Complete (2 + 1)-dimensional Ricci flow spacetimes

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Abstract

Ricci flow spacetimes were introduced by Kleiner & Lott as a way to describe Ricci flow through singularities, and have since been used elsewhere in the literature, prompting the question of their rigidity. In (2 + 1)-dimensions, we show that every complete and sufficiently regular spacetime must be a cylindrical spacetime. That is, if the metric is complete on each spatial slice, after imposing a necessary continuity condition, we can conclude that every spatial slice must be diffeomorphic to a fixed surface, and the Ricci flow spacetime is isometric to a classical Ricci flow on this surface.

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1 Introduction

1.1 Motivation

Given a smooth manifold $M^n$ equipped with a smooth family of Riemannian metrics $g(t)$ for $t \in (0, T)$, we say that $g(t)$ is a Ricci flow on $M \times (0, T)$, if the metrics solve the equation

$$\frac{\partial g}{\partial t}(t) = -2 \text{Ric} g(t),$$

(1)

at every point in $M \times (0, T)$. Since equation (1) is local, we could instead ask for a family of metrics defined only on an open subset of $M \times (0, T)$, and which solve equation (1) on this open subset.

**Definition 1.1** (Spacetime within an ambient space). Given a smooth manifold $M^n$, we say that $M$ is a spacetime in the ambient space $M \times (0, T)$ if

- $M$ is an open subset of $M \times (0, T)$ equipped with the product topology.
- The spatial slice of $M$ at time $t \in (0, T)$,

$$\mathcal{M}_t := \{x \in M : (x, t) \in \mathcal{M}\},$$

is non-empty, for all $t \in (0, T)$.

**Definition 1.2.** Let $\mathcal{M}$ be a spacetime in $M \times (0, T)$. A Ricci flow on $\mathcal{M}$ is a smooth family of Riemannian metrics $g(t)$ on $\mathcal{M}_t$, for each $t \in (0, T)$, satisfying equation (1) at every point in $\mathcal{M}$. We call such a Ricci flow complete if $g(t)$ is a complete metric on the spatial slice $\mathcal{M}_t$, for all $t \in (0, T)$.

The following should be considered as the motivating question behind this paper:

*Which spacetimes $\mathcal{M}$ in a given ambient space $M \times (0, T)$ admit complete Ricci flows?*

**Example 1.3.** Suppose that $M^2$ is a surface. Let $N \subseteq M$ be a fixed subsurface. If $\mathcal{M}$ is chosen so that $\mathcal{M}_t = N$ for every $t \in (0, T)$, then $\mathcal{M}$ does admit a complete Ricci flow. One way to see this would be to fix a metric $g_0$ on $N$ (with large enough volume depending on the conformal type of $N$) and use the existence result of Giesen & Topping [GT11] to find an instantaneously complete Ricci flow on $N \times [0, T)$ starting from this initial data. Topologically, these spacetimes looks like a cylinder.

A priori, we may guess that any spacetime that admits a complete Ricci flow should look like a cylinder. Unlike for closed manifolds however, the topology of non-closed manifolds can change within a complete Ricci flow without encountering a singularity.
Example 1.4. Fix $t_0 > 0$ and a point $p \in \mathbb{T}^2 := S^1 \times S^1$ in the torus. We choose our ambient space to be $\mathbb{T}^2 \times (0, \infty)$, and our spacetime to be the open subset $\mathcal{M} := \mathbb{T}^2 \times (0, \infty) \setminus \{p\} \times (0, t_0]$. Our spatial slices are

$$\mathcal{M}_t := \begin{cases} \mathbb{T}^2 \setminus \{p\} & : \forall t \in (0, t_0], \\ \mathbb{T}^2 & : \forall t > t_0. \end{cases}$$

If we take $g(t)$ to be a homeothetically expanding complete hyperbolic metric on $\mathcal{M}_t$ for $t \in (0, t_0]$, we can then cap off the hyperbolic cusp at $p$ after time $t_0$, to give the complete contracting cusp Ricci flow $g(t)$ on $\mathcal{M}_t$, for $t \in (t_0, \infty)$ [Top12, Theorem 1.2].

Example 1.5. Consider $\mathcal{M} \subseteq \mathbb{C} \times (0, 1)$ to be some subset of the ambient space whose spatial slice varies in time. For example, we could choose $\mathcal{M}_t = D_{2+t}$, the disk centred at the origin of radius $2 + t$ (with respect to the Euclidean metric), for all $t \in (0, 1)$. Can we find a complete Ricci flow on this spacetime $\mathcal{M}$? What about the case that $\mathcal{M}_t = D_{2-t}$, for all $t \in (0, 1)$?

We shall show that the conical spacetimes mentioned in Example 1.5 do not admit complete Ricci flows. See Corollary 1.17.

Remark. Unlike what we have considered so far, there are multiple ways in the literature to formulate a Ricci flow on a changing underlying manifold without reference to an ambient space. In this paper, we will use the language of a Ricci flow spacetime, first formulated in [KL17]. By using this formulation, it provides immediate context for the results in this paper. However, when working with this more general definition, one should always keep in mind the easy to visualise examples discussed above, where our spacetime is an open subset of a larger ambient space. Nothing is lost by doing so, as we will ultimately reduce our more general Ricci flow spacetime to one within an ambient space by virtue of an embedding lemma (see Section 3).
1.2 Set-up

Given a smooth surface equipped with a smooth metric, there exists a unique instantaneously complete Ricci flow on the surface starting from this smooth metric \([GT11],[Top15]\). On the other hand, in higher dimensions there are smooth manifolds equipped with smooth metrics from which we do not expect to be able to start the Ricci flow.\(^1\) However, imposing curvature bounds on our initial metric leads to the following well-known short-time existence conjecture in 3-dimensions.\(^2\)

**Conjecture.** Let \((M^3,g_0)\) be a 3-dimensional complete Riemannian manifold with non-negative Ricci curvature \(\text{Ric}(g_0) \geq 0\). Then there exists \(T > 0\) and a smooth family of complete metrics \(g(t)\) on \(M \times [0,T)\) such that \(g(0) = g_0\) and \(g(t)\) is a solution to the Ricci flow equation (1) on \(M \times (0,T)\).

A partial resolution to this conjecture was given by Lai by considering Ricci flow spacetimes. Recall the following definition \([KL17]\).

**Definition 1.6.** A Ricci flow spacetime is a tuple \((M^{n+1},t,\partial_t,g)\) where

- \(M\) is a smooth and connected \((n+1)\)-manifold (without boundary).
- The time function \(t\) is a smooth submersion \(t : M \to I\), for some open interval \(I \subseteq \mathbb{R}\).
- \(\partial_t\) is a smooth vector field on \(M\) satisfying \(\partial_t(t) \equiv 1\).
- \(g\) is a smooth inner product on the bundle \(T_M^{\text{spat}} := \ker(dt)\), such that its restriction \(g(t)\) to the time slice \(M_t = t^{-1}(t)\) is a Riemannian metric, for all \(t \in I\).
- \(g\) and \(\partial_t\) satisfy the Ricci flow spacetime equation \(\mathcal{L}_{\partial_t}g = -2\text{Ric}(g)\).

A Ricci flow spacetime is said to be complete if \(g(t)\) is a complete Riemannian metric on \(M_t\), for every \(t \in I\).

**Example 1.7** (Spacetimes within an ambient space). Suppose \(M^n\) is a connected smooth manifold. An important class of easy to visualise examples of Ricci flow spacetimes are ones lying inside the cylinder \(M^n \times \mathbb{R}\), mentioned above in Subsection 1.1. More precisely, suppose \(M^{n+1}\) is an open subset of \(M^n \times I\) (equipped with the product topology), \(t\) is the restriction of the standard projection \(t : M^n \times I \to I\), and \(\partial_t\) is the restriction of \(\partial_{\mathbb{R}}\). Since we specify that the time function is a submersion, each spatial slice \(M_t \subseteq M\) is a non-empty open subset of \(M\), for \(t \in I\). The Ricci flow spacetime equation then corresponds to the metrics \(g(t)\) locally satisfy the usual Ricci flow equation (1). We say that we have a Ricci flow spacetime \((M^{n+1},g)\) in the ambient space \(M^n \times I\). If \(M^{n+1}\) is equal to the entire ambient space, then we say it is a *cylindrical* spacetime and denote it by \((M^n \times I, g)\).

**Definition 1.8.** A pair of Ricci flow spacetimes \((M^{n+1},t,\partial_t,g), (N^{n+1},s,\partial_s,G)\) are isomorphic if \(t(M) = I = s(N)\), and there exists a smooth diffeomorphism \(\Phi : M \to N\), such that

\(^1\)For example, consider \(S^2 \times \mathbb{R}\) equipped with a metric so that it looks geometrically like a countable collection of 3-spheres connected by thinner and thinner necks of increasing length \([Top20]\).

\(^2\)It is unclear who the conjecture is originally attributed to, although the conjecture arises naturally from the work of Shi in \([Shi89]\).
(i) For each \( t \in I \), the restriction \( \Phi : \mathcal{M}_t \to \mathcal{N}_t \) is a diffeomorphism;
(ii) \( t = s \circ \Phi \);
(iii) \( \Phi_\ast(\partial_t) = \partial_s \);
(iv) \( g(t) = \Phi_\ast(G(t)) \).

In this case, we write \((\mathcal{M}^{n+1}, t, \partial_t, g) \cong (\mathcal{N}^{n+1}, s, \partial_s, G)\).

Returning to the conjecture, in [Lai20], Lai constructed a Ricci flow spacetime \((\mathcal{M}^{3+1}, t, \partial_t, g)\) containing within it a Ricci flow on \( M \times (0, T) \). That is, after restricting to a suitable subset of \( \mathcal{M} \), the corresponding Ricci flow spacetime is isomorphic to a cylindrical spacetime \((M^3 \times (0, T), g)\). Moreover, this Ricci flow \( g(t) \) on \( M \times (0, T) \) extends to a smooth family of metrics on \( M \times [0, T] \) with \( g(0) = g_0 \).

Although this Ricci flow \( g(t) \) on \( M \times [0, T] \) may not necessarily be complete, it lies within the larger Ricci flow spacetime \((\mathcal{M}^{3+1}, t, \partial_t, g)\) which does satisfy a completeness like property.\(^3\)

We are therefore motivated to ask what the structure of such a Ricci flow spacetime can be. For example, if we could show that the Ricci flow spacetime \((\mathcal{M}^{3+1}, t, \partial_t, g)\) constructed by Lai was in fact a complete and cylindrical spacetime itself, this would lead to a full resolution of the short-time existence conjecture stated above.

In order to say something about the structure of \( \mathcal{M} \), we need to look at how points inside different timeslices are associated to one another under the flow of the vector field \( \partial_t \). Recall the following definition also taken from [KL17].

**Definition 1.9.** Let \((\mathcal{M}, t, \partial_t, g)\) be a Ricci flow spacetime. The worldline of a point \( x \in \mathcal{M} \) is the maximal integral curve \( I_x : \mathcal{M} \to \mathcal{M}, t \mapsto x(t) \) of \( \partial_t \) that passes through \( x \) at time \( t(x) \in I_x \).

More generally, for a subset \( U \subseteq \mathcal{M}_s \) of some timeslice \((s \in I)\), for each \( t \in I \) we set
\[
U(t) := \{ x(t) \in \mathcal{M}_t : x \in U, t \in I_x \},
\]
and denote those times where the flow lines exist for all points in \( U \) by \( I_U := \bigcap_{x \in U} I_x \).

**Example 1.10.** Suppose \((\mathcal{M}^{n+1}, g)\) is a Ricci flow spacetime in the ambient space \( M^n \times I \). Then, for any point \((x_0, t_0) \in \mathcal{M} \subseteq M \times I\), the interval \( I_{(x_0, t_0)}\) is the connected component of \( \mathcal{M} \cap \{ \{x_0\} \times I \} \) containing \( t_0 \), and \( x_0(t) = (x_0, t) \) for each \( t \in I_{(x_0, t_0)}\). Similarly, given a subset \( U \subseteq \mathcal{M} \), we have that \( U(t) = \mathcal{M} \cap (U \times \{t\}) \), for any \( t \in I \). In the special case \((M^n \times I, g)\) is cylindrical, \( I_{(x_0, t_0)} = I \) for every \((x_0, t_0) \in M \times I\), and \( U(t) = U \times \{t\} \) for every \( U \subseteq \mathcal{M} \).

Although we are interested in \((3+1)\)-dimensional Ricci flow spacetimes, we shall consider the \((2+1)\)-dimensional case as a concrete starting point. The following theorem shows that, for a complete \((2+1)\)-dimensional Ricci flow spacetime, worldlines always persist until the final time.

**Theorem 1.11 (Complete spacetimes are expanding).** Let \((\mathcal{M}^{2+1}, t, \partial_t, g)\) be a complete Ricci flow spacetime with \( I = (0, T) \). Then \( \mathcal{M} \) is expanding. That is, the vanishing times
\[
T_x := \sup I_x = T, \quad \forall x \in \mathcal{M}.
\]

\(^3\)More precisely, the spacetime is forward 0-complete and weakly backward 0-complete. See [Lai20] for the precise definitions.
As seen in Example 1.4, we cannot expect a general complete Ricci flow spacetime to be cylindrical. This leads us to formulate the following regularity hypotheses that we shall impose on our spacetimes.

**Definition 1.12** (Continuous spacetime). Let \((\mathcal{M}^{n+1}, t, \partial_t, g)\) be a Ricci flow spacetime. For any \(s, t \in I\), define the temporal closure of the time slice \(\mathcal{M}_s\) at time \(t\) to be

\[
\overline{\mathcal{M}_s(t)} := \{ x \in \mathcal{M}_t : s \in I_x \} \supseteq \mathcal{M}_s(t).
\]

We say that the Ricci flow spacetime is **continuous** if

\[
(\overline{\mathcal{M}_s(t)})^\circ = \mathcal{M}_s(t), \quad \forall s, t \in I.
\]

This condition prevents cusps from being capped off as in Example 1.4 (see Lemma 3.5). For the next condition, recall the following definition of a time-preserving path \([KL17]\).

**Definition 1.13.** Let \((\mathcal{M}, t, \partial_t, g)\) be a Ricci flow spacetime. Suppose \(J \subseteq I\) is an interval and let \(\eta : J \to \mathcal{M}\) be a smooth path. We say that \(\eta\) is **time-preserving** if \(t \circ \eta(t) = t, \forall t \in J\).

Given a point in spacetime, we may ask how far backwards in time we can see from that point. More precisely, consider every time-preserving path ending at this point, and then take the infimum of their starting times.

**Definition 1.14** (Hindsight function). Let \((\mathcal{M}, t, \partial_t, g)\) be a Ricci flow spacetime. Define the **hindsight function** \(h : \mathcal{M} \to \overline{\mathbb{R}}\) by

\[
h(x) := \inf \{ s \in I : \text{there exists a time-preserving path } \eta : [s, t(x)] \to \mathcal{M} \text{ with } \eta \circ t(x) = x \}.
\]

The following condition, in conjunction with continuity, rules out any auxiliary data from being added at positive time.

**Definition 1.15** (Initially determined). Let \((\mathcal{M}, t, \partial_t, g)\) be a Ricci flow spacetime with \(\inf I = 0\). \(\mathcal{M}\) is said to be **initially determined** if \(h \equiv 0\), where \(h\) denotes the hindsight function (see Definition 1.14).

With these extra hypotheses, complete \((2 + 1)\)-dimensional spacetimes are in fact cylindrical.

**Theorem 1.16.** Let \((\mathcal{M}^{2+1}, t, \partial_t, g)\) be a complete, continuous and initially determined Ricci flow spacetime with \(I = (0, T)\). Then \(I_x = (0, T)\), for every \(x \in \mathcal{M}\).
Corollary 1.17 (Rigidity result). Let $(\mathcal{M}^{2+1}, t, \partial_t, g)$ be a complete, continuous and initially determined Ricci flow spacetime with $I = (0, T)$. Then there exists a connected smooth surface $M^2$ such that $(\mathcal{M}^{2+1}, t, \partial_t, g)$ is isomorphic to a complete Ricci flow on $M \times (0, T)$. That is, there exists a smooth diffeomorphism $\Phi : \mathcal{M} \to M \times (0, T)$ such that

(i) For each $t \in (0, T)$, the restriction $\Phi : \mathcal{M}_t \to M \times \{t\}$ is a diffeomorphism;
(ii) $t = t \circ \Phi$, where $t : M \times (0, T) \to (0, T)$ denotes the standard projection map;
(iii) $\Phi_*(\partial_t) = \frac{\partial}{\partial t}$;
(iv) $\Phi_*(g(t))$ is a complete Ricci flow on $M \times (0, T)$.

1.3 Outline of the paper

In Section 2, we use a Harnack estimate to show that spacetimes with connected spatial slices are expanding. We then decompose any connected spacetime into the union of such spacetimes to show Theorem 1.11.

In Section 3, we show that expanding spacetimes can be embedded into a larger ambient space. We then use the ambient space to define a global conformal structure on our spacetime, allowing us to significantly simplify Theorem 1.16.

In Section 4, we further reduce Theorem 1.16 to complete and conformal Ricci flows on spacetimes within the unit disk. We then formulate a geometric condition equivalent to continuity for such spacetimes, which we use to then prove a comparison principle.

In Section 5, we introduce the idea of an initial time blow-up. Taking any non-atomic Radon measure on a Riemann surface, we can start the Ricci flow from this measure [TY21]. By looking at larger and larger parabolic rescalings of this flow away from the support of the initial measure, in the limit at time zero, we have a hyperbolic metric. We combine this with the comparison principle from Section 4 to show Theorem 1.16.

1.4 Preliminaries

Since $\partial_t$ is a smooth vector field on $\mathcal{M}$, we can consider its maximal flow $\{(x, t) \in \mathcal{M} \times I : t \in I_x\} \to \mathcal{M}$, mapping $(x, t) \mapsto x(t)$. The following are standard properties of the flow.\(^4\)

(a) The set of points we can flow for time $t$, $\mathcal{M}^{(t)} := \{x \in \mathcal{M} : t(x) + t \in I_x\}$, is an open subset of $\mathcal{M}$, for every $t \in \mathbb{R}$.

(b) The map $\mathcal{M}^{(t)} \to \mathcal{M}^{(-t)}$, sending $x \mapsto x(t(x) + t)$, is a smooth diffeomorphism.

For any $s, t \in I$, consider flowing all of the points in spatial slice at time $s$ into the spatial slice at time $t$. This is $\mathcal{M}_s(t)$ in the notation from Definition 1.9. Similarly, consider $\mathcal{M}_t(s)$, which we get from flowing all of the points in the spatial slice at time $t$, into the spatial slice at time $s$.

Since $\mathcal{M}_s(t) = \mathcal{M}_t \cap \mathcal{M}^{(s-t)}$, (a) tells us that $\mathcal{M}_s(t)$ is open in $\mathcal{M}_t$. Similarly $\mathcal{M}_t(s)$ is open in $\mathcal{M}_s$. Then by (b), we have the smooth diffeomorphism $\mathcal{M}_t(s) \to \mathcal{M}_s(t)$, mapping

\(^4\)See [Lee13] for details.
$x \mapsto x(t)$. In particular, can conclude that the flow of $\partial_t$ preserves open subsets within our spatial slices:

**Lemma 1.18.** If $U \subseteq M_s$ is open for some $s \in I$, then $U(t) \subseteq M_t$ is open, $\forall t \in I$.

*Proof.* $U \cap M_t(s)$ is open in $M_s$, whose image under the diffeomorphism $M_t(s) \to M_s(t)$, $x \mapsto x(t)$, is $U(t)$. Since $M_s(t)$ is open in $M_t$, the result follows. □

Fix an open subset of a time slice, $U \subseteq M_s$, for some $s \in I$. Given an interval $J \subseteq I$, we define the parabolic cylinder

$$U(J) := \bigcup_{t \in J} U(t) \subseteq M.$$  \hfill (2)

For $x \in M$ and $r > 0$, consider the special case where $U$ is the ball centred at $x$ of radius $r$

$$B(x, r) := B_{g(t(x))}(x, r) \subseteq M_{t(x)}.$$  

In the language of equation (2), we set

$$C(x, r) := [B(x, r)](t(x) - r^2, t(x) + r^2),$$

to be the parabolic cylinder centred at $x$ of radius $r$.

Given a general parabolic cylinder $U(J)$, we say that it is unscathed if $J \subseteq I_U$ (see Definition 1.9). Since $t$ is a smooth submersion, for any $x \in M$, there exists smooth coordinates $(x_1, \cdots, x_n, t)$ locally around $x$, so that $\partial_i \cdot x_i \equiv 0$ for $i \in \{1, \ldots, n\}$. In particular, we can choose $r > 0$ sufficiently small so that the parabolic cylinder $C(x, r)$ is unscathed.

**Lemma 1.19.** Fix $s \in I$. For any open subset $U \subseteq M_s$, and any open sub-interval $J \subseteq I$, the parabolic cylinder $U(J)$ is open in $M$.

*Proof.* Fix $x \in U(J)$. For $r$ sufficiently small, the parabolic cylinder $C(x, r)$ is unscathed and open in $M$. Since $U(t(x))$ is open in $M_{t(x)}$ and $J$ is open in $I$, shrinking $r$ if necessary, $C(x, r) \subseteq U(J)$. □

**Lemma 1.20.** Fix $s \in I$. Suppose $K \subseteq M_s$. Then there exists an open sub-interval $J \subseteq I$ containing $s$, such that $K(J)$ is unscathed.

*Proof.* Cover $K$ by sufficiently small, unscathed parabolic cylinders centred at points in $K$, and use the compactness of $K$. □

Consider now the map $\Psi : U(J) \subseteq M \to U \times J$, given by the inverse of the flow lines of $\partial_t$

$$\Psi(x) := (x(s), t(x)), \quad \forall x \in U(J).$$

This is a diffeomorphism onto it’s image $\Psi(U(J)) = \{(x, t) \in U \times J : t \in I_s\}$. Note that $(\Psi)_*(\partial_t) = \frac{\partial}{\partial t}$, and the push forward of the metric satisfies the usual Ricci flow equation

$$\frac{\partial}{\partial t} \Psi_g = \Psi_*(\mathcal{L}_{\partial_t} g) = \Psi_*(-2 \text{Ric}_g) = -2 \text{Ric}_\Psi(g), \quad \text{on } \Psi(U(J)).$$

We call such a map, $\Psi : U(J) \subseteq M \to U \times J$, cylindrical coordinates on $U(J)$. 8
Complete Ricci flow spacetimes are expanding

The aim of this section is to prove Theorem 1.11. The key idea in the proof is Lemma 2.5, which states that the vanishing times of points is locally constant within each spatial slice of our spacetime. To show Lemma 2.5 we require bounds on our metric along the worldlines. This takes the form of a Harnack estimate.

2.1 A simple Harnack estimate for spacetimes

This short subsection introduces a simple Harnack estimate for complete Ricci flow spacetimes. Recall that if \( \inf I = 0 \) and the hindsight function \( h \equiv 0 \), then we say that the spacetime \( M \) is initially determined.

The following is Chen’s lower scalar curvature bound [Che09, Proposition 2.1] adapted to Ricci flow spacetimes. Lemma B.1 follows from a slight modification of the original proof, the details of which are in the appendix.

**Lemma B.1** (Scalar curvature lower bound). \( \forall \delta \in (0, \frac{2}{n}) \), \( \exists C(\delta, n) > 0 \) with the following property. Let \( (M^{n+1}, t, \partial_t, g) \) be a Ricci flow spacetime. Fix \( [t_1, t_2] \subseteq I \) and \( \eta : [t_1, t_2] \rightarrow M \) a time-preserving path. Let \( \pi : T M \rightarrow T M^{\text{spat}} \) denote the spatial projection, and suppose there are constants \( r_0, K > 0 \) and \( A \geq 2 + 24(n - 1)r_0^{-2}t_2 \) such that

- \( B_{g(t)}(\eta(t), Ar_0) \subseteq M_t \), for every \( t \in [t_1, t_2] \);
- \( \operatorname{Ric}(g(t)) \leq (n - 1)r_0^{-2} \) on \( B_{g(t)}(\eta(t), r_0) \), for every \( t \in [t_1, t_2] \);
- \( R_{g(t_1)} \geq -K \) on \( B_{g(t_1)}(\eta(t_1), Ar_0) \);
- \( |\pi \circ \eta'(t)| \leq r_0^{-1}, \) for every \( t \in [t_1, t_2] \).

Then for each \( t \in [t_1, t_2] \) and \( x \in B_{g(t)}(\eta(t), \frac{3Ar_0}{4}) \), we have

\[
R_{g(t)}(x) \geq \min\{ -\frac{1}{(\frac{2}{n} - \delta)(t - t_1) + \frac{1}{K}}, -\frac{C}{A^2r_0^2} \}.
\]

As a corollary to Lemma B.1, we also have a lower scalar curvature bound for complete Ricci flow spacetimes (see [Che09, Corollary 2.3] for the comparable result on cylindrical spacetimes).

**Corollary 2.1** (Scalar curvature lower bound in complete spacetimes). Let \( (M^{n+1}, t, \partial_t, g) \) be a complete Ricci flow spacetime. Then for any \( x \in M \) we have

\[
R_{g(t(x))}(x) \geq -\frac{n}{2(t(x) - h(x))}.
\]

If \( M \) is initially determined, this simplifies to

\[
R_{g(t(x))}(x) \geq -\frac{n}{2t(x)}.
\]
Proof. Fix $x ∈ M$ and choose $s ∈ (κ(x), t(x))$. Then by the definition of $κ$ we can find a time-preserving path $η : [s, t(x)) → M$ such that $η ◦ t(x) = x$. For $r_0$ sufficiently small, $\text{Ric}(g(t)) ≤ (n - 1)r_0^{-2}$ on $B_{g(t)}(η(t), r_0)$ for every $t ∈ [s, t(x))$, and $\text{lim}_{t → x} g(t(x)) ≤ r_0^{-1}$ for every $t ∈ [s, t(x))$. For any $A > 0$, and any $t ∈ [s, t(x))$, $B_{g(t)}(η(t), Ar_0) ⊆ M_t$ follows from the completeness of the metric $g(t)$. By compactness, there exists some lower bound $R_{g(s)} ≥ -K$ on $B_{g(s)}(η(s), Ar_0)$. So, for each $δ ∈ (0, \frac{2}{n})$, choosing $A$ sufficiently large, Lemma B.1 implies the existence of $C > 0$ such that

$$R_{g(t(x))}(x) ≥ \min\{-\frac{1}{(\frac{2}{n} - δ)(t(x) - s)} + \frac{C}{A^2}, -\frac{1}{(\frac{2}{n} - δ)(t(x) - s)}\} ≥ \min\{-\frac{1}{(\frac{2}{n} - δ)(t(x) - s)}, -\frac{C}{A^2}\}. \tag{3}$$

Taking $A$ sufficiently large, we conclude that $R_{g(t(x))}(x) ≥ -\frac{1}{(\frac{2}{n} - δ)(t(x) - s)}$. The Corollary is finished by taking $δ ↘ 0$ and $s ↘ h(x)$.

Recall that in dimension $n = 2$, Ricci flow preserves the conformal class of the metric. Within a Ricci flow spacetime $(M^{2+1}, t, \partial_t, g)$, consider cylindrical coordinates $Ψ : U(I) ⊆ M → U × I$, for some open set $U ⊆ M_s$, $s ∈ I$.

Shrinking $U$ if necessary, we can assume that $U$, equipped with the metric $Ψ^∗(g(s))$, admits isothermal coordinates $(x, y)$, so that $Ψ^∗(g(s))$ is conformally equivalent to $(dx^2 + dy^2)$ on $U × \{s\}$. Since $Ψ^∗(g)$ solves the usual Ricci flow equation on $Ψ(U(I)) ⊆ U × I$, we see that

$$Ψ^∗(g) = u(dx^2 + dy^2) \quad \text{on } Ψ(U(I)), \tag{4}$$

where $u : Ψ(U(I)) → (0, ∞)$ is a smooth function satisfying the logarithmic fast diffusion equation (LFDE)

$$\frac{∂u}{∂t} = Δ \log u. \tag{5}$$

Working locally, we can combine the lower scalar curvature bound with the LFDE to give the following simple Harnack estimate.

Definition 2.2. Given a smooth manifold $M$, a subset $Γ ⊆ M$ is a smooth arc in $M$ if it has a smooth and regular parameterisation $γ : J → M$, for some interval $J ⊆ \mathbb{R}$.

$$Γ = \{γ(s) : s ∈ J\}.$$ 

If we can take $J = [0, 1]$, we say that $Γ$ is a compact smooth arc.

Lemma 2.3 (Simple Harnack estimate). Let $(M^{2+1}, t, \partial_t, g)$ be a complete Ricci flow spacetime with $I = (0, T)$. For any $x_0 ∈ M_{t_0}$, choose $r > 0$ sufficiently small such that the ball $B := B(x_0, r) ⊆ M_{t_0}$ admits isothermal coordinates. Let $Γ$ be a smooth arc in $B$. Then

$$ℓ_g(Γ(t)) ≤ \sqrt{\frac{t - h(x_0)}{t_0 - h(x_0)}} · ℓ_g(Γ), \quad ∀t ∈ [t_0, T),$$

where $ℓ_g(Γ(t))$ denotes the length of $Γ(t) ⊆ M_t$ with respect to the metric $g(t)$. If $M$ is initially determined, this simplifies to

$$ℓ_g(Γ(t)) ≤ \sqrt{\frac{t}{t_0}} · ℓ_g(Γ), \quad ∀t ∈ [t_0, T).$$
Proof. As we did above, there exist cylindrical coordinates \( \Psi : B(I) \subseteq \mathcal{M} \rightarrow B \times I \), so that \( \Psi_r(g) \) satisfies (4) for some \( u : \Psi(B(I)) \rightarrow (0, \infty) \) solving the LFDE (5). Since
\[
\Delta \log u = -u \cdot R_{\Psi_r} = -u \cdot \Psi_r(R_g),
\]
we can use Corollary 2.1 to bound the time derivative of our conformal factor
\[
\frac{\partial u}{\partial t}(z,t) = -u(z,t) \cdot R_g(\Psi^{-1}(z,t)) \leq \frac{u(z,t)}{t - h \circ \Psi^{-1}(z,t)} = \frac{u(z,t)}{t - h(x_0)}, \quad \forall (z,t) \in \Psi(B(I)).
\]
Note that \( (z,t) \in \Psi(B(I)) \iff [t_0,t] \subseteq I_z \). So we can integrate the above to get
\[
u(z,t) \leq \left( \frac{t - h(x_0)}{t_0 - h(x_0)} \right) u(z,t_0), \quad \forall (z,t) \in \Psi(B(I)).
\]
Let \( \Gamma \) be the image of the smooth map \( \gamma : J \rightarrow B \). Recall, the set \( \Gamma(t) \) denotes the collection of points \( \{x(t) \in \mathcal{M}_t : x \in \Gamma, t \in I_x\} \) (see Definition 1.9). We shall consider those points in \( \Gamma \) which persist until time \( t \),
\[
[\Gamma(t)](s) = \{x \in \Gamma : t \in I_x\} \subseteq \Gamma
\]
and let
\[
J_t := \gamma^{-1}([\Gamma(t)](s)) \subseteq J.
\]
As \( [\Gamma(t)](s) \) is open in \( \Gamma \), \( J_t \) is open in \( J \). Since \( \Psi(\Gamma(t)) = [\Gamma(t)](s) \times \{t\} \), we have
\[
\ell_g(\Gamma(t)) = \ell_{\Psi_r(g)}([\Gamma(t)](s) \times \{t\}) = \int_{J_t} u(\gamma(s), t_0) \frac{1}{2} |\gamma'(s)| \, ds,
\]
where \(|\cdot|\) is the size of a vector with respect to our local isothermal coordinates. In particular
\[
\ell_g(\Gamma(t)) = \int_{J_t} u(\gamma(s), t_0) \frac{1}{2} |\gamma'(s)| \, ds
\]
\[
\leq \sqrt{\frac{t - h(x_0)}{t_0 - h(x_0)}} \cdot \int_{J_t} u(\gamma(s), t_0) \frac{1}{2} |\gamma'(s)| \, ds
\]
\[
\leq \sqrt{\frac{t - h(x_0)}{t_0 - h(x_0)}} \cdot \ell_g(\Gamma)
\]
\[
= \sqrt{\frac{t - h(x_0)}{t_0 - h(x_0)}} \cdot \ell_g(\Gamma).
\]

Remark. In higher dimensions, the same reasoning applied to the evolution equation for the volume form
\[
\mathcal{L}_{\partial_t}dV_g = -R_gdV_g,
\]
gives an analogous inequality, and hence a local upper bound on volume growth.

By piecing together the above lemma locally, we have the same result for any smooth arc within a spatial slice.
Lemma 2.4. Let \((\mathcal{M}^{2+1}, t, \partial_t, g)\) be a complete Ricci flow spacetime with \(I = (0,T)\). Suppose \(\Gamma\) is a smooth arc in \(\mathcal{M}_{t_0}\). Then
\[
\ell_g(\Gamma(t)) \leq \sqrt{\frac{t - h(x_0)}{t_0 - h(x_0)}} \cdot \ell_g(\Gamma), \quad \forall t \in [t_0, T).
\]
where \(\ell_g(\Gamma(t))\) denotes the length of \(\Gamma(t)\) with respect to the metric \(g(t)\). If \(\mathcal{M}\) is initially determined, this simplifies to
\[
\ell_g(\Gamma(t)) \leq \sqrt{\frac{t}{t_0}} \cdot \ell_g(\Gamma), \quad \forall t \in [t_0, T).
\]

Proof. We aim to apply the Harnack estimate to small balls covering \(\Gamma\). Let \(\Gamma\) be the image of the smooth map \(\gamma: J \to B\). For each \(s \in J\), choose \(r_s > 0\) sufficiently small such that the ball \(B_s := B(\gamma(s), r_s)\) satisfies the hypothesis of Lemma 2.3. Consider the open cover \(\gamma^{-1}(B_s)\) of \(J\). As \(\mathbb{R}\) is locally compact, write \(J\) as a union of compact intervals \(K_i\) for \(i \in \mathbb{N}\), overlapping only at their endpoints. Applying the Lebesgue number lemma to each \(K_i\), we can find a finite number of compact intervals \(J_{i,l}\) such that
\[
K_i = \bigcup_l J_{i,l}
\]
and with the additional property that \(J_{i,l} \subseteq \gamma^{-1}(B_{s_{i,l}})\), for some \(s_{i,l} \in J\). Then, applying Lemma 2.3 to each of the balls \(B_{s_{i,l}}\), we conclude that
\[
\ell_g(\Gamma(t_1)) = \sum_{i,l} \ell_g(\Gamma(t_1) \cap [\gamma(J_{i,l})](t_1))
\leq \sqrt{\frac{t_1 - h(x_0)}{t_0 - h(x_0)}} \sum_{i,l} \ell_g(\Gamma \cap \gamma(J_{i,l}))
= \sqrt{\frac{t_1 - h(x_0)}{t_0 - h(x_0)}} \cdot \ell_g(\Gamma).
\]

2.2 Vanishing times are locally constant

Let \(\Gamma \subseteq \mathcal{M}_s\) be a compact smooth arc with the spatial slice at time \(s \in I\). Recall, in Definition 1.9, we defined the interval \(I_\Gamma = \cap_{x \in \Gamma} I_x\). Since \(\Gamma\) is compactly contained in \(\mathcal{M}_s\), \(s\) is in the interior of \(I_\Gamma\) by Lemma 1.19. Let \(T_\Gamma = \sup I_\Gamma > s\), be the vanishing time of the arc. The following lemma shows that along a compact smooth arc \(\Gamma\), the vanishing times of all of the points within the arc are the same.

Lemma 2.5. Let \((\mathcal{M}^{2+1}, t, \partial_t, g)\) be a complete Ricci flow spacetime with \(I = (0,T)\). Fix \(s \in I\) and let \(\Gamma \subseteq \mathcal{M}_s\) be a compact smooth arc. Then
\[
T_x = T_\Gamma, \quad \forall x \in \Gamma.
\]

Proof. We can assume that \(T_\Gamma < T\), otherwise we would have \(T_x = T_\Gamma = T\), for any \(x \in \Gamma\). Let \(\Gamma\) be the image of the smooth map \(\gamma: [0,1] \to \mathcal{M}_s\). Consider those points in \(\Gamma\) which persist past the vanishing time of \(\Gamma\), \([\Gamma(T_\Gamma)](s) \subseteq \Gamma\). As in Lemma 2.3, we look at the preimage of this subset under our parameterisation
\[
J' := \gamma^{-1}([\Gamma(T_\Gamma)](s)) \subseteq [0,1].
\]
We note that $J'$ is open in $[0,1]$. If $J'$ is non-empty, choose $J'' \subset J'$ to be a non-empty connected component. We use Lemma 2.3 to show that $J''$ contains its infimum and supremum.

**Claim.** $\inf J'', \sup J'' \in J''$.

**Proof of claim.** Let $s := \sup J''$. Choose an increasing sequence $s_j \in J''$ such that $s_j \nearrow s$, and hence $\gamma(s_j) \to \gamma(s)$ in $\mathcal{M}_s$, as $j \to \infty$. Since the metric is smooth,

$$\ell_g(\gamma([s_j,s])) \to 0, \quad \text{as } j \to \infty. \quad (6)$$

In particular, by the Harnack estimate (2.3), the sequence $[\gamma(s_j)](T_\Gamma)$ is Cauchy in $\mathcal{M}_{T_\Gamma}$. As $g(T_\Gamma)$ is complete, there exists a limit $z \in \mathcal{M}_{T_\Gamma}$. Choose $r > 0$ such that the parabolic cylinder $C(z,r)$ is unscathed. In particular, $g$ is smooth, for some $\tau \in (T_\Gamma - r^2, T_\Gamma) \cap (s, T_\Gamma)$, we deduce that $[\gamma(s_j)](\tau) \to z(\tau)$ as $j \to \infty$. Finally, applying the Harnack estimate (2.3) again to equation (6), we see that $[\gamma(s_j)](\tau) \to [\gamma(s)](\tau)$ as $j \to \infty$. So $[\gamma(s)](\tau) = z(\tau)$, or $z = [\gamma(s)](T_\Gamma)$, which implies $s \in J''$. The argument for $\inf J''$ is the same. \(\square\)

By the above claim, $J''$ must be the entire interval $[0,1]$, and $[\Gamma(T_\Gamma)](s) = \Gamma$. This gives a contradiction to the value of $T_\Gamma$ as $\Gamma(T_\Gamma) \in \mathcal{M}_{T_\Gamma}$. Therefore $J'$ is empty, and all points in $\Gamma$ vanish at time $T_\Gamma$. \(\square\)

**Corollary 2.6.** Let $(\mathcal{M}^{2+1}, t, \partial_t, g)$ be a complete Ricci flow spacetime with $I = (0,T)$. Suppose $U \subseteq \mathcal{M}_s$ is connected. Then $T_x = T_y$, for all $x,y \in U$.

### 2.3 Spatially-connected spacetimes are expanding

**Definition 2.7.** Let $(\mathcal{M}, t, \partial_t, g)$ be a connected Ricci flow spacetime. We say that $\mathcal{M}$ is spatially-connected if the time slices $\mathcal{M}_t$ are connected, for all $t \in I$.

Under the additional assumption that $\mathcal{M}$ is spatially-connected, we have a well-defined vanishing time for each spatial slice. If our spacetime was not expanding, then $\mathcal{M}_s(t) = \emptyset$ for some $s < t \in I$. We can now use this to contradict our assumption that our spacetime is connected.

**Theorem 2.8** (Complete, spatially-connected spacetimes are expanding). Let $(\mathcal{M}^{2+1}, t, \partial_t, g)$ be a complete, spatially-connected Ricci flow spacetime with $I = (0,T)$. Then it is expanding. That is, the vanishing times

$$T_x = T, \quad \forall x \in \mathcal{M}.$$

**Proof.** By Corollary 2.6, each spatial slice $\mathcal{M}_s$ has extinction time $T_s \in (s,T]$ for each $s \in (0,T)$. That is, $T_x = T_s$ for all $x \in \mathcal{M}_s$. If $T_s < T$ for some $s \in (0,T)$, then $T_t = T_s$ for all $t \in (s,T_s)$. Choosing any point $x \in \mathcal{M}_{T_s}$, we can pick $\delta > 0$ sufficiently small so that $T_s \pm \delta \in I_x$. This leads to the obvious contradiction

$$T_s + \delta \leq T_x = T_{x(T_s - \delta)} = T_s. \quad \square$$
2.4 Decomposing a spacetime

Finally, we show that any complete Ricci flow spacetime can be decomposed into a collection of complete and spatially-connected Ricci flow spacetimes.

**Definition 2.9.** Let \((\mathcal{M}^{n+1}, t, \partial_t, g)\) be a Ricci flow spacetime and let \(\eta : J \to \mathcal{M}\) be a time-preserving path. Define \(\mathcal{M}^\eta \subseteq \mathcal{M}\) by setting \(\mathcal{M}^\eta_t\) to be the connected component of \(\mathcal{M}_t\) containing \(\eta(t)\), for each \(t \in J\).

**Lemma 2.10.** Let \((\mathcal{M}^{n+1}, t, \partial_t, g)\) be a complete Ricci flow spacetime with \(I = (0, T)\). Fix \((t_0, t_1) \subseteq (0, T)\). If \(\eta : (t_0, t_1) \to \mathcal{M}\) a time-preserving path, then the restriction of \(t, \partial_t\) and \(g\) to \(\mathcal{M}_t^\eta\) is a complete and spatially-connected Ricci flow spacetime.

**Proof.** In order to show that \((\mathcal{M}^\eta, t, \partial_t, g)\) is a Ricci flow spacetime, it suffices to prove that \(\mathcal{M}^\eta\) is an open subset of \(\mathcal{M}\). Fix \(s \in (t_0, t_1)\) and \(x \in \mathcal{M}_t^\eta\). Since \(g(s)\) is complete, \(B := B(x, 1) \subseteq \mathcal{M}_t^\eta\). Since \(\mathcal{M}_t^\eta\) is path-connected, there exists a continuous path \(\rho : [0, 1] \to \mathcal{M}_t^\eta\) from \(x = \rho(0)\) to \(\eta(s) = \rho(1)\). Using the continuity of \(\eta\), there exists \(\delta > 0\) such that \((s - \delta, s + \delta) \subseteq (t_0, t_1)\), with \(t_0 \in I_{\eta(t)}\) for any \(t \in (s - \delta, s + \delta)\). As such, let

\[
K := [\eta((s - \delta, s + \delta))](s) \cup B \cup \rho([0, 1]) \subseteq \mathcal{M}_t^\eta.
\]

By compactness, after possibly shrinking \(\delta\), we can assume that the parabolic cylinder \(K((s - \delta, s + \delta))\) is unscathed. In particular, the unscathed parabolic cylinder \(B((s - \delta, s + \delta))\) lies within \(\mathcal{M}^\eta\). This shows that \(\mathcal{M}^\eta\) is open in \(\mathcal{M}\). Finally, we note that \(\mathcal{M}_t^\eta\) is a closed subset of the complete space \((\mathcal{M}_t, g(t))\) for each \(t \in (t_0, t_1)\), so the Ricci flow spacetime \((\mathcal{M}^\eta, t, \partial_t, g)\) is also complete.

The following lemma shows that, if we have two time-preserving paths starting from the same point, then the corresponding spacetimes we construct from them agree.

**Lemma 2.11.** Let \((\mathcal{M}^{2+1}, t, \partial_t, g)\) be a complete Ricci flow spacetime with \(I = (0, T)\). Let \(\eta_i^t : [t_0, t_1) \to \mathcal{M}\), \(i = 1, 2\), be two time-preserving paths such that \(\eta_1^t(t_0) = \eta_2^t(t_0)\). Consider the restrictions on the interior \(\eta_i := \eta_i^{|(t_0, t_1)}\). Then \(\mathcal{M}^{\eta_1} = \mathcal{M}^{\eta_2}\).

**Proof.** Let \(x_0 := \eta_1^t(t_0) = \eta_2^t(t_0)\), and denote the domain of the worldline of \(x_0\) within the spacetime \(\mathcal{M}^{\eta_i}\) by \(I^{(i)} \subseteq (t_0, t_1)\), for \(i = 1, 2\).

Choose \(r > 0\) sufficiently small such that the parabolic cylinder \(C(x_0, r)\) is unscathed. From this, we can conclude that \(\eta_i(s)\) is path-connected to \(x_0(s)\) in \(\mathcal{M}_s\), for any \(s \in (t_0, t_0 + r^2)\). In particular, \(x_0(s) \in \mathcal{M}_s^{\eta_i}\) for any \(s \in (t_0, t_0 + r^2)\). As \(\mathcal{M}^{\eta_i}\) is complete and spatially-connected, by Theorem 2.8, it is expanding. Thus, \(I^{(1)} = (t_0, t_1)\).

Repeating the same argument with \(i = 2\), we have \(I^{(2)} = (t_0, t_1)\) also. So, for any \(t \in (t_0, t_1)\), \(\mathcal{M}_t^{\eta_1}\) and \(\mathcal{M}_t^{\eta_2}\) are connected components of \(\mathcal{M}_t\), both containing \(x_0(t)\). Hence they agree.

Although our spacetime \(\mathcal{M}\) is path-connected, it is not necessarily path-connected with time-preserving paths. Instead, the following corollary shows that any two points in \(\mathcal{M}\) can be connected by a finite number of concatenated time-preserving paths with alternating orientations.
Corollary A.2. Let \((\mathcal{M}^{n+1}, t, \partial_t, g)\) be a Ricci flow spacetime. Suppose \(x, x' \in \mathcal{M}\). Then there exists \(m \in \mathbb{N}\), a collection of points \(x_0, \ldots, x_m, y_1, \ldots, y_m \in \mathcal{M}\), and a collection of time-dependent paths \(\eta_i : [t(x_{i-1}), t(y_i)] \to \mathcal{M}\), \(\gamma_i : [t(x_i), t(y_i)] \to \mathcal{M}\) for \(i = 1, \ldots, m\), such that

- \(x_0 = x\), and \(x_m = x'\).
- \(\eta_i \circ t(x_{i-1}) = x_{i-1}\) and \(\eta_i \circ t(y_i) = y_i\), for each \(i \in \{1, \ldots, m\}\).
- \(\gamma_i \circ t(x_i) = x_i\) and \(\gamma_i \circ t(y_i) = y_i\), for each \(i \in \{1, \ldots, m\}\).

In light of Lemma 2.11 and Theorem 2.8, we can improve this corollary for complete spacetimes.

Lemma 2.12. Let \((\mathcal{M}^{2+1}, t, \partial_t, g)\) be a complete Ricci flow spacetime with \(I = (0, T)\). Suppose \(x, x' \in \mathcal{M}\). Then there exists \(y \in \mathcal{M}\) and time-dependent paths \(\eta : [t(x), t(y)] \to \mathcal{M}\), \(\gamma : [t(x'), t(y)] \to \mathcal{M}\) such that \(\eta(t(x)) = x\), \(\gamma(t(x')) = x'\), and \(\eta(t(y)) = \gamma(t(y)) = y\). That is, in the language of A.2, we can always choose \(m = 1\).

Proof. Suppose \(m \in \mathbb{N}\) is chosen to be the smallest possible value so that the paths from A.2 exist, and assume \(m > 1\). Without loss of generality, we can assume \(t(y_1) \leq t(y_2)\). Choose \(r > 0\) sufficiently small so that the parabolic cylinders \(C(y_1, r)\) and \(C(\eta_2 \circ t(y_1), r)\) are unscathed. Since \(\gamma_1 : [t(x_1), t(y_1)] \to \mathcal{M}\) and \(\eta_2 : [t(x_1), t(y_2)] \to \mathcal{M}\) are such that \(\gamma_1 \circ t(x_1) = \eta_2 \circ t(x_1) = x_1\), by Lemma 2.11, after restricting these paths to \((t(x_1), t(y_1))\), we have \(\mathcal{M}^{n_1} = \mathcal{M}^{n_2}\). In particular, we can find a time-preserving path from a point in \(C(y_1, r)\) to a point in \(C(\eta_2 \circ t(y_1), r)\). We then modify the end of the path \(\eta_1 : [t(x_0), t(y_1)] \to \mathcal{M}\) and the start of the path \(\eta_2 : [t(y_1), t(y_2)] \to \mathcal{M}\) to connect up with our time-preserving path between the parabolic cylinders, to give a new time-preserving path \(\eta : [t(x_0), t(y_2)] \to \mathcal{M}\) such that \(\eta(t(x_0)) = x_0\) and \(\eta(t(y_2)) = y_2\). This shows that we can reduce the value of \(m\) by at least one, which is a contradiction. Therefore \(m = 1\).

Equipped with the above lemma, we can complete the proof that connected spacetimes are expanding.

Theorem 1.11 (Complete spacetimes are expanding). Let \((\mathcal{M}^{2+1}, t, \partial_t, g)\) be a complete Ricci flow spacetime with \(I = (0, T)\). Then \(\mathcal{M}\) is expanding. That is, the vanishing times

\[ T_x := \sup I_x = T, \quad \forall x \in \mathcal{M}. \]

Proof of theorem 1.11. Suppose there exists \(x \in \mathcal{M}\) such that its extinction time \(T_x < T\). Choose any \(x' \in \mathcal{M}\) such that \(t(x') > T_x\). Applying Lemma 2.12 to the pair of points \(x, x' \in \mathcal{M}\), there exists \(T' \in [t(x'), T)\) and a time-preserving path \(\eta : [t(x), T'] \to \mathcal{M}\) such that \(\eta(t(x)) = x\). Restricting \(\eta\) to the interior \((t(x), T')\) and applying Lemma 2.10, we have that \(\mathcal{M}^{n}\) is a complete and spatially-connected Ricci flow spacetime. Note that for small \(r > 0\), the parabolic cylinder \(C(x, r)\) is unscathed, and hence \(x(s) \in \mathcal{M}^{n}\) for \(s \in (t(x), t(x) + r^2)\). Applying Theorem 2.8, \(\mathcal{M}^{n}\) is expanding. Therefore, inside of \(\mathcal{M}^{n}\), the extinction time of the point \(x\) is \(T'\), from which we can deduce that inside of the spacetime \(\mathcal{M}\), the point \(x\) has extinction time \(T_x \geq T' > T_x\). This is a contradiction.
3 Embedding spacetimes within an ambient space

As we shall see later, an ambient space for a Ricci flow spacetime can be a useful tool. In the case that our Ricci flow spacetime is expanding, we can define a map which looks like a global cylindrical coordinate chart. This will give an embedding (up to some non-final time) of our spacetime into an ambient space.

**Lemma 3.1.** Let $(\mathcal{M}^{n+1}, t, \partial_t, g)$ be an expanding Ricci flow spacetime with $I = (0, T)$. For each $\tau \in (0, T)$, there exists a smooth map

$$\Phi : \mathcal{M}_{(0, \tau)} \rightarrow \mathcal{M}_\tau \times (0, \tau),$$

such that

(i) $\Phi$ is a diffeomorphism onto its image.

(ii) For each $t \in (0, \tau)$, restricting to the spatial slice $\Phi : \mathcal{M}_t \rightarrow \mathcal{M}_\tau \times \{t\}$ is a diffeomorphism onto its image.

(iii) If $t : \mathcal{M} \times (0, \tau) \rightarrow (0, \tau)$ denotes the standard projection, then $t = t \circ \Phi$ and $\Phi_* (\partial_t) = \frac{\partial}{\partial t}$.

(iv) $\Phi^*(g)$ is a solution to the usual Ricci flow equation (1) on $\Phi(\mathcal{M}_{(0, \tau)})$.

**Proof.** Since $\mathcal{M}$ is expanding, for any $s \in (0, \tau)$, we have the smooth embedding

$$\mathcal{M}_s \hookrightarrow \mathcal{M}, \quad x \mapsto x(\tau).$$

Define the map $\Phi : \mathcal{M}_{(0, \tau)} \hookrightarrow \mathcal{M}_\tau \times (0, \tau)$ by

$$\Phi(x) := (x(\tau), t(x)).$$

$\Phi$ is smooth, with smooth inverse $(p, t) \mapsto p(t)$. In particular, conditions (i) and (ii) are satisfied. Directly from the definition of $\Phi$, we see that $t \circ \Phi = t$. Moreover, since worldlines are integral curves of $\partial_t$, the identity $\partial_t \cdot t \equiv 1$ implies $\Phi_* \partial_t = \frac{\partial}{\partial t}$. Finally we have the equality

$$\frac{\partial}{\partial t} \Phi^*(g) = L_{\Phi_* (\partial_t)} \Phi^*(g) = \Phi_* (L_{\partial_t} g) = \Phi_* (-2 \text{Ric}(g)) = -2 \text{Ric}(\Phi^*(g)).$$

\[ \square \]

3.1 Existence of an ambient space for expanding spacetimes

The previous lemma shows that for an expanding Ricci flow spacetime, we have an embedding locally in time. The following theorem extends this to all times.

**Theorem 3.2.** Let $(\mathcal{M}^{n+1}, t, \partial_t, g)$ be an expanding Ricci flow spacetime with $I = (0, T)$. Then there exists a smooth connected manifold $\mathcal{M}^n$ and a smooth map

$$\Phi : \mathcal{M} \hookrightarrow \mathcal{M} \times (0, T),$$

such that

(i) $\Phi$ is a diffeomorphism onto its image;
(ii) For each \( t \in (0, T) \), restricting to the spatial slice \( \Phi : M_t \hookrightarrow M \times \{ t \} \) is a diffeomorphism onto its image;

(iii) If \( t : M \times (0, T) \to (0, T) \) denotes the standard projection, then \( t = t \circ \Phi \) and \( \Phi_*(\partial_t) = \frac{\partial}{\partial t} \);

(iv) \( \Phi_*(g) \) is a solution to the usual Ricci flow equation (1) on \( \Phi(M) \).

That is, \( (M^{n+1}, t, \partial_t, g) \) is isomorphic to a spacetime inside the ambient space \( M \times (0, T) \).

Proof. We first construct \( M \). Choose any increasing sequence \( t_i \not\to T \) in \( (0, T) \), and for \( i \leq j \), consider the embeddings \( M_{t_i} \hookrightarrow M_{t_j}, x \mapsto x(t_j) \). Choose \( M \) to be the direct limit

\[
M = \lim_{i \to \infty} M_{t_i} := \bigcup_{i \in \mathbb{N}} M_{t_i}/\sim,
\]

where \( x \sim y \) iff \( x(t) = y \) for some \( t \in I \), and with canonical maps \( f_i : M_{t_i} \to M, x \mapsto [x] \). It is straight forward to show that \( M \) is connected, and can be equipped with a smooth atlas so that the canonical maps \( f_i : M_{t_i} \hookrightarrow M \) are smooth embeddings C.3. For each \( i \in \mathbb{N} \), we combine the map we get by choosing \( t_i \in (0, T) \) in Lemma 3.1 and the canonical map \( f_i \) to get the well-defined map \( \Phi^i : M_{(0,t_i)} \to M \times (0, t_i) \)

\[
\Phi^i(x) := (f_i \circ x(t_i), t(x)).
\]

Suppose \( i \leq j \) and \( x \in M_{(0,t_i)} \). Since \( x(t_j) = (x(t_i))(t_j) \), we have that

\[
f_j \circ x(t_j) = f_i \circ x(t_i),
\]

and \( \Phi^j \) is an extension of the function \( \Phi^i \). Therefore, we can piece the functions \( \{ \Phi^i : i \in \mathbb{N} \} \) together, giving the well-defined function \( \Phi : M \to M \times (0, T) \). The properties of \( \Phi \) follow from the properties of the embeddings in Lemma 3.1.

Due to Theorem 1.11, we can apply Theorem 3.2 to any complete \((2 + 1)\)-dimensional Ricci flow spacetime. The following corollary shall be used implicitly from now on.

**Corollary 3.3.** Let \( (M^{2+1}, t, \partial_t, g) \) be a complete Ricci flow spacetime with \( I = (0, T) \). Then there exists a connected smooth ambient surface \( M^2 \) such that our spacetime is isomorphic to a complete Ricci flow spacetime \( (M^{2+1}, g) \) in \( M^2 \times (0, T) \). Moreover, \( M \) is chosen as small as possible. That is, up to isomorphism, we can assume that

(i) \( M \) is an open subset of \( M \times (0, T) \) equipped with the product topology;

(ii) \( t \) is the restriction of the standard projection map \( t : M \times (0, T) \to (0, T) \) to \( M \);

(iii) \( \partial_t \) is the restriction of the vector field \( \frac{\partial}{\partial t} \) to \( M \);

(iv) \( g \) solves the Ricci flow equation (1) on \( M \);

(v) \( M = \bigcup_{t \in (0,T)} M_t \) and \( M \) is expanding: \( M_{t_1} \subseteq M_{t_2} \) for every \( 0 < t_1 \leq t_2 < T \).
3.2 Continuity within an ambient space

Now that we have an ambient space, we have the following simplification for the definition of a spacetime being continuous.

**Lemma 3.4.** Let \((\mathcal{M}^{2+1}, g)\) be a complete Ricci flow spacetime in \(M \times (0, T)\). Then the spacetime is continuous (see Definition 1.12) iff

\[
\mathcal{M}_s = \left( \bigcap_{t > s} \mathcal{M}_t \right) ^ \circ \subseteq M, \quad \forall s \in (0, T).
\]

**Proof.** Since \(\mathcal{M}\) is expanding, for \(t \leq s\), we have the continuity criteria for free:

\[
\mathcal{M}_s(t) = \mathcal{M}_t = \left( \mathcal{M}_s(t) \right) ^ \circ,
\]
and so we only need to consider the case \(s < t\). Using again that \(\mathcal{M}\) is expanding, as subsets of \(M\), we see that \(\mathcal{M}_s(t) = \mathcal{M}_s\), and

\[
\mathcal{M}_s(t) = \{ x \in M : (s, t] \subseteq I_x \} = \bigcap_{t > s} \mathcal{M}_t.
\]

Since each spatial slice is open in \(M\), our original definition of continuity reduces to (8). □

The following lemma shows that for complete and continuous spacetimes, isolated punctures cannot be added to the spatial slices (see Example 1.4).

**Lemma 3.5.** Suppose \((\mathcal{M}^{2+1}, g)\) is a complete and continuous Ricci flow spacetime in \(M \times (0, T)\). Consider a point \(x \in M \setminus \mathcal{M}_s\) laying outside of the spatial slice of \(\mathcal{M}\) at some time \(s \in (0, T)\). Suppose \(x\) is an isolated point of \(M \setminus \mathcal{M}_s\). Then \(x\) is never in \(\mathcal{M}\). That is, \((x, t) \notin \mathcal{M}\), for all \(t \in (0, T)\).

**Proof.** Since \(x\) is an isolated point of the complement, there exists an open neighbourhood \(x \in U \subseteq M\) such that the punctured neighbourhood \(U \setminus \{x\}\) is contained in \(\mathcal{M}_s\). As \(\mathcal{M}\) is expanding, \(U \setminus \{x\}\) \(\subseteq \mathcal{M}_t\), for all \(t \in [s, T)\). If the lemma is false, we have a well defined first time \(t_0\) that \(x\) enters our spacetime

\[
t_0 := \inf\{t \in (s, T) : x \in \mathcal{M}_t\} \geq s.
\]

Note that \(U \subseteq \mathcal{M}_t\) for every \(t \in (t_0, T)\), which by continuity implies

\[
U \subseteq \left( \bigcap_{t > t_0} \mathcal{M}_t \right) ^ \circ = \mathcal{M}_{t_0}.
\]

In particular, \(p \in \mathcal{M}_{t_0}\). This contradicts the definition of \(t_0\) as \(\mathcal{M}\) is open in \(M \times (0, T)\). □

**Corollary 3.6.** Suppose \((\mathcal{M}^{2+1}, g)\) is a complete and continuous spacetime in \(M \times (0, T)\). Then for any times \(0 < t_1 < t_2 < T\), \(\mathcal{M}_{t_2} \setminus \mathcal{M}_{t_1}\) contains no isolated points.

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3.3 Lifting a spacetime

Let \((\mathcal{M}^{2+1}, g)\) be a complete Ricci flow spacetime inside the ambient space \(M \times (0, T)\). Suppose \(p : X \to M\) is a covering map from a connected surface \(X\). We define the lifted spacetime \((\mathcal{M}', g')\) inside of \(X \times (0, T)\) via:

\[
\mathcal{M}'_t := p^{-1}(\mathcal{M}_t), \quad g'(t) := p^*(g(t)), \quad \forall t \in (0, T).
\]

(9)

In the following lemma, we first show that what we constructed above is in fact a well-defined Ricci flow spacetime. Moreover, we show that the lifted spacetime inherits continuity and being initially determined from the original spacetime.

**Lemma 3.7.** If \((\mathcal{M}^{2+1}, g)\) is a complete Ricci flow spacetime inside the ambient space \(M \times (0, T)\), and \(p : X \to M\) is a covering map, with \(X\) a connected surface, then \((\mathcal{M}', g')\) as defined in equation (9) is a complete Ricci flow spacetime in \(X \times (0, T)\). Moreover, if \((\mathcal{M}, g)\) is continuous (initially determined), then \((\mathcal{M}', g')\) is continuous (initially determined).

*Proof.* We must first show that \((\mathcal{M}', g')\) is a spacetime in \(X \times (0, T)\). For any \((x, t) \in \mathcal{M}'\), since \(\mathcal{M}\) is open inside \(M \times (0, T)\), there exists \(U \subseteq M\) open and \(\delta > 0\) such that \((p(x), t) \in U \times (t - \delta, t + \delta) \subseteq \mathcal{M}\), and hence \((x, t) \in p^{-1}(U) \times (t - \delta, t + \delta) \subseteq \mathcal{M}'\). Since \(p\) is continuous, this neighbourhood is open in \(X \times (0, T)\), and so \(\mathcal{M}' \subseteq X \times (0, T)\) is open. It is also clear that

\[
X = p^{-1}(\mathcal{M}) = p^{-1}\left( \bigcup_{t \in (0, T)} \mathcal{M}_t \right) = \bigcup_{t \in (0, T)} p^{-1}(\mathcal{M}_t) = \bigcup_{t \in (0, T)} \mathcal{M}'_t,
\]

and \(\mathcal{M}'\) is expanding: \(\mathcal{M}'_{t_1} = p^{-1}(\mathcal{M}_{t_1}) \subseteq p^{-1}(\mathcal{M}_{t_2}) = \mathcal{M}'_{t_2}\) for every \(0 < t_1 \leq t_2 < T\).

For any \((x_0, t_0), (x_1, t_1) \in \mathcal{M}'\), there exists a continuous path \(\gamma : [0, 1] \to X\) such that \(\gamma(0) = x_0 \quad \text{and} \quad \gamma(1) = x_1\). Since \(\gamma([0, 1]) \subseteq X\) is compact and the set \(\{\mathcal{M}'_t : t \in (0, T)\}\) form a nested open cover of \(X\), there exists some \(\tau \in (0, T)\) such that \(t_0, t_1 \leq \tau\), and \(\gamma : [0, 1] \to \mathcal{M}'\). By concatenating \(\gamma\) with the worldlines of \(x_0 \text{ and } x_1\), we see that \(\mathcal{M}'\) is also connected. Directly from the definition of \(g'\), we also see that \((\mathcal{M}', g')\) is complete. This finishes the first part of the statement.

If \(\mathcal{M}\) is continuous, since \(p\) is a local homeomorphism, for any \(s \in (0, T)\) we have

\[
\left( \bigcap_{t > s} \mathcal{M}'_t \right)^\circ = \left( p^{-1}\left( \bigcap_{t > s} \mathcal{M}_t \right) \right)^\circ = p^{-1}\left( \left( \bigcap_{t > s} \mathcal{M}_t \right)^\circ \right) = p^{-1}(\mathcal{M}_s) = \mathcal{M}'_s,
\]

and \(\mathcal{M}'\) is also continuous. Similarly, suppose \(\mathcal{M}\) is initially determined. For any \(x \in \mathcal{M}'\) and \(\epsilon > 0\), there exists a smooth time-preserving path \(\eta : [\epsilon, t(x)] \to \mathcal{M}\), such that \(\eta \circ t(x) = p(x)\). As the restriction \(p : \mathcal{M}' \to \mathcal{M}\) is a covering map, there exists a unique lift of \(\eta\) to a smooth time-preserving path \(\eta' : [\epsilon, t(x)] \to \mathcal{M}'\), such that \(\eta' \circ t(x) = x\). Taking \(\epsilon \searrow 0\), we conclude that \(\eta'(x) = 0\) for any \(x \in \mathcal{M}'\), and \(\mathcal{M}'\) is also initially determined. \(\square\)

Recall that we are aiming to show Theorem 1.16. Suppose \((\mathcal{M}^{2+1}, g)\) is a complete, continuous and initially determined spacetime inside \(M \times (0, T)\). Since \(M\) is connected, it has connected universal cover \(X \to M\). Lemma 3.7 then tells us that the lifted spacetime is also complete, continuous and initially determined. Moreover, if the lifted spacetime is cylindrical, \(\mathcal{M}' = \ldots\)
$X \times (0, T)$, then the original spacetime would also be cylindrical, $\mathcal{M} = M \times (0, T)$, since the covering map is surjective onto $M$. Therefore, to prove Theorem 1.16, it suffices to consider the case when $M$ is simply connected.

3.4 Conformal structures

Let $(\mathcal{M}^{2+1}, g)$ be a complete Ricci flow spacetime inside the ambient space $M \times (0, T)$. From the previous subsection, we may assume that $M$ is simply connected.

Given a conformal structure on $M$, each spatial slice $\mathcal{M}_t$ for $t \in (0, T)$ inherits a conformal structure as a subset of $M$. Our Ricci flow $g(t)$ on $\mathcal{M}$ is then said to be conformal if this inherited conformal structure on $\mathcal{M}_t$ agrees with $g(t)$ for all $t \in (0, T)$. Fix $0 < t_1 < t_2 < T$ and consider the conformal structures on $\mathcal{M}_{t_i}$ determined by the metrics $g(t_i)$, for $i = 1, 2$.

Recall that on an orientable surface, a choice of metric and hence its corresponding conformal class is equivalent to a choice of complex structure on the surface. Since our ambient surface $M$ is simply connected, it is orientable, and hence each spatial slice $\mathcal{M}_t$ is orientable too. As such, the conformal structure on $\mathcal{M}_{t_1}$ defines a complex structure. Let $U$ be a complex coordinate chart on $\mathcal{M}_{t_1}$. Writing our Ricci flow locally as in (4), we see that $U$ is also a complex chart on $\mathcal{M}_{t_2}$, and therefore the complex structure on $\mathcal{M}_{t_1}$ agrees with the complex structure it inherits when viewed as a subspace of $\mathcal{M}_{t_2}$. For each $t \in (0, T)$, consider the complex coordinate charts defined on $\mathcal{M}_t$ by the metric $g(t)$. We have shown that taking the union of all such charts gives a well-defined complex atlas on $M$, and with respect to this conformal structure, $g(t)$ is a conformal Ricci flow.

Lemma 3.8. Let $(\mathcal{M}^{2+1}, g)$ be a complete Ricci flow spacetime in $M^2 \times (0, T)$, with $M = \bigcup_{t \in (0, T)} \mathcal{M}_t$. If $M$ is orientable, then there exists a conformal structure on $M$ such that $g(t)$ is a conformal Ricci flow.

In light of this, we can assume that our ambient space is a Riemann surface whose conformal structure is compatible with the Ricci flow $g(t)$. Combined with the assumption that $M$ is simply connected and the uniformisation theorem, we have reduced Theorem 1.16 to proving the following.

Theorem 3.9. Suppose $(\mathcal{M}^{2+1}, g)$ is a complete, continuous and initially determined spacetime in $M^2 \times (0, T)$ with $M = \bigcup_{t \in (0, T)} \mathcal{M}_t$, and where $M^2$ is either the disk, plane or sphere equipped with their standard conformal structures. Suppose further that $g(t)$ is a conformal Ricci flow on $\mathcal{M}$. Then $\mathcal{M} = M \times (0, T)$.

4 Spacetimes in the disk

In the previous section, we reduced our theorem to the special case that our Ricci flow is conformal on a spacetime lying in either the disk, plane or sphere. We can simply this further to the case that the spacetime lies in the disk.

Definition 4.1 (Hyperbolic surface). Given any (possibly disconnected) Riemann surface $N$, we say that $N$ is hyperbolic if each of its connected components has universal cover the disk $D$, or equivalently, if $N$ admits a smooth conformal complete metric of constant curvature $-1$. 

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Theorem 4.2. Suppose \((\mathcal{M}^{2+1}, g)\) is a complete, continuous and initially determined spacetime in \(D \times (0, T)\) with \(D = \cup_{t \in (0, T)} \mathcal{M}_t\), and where \(D\) is the disk equipped with its hyperbolic conformal structure. Suppose further that \(g\) is a conformal Ricci flow on \(\mathcal{M}\). Then \(\mathcal{M} = D \times (0, T)\).

Proof (Theorem 4.2 \(\implies\) Theorem 3.9). Suppose that the theorem fails and there exists \(0 < t_1 < t_2 < T\) such that \(\mathcal{M}_{t_1} \subsetneq \mathcal{M}_{t_2}\). By our assumption, \(\mathcal{M}\) is either the plane or the sphere. By Corollary 3.6, \(\mathcal{M}_{t_2} \setminus \mathcal{M}_{t_1} \subseteq M \setminus \mathcal{M}_{t_1}\) contains no isolated points, and so \(\mathcal{M}_{t_1}\) must be hyperbolic by Lemma C.4. \(\mathcal{M}_{t_2}\) cannot be hyperbolic, as otherwise we could lift each connected component of the spacetime \(\mathcal{M}_{(0,t_2)}\) into \(D \times (0, t_2)\) and apply Theorem 4.2 to conclude its lift is cylindrical, which would then imply that \(\mathcal{M}_{(0,t_2)}\) is cylindrical, contradicting our assumption that \(\mathcal{M}_{t_1} \subsetneq \mathcal{M}_{t_2}\). Since \(\mathcal{M}_{t_2}\) is a non-hyperbolic subset of \(\mathcal{M}\), we can again apply Lemma C.4 and Corollary 3.6 to deduce that \(\mathcal{M}_{t_2} = \mathcal{M}\). Consider the first time at which the spatial slices are not hyperbolic:

\[ t_0 := \inf\{ t \in [t_1, t_2] : \mathcal{M}_t \text{ is not hyperbolic} \}. \]

Using Corollary 3.6 yet again, we have that \(\mathcal{M}_t = \mathcal{M}\) for all \(t > t_0\), and hence by continuity, \(\mathcal{M}_{t_0} = \mathcal{M}\). Finally, we note that each connected component of \(\mathcal{M}_{(0,t_0)}\) can be lifted into \(D \times (0, t_0)\). Applying the same reasoning as before, \(\mathcal{M}_{(0,t_0)}\) must be cylindrical, and \(\mathcal{M}_t = \mathcal{M}_{t_1}\) for all \(t \in (t_1, t_0)\). From this we deduce the contradiction

\[ \mathcal{M}_{t_2} = \mathcal{M} = \mathcal{M}_{t_0} = \bigcup_{t < t_0} \mathcal{M}_t = \mathcal{M}_{t_1}. \]

4.1 A comparison principle

From now on, we may assume that \((\mathcal{M}^{2+1}, g)\) is a complete, connected, continuous and initially determined spacetime in \(D \times (0, T)\) with \(D = \cup_{t \in (0, T)} \mathcal{M}_t\), and that the Ricci flow is conformal with respect to the standard conformal structure on \(D\). In particular, we can write our metrics \(g(t)\) in the form

\[ g(t) = v(z, t)|dz|^2, \]

where \(|dz|^2\) denotes the standard flat metric on the disk, and \(v : \mathcal{M} \to (0, \infty)\) is some smooth solution to the LFDE (5).

Using the conformal structure on each spatial slice, we see that each slice admits a unique complete hyperbolic metric \(h(t)\) (by Lemma C.4). Writing \(h(t) = H(z, t)|dz|^2\), since \(\mathcal{M}\) is expanding, the Schwarz lemma C.2 gives the inequality \(h(t) \leq h(s)\) for any \(0 < s < t < T\). In particular, the conformal factors \(H(\cdot, t)\) are decreasing in \(t\).

Since the metrics \(g(t)\) are complete on \(\mathcal{M}_t\) and have the lower scalar curvature bound from Corollary 2.1, we can use the Schwarz lemma C.2 again to get

\[ v(z, t) \geq 2t \cdot H(z, t), \quad \forall (z, t) \in \mathcal{M}, \]  

where \(H : \mathcal{M} \to (0, \infty)\) is the conformal factor of the complete hyperbolic metrics \(h(t)\) on \(\mathcal{M}\) defined above.
We want a comparison principle between our solution $v$ to the LFDE, and any other solution $u$ to the LFDE on $M$. In order to do so, we want a lower bound on $v$ that behaves like a proper function. We define the correct notion of proper in the parabolic setting below.

**Definition 4.3** (Parabolically proper). Let $(\mathcal{M}, t, \partial_t, g)$ be a Ricci flow spacetime with $I = (0, T)$. A function $f : \mathcal{M} \to \mathbb{R}$ is parabolically proper if

$$\mathcal{M}_{(t, T-\epsilon)} \cap f^{-1}(I) \subseteq \mathcal{M}, \quad \forall I \subseteq \mathbb{R}, \forall \epsilon > 0.$$ 

The following lemma shows that, for a complete spacetime in the disk, the conformal factors $H$ being a parabolically proper map is an equivalent way to characterise our spacetime being continuous. In particular, due to equation (10), $H$ is an appropriate lower bound to take.

**Lemma 4.4.** Let $(\mathcal{M}^{2+1}, g)$ be a complete Ricci flow spacetime in $D \times (0, T)$, and let $H : \mathcal{M} \to (0, \infty)$ be the conformal factors of the complete hyperbolic metrics $h(t)$ on $\mathcal{M}$. Then $\mathcal{M}$ is continuous iff $H : \mathcal{M} \to (0, \infty)$ is parabolically proper (see Definition 4.3).

*Proof.* If $\mathcal{M}$ is not continuous, there exists $s \in (0, T)$ and a point $p \in (\bigcap_{t>s} \mathcal{M}_s)^c \setminus \mathcal{M}_s$. In particular, with respect to the background metric $|dz|^2$, there exists $r > 0$ such that the ball $B(p, r) \subseteq \mathcal{M}_t$, for every $t > s$. By the Schwarz lemma C.2, $H(p, t) \leq \frac{4}{r}$, for every $t > s$. That is, for any sequence of times $t_n \downarrow s$, and for any $\epsilon > 0$ sufficiently small, we have the sequence of points $(p, t_n) \in \mathcal{M}_{(t, T-\epsilon)}$ such that $(p, t_n) \to (p, s) \in \partial \mathcal{M}$, but $H(p, t_n)$ is uniformly bounded for all $n$. That is, $H$ isn’t parabolically proper.

Conversely, suppose $\mathcal{M}$ is continuous and fix $\epsilon > 0$. Let $(z_n, t_n) \in \mathcal{M}_{(t, T-\epsilon)}$ be any sequence such that $(z_n, t_n) \to (z, s) \in \partial \mathcal{M}$. If we can show that the sequence $H(z_n, t_n)$ diverges then we are done. Passing to a subsequence, we may assume that the sequence of times $t_n$ is monotone. Suppose $t_n \not\nearrow s$. As $\mathcal{M}$ is expanding, $z_n \in \mathcal{M}_s$ with $H(z_n, s) \leq H(z_n, t_n)$ for every $n$. In particular, $z_n \in \mathcal{M}_s$ with $z_n \to z \in \partial \mathcal{M}_s$. As $H(\cdot, s) : \mathcal{M}_s \to (0, \infty)$ is a proper map, $H(z_n, s)$ is unbounded, and hence so is $H(z_n, t_n)$. Instead we assume that $t_n \nearrow s$. For any fixed $m \in \mathbb{N}$, using that $\mathcal{M}$ is expanding, we have that $z_n \in \mathcal{M}_{t_m}$ with $H(z_n, t_m) \leq H(z_n, t_n)$ for any $n \geq m$. So, for each $m \in \mathbb{N}$, we have a sequence $z_n \in \mathcal{M}_{t_m}$ with $z_n \to z$. If $z \in \partial \mathcal{M}_{t_m}$, then using again that $H(\cdot, t_m) : \mathcal{M}_{t_m} \to (0, \infty)$ is a proper map, we would deduce that $H(z_n, t_m)$ diverges, so $H(z_n, t_n)$ diverges. This leads us to assume that $z \in \mathcal{M}_{t_m}$ for every $m \in \mathbb{N}$. By the continuity of each $H(\cdot, t_m) : \mathcal{M}_{t_m} \to (0, \infty)$, we have

$$H(z, t_m) = \lim_{n \to \infty} H(z_n, t_m) \leq \limsup_{n \to \infty} H(z_n, t_n),$$

and hence

$$\limsup_{n \to \infty} H(z_n, t_n) \leq \limsup_{n \to \infty} H(z_n, t_n).$$

From the above equation, it suffices to show that $H(z_n, t_n) \not\to \infty$ as $n \to \infty$. Since $\mathcal{M}$ is continuous, $z \in (\bigcap_{t>s} \mathcal{M}_t) \setminus \mathcal{M}_s = (\bigcap_{t>s} \mathcal{M}_t) \setminus (\bigcap_{t>s} \mathcal{M}_t)^c$. So there exists $\epsilon_n \searrow 0$ such that the ball of radius $\epsilon_n$ centred at $z$ is contained in our spatial slice, $B(z, \epsilon_n) \subseteq \mathcal{M}_t$, but some point in the boundary of the ball $b_n \in \partial B(z, \epsilon_n)$ isn’t in our spatial slice $b_n \notin \mathcal{M}_n$. Note that all of the spatial slices are contained within the disk and hence at each time $t_n$, each spatial slice is contained within the punctured ball of radius 2 centred at $b_n$

$$\mathcal{M}_{t_n} \subseteq B^\times(b_n, 2), \quad \forall n \in \mathbb{N}.$$
Consider the complete hyperbolic metric on this punctured ball \( B^\times(b_n, 2) \)

\[
h_n^\times(z) := \frac{1}{|z - b_n|^2(- \log(2|z - b_n|))^2} |dz|^2.
\]

By the Schwarz lemma C.2, \( h(t_n) \geq h_n^\times \) on \( \mathcal{M}_{t_n} \). In particular, we have that

\[
H(z, t_n) \geq \frac{1}{\epsilon_n^2(- \log(2\epsilon_n))^2}, \quad \forall n \in \mathbb{N},
\]

and thus \( H(z, t_n) \to \infty \) as \( n \to \infty \), concluding the proof.

Using Lemma 4.4, we now have a comparison principle for solutions to the LFDE on spacetimes in the disk.

**Lemma 4.5 (Comparison principle).** Let \( (\mathcal{M}^{2+1}, g) \) be a complete, continuous and initially determined spacetime in \( D \times (0, T) \). Let \( u \in C^{2,1} (\mathcal{M}) \) be a bounded solution to the LFDE with \( u > 0 \), and \( v \in C^{2,1} (\mathcal{M}) \) be the solution to the LFDE such that \( g(t) = v(z, t) \cdot |dz|^2 \). If for some \( s \in (0, T) \) we have \( v > u \) on \( \mathcal{M}(0, s) \), then \( v \geq u \) on \( \mathcal{M} \).

**Proof.** Let \( C := \sup_{\mathcal{M}} u < \infty \) and \( \epsilon \in (0, s) \). Consider the region

\[
\mathcal{M}': = \{(x, t) \in \mathcal{M} : t \in (\epsilon, T - \epsilon), H(x, t) < \frac{C + 1}{2\epsilon}\}.
\]

Since \( \mathcal{M} \) is continuous, by Lemma 4.4 \( H \) is parabolically proper, and hence \( \mathcal{M}' \subset \mathcal{M} \). We note that \( u, v \in C^{2,1} (\mathcal{M}') \). On the region \( \mathcal{M}(\epsilon, T-\epsilon) \setminus \mathcal{M}' \), by the Schwarz lemma C.2, we have

\[
v(z, t) \geq 2t \cdot H(z, t) \geq 2\epsilon \cdot H(z, t) \geq C + 1 > u(z, t).
\]

Since \( \epsilon \in (0, s) \), we conclude that \( v > u \) on the parabolic boundary \( \partial_P \mathcal{M}' \). Applying the usual comparison principle for the LFDE to \( u \) and \( v \) on the region \( \mathcal{M}' \) (Lemma B.6), we deduce that \( v \geq u \) on \( \mathcal{M}(0, T-\epsilon) \). To finish the proof take \( \epsilon \searrow 0 \).

### 5 Initial time blow-ups

We now aim to find suitable lower barriers for our Ricci flow. In order to do so, we must make a small diversion into Ricci flows on Riemann surfaces starting from measures.

Let \( (M, g(t))_{t \in (0, T)} \) be a Riemann surface equipped with a complete and conformal Ricci flow. Since the flow is complete, we have the lower scalar curvature bound \( R_{g(t)} \geq -\frac{1}{t} \), and hence \( t \mapsto g(t) \) is decreasing. It is then immediate that we have a well-defined (potentially infinite) limiting metric as time approaches zero.

**Definition 5.1 (Initial time blow-up).** Let \( (M, g(t))_{t \in (0, T)} \) be a Riemann surface equipped with a complete and conformal Ricci flow. The initial time blow-up of \( g(t) \) is the (possibly infinite) conformal metric \( \hat{g} \) on \( M \), defined by

\[
\hat{g} := \lim_{t \searrow 0} \frac{g(t)}{2t}.
\]
Remark. Suppose $\phi : N \to M$ is a local biholomorphism. Then if $g(t)$ is a complete and conformal Ricci flow on $M$ with initial time blow-up $\widehat{g}$, we have that $\phi^*(g(t))$ is a complete and conformal Ricci flow on $N$, and its initial time blow-up is $(\phi^* \widehat{g}) = \phi^*(\widehat{g})$.

Given a connected Riemann surface $M$, denote the collection of all non-atomic, non-zero Radon measures on $M$ by $\mathcal{R}(M)$. Due to the work of Topping & Yin in [TY21], these are precisely those measures which can be smoothed out using Ricci flow. To be more precise, given any $\mu \in \mathcal{R}(M)$, there exists a complete and conformal Ricci flow $g(t)$ starting weakly from $\mu$ (see Theorem B.2 for the full statement). The following lemma shows that, away from the support of the initial measure, the initial time blow-up is a hyperbolic metric.

**Lemma 5.2.** Let $g(t)$ be a complete and conformal Ricci flow on $M \times (0, T)$ starting weakly from some $\mu \in \mathcal{R}(M)$. Then the initial time blow-up $\widehat{g}$ of $g(t)$ is a well-defined smooth metric on $M \setminus \text{supp}(\mu)$ with constant Gaussian curvature $K_{\widehat{g}} = -1$.

**Proof.** Consider the parabolically rescaled Ricci flows $(M, g_m(t)_{t \in (0, mT)})$ given by

$$g_m(t) := mg(tm^{-1}), \quad \forall m \in \mathbb{N},$$

so that $g_1(t) \equiv g(t)$. We note that $g_m(t)$ is a complete and conformal Ricci flow starting weakly from the measure $m \cdot \mu \in \mathcal{R}(M)$. Given a local complex coordinate $z$ on $M$, our metrics are given locally by $g_m(t) = u_m(z, t)|dz|^2$ with the conformal factors $u_m$ satisfying the relation

$$u_m(z, t) := m u_1(z, tm^{-1}).$$

By the monotonicity of $t \mapsto t^{-1} u_1(\cdot, t)$, we have that $u_m(z, t)$ is an increasing sequence in $m$, for any fixed $(z, t)$. In particular, we have the uniform lower bound $u_1(z, t)$ on our sequence of conformal factors. Moreover, we have local uniform upper bounds on our sequence $u_m$ in $M \setminus \text{supp}(\mu)$ by Lemma B.4. Combining the lower and upper bounds with standard parabolic theory, we deduce local $C^k$-bounds on our sequence of conformal factors, for any $k \in \mathbb{N}$. Finally, by Arzela-Ascoli, our sequence of Ricci flows $g_m(t)$ converges locally smoothly to some limiting eternal Ricci flow $(M \setminus \text{supp} \mu, g_\infty(t)_{t \in (0, \infty)})$. For any fixed $t > 0$, we note that

$$g_\infty(t) := \lim_{m \to \infty} g_m(t) = \lim_{m \to \infty} 2t \cdot \frac{g(tm^{-1})}{2tm^2} = 2t \cdot \widehat{g},$$

and so $g_\infty(t) = 2t \cdot \widehat{g}$. Substituting this into the Ricci flow equation

$$2 \cdot \widehat{g} = \frac{\partial g_\infty}{\partial t} = -2 \text{Ric}(g_\infty) = -2K_{\widehat{g}} \cdot \widehat{g},$$

and hence $K_{\widehat{g}} \equiv -1$. \qed

**Proof of Theorem 4.2**

We now have all of the necessary ingredients to prove Theorem 4.2, and hence Theorem 1.16.

**Theorem 4.2.** Suppose $(M^{2+1}, g)$ is a complete, continuous and initially determined spacetime in $D \times (0, T)$ with $D = \cup_{t \in (0, T)} M_t$, and where $D$ is the disk equipped with its hyperbolic conformal structure. Suppose further that $g$ is a conformal Ricci flow on $M$. Then $\mathcal{M} = D \times (0, T)$. 

Proof. Fix \( t_1 \in (0, T) \). By the Cantor-Bendixson theorem [Kec95], we can partition the set \( \overline{D} \setminus \mathcal{M}_{t_1} = P \sqcup X \) into a perfect set \( P \) and a scattered set \( X \). In particular we have the inclusions:
\[
\mathcal{M}_{t_1} \subseteq \mathcal{M}_{t_2} \subseteq \mathcal{M}_{t_1} \sqcup P \sqcup X, \quad \forall t_2 \in (t_1, T).
\]
(11)
Due to a result of Hebert & Lacey [HL68], as \( P \) is a compact perfect subset of the plane, there exists a Radon measure \( \mu \in \mathcal{R}(\mathbb{C}) \) such that \( \text{supp}(\mu) = P \) (see Lemma C.1). Let \( G(t) = u(z,t) \cdot |dz|^2 \) be a complete conformal Ricci flow on \( \mathbb{C} \times (0, T) \) starting weakly from \( \mu \in \mathcal{R}(\mathbb{C}) \) with initial time blow-up \( \tilde{G} \). For any \( \lambda > 0 \), we note that the parabolically rescaled Ricci flow \( G_\lambda(t) := \lambda G(\frac{t}{\lambda}) = \lambda u(z,\frac{t}{\lambda})|dz|^2 \) is a complete conformal Ricci flow on \( \mathbb{C} \times (0, \lambda T) \) starting weakly from \( \lambda \mu \in \mathcal{R}(\mathbb{C}) \), with the same initial time blow-up \( \tilde{G} \).
Fix \( t \in (0, t_1) \). By the monotonicity of \( t \mapsto G_\lambda(t)/t \), \( G_\lambda(t) \leq 2t \cdot \tilde{G} \) within \( \mathbb{C} \setminus P \). Since \( \mathcal{M} \) is expanding, \( \mathcal{M}_t \subseteq \mathcal{M}_{t_1} \subseteq \mathcal{C} \setminus P \), and \( \tilde{G} \) is defined on all of \( \mathcal{M}_t \). Applying Lemma 5.2 and the Schwarz Lemma C.2, we deduce that \( \tilde{G} \leq h(t) \) on \( \mathcal{M}_t \). Note that, if we had equality between the metrics \( G \) and \( h(t) \) at any point inside of \( \mathcal{M}_t \), then this would imply that the subharmonic function \( \log(h(t)/\tilde{G}) \) has a minimum in its interior, and hence by the elliptic maximum principle, \( \tilde{G} \equiv h(t) \) in \( \mathcal{M}_t \). This would contradict the fact that \( \tilde{G} \) is defined on a larger domain. Therefore, we have the strict inequality \( \tilde{G} < h(t) \) on \( \mathcal{M}_t \). Finally, we can use equation (10) together with the previous inequalities to deduce that
\[
\frac{G_\lambda(t)}{2t} \leq \tilde{G} < h(t) \leq \frac{g(t)}{2t}, \quad \forall t \in (0, t_1).
\]
Therefore, \( g > G_\lambda \) on \( \mathcal{M}_{(0,t_1)} \). As \( G_\lambda \) is bounded on \( \mathcal{M} \), we can then apply Lemma 4.5 to conclude that \( g \geq G_\lambda \) on all of \( \mathcal{M} \), for any \( \lambda > 0 \). Suppose at some later time \( t_2 \in (t_1, T) \) we have that \( \mathcal{M}_{t_2} \cap P \neq \emptyset \). That is, there exists some point \( p \in P \) such that \( (p, t_2) \in \mathcal{M} \). Since \( \mathcal{M} \) is open, there exists some small \( r > 0 \) such that the conformal ball \( B(p,r) \in \mathcal{M}_{t_2} \). By compactness, \( \text{Vol}_{g(t_2)}(B(p,r)) < \infty \). We now derive a contradiction by showing that the volume of this ball with respect to the metric \( G_\lambda(t_2) \) blows up to infinity as \( \lambda \to \infty \). Indeed, \( \mu(B(p,r)) > 0 \) by the definition of \( p \in P = \text{supp}(\mu) \). Using that the volume of a ball can’t decrease too rapidly in a Ricci flow (see Lemma B.5), we have that
\[
\text{Vol}_{G(t)}(B(p,r)) \geq \mu(B(p,r)) - Ct, \quad \forall t \in \left(0, T \wedge \frac{\mu(B(p,r))}{4\pi}\right),
\]
which means for \( \lambda \) sufficiently large
\[
\text{Vol}_{G_\lambda(t_2)}(B(p,r)) = \lambda \cdot \text{Vol}_{G_\lambda(t_2)}(B(p,r)) \geq \lambda \cdot \left( \mu(B(p,r)) - \frac{Ct_2}{\lambda} \right) \geq \frac{\lambda}{2} \cdot \mu(B(p,r)).
\]
Taking \( \lambda \searrow \infty \), this contradicts the volume of the ball with respect to \( g(t_2) \) being finite. Therefore, we have shown that \( \mathcal{M}_{t_2} \cap P = \emptyset \), which means that (11) reduces to
\[
\mathcal{M}_{t_1} \subseteq \mathcal{M}_{t_2} \subseteq \mathcal{M}_{t_1} \sqcup X, \quad \forall t_2 \in (t_1, T).
\]
Since \( \partial D \subseteq \mathbb{C} \) is a perfect subset of the plane, we see that \( X \subseteq D \). If \( X \) was non-empty, then because it is a scattered set, it must contain an isolated point, which would contradict Corollary 3.6. We can therefore conclude that \( X \) must be empty, and that \( \mathcal{M}_{t_1} = \mathcal{M}_{t_2} \). \( \square \)
A Topology of spacetimes

Let \((\mathcal{M}, t, \partial_t, g)\) be a Ricci flow spacetime. Given any point \(x \in \mathcal{M}\), we define the set \(U_x^-\) to be the collection of all points at earlier times which can be connected to \(x\) by a time-preserving path:

\[
U_x^- := \{ y \in \mathcal{M} : t(y) < t(x), \exists \eta : [t(y), t(x)] \to \mathcal{M} \text{ time-preserving, with } \eta \circ t(y) = y, \eta \circ t(x) = x \}.
\]

Similarly, we define \(U_x^+\) to be the points at later times connected to \(x\) by time preserving paths:

\[
U_x^+ := \{ y \in \mathcal{M} : t(y) < t(x), \exists \eta : [t(y), t(x)] \to \mathcal{M} \text{ time-preserving, with } \eta \circ t(x) = x, \eta \circ t(y) = y \}.
\]

Since every point in our spacetime admits a small unscathed parabolic cylinder around it, we deduce that the sets \(U_x^\pm\) are both non-empty and open in \(\mathcal{M}\).

Fixing \(x \in \mathcal{M}\), we can now union open sets of the form defined above in an iterative way to see which points in \(\mathcal{M}\) can be joined to \(x\) by a string of time-preserving paths. Define \(U_1^+ := U_x^+\) and \(U_1^- := \bigcup_{y \in U_1^+} U_y^-\). For each \(m \in \mathbb{N}\), we recursively set

\[
U_m^+ := \bigcup_{y \in U_{m-1}^+} U_y^+, \quad U_m^- := \bigcup_{y \in U_{m}^+} U_y^-.
\]

Finally, let \(U^- := \bigcup_{m \in \mathbb{N}} U_m^- \subseteq \mathcal{M}\). The following Lemma shows that spacetimes are path-connected by strings of time-preserving paths.

**Lemma A.1.** Let \((\mathcal{M}, t, \partial_t, g)\) be a Ricci flow spacetime. Fix \(x \in \mathcal{M}\) and let \(U^-\) be the open subset of \(\mathcal{M}\) defined above. Then \(\mathcal{M} = U^-\).

**Proof.** Since \(U^-\) is open and non-empty, it suffices to show that \(U^-\) is closed in \(\mathcal{M}\). As such, consider a sequence \(x_n \in U^-\) such that \(x_n \to x_\infty \in \mathcal{M}\). For \(r > 0\) sufficiently small, the parabolic cylinder \(C(x_\infty, r)\) is unscathed in \(\mathcal{M}\). For \(n\) sufficiently large, \(x_n\) lies inside this parabolic ball. Then, for this large value of \(n\) we have that either

\[
t(x_n) > t(x_\infty) \implies x_\infty \in U_{x_n}^-,
\]

\[
t(x_n) \leq t(x_\infty) \implies \exists y \in U_{x_n}^+ \text{ with } x_\infty \in U_y^-.
\]

So, as \(x_n \in U^-\), there is some \(m \in \mathbb{N}\) such that \(x_n \in U_m^-\), and hence by the above, \(x_\infty \in U_{m+1}^- \subseteq U^-\) as required. \(\square\)

The following corollary is a useful consequence of the above Lemma that we used in Section 2.

**Corollary A.2.** Let \((\mathcal{M}^{n+1}, t, \partial_t, g)\) be a Ricci flow spacetime. Suppose \(x, x' \in \mathcal{M}\). Then there exists \(m \in \mathbb{N}\), a collection of points \(x_0, \ldots, x_m, y_1, \ldots, y_m \in \mathcal{M}\), and a collection of time-dependent paths \(\eta_i : [t(x_{i-1}), t(y_i)] \to \mathcal{M}\), \(\gamma_i : [t(x_i), t(y_i)] \to \mathcal{M}\) for \(i = 1, \ldots, m\), such that

- \(x_0 = x, \text{ and } x_m = x'\).
- \(\eta_i \circ t(x_{i-1}) = x_{i-1} \text{ and } \eta_i \circ t(y_i) = y_i, \text{ for each } i \in \{1, \ldots, m\}\).
- \(\gamma_i \circ t(x_i) = x_i \text{ and } \gamma_i \circ t(y_i) = y_i, \text{ for each } i \in \{1, \ldots, m\}\).
B Ricci flow

B.1 Local lower bounds on scalar curvature

The following is the proof of the lower scalar curvature bound used in Section 2. It is a result of Chen’s [Che09, Proposition 2.1] adapted to Ricci flow spacetimes. The proof presented here is a modification of the original proof, where now the basepoint of the balls is allowed to vary smoothly in time.

Lemma B.1 (Scalar curvature lower bound). ∀δ ∈ (0, \frac{2}{n}), ∃C(δ, n) > 0 with the following property. Let (M^{n+1}, t, \partial_t, g) be a Ricci flow spacetime. Fix [t_1, t_2] ⊆ I and η : [t_1, t_2] → M a time-preserving path. Let π : TM → T^{\ast}M denote the spatial projection, and suppose there are constants r_0, K > 0 and A ≥ 2 + 24(n − 1)r_0^{-2}t_2 such that

- B_{g(t)}(η(t), Ar_0) ⊆ M_t, for every t ∈ [t_1, t_2];
- Ric(g(t)) ≤ (n − 1)r_0^{-2} on B_{g(t)}(η(t), r_0), for every t ∈ [t_1, t_2];
- R_{g(t)}(t_1) ≥ −K on B_{g(t_1)}(η(t_1), Ar_0);
- |π ◦ η'(t)|_{g(t)} ≤ r_0^{-1}, for every t ∈ [t_1, t_2].

Then for each t ∈ [t_1, t_2] and x ∈ B_{g(t)}(η(t), \frac{3Ar_0}{4}), we have

\[ R_{g(t)}(x) ≥ \min\{-\frac{1}{(\frac{2}{n} - \delta)(t - t_1) + \frac{1}{K}}, -\frac{C}{A^2 r_0^2}\}. \]

Proof of B.1. Fix a smooth decreasing function φ : R → R with φ ≡ 1 on (−∞, \frac{7}{3}] and φ ≡ 0 on [1, ∞). For points x, y ∈ M in the same time slice s = t(x) = t(y), let d_g(x, y) := d_{g(s)}(x, y) denote the distance between x and y in (M_s, g(s)). Define u : M_{[t_1, t_2]} → R by

u(x) := φ\left(\frac{d_g(x, η ◦ t(x)) + 3(n − 1)r_0^{-1}t(x)}{Ar_0}\right) \cdot R_{g(t(x))}(x).

In the region where d_g is smooth, a direct calculation yields

\[ \Box u = \frac{Rd_{g}''}{Ar_0} \cdot (\Box d_{g} + \pi ◦ η' \cdot \nabla d_{g} + 3(n − 1)r_0^{-1} \nabla R − \frac{Rd_{g}''}{A^2 r_0^2} + 2φ|\text{Ric}|^2), \tag{12} \]

where \[ \Box := ∂_t − Δ_g \] denotes the heat operator on M, and \nabla d_g denotes the gradient of the function M_{t(x)} → R, y → d_g(x, y), evaluated at the point y = η ◦ t(x).

Fix x ∈ M_{(t_1, t_2)}. If the distance d_g(x, η ◦ t(x)) ≤ r_0, then \[ \frac{d_g(x, η ◦ t(x)) + 3(n − 1)r_0^{-1}t(x)}{Ar_0} \leq \frac{5}{8}, \] and hence \[ φ' \left(\frac{d_g(x, η ◦ t(x)) + 3(n − 1)r_0^{-1}t(x)}{Ar_0}\right) \cdot R_{g(t(x))}(x) = 0. \] Otherwise, the distance d_g(x, η ◦ t(x)) > r_0, and we have the lower bound (in the barrier sense) from [Per02, Lemma 8.3]

\[ \Box d_g(x, η ◦ t(x)) + \pi ◦ η'(t(x)) \cdot \nabla d_g(x, η ◦ t(x)) ≥ -\frac{8(n − 1)}{3r_0}. \]

In particular, on all of M_{(t_1, t_2)} we have the lower bound (in the barrier sense)

\[ \Box d_g + \pi ◦ η' \cdot \nabla d_g + 3(n − 1)r_0^{-1} ≥ 0. \tag{13} \]
For each $s \in [t_1, t_2]$ define

$$u_0(s) := \inf_{x \in M_s} u(x).$$

Assume for some fixed $t_0 \in (t_1, t_2)$ that $u_0(t_0) < 0$. Then, since $u$ is continuous on the compact set $B_{\eta(t_0)}(\eta(t_0), Ar_0)$ and vanishes outside of it, the minimum is attained at some point $x_0 \in M_t$. By continuity, for any point $y$ in a sufficiently small neighbourhood of $x_0$, $R_{\eta(t_0)}(y) < 0$, and so $\phi' \left( \frac{d_\eta(y, \eta(t_0)) + 3(n-1)r_0^{-1}}{Ar_0} \right) \cdot R_{\eta(t_0)}(y) \geq 0$. Combining this with (13), we have

$$R\phi' \cdot (\Box d_\eta + \pi \circ \eta' \cdot \nabla d_\eta + 3(n-1)r_0^{-1}) \geq 0,$$

in a neighbourhood $x_0$. Therefore, in a spacetime neighbourhood of $x_0$, equation (12) implies

$$\Box u \geq -2\phi \cdot \nabla R - \frac{R\phi''}{A^2 r_0^2} + \frac{2n}{n} \phi R^2,$$

in the barrier sense. Moreover, using that $\phi > 0$ near $x_0$, within a possibly smaller neighbourhood of $x_0$, at smooth points of $d_\eta$, we have

$$\nabla R = \frac{\nabla u}{\phi} - \frac{R}{\phi} \nabla \phi, \quad |\nabla \phi|^2 = \frac{(\phi')^2}{(Ar_0)^2}.$$

Therefore, on this small neighbourhood of $x_0$, the differential inequality

$$\Box u \geq -2\phi \cdot \nabla \phi \cdot \nabla u + \frac{1}{(Ar_0)^2} \left( 2\phi' \right) \phi R^2 + \frac{2n}{n} \phi R^2,$$

(15)

holds in the barrier sense. With our choice of $\phi$, we can ensure that we have the bound $\frac{2n}{n} \phi' \phi^2 \leq C' \phi^2$, for some $C' > 0$ depending only on $\phi$. Using Peter-Paul, we also have the inequality

$$\frac{C' R_{\phi}^2}{(Ar_0)^2} \leq \frac{2}{\delta} \left( \frac{C'}{(Ar_0)^2} \right)^2 + \frac{\delta}{2} \phi R^2 = \frac{\delta}{2} \left( \left( \frac{C}{(Ar_0)^2} \right)^2 + \phi R^2 \right),$$

where $C$ now depends on $\delta$. Therefore, we can simplify equation (15) to the differential inequality

$$\Box u \geq -\frac{2\phi}{\phi} \nabla \phi \cdot \nabla u + \left( \frac{2}{n} - \frac{\delta}{2} \right) \phi R^2 - \frac{\delta}{2} \left( \frac{C}{(Ar_0)^2} \right)^2.$$  

Applying the maximum principle, we conclude that

$$\liminf_{h \searrow 0} \frac{u_0(s + h) - u_0(s)}{h} \geq \left( \frac{2}{n} - \frac{\delta}{2} \right) \phi R^2 - \frac{\delta}{2} \left( \frac{C}{(Ar_0)^2} \right)^2$$

$$\geq \left( \frac{2}{n} - \delta \right) u_0^2(s) + \frac{\delta}{2} \left( u_0(s)^2 - \frac{C}{(Ar_0)^2} \right)^2.$$  

Integrating this up, we have the desired inequality. \qed
B.2 2D Ricci flow

In Section 5 we stated an existence theorem for Ricci flows starting from non-atomic Radon measures. The following is [TY21, Theorem 1.2].

**Theorem B.2** (Existence of Ricci flows smoothing out measures [TY21]). Suppose $M$ is a connected Riemann surface and $\mu \in R(M)$ is any non-zero, non-atomic Radon measure on $M$. Define

$$T := \begin{cases} \frac{\mu(M)}{4\pi} : & \text{if } M = \mathbb{C} \\ \frac{\mu(M)}{8\pi} : & \text{if } M = S^2 \\ \infty : & \text{otherwise} \end{cases}$$

(17)

Then there exists a smooth complete conformal Ricci flow $g(t)$ on $M \times (0, T)$ starting weakly from $\mu$. That is, the Riemannian volume measure $\mu_{g(t)} \rightharpoonup \mu$ as $t \downarrow 0$.

The following is the $L^1 - L^\infty$ smoothing result for solutions to Ricci flow [TY21, Theorem 2.1] generalised slightly for weak initial data.

**Theorem B.3** ($L^1 - L^\infty$ smoothing). Suppose $g(t) = u(z,t)|dz|^2$ is a smooth conformal Ricci flow on the ball $B_{3r} \times (0, T)$, for some $r > 0$. Suppose there exists a non-atomic, non-zero Radon measure $\mu \in R(B_{3r})$ such that $g(t)$ starts weakly from $\mu$. If $t \in (0, T)$ satisfies $t > \frac{\mu(B_{2r})}{2\pi}$ then

$$\sup_{B_r} u(t) \leq C_0 r^{-2} t,$$

where $C_0 < \infty$ is universal.

**Proof.** Since $\mu$ is Radon, for some $\delta > 0$, $\frac{\mu(B_{2r} + \delta)}{2\pi} < t$. Choose a cut off function $f$ such that $f \equiv 1$ on $B_{2r}$ and $\text{supp}(f) \subseteq B_{2r+\delta}$. For $\epsilon > 0$ sufficiently small

$$\frac{\text{Vol}_{g(\epsilon)}(B_{2r})}{2\pi} \leq \frac{\int f \, d\mu_{g(\epsilon)}}{2\pi} \leq \frac{\int f \, d\mu}{2\pi} + \left( t - \frac{\mu(B_{2r+\delta})}{2\pi} \right) \leq t.$$

Therefore applying the estimate in the smooth case [TY21, Theorem 2.1] to the Ricci flow on $B_{2r} \times (\epsilon, T)$ gives

$$\sup_{B_r} u(\epsilon + t) \leq C_0 r^{-2} t.$$

Since $u$ is smooth away from zero, taking $\epsilon$ to zero gives the result. \qed

During the proof of Lemma 5.2 in Section 5, we used a uniform upper bound on our conformal factors away from the support of our initial measure. The following Lemma is a direct consequence of the $L^1 - L^\infty$ smoothing result.

**Lemma B.4.** Let $(M, g(t))_{t \in (0, T)}$ be a Riemann surface equipped with a smooth complete conformal Ricci flow. Suppose there exists a non-atomic, non-zero Radon measure $\mu \in R(M)$ such that $g(t)$ starts weakly from $\mu$. Then the initial time blow-up $\hat{g}$ of $g(t)$ is a well-defined smooth metric on $M \setminus \text{supp}(\mu)$. 

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Proof. For any point \( p \in M \setminus \text{supp}(\mu) \), choose a complex coordinate \( z \) on a neighbourhood of this point and choose \( r > 0 \) sufficiently small such that \( B_{3r} \subseteq M \setminus \text{supp}(\mu) \), where \( B_{3r} \) denotes the ball centred at \( p \) radius \( 3r \) with respect to the metric \( |dz|^2 \). Write \( g(t) = u(t)|dz|^2 \) on \( B_{3r} \). Since \( \mu(B_{2r}) = 0 \), the \( L^1 - L^\infty \) smoothing result \( B.3 \) will apply at all positive times

\[
\sup_{B_r} u(t) \leq C_0 r^{-2} t, \quad \forall t \in (0, T).
\]

Hence

\[
\hat{u}(p) = \lim_{t \to 0} \frac{u(p, t)}{2t} \leq C_0 r^{-2} < \infty.
\]

The following is another result from Topping & Yin [TY21, Lemma 3.1], which stops complete Ricci flows from losing volume too quickly within a conformal ball.

**Lemma B.5** (Control on volume decay [TY21]). Let \( g(t) \) be a smooth instantaneously complete conformal Ricci flow on \( \mathbb{C} \times (0, T) \) starting weakly from \( \mu \in \mathcal{R}(\mathbb{C}) \). Fix \( 0 < r < R < \infty \) and suppose \( B_r \subseteq B_R \) are concentric balls in the plane. Then there exists an explicit constant \( C := 4\pi \cdot \frac{1 + (r/R)^2}{1 - (r/R)^2} \), such that

\[
\mu(B_r) \leq \text{Vol}_{g(t)}(B_R) + Ct, \quad \forall t \in \left(0, T \wedge \frac{\mu(B_r)}{4\pi}\right).
\]

Proof. We first deal with the case \( g(t) \) is smooth up to time zero. For each \( n \in \mathbb{N} \) choose \( r_n \in (r, R) \) such that \( r_n \wedge r \) and define a new smooth metric \( g_n \) on \( \mathbb{C} \) such that

- \( g_n \equiv g(0) \) on \( B_r \),
- \( g_n \leq g(0) \) on \( \mathbb{C} \),
- \( g_n \leq \frac{1}{n} H_{r_n} \) on \( \mathbb{C} \setminus B_{r_n} \),

where \( H_r \) denotes the complete hyperbolic metric on \( \mathbb{C} \setminus B_r \). Let \( g_n(t) \) be the instantaneously complete Ricci flow on \( \mathbb{C} \) starting from \( g_n \). Since \( \text{Vol}_{g_n}(\mathbb{C}) \geq \text{Vol}_{g(0)}(B_r) =: v_0 \), by the existence theorem for Ricci flows B.2, we have that each of the \( g_n(t) \) exist for \( t \in (0, \frac{v_0}{4\pi}) \). Using [GT11, Corollary 3.3], we can deduce that \( g_n(t) \leq g(t) \) on \( \mathbb{C} \times (0, T \wedge \frac{v_0}{4\pi}) \). In particular, we have that for each \( t \in (0, T \wedge \frac{v_0}{4\pi}) \),

\[
\text{Vol}_{g(t)}(B_R) \geq \text{Vol}_{g_n(t)}(B_R) = \text{Vol}_{g_n(t)}(\mathbb{C}) - \text{Vol}_{g_n(t)}(\mathbb{C} \setminus B_R)
\geq v_0 - 4\pi t - \left(\frac{1}{n} + 2t\right)\text{Vol}_{H_{r_n}}(\mathbb{C} \setminus B_R).
\]

By a direct calculation

\[
\text{Vol}_{H_r}(\mathbb{C} \setminus B_R) = \frac{4\pi r^2}{R^2 - r^2},
\]

and so taking \( n \to \infty \) we have that, for each \( t \in (0, T \wedge \frac{v_0}{4\pi}) \)

\[
\text{Vol}_{g(t)}(B_R) \geq \text{Vol}_{g(0)}(B_R) - Ct. \tag{18}
\]
This deals with the case \( g \) is initially smooth. For the general case of weak initial data, fix \( \epsilon > 0 \). Since \( \mu \) is Radon, there exists \( K \in B_r \) such that \( \mu(K) \geq \mu(B_r) - \epsilon \). Choosing a test function \( f \) with support in \( B_r \) and equal to 1 on \( K \), we have for \( \delta \) sufficiently small

\[
\text{Vol}_{g(\delta)}(B_r) \geq \int f \, d\mu_{g(\delta)} \geq \int f \, d\mu - \epsilon \geq \mu(K) - \epsilon \geq \mu(B_r) - 2\epsilon.
\]  

(19)

For any fixed \( t \in (0, T \wedge \frac{\mu(B_r)}{4\pi}) \), we can choose \( \delta \in (0, T) \) sufficiently small such that equation (19) holds as well as \( t \in (\delta, T \wedge \frac{\text{Vol}_{g(\delta)}(B_r)}{4\pi}) \). Applying (18) to \( g(t) \) on \( \mathbb{C} \times [\delta, T) \) gives

\[
\mu(B_r) \leq \text{Vol}_{g(\delta)}(B_r) + 2\epsilon \leq \text{Vol}_{g(t)}(B_R) + Ct + 2\epsilon.
\]

Taking \( \epsilon \searrow 0 \) finishes the proof. \( \square \)

Recall that the locally defined conformal factors of a 2-dimensional Ricci flow solve the LFDE (5). In Section 4 we proved a stronger comparison principle for spacetimes by using the usual comparison principle for the LFDE stated below.

**Lemma B.6 (Direct comparison principle).** Let \( \mathcal{M} \) be a compactly contained, open subset of \( D \times (0, T) \), and let \( u,v \in C^{2,1}(\overline{\mathcal{M}}) \) be solutions to the LFDE (5) with \( u,v > 0 \). If \( v > u \) on the parabolic boundary \( \partial_P \mathcal{M} \), then \( v \geq u \) on \( \mathcal{M} \).

**Proof.** We modify the argument used by Giesen in [Gie12].

By compactness, we can choose \( \delta > 0 \) such that \( v \geq u + \delta \) on \( \partial_P \mathcal{M} \). For any \( \epsilon > 0 \), consider

\[
\mathcal{M}(\epsilon) := \{(z,t) \in D \times (0, T) : (z,\epsilon^{-1}\log(1 + \epsilon t)) \in \mathcal{M}\},
\]

and the modified function

\[
v_\epsilon(z,t) := (1 + \epsilon t) \cdot v(z,\epsilon^{-1}\log(1 + \epsilon t)), \quad \forall (z,t) \in \overline{\mathcal{M}(\epsilon)}.
\]

This modification makes \( v_\epsilon \) a strict supersolution to the LFDE on \( \mathcal{M}(\epsilon) \):

\[
(\partial_t v_\epsilon - \Delta \log(v_\epsilon))(z,t) = \epsilon \cdot v(z,\epsilon^{-1}\log(1 + \epsilon t)) > 0, \quad \forall (z,t) \in \mathcal{M}(\epsilon).
\]

Since \( u \) and \( v \) are continuous on \( \overline{\mathcal{M}} \) compact, they are uniformly continuous. In particular, from the inequality

\[
0 < t - \epsilon^{-1}\log(1 + \epsilon t) \leq T - \epsilon^{-1}\log(1 + \epsilon T), \quad \forall t \in (0, T),
\]

we see that \( |t - \epsilon^{-1}\log(1 + \epsilon t)| \) converges to zero uniformly in \( t \in (0, T) \) as \( \epsilon \searrow 0 \). Thus, for \( \epsilon \) sufficiently small, \( |v(z,t) - v(z,\epsilon^{-1}\log(1 + \epsilon t))| < \delta/2 \), and \( |u(z,t) - u(z,\epsilon^{-1}\log(1 + \epsilon t))| < \delta/2 \), whenever \( (z,t), (z,\epsilon^{-1}\log(1 + \epsilon t)) \in \overline{\mathcal{M}} \).

We now consider those points lying in both \( \mathcal{M} \) and the shifted spacetime \( \mathcal{M}(\epsilon) \).

**Claim.** Define \( M(\epsilon) = \mathcal{M} \cap \mathcal{M}(\epsilon) \). For \( \epsilon > 0 \) sufficiently small, \( v_\epsilon \geq u \) on \( \partial_P (M(\epsilon)) \).

**Proof of Claim.** Fix \( (z,t) \in \partial_P (M(\epsilon)) \). Note that \( \partial_P (M(\epsilon)) \subseteq (\partial_P \mathcal{M}) \cup (\partial_P (\mathcal{M}(\epsilon))) \). This splits our analysis into two cases:
(I) \((z,t) \in \partial P M\). In this case we have the inequality
\[
v_{\epsilon}(z,t) \geq v(z,t) - |v_{\epsilon}(z,t) - v(z,t)| \\
\geq u(z,t) + \delta - |\epsilon t| \cdot |v(z,\epsilon^{-1} \log(1 + \epsilon t)) - v(z,\epsilon^{-1} \log(1 + \epsilon t)) - v(z,t)| \\
\geq u(z,t) + \delta - \epsilon \cdot T \cdot \|v\|_{\infty} - \frac{\delta}{2} \geq u(z,t),
\]

for any \(\epsilon\) sufficiently small.

(II) \((z,t) \in \partial P (M(\epsilon))\). Note that, this implies \((z,\epsilon^{-1} \log(1 + \epsilon t)) \in \partial P M\), and therefore we have the inequality
\[
v_{\epsilon}(z,t) \geq v(z,\epsilon^{-1} \log(1 + \epsilon t)) \\
\geq u(z,\epsilon^{-1} \log(1 + \epsilon t)) + \delta \\
\geq u(z,t) + \delta - |u(z,\epsilon^{-1} \log(1 + \epsilon t)) - u(z,t)| \\
\geq u(z,t) + \delta - \frac{\delta}{2} \geq u(z,t),
\]

for any \(\epsilon\) sufficiently small.

\[\Box\]

**Claim.** \(v_{\epsilon} \geq u\) on \(M(\epsilon)\).

**Proof of claim.** Let \(M_t\) denote \(M(\epsilon) \cap (D \times \{t\})\) for each \(t \in (0, T)\). If the claim doesn’t hold, then for \(n \in \mathbb{N}\) sufficiently large, the times
\[
t_n := \inf\{t \in (0, T) : \min_{M_t} (v_{\epsilon} - u)(z,t) \leq -1/n\} \in (0, T),
\]

are well-defined. As \(\overline{M_{t_n}}\) is compact, there exists \(z_n \in \overline{M_{t_n}}\) such that \((v_{\epsilon} - u)(z_n, t_n) = -1/n\). By the previous claim, \((v_{\epsilon} - u) \geq 0\) on the parabolic boundary \(\partial P (M(\epsilon))\), and hence \(z_n \in M_{t_n}\). Since the point \((z_n, t_n)\) is in the interior, we have the inequalities
\[
(v_{\epsilon} - u) = -1/n, \quad \Delta (v_{\epsilon} - u) \geq 0, \quad \partial_t (v_{\epsilon} - u) \leq 0,
\]

and therefore
\[
0 < \epsilon v(z_n, \epsilon^{-1} \log(1 + t_n \epsilon)) \\
\leq (\partial_t v_{\epsilon} - \Delta \log(v_{\epsilon}))(z_n, t_n) - (\partial_t u - \Delta \log(u))(z_n, t_n) \\
\leq \Delta \log(u) - \Delta \log(v_{\epsilon}) \\
= \left(\frac{\Delta u}{u} - \frac{\|\nabla u\|^2}{u^2}\right) - \left(\frac{\Delta v_{\epsilon}}{v_{\epsilon}} - \frac{\|\nabla v_{\epsilon}\|^2}{v_{\epsilon}^2}\right) \\
= \frac{1}{v_{\epsilon}} \Delta (u - v_{\epsilon}) + \left(\frac{1}{u} - \frac{1}{v_{\epsilon}}\right) \Delta u + \|\nabla u\|^2 \left(\frac{1}{v_{\epsilon}^2} - \frac{1}{u^2}\right) \\
\leq \frac{1}{n} \left(\frac{|\Delta u|}{u v_{\epsilon}} + \|\nabla u\|^2 \left(\frac{(u + v_{\epsilon})}{u^2 v_{\epsilon}^2}\right)\right).
\]
Since $u$ and $v$ are uniformly bounded away from zero, and $u, v \in C^{2,1}(\overline{M})$, there exists $C < \infty$, independent of $n$, such that
\[
0 < \epsilon \leq \frac{1}{v(z_n, \epsilon^{-1} \log(1 + t_n \epsilon))} \cdot \left( \frac{1}{n} \left( \frac{|\Delta u|}{u v} + |\nabla u|^2 \left( \frac{(u + v)}{u^2 v^2} \right) \right) \right) \leq \frac{C}{n}.
\]

Taking $n \to \infty$, this gives a contradiction. \qed

Fix $(z, t) \in \mathcal{M}$. Recall that $|t - \epsilon^{-1} \log(1 + \epsilon t)|$ converges to zero uniformly in $t \in (0, T)$ as $\epsilon \searrow 0$. Therefore, as $\mathcal{M}$ is open, for any $\epsilon$ sufficiently small, $(z, t) \in M(\epsilon)$. Moreover, by continuity, $v(\epsilon, z, t)$ converges to $v(z, t)$ as $\epsilon \searrow 0$, and we can conclude that $v \geq u$ on all of $\mathcal{M}$. \qed

\section{Miscellaneous}

\textbf{Lemma C.1} ([HL68], Corollary 2.8). If $X$ is a perfect and compact subset of a Riemann surface $M$, then there exists a non-atomic Radon measure $\mu \in \mathcal{R}(M)$ with $\text{supp}(\mu) = X$.

Throughout Sections 4 & 5 we referred to the following result.

\textbf{Lemma C.2} (Schwarz Lemma [Yau73]). Let $(M_1, g_1), (M_2, g_2)$ be Riemannian surfaces. If

1. $(M_1, g_1)$ is complete,
2. the Gaussian curvature $K(g_1) \geq -a_1$, for some $a_1 \geq 0$,
3. $K(g_2) \leq -a_2$, for some $a_2 > 0$,

then any conformal map $f : M_1 \to M_2$ satisfies the inequality.

\[
f^*(g_2) \leq \frac{a_1}{a_2} g_1.
\]

The following Lemma is used in Section 3 when constructing the ambient space for our spacetime. It states that if you have a sequence of $n$-dimensional manifolds which embed into one another, then there is a well-defined smooth $n$-dimensional manifold that all of these manifolds embed into.

\textbf{Lemma C.3} (Direct limit of embedded manifolds). Let $\{M_i\}_{i \in \mathbb{N}}$ be a smooth family of manifolds which embed smoothly into one another $M_i \hookrightarrow M_j$, for $i \leq j$. Then the direct limit:

\[
M = \lim_{\to} M_i := \bigsqcup_{i \in \mathbb{N}} M_i / \sim,
\]

where $x \sim y$ iff $x = y$ within an embedding, can be made into a smooth manifold. Moreover, the canonical maps $f_i : M_i \hookrightarrow M$, $x \mapsto [x]$ will be smooth embeddings.

\textbf{Proof}. Equip the set $M$ with the final topology, so that $U \subseteq M$ is open iff $f_i^{-1}(U)$ is open, for all $i \in \mathbb{N}$. In particular, each $f_i$ is now a continuous injection. Let $\psi_{ij}$ denote the embedding $M_i \hookrightarrow M_j$. Note that

\[
f_i = f_j \circ \psi_{ij}, \quad \forall i \leq j.
\]
In particular, for any open subset \( U \subseteq M_i \), we have

\[
\begin{align*}
    f_j^{-1} \circ f_i(U) &= \psi_{ji}^{-1} \circ f_i^{-1} \circ f_i(U) = \psi_{ji}^{-1}(U), \quad \text{if } j < i \\
    f_j^{-1} \circ f_i(U) &= f_j^{-1} \circ f_j \circ \psi_{ij}(U) = \psi_{ij}(U), \quad \text{if } j \geq i.
\end{align*}
\]

That is, \( f_i(U) \) is open in \( M \) and \( f_i \) is an embedding.

Choosing a countable base \( \{ B_{ij} : j \in \mathbb{N} \} \) of \( M_i \) for each \( i \in \mathbb{N} \), the collection

\[
    \mathcal{B} := \{ f_i(B_{ij}) \subseteq M : i, j \in \mathbb{N} \},
\]

is then a countable base of \( M \). \( M \) being Hausdorff follows from the embedded subspaces \( M_i \) being Hausdorff. Finally, for each \( j \in \mathbb{N} \), consider the atlas \( \{ c_\alpha : U_\alpha \subset M_j \hookrightarrow \mathbb{R}^n \}_{\alpha \in A_j} \) on \( M_j \). Define the new collection of charts \( \{ \tilde{c}_\alpha : \tilde{U}_\alpha \subset M \hookrightarrow \mathbb{R}^n \}_{\alpha \in \tilde{A}_j} \) by

\[
    \tilde{U}_\alpha := f_j(U_\alpha), \quad \tilde{c}_\alpha := c_\alpha \circ f_j^{-1}.
\]

The union over \( j \in \mathbb{N} \) of all such charts gives a well-defined smooth atlas on \( M \). With respect to this smooth structure, the canonical maps are smooth.

The following is a simple application of the uniformisation theorem needed in Sections 4 & 5.

**Lemma C.4.** Let \( M \) be the sphere \( S^2 \) or the complex plane \( \mathbb{C} \) equipped with its standard conformal structure. Consider a Riemann surface \( A \subseteq M \) given by some open subset. If the complement of \( A \) in \( M \) contains no isolated points, then \( A \) is hyperbolic.

**Proof.** Since we can identify the once-punctured sphere with the plane, it suffices to consider the case \( M = \mathbb{C} \). Consider a connected component \( \tilde{A} \) of \( A \). If the complement of \( A \) (and hence \( \tilde{A} \)) in \( \mathbb{C} \) has no isolated points, then \( \mathbb{C} \setminus \tilde{A} \) contains at least 2 points, and so by the planar uniformisation theorem \([Gol69]\), there exists a holomorphic covering \( D \to \tilde{A} \).

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