Gravitational Quantum Well as an Effective Quantum Heat Engine

Jonas F. G. Santos

*Federal University of ABC - Santo André - São Paulo - Brazil*

In this work we use the gravitational quantum well to model an effective two level system and perform two thermodynamic cycles, the iso-gravitational and the iso-energetic ones. It is shown that the iso-gravitational is independent of a scale parameter whereas the iso-energetic has a dependence on the eigenstates chosen to form the cycle. We also obtain an equivalent equation for the iso-energetic cycle which is similar to the equation of state for an isothermal process of an ideal gas. This equation reinforces the concept of an energy bath, where the temperature is replaced by the energy into the expression of efficiency.

jonas.floriano@ufabc.edu.br

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I. INTRODUCTION

Quantum heat engines are devices which transform an amount of heat (thermal energy) into work (mechanical energy) in a scale where the quantum effects are relevant and can be useful in some way to improve this conversion of energy [1–3]. In these cases, the working substance is a quantum system such as a single [4] or many [5] harmonic oscillators, single-atom [6], single-ion [7], vacuum forces through the Casimir interaction [8], quantum rotors [9], or a two level system (TLS) normally characterized as the spin of some molecule [10,11] or atom. For the cases where the thermal reservoirs are considered classical, the efficiency of these engines surprisingly does not surpasses the classical limit of Carnot, \( \eta_{\text{Carnot}} = 1 - T_C/T_H \), where \( T_C \) and \( T_H \) are the temperatures of the cold and hot reservoirs, respectively [1]. The same bound is no longer applicable when the reservoirs are quantum in some level such as, for instance, when the working substance interacts with a squeezed thermal reservoir [12,13], when there is a protocol to extract the so-called imperfect work [14], or in the case of a heat engine based on off-resonant light interaction [15], showing the role of quantum effects in the conversion of energy.

Another possibility of changing the reservoirs such that we move from a classical to a quantum perspective was introduced by Bender [16]. He considered the concept of an energy reservoir in substitution of a thermal reservoir in the quantum heat engine. Thermal reservoirs are characterized by a defined temperature associated with a stroke where the working substance interacts exchanging heat. On the other hand, the energy bath is such that during the correspondent stroke the expectation value of the Hamiltonian, \( \langle \hat{H} \rangle \), of the working substance is kept fixed [17]. Such a stroke is called iso-energetic. The applicability of iso-energetic strokes has been extended to non-relativistic regimes such as in the case of a single particle confined into a cylindrical potential and submitted to an external magnetic field [18], to the non-commutative version of quantum mechanics [19], in the relativistic regime of a single-particle Dirac spectrum [20], and for the Habl model [17].

On the basis of two level systems, the iso-energetic cycles can be effectively modeled by considering the two first states of some particular system provided that we have sufficiently control in order to avoid that others states are occupied. Once this condition is fulfilled, thermodynamic cycles can be performed. In Refs. [16,22], the authors considered an iso-energetic stroke where the length of an infinity square well is quasi-statically changed from \( L \) to \( L + \Delta L \). The relevant point here is the existence of a length scale that can be changed by using the variation of some external agent in an iso-energetic stroke.

Based on the argument above, the gravitational quantum well (GQW) is a suitable system where the iso-energetic stroke can be tested by using the intensity of the gravitational interaction as an external field. The GQW system is important from the fact that it is possible to obtain bound states for a particle coupled to the gravitational field. In Ref. [21], the spacial distribution of ultracold neutrons coupled to Earth by gravitational interaction were experimentally measured and were consistent with the theoretical result by using the Wigner function. Moreover, there are several recent studies employing the GQW system as a base of test for generalizations of quantum theory, for instance, in the case of a deformed Heisenberg algebra [23–25]. Our basic idea is to use the GQW as an effective two level system by considering the two first eigenstates, given by the Airy functions. Introducing a gravitational length scale, \( \ell_g = (\hbar^2/2m^2g)^{1/3} \), we consider two different thermodynamic cycles based on the GQW system. The iso-gravitational cycle, composed by two iso-gravitational and two iso-entropic strokes, and the iso-energetic cycle, composed by two iso-energetic and two iso-entropic strokes. From an experimental perspective, recent development in nanoscale experimental techniques made it possible to engineer realistic quantum heat engines [26], for instance, using a single molecule junctions [27], single-level quantum dot [28], optomechanical systems [29] etc. Such a nanoscale quantum heat engines demonstrate the fine control in order to justify the experimental realization of an iso-energetic cycle.

This work is organized as follows. In the next section we review the eigenvalue equation for the Hamiltonian of a particle coupled to a linear potential and particularize for the GQW system. The section [III] is devoted to describe in detail the two cycles and discuss the analytical expressions for the efficiencies. We generalize the possibility of generating a two level system with any pair of eigenstates of the GQW system in section [IV]. In section [V] we extend our analyzes in order to obtain an analogous of equation of state for an iso-energetic stroke and compare it with the same expression for an isothermal process for an ideal gas. We present our conclusions and final considerations in section [VI].

II. REVIEW ON THE QUANTUM MECHANICS FOR LINEAR POTENTIAL

Here we will review the important aspects of the system composed by a particle of mass \( m \) into a linear potential and then restrict the solutions to the particular case of the gravitational quantum well. Consider a system described by the one-dimensional Hamiltonian,
\[
H(x, p) = \frac{p^2}{2m} + Cx, \tag{1}
\]

where \(C\) is an arbitrary constant such that the product \(Cx\) has dimension of energy. The eigenstates and eigenenergies are obtained by solving the eigenvalue equation,

\[
H \psi_n = E_n \psi_n \tag{2}
\]

\[
\frac{d^2 \psi}{dx^2} + \frac{2m}{\hbar^2} (E_n - C x) \psi_n = 0. \tag{3}
\]

If we define a new variable,

\[
\xi = \left( x - \frac{E_n}{C} \right) \left( \frac{2mC}{\hbar^2} \right)^{1/3},
\]

Eq. (3) can be written in the form,

\[
\frac{d^2 \psi_n}{d\xi^2} + \xi \psi_n = 0, \tag{4}
\]

which is the Airy equation whose solutions are called Airy functions. The general form for these solutions are \[30\],

\[
\psi(\xi) = D_1 A_1(-\xi) + D_2 B_1(-\xi), \tag{5}
\]

but as \(B_1(-\xi)\) diverges for \(x > 0\), the physical meaning of \(\psi(\xi)\) imposes that \(D_2 = 0\). The constant \(D_1\) is found by normalizing the wave function,

\[
\int_{-\infty}^{\infty} \psi(\xi) \psi^*(\xi') d\xi = \delta(E - E'), \tag{6}
\]

resulting in \(D_1 = (2m)^{1/3}/\hbar^{2/3}C^{1/6}\), and the eigenstates can be written as,

\[
\psi_n(x) = \frac{(2m)^{1/3}}{\hbar^{2/3}C^{1/6}} A_1 \left[ - \frac{(2mC)^{1/3}}{\hbar^{2/3}} \left( x - \frac{E_n}{C} \right) \right]. \tag{7}
\]

In our case, we are interested in the gravitational quantum well (GQW), i.e., when \(C = mg\). In this case,

\[
\psi_n(x) = \frac{(2m)^{1/3}}{\hbar^{2/3}mg^{1/6}} A_1 \left[ - \frac{(2m^2g)^{1/3}}{\hbar^2} \left( x - \frac{E_n}{mg} \right) \right], \tag{8}
\]

where the eigenenergies are obtained by imposing that \(\psi_n(x = 0) = 0\), which results in,

\[
E_n = - \left( \frac{mg^2\hbar^2}{2} \right)^{1/3} a_n, \tag{9}
\]

where \(a_n\) are the zeroes of the Airy functions.

The GQW system is an important one because it possesses bound states due to the gravitational coupling and has been tested in laboratory by using ultracold neutrons (UCN) \[21\], where the spatial distribution was experimentally obtained and it agrees with the theoretical results via Wigner function of the system.
III. THE GQW AS AN EFFECTIVE TWO LEVEL SYSTEM

In this section we will model the GQW Hamiltonian in an effective two level system. This was originally done in Ref. [18], where it was considered a particle confined into a cylindrical potential and under the action of an external magnetic field. In order to clarify the physical meaning of the effective two level system and the strokes involved, we will rewrite the eigenenergies (9) as,

\[ E_n(g) = -\hbar \Omega_{g_0} \left( \frac{g}{g_0} \right)^{2/3} \alpha_n, \]

where \( \Omega_{g_0} = (mg_0^2/2h)^{1/3} \) has unity of frequency and \((g/g_0)^{2/3}\) is a dimensionless quantity. Thus the energy is dependent explicitly on the intensity of the gravitational field.

We are interested in two different cycles as illustrated in Fig.1. The first is the iso-energetic cycle and was originally proposed by Bender [16], having been studied in different contexts [17–19]. It is composed by two iso-entropic and two iso-gravitational strokes. A similar type of cycle was performed in [19] in the case of an external magnetic field. The iso-energetic process is performed theoretically replacing the thermal bath model by an energy bath one [17] and has the property that during the process the expectation value of the Hamiltonian is kept fixed. The second cycle will be called here of iso-gravitational whose is composed by two iso-entropic and two iso-gravitational strokes. A similar type of cycle was perfomed in [19] in the case of an external magnetic field. The iso-gravitational stroke is a new one introduced theoretically here, where the gravitational field is kept fixed during the stroke, with the system performing a transition from the energy \(E_n(g_1) \to E_m(g_1)\). We will start our analysis studying the iso-gravitational cycle because it is mathematically simpler.

A. The Iso-Gravitational Cycle

This cycle is composed by two iso-entropic and two iso-gravitational strokes as depicted in Fig.1. The system starts in the ground state \(\psi_1(g_1)\) with energy \(E_1(g_1)\) and is quasi-statically moved to the first excited state \(\psi_2(g_1)\) with energy \(E_2(g_1)\). As in this stroke the gravitational field, the only external agent, is kept fixed, the work performed on the system is zero and the change in energy is exclusively called of heat and given by,

\[ \langle Q \rangle_{1\rightarrow 2} = E_2(g_1) - E_1(g_1) \]
\[ = -\hbar \Omega_{g_0} \left( \frac{g_1}{g_0} \right)^{2/3} (a_1 - a_2) < 0. \]  

(11)

The second stroke is an iso-entropic expansion and there is no heat exchange. However the system is driven from \(\ell_{g_1}\) to \(\ell_{g_2}\), and then we can define an expansion coefficient \(\alpha \equiv \ell_{g_2}/\ell_{g_1}\). The third stroke is an iso-gravitational one such that it moves the system from the state \(\psi_2(g_2)\) to \(\psi_1(g_2)\) with \(g_2 < g_1\). The heat exchange is then given by,

\[ \langle Q \rangle_{3\rightarrow 4} = E_1(g_2) - E_2(g_1) \]
\[ = -\hbar \Omega_{g_0} \left( \frac{g_1}{g_0} \right)^{2/3} \frac{1}{\alpha^2} (a_2 - a_1) > 0, \]  

(12)

where it was used the definition of \(\alpha\). The last stroke is an iso-entropic compression and again there is no heat exchange. By observing the convention of signal of heat exchange, that is, it is positive when absorbed and negative when released by the system, we can define the thermodynamic efficiency as,

\[ \eta = \frac{\langle W \rangle}{\langle Q \rangle_{3\rightarrow 4}} = 1 - \frac{\langle Q \rangle_{1\rightarrow 2}}{\langle Q \rangle_{3\rightarrow 4}} = 1 - \frac{1}{\alpha^2}, \]

(13)

that is, the efficiency of an iso-gravitational cycle does not depend of the particular choice of \(g_0\), and becomes close to one when \(\alpha\) is very large, which physically means an extremelly difference between \(g_1\) and \(g_2\). Thus, in a realistic point of view, the efficiency of this type of cycle will be very small in practice. The result in Eq. (13) is in agreement with the same obtained in Ref. [31], where the quantum Otto cycle was considered for a single particle into a one-dimensional box and is discussed from an entropy production perspective in Ref. [32].
B. The Iso-Energetic Cycle for the GQW

By analogy with other models which use an iso-energetic stroke to build a quantum heat engine \[17\,19\], the iso-energetic cycle based on the gravitational quantum well is depicted in Fig. 1, and it is composed by two iso-energetic and two iso-entropic strokes. Again, it will be assumed that the iso-energetic strokes are performed by changing the value of the gravitational field quasi-statically in order to keep the expectation value of the Hamiltonian constant. This requirement defines an energy bath. By considering the average energy, \( \langle E \rangle = \langle \hat{H} \rangle \), one has,

\[
\langle E \rangle = \sum_n p_n(g)E_n(g),
\]

(14)

where we have written explicitly the dependence on the intensity of \( g \). The change in the average energy by considering quasi-static strokes which depend exclusively on \( g \) is given by \[17\],

\[
\delta \langle E \rangle = \sum_n E_n(g) \frac{\partial}{\partial g} p_n(g) + \sum_n p_n(g) \frac{\partial}{\partial g} E_n(g),
\]

(15)

where it was defined the quantities,

\[
\delta \langle Q \rangle = E_n(g) \frac{\partial}{\partial g} p_n(g),
\]

(16)

\[
\delta \langle W \rangle = \sum_n p_n(g) \frac{\partial}{\partial g} E_n(g).
\]

(17)

Here, it is important to note that, although Eq. \[15\] reflects the first law of thermodynamics, the quantity \( \delta \langle Q \rangle \) is traditionally associated with a well defined temperature of the system when in contact with a thermal reservoir. As this is not the case for an iso-energetic stroke, where the system is in contact with an energy reservoir, \( \delta \langle Q \rangle \) is known as the energy exchange or simply the heat exchange for convenience of language, while \( \delta \langle W \rangle \) is the work done/performed by/on the system.

By taking into account that work is done when \( p_n(g) \) is kept fixed we can explicitly obtain an expression for the work as been,

\[
\langle W \rangle_{k \rightarrow \ell} = \int_{g_k}^{g_\ell} p_n(g) \frac{\partial E_n(g)}{\partial g} dg |_{p_n(g) \text{ fixed}}
\]

\[
= p_n(g_k)(E_n(g_\ell) - E_n(g_k)).
\]

(18)

For the iso-energetic stroke, the heat exchange between system-reservoir can be obtained analytically from \[16\] and, by considering that the system starts the cycle in the ground state with \( p_1(g_1) = 1 \), and performs a maximal expansion and maximal compression, one has \[18\,19\],

\[
\langle Q \rangle_{1 \rightarrow 2} = E_1(g_1) \ln \left[ \frac{E_1(g_2) - E_2(g_2)}{E_1(g_1) - E_2(g_1)} \right]
\]

\[
+ \int_{g_1}^{g_2} \frac{E_1 \frac{dE_2}{dg} - E_2 \frac{dE_1}{dg}}{E_1(g) - E_2(g)} dg,
\]

(19)

for maximal expansion and,

\[
\langle Q \rangle_{3 \rightarrow 4} = E_2(g_3) \ln \left[ \frac{E_2(g_4) - E_1(g_4)}{E_2(g_3) - E_1(g_3)} \right]
\]

\[
+ \int_{g_3}^{g_4} \frac{E_2 \frac{dE_1}{dg} - E_1 \frac{dE_2}{dg}}{E_2(g) - E_1(g)} dg,
\]

(20)
for the maximal compression.

With the analytical expressions for work and heat, let us now describe the iso-energetic cycle for the GQW in detail. The working substance starts in the ground state \( \psi_1(g_1) \) and with energy \( E_1(g_1) \). The first stroke is an iso-energetic expansion from \( \ell_{g_1} \) to \( \ell_{g_{III}} \). Considering the maximal expansion and defining an expansion coefficient \( c_1 \equiv \ell_{g_{III}}/\ell_{g_1} \) the iso-energetic stroke leads to,

\[
E_1(g_1) = E_2(g_{III}),
\]

which results in,

\[
c_1 = \frac{a_2}{a_1},
\]

For the eigenenergies given by (10), the second term on right in (19) and (20) vanish and for the iso-energetic expansion the heat exchange is given by,

\[
\langle Q \rangle_{I \rightarrow III} = E_1(g_1) \ln \left[ \frac{a_1}{a_2} \right].
\]

The next stroke is an iso-entropic compression characterized by \( p_2(g_{III}) = p_2(g_{IV}) = 1 \). As in the iso-gravitational cycle, it will be convenient to define a compression coefficient expansion \( \alpha \equiv \ell_{g_{IV}}/\ell_{g_{III}} \). The work performed in this stroke can be easily obtained using (18). The third stroke is an iso-energetic compression from \( \ell_{g_{III}} \) to \( \ell_{g_{IV}} \). By defining a compression coefficient, \( c_3 \equiv \ell_{g_{IV}}/\ell_{g_{III}} \), the iso-energetic condition implies in,

\[
E_2(g_{III}) = E_1(g_{IV}),
\]

which results in,

\[
c_3 = \frac{a_1}{a_2},
\]

By solving Eq. (20) for the conditions above one obtains,

\[
\langle Q \rangle_{III \rightarrow IV} = E_2(g_1/(c_1 \alpha) \ln \left[ \frac{a_2}{a_1} \right].
\]

To complete the iso-energetic cycle, an iso-entropic stroke is performed from \( \ell_{g_{IV}} \) to \( \ell_{g_1} \), such that the work performed here can be again obtained using (18). The thermodynamic efficiency of this cycle is given by,

\[
\eta = 1 - \frac{\langle Q \rangle_{III \rightarrow IV}}{\langle Q \rangle_{I \rightarrow III}} = 1 - \frac{a_1}{a_2} \frac{1}{\alpha^2}.
\]

The efficiency for the iso-gravitational and iso-energetic cycles are depicted in Fig. 2. From Eq. (27), it can be observed that the ratio \( a_1/a_\ell \) is the lowest possible when \( \ell = 2 \), which means that one can, in principle, improve the efficiency of the iso-energetic cycle by modeling the effective two level system considering the first excited state and other states with \( \ell > 2 \), but is not possible it surpasses the unit, a physical meaning that must be fulfilled. Another point concerning the iso-energetic cycle is that \( \alpha = \ell_{g_{III}}/\ell_{g_1} = (g_{III}/g_{I})^{1/3} \) can be arbitrarily large, resulting this way in a higher value of \( \eta \). However, it is important to stress that the real value of the efficiency is limited by the length scale of the system, i. e., \( \ell_0 \).

**IV. GENERAL ISO-ENERGETIC CYCLE FOR ANY PAIR OF EIGENSTATES**

Here we will generalize our analysis of section III for the case of arbitrary pair of eigenstates. The idea is to show that it is possible, in principle, to model an iso-energetic cycle for the GQW system with any pair of eigenstates \( \psi_k(g_i) \)
and $\psi_\ell(g_j)$ and thus obtain a general relation for the efficiency. From Eqs. \([11]\) and \([12]\), the heat exchange between the system and the energy reservoirs for general eigenstates are given by,

$$
\langle Q \rangle_{n \rightarrow m} = E_n(g_a) \ln \left[ \frac{E_n(g_b) - E_m(g_b)}{E_n(g_a) - E_m(g_a)} \right],
$$

(28)

$$
\langle Q \rangle_{m \rightarrow n} = E_m(g_c) \ln \left[ \frac{E_m(g_d) - E_n(g_d)}{E_m(g_c) - E_n(g_c)} \right],
$$

(29)

where we have introduced a general notation such that $g_b < g_a$ and $g_c < g_d$, with $E_n$ and $E_m$ being the corresponding energies to the first and second arbitrary eigenstates, respectively. Note that the second term on right in \([19]\) and \([20]\) vanishes independent of the choice of the eigenstates. The Fig. 3 illustrates the first five eigenenergies of the GQW system in order to show the possibility of generating an effective two level system by considering any pair of eigenstates.

By solving these expressions using the eigenenergies \([10]\), one has,

$$
\langle Q \rangle_{n \rightarrow m} = E_n(g_a) \ln \left[ \frac{a_n}{a_m} \right],
$$

(30)

$$
\langle Q \rangle_{m \rightarrow n} = E_m(g_c) \ln \left[ \frac{a_m}{a_n} \right],
$$

(31)

with $a_{n(m)}$ being the root of the Airy function in \([8]\). Thus, through the thermodynamic efficiency, one obtains a general expression for an iso-energetic cycle for the GQW system which is valid for any pair of eigenstates and is given by,

$$
\eta_{\text{General}} = 1 - \frac{E_m(g_c)}{E_n(g_a)}.
$$

(32)

The equation \([32]\) is general and is in complete agreement with the analogous result obtained in Ref. \([34]\), which introduces the notion of energy reservoir. We stress that this equation is equivalent to the Carnot efficiency in the sense that the energy assumes the role of the temperature and thus it is analogous to the classical thermodynamic result. Note that we find this result without any assumptions on the expansion coefficient during the process. The only requirement is that the strokes are performed quasi-statically.

V. EQUATION OF STATE FOR GQW ISO-ENERGETIC CYCLE

Now, we want to describe the iso-energetic stroke for the GQW system through an equation of state. First of all, using the length scale for the gravitational well, $\ell_g = (\hbar^2 / 2m^2g)^{1/3}$, the eigenenergies can be written as follow,

$$
E_n(g) = -\frac{\hbar^2}{2m\ell_g^2} \alpha_n.
$$

(33)

During the first stroke, the iso-energetic expansion, the work done can be attributed to a force-like external agent on the wall of the gravitational well. The contribution to this force from the nth energy eigenstate is defined here as,

$$
f_n = -\frac{\hbar^2}{m\ell_g^3} \alpha_n,
$$

(34)

such that the force $F$ is given by the expectation value,

$$
F = \sum_n p_n f_n.
$$

(35)

Using the Eq. \([14]\) for the expectation value of energy and considering that the working substance starts the cycle in $E_1(g_a)$, one has that $\sum_n p_n E_n(g_a) = E_1(g_a)$, thus we get,
\[ F \ell_g = 2E_1(g_a), \]  

(36)

where, multiplying both sides for \( \ell_g^2 \), one has,

\[ P \ell_g^3 = 2E_1(g_a), \]  

(37)

having defined \( P = F/\ell_g^2 \) as an analogous of pressure and \( \ell_g^3 \) as an equivalent of volume. Equation (37) is similar to the corresponding equation of state for an isothermal process of a classical ideal gas when the analogies \( 2E_1(g_a) \leftrightarrow k_BT \) and \( \ell_g^3 \leftrightarrow V \) are performed. This result reinforces the conceptual definition of thermal bath summarized in the Eq. 32 which replaces the temperatures by the eigenenergies in the efficiency.

VI. CONCLUSIONS

In this work we explore the possibility of modeling the gravitational quantum well as an effective two level system. By considering two cycles for the TLS, the iso-gravitational and the iso-energetic ones, it was possible to verify analytically how the thermodynamics efficiency behaves when varying the iso-entropic expansion coefficient, \( \alpha \), defined in terms of a length scale for the GQW system. The natural length for the system is \( \ell_0 = (\hbar^2/2m^2g_0)^{1/3} \), which is approximately 5.87 \( \mu m \) if we consider the mass of the neutron and the gravitational acceleration of the Earth, that is, greater than the typical size of a semiconductor quantum dot [33], \( \ell_d = 70 \) nm, evidencing, in principle, the physical realization of the model presented here. Another motivation to perform experimentally the iso-gravitational and the iso-energetic cycles, is given in [11], where it was demonstrated a high level of control in implementing a quantum heat engine based on a spin-1/2 system in a nuclear magnetic resonance apparatus. However, due to the extremely weak intensity of the gravitational field the efficiency shall be very small. In this point, it is important to mention that the general scheme developed here is useful also, for instance, the electric field, which can be conveniently controlled in the laboratory in terms of intensity and the particle confined into the GQW could be an electron.

We also obtain a general expression for an iso-energetic cycle generated with any pair of eigenstates of the gravitational quantum well system and we have shown that the equation is in concordance with the obtained in Ref. [34], where the concept of energy bath was introduced. This result shows that the efficiency of the iso-energetic cycle for the GQW system will depend only on the ratio of the two roots of the Airy function and can be modified depending on the distance between the two chosen eigenstates.

As a final result, we derive a relevant relation for the iso-energetic stroke, an analogous equation of state for this process, which is similar to the equation for the isothermal process of an ideal gas. This relation reinforces the substitution of the temperature of the heat bath by the energy concept in the energy bath. Finally we believe that this application of the gravitational quantum well as a base for quantum heat engines could be useful to encourage the application of quantum thermodynamics concepts to this system such as the possibility of generating entangled states or non-equilibrium strokes.

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Figure 1. (Color Online): Iso-gravitational and iso-energetic cycles for the effective two level system modeled by the ground and first excited states of the GQW system. The iso-gravitational cycle involves two iso-gravitational and two iso-entropic strokes (Latin numbers) and the iso-energetic cycles is composed by two iso-energetic and two iso-entropic strokes (Arabic numbers). It was considered $g_0 = 10 \text{m/s}^2$.

Figure 2. (Color Online): Thermodynamic efficiency for the iso-gravitational (red) and iso-energetic (blue) cycles for gravitational quantum well as an effective two level system.

Figure 3. (Color Online): First five eigenenergies of the GQW system to illustrate the possibility of generating an effective two level system with any pair of eigenstates. The cycle can be performed with any two arbitrary states. It was considered $g_0 = 10 \text{m/s}^2$. 