Vortices in rotating Bose gas interacting via finite range Gaussian potential in a quasi-two-dimensional harmonic trap

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(Dated: May 26, 2022)
Abstract

A system of harmonically trapped $N=16$ spin-0 bosons confined in quasi-2D symmetrical $x-y$ plane interacting via a finite range repulsive Gaussian potential is studied under an externally impressed rotation to an over all angular velocity $\Omega$ about the $z-$axis. The exact diagonalization (ED) of $n \times n$ many-body Hamiltonian matrix in a given subspace of quantized total angular momentum $0 \leq L_z \leq 4N$ is performed using Davidson algorithm. For $N = 16$ and $L_z = 32$, the dimensionality of the Hilbert space turns out to be $n = 384559$. With increase in the interaction range in the Gaussian potential, the active Fock space size gets reduced and hence computation becomes more feasible compared to the zero-range $\delta$-function potential. Following this idea, we considered the interaction-range parameter $\sigma = 0.30, 0.50$ and $0.75$ to study the finite-range effects on the many-body ground state. The trap velocity $\Omega$ being the Langrange multiplier associated with the angular momentum $L_z$ for the rotating systems, the $L_z-\Omega$ phase diagram (or stability line) is drawn which determines the critical angular velocities, $\Omega_{c_i}, i=0,1,2...$, at which, for a given angular momentum $L_z$, the system goes through a quantum phase transition. Further with increase in interaction range $\sigma$, the quantum mechanical coherence extends over more and more particles in the system resulting in an enhanced stability of the $i^{th}$ vortical state with angular momentum $L_z(\Omega_{c_i})$ leading to a delayed onset of the the next vortical state $L_z(\Omega_{c_{i+1}})$ at a higher value of the next critical angular velocity $\Omega_{c_{i+1}}$. There is an increase in the critical angular velocity ($\Omega_{c_i}, i=1,2,3...$) and in the largest condensate fraction $\lambda_1$, calculated using single particle reduced density matrix(SPRDM) eigen-values with increase in the interaction range $\sigma$. We calculated the von-Neumann quantum entropy ($S_1$), degree of condensation ($C_d$) and the conditional probability density (CPDs). There is no significant change in von-Neumann entropy $S_1$ and the degree of condensation $C_d$ in slow-rotating gas in the region $0 \leq L_z \leq (L_z = N)$. However, for higher angular momentum, $L_z \geq 2N$, with increase in interaction range $\sigma$, small variations in $S_1$ and $C_d$ are observed. We plot the isosurface CPDs
plots for conditional probability and hence study the nucleation of vortex states, which is one of the signature of rotating bose gas, we report that with the increase in interaction range $\sigma$, the probability density starts accumulation at the edge of vortices. Accumulation of density at the surfaces of sphere in the case of Bose-condensate in the bubble trap has been discussed here [1]. We also observed that the vortices entry started for the higher $\sigma$ followed by the lower. We plotted the CPDs to study $p = 1, 2, 3, 4, 5$ fold symmetrical vortex lattices. Concluding we suggest the modification of single particles basis state with the increase in $\sigma$ at $L_z = 32, \sum_{i=1,3}(n_i, m_i)(\sigma = 0.30) [(2, 2), (0, 0), (4, 4)], \sum_{i=1,3}(n_i, m_i)(\sigma = 0.50) [(2, 2), (0, 0), (4, 4)]$ and $\sum_{i=1,3}(n_i, m_i)(\sigma = 0.75) [(2, 2), (3, 3), (1, 1)]$.

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I. INTRODUCTION

The achievement of Bose-Einstein condensates (BECs) in harmonically trapped atomic vapours has stimulated explosively growing interest in the system of interacting bosons [2–11] and for the parabolic trap is studied for contact potential. Theoretical studies in quasi-two dimensions are mainly focused on the rotating properties and vortex structure in BEC. The presence of vortex states is a signature of macroscopic phase coherence state moreover vortices are the signature of superfluidity of the Bose Einstein condensate state. Quantized vortices have been studied for the superfluid $^4$He as the topological defects behaviour of superfluid [12]. For weakly interacting bose gas, vortex states are not stable and becomes unstable with the halt of externally impressed rotation, such state can’t be the superfluid, on the other hand when interaction and trap effects become comparable, we find interesting symmetry breaking (SB) instability which modify vortex core and induce precession about the axis of trap[13, 14]. Experiments with Alkali atoms such as $^{23}$Na, $^{39}$K, $^{87}$Rb and $^7$Li (attractive interaction[15]) confined in magnetic traps and cooled down to 100 nK have been achieved in laboratories have sustained the interest in the BECs [16–18]. By using the focused laser beam, Madison et.al has stirred the $^{87}$ Rb atom to observed the vortex at stirred angular velocity exceeding the critical angular velocity $\Omega_c$, till the four vortices presented simultaneously [19]. In the usual ultra-cold atomic gases experiments, the atom-atom interaction is well modelled by the zero-contact $\delta$ potential i.e. $\sigma \rightarrow 0$ [20]. For 1D study the contact zero range $\delta$ potential is a suitable choice for contact interactions but for higher dimensions the achieving the convergence of single particle basis states becomes painful task, finite range Gaussian potential is appropriate in terms of convergence. The sign of finite range Gaussian potential is taken to be positive to make the interaction repulsive[21–23]. For numerical studies, in finite range Gaussian potential we have some advantages in tuning
the inter-particle interaction range $\sigma$, to study the finite range effects. The finite range potential, can be expanded within the finite numbers of single particles basis state in contrast to the zero-range $\delta$ function, which significantly facilitate the convergence of numerical results, are among the few advantages of finite range Gaussian interaction ($\sigma \to \infty$) over contact potential ($\sigma \to 0$). In the present study, we have tuned the interaction range parameter of the finite range Gaussian function in the range from 0.3 to 0.75 within the slow to fast rotating angular momentum in the range $0 \leq L_z \leq 4N$ along the rotation axis $-z$. For some results we studied $0 \geq \sigma \leq 0.80$ for $L_z = 32$. Our focussed is in the study of the finite range interaction. Most of the theoretical work in the literature have focused on the “lowest-Landau-level” (LLL) approximation, in which single particle orbital states with the radial quantum $n_r = 0$, while single particle angular momentum quantum number $m$ takes positive values (with respect to trap angular velocity $\Omega$ along the rotation axis $-z$). This reduces the dimensionality of the Hilbert space significantly [24–27]. In moderately to strongly interacting regime and slowly rotating system, beyond LLL approximation with $n_r = 0, 1 \cdots$ while $m$ is allowed to take both positive and negative values $\pm m$, is considered a better variational approximation over LLL approximation [30]. In fast rotating regime, both LLL and beyond LLL approximations give comparable ground state energy $E_{0}^{lab}$ and critical angular velocity $\Omega_c$, as is in Tab. VIII. For two-particles interaction range other then $\sigma \to 0$, the Dirac $\delta$-function is not suitable choice for two-body interaction in exact theories. The exact diagonalization of the Hamiltonian matrix for the finite N-body system yields an exact solution within a finite number of single particle basis. For the numerical feasibility it is necessary to select the single particles basis out of many particle basis to formulate the variational wave-basis. The usual contact $\delta$-potential $g_2 \delta_{r,r'} = g_2 \sum_n u_n^*(r)u_n(r')$, where sum runs over all infinite single particle basis states, therefore is not suitable for the numerically carry out exact diagonalization. Therefore the choice for the finite range smooth Gaussian
potential in Eq. (2) over usual zero-contact range $\delta$-potential in Eq. (3) is preferable which can be expanded within the finite number of single particle basis states. [28, 29]. In order to construct the many-particle basis states, we go beyond the LLL so as to select the single particle basis state that includes the radial quantum number $n_r = 0, 1$ and the single particle angular momentum quantum number $m$ of either sign [30].

We explore the physics of various stable and unstable states in subspace of quantized total angular momentum $L_z$ and their transition to different vortex patterns. For fast rotating case, in the orbital angular momentum range $2N > L_z \leq 4N$, higher p-fold symmetry started to emerge, $p = 3, 4, 5,...$ we have triangular lattice for $L_z = 35$, as shown in Fig. 14(f) and $L_z = 36$, in Fig. 14(g), 15(g) and 16(h). As angular momentum $L_z$ grows, the vortex lattices arranged themselves in the patterns of Abrikosov lattices [31, 32]. In the interaction regime, we can not ignore in contribution of kinetic energy even for larger $N$, since it determines the structure of vortex core. In particular the balance between the kinetic and the interaction energy fixes a typical distance over which the condensate wave function can heal [33].

This paper is organised as follows: In Sec. II, we derive a general Hamiltonian equation for $N$ trapped spin-0 particles interacting via a normalized repulsive Gaussian-shaped two-body interacting potential in quasi-2D harmonic trap exposed to an over-all rotation about an axis of the symmetric plane. Discussed the need to go beyond LLL and set-up many-body wavefunction out of single partile basis state to carry-out the diagonalization of Hamiltonian. In Sec. III, where present our findings followed by the brief introduction about the quantity of interests like the critical angular velocity $\Omega_c$, single particle reduced density matrix (SPRDM) using variational wavefunction and follows the von-Neumann entropy $S_1$, degree of condensation $c_d$ and lastly the conditional probability density (CPD) to study the spatial correlation of condensed bose gas. In Sec. V, we present the conclusion.
II. THE HAMILTONIAN

We consider a system of \( N = 16 \) spin-0 bosons, each of mass \( M \) in an external harmonic trap, which are interacting via a normalized Gaussian potential in a \( x-y \) symmetric two-dimensional plane with an externally impressed rotation \( \Omega = \Omega \hat{z} \) about \( z \)-axis, has been studied in the recent years \([28, 34–37]\) are the bench mark for our study. The Hamiltonian of this system in co-rotating frame can be written as,

\[
\hat{H}^{\text{rot}} = \hat{H}^{\text{lab}} - \hat{\Omega} \cdot \hat{L}^{\text{lab}}
\]

here, \( \Omega \) ia the angular velocity and \( L \) is the angular momentum along the rotation \( z \)-axis.

\[
\hat{H}^{\text{lab}} = \sum_{i=1}^{N} \left[ \frac{\hat{p}_{i}^{2}}{2M} + \frac{1}{2} M \omega_\perp^2 r_{i}^2 \right] + \frac{1}{2} \sum_{i \neq j}^{N} U(r_i, r'_j)
\]

the first two terms in Hamilton Eq. (1), corresponds to the kinetic and trap potential energy respectively and the third term is inter-atomic interaction potential. terms used in the above equation has been discussed in the coming sections.

A. INTERACTION POTENTIAL

The pairwise interaction \( U(r, r') \) is assumed to be Gaussian of the form,

\[
U(r, r') = \frac{\tilde{g}_2}{2\pi \sigma^2} \exp \left[ -\frac{(r - r')^2}{2\sigma^2} \right]
\]

where, \( \tilde{g}_2 = \frac{4\pi \hbar^2 a_s}{M} \) having units of energy \( \times \) volume and in dimensionless units it becomes \( g_2 = \frac{4\pi a_s}{a_\perp} \) is interaction strength.

where, \( \sigma \) (scaled as \( a_\perp \)) is the two particles interaction range and dimensionless interaction strength \( g_2 = \frac{4\pi a_s}{a_\perp} \), in terms of s-wave scattering length \( a_s = 1000a_0 \), where,
$a_0 = 0.05292$ nm, is Bohr radius. The exact diagonalization study of many-body Hamiltonian matrix is carried out in the beyond Landau Level approximation to obtain the low-lying energy spectrum of ultracold Bose gas in a harmonic trap. Our scheme includes both Lowest Landau Level and beyond Lowest Landau Level approximation with single particle angular momentum $m$ with either sign to construct the many body basis states. where, $\sigma$ being the range of two-body interaction and $g_2$ measures the interaction strength. In the limit $\sigma \to 0$, the above Gaussian potential reduces to the usual zero range-contact $\delta$ potential of the form

$$U(r, r') = g_2 \delta_{rr'}.$$  \hfill (3)

have been used widely in literatures.

**B. TRAPPING POTENTIAL**

Harmonic trapping along the symmetric $x - y$ plane

$$V(r) = \frac{1}{2} M(\omega_{\perp}^2 r_\perp^2 + \omega_z^2 z^2)$$ \hfill (4)

where, $r_\perp = \sqrt{x^2 + y^2}$, normal radius from the axis of rotation and $\omega_{\perp}, \omega_z$ are the radial and axial trap frequency respectively of harmonic confinement. According to these trapping frequencies, here we are writing the corresponding harmonic oscillators lengths $a_\perp = \sqrt{\frac{\hbar}{M \omega_{\perp}}}$ and $a_z = \sqrt{\frac{\hbar}{M \omega_z}}$, radial and axial trapping lengths respectively. Further rewriting the above Eq. (4) as,

$$V(r) = \frac{1}{2} M \omega_{\perp}^2 \left( r_\perp^2 + \lambda_z^2 z^2 \right)$$ \hfill (5)

where, $\lambda_z^2 = \frac{\omega_z^2}{\omega_{\perp}^2}$ is the anisotropic parameter, corresponds to different trap geometries such that

- $\lambda_z \ll 1$, yields an elongated cigar-shaped condensate (prolate).
\( \lambda_z \gg 1 \), yields a flattened disk shaped/ pancake condensate (oblate).

\( \lambda_z = 1 \), yields a spherical shape condensate.

C. THE SINGLE PARTICLE ENERGY SPECTRUM

The N-body variational wavefunction \( \Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N) \) is constructed as the linear sum of single particle basis-functions \( u_{n,m,n_z}(\mathbf{r}) \), chosen to be the eigenfunctions of the non-interacting (single-particle) Hamiltonian.

\[
H_{sp} = \frac{1}{2} \left( -\frac{\hbar^2}{m} \nabla^2 + m\omega^2 r^2 \right) - \Omega l_z \\
+ \frac{1}{2} \left( -\frac{\hbar^2}{m} \nabla^2_z + \lambda_z^2 z^2 \right)
\]

(6)

the above Eq. (6) is identified as the harmonic oscillator Hamiltonian in a rotating frame with \( l_z \) being single particle angular momentum along the rotation z-axis. The well know harmonic oscillator eigensolution is of the form \( H_{sp} u_{n, m, n_z}(r_{\perp}, \alpha_{\perp}, \phi) = \epsilon_{n, m, n_z} u_{n, m, n_z}(r_{\perp}, \alpha_{\perp}, \phi) \) has the following form;

\[
\epsilon_{n, m, n_z} = (2n + |m| + 1)\hbar \omega_{\perp} + (n_z + 1/2)\hbar \omega_z
\]

\[
u_{n, m, n_z}(r_{\perp}, \alpha_{\perp}, \phi) = \sqrt{\frac{\lambda_z / \pi^3}{2^n n_z! (n_r + |m|)!}} (\frac{1}{\sqrt{\lambda_z}})^{|m|} \\
\times e^{-\frac{1}{2}(r_{\perp}^2 + \lambda_z z^2)} e^{im\phi} \\
\times L_{n_r}^{|m|} (r_{\perp}^2 \alpha_{\perp}^2) H_{n_z}(\sqrt{\lambda_z} z)
\]

(7)
where, \(n_r\) is radial quantum number, single particle angular momentum quantum number \(m = 0, \pm 1, \pm 2..\) and axial-quantum number \(n_z\). The \(L_{nr}^{m}(r^2 \alpha^2_{\perp})\) and \(H_{nz}(\sqrt{\lambda_z} z)\) are associated Laguerre polynomial and is Hermite polynomial respectively. Our system of quasi 2D in which we assumed such that there is no excitation in stiffer \(z\)-axis and hence setting the \(n_z = 0\) reduced the Eq. (7)

\[
u_{n_r,m}(r_{\perp} \alpha_{\perp}, \phi) = \sqrt{\frac{\sqrt{\lambda_z}/\pi^3 \alpha_{\perp}^2 n_r!}{(n_r + |m|)!}} (r_{\perp} \alpha_{\perp})^{|m|} \\
\times e^{-\frac{1}{2}(r^2 \alpha^2_{\perp} + \lambda_z z^2)} e^{im\phi} \\
\times L_{nr}^{m}(r_{\perp} \alpha^2_{\perp})
\]

(8)

here, \(\alpha_z\) is inverse harmonic oscillator length \(= 1/a_{\perp}\). Quasi 2D harmonic oscillator energy turned out to be,

\[
\epsilon_{n_z,n_r,m} = (2n_r + |m| + 1)\hbar \omega_{\perp} + 1/2\hbar \omega_{z}
\]

(9)

to study the LLL approximation we set \(n_r = 0\) for which single particle angular momentum takes the values \(m \geq 0\) and the choice for \(n_r = 0,1\) corresponds to \(m = \pm 1, \pm 2..\) termed as the beyond LLL approximation in the above Eq. (8). The diagonalization of Hamiltonian has been performed in the subspace of \(L_z\). For bosons, the N-body variational wave function can be expanded in the form of symmetrize product of single particles basis states i.e. \(\Psi(r_1, r_2, .. r_N) = \sum_{\nu} C_{\nu}\Phi_{\nu}(r_1, r_2, .. r_N),\)

\[
\Phi_{\nu} = \frac{\sum_{P} P \prod_{k=0} \frac{\sum_{i=0}^{k} u_{k}(r_i)}{\sqrt{\nu_{k}!}}}{\sqrt{N!}}
\]

(10)

where, \(C_{\nu}\) are variational coefficients and \(P\) is the permutation of N particle coordinates where as the sum denotes the symmetrization against the particles exchange
referred as Fock terms. The many-body wavefunction $\Phi_\nu(r_1, r_2, \ldots, r_N)$, is labelled by the many-body index $\nu \equiv (\nu_0, \nu_1, \ldots, \nu_k)$ are the set of single particles quantum numbers $j \equiv (n_z, n, m)$ and their respective occupancies $\nu_j$. Satisfying the given constraints $\sum_{j=0}^{k} \nu_j = N$ and $\sum_{j=0}^{k} \nu_j m_j = L_z$. The single particle angular momentum quantum number $m_j$, total number of particles $N$ and the total angular momentum quantum number $L_z$. Calculation of the interaction co-efficient using direct technique is not much favorable over the second quantization, which have been developed for the calculation of matrix elements of the operators between the wavefunction of the form Eq. (10). It is point where we should switch to the second quantization formalism where the occupation numbers becomes the variable of state. Rewriting the Eq. (10), in the second quantization form in the occupation numbers $\nu_j$ notation.

$$|\Phi_\nu\rangle \equiv \prod_{j=0}^{k} \frac{1}{\sqrt{\nu_j!}} \left(\hat{b}_j^{\dagger}\right)^{\nu_j} |\text{vac}\rangle \equiv |\nu_0 \nu_1 \cdots \nu_j \cdots \nu_k\rangle \quad (11)$$

Satisfying the given constraints $\sum_{j=0}^{k} \nu_j = N$ and $\sum_{j=0}^{k} \nu_j m_j = L_z$. The single particle angular momentum quantum number $m_j$, total number of particles $N$ and the total angular momentum quantum number $L_z$ with $j \equiv (n_z, n, m)$. With these constraints the size of active Fock space of orders $10^5$ is constructed to carry out the numerical calculation on the small workstation of 3.00 GHz processor. and hence the many-body Hamiltonian Eq. (1), can be written in the following forms

$$H^{lab} = \sum_{i,j} H^{sp}_{ij} \hat{b}_i^{\dagger} \hat{b}_j + \frac{1}{2} \sum_{i,j,k,l} \langle i, j | U^{kj}_{il} | k, l \rangle \hat{b}_i^{\dagger} \hat{b}_j^{\dagger} \hat{b}_l \hat{b}_k \quad (12)$$

where $\hat{b}^{\dagger} (\hat{b})$, creates(destroys) an atom in the single particle states $\Phi_\nu$. The first term in the second quantized form of Hamiltonian is one-body matrix-elements in Eq. (12), corresponds to the diagonal terms as can be think of eigen-values $\sum_i \epsilon_i N_i$, and the second one is the interaction Hamiltonian in the form of two-body matrix-elements.

In this article for $N=16$ spin-0 bosons, we have calculated the ground state energy-spectrum for total angular momentum, slow to fast rotation, in the range $0 \leq L_z \leq$
The diagonalization of Hamiltonian matrix is performed for each of subspace angular momentum basis state $L_z$ separately. In the single particle basis state $u_{n,m}$, the single particle angular momentum quantum number is chosen as: $m = l_z - n_b, l_z - n_b + 1, \ldots, m = l_z - n_b - 1, l_z + n_b$. Where, $l_z$ is defined in the terms of greatest integer function and $n_b = 3, 4$ as the size of single particle basis state depending on the available numerical resources and interaction strength. If we consider $N = 16$ and $L_z = 32$ following the scheme, we define a greatest integer function $l_z = [L_z/N] = 2$, is defined as the $[x]$, for a given real $x$, the integer is either less or equal to itself. The example presented here, we choose $n_r = 0, 1$ and $n_b = 3$ for numerical feasibility, the single particles angular momentum takes the values $m = -1, 0, 1, 2, 3, 4$ and the $u_{n,m}$ single particle basis state set turns out to be the following,

$$\{u_{0,0}, u_{1,1}, u_{2,2}, u_{3,3}, u_{4,4}, u_{5,5}, u_{1,-1}, u_{2,0}, u_{3,1}, u_{4,2}, u_{5,3}, u_{3,-1}\}.$$

These single particle basis states are selected for the construction of the many-particle basis state $\Phi_\nu$, in the variational trial wave function $\Psi = \sum_\nu C_\nu \Phi_\nu$ of the system for each total angular momentum $L_z$ in the similar manner for the entire range $0 \leq L_z \leq 4N$.

III. RESULTS FOR A SYSTEM OF FIXED PARTICLES

The results and discussions are presented here for $N = 16$ spin-0 bosons of $^{87}$Rb atom confined to a quasi 2D anisotropic harmonic trap with anisotropic in ratio $\lambda_z = \frac{\omega_z}{\omega_\perp} = \sqrt{8}$ with radial frequency $\omega_\perp = 2\pi \times 220$ Hz. This choice of radial frequency corresponds to the trap length $a_\perp = \sqrt{\hbar/(M\omega_\perp)} = 0.727\mu m$. The particle-particle dimensionless interaction strength turns out to be $g_2 = 0.9151$ corresponding to the s-wave scattering length $a_s = 1000a_0$, where $a_0 = 0.05292$ nm is the Bohr
radius. The condensate has a extension along the vertical axis of dimension $a_z = \sqrt{\hbar/M_\omega z} = a_\perp \lambda_2^{-1/2}$ and its dynamics along this axis is assumed to frozen. For the many-body system under study, the characteristic energy scale is determined in terms of dimensionless parameters $(Na_\perp/a_\perp)$ corresponding to experimental situation [4].

With the choice of above system parameters, we are dealing with the dilute bose-gas in the limit $\bar{n}a_3^3 = 6.171 \times 10^{-3} \ll 1$ with $\bar{n} \equiv N/a_\perp^3$ being the density of bose-gas. Moreover, the size of the system $a_\perp = 0.727 \mu m$ is larger than the healing length $\xi = \frac{1}{\sqrt{8\pi\bar{n}a_\perp}} = 0.1343$, the minimum characteristic length at which the wave function tends to its bulk value when subjected to the localized perturbation [38].

The characteristics exhibited by our model may, thus, be expected to be suitable for the bulk superfluid [39]. Here we present our study for a range of orbital angular momentum $0 \leq L_z \leq 4N$ and interaction range $\sigma = 0.3, 0.5$ and 0.75.

### A. CRITICAL ANGULAR VELOCITY

For the considered system of $N=16$ spin-0 bosons interacting via a finite range Gaussian potentian, the ground state is obtained by minimizing the free energy function $F(T, V, N) = E(S, V, N) - TS$ given by $\exp(-\beta F) = \text{Tr}[\exp\{-\beta(\hat{H}^{\text{lab}} - \Omega\hbar\hat{L}^{\text{lab}}_z)\}]$ which at $T = 0$ becomes $E(S = 0, V, N) = \langle \Psi_0 | (\hat{H}^{\text{lab}} - \Omega\hbar\hat{L}^{\text{lab}}_z) | \Psi_0 \rangle = \langle \hat{H}^{\text{lab}} \rangle - \hbar\Omega \langle \hat{L}^{\perp}_z \rangle$ where $\Psi_0$ is the is the simultaneous eigenstate of the Hamiltonian and total angular momentum obtained variationally with total angular momentum $L_z$ in non-rotating frame. Therefore the many-body Hamiltonian $\hat{H}^{\text{lab}}$ in Eq. (1) is diagonalized in the given subspace of total angular momentum $L_z$ to obtained the minimized energy $E^{\text{rot}}(L_z, \Omega, g_2) = E^{\text{lab}}_0(L_z, \Omega, g_2) - \hbar\Omega L^{\text{lab}}_z$ in the co-rotating frame. This can be seen as the minimization of energy $\langle \hat{H}^{\text{lab}} \rangle = \langle E^{\text{lab}}_0 \rangle$ subjected to the constraint that the system has the angular momentum expectation $\hat{L}^{\text{lab}}_z$ with the identification that the angular velocity $\Omega$ is the Lagrange multiplier. In the absence of external rotation,
FIG. 1. (Color online) Gives the plot for angular velocity $\Omega$ vs the total orbital angular momentum state $L_z$ along rotation axis, for different interaction range $\sigma$ in the beyond LLL approximation ($n_r = 1$). The horizontal plateaus represents the stability of $L_z$ for the $\Omega_{c_i}$ and the vertical steps shows the quantum jumps along the different vortical symmetry stable states. In units of $\omega_\perp$, $\hbar$, $\sqrt{\hbar^2/m\omega}$ respectively.

the angular momentum state $L_z = 0$ corresponds to ground state of the system. When system is subjected to the externally impressed rotation, the non zero angular momentum states $L_z \neq 0$ successively becomes the next ground state. Thus For higher angular velocity $\Omega_{c_{(i+1)}}$, the corresponding angular momentum state becomes
FIG. 2. (Color online) Gives the plot for angular velocity $\Omega$ vs the total orbital angular momentum state $L_z$ along rotation axis, for different interaction range $\sigma$ in the LL approximation $(n_r = 0)$. The horizontal plateaus represents the stability of state $L_z$ for the $\Omega_{c_i}$ and the vertical steps shows the quantum jumps along the different vortical symmetry stable states. In units of $\omega_\perp$, $\hbar$, $\sqrt{\hbar \omega}$ respectively.

the ground state in energy. Following this scheme we can have a set of critical angular velocities $\{\Omega_{c_i}(L_z, U(r)), i = 1, 2, 3...\}$ at which there is an abrupt quantum jump in the quantized angular momentum $L_z$ of the rotating system given by the
below expression Eq.(13).

\[
\Omega_c \left( L_z, U(r) \right) = \frac{E_{0}^{lab} (L_z, g_2) - E_{0}^{lab} (L_z^{(i-1)}, g_2)}{L_z^i - L_z^{(i-1)}}
\]

(13)

where, \( E_{0}^{lab} (L_z, g_2, \sigma, ) \) is interacting ground state energy in laboratory frame under rotation \( L_z^i (\Omega) \) of i-th particle angular momentum, interaction range \( \sigma \) and interaction strength \( g_2 \). The stability line in Fig. 1 has a series of discrete jumps at the critical angular velocities \{\Omega_{c_i}, i = 1, 2, \ldots\} and corresponding plateaus at \( L_z^i \) with increase in angular momentum in the range \( 0 \leq L_z \leq 4N \). We observe that the angular momentum state \( L_z = 0 \) remains the ground state of the system till the angular velocity attains the first critical value \( \Omega_{c_1} \). As the system is subjected to further rotation, higher angular momentum states \( L_z > 0 \), which minimize the free energy \( E^{lab} - \hbar L \Omega \) in the co-rotating frame, become the successive ground states of the system at a series of critical angular velocities \{\Omega_{c_i}, i = 1, 2, \ldots\}. We note that for a given value of interaction strength \( g_2 \), the respective critical angular velocities increase for increasing values of the interaction range \( \sigma \) in slowly to moderately rotating regime e.g. \( \Omega_{c_1} (\sigma = 0.3) < \Omega_{c_1} (\sigma = 0.50) < \Omega_{c_1} (\sigma = 0.75) \) as seen in Fig. 1. This can be understood as follows: with increase in interaction range \( \sigma \), the quantum mechanical coherence extends over more and more particles in the system resulting in an enhanced stability of the \( i^{th} \) vortical state with angular momentum \( L_z (\Omega_{c_i}) \) leading to a delayed onset of the next vortical state \( L_z (\Omega_{c_{i+1}}) \) at a higher value of the next critical angular velocity \( \Omega_{c_{i+1}} \). We observe jumps in the total angular momentum state of the ground-state at characteristic critical angular velocities \{\Omega_{c_i} (L_z), i = 1, 2, \ldots\}. For the non-rotating \( L_z = 0 \) state, the critical angular velocity for various values of \( \sigma \) are: \( \Omega_{c_1}^{\sigma=0.30} = 0.774 \), \( \Omega_{c_1}^{\sigma=0.50} = 0.784 \) and \( \Omega_{c_1}^{\sigma=0.75} = 0.818 \). The plateaus in
the \( L_z \) state are found at \( \Omega \to \Omega_{c_i} \). These ground state in \( L_z \) (plateaus) are being maintained till the entry of next vortical state at \( L_z \left( \Omega_{c_{i+1}} \right) \). Second quantum jump for the single vortex state at \( L_z = 16 = N \) for the various values of \( \sigma \) are: \( \Omega_{c_2}^{\sigma=0.30} = 0.907, \Omega_{c_2}^{\sigma=0.50} = 0.914 \) and \( \Omega_{c_2}^{\sigma=0.75} = 0.925 \). Thus obtained a series of quantum jumps and corresponding vortical states at the critical angular velocities with decreased in plateaus size and step length for a given interaction range \( \sigma \) and strength \( g_2 \). Similar observations is also reported for few-body system in the LLL approximation for \( \delta \) potential [41]. This is also noted that \( \Omega_{c_i} (L_z, g_2, \sigma) \) decreases with increase in number of particles and interaction strength \( g_2 \) [30]. Complete series of quantum jumps and plateaus for \( L_i^1 (\Omega_{c_i}) \) state for different interaction range \( \sigma \) is given here for the system under study, can be seen in Fig. 1,

\[
L_i^1 (\sigma = 0.30) = 0, 16, 28, 32, 35, 36, 48, 52, 60. \\
L_i^1 (\sigma = 0.50) = 0, 16, 28, 32, 36, 48, 52, 60. \\
L_i^1 (\sigma = 0.75) = 0, 16, 28, 30, 32, 36, 39, 48, 52, 60.
\]

similar trends are reported in these articles [40–42] for \( \delta \) potential in the Lowest Landau Level (LLL) approximation. We can connect these \( \Omega_{c_i} (L_z, g_2, \sigma) \{i = 1, 2, 3 \cdots \} \) to the vortex nucleation, degree of condensation and to the von-Nuemann quantum entropy. Number of vortical states increase with increase in the number of micro-plateaus and it is proportional to the vorticity \( m \) as indicated in Tab. III,IV and V. Few more higher order \( p \)-fold symmetrical vortices for different \( \sigma \) along with \( L_z (\Omega_{c_i}) \) for the system in slow rotating regime \( 0 \leq L_z \leq 2N \) are discussed here, \( p = 2 \) fold-vortical states appear at \( L_{28}^{(2)} (\sigma = 0.30) = L_{28}^{(2)} (\sigma = 0.50) = L_{28}^{(2)} (\sigma = 0.75) \). In the moderately rotating angular momentum \( L_z \), \( p = 3 \) fold symmetrical vortex appears at \( L_{32}^{(3)} (\sigma = 0.30) = L_{32}^{(3)} (\sigma = 0.50) = L_{32}^{(3)} (\sigma = 0.75) \) and the \( p = 4 \) fold at \( L_{35}^{(4)} (\sigma = 0.30) = L_{35}^{(4)} (\sigma = 0.50) = L_{35}^{(4)} (\sigma = 0.75) \). The first and second \( p \) fold vortices appeared at the same \( L_z (\Omega_{c_i}) \) irrespective of interaction range \( \sigma \) however higher
vortices has different $\sigma$ and angular momentum states $L_z \ (\Omega_{ci})$ in the selected region. As we mentioned above that the transition of ground state angular momentum from zero to state $L_z = N$ corresponds to the first on-centered vortex and $L_z > 16$ corresponds to the off-centred multi-votical states, detailed in section IV C 2. The single particle wave-function constructed with odd $m$ satisfied the Jastrow form defined quantum Hall effect [43]. The each ground state corresponding to $\Omega_{ci} \ (L_z, g_2, \sigma)$ are referred as the stable phase-coherent vortices [44, 45]. In ultra-fast rotating regime, the critical angular velocity $\Omega_{ci} \ (L_z, g_2, \sigma)$ becomes independent if interaction range and have coinciding plateaus with same trap height where the rotation motion dominates over the finite range interaction. These are finite range effects in rotating bose-gas, quantum mechanically stable phase-coherent vortical states changed with the interaction range $\sigma$ in co-rotating frame. The depletion in the condensate is observed to be increased with the increase in $L_z$ as the scattering of particles increased.

**B. SINGLE PARTICLE REDUCED DENSITY MATRIX**

The single particle reduced density matrix (SPRDM) is an useful tool to study the BECs of the finite system and more then that it is a criterion for the BECs of interacting gas. Since the experimental realization of BECs in dilute gases, efforts has been made to characterize it microscopically. For non-interacting gases, one of the criteria is that the occupation number for one of the single particle energy level should be macroscopic. For interacting gases, criterion was suggested Penrose[46] and Landau[47] later elaborated by Penrose-Onsager[48, 49] and Yang[50]. For many-body problem, SPRDM provides information about dipole moment, absorption spectra and momentum distribution etcetera. To calculate the SPRDM, $\rho_1(r, r')$ we integrate out $N - 1$ co-ordinates of varitionally obtained many-body ground state
wave-function $\Psi_0(r_1, r_2, \ldots, r_N)$ through the exact-diagonalization is defined as

$$
\rho_1(r, r') \equiv \int \int \ldots \int d r_2 d r_3 \ldots d r_N
\Psi_0^*(r, r_2, r_3, \ldots, r_N) \Psi_0(r', r_2, r_3, \ldots, r_N)
= \sum_n \sum_m \sum_{n_z} \sum_{n'_{m_z}} \sum_{n'_{m'_z}} \rho_{nmn_z, n'm'n'_z}
\times u_{n, m, n_z}^*(r) u_{m', n_z'}(r')
$$

(14)

where, $\rho_{nmn_z, n'm'n'_z} = c_{nmn_z} c_{n'm'n'_z}$

for harmonically trapped finite systems, such as in our case, this Eq. (14), can be expended in the forms of eigenfunctions,

$$
\rho_1(r, r') = \sum_{\mu=1,2,3,\ldots} \lambda_{\mu} \chi_{\mu}(r) \chi_{\mu}(r')
$$

(15)

where, $\{\lambda_{\mu}\}$ which are eigenvalues(since the density matrix is Hermitian and positive definite so its eigenvalues are positive and real, normalized to $\sum_{\mu} \lambda_{\mu} = 1$) and $\{\chi_{\mu}(r)\}$ is corresponding eigenvectors of $\rho_1(r, r')$. The single particle density profile of the system is given by the diagonal part of SPRDM, $\rho_1(r, r) \equiv \rho(r)$. If state is pure condensate, then in this case only one largest eigenvalue ($\lambda = 1$) remains significant i.e. von-Neumann entropy $S_1 \to 0$, which will be discussed in details in coming sections. In the case of fragmented condensate more then one eigenvalues of SPRDM $\lambda_1 \approx \lambda_2 \approx \lambda_3 \approx \cdots$ are significantly comparable, where indices are the labelled as quantum numbers corresponding the fragmented condensate. The fragmentation becomes a possible reality in the present case of finite number interacting dilute alkali vapours in harmonically trapped system of different parameters like interaction strength $g_2$, interaction range $\sigma$, orbital angular momentum $L_z$ or angular velocity $\Omega$, may favours the a commutation relation $[\hat{N}, \hat{\Theta}] = \hbar$, between the particles number $\hat{N}$.
and phase operator $\hat{\Theta}$. Our observations suggested that for $\sigma = 0.75$, the $\lambda_1 \equiv \lambda_2$ at $L_z(\sigma = 0.75) = 12, 23, 35, 46, \text{and} 58$ shown in Fig. 8. The fragmented ground state can also be viewed as the expectation over symmetry broken ground states [51, 52].

For our quasi-2D systems, the z-dimension in SPRDM has been traced out and hence setting $n_z = 0$, Eq. (15) reduces to

$$\rho_1(r_\perp, \phi; r'_\perp, \phi) = \sum_{\mu=1,2,3,\ldots} \lambda_{\mu} \chi_{\mu}^*(r_\perp, \phi; r'_\perp, \phi) \chi_{\mu}(r_\perp, \phi; r'_\perp, \phi)$$

(16)

where, $\mu \equiv \{n, m, n_z\}$.

$$\chi_{\mu}(r_\perp, \phi) = \sum_n c_{n,m,0}^\mu u_{n,m,0}(r, \phi)$$

where $|c_{n,m,0}^\mu|^2$ in the weight function. SPRDM eigenvectors can be redefined in terms of phase factor single particle basis function as, becomes,

$$\chi_{\mu}(r_\perp, \phi) = \sum_n \left(c_{n,m,0}^\mu f_{n,m,\mu}(r_\perp)\right) e^{im_\mu \phi}$$

where, $n = 2n_r + |m|$ and $\mu$ takes values in steps of $m_\mu$ with difference of two ($n = 0, m_1 = 0$) and ($n = 2, m_2 = 0$) which can be seen from Tables III, IV and V, where vorticity $m_\mu$ remains constant for the quantum number $n$. The constraint on total angular momentum as the empirical sum rule follows,

$$L_z = N \sum_\mu m_\mu \lambda_\mu$$

(17)
for a harmonically rotating trap. where, \( m_\mu \) can takes both sign, 0, \( \pm 1, \pm 2 \ldots \), the angular momentum is conserved in fragmented condensate, for a rotating system in a harmonic trap Eq. (17) and (17) have been verified through our computational schemes.

The eigenvector \( \chi_1(r) = \sum c_n^1 u_n(r) \) which is the macroscopic parameter in the mean-field Gross-Pitaevskii systems. From Fig. 6-8, the largest condensate fraction i.e. \( \lambda_1 \) at \( L_z \) for \( \sigma \) is listed below. for details see Fig. 9. The condensate depletes

| \( L_z \) | \( \lambda_1(\sigma = 0.30) \) | \( \lambda_1(\sigma = 0.50) \) | \( \lambda_1(\sigma = 0.75) \) |
|--------|--------------------|--------------------|--------------------|
| 0      | 0.989              | 0.993              | 0.997              |
| 16     | 0.875              | 0.879              | 0.883              |
| 32     | 0.435              | 0.481              | 0.639              |
| 48     | 0.484              | 0.479              | 0.468              |
| 60     | 0.493              | 0.501              | 0.505              |

with increase in \( L_z \) for a given interaction range \( \sigma \) and increases with the increase in the \( \sigma \).
FIG. 3. (Color online) For $N=16$ bosons, the von-Nuemann entropy $S_1$ vs $L_z$, total orbital angular momentum in the range of $0 \leq L_z \leq 4N$ for the interaction range $\sigma = 0.30, 0.50$ and 0.75 and interaction strength $g_2 = 0.9151$ of the Gaussian potential Eq. (1).
FIG. 4. (Color online) For N=16 bosons, the Degree of condensation $C_d$ vs $L_z$, total orbital angular momentum in the range of $0 \leq L_z \leq 4N$ for the interaction range $\sigma = 0.30, 0.50$ and 0.75 and interaction strength $g_2 = 0.9151$ of the Gaussian potential Eq. (1).
FIG. 5. (Color online) Gives the plot for von-Neumann entropy $S_1$ and degree of condensation $C_d$ vs interaction range $\sigma$ for the orbital angular momentum state $L_z = 32$ along rotation axis, and 2-D interaction strength $g_2 = 0.9151$. 
FIG. 6. (Color online) For $N=16$ bosons, plot for condensate fraction $\lambda_\mu$ vs total orbital angular momentum state $L_z$ about the rotation axis, such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$, for interaction range $\sigma = 0.30$, 2-D interaction strength $g_2 = 0.9151$, in the regime $0 \leq L_z \leq 4N$. 
FIG. 7. (Color online) For $N=16$ bosons, plot for condensate fraction $\lambda_\mu$ vs total orbital angular momentum state $L_z$ about rotation axis, such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$, for interaction range $\sigma = 0.50$, 2-D interaction strength $g_2 = 0.9151$, in the regime $0 \leq L_z \leq 4N$. 

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FIG. 8. (Color online) For $N=16$ bosons, plot for condensate fraction $\lambda_\mu$ vs $L_z$, total orbital angular momentum about rotation axis, such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$, for interaction range $\sigma = 0.75$, 2-D interaction strength $g_2 = 0.9151$, in the regime $0 \leq L_z \leq 4N$. 
FIG. 9. (Color online) For $N=16$ bosons, plot for condensate fraction $\lambda_1$ vs total orbital angular momentum state $L_z$ about the rotation axis, for different interaction range $\sigma$, 2-D interaction strength $g_2 = 0.9151$, in the regime $0 \leq L_z \leq 4N$.

IV. MANY-BODY QUANTUM CORRELATION

To further probe the systems we calculate the some other quantities of interest like von-Nuemann entropy degree of condensation and conditional probability density (CPD). Quantum correlation has been studied extensively to study the quantum nature of the systems [53, 54]. We calculated the SPRDM in Eq. (14), to calculate
A. QUANTUM ENTROPY

To study the quantum correlation of the many-body ground state, we calculated the von-Neumann entanglement entropy. It has been defined in different ways in thermodynamics, it has very well known form as $S = k_B \log \Omega$, where $k_B$ is Boltzmann constant and $\Omega$ is canonical ensemble. In terms of reduced density matrix it is defined as [55–58]

$$S_1 = -\text{Tr} \left( \hat{\rho} \ln \hat{\rho} \right)$$  \hspace{1cm} (18)

where, $\hat{\rho}$ is the single particle reduced density matrix obtained through the SPRDM in Eq. (14). In terms of the eigenvalues $\{\lambda_\mu\}$ of SPRDM calculated in the earlier section, the von-Nuemann entanglement entropy explicitly becomes

$$S_1 = -\sum_\mu \lambda_\mu \ln \lambda_\mu$$  \hspace{1cm} (19)

in the subspaces of total angular momentum $L_z$. 

the von-Neumann entropy defined in Eq. (19), Degree of condensation in Eq. (20) and CPD in Eq. (21).
FIG. 10. (Color online) CPD contour plots for \( N=16 \) spin-0 bosons in subspace of total angular momentum \( 0 \leq L_z \leq N \) with interaction strength \( g_2 = 0.9151 \) and different interaction range parameter \( \sigma \) in the finite range smooth Gaussian potential in Eq. (2). These contour plots are iso-surface density profile viewed along the rotation axis-\( z \). The reference point is chosen relatively large \( r_0 = (x_0, y_0) = (3, 0) \) in the units of \( a_\bot \). Yellow represents the highest probability density region falling off to the least in blue region shown in the vertical color bar.
FIG. 11. (Color online) CPD contour plots for N=16 spin-0 bosons in subspace of total angular momentum $L_z = 32$ with interaction strength $g_2 = 0.9151$ and different interaction range parameter $\sigma$ in the finite range smooth Gaussian potential in Eq. (2). These contour plots are iso-surface density profile viewed along the rotation axis-$z$. The reference point is chosen relatively large $r_0 = (x_0, y_0) = (3, 0)$ in the units of $a_\perp$. Yellow represents the highest probability density region falling off to the least in blue region shown in the vertical color bar.
FIG. 12. (Color online) CPD contour plots for $N=16$ spin-0 bosons in subspace of total angular momentum $L_z = 32$ with different interaction strength $g_2$ and interaction range parameter $\sigma$ in the finite range smooth Gaussian potential in Eq. (2). These contour plots are iso-surface density profile viewed along the rotation axis-z. The reference point is chosen relatively large $r_0 = (x_0, y_0) = (3, 0)$ in the units of $a_\perp$. Patterns of vortex nucleartion as function of $g_2$. Yellow represents the highest probability density region falling off to the least in blue region shown in the vertical color bar.
B. DEGREE OF CONDENSATION

The degree of condensation is being calculated using the eigenvalues \( \lambda_\mu \), obtained from the one particle reduced density matrix (SPRDM) in Eq. (14).

\[
C_d = \lambda_1 - \bar{\lambda}
\]  

(20)

where, 
\[
\bar{\lambda}_1 = \frac{1}{q-1} \sum_{\mu=2}^{q} \lambda_\mu
\]
is the arithmetic mean of the (N-1) eigenvalues. In the absence of condensation, the eigenstates are equally occupied, in such case the condensate would be zero[59]. In Fig. 3 and 4 we drawn two graphs to do a comparative study for von-Nueman entropy \( S_1 \) and degree of condensation \( C_d \). For non rotating case i.e. \( L_z = 0 \), the entropy is minimum \( S_1(\sigma) \rightarrow 0 \) while the degree of condensation \( C_d(\sigma) \rightarrow 1 \) is maximum irrespective of interaction range \( \sigma \), this is one of the signature of perfectly ordered state. As well for \( L_z = 0 \) the condensate fraction occupancy is maximum \( \lambda_1 \rightarrow 1 \) and all other \( \lambda_{(\mu \geq 2)} \rightarrow 0 \) in the ground state shown in Fig. 9. As \( L_z \) grows the condensate started depleting, more then one \( \lambda_\mu \) acquires non-zero values hence \( S_1 \) increases. One of the effects of stirring the condensate gives rise to vortices, which start to appear at the critical angular velocity \( \Omega_c \), which is one of the most important signature of superfluidity will be discussed in details in coming sections. The \( S_1 \) is slowly increasing with the increase in \( L_z \) till \( L_z = 11 \) and then starts decreasing till the first vortex state \( L_z = N \) appears has an oscillatory nature of varying amplitude throughout the entire range \( 0 \leq L_z \leq 4N \). There are minimum at every vortical state \( L_z(\Omega_{C_i}) \). There are few local minima in \( S_1 \) appeared for the different interaction range \( \sigma \) against \( L_z \) are listed here,

\[
L_z^{S_1}(\sigma = 0.30) = 0, 16, 19, 24, 26, 28, 30, 36, 39, 44, 52, 55, 60.
\]

\[
L_z^{S_1}(\sigma = 0.50) = 0, 16, 19, 24, 26, 28, 30, 36, 39, 44, 48, 52, 55, 60.
\]

\[
L_z^{S_1}(\sigma = 0.75) = 0, 16, 19, 22, 24, 26, 28, 30, 32, 36, 39, 42, 45, 48, 52, 56, 60.
\]
Fig. 1 belongs to these bold numbers correspond to the $L_z(\Omega_{c_i})$, $i = 1, 2, ...$. Therefore the von-Neumann entropy $S_1$ in Fig. 3 can also be studied in terms of stability-curve in Fig. 1. We can see clearly the finite range effects in the highly rotating regime, for example $S_{L_z=32}^{1\{\sigma = 0.30\} \{1.540\}} > S_{L_z=32}^{1\{\sigma = 0.50\} \{1.472\}} > S_{L_z=32}^{1\{\sigma = 0.75\} \{1.187\}}$. If we observe further, at $L_z = 22, 42, 50$ and $56$, a similar trend with the increase in interaction range occurs $S_{L_z=32}^{1\{\sigma = 0.30\} \{1.540\}} > S_{L_z=32}^{1\{\sigma = 0.50\} \{1.472\}} > S_{L_z=32}^{1\{\sigma = 0.75\} \{1.187\}}$. We summarized that the $L_z = 32$ is a vortex state must have minimum $S_1$ to become a stable state which is only greatly favoured by the largest $\sigma = 0.75$. For non-vortical states at $L_z = 40$ and $50$ the higher $\sigma$ has minimum $S_1$.

The another quantity of interest $C_d$ defined in Eq. (20) can be defined in terms of von-Neumann entropy $S_1$ defined in Eq. (19) which is sensitive to the loss of macro-occupation. To ascertain the relation between $C_d$ and $S_1$, we compare the plots in Fig. 4 and 3. We can see that the effects shown by the both graphs are counter contrast in nature. For non rotating case i.e. $L_z = 0$, the degree of condensation $C_d$ is maximum while the $S_1$ is minimum irrespective of interaction range $\sigma$. As rotation grows, $C_d$ has peak and $S_1$ has a dip at $L_z(\Omega_{c_i})$ as both shows competing effects. Minimum $S_1$ and maximum $C_d$ corresponds maximum may-body quantum correlation among the particles and maximally condensate states. Dip in $S_1$ and peak in $C_d$ can also be understood in the terms of vortex nucleation, first peak in $C_d$ and the corresponding dip corresponds in $S_1$ to the first vortex state $N = L_z$ shown in iso-surface CPD plots in Fig 10(p), 10(q) and 10(r) with vorticity $m_1 = 1$. At $L_z = 32$, there is another dip in $S_1$ and peak in $C_d$ corresponds to the higher $p = 2$ fold symmetrical vortex state with vorticity $m_1 = 2$ shown in the Fig. 14(e), 15(e) and 16(f) and $p = 5$ fold symmetrical vortices with vorticity $m_1 = 5$ are found at $L_z = 60$ corresponding the dip in $S_1$ and peak in $C_d$ is shown here in Fig. 14(j), 15(j) and 16(l). The von-Neumann quantum entanglement entropy $S_1$ and degree of condensation $C_d$ is listed
for the entire angular momentum state in range $0 \leq L_z \leq 4N$ in the Table VIII, ?? and ??, for different values of interaction range $\sigma = 0.30, 0.50$ and $0.75$ respectively. These Tables contain few details columnwise, (i) orbital angular momentum $L_z$ about the axis of rotation, (ii) ground state energy $E_{0}^{Lab}(L_z)$ in laboratory frame, (iii) critical angular $\Omega_k(L_z^i)$, $i = 1, 2, 3...$, discrete rotational symmetry, denoted by integer $p$, largest macroscopic occupation $\lambda_1$(macroscopic eigenvalue) $(\lambda_2, \lambda_3)$ (iv) corresponding single particle quantum numbers $n_1m_1$ ($(n_2m_2), (n_3m_3)$). (v) degree of condensation $C_d$ and the last (vi) the von-Nuemann entropy $S_1$. Further probing the system at second vortex state for angular momentum $L_z = 32(\sigma = 32)$, we plot Fig. 5, the von-Neumann entropy $S_1$ (degree of condensation $C_d$) decreases (increases) with the increase in the interaction range $\sigma$. In iso-surface CPD plots in Fig. 11 to observe the nucleation of vortices as the function of $\sigma$. The accumulation of probability density at the edge of vortices can be seen with increase in the interaction range $\sigma$. We compare the nucleation of vortex state around the second vortex in Fig. 17 and conclude this with the complete results in Tab. II.
FIG. 13. (Color online) CPD contour plots for $N=16$ spin-0 bosons in subspace of total angular momentum $L_z = 32$ in LLL approximation $n_r = 0$ with interaction strength $g_2 = 0.9151$ and different interaction range parameter $\sigma$ in the finite range smooth Gaussian potential in Eq. (2). These contour plots are iso-surface density profile viewed along the rotation axis-$z$. The reference point is chosen relatively large $r_0 = (x_0, y_0) = (3, 0)$ in the units of $a_\perp$. Yellow represents the highest probability density region falling off to the least in blue region shown in the vertical color bar.
FIG. 14. CPD contour plots for N=16 spin-0 bosons in subspace of total angular momentum $0 \leq L_z \leq 4N$ with interaction strength $g_2 = 0.9151$ and interaction range parameter $\sigma = 0.30$ in the finite range smooth Gaussian potential in Eq. (2). These contour plots are iso-surface density profile viewed along the rotation axis-z with p-fold symmetry ($p = 1, 2, 3, 4, \ldots$). The reference point is chosen relatively large $r_0 = (x_0, y_0) = (3, 0)$ in the units of $a_\perp$ for (14(a)-14(e)) and at (1.5,0) for (14(f)-14(j)). Yellow represents the highest probability density region falling off to the least in blue region shown in the vertical color bar.
FIG. 15. (Color online) CPD contour plots for $N=16$ spin-0 bosons in subspace of total angular momentum $0 \leq L_z \leq 4N$ with interaction strength $g_2 = 0.9151$ and interaction range parameter $\sigma = 0.50$ in the finite range smooth Gaussian potential in Eq. (2). These contour plots are iso-surface density profile viewed along the rotation axis-$z$ with $p$-fold symmetry ($p = 1, 2, 3, 4, \ldots$). The reference point is chosen relatively large $r_0 = (x_0, y_0) = (3, 0)$ in the units of $a_\perp$ for (15(a)-15(e)) and at (1.5,0) for (15(f)-15(j)). Yellow represents the highest probability density region falling off to the least in blue region shown in the vertical color bar.
| $L_z$ | $E_{0}^{lab} (\sigma = 0.3)$ | $E_{0}^{lab} (\sigma = 0.5)$ | $E_{0}^{lab} (\sigma = 0.75)$ | $E_{0}^{lab} (\sigma = 0.3)$ | $E_{0}^{lab} (\sigma = 0.5)$ | $E_{0}^{lab} (\sigma = 0.75)$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0    | 46.864          | 46.359          | 45.368          | 49.132          | 47.871          | 46.080          |
| 16   | 59.252          | 58.904          | 58.463          | 59.613          | 59.164          | 58.634          |
| 28   | 70.133          | 69.867          | 69.560          | 70.310          | 69.988          | 69.637          |
| 32   | 73.831          | 73.593          | 73.318          | 73.950          | 73.674          | 73.375          |
| 48   | 88.890          | 88.738          | 88.602          | 88.956          | 88.781          | 88.630          |
| 60   | 100.635         | 100.487         | 100.358         | 100.650         | 100.495         | 100.362         |
| 64   | 104.613         | 104.464         | 104.315         | 104.637         | 104.480         | 104.326         |

TABLE I. In this Table we note the ground state energy $E_0(\sigma)$ for Landau level approximation ($n_r = 0$) and for beyond Lowest level approximation ($n_r = 1$) in the subspace of total angular momentum $L_z$. 
FIG. 16. CPD contour plots for $N=16$ spin-0 bosons in subspace of total angular momentum $0 \leq L_z \leq 4N$ with interaction strength $g_2 = 0.9151$ and interaction range parameter $\sigma = 0.75$ in the finite range smooth Gaussian potential in Eq. (2). These contour plots are iso-surface density profile viewed along the rotation axis-z with $p$-fold symmetry ($p = 1, 2, 3, \ldots$). The reference point is chosen relatively large $r_0 = (x_0, y_0) = (3, 0)$ in the units of $a_\perp$ for (16(a)-16(f)) and at $(1.5,0)$ for (16(g)-16(l)). Yellow represents the highest probability density region falling off to the least in blue region shown in the vertical color bar.
FIG. 17. (Color online) For N=16 bosons, CPD isosurface contour plots for different total orbital angular momentum $L_z$ for a fixed interaction range $\sigma = 0.75$ and strength $g_2 = 0.9151$. The reference point is chosen at $r_0 = (3, 0)$ in the units of $a_\perp$. Probability density strength is shown in vertical color bar.
C. CONDITIONAL PROBABILITY DENSITY

To further probe the ground state property of weakly interaction bose gas, we calculate the Conditional Probability Density (CPDs). The CPDs is the measures of an intrinsic density distribution of bosons which makes it a experimental observable quantity over the circularly symmetric density distribution. This become an important tool to measure the boson density in the condensed bose gas where the circular symmetry is broken spontaneously. It is noted that CPDs is also used for measure of electron correlations in doubly excited helium like atoms [60, 61] and in quantum dot to study the formation of Wigner molecules[62, 63]. The CPDs $\rho(r, r_0)$ is defined as the probability of finding one boson at position $r$ given that the other particle is at $r_0$ is mathematically defined as. [24, 64].

$$\rho(r, r_0) = \frac{\sum_{i \neq j} \langle \Psi | \delta(r - r_i)\delta(r_0 - r_j) | \Psi \rangle}{(N - 1) \sum_j \langle \psi | \delta(r_0 - r_j) | \Psi \rangle}$$

(21)

where, $| \Psi \rangle$ represent the ground state single particle wave function obtained from the exact diagonalization and $r_0$ is the reference point in the x-y plane. Choice of reference $r_0 = (x_0, y_0)$ (in units of $a_\perp$) is chosen purposefully, for smaller orbital angular momentum $0 \leq L_z \leq 2N$, is given a significantly large value so that the symmetry breaking should be visible i.e. $(3, 0)$ and for $2N \leq L_z \leq 4N$, as condensate starts depletion and CPD countor starts shrinking inward hence the choice of reference is reduced to $(1.5, 0)$, for large number of particles, CPD contour becomes independent of $r_0$ [14] however [66] disagree with this.
1. SINGLE VORTEX STATE

In this section, we are confined in orbital angular momentum range \(0 \leq L_z < N\). In Fig. 10, we have taken \(L_z = 0, 12, 13, 14, 15\) and 16 for different interaction range \(\sigma = 0.30, 0.50\) and 0.75 have observed that the reference point plays an important role in the view-point of CPD symmetry. The feature is not visible unless the reference is not chosen significantly larger. The non-rotating orbital angular momentum state \(L_z = 0\) has a Gaussian symmetry for all \(\sigma\) despite increase in the probability density strength with increase in \(\sigma\) from 2.5 to 3.0 (shown in vertical bar) in Fig. 10(a), 10(b) and 10(c). There is decrease in probability density strength with increase in the angular momentum as the condensate deplete with increasing in \(L_z\). With the increase in \(L_z\) the Gaussian symmetry has diminished and envelope has started moving inward from the periphery, can be viewed in Fig. 10(d), 10(e) and 10(f). The observed broken symmetry starts emerging from \(l_z(\sigma = 0.30, 0.50, 0.75) = 0.75 = 12/16, 0.81 = 13/16\) unless and is not visible until the reference point \(r_0\) is not chosen relatively large, in our case \((3, 0)\). The symmetry breaking first occurred for the large interaction range \(\sigma = 0.75\) followed by the smaller \(\sigma = 0.30\) can be seen at \(l_z(\sigma = 0.30, 0.50, 0.75) = 0.88 = 14/16\) where the first signature of vortex nucleation appears onwards in Fig. 10(j), 10(k) and 10(l). The complete central vortex appears at \(l_z = 1\), in Fig. 10(p), 10(q) and 10(r) corresponding to first critical angular velocity \(\Omega_{c_1}\), shown in the Fig. 1, mimimum in von-Neumann entropy \(S_1\) shown in Fig. 3 and peak in the degree of condensation \(C_d\) shown in the Fig. 4 as well in condensate fraction \(\lambda_1\) in Fig. 6, 7 and 8. The quantized vortex has same value (vorticity) \(m_1 = 1\) for the quantum number \(n = 1, 3\), shown in Table III, IV and V. In conclusion we say that the larger interaction range range \(\sigma\) facilitates the nucleation of vortex. Emergence of single vortex state is studied by \([65]\) for \(\delta\) in LLL approximation for harmonic trap.
2. **MULTIPLE VORTEX STATE**

We presented the study of single vortex state \( N = L_z \) in the earlier section for all three interaction ranges in CPDs contour in Fig. 10 now we are going to present the study for higher vortices in the orbital angular momentum range \( N \geq L_z \leq 4N \) with interaction range paramter \( \sigma \) as we observed that the emergece of profiles first appears for higher \( \sigma \). Interaction range \( \sigma \) is an important agent to play with in order to understand the many-body effects of Bose-Einstein condensate through vortices nucleation. \( N = 16 \) is sufficient number of particles to study the many-body effect on higher vortices to study the Bose-Einstein condensation for different interaction ranges.

As angular momentum increased beyond \( L_z = N \) some additional vortices started to appear and arranged themself into the regular lattice patterns as shown in Fig. 14, 15 and 16 includes some unstable stable vortices for all three interaction range \( \sigma = 0.30, 0.50 \) and 0.75. The orbital angular momentm \( L_z = 26 \) is unstable state with single vortex for all values of \( \sigma \) shown in Fig. 14(c), 15(c) and 16(c) referring to the stability curve shown in the Fig. 1 correspondig dip in von-Neumann entropy \( S_1 \) shown in Fig. 3, peak in the degree of condensation \( C_d \) as shown in the Fig. 4 and in condensate fraction \( \lambda_1 \) in Fig. 9. The second stable vortex state \( (L_z = 28) \), appears at the critical velocity \( \Omega_c^2 \), has \( p = 2 \) fold symmetry. The quantized vortex has same value (vorticity) \( m_1 = 2 \) for the quantum number \( n = 2, 4 \), shown in Table III, IV and V a quantum jump, and the isosurface CPD contour in the Fig. 14(d), 15(d) and 16(d). For the \( \sigma = 0.75 \), in the stability curve Fig 1, appears at the orbital angular momentum \( L_z = 30 \) is stable vertex state which is absent for \( \sigma = 0.30 \) and 0.50, further a peak can be clearly seen in the condensate fraction \( \lambda_1 \) a dip in the von-Neuman entropy \( , \) peak in the degree of condensation \( C_d \) and the 2-fold symmetry isosurface in CPDs contour shown in Fig. 16(e), next \( p = 2 \) fold symmetrical vortices
are set to be at the orbital angular momentum \( L_z = 32 \) for all three interaction range \( \sigma = 0.30, 0.50 \) and 0.75 as shown in Fig. 14(e), 15(e) and 16(f). At this point we need to check the finite range effects, we observed that with increase in the parameter \( \sigma \) condensate fraction and degree of condensation has been increased as shown in Fig. 9 and 4 and corresponding decrease in the von-Neumann entropy \( S_1 \) is also reported in Fig. 3. We plot the CPDs for \( 0 \leq \sigma \leq 0.80 \) to further focus the change of state with the change of \( \sigma \) here are few more CPDs contour plots shown here in Fig. 11.

In Fig 17 for a fixed interaction range \( \sigma = 0.75 \) and in the subspace of total angular \( 31 \geq L_z \leq 33 \) the accumulation of particles density at the core of vortices has been reported in Fig. 17(b). Here in Fig. 11, we study the differences in isosurface CPD contour. With the increase in the interaction range \( \sigma \), emergence of stronger and equal depth vortices peak and the formation of hill of higher density around the vortices. One more thing to record is that, with the increase in \( \sigma \), nucleated vortices are going through signature change despite being the stable state (not sure for \( \sigma = 0.0, 0.1, 0.2, 0.4, 0.60, 0.70 \) and 0.80). Complete results are written in the Table. II

For fast rotating case, in the orbital angular momentum range \( 2N > L_z \leq 4N \), higher \( p = 3, 4, 5 \)...-fold symmetry started to emerge. we have triangular lattice for \( L_z = 35 \), 14(f) and \( L_z = 36 \), 14(g), 15(g) and 16(h). The vortices have arranged in a regular arrangements of Abrikosov lattices [31, 32]. Square vortic lattices are shown in Fig. 14(h), 14(i), 15(h), 15(i) and 16(j), 16(k) and pentagonal vortic lattices 14(j), 15(j) and 16(l) appears for the orbital angular momentum \( L_z = 48, 52 \) and 60 for all three interaction range parameter \( \sigma \) in accordance with the stability graph in Fig. 1. For interaction range \( \sigma = 0.75 \) and \( L_z = 35 \) is not the critical state in contrast to other \( \sigma = 0.30 \) and 0.50 we do this plot for the comparative study.
TABLE II. For N=16 bosons, interacting via finite range Gaussian potential in Eq. (1) with interaction strength $g_2 = 0.9151$ and different interaction range $\sigma$. With their vorticity $(m_\mu) (\mu = 1, 2, 3)$ are reported from SPRDM in Eq. (14) in given subspaces of the total angular momentum $L_z = 32$ along with von-Nuemann entropy $S_1$ and ground state energy $E_{0}^{Lab}(L_z)$. Here the state has been changed with the increase in $\sigma$.

| $\sigma$ | $n_1,m_1$ | $n_2,m_2$ | $n_3,m_3$ | $S_1$ | $E_{0}^{Lab}(L_z)$ |
|----------|----------|----------|----------|------|-------------------|
| 0.00     | 2,2      | 0,0      | 4,4      | 1.56 | 74.04561          |
| 0.10     | 2,2      | 0,0      | 4,4      | 1.56 | 74.01746          |
| 0.20     | 2,2      | 0,0      | 4,4      | 1.55 | 73.94022          |
| 0.30     | 2,2      | 0,0      | 4,4      | 1.54 | 73.83171          |
| 0.40     | 2,2      | 0,0      | 4,4      | 1.52 | 73.71143          |
| 0.50     | 2,2      | 0,0      | 4,4      | 1.47 | 73.59262          |
| 0.60     | 2,2      | 0,0      | 4,4      | 1.39 | 73.47973          |
| 0.70     | 2,2      | 0,0      | 3,3      | 1.26 | 73.37109          |
| 0.75     | 2,2      | 3,3      | 1,1      | 1.19 | 73.31758          |
| 0.80     | 2,2      | 1,1      | 3,3      | 1.11 | 73.26460          |

V. SUMMARY AND CONCLUSION

For a system of N= 16 spin-0 bosons interacting via a finite range normalized Gaussian potential in an harmonic trap. Our scheme suggests that for many-body system in finite range has explicit dependency of the range $\sigma$ and strength $g_2$. We explicitly shown that the critical angular velocity $\{\Omega_{c}, i = 1, 2 \cdots\}$ increases with increase in interaction range. Apart from this, length and height of plateaus aslo depends on the finite range $\sigma$ for a given parameters shown in the stability graph in Fig.1. In the Condensate fraction $\lambda_1$, the finite range effects are dominant in the higher angular momentum regime, $N < L_z < 4N$. Fig.9 as well as the von-Neumann entropy and degree of condensation too shows variation for higher $L_z$ as shown in Fig. 3 and 4 with this we conclude that the vortex entry takes place first for higher $\sigma$ followed by the smaller $\sigma$. 

TABLE III. For \( N=16 \) bosons, the many-body ground state energies in Lab frame \( E_{0}^{lab}(L_{i}^{z}) \), interacting via finite range Gaussian potential in Eq. (1) with interaction strength \( g_{2} = 0.9151 \) and interaction range \( \sigma = 0.30 \). Three largest macroscopic eigenvalues \( \lambda_{1} > \lambda_{(\mu \geq 2)} \), weight \( \left| c_{n,m}^{1} \right|^{2} \) of the corresponding states with their quantum numbers \( (n, m) \) are reported from SPRDM in Eq. (14) respectively along with critical angular velocity \( \Omega_{c}(L_{i}^{z}) \), \( i = 1, 2, 3, \ldots \), in given subspaces of the total angular momentum \( L_{i}^{z} \).

| \( L_{i}^{z} \) | \( E_{0}^{lab}(L_{i}^{z}) \) | \( \Omega_{c}(L_{i}^{z}) \) | \( (n, m) \) | \( \left| c_{n,m}^{1} \right|^{2} \) | \( \lambda_{1} \) |
|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0              | 46.86428        | 0.0000          | 0, 0            | 0.97593         | 0.98949         |
|                |                 |                 | 2, 0            | 0.21809         |                 |
| 16             | 59.25225        | 0.77425         | 1, 1            | 0.99542         | 0.87484         |
|                |                 |                 | 3, 1            | 0.09558         |                 |
| 28             | 70.13310        | 0.90674         | 2, 2            | 0.99684         | 0.61063         |
|                |                 |                 | 4, 2            | 0.07941         |                 |
| 32             | 73.83171        | 0.92465         | 2, 2            | 0.99814         | 0.43495         |
|                |                 |                 | 4, 2            | 0.06089         |                 |
| 35             | 76.62143        | 0.92990         | 3, 3            | 0.99786         | 0.32727         |
|                |                 |                 | 5, 3            | 0.06540         |                 |
| 36             | 77.55398        | 0.93255         | 3, 3            | 0.99793         | 0.48498         |
|                |                 |                 | 5, 3            | 0.06429         |                 |
| 48             | 88.89032        | 0.94469         | 4, 4            | 0.99900         | 0.48443         |
|                |                 |                 | 6, 4            | 0.04476         |                 |
| 52             | 92.79163        | 0.97533         | 4, 4            | 0.99916         | 0.54632         |
|                |                 |                 | 6, 4            | 0.04103         |                 |
| 60             | 100.63525       | 0.98045         | 5, 5            | 1.0000          | 0.49277         |
TABLE IV. For $N=16$ bosons, the many-body ground state energies in Lab frame $E_{lab}^{(L_z)}$, interacting via finite range Gaussian potential in Eq. (1) with interaction strength $g_2 = 0.9151$ and interaction range $\sigma = 0.50$. Three largest macroscopic eigenvalues $\lambda_1 > \lambda_{(\mu \geq 2)}$, weight $|c_{n,m_1}^{\mu}|^2$ of the corresponding states with their quantum numbers $(n, m_1)$ are reported from SPRDM in Eq. (14) respectively along with critical angular velocity $\Omega_c (L_z^i)$, $i = 1, 2, 3, \ldots$, in given subspaces of the total angular momentum $L_z^i$.

| $L_z^i$ | $E_{lab}^{(L_z^i)}$ | $\Omega_c (L_z^i)$ | $(n, m_1)$ | $|c_{n,m_1}^{\mu}|^2$ | $\lambda_1$ |
|--------|---------------------|---------------------|-----------|------------------------|-----------|
| 0      | 46.35902            | 0.0000              | 0, 0      | 0.98153                | 0.99292   |
|        |                     |                     | 2, 0      | 0.19131                |           |
| 16     | 58.90377            | 0.78405             | 1, 1      | 0.99650                | 0.87949   |
|        |                     |                     | 3, 1      | 0.08360                |           |
| 28     | 69.86716            | 0.91362             | 2, 2      | 0.99773                | 0.61129   |
|        |                     |                     | 4, 2      | 0.06737                |           |
| 32     | 73.59262            | 0.93136             | 2, 2      | 0.99866                | 0.48124   |
|        |                     |                     | 4, 2      | 0.05178                |           |
| 36     | 77.34967            | 0.93926             | 3, 3      | 0.99846                | 0.48347   |
|        |                     |                     | 5, 3      | 0.05554                |           |
| 48     | 88.73774            | 0.94901             | 4, 4      | 0.99923                | 0.47930   |
|        |                     |                     | 6, 4      | 0.03931                |           |
| 52     | 92.64065            | 0.97573             | 4, 4      | 0.99935                | 0.54190   |
|        |                     |                     | 2, 0      | 0.02151                |           |
| 60     | 102.50076           | 0.98079             | 6, 6      | 1.0000                 | 0.40369   |

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TABLE V. For \( N = 16 \) bosons, the many-body ground state energies in Lab frame \( E_{0}^{lab}(L_{i}^{z}) \), interacting via finite range Gaussian potential in Eq. (1) with interaction strength \( g_{2} = 0.9151 \) and interaction range \( \sigma = 0.75 \). Three largest macroscopic eigenvalues \( \lambda_{1} > \lambda_{(\mu \geq 2)} \), weight \( |c_{n,m_{1}}^{1}|^{2} \) of the corresponding states with their quantum numbers \((n, m_{1})\) are reported from SPRDM in Eq. (14) respectively along with critical angular velocity \( \Omega_{c}(L_{i}^{z}) \), \( i = 1, 2, 3, \ldots \), in given subspaces of the total angular momentum \( L_{i}^{z} \).

| \( L_{i}^{z} \) | \( E_{0}^{lab}(L_{i}^{z}) \) | \( \Omega_{c}(L_{i}^{z}) \) | \( (n, m_{1}) \) | \( |c_{n,m_{1}}^{1}|^{2} \) | \( \lambda_{1} \) |
|---|---|---|---|---|---|
| 0  | 45.36785 | 0.0000 | 0, 0 | 0.98978 | 0.99664 |
|    |           |        | 2, 0 | 0.14263 |        |
| 16 | 58.46294 | 0.81844 | 1, 1 | 0.99650 | 0.87949 |
|    |           |        | 3, 1 | 0.08360 |        |
| 28 | 69.55952 | 0.92471 | 2, 2 | 0.99854 | 0.62821 |
|    |           |        | 4, 2 | 0.05405 |        |
| 30 | 71.43247 | 0.93648 | 2, 2 | 0.99886 | 0.67711 |
|    |           |        | 4, 2 | 0.04779 |        |
| 32 | 73.31758 | 0.94256 | 2, 2 | 0.99912 | 0.63923 |
|    |           |        | 4, 2 | 0.04202 |        |
| 36 | 77.12801 | 0.95262 | 3, 3 | 0.99894 | 0.50812 |
|    |           |        | 5, 3 | 0.04597 |        |
| 39 | 79.98856 | 0.95352 | 3, 3 | 0.99914 | 0.53684 |
|    |           |        | 5, 3 | 0.04148 |        |
| 48 | 88.60174 | 0.95702 | 4, 4 | 0.99942 | 0.46760 |
|    |           |        | 6, 4 | 0.03406 |        |
| 52 | 93.51604 | 0.97464 | 4, 4 | 0.99960 | 0.32485 |
|    |           |        | 6, 4 | 0.02834 |        |
| 60 | 100.35792| 0.98220 | 5, 5 | 1.0000  | 0.50544 |
TABLE VI: Many-body ground state energies in Lab frame, $E^\text{lab}_0$, for $N=16$ bosons interacting via finite range Gaussian potential given in Eq. (1), with interaction strength $g_2 = 0.9151$ and interaction range $\sigma = 0.30$. The three largest macroscopic eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$ of the SPRDM, corresponding single particle quantum numbers $(n_1, m_1)$, $(n_2, m_2)$ and $(n_3, m_3)$ respectively, degree of condensation $C_d$ and von-Nuemann entropy $S_1$ are calculated from SPRDM in Eq. (14) along with critical angular velocity $\Omega_c$ and $p$-fold symmetry in the total angular momentum space ranging from $0 \leq L_z \leq 4N$ are noted below in beyond LLL $n_r = 1$.

| $L_z$ | $E^\text{lab}_0(L_z)$ | $\Omega_c$ | $p$ | $(n_1, m_1)$ | $\lambda_1$ | $(n_2, m_2)$ | $\lambda_2$ | $(n_3, m_3)$ | $\lambda_3$ | $C_d$ | $S_1$ |
|------|----------------|--------|----|--------------|-----------|--------------|-----------|--------------|-----------|------|------|
| 0    | 46.86428       | 0.0    | -  | 0,0          | 0.98949   | 1,-1         | 0.00378   | 1,1          | 0.00378   | 0.98833 | 0.07430 |
| 1    | 47.87008       | 0,0    | 1,1| 0.92247      | 0.06723   | 1,-1         | 0.00643   | 0.91385      | 0.31638   |        |      |
| 2    | 48.63374       | 0,0    | 2,2| 0.91733      | 0.05309   | 1,1          | 0.02279   | 0.90814      | 0.36496   |        |      |
| 3    | 49.32981       | 0,0    | 3,3| 0.91477      | 0.05037   | 1,1          | 0.01778   | 0.90530      | 0.39136   |        |      |
| 4    | 50.25820       | 0,0    | 1,1| 0.83190      | 0.09654   | 2,2          | 0.03397   | 0.81322      | 0.64360   |        |      |
| 5    | 51.02044       | 0,0    | 1,1| 0.81554      | 0.07699   | 2,2          | 0.06578   | 0.79505      | 0.69920   |        |      |
| 6    | 51.76595       | 0,0    | 3,3| 0.80862      | 0.07690   | 2,2          | 0.07077   | 0.78736      | 0.71579   |        |      |
| 7    | 52.59151       | 0,0    | 1,1| 0.70849      | 0.17101   | 2,2          | 0.07419   | 0.67610      | 0.90531   |        |      |
| 8    | 53.34973       | 0,0    | 1,1| 0.66720      | 0.19594   | 2,2          | 0.08829   | 0.63022      | 0.97283   |        |      |
| 9    | 54.10657       | 0,0    | 1,1| 0.61417      | 0.24315   | 2,2          | 0.09162   | 0.57131      | 1.03552   |        |      |
| 10   | 54.86196       | 0,0    | 1,1| 0.54235      | 0.31429   | 2,2          | 0.10388   | 0.49151      | 1.07763   |        |      |
| 11   | 55.60626       | 0,0    | 1,1| 0.47917      | 0.37575   | 2,2          | 0.10950   | 0.42130      | 1.09831   |        |      |
| 12   | 56.34683       | 1,1    | -  | 0.44883      | 0.040950  | 2,2          | 0.11146   | 0.38759      | 1.09055   |        |      |

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|   |       |     |     |     |     |     |     |     |
|---|-------|-----|-----|-----|-----|-----|-----|-----|
| 13 | 57.08395 | 1,1 | 0.53212 | 0.00 | 0.33439 | 2,2 | 0.10924 | 0.48013 | 1.04701 |
| 14 | 57.81826 | 1,1 | 0.62887 | 0.00 | 0.25247 | 2,2 | 0.10044 | 0.58764 | 0.95374 |
| 15 | 58.54967 | 1,1 | 0.74481 | 0.00 | 0.16083 | 2,2 | 0.08223 | 0.71646 | 0.78072 |
| 16 | 59.25225 | 1,1 | 0.87484 | 0.00 | 0.06268 | 2,2 | 0.05460 | 0.86347 | 0.49646 |
| 17 | 60.25409 | 1,1 | 0.72000 | 2,2 | 0.14957 | 0.00 | 0.11366 | 0.69455 | 0.85252 |
| 18 | 61.23338 | 1,1 | 0.62690 | 2,2 | 0.19541 | 0.00 | 0.14181 | 0.59298 | 1.04460 |
| 19 | 62.05475 | 1,1 | 0.73381 | 2,2 | 0.10344 | 0.00 | 0.09326 | 0.70961 | 0.92369 |
| 20 | 63.00573 | 1,1 | 0.54544 | 2,2 | 0.22910 | 0.00 | 0.15422 | 0.50412 | 1.20600 |
| 21 | 63.93098 | 1,1 | 0.37760 | 2,2 | 0.33382 | 0.00 | 0.21138 | 0.32102 | 1.32639 |
| 22 | 64.79399 | 2,2 | 0.36431 | 1,1 | 0.30116 | 0.00 | 0.23872 | 0.30652 | 1.36787 |
| 23 | 65.70191 | 2,2 | 0.40268 | 1,1 | 0.27154 | 0.00 | 0.22220 | 0.34837 | 1.37355 |
| 24 | 66.54940 | 2,2 | 0.53143 | 0.00 | 0.28750 | 1,1 | 0.08470 | 0.48884 | 1.19590 |
| 25 | 67.49024 | 2,2 | 0.46150 | 0.00 | 0.21617 | 1,1 | 0.18994 | 0.41254 | 1.37520 |
| 26 | 68.31986 | 2,2 | 0.58021 | 0.00 | 0.24529 | 4,4 | 0.07453 | 0.54205 | 1.15945 |
| 27 | 69.31222 | 2,2 | 0.50609 | 0.00 | 0.19009 | 1,1 | 0.14434 | 0.46119 | 1.34971 |
| 28 | 70.13310 | 2,2 | 0.61063 | 0.00 | 0.20011 | 4,4 | 0.08265 | 0.57524 | 1.15051 |
| 29 | 71.16949 | 2,2 | 0.47328 | 0.00 | 0.19354 | 3,3 | 0.17233 | 0.42540 | 1.38655 |
| 30 | 71.99933 | 2,2 | 0.61755 | 0.00 | 0.16475 | 4,4 | 0.10182 | 0.58278 | 1.16531 |
| 31 | 73.03732 | 2,2 | 0.42735 | 3,3 | 0.20508 | 0.00 | 0.19018 | 0.37529 | 1.44184 |
| 32 | 73.83171 | 2,2 | 0.43495 | 0.00 | 0.19058 | 4,4 | 0.14261 | 0.38358 | 1.53990 |
| 33 | 74.77374 | 3,3 | 0.26996 | 0.00 | 0.24653 | 2,2 | 0.22551 | 0.20360 | 1.67774 |
| 34 | 75.70208 | 3,3 | 0.26981 | 2,2 | 0.23826 | 0.00 | 0.23264 | 0.20343 | 1.66348 |
| 35 | 76.62143 | 3,3 | 0.32727 | 0.00 | 0.24476 | 2,2 | 0.18294 | 0.26611 | 1.62796 |
| 36 | 77.55398 | 3,3 | 0.48498 | 0.00 | 0.24302 | 2,2 | 0.10486 | 0.43816 | 1.41080 |
| 37 | 78.52752 | 3,3 | 0.28141 | 0.00 | 0.21992 | 2,2 | 0.20067 | 0.21608 | 1.70601 |
|   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|
| 38 | 79.45758 | 3,3 | 0.36911 | 0,0 | 0.21714 | 2,2 | 0.15734 | 0.31176 | 1.61093 |
| 39 | 80.39032 | 3,3 | 0.47656 | 0,0 | 0.20374 | 4,4 | 0.12276 | 0.42898 | 1.46202 |
| 40 | 81.37679 | 0,0 | 0.24484 | 4,4 | 0.22016 | 3,3 | 0.19725 | 0.17619 | 1.72851 |
| 41 | 82.34432 | 3,3 | 0.32571 | 0,0 | 0.19444 | 4,4 | 0.15778 | 0.26441 | 1.67173 |
| 42 | 83.28308 | 3,3 | 0.35277 | 0,0 | 0.19483 | 4,4 | 0.16750 | 0.29393 | 1.64412 |
| 43 | 84.23940 | 4,4 | 0.33911 | 0,0 | 0.24764 | 3,3 | 0.16590 | 0.27903 | 1.61433 |
| 44 | 85.17724 | 4,4 | 0.44093 | 0,0 | 0.26281 | 5,5 | 0.10992 | 0.39011 | 1.46003 |
| 45 | 86.19175 | 4,4 | 0.25767 | 0,0 | 0.24947 | 5,5 | 0.23945 | 0.19018 | 1.65250 |
| 46 | 87.16767 | 4,4 | 0.28202 | 0,0 | 0.19966 | 5,5 | 0.19827 | 0.21675 | 1.68982 |
| 47 | 88.09247 | 4,4 | 0.37420 | 0,0 | 0.21216 | 5,5 | 0.16681 | 0.31731 | 1.57529 |
| 48 | 88.89032 | 0.94469 | 4 | 4,4 | 0.48443 | 0,0 | 0.22052 | 3,3 | 0.10773 | 0.43757 | 1.48350 |
| 49 | 89.92046 | 4,4 | 0.32123 | 0,0 | 0.21793 | 5,5 | 0.17497 | 0.25953 | 1.69896 |
| 50 | 90.90461 | 4,4 | 0.21761 | 0,0 | 0.20342 | 3,3 | 0.20182 | 0.14648 | 1.83417 |
| 51 | 91.86018 | 4,4 | 0.28196 | 3,3 | 0.21103 | 0,0 | 0.18267 | 0.21668 | 1.76550 |
| 52 | 92.79163 | 0.97533 | 4 | 4,4 | 0.54632 | 0,0 | 0.17165 | 3,3 | 0.09413 | 0.50507 | 1.40655 |
| 53 | 93.81503 | 4,4 | 0.29268 | 5,5 | 0.19396 | 0,0 | 0.17741 | 0.22837 | 1.75579 |
| 54 | 94.79663 | 5,5 | 0.23017 | 0,0 | 0.21764 | 4,4 | 0.19550 | 0.16018 | 1.79180 |
| 55 | 95.76346 | 5,5 | 0.40131 | 0,0 | 0.24870 | 6,6 | 0.13557 | 0.34689 | 1.59887 |
| 56 | 96.74138 | 6,6 | 0.36293 | 1,1 | 0.22038 | 0,0 | 0.11831 | 0.30501 | 1.72438 |
| 57 | 97.74751 | 6,6 | 0.31343 | 5,5 | 0.17286 | 0,0 | 0.16517849 | 0.25101 | 1.81372 |
| 58 | 98.73114 | 6,6 | 0.26638 | 5,5 | 0.22223 | 0,0 | 0.17347 | 0.19969 | 1.81821 |
| 59 | 99.68387 | 5,5 | 0.33968 | 0,0 | 0.20227 | 6,6 | 0.19941 | 0.27965 | 1.68349 |
| 60 | 100.63525 | 0.98045 | 5 | 5,5 | 0.49277 | 0,0 | 0.21973 | 6,6 | 0.13320 | 0.44666 | 1.42236 |
| 61 | 101.62990 | 6,6 | 0.45397 | 1,1 | 0.22843 | 5,5 | 0.09139 | 0.40433 | 1.55209 |
| 62 | 102.64995 | 6,6 | 0.41259 | 0,0 | 0.14677 | 5,5 | 0.10603 | 0.35918 | 1.69092 |
TABLE VII: Many-body ground state energies in Lab frame, $E_{0}^{\text{lab}}$, for $N=16$ bosons interacting via finite range Gaussian potential given in Eq. (1), with interaction strength $g_{2} = 0.9151$ and interaction range $\sigma = 0.50$. The three largest macroscopic eigenvalues $\lambda_{1} > \lambda_{2} > \lambda_{2}$ of the SPRDM, corresponding single particle quantum numbers $(n_{1}, m_{1})$, $(n_{2}, m_{2})$ and $(n_{3}, m_{3})$ respectively, degree of condensation $C_{d}$ and von-Nuemann entropy $S_{1}$ are calculated from SPRDM in Eq. (14) along with critical angular velocity $\Omega_{c}$ and $p-$fold symmetry in the total angular momentum space ranging from $0 \leq L_{z} \leq 4N$ are noted below in beyond LLL $n_{r} = 1$.

| $L_{z}$ | $E_{0}^{\text{lab}}(L_{z})$ | $\Omega_{c}$ | $p$ | $(n_{1}, m_{1})$ | $\lambda_{1}$ | $(n_{2}, m_{2})$ | $\lambda_{2}$ | $(n_{3}, m_{3})$ | $\lambda_{3}$ | $C_{d}$ | $S_{1}$ |
|--------|-----------------------------|-------------|-----|-----------------|----------------|-----------------|----------------|----------------|----------------|----------|--------|
| 0      | 46.35902                    | 0           | -   | 0.0             | 0.99292        | 1, -1           | 0.00278        | 1, 1           | 0.00278        | 0.99213   | 0.05181 |
| 1      | 47.36087                    | 0           | 0.0 | 0.92678         | 1.1            | 0.06622         | 1, -1           | 0.00485        | 0.91865        | 0.29272   |
| 2      | 48.08484                    | 0           | 0.0 | 0.92363         | 2, 2           | 0.05542         | 1, 1           | 0.01677        | 0.91515        | 0.33062   |
| 3      | 48.81060                    | 0           | 0.0 | 0.91811         | 3, 3           | 0.05051         | 1, 1           | 0.01665        | 0.90902        | 0.37195   |
| 4      | 49.73297                    | 0           | 0.0 | 0.83691         | 1, 1           | 0.08557         | 2, 2           | 0.05098        | 0.81879        | 0.62421   |
| 5      | 50.48376                    | 0           | 0.0 | 0.82582         | 2, 2           | 0.06967         | 1, 1           | 0.06365        | 0.80646        | 0.66569   |
| 6      | 51.25036                    | 0           | 0.0 | 0.81479         | 3, 3           | 0.07857         | 1, 1           | 0.06500        | 0.79422        | 0.69206   |
| 7      | 52.08038                    | 0           | 0.0 | 0.71652         | 1, 1           | 0.15698         | 2, 2           | 0.08601        | 0.68503        | 0.88604   |
| 8      | 52.84751                    | 0           | 0.0 | 0.67944         | 1, 1           | 0.17796         | 2, 2           | 0.09439        | 0.64382        | 0.95372   |
| 9      | 53.62127                    | 0           | 0.0 | 0.62186         | 1, 1           | 0.23161         | 2, 2           | 0.09790        | 0.57984        | 1.02310   |

54
|   | 54.39106 | 0.0 | 0.54967 | 1,1 | 0.30222 | 2,2 | 0.11176 | 0.49964 | 1.06854 |
|---|----------|-----|---------|-----|---------|-----|---------|---------|---------|
| 11| 55.15183 | 0.0 | 0.48501 | 1,1 | 0.36641 | 2,2 | 0.11592 | 0.42779 | 1.09168 |
| 12| 55.90975 | 1,1 | 0.44355 | 0.0 | 0.41282 | 2,2 | 0.11674 | 0.38172 | 1.08372 |
| 13| 56.66452 | 1,1 | 0.53037 | 0.0 | 0.33575 | 2,2 | 0.11280 | 0.47818 | 1.03877 |
| 14| 57.41706 | 1,1 | 0.63071 | 0.0 | 0.25192 | 2,2 | 0.10200 | 0.58967 | 0.94210 |
| 15| 58.16768 | 1,1 | 0.74901 | 0.0 | 0.15910 | 2,2 | 0.08204 | 0.72113 | 0.76514 |
| 16| 58.90377 | 0.78405 | 1,1 | 0.87949 | 0.0 | 0.06083 | 2,2 | 0.05387 | 0.86853 | 0.47622 |
| 17| 59.90512 | 1,1 | 0.72520 | 2,2 | 0.14895 | 0.0 | 0.11138 | 0.70022 | 0.83373 |
| 18| 60.87598 | 1,1 | 0.65539 | 2,2 | 0.16395 | 0.0 | 0.13487 | 0.62406 | 1.02094 |
| 19| 61.70337 | 1,1 | 0.74832 | 2,2 | 0.09687 | 0.0 | 0.08710 | 0.72544 | 0.88665 |
| 20| 62.66299 | 1,1 | 0.56506 | 2,2 | 0.21911 | 0.0 | 0.14556 | 0.52552 | 1.17825 |
| 21| 63.59491 | 1,1 | 0.40591 | 2,2 | 0.31222 | 0.0 | 0.20091 | 0.35190 | 1.32012 |
| 22| 64.46708 | 1,1 | 0.41047 | 2,2 | 0.29314 | 0.0 | 0.19153 | 0.35688 | 1.35405 |
| 23| 65.38521 | 2,2 | 0.38009 | 1,1 | 0.30566 | 0.0 | 0.20860 | 0.32373 | 1.37516 |
| 24| 66.25488 | 2,2 | 0.52244 | 0.0 | 0.27979 | 1,1 | 0.27979 | 0.47903 | 1.21842 |
| 25| 67.19100 | 2,2 | 0.43787 | 1,1 | 0.21557 | 0.0 | 0.20891 | 0.38676 | 1.39541 |
| 26| 68.04368 | 2,2 | 0.57653 | 0.0 | 0.24110 | 4,4 | 0.07248 | 0.53803 | 1.17032 |
| 27| 69.02539 | 2,2 | 0.49911 | 0.0 | 0.18611 | 1,1 | 0.15505 | 0.45357 | 1.35454 |
| 28| 69.86716 | 0.91362 | 2,2 | 0.61129 | 0.0 | 0.19537 | 4,4 | 0.07853 | 0.57595 | 1.15285 |
| 29| 70.88586 | 2,2 | 0.47026 | 0.0 | 0.18210 | 3,3 | 0.17080 | 0.42211 | 1.39336 |
| 30| 71.73464 | 2,2 | 0.63313 | 0.0 | 0.15321 | 3,3 | 0.09048 | 0.59978 | 1.14251 |
| 31| 72.76240 | 2,2 | 0.44022 | 3,3 | 0.20842 | 0.0 | 0.17746 | 0.38933 | 1.42515 |
| 32| 73.59262 | 0.93136 | 2,2 | 0.48124 | 0.0 | 0.16906 | 4,4 | 0.13125 | 0.43408 | 1.47162 |
| 33| 74.54991 | 2,2 | 0.30196 | 0.0 | 0.21557 | 3,3 | 0.19393 | 0.23851 | 1.68952 |
| 34| 75.48376 | 3,3 | 0.25519 | 2,2 | 0.25365 | 0.0 | 0.22252 | 0.18748 | 1.66507 |
|   | 35 76.41198 | 3,3 0.31683 0,0 0.23683 2,2 0.19349 0.25472 1.63755 |
|---|-------------|--------------------------------------------------|
|   | 36 77.34967 | 0.93926 3,3 0.48347 0,0 0.23920 2,2 0.18098 0.43652 1.41417 |
|   | 37 78.32315 | 3,3 0.27538 0,0 0.21411 2,2 0.20491 0.43652 1.45707 |
|   | 38 79.26053 | 3,3 0.36638 0,0 0.21275 2,2 0.16164 0.30878 1.61267 |
|   | 39 80.19670 | 3,3 0.47883 0,0 0.20164 4,4 0.12226 0.43146 1.45707 |
|   | 40 81.18208 | 0,0 0.23981 4,4 0.22654 3,3 0.18993 0.17070 1.73263 |
|   | 41 82.14866 | 3,3 0.32007 0,0 0.19086 4,4 0.16184 0.25826 1.67613 |
|   | 42 83.09047 | 3,3 0.38254 0,0 0.18530 4,4 0.16036 0.32641 1.61470 |
|   | 43 84.05134 | 4,4 0.33797 0,0 0.24222 3,3 0.17261 0.27778 1.61750 |
|   | 44 84.99470 | 4,4 0.43838 0,0 0.25920 5,5 0.10867 0.38733 1.46766 |
|   | 45 86.00803 | 4,4 0.25919 0,0 0.24227 5,5 0.23546 0.19184 1.66284 |
|   | 46 86.97846 | 4,4 0.27755 5,5 0.19614 0,0 0.19235 0.21188 1.69785 |
|   | 47 87.91174 | 4,4 0.37577 0,0 0.20809 5,5 0.16238 0.31902 1.57633 |
|   | 48 88.73774 | 0.94901 4,4 0.47930 0,0 0.21845 3,3 0.11075 0.43196 1.49457 |
|   | 49 89.76270 | 4,4 0.32100 0,0 0.21662 5,5 0.17762 0.25927 1.69756 |
|   | 50 90.74682 | 4,4 0.23463 3,3 0.20335 0,0 0.19545 0.16505 1.82869 |
|   | 51 91.70336 | 4,4 0.29217 3,3 0.21128 0,0 0.17811 0.22782 1.75395 |
|   | 52 92.64065 | 0.97573 4,4 0.54190 0,0 0.17028 3,3 0.09559 0.50026 1.41701 |
|   | 53 93.65995 | 4,4 0.30013 5,5 0.19670 0,0 0.17472 0.23651 1.74607 |
|   | 54 94.64215 | 5,5 0.22788 0,0 0.21393 3,3 0.21335 0.15769 1.78591 |
|   | 55 95.61163 | 5,5 0.41169 0,0 0.2505 6,6 0.12700 0.35821 1.57743 |
|   | 56 96.59324 | 6,6 0.32819 1,1 0.19334 0,0 0.12312 0.26711 1.79390 |
|   | 57 97.59379 | 6,6 0.27007 5,5 0.22202 0,0 0.17065 0.20371 1.82421 |
|   | 58 98.57570 | 6,6 0.25901 5,5 0.22335 0,0 0.17368 0.19165 1.81728 |
|   | 59 99.53192 | 5,5 0.34524 0,0 0.20254 6,6 0.19324 0.28572 1.67410 |
TABLE VIII: Many-body ground state energies in Lab frame, $E_{0}^{lab}$, for N=16 bosons interacting via finite range Gaussian potential given in Eq. (1), with interaction strength $g_2 = 0.9151$ and interaction range $\sigma = 0.75$. The three largest macroscopic eigenvalues $\lambda_1 > \lambda_2 > \lambda_2$ of the SPRDM, corresponding single particle quantum numbers $(n_1,m_1)$, $(n_2,m_2)$ and $(n_3,m_3)$ respectively, degree of condensation $C_d$ and von-Neumann entropy $S_1$ are calculated from SPRDM in Eq. (14) along with critical angular velocity $\Omega_c$ and $p$–fold symmetry in the total angular momentum space ranging from $0 \leq L_z \leq 4N$ are noted below in beyond LLL $n_r = 1$.

| $L_z$ | $E_{0}^{lab}(L_z)$ | $\Omega_c$ | $p$ | $(n_1,m_1)$ | $\lambda_1$ | $(n_2,m_2)$ | $\lambda_2$ | $(n_3,m_3)$ | $\lambda_3$ | $C_d$ | $S_1$ |
|------|------------------|-----------|-----|------------|-----------|------------|-----------|------------|-----------|------|-----|
| 0    | 45.36785         | 0.0       | -   | 0,0        | 0.99664   | 1,1        | 0.00146   | 1,1        | 0.00146   | 0.99627 | 0.02637 |
| 1    | 46.36791         | 0.0       | -   | 0,0        | 0.93204   | 1,1        | 0.06462   | 1,1        | 0.00263   | 0.92448 | 0.26448 |
| 2    | 47.08297         | 0.0       | -   | 0,0        | 0.92901   | 2,2        | 0.05695   | 1,1        | 0.01227   | 0.92412 | 0.29870 |
| 3    | 47.87304         | 0.0       | -   | 0,0        | 0.92162   | 3,3        | 0.05066   | 1,1        | 0.01533   | 0.91291 | 0.35101 |
| 4    | 48.77732         | 0.0       | -   | 0,0        | 0.84923   | 2,2        | 0.08871   | 1,1        | 0.05363   | 0.83248 | 0.55652 |
| 5    | 49.56409         | 0.0       | -   | 0,0        | 0.83417   | 2,2        | 0.07245   | 1,1        | 0.05350   | 0.81574 | 0.63424 |
| 6    | 50.38120         | 0.0       | -   | 0,0        | 0.81743   | 3,3        | 0.07690   | 1,1        | 0.06302   | 0.79714 | 0.67842 |
|   | 51.22436 | 0.0 | 0.72779 | 2.2 | 0.13523 | 2.2 | 0.10344 | 0.69755 | 0.85740 |
|---|----------|-----|---------|-----|---------|-----|---------|---------|---------|
| 8 | 52.03192 | 0.0 | 0.68952 | 1.1 | 0.16253 | 2.2 | 0.10159 | 0.65502 | 0.93344 |
| 9 | 52.84731 | 0.0 | 0.62580 | 1.1 | 0.22300 | 2.2 | 0.10954 | 0.58423 | 1.00928 |
|10 | 53.65663 | 0.0 | 0.55727 | 1.1 | 0.28850 | 2.2 | 0.12212 | 0.50808 | 1.05753 |
|11 | 54.46197 | 0.0 | 0.48999 | 1.1 | 0.35737 | 2.2 | 0.12412 | 0.43333 | 1.08301 |
|12 | 55.26553 | 1.1 | 0.43835 | 0.0 | 0.41559 | 2.2 | 0.12329 | 0.37594 | 1.07531 |
|13 | 56.06727 | 1.1 | 0.52917 | 0.0 | 0.33649 | 2.2 | 0.11684 | 0.47686 | 1.02898 |
|14 | 56.86801 | 1.1 | 0.63353 | 0.0 | 0.25064 | 2.2 | 0.10350 | 0.59281 | 0.92839 |
|15 | 57.66821 | 1.1 | 0.75436 | 0.0 | 0.15669 | 2.2 | 0.08146 | 0.72707 | 0.74679 |
|16 | 58.46294 | 0.81844 | 2.2 | 0.14918 | 0.0 | 0.10962 | 0.70447 | 0.81661 |
|17 | 59.46389 | 1.1 | 0.72910 | 2.2 | 0.13243 | 0.0 | 0.12793 | 0.65466 | 0.98546 |
|18 | 60.41157 | 1.1 | 0.68344 | 2.2 | 0.09034 | 0.0 | 0.08000 | 0.74246 | 0.84582 |
|19 | 61.26707 | 1.1 | 0.76392 | 2.2 | 0.21401 | 0.0 | 0.13675 | 0.54387 | 1.14812 |
|20 | 62.23797 | 1.1 | 0.58188 | 2.2 | 0.27007 | 0.0 | 0.18285 | 0.40535 | 1.31036 |
|21 | 63.16968 | 1.1 | 0.45490 | 2.2 | 0.22401 | 0.0 | 0.14461 | 0.47531 | 1.27725 |
|22 | 64.05903 | 1.1 | 0.51903 | 2.2 | 0.34470 | 0.0 | 0.18875 | 0.30431 | 1.36103 |
|23 | 64.99627 | 2.2 | 0.36228 | 1.1 | 0.26090 | 1.1 | 0.13135 | 0.46831 | 1.24812 |
|24 | 65.89361 | 2.2 | 0.51307 | 0.0 | 0.26644 | 0.0 | 0.18659 | 0.35229 | 1.40786 |
|25 | 66.82809 | 2.2 | 0.40627 | 1.1 | 0.22595 | 1.1 | 0.08725 | 0.45148 | 1.17153 |
|26 | 67.71570 | 2.2 | 0.58305 | 0.0 | 0.18282 | 0.0 | 0.16587 | 0.43545 | 1.34291 |
|27 | 68.68383 | 2.2 | 0.49899 | 1.1 | 0.17685 | 1.1 | 0.07645 | 0.59452 | 1.13237 |
|28 | 69.55952 | 0.92471 | 2.2 | 0.05003 | 1.1 | 0.15967 | 3.3 | 0.14859 | 0.45458 | 1.35407 |
|29 | 70.55437 | 2.2 | 0.67711 | 0.0 | 0.12229 | 1.1 | 0.07103 | 0.64776 | 1.06564 |
|30 | 71.43247 | 0.93648 | 2.2 | 0.48912 | 3.3 | 0.20769 | 0.0 | 0.14204 | 0.44268 | 1.35549 |
|   | 73.31758 | 0.94256 | 2  | 2,2 | 0.63923 | 3,3 | 0.09346 | 1,1 | 0.09329 | 0.60644 | 1.18693 |
|---|---------|--------|----|-----|--------|----|---------|----|---------|---------|---------|
| 36| 77.12801| 0.95261| 3  | 3,3 | 0.50812 | 0,0 | 0.23021 | 2,2 | 0.10629 | 0.46340 | 1.36944 |
| 37| 78.10569|        |    |     |        |     |         |    |         |         |         |
| 38| 79.05078|        |    |     |        |     |         |    |         |         |         |
| 39| 79.98856| 0.95352| 3  | 3,3 | 0.53684 | 0,0 | 0.18769 | 4,4 | 0.09989 | 0.49474 | 1.35894 |
| 40| 80.98327|        |    |     |        |     |         |    |         |         |         |
| 41| 81.94333|        |    |     |        |     |         |    |         |         |         |
| 42| 82.88402|        |    |     |        |     |         |    |         |         |         |
| 43| 83.86494|        |    |     |        |     |         |    |         |         |         |
| 44| 84.82073|        |    |     |        |     |         |    |         |         |         |
| 45| 85.81515|        |    |     |        |     |         |    |         |         |         |
| 46| 86.78861|        |    |     |        |     |         |    |         |         |         |
| 47| 87.73690|        |    |     |        |     |         |    |         |         |         |
| 48| 88.60174| 0.95702| 4  | 4,4 | 0.46760 | 0,0 | 0.21079 | 3,3 | 0.11811 | 0.41920 | 1.52696 |
| 49| 89.62116|        |    |     |        |     |         |    |         |         |         |
| 50| 90.59611|        |    |     |        |     |         |    |         |         |         |
| 51| 91.55611|        |    |     |        |     |         |    |         |         |         |
| 52| 92.50030| 0.97464| 4  | 4,4 | 0.53680 | 0,0 | 0.16591 | 3,3 | 0.09784 | 0.49469 | 1.43474 |
| 53| 93.51604|        |    |     |        |     |         |    |         |         |         |
| 54| 94.50029|        |    |     |        |     |         |    |         |         |         |
| 55| 95.47868|        |    |     |        |     |         |    |         |         |         |
| 56| 96.44362|        |    |     |        |     |         |    |         |         |         |
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