Efficient constructions of convex combinations for 2-edge-connected subgraphs on fundamental classes

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Abstract

We present coloring-based algorithms for tree augmentation and use them to construct convex combinations of 2-edge-connected subgraphs. This classic tool has been applied previously to the problem, but our algorithms illustrate its flexibility, which—in coordination with the choice of spanning tree—can be used to obtain various properties (e.g., 2-vertex connectivity) that are useful in our applications.

We use these coloring algorithms to design approximation algorithms for the 2-edge-connected multigraph problem (2ECM) and the 2-edge-connected spanning subgraph problem (2ECS) on two well-studied types of LP solutions. The first type of points, half-integer square points, belong to a class of fundamental extreme points, which exhibit the same integrality gap as the general case. For half-integer square points, the integrality gap for 2ECM is known to be between $\frac{6}{5}$ and $\frac{4}{3}$. We improve the upper bound to $\frac{9}{7}$. The second type of points we study are uniform points whose support is a 3-edge-connected graph and each entry is $\frac{2}{3}$. Although the best-known upper bound on the integrality gap of 2ECS for these points is less than $\frac{4}{3}$, previous results do not yield an efficient algorithm. We give the first approximation algorithm for 2ECS with ratio below $\frac{4}{3}$ for this class of points.

1 Introduction

Due, at least in part, to its similarities and connections to the traveling salesman problem (TSP), the 2-edge-connected spanning multigraph problem (2ECM) is a well-studied problem in the areas of combinatorial optimization and approximation algorithms, and the two problems have often been studied alongside each other. Let $G = (V, E)$ be a graph with edge weights $w \in \mathbb{R}_+^E$. TSP is the problem of finding a minimum weight connected spanning Eulerian multigraph of $G$ (henceforth a tour of $G$). Note that a tour is Eulerian and connected,
which implies that it is also 2-edge-connected. 2ECM is the problem of finding a minimum weight 2-edge-connected spanning multigraph of $G$ (henceforth a 2-edge-connected multigraph of $G$) and is a relaxation of TSP. A well-studied relaxation for both TSP and 2ECM on a graph $G = (V, E)$ is as follows.

$$
\min \ wx \\
\text{subject to: } x(\delta(S)) \geq 2 \quad \text{for } \emptyset \subset S \subset V \\
x \geq 0.
$$

Let LP($G$) be the feasible region of this LP. The integrality gap $\alpha_{LP}^{2EC}$ is defined as

$$
\sup_{G,w} \min \{wx : x \text{ is the incidence vector of a 2-edge-connected multigraph of } G\} \\
\min \{wx : x \in \text{LP}(G)\}.
$$

Alternatively, $\alpha_{LP}^{2EC}$ is the smallest number such that for any graph $G$ and for any $x \in \text{LP}(G)$, vector $\alpha_{LP}^{2EC} x$ dominates a convex combination of 2-edge-connected multigraphs of $G$ (see [Goe95] or Theorem 1 of [CV04]). Wolsey’s analysis of Christofides’ algorithm shows that $\alpha_{LP}^{2EC} \leq \frac{3}{2}$ [Chr76, Wol80], which is currently the best-known approximation factor for 2ECM.

This seems strange since Christofides’ algorithm finds tours, which are more constrained than 2-edge-connected multigraphs.

Stated as a potentially easier-to-prove variant of the famous four-thirds conjecture for TSP, it has been conjectured that $\alpha_{LP}^{2EC} \leq \frac{4}{3}$ (e.g., Conjecture 2 in [CR98], Conjecture 1 in [ABE06] and Conjecture 4 in [BC11]). However, in contrast to the four-thirds conjecture, the largest lower bound only shows that $\alpha_{LP}^{2EC} \geq \frac{6}{5}$ [ABE06]. Based on this lower bound and computational evidence, Alexander, Boyd and Elliott-Magwood proposed the following stronger conjecture (Conjecture 6 in [ABE06]), to which we will refer as the six-fifths conjecture.

**Conjecture 1.** If $x \in \text{LP}(G)$, then $\frac{6}{5} x$ dominates a convex combination of 2-edge-connected multigraphs of $G$. In other words, $\alpha_{LP}^{2EC} \leq \frac{6}{5}$.

Despite lack of progress in the general case, there has been some progress towards validating this conjecture for special cases. For example, in the unweighted case (when $w_e = 1$ for all $e \in E$), there has been some success in beating the factor of $\frac{3}{2}$ [SV14, BFS16, BIT13, Tak16]. Another important special case is half-integer points, which are conjectured to exhibit the largest gap for TSP (e.g., see [SWZ14, BS21]). Carr and Ravi proved that $\alpha_{2EC}^{LP} \leq \frac{4}{3}$ if the optimal solution to $\min_{x \in \text{LP}(G)} wx$ is half-integer [CR98]. More specifically, they proved that if $x \in \text{LP}(G)$ and $x_e$ is a multiple of $\frac{1}{2}$ for $e \in E$, then $\frac{4}{3} x$ dominates a convex combination of 2-edge-connected multigraphs of $G$. Very recently, Boyd et al. turned this proof into a polynomial-time algorithm [BCC+20].

In this paper, we focus on two well-studied special cases of feasible solutions for LP($G$)
both of which, directly or indirectly, can be viewed as special cases of half-integer points. Let $G = (V, E)$ be a 3-edge-connected graph. Observe that $\frac{2}{5}x^E \in \text{LP}(G)$. Such solutions can be reduced to half-integer solutions (see proof of Theorem 3.1) and provide a lower bound on the integrality gap for half-integer solutions (see Theorem 4.10 [Had20]). Conjecture 1 implies that $\frac{4}{5}x^E$ can be written as a convex combination of 2-edge-connected multigraphs of $G$. In fact, this was proved by Boyd and Legault, who actually proved the stronger statement that $\frac{4}{5}x^E$ dominates a convex combination of 2-edge-connected subgraphs of $G$ (a subgraph has at most one copy of each edge) [BL17]. The factor of $\frac{4}{5}$ was subsequently improved even further to $\frac{7}{9}$ [Leg17].

However, these proofs do not yield polynomial-time (approximation) algorithms for the 2ECS problem, which is the problem of finding a minimum weight 2-edge-connected spanning subgraph of $G$ (henceforth a 2-edge-connected subgraph of $G$). This gives rise to the following natural problem: Find a small positive rational number $\alpha$ such that for a 3-edge-connected graph $G = (V, E)$, the vector $\alpha x^E$ dominates a convex combination of 2-edge-connected subgraphs of $G$ and this convex combination can be found in polynomial time. The best-known answer to this question is $\frac{8}{9}$ [CJR99, HNR21]. (If we allow 2-edge-connected multigraphs instead of 2-edge-connected subgraphs, the best-known answer to this question is $\frac{15}{17}$ [HNR21].) In this paper, we improve this factor.

**Theorem 1.1.** Let $G = (V, E)$ be a 3-edge-connected graph. The vector $\frac{7}{8}x^E$ dominates a convex combination of 2-edge-connected subgraphs of $G$. Moreover, this convex combination can be found in time polynomial in the size of $G$.

We can improve this factor to $\frac{41}{47}$ (Theorem 3.10) when we are finding convex combination of multigraphs (i.e., when we allow doubled edges). One consequence of Theorem 1.1 is a polynomial-time algorithm to write $\left(\frac{6}{5} + \frac{1}{120}\right)x$ as a convex combination of 2-edge-connected multigraphs when $x$ is a half-integer triangle point (see Theorem 5.1 in [Had20]). Half-integer triangle points were introduced in [BC11] as a class of points for which the conjectured lower bound of $\frac{4}{3}$ for integrality gap of TSP with the subtour elimination relaxation is achieved. Later [ABE06] showed that this class also achieves the conjectured lower bound of $\frac{6}{5}$ for $\alpha_{2\text{EC}}^{\text{LP}}$. Boyd and Legault [BL17] showed that $\frac{6}{5}$ is also an upper bound on $\alpha_{2\text{EC}}^{\text{LP}}$ when restricted to half-integer triangle points. However, their proof does not yield an (efficient) approximation algorithm. We also remark that very recently, Boyd et al., gave an efficient algorithm to write $\frac{7}{8}x^E$ as a convex combination of 2-edge-connected multigraphs when $G = (V, E)$ is 3-edge-connected [BCC+20].

Another approach to the six-fifths conjecture is to consider so-called fundamental extreme points introduced by Carr and Ravi [CR98] and further developed by Boyd and Carr [BC11]. A Boyd-Carr point is a point $x$ of LP($G$) that satisfies the following conditions.

(i) The support graph of $x$ is cubic and 3-edge-connected.
(ii) For each vertex, there is exactly one incident edge $e$ such that $x_e = 1$ (a 1-edge.)

(iii) The fractional edges form disjoint 4-cycles.

Boyd and Carr proved that in order to bound $\alpha_{2EC}$ (e.g., to prove the six-fifths conjecture), it suffices to prove a bound for Boyd-Carr points \cite{BC11}. A generalization of Boyd-Carr points are square points, which are obtained by replacing each 1-edge in a Boyd-Carr point by an arbitrary-length path of 1-edges. Half-integer square points are particularly interesting for various reasons. For every $\epsilon > 0$, there is a half-integer square point $x$ such that $(\frac{6}{5} - \epsilon)x$ does not dominate a convex combination of 2-edge-connected multigraphs in the support of $x$. In other words, the lower bound for $\alpha_{2EC}$ is achieved for half-integer square points. (This specific square point is discussed in Section \ref{half}) Furthermore, half-integer square points also demonstrate the lower bound of $\frac{4}{3}$ for the integrality gap of TSP with respect to the Held-Karp relaxation \cite{BS21}.

Recently, Boyd and Sebő initiated the study of improving upper bounds on the integrality gap for these classes and presented a $\frac{10}{7}$-approximation algorithm (and upper bound on the integrality gap) for TSP in the special case of half-integer square points. They pointed out that, despite their significance, not much effort has been expended on improving bounds on the integrality gaps for these classes of extreme point solutions.

In this paper, we focus on this class of solutions and we improve the best-known upper bound on $\alpha_{2EC}$ for half-integer square points. The best previously-known upper bound on $\alpha_{2EC}$ for half-integer square points is $\frac{4}{3}$, which follows from the aforementioned bound of Carr and Ravi on all half-integer points \cite{CR98}. We note that there is also a simple $\frac{4}{3}$-approximation algorithm using the observation from \cite{BS21} that the support of a square point is Hamiltonian. Our main result is to improve this factor.

**Theorem 1.2.** Let $x$ be a half-integer square point. Then $\frac{6}{5}x$ dominates a convex combination of 2-edge-connected multigraphs in $G_x$, the support graph of $x$. Moreover, this convex combination can be found in polynomial time.

1.1 Overview of our methods

A common approach to TSP and 2ECM is to choose a spanning tree that is specially tailored to a particular type of cheap augmentation (e.g., Gao trees for path-TSP \cite{Gao13}, max-entropy spanning trees for asymmetric TSP \cite{AGM+17} and rainbow 1-trees for TSP \cite{BS21}). Our algorithms fall within this general paradigm. To construct our spanning trees and augmentations, we use the following three key ingredients: (i) gluing solutions over 3-edge cuts, (ii) rainbow spanning tree decompositions and (iii) the top-down coloring framework for tree augmentation. The first ingredient, gluing solutions for the 2-edge-connected subgraph problem over 3-edge cuts, was introduced by Carr and Ravi $\frac{2}{3}$ \cite{CR98} and has by now become
a standard tool [BL17, Leg17]. In each of these works, a common pitfall is that the number of times the gluing procedure is applied is not provably polynomial, leading to inefficient algorithms. We take a different approach to ensure a polynomial running time. While we do use gluing in the proof of Theorem 1.1, we use it more sparingly (i.e., only over proper 3-edge cuts that appear in the initial input graph and therefore only a polynomial number of times) and with the sole objective of assuming that the input graph is essentially 4-edge-connected.

The second ingredient is rainbow spanning tree decompositions, which were introduced by Broersma and Li [BL97] and recently employed by Boyd and Sebő who used it to control the parity of cuts of spanning trees over certain 4-edge cuts in half-integer square points [BS21]. In this paper, we apply this tool in a new way and for a completely different purpose (unrelated to parity), highlighting its flexibility and usefulness. Roughly speaking, this tool allows the edges of a graph to be partitioned (subject to certain constraints) so that only one edge from each set in the partition belongs to a spanning tree. Hypothetically, this powerful decomposition tool could be applied to control properties of the spanning trees output in the convex combination of $x \in \text{LP}(G)$ in, for example, an implementation of the best-of-many Christofides’ algorithm. However, exactly which properties can be obtained and how to use such properties is not yet clear; the power of this tool is likely still far from being fully realized.

In this paper, we use it to obtain spanning trees in which few pairs of adjacent vertices in the graph are both leaves of the tree.

The third ingredient is coloring-based algorithms for tree augmentation. Such algorithms were recently studied by Iglesias and Ravi who introduced a top-down coloring algorithm for tree augmentation [IR17], which generalized results of Cheriyan, Jordan and Ravi [CJR99]. A straightforward application of the algorithm of Iglesias and Ravi can be used to prove Theorem 1.1 with a factor $\frac{8}{9}$. In this paper, we significantly extend this coloring-based framework. We prove Theorem 1.1 by demonstrating that certain properties of the graph (e.g., essentially 4-edge-connected) and the spanning tree (e.g., the leaf structure obtained via rainbow decomposition) can be used to design more careful coloring rules which results in smaller tree augmentations. In a key step in the proof of Theorem 1.2 we tailor the coloring rules to obtain a convex combination of 2-vertex-connected subgraphs with minimum degree three. The objective here is to construct convex combinations which use the half-edges in the half-integer square points sparingly. The property of 2-vertex connectivity and the fact that the complement of a subgraph forms a matching crucially allows us to be more parsimonious with the half-edges when constructing the convex combinations. Thus, we demonstrate that this coloring framework is a flexible and therefore powerful tool in designing approximation algorithms for 2ECM and related problems, and we believe it likely has further applications.
2 Preliminaries and tools

In this paper, a graph may contain parallel edges. We work with multisets of edges of $G = (V, E)$. For a multiset $F$ of $E$, the submultigraph induced by $F$ (henceforth, we simply call $F$ a multigraph of $G$) is the graph with the same number of copies of each edge as $F$. A subgraph of $G$ has at most one copy of each edge in $E$. The incidence vector of multigraph $F$ of $G$, denoted by $\chi^F$ is a vector in $\mathbb{R}^E$, where $\chi^F_e$ is the number of copies of $e$ in $F$. For multigraphs $F$ and $F'$ of $E$, we define $F + F'$ to be the multigraph that contains $\chi^F_e + \chi^{F'}_e$ copies of each edge $e \in E$. For a subset $S \subseteq V$ of vertices, let $\delta(S)$ be the edges in $E$ with one endpoint in $S$ and one endpoint not in $S$. For a subgraph $F$ of $G$, $\delta_F(S) = \delta(S) \cap F$.

If multigraph $F$ spans $V$ and is 2-edge-connected, we say $F$ is a 2-edge-connected spanning multigraph of $G$ (or a 2-edge-connected multigraph of $G$ for brevity). If in addition, $F$ is a subgraph, we say $F$ is a 2-edge-connected subgraph of $G$. Let $\mathcal{F}$ be a collection of multigraphs of $G$. We say $\sum_{i=1}^k \lambda_i \chi^{F_i}$ is a convex combination of $k$ multigraphs in $\mathcal{F}$ if $\lambda_i > 0$ for all $i \in \{1, \ldots, k\}$ and $\sum_{i=1}^k \lambda_i = 1$. Let $x$ be a vector in $\mathbb{R}^E$. We say $x$ can be written as convex combination of multigraphs in $\mathcal{F}$ if additionally $x = \sum_{i=1}^k \lambda_i \chi^{F_i}$, and $\lambda_i$ and $F_i$ for $i \in \{1, \ldots, k\}$ can be found in polynomial time in the size of $x$. Here by the size of $x$ we refer to $|E|$ (i.e., the number of edges in the support of $x$). For a vector $x, y \in \mathbb{R}^E$ we say $y$ dominates $x$ if $y_e \geq x_e$ for $e \in E$. For any $x \leq y$, if $x$ can be written as a convex combination of multigraphs in $\mathcal{F}$, then we say $y$ dominates a convex combination of multigraphs in $\mathcal{F}$. Moreover, for $x \leq y \leq 1$ ($x \leq y \leq 2$, respectively), if $x$ can be written as a convex combination of 2-edge-connected subgraphs (multigraphs, respectively), then $y$ can be written as a convex combination of 2-edge-connected subgraphs (multigraphs, respectively). Thus, in our proofs we sometimes use the phrase “dominates a convex combination” in place of “written as a convex combination” when the context is appropriate.

The support graph of $x$, denoted by $G_x$ is the graph induced on $G$ by $E_x = \{e \in E : x_e > 0\}$. Vector $x$ is half-integer if $x_e$ is a multiple of $\frac{1}{2}$ for all $e \in E$. We say that an edge $e \in G_x$ is a 1-edge if $x_e = 1$. Similarly, an edge is a half-edge if $x_e = \frac{1}{2}$.

Finally, we say graph $G$ is $k$-edge-connected if $|\delta(S)| \geq k$ for all $\emptyset \subset S \subset V$, and is essentially $k + 1$-edge-connected if additionally $|\delta(S)| \geq k + 1$ for all $S \subset V$ with $2 \leq |S| \leq |V| - 2$.

Next, we introduce some key tools.

2.1 Cycle covers

A cycle cover $\mathcal{C}$ of graph $G = (V, E)$ is a subgraph of $G$ where every vertex $v \in V$ belongs to exactly one cycle in $\mathcal{C}$. We now present some (well-known) observations pertaining to cycle covers that we will use.

Observation 2.1. Let $G = (V, E)$ be a 2-edge-connected cubic graph. Then vector $\frac{2}{3} \chi^F$ can
be written as a convex combination of cycle covers of \( G \) where each cycle cover covers the 3-edge cuts of \( G \).

**Proof.** The vector \( \frac{1}{3} \chi_E \) belongs to the perfect matching polytope (see Theorem 4 in [NPS81]). Thus, this vector can be written as the convex combination of at most \( |E| + 1 \) perfect matchings via Carathéodory’s Theorem [Car11]. Moreover, each of these perfect matchings intersects each 3-edge cut of \( G \) in exactly one edge. (This fact has been used many times and can be considered folklore; see [KN05] for an application.) In a cubic graph, the complement of a perfect matching is a cycle cover. This proves the observation.

Notice that when we say a cycle cover covers a 3-edge cut, we mean that the cycle cover intersects this cut in exactly two edges. One useful property of a cycle cover that covers all 3-edge cuts of a graph is that it does not contain any cycle with length less than four (i.e., it does not contain a triangle).

**Observation 2.2.** Let \( G = (V, E) \) be 3-edge-connected cubic graph and \( C \) be a cycle cover of \( G \) that covers all 3-edge cuts of \( G \). Then the vector \( x = \chi_E - \frac{1}{2} \chi_C \) belongs to \( \text{LP}(G) \). Moreover, \( x(\delta(v)) = 2 \) for all \( v \in V \).

**Proof.** Take \( \emptyset \subset U \subset V \). If \( |\delta(U)| \geq 4 \), then clearly \( x(\delta(U)) \geq 2 \). Otherwise, \( |\delta(U)| = 3 \). In this case, since exactly two edges in \( \delta(U) \) belong to \( C \), there is one edge \( e \in \delta(U) \) with \( x_e = 1 \). Hence, \( x(\delta(U)) = 2 \). Therefore, \( x \in \text{LP}(G) \).

**Theorem 2.3** ([BIT13]). Let \( G = (V, E) \) be a 3-edge-connected cubic graph. We can in polynomial time find a cycle cover \( C \) of \( G \) such that for every \( S \subseteq V \) for which \( |\delta(S)| \leq 4 \), we have \( C \cap \delta(S) \neq \emptyset \) (i.e., cycle cover \( C \) covers all 3-edge cuts and 4-edge cuts of \( G \)).

### 2.2 Gluing over 3-edge cuts

A tool introduced by Carr and Ravi and frequently used for constructing convex combinations of 2-edge-connected subgraphs is **gluing solutions over 3-edge cuts** [CR98]. This allows us to focus on essentially 4-edge-connected graphs, as stated in the next lemma whose full proof can be found in Appendix A.

**Theorem 2.4.** For \( \alpha \in [0, 1] \), the following two statements are equivalent.

1. For an essentially 4-edge-connected graph \( G = (V, E) \) and any cycle cover \( C \) of \( G \), the vector \( \chi_E - \alpha \chi_C \) can be written as a convex combination of 2-edge-connected subgraphs of \( G \).

2. For a 3-edge-connected graph \( G = (V, E) \) and any cycle cover \( C \) of \( G \) that covers all the 3-edge cuts of \( G \), the vector \( \chi_E - \alpha \chi_C \) can be written as a convex combination of 2-edge-connected subgraphs of \( G \).
2.3 Rainbow 1-tree decomposition

Given a graph $G = (V, E)$, a 1-tree $T$ of $G$ is a connected spanning subgraph of $G$ containing $|V|$ edges, where the vertex labeled 1 has degree exactly two and $T \setminus \delta(1)$ is a spanning tree on $V \setminus \{1\}$. Boyd and Sebő proved the following theorem (see Theorem 5 in [BS21]). In fact, they showed that the relevant decomposition can be found in time polynomial in the size of graph $G$.

**Theorem 2.5** ([BS21]). Let $x \in \text{LP}(G)$ be half-integer, $x(\delta(v)) = 2$ for all $v \in V$, and $\mathcal{P}$ be a partition of the half-edges into pairs. Then $x$ can be written as a convex combination of 1-trees of $G$ such that each 1-tree contains exactly one edge from each pair in $\mathcal{P}$.

To prove Theorem 1.1 we construct 2-edge connected subgraphs by augmenting 1-trees. However, it is often easier to think of augmenting spanning trees. We will use the term *connector* to refer to a subgraph of $G$ that is either a spanning tree or a 1-tree of $G$. For a 1-tree $K$, let $C_K$ denote the unique cycle in $K$. Note that $K/C_K$ is a spanning tree $T$ of $G/C_K$ that is obtained when we contract that cycle to a single vertex. We let the vertex corresponding to the contracted $C_K$ be the root $r$ of $T$. In a connector, each vertex with degree 1 is a leaf. The exception to this is that the root of a spanning tree is not a leaf (even if it has degree 1). We have the following useful facts.

**Observation 2.6.** Let $G = (V, E)$ be a graph, let $K$ be a 1-tree of $G$ with cycle $C_K$ and let $T = K/C_K$ be a spanning tree with root $r$ (where $r$ corresponds to the contracted $C_K$).

1. Let $F \subset E(G/C_K) \setminus T$ and suppose that $T + F$ is 2-edge connected. Then $K + F$ is 2-edge connected.
2. A vertex $v \in V$ is a leaf in $K$ iff it is a leaf in $T$.

2.4 Tree augmentation and the top-down coloring framework

We now describe the *top-down coloring* framework which is key to proving both of our main results. Consider a graph $G = (V, E)$ and a spanning tree $T$ of $G$. Let $L = E \setminus T$ be the set of links, and let $w \in \mathbb{R}^L_{\geq 0}$ be a weight vector. The tree augmentation problem asks for the minimum weight $F \subseteq L$ such that $T + F$ is 2-edge-connected (i.e., $F$ is a feasible augmentation for $T$). For a link $\ell \in L$, denote by $P_\ell$ the unique path between the endpoints of $\ell$ in $T$. For an edge $e \in T$, we say $\ell \in \text{cov}(e)$ if $e \in P_\ell$.

For $q \in \mathbb{Z}_+$ and vector $\rho \in \mathbb{Z}^L_+$, where $\rho = (p_1, \ldots, p_L)$ and $p_\ell \leq q$ for all $\ell \in L$, a $(\rho, q)$-coloring of $L$ is a function $\gamma : L \to 2^{\{c_1, \ldots, c_q\}}$ where $\gamma(\ell)$ has size at most $p_\ell$ for $\ell \in L$. In other words, a $(\rho, q)$-coloring of $L$ assigns at most $p_\ell$ different colors from a set of $q$ available colors to each link $\ell \in L$. Although $\gamma$ is defined to be a function on $L$, we abuse notation...
and for \( e \in T \), we let \( \gamma(e) = \bigcup_{\ell \in \text{cov}(e)} \gamma(\ell) \) denote the set of (distinct) colors edge \( e \in T \) has received in the coloring \( \gamma \). For a \((\rho,q)\)-coloring \( \gamma \) of \( L \), an edge \( e \in T \) and \( i \in \{1,\ldots,q\} \), we say \( e \) received color \( c_i \) if \( c_i \in \gamma(e) \), otherwise, we say \( e \) is missing color \( c_i \). We denote the set of colors an edge \( e \) is missing by \( \gamma(e) \) which is defined by \( \{c_1,\ldots,c_q\} \setminus \gamma(e) \).

**Definition 2.7.** Let \( \gamma \) be a \((\rho,q)\)-coloring of \( L \). We say \( \gamma \) is \( T \)-admissible if for each edge \( e \in T \), \( e \) has received all \( q \) colors (i.e., for each \( e \in T \), we have \( \gamma(e) = \{c_1,\ldots,c_q\} \)).

We note that Observations 2.8, 2.9 and 2.10 are from [IR17], but we present them here using our notation.

**Observation 2.8.** Let \( T \) be a spanning tree of \( G = (V,E) \) and \( L = E \setminus T \) be the set of links. If there exists a \( T \)-admissible \((\rho,q)\)-coloring of \( L \), namely \( \gamma \), then the vector \( z \in \mathbb{R}^L \), where \( z_\ell = \frac{p_\ell}{q} \) for \( \ell \in L \), dominates a convex combination of feasible augmentations of \( T \). Moreover, given \( \gamma \), this convex combination can be found in polynomial time.

**Proof.** For \( i \in \{1,\ldots,q\} \), let \( A_i = \{\ell \in L : c_i \in \gamma(\ell)\} \). By the definition of \( T \)-admissibility, for each \( e \in T \) and each color \( c_i \in \{c_1,\ldots,c_q\} \) there is at least one link \( \ell \in L \cap \text{cov}(e) \) with \( c_i \in \gamma(\ell) \). Hence, for each \( i \in \{1,\ldots,q\} \), \( A_i \) is a feasible augmentation for \( T \). Moreover, a link \( \ell \) is in at most \( p_\ell \) of \( \{A_1,\ldots,A_q\} \) since a link \( \ell \) is colored with at most \( p_\ell \) colors. Finally, observe that \( \sum_{i=1}^q \frac{1}{q} \lambda^{A_i} \leq z \). \( \square \)

Now we are almost ready to define a top-down \((\rho,q)\)-coloring algorithm for finding a \( T \)-admissible \((\rho,q)\)-coloring of the links \( L \). We first need to introduce some more terminology. If we choose a vertex \( r \in V \) to be the root of tree \( T \), we can think of \( T \) as an arborescence, with all edges oriented away from the root. For a link \( \ell = uv \) in \( L \), a least common ancestor of \( u \) and \( v \), denoted by \( \text{LCA}(\ell) \), is the vertex \( s \) that has edge-disjoint directed paths to \( u \) and \( v \) in \( T \). An edge \( e \in T \) is an ancestor of \( f \in T \) if there is a directed path containing both \( e \) and \( f \). (Note that \( e \) is an ancestor of itself.) By \( \text{LCA} \) order, we mean the partial ordering of the links according to their LCAs (i.e., if \( \text{LCA}(\ell) \) is higher than \( \text{LCA}(\ell') \), then \( \ell < \ell' \) in the partial order). For a link \( \ell = uv \) where \( s = \text{LCA}(\ell) \), we use \( \mathcal{L}_\ell \) to denote the edges in \( T \) on the path from \( s \) to \( u \) and \( \mathcal{R}_\ell \) for the edges in \( T \) on the path from \( s \) to \( v \).

In each iteration of a top-down \((\rho,q)\)-coloring algorithm, we choose a link \( \ell \in L \) and color \( \ell \) with \( p_\ell \) different colors from a set of \( q \) available colors, \( \{c_1,\ldots,c_q\} \). There are two key requirements: (i) the links are chosen in any order that respects the \( \text{LCA} \) order, and (ii) we continue until all links are colored.

After each iteration of a top-down \((\rho,q)\)-coloring algorithm, we have a \((\rho,q)\)-coloring of the links, and if some links are not yet colored, we sometimes refer to this as a partial \((\rho,q)\)-coloring of the links. For a partial coloring of the links, \( e \in T \) and \( i \in \{1,\ldots,q\} \), we say color \( c_i \) is missing for edge \( e \) if no links in \( \text{cov}(e) \) have been colored with \( c_i \). When coloring
link $\ell$ with $c_i$ as one of its $p_\ell$ colors, we say $e$ receives a new color $c_i$ or color $c_i$ is new for edge $e$ if $e \in P_\ell$ and edge $e$ was missing $c_i$ before this iteration of the algorithm.

**Observation 2.9.** Consider a partial $(\rho, q)$-coloring $\gamma$ of $L$ produced after some iterations of a top-down $(\rho, q)$-coloring algorithm. For an edge $e \in T$, let $\ell$ be any link in $\text{cov}(e)$. If $\ell$ is colored in $\gamma$, then $e$ has received at least $p_\ell$ colors (i.e., $|\gamma(e)| \geq p_\ell$).

**Observation 2.10.** Consider a partial $(\rho, q)$-coloring $\gamma$ of $L$ produced after some iterations of a top-down $(\rho, q)$-coloring algorithm. Consider $e, f \in T$ where $e$ is an ancestor of $f$. For any link $\ell \in \text{cov}(e) \cap \text{cov}(f)$ that is not colored in $\gamma$, any color given to $\ell$ that is new for $e$ is also new for $f$.

Now consider a partial $(\rho, q)$-coloring $\gamma$ of $L$ produced after some iterations of a top-down $(\rho, q)$-coloring algorithm. Let $\ell \in L$ be a link that is not yet colored in $\gamma$ and let $P \in \{R_\ell, L_\ell\}$. Then $P = e_1, \ldots, e_k$ where $e_k$ is the lowest edge in the path $P$. By Observation 2.10 we have $\gamma(e_k) \subseteq \gamma(e_{k-1}) \subseteq \ldots \subseteq \gamma(e_1)$, or $\gamma(e_1) \subseteq \ldots \subseteq \gamma(e_k)$. We define the top $i$ missing colors for path $P$, denoted by $\bar{\gamma}_i(P)$, to be a set of size $i$ where $\bar{\gamma}(e_{j-1}) \subseteq \bar{\gamma}_i(P) \subseteq \bar{\gamma}(e_j)$ where $e_j$ is the highest edge in $P$ with $|\bar{\gamma}(e_j)| \geq i$. If no such $e_j$ exists, it must be that $|\bar{\gamma}_i(e_k)| < i$ and we define $\bar{\gamma}_i(P)$ to be $\bar{\gamma}(e_k)$ (in which case $|\bar{\gamma}_i(P)| < i$).

A $(\rho, q)$-coloring algorithm is $T$-admissible if the final $(\rho, q)$-coloring of $L$ (i.e., after the last iteration) is $T$-admissible. Throughout this paper, we prove that a top-down coloring algorithm is $T$-admissible by showing that after the iteration in the algorithm when all the links covering an edge $e \in T$ are colored, the edge $e$ has received all $q$ colors.

### 2.4.1 A simple application of the top-down coloring algorithm

To illustrate the utility of the top-down coloring framework, we show how it can be used to state a short proof of a theorem of DeVos, Johnson and Seymour [DJS03]. Here, the key fact is that for each spanning tree $T$, a $T$-admissible top-down $(\rho, q)$-coloring algorithm produces only $q$ feasible augmentations.

**Theorem 2.11 (DJS03).** Let $G = (V, E)$ be a 3-edge-connected graph. Then there exists a partition of $E$ into sets $\{X_1, X_2, \ldots, X_9\}$ (where $X_i$ is allowed to be empty) such that the graph $G_i = (V, E \setminus X_i)$ is 2-edge-connected for $i \in \{1, \ldots, 9\}$.

Before we can prove Theorem 2.11 we need to prove the following claim, which directly follows from [IR17].

**Claim 1.** Let $G = (V, E)$ be a 3-edge-connected graph, let $T$ be a spanning tree of $G$ with root $r$, and let $L = E \setminus T$. Then there is a $T$-admissible top-down $(2\chi^L, 3)$-coloring algorithm to color the links in $L$.

**Proof.** Let $\gamma$ be the $(2\chi^L, 3)$-coloring of $L$ we maintain. At the start, we have $\gamma(\ell) = \emptyset$ for all $\ell \in L$. To color link $\ell = uv$, we use the following coloring rule.
**Coloring Rule:** Give link \( \ell \) colors \( \bar{\gamma}_1(\mathcal{L}_\ell) \) and \( \bar{\gamma}_1(\mathcal{R}_\ell) \). If either is empty, give \( \ell \) an arbitrary color that \( \ell \) does not already have.

We now prove that this top-down coloring algorithm is \( T \)-admissible. Consider an \( e \in T \). If \( e \) is an edge in \( T \), then since the graph is 3-edge-connected we have \( |\text{cov}(\ell)| \geq 2 \). Let \( \ell_1, \ell_2 \) be two of the links in \( \text{cov}(e) \) with the highest LCAs.

When coloring \( \ell_1 \), edge \( e \) receives two new colors by Observation 2.9. Now consider the iteration in which the algorithm colors \( \ell_2 \). At the time of coloring \( \ell_2 \), the top-down coloring algorithm that we described above will give \( \ell_2 \) at least one color that an ancestor of \( e \) is missing since \( e \) is either in \( \mathcal{R}_{\ell_2} \) or \( \mathcal{L}_{\ell_2} \). By Observation 2.10, we can conclude that \( e \) receives a new color after coloring \( \ell_2 \). Thus, after we have colored link \( \ell_2 \), edge \( e \) has received at least \( 2 + 1 = 3 \) colors.

**Proof of Theorem 2.11** From the theorem of Nash-Williams [NW61], we know that \( 2G \) contains three edge-disjoint spanning trees of \( G \). Call these trees \( T_1, T_2 \) and \( T_3 \). Observe that each edge in \( E \) is absent from at least one of the three spanning trees. For each \( i \in \{1, 2, 3\} \), we want to show that there is an admissible top-down \((2\chi^L, 3)\)-coloring algorithm for \( T_i \) and \( L_i = E \setminus T_i \). Since \( G \) is 3-edge-connected, we can apply Claim 1. Observe that each link receives two colors and the algorithm uses three colors in total.

For each \( i \in \{1, 2, 3\} \), we obtain three augmentations \( A^j_i \subset L_i \) for \( j \in \{1, 2, 3\} \) such that \( A^j_i \cup T_i \) is 2-edge-connected. The set \( A^j_i \) contains all links in \( L_i \) that received color \( j \) as one of their two colors. Let \( X^j_i = L_i \setminus A^j_i \) be the set of links in \( L_i \) that did not receive color \( j \). Then for each \( e \in L_i, e \) belongs to \( X^j_i \) for some \( j \in \{1, 2, 3\} \). Since each edge \( e \in E \) belongs to \( L_i \) for some \( i \in \{1, 2, 3\} \), we conclude that each edge \( e \in E \) belongs to at least one of the nine sets \( X^j_i \) for \( i, j \in \{1, 2, 3\} \).

The top-down coloring framework might have further applications for problems in which the objective is to obtain a convex combination of few subgraphs. Such problems were recently explored by Hörsch and Szigeti [HS21].

### 3 Uniform cover for 2-edge-connected subgraphs

Boyd and Legault [BL17] showed that to prove Theorem 1.1 it suffices to prove it for all cubic 3-edge-connected graphs (See Lemma 2.2 of [BL17]).

**Theorem 3.1.** Let \( G = (V, E) \) be a 3-edge-connected cubic graph. Then \( \frac{7}{8} \chi^E \) can be written as a convex combination of 2-edge-connected subgraphs of \( G \).

Recall that we say a vector can be written as a convex combination of subgraphs if the convex multipliers and the respective subgraphs can be constructed in polynomial time. In order to prove Theorem 3.1 we prove the following theorem.
Theorem 3.2. Let $G = (V,E)$ be a 3-edge-connected cubic graph and let $C$ be a cycle cover of $G$ covering all 3-edge cuts of $G$. The vector $\chi^E - \frac{3}{10} \chi^C$ can be written as a convex combination of 2-edge-connected subgraphs of $G$.

Proof of Theorem 3.1. Let $\sum_{i=1}^{k} \lambda_i \chi^C_i = \frac{2}{3} \chi^E$ be the convex combination of cycle covers obtained via Observation 2.1. By Theorem 3.2, for each $i \in \{1, \ldots, k\}$ we can write $\chi^E - \frac{3}{10} \chi^C_i$ as convex combination of 2-edge-connected subgraphs of $G$. Hence, $\sum_{i=1}^{k} \lambda_i (\chi^E - \frac{3}{10} \chi^C_i) = \chi^E - \frac{1}{5} \chi^E = \frac{7}{8} \chi^E$, and we conclude that $\frac{7}{8} \chi^E$ can be written as a convex combination of 2-edge-connected subgraphs of $G$. \hfill $\square$

Furthermore, applying Theorem 2.4 we can focus on proving the next lemma, which implies Theorem 3.2.

Lemma 3.3. Let $G = (V,E)$ be an essentially 4-edge-connected cubic graph and $C$ be a cycle cover of $G$. The vector $\chi^E - \frac{3}{10} \chi^C$ can be written as a convex combination of 2-edge-connected subgraphs of $G$.

Our approach to proving Lemma 3.3 is based on the top-down coloring framework introduced in Section 2.4. This allows us to avoid gluing completely when dealing with an essentially 4-edge-connected cubic graph (in contrast to [BL17], [Leg17]). In particular, in an essentially 4-edge-connected graph, if we consider any spanning tree $T$, then any edge $e \in T$ that is not incident to a leaf vertex is covered by at least three links (i.e., $|\text{cov}(e)| \geq 3$), as opposed to only two links if the graph is only 3-edge-connected. Therefore, assigning fewer colors to each link still satisfies the requirements of the top-down coloring algorithm for most of the edges in $T$. The problematic links are those that are incident to two leaves, since we cannot satisfy the color requirements of both adjacent tree edges using fewer colors on these links. These problematic links (called leaf-matching links) must be assigned more colors. Using a specially designed rainbow 1-tree decomposition, we can ensure that there are actually few such problematic links. We now present some necessary definitions.

Definition 3.4. Let $K$ be a connector of a graph $G = (V,E)$ and let $L = E \setminus K$ denote the set of links. We say an edge $e = uv \in L$ is a leaf-matching link for $K$ if both $u$ and $v$ are leaves in $K$. We denote by $\text{LM}(G,K)$ the set of leaf-matching links for $K$ in $G$.

The following lemma shows that using the top-down coloring algorithm we can find feasible augmentations that are “cheap” when there are few leaf-matching links.

Lemma 3.5. Let $H = (U,F)$ be an essentially 4-edge-connected graph and let $T$ be a spanning tree of $H$ with root $r$. Then we can find a $(\chi^{\text{LM}(H,T)} + 3 \chi^L, 5)$-coloring of $L = F \setminus T$ in polynomial time.
Lemma 3.5 is not strong enough to prove Lemma 3.3, but its proof helps illustrate our tools and techniques. The next lemma states that we can in fact find a convex combination of 1-trees with few leaf-matching links (i.e., each edge in $C$ is a leaf-matching link in at most $\frac{1}{10}$ fraction of the convex combination).

**Lemma 3.6.** Let $G = (V, E)$ be a 3-edge-connected cubic graph and let $C$ be a cycle cover of $G$. The vector $\chi^E - \frac{1}{2} \chi^C$ can be written as a convex combination of 1-trees $\{K_1, \ldots, K_k\}$ of $G$. Moreover, this convex combination (i.e., $\sum_{i=1}^k \lambda_i \chi^{K_i}$, where $\lambda_i \in [0, 1]$ and $\sum_{i=1}^k \lambda_i = 1$) has the following two properties: (i) $\sum_{i \in \text{LM}(G, K_i)} \lambda_i \leq \frac{1}{10}$ for $e \in C$, (ii) the links in $\text{LM}(G, K_i)$ are vertex-disjoint.

We show how to use property (i) from Lemma 3.6 to obtain a convex combination with a weaker bound than that proved in Lemma 3.3. For an essentially 4-edge-connected cubic graph $G = (V, E)$, let $C$ be a cycle cover and let $y = \chi^E - \frac{1}{2} \chi^C$. Then we show that $\chi^E - \frac{9}{50} \chi^C$ can be written as a convex combination of 2-edge-connected subgraphs of $G$. Let $\sum_{i=1}^k \lambda_i \chi^{K_i}$ be the convex combination of $y$ obtained via Lemma 3.6. For $i \in \{1, \ldots, k\}$, let $H_i = G/C_{K_i} = (U_i, F_i)$, $T_i = K_i/C_i$, and define $\rho^i = \chi^\text{LM}(H_i, T_i) + \frac{3}{5} \chi^{F_i \setminus T_i}$. Lemma 3.5 implies that we can find a $(\rho^i, 5)$-coloring of $F_i \setminus T_i$ for $i \in \{1, \ldots, k\}$. By Observation 2.8, we have $\rho^i = \sum_{j=1}^5 \frac{1}{5} \chi^A_j$ where $A_j$ is a feasible augmentation for $T_i$ and hence also for $K_i$.

Let $z_j^i = \chi^{K_i + A_j}$ be the characteristic vector of the corresponding 2-edge-connected subgraph of $G$. Then, we have

$$z^i = \sum_{j=1}^5 \frac{1}{5} z_j^i = \sum_{j=1}^5 \frac{1}{5} \chi^{K_i + A_j} = \chi^{K_i} + \sum_{j=1}^5 \frac{1}{5} \chi^{A_j} = \chi^{K_i} + \frac{3}{5} \chi^{L_i} + \frac{1}{5} \chi^\text{LM}(H_i, T_i).$$

Notice that $\sum_{i=1}^k \lambda_i \chi^{L_i} \leq \frac{1}{2} \chi^C$. Moreover, by property (i) of Lemma 3.6, $\sum_{i=1}^k \lambda_i \chi^\text{LM}(H_i, T_i) \leq \frac{1}{10} \chi^C$. Next, we claim that the vector $z$ can be written as a convex combination of 2-edge-connected subgraphs of $G$.

$$z = \sum_{k=1}^k \lambda_k z^i = \sum_{i=1}^k \lambda_i \left( \chi^{K_i} + \frac{3}{5} \chi^{L_i} + \frac{1}{5} \chi^\text{LM}(H_i, T_i) \right)$$

$$= \sum_{i=1}^k \lambda_i \chi^{K_i} + \frac{3}{5} \sum_{i=1}^k \lambda_i \chi^{L_i} + \frac{1}{5} \sum_{i=1}^k \lambda_i \chi^{\text{LM}(H_i, T_i)}$$

$$\leq (\chi^E - \frac{1}{2} \chi^C) + \frac{3}{10} \chi^C + \frac{1}{50} \chi^C = \chi^E - \frac{9}{50} \chi^C.$$

This is slightly worse than the factor promised in Lemma 3.3. We show that by paying a bit more on non leaf-matching links and exploiting a different property of the leaf-matching
Let \( H = (U, F) \) be an essentially 4-edge-connected graph and let \( T \) be a spanning tree of \( H \) with root \( r \). If the edges in \( \text{LM}(H, T) \) are vertex-disjoint, then we can find a \( T \)-admissible \((5\chi^L, 8)\) coloring of \( L = F \setminus T \) in polynomial time.

The proof of Lemma 3.7 is based on a top-down coloring algorithm and is deferred to Section 3.1.

Proof of Lemma 3.7. Let \( y = \chi^E - \frac{1}{2} \chi^C \). Let \( \sum_{i=1}^k \lambda_i \chi^{K_i} \) be the convex combination of vector \( y \) obtained via Lemma 3.6. We now set \( H_i = G/C_{K_i} = (U_i, F_i) \) and \( T_i = K_i/C_{K_i} \) (recall that \( C_{K_i} \) is the unique cycle in the 1-tree \( K_i \)). By Lemma 3.7 we can find a \((5\chi^{F_i \setminus T_i}, 8)\)-coloring of \( L_i = F_i \setminus T_i \) for \( i \in \{1, \ldots, k\} \). By Observation 2.8, we have \( \sum_{i=1}^8 \lambda_i \chi^{L_i} \leq \chi^E - \frac{1}{2} \chi^C \). Next, we claim that the vector \( z \) can be written as a convex combination of 2-edge-connected subgraphs of \( G \).

\[
\begin{align*}
z^i &= \sum_{j=1}^8 \frac{1}{8} \chi^{K_i + A_j^i} = \chi^{K_i} + \sum_{j=1}^8 \frac{1}{8} \chi^{A_j^i} = \chi^{K_i} + \frac{5}{8} \chi^{L_i}.
\end{align*}
\]

Notice that \( \sum_{i=1}^k \lambda_i \chi^{L_i} \leq \frac{1}{2} \chi^C \). Next, we claim that the vector \( z \) can be written as a convex combination of 2-edge-connected subgraphs of \( G \).

\[
\begin{align*}
z &= \sum_{k=1}^k \lambda_i z^i = \sum_{i=1}^k \lambda_i (\chi^{K_i} + \frac{5}{8} \chi^{L_i}) = \sum_{i=1}^k \lambda_i \chi^{K_i} + \frac{5}{8} \sum_{i=1}^k \lambda_i \chi^{L_i} \\
&\leq (\chi^E - \frac{1}{2} \chi^C) + \frac{5}{16} \chi^C = \chi^E - \frac{3}{16} \chi^C.
\end{align*}
\]

This concludes the proof of Lemma 3.3.

### 3.1 Coloring algorithms: Proofs of Lemmas 3.5 and 3.7

Lemma 3.5. Let \( H = (U, F) \) be an essentially 4-edge-connected graph and let \( T \) be a spanning tree of \( H \) with root \( r \). Then we can find a \((\chi^{\text{LM}(H, T)} + 3\chi^L, 5)\) coloring of \( L = F \setminus T \) in polynomial time.

Proof. Let \( \rho = \chi^{\text{LM}(H, T)} + 3\chi^L \). We show that there is \( T \)-admissible top-down \((\rho, 5)\) coloring algorithm of the links in \( L \).

Let \( \gamma \) be the \((\rho, 5)\)-coloring of \( L \) that we maintain. Initially, we have \( \gamma(\ell) = \emptyset \) for \( \ell \in L \). Suppose we want to color link \( \ell = uv \) at some iteration of the algorithm.
**Coloring Rule:** Give $\ell$ the colors in $\bar{\gamma}_2(L_\ell)$ if $u$ is a leaf in $T$. If $u$ is not a leaf, give $\bar{\gamma}_1(L_\ell)$ to $\ell$. Similarly, give $\ell$ the colors in $\bar{\gamma}_2(R_\ell)$ if $v$ is a leaf in $T$, and if $v$ is not a leaf, give $\bar{\gamma}_1(R_\ell)$ to $\ell$. If $\ell$ has fewer than three distinct colors, give it arbitrary colors that it does not already have.

We now prove that this top-down coloring algorithm is $T$-admissible. Consider an $e \in T$. If $e$ is an internal edge of $T$ (not incident on any leaf), then since the graph is essentially 4-edge-connected we have $|\text{cov}(e)| \geq 3$. Let $\ell_1, \ell_2, \ell_3$ be three of the links in $\text{cov}(e)$ with the highest LCAs. When coloring $\ell_1$, edge $e$ receives three new colors by Observation 2.9. Now consider the iteration in which the algorithm colors $\ell_i$ for some $i \in \{2, 3\}$. At the time of coloring $\ell_i$, the top-down coloring algorithm that we described above will give $\ell_i$ at least one color that an ancestor of $e$ is missing since $e$ is either in $R_{\ell_i}$ or $L_{\ell_i}$. By Observation 2.10, we can conclude that $e$ receives a new color after coloring $\ell_i$. Thus, after we have colored link $\ell_3$, edge $e$ has received at least $3 + 1 + 1 = 5$ colors.

If $e$ is incident to a leaf, then $|\text{cov}(e)| \geq 2$. Let $\ell_1, \ell_2$ be two of the links in $\text{cov}(e)$ with the highest LCAs. When coloring $\ell_1$, edge $e$ receives three new colors by Observation 2.9. When coloring $\ell_2$, receives two colors that $e$ is missing by Observation 2.10. So in total $e$ receives at least $3 + 2 = 5$ colors. This concludes the proof of $T$-admissibility of the coloring algorithm.

Finally notice that each link in $L$ receives at least three colors by construction. Moreover, if $\ell \in \text{LM}(H, T)$, then $\ell$ is colored with at most four colors. Therefore, the number of colors given to a link $\ell$ is at most $p_\ell$ as desired.

To prove Lemma 3.7, we need a different strategy to handle the leaf-matching links. In fact, there is only one case in which coloring a leaf-matching link is problematic, which we describe next. Recall that the top-down coloring algorithm colors the links in any order that respects the partial order according to their LCAs.

**Definition 3.8.** Consider $\ell \in \text{LM}(H, T)$ where $\ell = uv$. Let $\ell_u$ be the (only) other link that is incident on $u$ and $\ell_v$ be the (only) other link incident on $v$. If $\ell$ is colored after both $\ell_u$ and $\ell_v$, then we say that link $\ell$ is a bad link.

For example, suppose vertex $u$ and $v$ each have degree three in $H$. If the LCA of $\ell$ is lower than that of either $\ell_u$ or $\ell_v$, then $\ell$ is a bad link. We call such links “bad” for the following reason. Suppose For $p, q \in \mathbb{Z}_+$, suppose that we have a partial $(p \cdot \chi^L, q)$-coloring $\gamma$ of $L$ obtained during some iteration of a top-down coloring algorithm. Suppose that $u$ and $v$ each have degree three, and suppose that both $\ell_u$ and $\ell_v$ are both colored in $\gamma$, but link $\ell$ has not yet been colored. Before we color link $\ell$, the leaf edges $e_u$ and $e_v$ (a leaf edge is the unique edge in $T$ incident to a leaf) adjacent to $\ell_u$ and $\ell_v$, respectively, are each missing $q - p$ colors. If these two sets of missing colors are disjoint and $p < 2(q - p)$, then we will not be able to

---

1If $p \geq 2(q - p)$, then $p/q \geq 2/3$, which is not small enough for our applications.
color the link $\ell$ with $p$ colors so that $\ell_u$ and $\ell_v$ receive all $q$ colors.

To address this issue, consider the case in which $\ell$ is a leaf-matching ink and our algorithm colors the links $\ell_u, \ell_v, \ell$ in this order. When we color $\ell_v$, we want the respective set of $p$ colors to sufficiently overlap with the set of $p$ colors already assigned to $\ell_u$; in other words, we want the set of colors missed by $e_u$ and $e_v$ to overlap. This way, we will be able to ensure that $e_u$ and $e_v$ receive all $q$ colors when we finally color the link $\ell$ with $p$ colors. If all leaf-matching links are vertex-disjoint, then notice that $\ell_u$ and $\ell_v$ are not leaf-matching links. Furthermore, link $\ell_u$ will share an endpoint with at most one leaf-matching link, which in this case is $\ell$.

If link $uv$ is a leaf-matching link, then we say $u$ and $v$ are leaf-mates. This is the intuition behind the proof of Lemma 3.7, which we now present.

**Lemma 3.7.** Let $H = (U, F)$ be an essentially 4-edge-connected graph and let $T$ be a spanning tree of $H$ with root $r$. If the edges in $\text{LM}(H, T)$ are vertex-disjoint, then we can find a $T$-admissible $(5\chi_L, 8)$ coloring of $L = F \setminus T$ in polynomial time.

**Proof.** We introduce a top-down $(5\chi_L, 8)$-coloring algorithm of $L$, and we then prove that it is $T$-admissible.

Since this is a top-down coloring algorithm, we sort the links by the height of their LCA. When we color a link $\ell$, we give it five different colors before moving to the next link. Hence, the algorithm runs in $|L|$ iterations. After each iteration $i \in \{1, \ldots, |L|\}$ of the algorithm, we have a partial $(5\chi_L, 8)$ coloring of $L$, namely $\gamma^i$.

We show that our coloring algorithm will maintain two additional invariants:

(a) For any coloring $\gamma^i$, an edge $e$ can only miss $8, 3, 1$, or $0$ colors for $i \in \{1, \ldots, |L|\}$.

(b) If both $u$ and $v$ have degree three in $H$, and if $\ell = uv$ is a leaf-matching link for $T$, then in any coloring $\gamma^i$ for which both $e_u$ and $e_v$ are missing a color, they miss a common color in $\gamma^i$. (For leaves $u$ and $v$ in $T$, let $e_u$ and $e_v$ be the leaf edges in $T$ incident on $u$ and $v$, respectively.)

Suppose we are performing iteration $i + 1$ of the algorithm and we want to color link $\ell = uv$.

**Coloring Rules:** Depending on $u$ and $v$ we will do one of the following. We classify the root as an internal vertex.

Case 1. If both $u$ and $v$ are internal vertices in $T$, then give $\ell$ all colors in $\bar{\gamma}^i(L_\ell) \cup \bar{\gamma}^i(R_\ell)$. At this point $\ell$ will have at most four colors. Give a color that $\ell$ does not already have until it has five distinct colors.

Case 2. If $u$ is a leaf in $T$ and $v$ is an internal vertex of $T$, then we consider two cases.
Case 2a: If $\ell_{uv}$ is a bad link and link $\ell_w$ is already colored (where $\ell_{uw}$ is the link between $u$ and $w$, and $\ell_w$ be the other link incident on $w$), then we choose five colors $C'$ for $\ell$ in the following way. Let $\gamma^i(\ell_w)$ is its set of five colors already assigned to $\ell_w$. By Claim 5 we can choose five colors $C'$ such that $\gamma_1^i(\mathcal{L}_\ell) \in C'$, $\gamma_1^i(\mathcal{R}_\ell) \in C'$, $|C' \cap \gamma_3^i(\mathcal{L}_\ell)| \geq 2$, $|C' \cap \gamma_3^i(\mathcal{R}_\ell)| \geq 2$, and $|C' \cap \gamma^i(\ell_w)| \geq 3$. (Specifically, let $a = \gamma_1^i(\mathcal{L}_\ell), b = \gamma_1^i(\mathcal{R}_\ell), A = \gamma_3^i(\mathcal{L}_\ell), B = \gamma_3^i(\mathcal{R}_\ell), C_5 = \gamma^i(\ell_w)$ and $S = C'$.)

Case 2b: Otherwise (i.e., if $\ell_{uw}$ is a bad link and $\ell_w$ is not already colored, or if $\ell_{uw}$ is not a bad link), give $\ell$ the colors from $\gamma_2^i(\mathcal{R}_\ell)$ and all colors in $\gamma_3^i(\mathcal{L}_\ell)$. If $\ell$ has fewer than five distinct colors, we give it any color it does not already have until it has five distinct colors.

Case 3. If both $u$ and $v$ are leaves in $T$, then we consider two cases. Let $e_u$ and $e_v$ be the edges in the tree incident on $u$ and $v$, respectively.

Case 3a: If both $u$ and $v$ have degree three in $H$, then by invariant (b), if $e_u$ and $e_v$ are each missing at least one color, then there is a color $c$ that both $e_u$ and $e_v$ are missing. We first give color $c$ to $\ell$. Then we give colors $\gamma_3^i(\mathcal{L}_\ell) \setminus \{c\}$ and $\gamma_3^i(\mathcal{R}_\ell) \setminus \{c\}$ to $\ell$.

Case 3b: If at least one vertex has degree greater than three in $H$ (say $v$), then give colors $\gamma_3^i(\mathcal{L}_\ell)$ and $\gamma_2^i(\mathcal{R}_\ell)$ to $\ell$.

**Claim 2.** The above top-down coloring algorithm preserves invariant (a).

**Proof.** We proceed by induction on the iteration of the above top-down coloring algorithm. It is easy to see for $\gamma^0$ the invariant holds. So we assume the invariant holds before the iteration $i$ in which we color link $\ell = uv$. Consider an edge $e \in P_\ell$, and assume without loss of generality $e \in \mathcal{R}_\ell$. By the induction hypothesis, $e$ is missing 8, 3, 1 or 0 colors before coloring $\ell$. If $e$ is missing eight colors, all the colors we give to $\ell$ are new for $e$, hence after coloring $\ell$, $e$ will miss three colors. So suppose $e$ is missing three colors before we color link $\ell$. But notice in all coloring rules $\ell$ will be colored with at least two colors from $\gamma_3^i(\mathcal{R}_\ell)$. This means that after coloring $\ell$, edge $e$ will miss at most one color. So invariant (a) holds after coloring $\ell$. ◊

Next, we show that invariant (b) also holds after coloring $\ell$.

**Claim 3.** The above top-down coloring algorithm preserves invariant (b).

**Proof.** Again, we proceed by induction. We assume the invariant holds before the iteration in which we color link $\ell = uv$. If neither $u$ nor $v$ have leaf-mates, then the invariant holds after coloring link $\ell$. Thus, either (i) $\ell$ is leaf-matching or (ii) without loss of generality, $u$ is a leaf and has a leaf-mate and $v$ is an internal vertex.
Suppose $\ell$ is a leaf-matching link for $T$. If either $u$ or $v$ have degree greater than three in $H$, then the invariant holds after we color $\ell$. So assume both $u$ and $v$ have degree three in $H$. Let $e_u$ and $e_v$ be the leaf edges incident on $u$ and $v$, respectively. Also let $\ell_u$ and $\ell_v$ be the other links incident on $u$ and $v$, respectively. Since leaf-matching links for $H$ are disjoint, neither $\ell_u$ nor $\ell_v$ is leaf-matching. If $\ell$ is not a bad link, then $\ell$ is colored before either $\ell_u$ or $\ell_v$. Before we color $\ell$, either $e_u$ or $e_v$ is missing eight colors. After we color $\ell$, either $e_u$ and $e_v$ are missing the same three colors, or one is missing three colors and the other is missing zero colors. Otherwise, $\ell$ is a bad link. Now, consider the case in which $\ell$ is colored after both $\ell_u$ and $\ell_v$ have already been colored. Since both $e_u$ and $e_v$ are missing a common color, after coloring $\ell$, either $e_u$ and $e_v$ are missing zero colors.

Now consider the case in which $u$ is a leaf in $T$ and $v$ is an internal vertex of $T$. Suppose $u$ has leaf-mate $w$ adjacent to link $\ell_w$ (which is not a leaf-matching link). Moreover, we can assume that both $u$ and $w$ have degree three in $H$. If $\ell_w$ is to be colored after $\ell$, then $e_w$ is missing eight colors both before and after coloring $\ell$. Therefore, clearly there is a color that both $e_u$ and $e_w$ are missing after coloring $\ell$. Now, consider the remaining case: assume that $\ell_w$ was colored before $\ell$ in the partial coloring. Then when coloring $\ell$ the coloring rule is that of Case 2a. This rule ensures that the set of colors we give to $\ell$ has three common elements with the set of colors we gave to $\ell_w$. After coloring $\ell$, the set of the colors that $e_u$ and $e_w$ received are exactly the colors in $\ell$ and $\ell_w$, respectively. In addition $e_u$ and $e_w$ each miss exactly three colors in this partial coloring. Therefore, the set of colors $e_u$ is missing is not disjoint from the colors that $e_w$ is missing, and both $e_u$ and $e_w$ are missing a common color.

**Claim 4.** The above top-down coloring algorithm is $T$-admissible.

*Proof.* We now prove admissibility. Let $e$ be an edge in $T$. First assume $|\text{cov}(e)| \geq 3$. So there are at least three links $\ell_1$, $\ell_2$, and $\ell_3$ in $\text{cov}(e)$ labeled by their LCA ordering. When the algorithm colors $\ell_1$ since edge $e$ is missing all eight colors before coloring $\ell_1$ and all the five colors we use for $\ell_1$ are distinct, edge $e$ receives five new colors by Observation 2.9. Later, the algorithm colors $\ell_2$ and $e$ receives at least two more new colors. This is because of the following: in every case of the coloring rules, ancestors of edge $e$ receive at least two new colors. By Observation 2.10 both these colors are new for $e$. With a similar argument, when $\ell_3$ is colored, if $e$ is still missing a color, it receives its final missing color.

If on the other hand we have $|\text{cov}(e)| = 2$, edge $e$ is a leaf edge. Let $\ell_1$ and $\ell_2$ be the two links that are covering $e$ labeled by the LCA ordering. When $\ell_1$ is colored, $e$ receives five new colors since all colors are new for $e$. At the iteration that we color $\ell_2$, the algorithm either applies a rule in Case 2 or in Case 3. In both cases, three different missing colors from ancestors of $e$ are given to $\ell_2$. Hence, by Observation 2.10 edge $e$ receives the three missing colors.  

\[ \diamond \]
In order to finish the proof we just need to prove the following claim.

Claim 5. Let $C$ denote a set of eight distinct colors. Let $a, b \in C$ and let $A, B, C_5 \subset C$ such that $a \in A, b \in B$ and $|A| = |B| = 3$ and $|C_5| = 5$. Then we can find $S \subset C$ such that $|S| = 5$ and

1. $a \in S$ and $b \in S$,
2. $|S \cap A| \geq 2$,
3. $|S \cap B| \geq 2$, and
4. $|S \cap C_5| \geq 3$.

Proof. If $|A \cap B| = 0$, then observe that $|(A \cup B) \cap C_5| \geq 3$. If $|(A \cup B) \cap C_5| = 3$, then set $S = (A \cup B) \setminus c$ where $c \neq a, b$ and $c \notin C_5$. If $|(A \cup B) \cap C_5| \geq 4$, then set $S = (A \cup B) \setminus c$ where $c \neq a, b$.

If $|A \cap B| = 1$, then if $|(A \cup B) \cap C_5| \geq 3$, let $S = A \cup B$. So assume $|(A \cup B) \cap C_5| = 2$. Then $A \cup B$ contains a color $c$ such that $c \neq a, b$ and $c \notin C_5$. Let $S = (A \cup B) \setminus c$ and add an arbitrary new color from $C_5$ to $S$.

If $|A \cap B| = 2$, then if $|(A \cup B) \cap C_5| \geq 2$, let $S = A \cup B$ and add an arbitrary new color from $C_5$. If $|(A \cup B) \cap C_5| = 1$, then there is some color $c \in A \cup B$ such that $c \neq a, b$ and $c \notin C_5$. Let $S = (A \cup B) \setminus c$ and add two new colors from $C_5$ to $S$.

If $|A \cap B| = 3$, then let $c_1, c_2$ and $c_3$ be any three colors in $C_5 \setminus \{a, b\}$. Set $S = \{a, b, c_1, c_2, c_3\}$.

This concludes the proof. ☐

3.2 1-trees with few leaf-matching links: Proof of Lemma 3.6

Lemma 3.6. Let $G = (V, E)$ be a 3-edge-connected cubic graph and let $C$ be a cycle cover of $G$. The vector $\chi^E - \frac{1}{2}\chi^C$ can be written as a convex combination of 1-trees $\{K_1, \ldots, K_k\}$ of $G$. Moreover, this convex combination (i.e., $\sum_{i=1}^{k} \lambda_i K_i$ where $\lambda_i \in [0, 1]$ and $\sum_{i=1}^{k} \lambda_i = 1$) has the following two properties: (i) $\sum_{e \in \text{LM}(G, K_i)} \lambda_i \leq \frac{1}{10}$ for $e \in C$, (ii) the links in $\text{LM}(G, K_i)$ are vertex-disjoint.

Proof. Let $C_{\text{odd}}$ denote the collection of odd cycles in $C$. For each cycle $C$ in $C$, consider an arbitrary partition of the edges into adjacent pairs, leaving at most one edge $e_C$ unpaired if $C$ has odd length. Since the number of odd length cycles is even (because $G$ is a cubic graph and has an even number of vertices), we can arbitrarily pair the edges in the set $\{e_C\}_{C \in C_{\text{odd}}}$. By Observation 2.2, we have $y = \chi^E - \frac{1}{2}\chi^C \in \text{LP}(G)$ and $y(\delta(v)) = 2$ for all $v \in V$. Thus, if we apply Theorem 2.5, we find a set of 1-trees $\{K_1, \ldots, K_k\}$ such that each 1-tree uses exactly one edge from each pair. Notice that each edge in $E \setminus C$ belongs to $K_i$ for $i \in \{1, \ldots, k\}$. 

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Observe that for any such partition of edges in $C$, which pairs at most one edge $e_C$ from an odd cycle $C$ with an edge in another odd cycle, we have $\text{LM}(G, K_i) \subseteq \{e_C\}_{C \in \text{odd}}$. Since the set of edges $\{e_C\}_{C \in \text{odd}}$ are vertex disjoint, any such partition of $C$ results via Theorem 2.5 in a set of 1-trees that satisfy property (ii). Moreover, notice that if $C \in C_{\text{odd}}$ is a triangle, then $e_C$ will never be a leaf-matching link for any $K_i$.\footnote{We use Lemma 3.6 to prove Lemma 3.3 which assumes that $G$ is essentially 4-edge-connected. A cycle cover of such a graph cannot contain a triangle. However, we choose to state Lemma 3.6 using the fewest possible assumptions.}

To show property (i), for each odd cycle in $C_{\text{odd}}$ with length at least five, we choose five edges from this odd cycle and label them $e^j_C$ for each $j \in \{1, \ldots, 5\}$. For a triangle $C$ in $C_{\text{odd}}$, we choose a single edge to be $e_C$ and set $e^j_C$ to be equal to this edge for all $j \in \{1, \ldots, 5\}$. Next we construct five partitions of edges in $C$: for $j \in \{1, \ldots, 5\}$ only edges in $\{e^j_C\}_{C \in C_{\text{odd}}}$ are paired with an edge in another cycle. Then we apply Theorem 2.5 five times, one for each of the five partitions. Let $\{K^j_1, \ldots, K^j_{k_j}\}$ denote the 1-trees in the convex combination obtained for the $j^{th}$ partition. Note that if cycle $C$ has length at least five, then $e^j_C$ is leaf-matching at most half the time in this convex combination. Thus, the union of the 1-trees constructed for these five partitions satisfy properties (i) and (ii).

We remark that if the cycle cover $C$ of $G$ contains only even cycles (e.g., when $G$ is 3-edge-colorable), then the 1-trees found via Lemma 3.6 have no leaf-matching links. For such graphs, we can write $\chi^E - \frac{2}{15}\chi^C$ as a convex combination of 2-edge-connected subgraphs. This yields the following theorem, whose complete proof can be found in Appendix B.

**Theorem 3.9.** Let $G = (V, E)$ be a 3-edge-connected, cubic, 3-edge-colorable graph. Then $\frac{13}{15}\chi^E$ can be written as a convex combination of 2-edge-connected subgraphs of $G$.

### 3.3 Improved bounds for multigraphs

The ideas used to prove Theorem 3.1 can be combined with the fact that 3-edge-connected cubic graphs have a cycle cover covering all 3-edge cuts and 4-edge cuts \cite{BIT13} to improve the factor of $\frac{7}{8}$ when we are allowed to double edges.

**Theorem 3.10.** Let $G = (V, E)$ be a 3-edge-connected cubic graph. The vector $\frac{41}{47}\chi^E$ can be written as a convex combination of 2-edge-connected multigraphs of $G$.

This improves over the factor of $\frac{15}{17}$ in \cite{HNR21}. We remark that Theorem 3.10 implies that for any cost vector on a 3-edge-connected cubic graph that is optimized by the vector $\frac{2}{3}\chi^E$, there is a 1.309-approximation algorithm for 2ECS. Such cost vectors include unit costs on 3-edge-connected cubic graphs for which a 1.2-approximation algorithm is known for 2ECS \cite{BFS16}, and node-weighted costs on 3-edge-connected cubic graph for which a 1.3-approximation algorithm is known for 2ECS \cite{HNR21}.
To prove Theorem 3.10, we first apply Theorem 2.3 to obtain a cycle cover $C$ of $G$ that covers all 3- and 4-edge cuts of $G$.

Lemma 3.11. Let $G = (V, E)$ be a 3-edge-connected cubic graph and $C$ be a cycle cover of $G$ that covers 3-edge cuts and 4-edge cuts of $G$. Then $\frac{3}{5}\chi^E + \frac{2}{5}\chi^C$ can be written as a convex combination of 2-edge-connected multigraphs of $G$.

Proof. Consider graph $H = G/C$. Notice that $H$ is 5-edge-connected, which means that the vector $\frac{2}{5}\chi^E(H)$ for $H$ is in LP($H$). The polyhedral proof of Christofides algorithm implies that the vector $\frac{3}{5}\chi^E(H)$ can be written as a convex combination of 2-edge-connected multigraphs of $H$, namely $F_1, \ldots, F_k$. Notice that for $i \in \{1, \ldots, k\}$, the set of edges $C + F_i$ induces a 2-edge-connected multigraph on $G$. \hfill $\square$

Proof of Theorem 3.10. From Lemma 3.2, we have that $\chi^E - \frac{3}{16}\chi^C$ can be written as convex combination of 2-edge-connected subgraphs of $G$. Combining this with Lemma 3.11 we have

\[ \frac{32}{47}(\chi^E - \frac{3}{16}\chi^C) + \frac{15}{47}(\frac{3}{5}\chi^E + \frac{2}{5}\chi^C) = \frac{41}{47}\chi^E . \] (3.1)

4 2ECM for half-integer square points

In this section, our goal is to prove the following theorem.

Theorem 1.2. Let $x$ be a half-integer square point. Then $\frac{9}{7}x$ dominates a convex combination of 2-edge-connected multigraphs in $G_x$, the support graph of $x$. Moreover, this convex combination can be found in polynomial time.

We will use the following theorem due to Boyd and Sebő [BS21].

Theorem 4.1 ([BS21]). Let $x$ be a half-integer square point. The graph $G_x$ has a Hamiltonian cycle that contains all the 1-edges of $x$ and opposite half-edges from each half-square in $G_x$. Moreover, this Hamiltonian cycle can be found in time polynomial in the size of $G_x$.

Let $H$ be such a Hamiltonian cycle of $G_x$. For simplicity, let $A$ be the set of 1-edges of $G_x$, $B$ be the set of half-edges of $G_x$ that are in $H$, and $C$ be the half-edges of $G_x$ that are not in $H$. Thus, the incidence vector of $H$ is

$\chi^H_e = \begin{cases} 1 & \text{if } e \in A; \\ 1 & \text{if } e \in B; \\ 0 & \text{if } e \in C. \end{cases}$
In order to use $H$ as part of a convex combination in proving Theorem 1.2, we need to be able to save on edges in $B$. To this end, we introduce the following definitions.

**Definition 4.2.** For $\alpha > 0$, let $r^{\alpha,x}$ to be the vector in $\mathbb{R}^{E(G_x)}$ where

$$
 r^{\alpha,x}_e = \begin{cases}
 1 + \alpha & \text{if } e \in A; \\
 \frac{1}{2} & \text{if } e \in B; \\
 1 - \alpha & \text{if } e \in C.
\end{cases}
$$

**Definition 4.3.** Let $G = (V,E)$ be a graph. We say property $P(G,\alpha)$ holds if the vector $\alpha \chi^E$ can be written as a convex combination of matchings $\{M_1, \ldots, M_k\}$ of $G$ such that $G_i' = (V,E \setminus M_i)$ are 2-vertex-connected subgraphs of $G$.

Let $G_x$ be the support graph of a half-integer square point, and let $G = (V,E)$ be the 4-regular 4-edge-connected graph obtained from $G_x$ by replacing each path of 1-edges with a single 1-edge and contracting all of its half-squares.

**Lemma 4.4.** If $P(G,\alpha)$ holds for the graph $G$ obtained from $G_x$, then the vector $r^{\alpha,x}$ can be written as a convex combination of 2-edge-connected multigraphs of $G_x$.

Lemma 4.4 will be proved in Section 4.1. It is clear that $P(G,0)$ holds. By Lemma 4.4, the vector $r^{0,x}$ dominates a convex combination of 2-edge-connected multigraphs of $G_x$. Hence any convex combination of vectors $r^{0,x}$ and $\chi^H$ also dominates a convex combination of 2-edge-connected multigraphs. Thus, $\frac{2}{3}r^{0,x} + \frac{1}{3}\chi^H$ dominates a convex combination of 2-edge-connected multigraphs of $G_x$. We have $\frac{2}{3}r^{0,x} + \frac{1}{3}\chi^H \leq \frac{4}{3}x$. To go beyond $\frac{4}{3}$, we need to use the half-edges less and therefore we need to account for this by sometimes doubling 1-edges. The property $P(G,\alpha)$ will allow us to double all the 1-edges in $G_x$ that belong to a particular matching in $G$ (i.e., an $\alpha$-fraction of the 1-edges). In this section, our main goal is to prove the following lemma.

**Theorem 4.5.** For any 4-regular, 4-edge-connected graph $G$, $P(G, \frac{1}{10})$ holds.

By Lemma 4.4, we have the following corollary.

**Corollary 4.6.** For a half-integer square point $x$, the vector $r^{\frac{1}{10},x}$ dominates a convex combination of 2-edge-connected multigraphs of $G_x$ and this convex combination can be found in time polynomial in the size of $G_x$.

From Corollary 4.6, the proof of Theorem 1.2 is easy. Obviously any convex combination of $r^{\frac{1}{10},x}$ and $\chi^H$ also dominates a convex combination of 2-edge-connected multigraphs of $G_x$. Observe that $G$ is Eulerian and is therefore 4-edge-connected since the corresponding Boyd-Carr point is 3-edge-connected.
Consider the combination $\frac{5}{7}r^{x} + \frac{2}{7}H$. It is easy to see this convex combination is dominated by $\frac{9}{10}x$.

It remains to prove Lemma 4.4 and Theorem 4.5. We will prove Lemma 4.4 in Section 4.1 where we describe how to construct the convex combination. Regarding Theorem 4.5, note that $P(G, \frac{1}{10})$ is equivalent to saying that the vector $\frac{9}{10}x$ can be written as a convex combination of 2-vertex-connected subgraphs of minimum degree three. This equivalent statement will be proved using Lemma 4.7.

**Lemma 4.7.** Let $G$ be a 4-regular 4-edge-connected graph. Let $T$ be a spanning tree of $G$ such that $T$ does not have any vertex of degree four. The vector $y \in \mathbb{R}^{G}$, where $y_{e} = \frac{4}{5}$ for $e \notin T$ and $y_{e} = 0$ for $e \in T$, dominates a convex combination of edge sets $\{F_{1}, \ldots, F_{k}\}$ such that $T + F_{i}$ is a 2-vertex-connected subgraph of $G$ where each vertex has degree at least three in $T + F_{i}$ for $i \in \{1, \ldots, k\}$.

The proof of Lemma 4.7 can be found in Section 4.2. In order to prove this lemma, we need a way to reduce vertex connectivity to edge-connectivity, which is done in Section 4.2.1. The main tool in the proof of Lemma 4.7 is a top-down coloring algorithm, which is detailed in Section 4.2.2. From Lemma 4.7, one can easily prove Theorem 4.5.

**Proof of Theorem 4.5.** Consider a half-integer square point $x$. Let $G = (V, E)$ be the graph obtained from $G_{x}$ by replacing each path of 1-edges with a single 1-edge and contracting all the half-squares in $G_{x}$. Graph $G$ is 4-regular and 4-edge-connected, hence $G$ has two edge-disjoint spanning trees $T_{1}$ and $T_{2}$ [NW61]. Notice that $T_{1}$ and $T_{2}$ cannot have any vertex of degree four, since for all vertices $v \in V$, we have $\delta_{T_{1}}(v) \geq 1$ and $\delta_{T_{2}}(v) \geq 1$ while $\delta_{T_{1}}(v) + \delta_{T_{2}}(v) \leq 4$. Hence, by Lemma 4.7 we can write vector $z^{i} \in \mathbb{R}^{G}$, with $z^{i}_{e} = 1$ for $e \in T_{i}$, and $z^{i}_{e} = \frac{4}{5}$ for $e \notin T_{i}$ as a convex combination of 2-vertex-connected subgraphs of $G$ where every vertex has degree at least three, for $i \in \{1, 2\}$. Now consider $\frac{1}{2} \cdot z^{1} + \frac{1}{2} \cdot z^{2}$: it dominates a convex combination of 2-vertex-connected subgraphs of $G$ where every vertex has degree at least three. Also, $\frac{1}{2} \cdot z^{1} + \frac{1}{2} \cdot z^{2}$ is the vector $\frac{9}{10}x$. This concludes the proof, since the complement of the solutions in the convex combination form the desired convex combination of matchings.

### 4.1 From matching to 2ECM: Proof of Lemma 4.4

Recall that $G_{x}$ is the support graph of a half-integer square point $x$, and $G = (V, E)$ is the 4-regular 4-edge-connected graph obtained from $G_{x}$ by replacing each path of 1-edges with a single 1-edge and contracting all of its half-squares. The definition of vector $r^{x}$ can be found in Definition 4.2 and the definition of edge sets $A, B$ and $C$ can be found directly before.

**Lemma 4.4.** If $P(G, \alpha)$ holds for the graph $G$ obtained from $G_{x}$, then the vector $r^{x}$ can be written as a convex combination of 2-edge-connected multigraphs of $G_{x}$.
Additionally, we assign an arbitrary orientation to each edge so that we will have at least one half-edge in \( F_i \) that is incident on \( v \) for each \( i \in \{1, \ldots, k\} \). Specifically, for each \( i \in \{1, \ldots, k\} \), we create two 2-edge-connected multigraphs \( F^i_1 \) and \( F^i_2 \), as follows. Notice that each edge in \( M_i \) corresponds to a 1-edge (an edge in \( A \)) in \( G_x \). For each \( e \in M_i \) we add two copies of the 1-edge corresponding to \( e \) in \( G_x \) to \( F^i_1 \) and \( F^i_2 \). For each \( e \notin M_i \) we add one copy of the 1-edge corresponding to \( e \) in \( G_x \) to \( F^i_1 \) and \( F^i_2 \). Additionally, we assign an arbitrary orientation to each edge \( e \in M_i \). For each edge \( e \in M_i \), there are two squares \( Q_1 \) and \( Q_2 \) incident on \( e \). We say \( e \to Q_1 \) and \( e \leftarrow Q_2 \) if \( e \) is oriented from the endpoint in \( Q_2 \) towards the endpoint in \( Q_1 \).

Consider a half-square \( Q \) with vertices \( u_1, u_2, u_3 \) and \( u_4 \) in \( G_x \). There are four 1-edges incident on \( Q \), namely \( f_j \) for \( j \in \{1, \ldots, 4\} \), where \( f_j \) is incident to \( u_j \). Since \( M_i \) is a matching in \( G \), at most one of \( \{f_1, f_2, f_3, f_4\} \) belongs to \( M_i \). If one of \( \{f_1, \ldots, f_4\} \) is in \( M_i \) we can assume without loss of generality that \( f_1 \in M_i \). If \( f_1 \to Q \), then we add to \( F^i_1 \) the two half-edges in \( Q \) that are not incident on \( u_1 \). If \( f_1 \leftarrow Q \), then we add to \( F^i_1 \) the two half-edges in \( Q \) that are incident to \( u_1 \) together with the other half-edge in \( Q \cap C \). For \( F^i_2 \) we do the opposite: If \( f_1 \leftarrow Q \), then we add to \( F^i_2 \) the two half-edges in \( Q \) that do not have as endpoint \( u_1 \), and if \( f_1 \to Q \), then we add to \( F^i_2 \) the two half-edges in \( Q \) that are not incident to \( u_1 \) together with the other half-edge in \( Q \cap C \). See Figure 1 for an illustration. If none of \( \{f_1, \ldots, f_4\} \) belong to \( M_i \), we add both edges in \( C \cap Q \) to \( F^i_1 \) and \( F^i_2 \). We also arbitrarily choose an edge in \( Q \cap B \) to add to \( F^i_1 \) and add the other edge in \( Q \cap B \) to \( F^i_2 \).

![Figure 1](image.png)

Figure 1: Solid edges belong to \( B \) and dashed edges belong to \( C \). The directed edge belongs to the matching. Thick edges represent those half-edges that are added to \( F^i_1 \) and \( F^i_2 \), respectively.

We conclude this proof with the following two key claims.

**Claim 6.** The graphs induced on \( G_x \) by edge sets \( F^i_1 \) and \( F^i_2 \) are 2-edge-connected multigraphs of \( G_x \) for \( i \in \{1, \ldots, k\} \).

**Proof.** Since the construction of \( F^i_1 \) and \( F^i_2 \) are symmetric, it is enough to show this only for \( F^i_1 \). First notice that for every vertex \( v \in G_x \), we have \( |F^i_1 \cap \delta(v)| \geq 2 \). Let \( e \) be the 1-edge incident on \( v \). If \( e \in M_i \), then we have two copies of \( e \) in \( F^i_1 \) so we are done. If \( e \notin M_i \), then \( F^i_1 \) contains only one copy of \( e \). However, by construction, in the half-square that contains \( v \), we will have at least one half-edge in \( F^i_1 \) that is incident to \( v \).
We proceed by showing that for every set of edges \( D \) in \( G_x \) that forms a cut (i.e., whose removal disconnects the graph \( G_x \)), we have \(|D \cap F_1| \geq 2\). Clearly, if \( D \) contains two or more 1-edges, since \( F_1 \) contains all the 1-edges, we have \(|D \cap F_1| \geq 2\). So assume \(|D \cap A| = 1\); \( D \) contains exactly one 1-edge \( e \) of \( G_x \). If \( e \in M_i \), we are done as the matching will take two copies of \( e \). Thus, we may assume \( e \notin M_i \). Notice that for any edge cut \( D \), \( D \) contains either zero or two edges from every half-square. Hence, we can pair up the half-edges in \( D \). Let \( e_1, \ldots, e_n, f_1, \ldots, f_m \) and \( e'_1, \ldots, e'_n, f'_1, \ldots, f'_m \) be the half-edges in \( D \) such that \( e_j \) and \( e'_j \) belong to the same half-square and are opposite edges, and \( f_j \) and \( f'_j \) belong to the same half-square and share an endpoint. Notice that while we can have \( m = 0 \) or \( n = 0 \), it must be the case that \( n + m \geq 1 \), since \( G_x \) is 2-edge-connected and hence \( D \) must contain two edges from at least one half-square. Note that \( D \cap F_1 \) contains edge \( e \). For a contradiction, suppose that \(|D \cap F_1| = 1\). In this case, we must have \( n = 0 \) since in our construction we take at least one half-edge from every pair of opposite half-edges. (In other words, if \( n \geq 1 \), then \( D \) and \( F_1 \) must have at least one half-edge in common.) For \( j \in \{1, \ldots, m\} \), let \( u_j \) be the endpoint that \( f_j \) and \( f'_j \) share and let \( g_j \) be the 1-edge incident to \( u_j \). Notice that \( D' = e \cup \bigcup_{j=1}^{m} g_j \) forms a cut in \( G_x \) that contains \( e \). Thus, \( D' \) is also a cut in \( G \). This implies that there is an edge \( g_j \) for some \( j \in \{1, \ldots, m\} \) such that \( g_j \notin M_i \). Otherwise, \( e \) is the unique edge of cut \( D' \) that is not in \( M_i \). This means that \( G'_i = (V, E \setminus M_i) \) has a cut with only one edge, which implies that it is not 2-vertex-connected. Since \( g_j \notin M_i \), by construction \( F_i \) contains an edge in the half-square that contains \( u_j \). This implies that \(|F_i \cap \{f_j, f'_j\}| \geq 1\), which is a contradiction to the assumption that \(|D \cap F_1| = 1\). (See Figure 2.)

![Figure 2: Edges in the cuts D and D'.](image_url)

Finally, assume that \( D \) does not contain any 1-edges. In this case, let \( e_1, \ldots, e_n, f_1, \ldots, f_m \) and \( e'_1, \ldots, e'_n, f'_1, \ldots, f'_m \) be the half-edges in \( D \) such that \( e_j \) and \( e'_j \) belong to the same half-square and are opposite edges, and \( f_j \) and \( f'_j \) belong to the same half-square and share one endpoint. Notice that we can have \( m = 0 \) or \( n = 0 \) but \( n + m \geq 2 \), because \( D \) must contain edges from at least two half-squares (since \( G_x \) is 2-vertex connected). For \( j \in \{1, \ldots, m\} \) let \( u_j \) be the endpoint that \( f_j \) and \( f'_j \) share and \( g_j \) be the 1-edge incident on \( u_j \). If \( n = 0 \), then \( D' = \bigcup_{j=1}^{m} g_j \) forms a cut in \( G \). Hence, there are two edges \( g_j \) and \( g_j' \) such that \( g_j, g_j' \notin M_i \). This implies that \(|F_i \cap \{f_j, f'_j\}| \geq 1\), and \(|F_i \cap \{f_j, f'_j\}| \geq 1\). Therefore, \(|D \cap F_i| \geq 2\). If \( n = 2 \), then by construction \(|F_i \cap \{e_1, e'_1\}| \geq 1\), and \(|F_i \cap \{e_2, e'_2\}| \geq 1\), so we have the result. It only remains to consider the case when \( n = 1 \). Notice as before we have \(|F_i \cap \{e_1, e'_1\}| \geq 1\).
If there is $g_j$ for some $j \in \{1, \ldots, m\}$ such that $g_j \notin M_i$, then we have $|F_i^1 \cap \{f_j, f'_j\}| \geq 1$ in which case we are done. Thus, we may assume $g_j \in M_i$. Let $Q$ be the half-square that contains $e_1$ and $e'_1$. In $G'_i = (V, E \setminus M_i)$ the vertex corresponding to $Q$ will be a cut vertex, which is a contradiction.

\[ \square \]

Now we conclude the proof by proving the second and last claim.

**Claim 7.** Let $r = \sum_{i=1}^{k} \frac{\lambda_i}{2} \chi^{F_i^1} + \sum_{i=1}^{k} \frac{\lambda_i}{2} \chi^{F_i^2}$. We have $r_e = 1 + \alpha$ for $e \in A$, $r_e = \frac{1}{2}$ for $e \in B$, and $r_e = 1 - \alpha$ for $e \in C$, i.e. $r = r^x, \alpha$.

**Proof.** Let $e \in A$ (a 1-edge in $G_x$). We have \( \sum_{i \in [k]} e \in M_i \lambda_i = \alpha \). Therefore,

\[
\sum_{i=1}^{k} \frac{\lambda_i}{2} \chi^{F_i^1} + \sum_{i=1}^{k} \frac{\lambda_i}{2} \chi^{F_i^2} = \sum_{i \in [k]} e \in M_i \frac{2\lambda_i}{2} + \sum_{i \in [k]} e \notin M_i \frac{\lambda_i}{2} = \sum_{i \in [k]} e \notin M_i \frac{\lambda_i}{2} = 1 + \alpha.
\]

Now consider a half-edge $e \in B$. Let $f$ and $g$ be the 1-edges incident on the endpoints of $e$. If $f \in M_i$ and $f$ is incoming to $e$, then $e \notin F_i^1$ and $e \in F_i^2$, otherwise if $f \in M_i$ and $f$ is outgoing of $e$, then $e \in F_i^1$ and $e \notin F_i^2$. This means that if $f \in M_i$, then $\frac{\lambda_i}{2} \chi^{F_i^1} + \frac{\lambda_i}{2} \chi^{F_i^2} = \frac{\lambda_i}{2}$. Similarly, if $g \in M_i$, we have $\frac{\lambda_i}{2} \chi^{F_i^1} + \frac{\lambda_i}{2} \chi^{F_i^2} = \frac{\lambda_i}{2}$. Notice that if $f \in M_i$, then $g \notin M_i$, since in $G$, edges $f$ and $g$ share an endpoint and $M_i$ is a matching.

Now, assume $f, g \notin M_i$. Let $f', g'$ be the other 1-edges incident on the square $Q$ that contains $e$. If $f' \in M_i$, then if $f'$ is incoming to $Q$, then $e \in F_i^1$ and $e \notin F_i^2$. If $f'$ is outgoing from $Q$, then $e \notin F_i^1$ and $e \in F_i^2$. In both case, $\frac{\lambda_i}{2} \chi^{F_i^1} + \frac{\lambda_i}{2} \chi^{F_i^2} = \frac{\lambda_i}{2}$. Similarly, if $g' \in M_i$. If $f, g, f', g' \notin M_i$, then exactly one of $F_i^1$ and $F_i^2$ will contain $e$. Hence, $\frac{\lambda_i}{2} \chi^{F_i^1} + \frac{\lambda_i}{2} \chi^{F_i^2} = \frac{\lambda_i}{2}$.

We have,

\[
\sum_{i=1}^{k} \frac{\lambda_i}{2} \chi^{F_i^1} + \sum_{i=1}^{k} \frac{\lambda_i}{2} \chi^{F_i^2} = \sum_{i=1}^{k} \frac{\lambda_i}{2} = 1/2.
\]

Now consider edge $e \in C$. Let $Q$ be the square in $G_x$ that contains $e$. Let $f, g, f', g'$ be the 1-edges incident on $Q$ such that $f, g$ are the 1-edges that are incident on the endpoints of $e$. If $f \in M_i$ and $f$ is incoming to $Q$, then $e \notin F_i^1$. Also, if $g \in M_i$ and $g$ is incoming to $Q$, then $e \notin F_i^1$. In all other cases $e \in F_i^1$. Similarly, if $f \in M_i$ and $f$ is outgoing from $Q$, then $e \notin F_i^2$. 
Also, if \( g \in M_i \) and \( g \) is outgoing from \( Q \), then \( g \not\in F_i^2 \). In all other case \( e \in F_i^1 \). We conclude

\[
\sum_{i=1}^{k} \frac{\lambda_i}{2} e_i^1 + \sum_{i=1}^{k} \frac{\lambda_i}{2} e_i^2 = \frac{1}{2} - \sum_{i \in k: f \in M_i, f \rightarrow Q} \frac{\lambda_i}{2} - \sum_{i \in k: g \in M_i, g \rightarrow Q} \frac{\lambda_i}{2} + \frac{1}{2} - \sum_{i \in k: f \in M_i, f \leftarrow Q} \frac{\lambda_i}{2} - \sum_{i \in k: g \in M_i, g \leftarrow Q} \frac{\lambda_i}{2} = 1 - \sum_{i \in k: f \in M_i} \frac{\lambda_i}{2} - \sum_{i \in k: g \in M_i} \frac{\lambda_i}{2} = 1 - \alpha.
\]

This concludes the proof.

\[\Diamond\]

### 4.2 A top-down coloring approach: Proof of Lemma 4.7

In this section we prove Lemma 4.7.

**Lemma 4.7.** Let \( G \) be a 4-regular 4-edge-connected graph. Let \( T \) be a spanning tree of \( G \) such that \( T \) does not have any vertex of degree four. The vector \( y \in \mathbb{R}^G \), where \( y_e = \frac{4}{5} \) for \( e \not\in T \) and \( y_e = 0 \) for \( e \in T \), dominates a convex combination of edge sets \( \{F_1, \ldots, F_k\} \) such that \( T + F_i \) is a 2-vertex-connected subgraph of \( G \) where each vertex has degree at least three in \( T + F_i \) for \( i \in \{1, \ldots, k\} \).

In order to prove this lemma, we need a way to reduce vertex connectivity to edge-connectivity to be able to employ the top-down coloring approach.

#### 4.2.1 Reducing 2-vertex connectivity to 2-edge connectivity

We now present an approach to reduce vertex connectivity to edge-connectivity. Let \( G = (V, E) \) be a 4-regular 4-edge-connected graph. Note that \( G \) must be 2-vertex-connected. Let \( T \) be a spanning tree of \( G \) such that \( T \) does not have any vertices of degree four and let \( L = E \setminus T \) be the set of links.

For a link \( \ell \) in \( L \), let \( P_\ell \) be the set of edges in \( T \) on the unique path in \( T \) between the endpoints of \( \ell \). For \( e \in T \), let \( \text{cov}(e) \) be the set of links \( \ell \) such that \( e \in P_\ell \). Since \( G \) is 4-edge-connected, \( |\text{cov}(e)| \geq 3 \) for all \( e \in T \).

**Definition 4.8.** The subdivided graph \( G' = (V', E') \) of \( G \) is the graph in which each edge \( e = uw \) of \( T \) is subdivided into \( u_e \) and \( v_e, w \). Then \( T' \) is a spanning tree of \( G' \) in which for each edge \( uw \in T \), we include both \( u_e \) and \( v_e, w \) in \( T' \). We define \( L' = E' \setminus T' \) as follows. For each link \( \ell \in L \), we make a link \( \ell' \in L' \) as follows. Let \( u \) be an endpoint of \( \ell \).
1. If \( u \) is a leaf of \( T \), then \( u \) is an endpoint of \( \ell' \).

2. If \( u \) is an internal vertex, let \( e \) be the edge in \( P_\ell \) such that \( u \) is also an endpoint of \( e \).
   (Note that there is only one such \( e \), since \( P_\ell \) is a unique path and \( e \) is the first, or last, edge in \( P_\ell \).) Then \( v_e \) is the endpoint of \( \ell' \).

The procedure outlined in Definition 4.8 defines a bijection between links in \( L \) and \( L' \). Thus, for every set of links \( F' \subseteq L' \), we let \( F \subseteq L \) denote the corresponding set of links. We use this bijection to go from 2-edge-connectivity to 2-vertex-connectivity.

**Lemma 4.9.** Let \( G = (V, E) \) be a 4-regular 4-edge-connected graph, let \( T \) be a spanning tree of \( G \) with maximum degree three, and let \( L = E \setminus T \). Let \( G' = (V', E') \) be a subdivided graph with spanning tree \( T' \) and links \( L' = E' \setminus T' \). We have

- For any \( F' \subseteq L' \) such that \( T' + F' \) is 2-edge-connected, \( T + F \) is 2-vertex-connected.
- For every edge \( e' \in T' \), there are at least two links \( \ell'_1, \ell'_2 \in L' \) such that \( \ell'_1, \ell'_2 \in \text{cov}(e') \).

**Proof.** Let us show that this reduction satisfies the first property. Suppose for contradiction that there is \( F' \subseteq L' \) such that \( T' + F' \) is 2-edge-connected, but the corresponding set of links \( F \), is such that \( T + F \) has a cut-vertex, namely \( u \). Clearly \( u \) cannot be a leaf of \( T \), since \( T - u \) is a connected graph. Similarly, \( r \neq u \). Hence, we can assume that \( u \) is an internal vertex of \( T \).

Since \( u \) is a cut-vertex of \( T + F \), we can partition \( V \setminus \{u\} \) into \( S_1 \) and \( S_2 \) such that there is no edge in \( T + F - \delta(u) \) that has one endpoint in \( S_1 \) and one endpoint in \( S_2 \). Let \( \delta_T(u) \) be the set of edges in \( T \) incident on \( u \). Since \( u \) is an internal vertex of \( T \), we have \( 2 \leq |\delta_T(u)| \leq 3 \). Suppose \( u \) is adjacent to \( v \) in \( T \). Label the \( vu \) edge in \( T \) with \( e \). Assume first that \( |\delta_T(u)| = 2 \): let \( f \) be the other edge incident to \( u \) in \( T \). There is no link \( \ell' \in F' \) such that \( \ell' \) covers the edge \( uv_f \), because such a link \( \ell' \) corresponds to a link in \( \ell \in L \) that has one endpoint in \( S_1 \) and other in \( S_2 \). Now, assume \( |\delta_T(u)| = 3 \): let \( f_1 \) and \( f_2 \) be the edges incident to \( u \) (besides \( e \)) in \( T \). Let \( w_1 \) and \( w_2 \) be the endpoints of \( f_1 \) and \( f_2 \) other than \( u \). Again, let \( S_1 \) and \( S_2 \) be a partition of \( V \setminus \{u\} \) such that no edge in \( T + F - \delta(u) \) that has one end in \( S_1 \) and other in \( S_2 \). Without loss of generality, assume \( v \in S_1 \) and \( w_1, w_2 \in S_2 \). Consider edge \( v_ee \) in \( T' \): if there is a link \( \ell' \in L' \) covering \( v_ee \), then the link \( \ell \) corresponding to \( \ell' \) has one end in \( S_1 \) and the other in \( S_2 \). Hence, we get a contradiction.

Now we show the second property holds: for each edge \( e' \in T' \), there are at least two links \( \ell, \ell' \in L' \) that are in \( \text{cov}(e') \). Suppose there is an edge \( e' \) such that \( e' \) does not have this property. Edge \( e' \) corresponds to one part of a subdivided edge \( e \) in the tree \( T \). Let \( v \) and \( v_e \) be the endpoints of \( e' \).
First, notice that if \( v \) is a leaf, then there are three links in \( \ell \) that cover edge \( e \) in \( T \), and all these links will cover \( e' \) in the new instance as we do not change the leaf endpoints. Thus we may assume that \( v \) is not a leaf.

If \( v \) has degree two in \( T \), then let edge \( f \) be the other edge incident to \( v \), as shown in Figure 3a. Let \( \ell \) be a link in \( L \) such that \( e \) and \( f \) are both covered by \( \ell \). If \( \ell' \in L' \) is the link corresponding to \( \ell \), then \( \ell' \) covers \( e' \). Hence we can suppose there is at most one link \( \ell_1, \ldots, \ell_4 \) such that \( \ell_1, \ell_2 \) cover \( f \) and \( \ell_3, \ell_4 \) cover \( e \). But then vertex \( v \) has degree six in \( G \) as every link that covers \( e \) and does not cover \( f \) or vice versa must have \( v \) as an endpoint. Thus, we may assume that \( v \) has degree three in \( T \), which means \( v \) is incident to edges \( e, f \) and \( g \) in \( T \), as shown in Figure 3b.

![Figure 3: Different cases for the second property in Lemma 4.9.](image)

4.2.2 The top-down coloring algorithm

We want to find a set of links \( F' \subset L' \) such that i) \( T' + F' \) is 2-edge-connected, and ii) each vertex in \( T + F \) has degree at least three. Now we expand our terminology for a top-down coloring algorithm to address these additional requirements. For each \( \ell' \in L' \), where \( \ell \) is the link in \( L \) corresponding to \( \ell' \), we define end(\( \ell' \)) to be the two endpoints of \( \ell \) in \( G \).

For a \((\rho,q)\)-coloring of \( L' \), namely \( \gamma \), we say that \( v \) in \( G \) has received a color \( c \) in \( \gamma \) if there is \( \ell' \) such that \( v \in \text{end}(\ell') \) and \( c \in \gamma(\ell') \). We say a vertex \( v \) received a color \( c \) twice in \( \gamma \), if there are two links \( \ell' \) and \( \ell'' \) such that \( v \in \text{end}(\ell') \) and \( v \in \text{end}(\ell'') \) and both \( c \in \gamma(\ell') \) and \( c \in \gamma(\ell'') \). Similarly, we say \( v \) is missing color \( c \) in \( \gamma \), if there is no link \( \ell' \) such that \( v \in \text{end}(\ell') \) with \( c \in \gamma(\ell') \). Moreover, we say \( v \) is missing a color \( c \) for the second time in \( \gamma \), if there is exactly one link \( \ell' \) with \( v \in \text{end}(\ell') \) with \( c \in \gamma(\ell') \).

**Lemma 4.10.** Let \( G = (V, E) \) be a 4-regular 4-edge-connected graph and let \( T \) be a spanning tree of \( G \) with maximum degree three. Let \( G' = (V', E') \) and \( T' \) be the subdivided graph and
spanning tree. Then there is $T'$-admissible $(4\chi^L, 5)$-coloring of $L' = E' \setminus T'$, namely $\gamma$ such that for a vertex $v$ of $G$, i) if $v$ has degree two in $T$, then $v$ receives all the five colors in $\gamma$, and ii) if $v$ is a degree one vertex in $T$, then $v$ receives all the five colors twice in $\gamma$.

Proof. We construct $\gamma$ using the top-down coloring algorithm.

Let $\gamma$ be the $(4\chi^L, 5)$-coloring of $L'$ that we maintain. Initially, we have $\gamma(\ell') = \emptyset$ for $\ell' \in L'$. Suppose we want to color link $\ell'$ at some iteration of the algorithm. Let $u', v'$ be the endpoints of $\ell'$ in $G'$. Let $s'$ be the LCA of $\ell'$ in $T'$. Let $\mathcal{L}_{\ell'}$ be the $s'u'$-path in $T'$ and $\mathcal{R}_{\ell'}$ be the $s'v'$-path in $T'$. Let $\text{end}(\ell') = \{u, v\}$.

Coloring Rules:

1. If there is a color $c$ that $u$ has not received we set one color on $\ell'$ to be $c$. If $u$ is not missing a color, but missing a color $c$ for the second time, give color $c$ to $\ell'$.

2. If there is a color $c$ that $v$ has not received we set one color on $\ell'$ to be $c$. If $v$ is not missing a color, but missing a color $c$ for the second time, give color $c$ to $\ell'$.

3. Give color $\tilde{\gamma}_1(\mathcal{L}_{\ell'})$ to $\ell'$. If there is no such color and vertex $u$ is missing a color $c$ for the second time, give color $c$ to $\ell'$.

4. Give color $\tilde{\gamma}_1(\mathcal{R}_{\ell'})$ to $\ell'$. If there is no such color and vertex $v$ is missing a color $c$ for the second time, give color $c$ to $\ell'$.

5. If after applying all the above four rules, $\ell'$ still has fewer than four distinct colors, give $\ell'$ any color that it does not already have until $\ell'$ has four different colors.

First we show that the top-down coloring algorithm above is $T'$-admissible. Consider an edge $e'$ in $T'$. We know by Lemma 4.9 that there are links $\ell'$ and $\ell''$ in $L'$ such that $\ell', \ell'' \in \text{cov}(e')$. Without loss of generality, suppose that $\ell'$ has a higher LCA. After we color $\ell'$, $e'$ has received at least four colors (Observation 2.9). When we color $\ell''$ (using Rule 3 or 4) we give at least one new color to $e'$ so it receives all the five colors (Observation 2.10). Therefore, the coloring algorithm is $T'$-admissible.

Now, we show the extra properties hold as well. Consider a vertex $v$ of degree two in $T$. Notice that since $G$ is 4-regular, there are at least two links $\ell'$ and $\ell''$ such that $v \in \text{end}(\ell')$ and $v \in \text{end}(\ell'')$. At the iteration the algorithm colors $\ell'$, vertex $v$ receives four new colors, and later when the algorithm color $\ell''$, vertex $v$ receives its fifth missing color (by Rule 1 or 2).

Finally, assume $v$ is a vertex of degree one in $T$. This implies that $v'$ is also a degree one vertex in $T'$ (since in the reduction we do not change the endpoints for degree one vertices). Let $e'_{v'}$ be the leaf edge in $T'$ incident on $v'$. By 4-regularity there are three links $\ell'_1, \ell'_2, \ell'_3$ labeled in LCA order such that $v \in \text{end}(\ell'_i)$ for $i \in \{1, 2, 3\}$. In the iteration that $\ell'_1$ is colored,
Lemma 4.7 follows immediately from Lemmas 4.9 and 4.10.

4.3 Hard to round half-integer square points

As discussed in the introduction, \( \alpha_{\text{LP}}^{2\text{EC}} \geq \frac{6}{5} \). An example achieving this lower bound is given in [ABE06]. However, a more curious instance is the \( k \)-donut. A \( k \)-donut point for \( k \in \mathbb{Z}, k \geq 2 \), is a graph \( G_k = (V_k, E_k) \) that has \( k \) half-squares arranged around a cycle, and the squares are joined by paths consisting of \( k \) 1-edges. (See Figure 4 for an illustration of the 4-donut.)

Define the edge cost \( c_e \) of each half-edge in the outer cycle and the inner cycle to be 2. All other half-edges have cost 1. All the 1-edges have cost \( \frac{1}{k} \). Therefore, \( \sum_{e \in E} c_e x_e = 5k \), while the optimal solution is \( 6k - 2 \). We note that this instance was discovered by the authors of [CR98], but due to the page limit of their conference paper they did not present it and just mentioned that they know a lower bound. Recently, Boyd and Sebő used \( k \)-donut points with different costs to show a new instance that achieves a lower bound of \( \frac{4}{3} \) for the integrality gap of 2ECM and TSP, and we attribute the term “\( k \)-donut” to them [BS21]. Notice that if \( x \) is the \( k \)-donut point, then \( P(G, \frac{1}{4} - \frac{1}{2k}) \) holds. This implies that \( z = \frac{1}{5}x^H + \frac{4}{5}r^{\frac{1}{2}} - \frac{1}{2\kappa} \) is a convex combination of 2-edge-connected multigraphs of \( G_x \). We have \( z_e = \frac{9}{5} - \frac{2}{3k} \) for \( e \in A \).
\[ z_e = \frac{3}{5} \text{ for } e \in B, \text{ and } z_e = \frac{3}{5} + \frac{2}{5k} \text{ for } e \in C. \] As \( k \to \infty \), this approaches \( \frac{6}{5}x \) and thus shows that our approach can verify the six-fifths conjecture for \( k \)-donut points. We conclude with the following corollary of Theorem 1.2.

**Corollary 4.11.** The integrality gap \( \alpha_{2EC}^{LP(G)} \) is between \( \frac{6}{5} \) and \( \frac{9}{7} \) for half-integer square points.

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A Proof of Theorem 2.4

The main tool used to prove Theorem 2.4 is *gluing solutions over 3-edge cuts* [CR98], which was introduced by Carr and Ravi [CR98] and has been used frequently for constructing...
convex combinations of 2-edge-connected subgraphs \cite{CR98, BL17, Leg17} and more recently for gluing tours (see Proposition 3.2 in \cite{HN20}).

We need the following definitions.

**Definition A.1.** A proper 3-edge cut of $G$ is a set $S \subset V$ such that $|\delta(S)| = 3$, $|S| \geq 2$ and $|V \setminus S| \geq 2$.

**Definition A.2.** For a 3-edge-connected graph $G = (V, E)$ and a proper 3-edge cut $S \subset V$, we define the graph $G_S$ to be the graph obtained after we contract the set $V \setminus S$ to a single vertex.

Lemmas A.3 and A.7 appear in different forms in \cite{CR98, BL17, Leg17}, always with the purpose of reducing to the problem on essentially 4-edge-connected graphs.

**Lemma A.3.** Let $G = (V, E)$ be a 3-edge-connected graph and $x \in [0, 1]^E$. Let $S$ be a 3-edge cut of $G$. Define $x^S$ and $\bar{x}^S$ to be vector $x$ restricted to the edges in $G_S$ and $G_\bar{S}$, respectively. If $x^S$ and $\bar{x}^S$ can be written as a convex combination of 2-edge-connected subgraphs of $G_S$ and $G_\bar{S}$, respectively, then $x$ can be written as convex combination of 2-edge-connected subgraphs of $G$.

**Proof.** By the assumption, vector $x^S$ can be written as a convex combination of 2-edge-connected subgraphs of $G_S$: $x^S_e = \sum_{i=1}^k \theta_i \chi^F_{e_i}$ for $e \in E(G_S)$. The same holds for $G_\bar{S}$: $\bar{x}^S_e = \sum_{i=1}^k \theta_i \chi^F_{\bar{e}_i}$ for $e \in E(G_\bar{S})$.

Note that $\delta(X_S) = \delta(X_\bar{S}) = \{a, b, c\}$, and hence $x^S = \bar{x}^S = x_e$ for $e \in \{a, b, c\}$. Let $\lambda_{a,b}$ be the sum of all $\lambda_i$'s where $F^{ij}_i$ contains exactly the two edges $a$ and $b$ from $\delta(X_S)$. Define $\lambda_{a,c}, \lambda_{b,c}$, and $\lambda_{a,b,c}$ analogously. Notice that these are the only possible outcomes since a 2-edge-connected subgraphs contains at least two edges from the cut around any vertex. Hence, $\lambda_{a,b} + \lambda_{a,c} + \lambda_{b,c} + \lambda_{a,b,c} = 1$. Also

\[
\begin{align*}
\lambda_{a,b} + \lambda_{a,c} + \lambda_{a,b,c} &= x_a, \\
\lambda_{a,b} + \lambda_{b,c} + \lambda_{a,b,c} &= x_b, \\
\lambda_{a,c} + \lambda_{b,c} + \lambda_{a,b,c} &= x_c.
\end{align*}
\]

This system of equations has a unique solution: $\lambda_{a,b} = x_a + x_b + x_c - 2$, $\lambda_{b,c} = 1 - x_a$, $\lambda_{a,b} = 1 - x_c$, and $\lambda_{a,c} = 1 - x_b$. Similarly, we can define and show that $\theta_{a,b,c} = x_a + x_b + x_c - 2$, $\theta_{b,c} = 1 - x_a$, $\theta_{a,b} = 1 - x_c$, and $\theta_{a,c} = 1 - x_b$.

So we have $\lambda^h = \theta^h$ for $h \in \{\{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. This allows us to glue the two convex combinations in the following way: suppose $F^{ij}_i$ and $F^{ij}_j$ use the same edges from $\{a, b, c\}$. Now we glue $\sum_{i=1}^k \lambda_i F^{ij}_i$ and $\sum_{i=1}^k \theta_i F^{ij}_j$ as follows. Let $\sigma_{ij} = \min\{\lambda_i, \theta_j\}$, and $F^{ij} = F^{ij}_i + F^{ij}_j$. Update $\lambda_i$ and $\theta_j$ by subtracting $\sigma_{ij}$ from both, and continue. The arguments
in the lemma ensure that we can find the \(i\) and \(j\) pair until all the remaining \(\lambda_i\) and \(\theta_j\) multipliers are zero. The convex combination with multipliers \(\sigma_{ij}\) and 2-edge-connected subgraphs \(F^{ij}\) is equal to \(x_e\) on every edge in \(E(G)\). Note that the number of subgraphs in the set \(\{F^{ij}\}\) (i.e., the support of the new convex combination) is at most \(k + \bar{k}\). Assuming that the size of the support of the convex combinations for both \(G_S\) and \(G_{\bar{S}}\) are polynomial in the size of the respective graphs, then the total number of subgraphs in the convex combination produced for \(G\) is polynomial.

The next two observations are needed for the proof of the next lemma.

**Observation A.4.** Let \(U \subseteq V\) be a minimal proper 3-edge-cut in \(G\) (i.e., for \(S \subset U\), the cut defined by \(S\) is not a proper 3-edge cut in \(G_U\)). Then \(G_U\) does not contain any proper 3-edge cuts.

**Proof.** Suppose for contradiction that there is \(S \subset V(G_U)\) that is a proper 3-edge cut of \(G_U\). This is a contradiction to minimality of \(U\) since \(S\) constitutes a proper 3-edge cut in \(G\) as well.

**Observation A.5.** Suppose that \(G\) has \(t\) proper 3-edge cuts. Let \(U\) be a proper 3-edge cut of \(G\). Then the number of proper 3-edge cuts in \(G_U\) is at most \(t - 1\).

**Proof.** There is correspondence between the proper 3-edge cuts of \(G_U\) and \(G\): each proper 3-edge cut in \(G_U\) corresponds to a proper 3-edge cut in \(G\). However, clearly, \(U\) is not a proper 3-edge cut of \(G_U\). This implies that \(G_U\) can have at most \(t - 1\) proper 3-edge cuts.

**Definition A.6.** Let \(G = (V, E)\) be a 3-edge-connected graph. Define \(G\) to be the collection of graphs obtained from \(G\) by iteratively contracting an arbitrary proper 3-edge cut and contracting it into a single vertex until the graph becomes essentially 4-edge-connected.

**Lemma A.7.** Let \(G = (V, E)\) be a 3-edge-connected graph and \(x \in [0, 1]^E\). The following two statements are equivalent.

1. For any \(G' \in G\), vector \(x\) restricted to the entries of \(E(G')\) can be written as a convex combination of 2-edge-connected subgraphs of \(G'\).

2. Vector \(x\) can be written as a convex combination of 2-edge-connected subgraphs of \(G\).

**Proof.** To show that 2. \(\Rightarrow\) 1., we observe that every 2-edge-connected subgraph of \(G\) can be mapped to a 2-edge-connected subgraph of \(G'\) by considering only the subset of edges that belong to \(G'\). Moreover, a convex combination corresponding to the vector \(x\) can be mapped to a convex combination for the vector \(x'\), where \(x'\) contains only entries corresponding to edges in \(G'\). Now we show the other direction, namely 1. \(\Rightarrow\) 2.
Suppose $G$ contains $t$ proper 3-edge cuts. We assume 1. and we prove the statement 2. by induction on $t$. If $t = 0$, then $G$ does not contain a proper 3-edge cut. In this case, $G \in \mathcal{G}$ and we are done. If $t \geq 1$, find a minimal 3-edge cut $U$ of $G$. By Observation A.4, $G_U$ does not contain any proper 3-edge cuts and by Observation A.5, $\bar{G}_U$ contains at most $t - 1$ proper 3-edge cuts. Since $G_U$ belongs to $\mathcal{G}$, we can write $x$ restricted to $E(G_U)$ as a convex combination of 2-edge-connected subgraphs of $G_U$. By induction, since $\bar{G}_U$ has at most $t - 1$ proper 3-edge cuts, we can write $x$ restricted to the entries of $E(\bar{G}_U)$ as a convex combination of 2-edge-connected subgraphs of $\bar{G}_U$. Then we can apply Lemma A.3 to find a convex combination of 2-edge-connected subgraphs for $G$.

Moreover, since $t$ is upper bounded by a polynomial in $|V|$, this shows that the support of the convex combination for $G$ has polynomial size. Note that a minimal proper 3-edge cut of $G$ can be found in polynomial time, since it is a minimum cut of $G$ and there are only polynomially many minimum cuts of $G$.

We will use the following observation in the proof of Theorem 2.4.

**Observation A.8.** Let $G = (V, E)$ be a 3-edge-connected graph and $\mathcal{C}$ be a cycle cover of $G$ that covers the 3-edge cuts of $G$. Let $\emptyset \subset S \subset V$ be a proper 3-edge cut of $G$ (i.e., $|\delta(S)| = 3$) such that $|\delta(S) \cap \mathcal{C}| = 2$. Then the graph $G_S$ is 3-edge-connected and $\mathcal{C}$ restricted to $E(G_S)$ is a cycle cover of $G_S$.

**Theorem 2.4.** For $\alpha \in [0, 1]$, the following two statements are equivalent.

1. For an essentially 4-edge-connected graph $G = (V, E)$ and any cycle cover $\mathcal{C}$ of $G$, the vector $\chi^E - \alpha \chi^\mathcal{C}$ can be written as a convex combination of 2-edge-connected subgraphs of $G$.

2. For a 3-edge-connected graph $G = (V, E)$ and any cycle cover $\mathcal{C}$ of $G$ that covers all the 3-edge cuts of $G$, the vector $\chi^E - \alpha \chi^\mathcal{C}$ can be written as a convex combination of 2-edge-connected subgraphs of $G$.

**Proof.** Suppose we are given a 3-edge-connected graph $G = (V, E)$ and a cycle cover $\mathcal{C}$ that covers all the 3-edge cuts of $G$. Let $x = \chi^E - \alpha \chi^\mathcal{C}$ for some $\alpha \in [0, 1]$. For each $G' \in \mathcal{G}$, we apply Observation A.8 which yields cycle cover $\mathcal{C}'$ (i.e., the cycle cover $\mathcal{C}$ restricted to the edges of $G'$). By Lemma A.7 we can conclude that statements 1. and 2. are equivalent.

**B 3-Edge-colorable cubic graphs: Proof of Theorem 3.9**

We now give the complete proof for the following theorem.
Theorem 3.9. Let $G = (V, E)$ be a 3-edge-connected, cubic, 3-edge-colorable graph. Then $\frac{13}{15} \chi^E$ can be written as a convex combination of 2-edge-connected subgraphs of $G$.

Note that while deciding whether a graph is 3-edge-colorable is NP-complete, there are special cases in which it is easy to find a 3-edge coloring, for example in the case of a bipartite, cubic graph. Throughout this section, we assume that we are given a 3-edge coloring of the input graph $G = (V, E)$. We will consider three cycle covers of $G$ where each cycle cover consists of all the edges on two of the colors. In to prove Theorem 3.9 we prove the following theorem.

Theorem B.1. Let $G = (V, E)$ be a 3-edge-connected cubic, 3-edge-colorable graph and let $\mathcal{C}$ be a cycle cover of $G$ consisting of all edges of two colors. Then the vector $\chi^E - \frac{1}{5} \chi^C$ can be written as a convex combination of 2-edge-connected subgraphs of $G$.

Proof of Theorem 3.9. Let $\sum_{i=1}^{3} \lambda_i \chi^{C_i} = \frac{2}{3} \chi^E$ be the convex combination of three cycle covers, each one consisting of two edge colors. By Theorem 3.2, for each $i \in \{1, \ldots, 3\}$ we can write $\chi^E - \frac{1}{5} \chi^{C_i}$ as convex combination of 2-edge-connected subgraphs of $G$. Hence, $\sum_{i=1}^{3} \lambda_i (\chi^E - \frac{1}{5} \chi^{C_i}) = \chi^E - \frac{2}{15} \chi^E = \frac{13}{15} \chi^E$, and we conclude that $\frac{13}{15} \chi^E$ can be written as a convex combination of 2-edge-connected subgraphs of $G$.

Using the following observation, we can apply Theorem 2.4. This allows us to focus on proving Lemma B.3 which implies Theorem B.1.

Observation B.2. Let $G = (V, E)$ be a 3-edge-connected, 3-edge-colorable graph and $\mathcal{C}$ be a cycle cover of $G$ that consists of all edges of two colors. Then (i) $\mathcal{C}$ covers the 3-edge cuts of $G$, (ii) $\mathcal{C}$ consists of only even-length cycles, (iii) $G_S$ is 3-edge colorable (i.e., the 3-edge coloring of $G$ restricted to $E(G_S)$ is a 3-edge coloring of $G_S$), and (iv) $\mathcal{C}$ restricted to $E(G_S)$ is also a cycle cover of $G_S$ on two colors.

Lemma B.3. Let $G = (V, E)$ be an essentially 4-edge-connected cubic graph and $\mathcal{C}$ be a cycle cover of $G$ that consists of only even-length cycles. Then the vector $\chi^E - \frac{1}{5} \chi^C$ can be written as a convex combination of 2-edge-connected subgraphs of $G$.

Proof. Let $y = \chi^E - \frac{1}{5} \chi^C$. Let $\sum_{i=1}^{k} \lambda_i \chi^{K_i}$ be the convex combination of vector $y$ obtained via Lemma 3.6. We now set $H_i = G/C_{K_i} = (U_i, F_i)$ and $T_i = K_i/C_{K_i}$ (recall that $C_{K_i}$ is the unique cycle in the 1-tree $K_i$). Notice that since $\mathcal{C}$ contains only even cycles, there are no leaf-matching links for $K_i$ in $G$, and consequently $LM(H_i, T_i)$ is empty. Thus, by Lemma 3.5 we can find a $(3\chi^{F_i \setminus T_i}, 5)$-coloring of $L_i = F_i \setminus T_i$ for $i \in \{1, \ldots, k\}$. By Observation 2.8 we have $\frac{3}{5} \chi^{L_i} = \sum_{j=1}^{5} \frac{1}{5} \chi^{A_{i,j}}$ where $A_{i,j}$ is a feasible augmentation for $T_i$ and therefore for $K_i$ (by Observation 2.6). In other words, $K_i + A_{i,j}$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, 5\}$
is a 2-edge-connected subgraph of $G$. Let $z^i_j = \chi^{K_i+A_j}$ be the characteristic vector of the corresponding 2-edge-connected subgraph of $G$. Then, we have:

$$z^i = \sum_{j=1}^{5} \frac{1}{5} z^j_j = \sum_{j=1}^{5} \frac{1}{5} \chi^{K_i+A_j} = \chi^{K_i} + \sum_{j=1}^{5} \frac{1}{5} \chi^{A_j} = \chi^{K_i} + \frac{3}{5} \chi^{L_i}.$$

Notice that $\sum_{i=1}^{k} \lambda_i \chi^{L_i} \leq \frac{1}{2} \chi^C$. Next, we claim that the vector $z$ can be written as a convex combination of 2-edge-connected subgraphs of $G$.

$$z = \sum_{k=1}^{k} \lambda_i z^i = \sum_{i=1}^{k} \lambda_i (\chi^{K_i} + \frac{3}{5} \chi^{L_i}) = \sum_{i=1}^{k} \lambda_i \chi^{K_i} + \frac{3}{5} \sum_{i=1}^{k} \lambda_i \chi^{L_i}$$

$$\leq (\chi^E - \frac{1}{2} \chi^C) + \frac{3}{10} \chi^C = \chi^E - \frac{1}{5} \chi^C.$$

This concludes the proof of Lemma 3.3.