Path integral measure for first order and metric gravities

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Abstract: The equivalence between the path integrals for first order gravity and the standard torsion-free, metric gravity in 3 + 1 dimensions is analyzed. Starting with the path integral for first order gravity, the correct measure for the path integral of the metric theory is obtained.

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1. Introduction

Gravitation as originally formulated by Einstein is a field theory for the metric. The Einstein-Hilbert action,

\[ I_{EH}[g] = \int \sqrt{g^{(4)}} R d^4 x, \]

is the only functional of the metric, up to a cosmological constant, whose variation yields second order field equations for \( g_{\mu\nu} \) in four dimensions.

This action can also be written in terms of the local orthonormal basis of the (co)tangent space (vierbein), \( e^a \), and the spin connection \( \omega^{ab} \), as

\[ I_{EH}[e, \omega] = \frac{1}{32\pi G} \int_M R^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd} \]  \hspace{1cm} (1.1)

where \( R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega^{cb} \) is the curvature two form, related to the Riemann tensor by
\[ R^{ab} = \frac{1}{2} R^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu, \]
which contains only up to first derivatives of the fields.

Extremizing the action yields only first order differential equations for these fields. The resulting equations are equivalent to Einstein’s second order field equations for the metric. Thus, this first order action describes the same classical system as General Relativity [1, 2, 3]. The first order formulation is similar to the Palatini approach, where the inverse metric does not enter in the action, and the metric and affine connection are varied independently. In both cases the vanishing of the torsion tensor is not postulated but is a consequence of the field equations.

The on shell equivalence between metric gravity and the first order theory in four dimensions is easily recognized [4, 5, 6]. The purpose of this note is to establish the off shell equivalence as well by comparing the path integrals for the first order and the metric formulations of the Einstein-Hilbert theory in four dimensions.
The problem of quantizing the gravitational field has been extensively discussed over the past seventy years. Different attempts to turn the metric into quantum field have led to uncontrollable divergences. In the past twenty years some conceptually different ideas have been put forward (see, e.g., [7]). In some sense, gravity can be viewed as an effective low energy remnant of the fundamental string [8]. Alternatively, spacetime can be construed as a tapestry made out of fundamental loops roughly $10^{-33} \text{cm}$ in length [9, 10], etc.

Regardless of what the ultimate picture might be, in some approximate sense, quantum gravity might be represented through a path integral for some fundamental local field, be it a gauge connection, the vierbein or the metric. One can then ask whether the different formulations are equivalent to each other or not. Here we show that the path integral for the first order theory is formally the same as the one for the metric formalism. Any attempt, however, to prove or disprove that either formalism is renormalizable—which, by the way, has not been analyzed in the first order form—is beyond the scope of this work.

2. Metric and first order actions

In this section we review the expressions of the path integrals for the Einstein-Hilbert theory, both in the standard metric form (see, e.g., [11]) and in the first order formalism.

2.1 Second order (metric) gravity

In the standard metric formulation the torsion is set identically to zero. The Hamiltonian action, [12, 13, 14] reads

$$I[g, \pi_g] = \int d^4x (\dot{g}_{ij} \pi^{ij} - NH - N^i H_i), \quad (2.1)$$

where $g_{ij}$ and $\pi^{ij} = G^{ijmn}(g) \dot{g}_{ij}$ are the phase space coordinates, and

$$G^{ijmn}(g) = g^{1/2} \left[ g^{im} g^{jn} + g^{in} g^{jm} - 2 g^{ij} g^{mn} \right], \quad (2.2)$$

is the supermetric (here $g \equiv \det g_{ij}$). The Lagrange multipliers $N$ and $N^i$ enforce the first class constraints $H \approx 0$ and $H_i \approx 0$, respectively. There are 6 q’s, 6 p’s, which together with the 4 first class constraints yield 2 propagating degrees of freedom. The path integral reads

$$Z_g = \int [Dg_{ij}] [D\pi^{ij}] [DN] [DN^i] \exp \left[ \frac{i}{\hbar} I[g, \pi_g] \right] \times [\text{Ghosts}], \quad (2.3)$$

where “[Ghosts]” represents the measure for the ghost and antighosts needed to fix the diffeomorphism invariance of the theory. This measure has been extensively discussed in the literature and different proposals have been advanced [11, 13, 14]. We shall not consider this contribution to the path integral measure here. Since both the metric and the first order theories share the same invariance under diffeomorphisms, we shall assume that, for a given prescription for [Ghosts] in one formulation there is a corresponding equivalent in the other.
After integrating out the momenta $\pi^{ij}$, Eq. (2.3) can be rewritten (modulo ghosts terms) as

$$Z_g = \int \left[ \prod_{ij} \mathcal{D}g_{ij} \right] \prod_i \mathcal{D}N_i \left[ \sqrt{\text{det}(NG_{ijmn})} \right] \exp \left[ \frac{i}{\hbar} \int_M \sqrt{g^{(4)}} R \, d^4x \right]. \tag{2.4}$$

### 2.2 First order gravity

Two descriptions of first order gravity can be considered, the $e$-frame and the $\omega$-frame [17, 18]. Both frames are related by a canonical transformation and therefore have the same classical action, modulo boundary terms. In the $e$-frame the fields are the vierbein $e^a_k$ and its conjugate momentum $P^a_k$, and only first class constraints are present. In the $\omega$-frame the action is a functional of the spin connection and its conjugate momentum, and both, first and second class constraints appear. In [18], both frames were shown to be quantum mechanically equivalent.

The field equations obtained extremizing Eq. (1.1) with respect to $e^a$ are

$$\varepsilon_{abcd} R^{bc} \wedge e^d = 0, \tag{2.5}$$

which are equivalent to the usual Einstein equations. Varying with respect to $\omega_{ab}$, yields

$$\varepsilon_{abcd} T^c \wedge e^d = 0. \tag{2.6}$$

Here

$$T^a = de^a + \omega^a_b e^b, \tag{2.7}$$

is the torsion two-form, related to the torsion tensor ($T^a = \frac{1}{2} T^a_{\mu\nu} dx^\mu dx^\nu$). Note that, as in the Palatini formalism, Eq. (2.6) implies $T^a = 0$. In the metric formalism, instead, $T^a$ is assumed to be identically zero, and $\omega^{ab}$ is not assumed to be an independent field. This means that although the two formalism give the same classical equations, they need not define equivalent quantum theories.

### 3. The $e$ frame

In coordinates $(t, x^i)$, the canonical action in 3+1 dimensions in $e$-frame reads [18]

$$I[e, P_e] = \int d^4x (\dot{e}^a_k P^k_a - \omega^a_b J_{ab} - NH_\perp - N^i H_i). \tag{3.1}$$

Here $e^a_j$ is the canonical coordinate and its conjugate momentum is

$$P^j_d := \Omega_{d,ab}^{jk} \omega^{ab}_k, \tag{3.2}$$

where $\Omega$ is the symplectic form

$$\Omega_{d,ab}^{jk} = 2\varepsilon_{abcd} \omega^a_k \omega^b_j e^c. \tag{3.3}$$

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1. In the variation leading to (2.7), an integration by parts was performed. This usually brings in a boundary term which here is assumed to vanish by virtue of some appropriate boundary conditions. This is the case in asymptotically flat spacetimes, or more generally, if $\omega$ is held fixed at infinity.
The Lagrange multipliers $\omega^{ab}_{i}$, $N$ and $N^i$ correspond to the the first class generators of Lorentz transformations and diffeomorphisms, respectively. In this frame, the phase space has 12 $q$s and 12 $p$s, there are 10 first class (and no second class) constraints. This gives again 2 propagating degrees of freedom, as in the metric formalism. The resulting path integral in this frame is therefore given by

$$Z_e = \int [De_a^i][DP_a^k][D\omega^{ab}_i][DN][DN^i] \det(M_{\alpha\beta}) \exp \left[ \frac{i}{\hbar} I[e, P_e] \right],$$

(3.4)

where $M_{\alpha\beta}$ is the matrix of Poisson brackets

$$M_{\alpha\beta} = \{F_\alpha, \varphi_\beta\}^*,$$

(3.5)

where $F_\alpha = (\sigma_\bot, \sigma_i, \sigma_{ab})$ are the gauge fixing conditions for the first class constraints $(H_\bot, H_i, J_{ab})$ respectively.

In the $\omega$-frame there are 18 $q$s and 18 $p$s, there are also 10 first class and 12 second class constraints, which also yields 2 propagating degrees of freedom as well. It was shown in [18] that the path integrals in the two frames are equal,

$$Z_\omega = Z_e = Z_{\text{First Order}}.$$ 

(3.6)

4. Field redefinitions

It is clear from (3.2) that the momentum $P^i_a$ is essentially proportional to the spin connection. The connection contains a part which is determined by the vierbein, and a torsion-dependent part. This means that, in the first order formulation, the torsion tensor is a function of the momentum canonically conjugate to the vierbein. Thus, the first step to establish the relation between the path integral of the first order theory (3.4) and that of the metric form (2.4), can be to separate the metric from the nonmetric (torsional) components of the spin connection. The torsion tensor (2.7) can be solved for $\omega^{ab}$, expressing the spin connection as

$$\omega^{ab}_\mu = \bar{\omega}^{ab}_\mu(e) + K^{ab}_\mu(e, T),$$

where $K^{ab}_\mu$ is the contorsion tensor and $\bar{\omega}$ is a torsion-free connection, that is,

$$de^a + \bar{\omega}^a_b \wedge e^b = 0.$$ 

Consequently, the momentum $P^j_c$ can be decomposed into a term depending on the vierbein and a projection of contorsion as $P^j_c = \Omega^{j}_c a b (\bar{\omega}^{ab}_i(e) + K^{ab}_i(e, T))$, which can also be written as

$$P^j_c = \Omega^{j}_c a b \bar{\omega}^{ab}_i(e) + K^j_c(e, T),$$

(4.1)

where all the torsional dependence is contained in the new canonical momenta $K^{i}_a$. Since $\bar{\omega}$ has vanishing Poisson bracket with $e$, the canonical measure in the $e$-$P_e$ phase space can be directly expressed in the $e$-$K$ space as

$$[De^a_i][DP^j_a] = [De^a_i][DK^j_a].$$

(4.2)
On the other hand, decomposing the frame basis $e_a^\mu$ along the spatial directions $e_a^j$ and the timelike normal $\eta_a$, the Lagrange multipliers $N$ and $N^i$ can be written as

\begin{align}
N(e) &= \eta_a e_a^0, \\
N^i(e) &= E^0_a e_a^0,
\end{align}
where $\eta_a e_a^a = -1$, $\eta_a e_a^i = 0$, and $E^i_a e_a^i = \delta^i_j$. Now the measure $[DN][DN^i]$ can be shown to be $-\sqrt{g^{(3)}}[De_a^0]$, where $g^{(3)} = \det(g_{ij})$ and $g_{ij} = e^a_i e^b_j \eta_{ab}$. Thus, the integration measure in (3.4) reads

\[ [De^a_k][DP^k_a][D\omega_t^{ab}][DN][DN^i] = -\sqrt{g^{(3)}}[De_a^0][DK^i_a][DK^{ab}_0], \tag{4.5} \]

where $\omega_t^{ab} \equiv K_0^{ab}$. In these new variables, the angular momentum is

\[ J_{ab} = K^i_a e_{bj} - K^i_b e_{aj}. \]

The 12 components of $K^i_a$ can be projected also along spatial and normal directions,

\[ K^i_a = K^i \eta_a + (\kappa^{(ij)} + \kappa^{[ij]}) e_{aj}, \tag{4.6} \]

where $\kappa^{(ij)}$ is symmetric and $\kappa^{[ij]}$ antisymmetric. The angular momentum constraint can be written in terms of $K^i$ and $\kappa^{[ij]}$ only, as

\[ J_{ab} = K^i (\eta_a e_{bi} - \eta_b e_{ai}) + \kappa^{[ij]} (e_{aj} e_{bi} - e_{bj} e_{ai}). \tag{4.7} \]

It is straightforward to invert this relation, writing $K^i$ and $\kappa^{[ij]}$ in terms of $J_{ab}$,

\[ K^i = -\eta^a E^{bij} J_{ab} = -g^{jk} \eta^a e_k^b J_{ab} \]

\[ \kappa^{[ij]} = E^{ai} E^{bj} J_{ab} = g^{ip} g^{jq} e^a_i e^b_j J_{ab}. \tag{4.8} \]

The EH action, written in terms of these fields, becomes

\[ I = \int_M \left( \sqrt{g^{(4)} R - K_0^{ab} J_{ab} - N \kappa^{[ij]} G_{ijmn} \kappa^{(mn)} + f(J_{ab})} \right) d^4x, \tag{4.9} \]

where $G_{ijmn}$ is the inverse of the supermetric defined in (2.2) and $f(J_{ab})$ is a functional which vanishes for $J_{ab} = 0$. In terms of these new fields, the measure in the $e$-$K$ space becomes

\[ [De^a_i][DK^i_a] = \det(D) [De^a_i][DJK_{cd}][DK^{(mn)}], \tag{4.10} \]

where $D$ is the Jacobian matrix

\begin{align*}
D^M_N &= \left[ \partial K^i_a / \partial \left( J_{cd}, \kappa^{(mn)} \right) \right] \\
&= \left[ g^{jp} g^{jq} e^a_i e^d_j e_{aj} - \eta^a e^d_j g^{jk} \eta_a, \frac{1}{2} (\delta^i_m e_{an} + \delta^i_n e_{am}) \right]. \tag{4.11} \end{align*}

where the indexes stand for the 12 combinations $\textbf{M} = [i_a]$ and $\textbf{N} = [cd]_{(mn)}$. It is straightforward to show that $\det(D) = (\text{Const})$, which can be confirmed by observing that under $e^a_j \rightarrow \lambda e^a_j$, determinant of $D$ remains unchanged.
In order to make contact with the path integral in the second order formulation, three more steps are in order: first, integrating over the Lagrange multiplier $K_{ab}$ in (4.9) produces a $\delta(J_{ab})$ which makes $f(J_{ab})$ drop out from the action and eliminates the integration over $J_{cd}$. Second, integrating over $\kappa^{(ij)}$ yields a Gaussian form and brings down a factor $[\det(NG_{ijmn})]^{(-1/2)}$.

Finally, since Lorentz symmetry is not present in the metric theory, it should be frozen out of the 16 components of $e^a_\mu$, replacing them by their Lorentz invariant components (metric) and six Lorentz rotation coefficients. This can be done expressing the vierbein in the form

$$e^a_\mu = U(x)^a_b \hat{e}^b_\mu,$$  \hspace{1cm} (4.12)

where $\hat{e}^b_\mu$ is a fixed vierbein and $U(x)^a_b$ corresponds to a local Lorentz transformation. Here we shall assume that $e^a_\mu$ is globally defined at least in each spatial section $\Sigma_{t=t_0}$. This means that there is a gauge (choice of $U(x)^a_b$) such that the vierbein is equal to $\hat{e}^a_\mu$ throughout $\Sigma_{t=t_0}$.

In terms of the group parameters $\lambda^{ab} = -\lambda^{ba}$ the local Lorentz rotations read

$$U^a_b = \delta^a_b + (\lambda L)^a_b + \frac{1}{2}(\lambda L)^c_a (\lambda L)^e_b + \ldots,$$

where $L$ are the generators of $SO(3,1)$ in the vector representation, expressed as $\eta^{cd} \delta^a_d - \eta^{db} \delta^a_b$. Thus, the 16 components $e^a_\mu$ are described by 10 fields corresponding to the rotational invariant part of the representative vierbein $\hat{e}^b_\mu$, which can be identified with the metric $g_{\mu\nu} = \eta^{ab} \hat{e}^a_\mu \hat{e}^b_\nu$, and the 6 variables $\lambda^{ab}$.

Varying the expression (4.12) with respect to $\lambda^{cd}$ and $g_{\mu\nu}$ yields

$$\delta e^a_\mu = \frac{\delta U^a_b}{\delta \lambda^{cd}} E^{b\nu} g_{\mu\nu} \delta \lambda^{cd} + U^a_b E^{b\nu} \delta g_{\mu\nu}. \hspace{1cm} (4.13)$$

The measure of integration over the vierbein can be written as

$$[De^a_\mu] = [D(\lambda^{cd})][D(g_{\alpha\beta})] \det B, \hspace{1cm} (4.14)$$

where $B$ stands for $16 \times 16$ matrix

$$B^M_N = \hat{E}^\nu_b \left[ g_{\mu\nu} \frac{\delta U^{ab}}{\delta \lambda^{cd}}, U^{ab} \frac{1}{2} \left( \delta^{\alpha}_{\mu} \delta^{\beta}_{\nu} + \delta^{\beta}_{\mu} \delta^{\alpha}_{\nu} \right) \right]. \hspace{1cm} (4.15)$$

The assumption that (4.12) be globally defined implies that diffeomorphisms and Lorentz rotations can be performed independently. Consequently, one can fix the Lorentz frame by choosing $U^a_b = \delta^a_b$ globally, so it is always possible to select the gauge condition by fixing $\lambda = 0$ everywhere, say. This yields

$$\left. \frac{\delta U^{ab}}{\delta \lambda^{cd}} \right|_{\lambda=0} = \delta^{ab}_{cd},$$

which implies

$$B^M_N \big|_{\lambda=0} = \frac{1}{2} \hat{E}^\nu_b \left[ g_{\mu\nu} \delta^{ab}_{cd}, \eta^{ab} \left( \delta^{\alpha}_{\mu} \delta^{\beta}_{\nu} + \delta^{\beta}_{\mu} \delta^{\alpha}_{\nu} \right) \right].$$
It is straightforward to check from (4.15) that \( \det(B) = (\text{const}) \times (g^{(4)})^{-1/2} \). Thus, the path integral (3.4) of the first order theory can finally be written, up to a multiplicative constant, as

\[
Z_{\text{First Order}} = \int \frac{D[g_{ij}]D[N][DN^i]}{\sqrt{\det(N G_{ijmn})}} \times \exp \left[ \frac{i}{\hbar} \int_M \sqrt{g^{(4)}} R \ d^4x \right],
\]

which coincides with the second order expression, as expected (here we have used the fact that \( \det(D) = (\text{constant}) \)).

5. Summary and prospects

We have shown that the path integral for the Einstein-Hilbert action in four dimensions and with vanishing cosmological constant is formally identical for the first order (vierbein) formulation as for the second order (metric) theory. However, as the steps of the proof depend critically on several features peculiar to the four dimensional EH action, it is likely to fail in more general settings.

**LL theories.** For spacetime dimensions \( D > 4 \) there exist a family of sensible theories, including higher powers of curvature but no explicit torsion in the action, with second order equations for the metric, that generalize General Relativity. These are the so-called Lanczos-Lovelock theories [19, 20]. For them there are also two versions: a first order and a metric one. But, unlike the EH case, these two formulations are not classically equivalent for every field configuration. In particular, there are configurations in which the first order formulation might allow nonvanishing classical torsion [21], whereas the second order version always assumes zero torsion. However, the configurations where the two theories are classically inequivalent form a set of measure zero in the space of solutions and may be ignored in generic backgrounds.

**Torsion as momentum.** In the second order formulation, the torsion is assumed to be identically zero and therefore never varied in the action or integrated over in the path integral. In contrast, we observe that in the first order approach, torsion is not only allowed to vary, it is necessary since it represents the canonical momentum conjugate to the vierbein (c.f. Eq. (4.2)).

**Degrees of freedom.** The off shell equivalence is probably still true for the LL gravity theories. This is because the torsion-free LL theories have the same degrees of freedom as the standard Einstein-Hilbert system (see e.g., [22]). On the other hand, the proof is unlikely to go through for more general actions explicitly involving torsion in the Lagrangian. Theories of this type were discussed in [23], and were shown to possess more degrees of freedom in general [24, 25, 21].

**Adding a cosmological constant.** As mentioned above, the boundary terms that arose in calculations vanished provided a flat asymptotic conditions are assumed. This implies that the proof remains valid for the theory defined on an open domain in the presence of a cosmological constant. However, the case \( \Lambda \neq 0 \), where the boundary terms at infinity could diverge, should be analyzed more carefully to see whether a similar picture as that for the asymptotically flat case can be drawn.
The character of the equivalence. The proof of quantum equivalence presented here is in any case formal. Principally, because the equality is between two expressions which no one knows how to unambiguously evaluate, interpret or use to predict any experiment.

In view of this plethora of possibilities it would be interesting to extend this work, establishing the path ordered integral to some these alternative theories of gravity.

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