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Hydrodynamic fluctuation-induced forces in incompressible fluid layers

6th October 2013

Abstract We study the effects of thermal fluctuations in a fluid layer confined between two planar hard walls with no-slip boundary conditions. Hydrodynamic interaction forces between the walls exhibit random fluctuations and can be calculated using the stochastic Navier-Stokes formalism of Landau and Lifshitz. Previous studies indicate that the mean value of these forces vanishes for a compressible fluid but takes a finite value for an incompressible fluid. We derive a general expression for the root-mean-square value of the fluctuation-induced pressure acting on the walls for a compressible fluid. We then show that, in the incompressible limit, this expression reduces to a simple analytic form that decays much more weakly than the mean value of the pressure itself. Our results predict that the pressure fluctuations dominate at large separations and therefore may give a more convenient measure than the mean pressure for the experimental investigation of hydrodynamic fluctuation effects.

Keywords Hydrodynamics · Fluctuations · Casimir Force · Navier-Stokes

PACS 47.35.-i · 05.40.-a · 05.20.Jj

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1 Introduction

Quantum and thermal field fluctuations in confined geometries that impose specific boundary conditions result in long-range Casimir-like interactions [2]. In the case of electromagnetic fields these confinement effects lead to van der Waals' interactions that can be derived within the general quantum electrodynamics framework [1]. The original Lifshitz derivation, however, starts with stochastic electrodynamics as formulated by Rytov [9], where Maxwell’s equations are augmented by fluctuating sources. Analogously, stochastic dissipative hydrodynamic equations have been proposed by Landau and Lifshitz [8], where the Navier-Stokes equations are augmented by fluctuating stresses.

Based on this analogy Jones [7] investigated the possible existence of a long range effective force acting between small hard inclusions in a fluid. He showed that these interactions can exist and would be strongest in incompressible fluids, while being considerably weakened by the finite compressibility. Later Chan and White [3], in the context of two planar hard walls immersed in a fluid, showed that the effective inter-surface force in the case of an incompressible fluid takes an inverse-cubic form as a function of the distance between the walls, \( L \). This result is similar to the classical thermal Casimir force, i.e. \( k_B T / L^3 \), but with a positive sign. Hydrodynamic fluctuations in a fluid therefore give rise to a repulsive force between two identical hard walls. (The existence of a repulsive force in the context of van der Waals' interactions was originally proposed by Dzyaloshinski et al. [5].) Chan and White argued, however, that because of the finite compressibility of fluids in the high-frequency limit, the hydrodynamic fluctuation forces should vanish for classical compressible fluids (although they suggest that quantum effects could generate a non-zero force, even in compressible fluids) [3]. Subsequent developments failed to conclusively prove either point of view [6, 10, 11] and thus this problem remains in need of reassessment.

While the focus of previous endeavours were the calculations of the effective interactions, i.e., the mean forces between hard inclusions or bounding surfaces, we are interested in their fluctuations. That is, how do boundary conditions and the statistical properties of the fluctuating stresses affect the correlators of the forces (or equivalently, the pressures) on the boundaries?

We formulate a general approach to this problem by considering a fluid of arbitrary compressibility bounded between two plane-parallel hard walls with no-slip boundary conditions. Thermal fluctuations lead to spatiotemporal variations in the pressure and velocity fields that can be calculated using the stochastic Navier-Stokes formalism of Landau and Lifshitz [8]. We derive an expression for the variance of the fluctuation-induced pressure acting on the walls and show that this expression reduces to a simple analytic form with a long-range dependence on the separation between the walls in the incompressible limit. For separations much smaller than the plate sizes, the mean pressure acting on the plates decays with the inverse plate separation cubed. The root-mean-square fluctuations, however, decay with the plate separation to the power three halves. Therefore the root-mean-square of the fluctuation-induced pressure decays much more weakly than the mean pressure and dominates at large separations. In other words, the relative
size of force fluctuations (or “errorbars” in possible measurements of the hydrodynamic fluctuation forces) grow with the separation and can therefore provide a more convenient candidate for experimental investigation of such forces than the mean force itself.

In the next section we outline the stochastic formalism of Landau and Lifshitz and the strategy of our calculation. Sections 3 to 5 present the main steps of the calculation in detail for the general case of compressible fluids. In Section 6 we restrict our attention to incompressible fluids and derive the main result of this work: the variance of fluctuation-induced pressures. We conclude in Section 7.

2 Formalism

We consider the hydrodynamic fluctuations in a Newtonian fluid, neglecting any heat transfer effects for simplicity. We write the Landau-Lifshitz stochastic equations as [8]

\[ \eta \nabla^2 \mathbf{v} + \left( \frac{\eta}{3} + \zeta \right) \nabla (\nabla \cdot \mathbf{v}) - \nabla p - \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = - \nabla \cdot \mathbf{S}, \]  

(1)

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \]  

(2)

Here \( \mathbf{v} = \mathbf{v}(\mathbf{r}, t), p = p(\mathbf{r}, t) \) and \( \rho = \rho(\mathbf{r}, t) \) are the macroscopic velocity, pressure and density fields respectively and \( \eta \) and \( \zeta \) are the shear and bulk viscosity coefficients. The randomly-fluctuating microscopic degrees of freedom are driven by the random stress tensor \( \mathbf{S} \), which has Gaussian properties

\[ \left \langle \tilde{S}_{ij}(\mathbf{r}; \omega) \right \rangle = 0, \]

\[ \left \langle \tilde{S}_{kl}(\mathbf{r}; \omega) \tilde{S}_{mn}^*(\mathbf{r}'; \omega') \right \rangle = \frac{k_B T}{\pi} \left[ \eta (\delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm}) \right. \]

\[ \left. - \left( \frac{2 \eta}{3} - \zeta \right) \delta_{kl} \delta_{mn} \right] \delta(\mathbf{r} - \mathbf{r}') \delta(\omega + \omega'). \]  

(4)

The angled brackets denote an equilibrium ensemble average at temperature \( T \) and here \( k_B \) is Boltzmann’s constant. We do not include any possible relaxation effects, which would formally correspond to frequency dependent viscosities, but these can be easily incorporated [8]. We have expressed the random stress tensor in the frequency representation, denoting the frequency Fourier transform by a tilde:

\[ \tilde{f}(\omega) = \int dt \, e^{i \omega t} f(t). \]  

(5)

\[ ^1 \text{ We note the typographic error in Equation 2.1 of [3], a missing occurrence of the field } \mathbf{v}. \]
We note at this point that for the incompressible limit, which will eventually be of concern in this paper, the random stress tensor correlation function, Equation (4), is just

$$\langle \tilde{S}_{kl}(r; \omega) \tilde{S}_{mn}(r'; \omega') \rangle = \frac{k_B T}{\pi \eta} \delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm} \delta(r-r') \delta(\omega+\omega').$$  (6)

In this work, we shall focus on the specific case of a fluid layer confined between two hard walls (half-spaces) with plane-parallel surfaces located at separation distance $L$. We assume no-slip boundary conditions on contact surfaces. The net force acting on each wall is obtained by integrating the hydrodynamic stress tensor, $\sigma_{ij} = \sigma_{ij}(r, t)$, over its bounding surface, $\Gamma$, in the usual manner:

$$\langle F_i(t) \rangle = \int_{\Gamma} \langle \sigma_{ij}(r, t) \rangle \, dA_j,$$  (7)

where we have

$$\sigma_{ij} = \eta \left[ \partial_i v_j + \partial_j v_i \right] - \left[ \left( \frac{2 \eta}{3} - \zeta \right) \partial_k v_k + p \right] \delta_{ij}.$$  (8)

The net pressure is then given by the net force per unit area.

We immediately see from Equation (3) that any net force must arise from non-linear terms in the stochastic Landau-Lifshitz equations because the stress tensor is linear in velocity and pressure fields. In this work we will examine the equal-time pressure-pressure correlation function in the planar geometry of a fluid layer bounded by two parallel plates. For this geometry we are concerned only with the pressure acting perpendicular to the boundaries and consider the correlation function

$$\langle P(z, t) P(z', t) \rangle = \frac{1}{\Gamma^2} \int_{\Gamma} \int_{\Gamma'} \langle \sigma_{3k}(r, t) \sigma_{3l}(r', t) \rangle \, dA_k \, dA_l,$$  (9)

where the prefactor $\Gamma^2$ is just the square of the area of integration associated with each surface integral. We have labelled the direction perpendicular to the plates as $i = j = 3$ and used the symmetry of the planar geometry to reduce the spatial dependence of the left hand side to a function of only the coordinate $(z)$ perpendicular to the plates. Here, and throughout this work, we use the summation convention: we sum over all repeated indices. In general we refer to $\langle P(z, t) P(z', t) \rangle$ as the pressure-pressure correlation function, but this quantity should be distinguished from the two-point correlation function of the pressure field $\langle p(r, t) p(r', t) \rangle$.

We now calculate these correlation functions in the following way:

1. express the net pressure in terms of the linearised velocity and pressure fields;
2. express the correlation functions of velocity and pressure fields in terms of Green functions of the linearised stochastic equations above;
3. obtain the Green functions; and
4. integrate the resulting expressions over the boundaries of the liquid.
Before proceeding further, we should note that above form of fluctuating hydrodynamics is obviously similar to the Rytov fluctuating electrodynamics [9], where the basic equations for the electric and magnetic field turn out to be

\[ \nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}, \]  
\[ \nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} + \frac{\partial \mathbf{K}(\mathbf{r}, t)}{\partial t}, \]

and the fluctuating random polarization \( \mathbf{K}(\mathbf{r}, t) \) again has Gaussian properties:

\[ \langle \tilde{K}_i(\mathbf{r}; \omega) \rangle = 0, \]
\[ \langle \tilde{K}_i(\mathbf{r}; \omega) \tilde{K}_j^*(\mathbf{r}'; \omega') \rangle = k_B T \frac{\varepsilon''(\omega)}{\omega} \delta_{ij} \delta(\mathbf{r} - \mathbf{r}') \delta(\omega + \omega'). \]

Here we have assumed a non-local dielectric response in time that is given by the dielectric permittivity \( \varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega) \). The similarity between Equations (1)–(4) and Equations (10)–(13) is obvious.

Thus the present general approach to hydrodynamics is very close to the analysis of the electromagnetic problem in the original paper by Lifshitz [9], provided one fully takes into account the basic differences between the Maxwell equations and the Navier-Stokes equations [3]: the former are linear in the fields with stresses quadratic in the field, while the latter are non-linear in the fields with stresses linear in the fields. This discrepancy between the two leads to some important differences and precludes a direct portability of results from the electrodynamic domain to the hydrodynamic domain.

3 Relating net pressure correlation functions to field correlation functions

For vanishing random stress tensor, the equilibrium solution of Equations (1) and (2) is \( \mathbf{v} = 0, \ p = p_0 \) and \( \rho = \rho_0 \), corresponding to a fluid at rest at constant temperature with uniform pressure, \( p_0 \), and density, \( \rho_0 \). The random stress tensor, \( \mathbf{S} \), is of order \( k_B T \) and consequently macroscopically small, so that the corresponding fluctuations in the velocity, pressure and density fields are also macroscopically small.

This allows us to introduce a linearised treatment of the Landau-Lifshitz equations, by setting \( \mathbf{v} = \mathbf{v}^{(1)}, \ p = p_0 + p_0^{(1)} \) and \( \rho = \rho_0 + \rho^{(1)} \), where the superscript \( (1) \) denotes a term of order \( \mathbf{S} \). In view of the comments preceding Equation (9), however, we see that we must include higher order contributions to account for the non-linear nature of any net hydrodynamic forces. We therefore seek solutions of Equations (1) and (2) of the form

\[ \mathbf{v} = \sum_{n=0}^{\infty} \mathbf{v}^{(n)}, \quad p = \sum_{n=0}^{\infty} p^{(n)}, \quad \rho = \sum_{n=0}^{\infty} \rho^{(n)}, \]  

(14)
where \( n \) denotes each order in the dimensionless expansion parameter, \( g \), which is given by
\[
g^2 = \frac{k_B T \sqrt{\rho_0 \rho_0^3}}{\eta_0^2}.
\] (15)

The zeroth order values are \( v^{(0)} = 0 \), \( p^{(0)} = p_0 \) and \( \rho^{(0)} = \rho_0 \).

We assume local equilibrium; this allows us to relate the density and pressure by a virial expansion
\[
\delta p = c_0^2 \delta \rho + \frac{1}{2} b_0 (\delta \rho)^2.
\] (16)

In this case
\[
c_0^2 = \frac{\partial p}{\partial \rho_0}
\] (17)

is the speed of sound, the first virial coefficient is
\[
b_0 = \frac{\partial^2 p}{\partial \rho^2}
\] (18)

and the derivatives in these expressions are taken adiabatically. Equations (14) to (18) allow us to relate the pressure and density as
\[
p^{(1)}(\omega) = c_0^2 \rho^{(1)}(\omega).
\] (19)

At leading order in the frequency representation the linearised Landau-Lifshitz equations are\(^2\)
\[
\eta \nabla^2 v^{(1)}(\omega) + \left( \frac{\eta}{3} + \zeta \right) \nabla (\nabla \cdot v^{(1)}(\omega)) - \nabla p^{(1)}(\omega) + i \omega \rho_0 v^{(1)}(\omega) = -\nabla \cdot S(\omega),
\] (20)

\[
\nabla \cdot v^{(1)}(\omega) - \frac{i \omega}{\rho_0 c_0^2} p^{(1)}(\omega) = 0.
\] (21)

The analogous next-to-leading order equations are given in [3] and [7], but we do not require these equations for this work.

We now expand the hydrodynamic stress tensor in powers of the random stress tensor,
\[
\sigma_{ij}(r, t) = \sum_{n=0}^{\infty} \sigma_{ij}^{(n)}(r, t),
\] (22)

where the zeroth order value is \( \sigma_{ij}^{(0)} = -\rho_0 \delta_{ij} \). We substitute our expansion for the hydrodynamic stress tensor into our expression for the pressure-pressure correlation function, Equation (9), to obtain
\[
\langle P(z, t) P(z', t) \rangle^{(2)} = \frac{1}{T^2} \int \int \left[ \sigma_{3k}^{(0)}(r, t) \langle \sigma_{3l}^{(2)}(r', t) \rangle + \sigma_{3l}^{(0)}(r', t) \langle \sigma_{3k}^{(2)}(r, t) \rangle + \langle \sigma_{3k}^{(1)}(r, t) \sigma_{3l}^{(1)}(r', t) \rangle \right] dA_k dA'_l
\] (23)

\(^2\) Here we note a typographic error in Equation 2.11 of [3], corresponding to a factor of \( 1/\rho_0 \) missing from Equation (21) above.
at second order in the random stress tensor. Ref. [3] gives the second order averages, albeit not explicitly. The contribution from the velocity field vanishes,
\[ \langle v_i^{(2)}(\mathbf{r}, t) \rangle = 0. \] (24)

Thus the second-order average is
\[ \langle \sigma_{ij}^{(2)} \rangle = - \langle p^{(2)} \rangle \delta_{ij}. \] (25)

For incompressible fluids the authors of [3] calculate the result in terms of the static pressure field, \( P(\mathbf{r}) \), as
\[ P(\mathbf{r}) \delta(\omega) = \langle p^{(2)}(\mathbf{r}, \omega) \rangle. \] (26)

The pressure field in the plane-parallel geometry considered here (Section 2) is only a function of the \( z \)-coordinate, perpendicular to the parallel boundaries, and is given by
\[ P(z) = P_0 + \frac{k_B T}{4\pi} \int_{1/L'}^{\infty} dk k^2 \frac{\cosh[k(2z - L)]}{\sinh(kL)}. \] (27)

with \( P_0 \) a constant of integration. We have introduced a small momentum cutoff, \( 1/L' \), that corresponds to the lateral size of the area over which the pressure is acting on the plate boundaries. The divergence in the pressure at \( z = 0 \) and \( z = L \) is spurious and arises because we have formally integrated the momentum to infinity, even though we expect the hydrodynamic picture to break down at microscopic scales. Introducing an ultraviolet momentum cutoff regularises the momentum integral and leads to a finite pressure at the plate boundaries. The disjoining pressure at each plate boundary is finite and independent of the cutoff.

Therefore, in order to calculate the full expression in Equation (23), we need only calculate the contribution from
\[ \Pi(z, z', t) \equiv \frac{1}{T^2} \iint_{I^2} \langle \sigma_{3k}^{(1)}(\mathbf{r}, t)\sigma_{3l}^{(1)}(\mathbf{r}', t) \rangle \ dA_k dA_l. \] (28)

In terms of the first order velocity and pressure fields, the first order hydrodynamic stress tensor is
\[ \sigma_{ij}^{(1)}(\mathbf{r}, t) = \eta \left( \nabla_i v_j^{(1)}(\mathbf{r}, t) + \nabla_j v_i^{(1)}(\mathbf{r}, t) \right) - \left[ \left( \frac{2\eta}{3} - \zeta \right) \nabla_k v_k^{(1)}(\mathbf{r}, t) + p^{(1)}(\mathbf{r}, t) \right] \delta_{ij}. \] (29)

From Equations (28) and (29) we immediately see that we must then calculate the three correlation functions
\[ \langle v_i^{(1)}(\mathbf{r}, t)v_j^{(1)}(\mathbf{r}', t) \rangle, \langle v_i^{(1)}(\mathbf{r}, t)p^{(1)}(\mathbf{r}', t) \rangle \quad \text{and} \quad \langle p^{(1)}(\mathbf{r}, t)p^{(1)}(\mathbf{r}', t) \rangle. \] (30)
We obtain the expression for the pressure-pressure correlation function in terms of the field-field correlation functions by substituting Equation (29) into Equation (28). The resulting expression can be considerably simplified by considering the symmetry of the two-wall geometry and by imposing no-slip boundary conditions at the two infinite boundaries. These considerations, and some straightforward but lengthy algebra, ultimately lead to a simplified expression for the pressure-pressure correlation function in terms of the field-field correlation functions, which is given by:

\[
\Pi(z, z', t) = \left\langle p(1)(z, t)p(1)(z', t) \right\rangle + \left( \frac{4\eta}{3} + \zeta \right)^2 \nabla_3 \nabla'_3 \left\langle v_3^{(1)}(z, t)v_3^{(1)}(z', t) \right\rangle
\]

\[
- \left( \frac{4\eta}{3} + \zeta \right) \left[ \nabla_3 \left\langle v_3^{(1)}(z, t)p(1)(z', t) \right\rangle - \nabla'_3 \left\langle p(1)(z, t)v_3^{(1)}(z', t) \right\rangle \right].
\] (31)

To evaluate this expression for the pressure-pressure correlation function (23) we now relate the field-field correlation functions to the Green functions of the stochastic Landau-Lifshitz equations.

4 Relating field correlation functions to Green functions

We solve for the Green functions of the stochastic Landau-Lifshitz equations by introducing a formal four-vector notation:

\[
V_\alpha = \begin{pmatrix} v(\omega) \\ p(\omega) \end{pmatrix}, \quad S_\alpha = \begin{pmatrix} -\nabla \cdot S(\omega) \\ 0 \end{pmatrix},
\] (32)

where the index \( \alpha \) takes values in \((1, 2, 3, 4)\). In general, we use the convention that Greek indices run from one to four, whilst Latin indices run from one to three. Note that we do not include a stochastic stress in the fourth component of the four-vector \( S_\alpha \), which corresponds to the right hand side of the continuity equation (Equation (2)), in contrast with Ref. [7]. Equations (20) and (21) are naturally represented by a tensor differential operator, \( D^{\alpha\beta} \), acting on the four-vector \( V_\alpha \) as follows:

\[
D^{\alpha\beta}(r, \omega)V_\beta(r, \omega) = S_\alpha(r, \omega).
\] (33)

The “Green functions” of this differential operator are defined by

\[
D_{\beta\gamma}(r, \omega)G_{\gamma\beta}(r, \omega'') = \delta_{\beta\gamma}\delta(r - r''),
\] (34)

\[
D_{4\beta}(r, \omega)G_{\beta4}(r, \omega'') = 0.
\] (35)

We note that the vanishing right hand side of Equation (35) implies that the elements \( G_{\beta4} \) are not true Green functions, but this nomenclature is convenient for our purposes here. The solutions of the inhomogeneous Equation (20) take the form

\[
V_\alpha(r, \omega) = \int G_{\alpha\gamma}(r, \omega'') S_\gamma(r'', \omega) \, dr''.
\] (36)
We may now relate the correlation functions of Equation (30) to the Green functions, \( G_{\alpha\beta} \), via the formal solution of Equation (36) and the stochastic properties of the random stress tensor, Equations (3) and (4). The resulting expressions are

\[
\left\langle v_i^{(1)}(r, t) v_j^{(1)}(r', t) \right\rangle = -\frac{k_B T}{\pi} \int d\omega' \int d\omega'' G_{ij}(r, r'', \omega') \times 
\left[ \eta \nabla''^2 G_{jl}^{\alpha\beta}(r', r'', \omega') + \left\{ \frac{\eta}{3} + \zeta \right\} \nabla''_j \nabla''_l G_{jk}^{\alpha\beta}(r', r'', \omega') \right],
\]

\[
\left\langle p_i^{(1)}(r, t) v_j^{(1)}(r', t) \right\rangle = -\frac{k_B T}{\pi} \int d\omega' \int d\omega'' G_{ij}^{*\alpha\beta}(r', r'', \omega') \times 
\left[ \eta \nabla''^2 G_{4j}(r, r'', \omega') + \left\{ \frac{\eta}{3} + \zeta \right\} \nabla''_j \nabla''_l G_{4k}(r, r'', \omega') \right],
\]

\[
\left\langle p_i^{(1)}(r, t) p_j^{(1)}(r', t) \right\rangle = -\frac{k_B T}{\pi} \int d\omega' \int d\omega'' G_{4i}(r', r'', \omega') \times 
\left[ \eta \nabla''^2 G_{4j}(r', r'', \omega') + \left\{ \frac{\eta}{3} + \zeta \right\} \nabla''_j \nabla''_l G_{4k}(r', r'', \omega') \right].
\]

A full derivation is given in \[7\].

Our next task is to solve for the Green functions, \( G_{\alpha\beta} \). Before we do that, however, we may use the symmetry of the two-wall geometry to make a little progress. Translation invariance in the two directions perpendicular to the plane of the boundaries at \( z = 0 \) and \( z = L \), labelled by 1 (the \( x \)-direction) and 2 (the \( y \)-direction) respectively, prompts us to search for Green function components \( G_{\alpha\beta} \), via the formal solution of Equation (36) and the stochastic properties of the Green function components \( G_{4\alpha} \), Equation (34), becomes

\[
\delta(r - r'') = \frac{1}{(2\pi)^2} \int d^2 k e^{\mathbf{i} \mathbf{k} \cdot (\mathbf{s} - \mathbf{s}'')} \tilde{G}_{\alpha\beta}(z, z''; \mathbf{k}; \omega),
\]

where \( \mathbf{k} = (k_x, k_y) \) and \( \mathbf{s} = (x, y) \). Then the Dirac delta function in the defining equation for the Green function components \( G_{4\alpha} \), Equation (34), becomes

\[
\delta(r - r'') = \frac{1}{(2\pi)^2} \int d^2 \mathbf{k} e^{\mathbf{i} \mathbf{k} \cdot (\mathbf{s} - \mathbf{s}'')} \delta(z - z'').
\]

Substituting the expression for \( G_{\alpha\beta} \) in Equation (40) into Equations (37) to (39) leads to integrals whose only dependence on the \((x'', y'')\) coordinates is of the form

\[
\frac{1}{L'^2} \int_0^{L'} dx'' \int_0^{L'} dy'' e^{i(k_x x'' + k_y y'')},
\]

where the prefactor is a direct consequence of the prefactor in Equation (28), included to normalise the result to unit area. Such integrals generate delta functions for the \( x \) and \( y \) components of the momentum, which greatly simplifies the evaluation of the double momentum integral. Furthermore, all
derivatives of the Green functions in the \(x\) and \(y\) directions will vanish. We can therefore write Equations (37) to (39) as

\[
\left\langle v_i^{(1)}(z,t) v_j^{(1)}(z',t) \right\rangle = -\frac{k_B T}{4\pi^3 L^2} \int \mathrm{d}\omega' \int \mathrm{d}z'' \left[ \eta \tilde{G}_{ij}(z,z'';0;\omega') \nabla^2_{z''} \tilde{G}^*_{ij}(z',z'';0;\omega') \right] \\
\quad \times \left[ \left( \frac{\eta}{3} + \zeta \right) \tilde{G}_{ij}(z,z'';0;\omega') \nabla^2_{z''} \tilde{G}^*_{ij}(z',z'';0;\omega') + \left( \frac{\eta}{3} + \zeta \right) \tilde{G}_{ij}(z,z'';0;\omega') \nabla^2_{z''} \tilde{G}^*_{ij}(z',z'';0;\omega') \right],
\]

(43)

\[
\left\langle p_i^{(1)}(z,t) v_j^{(1)}(z',t) \right\rangle = -\frac{k_B T}{4\pi^3 L^2} \int \mathrm{d}\omega' \int \mathrm{d}z'' \left[ \eta \tilde{G}_{ij}(z,z'';0;\omega') \nabla^2_{z''} \tilde{G}^*_{ij}(z',z'';0;\omega') \right] \\
\quad \times \left[ \left( \frac{\eta}{3} + \zeta \right) \tilde{G}_{ij}(z,z'';0;\omega') \nabla^2_{z''} \tilde{G}^*_{ij}(z',z'';0;\omega') \right],
\]

(44)

\[
\left\langle p_i^{(1)}(z,t) p_j^{(1)}(z',t) \right\rangle = \frac{k_B T}{4\pi^3 L^2} \int \mathrm{d}\omega' \int \mathrm{d}z'' \left[ \eta \tilde{G}_{ij}(z,z'';0;\omega') \nabla^2_{z''} \tilde{G}^*_{ij}(z',z'';0;\omega') \right] \\
\quad \times \left[ \left( \frac{\eta}{3} + \zeta \right) \tilde{G}_{ij}(z,z'';0;\omega') \nabla^2_{z''} \tilde{G}^*_{ij}(z',z'';0;\omega') \right].
\]

(45)

The important point to note is that there is no dependence of these correlation functions on the \(x\) and \(y\) coordinates, that is, the directions parallel to the infinite boundaries. This is a direct result of our assumption of translational invariance in the \((x,y)\)-plane.

The final step before we attempt to obtain explicit expressions for the Green functions is to impose the no-slip boundary conditions, which correspond to

\[
\tilde{G}_{1\gamma}(z,z'';0;\omega') = 0, \quad \tilde{G}_{2\gamma}(z,z'';0;\omega') = 0, \quad \text{and} \quad \tilde{G}_{3\gamma}(z,z'';0;\omega') = 0
\]

at \(z = 0\) and \(z = L\). Thus both correlation functions that involve the velocity field vanish if the velocity field is evaluated at either boundary, that is at \(z = 0, L\) and \(z' = 0, L\).

5 Solving for the Green functions

Using the formal four-vector notation that we introduced in Section 4, we write the differential operator \(D_{\alpha\beta}(r,\omega)\) as

\[
D_{\alpha\beta} = \begin{pmatrix}
\eta \nabla^2 + i\omega \rho_0 & 0 & 0 & \chi(\omega) \partial / \partial x \\
0 & \eta \nabla^2 + i\omega \rho_0 & 0 & \chi(\omega) \partial / \partial y \\
0 & 0 & \eta \nabla^2 + i\omega \rho_0 & \chi(\omega) \partial / \partial z \\
\partial / \partial x & \partial / \partial y & \partial / \partial z & -\kappa(\omega)
\end{pmatrix}
\]

(47)
where we have defined
\[ \chi(\omega) = \left( \frac{\eta}{3} + \zeta \right) \kappa(\omega) - 1, \quad \text{and} \quad \kappa(\omega) = \frac{i\omega}{\rho_0 c_0}. \] (48)

In the momentum representation, the four equations that we must solve are, using the geometry of the two-wall system and Equation (40),
\[
\eta \left( \frac{\partial^2}{\partial z^2} - k^2 \right) + i\omega \rho_0 \left[ \tilde{G}_{1\gamma} + ik_x \chi(\omega) \tilde{G}_{4\gamma} = \delta_{1\gamma} \delta(z - z''), \right. (49)
\]
\[
\eta \left( \frac{\partial^2}{\partial z^2} - k^2 \right) + i\omega \rho_0 \left[ \tilde{G}_{2\gamma} + ik_y \chi(\omega) \tilde{G}_{4\gamma} = \delta_{2\gamma} \delta(z - z''), \right. (50)
\]
\[
\eta \left( \frac{\partial^2}{\partial z^2} - k^2 \right) + i\omega \rho_0 \left[ \tilde{G}_{3\gamma} + \chi(\omega) \frac{\partial}{\partial z} \tilde{G}_{4\gamma} = \delta_{3\gamma} \delta(z - z''), \right. (51)
\]
\[ ik_x \tilde{G}_{1\gamma} + ik_y \tilde{G}_{2\gamma} + \frac{\partial}{\partial z} \tilde{G}_{3\gamma} - \kappa(\omega) \tilde{G}_{4\gamma} = 0, \] (52)

where we have suppressed the arguments of the Green functions for clarity.

We cast these equations into a more tractable form using the transformations related to those introduced in Ref. [3]. We define
\[
\tilde{A}_\gamma(z, z''; k; \omega) = \eta \left( k_x \tilde{G}_{1\gamma}(z, z''; k; \omega) + k_y \tilde{G}_{2\gamma}(z, z''; k; \omega) \right), \] (53)
\[
\tilde{B}_\gamma(z, z''; k; \omega) = -\chi(\omega) \tilde{G}_{4\gamma}(z, z''; k; \omega), \] (54)
\[
\tilde{C}_\gamma(z, z''; k; \omega) = \eta \left( k_y \tilde{G}_{1\gamma}(z, z''; k; \omega) - k_x \tilde{G}_{2\gamma}(z, z''; k; \omega) \right), \] (55)
\[
\tilde{F}_\gamma(z, z''; k; \omega) = i\kappa \eta \tilde{G}_{3\gamma}(z, z''; k; \omega), \] (56)

where
\[ K_\gamma = (k_x \delta_{1\gamma} + k_y \delta_{2\gamma}), \quad q^2 = k^2 - \frac{i\omega \rho_0}{\eta} \quad \text{and} \quad k = |k|. \] (57)

Again suppressing the arguments of the Green functions, Equations (49) to (52) are now
\[
\left( \frac{\partial^2}{\partial z^2} - q^2 \right) \tilde{A}_\gamma - ik^2 \tilde{B}_\gamma = K_\gamma \delta(z - z''), \] (58)
\[
\left( \frac{\partial^2}{\partial z^2} - q^2 \right) \tilde{C}_\gamma = (k_y \delta_{1\gamma} - k_x \delta_{2\gamma}) \delta(z - z''), \] (59)
\[
\left( \frac{\partial^2}{\partial z^2} - q^2 \right) \tilde{F}_\gamma - ik \frac{\partial}{\partial z} \tilde{B}_\gamma = ik \delta_{3\gamma} \delta(z - z''), \] (60)
\[
\frac{\partial}{\partial z} \tilde{F}_\gamma - k \tilde{A}_\gamma - i\kappa(\omega) \tilde{B}_\gamma = 0, \] (61)

where
\[ \beta(\omega) = -\frac{\kappa(\omega) \eta}{\chi(\omega)}. \] (62)
The corresponding boundary conditions are just
\[ \tilde{A}_\gamma(z, z''; k; \omega') = 0, \ \tilde{C}_\gamma(z, z''; k; \omega') = 0, \ \text{and} \ \tilde{F}_\gamma(z, z''; k; \omega') = 0, \]
(63)
at \( z = 0, L \). We present a complete derivation of the solutions in Appendix A. Here, however, we can see from Equation (31) that the only components we require are
\[ \tilde{G}_3^\gamma(z, z''; 0; \omega) = m_0 \eta (m_0^2 - q_0^2) \left\{ b_{1\gamma}|_{k=0} e^{-m_0 z} + b_{2\gamma}|_{k=0} e^{m_0(z-L)} \right\} \]
(64)
\[ \tilde{G}_4^\gamma(z, z''; 0; \omega) = -\frac{1}{\lambda} \left\{ b_{1\gamma}|_{k=0} e^{-m_0 z} + b_{2\gamma}|_{k=0} e^{m_0(z-L)} \right\} \]
(65)
\[ + \frac{i \delta_{\beta}}{2(\beta(\omega) - 1)} \text{sgn}(z - z'') e^{-m_0|z-z'|} \}
Here there are eight constants of motion, \( b_{1\gamma}, b_{2\gamma}, f_{1\gamma}, f_{2\gamma} \), but only two are non-zero:
\[ b_{13}|_{k=0} = \frac{\sinh(m_0(z'' - L))}{2(\beta(\omega) - 1) \sinh(m_0 L)}, \]
(66)
\[ b_{23}|_{k=0} = \frac{\sinh(m_0 z'')}{2(\beta(\omega) - 1) \sinh(m_0 L)} \]
(67)
In the above expressions we have introduced
\[ m_0^2|_{k=0} = -\omega^2 \left[ \frac{2}{c_0^2} - \left( \frac{4}{3} \eta + \zeta \right) \frac{i \omega}{\rho_0} \right]^{-1}. \]
(68)
We have confirmed that in the infinite separation limit our results reproduce the Green function expressions given in [7]. This serves as a useful cross-check of our solutions.

Combining Equations (31), (43) to (45), and (64) to (68), we now have all the ingredients we need to evaluate the pressure-pressure correlation function in Equation (23).

6 Incompressible fluids

While all results derived heretofore were general, valid for compressible as well as incompressible fluids, we now restrict ourselves to the latter case. In the incompressible limit we expect
\[ \nabla \cdot \mathbf{v}^{(n)} = 0 \]
(69)
for all \( n \), i.e. at all orders in the random stress tensor expansion. This can be seen, for the relevant case of \( n = 1 \), from the infinite adiabatic speed of sound limit of Equation (21). The immediate consequence is a considerable
simplification of Equation (31), since any term involving $\nabla_3 v_3^{(1)}(z, t)$ will vanish. This follows from the symmetries of the two-wall geometry, which imply $\nabla_1 v_1^{(1)}(z, t) = \nabla_2 v_1^{(2)}(z, t) = 0$. Thus the only contribution to Equation (31) in the incompressible limit is

$$\Pi(z, z', t) = \langle p^{(1)}(z, t)p^{(1)}(z', t) \rangle. \quad (70)$$

We have confirmed by explicit calculation that the other correlation functions, given in Equations (37) and (38), vanish as expected. This serves as an important check of our results.

We must also take into account the modification of Equation (4) to (6) in our derivation of the expression for the correlation functions in terms of the Green functions. In the incompressible limit Equation (45) becomes

$$\Pi(z, z', t) = \frac{k_B T \eta_0}{2\pi^3 L^2} \int d\omega' \int dz'' \nabla''_3 G_{43}(z, z''; 0; \omega') \nabla''_3 G^*(z', z''; 0; \omega'), \quad (71)$$

where we have accounted for the fact that the only non-vanishing component of the Green function is $\tilde{G}_{43}(z, z''; 0; \omega')$, as discussed above.

We see from Equations (48), (62), and (68) that, in the incompressible limit,

$$\lim_{c_0 \to \infty} m_0 = 0, \quad \lim_{c_0 \to \infty} \chi(\omega) = -1, \quad \lim_{c_0 \to \infty} \beta(\omega) = 0 \quad (72)$$

and therefore Equations (65) to (67) simplify to

$$\lim_{c_0 \to \infty} \tilde{G}_{43}(z, z''; 0; \omega) = \frac{1}{2} \left[ 1 - \text{sgn}(z - z'') \right] - \frac{z''}{L}. \quad (73)$$

The derivative we require is just

$$\nabla''_3 \left[ \lim_{c_0 \to \infty} \tilde{G}_{43}(z, z''; 0; \omega) \right] = \delta(z - z'') - \frac{1}{L}. \quad (74)$$

The pressure field correlation function is thus

$$\Pi(z, z', t) = \frac{k_B T \eta_0}{2\pi^3 L^2} \int d\omega' \int dz'' \left( \delta(z - z'') - \frac{1}{L} \right) \left( \delta(z' - z'') - \frac{1}{L} \right). \quad (75)$$

Carrying out the trivial $z''$ integrals leads to

$$\Pi^{(\text{in})}(z, z', t) = \frac{k_B T \eta_0}{2\pi^3 L^2} \left( \delta(z - z') - \frac{1}{L} \right) \int d\omega'. \quad (76)$$

While the first term in the above equation stems directly from the assumption of the statistical properties of the fluctuating stresses, Equation 4, the second term is non-trivial and describes long-range correlations between the two surfaces immersed in the fluid. This is the confinement effect that bears a direct similarity with the Casimir interactions. The frequency integral in Equation (76) is the fingerprint of the uncorrelated nature of temporal fluctuations in the pressure field.
We are now in a position to evaluate the algebraic expression for the final result in the incompressible limit. Combining Equations (23), (25)–(27), and (76) we have

\[ \langle \mathcal{P}(z, t)\mathcal{P}(z', t) \rangle = 2p_0P_0 + \frac{k_BT\eta_0}{2\pi^3 L'^2} \left( \delta(z - z') - \frac{1}{L} \right) \int d\omega' \]

\[ + \frac{k_BTp_0}{4\pi} \int_{L'/L}^{\infty} dk k^2 \frac{\cosh[k(2z - L)] + \cosh[k(2z' - L)]}{\sinh(kL)}. \] (77)

Here \( p_0 = p^{(0)} \) is the zeroth-order pressure field, corresponding to the equilibrium solution of Equations (1) and (2), and \( P_0 \) is the constant of integration in the solution of \( P(z) \) given in Equation (27).

The difference in the statistical variance of the pressure acting on one of the plates (located, e.g., at \( z = L \)) is

\[ \Delta\langle \mathcal{P}(t)\mathcal{P}(t) \rangle \equiv \langle \mathcal{P}(L, t)\mathcal{P}(L, t) \rangle - \lim_{L \to \infty} \langle \mathcal{P}(L, t)\mathcal{P}(L, t) \rangle, \] (78)

which we can evaluate to give

\[ \Delta\langle \mathcal{P}(t)\mathcal{P}(t) \rangle = \frac{k_BT}{2\pi} \left( p_0 \int_{1/L'}^{\infty} dk k^2 [\coth(kL) - 1] - \frac{\eta_0}{\pi^2 L'^2 L} \int d\omega' \right) \]

\[ - \frac{k_BT}{2\pi L'^3} \left\{ \frac{\eta_0}{\pi^2 \lambda^2} \int d\omega' - \frac{p_0}{6} \left( 3\text{Li}_3 \left( e^{2/\lambda} \right) \right) \right. \]

\[ + \frac{2}{\lambda} \left[ \frac{1}{\lambda} \left( 2 - 3i\pi - 3 \log \left( e^{2/\lambda} - 1 \right) \right) - 3\text{Li}_2 \left( e^{2/\lambda} \right) \right] \right\}. \] (79)

Here we have defined \( \lambda = L'/L \) to be the ratio of the side length of the area of integration on the plates to the plate separation and \( \text{Li}_n(x) = \sum_{j=1}^{\infty} x^j/j^n \) is the polylogarithm function. This is our final result: an analytic expression for the variance of the net pressure acting on a plane boundary due to the presence of a second plane boundary at distance \( L \).

We may now consider two relevant limits of our result: the limit in which the size of the plates is much greater than the plate separation \( \lambda \to \infty \), and the limit \( \lambda \ll 1 \). The second limit must be interpreted with caution: to express the Green functions in the form of Equation (40) we assumed translational invariance in the directions parallel to the plane boundaries. The limit \( \lambda \ll 1 \), while existing in a formal sense, does not satisfy the translational invariance assumption and one therefore expects boundary effects to come into play in this case.

In the first limit, we have

\[ \Delta\langle \mathcal{P}(t)\mathcal{P}(t) \rangle \overset{\lambda \to \infty}{=} \frac{k_BTp_0}{4\pi L^3} \zeta(3), \] (80)

where \( \zeta(\alpha) = \sum_{j=1}^{\infty} j^{-\alpha} \), for \( \text{Re}(\alpha) > 0 \), is the Riemann zeta function.

Therefore, for an incompressible fluid between two parallel plates with size much greater than their separation \( L \), there is a net repulsive pressure acting on the plates. This net pressure has a mean

\[ \Delta\langle \mathcal{P}(t) \rangle \overset{\lambda \to \infty}{=} \frac{k_BT}{8\pi L^3} \zeta(3), \] (81)
and a variance
\[ \Delta \langle P(t)P(t) \rangle \xrightarrow{\lambda \to \infty} \frac{k_B T p_0}{4\pi L^3} \zeta(3). \] (82)

This gives a typical root-mean-square value of
\[ \Delta P_{\text{rms}}(t) \xrightarrow{\lambda \to \infty} \sqrt{\frac{k_B T p_0 L^3}{4\pi^2}}. \] (83)

Hence the relative magnitude of fluctuations in the pressure that acts on the walls grows with the wall separation as
\[ \frac{\Delta P_{\text{rms}}(t)}{\Delta \langle P(t) \rangle_{\text{in}}} \xrightarrow{\lambda \to \infty} L^{3/2}. \] (84)

Although we must keep in mind that the \( \lambda \ll 1 \) limit breaks translational invariance, we may formally investigate the small \( \lambda \) behaviour by first changing integration variables to \( u = kL \) to obtain
\[ \Delta \langle P(t)P(t) \rangle = \frac{k_B T}{2\pi L^3} \left( p_0 \int_{1/\lambda}^{\infty} du u^2 \left[ \coth(u) - 1 \right] - \frac{\eta_0}{\pi^2 \lambda^2} \int d\omega' \right). \] (85)

In the limit of vanishing \( \lambda \), the region of integration over \( u \) is restricted to large \( u \) and the leading order behaviour is
\[ \int_{1/\lambda}^{\infty} du u^2 \left[ \coth(u) - 1 \right] \simeq \int_{1/\lambda}^{\infty} du u^2 e^{-2u}. \] (86)

Clearly the second term in Equation (85) dominates in this case and we obtain
\[ \Delta \langle P(t)P(t) \rangle \xrightarrow{\lambda \ll 1} \frac{k_B T \eta_0}{2\pi^2 L^3} \frac{1}{\lambda} \int d\omega'. \] (87)

Formally the frequency integral runs from negative infinity to positive infinity and is therefore divergent. This divergence arises because we evaluate the correlation function at equal times and is a direct consequence of the time dependence for the stochastic stress. In reality the hydrodynamic picture breaks down at microscopic scales. We therefore impose a microscopic cutoff, \( \Omega \), to regularise this integral. As we noted earlier for Equation (76), the frequency integral in Equation (87) represents the uncorrelated nature of temporal fluctuations in the pressure field.

Thus, for an incompressible fluid between two parallel plates with size much smaller than their separation, there is a net pressure that vanishes exponentially with the plate separation:
\[ \Delta \langle P(t) \rangle \xrightarrow{\lambda \ll 1} \frac{k_B T}{16\pi L^3} e^{-2/\lambda}. \] (88)

The average hydrodynamic pressure thus decays towards zero exponentially fast. The fluctuations in this pressure are given by
\[ \Delta \langle P(t)P(t) \rangle \xrightarrow{\lambda \ll 1} - \frac{k_B T \eta_0 \Omega}{\pi^3 L^3} \frac{1}{\lambda}. \] (89)
where we note that the scaling behaviour is now $1/L^2$, rather than the $1/L^3$ decay of the $\lambda \gg 1$ regime.

In this case, the relative magnitude of fluctuations in the pressure acting on the walls diverges exponentially with their separation as

$$\frac{\Delta P_{\text{rms}}(t)}{\Delta \langle P(t) \rangle_{(\text{ln})}} \sim \frac{L^{3/2}}{\lambda^{3/2}} e^{2/\lambda} = L^{3/2} e^{2/\lambda}.$$  \hspace{1cm} (90)

Again one needs to bear in mind that the integration limits have to be consistent with the basic assumption of the continuum description and that the $\lambda \ll 1$ limit breaks our assumption of translational symmetry.

7 Conclusions

We have calculated the pressure-pressure correlation function for an incompressible fluid confined between two hard boundaries. For this case the average pressure has two limits that can be evaluated explicitly. In the limit of small separations or large bounding surface area the hydrodynamic pressure is non-vanishing and Casimir-like, decaying as the third power of the separation between the surfaces, as already found in [3]. In the opposite limit the hydrodynamic pressure tends to zero exponentially fast.

We have also evaluated the corresponding fluctuations in the induced pressure fields. They appear very different from the electromagnetic (thermal) Casimir case [4]. In the limit of the plate separation much smaller than the plate size the fluctuations of the pressure at the bounding surface scale with the separation in the same way as the average, i.e. with the inverse separation cubed, whereas in the second case they decay inversely with the separation between the surfaces. The second limit breaks the assumption of translational invariance in the directions parallel to the boundaries and must be interpreted with caution. The implications of the first limit are clear, however: the relative fluctuations of the induced pressure scale as the separation to the power three halves and therefore dominate as the separation increases. In other words, the width of the hydrodynamic pressure distribution on the bounding surfaces decays anomalously slowly with separation between the bounding surfaces and potentially provides a more convenient candidate for the experimental investigation of such fluctuation-induced forces than the mean force itself.

Our derivation concentrated on the incompressible case. Chan and White argued that for compressible fluids, the pressure field due to hydrodynamic fluctuations between hard boundaries vanishes [3]. This was surmised to be true based on an analogy with electromagnetic field, where frequency integrals over the real axis can be extended into a contour integral in the upper half of the complex plain. That procedure does not seem to be self-evident since the hydrodynamic description breaks down in the very limit where hydrodynamic consequences are purported to be evaluated. At this point it is not yet clear whether the fluctuations in the pressure remain finite even if the mean pressure vanishes for compressible fluids.
A Green function solutions

Here we solve the Green function equations, Equations (49) to (52). As a reminder, these are

\[
\begin{align*}
\eta \left( \frac{\partial^2}{\partial z^2} - k^2 \right) + i \omega \rho_0 \tilde{G}_{1\gamma} + ik_\gamma \chi(\omega) \tilde{G}_{2\gamma} &= \delta_1 \gamma (z - z''), \\
\eta \left( \frac{\partial^2}{\partial z^2} - k^2 \right) + i \omega \rho_0 \tilde{G}_{2\gamma} + ik_\gamma \chi(\omega) \tilde{G}_{2\gamma} &= \delta_2 \gamma (z - z''), \\
\eta \left( \frac{\partial^2}{\partial z^2} - k^2 \right) + i \omega \rho_0 \tilde{G}_{3\gamma} + \chi(\omega) \frac{\partial}{\partial z} \tilde{G}_{4\gamma} &= \delta_3 \gamma (z - z''), \\
i k_\gamma \tilde{G}_{1\gamma} + i k_\gamma \tilde{G}_{2\gamma} + \frac{\partial}{\partial z} \tilde{G}_{3\gamma} - \kappa(\omega) \tilde{G}_{4\gamma} &= 0.
\end{align*}
\]  

We would like to cast these into a more helpful form, so we take combinations of these equations as follows. Defining

\[
\begin{align*}
\tilde{A}_\gamma(z, z''; k; \omega) &= \eta \left( k_\gamma \tilde{G}_{1\gamma}(z, z''; k; \omega) + k_\gamma \tilde{G}_{2\gamma},(z, z''; k; \omega) \right), \\
\tilde{B}_\gamma(z, z''; k; \omega) &= -\chi(\omega) \tilde{G}_{4\gamma}(z, z''; k; \omega), \\
\tilde{C}_\gamma(z, z''; k; \omega) &= \eta \left( k_\gamma \tilde{G}_{1\gamma}(z, z''; k; \omega) - k_\gamma \tilde{G}_{2\gamma}(z, z''; k; \omega) \right), \\
\tilde{F}_\gamma(z, z''; k; \omega) &= i k_\gamma \tilde{G}_{3\gamma}(z, z''; k; \omega),
\end{align*}
\]

the linear combination of Equations \((k_\gamma(91)+k_\gamma(92))\) leads to

\[
\left( \frac{\partial^2}{\partial z^2} - q^2 \right) \tilde{A}_\gamma(z, z''; k; \omega) - i k^2 \tilde{B}_\gamma(z, z''; k; \omega) = K_\gamma \delta(z - z''),
\]

with

\[
K_\gamma = (k_\gamma \delta_1 \gamma + k_\gamma \delta_2 \gamma)
\]

and

\[
q^2 = k^2 - \frac{i \omega \rho_0}{\eta}
\]

whilst the second linear combination of Equations \((k_\gamma(91)-k_\gamma(92))\) gives

\[
\left( \frac{\partial^2}{\partial z^2} - q^2 \right) \tilde{C}_\gamma(z, z''; k; \omega) = (k_\gamma \delta_1 \gamma - k_\gamma \delta_2 \gamma) \delta(z - z'').
\]

To obtain the third equation we multiply Equation (93) by \(i k = i \sqrt{K}\):

\[
\left( \frac{\partial^2}{\partial z^2} - q^2 \right) \tilde{F}_\gamma(z, z''; k; \omega) - i k \frac{\partial}{\partial z} \tilde{B}_\gamma(z, z''; k; \omega) = i k \delta_3 \gamma \delta(z - z'').
\]

Finally we multiply Equation (94) by \(i k \eta\) to obtain

\[
\frac{\partial}{\partial z} \tilde{F}_\gamma(z, z''; k; \omega) - k \tilde{A}_\gamma(z, z''; k; \omega) - i k \beta(\omega) \tilde{B}_\gamma(z, z''; k; \omega) = 0,
\]

where

\[
\beta(\omega) = -\frac{\kappa(\omega) \eta}{\chi(\omega)}.
\]
The boundary conditions for the modified Green functions are

\[ \bar{A}_\gamma, \bar{C}_\gamma, \bar{F}_\gamma = 0, \quad \text{at} \quad z = 0, L. \tag{105} \]

To actually solve Equations (99) to (103), our first step is to tackle the uncoupled equation for \( \bar{C}_\gamma \), Equation (101), which can be solved immediately:

\[ \bar{C}_\gamma(z, z''; k; \omega) = c_{1\gamma}e^{-qz} + c_{2\gamma}e^{qz} - \frac{e^{-q|z-z''|}}{2q}(k_0\delta_{1\gamma} - k_0\delta_{2\gamma}). \tag{106} \]

Here \( c_{1\gamma} \) and \( c_{2\gamma} \) are constants of integration, which we can find by imposing the boundary conditions \( \bar{C}_\gamma = 0 \) at \( z = 0 \) and \( z = L \).

We obtain \( \bar{B}_\gamma \) by: differentiating Equation (102) and then eliminating \((\partial \bar{F}_\gamma / \partial z)\) using Equation (103), to give an equation in \( \bar{A}_\gamma \) and \( \bar{B}_\gamma \). We then eliminate \( \bar{A}_\gamma \) using Equation (99), leaving

\[ \left( \frac{\partial^2}{\partial z^2} - m^2 \right) \bar{B}_\gamma(z, z''; k; \omega) = \frac{1}{\beta(\omega) - 1} \left[ iK_\gamma + \delta_{3\gamma} \frac{\partial}{\partial z} \right] \delta(z - z''), \tag{107} \]

where

\[ m^2 = k^2 - \omega^2 \left[ c_0 - \left( \frac{4}{3}q + \zeta(\omega) \right) \frac{i\omega}{\rho_0} \right]^{-1}. \tag{108} \]

The solution is:

\[ \bar{B}_\gamma(z, z''; k; \omega) = b_{1\gamma}e^{-mxz} + b_{2\gamma}e^{m(z-L)} \]

\[ - \frac{1}{2m(\beta(\omega) - 1)} \left( iK_\gamma - m\delta_{3\gamma} \text{sign}(z-z'') \right) e^{-m|z-z'|}. \tag{109} \]

We now have a solution for \( \bar{B}_\gamma \), which we can differentiate with respect to \( z \) and substitute into Equation (60) to obtain a second-order uncoupled differential equation for \( \bar{F}_\gamma \):

\[ \left( \frac{\partial^2}{\partial z^2} - q^2 \right) \bar{F}_\gamma(z, z''; k; \omega) = ik \left\{ m \left( b_{2\gamma}e^{m(z-L)} - b_{1\gamma}e^{-mxz} \right) \right. \]

\[ + \frac{\beta(\omega)\delta_{3\gamma}}{\beta(\omega) - 1} \delta(z - z'') + \frac{e^{-m|z-z''|}}{2(\beta(\omega) - 1)} \left( iK_\gamma \text{sign}(z-z'') - m\delta_{3\gamma} \right) \left\{ iK_\gamma \text{sign}(z-z'') - m\delta_{3\gamma} \right\}, \]

with general solution

\[ \bar{F}_\gamma(z, z''; k; \omega) = f_{1\gamma}e^{-qz} + f_{2\gamma}e^{qz} + \frac{i\eta k}{m^2 - q^2} \left( b_{2\gamma}e^{m(z-L)} - b_{1\gamma}e^{-mxz} \right) \]

\[ - \frac{1}{2(\beta(\omega) - 1)} \left\{ e^{-q|z-z''|} \left( \frac{\beta(\omega)}{q} + \frac{m^2}{q(q^2 - m^2)} \right) i\delta_{3\gamma} \right. \]

\[ + \frac{K_\gamma}{q^2 - m^2} \text{sign}(z-z'') \left. - \frac{e^{-m|z-z''|}}{q^2 - m^2} \left( i\delta_{3\gamma} + K_\gamma \text{sign}(z-z'') \right) \right\}, \tag{110} \]

which includes two further constants of integration \( f_{1\gamma} \) and \( f_{2\gamma} \).

\(^3\) Note that this equation corrects a typographic error in Equation 3.13 of [3].
Finally we use our expressions for $\tilde{F}$ and $\tilde{B}$ to obtain $\tilde{A}_\gamma$ from Equation (103):

$$
\tilde{A}_\gamma(z, z''; k; \omega) = i \left( \frac{m^2}{m^2 - q^2} - \beta(\omega) \right) \left( b_{1\gamma} e^{-mz} + b_{2\gamma} e^{m(z - L)} \right) + \frac{q}{k} \left( f_{2\gamma} e^{qz} - f_{1\gamma} e^{-qz} \right) + \frac{K_\gamma}{2(\beta(\omega) - 1)} \left[ q e^{-qz} - q e^{-qz''} \right] + \left( \frac{m}{m^2 - q^2} - \frac{\beta(\omega)}{m} \right) e^{-m|z - z'|} - 2 \left( 1 + \frac{1}{m^2 - q^2} \right) \delta(z - z') \right)
+ \frac{i \delta_{3\gamma}}{2(\beta(\omega) - 1)} \left[ \frac{m^2 e^{-m|z - z'|}}{m^2 - q^2} + \left( \beta(\omega) - \frac{m^2}{m^2 - q^2} \right) e^{-q|z - z'|} - \beta(\omega) \right].
$$

We are now in a position to solve for the integration constants $b_{1\gamma}$, $b_{2\gamma}$, $f_{1\gamma}$ and $f_{2\gamma}$ by imposing the boundary conditions $\tilde{A}_\gamma = 0$ and $\tilde{F}_\gamma = 0$ at $z = 0$ and $z = L$. We do not require full expressions for all the constants of motion, which are complicated functions of $z'$, and too unwieldy to reproduce in full here. Instead we list below only the constants of motion that are required, for the special case of interest in this work: that of vanishing two-momentum, $k = 0$.

First we note that there are four trivial constants of motion (for arbitrary values of the two-momentum $k$),

$$
b_{14} = b_{24} = f_{14} = f_{24} = 0,
$$
from which we can deduce

$$
\tilde{A}_4(z, z''; k; \omega) = \tilde{B}_4(z, z''; k; \omega) = \tilde{G}_4(z, z''; k; \omega) = \tilde{F}_4(z, z''; k; \omega) = 0
$$
and thus

$$
\tilde{G}_{4\gamma}(z, z''; k; \omega) = 0.
$$

The only Green functions that we require are in fact

$$
\tilde{G}_{4\gamma}(z, z''; k; \omega) = -\frac{1}{\chi} \left\{ b_{1\gamma} e^{-mz} + b_{2\gamma} e^{m(z - L)} \right\} - \frac{1}{2m(\beta(\omega) - 1)} \left( iK_\gamma - m\delta_{3\gamma} \text{sign}(z - z'') \right) e^{-m|z - z''|}
$$

$$
\tilde{G}_{3\gamma}(z, z''; k; \omega) = \frac{1}{\eta} \left\{ \frac{m}{m^2 - q^2} \left( b_{2\gamma} e^{m(z - L)} - b_{1\gamma} e^{-mz} \right) - \frac{i}{k} \left( f_{1\gamma} e^{-qz} + f_{2\gamma} e^{qz} \right) + \frac{1}{2(\beta(\omega) - 1)(q^2 - m^2)} \left[ e^{-q|z - z''|} \right. \\
\times \left. \left( i\frac{k^2}{q} \delta_{3\gamma} + K_\gamma \text{sign}(z - z'') \right) - e^{-m|z - z''|} \left( \text{sign}(z - z'') \right) \right] \right\},
$$

where we have used the identity

$$
m^2(\beta(\omega) - 1) - q^2\beta(\omega) = -k^2
$$

to simplify $\tilde{G}_{3\gamma}(z, z''; k; \omega)$.

The constants of motion $b_{11}$, $b_{12}$, $b_{21}$ and $b_{22}$ all vanish for $k = 0$. Therefore the only non-zero component of $\tilde{G}_{4\gamma}(z, z''; k; \omega)$ is the $\gamma = 3$ component. From our expressions for the force-force and field-field correlators, Equations (31) and (43) to (45) we can further deduce that we only need the $\gamma = 3$ component of $\tilde{G}_{3\gamma}(z, z''; k; \omega)$ as well. The only non-zero constants of motion are

$$
b_{13}|_{k = 0} = \frac{\sinh(m_0(z'' - L))}{2(\beta(\omega) - 1)\sinh(m_0 L)}, \quad b_{23}|_{k = 0} = \frac{\sinh(m_0 z'')}{2(\beta(\omega) - 1)\sinh(m_0 L)}.
$$
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