Cohomology of holomorphic line bundles and Hodge symmetry on Oeljeklaus–Toma manifolds

Hisashi Kasuya

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Abstract

We prove the Hodge symmetry type result on the Dolbeault cohomology of Oeljeklaus–Toma manifolds with values in the direct sum of holomorphic line bundles. Consequently, we show the vanishing and non-vanishing of Dolbeault cohomology of Oeljeklaus–Toma manifolds with values in holomorphic line bundles.

Keywords
Oeljeklaus–Toma manifold · Solvmanifold · Dolbeault cohomology · Holomorphic Line bundle

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1 Introduction

For positive integers $s, t$, let $K$ be a finite extension field of $\mathbb{Q}$ of degree $s + 2t$ admitting embeddings $\sigma_1, \ldots, \sigma_s, \sigma_{s+1}, \ldots, \sigma_{s+2t}$ into $\mathbb{C}$ such that $\sigma_1, \ldots, \sigma_s$ are real embeddings and $\sigma_{s+1}, \ldots, \sigma_{s+2t}$ are complex ones satisfying $\sigma_{s+i} = \overline{\sigma_{s+i+t}}$ for $1 \leq i \leq t$. Denote by $\mathcal{O}_K$ the ring of algebraic integers of $K$. In [9], for a free subgroup $U$ of rank $s$ in the group of units in $\mathcal{O}_K$ satisfying certain conditions related to the embeddings $\sigma_1, \ldots, \sigma_s, \sigma_{s+1}, \ldots, \sigma_{s+2t}$, Oeljeklaus and Toma construct an $(s + t)$-dimensional complex manifold $X(K, U)$ whose fundamental group is the semi-direct product $U \rtimes \mathcal{O}_K$ of finitely generated free abelian groups $U$ and $\mathcal{O}_K$. We call this complex manifold an Oeljeklaus–Toma (OT) manifold. For any OT-manifold $X(K, U)$, the Hodge symmetry

$$\dim H^{p,q}(X(K, U)) = \dim H^{q,p}(X(K, U))$$
on the Dolbeault cohomology does not hold (see [9, Proposition 2.5]). Hence every OT-manifold $X(K, U)$ is a non-Kähler complex manifold. Meanwhile, in [11], Otiman and Toma show that the Hodge decomposition

$$\dim H^r(X(K, U), \mathbb{C}) = \sum_{p+q=r} \dim H^{p,q}(X(K, U))$$

holds for any OT-manifold $X(K, U)$.

Consider the set $\text{Hom}(U, \mathbb{C}^*)$ of group homomorphisms from $U$ to $\mathbb{C}^*$. Corresponding representations of the fundamental group of $X(K, U)$ to flat bundles over $X(K, U)$, this set is identified with a set $\mathcal{A}(U)$ of isomorphism classes of flat complex line bundles over $X(K, U)$. For $E \in \mathcal{A}(U)$, we consider the de Rham cohomology $H^*(X(K, U), E)$ with values in $E$. We have the following result.

**Proposition 1.1** For any integer $r$, we have

$$\dim \bigwedge^r \mathbb{C}^{2s+2t} = \sum_{E \in \mathcal{A}(U)} \dim H^r(X(K, U), E).$$

As explained in [5], every OT-manifold is diffeomorphic to a solvmanifold. This proposition is a consequence of the cohomology computation of solvmanifolds in [7].

Regarding each $E \in \mathcal{A}(U)$ as a holomorphic line bundle over $X(K, U)$, we consider the Dolbeault cohomology $H^{p,q}(X(K, U), E) \equiv H^q(X(K, U), \Omega^p(E))$. In this paper, we prove the following Hodge symmetry type result.

**Theorem 1.2** For any integers $p, q$, we have

$$\dim \bigwedge^p \mathbb{C}^{s+l} \otimes \bigwedge^q \mathbb{C}^{s+l} = \sum_{E \in \mathcal{A}(U)} \dim H^{p,q}(X(K, U), E).$$

More precisely, we can give explicit harmonic representatives of $\bigoplus_{\mathcal{A}(U)} H^{p,q}(X(K, U), E)$. In particular, we know the vanishing or non-vanishing of the cohomology $H^{p,q}(X(K, U), E)$ for any $E \in \mathcal{A}(U)$. For a positive integer $s$, we denote $[s] = \{1, 2, \ldots, s\}$ and we call a subset in $[s]$ with the natural order a multi-index.

**Theorem 1.3** Let $E$ be a flat complex line bundle over an OT-manifold $X(K, U)$ corresponding to $\rho \in \text{Hom}(U, \mathbb{C}^*)$. Then

$$H^{p,q}(X(K, U), E) \neq 0$$

if and only if for some multi-indices $I \subset [s]$, $K, L \subset [t]$ with $|I| + |K| = p$ and $|L| \leq q$, we have

$$\rho(u) = \prod_{i \in I} \sigma_i(u) \prod_{k \in K} \sigma_{s+k}(u) \prod_{l \in L} \sigma_{s+l+l}(u)$$

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for any \( u \in U \). If
\[
\rho(u) = \prod_{i \in I} \sigma_i(u) \prod_{k \in K} \sigma_{s+k}(u) \prod_{l \in L} \sigma_{s+t+l}(u)
\]
for any \( u \in U \), then we have
\[
\dim H^{p, q}(X(K, U), E) \geq \left( \begin{array}{c} s \\ q - |L| \end{array} \right)
\]
where \( \left( \begin{array}{c} n \\ k \end{array} \right) \) means the number of \( k \)-combinations.

Obviously we have the following consequence.

**Corollary 1.4** Let \( E \) be a flat complex line bundle over an OT-manifold \( X(K, U) \) corresponding to \( \rho \in \text{Hom}(U, \mathbb{C}^*) \). Then, \( E \) admits a non-zero holomorphic section if and only if \( \rho \) is trivial.

As a corollary of Theorem 1.3, we also obtain [1, Theorem 3.1] (see Remark 5.3).

**Remark 1.5** Our main results are obtained by using the solvmanifold presentations of OT manifolds. We give statements in terms of solvmanifolds (Proposition 3.1, Theorems 4.1 and 5.1) implying Proposition 1.1, Theorems 1.2 and 1.3.

We notice that Theorem 1.2 is deeper than Proposition 1.1. In [5], we also compute the cohomology of complex parallelizable solvmanifolds with values in holomorphic vector bundles. OT-manifolds do not admit non-zero holomorphic vector field [9, Proposition 2.5] and they are far from complex parallelizable solvmanifolds. To prove Theorem 1.2, we use the result in [11] given by the analysis on Cousin groups.

### 2 Cohomology of solvmanifolds

Let \( G \) be a simply connected real solvable Lie group with a lattice \( \Gamma \) and \( \mathfrak{g} \) the Lie algebra of \( G \). Denote by \( A^*(\Gamma \backslash G) \) the de Rham complex of the solvmanifold \( \Gamma \backslash G \). We identify the de Rham complex \( A^*(\Gamma \backslash G) \) of \( \Gamma \backslash G \) with the subcomplex of the de Rham complex \( A^*(G) \) of \( G \) consisting of the left-\( \Gamma \)-invariant differential forms. Consider the cochain complex \( \bigwedge \mathfrak{g}^* \) of the Lie algebra \( \mathfrak{g} \). We identify \( \bigwedge \mathfrak{g}^* \) with the subcomplex of the de Rham complex \( A^*(G) \) of \( G \) consisting of the left-\( G \)-invariant differential forms. Hence, we consider \( \bigwedge \mathfrak{g}^* \) as a subcomplex of \( A^*(\Gamma \backslash G) \).

Let \( N \) be the nilradical (i.e. maximal connected nilpotent normal subgroup) of \( G \). Denote by \( A_{(G, N)} = \{ \alpha \in \text{Hom}(G, \mathbb{C}^*) \mid |\alpha|_N = 1 \} \). For \( \alpha \in A_{(G, N)} \), we consider the \( \Gamma \)-action on \( G \times \mathbb{C} \) so that for \( \gamma \in \Gamma \) and \( (g, c) \in G \times \mathbb{C} \), \( \gamma \cdot (g, c) = (\gamma g, \alpha(\gamma)c) \). Define the flat complex line bundle \( E_\alpha = \Gamma \backslash (G \times \mathbb{C}) \) over the solvmanifold \( \Gamma \backslash G \). We have the global section \( v_\alpha \) induced by the section \( (g, \alpha(g)) \) of \( G \times \mathbb{C} \). Denote by \( A^*(\Gamma \backslash G, E_\alpha) \) the de Rham complex with values in the line bundle \( E_\alpha \). Since \( v_\alpha \) trivializes \( E_\alpha \), we have \( A^*(\Gamma \backslash G, E_\alpha) = A^*(\Gamma \backslash G) \otimes \langle v_\alpha \rangle \) and \( dv_\alpha = \alpha^{-1} d\alpha v_\alpha \). We have the subcomplex \( \bigwedge g_\mathbb{C}^* \otimes \langle v_\alpha \rangle \subset A^*(\Gamma \backslash G, E_\alpha) = A^*(\Gamma \backslash G) \otimes \langle v_\alpha \rangle \). The cochain
complex $\wedge \mathfrak{g}^*_C \otimes \langle v_\alpha \rangle$ is the cochain complex of the Lie algebra $\mathfrak{g}$ associated with the representation $\alpha \in \text{Hom}(G, \mathbb{C}^*)$.

We define $\mathcal{A}_{(G, N)}(\Gamma)$ by the set $\{E_\alpha\}$ of all the isomorphism classes of flat line bundles $E_\alpha$ associated with $\alpha \in \mathcal{A}_{(G, N)}$. This set is identified with the set $\{\alpha|_\Gamma \in \text{Hom}(\Gamma, \mathbb{C}^*) \mid \alpha \in \mathcal{A}_{(G, N)}\}$. Consider the direct sum

$$\bigoplus_{E_\alpha \in \mathcal{A}_{(G, N)}(\Gamma)} A^*(\Gamma \backslash G, E_\alpha).$$

By the natural isomorphisms $E_\alpha \otimes E_\beta \cong E_{\alpha\beta}$ for $\alpha, \beta \in \mathcal{A}_{(G, N)}$, this direct sum is a differential graded algebra. By the above argument, we have the inclusion

$$\bigoplus_{\alpha \in \mathcal{A}_{(G, N)}} \wedge \mathfrak{g}^*_C \otimes \langle v_\alpha \rangle \subset \bigoplus_{E_\alpha \in \mathcal{A}_{(G, N)}(\Gamma)} A^*(\Gamma \backslash G, E_\alpha).$$

We have a simply connected nilpotent subgroup $C \subset G$ such that $G = C \cdot N$ (see [2, Proposition 3.3]). Since $C$ is nilpotent, the map

$$\Phi : C \ni c \mapsto (\text{Ad}_c)_* \otimes \alpha(c) \in \text{Aut}\left(\wedge \mathfrak{g}^*_C \otimes \langle v_\alpha \rangle\right)$$

is a homomorphism where $(\text{Ad}_c)_*$ is the semi-simple part of the Jordan decomposition of the adjoint operator. We denote by

$$\left(\wedge \mathfrak{g}^*_C \otimes \langle v_\alpha \rangle\right)^{\Phi(C)}$$

the subcomplex of $\wedge \mathfrak{g}^*_C \otimes \langle v_\alpha \rangle$ consisting of the $\Phi(C)$-invariant elements. [7, Theorem 1.4, Lemma 5.2] says that the inclusion

$$\bigoplus_{\alpha \in \mathcal{A}_{(G, N)}} \left(\wedge \mathfrak{g}^*_C \otimes \langle v_\alpha \rangle\right)^{\Phi(C)} \subset \bigoplus_{E_\alpha \in \mathcal{A}_{(G, N)}(\Gamma)} A^*(\Gamma \backslash G, E_\alpha)$$

induces a cohomology isomorphism.

We have a basis $X_1, \ldots, X_n$ of $\mathfrak{g}_C$ such that $(\text{Ad}_c)_s = \text{diag}(\alpha_1(c), \ldots, \alpha_n(c))$ for all $c \in C$. Let $x_1, \ldots, x_n$ be the basis of $\mathfrak{g}_C^*$ which is dual to $X_1, \ldots, X_n$. By $G = C \cdot N$, we have $G/N = C/C \cap N$ and hence we have $\mathcal{A}_{(G, N)} = \mathcal{A}_{C, C \cap N} = \{\alpha \in \text{Hom}(C, \mathbb{C}^*) \mid \alpha|_{C \cap N} = 1\}$. For a multi-index $I = \{i_1, \ldots, i_p\} \subset [n]$ we write $x_I = x_{i_1} \wedge \cdots \wedge x_{i_p}$, and $\alpha_I = \alpha_{i_1} \cdots \alpha_{i_p}$. We consider the basis

$$\{x_I \otimes v_\alpha\}_{I \subset [n]}$$

of $\wedge \mathfrak{g}^*_C \otimes \langle v_\alpha \rangle$. Since the action

$$\Phi : C \rightarrow \text{Aut}\left(\wedge \mathfrak{g}^*_C \otimes \langle v_\alpha \rangle\right)$$

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is the semi-simple part of \((\text{Ad} \otimes \alpha)\)|, we have

\[
\Phi(a)(x_I \otimes v_\alpha) = \alpha_I^{-1} \alpha x_I \otimes v_\alpha.
\]

Hence we have

\[
\bigoplus_{\alpha \in \mathcal{A}(G, N)} \left( \bigwedge g^*_C \otimes \langle v_\alpha \rangle \right)^{\Phi(C)} = \langle x_I \otimes v_{\alpha_I} \rangle_I \subset \{n\} = \bigwedge \{ x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n} \}.
\]

It is known that the differential graded algebra

\[
\bigwedge \{ x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n} \}
\]

is identified with the cochain complex of certain nilpotent Lie algebra determined by the solvable Lie algebra \(g\) (see [7, Remark 4] and [6]).

### 3 de Rham and Dolbeault cohomology of certain solvmanifolds

Let \(s, t\) be positive integers. We consider the semi-direct product \(G = \mathbb{R}^s \ltimes \phi (\mathbb{R}^s \oplus \mathbb{C}^t)\) of real abelian Lie groups \(\mathbb{R}^s\) and \(\mathbb{R}^s \oplus \mathbb{C}^t\) given by the homomorphism \(\phi : \mathbb{R}^s \to \text{Aut}(\mathbb{R}^s \oplus \mathbb{C}^t)\) so that

\[
\phi(x)(y, z) = \left( e^{x_1} y_1, \ldots, e^{x_s} y_s, e^{\psi_1(x)} z_1, \ldots, e^{\psi_t(x)} z_t \right)
\]

for \(x = (x_1, \ldots, x_s) \in \mathbb{R}^s\), \((y, z) = (y_1, \ldots, y_s, z_1, \ldots, z_t) \in \mathbb{R}^s \oplus \mathbb{C}^t\) and some non-zero linear functions \(\psi_1, \ldots, \psi_t : \mathbb{R}^s \to \mathbb{C}\). \(G\) is a simply connected solvable Lie group. Suppose we have lattices \(\Lambda \subset \mathbb{R}^s\) and \(\Delta \subset \mathbb{R}^s \oplus \mathbb{C}^t\) so that for every \(\lambda \in \Lambda\) the automorphism \(\phi(\lambda)\) on \(\mathbb{R}^s \oplus \mathbb{C}^t\) preserves \(\Delta\). Then the subgroup \(\Gamma = \Lambda \ltimes \phi \Delta \subset G\) is a cocompact discrete subgroup of \(G\). Consider the solvmanifold \(\Gamma \setminus G\).

We have a basis

\[
\begin{align*}
dx_1, \ldots, dx_s, e^{-x_1} dy_1, \ldots, e^{-x_s} dy_s, \\
e^{-\psi_1(x)} dz_1, \ldots, e^{-\psi_t(x)} dz_t, e^{-\overline{\psi}_1(x)} d\overline{z}_1, \ldots, e^{-\overline{\psi}_t(x)} d\overline{z}_t
\end{align*}
\]

of \(g^*_C = g^* \otimes \mathbb{C}\).

On \(G = \mathbb{R}^n \ltimes \phi (\mathbb{R}^s \oplus \mathbb{C}^t)\), the nilradical \(N = \mathbb{R}^s \oplus \mathbb{C}^t\) and we take the subgroup \(C = \mathbb{R}^s \subset G\) so that \(C \cdot N = G\). We apply the last section to the solvmanifold \(\Gamma \setminus G\).

By the last section, defining

\[
V = \begin{pmatrix}
dx_1, \ldots, dx_s, \\
e^{-x_1} dy_1 \otimes v_{e^{x_1}}, \ldots, e^{-x_s} dy_s \otimes v_{e^{x_s}}, \\
e^{-\psi_1(x)} dz_1 \otimes v_{e^{\psi_1(x)}}, \ldots, e^{-\psi_t(x)} dz_t \otimes v_{e^{\psi_t(x)}}, \\
e^{-\overline{\psi}_1(x)} d\overline{z}_1 \otimes v_{e^{\overline{\psi}_1(x)}}, \ldots, e^{-\overline{\psi}_t(x)} d\overline{z}_t \otimes v_{e^{\overline{\psi}_t(x)}}
\end{pmatrix}
\]

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where we regard $dx_1, \ldots, dx_s$ as 1-forms with values in the trivial line bundle, we have the inclusion

$$\wedge V \subset \bigoplus_{E_a \in A_{(G, N)}(\Gamma)} A^*(\Gamma \setminus G, E_a)$$

which induces a cohomology isomorphism. We notice that we can identify $A_{(G, N)}$ with the set $\text{Hom}(\mathbb{R}^s, \mathbb{C}^*)$ of Lie group homomorphisms from $\mathbb{R}^s$ to $\mathbb{C}^*$ and the set $A_{(G, N)}(\Gamma)$ is equal to the set of isomorphism classes of flat complex line bundles over $\Gamma \setminus G$ given by homomorphisms in $\text{Hom}(\Lambda, \mathbb{C}^*)$. We can easily check that the differential on $V$ is 0. Hence we have the following:

**Proposition 3.1** We have an isomorphism

$$\wedge V \cong \bigoplus_{E_a \in A_{(G, N)}} H^*(\Gamma \setminus G, E_a).$$

Regarding 1-forms

$$\alpha_1 = dx_1 + \sqrt{-1} e^{-x_1} dy_1, \ldots, \alpha_s = dx_s + \sqrt{-1} e^{-x_s} dy_s,$$

$$\beta_1 = e^{-\psi_1(x)} dz_1, \ldots, \beta_t = e^{-\psi_t(x)} dz_t$$

as $(1, 0)$-forms on $\Gamma \setminus G$, we have a left-$G$-invariant almost complex structure $J$ on $\Gamma \setminus G$. We can easily check that $J$ is integrable. We consider the Dolbeault complex $(A^{*,*}(\Gamma \setminus G), \bar{\partial})$ of the complex manifold $(\Gamma \setminus G, J)$. We have

$$\bar{\partial} \alpha_i = -\frac{1}{2} \bar{\alpha}_i \wedge \alpha_i, \quad \bar{\partial} \beta_i = -\frac{1}{2} \bar{\psi}_i(\bar{\alpha}) \wedge \beta_i \quad \text{and} \quad \bar{\partial} \bar{\beta}_i = -\frac{1}{2} \bar{\psi}_i(\bar{\alpha}) \wedge \bar{\beta}_i$$

where $\psi_i(\bar{\alpha})$ and $\bar{\psi}_i(\bar{\alpha})$ are $(0, 1)$-forms associated with linear functions $\psi_i(x)$ and $\bar{\psi}_i(x)$ by putting $x = \bar{\alpha} = (\bar{\alpha}_1, \ldots, \bar{\alpha}_s)$.

**Remark 3.2** We consider the holomorphic tangent bundle $\Theta$ and holomorphic cotangent bundle $\Omega^1$ of $(\Gamma \setminus G, J)$. Then, $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t$ is a global $C^\infty$-frame of $\Omega^1$. Hence we have an isomorphism

$$\Omega^1 \cong E_{e^{-x_1}} \oplus \cdots \oplus E_{e^{-x_s}} \oplus E_{e^{-\psi_1(x)}} \oplus \cdots \oplus E_{e^{-\psi_t(x)}}$$

of holomorphic vector bundles. By this, we also have

$$\Theta \cong E_{e^{x_1}} \oplus \cdots \oplus E_{e^{x_s}} \oplus E_{e^{\psi_1(x)}} \oplus \cdots \oplus E_{e^{\psi_t(x)}}.$$
where we regard $\bar{\alpha}_1, \ldots, \bar{\alpha}_s$ as 1-forms with values in the trivial line bundle. We consider the subspace

$$\wedge^p W_1 \otimes \wedge^q W_2 \subset \bigoplus_{E_\alpha \in \mathcal{A}(G,N)(\Gamma)} A^{p,q}(\Gamma \setminus G, E_\alpha).$$

Define the left-$G$-invariant Hermitian metric

$$h_G = \alpha_1 \cdot \bar{\alpha}_1 + \cdots + \alpha_s \cdot \bar{\alpha}_s + \beta_1 \cdot \bar{\beta}_1 + \cdots + \beta_t \cdot \bar{\beta}_t.$$ 

Define the Hermitian metric $h_\alpha$ on each $E_\alpha \in \mathcal{A}(G,N)$ so that $h_\alpha(v_\alpha, v_\alpha) = 1$. We notice that for $\alpha, \alpha' \in \mathcal{A}(G,N)$, if $E_\alpha = E_{\alpha'}$, then $h_\alpha = h_{\alpha'}$ since $E_\alpha = E_{\alpha'}$ if and only if $\alpha|_{\Gamma} = \alpha'|_{\Gamma}$ and hence $\alpha^{-1} \alpha'$ is unitary. We consider the Hodge star operator $\bar{\Psi}: A^{p,q}(\Gamma \setminus G, E_\alpha) \to A^{s+t-p,s+t-q}(\Gamma \setminus G, E_\alpha^*)$ associated with this metric. Then we have

$$\bar{\Psi}(\alpha_I \wedge \bar{\alpha}_J \wedge \beta_K \wedge \bar{\beta}_L \otimes v_e^{\psi_{IKL}(x)}) = \pm \alpha_{\tilde{I}} \wedge \bar{\alpha}_{\tilde{J}} \wedge \beta_{\tilde{K}} \wedge \bar{\beta}_{\tilde{L}} \otimes v_e^{-\psi_{IK\tilde{L}(x)}}$$

where for some multi-indices $I, J \subset [s], K, L \subset [t]$ we write

$$\Psi_{IKL}(x) = \sum_{j \in I} x_j + \sum_{k \in K} \psi_k(x) + \sum_{l \in L} \bar{\psi}_l(x),$$

$\tilde{I} = [s] - I, \tilde{J} = [s] - J, \tilde{K} = [t] - K$ and $\tilde{L} = [t] - L$. Since $G$ admits a lattice $\Gamma$, $G$ is unimodular (see [12, Remark 1.9]). This implies

$$\exp(\Psi_{[s][t][\tilde{I}]}(x)) = \exp\left(\sum_{i \in [s]} x_i + \sum_{k \in [r]} \psi_k(x) + \sum_{l \in [t]} \bar{\psi}_l(x)\right) = 1.$$ 

Thus

$$\exp(\Psi_{\tilde{I} \tilde{K} \tilde{L}}(x)) = \exp(-\Psi_{IKL}(x)).$$

By this, we can say that the Hodge star operator $\bar{\Psi}$ preserves the space $\wedge W_1 \otimes \wedge W_2$ (compare [8, Lemma 2.3]).

We can easily check that the Dolbeault operator on $W_1$ and $W_2$ is 0. Hence, $\wedge W_1 \otimes \wedge W_2$ consists of harmonic forms associated with the Dolbeault operator. This implies the following result.

**Proposition 3.3** We have an injection

$$\wedge^p W_1 \otimes \wedge^q W_2 \hookrightarrow \bigoplus_{E_\alpha \in \mathcal{A}(G,N)(\Gamma)} H^{p,q}(\Gamma \setminus G, E_\alpha)$$

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hence we have
\[ \dim \bigwedge^p \mathbb{C}^{s+t} \otimes \bigwedge^q \mathbb{C}^{s+t} \leq \sum_{E_a \in \mathcal{A}(G, N) \,(\Gamma)} \dim H^{p,q}(\Gamma \backslash G, E_a). \]

4 Oeljeklaus–Toma manifolds

For positive integers \( s, t \), let \( K \) be a finite extension field of \( \mathbb{Q} \) of degree \( s + 2t \) admitting embeddings \( \sigma_1, \ldots, \sigma_s, \sigma_{s+1}, \ldots, \sigma_{s+2t} \) into \( \mathbb{C} \) such that \( \sigma_1, \ldots, \sigma_s \) are real embeddings and \( \sigma_{s+1}, \ldots, \sigma_{s+2t} \) are complex ones satisfying \( \sigma_{s+i} = \overline{\sigma_{s+i}} \) for \( 1 \leq i \leq t \). Let \( \mathcal{O}_K \) be the ring of algebraic integers of \( K \), \( \mathcal{O}_K^* \) the group of units in \( \mathcal{O}_K \) and

\[ \mathcal{O}_K^{s+} = \{ a \in \mathcal{O}_K^* : \sigma_i(a) > 0 \text{ for all } 1 \leq i \leq s \}. \]

Define \( \sigma : \mathcal{O}_K \to \mathbb{R}^s \times \mathbb{C}^t \) by

\[ \sigma(a) = (\sigma_1(a), \ldots, \sigma_s(a), \sigma_{s+1}(a), \ldots, \sigma_{s+t}(a)) \]

for \( a \in \mathcal{O}_K \). Define \( l : \mathcal{O}_K^{s+} \to \mathbb{R}^{s+t} \) by

\[ l(a) = (\log |\sigma_1(a)|, \ldots, \log |\sigma_s(a)|, 2 \log |\sigma_{s+1}(a)|, \ldots, 2 \log |\sigma_{s+t}(a)|) \]

for \( a \in \mathcal{O}_K^{s+} \). Then by Dirichlet’s units theorem, \( l(\mathcal{O}_K^{s+}) \) is a lattice in the vector space \( L = \{ x \in \mathbb{R}^{s+t} \mid \sum_{i=1}^{s+t} x_i = 0 \} \). Consider the projection \( p : L \to \mathbb{R}^s \) given by the first \( s \) coordinate functions. Then we have a subgroup \( U \) with the rank \( s \) of \( \mathcal{O}_K^{s+} \) such that \( p(l(U)) \) is a lattice in \( \mathbb{R}^s \). Write \( l(U) = \mathbb{Z} v_1 \oplus \cdots \oplus \mathbb{Z} v_s \) for generators \( v_1, \ldots, v_s \) of \( l(U) \). For the standard basis \( e_1, \ldots, e_{s+t} \) of \( \mathbb{R}^{s+t} \), we have a regular real \( s \times s \)-matrix \( (a_{ij}) \) and \( s \times t \) real constants \( b_{jk} \) such that

\[ v_i = \sum_{j=1}^{s} a_{ij} \left( e_j + \sum_{k=1}^{t} b_{jk} e_{s+k} \right) \]

for any \( 1 \leq i \leq s \). Consider the complex upper half plane \( H = \{ z \in \mathbb{C} \mid \text{Im } z > 0 \} = \mathbb{R} \times \mathbb{R}_{>0} \). We have the left action of \( U \rtimes \mathcal{O}_K \) on \( H^s \times \mathbb{C}^t \) such that

\[ (a, b) \cdot (x_1 + \sqrt{-1} y_1, \ldots, x_s + \sqrt{-1} y_s, z_1, \ldots, z_t) = (\sigma_1(a)x_1 + \sigma_1(b) + \sqrt{-1} \sigma_1(a)y_1, \ldots, \sigma_s(a)x_s + \sigma_s(b) + \sqrt{-1} \sigma_s(a)y_s, \]

\[ \sigma_{s+1}(a)z_1 + \sigma_{s+1}(b), \ldots, \sigma_{s+t}(a)z_t + \sigma_{s+t}(b)) \]

In [9] it is proved that the quotient \( X(U, K) = U \rtimes \mathcal{O}_K \backslash H^s \times \mathbb{C}^t \) is compact. Actually we have the real fiber bundle \( X(U, K) \to U \backslash (\mathbb{R}_{>0})^s \) with the fiber...
σ(𝒪_K)((ℝ^s × ℂ') and both the base U \ ((ℝ_{>0})^s and the fiber σ(𝒪_K)((ℝ^s × ℂ') are real tori. We call this complex manifold an Oeljeklaus–Toma (OT) manifold.

As in [5], we present OT-manifolds as solvmanifolds considered in the last section. For a ∈ U and (x_1, ..., x_s) = p(l(a)) ∈ p(l(U)), since l(U) is generated by the basis v_1, ..., v_s as above, l(a) is a linear combination of e_1 + ∑_{k=1}^{t} b_{1k} e_{s+k}, ..., e_s + ∑_{k=1}^{t} b_{sk} e_{s+k} and hence we have

\[l(a) = \sum_{i=1}^{s} x_i (e_i + \sum_{k=1}^{t} b_{ik} e_{s+k}) = (x_1, ..., x_s, \sum_{i=1}^{s} b_{1i} x_i, ..., \sum_{i=1}^{s} b_{si} x_i) .\]

By 2 log |σ_{s+k}(a)| = ∑_{i=1}^{s} b_{ik} x_i, we can write

\[σ_{s+k}(a) = e^{\frac{1}{2}} \sum_{i=1}^{s} b_{ik} x_i + \sqrt{-1} \sum_{i=1}^{s} c_{ik} x_i \]

for some c_{ik} ∈ ℝ. We consider the Lie group G = ℝ^s × φ (ℝ^s × ℂ') with

\[φ(x_1, ..., x_s) = \text{diag}(e^{x_1}, e^{x_2}, e^{ψ_1(x)}, e^{ψ_2(x)})\]

where ψ_k = \frac{1}{2} ∑_{i=1}^{s} b_{ik} x_i + \sqrt{-1} ∑_{i=1}^{s} c_{ik} x_i. Then for (x_1, ..., x_s) ∈ p(l(U)), we have

\[φ(x_1, ..., x_s)(σ(𝒪_K)) ⊂ σ(𝒪_K).\]

Write p(l(U)) = Λ and σ(𝒪_K) = Δ. Then, via the diffeomorphism

\[H^s × ℂ' \ni (y_1 + \sqrt{-1} w_1, ..., y_s + \sqrt{-1} w_s, z_1, ..., z_t) \mapsto (\log(w_1), ..., \log(w_s), -y_1, ..., -y_s, z_1, ..., z_t) ∈ ℝ^s × φ (ℝ^s × ℂ'), \]

the OT-manifold X(U, K) = U × 𝒪_K \ H^s × ℂ' is identified with a complex solvmanifold (Γ\G, J) of the form as in the last section.

**Theorem 4.1** Define W_1 and W_2 as (3.1). An isomorphism

\[∧^p W_1 ⊗ ∧^q W_2 ∼= \bigoplus_{E_α ∈ A(G, N)(Γ)} H^{p,q}(Γ\G, E_α)\]

holds. Hence we have

\[\dim ∧^p C^{s+t} ⊗ ∧^q C^{s+t} = \sum_{E_α ∈ A(G, N)(Γ)} \dim H^{p,q}(Γ\G, E_α).\]

**Proof** We regard H^{p,q}(Γ\G, E_α) as the sheaf cohomology H^q(Γ\G, Ω^p(E_α)). We consider the real fiber bundle π: Γ\G → Λ\R^s over the real torus Λ\R^s with the real torus fiber Δ\(R^s × ℂ'). This fiber bundle is identified with the fiber bundle
We have

\[ X(U, K) \to U \setminus (\mathbb{R}_{>0})^s \] with the fiber \( \sigma(\emptyset_K) \setminus (\mathbb{R}^s \times C^t) \). Consider the Leray spectral sequence \( E^{a,b}_{k} (\Omega^p(E_\alpha)) \) associated with the map \( \pi : \Gamma \setminus G \to \Lambda \setminus \mathbb{R}^s \) and the sheaf \( \Omega^p(E_\alpha) \). Then \( E^{a,b}_{2} (\Omega^p(E_\alpha)) = H^a(\Lambda \setminus \mathbb{R}^s, R^b \pi_* \Omega^p(E_\alpha)) \) and \( E^{a,b}_{2} (\Omega^p(E_\alpha)) \) converges to \( H^{a+b}(\Gamma \setminus G, \Omega^p(E_\alpha)) \). The sheaf \( R^b \pi_* \Omega^p(E_\alpha) \) over \( \Lambda \setminus \mathbb{R}^s \) is the sheafification of the pre-sheaf such that each open set \( O \subset \Lambda \setminus \mathbb{R}^s \) corresponds to the vector space \( H^b(\pi^{-1}(O), \Omega^p(E_\alpha)) \). Since the flat bundle \( E_\alpha \) corresponds to a homomorphism \( \alpha : \Lambda \to \text{Hom}(A, \mathbb{C}^s) \), as a sheaf on \( \Gamma \setminus G, E_\alpha \) is constant on \( \pi^{-1}(O) \) for sufficiently small open set \( O \subset \Lambda \setminus \mathbb{R}^s \). Thus, we have \( R^b \pi_* \Omega^p(E_\alpha) \cong (R^b \pi_* \Omega^p) \otimes \tilde{E}_{\alpha|\Lambda} \) where \( \tilde{E}_{\alpha|\Lambda} \) is the local system on \( \Lambda \setminus \mathbb{R}^s \) corresponding to \( \alpha \in \text{Hom}(\Lambda, \mathbb{C}^s) \). [11, Lemma 4.3] says that \( R^b \pi_* \Omega^p \) is a local system on the torus \( \Lambda \setminus \mathbb{R}^s \) so that locally \( R^b \pi_* \Omega^p \) is isomorphic to \( \wedge^p \mathbb{C}^{s+t} \otimes \wedge^b \mathbb{C}^r \) and \( R^b \pi_* \Omega^p \) corresponds to a diagonal representation of \( \Lambda = p(l(U)) \cong U \). Let \( E \) be a local system on \( \Lambda \setminus \mathbb{R}^s \) corresponding to a 1-dimensional complex representation of \( \Lambda \). It is well known that \( H^*(\Lambda \setminus \mathbb{R}^s, E) \cong \wedge \mathbb{C}^s \) for trivial \( E \) otherwise \( H^*(\Lambda \setminus \mathbb{R}^s, E) = 0 \). By this for any \( E \), we have

\[
\bigoplus_{\beta \in \text{Hom}(\Lambda, \mathbb{C}^s)} H^*(\Lambda \setminus \mathbb{R}^s, E \otimes \tilde{E}_{\beta}) = H^*(\Lambda \setminus \mathbb{R}^s, E \otimes E^{-1}) \oplus \bigoplus_{\beta \in \text{Hom}(\Lambda, \mathbb{C}^s)} H^*(\Lambda \setminus \mathbb{R}^s, E \otimes \tilde{E}_{\beta}) \cong \wedge \mathbb{C}^s.
\]

Thus, identifying \( A_{(G,N)}(\Gamma) \) with \( \text{Hom}(\Lambda, \mathbb{C}^s) \), we have

\[
\bigoplus_{E_\alpha \in A_{(G,N)}(\Gamma)} E_{2}^{a,b}(\Omega^p(E_\alpha)) \cong \bigoplus_{\beta \in \text{Hom}(\Lambda, \mathbb{C}^s)} H^a(\Lambda \setminus \mathbb{R}^s, (R^b \pi_* \Omega^p) \otimes \tilde{E}_{\beta}) \cong \wedge^a \mathbb{C}^s \otimes \wedge^b \mathbb{C}^r.
\]

We have

\[
\bigoplus_{a+b=q} \bigoplus_{E_\alpha \in A_{(G,N)}(\Gamma)} E_{2}^{a,b}(\Omega^p(E_\alpha)) \cong \wedge^p \mathbb{C}^{s+t} \otimes \wedge^q \mathbb{C}^{s+t}.
\]

By Proposition 3.3, we have

\[
\sum_{a+b=q} \sum_{E_\alpha \in A_{(G,N)}(\Gamma)} \dim E_{2}^{a,b}(\Omega^p(E_\alpha)) = \dim \wedge^p \mathbb{C}^{s+t} \otimes \wedge^q \mathbb{C}^{s+t} \leq \sum_{E_\alpha \in A_{(G,N)}(\Gamma)} \dim H^{p,q}(\Gamma \setminus G, E_\alpha).
\]

Since \( E_{r}^{a,b}(\Omega^p(E_\alpha)) \) converges to \( H^{a+b}(\Gamma \setminus G, \Omega^p(E_\alpha)) \cong H^{p,a+b}(\Gamma \setminus G, E_\alpha) \), the converse inequality holds by the standard argument on the spectral sequence. Thus, the Leray spectral sequence \( E^{a,b}_{k}(\Omega^p(E_\alpha)) \) degenerates at \( E_2 \)-term and so we have

\[
\dim \wedge^p \mathbb{C}^{s+t} \otimes \wedge^q \mathbb{C}^{s+t} = \sum_{E_\alpha \in A_{(G,N)}(\Gamma)} \dim H^{p,q}(\Gamma \setminus G, E_\alpha).
\]

Hence the injection in Proposition 3.3 is an isomorphism. \( \square \)

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Since $U \cong p(l(U)) = \Lambda$ is embedded in $\mathbb{R}^s$ as a lattice, the set $\mathcal{A}_{(G, N)}(\Gamma)$ is equal to the set $\mathcal{A}(U)$ of isomorphism classes of flat complex line bundles over $X(K, U) = \Gamma \backslash G$ given by homomorphisms in $\text{Hom}(U, \mathbb{C}^*)$. We have the following consequence of Proposition 3.3.

**Corollary 4.2** For any integer $r$, we have

$$\dim \wedge^r \mathbb{C}^{2s+2t} = \sum_{E \in \mathcal{A}(U)} \dim H^r(X(K, U), E).$$

**Remark 4.3** See [4] for the de Rham cohomology of OT-manifolds with values in trivial and some specific flat line bundles.

Theorem 4.1 gives the following statement.

**Corollary 4.4** For any integers $p$, $q$, we have

$$\dim \wedge^p \mathbb{C}^s \otimes \wedge^q \mathbb{C}^s = \sum_{E \in \mathcal{A}(U)} \dim H^{p, q}(X(K, U), E).$$

### 5 Cohomology of holomorphic line bundles over OT-manifolds: vanishing and non-vanishing

For each $E_\alpha \in \mathcal{A}_{(G, N)}(\Gamma)$, by Theorem 4.1, we have

$$H^{p, q}(\Gamma \backslash G, E_\alpha) \cong \left\{ \alpha_I \wedge \bar{\alpha}_J \wedge \beta_K \wedge \bar{\beta}_L \otimes \psi_{1,K}^{(s)} \mid |I| + |K| = p \text{ and } |J| + |L| = q \right\}.$$

**Corollary 5.1** $H^{p, q}(\Gamma \backslash G, E_\alpha) \neq 0$ if and only if for some multi-indices $I \subset [s]$, $K, L \subset [t]$ with $|I| + |K| = p$ and $|J| \leq q$, $E_\alpha = E_{e^{\psi_{1,K}^{(s)}}}$. If $E_\alpha = E_{e^{\psi_{1,K}^{(s)}}}$, then we have

$$\dim H^{p, q}(\Gamma \backslash G, E_\alpha) \geq \binom{s}{q - |L|},$$

where $\binom{n}{k}$ means the number of $k$-combinations.

We notice that $E_\alpha = E_{e^{\psi_{1,K}^{(s)}}}$ if and only if $\alpha(x) = e^{\psi_{1,K}^{(s)}}(x)$ for any $x \in \Lambda$. For the trivial $E_\alpha$, this corollary gives [8, Corollary 3.5].

For $u \in U$ with $x = p(l(u)) \in p(l(U))$ we have $\sigma_i(u) = e^{x_i}$ for $1 \leq i \leq s$, $\sigma_{s+k}(u) = e^{\psi_{k}^{(s)}}$ and $\sigma_{s+t+k}(u) = e^{\bar{\psi}_{k}^{(s)}}$ for $1 \leq k \leq s$. Hence we have

$$e^{\psi_{1,K}^{(s)}} = \prod_{i \in I} \sigma_i(u) \prod_{j \in K} \sigma_{s+k}(u) \prod_{l \in L} \sigma_{s+t+l}(u).$$
Corollary 5.2 Let $E$ be a flat complex line bundle over an OT-manifold $X(K, U)$ corresponding to $\rho \in \text{Hom}(U, C^*)$. Then

$$H^{p,q}(X(K, U), E) \neq 0$$

if and only if for some multi-indices $I \subset [s]$, $K, L \subset [t]$ with $|I| + |K| = p$ and $|L| \leq q$, we have

$$\rho(u) = \prod_{i \in I} \sigma_i(u) \prod_{k \in K} \sigma_{s+k}(u) \prod_{l \in L} \sigma_{s+l+t+i}(u)$$

for any $u \in U$. If

$$\rho(u) = \prod_{i \in I} \sigma_i(u) \prod_{k \in K} \sigma_{s+k}(u) \prod_{l \in L} \sigma_{s+l+t+i}(u)$$

for any $u \in U$, then we have

$$\dim H^{p,q}(X(K, U), E) \geq \left( s - t - |L| \right).$$

Remark 5.3 This statement implies [1, Theorem 3.1]. Actually, for $p = 0$ and $q = 1$, $H^{0,1}(X(K, U), E) \neq 0$ if and only if $\rho$ is trivial or $\rho = \sigma_{s+t+1}, \ldots, \sigma_{s+2t}$. This seems different from [1, Theorem 3.1]. But we may remark that we use the left action but on the other hand in [1] the right action is used. The correspondence between the right-quotient and left-quotient is given by the inverse.

Example 5.4 We consider the case $t = 1$. In this case, any $U$ is a finite-index subgroup of $O^*_K$. For our solvmanifold presentation $\Gamma \backslash G$ of an OT-manifold $X(K, U)$, we can write

$$\psi_1(x) = -\frac{1}{2} (x_1 + \cdots + x_s) + \sqrt{-1} \varphi(x)$$

for some real linear function $\varphi(x)$. Thus, for for multi-indices $I \subset [s]$, $K, L \subset [t = 1]$, we have

$$\psi_{IKL}(x) = \begin{cases} 
\sum_{i \in I} x_i & (K = L = \emptyset), \\
\frac{1}{2} \sum_{i \in I} x_i - \frac{1}{2} \sum_{i \in I} x_i + \sqrt{-1} \varphi(x) & (K = \{1\}, L = \emptyset), \\
\frac{1}{2} \sum_{i \in I} x_i - \frac{1}{2} \sum_{i \in I} x_i - \sqrt{-1} \varphi(x) & (K = \emptyset, L = \{1\}), \\
-\sum_{i \in I} x_i & (K = \{1\}, L = \{1\}).
\end{cases}$$

We can say that $E_{\psi_{IKL}^{(s)}} = E_{\psi_{I'K'L'}^{(s)}}$ if and only if $(I, K, L) = (I', K', L')$, $(I, K, L) = (\emptyset, \emptyset, \emptyset)$ and $(I', K', L') = ([s], \{1\}, \{1\})$ or $(I, K, L) = ([s], \{1\}, \{1\})$ and $(I', K', L') = (\emptyset, \emptyset, \emptyset)$. 

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We compute $H^{p,q}(\Gamma \setminus G, E_{e^{iK\mathcal{T}(s)}})$ for each $I$, $K$, $L$ so that $E_{e^{iK\mathcal{T}(s)}}$ is non-trivial, i.e. $(I, K, L) \neq (\emptyset, \emptyset, \emptyset), ([s], [1], [1])$. If $K = L = \emptyset$,

$$H^{p,q}(\Gamma \setminus G, E_{e^{iK\mathcal{T}(s)}}) = \begin{cases} \langle \alpha_I \rangle \wedge \wedge^q \langle \overline{\alpha}_1, \ldots, \overline{\alpha}_s \rangle & (p = |I|), \\ 0 & \text{(otherwise)}. \end{cases}$$

If $K = [1]$, $L = \emptyset$,

$$H^{p,q}(\Gamma \setminus G, E_{e^{iK\mathcal{T}(s)}}) = \begin{cases} \langle \alpha_I \rangle \wedge \wedge^q \langle \overline{\alpha}_1, \ldots, \overline{\alpha}_s \rangle & (p = |I| + 1), \\ 0 & \text{(otherwise)}. \end{cases}$$

If $K = \emptyset$, $L = [1]$,

$$H^{p,q}(\Gamma \setminus G, E_{e^{iK\mathcal{T}(s)}}) = \begin{cases} \langle \alpha_I \rangle \wedge \wedge^{q-1} \langle \overline{\alpha}_1, \ldots, \overline{\alpha}_s \rangle \wedge \langle \overline{\beta}_1 \rangle & (p = |I|), \\ 0 & \text{(otherwise)}. \end{cases}$$

If $K = [1]$, $L = [1]$,

$$H^{p,q}(\Gamma \setminus G, E_{e^{iK\mathcal{T}(s)}}) = \begin{cases} \langle \alpha_I \rangle \wedge \wedge^{q-1} \langle \overline{\alpha}_1, \ldots, \overline{\alpha}_s \rangle \wedge \langle \overline{\beta}_1 \rangle & (p = |I| + 1), \\ 0 & \text{(otherwise)}. \end{cases}$$

In particular for any $I$, $K$, $L$ with $(I, K, L) \neq (\emptyset, \emptyset, \emptyset), ([s], [1], [1])$,

$$H^{0,q}(\Gamma \setminus G, E_{e^{iK\mathcal{T}(s)}}) = 0.$$

**Remark 5.5** In this case, the equality

$$\dim H^{p,q}(\Gamma \setminus G, E_{e^{iK\mathcal{T}(s)}}) = \binom{s}{q - |L|}$$

holds (see the inequality as in Corollary 5.1).

As noted in Remark 3.2, we have

$$\Theta \cong E_{e^{s_1}} \oplus \cdots \oplus E_{e^{s_r}} \oplus E_{e^{\varphi_1(s)}} \oplus \cdots \oplus E_{e^{\varphi_t(s)}}.$$

Hence, for $0 < p \leq s + 1$, we have

$$H^{0,q}(\mathcal{M}, \wedge^p \Theta) \cong \bigoplus_{|I| + |K| = p \atop L = \emptyset} H^{0,q}(\Gamma \setminus G, E_{e^{iK\mathcal{T}(s)}}) = 0.$$

This means the following statement.
**Proposition 5.6** For an OT-manifold $X(K, U)$ with $t = 1$, we have

$$H^{0, q}(X(K, U), \Lambda^p \Theta) = 0.$$ 

Since $H^{0, 1}(X(K, U), \Theta) = 0$, every OT-manifold $X(K, U)$ with $t = 1$ is rigid (cf. [1]). Moreover by $H^{0, 0}(X(K, U), \Lambda^2 \Theta) = 0$, $X(K, U)$ with $t = 1$ does not admit non-zero holomorphic Poisson structure.

Generalized complex structures are geometric structures including complex structures and symplectic structures as special cases introduced by Hitchin and developed by Gualtieri [3]. It is known that small deformations of a complex structure regarded as a generalized complex structure can be controlled by the following three parameters (see [3, Section 5]):

- holomorphic Poisson structures;
- usual deformations of the complex structure;
- $B$-field transformations.

Thus, for every OT-manifold $X(K, U)$ with $t = 1$, every small deformation of the complex structure on $X(K, U)$ regarded as a generalized complex structure is given by a $B$-field transformation.

**Example 5.7** In [10, Section 3.1], Otiman gives a field $K$ and a subgroup $U \subset O_K^{*+}$ such that $s = t = 2$ and for any $u \in U$, $\sigma_1(u)\sigma_3(u)\sigma_5(u) = \sigma_2(u)\sigma_4(u)\sigma_6(u) = 1$. By these relations, we can write

$$\psi_1(x) = -\frac{1}{2} x_1 + \sqrt{-1} \varphi_1(x), \quad \psi_2(x) = -\frac{1}{2} x_2 + \sqrt{-1} \varphi_2(x)$$

for some real linear functions $\varphi_1(x), \varphi_2(x)$. Thus, for example, we compute

$$H^{p, q}(\Gamma \backslash G) = \begin{cases} \wedge^q \langle \alpha_1, \alpha_2 \rangle & (p = 0), \\ \langle \alpha_1 \wedge \beta_1 \wedge \beta_2 \rangle \wedge \Lambda^{q-1} \langle \alpha_1, \alpha_2 \rangle & (p = 2, q \geq 1), \\ \langle \alpha_2 \wedge \beta_1 \wedge \beta_2 \wedge \beta_2 \rangle \wedge \Lambda^{q-2} \langle \alpha_1, \alpha_2 \rangle & (p = 4, q \geq 2), \\ 0 & (\text{otherwise}) \end{cases}$$

and

$$H^{p, q}(\Gamma \backslash G, E_{e^1}) = \begin{cases} \langle \alpha_1 \rangle \wedge \Lambda^q \langle \alpha_1, \alpha_2 \rangle & (p = 1), \\ \langle \alpha_1 \wedge \alpha_2 \wedge \beta_2 \wedge \beta_2 \rangle \wedge \Lambda^{q-1} \langle \alpha_1, \alpha_2 \rangle & (p = 3, q \geq 1), \\ 0 & (\text{otherwise}) \end{cases}$$

Hence, for $I = K = J = \{1\}$ and $q \geq 2$, we have

$$H^{2, q}(\Gamma \backslash G, E_{e^1 K \Gamma^I}) = H^{2, q}(\Gamma \backslash G) = 2 \left( \begin{array}{c} 2 \\ q - 1 \end{array} \right) > \left( \begin{array}{c} 2 \\ q - 1 \end{array} \right).$$
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