Regular Trace Formula of Eigenvalues of a Discontinuous One-point Boundary Value Problem with Retarded Argument

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Abstract

In this study, we found a regular trace formula for the eigenvalues of the boundary value problem, which we created with the second-order differential equation with eigen parameter and discontinuity at \( x = \frac{\pi}{2} \), which is an interior point of the finite range \([0, \pi]\), and boundary conditions that also contain eigen parameter, and interface conditions.

Keywords: Sturm-Liouville, trace, regular trace formula, differential equation with retarded argument.

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1 Introduction

The purpose of this study is to obtain the regular trace formula for the eigenvalues of the boundary value problem formed by the second order differential equation contains a eigen parameter

\[
y''(x) + q(x)y(x - \Delta(x)) = -\mu^2 y(x)
\]

on \([0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]\), with boundary conditions depend on a eigen parameter

\[
(\mu a'_1 + a_1)y(0) - (\mu a'_2 + a_2)y'(0) = 0
\]
\[ \cos by(\pi) + \mu \sin by'(\pi) = 0 \] (3)

and with interface conditions
\[ y\left(\frac{\pi}{2} - 0\right) = \delta y\left(\frac{\pi}{2} + 0\right) \] (4)
\[ y'\left(\frac{\pi}{2} - 0\right) = \delta y'\left(\frac{\pi}{2} + 0\right). \] (5)

\( q(x) \) is the real-valued continuous function on both the intervals \([0, \frac{\pi}{2})\) and \((\frac{\pi}{2}, \pi]\)
and \( q(x) \) has finite limits
\[
\begin{cases}
q\left(\frac{\pi}{2} + 0\right) = \lim_{x \to \frac{\pi}{2} + 0} q(x) \\
q\left(\frac{\pi}{2} - 0\right) = \lim_{x \to \frac{\pi}{2} - 0} q(x)
\end{cases}
\] (6)

and \( \Delta(x) \geq 0 \) is the real-values function and continuous on \([0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]\)
and has finite limits
\[
\begin{cases}
\Delta\left(\frac{\pi}{2} + 0\right) = \lim_{x \to \frac{\pi}{2} + 0} \Delta(x) \\
\Delta\left(\frac{\pi}{2} - 0\right) = \lim_{x \to \frac{\pi}{2} - 0} \Delta(x),
\end{cases}
\] (7)

Moreover,
\[
\begin{cases}
x - \Delta(x) \geq 0 \quad \text{if} \quad x \in \left[0, \frac{\pi}{2}\right) \\
x - \Delta(x) \geq \frac{\pi}{2} \quad \text{if} \quad x \in \left(\frac{\pi}{2}, \pi]\n\end{cases}
\]

\( \mu \) is a eigen parameter, \( a'_{1}, a_{1}, a'_{2}, a_{2}, \delta \neq 0 \) are arbitrary real numbers.

The scholars have long been interested in the theory of regular trace of ordinary differential operators. Gelfand and Levitan [1] firstly obtained the trace formula for the Sturm-Liouville differential equation. After this study mathematicians were interested in developing trace formulas for different differential operators. In the survey article [2], the whole history of the regular trace of the linear operators is given. The regular trace formulas for differential equations are found [3–7] and however, there are a small number of works on the regular trace for the differential equations with retarded argument.

The boundary value problems with a discontinuity condition inside the interval have many applications such as mathematics, mechanics, radio electronics, geophysics and other fields of science. Furthermore, [8] contains several physical applications of such problems. Boundary value problems involving interface conditions are commonly encountered in the theory of heat and mass transfer in a wide range of physical transfer problems.

Due to the demands of modern technology, engineering, and physics, the problems with interface conditions have grown in importance in recent years. The retarded differential equations, which are one of the differential classes, can be continuous in the interval in they are defined or have discontinuity at one or more interior point of this interval. This type of problem has been addressed in many studies [8–13].
In our study, we created our problem by adding the eigen parameter-containing conditions to our equation with a discontinuity at one point, as well as the interface conditions in order not to complicate the solution of discontinuity-related irregularities, and we have obtained a regular trace formula for the eigenvalues of this problem

\[
\mu_2^2 - \mu_0^2 + \sum_{n=1}^{\infty} \left( \mu_2^2 - \mu_0^2 - 2(n - 1)^2 + \frac{4}{\pi} \left( -\frac{\alpha_1'}{\alpha_2} + P(\mu, \Delta(\theta)) + Q(\mu, \Delta(\theta)) \right) \right) = -\frac{2}{\pi} \left( -\frac{\alpha_1'}{\alpha_2} + P(0, \Delta(\theta)) + Q(0, \Delta(\theta)) \right)
- \left( -\frac{\alpha_1'}{\alpha_2} + P(0, \Delta(\theta)) + Q(0, \Delta(\theta)) \right)^2
+ \left( \frac{\alpha_1'}{\alpha_2} + R(0, \Delta(\theta)) + S(0, \Delta(\theta)) \right)^2 + O \left( \frac{1}{\mu_0^2} \right).
\]

We think that this study will make new contributions to the problems in the field of quantum statistics and technology.

2 Transform of Boundary Value Problem to Integral Equation

Let \( \omega_1(x, \mu) \) and \( \omega_2(x, \mu) \) be the solutions of our quadratic equation with discontinuity at \( x = \frac{\pi}{2} \). \( \omega_1(x, \mu) \) and \( \omega_1'(x, \mu) \) satisfy the following conditions

\[
\begin{align*}
\omega_1(0, \mu) &= \mu a_2' + a_2 \\
\omega_1'(0, \mu) &= \mu a_1' + a_1
\end{align*}
\]

on \([0, \frac{\pi}{2}]\). Under these conditions, \( \omega_1(x, \mu) \) is the unique solution of our equation on \([0, \frac{\pi}{2}]\).

We can express \( \omega_2(x, \mu) \) which is the solution of our equation, using interface conditions, depending on \( \omega_1(x, \mu) \) in the interval \([\frac{\pi}{2}, \pi]\) as follows:

\[
\begin{align*}
\omega_2(\frac{\pi}{2}, \mu) &= \delta^{-1} \omega_1(\frac{\pi}{2}, \mu) \\
\omega_2'(\frac{\pi}{2}, \mu) &= \delta^{-1} \omega_1'(\frac{\pi}{2}, \mu).
\end{align*}
\]

With these conditions, \( \omega_2(x, \mu) \) is the unique solution of our equation on the interval \([\frac{\pi}{2}, \pi]\).

As a result, the unique solution of our equation defined in the interval \([0, \frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]\) which satisfies the first of the boundary condition and interface conditions, is as follows

\[
\omega(x, \mu) = \begin{cases} 
\omega_1(x, \mu), & x \in [0, \frac{\pi}{2}] \\
\omega_2(x, \mu), & x \in (\frac{\pi}{2}, \pi].
\end{cases}
\]

The solution \( \omega(x, \mu) \) defined above is nontrivial solution of the second order differential equation (1) that satisfies boundary condition (2) and interface conditions (4) – (5).
Lemma 2.1. Let $\omega(x, \mu)$ be a solution of the second order differential equation (1) and $\mu > 0$. Thus providing the following integral equations:

$$
\omega_1(x, \mu) = \cos \mu x \left( \mu a_2' + a_2 \right) + \frac{\sin \mu x}{\mu} (\mu a_1' + a_1) - \frac{1}{\mu} \int_0^x q(\theta) \sin \mu (x - \theta) \omega_1(\theta - \Delta(\theta), \mu) d\theta \tag{11}
$$

and

$$
\omega_2(x, \mu) = \frac{1}{\mu} \sin \mu (x - \pi/2) \omega_1'(\pi/2, \mu) + \frac{\mu}{\pi} \cos \mu (x - \pi/2) \omega_1(\pi/2, \mu) - \frac{1}{\mu} \int_{\pi/2}^x q(\theta) \cos \mu (x - \theta) \omega_2(\theta - \Delta(\theta), \mu) d\theta. \tag{12}
$$

Proof. To prove this, it is enough to substitute $-\omega_1''(x, \mu) - \mu^2 \omega_1(x, \mu)$ and $-\omega_2''(x, \mu) - \mu^2 \omega_2(x, \mu)$ instead of $q(\theta) \omega_1(\theta - \Delta(\theta), \mu)$ and $q(\theta) \omega_2(\theta - \Delta(\theta), \mu)$ in integrals in equations (11) and (12), respectively, and integrate by parts.

Let’s call the equation we obtained by considering $\omega(x, \mu)$ in (2) as the characteristic equation

$$
F(\mu) = \cos b\omega(\pi, \mu) + \mu \sin b\omega'(\pi, \mu) = 0. \tag{13}
$$

The eigenvalues of our problem (1) – (5) coincides with the roots of $F(\mu)$, and the eigenvalues are simple. $F(\mu)$ is determined with equations (11), (12) and their derivatives.

$$
\omega_1'(x, \mu) = -\mu \sin \mu x (\mu a_2' + a_2) + \cos \mu x (\mu a_1' + a_1) - \int_0^x q(\theta) \cos \mu (x - \theta) \omega_1(\theta - \Delta(\theta), \mu) d\theta \tag{14}
$$

and

$$
\omega_2'(x, \mu) = \frac{1}{\mu} \cos \mu (x - \pi/2) \omega_1'(\pi/2, \mu) - \mu \sin \mu (x - \pi/2) \omega_1(\pi/2, \mu) - \int_{\pi/2}^x q(\theta) \cos \mu (x - \theta) \omega_2(\theta - \Delta(\theta), \mu) d\theta. \tag{15}
$$

Then we have:
\[
F(\mu) = \frac{(\mu a'_1 + a_1)}{\mu b} \cos b \sin \mu \pi + \frac{(\mu a'_2 + a_2)}{\theta} \cos b \cos \mu \pi
- \frac{\cos b}{\mu b} \int_0^{\pi/2} q(\theta) \sin (\mu \pi - \theta) \omega_1(\theta - \Delta(\theta), \mu) d\theta
- \frac{\cos b}{\mu \pi} \int_0^{\pi/2} q(\theta) \sin (\mu \pi - \theta) \omega_2(\theta - \Delta(\theta), \mu) d\theta
+ \frac{\mu(\mu a'_1 + a_1)}{\delta} \sin b \cos \mu \pi
+ \frac{\mu^2(\mu a'_2 + a_2)}{\delta} \sin b \sin \mu \pi
- \frac{\mu \sin b}{\delta} \int_0^{\pi/2} q(\theta) \cos (\mu \pi - \theta) \omega_1(\theta - \Delta(\theta), \mu) d\theta
- \mu \sin b \int_0^{\pi/2} q(\theta) \cos (\mu \pi - \theta) \omega_2(\theta - \Delta(\theta), \mu) d\theta
= 0.
\]

Let's use the sequential approximation of \(\omega_1(x, \mu)\) and \(\omega_2(x, \mu)\) in this last equation of \(F(\mu)\). For this we write the sequential approximation:

\[
\omega_1(x, \mu) = \frac{\sin \mu x}{\mu b} (\mu a'_1 + a_1) + \cos \mu x (\mu a'_2 + a_2)
+ \frac{\mu a'_1 + a_1}{2\mu^2} \int_0^\pi q(\theta) \cos (\mu(x - \Delta(\theta))) d\theta
- \frac{\mu a'_2 + a_2}{2\mu^2} \int_0^\pi q(\theta) \cos (\mu(x - 2\theta + \Delta(\theta))) d\theta
- \mu \sin b \int_0^{\pi/2} q(\theta) \cos (\mu(x - \Delta(\theta))) d\theta
- \mu \sin b \int_0^{\pi/2} q(\theta) \cos (\mu(x - 2\theta + \Delta(\theta))) d\theta.
\]

(16)

\[
\omega_2(x, \mu) = \frac{\sin \mu x}{\mu b} (\mu a'_1 + a_1) + \frac{\cos \mu x}{\mu b} (\mu a'_2 + a_2)
+ \frac{\mu a'_1 + a_1}{2\mu^2} \int_0^\pi q(\theta) \cos (\mu(x - \Delta(\theta))) d\theta
- \frac{\mu a'_2 + a_2}{2\mu^2} \int_0^\pi q(\theta) \cos (\mu(x - 2\theta + \Delta(\theta))) d\theta
- \mu \sin b \int_0^{\pi/2} q(\theta) \cos (\mu(x - \Delta(\theta))) d\theta
- \mu \sin b \int_0^{\pi/2} q(\theta) \cos (\mu(x - 2\theta + \Delta(\theta))) d\theta.
\]

(17)

Then we have,
By making trigonometric transformations and some adjustments, we find:

\[
F(\mu) = \left(\frac{\mu a' + a_1}{b^2}\right) \cos b \sin \mu \pi + \left(\frac{\mu a' + a_2}{\delta}\right) \cos b \cos \mu \pi
\]

\[
- \cos \left(\frac{\mu a' + a_1}{b^2}\right) \int_0^\pi q(\theta) \sin \mu (\pi - \theta) \sin \mu (\theta - \Delta(\theta)) d\theta
\]

\[
- \cos \left(\frac{\mu a' + a_2}{\delta}\right) \int_0^\pi q(\theta) \sin \mu (\pi - \theta) \cos \mu (\theta - \Delta(\theta)) d\theta
\]

\[
- \cos \left(\frac{\mu a' + a_1}{2b^2}\right) \int_0^\pi q(\theta) \sin \mu (\pi - \theta) \int_0^{\theta - \Delta(\theta)} q(t_1) \cos \mu (\theta - \Delta(t_1)) dt_1 d\theta
\]

\[
- \cos \left(\frac{\mu a' + a_2}{2\delta}\right) \int_0^\pi q(\theta) \sin \mu (\pi - \theta) \int_0^{\theta - \Delta(\theta)} q(t_1) \sin \mu (\theta - \Delta(t_1)) dt_1 d\theta
\]

\[
+ \cos \left(\frac{\mu a' + a_1}{2b^2}\right) \int_0^\pi q(\theta) \sin \mu (\pi - \theta) \int_0^{\theta - \Delta(\theta)} q(t_1) \cos \mu (\theta - \Delta(t_1)) dt_1 d\theta
\]

\[
+ \cos \left(\frac{\mu a' + a_2}{2\delta}\right) \int_0^\pi q(\theta) \sin \mu (\pi - \theta) \int_0^{\theta - \Delta(\theta)} q(t_1) \sin \mu (\theta - \Delta(t_1)) dt_1 d\theta
\]

\[
+ \mu \left(\frac{\mu a' + a_1}{\delta}\right) \sin b \cos \mu \pi - \mu \left(\frac{\mu a' + a_2}{\delta}\right) \sin b \cos \mu \pi
\]

\[
- \sin \left(\frac{\mu a' + a_1}{b^2}\right) \int_0^\pi q(\theta) \cos \mu (\pi - \theta) \sin \mu (\theta - \Delta(\theta)) d\theta
\]

\[
- \sin \left(\frac{\mu a' + a_2}{\delta}\right) \int_0^\pi q(\theta) \cos \mu (\pi - \theta) \cos \mu (\theta - \Delta(\theta)) d\theta
\]

\[
+ \sin \left(\frac{\mu a' + a_1}{2b^2}\right) \int_0^\pi q(\theta) \cos \mu (\pi - \theta) \int_0^{\theta - \Delta(\theta)} q(t_1) \cos \mu (\theta - \Delta(t_1)) dt_1 d\theta
\]

\[
+ \sin \left(\frac{\mu a' + a_2}{2\delta}\right) \int_0^\pi q(\theta) \cos \mu (\pi - \theta) \int_0^{\theta - \Delta(\theta)} q(t_1) \sin \mu (\theta - \Delta(t_1)) dt_1 d\theta
\]

\[
= 0.
\]
\[ F(\mu) = -\mu^3 a_2' \sin b \sin \mu \pi + \frac{\mu a_1^2}{3} \cos b \cos \mu \pi + \frac{a_1}{3} \sin b \cos \mu \pi \]
\[ + \frac{a_1}{24} \int_0^\pi q(\theta) \sin \mu \Delta(\theta) \cos \mu \pi d\theta \]
\[ - \frac{a_1}{24} \int_0^\pi q(\theta) \sin (2\theta - \Delta(\theta)) \cos \mu \pi d\theta \]
\[ - \frac{a_2}{24} \int_0^\pi q(\theta) \cos \mu \Delta(\theta) \cos \mu \pi d\theta \]
\[ - \frac{a_2}{24} \int_0^\pi q(\theta) \cos \mu (2\theta - \Delta(\theta)) \cos \mu \pi d\theta \]
\[ - \frac{a_2}{24} \int_0^\pi q(\theta) \cos \mu \Delta(\theta) \sin \mu \pi d\theta \]
\[ - \frac{a_2}{24} \int_0^\pi q(\theta) \cos \mu (2\theta - \Delta(\theta)) \sin \mu \pi d\theta \]
\[ + \frac{a_2}{24} \int_0^\pi q(\theta) \sin \mu \Delta(\theta) \cos \mu \pi d\theta \]
\[ - \frac{a_2}{24} \int_0^\pi q(\theta) \sin \mu (2\theta - \Delta(\theta)) \cos \mu \pi d\theta \]
\[ - \frac{a_2}{24} \int_0^\pi q(\theta) \sin \mu \Delta(\theta) \sin \mu \pi d\theta \]
\[ - \frac{a_2}{24} \int_0^\pi q(\theta) \sin \mu (2\theta - \Delta(\theta)) \sin \mu \pi d\theta \]
\[ = 0. \]

Now, let us denote
\[ P(\mu, \Delta(\theta)) = \frac{1}{2} \int_0^\pi q(\theta) \cos \mu \Delta(\theta) d\theta \]
\[ Q(\mu, \Delta(\theta)) = \frac{1}{2} \int_0^\pi q(\theta) \cos \mu (2\theta - \Delta(\theta)) d\theta \]
\[ R(\mu, \Delta(\theta)) = \frac{1}{2} \int_0^\pi q(\theta) \sin \mu \Delta(\theta) d\theta \]
\[ S(\mu, \Delta(\theta)) = \frac{1}{2} \int_0^\pi q(\theta) \sin \mu (2\theta - \Delta(\theta)) d\theta \]

and we write
\[ F(\mu) = -\mu^3 a_2' \sin b \sin \mu \pi + \frac{\mu^2}{3} (a_1^2 \sin b - a_2' \sin b) P(\mu, \Delta(\theta)) \]
\[ - a_2' \sin b Q(\mu, \Delta(\theta)) \cos \mu \pi \]
\[ + \frac{\mu}{3} (-a_2' \sin b - a_2' \sin b R(\mu, \Delta(\theta)) - a_2' \sin b S(\mu, \Delta(\theta)) \sin \mu \pi \]
\[ + O(\mu e^{\frac{Im \mu}{\pi}}) \]
# A Formula For The Regular Trace

Let’s denote the first term in the last equation of $F(\mu)$ by $F_0(\mu)$

$$F_0(\mu) = -\frac{\mu^3 a'_2}{\delta} \sin b \sin \mu \pi.$$  \hspace{1cm} (18)

Let’s show the zeros of $F_0(\mu)$ except the zeros $\mu_0 = \mu_{0\pm 1} = 0$ with multiplicity of 4, as follows [similar to [12]]

$$\mu_0^0 = \left\{ \begin{array}{ll}
n - 1, & n \geq 1 \\
n + 1, & n \leq 1 \end{array} \right.$$  

Here $n \in \mathbb{N} \cup \{0\}$. Let’s denote circles of radius $\varepsilon$ with centers at $\mu_0$ points by $\Gamma_n$. From the last equation of $F(\mu)$ and the equation of $F_0(\mu)$, on the contour $\Gamma_n$, we have

$$\frac{F(\mu)}{F_0(\mu)} = 1 + \frac{1}{\mu} \left\{ -\frac{a'_2}{a_2} + P(\mu, \Delta(\theta)) + Q(\mu, \Delta(\theta)) \right\} \cot \mu \pi$$

$$+ \frac{a'_2}{a_2} + R(\mu, \Delta(\theta)) + S(\mu, \Delta(\theta))$$

$$+ O \left( \frac{1}{\mu^2} \right).$$ \hspace{1cm} (19)

Expanding $\ln \frac{F(\mu)}{F_0(\mu)}$ by the Maclaurin formula, we obtain

$$\ln \frac{F(\mu)}{F_0(\mu)} = \frac{1}{\mu} \left\{ -\frac{a'_2}{a_2} + P(\mu, \Delta(\theta)) + Q(\mu, \Delta(\theta)) \right\} \cot \mu \pi$$

$$+ \frac{a'_2}{a_2} + R(\mu, \Delta(\theta)) + S(\mu, \Delta(\theta))$$

$$- \frac{1}{2\mu^2} \left( -\frac{a'_2}{a_2} + P(\mu, \Delta(\theta)) + Q(\mu, \Delta(\theta)) \right)^2 \cot^2 \mu \pi$$

$$- \frac{1}{2\mu^2} \left( \frac{a'_2}{a_2} + R(\mu, \Delta(\theta)) + S(\mu, \Delta(\theta)) \right)^2$$

$$+ \frac{1}{\mu^2} \left(-\frac{a'_2}{a_2} + P(\mu, \Delta(\theta)) + Q(\mu, \Delta(\theta)) \right)$$

$$\times \left( \frac{a'_2}{a_2} + R(\mu, \Delta(\theta)) + S(\mu, \Delta(\theta)) \right) \cot \mu \pi + O \left( \frac{1}{\mu^2} \right).$$

Using the Rouche theorem it follows that $F(\mu)$ has the same number of zeros inside the contour as $F_0(0)$. Then, we have $\mu_n = \mu_0^0 + \varepsilon_n$ for sufficiently large $n$, where $|\varepsilon_n| < \frac{\pi}{4}$. Substituting into we get $\varepsilon_n = O \left( \frac{1}{n^2} \right)$. We continue making $\mu_n$ more precise. Considering the residue theorem, we obtain the asymptotic expression of the eigenvalues as follows:

$$\mu_n - \mu_0^0 = -\frac{1}{2\pi i} \oint_{C_n} \ln \frac{F(\mu)}{F_0(\mu)} d\mu$$

$$= -\frac{1}{2\pi i} \oint_{C_n} \left( -\frac{a'_2}{a_2} + P(\mu, \Delta(\theta)) + Q(\mu, \Delta(\theta)) \right) \cot \mu \pi d\mu$$

$$- \frac{1}{2\pi i} \oint_{C_n} \left( \frac{a'_2}{a_2} + R(\mu, \Delta(\theta)) + S(\mu, \Delta(\theta)) \right) \frac{1}{\mu} d\mu + O \left( \frac{1}{n^2} \right)$$

$$= -\frac{1}{\mu_0^0 \pi} \left\{ -\frac{a'_1}{a_1} + P(n, \Delta(\theta)) + Q(n, \Delta(\theta)) \right\} + O \left( \frac{1}{n^2} \right)$$
Thus, we have proved Theorem 1, which we have stated below.

**Theorem 3.1.** When \( n \) approaches infinity, the eigenvalues of the boundary value problem, we have created with (1) differential equation, (2)-(3) boundary conditions and (4)-(5) interface conditions, are expressed by the following asymptotic formula

\[
\mu_n = \mu_n^0 - 1 \frac{1}{\mu_n^0} \left\{ -\frac{a_1}{a_2} + P(n, \Delta(\theta)) + Q(n, \Delta(\theta)) \right\} + O \left( \frac{1}{n^2} \right). \tag{20}
\]

**Theorem 3.2.** The regular trace formula for the eigenvalues of the (1) – (5) problem is obtained with the \( P(\mu, \Delta(\theta)), Q(\mu, \Delta(\theta)), R(\mu, \Delta(\theta)) \) and \( S(\mu, \Delta(\theta)) \) integrals we have determined, as follows

\[
\mu_n^2 + \mu_n^3 + \sum_{n=1}^{\infty} \left( \mu_n^2 + \mu_n^3 - 2(n-1)^2 + \frac{4}{n} \left( -\frac{a_1}{a_2} + P(\mu, \Delta(\theta)) + Q(\mu, \Delta(\theta)) \right) \right)
\]

\[
= -\frac{2}{\pi} \left( -\frac{a_1}{a_2} + P(0, \Delta(\theta)) + Q(0, \Delta(\theta)) \right)
\]

\[
- \left( -\frac{a_1}{a_2} + P(0, \Delta(\theta)) + Q(0, \Delta(\theta)) \right)^2
\]

\[
+ \left( \frac{a_2}{\pi} + R(0, \Delta(\theta)) + S(0, \Delta(\theta)) \right)^2 + O \left( \frac{1}{N_0} \right). \tag{21}
\]

**Proof.** \( N_0 \) is an integer and denote by \( \Gamma_{N_0} \) the counterclockwise square contours EFGH with \( E = (N_0 - 1 + \varepsilon)(1 - i), F = (N_0 - 1 + \varepsilon)(1 + i), G = (N_0 - 1 + \varepsilon)(-1 + i), H = (N_0 - 1 + \varepsilon)(-1 - i) \). Asymptotic formula of \( \mu_n \) implies that for all sufficiently large \( N_0 \), the numbers \( \mu_n \), with \( |n| \leq N_0 \) are inside \( \Gamma_{N_0} \). The numbers \( \mu_n \), with \( |n| > N_0 \) are outside \( \Gamma_{N_0} \). Obviously \( \mu_n^0 \) do not lie on the contour \( \Gamma_{N_0} \). It follows that

\[
\sum_{\Gamma_{N_0}} \mu_n^2 + (\mu_n^0)^2 = \mu_n^2 + \mu_n^3 + \sum_{n=1}^{N_0} \left( \mu_n^2 + \mu_n^3 - 2(n-1)^2 \right)
\]

\[
= -\frac{1}{2\pi i} \oint_{\Gamma_{N_0}} \frac{F(\mu)}{\mu} d\mu
\]

\[
= -\frac{1}{2\pi i} \oint_{C_0} 2 \left( -\frac{a_1}{a_2} + P(\mu, \Delta(\theta)) + Q(\mu, \Delta(\theta)) \right) \cot \mu d\mu
\]

\[
- \frac{1}{2\pi i} \oint_{C_0} 2 \left( \frac{a_2}{\pi} + R(\mu, \Delta(\theta)) + S(\mu, \Delta(\theta)) \right) d\mu
\]

\[
+ \frac{1}{2\pi i} \oint_{C_0} \left( \frac{a_2}{\pi} + R(\mu, \Delta(\theta)) + S(\mu, \Delta(\theta)) \right)^2 \frac{\cot \mu}{\mu} d\mu
\]

\[
+ \frac{1}{2\pi i} \oint_{C_0} \left( \frac{a_2}{\pi} + R(\mu, \Delta(\theta)) + S(\mu, \Delta(\theta)) \right) \frac{2}{\mu} d\mu
\]

\[
\times \left( \frac{a_2}{\pi} + R(\mu, \Delta(\theta)) + S(\mu, \Delta(\theta)) \right) \frac{\cot \mu}{\mu} d\mu + O \left( \frac{1}{N_0} \right).
\]
And we get

\[
\mu_0^2 + \mu_0^2 + \sum_{n=1}^{N_0} \left( \mu_n^2 + \mu_{-n}^2 - 2(n-1)^2 \right) \\
= -2 \left( -\frac{a_1}{\alpha_2} + P(n, \Delta(\theta)) + Q(n, \Delta(\theta)) \right) \left( \frac{2(N_0-1)+1}{\pi} \right) \\
- \frac{2}{\pi} \left( -\frac{a_1}{\alpha_2} + P(0, \Delta(\theta)) + Q(0, \Delta(\theta)) \right) \\
- \left( -\frac{a_1}{\alpha_2} + P(0, \Delta(\theta)) + Q(0, \Delta(\theta)) \right)^2 \\
+ \left( \frac{a_2}{\alpha_2} + R(0, \Delta(\theta)) + S(0, \Delta(\theta)) \right)^2 + O \left( \frac{1}{N_0} \right).
\]

again by residue calculation. So, this last equation implies the following equation

\[
\mu_{-0}^2 + \mu_0^2 + \sum_{n=1}^{\infty} \left( \mu_n^2 + \mu_{-n}^2 - 2(n-1)^2 + \frac{1}{\pi} \left( -\frac{a_1}{\alpha_2} + P(\mu, \Delta(\theta)) + Q(\mu, \Delta(\theta)) \right) \right) \\
= -\frac{2}{\pi} \left( -\frac{a_1}{\alpha_2} + P(0, \Delta(\theta)) + Q(0, \Delta(\theta)) \right) \\
- \left( -\frac{a_1}{\alpha_2} + P(0, \Delta(\theta)) + Q(0, \Delta(\theta)) \right)^2 \\
+ \left( \frac{a_2}{\alpha_2} + R(0, \Delta(\theta)) + S(0, \Delta(\theta)) \right)^2 + O \left( \frac{1}{N_0} \right).
\]

Thus, by approximating \( N_0 \) to \( \infty \), we get the regular trace formula (21) which we want, and our proof is complete. \( \square \)

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