JACOB’S LADDERS AND THE ASYMPTOTIC FORMULA FOR SHORT AND MICROSCOPIC PARTS OF THE HARDY-LITTLEWOOD INTEGRAL OF THE FUNCTION \(|\zeta(1/2 + it)|^4\)

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Abstract. The elementary geometric properties of Jacob’s ladders of the second order lead to a class of new asymptotic formulae for short and microscopic parts of the Hardy-Littlewood integral of \(|\zeta(1/2 + it)|^4\). These formulae cannot be obtained by methods of Balasubramanian, Heath-Brown and Ivic.

1. Formulation of the Theorem

1.1. Let us remind that Hardy and Littlewood started to study the following integral in 1922

\[
\int_1^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 dt = \int_1^T Z^4(t) dt,
\]

where

\[
Z(t) = e^{i\vartheta(t)} \zeta \left( \frac{1}{2} + it \right), \quad \vartheta(t) = -\frac{1}{2} t \ln \pi + \text{Im} \ln \Gamma \left( \frac{1}{4} + \frac{t}{2} \right),
\]

and they derived the following estimate (see [2], pp. 41, 59, [14], p. 124)

\[
\int_1^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 dt \sim O(T \ln^4 T).
\]

Let us remind furthermore that Ingham, in 1926, has derived the first asymptotic formula

\[
\int_1^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 dt = \frac{1}{2\pi^2} T \ln^4 T + O(T \ln^3 T)
\]

(see [3], p. 277, [14], p.125). In 1928 Titchmarsh has discovered a new treatment to the integral (1.1)

\[
\int_0^T Z^4(t) e^{-\delta t} dt \sim \frac{1}{2\pi^2 \delta} \ln^4 \frac{1}{\delta} \Rightarrow \int_1^T Z^4(t) dt \sim \frac{1}{2\pi^2} T \ln^4 T
\]

(see [14], pp. 136, 143). Let us remind, finally, the Titchmarsh-Atkinson formula (see [14], p. 145)

\[
\int_0^T Z^4(t) e^{-\delta t} dt = \frac{1}{\delta} \left( A \ln^4 \frac{1}{\delta} + B \ln^3 \frac{1}{\delta} + C \ln^2 \frac{1}{\delta} +
\right.
\]
\[
\left. + D \ln \frac{1}{\delta} + E \right) + O \left( \frac{1}{\delta} \right)^{13/14 + \epsilon}, \quad A = \frac{1}{2\pi^2}.
\]

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which improved the Titchmarsh formula (1.4), and the Ingham - Heath-Brown formula (see [4], p. 129)

\[
\int_0^T Z^4(t)dt = T \sum_{K=0}^4 C_K \ln^{4-K} T + O(T^{7/8+\epsilon}), \quad C_0 = \frac{1}{2\pi^2}
\]

which improved the Ingham formula (1.3).

1.2. It is clear that the asymptotic formulae for short and microscopic parts

\[
\int_{T+U}^{T+U_0} \left| \frac{\zeta}{2} + it \right|^4 dt
\]

of the Hardy-Littlewood integral (1.1) cannot be obtained by methods which lead to the results (1.2)-(1.6). It is proved in this paper that the Jacob’s ladders of the second order \( \phi_2(T) \) (see [12]) lead to new asymptotic formulae in this direction.

Let us remind our formula (see [12], (5.11))

\[
Z^4(t) = \frac{1}{2\pi^2} \left\{ 1 + O \left( \frac{(\ln \ln T)^2}{\ln T} \right) \right\} \ln^4 T \frac{d\phi_2(t)}{dt},
\]

\( t \in [T, T+U_0], \quad U_0 = T^{13/14+2\epsilon} \).

Then from (1.8) the multiplicative asymptotic formula for short and microscopic parts (1.7) of the Hardy-Littlewood integral (1.1) follows (compare [7], (1.2)).

**Theorem.**

\[
\int_{T}^{T+U} Z^4(t)dt = \frac{1}{2\pi^2} \left\{ 1 + O \left( \frac{(\ln \ln T)^2}{\ln T} \right) \right\} U \ln^4 T \tan[\alpha_2(T, U)],
\]

\( U \in (0, U_0], \quad U_0 = T^{13/14+2\epsilon} \),

where \( \alpha_2 \) is the angle of the chord of the curve \( y = \phi_2(T) \) that binds the points \( [T, \phi_2(T)], [T+U, \phi_2(T+U)] \).

**Remark 1.** The small improvements of the exponent 13/14 of the type 13/14 \( \rightarrow 8/9 \rightarrow \ldots \) are irrelevant in this question.

This paper is a continuation of the series of papers [5]-[13].

2. SOME CANONICAL EQUIVALENCES

2.1. Let us remind that

\[
\tan[\alpha_2(T, U_0)] = 1 + O \left( \frac{1}{\ln T} \right)
\]

is true (see [12], (5.6)). Then, similarly to [7], 2.1, we call the chord binding the points \( [T, \phi_2(T)], [T+U_0, \phi_2(T+U_0)] \) of the Jacob’s ladder \( y = \phi_2(T) \) the fundamental chord (compare [7]).

Let us consider the set of all segments \( [M, N] \subset [T, T+U_0] \).

**Definition.** The chord binding the points

\( [N, \phi_2(N)], [M, \phi_2(M)], [M, N] \subset [T, T+U_0] \).
such that the property
\begin{equation}
\tan[\alpha_2(N, M - N)] = 1 + o(1), \ T \to \infty 
\end{equation}
is fulfilled, is called the \textit{almost parallel chord} to the fundamental chord. This property will be denoted by the symbol $/\fat/ 2$, (comp. [7]).

Now, we obtain the following corollary from (1.9) and (2.2).

\textbf{Corollary 1.} Let $[M, N] \subset [T, T + U_0]$. Then
\begin{equation}
\frac{1}{M - N} \int_N^M Z^4(t)dt \sim \frac{1}{2\pi^2} \ln^4 T \ \Leftrightarrow \ /\fat/ 2.
\end{equation}

\textit{Remark 2.} We see that the analytic property
\begin{equation}
\frac{1}{M - N} \int_N^M Z^4(t)dt \sim \frac{1}{2\pi^2} \ln^4 T
\end{equation}
is equivalent to the geometric property $/\fat/ 2$ of Jacob’s ladder $y = \varphi_2(T)$ of the second order.

2.2. Next, similarly to the case of the paper [7], the following corollary is obtained from our Theorem.

\textbf{Corollary 2.} There is a continuum of intervals $[M, N] \subset [T, T + U_0]$ such that the asymptotic formula
\begin{equation}
\int_N^M Z^4(t)dt \sim \frac{1}{2\pi^2} (M - N) \ln^4 T
\end{equation}
holds true.

\textit{Remark 3.} Especially, there is a continuum of intervals $[N, M] : 0 < M - N < 1$, such that the asymptotic formula (2.4) is true (this follows from the elementary mean-value theorem of differentiation).

3. \textbf{On microscopic parts of the Hardy-Littlewood integral (1.1) in neighborhoods of zeroes of the function $\zeta(1/2 + iT)$}

Let $\gamma, \gamma'$ be a pair of neighboring zeroes of the function $\zeta(1/2 + iT)$. The function $\varphi_2(T)$ is necessarily convex on some right neighborhood of the point $T = \gamma$, and this function is necessarily concave on some left neighborhood of the point $T = \gamma'$. Therefore, there exists a minimal value $\rho \in (\gamma, \gamma')$ such that $[\rho, \varphi_2(\rho)]$ is the point of inflection of the curve $y = \varphi_2(T)$. At this point, by the properties of the Jacob’s ladders, we have $\varphi'_2(\rho) > 0$. Let furthermore $\beta = \beta(\gamma, \rho)$ be the angle of the chord binding the points
\begin{equation}
[\gamma, \varphi_2(\gamma)], \ [\rho, \varphi_2(\rho)].
\end{equation}

Then we obtain by Theorem (compare [7])

\textbf{Corollary 3.} For every sufficiently big zero $T = \gamma$ of the function $\zeta(1/2 + iT)$ the following formulae describing microscopic parts (1.7) of the Hardy-Littlewood integral (1.1) hold true.
(A) a continuum of asymptotic formulae

\[
\int_{\gamma}^{\gamma+U} Z^4(t)dt \sim \frac{\tan \alpha}{2\pi^2} U \ln^4 \gamma, \quad \gamma \to \infty,
\]

\[\alpha \in (0, \beta(\gamma, \rho)), \quad U = U(\gamma, \alpha) \in (0, \rho - \gamma),\]

where \(\alpha = \alpha(\gamma, U)\) is the angle of the rotating chord binding the points \([\gamma, \varphi_2(\gamma)], [\gamma + U, \varphi_2(\gamma + U)]\),

(B) a continuum of asymptotic formulae for a chord parallel to the chord given by the points (3.1)

\[
\int_{M}^{N} Z^4(t)dt \sim \frac{\tan[\beta(\gamma, \rho)]}{2\pi^2} (M - N) \ln^4 \gamma, \quad \gamma < N < M < \rho.
\]

**Remark 4.** Let us remind that if the Riemann conjecture is true then the Littlewood estimate

\[
\gamma' - \gamma < -\frac{A}{\ln \ln \gamma} \to 0, \quad \gamma \to \infty
\]

takes place (a simple consequence of the estimate \(S(T) = O(\ln T/\ln \ln T)\), see \[14\], p. 296).

4. **Second class of formulae for parts (1.7) of the Hardy-Littlewood integral (1.1) beginning in zeroes of the function \(\zeta(1/2 + iT)\)**

Let \(T = \gamma, \bar{\gamma}\) be a pair of zeroes of the function \(\zeta(1/2 + iT)\), where \(\bar{\gamma}\) obeys the following conditions (compare \[7\])

\[
\bar{\gamma} = \gamma + \gamma^{13/14 + 2\epsilon} + \Delta(\gamma), \quad 0 \leq \Delta(\gamma) = O(\gamma^{1/4 + \epsilon})
\]

(see the Hardy-Littlewood estimate for the distance between the neighboring zeroes \[1\], pp. 125, 177-184). Consequently

\[
U(\gamma) = \gamma^{13/14 + 2\epsilon} + \Delta(\gamma) \sim \gamma^{13/14 + 2\epsilon}, \quad \gamma \to \infty.
\]

For the chord that binds the points

\[
[\gamma, \varphi_2(\gamma)], \quad [\bar{\gamma}, \varphi_2(\bar{\gamma})]
\]

we obtain (similarly to \[12\], (5.6))

\[
\tan[\alpha_2(T, U)] = 1 + O\left(\frac{1}{\ln T}\right).
\]

The continuous curve \(y = \varphi_2(T)\) lies below the chord given by the points (4.2) on some right neighborhood of the point \(T = \gamma\), and this curve lies above that chord on some left neighborhood of the point \(T = \bar{\gamma}\). Therefore there exists a common point of the curve and the chord. Let \(\bar{\rho} \in (\gamma, \bar{\gamma})\) be such a common point that is the closest one to the point \([\gamma, \varphi_2(\gamma)]\). Then we obtain from our Theorem (compare \[7\]) the next corollary.

**Corollary 4.** For every sufficiently big zero \(T = \gamma\) of the function \(\zeta(1/2 + iT)\) we have the following formulae for the parts (1.7) of the Hardy-Littlewood integral (1.1)
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(A) a continuum of asymptotic formulae for the rotating chord

\[ \int_{\gamma}^{\gamma+U} Z^4(t) dt \sim \frac{\tan \alpha}{2\pi^2} U \ln^4 \gamma, \quad \tan \alpha \in [\eta, 1-\eta], \]

where \( \alpha \) is the angle of the rotating chord binding the points \([\gamma, \varphi_2(\gamma)]\) and \([\gamma+U, \varphi_2(\gamma+U)]\), and \( 0 < \eta \) is an arbitrary small number.

(B) a continuum of asymptotic formulae for the chords parallel to the chord binding the points (4.2), (see (4.3))

\[ \int_{M}^{N} Z^4(t) dt \sim \frac{1}{2\sqrt{3}\pi^2} (M-N) \ln^4 \gamma, \quad \gamma \leq N < M \leq \bar{\rho}. \]

Remark 5. For example, in the case \( \alpha = \pi/6 \) we have from (4.4)

\[ \int_{\gamma}^{\gamma+U} Z^4(t) dt \sim \frac{1}{2\sqrt{3}\pi^2} U \ln^4 \gamma, \quad U = U\left(\gamma, \frac{\pi}{6}\right). \]

Remark 6. It is obvious that (see (4.4))

\[ U(\gamma, \alpha) < T^{7/8+2\epsilon}. \]

Moreover, the following is also true

\[ U(\gamma, \alpha) < T^{\omega+2\epsilon}, \quad \omega < \frac{7}{8}, \]

where \( \omega \) is an arbitrary improvement of the exponent 7/8 which will be proved.

Remark 7. The asymptotic formulae (1.9), (2.3), (2.4), (3.2), (3.3), (4.4), (4.5) cannot be derived within complicated methods of Balasubramanian, Heath-Brown and Ivic (compare [4]).

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