s one \footnote{1}. It is known that if there is no center in \( \hat{H} \), B-type Kähler potentials are strictly \( G \)-invariant and are not independent of A- or C-type invariants \footnote{8}. In the \( O(N) \) model, this is the case and B-type Kähler potentials are strictly \( G \)-invariant. Nevertheless, in this section we show that if and only if the orbit is open, does the Kähler manifold, \( G^C/\hat{H} \), have a compact Kähler submanifold, \( G^C/\tilde{H} \), and B-type Kähler potentials on \( G^C/\hat{H} \) reduce to those on \( G^C/\tilde{H} \), which is \( G \)-invariant up to a Kähler transformation.

\footnote{12} \( \eta' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \) also satisfies the projection condition. However, from a relation \( \eta V \subset \eta' V \), we can not construct C-type invariants. In the case of B-types, which is discussed in the next section, B-type Kähler potentials constructed by using \( \eta \) and \( \eta' \) are not independent as in the case of pure realizations \footnote{10}. See the footnote 15 in App. A. Therefore, we do not need \( \eta' \).
4.1 Closed orbits

In this subsection, we consider the closed orbit. By using \( \eta \) projections in Eq. (3.6), B-type Kähler potentials can be constructed as \[ K_{Bi} = \log \det_{\eta_i} \xi^\dagger \xi, \] (4.1)

where \( \det_{\eta_i} \) denotes a determinant in the \( \eta_i \) projected matrix.\(^{13}\) Since there is no center in \( \hat{H} \), they are strictly \( G \)-invariant, contrary to the pure realization cases. Actually,

\[ V_1 = \det_{\eta_i} \xi^\dagger \xi = \tilde{v}^\dagger \xi^\dagger \xi \tilde{v} / |\tilde{v}|^2 \] (4.2)

is not independent of the A-type invariant obtained in the last section. Moreover, \( K_{B2} \) is also expected not to be independent of the C-type invariant. (But we do not calculate an explicit form.) Although the B-types are strictly \( G \)-invariant and are not independent of the A- or C-type invariants, we investigate these in more detail.

First we consider the case at the symmetric point. Since all broken generators are mixed-type and there is no pure-type multiplet, one can find no compact submanifold, which, if any, would be parameterized by pure-type multiplets. Moreover, all broken generators belong to one representation, \( N - 1 \) of \( H = O(N - 1) \). Therefore, it is impossible to find any submanifold where some of the broken generators are changed to unbroken generators.

Secondly, we consider the non-symmetric points. At the non-symmetric points, the supersymmetric vacuum alignment occurs, and there appear \( N - 2 \) pure-type multiplets and one mixed-type multiplet. The pure-type multiplets belong to \( N - 2 \) representation of \( H' = O(N - 2) \); on the other hand, the mixed-type multiplet is a singlet. Therefore, one may consider that there is a compact submanifold, but we show that this is not true.

\(^{13}\) Since, under a \( g \in G \) transformation, the coset representative, \( \xi \), is transformed as \( \xi \xrightarrow{g} \xi' = g \xi \hat{h}^{-1} \), where \( \hat{h} \in \hat{H} \), \( K_{Bi} \) is transformed as \( K_{Bi} \xrightarrow{g} K_{Bi} + \log \det_{\eta_i} \hat{h}^{-1} + \log \det_{\eta_j} \hat{h}^{-1} \). Here, we have used the projection conditions (3.3). Last two terms are changed to space-time total derivatives by the superspace integral. This can be understood as a Kähler transformation.
If there would be a compact submanifold, only one possibility is that it were a manifold parameterized by pure-type multiplets without one mixed-type multiplet. We, thus, change the mixed-type broken generator $Q \overset{\text{def}}{=} X_{N-1}$ to an unbroken generator and define new unbroken and broken generators as

$$\tilde{\mathcal{H}} = \{H_{ss}, B_i, Q\}, \quad G^C - \tilde{\mathcal{H}} = \{Z_i\}, \quad (4.3)$$

where $H_{ss}$ is a Lie algebra of $H' = O(N-2)$ and $Z_i$ and $B_i$ are given in Eqs. (2.13) and (2.14). Since $Q$ commutes with $H_{ss}$, it is a center in $\{H_{ss}, Q\}$. But, commutation relations of $Q$ with $Z_i$ and $B_i$ are

$$[Q, Z_i] = -iB_i, \quad [Q, B_i] = iZ_i. \quad (4.4)$$

Note that $Z_i$ and $B_i$ do not carry definite charges of $Q$. Moreover, $\tilde{\mathcal{H}}$ can not constitute a closed algebra. Thus, the submanifold $G^C/\tilde{\mathcal{H}}$ can not be considered to be a coset space; but, since we can define it by using broken generators $Z_i$, we continue the argument. The relation of coset representative $\xi$ of $G^C/\tilde{\mathcal{H}}$ and the corresponding quantity (but not coset representative) $\zeta$ of $G^C/\tilde{H}$ is

$$\xi = \exp(i\varphi \cdot Z + i\Phi Q) \in G^C/\tilde{H}'$$

$$= \exp(i\varphi \cdot Z) \exp(i\Phi Q)\tilde{h}^{-1}$$

$$= \zeta \exp(i\Phi Q)\tilde{h}^{-1} \quad (4.5)$$

where $\varphi^i$ in $\zeta \overset{\text{def}}{=} \exp(i\varphi \cdot Z)$ parameterize $G^C/\tilde{H}$. Here, $\tilde{h} \in \tilde{H}$ is needed, since the commutation relations of $Z_i$ and $Q$ include unbroken generators $B_i \in \tilde{\mathcal{H}}$ as Eq. (1.3). If projections $\eta'_i = g_0\eta g_0^{-1}$ on $G^C/\tilde{H}'$ would satisfy $\eta'\tilde{H}\eta' = \tilde{H}\eta'$, they could be considered as also being projections on $G^C/\tilde{H}$, and the Kähler potentials (4.1) could reduce to Kähler potentials on $G^C/\tilde{H}$. But, unfortunately, they do not satisfy the projection conditions on $G^C/\tilde{H}$, Eq. (A.3). We, thus, conclude that Kähler potentials (4.1) on $G^C/\tilde{H}$ do not reduce to those on $G^C/\tilde{H}$. In fact, the above argument at the non-symmetric points could be concluded by the arguments at the symmetric point: Even if the B-types reduced to those on any compact submanifold at the non-symmetric point, they could not connect to the symmetric point smoothly by any $G^C$-transformation.
In the case of the closed orbit, we do not need B-type invariants, since they are not independent of A- or C-type invariants. In the next section we show that this feature is quite different in the open orbit.

4.2 Open orbit

In the open orbit, there is no vacuum alignment. There are \( N-1 \) pure-type generators and one mixed-type generator. To consider a compact submanifold, as is done in the closed orbit, we change the mixed-type broken generator, \( Q = X_{N-1} \), to an unbroken generator. New unbroken and broken generators are

\[
\tilde{\mathcal{H}} = \{ H_{ss}, B_i, Q \}, \quad \mathcal{G}^C - \tilde{\mathcal{H}} = \{ Z_i \},
\]

where \( Z_i \) and \( B_i \) are given in Eq. (2.17). Although \( H_{ss} \) and \( Q \) are the same as in the closed-orbit case and \( Q \) is a center in \( \{ H_{ss}, Q \} \), non-Hermitian broken and unbroken generators, \( Z_i \) and \( B_i \), are different from the closed orbit. Namely, commutation relations between, \( Z_i \) and \( B_i \), and \( Q_i \) are

\[
[Q, Z_i] = Z_i, \quad [Q, B_i] = -B_i.
\]

(4.7)

Compare these equations with Eq. (4.4). Different from the closed-orbit case, \( Z_i \) and \( B_i \) carry definite charges this time. From the second equation, \( \tilde{\mathcal{H}} \) becomes an algebra and \( \mathcal{G}^C / \tilde{\mathcal{H}} = O(N)^C / O(N-2)^C \times U(1)^C \) and \( B \simeq O(N) / O(N-2) \times U(1) \), where \( B \) denotes the Borel group, is a coset space. Since we obtain \( [\tilde{\mathcal{H}}, B_i] \sim B_j \), \( B_i \) constitutes a Borel subalgebra not only in \( \tilde{\mathcal{H}} \), but also in \( \hat{\mathcal{H}} \). By using the first equation of (4.7) and \( [Z_i, Z_j] \sim Z_k \), the relation between the coset representative \( \xi \) of \( \mathcal{G}^C / \hat{\mathcal{H}} \) and the one \( \zeta \) of compact homogeneous submanifold \( \mathcal{G}^C / \tilde{\mathcal{H}} \) can be obtained as

\[
\xi = \exp(i\tilde{\varphi} \cdot Z + i\Phi Q) \in \mathcal{G}^C / \hat{\mathcal{H}}
\]

\[
= \exp(i\varphi \cdot Z) \exp(i\Phi Q)
\]

\[
= \zeta \exp(i\Phi Q).
\]

(4.8)

The coordinate transformation \( \tilde{\varphi} = \tilde{\varphi}(\Phi, \varphi) \) can be calculated by using the Baker-Campbell-Hausdorff formula, but we do not need an explicit representation. If we
define

\[ V = \det_\eta \xi^\dagger \xi, \quad (4.9) \]

it is strictly $G$-invariant, since it is not independent of A-type invariant, $K_A$, constructed in the last section: $V = \vec{v}^\dagger \xi^\dagger \vec{v} / ||\vec{v}||^2$. Therefore a B-type Kähler potential,

\[ K_B = c \log V = c \log \det_\eta \xi^\dagger \xi, \quad (4.10) \]

where $c$ is a real constant, is also strictly $G$-invariant. This situation is the same as closed orbits. However, in this case, the $\eta$ projection on $G^C/\tilde{H}$ also satisfies the projection conditions on the submanifold $G^C/\tilde{H}$, Eq. (A.6). Hence, by using Eq. (4.8), $V$ can be calculated as

\[ V = \det_\eta \xi^\dagger \xi = \det (\eta \xi^\dagger \xi \eta) \]
\[ = \det [\eta \exp(-i\Phi^\dagger Q)\zeta^\dagger \zeta \exp(i\Phi Q)\eta] \]
\[ = \det [\eta \exp(-i\Phi^\dagger Q)\eta \zeta^\dagger \zeta \eta \exp(i\Phi Q)\eta] \]
\[ = \det [(\eta \exp(-i\Phi^\dagger Q)\eta)(\eta \zeta^\dagger \zeta \eta)(\eta \exp(i\Phi Q)\eta)] \]
\[ = U \exp[i(\Phi^\dagger - \Phi)] \quad (4.11) \]

where $U \overset{\text{def}}{=} \det_\eta \xi^\dagger \zeta$ is a corresponding quantity in $G^C/\tilde{H}$. Here, we have used the projection condition (A.6) in the third line and a formula, $\log \det_\eta A = \text{tr} (\eta \log A)$, and $\text{tr}(\eta Q) = -1$ in the fourth line. From this equation, the Kähler potential can be calculated as

\[ K_B = c \log U + ic(\Phi^\dagger - \Phi). \quad (4.12) \]

The last term changes to the space-time total derivative by the superspace integral and $K_B$ reduces to a Kähler potential $K_{B0}$ of the compact submanifold $G^C/\tilde{H}$,

\[ K_{B0} = c \log U = c \log \det_\eta \xi^\dagger \zeta = c \log \left(1 + |\varphi|^2 + \frac{1}{4} \varphi^2 \varphi^\dagger \right). \quad (4.13) \]

The last explicit form is calculated in App. A. Note that, although $K_B$ is strictly $G$-invariant, $K_{B0}$ is $G$-invariant up to a Kähler transformation. $G^C/\tilde{H}$ is parameterized by scalar parts of pure-type multiplets without any mixed-type multiplet. This
realization is called a pure realization. Hence, we conclude that the pure realization is embedded in the open orbit, but not in the closed orbit.

In the open orbit, the most general effective Kähler potential can be written as a sum of the B-type Kähler potential, $K_{B0}$ in Eq. (4.13), on a compact submanifold $G^C/\tilde{H} \simeq O(N)/O(N - 2) \times U(1)$, and an arbitrary function of strictly $G$-invariant, $K_A$ in Eq. (3.2). Of course, if we choose the arbitrary function as $f(x) = c \log x$ in the latter, it becomes a sum of the former and the space-time total derivative. In this sense, the former can be included in the latter.

Before closing this section, we discuss a relation with the broken center models considered by Buchmüller and Ellwanger [14, 15, 16]. The pure realizations have the compact homogeneous target manifold $G^C/\tilde{H}$. In compact homogeneous Kähler manifolds, there exist a homeomorphism $G^C/\tilde{H} \simeq G/H = G/H_{ss} \times U(1)^n$. Here, $H_{ss}$ is a semisimple subgroup of $\tilde{H}$ or $H$, and there are $n$ centers in $\tilde{H}$ or $H$. The Kähler potential is written as a sum of $n$ B-type Kähler potentials [10]. Buchmüller and Ellwanger [14] have considered models where $m (\leq n)$ centers in $\tilde{H}$ are broken by hand from pure realizations. It was shown that $m$ linear combinations of B-type Kähler potentials are strictly $G$-invariant and that the Kähler potential can be written as a sum of $n$ B-type Kähler potentials and an arbitrary function of $m$ strictly $G$-invariants. Since these models are not pure realizations, they were expected to have linear model origins. However, it was not known whether they have linear model origins.

In the open orbit, we have shown that if one center in $\tilde{H}$ is unbroken by hand, it reduces to the compact homogeneous manifold $O(N)/O(N - 2) \times U(1)$. This can be considered as being an inverting procedure of Buchmüller and Ellwanger. We, thus, have been able to find that special case (the case when one center is broken: $n = m = 1$) of the broken-center models has an open orbit as a linear-model origin. A question as to whether the general cases of broken center models have linear model origins is discussed in the next section.
5 Conclusions and Discussions

The target spaces of the nonlinear sigma models, which have linear model origins, are obtained as $G^C$-orbits of the vacuum. In the $O(N)$ model, there are closed orbits and an open orbit, depending on the value of the $G^C$-invariant. (They are the orbit I and III in Fig. 1.) Both kinds of orbits can be written as complex coset spaces: $G^C/H = O(N)^C/O(N - 1)^C$ and $G^C/H = O(N)^C/O(N - 2)^C \otimes B$ respectively. On the closed orbits, the vacuum alignment occurs (as Table 1), and the numbers of NG and QNG bosons change, with the total number remain unchanged, at various vacua. These two orbits are similar, except near to the origin, as in Fig. 1. Actually, the numbers of the pure-type multiplets and the mixed-type multiplets are the same at the non-symmetric points of the closed orbit and the generic points on the open orbit. (See second line of Table 1 and Table 2.) However, we have shown that, in the open orbit, the non-Hermitian unbroken generators, $B_i$ in Eqs. (2.17) or (2.20), constitute a Borel subalgebra in the complex isotropy $\hat{H}$, but in the closed orbit, $B_i$ in Eq. (2.13) do not constitute it. This difference plays a crucial role on the both orbits.

In the nonlinear realization with a linear model origin, the B-type Kähler potentials are strictly $G$-invariant and are not independent of A- or C-type invariants. To find a compact Kähler manifold as a submanifold, we have changed the mixed-type generator, $Q = X_{N-1}$, to an unbroken generator by hand in both orbits. In the closed orbit, $Z_i$ and $B_i$ does not carry definite charges of $Q$ as Eq. (4.4); on the other hand, in the open orbit, $Z_i$ and $B_i$ carry definite opposite charges of $Q$ as Eq. (4.7). Moreover, in the closed orbit, the $\eta$-projection on the full manifold does not satisfy the perjection conditions (A.6) on the compact submanifold; on the other hand, in the open orbit, it does satisfy the perjection conditions on the compact submanifold. We, thus, have found that any compact manifold is not embedded in the closed orbit, but it is embedded in the open orbit. From these differences, we have showed that the B-type Kähler potentials of the closed orbit are still just strictly $G$-invariant, and, on the other hand, that the B-type Kähler potential on the open orbit reduces to one of the compact homogeneous Kähler submanifold,
$G^C/\tilde{H} = O(N)^C/O(N-2)^C \times U(1)^C \wedge B \simeq O(N)/O(N-2) \times U(1)$, whose Kähler potential is $G$-invariant up to a Kähler transformation.

It was known that a Kähler potential of a compact Kähler manifold can be written as a sum of B-type Kähler potentials. We may strengthen this property: It seems that even when the target manifold is non-compact, the B-type Kähler potentials can essentially live on compact Kähler manifolds (embedded in the target space). We can also say that B-type Kähler potentials can automatically find a compact homogeneous Kähler manifold as a submanifold in the full target manifold.

Here we have discussions. These results can be generalized to more general models. First, we discuss closed orbits. It is known that a closed orbit has a symmetric point, such that a equation $\hat{H} = H^C$ is established. The maximal realization occurs at the symmetric point, there is no pure-type multiplet and one can not find any compact submanifold. Moreover, some mixed-type multiplets belong to non-singlet of $H$; hence, a combination with our previous results \cite{20} leads to a supersymmetric vacuum alignment. The pure-type generators which arise by vacuum alignment are all non-Borelian, since the $G^C$-transformation does not change the commutation relations. We, thus, can conclude that, on the closed orbits, there is no Kähler potential $G$-invariant up to a Kähler transformation.

Secondly, we discuss the open orbits. We do not know whether the open orbits have a vacuum alignment. However it does not seem that open orbits have a vacuum alignment, based on some examples (the open orbit of the $O(N)$ model in this paper and $U(N)$ with $N$ in the previous paper \cite{20}). If this is true, all pure-type generators constitute a Borel subalgebra in $\mathcal{H}$. Since there is no vacuum alignment, from our previous results \cite{20}, it can be concluded that all mixed-type generators are $H$-singlets. (Let its number to be $n$.) Hence, if we change these mixed-type generators to be unbroken, non-Hermitian unbroken generators constitute a Borel subalgebra in a complex isotropy which includes a complexification of $H_{ss} \times U(1)^n$. (From now on we write $H$ as $H_{ss}$.) But, to find a compact homogeneous manifold, we must change some off-diagonal charged broken generators to be unbroken. (This will show that there is no linear-model origin of the broken-center models considered by Buchmüller and Ellwanger \cite{14, 13, 16}, except for the case where one center is broken, considered
in this paper.) This brings an arbitrariness of a choice of $\hat{H}$, with $H_{ss} \times U(1)^n$ being fixed. This is just a choice of a $G$-invariant complex structure [10, 11, 12]. Namely, any compact homogeneous Kähler submanifold, $G^C/\hat{H} \simeq G/H_{ss} \times U(1)^n$ ($n \geq 2$), have such the arbitrariness [10, 11, 12]. Hence, it seems that we must add B-type Kähler potentials of all compact Kähler submanifolds (even same real manifolds, $G^C/\hat{H} \simeq G/H_{ss} \times U(1)^n$, with different invariant complex structures) to its Kähler potential. We can show that they correspond to the choices of $\eta$ projections on $G^C/\hat{H}$. These aspects will be reported in the near future [29].

Finally, we give some comments concerning to applications. Since we show that a pure realization is embedded in the open orbit, we can apply this phenomenon to seeking linear-model origins of pure realizations [10]–[13] by introducing a proper gauging. We hope that such investigation reaches to linear origins of the coset unification models [4].

Since open orbits have pure-type multiplets anywhere, namely there exist no point such that $\hat{H} = H^C$ can be established and that a maximal realization can occur. Hence, if we gauge $G$, there is no point such that vector multiplets can get mass supersymmetrically by the supersymmetric Higgs mechanism. This brings about a spontaneous breaking of supersymmetry [23, 27, 31]. These phenomena may be applied to dynamical supersymmetry breaking [2, 30].

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A  Kähler potential on $O(N)/O(N - 2) \times U(1)$

Itoh, Kugo and Kunitomo have given the method to construct a Kähler potential of an arbitrary compact homogeneous Kähler manifold, $G^C/\hat{H} \simeq G/H$ [10]. In this appendix, we construct a Kähler potential of $G^C/\hat{H} = O(N)/O(N - 2) \times U(1)$ by using their methods. We work in the changed basis [2;20]. Broken and unbroken
generators are
\[
\mathcal{G}^C - \mathcal{H} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}, \quad \bar{\mathcal{H}} = \begin{pmatrix} \bar{Q} & \mathcal{H}^C_{SS} \\ B & \bar{Q} \end{pmatrix}.
\] (A.1)

A distinct point from the open orbit is that \( Q = X_{N-1} \) is an unbroken generator. Compare these equations with Eq. (2.21). Let a representative of the coset space \( O(N)/O(N-2) \times U(1) \) as \( \zeta \). From the equation
\[
i\varphi \cdot Z = \begin{pmatrix} i\varphi^T \\ 0 & -i\varphi \end{pmatrix},
\] (A.2)
the representative \( \zeta \) can be calculated explicitly as
\[
\zeta = \exp(i\varphi \cdot Z) = \begin{pmatrix} 1 & i\varphi^T & \frac{1}{2}\varphi^2 \\ 1 & -i\varphi \\ 1 \end{pmatrix}.
\] (A.3)

There exists one center \( Q \) in \( \mathcal{H} \) and broken generators carry positive charge: \([Q, Z_i] = Z_i\).\(^{14}\) The fundamental representation \( \mathbf{N} \) can be decomposed as
\[
\mathbf{N} = \mathbf{1}^1 \oplus (\mathbf{N} - 2)^0 \oplus \mathbf{1}^{-1},
\] (A.4)
in the \( H \) representation, where the subscripts represent the charges under \( Q \). Therefore, there is only one independent projection.\(^{15}\)
\[
\eta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\] (A.5)

\(^{14}\) For a convention, sines of charges carried by broken and unbroken generators are opposite to those in Ref. [10]. Therefore the projections are also opposite.

\(^{15}\) Although \( \eta' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \) also satisfy Eq. (A.6), it is not independent of \( \eta \), since there is only one center in \( H \). In the case of the pure realization, there exist independent projections as many as centers in \( H \). [10].
which satisfies
\[ \eta^\dagger = \eta, \; \eta \tilde{H} \eta = \tilde{H} \eta, \; \eta^2 = \eta. \] (A.6)

Note that \( \eta \) in Eq. (A.5) coincides with \( \eta \) on the open orbit, Eq. (3.9). The second condition is satisfied, since all generators in \( \tilde{H} \) carry negative or zero charge:
\[ [Q, B_i] = -B_i, \; [Q, H_8] = 0, \text{ where } H_8 \in \mathcal{H}_{ss} \] 10. By using this projection, the Kähler potential can be calculated as 8,10
\[ K = \log \det_\eta \xi^\dagger \zeta = \log \left( 1 + |\varphi|^2 + \frac{1}{4} \varphi^2 \varphi^\dagger 2 \right). \] (A.7)

This explicit form can be found in Refs. 15, 16.

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