Nonperturbative improvement of SU(2) lattice gauge theory with adjoint or fundamental flavors

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Abstract: SU(2) gauge theory with two fermions transforming under the adjoint representation may appear conformal or almost conformal in the infrared, and is one of the candidate theories for building models for technicolor. Early lattice Monte Carlo studies of this model have used unimproved Wilson fermion formulation, which can be expected to have large lattice cutoff effects. In this paper we present the calculation of the $O(a)$ improved lattice Wilson-clover action of the theory. The Sheikholeslami-Wohlert coefficient has been determined non-perturbatively, and various boundary improvement terms, needed for the Schrödinger functional formalism, have been calculated in perturbation theory. For comparison, we have also determined the improvement coefficients for SU(2) gauge theory with two fundamental representation fermions. The calculation paves way for more accurate lattice Monte Carlo analyses of the theory in the future.

Keywords: Lattice field theory, Conformal field theory.
1. Introduction

Quantum field theories with nontrivial infrared fixed points of the $\beta$- function have recently been studied due to their applications in beyond Standard Model model building. In these theories the coupling runs when probed at very short distances, but becomes a constant over some energy range in the infrared and the theory appears conformal. One of the phenomenological connections is the unparticle \cite{1, 2, 3}, i.e. the possibility of a fully conformal sector coupled only weakly to the Standard Model through effective operators at low energies. Another phenomenological motivation to study theories which either feature an infrared fixed point or are, in theory space, close to one which does, originates from technicolor (TC) and the associated extended technicolor (ETC) models. These models were devised to explain the mass patterns of the Standard Model gauge bosons and fundamental fermions without the need to introduce a fundamental scalar particle \cite{4, 5, 6, 7}.

Early TC models, based on a technicolor sector straightforwardly extrapolated from a QCD-like strongly interacting theory, lead to too large flavor changing neutral currents due to the extended technicolor interactions. The problems of these simple TC models are solved in so called walking technicolor theories \cite{8, 9, 10, 11}. These theories are quasi-conformal, i.e. the evolution of the coupling constant is, over a wide range of energy, governed by an attractive quasi-stable infrared fixed point at strong coupling.

\begin{verbatim}
\includegraphics{1}
\end{verbatim}
To build walking TC models one needs to tune the gauge and matter degrees of freedom so that the desired quasi-conformality arises. To achieve this in SU($N$) gauge theory with fermions in the fundamental representation several $O(10)$ Dirac flavors are required. These contribute to the precision parameter $S$, which becomes too large to be compatible with the current observations. To obtain enough screening, as required for quasi-conformality, but with smaller number of flavors, one considers fermions in higher representations. It has been suggested [12] that an ideal candidate for minimal walking technicolor theory would be the one with just two (techni)quark flavors in the two-index symmetric representation of SU(2) or SU(3).

Reliable quantitative studies of the models, especially evaluating the $\beta$-functions, require lattice Monte Carlo simulations. There are several recent studies of both SU(2) [13, 14, 15, 16, 17, 18, 19, 20, 21, 22] and SU(3) [23, 24, 25, 26, 27] gauge theories with two-index symmetric representation fermions. For related studies in QCD-like theories with fundamental representation fermions see [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40].

In this paper we consider the case of SU(2) gauge fields with two fermions in the two-index symmetric representation, which, for SU(2), is equivalent to the adjoint representation. So far the lattice studies of this theory have been performed using unimproved Wilson fermion action and are hence subject to large $O(a)$ lattice artifacts. In this paper we present the computation of $O(a)$-improvement. This is a generalisation of the program used earlier to compute the improved action for two fundamental representation fermions in SU(3) gauge theory [41, 42, 43, 44, 45, 46]. The early results of this calculation have been presented in refs. [47, 48].

The Wilson fermion action can be improved for on-shell quantities by adding the well-known clover term. We tune the coefficient of the clover term (Sheikholeslami-Wohlert coefficient [49]) non-perturbatively, using the Schrödinger functional method. For the measurement of the coupling constant we also need the improvement coefficients of certain boundary terms. This computation is done using perturbative analysis. For comparison, we also calculate the improvement for SU(2) gauge theory with two flavors of fundamental representation fermions.\footnote{Non-perturbative improvement of the clover term has been recently published for SU(3) gauge field theory with 2-index symmetric (sextet) fermions, using the HYP-smeared link clover action [50].}

The paper is structured so that in section 2 we first recall the basics of the model as well as of the lattice formulation we use. In section 3 we present our perturbative results for the boundary terms and nonperturbative results for the improvement coefficients are presented in section 4. In section 5 we conclude and outline the directions of our future work.

2. Lattice formulation: the model and $O(a)$ improvement

We study SU(2) gauge theory with two different matter contents: two mass-degenerate flavors of Dirac fermions either in the adjoint or in the fundamental representation. The
continuum theory in Euclidean spacetime is defined by

\[ \mathcal{L} = \frac{1}{2} \text{Tr} F_{\mu\nu} F_{\mu\nu} + \sum_\alpha \bar{\psi}_\alpha (i\gamma_\mu \partial^\mu + m) \psi_\alpha \]  

(2.1)

where \( F_{\mu\nu} \) is the usual SU(2) field strength, and the gauge covariant derivative is

\[ D_\mu \psi = \left( \partial_\mu - ig A_\mu^a T^a \right) \psi \]  

(2.2)

where \( a = 1, 2, 3 \) and the generators \( T^a \) are taken either in the fundamental \( (T^a = \sigma^a / 2) \) or in the adjoint representation \( ([T^a]^{bc} = -i\epsilon^{abc}) \). The summation in Eq. (2.1) is over \( \alpha = u, d \).

Our main goal in this work is to establish nonperturbative \( \mathcal{O}(a) \) improved lattice implementation of these theories. While the improvement has been discussed in detail in existing literature for SU(3) gauge field with fundamental fermions, the studies of adjoint flavors require some alterations. Hence we find it necessary and useful to repeat essential parts of the analysis in detail here.

First recall the usual \( \mathcal{O}(a) \) improvement obtained by Sheikholeslami and Wohlert [49]. The lattice action, split to the gauge and fermionic parts \( S_G \) and \( S_F \), is

\[ S_0 = S_G + S_F. \]  

(2.3)

Here we use the standard Wilson plaquette gauge action

\[ S_G = \beta_L \sum_{x: \mu < \nu} \left( 1 - \frac{1}{2} \text{Tr} P_{x:\mu\nu} \right) \]  

(2.4)

where \( \beta_L = 4/g_0^2 \) and the plaquette is written in terms of the SU(2) fundamental representation link matrices \( U_\mu(x) \), which act as parallel transporters between sites \( x \) and \( x + a\hat{\mu} \):

\[ P_{x:\mu\nu} = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\nu^\dagger(x + a\hat{\nu}) U_\mu^\dagger(x). \]  

(2.5)

The Wilson fermion action, \( S_F \), for \( N_f \) (degenerate) Dirac fermions in the fundamental or adjoint representation of the gauge group is

\[ S_F = a^4 \sum_\alpha \sum_x \bar{\psi}_\alpha(x) (iD + m_{q,0} \mathbb{1}) \psi_\alpha(x), \]  

(2.6)

where the usual Wilson-Dirac operator is

\[ D = \frac{1}{2} (\gamma_\mu (\nabla_\mu^s + \nabla_\mu) - a \nabla_\mu^s \nabla_\mu), \]  

(2.7)

involving the gauge covariant lattice derivatives \( \nabla_\mu \) and \( \nabla_\mu^s \) defined as

\[ \nabla_\mu \psi(x) = \frac{1}{a} [\bar{U}_\mu(x) \psi(x + a\hat{\mu}) - \psi(x)], \]  

(2.8)

\[ \nabla_\mu^s \psi(x) = \frac{1}{a} [\psi(x) - \bar{U}_\mu^{-1}(x - a\hat{\mu}) \psi(x - a\hat{\mu})], \]  

(2.9)
Here, the link variables are the usual ones, \( \tilde{U}_\mu(x) = U_\mu(x) \), for fermions in the fundamental representation while for the adjoint representation they are
\[
\tilde{U}_{\mu}^{ab}(x) = 2 \text{Tr} \left( T^a U_\mu(x) T^b U_\mu^\dagger(x) \right),
\]  
where \( T^a, a = 1, 2, 3 \), are the generators of the fundamental representation, normalised as \( \text{Tr} T^a T^b = \frac{1}{2} \delta^{ab} \). We note that in the adjoint representation the elements of \( \tilde{U} \)-matrices are real and \( \tilde{U}^{-1} = \tilde{U}^\dagger \).

The lattice action \((2.3)\) is parametrised with two dimensionless parameters, \( \beta_L = 4/g_{\text{bare}}^2 \) and \( \kappa = 1/[8 + 2am_{q,0}] \). The parameter \( \kappa \) is related to the fermion mass. In the continuum limit \( a^4 \sum_x \to \int d^4x \) as \( a \to 0 \), and the leading order contribution from \((2.3)\) yields the continuum action while the terms of higher order in \( a \) will be suppressed; these terms are generically termed “lattice artifacts”. Since gauge invariance forbids any contribution from dimension five operators to the gauge action, only the fermion action here is subject to lattice artifacts at \( \mathcal{O}(a) \). These are removed (for on-shell quantities) by considering the improved action
\[
S_{\text{impr}} = S_0 + \delta S_{\text{sw}},
\]
\[
\delta S_{\text{sw}} = a^5 \sum_x c_{\text{sw}} \bar{\psi}(x) \frac{i}{4} \sigma_{\mu\nu} F_{\mu\nu}(x) \psi(x)
\]  
and tuning the Sheikholeslami-Wohlert coefficient \( c_{\text{sw}} \) at each \( \beta_L \) so that the \( \mathcal{O}(a) \) effects in on-shell quantities cancel; to lowest order in perturbation theory \( c_{\text{sw}} = 1 \) \[15\]. Here \( \sigma_{\mu\nu} = i[\gamma_{\mu}, \gamma_{\nu}]/2 \) and \( F_{\mu\nu}(x) \) is the “clover term”, lattice field strength tensor in the appropriate representation symmetrized over the four \( \mu, \nu \)-plane plaquettes which include the point \( x \).

Because our aim in future work is to measure the evolution of the gauge coupling constant using the Schrödinger functional method, we also need to consider the improvement of the action at the special Schrödinger functional boundary conditions. Schrödinger functional method is also used in this work for measuring \( c_{\text{sw}} \), but for this the boundary improvement is not necessary.

We consider a system of size \( L^3 \times T \), with periodic boundary conditions to the spatial directions and with Dirichlet boundary conditions for the gauge fields to the time direction:
\[
U_k(x_0 = 0) = W(k), \quad U_k(x_0 = T) = W'(k),
\]  
where \( k = 1, 2, 3 \); the explicit form of the boundary fields will be discussed later. For the measurement of the coupling constant the boundary gauge fields are chosen so that they lead to a constant background chromoelectric field. Due to the frozen boundaries there now exists \( \mathcal{O}(a) \) contribution to the gauge part of the action, and to account for these we consider
\[
S_{G,\text{impr}} = \frac{\beta_L}{4} \sum_p w(p) \text{tr}(1 - U(p)),
\]  
where the weights \( w(p) \) are equal to 1 for plaquettes in the bulk, \( w(p) = c_s/2 \) for spatial plaquettes at \( x_0 = 0 \) and \( T \) and \( w(p) = c_t \) for time-like plaquettes attached to a boundary.
plane. The parameters $c_s$ and $c_t$ are tuned to reduce the $O(a)$ boundary contributions.\footnote{Recall that gauge invariance guarantees that there are no $O(a)$ contributions to the gauge action in the bulk, and hence the boundary terms controlled by $c_s$ and $c_t$ are the only ones which arise to $O(a)$ in the gauge action.}

To leading order in perturbation theory $c_t = c_s = 1$. For the electric background field which we consider the terms proportional to $c_s$ do not contribute.

The boundary values of the fermion fields are set as

$$
P_+ \psi(x_0 = 0, x) = \rho(x), \quad P_- \psi(x_0 = T, x) + \rho'(x),$$

\begin{equation}
(2.15)
\end{equation}

with similar definitions on the conjugate fields. The projection operators are $P_\pm = \frac{1}{2}(1 \pm \gamma_0)$. The boundary fields $\rho, \rho'$ are source fields for correlation functions, and they are set to zero when generating configurations in simulations. In the spatial directions it is customary to introduce a “twist” for the phase of the fermion fields \cite{12}:

$$
\psi(x + L \hat{k}) = e^{i \theta_k} \psi(x), \quad \bar{\psi}(x + L \hat{k}) = \bar{\psi}(x) e^{-i \theta_k}.
$$

\begin{equation}
(2.16)
\end{equation}

In this work we use $\theta_k = \pi/5$ throughout. The twist, together with the Dirichlet boundary conditions, regulates the fermion matrix so that simulations at zero fermion masses become possible.

The improved lattice action is now given by

$$
S_{\text{impr}} = S_{G,\text{impr}} + S_F + \delta S_{sw} + \delta S_{F,b}.
$$

\begin{equation}
(2.17)
\end{equation}

Now the Sheikholeslami-Wohlert term only accounts for the bulk,

$$
\delta S_{cw} = a^5 \sum_{x_0=a}^{T-a} \sum_{x} c_{sw} \bar{\psi}(x) \frac{i}{4} \sigma_{\mu \nu} F_{\mu \nu}(x) \psi(x),
$$

\begin{equation}
(2.18)
\end{equation}

while the boundary effects are captured by $\delta S_{F,b}$. This counterterm has two contributions, controlled by parameters denoted by $\tilde{c}_s$ and $\tilde{c}_t$. The term proportional to $\tilde{c}_s$ is

$$
\delta S_{\tilde{c}_s} = a^4 (\tilde{c}_s - 1) \sum_{x} \left[ \frac{1}{2} \bar{\psi}(0, x) P_- \gamma_k (\nabla_k^* + \nabla_k) P_+ \psi(0, x) \\
+ \frac{1}{2} \bar{\psi}(L, x) P_+ \gamma_k (\nabla_k^* + \nabla_k) P_- \psi(L, x) \right]
$$

\begin{equation}
(2.19)
\end{equation}

and it clearly vanishes if we set fermionic fields to zero on the boundaries.

So, similarly to the gauge action, only the term proportional to $\tilde{c}_t$ contributes, and this contribution is given by

$$
\delta S_{F,b} = a^4 \sum_{x} (\tilde{c}_t - 1) \frac{1}{a} \bar{\psi}(x) \psi(x) (\delta(x_0 - a) + \delta(x_0 - (L - a))).
$$

\begin{equation}
(2.20)
\end{equation}

This can be seen as a correction to the bare mass term at $x_0 = a$ and $x_0 = L - a$, hence accounted for by the modification

$$
m_{q,0} \mapsto m_{q,0} + (\tilde{c}_t - 1) (\delta_{t,a} + \delta_{t,L-a}).
$$

\begin{equation}
(2.21)
\end{equation}
It is known that $\tilde{c}_t = 1$ to leading order.\footnote{This is so because free Wilson fermions are not subject to $O(a)$ artifacts.}

Hence, to obtain $O(a)$ improvement we need to determine the parameters $c_t$, $\tilde{c}_t$ and $c_{sw}$ in the action (2.17). The parameters $c_t$ and $\tilde{c}_t$ are determined perturbatively as will be described in the following section. The parameter $c_{sw}$ is determined nonperturbatively, and this will be determined in section 4.

3. Perturbative analysis of the boundary improvement

As explained in the previous section, due to the Dirichlet boundary conditions associated with the Schrödinger functional formalism, we are led to counteract $O(a)$ lattice artifacts on the boundaries both in the gauge and fermion parts of the action. In this section we describe in detail the analysis of the required counterterms. Although we are mostly interested in matter fields in fundamental or adjoint representation of SU(2) gauge group, we will present the results applicable also for higher representations of SU(3) since these are relevant for the current developments in the studies of these theories on the lattice.

In principle there exists four counterterms associated with the spatial links in the boundary and with temporal links connected to the boundary. Due to the specific form of the background field we have chosen, only two of these are needed and these are denoted by $c_t$ and $\tilde{c}_t$. These boundary coefficients have a perturbative expansion of the form

$$c_x = 1 + c_x^{(1)} g_0^2 + O(g_0^4).$$

Our goal is to determine $\tilde{c}_t$ and $c_t$ to one-loop order in perturbation theory.

3.1 Coefficient $\tilde{c}_t^{(1)}$

We follow the analysis performed in \cite{14} for the fundamental representation. The result of \cite{14} is

$$\tilde{c}_t^{(1)} = -0.0135(1) C_F,$$

and this generalizes to other fermion representations simply by replacing the fundamental representation Casimir operator $C_F$ with Casimir operator $C_R$ of the representation $R$ under consideration. This is so because the relevant correlations functions are proportional to the diagrams presented in figure \[\text{fig:diagram}\], which all include the color factor $\sum_a (T^a)^2 = C_R$. Thus it can be shown that also $\tilde{c}_t^{(1)} \propto C_R$.

In the case of fundamental fermions the original result of \cite{14} is directly applicable with $C_F = (N_c^2 - 1)/(2N_c) = 3/4$ for $N_c = 2$. For the other case we have fermions transforming in the adjoint representation of SU(2), for which the Casimir invariant is $C_A = 2$. The results for different gauge groups and fermion representations are shown in table \[\text{table}\].

3.2 Coefficient $c_t^{(1)}$

The coefficient $c_t^{(1)}$ can be split into gauge and fermionic parts

$$c_t^{(1)} = c_t^{(1,0)} + c_t^{(1,1)} N_f.$$
Figure 1: Diagrams contributing to the calculation of $\tilde{c}_t^{(1)}$. The shaded blob on each diagram indicates the insertion of the operator $\Gamma_x = \{1, \gamma_5\}$.

The contribution $c_t^{(1,0)}$ is entirely due to gauge fields and has been evaluated in [51] for SU(2) and in [52] for SU(3). The fermionic contribution $c_t^{(1,1)}$ to $c_t$ has been evaluated for fundamental fermions in [43] both for SU(2) and SU(3). We have extended these computations for SU(2) and SU(3) gauge theory with higher representation fermions and for SU(4) gauge theory with fundamental representation fermions.

The method we have used is the same as the one presented in [43], with two exceptions. First, the boundary fields have to be transformed to the desired fermion representation. Generally the boundary fields have to be transformed to the desired fermion representation. Generally the boundary fields are of the form

\[ U(x, k)|_{x_0=0} = \exp(aC_k), \quad U(x, k)|_{x_0=L} = \exp(aC'_k), \]

where

\[ C_k = \frac{i}{L} \text{diag}(\phi_1, \ldots, \phi_n), \quad C'_k = \frac{i}{L} \text{diag}(\phi'_1, \ldots, \phi'_n), \]

and $n$ is the dimension of the representation. The transformed boundary fields are obtained from the fundamental representation counterparts for adjoint representation via (2.10). After the transformation one simply diagonalizes the resulting matrices and ends up with a matrix of the form

\[ \text{diag} \left( \exp[i\phi_1^A], \ldots, \exp[i\phi_n^A] \right), \]

where $\phi_i^A$ give the adjoint representation boundary fields

\[ C_k^A = \frac{i}{L} \text{diag}(\phi_1^A, \ldots, \phi_n^A), \quad C'_k = \frac{i}{L} \text{diag}(\phi'_1^A, \ldots, \phi'_n^A). \]

For the symmetric representation the components of the boundary fields $C_k^S$ and $C'_k^S$ can be obtained by taking all the symmetric combinations of $\phi_i$. For SU(3) sextet repre-
sentation this is
\begin{align*}
\phi_1^S &= \phi_1 + \phi_1, \\
\phi_2^S &= \phi_1 + \phi_2, \\
\phi_3^S &= \phi_1 + \phi_3, \\
\phi_4^S &= \phi_2 + \phi_2, \\
\phi_5^S &= \phi_2 + \phi_3, \\
\phi_6^S &= \phi_3 + \phi_3.
\end{align*}
(3.8)

The other crucial note comes from the normalization in the calculation of \( c_t^{(1,1)} \). Using the Schrödinger functional scheme and taking the lattice action with constant background field as an effective action \( \Gamma_0 \), the running coupling is defined via
\begin{equation}
\bar{g}^2 = \frac{\partial \eta \Gamma_0}{\partial \eta \Gamma_0}.
\end{equation}
(3.9)

The boundary fields \( C_k \) and \( C'_k \) are functions of the parameter \( \eta \) so the running coupling is given by the change of the system as the boundary fields are altered. The effective action \( \Gamma \) is to one loop order in perturbation theory
\begin{equation}
\Gamma = g_0^{-2} \Gamma_0 + \Gamma_1 + \mathcal{O}(g_0^2),
\end{equation}
(3.10)
so the running coupling can be written, as a function of the bare coupling \( g_0 \), in the form
\begin{equation}
\bar{g}^2 = g_0^2 \left( 1 - g_0^2 \frac{\partial \eta \Gamma_1}{\partial \eta \Gamma_0} + \mathcal{O}(g_0^6) \right).
\end{equation}
(3.11)

On small lattice spacings \( a \), the one loop correction \( \Gamma_1 \) diverges. This leads to renormalization of the lattice coupling, which is given in terms of the bare coupling as
\begin{equation}
g^2_{\text{lat}} = g_0^2 + z_1 g_0^4 + \mathcal{O}(g_0^6),
\end{equation}
(3.12)
where \( z_1 = 2b_0 \ln(a\mu) \) and
\begin{equation}
b_0 = \frac{1}{(4\pi)^2} \left( \frac{11}{3} C_A - \frac{4}{3} T(R) N_F \right)
\end{equation}
(3.13)
is the coefficient in one loop beta function. Now we can write the running coupling as a function of the renormalized coupling
\begin{equation}
\bar{g}^2 = g^2_{\text{lat}} \left[ 1 - g^2_{\text{lat}} \left( \frac{\partial \eta \Gamma_1}{\partial \eta \Gamma_0} + z_1 \right) \right] + \mathcal{O}(g^6_{\text{lat}}).
\end{equation}
(3.14)

The one loop correction to the effective action \( \Gamma_1 \) can also be written as
\begin{equation}
\Gamma_1 = \frac{1}{2} \ln \det \Delta_1 - \ln \det \Delta_0 - \frac{1}{2} \ln \det \Delta_2,
\end{equation}
(3.15)
where the operators \( \Delta_0 \) and \( \Delta_1 \) are related to the gauge fixing and pure gauge part of the action and the operator \( \Delta_2 = [(D_{\text{sw}} + m_0)\gamma_5]^2 \) is related to the fermionic part of the action. The operator \( D_{\text{sw}} \) is the lattice Dirac operator that includes the Sheikholeslami-Wohlert term. Now for the calculation of \( \Delta_2 \) one needs to transform the boundary fields
to the appropriate representation. However in the calculation of $\Gamma_0$ one needs to keep the boundary fields in the fundamental representation. This is so because the pure gauge part of $c_t^{(1)}$ should be independent of the representation of the fermions and this can only be achieved if the boundary fields in $\Gamma_0$ are kept in the fundamental representation. Also this produces the expected behavior for the series expansion of

$$\left( \frac{\partial_1 \Gamma_1}{\partial_1 \Gamma_0} + z_1 \right).$$

(3.16)

With these remarks, the numerical calculation is straightforward. The results for the nonzero improvement coefficients are tabulated in table 1. The numbers beyond the fundamental representation are new, while those for the fundamental representation provide a good check on our computations. For the application to minimal walking technicolor, the relevant numbers are the ones on the second line of table 1.

Our results are consistent with the generic formula

$$c_t^{(1,1)} \approx 0.019141(2 T(R)), \quad (3.17)$$

where $T(R)$ is the normalization of the representation $R$, defined as $\text{Tr}(T^a_R T^b_R) = T(R) \delta^{ab}$. For the details of the numerical method used to determine coefficient $c_t^{(1,1)}$, we refer to the original literature where the method was developed and applied first for the pure gauge theory case in [51], and later for fundamental representation fermions in [51, 43].

| $N_c$ | rep. | $c_t^{(1,0)}$ | $c_t^{(1,1)}$ | $\tilde{c}_t^{(1)}$ |
|------|------|-------------|-------------|-------------|
| 2    | 2    | -0.0543(5)  | 0.0192(2)   | -0.0101(3)  |
| 2    | 3    | -0.0543(5)  | 0.075(1)    | -0.0270(2)  |
| 3    | 3    | -0.08900(5) | 0.0192(4)   | -0.0180(1)  |
| 3    | 8    | -0.08900(5) | 0.113(1)    | -0.0405(3)  |
| 3    | 6    | -0.08900(5) | 0.0946(9)   | -0.0450(3)  |
| 4    | 4    |             | 0.0192(5)   | -0.0253(2)  |

Table 1: The nonzero improvement coefficients for Schrödinger functional boundary conditions with electric background field for various gauge groups and fermion representations.

We have also plotted our results of $c_t^{(1,1)}$ scaled with $1/(2T(R))$ against (3.17) in figure 2. Although we did not achieve the accuracy of the original work [43], our results are fully compatible for fundamental representation fermions. The figure also clearly indicates that $c_t^{(1,1)}$ scales with $2T(R)$.

4. Non-perturbative tuning

The continuum physics we are interested in corresponds to massless fermions, so we need to simulate at zero physical quark mass. With Wilson fermions the bare quark mass is additively renormalized, and the zero of the physical quark mass corresponds to tuning the bare quark mass to a critical value, $m_0 = m_{cr}$. This tuning is done nonperturbatively and allows for determination of the improvement coefficient $c_{sw}$ simultaneously. Here we describe
the calculation of $c_{sw}$ for $N_f = 2$ flavors of SU(2) fundamental and adjoint representation fermions.

In these simulations the fermion fields have the boundary conditions given in Eqs. (2.13, 2.16). For the fundamental representation fermions we fix the gauge field Dirichlet boundary conditions at $x_0 = 0$ and $x_0 = T$: 

$$U_k(x_0 = T) = \exp(iC'), \quad C' = -\frac{\pi a\sigma^3}{4L},$$  

$$U_k(x_0 = 0) = \exp(iC), \quad C = -\frac{3\pi a\sigma^3}{4L},$$  

for $k = 1, 2, 3$. Because the boundary link matrices commute, we call these boundary conditions Abelian, in contrast to the non-Abelian (non-commuting) ones defined below. The physical quark mass is defined via the partial conservation of the axial current (PCAC) relation,

$$M(x_0) = \frac{1}{2} \frac{(\partial_0^* + \partial_0)f_A(x_0) + c_A a\partial_0^*\partial_0 f_P(x_0)}{f_P(x_0)} \equiv r(x_0) + c_As(x_0),$$  

where

$$A_\mu^a = \bar{\psi}(x)\gamma_\mu\gamma_5\frac{1}{2}\sigma^a\psi(x),$$  

$$P^a = \bar{\psi}(x)\gamma_5\frac{1}{2}\sigma^a\psi(x),$$  

$$f_A(x_0) = -a^6 \sum_{y,z} \langle A_0^a(x)\bar{\zeta}(y)\gamma_5\frac{1}{2}\sigma^a\zeta(z) \rangle,$$  

$$f_P(x_0) = -a^6 \sum_{y,z} \langle P^a(x)\bar{\zeta}(y)\gamma_5\frac{1}{2}\sigma^a\zeta(z) \rangle.$$  

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**Figure 2:** Our results of $c_t^{(1,1)}$ scaled with $2T(R)$ compared with conjectured value of $c_t^{(1,1)}/(2T(R))$. 

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Another set of correlation functions, $f'_A$ and $f'_P$, is defined via

$$f'_A(T - x_0) = -a^6 \sum_{y,z} \langle A^\alpha(x) \bar{\zeta}(y) \gamma_5 \frac{1}{2} \sigma^\alpha \zeta(z) \rangle,$$

(4.8)

$$f'_P(T - x_0) = -a^6 \sum_{y,z} \langle P^\alpha(x) \bar{\zeta}(y) \gamma_5 \frac{1}{2} \sigma^\alpha \zeta(z) \rangle.$$

(4.9)

The bare mass is tuned so that $M(T/2)$ vanishes. The $c_{sw}$ term is tuned simultaneously using mass measurements at a different point in the bulk looking for variations of the order of the lattice spacing. Defining $M'$ with obvious replacements of primes, it follows that the quantity

$$\Delta M(x_0) = M(x_0) - M'(x_0)$$

(4.10)

vanishes up to corrections of $O(a^2)$ if both $c_{sw}$ and $c_A$ have their proper values. In order to recover the correct tree level behaviour we fix these quantities $M$ and $\Delta M$ two their tree level values, measured by from a cold gauge configuration with $\kappa_c = 0.125$. This gives a small correction to the relations:

$$\Delta M(x_0) = M(x_0) - M'(x_0) - \delta = 0, M(x_0) = \delta_M$$

(4.11)

However, for the adjoint representation fermions there are complications which significantly reduce the effectiveness of the above method. Using Eq. (2.10) we immediately notice that the Abelian boundary matrices (4.1,4.2) are transformed into form

$$\tilde{U}_k = \begin{pmatrix} \ldots & \ldots & 0 \\ \ldots & \ldots & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(4.12)

Thus, there is a component of the adjoint representation color vector which simply does not see the background field. This feature is independent of the color structure chosen for the boundary conditions. It turns out that regardless of how the fermion sources or the constant boundary conditions are chosen, at long distances the correlation functions behave as if there is no background field. In other words, the adjoint fermion correlation functions “see” the background electric field only at short distances. This significantly reduces the effectiveness of the background field method for tuning $c_{sw}$.

This effect can be improved by using boundary conditions which maximize the difference between the two boundaries. We use the following asymmetric ”non-Abelian” boundary conditions: links at the upper $x_0 = T$ boundary are chosen to be trivial

$$U(x_0 = T, k) = I$$

(4.13)

and at the lower boundary $x_0 = 0$ we use

$$U(x_0 = 0, k) = \exp(aC_k), \quad C_k = \frac{\pi \tau^k}{2 i L}.$$}

(4.14)

This creates a strong chromomagnetic field at $x_0 = 0$ boundary. These boundary conditions do not fully cure the problem, but nevertheless provide enough leverage so that the PCAC mass relation can be used to tune $c_{sw}$. 
Figure 3: Left: Fundamental representation fermion mass $aM(x_0)$ measured from the classical gauge field configuration satisfying the Abelian boundary conditions (4.1, 4.2) on a $8^3 \times 16$-lattice. Bare mass is $am_0 = 0.01$, which is also $aM$ in the continuum limit. Inclusion of the clover term ($c_{sw} = 1$) significantly reduces the cutoff effects. Right: $aM(x_0)$ for adjoint representation fermions and for the Abelian boundary conditions (4.1), (4.2), and for the non-Abelian boundary conditions (4.13, 4.14). Here the correlation functions between $c_{sw} = 0$ and $c_{sw} = 1$ differ significantly at long distances only for the non-Abelian boundary conditions.

This behaviour can be demonstrated already at the classical level: in figure 3 we show the PCAC fermion mass (4.3), measured using the classical minimum action gauge field configuration which satisfies the appropriate boundary conditions. The bare fermion mass has been set to $am_0 = 0.01$, and, in the absence of the background field or lattice cutoff effects, the PCAC measurement would yield precisely this value. However, with finite lattice spacing the non-trivial classical background field gives rise to cutoff effects, which moves the PCAC mass away from $aM = 0.01$. For the fundamental representation fermions and the Abelian boundary conditions (4.1), (4.2), (left panel in figure 3), we can observe that setting $c_{sw} = 0$ (non-improved standard Wilson fermions) the measured mass values are far from the continuum limit, whereas using $c_{sw} = 1$ (the correct value at the classical level) these effects are strongly reduced.

For the adjoint representation fermions the behaviour is very different, as shown on the right panel of fig. 3: using the Abelian boundary conditions the measured masses $aM(x_0)$ rapidly approach $0.01/a$ as $x_0$ increases, for both $c_{sw} = 0$ or 1. This indicates that the correlation function lacks the sensitivity to $c_{sw}$ and cannot be used for tuning it to the correct value.

On the other hand, with the non-Abelian boundary conditions (4.13, 4.14) the correlation function remains sensitive to the value of $c_{sw}$ to longer distances. The sensitivity remains in the mass asymmetry $\Delta M(x_0)$, (4.10), which can now be used to tune $c_{sw}$. We note that these boundary conditions are useful only for determining $c_{sw}$, not for evaluating the coupling constant.
In order to remove the dependence on \( c_A \), for fundamental fermions, we consider
\[
M(x_0, y_0) = r(x_0) - s(x_0) \frac{r(y_0) - r'(y_0)}{s(y_0) - s'(y_0)},
\]
which coincides with \( M(x_0) \) up to \( \mathcal{O}(a^2) \) corrections and is independent of \( c_A \). With adjoint fermions this quantity suffers from large statistical fluctuations and is not useful. Instead we simply consider the quantity \( M(x_0) \) and fix \( c_A \) to its perturbative value \([44]\)
\[
c_A = -0.00567(1) C_R g^2 + O(g^4).
\]
We then measure \( c_A \) separately to confirm the validity of our choice.

In order to evaluate \( c_{sw} \) we used the following routine: we choose lattice volume \( L^3 \times T = 8^3 \times 16 \) for both fundamental and adjoint representation fermions, and a set of values of the lattice coupling \( \beta \). For fundamental fermions we measure \( M = M(T/2, T/4) \) and \( \Delta M = \Delta M(3T/4, T/4) \). For adjoint fermions we measure \( M = M(T/2) \) and \( \Delta M = \Delta M(3T/4) \) fixing \( c_A \) to its perturbative value.

1. For a given \( \beta \), we choose initial \( c_{sw} \) (typically extrapolating from results obtained with previous values of \( \beta \)).

2. We choose a couple of values for \( \kappa \), and determine by interpolation the critical value \( \kappa_c(\beta, c_{sw}) \) where the fermion mass \( M \) is equal to the tree level value.

3. Once we have an estimate of the critical \( \kappa \), we choose a new value for \( c_{sw} \) and repeat the search of \( \kappa_c \).

4. At the same time, we measure \( \Delta M(c_{sw}) \). Now we can linearly interpolate/extrapolate in \( c_{sw} \) so that \( \Delta M \) vanishes, obtaining the desired value of \( c_{sw} \). Using simulations at this final \( c_{sw} \) we can relocate the critical \( \kappa \), if desired, and verify the results of the interpolation.

The above tuning is done at small \( L/a \), and the results are applied for all lattice sizes since the \( L/a \) dependence is expected to be weak. Furthermore, we only consider a range of \( \beta \) and fit the critical values to an interpolating function to obtain \( m_c(\beta L) \) and \( c_{sw}(\beta L) \).

### 4.1 Measurement of \( c_{sw} \)

In figures 4 and 5 we show our results for the clover coefficient \( c_{sw} \) for both fundamental and adjoint representations. The values of \( \beta \) used are \( \beta = 2.5, 3, 4, 5, 6, 8 \), and also \( \beta = 2.25 \) and 10 for the adjoint representation. To clarify the tuning method we provide the measurements of \( M \) and \( \Delta M \) with adjoint fermions in table 2. In tables 3 and 4 we give our results for \( c_{sw} \) for fundamental and adjoint fermions respectively.

Finally, the measured values for \( c_{sw} \) can be fitted with a rational interpolating expression, which can used in simulations for this range of \( \beta \)-values. For fundamental representation fermions we use the perturbative 1-loop result \( c_{sw} = 1 + 0.1551g^2 + O(g^4) \) \([44]\) to constrain the fit:
\[
c_{sw} = \frac{1 - 0.090254g^2 - 0.038846g^4 + 0.028054g^6}{1 - (0.1551 + 0.090254)g^2}.
\]
Figure 4: $c_{sw}$ for two flavors of fundamental representation fermions. The solid line is the interpolating fit, Eq. (4.17), and the dashed line is the 1-loop perturbative value.

Figure 5: $c_{sw}$ for two flavors of adjoint representation fermions, with the interpolating fit, Eq. (4.18).

For the adjoint representation the perturbative result is not known, and we obtain the fit result

$$c_{sw} = \frac{1 + 0.032653g^2 - 0.002844g^4}{1 - 0.314153g^2}. \tag{4.18}$$

In both cases the interpolating fits are valid for $\beta \gtrsim 2.5$. For the adjoint fermions it is difficult to reach smaller $\beta$-values because $c_{sw}$ grows rapidly, and while we were able to reach $\beta = 2.25$ the errors were too large to constrain the fit further.
When measuring $c_{sw}$ for adjoint fermions we chose to keep the coefficient $c_A$ at the perturbative value. In figure 8 we show how the choice of $c_A$ affect $\Delta M (3T/4)$ with certain choice of parameters. Typically $c_A$ is between $-0.005$ and $-0.01$ at the range of $\beta$ we explored. We see that even differences of this order have small effect to $\Delta M$. 

| $\beta$ | $c_{sw}$ | $\kappa$ | $aM$     | $a\Delta M$ |
|--------|----------|----------|----------|-------------|
| 10     | 1.16     | 0.1302552| 0.00020(7)| $-0.0004(1)$|
| 10     | 1.17208  | 0.1301818| 0.00114(7)| $-0.0002(1)$|
| 10     | 1.1774   | 0.1301818| 0.00050(7)|  0.0001(1) |
| 10     | 1.17915  | 0.13017157| 0.00037(10)| $0.0002(2)$ |
| 8      | 1.2      | 0.13171  | $-0.00156(8)$| $-0.0004(2)$|
| 8      | 1.225    | 0.13154  |  0.00031(8)| $-0.0001(2)$|
| 8      | 1.227    | 0.1315265|  0.00035(8)|  0.0000(2) |
| 8      | 1.23     | 0.1315265| $-0.00018(9)$|  0.0000(2) |
| 8      | 1.25     | 0.1315265|  0.00003(8)|  0.0003(2) |
| 6      | 1.28     | 0.1340604| $-0.00054(7)$| $-0.0007(1)$|
| 6      | 1.3      | 0.133903 |  0.00034(8)| $-0.0003(1)$|
| 6      | 1.3135   | 0.1338131|  0.00055(8)| $-0.0001(1)$|
| 6      | 1.3143   | 0.1338131|  0.0002510|  0.0001(1) |
| 6      | 1.33     | 0.1338131| $-0.00280(8)$|  0.0005(1) |
| 5      | 1.3      | 0.1363278|  0.0006(1)|  $-0.0015(3)$|
| 5      | 1.4      | 0.1356033|  0.0007(1)|  $-0.0003(3)$|
| 5      | 1.4058   | 0.136       | $-0.0130(2)$|  0.0000(3) |
| 5      | 1.5      | 0.1348774  |  0.0007(1)|  0.0014(3) |
| 4      | 1.45     | 0.1391039 |  0.0012(2)|  $-0.0008(3)$|
| 4      | 1.522    | 0.1385882 | $-0.0024(2)$|  0.0001(2) |
| 4      | 1.6      | 0.1378078 |  0.0004(2)|  0.0008(2) |
| 3      | 1.6      | 0.145311  |  0.0002(2)|  $-0.0022(4)$|
| 3      | 1.75     | 0.1435289 |  0.0038(2)|  $-0.0005(3)$|
| 3      | 1.834    | 0.1426551 |  0.0018(2)|  $-0.0006(4)$|
| 3      | 1.9      | 0.1419574 |  0.0009(3)|  0.0002(4) |
| 3      | 2.1      | 0.1400727 |  0.0082(2)|  0.0016(3) |
| 2.5    | 1.5      | 0.1540744 |  0.0021(4)|  $-0.023(5)$|
| 2.5    | 2        | 0.147733  | $-0.0036(3)$|  $-0.0005(4)$|
| 2.5    | 2.5      | 0.141683  |  0.0015(2)|  0.0005(4) |
| 2.5    | 2.7      | 0.139561  | $-0.0025(2)$|  0.0027(9) |
| 2.25   | 1.5      | 0.1590893 |  0.0306(3)|  $-0.0019(6)$|
| 2.25   | 2.3      | 0.147733  | $-0.0004(3)$|  $-0.0004(5)$|
| 2.25   | 2.5      | 0.141683  |  0.0033(3)|  $-0.0002(4)$|

Table 2: Results for the quark mass $M$ and $\Delta M$ with two fermions in the adjoint representation

4.2 Non-Perturbative measurement of $c_A$

When measuring $c_{sw}$ for adjoint fermions we chose to keep the coefficient $c_A$ at the perturbative value. In figure 8 we show how the choice of $c_A$ affect $\Delta M (3T/4)$ with certain choice of parameters. Typically $c_A$ is between $-0.005$ and $-0.01$ at the range of $\beta$ we explored. We see that even differences of this order have small effect to $\Delta M$. 


Table 3: Results for $c_{sw}$ with two flavors of fermions in the fundamental representation

| $\beta$ | $c_{sw}$   | $\beta$ | $c_{sw}$   |
|---------|------------|---------|------------|
| 8       | 1.082(2)   | 4       | 1.190(8)   |
| 6       | 1.109(3)   | 3       | 1.309(13)  |
| 5       | 1.145(8)   | 2.5     | 1.430(19)  |

Table 4: Results for $c_{sw}$ with two flavors of fermions in the adjoint representation

| $\beta$ | $c_{sw}$   | $\beta$ | $c_{sw}$   |
|---------|------------|---------|------------|
| 10      | 1.159(3)   | 4       | 1.476(17)  |
| 8       | 1.197(8)   | 3       | 1.805(23)  |
| 6       | 1.291(3)   | 2.5     | 2.059(74)  |
| 5       | 1.376(9)   | 2.25    | 2.593(215) |

Figure 6: The dependence of $\Delta M(3T/4)$ of $c_A$. The measurement was done with $\beta = 4$ and $c_{sw} = 1.522$.

To verify the accuracy of our choice we have also estimated a non-perturbative value for $c_A$. For this we have used the same improvement condition as in [45]. We do simulations with two different values of the fermion phase $\theta$ in the boundary conditions $2.16$, using the measured values of $c_{sw}$ and $\kappa_c$ above. Without any discretisation errors the difference in the measured masses should be equal to the tree level value. Requiring that this condition is met, we can find an estimate of $c_A$.

From two simulations with $\theta = 0$ and $\theta = \pi/2$ we calculated the discretisation effect

$$\Delta M(c_A)' = M(x_0 = 8; \theta = 0, c_A) - M(x_0 = 8; \theta = \pi/2, c_A) - \delta,$$

(4.19)

where $\delta$ is the tree level value of the difference. It is similar to the tree level correction in equation [1.11] and is relatively small. These simulations were done using a trivial boundary condition, where all the boundary matrices were set to unity. Depending on the lattice coupling between 2000 and 35000 trajectories were performed for each value of $\theta$. 
Figure 7: $c_A$ for two flavors of adjoint fermions. The solid line is the interpolating fit, Eq. (4.17), and the dashed line is the 1-loop perturbative value.

As the quark mass, and therefore $\Delta M'$ is simply linearly dependent on $c_A$, we can measure $\Delta M'$ for two of values of $c_A$ to find the correct value where $\Delta M' = 0$.

$$
\begin{array}{|c|c|}
\hline
\beta & c_A \\
\hline
10 & -0.0043(5) \\
8 & -0.0056(4) \\
6 & -0.0053(6) \\
5 & -0.0087(5) \\
\hline
\end{array}
$$

Table 5: Results for $c_A$

The results for $c_A$ are given in table 5 and depicted in figure 7. We see that in the region where we have measured $c_{sw}$ and $c_A$, it is justified to use the perturbative value for $c_A$.

5. Conclusions and outlook

We have calculated $O$-improvement of SU(2) gauge theory with two Wilson fermions in the fundamental or adjoint representation. The main results are the non-perturbative evaluation of the Sheikhholeslami-Wohlert clover coefficient $c_{sw}$ and the perturbative calculation of the boundary improvement terms needed for full improvement in the Schrödinger functional formalism. The result for $c_{sw}$ is generally applicable to lattice simulations of these theories. We also verified that the axial current improvement coefficient $c_A$ is well described by the 1-loop perturbative formula in the range of lattice spacings studied. In addition to the perturbative results on SU(2) gauge theory and adjoint fermions, we obtained results also for SU(3) and adjoint or sextet fermions which will be useful also for other groups studying these theories.
The main application for the improved action is more accurate lattice Monte Carlo analyses of the candidate theory for minimal walking technicolor, SU(2) gauge theory with two adjoint representation fermions. The boundary improvement terms permit improved measurement of the evolution of the coupling constant with the Schrödinger functional scheme. Indeed, in earlier unimproved analyses [17, 19] significant cutoff effects were observed at coarse lattices. The measurement of the coupling with the improved action is left for future work.

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