GENERALIZED FOCK SPACE AND FRACTIONAL DERIVATIVES WITH APPLICATIONS TO UNIQUENESS OF SAMPLING AND INTERPOLATION SETS

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Abstract. In this paper we introduce a Fock space related to derivatives of Gelfond-Leontiev type, a class of derivatives which includes many classic examples like fractional derivatives or Dunkl operators. For this space we establish a modified Bargmann transform as well as density theorems for sampling and interpolation. These density theorems allow us to establish lattice conditions for the construction of frames arising from integral transforms which are linked by the modified Bargmann transform with the Fock space.

1. Introduction

One principal problem in modern signal and image processing consists the construction of frames arising from discretization of integral transforms (e.g. wavelet and Gabor systems). This problem has its origins in quantum mechanics and in information theory (see J. von Neumann, D. Gabor) where one aims to represent functions in terms of time-frequency atoms which have a minimal support in the time-frequency plane. For practical applications the continuous transforms are substituted by discrete systems in form of frames which represent a generalization of biorthogonal bases. Usually, frames are obtained by discretizing the parameter space, which leads to the problem of finding conditions for a lattice in the parameter space to be dense enough to create a frame. In the case of Gabor systems K. Gröchenig and Y. Lyubarskii established a machinery which allows to find lattices for the construction of Gabor frames with Hermite functions as window functions by connecting the Gabor system with the standard orthonormal basis in the Fock space via the Bargmann transform (see [10, 11]). This reduces the problem of finding lattice constants for the frame parameters to the problem of sets of interpolation and uniqueness of entire functions belonging to the Fock space. This last problem has been studied in detail by K. Seip and co-authors, see [19, 20, 5, 1]. But, as we will show in the present paper, the proposed method is much more general than the case of Gabor frames seems to indicate. While the classic setting of the Fock space is linked to the classic derivative and multiplication operators we will consider the case of Gelfond-Leontiev derivatives in this paper. This type of derivatives includes many important examples as special cases, like fractional derivatives of Caputo or Riemann-Liouville type (the latter via changing of the ground state) or difference-differential operators linked to finite reflection groups, also known as Dunkl operators. While the former is being applied in a variety of areas, like fractional mechanics or grey noise analysis in stochastic processes, the latter appear in the study of Calogero-Sutherland-Moser models for n-particle systems. For this type of operators we are going to construct and study the corresponding Fock space and the connected Bargmann transform. This will allow us to establish the necessary density theorems for sampling and interpolation sequences in these Fock spaces. To show the applicability of the Gröchenig-Lyubarskii theory we are going to establish lattice conditions for the existence of frames for the corresponding integral transform like in the classic case of the Gabor transform. Furthermore, given the...
importance of the class of operators under consideration we can see applications in quantum mechanics, stochastic analysis, signal processing or other fields, not only for the discussion of frame construction, but also for representation of coherent states in terms of position and momentum operators.

The paper is organized as follows. In Section 2 we recall the definition of a derivative of Gelfond-Leontiev type and present some important examples. In Section 3 we are going to discuss the corresponding Fock space. In Section 4 we present the Bargmann transform in this setting and the density theorems for sampling and interpolation sequences. We prefer to move the necessary proofs into its own Section 5. In the last Section we will present the application of our density theorems to obtain lattice densities for the construction of frames of the corresponding integral transforms.

2. Preliminaries

We are going to study Fock spaces related to Gelfond-Leontiev operators. To this end let us start with the definition of operators of generalized differentiation and integration with respect to a given entire function.

2.1. Generalized fractional derivatives. Since we are going to be interested in Fock spaces we are going to consider Gelfond-Leontiev operators with respect to an entire function. Before looking into that we want to remark that one could also consider functions analytic in a disk which would lead to Hardy spaces instead of Fock spaces.

Definition 2.1. Let

\[ \varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^k, \]

be an entire function with order \( \rho > 0 \) and degree \( \sigma > 0 \), that is, such that \( \lim_{k \to \infty} k^\frac{1}{\rho} \sqrt{\varphi_k} = (\sigma \rho)^\frac{1}{\rho} \).

We define the Gelfond-Leontiev (G-L) operator of generalized differentiation with respect to \( \varphi \), denoted as \( D_{\varphi} \), as the operator acting on an analytic function \( f(z) = \sum_{k=0}^{\infty} a_k z^k, \) \( |z| < 1 \), as

\[ f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \mapsto \quad D_{\varphi} f(z) = \sum_{k=1}^{\infty} a_k \frac{\varphi_{k-1}}{\varphi_k} z^{k-1}. \]

Hence, under the condition on \( \varphi \) that \( \limsup_{k \to \infty} \sqrt[k]{\frac{\varphi_{k-1}}{\varphi_k}} = 1 \) by the Cauchy-Hadamard formula we have that the series in (2) inherit the same radius of convergence \( R > 0 \) of the original series \( f \).

Also, we like to point out that the function \( \varphi \) acts as a replacement of the exponential function for the Gelfond-Leontiev operator of generalized differentiation. Indeed, \( D_{\varphi} \varphi = \varphi \).

Let us take a look at some examples. The first example is just the classic derivative.

Example 2.1. For \( \varphi(z) = e^z \), with \( \varphi_k = 1/\Gamma(k+1) \) for \( k = 0, 1, 2, \ldots \). We get:

\[ D_{\varphi} f(z) = D_{\varphi} \left( \sum_{k=0}^{\infty} a_k z^k \right) = \sum_{k=1}^{\infty} a_k k z^{k-1} = \partial_z f(z), \]

since \( \frac{\varphi_{k-1}}{\varphi_k} = \frac{k!}{(k-1)!} = k \).
The next example is a classic example of a fractional derivative.

**Example 2.2.** Let \( \varphi \) be the Mittag-Leffler function defined as:

\[
E_{\frac{\rho}{\mu},\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\mu + \frac{k}{\rho}\right)}, \quad \rho > 0, \ \mu \in \mathbb{C}, \ \text{Re}(\mu) > 0,
\]

with coefficients \( \varphi_k = \frac{1}{\Gamma\left(\mu + \frac{k}{\rho}\right)} \). The operator (2) becomes then the Dzrbashjan-Gelfond-Leontiev operator:

\[
D_{\rho,\mu} f(z) = \sum_{k=1}^{\infty} a_k \frac{\Gamma\left(\mu + \frac{k}{\rho}\right)}{\Gamma\left(\mu + \frac{k-1}{\rho}\right)} \frac{\Gamma\left(\mu + \frac{k+1}{\rho}\right)}{\Gamma\left(\mu + \frac{k}{\rho}\right)} z^{k-1}.
\]

That the range of possibilities for fractional derivatives is much larger can be seen in the next example.

**Example 2.3.** For \( \varphi_k = \frac{b_0 k + 1}{\Gamma\left(\frac{k}{\gamma}\right)} \) for Re\( (a) > 0, b > 0, \) and \( k = 0, 1, 2, \ldots \), we have

\[
D_\varphi f(z) = \sum_{k=1}^{\infty} a_k \left[ \Gamma\left(\frac{k+1}{\gamma}\right) \right] z^{k-1} = \sum_{k=1}^{\infty} a_k \frac{\Gamma\left(\frac{k+1}{\gamma}\right)}{\Gamma\left(\frac{k}{\gamma}\right)} z^{k-1}.
\]

We remark that for \( a = b = 1 \) we have the usual case \( \varphi(z) = e^z \).

Although in this work we consider entire function, we can also give these additional examples with functions that are not entire.

**Example 2.4.** For \( \varphi(z) = \frac{1}{1 - z} \), \(|z| < 1\), we have \( \varphi_k = 1 \) and so

\[
D_\varphi f(z) = \left(\sum_{k=0}^{\infty} a_k z^k\right) = \sum_{k=1}^{\infty} a_k z^{k-1} = \frac{f(z) - f(0)}{z}.
\]

In this case, the G-L operator is also known as backward-shift operator.

**Example 2.5.** Consider \( \varphi(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha)(k+1)} \), for a fixed \( \alpha \in \mathbb{N} \). We get

\[
D_\varphi f(z) = \sum_{k=1}^{\infty} a_k \left[ \frac{\Gamma(\alpha)(k+1)}{\Gamma(\alpha)(k)} \right] z^{k-1}.
\]

For \( \alpha = 1 \) we have \( D_\varphi f(z) = \sum_{k=1}^{\infty} ka_k c_k z^{k-1} \), where \( c_k \) is a constant depending on the Euler-Mascheroni constant (see example 4.2 given later on).

The examples above are examples of fractional derivatives, including Caputo and Riemann-Liouville derivatives, the latter being obtained by considering the GL-derivative as an operator acting on the ground state \( z^{1-\delta} \). But the class of GL derivatives is much broader. Let us now consider an example of a G-L derivative whose connection with G-L derivatives may not be so well-known.

2.2. **Dunkl operators.** Another important example of a generalized differentiation operator of Gelfond-Leontiev type is the case of Dunkl operators, also called differential-difference operators linked to a finite reflection group.
These operators are introduced as follows. Given a non-zero vector $\nu \in \mathbb{R}^n$ let $\sigma_\nu(x)$ denote the reflection of a given vector $x \in \mathbb{R}^n$ on the hyperplane orthogonal to $\nu$. A root system $R$ is a finite set of non-zero vectors in $\mathbb{R}^n$ such that $\sigma_\nu R = R$ for all $\nu \in R$. A positive subsystem $R_+$ is any subset of $R$ that contains the vector $\nu_0$. This implies that $R_+$ and $-R_+$ are separated by a hyperplane passing through the origin.

A Coxeter group (or finite reflection group) $G$ is a group generated by the reflections $\sigma_\nu, \nu \in R$ thus, it is a subgroup of the orthogonal group $O(n)$. Standard examples are the groups $A_{n-1}$ and $B_n$ (see e.g. [18], [19]). A multiplicity function $\kappa_\nu$ is a $G$-invariant complex-valued function defined on $R$, i.e., $\kappa_\nu = \kappa_{g\nu}$ for all $g \in G$. For a chosen positive subsystem $R_+$ we introduce the index

$$\gamma_\kappa = \sum_{\nu \in R_+} \kappa_\nu,$$

and the weight function

$$h_\kappa(x) = \prod_{\nu \in R_+} | \nu, x |^{\kappa_\nu},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$.

For each fixed positive subsystem $R_+$ and multiplicity function $\kappa_\nu$ we have, as invariant operators, the Dunkl operators (or differential-difference operators):

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\nu \in R_+} \kappa_\nu \frac{f(x) - f(\sigma_\nu x)}{\langle x, \nu \rangle^\nu_j}.$$

Associated to these is the intertwining operator which allows to interchange Dunkl derivatives with the usual partial derivatives. Let $\Pi$ denote the space of homogeneous polynomials. Furthermore, let $\Pi_k$ denote the space of homogeneous polynomials of degree $k$.

**Lemma 2.1** ([18]). If the multiplicity function $\kappa$ is such that $\cap_j \ker T_j = \mathbb{C}$ then it exists a unique positive linear isomorphism $V_\kappa : \Pi \rightarrow \Pi$, denoted as intertwining operator, which satisfies

1. $V_\kappa(\Pi_k) \subseteq \Pi_k$;
2. $V_\kappa|_{\Pi_0} = \text{id}$;
3. $T_j V_\kappa = V_\kappa \partial_j$, with $V_\kappa(1) = 1$.

This means that we can express the Dunkl operators in terms of generalized differentiation operators with respect to the function $\varphi(z) = V_\kappa(e^z)$.

For instance, in the rank-one case we have

$$V_\kappa(z^{2n}) = \frac{(\frac{1}{2})_n}{(\kappa + \frac{1}{2})_n} z^{2n}, \quad V_\kappa(z^{2n+1}) = \frac{(\frac{1}{2})_{n+1}}{(\kappa + \frac{1}{2})_{n+1}} z^{2n+1},$$

where $(a)_0 = 1$, and $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, $\Re(a) > 0$, denotes the Pochhammer symbol, or rising factorial. This leads to the function

$$\varphi(z) = e^z F_1(\kappa, 2\kappa + 1; -2z),$$

with

$$\varphi_{2n} = \frac{(\frac{1}{2})_n}{(2n)!} (\kappa + \frac{1}{2})_n \quad \text{and} \quad \varphi_{2n+1} = \frac{(\frac{1}{2})_{n+1}}{(2n+1)!} (\kappa + \frac{1}{2})_{n+1}. $$
3. Fractional Fock space

The classical Bargmann-Fock space $\mathcal{F}$ is the set of all entire functions $f$ such that

$$\langle f, f \rangle = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dxdy < \infty.$$  

Hence, $\mathcal{F}$ can be seen as the reproducing kernel Hilbert space with reproducing kernel given by

$$k(z, w) = e^{zw}.$$  

The Fock space links to quantum mechanics through the Schrödinger equation which describes the evolution of the state of the system by means of the Hamiltonian. In this space the momentum $P$ and position $Q$ operators, which describe the observables, and are related by canonical commutation relations as well as by duality. The Fock space is the unique Hilbert space of entire functions in which the momentum operator coincides with the classic derivative while the position operator is the multiplicative operator. This establishes a framework for other similar characterizations of spaces of analytic functions such as the Hardy space and Dirichlet space which was done in previous work (N. Alpay [3]).

Here we are interested in the Fock space related to our G-L derivative, which we are going to introduce next. Although $\varphi$ is an entire function with complex coefficients, in the following we additionally assume $\varphi$ to have positive coefficients, that is, $\varphi_n > 0$, $\forall n$, in order to ensure positivity of the measure. In fact, it will allow us to obtain a probability measure similar to the classic case.

3.1. Inner product. Given two entire functions $f(z) = \sum_{k=0}^{\infty} f_k z^k$, $g(z) = \sum_{k=0}^{\infty} g_k z^k$, we consider the following Hilbert spaces

(i) the fractional space $\ell^2_\varphi$ of the sequences $f \sim (f_k)_{k=0}^{\infty}$ and weighted inner product

$$\langle f, g \rangle_{2,\varphi} = \sum_{k=0}^{\infty} \frac{f_k g_k}{\varphi_k};$$

(ii) the fractional Fock space $\mathcal{F}_\varphi$ endowed with the weighted inner product

$$\langle f, g \rangle_{\mathcal{F},\varphi} = \frac{1}{\pi} \int_{\mathbb{C}} \overline{f(z)} g(z) K_\varphi(-|z|^2) dxdy,$$

and where $K_\varphi$ denotes the weight function. We remark that in the classical case $K_\varphi(-|z|^2)$ is the Gaussian $e^{-|z|^2/2}$.

These weighted inner products should be related by the identity $\langle f, g \rangle_{2,\varphi} = \langle f, g \rangle_{\mathcal{F},\varphi}$, i.e.

$$\langle f, g \rangle_{2,\varphi} = \sum_{k=0}^{\infty} \frac{f_k g_k}{\varphi_k} = \frac{1}{\pi} \int_{\mathbb{C}} \overline{f(z)} g(z) K_\varphi(-|z|^2) dxdy = \langle f, g \rangle_{\mathcal{F},\varphi}.$$
For \( f(z) = z^k \) and \( g(z) = z^n \) we have
\[
\delta_{n,k} / \varphi_n = \langle z^k, z^n \rangle_{F, \varphi} = \frac{1}{\pi} \int_{C} z^k z^n K_\varphi(-|z|^2) dxdy,
\]
\[
= \frac{1}{\pi} \int_0^\infty \int_0^{2\pi} r^{k+n} e^{i(n-k)\theta} K_\varphi(-r^2) d\theta dr,
\]
\[
= 2\delta_{n,k} \int_0^\infty r^{k+n+1} K_\varphi(-r^2) dr.
\]
This leads us to
\[
\frac{1}{\varphi_n} = 2 \int_0^\infty r^{2n+1} K_\varphi(-r^2) dr,
\]
\[
= \int_0^\infty x^n K_\varphi(-x) dx, \quad (x = r^2),
\]
\[
(9)
\]
where \( \mathcal{M} \) denotes the Mellin transform of the weight function \( \tilde{K}_\varphi(x) := K_\varphi(-x) \) evaluated at the point \( n + 1 \). This reduces the determination of the measure \( K_\varphi(-|z|^2) dxdy \) either to an inversion of the Mellin transform or to a Stieltjes moment problem. Of course, a sufficient condition for the determination of \( K_\varphi \) consists in the Carleman condition
\[
\sum_{n=1}^\infty \frac{1}{\varphi_n^n} = +\infty.
\]

Let us first consider the classic case as an example:

**Example 3.1.** Again, for \( \varphi(z) = e^z \), we have
\[
\frac{1}{\varphi_n} = n! = \int_0^\infty x^n K_\varphi(-x) dx = \mathcal{M}(\tilde{K}_\varphi)(n + 1).
\]
Moreover, as
\[
n! = \Gamma(n + 1) = \int_0^\infty x^n e^{-x} dx
\]
we identify the weight function as
\[
K_\varphi(x) = e^x,
\]
and leading to the classic inner product in the Fock space
\[
\langle f, g \rangle_{F, \varphi} = \frac{1}{\pi} \int_{C} \overline{f(z)} g(z) e^{-|z|^2} dxdy.
\]

**Example 3.2.** When \( K_\varphi(z) = E_{\frac{1}{2},\mu}(z) \), that is, the Mittag-Leffler function defined as in [3] the weighted inner product would be given by [3]
\[
\langle f, g \rangle_{F, \varphi} = \frac{1}{\pi} \int_{C} \overline{f(z)} g(z) E_{\frac{1}{2},\mu}(-|z|^2) dxdy.
\]

**Example 3.3.** Consider \( \varphi(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(n)(k+1)} \), for some fixed \( n \in \mathbb{N} \) with coefficients \( \varphi_k = \frac{1}{\Gamma(n)(k+1)} \).
Using the Mellin Transform we get the weight function
\[
K_\varphi(-|z|^2) = 2 e^{-|z|^2} \ln^n |z|.
\]
We can now discuss the multiplication and derivative operators in this Fock space $F$. As usual the multiplication operator is given by

$$M_z f(z) := zf(z) = \sum_{k=0}^{\infty} f_k z^{k+1} = f_0 z + f_1 z^2 + f_2 z^3 + \cdots$$

defined over the domain $\text{Dom}(M_z) = \{ F \in F : zF \in F \}$. One can observe that $M_z$ induces a shift in $\ell^2_\varphi$.

Its dual $M_z^*$ is defined by

$$\langle \langle M_z f, g \rangle \rangle_{F, \varphi} = \langle \langle f, M_z^* g \rangle \rangle_{F, \varphi}.$$

This can be easily calculated by passing to the space $\ell^2_\varphi$:

$$\langle \langle M_z f, g \rangle \rangle_{F, \varphi} = \langle f, M_z^* g \rangle_{\ell^2_\varphi} = \langle \sum_{k=0}^{\infty} f_k z^{k+1}, \frac{\varphi_k}{\varphi_{k+1}} \rangle_{\ell^2_\varphi} = \langle f, M_z^* g \rangle_{\ell^2_\varphi} = \langle \langle f, M_z^* g \rangle \rangle_{F, \varphi}.$$

Therefore, the dual $M_z^*$ is defined over the domain $\text{Dom}(M_z^*) = \{ F \in F : D_\varphi F \in F \}$ by

$$M_z^* g(z) := \sum_{k=0}^{\infty} g_k z^{k+1} \frac{\varphi_{k+1}}{\varphi_k} = g_1 \frac{\varphi_0}{\varphi_1} + g_2 \frac{\varphi_1}{\varphi_2} z^2 + g_3 \frac{\varphi_2}{\varphi_3} z^3 + \cdots = D_\varphi g(z).$$

It is also an easy task to prove that this Fock space is the unique space where the associated GL-derivative is the dual operator to the multiplication operator.

We also need to point out that an orthonormal basis $\{ e_n \}$ for our Fock space $F$ is given by $e_n(z) = \sqrt{\varphi_n} z^n$. By looking at the action of $M_z$ and $D_\varphi$ on the orthonormal basis it follows that $\text{Dom}(M_z^*) = \text{Dom}(M_z)$.

We are now going to study the reproducing kernel property of our Fock space. We recall that the weight $K_\varphi(-|z|^2)$ has to satisfy the property for $K_\varphi$

$$\frac{1}{\varphi_n} = \mathcal{M}(K_\varphi(-\cdot))(n + 1),$$

whereas $\mathcal{M}$ denotes the Mellin transform. This relation also induces a discrete reproducing kernel given by

$$k_\varphi(z, w) := \varphi(\overline{w} z) = \sum_{n=0}^{\infty} \varphi_n (\overline{w} z)^n,$$

and we define the corresponding reproducing kernel Hilbert space as

$$\mathcal{H}(k_\varphi) := \left\{ f(z) = \sum_{n=0}^{\infty} f_n z^n \text{ entire : } \| f \|_{2, \varphi}^2 = \sum_{n=0}^{\infty} |f_n|^2 \frac{1}{\varphi_n} < \infty \right\}.$$

For all $f \in \mathcal{H}(k_\varphi)$ we have

$$f(z) = \langle k_\varphi(z, \cdot), f \rangle_{2, \varphi}.$$

From the Cauchy Schwarz Inequality $|\langle f, g \rangle_{2, \varphi}| \leq \| f \|_{2, \varphi} \| g \|_{2, \varphi}$ and $f(z) = \langle k_\varphi(z, \cdot), f \rangle_{2, \varphi}$, we have

$$|f(z)| \leq \| f \|_{2, \varphi} \| k_\varphi(z, \cdot) \|_{2, \varphi}.$$
Recall here that \( \varphi \) as in (1) is an entire function with order \( \rho > 0 \) and degree \( \sigma > 0 \). Hence, we can estimate the norm of the discrete reproducing kernel since \( \|k_{\varphi}(z, \cdot)\|_{2,\varphi}^2 = \sum_{n=0}^{\infty} \varphi_n |z|^{2n} \). We obtain

\[
\|k_{\varphi}(z, \cdot)\|_{2,\varphi}^2 \leq \varphi(r^2) \leq e^{\sigma r^2}, \quad |z| \leq r,
\]

where \( r > 0 \). Therefore, for every \( f \in \mathcal{H}(k_{\varphi}) \) one has

\[
|f(z)| \leq \|f\|_{2,\varphi} e^{\frac{\sigma r^2}{2}}, \quad |z| \leq r,
\]

that is to say, \( f \) is an entire function of order \( 2\rho \) and degree \( 2\sigma \).

We look now into the continuous kernel associated to \( k_{\varphi} \). Combining \( \langle f, g \rangle_{2,\varphi} = \langle \langle f, g \rangle \rangle_{F,\varphi} \) with the reproducing kernel property (15) we obtain

\[
f(z) = \langle k_{\varphi}(z, \cdot), f \rangle_{2,\varphi} = \frac{1}{\pi} \int_{\mathbb{C}} \overline{k_{\varphi}(z, w)} f(w) K_{\varphi}(-|w|^2) dxdy \quad (w = x + iy)
\]

\[
= \frac{1}{\pi} \int_{\mathbb{C}} \overline{\varphi(w)} f(w) K_{\varphi}(-|w|^2) dxdy =: \mathbb{K}_{\varphi}(z, w) f(w) K_{\varphi}(-|w|^2) dxdy,
\]

(18)

where \( \mathbb{K}_{\varphi}(z, w) := \varphi(\overline{w}) \) denotes the continuous reproducing kernel with respect to the weighted measure \( d\mu(w) = K_{\varphi}(-|w|^2) dxdy \).

**Example 3.4.** For \( \varphi(z) = e^z \), we obtain

\[
\mathbb{K}_{\varphi}(z, w) = e^{\overline{w}z} = \varphi(\overline{z}w),
\]

with the Gaussian weighted measure \( d\mu(z) = e^{-|z|^2} dxdy \).

Using our reproducing kernel we have the following characterization of bounded operators on \( F \)

**Theorem 3.1.** Let \( T \) be a bounded operator on \( F \) and \( \mathbb{K}_{\varphi,T}(w, \cdot) = T^*\langle \mathbb{K}_{\varphi}(\overline{w}, \cdot) \rangle(w) \). Then \( \mathbb{K}_{\varphi,T} \) has the following properties

(1) \( \mathbb{K}_{\varphi,T} \) is an entire function on \( \mathbb{C}^2 \).
(2) \( \mathbb{K}_{\varphi,T}(\cdot, w) \in \mathcal{F} \) for all \( w \) and \( \mathbb{K}_{\varphi,T}(z, \cdot) \in \mathcal{F} \) for all \( z \).
(3) \( |\mathbb{K}_{\varphi,T}(z, w)| \leq K_{\varphi}(|z|^2) K_{\varphi}(|w|^2) \|T\| \).
(4) \( TF(z) = \int_{\mathbb{R}^2} \mathbb{K}_{\varphi,T}(w) F(w) K_{\varphi}(-|w|^2) dxdy \) for all \( F \in \mathcal{F} \) and \( z \in \mathbb{C} \).

The proof is a straightforward adaptation of the proof of Proposition (1.68) in [7]. In particular, this means that any bounded operator is determine by the action of its adjoint on the reproducing kernel \( T_w^* \varphi(\overline{z}w) \)

4. **Generalized Bargmann transform**

One of the important links of the Fock space to applications is given by the Bargmann transform which allows to transform problems over the space \( L^2(\mathbb{R}) \) into problems over the Fock spaces which is also closely linked to the Bargmann-Fock representations of the Weyl-Heisenberg group.
It is well known that the system of Hermite functions $h_n : \mathbb{R} \to \mathbb{R}$ given by
\[
h_n(x) = \frac{1}{\sqrt{\pi n!}} H_n(x) e^{-x^2/2}, \quad n \in \mathbb{N}_0
\]
where $H_n$ denote the Hermite polynomials, forms an orthonormal basis in $L^2(\mathbb{R})$. If we map each $h_n = h_n(x)$ into $\frac{z^n}{\sqrt{n!}}$ we get the so-called Bargmann transform:
\[
B : L^2(\mathbb{R}) \to F
\]
given by
\[
Bf(z) = \int_{\mathbb{R}} \left( \sum_{n=0}^{\infty} \frac{h_n(x)}{\sqrt{n!}} \frac{z^n}{\sqrt{n!}} \right) f(x) dx, \quad f \in L^2(\mathbb{R}).
\]
The calculation of the sum of this series gives the well-known formula for the kernel of the Bargmann transform $k(x, z) = 2^{1/4} e^{2\pi xz - \pi x^2 - (\pi/2)z^2}$ so that in closed form the Bargmann transform appears as a double version of the Weierstraß transform. It also was as an obvious consequence that the Bargmann transform is a unitary isomorphic mapping between $L^2(\mathbb{R})$ and the Fock space $\mathcal{F}$.

This mapping provides a large number of applications including mapping a windowed Fourier transform of a signal in $L^2(\mathbb{R})$ with a window given by a Hermite function into an analytic function belonging to the Fock space (see [10, 12]). Moreover, it also allows to consider the pre-image of the annihilation and creation operators in the Fock space, given by $\partial_z$ and $M_z$, as operators over $L^2(\mathbb{R})$ which in the classic case turns out to be the classic position and momentum operators together with their corresponding coherent states.

4.1. **Generalized Bargmann transform.** In our case we consider the modified Bargmann transform which maps $h_n = h_n(x)$ into $\sqrt{\varphi_n} z^n$. This correspondence allow us to link the multiplication and derivative operators in the fractional Fock space with creation and annihilation operators in the classic $L^2$-space which leads different types of coherent states such as the squeezed coherent states.

Consider the modified Bargmann transform $\mathcal{B} : L^2(\mathbb{R}) \to F_{\varphi}$ given by
\[
\mathcal{B}f(z) = \int_{\mathbb{R}} \left( \sum_{n=0}^{\infty} \frac{h_n(x)}{\sqrt{\varphi_n} z^n} \right) f(x) dx = \sum_{n=0}^{\infty} f_n \sqrt{\varphi_n} z^n,
\]
with $f_n = \int_{\mathbb{R}} h_n(x)f(x) dx = \langle h_n, f \rangle_{L^2(\mathbb{R})}$.

Since by construction $\mathcal{B}$ is a unitary operator we have $\mathcal{B}^* = \mathcal{B}^{-1}$. Hence, the inverse of the Bargman transform, $\mathcal{B}^{-1}$, can be found in the following manner:
\[
\langle \mathcal{B}g, F \rangle_{F_{\varphi}} = \int_{\mathbb{C}} \left[ \int_{\mathbb{R}} \left( \sum_{n=0}^{\infty} \frac{h_n(t)}{\sqrt{\varphi_n} z^n} \right) g(t) dt \right] F(z) d\mu(z)
\]
\[
= \int_{\mathbb{R}} g(t) \sum_{n=0}^{\infty} \left( \int_{\mathbb{C}} \sqrt{\varphi_n} z^n F(z) d\mu(z) \right) h_n(t) dt
\]
\[
= \langle g, \mathcal{B}^{-1} F \rangle_{L^2(\mathbb{R})}.
\]
By considering the action of the multiplication and G-L operators, $M_z$ and $D_\varphi$, on $\tilde{B}f$ we get

$$M_z\tilde{B}f(z) = z\tilde{B}f(z) = \sum_{n=0}^{\infty} z^{n+1} f_n \varphi_n = \sum_{n=1}^{\infty} z^n f_{n-1} \sqrt{\varphi_{n-1}} = \sum_{n=1}^{\infty} z^n f_{n-1} \sqrt{\varphi_n - \varphi_{n-1}},$$

as well as

$$D_\varphi\tilde{B}f = \sum_{n=1}^{\infty} z^{n-1} f_n \sqrt{\varphi_{n-1}} \varphi_n = \sum_{n=1}^{\infty} z^{n-1} f_{n-1} \sqrt{\varphi_{n-1}} \varphi_n.$$

Thus, we have

$$a^* \left( \sum_{n=0}^{\infty} h_n f_n \right) = \sum_{n=0}^{\infty} \sqrt{\frac{\varphi_n}{\varphi_{n+1}}} h_{n+1} f_n, \quad a \left( \sum_{n=0}^{\infty} h_n f_n \right) = \sum_{n=1}^{\infty} \sqrt{\frac{\varphi_{n-1}}{\varphi_n}} h_{n-1} f_n.$$

This gives us the action of the raising and lowering operators $a^*$ and $a$ on the Hermite functions as

$$a^* h_{n-1} = \sqrt{\frac{\varphi_{n-1}}{\varphi_n}} h_n, \quad ah_n = \sqrt{\frac{\varphi_{n-1}}{\varphi_n}} h_{n-1}, \quad n = 1, 2, \ldots$$

Let us emphasize that $\tilde{B}$ acts now as an intertwining operator in the following way

$$\tilde{B}af = D_\varphi\tilde{B}f, \quad \tilde{B}a^* f = z\tilde{B}f.$$

Let us give two concrete examples. The first example is not really correct in our setting since the involved function $\varphi$ is not entire and, hence, the transform is not linked to the Fock space, but to the Hardy space. However, it provides the simple case in which the operators $a$ and $a^*$ appear as forward and backward shift operators, where the corresponding integral transform maps only into the space of analytic functions over the unit disk.

**Example 4.1.** Let $\varphi(z) = \frac{1}{1-z^2}$, where $\varphi_k = 1, k = 0, 1, 2, \ldots$. Then from (19) we have:

$$a^* h_{n-1} = h_n, \quad ah_n = h_{n-1}.$$

Let us now consider a case which fits into our setting.

**Example 4.2.** Let us consider $\varphi(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)}$, where $\varphi_n = \frac{1}{\Gamma(n+1)}$, $n = 0, 1, 2, \ldots$. Recall now that $\Gamma(x+1) = x\Gamma(x)$, $x > 0$. Using the Digamma function $\psi$ we have

$$\psi(x) = \frac{d}{dx} \ln[\Gamma(x)] = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0.$$

The Digamma function is related to the harmonic numbers $H_0 = 1, H_n = \sum_{k=1}^{n} \frac{1}{k}$, $n \in \mathbb{N}$, by

$$\psi(n) = -\gamma + H_{n-1} \Rightarrow \Gamma'(n+1) = \Gamma(n+1)(-\gamma + H_n) = n!(-\gamma + H_n),$$

where $\gamma$ is the Euler–Mascheroni constant. Hence,

$$\frac{\varphi_{n-1}}{\varphi_n} = \frac{n!}{(n-1)!} \left[ \frac{-\gamma + H_n}{-\gamma + H_{n-1}} \right] := nc_n, \quad n = 1, 2, \ldots$$

where the constants $c_n := \frac{[-\gamma + H_n]}{[-\gamma + H_{n-1}]}$ are such that $\lim_{n \to \infty} c_n = 1$. So we get:

$$a^* h_{n-1} = \sqrt{n-1} c_n h_n, \quad ah_n = \sqrt{n-1} c_n h_{n-1}.$$
4.2. Density Theorems for Sampling and Interpolations. Using the general framework of the Fock spaces with respect to the Gelfond-Leontiev operator of generalized differentiation, we extend density results of K. Seip [19] to our setting. This will later on allow us to obtain lattice conditions for frames arising from the corresponding integral transform over $L^2(\mathbb{R})$.

We recall the following definitions for sampling and interpolation sets that include the notion of a weight function $K_\phi$ in order to match our setting.

A discrete set $\Gamma = \{z_j | j \in J\}$ is a sampling set of $\mathcal{F}_\phi$ if it satisfies an appropriated frame condition, that is, if there exists $0 < A \leq B < \infty$ such that

\[
A \|f\|^2_\phi \leq \sum_{j \in \Lambda} K_\phi(-|z_j|^2)|f(z_j)|^2 \leq B \|f\|^2_\phi, \quad \text{for all } f \in \mathcal{F}_\phi.
\]

The set $\Gamma$ is an interpolation set of $\mathcal{F}_\phi$ if for every $\ell^2$-sequence $(a_j)_{j \in \Lambda}$ satisfying to the growth condition $\sum_{j \in \Lambda} |a_j|^2 K_\phi(-|z_j|^2) < \infty$, there exists $f \in \mathcal{F}_\phi$ such that $f(z_j) = a_j$, $j \in J$.

For a uniformly discrete set $\Gamma$, using Landau’s generalizations of Beurling densities we define the upper and lower uniform densities respectively by:

\[
D^+(\Gamma) = \lim_{r \to \infty} \sup_n \frac{n^+(r)}{2\pi r^2} \quad \text{and} \quad D^-(\Gamma) = \lim_{r \to \infty} \inf_n \frac{n^-(r)}{2\pi r^2}
\]

where $n^\pm$ represent the smallest and largest number of points of $\Gamma$ in a translate $rI$, where $I$ is a fixed compact set of measure 1. Then for uniform discrete sets $\Gamma, \Gamma'$ we can extend the following theorems to our framework of generalized Fock space as follows:

First of all, we have the following characterization of an interpolation set:

**Theorem 4.1.** $\Gamma$ is a set of interpolation for $\mathcal{F}_\phi$ if and only if $D^+(\Gamma) < \beta_\phi$.

This theorem helps us to build $\Gamma$, and then we have the following theorems characterizing a sampling set.

**Theorem 4.2.** $\Gamma$ is a set of sampling for $\mathcal{F}_\phi$ if and only if it can be expressed as a finite union of uniformly discrete sets and contains a subset $\Gamma'$ s.t. $D^-(\Gamma') > \beta_\phi$.

**Theorem 4.3.** $\Gamma$ is a sampling set for $\mathcal{F}_\phi$ if and only if it can contains a subset $\Gamma'$ and $D^+(\Gamma') > \beta_\phi$.

Proofs of these theorems are given in the following section.

5. Proofs of main theorems

While the proofs for Theorems 4.2, 4.1, and 4.3 follow similar steps as in [19] and [20], they require a new analogue for the Weierstrass-$\sigma$ function endowed with the Gelfond-Leontiev operator of generalized differentiation.

Let $\Lambda = \{\lambda_{m,n} : \lambda_{m,n} = \lambda(m + in), m, n \in \mathbb{Z}, \lambda > 0\}$ be a square lattice. Let us consider a lattice $\Gamma = \{z_{m,n}, m, n \in \mathbb{Z}\}$ which is uniformly closed to $\Lambda$, i.e. for which there exist constants $Q$ and $q(\Gamma)$ such that

$|z_{m,n} - \lambda_{m,n}| < Q$.
and

\[ q(\Gamma) = \inf_{(m,n) \neq (k,l)} |z_{m,n} - z_{k,l}| > 0. \]

In the classic case the Weierstrass-\(\sigma\) function associated to the original square lattice \(\Lambda\) is given by

\[ \sigma(z; \Lambda) := z \prod_{(m,n) \neq (0,0)} \left( 1 - \frac{z}{\lambda_{m,n}} \right) e^{\left( \frac{z^2}{\lambda_{m,n}} + \frac{1}{2} \frac{z^3}{\lambda_{m,n}^2} \right)} = z \prod_{(m,n) \neq (0,0)} E_{2,\sigma}(z; m, n), \]

where

\[ E_{2,\sigma}(z; m, n) = \left( 1 - \frac{z}{\lambda_{m,n}} \right) e^{\left( \frac{z^2}{\lambda_{m,n}} + \frac{1}{2} \frac{z^3}{\lambda_{m,n}^2} \right)}. \]

In this work, we wish to construct a function which generalizes the Weierstrass-\(\sigma\) function.

In order to obtain such a function we assume \(\varphi_0 = 1\). Given

\[ \varphi(z) = 1 + \sum_{k=1}^{\infty} \varphi_k z^k, \]

we define the auxiliar entire function \(\psi(z) = \sum_{n=1}^{\infty} \psi_n z^n\), such that the corresponding fractional Weierstrass-\(\sigma\) factor \(E(z)\) satisfies

\[ E(z) = (1 - z) \varphi(z \psi_1 + z^2 \psi_2). \]

Developing we get

\[
E(z) = (1 - z) \sum_{n=0}^{\infty} \varphi_n z^n \left( \psi_1 + z \psi_2 \right)^n \\
= (1 - z) \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \varphi_n \psi_1^{n-k} \psi_2^k z^{n+k} \\
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \varphi_n \psi_1^{n-k} \psi_2^k z^{n+k} - \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \varphi_n \psi_1^{n-k} \psi_2^k z^{n+k+1} \\
= \sum_{m=0}^{[m/2]} \sum_{k=0}^{m} \binom{m-k}{k} \varphi_{m-k} \psi_1^{m-2k} \psi_2^k z^m - \sum_{m=1}^{[m-1/2]} \sum_{k=0}^{m-1} \binom{m-1-k}{k} \varphi_{m-1-k} \psi_1^{m-2k} \psi_2^k z^m \\
= \psi_0 + \sum_{m=1}^{[m/2]} \sum_{k=0}^{m} \binom{m-k}{k} \varphi_{m-k} \psi_1^{m-2k} \psi_2^k - \sum_{m=1}^{[m-1/2]} \sum_{k=0}^{m-1} \binom{m-1-k}{k} \varphi_{m-1-k} \psi_1^{m-2k} \psi_2^k \]

(24)

Hence, we obtain (recall \(\varphi_0 = 1\))

\[ E(z) = 1 + (\varphi_1 \psi_1 - 1)z + (\varphi_2 \psi_1^2 + \varphi_1 \psi_2 - \varphi_1 \psi_1)z^2 \]

(25)

\[ + (\varphi_3 \psi_1^3 + 2 \varphi_2 \psi_1 \psi_2 - \varphi_2 \psi_1^2 - \varphi_1 \psi_2)z^3 + \cdots = \Omega(z)z^3 \]

We further impose the coefficients of \(z\) and \(z^2\) to be zero, that is

\[ \psi_1 = \frac{1}{\varphi_1}, \quad \psi_2 = \frac{\varphi_2^2 - \varphi_2}{\varphi_1^3}, \]

(26)
so that we get \(|1 - E(z)| = |\Omega(z)||z|^3\), where \(\Omega(z)\) is the remainder. In the unit disk \(|z| < 1\) we get the inequality:

\[
|1 - E(z)| \leq |\Omega(z)|,
\]

as desired.

We obtain a \(\sigma\)-function of the form

\[
\sigma(z; \Lambda) := z \prod_{(m,n) \neq (0,0)} \left(1 - \frac{z}{\lambda_{m,n}}\right)^\varphi \left(\psi_1 \frac{z}{\lambda_{m,n}} + \psi_2 \frac{z^2}{\lambda_{m,n}^2}\right),
\]

and to which we associated the auxiliar function \(g\) given by

\[
g(z; \Gamma) := (z - z_{00}) \prod_{(m,n) \neq (0,0)} \left(1 - \frac{z}{z_{m,n}}\right)^\varphi \left(\psi_1 \frac{z}{z_{m,n}} + \psi_2 \frac{z^2}{\lambda_{m,n}^2}\right),
\]

with factors

\[
E_\Lambda(z; m, n) = \left(1 - \frac{z}{\lambda_{m,n}}\right)^\varphi \left(\psi_1 \frac{z}{\lambda_{m,n}} + \psi_2 \frac{z^2}{\lambda_{m,n}^2}\right), \quad E_\Gamma(z; m, n) = \left(1 - \frac{z}{z_{m,n}}\right)^\varphi \left(\psi_1 \frac{z}{z_{m,n}} + \psi_2 \frac{z^2}{\lambda_{m,n}^2}\right).
\]

When clear from the context, we will abbreviate to \(E(z) := E_\Gamma/\Lambda(z; m, n)\).

We can further develop \(\Omega = \Omega(z)\) as

\[
\Omega(z) = \sum_{m=3}^{\infty} \left[ \sum_{k=0}^{\lfloor (m-2)/2 \rfloor} \binom{m-1}{k} \varphi_2^{m-2k} \psi_2^{k} - \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-1}{k} \varphi_1^{m-2k} \psi_1^{k} \right] z^{m-3}
\]

\[
= \sum_{m=0}^{\infty} \left[ \sum_{k=0}^{\lfloor (m+3)/2 \rfloor} \binom{m+2}{k} \varphi_2^{m+2k} \psi_2^{k} - \sum_{k=0}^{\lfloor (m+2)/2 \rfloor} \binom{m+2}{k} \varphi_1^{m+2k} \psi_1^{k} \right] z^{m}
\]

\[
= \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n+2} \binom{2n+2}{k} \varphi_2^{2n+2k} \psi_2^{k} - \sum_{k=0}^{n+1} \binom{2n+1}{k} \varphi_1^{2n+1+k} \psi_1^{k} \right] z^{2n+1}
\]

\[
+ \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n+2} \binom{2n+2}{k} \varphi_1^{2n+2+k} \psi_1^{k} - \sum_{k=0}^{n+1} \binom{2n+1}{k} \varphi_2^{2n+1+k} \psi_2^{k} \right] z^{2n+1}.
\]
For $|z| \leq 1$ we have for $\Omega = \Omega(z)$ an estimate in terms of the coefficients of $\varphi_n, \psi_1,$ and $\psi_2$.  

$$E(z) = \varphi(z\psi_1 + z^2\psi_2) - z\varphi(z\psi_1 + z^2\psi_2) = \sum_{n=0}^{\infty} \varphi_n z^n (1 - z)(\psi_1 - \psi_2 z)^n$$

$$\Rightarrow \quad E(z) - 1 = -z + \varphi_1 z(1 - z)(\psi_1 - \psi_2 z) + \varphi_2 z^2(1 - z)(\psi_1 - \psi_2 z)^2 + \sum_{n=3}^{\infty} \varphi_n z^n (1 - z)(\psi_1 - \psi_2 z)^n$$

$$= (-\varphi_1 \psi_2 + 2\varphi_2 \psi_1 \psi_2 - \varphi_2 \psi_1^2) z^3 + (\varphi_2 \psi_1 \psi_2^2 - 2\varphi_2 \psi_1 \psi_2) z^4 - \varphi_2 \psi_2^2 z^5 + \sum_{n=3}^{\infty} \varphi_n z^n (1 - z)(\psi_1 - \psi_2 z)^n,$$

and hence, for $|z| \leq 1$ we obtain

$$|\Omega(z)| \leq |\varphi_1 \psi_2 - 2\varphi_2 \psi_1 \psi_2 + \varphi_2 \psi_1^2| + |\varphi_2 \psi_1 \psi_2^2 - 2\varphi_2 \psi_1 \psi_2| + |\varphi_2 \psi_2^2| + 2 \sum_{n=3}^{\infty} |\varphi_n|(|\psi_1| + |\psi_2|)^n. \quad (29)$$

With the restrictions on $\varphi_1$ and $\varphi_2$ such that $|\psi_1| + |\psi_2| < R,$ where $R$ is the radius of convergence of $\sum \varphi_n z^n,$ we obtain the lower bound for $R$ given by:

$$R_L = |\psi_1| + |\psi_2| = \frac{2\varphi_2 - \varphi_1}{\varphi_3} \quad \text{and} \quad R_U = \limsup_{n \to \infty} \frac{1}{|\varphi_n|^{1/n}}. \quad (30)$$

while the upper bound for $R$ is given by

$$R_U = \limsup_{n \to \infty} \frac{1}{|\varphi_n|^{1/n}}. \quad (31)$$

**Example 5.1.** For $\varphi(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, that is, $\varphi_n = \frac{1}{\Gamma(n+1)} = \frac{1}{n!}$, we obtain $\psi_1 = 1$ and $\psi_2 = \frac{1^2 - 1/2}{1^3} = \frac{1}{2}$. Hence, for the Weierstrass factor we have

$$E(z) = (1 - z)e^{z + \frac{z^2}{2}},$$

with lower and upper bounds given by (30) and (31) as $R_L = \frac{3}{2}$ and $R_U = \limsup_{n \to \infty} \frac{1}{(n!)^{1/n}} \to \infty.$

**Example 5.2.** For $\varphi(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ (not an entire function), we have $\psi_1 = 1$, and $\psi_2 = 0$, which leads to

$$E(z) = (1 - z)\varphi(z\psi_1 + z^2\psi_2) = 1$$

and, thus, $\Omega(z) = 0$. Also, our bounds for $R$ are $R_L = (2 - 1)/1 = 1$ and $R_U = \limsup_{n \to \infty} \frac{1}{(n!)^{1/n}} = 1$. This example is also an example of an extreme case where one has $R_U = R_L = R = 1$.

**Example 5.3.** Consider the Mittag-Leffler function defined in Example 2.2

$$\varphi(z) = E_{\frac{1}{\rho}, \mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\mu + \frac{k}{\rho}\right)}$$

for $\rho > 0, \text{Re}(\mu) > 0.$
Then, and taking into account the normalization factor in (22), we have \( \psi_1 = \frac{\Gamma(\mu + \frac{1}{\rho})}{\Gamma(\mu)} \) and \( \psi_2 = \frac{\Gamma(\mu + \frac{1}{\rho})}{\Gamma(\mu + \frac{1}{\rho})} \). Replacing these values in (23) we obtain

\[
E(z) = 1 + z^3 \Omega(z) = \Gamma(\mu)(1 - z)E_{\mu,\mu} \left( \frac{\Gamma(\mu + \frac{1}{\rho})}{\Gamma(\mu)} z + \frac{\Gamma(\mu + \frac{1}{\rho})}{\Gamma(\mu + \frac{2}{\rho})} \Gamma(\mu) - \Gamma^3(\mu + \frac{1}{\rho}) z^2 \right).
\]

Moreover, the lower bound is then given by

\[
R_L = 2 \frac{\Gamma(\mu + \frac{1}{\rho})}{\Gamma(\mu)} - \frac{\Gamma^3(\mu + \frac{1}{\rho})}{\Gamma(\mu + \frac{2}{\rho})} \Gamma^2(\mu).
\]

For the upper bound, and using Stirling’s approximation \( \Gamma(x + 1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \), we have for the normalized coefficients of \( \varphi \)

\[
|\Gamma(\mu)\varphi_n|^\frac{1}{n} = \left| \frac{\Gamma(\mu)}{\Gamma(\mu + n\frac{1}{\rho})} \right|^{\frac{1}{n}} = |\Gamma(\mu)|^{\frac{1}{n}} |\Gamma(\mu + n\frac{1}{\rho})|^{-\frac{1}{n}}.
\]

As \( |\Gamma(\mu)| > 0 \) we only analyse the behaviour of the last term.

\[
\left| \Gamma \left( \mu + \frac{n}{\rho} \right) \right|^{\frac{1}{n}} \sim \left| \sqrt{2\pi \left( \mu + \frac{n}{\rho} - 1 \right)} \left( \frac{\mu + \frac{n}{\rho} - 1}{e} \right)^{\left( \mu + \frac{n}{\rho} - 1 \right)} \right|^{\frac{1}{n}}
\]

\[
= \left| \sqrt{2\pi \left( \mu + \frac{n}{\rho} - 1 \right)} \right|^{\frac{1}{n}} \left| \left( \frac{\mu + \frac{n}{\rho} - 1}{e} \right)^{\left( \mu + \frac{n}{\rho} - 1 \right)} \right|^{\frac{1}{n}} \to \infty \quad \text{as} \quad n \to \infty.
\]

Therefore, the upper bound is given by \( R_U = \limsup_{n \to \infty} |\varphi_n|^\frac{1}{n} = \infty \).

**Example 5.4.** Let \( K_\varphi(x) = e^{-a(-x)^b} \) for \( x > 0 \), where \( b > 0 \), \( \text{Re}(a) > 0 \). Then, we have \( \varphi(z) = \sum_{n=0}^\infty \varphi_n z^n \), where

\[
\varphi_n = \frac{ba^{n+1} - b}{\Gamma(\frac{n+1}{b})}, \quad n = 0, 1, 2, \ldots
\]

We have (again taking into account the normalization factor)

\[
\varphi_1 = \frac{a^{\frac{b}{2}} \Gamma \left( \frac{1}{b} \right)}{\Gamma \left( \frac{2}{b} \right)}, \quad \varphi_2 = \frac{a^{\frac{b}{2}} \Gamma \left( \frac{1}{b} \right)}{\Gamma \left( \frac{2}{b} \right)}.
\]

This gives

\[
E(z) = \frac{\Gamma \left( \frac{1}{b} \right)}{ba^b} (1 - z) \exp \left[ -a \left( -\frac{\Gamma \left( \frac{2}{b} \right)}{a^{\frac{b}{2}} \Gamma \left( \frac{1}{b} \right)} z - 2\frac{\Gamma \left( \frac{3}{b} \right)}{a^{\frac{b}{2}} \Gamma \left( \frac{2}{b} \right) \Gamma \left( \frac{3}{b} \right)} - \Gamma^3 \left( \frac{2}{b} \right) z^2 \right) \right],
\]
with the following lower bound

$$R_L = 2 \frac{\Gamma(\frac{2}{b})}{a^2 \Gamma(\frac{1}{b})} - \frac{\Gamma^3(\frac{2}{b})}{a^2 \Gamma(\frac{2}{b}) \Gamma^2(\frac{1}{b})}.$$  

Using Stirling’s approximation $\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$, we have

$$|\varphi_n|^{\frac{1}{n}} = \left| \frac{b a^{n+1}}{\Gamma\left(\frac{n+1}{b}\right)} \right|^{\frac{1}{n}} = \left| b a^{n+1} \right|^{\frac{1}{n}} \left| \Gamma\left(\frac{n+1}{b}\right) \right|^{-\frac{1}{n}}$$

and again

$$\left| \Gamma\left(\frac{n+1}{b}\right) \right|^{\frac{1}{n}} \sim \left( \sqrt{2\pi \left(\frac{n+1}{b} - 1\right)} \left(\frac{n+1}{b} - 1\right) \left(\frac{n+1}{b} - 1\right) \right)^\frac{1}{n} \rightarrow \infty, \quad n \rightarrow \infty,$$

so that we obtain $R_U = \infty$.

Example 5.5. Consider $\varphi(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma^{(n)}(k+1)}$, for some fixed $n \in \mathbb{N}$, associated to the measure $K_{\varphi}(x) = e^{-x} \ln^n |x|, x > 0$. Hence, we obtain for the normalized coefficients

$$\tilde{\varphi}_1 = \frac{\Gamma^{(n)}(2)}{\Gamma^{(n)}(1)}, \quad \tilde{\varphi}_2 = \frac{\Gamma^{(n)}(3)}{\Gamma^{(n)}(1)},$$

and

$$E(z) = (1 - z) \Gamma^{(n)}(1) \varphi \left( \frac{\Gamma^{(n)}(1)}{\Gamma^{(n)}(2)} z + \frac{\Gamma^{(n)}(1)\Gamma^{(n)}(2) - \Gamma^{(n)}(3)(\Gamma^{(n)}(1))^2}{(\Gamma^{(n)}(2))^3} z^2 \right).$$

Hence, we obtain the lower bound for $|\Omega(z)|$ as

$$R_L = 2 \frac{\Gamma^{(n)}(1)}{\Gamma^{(n)}(2)} - \frac{\Gamma^{(n)}(3)(\Gamma^{(n)}(1))^2}{(\Gamma^{(n)}(2))^3}.$$

In order to find the upper limit we need to consider the Digamma function $\phi$ like in Example 4.2

$$R_U = \lim_{k \to \infty} \sup_k \frac{1}{\Gamma^{(n)}(k+1)}.$$

For $n = 1$ we have

$$|\Gamma'(k+1)|^{\frac{1}{k}} = |k!(-\gamma + H_k)|^{\frac{1}{k}} = (k!)^{\frac{1}{k}}|(-\gamma + H_k)|^{\frac{1}{k}} \rightarrow \infty$$

Since the dominating term satisfies $\lim_{k \to \infty} (k!)^{\frac{1}{k}} \rightarrow \infty$, we get $R_U = \infty$ when $n = 1$. 
For a general \( n > 1 \), using the fact that
\[
\Gamma^{(n)}(x) = \int_0^\infty t^{x-1} e^{-t} \ln^n(t) dt
\]
we have
\[
RU = \limsup_{k \to \infty} \frac{1}{|\Gamma^{(n)}(k)|^{\frac{1}{k}}} = \limsup_{k \to \infty} \left( \frac{1}{\int_0^\infty t^{k-\frac{1}{k} \ln^n(t) \frac{1}{k} dt} \right)^{-1}
\]
\[
\geq \limsup_{k \to \infty} \left( \int_0^\infty |t^{k-\frac{1}{k} \ln^n(t) \frac{1}{k} dt} \right)^{-1}
\]
\[
\to \frac{1}{\int_0^\infty t dt} \to \infty.
\]
Therefore \( RU = \infty \), for all \( n \geq 1 \).

**Remark 5.1.** For every function \( \varphi \) we can, with some appropriate restrictions, associate a counterpart of the Weierstrass factors
\[
\prod_{m,n} (1 - \frac{z}{\lambda_{mn}}) \varphi \left( \psi_1 \frac{z}{\lambda_{mn}} + \psi_2 \frac{z^2}{\lambda_{mn}} \right).
\]
This allows us to find classes of entire functions with preassigned zeros associated to \( \varphi \).

Now, using our fractional Weierstrass factors we can obtain the following estimate for the function \( g \).

**Lemma 5.1.** For the function \( g = g(\cdot; \Gamma) \) we have the inequality
\[
|K_\varphi (-|z|^2) g(z)| \geq cd(z, \Lambda)
\]
with a positive constant \( c \).

**Proof.** Using \( |1 - E(z)| \leq |z|^3 |\Omega(z)| \) from (27) and the triangle inequality we have
\[
|1| - |E(z)| \leq |1 - E(z)| \leq |z|^3 |\Omega(z)| \implies |E(z)| \geq |z|^3 |\Omega(z)| - 1.
\]
Then for a constant \( c > 0 \) we have
\[
|K_\varphi (-|z|^2) g(z)| = |K_\varphi (-|z|^2) (z - z_0) \prod_{m,n} E_2(z)|
\]
\[
= |K_\varphi (-|z|^2)| |(z - z_0)| \prod_{m,n} |E_2(z)|
\]
\[
\geq |K_\varphi (-|z|^2)| \prod_{m,n} (|z|^3 |\Omega(z)| - 1) |(z - z_0)|
\]
\[
= cd(z, \Lambda).
\]
\( \square \)
In addition we can also give a bound for \( g(z) \) who by (29) is independent of \( z \). Therefore, from (27) and (29) in (28) combined with the triangle inequality we obtain the following independent bound of \( g(z) \):

\[
|g(z)| = |(z - z_{00})| \prod_{m,n} E_2(z),
\]

(32)

\[
\leq |(z - z_{00})| \prod_{m,n} |(1 + z^3 \Omega(z))| \leq |z - z_{00}| \Omega(z)^N z^{3N} + \beta_1(z) = |\Omega(z)^N z^{3N+1} + \beta_2(z)|,
\]

where \( N \) is a constant equal to the highest degree when performing the product \( \prod_{m,n} (z^3 \Omega(z) + 1) \). As we only consider the leading term when looking for the upper bound, we denote by \( \beta_1(z) \) and \( \beta_2(z) \) the expressions depending on \( z \) which contain all terms with lower degrees than the leading term.

We also need a lower estimate for our generalized \( \sigma \)–function:

**Lemma 5.2.** For the generalized Weierstrass \( \sigma \)–function we have the estimate

\[
K_\varphi(-|z|^2)\sigma(z) \geq Cd(z, \Lambda).
\]

with a constant \( C > 0 \).

**Lemma 5.3.** Let \( \Gamma \) be uniformly close to the square lattice of density \( \beta_\varphi \) and \( K_\varphi \) be non-zero and monotonic. Then, there exists constants \( c, c_1, \) and \( c_2 \) such that for all \( z \) we have

\[
c_1 \gamma(z) e^{-c|z|\log|z|} \cdot \text{dist}(z, \Gamma) \leq |K_\varphi(-|z|^2)g(z)| \leq c_2 \gamma(z) e^{c|z|\log|z|}
\]

where

\[
\gamma(z) = \begin{cases} 
1, & \text{growth (}K_\varphi\text{) < growth (}e^z\text{)} \\
|K_\varphi(z)|, & \text{otherwise}
\end{cases}
\]

and for every \( z_{mn} \in \Gamma \) we have

\[
|K_\varphi(-|z_{mn}|^2)g'(z_{mn})| \geq c_1 e^{-c|z_{mn}|\log|z_{mn}|}.
\]

**Proof.** In a similar spirit to the proof in [20] [Page 3, Lemma 2.2] , we write

\[
g(z) = \sigma(z) d(z, \Lambda) h(z) \quad \Rightarrow \quad h(z) = \frac{g(z)}{\sigma(z)} \frac{d(z, \Lambda)}{d(z, \Gamma)}.
\]

We estimate \( h(z) \) by factorizing and estimating each of the components of

\[
h(z) = h_1(z) h_2(z),
\]

where

\[
h_1(z) = \frac{d(z, \Lambda) z - z_{00}}{d(z, \Gamma)} \prod_{m,n} \left( \frac{1 - \frac{z}{\lambda_{m,n}}}{1 - \frac{z}{\lambda_{m,n}}} \right).
\]

(35)

and

\[
h_2(z) = \prod_{m,n} \varphi \left( \psi_1 \frac{z}{\lambda_{m,n}} + \psi_2 \frac{z^2}{\lambda_{m,n}} \right) = \prod_{|z_{mn}| > 2|z|} \ldots \prod_{|z_{mn}| \leq 2|z|} \ldots.
\]

(36)
We estimate $h(z)$ for a non-zero monotonic $\varphi > 0$, for $|z_{m,n}| \leq 2|z|$ by finding $a$, $b$ such that
\[
\varphi \left( \psi_1 \frac{z}{z_{m,n}} + \psi_2 \frac{z^2}{z_{m,n}} \right) \leq a, \quad \varphi \left( \psi_1 \frac{z}{z_{m,n}} + \psi_2 \frac{z^2}{z_{m,n}} \right) \geq b.
\]

Since $\varphi$ is non-zero and monotonic, it would be sufficient to find the value of $z$ for which $\varphi(z)$ is minimum and maximum, respectively, by solving the quadratic equation for $z$. We get $z_{\min} = \frac{\lambda_{m,n}}{\lambda_{m,n}} = -1$, and $z_{\max} = -\frac{\lambda_{m,n}}{z_{m,n}}$. So
\[
\prod_{|z_{m,n}| \leq 2|z|} \frac{\varphi \left( \psi_1 \frac{z}{z_{m,n}} + \psi_2 \frac{z^2}{z_{m,n}} \right)}{\varphi \left( \psi_1 \frac{z}{z_{m,n}} + \psi_2 \frac{z^2}{z_{m,n}} \right)} \geq \prod_{|z_{m,n}| \leq 2|z|} \frac{\varphi (z_{\min})}{\varphi (z_{\max})}.
\]

For the case $|z_{m,n}| > 2|z|$ we use the same idea since $K_{\varphi}$ is non-zero and monotonic to estimate that
\[
\prod_{|z_{m,n}| > 2|z|} \frac{\varphi \left( \frac{z}{z_{m,n}} \right)}{\varphi \left( \frac{z}{z_{m,n}} \right)} \geq \prod_{|z_{m,n}| > 2|z|} \frac{\varphi (z'_{\min})}{\varphi (z'_{\max})},
\]
where $z'_{\max}$ and $z'_{\min}$ are the $z \in \{z_{m,n} : z_{m,n} > |z|\}$ that give the maximum and minimum of $\varphi(z)$, respectively.

Therefore, we have the bound for $|h_2(z)| \geq C \varphi(...).$ The estimate of $h_1(z)$ is given in [20] [Page 4, the term with $|h_2(z)|$], and since it is independent of the choice of $\varphi$ we obtain
\[
|h_2(z)| \geq C e^{-c|z|\log |z|}.
\]

We note that, that if the growth of $\varphi$ is slower than the growth of the exponential, we simply use the estimate of $h_1(z)$, which explains the function $\gamma(z)$ from [21] in the inequality (33).

From this we get now
\[
|K_{\varphi}(-|z|^2)g(z)| = \left| K_{\varphi}(-|z|^2) \frac{\sigma(z)}{d(z,\Gamma)} d(z,\Gamma) h(z) \right| \geq |K_{\varphi}(-|z|^2) \frac{\sigma(z)}{d(z,\Gamma)} d(z,\Gamma) | C \gamma(z) e^{-c|z|\log |z|}.
\]

With the estimate $|K_{\varphi}(-|z|^2)\sigma(z)| \geq \tilde{C} d(z,\Gamma)$ we obtain the result. This completes the proof. \(\square\)

**Example 5.6.** Consider the trivial example with $K_{\varphi}(z) = e^z$ in which we are back in the case of [19] and have the inequality
\[
c_1 e^{-c|z|\log |z|}, \text{ dist}(z,\Gamma) \leq |e^{-|z|^2} g(z)| \leq c_2 e^{|z|\log |z|},
\]
in which $c_1, c_2, c_3$ depend on $q(\Gamma)$ and $Q$ as previously defined.

Using Lemma 5.3, we get the following Lagrange-type interpolation formula:

**Lemma 5.4.** Let $\Gamma = \{z_{m,n}\}$ be a uniformly close to the square lattice $\Lambda$, of density $\beta_\varphi$, and let $g$ be our fractional $\sigma$-function associated to $\Gamma$. If $\alpha < \beta_\varphi$, then we have
\[
f(z) = \sum_{m,n} \frac{f(z_{m,n})}{g'(z_{m,n})} \frac{g(z)}{z - z_{m,n}}, \quad z_{m,n} \in \Gamma,
\]
with uniform convergence on compact sets for $f \in \mathcal{F}^\infty$.

The proof follows the same lines as the proof of Lemma 3.1 in [20].
Proof. (for Theorem 4.1 and Theorems 4.2)

Beurling ([4], p.356) showed that for a square lattice \( \Lambda \) of density \( \frac{\alpha}{\pi} \), it is possible to find a subset \( R \subseteq \Gamma \) uniformly closed to the lattice with \( d < \frac{\alpha}{\pi} \leq d_1 \) which implies that one can replace \( \Gamma \) with \( R \) and always assume \( \Gamma \) to be uniformly close to the square lattice of density \( \beta_\varphi = D^- (\Gamma) \). The same holds in our case and, therefore, we can assume \( \Gamma \) to be uniformly close to the square lattice of density \( \beta_\varphi = D^- (\Gamma) \).

Consider now the translation operator \( T_a \)

\[
(T_a f)(z) := \sqrt{\frac{K_\varphi(-|z-a|^2)}{K_\varphi(-|z|^2)}} f(z-a). 
\]  

We note that this translation acts isometrically in \( F_\varphi \), since taking \( w = z - a \) we have

\[
\int_{\mathbb{C}} |T_a f(z)|^2 d\mu(z) = \int_{\mathbb{C}} \left| \sqrt{\frac{K_\varphi(-|z-a|^2)}{K_\varphi(-|z|^2)}} f(z-a) \right|^2 dK_\varphi(-|z|^2) dxdy,
\]

\[
= \int_{\mathbb{C}} \left| \frac{K_\varphi(-|w|^2)}{K_\varphi(-|w+a|^2)} \right|^2 |f(w)|^2 dK_\varphi(-|w+a|^2) dxdy,
\]

\[
= \int_{\mathbb{C}} |K_\varphi(-|w|^2)|^2 |f(w)|^2 dxdy,
\]

\[
= \int_{\mathbb{C}} |f(w)|^2 d\mu(w).
\]

That these translation operators are isometrically invariant implies that \( \Gamma + z \) is a sampling set if and only if \( \Gamma \) is a sampling set. The same holds true for interpolation.

In particular, we get

\[
\int_{\mathbb{R}^2} |K_\varphi(-|w|^2)| |f(w)|^2 dxdy = \sum_{k,l} \int_{\mathbb{R}^2} |K_\varphi(-|w|^2)| |T_{\lambda_{k,l}} f(w)|^2 dxdy
\]

with \( R = \{ z = x + iy : |x| < \frac{1}{2} \sqrt{1/\alpha}, |y| < \frac{1}{2} \sqrt{1/\alpha} \} \) and \( \alpha < \beta_\varphi \).

In order to estimate the terms \( |T_{\lambda_{k,l}} f(w)|^2 \) on the right-hand side we use Lemma 5.4 with \( a = \lambda_{k,l} \).

This gives us

\[
(T_a f)(w) = \sum_{m,n} \frac{(T_a f)(z_{m,n} + a)}{g'_a(z_{m,n})} \frac{g_a(w)}{w - z_{m,n} - a}.
\]

where the function \( g_a(z) = g(z; \Gamma + a) \) is related to the lattice \( \Gamma + a \) instead of the function \( g \) where we sum over the lattice \( \Gamma \) (c.f. (28)).

Cauchy-Schwarz inequality implies

\[
|T_{\lambda_{k,l}} f(w)|^2 \leq \left( \sum_{m,n} \frac{(T_a f)(z_{m,n} + a)^2 |K_\varphi(-|z_{m,n}|^2)|}{K_\varphi(-|z_{m,n}|^2)|g'_a(z_{m,n})|^2} \right) \left( \sum_{m,n} \frac{|g_a(w)|^2}{|w - z_{m,n} - a|^2} \right). 
\]
Using Lemma 5.3 we obtain
\[ |T_{\lambda_{k,l}}f(w)|^2 \leq C_1 \left( \sum_{m,n} (|T_a f)(z_{m,n} + a)|^2 K_\varphi(-|z_{m,n}|^2) e^{-c|z_{m,n}|\ln|z_{m,n}|} \right) \left( \sum_{m,n} \frac{|g_a(w)|^2}{|w - z_{m,n} - a|^2} \right) \]
which leads to
\[ \iint_{\mathbb{R}} |T_{\lambda_{k,l}}f(w)|^2 K_\varphi(-|w|^2) |dx dy| \]
\[ \leq C_1 \left( \sum_{m,n} (|T_a f)(z_{m,n} + a)|^2 K_\varphi(-|z_{m,n}|^2) e^{-c|z_{m,n}|\ln|z_{m,n}|} \right) \left( \sum_{m,n} \int_{\mathbb{R}} \frac{|g_a(w)|^2}{|w - z_{m,n} - a|^2} K_\varphi(-|w|^2) |dx dy| \right) \]
Collecting everything together we get
\[ \iint_{\mathbb{C}} |f(z)|^2 K_\varphi(-|z|^2) |dx dy| \leq C_2 \sum_{m,n} |f(z_{m,n})|^2 K_\varphi(-|z_{m,n}|^2) < \infty, \]
to prove Theorem 4.2 in which it is sufficient to verify the left hand side of the inequality in Equation (21).

One could prove Theorem 4.3 in a similar manner.

To prove Theorem 4.1 we write \( f \) as:
\[ f(z) = \sum_{m,n} a_{mn} \sqrt{K_\varphi\left(-|z - a|^2\right) g_{-z_{mn}}(z - z_{mn})}, \]
Then, the interpolation problem is solved directly from the above formula using the translation operator (38) and Lemma (5.3) and this completes the proofs.

6. Applications

Gabor Frames first developed by D. Gabor in 1946 in the area of information theory [8], and J. Von Neumann in quantum mechanics at the same time. Gabor Frames are used to establish frames for function systems depending on continuous parameters in modern signal and image processing. For Gabor systems, Gröchenig and Lyubarskii established a construction method with Hermite functions as window functions, which allows to connect the Gabor system with an orthonormal system in the Fock space via the Bargmann transform. This allows to reduce the problem of lattice constants for the frame parameters to the one of proving the uniqueness of sets of entire functions in the Fock space.

An application would be to perform a similar investigation of Gabor frames and sufficient conditions to form a frame in \( L^2(\mathbb{R}) \). In a similar spirit to the one done by K. Gröchenig and Y. Lyubarskii (see [10, 11]) based on Hermite functions \( h_n \), but in our framework of generalized differentiation through the generalized Bargmann transform and generalized Weierstrass-\( \sigma \) function defined in Sections 4 and 5, respectively.

Let us recall the necessary results in the following theorem.

**Theorem 6.1.** For the modified Bargmann transform \( \tilde{B} : L^2(\mathbb{R}) \to \mathcal{F}_\varphi \) we have:

(i) \( \tilde{B} \) is a unitary mapping.
(ii) $\mathcal{F}_\varphi$ is a RKHS with reproducing kernel $k(z, w) = \varphi(z\bar{w})$ and $F(z) = \langle F, k(z, \cdot) \rangle_{\mathcal{F}_\varphi}$.

(iii) Let $h_n(t) = e^{\pi t^2} \frac{d^n}{dt^n}(e^{-2\pi t^2})$ denote the $n$–th Hermite function, then $\hat{\varphi}_n(z) = \pi^{\frac{n}{2}}(\varphi_n)^{-\frac{1}{2}}z^n$.

We remark that (i) and (iv) are obtained in Section 4.1 (ii) is obtained in Section 3.1 in particular, Equation (18).

Based on this theorem we can now consider the construction of frames for integral transforms of the type

$$Vf(z) = \langle k(z, \cdot), f \rangle_{L^2(\mathbb{R})}$$

with $f \in L^2(\mathbb{R})$ and kernels of the form

$$k(z, t) = \sum_{j=0}^{\infty} c_j (\overline{z}a^* + za)^j h_0(t)$$

depending on a complex parameter $z$, with coefficients $c_j$ such that the series converges in $L_2(\mathbb{R})$ with respect to $x$ and converges uniformly on compact sets with respect to $z$. Let us remark that a special case of such transforms is the Gabor transform:

$$V_{h_n} f(z) = \langle \pi z h_n, f \rangle_{L^2(\mathbb{R})} = \frac{e^{i\pi xy}}{\sqrt{\pi^n \varphi_n}} \frac{1}{\varphi(\sqrt{2})} \sum_{k=0}^{n} \binom{n}{k} (-\pi z)^k D^k \varphi(z)$$

with $z = x + iy$, which we get in the case where the operators $a$ and $a^*$ generate the Heisenberg algebra, that is $[a, a^*] = I$.

Using our generalized Bargmann transform $\hat{B} : L^2(\mathbb{R}) \to \mathcal{F}_\varphi$ as $F = \hat{B} f$ (c.f. Section 4) we have

$$V f(z) = \langle k(z, \cdot), f \rangle_{L^2(\mathbb{R})} = \langle \hat{B}k(z, \cdot), F \rangle_{\mathcal{F}_\varphi} = \langle \sum_{j=0}^{\infty} c_j (\overline{z}M_w + zD_\varphi)^j 1, F \rangle_{\mathcal{F}_\varphi} = \langle \varphi(\overline{z}w), \frac{1}{\varphi(\overline{z}w)} \sum_{j=0}^{\infty} c_j (\overline{z}M_w + zD_\varphi)^j 1 \rangle_{\mathcal{F}_\varphi} F(z) = \frac{1}{\varphi(|z|^2)} \sum_{j=0}^{\infty} c_j (\overline{z}M_w + zD_\varphi)^j 1)(z) F(z)$$

For an invertible matrix $C$ and a lattice $\Lambda = CZ^2$, we define $\mathcal{G}(k, \Lambda) = \{k(z, \cdot) : z \in \Lambda\}$. $\mathcal{G}(k, \Lambda)$ is a frame if there exists constants $A, B$ with $0 < A \leq B < \infty$ such that

$$A\|f\|_{L^2(\mathbb{R})}^2 \leq \sum_{z \in \Lambda} |\langle f, k(z, \cdot) \rangle_{L^2(\mathbb{R})}|^2 \leq B\|f\|_{L^2(\mathbb{R})}^2, \quad \forall f \in L^2(\mathbb{R}).$$

Here, we are going to adapt the classic scheme by Gröchenig and Lyubarskii [10] [11] to our setting. Of course, we recover the original case if we choose $\varphi(z) = e^z$ and $k(z, t) = \pi \lambda h_n(t) = e^{2\pi i yt}h_n(t-x)$.

The following results will provide sufficient conditions for $\mathcal{G}(k, \Lambda)$ to form a frame and describe its structure. Since in the previous sections we created the same machinery as in the case of Gabor frames
with Hermite windows [10] we can transfer the proofs to our case directly, for Gabor frame that \( \mathcal{G}(k, \Lambda) \) is a Bessel sequence.

**Theorem 6.2.** If \( \mathcal{G}(k, \Lambda) \) is a frame for \( L^2(\mathbb{R}) \) then the size of \( \Lambda \) satisfies \( s(\Lambda) = |\det(C)| \leq 1 \), where \( s(\cdot) \) is the size of \( \Lambda \) obtained from the sympletic area of a cell (see also [16]).

The density of \( \Lambda \) is denoted by \( d(\Lambda) = s(\Lambda)^{-1} \) and equals to the Beurling density. The adjoint lattice is denoted by \( \Lambda^o = s(\Lambda)^{-1} \Lambda \) with size \( s(\Lambda^o) = s(\Lambda)^{-1} \).

**Theorem 6.3.** Consider a lattice \( \Lambda \subset \mathbb{R}^2 \) with adjoint \( \Lambda^0 \). The following statements are equivalent

(i) \( \mathcal{G}(k, \Lambda) \) is a frame for \( L^2(\mathbb{R}) \).

(ii) There exists a function \( \gamma \in L^2(\mathbb{R}) \) such that \( \mathcal{G}(\gamma, \Lambda) \) is a Bessel sequence in \( L^2(\mathbb{R}) \) and the inner product for all \( \mu \in \Lambda^0 \) satisfies \( \langle k(\mu, \cdot), \gamma \rangle_{L^2(\mathbb{R})} = \delta_{\mu,0} \) for all \( \mu \in \Lambda^0 \).

The following theorem gives us a sufficient condition to form a frame in \( L^2(\mathbb{R}) \).

**Theorem 6.4.** Let \( n \in \mathbb{Z}, n \geq 0 \). If the size of \( \Lambda \) satisfies \( s(\Lambda) < (n+1)^{-1} \) then \( \mathcal{G}(k, \Lambda) \) is a frame for \( L^2(\mathbb{R}) \).

The following is an example of these results with the Dunkl operator from Subsection 2.2. It is based on the Fock space representation of the Dunkl-Gabor transform (see, e.g. [15]). We remark that this representation is easy to obtain from [7] [10], since in this case the group generated by the multiplication operator and the Dunkl operator is still the Heisenberg group.

**Example 6.1.** From Subsection 2.2 we have that the Dunkl operator in the rank one case can be written as

\[
T_1 f(x) = \frac{\partial f}{\partial x}(x) + \kappa \frac{f(x) - f(-x)}{x},
\]

associated to

\[
\varphi(z) := e^z L_1(k, 2k + 1; -2z),
\]

with

\[
\varphi_{2n} = \frac{(\frac{1}{2})_n}{(2n)! (\kappa + \frac{1}{2})_n} \quad \text{and} \quad \varphi_{2n+1} = \frac{(\frac{1}{2})_{n+1}}{(2n+1)! (\kappa + \frac{1}{2})_{n+1}},
\]

and where \((a)_0 = 1 \) and \((a)_n = a(a+1) \cdots (a+n-1)\), for \( a \in \mathbb{R} \setminus \mathbb{Z}^- \).

\[
\varphi \left( |z|^2 \right) = \sum_{j=0}^\infty \varphi_j |z|^{2j} = \sum_{k=0}^\infty \left[ \varphi_{2k} |z|^{4k} + \varphi_{2k+1} |z|^{4k+2} \right]
\]

\[
= \sum_{k=0}^\infty \left[ \frac{|z|^{4k}}{(2k+1)(2k)!} + \frac{|z|^{4k+2}}{(4k+6)(2k+1)!} \right].
\]

(40)

This leads to the following representation of the Dunkl-Gabor transform

\[
V_{h_n} f(z) = \langle f, \pi_z h_n \rangle = \frac{e^{i \pi x y}}{\sqrt{\varphi_n}} \frac{1}{\varphi \left( \frac{|z|^2}{2} \right)} \sum_{k=0}^n \binom{n}{k} (-\pi \bar{z})^k D^k \varphi F(z)
\]

with \( z = x + iy \).
By Theorem 6.4, choosing \( z \) from a lattice \( \Lambda \) of size less than \((n+1)^{-1}\) will give us a frame.

Another example is given in the following considerations.

**Example 6.2.** Let \( A \) be an operator acting on the Fock space \( \mathcal{F} \). Then, this action is given by

\[
AF(z) := \langle \varphi(\bar{z}), AF \rangle_{\mathcal{F}, \varphi} = \langle A_w^* \varphi(\bar{z}), F \rangle_{\mathcal{F}, \varphi} = \langle B^{-1}(A_w^* \varphi(\bar{z})), f \rangle_{L^2(\mathbb{R})} = \langle k(z; A, f) \rangle_{L^2(\mathbb{R})},
\]

for all \( F = Bf \), where \( f \in L^2(\mathbb{R}) \), and \( k(t; z, A) := B^{-1}(A_w^* \varphi(\bar{z}))(t) \).

Consider now \( \Lambda \) of the form

\[
AF(w) := \sum_{k=0}^{n} \binom{n}{k} (-\pi)^k D^k \varphi F(w).
\]

We obtain for its adjoint

\[
A^*G(w) = \sum_{k=0}^{n} \binom{n}{k} (-\pi D\varphi)^k [w^k G(w)] = \sum_{k=0}^{n} \binom{n}{k} (-\pi)^k D^k \varphi [w^k G(w)].
\]

When \( G(w) = \varphi(\bar{z}w) \) we get

\[
D^k \varphi \left( w^k \varphi(\bar{z}w) \right) = D^k \varphi \left( \sum_{j=0}^{\infty} \varphi_j \bar{z}^j w^{j+k} \right) = \sum_{j=0}^{\infty} \varphi_j \bar{z}^j D^k \varphi (w^{j+k}).
\]

Computing these derivatives, we obtain

\[
D^k \left( w^{j+k} \right) = \frac{\varphi_{j+k-1}}{\varphi_{j+k}} D^{k-1} \left( w^{j+k-1} \right) = \frac{\varphi_{j+k-2}}{\varphi_{j+k}} D^{k-2} \left( w^{j+k-2} \right) = \ldots = \frac{\varphi_j}{\varphi_{j+k}} w^j.
\]

Hence, we get

\[
A^* \varphi(\bar{z}w) = \sum_{k=0}^{n} \binom{n}{k} (-\pi)^k \left( \sum_{j=0}^{\infty} \frac{\varphi_j^2}{\varphi_{j+k}} (\bar{z}w)^j \right).
\]

This means that we can consider the transform

\[
V_{h_n} f(z) := \langle k(z; A, f) \rangle_{L^2(\mathbb{R})}
\]

with kernel \( k(t; z, A) = \sum_{k=0}^{n} \binom{n}{k} (-\pi)^k \left( \sum_{j=0}^{\infty} \frac{\varphi_j^2}{\varphi_{j+k}} h_j(t) \right) \).

By Theorem 6.4, we have that a lattice \( \Lambda \) gives us a frame if \( s(\Lambda) < (n+1)^{-1} \). By Theorem 6.5, we have now that there exists a \( \gamma \) such that \( G(\gamma, \Lambda) \) is a Bessel sequence in \( L^2(\mathbb{R}) \) and the inner product satisfies \( \langle k(\cdot; \mu, A), \gamma \rangle_{L^2(\mathbb{R})} = \delta_{\mu,0} \) for all \( \mu \) in the adjoint lattice.

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