Super-resolution for a point source better than $\lambda/500$ using positive refraction

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Abstract. Leonhardt (2009 New J. Phys. 11 093040) demonstrated that the two-dimensional (2D) Maxwell fish eye (MFE) lens can focus perfectly 2D Helmholtz waves of arbitrary frequency; that is, it can transport perfectly an outward (monopole) 2D Helmholtz wave field, generated by a point source, towards a ‘perfect point drain’ located at the corresponding image point. Moreover, a prototype with $\lambda/5$ super-resolution property for one microwave frequency has been manufactured and tested (Ma et al 2010 arXiv:1007.2530v1; Ma et al 2010 New J. Phys. 13 033016). However, neither software simulations nor experimental measurements for a broad band of frequencies have yet been reported. Here, we present steady-state simulations with a non-perfect drain for a device equivalent to the MFE, called the spherical geodesic waveguide (SGW), which predicts up to $\lambda/500$ super-resolution close to discrete frequencies. Out of these frequencies, the SGW does not show super-resolution in the analysis carried out.

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1. Introduction

‘Super-resolution’ stands for the capacity of an optical system to produce images with details below the classic Abbe diffraction limit. In the last decade, super-resolution has been shown experimentally [1, 2] with devices made of left-handed materials (i.e. materials with negative dielectric and magnetic constants) [3, 4]. Unfortunately, high absorption and small (wavelength scale) source-to-image distance are both present in these experiments. In fact, it was claimed in [5] that this is inevitable with real negative refraction materials, although recent arguments rebut it [6]. Nevertheless, these devices have been claimed to reach the limit case of infinite super-resolution theoretically [3], known as ‘perfect imaging’ (which stands for the capacity of an optical system to produce images with details not limited by the wavelength of light). This perfect imaging concept should not be confused with the classic perfect imaging definition of geometrical optics [9]. In this paper, we will only use the former definition.

An alternative device for perfect imaging has recently been proposed [7, 8]: the Maxwell fish eye (MFE) lens. Unlike previous perfect imaging devices, MFE uses materials with a positive, isotropic refractive index distribution. This device is very well known in the framework of geometrical optics because it is an absolute instrument [9], so every object point has a stigmatic image point.

Leonhardt [7] analyzed Helmholtz wave fields in the MFE lens in two dimensions (2D). These Helmholtz wave fields describe TE-polarized modes in a cylindrical MFE, i.e. modes in which the electric field vector points orthogonally to the cross section of the cylinder. Leonhardt found a family of Helmholtz wave fields that have monopole asymptotic behavior at an object point as well as at its stigmatic image point. Each one of these solutions describes a wave propagating from the object point to the image point. It coincides asymptotically with an outward (monopole) Helmholtz wave at the object point, as generated by a point source, and with an inward (monopole) wave at the image point, as it was sunk by an ‘infinitely well localized drain’ (which we call a ‘perfect point drain’). This perfect point drain absorbs the incident wave, with no reflection or scattering. This result has also been confirmed via a different approach [10].

The physical significance of a passive perfect point drain has been controversial [11–20]. In [7, 10], the perfect point drain was not physically described, but only considered as a
Figure 1. Spherical guide wave (SGW) analyzed in this paper. The SGW is bounded by two spherical shells made of perfect conductors. On the left, the dark gray area is the inner metallic sphere and the light gray area is the outer one. On the right, the electric field inside the guide for point source and perfect point drain placed on opposite poles.

mathematical entity (a point drain is represented by Dirac-delta as the point source). However, a rigorous example of a passive perfect point drain for the MFE has recently been found, clarifying the controversy [21]. It consists of a dissipative region whose diameter tends towards zero and whose complex permittivity \( \varepsilon \) takes a specific value depending on the size of the drain as well as the operation frequency.

Two sets of experiments have recently been carried out to support the super-resolution capability in the MFE. In the first one, super-resolution with positive refraction has been demonstrated for the very first time at a microwave frequency (\( \lambda = 3 \text{ cm} \)) [22, 23]. In this experimental setup, a 2D MFE medium was assembled as a planar waveguide 5 mm thick with concentric layers of copper circuit board and dielectric fillers forming the desired refractive index profile of the MFE. This profile is made only up to a diameter of 10 cm index and limited by a metallic mirror, shaping a device called an MFE mirror [7, 23] that has ideal properties similar to those of the MFE. Sources and drains were built as coaxial probes, inserted through the bottom plate. The experimental results showed that two sources with a distance of \( \lambda/5 \) from each other (where \( \lambda \) denotes the local wavelength \( \lambda = \lambda_0/n \)) could be resolved with an array made up of ten drains spaced \( \lambda/20 \), which exceeded the \( \sim \lambda/2.5 \) classic diffraction limit [22]. Results with closer sources were not reported, but it should be noted that this experiment was limited to the resolution of the array of drains.

The second set of experiments has been carried out for the near-infrared frequency (\( \lambda = 1.55 \text{ \mu m} \)), but resolution below the diffraction limit was not found [24]. In this case, the planar waveguide is filled with a medium made of nano-cylinders of silicon with a different radius and separation between them, so they emulate the MFE distribution. The authors assume that the failure in the experimental demonstration is due to manufacturing flaws in the prototype.

We show here the results of the analysis for microwaves of the spherical geodesic waveguide (SGW), a device suggested in [25] for perfect imaging with waves (see figure 1). The SGW is a spherical waveguide filled with a non-magnetic material and isotropic refractive
index distribution proportional to $1/r$ ($\varepsilon = (r_0/r)^2$ and $\mu = 1$), $r$ being the distance to the center of the spheres. Transformation optics theory [26] proves that the TE-polarized electric modes of the cylindrical MFE [7] are transformed into radial-polarized modes in the SGW, so both have the same imaging properties. Our simulation only considers steady state, i.e. does not incorporate transients.

When the waveguide thickness is small enough, the variation of the refractive index within the two spherical shells can be ignored, resulting in a constant refractive index within the waveguide. This is obviously very attractive from the practical point of view. The waveguide can be manufactured with just two concentric metallic spheres separated at a distance much less than the radius and constant refraction index between them. Two devices (one with gradient index $\propto 1/r$ and another with constant index) have been analyzed in this paper, with the source and the drain implemented with coaxial probes. This drain, which is not the perfect drain described previously, will be detailed later in section 3.

The equivalence between the MFE and SGW is reviewed in section 2. The microwave circuit model of that spherical waveguide is described in section 3. Results showing up to $\lambda/500$ super-resolution close to a specific set of frequencies are given in section 4. In section 5, we present a comparative analysis between the results described in section 4 and the experimental results of the prototypes presented in [22–24]. The conclusions are given in section 6.

2. Maxwell fish eye lens and spherical geodesic waveguide

An MFE is a lens with the following refraction index distribution:

$$n(\rho) = \frac{2n_0}{1 + (\rho/a)^2},$$

(1)

where $\rho$ is the distance to the origin, which in the 2D case is $\rho^2 = x^2 + y^2$. Within the geometrical optics framework, the rays emitted from an arbitrary object point $(x_0, 0)$ will be stigmatically imaged onto its image point $(-a^2/x_0, 0)$. The wave propagating from an object point $(x_0, 0)$ onto its image point $(-a^2/x_0, 0)$ found by Leonhardt [7] (which is known as the forward running-wave Legendre function [27]) is given by

$$E(x, y, z) = E_z(x, y)z = A \left(P_v(\zeta) + i \frac{2}{\pi} Q_v(\zeta)\right) z,$$

$$\left(\frac{a}{n_0}k_0\right)^2 = v(v + 1), \quad k_0 = 2\pi f \sqrt{\mu_0 \varepsilon_0},$$

(2)

$$\zeta = \frac{\left|z'\right|^2 - a^2}{\left|z'\right|^2 + a^2}, \quad z' = \frac{z - x_0}{z(x_0/a^2) + 1}, \quad z = x + iy.$$  

Here $A$ is a complex constant, and $P_v, Q_v$ are the Legendre functions [28]. The field $E_z$ diverges at the points $(x_0, 0)$ and $(-a^2/x_0, 0)$, i.e. at both the source (located at the object point) and the drain (located at the image point). Besides the point source, this solution requires a perfect point drain, which is a theoretical concept that can be modeled as an infinitely small region centered around the image point and with a particular complex permittivity distribution [21].

Using transformation optics it was proven that the fields given by equation (2) in a 2D MFE are transformed into radial fields in the SGW filled with a refractive index medium with law

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\( n(r) = an_0/r \) (where \( r^2 = x^2 + y^2 + z^2 \)), see figure 1. The radial field \( E(r, \theta, \phi) = E_r(r, \theta, \phi)r \) in the SGW is related to the MFE field (equation (2)) of the corresponding point by

\[
E_r(r, \theta, \varphi) = E_z(\zeta), \quad \zeta = \cos \theta.
\]

(3)

Corresponding points in the MFE and SGW are related by a stereographic projection. Source and drain points \((x_0, 0)\) and \((-\frac{a^2}{x_0}, 0)\) are transformed into opposite poles of the SGW. The forward running-wave Legendre function is transformed into a wave with rotational symmetry with respect to the line passing through the object and image points, as shown in figure 1. The perfect point drain complex permittivity distribution is transformed into another one.

When the drain of the SGW is not perfect but still rotationally symmetric, the field can be expressed as follows [21]:

\[
E_r(r, \theta, \varphi) = E_r(\zeta) = AF_\nu(\zeta) + BR_\nu(\zeta),
\]

(4)

where

\[
F_\nu(\zeta) = P_\nu(\zeta) + i\frac{2}{\pi}Q_\nu(\zeta),
\]

\[
R_\nu(\zeta) = P_\nu(\zeta) - i\frac{2}{\pi}Q_\nu(\zeta).
\]

(5)

\( F_\nu \) is the forward running-wave Legendre function and \( R_\nu \) is the reverse one [27]. \( \zeta \) is given in equation (3).

3. Microwave circuit and parameters of the simulation

The SGW is bounded by two spherical shells made of conductors. The media between shells has the refractive index distribution proportional to \( 1/r \), as said before. Two coaxial probes loaded with their characteristic impedance have been used to simulate the source and the drain in the SGW. The microwave prototype described in [22, 23] was made in the same way. We shall call them the source port and the drain port, respectively. This drain port does not emulate a perfect point drain; that is, it causes a reversed wave \((B \neq 0\) in equation (4)) (i.e. there is no full absorption of the forward wave). Using this non-perfect drain port is interesting because it is easier to manufacture than the perfect drain [21].

Figure 2 shows the cross section of the SGW with the two coaxial probes simulating the point source and the point drain. The radiation is injected through the source port, guided between the spheres, and may (or not) be extracted from the SGW through the drain port. When the angle \( \theta_d = \pi \), the drain port is located at the image point and the fields will have rotational symmetry. For \( \theta_d \neq \pi \), the rotational symmetry is broken.

The frequencies used in the analysis are low enough that only transversal electromagnetic modes (TEMs) propagate in the coaxial cables. Therefore, the complete system can be analyzed as a microwave circuit using the classic scattering matrix \( S \) [29]. Figure 3 shows the equivalent circuit. The definition of the different electrical variables is as follows:

- \( V_s^+, I_s^+ \): voltage and current waves in the source port propagating towards the SGW.
- \( V_s^-, I_s^- \): voltage and current waves in the source port propagating from the SGW.
- \( V_d^+, I_d^+ \): voltage and current waves in the drain port propagating towards the SGW.
- \( V_d^-, I_d^- \): voltage and current waves in the drain port propagating from the SGW.
- \( Z_0 \): characteristic impedance of the coaxial lines.

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Figure 2. Cross section of the two coaxial lines and the spherical waveguide (SGW). The power is injected through the source port. The radiation is guided between the spheres and may be extracted at the drain port. Both ports are geometrically identical. The drain port is loaded with its characteristic impedance $Z_0$.

Figure 3. Microwave circuit made up of the SGW with two ports. It is completely characterized by the matrix $S$.

The matrix $S$ of the SGW is defined as follows:

$$
\begin{bmatrix}
V_s^- \\
V_d^-
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix}
\begin{bmatrix}
V_s^+ \\
V_d^+
\end{bmatrix}.
$$

(6)

When the drain port is matched with its characteristic impedance $Z_0$, there is no reflected wave in this coaxial line and thus the voltage and current waves $V_d^+$ and $I_d^+$ are null:

$$
V_d^- = S_{21}V_s^+, \quad V_s^- = S_{11}V_s^+.
$$

(7)

The powers injected through the source port, $P_I$, transmitted to the drain port, $P_T$, and reflected by it, $P_R$, are

$$
P_I = \frac{1}{2} \frac{|V_s^+|^2}{Z_0}, \quad P_T = P_I|S_{21}|^2, \quad P_R = P_I(1 - |S_{21}|^2).
$$

(8)
When the drain port is located at the source’s image point, and it is loaded with its characteristic impedance, it does not behave as the perfect drain designed in [21], because with this load the electric field $E_r(\theta)$ is given by equation (4) with $A \neq 0$ and $B \neq 0$, i.e. both forward and reverse waves exist.

The simulation has been performed using COMSOL and CST Microwave Studio with the following geometrical parameters (see figure 2):

$$D_e = 10 \text{ mm}, \quad D_i = 5 \text{ mm}, \quad L = 20 \text{ mm}, \quad R_M = 1005 \text{ mm}, \quad R_m = 1000 \text{ mm}. \quad (9)$$

The frequency range being analyzed goes from 0.2 to 0.4 GHz ($\lambda$ between 0.75 and 1.5 m), well below the cut-off frequency of the next higher order mode in the coaxial cables (which approximately is $(2c/\pi)/(D_e + D_i) = 112.7 \text{ GHz}$ [29]).

The refractive index between the spherical shells is $n(r) = an_0/r$. By selecting $a = R_M$ and $n_0 = 1$, we obtain $n(r) = R_M/r$ and from the second row in equation (2):

$$\left(R_Mk_0\right)^2 = \nu(\nu + 1). \quad (10)$$

Since $R_M/R_m \approx 1$, we have approximated $n = 1$ within the waveguide for the CST Microwave Studio model. Both the gradient and constant index cases have been analyzed with the COMSOL package. There is no significant difference in the results (there is a small difference < 0.01% in the notch frequency for the two refractive index distributions, which may be caused by numerical errors). The port radius is less than $\lambda/100$ for the analyzed frequencies and the ratio $D_e/R_m = 1$ with the aim of properly modeling the point nature of both the source and the drain. The coaxial lines are 20 mm long, which is enough to guarantee that the evanescent modes in the coaxial lines are negligible at their ends.

Special care has been taken to define the mesh of the system. In the CST Microwave Studio we used the auto-scaling option, while in COMSOL we selected user-defined grids. Figure 4 shows an example of the mesh used in COMSOL.

4. Power transmitted for different drain port positions and different frequencies

We have made several simulations to analyze the imaging properties of the system. We have used $|S_{21}|^2$ to determine the sensitivity of the transmitted power $P_T$. Note that the power transmitted to the drain port ($P_T$) per unit of power injected through the source port ($P_I$) is $|S_{21}|^2$, see equation (8)).
Figure 5. $|S_{21}|^2$ as a function of frequency when the drain and source ports are at opposite poles. The peaks occur at the Schumann resonance frequencies.

Figure 6. $|S_{21}|^2$ versus frequency when the drain port is shifted $\lambda/30$ ($\lambda = 1$ m) from the source port antipode. The results are similar to those presented in figure 5 in accordance with the classic prediction, except for the very narrow notches near the Schumann frequencies.

4.1. $|S_{21}|^2$ as a function of frequencies for different drain port positions

Figure 5 shows $|S_{21}|^2$ for a frequency range between 0.2 and 0.4 GHz when the drain port is placed at the source’s image point; that is, $\theta = 0$ for the source port and $\theta = \pi$ for the drain port. There are peaks of $|S_{21}|^2$ indicating total transmission from the source port towards the drain port. These peaks occur at the so-called Schumann resonance frequencies of the spherical systems (see, e.g., p 374 in [30]), which correspond to integer values of $v$ in equation (10). These peaks are characteristics of resonators such as the Fabry–Pérot one (see, e.g., [31]).

Figure 6 shows $|S_{21}|^2$ when the drain port is shifted $\lambda/30$ (for $\lambda = 1$ m corresponding to 0.3 GHz) away from the source port antipode. Although the results are extremely similar, narrow notches in the transmission very close to the Schumann frequencies occur. These notches widen when the drain port is shifted further from the image point of the source, but the null of $|S_{21}|^2$...
Figure 7. Detail of $|S_{21}|^2$ as a function of the frequency in a narrow band around a notch frequency for different drain port positions (the corresponding shift on the inner sphere of the SGW between the center of the drain port and the source port antipode has been used for labeling). There is no notch in the transmission when the drain port is at the source port antipode (no shift). The notch frequency is $f = 0.2606873$ GHz ($\nu = 4.996$). The nearest Schumann frequency is $f = 0.26086609$ GHz ($\nu = 5$), which is out of the range of this figure.

remains fixed, as can be seen better in figure 7. The frequencies corresponding to these nulls will be called here notch frequencies.

Figure 7 shows $|S_{21}|^2$ for different drain port positions in a very narrow band in the neighborhood of the notch frequency corresponding to the second peak in figure 6 (for which $\nu = 5$). The label of each curve indicates the distance between the center of the drain port and the source port antipode. The black curve corresponds to the drain port placed in the source port antipode (it looks flat because of the high zoom in the frequency axis). The other curves correspond to different shifts of the drain port. The shifts are in all cases much smaller than wavelength (from $\lambda/33$ to $\lambda/500$ with $\lambda = 1.15084047$ m that corresponds to $f = 0.2606873$ GHz, see figure 7). These results are quite surprising, since close to a specific frequency the power transmitted to the drain port suddenly reduces to a value near zero.

4.2. $|S_{21}|^2$ as a function of drain port shift for different frequencies

Since $|S_{21}|^2$ is proportional to the transmitted power, the graph representing $|S_{21}|^2$ versus the drain port shift (figure 8) is equivalent to the point-spread-function (PSF) commonly used in optics. This equivalence may seem surprising since the PSF is defined as the square of the electric field amplitude calculated in the absence of absorbers in the image space, and $|S_{21}|^2$ is defined in terms of the power transmitted to an absorber (the drain port). However, the equivalence comes from the fact that, in optics, the detection at the image is assumed to be made with a sensor that does not perturb the free-space fields; and even if it does perturb the fields, it is assumed that the sensor signal is nevertheless proportional to the field amplitude (or its square, which is the PSF). In section 5, we will discuss that this assumption of negligible perturbation does not apply to the SGW, and it is crucial to understand its super-resolution properties.
Figure 8. $|S_{21}|^2$ as a function of the drain port shift for a frequency near a notch one (red curve) and for a frequency far from a notch one (blue curve).

Figure 9. $|S_{21}|^2$ as a function of the drain port shift for different frequencies that present a super-resolution of between $\lambda/30$ and $\lambda/500$.

Figure 8 shows $|S_{21}|^2$ versus the drain port shift for two frequencies, normalized to the value of $|S_{21}|^2$ when the drain port is at antipode of the source (see figure 5). The blue curve corresponds to $f = 0.2847 \, \text{GHz}$, i.e. far from a notch frequency ($\nu = 5.5$). This blue curve is indistinguishable from the energy density distribution in the SGW when there is no drain port (see [12, 22] and the annex in [21]), which is the classic diffraction limit (it corresponds to $A = B$ in equation (4), and it is proportional to the function $(P_\nu(\cos \theta))^2$, where $\theta R_{\text{min}}$ is equal to the shift).

Let us define ‘resolution’ as the arc length (in wavelength units) that a drain port needs to be shifted, so $|S_{21}|^2$ drops to 10% (not far from the Rayleigh criteria in optics, which refers to the first null). With this definition, the diffraction limited resolution given by the blue curve is $\lambda/3.45$.

The red curve corresponds to notch frequency $f = 0.26068741 \, \text{GHz}$ ($\nu = 4.996$), which clearly shows a much better resolution.
Figure 10. Bandwidth as a function of the resolution. The abscissa axis shows $N$, meaning that the resolution is better than $\lambda/N$.

Figure 9 is a blow-up of figure 8 in the upper neighborhood of a notch frequency. The graph for frequencies slightly below the notch frequency is similar. Note that figure 9 shows the same information as figure 7 but plotting $|S_{21}|^2$ versus the drain port shift (expressed in units of $\lambda$) and using the frequency as a parameter.

From the orange to the red curves, increasing resolutions are achieved: 0.03 $\lambda$ (i.e. $\lambda/33$) for the orange to $\lambda/500$ for the red. The latter, whose frequency $f = 0.26068741$ GHz corresponds to $\nu = 4.99636$, is the highest resolution that we have obtained. Computations for frequencies near the notch frequency show essentially null $|S_{21}|^2$ values for shifts $>\lambda/500$ (as in the red line in the picture). $|S_{21}|^2$ values for shifts below $\lambda/500$ (except for no shift or shifts very near zero) and frequencies near a notch frequency are inconsistent (the solver did not converge to a single solution due to numerical errors). It seems that Leonhardt’s assertion of infinite resolution (i.e. perfect imaging) may occur for the discrete notch frequencies in the SGW, although the aforementioned inconsistencies have prevented us from numerically predicting resolutions beyond $\lambda/500$.

The $\lambda/500$ resolution is achieved only for a narrow bandwidth ($\approx 20$ Hz, which is much smaller than the notch frequency $\approx 0.3$ GHz). If larger bandwidths are needed, lower resolutions (but still sub-wavelength) may be achieved. Figure 10 shows the bandwidth versus $N$, meaning that the resolution is better than $\lambda/N$. The bandwidth has been calculated as $f_{\text{max}} - f_{\text{min}}$ with $f_{\text{max}}$ and $f_{\text{min}}$ fulfilling $|S_{21}(f_{\text{max}})|^2 = |S_{21}(f_{\text{min}})|^2 = 0.1$, using the information of the curves in figure 7 and similar curves. The linear dependence shown in figure 10 (slope $-2$) reveals that the product $N^2 \times$ bandwidth is constant in the range analyzed here.

5. Discussion

The traditional diffraction limit is obtained in the analysis of focal regions in free space. Although in practical applications a detector is placed in the focal plane, it is implicitly assumed that the limit stands even if the detector is perturbing the field, probably considering that such a perturbation will be small. However, the analysis here proves that it is possible to exceed that limit if the field is strongly perturbed by the detector. Computer simulations have shown that super-resolution effectively occurs in the SGW (even when the drain is not perfect, i.e. even when there is a wave reflected at the drain) if the frequency is close to one of some selected frequencies called notch frequencies. However, far from these frequencies, the classic resolution
Table 1. Electric field module (in V m\(^{-1}\)) for two positions of the drain port and two frequencies: one far from (left) and one near a notch frequency. Incident power of the TEM mode at the source port is 1 W.

|                       | Far from notch frequency | Near notch frequency |
|-----------------------|--------------------------|----------------------|
| No shift              | ![Image](image1.png)     | ![Image](image2.png) |
| \(\lambda/75\) shift | ![Image](image3.png)     | ![Image](image4.png) |

Transformation optics theory has shown that the 2D MFE system and the SGW are equivalent [25] for the field modes considered here, and so we conclude that super-resolution in the MFE with the drains considered here only occurs near notch frequencies. Schumann frequencies are those for which \(\nu\) is an integer (see equation (2)) and are very close to notch frequencies. One rough way of measuring how close a frequency is to a notch frequency is just by evaluating the fractional part of \(\nu\). This variable takes the value \(\nu_1 = 9.98\) for the microwave experiment referred to in [22, 23]; it is quite close to the Schumann frequency corresponding to \(\nu = 10\), while in the optical experiment referred to in [24] the value is \(\nu_2 = 96.78\), which is not that close. According to our definition of resolution, the microwave experiment would get \(\lambda/4\) (the reported resolution in [22, 23] is \(\lambda/5\), with a slightly different definition of resolution). We think that the main reason why only the microwave experiment showed super-resolution is because the frequency chosen was close to a notch frequency.

When a perfect drain is placed at the source’s image point in the MFE, the forward wave is absorbed perfectly [7, 21]. However, the impedance that we used as a load, which is the characteristic impedance of the coaxial cable, does not make the drain port a perfect drain. We conjecture that this perfect drain can be achieved with a specific load impedance that causes the reverse wave inside the SGW (\(B = 0\) in equation (4)) to disappear.

We do not have a theoretical explanation of the special role of notch frequencies. Several other questions remain open and are outside the scope of this paper. In particular, would a perfect drain provide super-resolution for all frequencies or would the notch frequencies again
have a special role? Is there a load impedance such that the drain port behaves like a perfect point drain? What happens when multiple source and drain ports are used?

Leonhardt in [7, 8] suggested that MFE should produce perfect imaging for any frequency using perfect drains. The experiments in [22, 23], like our simulation here, used coaxial probes, which are not perfect drains, but apart from this difference, Leonhardt’s suggestion could be criticized. Leonhardt assumed that the ability of the MFE to propagate the wave generated by a point source up to a perfect point drain was enough to guarantee perfect imaging. This does not seem to be sufficient, since it does not provide information on how much power the drain will absorb when it is displaced out of the image point, which is the case analyzed here.

6. Conclusions

Simulations of the spherical waveguide (SGW) showing super-resolution up to $\lambda/500$ at microwave frequencies have been presented. The simulations presented here prove that super-resolution exists in an SGW and MFE lens within a narrow band around discrete frequencies (which we called notch frequencies) as shown in figures 6 and 7. Two experimental prototypes have recently been manufactured in order to prove the super-resolution property of MFE. Our results are consistent with the two experimental setups, because one prototype that showed the super-resolution was tested for a frequency very close to a notch one, while the other prototype that did not show up any super-resolution response was tested at a frequency far from a notch one.

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