Green function for the Poisson equation in a general case of astrophysical interest.

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Abstract

In present paper we suggest exact solution of the Poisson problem which appears in frequently addressed applications regarding calculation of the gravitational potential of spiral galaxies. We suggest an analytical solution for the problem in cylindrical coordinates by using the finite integral transform technique. The final solution is presented as expansion on the eigenfunctions of the corresponding Sturm-Liouville problem. Green function of the problem is constructed.

Keywords: Integral transforms; Poisson Equation; Gravitational Potential; Spiral Galaxies.

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1 Introduction

In a wide variety of astrophysical applications there are a great number of extremely important problems which can be reduced to the Poisson’s problem in cylindrical coordinates. These problems appear in the modeling of the galaxies, accretion discs, when we are interested in the reconstruction of the gravitational potential of the object under consideration by using an observational data. As an example the method of determining of the spiral galaxies thickness by solution of the Poisson equation should be mentioned [1,2]. Unfortunately all these solutions deal with very particular cases of the density distribution function and for this reason have some restrictions in applying these models to real galaxies. The mentioned model of Peng [1,2], for example, is based on the Parenago’s density distribution along the z-direction \( \rho_h(z) \sim \exp(-\alpha|z|) \) (they suppose that the density distribution function can be factorized, i.e. \( \rho(r, \phi, z) = \rho_\sigma(r, \phi) \rho_h(z) \)). But this function can not be considered as satisfactory choice because 1) it does not have a defined derivative on the galaxy plane \( z = 0 \) and 2) it was obtained from the barometric formula which suppose (in thermalized
case) the presence of a constant, solid gravitational mass for \( z < 0 \). As one can see it is not quite right in the case of a galaxy, where the gravitational potential in a particular point is defined by the density distribution which in turn should not be written from simple barometric relation. For this reasons the model mentioned above needs to be generalized.

Another important application which leads to the problem under consideration is the calculation of the gravitational potential and field of velocities of stars in the presence of so-called dark matter (DM). See for example [3] and references therein. By taking into account a great impact that the phenomena of DM produces to the modern theoretical physics, the need for the accurate analytical solution of the Poisson’s equation for gravitational potential of thin oblate ellipsoid, characterized by more sophisticated density distribution function becomes clear.

Up to now the Poisson’s problem solutions were suggested for rather small class of the density distribution functions. The purpose of present paper is to fill this gap and suggest the Green function for the Poisson’s problem for the most general case of an arbitrary density distribution function.

On the one hand if we are interested in the DM distribution, the final solution is approximately no depends on the spiral structure of the galaxy, so the angular variable can be omitted in first approximation as insignificant [3] and on the other hand 2D solution is important because it appears in many applications and allow simple usage, analysis and interpretation. By taking into account all mentioned above, we consider 2D problem because of its great importance.

In present paper we solve the 2D Poisson’s problem in cylindric coordinates by using the finite integral transform technique (FITT) [4, 5, 6, 7, 8, 9] first time developed by Grinberg [4] (in this paper we will refer this method as Grinberg’s method). The final solution is presented as expansion on the eigenfunctions of the corresponding Sturm-Liouville problem. Green function of the problem is constructed.

## 2 Gravitational Potential of Spiral Galaxies

Consider a symmetrical, flat, spinning, disc-like oblate object characterized by the density distribution \( \rho(r, \phi, z) \). In this case the equation of Poisson \( \nabla^2 u = \kappa \rho \) in cylindrical coordinates for gravitational potential \( u \) is:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = \kappa \rho(r, \phi, z). \tag{1}
\]

As it was mentioned above, the spiral structure of the galaxy is approximately insignificant in the problems mentioned above (variation of the stars velocity due to the presence of the spiral structure is more than order of magnitude smaller if compared with its unperturbed value [3]) and for this reason here we can consider only two variables \( r \) and \( z \), by setting \( \phi = \text{conts} \). In this
case the equation of Poisson becomes:

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} = \kappa \rho(r, z) , \]  

(2)

where \( \rho(r, z) \) is the surface density of the matter.

This equation also should be supplied by following boundary conditions:

\[ \frac{\partial u}{\partial r} \bigg|_{r=0} = 0 , \quad \frac{\partial u}{\partial r} \bigg|_{r=a} = -\frac{V_0^2}{a} g(z) , \]  

(3)

\[ \frac{\partial u}{\partial z} \bigg|_{z=0} = 0 , \quad u \bigg|_{z=\infty} < \infty , \]  

(4)

where \( V_0 \) is the experimentally measured velocity of stars located at distance \( a \) from center of galaxy. To resolve the problem (2),(3),(4), the method of Grinberg (FIT method) \[4,5,6,7,8,9\] is applied. In consequence with the technic we should firstly define and resolve the Sturm-Liouville problem (SLP) to obtain the complete set of eigenfunctions in which the final solution can be expanded. The corresponding homogeneous equation reads:

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = -\frac{\partial^2 u}{\partial z^2} = -\lambda . \]  

(5)

Let \( u(r, z) = R(r)Z(z) \), in this case we obtain corresponding SLP for \( R(r) \) which actually is the Bessel equation:

\[ (rR')' + \lambda rR = 0 , \]  

(6)

completed by the homogeneous boundary conditions:

\[ R' \bigg|_{r=0} = 0 ; \quad R' \bigg|_{r=a} = 0 . \]  

(7)

We note here that the SLP based on the \( Z(z) \) is not regular one and for this reason the problem (6),(7) should be favored. As one can see the equation (6) and boundary conditions (7) do form the SLP. For zeroth eigenvalue it has solution:

\[ \lambda_0 = 0 ; \quad R_0 = 1 . \]  

(8)

In the case when the eigenvalue \( \lambda \neq 0 \), we obtain the equation of Bessel, which has the radial solution for this problem expressed with functions of Bessel:

\[ \lambda_n = \left( \frac{\gamma_n}{a} \right)^2 ; \quad R_n = J_0(\sqrt{\lambda_n}r) , \]  

(9)

where the roots \( \gamma_n = (\sqrt{\lambda_n}a) \) satisfy the following transcendent equation:

\[ J_1(\gamma_n) = 0 . \]  

(10)
For reader’s convenience we also write here the absolute values of the eigenfunctions (we will use them to construct the final solution of the problem):

$$\|R_n\|^2 = \int_0^a R_n R_n r dr . \quad (11)$$

For zeroth eigenvalue $n = 0$ we have

$$\|R_0\|^2 = \frac{a^2}{2} , \quad (12)$$

and for $n \neq 0$ this value is

$$\|R_n\|^2 = \frac{a^2}{2} J_0^2(\gamma_n). \quad (13)$$

It is well known that the eigenfunctions of problem of Sturm-Liouville (6),(7) form a complete set of functions in Hilbert space. For this reason if the function $u(r, z)$ comply the Dirichlet condition within interval $[0, a]$, it can be expanded into the series of Dini:

$$u(r, z) = \sum_{n=0}^{\infty} C_n(z) R_n(r) = C_0(z) + \sum_{n=0}^{\infty} C_n(z) J_0(\sqrt{\lambda_n} r), \quad (14)$$

where

$$C_n(z) = \frac{\bar{u}_n(z)}{\|R_n\|^2}, \quad (15)$$

and the transformed potential function $\bar{u}_n(z)$ is given by integral

$$\bar{u}_n(z) = \int_0^a u(r, z) R_n(r) r dr. \quad (16)$$

To find the transformed potential function $\bar{u}_n(z)$, in consequence with the Grinberg’s method, we should transform initial equation (2) and the boundary conditions (4). Transformed equations can be easily obtained by using the boundary conditions (3) and (7). They can be written as follows:

$$\frac{d^2 \bar{u}_n(z)}{dz^2} - \lambda_n \bar{u}_n(z) = F_n(z) , \quad (17)$$

where

$$F_n(z) = \kappa \rho_n(z) + V_0^2 J_0(\gamma_n) g(z) , \quad (18)$$

and the transformed surface density $\rho_n(z)$ is determined by relation

$$\rho_n(z) = \int_0^a \rho(r, z) R_n(r) r dr . \quad (19)$$

Transformed boundary conditions are:
\[ \bar{u}_n |_{z=\infty} < \infty; \quad \frac{\partial \bar{u}_n}{\partial z} |_{z=0} = 0. \quad (20) \]

The equations (17) with the boundary conditions (20) form the problem to obtain the transformed potential function \( \bar{u}_n(z) \).

Now we should consider two particular cases. The first one corresponds to the zeroth eigenvalue, when \( n = 0, \lambda_0 = 0 \) and \( R_0 = 1 \). In this case the equation (17) become:

\[ \bar{u}_0''(z) = F_0(z). \quad (21) \]

By taking into account the symmetry of the problem in respect to the \( z \) variable, one can write the solution of the equation (21) which satisfy the boundary conditions (20):

\[ \bar{u}_0(z) = 4 \int_0^z \int_0^{z'} F_0(z) d\zeta'' d\zeta' + B_0 = \]

\[ = 4 \pi \int_0^z \int_0^{z'} \rho(r, z'') r d\zeta'' d\zeta' + 4V_0^2 \int_0^z \int_0^{z'} g(z'') d\zeta'' d\zeta' + B_0. \quad (22) \]

Consider now the case when \( n \neq 0 \) i.e. \( \lambda_n \neq 0 \). Solution of the equation (17) can be written as

\[ \bar{u}_n(z) = C_1(z) e^{-\sqrt{\lambda_n}z} + C_2(z) e^{\sqrt{\lambda_n}z}, \quad (23) \]

where \( e^{\sqrt{\lambda_n}z} \) and \( e^{-\sqrt{\lambda_n}z} \) are general solutions of the corresponding homogeneous equations

\[ \frac{d^2 \bar{u}_n(z)}{dz^2} - \lambda_n \bar{u}_n(z) = 0, \quad (24) \]

and coefficients \( C_1(z) \) and \( C_2(z) \) are some functions to be determined. The functions \( C_1(z) \) and \( C_2(z) \) can be obtained immediately:

\[ C_1(z) = -\frac{1}{\sqrt{\lambda_n}} \int_0^z F_n(z') e^{\sqrt{\lambda_n}z'} d\zeta' \quad (25) \]

and

\[ C_2(z) = \frac{1}{\sqrt{\lambda_n}} \int_0^z F_n(z') e^{-\sqrt{\lambda_n}z'} d\zeta'. \quad (26) \]

So, the solution of the problem (17),(20) for the case \( \lambda_n \neq 0 \) i.e. \( n \neq 0 \) can be written as:

\[ \bar{u}_n(z) = \frac{1}{\sqrt{\lambda_n}} \left[ e^{\sqrt{\lambda_n}z} \int_0^z F_n(z') e^{-\sqrt{\lambda_n}z'} d\zeta' - e^{-\sqrt{\lambda_n}z} \int_0^z F_n(z') e^{\sqrt{\lambda_n}z'} d\zeta' \right], \quad (27) \]

where \( F_n(z') \) is given by (18), and
\[ \rho_n(z) = \int_0^a \rho(r,z)R_n(r)rdr. \]  

(28)

Now we are ready to construct the Green function and write the final expression for the gravitational potential \( u(r, z) \). The solution of the problem under consideration described by equation (2) and boundary conditions (3) and (4), can be written as follow:

\[
    u(r, z) = \bar{u}_0(r, z) \frac{R_0}{\|R_0\|_r^2} + \sum_{n=1}^{\infty} \bar{u}_n(r, z) \frac{R_n(r)}{\|R_n(r)\|_r^2}.
\]

(29)

By substituting (8,9,12,13,22,27) into (29) we obtain final solution of our problem:

\[
    u(r, z) = \frac{8}{a^2} \int_0^z \int_0^{z'} F_0(z'')d z'' dz' + \frac{2B_0}{a^2} + \frac{2}{a^2} \sum_{n=1}^{\infty} \bar{u}_n(z) J_0(\sqrt{\lambda_n} r) J_2(\gamma_n) J_0(\sqrt{\lambda_n} r) J_2(\gamma_n).
\]

(30)

where \( \bar{u}_n(r, z) \) is given by the relation (27).

Expression (30) describes the gravitational potential for a spiral galaxy in the case of an arbitrary distribution of the density function \( \rho(r, z) \).

3 Discussion

Let us apply the final result (30) to some important particular cases of the widely used density functions \( \rho(r, z) \).

Suppose that function \( \rho(r, z) \) can be factorized \( \rho(r, z) = \rho_r(r) \rho_h(z) \), and \( \rho_h(z) = \beta \exp(-\beta|z|) \) is the Parenago’s density distribution along the z-direction, where \( \beta = 1/z_0 \), and \( z_0 \) is the half depth of the disk.

In this case the integrations over \( z \) in (27) and (30) can be carried out analytically and we obtain:

\[
    u(r, z) = \frac{8}{a^2} \left\{ \frac{\beta \rho_{r_0}(a)}{\beta} \left[ e^{-\beta z} - 1 + \beta z \right] \right\} + \frac{2B_0}{a^2} + \frac{2}{a^2} \sum_{n=1}^{\infty} \bar{u}_n(z) \frac{J_0(\sqrt{\lambda_n} r) J_2(\gamma_n)}{J_0(\gamma_n)}. \]

(31)

where

\[
    \rho_{r_0}(a) = \int_0^a \rho_r(r')r'dr',
\]

(32)
\[ \rho_{rn}(a) = \int_0^a \rho_r(r') J_0(\gamma_n \frac{r'}{a}) r' dr'. \] (33)

Expressions (31), (32) and (33) suggest solution of the Poisson problem written in cylindrical coordinates for a particular case of the Parenago’s density distribution along the z-direction.

First integration over \( r' \) in (31) can be carried out analytically in some commonly used cases of the density distribution function \( \rho(r) \).

a) Plummer [10], [11] distribution function

\[ \rho(r) = \frac{3M}{4\pi b^2} \left(1 + \frac{r^2}{b^2}\right)^{-5/2}, \] (34)

where \( b \) is half-depth of the galaxy. Integrating (32) we obtain:

\[ \rho_{r0}(a) = \frac{M}{4\pi} \left[1 - \left(1 + \frac{a^2}{b^2}\right)^{-3/2}\right]. \] (35)

b) Plummer-Kuzmin model, also known as “Toomre’s model 1.” [11] is defined by the distribution function

\[ \rho(r) = \frac{Mb}{2\pi(r^2 + a^2)^{3/2}}, \] (36)

where \( a \) is truncation radius. In this case the integral (32) became

\[ \rho_{r0}(a) = \frac{Mb \sqrt{2} - 1}{2\pi a \sqrt{2}}. \] (37)

These relations suggest solutions of the problem for some important particular cases of density distribution.

4 Conclusions

In present paper we consider the Poisson problem in cylindrical coordinates, which often arises in calculation of the gravitational potential of spiral galaxies.

By the finite integral transform technique, an exact analytic solution of the problem is obtained for an arbitrary mass distribution function.

Green function of the problem is written out. As an example, solutions for widely used in astrophysics density functions, namely the functions of Parenago, Plummer and Plummer-Kuzmin, are suggested. The obtained expressions make it possible to avoid or reduce the cumbersome numerical calculations and allow a clear interpretation of the results.

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