A generalization of Hall’s theorem on hypercenter

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Abstract

Let $\sigma$ be a partition of the set of all primes and $\mathcal{F}$ be a hereditary formation. We described all formations $\mathcal{F}$ for which the $\mathcal{F}$-hypercenter and the intersection of weak $K$-$\mathcal{F}$-subnormalizers of all Sylow subgroups coincide in every group. In particular the formation of all $\sigma$-nilpotent groups has this property. With the help of our results we solve a particular case of L.A. Shemetkov’s problem about the intersection of $\mathcal{F}$-maximal subgroups and the $\mathcal{F}$-hypercenter. As corollaries we obtained P. Hall’s and R. Baer’s classical results about the hypercenter. We proved that the non-$\sigma$-nilpotent graph of a group is connected and its diameter is at most 3.

Keywords: Finite group; $\sigma$-nilpotent group; hereditary formation; $K$-$\mathcal{F}$-subnormal subgroup; $\mathcal{F}$-hypercenter; non-$\mathcal{F}$-graph of a group.

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1 Introduction

Throughout this paper, all groups are finite; $G$ and $p$ always denote a finite group and a prime respectively. The notion of the hypercenter of a group naturally appears with the definition of nilpotency of a group through upper central series. R. Baer [4] introduced and studied the analogue of hypercenter for the class of all supersoluble groups. B. Huppert [18] considered the $\mathcal{F}$-hypercenter where $\mathcal{F}$ is a hereditary saturated formation. L.A. Shemetkov [28] extended the notion of $\mathcal{F}$-hypercenter for graduated formations. The $\mathcal{F}$-hypercenter for formations of algebraic systems (including finite groups) was suggested in [31].

Recall that a chief factor $H/K$ of $G$ is called $X$-central (see [31, p. 127–128]) in $G$ provided $(H/K) \rtimes (G/C_G(H/K)) \in X$. A normal subgroup $N$ of $G$ is said to be $X$-hypercentral in $G$ if $N = 1$ or $N \neq 1$ and every chief factor of $G$ below $N$ is $X$-central. The symbol $Z_X(G)$ denotes the $X$-hypercenter of $G$, that is, the product of all normal $X$-hypercentral in $G$ subgroups. According to [31, Lemma 14.1] $Z_X(G)$ is the largest normal $X$-hypercentral subgroup of $G$. If $X = \mathcal{N}$ is the class of all nilpotent groups, then $Z_{\mathcal{N}}(G)$ is the hypercenter $Z_\infty(G)$ of $G$.

One of the first characterizations of the hypercenter was obtained by P. Hall [16]. He proved that the hypercenter of a group coincides with the intersection of normalizers of all its Sylow subgroups. P. Schmid [27] proved the analogue of Hall’s result in profinite groups. There were generalizations of P. Hall’s theorem in terms of intersections of normalizers of $\pi_i$-maximal subgroups [23] or Hall $\pi_i$-subgroups [17] where $\pi_i$ belongs to some partition $\sigma$ of $\mathbb{P}$ (see Corollaries 1.2 and 1.3).

These results are the part of research project in which the $\mathcal{F}$-hypercenter and its generalizations are used as descriptors for characterising some structural properties of the group. A useful tool that provides a suitable language in this direction is the theory of formations. Nowadays this project is actively developing by many researchers (for example [2, 5, 17] and [15, Chapter 1]). As part of the above mentioned project, the aim of our paper is to describe all hereditary (not necessary saturated) formations $\mathcal{F}$ for which the analogue of Hall’s result holds for the

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\( \mathfrak{F} \)-hypercenter and find the applications of this result. To formulate our results we need the following definitions.

Let \( \mathfrak{F} \) be a formation. O.H. Kegel \cite{Kegel} introduced the formation generalization of subnormality. Recall \cite[Definition 6.1.4]{Kegel} that a subgroup \( H \) of \( G \) is called \( K^-\mathfrak{F} \)-subnormal in \( G \) if there is a chain of subgroups \( H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G \) with \( H_{i-1} \trianglelefteq H_i \) or \( H_i/\text{Core}_{H_i}(H_{i-1}) \in \mathfrak{F} \) for all \( i = 1, \ldots, n \). Denoted by \( H \text{-}K^-\mathfrak{F}\text{-sn} \in G \). If \( \mathfrak{F} = \mathfrak{N} \), then the notions of \( K^-\mathfrak{N} \)-subnormal and subnormal subgroups coincide.

R.W. Carter \cite{Carter} and C.J. Graddon \cite{Graddon} studied subnormalizers and \( \mathfrak{N} \)-subnormalizers respectively. Note that the subnormalizer of a Sylow subgroup always exists and coincides with its normalizer. For an arbitrary subgroup a subnormalizer or \( \mathfrak{N} \)-subnormalizer may not exist. A. Mann \cite{Mann} suggested the concept of a weak subnormalizer that always exists but may be not unique. A subgroup \( T \) of \( G \) is called a weak subnormalizer of \( H \) in \( G \) if \( H \) is subnormal in \( T \) and if \( H \) is subnormal in \( M \leq G \) and \( T \leq M \), then \( T = M \). Here we introduced its generalization.

**Definition 1.** Let \( \mathfrak{F} \) be a formation. We shall call a subgroup \( T \) of \( G \) a weak \( K^-\mathfrak{F} \)-subnormalizer of \( H \) in \( G \) if \( H \) is \( K^-\mathfrak{F} \)-subnormal in \( T \) and if \( H \) is \( K^-\mathfrak{F} \)-subnormal in \( M \leq G \) and \( T \leq M \), then \( T = M \).

It is clear that a weak \( K^-\mathfrak{F} \)-subnormalizer always exists. Note that the notions of weak subnormalizer and \( K^-\mathfrak{N} \)-subnormalizer coincide. See \cite[Example 14.12]{Skiba} for an example of a group that has a subgroup without an unique weak subnormalizer.

Let \( \pi = \{ \pi_i \mid i \in I \} \) be a partition of the set \( \mathbb{P} \) of all primes. According to A.N. Skiba \cite{Skiba}, a group \( G \) is called \( \sigma \)-nilpotent if \( G \) has a normal Hall \( \pi_i \)-subgroup for every \( i \in I \) with \( \pi(G) \cap \pi_i \neq \emptyset \). The class of all \( \sigma \)-nilpotent groups is denoted by \( \mathfrak{N}_\sigma \). This class is a very interesting generalization of the class of nilpotent groups and widely studied, applied and are part of the actively developing nowadays \( \sigma \)-method, i.e. the studying the properties of a group that depends on the given partition \( \sigma \) (for example, see \cite{Carter, Graddon, Mann, Skiba}). The class \( \mathfrak{N} \) of all nilpotent groups coincides with the class \( \mathfrak{N}_\sigma \) for \( \sigma = \{ \{ p \} \mid p \in \mathbb{P} \} \).

Recall \cite[Example 2.2.12]{Kegel} that \( \times_{\pi_i} \mathfrak{F}_{\pi_i} = ( G = \times_{\pi_i} \text{O}_{\pi_i}(G) \mid \text{O}_{\pi_i}(G) \in \mathfrak{F}_{\pi_i} ) \) is a hereditary formation where \( \mathfrak{F}_{\pi_i} \) is a hereditary formation with \( \pi(\mathfrak{F}_{\pi_i}) = \pi_i \) for all \( i \in I \). The main result of this paper is

**Theorem 1.** Let \( \mathfrak{F} \) be a hereditary formation. The following statements are equivalent:

1. The intersection of all weak \( K^-\mathfrak{F} \)-subnormalizers of all cyclic primary subgroups coincides with the \( \mathfrak{F} \)-hypercenter in every group.
2. The intersection of all weak \( K^-\mathfrak{F} \)-subnormalizers of all Sylow subgroups coincides with the \( \mathfrak{F} \)-hypercenter in every group.
3. There is a partition \( \pi = \{ \pi_i \mid i \in I \} \) of \( \mathbb{P} \) such that the \( \mathfrak{F} \)-hypercenter coincides with the \( \sigma \)-nilpotent-hypercenter in every group.
4. There is a partition \( \pi = \{ \pi_i \mid i \in I \} \) of \( \mathbb{P} \) such that \( \mathfrak{F} = \times_{\pi_i} \mathfrak{F}_{\pi_i} \), where \( \mathfrak{F}_{\pi_i} \) is a hereditary formation with \( \pi(\mathfrak{F}_{\pi_i}) = \pi_i \) and \( \mathfrak{F}_{\pi_i} \) coincides with the class of all \( \pi_i \)-groups for all \( i \in I \) with \( |\pi_i| \geq 2 \).

**Remark 1.** As follows from \cite[Theorem]{Skiba} formations from (4) of Theorem 1 are lattice formations.

**Corollary 1.1.** Let \( \pi = \{ \pi_i \mid i \in I \} \) be a partition \( \mathbb{P} \), \( G \) be a group and \( \mathcal{M} \) be a set of maximal \( \pi_i \)-subgroups of \( G \), \( \pi_i \in \sigma \), such that
(a) if \( H \in \mathcal{M} \), then \( H^x \in \mathcal{M} \) for every \( x \in G \);

(b) for every Sylow subgroup \( P \) of \( G \) there is \( H \in \mathcal{M} \) with \( P \leq H \).

Then the intersection of normalizers in \( G \) of all subgroups from \( \mathcal{M} \) is \( Z_{\mathcal{M}}(G) \).

**Corollary 1.2** ([23, Corollary 3.7]). Let \( \sigma = \{ \pi_i \mid i \in I \} \) be a partition \( \mathbb{P} \). The intersection of normalizers of all \( \pi_i \)-maximal subgroups of \( G \), \( \pi_i \in \sigma \), is \( Z_{\mathcal{M}}(G) \).

**Corollary 1.3** ([17, Theorem B(ii)]). Let \( \sigma = \{ \pi_i \mid i \in I \} \) be a partition \( \mathbb{P} \). Assume that a group \( G \) possesses a set \( \mathcal{H} \) of Hall subgroups such that \( \mathcal{H} \) contains exactly one Hall \( \pi_i \)-subgroup of \( G \) with \( \pi_i \cap \pi(G) \neq \emptyset \). Then

\[
\bigcap_{x \in G} \bigcap_{H \in \mathcal{H}} N_G(H^x) = Z_{\mathcal{M}}(G).
\]

**Corollary 1.4** (P. Hall [16]). The intersection of all normalizers of Sylow subgroups is the hypercenter in every group.

**Corollary 1.5.** The intersection of all weak subnormalizers of cyclic primary subgroups is the hypercenter in every group.

**Corollary 1.6** ([23 Theorem 3.1(2)]). Let \( \sigma = \{ \pi_i \mid i \in I \} \) be a partition \( \mathbb{P} \). A \( \pi_i \)-element belongs to \( Z_{\mathcal{M}}(G) \) iff its permutes with all \( \pi_i' \)-elements of a group \( G \).

**Corollary 1.7** (R. Baer [3, 5, Theorem 1(ii)]). Let \( p \) be a prime. A \( p \)-element belongs to \( Z_\infty(G) \) iff its permutes with all \( p' \)-elements of a group \( G \).

## 2 Preliminaries

The notation and terminology agree with [8] and [12]. We refer the reader to these books for the results about formations.

Recall that a formation is a class of groups which is closed under taking epimorphic images and subdirect products. A formation \( \mathcal{F} \) is called hereditary if \( H \in \mathcal{F} \) whenever \( H \leq G \in \mathcal{F} \); saturated if \( G \in \mathcal{F} \) whenever \( G/\Phi(N) \in \mathcal{F} \) for some normal subgroup \( N \) of \( G \).

**Lemma 1** ([33, Lemma 2.5]). The class of all \( \sigma \)-nilpotent groups is a hereditary saturated formation.

The following two lemmas follow from [8, Lemmas 6.1.6 and 6.1.7].

**Lemma 2.** Let \( \mathcal{F} \) be a formation, \( H \) and \( R \) be subgroups of \( G \) and \( N \trianglelefteq G \).

(1) If \( H \triangleleft_{\mathcal{F}} G \), then \( HN/N \triangleleft_{\mathcal{F}} K \triangleleft_{\mathcal{F}} G/N \).

(2) If \( H/N \triangleleft_{\mathcal{F}} K \triangleleft_{\mathcal{F}} G/N \), then \( H \triangleleft_{\mathcal{F}} G \).

(3) If \( H \triangleleft_{\mathcal{F}} G \) and \( R \triangleleft_{\mathcal{F}} G \), then \( H \cap R \triangleleft_{\mathcal{F}} G \).

**Lemma 3.** Let \( \mathcal{F} \) be a hereditary formation, \( H \) and \( R \) be subgroups of \( G \).

(1) If \( H \triangleleft_{\mathcal{F}} G \), then \( H \cap R \triangleleft_{\mathcal{F}} G \).

(2) If \( H \triangleleft_{\mathcal{F}} G \) and \( R \triangleleft_{\mathcal{F}} G \), then \( H \cap RK \triangleleft_{\mathcal{F}} G \).

The following lemma directly follows from Lemma 2.
Lemma 4. Let $\mathcal{F}$ be a formation, $H$ and $R$ be subgroups of $G$ and $N \leq G$. If $H$ a $\mathcal{F}$-sn $R$, then $HN$ a $\mathcal{F}$-sn $RN$.

Recall that $\mathbb{F}_p$ denotes a field with $p$ elements. The following result directly follows from [12, B, Theorem 10.3].

Lemma 5. If $O_p(G) = 1$ and $G$ has a unique minimal normal subgroup, then $G$ has a faithful irreducible module over $\mathbb{F}_p$.

In [30] L.A. Shemetkov posed the problem to describe the set of formations $\mathcal{F}$ having the following property

$$\mathcal{F} = (G \mid \text{every chief factor of } G \text{ is } \mathcal{F}\text{-central}) = (G \mid G = Z_\mathcal{F}(G)).$$

This class of formations contains saturated (local) and solubly saturated (composition or Baer-local) formations and other. Shortly we shall call formations from this class $Z$-saturated. In [7] A. Ballester-Bolinches and M. Pérez-Ramos showed that for a formation $\mathcal{F}$ the class

$$Z\mathcal{F} = (G \mid G = Z_\mathcal{F}(G))$$

is a formation and $\mathcal{F} \subseteq Z\mathcal{F} \subseteq E_\Phi \mathcal{F}$.

Let $\mathcal{F}$ be a hereditary formation. In [24] and [34] the classes $w\mathcal{F}$ and $v^*\mathcal{F}$ of all groups all whose Sylow and cyclic primary subgroups respectively are $K$-$\mathcal{F}$-subnormal were studied. From the results of these papers follows

Proposition 1. If $\mathcal{F}$ is a hereditary formation, then $w\mathcal{F}$ and $v^*\mathcal{F}$ are hereditary formations and $\mathcal{F} \cup \mathcal{F} \subseteq w\mathcal{F} \subseteq v^*\mathcal{F}$.

Recall that a Schmidt group $G$ is a non-nilpotent group all whose proper subgroups are nilpotent. It is well known that $\pi(G) = \{p, q\}$ and $G$ has a unique normal Sylow subgroup. Recall [35] that a Schmidt $(p, q)$-group is a Schmidt group with a normal Sylow $p$-subgroup. An $N$-critical graph $\Gamma_{Nc}(G)$ of a group $G$ [35, Definition 1.3] is a directed graph on the vertex set $\pi(G)$ of all prime divisors of $|G|$ and $(p, q)$ is an edge of $\Gamma_{Nc}(G)$ if $G$ has a Schmidt $(p, q)$-subgroup. An $N$-critical graph $\Gamma_{Nc}(\mathcal{X})$ of a class of groups $\mathcal{X}$ [35, Definition 3.1] is a directed graph on the vertex set $\pi(\mathcal{X}) = \cup_{G \in \mathcal{X}} \pi(G)$ such that $\Gamma_{Nc}(\mathcal{X}) = \cup_{G \in \mathcal{X}} \Gamma_{Nc}(G)$.

Proposition 2 ([35, Theorem 5.4]). Let $\mathcal{X} = \{\pi_i \mid i \in I\}$ be a partition of the vertex set $V(\Gamma_{Nc}(\mathcal{X}))$ such that for $i \neq j$ there are no edges between $\pi_i$ and $\pi_j$. Then every $\mathcal{X}$-group is the direct product of its Hall $\pi_k$-subgroups, where $k \in \{i \in I \mid \pi(G) \cap \pi_k \neq \emptyset\}$.

3 The proof of Theorem 1 and its corollaries

The proof of Theorem 1 is rather complicated and require various preliminary results and definitions. A subgroup $U$ of $G$ is called $\mathcal{X}$-maximal in $G$ provided that (a) $U \in \mathcal{X}$, and (b) if $U \leq V \leq G$ and $V \in \mathcal{X}$, then $U = V$. Let $\int_{\mathcal{X}}(G)$ denotes the intersection of all $\mathcal{X}$-maximal subgroups of $G$ [32].

Proposition 3. Let $\mathcal{F}$ be a hereditary formation. Then

1. [2, Lemma 2.4] $Z_\mathcal{F}(G) \cap H \leq Z_\mathcal{F}(H)$ for every subgroup $H$ of a group $G$.
2. $Z_\mathcal{F}(G) = Z_\mathcal{F}(Z_\mathcal{F}(G))$ for every group $G$.
3. Assume that $H$ is an $\mathcal{F}$-subgroup of a group $G$. If $\mathcal{F}$ is $Z$-saturated, then $HZ_\mathcal{F}(G) \in \mathcal{F}$. In particular $Z_\mathcal{F}(G) \leq \int_{\mathcal{F}}(G)$ for every group $G$. 

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Proof. (2) From (1) it follows that $Z_{\mathfrak{F}}(G) = Z_{\mathfrak{F}}(G) \cap Z_{\mathfrak{F}}(G) \leq Z_{\mathfrak{F}}(Z_{\mathfrak{F}}(G)) \leq Z_{\mathfrak{F}}(G)$. Thus $Z_{\mathfrak{F}}(G) = Z_{\mathfrak{F}}(Z_{\mathfrak{F}}(G))$.

(3) From (1) it follows that $Z_{\mathfrak{F}}(G) \leq Z_{\mathfrak{F}}(G) \cap HZ_{\mathfrak{F}}(G) \leq Z_{\mathfrak{F}}(HZ_{\mathfrak{F}}(G))$. Since the group $HZ_{\mathfrak{F}}(G)/Z_{\mathfrak{F}}(G) \in \mathfrak{F}$, we see that $HZ_{\mathfrak{F}}(G)/Z_{\mathfrak{F}}(HZ_{\mathfrak{F}}(G)) \in \mathfrak{F}$. Hence $HZ_{\mathfrak{F}}(G) = Z_{\mathfrak{F}}(HZ_{\mathfrak{F}}(G)) \in Z_{\mathfrak{F}} = \mathfrak{F}$.

Let $M$ be an $\mathfrak{F}$-maximal subgroup of $G$. Then $MZ_{\mathfrak{F}}(G) \in \mathfrak{F}$. It means that $MZ_{\mathfrak{F}}(G) = M$. Thus $Z_{\mathfrak{F}}(G) \leq \text{Int}_{\mathfrak{F}}(G)$. \hfill $\square$

The following result plays the key role in the proof of Theorem 1.

**Proposition 4.** Let $\mathfrak{F}$ be a formation.

(1) $Z_{Z_{\mathfrak{F}}(G)}(G) = Z_{\mathfrak{F}}(G)$ holds for every group $G$.

(2) Assume that $\mathfrak{F}$ is hereditary. A subgroup $H$ is $K-\mathfrak{F}$-subnormal in a group $G$ iff it is $K-Z_{\mathfrak{F}}$-subnormal in $G$.

Proof. (1) Let $H/K$ be a chief factor of a group $G$. Now $(H/K) \times G/C_{G}(H/K)$ is a primitive group. It means that the $\mathfrak{F}$-hypercenter is defined by the set of all primitive $\mathfrak{F}$-groups. According to $(\ref{lem:hypercenter})$, $\mathfrak{F} \subseteq Z_{\mathfrak{F}} \subseteq E_{\Phi}Z_{\mathfrak{F}}$. It means that every $Z_{\mathfrak{F}}$-group $G$ with $\Phi(G) = 1$ belongs $\mathfrak{F}$. Thus the sets of all primitive $\mathfrak{F}$-groups and $Z_{\mathfrak{F}}$-groups coincide. Hence $Z_{Z_{\mathfrak{F}}}(G) = Z_{\mathfrak{F}}(G)$.

(2) Note that $Z_{\mathfrak{F}}$ is a hereditary formation by Statement (1) of Proposition 3. Since $\mathfrak{F}$ is a hereditary formation, we see that $H$ is a $K-\mathfrak{F}$-subnormal subgroup of a group $G$ if and only if there is a chain of subgroups $H = H_{0} \subseteq H_{1} \subseteq \cdots \subseteq H_{n} = G$ with $H_{i} \subseteq H_{i+1}$ or $H_{i}/\text{Core}_{H}(H_{i-1}) \in \mathfrak{F}$ and $H_{i-1}$ is a maximal subgroup of $H_{i}$ for all $i = 1, \ldots, n$. It means that $K-\mathfrak{F}$-subnormality is defined by the set of all primitive $\mathfrak{F}$-groups for a hereditary formation $\mathfrak{F}$. As we have already mentioned the sets of all primitive $\mathfrak{F}$-groups and $Z_{\mathfrak{F}}$-groups coincide. Thus a subgroup is $K-\mathfrak{F}$-subnormal in a group $G$ iff it is $K-Z_{\mathfrak{F}}$-subnormal in $G$. \hfill $\square$

The next step in the proof of Theorem 1 is to characterize the intersections $S_{\mathfrak{F}}(G)$ and $C_{\mathfrak{F}}(G)$ of all weak $K-\mathfrak{F}$-subnormalizers of all Sylow and all cyclic primary subgroups of $G$ respectively.

**Proposition 5.** Let $\mathfrak{F}$ be a hereditary formation.

(1) $S_{\mathfrak{F}}(G)$ is the largest subgroup among normal subgroups $N$ of $G$ with $P K-\mathfrak{F}$-sn $PN$ for every Sylow subgroup $P$ of $G$.

(2) $C_{\mathfrak{F}}(G)$ is the largest subgroup among normal subgroups $N$ of $G$ with $C K-\mathfrak{F}$-sn $CN$ for every cyclic primary subgroup $C$ of $G$.

Proof. (1) Let $N \leq G$ with $P K-\mathfrak{F}$-sn $PN$ for every Sylow subgroup $P$ of $G$. If $S$ is a weak $K-\mathfrak{F}$-subnormalizer of $P$ in $G$, then $PN K-\mathfrak{F}$-sn $SN$ by Lemma 4. Hence $P K-\mathfrak{F}$-sn $SN$ by (3) of Lemma 2. Now $SN = S$ by the definition of a weak $K-\mathfrak{F}$-subnormalizer. Thus $N \leq S_{\mathfrak{F}}(G)$.

From the other hand, since $\mathfrak{F}$ is a hereditary formation and $PS_{\mathfrak{F}}(G)$ lies in every weak $K-\mathfrak{F}$-subnormalizer of every Sylow subgroup $P$ of $G$, we see that $P K-\mathfrak{F}$-sn $PS_{\mathfrak{F}}(G)$ for every Sylow subgroup $P$ of $G$ by Lemma 5. Thus $S_{\mathfrak{F}}(G)$ is the largest normal subgroup $N$ of $G$ with $P K-\mathfrak{F}$-sn $PN$ for every Sylow subgroup $P$ of $G$.

The proof of (2) is the same. \hfill $\square$

The connections between the previous steps are shown in the following proposition:

**Proposition 6.** Let $\mathfrak{F}$ be a hereditary formation. Then $\text{w}Z_{\mathfrak{F}}$ and $\nu Z_{\mathfrak{F}}$ are hereditary $Z$-saturated formations and $\text{Int}_{\text{w}Z_{\mathfrak{F}}}(G) = S_{\mathfrak{F}}(G) \leq C_{\mathfrak{F}}(G) = \text{Int}_{\nu Z_{\mathfrak{F}}}(G)$ holds for every group $G$. 
Proof. Note that $v^*\mathfrak{F}$ and $\overline{\mathfrak{F}}$ are hereditary formations by Proposition 3. Assume that $\overline{\mathfrak{F}}$ is not a $Z$-saturated formation. Let chose a minimal order group $G$ from $Z(\overline{\mathfrak{F}}) \setminus \overline{\mathfrak{F}}$. From Proposition 3 it follows that $Z\overline{\mathfrak{F}}$ is a hereditary formation. So $G$ is $\overline{\mathfrak{F}}$-critical. Now $|\pi(G)| > 1$ by Proposition 1. From $Z\overline{\mathfrak{F}} \subset Z\overline{\mathfrak{F}} \subseteq E_{\overline{\mathfrak{F}}}$ it follows that $\Phi(\overline{\mathfrak{F}}) \neq 1$ and $G/\Phi(\overline{\mathfrak{F}}) \in \overline{\mathfrak{F}}$. Let $P$ be a Sylow subgroup of $G$. Then $P\Phi(G) < G$ and $P\Phi(G) \in \overline{\mathfrak{F}}$. Hence $P$ is $\overline{\mathfrak{F}}$-critcal. Therefore $P\Phi(G) \in \overline{\mathfrak{F}}$ it follows that $\Phi(\overline{\mathfrak{F}})/\Phi(G) K-\overline{\mathfrak{F}}$-sn $G/\Phi(G)$. Thus $P K-\overline{\mathfrak{F}}$-sn $G$. It means that $G \in \overline{\mathfrak{F}}$, a contradiction. Thus $\overline{\mathfrak{F}}$ is a $Z$-saturated formation. The proof for $v^*\mathfrak{F}$ is the same.

Note that $\mathfrak{N} \subseteq v^*\mathfrak{F}$ by Proposition 1. Hence $C\text{Int}_{v^*\mathfrak{F}}(G) \in v^*\mathfrak{F}$ for every cyclic primary subgroup $C$ of $G$. Therefore $\mathfrak{N}$ $K-\overline{\mathfrak{F}}$-sn $C\text{Int}_{v^*\mathfrak{F}}(G)$ for every cyclic primary subgroup $C$ of $G$. Thus $\text{Int}_{v^*\mathfrak{F}}(G) \leq C\overline{\mathfrak{F}}(G)$ by (2) of Proposition 5.

From the other hand let $M$ be a $v^*\mathfrak{F}$-maximal subgroup of $G$ and $C$ be a cyclic primary subgroup of $MC\overline{\mathfrak{F}}(G)$. Since $MC\overline{\mathfrak{F}}(G)/C\overline{\mathfrak{F}}(G) \in v^*\mathfrak{F}$, we see that $C\overline{\mathfrak{F}}(G)C/C\overline{\mathfrak{F}}(G) K-\overline{\mathfrak{F}}$-sn $MC\overline{\mathfrak{F}}(G)/C\overline{\mathfrak{F}}(G)$. Hence $C\overline{\mathfrak{F}}(G)C$ $K-\overline{\mathfrak{F}}$-sn $MC\overline{\mathfrak{F}}(G)$ by (2) of Lemma 2. Note that $C K-\overline{\mathfrak{F}}$-sn $MC\overline{\mathfrak{F}}(G)$ by Proposition 5. So $C K-\overline{\mathfrak{F}}$-sn $MC\overline{\mathfrak{F}}(G)$ by (3) of Lemma 2. Thus $MC\overline{\mathfrak{F}}(G) \in v^*\mathfrak{F}$ by the definition of $v^*\mathfrak{F}$. Hence $MC\overline{\mathfrak{F}}(G) = M$. Therefore $C\overline{\mathfrak{F}}(G) \leq \text{Int}_{v^*\mathfrak{F}}(G)$. Thus $\text{Int}_{v^*\mathfrak{F}}(G) = C\overline{\mathfrak{F}}(G)$. The proof of that equality $\text{Int}_{v^*\mathfrak{F}}(G) = S\overline{\mathfrak{F}}(G)$ holds in every group is the same.

Since every cyclic primary subgroup is subnormal in some Sylow subgroup, we see that $P$ $K-\overline{\mathfrak{F}}$-sn $PS\overline{\mathfrak{F}}(G)$ for every cyclic primary subgroup $P$ of $G$. So $S\overline{\mathfrak{F}}(G) \leq C\overline{\mathfrak{F}}(G)$ holds for every group $G$ by Proposition 5.

Proof of Theorem 1. (1) $\Rightarrow$ (2). Since $\mathfrak{F} \subseteq \overline{\mathfrak{F}}$ by Proposition 1 we see that $Z\overline{\mathfrak{F}}(G) \leq Z\overline{\mathfrak{F}}(G)$ for every group $G$. Note that $Z\overline{\mathfrak{F}}(G) \leq \text{Int}_{\overline{\mathfrak{F}}}(G)$ for every group $G$ by (3) of Proposition 3 and Proposition 6. According to Proposition 6 $S\overline{\mathfrak{F}}(G) = \text{Int}_{\overline{\mathfrak{F}}}(G)$ and $S\overline{\mathfrak{F}}(G) \leq C\overline{\mathfrak{F}}(G)$ for every group $G$. From these and (1) it follows that

$$Z\overline{\mathfrak{F}}(G) \leq Z\overline{\mathfrak{F}}(G) \leq \text{Int}_{\overline{\mathfrak{F}}}(G) = S\overline{\mathfrak{F}}(G) \leq C\overline{\mathfrak{F}}(G) = Z\overline{\mathfrak{F}}(G)$$

for every group $G$. Thus $Z\overline{\mathfrak{F}}(G) = S\overline{\mathfrak{F}}(G)$ for every group $G$.

(2) $\Rightarrow$ (3). The proof consists of the following steps:

(a) We may assume that $\mathfrak{N} \subseteq \mathfrak{F}$ is $Z$-saturated.

According to Proposition 4 Statements (2) and (3) mean the same for $\mathfrak{F}$ and $Z\mathfrak{F}$. Note that $Z\mathfrak{F} = Z(Z\mathfrak{F})$ by Proposition 4. Therefore without lose of generality we may assume that $\mathfrak{F}$ is $Z$-saturated in the proof of (2) $\Rightarrow$ (3). Since in every nilpotent group every Sylow subgroup is subnormal and $Z\mathfrak{F} = \mathfrak{F}$ we see that $\pi(\mathfrak{F}) = \mathfrak{P}$ and $\mathfrak{N} \subseteq \mathfrak{F}$.

(b) Assume that a group $G$ has faithful irreducible module $L$ over $\mathfrak{P}$, $T = L \rtimes G$ and $L \leq S\overline{\mathfrak{F}}(T)$. Then $G \in \mathfrak{F}$.

Note that $L \leq S\overline{\mathfrak{F}}(G) = Z\overline{\mathfrak{F}}(T)$. Hence $L \rtimes (T/C\overline{\mathfrak{F}}(L)) \in \mathfrak{F}$, $G \cong T/C\overline{\mathfrak{F}}(L) \in \mathfrak{F}$, the contradiction.

(c) Let $\pi(p) = \{q \in \mathfrak{P} \mid (p,q) \in \Gamma_{N_c}(\mathfrak{F})\} \cup \{p\}$. Then $\mathfrak{F}$ contains every $q$-closed $\{p,q\}$-group for every $q \in \pi(p)$.

Assume the contrary. Let $G$ be a minimal order counterexample. Since $\mathfrak{F}$ and the class of all $q$-closed groups are hereditary formations, we see that $G$ is an $\mathfrak{F}$-critical group, $G$ has a unique minimal normal subgroup $N$ and $G/N \in \mathfrak{F}$. Let $P$ be a Sylow $p$-subgroup of $G$. If $NP < G$, then $NP \in \mathfrak{F}$. Hence $P K-\mathfrak{F}$-sn $PN$ and $PN/N K-\mathfrak{F}$-sn $G/N$. From Lemma 2 it follows that $P K-\mathfrak{F}$-sn $G$. Since $G$ is $q$-closed $\{p,q\}$-group, we see that every Sylow subgroup of $G$ is $K-\mathfrak{F}$-subnormal. So $G \in Z\mathfrak{F} = \mathfrak{F}$, a contradiction.

Now $N$ is a Sylow $q$-subgroup and $O_p(G) = 1$. By Lemma 5 $G$ has a faithful irreducible module $L$ over $\mathfrak{P}$. Let $T = L \rtimes G$. Therefore for every chief factor $H/K$ of $NL$ a group
(H/K) \rtimes C_{NL}(H/K) is isomorphic to one of the following groups $Z_p, Z_q$ and a Schmidt $(p,q)$-group with the trivial Frattini subgroup. Note that all these groups belong $\mathfrak{F}$. So $NL \in Z\mathfrak{F} = \mathfrak{F}$. Note that $L \leq O_p(T)$. Hence $L \leq S_\mathfrak{F}(T)$ by Proposition 5. Thus $G \in \mathfrak{F}$ by (b), a contradiction.

From (c) it follows that

(d) $\Gamma_{NC}(\mathfrak{F})$ is undirected, i.e. $(p,q) \in \Gamma_{NC}(\mathfrak{F})$ iff $(q,p) \in \Gamma_{NC}(\mathfrak{F})$.

(e) Let $p, q$ and $r$ be different primes. If $(p,r), (q,r) \in \Gamma_{NC}(\mathfrak{F})$, then $(p,q) \in \Gamma_{NC}(\mathfrak{F})$.

Note that the cyclic group $Z_q$ of order $q$ has a faithful irreducible module $P$ over $\mathbb{F}_p$ by Lemma 3. Let $G = P \rtimes Z_q$. Then $G$ has a faithful irreducible module $R$ over $\mathbb{F}_q$ by Lemma 5. Let $T = R \times G$. From (c) it follows that $\mathfrak{F}$-contains all $r$-closed $(p,r)$-groups and $(q,r)$-groups. Hence $R \leq S_\mathfrak{F}(T)$ by Proposition 5. Thus $G \in \mathfrak{F}$ by (b). Note that $G$ is a Schmidt $(p,q)$-group. It means that $(p,q) \in \Gamma_{NC}(\mathfrak{F})$ by the definition of $N$-critical graph.

(f) $\mathfrak{F} = \mathfrak{N}_\sigma$ for some partition $\sigma$ of $\mathbb{P}$.

From (d) and (e) it follows that $\Gamma_{NC}(\mathfrak{F})$ is a disjoint union of complete (directed) graphs $\Gamma_i$, $i \in I$. Let $\pi_i = V(\Gamma_i)$. Then $\sigma = \{ \pi_i | i \in I \}$ is a partition of $\mathbb{P}$. From Proposition 2 it follows that every $\mathfrak{F}$-group $G$ has normal Hall $\pi_i$-subgroups for every $i \in I$ with $\pi_i \cap \pi(\sigma) \neq \emptyset$. So $G$ is $\sigma$-nilpotent. Hence $\mathfrak{F} \subseteq \mathfrak{N}_\sigma$.

Let show that the class $\mathfrak{G}_{\pi_i}$ of all $\pi_i$-groups is a subset of $\mathfrak{F}$ for every $i \in I$. It is true if $|\pi_i| = 1$. Assume now $|\pi_i| > 1$. Suppose the contrary and let a group $G$ be a minimal order group from $\mathfrak{G}_{\pi_i} \setminus \mathfrak{F}$. Then $G$ has a unique minimal normal subgroup, $\pi(G) \subseteq \pi_i$ and $|\pi(G)| > 1$. Note that $O_p(G) = 1$ for some $q \in \pi(G)$. Hence $G$ has a faithful irreducible module $N$ over $\mathbb{F}_q$ by Lemma 5. Let $T = N \times G$. Hence $NP \in \mathfrak{F}$ for every Sylow subgroup $P$ of $T$ by (c). Now $N \leq S_\mathfrak{F}(T)$ by Proposition 5. So $G \in \mathfrak{F}$ by (b), a contradiction.

Since a formation is closed under taking direct products, we see that $\mathfrak{N}_\sigma \subseteq \mathfrak{F}$. Thus $\mathfrak{F} = \mathfrak{N}_\sigma$.

(3) $\Rightarrow$ (1). Recall that the class of all $\sigma$-nilpotent groups is saturated. Hence it is $\mathfrak{S}$-saturated. According to Proposition 4 Statements (3) and (1) mean the same for $\mathfrak{S}$ and $Z\mathfrak{S}$. Hence we may assume that $\mathfrak{F} = \mathfrak{N}_\sigma$ for some partition $\sigma = \{ \pi_i | i \in I \}$ of $\mathbb{P}$. Then $\mathfrak{N}_\sigma$ has the lattice property for $K$-$\mathfrak{S}$-subnormal subgroups (see [33] Lemma 2.6(3)] or [33] Chapter 3].

According to [24] Theorem B and Corollary E.2) $\nu^*\mathfrak{S} = \mathfrak{F}$. By [32] Theorem A and Proposition 4.2) $\text{Int}_{\mathfrak{S}}(G) = Z_{\mathfrak{S}}(G)$ holds for every group $G$. By Proposition 5 $C_{\mathfrak{S}}(G) = \text{Int}_{\mathfrak{S}}(G) = Z_{\mathfrak{S}}(G)$ for every group $G$.

(3) $\Rightarrow$ (4) Statement (3) means that $Z\mathfrak{S} = \mathfrak{N}_\sigma$ and $\pi(\mathfrak{S}) = \pi(Z\mathfrak{S}) = \mathbb{P}$. From $\mathfrak{S} \subseteq Z\mathfrak{S}$ it follows that $\mathfrak{S} = \times_{i \in I} \mathfrak{S}_{\pi_i}$, where $\mathfrak{S}_{\pi_i}$ is a hereditary formation with $\pi(\mathfrak{S}_{\pi_i}) = \pi_i$.

Assume that $\pi_i \in \sigma$ and $|\pi_i| \geq 2$. Let choose a minimal order $\pi_i$-group $G$ from $Z\mathfrak{S} \setminus \mathfrak{S}_{\pi_i}$. Since $Z\mathfrak{S} = \mathfrak{N}_\sigma$ and $\mathfrak{S}_{\pi_i} = \mathfrak{G}_{\pi_i} \cap \mathfrak{S}_{\pi_i}$ are formations, we see that $G$ has a unique minimal normal subgroup $N$. From $|\pi_i| \geq 2$ it follows that there exists $p \in \pi_i$ such that $N$ is not a $p$-group. Therefore $G$ has a faithful irreducible module $V$ over $\mathbb{F}_p$ by Lemma 5. Let $T = V \times G$. Since $T$ is a $\pi_i$-group, $T \in \mathfrak{N}_\sigma = Z\mathfrak{S}$. Hence $R = V \times (T/C_T(V)) \in \mathfrak{S} \cap \mathfrak{G}_{\pi_i} = \mathfrak{S}_{\pi_i}$ and $T/C_T(V) \simeq G$. Now $G \in \mathfrak{S}_{\pi_i}$ as a quotient group of $R$, a contradiction. It means that $\mathfrak{S} \cap \mathfrak{G}_{\pi_i} = Z\mathfrak{S} \cap \mathfrak{G}_{\pi_i} = \mathfrak{S}_{\pi_i}$.

(4) $\Rightarrow$ (3) Assume that $Z_{\mathfrak{S}}(G) \neq Z_{\mathfrak{N}_\sigma}(G)$ for some group $G$. It means that there exists a primitive $\mathfrak{N}_\sigma$-group $H$ with $H \notin \mathfrak{F}$. Since $H$ is a primitive $\mathfrak{N}_\sigma$-group, we see that $H$ is a $\pi_i$-group for some $i \in I$. If $|\pi_i| \geq 2$, then $H \in \mathfrak{G}_{\pi_i} \subseteq \mathfrak{F}$, a contradiction. Hence $|\pi_i| = 1$. So $H$ is a $p$-group for some $p \in \mathbb{P}$. Therefore $H$ is a cyclic group of order $p$. Thus $H \in \mathfrak{F}$, the final contradiction.

Proof of Corollary 5.7. Let $D$ be the intersection of normalizers in $\mathcal{M}$ of all subgroups from $\mathcal{M}$. From (a) it follows that $D \leq G$. Let $P$ be a Sylow subgroup of $G$ and $H$ be a subgroup from $\mathcal{M}$ with $P \leq H$. Note that $H \in \mathfrak{N}_\sigma$. Now $P - K - \mathfrak{N}_\sigma - sn H \leq \mathfrak{N}_\sigma$ and $P - K - \mathfrak{N}_\sigma - sn H$. Hence $P - K - \mathfrak{N}_\sigma - sn PD$ by Lemma 5. It means that $D - K - \mathfrak{N}_\sigma$-subnormalizes all Sylow subgroups of $G$. Thus $D \leq S_{\mathfrak{N}_\sigma}(G)$ by Proposition 5.

From the proof of Theorem 1 it follows that $S_{\mathfrak{N}_\sigma}(G) = Z_{\mathfrak{N}_\sigma}(G) = \text{Int}_{\mathfrak{N}_\sigma}(G)$. Let $H \in \mathcal{M}$. Now $HS_{\mathfrak{N}_\sigma}(G) \in \mathfrak{N}_\sigma$. Since $H$ is a $\pi_i$-maximal subgroup of $G$, $H$ is a $\pi_i$-maximal subgroup of
4 Applications

R. Baer [3] proved that the hypercenter of a group coincides with the intersection of all its maximal nilpotent subgroups. L. A. Shemetkov posed a question at the Gomel Algebraic Seminar in 1995 that can be formulated in the following way: For what non-empty (normally) hereditary (solubly) saturated formations \( \mathfrak{F} \) does the intersection of all \( \mathfrak{F} \)-maximal subgroups coincide with the \( \mathfrak{F} \)-hypercenter in every group? A. N. Skiba [32] answered on this question for hereditary saturated formations \( \mathfrak{F} \) (for the soluble case, see also J. C. Beidleman and H. Heineken [9]). From Theorem 1 follows a solution of this question for a family of hereditary not necessary saturated formations.

**Theorem 2.** Let \( \mathfrak{F} \) be a hereditary formation.

1. \( \mathfrak{F} = \mathfrak{M}_\mathfrak{F} \) if and only if \( S_{\mathfrak{F}}(G) = \text{Int}_{\mathfrak{F}}(G) \) holds for every group.

2. \( \mathfrak{F} = v^* \mathfrak{F} \) if and only if \( C_{\mathfrak{F}}(G) = \text{Int}_{\mathfrak{F}}(G) \) holds for every group.

3. Assume that \( \mathfrak{F} = \mathfrak{M}_\mathfrak{F} \) or \( \mathfrak{F} = v^* \mathfrak{F} \). Then \( Z_{\mathfrak{F}}(G) = \text{Int}_{\mathfrak{F}}(G) \) holds for every group if and only if there is a partition \( \sigma \) of \( \mathcal{P} \) such that \( \mathfrak{F} \) is the class of all \( \sigma \)-nilpotent groups.

**Proof.** From Proposition 6 it follows that \( S_{\mathfrak{F}}(G) = \text{Int}_{\mathfrak{M}_\mathfrak{F}}(G) \). Now (1) follows from the fact that \( \text{Int}_{\mathfrak{F}}(G) = \text{Int}_{\mathfrak{M}_\mathfrak{F}}(G) \) holds for every group if and only if \( \mathfrak{F} = \mathfrak{M}_\mathfrak{F} \). The proof of (2) is the same.

(3) Assume that \( \mathfrak{F} = \mathfrak{M}_\mathfrak{F} \). Now \( \mathfrak{F} \) is \( Z \)-saturated by Proposition 6 and \( \text{Int}_{\mathfrak{F}}(G) = \text{Int}_{\mathfrak{M}_\mathfrak{F}}(G) \) holds for every group \( G \). From Proposition 6 it follows that \( S_{\mathfrak{F}}(G) = \text{Int}_{\mathfrak{M}_\mathfrak{F}}(G) \) holds for every group \( G \). Now \( \text{Int}_{\mathfrak{F}}(G) = Z_{\mathfrak{F}}(G) \) holds for every group if and only if \( S_{\mathfrak{F}}(G) = Z_{\mathfrak{F}}(G) \) holds for every group \( G \). From (3) of Theorem 1 it follows that the last equality holds for every group if and only if there is a partition \( \sigma \) of \( \mathcal{P} \) such that \( Z_{\mathfrak{F}} = \mathcal{N}_\sigma \). Hence \( \mathfrak{F} = \mathcal{N}_\sigma \). From this theorem is also follows that \( \mathcal{N}_\sigma = \mathfrak{M}_\sigma \).

The proof of (3) for \( \mathfrak{F} = v^* \mathfrak{F} \) is the same.

**Remark 2.** There is a rather important family of not necessary saturated hereditary formations \( \mathfrak{F} \) with \( v^* \mathfrak{F} = \mathfrak{F} \) and \( \mathfrak{M}_\mathfrak{F} = \mathfrak{F} \). Recall that a formation \( \mathfrak{F} \) has the Shemetkov property if every \( \mathfrak{F} \)-critical group is either a Schmidt group of a cyclic group of prime order. The family of hereditary formations with the Shemetkov property contains non-saturated formations (see [8, Chapter 6.4]). For example let \( \mathfrak{F} \) be a class of groups all whose Schmidt subgroups are Schmidt \((p, q)\)-groups for \((p, q) \in \{(2, 3), (3, 2), (5, 2)\}\). Then \( \mathfrak{F} \) has the Shemetkov property by [35, Theorem 3.5] and \( \pi(\mathfrak{F}) = \mathcal{P} \). Let \( G \) be the alternating group of degree 5. Hence \( G \in \mathfrak{F} \).
According to [14] there is a Frattini $\mathbb{F}_3G$-module $T$ which is faithful for $G$. By the Gaschütz theorem (see [12] Appendix $\beta$), there exists a Frattini extension $T \twoheadrightarrow R \twoheadrightarrow G$ such that $T \cong \Phi(R)$ and $R/\Phi(R) \cong G$. Let $K/\Phi(R)$ be a cyclic subgroup of $G/\Phi(G)$ of order 5. Since $T$ is faithful for $G$, we see that $K$ is a non-nilpotent group with a normal Sylow 3-subgroup. Hence it contains a Schmidt (5,3)-subgroup. It means that $G \not\cong \mathfrak{S}$, i.e. $\mathfrak{S}$ is not saturated.

As follows from [24] [34] and [25] Corollaries 3.9 and 3.10 $v^*\mathfrak{S} = \mathfrak{S}$ and $\mathfrak{S}^* = \mathfrak{S}$ for every hereditary formation $\mathfrak{S}$ with the Shemetaev property and $\pi(\mathfrak{S}) = \mathbb{P}$.

Let give another application of Theorem [1]. Recall that a formation $\mathfrak{S}$ is called regular [26], if for every group $G$ holds

$$\mathcal{I}_\mathfrak{S}(G) = \{ x \in G \mid \langle x, y \rangle \in \mathfrak{S} \ \forall y \in G \} = \text{Int}_\mathfrak{S}(G).$$

The regular formations of soluble groups were studied in [26]. Here we give examples of such formations of non-necessary soluble groups.

Recall (see [20]) that the non-$\mathfrak{S}$-graph $\Gamma_\mathfrak{S}(G)$ of a group $G$ is the graph whose vertex set is $G \setminus \mathcal{I}_\mathfrak{S}(G)$ and two vertices $x$ and $y$ are connected if $\langle x, y \rangle \not\in \mathfrak{S}$. This type of graphs can be traced back to P. Erdős who considered non-commuting (non-abelian) graph. A. Abdollahi and M. Zarrin [1] asked to find the bounds for diameters of non-nilpotent graphs. The final answer on this question was obtained by A. Lucchini and D. Nemmi [21].

**Theorem 3.** The formation of all $\sigma$-nilpotent groups is regular and $\mathcal{I}_{\mathfrak{N}_\sigma}(G) = \mathcal{Z}_{\mathfrak{N}_\sigma}(G)$ holds for every group $G$. Moreover the graph $\Gamma_{\mathfrak{N}_\sigma}(G)$ is connected and $\text{diam}(\Gamma_{\mathfrak{N}_\sigma}(G)) \leq 3$ for every group $G$.

**Proof.** Let $x \in G$. Denote by $G_p$ and $x_p$ a Sylow $p$-subgroup of $G$ and $x|_G/|G_p|$ respectively. Note that if $\langle x_p, y \rangle \not\in \mathfrak{N}_\sigma$, then $\langle x, y \rangle \not\in \mathfrak{N}_\sigma$.

1. $\mathfrak{N}_\sigma$ is regular and $\mathcal{I}_{\mathfrak{N}_\sigma}(G) = \mathcal{Z}_{\mathfrak{N}_\sigma}(G)$ holds for every group $G$.

Let $y \in G$. Then $\langle y \rangle \in \mathfrak{N}_\sigma$. It means that $\langle y \rangle \mathcal{Z}_{\mathfrak{N}_\sigma}(G) \subseteq \mathfrak{N}_\sigma$. Hence $\langle x, y \rangle \in \mathfrak{N}_\sigma$ for all $x \in \mathcal{Z}_{\mathfrak{N}_\sigma}(G)$ and $y \in G$. It means that $\mathcal{Z}_{\mathfrak{N}_\sigma}(G) \subseteq \mathcal{I}_{\mathfrak{N}_\sigma}(G)$.

Let $x \in \mathcal{I}_{\mathfrak{N}_\sigma}(G)$. Note that $x = \prod_{p \in \pi(G)} x_p$. From $\langle x_p, y \rangle \leq \langle x, y \rangle \in \mathfrak{N}_\sigma$ it follows that $x_p \in \mathcal{I}_{\mathfrak{N}_\sigma}(G)$ for all $p \in \pi(G)$.

Let $q \in \pi(G)$. Since $\sigma$ is a partition of $\mathbb{P}$, there exists a unique $\pi_i \in \sigma$ with $q \in \pi_i$. Let $y$ be a $\pi_i$-element of $G$. Now $\langle x_q, y \rangle \in \mathfrak{N}_\sigma$. It means that $x_q y = y x_q$. So a $\pi_i$-element $x_q$ permutes with all $\pi_i$-elements of $G$. Thus $x_q \in \mathcal{Z}_{\mathfrak{N}_\sigma}(G)$ by Corollary [1,3]. Therefore $x \in \mathcal{Z}_{\mathfrak{N}_\sigma}(G)$. So $\mathcal{I}_{\mathfrak{N}_\sigma}(G) \subseteq \mathcal{Z}_{\mathfrak{N}_\sigma}(G)$. Hence $\mathcal{I}_{\mathfrak{N}_\sigma}(G) = \mathcal{Z}_{\mathfrak{N}_\sigma}(G) = \text{Int}_{\mathfrak{N}_\sigma}(G)$. Thus $\mathfrak{N}_\sigma$ is regular.

2. $\Gamma_{\mathfrak{N}_\sigma}(G)$ is connected and $\text{diam}(\Gamma_{\mathfrak{N}_\sigma}(G)) \leq 3$ for every group $G$.

If $|G \setminus \mathcal{I}_{\mathfrak{N}_\sigma}(G)| < 2$, then there is nothing to prove. So we may assume that $|G \setminus \mathcal{I}_{\mathfrak{N}_\sigma}(G)| \geq 2$. Assume that $G$ is a counterexample to (2). Hence there are elements $x, y \in G$ such that they are not connected or the lengths of all paths connecting them are greater than 3.

If $x \in \mathcal{I}_{\mathfrak{N}_\sigma}(G)$ for all $p \in \pi(G)$, then $x = \prod_{p \in \pi(G)} x_p \in \mathcal{Z}_{\mathfrak{N}_\sigma}(G) = \mathcal{I}_{\mathfrak{N}_\sigma}(G)$, a contradiction.

It means that there exist $p, q \in \pi(G)$ with $x_p y_q \not\in \mathcal{I}_{\mathfrak{N}_\sigma}(G)$. Hence there exist $\pi_i, \pi_j \in \sigma$, $\pi_i$-element $w$ and $\pi_j$-element $z$ with $p \not\in \pi_j$, $q \not\in \pi_j$, $\langle w, x \rangle \not\in \mathfrak{N}_\sigma$ and $\langle y, z \rangle \not\in \mathfrak{N}_\sigma$.

If $\langle w, z \rangle \not\in \mathfrak{N}_\sigma$, then $\langle x, w, z, y \rangle$ is the path connecting $x$ and $y$ and its length is not greater than 3, a contradiction. Now $\langle w, z \rangle \in \mathfrak{N}_\sigma$. Assume that $i \neq j$. So $w z = z w$ and $\langle z w \rangle = \langle z, w \rangle$. Now $\langle x, w, z, y \rangle$ is the path connecting $x$ and $y$ of length 2, a contradiction. So $i = j$. If $\langle x_p, z \rangle \not\in \mathfrak{N}_\sigma$, then $\langle x, z \rangle$ is the path connecting $x$ and $y$ of length 2, a contradiction. Hence $\langle x_p, z \rangle \in \mathfrak{N}_\sigma$. Since $p \not\in \pi_i = \pi_j$, we see that $x_p z = x_p$ and $\langle x_p \rangle = \langle z, x_p \rangle$. Now $\langle x, w, x, z, y \rangle$ is the path connecting $x$ and $y$ and its length is not greater than 3, the final contradiction.

**Corollary 3.1** ([21] Theorem 1.1]). $\Gamma_{\mathfrak{N}}(G)$ is connected and $\text{diam}(\Gamma_{\mathfrak{N}}(G)) \leq 3$ for every group $G$.

**Corollary 3.2** ([1] Theorem 5.1]). $\Gamma_{\mathfrak{N}}(G)$ is connected and $\text{diam}(\Gamma_{\mathfrak{N}}(G)) \leq 6$ for every group $G$.  

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