A note on an extension of Gelfond’s constant

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Abstract

The aim of this note is to provide a natural extension of Gelfond’s constant \( e^\pi \) using a hypergeometric function approach. An extension is also found for the square root of this constant. A few interesting special cases are presented.

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1. Introduction

In mathematics, Gelfond’s constant, which is named after Aleksandr Gelfond, is given by \( e^\pi \). Like both \( e \) and \( \pi \), this constant is a transcendental number. The decimal expansion of Gelfond’s constant is

\[ e^\pi = 23.140692632779 \ldots \]

and its continued fraction representation is given in [4, A039661].

This number has a connection to the Ramanujan constant \( e^{\pi \sqrt{163}} = (e^\pi)^{\sqrt{163}} \). It is worth noting that this last number is almost an integer:

\[ e^{\pi \sqrt{163}} \simeq 640320^3 + 744. \]

A geometrical occurrence of Gelfond’s constant arises in the sum of even-dimension unit spheres with volume \( V_{2n} = \pi^n/n! \). Then

\[ \sum_{n=0}^{\infty} V_{2n} = e^\pi. \]

There are several ways of expressing Gelfond’s constant, some of which are enumerated below:

\[ e^\pi = (i^\frac{1}{2})^{-2} \quad (i = \sqrt{-1}); \]
\[ e^\pi = \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \right)^{-4s}, \quad s = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}; \]
\[ e^\pi = \left( \prod_{k=1}^{\infty} k^{-\mu(k)/k} \right)^{\sigma}, \quad \sigma = \sqrt{6} \text{Li}_2(1), \]
where \( \mu(k) \) is the Möbius function and \( \text{Li}_n(x) \) is the polylogarithm function;

\[ e^\pi = aF_1(-; \frac{1}{2}; \pi^2/4) + \pi aF_1(-; \frac{3}{2}; \pi^2/4), \]
where \( aF_1(-; a; z) \) is a generalised hypergeometric function that can be expressed in terms of modified \( I \)-Bessel functions of order \( \pm \frac{1}{2} \); and finally

\[ e^\pi = 2F_1(i, -i; \frac{1}{2}, 1) + 2 \cdot 2F_1(\frac{1}{2} + i, \frac{1}{2} - i; \frac{3}{2}, 1), \] (1.1)
where \( 2F_1(a, b; c; z) \) is the well-known Gauss hypergeometric function [2, p. 384].

The result (1.1) can be easily established by making use of the classical Gauss summation theorem

\[ 2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \] (1.2)
provided \( \Re(c - a - b) > 0. \)

The natural extension of the summation theorem (1.2) to the \( 3F_2 \) hypergeometric series is available in the literature [3], which we shall write in the following manner:

\[ 3F_2 \left( \begin{array}{c} a, b, d+1 \\ c+1, d \end{array} ; 1 \right) = \frac{\Gamma(c+1)\Gamma(c - a - b)}{\Gamma(c - a + 1)\Gamma(c - b + 1)} \left( c - a - b + \frac{ab}{d} \right) \] (1.3)
provided \( d \neq 0, -1, -2, \ldots \) and \( \Re(c - a - b) > 0. \) The aim of this note is to provide a natural extension of Gelfond’s constant (1.1), and also its square root, with the help of the result (1.3). A few interesting results closely related to Gelfond’s constant and its square root are also given.

### 2. Extension of Gelfond’s constant

The natural extension of Gelfond’s constant to be established here is given in the following theorem.

**Theorem 1** For \( d_1, d_2 \neq 0, -1, -2, \ldots \), the following result holds true:

\[ e^\pi \left( \frac{1}{5d_1} + \frac{15}{32d_2} + \frac{23}{80} \right) + e^{-\pi} \left( \frac{1}{5d_1} - \frac{15}{32d_2} - \frac{7}{80} \right) \]
\[ = 3F_2 \left( \begin{array}{c} i, -i, d_1 + 1 \\ \frac{3}{2}, d_1 \end{array} ; 1 \right) + 2 \cdot 3F_2 \left( \begin{array}{c} \frac{1}{2} + i, \frac{1}{2} - i, d_2 + 1 \\ \frac{3}{2}, d_2 \end{array} ; 1 \right). \] (2.1)

**Proof.** The derivation of (2.1) follows from application of the summation formula (1.3). We have

\[ 3F_2 \left( \begin{array}{c} i, -i, d_1 + 1 \\ \frac{3}{2}, d_1 \end{array} ; 1 \right) = (e^\pi + e^{-\pi}) \left( \frac{1}{10} + \frac{1}{5d_1} \right) \]
and
\[ 3F_2\left( \frac{1}{2} + i, \frac{1}{2} - i, d_2 + 1 ; 1 \right) = (e^\pi - e^{-\pi}) \left( \frac{3}{32} + \frac{15}{64d_2} \right). \]

Insertion of these summations into the right-hand side of (2.1) then yields the result asserted by the theorem. □

3. Corollaries

In this section, we mention some interesting special cases of our main result in (2.1).

**Corollary 1**  In (2.1), if we take \( d_1 = 2/(5n - 1) \) and \( d_2 = 15/(2(8n - 3)) \) for positive integer \( n \), then we obtain after a little calculation the following result:

\[ ne^\pi = 3F_2\left( \frac{i}{2}, -i, \frac{5n+1}{2}, \frac{2}{5n-1} ; 1 \right) + 2 \times 3F_2\left( \frac{1}{2} + i, \frac{1}{2} - i, \frac{16n+9}{2(8n-3)} ; 1 \right). \] (3.1)

In particular, when \( n = 1 \) we recover Gelfond’s constant (1.1). For \( n = 2, 3 \) we find respectively the following results related to (1.1):

\[ 2e^\pi = 3F_2\left( \frac{i}{2}, -i, \frac{11}{2}, \frac{9}{2} ; 1 \right) + 2 \times 3F_2\left( \frac{1}{2} + i, \frac{1}{2} - i, \frac{41}{20} ; 1 \right) \] (3.2)

and

\[ 3e^\pi = 3F_2\left( \frac{i}{2}, -i, \frac{8}{2}, \frac{1}{2} ; 1 \right) + 2 \times 3F_2\left( \frac{1}{2} + i, \frac{1}{2} - i, \frac{14}{11} ; 1 \right). \] (3.3)

**Corollary 2**  In (2.1), if we take \( d_1 = 2/(5n - 1) \) and \( d_2 = -15/(2(8n + 3)) \) for positive integer \( n \), then we obtain after a little calculation the following result:

\[ ne^{-\pi} = 3F_2\left( \frac{i}{2}, -i, \frac{5n+1}{2}, \frac{2}{5n-1} ; 1 \right) + 2 \times 3F_2\left( \frac{1}{2} + i, \frac{1}{2} - i, \frac{16n-9}{2(8n+3)} ; 1 \right). \] (3.4)

In particular, for \( n = 1, 2, 3 \) we find respectively the following results:

\[ e^{-\pi} = 2F_1\left( \frac{i}{2}, -i, \frac{1}{2}, 1 ; 1 \right) \] (3.5)

\[ 2e^{-\pi} = 3F_2\left( \frac{i}{2}, -i, \frac{1}{2}, \frac{9}{2} ; 1 \right) + 2 \times 3F_2\left( \frac{1}{2} + i, \frac{1}{2} - i, \frac{43}{38} ; 1 \right) \] (3.6)

and

\[ 3e^{-\pi} = 3F_2\left( \frac{i}{2}, -i, \frac{1}{2}, \frac{1}{2} ; 1 \right) + 2 \times 3F_2\left( \frac{1}{2} + i, \frac{1}{2} - i, \frac{13}{18} ; 1 \right). \] (3.7)
Corollary 3  In (2.1), if we take \( d_1 = 1/(2(10n - 1)) \) and \( d_2 = -5/2 \) for positive integer \( n \), then we obtain after a little calculation the following result:

\[
n(e^\pi + e^{-\pi}) = 3F_2\left( \frac{i, -i, \frac{2n-1}{2(10n-1)}}{2}; \frac{1}{2}, 1 \right) + 2 \cdot 3F_2\left( \frac{1}{2} + i, \frac{1}{2} - i, -\frac{3}{2}; \frac{1}{2}, 1 \right).
\]

(3.8)

In particular, for \( n = 1, 2, 3 \) we find respectively the following results:

\[
e^\pi + e^{-\pi} = 3F_2\left( \frac{i, -i, \frac{19}{18}}{3}; \frac{1}{2}, 1 \right) + 2 \cdot 3F_2\left( \frac{1}{2} + i, \frac{1}{2} - i, -\frac{3}{2}; \frac{1}{2}, 1 \right),
\]

(3.9)

\[
2(e^\pi + e^{-\pi}) = 3F_2\left( \frac{i, -i, \frac{39}{38}}{3}; \frac{1}{2}, 1 \right) + 2 \cdot 3F_2\left( \frac{1}{2} + i, \frac{1}{2} - i, -\frac{3}{2}; \frac{1}{2}, 1 \right)
\]

(3.10)

and

\[
3(e^\pi + e^{-\pi}) = 3F_2\left( \frac{i, -i, \frac{59}{38}}{3}; \frac{1}{2}, 1 \right) + 2 \cdot 3F_2\left( \frac{1}{2} + i, \frac{1}{2} - i, -\frac{3}{2}; \frac{1}{2}, 1 \right).
\]

(3.11)

Similarly other results can be obtained.

4. The square root of Gelfond’s constant: \( e^{\pi/2} \)

Expressions for the square root of Gelfond’s constant are:

\[ e^{\pi/2} = i^{-i}; \]

\[
e^{\pi/2} = \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \right)^{-2s}, \quad s = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1};
\]

\[
e^{\pi/2} = _2F_1\left( i, -i, \frac{1}{2}; \frac{1}{2} \right) + \sqrt{2} _2F_1\left( \frac{1}{2} + i, \frac{1}{2} - i, \frac{1}{2}; \frac{1}{2} \right)
\]

(4.1)

together with the inverse expression

\[
e^{-\pi/2} = _2F_1\left( i, -i, \frac{1}{2}; \frac{1}{2} \right) - \sqrt{2} _2F_1\left( \frac{1}{2} + i, \frac{1}{2} - i, \frac{1}{2}; \frac{1}{2} \right).
\]

(4.2)

The results in (4.1) and (4.2) can be obtained by evaluating the first hypergeometric function by the second Gauss theorem and the second hypergeometric function by Bailey’s theorem viz.

\[
_2F_1\left( \frac{a}{2}, \frac{b}{2}; \frac{1}{2}(a+b+1); \frac{1}{2} \right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}.
\]

\[
_2F_1\left( \frac{a-1}{2}, \frac{1}{2}; \frac{1}{2}c+\frac{1}{2}; \frac{1}{2} \right) = \frac{\Gamma\left(\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)}.
\]

We now derive the analogue of Theorem 1 by making use of the extension of the second Gauss and Bailey’s theorems applied to \( 3F_2 \) series. These are given by [1]:

\[
_3F_2\left( \frac{a, b, d+1}{2}, \frac{d}{2}(a+b+3), d; \frac{1}{2} \right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}.
\]
When $n$ and $d$ are provided

For Theorem 2

Corollary 4

\[
\begin{aligned}
\text{If in (4.5) we take } &\left[\begin{array}{c}
\frac{1}{2} (a+b-1)-ab/d \\
\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)
\end{array}\right] + \frac{(a+b+1)/d-2}{\Gamma\left(\frac{1}{2}a\right) \Gamma\left(\frac{1}{2}b\right)} \\
\text{and}
\end{aligned}
\]

\[
\begin{aligned}
3F2\left(a, 1-a, d+1 \\
c+1, d 
\end{aligned} : \frac{1}{2}\right) = 2^{-c} \Gamma\left(\frac{1}{2}c\right) \Gamma\left(c+1\right)
\]

\[
\begin{aligned}
\times \left\{\frac{2/d}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}d\right) \Gamma\left(\frac{1}{2}c-\frac{1}{2}d+\frac{1}{2}\right)} + \frac{1-(c/d)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}d+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}c-\frac{1}{2}d+\frac{1}{2}\right)} \right\},
\end{aligned}
\]

provided $d \neq 0, -1, -2, \ldots$. Then we have the following theorem:

**Theorem 2** For $d_1, d_2 \neq 0, -1, -2, \ldots$, the following result holds true:

\[
e^{\pi/2} \left(\frac{1}{10d_1} + \frac{3}{16d_2} + \frac{27}{40}\right) + e^{-\pi/2} \left(\frac{3}{10d_1} - \frac{21}{16d_2} + \frac{11}{40}\right)
\]

\[
= 3F2\left(i, -i, \frac{d_1}{3}, \frac{1}{2} \right) + \sqrt{2} \times 3F2\left(\frac{1}{2} + i, \frac{1}{2}, -i, \frac{d_2}{3}, \frac{1}{2}\right). \quad (4.5)
\]

**Proof.** In the first $3F2$ series use (4.3) and in the second $3F2$ series use (4.4) together with standard properties of the gamma function. □

**Corollary 4** If in (4.5) we take $d_1 = 1/(7n-5)$ and $d_2 = 15/(24n-14)$ for positive integer $n$ then we find

\[
ne^{\pi/2} = 3F2\left(i, -i, \frac{7n-4}{3}, \frac{1}{2} \right) + \sqrt{2} \times 3F2\left(\frac{1}{2} + i, \frac{1}{2}, -i, \frac{24n+1}{24n-14}, \frac{1}{2}\right). \quad (4.6)
\]

When $n = 1$ we recover (4.1). For $n = 2, 3$ we find respectively the following results:

\[
\begin{aligned}
2e^{\pi/2} &= 3F2\left(i, -i, \frac{10}{3}, \frac{1}{2} \right) + \sqrt{2} \times 3F2\left(\frac{1}{2} + i, \frac{1}{2}, -i, \frac{49}{153}, \frac{1}{2}\right) \quad (4.7)
\end{aligned}
\]

and

\[
3e^{\pi/2} = 3F2\left(i, -i, \frac{17}{3}, \frac{1}{2} \right) + \sqrt{2} \times 3F2\left(\frac{1}{2} + i, \frac{1}{2}, -i, \frac{27}{80}, \frac{1}{2}\right). \quad (4.8)
\]

Similarly other results can be obtained.

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