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A POSTERIORI ERROR ESTIMATES FOR TIME-DEPENDENT
REACTION-DIFFUSION PROBLEMS BASED ON THE
PAYNE–WEINBERGER INEQUALITY

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Abstract. We consider evolutionary reaction-diffusion problem with mixed
Dirichlet–Robin boundary conditions. For this class of problems, we derive
two-sided estimates of the distance between any function in the admissible
energy space and exact solution of the problem. The estimates (majorants and
minorants) are explicitly computable and do not contain unknown functions
or constants. Moreover, it is proved that the estimates are equivalent to the
energy norm of the deviation from the exact solution.

1. Problem statement. Let \( \Omega \subset \mathbb{R}^d \) be a bounded connected domain with Lipchitz continuous boundary \( \partial \Omega \), which consists of two measurable non-intersecting parts \( \Gamma_D \) and \( \Gamma_R \) associated with the Dirichlet and Robin boundary conditions, respectively. By \( Q_T \) we denote the space-time cylinder \( Q_T := \Omega \times (0, T) \), \( T > 0 \), and \( S_T := \partial \Omega \times [0, T] = (\Gamma_D \cup \Gamma_R) \times [0, T] \). The parts of \( S_T \) related to \( \Gamma_D \) and \( \Gamma_R \) are denoted by \( S_D \) and \( S_R \), respectively.

We consider the classical reaction-diffusion initial boundary value problem: find \( u(x, t) \) and \( p(x, t) \) such that

\[
\begin{align*}
  u_t - \nabla \cdot p + \lambda u &= f, & (x, t) &\in Q_T, \\
  p &= A \nabla u, & (x, t) &\in Q_T, \\
  u(x, 0) &= \varphi, & x &\in \Omega, \\
  u &= 0, & (x, t) &\in S_D, \\
  p \cdot n + \sigma u &= g, & (x, t) &\in S_R,
\end{align*}
\]

where \( n \) denotes the vector of unit outward normal to \( \partial \Omega \), and

\[
f(x, t) \in L^2(Q_T), \quad \varphi(x) \in L^2(\Omega), \quad g(x, t) \in L^2(0, T; L^2(S_R)).
\]

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The function $\lambda$ entering the reaction part of (1) is a non-negative bounded function, which values may vary from very small (or zero) to large values in different parts of the domain. The function $\sigma(s,t)$ is a bounded function defined on $\Gamma_R$. We assume that for any $(x,t) \in Q_T$ the matrix $A$ is symmetric and satisfies the condition

$$\nu_1|\xi|^2 \leq A(x,t)\xi \cdot \xi \leq \nu_2|\xi|^2, \quad \xi \in \mathbb{R}^d, \quad 0 < \nu_1 \leq \nu_2 < \infty.$$  \hspace{1cm} (6)

By $\|\cdot\|_\Omega$ and $\|\cdot\|_{Q_T}$, we denote the standard norms in $L^2(\Omega)$ and $L^2(Q_T)$, respectively. $L^{2,1}(Q_T)$ is the space of functions $g(x,t)$ with the finite norm $\int_0^T \|g(\cdot,t)\|_\Omega \, dt$.

$\dot{H}^{1}(Q_T)$ is a subspace of $H^1(Q_T)$, which contains functions satisfying (3), $H^{1,0}(Q_T) := L^2(0,T;H^1(\Omega))$, and $V_2(Q_T) := H^{1,0}(Q_T) \cap L^\infty(0,T;L^2(\Omega))$. The space $V_2^{1,0}(Q_T) := H^{1,0}(Q_T) \cap C(0,T;L^2(\Omega))$ is a subspace of $V_2(Q_T)$ with functions possessing $L^2$-traces defined for a.a. $t \in [0,T]$.

The generalized solution of (1)–(4) is defined as a function $u(x,t) \in V_2^{1,0}(Q_T)$, satisfying the integral identity

$$\int_{\Omega} \left( u(x,T)\eta(x,T) - u(x,0)\eta(x,0) \right) \, dx - \int_{Q_T} u_\eta \, dx \, dt + \int_{Q_T} A\nabla u \cdot \nabla \eta \, dx \, dt + \int_{S_R} \sigma u\eta \, d\nu + \int_{Q_T} \lambda u\eta \, dx \, dt = \int_{Q_T} f\eta \, dx \, dt + \int_{S_R} g\eta \, d\nu, \quad \forall \eta \in \dot{H}^{1}(Q_T). \hspace{1cm} (7)$$

Classical solvability results (see, e.g., [4, 5, 3]) guarantee that $u$ exists and is unique in $V_2^{1,0}(Q_T)$.

Assume that $v \in \dot{H}^{1}(Q_T)$ is an approximation of $u$. Our goal is to deduce explicitly computable and realistic estimates of the distance between $u$ and $v$. In other words, we wish to quantify neighborhoods of the exact solution in terms of local topology equivalent to the natural energy norm. More precisely, we introduce the measure

$$[u - v]^2_{(\nu,\theta,\zeta,\chi)} = \nu \|\nabla(u - v)\|_A^2 + \|\theta(u - v)\|_{Q_T}^2 + \zeta \|\lambda(u - v)(\cdot,T)\|_A^2 + \chi \|\sqrt{\sigma}(u - v)\|_{S_R}^2,$$  \hspace{1cm} (8)

where $\nu$, $\theta$, $\zeta$ and $\chi$ are certain positive weights (balancing different components of the error). They can be selected in different ways so that (8) presents a collection of different error measures. Here,

$$\|\tau\|_{A}^2 := \int_{Q_T} A\tau \cdot \tau \, dx \, dt, \hspace{1cm} (9)$$

henceforth, we also use the norms

$$\|\tau\|_{A}^2 := \int_{\Omega} A\tau \cdot \tau \, dx, \quad \|\tau\|_{A^{-1}}^2 := \int_{\Omega} A^{-1}\tau \cdot \tau \, dx, \quad \|\tau\|_{A^{-1}}^2 := \int_{Q_T} A^{-1}\tau \cdot \tau \, dx \, dt.$$  

In Theorem 2.1, we derive a fully computable and guaranteed upper bound of $e = u - v$ (for this purpose we use the method originally introduced in [12]). In [15], this method was applied to problems with convection, and in [9] guaranteed error majorants were derived for the Stokes problem. In Section 2, we combine this approach with the technique suggested in [14] for the stationary reaction-diffusion
problem, which yields efficient bounds of the distance to the exact solution (error majorants) for problems with strongly changing reaction function.

The majorant presented in Theorem 2.1 contains the constant \( C_{FΩ} \) in the Friedrichs type inequality (18). If \( S_T = S_D \), then this constant (or a guaranteed upper bound of it) is easy to find. However, in the case of mixed boundary conditions and complicated domains, finding \( C_{FΩ} \) may cause a serious problem. Therefore, in Theorems 2.2 and 2.3, we derive another upper bounds, which are based on decomposition of \( Ω \) into a collection of non-overlapping convex sub-domains. By means of a technique close to that has been used in [13] for elliptic problem, we deduce majorants, which involve only constants in the Poincare type inequalities. For convex domains these constants are easy to estimate due to the well known result of Payne and Weinberger [11] (with correction of Bebendorf [2]). Therefore, we obtain a fully computable error majorant (13), which involves only known data and constants. In Subsection 2.2, we prove that it is equivalent to the distance to the exact solution measured in terms of the combined (primal-dual) norm.

An advanced form of the majorant (which is sharper than those in Theorems 2.1, 2.2, and 2.3 but has a more complicated structure) is derived in Section 3. In Subsection 3.2, it is shown that the advanced majorant is equivalent to the distance to the exact solution measured in terms of the primal energy norm. A guaranteed and fully computable lower bound of the error is derived in Theorem 4.1. The minorant (87) also contains only known data and can be computed directly. Finally, we note that the practical efficiency of estimates similar to those derived in this paper has been recently tested and confirmed in [7].

2. Majorants of the deviation from \( u \). In this section, we deduce the first (and the simplest) form of the functional, which provides a guaranteed and fully computable upper bound of the deviation (error) \( e = u - v \) for any function \( v \in \hat{H}^1(Q_T) \) and the solution \( u \). From (7), it follows that

\[
\int_{Ω}(e(x, T)η(x, T) - e(x, 0)η(x, 0)) \, dx - \int_{Q_T} eη \, dxdt + \int_{Q_T} A\nabla e \cdot \nabla η \, dxdt + \int_{Q_T} λeη \, dxdt + \int_{S_R} σeη \, dsdt = \int_{Q_T} (f - v_1 - λv) \, η \, dxdt - \int_{S_R} A\nabla v \cdot \nabla η \, dxdt + \int_{S_R} (g - σv)η \, dsdt.
\]

Since \( e \in \hat{H}^1(Q_T) \), we can set \( η = e \), use the relation

\[
\int_{Ω}(e^2(x, T) - e^2(x, 0)) \, dx - \int_{Q_T} ee \, dxdt = \frac{1}{2} \left( \|e(·, T)\|_{Ω}^2 - \|e(·, 0)\|_{Ω}^2 \right), \quad (10)
\]

and obtain

\[
\frac{1}{2} \|e(·, T)\|_{Ω}^2 + \|\nabla e\|_{Q_T}^2 + \int_{Q_T} λe^2 \, dxdt + \int_{S_R} σe^2 \, dsdt = \int_{Q_T} (f - v_1 - λv) \, e \, dxdt - \int_{Q_T} A\nabla v \cdot \nabla e \, dxdt + \int_{S_R} (g - σv)e \, dsdt + \frac{1}{2} \|e(·, 0)\|_{Ω}^2. \quad (11)
\]

This relation is a form of the ‘energy-balance’ identity in terms of deviations. It plays an important role in subsequent analysis. Next, we introduce an additional
variable $y \in Y^*_\text{div} (Q_T)$, where

$$Y^*_\text{div} (Q_T) := \left\{ y \in L^2(\Omega) \mid \text{div } y \in L^2(\Omega), \; y \cdot n \in L^2(T_R) \text{ for a.a. } t \in (0, T) \right\}. \quad (12)$$

**Theorem 2.1.** (i) For any $v \in \tilde{H}^1(Q_T)$ and $y \in Y^*_\text{div} (Q_T)$ the following inequality holds:

$$(2 - \delta) \| \nabla e \|_A^2 + \left(2 - \frac{1}{\gamma}\right) \| \sqrt{\lambda} c \|_{Q_T}^2 + \| e(\cdot, T) \|_{A^t}^2 + 2 \sqrt{\sigma e} \|_{S_R}^2 =: [e]^2_{(v, \vartheta, \zeta, \chi)} \leq \overline{M}_1^2(v, y; \delta, \gamma, \mu) := \| e(\cdot, 0) \|_{A^t}^2 + \int_0^T \left( \gamma \left\| \frac{\mathcal{R}_f(v, y)}{\sqrt{\lambda}} \right\|_\Omega^2 + \alpha_1(t) \frac{C_F^2}{\nu_1} \| \mathcal{R}_{f,1-\mu}(v, y) \|_{A^t}^2 + \alpha_2(t) \| \mathcal{R}_d(v, y) \|_{A^t-1}^2 + \alpha_3(t) \frac{C_F^2}{\nu_1} \| \mathcal{R}_b(v, y) \|_{T_R}^2 \right) dt, \quad (13)$$

where $\delta \in (0, 2]$, $\gamma \geq 1$, $\mu \in [0, 1]$,

$$\mathcal{R}_f(v, y) := f - v_t - \lambda v + \text{div } y, \quad (14)$$

$$\mathcal{R}_{f, \mu}(v, y) := \mu \mathcal{R}_f, \quad \mathcal{R}_{f,1-\mu}(v, y) := (1 - \mu) \mathcal{R}_f, \quad (15)$$

$$\mathcal{R}_d(v, y) := y - A \nabla v, \quad (16)$$

$$\mathcal{R}_b(v, y) := g - \sigma v - y \cdot n, \quad (17)$$

$C_F^2$ is the constant in the Friedrichs’ inequality

$$\| \eta \|_\Omega \leq C_F \| \nabla \eta \|_\Omega, \quad \forall \eta \in \tilde{H}^1(\Omega), \quad (18)$$

$C_\Gamma$ is the constant in the trace inequality related to the Robin part of the boundary

$$\| \eta \|_{T_R} \leq C_\Gamma \| \nabla \eta \|_\Omega, \quad \forall \eta \in H^1(\Omega), \quad (19)$$

$\nu = 2 - \delta$, $\theta = \sqrt{(2 - \frac{1}{\gamma})}$, $\lambda, \; \zeta = 1$, $\chi = 2$ are positive weights, and $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$ are positive scalar-valued functions satisfying the relation

$$\frac{1}{\alpha_1(t)} + \frac{1}{\alpha_2(t)} + \frac{1}{\alpha_3(t)} = \delta. \quad (20)$$

(ii) For any $\delta \in (0, 2]$, $\gamma \geq 1$, and $\mu \in [0, 1]$, the lower bound of the variation problem generated by the majorant

$$\inf_{v \in \tilde{H}^1(Q_T)} \inf_{y \in Y^*_\text{div} (Q_T)} \overline{M}_1^2(v, y; \delta, \gamma, \mu) \quad (21)$$

is zero, and it is attained if and only if $v = u$ and $y = A \nabla u$.

**Proof.** (i) We transform the right-hand side of (11) by means of the relation

$$\int_{Q_T} \text{div } y \eta \; dx \; dt + \int_{Q_T} y \cdot \nabla \eta \; dx \; dt = \int_{S_R} y \cdot n \; ds \; dt,$$
which yields

\[
\frac{1}{2} \| e(\cdot, T) \|_{\Omega}^2 + \| \nabla e \|_A^2 + \int_{S_R} \sigma e^2 \, ds + \int_{Q_T} \lambda e^2 \, dx dt = \mathcal{I}_f + \mathcal{I}_d + \mathcal{I}_b + \frac{1}{2} \| e(\cdot, 0) \|_{\Omega}^2, \tag{22}
\]

where

\[
\mathcal{I}_f = \int_{Q_T} \mathcal{R}_f e \, dx dt, \quad \mathcal{I}_d = \int_{Q_T} \mathcal{R}_d \cdot \nabla e \, dx dt, \quad \mathcal{I}_b = \int_{S_R} \mathcal{R}_b e \, ds dt. \tag{23}
\]

By means of the Hölder inequality, we find that

\[
\mathcal{I}_d = \int_{Q_T} \mathcal{R}_d \cdot \nabla e \, dx dt \leq \int_0^T \| \mathcal{R}_d \|_{A^{-1}} \| \nabla e \|_A \, dt \tag{24}
\]

and

\[
\mathcal{I}_b = \int_{S_R} \mathcal{R}_b e \, ds dt \leq \int_0^T \| \mathcal{R}_b \|_{\Gamma_R} \| e \|_{\Gamma_R} \, dt \leq \int_0^T \| \mathcal{R}_b \|_{\Gamma_R} \frac{C_{\text{tr}}}{\sqrt{\nu_1}} \| \nabla e \|_A \, dt, \tag{25}
\]

where \( \nu_1 \) is the constant in (6). Let \( \mu(x, t) \) be a real-valued function taking values in \([0, 1]\). Next, we estimate the term \( \mathcal{I}_f \) as follows:

\[
\mathcal{I}_f \leq \int_0^T \left( \| \mathcal{F}_{f, \mu} \|_{\Omega} \| \sqrt{\lambda} e \|_{\Omega} + \frac{C_{F\Omega}}{\sqrt{\nu_1}} \| \mathcal{F}_{f,1-\mu} \|_{\Omega} \| \nabla e \|_A \right) \, dt. \tag{26}
\]

In [14], this decomposition was used in order to overcome difficulties arising in the stationary problem if \( \lambda \) is small (or zero) in some parts of the domain and large in another (more detailed study of this form of the majorant can be found in [10] and [6]).

By combining (24)–(26), we obtain

\[
\frac{1}{2} \| e(\cdot, T) \|_{\Omega}^2 + \| \nabla e \|_A^2 + \int_{S_R} \sigma e^2 \, ds + \int_{Q_T} \lambda e^2 \, dx dt \leq \frac{1}{2} \| e(\cdot, 0) \|_{\Omega}^2 + \\
\int_0^T \left( \| \mathcal{F}_{f, \mu} \|_{\Omega} \| \sqrt{\lambda} e \|_{\Omega} + \frac{C_{F\Omega}}{\sqrt{\nu_1}} \| \mathcal{F}_{f,1-\mu} \|_{\Omega} \| \nabla e \|_A + \| \mathcal{R}_d \|_{A^{-1}} \| \nabla e \|_A + \| \mathcal{R}_b \|_{\Gamma_R} \frac{C_{\text{tr}}}{\sqrt{\nu_1}} \| \nabla e \|_A \right) \, dt. \tag{27}
\]

The second term in the right-hand side of (27) is estimated by the Young–Fenchel inequality

\[
\int_0^T \| \mathcal{F}_{f, \mu} \|_{\omega} \| \sqrt{\lambda} e \|_{\Omega} \, dt \leq \int_0^T \left( \frac{\gamma}{2} \| \mathcal{F}_{f, \mu} \|_{\omega}^2 + \frac{1}{2\gamma} \| \sqrt{\lambda} e \|_{\Omega}^2 \right) \, dt, \tag{28}
\]
Thus, we see that
\begin{align*}
\int_0^T \frac{C_{fQ}}{\sqrt{\nu_1}} \| \mathcal{R}_{f,1-\mu} \|_A \| \nabla e \|_A \, dt & \leq \int_0^T \left( \frac{\alpha_1(t)}{2} \frac{C_{fQ}^2}{\sqrt{\nu_1}} \| \mathcal{R}_{f,1-\mu} \|_Q^2 + \frac{1}{2\alpha_1(t)} \| \nabla e \|_A^2 \right) \, dt, \\
\int_0^T \| \mathcal{R}_d \|_{A^{-1}} \| \nabla e \|_A \, dt & \leq \int_0^T \left( \frac{\alpha_2(t)}{2} \| \mathcal{R}_d \|_{A^{-1}}^2 + \frac{1}{2\alpha_2(t)} \| \nabla e \|_A^2 \right) \, dt,
\end{align*}
and
\begin{align*}
\int_0^T \| \mathcal{R}_b \|_{\Gamma_R} \frac{C_{\mathcal{R}_b}}{\sqrt{\nu_1}} \| \nabla e \|_A \, dt & \leq \int_0^T \left( \frac{\alpha_3(t)}{2} \frac{C_{\mathcal{R}_b}^2}{\sqrt{\nu_1}} \| \mathcal{R}_b \|_{\Gamma_R}^2 + \frac{1}{2\alpha_3(t)} \| \nabla e \|_A^2 \right) \, dt.
\end{align*}
Here, \( \alpha_1(t) \), \( \alpha_2(t) \), and \( \alpha_3(t) \) are functions satisfying the relation (20). Then, the estimate (13) follows from (28)–(31).

(ii) Existence of the pair \((v, y) \in H^1(Q_T) \times Y_{\text{div}}^*(Q_T)\) minimizing the functional \( \overline{\mathcal{M}}_1^2(v, y; \delta, \gamma, \mu) \) can be proven straightforwardly. Indeed, let \( v = u \) and \( y = A \nabla u \). Since \( \text{div} \ (A \nabla u) \in L^2(Q_T) \), we see that \( y \in Y_{\text{div}}^*(Q_T) \). In this case, according to (1)–(4),
\begin{align*}
e(x, 0) &= (u - v)(x, 0) = \varphi(x) - v(x, 0) = 0, \\
\mathcal{R}_f(u, A \nabla u) &= f - u_t - \lambda u + \text{div} \ A \nabla u = 0, \\
\mathcal{R}_d(u, A \nabla u) &= A \nabla u - A \nabla u = 0, \\
\mathcal{R}_b(v, y) &= g - \sigma v - A \nabla u \cdot n = 0,
\end{align*}
Thus, we see that \( \overline{\mathcal{M}}_1^2(v, y; \delta, \gamma, \mu) = 0 \) and, therefore, the exact lower bound of \( \overline{\mathcal{M}}_1^2(v, y; \delta, \gamma, \mu) \) is attained on the pair presenting the exact solution of (1)–(4).

Assume that \( \overline{\mathcal{M}}_1^2(v, y; \delta, \gamma, \mu) = 0 \), which means that for a.a. \((x, t) \in Q_T\) the following relations hold:
\begin{align*}
y &= A \nabla v & \text{a.a.} \ (t, x) & \in Q_T, \\
f - v_t - \lambda v + \text{div} \ y &= 0 & \text{a.a.} \ (t, x) & \in Q_T, \\
v(x, 0) &= \varphi(x) & \text{a.a.} \ x & \in \Omega, \\
v &= 0 & \text{a.a.} \ (t, x) & \in S_D, \\
y \cdot n + \sigma v &= g & \text{a.a.} \ (t, x) & \in S_R.
\end{align*}
From (33)–(36), it follows that
\begin{equation}
\int_{Q_T} (f - v_t - \lambda v) \eta \, dx dt - \int_{Q_T} y \cdot \nabla \eta \, dx dt + \int_{S_N} g \eta \, ds dt = 0, \quad \forall \eta \in H^1(Q_T).
\end{equation}
In view of (32), this relation is equivalent to (7), whence it follows that \( v = u \) and \( y = A \nabla u \).

**Remark 1.** We see that \( \overline{\mathcal{M}}_1^2(v, y; \delta, \gamma, \mu) \) depends on a collection of parameters, which can be selected within certain admissible sets. Varying \( \delta \) and \( \gamma \) allows us to obtain estimates for different error measures. By selecting the functions \( \alpha_i \) and \( \mu \), we find the best possible value of the majorant. This fact is beneficial for practical applications because we can select values of the parameters in an optimal way for...
a concrete problem. In particular, \( \mu \) can be set to 0 and 1. For these two cases, we use the abridged notation \( \overline{M}^2_{I, \mu=0} \) and \( \overline{M}^2_{I, \mu=1} \):

\[
\overline{M}^2_{I, \mu=0} := \| u \cdot 0 \|_{\Omega}^2 + \int_0^T \left( \alpha_1(t) \frac{C^2 \nu_1}{\nu_1} \| \mathcal{R}_f \|_{\Omega}^2 + \alpha_2(t) \| \mathcal{R}_d \|_{\Lambda - 1}^2 + \alpha_3(t) \frac{C^2 \nu_1}{\nu_1} \| \mathcal{R}_b \|_{\Gamma R}^2 \right) \, dt
\]

and

\[
\overline{M}^2_{I, \mu=1} := \| u \cdot 0 \|_{\Omega}^2 + \int_0^T \left( \gamma \| \mathcal{R}_f \|_{\Omega}^2 + \alpha_2(t) \| \mathcal{R}_d \|_{\Lambda - 1}^2 + \alpha_3(t) \frac{C^2 \nu_1}{\nu_1} \| \mathcal{R}_b \|_{\Gamma R}^2 \right) \, dt.
\]

The majorant \( \overline{M}^2_{I, \mu=0} \) is well adapted to problems, in which \( \lambda \) is small or zero (so that the impact of the reaction term is insignificant). In such type problems, we should avoid the term \( \| \mathcal{R}_f(u, v, y) \|_{\Omega}^2 \), which makes the whole estimate sensitive to the residual \( \mathcal{R}_f(v, y) \) and may lead to a considerable overestimation of the error. The estimate \( \overline{M}^2_{I, \mu=1} \) is useful if \( \lambda \) is not small and may attain large values in some parts of \( \Omega \). If \( \lambda \) reaches both small (or zero) and large values, then the combined estimate (13) is preferable.

2.1. Estimates based upon domain decomposition. The majorant defined by (13) contains the Friedrichs constant \( C_{F\Omega} \) and the trace constant \( C_{T\Omega} \). If \( \Omega \) has a complicated geometry, then finding these constants (or guaranteed bounds of them) may not be an easy task. Below we suggest the method, which allows to overcome this difficulty. It is based on domain decomposition and leads to the estimates with a different set of constants (a consequent discussion of this method for elliptic problems can be found in [13]).

Assume that \( \Omega \) is decomposed into a set of sub-domains

\[
\overline{\Omega} = \bigcup_{i=1,...,N} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j.
\]

We use the Poincare inequalities

\[
\| \bar{w} \|_{\Omega_i} \leq C_{P\Omega_i} \|\nu \|_{\Omega_i}, \quad i = 1, ..., N, \quad \forall w \in H^1(\Omega),
\]

where \( \bar{w} = w - \{ |w| \}_\Omega \), and \( \{ |w| \}_\Omega \) denotes the mean value of \( w \) on \( \Omega \). If all \( \Omega_i \) are convex, then \( C_{P\Omega_i} \) can be estimated from the above by the quantity \( \text{diam} \Omega_i / \pi \) (see [11]). We use this fact in order to represent the majorant in a somewhat different form. In further analysis, we assume (for the sake of simplicity only) that \( S_T = S_D \) and \( \varphi(x) = v(x, 0) \).

Theorem 2.2. For any \( v \in \dot{H}^1(Q_T) \) and \( y \in Y_{\text{div}}^*(Q_T) \) the following inequality holds:

\[
(2 - \delta) \| \nabla \hat{e} \|_{\Lambda}^2 + \left( 2 - \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \| \sqrt{\lambda} \hat{e} \|_{Q_T}^2 + \| e(\cdot, T) \|_{\Omega}^2 =: [e]_{(\nu, \theta, \zeta)} = \overline{M}^2_{I, N} := \int_0^T \left( \frac{\rho_1}{\nu_1} \frac{\| \mathcal{R}_f(u, v, y) \|_{\Omega}^2}{\sqrt{\lambda}} + \rho_2 R_{I, 1}(t) + \alpha_1(t) R_{I, 2}(t) + \alpha_2(t) \| \mathcal{R}_d(v, y) \|_{\Lambda - 1}^2 \right) \, dt.
\]
where $\delta \in (0, 2]$, $\rho_1 \geq \frac{1}{2 - \frac{1}{\rho_2}}$, $\mu \in [0, 1]$, $\mathcal{R}_f, \mu(v, y)$ and $\mathcal{R}_d(v, y)$ are defined in (15) and (16), respectively, and

$$R_{1,1}(t) := \sqrt{\sum_{i=1}^{N} \frac{|\Omega_i|}{\lambda_i} \left( \left\{ |\mathcal{R}_f,1-\mu| \right\}_{\Omega_i} \right)^2}, \quad R_{1,2}(t) := \sqrt{\sum_{i=1}^{N} \frac{C_{\rho_i}}{\nu} ||\mathcal{R}_f,1-\mu||_{\Omega_i}^2}.$$ 

Here, $\lambda_i = \min_{x \in \Omega_i} \lambda(x, t)$ for a.a. $t \in [0, T]$, $\nu = 2 - \delta$, $\theta = \sqrt{\left(2 - \frac{1}{\rho_1} - \frac{1}{\rho_2}\right)} \lambda$, and $\zeta = 1$, and $\alpha_1(t)$, $\alpha_2(t)$ are positive scalar-valued functions satisfying the relation $\frac{1}{\alpha_1(t)} + \frac{1}{\alpha_2(t)} = \delta$.

Proof. Consider the integral identity (22). The term $\mathcal{J}_f$ can be represented as

$$\mathcal{J}_f = \int_{Q_T} \mathcal{R}_f, \mu \, dx \, dt + \int_{Q_T} \mathcal{R}_f,1-\mu \, dx \, dt = \mathcal{J}_f^\mu + \mathcal{J}_f^{1-\mu}. \quad (41)$$

$\mathcal{J}_f^\mu$ is estimated as

$$\mathcal{J}_f^\mu \leq \int_0^T \left\| \frac{\mathcal{F}_f, \mu}{\sqrt{\lambda}} \right\|_{\Omega_i} \left\| \sqrt{\lambda_e} \right\|_{\Omega_i} \, dt. \quad (42)$$

By means of the Hölder inequality, for $\mathcal{J}_f^{1-\mu}$ we have

$$\mathcal{J}_f^{1-\mu} = \int_0^T \left( \sum_{i=1}^{N} \int_{\Omega_i} \mathcal{F}_f,1-\mu \, dx + \sum_{i=1}^{N} \left\{ |\mathcal{F}_f,1-\mu| \right\}_{\Omega_i} \int_{\Omega_i} e \, dx \right) \, dt \leq \int_0^T \sum_{i=1}^{N} \frac{\sqrt{|\Omega_i|}}{\sqrt{\lambda_i}} \left\{ |\mathcal{F}_f,1-\mu| \right\}_{\Omega_i} \left\| \sqrt{\lambda_e} \right\|_{\Omega_i} \, dt. \quad (43)$$

where $\lambda_i = \min_{x \in \Omega_i} \lambda(x, t)$ for a.a. $t \in [0, T]$. Each of the terms on the right-hand side of (43) can be estimated as follows:

$$\int_0^T \sum_{i=1}^{N} \int_{\Omega_i} \mathcal{F}_f,1-\mu \, dx \, dt \leq R_{1,2} \left\| \nabla e \right\|_{\lambda} \, dt, \quad (44)$$

$$\int_0^T \sum_{i=1}^{N} \frac{\sqrt{|\Omega_i|}}{\sqrt{\lambda_i}} \left\{ |\mathcal{F}_f,1-\mu| \right\}_{\Omega_i} \left\| \sqrt{\lambda_e} \right\|_{\Omega_i} \, dt \leq \int_0^T R_{1,1} \left\| \sqrt{\lambda_e} \right\|_{\Omega_i} \, dt. \quad (45)$$

At last, using the Young–Fenchel inequality, we obtain the following estimates

$$\int_0^T \left\| \frac{\mathcal{F}_f, \mu}{\sqrt{\lambda}} \right\|_{\Omega_i} \left\| \sqrt{\lambda_e} \right\|_{\Omega_i} \, dt \leq \int_0^T \left( \frac{\rho_1}{2} \left\| \mathcal{F}_f, \mu \right\|_{\Omega_i}^2 + \frac{1}{2 \rho_1} \left\| \sqrt{\lambda_e} \right\|_{\Omega_i}^2 \right) \, dt, \quad (46)$$
These integral type conditions do not lead to essential technical difficulties provided positive scalar-valued functions satisfying the relation
\[ \| \nabla e \|_\Omega \leq \left( \frac{\rho^2}{2} R_{1,1}^2 + \frac{1}{2\rho^2} \| \sqrt{\lambda} \|_\Omega^2 \right) \]  \( \text{(47)} \)

\[ \int_0^T R_{1,2} \| \nabla e \|_A dt \leq \int_0^T \left( \frac{\alpha_1(t)}{2} R_{1,2}^2 + \frac{1}{2\alpha_1(t)} \| \nabla e \|_A^2 \right) dt, \]  \( \text{(48)} \)

and, analogously,
\[ \int_0^T \| \mathcal{R}_d(v, y) \|_{A^{-1}} \| \nabla e \|_A dt \leq \int_0^T \left( \frac{\alpha_2(t)}{2} \| \mathcal{R}_d \|_{A^{-1}}^2 + \frac{1}{2\alpha_2(t)} \| \nabla e \|_A^2 \right) dt. \]  \( \text{(49)} \)

By combining (46)–(49), we obtain (40).

Consider a special case, which arises if we impose additional conditions, namely,
\[ \left\{ \left\{ \mathcal{R}_{f,1-\mu}(v, y) \right\} \right\}_{\Omega_i} = 0, \quad i = 1, \ldots, N, \quad \text{for a.a. } t \in [0, T], \]  \( \text{(50)} \)

where \( \mu \) is inherited from (13). Since the functions \( y \) and \( \mu \) are in our disposal, these integral type conditions do not lead to essential technical difficulties provided that \( N \) is not too large. Now, (40) can be represented in a simpler form.

**Theorem 2.3.** If (50) is satisfied, then for any \( v \in \dot{H}^1(Q_T) \) and \( y \in Y_{\text{div}}^*(Q_T) \)
\[ (2 - \delta) \| \nabla e \|_A^2 + \left( 2 - \frac{1}{\gamma} \right) \| \sqrt{\lambda} \|_{Q_T}^2 + \| e(\cdot, T) \|_{\Omega_i}^2 =: [e]_T^2 \leq \mathcal{M}_{1,N}^T := \int_0^T \left[ \left\{ \frac{\| \mathcal{R}_{f,\mu}(v, y) \|_\Omega}{\sqrt{\lambda}} \right\}^2 + \alpha_1(t) R_{1,1}^2(t) + \alpha_2(t) \| \mathcal{R}_d(v, y) \|_{A^{-1}}^2 \right] \]  \( \text{(51)} \)

where \( \delta \in (0, 2), \gamma \geq \frac{1}{2}, \mu \in [0, 1], \mathcal{R}_{f,\mu}(v, y) \) and \( \mathcal{R}_d(v, y) \) are defined in (15) and (16), respectively, and
\[ R_{1}(t) := \sqrt{\sum_{i=1}^{N} \frac{C_{P_{d,i}}^2}{\nu_1} \| \mathcal{R}_{f,1-\mu} \|_{\Omega_i}}, \]
\[ \nu = 2 - \delta, \theta = \sqrt{\left( 2 - \frac{1}{\gamma} \right) \lambda}, \] and \( \zeta = 1 \) are positive weights, and \( \alpha_1(t), \alpha_2(t) \) are positive scalar-valued functions satisfying the relation \( \frac{1}{\alpha_1(t)} + \frac{1}{\alpha_2(t)} = \delta. \)

**Proof.** If (50) holds, then,
\[ \mathcal{J}_{f}^{1-\mu} = \int_0^T \sum_{i=1}^{N} \int_{\Omega_i} \mathcal{R}_{f,1-\mu} e dx dt = \int_0^T \sum_{i=1}^{N} \int_{\Omega_i} \mathcal{R}_{f,1-\mu} e dx dt. \]  \( \text{(52)} \)

Therefore, using (39), we obtain
\[ \mathcal{J}_{f}^{1-\mu} \leq \int_0^T R_{1} \| \nabla e \|_A dt. \]  \( \text{(53)} \)
By means of the Young–Fenchel inequality, we deduce

$$
\int_0^T \left\| \mathcal{R}_{f, \mu} \right\|_{\Omega} \left\| \sqrt{\mathcal{N}} \right\|_{\Omega} dt \leq \int_0^T \left( \frac{\gamma}{2} \left\| \mathcal{R}_{f, \mu} \right\|_{\Omega}^2 + \frac{1}{2\gamma} \left\| \sqrt{\mathcal{N}} \right\|_{\Omega}^2 \right) dt \quad (54)
$$

and

$$
\int_0^T R_t \left\| \nabla e \right\|_A^2 dt \leq \int_0^T \left( \frac{\alpha_1(t)}{2} R_t^2 + \frac{1}{2\alpha_1(t)} \left\| \nabla e \right\|_A^2 \right) dt. \quad (55)
$$

The term $\mathcal{I}_d$ is estimated analogously to the method used in proof of Theorem 2.1:

$$
\mathcal{I}_d \leq \int_0^T \left\| \mathcal{R}_d \right\|_{A^{-1}} \left\| \nabla e \right\|_A dt \leq \int_0^T \left( \frac{\alpha_2(t)}{2} \left\| \mathcal{R}_d \right\|_{A^{-1}}^2 + \frac{1}{2\alpha_2(t)} \left\| \nabla e \right\|_A^2 \right) dt. \quad (56)
$$

Therefore, (54)–(56) yield the estimate (51). \qed

2.2. Two-sided estimates for combined norms. In modern numerical methods (e.g., in various mixed finite element schemes) the approximations are generated for both primal and dual components of the solution. We note that this concept is perfectly motivated by physical arguments because primal and dual components often reflect physically meaningful parts of the solution. By following this idea, we now consider the solution of (1)–(4) as a pair $(u, p) \in V_1^0(Q_T) \times Y_{\text{div}}(Q_T)$. In order to measure the deviation of the approximation $(v, y) \in \hat{H}^1(Q_T) \times Y_{\text{div}}(Q_T)$ from $(u, p)$, we use the combined primal-dual norm

$$
\left\| [(u, p) - (v, y)] \right\|_{(\nu, \theta, \zeta, \chi)}^2 := \nu \left\| \nabla e \right\|_A^2 + \theta \left\| (y - p) \right\|_{A^{-1}}^2 + 
\zeta \left\| \text{div} (p - y) - (u - v) \right\|_{Q_T}^2 + \chi \left\| e(\cdot, T) \right\|_{\Omega}^2. \quad (57)
$$

Let $\lambda = 0$, $S_N = S_D$, and $\varphi(x) = v(x, 0)$. Then, from Theorem 2.1 (with $\beta = \text{const}$, $\delta = 1$, and $\mu = 0$) the estimate can be written in the form

$$
\left\| \nabla e \right\|_A^2 + \left\| e(\cdot, T) \right\|_{\Omega}^2 \leq M_1^2 := (1 + \beta) \left\| y - A\nabla v \right\|_{A^{-1}}^2 + \left( 1 + \frac{1}{\beta} \right) \frac{C_{F1}}{v_1} \left\| f - v_t + \text{div} y \right\|_{Q_T}^2. \quad (58)
$$

Since $p = A\nabla u$, we reform the right-hand side of (58) as follows:

$$
M_1^2 \leq (1 + \beta) \left( \left\| \nabla (u - v) \right\|_A^2 + \left\| y - p \right\|_{A^{-1}}^2 \right) + \left( 1 + \frac{1}{\beta} \right) \frac{C_{F1}}{v_1} \left\| f - v_t + \text{div} y \right\|_{Q_T}^2. \quad (59)
$$
By using (1), we find that

\[
(1 + \beta) \left( \| \nabla (u - v) \|_A^2 + \| y - p \|_{A^{-1}}^2 \right) + \left( 1 + \frac{1}{\beta} \right) \frac{C_{\mathcal{F} \Omega}^2}{\nu_1} \| f - v_t + \text{div} \ y \|_{\mathcal{Q}_T}^2 \leq \\
(1 + \beta) \left( \| \nabla (u - v) \|_A^2 + \| y - p \|_{A^{-1}}^2 \right) + \left( 1 + \frac{1}{\beta} \right) \frac{C_{\mathcal{F} \Omega}^2}{\nu_1} \| \text{div} (p - y) - (u - v)_t \|_{\mathcal{Q}_T}^2 + \\
\| e(\cdot, T) \|_{\Omega}^2 = \| (u, p) - (v, y) \|_C^2, \quad (60)
\]

where \( \tilde{\nu} = \tilde{\theta} = (1 + \beta), \tilde{\zeta} = \left( 1 + \frac{1}{\gamma} \right) \frac{C_{\mathcal{F} \Omega}^2}{\nu_1}, \) and \( \tilde{\chi} = 1. \) Next, by combining the first two terms, applying (58), and, finally, adding and subtracting \( A \nabla v \) in the third term, we obtain

\[
\| (u, p) - (v, y) \|_C^2 \leq \max \left\{ 1, (1 + \beta) \right\} \left( \| e(\cdot, T) \|_{\Omega}^2 + \| \nabla (u - v) \|_A^2 + \right. \\
(1 + \beta) \| y - A \nabla v + A \nabla v - p \|_{A^{-1}}^2 + \left( 1 + \frac{1}{\beta} \right) \frac{C_{\mathcal{F} \Omega}^2}{\nu_1} \| f - v_t + \text{div} \ y \|_{\mathcal{Q}_T}^2 \leq \\
\max \left\{ 1, (1 + \beta) \right\} \left( (1 + \beta) \| y - A \nabla v \|_{A^{-1}}^2 + \left( 1 + \frac{1}{\beta} \right) \frac{C_{\mathcal{F} \Omega}^2}{\nu_1} \| f - v_t + \text{div} \ y \|_{\mathcal{Q}_T}^2 \right) + \\
(1 + \beta) \left( \| y - A \nabla v \|_{A^{-1}}^2 + \| A \nabla v - p \|_{A^{-1}}^2 \right) + \left( 1 + \frac{1}{\beta} \right) \frac{C_{\mathcal{F} \Omega}^2}{\nu_1} \| f - v_t + \text{div} \ y \|_{\mathcal{Q}_T}^2. \quad (61)
\]

Hence, we obtain the double inequality

\[
\mathbf{M}_1 \leq \| (u, p) - (v, y) \|_C^2 \leq \left( \max \left\{ 1, (1 + \beta) \right\} + \beta + 2 \right) \mathbf{M}_1, \quad (62)
\]

which shows that the majorant is equivalent to the combined primal-dual error norm. In other words, \( \mathbf{M}_1 \) (which contains only known functions and parameters) adequately reflects the distance from \((v, y) \in H^1(Q_T) \times Y_{\text{div}}^*(Q_T)\) to the exact solution \((u, p)\). In particular, this means that if \((u_h, p_h)\) is the sequence of approximations computed on a certain set of meshes \( \mathcal{F}_h \), which converges to \((u, p)\) with the rate \( h^\alpha \), then the values of the majorant tend to zero with the same rate.

3. An advanced form of the majorant.

**Theorem 3.1.** (i) For any \( v, w \in H^1(Q_T) \) and \( y \in Y_{\text{div}}^*(Q_T) \) the following estimate holds:

\[
(2 - \delta) \| \nabla e \|_A^2 + \left( 2 - \frac{1}{\gamma} \right) \| \nabla e \|_{\mathcal{Q}_T}^2 + \left( 1 - \frac{1}{\epsilon} \right) \| e(\cdot, T) \|_{\Omega}^2 + 2 \| \sqrt{\sigma} e \|_{\mathcal{S}_{R}}^2 =:
\]

\[
|e|_{v, \theta, \zeta, \chi} \leq \mathbf{M}_1^2 (v, y, w; \delta, \epsilon, \gamma, \mu) := \| w(\cdot, T) \|_{\Omega}^2 + 2L(v, w) + l(w, v)
\]

\[
\int_0^T \left( \gamma \left\| \mathcal{R}_{1, \mu}(v, y, w) \right\|_{\Omega}^2 + \alpha_1(t) \frac{C_{\mathcal{F} \Omega}^2}{\nu_1} \| \mathcal{R}_{1, 1-\mu}(v, y, w) \|_{\Omega}^2 + \\
\alpha_2(t) \| \mathcal{R}_{2}(v, y, w) \|_{A^{-1}}^2 + \alpha_3(t) \frac{C_{\mathcal{F} \Omega}^2}{\nu_1} \| \mathcal{R}_{3}(v, y, w) \|_{\Gamma_R}^2 \right) dt, \quad (63)
\]
where $\delta \in (0, 2]$, $\gamma \geq \frac{1}{2}$, $\epsilon \geq 1$, and $\mu \in [0, 1]$,

\[
L(v, w) = \int_{Q_T} \left( v_t w + A \nabla v \cdot \nabla w + \lambda v w - f w \right) \, dx dt - \int_{S_R} \left( g - \sigma v \right) w \, ds dt,
\]

\[
l(v, w) = \int_{\Omega} |v(x, 0) - \varphi(x)|^2 - 2w(x, 0)(\varphi(x) - v(0, x)) \, dx,
\]

and

\[
\mathcal{R}_1(v, y, w) := f - (v + w)t - \lambda(v - w) + \text{div} y,
\]

\[
\mathcal{R}_{1, \mu}(v, y, w) := \mu \mathcal{R}_1(v, y, w), \quad \mathcal{R}_{1, 1-\mu}(v, y, w) := (1 - \mu) \mathcal{R}_1(v, y, w),
\]

\[
\mathcal{R}_2(v, y, w) := y - A \nabla (v - w),
\]

\[
\mathcal{R}_3(v, y, w) := g - \sigma (v - w) - y \cdot n,
\]

\[
\nu = 2 - \delta, \theta = \sqrt{\left(2 - \frac{1}{\gamma}\right)} \lambda, \quad \zeta = 1 - \frac{1}{2}, \quad \chi = 2 \text{ are positive weights, and } \alpha_1(t), \alpha_2(t), \text{ and } \alpha_3(t) \text{ are positive function satisfying (20).}
\]

(ii) For any $\delta \in (0, 2]$, $\gamma \geq \frac{1}{2}$, $\epsilon \geq 1$, and $\mu \in [0, 1]$ the lower bound of the variation problem

\[
\inf_{v, w \in \mathcal{H}^0_\Omega(v, y, w; \delta, \epsilon, \gamma, \mu)}
\]

\[
y \in Y^*_{\text{div}}(Q_T)
\]

is zero, and it is attained if and only if $v = u$, $y = A \nabla u$, and $w = 0$.

**Proof:** (i) We rewrite the right-hand side of (11) by inserting functions $w \in H^1(Q_T)$ and $y \in Y^*_{\text{div}}(Q_T)$, which implies the following relation

\[
\frac{1}{2} \| e(\cdot, T) \|^2_\Omega + \| \nabla e \|^2_A + \int_{S_R} \sigma e^2 \, ds dt + \int_{Q_T} \lambda e^2 \, dx dt =
\]

\[
\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \int_{S_R} (g - \sigma v - y \cdot n) e \, ds dt + \frac{1}{2} \| e(\cdot, 0) \|^2_\Omega,
\]

where

\[
\mathcal{I}_1 = \int_{Q_T} \mathcal{R}_1 e \, dx dt, \quad \mathcal{I}_2 = \int_{Q_T} \mathcal{R}_2 \cdot \nabla e \, dx dt, \quad \mathcal{I}_3 = \int_{Q_T} ((w_t - \lambda)e - A \nabla w \cdot \nabla e) \, dx dt.
\]

(72)

The term $\mathcal{I}_3$ can be rewritten as

\[
\mathcal{I}_3 = L(v, w) + \int_{\Omega} \left( e(x, T)w(x, T) - e(x, 0)w(x, 0) \right) \, dx + \int_{S_R} \sigma w e \, ds dt.
\]

(73)

By combining (71) and (73), we obtain

\[
\frac{1}{2} \| e(\cdot, T) \|^2_\Omega + \| \nabla e \|^2_A + \int_{S_R} \sigma e^2 \, ds dt + \int_{Q_T} \lambda e^2 \, dx dt = \mathcal{I}_1 + \mathcal{I}_2 + L(v, w)+
\]

\[
\int_{S_R} \mathcal{R}_3 e \, ds dt + \int_{\Omega} e(x, T)w(x, T) \, dx + \int_{\Omega} \left( \frac{1}{2} e^2(x, 0) - e(x, 0)w(x, 0) \right) \, dx,
\]

(74)
Using the same technique as in Section 2, the right-hand side of (74) can be estimated the following way:

\[
\int_{\Omega} e(x, T) w(x, T) \, dx \, dt \leq \frac{1}{2\epsilon} \|e(\cdot, T)\|_{Q_T}^2 + \frac{\epsilon}{2} \|w(\cdot, T)\|_{Q_T}^2, \tag{75}
\]

\[
\int_0^T \left\| \mathcal{S}_{1, \mu} \right\|_\Omega \|\sqrt{\lambda} e\|_{Q_T}^2 \, dt \leq \int_0^T \left( \frac{\gamma}{2} \left\| \mathcal{S}_{1, \mu} \right\|_\Omega \|\sqrt{\lambda} e\|_{Q_T}^2 + \frac{1}{2\gamma} \right) \, dt, \tag{76}
\]

\[
\int_0^T \frac{C_{\mathcal{F}}}{\nu_1} \left\| \mathcal{S}_{1,1-\mu} \right\|_A \|\nabla e\|_A \, dt \leq \int_0^T \left( \frac{\alpha_1(t) C_{\mathcal{F}}}{\nu_1} \left\| \mathcal{S}_{1,1-\mu} \right\|_\Omega \|\nabla e\|_A^2 + \frac{1}{2\alpha_1(t)} \right) \, dt, \tag{77}
\]

\[
\int_0^T \left\| \mathcal{S}_{2, A^{-1}} \right\|_A \|\nabla e\|_A \, dt \leq \int_0^T \left( \frac{\alpha_2(t)}{2} \left\| \mathcal{S}_{2, A^{-1}} \right\|_\Omega \|\nabla e\|_A^2 + \frac{1}{2\alpha_2(t)} \right) \, dt, \tag{78}
\]

\[
\int_0^T \left\| \mathcal{S}_{3, \Gamma_R} \right\|_{\Gamma_R} \|\nabla e\|_A \, dt \leq \int_0^T \left( \frac{\alpha_3(t) C_{\mathcal{S}_{\Gamma}}}{{\nu}_1} \left\| \mathcal{S}_{3, \Gamma_R} \right\|_\Omega \|\nabla e\|_A^2 + \frac{1}{2\alpha_3(t)} \right) \, dt, \tag{79}
\]

where \( \gamma \geq 1, \epsilon \geq 1, \) and \( \alpha_1(t), \alpha_2(t), \) and \( \alpha_3(t) \) are functions satisfying (20). Thus, by combination of (75)–(79), we obtain the required estimate (63).

(ii) This item is proven by the same arguments as in Theorem 2.1.

3.1. An advanced majorant based upon domain decomposition. Now, we deduce an advanced versions of the estimates (40) and (51). Let (38) hold. First, we consider the case where \( \lambda \) is not small (or zero). Assume (for the sake of simplicity only) that \( S_T = S_D \). Then, we have the following result.

**Theorem 3.2.** For any \( v, w \in \dot{H}^1(Q_T) \) and \( y \in Y^*_{\text{div}} (Q_T) \) we obtain the estimate

\[
(2 - \delta) \|\nabla e\|_A^2 + \left( 2 - \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \|\sqrt{\lambda} e\|_{Q_T}^2 + \left( 1 - \frac{1}{\epsilon} \right) \|e(\cdot, T)\|_{\Omega}^2 + 2 \|\sqrt{\sigma} e\|_{\mathcal{S}_0}^2 =: [e]_{\nu, \theta, \zeta, \chi}^2 \leq \left( \mathcal{M}_{\text{IH}}^2 \right) + 2L(v, w) + I(v, w) + \int_0^T \left( \rho_1 \left\| \mathcal{S}_{1, \mu} (v, y, w) \right\|_\Omega^2 + \rho_2 R_{1,1}(t) + \alpha_1(t) R_{1,2}(t) + \alpha_2(t) \|\mathcal{S}_{2, v, y, w}\|_A^{-1} \right) \, dt,
\]

where \( \delta \in (0, 2], \rho_1 \geq \frac{1}{2 - \frac{1}{\rho_1}}, \epsilon \geq 1, \) and \( \mu \in [0, 1], \mathcal{S}_{1, \mu} (v, y, w) \) and \( \mathcal{S}_{2, v, y, w} \) are defined by (67) and (68), respectively, and

\[
R_{1,1}(t) := \sum_{i=1}^N \frac{\Omega_i}{\lambda_i} \left\| \mathcal{S}_{1,1-\mu} \right\|_{\Omega_i}^2, \quad R_{1,2}(t) := \sum_{i=1}^N \frac{C_{\mathcal{F}_{\Omega_i}}}{\nu_1} \left\| \mathcal{S}_{1,1-\mu} \right\|_{\Omega_i}^2.
\]

Here, \( \lambda_i = \min_{x \in \Omega_i} \lambda(x, t) \) for a.a. \( t \in [0, T], \) and \( \nu = 2 - \delta, \theta = \sqrt{\left( 2 - \frac{1}{\rho_1} - \frac{1}{\rho_2} \right)} \chi, \zeta = 1 - \frac{1}{\rho_1}, \chi = 2, \) and \( \alpha_1(t), \alpha_2(t) \) are positive functions satisfying the relation

\[
\frac{1}{\sigma_1(t)} + \frac{1}{\sigma_2(t)} = \delta.
\]
For problems, in which $\lambda$ can attain small or zero values we deduce another estimate. Assume that

$$\left\{ [\mathcal{A}_{1,1 - \mu} (v, y, w)] \right\}_{\Omega} = 0, \quad i = 1, \ldots, N, \quad \text{for a.a. } t \in [0, T].$$  \tag{80}$$

**Theorem 3.3.** (i) If (80) holds, then for $v, w \in \dot{H}^1(Q_T)$ and $y \in Y^+_{\text{div}} (Q_T)$

$$(2 - \delta) \| \nabla e \|^2_A + \left( 2 - \frac{1}{\gamma} \right) \| \sqrt{\lambda e} \|^2_{Q_T} + \left( 1 - \frac{1}{\epsilon} \right) \| e (\cdot, T) \|^2_{\Omega} + 2 \| \sqrt{\sigma e} \|^2_{S_R} =:$$

$$[e]^2_{(\nu, \nu, \alpha, \mu)} \leq \bar{M}^2_{11, N} := \epsilon \| w(x, T) \|^2_{\Omega} + 2L(v, w) + l(v, w) +$$

$$\int_0^T \left( \gamma \frac{\mathcal{A}_{1, \mu} (v, y, w)}{\sqrt{\lambda}} \right)_{\Omega}^2 + \alpha_1 (t) R_{11}^2 (t) + \alpha_2 (t) \| \mathcal{A}_{2} (v, y, w) \|^2_{A - 1} \right) dt,$$

where $\delta \in (0, 2], \gamma \geq \frac{1}{2}, \epsilon \geq 1,$ and $\mu \in [0, 1], \mathcal{A}_{1, \mu} (v, y, w)$ and $\mathcal{A}_{2} (v, y, w)$ are defined by (67) and (68), respectively, and

$$R_{11} (t) := \sqrt{\sum_{i=1}^N \frac{C_{P, i}^2}{\nu_1} \| \mathcal{A}_{1,1 - \mu} \|^2_{\Omega}}, \tag{81}$$

$$\nu = 2 - \delta, \quad \theta = \sqrt{\left( 2 - \frac{1}{\gamma} \right) \lambda}, \quad \zeta = 1 - \frac{1}{\epsilon}, \quad \chi = 2, \quad \text{and } \alpha_1 (t), \alpha_2 (t) \text{ are positive functions satisfying the relation } \frac{1}{\alpha_1 (t)} + \frac{1}{\alpha_2 (t)} = \delta.$$

Theorems 3.2 and 3.3 can be proven by combining arguments used in Theorems 2.2 and 2.3. Since proofs do not contain principally new items, we omit these details.

### 3.2. Equivalence of $[e]^2_{(\nu, \nu, \alpha, \mu)}$ and $\bar{M}^2_{11}$

We aim to show that the advanced form of the majorant does not lead to an uncontrollable overestimation of the actual value of the norm (8). For this purpose, we estimate $\bar{M}^2_{11}$ from above and show that this upper bound is equivalent to the error norm. Henceforth, we assume that $S_T = S_D$, $\beta = \text{const}$ and $\mu = 0$. As before, these assumption are introduced for the sake of simplicity only. Similar estimates for the problems with mixed boundary conditions and variable coefficients can be deduced by arguments close to those presented below.

Assume that $y = A \nabla u \in Y^+_{\text{div}} (Q_T)$ and $w = u - v = e$, then

$$\mathcal{A}_1 (v, A \nabla u, e) = f - (v + e)_t - \lambda (v - e) + \text{div} (A \nabla u) = 2\lambda e,$$

$$\mathcal{A}_2 (v, A \nabla u, e) = A \nabla u - A \nabla (v - e) = 2A \nabla e. \tag{82}$$

The functional (64) can be represented as follows:

$$L(v, e) = \int_{Q_T} \left( v_t e + A \nabla v \cdot \nabla e + \lambda v e - fe \right) dx dt =$$

$$\int_{Q_T} \left( u_t e + A \nabla u \cdot \nabla e + \lambda u e - fe \right) dx dt - \int_{Q_T} \left( A \nabla e \cdot \nabla e + \epsilon_1 e + \lambda e^2 \right) dx dt. \tag{83}$$
In view of (1), the first term in the right-hand side of (83) vanishes, and we find that
\[ L(v, e) = - \int_{Q_T} (A \nabla e \cdot \nabla e + e \epsilon e + \lambda e^2) \, dx \, dt. \tag{84} \]

Next,
\[ l(v, e) = \int_{\Omega} (|v(x, 0) - \varphi(x)|^2 - 2e(x, 0)(\varphi(x) - v(0, x))) \, dx = -\|e(x, 0)\|^2_{\Omega}. \tag{85} \]

Let \( \frac{4(\beta+1)}{\delta} = \varphi \), then by means of (10) and (85), we obtain the estimate
\[ M_2^{II} \leq (\varphi - 2) \|\nabla e\|_A^2 + (\frac{\varphi}{\beta} - 2) \|\sqrt{\lambda} e\|_{Q_T}^2 + \epsilon \|e(\cdot, T)\|^2_{\Omega} - 2 \int_{Q_T} e \epsilon \, dx \, dt \leq \]
\[ (\varphi - 2) \|\nabla e\|_A^2 + (\frac{\varphi}{\beta} - 2) \|\sqrt{\lambda} e\|_{Q_T}^2 + (\epsilon - 1) \|e(\cdot, T)\|^2_{\Omega}. \]

By setting \( \hat{\delta} = 2 - \delta \), we have
\[ M_2^{II} \leq \frac{2\hat{\delta}}{\delta} \left( 1 + \frac{2\beta}{\delta} \right) \|\nabla e\|_A^2 + \frac{2\delta}{\delta} \left( 1 + \frac{2}{\beta} \right) \|\sqrt{\lambda} e\|_{Q_T}^2 + (\epsilon - 1) \|e(\cdot, T)\|^2_{\Omega}. \]

Therefore, for any \( v \in \hat{H}^1(Q_T) \) we arrive at two-sided estimates
\[ [e]^2_{(\hat{\nu}, \hat{\theta}, \hat{\zeta})} := \hat{\delta} \|\nabla e\|_A^2 + \hat{\gamma} \|\sqrt{\lambda} e\|_{Q_T}^2 + \hat{\epsilon} \|e(\cdot, T)\|^2_{\Omega} \leq M_2^{II} \leq \]
\[ [e]^2_{(\tilde{\nu}, \tilde{\theta}, \tilde{\zeta})} := \tilde{\delta} \|\nabla e\|_A^2 + \tilde{\gamma} \|\sqrt{\lambda} e\|_{Q_T}^2 + \tilde{\epsilon} \|e(\cdot, T)\|^2_{\Omega} \leq \mathcal{C} [e]^2_{(\hat{\nu}, \hat{\theta}, \hat{\zeta})}, \tag{86} \]

where
\[ \hat{\gamma} = 2, \quad \hat{\epsilon} = \frac{\epsilon - 1}{\epsilon} = \frac{\epsilon}{\tilde{\epsilon}}, \quad \hat{\delta} = \frac{2\delta}{\delta} \left( 1 + \frac{2}{\delta} \right), \quad \hat{\gamma} = \frac{2\delta}{\delta} \left( 1 + \frac{2}{\beta \delta} \right), \quad \hat{\epsilon} = \epsilon - 1, \]

and
\[ \mathcal{C} = \max \left\{ \frac{\delta}{\delta} \left( 1 + \frac{2}{\delta} \right), \frac{\delta}{\delta} \left( 1 + \frac{2}{\beta \delta} \right), \epsilon \right\}. \]

The relation (86) shows that the quantity \( M_2^{II} \) is equivalent to the energy type measure of the error. This means that the advanced majorant reliably controls deviations from \( u \) in terms of the norm (8).

4. **A lower bound of the deviation from \( u \).** Computable minorants of the deviations from exact solutions of partial differential equations provide useful information, which allows us to judge on the quantity of the error majorants. For elliptic problems having an variational formulation, the minorant of the errors can be derived fairly easily by means of the variational arguments (see [13, 8]). Below, we derive minorants for the considered class of evolutionary problem with the help of a different technique.
Theorem 4.1. Let $v, \eta \in \dot{H}^1(Q_T)$, then, the following estimate holds:

$$
M^2(\eta, v; \kappa) := \sup_{\eta \in \dot{H}^1} \left\{ \frac{\kappa_1}{2} \| \nabla e \|^2_A + \left\| \frac{\kappa_2 + \kappa_3}{2} e \right\|_{Q_T}^2 + \frac{\kappa_4}{2} \| e(x, T) \|^2_\Omega + \frac{\kappa_5}{2} \| \sqrt{\sigma} \|^2_{S_R} \right\} \leq \left[ e \right]^2_{(\mathcal{L_2}, \mathcal{L}_2)} :=
$$

$$
\frac{\kappa_1}{2} \| \nabla e \|^2_A + \left\| \frac{\kappa_2 + \kappa_3}{2} e \right\|_{Q_T}^2 + \frac{\kappa_4}{2} \| e(x, T) \|^2_\Omega + \frac{\kappa_5}{2} \| \sqrt{\sigma} \|^2_{S_R}, \quad (87)
$$

where

$$
G_{v,1}(\nabla \eta) = \int_{Q_T} \left( - \nabla \eta \cdot A \nabla v - \frac{1}{2\kappa_1} |\nabla \eta|^2 \right) \, dx dt,
$$

$$
G_{v,2}(\eta_t) = \int_{Q_T} \left( \eta_t v - \frac{1}{2\kappa_2} |\eta_t|^2 \right) \, dx dt,
$$

$$
G_{v,3}(\eta) = \int_{Q_T} \lambda \left( - v \eta - \frac{1}{2\kappa_3} |\eta|^2 \right) \, dx dt,
$$

$$
G_{v,4}(\eta(x, T)) = \int_{\Omega} \left( - v(x, T) \eta(x, T) - \frac{1}{2\kappa_4} |\eta(x, T)|^2 \right) \, dx,
$$

$$
G_{v,5}(\eta(s, t)) = \int_{S_R} \sigma \left( - v \eta - \frac{1}{2\kappa_5} |\eta|^2 \right) \, ds dt,
$$

and

$$
F_{f,g\varphi}(\eta) = \int_{Q_T} f \eta \, dx dt + \int_{S_R} g \eta \, ds dt + \int_{\Omega} \varphi(x) \eta(x, 0) \, dx,
$$

where $\nu = \frac{\kappa_1}{2}$, $\theta = \sqrt{\frac{\kappa_2 + \kappa_3}{2}}$, $\zeta = \frac{\kappa_4}{2}$, $\chi = \frac{\kappa_5}{2}$, and $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5 > 0$.

Proof. It is not difficult to see that

$$
\sup_{\eta \in \dot{H}^1(Q_T)} \left\{ \int_{Q_T} \left( \nabla \eta \cdot A \nabla e - \frac{1}{2\kappa_1} |\nabla \eta|^2 - \eta_t e - \frac{1}{2\kappa_2} |\eta_t|^2 + \lambda \left( e \eta - \frac{1}{2\kappa_3} |\eta|^2 \right) \right) \, dx dt + \int_{\Omega} \left( e(x, T) \eta(x, T) - \frac{1}{2\kappa_4} |\eta(x, T)|^2 \right) \, dx + \int_{S_R} \sigma \left( e \eta - \frac{1}{2\kappa_5} |\eta|^2 \right) \, ds dt \right\} \leq
$$

$$
\sup_{\eta \in \dot{H}^1(Q_T)} \int_{Q_T} \left( \nabla \eta \cdot A \nabla e - \frac{1}{2\kappa_1} |\nabla \eta|^2 \right) \, dx dt + \sup_{\eta \in \dot{H}^1(Q_T)} \int_{Q_T} \left( - \eta_t e - \frac{1}{2\kappa_2} |\eta_t|^2 \right) \, dx dt + \sup_{\eta \in \dot{H}^1(Q_T)} \lambda \left( e \eta - \frac{1}{2\kappa_3} |\eta|^2 \right) \, dx dt + \sup_{\eta \in \dot{H}^1(Q_T)} \int_{Q_T} \left( e(x, T) \eta(x, T) - \frac{1}{2\kappa_4} |\eta(x, T)|^2 \right) \, dx dt + \sup_{\eta \in \dot{H}^1(S_R)} \int_{S_R} \sigma \left( e \eta - \frac{1}{2\kappa_5} |\eta|^2 \right) \, ds dt.
$$
Since

\[
\sup_{\eta \in H^1(Q_T)} \left\{ \int_{Q_T} \left( \nabla \eta \cdot A \nabla (u - v) - \frac{1}{2\kappa_1} |\nabla \eta|^2 \right) \, dx \, dt \right\} \leq \frac{\kappa_1}{2} \| \nabla e \|_{L^2}^2,
\]

\[
\sup_{\eta \in H^1(Q_T)} \left\{ \int_{Q_T} \left( - \eta_t - \frac{1}{2\kappa_2} |\eta_t|^2 \right) \, dx \, dt \right\} \leq \frac{\kappa_2}{2} \| e \|_{L^2}^2,
\]

\[
\sup_{\eta \in H^1(Q_T)} \left\{ \int_{Q_T} \lambda \left( \frac{1}{2\kappa_3} |\eta|^2 \right) \, dx \, dt \right\} \leq \frac{\kappa_3}{2} \| \sqrt{\lambda} e \|_{L^2}^2,
\]

\[
\sup_{\eta(x,T) \in H^1(\Omega)} \left\{ \int_{\Omega} \left( e(x,T) \eta(x,T) - \frac{1}{2\kappa_4} |\eta(x,T)|^2 \right) \, dx \right\} \leq \frac{\kappa_4}{2} \| e(x,T) \|_{L^2}^2,
\]

we find that from one hand

\[
\sup_{\eta \in H^1(Q_T)} \left\{ \int_{Q_T} \left( \nabla \eta \cdot A \nabla e - \frac{1}{2\kappa_1} |\nabla \eta|^2 - \eta_t e - \frac{1}{2\kappa_2} |\eta_t|^2 + \lambda \left( \frac{1}{2\kappa_3} |\eta|^2 \right) \right) \, dx \, dt \right\} + \int_{\Omega} \left( e(x,T) \eta(x,T) - \frac{1}{2\kappa_4} |\eta(x,T)|^2 \right) \, dx + \int_{S_R} \left| \sigma \left( e \eta - \frac{1}{2\kappa_5} |\eta|^2 \right) \right| \, ds \, dt \right\} \leq \left[ e \right]_{\mathcal{W}_{\Omega}^2, \mathcal{W}_{\Omega}^1} := \frac{\kappa_1}{2} \| \nabla e \|_{L^2}^2 + \left\| \frac{\kappa_2 + \kappa_3 \lambda}{2} e \right\|_{L^2} + \frac{\kappa_4}{2} \| e(x,T) \|_{L^2}^2 + \frac{\kappa_4}{2} \| e(x,t) \|_{L^2}. \tag{91}
\]

From another hand, (by using (7)) we see that for any \( \eta \) the functional

\[
\sup_{\eta \in H^1(Q_T)} \left\{ \int_{Q_T} \left( \nabla \eta \cdot A \nabla e - \frac{1}{2\kappa_1} |\nabla \eta|^2 - \eta_t e - \frac{1}{2\kappa_2} |\eta_t|^2 + \lambda \left( \frac{1}{2\kappa_3} |\eta|^2 \right) \right) \, dx \, dt \right\} + \int_{\Omega} \left( e(x,T) \eta(x,T) - \frac{1}{2\kappa_4} |\eta(x,T)|^2 \right) \, dx + \int_{S_R} \left| \sigma \left( e \eta - \frac{1}{2\kappa_5} |\eta|^2 \right) \right| \, ds \, dt \right\} = \sum_{i=1}^5 G_{\psi,i} + F_{f,y}(\eta), \tag{92}
\]

generates the lower bound of the error norm defined in the right-hand side of the inequality (91). \( \square \)

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