Aldaz–Kounchev–Render Operators and Their Approximation Properties

Ana-Maria Acu, Stefano De Marchi, and Ioan Raşa

Abstract. The approximation properties of the Aldaz–Kounchev–Render (AKR) operators where investigated in several papers. We improve some existing quantitative results concerning these approximation properties. Moreover, we describe classes of functions for which these operators approximate better than the classical Bernstein operators and classes of functions for which Bernstein operators approximate better than AKR operators. The new results, in particular involving monotonic convergence and Voronovskaja type formulas, are then extended to the bivariate case on the square $[0,1]^2$ and compared with other existing results. Several numerical examples, illustrating the relevance and supporting the theoretical findings, are presented.

Mathematics Subject Classification. 41A36, 41A15, 65D30.

Keywords. Aldaz–Kounchev–Render operators, Bernstein operator, convex functions, tensor product.

1. Introduction

The functions fixed by the positive linear operators $L_n$, $n \geq 1$, encode important information about the operators. Algebraically, because they are eigenfunctions associated with the eigenvalue 1. Analytically, because they have impact on the approximation properties of $L_n$: in particular, they determine to a large extent the structure of the Voronovskaja operator associated with the sequence $(L_n)_{n \geq 1}$. And, very important for CAGD, the shape preserving properties of the operators are intimately related with the fixed functions (see [7]).
Generally speaking, given a classical sequence \((L_n)_{n \geq 1}\) the functions fixed by \(L_n\) are well-known. In the last decades an inverse problem was raised: given some functions, construct a sequence of positive linear operators \((L_n)_{n \geq 1}\) (an approximation process) such that each \(L_n\) fixes the given functions. For example, the operators constructed by King \[22\] on \(C[0,1]\) preserve the functions \(1\) and \(x^2\). Operators on \(C[0,1]\) preserving \(1\) and a given function \(\tau\) were constructed in \[15,19\]. There exists a rich literature devoted to this topic (see e.g., \[1,4,6,20,26\] and the references therein) Aldaz et al. \[7\] introduced a sequence of Bernstein type polynomial operators on \(C[0,1]\) which preserve \(1\) and \(x^j\), \(j \in \mathbb{N}\) being given. Several papers were subsequently devoted to their study. In particular, the Voronovskaja formula conjectured in \[16\] was proved in \[13\] (see also \[3,17,18\]).

The aim of this paper is twofold. On one hand, we introduce the bivariate AKR operators on \(C([0,1]^2)\) and investigate some of their approximation properties. On the other hand, we compare, in the univariate case and also in the bivariate case, the properties of Bernstein and AKR operators. In particular, we describe classes of functions on which AKR operators approximate better than Bernstein operators, and classes of functions on which the approximation given by Bernstein operators is better that the approximation provided by AKR operators.

We recall a definition which is needed in order to introduce AKR operators on \(C([0,1]^2)\).

**Definition 1.1.** Let \(I, J\) be arbitrary compact intervals of the real axis, and let \(L : C(I) \to C(I)\), \(M : C(J) \to C(J)\) be discretely defined operators

\[
L f(x) = \sum_{i=0}^{n} f(x_i)p_i(x), \quad f \in C(I),
\]

and

\[
M f(y) = \sum_{k=0}^{m} f(y_k)q_k(y), \quad f \in C(J),
\]

where \(x_i \in I\), \(y_k \in J\) are mutually distinct, and \(p_i \in C(I)\), \(q_j \in C(J)\).

Let \((x,y) \in I \times J\). The parametric extensions of \(L\) and \(M\) to \(C(I \times J)\) are given by

\[
xL f(x,y) = \sum_{i=0}^{n} f(x_i,y)p_i(x)
\]

and

\[
yM f(x,y) = \sum_{k=0}^{m} f(x,y_k)q_k(y).
\]
The tensor product of $L$ and $M$ is given by

$$Tf(x, y) := (xL \circ_y M) f(x, y) = \sum_{i=0}^{n} \sum_{k=0}^{m} f(x_i, y_k) p_i(x) q_k(y), \ f \in C(I \times J).$$

Section 2 contains results concerning AKR operators on $C[0, 1]$. In particular, we investigate their monotonic convergence in relation with the Voronovskaja formula. A class of functions for which they approximate better than Bernstein operators is described, as well as a class of functions for which Bernstein operators provide a better approximation. Section 3 is devoted to Bernstein operators on $C([0, 1]^2)$. Proposition 3.1 improves a result concerning the approximation by these operators. The corresponding result involving AKR operators is contained in Proposition 4.1. In Sect. 4 we introduce AKR operators on $C([0, 1]^2)$ as tensor products of univariate AKR operators. We formulate a conjecture concerning their Voronovskaja formula and extend the results from univariate case to the bivariate case.

The analytic description is accompanied by numerical and graphical experiments. Some implementation details can be found in Sect. 4.1.

Throughout the paper we use the notation $e_i(x) = x^i, \ i = 0, 1, \ldots$ and $\| \cdot \|$ stands for the supremum norm.

2. Bernstein-Type Operator of Aldaz, Kounchev and Render

Let $B_n : C[0, 1] \to C[0, 1]$ be the classical Bernstein operator defined as

$$B_n f(x) = \sum_{i=0}^{n} f \left( \frac{i}{n} \right) p_{n,i}(x),$$

where $p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \ x \in [0, 1]$.

Starting from these operators, during recent years some modifications have been considered. One of them was introduced by King [22] in order to obtain linear positive operators which preserve the functions $e_0$ and $e_2$. A slight extension of King operators was considered by Cárdenas-Morales et al. in [14] where a sequence of operators $B_{n,\alpha}$ that preserve $e_0$ and $e_2 + \alpha e_1, \ \alpha \in [0, +\infty)$ was introduced. Using a continuous strictly increasing function $\tau$ defined on $[0, 1]$ with $\tau(0) = 0$ and $\tau(1) = 1$, $\tau'(x) > 0, \ x \in [0, 1]$, Cárdenas-Morales et al. (see [15,19]) introduced a modification of the Bernstein operator which preserves the functions $e_0$ and $\tau$.

For $j > 1, \ j \in \mathbb{N}$ fixed and $n \geq j$, Aldaz, Kounchev and Render [7] introduced a polynomial operator $B_{n,j} : C[0, 1] \to C[0, 1]$ that fixes $e_0$ and $e_j$. The operator is explicitly given by

$$B_{n,j} f(x) = \sum_{k=0}^{n} f \left( t_{n,k}^j \right) p_{n,k}(x),$$
Figure 1. Nodes of $B_{n,j}$ and $B_n$

where

$$t_{n,k}^j = \left( \frac{k(k-1)\ldots(k-j+1)}{n(n-1)\ldots(n-j+1)} \right)^{1/j}.$$  

For $n = 10$ and $j = 2$ in Fig. 1 the nodes of AKR operator and Bernstein operator, respectively, are illustrated graphically.

Next we determine a class of functions for which the approximation by the AKR operators is better than the approximation by the Bernstein operators. In order to describe this class of functions, in the sequel we will recall some notions from the literature (see [7]).

We say that $(f_0,f_1)$ is a Haar pair if $f_0$ is strictly positive and $f_1/f_0$ strictly increasing.

**Definition 2.1** (see [21, p. 280]). A function $f: [a, b] \to \mathbb{R}$ is called $(f_0,f_1)$-convex if for all $x_0,x_1,x_2 \in [a,b]$ with $x_0 < x_1 < x_2$, the determinant

$$\det \begin{pmatrix} f_0(x_0) & f_0(x_1) & f_0(x_2) \\ f_1(x_0) & f_1(x_1) & f_1(x_2) \\ f(x_0) & f(x_1) & f(x_2) \end{pmatrix}$$

is non-negative.

We will use the following characterization of $(f_0,f_1)$-convexity given in [12, Theorem 5]:

**Theorem 2.1** [12]. Let $(f_0,f_1)$ be a Haar pair and let $I := (f_1/f_0)([a,b])$. Then $f \in C[a,b]$ is $(f_0,f_1)$-convex if and only if $(f/f_0) \circ (f_1/f_0)^{-1} \in C(I)$ is convex in the standard sense.

Let $B_n$ be a Bernstein type operator defined as

$$B_n f(x) = \sum_{k=0}^{n} f(t_{n,k}) \alpha_{n,k} p_{n,k}(x), \ f \in C[a,b], x \in [a,b],$$
where $\alpha_{n,k} > 0$, $t_{n,k} \in [a, b]$, $k = 1, \ldots, n$.

The next result generalizes the well known inequality verified by the classical Bernstein operator

$$B_n f \geq f$$

for all convex functions $f \in C[0, 1]$.

**Theorem 2.2** [7, Th.15]. Assume that for some $n \geq 1$, there is a Bernstein type operator $B_n$ fixing $f_0$ and $f_1$. Then for every $(f_0, f_1)$-convex function $f \in C[a, b]$ we have $B_n f \geq f$.

In what follows let $j \geq 2$ and

$K_j^{[1]} := \{ f \in C[0, 1] \mid f \text{ is increasing}, g(x) := f(x^{1/j}) \text{ is convex on } [0, 1] \}$.

**Proposition 2.1.** Let $f \in K_j^{[1]}$. Then

$$f \leq B_{n,j} f \leq B_n f.$$  \hspace{1cm} (2.1)

**Proof.** It is easy to verify that

$$t_{n,k}^j \leq \frac{k}{n}, \quad k = 0, \ldots, n, \quad j \geq 2.$$

Since $f$ is increasing, we get $f(t_{n,k}^j) \leq f \left( \frac{k}{n} \right)$, therefore

$$B_{n,j} f \leq B_n f.$$  \hspace{1cm} (2.2)

We consider $f_0 = e_0$ and $f_1 = e_j$ in Theorem 2.1. Then $(f_0, f_1)$ is a Haar pair and $(f/f_0) \circ (f_1/f_0)^{-1}(x) = f(x^{1/j})$ is convex. Applying Theorem 2.2 (for $B_n = B_{n,j}$) we get $B_{n,j} f \geq f$. Combined with (2.2) this proves the inequality (2.1). \hfill $\square$

Let $\Omega$ be the set of the functions $\omega$ such that

1. $\omega \in C[0, 1] \cap C^1(0, 1)$,
2. $\omega(x) \geq 0$, $x \in [0, 1]$,
3. $\omega'(x) \geq 0$, $x \in (0, 1]$,
4. there exists $\lim_{x \to 0} x^{j-1} \omega'(x) \in \mathbb{R}$.

**Theorem 2.3.** The following statements are equivalent:

1. $f \in C^2[0, 1] \cap K_j^{[1]}$,
2. $f \in C^2[0, 1]$, $f'(x) \geq 0$, $xf''(x) - (j - 1)f'(x) \geq 0$, $x \in [0, 1]$,
3. There exists $\varphi \in \Omega$ such that $f(x) = f(0) + \int_0^x t^{j-1} \varphi(t) dt$, $x \in [0, 1]$.

**Proof.** To prove that (a) and (b) are equivalent let $f \in C^2[0, 1]$ and $g(x) := f(x^{1/j})$, $x \in [0, 1]$. Then $f \in K_j^{[1]}$ if and only if $f'(x) \geq 0$ and $g''(x) \geq 0$, $x \in (0, 1]$. But $g''(x) \geq 0$ for $0 < x \leq 1$ is equivalent to $xf''(x) - (j - 1)f'(x) \geq 0$, $x \in [0, 1]$. Therefore, we get $f \in K_j^{[1]}$. The equivalence (a) $(\Rightarrow)$ (c) follows from (a) and (b). To prove (c) $(\Rightarrow)$ (a) let $\varphi \in \Omega$ such that $f(x) = f(0) + \int_0^x t^{j-1} \varphi(t) dt$, $x \in [0, 1]$. Then $f(x) = f(0) + \int_0^x t^{j-1} \varphi(t) dt$, $x \in [0, 1]$. But $g''(x) \geq 0$ for $0 < x \leq 1$ is equivalent to $xf''(x) - (j - 1)f'(x) \geq 0$, $x \in [0, 1]$. Thus, we get $f \in K_j^{[1]}$. \hfill $\square$
In order to prove that (b) implies (c) let
\[ f \in C^2[0, 1], \quad f'(x) \geq 0, \quad xf''(x) - (j - 1)f'(x) \geq 0, \quad x \in [0, 1]. \]
Set \( \varphi(x) = \frac{f'(x)}{x^{j-1}}, \) \( 0 < x \leq 1. \) Then for \( x \in (0, 1] \) we have
\[ \varphi'(x) = \frac{xf''(x) - (j - 1)f'(x)}{x^j} \geq 0. \]
Moreover, \( \varphi \geq 0 \) and \( \varphi \) is increasing on \( (0, 1] \). This implies the existence of \( l := \lim_{x \to 0} \varphi(x) \in [0, \infty). \)

Define
\[
\varphi(x) := \begin{cases} 
\frac{f'(x)}{x^{j-1}}, & x \in (0, 1], \\
l, & x = 0.
\end{cases}
\]
Then \( \varphi \in C[0, 1] \cap C^1(0, 1]. \)

From (b) we see that \( f'(0) = 0. \) Now \( \lim_{x \to 0} \frac{f'(x)}{x} = f''(0) \) and consequently
\[ \lim_{x \to 0} x^{j-1}\varphi'(x) = (2 - j)f''(0) \in \mathbb{R}. \]

From \( f'(t) = t^{j-1}\varphi(t), \) \( t \in [0, 1], \) we get \( f(x) = f(0) + \int_0^x t^{j-1}\varphi(t)dt, \) \( x \in [0, 1], \) and (c) is proved.

It remains to prove that (c) implies (b). Let \( f(x) = f(0) + \int_0^x t^{j-1}\varphi(t)dt, \) \( x \in [0, 1], \) with \( \varphi \in \Omega. \) Then \( f'(x) = x^{j-1}\varphi(x), \) \( x \in [0, 1]. \) The function \( f' \) is continuous on \([0, 1]\) and
\[ f''(x) = (j - 1)x^{j-2}\varphi(x) + x^{j-1}\varphi'(x) \text{ on } (0, 1]. \] (2.3)

Using (2.3) and (iv) we infer that there exists \( f''(0) := \lim_{x \to 0} f''(x) \in \mathbb{R}. \)

Therefore \( f \in C^2[0, 1]. \) From \( f'(x) = x^{j-1}\varphi(x) \) we see that \( f'(x) \geq 0, \) \( x \in [0, 1]. \)

Using (2.3) we get
\[ xf''(x) - (j - 1)f'(x) \geq 0, \quad x \in (0, 1]. \]
By continuity this is true also for \( x = 0, \) and now (b) is proved. \( \square \)

**Theorem 2.4.** Let \( f \in C^2[0, 1]. \) If
\[ f'(x) \geq 0 \quad \text{and} \quad xf''(x) - (j - 1)f'(x) \geq 0, \quad 0 \leq x \leq 1, \] (2.4)
then
\[ f \leq B_{n,j}f \leq B_nf. \]

**Proof.** According to (2.4) and Theorem 2.3, \( f \in K_j^{[1]} \) and now is sufficient to apply Proposition 4.1. \( \square \)
Theorem 2.5. For \( f \in C^2[0,1] \) the statements
\[
x f''(x) - (j - 1)f'(x) \geq 0, \quad x \in [0,1],
\]
and
\[
B_{n,j} f(x) \geq f(x), \quad x \in [0,1], \quad \text{for all } n \in \mathbb{N},
\]
are equivalent.

Proof. Recall the Voronovskaja formula for \( B_{n,j} \). If \( x \in (0,1) \) and there exists the finite \( f''(x) \), then
\[
\lim_{n \to \infty} n (B_{n,j} f(x) - f(x)) = \frac{1-x}{2} [xf''(x) - (j - 1)f'(x)]. \tag{2.5}
\]
Combining (2.5) and Theorem 2.4 proves the theorem. \( \square \)

Corollary 2.1. Let \( \varphi \in C^1[0,1] \) such that \( \varphi(x) \geq 0, \varphi'(x) \geq 0, \quad x \in [0,1] \), and let \( f(x) = \int_0^x t^{j-1} \varphi(t)dt, \quad x \in [0,1] \). Then
\[
f \leq B_{n,j} f \leq B_n f.
\]

Proof. Let us remark that \( \varphi \in \Omega \). Now Theorem 2.3 shows that \( f \in K_j^1 \) and an application of Proposition 4.1 concludes the proof. \( \square \)

Example 2.1. Let \( \varphi(t) = \sin \frac{\pi t}{2}, \quad t \in [0,1] \). Using Corollary 2.1 with \( j = 2 \), we determine the function \( f(x) = \int_0^x t \varphi(t)dt = -2 + \frac{\pi x}{2} \cos \frac{\pi x}{2} + \frac{4}{\pi^2} \sin \frac{\pi x}{2} \). In Fig. 2 it can be seen that for this function the approximation by the operator \( B_{5,2} \) is better than the approximation by the Bernstein operator \( B_5 \). Moreover, if we denote
\[
E_{AKR}(f; n, j) = \|B_{n,j}f - f\| \quad \text{and} \quad E_B(f; n) = \|B_n f - f\|,
\]
the approximation error by AKR operator and the Bernstein operator, in Table 1 we present \( E_{AKR}f \) and \( E_Bf \) for certain values of \( n \).

Example 2.2. Take \( \varphi(t) = e^t, \quad t \in [0,1] \). For \( j = 5 \) Corollary 2.1 gives the function \( f(x) = \int_0^x t^4 e^t dt = -24 + (x^4 - 4x^3 + 12x^2 - 24x + 24)e^x \). Figure 3 shows that AKR operator approximates the function better than the Bernstein operator.

In the next result we mention a class of functions for which the approximation by the Bernstein operators is better than the approximation by AKR operators.

Proposition 2.2. Let \( f \in C[0,1] \) be a decreasing and convex function. Then
\[
B_{n,j} f \geq B_n f \geq f. \tag{2.6}
\]
Figure 2. Plots of $f$, $B_{5,2}f$ and $B_5f$

Table 1. Error of approximation

| n | 5  | 10 | 20  | 30 | 40 | 50   | 60   |
|---|----|----|-----|----|----|------|------|
| $E_{Bf}$ | 0.0140 | 0.0070 | 0.0035 | 0.0023 | 0.0017 | 0.0014 | 0.0012 |
| $E_{AKRf}$ | 0.0309 | 0.0159 | 0.0081 | 0.0054 | 0.0041 | 0.0033 | 0.0027 |

Figure 3. Graph of $f$, $B_{5,5}f$ and $B_5f$
Proof. Since the function $f$ is decreasing and $t^j_{n,k} \leq \frac{k}{n}$, $k = 0, \ldots, n$, $j \geq 2$, we get $f(t^j_{n,k}) \geq f\left(\frac{k}{n}\right)$, therefore $B_{n,j}f \geq B_nf$. But, it is well known that $B_nf \geq f$ for each convex function $f$. This leads to the inequalities (2.6). \qed

Example 2.3. Let $f(x) = \cos^2 \left(\frac{\pi}{4}(x + 1)\right)$, $x \in [0, 1]$. Note that $f$ is a decreasing and convex function and $f \in C[0, 1]$. In Fig. 4 are represented graphically the functions $f$, $B_{n,j}f$ and $B_nf$ for $n = 10$ and $j = 2$. Note that in this case the approximation by the Bernstein operator is better than the approximation by AKR operator. If we consider the notation introduced in Example 2.1 for the approximation error we have

$$E_{AKR}(f; 10, 2) = 0.0450, \; E_B(f; 10) = 0.0118.$$ 

3. Application to Bernstein Operators

For $f \in C([0, 1]^2)$, the tensor product of $B_n, B_m$ is given by

$$B_{n,m}f(x, y) := (xB_n \circ yB_m)f(x, y) = \sum_{i=0}^{n} \sum_{k=0}^{m} f\left(\frac{i}{n}, \frac{k}{m}\right) p_{n,i}(x)p_{m,k}(y). \quad (3.1)$$

Denote by $C^{p,q}([0, 1]^2)$ the space of all real valued functions defined on $[0, 1]^2$ and having continuous partial derivatives of order $p$ (with respect to the first variable), respectively $q$ (with respect to the second variable).
Let \( f \in C^{2,2}([0,1]^2) \). The following approximation formula for the bivariate Bernstein operators \( B_{n,m} \) was obtained in [2]

\[
|f(x,y) - B_{n,m}f(x,y)| \leq \frac{3}{2} \left[ \frac{x(1-x)}{n} \|f^{(2,0)}\| + \frac{y(1-y)}{m} \|f^{(0,2)}\| \right]. \tag{3.2}
\]

Motivated by the well-known results for the classical Bernstein operators, namely

\[
|f(x) - B_{n}f(x)| \leq \frac{1}{2} \|f''\| \frac{x(1-x)}{n}, \tag{3.3}
\]

in [2] the constant \( 3/2 \) was improved and the following approximation formula was obtained (see also [11, Theorem 2.3])

\[
|f(x,y) - B_{n,m}f(x,y)| \leq \frac{x(1-x)}{2n} \|f^{(2,0)}\| + \frac{y(1-y)}{2m} \|f^{(0,2)}\| + \frac{x(1-x)y(1-y)}{4nm} \|f^{(2,2)}\|. \tag{3.4}
\]

In the following we obtain a new result that improves (3.2) and (3.4).

**Proposition 3.1.** Let \( f \in C^{2,2}([0,1]^2) \). Then

\[
|f(x,y) - B_{n,m}f(x,y)| \leq \frac{1}{2} \left[ \frac{x(1-x)}{n} \|f^{(2,0)}\| + \frac{y(1-y)}{m} \|f^{(0,2)}\| \right].
\]

**Proof.** Using the estimate (3.3) we get

\[
|f(x,y) - B_{n,m}f(x,y)|
\leq |f(x,y) - xB_{n}f(x,y)| + |xB_{n}f(x,y) - (xB_{n} \circ yB_{m})f(x,y)|
\leq |f(x,y) - xB_{n}f(x,y)| + xB_{n} |f - yB_{m}f| (x,y)
\leq \frac{x(1-x)}{2n} \|f^{(2,0)}\| + xB_{n} \left( \frac{y(1-y)}{2m} \|f^{(0,2)}\| \right)
\leq \frac{1}{2} \left[ \frac{x(1-x)}{n} \|f^{(2,0)}\| + \frac{y(1-y)}{m} \|f^{(0,2)}\| \right]. \quad \square
\]

4. Bivariate Bernstein-Type Operator of Aldaz, Kounchev and Render

Let \( B_{n,j}, B_{m,j} : C(0,1) \to C(0,1) \) be the AKR operators and \((x,y) \in [0,1]^2\). Then, for \( f \in C([0,1]^2) \), the tensor product of AKR operators is given by

\[
B_{n,m,j}f(x,y) = \sum_{i=0}^{n} \sum_{k=0}^{m} \sum_{j} f(t_{n,i}^j, t_{m,k}^j) p_{n,i}(x)p_{m,k}(y), \, (x,y) \in [0,1]^2. \tag{4.1}
\]

Since \( B_{n,j} \) preserves the functions \( 1, x^j \) and \( B_{m,j} \) preserves the functions \( 1, y^j \) defined on \([0,1]\), it follows immediately that \( B_{n,m,j} \) preserves the functions \( 1, x^j, y^j \) defined on \([0,1]^2\).
The nodes of the operators $B_{10,10,2}$ and $B_{10,10}$, respectively, are presented graphically in Fig. 5.

**Proposition 4.1.** Let $f \in C^{2,2}([0, 1]^2)$. Then

\[
|f(x, y) - B_{n,m,j}f(x, y)| \leq \frac{x(1-x)}{2n} \|f^{(2,0)}\| + \frac{y(1-y)}{2m} \|f^{(0,2)}\| + \frac{j-1}{n} \|f^{(1,0)}\| + \frac{j-1}{m} \|f^{(0,1)}\|.
\]

**Proof.** In [5] the following estimate of the difference between the AKR and Bernstein operators was obtained

\[
|B_n f(x) - B_{n,j} f(x)| \leq \omega_1 \left( f, \frac{j-1}{n} \right) \leq \frac{j-1}{n} \|f'\|, \quad f \in C^1[0, 1]. \tag{4.2}
\]

From (3.3) and (4.2) we obtain

\[
|f(x) - B_{n,j} f(x)| \leq |f(x) - B_n f(x)| + |B_n f(x) - B_{n,j} f(x)|
\]

\[
\leq \frac{x(1-x)}{2n} \|f''\| + \frac{j-1}{n} \|f'\|. \tag{4.3}
\]

Using (4.3) it follows that

\[
|f(x, y) - B_{n,m,j} f(x, y)| \leq |f(x, y) - x B_{n,j} f(x, y)|
\]

\[
+ |x B_{n,j} f(x, y) - x B_{n,j} \circ y B_{m,j} f(x, y)|
\]

\[
\leq |f(x, y) - x B_{n,j} f(x, y)| + x B_{n,j} (|f - y B_{m,j} f|)(x, y)
\]

\[
\leq \frac{x(1-x)}{2n} \|f^{(2,0)}\| + \frac{j-1}{n} \|f^{(1,0)}\| + \frac{y(1-y)}{2m} \|f^{(0,2)}\| + \frac{j-1}{m} \|f^{(0,1)}\|.
\]

This concludes the proof. \[\square\]
Define
\[ K_j^{[2]} := \left\{ f \in C([0,1]^2) \mid f(\cdot, y) \in K_j^{[1]}, f(x, \cdot) \in K_j^{[1]}, x, y \in [0,1] \right\}. \]

**Theorem 4.1.** If \( f \in K_j^{[2]} \), then
\[ f \leq B_{n,m,j}f \leq B_{n,m}f. \tag{4.4} \]

**Proof.** Let \( f \in K_j^{[2]} \). Then \( f(\cdot, y) \in K_j^{[1]}, y \in [0,1] \). Using Proposition 4.1 we obtain
\[ f(x, y) \leq xB_{n,j}f(x, y), \quad x, y \in [0,1]. \tag{4.5} \]

From (4.5) it follows that
\[ yB_{m,j}f(x, y) \leq B_{n,m,j}f(x, y). \tag{4.6} \]

On the other hand \( f(x, \cdot) \in K_j^{[1]} \) and the same Proposition 4.1 tells us that
\[ f(x, y) \leq yB_{m,j}f(x, y). \tag{4.7} \]

Combining (4.6) and (4.7) we get
\[ f(x, y) \leq B_{n,m,j}f(x, y), \quad x, y \in [0,1]. \tag{4.8} \]

Remember that \( f(x, \cdot) \) and \( f(\cdot, y) \) are increasing functions for all \( x, y \in [0,1] \).

Moreover,
\[ t_{n,i}^j \leq \frac{i}{n}, \quad t_{m,k}^j \leq \frac{k}{m}. \]

It follows that
\[ f\left(t_{n,i}^j, t_{m,k}^j\right) \leq f\left(\frac{i}{n}, \frac{k}{m}\right), \tag{4.9} \]

for all \( i = 0, \ldots, n, k = 0, \ldots, m \).

From (3.1), (4.1) and (4.9) we get
\[ B_{n,m,j}f(x, y) \leq B_{n,m}f(x, y), \quad x, y \in [0,1]. \tag{4.10} \]

Now (4.4) is a consequence of (4.8) and (4.10). \( \square \)

In relation with Voronovskaja formula for \( B_{n,j} \) (see Remark 2.5) we state here a conjecture about the Voronovskaja formula for \( B_{n,n,j} \).

**Conjecture 4.1.** Suppose that \((x, y) \notin \{(0,0), (0,1), (1,0)\}, \, f \in C([0,1]^2) \) and the partial derivatives \( f^{(2,0)}, f^{(0,2)} \) exist and are finite at \((x, y)\). Then,
\[ \lim_{n \to \infty} n(B_{n,n,j}f(x, y) - f(x, y)) = Uf(x, y) + Vf(x, y), \]

where
\[ Uf(x, y) := \frac{x(1-x)}{2}f^{(2,0)}(x, y) - \frac{j - 1}{2}(1-x)f^{(1,0)}(x, y) \]

and
\[ Vf(x, y) := \frac{y(1-y)}{2}f^{(0,2)}(x, y) - \frac{j - 1}{2}(1-y)f^{(0,1)}(x, y). \]
Hopefully this Voronovskaja type formula could be proved by using either the classical approach (based on the study of moments), or some newly invented techniques (see, e.g., [23,24,27] and the references therein). We will return to this problem in a forthcoming paper, where we will compare our results with other existing ones.

**Remark 4.1.** Let \( f \in C^{2,2}([0,1]^2) \). According to Theorem 2.3 if \( Uf \geq 0 \) and \( Vf \geq 0 \), then \( f \in K_j^{[2]} \) and consequently Theorem 4.1 shows that \( B_{n,n,j} f \geq f, \ n \geq 1 \).

If Conjecture 4.1 is valid and \( B_{n,n,j} f \geq f, \ n \geq 1 \), then \( Uf + Vf \geq 0 \). Is it true that if \( Uf + Vf \geq 0 \), then \( B_{n,n,j} f \geq f, \ n \geq 1 \)?

As functions from \( K_j^{[1]} \), \( f(\cdot,y) \) and \( f(x,\cdot) \) are characterized in Theorem 2.3 with \( \varphi \in C^1[0,1], \varphi(x) \geq 0, \varphi'(x) \geq 0, x \in [0,1] \). Suppose that \( f \in C^{2,2}([0,1]^2) \). Using convenient notation we have

\[
f(x, y) = f(0, y) + \int_0^x t^{j-1} \varphi(t, y) dt \tag{4.11}
\]

and

\[
f(x, y) = f(x, 0) + \int_0^y s^{j-1} \psi(x, s) ds, \tag{4.12}
\]

where

\[
\varphi \in C^{1,1}([0,1]^2), \varphi \geq 0, \varphi^{(1,0)} \geq 0, \tag{4.13}
\]

\[
\psi \in C^{1,1}([0,1]^2), \psi \geq 0, \psi^{(0,1)} \geq 0. \tag{4.14}
\]

Due to (4.11) and (4.12) we need the compatibility condition

\[
f(x, y) = f(0, y) + \int_0^x t^{j-1} \varphi(t, y) dt = f(x, 0) + \int_0^y s^{j-1} \psi(x, s) ds. \tag{4.15}
\]

Taking in (4.15) the derivative with respect to \( x \) and then with respect to \( y \) we get

\[
t^{j-1} \varphi^{(0,1)}(t, s) = s^{j-1} \psi^{(1,0)}(t, s). \tag{4.16}
\]

Conversely, we will show that if (4.16) is fulfilled then (4.11) and (4.12) give us a function \( f(x, y) \) for which the compatibility condition (4.15) is fulfilled.

In (4.16) take the integral with respect to \( t \) on the interval \([0, x]\) and then the integral with respect to \( s \) on the interval \([0, y]\). This yields

\[
\int_0^x t^{j-1} \varphi(t, y) dt + \int_0^y s^{j-1} \psi(0, s) ds = \int_0^y s^{j-1} \psi(x, s) ds + \int_0^x t^{j-1} \varphi(t, 0) dt =: f(x, y). \tag{4.17}
\]

It is easy to check that the above function \( f(x, y) \) satisfies (4.15). To resume, we have proved
Theorem 4.2. Let $\phi$ and $\psi$ satisfying (4.13), (4.14) and (4.16). The function $f$ given by (4.17) is in $K_{j}^{[2]}$ and satisfies

$$ f \leq B_{n,m,j}f \leq B_{n,m}f. $$

Example 4.1. Let $\phi(x,y) = \psi(x,y) = h(x^j + y^j)$, $h \in C^1[0,2]$, $h \geq 0$, $h' \geq 0$. We have $x^{j^{-1} \phi(0,1)}(x,y) = y^{j^{-1} \psi(1,0)}(x,y)$, and conditions (4.13), (4.14), (4.16) are verified.

The function $f$ given by (4.17) is in this case

$$ f(x,y) = \frac{1}{j} \int_0^{x^j + y^j} h(u)du. $$

Example 4.2. Let $\phi(x,y) = 2y^2e^{x^2y^2}$ and $\psi(x,y) = 2x^2e^{x^2y^2}$. For $j = 2$ the conditions (4.13), (4.14), (4.16) are verified. Therefore, from (4.17) we get

$$ f(x,y) = \int_0^x t^{j^{-1}} \phi(t,y)dt = e^{x^2y^2} - 1. $$

For $n = 3$, $m = 4$ and $j = 2$ Figs. 6 and 7 illustrate the inequalities $f \leq B_{3,4,2}f \leq B_{3,4}$ (see Theorem 4.2).

For $j = 2$ in Table 2 we present the error of approximation for AKR operator ($E_{AKR}f$) and Bernstein operator ($E_Bf$) for certain values of $n$ and $m$. Note that in this case the approximation by AKR operator is better than the approximation by the Bernstein operator.

Next we present two methods in order to construct a function $f \in C([0,1]^2)$ for which the AKR operator acts better than the Bernstein operator.
I. Let $\omega(u, v)$ be such that $\omega(\cdot, v_0)$ and $\omega(u_0, \cdot)$ are increasing and convex, for all $u_0, v_0 \in [0, 1]$. For $x, y \in [0, 1]$ let $a(x)$ and $b(y)$ be increasing and convex functions. Define $g(x, y) = \omega(a(x), b(y))$. It can be verified immediately that $g(\cdot, y_0)$ and $g(x_0, \cdot)$ are increasing and convex, for all $x_0, y_0 \in [0, 1]$. Then, we can consider $f(x, y) := g(x^j, y^j)$ for which AKR operator acts better than Bernstein operator.

**Example 4.3.** Let $\omega(u, v) = \tan\left(\frac{\pi}{4}uv\right)$, $0 \leq u, v \leq 1$, $a(x) = 2^x - 1$, $b(y) = y^3$. Then, for $f(x, y) = \tan\left(\frac{\pi}{4}(2^x - 1)y^3\right)$ the approximation by AKR operator is better than the approximation by the Bernstein operator.

For $n = 3$, $m = 4$ and $j = 2$ Figs. 8 and 9 illustrate the inequalities $f \leq B_{3,4,2}f \leq B_{3,4}$.

II. We want to dispose of two functions $\varphi$ and $\psi$ satisfying conditions (4.13), (4.14) and (4.16). Choose $\varphi \in C^{1,2}([0, 1]^2)$ such that

$$\varphi \geq 0, \varphi^{(1,0)} \geq 0, \varphi^{(0,1)} \geq 0, y\varphi^{(0,2)} - (j - 1)\varphi^{(0,1)} \geq 0.$$

**Table 2. Error of approximation**

| $n = m$ | 10  | 20  | 30  | 40  | 50  | 60  |
|--------|-----|-----|-----|-----|-----|-----|
| $E_Bf$ | 0.1057 | 0.0516 | 0.0342 | 0.0255 | 0.0204 | 0.0169 |
| $E_{AKR}f$ | 0.0449 | 0.0215 | 0.0142 | 0.0106 | 0.0084 | 0.0070 |
Moreover, assume that the function
\[ \psi(x, y) := y^{1-j} \int_0^x t^{j-1} \varphi^{(0,1)}(t, y) dt \] (4.19)
is in \( C^{1,1}([0,1]^2) \). It is easy to verify that the conditions (4.13), (4.14) and (4.16) are indeed satisfied.
As far as conditions (4.18) are concerned we can start with a function \( \tau \in C^{1,1}( [0,1]^2) \) subject to the conditions
\[
\tau \geq 0, \quad \tau^{(1,0)} \geq 0, \quad \tau^{(0,1)} \geq 0,
\] (4.20)
and then construct
\[
\varphi(x,y) = \int_0^y s^{j-1} \tau(x,s) ds.
\] (4.21)
It will satisfy (4.18). From (4.19) we get
\[
\psi(x,y) = \int_0^x t^{j-1} \tau(t,y) dt.
\] (4.22)

**Conclusion 4.1.** Starting with \( \tau \) satisfying (4.20), the Eqs. (4.21), (4.22), (4.17) yield the function
\[
f(x,y) = \int_0^x \int_0^y t^{j-1}s^{j-1} \tau(t,s) dt ds,
\] (4.23)
which is approximated by AKR operator better than by Bernstein operator, in the sense that
\[
f \leq B_{n,m,j} f \leq B_{n,m} f.
\] (4.24)

In Example 4.2, \( \tau(x,y) = 4(1 + x^2 y^2)e^{x^2 y^2} \).

**Example 4.4.** Let \( \tau(t,s) = \sin \frac{\pi (t+s)}{4} \). The function \( f \) given by (4.23) is
\[
f(x,y) = \frac{1}{\pi^4} \left[ -64\pi(y+x) \cos(\pi(y+x)/4) + (-16\pi^2 xy + 256) \sin(\pi(y+x)/4) \\
+ 64\pi x \cos(\pi x/4) + 64\pi y \cos(\pi y/4) - 256 \sin(\pi x/4) - 256 \sin(\pi y/4) \right].
\]
Figs. 10 and 11 show that \( f \leq B_{4,4,2} f \leq B_{4,4} f \). This is an illustration of the inequalities (4.24).

4.1. On Computing with Bernstein and AKR Operators

Here we briefly describe some implementation details, which are useful for people interested in reproducing the results of the paper and/or wish to get more computational insights. In GitHub at the link

https://github.com/demarchi17/Bernstein-and-AKR-Operators

there is the Matlab script AldazBernstein1d2d.m that allows to construct both operators on \( I = [0,1] \) and in the unit square \( Q = [0,1] \times [0,1] \). In the interval we have considered the function \( f(x) = -\frac{2}{\pi} x \cos \frac{\pi x}{2} + \frac{4}{\pi^2} \sin \frac{\pi x}{2} \) (see Example 2.1) while in the unit square the 3 functions \( f(x,y) \) of Examples 4.2, 4.3 and 4.4.

On input, the user is asked to provide the higher degree \( N \) up to which to approximate the functions and the AKR index \( j \). Then, a loop on the degree \( n \) gives the results presented in Figs. 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11 (see the corresponding Examples for details). More in details, the script allows:
• To compute Bernstein and AKR points in $I$ and $Q$;
• To construct Bernstein $B_n$ and AKR operators $B_{n,m,j}$ and evaluate them on a suitable and larger grid, say $X_M$ where $M = (n + 1)^2$ (constructed via meshgrid on the unit square);
• In $[0, 1]$ to compute the relative 2-norm error

$$E_r := \frac{\|f - B\|_2}{\|f\|_2}$$
where $B$ is one of the operators and $f$ one of the functions previously considered;

- In $[0,1]^2$ to compute the errors $B_{n,m,j}f - f$ and $B_{n,m}f - B_{n,m,j}f$
- To make the plots.

The script available at GitHub is downloadable and the interested readers can play and tell us if there are improvements and bugs.

5. Conclusions and Further Work

The AKR operators have been subject of intense research. They are important in Approximation Theory, where rate of convergence and Voronovskaja formula play a significant role. Their shape preserving properties are useful in CAGD.

Our paper has two aims. On one hand, we introduce the bivariate version of the AKR operators on $C([0,1]^2)$ and investigate some approximation properties of them. An estimate of the rate of approximation by these operators is given in Proposition 4.1. It is similar to the result provided in Proposition 3.1, which represents an improvement of other results from the literature. Conjecture 4.1 is concerned with a Voronovskaja type formula for the bivariate AKR operators. We will try to prove it either with classical methods or with some recent techniques described in [23,24,27]. This will be the subject of a possible new article. Moreover, in the present paper we investigate the monotonic convergence of univariate and bivariate AKR operators in relation with the Voronovskaja formula. On the other hand we compare, in the univariate case and also in the bivariate case, the approximation provided by AKR operators with that provided by Bernstein operators, in the spirit of [8,9]. More precisely, we describe families of functions which are better approximated by AKR operators and families of functions which are better approximated by Bernstein operators. Numerical and graphical experiments illustrate the theoretical results.

Theorem 22 and Theorem 24 in [7] exhibit shape preserving properties of AKR operators in the univariate setting. See also [8,9]. It is known that the usual convexity is not generally invariant under the bivariate Bernstein operators. Families of convex functions for which the convexity is preserved by these operators are described in [10, Section 3.4]. The monotonicity of the sequence $(B_{n,j}f)_{n \geq 1}$ for generalized convex functions $f$ is presented in [7, Theorem 19]. In the multivariate case this kind of monotonicity is investigated in [10, Section 3.5]. The preservation of convexity and the above mentioned kind of monotonicity under the bivariate AKR operators deserve to be investigated, together with their applications to CAGD. We intend to develop the study of these topics.
Acknowledgements

This work has been accomplished within the Rete Italiana di Approssimazione and the UMI Group “Teoria dell’Approssimazione e Applicazioni”. The first author has been supported by the INdAM-GNCS Visiting Professors program 2021. The second author has been also partially supported by the Erasmus+ Programme for Teaching Staff Mobility between the Universities of Padova and Sibiu.

Funding Funding is provided by INdAM Gruppo Nazionale per il Calcolo Scientifico.

Data Availability No data were used to support this study.

Declarations

Conflict of interest The authors declare no competing financial interests.

References

[1] Acar, T., Aral, A., Cárdenas-Morales, D., Garrancho, P.: Szász-Mirakyan type operators which fix exponentials. Results Math. 72, 1393–1404 (2017)
[2] Acu, A.M., Gonska, H.: Composite Bernstein cubature. Banach J. Math. Anal. 10(2), 235–250 (2016)
[3] Acu, A.M., Gonska, H., Heilmann, M.: Remarks on a Bernstein-type operator of Aldaz, Kounchev and Render. J. Numer. Anal. Approx. Theory 50(1), 3–11 (2021)
[4] Acu, A.M., Mduţa, A.I., Raşa, I.: Voronovskaya type results and operators fixing two functions. Math. Model. Anal. 26(3), 395–410 (2021)
[5] Acu, A.M., Raşa, I.: New estimates for the differences of positive linear operators. Numer. Algorithms 73(3), 775–789 (2016)
[6] Acu, A.M., Tachev, G.: Yet Another New Variant of Szász-Mirakyan Operator. Symmetry 13, 2018 (2021)
[7] Aldaz, J.M., Kounchev, O., Render, H.: Shape preserving properties of generalized Bernstein operators on extended Chebyshev spaces. Numer. Math. 114(1), 1–25 (2009)
[8] Aldaz, J.M., Kounchev, O., Render, H.: Bernstein operators for extended Chebyshev systems. Appl. Math. Comput. 217(2), 790–800 (2010)
[9] Aldaz, J.M., Render, H.: Optimality of generalized Bernstein operators. J. Approx. Theory 162, 1407–1416 (2010)
[10] Altomare, F., Cappelletti Montano, M., Leonessa, V., Raşa, I.: Markov operators, positive semigroups and approximation processes, Walter de Gruyter, Berlin, Munich, Boston (2014)
[11] Brbosu, D., Pop, O.: On the Bernstein bivariate approximation formula. Carpath. J. Math. 24(3), 293–298 (2008)
[12] Bessenyei, M., Páles, Z.: Hadamard-type inequalities for generalized convex functions. Math. Inequal. Appl. 6, 379–392 (2003)

[13] Birou, M.: A proof of a conjecture about the asymptotic formula of a Bernstein type operator. Results Math. 72, 1129–1138 (2017)

[14] Cárdenas-Morales, D., Garrancho, P., Muñoz-Delgado, F.J.: Shape preserving approximation by Bernstein-type operators which fix polynomials. Appl. Math. Comput. 182, 1615–1622 (2006)

[15] Cárdenas-Morales, D., Garrancho, P., Rasa, I.: Bernstein-type operators which preserve polynomials. Comput. Math. Appl. 62, 158–163 (2011)

[16] Cárdenas-Morales, D., Garrancho, P., Rasa, I.: Asymptotic formulae via a Korovkin-Type result. Abstr. Appl. Anal. 217464, 12 (2012)

[17] Finta, Z.: A quantitative variant of Voronovskaja’s theorem for King-type operators. Constr. Math. Anal. 2(3), 124–129 (2019)

[18] Gavrea, I., Ivan, M.: Complete asymptotic expansions related to conjecture on a Voronovskaja-type theorem. J. Math. Anal. Appl. 458(1), 452–463 (2018)

[19] Gonska, H., Piţul, P., Raşa, I.: General King-type operators. Result Math. 53, 279–286 (2009)

[20] Gupta, V., Agrawal, D.: Convergence by modified Post-Widder operators. RACSAM 113, 1475–1486 (2019)

[21] Karlin, S.: Total positivity, vol. 1. Stanford University Press, Standford (1968)

[22] King, J.P.: Positive linear operators which preserve $x^2$. Acta Math. Hungar. 99(3), 203–208 (2003)

[23] Popa, D.: An intermediate Voronovskaja type theorem. Rev. R. Acad. Cienc. Exactas. Fis. Nat. Ser. A Mat. RACSAM 113(3), 2421–2429 (2019)

[24] Popa, D.: Voronovskaja type results and their applications. Results Math. 77, 15 (2022)

[25] Pltnea, R.: On some constants in approximation by Bernstein operators. Gen. Math. 16(4), 137–148 (2008)

[26] Yilmaz, O.G., Gupta, V., Aral, A.: A note on Baskakov-Kantorovich type operators preserving $e^{-x}$. Math. Meth. Appl. Sci. 43(13), 7511–7517 (2020)

[27] Xiang, J.X.: Voronovskaja-type theorem for modified Bernstein operators. J. Math. Anal. Appl. 495, 124728 (2021)

Ana-Maria Acu
Department of Mathematics and Informatics
Lucian Blaga University of Sibiu
Sibiu
Romania
e-mail: anamaria.acu@ulbsibiu.ro
Stefano De Marchi
Department of Mathematics “Tullio Levi-Civita”
University of Padova
Padova
Italy
e-mail: stefano.demarchi@unipd.it

and

INdAM Gruppo Nazionale di Calcolo Scientifico
Rome
Italy

Ioan Rasa
Department of Mathematics, Faculty of Automation and Computer Science
Technical University of Cluj-Napoca
Str. Memorandumului nr. 28
400114 Cluj-Napoca
Romania
e-mail: ioan.rasa@math.utcluj.ro

Received: July 20, 2022.
Accepted: November 8, 2022.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.