Equiangular Lines and Covers of the Complete Graph

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Abstract

The relation between equiangular sets of lines in the real space and distance-regular double covers of the complete graph is well known and studied since the work of Seidel and others in the 70’s. The main topic of this paper is to continue the study on how complex equiangular lines relate to distance-regular covers of the complete graph with larger index. Given a set of equiangular lines meeting the relative (or Welch) bound, we show that if the entries of the corresponding Gram matrix are prime roots of unity, then these lines can be used to construct an antipodal distance-regular graph of diameter three. We also study in detail how the absolute (or Gerzon) bound for a set of equiangular lines can be used to derive bounds of the parameters of abelian distance-regular covers of the complete graph.

1 Introduction

We explore the rich relation between equiangular lines in a real or complex vector space and covers of the complete graph. In our journey, we will link these concepts to other structures and translate properties across the different topics.

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Equiangular lines have been studied for a long a time, but there has been a recent surge in interest since the connection with quantum information theory was established (see, for example, Appleby [1] or Scott and Grassl [13]). A set of equiangular lines meeting the so-called absolute (or Gerzon) bound is also known as a symmetric, informationally complete, positive operator valued measured (SIC-POVM), and the problem of constructing SIC-POVMs is a major problem both in quantum information theory and in combinatorics. Also another important connection is the well known correspondence between equiangular lines and Seidel matrices, and that a set of lines meets the so-called relative (or Welch) bound if and only if the corresponding Seidel matrix has only two distinct eigenvalues. These structures are also studied in frame theory, and sets of lines meeting the relative bound are called equiangular tight frames.

In this paper, we study the relation between Seidel matrices and simple graphs that are covers of the complete graph. If a Seidel matrix has only two eigenvalues and if its entries are prime roots of unity, then we will show that such matrix implies the existence of distance-regular covers of the complete graph whose automorphism group satisfies certain properties, namely, that the covers are cyclic (in the sense defined by Godsil and Hensel [6]). This is a natural generalization of the well known correspondence between real tight frames and regular two-graphs.

Furthermore, we will use this relation to derive bounds on the defining parameters of such graphs using the absolute bound for equiangular lines. It turns out that the existence of certain graphs could give \( d^2 \) equiangular lines in \( \mathbb{C}^d \). We give the parameter sets of these graphs, and analyse the real case similarly.

## 2 Background on lines

A set of lines spanned by vectors \( x_1, ..., x_n \) in \( \mathbb{C}^d \) (or \( \mathbb{R}^d \)) is a set of complex (or real) equiangular lines if there is \( \alpha \in \mathbb{R} \) such that, for all \( i \) and \( j \),

\[
|\langle x_i, x_j \rangle| = \alpha.
\]

We will call \( \alpha \) the angle between two lines.

Upon associating each line determined by \( x_i \) with the corresponding orthogonal projection given by \( x_i x_i^* \), a set of equiangular lines is precisely the
same thing as a 1-regular quantum design of degree 1. Such design will be called a tight frame if

\[ x_1 x_1^* + \ldots + x_d x_d^* = \frac{n}{d} I. \]

2.1 Theorem (Relative bound, Van Lint and Seidel [14]). If there is a set of \( n \) equiangular lines in dimension \( d \), and if the angle of the set is \( \alpha \), then

\[ \alpha^2 \geq \frac{n - d}{(n - 1)d}. \]

Equality holds if and only if the set of lines corresponds to a tight frame.

This result is sometimes called the Welch bound (see [15]).

Note that in the theorem above, it is irrelevant whether the set of lines is in a real or a complex space. The bound below however distinguishes between the two cases.

2.2 Theorem (Absolute bound, Gerzon (private communication to Seidel and Lemmens, see [12])). If there is a set of \( n \) equiangular lines in \( \mathbb{C}^d \), then

\[ n \leq d^2. \]

If there is a set of \( n \) equiangular lines in \( \mathbb{R}^d \), then

\[ n \leq \binom{d + 1}{2}. \]

In either case, if equality holds, then the set of lines corresponds to a tight frame, and therefore the relative bound holds with equality.

Equiangular sets of lines are equivalent to other combinatorial structures, which we describe below. A Seidel matrix is a Hermitian matrix whose diagonal entries are zeros and off-diagonal entries have absolute value one.

2.3 Theorem (Lemmens and Seidel [12]). Let \( S \) be an \( n \times n \) Seidel matrix. Let \( m_\tau \) be the multiplicity of its least eigenvalue \( \tau \), and \( m_\theta \) the multiplicity of its largest eigenvalue \( \theta \). Then

- the matrix \( I - (1/\tau)S \) is the Gram matrix of a set of \( n \) equiangular lines in dimension \( n - m_\tau \), and
• the matrix $I - (1/\theta)S$ is the Gram matrix of a set of $n$ equiangular lines in dimension $n - m_\theta$.

Moreover, a set of $n$ equiangular lines over $\mathbb{C}^d$ with angle $\alpha$ meets the equality in the relative bound if and only if the corresponding Seidel matrix (up to a sign) has precisely two distinct eigenvalues, and they are

$$\frac{1}{\alpha}, \quad \frac{n - d}{\alpha d}$$

with multiplicities $n - d$ and $d$, respectively.

### 3 Background on covers

For more details about the content of this section, we refer the reader to Godsil and Hensel [6].

An $r$-fold cover of $K_n$ is a simple graph $X$ on $rn$ vertices satisfying the following two properties:

1. There is a partition of the vertex set of $X$ into $n$ sets of $r$ vertices each, to be called fibres, such that no two vertices in the same fibre are connected.

2. There is a perfect matching between any two fibres.

In some cases, an $r$-fold cover of $K_n$ will enjoy the property of being distance-regular. When this happens, the distance-regular graph will be antipodal and all of its intersection parameters will be determined by $n$, $r$ and a third parameter, typically denoted by $c$, which counts the number of common neighbours of two vertices at distance two. We will shortly refer to such a graph as an $(n, r, c)$-DRACKN (standing for distance-regular antipodal cover of $K_n$). Conversely, one can show that all antipodal distance-regular graphs of a diameter 3 are covers of a complete graph. It is straightforward to show that $n - 1$ and $-1$ are eigenvalues of these graphs with respective multiplicities 1 and $n - 1$. There are another two distinct eigenvalues, of opposing signs, that are typically called $\theta$ and $\tau$ with the convention that $\theta > 0$ and $\tau < 0$. If $\delta = n - rc - 2$, it follows that

$$\theta = \frac{\delta + \sqrt{\delta^2 + 4(n - 1)}}{2} \quad \text{and} \quad \tau = \frac{\delta - \sqrt{\delta^2 + 4(n - 1)}}{2} \quad (3.1)$$
The problem of determining which parameter sets correspond to an actual distance-regular graph has been attacked for decades, and despite the many efforts, its full solution most likely will not be seen in the near future. This is the case even for distance-regular graphs of small diameter and constrictive structural properties, such as DRACKNs. However, we can compile a list of fairly restrictive non-trivial conditions that \( n, r \) and \( c \) must satisfy in order to correspond to a graph.

### 3.1 Theorem

Let \( X \) be an \((n, r, c)\)-DRACKN with distinct eigenvalues \( n-1, \theta, -1 \) and \( \tau \), and with \( n \geq 2, r \geq 2, c \geq 1 \). Recall that \( \delta = n - rc - 2 \). Then the parameters of \( X \) satisfy the following conditions.

(a) \( 1 \leq c(r - 1) \leq n - 2 \leq c(2r - 1) - 2 \).

(b) The multiplicities of \( \theta \) and \( \tau \) satisfy

\[
\begin{align*}
m_\theta &= \frac{n(r - 1)\tau}{r - \theta} \quad \text{and} \quad m_\tau = \frac{n(r - 1)\theta}{\theta - \tau},
\end{align*}
\]

and these ratios must be integers.

(c) If \( \delta \neq 0 \), then \( \theta \) and \( \tau \) are integers.

(d) If \( \delta = 0 \), then \( \theta = -\tau = \sqrt{n - 1} \).

(e) If \( n \) is even, then \( c \) is even.

(f) If \( c = 1 \), then \((n - r)\) divides \( n - 1 \), \((n - r)(n - r + 1)\) divides \( rn(n - 1) \), and \((n - r)^2 \leq n - 1 \).

(g) If \( r > 2 \), then \( \theta^3 \geq n - 1 \).

(h) Suppose \( \theta \neq 1, \tau \neq -1 \), and \( \theta^3 \neq n - 1 \). For \( r > 2 \), we have

\[
\begin{align*}

rn &\leq \frac{1}{2} m_\theta (m_\theta + 1), \quad rn \leq \frac{1}{2} m_\tau (m_\tau + 1).
\end{align*}
\]

For \( r = 2 \), we have

\[
\begin{align*}
n &\leq \frac{1}{2} m_\theta (m_\theta + 1), \quad n \leq \frac{1}{2} m_\tau (m_\tau + 1).
\end{align*}
\]

(i) Let \( r > 2 \) and \( \beta \in \{ \theta, \tau \} \) be an integer. If \( n > m_\beta - r + 3 \), then \( \beta + 1 \) divides \( c \).
An arc function of index $r$ over $K_n$ is a function $f$ from the arcs of $K_n$ to the symmetric group $\text{Sym}(r)$ satisfying $f(u,v)^{-1} = f(v,u)$. Arc functions are equivalent to covers in a very natural way. In an $r$-fold cover of $K_n$, a matching from the fibre corresponding to a vertex $u$ of $K_n$ to the fibre corresponding to a vertex $v$ of $K_n$ can be seen as a permutation on $r$ elements that is precisely equal to $f(u,v)$. Without loss of generality, we can always suppose that an arc function $f$ will be equal to the identity permutation when evaluated over the edges of a spanning tree. When this happens, $f$ is called a normalized arc function.

Let $\langle f \rangle$ be the permutation group generated by the images of $f$ over all arcs of $K_n$. An $r$-fold cover of $K_n$ determined by a normalized arc function $f$ is called regular if $\langle f \rangle$ is regular, and moreover abelian if $\langle f \rangle$ is an abelian group. If $\langle f \rangle$ is a cyclic group, we say that the cover is cyclic. Note that a cover is regular if and only if $|\langle f \rangle| = r$. Finally, the automorphism group of the cover that fixes each fibre as a set is regular if and only if $\langle f \rangle$ is regular, and in this case these are isomorphic groups.

Consider the square matrix whose rows and columns are indexed by the vertices of $K_n$, and whose entry $(u,v)$ is equal to the permutation $f(u,v)$ and diagonal entries are equal to 0. We denote this matrix by $A(K_n)^f$.

Let $\varphi$ be an $s$-dimensional representation of $\langle f \rangle$. Let $A(K_n)^{\varphi(f)}$ stand for the matrix obtained from $A(K_n)^f$ by replacing each of its entries by an $s \times s$ permutation matrix corresponding to their image under $\varphi$, and the diagonal entries by $s \times s$ blocks of 0.

If $X$ is a regular cover defined by $f$ and $\phi$ is the regular representation of $\langle f \rangle$, then

$$A(K_n)^{\phi(f)} = A(X).$$

We summarize in the next theorem some important facts relating covers, representations and linear algebra. This and more can be found in Godsil and Hensel [6, Sections 8 and 9].

The first part is an immediate consequence of the well known expression for the eigenvectors of the regular representation of abelian groups in terms of linear characters. The second is a restatement of [6, Corollary 7.5].

**3.2 Theorem.** Let $X$ be a connected abelian $r$-fold cover of $K_n$ determined by a normalized arc function $f$. Let $\phi_1, \ldots, \phi_r$ be the linear characters of $\langle f \rangle$. 


Then $A(X)$ is similar to the matrix

$$
\begin{pmatrix}
A(K_n)^{\phi_1(f)} & A(K_n)^{\phi_2(f)} & \ldots & A(K_n)^{\phi_r(f)}
\end{pmatrix}.
$$

Moreover, $X$ is an $(n, r, c)$-DRACKN if and only if the minimal polynomial of each matrix in $\{A(K_n)^{\phi_i(f)} : i = 2, \ldots, r\}$ is

$$
x^2 - (n - rc - 2)x - (n - 1).
$$

## 4 Lines from covers

A consequence of Theorem 3.2 is that the existence of abelian DRACKNs implies the existence of sets of complex equiangular lines meeting the relative bound.

### 4.1 Theorem

Let $X$ be an abelian $(n, r, c)$-DRACKN defined by a symmetric arc function $f$. Suppose the eigenvalues of $X$ are $n - 1, \theta, -1$ and $\tau$, with respective multiplicities $1, m_\theta, n - 1$ and $m_\tau$. Let $\phi$ be a non-trivial character of $\langle f \rangle$. Then $A(K_n)^{\phi(f)}$ is a Seidel matrix with precisely two distinct eigenvalues, $\theta$ and $\tau$, and therefore:

(a) There are $n$ complex equiangular lines in dimension $(n - \frac{m_\theta}{r - 1})$ meeting the relative bound.

(b) There are $n$ complex equiangular lines in dimension $(n - \frac{m_\tau}{r - 1})$ meeting the relative bound.

**Proof.** Let $\phi_1, \ldots, \phi_r$ be the linear characters of $\langle f \rangle$. If $\phi_1$ is the trivial character, then $A(K_n)^{\phi_1(f)} = A(K_n)$, therefore its spectrum is $(n - 1)^{(1)}$ and $(-1)^{(n-1)}$. By Theorem 3.2 this implies that the spectrum of each matrix $A(K_n)^{\phi_k(f)}$, for $k = 2, \ldots, r$, contains only multiple copies of $\theta$ or $\tau$. Because these matrices have trace 0, the multiplicities of $\theta$ and $\tau$ in each one of them do not depend on $k$, and will be equal to $m_\theta/(r - 1)$ and $m_\tau/(r - 1)$ respectively. The result now follows from Theorem 2.3.

We present here some examples of infinite families of abelian covers.
Symplectic covers

This construction generalizes the so-called Thas - Somma construction. Let $p$ be a prime, and let $V$ and $U$ be vector spaces over $GF(p)$ of respective dimensions $m$ and $s$. Let $B$ be a $GF(p)$-linear alternating form from $V \times V$ to $U$, with the extra property that for each $a \in V$, the linear mapping $B_a : V \to U$ defined as $B_a(v) = B(a, v)$ is surjective. We define a graph $X(B)$ on the vertex set $V \times U$ where adjacency between distinct vertices is defined by $(v, a) \sim (w, b) \iff B(v, w) = a - b$.

It follows that $X(B)$ is a $(p^m, p^s, p^m-s)$-DRACKN. If $f$ is the normalized arc function defining $X(B)$, it is easy to see that $\langle f \rangle \cong Z_p^s$, hence $X(B)$ is abelian. Godsil [8] describes a way of constructing such symplectic forms whenever there exists an $s$-dimensional space of invertible $m \times m$ skew-symmetric matrices over $GF(p)$. In particular, this always exists when $s = 1$ and $m$ is even.

De Caen and Fon-der-Flaass construction

Let $V$ be a $d$-dimensional vector space over $GF(2^t)$. Consider a skew product $\ast$ of $V$, that is, a bilinear mapping $\ast : V^2 \to V$ such that $x \mapsto x \ast x$ is a bijection and $x \ast y = y \ast x$ if and only if $x$ and $y$ are linearly dependent. Let $S = ((s_{ij}))_{i,j \in F}$ be a symmetric latin square filled with the elements of $GF(2^t)$. Construct a graph on the vertex set $V \times F \times V$ and such that $(a, i, \alpha)$ and $(b, j, \beta)$ are adjacent if and only if

$$\alpha + \beta = a \ast b + b \ast a + s_{ij}(a \ast a + b \ast b).$$

This graph is a $(2^{t(d+1)}, 2^{td}, 2^t)$-DRACKN, and by construction it also follows that it is an abelian cover whose automorphism group fixing each fibre is isomorphic to $Z_{2^d}$. De Caen and Fon-der-Flaas [3] also remark that skew products exist if and only if $d$ is odd.

Generalized Hadamard matrices

Let $X$ be an abelian $(n, r, c)$-DRACKN determined by a normalized arc function $f$ such that $\delta = -2$. If $\phi$ is a character of $\langle f \rangle$, let $S = A(K_n)^{\phi(f)}$ (recall the notation introduced after Theorem 3.1). It follows from Theorem 3.2 that

$$S^2 = (n-1)I + \delta S,$$
and so because $\delta = -2$, we have that $(S + I)^2 = nI$. The matrix $S + I$ is therefore an Hermitian Butson-type Hadamard matrix with constant diagonal.

Let $G = \langle f \rangle$ and $e$ be the identity of $G$. Consider the group ring $\mathbb{Z}[G]$. Given a subset $S$ of $G$, we use the notation

$$S = \sum_{g \in S} g,$$

where the sum is the ring sum. Consider the matrix $H = A(K_n)f + eI$ over $\mathbb{Z}[G]$. Note that

$$H^2 = nI + c \mathcal{G} (J - I),$$

therefore $H$ is a generalized Hadamard matrix $GH(r, c)$ (note that $n = rc$) over the group $G$ (of order $r$). See Colbourn and Dinitz [4, Chapter V.5] for more details.

The paragraph above shows that any abelian $(rc, r, c)$-DRACKN implies the existence of a self-adjoint $GH(r, c)$ with constant diagonal, and [6, Corollary 7.5] states precisely the opposite.

In a recent paper, Klin and Pech [11] derived many new constructions of abelian DRACKNs based on these generalized Hadamard matrices. In particular, they showed in [11, Theorem 5.6] that any $n \times n$ generalized Hadamard matrix over a group $G$ implies the existence of a $n^2 \times n^2$ self-adjoint generalized Hadamard matrix with constant diagonal over $G$, and by the remarks above, those are equivalent to abelian DRACKNs with $\delta = -2$. The table below, extracted from [11, page 227], contains a list of the parameter sets of DRACKNs that are obtained from known generalized Hadamard matrices.
\[
\begin{array}{|c|c|}
\hline
(n, r, c) & \text{conditions} \\
\hline
(p^{m2^t}, p^n, p^{m2^t-n}) & m \geq n \geq 1, \ p \ \text{prime}, \ t > 0 \\
\hline
(2^t p^{m2^t}, p^n, p^{m2^t-n}) & m \geq n \geq 1, \ p \ \text{prime}, \ t > 0 \\
\hline
(4^t p^{m2^t}, p^n, 4^t p^{m2^t-n}) & m \geq n \geq 1, \ p \ \text{prime}, \ t > 0 \\
\hline
(8^t q^{2^t}, q, 8^t p^{2^t-1}) & 19 < q < 200, \ q \ \text{prime power}, \ t > 0 \\
\hline
(8^t p^{2^t}, p, 8^t p^{2^t-1}) & 19 < p, \ p \ \text{prime}, \ t > 0 \\
\hline
(k^{2^t} q^{2^t}, q, k^{2^t} q^{2^t-1}) & q > ((k - 2)2^{k-2}), \ \exists \ \text{Hadamard matrix of order} \ k, \ t > 0 \\
\hline
(45, 3, 12) & \text{due to Klin and Pech, coming from the Foster graph.} \\
\hline
(144, 4, 36) & \text{discovered by J. Seberry.} \\
\hline
\end{array}
\]

5 Covers from lines

In Section 4, we showed how to construct equiangular lines using an abelian DRACKN. On the opposite direction, it is well known that real equiangular lines can be used to construct regular two-graphs, as we will briefly explain. Then, we generalize this result, showing how a set of complex equiangular lines can be used to construct an abelian DRACKN of index larger than two.

Any given \((n, 2, c)\)-DRACKN is necessarily abelian. These graphs are equivalent to the so-called and well studied regular two-graphs. The Seidel matrix \(S\) of any set of real equiangular lines has off diagonal entries equal to +1 or −1, and upon replacing

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\text{ by } \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\text{ and } -1 \text{ by } \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\]

it is easy to see that the new \(2n \times 2n\) matrix is the adjacency matrix of a 2-fold cover of \(K_n\), say \(X\). If the relative bound is met on the original set of lines, then \(S\) has only two distinct eigenvalues, and so \(X\) will be an abelian DRACKN. This describes the correspondence between real equiangular tight frames and regular two-graphs. The classification of regular two-graphs has
received a considerable amount of attention in the past 40 years or so. See Godsil and Royle [7, Chapter 11] or Brouwer and Haemers [2, Chapter 10] for more information.

We will now describe a way of constructing cyclic DRACKNs of index larger than two from sets of equiangular lines satisfying certain properties.

5.1 Theorem. Suppose $x_1, \ldots, x_n$ is a set of complex equiangular lines in $\mathbb{C}^d$ with angle $\alpha$ and Gram matrix $G$. Suppose they satisfy the following two properties:

(i) This set of lines meets the relative bound, and hence

$$\alpha^2 = \frac{n-d}{(n-1)d}.$$  

(ii) All off-diagonal entries of the matrix

$$S = \frac{1}{\alpha}(G - I)$$

are $r$-th roots of unity, where $r$ is a prime.

Then there exists a cyclic $(n, r, c)$-DRACKN, where

$$c = \frac{1}{r} \left( (n-2) + \frac{2d-n}{\alpha d} \right).$$

Proof. Let $C_r$ be the multiplicative group of the $r$-th roots of unity. Let $\varphi$ be a representation of $C_r$ of degree $k$, and let $S^\varphi$ be the matrix obtained from $S$ by replacing each diagonal entry by a $k \times k$ block of 0s, and each off-diagonal entry by its image under $\varphi$. Naturally, if $\phi$ is the regular representation of $C_r$, $S^\phi$ is the adjacency matrix of a graph $X$, and our goal is to show that $X$ is a DRACKN.

Let $\phi_1, \ldots, \phi_r$ be the linear characters of $C_r$, satisfying

$$\phi_k(e^{2\pi i/r}) = e^{(k-1)2\pi i/r}.$$  

Clearly $A(K_n) = S^{\phi_1}$ and $S = S^{\phi_2}$. We claim that, for $j = 2, \ldots, r$, the minimal polynomials of the matrices $S^{\phi_j}$ are all equal.

In fact, let $\Phi_r(x)$ be the $r$-th cyclotomic polynomial, and because $r$ is prime, we have $\Phi(x) = x^{r-1} + x^{r-2} + \ldots + x + 1$. Since

$$\mathbb{Q}(e^{2\pi i/r}) \cong \mathbb{Q}[x]/(\Phi_r(x)),$$  

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we can see the matrix $S$ as a matrix whose off-diagonal entries are powers of the indeterminate $x$ subject to the relation $\Phi(x) = 0$. Thus, if $m(y)$ is the minimal polynomial of $S$, then $m(S^{\phi_j})$ will vanish because $\phi_j(e^{2\pi i/r})$ is also a root of $\Phi_r(x)$ for all $j$. From Theorem \[2.3\] and condition (i), $m(y)$ has degree two and $S^{\phi_j}$ is not a multiple of the identity matrix for any $j$, hence the claim follows.

Moreover, the trace of $S^{\phi_j}$ is equal to 0 for all $j$, therefore all these matrices with $j \geq 2$ are cospectral. Because the eigenvectors of the regular representation of an abelian group are its characters, it follows that $S^\phi$ is similar to the block diagonal matrix whose blocks are the matrices $S^{\phi_j}$, with $j = 1, ..., r$. All together, and from the expression for the eigenvalues of $S$ given by Theorem \[2.3\] we have shown that $X$ is an $r$-fold cover of $K_n$ with spectrum given by

$$n - 1^{(1)} , \left(\frac{n-d}{\alpha d}\right)^{d(r-1)} , -1^{(n-1)} , \left(-\frac{1}{\alpha}\right)^{(r-1)(n-d)}.$$ 

By Godsil and Hensel \[6\] Lemma 7.1 and because $r$ is prime, it follows that $X$ is connected.

Finally, by Theorem \[3.2\] we have that $X$ is a DRACKN with the given parameters. \[\square\]

Unfortunately, the only examples we know of sets of lines satisfying the conditions of Theorem \[5.1\] are those constructed from known abelian DRACKNs. Recently, Fickus et al. (see \[5\] and \[10\]) showed how to construct sets of equiangular lines meeting the relative bound based on previous sets and combinatorial designs. Using their construction, one is almost capable of obtaining a new set of lines satisfying the conditions of Theorem \[5.1\]. The only problem is that certain entries of the matrix $S$ will be equal to $-1$, and thus not an $r$-th root of unity for any $r$ prime other than $r = 2$. In this case, the corresponding abelian DRACKNs coming from certain designs were already known (see Goethals and Seidel \[9\]).

## 6 Feasibility conditions for DRACKNs

In Theorem \[3.1\] we presented feasibility conditions for the parameter sets of DRACKNs. In this section, we work out some extra feasibility conditions that the parameters of an abelian DRACKN must satisfy. Our main tool will be to
use the absolute bound for a set of equiangular lines to find bounds on the parameters of abelian DRACKNs. We will then study the extreme cases. For instance, we find, to our surprise, that there are some feasible parameter sets of abelian DRACKNs that would give \( d^2 \) equiangular lines in \( \mathbb{C}^d \).

We begin by pointing out an immediate corollary of two results due to Godsil and Hensel.

**6.1 Theorem** ("quotienting", \cite{6}, Lemma 6.2). Let \( X \) be an abelian \((n, r, c)\)-DRACKN determined by a normalized arc function \( f \). Let \( H \) be a subgroup of \( \langle f \rangle \) of size \( t \). Then the partition induced by the orbits of \( H \) in each fibre is equitable, and therefore there is an abelian \((n, r/t, tc)\)-DRACKN obtained as a quotient by this partition.

**6.2 Theorem** (\cite{6}, Theorem 9.2). If \( X \) is a cyclic \((n, r, c)\)-DRACKN with \( r > 2 \), then \( r \) divides \( n \).

If \( p \) is a prime that divides the order of a group \( G \), then there is a cyclic subgroup of \( H \) of order \( p \). Using this, we obtain the corollary below, which in particular implies that DRACKNs with \( \delta = 0 \) and \( r \) not a power of two cannot be abelian.

**6.3 Corollary.** If \( X \) is an abelian \((n, r, c)\)-DRACKN, then any odd prime that divides \( r \) also divides \( n \).

Now we show how to translate the absolute bound for a set of equiangular lines into some extra feasibility conditions for the parameter sets of abelian DRACKNs. To simplify the notation, when \( r \) is given by the context and \( m \) is an integer, let

\[
\overline{m} = \frac{m}{r - 1}.
\]

Throughout the following results, the parity of \( r \) will play an important role. The reason is that every abelian group of even order has a real-valued linear character. The corresponding Seidel matrix of this character is a matrix with real entries, and hence corresponds to a set of real equiangular lines. Hence if \( r \) is even, the parameters of the DRACKN will be subject to the absolute bound for real lines, and therefore the bounds will be more restrictive.

**6.4 Lemma.** Let \( X \) be an abelian \((n, r, c)\)-DRACKN with distinct eigenvalues \( n - 1 > \theta > -1 > \tau \). If \( r \) is even, then

\[
-\frac{1}{2} \sqrt{(n - 1) \left( \sqrt{8n + 1} - 3 \right)} \leq \tau \leq -\sqrt{\frac{1}{2} \left( \sqrt{8n + 1} + 3 \right)}.
\]
Moreover, if the lower bound is tight, then there is a set of real equiangular lines of size \( \binom{mτ+1}{2} \) in dimension \( mτ \), and if the upper bound is tight, then there is a set of real equiangular lines of size \( \binom{mθ+1}{2} \) in dimension \( mθ \).

If \( r \) is odd, then

\[
- (\sqrt{n} - 1) \sqrt{\sqrt{n} + 1} \leq \tau \leq -\sqrt{\sqrt{n} + 1}.
\]

Moreover, if the lower bound is tight, then there is a set of complex equiangular lines of size \( mτ^2 \) in dimension \( mτ \), and if the upper bound is tight, then there is a set of complex equiangular lines of size \( mθ^2 \) in dimension \( mθ \).

Proof. We explicitly prove the second lower bound. The other three bounds are very similar. By Theorem 4.1, the existence of the \textsc{drackn} implies the existence of \( n \) complex equiangular lines in dimension \( mτ \). By the absolute bound [2.2]

\[
n \leq \overline{mτ}^2.
\]

From the expression for the multiplicities given by Theorem 3.1,

\[
\overline{mτ} = \frac{n}{1 + \tau^2/(n - 1)}
\]

which is strictly decreasing in \(|τ|\), and is greater than or equal to \( \sqrt{n} \) if and only if

\[
|τ| \leq (\sqrt{n} - 1) \sqrt{\sqrt{n} + 1}.
\]

Since \( τ < 0 \), the lower bound follows.

Below, we show that the extreme cases in the bounds of the lemma above can be conveniently parametrized.

6.5 Theorem. Let \( X \) be an abelian \((n, r, c)\)-\textsc{drackn} with \( r \) even. It gives a set of real equiangular lines meeting the absolute bound if and only if, for some positive integer \( t \) or \( t = \sqrt{5} \), one of the following cases holds.
Let $X$ be an abelian $(n, r, c)$-DRACKN with $r$ odd. It gives a set of complex equiangular lines meeting the absolute bound if and only if, for some positive integer $t$, one of the following cases holds.

| Parameter | case (I.a) | case (I.b) |
|-----------|------------|------------|
| $n$       | $\frac{1}{2}(t^2 - 2)(t^2 - 1)$ | $\frac{1}{2}(t^2 - 2)(t^2 - 1)$ |
| $rc$      | $\frac{1}{2}(t + 1)^3(t - 2)$      | $\frac{1}{2}(t - 1)^3(t + 2)$      |
| $\delta$  | $-\frac{1}{2}t(t^2 - 5)$            | $\frac{1}{2}t(t^2 - 5)$            |
| $\theta$  | $t$                                    | $\frac{1}{2}t(t^2 - 3)$            |
| $\tau$    | $-\frac{1}{2}t(t^2 - 3)$            | $-t$                                |
| $\overline{m}_\theta$ | $\frac{1}{2}(t^2 - 2)(t^2 - 3)$     | $t^2 - 2$                          |
| $\overline{m}_\tau$ | $t^2 - 2$                              | $\frac{1}{2}(t^2 - 2)(t^2 - 3)$   |

Proof. It is trivial to check that if either of the cases hold, then the absolute bound is satisfied with equality. In the cases (a), we have a maximum sized set of lines in dimension $\overline{m}_\tau$, whereas in cases (b), the dimension of the lines is given by $\overline{m}_\theta$.

For the converse, note that each case corresponds to equality being achieved in either the lower or the upper bound in either of the cases of Lemma 6.4.
We show the case (I.a) of the converse. The other three cases are similar. Suppose the lower bound in the case where \( r \) is even in Lemma 6.4 is tight. Thus
\[
\tau = -\frac{1}{2} \sqrt{(n-1) \left( \sqrt{8n+1} - 3 \right)}.
\] (6.1)

From (3.1), it follows that \( \delta = \tau - (n-1)/\tau \). If \( \delta = 0 \), then \( \tau = -\sqrt{n-1} \).

Plugging it into (6.1) gives \( n = 6 \). The remaining parameters can be computed accordingly and fit the expressions in case (I.a) with \( t = \sqrt{5} \). If \( \delta \neq 0 \), then by Theorem 3.1 both eigenvalues \( \theta \) and \( \tau \) are integers. Let \( t = \theta \in \mathbb{Z}^+ \).

From (3.1), we have \( \theta \tau = -(n-1) \). Coupling with (6.1), it follows that
\[
n = \frac{1}{2} (t^2 - 2)(t^2 - 1).
\]
The remaining parameters can be calculated accordingly. \( \Box \)

Now we proceed to unfold various cases presented in Theorem 6.5. Our final goal is to present a complete list of parameter sets which satisfy all known feasibility conditions given by Theorem 3.1 and Corollary 6.3. For the reasons presented in the beginning of Section 5, we will skip the case \( r = 2 \).

### 6.1 Case I.a

#### 6.6 Theorem. If \( r \geq 4 \), then the only feasible parameter set \((n, r, c)\) for an abelian DRACKN corresponding to set of real equiangular lines in dimension \( m^\tau \) of maximum size is the \((28, 4, 8)\).

**Proof.** Suppose there is a \( t \) such that \((n, r, c)\) and the other parameters are given as in case (I.a). Because \( rc > 0 \), \( t > 2 \). If \( t = \sqrt{5} \), then in view of Theorem 3.1 condition (f), the only possible parameter set is \((6, 2, 2)\). So suppose \( t \geq 3 \). By condition (h) in Theorem 3.1,
\[
t^3 = \theta^3 \geq n - 1 = \frac{1}{2} (t^2 - 2)(t^2 - 1) - 1,
\]
which gives \( t \geq 3 \). Therefore \( t = 3 \). The corresponding parameters are
\[
n = 28, \quad rc = 32, \quad \theta = 3, \quad \tau = -9.
\]
Thus \( r \in \{4, 8, 16, 32\} \). If \( r > 4 \), then \( c \leq 4 \), and so condition (a) of Theorem 3.1 is not satisfied. Therefore \( r = 4 \). \( \Box \)
6.2 Case II.a

When $t = 2$, an abelian $(9, 3, 3)$-DRACKN corresponding to case (II.a) above exists. It can be obtained via the Thas - Somma construction described in the end of Section 4. The other cases are covered by the result below.

6.7 Corollary. The only parameter set $(n, r, c)$ corresponding to an abelian DRACKN that gives a set of complex equiangular lines in dimension $\frac{m}{r}$ of maximum size is the $(9, 3, 3)$.

Proof. By Theorem 3.1 if $r > 2$, then
\[ \theta^3 \geq n - 1. \]
Plugging in $n = (t^2 - 1)^2$ and $\theta = t$ yields
\[ t^2 - t - 2 \leq 0, \]
which restricts $t$ to 2.

6.3 Case I.b

6.8 Theorem. Suppose that for some positive integer $t$, parameters $(n, r, c)$ satisfy
\[ n = \frac{1}{2}(t^2 - 2)(t^2 - 1) \quad \text{and} \quad rc = \frac{1}{2}(t - 1)^3(t + 2), \]
and $r$ is an even integer. These parameters satisfy all the feasibility conditions listed in Theorem 3.1 and in Corollary 6.3 for the existence of a corresponding abelian DRACKN if and only if all of the following hold.

(1) $t \geq 3$ and is not divisible by four.

(2) $c \geq 2$.

(3) If $r \leq (1/2)(t^2 + 1)$, then $r$ divides $t - 1$.

(4) If $t$ is odd then $c$ is even.

(5) Any odd prime that divides $r$ must divide $t - 1$ as well.

Proof. First we show that conditions (1) - (5) are necessary.
(1) If $t = 2$, then $n = 3$ and $r = 2$. Hence $t \geq 3$. On the other hand, $rc$ is even if and only if $t$ is not divisible by four.

(2) Assume that $c = 1$. Then condition (f) in Theorem 3.1 says that
\[(n - rc)^2 = (n - r)^2 \leq n - 1.\]
However
\[4 \left( (n-r)^2 - (n-1) \right) = \left( t^2 - 2 \right) (t^2 - 1) - (t-1)^3 (t+2)^2 - 2(t^2 - 2)(t^2 - 1) + 4\]
is positive for all $t \geq 3$, contradicting our assumption. Thus $c \geq 2$.

(3) With the given parameters, note that
\[n > m_\theta - r + 3\]
and $t \geq 3$ if and only if
\[r \leq \frac{1}{2}(t^2 + 1).\]
It also follows from the given parameters that
\[(\theta + 1)(t - 1) = rc,\]
so $(\theta + 1)$ divides $c$ if and only if $r$ divides $(t - 1)$. Thus condition (i) of Theorem 3.1 implies (3).

(4) Since
\[n = \frac{1}{2}(t^2 - 2)(t^2 - 1)\]
we see that $n$ is even if and only if $t$ is odd. By condition (e) in Theorem 3.1 if $t$ is odd, then $n$ is even and so $c$ is even.

(5) This is the statement of Corollary 6.3.
Now we proceed to show that condition (1) - (4) are sufficient to guarantee all conditions in Theorem 3.1.

(a) The first bound is satisfied as $r \geq 2$ and $c \geq 2$. The second is equivalent to $(\delta + c) \geq 0$, which is also true since
\[\delta = \frac{1}{2}t(t^2 - 5)\]
is positive whenever \( t \geq 3 \). The third bound is equivalent to

\[
  r \geq \frac{rc}{2rc - n}.
\]

From the given parameters,

\[
  \frac{rc}{2rc - n} = \frac{(t - 1)^2(t + 2)}{t^3 - t^2 - 4t + 6},
\]

and it is easy to see that for all \( t \geq 3 \), we have

\[
  2 \geq \frac{(t - 1)^2(t + 2)}{t^3 - t^2 - 4t + 6}.
\]

Since \( r \geq 2 \), the third bound in condition (a) is satisfied.

(b)-(f) These conditions are trivially satisfied given the parametrization of \( n \), \( r \) and \( c \) in terms of \( t \) and condition (1)-(4).

(g) Note that

\[
  \theta^3 - (n - 1) = \frac{1}{8} t^3(t^2 - 3)^3 - \frac{1}{2} (t^2 - 2)(t^2 - 1) + 1
  = \frac{1}{8} (t - 1)t^2(t^2 - 3)(t(t + 1)(t^2 - 5) + 4)
\]

which is positive for \( t \geq 3 \).

(h) If \( r = 2 \), the absolute bound for lines is equivalent to condition (h), as \( \overline{m_\theta} = m_\theta \) and \( \overline{m_r} = m_r \). If \( r > 2 \), then \( r \geq 4 \). The condition

\[
  rn \leq \frac{1}{2} m_\theta(m_\theta + 1)
\]

is equivalent to

\[
  (r^2 - 3r + 1)m_\theta \geq r - 1,
\]

which holds for all \( r \geq 4 \). Since \( m_r > m_\theta \) for \( t \geq 3 \), the remaining condition in terms of \( m_r \) is also satisfied.

(i) We showed that (3) is equivalent to (i) with \( \beta = \theta \). If \( r \geq 4 \) and

\[
  n > m_r - r + 3,
\]

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then
\[ r < \frac{2(t^4 - 4t^2 + 1)}{(t^2 - 1)(t^2 - 4)}. \]

But the right hand side is less than four for all \( t \geq 3 \), hence condition (i) is vacuously satisfied when \( \beta = \tau \).

We show below the first ten feasible parameter sets corresponding to the theorem above. It is not known whether any of these parameter sets correspond to an actual graph.

| \( n \) | \( r \) | \( c \) | \( \delta \) | \( \theta \) | \( \tau \) | \( m_\theta \) | \( m_\tau \) |
|-------|------|-----|-----|------|-----|-------|-------|
| 276   | 4    | 56  | 50  | 55   | -5  | 69    | 759   |
| 276   | 16   | 14  | 50  | 55   | -5  | 345   | 3795  |
| 1128  | 6    | 162 | 154 | 161  | -7  | 235   | 5405  |
| 1128  | 54   | 18  | 154 | 161  | -7  | 2491  | 57293 |
| 1128  | 162  | 6   | 154 | 161  | -7  | 7567  | 174041|
| 1128  | 486  | 2   | 154 | 161  | -7  | 22795 | 524285|
| 3160  | 4    | 704 | 342 | 351  | -9  | 237   | 9243  |
| 3160  | 8    | 352 | 342 | 351  | -9  | 553   | 21567 |
| 3160  | 64   | 44  | 342 | 351  | -9  | 4977  | 194103|
| 3160  | 128  | 22  | 342 | 351  | -9  | 10033 | 391287|

Table 1.

Note that if there is an abelian DRACKN with any of the parameters \((n, r, c)\) above, then there is a regular two-graph with parameters \((n, 2, cr/2)\).
6.4 Case II.b

6.9 Lemma. Let $X$ be an abelian $(n, r, c)$-DRACKN such that for some positive integer $t$, we have

$$n = (t^2 - 1)^2, \quad rc = (t - 1)^2(t^2 + t - 1).$$

Suppose $r \geq 3$. Then $r$ divides $t - 1$.

Proof. Suppose that there is a prime $p$ that divides both $r$ and $t^2 + t - 1$. Since $r$ is odd, $p$ is odd. By Corollary 6.3, $p$ divides $n$, and $n = (t^2 - 1)^2$. So $p$ divides both $t^2 + t - 1$ and $t^2 - 1$, hence $p$ divides $t$, a clear contradiction. Thus, because

$$rc = (t - 1)^2(t^2 + t - 1),$$

it follows that $r$ divides $(t - 1)^2$. Hence $r \leq t^2$, and this immediately implies that

$$n > m_\theta - r + 3.$$

By condition (i) of Theorem 3.1, we now have that $\theta + 1$ divides $c$. Since

$$(t - 1)(\theta + 1) = rc,$$

this is equivalent to saying that $r$ divides $t - 1$. \hfill \square

The above theorem shows that for $r \geq 3$, the condition that $r$ divides $t - 1$ is necessary for Theorem 3.1 to hold. Now we show that this is also sufficient, apart from some lower bounds on $t$, $r$ and $c$.

6.10 Theorem. Suppose that there is a positive integer $t$ defining a parameter set $(n, r, c)$ corresponding to case (II.b) of Theorem 6.5, and so with $r$ odd. This parameter set satisfies all the feasibility conditions in Theorem 3.1 and Corollary 6.3 if and only if $t \geq 3$, $c \geq 2$, 3 does not divide $r$ and $r$ divides $t - 1$.

Proof. First we show that the conditions on $t$, $c$ and $r$ are necessary. If $c = 1$, then condition (f) of Theorem 3.1 says that

$$(n - r)^2 \leq n - 1.$$ 

Note that

$$(n - r)^2 - (n - 1) = (n - rc)^2 - (n - 1) = (t - 1)^4(t + 2)^2 - (t^2 - 2)t^2.$$
It is easy to see that this is positive for all \( t \geq 2 \), hence \( c > 1 \). Moreover, if \( t = 2 \), then \( rc = 5 \), and this could only occur with \( c = 1 \). Hence it also follows that \( t \geq 3 \).

Now \( \theta = (t^2 - 2)t \), so clearly \( \theta^3 \neq n - 1 \). Hence condition (h) of Theorem 3.1 can be applied, and in particular

\[
rn \leq \frac{1}{2}m_\theta(m_\theta + 1).
\]

If \( r = 3 \), this reduces to \( n \leq \sqrt{n} \), a contradiction. If \( 3 \) divides \( r \), then we could use Theorem 6.1 to construct an abelian DRACKN with the same values of \( n \) and \( \delta \), hence still meeting the absolute bound with equality, but of index equal to 3, an absurd. Therefore \( 3 \) cannot divide \( r \).

For the converse, we must check that all conditions in Theorem 3.1 are satisfied.

The bounds on condition (a) can be proved similarly to Theorem 6.8. Conditions (b) - (d) are trivially satisfied, as well as condition (f). To see (e), note that \( n \) is even if and only if \( t \) is odd, and since

\[
rc = (t - 1)^2(t^2 + t - 1)
\]

and \( r \) divides \( t - 1 \), it follows that \( t - 1 \) divides \( c \). So if \( n \) is even, then \( c \) is even.

As we already mentioned, \( \theta^3 > n - 1 \), hence condition (g) is satisfied. To see (h), note that if \( r \geq 4 \), then \( r < (1/2)(r - 1)^2 \), hence

\[
rn \leq \frac{1}{2}(r - 1)^2n + \frac{1}{2}(r - 1)\sqrt{n},
\]

which is equivalent to \( rn \leq (1/2)m_\theta(m_\theta + 1) \). Since \( m_r > m_\theta \), we have that both inequalities of condition (h) are satisfied.

For condition (i), note that \( \theta + 1 \) divides \( c \) if and only if \( r \) divides \( t - 1 \), so the case \( \beta = \theta \) offers no risk. For the other case, note that

\[
\overline{m}_r = n - \overline{m}_\theta = n - \sqrt{n}.
\]

Since \( r \geq 4 \), we have

\[
m_r - r + 3 - n = \overline{m}_r(r - 1) - r + 3 - n
\]

\[
= r(n - \sqrt{n} - 1) - (2n - \sqrt{n} - 3)
\]

\[
\geq 4(n - \sqrt{n} - 1) - (2n - \sqrt{n} - 3)
\]

\[
= 2n - 3\sqrt{n} - 1.
\]
Because $t \geq 3$, we have $n \geq 64$, hence $n < m_\tau - r + 3$. Thus condition (i) is vacuously satisfied in this case.

Finally, Corollary 6.3 follows from the fact that $r$ divides $t - 1$ and thus $r$ divides $n$. □

We show on Table 2 the first ten feasible parameter sets corresponding to the Theorem above. It is not known whether any of these parameter sets corresponds to an actual graph.

| $n$  | $r$ | $c$ | $\delta$ | $\theta$ | $\tau$ | $m_\theta$ | $m_\tau$ |
|------|-----|-----|----------|----------|--------|-------------|----------|
| 1225 | 5   | 205 | 198      | 204      | -6     | 140         | 4760     |
| 3969 | 7   | 497 | 488      | 496      | -8     | 378         | 23436    |
| 14400| 5   | 2620| 1298     | 1309     | -11    | 480         | 57120    |
| 20449| 11  | 1705| 1692     | 1704     | -12    | 1430        | 203060   |
| 38025| 13  | 2717| 2702     | 2716     | -14    | 2340        | 453960   |
| 50176| 7   | 6692| 3330     | 3345     | -15    | 1344        | 299712   |
| 65025| 5   | 12195| 4048    | 4064     | -16    | 1020        | 259080   |
| 104329| 17 | 5797 | 5778    | 5796     | -18    | 5168        | 1664096  |
| 159201| 19 | 7961 | 7940    | 7960     | -20    | 7182        | 2858436  |
| 193600| 5  | 36880| 9198    | 9219     | -21    | 1760        | 772640   |

Table 2.

7 Final comments and open problems

Warwick de Launey stated a conjecture in [4, V.5.18.1] that there are no generalized Hadamard matrices over non-prime-power order groups. If this conjecture is true, it would imply that the index of all abelian DRACKNs with $\delta = -2$ is a prime power. Conversely, if one proves such statement, then De Launey's conjecture will be settled for abelian groups.
A remarkable feature of the feasible parameter sets appearing in Tables 1 and 2 is that $\delta$ increases arbitrarily. Most known DRACKNs occur with $\delta \in \{-2, 0, 2\}$, hence suggesting that these graphs might be very hard to construct.

We are particularly interested in finding a set of equiangular lines satisfying the hypothesis of Theorem 5.1 which yields the construction of a new abelian DRACKN. Likewise, it would be very interesting to find a construction for an abelian DRACKN whose parameters appear in Table 2, thus constructing a set of $d^2$ equiangular lines in $\mathbb{C}^d$.

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