Towards a stochastic multi-point description of turbulence

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Abstract. It has been found in previous work that the multi-scale statistics of homogeneous isotropic turbulence can be described by a stochastic ‘cascade’ process of the velocity increment from scale to scale, which is governed by a Fokker–Planck equation. We show in this paper how this description can be extended to obtain the complete multi-point statistics of the velocity field. We extend the stochastic cascade description by conditioning on the velocity value itself and find that the corresponding process is also governed by a Fokker–Planck equation, which contains as a leading term a simple additional velocity-dependent coefficient in the drift function. Taking into account the velocity dependence of the Fokker–Planck equation, the multi-point statistics in the inertial range can be expressed by the two-scale statistics of velocity increments, which are equivalent to the three-point statistics of the velocity field. Thus, we propose a stochastic three-point closure for the velocity field of homogeneous isotropic turbulence.

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1. Introduction

There has been extensive research activity aimed at achieving a stochastic description of the complexity of turbulent velocity fields. The central focus has been directed at the case of homogeneous isotropic turbulence in the hope of simplifying this attempt. In recent decades, many works on the statistical properties of velocity increments

$$\xi(r) = U(x + r) - U(x)$$

have been published. Two-point statistics with respect to the increments are given by the probability density function (PDF) $p[\xi(r)]$. The knowledge of $p[\xi(r)]$ is equivalent to the knowledge of all structure functions $S_n(r) = \langle \xi(r)^n \rangle$ (cf [1]).

By focusing on the statistics of increments, the actual values of the velocity $U(x)$ are somehow filtered out. However, there is experimental evidence that the velocity increments statistically depend on the velocity itself [2]–[4]. In other words, the ‘sweeping decorrelation hypothesis’, which implies statistical independence of small-scale quantities such as velocity increments of the large-scale excitation represented by the velocity itself, is in general not valid for turbulence. Hosokawa [5] takes the quantity $\xi_l(r) := U(x + r) + U(x)$ as a measure of the large-scale dynamics of turbulence. He shows that for homogeneous turbulence, the assumption of statistical independence of the small-scale increment $\xi$ and the large-scale quantity $\xi_l$ is inconsistent with the existence of a non-vanishing third-order structure function $\langle \xi(r)^3 \rangle \neq 0$. This argument also rules out the sweeping decorrelation hypothesis. Thus, by focusing exclusively on velocity increments, some information about the complexity of turbulence is lost.

In this paper, we go beyond these findings regarding the interdependence of velocity increments (i.e. single-scale statistics) and velocities (i.e. single-point statistics) by investigating the relation between multi-scale and multi-point statistics. We are interested in these quantities because they allow a more profound characterization of the velocity field than increment statistics $p[\xi(r)]$, which include only a single scale.

In recent decades, various attempts have been made to describe turbulence by using scale-dependent stochastic models such as the multifractal model (cf [1, 6, 7]), which, in modified form, has also been applied to financial data (cf [8]–[12]). Yakhot [13] presents arguments from dynamic theory that are consistent with the multifractal approach, showing that the PDF of the small-scale velocity fluctuations includes information about the large-scale dynamics.

Friedrich and Peinke [14, 15] suggested a different scale-dependent description of turbulence, based on a Fokker–Planck equation for the stochastic process of the velocity increment $\xi$ as a function of scale $r$. This approach is based on the experimental evidence that the stochastic process of $\xi(r)$ has the Markov property for step sizes $\Delta r$ larger than the so-called Einstein–Markov coherence length $l_{EM}$, which is of the order of the Taylor microscale, $l_{EM} \approx \lambda$ [16]. The method gives access to the multi-scale statistics expressed by the joint $N$-scale PDF $p[\xi(r_1), \xi(r_2), \ldots, \xi(r_N)]$ for $N$ differently chosen scales $r_j$, which are separated by at least one Einstein–Markov length. The relevant information about the stochastic process, i.e. the coefficients of the Fokker–Planck equation, can be estimated directly from given experimental data (cf [17]–[22]).

2 Different types of brackets, (·) and [·], are used to facilitate reading.
Nawroth and Peinke [23] noted that this multi-scale description can also be used for the generation of synthetic time series with the same statistical multi-scale properties as, for example, a given turbulent velocity time series, or for the prediction of financial time series [24]. They also note that it is necessary to take into account the statistics of the velocity itself in order to obtain stationary synthetic data [23]. In this work, we want to go beyond this approach and show precisely how the joint \( N \)-scale PDF \( p[\xi(r_1), \xi(r_2), \ldots, \xi(r_N)] \) is related to the joint \((N + 1)\)-point PDF \( p[U(x), U(x + r_1), \ldots, U(x + r_N)] \). As a consequence of the Markov properties of the velocity increments, the multi-point statistics can be obtained based on the knowledge of the velocity field at three points. Thus, we propose a stochastic three-point closure for homogeneous isotropic turbulence.

This paper is organized as follows. In section 2, we examine the relationship between multi-scale and multi-point statistics. We derive a velocity-dependent Fokker–Planck equation for the velocity increments, which gives access to the multi-point statistics. In section 3, we present experimental evidence for the validity of such a stochastic description, and examine the empirical velocity dependence of the Fokker–Planck equation. Section 4 concludes the paper.

2. Multi-scale and multi-point statistics

In the following, we use the notation \( u := U - \bar{U} \) for the velocity fluctuation, where \( U \) is the velocity and \( \bar{U} \) is the mean flow velocity. We also use the short-hand notations \( u_i := u(x_i) \) and \( u_{i+j} := u(x_i + r_j) \), and for the velocity increment \( \xi \) at scale \( r_j \) we write

\[
\xi_j \equiv \xi(r_j) = u(x_i + r_j) - u(x_i) .
\] (1)

The starting point of this paper is the finding that the \( N \)-scale statistics of velocity increments can be expressed by two-scale conditional probability densities \( p(\xi_j | \xi_{j+1}) \). More precisely, in several papers, cf [18]–[22], it has been shown by experimental evidence that the stochastic process of the velocity increment \( \xi \) in scale \( r \) has the Markov property

\[
p(\xi_j | \xi_{j+1}, \xi_{j+2}, \ldots, \xi_{j+N}) = p(\xi_j | \xi_{j+1}) ,
\] (2)

where the process direction has been chosen from large to small scales, by sorting the scale variables \( r_j \) as \( r_1 < r_2 < \cdots < r_N \). This equation is usually studied for \( N = 2 \), as an adequate simplification for finite data sets [14],

\[
p(\xi_j | \xi_{j+1}, \xi_{j+2}) = p(\xi_j | \xi_{j+1}).
\] (3)

We note that (2) and (3) hold only for step sizes \( \Delta r := r_{j+1} - r_j \) that are larger than the so-called Einstein–Markov coherence length \( l_{EM} \), which is of the order of magnitude of the Taylor microscale \( \lambda \) [16, 25]. If (2) holds, the \( N \)-scale joint PDF of the velocity increments can be expressed by a product of conditional PDFs,

\[
p(\xi_1, \ldots, \xi_N) = p(\xi_1 | \xi_2) p(\xi_2 | \xi_3) \ldots p(\xi_{N-1} | \xi_N) p(\xi_N) .
\] (4)

As the next step we consider the \((N + 1)\)-point statistics that can be expressed by

\[
p(u_i, u_{i+1}, \ldots, u_{i+N}) = p(\xi_1, \ldots, \xi_N, u_i) = p(\xi_1, \ldots, \xi_N | u_i) \cdot p(u_i),
\] (5)
as can be easily seen, since the increments $\xi_j$ of this equation are calculated from the velocity values that appear on the left-hand side. Let us assume that the Markov property of the interscale process is conserved when the process is conditioned on the velocity, i.e. let us assume that

$$p(\xi_j | \xi_{j+1}, \xi_{j+2}, \ldots, \xi_N, u_i) = p(\xi_j | \xi_{j+1}, u_i),$$

which can be simplified to a sufficient condition for finite data,

$$p(\xi_j | \xi_{j+1}, u_i) = p(\xi_j | \xi_{j+1}, u_i).$$  \hspace{1cm} (6)

Equation (6) implies the following factorization of the multi-point joint PDF:

$$p(u_i, u_{i+1}, \ldots, u_{i+N}) = p(\xi_1 | \xi_2, u_i) \ldots p(\xi_{N-1} | \xi_N, u_i) p(\xi_N | u_i) p(u_i).$$  \hspace{1cm} (7)

The evolution of the conditional PDFs of (8) in scale $r_j$ can be expressed by a Kramers–Moyal expansion [26],

$$-r_j \frac{\partial}{\partial r_j} p(\xi_j | \xi_k, u_i) = \sum_{q=1}^{\infty} \left( -\frac{\partial}{\partial \xi_j} \right)^q \left[ D^{(q)}(\xi_j, r_j, u_i) p(\xi_j | \xi_k, u_i) \right],$$

where $r_k > r_j$, with the Kramers–Moyal coefficients

$$D^{(q)}(\xi_j, r_j, u_i) = \lim_{q \to 0} \frac{r_j}{q!} \int \left[ (\xi_j'(r_j - \delta r, u_i) - \xi_j(r_j, u_i))]^q \right] \xi_j'$$

$$= \lim_{q \to 0} \frac{r_j}{q!} \int_{-\infty}^{\infty} (\xi_j - \xi_j')^q p(\xi_j'(r_j - \delta r, u_i) | \xi_j(r_j, u_i)] \xi_j'. \hspace{1cm} (10)$$

Note that in contrast to the usual notation, we multiplied both sides of (9) by $r_j$, and we have a negative sign on the left-hand side due to the process direction from large to small scales. If the Kramers–Moyal coefficient of fourth order, $D^{(4)}$, vanishes, the expansion truncates after the second term and becomes a Fokker–Planck equation [26],

$$-r_j \frac{\partial}{\partial r_j} p(\xi_j | \xi_k, u_i) = -\frac{\partial}{\partial \xi_j} \left[ D^{(1)}(\xi_j, r_j, u_i) p(\xi_j | \xi_k, u_i) \right] + \frac{\partial^2}{\partial \xi_j^2} \left[ D^{(2)}(\xi_j, r_j, u_i) p(\xi_j | \xi_k, u_i) \right].$$

\hspace{1cm} (11)

In the same way, a velocity-independent Fokker–Planck equation can be derived from (4),

$$-r_j \frac{\partial}{\partial r_j} p(\xi_j | \xi_k) = -\frac{\partial}{\partial \xi_j} \left[ \tilde{D}^{(1)}(\xi_j, r_j) p(\xi_j | \xi_k) \right] + \frac{\partial^2}{\partial \xi_j^2} \left[ \tilde{D}^{(2)}(\xi_j, r_j) p(\xi_j | \xi_k) \right]. \hspace{1cm} (12)$$

It has been shown for homogeneous isotropic turbulence that the coefficient $\tilde{D}^{(4)}(\xi_j, r_j)$ can in fact be neglected [18, 20]. Therefore, the Fokker–Planck equation (12) is an adequate description of the interscale process of the velocity increments, giving access to the $N$-scale joint PDF of (4).

In order to step from the description of multi-scale to multi-point statistics, we have to examine the validity of (7), check whether or not the coefficient $D^{(4)}(\xi_j, r_j, u_i)$ can be neglected and study the empirical dependence of the Fokker–Planck equation (11) on the velocity $u_i$. This will be done in section 3.

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3 We note that equations like $p(u_i, \ldots) = p(\xi_i, \ldots)$ should actually be written as $p(u_i, \ldots) = \tilde{p}(\xi_i, \ldots)$. For convenience, we use a simplified notation where different arguments of $p(\cdot)$ imply different functions.
3. Experimental results

We analyze hot-wire measurement data from three different flow types at different Reynolds numbers. The first flow type is the wake of a cylinder with diameter \( D = 2 \text{ cm} \) in a wind tunnel at downstream distance \( x = 100D \), for Taylor microscale Reynolds numbers \( R = 338 \) over the range 86–338 [27]. We use the data at \( R = 338 \) for the most detailed analysis. These data are characterized by a dissipation scale \( \eta = 0.10 \text{ mm} \), Taylor microscale \( \lambda = 3.7 \text{ mm} \) and integral scale \( L = 119 \text{ mm} \). The second flow type is an axisymmetric air free jet at 145 nozzle-diameters distance from the nozzle, at \( R = 190 \), with \( \eta = 0.25 \text{ mm} \), \( \lambda = 6.6 \text{ mm} \) and \( L = 67 \text{ mm} \). These data are described in detail in [18]. Finally, we also analyze data from the flow behind a so-called space-filling fractal square grid, described in detail in [28], measured at different positions in the decay region of the turbulence in the wind tunnel, for different flow speeds, at Reynolds numbers \( R = 175 \) and 740. The cylinder wake and free jet data sets have a total length of \( 2.5 \times 10^6 \) values each, and the fractal grid data sets have a length of \( 3 \times 10^6 \) values each. We analyze the streamwise velocity component of the cylinder wake and fractal grid data, which has been measured with cross-wire probes. The free jet data have been measured with a single wire.

We examine the dependence of the conditional PDFs of the velocity increments on the velocity itself by comparing both sides of the equation

\[
p(\xi_1|\xi_2, u_i) = p(\xi_1|\xi_2).
\]

Note that if equation (13) holds, the Fokker–Planck equations (11) and (12) will be identical. In figure 1, we compare the contour plots of both conditional distributions of (13) for the free jet at scales \( r_1 = 2\lambda \approx L/4 \) and \( r_2 = 2r_1 \) for two different values of \( u_i \). The distributions agree almost perfectly for \( u_i = 0 \), while for a larger velocity, \( u_i = 2\sigma_u \) (where \( \sigma_u = \sqrt{(u^2)} \)), the distribution \( p(\xi_1|\xi_2, u_i) \) is slightly shifted downward with respect to the other distribution \( p(\xi_1|\xi_2) \). According to the errors of \( p(\xi_1|\xi_2, u_i) \) shown in this figure, the shift is statistically significant.\(^4\) Such a shift of the conditional PDF for velocities \( u_i \neq 0 \) cannot be observed at any scale \( r_i \) for all the examined flows. As we will see below, this shift will lead to a velocity-dependent term in the drift function of the Fokker–Planck equation.

In section 2, we argued that the multi-point statistics might be obtained from the velocity-dependent Fokker–Planck equation (11) of the interscale process. A necessary condition for this description to hold is the validity of (7), which states that the Markov property of the interscale process is conserved under the additional condition of \( u_i \). In order to verify the validity of (7), we apply the (Mann–Whitney–) Wilcoxon test [18, 29, 30], which tests whether or not two samples of different sizes have the same statistical distribution. Since the two distributions of (7) are necessarily of different sizes, the Wilcoxon test is an appropriate method for estimating its validity. The test is described in detail in the appendix. Figure 2 shows the results of the Wilcoxon test for (7) for different values of \( u_i \) and \( \xi_{j+2} \). In the present implementation of the Wilcoxon test, a statistical test value \( \langle \Delta Q^* \rangle \) is computed, which must be close to 1 for

\(^4\) The errors shown in figure 1 are calculated in the following way: the original data are divided into ten subsets of equal length, the conditional probability densities are estimated for all ten subsets, and the variance of the ten results, divided by \( \sqrt{10} \), is taken as the error at each point. We should note that such error estimates can only serve as a rough estimate of the statistical significance of the shift of the PDFs. With the Wilcoxon test, described in the appendix, it can be shown that (13) is only valid as an approximation for very small scales \( \lambda_{EM} \ll r_1 \ll L \), and not for very large values of \( |u_i| \approx 2\sigma_u \). To save space, these results have not been presented in this paper.
Figure 1. Contour plots of the conditional distributions \( p(\xi_1|\xi_2) \) (black) and \( p(\xi_1|\xi_2, u_i) \) (red) with \( \xi_j \equiv \xi(r_j) \), for scales \( r_1 = 2\lambda \approx L/4 \) and \( r_2 = 4\lambda \approx L/2 \), for \( u_i = 0 \pm \sigma_u/4 \) (a) and \( u_i = 2\sigma_u \pm \sigma_u/4 \) (c). In (b) and (d), cuts through the contour plots at fixed values of \( \xi_2 \) (along the vertical straight lines in (a) and (b), respectively) are shown. Free jet.

Figure 2. Wilcoxon test of (7) for different values of \( u_i \) and \( \xi_{j+2} \), as indicated in the legend (\( \xi_{j+2} = 0 \) is short for \( \xi_{j+2} = 0 \pm \sigma_u/8 \), and \( \xi_{j+2} = \pm \sigma_u \) is short for \( \xi_{j+2} = (-)\sigma_u \pm \sigma_u/6 \)). Cylinder wake at \( R_\lambda = 338 \).
acceptance of the hypothesis expressed by (7). The values \( \langle \Delta Q^* \rangle \) in figure 2 are in fact identical to those for 1, except for some inevitable scattering, for large enough scale distances \( \Delta r > l_{\text{EM}} \approx 0.6\lambda \). Thus, the Markov property of the interscale process is conserved under the additional condition of \( u_i \), and equations (7) and (8) do apply.

The drift and diffusion functions of (12) for homogeneous isotropic turbulence are approximately linear and second-order functions in \( \xi \), as was found previously in [18, 19]:

\[
\tilde{D}^{(1)}(\xi, r) = -\tilde{d}_{11}(r)\xi, \quad (14a)
\]
\[
\tilde{D}^{(2)}(\xi, r) = \tilde{d}_{20}(r) - \tilde{d}_{21}(r)\xi + \tilde{d}_{21}(r)\xi^2. \quad (14b)
\]

Here and for the remaining part of the paper, we skip the indices in \( \xi_j, r_j \) and \( u_j \) for simplicity. The velocity increments \( \xi \) are given in units of their standard deviation in the limit \( r \to \infty \), \( \sigma_\infty \), which is identical to \( \sqrt{2}\sigma_u = \sqrt{2}(u^2) \) [18]. This normalization allows a comparison of the Kramers–Moyal coefficients of different flows. As pointed out in [18], it is known that a diffusion function that is constant in \( \xi \) has Gaussian solutions, whereas the additional \( \xi \)-dependent terms present in (14b) are responsible for intermittency effects and anomalous scaling of the structure functions.

In the following, we examine the velocity dependence of the drift and diffusion functions. Figure 3 shows exemplary drift functions \( D^{(1)}(\xi, r, u) \) of the cylinder wake at \( R_s = 338 \) for different values of \( u \). For \( u \neq 0 \), the drift functions are essentially shifted in the vertical direction in comparison to the drift for \( u = 0 \), while their slopes do not depend visibly on \( u \). The diffusion functions, also shown in figure 3, do not depend significantly on \( u \). Furthermore, the fourth-order Kramers–Moyal coefficient \( D^{(4)}(\xi, r, u) \) in figure 3 is zero within the estimation errors. Thus, the description of the stochastic process by a Fokker–Planck equation is valid, and the velocity-dependent drift and diffusion functions can be approximately described by first- and second-order polynomials in \( \xi \),

\[
D^{(1)}(\xi, r, u) = d_{10}(r, u) - d_{11}(r)\xi, \quad (15a)
\]
\[
D^{(2)}(\xi, r) = d_{20}(r) - d_{21}(r)\xi + d_{21}(r)\xi^2. \quad (15b)
\]

The most significant difference to (14a) and (14b) is the presence of the additional velocity-dependent term \( d_{10} \) in the drift function. The coefficient \( d_{10} \) is the leading velocity-dependent term, which can also be seen from the shift of the conditional PDFs in figures 1(c) and (d).

We presented the drift functions in figure 3 at a relatively large scale \( r = 20\lambda \), where the vertical shift for \( u \neq 0 \) is relatively strong, and the diffusion functions at a relatively small scale \( r = 3\lambda \), where the quadratic shape of the diffusion is quite pronounced (they become flatter with increasing scale). The same observations—a shift of the drift and no significant change in the diffusion for different values of \( u \)—can be made at any scale. Possible further dependences on \( u \), like the weak ‘bending’ observed in the drift in figure 3(a) for very large values of \( |\xi| \), are within estimation errors and are therefore not statistically significant for the examined data. They are considered in the following as possible higher-order effects.

The \( r \)-dependence of \( d_{10} \) for different values of \( u \) is shown in figure 4. For the cylinder wake and fractal grid in figures 4(a) and (c), the dependences of \( d_{10} \) on both \( r \) and \( u \) are quite similar and approximately symmetric in \( u \), \( d_{10}(r, u) \approx -d_{10}(r, -u) \). In contrast, \( d_{10} \) is not symmetric
Figure 3. Drift and diffusion functions $D^{(1)}(\xi, r, u)$ and $D^{(2)}(\xi, r, u)$ and the Kramers–Moyal coefficient $D^{(4)}(\xi, r, u)$ for different values of $u$. $D^{(1)}$ is shown for $r_j = 20\lambda$ and $D^{(2)}$ and $D^{(4)}$ for $r = 3\lambda$. Cylinder wake at $R_\lambda = 338$. 

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Figure 4. Coefficient $d_{10}$ from (15a) as a function of $r$ for different values of the velocity $u$, given in the legend. The dotted lines are fits according to (16). (a) Cylinder wake at $R_{\lambda} = 338$; (b) free jet at $R_{\lambda} = 190$; (c) fractal square grid at $R_{\lambda} = 740$.

in $u$ for the free jet in figure 4(b). In all cases, $d_{10}$ can be approximated by a second-order polynomial in $r$,

$$d_{10}(r, u) = d_{101}(u) \frac{r}{\lambda} + d_{102}(u) \left( \frac{r}{\lambda} \right)^2. \tag{16}$$

The dependence of the coefficients $d_{101}$ and $d_{102}$ on the velocity $u$ is shown in figure 5. We find that the parameters $d_{101}$ and $d_{102}$ depend on the flow geometry, as well as on the Reynolds number. In the cases of the cylinder wake and fractal grid, the term $d_{101}$ is almost linear in $u$ for low Reynolds numbers, and becomes more S-shaped for higher Reynolds numbers (figures 5(a) and (c)); the term $d_{102}$ is approximately linear in $u$, with Reynolds-number-dependent slopes in the case of the cylinder wake (figures 5(d) and (f)). For the free jet, neither $d_{101}$ nor $d_{102}$ is linear in $u$, and $d_{101}$ is also strongly asymmetric (figures 5(b) and (e)).

Figure 6 shows the coefficients $d_{11}$, $d_{20}$, $d_{21}$ and $d_{22}$ of equations (15a) and (15b) as functions of $r$ for different values of $u$ for the cylinder wake data at $R_{\lambda} = 338$. Reliable
estimates of these coefficients for values of \(|u| > \sigma_u\) cannot be obtained due to the small number of available data for large velocity fluctuations. None of the coefficients shows a similarly systematic dependence on \(u\) as the coefficient \(d_{10}\) in figure 4. The slope \(d_{11}\) of the drift function only deviates for \(u \approx -\sigma_u\) at large scales \(r\), and the leading order coefficient of the diffusion function, \(d_{20}\), practically does not depend on \(u\) at all. The coefficients \(d_{21}\) and \(d_{22}\) are scattered due to the large estimation errors, especially at large scales. Figure 6(d) suggests that \(d_{22}\) decreases—and therefore the diffusion function \(D^2(\xi)\) becomes flatter—for smaller values of \(|u|\), which would correspond to a more Ornstein–Uhlenbeck-like process in the vicinity of \(u = 0\). However, we should note that the dependences of \(d_{21}\) and \(d_{22}\) on \(u\) are relatively small at scales \(r < 5l_{EM}\) and that these coefficients are more difficult to estimate than \(d_{10}, d_{11}\) and \(d_{20}\).

Figure 6 also shows the coefficients \(\tilde{d}_{11}, \tilde{d}_{20}, \tilde{d}_{21}\) and \(\tilde{d}_{22}\) of equations (14a) and (14b) as solid lines. While the coefficients \(d_{20}\) and \(\tilde{d}_{20}\) are identical at all scales, the coefficients \(d_{11}\) and \(d_{22}\) are different from \(\tilde{d}_{11}\) and \(\tilde{d}_{22}\), respectively, at large scales \(r\). The difference between the slopes of the drift functions, \(\tilde{d}_{11}\) and \(\tilde{d}_{11}\), at large scales \(r\) can also be observed in figure 3.
Figure 6. Coefficients $d_{11}$, $d_{20}$, $d_{21}$ and $d_{22}$ from (15a) and (15b) as a function of $r$ for different values $u$, given in the legend. The solid lines represent the coefficients $\tilde{d}_{11}$, $\tilde{d}_{20}$, $\tilde{d}_{21}$ and $\tilde{d}_{22}$ from (14a) and (14b). Cylinder wake at $R_\lambda = 338$.

4. Conclusions

It was found in [14] that the stochastic cascade process of velocity increments from scale to scale can be described by a Fokker–Planck equation. This description gives access to the joint multi-scale PDF $p[\xi(r_1), \xi(r_2), \ldots, \xi(r_N)]$ of the velocity increments $\xi$ at the scales $r_j$. The structure functions $S_n(r) \equiv \langle \xi(r)^n \rangle$ can also be obtained from this description (see also [15], [17]–[22]).

In this paper, we showed how this method can be extended in order to obtain the joint multi-point PDF $p[U(x_1), U(x_2), \ldots, U(x_N)]$ of the velocity $U$ at the points $x_i$. This description is more complete than the multi-scale description, since it takes into account the velocity dependence of the small-scale statistics. The multi-point statistics can be obtained from a Fokker–Planck equation for the conditional PDF $p[\xi(r_1)|\xi(r_2), u]$, where $u$ is the fluctuating velocity. The Fokker–Planck equation follows from the Markov property of the underlying stochastic process and from the experimental observation that the fourth-order Kramers–Moyal coefficient can be neglected.

The Fokker–Planck equation for the multi-point statistics differs from the Fokker–Planck equation for the multi-scale statistics mainly by the presence of a simple additional term in the...
drift function. This term, $d_{10}(r, u)$, represents a vertical, velocity-dependent shift of the drift function. It implies that, for example, $\xi(r_2) = 0$ and $u \gg 0$, then $\xi(r_1)$ at the scale $r_1 < r_2$ is likely to be negative. This is reasonable because $\xi(r_2)$ and $\xi(r_1)$ are to some extent independent and, loosely speaking, the increments have the tendency to drive the velocity signal back to zero, since it is stationary at large scales $r \gg L$. The shift of the drift function corresponds to the shift of the conditional PDF $p[\xi(r_1)|\xi(r_2), u]$, observed in figure 1.

It was found in previous work that the coefficients of the Fokker–Planck equation for the interscale process are not universal, but depend on the Reynolds number [19] and/or flow geometry [22]. We have now found a similar result for the velocity-dependent coefficient $d_{10}$, which depends on the flow type and, at least in the case of the cylinder wake, also on the Reynolds number. On the basis of the examined data sets with lengths of up to $10^7$, we cannot make definite statements on the velocity dependence of the other coefficients.

The coefficients of the Fokker–Planck equation can be estimated directly from the measured data. With the knowledge of these coefficients, the $N$-point statistics of the velocity field are given by the three-point statistics $p[u(x + r_1)|u(x + r_2), u(x)]$, which are equivalent to $p[\xi(r_1)|\xi(r_2), u(x)]$. Thus, a stochastic three-point closure for the turbulent velocity is given.

The analysis presented in this paper is restricted to a single velocity component, but in principle it can be extended to a velocity vector with three components. The drift and diffusion functions would then become tensors $D^{(1)}(\xi, r, u)$ and $D^{(2)}_{ij}(\xi, r, u)$, which contain coupling terms between the different velocity (increment) components. Siefert and Peinke [21] investigated the Fokker–Planck equation for two components of the velocity increment $\xi$, and found that the drift function decouples, while the diffusion function contains non-vanishing coupling terms between longitudinal and transversal velocity increments. These coupling terms were found to have simple functional forms, and thus the analysis can be easily extended to more than one velocity component. However, such an analysis requires more data—Siefert and Peinke [21] needed data of length $10^8$ to perform the two-dimensional analysis without conditioning on the velocity—and is therefore left to future studies. With the knowledge of (or appropriate assumptions about) the coupling coefficients, it would be possible to extend the method of time-series generation proposed in [23, 24]—with the additional conditioning on the velocity proposed in the present paper—to the generation of synthetic turbulent velocity signals with three components.

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Appendix. The (Mann–Whitney–)Wilcoxon test

The null hypothesis to be tested by the Wilcoxon test [18, 29, 30] is that the PDFs $p(x)$ and $\tilde{p}(y)$ of the stochastic variables $x$ and $y$ are identical. In the case of (3), for example, the two stochastic variables are $x = \xi_j|\xi_{j+1}$ and $y = \xi_j|\xi_{j+1}, \xi_{j+2}$. Two samples $x_1, \ldots, x_N$ and $y_1, \ldots, y_M$ of independent realizations of the variables are taken from the data. We take values that are separated by one integral length to be sufficiently independent of each other for the purpose of the test, in order not to reduce the sample sizes too much. Then, the number of values $x_n$ with
\( x_n < y_m \) is counted for each \( y_m \) and summed over \( m \):

\[
Q = \sum_{m=1}^{M} \sum_{n=1}^{N} z_{mn}, \quad z_{mn} = \begin{cases} 
1 & : x_n < y_m, \\
0 & : x_n \geq y_m.
\end{cases}
\] (A.1)

Under the null hypothesis, the quantity \( Q \) is Gaussian distributed with mean value \( \mu_0(N, M) = NM/2 \), and for \( N, M > 25 \), standard deviation \( \sigma_0(N, M) = \sqrt{NM(N+M+1)/12} \). Also under the null hypothesis, it follows that the quantity

\[
\Delta Q = \frac{|Q - \mu_0(N, M)|}{\sigma_0(N, M)},
\] (A.2)

which is the absolute value of a standard normal distributed variable, has a mean value of \( \sqrt{2/\pi} \).

In the present implementation of the test, the quantity

\[
\Delta Q^* = \Delta Q / \sqrt{2/\pi}
\] (A.3)

is calculated for a fixed value of \( \xi_{j+2} \) for a total of 100 bins for \( \xi_{j+1} \), which span the complete range of \( \xi_{j+1} \). The mean value \( \langle \Delta Q^* \rangle \) is calculated by taking the average over the 100 values of \( \Delta Q^* \). Under the null hypothesis of the Markov properties, the expectation value of \( \Delta Q^* \) is 1.

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