Relationships between the decoherence-free algebra and the set of fixed points

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Abstract

We show that, for a Quantum Markov Semigroup (QMS) with a faithful normal invariant state, the atomicity of the decoherence-free subalgebra and environmental decoherence are equivalent. Moreover, we characterize the set of reversible states and explicitly describe the relationship between the decoherence-free subalgebra and the fixed point subalgebra for QMSs with the above equivalent properties.

1 Introduction

Starting from the fundamental papers of Gorini-Kossakowski-Sudarshan \cite{19} and Lindblad \cite{23} the structure of uniformly continuous quantum Markov semigroups (QMS) $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$, or, in the physical terminology, quantum dynamical semigroups, and their generators, has been the object of significant attention.

The increasing interest in mathematical modelling of decoherence, coherent quantum computing and approach to equilibrium in open quantum systems continues to motivate investigation on special features of QMS. Special attention is paid to subalgebras or subspaces where irreversibility and dissipation disappear (see \cite{2, 8, 7, 13, 14, 22, 29} and the references therein). States leaving in such subspaces are promising candidates for storing and manipulating quantum information.

The decoherence-free subalgebra, where completely positive maps $\mathcal{T}_t$ of the semigroup act as automorphisms, and the set of fixed points, which is a subalgebra when there exists a faithful invariant state, also allow us to gain insight into the structure of a QMS, its invariant states and environment induced decoherence. Indeed, its structure as a von Neumann algebra, has important consequences on the structure and the action of the whole QMS. Recently, we showed in \cite{13} that, if the decoherence-free subalgebra of a uniformly continuous QMS is atomic, it induces a decomposition of the system into its noiseless and purely dissipative parts, determining the structure of invariant states, as well as decoherence-free subsystems and subspaces \cite{29}. In
particular, we provided a full description of invariant states extending known ones in the finite dimensional case [6].

In this paper we push further the analysis of uniformly continuous QMS with atomic decoherence-free subalgebra and a faithful invariant state proving a number new results we briefly list and outline below.

1. Environment induced decoherence ([7, 9, 11]) holds if and only if $N(T)$ is atomic. In this case the decoherence-free subalgebra is generated by the set of eigenvectors corresponding to modulus one eigenvalues of the completely positive maps $T_t$, namely, in an equivalent way, by the eigenvectors with purely imaginary eigenvalue of the generator (Theorem 12).

2. The decoherence-free subalgebra and the set of so-called reversible states, i.e. the linear space generated by eigenvectors corresponding to modulus $1$ eigenvalues of predual maps $T^*_t$ are in the natural duality of a von Neumann algebra with its predual (Theorem 16). Moreover, Theorems 18 and 20 explicitly describe the structure of reversible states.

3. We find a spectral characterization of the decomposition of the fixed point algebra (Theorem 24). Moreover, the exact relationship between $F(T)$ and $N(T)$ (Theorems 23 and 25) is established in an explicit and constructive way allowing one to find the structure of each one from the structure of the other.

Loosely speaking one can say that, for QMSs with a faithful invariant state, the same conclusions can be drawn replacing finite dimensionality of the system Hilbert space by atomicity of the decoherence-free subalgebra.

Counterexamples (Examples 10 and 17) show that, in general, the above conclusions may fail if for QMSs without faithful normal invariant states.

The above results, clarify then the relationships between the atomicity of the decoherence-free subalgebra, environmental decoherence, ergodic decomposition of the trace class operators, and the structure of fixed points.

In particular the first result implies that the decomposition induced by decoherence coincides with the Jacobs-de-Leeuw-Glickerberg (JDG) splitting. Such decomposition was originally introduced for weakly almost periodic semigroups and generalized to QMSs on von Neumann algebras in [24, 21] at all. It is among the most useful tools in the study of the asymptotic behavior of operators semigroups on Banach spaces or von Neumann algebras. Indeed, it provides a decomposition of the space into the direct sum of the space generated by eigenvectors of the semigroup associated with modulus $1$ eigenvalues, and the remaining space, called stable, consisting of all vectors whose orbits have $0$ as a weak cluster point. Under suitable conditions, we obtain the convergence to $0$ for each vector belonging to the stable space.

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On the other hand, the ergodic decomposition of trace class operators (which is a particular case of the JDS splitting applied to the predual of $T$), allows one, for instance, to determine reversible subsystems by spectral calculus. Determining reversible states, in particular, is an important task in the study of irreversible (Markovian) dynamics because these states retain their quantum features that are exploited in quantum computation (see [3, 29] and the references therein). More precisely, reversible (or rotating) and invariant states of a quantum channel (acting on $M_n(\mathbb{C})$ for some $n > 1$) allow to classify kinds of information that the process can preserve. When the space is finite-dimensional and there exists a faithful invariant state, the structure of these states can be easily found (see e.g. Lemma 6 and Section V in [8], and Theorems 6.12, 6.16 in [31]) through the decomposition of $\mathcal{N}(T)$ and the algebra of fixed points $\mathcal{F}(T)$ in “block diagonal matrices”, i.e. in their canonical form given by the structure theorem for matrix algebras (see Theorem 11.2 in [28]). Since the same decomposition holds for atomic von Neumann algebras, in this paper we generalize these results to uniformly continuous QMSs acting on $\mathcal{B}(h)$ with $h$ infinite-dimensional.

The paper is organized as follows. In Section 2 we collect some notation and known results on the structure of norm-continuous QMS with atomic decoherence-free subalgebra and the structure of their invariant states. In Section 3, after recalling some known results from [11] on the relationship between EID and Jacobs-de Leeuw-Glickeberg decomposition, we prove the main result of this paper: the equivalence between EID and atomicity of the decoherence free subalgebra. The predual algebra of the decoherence-free subalgebra is characterized in Section 4 as the set of reversible states. Finally, in Section 5 we study the structure of the set of fixed points of the semigroup and its relationships with the decomposition of $\mathcal{N}(T)$ when this algebra is atomic.

2 The structure of the decoherence-free algebra

Let $h$ be a complex separable Hilbert space and let $\mathcal{B}(h)$ the algebra of all bounded operators on $h$ with unit $\mathbb{1}$. A QMS on $\mathcal{B}(h)$ is a semigroup $\mathcal{T} = (T_t)_{t \geq 0}$ of completely positive, identity preserving normal maps which is also weakly* continuous. In this paper we assume $\mathcal{T}$ uniformly continuous i.e.

$$\lim_{t \to 0^+} \sup_{\|x\| \leq 1} \|T_t(x) - x\| = 0,$$

so that there exists a linear bounded operator $\mathcal{L}$ on $\mathcal{B}(h)$ such that $T_t = e^{t\mathcal{L}}$. The operator $\mathcal{L}$ is the generator of $\mathcal{T}$, and it can be represented in the well-known (see
Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form as

\[ \mathcal{L}(x) = i[H, x] - \frac{1}{2} \sum_{\ell \geq 1} (L_\ell^* L_\ell x - 2L_\ell^* x L_\ell + x L_\ell^* L_\ell), \]  

(1)

where \( H = H^* \) and \( (L_\ell)_{\ell \geq 1} \) are bounded operators on \( \mathfrak{h} \) such that the series \( \sum_{\ell \geq 1} L_\ell^* L_\ell \) is strongly convergent and \([\cdot, \cdot]\) denotes the commutator \( [x, y] = xy - yx \). The choice of operators \( H \) and \( (L_\ell)_{\ell \geq 1} \) is not unique, but this will not create any inconvenience in this paper. More precisely, we have the following characterization (see [26], Proposition 30.14 and the discussion below the proof of Theorem 30.16).

Theorem 1. Let \( \mathcal{L} \) be the generator of a uniformly continuous QMS on \( \mathcal{B}(\mathfrak{h}) \). Then there exist a bounded selfadjoint operator \( H \) and a sequence \( (L_\ell)_{\ell \geq 1} \) of elements in \( \mathcal{B}(\mathfrak{h}) \) such that:

1. \( \sum_{\ell \geq 1} L_\ell^* L_\ell \) is strongly convergent,
2. if \( \sum_{\ell \geq 0} |c_\ell|^2 < \infty \) and \( c_0 \mathbb{I} + \sum_{\ell \geq 1} c_\ell L_\ell = 0 \) for scalars \( (c_\ell)_{\ell \geq 0} \) then \( c_\ell = 0 \) for every \( \ell \geq 0 \),
3. \( \mathcal{L}(x) = i[H, x] - \frac{1}{2} \sum_{\ell \geq 1} (L_\ell^* L_\ell x - 2L_\ell^* x L_\ell + x L_\ell^* L_\ell) \) for all \( x \in \mathcal{B}(\mathfrak{h}). \)

We recall that, for an arbitrary von Neumann algebra \( \mathcal{M} \), its predual space \( \mathcal{M}_* \) is the space of \( w^* \)-continuous functionals on \( \mathcal{M} \) (said normal). It is a well-known fact that for all \( \omega \in \mathcal{M}_* \), there exists \( \rho \in \mathcal{I}(\mathfrak{h}) \), the space of trace-class operators, such that \( \omega(x) = \text{tr} \,(\rho x) \) for all \( x \in \mathcal{M} \). In particular, if \( \omega \) is a positive and \( ||\omega|| = 1 \), it is called state, and \( \rho \) is positive with \( \text{tr} \,(\rho) = 1 \), i.e. \( \rho \) is a density. If \( \mathcal{M} = \mathcal{B}(\mathfrak{h}) \), every normal state \( \omega \) has a unique density \( \rho \). Therefore, in this case, we can identify them. Finally, \( \rho \) is faithful if \( \text{tr} \,(\rho x) = 0 \) for a positive \( x \in \mathcal{B}(\mathfrak{h}) \) implies \( x = 0 \) (see [28], Definition 9.4).

Given a \( w^* \)-continuous operator \( S : \mathcal{M} \to \mathcal{M} \), we can define its predual map \( S_* : \mathcal{M}_* \to \mathcal{M}_* \) as \( S_*(\omega) = \omega \circ S \).

In particular, for \( \mathcal{M} = \mathcal{B}(\mathfrak{h}) \), by considering the predual map of every \( \mathcal{T}_t \), we obtain the predual semigroup \( \mathcal{T}_* = (\mathcal{T}_{*t})_t \) satisfying

\[ \text{tr} \,(\mathcal{T}_{*t}(\rho)x) = \text{tr} \,(\rho \mathcal{T}_{*t}(x)) \quad \forall \rho \in \mathcal{I}(\mathfrak{h}), \ x \in \mathcal{B}(\mathfrak{h}). \]

The decoherence-free (DF) subalgebra of \( \mathcal{T} \) is defined by

\[ \mathcal{N}(\mathcal{T}) = \{ x \in \mathcal{B}(\mathfrak{h}) : \mathcal{T}_t(x^* x) = \mathcal{T}_t(x)^* \mathcal{T}_t(x), \ \mathcal{T}_t(xx^*) = \mathcal{T}_t(x) \mathcal{T}_t(x)^* \ \forall t \geq 0 \}. \]  

(2)

It is a well known fact that \( \mathcal{N}(\mathcal{T}) \) is the biggest von Neumann subalgebra of \( \mathcal{B}(\mathfrak{h}) \) on which every \( \mathcal{T}_t \) acts as a \( \ast \)-homomorphism (see e.g. Evans [16] Theorem 3.1). Moreover, the following facts hold (see [14] Proposition 2.1).
Proposition 2. Let $\mathcal{T}$ be a QMS on $\mathcal{B}(\mathfrak{h})$ and let $\mathcal{N}(\mathcal{T})$ be the set defined by (2). Then

1. $\mathcal{N}(\mathcal{T})$ is invariant with respect to every $\mathcal{T}_t$,
2. $\mathcal{N}(\mathcal{T}) = \{\delta_H^{(n)}(L_k), \delta_H^{(n)}(L_k^*) : n \geq 0\}'$, where $\delta_H(x) := [H, x]$,
3. $\mathcal{T}_t(x) = e^{iH t} x e^{-iH t}$ for all $x \in \mathcal{N}(\mathcal{T})$, $t \geq 0$,
4. if $\mathcal{T}$ possesses a faithful normal invariant state, then $\mathcal{N}(\mathcal{T})$ contains the set of fixed points $\mathcal{F}(\mathcal{T}) = \{L_k, L_k^*, H : k \geq 1\}'$.

In addition, if the QMS is uniformly continuous, its action on $\mathcal{N}(\mathcal{T})$ is bijective.

Theorem 3. If $\mathcal{T}$ is a uniformly continuous QMS, then $\mathcal{N}(\mathcal{T})$ is the biggest von Neumann subalgebra on which every map $\mathcal{T}_t$ acts as a $\ast$-automorphism.

Proof. The restriction of every $\mathcal{T}_t$ to $\mathcal{N}(\mathcal{T})$ is clearly injective thanks to item 3 of Proposition 2. Now, given $x \in \mathcal{N}(\mathcal{T})$ and $t > 0$, we have to prove that $x = \mathcal{T}_t(y)$ for some $y \in \mathcal{N}(\mathcal{T})$.

First of all note that, since the QMS is norm continuous, it can be extended to norm continuous group $(\mathcal{T}_t)_{-\infty < t < +\infty}$ of normal maps on $\mathcal{B}(\mathfrak{h})$, and, by analyticity in $t$, $\mathcal{T}_{-t}(z) \in \mathcal{N}(\mathcal{T})$ for all $t > 0$ and $z \in \mathcal{N}(\mathcal{T})$, and formula $\mathcal{T}_{-t}(z) = e^{-itH} z e^{itH}$ holds. $\square$

As we said in the introduction, we will study the relationships between the structure of $\mathcal{N}(\mathcal{T})$ and other problems in the theory of uniformly continuous QMSs in which the atomicity of $\mathcal{N}(\mathcal{T})$ plays a key role.

First of all, as shown in [13], the structure of the decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ gives information on the whole QMS. Let us recall these results. Assume that $\mathcal{N}(\mathcal{T})$ is an atomic algebra, that is, there exists an (at most countable) family $(p_i)_{i \in I}$ of mutually orthogonal non-zero projections, which are minimal projections in the center of $\mathcal{N}(\mathcal{T})$, such that $\sum_{i \in I} p_i = 1$ and each von Neumann algebra $p_i \mathcal{N}(\mathcal{T}) p_i$ is a type I factor. In that case, the subalgebra $\mathcal{N}(\mathcal{T})$ can be decomposed as

$$\mathcal{N}(\mathcal{T}) = \oplus_{i \in I} p_i \mathcal{N}(\mathcal{T}) p_i.$$  \hspace{1cm} (3)

The properties of the projections $p_i$ imply their invariance under the semigroup and more generally the $\mathcal{T}_t$-invariance of each factor $p_i \mathcal{N}(\mathcal{T}) p_i$. Moreover, each $p_i \mathcal{N}(\mathcal{T}) p_i$ is a type I factor acting on the Hilbert space $p_i \mathfrak{h}$; thus, there exist two countable sequences of Hilbert spaces $(k_i)_{i \in I}$, $(m_i)_{i \in I}$, and unitary operators $U_i : p_i \mathfrak{h} \to k_i \otimes m_i$ such that

$$U_i p_i \mathcal{N}(\mathcal{T}) p_i U_i^* = \mathcal{B}(k_i) \otimes 1_{m_i}, \quad U_i p_i \mathcal{B}(\mathfrak{h}) p_i U_i^* = \mathcal{B}(k_i \otimes m_i).$$  \hspace{1cm} (4)
Therefore, defining \( U = \oplus_{i \in I} U_i \), we obtain a unitary operator \( U : \mathcal{H} \rightarrow \oplus_{i \in I} (k_i \otimes m_i) \) such that

\[
UN(T)U^* = \oplus_{i \in I} (\mathcal{B}(k_i) \otimes \mathbb{1}_{m_i}).
\]

(5)

As a consequence, we find the following result.

**Theorem 4.** \( \mathcal{N}(T) \) is an atomic algebra if and only if there exist two countable sequences of separable Hilbert spaces \((k_i)_{i \in I}, (m_i)_{i \in I}\) such that \( h = \oplus_{i \in I} (k_i \otimes m_i) \) (up to a unitary operator) and \( \mathcal{N}(T) = \oplus_{i \in I} (\mathcal{B}(k_i) \otimes \mathbb{1}_{m_i}) \) (up to an isometric isomorphism).

In this case:

1. for every GSKL representation of \( \mathcal{L} \) by means of operators \( H, (L_i)_{i \geq 1} \), we have

\[
L_i = \oplus_{i \in I} \left( \mathbb{1}_{k_i} \otimes M^{(i)}_{k_i} \right)
\]

for a collection \( (M^{(i)}_{k_i})_{i \geq 1} \) of operators in \( \mathcal{B}(m_i) \), such that the series \( \sum_{i \geq 1} M^{(i)}_{k_i} M^{(i)}_{k_i} \) strongly convergent for all \( i \in I \), and

\[
H = \oplus_{i \in I} \left( K_i \otimes \mathbb{1}_{m_i} + \mathbb{1}_{k_i} \otimes M^{(i)}_{0} \right)
\]

for self-adjoint operators \( K_i \in \mathcal{B}(k_i) \) and \( M^{(i)}_{0} \in \mathcal{B}(m_i), i \in I \),

2. defining on \( \mathcal{B}(m_i) \) the GKSL generator \( \mathcal{L}^{m_i} \) associated with operators \( \{M^{(i)}_{0}, M^{(i)}_{k_i}: \ell^{(i)} \geq 1\} \), we have

\[
T_i(x_i \otimes y_i) = e^{itK_i}x_i e^{-itK_i} \otimes T^{m_i}(y_i)
\]

for all \( t \geq 0, x_i \in \mathcal{B}(k_i) \) and \( y_i \in \mathcal{B}(m_i) \), where \( T^{m_i} \) is the QMS generated by \( \mathcal{L}^{m_i} \),

3. if there exists a faithful normal invariant state, then the QMS \( T^{m_i} \) is irreducible, possesses a unique invariant state \( \tau_m \), which is also faithful, and we have \( \mathcal{F}(T^{m_i}) = \mathcal{N}(T^{m_i}) = \mathbb{C}\mathbb{1}_{m_i} \). Moreover, for all \( i \in I \), \( K_i \) has pure point spectrum.

**Proof.** The proof of the necessary condition is given in Theorems 3.2 and 4.1, Proposition 4.3 and Lemma 4.2 in [13]. Conversely, given two countable sequences of Hilbert spaces \((k_i)_{i}, (m_i)_{i}\), such that \( h = \oplus_{i \in I} (k_i \otimes m_i) \) up to a unitary operator, and \( \mathcal{N}(T) = \oplus_{i \in I} (\mathcal{B}(k_i) \otimes \mathbb{1}_{m_i}) \) (up to the corresponding isometric isomorphism), set \( p_i \) the orthogonal projection onto \( k_i \otimes m_i \). Then we obtain a family \( (p_i)_{i \in I} \) of mutually orthogonal non-zero projections, which are minimal projections in the center of \( \mathcal{N}(T) \), such that \( \sum_{i \in I} p_i = \mathbb{1} \) and each von Neumann algebra \( p_i \mathcal{N}(T) p_i = \mathcal{B}(k_i) \otimes \mathbb{1}_{m_i} \) is a type I factor.

\( \square \)
Now, we recall a characterization of the atomicity of a von Neumann algebra in terms of the existence of a normal conditional expectation, i.e. a weakly*-continuous norm one projection (see [30] Theorem 5).

To this end we will use that, given \( x \in h \) and \( k \) complex separable Hilbert spaces and \( \sigma \) a normal state on \( k \), there always exists (see e.g. Exercise 16.10 in [26]) a normal completely positive linear map \( E_\sigma : B(h \otimes k) \to B(h) \) satisfying

\[
\text{tr}(E_\sigma(X)\eta) = \text{tr}(X(\eta \otimes \sigma)) \quad \forall X \in B(h \otimes k), \ \eta \in \mathcal{I}(h).
\]

Since \( E_\sigma \) is positive and \( E_\sigma \circ E_\sigma = E_\sigma \), it is a normal conditional expectation called \( \sigma \)-conditional expectation.

**Theorem 5** (Tomiyama). Let \( \mathcal{M} \) be a von Neumann algebra acting on the Hilbert space \( h \). Then \( \mathcal{M} \) is atomic if and only if \( \mathcal{M} \) is the image of a normal conditional expectation \( E : B(h) \to \mathcal{M} \).

**Proof.** If there exists a normal conditional expectation \( E : B(h) \to \mathcal{M} \) onto \( \mathcal{M} \), then \( \mathcal{M} \) is atomic by Proposition 3 and Lemma 5 in [20], being \( B(h) \) atomic and semifinite.

On the other hand, if \( \mathcal{M} \) is atomic, let \( (p_i) \) be a sequence of orthogonal projections in the center of \( \mathcal{M} \) such that \( p_i \mathcal{M} p_i \) is a type I factor; moreover, assume \( p_i \mathcal{M} p_i \simeq B(k_i) \otimes 1 \mathbb{L} m_i \) with \( (k_i) \) and \( (m_i) \) be two sequences of complex and separable Hilbert spaces such that \( h \simeq \oplus_i (k_i \otimes m_i) \). Set \( (\sigma_i)_i \) a sequence of normal states on \( \mathcal{M} \) such that \( \sigma_i(x) = \mathcal{E}_{\sigma_i}(x) \otimes 1 \mathbb{L} m_i \) for all \( x \in B(k_i \otimes m_i) \).

\[
E(x) := \sum_i \pi(x_{ii}) \quad \text{for } x = (x_{ij})_{ij} \in B(\oplus_i (k_i \otimes m_i)) \simeq B(h)
\]
gives a normal conditional expectation onto \( \mathcal{M} \). \( \square \)

**Corollary 6.** Let \( \mathcal{M} \) be an atomic von Neumann algebra acting on \( h \) and let \( \mathcal{N} \subseteq \mathcal{M} \) be a von Neumann subalgebra. If there exists a normal conditional expectation \( E : \mathcal{M} \to \mathcal{N} \) onto \( \mathcal{N} \), then \( \mathcal{N} \) is atomic.

**Proof.** By Tomiyama’s Theorem we know that, since \( \mathcal{M} \) is atomic, it is the image of a normal conditional expectation \( \mathcal{F} : B(h) \to \mathcal{M} \) (see also the proof of Theorem 5 in [30]). Therefore, the map \( \mathcal{E} \circ \mathcal{F} : B(h) \to \mathcal{N} \) is a normal conditional expectation onto \( \mathcal{N} \). Indeed, since \( \mathcal{N} = \text{Ran } \mathcal{E} \) is contained in \( \mathcal{M} = \text{Ran } \mathcal{F} \), for \( x \in B(h) \) we have

\[
(\mathcal{E} \circ \mathcal{F})(\mathcal{E} \circ \mathcal{F})x = \mathcal{E}^2(\mathcal{F}(x))) = (\mathcal{E} \circ \mathcal{F})(x),
\]
i.e. \( \mathcal{E} \circ \mathcal{F} \) is a projection. Therefore, \( \|\mathcal{E} \circ \mathcal{F}\| \geq 1 \). On the other hand, since \( \mathcal{E} \) and \( \mathcal{F} \) are norm one operators, we clearly obtain \( \|\mathcal{E} \circ \mathcal{F}\| = 1 \). The normality of \( \mathcal{E} \circ \mathcal{F} \) is evident, and so we can conclude that the algebra \( \mathcal{N} \) is atomic by Tomiyama Theorem. \( \square \)
Remark 7. Theorem 5 is a simplified version of Theorem 5 in [30], given in terms of atomicity of the subalgebra \( \mathcal{M} \). Moreover, Corollary 4 generalizes to atomic algebras one implication of the same theorem. In particular, we give easier proofs of these results.

In the following we assume the existence of a faithful normal invariant state; note that, in general, this condition is not necessary for the decoherence-free subalgebra to be atomic. This is always the case for any QMS acting on a finite dimensional algebra. However, we show the following example that will be useful later.

Example 8. Let \( h = \mathbb{C}^3 \) with the canonical orthonormal basis \((e_i)_{i=1,2,3}\) and \( \mathcal{B}(h) = M_3(\mathbb{C}) \). We consider the operator \( \mathcal{L} \) on \( M_3(\mathbb{C}) \) given by

\[
\mathcal{L}(x) = i \omega \left( |e_1\rangle\langle e_1|, x - \frac{1}{2} (|e_3\rangle\langle e_3|x - 2|e_2\rangle\langle e_2|x\langle e_2| + x|e_3\rangle\langle e_3|) \right)
\]

for all \( x \in M_3(\mathbb{C}) \), with \( \omega \in \mathbb{R} \), \( \omega \neq 0 \). Clearly \( \mathcal{L} \) is written in the GKSL form with \( H = \omega|e_1\rangle\langle e_1| \) and \( L = |e_2\rangle\langle e_3| \), and so it generates a uniformly continuous QMS \( T = (T_t)_{t \geq 0} \) on \( M_3(\mathbb{C}) \).

An easy computation shows that any invariant functional has the form

\[
a|e_1\rangle\langle e_1| + b|e_2\rangle\langle e_2|
\]

for some \( a, b \in \mathbb{C} \), and so the semigroup has no faithful invariant states. Since \([H,L] = 0\), by item 2 of Proposition 4 we have \( \mathcal{N}(T) = \{ L, L^* \}' \), so that an element \( x \in \mathcal{B}(h) \) belongs to \( \mathcal{N}(T) \) if and only if

\[
\begin{align*}
|xe_2\rangle\langle e_3| &= |e_2\rangle\langle x^*e_3|, \quad \text{i.e.} \quad xe_2 = x_{33}e_2 \\
|xe_3\rangle\langle e_2| &= |e_3\rangle\langle x^*e_2|,
\end{align*}
\]

\[
x_{31} = x_{21} = 0
\]

Therefore we get

\[
\mathcal{N}(T) = \{ x_{11}|e_1\rangle\langle e_1| + x_{22} (|e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|) \mid x_{11}, x_{22} \in \mathbb{C} \},
\]

i.e. \( \mathcal{N}(T) \) is isometrically isomorphic to the atomic algebra \( \mathbb{C} \oplus \mathbb{C}p \), where \( p \) denotes the identity matrix in \( M_2(\mathbb{C}) \).

3 Atomicity of \( \mathcal{N}(T) \) and decoherence

In this section, we explore the relationships between the atomicity of \( \mathcal{N}(T) \) and the property of environmental decoherence, under the assumption of the existence of a faithful normal invariant state \( \rho \). Following [11], we say that there is environment induced decoherence (EID) on the open system described by \( T \) if there exists a \( T_t \)-invariant and \(*\)-invariant weakly* closed subspace \( \mathcal{M}_2 \) of \( \mathcal{B}(h) \) such that:
(EID1) $\mathcal{B}(h) = \mathcal{N}(\mathcal{T}) \oplus \mathcal{M}_2$ with $\mathcal{M}_2 \neq \{0\}$.

(EID2) $w^* - \lim_{t \to \infty} T_t(x) = 0$ for all $x \in \mathcal{M}_2$.

Unfortunately, if EID holds and $h$ is infinite-dimensional, it is not clear if the space $\mathcal{M}_2$ is uniquely determined. However, $\mathcal{M}_2$ is always contained in the $\mathcal{T}$-invariant and $*$-invariant closed subspace

$$
\mathcal{M}_0 = \left\{ x \in \mathcal{B}(h) : w^* - \lim_{t \to \infty} T_t(x) = 0 \right\}.
$$

In [13] we showed that, if $\mathcal{N}(\mathcal{T})$ is atomic, then EID holds (see Theorem 5.1) and, in particular, $\mathcal{N}(\mathcal{T})$ is the image of a normal conditional expectation $\mathcal{E} : \mathcal{B}(h) \to \mathcal{N}(\mathcal{T})$ compatible with the faithful state $\rho$ (i.e. $\rho \circ \mathcal{E} = \rho$) and such that

$$
\ker \mathcal{E} = \mathcal{M}_2 = \left\{ x \in \mathcal{B}(h) : \text{tr}(\rho xy) = 0 \ \forall y \in \mathcal{N}(\mathcal{T}) \right\}.
$$

(7) (see Theorem 19 in [11]). In the following we will show that, if $\mathcal{N}(\mathcal{T})$ is atomic, the decomposition unique, i.e. the only way to realize it, is taking $\mathcal{M}_2$ given by (7) (see Theorem 12 and Remark 13.2).

Moreover, in this case, we will study the relationships of such a decomposition with another famous asymptotic splitting of $\mathcal{B}(h)$, called the Jacobs-de Leeuw-Glickberg splitting: this comparison is very natural since the decomposition $\mathcal{B}(h) = \mathcal{N}(\mathcal{T}) \oplus \mathcal{M}_2$ is clearly related too to the asymptotic properties of the semigroup.

We recall that, since there exists $\rho$ faithful invariant, the Jacobs-de Leeuw-Glickberg splitting holds (see e.g. Corollary 3.3 and Proposition 3.3 in [21]) and is given by $\mathcal{B}(h) = \mathcal{M}_r \oplus \mathcal{M}_s$ with

$$
\mathcal{M}_r := \text{span}^{w^*} \left\{ x \in \mathcal{B}(h) : T_t(x) = e^{it\lambda} x \text{ for some } \lambda \in \mathbb{R}, \ \forall t \geq 0 \right\}
$$

(8)

$$
\mathcal{M}_s := \left\{ x \in \mathcal{B}(h) : 0 \in \left\{ T_t(x) \right\}_{t \geq 0}^{w^*} \right\}.
$$

(9)

Moreover, in this case $\mathcal{M}_r$ is a von Neumann algebra.

The relationship between the decomposition induced by decoherence and the Jacobs-de Leeuw-Glickberg splitting is given by the following result (see Proposition 31 in [11]).

**Proposition 9.** If there exists a faithful normal invariant state $\rho$, then the following conditions are equivalent:

1. EID holds with $\mathcal{M}_2 = \mathcal{M}_0$ and the induced decomposition coincides with the Jacobs-de Leeuw-Glickberg splitting,
2. $\mathcal{N}(\mathcal{T}) \cap \mathcal{M}_s = \{0\},$
3. $\mathcal{N}(\mathcal{T}) = \mathcal{M}_r.$
Moreover, if one of the previous conditions holds, then \( \mathcal{N}(\mathcal{T}) \) is the image of a normal conditional expectation \( \mathcal{E} \) compatible with \( \rho \) and such that \( \text{Ker} \, \mathcal{E} = M_0 = \mathcal{M}_s \).

Clearly, if \( \mathcal{M}_r \) is not an algebra, it does not make sense to pose the problem to understand if it coincides with \( \mathcal{N}(\mathcal{T}) \). In particular, this could happen when \( \mathcal{T} \) has no faithful invariant states, as the following example shows.

**Example 10.** Let us consider the uniformly continuous QMS \( \mathcal{T} \) on \( M_3(\mathbb{C}) \) defined in Example \[. We have already seen that \( \mathcal{T} \) does not posses faithful invariant states, and

\[
\mathcal{N}(\mathcal{T}) = \{ x_{11}|e_1\rangle\langle e_1| + x_{22}(|e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|) \mid x_{11}, x_{22} \in \mathbb{C} \}.
\]

We want now to find the space \( \mathcal{M}_r \), generated by eigenvectors of \( \mathcal{L} \) corresponding to purely imaginary eigenvalues. Easy computations show that we have \( \mathcal{L}(x) = i\lambda x \) for some \( \lambda \in \mathbb{R} \) if and only if

\[
i\lambda \sum_{i,j=1}^{3} x_{ij}|e_i\rangle\langle e_j| = i \omega \sum_{j=1}^{3} (x_{1j}|e_1\rangle\langle e_j| - x_{j1}|e_j\rangle\langle e_1|)
- \frac{1}{2} \left( \sum_{j=1}^{3} x_{3j}|e_3\rangle\langle e_j| - 2x_{22}|e_3\rangle\langle e_3| + \sum_{i=1}^{3} x_{i3}|e_i\rangle\langle e_3| \right),
\]

e.g., in the case when \( \lambda = 0 \), \( x_{ij} = 0 \) for \( i \neq j \) and \( x_{22} = x_{33} \), and, in the case \( \lambda \neq 0 \), if and only if the following identities hold

\[
x_{11} = 0, \quad x_{22} = 0, \quad x_{12}(\omega - \lambda) = 0,
x_{13} \left( -\frac{1}{2} + i(\omega - \lambda) \right) = 0, \quad x_{21}(\omega + \lambda) = 0, \quad x_{23}(\frac{1}{2} + i\lambda) = 0,
x_{31} \left( \frac{1}{2} + i(\omega + \lambda) \right) = 0, \quad x_{32}(\frac{1}{2} + i\lambda) = 0, \quad x_{33} (1 + i\lambda) = x_{22}.
\]

Since \( \omega \) and \( \lambda \) belong to \( \mathbb{R} \) this is equivalent to have either \( x = x_{12}|e_1\rangle\langle e_2| \) and \( \lambda = \omega \), or \( x = x_{21}|e_2\rangle\langle e_1| \) and \( \lambda = -\omega \). Therefore, we can conclude that

\[
\mathcal{M}_r = \left\{ \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & x_{22} \end{pmatrix} : x_{11}, x_{22}, x_{12}, x_{21} \in \mathbb{C} \right\}.
\]

In particular, \( \mathcal{M}_r \) is not an algebra and it is strictly bigger that \( \mathcal{N}(\mathcal{T}) \).

**Remarks 11.** 1. Note that if \( \mathcal{M}_r \) is contained in \( \mathcal{N}(\mathcal{T}) \), then it is a *-algebra. Indeed, if \( \mathcal{M}_r \subseteq \mathcal{N}(\mathcal{T}) \), taken \( x, y \in \mathcal{M}_r \) such that \( \mathcal{T}_t(x) = e^{i\lambda t}x \) and \( \mathcal{T}_t(y) = e^{i\mu t}y \) for some \( \lambda, \mu \in \mathbb{R} \) and any \( t \), by property 3 in Proposition \[ we have

\[
\mathcal{T}_t(x^{*}y) = \mathcal{T}_t(x)^{*}\mathcal{T}_t(y) = e^{i(\mu - \lambda)t}x^{*}y \quad \forall \ t \geq 0.
\]
As a consequence $x^*y$ belongs to $\mathfrak{M}_r$.

2. If $h$ is finite-dimensional, then also the opposite implication is true. Indeed, if $\mathfrak{M}_r$ is a $*$-algebra, given $x \in B(h)$ such that $T_t(x) = e^{it\lambda}x$, $\lambda \in \mathbb{R}$, we have $T_t(x^*)T_t(x) = x^*x$. Then, by the Schwarz inequality, $T_t(x^*x) \geq x^*x$ for all $t \geq 0$. Set $T'_t := T_{t|_{\mathfrak{M}_r}}$. Since in this case $T$ is a strongly continuous semigroup, by definition of $\mathfrak{M}_r$ and by Corollary 2.9, Chapter V, of [15], the strong operator closure of $\{T'_t : t \geq 0\}$ is a compact topological group of operators in $B(\mathfrak{M}_r)$. Hence, $(T'_t)^{-1}$ is the limit of some net $(T'_{t_\alpha})$ and so $(T'_t)^{-1}$ is a positive operator. Since $x^*x \in \mathfrak{M}_r$, for all $\alpha$ we have $T'_{t_\alpha}(x^*x) \geq x^*x$, and so $(T'_t)^{-1}(x^*x) \geq x^*x$. On the other hand,

$$(T'_t)^{-1}(T'_t(x^*x)) = x^*x \geq (T'_t)^{-1}(x^*x).$$

Therefore, $(T'_t)^{-1}(x^*x) = x^*x$ and this implies $T_t(x^*x) = x^*x = T_t(x)^*T_t(x)$. Similarly we can prove the equality $T_t(xx^*) = xx^* = T_t(x)T_t(x)^*$, and so $x$ belongs to $\mathcal{N}(T)$.

Now we are able to prove one of the central results of this paper.

**Theorem 12.** Assume that there exists a faithful normal invariant state $\rho$. Then $\mathcal{N}(T)$ is atomic if and only if EID holds with $\mathcal{N}(T) = \mathfrak{M}_r$ and $\mathcal{M}_2 = \mathcal{M}_0$.

**Proof.** If $\mathcal{N}(T)$ is atomic, then EID holds by Theorem 5.1 in [13]. It remains to prove that $\mathcal{N}(T) = \mathfrak{M}_r$ and $\mathcal{M}_2 = \mathcal{M}_0$. The atomicity implies $\mathcal{N}(T) = \oplus_{i \in I} (B(k_i) \otimes \mathbb{1}_{m_i})$ up to a unitary isomorphism. Let $x = \sum_{i \in I} (x_i \otimes \mathbb{1}_{m_i})$ be in $\mathcal{N}(T) \cap \mathfrak{M}_s$, with $x_i \in B(k_i)$ for every $i \in I$, and assume $w^* - \lim_{\alpha} T_{t_\alpha}(x) = 0$. Given $u_i, v_i \in k_i$ and $\tau_i$ an arbitrary state on $m_i$, by Theorem 4

$$\text{tr}\left((|u_i\rangle \langle v_i| \otimes \tau_i)T_{t_\alpha}(x)\right) = \langle v_i, e^{it_\alpha K_i}x ie^{-it_\alpha K_i}u_i\rangle. \quad (10)$$

Choosing $u_i$ and $v_i$ such that $K_iu_i = \lambda_i u_i$ and $K_iv_i = \mu_i v_i$, $\lambda_i, \mu_i \in \mathbb{R}$, equation (10) becomes

$$\text{tr}\left((|u_i\rangle \langle v_i| \otimes \tau_i)T_{t_\alpha}(x)\right) = e^{it_\alpha (\mu_i - \lambda_i)} \langle v_i, x_i u_i\rangle,$$

so that $\langle v_i, x_i u_i\rangle = 0$, i.e. $x_i = 0$ because the eigenvectors of $K_i$ from an orthonormal basis of $k_i$ (see item 3 in Theorem 4). This proves the equality $\mathcal{N}(T) \cap \mathfrak{M}_s = \{0\}$. So we can conclude thanks to item 3 of Proposition 4.

Conversely, if EID holds with $\mathcal{N}(T) = \mathfrak{M}_r$ and $\mathcal{M}_2 = \mathcal{M}_0$, by Proposition 4 there exists a normal conditional expectation $E : B(h) \to \mathcal{N}(T)$ onto $\mathcal{N}(T)$, compatible to $\rho$. Therefore, $\mathcal{N}(T)$ is atomic thanks to Theorem 5. \hfill \Box

**Remarks 13.** As a consequence of Theorem 12 and Proposition 4 the following facts hold:

1. $\mathcal{N}(T)$ is atomic if and only if $\mathcal{N}(T) \cap \mathfrak{M}_s = \{0\}$, if and only if $\mathcal{N}(T) = \mathfrak{M}_r$, i.e $\mathcal{N}(T)$ is generated by eigenvectors of $\mathcal{L}$ corresponding to purely imaginary
eigenvalues.
Moreover, in this case we also have $\mathcal{N}(\mathcal{T}) \cap \mathcal{M}_0 = \{0\}$, being $\mathcal{M}_0 \subseteq \mathfrak{M}$: this means that, assuming $\mathcal{N}(\mathcal{T})$ atomic and the existence of a faithful invariant state, the situation is similar to the finite-dimensional case, i.e $\mathcal{N}(\mathcal{T})$ does not contain operators going to 0 under the action of the semigroup.

2. If $\mathcal{N}(\mathcal{T})$ is atomic and $\mathcal{F}(\mathcal{T}) = \mathbb{C} \mathbb{1}$, the semigroup satisfies the following properties given by non-commutative Perron-Frobenius Theorem (see e.g. Propositions 6.1 and 6.2 in [3], Theorem 2.5 in [4]):

- the peripheral point spectrum $\sigma_p(\mathcal{T}_t) \cap \mathbb{T}$ of each $\mathcal{T}_t$ is a subgroup of the circle group $\mathbb{T}$,
- given $t \geq 0$, each peripheral eigenvalue $\alpha$ of $\mathcal{T}_t$ is simple and we have $\sigma_p(\mathcal{T}_t) \cap \mathbb{T} = \alpha(\sigma_p(\mathcal{T}_t) \cap \mathbb{T})$,
- the restriction of $\rho$ to $\mathcal{N}(\mathcal{T})$ is a trace.

As a consequence, the peripheral point spectrum of each $\mathcal{T}_t$ is the cyclic group of all $h$-roots of unit for some $h \in \mathbb{N}$.

3. If $\mathcal{N}(\mathcal{T})$ is atomic, the decomposition induced by decoherence is uniquely determined. This fact follows from Proposition 5 in [11], since we have $\mathcal{N}(\mathcal{T}) \cap \mathfrak{M}_0 = \{0\}$.

4. Note that Theorem 12 does not exclude the possibility to have a QMS $\mathcal{T}$ displaying decoherence with $\mathcal{N}(\mathcal{T})$ a non-atomic type I algebra. Clearly, in this case, we will get $\mathcal{N}(\mathcal{T}) \supseteq \mathfrak{M}_r$ or $\mathcal{M}_2 \subsetneq \mathcal{M}_0$.

Remark 14. In [25] the authors prove that EID holds when the semigroup commutes with the modular group associated with a faithful normal invariant state. However, our result in Theorem 12 is stronger since we find the equivalence between EID and the atomicity of $\mathcal{N}(\mathcal{T})$, which is a weaker assumption of the commutation with the modular group. In fact, it can be shown (see [27], section 3) that commutation with the modular group implies atomicity of $\mathcal{N}(\mathcal{T})$. Moreover, it is not difficult to find an example of a QMS on $\mathcal{B}(\mathfrak{h})$, with $\mathfrak{h}$ finite dimensional which does not commute with the modular group. Its decoherence-free subalgebra, as any finite dimensional von Neumann algebra, will be atomic.

4 Structure of reversible states

In this section, assuming $\mathcal{N}(\mathcal{T})$ atomic and the existence of a faithful invariant state $\rho$, we study the structure of reversible states, i.e. states belonging to the vector space

$$\mathcal{R}(\mathcal{T}_t) : = \overline{\text{span}}\{\sigma \in \mathcal{I}(\mathfrak{h}) : \mathcal{T}_t(\sigma) = e^{it\lambda} \sigma \text{ for some } \lambda \in \mathbb{R}, \forall t \geq 0\} \quad (11)$$

$$= \overline{\text{span}}\{\sigma \in \mathcal{I}(\mathfrak{h}) : \mathcal{L}_t(\sigma) = i\lambda \sigma \text{ for some } \lambda \in \mathbb{R}\}. \quad (12)$$
In particular we will prove that $\mathcal{R}(T_s)$ is the predual of the decoherence-free algebra $\mathcal{N}(T)$.

To this end, we recall the following result which is a version of the Jacobi-De Leeuw-Glicksberg theorem for strongly continuous semigroup (see Propositions 3.1, 3.2 in [24] and Theorem 2.8 in [15]).

\textbf{Theorem 15.} If there exists a normal density $\rho \in \mathcal{H}$ satisfying
\begin{equation}
\text{tr} \left( \rho (T_t(x)^* T_t(x)) \right) \leq \text{tr} \left( x^* x \right) \quad \forall \ x \in \mathcal{B}(h), \ t \geq 0,
\end{equation}
then we can decompose $\mathcal{H}$ as
\begin{equation}
\mathcal{H} = \mathcal{R}(T_s) \oplus \{ \sigma \in \mathcal{H} : 0 \in \{ T_t(\sigma) \}_{t \geq 0} \}. \tag{14}
\end{equation}

Since each faithful invariant state clearly fulfills (13), we obtain the splitting given by equation (14).

On the other hand, denoting by $\perp A$ the vector space $\{ \sigma \in \mathcal{H} : \text{tr}(\sigma x) = 0 \ \forall \ x \in A \}$ for all subset $A$ of $\mathcal{B}(h)$, the atomicity of $\mathcal{N}(T)$ ensures the following facts:

(F1). $\mathcal{B}(h) = \mathcal{N}(T) \oplus \mathcal{M}_0$ with $\mathcal{N}(T) = \mathcal{M}^*, \mathcal{M}_0 = \mathcal{R} \mathcal{E}$ and $\mathcal{M}_0 = \mathcal{M}^* = \text{Ker} \mathcal{E}$, where $\mathcal{E} : \mathcal{B}(h) \to \mathcal{N}(T)$ is a conditional expectation compatible with the faithful state $\rho$ (see Theorem 12 and Proposition 9);

(F2). $\mathcal{H} = \perp \mathcal{M}_0 \oplus \perp \mathcal{N}(T)$ with
\begin{align*}
\perp \mathcal{M}_0 &= \text{Ran} \mathcal{E}_s \simeq \mathcal{N}(T)_s, \quad \perp \mathcal{N}(T) = \text{Ker} \mathcal{E}_s \simeq \mathcal{M}_{2s}.
\end{align*}

Moreover each $T_{st}$ acts as a surjective isometry on $\perp \mathcal{M}_0$, and $\lim_t T_{st}(\sigma) = 0$ for all $\sigma \in \perp \mathcal{N}(T)$ (see Theorem 10 in [14]).

As a consequence, every state $\omega \in \mathcal{N}(T)_s$ is represented by a unique density $\sigma$ in $\perp \mathcal{M}_0$, and, in this case, we write $\omega = \omega_\sigma$ to mean that $\omega(x) = \text{tr}(\sigma x)$ for all $x \in \mathcal{N}(T)$. Therefore, if we denote by $S = (S_t)_{t \geq 0}$ the restriction of $T$ to $\mathcal{N}(T)$, we have
\begin{equation}
(S_t \omega_\sigma)(x) = \omega_\sigma(T_t(x)) = \text{tr} \left( \sigma e^{itH} x e^{-itH} \right) = \text{tr} \left( \mathcal{E}_s(e^{-itH} \sigma e^{itH}) x \right)
\end{equation}
for all $x = \mathcal{E}(x) \in \mathcal{N}(T)$, concluding that $S_t \omega_\sigma$ is represented by the density $\mathcal{E}_s(e^{-itH} \sigma e^{itH}) \in \perp \mathcal{M}_0$. In an equivalent way, we have
\begin{equation}
T_{st}(\sigma) = \mathcal{E}_s(e^{-itH} \sigma e^{itH}) \quad \forall \ \sigma \in \perp \mathcal{M}_0. \tag{15}
\end{equation}

\textbf{Theorem 16.} If $\mathcal{N}(T)$ is atomic and there exists a faithful invariant state, then
\begin{equation*}
\mathcal{R}(T_s) = \perp \mathcal{M}_0 = \{ \sigma \in \mathcal{H} : T_{st}(\sigma) = \mathcal{E}_s(e^{-itH} \sigma e^{itH}) \ \forall \ t \geq 0 \} \simeq \mathcal{N}(T)_s,
\end{equation*}
for every Hamiltonian $H$ in a GKSL representation of the generator of $T$.\textsuperscript{13}
Proof. The inclusion $\perp M_0 \subseteq \{ \sigma \in \mathcal{J}(h) : \mathcal{T}_{st}(\sigma) = \mathcal{E}_s(e^{-itH}e^{itH}) \quad \forall \ t \geq 0 \}$ follows from the previous discussion. On the other hand, if we have $\mathcal{T}_{st}(\sigma) = \mathcal{E}_s(e^{-itH}e^{itH})$ for all $t \geq 0$, taking $t = 0$ we get $\sigma = \mathcal{E}_s(\sigma)$, i.e. $\sigma$ belongs to $\perp M_0$.

Now, given $\sigma \in \mathcal{R}(\mathcal{T}_s)$ such that $\mathcal{T}_{st}(\sigma) = e^{it\lambda} \sigma$ for all $t \geq 0$, $\lambda \in \mathbb{R}$, we have

$$\text{tr} (\sigma x) = \lim_{t \to \infty} \text{tr} (\sigma x) = \lim_{t \to \infty} \text{tr} (\mathcal{T}_{st}(\sigma)e^{-it\lambda} x) = \lim_{t \to \infty} e^{-it\lambda} \text{tr} (\sigma \mathcal{T}_t(x)) = 0$$

for all $x \in M_0$, so that $\sigma$ belongs to $\perp M_0$. This proves that $\mathcal{R}(\mathcal{T}_s)$ is contained in $\perp M_0$.

In order to prove the opposite inclusion it is enough to show that $\perp N(\mathcal{T})$ contains $\{ \sigma \in \mathcal{J}(h) : 0 \in \{ \mathcal{T}_{st}(\sigma) \}_{t \geq 0} \}$, since we have

$$\mathcal{J}(h) = \mathcal{R}(\mathcal{T}_s) \oplus \{ \sigma \in \mathcal{J}(h) : 0 \in \{ \mathcal{T}_{st}(\sigma) \}_{t \geq 0} \} = \perp M_0 \oplus \perp N(\mathcal{T})$$

by equation (14), item (F2) and Theorem 12.

So, let $\sigma \in \mathcal{J}(h)$ such that $0 \in \{ \mathcal{T}_{st}(\sigma) \}_{t \geq 0}$; given $(t_0)$ with $w - \lim_{t} \mathcal{T}_{st}(\sigma) = 0$ and $x \in M_r$ such that $\mathcal{T}_t(x) = e^{it\lambda} x$ for some $\lambda \in \mathbb{R}$, we have

$$\text{tr} (\sigma x) = \lim_{t} \text{tr} (\sigma e^{-it\lambda} \mathcal{T}_{st}(\sigma)) = \lim_{t} e^{-it\lambda} \text{tr} (\sigma \mathcal{T}_{st}(\sigma)x) = 0.$$

This means that $\sigma$ belongs to $\perp N(\mathcal{T})$ by Theorem 12.

In general, when there does not exist a faithful invariant state $\mathcal{R}(\mathcal{T}_s)$ could be different from $\mathcal{N}(\mathcal{T})$, as we can see in Example 17.

**Example 17.** Let us consider a generic Quantum Markov Semigroup with $\mathbb{C}^3$, more precisely the uniformly continuous QMS generated by

$$\mathcal{L}(x) = G^* x + \sum_{j=1,2} L_{3j} x L_{3j} + xG$$

where

$$G = \left( \frac{-\gamma_{33}}{2} + i\kappa_3 \right) |e_3\rangle \langle e_3|, \quad L_{3j} = \sqrt{\gamma_{3j}} |e_j\rangle \langle e_3| \quad \text{for } j = 1, 2,$$

with $\kappa_3 \in \mathbb{R}, \gamma_{3j} > 0$ for $j = 1, 2$, and $\gamma_{33} = -\gamma_{31} - \gamma_{32}$. We know that, the restriction of $\mathcal{L}$ to the diagonal matrices is the generator of a continuous time Markov chain $(X_t)_t$ with values in $\{1, 2, 3\}$. For more details see [12].

Since 1 and 2 are absorbing states for $(X_t)_t$ and 3 is a transient state, by Proposition 2 in [12] we know that any invariant state of $\mathcal{T}$ is supported on $\text{span}\{e_1, e_2\}$. In particular, this implies there is no faithful invariant state.

Moreover, Theorem 8 in [12] gives $\mathcal{N}(\mathcal{T}) = \mathbb{C} \mathbb{I}$, since the absorbing states are accessible from 3. As a consequence, $\mathcal{N}(\mathcal{T})_* = \mathbb{C} \mathbb{I}$.

On the other hand, since 1 is absorbing, the state $|e_1\rangle \langle e_1|$ is invariant, and so it belongs in particular to $\mathcal{R}(\mathcal{T}_s)$. Therefore, we have $\mathcal{R}(\mathcal{T}_s) \neq \mathcal{N}(\mathcal{T})_*$. 

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We can now give the structure of reversible states when $\mathcal{N}(\mathcal{T})$ is a type $I$ factor.

**Theorem 18.** Let $\mathcal{T}$ be a QMS on $\mathcal{B}(k \otimes m)$ with a faithful invariant state $\rho$ and $\mathcal{N}(\mathcal{T}) = \mathcal{B}(k) \otimes \mathbb{I}_m$ and let $\tau_m$ be the unique invariant state of the partially traced semigroup $\mathcal{T}^m$ defined in Theorem 4 item 2. Then a state $\eta$ belongs to $\mathcal{R}(\mathcal{T}) \simeq \mathcal{N}(\mathcal{T})^*$ if and only if

$$\eta = \sigma \otimes \tau_m$$  \hspace{1cm} (16)

for some state $\sigma$ on $\mathcal{B}(k)$.

**Proof.** Let $(e_j)_{j \geq 1}$ be an orthonormal basis of eigenvectors of $K$ so that $Ke_j = \kappa_j e_j$ for some $\kappa_j \in \mathbb{R}$. Given a state $\eta$, we can write

$$\eta = \sum_{j,k \geq 1} |e_j\rangle \langle e_k| \otimes \eta_{jk}$$

with $\eta_{jk}$ trace class operator on $m$, so that

$$\mathcal{T}_{st}(\eta) = \sum_{j,k \geq 1} e^{i(\kappa_k - \kappa_j)t}|e_j\rangle \langle e_k| \otimes \mathcal{T}_{st}^m(\eta_{jk})$$

by Theorem 3. Therefore, by the linear independence of operators $|e_j\rangle \langle e_k|$, we have

$$\mathcal{T}_{st}(\eta) = e^{it\lambda} \eta$$

for some $\lambda \in \mathbb{R}$ if and only if

$$\mathcal{T}_{st}^m(\eta_{jk}) = e^{it(\lambda - \kappa_k + \kappa_j)} \eta_{jk} \quad \forall j, k,$$

i.e. if and only if each $\eta_{jk}$ belongs to $\mathcal{R}(\mathcal{T}^m)$.

Now, by item 3 of Theorem 4 we know that $\mathcal{T}^m$ has a unique (faithful) invariant state $\tau_m$ and $\mathcal{N}(\mathcal{T}^m) = \mathcal{F}(\mathcal{T}^m)$; hence, Theorem 16 and Proposition 21 give $\mathcal{R}(\mathcal{T}^m) \simeq \mathcal{N}(\mathcal{T}^m)^* = \mathcal{F}(\mathcal{T}^m)^* = \mathcal{F}(\mathcal{T}^m) = \text{span}\{\tau_m\}$. As a consequence, we can conclude that $\eta$ belongs to $\mathcal{N}(\mathcal{T})^*$ if and only if $\eta_{jk} = \text{tr}(\eta_{jk}) \tau_m$ for all $j, k \geq 1$, i.e. if and only if

$$\eta = \sum_{j,k} (\text{tr}(\eta_{jk}) |e_j\rangle \langle e_k|) \otimes \tau_m = \sigma \otimes \tau_m$$

with $\sigma := \sum_{j,k} \text{tr}(\eta_{jk}) |e_j\rangle \langle e_k|$. This is enough to prove the statement since $\mathcal{R}(\mathcal{T})$ is the vector space generated by eigenstates of $\mathcal{T}_{st}$ corresponding to modulo 1 eigenvalues. \hfill \Box

If $\mathcal{N}(\mathcal{T})$ is not a type I factor, but it is atomic, we can obtain a similar result using that reversible states are “block-diagonal”, as the following proposition shows.

**Proposition 19.** Assume $\mathcal{N}(\mathcal{T})$ atomic. Let $\mathcal{T}$ be a QMS with a faithful invariant state and let $\mathcal{N}(\mathcal{T})$ as in (3) with $(p_i)_{i \in I}$ minimal projections in the center of $\mathcal{N}(\mathcal{T})$. Then $p_i \sigma p_j = 0$ for all $i \neq j$ and for all reversible state $\sigma \in \mathcal{R}(\mathcal{T})$. 

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Proof. Let \( \sigma \in \mathcal{R}(T_*) \) such that \( T_*(\sigma) = e^{i\lambda} \sigma \) for some \( \lambda \in \mathbb{R} \). Since \( \mathcal{R}(T_*) = \perp \mathcal{M}_0 \), by (F1) we have \( \text{tr}(\sigma x) = \text{tr}(\sigma x_1) \) for all \( B(\mathcal{H}) \ni x = x_1 + x_2 \) with \( x_1 \in \mathcal{N}(T) \) and \( x_2 \in \mathcal{M}_0 \). Moreover, the property \( T_*(p_k) = p_k \) for all \( k \in I \) and \( t \geq 0 \) gives

\[
T_*(p_i \sigma p_j) = p_i T_*(\sigma) p_j = e^{it\lambda} p_i \sigma p_j \quad \forall \ i, j \in I,
\]

so that each \( p_i \sigma p_j \) belongs to \( \mathcal{R}(T_*) \simeq \mathcal{N}(T)_* \). Therefore, since \( \text{tr}(p_i \sigma p_j x) = \text{tr}(\sigma p_j x p_i) = 0 \) for all \( x \in \mathcal{N}(T) = \bigoplus_{i \in I} p_i \mathcal{N}(T)p_i, i \neq j \), we obtain that \( p_i \sigma p_j = 0 \) for all \( i \neq j \).

As a consequence, by Theorem 4 we have the following characterization of reversible states

**Theorem 20.** Assume \( \mathcal{N}(T) \) atomic and suppose there exists a faithful \( T \)-invariant state. Let \( (p_i)_{i \in I}, (k_i)_{i \in I}, (m_i)_{i \in I} \) be as in Theorem 4. A state \( \eta \) belongs to \( \mathcal{R}(T_*) \) if and only if it can be written in the form

\[
\eta = \sum_{i \in I} \text{tr}(\eta p_i) \sigma_i \otimes \tau_{m_i}
\]

where, for every \( i \in I \),

1. \( \tau_{m_i} \) is the unique \( T^{m_i} \)-invariant state which is also faithful,
2. \( \sigma_i \) is a density on \( k_i \).

We have thus derived the general form of reversible states starting from the structure of the atomic decoherence-free algebra \( \mathcal{N}(T) \). This is a well known fact for a completely positive and unitary map (i.e. a channel) on a finite-dimensional space (see e.g. Theorem 6.16 in [31] and section V in [8]), but the proof of this result is not generalizable to the infinite dimensional case since it is based on a spectral decomposition of the channel based on eigenvectors.

5 Relationships with the structure of fixed points

In this section we investigate the structure of the set \( \mathcal{F}(T) \) of fixed points of the semigroup and its relationships with the decomposition of \( \mathcal{N}(T) \) given in the previous section.

First of all we prove the atomicity of \( \mathcal{F}(T) \) and relate this algebra with the space of invariant states. Really, the reader can find the proof of these results in [17]. We report them for sake of completeness.

**Proposition 21.** If there exists a faithful normal invariant state, then \( \mathcal{F}(T) \) is an atomic algebra and \( \mathcal{F}(T)_* \) is isomorphic to the space \( \mathcal{F}(T_*) \) of normal invariant functionals.
Proof. Since there exists a faithful invariant state, by Theorem 2.1 in [17] and in [18], \( \mathcal{F}(T) \) is the image of a normal conditional expectation \( \mathcal{E} : \mathcal{B}(h) \to \mathcal{F}(T) \) given by

\[
\mathcal{E}(x) = w^* - \lim_{\lambda \to 0} \lambda \int_0^\infty e^{-\lambda t} T_t(x) \, dt = w^* - \lim_{t \to +\infty} \frac{1}{t} \int_0^t T_s(x) \, ds.
\]

(17)

Hence, \( \mathcal{F}(T) \) is atomic by Theorem 5, the range of the predual operator \( \mathcal{E}^* \) coincides with \( \bot \ker \mathcal{E} \) and it is isomorphic to \( \mathcal{F}(T)^* \) through the map

\[
\text{Ran} \mathcal{E}^* = \bot \ker \mathcal{E} \ni \sigma \mapsto \sigma \circ \mathcal{E} \in \mathcal{F}(T)^*.
\]

Moreover, we clearly have \( \text{Ran} \mathcal{E}^* = \mathcal{F}(T) \) (see also Corollary 2.2 in [17]).

Therefore, assuming the existence of a faithful invariant state, we can find a countable set \( J \), and two sequences \( (s_j)_j \in J \), \( (f_j)_j \in J \) of separable Hilbert spaces such that

\[
h \simeq \bigoplus_{j \in J} (s_j \otimes f_j) \quad \text{(unitary equivalence)} \quad \text{(18)}
\]

\[
\mathcal{F}(T) \simeq \bigoplus_{j \in J} \left( \mathcal{B}(s_j) \otimes \mathbb{1}_{f_j} \right), \quad \text{(*-isomorphism isometric)} \quad \text{(19)}
\]

where \( \mathbb{1}_{f_j} \) denote the identity operator on \( f_j \).

Even if the decomposition (19) is given up to an isometric isomorphism, for sake of simplicity we will identify \( h \) with \( \bigoplus_{j \in J} (s_j \otimes f_j) \) and \( \mathcal{F}(T) \) with \( \bigoplus_{j \in J} \left( \mathcal{B}(s_j) \otimes \mathbb{1}_{f_j} \right) \).

Now we can state for \( \mathcal{F}(T) \) a similar result to Theorem 4. Note that, it has already been proved for a quantum channel on a matrix algebra in [8] Lemma 6, and in [31] Theorems 6.12 and 6.14. Here, we extend this in the infinite-dimensional framework.

**Theorem 22.** Assume there exists a faithful normal invariant state. Let \( (s_j)_j \in J \) and \( (f_j)_j \in J \) be two countable sequences of Hilbert spaces such that (18) and (19) hold. Then we have the following facts:

1. for every GKSRL representation (11) of the generator \( L \) by means of operators \( L_\ell, H \), we have

\[
L_\ell = \bigoplus_{j \in J} \left( \mathbb{1}_{s_j} \otimes N_{\ell j} \right) \quad \forall \, \ell \geq 1,
\]

\[
H = \bigoplus_{j \in J} \left( \lambda_j \mathbb{1}_{s_j} \otimes \mathbb{1}_{f_j} + \mathbb{1}_{s_j} \otimes M_0^{(j)} \right),
\]

where \( N_{\ell j} \) are operators on \( f_j \) such that the series \( \sum_{\ell} (N_{\ell j})^* N_{\ell j} \) are strongly convergent for all \( j \in J \), \( (\lambda_j)_{j \in J} \) is a sequence of real numbers, and every \( M_0^{(j)} \) is a self-adjoint operator on \( f_j \);
2. \( \mathcal{T}_i(x \otimes y) = x \otimes \mathcal{T}_i^f(y) \) for all \( x \in \mathcal{B}(s_j) \) and \( y \in \mathcal{B}(f_j) \), for \( j \in J \), where \( \mathcal{T}_i^f \) is the QMS on \( \mathcal{B}(f_j) \) generated by \( \mathcal{L}^f \), whose GKSL representation is given by \( \{ N^{(j)}_\ell, N^{(j)}_0 : \ell \geq 1 \} \);

3. every \( \mathcal{T}_i^f \) is irreducible and possesses a unique (faithful) normal invariant state \( \tau_i \);

4. every invariant state \( \eta \) has the form \( \eta = \sum_{j \in J} \sigma_j \otimes \tau_i \) with \( \sigma_j \) an arbitrary positive trace-class operator on \( s_j \) such that \( \sum_{j \in J} \text{tr} (\sigma_j) = 1 \).

Proof. Since (19) holds, like in the proof of Theorems 3.1, 3.2 in [13] there exist operators \( (N^{(j)}_\ell)_{\ell} \) on \( \mathcal{B}(f_j) \) such that \( L_\ell = \oplus_{j \in J} \left( \mathbb{1}_{s_j} \otimes N^{(j)}_\ell \right) \) for all \( \ell \geq 1 \) and \( j \in J \). Now, if \( p_j \) is the orthogonal projection onto \( s_j \otimes f_j \), we have \( H = \sum_{l,m} p_l H_{lm} p_m \) with \( H_{lm} : h_{s_m} \otimes h_{f_l} \to h_{s_l} \otimes h_{f_l} \), and \( H^{(j)}_{lm} = H_{ml} \) for all \( l,m \in J \). Since every \( x = \oplus_{j \in J} (x_j \otimes \mathbb{1}_{f_j}) \in \mathcal{F}(\mathcal{T}) \) commutes with \( H \) we get \( 0 = [x, H] \), i.e.

\[
0 = \sum_j p_j [x_j \otimes \mathbb{1}_{f_j}, H_{jj}] p_j + \sum_{j \neq m} p_j \left( (x_j \otimes \mathbb{1}_{f_j}) H_{jm} - H_{jm} (x_m \otimes \mathbb{1}_{f_m}) \right) p_m,
\]

which implies

\[
[x_j \otimes \mathbb{1}_{f_j}, H_{jj}] = 0 \quad \forall j \in J, \quad (x_j \otimes \mathbb{1}_{f_j}) H_{jm} = H_{jm} (x_m \otimes \mathbb{1}_{f_m}) \quad \forall j \neq m.
\]

The first condition is equivalent to have \( H_{jj} = \lambda_j \mathbb{1}_{s_j} \otimes \mathbb{1}_{f_j} + \mathbb{1}_{s_j} \otimes N^{(j)}_0 \) for some \( N^{(j)}_0 \in \mathcal{B}(f_j) \) and \( \lambda_j \in \mathbb{R} \); the second one gives \( H_{jm} = 0 \) for all \( j \neq m \), and so we obtain item 1.

Item 2 trivially follows. The proof of items 3 and 4 are similar to the ones of Theorem 4.1 and 4.3, respectively, in [13].

We want now to understand the relationships between decompositions (5) and (19) making use of the notations introduced in Theorems 1 and 22. In particular, in Theorem [23] we find a spectral characterization of the decomposition of the fixed point algebra, up to an isometric isomorphism. Indeed, in this representation, the spaces \( s_j \) undergoing trivial evolutions are the eigenspaces of suitable Hamiltonians \( K_i \) corresponding to their different eigenvalues.

First of all we introduce the following notation: for every \( i \in I \) denote by

\[
\sigma(K_i) := \{ \kappa^{(i)}_j : j \in J_i \}
\]

with \( \kappa^{(i)}_j \neq \kappa^{(i)}_l \) for \( j \neq l \) in \( J_i \), the (pure point) spectrum of the Hamiltonian \( K_i \in \mathcal{B}(k_i) \) for some at most countable set \( J_i \subseteq \mathbb{N} \). Note that, if \( \mathcal{T} \) has a faithful normal invariant state, then \( \sigma(K_i) \) is exactly the spectrum of \( K_i \) thanks to Theorem 4.
Without loss of generality we can choose the family \( \{ J_i : i \in I \} \) such that \( J_h \cap J_l = \emptyset \) whenever \( h \neq l \). In this way, set
\[
J := \cup_{i \in I} J_i,
\]
for \( j \in J \) there exists a unique \( i \in I \) such that \( j = j_i \in J_i \).

**Theorem 23.** Assume \( \mathcal{N}(T) \) atomic and let \( \mathcal{N}(T) = \oplus_{i \in I} (\mathcal{B}(k_i) \otimes \mathbb{1}_{m_i}) \) with \( (k_i)_i, (m_i)_i \), two countable sequences of Hilbert spaces such that \( h = \oplus_{i \in I} (k_i \otimes m_i) \). If there exists a faithful normal invariant state, up to an isometric isomorphism we have
\[
\mathcal{F}(T) = \oplus_{j \in J} (\mathcal{B}(s_j) \otimes \mathbb{1}_{f_j})
\]
with \( J \) defined in (21), and
\[
s_j = s_{j_i} := \text{Ker} \left( K_i - \kappa_j^{(i)} \mathbb{1}_{k_i} \right), \quad f_j = f_{j_i} := m_i \quad \forall j_i \in J_i, \ i \in I.
\]

**Proof.** By considering the spectral decomposition \( K_i = \sum_{j \in J_i} \kappa_j^{(i)} q_{ji} \) with \( (q_{ji})_{j \in J_i} \), mutually orthogonal projections such that
\[
q_{ji} k_i = \text{Ker} \left( K_i - \kappa_j^{(i)} \mathbb{1}_{k_i} \right) =: s_{ji}
\]
and \( \sum_{j \in J_i} q_{ji} = \mathbb{1}_{m_i} \), we immediately obtain
\[
k_i \otimes m_i = (\oplus_{j \in J_i} s_{j_i}) \otimes m_i = \oplus_{j \in J_i} (s_{j_i} \otimes f_{j_i})
\]
by setting \( f_{j_i} := m_i \) for all \( j \in J_i \). Therefore, by definition of \( J \), since every \( j \in J \) belongs to a unique \( J_i \), we have
\[
h = \oplus_{i \in I} (k_i \otimes m_i) = \oplus_{i \in I} \oplus_{j \in J_i} (s_{j_i} \otimes f_{j_i}) = \oplus_{j \in J} (s_j \otimes f_j).
\]

In order to conclude the proof we have to show equality (22). Given \( x \in \mathcal{F}(T) \subseteq \mathcal{N}(T) \) (see item 4 in Proposition 4), we can write \( x = \oplus_{i \in I} (x_i \otimes \mathbb{1}_{m_i}) \) with \( (x_i)_{i \in I} \subseteq \mathcal{B}(k_i) \), and so, by Theorem 4 we have
\[
x = \oplus_{i \in I} (x_i \otimes \mathbb{1}_{m_i}) = T_i(\oplus_{i \in I} (x_i \otimes \mathbb{1}_{m_i})) = \oplus_{i \in I} (e^{itK_i} x_i e^{-itK_i} \otimes \mathbb{1}_{m_i}).
\]
Consequently, \( x_i = e^{itK_i} x_i e^{-itK_i} \) for all \( i \in I \), i.e. every \( x_i \) commutes with \( K_i \), and then with each projection \( q_{ji} \) with \( j \in J_i \). This means that each \( x_i = \oplus_{j \in J_i} q_{ji} x_i q_{ji} \) belongs to the algebra \( \oplus_{j \in J_i} \mathcal{B}(k_i) q_{ji} = \oplus_{j \in J_i} \mathcal{B}(q_{ji} k_i) = \oplus_{j \in J_i} \mathcal{B}(s_j) \), so that \( x \) is in \( \oplus_{j \in J} (\mathcal{B}(s_j) \otimes \mathbb{1}_{f_j}) \).

On the other hand, given \( i \in I \) and \( j \in J_i \), for \( u, v \in s_j = \text{Ker}(K_i - \kappa_j^{(i)} \mathbb{1}_{k_i}) \) we get
\[
T_i(|u \rangle \langle v| \otimes \mathbb{1}_{m_i}) = |e^{itK_i} u \rangle \langle e^{itK_i} v| \otimes \mathbb{1}_{m_i} = |u \rangle \langle v| \otimes \mathbb{1}_{m_i}, \quad \forall t \geq 0,
\]
and
\[
g_i := \text{Ker} \left( K_i - \kappa_j^{(i)} \mathbb{1}_{k_i} \right) \subseteq \text{Ker}(K_i - \kappa_j^{(i)} \mathbb{1}_{k_i}) \subseteq \text{Ker}(K_i), \quad \forall i \in I,
\]

where \( g_i \) is the unique common invariant state of \( (K_i)_{i \in I} \) and the family \( \{ g_i \} \) is defined in (20).

Finally, we show that \( \mathcal{F}(T) \subseteq \mathcal{N}(T) \). Given \( x \in \mathcal{F}(T) \), by Theorem 4 we have
\[
x = \oplus_{i \in I} (x_i \otimes \mathbb{1}_{m_i}) = T_i(\oplus_{i \in I} (x_i \otimes \mathbb{1}_{m_i})) = \oplus_{i \in I} (e^{itK_i} x_i e^{-itK_i} \otimes \mathbb{1}_{m_i}),
\]
for all \( t \geq 0 \). Therefore, \( x \) is in \( \mathcal{N}(T) \) and \( \mathcal{F}(T) \subseteq \mathcal{N}(T) \).
where \( u \) and \( v \) are eigenvectors of \( K_i \) associated with the same eigenvalue \( \kappa_j^{(i)} \). Since \( \ker(K_i - \kappa_j^{(i)} \mathbb{I}_{k_i}) \) is generated by elements of the form \(|u\rangle \langle v|\), and the net \( (T(z))_t \) is uniformly bounded for all \( z \), we obtain the inclusion \( \mathcal{B}(s_j) \otimes \mathbb{I}_{m_i} \subseteq \mathcal{F}(T) \) for all \( j \in J_i, i \in I \), i.e. \( \mathcal{B}(s_j) \otimes \mathbb{I}_{t_j} \subseteq \mathcal{F}(T) \) for all \( j \in J = \bigcup_{i \in I} J_i \).

Theorem below shows how we can derive an "atomic decomposition" of \( \mathcal{F}(T) \) from one of \( \mathcal{N}(T)'s \). We want now to analyze the opposite procedure.

Assuming \( \mathcal{F}(T) = \bigoplus_{j \in J} (\mathcal{B}(s_j) \otimes \mathbb{I}_{t_j}) \) with \( (s_j)_j, (f_j)_j \) two countable sequences of Hilbert spaces such that \( h = \bigoplus_{j \in J} (s_j \otimes f_j) \), and using notations of Theorem \( 22 \) we set an equivalence relation on \( J \) in the following way:

**Definition 24.** Given \( j, k \in J \), we say that \( j \) is in relation with \( k \) (and write \( j \sim k \)) if there exist a complex separable Hilbert space \( m \) and unitary isomorphisms

\[
V_j : f_j \to m, \quad V_k : f_k \to m
\]

such that operators \( \{V_j N_j^{(j)} V_j^*, V_j N_0^{(j)} V_j^* : l \geq 1\} \) and \( \{V_k N_k^{(k)} V_k^*, V_k N_0^{(k)} V_k^* : l \geq 1\} \) give the same Lindbladian operator on \( \mathcal{B}(m) \).

We obtain in this way an equivalence relation which induces a partition of \( J \), \( J = \bigcup_{n \in I} I_n \), for some finite or countable set \( I \subseteq \mathbb{N} \), where each \( I_n \) is an equivalence class with respect to \( \sim \).

**Theorem 25.** Assume that there exists a faithful normal invariant state and \( \mathcal{N}(T) \) atomic. Let \( \mathcal{F}(T) = \bigoplus_{j \in J} (\mathcal{B}(s_j) \otimes \mathbb{I}_{t_j}) \) with \( (s_j)_j, (f_j)_j \) two countable sequences of Hilbert spaces such that \( h = \bigoplus_{j \in J} (s_j \otimes f_j) \), and let \( \{I_n : n \in I\} \) be the set of equivalence classes of \( J \) with respect to the relation \( \sim \). Then \( \mathcal{N}(T) \) is isometrically isomorphic to the direct sum \( \bigoplus_{n \in I} (\mathcal{B}(k_n) \otimes \mathbb{I}_{m_n}) \) with

\[
k_n := \bigoplus_{j \in I_n} s_j, \quad m_n := V_j f_j \quad \forall j \in I_n, \quad (25)
\]

where \( V_j \)'s are the unitary isomorphisms given in \( (24) \).

**Proof.** Given \( n \in I \), by definition of \( m_n \) we can define a unitary operator by setting

\[
U_n : \bigoplus_{j \in I_n} (s_j \otimes f_j) \to (\bigoplus_{j \in I_n} s_j) \otimes m_n = k_n \otimes m_n
\]

\[
\bigoplus_{j \in I_n} (u_j \otimes z_j) \mapsto \sum_{j \in I_n} \left( u_j^{(j)} \otimes V_j z_j \right)
\]

where \( u_j^{(j)} \) in \( \bigoplus_{i \in I_n} h_k \) denotes the vector

\[
u_j := \begin{cases} 0 & \text{if } i \neq j \\ u_j & \text{if } i = j \end{cases}
\]
Now, since
\[ h = \oplus_{j \in J} (s_j \otimes f_j) = \oplus_{n \in I} (\oplus_{j \in I_n} (s_j \otimes f_j)) \]
by the equality \( J = \bigcup_{n \in I} I_n \), by setting \( U := \oplus_{n \in I} U_n \) we get a unitary operator
\( U : h \to \oplus_{n \in I} (k_n \otimes m_n) \) such that
\[ U \mathcal{F}(T) U^* = \oplus_{n \in I} \left( (\oplus_{j \in I_n} \mathcal{B}(s_j)) \otimes \mathbb{I}_{m_n} \right). \]
In order to conclude the proof we have to show that
\[ U \mathcal{N}(T) U^* = \oplus_{n \in I} \left( \mathcal{B}(\oplus_{j \in I_n} s_j) \otimes \mathbb{I}_{m_n} \right). \]
To this end recall that, by Theorem 22, the operators \((L_\ell)_\ell\), \( H \) in a GKSL representation of the generator \( \mathcal{L} \) can be written as
\[ L_\ell = \oplus_{j \in J} \left( \mathbb{I}_{s_j} \otimes N_\ell^{(j)} \right) \quad \forall \ell \geq 1, \quad H = \oplus_{j \in J} \left( \lambda_j \mathbb{I}_{s_j} \otimes \mathbb{I}_{f_j} + \mathbb{I}_{s_j} \otimes N_0^{(j)} \right) \]
with \((N_\ell^{(j)})_{\ell \geq 1} \subseteq \mathcal{B}(s_j)\), \( N_0^{(j)} = (N_0^{(j)})^* \in \mathcal{B}(f_j)\) and \( \lambda_j \in \mathbb{R} \) for all \( j \in J \); moreover, by definition of \( I_n \), we can choose operators \((M_\ell^{(n)})_\ell\) and \((M_0^{(n)}) = (M_0^{(n)})^* \in \mathcal{B}(m_n)\) such that \( \{M_\ell^{(n)}, M_0^{(n)} : \ell \geq 1\} \) is a GKSL representation of the the generator \( \mathcal{L}_m^{(n)} \) of a QMS \( \mathcal{T}^{mn} \) on \( \mathcal{B}(m_n) \) equivalent to \( \{V_j N_\ell^{(j)} V_j^*, V_j N_0^{(j)} V_j^* : \ell \geq 1\} \) for all \( j \in I_n \). Therefore, the operators
\[ L'_\ell := \oplus_{n \in I} \left( \oplus_{j \in I_n} \left( \mathbb{I}_{s_j} \otimes V_j^* M_\ell^{(n)} V_j \right) \right) \]
\[ H' := \oplus_{n \in I} \left( \oplus_{j \in I_n} \left( \lambda_j \mathbb{I}_{s_j} \otimes \mathbb{I}_{f_j} \right) + \oplus_{j \in I_n} \left( \mathbb{I}_{s_j} \otimes V_j^* M_0^{(n)} V_j \right) \right) \]
clearly give the same GKSL representation of \( \{L_\ell, H : \ell \geq 1\} \). Moreover we have
\[ U L'_\ell U^* = \oplus_{n \in I} U_n \left( \oplus_{j \in I_n} \left( \mathbb{I}_{s_j} \otimes V_j^* M_\ell^{(n)} V_j \right) \right) U_n^* = \oplus_{n \in I} \left( \mathbb{I}_{k_n} \otimes M_\ell^{(n)} \right) \]
\[ U H' U^* = \oplus_{n \in I} U_n \left( \oplus_{j \in I_n} \left( \lambda_j \mathbb{I}_{s_j} \otimes \mathbb{I}_{f_j} \right) + \oplus_{j \in I_n} \left( \mathbb{I}_{s_j} \otimes V_j^* M_0^{(n)} V_j \right) \right) U_n^* \]
\[ = \oplus_{n \in I} \left( K_n \otimes \mathbb{I}_{m_n} + \mathbb{I}_{k_n} \otimes M_0^{(n)} \right) \]
with \( K_n := \oplus_{j \in I_n} \lambda_j \mathbb{I}_{s_j} = K_n^* \in \mathcal{B}(k_n) \), so that
\[ U \delta_\mathcal{H}^m(L'_\ell U^*) = \delta_\mathcal{H}^m(U L'_\ell U^*) = \oplus_{n \in I} \left( \mathbb{I}_{k_n} \otimes \delta_\mathcal{H}^m(M_\ell^{(n)}) \right) \]
\[ U \delta_\mathcal{H}^m(L'_\ell U^*) = \delta_\mathcal{H}^m(U L'_\ell U^*) = \oplus_{n \in I} \left( \mathbb{I}_{k_n} \otimes \delta_\mathcal{H}^m(M_0^{(n)}) \right). \]
for all $m \geq 0$. Therefore, since $\mathcal{U}\mathcal{N}(\mathcal{T})U^* = \{U\delta_{H_1}(L^*_{t})U^*, U\delta_{H_1}(L^*_{0})U^* : m \geq 0\}'$ by item 2 of Proposition 2 we obtain that an operator $x \in \mathcal{B}(\bigoplus_{n \in I} (k_n \otimes m_n))$ belongs to $\mathcal{U}\mathcal{N}(\mathcal{T})U^*$ if and only if it commutes with

$$\bigoplus_{n \in I} \left( \mathbb{I}_{k_n} \otimes \delta^m_{\mathcal{L}_{t}^{(n)}}(M^{(n)}_{t}) \right), \quad \bigoplus_{n \in I} \left( \mathbb{I}_{k_n} \otimes \delta^m_{\mathcal{L}_{t}^{(n)}}(M^{(n)*}_{t}) \right)$$

for every $m \geq 0$.

Now, let $q_n := \mathbb{I}_{k_n} \otimes \mathbb{I}_{m_n}$ the orthogonal projection onto $k_n \otimes m_n$. We clearly have $q_n \in U\mathcal{F}(\mathcal{T})U^*$ and $\sum_{n \in I} q_n = \mathbb{1}$. Therefore, the algebra $q_n \mathcal{B}(\mathcal{H})q_n = \mathcal{B}(k_n \otimes m_n)$ is preserved by the action of every map $\mathcal{T}$ satisfying $\mathcal{N}(\mathcal{T}) = q_n \mathcal{U}\mathcal{N}(\mathcal{T})U^* q_n$, where $\mathcal{N}(\mathcal{T})$ is the decoherence-free algebra of $\mathcal{T}$. Note that, since each $q_n$ commutes with every $UL^*_t U^*$, $UL^*_t U^*$ and $UH^* U^*$, a GKSL representation of the generator $\mathcal{L}^{(n)}$ of $\mathcal{T}^{(n)}$ is given by operators

$$q_n UH^* U^* q_n = K_n \otimes \mathbb{I}_{m_n} + \mathbb{I}_{k_n} \otimes M_{t}^{(n)} \quad q_n UL^*_t U^* q_n = \mathbb{I}_{k_n} \otimes M_{t}^{(n)};$$

so that

$$\mathcal{T}_t(x \otimes y) = e^{itK_n} e^{-itK_n} \otimes \mathcal{T}_t^{m_n}(y) \quad \forall x \in \mathcal{B}(k_n), y \in \mathcal{B}(m_n)$$

and

$$\mathcal{N}(\mathcal{T})^{(n)} = \{ \mathbb{I}_{k_n} \otimes \delta^m_{\mathcal{L}_{t}^{(n)}}(M_{t}^{(n)}), \mathbb{I}_{k_n} \otimes \delta^m_{\mathcal{L}_{t}^{(n)}}(M_{t}^{(n)*}) : \ell \geq 1, m \geq 0\}'$$

$$= \mathcal{B}(k_n) \otimes \mathcal{N}(\mathcal{T})^{m_n}. \quad (26)$$

Since

$$\mathcal{U}\mathcal{N}(\mathcal{T})U^* = \bigoplus_{n,m \in I} q_n \mathcal{U}\mathcal{N}(\mathcal{T})U^* q_m$$

and equation (26) holds, to conclude the proof we have to show that

$$q_n \mathcal{U}\mathcal{N}(\mathcal{T})U^* q_m = \{0\} \quad \forall n \neq m, \quad \text{and} \quad \mathcal{N}(\mathcal{T})^{m_n} = \mathbb{C}\mathbb{I}_{m_n}.$$ 

So, let $x \in \mathcal{U}\mathcal{N}(\mathcal{T})U^*$ and consider $n, m \in I$ with $n \neq m$. Since the net $(\mathcal{T}_t(q_n x q_m))_{t \geq 0}$ is bounded in norm and the unit ball is weakly* compact, there exists a weak* cluster point $y$ such that $y = w^* - \lim_n \mathcal{T}_t(q_n x q_m)$. Therefore, for $\sigma \in \mathcal{R}(\mathcal{T}_t)$ with $\mathcal{T}_t(\sigma) = e^{it\lambda}\sigma$ for some $\lambda \in \mathbb{R}$, we have

$$\text{tr}(\sigma y) = \lim_{t \to 0} \text{tr}(\sigma \mathcal{T}_t(q_n x q_m)) = \lim_{t \to 0} e^{it\lambda}\text{tr}(\sigma q_n x q_m).$$

Now, if $\lambda \neq 0$ this implies $\text{tr}(\sigma q_n x q_m) = 0 = \text{tr}(\sigma y)$; otherwise, since $\sigma$ is an invariant state, we automatically have $q_n \sigma q_n = 0$ for $n \neq m$ by Theorem 22 so that $\text{tr}(\sigma y)$ is
such that $0 \in \{ T_{st} (\eta) \}_{t \geq 0}$, up to passing to generalized subsequences we have

$$\text{tr} (\eta y) = \lim_\alpha \text{tr} (\eta T_{t_\alpha} (\eta q_n x q_m)) = \lim_\alpha \text{tr} (T_{st_\alpha} (\eta) q_n x q_m) = 0.$$ 

We can then conclude that $\text{tr} (\sigma y) = 0$ for all $\sigma \in \mathcal{R}(\mathcal{T}_s)$. On the other hand, taking $\eta$ such that $0 \in \{ T_{st} (\eta) \}_{t \geq 0}$, up to passing to generalized subsequences we have

$$\text{tr} (\eta y) = \lim_\alpha \text{tr} (\eta T_{t_\alpha} (\eta q_n x q_m)) = \lim_\alpha \text{tr} (T_{st_\alpha} (\eta) q_n x q_m) = 0.$$ 

Moreover, since $\mathcal{N}(\mathcal{T})$ is atomic by Corollary 12 the general theory of von Neumann algebras (see e.g. [28]) says that each algebra $\mathcal{N}(\mathcal{T}^{(n)}) = \mathcal{B}(k_n) \otimes \mathcal{N}(\mathcal{T}^{(m)})$ is atomic too, and consequently $\mathcal{N}(\mathcal{T}^{m_n})$ is atomic. Finally, recalling that, by construction, $\mathcal{T}^{m_n} = \mathcal{T}_{I_j}^{(s_j)}$ for all $j \in I_n$ with $\mathcal{T}_{I_j}^{(s_j)}$ an irreducible QMS having a faithful invariant state (see Theorem 22), we get $\mathcal{N}(\mathcal{T}^{m_n}) = \mathcal{C} \mathds{1}_{m_n}$ thanks to Proposition 4.3 in [13].

**Remark 26.** Theorem 25 provides in particular a way to pass from the decomposition of GKSL operators $H$, $(L_{i})_i$ of $\mathcal{L}$ according to the splitting of $h = \oplus_{j \in J} (s_j \otimes f_j)$ associated with the atomic algebra $\mathcal{F}(\mathcal{T})$, to the other one decomposition with respect to the splitting $h = \oplus_{n \in I} (k_n \otimes m_n)$ associated with $\mathcal{N}(\mathcal{T})$.

More precisely, if $\mathcal{F}(\mathcal{T}) = \oplus_{j \in J} (\mathcal{B}(s_j) \otimes \mathds{1}_{f_j})$ and

$$L_{\ell} = \oplus_{j \in J} \left( \mathds{1}_{s_j} \otimes \mathcal{N}^{(j)}_{\ell} \right) \quad \forall \ell \geq 1, \quad H = \oplus_{j \in J} \left( \lambda_j \mathds{1}_{s_j} \otimes \mathds{1}_{f_j} + \mathds{1}_{s_j} \otimes \mathcal{N}^{(j)}_0 \right)$$

with $(\mathcal{N}^{(j)}_{\ell})_{\ell \geq 1} \subseteq \mathcal{B}(s_j)$, $\mathcal{N}^{(j)}_0 \subseteq \mathcal{B}(f_j)$ and $\lambda_j \in \mathbb{R}$ for all $j \in J$, then we can decompose $H$, $(L_{\ell})_\ell$ with respect to the splitting $h = \oplus_{n \in I} (k_n \otimes m_n)$ given by (25) as follows:

$$L_{\ell} = \oplus_{n \in I} \left( \mathds{1}_{k_n} \otimes M_{\ell}^{(n)} \right), \quad H = \oplus_{n \in I} \left( K_n \otimes \mathds{1}_{m_n} + \mathds{1}_{k_n} \otimes M_0^{(n)} \right),$$

where $(M_{\ell}^{(n)})_{\ell \geq 1}$, $M_0^{(n)} = M_0^{(n)}$ are operators in $\mathcal{B}(m_n)$ giving the same GKSL representation of $\mathcal{N}^{(j)}_{\ell \geq 1}$, $\mathcal{N}^{(j)}_0$ for all $j \in I_n$, and every $K_n$ is the self-adjoint operator on $\mathcal{B}(k_n)$ having $(\lambda_j)_{j \in I_n}$ as eigenvalues and $(s_j)_{j \in I_n}$ as corresponding eigenspaces.

**Acknowledgements.** The financial support of MIUR FIRB 2010 project RBFR10COAQ “Quantum Markov Semigroups and their Empirical Estimation” is gratefully acknowledged.

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