Effects of hydrodynamic and initial longitudinal fluctuations on rapidity decorrelation of collective flow

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Abstract
We investigate the interplay between hydrodynamic fluctuations and initial longitudinal fluctuations for their effects on the rapidity decorrelation of collective flow in high-energy nuclear collisions. We use a (3+1)-dimensional integrated dynamical model in which we combine initial conditions with longitudinal fluctuations, fluctuating hydrodynamics and hadronic cascades. We analyse the factorisation ratio in the longitudinal direction to study the effect of these fluctuations on the rapidity decorrelation. We find an essential difference between the effects of the hydrodynamic fluctuations and the initial longitudinal fluctuations in the centrality dependence of the factorisation ratios. A combination of the hydrodynamic fluctuations and the initial longitudinal fluctuations leads to reproduction of the centrality dependence of the second-order factorisation ratio, \( r_2(\eta_p^a, \eta_p^b) \), measured by the CMS Collaboration. Our model also qualitatively describes the centrality dependence of the third-order factorisation ratio, \( r_3(\eta_p^a, \eta_p^b) \). These results demonstrate the importance of the hydrodynamic fluctuations, as well as the initial longitudinal fluctuations, in understanding the longitudinal dynamics of high-energy nuclear collision reactions.

Keywords: quark–gluon plasma, relativistic fluctuating hydrodynamics, factorisation ratios

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High-energy nuclear collision experiments performed at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory and the Large Hadron Collider (LHC) at CERN aim at understanding the bulk and transport properties of the deconfined nuclear matter, the quark–gluon plasma (QGP) [1]. One of the major discoveries at RHIC is the large magnitude of the second-order azimuthal anisotropy [2, 3, 4, 5, 6, 7], also known as the elliptic flow [8]. The elliptic flow turned out to be consistent with the results from ideal hydrodynamic models [9, 10, 11, 12, 13, 14], which led to the development of later sophisticated dynamical models based on hydrodynamics including viscosity [15, 16, 17, 18, 19, 20, 21, 22] and hydrodynamic fluctuations [23, 24, 25, 26, 27, 28]. The large elliptic flow was observed also at LHC [29, 30, 31, 32], and later, higher-order anisotropic flows have been systematically measured at RHIC [33, 34] and LHC [30, 35, 36].

To understand collective flow phenomena more comprehensively, the correlation of the anisotropic flow has been studied through, e.g., the factorisation ratio which was initially proposed as a function of the transverse momentum [37] and extended in the longitudinal direction [38]. The longitudinal factorisation ratios are widely measured in experiments [38, 39, 40, 41, 42], where the rapidity decorrelation is observed as the factorisation breakdown. The longitudinal dynamics carries more information on the model-specific fluctuations that are independent of the geometric origin of the initial nucleon distributions, and thus the longitudinal factorisation ratio is one of the good measures for the model discrimination. These factorisation ratios are studied with various initialisation models [43, 44, 45, 46, 47, 48] and longitudinal-fluctuation mechanisms [49, 50, 51, 52, 25, 53, 54, 28]. Although these models exhibit the factorisation breakdown, none has quantitatively described all the measurements, including the centrality dependence, different harmonic orders and the collision energy dependence, in a single model setup. In our previous study with the fluctuating hydrodynamic model [28], we have shown the significance of the hydrodynamic fluctuations in understanding the longitudinal dynamics while the initial fluctuations of longitudinal profiles were missing there. Hydrodynamic fluctuations are thermal fluctuations of the hydrodynamic description whose power is determined by the fluctuation–dissipation relation [55, 56, 57, 58, 59, 60, 61]. Besides, the initial-state longitudinal fluctuations also play an important role in the final-state decorrelations [49, 50, 51, 52, 53, 54]. Therefore, in this Letter, we investigate both effects of the longitudinal fluctuations in the initial stage and the hydrodynamic fluctuations in the hydrodynamic stage on the rapid-
ity decorrelation.

In this Letter, we employ the integrated dynamical model with hydrodynamic fluctuations [62, 28], where the causal fluctuating hydrodynamic code rfh [24] is combined with the initialisation model and the cascade model JAM [63] with the prescription described in Ref. [64]. For the initialisation model, we newly implement the initial longitudinal fluctuations in the Monte-Carlo version of the Glauber model. The constitutive equations for the shear-stress tensor, $\pi^{\mu\nu}$, in the causal fluctuating hydrodynamics are chosen as [24, 61, 65]

$$\tau_0 \Delta^{\mu\nu}_{\alpha\beta} u^\lambda \partial_{\lambda} \pi^{\alpha\beta} + \pi^{\mu\nu} \left( 1 + \frac{4}{3} \tau_0 \partial_{\lambda} u^\lambda \right) = 2\eta \Delta^{\mu\nu}_{\alpha\beta} \partial^\alpha u^\beta + \xi^{\mu\nu},$$

(1)

where $\eta$ and $\tau_0$ are the shear viscosity and the relaxation time, respectively. The tensor $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is a projector for four-vectors onto the components transverse to the flow velocity $u^\mu$ where the sign convention for the metric is $g^{\mu\nu} = \text{diag}(+, -, -, -)$. The tensor $\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2} (\Delta^\mu_{\alpha} \Delta^\nu_{\beta} + \Delta^\mu_{\beta} \Delta^\nu_{\alpha}) - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta}$ is a projector for second-rank tensors onto the symmetric and traceless components transverse to the flow velocity. The symbol $\partial_{\mu}$ denotes the covariant derivative. The noise term $\xi^{\mu\nu}$ represents the hydrodynamic fluctuations and obeys the fluctuation–dissipation relation. In the Milne coordinates $(\tau, \eta_s, \bm{x}_\perp) := (\sqrt{t^2 - z^2}, \tanh^{-1}(z/t), x, y)$, the fluctuation–dissipation relation is written as

$$\langle \xi^{\mu\nu}(\tau, \eta_s, \bm{x}_\perp) \xi^{\alpha\beta}(\tau', \eta'_s, \bm{x}'_\perp) \rangle = 4\eta T \Delta^{\mu\nu}_{\alpha\beta} \cdot \frac{1}{\tau} \delta(\tau - \tau') \delta(\eta_s - \eta'_s) \delta(\bm{x}_\perp - \bm{x}'_\perp),$$

(2)

where $T$ is the temperature, and the angle brackets mean ensemble average, which is associated with the event average in the context of the high-energy nuclear collisions. Here, the Lorentz indices in Eq. (2) represent $\tau$, $\eta_s$, $x$ or $y$. In the actual calculations, we introduce the spatial cutoff to the hydrodynamic fluctuations by convoluting the Gaussian profile of widths $\lambda_\eta$ and $\lambda_\perp$ in the longitudinal and transverse directions, respectively. This effectively replaces the spatial delta function in Eq. (2) by the Gaussian of width $2\lambda_\eta$ and $2\lambda_\perp$. Smaller cutoff parameters result in larger effects of the hydrodynamic fluctuations. It is noted here that these cutoff parameters are specified in the coordinate space rather than in the momentum space in this Letter.

In the hydrodynamic simulations, the initial entropy density distributions, $s(\tau_0, \eta_s, \bm{x}_\perp)$, are needed at a fixed initial time $\tau_0$. For the event-by-event initial profiles, we utilise the Monte-Carlo version of the Glauber
(MC-Glauber) model \[66, 64\] combined with a general-purpose event generator, \textsc{Pythia} \[67\]. For each binary collision, we generate hadrons in a p+p collision using \textsc{Pythia}. Here, we neglect the difference between p+p and p+n / n+n binary collisions that happen in real nuclear collisions for simplicity. If we would simply sum up all the hadrons from the binary collisions, the multiplicity would scale with the number of the binary collisions, \(N_{\text{coll}}\), which would be plausible only in high-transverse-momentum (high-\(p_T\)) regions. On the other hand, yields of low-transverse-momentum (low-\(p_T\)) hadrons are expected to scale with the number of participants, \(N_{\text{part}}\). To embody this scaling behaviour throughout the whole transverse momentum regions, we perform a rejection sampling for these generated hadrons with the momentum-dependent acceptance probability \(w(Y, p_T)\) \[68, 69\]:

\[
w(p_T, Y) = w(Y) \times \frac{1}{2} \left[ 1 - \tanh \left( \frac{p_T - p_{T0}}{\Delta p_T} \right) \right] + \frac{1}{2} \left[ 1 + \tanh \left( \frac{p_T - p_{T0}}{\Delta p_T} \right) \right],
\]

\[
w(Y) = \frac{Y_b + Y}{2Y_b} \left( 1 + \frac{1}{n_A} \right) + \frac{Y_b - Y}{2Y_b} \left( 1 - \frac{1}{n_B} \right).
\]

Here, \(p_T\) and \(Y\) are the transverse momentum and the rapidity of the hadron, respectively. We introduce the parameters \(p_{T0}\) and \(\Delta p_T\) to smoothly separate low- and high-\(p_T\) regions into the first and second terms, respectively, so that the total number of accepted hadrons scales with \(N_{\text{part}}\) (\(N_{\text{coll}}\)) in the low- (high-) \(p_T\) region. The symbol \(Y_b\) denotes the beam rapidity, and \(n_A\) (\(n_B\)) is the number of binary collisions that the nucleon of the current binary collision at positive (negative) beam rapidity experiences. The scaling with \(N_{\text{part}}\) is implemented by the function \(w(Y)\). This function \(w(Y)\) also brings rapidity-dependent yields of hadrons \[70, 71, 72\], which shares the idea with that of the wounded nucleon model \[73\]. The free parameters \(p_{T0}\) and \(\Delta p_T\) are later tuned to reproduce the centrality dependence of the charged-particle multiplicity.

We assume the initial entropy-density distribution is proportional to the number distribution of the accepted hadrons:

\[
s(\tau_0, \eta, x_\perp) = \frac{K}{\tau_0} \sum_i \frac{1}{\sqrt{2\pi \sigma_\eta^2}} \frac{1}{2\pi \sigma_\perp^2} \exp \left[ -\frac{(x_\perp - x_{\perp i})^2}{2\sigma_\perp^2} - \frac{(\eta - \eta_i)^2}{2\sigma_\eta^2} \right],
\]

where the model parameter \(K\) controls the overall normalisation. The longitudinal position of the \(i\)-th hadron at the initial time \(\tau_0\) is determined as
\[ \eta^i_b = Y^i \]  

The initial transverse position of the \( i \)-th hadron \( \mathbf{x}^i_\perp \) is randomly sampled in the uniform disk of the radius \( \sqrt{\sigma_{\text{NN}}^{\text{in}}/\pi} \) and the centre \( \mathbf{x}^i_{\text{centre}} \). Here, \( \sigma_{\text{NN}}^{\text{in}}(\sqrt{s_{\text{NN}}}) \) is the inelastic-scattering cross section of nucleons, and we place the centre at

\[
\mathbf{x}^i_{\text{centre}} = \frac{\mathbf{x}^i_A + \mathbf{x}^i_B}{2} + \frac{\mathbf{x}^i_A - \mathbf{x}^i_B}{2Y_b^i} \eta^i_b, \tag{6}
\]

where \( \mathbf{x}^i_A (\mathbf{x}^i_B) \) is the position of the associated nucleon at the positive (negative) beam rapidity. Here we assumed that a hadron is produced around a line in the Milne coordinates connecting the two nucleons of the associated binary collision.

We put the Bjorken scaling solution \([74]\) for the initial flow velocity \( u^\mu(\tau_0, \eta_b, \mathbf{x}^\perp) = (u^\tau, u^\eta, u^x, u^y) = (1, 0, 0, 0) \), which means that we ignore the fluctuations of initial flows in this study. For the equation of state, we employ a lattice-based model, \( s95p-v1.1 \) \([75]\), in which the list of hadrons in the hadron resonance gas model is taken from the cascade model \( \text{JAM} \) \([63]\). At the switching temperature \( T_{\text{sw}} \), we change the description from the macroscopic hydrodynamics to the microscopic kinetic theory using the Cooper–Frye formula \([76]\). The subsequent space-time evolution of hadrons is described by using the cascade model \( \text{JAM} \) \([63]\). Further details of the integrated dynamical model with fluctuating hydrodynamics can be found in Refs. \([24, 61, 28]\).

Using this model, we perform simulations for Pb+Pb collisions at \( \sqrt{s_{\text{NN}}} = 2.76 \text{ TeV} \). For the reference setup to see the effects of hydrodynamic fluctuations, we also perform the simulations of viscous hydrodynamics by turning off the hydrodynamic fluctuations. For each hydrodynamic model (fluctuating hydrodynamics and viscous hydrodynamics), we gain 4 000 minimum-bias hydrodynamic events. For each hydrodynamic event, we perform 100 independent particlisation and hadronic cascade simulations to reduce the computational cost. In total we obtain the 400 000 (= 4 000 \times 100) events for the subsequent analyses.

Let us here summarise the parameters of the present study. Following the previous calculations \([62, 64]\), we set the specific shear viscosity \( \eta/s = 1/4\pi \) \([77]\), the relaxation time \( \tau_\pi = 3/4\pi T \) \([78, 65]\), the initial proper time \( \tau_0 = 0.6 \text{ fm} \) and the switching temperature \( T_{\text{sw}} = 155 \text{ MeV} \). We fix the widths of the hadron profile as \( \sigma_\perp = 0.3 \text{ fm} \) and \( \sigma_\eta = 0.3 \text{ at present} \). We tune initial parameters \( K, p_{T0} \) and \( \Delta p_T \) for each hydrodynamic model to reproduce centrality dependence of charged-particle multiplicity measured
by the ALICE Collaboration \cite{79}. For the fluctuating hydrodynamics, we tune the cutoff parameters $\lambda_\perp$ and $\lambda_\eta$ to roughly reproduce the factorisation ratio $r_2(\eta^a_\p, \eta^b_\p)$ measured by the CMS Collaboration \cite{38}. In the present study we assume $\lambda_\perp/\text{fm} = \lambda_\eta$ for simplicity. These parameters for each hydrodynamic model are summarised in Table 1. The parameter $K$ controls the overall magnitude of multiplicity per participant pair. On the other hand, the parameter $p_{T0}$ controls not only the overall magnitude but also the slope of multiplicity per participant pair. The parameter $p_{T0}$ is slightly larger in fluctuating hydrodynamics because the entropy production in the hydrodynamic stage in the peripheral collisions is larger with fluctuating hydrodynamics \cite{28}.

| Model               | $\lambda_\perp$ (fm) | $\lambda_\eta$ | $K$ | $p_{T0}$ (GeV) | $\Delta p_T$ (GeV) |
|---------------------|----------------------|----------------|-----|----------------|-------------------|
| Viscous hydro       | N/A                  | N/A            | 4.8 | 1.75           | 1.0               |
| Fluc. hydro-$\lambda$ | 2.0                  | 2.0            | 4.8 | 1.80           | 1.0               |

Figure 1: (Colour Online) Charged-hadron multiplicity normalised by the number of the participant pair $(dN_{\text{ch}}/d\eta)/ (N_{\text{part}}/2)$, as a function of the number of participants. Results from viscous hydrodynamics with initial longitudinal fluctuations (open circle) and fluctuating hydrodynamics with initial longitudinal fluctuations (filled triangle) are compared with experimental data (open diamond) obtained by the ALICE Collaboration \cite{79}.

Figure 1 shows $N_{\text{part}}$ dependence of charged-hadron multiplicity $dN_{\text{ch}}/d\eta_\p$ per participant pair $N_{\text{part}}/2$ at midrapidity $|\eta_\p| < 0.5$ in Pb+Pb collisions.
at $\sqrt{s_{\text{NN}}} = 2.76$ TeV. Within our initialisation model, the violation of $N_{\text{part}}$ scaling in multiplicity in Fig. 1 is described by the initial (semi-)hard components introduced through the acceptance probability Eq. (3).

The initial longitudinal fluctuations and the hydrodynamic fluctuations, which follow Eqs. (1) and (5), respectively, randomly disturb the correlations such as alignment of the event planes along rapidity. We analyse factorisation ratios [37] to see the effects of these fluctuations on (de-)correlation of event-plane angles along rapidity. The factorisation ratio in the longitudinal direction is defined as

$$r_n(\eta^a_p, \eta^b_p) = \frac{V_n \Delta (-\eta^a_p, \eta^b_p)}{V_n \Delta (\eta^a_p, \eta^b_p)}, \quad V_n \Delta = \langle \cos(n \Delta \phi) \rangle. \tag{7}$$

Here $V_n \Delta$ is the Fourier coefficients of two-particle correlation functions at the $n$-th order and $\Delta \phi$ is a difference of the azimuthal angles between two charged hadrons in separated pseudorapidity regions, $\eta^a_p$ and $\eta^b_p$. If one could factorise the two-particle correlation functions in the denominator and numerator in Eq. (7) into two anisotropic flows with corresponding momentum regions, e.g., $V_n \Delta (\eta^a_p, \eta^b_p) = v_2(\eta^a_p)v_2(\eta^b_p)$, the resultant factorisation ratio would be equal to unity in symmetric collisions. Contrarily, when the event-plane angle and the flow magnitude fluctuate as functions of pseudorapidity, the ratio becomes smaller than unity in general because one cannot factorise the two-particle correlation functions. This decrease of the factorisation ratio from the unity is called the flow decorrelation in the longitudinal direction.

In the following analyses, the two-particle correlation function is calculated by changing the rapidity region of the hadrons within $0 < \eta^a_p < 2.5$ while fixing the region of the reference hadrons to be $3.0 < \eta^b_p < 4.0$ following the experimental setup by the CMS Collaboration [38].

Figure 2 shows the factorisation ratio in the longitudinal direction, $r_2(\eta^a_p, \eta^b_p)$, in Pb+Pb collisions at $\sqrt{s_{\text{NN}}} = 2.76$ TeV for 0–5% and 20–30% centralities compared with the experimental data by the CMS Collaboration [38]. The cutoff parameters $\lambda_\perp/\text{fm} = \lambda_\eta = 2.0$ in Table 1 have been here determined to reproduce the experimental results of $r_2(\eta^a_p, \eta^b_p)$ of the 20–30% centrality. These values are larger than $\lambda_\perp/\text{fm} = \lambda_\eta = 1.0$ and 1.5 (parameters $\lambda_1.0$ and $\lambda_1.5$, respectively) of the previous work [28]. This is because the effect of newly introduced initial longitudinal fluctuations needs to be compensated by reducing the effect of the hydrodynamic fluctuations to keep the factorisation ratios the same.
Figure 2: Factorisation ratio $r_2(\eta^a_p, \eta^b_p)$ in Pb+Pb collisions at $\sqrt{s_{\text{NN}}} = 2.76$ TeV for (a) 0–5% and (b) 20–30% centralities. The rapidity region for reference is $3.0 < \eta^b_p < 4.0$. Open circles and filled triangles are results from the viscous and fluctuating hydrodynamic models, respectively. The experimental data of $r_2(\eta^a_p, \eta^b_p)$ from the CMS Collaboration [38] are shown by open diamonds.

The experimental data is close to unity at small $\eta^a_p$ and decreases with increasing $\eta^a_p$. The factorisation ratio $r_2(\eta^a_p, \eta^b_p)$ from the viscous hydrodynamic model with initial longitudinal fluctuations decreases with increasing $\eta^a_p$. However, the decorrelation is weaker than experimental data in particular at the large rapidity gap $\eta^a_p \sim 1.5–2.5$. Since viscous hydrodynamics tends to keep the long-range correlation in the rapidity direction [28], the decreasing behaviour of $r_2(\eta^a_p, \eta^b_p)$ can be attributed to the initial longitudinal fluctuations [54, 51, 50, 52, 53]. In the fluctuating hydrodynamic model, the decorrelation is stronger than in the viscous hydrodynamic model, which is the same trend as in the previous analysis without the initial longitudinal fluctuations in Ref. [28].

Figures 3 (a) and (b) compare the centrality dependence of $r_2(\eta^a_p, \eta^b_p)$ from hydrodynamic models with and without initial longitudinal fluctuations, respectively. We find that the experimental data of the centrality dependence of $r_2(\eta^a_p, \eta^b_p)$ can only be reproduced by the model with both the hydrodynamic fluctuations and the initial longitudinal fluctuations. This can be understood from the different centrality dependence between the effects of the initial longitudinal fluctuations and the hydrodynamic fluctuations.

Within our model, the genuine initial longitudinal fluctuations tend to decrease $r_2(\eta^a_p, \eta^b_p)$ in central collisions (particularly in 0–10% centrality), as
Figure 3: (Colour Online) Centrality dependence of factorisation ratio $r_2(\eta_a, \eta_b)$ in Pb+Pb collisions at $\sqrt{s_{NN}} = 2.76$ TeV. The rapidity regions of two-particle correlation functions are taken as $2.0 < \eta_a < 2.5$ and $3.0 < \eta_b < 4.0$. (a) The results from hydrodynamic models with initial longitudinal fluctuations are compared with the experimental data [38]. (b) The results from hydrodynamic model without initial longitudinal fluctuations (taken from Fig. 6 in Ref. [28]) are also shown for comparison. In the right panel (b), we show the result from viscous hydrodynamics and two results from the fluctuating hydrodynamics with different cutoff parameters $\lambda_{1.5}$ (filled triangle) and $\lambda_{1.0}$ (open triangle), which correspond to the cutoff $\lambda_{1}/\text{fm} = \lambda_n = 1.5$ and 1.0, respectively [see Ref. [28] for the other parameters used in panel (b)]. The symbols are the same as in Fig. 2 for other plots.
seen in the open circles in Fig. 3 (a). This is because the flow correlation along the rapidity direction can be easily broken in the central collisions as the magnitude of the elliptic flow driven by the collision geometry is small there. On the other hand, the genuine hydrodynamic fluctuations tend to decrease $r_2(\eta^a_p, \eta^b_p)$ in both central collisions (0–10% centrality) and peripheral collisions (40–60% centrality) as seen in Fig. 3 (b). The mechanism of the decorrelation in central collisions is the same as with the initial longitudinal fluctuations. The decorrelation in the peripheral collisions can be attributed to the nature of the hydrodynamic fluctuations being significant in small and short-lived systems, i.e., the magnitude of the averaged thermal fluctuations at the linear order scales as $(Vt)^{-1/2}$ where $V$ and $t$ are the typical volume and lifetime of the system. If we do not take account of initial longitudinal fluctuations, the factorisation ratio $r_2(\eta^a_p, \eta^b_p)$ of the experimental data can be well fitted by parameter $\lambda_1.0$ (open triangles) and $\lambda_1.5$ (filled triangles) in central collisions and peripheral collisions, respectively, but the overall centrality dependence cannot be reproduced by a single cutoff parameter.

Neither the fluctuating hydrodynamics without the initial longitudinal fluctuations with fixed $\lambda$ parameters nor the viscous hydrodynamics with the initial longitudinal fluctuations can reproduce the centrality dependence of the experimental data. Thus we conclude that both the initial longitudinal fluctuations and the hydrodynamic fluctuations must be taken into consideration in understanding the centrality dependence of the factorisation ratios.

![Figure 4: (Colour Online) Centrality dependence of factorisation ratio $r_3(\eta^a_p, \eta^b_p)$ in Pb+Pb collisions at $\sqrt{s_{NN}} = 2.76$ TeV. The rapidity regions of two-particle correlation functions are taken as $2.0 < \eta^a_p < 2.5$ and $3.0 < \eta^b_p < 4.0$. The symbols are the same as in Fig. 1. The experimental data are obtained by the CMS Collaboration [38].]
Figure 4 shows the centrality dependence of $r_3(\eta^a_P, \eta^b_P)$ in Pb+Pb collisions at $\sqrt{s_{NN}} = 2.76$ TeV. The factorisation ratio $r_3(\eta^a_P, \eta^b_P)$ in experimental data depends mildly on centrality. This is because the triangular flow is dominated by fluctuations in symmetric collision systems. The factorisation ratio $r_3(\eta^a_P, \eta^b_P)$ from the viscous hydrodynamic model with initial longitudinal fluctuations decreases with increasing centrality percentage and is larger than the experimental data in all centralities. In contrast, $r_3(\eta^a_P, \eta^b_P)$ obtained from the fluctuating hydrodynamic model has less centrality dependence and is also close to experimental data. Although more sophisticated modelling would be required to perfectly reproduce the experimental data of both $r_2(\eta^a_P, \eta^b_P)$ and $r_3(\eta^a_P, \eta^b_P)$ simultaneously, the centrality dependences of $r_2(\eta^a_P, \eta^b_P)$ and $r_3(\eta^a_P, \eta^b_P)$ are better reproduced by including both the hydrodynamic fluctuations and the initial longitudinal fluctuations. These results suggest that considering either the initial longitudinal fluctuations or the hydrodynamic fluctuations is insufficient in understanding the decorrelation of anisotropic flows in the longitudinal directions and thus that considering both simultaneously in a model is important.

In this Letter, we investigated the effects of the longitudinal fluctuations in the initial stage and the hydrodynamic fluctuations in the expansion stage on the longitudinal factorisation ratios to understand the rapidity decorrelation of anisotropic flows. We employed an integrated dynamical model which consists of the initialisation model for fluctuations in both longitudinal and transverse profiles, the hydrodynamic model, rfh, for causal hydrodynamic fluctuations and dissipations, and the hadronic cascade model, JAM, for the final-state interactions. We performed simulations of Pb+Pb collisions at $\sqrt{s_{NN}} = 2.76$ TeV using the integrated dynamical model with or without the hydrodynamic fluctuations for comparison. We fixed the model parameters so that our model fairly reproduces the experimental data of the centrality dependence of charged-hadron multiplicity normalised by the number of participants. With these model settings, we calculated the factorisation ratio $r_n(\eta^a_P, \eta^b_P)$ ($n = 2, 3$) in the longitudinal direction and its centrality dependence. We confirmed that both the initial longitudinal fluctuations and the hydrodynamic fluctuations decrease the factorisation ratio, which is consistent with the previous studies. However, we found that the centrality dependence of the effects of hydrodynamic fluctuations is different from those of initial longitudinal fluctuations. We reproduce the centrality dependence of the factorisation ratio at the second order, $r_2(\eta^a_P, \eta^b_P)$, from the CMS collaboration only after including both the initial longitudinal and the hydrodynamic fluctuations.
fluctuations. We also qualitatively described the centrality dependence of the factorisation ratio at the third order, \( r_3(\eta^a_p, \eta^b_p) \), with these two fluctuations. These analyses show that incorporating both the initial longitudinal and the hydrodynamic fluctuations simultaneously in a dynamical model is the key to quantitatively understanding the decorrelation dynamics in the longitudinal direction.

**Acknowledgement**

This work was supported by JSPS KAKENHI Grant Numbers JP18J22227 (A.S.) and JP19K21881 (T.H.). K.M. was supported by the NSFC under Grant No. 11947236.

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