A Laplace principle
for a stochastic wave equation
in spatial dimension three

by

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Abstract: We consider a stochastic wave equation in spatial dimension three, driven by a Gaussian noise, white in time and with a stationary spatial covariance. The free terms are nonlinear with Lipschitz continuous coefficients. Under suitable conditions on the covariance measure, Dalang and Sanz-Solé [7] have proved the existence of a random field solution with Hölder continuous sample paths, jointly in both arguments, time and space. By perturbing the driving noise with a multiplicative parameter $\varepsilon \in [0, 1]$, a family of probability laws corresponding to the respective solutions to the equation is obtained. Using the weak convergence approach to large deviations developed in [10], we prove that this family satisfies a Laplace principle in the Hölder norm.

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1 Introduction

We consider the stochastic wave equation in spatial dimension $d = 3$

\[
\begin{align*}
\left( \frac{\partial^2}{\partial t^2} - \Delta \right) u(t, x) &= \sigma(u(t, x)) \dot{F}(t, x) + b(u(t, x)), \quad t \in [0, T], \\
u(0, x) &= v_0(x), \\
\frac{\partial}{\partial t} u(0, x) &= \tilde{v}_0(x), \quad x \in \mathbb{R}^3,
\end{align*}
\]

where $\Delta$ denotes the Laplacian on $\mathbb{R}^3$. The coefficients $\sigma$ and $b$ are Lipschitz continuous functions and the process $F$ is the formal derivative of a Gaussian random field, white in time and correlated in space. More precisely, for any $d \geq 1$, let $\mathcal{D}(\mathbb{R}^{d+1})$ be the space of Schwartz test functions and let $\Gamma$ be a non-negative and non-negative definite tempered measure on $\mathbb{R}^d$. Then, on some probability space, there exists a Gaussian process $F = (F(\varphi), \varphi \in \mathcal{D}(\mathbb{R}^{d+1}))$ with mean zero and covariance functional

\[
E(F(\varphi)F(\psi)) = \int_{\mathbb{R}^+} ds \int_{\mathbb{R}^d} \Gamma(dx)(\varphi(s) \ast \tilde{\psi}(s))(x),
\]

where $\tilde{\psi}(s)(x) = \psi(s)(-x)$ and the notation “$\ast$” means the convolution operator. As has been proved in [5], the process $F$ can be extended to a martingale measure $M = (M_t(A), t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d))$, where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the set of bounded Borel sets of $\mathbb{R}^d$.

For any $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$, define the inner product

\[
\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^n} \Gamma(dx)(\varphi \ast \tilde{\psi})(x)
\]

and denote by $\mathcal{H}$ the Hilbert space obtained by the completion of $\mathcal{D}(\mathbb{R}^n)$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Using the theory of stochastic integration with respect to martingale measures (see for instance [16]), the stochastic integral $B_t(h) := \int_0^t ds \int_{\mathbb{R}^d} h(y)M(ds, dy)$ is well defined, and for any $h \in \mathcal{H}$ with $\|h\|_{\mathcal{H}} = 1$, the process $(B_t(h), t \in [0, T])$ is a standard Wiener process. In addition, for any fixed $t \in [0, T]$, the mapping $h \to B_t(h)$ is linear. Thus, the process $(B_t, t \in [0, T])$ is a cylindrical Wiener process on $\mathcal{H}$ (see [9] for a definition of this notion). Let $(e_k, k \geq 1)$ be a complete orthonormal system of $\mathcal{H}$. Clearly, $B_k(t) := \int_0^t ds \int_{\mathbb{R}^d} e_k(y)M(ds, dy), \quad k \geq 1$, defines a sequence of independent, standard Wiener processes and we have the representation

\[
B_t = \sum_{k \geq 1} B_k(t)e_k.
\]
Let \( \mathcal{F}_t, t \in [0, T] \), be the \( \sigma \)-field generated by the random variables \( (B_k(s), s \in [0, t], k \geq 1) \). \( (\mathcal{F}_t) \)-predictable processes \( \Phi \in L^2(\Omega \times [0, T]; \mathcal{H}) \) can be integrated with respect to the cylindrical Wiener process \( (B_t, t \in [0, T]) \) and the stochastic integral \( \int_0^t \Phi(s)dB_t \) coincides with the Itô stochastic integral with respect to the infinite dimensional Brownian motion \( (B_k(t), t \in [0, T], k \geq 1), \sum_{k \geq 1} \int_0^t \langle \Phi(s), e_k \rangle_H dB_k(t) \).

We shall consider the mild formulation of equation (1),

\[
\begin{align*}
  u(t, x) &= w(t, x) + \sum_{k \geq 1} \int_0^t \langle G(t-s, x-\cdot)\sigma(u(s, \cdot)), e_k \rangle_H dB_k(s) \\
  &+ \int_0^t [G(t-s) \ast b(u(s, \cdot))] (x) ds,
\end{align*}
\]

\( t \in [0,T], x \in \mathbb{R}^3 \). Here

\[
w(t, x) = \left( \frac{d}{dt} G(t) \ast v_0 \right) (x) + (G(t) \ast \tilde{v}_0)(x),
\]

and \( G(t) = \frac{1}{4\pi t} \sigma_t \), where \( \sigma_t \) denotes the uniform surface measure (with total mass \( 4\pi t^2 \)) on the sphere of radius \( t \).

Throughout the paper, we will consider the following set of assumptions.

\[\text{(H)} \]

1. The coefficients \( \sigma, b \) are real Lipschitz continuous functions.

2. The spatial covariance measure \( \Gamma \) is absolutely continuous with respect to Lebesgue measure and the density is \( f(x) = \varphi(x)|x|^{-\beta}, x \in \mathbb{R}^3 \{0\} \). The function \( \varphi \) is bounded and positive, \( \varphi \in C^1(\mathbb{R}^3), \nabla \varphi \in C^0_0(\mathbb{R}^3) \) (the space of bounded and Hölder continuous functions with exponent \( \delta \in [0, 1] \)) and \( \beta \in [0, 2[. \)

3. The initial values \( v_0, \tilde{v}_0 \) are bounded and such that \( v_0 \in C^2(\mathbb{R}^3), \nabla v_0 \) is bounded and \( \Delta v_0 \) and \( \tilde{v}_0 \) are Hölder continuous with degrees \( \gamma_1, \gamma_2 \in [0, 1], \) respectively.

We remark that the assumptions on \( \Gamma \) imply

\[
\sup_{t \in [0, T]} \int_{\mathbb{R}^3} |\mathcal{F}(G(t))(\xi)|^2 \mu(d\xi) < \infty,
\]

where \( \mathcal{F} \) denotes the Fourier transform operator and \( \mu = \mathcal{F}^{-1} \Gamma. \) This is a relevant condition in connection with the definition of the stochastic integral with respect to the martingale measure \( M ([4]). \)
The set of hypotheses (H) are used in Chapter 4 of [7] to prove a theorem on existence and uniqueness of solution to equation (4) and the properties of the sample paths. More precisely, under a slightly weaker set of assumptions than (H) (not requiring boundedness of the functions \(v_0, \tilde{v}_0, \nabla v_0\)), Theorem 4.11 in [7] states that for any \(q \in [2, \infty], \alpha \in [0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}]\), there exists \(C > 0\) such that for \((t, x), (\bar{t}, y) \in [0, T] \times D\),

\[
E(|u(t, x) - u(\bar{t}, y)|^q) \leq C(|t - \bar{t}| + |x - y|)^{\alpha q},
\]

where \(D\) is a fixed bounded domain of \(\mathbb{R}^3\). Consequently, a.s., the stochastic process \((u(t, x), (t, x) \in [0, T] \times D)\) solution of (4) has \(\alpha\)-Hölder continuous sample paths, jointly in \((t, x)\).

The reason for strengthening the assumptions of [7] is to ensure that

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}^3} |w(t, x)| < \infty
\]

(see Hypothesis 4.1 and Lemma 4.2 in [8]), a condition that is needed in the proof of Theorem 2.3 below. This is in addition to (4.19) in [7], which provides an estimate of a fractional Sobolev norm of the function \(w\).

We notice that in [7], the mild formulation of equation (1) is stated using the stochastic integral developed in [6]. Recent results by Dalang and Quer-Sardanyons (see [8], Proposition 2.11 and Proposition 2.6 (b)) show that this formulation is equivalent to (4).

In this paper, we consider the family of stochastic wave equations

\[
u^\varepsilon(t, x) = w(t, x) + \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \langle G(t - s, x - \cdot) \sigma(u(s, \cdot)), e_k \rangle_H dB_k(s)
\]

\[
+ \int_0^t [G(t - s) * b(\nu^\varepsilon(s, \cdot))](x) ds
\]

\(\varepsilon \in \{0, 1\}\), and we establish a large deviation principle for the family \((\nu^\varepsilon, \varepsilon \in \{0, 1\})\) in a Polish space closely related to \(C^\alpha([0, T] \times D)\), the space of functions defined on \([0, T] \times D\), Hölder continuous jointly in its two arguments, of degree \(\alpha \in I\), where

\[
I := \left[0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}\right].
\]

To formulate the large deviation principle, we should consider a Polish space carrying the probability laws of the family \((\nu^\varepsilon, \varepsilon > 0)\). This cannot be \(C^\alpha([0, T] \times D)\), since this space is not separable. Instead, we consider the
space $C^{\alpha',0}([0,T] \times D)$ of Hölder continuous functions $g$ of degree $\alpha' < \alpha$, with modulus of continuity

$$O_g(\delta) := \sup_{|t-s|+|x-y|<\delta} \frac{|g(t,s) - g(s,y)|}{(|t-s| + |x-y|)^{\alpha'}}$$

satisfying $\lim_{\delta \to 0^+} O_g(\delta) = 0$. This is a Banach space and $C^{\alpha}([0,T] \times D) \subset C^{\alpha',0}([0,T] \times D)$.

In the sequel, we shall denote by $(\mathcal{E}_\alpha, \|\cdot\|_\alpha)$ the Banach space $C^{\alpha,0}([0,T] \times D)$ endowed with the Hölder norm of degree $\alpha$, and consider values of $\alpha \in \mathcal{I}$.

Let $\mathcal{H}_T = L^2([0,T]; \mathcal{H})$. For any $h \in \mathcal{H}_T$, we consider the deterministic evolution equation

$$V^h(t,x) = w(t,x) + \int_0^t \langle G(t-s, x-\cdot), \sigma(V^h(s,\cdot)), h(s,\cdot) \rangle_{\mathcal{H}} ds + \int_0^t \left[ G(t-s) * b(V^h(s)) \right](x) \, ds.$$  \hspace{1cm} (9)

The second term on the right-hand side of this equation can be written as

$$\sum_{k \geq 1} \int_0^t \langle G(t-s, x-\cdot), \sigma(V^h(s,\cdot)), e_k \rangle_{\mathcal{H}} h_k(s) \, ds,$$

with $h_k(t) = \langle h(t), e_k \rangle_{\mathcal{H}}$, $t \in [0,T]$, $k \geq 1$.

Existence and uniqueness of solution of equation (9) can be proved in a similar (but easier) way than for (4). This will be obtained in the next section as a by-product of Theorem 2.3, where it is also proved that $V^h \in \mathcal{E}_\alpha$. We will denote by $\mathcal{G}^0 : \mathcal{H}_T \rightarrow \mathcal{E}_\alpha$ the mapping defined by $\mathcal{G}^0(h) = V^h$.

For any $f \in \mathcal{E}_\alpha$ define

$$I(f) = \inf_{h \in \mathcal{H}_T: \mathcal{G}^0(h) = f} \left\{ \frac{1}{2} \|h\|_{\mathcal{H}_T}^2 \right\}$$

and for any $A \subset \mathcal{E}_\alpha$, $I(A) = \inf\{I(f), f \in A\}$.

The main result of this paper is the following theorem.

**Theorem 1.1** Assume that the set of hypotheses (H) are satisfied. Then, the family $\{v^\varepsilon, \varepsilon \in [0,1]\}$ given by (8) satisfies a large deviation principle on
$E_{\alpha}$ with rate function $I$ given by (10). That means, for any closed subset $F \in E_{\alpha}$ and any open subset $G \in E_{\alpha}$,

$$\limsup_{\varepsilon \to 0^+} \varepsilon \log P(u^\varepsilon \in F) \leq -I(F),$$

$$\liminf_{\varepsilon \to 0^+} \varepsilon \log P(u^\varepsilon \in G) \geq -I(G).$$

In the proof of this theorem, we will use the weak convergence approach to large deviations developed in [10]. An essential ingredient of this method is a variational representation for a reference Gaussian process (Brownian motion when studying diffusion processes, or different generalizations of infinite-dimensional Wiener process when dealing with stochastic partial differential equations). As it is shown in [2], a variational representation for an infinite-dimensional Brownian motion along with a transfer principle based on compactness and weak convergence, allow to derive a large deviation principle for some functionals of this process. This method has been applied in [3] to establish a large deviation principle to reaction-diffusion systems considered in [12] and also in several subsequent papers, for instance in [15], [11], [17]. We next give the ingredients for the proof of Theorem 1.1 based on this method.

**Variational representation of infinite dimensional Brownian motion**

Let $B = (B_k(t), t \in [0,T], k \geq 1)$ be a sequence of independent standard Brownian motions. Denote by $\mathcal{P}(l^2)$ the set of predictable processes belonging to $L^2(\Omega \times [0,T]; l^2)$ and let $g$ be a real-valued, bounded, Borel measurable function defined on $C([0,T]; \mathbb{R}^\infty)$. Then,

$$-\log E(\exp[-g(B)]) = \inf_{u \in \mathcal{P}(l^2)} E \left( \frac{1}{2}\|u\|_{L^2([0,T]; l^2)}^2 + g \left( B + \int_0^T u \right) \right) \quad (11)$$

(see Theorem 2 in [3]).

**Weak regularity**

Denote by $\mathcal{P}_H$ the set of predictable processes belonging to $L^2(\Omega \times [0,T]; \mathcal{H})$. For any $N > 0$, we define

$$\mathcal{H}^N_T = \{ h \in \mathcal{H}_T : \|h\|_{\mathcal{H}_T} \leq N \},$$

$$\mathcal{P}^N_H = \{ v \in \mathcal{P}_H : v \in \mathcal{H}^N_T, a.s. \},$$

and we consider $\mathcal{H}^N_T$ endowed with the weak topology of $\mathcal{H}_T$. 
For any \( v \in P^N_H, \varepsilon \in [0, 1] \), let \( u^\varepsilon,v \) be the solution to

\[
\begin{align*}
  u^\varepsilon,v(t, x) &= w(t, x) + \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \langle G(t - s, x - \cdot) \sigma(u^\varepsilon,v(s, \cdot)), e_k \rangle_H dB_k(s) \\
  &\quad + \int_0^t \langle G(t - s, x - \cdot) \sigma(u^\varepsilon,v(s, \cdot)), v(s, \cdot) \rangle_H ds \\
  &\quad + \int_0^t \left[ G(t - s) * b(u^\varepsilon,v(s, \cdot)) \right](x) ds.
\end{align*}
\]

We will prove in Theorem 2.3 that this equation has a unique solution and that \( u^\varepsilon,v \in E_\alpha \) with \( \alpha \in I \).

Consider the following conditions:

(a) The set \( \{V^h, h \in H^N_T\} \) is a compact subset of \( E_\alpha \), where \( V^h \) is the solution of (9).

(b) For any family \( (v^\varepsilon, \varepsilon > 0) \subset P^N_H \) which converges in distribution as \( \varepsilon \to 0 \) to \( v \in P^N_H \), as \( H^N_T \)-valued random variables, we have

\[
\lim_{\varepsilon \to 0} u^\varepsilon,v^\varepsilon = V^v,
\]

in distribution, as \( E_\alpha \)-valued random variables.

Here \( V^v \) stands for the solution of (9) corresponding to a \( H^N_T \)-valued random variable \( v \) (instead of a deterministic function \( h \)). The solution is a stochastic process \( \{V^h(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\} \) defined path-wise by (9).

According to [3], Theorem 6 applied to the functional \( \mathcal{G} : C([0, T]; \mathbb{R}^\infty) \to E_\alpha, \mathcal{G}(\sqrt{\varepsilon}B) := u^\varepsilon \) (the solution of (8)), and \( \mathcal{G}^0 : H_T \to E_\alpha, \mathcal{G}^0(h) := V^h \) (the solution of (9)), conditions (a) and (b) above imply the validity of Theorem 1.1.

2 Laplace principle for the wave equation

Following the discussion of the preceding section, the proof of Theorem 1.1 will consist of checking that conditions (a) and (b) above hold true. As we next show, both conditions will follow from a single continuity result. Indeed, the set \( H^N_T \) is a compact subset of \( H_T \) endowed with the weak topology (see [13], Chapter 12, Theorem 4). Thus, (a) can be obtained by proving that the mapping \( h \in H^N_T \to V^h \in E_\alpha \) is continuous with respect to the weak topology. For this, we consider a sequence \( (h_n, n \geq 1) \subset H^N_T \) and
\( h \in H_T^N \) satisfying \( \lim_{n \to \infty} \| h_n - h \|_w = 0 \), which means that for any \( g \in H_T \),
\[ \lim_{n \to \infty} \langle h_n - h, g \rangle_{H_T} = 0, \]
and we will prove that
\[ \lim_{n \to \infty} \| V^{h_n} - V^h \|_\alpha = 0. \] (13)

As for (b), we invoke Skorohod Representation Theorem and rephrase this condition as follows. On some probability space \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\), consider a sequence of independent Brownian motions \( \bar{B} = \{ \bar{B}_k, k \geq 1 \} \) along with the corresponding filtration \( (\bar{\mathcal{F}}_t, t \in [0,T]) \), where \( \bar{\mathcal{F}}_t \) is the \( \sigma \)-field generated by the random variables \( \bar{B}_k(s), s \in [0,t], k \geq 1 \). Furthermore, consider a family of \((\bar{\mathcal{F}}_t)\)-predictable processes \( (\bar{v}^\varepsilon, \varepsilon > 0, \bar{v}) \) belonging to \( L^2(\bar{\Omega} \times [0,T]; H) \) taking values on \( H_T^N, \; \bar{P} \) a.s., such that the joint law of \((v^\varepsilon, v, B)\) (under \( P \)) coincides with that of \((\bar{v}^\varepsilon, \bar{v}, \bar{B})\) (under \( \bar{P} \)) and such that,
\[ \lim_{\varepsilon \to 0} \| \bar{v}^\varepsilon - \bar{v} \|_w = 0, \; \bar{P} - a.s. \]
as \( H_T^N \)-valued random variables. Let \( \bar{u}^{\varepsilon, \bar{v}^\varepsilon} \) be the solution to a similar equation as (12) obtained by changing \( v \) into \( \bar{v}^\varepsilon \) and \( B_k \) into \( \bar{B}_k \). Then, we will prove that for any \( q \in [0, \infty[ \),
\[ \lim_{\varepsilon \to 0} \bar{E} \left( \| \bar{u}^{\varepsilon, \bar{v}^\varepsilon} - V^\bar{v} \|_\alpha^q \right) = 0, \] (14)
where \( \bar{E} \) denotes the expectation operator on \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\). Notice that, if in (12) we consider \( \varepsilon = 0 \) and \( v := h \in P_N^H \) deterministic, we obtain the equation satisfied by \( V^h \). Consequently, the convergence (13) can be obtained as a particular case of (14).

Therefore, we will focus our efforts on the proof of (14). In the sequel, we shall omit any reference to the bars in the notation, for the sake of simplicity.

According to Lemma A1 in [1], the proof of (14) can be carried out into two steps:

1. **Estimates on increments**
\[
\sup_{\varepsilon \geq 1} E \left( \left| \left[ u^{\varepsilon, \bar{v}^\varepsilon}(t,x) - V^\varepsilon(t,x) \right] - \left[ u^{\varepsilon, \bar{v}^\varepsilon}(r,z) - V^\varepsilon(r,z) \right] \right|^q \right) \\
\leq C |t-r| + |x-z|^{\alpha q}.
\] (15)

2. **Pointwise convergence**
\[
\lim_{\varepsilon \to 0} E \left( \left| u^{\varepsilon, \bar{v}^\varepsilon}(t,x) - V^\varepsilon(t,x) \right|^q \right) = 0.
\] (16)
Here, \( q \in [1, \infty], (t, x), (r, z) \in [0, T] \times D \) and \( \alpha \in \mathcal{I} \).

Before proving these facts, we will address the problem of giving a rigorous formulation of (12). As we have already mentioned, the stochastic integral with respect to \((B_k, k \geq 1)\) in (12) is equivalent to the stochastic integral 
\[
\int_0^t \int_{\mathbb{R}^3} G(t-s, x-y) \sigma(u^{x,y}(s,y)) M(ds, dy)
\]
considered in the sense of [6]. We recall that such an integral is defined for stochastic processes \( Z = (Z(s, \cdot), s \in [0, T]) \) with values in \( L^2(\mathbb{R}^3) \) a.s., adapted and mean-square continuous, and the integral

\[
\nu_{G,Z}^t(\ast) := \sum_{k \geq 1} \int_0^t \langle G(t-s, \ast - \cdot) Z(s, \cdot), e_k(s, \cdot) \rangle_{\mathcal{H}} dB_k(s)
\]

(17)
satisfies

\[
E \left( \| \nu_{G,Z}^t \|^2_{L^2(\mathbb{R}^3)} \right) = \int_0^t ds \int_{\mathbb{R}^3} d\xi E(\| \mathcal{F} Z(s)(\xi) \|^2) \int_{\mathbb{R}^3} \mu(d\eta) |\mathcal{F} G(t-s)(\xi-\eta)|^2.
\]

(18)

(see [6], Theorem 6).

As a function of the argument \( x \), for any \( v \in \mathcal{P}^N_H \), the path-wise integral

\[
\int_0^t \langle G(t-s, x-\cdot) \sigma(u^{x,\cdot}(s, \cdot)), v(s, \cdot) \rangle_{\mathcal{H}} ds,
\]
is also a well-defined \( L^2(\mathbb{R}^3) \)-valued random variable. Indeed, let \( Z \) be a stochastic process satisfying the hypotheses described before. Set

\[
\nu_{G,Z}^t(\ast) := \int_0^t \langle G(t-s, \ast - \cdot) Z(s, \cdot), v(s, \cdot) \rangle_{\mathcal{H}} ds.
\]

(19)

By Cauchy-Schwarz’ inequality applied to the inner product on \( \mathcal{H}_T \), we have

\[
\| \nu_{G,Z}^t \|^2_{L^2(\mathbb{R}^3)} \leq N^2 \int_{\mathbb{R}^3} dx \int_0^t ds \| G(t-s, x-\cdot) Z(s, \cdot) \|^2_{\mathcal{H}}
\]

\[
= N^2 \int_0^t ds \int_{\mathbb{R}^3} d\xi |\mathcal{F} Z(s)(\xi)|^2 \int_{\mathbb{R}^3} \mu(d\eta) |\mathcal{F} G(t-s)(\xi-\eta)|^2,
\]

where the last equality is derived following the arguments for the proof of Theorem 6 in [6]. We recall that this formula is firstly established for \( Z \) sufficiently smooth and by smoothing \( G \) by convolution with an approximation of the identity. The extension of the formula to the standing assumptions is done by a limit procedure.

From this, we clearly have

\[
E \left( \| \nu_{G,Z}^t \|^2_{L^2(\mathbb{R}^3)} \right) \leq N^2 \times \int_0^t ds \int_{\mathbb{R}^3} d\xi E(\| \mathcal{F} Z(s)(\xi) \|^2) \int_{\mathbb{R}^3} \mu(d\eta) |\mathcal{F} G(t-s)(\xi-\eta)|^2.
\]

(20)
Remark 2.1 Up to a positive constant, the $L^2(\Omega; L^2(\mathbb{R}^3))$-norm of the stochastic integral $v^t_{G,Z}$ and the path-wise integral $\nu^t_{G,Z}$ are bounded by the same expression.

Let $\mathcal{O}$ be a bounded or unbounded open subset of $\mathbb{R}^3$, $q \in [1, \infty]$, $\gamma \in [0, 1]$. We denote by $W^{\gamma,q}(\mathcal{O})$ the fractional Sobolev Banach space consisting of functions $\varphi : \mathbb{R}^3 \to \mathbb{R}$ such that

$$
\| \varphi \|_{W^{\gamma,q}(\mathcal{O})} := \left( \| \varphi \|_{L^q(\mathcal{O})}^q + \| \varphi \|_{W^{\gamma,q,q}(\mathcal{O})}^q \right)^{\frac{1}{q}} < \infty,
$$

where

$$
\| \varphi \|_{W^{\gamma,q,q}(\mathcal{O})} = \left( \int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \frac{|\varphi(x) - \varphi(y)|^q}{|x - y|^{3+\gamma}} \right)^{\frac{1}{q}}.
$$

For any $\varepsilon > 0$, we denote by $\mathcal{O}^\varepsilon$ the $\varepsilon$-enlargement of $\mathcal{O}$, that is,

$$
\mathcal{O}^\varepsilon = \{ x \in \mathbb{R}^3 : d(x, \mathcal{O}) < \varepsilon \}.
$$

In the proof of Theorem 2.3 below, we will use a smoothed version of the fundamental solution $G$ defined as follows. Consider a function $\psi \in C^\infty(\mathbb{R}^3; \mathbb{R}_+)$ with support included in the unit ball, such that $\int_{\mathbb{R}^3} \psi(x)dx = 1$. For any $t \in (0, 1]$ and $n \geq 1$, set

$$
\psi_n(t, x) = \left( \frac{n}{t} \right)^3 \psi \left( \frac{n}{t} x \right),
$$

and

$$
G_n(t, x) = (\psi_n(t, \cdot) * G(t))(x). \tag{21}
$$

Notice that, for any $t \in [0, T]$, supp $G_n(t, \cdot) \subset B_{t(1+\frac{1}{n})}(0)$.

Remark 2.2 Since $G_n(t)$ is smooth and has compact support, $v^t_{G_{n,z}}(x)$ is well-defined as a Walsh stochastic integral and this integral defines a random field indexed by $(t, x)$. By Burkholder’s inequality, for any $q \in [2, \infty[$,

$$
E \left( |v^t_{G_{n,z}}(x)|^q \right) \leq CE \left( \int_0^t \| G(t - s, x - \cdot)Z(s, \cdot) \|_{\mathcal{H}}^2 ds \right)^{\frac{q}{2}}.
$$

As for the path-wise integral $\nu^t_{G_{n,z}}(x)$, by applying Cauchy-Schwarz’ inequality to the inner product on $\mathcal{H}_T$, we have

$$
E \left( |\nu^t_{G_{n,z}}(x)|^q \right) \leq N^q E \left( \int_0^t \| G(t - s, x - \cdot)Z(s, \cdot) \|_{\mathcal{H}}^2 ds \right)^{\frac{q}{2}}.
$$

Hence, as in Remark 2.1, up to a constant, $L^q(\Omega)$-estimates for both type of integrals at fixed $(t, x) \in [0, T] \times \mathbb{R}^3$ coincide.
The following proposition is the analogue of Theorem 3.1 in [7] for the path-wise integral $\nu^t_{G, Z}$.

**Proposition 2.1** Fix $q \in ]3, \infty[$ and a bounded domain $O \subset \mathbb{R}^3$. Suppose that

$$\tau_q(\beta, \delta) := \left( \frac{2 - \beta}{2} \wedge \frac{1 + \delta}{2} \right) - \frac{3}{q} > 0$$

and fix $\gamma \in ]0, 1[, \rho \in ]0, \tau_q(\beta, \delta) \wedge \gamma[$. Let $\{Z_t, t \in [0, T]\}$ be a $L^2(\mathbb{R}^3)$-valued, $(\mathcal{F}_t)$-adapted, mean-square continuous stochastic process. Assume that for some fixed $t \in [0, T]$,

$$\int_0^t E\left( \|Z(s)\|^q_{W^{\gamma, q}(O_t-s)} \right) \, ds < \infty.$$ 

We have the following estimates:

$$E\left( \|\nu^t_{G, Z}\|^q_{L^q(O)} \right) \leq C \int_0^t E\left( \|Z(s)\|^q_{L^q(O_t-s)} \right) \, ds, \quad (22)$$

$$E\left( \|\nu^t_{G, n, Z}\|^q_{\rho, q, O} \right) \leq C \int_0^t E\left( \|Z(s)\|^q_{W^{\rho, q}(O_t-s)(1+\frac{1}{\rho})} \right) \, ds, \quad (23)$$

$$E\left( \|\nu^t_{G, Z}\|^q_{\rho, q, O} \right) \leq C \int_0^t E\left( \|Z(s)\|^q_{W^{\rho, q}(O_t-s)} \right) \, ds. \quad (24)$$

Consequently,

$$E\left( \|\nu^t_{G, Z}\|^q_{W^{\rho, q}(O)} \right) \leq C \int_0^t E\left( \|Z(s)\|^q_{W^{\rho, q}(O_t-s)} \right) \, ds. \quad (25)$$

**Proof.** By virtue of Remark 2.2, we see that the estimate (22) follows from the same arguments used in [7], Proposition 3.4. We recall that this proposition is devoted to prove an analogue property for the stochastic integral $\nu^t_{G, Z}$. In the very same way, (23) is established using the arguments of the proof of Proposition 3.5 in [7]. Then, as in [7], (24) is obtained from (23) by applying Fatou’s lemma. Finally, (25) is a consequence of (22), (24) and the definition of the fractional Sobolev norm $\| \cdot \|_{W^{\rho, q}(O)}$.

Next, we present an analogue of Theorem 3.8 [7] for the path-wise integral $\nu^t_{G, Z}$, which gives the sample path properties in the argument $t$ for this integral. As in Proposition 2.1, $O$ is a bounded domain in $\mathbb{R}^3$. 

10
Proposition 2.2 Consider a stochastic process \( \{Z_t, t \in [0,T]\} \), \((\mathcal{F}_t)\)-adapted, with values in \( L^2(\mathbb{R}^3) \), mean-square continuous. Assume that for some fixed \( q \in ]3, \infty[ \) and \( \gamma \in \left[ \frac{3}{q}, 1 \right[ \),

\[
\sup_{t \in [0,T]} E \left( \|Z(t)\|_{W^{\gamma,q}(\Omega^{T-t})}^q \right) < \infty.
\]

Then the stochastic process \( \{\nu^t_{G,Z}(x), t \in [0,T]\}, x \in \mathcal{O} \), satisfies

\[
\sup_{x \in \mathcal{O}} E \left( |\nu^t_{G,Z}(x) - \nu^\bar{t}_{G,Z}(x)|^q \right) \leq C|t - \bar{t}|^\rho q,
\]

for each \( t, \bar{t} \in [0,T] \), any \( \bar{q} \in ]2, q[ \), \( \rho \in ]0, \frac{3}{q} - \frac{2}{3} \) and \( \frac{1}{2} - \frac{2}{3} \).

Proof. We follow the same scheme as in the proof of [7], Theorem 3.8. To start with, we should prove an analogue of (3.26) in [7], with \( v^t \) for the expressions termed \( \nu^t \).

These are, up to a positive constant, the same upper bounds obtained in [7] for the stochastic integrals considered in the above mentioned reference. More precisely, assume \( 0 \leq t < \bar{t} \leq T \); by applying Cauchy-Schwarz’ inequality to the inner product on \( \mathcal{H}_T \), we obtain

\[
E \left( \left| \int_t^{\bar{t}} \langle G_n(\bar{t} - s, x - \cdot)Z(s, \cdot), v(s, \cdot) \rangle_{\mathcal{H}} \, ds \right|^q \right)^{\frac{1}{q}} \leq N^q E \left( \int_0^{\bar{t} - t} \|G_n(s, x - \cdot)Z(\bar{t} - s, \cdot)\|^2_{\mathcal{H}} \, ds \right)^{\frac{1}{2}}.
\]

\[
E \left( \left| \int_0^{t} \langle (G_n(\bar{t} - s, x - \cdot) - G_n(t - s, x - \cdot))Z(s, \cdot), v(s, \cdot) \rangle_{\mathcal{H}} \, ds \right|^q \right)^{\frac{1}{q}} \leq N^q E \left( \int_0^{t} \|G_n(\bar{t} - s, x - \cdot) - G_n(t - s, x - \cdot)Z(s, \cdot)\|^2_{\mathcal{H}} \, ds \right)^{\frac{1}{2}}.
\]

These are, up to a positive constant, the same upper bounds obtained in [7] for the expressions termed \( T^1_\alpha(t, \bar{t}, x) \) and \( T^2_\alpha(t, \bar{t}, x) \), respectively. After this remark, the proof follows the same arguments as in [7].

\[ \Box \]

For any \( t \in [0,T], \ a \geq 1 \), let \( K^D_\alpha(t) = \{y \in \mathbb{R}^3 : d(y, D) \leq a(T - t)\} \). For \( a = 1 \), we shall simply write \( K^D(t) \); this is the light cone of \( \{T\} \times D \).

In the next theorem, the statement on existence and uniqueness of solution, as well as (27), extend Theorem 4.3 in [8], while (28) and (29) are
extensions of the inequality (4.24) of Theorem 4.6 and (4.41) of Theorem 4.11 in [7], respectively. Indeed in the cited references, the results apply to Equation (4) while in the next theorem, they apply to Equation (12).

**Theorem 2.3** Assuming (H), the following statements hold true:

There exists a unique random field solution to (12), \( \{u^{\varepsilon,v}(t,x), (t,x) \in [0,T] \times \mathbb{R}^3 \} \), and this solution satisfies

\[
\sup_{\varepsilon \in [0,1]} \sup_{v \in P_N^{\mathcal{H}}} E \left( |u^{\varepsilon,v}(t,x)|^q \right) < \infty, \tag{27}
\]

\[
\sup_{\varepsilon \in [0,1]} \sup_{v \in P_N^{\mathcal{H}}} \sup_{t \in [0,T]} E \left( \|u^{\varepsilon,v}(t)\|_{W^{\alpha,q}(K^D(t))}^q \right) < \infty, \tag{28}
\]

for any \( q \in [2, \infty[ \), \( \alpha \in \mathcal{I} \).

Moreover, for any \( q \in [2, \infty[ \) and \( \alpha \in \mathcal{I} \), there exists \( C > 0 \) such that for \( (t,x), (\bar{t},y) \in [0,T] \times D \),

\[
\sup_{\varepsilon \in [0,1]} \sup_{v \in P_N^{\mathcal{H}}} E(|u^{\varepsilon,v}(t,x) - u^{\varepsilon,v}(\bar{t},y)|^q) \leq C(|t - \bar{t}| + |x - y|)^{\alpha q}. \tag{29}
\]

Thus, a.s., \( \{u^{\varepsilon,v}(t,x), (t,x) \in [0,T] \times D \} \) has Hölder continuous sample paths of degree \( \alpha \in \mathcal{I} \), jointly in \( (t,x) \).

**Proof.** For the sake of simplicity, we shall consider \( \varepsilon = 1 \) and write \( u^v \) instead of \( u^{\varepsilon,v} \).

We start by proving existence and uniqueness along with (27). For this, we will follow the method of the proof of [8], Theorem 4.3 (borrowed from [14], Theorem 1.2 and [4], Theorem 13). It is based on the Picard iteration scheme:

\[
u^v(0)(t,x) = w(t,x),
\]

\[
u^v(n+1)(t,x) = w(t,x) + \sum_{k \geq 1} \int_0^t \langle G(t-s,x-\cdot)\sigma(u^{v,(n)}(s,\cdot)), \varepsilon_k \rangle \mathcal{H} dB_k(s)
\]

\[
+ \int_0^t \langle G(t-s,x-\cdot)\sigma(u^{v,(n)}(s,\cdot)), v(s,\cdot) \rangle \mathcal{H} \, ds
\]

\[
+ \int_0^t [G(t-s) \ast b(u^{v,(n)}(s,\cdot))](x) \, ds, \quad n \geq 0. \tag{30}
\]

The steps of the proof are as follows. Firstly, we check that

\[
\sup_{v \in P_N^{\mathcal{H}}} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} E \left( |u^{v,(n)}(t,x)|^q \right) < \infty, \tag{31}
\]
and then
\[ \sup_{n \geq 0} \sup_{v \in \mathcal{P}_H} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} E \left( |u^{v,(n)}(t,x)|^q \right) < \infty. \] (32)

Secondly, by setting
\[ M_n(t) := \sup_{(s,x) \in [0,t] \times \mathbb{R}^3} E \left( |u^{v,(n+1)}(s,x) - u^{v,(n)}(s,x)|^q \right), \quad n \geq 0, \]
we prove
\[ M_n(t) \leq C \int_0^t M_{n-1}(s) \left( 1 + \int_{\mathbb{R}^3} \mu(d\xi)|FG(t-s)(\xi)|^2 \right) ds. \] (33)

With these facts, we conclude that \((u^{v,(n)}(t,x), n \geq 0)\) converges uniformly in \((t,x)\) in \(L^q(\Omega)\) to a limit \(u^v(t,x)\) which satisfies equation (12) with \(\varepsilon = 1\).

In comparison with the proof of Theorem 4.3 in [8], establishing (31)–(33) requires additionally the analysis of the term given by the path-wise integral
\[ I^{v,(n+1)} := \int_0^t \langle G(t-s,x-\cdot)\sigma(u^{v,(n)}(s,\cdot)), v(s,\cdot) \rangle_{\mathcal{H}} ds, \quad n \geq 0. \] (34)

This is done as follows. We assume that (31) holds true for some \(n \geq 0\). This is definitely the case for \(n = 0\) (see (7)). By applying Cauchy-Schwarz inequality on the Hilbert space \(\mathcal{H}_T\), and since \(\|v\|_{\mathcal{H}_T} \leq N\) a.s., we have
\[
E \left( \left\| \int_0^t \langle G(t-s,x-\cdot)\sigma(u^{v,(n)}(s,\cdot)), v(s,\cdot) \rangle_{\mathcal{H}} ds \right\|^q \right) 
\leq N^q E \left( \int_0^t \left\| G(t-s,x-\cdot)\sigma(u^{v,(n)}(s,\cdot)) \right\|^2_{\mathcal{H}} ds \right)^{\frac{q}{2}}.
\]

Notice that, by applying Burkholder’s inequality to the stochastic integral term in (30), we obtain
\[
E \left( \left\| \sum_{k \geq 1} \int_0^t \langle G(t-s,x-\cdot)\sigma(u^{v,(n)}(s,\cdot)), e_k \rangle_{\mathcal{H}} dB_k(s) \right\|^q \right) 
\leq CE \left( \int_0^t \left\| G(t-s,x-\cdot)\sigma(u^{v,(n)}(s,\cdot)) \right\|^2_{\mathcal{H}} ds \right)^{\frac{q}{2}}.
\]

Thus, as has already been mentioned in Remark 2.2, up to a positive constant, \(L^q(\Omega)\) estimates of the stochastic integral and of the path-wise integral \(I^{v,(n+1)}\) lead to the same upper bounds.

This simple but important remark yields the extension of properties (31)–(33), which are valid for Equation (4), as is proved in Theorem 4.3 in [8],
to Equation (12) with $\epsilon = 1$ and actually, for any $\epsilon \in [0, 1]$. In fact, those properties can be proved to hold uniformly in $\epsilon \in [0, 1]$.

Let us now argue on the validity of (28). We will follow the programme of Section 4.2 in [7], taking into account the new term

$$\int_0^t \langle G(t - s, x - \cdot)\sigma(u^n(s, \cdot)), v(s, \cdot) \rangle_{\mathcal{H}} \, ds$$

of Equation (12) (with $\epsilon = 1$) that did not appear in [7]. This consists of the following steps.

Firstly, we need an extension of Proposition 4.3 in [7]. This refers to an approximation of the localized version of (12) on a light cone. In the approximating sequence, the fundamental solution $G$ of the wave equation is replaced by the smoothed version $G_n$ defined in (21). Going through the proof of that Proposition, we see that for the required extension the term

$$M_n(t) := E\left(\|\nu^t_{G_n, Z} - \nu^t_{G, Z}\|_{L^q(K^P(t))}\right),$$

with $Z(s, y) = \sigma(u^n(s, y))1_{K^P(s)}(y)$, should be replaced by

$$\tilde{M}_n(t) := E\left(\|\nu^t_{G_n, Z} - \nu^t_{G, Z}\|_{L^q(K^P(t))}\right) + E\left(\|\nu^t_{G_n, Z} - \nu^t_{G, Z}\|_{L^q(K^P(t))}\right),$$

where we have used the notation introduced in (17), (19). Then we should prove that $\lim_{n \to \infty} \tilde{M}_n(t) = 0$. This is carried out by considering first the case $q = 2$. By Remark 2.1, it suffices to have $\lim_{n \to \infty} M_n(t) = 0$ for $q = 2$, and this fact is proved in [7], Proposition 4.3.

To extend the convergence to any $q \in [2, \infty]$, we must establish that for some fixed $n_0 > 0$,

$$\sup_{n \geq n_0} E\left(\|\nu^t_{G_n, Z}\|_{L^q(K^P(t))}\right) < \infty. \quad (35)$$

a result which holds true for $\nu^t_{G_n, Z}$. Once more, the first step in the proof of (35) consists in obtaining the upper bound

$$E\left(\|\nu^t_{G_n, Z}\|_{L^q(K^P(t))}\right) \leq CE \int_0^t ds \int_{K^P(t)} dx E\left(\|G_n(t - s, x - \cdot)Z(s, \cdot)\|_{\mathcal{H}}^q\right). \quad (36)$$

This follows easily by applying first Cauchy-Schwarz’ inequality to the inner product on $\mathcal{H}_T$ and then Hölder’s inequality. Once we have (36), we can obtain (35) by following the steps of the proof of Proposition 3.4 in [7].
The last ingredient for the proof of (28) consist of the extension of Theorem 4.6 in [7]. This requires the following additional arguments. Firstly, using similar notations as in that reference, we set
\[ R_{n,m}^{\gamma,D}(t) = E \left( \| u_{n,m}^{(m)}(t) \|_{W^{\gamma,q}(K^D(t))}^q \right), \]
where \( u_{n,m}^{(m)}(t, x) \) stands for the \( m \)-th Picard iteration of a similar equation as (12) with \( G \) replaced by the smoothed version \( G_n \). In comparison with [7], in order to check that \( \sup_{n,m \geq 1} R_{n,m}^{\gamma,D} < \infty \), we have to study the additional term
\[ T_{n,m}^{\gamma,D,3}(t) := E \left( \| u_{G_n,\sigma(u_{n,m})}^{(m)}(t) \|_{W^{\gamma,q}(K^D(t))}^q \right) \]
and more specifically, to check that
\[ T_{n,m}^{\gamma,D,3}(t) \leq C_1 + C_2 \int_0^t ds R_{n,m-1}^{\gamma,D}(s), \tag{37} \]
for some positive constants \( C_1, C_2 \).
This property holds true when \( T_{n,m}^{\gamma,D,3}(t) \) is replaced by
\[ E \left( \| u_{G_n,\sigma(u_{n,m})}^{(m)}(t) \|_{W^{\gamma,q}(K^D(t))}^q \right) \]
(see the arguments on page 42 of [7] based upon Proposition 3.5 of this reference). In a similar way, (37) follows from Proposition 2.1 and more precisely, from (23).
This completes the proof of (28).
An important consequence of (28) is the following. For any \( t > 0 \), a.s., the sample paths of \((u^{\epsilon,v}(t,x)1_{K_D(t)}(x), x \in \mathbb{R}^3)\) are \( \alpha \)-Hölder continuous with \( \alpha \in \mathcal{I} \). In addition, for any \( q \in [2, \infty[ \),
\[ \sup_{\epsilon \in [0,1]} \sup_{x \in \mathcal{P}^N x \in D} E \left( \| u^{\epsilon,v}(t,x) - u^{\epsilon,v}(t,y) \|_{K^D(t)}^q \right) \leq C |x - y|^{\alpha q}, \tag{38} \]
for any \( x, y \in K^D(t), \alpha \in \mathcal{I} \). Hence, in order to prove (29) it remains to establish that, for any \( q \in [2, \infty[ \) and \( \alpha \in \mathcal{I} \), there exists \( C > 0 \) such that for every \( t, \bar{t} \in [0, T] \),
\[ \sup_{\epsilon \in [0,1]} \sup_{x \in \mathcal{P}^N x \in D} E \left( \| u^{\epsilon,v}(t,x) - u^{\epsilon,v}(\bar{t},x) \|_{K^D(t)}^q \right) \leq C |t - \bar{t}|^{\alpha q}, \tag{39} \]
For this, we will follow the steps of Section 4.3 in [7] devoted to the analysis of the time regularity of the solution to (4) and get an extension of Theorem 4.10.
As in the first part of the proof, we consider the case \( \varepsilon = 1 \). The additional required ingredient consists of showing that

\[
E \left( \left| \int_0^t \langle G(t-s, x - \cdot) \sigma(u^n(s, \cdot)) 1_{KD(s)}(\cdot), v \rangle_{\mathcal{H}} \, ds \right|^q \right) - \int_0^{\bar{t}} \langle G(\bar{t}-s, x - \cdot) \sigma(u^n(s, \cdot)) 1_{KD(s)}(\cdot), v \rangle_{\mathcal{H}} \, ds \right| _H ds
\leq C |t - \bar{t}|^{\alpha q},
\]
(40)

uniformly in \( x \in D \).

Remark that the stochastic process

\[
\{ Z(s, y) := \sigma(u^n(s, y)) 1_{KD(s)}(y), (s, y) \in [0, T] \times \mathbb{R}^3 \},
\]
satisfies the assumptions of Proposition 2.2 with \( \mathcal{O} = D \) and arbitrarily large \( q \). This fact is proved in Theorem 4.10 in [7]. Thus, (40) follows from that Proposition.

Going through the arguments, it is easy to realize that for \( u^{v, \varepsilon} \), we can get uniform estimates in \( \varepsilon \in ]0, 1[ \) and \( v \in \mathcal{P}_H^N \), and therefore (39) holds true. This ends the proof of (29) and of the Theorem.

\[ \square \]

**Remark 2.3** In connection with conclusion (4.8) of Theorem 4.1 in [7], we notice that property (27) implies

\[
\sup_{\varepsilon \in ]0, 1[} \sup_{v \in \mathcal{P}_H^N} \sup_{t \in [0, T]} E \left( \| u^{v, \varepsilon}(t) \|_{L^q(KD(t))}^q \right) < \infty.
\]

The estimates on increments described in (15) are a consequence of (29). Indeed, as has been already mentioned, for any \( v \in \mathcal{P}_H^N \), the stochastic process \( V^v \) is the solution to the particular equation (12) obtained by setting \( \varepsilon = 0 \).

**Proposition 2.4** Assume (H). Consider a family \( (v^\varepsilon, \varepsilon > 0) \subset \mathcal{P}_H^N \) and \( v \in \mathcal{P}_H^N \) such that a.s.,

\[
\lim_{\varepsilon \to 0} \| v^\varepsilon - v \|_w = 0.
\]

Then, for any \((t, x) \in [0, T] \times D\) and any \( q \in [2, \infty[ , \)

\[
\lim_{\varepsilon \to 0} E \left( |u^{v, \varepsilon}(t, x) - V^v(t, x)|^q \right) = 0.
\]
(41)
Proof. We write
\[ u^{\varepsilon,v}(t,x) - V^v(t,x) = \sum_{i=1}^{4} T_i^{\varepsilon}(t,x), \]
with
\[ T_1^{\varepsilon}(t,x) = \int_0^t \left[ G(t-s) * \left( b(u^{\varepsilon,v}(s,\cdot)) - b(V^v(s,\cdot)) \right) \right](x) \, ds, \]
\[ T_2^{\varepsilon}(t,x) = \int_0^t \left\langle G(t-s,x-\cdot) \left[ \sigma(u^{\varepsilon,v}(s,\cdot)) - \sigma(V^v(s,\cdot)) \right] , v^\varepsilon(s,\cdot) \right\rangle_{\mathcal{H}} \, ds, \]
\[ T_3^{\varepsilon}(t,x) = \int_0^t \left\langle G(t-s,x-\cdot) \sigma(V^v(s,\cdot)) , v^\varepsilon(s,\cdot) - v(s,\cdot) \right\rangle_{\mathcal{H}} \, ds, \]
\[ T_4^{\varepsilon}(t,x) = \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \left\langle G(t-s,x-\cdot) \sigma(u^{\varepsilon,v}(s,\cdot)) , e_k \right\rangle_{\mathcal{H}} dB_k(s). \]

Fix \( q \in [2, \infty[. \) Hölder’s inequality with respect to the measure on \([0,t] \times \mathbb{R}^3\) given by \( G(t-s,dy)ds \), along with the Lipschitz continuity of \( b \) yield
\[ E(|T_1^{\varepsilon}(t,x)|^q) \leq \left( \int_0^t ds \int_{\mathbb{R}^3} G(s,dy) \right)^{q-1} \times \int_0^t \sup_{(r,z) \in [0,s] \times \mathbb{R}^3} E\left(|u^{\varepsilon,v}(r,z) - V^v(r,z)|^q\right) \left( \int_{\mathbb{R}^3} G(s,dy) \right) \, ds \]
\[ \leq C \int_0^t \sup_{(r,z) \in [0,s] \times \mathbb{R}^3} E\left(|u^{\varepsilon,v}(r,z) - V^v(r,z)|^q\right) \, ds \]

To study \( T_2^{\varepsilon}(t,x) \), we apply Cauchy-Schwarz’ inequality to the inner product on \( \mathcal{H} \) and then Hölder’s inequality with respect to the measure on \([0,t] \times \mathbb{R}^3\) given by \( |\mathcal{F} G(s)(\xi)|^2 ds \mu(d\xi) \). Notice that this measure can also we written as \( [G(s) * \tilde{G}(s)](x) \Gamma(dx) ds \). The Lipschitz continuity of \( \sigma \) along with (5) and
the property $\sup_{\varepsilon} \|v^{\varepsilon}\|_{H_T} \leq N$, imply

$$E (|T^\varepsilon_2(t,x)|^q) \leq E \left( \left( \int_0^t \|G(t-s,x-x) \left[ \sigma(u^{\varepsilon,v}(s,\cdot)) - \sigma(V^{v}(s,\cdot)) \right]\|^2_{H_T} \ ds \right)^{\frac{q}{2}} \right) \times \left( \int_0^t \|v^{\varepsilon}(s,\cdot)\|^2_{H_T} ds \right)^{\frac{q}{2}}$$

$$\leq CE \left( \int_0^t \|G(t-s,x-x) \left[ \sigma(u^{\varepsilon,v}(s,\cdot)) - \sigma(V^{v}(s,\cdot)) \right]\|^2_{H_T} \ ds \right)^{\frac{q}{2}}$$

$$\leq C \left( \int_0^t ds \int_{\mathbb{R}^3} \mu(d\xi)|FG(t-s)(\xi)|^2 \right)^{\frac{q}{2} - 1} \times \int_0^t \sup_{(r,z)\in[0,s]\times\mathbb{R}^3} E (|u^{\varepsilon,v}(r,z) - V^{v}(r,z)|^q) \left( \int_{\mathbb{R}^3} \mu(d\xi)|FG(s)(\xi)|^2 \right) ds$$

$$\leq C \int_0^t \sup_{(r,z)\in[0,s]\times\mathbb{R}^3} E (|u^{\varepsilon,v}(r,z) - V^{v}(r,z)|^q) \ ds.$$

For any $(t,x) \in [0,T] \times \mathbb{R}^3$, the stochastic process

$$\{G(t-s,x-y)\sigma(V^{v}(s,y)), (s,y) \in [0,T] \times \mathbb{R}^3\}$$

satisfies the property

$$\sup_{v \in \mathcal{P}_N^H} \sup_{s \in [0,T]} E (\|G(t-s,x-x)\sigma(V^{v}(s,\cdot))\|^q_{H_T}) < \infty. \quad (42)$$

Indeed, by applying (27) to the particular case $\varepsilon = 0$, we get

$$\sup_{v \in \mathcal{P}_N^H} \sup_{(t,x)\in[0,T] \times \mathbb{R}^3} E (|V^{v}(t,x)|^q) < \infty. \quad (43)$$

Then, we apply Hölder’s inequality with respect to the measure on $\mathbb{R}^3$ given by $|FG(t-s)(\xi)|^2 \mu(d\xi)$, along with the linear growth property of $\sigma$, and we obtain

$$E (\|G(t-s,x-x)\sigma(V^{v}(s,\cdot))\|^q_{H_T})$$

$$\leq C \left( \int_{\mathbb{R}^3} |FG(t-s)(\xi)|^2 \mu(d\xi) \right)^{\frac{q}{2}} \times \left( 1 + \sup_{(s,y)\in[0,T] \times \mathbb{R}^3} E (|V^{v}(s,y)|^q) \right).$$

With (5) and (43), we have (42).
From (42), it follows that \( \{ G(t-s,x-y)\sigma(V^v(s,y), (s,y) \in [0,T] \times \mathbb{R}^3) \} \) takes its values in \( \mathcal{H}_T \), a.s. Since \( \lim_{\varepsilon \to 0} \| v^\varepsilon - v \|_{\mathcal{H}} = 0 \), a.s.,

\[
\lim_{\varepsilon \to 0} \left| \int_0^t \langle G(t-s,x-\cdot)\sigma(V^v(s,\cdot)), v^\varepsilon(s,\cdot) - v(s,\cdot) \rangle_{\mathcal{H}} ds \right| = 0.
\]

Applying (42) and bounded convergence, we see that the above convergence takes place in \( L^q(\Omega) \) as well. Thus,

\[
\lim_{\varepsilon \to 0} E \left( |T^\varepsilon_{3}(t,x)|^q \right) = 0.
\]

By the \( L^q \) estimates of the stochastic integral and (27), we have

\[
E \left( \sum_{k \geq 1} \int_0^t \langle G(t-s,x-\cdot)\sigma(u^\varepsilon,v^\varepsilon(s,\cdot)), e_k \rangle_{\mathcal{H}} dB_k(s) \right)^q
\]

\[
= E \left( \int_0^t \| G(t-s,x-\cdot)\sigma(u^\varepsilon,v^\varepsilon(s,\cdot)) \|_{\mathcal{H}_T}^2 ds \right)^{\frac{q}{2}}
\]

\[
\leq \left( \int_0^t ds \int_{\mathbb{R}^3} \mu(d\xi) |\mathcal{F}G(t-s)(\xi)|^2 \right)^{\frac{q}{2}-1}
\]

\[
\times \int_0^t \left( 1 + \sup_{(r,z) \in [0,s] \times \mathbb{R}^3} E \left( |u^\varepsilon,v^\varepsilon(r,z)|^q \right) \right) \left( \int_{\mathbb{R}^3} \mu(d\xi) |\mathcal{F}G(s)(\xi)|^2 \right) ds
\]

\[
\leq C \int_0^t \left( 1 + \sup_{(r,z) \in [0,s] \times \mathbb{R}^3} E \left( |u^\varepsilon,v^\varepsilon(r,z)|^q \right) \right) ds
\]

\[
\leq C.
\]

This yields

\[
\lim_{\varepsilon \to 0} E \left( |T^\varepsilon_{3}(t,x)|^q \right) = 0.
\]

We end the proof of the Proposition by applying the usual version of Gronwall’s lemma.

Notice that we have actually proved the stronger statement

\[
\lim_{\varepsilon \to 0} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} E \left( |u^\varepsilon,v^\varepsilon(t,x) - V^v(t,x)|^p \right) = 0.
\]

\[
\square
\]

\textbf{Proof of Theorem 1.1.} As has been argued, it suffices to check the validity of (15) and (16). These statements follow from Theorem 2.3 and Proposition 2.4, respectively.

\[
\square
\]
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