New Potential-Based Bounds for Prediction with Expert Advice

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Abstract
This work addresses the classic machine learning problem of online prediction with expert advice. We consider the finite-horizon version of this zero-sum, two-person game.

Using verification arguments from optimal control theory, we view the task of finding better lower and upper bounds on the value of the game (regret) as the problem of finding better sub- and supersolutions of certain partial differential equations (PDEs). These sub- and supersolutions serve as the potentials for player and adversary strategies, which lead to the corresponding bounds. Our techniques extend in a nonasymptotic setting the recent work of Drenska and Kohn (J. Nonlinear Sci. 2019), which showed that the asymptotically optimal value function is the unique solution of an associated nonlinear PDE.

To get explicit bounds, we use closed-form solutions of specific PDEs. Our bounds hold for any fixed number of experts and any time-horizon \( T \); in certain regimes (which we identify) they improve upon the previous state-of-the-art.

For up to three experts, our bounds provide the asymptotically optimal leading order term. Therefore, we provide a continuum perspective on recent work on optimal strategies for the case of \( N \leq 3 \) experts.

We expect that our framework could be used to systematize and advance theory and applications of online learning in other settings as well.

1. Introduction

The classic machine learning problem of online prediction with expert advice (the expert problem) is a repeated two-person zero-sum game with the following structure. At each round, the predictor (player) uses guidance from a collection of experts with the goal of minimizing the difference (regret) between the player’s loss and that of the best performing expert in hindsight. The environment (adversary) determines the losses of each expert for that round. The player’s selection of the experts and the adversary’s choice of the loss for each expert are revealed to both parties, and this prediction process is repeated until the final round.

This problem arises in the context of applications of data science and machine learning in adversarial environments, such as aggregation of political polls (Roughgarden and
Schrijvers, 2017), portfolio allocation and trading (Agarwal et al., 2010; Dayri and Phadnis, 2016), cybersecurity (Truong et al., 2018) and cancer screening (Zhdanov et al., 2009; Morino et al., 2015). The experts framework also appears in other contexts where the data had no obvious distributional assumptions, such as neural architecture search (Nayman et al., 2019), online shortest path in graphs (Kalai and Vempala, 2005), signal processing (Singer and Kozat, 2010; Harrington, 2003), memory caching and energy saving (Gramacy et al., 2003; Helmbold et al., 2000). This framework has also been used to design approximation algorithms for provably hard off-line problems, such as the similarity mapping (Rakhlin et al., 2007). More broadly, the expert framework has been viewed as a meta-learning algorithm seeking to achieve the performance of the best among several constituent learning algorithms (robust model selection) (Bubeck, 2011).

The expert problem has several formulations, which reflect, among other things, differences in the flow of information, classes of loss functions, randomization of the strategies, as well as whether or not the regret is assessed in expectation. We will focus on the following representative definition of the expert problem, which mirrors (up to a trivial translation and rescaling of the loss) the version considered in recent work on optimal strategies (Gravin et al., 2016; Abbasi-Yadkori et al., 2017). However, we expect our approach to be broadly applicable in the expert setting.

**Prediction with expert advice:** At each period \( t \in [T] \) until the final time,

- the **player** determines which of the \( N \) experts to follow by selecting a discrete probability distribution \( p_t \in \Delta_N \);
- the **adversary** determines the allocation of losses to the experts by selecting a probability distribution \( a_t \) over the hypercube \([-1, 1]^N\); and
- the expert losses \( q_t \in [-1, 1]^N \) and the player’s choice of the expert \( I_t \in [N] \) are sampled from \( a_t \) and \( p_t \), respectively, and revealed to both parties.

We consider the finite horizon version, where the number of periods \( T \) is fixed and the regret is \( R_T(p, a) = \mathbb{E}_{p,a} \left[ \sum_{t \in [T]} (q_t)_{I_t} - \min_i \sum_{t \in [T]} (q_t)_i \right] \) where the joint distributions \( a = (a_t)_{t \in [T]} \) and \( p = (p_t)_{t \in [T]} \) refer to, respectively, the adversary and player strategies or simply the adversary and player. (In the literature survey that follows, we also mention work on the geometric stopping version of the game, where the final time \( T \) is not fixed but is rather random, chosen from the geometric distribution.)

Numerous strategies attain vanishing per round regret. For example, the exponentially weighted forecaster (Exp) strategy \( p^\epsilon \) provides the non-asymptotic upper bound for all \( a \):

\[
R_T(p^\epsilon, a) \leq \sqrt{2T \log N}.
\]

Also for all \( \epsilon > 0 \), there exist \( N \) and \( T \) sufficiently large, such that a randomized adversary \( a^\epsilon \) approaches that upper bound \((1 - \epsilon)\sqrt{2T \log N} \leq R_T(p, a^\epsilon) \) for all player strategies.\(^1\)

A **minmax optimal player** is a player strategy that minimizes the regret over all possible adversary strategies and a **minmax optimal adversary** is an adversary strategy that

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\(^1\) See Cesa-Bianchi et al. (1997) and Theorem 2.2 and 3.7 in Cesa-Bianchi and Lugosi (2006). These results are rescaled to apply to \([-1, 1]^N\), instead of \([0, 1]^N\), losses.
maximizes the regret over all possible player strategies. Thus, for instance, $p_e$ and $a_r$ are asymptotically minmax optimal.

Nonasymptotic minmax optimal strategies were determined explicitly using random walk methods (a) for $N = 2$ in the finite horizon and geometric settings, (b) for $N = 3$ in the geometric setting and, up to the leading order term, in the finite horizon setting (Cover, 1966; Gravin et al., 2016; Abbasi-Yadkori et al., 2017). For general $N$, minmax optimal strategies are given by a recursion, which can be computed by a dynamic program of size $O(T^N)$ (Gravin et al., 2016). However, the optimal strategies have not been determined explicitly.

In a related line of work, strategies that are optimal asymptotically (in $T$ or the mean of the geometric distribution $\frac{1}{\delta}$, as applicable) were determined by PDE-based methods. For $N = 2$, Zhu (2014) established that the value function in the fixed horizon problem is given by a solution of a 1D linear heat equation, which provides a continuous perspective on the earlier random walk characterization of the non-asymptotic problem. Drenska and Kohn (2019) showed that, for any fixed $N$, the value function, in the scaling limit of each of the fixed horizon and geometric problems, is the unique solution of an associated nonlinear PDE. The last reference also gave a closed-form solution of the geometric stopping PDE for $N = 3$; Bayraktar et al. (2019) determined a closed form solution of the geometric stopping PDE for $N = 4$.

Due to the complexity of determining minmax optimal strategies for an arbitrary fixed $N$, it is common to use potential functions to bound the regret above for all possible adversary strategies. For example, Exp could be viewed as a descent strategy for the entropy potential; the corresponding upper bound is obtained by bounding the evolution of this potential for all possible adversaries.

Rakhlin et al. (2012) proposed a principled way of deriving potential-based player strategies by bounding above the value function, conditional on the realized losses, in a manner that is consistent with its recursive minmax form. In the PDE setting, Rokhlin (2017) suggested using supersolutions of the asymptotic PDE as potentials for player strategies in the scaling limit. The present paper extends these ideas by applying related arguments to the original problem (not a scaling limit), and by providing numerous examples (including lower as well as upper bounds).

Adversary strategies have been commonly studied as random processes. For example, the randomized adversary $a^\tau$ mentioned above guarantees that the regret is given by the expectation of the maximum of $N$ i.i.d. Gaussians with mean zero and variance $T$. This guarantee is based on the central limit theorem and is therefore asymptotic in $T$. Nonasymptotic lower bounds have been established using random walk methods (Orabona and Pal, 2017; György et al.).

While the player and the adversary may use randomization in their strategies, the deterministic control paradigm fully describes the adversarial experts framework. Accordingly, in this paper we propose a control-based framework for designing provably robust and efficient strategies for the expert problem using sub- and supersolutions of certain PDEs.

Our principal contributions are the following:

1. The potential-based framework is extended to adversary strategies, leading to lower bounds (Section 3).
2. The bounds hold for any fixed number of experts and are nonasymptotic in $T$; their rate of convergence to the asymptotic (in $T$) value can be determined explicitly using error estimates similar to those applied to finite difference schemes in numerical analysis. (Theorems 1 and 3). For lower bounds, this rate is determined by the smoothness of the relevant potentials (Remark 2).

3. The task of finding better regret bounds reduces to the mathematical problem of finding better subsolutions and supersolutions of certain PDEs (See Equations (2) and (5)).

4. Our framework is based on elementary ”verification” arguments from the optimal control theory and does not rely on a scaling argument (Appendices A and B). Therefore, the final value function no longer needs to be homogeneous to satisfy the scaling property, which increases the range of possible applications of our methods.

5. To get explicit bounds, we use the classical solution of the linear heat equation with suitable diffusion factors as lower and upper bound potentials (Section 5).
   (a) The resulting lower bound is expressed as the expectation of the maximum of $N$ i.i.d. Gaussians with mean zero, and is therefore similar to the existing lower bounds of randomized strategies. However, the constant factor of the leading order term (i.e., the standard deviation of the Gaussians) is, to the best of our knowledge, state-of-the-art (Section 7).
   (b) Accordingly, the resulting lower bound improves the existing non-asymptotic lower bounds for general fixed $N$ and relatively large, but fixed, $T$ (Section 7).

6. To get another family of bounds, we introduce new upper and lower bound potentials using a closed-form solution of a nonlinear PDE based on the largest diagonal entry of the Hessian (Section 6).
   (a) For up to three experts, the lower and upper bounds for this potential provide a matching leading order term. Therefore, the corresponding strategies are min-max optimal at the leading order (Section 7).
   (b) The same leading order constant for three experts was determined in Abbasi-Yadkori et al. (2017) with the regret scaling as $4\sqrt{\frac{2}{9\pi}T} \pm O((\log T)^2)$ (for our $[-1, 1]^N$ loss function). Our PDE-based strategy, however, improves the guarantee with respect to the lower order (error) term: $4\sqrt{\frac{2}{9\pi}T} \pm O(\log T)$ (Section 7).
   (c) The resulting upper bound is tighter than the bound of the exponentially weighed average strategy for $N \leq 10$ and relatively large but fixed $T$ (Section 7).

7. Lastly, our framework leads to efficient strategies. For example, the explicit adversary strategies set forth in this paper do not require runtime computations involving the potential or its derivatives; moreover, those strategies are time independent. This illustrates the feasibility of the framework for high-dimensional problems.

We expect that our framework could be also used to systematize and advance theory and practice of online learning in other settings as well.
2. Notation

We will use the following notation. For a multi-index $I$, $\partial_I$ refers to the partial derivative and $d_I$ refers to the differential with respect to the spatial variable(s) in $I$, and $d^*_I$ refers to the differential with respect to all except the spatial variables in $I$. $D^3u[q,q,q]$ and $D^4u[q,q,q,q]$ denotes the 3-rd and 4-th derivative in the direction of $q$ given by the linear forms $\sum_{i,j,k} \partial_{ijk} u q_i q_j q_k$ and $\sum_{i,j,k,l} \partial_{ijkl} u q_i q_j q_k q_l$, respectively. Whenever the region of integration is omitted, it is assumed to be $\mathbb{R}^N$.

$[T]$ denotes the set $\{1, \ldots, T\}$ if $T \geq 1$ or $\{-T, \ldots, -1\}$ if $T \leq -1$. $\mathbb{1}$ is a vector in $\mathbb{R}^N$ with all components equal to 1, and $\mathbb{1}_S$ refers to the indicator function of the set $S$. $\Delta_N$ refers to a discrete probability distribution over $N$ outcomes. Whenever the feasible set of $q$ is omitted, it is assumed to be $[-1,1]^N$.

A classical solution of a partial differential equation (PDE) on a specified region is a solution such that all derivatives appearing in the statement of the PDE exist and are continuous on the specified region.

To bound the fixed horizon regret, we will apply the dynamic programming principle backwards from the final time. Since in this setting it is convenient to denote the time $t$ by nonpositive numbers such that the starting time is $T \leq -1$ and the final time is zero, we will use this convention in the remainder of this paper.

Let the vector $r_\tau = (q_\tau)_t 1 - q_\tau$ denote the player’s losses realized in round $\tau$ relative to those of each expert (instantaneous regret) and let the vector $x = \sum_{\tau < t} r_\tau$ denote the player’s cumulative losses realized before the outcome of round $t$ relative to those of each expert (cumulative regret or simply the regret).

3. Lower Bound

When the prediction process starts at given $x$ and $t$, the fixed horizon value function $v_a$ reflecting the worst-case (smallest) regret at the final time for a given adversary $a$ is constructed by a dynamic program (DP) backwards from the final time (functions constructed in this manner are referred to as Bellman functions).

$$v_a(x, 0) = \max_i x_i$$

$$v_a(x, t) = \min_{p_t} \mathbb{E}_{a_t, p_t} v_a(x + r_t, t + 1) \text{ for } t \leq -1$$

This reflects the fact that the optimal player $p_t$ against $a$ depends only on $t$ and the cumulative history represented by $x$, rather than the full history $(I_T, \ldots, I_{t-1}, q_T, \ldots, q_{t-1})$.

In the context of lower bounds in this paper, we will only consider those adversary strategies that assign the same probability to $q$ and $-q$ for all $q \in [-1,1]^N$ (symmetric strategies).

Note that $v_a(0, T) \leq R_T(a, p)$ for all $p$. To bound the regret below, we introduce the following potential function $u$, or simply potential.
Lower-bound potential $u$: We will use this term for a function $u : \mathbb{R}^N \times \mathbb{R}_{\leq 0} \to \mathbb{R}$, such that, for every $x \in \mathbb{R}^N$ and $t < 0$, there exists some symmetric probability distribution $a_t$ on $[-1,1]^N$ ensuring that $u$ is a classical solution of

$$
\begin{align*}
    u_t + \frac{1}{2} \mathbb{E}_a (D^2 u \cdot q, q) &\geq 0 \\
    u(x, 0) &= \max_i x_i \\
    u(x + c, t) &= u(x, t) + c
\end{align*}
$$

(2)

Adversary strategy $a$: Given $u$ as above, the associated strategy $a$ is: At each period $t = T, \ldots, -2$, the adversary selects a symmetric strategy $a_t$ such that (2) is satisfied at $(x_t, t+1)$, and, at $t = -1$, selects an arbitrary probability distribution over $[-1,1]^N$.

In this setting, as confirmed in Appendix A, the adversary can eliminate the first spatial derivative for all choices of $p$, and use (2) to control the sum of the second-order spatial derivative and the first-order time derivative. Finally, by controlling (i) the decrease of $u$ in the final round and (ii) the higher-order terms of its Taylor polynomial in the earlier rounds, this strategy attains the following lower bound.

**Theorem 1 (Lower bound)** If for all $x$, (i) $\min_{p \neq -1, p \neq -1} [u(x + r, 0)] - u(x, -1) \geq -C$, and (ii) for $t \leq -2$, $u_t(x, \cdot), D^2 u(\cdot, t + 1)$ are Lipschitz continuous, and for any $q_t$ sampled from $a_t$,

$$
\frac{1}{6} \ess sup_{y \in [x, x - q_t]} D^3 u(y, t + 1)[q_t, q_t, q_t] + \frac{1}{2} \ess sup_{\tau \in [t, t+1]} u_{tt}(x, \tau) \leq K(t)
$$

then $u(x, t) - E(t) \leq v_a(x, t)$, where $E(t) = C + \sum_{\tau=t}^{T-2} K(\tau)$.

**Remark 2 (Lower bound - Lipschitz continuous higher-order derivatives)** If $u$ has higher order Lipschitz continuous derivatives, they could be used to bound $v_a$. For example, if, for all $x$ and $t \leq -2$, $D^3 u(\cdot, t + 1)$ exists and is Lipschitz continuous, and for any $q_t$ sampled from $a_t$,

$$
-\frac{1}{24} \ess inf_{y \in [x, x - q_t]} D^4 u(y, t + 1)[q_t, q_t, q_t, q_t] + \frac{1}{2} \ess sup_{\tau \in [t, t+1]} u_{tt}(x, \tau) \leq K(t)
$$

then the conclusion of Theorem 1 still holds.

If a lower bound potential has higher-order Lipschitz continuous derivatives, the result in Remark 2 could be used to get better error estimates. For example, in the context of the heat potential discussed below, we use this result to bound the error uniformly in $T$.

**4. Upper Bound**

In parallel to the discussion above, the value function $v_p$ reflecting the worst-case (largest) regret at the final time inflicted on a given player $p$ is constructed by the following dynamic
As noted in connection with (1) as well, this reflects the fact that an optimal adversary \( a \) against \( p \) depends only on \( t \) and the cumulative history represented by \( x \), rather than the full history of losses and player's choices of experts in each period.

Note that \( R_T(a, p) \leq v_p(0, T) \) for all \( a \). To bound the regret above, we introduce a potential \( w \).

**Upper-bound potential \( w \):** We use this term for a function \( w : \mathbb{R}^N \times \mathbb{R}_{<0} \to \mathbb{R} \), which is nondecreasing as a function of each \( x_i \), and which is, for all \( x \in \mathbb{R}^N \), \( t < 0 \) and \( q \in [-1, 1]^N \), a classical solution of

\[
\begin{cases}
    w_t + \frac{1}{2} \langle D^2 w \cdot q, q \rangle \leq 0 \\
    w(x, 0) = \max_i x_i \\
    w(x + c \mathbf{1}, t) = w(x, t) + c
\end{cases}
\]  

**Player strategy \( p \):** Given \( w \) as above, the associated player strategy is: At each period \( t = T, \ldots, -2 \), the player selects \( p_t = \nabla w(x, t + 1) \), and, at \( t = -1 \), the player selects an arbitrary distribution in \( \Delta_N \).

Since \( w \) is nondecreasing as a function of each each \( x_i \) and \( \sum_i \partial_i w = 1 \) by linearity of \( w \) along \( \mathbf{1} \), \( p_t \in \Delta_N \).

In this setting, as confirmed in Appendix B, the player can eliminate the first spatial derivative for all choices of \( q \), and use (5) to control the sum of the second-order spatial derivative and the first-order time derivative. Thus, using an approach similar to that for Theorem 1 to control the increase of \( u \) in the final round and the remaining terms of the Taylor sum in the earlier rounds, \( p \) attains the following upper bound.

**Theorem 3 (Upper bound)** If for all \( x \), (i) \( \max_{a_{t-1}} \sum_{a_i} x_i - w(x + r_{t-1}, 0) - w(x, -1) \leq C \), and (ii) for \( t \leq -2 \), \( w_t(x, \cdot) \) and \( D^2 w(\cdot, t + 1) \) are Lipschitz continuous and for all \( q \in [-1, 1]^N \)

\[
\frac{1}{6} \text{ess inf}_{y \in [x, x - q]} D^3 w(y, t + 1)[q_t, q_t, q_t] + \frac{1}{2} \text{ess inf}_{t \in [t, t + 1]} w_{tt}(x, \tau) \geq -K(t)
\]

then \( v_p(x, t) \leq w(x, t) + E(t) \) where \( E(t) = C + \sum_{\tau=t}^{t-2} K(\tau) \).

**5. Heat Equation-Based Potentials**

In this subsection, we consider a specific potential \( u \) given by

\[
u(x, t) = \alpha \int e^{-\frac{||y||^2}{2\sigma^2}} \max_{k} (x_k - y_k) dy
\]  

\[\text{(6)}\]
where $\alpha = (2\pi \sigma^2)^{-\frac{N}{2}}$ and $\sigma^2 = -2\kappa t$. This potential is the classical solution, on $\mathbb{R}^N \times \mathbb{R}_{<0}$, of the following linear heat equation

$$\begin{align*}
\begin{cases}
    u_t + \kappa \Delta u = 0 \\
    u(x, 0) = \max_i x
\end{cases}
\end{align*}$$

To use $u$ as a lower bound potential, we need to find $\kappa$ such that

$$\kappa \Delta u \leq \frac{1}{2} \max_{q \in [-1, 1]^N} \langle D^2 u \cdot q, q \rangle$$

Since $\max$ is convex, $u$ is convex. Therefore, a maximum of this quadratic form is attained on the vertices of the hypercube $q \in \{\pm 1\}^N$.

The linearity of $\max_i x_i$ in the direction of $1$ confirms that $u(x + c1, t) = u(x, t) + c$. This implies that $\sum_i \partial_i u = 1$, $\partial_{ii} = -\sum \partial_{ij} u$, and therefore $D^2 u \cdot 1 = 0$. Also Appendix D confirms that $\partial_{ij} u < 0$ for $i \neq j$. Thus, we can consider $D^2 u$ to be the Laplacian $L(G)$ of an undirected weighted graph $G$ with $N$ vertices, and $\max_{q \in \{\pm 1\}^N} \langle D^2 u \cdot q, q \rangle = 4 \max \text{cut}(G)$ where $\max \text{cut}(G)$ is the maximum cut of $G$.

For an unweighted graph $G_u$ with $N$ vertices and $E = \text{Trace } (L_{G_u})$ edges, it is known that $\left(\frac{1}{2} + \frac{1}{2N}\right)E \leq \max \text{cut}(G_u)$ (Haglin and Venkatesan, 1991). In claim 4, we show a similar result for $D^2 u$. Note that in Claim 4 we only use the fact that $D^2 u$ is symmetric and has $1$ in the kernel, thus it is more general.

**Claim 4** If $D^2 u \cdot 1 = 0$, then $\kappa_h \Delta u \leq \frac{1}{2} \max_{q \in \{\pm 1\}^N} \langle D^2 u \cdot q, q \rangle$ for

$$\kappa_h = \begin{cases}
    1 & \text{if } N = 2 \\
    \frac{1}{2} + \frac{1}{2N} & \text{if } N \text{ is odd} \\
    \frac{1}{2} + \frac{1}{2N-2} & \text{otherwise}
\end{cases}$$

We define the adversarial $a^h$ to be a uniform distribution on the set of ”balanced cuts”

$$S = \begin{cases}
    \{ q \in \{-1, 1\}^N \mid \sum_{i=1}^N q_i = \pm 1 \} & \text{for } N \text{ odd} \\
    \{ q \in \{-1, 1\}^N \mid \sum_{i=1}^N q_i = 0 \} & \text{for } N \text{ even}
\end{cases}$$

which were used in the proof of Claim 4. The proof shows that $\frac{1}{2} \mathbb{E}_{a^h} \langle D^2 u \cdot q, q \rangle = \kappa_h \Delta u$. Therefore $a^h$ with the potential $u^h$ given by (6) with the diffusion factor (7) satisfies (2). We also observe that $a^h$ is symmetric because it is the uniform distribution over the symmetric set $S$.

**Heat-based adversary $a^h$:** At each $t = T, \ldots, -2$, the adversary samples $q_t$ uniformly from $S$, and at $t = -1$, the adversary selects an arbitrary probability distribution over $[-1, 1]^N$.

Note that this strategy does not require any runtime computations of $u^h$ or its derivatives; moreover $a^h$ does not depend on $x$ or $t$. 

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Next, we construct an upper bound. Appendix D confirms that $\partial_{ij}u < 0$ for $i \neq j$ and $\partial_{ii}u > 0$. Also the fact $\sum_i \partial_i u = 1$, as noted above implies that $\sum_{i,j} \partial_{ij}u = 0$. Therefore,

$$\frac{1}{2} \max_{q \in [-1,1]^N} \langle D^2u \cdot q, q \rangle \leq \frac{1}{2} \Delta u - \frac{1}{2} \sum_{i \neq j} \partial_{ij}u = \Delta u$$

Also we’ve proved $\partial_{ii}u \geq 0$, $\forall i \in [N]$ in Appendix D. This confirms that the potential $w^h$ given by (6) with $\kappa = 1$ satisfies (5) for all $q$.

**Heat-based player $p^h$:** At each $t = T, \ldots, -2$, the player selects $p^h_t = \nabla w^h(x, t + 1)$ and, at $t = -1$, the player selects an arbitrary distribution in $\Delta_N$.

In Appendix E, we compute the bounds on $u$ and its derivatives and using the Theorems above, confirm the following upper and lower bounds on the relevant value functions. Since the solution to the heat equation is smooth, we bound $E_{a^h}$ uniformly in $t$ using Remark 2.

**Example 1 (Heat potential bounds)**

(l.b) $u^h(x, t) - E_{a^h}(t) \leq v_{a^h}(x, t)$ where $v_{a^h}$ is the value function of $a^h$, $E_{a^h}$ is bounded uniformly in $t$ and $E_{a^h}(t) = O\left(N\sqrt{N}\right)$; and

(u.b.) $v_{p^h}(x, t) \leq w^h(x, t) + E_{p^h}(t)$ where $v_{p^h}$ is the value function of $p^h$ and $E_{p^h}(t) = O\left(\sqrt{N \log N} + \sqrt{N \log |t|}\right)$.

Since $u(0, T) = \sqrt{-2\kappa \mathbb{E}} \max_i G_i$ where $G$ is a Gaussian N-dimensional vector $N(0, I)$, the bounds on the value function lead to the following bounds on the regret

$$\sqrt{2\kappa T} \mathbb{E} \max_i G_i - O\left(N\sqrt{N}\right) \leq v_{a^h}(0, T) = \min_p R_T(a^h, p)$$

and

$$\max_a R_T(a, p^h) = v_{p^h}(0, T) \leq \sqrt{2T} \mathbb{E} \max_i G_i + O\left(\sqrt{N \log N} + \sqrt{N \log |T|}\right)$$

**6. Max Operator-Based Potentials**

In this section, we consider the potential $u$ given by the solution of

$$\begin{cases}
u_t + \kappa \max_i \partial^2_{ii}u = 0 \\ u(x, 0) = \max_i x
\end{cases} \quad (8)$$

The building blocks of $u$ are functions of the form $g(x, t) = \sqrt{-2\kappa t} f\left(\frac{x}{\sqrt{-2\kappa t}}\right)$, which are self-similar solutions of the linear 1D heat equation with the final value $g(x, 0) = |x|$. In this setting, we have

$$f(z) = \sqrt{\frac{2}{\pi}} e^{-z^2} + \text{erf}\left(\frac{z}{\sqrt{2}}\right) \quad \text{and} \quad \text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-s^2} ds. \quad (9)$$
As confirmed in Appendix F, \( f \) solves
\[
\begin{cases}
  f(z) = f''(z) + zf'(z) \\
  \lim_{|z| \to \infty} \frac{f(z)}{|z|} = 1.
\end{cases}
\]

Therefore, \( g(x, t) \) solves the one-dimensional linear heat equation on \( \mathbb{R} \times \mathbb{R}_{<0} \)
\[
\begin{cases}
  g_t + \kappa g_{xx} = 0 \\
  g(x, 0) = |x|
\end{cases}
\]

Like Bayraktar et al. (2019), for all \( x \in \mathbb{R}^N \), we will denote by \( \{x(i)\}_{i=1,\ldots,N} \) the ranked coordinates of \( x \), such that \( x(1) \geq x(2) \geq \ldots \geq x(N) \), and this allows us to define \( u \) globally in a uniform manner. In Appendix F, we confirm

**Claim 5** The classical solution of (8) on \( \mathbb{R}^N \times \mathbb{R}_{<0} \) is given by
\[
u(x, t) = \frac{1}{N} \sum_{i} x(i) + \sqrt{-2\kappa t} \sum_{l=1}^{N-1} c_l f(z_l)
\]
where \( z_l = \frac{1}{\sqrt{2\kappa t}} \left( \left( \sum_{n=1}^{l} x(n) \right) - lx(l+1) \right) \), \( f \) is given by (9) and \( c_l = \frac{1}{l(l+1)} \).

Since \( z_l \) does not change when a multiple of 1 is added to \( x \), we have \( u(x+c\mathbf{1}, t) = u(x, t) + c \). This implies that \( \sum_i \partial_i u = 1 \), and therefore \( D^2u \cdot \mathbf{1} = 0 \). The corresponding strategy \( a^m \) is given by

\[
\text{max adversary } a^m: \text{At each } t \in [T], \text{the adversary selects the distribution } a^m \text{ by assigning probability } \frac{1}{2} \text{ to each of } q^m \text{ and } -q^m \text{ where the entry of } q^m \text{ corresponding to the largest component of } x \text{ is set to } 1 \text{ and the remaining entries are set to } -1.
\]

Suppose \( x(1) = x_i \), in Appendix F.3 we confirm that \( \max_j \partial_j^2 u = \partial_i^2 u \), thus
\[
\langle D^2u \cdot (\pm q^m), \pm q^m \rangle = \langle D^2u \cdot (q^m + 1), \pm (q^m + 1) \rangle = 4 \partial_i u = 4 \max_j \partial_j u.
\]

Therefore, \( u^m \) given by (10) with \( \kappa = 2 \) satisfies (2) for the adversary strategy \( a^m \).

To determine an upper bound, we note that since \( f \) is convex, \( u \) is convex. Therefore, \( \max_{q \in \{-1,1\}^N} \langle D^2u \cdot q, q \rangle \) is attained at the vertices of the hypercube \( \{-1,1\}^N \).

Also from Appendix F, we see that \( D^2u \) has a special structure: \( \partial_{ik} u = \partial_{ji} u \) for all \( i, j < k \) and \( \partial_{ij} u \leq \partial_{ik} u \leq 0 \) for \( i < j < k \). In Appendix G, we use this structure to confirm that a class of simple rank-based strategies maximizes the quadratic form, and this allows us to obtain the following bound

**Claim 6** \( \max_{q \in \{-1,1\}^N} \langle D^2u \cdot q, q \rangle \leq \kappa_m \max_i \partial_i^2 u \) for
\[
\kappa_m = \begin{cases}
  \frac{N^2}{2(N-1)} & \text{for } N \text{ even} \\
  \frac{N+1}{2} & \text{for } N \text{ odd}
\end{cases}
\]

\(2\). This class includes, among others, the so-called comb strategy given in the ranked coordinates by \( (q^*)^{(i)} = 1 \) if \( i \) is odd and \( (q^*)^{(i)} = -1 \) if \( i \) is even.
Also in Appendix F.1 we have showed \( \partial_i u \geq 0, \forall i \in [N] \). Therefore, an upper bound potential \( w^m \) given by (10) with \( \kappa \) given by (11) satisfies (5) for all \( q \) and this yields the player strategy \( p^m \).

**max-potential player \( p^m \):** At each \( t = T, \ldots, -2 \), the player selects \( p_t = \nabla w^m(x, t + 1) \) and, at \( t = -1 \), the player selects an arbitrary \( p_{-1} \in \Delta_N \).

Since \( u \) is constructed by reflection of a smooth function whose first derivatives normal to the reflection boundary vanish, its third spatial derivatives are bounded almost everywhere on \( \mathbb{R}^N \) but are discontinuous at the reflection boundary. Therefore, in this setting, the tighter control of the lower bound error described in Remark 2, which we used for the heat potential, is not available. In Appendix H, we confirm the following bounds.

**Example 2 (max-based bounds)**

(l.b.) \( w^m(x, t) - E_{a^m}(t) \leq v_{a^m}(x, t) \) where \( v_{a^m} \) is the value function of \( a^m \) and \( E_{a^m}(t) = O(N \log |t|) \).

(u.b.) \( w^m(x, t) \leq v_{p^m}(x, t) + E_{p^m}(t) \) where \( v_{p^m} \) is the value function of \( p^m \) and \( E_{p^m}(t) = O(N \log |t|) \).

Since \( u(0, T) = \frac{2(N-1)}{N} \sqrt{\frac{2}{\pi} |T|} \), we obtain the following bounds on the regret

\[
\frac{2(N-1)}{N} \sqrt{\frac{2}{\pi} |T|} - O(N \log |t|) \leq v_{a^m}(x, t) \leq \min_p R_T(a^m, p)
\]

and

\[
\max_a R_T(a, p^m) \leq v_{p^m}(x, t) \leq \frac{2(N-1)}{N} \sqrt{\frac{\kappa_m}{\pi} |T|} + O(N \log |t|)
\]

7. Related work

Since

\[
\lim_{N \to \infty} \frac{1}{\sqrt{2 \log N}} \lim_{T \to -\infty} \frac{1}{\sqrt{-T}}[u^h(x, T) - E_{a^h}(T)] = 1
\]

the lower bound \( v_{a^h} \) is asymptotically the same as the upper bound given by Exp. Therefore, the adversary strategy \( a^h \) is asymptotically optimal in the limit where \( T \to -\infty \) first, and then \( N \to \infty \).

For a fixed \( N > 3 \), the state-of-the-art bound known to us that is asymptotic in \( T \) is given by the randomized adversary \( a^r \), which samples each \( q_i \) independently from a Radamacher random variable. This adversary attains the bound of \( \mathbb{E}_G \max G_i \).

\[\text{Note that} \quad \lim_{T \to -\infty} \frac{1}{\sqrt{-T}}[u^h(x, T) - E_{a^h}(T)] = \sqrt{2\kappa_h \mathbb{E}_G \max G_i} \]

3. See, e.g., Section 7.1 in György et al. and Theorem 3.7 in Cesa-Bianchi and Lugosi (2006). Those lower bounds are also rescaled to apply to \([-1,1]^N\), instead of \([0,1]^N\), losses.
Since $\kappa_h$ is strictly larger than $\frac{1}{2}$ for any fixed $N$, the lower bound attained by $a^h$ is tighter than the one attained by $a^r$.

When $N$ and $T$ are fixed, the bound for the strategy $a^r$ is obtained in closed form in Theorem 8 in Orabona and Pal (2017) by lower bounding the maximum of $N$ independent symmetric random walks of length $|T|$.

Another lower bound is given in Chapter 7 of György et al. for an adversary strategy $a^s$ constructed from a single random walk of length $|T|$. The lower bound attained by $a^s$, as rescaled for our losses, is

$$\sum_{j=0}^{M-1} \mathbb{E} \left[ \sum_{t=1}^{N} \sum_{1 \leq t \leq |T| \mod M = j} Z_t \right]$$

where $M = \lfloor \log_2 N \rfloor$ and each $Z_t$ is an independent Radamacher random variable. As noted in the same reference $\mathbb{E} \left[ \sum_{t=1}^{N} |Z_t| \right] \leq \sqrt{\frac{2N}{\pi}} \exp \left( \frac{1}{12n} - \frac{2}{6n+1} \right)$, and we will set the expected distance of each random walk to be equal to its upper bound for comparison purposes.

Note that $\mathbb{E} \max G_i = \int_{-\infty}^{\infty} t \frac{d}{dt} (\Phi(t))^N dt$ where $\Phi$ is the c.d.f. of the Gaussian random variable $N(0,1)$. Therefore, for comparison purposes, we evaluate the expectation of the maximum of Gaussian using numerical integration (integral function in MATLAB).

The strategy $a^s$ provides a tighter lower bound than our $a^h$ when $|T|$ is relatively small. However, as illustrated by Figure 1, as $|T|$ gets larger, our strategy $a^h$ improves the lower bound. (The lower bound given by Orabona and Pal (2017) is not shown because its value is negative for the given range of $T$ and $N$.)

Furthermore, when $N \leq 10$ and $|T|$ is relatively large, as illustrated by Figure 2, the max-based player $p^m$ and adversary $a^m$ and the heat-based player $p^h$ improve the upper and lower bounds guarantees given, respectively, by Exp and $a^s$ (the heat-based player $p^h$ also remains advantageous in this setting).

For up to three experts, the lower and upper bounds for the max potential provide a matching leading order term $4\sqrt{\frac{2}{\pi} |T|}$. Therefore, the corresponding strategies are minmax optimal at the leading order.

The same leading order constant for three experts was determined in Abbasi-Yadkori et al. (2017) with the regret scaling as $4\sqrt{\frac{2}{\pi} |T|} + O((\log |T|)^2)$ (for our $[-1,1]^N$ loss function). Our strategies $a^m$ and $p^m$ however, give improved guarantees with respect to the lower order (error) term: $4\sqrt{\frac{2}{\pi} |T|} + O(\log |T|)$.

8. Conclusions and Directions for Future Research

In this work, we establish that potentials can be used to design effective strategies leading to lower bounds as well as upper bounds. We also demonstrate that solutions of certain PDEs are good candidates for such potentials.

While this paper has focused on the fixed horizon version of the expert problem, our methods can be also applied to the geometric stopping version of that problem. This will be addressed in a separate publication.

Other interesting directions for exploration include the following:
Figure 1: This figure illustrates that the heat-potential based adversary $a^h$ is advantageous when $T$ is relatively large, but fixed. $C_N$ is such that $R_T(a,p) \geq C_N \sqrt{T}$ for all $p$.

1. Since the losses of all experts are revealed to the player, expert problems are categorized as so-called full information games. An ambitious direction would be to extend our framework to the setting where only the losses of the experts chosen by the player are revealed (adversarial bandits and other partial information games).

2. The expert problem is part of a broader class of online linear optimization problems (as the expectation is linear). Another potential research direction is to extend the framework to online linear optimization problem with decision sets other than the probability simplex and/or loss functions not restricted to a hypercube, as well as to online optimization problems with nonlinear loss functions.

3. Recently, Foster et al. (2018) introduced so-called Burkholder functions as potentials for a broad range of online learning problems. Since these functions are given by a dynamic program, the optimal control framework has promise as a tool for constructing such functions and developing two-sided bounds.

4. Lastly, in specific applications, expert problems have been endowed with additional structure, e.g., a proposed cybersecurity model assumes that only a subset of experts is adversarial (Truong et al., 2018). In other settings the player obtains side information (context). An interesting research direction would be to apply our framework to topical structured applications of the expert paradigm.
Figure 2: This figure illustrates that the max-based player $p^m$ and adversary $a^m$ and the heat-based player $p^h$ are advantageous when $N$ is small and $T$ is relatively large but fixed (the heat-based player $p^h$ remains advantageous in this setting). $C_N$ is such that for an adversary $a$ $R_T(a, p) \geq C_N \sqrt{-T}$ for all $p$ and for a given player $p$, $R_T(a, p) \leq C_N \sqrt{-T}$.

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Appendix A. Proof of Theorem 1 and Remark 2

Proof [of theorem 1]

Since $v_a$ is characterized by the dynamic program (1), we confirm that $u(x, t) - E(t) \leq v_a(x, t)$ by induction starting from the final time. The initial step follows from the equality of $v_a$ and $u$ at $t = 0$.

To prove the inductive step, as a preliminary result, we bound below the difference $\min E_{a_t, p_t} [u(x + r_t, t + 1)] - u(x, t)$ in terms of $C$ and $K(t)$.

At $t = -1$, the conditions of the theorem already provide,

$$\min_{p_{-1}} E_{a_{-1}, p_{-1}} [u(x + r_{-1}, 0)] - u(x, -1) \geq -C$$
For \( t \leq -2 \), we decompose the difference first with respect to the change of \( x \), holding \( t \) fixed, and with respect to \( t \), holding \( x \) fixed:

\[
\min_{p_t} \mathbb{E}_{p_t, a_t} [u(x + r_t, t + 1) - u(x, t)] = \min_{p_t} \mathbb{E}_{p_t, a_t} [u(x - q_t, t + 1) + (q_t) I_t] - u(x, t + 1) + u(x, t + 1) - u(x, t) = \mathbb{E}_{a_t} [u(x - q_t, t + 1) - u(x, t + 1)] + u(x, t + 1) - u(x, t)
\]

Here we eliminated the dependence on \( p \) using the fact that \( u(x + r_t, t + 1) = u(x - q_t, t + 1) + (q_t) I_t \) in the first equality and the fact that the expectation of \((q_t) I_t\) is zero by the symmetry of \( a_t \) in the second equality.

Since \( u(\cdot, t + 1) \) is \( C^2 \) with Lipschitz continuous second order derivatives, we use Taylor’s theorem with the integral remainder

\[
u(x - q_t, t + 1) = u(x, t + 1) - \nabla u(x, t + 1) \cdot q_t + \frac{1}{2} \langle D^2 u(x, t + 1) \cdot q_t, q_t \rangle - \int_0^1 D^3 u(x - \mu q_t, t + 1)[q_t, q_t, q_t] (1 - \mu)^2 d\mu
\]

Thus,

\[
u(x - q_t, t + 1) - u(x, t + 1) \geq - \nabla u(x, t + 1) \cdot q_t + \frac{1}{2} \langle D^2 u(x, t + 1) \cdot q_t, q_t \rangle - \frac{1}{6} \text{ess sup}_{y \in [x, x - q_t]} D^3 u(y, t + 1)[q_t, q_t, q_t]
\]

and similarly for \( u(\cdot, \cdot) \)

\[
u(x, t + 1) - u(x, t) \geq u_t(x, t + 1) - \frac{1}{2} \text{ess sup}_{\tau \in [t, t + 1]} u_{tt}(x, \tau)
\]

Also we use the condition on the potential \( u \)

\[
u_t(x, t + 1) + \frac{1}{2} \langle D^2 u(x, t + 1) \cdot q_t, q_t \rangle \geq 0
\]

Collecting the above inequalities and using the fact that \( a_t \) is symmetric we have

\[
\min_{p_t} \mathbb{E}_{p_t, a_t} [u(x + r_t, t + 1) - u(x, t)] \geq -K(t) = E(t + 1) - E(t)
\]

Finally, use the inductive hypothesis \( u(x + r_t, t + 1) - E(t + 1) \leq v_a(x + r_t, t + 1) \), and the dynamic program formulation of \( v_a \), we obtain

\[
u(x, t) - E(t) \leq u(x, t) + \min_{p_t} \mathbb{E}_{p_t, a_t} [u(x + r_t, t + 1) - u(x, t) - E(t + 1)] \leq \min_{p_t} \mathbb{E}_{p_t, a_t} [v_a(x + r_t, t + 1)] = v_a(x, t)
\]

The proof of Remark 2 is the same except that we expand \( u(x - q_t, t + 1) \) up to fourth order spatial derivatives and use the fact that \( \mathbb{E}_{a_t} D^3 u(x, t + 1)[q_t, q_t, q_t] = 0 \) by symmetry (\( q \) and \(-q \) have the same probability).
Appendix B. Proof of Theorem 3

Proof [of theorem 3] Since $v_p$ is characterized by the dynamic program (4), we confirm by induction that $v_p(x,t) \leq w(x,t) + E(t)$. The initialization is the same as in Appendix A, and the rest of the proof is similar. To prove the inductive step, we first note that

$$\max_{\alpha_1, \ldots, \alpha_n} \mathbb{E}[w(x + r_{\alpha_1}, 0)] - w(x, -1) \leq C.$$  

For $t \leq -2$, we decompose the difference as following

$$\max_{\alpha_t} \mathbb{E}_{p_t, \alpha_t} [w(x + r_t, t + 1) - w(x, t)]$$

$$= \max_{\alpha_t} \mathbb{E}_{p_t, \alpha_t} [w(x - q_t, t + 1) + \langle q_t \rangle_{t,t}] - w(x, t + 1) + w(x, t + 1) - w(x, t)$$

$$= \max_{\alpha_t} \mathbb{E}_{\alpha_t} [w(x - q_t, t + 1) - w(x, t + 1) + p_t \cdot q_t] + w(x, t + 1) - w(x, t)$$

(12)

where we applied the linearity along $1$ in the first equality.

Since $w(., t + 1)$ is $C^2$ with Lipschitz continuous second order derivatives, we again use Taylor’s theorem with the integral remainder

$$w(x - q_t, t + 1) = w(x, t + 1) - \nabla w(x, t + 1) \cdot q_t + \frac{1}{2} (D^2 w(x, t + 1) \cdot q_t, q_t)$$

$$- \int_0^1 D^3 w(x - \mu q_t, t + 1)[q_t, q_t, q_t] (1 - \mu)^2 d\mu$$

Thus

$$w(x - q_t, t + 1) - w(x, t + 1) + p_t \cdot q_t \leq \frac{1}{2} (D^2 w(x, t + 1) \cdot q_t, q_t)$$

$$- \frac{1}{6} \text{ess inf}_{y \in [x - q_t]} D^3 w(y, t + 1)[q_t, q_t, q_t]$$

Here we eliminated the dependence on $p$ using the fact that $p_t = \nabla w(x, t + 1)$, which gives the cancellation of $p_t \cdot q_t$ in (12) with $\nabla w(x, t + 1) \cdot q_t$ (13).

Similarly for $w(x, \cdot)$

$$w(x, t + 1) - w(x, t) \leq w_t(x, t + 1) - \frac{1}{2} \text{ess inf}_{r \in [t, t+1]} w_{rr}(x, \tau)$$

Also for potential $w$

$$w_t(x, t + 1) + \frac{1}{2} (D^2 w(x, t + 1) \cdot q_t, q_t) \leq 0$$

By collecting the above inequalities, we get

$$\max_{\alpha_t} \mathbb{E}_{p_t, \alpha_t} [w(x + r_t, t + 1) - w(x, t)] \leq K(t) = E(t) - E(t + 1)$$

Finally, using the inductive hypothesis $w(x + r, t + 1) + E(t + 1) \geq v_p(x + r, t + 1)$, and the dynamic program formulation of $v_p$, we obtain

$$w(x, t) + E(t) \geq w(x, t) + \max_{\alpha_t} \mathbb{E}_{p_t, \alpha_t} [w(x + r_t, t + 1) - w(x, t) + E(t + 1)]$$

$$\geq \max_{\alpha_t} \mathbb{E}_{p_t, \alpha_t} [v_p(x + r_t, t + 1)] = v_p(x, t)$$

$\blacksquare$
Appendix C. Proof of Claim 4

When \( N = 2 \), since \( D^u \) is symmetric and \( D^2u \cdot 1 = 0 \), it has the form

\[
D^2u = \begin{bmatrix}
a & -a \\
-a & a
\end{bmatrix}
\]

for some constant \( a \). It is straightforward to verify that \( \kappa = 1 \).

When \( N > 2 \), for any subset \( S \subset \{-1, 1\}^N \) and any \( \tilde{q} \in S \) we have

\[
\max_{q \in \{-1, 1\}^N} \langle D^2u \cdot q, q \rangle \geq \langle D^2u \cdot \tilde{q}, \tilde{q} \rangle
\]

We define

\[
S = \begin{cases}
\{q \in \{-1, 1\}^N | \sum_{i=1}^N q_i = \pm 1\}, & N \text{ odd} \\
\{q \in \{-1, 1\}^N | \sum_{i=1}^N q_i = 0\}, & N \text{ even}
\end{cases}
\]

then

\[
\max_{q \in \{-1, 1\}^N} \langle D^2u \cdot q, q \rangle \geq \frac{1}{|S|} \sum_{\tilde{q} \in S} \langle D^2u \cdot \tilde{q}, \tilde{q} \rangle = \langle D^2u, \frac{1}{|S|} \sum_{\tilde{q} \in S} \tilde{q}\tilde{q}^\top \rangle_F
\]

where \( \langle , \rangle_F \) is the Frobenius inner product.

Denote \( M = 11^\top \) then since \( S \) is permutation invariant we can write \( \frac{1}{|S|} \sum_{\tilde{q} \in S} \tilde{q}\tilde{q}^\top = (1 - \lambda)I + \lambda M \) for some constant \( \lambda \). Note that

\[
\frac{1}{|S|} \sum_{\tilde{q} \in S} \langle \tilde{q}\tilde{q}^\top, M \rangle_F = \begin{cases}
1, & N \text{ odd} \\
0, & N \text{ even}
\end{cases}
\]

which determines

\[
\lambda = \begin{cases}
-\frac{1}{N}, & N \text{ odd} \\
-\frac{1}{N-1}, & N \text{ even}
\end{cases}
\]

Using the fact that \( \langle D^2u, M \rangle_F = 0 \) we get

\[
\max_{q \in \{-1, 1\}^N} \langle D^2u \cdot q, q \rangle \geq \langle D^2u, \frac{1}{|S|} \sum_{\tilde{q} \in S} \tilde{q}\tilde{q}^\top \rangle_F = (1 - \lambda)\Delta u
\]

Plugging in the value of \( \lambda \) yields the desired result.
Appendix D. Derivatives of the Heat-Based Potential

In this section, we compute the spatial derivatives of the heat-equation solution (6) up to the fourth order, which will be used in Appendix E.

Note that $\max_k(x_k - y_k)$ is differentiable almost everywhere and

$$\partial_i \max_k(x_k - y_k) = \begin{cases} 1 & \text{if } x_i - y_i > \max_{j \neq i} (x_j - y_j) \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the first derivatives are

$$\partial_i u = \alpha \int e^{-\frac{||y||^2}{2\sigma^2}} \mathbb{1}_{x_i - y_i > \max_{j \neq i} x_j - y_j} dy = \alpha \int e^{-\frac{||x-y||^2}{2\sigma^2}} \mathbb{1}_{y_i > \max_{j \neq i} y_j} dy \geq 0$$

and the second pure derivatives are

$$\partial_{ij} u = -\frac{\alpha}{\sigma^2} \int e^{-\frac{||x-y||^2}{2\sigma^2}} (x_i - y_i) \mathbb{1}_{y_i > \max_{j \neq i} y_j} dy = -\frac{\alpha}{\sigma^2} \int e^{-\frac{||y||^2}{2\sigma^2}} y_i \mathbb{1}_{x_i - y_i > \max_{j \neq i} x_j - y_j} dy$$

$$= -\frac{\alpha}{\sigma^2} \int_{\mathbb{R}^{N-1}} e^{-\frac{\sum_{k \neq i} y_k^2}{2\sigma^2}} \int_{-\infty}^{x_i - \max_{j \neq i} x_j - y_j} e^{-\frac{y_i^2}{2\sigma^2}} y_i dy_i dy_j$$

Since $\int_{-\infty}^{x_i - \max_{j \neq i} x_j - y_j} e^{-\frac{y_i^2}{2\sigma^2}} y_i dy_i < 0$, we have $\partial_{ij} u > 0$.

The second mixed derivatives are

$$\partial_{ij} u = -\frac{\alpha}{\sigma^2} \int e^{-\frac{||x-y||^2}{2\sigma^2}} (x_j - y_j) \mathbb{1}_{y_i > \max_{k \neq i} y_k} dy = -\frac{\alpha}{\sigma^2} \int e^{-\frac{||y||^2}{2\sigma^2}} y_j \mathbb{1}_{x_i - y_i > \max_{k \neq i} x_k - y_k} dy$$

$$= -\frac{\alpha}{\sigma^2} \int_{\mathbb{R}^{N-1}} e^{-\frac{\sum_{k \neq i} y_k^2}{2\sigma^2}} \mathbb{1}_{x_i - y_i > \max_{k \neq i} x_k - y_k} \int_{x_j - x_i + y_i}^{\infty} e^{-\frac{y_j^2}{2\sigma^2}} y_j dy_j dy_j$$

Since $\int_{x_i - x_j + y_i}^{\infty} e^{-\frac{y_j^2}{2\sigma^2}} y_j dy_j > 0$, we have $\partial_{ij} u < 0$.

The third derivatives are

$$\partial_{iii} u = -\frac{\alpha}{\sigma^2} \int e^{-\frac{||x-y||^2}{2\sigma^2}} \left( 1 - \frac{(x_i - y_i)^2}{\sigma^2} \right) \mathbb{1}_{y_i > \max_{j \neq i} y_j} dy$$

$$= -\frac{\alpha}{\sigma^2} \int e^{-\frac{||y||^2}{2\sigma^2}} \left( 1 - \frac{y_i^2}{\sigma^2} \right) \mathbb{1}_{x_i - y_i > \max_{j \neq i} x_j - y_j} dy$$

$$\partial_{ijj} u = -\frac{\alpha}{\sigma^2} \int e^{-\frac{||x-y||^2}{2\sigma^2}} \left( 1 - \frac{(x_j - y_j)^2}{\sigma^2} \right) \mathbb{1}_{y_i > \max_{k \neq i} y_k} dy$$

$$= -\frac{\alpha}{\sigma^2} \int e^{-\frac{||y||^2}{2\sigma^2}} \left( 1 - \frac{y_j^2}{\sigma^2} \right) \mathbb{1}_{x_i - y_i > \max_{k \neq i} x_k - y_k} dy$$
when \( i, j, k \) are all distinct (assuming \( N \geq 3 \)),

\[
\partial_{ijk} u = \frac{\alpha}{\sigma^4} \int e^{-\frac{\|x-y\|^2}{2\sigma^2}} (x_j - y_j)(x_k - y_k) 1_{y_i > \max_{\ell \neq i, y_i}} dy
\]

\[
= \frac{\alpha}{\sigma^4} \int e^{-\frac{\|u\|^2}{2\sigma^2}} y_j y_k 1_{x_i - y_i > \max_{\ell \neq i, x_i - y_i}} dy
\]

\[
= \frac{\alpha}{\sigma^4} \int_{\mathbb{R}^{N-2}} e^{-\frac{\sum_{\ell \neq i, k} y_{\ell}^2}{2\sigma^2}} 1_{x_i - y_i > \max_{\ell \neq i, k} x_i - y_i} \int_{x_j - x_i + y_j}^{\infty} e^{-\frac{y_j^2}{2\sigma^2}} y_j dy_j \int_{x_k - x_i + y_i}^{\infty} e^{-\frac{y_k^2}{2\sigma^2}} y_k dy_k dy_{jk}
\]

Since \( \int_{x_j - x_i + y_j}^{\infty} e^{-\frac{y_j^2}{2\sigma^2}} y_j dy_j \int_{x_k - x_i + y_i}^{\infty} e^{-\frac{y_k^2}{2\sigma^2}} y_k dy_k > 0 \), we have \( \partial_{ijk} u > 0 \).

For fourth derivatives

\[
\partial_{iiii} u = \frac{\alpha}{\sigma^4} \int_{\mathbb{R}^N} e^{-\frac{\|x-y\|^2}{2\sigma^2}} (x_i - y_i)(3 - \frac{(x_i - y_i)^2}{\sigma^2}) 1_{y_i > \max_{\ell \neq i, y_i}} dy
\]

\[
= \frac{\alpha}{\sigma^4} \int_{\mathbb{R}^N} e^{-\frac{\|u\|^2}{2\sigma^2}} y_i (3 - \frac{y_i^2}{\sigma^2}) 1_{x_i - y_i > \max_{\ell \neq i, x_i - y_i}} dy
\]

\[
\partial_{iijj} u = \frac{\alpha}{\sigma^4} \int_{\mathbb{R}^N} e^{-\frac{\|x-y\|^2}{2\sigma^2}} (x_i - y_i)(1 - \frac{(x_j - y_j)^2}{\sigma^2}) 1_{y_j > \max_{\ell \neq j, y_j}} dy
\]

\[
= \frac{\alpha}{\sigma^4} \int_{\mathbb{R}^N} e^{-\frac{\|u\|^2}{2\sigma^2}} y_j (1 - \frac{y_j^2}{\sigma^2}) 1_{x_j - y_j > \max_{\ell \neq j, x_j - y_j}} dy
\]

\[
\partial_{ijii} u = \frac{\alpha}{\sigma^4} \int_{\mathbb{R}^N} e^{-\frac{\|x-y\|^2}{2\sigma^2}} y_i (3 - \frac{y_i^2}{\sigma^2}) 1_{x_j - y_j > \max_{\ell \neq j, x_j - y_j}} dy
\]

\[
\partial_{ijjj} u = \frac{\alpha}{\sigma^4} \int_{\mathbb{R}^N} e^{-\frac{\|x-y\|^2}{2\sigma^2}} y_j (1 - \frac{y_j^2}{\sigma^2}) 1_{x_j - y_j > \max_{\ell \neq j, x_j - y_j}} dy
\]

\[
\partial_{ijkl} u = \frac{\alpha}{\sigma^6} \int_{\mathbb{R}^{N-3}} e^{-\frac{\sum_{\ell \neq j, k, l} y_{\ell}^2}{2\sigma^2}} 1_{x_i - y_i > \max_{\ell \neq i, j, k, l} x_m - y_m} \left( \prod_{n=(j, k, l)}^{\infty} e^{-\frac{y_n^2}{2\sigma^2}} y_n dy_n \right) dy_{ijkl}
\]

Since \( \int_{x_i - x_j + y_j}^{\infty} e^{-\frac{y_j^2}{2\sigma^2}} y_j dy_j > 0 \), we have \( \partial_{ijkl} u < 0 \).

**Appendix E. Proof of Example 1**

In this section, we prove the error estimate for heat potential \( u \). Note that \( u \) is smooth we can use Remark 2 to bound the fourth order derivatives for lower bound.
We will first confirm that

$$|u(x_0, 0) - u(x_{-1}, -1)| \leq C_1$$

then for time derivatives, we have for all \(x\) and \(t \leq -2\)

$$\sup_{\tau \in [t, t+1]} |u_{tt}(x, \tau)| \leq \frac{C_2}{(-t - 1)^{\frac{3}{2}}}$$

For lower bound estimate we prove for \(q \sim \alpha^b\)

$$-\text{ess inf}_{y \in [x, x-q]} D^4 u(y, t + 1)[q, q, q, q] \leq \frac{2\sqrt{2N}}{(-\kappa(t + 1))^{\frac{5}{2}}(2\sqrt{6} + 3\sqrt{2N} + 4)}$$

For upper bound estimate we prove for any \(q \in [-1, 1]^N\)

$$-\text{ess inf}_{y \in [x, x-q]} D^3 u(y, t + 1)[q, q, q] \leq \frac{C_3}{-t - 1}$$

where

$$C_1 = 2 + \sqrt{2\kappa} \max_i Y_i$$
$$C_2 = \frac{\sqrt{\kappa}}{2\sqrt{2}} \mathbb{E} \left[ |N + 2 - \|Y\|^2| \max_i |Y_i| \right]$$
$$C_3 = \frac{1}{\kappa} \left( \frac{3}{\sqrt{2}} \sqrt{N + a \mathbb{E} \max_i |1 - Y_i^2|} \right)$$

where \(Y\) is a standard N-dimensional Gaussian random vector and \(a = 1\) for \(N = 2\) and \(2\) for \(N \geq 3\).

Note that for \(t \leq -2\)

$$\sum_{s=t}^{-2} \frac{1}{(-s - 1)^{\frac{3}{2}}} = \sum_{s=1}^{\left|t\right|-1} \frac{1}{s^{\frac{3}{2}}} \leq \log(-t - 1) + 1$$
$$\sum_{s=t}^{-2} \frac{1}{(-s - 1)} = \sum_{s=1}^{\left|t\right|-1} \frac{1}{s} \leq 3 - \frac{2}{\sqrt{-t - 1}}$$

For lower bound we take \(\kappa = \kappa_h\)

$$E(t) = C_1 + C_2(\frac{3}{2} - \frac{1}{\sqrt{-t - 1}}) + \frac{\sqrt{2N}}{6\kappa_h^\frac{5}{2}} \left( \frac{3}{2} - \frac{1}{\sqrt{-t - 1}} \right)(2\sqrt{6} + 3\sqrt{2N} + 4)$$

For upper bound we take \(\kappa = 1\)

$$E(t) = C_1 + C_2(\frac{3}{2} - \frac{1}{\sqrt{-t - 1}}) + \frac{C_2}{6}(\log(-t - 1) + 1)$$

To assess \(C_1\) numerically, we observe that \(\mathbb{E} \max_i Y_i\) has a closed-form expression for \(N \leq 5\), and can be estimated by numerical integration for larger \(N\). The asymptotically
optimal upper bound for this quantity is $\sqrt{2\log N}$, see, e.g., Lemmas A.12 and A.13 in Cesa-Bianchi and Lugosi (2006), and a sharper non-asymptotic upper bound for $N \geq 7$ is provided in DasGupta et al. (2014).

To assess $C_2$ numerically, using the fact that $\mathbb{E} \left[ |Y|^2 \right] = N$, $\mathbb{E} \left[ |Y|^4 \right] = N(N + 2)$ and $\mathbb{E} \max_i Y_i^2 \leq 2 \log N + 2\sqrt{\log N} + 1$ (e.g. Example 2.7 in Boucheron et al. (2013)).

we obtain:

$$
\mathbb{E} \left[ N + 2 - \|Y\|^2 \max_i |Y_i| \right] \leq \sqrt{\mathbb{E}(N + 2 - \|Y\|^2)^2 \max_i Y_i^2}
\leq 2(N + 2)(2\log N + 2\sqrt{\log N} + 1)
$$

To assess $C_3$ numerically, note that $\mathbb{E} \max_i |1 - Y_i^2| \leq \mathbb{E} \max_i Y_i^2 + 1$. Therefore, $C_1 = O(\sqrt{\log N})$, $C_2 = O(\sqrt{N \log N})$ and $C_3 = O(\sqrt{N})$.

### E.1 Final Time Step

We split the difference as follows

$$
u(x_0, 0) - u(x_{-1}, -1) = \max_i (x_0)_i - \max_i (x_{-1})_i + u(x_{-1}, 0) - u(x_{-1}, -1)
$$

Since $q_{-1} \in [-1, 1]^N$ we have $r_{-1} = (q_{-1})_{I_{-1}} \mathbb{1} - q_{-1} \in [-2, 2]^N$

$$
-2 \leq \max_i (x_0)_i - \max_i (x_{-1})_i \leq 2
$$

Since $- \max_i (x - y)_i \geq - \max_i x_i + \min_i y_i$, we have

$$
u(x_{-1}, 0) - u(x_{-1}, -1) = \alpha \int e^{-\frac{|y|^2}{2\sigma^2}} \max_i (x_{-1})_i - \max_i ((x_{-1})_i - y_i) dy
\geq \alpha \int e^{-\frac{|y|^2}{2\sigma^2}} \min_i y_i dy = -\sigma \mathbb{E} \max_i G_i
$$

where $\sigma = \sqrt{2\kappa}$ at $t = -1$. Thus we obtain

$$
u(x_0, 0) - u(x_{-1}, -1) \geq -2 - \sqrt{2\kappa} \mathbb{E} \max_i G_i
$$

Similarly, since $- \max_i (x - y)_i \leq - \max_i x_i + \max_i y_i$, we obtain

$$
u(x_0, 0) - u(x_{-1}, -1) \leq 2 + \sqrt{2\kappa} \mathbb{E} \max_i G_i
$$

### E.2 Bounds on $\nu_{tt}(x, \tau)$

It suffices to give an uniform upper bound of $|\nu_{tt}(x, \tau)|$ over all $x \in \mathbb{R}^N$ and $\tau \leq -1$.

Since

$$
\partial_{tt} u = \partial_t (-\kappa \Delta u) = -\kappa \Delta (\partial_t u) = \kappa^2 \Delta^2 u
$$

4. $\mathbb{E} \left[ \|Y\|^{2n} \right] = \int_0^{\infty} r^{2n} r^{n-1} e^{-r^2/2} dr / \int_0^{\infty} r^{n-1} e^{-r^2} dr$ can be computed explicitly using the properties of the Gamma function.
it suffices to bound $\Delta^2 u = \sum_{i,j} \partial_{iijj} u$. Using the formulas in Appendix D

$$\sum_{i,j} \partial_{iijj} u = \sum_i \partial_{iiii} u + \sum_{j \neq i} \partial_{ijij} u$$

$$= \frac{\alpha}{\sigma^4} \int_{\mathbb{R}^N} e^{-\frac{||y||^2}{2\sigma^2}} y_i (N + 2 - \frac{||y||^2}{\sigma^2}) 1_{x_i - y_i > \max_{k \neq i} x_k - y_k} dy$$

$$= \frac{\alpha}{\sigma^4} \int_{\mathbb{R}^N} e^{-\frac{||y||^2}{2\sigma^2}} (N + 2 - \frac{||y||^2}{\sigma^2}) \sum_i y_i 1_{x_i - y_i > \max_{k \neq i} x_k - y_k} dy$$

Combining above with the fact that $| \sum_i y_i 1_{x_i - y_i > \max_{k \neq i} x_k - y_k} | \leq \max_i |y_i|$

$$| \sum_{i,j} \partial_{iijj} u | \leq \frac{1}{\sigma^4} \mathbb{E}_{Y \sim N(0, I)} |N + 2 - ||Y||^2| \max_i |Y_i|$$

thus

$$|\partial_{tt} u| \leq \frac{1}{(-t - 1)^{\frac{3}{2}}} \frac{\sqrt{\kappa}}{2\sqrt{2}} \mathbb{E}_{Y \sim N(0, I)} |N + 2 - ||Y||^2| \max_i |Y_i|$$

**E.3 Upper Bound of $\text{--ess inf}_{y \in [x, x - q]} D^4 u(y, t + 1)[q, q, q, q]$ for $q \sim a^h$**

Since $q \sim a^h q$ is one of the vertices of the cube. It suffices to give an uniform lower bound for $D^4 u(x, t + 1)[q, q, q, q]$ over all $x \in \mathbb{R}^N$ and $q \in \{\pm 1\}^N$.

Note that, for $i, j, k$ distinct, from Appendix D we have

$$\partial_{iji} u + \partial_{ijj} u + \sum_{k \neq i,j} \partial_{ijkk} u = \frac{\alpha}{\sigma^4} \int_{\mathbb{R}^N} e^{-\frac{||y||^2}{2\sigma^2}} y_i \left( N + 2 - \frac{||y||^2}{\sigma^2} \right) 1_{x_i - y_i > \max_{m \neq i, j} x_k - y_k} dy$$

Also,

$$\sum_i \sum_j |\partial_{iji} u| \leq \sum_i \frac{\alpha}{\sigma^4} \int_{\mathbb{R}^N} e^{-\frac{||y||^2}{2\sigma^2}} \left| y_i \left( 3 - \frac{y_i^2}{\sigma^2} \right) \right| dy$$

$$\leq \frac{N}{\sigma^4} \mathbb{E}_{Y \sim N(0, I)} |Y(3 - Y^2)|$$
Since $\partial_{ijkl}u < 0$ for $i, j, k, l$ distinct (assuming $N \geq 4$) and $D^4u[1, 1, 1, 1] = 0$.

$$D^4u[q, q, q] \geq \sum_i \partial_i u_i + 3 \sum_{i \neq j} \partial_{iij}u + 2 \sum_{i \neq j} \partial_{ijj}u + \partial_{ijkl}u$$

$$\geq 2 \sum_i (\partial_{iij}u + \partial_{ijj}u)(q; q - 1) + 6 \sum_{i \neq j} \partial_{ijkl}u(q, q - 1) = -4 \sum_i (\partial_{iij}u + \partial_{ijj}u)(q; q - 1) + 6 \sum_{i \neq j} \partial_{ijkl}u(q, q - 1)$$

$$\geq -16 \sum_i \sum_j |\partial_{iij}u| - 24 \sum_i \sum_j |\partial_{ijkl}u|$$

$$\geq -\frac{16N}{\sigma^3} \mathbb{E}_{Y \sim N(0, 1)} |Y(3 - Y^2)| - 24 \frac{\alpha}{\sigma^2} \int_{\mathbb{R}^N} e^{-\frac{\|y\|^2}{2\sigma^2}} \sum_i |y_i (N + 2 - \|y\|^2)| \, dy$$

$$\geq -\frac{8N}{\sigma^3} \left(2\mathbb{E}_{G \sim N(0, 1)} |G(3 - G^2)| + 3\mathbb{E}_{Y \sim N(0, 1)} \sum_i |Y_i (N + 2 - \|Y\|^2)| \right)$$

$$\geq -\frac{2\sqrt{2N}}{(\kappa(t + 1))^2} (2\sqrt{6} + 3\sqrt{2N + 4})$$

For $N = 2, 3$ the calculation is similar.

**E.4 Upper Bound of $-\text{ess inf}_{y \in [x, x-q]} D^3u(y, t + 1)[q, q, q]$ for $q \in [-1, 1]^N$**

It suffices to give an uniform upper bound for $|D^3u(x, t + 1)[q, q, q]|$ over all $x \in \mathbb{R}^N$ and $q \in [-1, 1]^N$.

We start with

$$D^3u[q, q, q] = \sum_i (\partial_{ii}u_i q_i^2 + 3 \sum_{j \neq i} \partial_{ij}u_j q_i q_j) + \sum_i \sum_{j \neq i} \partial_{ijkl}u_i q_j q_k$$

$$= \sum_i (-2\partial_{ii}u_i q_i^2 + 3 \sum_{j} \partial_{ijj}u_j q_i) + \sum_i \sum_{j \neq i} \partial_{ijkl}u_i q_j q_k$$

We can derive the following identity by linearity of $u$ along $1$:

$$\sum_i \sum_{j \neq i} \partial_{ijkl}u = -\sum_i \sum_{j \neq i} (\partial_{ijj}u + \partial_{iij}u) = -2 \sum_i \sum_{j \neq i} \partial_{iij}u = 2 \sum_i \partial_{ii}u$$

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Using the fact that $\partial_{ijk}u > 0$ and this identity, for $N \geq 3,$

$$|D^3u[q, q, q]| \leq 2 \sum_i |\partial_{iii}u| + 3 \sum_i \left| \sum_j \partial_{ijj}u q_j^2 \right| + \sum_i \sum_{j \neq i} \sum_{k \neq i, j} \partial_{ijk}u$$

$$= 2 \sum_i |\partial_{iii}u| + 3 \sum_i \left| \sum_j \partial_{ijj}u q_j^2 \right| + 2 \sum_i \partial_{iii}u$$

$$\leq 3 \sum_i \left| \sum_j \partial_{ijj}u q_j^2 \right| + 4 \sum_i |\partial_{iii}u|$$

and for $N = 2,$

$$|D^3u[q, q, q]| \leq 2 \sum_i |\partial_{iii}u| + 3 \sum_i \left| \sum_j \partial_{ijj}u q_j^2 \right|$$

Using the formulas for third derivatives,

$$\sum_j \partial_{ijj}u q_j^2 = -\frac{cN}{\sigma^2} \int e^{-\frac{||y||^2}{2\sigma^2}} \left( \sum_j q_j^2 \left( 1 - \frac{y_j^2}{\sigma^2} \right) \right) \mathbb{1}_{x_i - y_i > \max_{k \neq i} x_k - y_k} dy$$

we have

$$\sum_i \sum_j \left| \partial_{ijj}u q_j^2 \right| \leq \frac{cN}{\sigma^2} \int e^{-\frac{||y||^2}{2\sigma^2}} \left( \sum_j q_j^2 \left( 1 - \frac{y_j^2}{\sigma^2} \right) \right) dy$$

$$= \frac{1}{\sigma^2} \mathbb{E}_{Y \sim N(0, I)} \left( \sum_j q_j^2 \left( 1 - Y_j^2 \right) \right)$$

Using Jensen’s inequality and the independence of $Y_j,$

$$\mathbb{E}_{Y \sim N(0, I)} \left( \sum_j q_j^2 \left( 1 - Y_j^2 \right) \right) \leq \sqrt{\mathbb{E}_{Y \sim N(0, I)} \left( \sum_j q_j^2 \left( 1 - Y_j^2 \right) \right)^2}$$

$$= \sqrt{Var \left( \sum_j q_j^2 Y_j^2 \right)} = \sqrt{2 \sum_j q_j^4 \leq \sqrt{2N}}$$

Also,

$$\sum_i |\partial_{iii}u| \leq \frac{\alpha}{\sigma^2} \int_{\mathbb{R}^N} e^{-\frac{||y||^2}{2\sigma^2}} \sum_i \left[ 1 - \frac{y_i^2}{\sigma^2} \right] \mathbb{1}_{x_i - y_i > \max_{k \neq i} x_k - y_k} dy$$

$$\leq \frac{\alpha}{\sigma^2} \int_{\mathbb{R}^N} e^{-\frac{||y||^2}{2\sigma^2}} \max_i \left[ 1 - \frac{y_i^2}{\sigma^2} \right] dy$$

$$= \frac{1}{\sigma^2} \mathbb{E}_{Y \sim N(0, I)} \max_i \left[ 1 - Y_i^2 \right]$$
Therefore, for all \( q \in [-1, 1]^N \),
\[
|D^3 u(x, t + 1)[q, q, q]| \leq \frac{1}{(-t - 1)} C_3
\]
where \( C_3 = \frac{1}{n} \left( \frac{3}{\sqrt{2}} \sqrt{N} + a \mathbb{E} \max_i |1 - Y_i^2| \right) \) and \( a = 1 \) for \( N = 2 \) and \( a = 2 \) for \( N \geq 3 \).

Appendix F. Proof of Claim 5

In this section we first compute the spatial derivatives of max potential \( u \) defined by (10) in each ranked sector \( \{x|x(1) > x(2) > \ldots > x(N)\} \) up to third order where \( \{1, \ldots, (N)\} \) is any permutation of \([N]\). Then we prove all the second order derivatives can be continuously extended to the boundary of sectors and the third order derivatives are defined almost everywhere and bounded. Thus \( u \) is \( C^2 \) with Lipschitz second order derivatives. Last we confirm it is the solution of (8).

F.1 Derivatives of the max-based potential

Note that
\[
f'(z) = \text{erf} \left( \frac{z}{\sqrt{2}} \right) \quad \text{and} \quad f''(z) = \sqrt{\frac{2}{\pi}} \exp \left( -\frac{z^2}{2} \right)
\]
Then for \( i \leq j \)
\[
\partial_{(i)} f(z_i) = \begin{cases} 
0 & \text{if } l + 1 < i \\
- \frac{l}{\sqrt{-2\kappa t}} f'(z_i) & \text{if } l + 1 = i \\
\frac{1}{\sqrt{-2\kappa t}} f'(z_i) & \text{if } l = i \\
\end{cases} \quad \text{and} \quad \partial_{(i)(j)} f(z_i) = \begin{cases} 
\frac{1}{\sqrt{-2\kappa t}^2} f''(z_i) & \text{if } j \leq l \\
\frac{l^2}{2\kappa t} f''(z_i) & \text{if } i = j = l + 1 \\
\frac{1}{2\kappa t} f''(z_i) & \text{if } i < j = l + 1 \\
0 & \text{if } j > l + 1
\end{cases}
\]

Therefore, the first derivatives are
\[
\partial_{(i)} u = \begin{cases} 
\frac{1}{N} + \frac{1}{\sqrt{2\kappa t}} \sum_{i=1}^{N-1} c_i f'(z_i) & \text{if } i = 1 \\
\frac{1}{N} + \frac{1}{\sqrt{2\kappa t}} \sum_{i=i}^{N-1} c_i f'(z_i) - (i - 1) c_{i-1} f'(z_{i-1}) & \text{if } i \geq 2
\end{cases}
\]
Since \( x(1) \geq x(2) \geq \ldots \geq x(N) \), we have \( 0 \leq z_1 \leq z_2 \leq \ldots \leq z_{N-1} \) and therefore \( 0 \leq f'(z_1) \leq f'(z_2) \leq \ldots \leq f'(z_{N-1}) \). As a consequence \( \partial_i u \geq 0, \forall i \in [N] \).

The second derivatives are
\[
\partial_{(i)(i)} u = \begin{cases} 
\frac{1}{\sqrt{2\kappa t}^2} \sum_{i=1}^{N-1} c_i f''(z_i) & \text{if } i = 1 \\
\frac{1}{\sqrt{2\kappa t}^2} \left( \sum_{i=i}^{N-1} c_i f''(z_i) + (i - 1)^2 c_{i-1} f''(z_{i-1}) \right) & \text{if } 2 \leq i \leq N - 1 \\
\frac{1}{\sqrt{2\kappa t}^2} (N - 1)^2 c_{N-1} f''(z_{N-1}) & \text{if } i = N
\end{cases}
\]
or for \( i < j \)
\[
\partial_{(i)(j)} u = \begin{cases} 
\frac{1}{\sqrt{2\kappa t}^2} \sum_{i=j}^{N-1} c_i f''(z_i) - (j - 1) c_{j-1} f''(z_{j-1}) & \text{if } j < N \\
\frac{1}{\sqrt{2\kappa t}^2} (N - 1) c_{N-1} f''(z_{N-1}) & \text{if } j = N
\end{cases}
\]
The third derivatives are

\[
\partial_{(i)(j)(k)} u = \begin{cases} 
\frac{1}{(2\pi t)^2} \sum_{l=1}^{N} c_l f'''(z_l) & \text{if } i = 1 \\
\frac{1}{(2\pi t)^2} \left( \sum_{l=i}^{N-1} c_l f'''(z_l) - (i-1)^3 c_{i-1} f'''(z_{i-1}) \right) & \text{if } 2 \leq i \leq N-1 \\
\frac{1}{2\pi t} (N-1)^3 c_{N-1} f'''(z_{N-1}) & \text{if } i = N
\end{cases}
\]

when \( i \neq j \),

\[
\partial_{(i)(j)(j)} u = \begin{cases} 
\frac{1}{(2\pi t)^2} \left( \sum_{l=j}^{N-1} c_l f'''(z_l) + (j-1)^2 c_{j-1} f'''(z_{j-1}) \right) & \text{if } i < j \leq N-1 \\
\frac{1}{(2\pi t)^2} (N-1)^2 c_{N-1} f'''(z_{N-1}) & \text{if } i < j = N \\
\frac{1}{(2\pi t)^2} \left( \sum_{l=i}^{N-1} c_l f'''(z_l) - (i-1) c_{i-1} f'''(z_{i-1}) \right) & \text{if } j < i \leq N-1 \\
\frac{1}{2\pi t} (N-1) c_{N-1} f'''(z_{N-1}) & \text{if } j < i = N
\end{cases}
\]

and when \( i < j < k \)

\[
\partial_{(i)(j)(k)} u = \begin{cases} 
\frac{1}{(2\pi t)^2} \left( \sum_{l=k}^{N-1} c_l f'''(z_l) - (k-1) c_{k-1} f'''(z_{k-1}) \right) & \text{if } k \leq N-1 \\
\frac{1}{2\pi t} (N-1) c_{N-1} f'''(z_{N-1}) & \text{if } k = N
\end{cases}
\]

F.2 \( u \) is \( C^2 \) with Lipschitz Continuous Second Order Derivatives

Since \( u \) is defined as reflection across the boundaries \( \{ x_k = x_{k+1} \} \) of the values on \( \{ x_1 \geq x_2 ... \geq x_N \} \), it suffices to confirm that the normal derivatives on those boundaries are 0. i.e \( \partial_k u = \partial_{k+1} u \) on \( \{ x_k = x_{k+1} \} \).

On the surface \( x_1 = x_2, z_1 = 0 \) and therefore \( \partial_1 u|_{z_1=0} = \partial_2 u|_{z_1=0} \). Also on the surface \( x_k = x_{k+1} \) for \( 2 \leq k \leq N-1, z_{k-1} = z_k \) and therefore,

\[
\partial_i u|_{z_{k-1}=z_k} = \partial_{k+1} u|_{z_{k-1}=z_k} = c_k f'(z_k) - (k-1) c_{k-1} f'(z_{k-1}) + k c_k f'(z_k) = 0
\]

This confirms that the derivatives normal to the surface \( x_k = x_{k+1} \) are equal on both sides for \( 1 \leq k \leq N-1 \). Thus \( u \) is \( C^2 \).

We next show \( u \) is not \( C^3 \). Suppose \( x_1 > x_2 > x_3 > x_4 ... > x_N \) then since \( u(x_1, x_2, x_3, ..., x_N) = u(x_1, x_3, x_2, ..., x_N) \) we have \( \partial_{23} u(x_1, x_2, x_3, x_N) = \partial_{32} u(x_1, x_3, x_2, x_N) = \partial_{32} u(x_1, x_3, x_2, x_N) \) however

\[
\partial_{23} u(x_1, x_2, x_3, x_N) = 3 f'''(z_2) - \frac{1}{2} f'''(z_1)
\]

which does not go to 0 when \( x \) approaches to \( \{ x_1 > x_2 = x_3 > ... > x_N \} \). This means \( \partial_{32} u \) can’t be continuously extended to the boundary \( \{ x_1 > x_2 = x_3 > ... > x_N \} \).

Finally we show the boundedness of third order derivatives. Note that for \( z \geq 0 \),

\[
-\sqrt{\frac{2}{e\pi}} \leq f'''(z) = -\sqrt{\frac{2}{e\pi}} ze^{-\frac{z^2}{2}} \leq 0
\]

From Appendix F.1 we have

\[
\begin{cases} 
\frac{1}{2\pi t} \sqrt{\frac{2}{e\pi}} \left( \frac{1}{i} - \frac{1}{N} \right) \leq \partial_{(i)(j)(j)} u \leq \frac{1}{(2\pi t)^2} \sqrt{\frac{2}{e\pi}} \left( \frac{i-1}{2} \right)^2 \\
\frac{1}{2\pi t} \sqrt{\frac{2}{e\pi}} \left( \frac{1}{j} - \frac{1}{N} \right) \leq \partial_{(i)(j)(j)} u \leq 0 & \text{if } i < j \\
\frac{1}{2\pi t} \sqrt{\frac{2}{e\pi}} \left( \frac{1}{j} - \frac{1}{N} \right) \leq \partial_{(i)(j)(j)} u \leq \frac{1}{(2\pi t)^2} \sqrt{\frac{2}{e\pi}} \frac{1}{j} & \text{if } i > j \\
\frac{1}{2\pi t} \sqrt{\frac{2}{e\pi}} \left( \frac{1}{k} - \frac{1}{N} \right) \leq \partial_{(i)(j)(k)} u \leq \frac{1}{(2\pi t)^2} \sqrt{\frac{2}{e\pi}} \frac{1}{k} & \text{if } i < j < k
\end{cases}
\]
F.3 \( u \) satisfies (8)

First, note that \( \lim_{z \to \infty} f(z) / z = 1 \), and therefore, the final value condition is satisfied.

\[
\lim_{t \to 0} u(x, t) = \frac{1}{N} \langle x, 1 \rangle + \sum_{i=1}^{N-1} \frac{1}{i(i+1)} \left( \left( \sum_{j=1}^{i} x(j) \right) - i x(i+1) \right)
\]

\[
= \frac{1}{N} \langle x, 1 \rangle + \sum_{j=1}^{N-1} x(j) \left( \sum_{i=j}^{N-1} \frac{1}{i(i+1)} \right) - \sum_{i=1}^{N-1} \frac{x(i+1)}{i+1}
\]

\[
= \frac{1}{N} \langle x, 1 \rangle + \sum_{j=1}^{N-1} x(j) \left( \frac{1}{j} - \frac{1}{N} \right) - \sum_{i=2}^{N} \frac{x(i)}{i}
\]

\[
= \frac{1}{N} \langle x, 1 \rangle + x(1) - \left( \sum_{j=1}^{N-1} \frac{x(j)}{N} \right) - \frac{x(N)}{N}
\]

\[
= x(1) = \max_i x_i
\]

Since \( x(1) \geq x(2) \geq ... \geq x(N) \), we have \( 0 \leq z_1 \leq z_2 \leq ... \leq z_{N-1} \) and, therefore,

\[
\sqrt{\frac{2}{\pi}} \geq f''(z_1) \geq f''(z_2) \geq ... \geq f''(z_{N-1}) \geq 0.
\]

This combines with a simple computation gives for \( i \leq N - 1 \),

\[
\partial_{ij} u - \partial_{i(i+1)+1} u = (1 - \frac{1}{i}) (f''(z_{i-1}) - f''(z_i)) \geq 0
\]

Therefore, \( \max_i \partial^2 u = \partial^2_{(1)} u = \partial^2_{(2)} u \).

Finally, \( u_t = -\frac{\sqrt{\pi}}{\sqrt{2t}} \sum_{l=1}^{N-1} c_l f''(z_l) \) and thus \( u_t + \kappa \partial^2_{(1)} u = 0 \).

Appendix G. Proof of Claim 6

In this section we prove for max potential \( u \), \( \max_{q \in \{\pm 1\}^N} \langle D^2 u \cdot q, q \rangle \) is obtained by a specific group of strategies and compute the optimal \( \kappa_m \) such that

\[
\frac{1}{2} \max_{q \in \{\pm 1\}^N} \langle D^2 u \cdot q, q \rangle \leq \kappa_m \max_i \partial_i u
\]

Without loss of generality we assume \( x_1 \geq x_2 \geq ... \geq x_N \). From Appendix F.1 we know for \( i < j \) \( \partial_{ij} u \) is a function of \( j \) alone, thus we denote \( a_j = -\partial_{ij} u \) for any \( i < j \). Also, notice that

\[
\partial_{ij} u = \frac{1}{\sqrt{-2\kappa t}} \left( \sum_{l=j}^{N-1} c_l f''(z_l) - (j-1)c_{j-1}f''(z_{j-1}) \right) \leq \frac{1}{\sqrt{-2\kappa t}} \frac{f''(z_j) - f''(z_{j-1})}{j} \leq 0
\]
and for $i < j < k$

$$\partial_{ij} u - \partial_{ik} u = \frac{1}{\sqrt{-2\kappa t}} \left( \left( \sum_{l=j}^{k-1} c_l f''(z_l) \right) - (j-1)c_{j-1} f''(z_{j-1}) + (k-1)c_{k-1} f''(z_{k-1}) \right)$$

$$\leq \frac{1}{\sqrt{-2\kappa t}} \left( f''(z_j) \left( \frac{1}{j} - \frac{1}{k} \right) - \frac{f''(z_{j-1})}{j} + \frac{f''(z_{j})}{k} \right) \leq 0$$

thus $a_2 \geq a_3 \geq \ldots \geq a_N \geq 0$.

**Theorem 7** For the max potential $u$ on $\{ x | x_1 \geq x_2 \geq \ldots \geq x_N \}$, $\max_{q \in \{\pm 1\}^N} \langle D^2 u \cdot q, q \rangle$ is obtained by strategies satisfying $q_{2i-1} + q_{2i} = 0$, $\forall 2i \leq N$. Specifically, comb strategy $q^c$ achieves the maximum.

**Proof** As noted previously, we can view $D^2 u$ as the Laplacian of an undirected weighted graph $G$ with $N$ vertices. The edge weight $w_{ij} = -\partial_{ij} u = a_j$ for $i < j$ and $a_2 \geq a_3 \ldots \geq a_N$. Also

$$\max_{q \in \{\pm 1\}^N} \langle D^2 u \cdot q, q \rangle = 4 \text{max}_\text{cut}(G)$$

Thus we convert the problem to finding the max cut for a special weighted graph, the theorem we proved below gives us the desired result. \qed

**Theorem 8** Consider an undirected graph with vertices $\{1, \ldots, N\}$ satisfying for any edge $(i, j)$ the weight depends on $\max(i, j)$, i.e. we can write $w_{ij} = a_j$ for $i < j$. Also suppose $a_2 \geq a_3 \ldots \geq a_N$, then the max cut, modulo permutations between vertices $(i, j)$ such that $a_i = a_j$, is any cut dividing $2i - 1$ and $2i$ for all $1 \leq i \leq \lfloor \frac{N}{2} \rfloor$.

**Proof** Without loss of generality, assume $a_2 > a_3 \ldots > a_N$. We use induction on $N$. For $N = 2$ and $N = 3$ it is straightforward to check that the max cut is any cut dividing 1 and 2.

For $N + 1$ points, we first prove the max cut must divide 1 and 2.

**Lemma 9** Any max cut must divide 1 and 2.

**Proof** [Proof of lemma 9] Assume a max cut doesn’t divide 1 and 2, denote

$$L = \{ i \in \{3, \ldots, N\} | i \text{ on the same side as 1 and 2} \}$$

$$R = \{ i \in \{3, \ldots, N\} | i \text{ on the other side} \}$$

by definition $R$ is nonempty.

Define $A_L = \sum_{j \in L} a_j$ and $A_R = \sum_{j \in R} a_j$. If $A_R < A_L + a_2$ then by moving 2 to $R$ the cut will get bigger since

$$T(\{1\} \cup L, \{2\} \cup R) = T(\{1, 2\} \cup L, R) + a_2 + A_L - A_R > T(\{1, 2\} \cup L, R)$$

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which is a contradiction.

So \( A_R \geq A_L + a_2 \). We denote \( p_i = \{2i - 1, 2i\} \), \( 2 \leq i \leq \left\lfloor \frac{N}{2} \right\rfloor \), if no \( p_i \) satisfies \( p_i \subset R \) then
\[
A_R - A_L \leq (a_3 - a_4) + (a_5 - a_6) + ... < a_2
\]
Thus we can assume \( p_k \) is the smallest set contained in \( R \). We prove that by moving \( 2 \) to \( R \) and \( 2k - 1 \) to \( L \) the cut will get bigger. Actually
\[
T(\{1, 2k - 1\} \cup L, \{2\} \cup R \setminus \{2k - 1\}) = T(\{1, 2\} \cup L, R) + a_2 + (|R_{2k-1}| - |L_{2k-1}|)a_{2k-1} + A_{L_{2k-1}} - A_{R_{2k-1}}
\]
where
\[
\begin{align*}
L_{2k-1} &= L \cap \{3, \ldots, 2k - 1\} \\
R_{2k-1} &= R \cap \{3, \ldots, 2k - 1\}
\end{align*}
\]
and \( A_{L_{2k-1}}, A_{R_{2k-1}} \) are defined under the same convention as \( A_L, A_R \).

By definition of \( k \) for any \( p_i \) such that \( 2 \leq i \leq k - 1 \), if one of the element is in \( R_{2k-1} \) then the other must be in \( L_{2k-1} \). Suppose \( R_{2k-1} \) contains elements of \( p_i_1, \ldots, p_i_{|R_{2k-1}|-1} \) and \( 2k - 1 \), then
\[
a_2 + (|R_{2k-1}| - |L_{2k-1}|)a_{2k-1} + A_{L_{2k-1}} - A_{R_{2k-1}} \geq a_2 + \sum_{j=1}^{|R_{2k-1}|-1} a_{2i_j} - \sum_{j=1}^{|R_{2k-1}|-1} a_{2i_j} = a_2 - 1
\]
\[
+ \left( \sum_{l \in L_{2k-1} \setminus \bigcup_{j=1}^{|R_{2k-1}|-1} p_i_j} a_l \right)
- (|L_{2k-1}| - |R_{2k-1}|)a_{2k-1}
\]
We can rearrange the sum in the first line
\[
a_2 + \sum_{j=1}^{|R_{2k-1}|-1} a_{2i_j} - \sum_{j=1}^{|R_{2k-1}|-1} a_{2i_j} = a_2 - 1
\]
\[
= (a_2 - a_{2i_1}) + (a_{2i_1} - a_{2i_2}) + \ldots + (a_{2i_{|R_{2k-1}|-1}} - a_{2k-1}) > 0
\]
Also notice that each \( a_l > a_{2k-1} \) for \( l \in L_{2k-1} \setminus \bigcup_{j=1}^{|R_{2k-1}|-1} p_i_j \) and
\[
|L_{2k-1} \setminus \bigcup_{j=1}^{|R_{2k-1}|-1} p_i_j| = |L_{2k-1}| - |R_{2k-1}| + 1
\]
the second line minus the third line is positive.
This confirms that the new cut is strictly better which is a contradiction.
Returning to the proof of theorem 8, denote

\[ S_i = \{ j \in \{3, ..., N \} | j \text{ on the same side as } i \} \]

for \( i = 1, 2 \).

Also denote \( T(A, B) \) as the total weights of edges between \( A \) and \( B \). Then

\[ T(\{1\} \cup S_1, \{2\} \cup S_2) = \sum_{i=3}^{N} a_i + T(S_1, S_2) \]

Thus \( (S_1, S_2) \) must be the max cut for \( \{3, ..., N\} \) as well. By induction hypothesis the max cut divides \( 2i - 1 \) and \( 2i \) for \( 2 \leq i \leq \left[ \frac{N}{2} \right] \).

Now we use theorem 7 to compute \( \kappa_m \). We use the same notation \( a_i \) same as above, since comb strategy \( q^c \) attains the maximum,

\[
\max (D^2 u \cdot q, q) = D^2 u \cdot q^c, q^c
\]

\[
= \begin{cases} 
\sum_{i=1}^{M-1} 4i(a_{2i} + a_{2i+1}) + 4Ma_{2k} & N = 2M \\
\sum_{i=1}^{M} 4i(a_{2i} + a_{2i+1}) & N = 2M + 1 
\end{cases}
\]

Notice that

\[
\max_i \partial_i u = \partial_{11} u = \sum_{i=2}^{N} a_i
\]

We take

\[
\kappa_m = \max_{a_2 \geq a_3 \geq \ldots \geq a_N \geq 0} \frac{1}{2} \frac{\langle D^2 u \cdot q^c, q^c \rangle}{\partial_{11} u}
\]

\[
= \begin{cases} 
\frac{N^2}{2(N-1)} & N \text{ even} \\
\frac{N+1}{2} & N \text{ odd} 
\end{cases}
\]

the max is obtained when \( a_2 = a_3 \ldots = a_N \).

**Appendix H. Proof of Example 2**

In this section, we prove the error estimate for max potential \( u \). Note that \( D^3 u \) is not Lipschitz continuous we can’t use Remark 2.

We will first confirm that

\[
-2 - 2 \sqrt{\frac{\kappa}{\pi}} \frac{N-1}{N} \leq u(x_0, 0) - u(x_{-1}, -1) \leq 2
\]

then for time derivatives, we have for all \( x \) and \( t \leq -2 \)

\[
- \frac{1}{(t-1)^{\frac{3}{2}}} \frac{N-1}{N} \sqrt{\frac{\kappa}{e^3 \pi}} \leq u_{tt} \leq \frac{1}{(t-1)^{\frac{3}{2}}} \frac{N-1}{N} \frac{\sqrt{\kappa}}{2\sqrt{\pi}}
\]
For lower bound estimate we prove for $q \sim a^m$

$$\text{ess sup}_{y \in [x,x+q]} D^3 u(y, t + 1)[\pm q, \pm q, \pm q] \leq \frac{4}{-\kappa(t + 1)} \frac{(N - 1)^2}{N} \sqrt{\frac{2}{e\pi}}$$

For upper bound estimate we prove for any $q \in [-1, 1]^N$

$$-\text{ess inf}_{y \in [x,x-q]} D^3 u(y, t + 1)[q,q,q] \leq \frac{1}{-2\kappa(t + 1)} \sqrt{\frac{2}{e\pi}} \frac{7}{2} N^2 - 8N + 5\log N + \frac{3}{2}$$

For lower bound we take $\kappa = 2$ thus

$$E(t) = 2 + 2\sqrt{\frac{2}{\pi}} + \frac{N - 1}{N} \sqrt{\frac{2}{e\pi}} \sum_{s=t}^{\infty} \frac{1}{(-s - 1)^{3/2}} + O(N) \sum_{s=t}^{\infty} \frac{1}{-s - 1} = O(N \log |t|)$$

For upper bound we take $\kappa = \kappa_m = O(N)$ thus

$$E(t) = 2 + O(\sqrt{N}) \sum_{s=t}^{\infty} \frac{1}{(-s - 1)^{3/2}} + O(N) \sum_{s=t}^{\infty} \frac{1}{-s - 1} = O(N \log |t|)$$

### H.1 Final time step

We split the difference as follows

$$u(x_0, 0) - u(x_{-1}, -1) = \max_i (x_0)_i - \max_i (x_{-1})_i + u(x_{-1}, 0) - u(x_{-1}, -1)$$

Since $q_{-1} \in [-1, 1]^N$ we have $r_{-1} = (q_{-1})_{l_{-1}, 1} - q_{-1} \in [-2, 2]^N$

$$-2 \leq \max_i (x_0)_i - \max_i (x_{-1})_i \leq 2$$

We have for any $x$

$$u(x, 0) - u(x, -1) = x(1) - \frac{1}{N} \sum_{l=1}^N x(l) - \sqrt{2\kappa} \sum_{l=1}^{N-1} c_l f(z_l) = \sqrt{2\kappa} \sum_{l=1}^{N-1} c_l z_l - \sqrt{2\kappa} \sum_{l=1}^{N-1} c_l f(z_l)$$

Since $-\sqrt{\frac{2}{\pi} z} \leq z - f(z) \leq 0$ for $z \geq 0$,

$$-2\sqrt{\frac{\kappa N - 1}{\pi N}} \leq u(x, 0) - u(x, -1) \leq 0$$

Therefore,

$$-2 - 2\sqrt{\frac{\kappa N - 1}{\pi N}} \leq u(x_0, 0) - u(x_{-1}, -1) \leq 2$$
H.2 Bound on \( u_{tt}(x, \tau) \)

We have

\[
\begin{align*}
   u_{tt} &= \frac{\sqrt{\kappa}}{2\sqrt{2(-\tau)^{3/2}}} \sum_{l=1}^{N-1} c_l f''(z_l) + \frac{\sqrt{\kappa}}{\sqrt{-2\tau}} \sum_{l=1}^{N-1} c_l f'''(z_l) \frac{z_l}{-2\tau} \\
   &= \frac{\sqrt{\kappa}}{2\sqrt{2(-\tau)^{3/2}}} \sum_{l=1}^{N-1} c_l \left( f''(z_l) + f'''(z_l)z_l \right) \\
   &= \frac{\sqrt{\kappa}}{2\sqrt{2(-\tau)^{3/2}}} \sum_{l=1}^{N-1} c_l \left( 1 - z_l^2 \right) \sqrt{\frac{2}{\pi}} e^{-\frac{z_l^2}{2}}
\end{align*}
\]

Note that for all \( z \), \(-2e^{-\frac{3z^2}{2}} \leq (1 - z^2) e^{-\frac{z^2}{2}} \leq 1\). Therefore

\[
\frac{1}{\sqrt{\kappa}} \frac{N-1}{N} \sqrt{\frac{2}{\pi}} \leq u_{tt} \leq \frac{1}{\sqrt{\kappa}} \frac{N-1}{2\sqrt{N}}
\]

H.3 Upper Bound of \( \text{ess sup}_{y \in [x,x-q]} D^3 u(y, t+1) [q, q, q] \) for \( q \sim a^m \)

Without loss of generality assume \( x_1 \geq x_2 \geq ... \geq x_N \), then \( q^m = (1, -1, ..., -1) \) and \( q \sim a^m \) could be either \( q^m \) or \(-q^m\). We give an upper bound of

\[
\text{ess sup}_{y \in [x,x+q^m]} D^3 u(y, t+1) [\pm q^m, \pm q^m, \pm q^m]
\]

Since \( u \) is linear along \( \bar{l} \)

\[
D^3 u(y, t+1) [\pm q^m, \pm q^m, \pm q^m] = D^3 u(y, t+1) [\pm (q^m + 1), \pm (q^m + 1), \pm (q^m + 1)] = \pm 8 \partial_{111} u(y, t+1)
\]

If \( q = -q^m \), then \( [x, x + q^m] \subset \{x|x_1 \geq x_2 \geq ... \geq x_N\} \). For \( y \in [x, x + q^m] \)

\[
D^3 u(y, t+1) [-q^m, -q^m, -q^m] = -8 \partial_{111} u(y, t+1) = -8 \partial_{(1)(1)(1)} u(y, t+1) \leq \frac{4}{-\kappa(t+1)} \sqrt{\frac{2}{\pi}} \frac{N-1}{N}
\]

If \( q = q^m \), suppose \( x_2 + 1 \geq x_3 + 1 ... ; x_k + 1 \geq x_1 - 1 \geq x_{k+1} + 1 ... \geq x_N + 1, k \) ranges from \( 1 \) to \( N \). We can accordingly partition \([x, x - q^m]\) into \( k \) subintervals \( I_1...I_k \) such that \( y_1 \) ranks \( l \)'s for \( y \in I_l \). Thus in this subinterval

\[
D^3 u(y, t+1) [q^m, q^m, q^m] = 8 \partial_{111} u(y, t+1) = 8 \partial_{(1)(1)(1)} u(y, t+1) \leq \frac{4}{-\kappa(t+1)} \frac{(l-1)^2}{l} \sqrt{\frac{2}{\pi e}}
\]

Summarizing above we have

\[
\text{ess sup}_{y \in [x,x+q^m]} D^3 u(y, t+1) [\pm q^m, \pm q^m, \pm q^m] \leq \frac{4}{-\kappa(t+1)} \frac{(N-1)^2}{N} \sqrt{\frac{2}{\pi e}}
\]
H.4 Upper Bound of $-\operatorname{ess} \inf_{y \in [x, x-q]} D^3 u(y, t + 1)[q, q, q]$ for $q \in [-1,1]^N$

It suffices to give an uniform upper bound for $|D^3 u(x, t + 1)[q, q, q]|$ over all $x \in \mathbb{R}^N$ and $q \in [-1,1]^N$.

From F.1 we have

\[
\left\{ \begin{array}{ll}
|\partial_{(1)(1)(1)} u| \leq \frac{1}{-2\kappa(t+1)} \sqrt{\frac{2}{e \pi} (1 - \frac{1}{N})} \\
|\partial_{(i)(i)(i)} u| \leq \frac{1}{-2\kappa(t+1)} \sqrt{\frac{2}{e \pi} (i^{-1})^2} & \text{if } i > 1 \\
|\partial_{(i)(j)(j)} u| \leq \frac{1}{-2\kappa(t+1)} \sqrt{\frac{2}{e \pi} (1 - \frac{1}{N})} & \text{if } i < j \\
|\partial_{(i)(j)(k)} u| \leq \frac{1}{-2\kappa(t+1)} \sqrt{\frac{2}{e \pi} \frac{1}{i}} & \text{if } i > j \\
|\partial_{(i)(j)(k)} u| \leq \frac{1}{-2\kappa(t+1)} \sqrt{\frac{2}{e \pi} \frac{1}{k}} & \text{if } i < j < k
\end{array} \right.
\]

Notice that for any $i, j, k$, $\partial_{(i)(j)(k)} u$ only depends on $\max(i, j, k)$, for $q \in [-1,1]^N$ we have

\[
|D^3 u(x, t + 1)[q, q, q]| \leq \sum_{i=1}^{N} \left| \partial_{(i)(i)(i)} u(x, t + 1) \right| + \sum_{i=2}^{N} \left( \sum_{j=1}^{i-1} q_j^2 \right) \left| \partial_{(i)(j)(j)} u(x, t + 1) \right|
+ \sum_{j=2}^{N} \left( \sum_{i=1}^{j-1} q_i \right) \left| \partial_{(i)(j)(j)} u(x, t + 1) \right| + 6 \sum_{k=3}^{N} \left( \sum_{i=1}^{k-1} q_i \right) \left( \sum_{j=1}^{k-1} q_j \right) \left| \partial_{(i)(j)(k)} u(x, t + 1) \right|
\leq \frac{1}{-2\kappa(t+1)} \sqrt{\frac{2}{e \pi} \left( N - 1 + \sum_{i=2}^{N} (1 - \frac{1}{i}) + \sum_{j=2}^{N} (j - 1) - \frac{1}{N} + 6 \sum_{k=3}^{N} \frac{(k - 1)^2}{k} \right)}
\leq \frac{1}{-2\kappa(t+1)} \sqrt{\frac{2}{e \pi} \left( \frac{7}{2} N^2 - 8N + 5 \log N + \frac{3}{2} \right)}
\]

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