Binding energy in two and three-body relativistic dynamics *

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Abstract

Two-body and three-body systems of scalar bosons are considered in the framework of covariant constraint dynamics. The reduced equation obtained after eliminating redundant degrees of freedom can be viewed as an eigenvalue equation for an observable which is intimately related with the relative motion. We display the connection of this observable with binding energy.

1 Introduction

In galilean mechanics, the binding energy of a bound state is just the energy of relative motion. It is an observable and can be expressed in terms of dynamical variables.

In the framework of special relativity, in contrast, the binding energy of a bound state is usually defined as the mass defect \( \sum m - M \) where \( M \) is the total mass and \( \sum m \) refers to the constituent masses. But it is still natural to look for some observable, depending on the relative degrees of freedom, that could precisely characterize the relative motion of the system. Of course, in case we succeed, a comparison of this quantity with \( \sum m - M \) is desirable.

There are many different formulations of few-body relativistic dynamics [1] [2]. Our approach is based upon coupled wave equations [3] [4] that can be interpreted as mass-shell constraints [5]. For scalar particles, these constraints deal with squared-mass operators that include an interaction term. In contrast, all the generators of the Poincaré algebra are free of interaction. In this report, we focus on scalar particles.

For two (resp. three) particles we start from a pair (resp. a triple) of coupled equations, for one wave function depending on two (resp. three) four-dimensional variables.
arguments. In this formulation, like in the Bethe-Salpeter (BS) equation, covariance is paid by redundant degrees of freedom.

But demanding that the total linear momentum four-vector \( P = \sum p \) has a sharp value \( k \), and solving the difference(s) of the constraint equations, it is possible to factorize out one (resp. two) degrees of freedom, and one is left with a reduced wave equation, in the form of an eigenvalue problem; solving this problem determines the total mass \( M = \sqrt{k \cdot k} \).

This procedure is straightforward for two-body systems \( \mathbb{3} \), but in the three-body case, it requires a nontrivial transformation \( \mathbb{3} \).

In both cases, by an elementary combination of the mass-shell constraints, one can exhibit a constant of the motion, say \( N = -\Lambda \) which is homogeneous to a squared mass, depends only on relative variables (therefore is translation invariant), includes the term of mutual interaction, and is intimately related with relative motion. Our purpose is to clarify the relationship between this observable and the mass defect. It seems natural to define slow relative motion by the condition that the eigenvalue \( \lambda \) of \( \Lambda \) is small (in absolute value) with respect to the squares of the constituent masses, \( m_a \) (this situation in turn implies that also \( |\lambda| \ll M^2 \)).

**Notation**

The dynamical variables associated with particle \( a \) are the four-dimensional conjugate quantities \( q_a, p_a \). Particle labels are \( a, b = 1, 2 \) (resp. \( a, b = 1, 2, 3 \)).

## 2 Two-Body System

For the class of models that we consider, the same interaction term \( 2V \) arises in both squared-mass operators. This scheme seems to accomodate practically all realistic interactions inspired from quantum field theory. Its relationship with either the quasi-potential approach \( \mathbb{4} \) \( \mathbb{5} \) or the BS equation \( \mathbb{4} \) has been demonstrated, and fermionic generalizations have been elaborated. We introduce relative variables; they are the four-vectors

\[
z = q_1 - q_2, \quad y = \frac{1}{2} (p_1 - p_2)
\]

We have the transverse parts

\[
\tilde{z} = z - \frac{z \cdot P}{P^2} P, \quad \tilde{y} = y - \frac{y \cdot P}{P^2} P,
\]

For arbitrary masses it is convenient to set

\[
\mu = \frac{1}{2} (m_1^2 + m_2^2), \quad \nu = \frac{1}{2} (m_1^2 - m_2^2)
\]

The reduced wave equation takes on the form

\[
(N + \lambda)\phi = 0
\]

with

\[
N = \tilde{y}^2 + 2V
\]
\[ \lambda = \frac{M^2}{4} + \frac{\nu^2}{M^2} - \mu \] (6)

The interaction term \( V \) has the dimension of a squared mass. It may essentially depend on \( \tilde{z}^2 \) and \( P^2 \) (possibly also on \( \tilde{y}^2, \tilde{z} \cdot \tilde{y}, y \cdot P \)). On shell and using the rest frame, we simply have \( \tilde{z}^2 = -z^2 \). Therefore (4) can be identified with a non-relativistic equation [10], at least when the ”potential” \( V \) does not depend on \( P^2 \). But realistic interactions may bear some dependence on \( P^2 \) (energy dependent potentials); in this case, (4) becomes a nonconventional eigenvalue problem. This complication has been discussed in the literature [8] [9].

**Weak Binding**

Eq. (6) can be solved for \( M^2 \). Insofar as \( \nu \) is not too large, we obtain

\[ M^2 = 2(\lambda + \mu) + 2\sqrt{(\lambda + \mu)^2 - \nu^2} \] (7)

For slow relative motion, \( |\lambda| \ll m_a^2 \), we can develop \( M^2 \) in powers of \( \lambda \). According to (3) we get

\[ M^2 = (m_1 + m_2)^2 + \lambda \frac{(m_1 + m_2)^2}{m_1 m_2} + O(\lambda^2) \]

hence

\[ m_1 + m_2 - M = -\frac{\lambda}{2m_0} + \cdots \] (8)

It is clear that slow relative motion corresponds to weak binding.

## 3 Three-Body System

The squared mass operators are \( p_n^2 + 2V \). This formulation aims at the elimination of two superfluous degrees of freedom.

For three-body systems it is difficult to find an interaction such that the mass-shell constraints are compatible among themselves, respect Poincaré invariance, reduce to three Klein-Gordon equations in the absence of interaction, and allow for eliminating two redundant degrees of freedom. All these requirements can be satisfied however, in a tractable manner, with help of a ”point transformation in momentum space”. In order to formulate this transformation, it is essential to introduce relative variables as follows [11].

Relative-particle indices are \( A, B = 2, 3 \). We define the four-vectors

\[ z_A = q_1 - q_A, \quad y_B = \frac{P}{3} - p_B \] (9)

The transverse part are \( \tilde{z}_A, \tilde{y}_B \) and \( \tilde{z}, \tilde{y} \) respectively with respect tp \( P \) and \( k \). In the rest frame we have \( \tilde{y}_A^2 = -q_A^2, \quad \tilde{z}_A^2 = -z_A^2, \) etc.

Our transformation [4] is characterized by

\[ (p_1 + p_A) \cdot (p_1 - p_A) = P \cdot (p'_1 - p'_A) \] (10)

and by the requirement that it leaves \( P \) and \( \tilde{y}_A \) invariant (the new relative momenta are of course \( \tilde{y}'_A = P/3 - p'_A \)).
This procedure generates a canonical transformation, giving rise to new configuration variables $z'_A$. The difference equations are mapped to $y'_A \cdot P = c_A\Psi$ where the constants $c_A$ are combinations of the squared constituent masses; the dependence of $\Psi$ on the new relative times is factorized out.

Here, for simplicity, we assume that $m_a = m$.

The sum of the mass-shell constraints yields the reduced equation

$$(9m^2 - M^2)\psi = 6N\psi$$

(11)

where now

$$N = -\Lambda = \hat{y}_2^2 + \hat{y}_3^2 + \hat{y}_2 \cdot \hat{y}_3 + 3V + P^2\Xi$$

(12)

In the momentum representation, the reduced wave function $\psi$ depends only on $\hat{y}_2, \hat{y}_3$. In view of the compatibility requirement, a closed form of the interaction is available only in terms of the new variables. A typical example would be a function of $\hat{z}'_2, \hat{z}'_3, P^2$.

The quantity $P^2\Xi$ has no counterpart in two-body systems. Here it stems from having added three constraints. Its exact expression in terms of the new variables amounts to solve a fourth-degree algebraic equation and would be extremely complicated (see details in [6]). Fortunately, it can be naturally expanded in powers of $1/P^2$, implying on the mass shell an expansion in powers of $1/M^2$. This makes our model more tractable when the constituent particles are light with respect to the total mass of the system.

When the constituent masses (although different from zero) are small with respect to the total mass $M$, that is $m^2 \ll M^2$, we can drop the last term in (12).

With this truncation, (11) is similar to a nonrelativistic equation, except for eventual complications resulting from a possible dependence of $N$ on $P^2$. Moreover, in this limit $z'_A$ differs very little from $\hat{z}'_A$, which allows for a weak form of cluster separability [6].

But when $m^2/M^2$ is not small enough, the on-shell expression of $P^2\Xi$ must be written as a Taylor expansion including several powers of $1/M^2$, say $M^2\Xi = \frac{1}{M^2} \Gamma$, where $\Gamma$ is regular for $M \to \infty$, say $\Gamma = \Gamma(0) + O(1/M^6)$ where

$$\Gamma(0) = (\hat{y}_2^2)^2 + (\hat{y}_3^2)^2 + (\hat{y}_2 \cdot \hat{y}_3)^2 + (\hat{y}_2^2 + \hat{y}_3^2) (\hat{y}_2 \cdot \hat{y}_3) - \hat{y}_2^2 \hat{y}_3^2$$

(13)

Note that $\Gamma$ is a positive operator and would survive in the absence of interaction. Irrespective of the shape of the potential, we expect that it provides a positive correction to the truncated expression of $N$. A rigorous statement, however, would require solving a nonconventional eigenvalue problem.

At first order in $1/M^2$ we have, in the rest frame

$$N\psi = (-y_2^2 - y_3^2 - y_2 \cdot y_3 + 3V + \frac{\Gamma(0)}{M^2})\psi$$

In the rest-frame $\Gamma(0)$ is bi-quadratic in $y$.

Let us evaluate the binding energy, in the general case described by (11). Equation (11) can be viewed as a (generalized) eigenvalue equation for $\Lambda$ with eigenvalue $\lambda$,
if we set $6\lambda = M^2 - 9m^2$. The mass defect is exactly

$$\sum m - M = 3m - \sqrt{9m^2 + 6\lambda}$$

It would increase together with the eigenvalue of $N$.

**Weak Binding**

It is noteworthy that weakly bound systems are not eligible for the light-constituent approximation. Indeed they have $m^2$ almost one order of magnitude smaller than $M^2$, but this is not enough for dropping terms like $O(1/M^2)$. Assuming that $|\lambda| \ll m^2$ we can develop the mass defect and find

$$3m - M = -\frac{\lambda}{m} + \frac{1}{6} \frac{\lambda^2}{m^3} + \cdots$$

(15)

Since we consider equal masses, then $m = 2m_0$ where $m_0$ is the reduced mass of either of particles 2, 3, with respect to particle 1. Note the analogy of (15) with (8).

4 Conclusion

The constraint formulation of relativistic two-body dynamics admits a well-understood contact with field theory, whereas constraint three-particle dynamics is still a field of recent investigation. Nevertheless, it is possible to present in parallel ways, for both cases, the relationship of binding energy with a remarkable observable $N$ which naturally arises in the reduced wave equation. This situation results from the fact that, in both cases, our basic equations involve a unique interaction term and are tailored for allowing elimination of the redundant variable(s) implied by manifest covariance.

Binding energy, defined as the mass defect, has a simple relationship with the eigenvalue of $N$. In the case of weak binding, these quantities become proportional through the constant factor $1/2m_0$.

The reduced wave equation can be compared and, to some extent, identified with a nonrelativistic equation: in a straightforward manner for two particles (with arbitrary masses), but only in the light-constituent limit for three particles (with equal masses); in three-body systems, weak binding does not correspond to a nonrelativistic form of the wave equation.

For applications, we plan to introduce spin and to improve the contact with other approaches [12].

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