GENERATORS OF ALGEBRAIC CURVATURE TENSORS
BASED ON A (2 1)-SYMMETRY

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Abstract. We consider generators of algebraic curvature tensors \( R \) which can be constructed by a Young symmetrization of product tensors \( U \otimes w \) or \( w \otimes U \), where \( U \) and \( w \) are covariant tensors of order 3 and 1. We assume that \( U \) belongs to a class of the infinite set \( \mathcal{S} \) of irreducible symmetry classes characterized by the partition \((2 1)\). We show that the set \( \mathcal{S} \) contains exactly one symmetry class \( \mathcal{S}_0 \in \mathcal{S} \) whose elements \( U \in \mathcal{S}_0 \) can not play the role of generators of tensors \( R \). The tensors \( U \) of all other symmetry classes from \( \mathcal{S} \setminus \{\mathcal{S}_0\} \) can be used as generators for tensors \( R \).

Using Computer Algebra we search for such generators whose coordinate representations are polynomials with a minimal number of summands. For a generic choice of the symmetry class of \( U \) we obtain lengths of 8 summands. In special cases these numbers can be reduced to the minimum 4. If this minimum occurs then \( U \) admits an index commutation symmetry. Furthermore minimal lengths are possible if \( U \) is formed from torsion-free covariant derivatives of alternating 2-tensor fields.

We apply ideals and idempotents of group rings \( \mathbb{C}[S_r] \) of symmetric groups \( S_r \), Young symmetrizers, discrete Fourier transforms and Littlewood-Richardson products. For symbolic calculations we used the Mathematica packages Ricci and PERMS.

1. Introduction

In [12, 14] we constructed and investigated generators of algebraic covariant derivative curvature tensors which contained 3-times covariant tensors \( U \) with a so-called \((2 1)\)-symmetry. Later we discovered that the same constructions can be applied in the simpler case of algebraic curvature tensors, too. In the present paper we carry out all constructions and investigations of [12, 14] for algebraic curvature tensors and join the results about these tensors to the results of [12, 14].

Let \( \mathcal{T}_r V \) be the vector space of the \( r \)-times covariant tensors \( T \) over a finite-dimensional \( K \)-vector space \( V \), \( K = \mathbb{R} \) or \( K = \mathbb{C} \). We assume that \( V \) possesses a fundamental tensor \( g \in \mathcal{T}_2 V \) (of arbitrary signature) which can be used for raising and lowering of tensor indices.

Definition 1.1. Let \( \mathcal{A}(V) \subset \mathcal{T}_4 V \) and \( \mathcal{A}'(V) \subset \mathcal{T}_5 V \) be the spaces of all algebraic curvature tensors and all algebraic covariant derivative curvature tensors,

1991 Mathematics Subject Classification. 53B20, 15A72, 05E10, 16D60, 05-04.
respectively, i.e. those tensors $\mathcal{R} \in \mathcal{A}(V), \mathcal{R}' \in \mathcal{A}'(V)$ which satisfy

\begin{align}
(1.1) \quad & \mathcal{R}(w, x, y, z) = -\mathcal{R}(w, x, z, y) = \mathcal{R}(y, z, w, x), \\
(1.2) \quad & \mathcal{R}(w, x, y, z) + \mathcal{R}(w, y, z, x) + \mathcal{R}(w, z, x, y) = 0, \\
(1.3) \quad & \mathcal{R}'(w, x, y, z; u) = -\mathcal{R}'(w, x, z, y; u) = \mathcal{R}'(y, z, w, x; u), \\
(1.4) \quad & \mathcal{R}'(w, x, y, z; u) + \mathcal{R}'(w, y, z, x; u) + \mathcal{R}'(w, z, x, y; u) = 0, \\
(1.5) \quad & \mathcal{R}'(w, x, y, z; u) + \mathcal{R}'(w, x, z, u; y) + \mathcal{R}'(w, x, u; y; z) = 0
\end{align}

for all $w, x, y, z \in V$.

$\mathcal{R}$ and $\mathcal{R}'$ have the symmetries of the Riemann tensor $R$ of a Levi-Civita connection $\nabla$ and the covariant derivative $\nabla R$.

Let $\mathcal{S}^p(V)$, $\Lambda^p(V)$ be the spaces of totally symmetric/alternating $p$-forms over $V$. Then the following tensors

\begin{align}
(1.6) \quad & \gamma(S)_{ijkl} := S_{il}S_{jk} - S_{ik}S_{jl}, \\
(1.7) \quad & \alpha(A)_{ijkl} := 2A_{ij}A_{kl} + A_{ik}A_{jl} - A_{il}A_{jk}, \\
(1.8) \quad & \hat{\gamma}(S, \hat{S})_{ijkl} := S_{il}\hat{S}_{jk} - S_{ik}\hat{S}_{jl} + S_{jk}\hat{S}_{ils} - S_{ik}\hat{S}_{jls},
\end{align}

are generators of $\mathcal{A}(V), \mathcal{A}'(V)$. P. Gilkey \[13\] pp.41–44, p.236] and B. Fiedler \[11\] \[12\] gave different proofs for

**Theorem 1.2.**

1. $\mathcal{A}(V) = \text{Span}_{S \in \mathcal{S}^2(V)}\{\gamma(S)\} = \text{Span}_{\Lambda \in \Lambda^2(V)}\{\alpha(\Lambda)\}.$
2. $\mathcal{A}'(V) = \text{Span}_{S \in \mathcal{S}^2(V), \hat{S} \in \mathcal{S}^2(V)}\{\hat{\gamma}(S, \hat{S})\}.$

The tensors $\gamma(S), \alpha(A)$ and $\hat{\gamma}(S, \hat{S})$ are expressions which arise from $S \otimes S$, $A \otimes A$, $\hat{S} \otimes S$ or $\hat{S} \otimes \hat{S}$ by a symmetrization

\begin{align}
(1.9) \quad & \gamma(S) = \frac{1}{12} y_t^* (S \otimes S), \quad \alpha(A) = \frac{1}{12} y_t^* (A \otimes A) \\
(1.10) \quad & \hat{\gamma}(S, \hat{S}) = \frac{1}{4} y_t^* (S \otimes \hat{S}) = \frac{1}{4} y_t^* (\hat{S} \otimes S)
\end{align}

where $y_t, y_t^*$ are the Young symmetrizers of the Young tableaux

\[
(1.11) \quad t = \begin{array}{c|c|c}
1 & 3 \\
2 & 4 
\end{array}, \quad t' = \begin{array}{c|c|c|c}
1 & 3 & 5 \\
2 & 4 
\end{array}
\]

(See \[11\] \[12\]. See also Section 2 for definitions.)

In the present paper we search for similar generators of $\mathcal{A}(V)$ and $\mathcal{A}'(V)$ which, however, are based on other product tensors. We use Boerner’s definition of symmetry classes for tensors $T \in \mathcal{T}_V$ by right ideals $\mathfrak{r} \subseteq \mathbb{K}[S_r]$ of the group ring $\mathbb{K}[S_r]$ of the symmetric group $S_r$ (see Section 2 and \[11\] \[12\] \[7\] \[10\]). On this basis we investigate the following...
Problem 1.3. We search for generators of $\mathcal{A}(V), \mathcal{A}'(V)$ which can be formed by a suitable symmetry operator from tensors

\begin{align*}
(1.12) & \quad \mathcal{A}(V) : U \otimes w, \ w \otimes U, \quad U \in T_3 V, \ w \in T_1 V, \\
(1.13) & \quad \mathcal{A}'(V) : U \otimes W, \ W \otimes U, \quad U \in T_3 V, \ W \in T_2 V,
\end{align*}

where $W$ and $U$ belong to symmetry classes of $T_2 V$ and $T_3 V$ which are defined by minimal right ideals $r \subset \mathbb{K}[S_2]$ and $\hat{r} \subset \mathbb{K}[S_3]$, respectively.

Here is a summary of our main results. The subject of the present paper is the determination of generators (1.12) of $\mathcal{A}(V)$. However, for comparing purposes we repeat also results concerning generators (1.13) of $\mathcal{A}'(V)$ which were proved in [12, 14].

Theorem 1.4. A solution of Problem 1.3 can be constructed at most from such products (1.12) or (1.13) whose factors $U \in T_3 V$, $W \in T_2 V$ lie in symmetry classes which belong\(^1\) to the following partitions of 3 or 2:

| product | partitions for $U, W$ |  |
|---------|-----------------------|---|
| $\mathcal{R} : U \otimes w, w \otimes U$ | (a) $U \leftrightarrow (2\ 1)$ |  |
| $\mathcal{R}' : U \otimes W, W \otimes U$ | (a') $U \leftrightarrow (3)$ and $W \leftrightarrow (2)$ | $U$ and $W$ symmetric |
| | (b') $U \leftrightarrow (2\ 1)$ and $W \leftrightarrow (2)$ | $W$ symmetric |
| | (c') $U \leftrightarrow (2\ 1)$ and $W \leftrightarrow (1\ 2)$ | $W$ skew-symmetric |

The case (a') of Theorem 1.4 is specified by Theorem 1.2(2) and (1.8) (see [12]). The cases (a), (b') and (c') of Theorem 1.4 lead to

Theorem 1.5. Let $r \subset \mathbb{K}[S_3]$ be a minimal right ideal belonging to the partition $(2\ 1) \vdash 3$ and let $\mathcal{T}_r$ be the symmetry class of tensors $U \in T_3 V$ that is defined by $r$. Then the following statements are equivalent:

1. $\mathcal{A}(V) = \text{Span}_{U \in \mathcal{T}_r, \ w \in T_1 V} \{ y_t^*(U \otimes w) \}$.
2. $\mathcal{A}'(V) = \text{Span}_{U \in \mathcal{T}_r, \ S \in S^2(V)} \{ y_t^*(U \otimes S) \}$.
3. $\mathcal{A}'(V) = \text{Span}_{U \in \mathcal{T}_r, \ A \in \Lambda^2(V)} \{ y_t^*(U \otimes A) \}$.
4. The right ideal $r$ is different from the right ideal $r_0 = f_0 \cdot \mathbb{K}[S_3]$ which is generated by the idempotent

\begin{equation}
(1.14) \quad f_0 := \frac{1}{2} \{ \text{id} - (1\ 3) \} - \frac{1}{6} y, \quad y := \sum_{p \in S_3} \text{sign}(p) \ p.
\end{equation}

Here $y_t$ and $y_t'$ are the Young symmetrizers of the Young tableaux (1.11).

The Statements (1), (2), (3) of Theorem 1.5 are independent of the order of the factors $U, S, A, w$ since the following Lemma holds.

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\(^1\)See Section 2 and [1, 2, 12] for the connection between partitions and symmetry classes or right ideals respectively.
Lemma 1.6. Let $T_r$ be the symmetry class and let $y_t$, $y_t'$ be the Young symmetrizers from Theorem 1.5. If $U \in T_r$, $S \in S^2(V)$, $A \in \Lambda^2(V)$ and $w \in T_1 V$, then it holds

\begin{align}
(1.15) \quad y_t(U \otimes w) & = y_t(w \otimes U), \\
(1.16) \quad y_t'(U \otimes S) & = y_t'(S \otimes U), \\
(1.17) \quad y_t'(U \otimes A) & = -y_t'(A \otimes U).
\end{align}

Remark 1.7. The set $S$ of symmetry classes $T_r$ considered in Theorem 1.5 is an infinite set. Theorem 1.5 says that exactly the tensors $U$ from symmetry classes $T_r \in S \setminus \{T_{r_0}\}$ yield generators of $\mathcal{A}(V)$ or $\mathcal{A}'(V)$ respectively.

The equivalence of the statements (2), (3), (4) of Theorem 1.5 was shown already in [12]. It is remarkable that we also have to exclude a single symmetry class from $S$ if we search for generators of $\mathcal{A}(V)$, and that the forbidden symmetry class is the same class $T_{r_0}$ which had to be excluded in the construction of generators of $\mathcal{A}'(V)$.

For the generators (1), (2), (3) of Theorem 1.5 we can also determine operators of the type $\alpha, \gamma, \hat{\gamma}$ which yield the coordinate representation of these generators. However, these operators depend now on the right ideal $r$ (or its generating idempotent $e$) that defines the symmetry class of $U$. And they yield no short expressions of 2, 3, or 4 terms but longer expressions. The search for shortest expressions of this type is a further subject of our paper. Some of our main results are collected in

Theorem 1.8. Consider the situation of Theorem 1.5 and assume $\dim V \geq 3$, $r \neq r_0$. Then it holds:

1. The coordinates of $y_t(U \otimes w)$, $y_t'(U \otimes S)$, $y_t'(U \otimes A)$ are sums of the following lengths

|        | $y_t(U \otimes w)$ | $y_t'(U \otimes S)$ | $y_t'(U \otimes A)$ |
|--------|--------------------|----------------------|----------------------|
| (a)    | 8                  | 16                   | 20                   |
| (b)    | 4                  | 12                   | 10                   |

2. There exist exactly 2 minimal right ideals $r \neq r_0$ of (21)$+3$ which lead to the minimal lengths of case (b) for all tensors $y_t(U \otimes w)$, $y_t'(U \otimes S)$ and $y_t'(U \otimes A)$.

3. If the coordinates of $y_t(U \otimes w)$, $y_t'(U \otimes S)$, $y_t'(U \otimes A)$ have the minimal lengths of case (b) then $U$ admits an index commutation symmetry.

Further results are given in Section 4.

Remark 1.9. The concept ”expression of minimal length” depends on the method which we use to reduce expressions (see Section 4, Procedure 4.3). In [14, Remark 3.9] we discuss a generalization of our reduction method which could possibly lead to a further decrease of the numbers in Theorem 1.8.

Examples of tensors $U$ with a $(21)$-symmetry from $S$ can be constructed from covariant derivatives of certain tensor fields. In [13, 14] we proved
Proposition 1.10. (Examples of $(2,1)$-symmetries)

Let $\nabla$ be a torsion-free covariant derivative on a $C^\infty$-manifold $M$, $\dim M \geq 2$. Further let $\psi \in S^2 M$, $\omega \in A^2 M$ be differentiable tensor fields of order 2 on $M$ which are symmetric or skew-symmetric, respectively. Then the infinite set $\mathcal{G}$ of symmetry classes contains 2 classes $\mathcal{T}_s, \mathcal{T}_a \in \mathcal{G}$ such that

$$\forall p \in M : (\nabla \psi - \text{sym}(\nabla \psi))|_p \in \mathcal{T}_s, \quad (\nabla \omega - \text{alt}(\nabla \omega))|_p \in \mathcal{T}_a.$$  

Here 'sym' denotes the symmetrization and 'alt' the anti-symmetrization of tensors.

More details can be found in Remark 2.12. For tensors (1.18) we obtained

Theorem 1.11. If we consider tensors $U \in \mathcal{T}_r$, $S \in S^2(V)$, $A \in \Lambda^2(V)$, $w \in \mathcal{T}_w V$ on a tangent space $V = T_p M$ of a differentiable manifold $M$, $\dim M \geq 3$, and form $U$ by one of the formulas (1.18), then we obtain the shortest lengths from Theorem 1.8 (1b) exactly in the following cases:

1. $y^r(U \otimes S)$ and $U = (\nabla \psi - \text{sym}(\nabla \psi))|_p$, $\psi \in \mathcal{T}_2 M$ symmetric,
2. $y^r(U \otimes w)$, $y^r(U \otimes S)$, $y^r(U \otimes A)$ and $U = (\nabla \omega - \text{alt}(\nabla \omega))|_p$, $\omega \in \mathcal{T}_2 M$ skew-symmetric.

Here is a brief outline to the paper. In Section 2 we give a summary of basic facts about symmetry classes, Young symmetrizers and discrete Fourier transforms. These tools are needed to obtain the infinite set $\mathcal{G}$ of symmetry classes for $U$. In Section 3 we prove the Theorems 1.4, 1.5 and Lemma 1.6 using the Littlewood-Richardson rule and computer calculations with group ring elements and tensor coordinates. In Section 4 we construct short coordinate representations for the tensors $y^r(U \otimes w)$ by determining and solving a complete system of linear identities for the tensors $U$. Furthermore we prove the Theorems 1.8 and 1.11 in this Section.

Many results were obtained by computer calculations by means of the Mathematica packages Ricci [24] and PERMS [8]. The Mathematica notebooks of these calculations are available at [5].

2. Symmetry classes, Young symmetrizers, discrete Fourier transforms

The vector spaces $\mathcal{A}(V)$ and $\mathcal{A}'(V)$ are symmetry classes in the sense of H. Boerner [11, p.127]. We denote by $\mathbb{K}[S_r]$ the group ring of a symmetric group $S_r$ over the field $\mathbb{K}$. Every group ring element $a = \sum_{p \in S_r} a(p) p \in \mathbb{K}[S_r]$ acts as so-called symmetry operator on tensors $T \in \mathcal{T}_r V$ according to the definition

$$(2.1) \quad (aT)(v_1, \ldots, v_r) := \sum_{p \in S_r} a(p) T(p(v_1), \ldots, p(v_r)), \quad v_i \in V.$$  

Equation (2.1) is equivalent to $(aT)_{i_1 \ldots i_r} = \sum_{p \in S_r} a(p) T_{i_{p(1)} \ldots i_{p(r)}}$.

Definition 2.1. Let $r \subseteq \mathbb{K}[S_r]$ be a right ideal of $\mathbb{K}[S_r]$ for which an $a \in r$ and a $T \in \mathcal{T}_r V$ exist such that $aT \neq 0$. Then the tensor set

$$(2.2) \quad \mathcal{T}_r := \{aT \mid a \in r, T \in \mathcal{T}_r V\}$$.
Lemma 2.2. If \(e\) is a generating idempotent of \(\mathfrak{r}\), then a tensor \(T \in T_rV\) belongs to \(T_e\) iff \(eT = T\). Thus we have \(T_e = \{ eT \mid T \in T_rV \}\).

Now we summarize tools from our Habilitationsschrift [7] (see also its summary [10]). We make use of the following connection between \(r\)-times covariant tensors \(T \in T_rV\) and elements of the group ring \(\mathbb{K}[S_r]\).

**Definition 2.3.** Any tensor \(T \in T_rV\) and any \(r\)-tuple \(b = (v_1, \ldots, v_r) \in V^r\) of \(r\) vectors from \(V\) induce a function \(T_b : S_r \to \mathbb{K}\) according to the rule

\[
T_b(p) := T(v_{p(1)}, \ldots, v_{p(r)}) , \quad p \in S_r.
\]

We identify this function with the group ring element \(T_b := \sum_{p \in S_r} T_b(p) p \in \mathbb{K}[S_r]\).

Obviously, two tensors \(S, T \in T_rV\) fulfill \(S = T\) iff \(S_b = T_b\) for all \(b \in V^r\). We denote by \(''\) the mapping \(\ast : a = \sum_{p \in S_r} a(p) p \mapsto a^* := \sum_{p \in S_r} a(p) p^{-1}\). Then the following important formula holds

\[
\forall T \in T_rV , \quad a \in \mathbb{K}[S_r] , \quad b \in V^r : \quad (aT)_b = T_b \cdot a^*.
\]

Now it can be shown that all \(T_b\) of tensors \(T\) of a given symmetry class lie in a certain left ideal of \(\mathbb{K}[S_r]\).

**Proposition 2.4.** Let \(e \in \mathbb{K}[S_r]\) be an idempotent. Then a \(T \in T_rV\) fulfils the condition \(eT = T\) iff \(T_b \in \mathfrak{l} := \mathbb{K}[S_r] \cdot e^*\) for all \(b \in V^r\), i.e. all \(T_b\) of \(T\) lie in the left ideal \(\mathfrak{l}\) generated by \(e^*\).

The proof follows easily from (2.4). Since a right ideal \(\mathfrak{r}\) defining a symmetry class and the left ideal \(\mathfrak{l}\) from Proposition 2.4 satisfy \(\mathfrak{r} = \mathfrak{l}^*\), we denote symmetry classes also by \(T_r\). A further result is

**Proposition 2.5.** If \(\dim V \geq r\), then every left ideal \(\mathfrak{l} \subseteq \mathbb{K}[S_r]\) fulfils \(\mathfrak{l} = L_{\mathbb{K}}\{ T_b \mid T \in T_r , \quad b \in V^r \}\). (Here \(L_{\mathbb{K}}\) denotes the forming of the linear closure.)

If \(\dim V < r\), then the \(T_b\) of the tensors from \(T_r\) will span only a linear subspace of \(\mathfrak{l}\) in general.

Important special symmetry operators are Young symmetrizers, which are defined by means of Young tableaux.

A Young tableau \(t\) of \(r \in \mathbb{N}\) is an arrangement of \(r\) boxes such that

1. the numbers \(\lambda_i\) of boxes in the rows \(i = 1, \ldots, l\) form a decreasing sequence \(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0\) with \(\lambda_1 + \ldots + \lambda_l = r\),
2. the boxes are fulfilled by the numbers \(1, 2, \ldots, r\) in any order.

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2See B. Fiedler [7] Sec.III.1 and B. Fiedler [9].
3See B. Fiedler [9] or B. Fiedler [7] Prop. III.2.5, III.3.1, III.3.4.
4See B. Fiedler [9] or B. Fiedler [7] Prop. III.2.6.
For instance, the following graphics shows a Young tableau of \( r = 16 \).

\[
\begin{array}{cccccc}
\lambda_1 = 5 & 11 & 2 & 5 & 4 & 12 \\
\lambda_2 = 4 & 9 & 6 & 16 & 15 \\
\lambda_3 = 4 & 8 & 14 & 1 & 7 \\
\lambda_4 = 2 & 13 & 3 \\
\lambda_5 = 1 & 10 \\
\end{array}
\]

\( = t \).

Obviously, the unfilled arrangement of boxes, the *Young frame*, is characterized by a partition \( \lambda = (\lambda_1, \ldots, \lambda_t) \vdash r \) of \( r \).

If a Young tableau \( t \) of a partition \( \lambda \vdash r \) is given, then the *Young symmetrizer* \( y_t \) of \( t \) is defined by\(^5\)

\[
y_t := \sum_{p \in \mathcal{H}_t} \sum_{q \in \mathcal{V}_t} \text{sign}(q) p \circ q
\]

where \( \mathcal{H}_t, \mathcal{V}_t \) are the groups of the *horizontal* or *vertical permutations* of \( t \) which only permute numbers within rows or columns of \( t \), respectively. The Young symmetrizers of \( \mathbb{K}[S_r] \) are essentially idempotent and define decompositions

\[
\mathbb{K}[S_r] = \bigoplus_{\lambda \vdash r} \bigoplus_{t \in \mathcal{S}T_\lambda} \mathbb{K}[S_r] \cdot y_t = \bigoplus_{\lambda \vdash r} \bigoplus_{t \in \mathcal{S}T_\lambda} y_t \cdot \mathbb{K}[S_r]
\]

of \( \mathbb{K}[S_r] \) into minimal left or right ideals. In (2.6), the symbol \( \mathcal{S}T_\lambda \) denotes the set of all standard tableaux of the partition \( \lambda \). *Standard tableaux* are Young tableaux in which the entries of every row and every column form an increasing number sequence.\(^6\)

Every *irreducible character* \( \chi : S_r \rightarrow \mathbb{C} \) of \( S_r \) induces a *centrally primitive idempotent* \( \chi := \chi^{(\text{id})} = \sum_{p \in S_r} \chi(p) p \) which generates a minimal two-sided ideal \( A_\chi := \mathbb{K}[S_r] \cdot \chi \). There is a *one-to-one correspondence* \( \chi \iff \lambda \) between the \( \chi \) and the partitions \( \lambda \vdash r \). For every \( \chi \) there exists a unique \( \lambda \vdash r \) such that

\[
A_\chi = \bigoplus_{t \in \mathcal{S}T_\lambda} \mathbb{K}[S_r] \cdot y_t = \bigoplus_{t \in \mathcal{S}T_\lambda} y_t \cdot \mathbb{K}[S_r]
\]

The set of all Young symmetrizers \( y_t \) which lie in \( A_\chi \) is equal to the set of all \( y_t \) whose tableau \( t \) has a frame \( \lambda \vdash r \). Furthermore two minimal left ideals \( I_1, I_2 \subseteq \mathbb{K}[S_r] \) or two minimal right ideals \( r_1, r_2 \subseteq \mathbb{K}[S_r] \) are *equivalent* iff they lie in the same ideal \( A_\chi \). Now we say that a symmetry class \( T_\chi \) *belongs to* \( \lambda \vdash r \) iff \( \mathfrak{r} \subseteq A_\chi \) and \( \chi \) corresponds to \( \lambda \). Then we write also \( A_\lambda \) for \( A_\chi \).

S.A. Fulling, R.C. King, B.G. Wybourne and C.J. Cummins showed in \[15\] that the symmetry classes of the Riemannian curvature tensor \( R \) and its *symmetrized covariant derivatives*

\[
(\nabla^{(\alpha)} R)_{ijkl;s_1 \ldots s_u} := \nabla_{(s_1} \nabla_{s_2} \cdots \nabla_{s_u)} R_{ijkl} = R_{ijkl; (s_1 \ldots s_u)}
\]

\(^5\)We use the convention \((p \circ q)(i) := p(q(i))\) for the product of two permutations \( p, q \).

\(^6\)About Young symmetrizers and Young tableaux see for instance \[11, 12, 15, 17, 19, 20, 25, 26, 27, 28, 29, 30\]. In particular, properties of Young symmetrizers in the case \( \mathbb{K} \neq \mathbb{C} \) are described in \[27\].

\((\ldots)\) denotes the symmetrization with respect to the indices \( s_1, \ldots, s_u \).
are generated by special Young symmetrizers. In the present paper we use only

**Theorem 2.6. (Fulling, King, Wybourne, Cummins)**

Let $y_t, y_{t'}$ be the Young symmetrizers of the standard tableaux

$$(2.9) \quad t = \begin{array}{ccc}
1 & 3 \\
2 & 4
\end{array}, \quad t' = \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4
\end{array}. $$

Then tensors $T \in T_4V, \hat{T} \in T_5V$ fulfil

$$(2.10) \quad T \in A(V) \iff \frac{1}{12} y_t^* T = T, \quad (2.11) \quad \hat{T} \in A'(V) \iff \frac{1}{24} y_{t'}^* \hat{T} = \hat{T}.$$ 

The group ring elements $\frac{1}{12} y_t^*, \frac{1}{24} y_{t'}^*$ are idempotents which are proportional to the essentially idempotent symmetrizers $y_t^*, y_{t'}^*$.

The group ring $K[S_3]$ decomposes into the minimal 2-sided ideals $a_{(3)}, a_{(21)}, a_{(13)}$. Whereas the 2-sided ideals $a_{(3)}, a_{(13)} \subset K[S_3]$ have dimension 1 and define consequently unique symmetry classes of $T_3V$, the 2-sided ideal $a_{(21)} \subset K[S_3]$ has dimension and contains an infinite set of minimal right ideals $\mathfrak{r}$ which lead to an infinite set of symmetry classes $T_\mathfrak{r}$ for tensors of $T_3V$. The set of generating idempotents for these right ideals $\mathfrak{r}$ is infinite, too. In [12] we used discrete Fourier transforms to determine a family of primitive generating idempotents of the above minimal right ideals $\mathfrak{r} \subset K[S_3]$.

We denote by $K^{d \times d}$ the set of all $d \times d$-matrices of elements of $K$.

**Definition 2.7.** A discrete Fourier transform for $S_r$ is an isomorphism

$$(2.12) \quad D : K[S_r] \to \bigotimes_{\lambda \vdash r} K^{d_\lambda \times d_\lambda}$$

according to Wedderburn’s theorem which maps the group ring $K[S_r]$ onto an outer direct product $\bigotimes_{\lambda \vdash r} K^{d_\lambda \times d_\lambda}$ of full matrix rings $K^{d_\lambda \times d_\lambda}$. We denote by $D_\lambda$ the natural projections $D_\lambda : K[S_r] \to K^{d_\lambda \times d_\lambda}$.

A discrete Fourier transform maps every group ring element $a \in K[S_r]$ to a block diagonal matrix

$$(2.13) \quad D : \quad a = \sum_{p \in S_r} a(p) p \mapsto \begin{pmatrix}
A_{\lambda_1} & 0 & & \\
& A_{\lambda_2} & & \\
& & \ddots & \\
& & & A_{\lambda_k}
\end{pmatrix}.$$ 

The matrices $A_\lambda$ are numbered by the partitions $\lambda \vdash r$. The dimension $d_\lambda$ of every matrix $A_\lambda \in K^{d_\lambda \times d_\lambda}$ can be calculated from the Young frame belonging to $\lambda \vdash r$ by means of the hook length formula. For $r = 3$ we have

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8A proof of this result of [15] can be found in [6, Sec.6], too. See also [14] for more details.

9The dimensions of the $a_{\lambda}$ can be calculated by means of the hook length formula.

10See M. Clausen and U. Baum [3, 4] for details about fast discrete Fourier transforms.
The inverse discrete Fourier transform is given by

Proposition 2.8. \(^{11}\) If \(D : \mathbb{K}[S_r] \to \bigotimes_{\lambda \vdash r} \mathbb{K}^{d_\lambda \times d_\lambda}\) is a discrete Fourier transform for \(\mathbb{K}[S_r]\), then we have for every \(a \in \mathbb{K}[S_r]\)

\[
\forall p \in S_r : \quad a(p) = \frac{1}{r!} \sum_{\lambda \vdash r} d_\lambda \text{trace}\left\{D_\lambda(p^{-1}) \cdot D_\lambda(a)\right\} = \frac{1}{r!} \sum_{\lambda \vdash r} d_\lambda \text{trace}\left\{D_\lambda(p^{-1}) \cdot A_\lambda\right\}.
\]

In our considerations we are interested in the matrix ring \(\mathbb{K}^{2 \times 2}\) which corresponds to the \((2,1)\)-equivalence class of minimal right ideals \(r \subset \mathbb{K}[S_3]\). In \([12]\) we proved

Proposition 2.9. Every minimal right ideal \(r \subset \mathbb{K}^{2 \times 2}\) is generated by exactly one of the following (primitive) idempotents

\[
Y := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad X_\nu := \begin{pmatrix} 1 & 0 \\ \nu & 0 \end{pmatrix}, \quad \nu \in \mathbb{K}.
\]

Using an inverse discrete Fourier transform we can determine the primitive idempotents \(\eta, \xi_\nu \in \mathbb{K}[S_3]\) which correspond to \(Y, X_\nu\) in (2.15). We calculated these idempotents by means of the tool InvFourierTransform of the Mathematica package PERMS [8] (see also [7, Appendix B]). This calculation can be carried out also by the program package SYMMETRICA [22, 23].

Proposition 2.10. Let us use Young’s natural representation\(^{12}\) of \(S_3\) as discrete Fourier transform. If we apply the Fourier inversion formula (2.14) to a \(4 \times 4\)-block matrix

\[
\begin{pmatrix} A_{(3)} & 0 \\ 0 & A_{(2,1)} \\ 0 & A_{(1,3)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

where \(A\) is equal to \(X_\nu\) or \(Y\) in (2.15), then we obtain the following idempotents of \(\mathbb{K}[S_3]\):

\[
X_\nu \quad \Rightarrow \quad \xi_\nu := \frac{1}{3} \left\{[1,2,3] + \nu[1,3,2] + (1-\nu)[2,1,3] \right. \\
\left. -\nu[2,3,1] + (-1+\nu)[3,1,2] - [3,2,1]\right\}
\]

\[
Y \quad \Rightarrow \quad \eta := \frac{1}{3} \left\{[1,2,3] - [2,1,3] - [2,3,1] + [3,2,1]\right\}.
\]

\(^{11}\)See M. Clausen and U. Baum [3, p.81].

\(^{12}\)Three discrete Fourier transforms \(2,4,12\) are known for symmetric groups \(S_r\): (1) Young’s natural representation, (2) Young’s seminormal representation and (3) Young’s orthogonal representation. See [11, 2, 20, 3]. A short description of (1) and (2) can be found in [7, Sec.I.5.2]. All three discrete Fourier transforms are implemented in the program package SYMMETRICA [22, 23]. PERMS [8] uses the natural representation. The fast DFT-algorithm of M. Clausen and U. Baum [3, 4] is based on the seminormal representation.
Remark 2.11. It is interesting to clear up the connection of the idempotents $\xi_\nu$ and $\eta$ with Young symmetrizers. A simple calculation shows that
\begin{equation}
\xi_0 = \frac{1}{3} y_{t_1}, \quad \eta = \frac{1}{3} y_{t_2}
\end{equation}
where $y_{t_1}$ and $y_{t_2}$ are the Young symmetrizers of the tableaux
\[
\begin{array}{c}
1 & 2 \\
3
\end{array}
\quad , \quad
\begin{array}{c}
1 \\
2 & 3
\end{array}.
\]

Remark 2.12. In [14, Thm 6.1] we showed that the symmetry classes $T_{r_s}, T_{r_a}$ of the tensors (1.18) correspond to the following values of the parameter $\nu$ in (2.17):
\begin{equation}
\xi_\nu \cdot K[S_3] = r_s \Leftrightarrow \nu = 0, \quad \xi_\nu \cdot K[S_3] = r_a \Leftrightarrow \nu = 2.
\end{equation}

3. Proof of the Theorems 1.4, 1.5 and of Lemma 1.6

3.1. Proof of Theorem 1.4. In [12] the statements of Theorem 1.4 about the products (a'), (b'), (c') were proved by an application of the Littlewood-Richardson rule. Using [12, Prop.2.10] and the arguments of [12, Sec.3.1] we can prove the assertion about the products (a) of Theorem 1.4 in the same way.

For tensor products $U \otimes w$ and $w \otimes U$ we have to investigate Littlewood-Richardson products $[\lambda][1] \sim [\lambda] \neq [1] \uparrow S_4$, $\lambda \vdash 3$. The three partitions (3), (2 1), (1 3) lead to the three Littlewood-Richardson products
\[
[3][1] \sim [4] + [3 1],
[2 1][1] \sim [3 1] + [2^2] + [2 1^2],
[1 3][1] \sim [2 1^2] + [1^4].
\]
But then we obtain by means of the arguments of [12, Sec.3.1], that the symmetry class of $U$ must belong to the partition (2 1) $\vdash 3$ since only the product $[2 1][1]$ contains a part $[2^2]$. The part $[2^2]$ describes a minimal left ideal $I \subset K[S_4]$ which lies in the same equivalence class of minimal left ideals as the left ideal $K[S_4] \cdot y_t$ generated by the Young symmetrizer $y_t$ from Theorem 2.6 with the Young frame $(2^2) \vdash 4$. Consequently, only for tensors $T = U \otimes w$ with a part $[2^2]$ a relation $0 \neq T_b \cdot y_t = (y_t T) b, b \in V^4$, or equivalently $y_t^*(U \otimes w) \neq 0$ is possible.

3.2. Proof of Theorem 1.5. The equivalence of the Statements (2), (3), (4) of Theorem 1.5 follows from [12, Thm.1.10]. The equivalence of (1) and (4) can also be proved by means of conclusions which were used in the proof of [12, Thm.1.10].

For a treatment of expressions $y_t^*(U \otimes w)$ we form the following elements of $K[S_4]$:
\begin{equation}
\sigma_\nu := y_t^* \cdot \xi_\nu, \quad \rho := y_t^* \cdot \eta,
\end{equation}
\begin{equation}
\xi_\nu \mapsto \xi_\nu' \in K[S_4], \quad \eta \mapsto \eta' \in K[S_4].
\end{equation}

Formula (3.2) denotes the embedding of $\xi_\nu, \eta \in K[S_3]$ into $K[S_4]$ which is induced by the mapping $S_3 \to S_4$, $[i_1, i_2, i_3] \mapsto [i_1, i_2, i_3, 4]$.

Using the Mathematica package PERMS we verified that

\text{See the references [20, 21, 19, 25, 26, 13, 16] for the Littlewood-Richardson rule. See also [14].}
If $\rho \neq 0$ and $\sigma_\nu \neq 0 \Leftrightarrow \nu \neq \frac{1}{2}$.

The calculation stored in the notebook \[5\], part1.nb\] yields (3.15) and the following 1-expression of 16 terms for the coordinates of $\nu$ proved in [14, Appendix A] by computer calculations using the Mathematica notebooks of PERMS-calculations for this proof can be found on the web page [5]. □

3.3. Proof of Lemma 1.6. The formulas (1.10) and (1.17) in Lemma 1.6 were proved in [14], Appendix A by computer calculations using the Mathematica packages Ricci [24] and PERMS [8]. A proof of (1.15) can be obtained in the same way. The calculation stored in the notebook [5], part1.nb] yield (1.15) and the following expression of 16 terms for the coordinates of $\frac{1}{12} y^*_i(U \otimes w)$:

$$\begin{align*}
\frac{1}{12} U_{jkl} w_i - \frac{1}{12} U_{jlk} w_i &+ \frac{1}{12} U_{ikj} w_i = \frac{1}{12} U_{lki} w_j + \frac{1}{12} U_{ikj} w_l - \frac{1}{12} U_{ljk} w_l + \\
\frac{1}{12} U_{kli} w_j - \frac{1}{12} U_{ilk} w_j &+ \frac{1}{12} U_{ijl} w_k + \frac{1}{12} U_{jli} w_k - \frac{1}{12} U_{kji} w_k + \\
\frac{1}{12} U_{ijkl} w_l &- \frac{1}{12} U_{jik} w_l = \frac{1}{12} U_{kji} w_l + \frac{1}{12} U_{kji} w_l
\end{align*}$$

(3.4)

4. Short formulas for algebraic curvature tensors $R$

4.1. The reduction method. In this section we begin to construct short coordinate representations of tensors $y^*_i(U \otimes w)$ considered in Theorem 1.5 and Lemma 1.6 Formula (3.4) represents the coordinates of $\frac{1}{12} y^*_i(U \otimes w)$ by a relatively long polynomial

$$\begin{align*}
\mathfrak{P}_{i_1 \ldots i_4} &:= \frac{1}{12} y^*_i(U \otimes w)_{i_1 \ldots i_4} = \sum_{p \in S_4} c_p U_{i_{p(1)}i_{p(2)}i_{p(3)}i_{p(4)}} w_{i_{p(1)}i_{p(2)}i_{p(3)}i_{p(4)}} , \ c_p \in \mathbb{K}
\end{align*}$$

in the coordinates of $U$ and $w$. In [14, Sec.3.4] we developed a method to reduce the length of the coordinate representation of tensors $y_\nu(U \otimes S)$ and $y_\xi(U \otimes A)$ from Theorem 1.5. We can also use this method for a reduction of the length of (4.1). Here is a summary of the method from [14, Sec.3.4].

A central role plays the following
Proposition 4.1. Let \( \mathfrak{r} \subset \mathbb{K}[S_r] \) be a d-dimensional right ideal that defines a symmetry class \( \mathcal{T}_\mathfrak{r} \) of tensors \( T \in \mathcal{T}_3 V \). If a basis \( \mathcal{B} = \{h_1, \ldots, h_a\} \) of the left ideal \( \mathfrak{l} = \mathfrak{r}^* \) is known, then the coefficients \( x_p \in \mathbb{K} \) for linear identities

\[
(4.2) \sum_{p \in S_r} x_p T_{i_p(1), \ldots, i_p(r)} = 0
\]
satisfied by the coordinates of the \( T \in \mathcal{T}_3 \) can be obtained from the linear \( (d \times r!)- \) equation system

\[
(4.3) \sum_{p \in S_r} h_i(p) x_p = 0 \quad (i = 1, \ldots, d).
\]

For the tensors \( U \) considered in Theorem 1.3 the system \((4.3)\) has a \((2 \times 6)\)-coefficient matrix since \( d = \dim \mathfrak{r} = 2 \) for the right ideal \( \mathfrak{r} \) defining the symmetry class of \( U \). Using discrete Fourier transforms and results from [7] we proved in [14] Lemma 4.2. The left ideals \( \mathbb{K}[S_3] \cdot \xi^*, \mathbb{K}[S_3] \cdot \eta^* \) given by the idempotents \((2.17), (2.18)\), possess bases which lead to the following coefficient matrices in \((4.3)\)

\[
(4.4) \xi_\nu \Rightarrow \frac{1}{9} \left( \begin{array}{ccccccc} 4 - 2\nu & -2 + 4\nu & 4 - 2\nu & -2 + 4\nu & -2 - 2\nu & -2 - 2\nu \\ -2 + 4\nu & 4 - 2\nu & -2 + 4\nu & 4 - 2\nu & -2 - 2\nu & -2 - 2\nu \\ 2 - 1 & -1 & 2 - 1 & -1 & 2 - 1 & -1 \\ 2 - 1 & -1 & 2 - 1 & -1 & 2 - 1 & -1 \end{array} \right).
\]

\[
(4.5) \eta \Rightarrow \frac{1}{9} \left( \begin{array}{ccccccc} -1 & 2 & -1 & 2 & -1 & -1 & 2 \\ 2 & -1 & -1 & 2 & -1 & -1 & 2 \\ -1 & 2 & -1 & 2 & -1 & -1 & 2 \\ 2 & -1 & -1 & 2 & -1 & -1 & 2 \end{array} \right).
\]

Here \( \nu \in \mathbb{K} \) is arbitrary. Further we use the following correspondence \( a \leftrightarrow p_a \) between the column index \( a \) in \((4.4), (4.5)\) and permutations \( p_a \in S_3 \):

\[
(4.6)
\begin{array}{cccccccc}
 a & 1 & 2 & 3 & 4 & 5 & 6 \\
p_a & [1, 2, 3] & [1, 3, 2] & [2, 1, 3] & [2, 3, 1] & [3, 1, 2] & [3, 2, 1] \\
\end{array}
\]

For \( a, b \in \{1, \ldots, 6\} \), \( a < b \), we denote by \( \mathcal{P}_{ab} \) the 2-set \( \mathcal{P}_{ab} \coloneqq \{p_a, p_b\} \) of permutations from \( S_3 \) which correspond to \( a, b \) via \((4.6)\). Furthermore we write \( \Delta_{\mathcal{P}_{ab}} \) for the determinant of the \((2 \times 2)\)-submatrix of \((4.4)\) or \((4.5)\) whose columns correspond to \( p_a, p_b \).

Procedure 4.3. Consider a symmetry class \( \mathcal{T}_\mathfrak{r} \) of tensors \( U \in \mathcal{T}_3 V \) defined by \( \xi_\nu \) or \( \eta \), and the corresponding equation system \((4.3)\) with coefficient matrix \((4.4)\) or \((4.5)\). Then carry out the following steps for every set \( \mathcal{P}_{ab} \) :

1. Check the condition \( \Delta_{\mathcal{P}_{ab}} \neq 0 \). If \( \Delta_{\mathcal{P}_{ab}} = 0 \), then skip the steps \((2), (3)\) for \( \mathcal{P}_{ab} \).
2. If \( \Delta_{\mathcal{P}_{ab}} \neq 0 \), then, for every \( \bar{p} \in S_3 \setminus \mathcal{P}_{ab} \), determine the solution \( x_p^{(\bar{p})} \) of \((4.3)\) which fulfills \( x_p^{(\bar{p})} = 1 \) and \( x_p^{(\bar{p})} = 0 \) for all \( p \in S_3 \setminus (\mathcal{P}_{ab} \cup \{\bar{p}\}) \).
3. Use the \( x_p^{(\bar{p})} \) of step \((2)\) to form identities

\[
(4.7) 0 = \sum_{p \in \mathcal{P}_{ab}} x_p^{(\bar{p})} U_{i_{p(1)}i_{p(2)}i_{p(3)}} + U_{i_{\bar{p}(1)}i_{\bar{p}(2)}i_{\bar{p}(3)}} \quad (\bar{p} \in S_3 \setminus \mathcal{P}_{ab}).
\]
(4) Interpret \( \{ i_1, i_2, i_3 \} \) as a permuted arrangement of a lexicographically ordered sequence \( \{ 1, 2, 3 \} \). Use (1.7) to express all coordinates \( U_{i_1 i_2 i_3} \), \( \tilde{p} \in S_3 \), by the coordinates \( U_{p(1)p(2)p(3)} \), \( p \in P_{ab} \), in (4.4).

For instance, let us consider (4.4). Then the set \( P_{12} = \{ [1, 2, 3], [1, 3, 2] \} \) leads to the determinant \( \Delta_{P_{12}}(\nu) = \frac{1}{27} (1 - \nu)(1 + \nu) \) which has the roots \( \nu_1 = 1 \) and \( \nu_2 = -1 \). For \( \nu \notin \{ 1, -1 \} \) we obtain the identities

\[
- \frac{\nu^2 - \nu - 1}{\nu^2 - \nu - 1} U_{ijk} + \frac{2\nu - 1}{\nu^2 - \nu - 1} U_{ikj} + U_{kij} = 0 \quad (4.8)
\]

\[
\frac{2\nu - 1}{\nu^2 - \nu - 1} U_{ijk} - \frac{\nu^2 - \nu + 1}{\nu^2 - \nu - 1} U_{ikj} + U_{jki} = 0 \quad \text{and}
\]

\[
\frac{\nu^2 - \nu + 1}{\nu^2 - \nu - 1} U_{ijk} + \frac{2\nu - 1}{\nu^2 - \nu - 1} U_{ikj} + U_{jik} = 0.
\]

There exist 15 subsets \( P_{ab} \) for \( U \) with respect to (4.4). The matrix (4.5) leads to 12 systems (4.8) because \( \Delta_{P_{16}} = \Delta_{P_{24}} = \Delta_{P_{15}} = 0 \) (see [11] Table 1).

4.2. Proof of Theorem 1.8. (1a). We carried out Procedure 4.3 in computer calculations using the Mathematica packages Ricci [24] and PERMS [8]. The Mathematica notebooks are available on the web page [5]. In all cases with \( \Delta_{P_{ab}} \neq 0 \) and \( \nu \neq \frac{1}{2} \) we obtained a reduction of (3.4) to 8 terms both for (4.4) and for (4.5). The roots of \( \Delta_{P_{ab}}(\nu) \) for (4.4) are given in [14] Table 2.

4.3. Proof of Theorem 1.8 (1b) and (2). For symmetry classes of tensors \( U \) described by \( \xi_1 \) and (4.3) a further reduction of the length of (3.4) is possible. When we use a system of linear identities (1.7) to reduce the length of (4.1), (3.4) then we obtain a sum with a structure

\[
(4.9) \quad \Psi_{i_1 \ldots i_4}^{red} = \sum_{q \in S_3} P_{q}^{P_{ab}}(\nu) Q_{q}^{P_{ab}}(\nu) U_{i_1 i_2 i_3 i_4} w_{i_1 i_2 i_3 i_4},
\]

where \( P_{q}^{P_{ab}}(\nu) \) and \( Q_{q}^{P_{ab}}(\nu) \) are polynomials, because the entries of (4.4) are polynomials. For instance, the identities (4.9) belonging to \( P_{12} \) transform (3.4) into

\[
(4.10) \quad \frac{1}{12} y_{ijkl}(U \otimes w)_{ijkl} = -\frac{1+2\nu}{12(-1+\nu)} U_{ijkl} w_i - \frac{1+2\nu}{12(-1+\nu)} U_{jikl} w_i - \frac{1+2\nu}{12(-1+\nu)} U_{ikjl} w_i + \frac{1+2\nu}{12(-1+\nu)} U_{ijkl} w_j + \frac{1-4\nu-4\nu^2}{12(-1+\nu)(1+\nu)} U_{ijkl} w_k + \frac{1-4\nu-4\nu^2}{12(-1+\nu)(1+\nu)} U_{ijkl} w_l.
\]

Now, for every \( P_{ab} \) we determine the set \( N_{P_{ab}} \) of all roots \( \nu \) of the \( P_{q}^{P_{ab}}(\nu) \) in (4.9) which are different from the roots of \( \Delta_{P_{ab}}(\nu) \), \( Q_{q}^{P_{ab}}(\nu) \) and from \( \nu = \frac{1}{2} \). If we set such a \( \nu \in N_{P_{ab}} \), into (4.9), the length of (4.9) will decrease. For example, (4.10) yield \( N_{P_{12}} = \{ 2 \} \). The root \( \nu = 2 \) reduces (4.10) to the 6 terms

\[
(4.11) \quad \frac{1}{4} U_{ijkl} w_i - \frac{1}{4} U_{ijkl} w_i - \frac{1}{4} U_{ikjl} w_j + \frac{1}{4} U_{ijlk} w_k - \frac{1}{4} U_{ijkl} w_l.
\]

Using Ricci [24] and PERMS [8], we determined all sets \( N_{P_{ab}} \) and all resulting length reductions of (4.9). Table 1 shows the results. The minimal length of \( \Psi_{i_1 \ldots i_4}^{red} \)
Table 1. The lengths of \( \mathfrak{P} \) for an \( U \) from a \( \xi_\nu \)-symmetry class, where \( \nu \) is an allowed root of a \( P_{q,ab}^\nu \).

| \( P_{ab} \) | \( N_{P_{ab}} \) | length of \( \mathfrak{P} \) | \( P_{ab} \) | \( N_{P_{ab}} \) |
|---|---|---|---|---|
| 12 | 2 | 6 | 25 | -1 |
| 13 | -1 | 6 | 26 | -1 |
| 14 | -1 | 6 | 26 | 2 |
| | 2 | 4 | | 6 |
| 15 | -1 | 4 | 34 | 2 |
| | 2 | 6 | | 4 |
| 16 | -1 | 4 | 35 | -1 |
| | 2 | 6 | | 2 |
| 17 | -1 | 6 | 36 | -1 |
| | 2 | 4 | | 2 |
| 18 | -1 | 6 | 45 | -1 |
| | 2 | 6 | | 2 |
| 19 | -1 | 6 | 46 | -1 |
| | 2 | 6 | | 2 |

which we found is equal to 4. For example, the set \( P_{16} \) lead to \( N_{P_{16}} = \{-1, 2\} \) and these two roots reduce \( \frac{1}{12} y^*_i(U \otimes w)_{ijkl} \) to the following 4 expressions

\[
\begin{align*}
\nu = -1 & \Rightarrow \frac{1}{4} U_{jkl} w_i - \frac{1}{4} U_{ikl} w_j - \frac{1}{4} U_{lji} w_k + \frac{1}{4} U_{kji} w_l \\
\nu = 2 & \Rightarrow \frac{1}{4} U_{lkj} w_i - \frac{1}{4} U_{lkj} w_j - \frac{1}{4} U_{ijl} w_k + \frac{1}{4} U_{ijl} w_l
\end{align*}
\]

Similar tables for \( y^*_i(U \otimes S) \) and \( y^*_i(U \otimes A) \) are given in [14]. They yield the minimal lengths 12 and 10 in Statement (1b) of Theorem 1.8. Furthermore Table 1 and the tables in [14] show that Statement (2) of Theorem 1.8 is valid exactly for the two right ideals \( \mathfrak{r} = \xi_\nu \cdot \mathbb{K}[S_3] \) which belong to \( \nu = -1 \) and \( \nu = 2 \). □

4.4. Proof of Theorem 1.8 (3) and Theorem 1.11. In [14, Sec.5] we proved that a symmetry class \( T_\mathfrak{r} \) of a minimal right ideal \( \mathfrak{r} \subset \mathbb{K}[S_3] \) admits an index commutation symmetry of the tensors \( U \in T_\mathfrak{r} \) iff \( \mathfrak{r} = \eta \cdot \mathbb{K}[S_3] \) or \( \mathfrak{r} = \xi_\nu \cdot \mathbb{K}[S_3] \) with \( \nu \in \{-1, 0, 1, 2\} \). The minimal lengths of Theorem 1.8 (1b) occured in following cases:

for \( y^*_i(U \otimes w) \), \( y^*_i(U \otimes A) \) \( \Rightarrow \mathfrak{r} = \xi_\nu \cdot \mathbb{K}[S_3] \) with \( \nu \in \{-1, 2\} \),

for \( y^*_i(U \otimes S) \) \( \Rightarrow \mathfrak{r} = \eta \cdot \mathbb{K}[S_3] \) or \( \mathfrak{r} = \xi_\nu \cdot \mathbb{K}[S_3] \), \( \nu \in \{-1, 0, 1, 2\} \).

This proves Theorem 1.8 (3). Furthermore, Theorem 1.11 follows from Remark 2.12. □

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