A Strongly Polynomial Algorithm for Finding a Shortest Non-zero Path in Group-Labeled Graphs

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Shortest Path Problem

**Input**  
$G = (V, E)$: Undirected Graph  
\[ \ell \in \mathbb{R}_{\geq 0}^E \text{: Edge Length, } s, t \in V \text{: Terminals} \]

**Goal**  
Find a shortest $s$–$t$ path $P$ in $G$
Shortest Path Problem

**Input**  \( G = (V, E) \): Undirected Graph

\( \ell \in \mathbb{R}^{E}_{\geq 0} \): Edge Length, \( s, t \in V \): Terminals

**Goal**  Find a shortest \( s–t \) path \( P \) in \( G \)

Solved by Dijkstra’s Algorithm
**Input**  \( G = (V, E) \): Undirected Graph

\( \ell \in \mathbb{R}^E_{\geq 0} \): Edge Length, \( s, t \in V \): Terminals

**Goal**  Find a shortest **odd** \( s-t \) path \( P \) in \( G \)

\( P' \)  \( \ell(P') = 40 \)
\( \#(\text{edges}) \text{ is even} \)
Shortest Odd Path Problem

Input \( G = (V, E) \): Undirected Graph

\[ \ell \in \mathbb{R}^E_{\geq 0} : \text{Edge Length}, \quad s, t \in V : \text{Terminals} \]

Goal Find a shortest odd \( s-t \) path \( P \) in \( G \)

Solved via Weighted Matching
Shortest Non-zero Path Problem

**Input**  \( G = (V, E) \): \( \Gamma \)-Labeled Graph \( (\Gamma: \text{Group}) \)

\[ \ell \in \mathbb{R}_{\geq 0}^E \text{: Edge Length, } s, t \in V \text{: Terminals} \]

**Goal** Find a shortest non-zero \( s-t \) path \( P \) in \( G \)

\[ \Gamma = (\mathbb{Z}, +) \]
Shortest Non-zero Path Problem

**Input** \( G = (V, E) \): \( \Gamma \)-Labeled Graph (\( \Gamma \): Group)

\[ \ell \in \mathbb{R}_{\geq 0}^E : \text{Edge Length, } s, t \in V : \text{Terminals} \]

**Goal** Find a shortest **non-zero** \( s \rightarrow t \) path \( P \) in \( G \)

\[ \psi_G(P') = 0 + 1 + (-2) + 1 = 0 \]

\[ \ell(P') = 40 \]

\[ \Gamma = (\mathbb{Z}, +) \]
Shortest Non-zero Path Problem

**Input** \( G = (V, E) \): \( \Gamma \)-Labeled Graph \((\Gamma: \text{Group})\)

\( \ell \in \mathbb{R}_{\geq 0}^E \): Edge Length, \( s, t \in V \): Terminals

**Goal** Find a shortest **non-zero** \( s-t \) path \( P \) in \( G \)

\( \psi_G(P'') = 0 + 1 + (-1) = 0 \)

\( \ell(P'') = 70 \)

\( \Gamma = (\mathbb{Z}, +) \)
Shortest Non-zero Path Problem

**Input**  \( G = (V, E) \): \( \Gamma \)-Labeled Graph  
\( \ell \in \mathbb{R}^E_{\geq 0} \): Edge Length,  
\( s, t \in V \): Terminals

**Goal**  Find a shortest **non-zero** \( s-t \) path \( P \) in \( G \)

\[
\psi_G(P) = 1 + (-1) + 1 = 1 \neq 0 \\
\ell(P) = 110
\]
Shortest Non-zero Path Problem

**Input**  
$G = (V, E)$: $\Gamma$-Labeled Graph  
($\Gamma$: Group)  
$\ell \in \mathbb{R}_{\geq 0}^E$: Edge Length,  
$s, t \in V$: Terminals

**Goal**  
Find a shortest **non-zero** $s$–$t$ path $P$ in $G$

**Thm.**  
Solved by $O(|V| \cdot |E|)$ Elementary Operations

[This Work]

- When $|\Gamma| = 2$, This Problem $\approx$ **Shortest Odd Path Problem**

- When $\Gamma \approx \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ (i.e., $\Gamma$ is finite & abelian),  
  **Randomized Pseudo-Poly** via **Permanent Computation**  
  [Kobayashi–Toyooka 2017]

- When $\Gamma \approx \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_k}$ ($p_i$: prime),  
  **Deterministic Strongly-Poly** via **Weighted Linear Matroid Parity**  
  [Y. 2016] + [Iwata–Kobayashi 2017]
Outline

• Algorithm Framework
  – Basic Idea
  – Auxiliary Problem (Shortest Unorthodox Path)
  – Main Lemma

• Key Structure: Lowest Blossoms
  – Detour yields a Shortest Unorthodox Path (SUP)
  – Shrinking preserves SUP Problem

• Conclusion
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Basic Idea

**Input**  $G = (V, E)$: $\Gamma$-Labeled Graph  ($\Gamma$: Group)

$\ell \in \mathbb{R}_{\geq 0}^E$: Edge Length,  $s, t \in V$: Terminals

**Goal**  Find a shortest **non-zero** $s$–$t$ path $P$ in $G$

1. Find a shortest $s$–$t$ path $P$ in $G$ by Dijkstra’s Algorithm
2. If $P$ is non-zero ($\psi_G(P) \neq 1_\Gamma$), then return $P$
3. Otherwise, find and return an $s$–$t$ path $Q$ in $G$ s.t. $\ell(Q)$ is minimized subject to $\psi_G(Q) \neq \psi_G(P)$
**Basic Idea**

**Input**  
$G = (V, E)$: $\Gamma$-Labeled Graph (\(\Gamma\): Group)  
$\ell \in \mathbb{R}_\geq 0^E$: Edge Length, $s, t \in V$: Terminals

**Goal**  
Find a shortest **non-zero** $s$–$t$ path $P$ in $G$

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![Diagram](image)
Auxiliary Problem for Main Task

**Input** \((G = (V, E), \ell, s, t)\): Original Input

- \(P\): Shortest \(s\)–\(t\) Path in \(G\)

**Goal** Find a shortest **unorthodox** \(s\)–\(t\) path \(Q\) in \(G\)

1. Find a shortest \(s\)–\(t\) path \(P\) in \(G\) by Dijkstra’s Algorithm
2. If \(P\) is non-zero \((\psi_G(P) \neq 1_\Gamma)\), then return \(P\)
3. Otherwise, find and return an \(s\)–\(t\) path \(Q\) in \(G\) s.t. \(\ell(Q)\) is minimized subject to \(\psi_G(Q) \neq \psi_G(P)\)
1. Find a shortest $s-t$ path $P$ in $G$ by Dijkstra’s Algorithm

Def. A Tree in which each $s-v$ path $P_v$ is shortest in $G$

Output

2. If $P$ is non-zero (i.e., $\psi_G(P) \neq 1$), then return $P$

3. Otherwise, find and return an $s-t$ path $Q$ in $G$ such that $\ell_Q$ is minimized subject to $\psi_G(Q) \neq \psi_G(P_t)$

Auxiliary Problem for Main Task

| Input | $(G = (V, E), \ell, s, t)$: Original Input |
|-------|------------------------------------------|
| $T = \bigcup_{v \in V} P_v$: Shortest Path Tree of $G$ rooted at $s$ |

Goal Find a shortest unorthodox $s-t$ path $Q$ in $G$
Finding a Shortest Unorthodox Path (SUP)

**Input** \((G = (V, E), \ell, s, t)\): Original Input

\(T = \bigcup_{v \in V} P_v\): Shortest Path Tree of \(G\) rooted at \(s\)

**Goal** Find a shortest unorthodox \(s\)–\(t\) path \(Q\) in \(G\)

1. Find a “NICE” non-zero cycle \(C\) \(\psi_{G}(C) \neq 1_{\Gamma}\)
2. If \(t\) is on \(C\), then return a Detour \(Q\) from \(P_t\) around \(C\)
3. Otherwise, shrink \(C\) into a single vertex \(b\), and recursively solve a small instance
Finding a Shortest Unorthodox Path (SUP)

**Input** \((G = (V, E), \ell, s, t)\): Original Input

\[ T = \bigcup_{v \in V} P_v \]: Shortest Path Tree of \(G\) rooted at \(s\)

**Goal** Find a shortest unorthodox \(s\–t\) path \(Q\) in \(G\)

1. Find a “NICE” non-zero cycle \(C\) \((\psi_G(C) \neq 1_\Gamma)\)
2. If \(t\) is on \(C\), then return a **Detour** \(Q\) from \(P_t\) around \(C\)
3. Otherwise, shrink \(C\) into a single vertex \(b\), and recursively solve a small instance

\[
\psi_G(Q) \neq \psi_G(P_t)
\]

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**Diagram:**
- **Input Graph:** \(G = (V, E)\), with \(s\) and \(t\) as start and end vertices.
- **Path Tree:** \(T = \bigcup_{v \in V} P_v\) represents the shortest paths to all vertices from \(s\).
- **Cycle:** \(C\) is a cycle that is not a single vertex.
- **Detour:** \(Q\) is a path that detours around \(C\) if \(t\) is on \(C\).
- **Recursive Case:** If \(t\) is not on \(C\), \(C\) is shrunk to a single vertex \(b\), and a small instance is solved recursively.
Finding a Shortest Unorthodox Path (SUP)

**Input** \((G = (V, E), \ell, s, t)\): Original Input
\[ T = \bigcup_{v \in V} P_v \]: Shortest Path Tree of \(G\) rooted at \(s\)

**Goal** Find a shortest unorthodox \(s\)–\(t\) path \(Q\) in \(G\)

1. Find a “NICE” non-zero cycle \(C\) \((\psi_G(C) \neq 1_\Gamma)\)
2. If \(t\) is on \(C\), then return a **Detour** \(Q\) from \(P_t\) around \(C\)
3. Otherwise, shrink \(C\) into a single vertex \(b\), and recursively solve a small instance
Main Lemma (Informal)

**Lem.** \( \exists C: \) Non-zero Cycle with a vertex \( b \in C \) s.t.
- For a vertex in \( C - b \), a detour \( Q \) around \( C \) is an SUP
- After shrinking \( C \) into \( b \), corresponding SUPs remain

Moreover, such \( C \) can be found in \( O(|E|) \) time
**On Computational Time**

**Lem.** \( \exists C: \text{Non-zero Cycle with a vertex } b \in C \text{ s.t.} \)
- For a vertex in \( C - b \), a detour \( Q \) around \( C \) is an SUP
- After shrinking \( C \) into \( b \), **corresponding SUPs** remain

Moreover, such \( C \) can be found in \( O(|E|) \) time

**Obs.** Shrinking occurs at most \((|V| - 2)\) times

**Cor.** An SUP can be found in \( O(|V| \cdot |E|) \) time (if exists)

![Diagram](image)
Outline

• Algorithm Framework
  – Basic Idea
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  – Main Lemma

• Key Structure: Lowest Blossoms
  – Detour yields a Shortest Unorthodox Path (SUP)
  – Shrinking preserves SUP Problem

• Conclusion
**Blossom**

**Def.** \( T = \bigcup_{v \in V} P_v \): Shortest Path Tree of \( G \) rooted at \( s \)

*\( C \) is a **Blossom***

\[
\exists e \in E - T \text{ s.t. } C \subseteq T + e \quad (\text{i.e., } C \text{ is a Fundamental Circuit})
\]

\[
\psi_G(C) \neq 1 \Gamma \quad (\iff \psi_G(P_u \ast e) \neq \psi_G(P_v))
\]

**Height:**
\[
\frac{1}{2} \left( \ell(P_u) + \ell(P_v) + \ell(e) \right)
\]
Lowest Blossom (LB)

Def. $T = \bigcup_{v \in V} P_v$: Shortest Path Tree of $G$ rooted at $s$

$C$ is a **Lowest Blossom**

\[\iff\]

- $\exists e \in E - T$ s.t. $C \subseteq T + e$ (i.e., $C$ is a Fundamental Circuit)
- $e \in \arg\min_{f = \{x, y\}} \{\ell(P_x) + \ell(P_y) + \ell(f) \mid \psi_G(P_x * f) \neq \psi_G(P_y)\}$

**Height:** $\frac{1}{2}(\ell(P_u) + \ell(P_v) + \ell(e))$

Minimized
Detour around LB yields SUP

**Def.** \( T = \bigcup_{v \in V} P_v \): Shortest Path Tree of \( G \) rooted at \( s \)

\( C \) is a **Lowest Blossom**

\[\iff\]

1. \( \exists e \in E - T \) s.t. \( C \subseteq T + e \) (i.e., \( C \) is a Fundamental Circuit)
2. \( e \in \arg\min \{ \ell(P_x) + \ell(P_y) + \ell(f) \mid \psi_G(P_x * f) \neq \psi_G(P_y) \} \)

**Lem.** For \( C - b \), every detour \( Q \) from \( T \) is an **SUP**
Lem. For $C - b$, every detour $Q$ from $T$ is an SUP

Fix an $s$–$t$ path $R$ with $\psi_G(R) \neq \psi_G(P_t)$

- $\exists f = \{x, y\} \in R$ s.t. $\psi_G(P_x \ast f) \neq \psi(P_y)$
  - $\ell(P_x) + \ell(P_y) + \ell(f) \geq \ell(P_t) + \ell(Q)$ (C is an LB)
- $\ell(R[s, x]) \geq \ell(P_x)$ ($P_x$ is shortest)
- $\ell(R[y, t]) \geq |\ell(P_y) - \ell(P_t)|$ (o/w, $\exists$shortcut for $T$)
Fix an $s$–$t$ path $R$ with $\psi_G(R) \neq \psi_G(P_t)$

- $\exists f = \{x, y\} \in R \text{ s.t. } \psi_G(P_x * f) \neq \psi(P_y)$

$s/w, \psi_G(R) = \prod_{f=\{x,y\} \in R} \left(\psi_G(P_x)^{-1} \cdot \psi_G(P_y)\right)$

$= \psi_G(P_s)^{-1} \cdot \psi_G(P_t) = \psi_G(P_t)$

Contradiction!
Detour around LB yields SUP

\textbf{Lem.} For $C - b$, every detour $Q$ from $T$ is an \textbf{SUP}

Fix an $s$–$t$ path $R$ with $\psi_G(R) \neq \psi_G(P_t)$

- $\exists f = \{x, y\} \in R$ s.t. $\psi_G(P_x * f) \neq \psi(P_y)$
  - $\ell(P_x) + \ell(P_y) + \ell(f) \geq \ell(P_t) + \ell(Q)$ (\(C\) is an LB)
Fix an $s$–$t$ path $R$ with $\psi_G(R) \neq \psi_G(P_t)$

- $\exists f = \{x, y\} \in R$ s.t. $\psi_G(P_x * f) \neq \psi(P_y)$
  $\rightarrow \ell(P_x) + \ell(P_y) + \ell(f) \geq \ell(P_t) + \ell(Q)$ (C is an LB)

- $\ell(R[s, x]) \geq \ell(P_x)$ ($P_x$ is shortest)

- $\ell(R[y, t]) \geq |\ell(P_y) - \ell(P_t)|$ (o/w, $\exists$ shortcut for $T$)
Detour around LB yields SUP

**Lem.** For $C - b$, every detour $Q$ from $T$ is an SUP

\[
\ell(R) \geq \ell(P_x) + \ell(f) + \ell(P_y) - \ell(P_t) \geq \ell(Q)
\]

\[
\ell(P_x) + \ell(P_y) + \ell(f) \geq \ell(P_t) + \ell(Q) \quad (C \text{ is an LB})
\]

- $\ell(R[s, x]) \geq \ell(P_x)$ ($P_x$ is shortest)
- $\ell(R[y, t]) \geq |\ell(P_y) - \ell(P_t)|$ (o/w, $\exists$ shortcut for $T$)
**Main Lemma (Informal)**

**Lem.** \(\exists C: \text{Non-zero Cycle with a vertex } b \in C \text{ s.t.} \)

- For a vertex in \(C - b\), a detour \(Q\) around \(C\) is an **SUP**.
- After shrinking \(C\) into \(b\), **corresponding SUPs** remain.
- Moreover, such \(C\) can be found in \(O(|E|)\) time.

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**Diagram:**
- **Detour** and **Shrink** operations illustrated with graphs and nodes.
- **Shortest Unorthodox** indication.
- **Recursion** notation.
Shrinking preserves SUP Problem

**Lem.** \( \forall R : \) Unorthodox \( s-t \) path in \( G \) intersecting \( C \),

\[ \exists R' : \) Unorthodox \( s-t \) path in \( G \) s.t. \( \ell(R') \leq \ell(R) \)

and \( R' \) remains in a shrunk form

\[ \psi_{\tilde{G}}(\tilde{f}_i; b \rightarrow y) := \psi_G(R_i \ast f) \]

\[ \tilde{\ell}(\tilde{f}_i) := \ell(R_i) + \ell(f) \]
Shrinking preserves SUP Problem

**Lem.** \( \forall R: \) Unorthodox \( s-t \) path in \( G \) intersecting \( C \), \\ \( \exists R': \) Unorthodox \( s-t \) path in \( G \) s.t. \( \ell(R') \leq \ell(R) \) and \( R' \) remains in a shrunk form

\( R: \) Unorthodox \( s-t \) path, \( x: \) Last Vertex intersecting \( P_b \cup C \)

**Case 1.** \( x \in C - b \) (Easy) \hspace{1cm} **Case 2.** \( x \in P_b \) (Complicated)
Shrinking preserves SUP Problem

**Lem.** \( \forall R: \) Unorthodox \( s \rightarrow t \) path in \( G \) intersecting \( C \),

\( \exists R': \) Unorthodox \( s \rightarrow t \) path in \( G \) s.t. \( \ell(R') \leq \ell(R) \)
and \( R' \) remains in a shrunk form

\( R: \) Unorthodox \( s \rightarrow t \) path, \( x: \) Last Vertex intersecting \( P_b \cup C \)

**Case 1.** \( x \in C - b \) (Easy)

\( \psi_G(P_x \ast R[x,t]) \neq \psi_G(P_t) \)
\[ \Downarrow \]
\( R' := P_x \ast R[x,t] \) is OK

- \( \ell(P_x) \leq \ell(R[s,x]) \)
Shrinking preserves SUP Problem

**Lem.** \( \forall R: \) Unorthodox \( s-t \) path in \( G \) intersecting \( C \),
\( \exists R': \) Unorthodox \( s-t \) path in \( G \) s.t. \( \ell(R') \leq \ell(R) \)
and \( R' \) remains in a shrunk form

**Case 1.** \( x \in C - b \) (Easy)

\[ \psi_G(P_x \ast R[x, t]) = \psi_G(P_t) \]
\[ \Downarrow \]
\( R' := Q_x \ast R[x, t] \) is OK

- \( \psi_G(Q_x) \neq \psi_G(P_x) \neq \psi_G(R[s, x]) \)
- \( \ell(Q_x) \leq \ell(R[s, x]) \)
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**Input**  \( G = (V, E): \Gamma\)-Labeled Graph  \((\Gamma: \text{Group})\)

\[ \ell \in \mathbb{R}_{\geq 0}^E: \text{Edge Length}, \ s, t \in V: \text{Terminals} \]

**Goal**  Find a shortest **non-zero** \( s-t \) path \( P \) in \( G \)

**Thm.**  Solved by \( O(|V| \cdot |E|) \) Elementary Operations
Conclusion

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**Thm.**  Solved by \( O(|V| \cdot |E|) \) Elementary Operations

- Dijkstra + Shrinking **Lowest Blossoms**
- Depending heavily on **Nonnegativity of Edge Length**

Q. How about a general input “without **Negative Cycle**”?  
[Unconstrained]  **Strongly-Poly** via **Weighted Matching**
[Parity Constrained]  Open