Fuzzy Phase Space and Space-time Structure as Approach to Quantization

S.N. Mayburov
Lebedev Inst. of Physics
Leninsky Prospect 53
Moscow, Russia, 117924
Email: mayburov@sci.lpi.msk.su

Abstract

The quantum space-time and the phase space of massive particles with Fuzzy Geometry structure investigated as the possible quantization formalism. In this model the state of nonrelativistic particle \( m \) corresponds to the element of fuzzy ordered set (Foset) - fuzzy point. Due to its partial ordering \( m \) space coordinate \( x \) acquires principal uncertainty \( \sigma_x \). It’s shown that Fuzzy Mechanics (FM) in 1+1 dimension is equivalent to the path integral formalism of nonrelativistic Quantum Mechanics.

1 Introduction

The properties of quantum space-time and its relation to Quantum Mechanics (QM) axiomatics is actively discussed now from the different angles [1, 2, 3]. The interest to it enforced by the indications that its structure at small (Plank) scale can be quite nontrivial [4, 5]. In particular, it was proposed that such fundamental geometric features like the metrics or topology are modified significantly in this case [5, 6]. Our work motivated largely by this ideas and we shall explore the possible insights prompted by Sets Theory, exploring the various set structures of space-time manifold \( M \). For example, in 1-dimensional Euclidean Geometry, the elements of its manifold \( X \) - the points \( x_i \) constitute the ordered set. Yet Sets Theory includes other kinds of sets which also permit to construct the consistent geometries. In this context we shall investigate Posets and the fuzzy ordered sets (Fosets); in this case their
elements are incomparable or weakly ordered relative to each other [7, 8]. Basing on Foset structure, the novel Fuzzy Geometry was constructed which will be studied here as the possible space-time geometry [9, 10, 11].

In nonrelativistic Classical Mechanics in 1-dimensional space \( X = R^1 \) the particle’s states corresponds to the material points \( x^m(t) \) ordered relative to \( R^1 \) set, i.e. relative to all its elements \( \{x_a\} \). In distinction on Fuzzy space manifold \( M^F \) it supposed that the particle’s states correspond to the fuzzy points \( b_m \) which smeared in \( R^1 \) space with an arbitrary dispersion \( \sigma_x \). In our approach the quantization regarded as the transfer from Classical ordered phase space to fuzzy one. In this paper as the simple model of such transition the quantization of nonrelativistic particle will be regarded; it will be argued that Fuzzy Geometry in 1-dimensional fuzzy space induces the particle’s dynamics which is equivalent to Schrödinger QM dynamics. Earlier it was shown that the fuzzy observables are the natural generalization of QM observables [12]. The ‘fuzzy lumps’ were applied in Quantum Gravity and Cosmology studies [14]. In the last years it was shown that some fuzzy sets features are appropriate also for Quantum Logics formalism ([13] and ref. therein).

Remind that in a partial ordered set (Poset) \( D = \{d_i\} \) beside the standard ordering relation between its elements \( d_k \leq d_l \) (or vice versa), the incomparability relation \( d_k \not\leq d_l \) is also permitted; it means that both \( d_k \leq d_l \) and \( d_l \leq d_k \) propositions are false in this case. Fuzzy relations can be regarded as the generalization of incomparability which introduces its continuous measure \( w \). To illustrate its meaning and other Foset properties, let’s consider the discrete Poset \( D^T \) which includes the subset of incomparable elements \( B = \{b_j\} \), and the ordered subset \( A = \{a_i\} \). Let’s concede that in \( A \) the elements indexes grow correspondingly to the elements order, so that for any \( i, a_i \leq a_{i+1} \). Any \( b_j \in B \) is incomparable at least to one \( a_i \in A \). The interval \( [d_e, d_f] \) is \( D^T \) subset, such that its maximal lower (upper) bound \( d_e, d_f \in A \), and any \( a_i, b_j \in [d_e, d_f] \), if \( d_e \leq a_i, b_j \leq d_f \). For the simplicity let’s consider \( B \) which includes only one element \( b_0 \). Let’s suppose that \( b_0 \in [a_{l+n}, a_{l+n+1}] \); \( n \geq 2 \), \( b_0 \) is incomparable with all \( [a_i, a_{i+n}] \) internal elements: \( b_0 \not\leq a_i \); iff \( l + 1 \leq i \leq l + n - 1 \), so that \( b_0 \) in some sense is ‘smeared’ inside \( [a_l, a_{l+n+1}] \) interval.

To introduce the measure of incomparability \( w \), let’s put in correspondence to each \( b_0, a_i \) pair the weight \( w_i^0 \geq 0 \) with the norm \( \sum w_i^0 = 1 \). The simplest example is the symmetric incomparability: \( w_i^0 = \frac{1}{n} \) for \( a_i \in [a_l, a_{l+n}] \) interval; \( w_i^0 = 0 \) outside of it; it can be interpreted as \( b_0 \) homogeneous smearing inside \( [a_l, a_{l+n}] \). If \( b_0 \) is ordered (localized), for example \( b_0 = a_i \), then \( w_i^0 = \delta_i j \). If \( w \) defined for all \( a_j, b_i \) pairs in \( D^T \), then \( D^T \) is Foset, and \( b_i \) are the fuzzy points. Note that in distinction from the regarded case, in general an arbitrary Foset isn’t necessarily Foset.

The continuous 1-dimensional Foset \( C^T \) can be defined in the same vein; the ordered subset \( A \) can be substituted by the continuous metricized (i.e. ordered) subset \( X \in C^T \) which is equivalent to the real numbers axe \( R^1 \). In the simplest case one can take the same \( B \), then \( B \in C^T \) is the discrete subset of fuzzy points, we shall
regard here only \( B = \{ b_0 \} \). The interval \([x_v, x_u]\) is defined analogously to the discrete case, the fuzzy \( b_0, x_a \) relations are described by the continuous distribution \( w^0(x_a) \geq 0 \) with the norm \( \int w^0 dx = 1 \). Below we shall call the fuzzy space \( C^T = B \cup X \) which is the direct sum of 1-dimensional Euclidean space and the discrete set of the fuzzy objects. For our study the following example will be important: let’s consider \( b_0 \) with \( w^0 \) support inside the system of noncrossing intervals \( E_x = \{ \bigcup \Delta_j, j = 1, ..., n \} \), then \( b_0 \) structure expressed by the relation (proposition):

\[
LP^b := b_0 \in E_x, \text{and} b_0 \in \Delta_1, \text{and} ..., \text{and} b_0 \in \Delta_n
\]

so that \( b_0 \) can’t be ascribed to any particular interval \( \Delta_j \), but only to their system \( E_x \) as the whole. Note that in Fuzzy Geometry \( w^0(x) \) doesn’t have any probabilistic (stochastic) meaning but only the topological and geometric one. The fuzzy structures to some extent are analogous to Orthomodular or BvN algebras which describes some Quantum Structures [13]. The remarkable analogy between the uncertainty of Fuzzy points coordinates and QM uncertainties was noticed already [10], but no proof of their equivalence was presented. Fuzzy Geometry formalism is the generalization of regarded examples and is reviewed elsewhere [10, 11]. In brief form the main results of present paper were published in [2]

### 2 Fuzzy Mechanics (FM) and Fuzzy States

Now we discuss the transition from Fuzzy geometry to Fuzzy mechanics (FM) which is analogous to the transition from Euclidean Geometry to Classical Mechanics. The particle’s state in Classical Mechanics is the ordered point \( \{ \vec{r}, \vec{p} \} \) in 6-dimensional phase space \( R^3 \ast R^3 \). We shall consider the fuzzy point in coordinate space and the description of its evolution by some state. In nonrelativistic case considered here the time \( t \) is the standard real parameter on the axe \( T \). We consider here 1-dimensional theory on \( R^1 \) and suppose that FM posess the invariance relative to the space and time shifts and also is invariant under space and time reflections analogously to Classical Mechanics; the particle evolution in FM is reversible [15]. In our theory we identify the nonrelativistic particle \( m \) with the fuzzy point \( b_0(t) \) in \( C^T \) manifold described above; it means that at any \( t \) \( m \) characterrized by the positive density \( w(x,t) \) in 1-dimensional space \( R^1 \). But its evolution, as will be shown below, can depends of more complicated correlations between several \( X \) points. We concede that \( m \) physical properties at the instant \( t \) in an arbitrary reference frame (RF) described by a fuzzy state \( |g(t)\rangle \), the used notation stresses its difference from Dirac quantum state \( |\psi\rangle \). The set of \( |g\rangle \) states \( M_s \) doesn’t supposed to be the linear space of any kind \( a \) prioriy, and one of our aims is to derive \( M_s \) structure. In this approach \( w(x, t) = F_N(g) \) is some functional of \( |g\rangle \), and \( |g\rangle \) have the positive constant norm: \( N = \int w dx = 1 \). Beside \( w(x) \), \( g \) can include the additional components \( \hat{g} \),
which will be used for the description of $m$ evolution parameters; below they will be related to such $m$ observables as the momentum $p$ and velocity $v_x$. We shall construct FM as the minimal theory i. e. at every stage it assumed that the number of degrees of freedom and theory parameters is as minimal as necessary for the theory consistency. Consequently we shall start from assuming $\tilde{g}$ to be the correlation field $\tilde{g} = \{g_{\mu}(x_1, ..., x_l)\}; \mu = 1, n$, where $g_{\mu}$ are some real functions, in the simplest case $\tilde{g} = \{g_{\mu}(x)\}$ can be the vector field. FM formalism is based mainly on geometric premises, analogously to General Relativity. In particular, the choice of $\tilde{g}$ components will be motivated by Fuzzy Geometry.

For the comparison of the observed effects, besides the nonstochastic (pure) fuzzy states we shall consider also the mixed fuzzy states $g^m$ which are the probabilistic ensembles of several fuzzy states $\{g_i\}$ with probabilities $P_i$ and are analogous to QM mixed states [15]. The measurement of $m$ observables in FM will be discussed in the final part of our paper; here we assume only that $x$ distributions $w(x,t)$ can be measured by some experimental procedure. In addition, analogously to QM it supposed that an arbitrary $m$ state can be prepared by some experimental procedure.

The evolution of any physical object can be described as the map of its initial state $g_0$ to the final $g(t)$; so the evolution of fuzzy point $m$ responds to the fuzzy map $\Xi_t\{g_0\} = |g(t)\}$. It’s instructive to start from the study of simple qualitative properties of such map. In particular, we consider the important effect of the sources smearing (SS) which is close analog of quantum interference. To illustrate this effect which is generic for FM, let’s study here 1-dimensional analog of the notorious two slits experiment (TSE) which widely used for QM foundations discussion [15, 21]. Here we regard the initial state $|g_0\}$, which support $E_x$ in some RF consists of $n_s$ noncrossing intervals (bins) $Dx_i$. After $g_0$ preparation at $t = 0$ presumably $m$ doesn’t interact with any other object and evolves freely. $g_0$ can be regarded as the source $S(g)$ for the future state - the signal $|g(t)\}$. The resulting density $w(x,t) = \Xi_t\{g_0\}$. The fuzzy map $\Xi_t$ in principle can project the internal fuzzy structure of the source $S(g_0)$ to the distribution of signal density $w(x,t)$, and due to it SS effect will appear. For the simplicity we shall consider only an infinitely small bins $Dx_i \to 0$, so that $w^0_s$ can be approximated as:

$$w^0(x) = \sum_{i=1}^{n_s} w^0_i \delta(x - x_i)$$  

(1)

Let’s consider first $n_s = 1$, $w^0_1 = 1$ and suppose that in this case FM evolution extends $g_0$ to $w(x,t)$ spread in some support $E_x(t)$, i.e. $w$ has the finite dispersion $\sigma_x(t)$. Then it can be described as:

$$w_1(x, t) = C_m(\tilde{g}^0, x - x_1, t)$$

where the function $C_m \geq 0$ is $w$ effective propagator which conserves the norm:

$$||w_1|| = \int C_m dx = 1$$
at any $t$. Consider now the situation when $n_s \neq 1$ and the signals from different sources don’t intersect practically. It’s possible, for example, if $w_i$ dispersions $\sigma_i^x(t)$ are small, and $|x_i - x_j| \to \infty$; then

$$w_s(x,t) \simeq \sum w_i(x,t)$$

From this properties, in particular, from $w$ norm conservation, it’s sensible to assume that $w_i(x,t)$ obeys to the relation:

$$w_i(x,t) = C_m(g^0_i, x - x_i, t)w^0_i$$

called here $w$-linearity. Consider now $n_s = 2$ case and suppose that $w_{1,2}(x,t)$ for emission from $Dx_{1,2}$ at some $t$ intersect largely, i.e. $|x_1 - x_2| \leq \sigma_x^1(t)$. What should be expected for the form of joint distribution $w(x,t)$ in that case ? If one prepares the statistical mixture of $g^0_i$ states $\{|g^m_0\}$, in that case the weight $w^0_i = P_i$ is the probability for $m$ to be in $Dx_i$, and in each individual event $m$ is emitted definitely by $Dx_1$ or $Dx_2$ at $t_0$, therefore for $g^m_0$ its structure is described by the relation (proposition):

$$LP^m := m \in Dx_1 \text{ or } m \in Dx_2$$

Consequently, the final $m$ distribution will be the additive sum

$$w^m(x,t) = w_1(x,t) + w_2(x,t) = \sum w^0_i C_m(g^0_i, x - x_i, t)$$

For the pure initial $m$ state $g_0$ the following proposition describes the source structure:

$$LP^p := m \in Dx_1 \text{ and } m \in Dx_2$$

It follows that $LP^m$ and $LP^p$ are incompatible:

$$LP^p \neq LP^m \text{ or } LP^e$$

for an arbitrary proposition $LP^e$ which describes also some $m$ signal. The incompatibility of $LP^p, LP^m$ indicates that the fuzzy source $S$ can’t be decomposed into the sum of the local nonintersecting sources $Dx_{1,2}$; the density $w_s(x,t)$ should have such form that it makes in principle impossible to represent $w_s$ as the sum of two components which describes the signals from $Dx_{1,2}$ sources. It should be maximally different from the mixture $w^m$, so the $w^m$ content in $w_s$ should be minimal, see lemma below. Therefore $w_s$ should include the nonlinear term $w_n$, for example, $w_n$ can be proportional to $\sqrt{w^0_1 w^0_2}$ feature of the fuzzy map.

Hence $g$ internal structure can be characterized by the parameter $l^g = 0,1$ for the mixed or pure states correspondingly. In both cases $w_s$ formulae for $n_s = 2$
should be applicable for an arbitrary \( w_i^0 \), in particular, if one of \( w_i^0 \to 0 \). Therefore it decomposed as:

\[
w_s(x,t) = w_1(x,t) + w_2(x,t) + l^g w_n(x,t) = w^m + l^g w_n
\]  

(2)

where \( w_n \) is FM nonlinear term. Due to \( w_s \) norm conservation, the resulting \( w_n \) should obey the constraint \( \int w_n dx = 0 \); it means that \( w_n \) oscillates around 0, and \( w_s \) around \( w^m \) correspondingly.

The importance of the statements formulated above demands to prove them formally:

**Lemma:** \( w_n(x,t) \) doesn’t contain any linear combination of \( w_i(x,t) \); because of that \( w^m \) content \( k_m \) in \( w_s \) is negligible. To prove the first proposition for \( n_s = 2 \) let’s suppose the opposite:

\[
w_n(x,t) = w'_n(x,t) + \sum B_i^l (x_i - x, t) w_i^0
\]

where \( w'_n \) - the true nonlinear component, \( B_i^l(z,t) \geq 0 \) are an arbitrary propagation functions. Then it means that :

\[
w_s(x,t) = \sum(C_m + B_i) w_i^0
\]

But this relation should be true also for \( w_i^0 = \delta_{il}; i = 1, 2 \), corresponding effectively to \( n_s = 1 \); in this case \( w_s = w^m \), and from that \( B_i = 0 \), so if \( w_n \neq 0 \), then \( w_n(x,t) \) is the nonlinear function of \( w_i^0 \). To demonstrate that \( w_n \neq 0 \) and \( k_m = 0 \), let’s rewrite \( w_s \) in the form:

\[
w_s(x,t) = k_m w^m(x,t) + w_a(x,t)
\]

for \( k_m \geq 0 \) and an arbitrary \( w_a(x,t) \geq 0 \) which describes the signal from the source \( S^a \) with an arbitrary structure \( LP^a \). The presence of \( w^m \) component in \( w_s \) implicates that: \( LP^p :L P^m .or. LP^a \), but it contradicts with the obtained \( LP^p, LP^m \) incompatibility. To demonstrate that \( |w_n| \) differs from 0, let’s consider its value inside \( w_s \) support \( E_s \), not including the points \( x_m \) in which \( w^m(x,t) = 0 \). If to assume that in the rest of \( E_s \) \( w_s(x) > 0 \) then \( w_s \) admits the solution with \( k_m > 0 \). To exclude it, it should exist at least one point \( x_e \) for which \( w_s(x_e) = 0 \), from that follows \( w_n(x_e) < 0 \). In other words the fuzzy map \( \Xi^f_i \) permits any \( w_s \) dependence on \( w_0 \) which conserve its norm, except the linear one, from which it should differ maximally. It’s natural to expect also that \( w_n \) is local term in a sense that

\[
w_n(x,t) = F[w_{1,2}(x,t)]
\]

notwithstanding the dependence on other \( g \) components and in our model we don’t assume it. However the weaker \( w_n \) locality property, which also follows from \( w_n \) nonlinearity, will be used: if at some \( x, t \) one of \( w_i(x,t) \to 0 \), then \( w_n(x,t) \to 0 \).
For illustration FM evolution can be compared with the classical diffusion [16]. Its state is described by $w(x, t)$ only, in this case for pointlike source in $x = 0$ one obtains:

$$w^D(x, t) = \frac{1}{2\sqrt{\pi k}} \exp -\frac{x^2}{4k^2}$$

where $k$ is the diffusion constant. Naturally for this process for any initial $w_n = 0$, yet $\sigma_x(t) \to \infty$ at $t \to \infty$. One should define also SS measure i.e. the criteria of signals separation - $R_{ss}$ for the evaluation of smearing rate; depending on it $R_{ss}$ can vary from 0 to 1. For $n_s = 2$ it depends on the rate of $w_1, w_2$ overlap:

$$R_w = \int \sqrt{w_1 w_2} dx$$

in any realistic situation $R_{ss} \leq R_w$, so to get the maximal SS it’s necessary that $R_w \to 1$ also. The general $R_{ss}$ ansatz is quite complicated [1], but $R_{ss}$ will be used in our formalism only in the asymptotic limits $R_{ss} \to 0$ or 1, in that case one can take $R_{ss} \simeq R_w$.

From this considerations we propose the simple toy-model of FM which helps to understand the main features of FM evolution. Let’s consider again TSE initial state $g_0$ at $t_0$ with $w^0$ located in $n_s = 2$ pointlike bins. As was argued, SS effects are generic for FM, and from that FM free evolution should be characterized by maximal SS, Hence it’s instructive to regard under which conditions it’s possible in FM. Obviously for $n_s = 2$ the maximal SS for an arbitrary, large $L_x = |x_1 - x_2|$ is achieved, if $w_i$ dispersion $\sigma_x^i(t)$ is as large as possible without violation of the theory consistency.

For example, if $w_i = w^D(x, t)$ of classical diffusion (but with resulting $w_n \neq 0$), then this property is obvious. In relativistic theory for an arbitrary localized $w_x$, the dispersion $\sigma_x^i(t)$ is restricted by the maximal signals velocity $c$, so that $\sigma^i_x \leq ct$. In 1-dimensions nonrelativistic theory nothing forbids to choose FM ansatz for $n_s = 1$ with $\sigma_x(t) \to \infty$ at finite $t$. In this case $w_i(x, t)$ should be Schwartz distribution (generalized function) for which at $x \to \pm \infty$, lim $w_i(x-x_i, t) \neq 0$ (or the limits don’t exist) [18]. This property is called $x$-limit condition and the class of distributions which obeys it is denoted $W^x$. In this case for $n_s = 2$ $R_{ss}$ can be independent of $L_x$, even for $L_x \to \infty$, so such theory doesn’t need the additional length parameters.

To sharpen our arguments let’s consider in this framework $n_s = 1$ case, so that $w_0^s(x) = \delta(x - x_0)$. The resulting $w_s(x, t)$ is the distribution and obeys $x$-limit condition, therefore $\bar{x}(t)$ and higher $x$-moments for it are undefined. Beside $x_0$, $\Xi^f_t$ can project to $w(x, t)$ $g_0$ internal structure. Yet as we assumed $g_0$ internal structure defined by its geometry, then for $n_s = 1$ $g_0$ has the trivial geometric structure of the ordered point $x_0$, which means that it has no internal structure at all, in fact. Therefore if only $x_0$ position defines $w_s(x, t)$ and $w_s(x, t) \neq 0$ at $|x| \to \infty$, then $w_s(x, t)$ should be a monotonously decreasing function of $\|x - x_0\|$. For example it
can be \( w_s \sim 1 + c \exp\left( -\frac{(x-x_0)^2}{d^2} \right) \); yet as this example illustrates, any such continuous monotonous function demands at least one length parameter \( d \) (which can also depend on \( t \)). Yet the minimal FM geometrical theory can be formulated without such fundamental length \( d \), in that case the only solution for such \( g_0 \) leads to \( w_s(x,t) = \text{const} \) at any \( t \). It assume that FM can include the length scaling - i.e. conformal invariance properties, and supports \( L_x \rightarrow \infty \) hypothesis. Meanwhile \( x_0 \) value should be eventually mapped to \( |g(t)| \), because for free FM evolution all \( g_0 \) parameters should be extracted from \( g(t) \). Thus, \( |g| \) should contain at least one more degree of freedom \( \hat{g} \) beside \( w(x) \), as was supposed above. Such \( g_0 \) free evolution at first sight seems quite exotic, remind yet that QM predicts the analogous evolution for the pointlike initial state \([17]\). Below this FM model prediction for \( n_s = 1 \) will not be used directly in the formalism construction, but it will be obtained eventually as the result of calculations.

Consider now \( n_s = 2 \) and the particular \( x_{1,2}, w_{1,2}^0 \), in our model \( w_s(x,t) \) of (2) obeys \( x \)-limit condition. Let’s choose the particular solution \( w_s(x,t) \) which responds to the maximal SS. Then \( w_s'(x,t) = w_s(x+a_x,t) \) also responds to it for an arbitrary \( a_x \), because \( R_{ss} \) depends of \( w_s \) form only, not of \( \bar{x} \). Therefore \( w_s' \) are also the solutions for \( g_0 \) evolution problem, if it depends only on \( w_0^0 \). If \( \sigma_x(t) \) is finite and \( \bar{x}(t) \) is well defined, then it unambiguously stipulated by the initial state and its dynamics; the example is the classical diffusion. But \( \bar{x}(t) \) and higher \( x \)-moments are undefined for \( w_s \) which obeys \( x \)-limit condition, and in that case only \( w_s \) form can depend on FM dynamics; thereon \( a_x \) value should be defined by the initial \( g_0 \) components, but the alternative solution with an arbitrary \( a_x \) is consistent also. This conclusion is especially obvious if \( w_i(x,t) \) are practically independent of \( x \), as our consideration of \( n_s = 1 \) supposes. \( w_s \) form can be characterized numerically by its fourier-transform and other methods which details are unimportant here \([20]\). It evidences that beside \( w_0(x) \), \( g_0 \) includes at least one more degree of freedom \( \hat{g} \). \( a_x \) depend on \( g \) both in \( x_1 \) and \( x_2 \), so if \( \hat{g} \) responds to \( a_x \) only, it is sensible to concede that it can be represented as the binary correlation \( K^f(x_1,x_2) \).

In the minimal theory \( K^f \) is an arbitrary real function of two variables, which is continuous or has the finite number of breaking points. Consequently, for \( n_s = 2 \)

\[ a_x = f(K^f); \]

if to choose the gauge: \( \forall x_c; K^f(x_c,x_c) = 0 \), then in 1-dimension one obtains for the fixed \( x_c \):

\[
K^f(x_d,x_c) = \int_{x_c}^{x_d} \frac{\partial K^f(\xi,x_c)}{\partial \xi} d\xi
\]

and from that:

\[
K^f(x_d,x_c) = K^f(x_d,x_c) - K^f(x_e,x_c)
\]

Therefore \( K^f \) is, in fact, the function of one observable: \( K^f(x,x_c) \rightarrow K(x) \). Because of it \( g \) can be regarded as the local field \( E^g(x) = \{w(x), K(x)\} \). To simplify the
evolution ansatz, we exploit the map \( \mathcal{OE}^g(x) \to g(x) \) which parameters are defined below. It transforms \( \mathcal{E}^g \) into the complex function \( g(x) = g_1(x) + ig_2(x) \), where \( g_{1,2} \) are the real functions. \( m \) density - \( w(x) = F_w(g(x)) \), where \( F_w \) is an arbitrary analytic function. If \( g \) is the local field, it’s natural to assume that \( w/g \) zero-equivalence holds: \( w(x) = 0 \iff g(x) = 0 \) and the same is true for \( w, g \) limits at \( x \to \infty \). In this case if \( x \)-limit condition fulfilled for \( w(x,t) \), then it’s true also for \( g(x,t) \) which should be also Schwartz distribution. Of course, the alternative reasons for the additional \( g \) degrees of freedom - \( K(x) \) appearance can exist, but in FM, as will be shown below, the proposed explanation is the most appropriate one. Note that even for the finite \( \sigma_x(t) \) such \( a_x \)-dependent ambiguous solution can exist in the theory but their ansatz will be more complicated [20].

Generally one should be careful with the interpretetation of \( w(x,t) \) distributions, as the measurable distributions of physical parameters, yet at this stage it’s admissible to regard them at the same ground as the standard, normalized distributions, as was demonstrated in [21]. Below we shall reconsider this problem. Regarded toy-model for \( m \) free evolution permit to assume that FM dynamics obeys the ‘principle of maximal fuzziness’ (or minimal ordering) which can be formulated as the following: at any \( t \) \( m \) state \( g(t) \) characterized by the density \( w(x,t) \) having the maximal \( x \) uncertainty \( \sigma_x(t) \) compatible with the initial \( g_0 \) structure. Consequently, for \( n_s = 1 \) \( g_0 \) has no internal structure and induces \( w(x,t) = \text{const} \) which has the maximal possible \( x \) uncertainty.

### 3 Particle Evolution in Fuzzy Mechanics

After the semiqualitative FM toy-model consideration we can turn to the fuzzy states \(|g⟩\) evolution formalism. From that discussion we concede that \( m \) state is described by a complex function \( g(x) \) for which \( w/g \) zero-equivalence holds. FM is assumed to be invariant under \( X,T \) shifts and it will be shown that this assumptions are enough to calculate \( m \) free evolution consitently. We shall assume also that for \( m \) free reversible evolution \( g(x,t) \) and its Fourier-transform \( \varphi(p,t) \) are continuous for \( t \geq t_0 \). We’ll argue that under this conditions \( g \) would evolve in accordance with Schrödinger free evolution operator \( U_s(t) \) and \( F_w(x,t) = |g(x,t)|^2 \). It would permit also to find the undefined \( O \) map parameters. In general \( g \) (or a state of any physical theory) free reversible evolution is described by the parameter-dependent operator \( U(t) \), so that: \( g(t) = U(t)g_0; \) \( U \), which correponds to \( \Xi^f_t \), isn’t supposed beforehand to be linear or unitary. However, it possesses the properties of group elements: \( \forall t_{1,2}; U(t_1 + t_2) = U(t_1)U(t_2) \), therefore for \( m \) free evolution \( U(t) = e^{-i\hat{H}_0 t} \) where \( \hat{H}_0 \) is an arbitrary constant operator [17]. Meanwhile the space shift operator \( V \) is equal to: \( V(a) = e^{ia\hat{p}} \), for the space shift of \( g(x,t) \); from that \( V(a) = e^{iap} \) when acting on \( \varphi(p,t) \) which is \( g(x,t) \) Fourier-transform [15]. The free evolution is
invariant relative to those $X$ shifts, because of it $U(t)$ commutes with $V(a)$ for the arbitrary $t,a$. It’s equivalent to the relation $[\dot{H}_0, p] = 0$, from which follows that $\dot{H}_0$ for $\varphi(p,t)$ is an arbitrary function of $p$: $\dot{H}_0 = F_0(p)$.

Consider now the initial state $g_0$ for $n_s = 1$: the w/g zero-equivalence at $t = t_0$ and the obvious condition $\varphi(p,t_0) \neq 0$ together result in $g_0(x) = c_0\delta(x - x_0)$ for $w_s^0 = \delta(x - x_0)$, where $c_0$ is an arbitrary complex number (below we settle $x_0 = 0, t_0 = 0$ for the simplicity). Then, from well-known $\delta(x - x_0)$ Fourier transform $\varphi_\delta(p) = e^{ipx_0}$ it follows:

$$\varphi(p,t) = c_0 U(t) \varphi_\delta = c_0 e^{-iF_0(p)t}$$

Let’s study under which conditions the transition $c_0 \delta(x) \to g(x,t)$ develops smoothly and continiuously. First, it demands that $g(x,t)$ constitutes $\delta$-sequence, so that $g(x,t_j) \to c_0 \delta(x)$ for any sequence $\{t_j\} \to +0$ [20]. It means that for an arbitrary function $\chi(x)$, which belongs to the class of main functions [16], one has:

$$I(\chi, t) = \int_{-\infty}^{\infty} \chi(x) g(x,t) dx \to \chi(0)$$

at $t \to +0$. It fulfilled only if $g(x,t)$ has $t = 0$ pole, so that $g(x,t)$ can be decomposed as: $g = g_s g_a$, where

$$g_s(x,t) = \frac{c_0}{f(t)} e^{i\gamma(z)},$$

with an arbitrary, complex $\gamma$; $f(t) \to 0$ at $t \to 0$ so that for the substitution $z = \frac{x}{f(t)}$ one has:

$$\int_{-\infty}^{\infty} g(z,t) f(t) dz \to 1$$

at $t \to +0$ [16]. $g_a$ is an arbitrary, nonsingular function with $g_a[zf(t), t] \to 1$ at $t \to +0$. Then under that conditions $g(x,t) \to \delta(x)$ at $t \to +0$. After $z$ substitution $g$ Fourier transform $\varphi$ alternatively can be represented as:

$$\varphi'(p,t) = c_0 \int_{-\infty}^{\infty} dz g_a[zf(t), t] e^{i\gamma(z) + izp} = \exp^{-i\Gamma[pf(t), t]}$$

(3)

Decomposing $g_a$ as the row in $t$, from the equivalence $\varphi(p,t) \simeq \varphi'(p,t)$ in the lowest $t$ order one obtains the equation:

$$\varphi(p,t) = e^{-iF_0(p)t} = \exp^{-i\Gamma[pf(t),0]}$$

(4)
from which follows $F_0(p) = \frac{p^2}{2m_0}$, $f(t) = d_t r^r$ with $rs = 1$, where $m_0, d_r$ are an arbitrary parameters. From that one finds $g_0(x, t) = 1$, $\Gamma(pf, t) = \Gamma(pf, 0)$, and the former equation holds true at any $t$. If $H_0 = F_0(p)$ is regarded as the free $m$ Hamiltonian, then from its symmetry properties the sensible $s$ values are only $s = 2n$, where $n$ are the natural numbers [15].

Let’s consider first the case $s = 2$, it follows that the free Hamiltonian is $H_0 = \frac{p^2}{2m_0}$ and $U(t)$ is the unitary operator for the real $m_0$. Therefore for $n_s = 1$ and $g_0(x) = e^{i\alpha_0} \delta(x - x_0)$ one obtains:

$$g(x, t) = G(x - x_0, t)e^{i\alpha_0} = \sqrt{\frac{m_0}{-i2\pi t}} e^{i\frac{m_0(x-x_0)^2}{2\pi} + \alpha_0}$$  \hspace{1cm} (5)

where for real, positive $m_0$ value the generalized function $G$ coincides with QM free particle propagator [21]; it defines $\gamma, f$ completely. Then in this formalism $m_0$ can be interpreted as the particle $m$ mass; note that for an imaginary $m_0$ such ansatz describes the classical diffusion. $g_0$ for the arbitrary $n_s$ can be written as :

$$g_0 = \sum_{i=1}^{n_s} \sqrt{w_i^0} \delta(x - x_i)e^{i\alpha_i^0}$$

where $\alpha_i^0 = K(x_i)$ are an arbitrary real constants. Obviously one can transfer from the sum with $n_s \to \infty$ to an arbitrary complex function for the initial state $g_0(x) = \sqrt{w_0(x)}e^{i\alpha_0(x)}$. In our formalism it evolves as:

$$g(x', t) = \int G(x' - x, t)g_0(x)dx = \sqrt{\frac{m_0}{-i2\pi t}} \int e^{i\frac{m_0(x'-x)^2}{2\pi}}g_0(x)dx$$ \hspace{1cm} (6)

which coincides with the free $g_0$ evolution in QM path integral formalism [21]. For such evolution ansatz one finds that the integral form $N_2 = \int |g(x, t)|^2 dx$ is time independent and can settle $N_2 = 1$, meanwhile in this case $F_w = |g|^2$ satisfies to $m$ flow conservation equation. Therefore $w(x) = F_w$ can be chosen as $m$ universal density, in particular, it permit to chose $c_0 = 1$.

Note that for $s = 2$, $g(x, t) \neq 0$ at $x \to \pm \infty$, i. e. satisfies to $x$-limit condition, as our minimal FM assumes. Yet it violated for free Hamiltonian with $s \geq 4$, in this case $g(x, t)$ asymptotics can be calculated [19] at $x \to \pm \infty$:

$$g(x, t) \simeq \frac{C_g}{t^2} \left( \frac{t}{x} \right)^{s-2} \exp i \frac{s-2}{2} m \frac{1}{2(x-1)^2} \frac{1}{x \pi(2-x)}$$

with $C_g$ - arbitrary constant. In particular, for $s = 4$, $|g| \sim \frac{1}{|x|^3}$. Therefore $g \to 0$ at $x \pm \infty$, so it contradicts to $x$-limit condition, and because of it the minimal FM assumptions are violated. In addition it can be shown that for $s \geq 4$ it’s impossible to construct $w(x) = F_w(g)$ as a nonnegative, local $g$ form which obeys the flow.
conservation. Therefore, all necessary conditions are fulfilled only for $s = 2$ (see also the calculations below). It turns out that the obtained $U(t)$ ansatz coincides with QM Schrödinger evolution operator $U_s(t)$ for the free $m$ evolution. Moreover, it agrees with the simple picture proposed in our FM toy-model. The analogous results for QM are obtained in the theory of the irreducible representations, but in that case they are based on more complicated axiomatics, which includes axiom of RFs Galilean invariance [15]. In distinction Galilean Invariance for $g$ states in different RFs wasn’t assumed in FM beforehand. It acknowledged in Quantum Physics that the classical massive objects, including physical RFs, can be regarded as the quantum objects in the limit $m \to \infty$ [15]. If such approach is correct in FM framework also, then regarding them as RFs, Galilean transformations for them can be derived from the obtained FM ansatz for $H_0$. Of course, this hypothesis needs further investigation, meanwhile in this approach it seems consistent. Obtained $F_c(g)$ for $n_s = 1$ gives $w(x, t) \sim \frac{1}{t}$ which describes constant $m$ density analogously to QM results for the initial state $g_0 = \delta(x)$ [21]. Therefore for $n_s = 2$

$$w = \frac{1}{t}(w_1^0 + w_2^0 + 2\sqrt{w_1^0 w_2^0} \cos[p_{12}(t)(x - \frac{x_1 + x_2}{2}) + \alpha_{12}]$$

where $p_{12}$ can be derived from (6). $w$ reproduces QM sources interference; it corresponds to the maximal SS with $R_{ss} = 1$ for $w_1^0 = w_2^0$. In physics formally the distributions has the meaning only as the functionals, so we should be careful with $w(x, t)$ interpretations. One can consider them also as the limit of the normalized observables distributions; for example, $w(x) = const$ is the limit of Gaussian with $\sigma_x \to \infty$. In this case $g(x, t)$ of (5) as the state originating from the narrow source can be substituted by the ansatz:

$$g'(x', t) = \int G(x' - x, t)e^{-\frac{x'^2}{\sigma^2_x}}dx$$

taken in the limit $\sigma_x \to 0$. Its main difference from $G(x, t)$ is that its norm $\int |g'(x', t)|^2dx' = 1$.

Now Hamilton formalism for FM can be formulated consistently. In our theory from $X$ space shift symmetry it follows that $m$ momentum is the operator $\hat{p} = i\frac{\partial}{\partial x}$ [15] in $x$-representation and the free Hamiltonian $\hat{H}_0 = \frac{\hat{p}^2}{2m_0}$. In FM the natural $U(t)$ evolution generalization for the $m$ potential interactions $V_m(x)$ is: $\hat{H} = \hat{H}_0 + V_m(x)$. From obtained relations it results in Schrödinger equation for $g$; the general path integral ansatz for $g$ can be obtained by means of Langrangian $L$ derived from $H$ for the given $V_e(x)$ [21]. The quantum phase $\alpha(x)$ properties acquires the natural description in FM framework: the real physical parameter is $K(x) \sim K^J(x, x')$ - the fuzzy correlation between $x, x'$, and $\alpha = K(x)$ is its local $x$ representation which is ambiguous up to $2n\pi$. 

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Any complex normalized function \( g(x) \) admits the orthogonal decomposition on \( |x_a\rangle = \delta(x - x_a) \), and \( |x_a\rangle \) set constitute the complete system [17]. Therefore \( \{g\} \) set \( M_s \) is equivalent to complex rigged Hilbert space \( \mathcal{H} \) with the scalar product \( g_1 \ast g_2 = \int g_1^* g_2 dx \). Consequently, our theory doesn’t need Superposition Principle as the independent axiom, its content follows from other FM axioms. In FM \( x \) is \( m \) observable and it’s sensible to suppose that \( \hat{p} \) and any Hermitian operator function \( \hat{Q}(x, p) \) is also \( m \) observable. For any such \( Q \) there are exists the corresponding complete system of orthogonal eigenvectors \( |q_a\rangle \) in \( \mathcal{H} \). It permit to assume that for FM measurements description QM reduction postulate for an arbitrary observable \( Q \) can be incorporated in FM copiously, so that the particular outcome of \( Q \) measurement \( q_i \) appears with the probability \( P = |\langle g|q_i\rangle|^2 \) [15].

Generalization of FM formalism on 3 dimensions is straightforward and will be regarded in the forthcoming paper; here we scratch only the main points. We assume that the signal \( g(x, t) \) from the point-like source \( w_0 = \delta(\vec{r} - vecr_0) \) possess the spherical symmetry. The correlation \( \bar{g} \) between two points \( \vec{r}_{1,2} \):

\[
K^f(\vec{r}_1, \vec{r}_2) = \int_l \frac{\partial K^f(\vec{r}, \vec{r}_2)}{\partial \vec{r}} d\vec{l}
\]

is supposed to be independent of the path \( l \) over which it calculated. Analogously to 1-dimensional case \( n_s = 2 \) we assume that for the state from two pointlike sources independently of the distance \( |\vec{r}_1 - \vec{r}_2| \) between them achieved the maximal SS (calculated over \( R^3 \)).

Note that Plank constant \( \hbar = 1 \) in our FM calibration alike it’s done in Relativistic QM; in any case it relates \( x, p \) scales in the regarded formalism [21]. We believe that the results obtained here can have the general meaning for QM axiomatics considerations independently of FM hypothesis. The proposed FM considers the nonrelativistic particle for which \( x \) is the fuzzy coordinate, yet from the symmetry of phase space one can choose any observable \( Q \) as the fundamental fuzzy coordinate and from this assumption to reconstruct FM formalism. It can be especially important in the relativistic case, where, in distinction from \( p, x \) can’t the proper observable [15]. Such approach, in principle, can be extended on any physical system. In particular, for the secondary quantization the numbers of certain particles \( N_c(p) \) can be regarded as the fuzzy values.

References

[1] S.Mayburov ‘Fuzzy Space-Time Geometry as Approach to Quantization’, Proc. of Quantum Foundations conference, (Vaxjio Univ. Press, Vaxjo, 2002), 233-242; hep-th 0210113
[2] S. Mayburov 'Fuzzy Geometry of Space-time and Quantum Dynamics' Proc. Steklov Math. Inst. 245, 154 (2004)

[3] Y. Aharonov, T. Kaufherr 'Reference Frames of Quantum Mechanics' Phys. Rev. D30, 368 - 381 (1984)

[4] S. Doplicher, K. Fredenhagen, K. Roberts 'Quantum Space-Time at Plank Scale' Comm. Math. Phys. 172, 187-199 (1995)

[5] C. Isham 'Introduction into Canonical Gravity Quantization' in: 'Canonical Gravity: from Classical to Quantum' Eds. J. Ehlers, H. Friedrich, Lecture Notes in Phys. 433, 11-27 (1994, Springer, Berlin)

[6] V. S. Vladimirov, I. V. Volovich 'P-adic Quantum Mechanics', Comm. Math. Phys. 123 - 132, 659 (1989)

[7] L. Zadeh 'Fuzzy Sets' Inform. and Control 8, 338 - 353 (1965)

[8] H. Bandemer, S. Gottwald 'Einfurlung in Fuzzy-Methoden' (Academie Verlag, Berlin, 1993)

[9] C. Zeeman, 'Topology of Brain and Visual Perception' in: 'Topology of 3-manifolds', ed. K. Fort, (Prentice-Hall, New Jersy, 1961), 240-256

[10] C. T. J. Dodson, 'Tangent Structures for Hazy Space' J. London Math. Soc., 2, 465 - 472 (1975)

[11] T. Poston 'Fuzzy Geometry', Manifold, 10, 25 (1971)

[12] T. Ali, G. Emch 'Fuzzy Observables in Quantum Mechanics' Journ. Math. Phys. 15, 176 - 182 (1974) ibid., 18, 219, (1977)

[13] J. Pykaz, 'Lukasiewics Operations on Fuzzy Sets' Found. Phys. 30, 1503 - 1519 (2000)

[14] M. Requardt, S. Roy 'Quantum Space-Time as Statistical Geometry of Fuzzy Lumps' Class. Quant. Grav 18, 3039 - 3058 (2001)

[15] J. M. Jauch, 'Foundations of Quantum Mechanics' (Adison-Wesly, Reading, 1968)

[16] V. S. Vladimirov 'Equation of Mathematical Physics' (Nauka, Moscow, 1971)

[17] F. Berezin, E. Shubin 'Schrödinger Equation' (Moscow, Nauka, 1985)

[18] L. Schwartz 'Methods Mathematique pour les Sciences Physique' (Hermann, Paris, 1961)
[19] M. Fedoriuk, 'Asymptotics: Integrals and raws' (Nauka, Moscow, 1987)

[20] R. Edwards 'Functional Analysis and Applications' (N-Y, McGraw-Hill, 1965)

[21] R. Feynman, A. Hibbs 'Quantum Mechanics and Path Integrals' (N-y, Mcgrow-Hill, 1965)