On rings of differential operators derived from automorphic forms

Atsuhiro Nagano

July 20, 2018

Abstract

We study linear ordinary differential equations which are analytically parametrized on Hermitian symmetric spaces and invariant under the action of symplectic groups. They are generalizations of the classical Lamé equation. Our main result gives a closed relation between such differential equations and automorphic forms for symplectic groups. Our study is based on techniques concerning with the monodromy of complex differential equations, the Baker-Akhiezer functions and algebraic curves attached to rings of differential operators.

Introduction

The main purpose of this paper is to study linear ordinary differential operators of a complex independent variable which are analytically parametrized on Hermitian symmetric domains and invariant under the action of symplectic groups. Our main result gives a closed relation between commutative rings of such differential operators and automorphic forms for symplectic groups.

Let us start the introduction with a typical example of the differential equations we study: the Lamé differential equation

\[
\left(-\frac{\partial^2}{\partial z^2} + B\wp(\Omega, z)\right)u = Xu,
\]

where \(B, X \in \mathbb{C}\) and \(\wp(\Omega, z)\) is the Weierstrass \(\wp\)-function with the double periods 1 and \(\Omega \in \mathbb{H} = \{z \in \mathbb{C}|\text{Im}(z) > 0\}\). The Lamé differential equation has the regular singular points at every \(z_0 \in \mathbb{Z} + \mathbb{Z}\Omega\). If \(B = \rho(\rho + 1)\), the characteristic exponents at every singular point are \(\rho + 1\) and \(-\rho\). When \(\rho \in \mathbb{Z}_{>0}\), a system of basis of the space of solutions of (0.1) is generated by \(\Lambda(z)\) and \(\Lambda(-z)\). Here, \(\Lambda(z) = \prod_{j=1}^{\rho} \frac{\sigma(\Omega, z + \kappa_j)}{\sigma(\Omega, z)} e^{-z(\Omega, \kappa_j)}\), where \(\sigma(\Omega, z)\) and \(\zeta(\Omega, z)\) are the classical Weierstrass functions and \(\kappa_j (j = 1, \ldots, \rho)\) can be calculated by \(X\) (for detail, see [WW]). We remark that \(\Lambda\) is a single-valued function of \(z\). However, for generic \(\rho \in \mathbb{C}\), the solutions of (0.1) are multivalued on \(\mathbb{C} - (\mathbb{Z} + \mathbb{Z}\Omega)\). The Lamé equation is an important topic in mathematics. For example, the periodic solutions of (0.1) is studied in many body theoretical physics. Also, via the double covering \(E \to \mathbb{P}^1(\mathbb{C})\), where \(E\) is an elliptic curve with the double periods 1 and \(\Omega\), the equation (0.1) gives a Fuchsian differential equation with an accessory parameter. Moreover, special types of (0.1) promoted a development of the theory of integrable systems and finite zone problems. For example, see [WW], [DMN], [MM] and [T]. In these researches, to the best of the author’s knowledge, the double periodicity of the coefficient \(\wp(\Omega, z)\) of (0.1) played an essential role.

Keywords: Ordinary Differential Operators ; Automorphic Forms ; Algebraic Curves.
Mathematics Subject Classification 2010: Primary 16S32 ; Secondary 47E05, 32N10, 14G35, 14H70, 33E05.
Running head: Differential operators derived from automorphic forms

Note: This is the corrected version of the published article DOI: 10.1007/s11785-017-0663-7. Typos and misleading phrases are corrected here. Especially, for simplicity, criteria in Section 1.5 and 2.7 are corrected using arithmetic genera of algebraic curves.
For our purpose, we focus on another important property of the Lamé equation: the coefficient \( \varphi(\Omega, z) \) satisfies the transformation law
\[
\varphi\left(\frac{a\Omega + b}{c\Omega + d}, \frac{z}{c\Omega + d}\right) = (c\Omega + d)^2 \varphi(\Omega, z)
\]
(0.2)
for any \( \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2, \mathbb{Z}) \). Due to (0.2), the Lamé equation becomes to be invariant under the action of the elliptic modular group \( SL(2, \mathbb{Z}) \). Namely, via the transformation \((\Omega, z, X) \mapsto (\Omega_1, z_1, X_1) = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \left(\begin{array}{c} \Omega \\ z \end{array}\right) + \left(\begin{array}{c} -b \\ c \end{array}\right)\), the differential equation (0.1) can be identified with \((-\frac{\partial^2}{\partial z_1^2} + B\varphi(\Omega_1, z_1))u = X_1u\). By the way, holomorphic functions on \( \mathbb{H} \) which are invariant under the action of \( SL(2, \mathbb{Z}) \) are called elliptic modular forms. The invariance between two differential equations suggests a strong and non-trivial relation between the Lamé equation and elliptic modular forms. Furthermore, elliptic modular forms are quite important in number theory (see [Sm1]). The author expects that the Lamé equation may have some effective applications in number theory.

Based on the above observation and expectation, we study a class of ordinary differential equations \( Pu = Xu \) of a complex variable \( z \) (for detail, see Definition 2.2). Here, the differential operator
\[
P = \frac{\partial^N}{\partial z^N} + a_2(\Omega, z)\frac{\partial^{N-2}}{\partial z^{N-2}} + a_3(\Omega, z)\frac{\partial^{N-3}}{\partial z^{N-3}} + \cdots + a_N(\Omega, z)
\]
is parametrized by \( \Omega \) of a product \( \mathbb{H}^n \) of the Siegel upper half planes and invariant under the action of a congruence subgroup \( \Gamma \) of the symplectic group. Such a class contains the Lamé equation because the action of the group \( \Gamma \) on \( \mathbb{H}^n \) is a natural extension of the action of \( SL(2, \mathbb{Z}) \) on \( \mathbb{H} \). In this paper, we study commutative rings of differential operators which commute with \( P \).

Here, we recall the importance of commutative rings of differential operators. Commutative rings of differential operators were firstly studied by Burchnall and Chaundy [BC]. In the later half of the 20th century, the relation between commutative rings of differential operators and algebraic curves was studied in the celebrated works of Krichever [K] and Mumford [Mm]. Their results are very important in the theory of integrable systems. Also, they yielded a substantial progress of the geometry of Riemann surfaces and abelian varieties. In fact, they were used to resolve the classical Riemann-Schottky problem for Riemann surfaces ([So], [KS]).

In this paper, we will give a relation between commutative rings of differential operators and automorphic forms. We study the structures of rings of differential operators which are invariant under the action of \( \Gamma \) and commute with the fixed differential operator \( P \). Such a ring will be denoted by \( D^P \) in Section 2. Our main result gives an isomorphism \( \chi : D^P \cong S^P \) of rings, where \( S^P \) is a ring of generating functions for sequences of automorphic forms for \( \Gamma \) (for the definition, see Definition 2.3 and 2.4). Here, we note that automorphic forms are natural extension of elliptic modular forms (see Definition 2.1). These rings are graded by the weight \( K \) induced from the action of \( \Gamma \): \( D^P = \bigoplus_{K=0}^{\infty} D^K_P, S^P = \bigoplus_{K=0}^{\infty} S^K_P \). The isomorphism \( \chi \) induces an isomorphisms among three vector spaces:
\[
D^K_P \xrightarrow{\chi} S^K_P \rightarrow W_K
\]
(see Theorem 2.6). Here, \( W_K \) is a vector space explicitly parametrized by automorphic forms for \( \Gamma \). Therefore, the structure of the ring \( D^P \) is closely related to the structure of the rings of automorphic forms.

For our study, we will use the Baker-Akhiezer functions. In [K], the Baker-Akhiezer functions give solutions of differential equations whose coefficients are smooth functions. However, for our purpose, it is natural to study differential equations whose coefficients have poles (precisely, see Remark 1.2). So, we need to modify the techniques of the Baker-Akhiezer functions for differential equations with some singularities. Section 1 will be devoted to such techniques. Namely, we will study the multivalued Baker-Akhiezer functions and its monodromy around singular points of \( P \).
In Section 2, we prove our main result. This is based on an invariance of the multivalued Baker-Akhiezer functions under the action of $\Gamma$, which is proved in Theorem 2.1. Moreover, we will see the following results:

- For fixed $P$ and an operator $Q \in D^P$, there exists an algebraic curve

$$\mathcal{R}_\Omega : \sum_{j,k} f_{j,k}(\Omega) X^j y^k = 0$$

such that $(X, Y) = (P, Q)$ gives a point of $\mathcal{R}_\Omega$. Here, the coefficients $f_{j,k}(\Omega)$ are automorphic forms for $\Gamma$ (see Theorem 2.2). Namely, from the differential operators $P$ and $Q$, we obtain a family of algebraic curves $\{\mathcal{R}_\Omega | \Omega \in \mathbb{H}\}$ parametrized on $\mathbb{H}$ via automorphic forms.

- If the coefficients of the fixed operator $P$ have poles in $z$-plane, the coefficients of $Q \in D^P$ can be multivalued functions of $z$ (for detail, see Proposition 1.3 and Theorem 2.2). However, if the genus of the algebraic curve $\mathcal{R}_\Omega$ is small enough, every coefficients of $Q \in D^P$ must be single-valued. We will have a sufficient criterion for $Q$ to be single-valued (see Theorem 2.8).

Throughout the paper, the Lamé differential equation is a prototype of our story. Via our new results between differential operators and automorphic forms, we have a simple interpretation of classical results of the Lamé equation via elliptic modular forms (Example 2.3, 2.4 and 2.5). This is an important example of our story.

Our results enable us to study differential equations based on the structures of rings of automorphic forms. In number theory, there are many famous generalizations of elliptic modular forms (for example, Siegel modular forms, Hilbert modular forms, etc.). Our results can be applied to such generalized forms also. The author expects that this paper may give a first step of the study of differential equations from the viewpoint of automorphic forms.

# 1 Commutative rings of differential operators with singularities and multivalued Baker-Akhiezer functions

## 1.1 Multivalued Baker-Akhiezer functions

In this subsection, we obtain the multivalued Baker-Akhiezer functions for the ordinary differential operator

$$P_z = \frac{d^N}{dz^N} + a_2(z)\frac{d^{N-2}}{dz^{N-2}} + a_3(z)\frac{d^{N-3}}{dz^{N-3}} + \cdots + a_N(z)$$

(1.1)

of the complex variable $z$. Here, we assume the coefficients $a_2(z), \cdots, a_N(z)$ are meromorphic functions of $z$. More precisely, we assume that $a_2(z), \cdots, a_N(z)$ are holomorphic on $\mathbb{C} - \mathcal{N}$, where $\mathcal{N}$ is the union of the sets of the poles of $a_j(z)$ ($j = 2, \cdots, N$).

**Remark 1.1.** If a differential operator $P^0_z = \frac{d^N}{dz^N} + a_0^0(z)\frac{d^{N-1}}{dz^{N-1}} + a_1^0(z)\frac{d^{N-2}}{dz^{N-2}} + \cdots + a_N^0(z)$ is given, by a gauge transformation $v P^0_z v^{-1}$ for some unit function $v = v(z)$, $P^0_z$ is transformed to $P_z$. So, in our study, we only consider the differential operator in the form (1.1) without loss of generality.

Let $\mathcal{X}$ be the universal covering of $\mathbb{C} - \mathcal{N}$. By taking a fixed point $w \in \mathbb{C} - \mathcal{N}$, any $s \in \mathcal{X}$ is represented by $s = (z, [\gamma])$, where $z \in \mathbb{C} - \mathcal{N}$, $\gamma$ is an arc in $\mathbb{C} - \mathcal{N}$ from $w$ to $z$ and $[\gamma]$ is the homotopy class of $\gamma$. We note that $z$ gives a local coordinate of $\mathcal{X}$.

**Proposition 1.1.** There exists the unique formal solution $\Psi((z, [\gamma]), w, \lambda)$ of the differential equation

$$P_z u = \lambda^N u$$

(1.2)
in the form
\[ \Psi((z, [\gamma]), w, \lambda) = \left( \sum_{s=0}^{\infty} \xi_s((z, [\gamma]), w) \lambda^{-s} \right) e^{\lambda(z-w)} \] (1.3)
such that
\[ \begin{align*}
\xi_0((z, [\gamma]), w) &\equiv 1, \\
\xi_s((w, [id]), w) &\equiv 0 \quad (s \geq 1).
\end{align*} \] (1.4)
Here, \( \xi_s \) are locally holomorphic functions of \((z, w)\).

**Proof.** In this proof, set \( a_0(z) \equiv 1, a_1(z) \equiv 0 \). Putting
\[ u = \left( \sum_{s=0}^{\infty} \eta_s(z) \lambda^{-s} \right) e^{\lambda(z-w)} \] to (1.2), we have
\[ \sum_{m=0}^{N} a_{N-m}(z) \sum_{l=0}^{m} \left( \frac{m!}{l!} \right) \left( \frac{\partial^{m-l}}{\partial z^{m-l}} \eta_s(z) \lambda^{-s} \right) e^{\lambda(z-w)} = \left( \sum_{s=0}^{\infty} \eta_s(z) \lambda^{N-s} \right) e^{\lambda(z-w)}. \]
Comparing the coefficients of \( \lambda^{-s_0} \), we have
\[ \sum_{m=0}^{N} a_{N-m}(z) \sum_{l=0}^{m} \left( \frac{m!}{l!} \right) \frac{\partial^{m-l}}{\partial z^{m-l}} \eta_{N+s_0}(z) = \eta_{N+s_0}(z). \] (1.5)
Since \( \eta_{N+s_0}(z) \) appears in the left hand side only when \( m = l = N \), the terms of \( \eta_{N+s_0}(z) \) is cancelled from the relation (1.5). The function \( \eta_{N+s_0-1}(z) \) and its derivation appear in (1.5) only when \( m = N \) and \( l = N - 1 \). Here, we used \( a_{N-1}(z) \equiv 0 \). Then, the equation (1.5) becomes to be
\[ N \frac{\partial}{\partial z} \eta_{N+s_0-1}(z) = \left( \text{a polynomial in} \frac{\partial^\nu}{\partial z^\nu} \eta(z) (l < N + s_0 - 1, \nu \in \mathbb{Z}_{\geq 0}) \right) \] (1.6)
By the integration of the relation (1.6) on the arc \( \gamma \in \mathbb{C} - \mathbb{N} \) whose start point is \( w \), we can obtain the expression of \( \eta_\mu(z) \) in terms of \( \eta_\nu(z) \) (\( \nu < \mu \)) and \( a_1(z) \). Especially, the condition that \( \eta_0(z, [\gamma]) \equiv 1 \) and \( \eta_s(w, [id]) = 0 \) (\( s \geq 1 \)) uniquely determines the sequence \( \{ \eta_s(z) \}_s \). Such functions \( \eta_s(z) \) give the required functions \( \xi_s((z, [\gamma]), w) \) (\( s \geq 0 \)).

From our construction given by the integration of the relation (1.6), we can see that \( \xi_s \) are locally holomorphic functions of \((z, w)\).

We call \( \Psi((z, [\gamma]), w, \lambda) \) of (1.3) the multivalued Baker-Akhiezer function for the equation (1.2).__

**Remark 1.2.** Krichever [K] studied ordinary differential equations whose coefficients are smooth functions of a real variable. Also, Mumford [M] studied differential equations whose coefficients are formal power series. For the purposes of their research, it is sufficient to study single-valued solutions of differential equations. However, for our main purpose of this paper, it is natural to study differential equations of a complex independent variables whose coefficients allow some singularities. In Section 2, we will consider the transformation \( z \mapsto z_1 = \frac{z}{\mu(j_{\alpha}(\Omega))} \), where \( j_{\alpha}(\Omega) \) is complex valued. In such cases, even if \( z \) is a real variable, \( z_1 \) is not always a real variable. Moreover, we will give results for a class of differential equations containing the Lamé equation. Since the Lamé equation has singularities, it is natural to study differential equations which admit singularities. This is the reason why we need the multivalued solution \( z \mapsto \Psi((z, [\gamma]), w, \lambda) \) of (1.3).

**Remark 1.3.** If \( a_2(z), \ldots, a_N(z) \) are holomorphic on the whole \( z \)-plane, we do not need to consider the universal covering \( X \) of \( \mathbb{C} - \mathbb{N} \). In this case, the function \( \Psi \) in the above theorem is given in the form
\[ \Psi(z, w, \lambda) = \left( \sum_{s=0}^{\infty} \xi_s(z, w) \lambda^{-s} \right) e^{\lambda(z-w)}. \]
Here, \( \Psi \) and \( \xi_s \) (\( s \geq 0 \)) are single-valued functions of \( z \in \mathbb{C} \).
For a differential operator (1.1), consider the differential equation

\[ P_z u = X u, \quad (1.7) \]

where \( X \in \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\} \). Let \( \lambda_1, \ldots, \lambda_N \) be the solutions of the equation \( \lambda^N = X \).

**Lemma 1.1.** For fixed \( w \in \mathbb{C} - \mathcal{N} \), the solutions \( \Psi((z, \gamma]), w, \lambda_j) \) \( (j = 1, \cdots, N) \) of (1.3) are linear independent for generic \( X \).

**Proof.** For \( \mu_1, \cdots, \mu_N \in \mathbb{C} \), suppose

\[ \sum_{j=1}^N \mu_j \Psi((z, \gamma]), w, \lambda_j) = 0 \quad (1.8) \]

holds for generic \( X \). Since the right hand side of the relation (1.8) is invariant under the permutation of \( \lambda_1, \cdots, \lambda_N \), together with the definition of \( \Psi \) of (1.3), we can assume that \( \mu_1 = \cdots = \mu_N = \mu \). Set \( E_s(z, w, X) = \sum_{j=1}^N \lambda_j^{-s} e^{\lambda_j(z-w)} \). For fixed \( z \) and \( w \), \( X \mapsto E_s(z, w, X) \) is a formal power series in \( X^{-1} \) and \( E_s(z, w, X) \) \( (s = 0, 1, \cdots) \) are linearly independent. The relation (1.8) becomes \( \sum_{s=0}^\infty \mu \xi_s((z, \gamma]), w) E_s(z, w, X) = 0 \) for generic \( X \). Therefore, \( \mu = 0 \) follows. \( \Box \)

**Proposition 1.2.** Let \( u = u((z, \gamma]), w) \) be a series given by the form

\[ u((z, \gamma]), w) = \left( \sum_{s=0}^\infty \eta_s(z, \gamma]) \lambda^{-s} \right) e^{\lambda(z-w)}, \quad (1.9) \]

where \( \eta_s(z, \gamma]) \) are analytic on \( \mathcal{X} \) and \( \lambda \) satisfies \( \lambda^N = X \). Then, \( u \) is a formal solution of the differential equation (1.7) if and only if \( u \) is given by

\[ u((z, \gamma]), w) = A(w, \lambda) \Psi((z, \gamma]), w, \lambda) \quad (1.10) \]

for generic \( \lambda \), where \( A(w, \lambda) \) does not depend on \( (z, \gamma]) \).

**Proof.** It is clear that \( u((z, \gamma]), w) \) of (1.10) is a solution of the differential equation (1.7).

Conversely, we assume that \( u((z, \gamma]), w) \) in the form (1.9) is a solution of the differential equation (1.7), where \( X = \lambda^N \). From Lemma 1.1, the space of solutions of (1.7) is generated by \( \Psi((z, \gamma]), w, \lambda_j) \) \( (j = 1, \cdots, N) \) for generic \( \lambda \). We can assume \( \lambda \) of (1.3) coincides with \( \lambda_j \) for some \( j \in \{1, \cdots, N\} \). Since the space \( \mathcal{X} \) is simply connected, \( (z, \gamma]) \mapsto \Psi((z, \gamma]), w, \lambda_j) \) \( (j = 1, \cdots, N) \) are single-valued on \( \mathcal{X} \). So, the solution in the form (1.9) must be an element of the 1-dimensional vector space generated by \( \langle \Psi((z, \gamma]), w, \lambda_j) \rangle \). Hence, \( u \) is given by the form (1.10). \( \Box \)

We will consider a differential operator

\[ Q_{(z, \gamma])} = b_0(z, \gamma]) \frac{d^M}{dz^M} + b_1(z, \gamma]) \frac{d^{M-1}}{dz^{M-1}} + \cdots + b_M(z, \gamma]). \quad (1.11) \]

Here, we assume that the coefficients \( b_k(z, \gamma]) \) \( (k = 0, \cdots, M) \) are multivalued analytic functions on \( \mathbb{C} - \mathcal{N} \). The operator \( Q_{(z, \gamma])} \) is defined on \( \mathcal{X} \).

From now on, we consider the action of the operator \( Q_{(z, \gamma])} \) on the function \( \Psi((z, \gamma]), w, \lambda) \). If \( P_z \) and \( Q_{(z, \gamma])} \) are commutative, we can apply Proposition 1.2 to \( Q_{(z, \gamma])} \Psi((z, \gamma]), w, \lambda) \). Therefore, it is natural to consider differential operator (1.11) whose coefficients are multivalued functions of \( z \) (for detail, see the proof of the next proposition).
**Proposition 1.3.** Let $P_z$ (resp. $Q_{(z,[\gamma])}$) be the differential operator of (1.17) (resp.). Then, $P_z$ and $Q_{(z,[\gamma])}$ are commutative if and only if the quotient

$$A(\lambda) = \sum_{s=-M}^{\infty} A_s \lambda^{-s}$$

(1.12)

for generic $\lambda$, where $\Psi$ is given in (1.3) and $A(\lambda)$ does not depend on $z, [\gamma]$ and $w$.

**Proof.** Suppose that $P_z$ and $Q_{(z,[\gamma])}$ are commutative. Then, $Q_{(z,[\gamma])} \Psi((z, [\gamma]), w, \lambda)$ gives a solution of the differential equation (1.17). Remark that $Q_{(z,[\gamma])} \Psi((z, [\gamma]), w, \lambda)$ is in the form (1.9) for some $\{\eta_s\}_s$. So, due to Proposition 1.2, we have

$$Q_{(z,[\gamma])} \Psi((z, [\gamma]), w, \lambda) = A(w, \lambda) \Psi((z, [\gamma]), w, \lambda),$$

(1.13)

for some $A(w, \lambda)$. Take any $w' \in \mathbb{C} - \mathcal{N}$. Then, $\Psi((z, [\gamma]), w', \lambda) e^{\lambda(w'-w)}$ has the form (1.8) and is a solution of (1.7). So, according to Proposition 1.2 again, there exist $B(w, \lambda)$ such that

$$\Psi((z, [\gamma]), w', \lambda) e^{\lambda(w'-w)} = B(w, \lambda) \Psi((z, [\gamma]), w, \lambda).$$

(1.14)

From (1.13) and (1.14),

$$A(w', \lambda) = \frac{Q_{(z,[\gamma])} \Psi((z, [\gamma]), w', \lambda)}{\Psi((z, [\gamma]), w', \lambda)} = \frac{Q_{(z,[\gamma])} (e^{\lambda(w'-w)} B(w, \lambda) \Psi((z, [\gamma]), w, \lambda))}{B(w, \lambda) \Psi((z, [\gamma]), w, \lambda) e^{\lambda(w'-w)}} = A(w, \lambda).$$

This shows that $A(w, \lambda)$ does not depend on the variable $w$. So, we set $A(\lambda) = A(w, \lambda)$. Hence, the relation (1.13) becomes to be

$$Q_{(z,[\gamma])} \left( \sum_{s=0}^{\infty} \xi_s \Psi((z, [\gamma]), w) \lambda^{-s} \right) e^{\lambda(w-w')} = \left( \sum_{s=0}^{\infty} A_s \lambda^{-s} \right) \left( \sum_{s=0}^{\infty} \xi_s \Psi((z, [\gamma]), w) \lambda^{-s} \right) e^{\lambda(w-w')}.$$  

(1.15)

Since $Q_{(z,[\gamma])}$ is a differential operator of rank $M$, a non-zero term which contains $\lambda^M$ appears in the left hand side of (1.15) as the higher term in $\lambda$. Therefore, considering the right hand side of (1.15), the series of $A(\lambda)$ must be in the form $A(\lambda) = \sum_{s=-M}^{\infty} A_s \lambda^{-s}$.

Conversely, we assume that the relation (1.13) holds. Then, we have $P_z Q_{(z,[\gamma])} \Psi((z, [\gamma]), w, \lambda) = P_z A(\lambda) \Psi((z, [\gamma]), w, \lambda) = \lambda M A(\lambda) \Psi((z, [\gamma]), w, \lambda)$. This is clearly equal to $Q_{(z,[\gamma])} P_z \Psi((z, [\gamma]), w, \lambda)$. Therefore, we have

$$[P_z, Q_{(z,[\gamma])}] \Psi((z, [\gamma]), w, \lambda) = 0.$$  

(1.16)

Here, the relation (1.16) means that the ordinary differential equation $[P_z, Q_{(z,[\gamma])}] u = 0$ has solutions $\{\Psi((z, [\gamma]), w, \lambda)\}_\lambda$ parametrized by $\lambda$. Since $\Psi$ is given by the form of (1.3), it follows that the differential operator $[P_z, Q_{(z,[\gamma])}]$ must be 0.

**Proposition 1.4.** Let $P_z$ be the differential operator of (1.17). Let $Q_{(z,[\gamma])}$ be the differential operator given by the form (1.17). If $P_z$ commutes with both $Q_{(z,[\gamma])}$ and $Q_{(z,[\gamma])}$, then $Q_{(z,[\gamma])}$ commutes with $Q_{(z,[\gamma])}$.

**Proof.** By the assumption and Proposition 1.3 there exist series $A^{(1)}(\lambda)$ and $A^{(2)}(\lambda)$ in $\lambda$ such that $Q_{(z,[\gamma])}^{(j)} \Psi((z, [\gamma]), w, \lambda) = A^{(j)}(\lambda) \Psi((z, [\gamma]), w, \lambda)$ ($j = 1, 2$) for $\Psi$ of (1.3). So, we have

$$[Q_{(z,[\gamma])}^{(1)}, Q_{(z,[\gamma])}^{(2)}] \Psi((z, [\gamma]), w, \lambda) = (A^{(1)}(\lambda) A^{(2)}(\lambda) - A^{(2)}(\lambda) A^{(1)}(\lambda)) \Psi((z, [\gamma]), w, \lambda) = 0.$$

As in the end of the proof of Proposition 1.3 we have $[Q_{(z,[\gamma])}^{(1)}, Q_{(z,[\gamma])}^{(2)}] = 0$. \qed
For the differential operator $P_z$ of (1.1), let $L(P_z, X)$ be the space of solutions of the differential equation $P_z u = X u$. Suppose $Q_{(z, [\gamma])}$ of (1.11) is a differential operator which commutes with $P_z$. Then, $Q_{(z, [\gamma])}$ defines a linear operator $Q_{[\gamma], X}$ on the vector space $L(P_z, X)$.

1.2 Monodromy

Let us take two arcs $\gamma$ and $\gamma'$ from $w$ to $z$ in $\mathbb{C} - \mathcal{N}$. Setting $\delta = \gamma^{-1} \cdot \gamma'$, $[\delta]$ gives an element of the fundamental group $\pi_1(\mathbb{C} - \mathcal{N})$. For $\lambda$ such that $\lambda^N = X$, since the coefficients of $P_z$ of (1.1) are single-valued, each $\Psi((z, [\gamma]), w, \lambda)$ and $\Psi((z, [\gamma']), w, \lambda)$ are solutions of the differential equation (1.7) for generic $X$. Based on Lemma [1.1] setting the vector

$$\Psi_v((z, [\gamma]), w, \lambda) = (\Psi((z, [\gamma]), w, \lambda_1), \ldots, \Psi((z, [\gamma]), w, \lambda_N)),$$

there exists a matrix $M([\delta], w, \lambda) \in GL(N, \mathbb{C})$ such that

$$\Psi_v((z, [\gamma']), w, \lambda) = \Psi_v((z, [\gamma]), w, \lambda)M([\delta], w, \lambda).$$

The matrix $M([\delta], w, \lambda)$ is called the monodromy matrix of $[\delta] \in \pi_1(\mathbb{C} - \mathcal{N})$ for the system $\Psi_v((z, [\gamma]), w, \lambda)$ of (1.17). We note that

$$r : \pi_1(\mathbb{C} - \mathcal{N}) \rightarrow GL(N, \mathbb{C}) \quad \text{given by} \quad [\delta] \mapsto M([\delta], w, \lambda)$$

is a homomorphism of groups.

Let $\gamma, \gamma'$ and $\delta$ be as above. Suppose $Q_{(z, [\gamma])}$ commutes with $P_z$ of (1.1). By considering the analytic continuation along the closed arc $\delta$, $Q_{(z, [\gamma'])}$ also commutes with $P_z$. For the linear operator $Q_{[\gamma], X}$ on $L(P_z, X)$, set

$$[\delta]^*(Q_{[\gamma], X}) = Q_{[\gamma'], X}.$$  

**Theorem 1.1.** Let $\gamma, \gamma', \delta, Q_{[\gamma], X}$ be as above.

1. The set of the eigenvalues of the linear operator $Q_{[\gamma], X}$ coincides with that of the linear operator $[\delta]^*(Q_{[\gamma], X})$ for generic $X$.

2. There exists $N_0 \in \mathbb{Z}_{>0}$ such that $([\delta]^*)^{N_0}(Q_{[\gamma], X}) = Q_{[\gamma], X}$ for any $[\delta] \in \pi_1(\mathbb{C} - \mathcal{N})$ and generic $X$.

**Proof.** Set $Q_{(z, [\gamma])}\Psi_v((z, [\gamma]), w, \lambda) = (Q_{(z, [\gamma])}\Psi((z, [\gamma]), w, \lambda_1), \ldots, Q_{(z, [\gamma])}\Psi((z, [\gamma]), w, \lambda_N))$. According to Proposition [1.3] together with the notation (1.17), we have

$$Q_{(z, [\gamma])}\Psi_v((z, [\gamma]), w, \lambda) = \Psi_v((z, [\gamma]), w, \lambda) \begin{pmatrix} A(\lambda_1) & \cdots & 0 \\ 0 & \ddots & A(\lambda_N) \end{pmatrix}.$$

For $[\delta] \in \pi_1(\mathbb{C} - \mathcal{N})$, using the monodromy matrix $M([\delta], w, \lambda)$ of (1.18), we have

$$Q_{(z, [\gamma'])}\Psi_v((z, [\gamma']), w, \lambda) = Q_{(z, [\gamma'])}\Psi_v((z, [\gamma']), w, \lambda)M([\delta], w, \lambda)^{-1}$$

$$= \Psi_v((z, [\gamma']), w, \lambda) \begin{pmatrix} A(\lambda_1) & \cdots & 0 \\ 0 & \ddots & A(\lambda_N) \end{pmatrix} M([\delta], w, \lambda)^{-1}$$

$$= \Psi_v((z, [\gamma']), w, \lambda)M([\delta], w, \lambda) \begin{pmatrix} A(\lambda_1) & \cdots & 0 \\ 0 & \ddots & A(\lambda_N) \end{pmatrix} M([\delta], w, \lambda)^{-1}. \quad (1.21)$$

Here, we used the fact that $Q_{(z, [\gamma'])}\Psi((z, [\gamma']), w, \lambda_j) = A(\lambda_j)\Psi((z, [\gamma']), w, \lambda_j)$ due to Proposition [1.3].
On the other hand, since $Q_{(z,[\gamma])}$ commutes with $P_z$, we can directly apply Proposition 1.3 to $Q_{(z,[\gamma])}$. Then, there exist $A'(\lambda_1), \cdots, A'(\lambda_N)$ such that

$$Q_{(z,[\gamma])}\Psi_v((z, [\gamma]), w, X) = \Psi_v((z, [\gamma]), w, \lambda) \begin{pmatrix} A'(\lambda_1) & \cdots & 0 \\ 0 & \cdots & A'(\lambda_N) \end{pmatrix}. \quad (1.22)$$

So, from 1.21 and 1.22, we have

$$\begin{pmatrix} A'(\lambda_1) & \cdots & 0 \\ 0 & \cdots & A'(\lambda_N) \end{pmatrix} = M([\delta], w, \lambda) \begin{pmatrix} A(\lambda_1) & \cdots & 0 \\ 0 & \cdots & A(\lambda_N) \end{pmatrix} M([\delta], w, \lambda)^{-1}. \quad (1.23)$$

This implies that the set of eigenvalues $\{A(\lambda_1), \cdots, A(\lambda_N)\}$ for $Q_{[\gamma],X}$ coincides with that of eigenvalues $\{A'(\lambda_1), \cdots, A'(\lambda_N)\}$ for $Q_{[\gamma],X}$.

(2) From the above (1), the correspondence $[\delta]$ of 1.20 induces a permutation of the eigenvalues. Therefore, setting $N_0 = N!$, we have

$$(([\delta]^*)^{N_0}Q_{(z,[\gamma])})\Psi_v((z, [\gamma]), w, \lambda) = \Psi_v((z, [\gamma]), w, \lambda) \begin{pmatrix} A(\lambda_1) & \cdots & 0 \\ 0 & \cdots & A(\lambda_N) \end{pmatrix} = Q_{(z,[\gamma])}\Psi_v((z, [\gamma]), w, \lambda).$$

So, for $j \in \{1, \cdots, N\}$, we have $(([\delta]^*)^{N_0}Q_{(z,[\gamma])}) - Q_{(z,[\gamma])}\Psi((z, [\gamma]), w, \lambda_j) = 0$. Therefore, by a similar argument to the end of the proof of Proposition 1.3 we obtain $([\delta]^*)^{N_0}Q_{(z,[\gamma])} = Q_{(z,[\gamma])}$. \quad \Box

**Corollary 1.1.** If $N$ is a finite set of $\mathbb{C}$, the coefficients $b_k((z, [\gamma])) \quad (k = 0, \cdots, M)$ of $Q_{(z,[\gamma])}$ of 1.20 are at most algebraic functions of $z$.

**Proof.** By Theorem 1.1 and the assumption of the corollary, the image of the correspondence $\pi_1(\mathbb{C} - N) \ni [\delta] \mapsto ([\delta]^*)$ of 1.20 is finite. This implies the assertion. \quad \Box

### 1.3 The algebraic curve $\mathcal{R}$

For generic $X \in \mathbb{C}$, we have the distinct $N$ values $\lambda_1, \cdots, \lambda_N$ such that $\lambda_j^N = X \quad (j = 1, \cdots, N)$. From Lemma 1.1, $\{\Psi((z, [\gamma]), w, \lambda_j) \mid j \in \{1, \cdots, N\}\}$ gives a system of basis of $\mathcal{L}(P_z, X)$. From Proposition 1.3, $\Psi((z, [\gamma]), w, \lambda_j)$ gives an eigenfunction with the eigenvalue $A(\lambda_j)$ of $Q_{[\gamma],X}$. Hence, the characteristic polynomial of $Q_{[\gamma],X}$ on $\mathcal{L}(P_z, X)$ is given by

$$\prod_{j=1}^{N} (Y - A(\lambda_j)). \quad (1.23)$$

**Lemma 1.2.** Let $\{C_i((z, [\gamma]), w, X)\}_{i=0,1,\cdots,N-1}$ be a system of basis of the vector space $\mathcal{L}(P_z, X)$ satisfying

$$\frac{\partial^r}{\partial z^r}C_i((z, [\gamma]), w, X) \bigg|_{(z, [\gamma])=(w, [id])} = \delta_{i,r}. \quad (1.24)$$

(1) For fixed $w \in \mathbb{C} - N$ and $(z, [\gamma]) \in \mathfrak{X}$, the correspondence $X \mapsto C_i((z, [\gamma]), w, X)$ gives a holomorphic function on $\mathbb{C} = \mathbb{P}^1(\mathbb{C}) - \{\infty\}$.

(2) For $Q_{(z,[\gamma])}$ of 1.17, the components of the representation matrix of the linear operator $Q_{[\gamma],X}$ for the system of basis $\{C_i((z, [\gamma]), w, X)\}_{i=0,1,\cdots,N-1}$ are given by polynomials in $X$ and special values of $a_j(z) \quad (j = 0, \cdots, N)$ and $b_j((z, [\gamma]), w) \quad (k = 0, \cdots, M)$.

**Proof.** (1) The solutions $C_i((z, [\gamma]), w, X)$ are given by solving the initial value problem for the differential equation 1.7. Hence, the correspondence $X \mapsto C_i((z, [\gamma]), w, X)$ is holomorphic.
(2) Since \( C_l((z, [\gamma]), w, X) \) \((l = 0, \ldots, N - 1)\) are solutions of the equation (1.14), we obtain
\[
\frac{\partial^N}{\partial z^N} C_l((z, [\gamma]), w, X) = X C_l((z, [\gamma]), w, X) - \sum_{k=0}^{N-1} a_k(z) \frac{\partial^k}{\partial z^k} C_l((z, [\gamma]), w, X).
\] (1.25)

By the way, since \( Q((z, [\gamma])) C_l((z, [\gamma]), w, X) \in \mathcal{L}(P_z, X) \), there exists constants \( c_{l,m}(w, X) \) \((l, m \in \{0, \ldots, N - 1\})\) for \( z \) such that
\[
Q((z, [\gamma])) C_l((z, [\gamma]), w, X) = \sum_{m=0}^{N-1} c_{l,m} C_m((z, [\gamma]), w, X). \quad \text{Due to (1.24), we have}
\]
\[
c_{l,m}(w, X) = \frac{\partial^m}{\partial z^m} Q((z, [\gamma])) C_l((z, [\gamma]), w, X) \bigg|_{((z, [\gamma])=(w, [id])).}
\] (1.26)

Using the relation (1.25), we can see that \( \frac{\partial^m}{\partial z^m} Q((z, [\gamma]), w, X) \bigg|_{((z, [\gamma])=(w, [id]))} \) are given by a polynomial in \( X \) and the special values \( a_k(w) \) for any \( r \in \mathbb{Z} \). So, according to (1.26), we can see that \( c_{l,m}(w, X) \) are given by polynomials in \( X \) and special values of \( a_j(z) \) and \( b_k((z, [\gamma]), w) \).

We note that \( \lambda_j \) \((j = 1, \ldots, N)\) are distinct solutions of the algebraic equation \( \lambda^N = X \). From (1.13), \( A(\Omega, \lambda_j) \) is a Laurent series in \( \lambda_j^{-1} \). Since the right hand side of (1.28) is symmetric in \( \lambda_j^{-1} \) \((j = 1, \ldots, N)\), the right hand side of (1.28) gives a Laurent series in \( X^{-1} \). Moreover, we have the following.

**Corollary 1.2.** The characteristic polynomial (1.25) defines a polynomial in \( X \).

**Proof.** We have a representation matrix of \( Q((z, [\gamma]), X) \) whose components are polynomial in \( X \) from the above lemma. Therefore, its characteristic polynomial is given by a polynomial in \( X \). \( \square \)

In the following, let \( F(X, Y) \) be the polynomial (1.23) in the variables \( X \) and \( Y \).

**Theorem 1.2.** The differential operators \( P_z \) of (1.14) and \( Q((z, [\gamma])) \) of (1.14) satisfy \( F(P_z, Q((z, [\gamma]))) = 0 \).

**Proof.** For generic \( X \in \mathbb{C} \) and letting \( \lambda \) be a solution of the equation of \( \lambda^N = X \), we have
\[
F(P_z, Q((z, [\gamma]))) \Psi((z, [\gamma]), w, \lambda) = F(X, Q((z, [\gamma]))) \Psi((z, [\gamma]), w, \lambda) = 0.
\]
Here, the last equality is due to the Hamilton-Cayley theorem. Then, the ordinary differential equation
\[
F(P_z, Q((z, [\gamma]))) \psi = 0 \text{ has a family } \{ \Psi((z, [\gamma]), w, \lambda) \}_{\lambda} \text{ of solutions with the parameter } \lambda.
\]
By a similar argument to the end of the proof of Proposition 1.3 the operator \( F(P_z, Q((z, [\gamma]))) \) is equal to 0. \( \square \)

The equation \( F(X, Y) = 0 \) defines an algebraic curve \( \mathcal{R} \). This curve should be in the form
\[
\mathcal{R} : \sum_{j,k} f_{j,k} X^j Y^k = 0 \quad (1.27)
\]

Let \( \mathcal{R} \) be the algebraic curve in Theorem 1.2. Let \( \pi : \mathcal{R} \to \mathbb{P}^1(\mathbb{C}) \) be the projection given by \( (X, Y) \mapsto X \). Let \( p_\infty \) be the point of \( \mathcal{R} \) corresponding to \( X = \infty \in \mathbb{P}^1(\mathbb{C}) \). Then, \( p_\infty \) is a ramification point of the mapping \( \pi \). We note that \( X_1 = \frac{1}{X} \) gives a complex coordinate around \( p_\infty \in \mathcal{R} \).

By the procedure of the algebraic curve \( \mathcal{R} \) and the covering \( \pi : \mathcal{R} \to \mathbb{P}^1(\mathbb{C}) \), Proposition 1.3 and Theorem 1.1(1) imply that any \( [\delta] \in \pi_1(\mathbb{C} - \mathcal{N}) \) induces the correspondence \( \sigma_{[\delta]} : \mathcal{R} \to \mathcal{R} \) given by
\[
p_j = (X, A(\lambda_j)) \mapsto \sigma_{[\delta]}(p_j) = p_k = (X, A(\lambda_k)), \quad \text{when}
\]
\[
([\delta]^*(Q((z, [\gamma]), X))) \Psi((z, [\gamma]), w, \lambda_j) = A(\lambda_k) \Psi((z, [\gamma]), w, \lambda_j). \quad (1.29)
\]
Hence, letting \( \text{Aut}(\pi) \) be the group of transformations for the covering \( \pi \), we have the homomorphism
\[
\pi_1(\mathbb{C} - \mathcal{N}) \to \text{Aut}(\pi)
\]
of groups given by \( [\delta] \mapsto \sigma_{[\delta]} \).
Lemma 1.3.}

(1) All coefficients of the operator $Q_{(z,[\gamma])}$ are single-valued on $\mathbb{C} - \mathcal{N}$ if and only if $\sigma_{[\delta]} = id$ for every $[\delta] \in \pi_1(\mathbb{C} - \mathcal{N})$.

(2) For $\lambda_j$ ($j = 1, \cdots, N$) satisfying $\lambda_j^N = X$, assume $A(\lambda_1), \cdots, A(\lambda_N)$ are distinct for generic $X$. Then, $\sigma_{[\delta]} = id$ if and only if

$$
\Psi((z, [\gamma]), w, \lambda_j) = \mu_j \Psi((z, [\gamma]), w, \lambda_j) \quad (j = 1, \cdots, N),
$$

where $\mu_j$ is a constant function of $z$.

**Proof.** (1) If all coefficients of $Q_{(z,[\gamma])}$ are single-valued, we have $[\delta]^{*}(Q_{[\gamma],X}) = Q_{[\gamma],X}$ for any $[\delta] \in \pi_1(\mathbb{C} - \mathcal{N})$. Then, by (1.28) and (1.29), we have $\sigma_{[\delta]} = id$.

Conversely, if $\sigma_{[\delta]} = id$ for any $[\delta] \in \pi_1(\mathbb{C} - \mathcal{N})$, from (1.28) and (1.29), we have

$$
[\delta]^{*}(Q_{[\gamma],X}) \Psi((z, [\gamma]), w, \lambda_j) = A(\lambda_j) \Psi((z, [\gamma]), w, \lambda_j) = Q_{[\gamma],X} \Psi((z, [\gamma]), w, \lambda_j)
$$

for generic $X$. So, by a similar argument to the proof of Proposition 1.3, we have $Q_{(z,[\gamma'])} = Q_{(z,[\gamma])}$, where $\gamma' = \gamma \cdot \delta$. Hence, the assertion holds.

(2) By the assumption, $\Psi((z, [\gamma]), w, \lambda_j)$ spans the 1-dimensional eigenspace for the eigenvalue $A(\lambda_j)$ of $Q_{[\gamma],X}$. Set $\gamma' = \gamma \cdot \delta$. If $\sigma_{[\delta]} = id$, from (1), we have $([\delta^{-1}]^{*})Q_{[\gamma'],X} = Q_{[\gamma],X}$ for any $[\delta] \in \pi_1(\mathbb{C} - \mathcal{N})$. This implies that

$$
Q_{z,[\gamma']} \Psi((z, [\gamma']), w, \lambda_j) = Q_{z,[\gamma']} \Psi((z, [\gamma']), w, \lambda_j) = A(\lambda_j) \Psi((z, [\gamma']), w, \lambda_j)
$$

for generic $X$, where $\lambda_j^N = X$. Here, we used Proposition 1.3. Therefore, $\Psi((z, [\gamma']), w, \lambda_j)$ is an eigenfunction for the eigenvalue $A(\lambda_j)$. So, $\Psi((z, [\gamma']), w, \lambda_j) \in (\Psi((z, [\gamma]), w, \lambda_j))_{\mathbb{C}}$ holds.

Conversely, if we have (1.30), then, due to Proposition 1.3

$$
Q_{z,[\gamma']} \Psi((z, [\gamma]), w, \lambda_j) = \mu_j^{-1} Q_{z,[\gamma']} \Psi((z, [\gamma]), w, \lambda_j)
$$

$$
\mu_j^{-1} A(\lambda_j) \Psi((z, [\gamma]), w, \lambda_j) = Q_{z,[\gamma]} \Psi((z, [\gamma]), w, \lambda_j)
$$

holds for generic $X$. Hence, as in (1), we have $\sigma_{[\delta]} = id$. \qed

### 1.4 The eigenfunction $\psi$

In this subsection, we use the same notation which we use in the previous subsection. Moreover, we suppose that

there exists $s$ ($s \geq -M$), where $N$ and $s$ are coprime, such that $A_s \neq 0$ \hspace{1cm} (1.31)

for $\{A_s\}$ of (1.12). Then, the operator $Q_{[\gamma],X}$ on $L(P_2, X)$ has $N$ distinct eigenvalues $A(\lambda_j)$ ($j = 1, \cdots, N$) in the sense of Proposition 1.3. Hence, the eigenspace for the eigenvalue $A_s(\lambda_j)$ is 1-dimensional.

Since $X$ is simply connected, for a general $X \in \mathbb{C}$ and $p \in \pi^{-1}(X) \subset \mathcal{R}$, we can take the unique eigenfunction on $X$:

$$
\psi((z, [\gamma]), w, p) = \sum_{l=0}^{N-1} h_l(w, p) C_l((z, [\gamma]), w, X),
$$

where $h_0(w, p) \equiv 1$. Here, $C_l((z, [\gamma]), w, X)$ ($l = 0, \cdots, N - 1$) are given in Lemma 1.2 and $h_l(w, p)$ does not depend on $z$.

**Lemma 1.3.** Let $\psi((z, [\gamma]), w, p)$ be the function of (1.32).

(1) For fixed $w \in \mathbb{C} - \mathcal{N}$, $p \mapsto h_l(w, p)$ gives a meromorphic function on $\mathcal{R} - \{p_\infty\}$.

(2) For fixed $w \in \mathbb{C} - \mathcal{N}$, the poles of $\mathcal{R} - \{p_\infty\} \ni p \mapsto \psi((z, [\gamma]), w, p) \in \mathbb{P}^1(\mathbb{C})$ do not depend on $(z, [\gamma]) \in \mathcal{X}$.

(3) Let $U_\infty \subset \mathbb{P}^1(\mathbb{C})$ be a sufficiently small neighborhood of $X = \infty$. Let $V \subset \mathbb{C} - \mathcal{N}$ be a sufficiently small and simply connected neighborhood of $w$. If $\pi(p) \in U_\infty - \{\infty\}$, $z \in V$ and $\gamma \subset V$, then $p \mapsto \psi((z, [\gamma]), w, p)$ is analytic and has an exponential singularity at $p = p_\infty$. \hspace{1cm} \text{10}
Proof. (1) We had the representation matrix $c(w, X) = (c_{jk}(w, X))$ of the linear operator $Q_{[\gamma]}$ on $L(P_z, X)$ for the system of basis $(C_1([z, \gamma]), [w, X])_{l=0, \cdots, N-1}$ of (1.24). Here, by Lemma (1.2), $c_{i,m}(w, X)$ are given by polynomials in $X$. Let $p \in \mathcal{R}$ be a point corresponding to $X \in \mathcal{C}$ and the eigenvalue $Y$. We can obtain $h_l(w, p)$ of (1.32) by solving the linear equation

$$c(w, X) \begin{pmatrix} h_0(w, p) \\ h_1(w, p) \\ \vdots \\ h_{N-1}(w, p) \end{pmatrix} = Y \begin{pmatrix} h_0(w, p) \\ h_1(w, p) \\ \vdots \\ h_{N-1}(w, p) \end{pmatrix},$$

where $h_0(w, p) = 1$. This implies that $h_l(w, p)$ $(l = 1, \cdots, N - 1)$ are given by rational functions of $X$ and $Y$. Therefore, $p \mapsto h_l(w, p)$ is meromorphic on $\mathcal{R}$.

(2) From Lemma (1.2) and the expression (1.32) of $\psi$, the poles of $\mathcal{R} - \{p_\infty\} \ni p \mapsto \psi([z, [\gamma]], w, p) \in \mathbb{P}^1(\mathbb{C})$ are coming only from the poles of $p \mapsto h_l(w, p)$ $(l = 1, \cdots, N - 1)$. These poles do not depend on $(z, [\gamma])$.

(3) From the procedure of the Riemann surface $\mathcal{R}$, we can take sufficiently small neighborhood $U_\infty$ such that the set $\pi^{-1}(X)$ consists $N$ distinct points for any $X \in U_\infty \setminus \{p_\infty\}$. Then, $p \in \mathcal{R} \setminus \{p_\infty\}$ such that $\pi(p) = X \in U_\infty$ corresponds to $(X, Y) = (X, A(\lambda_j))$ for $j = 1, \cdots, N$, where $\lambda_j^N = X$. Then, $\psi([z, [\gamma]], w, p)$ corresponds to $\Psi((z, [\gamma]), w, \lambda_j)$ of (1.34) and (1.32). So, from (1.3),

$$h_1(w, p) = \frac{\partial}{\partial z} \Psi((z, [\gamma]), w, \lambda_j) \bigg|_{(z, [\gamma]) = (w, [id])} = \lambda_j + O(\lambda_j^{-1}).$$

By the way, we take a sufficiently small and simply connected neighborhood $V \subset \mathcal{C} - \mathcal{N}$ of $w$. Let $x \in V$. We have the logarithmic derivative of $\psi$ at $w$:

$$\frac{\partial}{\partial z} \log \psi((z, [\gamma]), w, p) \bigg|_{(z, [\gamma]) = (w, [id])} = \frac{\partial^2 \psi((z, [\gamma]), w, p)}{\partial \psi((z, [\gamma]), w, p)} \bigg|_{(z, [\gamma]) = (w, [id])} = h_1(w, p).$$

By changing the base point $w$, which defines the universal covering $\mathfrak{X}$, to a point $x$ of the simply connected neighborhood $V$, we can regard $x \mapsto h_1(x, p)$ as a single-valued holomorphic function on $V$. So, if $z \in V$ and $\gamma \subset V$, then we locally have the expression

$$\psi((z, [\gamma]), w, p) = \exp \left( \int_x h_1(x, p) dx \right).$$

From (1.33) and (1.34), we have the assertion of (3). \qedhere

Remark 1.4. The expression (1.32) is valid only for sufficiently close $(x, [\gamma])$ to $(w, [id])$, because we used the change of the base point from $w$ to $x$. We note that $h_1(x, p)$ depends on the choice of the base point. Generically, $x \mapsto h_1(x, p)$ can be globally multivalued and the expression (1.32) does not holds.

For $X \in \mathcal{C}$, $\pi^{-1}(X) = p_1 + \cdots + p_N$ gives a divisor on $\mathcal{R}$. We set

$$G((z, [\gamma]), w, X) = \begin{pmatrix} \psi((z, [\gamma]), w, p_1) & \cdots & \psi((z, [\gamma]), w, p_N) \\ \frac{\partial}{\partial z} \psi((z, [\gamma]), w, p_1) & \cdots & \frac{\partial}{\partial z} \psi((z, [\gamma]), w, p_N) \\ \vdots & \cdots & \vdots \\ \frac{\partial^{N-1}}{\partial z^{N-1}} \psi((z, [\gamma]), w, p_1) & \cdots & \frac{\partial^{N-1}}{\partial z^{N-1}} \psi((z, [\gamma]), w, p_N) \end{pmatrix}^2.$$ (1.35)

for fixed $(z, [\gamma])$ and $w$. We note that $X \mapsto G((z, [\gamma]), w, X)$ is well-defined on $X \in \mathcal{C}$.

Lemma 1.4. For $(z, [\gamma]) \in \mathfrak{X}$ and $w \in \mathcal{C} - \mathcal{N}$, any poles of the function $\mathcal{R} - \{\infty\} \ni p \mapsto \psi((z, [\gamma]), w, p) \in \mathbb{P}^1(\mathbb{C})$ analytically depend on the base point $w$. They are not independent of $w$.

Proof. For a fixed base point $w$, if $q$ is a pole of $p \mapsto \psi((z, [\gamma]), w, p)$, by Lemma (1.3) (2), it holds that $\psi((z, [\gamma]), w, q) = \infty$ for any $(z, [\gamma]) \in \mathfrak{X}$. So, if we can take $q$ which is independent of $w$, we have
ψ((z, [γ]), w, q) = ∞ for any (z, [γ]) ∈ X and w ∈ C − N. This is a contradiction, because we have
ψ((w, [id]), w, q) = 1 by the definition (1.32) of ψ. So, any pole of p → ψ((z, [γ]), w, p) is not independent
of w. Moreover, from the proof of Lemma 1.3 (1), such poles are coming from the zeros of the common
denominator of h_1(w, p), · · · , h_{N−1}(w, p). We note that the common denominator is given by a polynomial
in X and Y analytically parametrized by w. So, poles analytically depend on w.

Lemma 1.5. Take w ∈ C − N and (z, [γ]) ∈ X. Then, the correspondence X → G((z, [γ]), w, X) of
(1.36) gives a rational function of X. Moreover, this rational function has a pole at X = ∞ of degree
N − 1.

Proof. Due to Lemma 1.3 (1) and the properties of determinant of (1.35), the correspondence X →
G((z, [γ]), w, X) defines a meromorphic function on C = P^1(C) − {∞}. Now, take a sufficiently small and
simply connected neighborhood V ⊂ C − N of w. Although p → ψ((z, [γ]), w, p) for z ∈ V and γ ⊂ V
has an exponential singularity at p∞ (Lemma 1.3 (3)), we will see that X → G((z, [γ]), w, X) is analytic
around X = ∞. Taking a sufficiently small neighborhood U∞, ψ of (1.32) is given by Ψ of (1.3) and
holomorphic on U∞ − {p∞}. Then, considering the properties of the determinant of (1.35), and the fact
that e^{λ_1(z−w)} · · · e^{λ_N(z−w)} = 1, we can see that G((z, [γ]), w, X) has the form

\( \left( \det \begin{pmatrix} 1 + O(λ_1^{-1}) & \cdots & 1 + O(λ_N^{-1}) \\ λ_1(1 + O(λ_1^{-1})) & \cdots & λ_N(1 + O(λ_N^{-1})) \\ \vdots & \cdots & \vdots \\ λ_1^{N−1}(1 + O(λ_1^{-1})) & \cdots & λ_N^{N−1}(1 + O(λ_N^{-1})) \end{pmatrix} \right)^2 \)  \hspace{1cm}  (1.36)

around X = ∞. Then, (1.36) is a symmetric series in λ_1, · · · , λ_N with the highest term of degree
2(0 + 1 + · · · + (N − 1)) = N(N − 1). Setting X_1 = \frac{1}{X}, X_1 gives a complex coordinate around X = ∞
and (1.36) gives a Laurent series in X_1. Due to Lemma 1.3 (1) and the assumption (1.31), applying the
Riemann extension theorem, (1.36) is holomorphic at X_1 = 0 and has a zero of degree \frac{N(N−1)}{N} = N − 1
for z ∈ V and γ ⊂ V. By the analytic continuation in terms of (z, [γ]) ∈ X, we have the assertion.

Theorem 1.4. Assume the condition (1.31). Suppose the algebraic curve R given by the defining equation
(1.27) is non-singular and of genus g. Then, for w ∈ C − N and (z, [γ]) ∈ X, the function

\( R − \{p_∞\} \ni p \mapsto ψ((z, [γ]), w, p) ∈ P^1(C) \)  \hspace{1cm}  (1.37)

has g poles.

Proof. Let κ be the number of poles of the function of (1.37). Since ψ is given by (1.32), together
with Lemma 1.3 (2), we can see that the set of the poles of p → ψ((z, [γ]), w, p) corresponds to that of
p → \frac{∂}{∂N}ψ((z, [γ]), w, p) (r ≥ 1). So, from the definition of the rational function X → G((z, [γ]), w, X) of
(1.35), the number of the poles of the function

\( P^1(C) − \{∞\} \ni X \mapsto G((z, [γ]), w, X) ∈ P^1(C) \)

is equal to 2κ. Together with Lemma 1.5, the number of poles of the function

\( P^1(C) \ni X \mapsto G((z, [γ]), w, X) ∈ P^1(C) \)  \hspace{1cm}  (1.38)

is equal to 2κ + N − 1. Since (1.35) is a rational function of the variable X, this function has 2κ + N − 1
zeros on \( P^1(C) − \{∞\} \).

On the other hand, from Lemma 1.4 and the fact that the ramification points of π are isolated points of
R, for generic base point w ∈ C − N, all poles of p → ψ((z, [γ]), w, p) are out of the set of the ramification
points of π. We fix such a base point w. From the definition (1.35), the function X → G((z, [γ]), w, X)
vanishes at X (≠ ∞) if and only if X is a branch point of the covering π. Letting e_p be the ramification
index of π at p ∈ R. From the property of determinants of matrices, the right hand side of (1.36)
has zeros of degree \(2(0 + 1 + (e_p - 1)) = e_p(e_p - 1)\) of a coordinate around \(p \in \mathcal{R}\). So, at \(X = \pi(p)\), \(X \mapsto G((z, [\gamma]), w, X)\) has zeros of degree \(\frac{e_p(e_p - 1)}{e_p} = e_p - 1\). Therefore, the degree of zeros of the function of (1.38) coincides with \(\sum_{p \in \mathcal{R} \setminus \{p_\infty\}} (e_p - 1)\). So, together with Lemma 1.5
\[
\sum_{p \in \mathcal{R}} (e_p - 1) = 2\kappa + 2N - 2.
\]
By the way, applying the Riemann-Hurwitz formula, we have
\[
\sum_{p \in \mathcal{R}} (e_p - 1) = (2g - 2) + N(2 - 0).
\]
By (1.39) and (1.40), we have \(\kappa = g\). Therefore, we have proved the assertion for generic \(w\). Since the number of poles of \(p \mapsto \psi((z, [\gamma]), w, p)\) on \(\mathcal{R}\) is analytically dependent on the variable \(w\), this is a constant function of \(w\). Thus, for every \(w\), the number of poles is equal to \(g\).

Next, we consider the case that the algebraic curve \(\mathcal{R}\) of (1.27) has singular points \(\mathcal{S} \subset \mathcal{R}\). We have a resolution of singularities \(\sigma : \tilde{\mathcal{R}} \to \mathcal{R}\). Here, \(\sigma\) is given by a composition \(\tilde{\mathcal{R}} = \mathcal{R}_l \to \mathcal{R}_{l-1} \to \cdots \to \mathcal{R}_0 = \mathcal{R}\) of blowing ups \(\sigma_{\nu} : \mathcal{R}_\nu \to \mathcal{R}_{\nu-1}\) for a singular point of multiplicity \(m_\nu \in \mathbb{Z}_{>0}\) \((\nu = 1, \cdots, \kappa)\). We have an \(N\) to 1 covering \(\pi \circ \sigma : \tilde{\mathcal{R}} \to \mathbb{P}^1(\mathbb{C})\). By considering the divisor \((\pi \circ \sigma)^{-1}(X)\) for \(X \in \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}\), we can define the function \(X \mapsto G((z, [\gamma]), w, X)\), also. By a similar argument of the proof of Theorem 1.4 and considering properties of the blowing ups (for example, see [G]), \(X \mapsto G((z, [\gamma]), w, X)\) has zeros, not only at the branch points of \(\pi \circ \sigma\), but also the images of \(\mathcal{S}\) under \(\pi\), where the sum of the orders of zeros is at most \(\sum_{\nu=1}^{l} m_\nu(m_\nu - 1)\). Applying the argument of the proof of Theorem 1.4 to the non-singular curve \(\tilde{\mathcal{R}}\), we have the following.

**Corollary 1.3.** Using the above notations and letting \(g\) be the genus of \(\mathcal{R}\), the function \(p \mapsto \psi((z, [\gamma]), w, p)\) has at most \(g + \sum_{\nu=1}^{l} \frac{m_\nu(m_\nu - 1)}{2}\) poles.

We note that \(\varpi(\mathcal{R}) = g + \sum_{\nu=1}^{l} \frac{m_\nu(m_\nu - 1)}{2}\) is called the arithmetic genus of the algebraic curve \(\mathcal{R}\). If \(\mathcal{R}\) is non-singular, \(g = \varpi(\mathcal{R})\) holds.

### 1.5 A criterion for single-valued differential operators

From Proposition 1.3 operators \(Q_{(z,[\gamma])}\) of (1.11), which commutes with \(P_z\) of (1.1), can be multivalued on \(\mathbb{C} - \mathcal{N}\). However, they are sometimes single-valued on \(\mathbb{C} - \mathcal{N}\). In this subsection, we give a criterion for such single-valued differential operators by applying the results of the eigenfunction \(\psi\) in the previous subsection.

**Theorem 1.5.** For the differential operators \(P_z\) and \(Q_{(z,[\gamma])}\), assume the condition (1.37). Suppose \(N\) is a prime number. Let \(\varpi(\mathcal{R})\) be the arithmetic genus of \(\mathcal{R}\). If \(\varpi(\mathcal{R}) < N\), every coefficient of \(Q_{(z,[\gamma])}\) is single-valued on \(\mathbb{C} - \mathcal{N}\).

**Proof.** By the assumption (1.37), the eigenvalues \(A(\lambda_1), \cdots, A(\lambda_N)\) are distinct. We have the eigenfunction \(\psi\) of (1.32). Due to Lemma 1.4 we can take the base point \(w\) such that there exist a pole \(q\) of the function \(p \mapsto \psi((z, [\gamma]), w, p)\) which is not a ramification point of the projection \(\pi : \mathcal{R} \to \mathbb{P}^1(\mathbb{C})\).

We assume that
\[
\text{there exists } [\delta_0] \in \pi_1(\mathbb{C} - \mathcal{N}) \text{ such that } \sigma_{[\delta_0]} \neq id.
\]
(1.41)
For generic $X \in \mathbb{P}^1(\mathbb{C}) - \{\infty\}$ where $X$ is not a branch point of $\pi$ and $\pi^{-1}(X)$ consists of $N$ distinct points $p_j = (X, A(\lambda_j))$ ($j = 1, \ldots, N$), there are $k_0, k_1 \in \{1, \ldots, N\}$ such that $k_0 \neq k_1$ and

$$\sigma_{[\delta]}^{-1}(p_{k_0}) = p_{k_1}. \quad (1.42)$$

From (1.28) and (1.29), (1.42) means that $Q_{\gamma, X} \Psi((z, [\gamma \cdot \delta_0]), w, \lambda_{k_0}) = A(\lambda_{k_1}) \Psi((z, [\gamma \cdot \delta_0]), w, \lambda_{k_0})$. Since the eigenvalues of $Q_{\gamma, X}$ are distinct, we obtain

$$\Psi((z, [\gamma \cdot \delta_0]), w, \lambda_{k_0}) = \text{const} \Psi((z, [\gamma]), w, \lambda_{k_1}) \quad (1.43)$$

for generic $(z, [\gamma])$ and $X$. Since $N$ is a prime number, by fixing the branch $\lambda$ of $\sqrt[2]{X}$ and letting $\zeta_N$ be the $N$-th root of the unity, we can suppose that $\lambda_{k_0} = \lambda$ and $\lambda_{k_1} = \zeta_N^l \lambda$ for some $l \in \{0, \ldots, N - 1\}$.

Recalling the form of $\Psi$ of (1.3), the equation (1.43) induces the relation

$$\left(\sum_{s=0}^{\infty} \xi_s((z, [\gamma \cdot \delta_0]), w) e^{\lambda(z-w)}\right) e^{\lambda(z-w)} = \text{const} \left(\sum_{s=0}^{\infty} \xi_s((z, [\gamma]), w) (\zeta_N^l e^\lambda(z-w))\right) e^{(\zeta_N^l e^\lambda)(z-w)} \quad (1.44)$$

for generic $(z, [\gamma])$ and $\lambda$. By substituting $\zeta_N^l \lambda$ for $\lambda$, we have

$$\left(\sum_{s=0}^{\infty} \xi_s((z, [\gamma \cdot \delta_0]), w) (\zeta_N^l \lambda)^{-s} e^{\lambda(z-w)}\right) e^{\lambda(z-w)} = \text{const} \left(\sum_{s=0}^{\infty} \xi_s((z, [\gamma]), w) (\zeta_N^2 e^\lambda(z-w))\right) e^{(\zeta_N^2 e^\lambda)(z-w)}$$

from (1.44). This means that it holds $\Psi((z, [\gamma \cdot \delta_0]), w, \lambda_{k_1}) = \text{const} \Psi((z, [\gamma]), w, \lambda_{k_1})$ for generic $(z, [\gamma])$ and $\lambda$, where $\lambda_2 = \zeta_N^l \lambda$. Setting $p_{k_2} = (X, A(\lambda_2))$, we have $\sigma_{[\delta]}^{-1}(p_{k_0}) = p_{k_2}$ because $\Psi((z, [\gamma \cdot \delta_0]), w, \lambda_{k_0}) = \text{const} \Psi((z, [\gamma]), w, \lambda_{k_2})$ holds. This implies that $\sigma_{[\delta]}^{-1}(p_{k_0}) = p_{k_2}$. Repeating this argument, putting $p_m = (X, A(\lambda_{k_m}))$ for $\lambda_{k_m} = \zeta_N^m \lambda$, we have

$$\sigma_{[\delta]}^{-1}(p_{k_0}) = p_{k_m} \quad (m = 0, \ldots, N - 1). \quad (1.45)$$

Since $N$ is a prime number and $A(\lambda)$ is given by the form (1.12), $p_{k_0}, \ldots, p_{k_{N-1}}$ are distinct and $\pi^{-1}(X) = \{p_{k_0}, \ldots, p_{k_{N-1}}\}$. Namely, (1.45) means that the action of the group $\langle \sigma_{[\delta]} \rangle$, which is generated by $\sigma_{[\delta]}$, is transitive on the fibre $\pi^{-1}(X)$ for generic $X$.

Recalling the eigenfunction $\psi$, (1.45) implies that

$$\psi((z, [\gamma \cdot \delta_0^m]), w, p) = \text{const} \psi((z, [\gamma]), w, \sigma_{[\delta]}^{-1}(p)) \quad (1.46)$$

for $m = 0, \ldots, N - 1$, if $\pi(p)$ is not a branch point of $\pi$. At the beginning of the proof, we took the pole $q$ of the function $p \mapsto \psi((z, [\gamma]), w, p)$ such that $\pi(q)$ is not a branch point. Since the poles of $p \mapsto \psi((z, [\gamma]), w, p)$ do not depend on $(z, [\gamma])$ (see Lemma 1.3 (2)), (1.46) yields that $\sigma_{[\delta]}^{-1}(q)$ for $m \in \{0, \ldots, N - 1\}$ are also poles. So, we have at least $N$ distinct poles of $p \mapsto \psi((z, [\gamma]), w, p)$.

However, due to the assumption, Theorem 1.3 and its corollary, we have at most $\sigma(\mathcal{R})(<N)$ poles of $p \mapsto \psi((z, [\gamma]), w, p)$. This is a contradiction. So, the assumption (1.41) is false. Therefore, $\sigma_{[\delta]} = \text{id}$ for any $[\delta] \in \pi_1(\mathbb{C} - N)$. According to Theorem 1.3 this means that all of the coefficients of $Q_{(z, [\gamma])}$ are single-valued.

\section{Differential equations with the action of the symplectic group}

\subsection{Preliminaries of automorphic forms}

For a commutative algebra $A$, we set $Sp(n, A) = \{\alpha \in \text{GL}(2n, A) | J \alpha J = J\}$, where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. The Siegel upper half plane $\mathbb{H}_n$ is given by $\mathbb{H}_n = \{\Omega \in M_n(\mathbb{C}) | \Omega = \Omega, \text{Im}(\Omega) > 0\}$. Here, $\text{Im}(\Omega) > 0$ means that the imaginary part of $\Omega$ is positive definite. If $n = 1$, $\mathbb{H}_1$ is the ordinary upper half plane.
\[ \mathbb{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \} \]. For \( \alpha \in \text{Sp}(n, \mathbb{R}) \) given by \( \alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), where \( A, B, C, D \in M_n(\mathbb{R}) \), we have the point \( \alpha(\Omega) = (A\Omega + B)(C\Omega + D)^{-1} \in \mathbb{H}_n \). Set \( j(\alpha, \Omega) = \det(C\Omega + D) \). We note that \( j_\alpha(\Omega) \neq 0 \).

To define automorphic forms, we will consider the case that the commutative ring \( A \) is given by a totally real field \( F \) such that \( [F : \mathbb{Q}] = g \). Let \( \varphi_1, \ldots, \varphi_g : F \to \mathbb{R} \) be distinct embeddings. Set \( \mathfrak{a} = \{ \varphi_1, \ldots, \varphi_g \} \). For any \( \alpha \in \text{Sp}(n, F) \), let \( \alpha^{\varphi_j} \) be the matrix whose components are given by the image of the components of \( \alpha \) under \( \varphi_j \). So, \( \mathfrak{a} \) embeds \( \text{Sp}(n, F) \) to \( \text{Sp}(n, \mathbb{R}) \) by \( \alpha \mapsto (\alpha^{\varphi_1}, \ldots, \alpha^{\varphi_g}) \).

From now on, we will identify \( \text{Sp}(n, F) \) with its image in \( \text{Sp}(n, \mathbb{R}) \) via this embedding. Then, for \( \alpha = (\alpha^{\varphi_1}, \ldots, \alpha^{\varphi_g}) \in \text{Sp}(n, F)^g \) and \( \Omega = (\Omega_1, \ldots, \Omega_g) \in \mathbb{H}_n^g \), we set

\[
\alpha(\Omega) = (\alpha^{\varphi_1}(\Omega_1), \ldots, \alpha^{\varphi_g}(\Omega_g)) \in \mathbb{H}_n^g.
\] (2.1)

For any \( \mathbb{C} \)-valued function \( f \) on \( \mathbb{H}_n^g \) and \( K \in \mathbb{Z} \), we set

\[
f|_{[\alpha]_K}(\Omega) = j_\alpha(\Omega)^{-K} f(\alpha(\Omega)),
\] (2.2)

where \( j_\alpha(\Omega) = \prod_{\nu=1}^g j(\alpha^{\varphi_\nu}(\Omega_\nu)) \). Throughout this paper, we use these notations.

Let \( \mathfrak{D}_F \) be the ring of integers of \( F \). For an ideal \( \mathfrak{c} \subset \mathfrak{D}_F \), we set \( \Gamma(\mathfrak{c}) = \{ \alpha \in \text{Sp}(n, \mathfrak{D}_F) | \alpha - I_{2n} \in \mathfrak{c}M(2n, \mathfrak{D}_F) \} \). For a group \( \Gamma \subset \text{Sp}(n, F) \), if there exists an ideal \( \mathfrak{c} \) such that \( \Gamma \) contains \( \Gamma(\mathfrak{c}) \) as a finite index subset, \( \Gamma \) is called a congruence subgroup of \( \text{Sp}(n, F) \).

**Definition 2.1.** Let \( \Gamma \subset \text{Sp}(n, F) \) be a congruence subgroup. If a function \( f \) on \( \mathbb{H}_n^g \) satisfies the following conditions (i), (ii) and (iii), we call \( f \) an automorphic form for \( \Gamma \) of weight \( K \).

(i) \( f \) is holomorphic on \( \mathbb{H}_n^g \).

(ii) \( f \) satisfies \( f|_{[\alpha]_K} = f \) for any \( \alpha \in \Gamma \).

(iii) When \( F = \mathbb{Q} \) and \( n = 1 \), \( f|_{[\alpha]_K}(\Omega) \) has a holomorphic Fourier expansion at cusps for any \( \alpha \in \text{SL}(2, \mathbb{Z}) \). Namely, \( f|_{[\alpha]_K}(\Omega) = \sum_{k=0}^{\infty} f_{\alpha, k} \exp \left( \frac{2\pi \sqrt{-1} k \Omega}{N_\alpha} \right) \), holds, where \( f_{\alpha, k} \in \mathbb{C} \) and \( N_\alpha \in \mathbb{Z}_{>0} \).

Here, ‘holomorphic’ means that the Fourier expansion does not have any terms for \( k < 0 \).

**Remark 2.1.** The case of \( F = \mathbb{Q} \) and \( n = 1 \) is an exceptional case. The condition (iii) is a growth condition for the cusps of \( \Gamma \). When \( F \neq \mathbb{Q} \) or \( n \geq 2 \), such a condition follows from the conditions (i) and (ii) (Koecher’s principle).

We note that automorphic forms of several variables are defined in various literature. Our definition above of automorphic forms is due to \( \text{Sm2} \). This definition seems general enough for applications because we can obtain important modular functions as reductions. For example, if \( \Gamma = \text{Sp}(n, \mathbb{Q}) \), then the corresponding automorphic forms are well-known Siegel modular forms. If \( F \neq \mathbb{Q} \) and \( n = 1 \), then the corresponding automorphic forms are Hilbert modular forms.

### 2.2 Differential operators with coefficients satisfying a transformation law

Let \( a_l(\Omega, z) \ (l = 2, \ldots, N) \) be a function of \( \Omega = (\Omega_1, \ldots, \Omega_g) \in \mathbb{H}_n^g \) and \( z \in \mathbb{C} \). We suppose that \( \Omega \to a_l(\Omega, z) \) is holomorphic for generic \( z \). Moreover, for fixed \( \Omega \), let \( z \to a_l(\Omega, z) \) be an analytic function with at most poles. We consider the cases that \( a_l(\Omega, z) \) satisfies the transformation law

\[
a_l \left( \alpha(\Omega), \frac{z}{j_\alpha(\Omega)} \right) = j_\alpha(\Omega)^l a_l(\Omega, z),
\] (2.3)

for \( \alpha \in \Gamma \). For fixed \( \Omega \in \mathbb{H}_n^g \), let \( \mathcal{N}_\Omega \subset \mathbb{C} = (z\text{-plane}) \) be the union of the sets of poles of the function \( a_j(\Omega, z) \ (j = 2, \ldots, n) \). Namely, for a fixed \( \Omega \in \mathbb{H}_n^g \), \( a_2(\Omega, z), \ldots, a_N(\Omega, z) \) are holomorphic functions of \( z \in \mathbb{C} - \mathcal{N}_\Omega \).
Remark 2.2. If \( n = 1 \), the action \((\Omega, z) \mapsto (\alpha(\Omega), \frac{z}{j_n(\Omega)})\) is equal to the action which defines the Jacobi forms of degree 1 (see [EZ]). However, if \( n \geq 2 \), our action is different from the action for Jacobi forms of higher degrees studied in [Z].

Lemma 2.1. For any \( \alpha \in \Gamma \) and \( \Omega \in \mathbb{H}_n^g \), \( z \in \mathbb{C} - N_\Omega \) if and only if \( \frac{z}{j_\alpha(\Omega)} \in \mathbb{C} - N_\alpha(\Omega) \).

Proof. Due to the transformation law \([2.3]\), it holds that

\[
\begin{align*}
  z \in \mathbb{C} - N_\Omega &\iff a_l(\Omega, z) \neq \infty \quad (l = 2, \ldots, N) \\
  &\iff a_l(\alpha(\Omega), \frac{z}{j_\alpha(\Omega)}) = j_\alpha(\Omega)^k a_l(\Omega, z) \neq \infty. \\
  &\iff \frac{z}{j_\alpha(\Omega)} \in \mathbb{C} - N_\alpha(\Omega).
\end{align*}
\]

Let \( X_\Omega \) be the universal covering of \( \mathbb{C} - N_\Omega \). For a fixed point \( w \in \mathbb{C} - N_\Omega \), any \( s \in X_\Omega \) is represented by \( s = (z, [\gamma]) \), where \( z \in \mathbb{C} - N_\Omega \), \( \gamma \) is an arc in \( \mathbb{C} - N_\Omega \) from \( w \) to \( z \) and \([\gamma]\) is the homotopy class of \( \gamma \). We note that \( z \) gives a local coordinate of \( X_\Omega \).

Let us consider the following ordinary differential operator of the independent variable \( z \):

\[
P_{\Omega, z} = \frac{\partial^N}{\partial z^N} + a_2(\Omega, z) \frac{\partial^{N-2}}{\partial z^{N-2}} + a_3(\Omega, z) \frac{\partial^{N-3}}{\partial z^{N-3}} + \cdots + a_N(\Omega, z).
\]

Set \((\Omega_1, z_1) = (\alpha(\Omega), \frac{z}{j_\alpha(\Omega)})\). Throughout this paper, we assume that \( \Omega \mapsto a_l(\Omega, z) \) are holomorphic for generic \( z \). Since \( \frac{\partial}{\partial z} = j_\alpha(\Omega) \frac{\partial}{\partial z} \) and \( a_l(\Omega_1, z_1) = j_\alpha(\Omega)^l a_l(\Omega, z) \), we have

\[
P_{\Omega_1, z_1} = j_\alpha(\Omega)^N P_{\Omega, z}.
\]

Definition 2.2. Let \( D_{\Omega, z} \) be a linear differential operator of \( z \) holomorphically parametrized by \( \Omega \in \mathbb{H}_n^g \). If

\[
D_{\Omega, z} = j_\alpha(\Omega)^K D_{\Omega, z}
\]

holds for \( \alpha \in \Gamma \), we call \( D_{\Omega, z} \) a differential operator of weight \( K \) with respect to the action of \( \Gamma \). We call \( D_{\Omega, z} u = 0 \) a linear differential equation of weight \( K \) with respect to the action of \( \Gamma \).

There exist important examples which satisfy the transformation law \([2.3]\).

Example 2.1. Let \( \Gamma \) be a congruence subgroup of \( SL(n, F) \). If \( f(\Omega) \) be an automorphic form of weight \( j \), then \( a_{j+k}(\Omega, z) = z^{-k} f(\Omega) \) satisfies the transformation law \([2.3]\) for \( l = j + k \) for any \( k \in \mathbb{Z}_{\geq 0} \). If \( k > 0 \), then \( N_\Omega \) = \{0\} holds.

Example 2.2. For a congruence subgroup \( \Gamma \subset SL(2, \mathbb{Z}) \), a weak Jacobi form \( \mathbb{H} \times \mathbb{C} \ni (\Omega, z) \mapsto f(\Omega, z) \in \mathbb{C} \) for \( \Gamma \subset SL(2, \mathbb{Z}) \) of weight \( K \) and level \( m \) is a holomorphic function with the following properties

(i) for any \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), \( f(\alpha(\Omega), \frac{z}{j_\alpha(\Omega)}) = j_\alpha(\Omega)^K \exp(-2\pi i \frac{m c z^2}{j_\alpha(\Omega)}) f(\Omega, z) \),

(ii) for any \( n_1, n_2 \in \mathbb{Z} \), \( f(\Omega, z + n_1 \Omega + n_2) = \exp(-\pi i (n_1^2 \Omega + 2n_2 z)) f(\Omega, z) \),

(iii) \( f \) has a Fourier expansion \( f(\Omega, z) = \sum_{n, l \in \mathbb{Z}} c_{n, l} \exp\left(\frac{2\pi \sqrt{-1} n \Omega}{N}\right) \exp(2\pi \sqrt{-1} n z) \) for some \( N \in \mathbb{Z} \).

Weak Jacobi forms are very important in number theory (see [EZ]). If \( f(\Omega, z) \) \((g(\Omega, z), \text{resp.})\) is a weak Jacobi form for \( \Gamma \) of weight \( K_1 \) \((K_2, \text{resp.})\) and level \( m \), then \( a_1(\Omega, z) = \frac{f(\Omega, z)}{g(\Omega, z)} \) satisfies the transformation law of \([2.3]\) for \( n = g = 1 \) and \( l = K_1 - K_2 \).
Example 2.3. As a special case of (2.2), we consider the Lamé differential operator

\[ P_{\Omega, z} = \frac{\partial^2}{\partial z^2} - B \varphi(\Omega, z), \]  

where \( \Omega \in \mathbb{H} \) and \( \varphi(\Omega, z) \) is the Weierstrass \( \varphi \)-function

\[ \varphi(\Omega, z) = \frac{1}{z^2} + \sum_{(n_1, n_2) \in \mathbb{Z}^2 - \{(0, 0)\}} \left( \frac{1}{(z - n_1 - n_2\Omega)^2} - \frac{1}{(n_1 + n_2\Omega)^2} \right). \]

We note that \( z \mapsto \varphi(\Omega, z) \) has poles of degree 2 at every \( z_0 \in \mathcal{N}_\Omega := \mathbb{Z} + \mathbb{Z}\Omega \).

Let \( \Gamma \) be the elliptic full-modular group \( SL(2, \mathbb{Z}) \). For any \( \alpha \in \Gamma \), we have

\[ \varphi\left( \alpha(\Omega), \frac{z}{j_\alpha(\Omega)} \right) = j_\alpha(\Omega)^2 \varphi(\Omega, z). \]

Especially, \( \varphi(\Omega + 1, z) = \varphi(\Omega, z) \) holds and \( \varphi \) has the Fourier expansion at cusps:

\[ \varphi(\Omega, z) = \pi^2 \left( \frac{1}{6} - 4 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \right) + \frac{\pi^2}{\sin^2(\pi z)} - 8\pi^2 \sum_{n=1}^{\infty} \cos(2n\pi z) \frac{nq^{2n}}{1 - q^{2n}}. \]

where \( q = \exp(2\pi \sqrt{-1} \Omega) \) (for detail, see [EMOF]). Therefore, in terms of Definition (2.2), \( P_{\Omega, z} \) of (2.7) is a differential operator of weight 2 with respect to the action of \( \Gamma = SL(2, \mathbb{Z}) \).

When \( \gamma : [0, 1] \to \mathbb{C} - \mathcal{N}_\Omega \) is an arc with \( \gamma(0) = w \) and \( \gamma(1) = z \), let \( \gamma_1 = \frac{z}{j_\alpha(\Omega)} \) be the arc given by \( \gamma_1(t) = \frac{\gamma(t)}{j_\alpha(\Omega)} \). By virtue of Lemma 2.1, \( \gamma_1 \) is an arc in \( \mathbb{C} - \mathcal{N}_\alpha(\Omega) \).

Theorem 2.1. Let \( P_{\Omega, z} \) be the differential operator of (2.4).

1. There exists the unique formal solution \( \Psi(\Omega, (z, [\gamma]), w, \lambda) \) of the differential equation

\[ P_{\Omega, z} u = \lambda^N u \]  

in the form

\[ \Psi(\Omega, (z, [\gamma]), w, \lambda) = \left( \sum_{s=0}^{\infty} \xi_s(\Omega, (z, [\gamma]), w)\lambda^{-s} \right)e^{\lambda(z-w)} \]  

such that

\[ \begin{cases} 
\xi_0(\Omega, (z, [\gamma]), w) = 1, \\
\xi_s(\Omega, (w, [id]), w) = 0 & (s \geq 1). 
\end{cases} \]  

(2) For any \( \alpha \in \Gamma \), it holds

\[ \Psi\left( \alpha(\Omega), \left( \frac{z}{j_\alpha(\Omega)}, \left[ \frac{\gamma}{j_\alpha(\Omega)} \right] \right), \frac{w}{j_\alpha(\Omega)}, j_\alpha(\Omega)\lambda \right) = \Psi(\Omega, (z, [\gamma]), w, \lambda). \]  

The function \( \xi_s(\Omega, (z, [\gamma]), w) \) in (2.7) satisfies the transformation law

\[ \xi_s\left( \alpha(\Omega), \left( \frac{z}{j_\alpha(\Omega)}, \left[ \frac{\gamma}{j_\alpha(\Omega)} \right] \right), \frac{w}{j_\alpha(\Omega)} \right) = j_\alpha(\Omega)^s \xi_s(\Omega, (z, [\gamma]), w). \]

Proof. (1) For fixed \( \Omega \in \mathbb{H}_g \), putting \( u = \left( \sum_{s=0}^{\infty} \eta_s(\Omega, z)\lambda^{-s} \right)e^{\lambda(z-w)} \) to (2.8), by the same argument as in the proof of Proposition 1.1, we can obtain

\[ N \frac{\partial}{\partial z} \eta_{N+s_0-1}(\Omega, z) = \left( \text{a polynomial in } \frac{\partial^\nu}{\partial z^\nu} \eta(\Omega, z) \right) \]  

\( (l < N + s_0 - 1, \nu \in \mathbb{Z}_{\geq 0}) \) and \( a_j(\Omega, z) \) defined over \( \mathbb{Z} \).
for any $s_0$. By the integration of the relation (2.13) on arc $\gamma \in \mathbb{C} - \mathcal{N}_\Omega$ whose start point is $w$, we can obtain the expression of $\eta_x(\Omega, z)$ in terms of $\eta_x(\Omega, z)$ ($\nu < \mu$) and $a_1(\Omega, z)$. From the conditions that $\eta_0 \equiv 1$ and $\eta_x(\Omega, (w, [id])) = 0$ ($s \geq 1$), we can determines the sequence $\{\eta_x(\Omega, z)\}_{s}$ uniquely. Such $\eta_x(\Omega, (z, [\gamma]))$ give the required functions $\xi_x(\Omega, (z, [\gamma]), w)$.

Moreover, since the coefficients $a_1(\Omega, z)$ of (2.4) are holomorphic functions of $\Omega$ and $\xi_x(\Omega, (z, [\gamma]), w)$ are determined by the construction via (2.13), $\xi_x(\Omega, (z, [\gamma]), w)$ are holomorphic functions of $\Omega \in \mathbb{H}_s^o$ for generic $(z, [\gamma]), w)$. Also, for fixed $\Omega$, $\xi_x(\Omega, (z, [\gamma]), w)$ are locally holomorphic functions of $(z, [\gamma]), w) \in \mathfrak{X}_\Omega \times (\mathbb{C} - \mathcal{N}_\Omega)$.

(2) We consider the transformation

$$(\Omega, (z, [\gamma]), w, \lambda) \mapsto (\Omega_1, (z_1, [\gamma_1]), w_1, \lambda_1) = \left(\alpha(\Omega), \left(\frac{z}{j_\alpha(\Omega)}, \left[\frac{\gamma}{j_\alpha(\Omega)}\right], \frac{w}{j_\alpha(\Omega)}, j_\alpha(\Omega)\lambda\right)\right). \quad (2.14)$$

By virtue of (2.13), the differential equation $P_{\Omega_1, z_1} u = \lambda^N u$ gives the same equation with $P_{\Omega, z} u = \lambda^N u$ under the correspondence (2.14). Since we have the uniqueness of the solution $\Psi$ in the form of (2.9) and the condition (2.11), we obtain (2.11). Then, we have

$$(\sum_{s=0}^{\infty} \xi_x(\Omega, (z, [\gamma]), w)\lambda^{-s}) e^{\lambda(z-w)} = (\sum_{s=0}^{\infty} \xi_x(\Omega, (z_1, [\gamma_1]), w_1)\lambda_1^{-s}) e^{\lambda_1(z_1-w_1)}. \quad (2.15)$$

By cancelling $e^{\lambda(z-w)} = e^{\lambda_1(z_1-w_1)}$ and comparing the coefficient of $\lambda^{-s} = \lambda_1^{-s}j_\alpha(\Omega)^s$, we have the transformation law (2.12).

### 2.3 Commutative differential operators with an action of a symplectic group (generic cases of $F \neq \mathbb{Q}$ or $n \geq 2$)

We consider the differential operator

$$Q_{\Omega,(z,[\gamma])} = b_0(\Omega, (z, [\gamma])) \frac{\partial^M}{\partial z^M} + b_1(\Omega, (z, [\gamma])) \frac{\partial^{M-1}}{\partial z^{M-1}} + \cdots + b_M(\Omega, (z, [\gamma])), \quad (2.16)$$

which commutes with the differential operator $P_{\Omega, z}$ of (2.4). Here, we assume that the coefficients $b_k(\Omega, (z, [\gamma]))$ ($k = 0, \cdots, M$) are locally analytic functions of $(z, [\gamma]) \in \mathfrak{X}_\Omega$.

**Theorem 2.2.** (1) Let $P_{\Omega, z}$ ($Q_{\Omega,(z,[\gamma])}$, resp.) be the differential operator of (2.4) (2.10), resp.) Then, $P_{\Omega, z}$ and $Q_{\Omega,(z,[\gamma])}$ are commutative if and only if the quotient $Q_{\Omega,(z,[\gamma])} \Psi(\Omega, (z, [\gamma]), w, \lambda)$ for $\Psi$ of (2.9) coincides with

$$A(\Omega, \lambda) = \sum_{s=-\infty}^{\infty} A_s(\Omega)\lambda^{-s} \quad (2.17)$$

for generic $\lambda$, where $A(\Omega, \lambda)$ does not depend on the variables $z$ and $w$.

(2) If $P_{\Omega, z}$ commutes with both $Q_{\Omega,(z,[\gamma])}^{(1)}$ and $Q_{\Omega,(z,[\gamma])}^{(2)}$, then $Q_{\Omega,(z,[\gamma])}^{(1)}$ commutes with $Q_{\Omega,(z,[\gamma])}^{(2)}$.

(3) If the differential operator $Q_{\Omega,(z,[\gamma])}$ is of weight $K$ with respect to the action of $\Gamma$, then the members of the sequence $\{A_s(\Omega)\}_{s}$ satisfy

$$A_s(\alpha(\Omega)) = j_\alpha(\Omega)^{K+s} A_s(\Omega). \quad (2.18)$$

**Proof.** (1) (2) These are proved by a similar argument to the proof of Proposition 1.3 and Proposition 1.4.

(3) We recall that $\Psi$ in (2.9) satisfies (2.11). Since $Q_{\Omega,(z,[\gamma])}$ is of weight $K$, we have $A(\Omega_1, \lambda_1) = j_\alpha(\Omega)^K A(\Omega, \lambda)$. Namely, we have

$$\sum_{s=-\infty}^{\infty} A_s(\Omega_1)\lambda_1^{-s} = j_\alpha(\Omega)^K \sum_{s=-\infty}^{\infty} A_s(\Omega)\lambda^{-s}. \quad (2.19)$$

By comparing the coefficients of $\lambda^{-s}$, the assertion follows. \qed
Theorem 2.3. For any \( j \in \{0, -1, \cdots, -M\} \), let \( A_j(\Omega) \) satisfy the transformation law
\[
A_j(\Omega) = j_\alpha(\Omega)^{K+j} A_j(\Omega) \tag{2.19}
\]
for any \( \alpha \in \Gamma \). If there exists a differential operator \( Q_{\Omega(z, [\gamma])} \) of rank \( M \) of weight \( K \) with respect to \( \Gamma \) satisfying
\[
Q_{\Omega(z, [\gamma])} \Psi(\Omega, (z, [\gamma]), w, \lambda) = A(\Omega, \lambda) \Psi(\Omega, (z, [\gamma]), w, \lambda), \tag{2.20}
\]
where \( A(\Omega, \lambda) = \sum_{s=-M}^\infty A_s(\Omega) \lambda^s \), then \( Q_{\Omega(z, [\gamma])} \) is uniquely determined only by given operator \( P_{\Omega,z} \) of (2.4) and the functions \( A_j(\Omega) \) \((j = 0, \cdots, -M)\). Here, \( A_s(\Omega) \) \((s \geq 1)\) are also uniquely determined only by \( P_{\Omega,z} \) and \( A_j(\Omega) \) \((j = 0, \cdots, -M)\).

Proof. Let \( \Psi(\Omega, (z, [\gamma]), w, \lambda) \) be the solution of (2.4) for the differential equation \( P_{\Omega,z} u = X u \), where \( X = \lambda^N \). Let \( \{A_s(\Omega)\}_s \) be the sequence satisfying the relation (2.18) and set \( A(\Omega, \lambda) = \sum_{s=-M}^\infty A_s(\Omega) \lambda^s \).

If there exists a differential operator \( Q_{\Omega(z, [\gamma])} \) satisfying (2.20), then \( Q_{\Omega(z, [\gamma])} \) is a differential operator of weight \( K \) with respect to \( \Gamma \) and commutes with \( P_{\Omega,z} \). Next, taking \( w' \in \mathbb{C} - N_\Omega \) and another solution \( \Psi(\Omega, (z, [\gamma]), w', \lambda) \), we suppose that there is an operator \( Q_{\Omega(z, [\gamma]),w'} \) such that
\[
Q_{\Omega(z, [\gamma]),w'} \Psi(\Omega, (z, [\gamma]), w', \lambda) = A(\Omega, \lambda) \Psi(\Omega, (z, [\gamma]), w', \lambda). \tag{2.21}
\]
As in Proposition 1.2 and Proposition 1.3 there exists \( B(\Omega, \lambda) \) such that \( \Psi(\Omega, (z, [\gamma]), w', \lambda) e^{\lambda(w' - w)} = B(\Omega, \lambda) \Psi(\Omega, (z, [\gamma]), w, \lambda) \). Therefore, by (2.20) and (2.21),
\[
\frac{Q_{\Omega(z, [\gamma]),w'} \Psi(\Omega, (z, [\gamma]), w', \lambda)}{\Psi(\Omega, (z, [\gamma]), w, \lambda)} = \frac{Q_{\Omega(z, [\gamma]),w} \Psi(\Omega, (z, [\gamma]), w', \lambda)}{\Psi(\Omega, (z, [\gamma]), w, \lambda)} = A(\Omega, \lambda) \frac{Q_{\Omega(z, [\gamma]),w} \Psi(\Omega, (z, [\gamma]), w, \lambda)}{\Psi(\Omega, (z, [\gamma]), w, \lambda)}.
\]
Therefore, we obtain
\[
(Q_{\Omega(z, [\gamma]),w} - Q_{\Omega(z, [\gamma]),w'}) \Psi(\Omega, (z, [\gamma]), w, \lambda) = 0. \]
By a similar argument to the end of the proof of Proposition 1.3, we can see that \( Q_{\Omega(z, [\gamma]),w} = Q_{\Omega(z, [\gamma]),w'} \). Hence, a differential operator \( Q_{\Omega(z, [\gamma]),w} \) satisfying (2.20) does not depend on the base point \( w \). So, we use the notation \( Q_{\Omega(z, [\gamma])} \) instead of \( Q_{\Omega(z, [\gamma]),w} \).

Now, we see that the differential operator \( Q_{\Omega(z, [\gamma])} \) and the series \( A(\Omega, \lambda) \) satisfying (2.20) are uniquely determined by \( P_{\Omega,z} \) and \( A_0(\Omega), \cdots, A_{-M}(\Omega) \). The relation (2.20) is equal to
\[
\sum_{s=0}^M \sum_{k=0}^s \sum_{\alpha=0}^k \binom{k}{\alpha} b_{M-k}(\Omega, (z, [\gamma])) \frac{\partial^\alpha}{\partial z^\alpha} \xi_s(\Omega, (z, [\gamma]), w) \lambda^{k-\alpha-s} = \left( \sum_{s=0}^\infty A_s(\Omega) \lambda^s \right) \left( \sum_{s=0}^\infty \xi_s(\Omega, (z, [\gamma]), w) \lambda^{-s} \right). \tag{2.22}
\]
We note that \( \{\xi_s\}_s \) is determined only by the given differential operator \( P_{\Omega,z} \) by Theorem 2.1 (1). For any \( j \in \{0, 1, \cdots, M\} \), recalling that \( \xi_0 \equiv 1 \) and taking the coefficients of the term for \( \lambda^{M-j} \) in the equation (2.22), we have
\[
b_j(\Omega, (z, [\gamma])) = \left( \text{a polynomial in } b_k(\Omega, (z, [\gamma])) \ (k \leq j - 1) \text{ and } \frac{\partial^{\nu'}}{\partial z^{\nu'}} \xi_s(\Omega, (z, [\gamma]), w) \right) \ (\nu \in \mathbb{Z}_{\geq 0})
\]
\[
= \left( \text{a polynomial in } A_t(\Omega) \ (t \leq j - M) \text{ and } \xi_s(\Omega, (z, [\gamma]), w) \right). \tag{2.23}
\]
From (2.22), we can obtain \( b_j(\Omega, (z, [\gamma])) \) \((j = 0, 1, \cdots, M)\) inductively. This shows that \( A_{-M}(\Omega), \cdots, A_0(M) \) and the differential operator \( P_{\Omega,z} \) uniquely determine \( Q_{\Omega(z, [\gamma])} \). Moreover, since \( \xi_0 \equiv 1 \) again, from the
coefficients of $\lambda^{-s}$ in the relation (2.22), we have
\[
A_s(\Omega) + \left( \text{a polynomial in } A_t(\Omega) \ (t < s) \text{ and } \frac{\partial^\nu}{\partial z^\nu} \xi_s(\Omega, (z, [\gamma]), w) \ (\nu \in \mathbb{Z}_{\geq 0}) \right)
\]
\[
= \left( \text{a polynomial in } b_k(\Omega, (z, [\gamma])) \ (0 \leq k \leq M) \text{ and } \xi_s(\Omega, (z, [\gamma]), w) \right),
\]
for any $s \geq -M$. Especially, $A_s(\Omega) \ (s \geq 1)$ are inductively determined by (2.23). Here, we note that such $A_s(\Omega) \ (s \geq 1)$ do not depend on $z$ and $w$ by virtue of Theorem 2.2 (1). Therefore, the assertion follows.

The following theorem gives a correspondence between automorphic forms and differential operators which commutes with $P_{\Omega,z}$ for generic cases of $F \neq \mathbb{Q}$ or $n \geq 2$.

**Theorem 2.4.** Suppose $F \neq \mathbb{Q}$ or $n \geq 2$. Let $P_{\Omega,z}$ of (2.14) \( (Q_{\Omega,(z,[\gamma])}) \) of (2.16), resp.) be differential operators studied in Theorem 2.2.

(1) If $A_j(\Omega) \ (j = 0, \cdots, -M)$ are automorphic forms of weight $K + j$ for $\Gamma$, then any coefficients $b_j(\Omega, (z, [\gamma]))$ of $Q_{\Omega,(z,[\gamma])}$ \( A_s(\Omega) \ (s \geq 1) \) of $A(\Omega, \lambda)$, resp.), which is derived from $P_{\Omega,z}$ and $A_j(\Omega) \ (j = 0, -1, \cdots, -M)$, give holomorphic functions $\Omega \mapsto b_j(\Omega, (z, [\gamma]))$ for generic $(z, [\gamma])$ (automorphic forms of weight $K + s$ for $\Gamma$, resp.).

(2) Conversely, if every coefficient $b_j(\Omega, (z, [\gamma]))$ of $Q_{\Omega,(z,[\gamma])}$ gives a holomorphic function $\Omega \mapsto b_j(\Omega, (z, [\gamma]))$ for generic $(z, [\gamma])$, then $A_s(\Omega) \ (s \geq -M)$, which are determined in the sense of Theorem 2.2, are automorphic forms for $\Gamma$.

**Proof.** (1) Recall Definition 2.1. From the assumption, $A_j(\Omega) \ (k \in \{0, \cdots, -M\})$ are holomorphic function of $\Omega \in \mathbb{H}_n^\alpha$ satisfying the transformation law (2.19). From Theorem 2.1 \( \xi_s(\Omega, (z, [\gamma]), w) \) are holomorphic of $\Omega \in \mathbb{H}_n^\alpha$ for generic $(z, [\gamma], w)$. So, due to the construction of $b_j(\Omega, (z, [\gamma]))$ via the relation (2.22), $b_j(\Omega, (z, [\gamma]))$ are holomorphic of $\Omega$ for generic $(z, [\gamma])$. Also, from (2.23), $A_s(\Omega) \ (s \geq 1)$ are also holomorphic in $\Omega$. Moreover, by Theorem 2.2, we obtain $A_s(\Omega) \ (s \geq 1)$ satisfying the transformation law (2.13). So, from Definition 2.1, $A_s(\Omega) \ (s \geq 1)$ are automorphic forms for $\Gamma$ of weight $K + s$.

(2) From Theorem 2.2 and Theorem 2.4, we only need to see that $\Omega \mapsto A_s(\Omega)$ are holomorphic under our assumption. However, we can see this property, because $A_0(\Omega), \cdots, A_{-M}(\Omega)$ are determined by $b_j(\Omega, (z, [\gamma]))$ via (2.24) and do not depend on $(z, [\gamma])$ and $w$. \( \square \)

### 2.4 Commutative differential operators with an action of a symplectic group (exceptional cases of $F = \mathbb{Q}$ and $n = 1$)

In this subsection, we consider exceptional cases of $F = \mathbb{Q}$ and $n = 1$ carefully. In such cases, we need to consider the action of $\alpha \in SL(2, \mathbb{Z})$, because automorphic forms for such exceptional cases require holomorphic Fourier expansion at cusps in the sense of Definition 2.1 (iii).

Recall that the set of poles of $P_{\Omega,z}$ of (2.1) is given by $N_\Omega$. If the coefficients $a_l(\Omega, z) \ (l = 2, \cdots, N)$ of $P_{\Omega,z}$ satisfy the transformation law (2.3) for $\alpha \in SL(2, \mathbb{Z})$, we have $N_\Omega = N_{\alpha(\Omega)}$ for any $\alpha \in SL(2, \mathbb{Z})$ by virtue of Lemma 2.1. However, the transformation law (2.3) for $\alpha \in SL(2, \mathbb{Z})$ generically does not hold. So, we need a bit delicate argument for holomorphic Fourier expansions at cusps. Let $j_\alpha(\Omega) \cdot (C - N_{\alpha(\Omega)})$ be the set $\{j_\alpha(\Omega)z \mid z \in C - N_{\alpha(\Omega)}\}$. If an arc $\gamma$ is in $j_\alpha(\Omega) \cdot (C - N_{\alpha(\Omega)})$, then $\gamma_1 = \frac{z}{j_\alpha(\Omega)}$ is in $C - N_{\alpha(\Omega)}$. We have the following lemma.

**Lemma 2.2.** Suppose $F = \mathbb{Q}$ and $n = 1$. For any $\alpha \in SL(2, \mathbb{Z})$, we suppose that the coefficients $a_l(\Omega, z)$ \( l = 2, \cdots, N \) have the holomorphic Fourier expansion at cusps:

\[
j_\alpha(\Omega)^{-1} a_l \left( \alpha(\Omega), \frac{z}{j_\alpha(\Omega)} \right) = \sum_{k \geq 0} \hat{a}_{l,a,k}(\xi) \exp \left( \frac{2\pi i k}{\xi} \right).
\]
\[
(2.25)
\]
where $\tilde{a}_{1,0,k}(z)$ are holomorphic functions of $z \in \mathbb{C} - \mathcal{N}_{\alpha}(\Omega)$ and $N_{1,k} \in \mathbb{Z}_{>0}$. Here ‘holomorphic’ means that the expression (2.27) does not contain any terms for $k < 0$. Then, the coefficients $\xi_s$ of the multivalued Baker-Akhiezer function $\Psi$ of (2.22) has a holomorphic Fourier expansion at cusps:

$$j_\alpha(\Omega)^{-s} \xi_s \left( \alpha(\Omega), \left( \frac{z}{j_\alpha(\Omega)}, \left[ \frac{\gamma}{j_\alpha(\Omega)} \right] \right), \frac{w}{j_\alpha(\Omega)} \right) = \sum_{k \geq 0} \xi_{s,\alpha,k} ((z, [\gamma]), w) \exp \left( \frac{2\pi \sqrt{-1} k \Omega}{N_{s,\alpha}} \right), \quad (2.26)$$

where $\xi_{s,\alpha,k} ((z, [\gamma]), w)$ are multivalued function on $j_\alpha(\Omega) \cdot (\mathbb{C} - \mathcal{N}_{\alpha}(\Omega))$.

Proof. For $\alpha \in SL(2, \mathbb{Z})$, we set $(\Omega_1, (z_1, [\gamma_1]), w_1) = \left( \alpha(\Omega), \left( \frac{z}{j_\alpha(\Omega)}, \left[ \frac{\gamma}{j_\alpha(\Omega)} \right] \right), \frac{w}{j_\alpha(\Omega)} \right)$. We prove the existence of the holomorphic Fourier expansions (2.26) of $\xi_s$ by a induction for $s$.

If $s = 0$, it is trivial. If $s = 1$, $\xi_1(\Omega_1, (z_1, [\gamma_1]), w_1)$ is determined by the integration of the relation

$$N \frac{\partial}{\partial z_1} \eta_1(\Omega_1, z_1) = -a_2(\Omega_1, z_1) \quad (2.27)$$
on the arc $\gamma_1 \subset \mathbb{C} - \mathcal{N}_{\Omega_1}$ (recall the proof of Theorem 2.1). Dividing (2.27) by $j_\alpha(\Omega)^2$, considering the relation $\frac{\partial}{\partial z_1} j_\alpha(\Omega) = j_\alpha(\Omega) \frac{\partial}{\partial z_1}$ and using the assumption (2.26), we have

$$N \frac{\partial}{\partial z_1} j_\alpha(\Omega)^{-1} \eta_1(\alpha(\Omega), \frac{z}{j_\alpha(\Omega)}) = \frac{-a_2(\Omega_1, z_1)}{j_\alpha(\Omega)^2} = - \sum_{k \geq 0} \tilde{a}_{2,\alpha,k}(z) \exp \left( \frac{2\pi \sqrt{-1} k \Omega}{N_{2,\alpha}} \right). \quad (2.28)$$

By integrating (2.28) on the arc $\gamma \subset j_\alpha(\Omega) \cdot (\mathbb{C} - \mathcal{N}_{\alpha}(\Omega))$, we have the holomorphic Fourier expansion at cusps for $\xi_1$ of (2.26).

Next, assume that we have the holomorphic Fourier expansion (2.26) of $\xi_s$ for $s = 0, 1, \ldots, s_0 - 1$ ($s_0 \geq 1$). We will obtain the holomorphic Fourier expansion of $\xi_{s_0}$. By the proof of Theorem 2.1 (1), especially the relation (2.13), $\xi_{s_0}(\Omega_1, (z_1, [\gamma_1]), w_1)$ is given by the integration of the relation

$$N \frac{\partial}{\partial z_1} \eta_{s_0}(\Omega_1, z_1) = H_{s_0} \left( \frac{\partial^\nu}{\partial z_1^\nu} \xi_m(\Omega_1, (z_1, [\gamma_1]), w_1), a_l(\Omega_1, z_1) \right) \quad (2.29)$$

Here, $H_{s_0} \left( \frac{\partial^\nu}{\partial z_1^\nu} \xi_m(\Omega_1, (z_1, [\gamma_1]), w_1), a_l(\Omega_1, z_1) \right)$ is a polynomial in $\frac{\partial^\nu}{\partial z_1^\nu} \xi_m(\Omega_1, (z_1, [\gamma_1]), w_1)$ ($m \leq s_0 - 1, \nu \in \mathbb{Z}_{\geq 0}$) and $a_l(\Omega_1, z_1)$ ($l = 2, \ldots, N$). By Theorem 2.1 (3), the polynomial $H_{s_0}$ is homogeneous of weight $s_0 + 1$ with respect to the action of $\Gamma$. This implies that, by dividing (2.29) by $j_\alpha(\Omega)^{s_0 + 1}$ for $\alpha \in SL(2, \mathbb{Z})$, the relation

$$N \frac{\partial}{\partial z} j_\alpha(\Omega)^{-s_0} \eta_{s_0}(\alpha(\Omega), \frac{z}{j_\alpha(\Omega)}) = H_{s_0} \left( \frac{\partial^\nu}{\partial z^\nu} j_\alpha(\Omega)^{-m} \xi_m(\Omega_1, (z_1, [\gamma_1]), w_1), j_\alpha(\Omega)^{-l} a_l(\Omega, z) \right), \quad (2.30)$$

holds similarly to the (2.26). By the assumption, the right hand side of (2.30) has the holomorphic Fourier expansion at cusps. So, by the integration of (2.30) on the arc $\gamma \subset j_\alpha(\Omega) \cdot (\mathbb{C} - \mathcal{N}_{\alpha}(\Omega))$, we have the holomorphic Fourier expansion (2.26) at cusps for $s_0$.

Hence, the assertion is proved.

\[ \square \]

Remark 2.3. If $\Gamma = SL(2, \mathbb{Z})$, the relation (2.3) holds for any $\alpha \in SL(2, \mathbb{Z})$. Then, from Lemma 2.4, $j_\alpha(\Omega) \cdot (\mathbb{C} - \mathcal{N}_{\alpha}(\Omega)) = (\mathbb{C} - \mathcal{N}_1)$ holds. So, in this case, we only need to consider multivalued functions on $\mathbb{C} - \mathcal{N}_1$. However, if $\Gamma \neq SL(2, \mathbb{Z})$, we need a detailed condition as we saw in Lemma 2.2.

Theorem 2.5. Suppose $F = \mathbb{Q}$ and $n = 1$. Let $P_{1,2}$ of (2.4) $(Q_{\Omega, (\Omega, [\gamma])})$ (2.16), resp.) be differential operators studied in Theorem 2.3. Moreover, assume that every coefficient $a_l(\Omega, z)$ ($l = 2, \ldots, N$) of the differential operator $P_{1,2}$ of (2.3) has a holomorphic Fourier expansion at cusps in the form (2.22).

(1) If $A_j(\Omega)$ ($j = 0, -1, \ldots, -M$) are automorphic forms of weight $K + j$ for $\Gamma$, then any coefficients $b_j(\Omega, (z, [\gamma]))$ of $Q_{\Omega, (\Omega, [\gamma])}$, which are derived from $A_j(\Omega)$ ($j = 0, -1, \ldots, -M$) in the sense of Theorem


For any \( \alpha \), the Fourier expansion of \( \Omega \) determined in the sense of Theorem 2.2, are automorphic forms for \( \Gamma \). Conversely, we suppose that every coefficient \( b_j(\Omega, (z, [\gamma])) \) have holomorphic Fourier expansions (2.31) at cusps. Then, \( A_s(\Omega) \) (\( s \geq -M \)), which are determined in the sense of Theorem 2.2 are automorphic forms for \( \Gamma \).

Proof. (1) Under the assumption, as in the proof of Theorem 2.3 we can see that \( \Omega \rightarrow b_j(\Omega, (z, [\gamma])) \) are holomorphic for \( (z, [\gamma]) \). We prove that \( b_j(\Omega, (z, [\gamma])) \) have holomorphic Fourier expansions (2.31) for any \( \alpha \in SL(2, \mathbb{Z}) \). The coefficients \( b_j(\Omega, (z, [\gamma])) \) are determined by the relation (2.23) inductively. For \( \alpha \in SL(2, \mathbb{Z}) \) and \( t = 0, \ldots, -M \), we have the holomorphic Fourier expansion

\[
 j_\alpha(\Omega)^{-t} A_t(\Omega) = \sum_{k \geq 0} \hat{A}_{t, k} \exp \left( \frac{2\pi \sqrt{-1} k \Omega}{N_{t, \alpha}} \right),
\]

by the assumption. We set \( (\Omega_1, (z_1, [\gamma_1]), w_1) = \left( \alpha(\Omega), \left( \frac{z}{j_\alpha(\Omega)}, \left[ \frac{\gamma}{j_\alpha(\Omega)} \right] \right), \frac{w}{j_\alpha(\Omega)} \right) \). From (2.23), it holds that

\[
 b_j(\Omega_1, (z_1, [\gamma_1])) = H_j^{\gamma}(b_m(\Omega_1, (z_1, [\gamma_1])), \frac{\partial^\nu}{\partial z_1^\nu} \xi_1(\Omega_1, (z_1, [\gamma_1]), w_1), A_t(\Omega_1)),
\]

where \( H_j^{\gamma}(b_m(\Omega_1, (z_1, [\gamma_1])), \frac{\partial^\nu}{\partial z_1^\nu} \xi_1(\Omega_1, (z_1, [\gamma_1]), w_1), A_t(\Omega_1)) \) is a polynomial in \( b_m(\Omega_1, (z_1, [\gamma_1])) \) \((m < j)\), \( \frac{\partial^\nu}{\partial z_1^\nu} \xi_1(\Omega_1, (z_1, [\gamma_1]), w_1) \) \((s, \nu \in \mathbb{Z}_{\geq 0})\) and \( A_t(\Omega_1) \) \((t = -M, \ldots, 0)\). By virtue of Theorem 2.2 (3), (2.33) is homogeneous of weight \( K + j - M \) with respect to the action of \( \Gamma \). This implies that, by dividing (2.23) by \( j_\alpha(\Omega)^{K+j-M} \) for \( \alpha \in SL(2, \mathbb{Z}) \) and considering \( \frac{\partial}{\partial z_1} = j_\alpha(\Omega) \frac{\partial}{\partial z_1} \), we obtain

\[
 j_\alpha(\Omega)^{-K-j+M} b_j(\Omega_1, (z_1, [\gamma_1])) = H_j^{\gamma}(j_\alpha(\Omega)^{-K-m+M} b_m(\Omega_1, (z_1, [\gamma_1])), \frac{\partial^\nu}{\partial z_1^\nu} j_\alpha(\Omega)^{-s} \xi_1(\Omega_1, (z_1, [\gamma_1]), w_1), j_\alpha(\Omega)^{-t} A_t(\Omega_1)),
\]

By virtue of the assumption and Lemma 2.2 \( \frac{\partial^\nu}{\partial z_1^\nu} j_\alpha(\Omega)^{-s} \xi_1(\Omega_1, (z_1, [\gamma_1]), w_1) \) (\( j_\alpha(\Omega)^{-t} A_t(\Omega_1) \)), resp. have Fourier expansions (2.30) (2.32), resp.). So, we can inductively obtain the Fourier expansions (2.31) of \( j_\alpha(\Omega)^{-K-j+M} b_j(\Omega_1, (z_1, [\gamma_1])) \).

Next, we will consider the Fourier expansion of \( A_s(\Omega) \) for \( s \geq 1 \). By (2.24), we obtain

\[
 A_s(\Omega_1) = H_s^{\gamma}(b_j(\Omega_1, (z_1, [\gamma_1])), \frac{\partial^\nu}{\partial z_1^\nu} \xi_1(\Omega_1, (z_1, [\gamma_1]), w_1), A_t(\Omega_1)),
\]

where \( H_s^{\gamma}(b_j(\Omega_1, (z_1, [\gamma_1])), \frac{\partial^\nu}{\partial z_1^\nu} \xi_1(\Omega_1, (z_1, [\gamma_1]), w_1), j_\alpha(\Omega)^{-t} A_t(\Omega_1)) \) is a polynomial in \( b_j(\Omega_1, (z_1, [\gamma_1])) \) \((0 \leq j \leq M)\), \( \frac{\partial^\nu}{\partial z_1^\nu} \xi_1(\Omega_1, (z_1, [\gamma_1]), w_1) \) \((s, \nu \in \mathbb{Z}_{\geq 0})\) and \( A_t(\Omega_1) \) \((t < s)\). We can see that the polynomial is homogeneous of weight \( s \) under the action of \( SL(2, \mathbb{Z}) \) also. Therefore, dividing (2.33) by \( j_\alpha(\Omega)^{s} \) \((\alpha \in SL(2, \mathbb{Z}))\) and using \( \frac{\partial}{\partial z_1} = j_\alpha(\Omega) \frac{\partial}{\partial z_1} \), we have

\[
 j_\alpha(\Omega)^{-s} A_s(\Omega_1) = H_s^{\gamma}(j_\alpha(\Omega)^{-K-j+M} b_j(\Omega_1, (z_1, [\gamma_1])), \frac{\partial^\nu}{\partial z_1^\nu} j_\alpha(\Omega)^{-s} \xi_1(\Omega_1, (z_1, [\gamma_1]), w_1), j_\alpha(\Omega)^{-t} A_t(\Omega_1)).
\]

So, we can also obtain the Fourier expansions of \( j_\alpha(\Omega)^{-s} A_s(\Omega) \) inductively.

(2) We only need to obtain the holomorphic Fourier expansions of \( A_s(\Omega) \) \((s \geq -M)\) at cusps. By the same argument with the latter of the proof of (1), we can obtain the Fourier expansion of \( j_\alpha(\Omega)^{-s} A_s(\Omega) \) inductively from the Fourier expansion (2.31).
2.5 A formulation via a ring of generating functions

In this subsection, we will give an interpretation of Theorems 2.2, 2.3, 2.4 and 2.5 using a ring of generating functions for sequences of automorphic forms.

Let $\Gamma \subset Sp(n, F)$ be a congruence subgroup and $M_K(\Gamma)$ be the vector space of automorphic forms for $\Gamma$ of weight $K$. It is well-known that $M_K(\Gamma) = \{0\}$ if $K < 0$. Let $M(\Gamma) = \bigoplus_{K=0}^{\infty} M_K(\Gamma)$ be the graded ring of automorphic forms for $\Gamma$. Let $V$ be the ring of formal Laurent series in $\lambda^{-1}$ over $M(\Gamma)$:

$$V = \{ \sum_{s=-M}^{\infty} A_s(\Omega)\lambda^{-s} \mid M \in \mathbb{Z}_{\geq 0}, A_s(\Omega) \in M(\Gamma) \}.$$ 

We take a subspace $R_K$ of $V$ defined by $R_K = \{ \sum_{s=-M}^{\infty} A_s(\Omega)\lambda^{-s} \in V \mid A_s(\Omega) \in M_{s+K}(\Gamma) \}$. Then, $R = \bigoplus_{K=0}^{\infty} R_K$ is a graded ring. The ring $R$ can be regarded as a ring of generating functions for sequences $\{A_s(\Omega)\}$ ($A_s(\Omega) \in M_{s+K}(\Gamma)$) of automorphic forms. We note that $R_0$ gives a subring of $R$.

Now, we define a vector space $D^P_K$ and a ring $D^P$ of differential operators. Let $P = P_{\Omega, z}$ of (2.24) be a differential operator of weight $N$ with respect to $\Gamma$. However, if $F = Q$ and $n = 1$, we additionally assume that the coefficients $a_i(\Omega, z)$ ($l = 0, \cdots, N$) have holomorphic Fourier expansions (2.25) at cusps.

**Definition 2.3.** For a fixed congruence subgroup $\Gamma(\subset Sp(n, F))$, set

$$\hat{D}^P_K = \{ Q = Q_{\Omega, (z, [\gamma])} \mid Q \text{ is given by (2.16); } Q \text{ commutes with } P; Q \text{ is of weight } K \text{ with respect to } \Gamma \}.$$ 

Then,

(i) if $F \neq Q$ or $n \geq 2$, set

$$D^P_K = \{ Q \in \hat{D}^P_K \mid b_j(\Omega, (z, [\gamma])) \text{ are holomorphic for generic } (z, [\gamma]) \}.$$ 

(ii) if $F = Q$ and $n = 1$, set

$$D^P_K = \{ Q \in \hat{D}^P_K \mid b_j(\Omega, (z, [\gamma])) \text{ are holomorphic for generic } (z, [\gamma]);$$

$$b_j(\Omega, (z, [\gamma])) \text{ have holomorphic Fourier expansions (2.25) at cusps} \}.$$ 

Set $D^P = \bigoplus_{K=0}^{\infty} D^P_K$. This is a commutative graded ring (see Theorem (2.37)).

Let $\chi : D^P \to R$ be a mapping given by

$$Q_{\Omega,(z,[\gamma])} = Q \mapsto A = A(\Omega, \lambda)$$

(2.37)

if $Q\Psi = A\Psi$ for $\Psi = \Psi(\Omega, (z, [\gamma]), w, \lambda)$ of (2.18) for generic $\lambda$. Theorem (2.3) implies that $\chi$ is an injective mapping. Moreover, if $Q_1, Q_2 \in D^P$ and $\chi(Q_j) = A_j$ ($j = 1, 2$), we have $(Q_1 + Q_2)\Psi = (A_1 + A_2)\Psi$ and $(Q_1 Q_2)\Psi = A_1 A_2 \Psi$. So, by a similar argument to the end of the proof of Proposition 1.3, we can see that $\chi(Q_1 + Q_2) = A_1 + A_2$, $\chi(Q_1 Q_2) = A_1 A_2$ and $\chi(1) = 1$ hold. Namely, $\chi$ gives an embedding $D^P \hookrightarrow R$ of rings.

**Definition 2.4.** Let $S^P_K$ be the vector space $\chi(D^P_K)(\subset R)$ over $\mathbb{C}$. Let $S^P$ be the graded ring $\chi(D^P) = \bigoplus_{K=0}^{\infty} S^P_K$.

For any $A = \sum_{s=-M}^{\infty} A_s(\Omega)\lambda^{-s} \in S^P_K$ ($A_{-M}(\Omega) \neq 0$), we put

$$\text{Prin}(A) = A_{-M}(\Omega)\lambda^M + \cdots + A_0(\Omega), \quad (A_{-M}(\Omega) \neq 0).$$

(2.38)
We call $Prin(A)$ the principal part of $A$. Set $W_K = Prin(S^p_K) = \bigoplus_{s=0}^K \mathcal{M}'_{K-s}(\Gamma)\lambda^s$. Here, $\mathcal{M}'_{K-s}(\Gamma)$ is a subspace of $\mathcal{M}_{K-s}(\Gamma)$. We have the following reformulation of our main results.

**Theorem 2.6.** The mapping $\chi: \mathcal{D}^p \to S^p$ given by (2.37) is an isomorphism of graded rings. For fixed $K$, the mapping $Prin: S^p_K \to W_K$ of (2.38) is an isomorphism of vector spaces over $\mathbb{C}$. Especially, the sequence

$$\mathcal{D}^p_K \xrightarrow{\chi} S^p_K \xrightarrow{Prin} W_K$$

gives isomorphisms of three vector spaces.

**Proof.** Due to Theorem 2.3 and 2.4, $Prin$ of (2.38) is a bijective mapping. So, $Prin$ is also an isomorphism of vector spaces over $\mathbb{C}$. \qed

**Remark 2.4.** We can naturally define $Prin$ on $S^p = \bigoplus_{K=0}^{\infty} S^p_K$. However, this does not give a homomorphism of rings.

From the proof of Theorem 2.3 and 2.4 we can see the following.

**Corollary 2.1.** An operator $Q \in \mathcal{D}^p_K$ is of rank $M$ if and only if $Prin(\chi(Q))$ is given by the form (2.39). Moreover, the leading coefficient of $Q \in \mathcal{D}^p_K$ is a constant number $c$ if and only if $M = K$ and $A_{-K}(\Omega) \in \mathcal{M}_0(\Gamma)$ is given by $A_{-K}(\Omega) \equiv c$.

Let $\mathcal{D}^p_{K,M}$ be the subspace consisting of differential operators $Q$ whose ranks are at most $M$. We have $\mathbb{C} = \mathcal{D}^p_{K,0} \subset \mathcal{D}^p_{K,1} \subset \cdots \subset \mathcal{D}^p_{K,K} = \mathcal{D}^p_K$.

From Theorem 2.6, any element of $Q \in \mathcal{D}^p_{K,M}$ is parametrized by the elements of the vector space $\mathcal{M}'_{K-M}(\Gamma) \oplus \cdots \oplus \mathcal{M}'_{K}(\Gamma)$. Then, $\mathcal{D}^p_{K,M}$ has $\sum_{s=0}^M \dim \mathcal{M}'_{K-s}(\Gamma)$ complex parameters.

**Corollary 2.2.**

$$\dim_{\mathbb{C}} \mathcal{D}^p_{K,M} = \bigoplus_{s=0}^M \dim_{\mathbb{C}} \mathcal{M}'_{K-s}(\Gamma).$$

Especially,

$$\dim_{\mathbb{C}} \mathcal{D}^p_K = \bigoplus_{s=0}^K \dim_{\mathbb{C}} \mathcal{M}'_{K-s}(\Gamma).$$

Anyway, if a differential operator $P_{\Omega,z}$ of weight $N$ with respect to the action of $\Gamma$ and automorphic forms $A_j(\Omega) \in \mathcal{M}'_{K+j}(\Gamma)$ ($j = 0, \ldots, -M$) are given, there is the unique differential operator $Q_{\Omega,(z,[\lambda])} = Q \in \mathcal{D}^p_{K,M}$ which commutes with $P = P_{\Omega,z}$. We set

$$Q_{\Omega,(z,[\lambda])}(P_{\Omega,z}; A_{-M}(\Omega)\lambda^M + \cdots + A_0(\Omega)) = \chi^{-1} \circ Prin^{-1}(A_{-M}(\Omega)\lambda^M + \cdots + A_0(\Omega)).$$

**Example 2.4.** In this example, we consider the special case for $B = 2$ of the Lamé operator of (2.39):

$$P_{\Omega,z} = \frac{\partial^2}{\partial z^2} = 2\wp(\Omega, z).$$

As we saw in Example 2.3, $P_{\Omega,z}$ is of weight 2 for $SL(2, \mathbb{Z})$.

Automorphic forms for $\Gamma = SL(2, \mathbb{Z})$ are called elliptic modular forms. Let $\mathcal{M}_k(\Gamma)$ be the vector space of elliptic modular forms of weight $k$. According to Theorem 2.4 and 2.5, elliptic modular forms $A_j(\Omega) \in \mathcal{M}_k(Gamma)$ ($j \in \{0, \ldots, -M\}$) determine a differential operator $Q_{\Omega,(z,[\lambda])}$ of rank $M$ of weight $K$ with respect to $\Gamma$, where $Q_{\Omega,(z,[\lambda])}$ commutes with $P_{\Omega,z}$. 

24
In the following, we shall consider a simple case of $K = 3$ and $M = 3$. In this case, modular forms must be quite simple: $A_{-3}(\Omega) \equiv \text{const} \in \mathcal{M}_0(\Gamma)$ and $A_{-j}(\Omega) \equiv 0 \in \mathcal{M}_{3+j}(\Gamma)$ ($j = 0, 1, 2$), because we have
\begin{equation}
\begin{cases}
\mathcal{M}_0(\Gamma) = \mathbb{C}, \\
\mathcal{M}_k(\Gamma) = \{0\} \quad (k = 1, 2, 3)
\end{cases}
\tag{2.40}
\end{equation}
(for detail, see [Sm1]). So, let us obtain the differential operator $Q_{\Omega,(z,[\gamma])} = Q_{\Omega,(z,[\gamma])}(P_{\Omega,z}; \lambda^3) = Q_{\Omega,(z,[\gamma])}(P_{\Omega,z}; \lambda^3 + 0\lambda^2 + 0\lambda + 0)$.

First, we calculate $\Psi$ of (2.2). As we saw in the proof of Proposition 2.3 and Theorem 2.5 we can determine $\{\xi_s(\Omega, (z, [\gamma]), w)\}$ inductively. Taking $w = \frac{1}{2} \not\in \mathbb{Z} + 2\Omega$, we have in fact
\begin{align}
\xi_0(\Omega, (z, [\gamma]), \Omega) &= 1, \\
\xi_1(\Omega, (z, [\gamma]), \Omega) &= -\zeta(\Omega, z) + \zeta_1(\Omega), \\
\xi_2(\Omega, (z, [\gamma]), \Omega) &= \frac{1}{2}\zeta^2(\Omega, z) + \frac{1}{2}\zeta_1^2(\Omega) - \zeta_1(\Omega)\zeta(\Omega, z) - \frac{1}{2}\varphi(\Omega, z) + \frac{1}{2}\varphi_1(\Omega), \\
&\quad \ldots,
\end{align}
where $\zeta(\Omega, z)$ is the Weierstrass $\zeta$-function
\[\zeta(\Omega, z) = \frac{1}{z} + \sum_{(n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}} \left( \frac{1}{z - n_1 - n_2\Omega} + \frac{1}{n_1 + n_2\Omega} + \frac{z}{(n_1 + n_2\Omega)^2} \right),\]
and $\zeta_1(\Omega) = \zeta(\Omega, \frac{1}{2})$ and $\varphi_1(\Omega) = \varphi(\Omega, \frac{1}{2})$. (We note that the right hand side of (2.41) satisfy the transformation law (2.12).)

From our data, set $A(\Omega, \lambda) = \text{Prim}^{-1}(\lambda^3) = \sum_{s=-3}^{\infty} A_s(\Omega)\lambda^{-s} = \lambda^3 + 0\lambda^2 + 0\lambda + 0 + A_1(\Omega)\lambda^{-1} + \ldots$. For $\Psi(\Omega, (z, [\gamma]), w, \lambda)$ given by (2.41), we can uniquely find the differential operator $Q_{\Omega,(z,[\gamma])} = \sum_{j=0}^{3} b_j(\Omega, (z, [\gamma])) \partial_{\Omega^{3-j}}$ satisfying $Q_{\Omega,(z,[\gamma])}\Psi(\Omega, (z, [\gamma]), w, \lambda) = A(\Omega, \lambda)\Psi(\Omega, (z, [\gamma]), w, \lambda)$. In fact, by a direct calculation as in the proof of Theorem 2.3 we can obtain $b_0(\Omega, (z, [\gamma])) = 1, b_1(\Omega, (z, [\gamma])) = 0, b_2(\Omega, (z, [\gamma])) = -3\varphi(\Omega, z)$ and $b_3(\Omega, (z, [\gamma])) = -\frac{3}{2}\varphi(\Omega, z)$. Therefore,
\begin{equation}
Q_{\Omega,(z,[\gamma])} = Q_{\Omega,(z,[\gamma])}(P_{\Omega,z}; \lambda^3) = \partial_{\lambda^3} - 3\varphi(\Omega, z) \frac{\partial}{\partial_{\varphi}(\Omega, z)} - \frac{3}{2} \left( \frac{\partial}{\partial_{\varphi}(\Omega, z)} \varphi(\Omega, z) \right)
\tag{2.42}
\end{equation}
is the differential operator we want. This is of weight 3 with respect to $\Gamma = SL(2, \mathbb{Z})$.

From Theorem 2.1, Corollary 1.3 and the fact (2.40), $Q_{\Omega,(z,[\gamma])}$ of (2.42) is the unique element of $D_3^P$ for $P = P_{\Omega,z}$ of (2.39) up to a constant factor.

We remark that the relation between such a Lamé operator $P_{\Omega,z}$ of (2.39) and $Q_{\Omega,(z,[\gamma])}$ of (2.42) were precisely studied from the viewpoint of integrable systems or physics (see [W], [DM] or [M]). Our result gives a new interpretation on this topic from the viewpoint of elliptic modular forms.

2.6 The family of algebraic curves $R_\Omega$

For $X \in \mathbb{C} = \mathbb{P}^1(C) \setminus \{\infty\}$ and $P_{\Omega,z}$ of (2.4), we consider the differential equation $P_{\Omega,z}u = Xu$ and its space of solutions $\mathcal{L}(P_{\Omega,z}, X)$. Suppose $Q_{\Omega,(z,[\gamma])}$ of (2.16) commutes with $P_{\Omega,z}$. Letting $\lambda_1, \ldots, \lambda_N$ be the disjoint solutions of $\lambda^N = X$, $\Psi(\Omega, (z, [\gamma]), w, \lambda_j)$ gives an eigenvector for the eigenvalue $A(\Omega, \lambda_j)$. Let $Q_{\Omega,(z,[\gamma])}$ be the linear operator derived from $Q_{\Omega,(z,[\gamma])}$ on $\mathcal{L}(P_{\Omega,z}, X)$. The characteristic polynomial of $Q_{\Omega,(z,[\gamma])}$ is given by
\begin{equation}
\prod_{j=1}^{N} (Y - A(\Omega, \lambda_j)).
\tag{2.43}
\end{equation}
Due to Corollary 1.2, \( F_\Omega(P_{\Omega,z}; Q_{\Omega,(z,[\gamma])}) = 0 \). From Theorem 1.2 it follows that \( F_\Omega(P_{\Omega,z}; Q_{\Omega,(z,[\gamma])}) = 0 \). We set

\[
F_\Omega(X,Y) = \sum_{j,k} f_{j,k}(\Omega) X^j Y^k.
\] (2.44)

**Theorem 2.7.** Let \( P = P_{\Omega,z} \) be the differential operator of (2.42). Take \( Q = Q_{\Omega,(z,[\gamma])} \in D_K^P \). Then, the coefficient \( f_{j,k}(\Omega) \) in (2.44) is an automorphic form of weight \( NK - N_j - Kk \) for \( \Gamma \).

**Proof.** From Theorem 2.6 we suppose that \( Q \in D_K^P \) is given by \( Q = Q_{\Omega,(z,[\gamma])}(P_{\Omega,z}; A_{-M}(\Omega)\lambda^M + \cdots + A_0(\Omega)) \). Then, \( \chi(Q) = \text{Prin}^{-1}(A_{-M}(\Omega)\lambda^M + \cdots + A_0(\Omega)) \) is given by a series \( A(\Omega, \lambda) = \sum_{s=-M}^{\infty} A_s(\Omega)\lambda^s \).

Due to Theorem 2.4 and 2.5, \( A_s(\Omega) \) \( (s \geq 1) \) are automorphic forms of weight \( s + K \). Since the set of automorphic forms is a ring, together with the argument in Section 1.3, the coefficients of the polynomial

\[
\prod_{j=1}^{N}(Y - A(\Omega, \lambda_j)) \text{ in } X \text{ and } Y, \text{ where } \lambda_j^N = X, \text{ are automorphic forms for } \Gamma.
\]

Since \( P_{\Omega,z} \) \( (Q_{\Omega,(z,[\gamma])}) \) is of weight \( N \) (\( K \)), resp. for the action of \( \Gamma \), we have the action of \( \alpha \in \Gamma \) given by \( (\Omega, X, Y) \mapsto (\alpha(\Omega), j_\alpha^N X, j_\alpha Y) = (\alpha(\Omega), X, Y) \). When we describe \( F_\Omega(X,Y) \) as in (2.44), we have

\[
f_{j,k}(\alpha(\Omega)) = j_\alpha(\Omega)^{NK - N_j - Kk} f_{j,k}(\Omega).
\]

Hence, by comparing the coefficients, \( f_{j,k}(\Omega) \) is of weight \( NK - N_j - Kk \).

By Theorem 2.6 and Theorem 2.7, the family \( \{ F_\Omega(X,Y) = 0 | \Omega \in \mathbb{H}_k \} \) of algebraic curves is uniquely determined by the differential operator \( P_{\Omega,z} \) and given principal part \( A_{-M}(\Omega)\lambda^M + \cdots + A_0(\Omega) \) of a Laurent series \( \text{Prin}^{-1}(A_{-M}(\Omega)\lambda^M + \cdots + A_0(\Omega)) \in S_K^P \). We denote such a family by \( \mathcal{F}(P_{\Omega,z}; A_{-M}(\Omega)\lambda^M + \cdots + A_0(\Omega)) \) whose members are \( R_\Omega = R(\Omega; P_{\Omega,z}; A_{-M}(\Omega)\lambda^M + \cdots + A_0(\Omega)) \).

**Example 2.5.** Let \( P_{\Omega,z} \) \( (Q_{\Omega,(z,[\gamma])}) \) \( (P_{\Omega,z}; \lambda^3) \), resp. be the operator of (2.39) ((2.42), resp.), as we saw in Example 2.4. We note that the \( \phi \)-function satisfies the Weierstrass equation

\[
\left( \frac{\partial}{\partial z} \psi(\Omega, z) \right)^2 = 4\psi^3(\Omega, z) - g_2(\Omega)\psi(\Omega, z) - g_3(\Omega),
\]

where \( g_2(\Omega) = \sum_{(n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{60}{(n_1 + n_2\Omega)^2} \text{ and } g_3(\Omega) = \sum_{(n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{140}{(n_1 + n_2\Omega)^3} \). It is well-known that \( g_2(\Omega) \in \mathcal{M}_4(\Gamma) \) and \( g_3(\Omega) \in \mathcal{M}_6(\Omega) \) (for detail, see [Sm1]). Using the relation (2.43), we can see that the defining equation \( F_\Omega(X,Y) = 0 \) of \( R_\Omega \) is given by

\[
F_\Omega(X,Y) = Y^2 - X^3 - \frac{g_2(\Omega)}{4} X - \frac{g_3(\Omega)}{4}.
\]

So, \( f_{0,2}(\Omega) = 1 \), \( f_{1,0}(\Omega) = \frac{g_2(\Omega)}{4} \) and \( f_{0,0}(\Omega) = \frac{g_3(\Omega)}{4} \). In this case, the family \( \mathcal{F}(P_{\Omega,z}; \lambda^3) \) consists of non-singular algebraic curves of genus 1.

Let \( \pi_\Omega : R_\Omega \to \mathbb{P}^1(\mathbb{C}) \) be the canonical projection given by \( (X,Y) \mapsto X \).

**2.7 A criterion for single-valued differential operators with actions of \( \Gamma \)**

In this subsection, we use the same notation with that of Section 2.4 and Section 2.5. Moreover, we assume

there exists \( s (s \geq -M) \), where \( N \) and \( s \) are coprime, such that \( A_s(\Omega) \neq 0 \) (2.46)

for \( \{ A_s(\Omega) \} \) of (2.44).
Remark 2.5. There are so many cases that the condition \((2.46)\) holds. For example, if \(N\) and \(M\) are coprime and \(Q_{\Omega, (z, [\gamma])}\) is given by \(Q_{\Omega, (z, [\gamma])}(P_{\Omega, z}; \lambda^M + A_{-M+1}(\Omega)\lambda^{M-1} + \cdots + A_0(\Omega))\) (namely the case of \(A_{-M}(\Omega) = 1\)), the condition \((2.46)\) is satisfied.

By a similar argument as in Section 1.4, we have the eigenfunction
\[
\psi(\Omega, (z, [\gamma]), w, p) = \sum_{l=0}^{N-1} h_l(\Omega, w, p) C_l(\Omega, (z, [\gamma]), w, p)
\]
of the operator \(Q_{\Omega, [\gamma], X}\) on \(L(P_{\Omega, z}, X)\). Here, \(C_l(\Omega, (z, [\gamma]), w, X) = \mathcal{L}(P_{\Omega, z}, X)\) such that
\[
\left. \frac{\partial^r}{\partial z^r} C_l(\Omega, (z, [\gamma]), w, X) \right|_{(z, [\gamma])=(w, [\alpha])} = \delta_{r, l}.
\]

We note that the function \(C_l(\Omega, (z, [\gamma]), w, X)\) of \((2.48)\) satisfies the transformation law
\[
C_l(\Omega_1, (z_1, [\gamma_1]), w_1, X_1) = \frac{1}{j_\alpha(\Omega)} C_l(\Omega, (z, [\gamma]), w, X),
\]
where \(\alpha \in \Gamma\) and \((\Omega_1, (z_1, [\gamma_1]), w_1, X_1)\) is given in \((2.14)\), because the equation \(P_{\Omega, z} u = X u\) coincides with \(P_{\Omega_1, z_1} u = X_1 u\) under the transformation \((\Omega, (z, [\gamma]), w, X) \to (\Omega_1, (z_1, [\gamma_1]), w_1, X_1)\) and it holds that
\[
\left. \frac{\partial^r}{\partial z^r} C_l(\Omega_1, (z_1, [\gamma_1]), w_1, X_1) \right|_{(z_1, [\gamma_1])=(w, [\alpha])} = \frac{1}{j_\alpha(\Omega)} \delta_{r, l}.
\]

If \(p = (X, Y) \in \mathcal{R}_\Omega\), set \(p_1 = (X_1, Y_1) = (j_\alpha(\Omega) N X, j_\alpha(\Omega) M Y)\). From \((2.44)\), \((X_1, Y_1) \in \mathcal{R}_{\Omega_1}\). The vector \(\left(h_0(\Omega, w, p), \cdots, h_{N-1}(\Omega, w, p)\right)\) admits a transformation law
\[
\begin{pmatrix}
h_0(\Omega_1, w_1, p_1) \\
h_1(\Omega_1, w_1, p_1) \\
\vdots \\
h_{N-1}(\Omega_1, w_1, p_1)
\end{pmatrix} = \begin{pmatrix}
h_0(\Omega, w, p) \\
j_\alpha(\Omega) h_1(\Omega, w, p) \\
\vdots \\
j_\alpha(\Omega) N^{-1} h_{N-1}(\Omega, w, p)
\end{pmatrix}.
\]

Theorem 2.8. Assume that \(N\) is a prime number and the differential operators \(P_{\Omega, z}\) of \((2.4)\) and \(Q_{\Omega, (z, [\gamma])}\) of \((2.10)\) satisfy the condition \((2.46)\). Suppose the arithmetic genus \(\mu(\mathcal{R}_\Omega)\) is smaller than \(N\) for any \(\Omega \in \mathbb{H}_n\), Then all coefficients of \(Q_{\Omega, (z, [\gamma])}\) are single-valued functions of \(z\).

Proof. As in Theorem 1.4 and Corollary 1.3, the function \(\mathcal{R}_\Omega \ni p \mapsto \psi(\Omega, (z, [\gamma]), w, p) \in \mathbb{P}^1(\mathbb{C})\) has at most \(\mu(\mathcal{R}_\Omega)\) poles. By our assumption, we have \(\mu(\mathcal{R}_\Omega) < N\) for any \(\Omega\). By a similar argument to the proof of Theorem 1.5, we can prove that every coefficient \(Q_{\Omega, (z, [\gamma])}\) are single-valued in \(z\).

The phenomenon that we saw in Example 2.3, 2.4 and 2.5 gives a typical example of the criterion of Theorem 2.8. Namely, the rank of the Lamé operator \(P_{\Omega, z}\) of \((2.39)\) is the prime number \(N = 2\), the commutative operator \(Q_{\Omega, (z, [\gamma])}\) of \((2.42)\) of rank \(M = 3\) is single-valued, where \(P_{\Omega, z}\) and \(Q_{\Omega, (z, [\gamma])}\) give a point of the non-singular curve \(\mathcal{R}_\Omega \in \mathcal{F}(P_{\Omega, z}; \lambda^3)\) of genus \(1 < 2 = N\).

Acknowledgment

This work is supported by The JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers “Mathematical Science of Symmetry, Topology and Moduli, Evolution of International Research Network based on OCAMI” and The Sumitomo Foundation Grant for Basic Science Research Project (No.150108).
References

[BC] J. L. Burchnall and T. W. Chaundy, *Commutative ordinary differential operators*, Proc. London Math. Soc. 21, 1922, 420-440.

[DMN] B. A. Dubrovin, V. B. Matveev and S. P. Novikov, *Non-linear equations of Korteweg-de Vries type, finite-zone linear operators and abelian varieties*, Uspekhi Math. Nauk 31, 1976, 55-136.

[EMOF] M. Erdelyi, W. Magnus, F. Oberhettinger and T. G. Francesco, *Higher transcendental functions*, McGraw-Hill, 1955.

[EZ] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Birkhäuser, Progress in Math. 55, 1985.

[G] P. A. Griffiths, *Interpolation to Algebraic Curves*, Translations of Math. Monographs 76 Amer. Math. Soc., 1989.

[K] I. Krichever, *Integration of nonlinear equations by the methods of algebraic geometry*, Funkt. Anal. Appl., 11, 1977, 12-26.

[KS] I. Krichever and T. Shiota, *Soliton equations and the Riemann-Schottky problem*, Adv. Lec. Math. 25, 2013, 205-258.

[M] M. Mulase, *Category of vector bundles on algebraic curves and infinite dimensional Grassmanianns*, Internat. J. Math, 1, 1990, 293-342.

[Mm] D. Mumford, *An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg deVries equation and related non-linear equations*,Intl. Symp. on Algebraic Geometry Kyoto, 1977, 115-153.

[MM] H. P. McKean and P. M. Moerbeke, *The spectrum of Hill’s equation*, Invent. Math. 30, 1975, 217-274.

[Sm1] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Prinston Univ. Press, 1971.

[Sm2] G. Shimura, *Arithmeticity of the special values of various zeta functions and the periods of Abelian integrals*, Sūgaku expositions 8 (1), 1995, 17-38.

[So] T. Shiota, *Characterization of Jacobian varieties in terms of soliton equations*, Invent. Math. 83 (2), 1986, 333-382.

[T] K. Takemura, *Analytic continuation of eigenvalues of Lamé operator*, J. Diff. Eq. 228 (1), 2004, 1-16.

[W] G. Wallenberg, *Über die Vertauschbarkeit homogener linearer Differentialausdrücke*, Archiv der Math. Phys. 4, 1903, 252-268.

[WW] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Cambridge University Press, 1962.

[Z] C. Ziegler, *Jacobi forms of higher degree*, Abh. Math. Sem. Univ. Hamburg 59, 1989, 191-224.

Atsuhira Nagano
Department of Mathematics
King’s College London
Strand, London, WC2R 2LS
The United Kingdom
(E-mail: atsuhira.nagano@gmail.com)