Chiral Magnetic Skyrmions with Arbitrary Topological Charge
("Skyrmionic Sacks")

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We show that the standard micromagnetic model with chiral interactions provides the existence of an infinite number of stable soliton solutions of any integer topological charge. A detailed description of the morphology of new skyrmions and corresponding energy dependencies are provided. Due to the generality of the model and presented approach it is expected that the variety of predicted particlelike states can be experimentally realized in many of the currently studied chiral magnets.

The existence of stable localized vortices in magnets was predicted in 1989 [1]. In the modern literature the authors typically refer to these objects as chiral magnetic skyrmions. In contrast to baby skyrmions [2, 3] and superconducting skyrmions [4], magnetic skyrmions are not explicitly or implicitly [5] related to the model for baryons proposed by T.H.R. Skyrme [6, 7]. Nevertheless, similar to other skyrmions such magnetic vortices also can be characterized by nonzero topological charge \(Q\) and in a certain sense they mimic the Skyrme mechanism of stabilization. The latter means that corresponding micromagnetic Hamiltonian contains competing terms corresponding to different power of spatial derivatives of the order parameter. The presence of such terms makes model of chiral magnets a rare exception [8] to the Hobart-Derrick theorem asserting that existence of stable localized solutions is forbidden in overwhelming majority of the models in nonlinear physics [9].

The research in this field has gotten a powerful impetus after the direct observation of magnetic skyrmions in cubic chiral magnets of B20-type with Transmission Electron Microscopy [10]. Later, the existence of magnetic skyrmions have been confirmed in many other materials [11, 12, 13, 14, 15]. Moreover, many new phenomenon as for instance the different approach for the nucleation of magnetic skyrmions [11, 16], the electric current induced motion of skyrmions [17, 18], attractive and repulsive inter-skyrmion interactions [19, 20] and many others have been reported recently. The possible application of magnetic skyrmions in spintronics is also under intensive study [21, 22, 23, 24, 25]. Nevertheless, despite the huge number of experimental and theoretical works published within last decade about chiral magnetic skyrmions one of the crucial question related to the diversity of possible skyrmion solutions has so far remained unexplored.

In contrast to some two-dimensional (2D) models as for instance baby Skyrme [2, 3] and isotropic ferromagnet [26], which possess a huge variety of solutions with arbitrary \(Q\), it was generally assumed that chiral interactions in magnetic systems restrict the diversity of skyrmions. Indeed, in the framework of standard 2D micromagnetic model with chiral Dzyaloshinskii-Moriya [27, 28] interaction (DMI) only one type of localized static solutions is known – the skyrmion with \(Q = −1\) [1] or its generalization representing Bogdanov-Hubert kπ-vortices [29] with alternating \(Q = −1\) and \(Q = 0\) for odd and even \(k\), respectively. In the literature the 2π-vortex with \(Q = 0\) is also known as skyrmionium [30, 31]. Moreover, recently in Ref. [32] it was reported that in frame of standard 2D micromagnetic model with isotropic DMI a stable antiskyrmion solution, which should in this case correspond to \(Q = 1\) can not exist. Note, everywhere in this Letter we follow the convention for topological charge sign proposed in Ref. [33]. Moreover, we follow the classification of skyrmion solutions given in Ref. [1] where it was emphasized that all solutions mutually transformed by trivial operations of reflection and/or rotation may be considered as solutions of the same class (see also [34]). Thereby, stable magnetic textures recently discovered in tetragonal Heusler alloy [14] and named “antskyrmion” also belong to the same class of solutions as other \(Q = −1\) skyrmions observed earlier. In such materials the above problem with diversity of solutions is absolutely the same but corresponding topological charge has an extra multiplier equal to \(-1\).

In this Letter it is shown that in fact the presence of chiral interactions does not prevent the diversity of skyrmionic solutions, but rather supports the stability of an infinite set of skyrmions of various morphology and different topological charge. The variety of discussed here solutions was simply overlooked earlier. The results presented in this Letter extend the theory of chiral magnetic skyrmions, suggest a new research direction for experimental physics of skyrmions, and may have significant impact on the practical applications of magnetic skyrmions.

Our approach is based on a general model of chiral magnet [35, 36] with continuous two-dimensional Hamiltonian [1], which contains three energy terms:

\[ E = \int \left( A \sum_i (\nabla n_i)^2 + D \mathbf{w}(n) + U(n_z) \right) dx dy, \]  

where \(n \equiv n(r)\) is a continuous unit vector field \((n_x^2 + n_y^2 + n_z^2 = 1)\) determining the direction of the magnetization \(\mathbf{M} = M_n \mathbf{n}, \mathbf{A}\) and \(D\) are the micromagnetic constants for exchange and DMI, respectively, and \(t\) is a thickness of the layer. The last term in (1) represents either the interaction with the external magnetic field \(\mathbf{B}_{\text{ext}}\) or the...
Zeeman energy term: $U_2 = B_{\text{ext}} M_0 (1 - n_z)$, or the energy density of uniaxial anisotropy $U_u = K (1 - n_z^2)$ or combination of both terms. The DMI term $w(n)$ is represented a linear combination of Lifshitz invariants

$$A_{ij}^{(k)} = n_i \frac{\partial n_j}{\partial r_k} - n_j \frac{\partial n_i}{\partial r_k},$$

which in turn is defined by the structural symmetry of the crystal.

Presented below results are valid for a very wide class of chiral magnets of various lattice symmetry: for the systems with Néel-type chiral modulations $w(n) = \Lambda_{x^2}^{(z)} + \Lambda_{y^2}^{(y)}$, for the tetragonal compounds of D24 symmetry $w(n) = \Lambda_{x^2}^{(x)} + \Lambda_{x^2}^{(z)}$ and for the crystals with bulk-type DMI and Bloch-type chiral modulations $w(n) = \Lambda_{x^2}^{(x)} + \Lambda_{x^2}^{(y)} + \Lambda_{x^2}^{(z)} = n \cdot (\nabla \times n)$. Note, for full compliance with our findings in the case of bulk-DMI the influence of $\Lambda_{x^2}^{(z)}$ term (playing crucial role in certain cases) should be effectively suppressed for instance by induced uniaxial anisotropy.

For convenience in the following discussion the length, the external magnetic field and energy are given in the dimensionless units of $L_D = 4 \pi A/|D|$ – the equilibrium period of helical spin spiral $[41, 42]$, $B_D = D^2/(2M_s A)$ – the critical field of the cone spiral transition into saturated state [42] and energy $E_\text{0} = 2A t$ respectively. Thus, the behavior of the system depends only on two dimensionless parameters

$$h = B_{\text{ext}}/B_D, \quad u = K/(M_s B_D).$$

The localized solutions are the excitations on the homogenous background meaning that $n(r) \rightarrow n_0$ for $|r| \rightarrow \infty$. Thus, the domain of definition of the order parameter $n(r)$ can be mapped to a sphere which can be associated with Riemann sphere $(R^2 \cup \{\infty\} \leftrightarrow S^2)$. The space of the order parameter $n$ is in turn also a sphere $S^2_{\text{spin}}$. The map $S^2 \rightarrow S^2_{\text{spin}}$ lead to homotopy classification of localized solutions in 2D with topological invariants related with integer index

$$Q = \frac{1}{4\pi} \int \left( n \cdot (\partial_x n \times \partial_y n) \right) dx dy. \quad (2)$$

For topologically non-trivial textures with $Q \neq 0$ a smooth transformation into homogeneous state $n(r) = n_0$ is impossible.

The earlier found stable solution for the skyrmion with $Q = -1$ and with energy below [1] and above [43] the saturated state as we shown below represent just one example of topologically non-trivial states, see Fig. 1. For the case of $u = 0$ and high magnetic fields $h \geq 2$ it was proved [33] that such skyrmion is energetically favorable among all other hypothetical textures with $Q \neq 0$. It will be shown below that such skyrmion with $Q = -1$ and skyrmion with $Q = 0$ are the key elements or “building blocks” of which the whole variety of other skyrmions is “constructed”.

The fact that stable solutions for skyrmions with $Q < -1$ and $Q > 0$ have been overlooked earlier may have the following explanation. First, a naive attempts to find $Q = -2$ skyrmion using numerical methods by placing a pair of $Q = -1$ particles as initial guess instead merging of skyrmions into one particle results in their scattering, i.e. so-called dichotomy [44]. The latter is because of well known effect of interparticle repulsion of chiral skyrmions [8]. Second, in contrast to isotropic ferromagnet [26] or baby Skyrmion model [2], equations for (1) does not contain axisymmetric solutions with arbitrary integer $Q$.

To find energetically stable solutions of the functional (1) we performed the direct energy minimization based on the nonlinear conjugate gradient method which in our implementation has been massively parallelized and optimized for NVIDIA CUDA architecture. In order to achieve the highest performance and stability of the energy minimization routine we use “Atlas” subroutine which automatically keep the constraint $n^2 = 1$, for details see Supplementary Material in Ref. [45]. We use finite-difference discretization scheme of the fourth-order on the simulated domain with a very dense mesh of $640 \times 640$ nodes. The values of variables $A$ and $D$ have been chosen such that parameter $L_D$ equals 32 inter-node distances. For the cases close to critical phenomena $L_D$ has been set to 52 inter-nodes distances.

Figure 1. (color online). Morphology of stable chiral skyrmions with topological charges $Q = -3, -2, ..., 2$. Top row of images (a) corresponds to isotropic magnet ($u = 0$) in external magnetic field applied perpendicular to the plane, $h = 0.65$; Bottom row of images (b) corresponds to the case of perpendicular anisotropy, $u = 1.3$ and zero external field, $h = 0$. All images are given in the same scale. Colors encode the direction of the $n$-vectors according to standard scheme [40]: black and white denote up and down spins respectively, red-green-blue reflect azimuthal angle with respect to $x$-axis.
One of the key features of our implementation of the code is the graphic user interface with interactive regime allowing in situ control of the magnetic configuration as well as easy way to construct a large variety of initial states. In particular, in the interactive regime one can flip of the spins inside a certain volume under the mouse pointer. This provides an efficient approach for construction of complex initial configurations composed of the domains with magnetization pointing either up or down. After a certain number of iterations of the energy minimization routine the initial configuration converges to one of the nearest energy minimum. Beside the calculation of standard termination criteria \cite{46} one can also perform an in situ examination of the stability by introducing small excitations and perturbations to the calculated configuration. Supplementary movies \cite{47} illustrate the process of manufacturing of the initial state for anticipated spin texture, relaxation and manipulation with magnetic configuration for parameters as in Fig. 3. To emphasize the isomorphism of systems with different Lifshitz invariants we tried to followed the same scenario and prepared three distinct movies corresponding to $C_{nv}$, $D_{2d}$ and $D_h$ cases.

To obtain $Q < 1$ solutions we put several distinct $Q = -1$ skyrmionic cores inside a “sack” representing a closed $2\pi$ domain wall which as well as skyrmion has no topological charge, $Q = 0$ (see Fig. 1). While such closed domain wall tends to shrink to the equilibrium size of skyrmionium, internal particles prevent such a shrinking due to the inter-particle repulsion. Similar to the effect of surface tension, the balance of external and internal pressures leads to the stability of such configuration. For $Q > 1$ the role of a “sack” plays closed $\pi$ domain wall having self topological charge, $Q = -1$. As a result the domain within the closed loop has magnetization opposite to the surrounding ferromagnetic state while vortices inside of it have a polarity in the cores as ferromagnetic vacuum and self topological charge $Q = 1$ for each vortex. The integration of the topological density (2) gives the total topological charge which is equal to the number of the “holes” minus one (see Fig. 1 for $Q = 1, 2$). Here we discuss the solution for skyrmions in the range of $Q = -10, -9, ..., 10$ only, however, it is easy to show that this series in general can be extend up to infinity.

The energy dependence of skyrmions is found to be a near-linear piecewise function of their topological charge, see Figs. 2, 3. Strictly speaking, the points do not lie on the straight lines, but show very weak deviations from the linear law similar to the baby Skyrme model \cite{2, 3}.

To illustrate the behavior of the system in isotropic case, Fig. 2, the value of parameter $h$ has been chosen to be above the value of elliptical instability \cite{48} and smaller than 0.8 to guarantee the thermodynamic stability of the $Q = -1$ skyrmion \cite{49}, i.e. $E_{Q=-1} < 0$. The right branch of the “spectrum” for $Q \geq -1$, Fig. 2, increases monotonically with $Q$. In contrast to that the left branch of the “spectrum” ($Q < -1$) demonstrates the opposite behavior and the energy decreases with $|Q|$. This feature reflects the fact that the global energy minimum corresponds to hexagonal skyrmion lattice \cite{49} and the left branch of “spectrum” tends to such a lattice as $Q \to -\infty$.

Significantly, the set of the solutions contains also the states with higher energies. In Fig. 2 we have shown the energies of only those states which corresponds to the smallest energy shift. Some of that solutions are shown in inset. Note, the $Q = -1$ skyrmion in inset of Fig. 2 corresponds to earlier known solution of $3\pi$-vortex \cite{29}. We suppose that the set of the solutions corresponding to the lowest energy states, open circles in Fig. 2, represents true minimizers in the corresponding topological sectors, however, exact mathematical proof for that is required.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig2.png}
\caption{(color online). The energy of skyrmions, $E$ as function of topological charge, $Q$ for the case of isotropic magnet, $u = 0$. Open circles are the lowest energy solutions for each particular $Q$, solid squares – solutions with higher energies but nearest to the lowest energy state. The straight lines are linear fit for corresponding sets of points.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig3.png}
\caption{(color online). The energy of skyrmions, $E$ as function of topological charge, $Q$ for the case of perpendicular anisotropy, $u = 0.65$ in external magnetic field $h = 0.3$ (a) and $h = 0.4$ (b). The slope of the dashed line is defined as $\tan(\alpha) = 4\pi$. The point marked by four thick arrows in (b) corresponds to the solution expected from the linear fit of the energy dependence and with morphology shown in inset in (a), but precise calculations shows instability of expected skyrmion. The morphology of stable skyrmion with $Q = -2$ is shown in inset.}
\end{figure}

For the case of nonzero out-of-plane uniaxial anisotropy, Fig. 3, we used the value $u = 0.65$ which is according to Ref. \cite{37} corresponds to the bilayer of PdFe on an Ir(111) single crystal substrat. We performed calculations for two values of $h$ for which $E_{Q=-1} > 0$. Both branches of the spectrum (Fig. 3) now demonstrate the same trend. Above certain values of $h$, the energy of some points of the left branch of spectrum become higher than the critical Dirichlet energy \cite{26, 50} shifted on the
corresponding energy of skyrmionium, see dashed line in Fig. 3(a),(b). This means that corresponding expected solutions become unstable, see for instance the marked point in Fig. 3(b). The pressure from the shell become too high and lead to the shrinking [50] of the internal $\pi$-vortices. This fact indicates that the system allows so-called dynamical blow-up phenomena [51], i.e. the divergence of the gradients of the order parameter for a finite time. To initiate this process it is enough to stabilize the initial state corresponding to $Q = -2$ case showed on Fig. 3(a) and then increase the external magnetic field. From that one may conclude that in the case of such critical phenomena, a certain solution of higher energy branch becomes the minimizer for corresponding $Q$, see Fig. 3(b).

In conclusion, we have shown that standard micromagnetic model with chiral Dzyaloshinskii-Moriya interaction allows an existence of chiral magnetic skyrmions with any integer topological charge. The morphology of new solutions with $Q < -1$ and $Q > 0$ are described in details and is found to be sufficiently differs from the systems where the inter-particle attraction naturally leads to the formation of clusters [4, 52, 53, 54]. The energies of new skyrmions are found to be comparable and controllable by external magnetic field. The latter suggests that direct observation of a large variety of new particle-like states presented here should be accessible in experiment. We suppose that the nucleation of new skyrmions possibly can be realized with the Scanning Tunneling Microscope equipped with the magnetic tip of special shape [55] or by means of time-dependent external magnetic field and current induced spin torque effect in geometrically confined systems. Found solitons can be considered as information bit carriers in a skyrmion racetrack memory and may extend a newly proposed concept of “two-particles” for binary data encoding, as skyrmion-bobber chains in cubic chiral magnets [25] or sequences of skyrmions and antiskyrmions in spatially anisotropic ultra thin films [34].

Finally, it has to be emphasized that found solutions with $|Q| > 1$ are natural high charge skyrmions of corresponding model and $Q=1$ solution could be considered as a new type of antiskyrmion. The solutions with $Q < -1$ have no relations to the clusters of distinct skyrmions which may appear under confinement as skyrmions embedded in some inhomogeneous phase [56] or skyrmions in small-size samples [57].

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