Abstract

There exist two different sorts of gaps in the nonsymmetric numerical additive semigroups finitely generated by a minimal set of positive integers \(\{d_1, \ldots, d_m\}\). The \(h\)-gaps are specific only for the nonsymmetric semigroups while the \(g\)-gaps are possessed by both, symmetric and nonsymmetric semigroups. We derive the generating functions for the corresponding sets of gaps, \(\Delta_H(d^m)\) and \(\Delta_G(d^m)\), and prove several statements on the minimal and maximal values of the \(h\)-gaps. Detailed description of both sorts of gaps is given for three special kinds of nonsymmetric semigroups: semigroups with maximal embedding dimension, semigroups of maximal and almost maximal length, and pseudo–symmetric semigroups.

Keywords: Nonsymmetric and symmetric semigroups, Frobenius problem.

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1 Introduction

In this paper we study the additive numerical semigroup in \(\mathbb{N} \cup \{0\}\) generated by a finite set of positive integers \(\{d_1, \ldots, d_m\}\) and the arrangement of the set of the integers which are unrepresentable by \(\{d_1, \ldots, d_m\}\). Such integers are also known as the gaps of the numerical semigroup. First, recall the main known facts [1]. Semigroup \(S(d_1, \ldots, d_m)\),

\[
S(d_1, \ldots, d_m) = \left\{ s \in \mathbb{N} \cup \{0\} \mid s = \sum_{i=1}^{m} x_i d_i, \quad x_i \in \mathbb{N} \cup \{0\} \right\},
\]

is said to be generated by the minimal set of \(m\) natural numbers \(d_1 < \ldots < d_m\), if there are no nonnegative integers \(b_{i,j}\) for which the following linear dependence holds:

\[
d_i = \sum_{j \neq i}^{m} b_{i,j} d_j, \quad b_{i,j} \in \{0, 1, \ldots\} \quad \text{for any} \quad i \leq m,
\]

and \(\gcd(d_1, \ldots, d_m) = 1\). It is classically known that \(d_1 \geq m\) [2]. For short we denote the tuple \((d_1, \ldots, d_m)\) by \(d^m\) where \(m\) is called the embedding dimension of the semigroup. Henceforth \(d^m\) will be a minimal generating set.
The least positive integer \( (d_1) \) belonging to \( S(d^m) \) is called the \textit{multiplicity}. The \textit{conductor} \( c(d^m) \) of \( S(d^m) \) is defined by \( c(d^m) := \min \{ s \in S(d^m) \mid s + N \cup \{0\} \subset S(d^m) \} \). Denote by \( \Delta(d^m) \) the complement of \( S(d^m) \) in \( N \), i.e. \( \Delta(d^m) = N \setminus S(d^m) \). It is the set of gaps. The cardinality (\#) of \( \Delta(d^m) \) is called the \textit{genus} of \( S(d^m) \),

\[
G(d^m) := \# \Delta(d^m) . \tag{1.3}
\]

Introduce the generating functions \( \Phi(d^m; z) \) for the set \( \Delta(d^m) \) of unrepresentable integers and \( H(d^m; z) \) for the set of representable integers

\[
\Phi(d^m; z) = \sum_{s \in \Delta(d^m)} z^s , \quad H(d^m; z) = \sum_{s \in S(d^m)} z^s . \tag{1.4}
\]

The latter is referred to as the \textit{Hilbert series}. The function \( \Phi(d^m; z) \) determines the Frobenius number and the genus as follows:

\[
F(d^m) = c(d^m) - 1 = \deg \Phi(d^m; z) , \quad G(d^m) = \Phi(d^m; 1) . \tag{1.5}
\]

In terms of the rings theory the entities \( F(d^m) \) and \( G(d^m) \) are also referred to as the \textit{Castelnuovo–Mumford regularity} \([3]\) and \textit{the length} \([1]\) of semigroups, respectively.

A semigroup \( S(d^m) \) is called \textit{symmetric} iff for any integer \( s \) the following condition holds:

\[
s \in S(d^m) \iff F(d^m) - s \not\in S(d^m) . \tag{1.6}
\]

Otherwise \( S(d^m) \) is called \textit{nonsymmetric}. \( F(d^m) \) and \( G(d^m) \) are related as follows:

\[
2G(d^m) = c(d^m) \text{ if } S(d^m) \text{ is symmetric, and } 2G(d^m) > c(d^m) \text{ otherwise.} \tag{1.7}
\]

Denote by \( t(d^m) \) the \textit{type} of the numerical semigroup \( S(d^m) \) which coincides with the cardinality of set \( S'(d^m) \) \([4]\), where

\[
S'(d^m) = \{ F_i \in \mathbb{Z} \mid F_i \not\in S(d^m) , F_i + s \in S(d^m) , s \in S(d^m) \setminus \{0\} , i = 1, \ldots, t(d^m) \} . \tag{1.8}
\]

Set \( S'(d^m) \) is not empty since \( F(d^m) \in S'(d^m) \) for any minimal generating set \( (d_1, \ldots, d_m) \).

Due to \([4]\), Proposition 2, semigroup \( S(d^m) \) is symmetric iff \( t(d^m) = 1 \).

### 2 The Two Sorts of Gaps in Numerical Semigroups

Due to \((1.6)\) set \( \Delta(d^m) \) can be decomposed into two sorts,

\[
\Delta(d^m) = \Delta_G(d^m) \cup \Delta_H(d^m) , \quad \Delta_G(d^m) \cap \Delta_H(d^m) = \emptyset , \tag{2.1}
\]

where the sets of the \textit{g–gaps} and the \textit{h–gaps} are defined, respectively, by

\[
\Delta_G(d^m) = \{ g \not\in S(d^m) \mid F(d^m) - g \in S(d^m) \} , \quad \# \Delta_G(d^m) = c(d^m) - G(d^m) , \tag{2.2}
\]

\[
\Delta_H(d^m) = \{ h \not\in S(d^m) \mid F(d^m) - h \not\in S(d^m) \} , \quad \# \Delta_H(d^m) = 2G(d^m) - c(d^m) . \tag{2.3}
\]

Figure 1: Initial part (black) of the semigroup generated by \((5,11,13)\). The arrows show the pairs of \( h \)-gaps. More details about this semigroup will be given later, in Example \([1]\).
Following [7] recall the relation between $\Phi (d^m; z)$ and the Hilbert series $H (d^m; z)$ of the graded subring $k[z^{d_1}, \ldots, z^{d_m}]$ associated with semigroup $S (d^m)$:

$$\Phi (d^m; z) = \frac{1}{1 - H (d^m; z)} , \quad H (d^m; z) = \frac{Q (d^m; z)}{\prod_{j=1}^{m} (1 - z^{d_j})} ,$$  \hspace{1cm} (2.4)

where $H (d^m; z)$ has a pole $z = 1$ of order 1. The numerator $Q (d^m; z)$ is a polynomial in $z$ of the form

$$Q (d^m; z) = 1 - \sum_{j=1}^{\beta_1 (d^m)} z^{C_{j,1}} + \sum_{j=1}^{\beta_2 (d^m)} z^{C_{j,2}} - \ldots \pm \sum_{j=1}^{\beta_{m-1} (d^m)} z^{C_{j,m-1}} .$$  \hspace{1cm} (2.5)

The summands of the powers $z^{C_{j,i}}$ in the last formula stand for the syzygies of different kinds and $C_{j,i}$, $C_{j+1,i} > C_{j,i}$, are the degrees of homogeneous basic invariants for the syzygies of the $i$th kind. The numbers of the terms $z^{C_{j,i}}$ in the summands are determined by the Betti numbers $\beta_i (d^m)$ which satisfy the relation

$$1 - \beta_1 (d^m) + \beta_2 (d^m) - \ldots \pm \beta_{m-1} (d^m) = 0 .$$  \hspace{1cm} (2.6)

It is also known [5], [6] that

$$\beta_{m-1} (d^m) = t (d^m) .$$  \hspace{1cm} (2.7)

**Theorem 1**

The generating functions for the sets $\Delta_H (d^m)$ and $\Delta_G (d^m)$ are given by

$$\sum_{h \in \Delta_H (d^m)} z^h = -H (d^m; z) - H \left( d^m; \frac{1}{z} \right) \cdot z^{F(d^m)} ,$$  \hspace{1cm} (2.8)

$$\sum_{g \in \Delta_G (d^m)} z^g = \frac{1}{1 - z} + H \left( d^m; \frac{1}{z} \right) \cdot z^{F(d^m)} .$$  \hspace{1cm} (2.9)

**Proof**  We have by (2.1)

$$\Phi (d^m; z) = \sum_{g \in \Delta_G (d^m)} z^g + \sum_{h \in \Delta_H (d^m)} z^h .$$  \hspace{1cm} (2.10)

Consider the transformation

$$\Phi \left( d^m; \frac{1}{z} \right) \cdot z^{F(d^m)} = \sum_{g \in \Delta_G (d^m)} z^{F(d^m)-g} + \sum_{h \in \Delta_H (d^m)} z^{F(d^m)-h} .$$  \hspace{1cm} (2.11)

However, according to (2.3),

$$F (d^m) - g \in S (d^m) , \quad F (d^m) - h \in \Delta_H (d^m) .$$  \hspace{1cm} (2.12)

Thence,

$$\Phi \left( d^m; \frac{1}{z} \right) \cdot z^{F(d^m)} = \sum_{s \in [0; F(d^m)]} z^s + \sum_{h \in \Delta_H (d^m)} z^h .$$  \hspace{1cm} (2.13)
Remark 1 The elements in $S(d^m)$ which are less than $F(d^m)$, i.e. $s \in S(d^m) \cap [0; F(d^m)]$, are also known as the nongaps of the semigroup. Their cardinality coincides with $\#\Delta_G(d^m)$.

Making summation of (2.10) and (2.13), we get
\[
\Phi(d^m; z) + \Phi(d^m; \frac{1}{z}) \cdot z^{F(d^m)} = \sum_{s \in [0; F(d^m)]} z^s + \sum_{g \in \Delta_G(d^m)} z^g + 2 \sum_{h \in \Delta_H(d^m)} z^h. \tag{2.14}
\]
So, we come to the generating functions of the set $\Delta_H(d^m)$,
\[
\sum_{h \in \Delta_H(d^m)} z^h = \Phi(d^m; z) + \Phi(d^m; \frac{1}{z}) \cdot z^{F(d^m)} - \sum_{k=0}^{F(d^m)} z^k. \tag{2.15}
\]
Substituting the 1st equality of (2.4) into (2.15) we obtain
\[
\sum_{h \in \Delta_H(d^m)} z^h = \frac{1}{1-z} - \frac{z^{F(d^m)+1}}{1-z} - \sum_{k=0}^{F(d^m)} z^k - H(d^m; z) - H\left(d^m; \frac{1}{z}\right) \cdot z^{F(d^m)}.
\]
Finally, combining (2.4), (2.10) and (2.8) we get formula (2.9). □

Corollary 1
The semigroup $S(d^m)$ is symmetric iff
\[
H(d^m; z) + H\left(d^m; \frac{1}{z}\right) \cdot z^{F(d^m)} = 0. \tag{2.17}
\]
Equations (2.8) and (2.9) allow us to formulate two new theorems. Let a semigroup $S(d^m)$ be given, and the Hilbert series $H(d^m; z)$ of its graded subring $k[z^{d_1}, \ldots, z^{d_m}]$ be given by (2.4) and (2.5). Define polynomial $V(d^m; z)$ by
\[
V(d^m; z) = (-1)^{m-1} z^\deg Q(d^m; z) Q\left(d^m; \frac{1}{z}\right) - Q(d^m; z). \tag{2.18}
\]

Theorem 2
The semigroup $S(d^m)$ is symmetric iff the polynomial $V(d^m; z)$ is vanishing and $S(d^m)$ is nonsymmetric iff $V(d^m; z)$ is divided by $\prod_{j=1}^{m} (1 - z^{d_j})$.

Proof Combining (2.18) with (2.8) and (2.4), we obtain:
\[
V(d^m; z) = \prod_{j=1}^{m} \left(1 - z^{d_j}\right) \sum_{h \in \Delta_H(d^m)} z^h. \tag{2.19}
\]
The statement of the Theorem follows immediately from this identity. □

Denote by $\min \Delta_H(d^m)$ and $\max \Delta_H(d^m)$ the least and the largest $h$–gaps of semigroup $S(d^m)$, respectively.
Theorem 3
Let a nonsymmetric semigroup $S(d^m)$ be given, and the Hilbert series $H(d^m; z)$ of its graded subring be given by (2.4), (2.5). Then $\min \Delta_H(d^m)$ is given by the minimum degree among the terms of polynomial $V(d^m; z)$, and $\max \Delta_H(d^m)$ is given by

$$\max \Delta_H(d^m) = \deg V(d^m; z) - \Sigma_m,$$

where $\Sigma_m = \sum_{j=1}^m d_j$.

Proof The term with minimum degree of the sum $\sum_{h \in \Delta_H(d^m)} z^h$ has to coincide with the corresponding term in the left hand side of (2.19). Thus, the minimal $h$–gap comes from the term with minimal degree in $V(d^m; z)$. On the other hand, the maximal degree among the terms of $V(d^m; z)$ satisfies:

$$\deg V(d^m; z) = \max \Delta_H(d^m) + \Sigma_m,$$

that concludes the proof. □

Remark 2 The term with degree $\min \Delta_H(d^m)$ has coefficient equal to $+1$, whereas the term with degree $\max \Delta_H(d^m) + \Sigma_m$ has coefficient equal to $(-1)^{m+1}$.

Corollary 2

$$\min \Delta_H(d^m) \leq \# \Delta_G(d^m),$$

Proof Since all the $h$–gaps are inside $\Delta_H$, its cardinality satisfies

$$\# \Delta_H(d^m) \leq \max \Delta_H(d^m) - \min \Delta_H(d^m) + 1.$$  

By the definition (2.3) we obtain

$$\max \Delta_H(d^m) = F(d^m) - \min \Delta_H(d^m),$$

and, by the second equality in (2.3), inequality (2.22) reads:

$$2G(d^m) - c(d^m) \leq F(d^m) - \min \Delta_H(d^m) - \min \Delta_H(d^m) + 1 = c(d^m) - 2 \min \Delta_H(d^m).$$  

(2.24)
Taking into account (2.2) we get finally (2.21). □

We finish this Section by discussing a question which is related to Theorem 1: does it exist a symmetric semigroup $S(p^n)$ with minimal generating set $p^n$ such that its total set of gaps $\Delta(p^n)$ coincides with the set of $g$–gaps $\Delta_G(d^m)$ of a given nonsymmetric semigroup $S(d^m)$?

Theorem 4
Let a nonsymmetric semigroup $S(d^m)$ be given with Hilbert series $H(d^m; z)$. It does not exist any symmetric semigroup $S(p^n)$ with minimal generating set $p^n$ such that $\Delta(p^n) = \Delta_G(d^m)$.

Proof Let, by way of contradiction, such symmetric semigroup $S(p^n)$ does exist, i.e. $\Delta(p^n) = \Delta_G(d^m)$. In accordance with (1.3), (2.4) and Theorem 1 we have

$$\frac{1}{1-z} + H(d^m; \frac{1}{z}) \cdot zF(d^m) = \frac{1}{1-z} - H(p^n; z).$$  

(2.25)
Being symmetric, semigroup $S(p^n)$ satisfies Corollary 1 and therefore (2.25) can be written as

$$H\left(d^m; \frac{1}{z}\right) \cdot z^{F(d^m)} = H\left(p^n; \frac{1}{z}\right) \cdot z^{F(p^n)},$$

(2.26)

which requires necessarily $F(d^m) = F(p^n)$ since both Hilbert series start with unity. Inverting $z \rightarrow z^{-1}$ in the remained equality (2.26), $H(d^m; z^{-1}) = H(p^n; z^{-1})$, and substituting the result into (2.25), we get

$$H(d^m; z) + H\left(d^m; \frac{1}{z}\right) \cdot z^{F(d^m)} = 0,$$

(2.27)

According to Corollary 1 we can conclude that semigroup $S(d^m)$ is symmetric. However, this contradicts our assumption about $S(d^m)$ and proves the theorem. \(\square\)

3 Low – Dimensional Cases

In this Section we apply Theorem 1 and Theorem 3 to numerical semigroups generated by two and three positive integers, respectively.

3.1 Semigroup $S(d^2)$

Semigroup $S(d^2)$ is always symmetric [9], and its Hilbert series reads

$$H\left(d^2; z\right) = \frac{1 - z^{d_1d_2}}{(1 - z^{d_1})(1 - z^{d_2})}.$$  

(3.1)

Simple calculation of (2.8) in accordance with Theorem 2 yields,

$$\sum_{h \in \Delta_H(d^2)} z^h = 0.$$  

(3.2)

3.2 Nonsymmetric Semigroup $S(d^3)$

Nonsymmetric semigroups $S(d^3)$ were studied by algebraic means in [10], [11], [12] and [13]. Recall its main results following [13].

Let $S(d_1, d_2, d_3) \subset \mathbb{Z}_+ \cup \{0\}$ be the additive nonsymmetric numerical semigroup finitely generated by the minimal set of positive integers $d_1 < d_2 < d_3$. Following Johnson [14] define the minimal relation for the triple $d^3 = (d_1, d_2, d_3)$:

$$a_{11}d_1 = a_{12}d_2 + a_{13}d_3, \quad a_{22}d_2 = a_{21}d_1 + a_{23}d_3, \quad a_{33}d_3 = a_{31}d_1 + a_{32}d_2,$$  

(3.3)

where

$$a_{jj} = \min \{v_{jj} \mid v_{jj} \geq 2, \ v_{jj}d_j = v_{jk}d_k + v_{jl}d_l, \ v_{jk}, v_{jl} \in \mathbb{Z}_+\},$$  

(3.4)

$$\gcd(a_{jj}, a_{jk}, a_{jl}) = 1, \quad \text{and} \quad (j, k, l) = (1, 2, 3), \ (2, 3, 1), \ (3, 1, 2).$$

The uniquely defined values of $v_{ij}, i \neq j$ which give $a_{ii}$ will be denoted by $a_{ij}, i \neq j$. The degeneracy of the matrix $(a_{ij})$ together with (3.4) results in strong equalities relating the matrix
elements $a_{ij}$ and the generators $d_k$: for any permutation of indices $(i, j, k)$, $i, j, k = 1, 2, 3$ the following identities hold \[ 13, 14 \]:

$$ a_{ii} = a_{ji} + a_{ki} , \quad a_{ii}a_{jj} = d_k + a_{ij}a_{ji} . \quad (3.5) $$

The Hilbert series $H(d^3; z)$, the type $t(d^3)$, the Frobenius number $F(d^3; z)$ and the genus $G(d^3)$ read \[ 13 \]:

$$ H(d^3; z) = \frac{Q(d^3; z)}{(1 - z^{d_1})(1 - z^{d_2})(1 - z^{d_3})} , \quad (3.6) $$

where

$$ Q(d^3; z) = 1 - \sum_{i=1}^{3} z^{a_{ii}d_i} + z^{1/2[(a,d) - J(d^3)]} + z^{1/2[(a,d) + J(d^3)]} , \quad t(d^3) = 2 , \quad (3.7) $$

$$ F(d^3) = \frac{1}{2} \left[ \langle a, d \rangle + J(d^3) \right] - \Sigma_3 , \quad G(d^3) = \frac{1}{2} \left( 1 + \langle a, d \rangle - \prod_{i=1}^{3} a_{ii} - \Sigma_3 \right) , \quad (3.8) $$

$$ J^2(d^3) = \langle a, d \rangle^2 - 4 \sum_{i>j}^{3} a_{ii}a_{jj}d_id_j + 4d_1d_2d_3 , \quad \langle a, d \rangle = \sum_{i=1}^{3} a_{ii}d_i . \quad (3.9) $$

We prove the following Theorem.

**Theorem 5**

*The minimal and the maximal values in $\Delta_H(d^3)$ are given by*

$$ \min_{\Delta_H(d^3)} = J(d^3) , \quad \max_{\Delta_H(d^3)} = F(d^3) - J(d^3) . \quad (3.10) $$

**Proof** Write the numerator $Q(d^3; z)$ in the following form:

$$ Q(d^3; z) = 1 - \sum_{i=1}^{3} z^{a_{ii}d_i} + z^{F(d^3) - J(d^3) + \Sigma_3} + z^{F(d^3) + \Sigma_3} , \quad (3.11) $$

that in accordance with \[ 2.13 \] gives polynomial $V(d^3; z)$

$$ V(d^3; z) = - \sum_{i=1}^{3} z^{F(d^3) + \Sigma_3 - a_{ii}d_i} + z^{J(d^3)} + \sum_{i=1}^{3} z^{a_{ii}d_i} - z^{F(d^3) + \Sigma_3} . \quad (3.12) $$

Write equation \[ 2.19 \] for the 3-dim case

$$ \sum_{h \in \Delta_H(d^3)} z^h = \frac{V(d^3; z)}{(1 - z^{d_1})(1 - z^{d_2})(1 - z^{d_3})} , \quad (3.13) $$

and notice that every term of the generating polynomial at the left hand side of \[ 3.13 \] has coefficient equal to 1. Moreover, the term of minimal degree of this polynomial has to coincide with the term having minimal degree in $V(d^3; z)$, and therefore such term has also coefficient equal to 1. By \[ 3.12 \], polynomial $V(d^3; z)$ has four candidates to have the minimal degree: $z^{J(d^3)}$ and $z^{a_{ii}d_i}$, $i = 1, 2, 3$.

We prove that the degrees of the last three terms always exceed $J(d^3)$. Making use of formula \[ 3.9 \] and relations \[ 3.3 \] and \[ 3.5 \], we obtain the following representation for $J(d^3)$, holding for any permutation $(i, j, k)$ of $(1, 2, 3)$:

$$ J(d^3) = |a_{ij}d_j - a_{kj}d_k| . \quad (3.14) $$
One can always suppose that $a_{ij} > a_{ji}$ up to permutation of indices $i, j, k$. Since all $a_{ij}$ are positive in the nonsymmetric case, we have by (3.14)

$$J(d^3) < a_{ij}d_j.$$  

(3.15)

However, by (3.5) we have $a_{ij}d_j = a_{jj}d_j - a_{kj}d_j$, or, in other words,

$$a_{ij}d_j < a_{jj}d_j.$$  

(3.16)

Combining the last inequality with (3.15) we get, for every $i = 1, 2, 3$,

$$J(d^3) < a_{ii}d_i.$$  

(3.17)

So, $J(d^3)$ is the minimal degree of the terms entering polynomial $V(d^3; z)$ and therefore, by comparing the left and right hand sides of (3.13), we have $\min \Delta_H(d^3) = J(d^3)$. The second part of (3.10) follows by (2.23). □

As example, consider the semigroup of Figure 1.

**Example 1** $\{d_1, d_2, d_3\} = \{5, 11, 13\}$

$$a_{ij} = \begin{pmatrix} 7 & 2 & 1 \\ 4 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \quad \{a_{11}d_1 = 35, a_{22}d_2 = 33, a_{33}d_3 = 26\}, \quad \left\{\begin{array}{l} F(5, 11, 13) = 19 \\ G(5, 11, 13) = 12 \\ J(5, 11, 13) = 2 \end{array}\right\}, \quad \left\{\begin{array}{l} S'(5, 11, 13) = \{17, 19\} \\ \Delta_S(5, 11, 13) = \{1, 3, 4, 6, 8, 9, 14, 19\} \\ \Delta_H(5, 11, 13) = \{2, 7, 12, 17\} \right\}$$

$$H(5, 11, 13; z) = \frac{Q(5, 12, 13; z)}{(1 - z^5)(1 - z^{11})(1 - z^{13})}, \quad Q(5, 11, 13; z) = 1 - z^{26} - z^{33} - z^{35} + z^{46} + z^{48},$$

$$\sum_{h \in \Delta_H} z^h = \frac{V(5, 11, 13; z)}{(1 - z^5)(1 - z^{11})(1 - z^{13})}, \quad V(5, 11, 13; z) = z^2 - z^{13} - z^{15} - z^{22} + z^{26} + z^{33} + z^{35} - z^{46}.$$  

One more statement can be proven for $\min \Delta_H(d^3)$ if we combine (2.22) and (2.23) with (3.8) and (3.10):

$$J(d^3) \leq c(d^m) - G(d^m) \quad \Rightarrow \quad J(d^3) \leq 1 + a_{11}a_{22}a_{33} - \Sigma_3.$$  

(3.18)

4 HIGH – DIMENSIONAL CASES

In this Section we extend our results about minimal and maximal values of the $h$–gaps to the numerical semigroups generated by four and more positive integers. The following Theorem indicates that for $m > 3$ the situation is much more difficult.

**Theorem 6** (115, Theorem 3)

The number $\#\{Q(d^m; z)\}$ of non–zero coefficients in the polynomials $Q(d^m; z)$ is not bounded by any function of $m$ for $m \geq 4$, although it is finite for every choice of the generators $d_i$.

However, the next theorem shows how the multiplicity $d_1$ and the embedding dimension $m$ of a numerical semigroup give an upper bound for $\#\{Q(d^m; z)\}$.

**Theorem 7** (113, Theorem 10)

The number of non–zero coefficients in the polynomial $Q(d^m; z)$, $m \geq 4$, is bounded and satisfies the inequality

$$\#\{Q(d^m; z)\} \leq d_12^m - 2(m - 1).$$  

(4.1)
Unfortunately, the Frobenius problem for nonsymmetric numerical semigroups $S(d^m), m \geq 4,$ is still open nowadays, although many numerical algorithms are developed \cite{16}. However, only the symmetric semigroups $S(d^4)$ are to our knowledge studied \cite{17}, \cite{11} by means of commutative algebra.

Recall the following basic entity for the $m$-dim semigroups facilitating further discussion: the Apéry set of semigroup $S(d^m)$ with respect to generator $d_j$ is the set \cite{18},

$$Ap(d^m; d_j) := \{ s \in S(d^m) \mid s - d_j \notin S(d^m) \} . \quad (4.2)$$

Hereafter, we’ll make worth of the Apéry set of semigroup with respect to its multiplicity $d_1$. This set has the following properties

$$\# Ap(d^m; d_1) = d_1, \quad F(d^m) = \max Ap(d^m; d_1) - d_1, \quad 0, d_2, d_3, \ldots, d_m \in Ap(d^m; d_1) . \quad (4.3)$$

The generating function $A_{d_1}(d^m; z)$ for the Apéry set $Ap(d^m; d_1)$ was given in \cite{13},

$$A_{d_1}(d^m; z) = \sum_{s \in Ap(d^m; d_1)} z^s = \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1})\Phi(d^m; z) , \quad (4.4)$$

where $\Phi(d^m; z)$ is the gaps generating function \cite{14}. Thus, combining \cite{2.4} and \cite{4.4} we get

$$Q(d^m; z) = \prod_{j=2}^{m} (1 - z^{d_j}) \cdot A_{d_1}(d^m; z) . \quad (4.5)$$

Notice that \cite{4.5} and the last containment in \cite{1.3} are sufficient to prove Theorem \cite{7}.

Denote by $D_i(d^m; d_1), 1 \leq i \leq m - 1,$ the sets of integers

$$D_i(d^m; d_1) = \{ d_{k_1} + \ldots + d_{k_i} \mid k_1, \ldots, k_i \in \{ 2, \ldots, m \} , \quad k_1 \neq k_2 \neq \ldots \neq k_i \} , \quad \text{i.e.} \quad (4.6)$$

$$D_1(d^m; d_1) = \{ d_k \mid k \in \{ 2, \ldots, m \} \} ,$$

$$D_2(d^m; d_1) = \{ d_k + d_l \mid k, l \in \{ 2, \ldots, m \} , \quad k \neq l \} , \quad \text{etc.}$$

Finally, the last set $D_{m-1}(d^m; d_1) = \{ \Sigma_m - d_1 \} \text{ consists of only one integer.}$

Construct, for every $i = 1, \ldots, m - 1,$ the direct sum of the sets

$$Ap(d^m; d_1) \oplus D_i(d^m; d_1) = \{ \alpha + \delta_i \mid \alpha \in Ap(d^m; d_1), \delta_i \in D_i(d^m; d_1) \} , \quad (4.7)$$

and denote by $B_i(d^m)$ the set of degrees of the terms $z^{C_{j,i}}$ for the syzygies of the $i$th kind in \cite{2.5}. By comparison of \cite{4.5} and \cite{2.5} we have

$$B_i(d^m) \subset Ap(d^m; d_1) \oplus D_i(d^m; d_1) , \quad (4.8)$$

that gives immediately the upper bounds for the Betti numbers, $\beta_i(d^m) \leq d_1(m_i - 1),$ which satisfy \cite{4.11}. In particular,

$$B_{m-1}(d^m) \subset Ap(d^m; d_1) \oplus \{ \Sigma_m - d_1 \} . \quad (4.9)$$

Notice that the elements of the set $B_{m-1}(d^m)$ are those elements of the set $Ap(d^m; d_1) \oplus \{ \Sigma_m - d_1 \}$ which are survived in $Q(d^m; z)$ after all cancellations in \cite{4.5}.

In order to prove the main result of this Section (next Theorem \cite{8} we investigate the relationship between two sets, $B_{m-1}(d^m)$ and $S'(d^m)$, which have equal cardinalities (see \cite{2.7}).
Enumerate the elements $F_i$ of the set $S'(d^m)$ in ascending order, $F_{i-1} < F_i$, $i = 1, \ldots, t(d^m)$, and write, according to (1.8),

$$F_j + d_1 \in S(d^m), \quad F_j \in \Delta(d^m).$$

(4.10)

Simple comparison of (4.10) with (4.11) gives

$$F_j + d_1 \in Ap(d^m; d_1), \quad j = 1, \ldots, t(d^m),$$

that implies

$$S'(d^m) \oplus \{\Sigma_m\} \subseteq Ap(d^m; d_1) \oplus \{\Sigma_m - d_1\}.$$

(4.12)

Consider what kind of elements of the set $S'(d^m) \oplus \{\Sigma_m\}$ is survived in $Q(d^m; z)$ after all cancellations in (4.5).

**Lemma 1**

*The set of values $F_j + \Sigma_m$ coincides with the set $B_{m-1}(d^m)$,*

$$S'(d^m) \oplus \{\Sigma_m\} = B_{m-1}(d^m).$$

(4.13)

**Proof** First, prove that elements $F_j + \Sigma_m$ are not cancelled in (4.5), or, in other words, $S'(d^m) \oplus \{\Sigma_m\} \subseteq B_{m-1}(d^m)$. According to (4.5) and (4.11), an element $F_j + \Sigma_m$ is the degree of a term of polynomial $Q(d^m; z)$ if it is not cancelled by a term (with opposite sign) of degree $y + \Sigma_m - \Sigma_{odd}$, where $y + d_1 \in Ap(d^m; d_1)$ and $\Sigma_{odd}$ is the sum of an odd number of generators different from $d_1$. Note that, by the same definition (4.2), $y \in \Delta(d^m)$. Let, by way of contradiction, $F_j + \Sigma_m$ be cancelled for some $j$. Then there exists $y$ such that

$$F_j + \Sigma_m = y + \Sigma_m - \Sigma_{odd} \quad \rightarrow \quad y = F_j + \Sigma_{odd}.$$

(4.14)

However, the last equality in (4.14) contradicts our assumptions: $y \in \Delta(d^m)$ and $F_j \in S'(d^m)$. Thus we have

$$S'(d^m) \oplus \{\Sigma_m\} \subseteq B_{m-1}(d^m).$$

(4.15)

Finally, combining equality (2.17) of both cardinalities, $\#S'(d^m)$ and $\#B_{m-1}(d^m)$, allows us to conclude the proof of Lemma 1 so obtaining the bijection:

$$C_{j,m-1} = F_j + \Sigma_m, \quad j = 1, \ldots, t(d^m).$$

(4.16)

□

**Lemma 2**

*The elements of the set $B_{m-1}(d^m)$ do not coincide with elements of any other set $B_{m-i}(d^m)$, $i \geq 2$.*

**Proof** By Lemma 1 the elements of the set $B_{m-1}(d^m)$ are given by (4.13). Because of cancellations in (4.5), an element of the set $B_{m-1}(d^m)$ could coincide only with an element of a set $B_{m-i}(d^m)$, for $i$ odd, $i \geq 3$. To exclude this, it is thus sufficient to repeat the proof, by way of contradiction, of the first part of Lemma 1 substituting $\Sigma_{odd}$ by $\Sigma_{even}$. □

Below we give a generalization of Theorem 5 for $m \geq 4$. Denote by $\Xi(d^m)$ the union of the sets $B_i(d^m)$ excluding its maximal element $\deg Q(d^m; z)$, i.e.

$$\Xi(d^m) = \cup_{i=1}^{m-1} B_i(d^m) \setminus \{\deg Q(d^m; z)\}.$$  

(4.17)
where \( \#I(d^m) = \#\{Q(d^m; z)\} - 2 \). Define also the complementary set \( I(d^m) \),

\[
I(d^m) := \{\deg Q(d^m; z) - \xi_k, \mid \xi_k \in I(d^m)\}, \quad \#I(d^m) = \#I(d^m).
\] (4.18)

**Lemma 3**

The set \( \Theta(d^m) \) of the degrees of all terms entering polynomial \( V(d^m; z) \) satisfies:

\[
\Theta(d^m) \subseteq I(d^m) \cup \overline{I(d^m)}.
\] (4.19)

**Proof** Combining (2.4) and (2.18) we obtain

\[
V(d^m; z) = \sum_{k=1}^{m-1} (-1)^{m-k+1} \sum_{j=1}^{\beta(m)} \left\{ z^{\deg Q(d^m; z) - C_{j,k}} + (-1)^{m} z^{C_{j,k}} \right\} - \sum_{j=1}^{\beta(m)} \left\{ 1 + (-1)^{m} z^{\deg Q(d^m; z)} \right\}.
\] (4.20)

The two last terms are cancelled with the similar terms existing in the left sums for \( k = m-1, j = \beta(m) \). The degrees of the remained terms in (4.20) are coming from both sets, \( I(d^m) \) and \( \overline{I(d^m)} \), that proves Lemma. \( \square \)

**Theorem 8**

Let a nonsymmetric semigroup \( S(d^m) \) be given, and the Hilbert series \( H(d^m; z) \) of its graded subring be given by (2.4), (2.5). Then \( \min \Delta_H(d^m) \) is equal to the difference of the degrees in the two last terms of polynomial \( Q(d^m; z) \),

\[
\min \Delta_H(d^m) = \deg Q(d^m; z) - \max I(d^m).
\] (4.21)

To prove this theorem we need the auxiliary Lemma 4.

**Lemma 4**

Let a nonsymmetric semigroup \( S(d^m) \) be given. Then

\[
F_{l(d^m)} = F(d^m), \quad \text{and} \quad F_{l(d^m) - 1} = \max \Delta_H(d^m).
\] (4.22)

**Proof** By definition (1.8), the Frobenius number \( F(d^m) \) is the maximal value in \( \Delta(d^m) \), and therefore, in accordance with (1.8), \( F(d^m) = \max S'(d^m) \). Hence, \( F_{l(d^m)} = F(d^m) \).

Let the element \( F_{l(d^m) - 1} \in S'(d^m) \) be given. Consider the difference \( x = F(d^m) - F_{l(d^m) - 1} \), and prove that \( x \in \Delta(d^m) \). Suppose, by way of contradiction, that \( x \in S(d^m) \). However, we have \( F_{l(d^m) - 1} + x = F(d^m) \in \Delta(d^m) \), that contradicts our assumption, \( F_{l(d^m) - 1} \in S'(d^m) \). Hence \( F(d^m) - F_{l(d^m) - 1} \in \Delta(d^m) \), that, according to (2.3), yields

\[
F_{l(d^m) - 1} \in \Delta_H(d^m).
\] (4.23)

Prove that \( F_{l(d^m) - 1} \) is the maximal value in \( \Delta_H(d^m) \). Suppose, by way of contradiction, that there exists a value \( y_1 \in \Delta_H(d^m) \) such that \( F_{l(d^m) - 1} < y_1 < F_{l(d^m)} \). Then, there exists necessarily \( y_2 \in S(d^m) \) such that \( y_1 + y_2 \in \Delta(d^m) \). Summarize the known facts as follows:

\[
y_1 \in \Delta_H(d^m), \quad F(d^m) - y_1 \in \Delta_H(d^m),
\]

\[
y_2 \in S(d^m), \quad y_1 + y_2 \in \Delta(d^m).
\]

(4.24)

(4.25)
First, suppose that $y_1 + y_2 \not\in \Delta_H (d^m)$, that implies
\[
F (d^m) - (y_1 + y_2) = (F(d^m) - y_1) - y_2 = s \in S (d^m) \rightarrow F(d^m) - y_1 = y_2 + s \in S (d^m)
\]
and contradicts (4.24).

Next, suppose that $y_1 + y_2 \in \Delta_H (d^m)$. However, this means that we have found another value $y_1 + y_2$ which satisfies
\[
F_t(d^m) - 1 < y_1 + y_2 < F_t(d^m).
\]

Further repetitions of this procedure contradict the fact that the interval $[F_t(d^m) - 1, F_t(d^m)]$ is finite and makes our assumption about a value $y_1 \in \Delta_H (d^m)$ such that $F_t(d^m) - 1 < y_1 < F_t(d^m)$ wrong. Thus, the 2nd equality in (4.31) is proven.

To complete the proof of Theorem 8 observe that, combining the 1st equality in (4.22) with equality (4.16), for $j = t (d^m)$, we obtain:
\[
F (d^m) = \deg Q (d^m; z) - \Sigma_m,
\]
(being $\deg Q (d^m; z) = C_{\beta_{m-1}, m-1}$).

Using equalities (2.23) and (4.26) we get
\[
\min \Delta_H (d^m) = \deg Q (d^m; z) - \Sigma_m - \max \Delta_H (d^m).
\]

Combining the 2nd equality in (4.22) with (4.16), we get
\[
\max \Delta_H (d^m) = \max \{B_{m-1} (d^m) \setminus \{\deg Q (d^m; z)\}\} - \Sigma_m.
\]

On the other hand, by Theorem 3 and Lemma 3
\[
\max \Delta_H (d^m) + \Sigma_m \text{ is either equal to } \max \Xi (d^m) \text{ or to } \max \Xi (d^m).
\]

Equation (4.28) says that $\max \Delta_H (d^m) + \Sigma_m$ belongs to set $\Xi (d^m)$, and, more precisely:
\[
\max \Delta_H (d^m) = \max \Xi (d^m) - \Sigma_m.
\]

Combining (4.27) with (4.29) we obtain the statement of the theorem.

As a spin-off of the proof of Theorem 8 we have a corollary concerning the maximal degrees of the terms entering the syzygies of different kinds in (2.5).

**Corollary 3**
\[
\max \{B_{m-1} (d^m) \setminus \{\deg Q (d^m; z)\}\} > \max \{B_{m-i} (d^m)\}, \quad i \geq 2.
\]

**Proof** By consequence of (4.28) and (4.29),
\[
\max \Xi (d^m) = \max \{B_{m-1} (d^m) \setminus \{\deg Q (d^m; z)\}\}.
\]

However by definition (4.17) of the set $\Xi (d^m)$, we have also $\max \Xi (d^m) \geq \max \{B_{m-i} (d^m)\}$, $i \geq 2$. On the other hand, by consequence of Lemma 2 the last inequality must be strict since $\max \{B_{m-1} (d^m) \setminus \{\deg Q (d^m; z)\}\} \not\in B_{m-i} (d^m)$ for any $i \geq 2$. Combining both these facts, we come to (4.30).
**Lemma 5**

Let a nonsymmetric semigroup \( S(d^m) \) be given. Then

\[
S'(d^m) \setminus \{F(d^m)\} \subseteq \Delta_H(d^m) .
\]

**Proof**  Let \( F_j \in S'(d^m), j \neq t(d^m) \). By definition (1.8), it belongs to the set \( \Delta(d^m) \). Suppose, by way of contradiction, that \( F_j \not\in \Delta_H(d^m) \). Then, there exists an element \( s = F(d^m) - F_j \) such that \( s \in \Delta \), and therefore \( F_j + s = F(d^m) \in \Delta(d^m) \). This contradicts the assumption that \( F_j \) belongs to \( S'(d^m) \). Hence \( F_j \in \Delta_H(d^m) \). □

**Lemma 5** has an important Corollary which would be made worth in Section 5.

**Corollary 4**

\[
\max \Delta_H(d^m) - \min \Delta_H(d^m) \geq t(d^m) - 2 , \quad (4.33)
\]

\[
\max \Delta_H(d^m) \geq \frac{1}{2} | F(d^m) + t(d^m) | - 1 , \quad \min \Delta_H(d^m) \leq \frac{1}{2} | F(d^m) - t(d^m) | + 1 . \quad (4.34)
\]

**Proof**  The proof is based on **Lemma 5** and follows from a chain of inequalities. Taking in mind \( \# \{S'(d^m) \setminus \{F(d^m)\}\} = t(d^m) - 1 \), we have

\[
\max \Delta_H(d^m) - \min \Delta_H(d^m) + 1 \geq \# \Delta_H(d^m) \geq t(d^m) - 1 ,
\]

that proves (4.33). Combining now (2.23) with (4.33) we arrive at (4.34) □

One more Corollary deals with the maximal and minimal values of the set \( \Xi(d^m) \).

**Corollary 5**

\[
\min \Xi(d^m) + \max \Xi(d^m) > \deg Q(d^m; z) . \quad (4.35)
\]

**Proof**  By Theorem 3 the minimal degree among the terms which contribute to \( V(d^m; z) \) is equal to \( \min \Delta_H(d^m) \). On the other hand, by Theorem 5 this degree is given by \( \deg Q(d^m; z) - \max \Xi(d^m) \). Since the degree \( \min \Xi(d^m) \) contributes also to the polynomial \( V(d^m; z) \), we conclude

\[
\deg Q(d^m; z) - \max \Xi(d^m) < \min \Xi(d^m) ,
\]

that proves Corollary. □

We finish this Section with one Example for the 4-dim numerical semigroup studied in [19]. The polynomial \( Q(d^m; z) \) was obtained by means of diagrammatic calculation on the set \( \Delta(d^m) \) developed for \( m = 3 \) and extended for \( m \geq 4 \) [13].

**Example 2** \( \{d_1, d_2, d_3, d_4\} = \{103, 133, 165, 228\} 

\[
\begin{align*}
 a_{11}d_1 &= 824, & F(103, 133, 165, 228) &= 1436, & \deg Q &= 2065 \\
a_{22}d_2 &= 1197, & G(103, 133, 165, 228) &= 840, & \max \Xi &= 2046 \\
a_{33}d_3 &= 825, & \Delta_H(103, 133, 165, 228) &= \{19, \ldots, 1417\}, & \min \Xi &= 824 \\
a_{44}d_4 &= 1368, & \# \{Q(103, 133, 165, 228; z)\} &= 26, & \min \Delta_H &= 2065 - 2046 \\
\end{align*}
\]

\[
S'(103, 133, 165, 228) = \{1145, 1316, 1355, 1374, 1417, 1436\}, \quad \beta_1 = 7, \quad \beta_2 = 12, \quad \beta_3 = 6 ,
\]

\[
H(103, 133, 165, 228; z) = \frac{Q(103, 133, 165, 228; z)}{(1 - z^{103})(1 - z^{133})(1 - z^{165})(1 - z^{228})}, \quad \Sigma_4 = 629 .
\]

\[
Q(103, 133, 165, 228; z) = 1 - z^{824} - z^{825} - z^{1077} - z^{1096} - z^{1197} - z^{1216} + z^{1319} + z^{1362} - z^{1368} + z^{1489} + z^{1508} + z^{1533} + z^{1546} + z^{1609} + z^{1737} + z^{1756} - z^{1774} + z^{1780} + z^{1881} + z^{1900} - z^{1945} - z^{1984} - z^{2003} - z^{2046} - z^{2065}
\]
5 Special Kinds of Numerical Semigroups

In this Section we focus on three special kinds of numerical semigroups, namely, the pseudo–symmetric semigroups, semigroups with maximal embedding dimension, and semigroups of maximal and almost maximal length.

5.1 Pseudo–symmetric semigroups

Following [1] and [4] call the nonsymmetric semigroup $S(d^m)$ pseudo–symmetric if

$$S'(d^m) = \left\{ \frac{F(d^m)}{2}, F(d^m) \right\}.$$ (5.1)

Notice that $F(d^m)$ is necessarily an even number and $t(d^m) = 2$. However the last equality alone is not enough to provide the pseudo–symmetricity of semigroup $S(d^m)$.

**Corollary 6**

Every pseudo–symmetric semigroup $S(d^m)$ satisfies:

$$\Delta_H(d^m) = \left\{ \frac{F(d^m)}{2} \right\}.$$ (5.2)

**Proof** By Lemma [4] we have $\max \Delta_H(d^m) = \frac{1}{2} F(d^m)$. On the other hand, according [2,23], we get $\min \Delta_H(d^m) = \frac{1}{2} F(d^m)$. Thus, $\#\Delta_H(d^m) = 1$ and (5.2) is proven. $\square$

**Remark 3** Notice that for all $m > 3$ there exist semigroups with $\#\Delta_H(d^m) = 1$ which are not pseudo–symmetric. However, the case $m = 3$ is a special one.

**Corollary 7**

Let a nonsymmetric semigroup $S(d^3)$ be given, and $\#\Delta_H(d^3) = 1$. Then $S(d^3)$ is pseudo–symmetric.

**Proof** By definition of the set $\Delta_H(d^m)$, the equality $\#\Delta_H(d^m) = 1$ implies that $\min \Delta_H(d^m) = \max \Delta_H(d^m) = \frac{1}{2} F(d^m)$. By Lemma [4] this value is equal to $F(d^m)$, $F(d^m)$ is pseudo–symmetric. $\square$

In the rest of this Section we study the pseudo–symmetric semigroups $S(d^3)$ in more details. We prove that all semigroups $S(d^3)$ with one $h$–gap are generated by a 3–parametric family of triples $\{d_1, d_2, d_3\}$ and give their parametric representation.

Recall the following basic representations of the conductor $c(d^3)$, the minimal value in $\Delta_H(d^3)$, the cardinality $\#\Delta_H(d^3)$ and generators $d_1, d_2$ and $d_3$ through the matrix elements $a_{ij}$ [13]:

$$c(d^3) = 1 + (a_{21} + a_{31})(a_{12} + a_{32}(a_{31} + a_{32})) - (a_{12}a_{23} + a_{23}a_{31} + a_{31}a_{12}) -$$

$$- (a_{21}a_{13} + a_{13}a_{32} + a_{32}a_{21}) - (a_{31}a_{32} + a_{12}a_{13} + a_{32}a_{21}) +$$

$$\max \{a_{12}a_{23}a_{31}, a_{21}a_{13}a_{32}\}.$$ (5.3)

$$\min \Delta_H(d^3) = |a_{12}a_{23}a_{31} - a_{21}a_{13}a_{32}|,$$ (5.4)

$$\#\Delta_H(d^3) = \min \{a_{12}a_{23}a_{31}, a_{21}a_{13}a_{32}\},$$ (5.5)

$$d_1 = a_{12}a_{23} + a_{32}a_{13} + a_{12}a_{13},$$ (5.6)
By definition of pseudo-symmetric semigroups and the 1st formula in (5.5) the elements of the Johnson matrix \((a_{ij})\) satisfy
\[
a_{12} = a_{23} = a_{31} = 1, \ a_{21}a_{13}a_{32} \geq 2, \tag{5.7}
\]
since at least one of three matrix elements \(a_{21}, a_{13}, a_{32}\) exceeds 1. According to the 2nd formulas in (5.4), (5.5) and (5.6) the generating triple \(d^3\) is given by:
\[
d_1 = 1 + a_{32}a_{13} + a_{13} , \quad d_2 = 1 + a_{13}a_{21} + a_{21} , \quad d_3 = 1 + a_{21}a_{32} + a_{32} . \tag{5.8}
\]
The straightforward calculation of \(c(d^3)\) and \(G(d^3)\) for pseudo-symmetric semigroup \(S(d^3)\) gives
\[
c(d^3) = 2a_{21}a_{13}a_{32} - 1, \quad G(d^3) = a_{21}a_{13}a_{32}, \quad \#\Delta^c_G(d^3) = a_{21}a_{13}a_{32} - 1,
\min\Delta^H(d^3) = \max\Delta^H(d^3) = a_{21}a_{13}a_{32} - 1. \tag{5.9}
\]
Note that the formula for \(\min\Delta^H(d^3)\) in (5.4) is in full agreement with (5.1) and (5.9).

Equations (5.8) define a nondegenerate mapping \(\{a_{32}, a_{13}, a_{21}\} \mapsto \{d_1, d_2, d_3\}\) since its Jacobian does not vanish,
\[
\det\left(\frac{\partial^2 d_i}{\partial a_{jk} \partial a_{pq}}\right) = 1 + a_{32} + a_{13} + a_{21} + a_{32}a_{13} + a_{13}a_{21} + a_{21}a_{32} + 2a_{32}a_{13}a_{21} \geq 14. \tag{5.10}
\]
Therefore all different (up to permutation) triples \(\{a_{32}, a_{13}, a_{21}\}\) give rise to different (up to permutation) triples \(\{d_1, d_2, d_3\}\).

Summarizing (5.7), (5.8) and Corollary 7 we formulate the following Theorem.

**Theorem 9**
A nonsymmetric semigroup \(S(d^3)\) is pseudo-symmetric iff there exist three positive integers \(a, b, c\) (\(abc \geq 2\)) such that \(d^3\) is given by:
\[
d_1 = 1 + ab + b , \quad d_2 = 1 + bc + c , \quad d_3 = 1 + ca + a. \tag{5.10}
\]

5.2 Semigroups with Maximal Embedding Dimension

In this Section we apply our results of Section 4 to numerical semigroups of a special kind, \(S(d_{MED}^m)\) where \(d_{MED}^m = (m, d_2, \ldots, d_m)\), called \([20]\) semigroups with maximal embedding dimension (MED). Their study is motivated by the following Theorem.

**Theorem 10** ([8], Proposition 1.14)
The numerical semigroup \(S(d_{MED}^m), m \geq 3\), is never symmetric.

There are many known results on the MED–semigroups \([8], [20]\):
\[
Ap(d_{MED}^m; m) = \{0, d_2, \ldots, d_m\}, \quad \#Ap(d_{MED}^m; m) = m, \quad t(d_{MED}^m) = m - 1, \quad \tag{5.11}
\]
\[
F(d_{MED}^m) = d_m - m, \quad G(d_{MED}^m) = \frac{1}{m}\sum_{k=2}^{m} d_k - \frac{m - 1}{2}, \quad \beta_k(d_{MED}^m) = k\left(\binom{m}{k+1}\right), \quad \tag{5.12}
\]
Note that the divisibility by \(m\) of the sum \(\sum_{k=2}^{m} d_k\) is guaranteed by the minimality of the set of generators, since \(d_i \neq d_j \pmod{m}, i \neq j \in \{2, \ldots, m\}\). The minimality of the set of generators implies also that
\[
d_m \leq (m - 1)d_2 - m. \tag{5.13}
\]
The sum of all Betti numbers of a MED–semigroup is less than the upper bound in Theorem 7

\[ \sum_{k=0}^{m-1} \beta_k (d^m_{MED}) = (m - 2)2^{m-1} + 2 < m 2^{m-1} - 2(m - 1). \]

The set \( S' (d^m_{MED}) \) is given by \( \{d_2 - m, \ldots, d_{m-1} - m, d_m - m \} \).

Simple structure of the MED–semigroups makes easy to study the properties of their \( h \)–gaps.

**Corollary 8**

\[
\begin{align*}
\min \Delta_H (d^m_{MED}) &= d_m - d_{m-1}, \\
\max \Delta_H (d^m_{MED}) &= d_{m-1} - m, \\
\#\Delta_H (d^m_{MED}) &= \frac{2}{m} \sum_{k=2}^{m} d_k - d_m, \\
\#\Delta_G (d^m_{MED}) &= d_m - \frac{1}{m} \sum_{k=2}^{m} d_k - \frac{m-1}{2}.
\end{align*}
\]

**Proof** Proof of (5.14) follows from Lemma 1 and Theorem 8. Proof of (5.15) follows from formulas (2.2) and (2.3) for cardinalities of \( \Delta_H (d^m) \) and \( \Delta_G (d^m) \) sets and formulas (5.12) for the Frobenius number and the genus of the MED–semigroups.

Formulas (5.14) and (5.15) allow us to formulate two more Corollaries.

**Corollary 9**

Let a numerical semigroup \( S (d^m_{MED}) \) be given. Then

\[ d_m \leq 2d_{m-1} - 2m + 3, \]

and equality in (5.16) holds iff \( \#\Delta_H (d^m_{MED}) = 1 \).

**Proof** By Theorem 10 the numerical semigroup \( S (d^m_{MED}) \) is nonsymmetric, and therefore, by Corollary 4 \( \max \Delta_H (d^m_{MED}) \geq \min \Delta_H (d^m_{MED}) + t (d^m) - 2 \). Substituting (5.11) and (5.14) in the last inequality we get (5.16). Due to the equivalence of equalities

\[ \max \Delta_H (d^m) = \min \Delta_H (d^m) \iff \#\Delta_H (d^m) = 1, \]

we conclude that the equality in (5.16) is attained iff \( \#\Delta_H (d^m_{MED}) = 1 \).

**Remark 4** Note that the equality in (5.16) can be fulfilled by nonpseudo–symmetric MED–semigroup for \( m \geq 4 \) (see Remark 3 in Section 5.1). The inequality (5.16) coincides with inequality (5.13) in the case \( m = 3 \).

The other Corollary deals with the cardinalities of sets \( \Delta_H (d^m_{MED}) \) and \( \Delta_G (d^m_{MED}) \).

**Corollary 10**

Let a numerical semigroup \( S (d^m_{MED}) \) be given. Then

\[ \#\Delta_H (d^m_{MED}) = \#\Delta_G (d^m_{MED}) , \]

iff

\[ 6 \sum_{k=2}^{m-1} d_k + m(m - 1) = 2(2m - 3)d_m. \]

**Proof** By Theorem 10 the numerical semigroup \( S (d^m_{MED}) \) is nonsymmetric, and therefore \( \Delta_H (d^m_{MED}) \) is nonempty. Equating both cardinalities \( \#\Delta_H (d^m_{MED}) \) and \( \#\Delta_G (d^m_{MED}) \) given in (5.15) we get (5.18). Vice versa, if (5.18) does hold for the MED–semigroup then by comparison with (5.15) we get \( \#\Delta_H (d^m_{MED}) = \#\Delta_G (d^m_{MED}) \).
Remark 5 By consequence of (5.18), in the case \( m = 3 \) equality (5.17) occurs iff

\[
d^3_{MED} = (3, 3k + 1, 3k + 2), \quad k \geq 1.
\]  

(5.19)

We finish this Section with one Example for the 5-dim numerical MED-semigroup. The polynomial \( Q(d^m; z) \) was obtained by means of diagrammatic calculation on the set \( \Delta(d^m) \).

Example 3 \( \{d_1, d_2, d_3, d_4, d_5\} = \{5, 7, 9, 11, 13\} \)

\[
\begin{align*}
    a_{11}d_1 & = 20 \quad F(5, 7, 9, 11, 13) = 8 \quad \deg Q = 53 \\
    a_{22}d_2 & = 14 \quad G(5, 7, 9, 11, 13) = 6 \quad \max \Xi = 51 \\
    a_{33}d_3 & = 18 \quad \Delta \Xi(5, 7, 9, 11, 13) = \{1, 3, 8\} \quad \min \Xi = 14 , \\
    a_{44}d_4 & = 22 \quad \Delta H(5, 7, 9, 11, 13) = \{2, 4, 6\} \quad \min \Delta H = 53 - 51 \\
    a_{55}d_5 & = 26 \quad \# \{Q(5, 7, 9, 11, 13; z)\} = 50
\end{align*}
\]

\( S'(5, 7, 9, 11, 13) = \{2, 4, 6, 8\} , \quad \beta_1 = 10 , \quad \beta_2 = 20 , \quad \beta_3 = 15 , \quad \beta_4 = 4 \),

\( H(5, 7, 9, 11, 13; z) = Q(5, 7, 9, 11, 13; z) = \frac{1 - z^5}{(1 - z^3)(1 - z^7)(1 - z^9)(1 - z^{11})(1 - z^{13})} , \quad \Sigma_5 = 45 \),

\( Q(5, 7, 9, 11, 13; z) = 1 - z^{14} + z^{16} - 2z^{18} - 2z^{20} - 2z^{22} + z^{23} - z^{24} + 2z^{25} - z^{26} + 3z^{27} + 4z^{29} + 4z^{31} + 3z^{33} - 3z^{34} + 2z^{35} - 2z^{36} + 3z^{37} - 3z^{38} - 3z^{40} - 3z^{42} - 2z^{44} - z^{46} + z^{47} + z^{49} + z^{51} + z^{53} \).

5.3 Semigroups of Maximal and Almost Maximal Length

In this Section we apply our results of Section 4 to another special kind of numerical semigroups, making use of important Theorem [4] on the structure of numerical semigroups. Recall Remark 4 on the number of nongaps of a semigroup.

Theorem 11 \( [4], \) Theorem 20

Let \( S(d^m) \) be a semigroup of type \( t(d^m) \) and let \( \# \Delta_G(d^m) \) be the number of elements in \( S(d^m) \) which are less than \( F(d^m) \). Then we have

\[
G(d^m) \leq \# \Delta_G(d^m) \cdot t(d^m).
\]

(5.20)

Notice that (5.20) together with (2.2) and (2.3) is equivalent to

\[
0 \leq \# \Delta_H(d^m) \leq \# \Delta_G(d^m) \cdot [t(d^m) - 1].
\]

(5.21)

Call the two extreme cases \( (5.21) \) semigroups of maximal length (ML), when \( \# \Delta_H(d^m) \) attains its maximal value, and semigroups of minimal length (ml), when \( \# \Delta_H(d^m) \) vanishes. The latter case corresponds exactly to symmetric semigroups.

Theorem 12 \( [5], \) Theorem 1; \( [6], \) Corollary at p. 339

Let \( S(d^m) \) be a numerical semigroup of type \( t(d^m) \). Then \( S(d^m_{ML}) \) has maximal length iff

\[
d^m_{ML} = \{m, mk + 1, \ldots, mk + m - 1\} , \quad k \geq 1.
\]

(5.22)

When \( \# \Delta_H(d^m) \) gets an intermediate values in the interval \( (0, \# \Delta_G(d^m) \cdot [t(d^m) - 1]) \), semigroup \( S(d^m) \) can be generated by sophisticated series of generators, similar to (5.22), or by a significant number of sporadic tuples [5]. In the meantime, the study of numerical semigroups with generic length are far from its completeness.
Theorem 14 ([5], Theorem 5; [6], Corollary 2)

Let \( \# \Delta_H (d_{AML}) = 1 \), or \( \# \Delta_H (d_{AML}) = \# \Delta_G (d_{AML}) \cdot [t(d_{AML}) - 1] - 1 \).

(5.23)

The semigroups of almost minimal length were discussed in Section 5.1. As for the semigroups of almost maximal length, several explicit results are known:

Lemma 6 ([6], Proposition at p. 345)

Let \( S(d_{AML}) \) be a numerical semigroup of almost maximal length and \( t(d_{AML}) \) its type. Then either

i) \( t(d_{AML}) = 2 \), and \( d_1 = 3 \) or \( 4 \); ii) \( t(d_{AML}) \geq 3 \), and \( d_1 = t(d_{AML}) + 1 \).

(5.24)

Theorem 13 ([5], Theorem 3)

Let \( S(d_m) \) be a numerical semigroup of type \( t(d_m) = 2 \). Then \( S(d_{AML}) \) has almost maximal length iff its generating tuple \( d_{AML} \) is one of the following:

i) \( d_{AML} = \{3, 3k + 2, 3k + 4\} \), \( k \geq 1 \); ii) \( d_{AML} = \{4, 5, 11\} \); iii) \( d_{AML} = \{4, 7, 13\} \).

(5.25)

Theorem 14 ([5], Theorem 5; [6], Corollary 2)

Let \( S(d_m) \) be a numerical semigroup of type \( t(d_m) \geq 3 \). Then \( S(d_{AML}) \) has almost maximal length iff

\[ d_{AML} = \{t_m + 1, k(t_m + 1) + t_m, k(t_m + 1) + (t_m + 2), k(t_m + 1) + (t_m + 3), \ldots, k(t_m + 1) + 2t_m\}. \]

We obtain two Corollaries on the semigroups of maximal and almost maximal length respectively.

Corollary 11

The numerical semigroups \( S(d_{ML}) \) satisfy

\[
\begin{cases}
F(d_{ML}) = mk - 1 \\
\# \Delta_H (d_{ML}) = mk - 2k \\
\# \Delta_G (d_{ML}) = k \\
\end{cases}, \quad \begin{cases}
\text{min } \Delta_H (d_{ML}) = 1 \\
\text{max } \Delta_H (d_{ML}) = mk - 2 \\
\end{cases}.
\]

(5.26)

Proof By Theorem 12 any ML–semigroup is a MED–semigroup; therefore, according to (5.12) and (5.14), we arrive at (5.26). \( \Box \)

Remark 6 By consequence of (5.26), there exists only one ML–semigroup with \( \# \Delta_H (d_{ML}) = 1 \) which is pseudo–symmetric, \( S(3, 4, 5) \).

Note that the AML–semigroup \( S(3, 3k + 2, 3k + 4) \) in Theorem 13 is also a MED–semigroup,

\[
\begin{cases}
F(3, 3k + 2, 3k + 4) = 3k + 1 \\
\# \Delta_H (3, 3k + 2, 3k + 4) = k \\
\# \Delta_G (3, 3k + 2, 3k + 4) = k + 1 \\
\end{cases}, \quad \begin{cases}
\text{min } \Delta_H (3, 3k + 2, 3k + 4) = 2 \\
\text{max } \Delta_H (3, 3k + 2, 3k + 4) = 3k - 1 \\
\end{cases}.
\]

(5.27)

while the other sporadic AML–semigroups (5.25) with \( t(d_{AML}) = 2 \) are not MED–semigroups,

\[
\begin{cases}
F(4, 5, 11) = 7 \\
\# \Delta_H (4, 5, 11) = 2 \\
\# \Delta_G (4, 5, 11) = 3 \\
\end{cases}, \quad \begin{cases}
\text{min } \Delta_H (4, 5, 11) = 1 \\
\text{max } \Delta_H (4, 5, 11) = 6 \\
\end{cases}.
\]

\[
\begin{cases}
F(4, 7, 13) = 10 \\
\# \Delta_H (4, 7, 13) = 3 \\
\# \Delta_G (4, 7, 13) = 4 \\
\end{cases}, \quad \begin{cases}
\text{min } \Delta_H (4, 5, 11) = 1 \\
\text{max } \Delta_H (4, 7, 13) = 9 \\
\end{cases}.
\]
Corollary 12
Any semigroup of almost maximal length $S(d_{AML}^m)$ with $t_m > 2$ satisfies

$$\begin{cases}
F(d_{AML}^m) = k(t_m + 1) + t_m - 1 \\
\#\Delta_H(d_{AML}^m) = (k + 1)t_m - k - 2 \\
\#\Delta_G(d_{AML}^m) = k + 1
\end{cases}, \quad \begin{cases}
\min \Delta_H(d_{AML}^m) = 1 \\
\max \Delta_H(d_{AML}^m) = k(t_m + 1) + t_m - 2
\end{cases} \quad (5.28)$$

Proof. By Theorem 14 the $S(d_{AML}^m)$–semigroup with $t_m > 2$ is the MED–semigroup, and therefore, according to (5.12) and (5.14), we arrive at (5.28). \qed

Remark 7 By consequence of (5.27) and (5.28), we get the known result \[1\]: there exists only one AML–pseudo–symmetric semigroup, $S(3, 5, 7)$.

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