The relation between effective action and vacuum energy in a kappa-deformed theory

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Abstract

In a quantum field with spacetime invariance governed by the Poincaré algebra the one-loop effective action is equal to the sum of zero modes frequencies, which is the vacuum energy of the field. The first Casimir invariant of the Poincaré algebra provides the proper time Hamiltonian in Schwinger’s proper time representation of the effective action. We consider here a massive neutral scalar field with spacetime invariance governed by the so called kappa-deformed Poincaré algebra. We show here that if in the kappa-deformed theory the first Casimir invariant of the algebra is also used as the proper-time hamiltonian the effective action appears with a real and an imaginary part. The real part is equal to half the sum of kappa-deformed zero mode frequencies, which gives the vacuum energy of the kappa-deformed field. In the limit in which the deformation disappears this real part reduces to half of the sum of zero mode frequencies of the usual scalar field. The imaginary part is proportional to the sum of the squares of the kappa-deformed zero mode frequencies. This part is a creation rate of field excitations in the situations in which it gives rise to a finite physically meaningful quantity. This is the case when the field is submitted to boundary conditions and properly renormalized, as we show in a related paper.

The $\kappa$-deformed Poincaré algebra is a quantum group (Hopf algebra) related to de Sitter and conformal algebras. It is a deformation of the Poincaré algebra, which is recovered when the limit $\kappa \to \infty$ is taken for the positive real parameter $\kappa$. Here we are not interested in the precise commutation relations which define the deformed algebra, but rather in its first Casimir invariant, which is given by

$$P^2 - (2\kappa)^2 \sinh^2(P_o/2\kappa) = -m^2,$$

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where $m^2$ is the value of the invariant in the chosen representation. This invariant reduces to $P^2 - P_o^2$ in the limit $\kappa \to \infty$, as it should be expected.

A $\kappa$-deformed field is defined in a spacetime whose symmetries are governed by the $\kappa$-deformed Poincaré algebra and accordingly reduces to a relativistic quantum field in the limit $\kappa \to \infty$. Since the continuous parameter $\kappa$ can be taken as large as we wish the deformed algebra is well suited for the investigation of possible small violations of relativistic symmetries. A $\kappa$-deformed quantum field exhibit non conservation of four momentum at interactions vertices and the phenomenom of creation of field excitations when submitted to boundary conditions. Those two distinct phenomena point to non conservation of energy, a possibility which should be faced once relativistic spacetime symmetries are broken by the deformation.

In a quantum field with spacetime invariance governed by the Poincaré algebra the one-loop effective action is determined by the first Casimir invariant of the algebra through Schwinger’s proper time representation of the effective action. We are interested here in the case of a scalar field, in which Schwinger’s representation is given by

$$W = -\frac{i}{2} \int_{s_o}^{\infty} \frac{ds}{s} \text{Tr} e^{-isH},$$

(2)

where $H = P^2 - P_o^2 + m^2$ and is obtained from the first Casimir invariant of the Poincaré algebra, namely: $P^2 - P_o^2$. The effective action $W$ is also given by the sum of zero modes frequencies. This sum gives the Casimir energy of the field when it is submitted to static boundary conditions. This sum can also give rise to an imaginary part describing the probability of creation of field excitations, as it occurs, e.g., in the case of moving boundary conditions or charged vacuum submitted to an external electric field. The method of calculating effective actions as sum of zero modes is a old one and the expression of the effective action as a sum of zero modes can actually be obtained directly from Schwinger’s proper-time representation [3].

The expression (2) for a $\kappa$-deformed scalar field requires

$$H = P^2 - (2\kappa)^2 \sinh^2(P_o/2\kappa) + m^2,$$

(3)

in accordance with the first Casimir invariant [1] of the $\kappa$-deformed Poincaré algebra. For the field submitted to static boundary conditions describing confinement between parallel plates this effective action has a real and imaginary part [1] [5] [6].

The real part is a Casimir energy of the $\kappa$-deformed field and actually reduces to the Casimir energy of the usual scalar field in the limit $\kappa \to \infty$. The imaginary part gives a probability of creation of field excitations which disappears in the limit $\kappa \to \infty$ or in the limit when the boundary conditions are turned off (infinite separation between the plates). This the above mentioned creation mechanism steaming from the $\kappa$-deformation and totally absent in the usual field theories. In order to further clarify these results we address here the relation between the effective action (2) and the sum of zero modes in the case of a $\kappa$-deformed scalar field. The
scalar field is submitted to boundary conditions in view of future applications of the results developed here. These results can be obtained in the absence of boundary conditions without further efforts.

Let us consider the $\kappa$-deformed scalar field submitted to Dirichlet boundary conditions on two large parallel plates of side $\ell$ and separation $a$ ($a \ll \ell$). We take the $OZ$-axis perpendicular to the plates and the eigenvalues of $P_z$ in (3) are accordingly given by $n\pi/a$ ($n \in \mathbb{N}$). We start by taking (3) into the effective action (2) with regularization parameters $\epsilon$ and $\nu$ to obtain:

$$W = -\frac{i}{2} \int_0^{\infty} \frac{ds}{s} \text{Tr} \exp\{-is[P^2 - (2\kappa)^2 \sinh^2(P_0/2\kappa) + m^2 - i\epsilon]\},$$

where $\epsilon$ is a positive infinitesimal and $\nu$ is large enough to render the integral well defined. By taking the boundary conditions in consideration the above expression acquires the form:

$$W = -\frac{i}{2} \frac{T\ell^2}{(2\pi)^3} \int_0^{\infty} \frac{ds}{s} s^\nu \sum_{n=1}^{\infty} \int d\omega \int dp_1 \int dp_2 \times \times \exp\{-is[p_1^2 + p_2^2 + (n\pi/a)^2 - (2\kappa)^2 \sinh^2(\omega/2\kappa) + m^2 - i\epsilon]\}$$

(5)

By taking the derivative of this expression in relation to $m^2$,

$$\frac{1}{T\ell^2} \frac{\partial W}{\partial m^2} = -\frac{1}{2} \frac{T\ell^2}{(2\pi)^3} \sum_{n=1}^{\infty} \int d\omega \int dp_1 \int dp_2 \int_0^{\infty} ds s^\nu \times \times \exp\{-is[p_1^2 + p_2^2 + (n\pi/a)^2 - (2\kappa)^2 \sinh^2(\omega/2\kappa) + m^2 - i\epsilon]\},$$

(6)

and using Euler’s Gamma function integral representation we arrive at:

$$\frac{1}{T\ell^2} \frac{\partial W}{\partial m^2} = -\frac{1}{2} \nu^1 \Gamma(\nu + 1)(2\kappa^2)^{-(\nu+1)} \sum_{n=1}^{\infty} \int \int \frac{dp_1 dp_2}{(2\pi)^2} \int \frac{d\omega}{2\pi} \left[ \cosh \left( \frac{\omega}{\kappa} \right) - \beta \right]^{-(\nu+1)},$$

(7)

where

$$\beta = 1 + \frac{1}{2\kappa^2}[p_1^2 + p_2^2 + (n\pi/a)^2 + m^2 - i\epsilon].$$

(8)

We retain for a while the regulator $\epsilon$ to perform the integration over $\omega$. We consider $\nu$ to be a positive integer to obtain:

$$\frac{1}{T\ell^2} \frac{\partial W}{\partial m^2} = \frac{\nu^1}{2} \Gamma(\nu + 1)(2\kappa^2)^{-(\nu+1)} \times \times \sum_{n=1}^{\infty} \int \int \frac{dp_1 dp_2}{(2\pi)^2} \frac{1}{2\pi} \kappa^\nu \partial^{\nu} \left[ \frac{2\pi i}{\sqrt{\beta^2 - 1}} - \frac{2}{\sqrt{\beta^2 - 1}} \log \left( \beta + \sqrt{\beta^2 - 1} \right) \right],$$

(9)
where we have finally taken $\epsilon \to 0$. From the definition (8) we have that $\partial W / \partial \beta = 2\kappa^2 \partial W / \partial m^2$
This identity can be used to eliminate the derivative in relation to the square mass in (9), which can be integrated in $\beta$ to become:

$$\frac{1}{2} \frac{W}{\ell^2} = - \frac{\kappa}{(2\kappa^2)^\nu} \sum_{n=1}^\infty \int \int \frac{dp_1 dp_2}{(2\pi)^3} \int_\infty^\beta d\beta \frac{\partial^{\nu}}{\partial m^{\nu}} \left[ \frac{\pi}{\sqrt{\beta^2 - 1}} + \frac{i}{\sqrt{\beta^2 - 1}} \log (\beta + \sqrt{\beta^2 - 1}) \right].$$ (10)

Now we understand that the $\nu$ derivatives in relation to $\beta$ has been taken in the integrand of (10). The resulting expression is a function of $\nu$ that we submit to an analytical continuation. In this way the limit $\nu \to 0$ can be taken after the subtraction of spurious terms in order to arrive at the physical quantities. The identification of spurious terms depends on the specific problem in consideration. Here we will be content in showing that (10) is properly regularized by the parameter $\nu$ and that the elimination of the regularization gives us the relation between the effective action (2) and the sum of zero modes, both without regularization. This relation is obtained by taking the value $\nu = 0$ in (10) to arrive at:

$$\frac{W}{\ell^2} = \sum_{n=1}^\infty \int \int \frac{dp_1 dp_2}{(2\pi)^2} \frac{1}{2} \omega(p_1, p_2, n) + \frac{i}{\pi \kappa} \sum_{n=1}^\infty \int \int \frac{dp_1 dp_2}{(2\pi)^2} \frac{1}{4} \omega^2(p_1, p_2, n)$$ (11)

where $\omega$ is the frequency given by the mass-shell condition derived from (11):

$$\omega(p_1, p_2, n) = 2\kappa \sinh^{-1} \left[ \frac{1}{2\kappa} \sqrt{p_1^2 + p_2^2 + (\pi n/a)^2 + m^2} \right].$$ (12)

If we prefer we can write (11) using box normalization in order to discretize all the components of momentum:

$$\frac{W}{T} = \sum_p \frac{1}{2} \omega_p + \frac{i}{\pi \kappa} \sum_p \frac{1}{4} \omega_p^2.$$ (13)

The expression (11) is our main result, which answers the question of what is the relation between the effective action (2) and the sum of zero modes in the case of a $\kappa$-deformed scalar field. The result (11) shows that the effective action $W$ as given in (2) has a real part given by the sum of half frequencies, as in the non-deformed case, although the frequencies to be summed in the present case are given by the $\kappa$-deformed expression (12). Contrary to the non-deformed case the effective action has also an imaginary part, which is given by a sum of squared half frequencies divided by the deformation parameter $\kappa$. In the limit $\kappa \to \infty$ in which the deformation disappears the imaginary part goes to zero and the real part goes to the sum of half non-deformed frequencies:

$$\lim_{\kappa \to \infty} \frac{W}{T} = - \sum_p \frac{1}{2} \sqrt{P^2 + m^2},$$ (14)
which is exactly what should be expected. The imaginary part in (11) is responsible for the creation of field excitations when the \( \kappa \)-deformed field is submitted to boundary conditions. This part is in total agreements with previously obtained results [8] and will be further investigated in a companion paper.

We are left now with the task of proving that the non-regularized result (13) is on firm ground. To this purpose we show that (10) is properly regularized by the parameter \( \nu \). This is a rather technical manipulation of inequalities and estimation of integrals that we present below, in a mode of appendix. We want to show that (10) is properly regularized, i.e., that there exists \( \nu \) such that the integral in (10) are well defined. Concerning the first term in the integrand of (10) we want to show that:

\[
\left| \frac{\partial^\nu}{\partial \beta^\nu} \frac{1}{\sqrt{\beta^2 - 1}} \right| \leq \left| \frac{\partial^\nu}{\partial \beta^\nu} \frac{1}{\beta - \beta^{-1}} \right|. \tag{15}
\]

By using the series expansions:

\[
\frac{\partial^\nu}{\partial \beta^\nu} \frac{1}{\sqrt{\beta^2 - 1}} = (-1)^\nu \sum_{j=0}^{\infty} \frac{(2j + \nu)!}{(2j)!^2} \beta^{-(2j+\nu+1)}, \tag{16}
\]

\[
\frac{\partial^\nu}{\partial \beta^\nu} \frac{1}{\beta - \beta^{-1}} = (-1)^\nu \sum_{j=0}^{\infty} \frac{(2j + \nu)!}{(2j)!} \beta^{-(2j+\nu+1)}, \tag{17}
\]

we reduce the verification of inequality (15) to the verification of the following simpler inequality:

\[
\frac{1}{(2j)!^2} \leq \frac{1}{(2j)!}. \tag{18}
\]

Since

\[
2^{2j} = \sum_{n=0}^{2j} \binom{2j}{n} = \binom{2j}{j} + \sum_{n=0}^{2j} \binom{2j}{n}, \tag{19}
\]

and all the quantities in the above equation are positive, we conclude that

\[
\binom{2j}{j} \leq 2^{2j}, \tag{20}
\]

which shows the correctness of (18) and thus the validity of the inequality (15). We now use this inequality in the real part of (10) to obtain:

\[
\left| \Re \left\{ -\frac{1}{\nu} W T \right\} \right| \leq \left| \frac{\kappa \pi}{(2\kappa^2)^\nu} \sum_{n=1}^{\infty} \int \int dp_1 dp_2 \frac{d\beta}{(2\pi)^3} \int_{r}^{\beta} d\beta \left| \frac{\partial^\nu}{\partial \beta^\nu} \frac{1}{\beta - \beta^{-1}} \right| \right|. \tag{21}
\]
We have now to calculate the $\nu$’nd derivative in (21), which result is given below:

$$\frac{\partial^{\nu}}{\partial \beta^{-\nu}} \frac{1}{\beta - \beta^{-1}} \left[ \frac{1}{(\beta + 1)^{\nu+1}} + \frac{1}{(\beta - 1)^{\nu+1}} \right].$$

(22)

Substituting this result in (21) we obtain:

$$\left| \Re \left\{ -\frac{1}{i^\nu} \frac{W}{T\ell^2} \right\} \right| \leq \left| \frac{\kappa \pi \nu!}{2(2\kappa^2)^\nu} \sum_{n=1}^{\infty} \int \int dp_1 dp_2 \int_{\infty}^{\beta} \frac{1}{(\beta + 1)^{\nu+1}} + \frac{1}{(\beta - 1)^{\nu+1}} \right|. \quad (23)$$

We see in this expression that the integrals $p_1$ and $p_2$ are well defined for $\nu \geq 1$. The integration over $\beta$ reduces the expression to:

$$\left| \Re \left\{ -\frac{1}{i^\nu} \frac{W}{T\ell^2} \right\} \right| \leq \left| \frac{\kappa \pi (\nu - 1)!}{(2\kappa^2)^\nu} \sum_{n=1}^{\infty} \int \int dp_1 dp_2 \frac{1}{(\beta + 1)^{\nu+1}} + \frac{1}{(\beta - 1)^{\nu+1}} \right|, \quad (24)$$

or,

$$\left| \Re \left\{ -\frac{1}{i^\nu} \frac{W}{T\ell^2} \right\} \right| \leq \left| \frac{\kappa \pi (\nu - 1)!}{(2\kappa^2)^\nu} \sum_{n=1}^{\infty} \int \int dp_1 dp_2 \frac{1}{(\beta + 1)^{\nu+1}} \right|. \quad (25)$$

From the definition (8) of $\beta$ we have

$$\beta = \frac{p_2^2}{2\kappa^2} + \alpha_n(a, m, \kappa),$$

where:

$$p_2^2 = p_1^2 + p_2^2,$$

and

$$\alpha_n(a, m, \kappa) = 1 + \frac{1}{2\kappa^2} \left( m^2 + \left( \frac{n\pi}{a} \right)^2 \right),$$

being clear that:

$$\alpha_n(a, m, \kappa) \geq 1.$$

We then write (25) as

$$\left| \Re \left\{ -\frac{1}{i^\nu} \frac{W}{T\ell^2} \right\} \right| \leq \left| \frac{\kappa \pi (\nu - 1)!}{(2\kappa^2)^\nu} \sum_{n=1}^{\infty} \int \int d\theta \int_{0}^{2\pi} dp_1 \int_{0}^{\infty} dp_\parallel \frac{p_\parallel}{(2\kappa^2)^\nu} + \alpha_n(a, m, \kappa) \right|. \quad (26)$$
If \( p_\parallel^2 / 2\kappa^2 + \alpha_n(a, m, \kappa) = x \), with \( \nu > 1 \) (to avoid divergences), we have:

\[
\int_0^\infty \frac{dp_\parallel}{\left[ \frac{p_\parallel^2}{2\kappa^2} + \alpha_n(a, m, \kappa) \right]^{\nu}} = -\frac{1}{\nu - 1} \left\{ \frac{1}{\nu} \left[ \alpha_n(a, m, \kappa) \right]^{\nu-1} \right\}_0^\infty = \frac{1}{\nu - 1} \left[ \alpha_n(a, m, \kappa) \right]^{\nu-1}.
\]

(27)

By assuming that \( \nu > 1 \) we can perform the integration on \( p_\parallel \) to obtain:

\[
\left| \Re \left\{ \frac{1}{i^\nu T l^2} \right\} \right| \leq \frac{\kappa(\nu - 2)!}{4\pi(2\kappa^2)^\nu} \sum_{n=1}^{\infty} \frac{1}{\alpha_n(a, m, \kappa)^{\nu-1}},
\]

(28)

or,

\[
\left| \Re \left\{ \frac{1}{i^\nu T l^2} \right\} \right| \leq \frac{\kappa(\nu - 2)!}{4\pi(2\kappa^2)^\nu} \sum_{n=1}^{\infty} \frac{1}{\left[ 1 + \frac{1}{2\kappa^2} \left( m^2 + \left( \frac{n\pi}{a} \right)^2 \right) \right]^{\nu-1}}.
\]

(29)

Right side of (29) has a finite value and is a representation of the Epstein function. This function has an analytical continuation \([9]\) and so we can choose any value for the variable \( \nu \). If \( 2(\nu - 1) > 1 \), or \( \nu > 3/2 \), we conclude that there exists \( \nu \) such that real part of the Schwinger’s effective action is regularized by the power \( s^\nu \) in (1).

Now we have to verify if the imaginary part of the effective action, equation (10), is already regularized.

\[
\Im \left\{ -\frac{1}{i^\nu T l^2} \right\} = \frac{\kappa}{(2\kappa^2)^\nu} \sum_{n=1}^{\infty} \int \int \frac{dp_1 dp_2}{(2\pi)^3} \int_0^\infty d\beta \frac{\partial^{\nu}}{\partial \beta^{\nu}} \log \left( \frac{\beta + \sqrt{\beta^2 - 1}}{\sqrt{\beta^2 - 1}} \right).
\]

(30)

As made for the real part, we have to maximize the imaginary part in (30). To make that, we try to find a function \( M : \beta \to M(\beta) \) that satisfy the inequality given by:

\[
\int_0^\beta d\beta \frac{\partial^{\nu}}{\partial \beta^{\nu}} \log \left( \frac{\beta + \sqrt{\beta^2 - 1}}{\sqrt{\beta^2 - 1}} \right) \leq \int_0^\beta d\beta \left| \frac{\partial^{\nu}}{\partial \beta^{\nu}} \log \left( \frac{\beta + \sqrt{\beta^2 - 1}}{\sqrt{\beta^2 - 1}} \right) \right| \leq \int_0^\beta d\beta \left| M(\beta) \right|
\]

Defining \( g(\beta) = 1/\sqrt{\beta^2 - 1} \), and \( h(\beta) = \log(\beta + \sqrt{\beta^2 - 1}) \), where, \( \partial h(\beta)/\partial \beta = g(\beta) \), the derivative of order \( \nu \) of a product of two functions can be calculated from:

\[
\frac{\partial^{\nu}}{\partial \beta^{\nu}} (g(\beta) h(\beta)) = \sum_{l=0}^{\nu} \left( \begin{array}{c} \nu \\ l \end{array} \right) \frac{\partial^{\nu-l} g(\beta)}{\partial \beta^{\nu-l}} \frac{\partial^l h(\beta)}{\partial \beta^l} =
\]

\[
= \frac{\partial^{\nu} g(\beta)}{\partial \beta^{\nu}} h(\beta) + \sum_{l=1}^{\nu} \left( \begin{array}{c} \nu \\ l \end{array} \right) \frac{\partial^{\nu-l} g(\beta)}{\partial \beta^{\nu-l}} \frac{\partial^l h(\beta)}{\partial \beta^l}
\]

(31)
Then, we can rewrite the (31) as:

\[ \frac{\partial^l h(\beta)}{\partial \beta^l} = \frac{\partial^{l-1} h(\beta)}{\partial \beta^{l-1}} = \frac{\partial^{l-1} g(\beta)}{\partial \beta^{l-1}} \]

Then, we can rewrite the (31) as:

\[ \frac{\partial^l (g(\beta)h(\beta))}{\partial \beta^l} = \frac{\partial^l g(\beta)}{\partial \beta^l} h(\beta) + \sum_{i=1}^{\nu} \left( \nu \right) \frac{\partial^{l-1} g(\beta)}{\partial \beta^{l-1}} \frac{\partial^{l-1} g(\beta)}{\partial \beta^{l-1}}, \]

and also knowing that,

\[ \left| \frac{\partial^l (g(\beta)h(\beta))}{\partial \beta^l} \right| \leq \left| \frac{\partial^l g(\beta)}{\partial \beta^l} h(\beta) \right| + \left| \sum_{i=1}^{\nu} \left( \nu \right) \frac{\partial^{l-1} g(\beta)}{\partial \beta^{l-1}} \right| \leq \left| \frac{\partial^{l-1} g(\beta)}{\partial \beta^{l-1}} \right|, \]

we are ready to search for functions \( g \) and \( h \) that maximize the imaginary part of the Casimir energy. The function \( g \) and its derivatives of the \( \nu \) order were already maximized by the function \( f: \beta \to f(\beta) = (\beta - \beta^{-1})^{-1} \). So, let’s analyze now the case of \( h \), verifying if exists \( \gamma \) such that:

\[ 0 \leq h(\beta) = \log(\beta + \sqrt{\beta^2 - 1}) \leq \log(\gamma \beta), \]

In other words,

\[ \gamma \geq 1 + \sqrt{\beta^2 - 1}. \]

in order to simplify and choose \( \gamma \) as being a numerical constant, we have to take the maximum of \( 1 + \sqrt{\beta^2 - 1} / \beta \) that is 2. So we conclude that,

\[ 0 \leq h(\beta) = \log(\beta + \sqrt{\beta^2 - 1}) < \log(2\beta) < \log(e^{2\beta}) = 2\beta \to \]

\[ \to h(\beta) = \log(\beta + \sqrt{\beta^2 - 1}) < 2\beta. \]

Then, \( \exists \ \sigma \ni \sigma \beta \geq \log(\beta + \sqrt{\beta^2 - 1}) \). We can now rewrite the inequality (32) as,

\[ \left| \frac{\partial^l \log(\beta + \sqrt{\beta^2 - 1})}{\partial \beta^l} \right| \leq \]

\[ \leq \left| \sigma \beta \frac{\partial^l \log(\beta + \sqrt{\beta^2 - 1})}{\partial \beta^l} \frac{1}{\beta - \beta^{-1}} \right| + \left| \sum_{i=1}^{\nu} \left( \nu \right) \frac{\partial^{l-1} \log(\beta + \sqrt{\beta^2 - 1})}{\partial \beta^{l-1}} \frac{1}{\beta - \beta^{-1}} \right| = |M(\beta)|, \]
using equation (22) we may write,

\[
\left| \frac{\partial^\nu \log(\beta + \sqrt{\beta^2 - 1})}{\partial \beta^\nu} \right| \leq \sigma \beta \frac{\nu!}{2} \left[ \frac{1}{(\beta + 1)^{\nu+1}} + \frac{1}{(\beta - 1)^{\nu+1}} \right] + \\
+ \sum_{l=1}^{\nu} \left( \nu \right) \left( \frac{l - 1)!}{2} \left[ \frac{1}{(\beta + 1)^l} + \frac{1}{(\beta - 1)^l} \right] \frac{(\nu - l)!}{2} \left[ \frac{1}{(\beta + 1)^{\nu-l+1}} + \frac{1}{(\beta - 1)^{\nu-l+1}} \right] = \\
= \sigma \beta \frac{\nu!}{2} \left[ \frac{1}{(\beta + 1)^{\nu+1}} + \frac{1}{(\beta - 1)^{\nu+1}} \right] + \\
+ \sum_{l=1}^{\nu} \frac{\nu!}{4l} \left[ \frac{1}{(\beta + 1)^{\nu+1}} + \frac{1}{(\beta - 1)^{\nu+1}} + \frac{1}{(\beta + 1)^l} + \frac{1}{(\beta - 1)^l} \right] < \\
< \sigma \beta \frac{\nu!}{(\beta - 1)^{\nu+1}} + \sum_{l=1}^{\nu} \frac{\nu!}{(\beta - 1)^{\nu+1}} = \frac{\nu!(\nu + \sigma \beta)}{(\beta - 1)^{\nu+1}}
\]

In this sense, we can assert that,

\[
\Im \left\{ -\frac{1}{i^\nu} \frac{\mathcal{W}}{T \ell^2} \right\} < \left| \frac{\kappa \nu!}{(2\kappa^2)^\nu} \sum_{n=1}^{\infty} \int \int \frac{dp_1 dp_2}{(2\pi)^3} \int_0^\beta d\beta \frac{(\nu + \sigma \beta)}{(\beta - 1)^{\nu+1}} \right|,
\]

or, considering \( \nu > 1 \),

\[
\Im \left\{ -\frac{1}{i^\nu} \frac{\mathcal{W}}{T \ell^2} \right\} < \left| \frac{\kappa \nu!}{(2\kappa^2)^\nu} \sum_{n=1}^{\infty} \int \int \frac{dp_1 dp_2}{(2\pi)^3} \left[ \frac{\sigma}{(\nu - 1)(\nu - 2)^{\nu+1}} + \frac{\sigma + \nu}{\nu(\beta - 1)^{\nu}} \right] \right|
\]

The integrals over \( p_1 \) and \( p_2 \) using the expression of \( \beta(p_1, p_2, n, a) \) given in (27) was already done, then:

\[
\Im \left\{ -\frac{1}{i^\nu} \frac{\mathcal{W}}{T \ell^2} \right\} < \left| \frac{\kappa \nu!}{(2\kappa^2)^\nu} \sum_{n=1}^{\infty} \frac{1}{(2\pi)^2} \left\{ \frac{\sigma}{(\nu - 1)(\nu - 2)} \left\{ \frac{\kappa}{2\kappa^2} \left[ m^2 + \left( \frac{2\pi}{a} \right)^2 \right] \right\}^{\nu-2} + \\
\frac{\sigma + \nu}{\nu(\nu - 1)} \left\{ \frac{1}{2\kappa^2} \left[ m^2 + \left( \frac{2\pi}{a} \right)^2 \right] \right\}^{\nu-1} \right\}
\]

Since these sums converge if \( \nu > 5/2 \), the regularization also works to the imaginary part.

The intersection of all the possible values for \( \nu \) is given from the inequality \( \nu > 5/2 \), which shows that there exists \( \nu \) such that the effective action can be regularized by a power regularization introduced in (4).

Despite the power regularization has been proved to be valid to calculate the Schwinger’s effective action, if we want to calculate the Casimir energy from \( \mathcal{E} = -\mathcal{W}/T \) and from the non-regularized sum of zero modes given by (13), it is convenient, for practical purposes, to use another type of regularization. A simple one given by \( \exp\{ -\epsilon|\mathbf{k}| \} \) would be enough.
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