Dynamics of the Born–Infeld dyons

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Abstract

The approach to the dynamics of a charged particle in the Born–Infeld nonlinear electrodynamics developed in [Phys. Lett. A 240 (1998) 8] is generalized to include a Born–Infeld dyon. Both Hamiltonian and Lagrangian structures of many dyons interacting with nonlinear electromagnetism are constructed. All results are manifestly duality invariant.

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1. Introduction

In a recent paper [1] it was shown how to describe the dynamics of a point charge in the Born–Infeld nonlinear electrodynamics [2] (see also [3]). There are several motivations to study this theory (see e.g. [1], [3], [4], [5], [6]). Moreover, it turns out that some natural objects in string theory, so called D-branes, are described by a kind of nonlinear Born–Infeld action [7].

The aim of the present Letter is to show that the results of [1] could be generalized in the case when a particle carries both electric and magnetic charges (dyon). We do so not only for an aesthetic purpose. It turns out that the Dirac idea of magnetically charged particles [8] (see [9] for a review) has led recently to very remarkable results in field and string theories (see e.g. [10] for an excellent introduction to this subject).

It is well known that Born–Infeld electrodynamics is duality invariant [11] (actually, this observation was already made by Schrödinger [12]). Therefore, in principle it should be possible to describe the dynamics of dyons (like in the Maxwell case). However, it turns out that previous attempts to the dynamics of charged particles in the Born–Infeld theory [13] (see [1] for the comparison of results obtained in [1] and long ago by Feenberg and Pryce [13]) are not consistent with the duality invariance of the underlying theory. We show that the approach proposed in [1] respects this invariance.

Moreover, we present a canonical formalism for a theory describing the dynamics of many Born–Infeld dyons. Both Hamiltonian and Lagrangian structures are constructed. To the best of our knowledge it is the first fully consistent canonical structure for classical electrodynamics of many point–like particles (including particles self–interaction).

We stress that one usually discusses classical dyons in a different context namely as static solutions to the non–Abelian Yang–Mills–Higgs models (see once more [9]). It is conjectured that this solutions (in the so called BPS limit [14]) play an important role in the nonperturbative quantum field theory [10]. Actually, it is possible to make a non–Abelian generalization of the Born–Infeld action and to investigate non–Abelian Born–Infeld–Higgs models. It was shown [15] that these models possess monopole (and dyon) solutions. The dyons considered in the present Letter are simply puted ”by hand”. However, there is a similarity between e.g. BPS mass formula for dyons in the non–Abelian theories and a Newton–like equation (12) of the present Letter. We shall comment it in the last section.

2. Dynamical condition

Let us briefly sketch the main result presented in [1]. The Born–Infeld nonlinear electrodynamics [2] is based on the following Lagrangian (we use the Heaviside-Lorentz system of units with the velocity of light $c = 1$):

$$
\mathcal{L}_{BI} := \sqrt{-\det(b\eta_{\mu\nu})} - \sqrt{-\det(b\eta_{\mu\nu} + F_{\mu\nu})} = b^2 \left(1 - \sqrt{1 - 2b^{-2}S - b^{-4}P^2}\right),
$$

where $\eta_{\mu\nu}$ denotes the Minkowski metric with the signature $(-,+,+,+)$ (the theory can be formulated in a general covariant way, however, in this paper we will consider only the flat Minkowski space-time). The standard Lorentz invariants $S$ and $P$ are defined by: $S =
\[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \text{ and } P = -\frac{1}{4} F_{\mu\nu} \ast F^{\mu\nu} \text{ (} \ast F^{\mu\nu} \text{ denotes the dual tensor). The arbitrary parameter “} b \text{” has a dimension of a field strength (Born and Infeld called it the absolute field) and it measures the nonlinearity of the theory. In the limit } b \to \infty \text{ the Lagrangian } L_{BI} \text{ tends to the Maxwell Lagrangian } S. \]

In the presence of an electrically charged matter one usually adds to (1) the standard electromagnetic interaction term “} j^{\mu} A_{\mu} \text{”. Then the complete set of field equations read:

\[
\frac{\partial}{\partial \mu} \ast F^{\mu\nu} = 0, \tag{2}
\]
\[
\frac{\partial}{\partial \mu} G^{\mu\nu} = -j^{\nu}, \tag{3}
\]

with \( G^{\mu\nu} := -2 \frac{\partial L_{BI}}{\partial F^{\mu\nu}} \). Equations (2)–(3) have formally the same form as Maxwell equations. What makes the theory effectively nonlinear are the constitutive relations, i.e. relations between inductions (\( D, B \)) and intensities (\( E, H \)):

\[
E(D, B) = \frac{1}{b^2 R} \left[ (b^2 + B^2) D - (DB)B \right], \tag{4}
\]
\[
H(D, B) = \frac{1}{b^2 R} \left[ (b^2 + D^2) B - (DB)D \right], \tag{5}
\]

with \( R := \sqrt{1 + b^{-2}(D^2 + B^2) + b^{-4}(D \times B)^2} \).

Now, let us assume that the external electric current \( j^{\mu} \) in (3) is produced by a point-like particle moving along the time-like trajectory \( \zeta \). The main idea of [1] (it was developed in the Maxwell case in [16]) was as follows: instead of solving very complicated distributional equations (2)–(3) on the entire Minkowski space-time \( M \) let us treat them as a boundary problem in the region \( M_\zeta := M - \{ \zeta \} \), i.e. outside the trajectory. Now, in order well to pose the problem we have to find an appropriate boundary condition which has to be satisfied along \( \zeta \), i.e. on the boundary \( M_\zeta \). Observe that in \( M_\zeta \) equations (2)–(3) are homogeneous.

To find this boundary condition we have analysed the asymptotic behaviour of the fields in the vicinity of a charged particle. The simplest way to do so is to use the particle’s rest frame. Let \( r \) denote the radial coordinate, i.e. a distance from a particle in its rest frame. Any vector field \( F = F(r) \) may be formally expanded in the vicinity of a charge:

\[
F(r) = \sum_n r^n F^{(n)}(r), \tag{6}
\]

where the vectors \( F^{(n)} \) do not depend on \( r \). The crucial observation is that the most singular part of \( D \) field behaves as

\[
D_{(-2)} = \frac{eA}{4\pi r}, \tag{7}
\]

where, due to the Gauss law, the monopole part of the \( r \)-independent function \( A \) equals 1. Note, that in the Maxwell case \( A \equiv 1 \). Using (7) it was shown [1] that:

\[
H \sim r^{-1}, \quad E \sim r^0, \quad B \sim r. \tag{8}
\]

Moreover, the \( E^{(0)} \) term is known explicitly:

\[
E^{(0)} = \frac{be}{|e| r}. \tag{9}
\]

Using these results the following theorem was proved [1]:
Theorem 1  Any regular solution of Born–Infeld field equations with point-like external current satisfies:

\[ E^T_{(1)} = \frac{be}{4|e|} \left( 3a - r^{-2}(ar)r \right), \]  

(10)

where \( E^T \) stands for the transversal part of \( E \) and \( a \) denotes the particle’s acceleration.

According to our “boundary philosophy” the formula (10) may be interpreted as a boundary condition for \( E \) field on \( \partial M_\zeta \) and the hyperbolicity of (2)–(3) implies:

Theorem 2  The mixed (initial-boundary) value problem for the Born–Infeld equations in \( M_\zeta \) with (10) playing the role of boundary condition on \( \partial M_\zeta \) has the unique solution.

The above theorem is no longer true when we consider a particle as a dynamical object. Choosing particle’s position \( q \) and velocity \( v \) as the Cauchy data for the particle’s dynamics let us observe that despite the fact that the time derivatives (\( \dot{D}, \dot{B}, \dot{q}, \dot{v} \)) of the Cauchy data are uniquely determined by the data themselves, the evolution of the composed system is not uniquely determined. Indeed, \( \dot{D} \) and \( \dot{B} \) are given by the field equations, \( \dot{q} = v \) and \( \dot{v} \) may be calculated from (11). Nevertheless, the initial value problem is not well posed: keeping the same initial data, particle’s trajectory can be modified almost at will. This is due to the fact, that now (10) plays no longer the role of boundary condition because we use it to as a dynamical equation to determine \( a \). Therefore a new boundary condition is necessary.

It was shown in [1] that this missing condition is implied by the conservation law of the total four–momentum for the “particle + field” system: \( \dot{p}^\mu = 0 \), where \( p^\mu \) stands for the four–momentum in a fixed laboratory frame. Due to the field equations only 3 among 4 equations are independent, i.e. the conservation of three–momentum

\[ \dot{p} = 0 \]  

(11)

implies the energy conservation: \( \dot{p}^0 = 0 \). Now, the formula (11) is equivalent to the following Newton-like equation:

\[ ma_k = \frac{|e|b}{3} A_k, \]  

(12)

where \( A_k \) is the dipole part of \( A \) (see (7) for the definition of \( A \)), i.e. \( \text{DP}(A) = A_k x^k / r \).

The above equation looks formally like a standard Newton equation. However, it could not be interpreted as the Newton equation because its r.h.s. is not \textit{a priori} given (it must be calculated from field equations).

To correctly interpret (12) we have to take into account (10). Now, calculating \( a \) in terms of \( E^T_{(1)} \) and inserting into (12) we obtain the following relation between \( E^T_{(1)} \) and \( D_{(-2)} \):

\[ \text{DP} \left( 4r_e^4 (E^T_{(1)})^r - \lambda_e (D_{(-2)})^r \right) = 0, \]  

(13)

where \( r_e := \sqrt{|e|/4\pi b} \) and \( \lambda_e := e^2/6\pi m \). \( (F)^r \) denotes the radial component of a 3–vector \( F \). The main result of [1] consists in the following

Theorem 3  Born–Infeld field equations supplemented by the dynamical condition (13) define perfectly deterministic theory, i.e. initial data for field and particle uniquely determine the entire evolution of the system.
3. Duality invariance

To show that the above theory can be generalized to include also Born–Infeld dyons let us introduce a complex notation:

\[ X := D + iB , \quad Y := E + iH . \] (14) (15)

It was shown [11] that the nonlinear electrodynamics is invariant under duality rotations:

\[ X \to e^{i\alpha} X , \quad Y \to e^{i\alpha} Y \] \hspace{1cm} (16)

if and only if the following relation is satisfied

\[ \text{Im} (XY) = 0 . \] \hspace{1cm} (17)

In the case of Born–Infeld theory the constitutive relation reads:

\[ Y(X, X) = \frac{1}{b^2 R} \left[ \left( b^2 + \frac{1}{2} X \overline{X} \right) X - \frac{1}{2} X^2 \overline{X} \right] \] \hspace{1cm} (18)

with duality invariant \( R = \sqrt{1 + b^{-2}X \overline{X} - \frac{1}{4} b^{-4} (X \times \overline{X})^2} \). One immediately shows that \( Y \) given by (18) satisfies (17) (it turns out that the duality invariance of the Born–Infeld theory was already observed by Schrödinger [12]).

Let us consider a dyon carrying electric and magnetic charges \( e \) and \( g \) respectively. Define the complex charge:

\[ Q := e + ig . \] \hspace{1cm} (19)

Now, instead of (7) we obviously have

\[ X(-2) = \frac{QA}{4\pi} \frac{r}{r} \] \hspace{1cm} (20)

and instead of relations (8) we have \( X \sim r^{-2} \) and \( Y \sim r^{-1} \). To obtain the duality invariant generalizations of (10) and (13) we proceed as follows: observe that \( \text{C} Y \) is duality invariant (\( \text{C} \) stands for the complex conjugation of \( Q \)). Therefore, its real and imaginary parts are also invariant. Using this invariance let us make the duality rotation \( Q' = e^{i\alpha} Q = e' \) (i.e. \( g' = 0 \)) and calculate

\[ \text{Re} (\text{C} Y) = eE + gH = e'E' . \] \hspace{1cm} (21)

But in the rotated frame (i.e. \( (e', 0) \)) we may use results of the previous section: formula (9) implies

\[ \text{Re} (\text{C} Y_{(0)}) = b|Q| \frac{r}{r} \] \hspace{1cm} (22)

and (10) leads to

\[ \text{Re} (\text{C} Y_{(1)}) = \frac{1}{4}|Q| \left( 3a - r^{-2}(ar)r \right) . \] \hspace{1cm} (23)
Now, instead of (12) we have duality invariant
\[ m a_k = \frac{|Q| b}{3} A_k , \]  
(24)
and finally the duality invariant dynamical condition reads:
\[ \text{DP} \left\{ \text{Re} \left[ Q \left( 4 r^4_0 (Y^T_{(i)})^r - \lambda_0 (X_{(-2)})^r \right) \right] \right\} = 0 , \]
(25)
where
\[ r^4_0 := r^4_e + r^4_g = \frac{e^2 + g^2}{(4\pi b)^2} = \frac{Q \overline{Q}}{(4\pi b)^2} , \]
(26)
\[ \lambda_0 := \lambda_e + \lambda_g = \frac{e^2 + g^2}{6\pi m} = \frac{Q \overline{Q}}{6\pi m} . \]
(27)

4. Canonical formulation

Now we show that the duality invariant dynamical condition (25) may be derived from the mathematically well defined variational principle. In the absence of a magnetic charge one could guess that such a principle should be based on the following Lagrangian:
\[ L_{\text{total}} = L_{\text{field}} + L_{\text{particle}} + L_{\text{int}} , \]
(28)
with \( L_{\text{field}} \) given by (1), \( L_{\text{particle}} = -m \sqrt{1 - v^2} \) and \( L_{\text{int}} = A_\mu j^\mu \). Varying \( L_{\text{total}} \) with respect to \( A_\mu \) one obviously gets field equations (2)–(3). The variation with respect to a particle’s trajectory leads to the standard Lorentz equation
\[ m a^\mu = e F^{\mu\nu} u_\nu . \]
(29)
However, despite the fact that \( F^{\mu\nu} \) is bounded, it is not regular at the particle’s position and, therefore, the r.h.s. of (29) is not well defined. This was already our motivation to find the mathematically well defined dynamical condition (13) which replaces ill defined equations of motion (29).

Observe that when \( g \neq 0 \) the situation is even worse. Actually, Lorentz equation (29) may be easily generalized to a duality invariant formula
\[ m a^\mu = (e F^{\mu\nu} + g*G^{\mu\nu}) u_\nu , \]
(30)
but now both \( F^{\mu\nu} \) and \( G^{\mu\nu} \) are highly singular near a particle.

It was shown in [17] how to construct a consistent variational principle for a pure electric monopole. Euler-Lagrange equations implied by this variational principle are perfectly equivalent to the dynamical condition (13). Moreover, it gives rise to a consistent Hamiltonian structure with well defined Poisson bracket. It turns out that this construction may be immediately generalized to include Born–Infeld dyon. However, due to the fact that one uses a particle’s rest frame the whole procedure can not be in a straightforward way generalized to a many particles case. In the present Letter we propose a new variational principle which is valid for an arbitrary (finite) number of Born–Infeld dyons. Moreover, it is much simpler that the one proposed in [17].
Let us consider $N$ Born–Infeld dyons: $(m_l, Q_l = e_l + ig_l)$; $l = 1, 2, ..., N$. The energy of the composed "$N$ dyons + field" system in a fixed inertial laboratory frame is given by:

$$H = \sum_{i=1}^{N} \sqrt{m_i^2 + p_i^2} + V(q_1, ..., q_N) , \quad (31)$$

where $p_i = m_i v_i / \sqrt{1 - v_i^2}$ denotes a "kinetic" momentum of an $i$-th dyon and the function

$$V(q_1, ..., q_N) := \int_{\{q_1, ..., q_N\}} d^3x T^{00} \quad (32)$$

defines the energy of the field configuartion $(X, \bar{X})$ ($T^{00}$ denotes corresponding component of an energy-momentum tensor of the Born–Infeld theory). The integral in (32) is defined on a punctured 3-dimensional (constant time) space where positions of $N$ dyons $\{q_1, ..., q_N\}$ are excluded. Moreover, in the phase space of our system, parameterized by $(q_i, p_i)$ in a "dyons sector" and by $(X, \bar{X})$ in a "field sector", define the following Poisson bracket:

$$\{F, G\} := \sum_{i=1}^{N} \frac{\partial F}{\partial q_i} \cdot \frac{\partial G}{\partial p_i} + \frac{1}{i} \int_{\{q_1, ..., q_N\}} d^3x \left[ \frac{\delta F}{\delta X} \cdot \nabla \times \frac{\delta G}{\delta \bar{X}} - (X \leftrightarrow \bar{X}) \right] - (F \leftarrow G) , \quad (33)$$

for any two functionals $F$ and $G$. One may prove that the formula (33) indeed defines a Poisson bracket (i.e. the Jacobi identity is satisfied). With this definition we have the following "commutation relations" between dyons and fields variables:

$$\{(q_i)^k, (p_j)_l\} = \delta_{ij} \delta^k_l , \quad (34)$$

$$\{X_k(x), \bar{X}_l(y)\} = 2i\epsilon_{klm} \partial^m \delta^3(x - y) \quad (35)$$

and remaining brackets vanish.

Let us treat $H$ given by (31) as the "dyons + fields" Hamiltonian $H = H(q_i, p_i, X, \bar{X})$ and look for the corresponding Hamilton equations. In the "field sector" everything is clear: fields equations

$$i\dot{X} = \{X, H\} = \nabla \times Y \quad (36)$$

supplemented by the Gauss law $\nabla \cdot X = 0$ are equivalent to the Born–Infeld field equations outside the dyons trajectories. Observe, that $Y$ is conjugated to $X$ via

$$Y = \frac{\delta H}{\delta X} . \quad (37)$$

Now, in the "dyon sector" one has obviously

$$\dot{q}_i = \{q_i, H\} = v_i . \quad (38)$$

The only nontrivial thing is to evaluate

$$\dot{p}_i = \{p_i, H\} = - \frac{\partial}{\partial q_i} V(q_1, ..., q_N) . \quad (39)$$
In the Appendix we show that (39) is equivalent to

\[ m_i(a_i)_k = |Q_i| b A_{(i)k}, \]  

where \( a_i \) denotes the acceleration of an \( i \)-th dyon in its rest frame and \( A_{(i)k} \) stands for a dipole part (its \( k \)-th component) of the function \( A_{(i)} \) which defines the behaviour of \( X_{(-2)} \) near an \( i \)-th dyon according to (20). Therefore, (40) is equivalent to the \( i \)-th dynamical condition. This way we have proved the following

**Theorem 4** The Hamiltonian (31) together with the Poisson bracket (33) define the consistent canonical structure of a system of \( N \) Born–Infeld dyons.

Let us observe that there is no "interaction term" in (31). All information about the interaction between dyons and the field is encoded in the boundary conditions for the field variables which have to be satisfied near dyons positions \( q_i \), i.e., on the multicomponent boundary of the punctured (constant time) space. From the point of view of dyons dynamics the function (32) plays the role of a potential energy stored in the "field sector".

Obviously, performing the Legendre transformation in the "dyons sector" one gets the corresponding Lagrange function

\[ L(q_i, v_i) = - \sum_{i=1}^{N} m_i \sqrt{1 - v_i^2} - V(q_1, ..., q_N). \]  

**Theorem 5** The Euler–Lagrange equations implied by \( L \) are equivalent to the \( N \) dynamical conditions for dyons dynamics.

The proof is straightforward. From the point of view of dyons dynamics the structure of (31) is evident: "kinetic energy – potential energy". But in the "field sector" (31) still generates the Hamiltonian dynamics because the field generator is given by (32). Therefore, (31) is a nice example of a "mixed generator" called in the analytical mechanics a Routhian function.

5. Concluding remarks

Finally, let us make few remarks:

1. Let us observe that the force in a Newton-like equation (24) does not depend on a sign of \( Q \) (contrary to the Lorentz equation (29) or (30)). It is a characteristic feature of the self-interaction force already present in the Lorentz–Dirac equation: \( ma^\mu = e F^\mu_\nu u_\nu + \lambda e (\dot{a}^\mu - a^2 u^\mu) \). The external force \( e F^\mu_\nu u_\nu \) does depend on a sign of \( e \) but a self-force proportional to \( \lambda e \) does not (\( \lambda e \sim e^2 \)).

2. The mass of dyon solution in the non–Abelian Yang–Mills–Higgs theory in the BPS limit is given by

\[ M_{BPS} = a |Q|, \]  

where \( a \) stands for the vacuum expectation value of the Higgs field (see [3], [14], [10]). Now, observe that the l.h.s. of Newton–like equation (12) contains purely mechanical quantities
mass \( m \) and acceleration \( a^k \), whereas its r.h.s. contains only electromagnetical quantities. The quantity \( b|Q|/3 \) looks formally like a BPS mass with \( a = b/3 \). With this identification (12) could be rewritten in a suggestive form:

\[
ma_k = M_{BPS} A_k .
\]

Observe, that duality invariant Lorentz equation (31) has no such a property. Of course we do not claim that this identification has any fundamental meaning. However, our observation is supported by the fact that in string theory the \( b \)-parameter of Born–Infeld action arises as a function of a vacuum expectation value of a dilaton field [2].

3. The remarkable feature of the Hamiltonian (31) and Lagrangian (41) is the absence of an interaction term. After removing the dyons positions the nontrivial topology of the space \( \mathbb{R}^3 - \{q_1, \ldots, q_N\} \) requires very nontrivial boundary conditions for the field variables at the multicomponent boundary \( \partial(\mathbb{R}^3 - \{q_1, \ldots, q_N\}) \). Therefore, in a sense, the interaction is implied by a space-time topology. Nevertheless, the above theory is not of the topological type (see e.g. [18]), i.e. it is not true that its action does not depend on a space–time metric.

4. It is interesting to note that the above feature is lost in the Maxwell case. This is due to the fact that in this case the self–energy of a point charged particle is infinite. To renormalize this theory one subtracts a Coulomb term \( X(-2) \) (with \( A = 1 \)) which produces this infinity. But this subtraction leads to a mixed term in the energy functional of the form \( X(-2) \cdot (X - X(-2)) \). This term is integrable (in a sense of the Cauchy principal value) and plays the role of a gauge–invariant interaction term [19]. Therefore, theorems 4 and 5 can not be applied to Maxwell electrodynamics. It turns out that these theorems are connected with an analytical structure of classical (in general nonlinear) electrodynamics of point–like objects. This point will be fully clarified in the next paper.

5. As in the Maxwell theory the consistency with quantum mechanics implies quantisation condition for dyons charges \( Q_i \). It turns out that the field dynamics does not play any role and the quantisation condition is the same as in the Maxwell case. Using e.g. methods of [3] one easily recovers duality invariant Zwanziger–Schwinger condition [20]:

\[
\text{Im}(Q_i Q_j) = 2\pi \hbar n_{ij} \tag{44}
\]

with \( n_{ij} \) integers.

Appendix

To prove the equivalence of (39) and (40) let us observe that for any field functional of the form

\[
\mathcal{F}(q_1, \ldots, q_N) = \int_{\{q_1, \ldots, q_N\}} f(X, \mathbf{X}) d^3x ,
\]

we have

\[
\frac{\partial}{\partial(q_i)^k} \mathcal{F}(q_1, \ldots, q_N) = -\{\mathcal{F}(q_1, \ldots, q_N), \mathcal{P}_k(q_i)\} ,
\]

where

\[
\mathcal{P}_k(q_i) := \int_{\{q_i\}} T^0_k d^3x \tag{A.3}
\]
denotes $k$—th component of field momentum outside the $i$—th dyon. This relation follows from the Poincaré algebra structure. Using (33) it is easy to prove that

$$\{V(q_1, \ldots, q_N), P_k(q_i)\} = \int_{\partial \Sigma_i} T^{00} n_k d\sigma,$$  \hfill (A.4)

where $\partial \Sigma_i$ denotes a boundary of $\Sigma_i := \mathbb{R}^3 - \{q_i\}$, $n_k$ stands for a unit normal to $\partial \Sigma_i$ and $d\sigma$ denotes its standard measure.

Now, $\partial \Sigma_i$ has two components. The boundary integral over the component at infinity vanishes but the one over the component surrounding $\{q_i\}$ gives exactly $|Q|bA_k/3$ (see [1]).

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## References

[1] D. Chruściński, Phys. Lett. A 240 (1998) 8.

[2] M. Born, Proc. Roy. Soc. A 143 (1934) 410;
M. Born and L. Infeld, Proc. Roy. Soc. A 144 (1934) 425.

[3] I. Białynicki-Birula and Z. Białynicka-Birula, Quantum Electrodynamics (Oxford: Pergamon) 1975;
I. Białynicki-Birula, Nonlinear Electrodynamics: Variations on a Theme of Born and Infeld, in Quantum Theory of Particles and Fields, Eds. B. Jancewicz and J. Lukierski, World Scientific, 1983, 31-48.

[4] T. Hagiwara, Nucl. Phys. B 189 (1981) 135;
T. Hagiwara, J. Phys. A: Math. Gen. 14 (1981) 3059.

[5] J. Plebański, Lectures on Non-linear Electrodynamics, Copenhagen: Nordita, 1968.

[6] D. Chruściński and J. Kijowski, C. R. Acad. Sci. Paris 324 Sérié IIb (1997) 435;
D. Chruściński and J. Kijowski, J. Phys. A: Math. Gen. 31 (1998) 269.

[7] E.S. Fradkin and A.A. Tseytlin, Phys. Lett. B 163 (1985) 123;
M. Perry and J.H. Schwarz, Interacting chiral gauge fields in six dimensions and Born–Infeld theory, Preprint hep-th/9611065;
G.W. Gibbons, Born–Infeld particles and Dirichlet p-branes, Preprint hep-th/9709027.

[8] P.A.M. Dirac, Proc. Roy. Soc. A 133 (1931) 60;
P.A.M. Dirac, Phys. Rev. 74 (1948) 817.

[9] P. Goddard and D. Olive, Rep. Prog. Phys. 41 (1978) 1357.
[10] D. Olive, Nucl. Phys. (Proc. Suppl.) B 58 (1997) 43.

[11] G.W. Gibbons and D.A. Rasheed, Nucl. Phys. B 454 (1995) 185;
    M.K. Gaillard and B. Zumino, Self-duality in nonlinear electromagnetism, Preprint
    hep-th/9705226.

[12] E. Schrödinger, Proc. Roy. Soc. A 150 (1935) 466.

[13] E. Feenberg, Phys. Rev. 47 (1935) 148;
    H.L.M. Pryce, Proc. Roy. Soc. A 155 (1936) 597.

[14] E.B. Bogomolny, Sov. J. Nucl. Phys. 24 (1976) 449;
    M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. 35 (1975) 760.

[15] A. Nakamura and K. Shiraishi, Hadronic Journal 14 (1991) 369;
    K. Shiraishi and S. Hirenzaki, Int. J. Mod. Phys. 6 (1991) 2635.

[16] J. Kijowski, Gen. Rel. Grav. 26 (1994) 167;
    J. Kijowski, Acta Phys. Polon. A 85 (1994) 771.

[17] D. Chruściński, Canonical formalism for the Born–Infeld particle, Preprint
    hep-th/9803100 (to appear in J. Phys. A).

[18] D. Birmingham, M. Blau and M. Rakowski, Phys. Rep. 209 (1991) 129.

[19] J. Kijowski and D. Chruściński, Gen. Rel. Grav. 27 (1995) 267;
    D. Chruściński and J. Kijowski, J. Geom. Phys. 20 (1996) 393;
    D. Chruściński, Hamiltonian structure for classical electrodynamics of a point particle,
    Preprint physics/9712014 (to appear in Rep. Math. Phys.).

[20] D. Zwanziger, Phys. Rev. 176 (1968) 1489;
    J. Schwinger, Science 165 (1969) 757.