Phase diagram and metastability of the Ising model on two coupled networks

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Abstract. We explore the cooperative behaviour and phase transitions of interacting networks by studying a simplified model consisting of Ising spins placed on the nodes of two coupled Erdős–Rényi random graphs. We derive analytical expressions for the free-energy of the system and the magnetization of each graph, from which the phase diagrams, the stability of the different states, and the nature of the transitions among them, are clearly characterized. We show that a metastable state appears discontinuously by varying the model parameters, yielding a region in the phase diagram where two solutions coexist. By performing Monte-Carlo simulations, we confirm the exactness of our main theoretical results and show that the typical time the system needs to escape from a metastable state grows exponentially fast as a function of the temperature, characterizing ergodicity breaking in the thermodynamic limit.

Keywords: metastable states, random graphs, networks, cavity and replica method, critical phenomena of socio-economic systems

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1. Introduction

Due to our increasing capability of collecting and manipulating large amounts of data, it has become common sense that many real-world systems are arranged in network structures [1]. Examples of problems defined on networks are abundant in physics, biology, and finance, ranging from the inference of ecological associations between microbial populations [2, 3] to the prediction of collapses in interbank networks [4].

The study of phase transitions and critical phenomena in networked systems constitutes an important research topic within the realm of complex networks [5]. Such phase transitions can be divided in two main classes: structural phase transitions, which refer to macroscopic changes in the architecture features of networks, and phase transitions emerging due to the cooperative behaviour of many entities interacting through the links of the network. Examples of structural phase transitions are the percolation and condensation transitions [5], while synchronization [6, 7] and formation of consensus in social systems [8] are typical examples of cooperative phenomena.

Spin models of statistical physics are prototypical in the study of the collective or cooperative behaviour of many interacting entities [9]. Since in this case the individual elements have a relatively simple mode of operation, one can usually deal with the intricate pattern of interconnections defining the network structure in a more detailed way [10], allowing to obtain qualitative conclusions about the collective behaviour of the system and the critical properties characterizing eventual phase transitions. In particular, the Ising model [9], where each elementary unit is represented by a binary variable, has been used to characterize the formation of consensus in social systems [11], opinion dynamics and social spreading phenomena (see [8] and references therein). In this context, each Ising spin represents an agent that is confronted with a binary decision or choice [12], which is taken based on the choice of the majority in its local neighbourhood. The graph connecting different spins reflects the social network structure, while the temperature mimics the uncertainties of the agents or their idiosyncratic beliefs.
More recently, it has been realized that many real-world networks frequently do not operate in isolation, but depend on the structure and dynamics of other networks [13, 14]. This is the typical situation, for instance, in infrastructure networks, where the communication, electric power stations, and transportation networks are coupled together [15], in such a way that failure of nodes in one network can lead to recursively disruption or malfunction of nodes on other networks, leading to a cascading of failures [16, 17]. Social networks are also commonly organized in modular or community structures [18], where the network is composed by sparsely coupled communities or subgroups, with individuals densely connected inside each community. Networks of mobile phone users [19] and of scientific collaborators [20] are typical examples of social networks with a modular structure.

The natural initial step to study the emergence of cooperative behaviour on coupled networks is to consider models with Ising spins. The effect of coupling two networks on the possible macroscopic states of the system has been considered in several works [21–27]. By performing Monte-Carlo simulations [22, 26, 27] and mean-field approximations [21, 23], it has been shown that two possible solutions in terms of the magnetization of each network coexist: the networks might be aligned (equal magnetizations) or anti-aligned (magnetizations with opposite signs), depending on the model parameters. While the aligned state corresponds to the formation of consensus in a social system, the anti-aligned state represents the coexistence of contrary opinions between two groups, which ultimately describes a polarized society. Since models with Ising spins give qualitative insights on real social systems, it is important to assess the robustness of the anti-aligned solution with respect to changes in the model parameters, to characterize the free-energy of the anti-aligned state in comparison to other solutions, and to study how its presence influences the dynamics of the system. Apart from Monte-Carlo simulations suggesting the presence of metastable configurations on the pathway of coupled networks to the equilibrium [26], not much is known about any of the aforementioned aspects. Phase diagrams illustrating the effect of the average connectivities on the macroscopic behaviour are also absent from previous works [21, 23]. These limitations stem from the naive mean-field approach employed in [21, 23], which leads to a set of fixed-point equations for the magnetizations of each network, valid strictly in the regime of large connectivities.

The aim of the present work is to fill this gap and fully explore the macroscopic behaviour of two interacting networks in contact with a source of thermal noise. We consider a model composed of Ising spins placed on the nodes of two interacting Erdős–Rényi random graphs, with a ferromagnetic coupling between any pair of spins. The model is simple enough to allow for a full analytical treatment and the results should yield valuable insights on the general behaviour of the Ising model on coupled networks. In other words, we expect our work serves as a benchmark for studying more sophisticated models.

By using the replica approach of disordered systems, we derive the exact expressions for the magnetizations of each network and for the free-energy of the system, which allows us to obtain complete phase diagrams that unveil the role of the topology on the coexistence between ferromagnetic and anti-aligned states in this model. In particular, the region where both solutions coexist is strongly suppressed by an increase of the number of links among the two networks (see figure 1). We also show that the
model displays a zero-temperature paramagnetic phase, essentially due to the low average connectivity within each network and between them. From the calculation of the free-energy of the system, we show that the anti-aligned solution is always metastable and it appears discontinuously as the model parameters are varied, clarifying the stability properties of the macroscopic states in the coexistence region [21, 23]. By means of Monte-Carlo simulations, we study the role of the metastability on the relaxation of the model to the equilibrium state by calculating the average time $\tau$ the system needs to escape from a metastable initial state. The results for $\tau(T)$ as a function of the temperature $T$ are described by the Vogel–Fulcher law

$$\ln \tau(T) \sim (T - T_0)^{-1},$$

where the temperature $T_0$ consistently converges, for increasing system size, to the instability temperature below which metastable states are present in the thermodynamic limit. In the context of formation of consensus in social systems, our results indicate that, under certain conditions, two social groups can coexist with opposite opinions for remarkably long times, even if all interactions favour their agreement. The exactness of our theoretical findings is supported by Monte-Carlo simulations.

In the next section we define the model of coupled random graphs in equilibrium with a thermal bath. In section 3 we present the main steps of the replica approach and the final analytical expressions for the magnetizations of each network and the free-energy of the system. The phase diagram and the stability of the macroscopic states are considered in section 4, while the results obtained through Monte-Carlo simulations are discussed in section 5. We present some final remarks and perspectives in the last section.

2. Random graph model of two coupled networks

The model is composed of $2N$ interacting Ising spins or state variables. The spins $\sigma_i = \pm 1$ ($i = 1, \ldots, N$) and $\tau_i = \pm 1$ ($i = 1, \ldots, N$) are coupled according to the following Hamiltonian

$$H(\sigma, \tau) = -J_\sigma \sum_{i<j}^N c_{ij}^\sigma \sigma_i \sigma_j - J_\tau \sum_{i<j}^N c_{ij}^\tau \tau_i \tau_j - U \sum_{i<j}^N c_{ij}^I (\tau_i \sigma_j + \tau_j \sigma_i),$$

(1)

where $\sigma = (\sigma_1, \ldots, \sigma_N)$ and $\tau = (\tau_1, \ldots, \tau_N)$. The sum $\sum_{i<j}^N(\ldots)$ runs over all distinct pairs of spins and the coupling strengths $(J_\sigma, J_\tau, U)$ are ferromagnetic.

The random variables $c_{ij}^\sigma$, $c_{ij}^\tau$ and $c_{ij}^I$ determine the topology of the model. We set $c_{ij}^\sigma = 1$ ($c_{ij}^\tau = 1$) if there is an edge between spins $\sigma_i$ ($\tau_i$) and $\sigma_j$ ($\tau_j$), and zero otherwise. The same definition applies to $c_{ij}^I$, which is responsible for the topology of connections among the two networks: we have $c_{ij}^I = 1$ if there is an edge between $\tau_i$ and $\sigma_j$ and between $\tau_j$ and $\sigma_i$, and $c_{ij}^I = 0$ otherwise. We consider the simplest network model, where these random variables are independently drawn from the distributions

$$P_\sigma(c_{ij}^\sigma) = \prod_{i<j} \left[ \frac{c_\sigma}{N} \delta(c_{ij}^\sigma, 0) + \left( 1 - \frac{c_\sigma}{N} \right) \delta(c_{ij}^\sigma, 1) \right],$$

(2)
$$P_{\tau}(c_{ij}) = \prod_{i<j} \left[ \frac{c_{\tau}}{N} \delta(c_{ij}^\tau, 0) + \left(1 - \frac{c_{\tau}}{N}\right) \delta(c_{ij}^\tau, 1) \right],$$

with $\delta$ representing the Kronecker delta. Essentially, the model is composed of two interacting Erdős–Rényi random graphs [28], where a given spin interacts with a random subset of spins within its own graph and with a random subset of spins belonging to the second graph. The parameter $c_\sigma > 0$ ($c_\tau > 0$) is the average number of neighbours per node that belong to graph-$\sigma$ ($\tau$), while $c_I$ controls the average number of edges per node connecting both graphs. In the limit $N \to \infty$, the number of edges per node within each network, namely $k_i^{(\sigma)} = \sum_{j=1,(\neq i)}^N c_i^\sigma$ and $k_i^{(\tau)} = \sum_{j=1,(\neq i)}^N c_i^\tau$, follows a Poisson distribution

$$p_\sigma(k) = \frac{c_k^\sigma \exp(-c_\sigma)}{k!}, \quad p_\tau(k) = \frac{c_k^\tau \exp(-c_\tau)}{k!}.$$  

The number of edges $k_i^{(I)} = \sum_{j=1,(\neq i)}^N c_{ij}^I$ connecting a node $i$ in a certain network to the nodes of the other network also follows a Poisson distribution

$$p_I(k) = \frac{c_k^I \exp(-c_I)}{k!}.$$  

The partition function of the system in equilibrium at temperature $T$ reads

$$Z = \sum_{\sigma,\tau} e^{-\beta H(\sigma,\tau)},$$

with $\beta = T^{-1}$. Our main objective consists in calculating the free-energy per spin in the thermodynamic limit $N \to \infty$. Besides giving access to macroscopic observables, such as the magnetization of each random graph, the free-energy allows us to clearly identify the presence of metastable states. Assuming that, in the limit $N \to \infty$, the free-energy per spin $f$ is a self-averaging quantity with respect to fluctuations in the random graph structure, we have that

$$f = -\lim_{N \to \infty} \frac{1}{2\beta N} \langle \ln Z \rangle,$$

in which $\langle \ldots \rangle$ denotes the average over the ensemble of random graphs, defined through equations (2)–(4).

### 3. The free-energy and the equations for the order-parameters

In order to calculate the ensemble average in equation (8), we employ the replica method [29]
\[ \langle \ln Z \rangle = \lim_{n \to 0} \frac{1}{n} \ln \langle Z^n \rangle. \]  

(9)

Initially, \( n \) is considered to be an integer and positive exponent. After the ensemble average in equation (9) has been evaluated and the limit \( N \to \infty \) has been taken, the analytical continuation \( n \to 0 \) yields the free-energy per spin. This is the standard strategy pursued in the replica approach [29]. Since the microscopic states in this model are not frustrated, replica symmetry yields exact results for the macroscopic behaviour of the system. We remark that the cavity method [32, 33], also known as belief propagation in information theory [34], provides an alternative tool to derive the same exact results as obtained in this section (see equations (27)–(29)).

The calculation of the replicated partition function \( \langle Z^n \rangle \) in the thermodynamic limit is analogous to previous models defined on random graphs [10, 30, 31], so that we just present here the main steps of the derivation. By computing the average over the random graph ensemble in equation (9) and then introducing the order-parameters

\[ P_1(\sigma) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\sigma, \sigma_i}, \quad \sigma = (\sigma_1, \ldots, \sigma_n), \]  

(10)

\[ P_2(\tau) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\tau, \tau_i}, \quad \tau = (\tau_1, \ldots, \tau_n), \]  

(11)

\[ P_{12}(\sigma, \tau) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\tau, \tau_i} \delta_{\sigma, \sigma_i}, \]  

(12)

we are able to decouple sites in the expression for \( \langle Z^n \rangle \), which can be recast in the integral form

\[ \langle Z^n \rangle \sim \int D\{P, \hat{P}\} \exp \left( Ng \left[ \{P, \hat{P}\} \right] \right), \]  

(13)

with the integration measure

\[ D\{P, \hat{P}\} \equiv \prod_{\sigma\tau} dP_1(\sigma) dP_2(\tau) dP_{12}(\sigma, \tau) d\hat{P}_1(\sigma) d\hat{P}_2(\tau) d\hat{P}_{12}(\sigma, \tau). \]  

(14)

The functional \( g \left[ \{P, \hat{P}\} \right] \) reads

\[
\begin{align*}
g \left[ \{P, \hat{P}\} \right] &= -\frac{1}{2} (c_\sigma + c_\tau + c_I) + \sum_\sigma P_1(\sigma) \hat{P}_1(\sigma) + \sum_\tau P_2(\tau) \hat{P}_2(\tau) + \sum_{\sigma\tau} P_{12}(\sigma, \tau) \hat{P}_{12}(\sigma, \tau) \\
&\quad + \frac{c_\sigma}{2} \sum_{\sigma\sigma'} P_1(\sigma) P_1(\sigma') \exp(\beta J_\sigma \sigma') \\
&\quad + \frac{c_\tau}{2} \sum_{\tau\tau'} P_2(\tau) P_2(\tau') \exp(\beta J_\tau \tau') \\
&\quad + \frac{c_I}{2} \sum_{\sigma\tau} \sum_{\sigma'\tau'} P_{12}(\sigma, \tau) P_{12}(\sigma', \tau') \exp[\beta U (\sigma \cdot \tau' + \sigma' \cdot \tau)] \\
&\quad + \ln \left[ \sum_{\sigma\tau} \exp[-i \hat{P}_1(\sigma) - i \hat{P}_2(\tau) - i \hat{P}_{12}(\sigma, \tau)] \right],
\end{align*}
\]

(15)
where the conjugate parameters $\hat{P}_1$, $\hat{P}_2$ and $\hat{P}_{12}$ have arisen from the integral representations of the Dirac delta functionals, used to introduce the order-parameters in the expression for $\langle Z^n \rangle$. From now on, the $n$-dimensional vector $\sigma (\tau)$ encodes the states of a single spin $\sigma_i (\tau_j)$ in the $n$ different replicas, as explicitly emphasized in equations (10) and (11). Unimportant factors, which give a vanishing contribution to the free-energy per spin in the limit $N \to \infty$, have been neglected in equation (13).

The function $\langle Z^n \rangle$ can now be evaluated through the saddle-point method. In the limit $N \to \infty$, the integral in equation (13) is dominated by the values of $\{P, \hat{P}\}$ that extremize the functional $g \{P, \hat{P}\}$. Substituting equations (9) in (8) and performing the limit $N \to \infty$ through the saddle-point method, we obtain a formal expression for the free-energy per spin

$$2\beta f = - \lim_{n \to 0} \frac{1}{n} g \{P, \hat{P}\},$$

where $\{P, \hat{P}\}$ refers, from now on, to the specific values that extremize $g \{P, \hat{P}\}$. The saddle-point equations that determine $\{P, \hat{P}\}$ are derived by taking functional derivatives of $g \{P, \hat{P}\}$ with respect to $\{P, \hat{P}\}$

$$P_1(\sigma) = \frac{1}{N} \sum_{\tau} \exp \left[ -i\hat{P}_1(\sigma) - i\hat{P}_2(\tau) - i\hat{P}_{12}(\sigma, \tau) \right],$$

$$P_2(\tau) = \frac{1}{N} \sum_{\sigma} \exp \left[ -i\hat{P}_1(\sigma) - i\hat{P}_2(\tau) - i\hat{P}_{12}(\sigma, \tau) \right],$$

$$P_{12}(\sigma, \tau) = \frac{1}{N} \exp \left[ -i\hat{P}_1(\sigma) - i\hat{P}_2(\tau) - i\hat{P}_{12}(\sigma, \tau) \right],$$

where $N$ is the normalization factor

$$N = \sum_{\sigma \tau} \exp \left[ -i\hat{P}_1(\sigma) - i\hat{P}_2(\tau) - i\hat{P}_{12}(\sigma, \tau) \right].$$

The conjugate parameters are given by

$$\hat{P}_1(\sigma) = ic_\sigma \sum_{\sigma'} P_1(\sigma') \exp (\beta J_s \sigma' \cdot \sigma),$$

$$\hat{P}_2(\tau) = ic_\tau \sum_{\tau'} P_2(\tau') \exp (\beta J_s \tau' \cdot \tau),$$

$$\hat{P}_{12}(\sigma, \tau) = ic_{\sigma \tau} \sum_{\sigma' \tau'} P_{12}(\sigma', \tau') \exp [\beta U (\sigma \cdot \tau' + \sigma' \cdot \tau)].$$

From equations (16)–(23), we see that the free-energy per spin is fully determined by the self-consistent equations (17)–(19) for the order-parameters.

In order to compute the limit $n \to 0$ in equation (16), one has to explicitly perform the sums over the replica Ising spins and unveil how $g \{P, \hat{P}\}$ depends on $n$, which is

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only possible if we make an assumption for the structure of the order-parameters. By 
considering there is one single thermodynamic state, each order-parameter is invariant 
with respect to permutations of the replica indexes \([31]\) and all information about 
the fluctuations of the local magnetizations lies in the distribution of effective fields 
\(h_i = \beta^{-1} \arctan(\langle S_i \rangle)\), where \(\langle S_i \rangle\) denotes the average of a generic spin \(S_i\) with respect 
to thermal and random graph fluctuations. The simplest form that fulfills replica sym-
metry is a function of the magnetizations only, namely \([10, 31]\)

\[
P_1(\sigma) = \int dh W_\sigma(h) \frac{\exp(\beta h \sum_{\alpha=1}^{n} \sigma_\alpha)}{[2 \cosh(\beta h)]^n}, \tag{24}
\]

\[
P_2(\tau) = \int dh W_\tau(h) \frac{\exp(\beta h \sum_{\alpha=1}^{n} \tau_\alpha)}{[2 \cosh(\beta h)]^n}, \tag{25}
\]

\[
P_{12}(\sigma, \tau) = \int du dv W_{\sigma\tau}(u, v) \frac{\exp(\beta u \sum_{\alpha=1}^{n} \sigma_\alpha + \beta v \sum_{\alpha=1}^{n} \tau_\alpha)}{[4 \cosh(\beta u) \cosh(\beta v)]^n}. \tag{26}
\]

The quantity \(W_\sigma(h) \ (W_\tau(h))\) is the distribution of effective fields on network-\(\sigma\) (\(\tau\)), independently of the configuration of effective fields in network-\(\tau\) (\(\sigma\)). The distributions \(W_\sigma(h)\) and \(W_\tau(h)\) are normalized, consistently with equations (10) and (11). The function \(W_{\sigma\tau}(u, v)\) is the joint distribution of effective fields in both networks, where the argument \(u\) (\(v\)) refers to the possible outcomes for the effective fields in network-\(\sigma\) (\(\tau\)). Besides the normalization of \(W_{\sigma\tau}(u, v)\), we have to supplement equation (26) with the conditions \(\int du W_{\sigma\tau}(u, v) = W_\tau(v)\) and \(\int dv W_{\sigma\tau}(u, v) = W_\sigma(u)\), which ensure the marginalization of \(P_{\sigma\tau}(\sigma, \tau)\) with respect to the spins of a given network, consistently 
with equations (10)–(12).

By substituting equations (24)–(26) in (16), computing the trace over the Ising 
spins, and performing the limit \(n \to 0\), we obtain the free-energy per spin

\[
f = c_\sigma \int dh dh' W_\sigma(h) W_{\sigma}(h') U_\delta(h, h'| J_\sigma) + c_\tau \int dh dh' W_\tau(h) W_{\tau}(h') U_\delta(h, h'| J_\tau) \\
+ 2c_\tau \int dh dh' W_\tau(h) W_{\sigma}(h') U_\delta(h, h'| U) - \frac{1}{2\beta} \sum_{k_\sigma, k_\tau=0}^{\infty} p_{\sigma}(k_\sigma) p_{\tau}(k_\tau) \\
\times \left( \prod_{n=1}^{k_\sigma} du_n W_\sigma(u_n) \right) \left( \prod_{m=1}^{k_\tau} dv_m W_\tau(v_m) \right) \ln \left[ \sum_{\gamma=\pm 1} G_\gamma(u_1, \ldots, u_{k_\sigma}, |J_\sigma|) G_\gamma(v_1, \ldots, v_{k_\tau}|U) \right] \\
- \frac{1}{2\beta} \sum_{k_\sigma, k_\tau=0}^{\infty} p_{\sigma}(k_\sigma) p_{\tau}(k_\tau) \int \left( \prod_{n=1}^{k_\sigma} du_n W_\sigma(u_n) \right) \left( \prod_{m=1}^{k_\tau} dv_m W_\tau(v_m) \right) \\
\times \ln \left[ \sum_{\gamma=\pm 1} G_\gamma(u_1, \ldots, u_{k_\sigma}, |J_\sigma|) G_\gamma(v_1, \ldots, v_{k_\tau}|U) \right], \tag{27}
\]

where
The magnetizations of each network are given by

\[ m_\sigma = \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i \rangle, \quad m_\tau = \frac{1}{N} \sum_{i=1}^{N} \langle \tau_i \rangle. \]  

(31)

In the present formalism, the magnetizations are obtained from the effective field distributions [31]:

\[ m_\sigma = \int dh W_\sigma(h) \tanh(\beta h), \]
\[ m_\tau = \int dh W_\tau(h) \tanh(\beta h). \]

Thus, once \( W_\sigma(h) \) and \( W_\tau(h) \) are determined from the solutions of equations (28) and (29), we can calculate the magnetizations of each network, obtain the phase diagrams, and probe the stability of the solutions through the free-energy.
4. Phase diagrams and metastability

For a general combination of model parameters, equations (28) and (29) cannot be analytically solved and one needs to employ a numerical approach. In this section we solve numerically equations (28) and (29) through the population dynamics method \cite{32,33}, from which the phase diagrams and the free-energy follow. In this numerical approach, the distributions $W_{\sigma}(\mathbf{h})$ and $W_{\tau}(\mathbf{h})$ are parametrized, respectively, by large sets of stochastic variables $\{h_i^{(\sigma)}\}_{i=1,...,N}$ and $\{h_i^{(\tau)}\}_{i=1,...,N}$, with $N$ denoting the population size. By choosing initial distributions $W_{\sigma}^{(0)}(\mathbf{h})$ and $W_{\tau}^{(0)}(\mathbf{h})$ for each network, the variables $\{h_i^{(\sigma)}\}_{i=1,...,N}$ and $\{h_i^{(\tau)}\}_{i=1,...,N}$ are consistently updated according to the arguments of the Dirac delta functions appearing in equations (28) and (29), until the empirical distribution obtained from each population of fields reaches its final, stationary form. Averages involving $W_{\sigma}(\mathbf{h})$ and $W_{\tau}(\mathbf{h})$ are evaluated by computing sample averages using, respectively, the collection of random variables $\{h_i^{(\sigma)}\}_{i=1,...,N}$ and $\{h_i^{(\tau)}\}_{i=1,...,N}$. We refer to \cite{34} for further details regarding this numerical method.

From the numerical solutions of equations (28) and (29), we have calculated the magnetizations $m_{\sigma}$ and $m_{\tau}$ of each network for different values of the model parameters. Three different solutions have been found: a paramagnetic state (P), with $m_{\sigma} = m_{\tau} = 0$; a ferromagnetic solution (F), where $m_{\sigma}m_{\tau} > 0$; and an anti-aligned state (AA), with $m_{\sigma}m_{\tau} < 0$. 

Figure 1. Phase diagram in the plane $(c_\sigma, c_\tau)$ for temperature $T = 0.001$, a value $U = 0.1$ for the coupling strength between the networks, and different values of the average connectivity $c_j$ between the networks. The model displays a ferromagnetic solution (F), an anti-aligned (AA) solution, and a paramagnetic (P) state. The ferromagnetic and the anti-aligned solutions coexist in the region marked with F+AA, where the AA solution is always metastable. These results are obtained through the numerical solution of equations (28) and (29) using the population dynamics method with $N = 5 \times 10^5$ and initial distributions $W_{\sigma}^{(0)}(\mathbf{h}) = \delta(h - 1)$ and $W_{\tau}^{(0)}(\mathbf{h}) = \delta(h + 1)$ (see the main text).
In order to discuss the phase diagrams and the stability of these macroscopic states, we set $J = J' = 1$ throughout this section. Figure 1 shows typical phase diagrams in the $(c_\sigma, c_\tau)$-plane for low temperatures and different values of $c_I$. The networks are weakly coupled with strength $U = 0.1$. For $J = J'$, the Hamiltonian is invariant with respect to the interchange of the adjacency matrix elements $c_{ij} \leftrightarrow c_{ij} \forall i, j$, which implies on the symmetry of the above phase diagram around the straight line $c_\sigma = c_\tau$. In the region F+AA of figure 1, the ferromagnetic solution coexists with the (metastable) anti-aligned state \[21, 23\], namely, both types of order are obtained from the numerical solution of equations (28) and (29), depending on the initial distributions $W_\sigma(h) = \delta(h - 1)$ and $W_\tau(h) = \delta(h + 1)$ (see the main text).

Figure 2. Critical coupling strength $U_c$ below which we find an anti-aligned solution from equations (28) and (29). The results are shown as a function of the average connectivity $c \equiv c_\sigma = c_\tau$ within each network, for different combinations of temperature $T$ and the average connectivity $c_I$ between the networks. We have rescaled all coupling constants (see equation (1)) by the common factor $c_{eff} = \frac{1}{3}(c_\sigma + c_\tau + c_I)$. These results are obtained through the numerical solution of equations (28) and (29) using the population dynamics method with $N = 5 \times 10^5$ and initial distributions $W_\sigma^0(h) = \delta(h - 1)$ and $W_\tau^0(h) = \delta(h + 1)$ (see the main text).

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$$c_\tau^2 \tanh^2 (\beta U) = [1 - c_\sigma \tanh (\beta J_\sigma)] [1 - c_\tau \tanh (\beta J_\tau)].$$ (32)

From the above equation, one concludes that the system exhibits a paramagnetic phase for $T = 0$, located in a region of the phase diagram where all average connectivities
Figure 3. Absolute value of the magnetization $|m|$ of a single network and free-energy per spin $f$ as functions of $c_\sigma = c_\tau$ for temperature $T = 0.001$ and ferromagnetic coupling $U = 0.1$. The values of the average connectivity $c_I$ between the networks are indicated on the graphs. The magnetizations of each network have the same absolute value for $c_\sigma = c_\tau$. The three different solutions of the phase diagram and the corresponding free-energies are shown here. These results are obtained through the population dynamics method with $N = 10^6$ and initial distributions $W(0)_\sigma = \delta(h - 1)$ and $W(0)_\tau = \delta(h + 1)$ (see the main text). (a) Magnetization. (b) Free-energy for $c_I = 0.5$. Must be smaller than one. This is consistent with figure 1, in which the P phase is absent for $c_I = 1$. The reason for the existence of this zero-temperature P phase is utterly topological, since for low connectivities the system is fragmented in a large number of finite non-interacting clusters [5].

Figure 2 complements the phase diagram of figure 1 by showing results for the critical coupling strength $U_c$ above which the anti-aligned solution is absent, considering different values of $T$ and $c_I$. The parameter $c \equiv c_\sigma = c_\tau$ is the average connectivity within each network. As can be noted, the increase of $T$ or $c_I$ has a detrimental effect on the existence of anti-aligned states. Below we study in more detail the effect of thermal fluctuations in the stability of such states. The curves in figure 2 converge to $c = c_{\text{perc}}(T)$ as $U \to 0$, where $c_{\text{perc}}(T)$ is the critical average connectivity above which a single random graph lies in a ferromagnetic state. We have that $c_{\text{perc}}(T) \simeq 1$ for the smallest temperature displayed in figure 2, consistent with standard results for the percolation transition in random graphs [5].

In figure 3(a) we show the absolute value of the magnetization of a single network along the straight line $c_\sigma = c_\tau$ of the phase diagram, considering initial distributions $W^{(0)}_\sigma(h) = \delta(h - 1)$ and $W^{(0)}_\tau(h) = \delta(h + 1)$ that yield $m_\sigma > 0$ and $m_\tau < 0$ in the region F+AA. As clearly shown, the transition between the paramagnetic and the F state is continuous, while the magnetization changes discontinuously along the boundary between the F region and the F+AA region. In this case, such discontinuity is not a signature of a first-order phase transition, but it simply reflects the sudden emergence of the metastable anti-aligned solution. The equilibrium magnetization, characterized by a continuous branch, is not shown when $c_\sigma = c_\tau$ lies in the F+AA region, since
equations (28) and (29) have been solved with initial distributions favouring the anti-aligned solution. The stability of the different solutions is characterized in figure 3(b), where we present the free-energy $f$ of each possible solution of equations (28) and (29) as a function of $c_\sigma = c_\tau$, for a single value of $c_I^6$. The main outcome is that the anti-aligned solution is always metastable, while the ferromagnetic solution is the stable macroscopic state, since it corresponds to the global minimum of the free-energy. We have checked many different combinations of model parameters and we did not find any qualitative changes in these stability properties. All results discussed in this section are also applied to the case where the couplings between the networks are anti-ferromagnetic. The difference is that, for $U < 0$, the ferromagnetic solution is metastable, while the anti-aligned solution is the stable macroscopic state. Apart from that, the phase diagrams remain unchanged, i.e. the phase boundaries for $U < 0$ are the same as those for $U > 0$.

5. Numerical simulations

In this section we compare our theoretical results with Monte-Carlo (MC) simulations using standard Metropolis dynamics of finite size systems with different system sizes. We also compute the typical time the system needs to escape from the metastable states. This is a way to go beyond the theoretical results and quantify the lifetime of the metastable states and have a better idea of their role on the relaxation of the system to equilibrium.

The theoretical results indicating a second order phase transition from a paramagnetic to a ferromagnetic phase, where metastable anti-aligned states appear for certain model parameters, are verified in the MC simulations. We have measured the magnetizations of each network in equilibrium or in metastable configurations, following a quasi-static heating protocol of the system prepared initially at zero temperature in a purely anti-aligned metastable state, where the spins in different networks have opposite directions. Simulations were done for $J_\sigma = J_\tau = U = 1/c_{\text{eff}}$, with $c_{\text{eff}} = \frac{1}{3}(c_\sigma + c_\tau + c_I)$, and two cases of average connectivities $c_\sigma = c_\tau = 10$ and $c_I = 1.0$, and $c_\sigma = c_\tau = 4$ and $c_I = 0.5$. Simulated system sizes range from $N = 400$ up to $N = 25,600$, where $N$ stands for the number of nodes in each graph. Equilibration times in each temperature are of order $10^5$ MC steps, and averages are taken from 100 different realizations of the random graphs, each realization contributing with 100 samples for each temperature.

Figure 4 shows the comparison between MC simulations and our theoretical results for the magnetization of each network as a function of the temperature. As can be noticed, finite size effects in MC simulations become remarkable as $T$ increases towards the instability temperature, above which the anti-aligned metastable state disappears. This is a purely dynamical effect in the heating protocol, due to the available thermal energy and finite energy barrier between the metastable state and the true thermodynamical equilibrium state. In spite of that, the overall agreement between our theory

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6 The free-energy of the paramagnetic solution is displayed only for model parameters within the region P, where the paramagnetic state is the only possible solution of equations (28) and (29). In regions F and F+AA, figure 3(b) exhibits only the free-energy of the nontrivial states.
Figure 4. Results for the magnetization $|m|$ of each network as a function of the temperature $T$ for coupling strengths $J_\sigma = J_r = U = 1/c_{\text{eff}}$, with $c_{\text{eff}} = \frac{1}{3}(c_\sigma + c_r + c_I)$. The magnetization of each graph has the same absolute value for $c_\sigma = c_r$. Each panel compares the theoretical results, derived from the solutions of equations (28) and (29), with Monte-Carlo simulations following a heating protocol started at $T = 0$ in the metastable anti-aligned state, for different systems sizes $N$, whose values are shown on each graph. Equations (28) and (29) are solved numerically using the population dynamics method with $N = 10^6$ and initial distributions $W^{(0)}_\sigma(h) = \delta(h - 1)$ and $W^{(0)}_r(h) = \delta(h + 1)$ (see the main text). (a) $c_\sigma = c_r = 10$ and $c_I = 1.0$. (b) $c_\sigma = c_r = 4$ and $c_I = 0.5$.

Figure 5. (a) Results obtained from Monte-Carlo simulations for the average time $\tau$ that it takes for the system to escape from a metastable anti-aligned state (see the main text) as a function of temperature. The values of the average connectivities are $c_\sigma = c_r = 10$ and $c_I = 1$, while the coupling strengths are given by $J_\sigma = J_r = U = 1/c_{\text{eff}}$, with $c_{\text{eff}} = \frac{1}{3}(c_\sigma + c_r + c_I)$. The different system sizes $N$ are indicated on the graph. Lines are fits to $\tau(T) = A \exp(E/(T - T_{\text{ins}}))$, and the vertical line corresponds to the theoretical result for the instability temperature $T_{\text{ins}} = 1.163$. (b) The upper figure displays the behaviour of $\tau$ as a function of $N$ for $T = 1.105$, where the line corresponds to an exponential fit (see the main text). The lower figure shows the system size dependence of the temperatures for which each fit in figure (a) gives a fixed average time $\tau$. The curves are fits of the form $T(\tau) = T_0 + a/\ln(bN)$, and the horizontal line is the instability temperature $T_{\text{ins}} = 1.163$ in the limit $N \rightarrow \infty$. 

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and MC simulations is excellent, with the simulation data consistently approaching the theoretical curves for increasing $N$.

We have also computed the average or typical time $\tau$ for the system to escape from a metastable anti-aligned state. By preparing the system in an initial configuration corresponding to the zero temperature anti-aligned solution, with $m_\sigma = 1$ and $m_\tau = -1$, the parameter $\tau$ counts the average number of MC steps that the system needs to reach a configuration in which one of the magnetizations changes sign. Averages are taken from 100 to 2000 different realizations of the random graphs. Figure 5(a) exhibits $\tau$ as a function of $T$ for the same combination of model parameters as in figure 4(a). The simplest temperature dependence of these results is well described, especially in the region where $\tau$ increases abruptly, by the Vogel–Fulcher law $\tau(T) = A \exp\left(\frac{E}{T-T_0}\right)$, where the parameters $A$, $E$ and $T_0$ depend on $N$.

Figure 5(a) strongly indicates that, in the limit $N \rightarrow \infty$, the typical time $\tau$ diverges as $T \rightarrow T_{\text{ins}}^+$, where $T_{\text{ins}}$ is the temperature above which metastable states are absent in the thermodynamic limit. Such divergent behaviour is confirmed in figure 5(b), where we show the exponential divergence of $\tau = B \exp(bN)$ as a function of the system size for $T = 1.105 < T_{\text{ins}}$, with resulting fitting parameters $B = 37(4)$ and $b = 0.00044(1)$. This typically characterizes a thermally activated process of crossing free-energy barriers [35], which is consistent with the mean-field character of our model, or with the fact that the free-energy barriers separating different macroscopic states are proportional to $N$. Thus, as long as $T < T_{\text{ins}}$ and $N \rightarrow \infty$, the system becomes trapped in the metastable states, once it is prepared in a configuration close to them.

The instability temperature can be extracted from the simulation data by inverting the fits in figure 5(a) to obtain $T(\tau)$, the temperatures at which, for a given size, the system takes on average $\tau$ MC steps to cross the free-energy barrier. Data for $\tau = 10^3$, $10^4$ and $10^5$ are shown in figure 5(b). The extrapolation of $T_0$ for $N \rightarrow \infty$ can be performed by fitting the curves with $T_0 + a/\log(bN)$. The resulting values for $T_0$ are 1.19(1), 1.20(1) and 1.22(1), respectively, for $\tau = 10^3$, $10^4$ and $10^5$, which agrees well with our theoretical result $T_{\text{ins}} = 1.163$, strictly valid for $N \rightarrow \infty$.

6. Final remarks

In this work we have studied the phase diagram and the existence of metastable states in a simple model of coupled networks. The model is composed of two coupled Erdős–Rényi random graphs with an Ising state variable or spin lying at each node. Each spin in a given graph has a finite number of ferromagnetic couplings within its own graph and ferromagnetic interactions with a finite subset of spins located on the other graph. The simplicity of this model has enabled us to exactly compute the magnetization of each network and the free-energy of the system in the thermodynamic limit, using the replica method of disordered systems. The main outcome of our work is the full characterization of the phase diagram and of the stability properties of the different macroscopic states. As we clearly illustrate through the computation of the free-energy, the ferromagnetic solution is the thermodynamic state, while the anti-aligned solution
magnetizations with opposite signs) is always metastable. The metastable solution appears through a discontinuous transition as a function of the model parameters, provided the average connectivity within each network is large enough and the temperature is sufficiently low.

We have estimated, through Monte-Carlo simulations, the average time $\tau$ the system needs to escape from a metastable configuration. Our results for $\tau(T)$ are well-described by the Vogel–Fulcher law, which tells us that there is a critical temperature below which $\tau$ diverges in the thermodynamic limit $N \to \infty$. Such ergodicity breaking stems from the mean-field character of our model, in which the free-energy barrier between the metastable and the stable macroscopic state diverges for $N \to \infty$. Our main theoretical results have been compared with Monte-Carlo simulations, showing a very good agreement.

We have also shown that the phase diagram, as defined in the space of the connectivities of each network, exhibits a low-temperature paramagnetic phase for very small average connectivities. This is explained by the fact that, in this sector of the phase diagram, the random graphs are fragmented in a large number of disconnected finite clusters that are unable to communicate. Finally, we point out that the present work paves the way to pursue a detailed study of the cooperative behaviour arising in coupled networks with different architectures [36, 37], such as modular and core-periphery structures. Work along these lines is underway.

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