Expontential Sums and Rank of Double Persymmetric Matrices Over $\mathbb{F}_2$  

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Résumé. Soit $K^2$ l'espace vectoriel de dimension 2 où $K$ dénote le corps des séries de Laurent formelles $\mathbb{F}_2((T^{-1}))$. Nous calculons en particulier des sommes exponentielles (dans $K^2$) de la forme

$$\sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU)$$

où $(t, \eta)$ est dans la boule unité de $K^2$.

Nous démontrons qu'elles dépendent uniquement du rang de matrices doubles persymétriques avec des entrées dans $\mathbb{F}_2$, c'est-à-dire des matrices de la forme $\hat{A} \hat{B}$ où $A$ est une matrice $s \times k$ persymétrique et $B$ une matrice $(s+m) \times k$ persymétrique (une matrice $[\alpha_{i,j}]$ est persymétrique si $\alpha_{i,j} = \alpha_{r,s}$ pour $i+j = r+s$). En outre, nous établissons plusieurs formules concernant des propriétés de rang de partitions de matrices doubles persymétriques, ce qui nous conduit à une formule récurrente du nombre $\Gamma_i^{s+m \times k}$ des matrices de rang $i$ de la forme $[\hat{A} \hat{B}]$. Nous déduisons de cette formule récurrente que si $0 \leq i \leq \min(s-1, k-1)$, le nombre $\Gamma_i^{s+m \times k}$ dépend uniquement de $i$. D'autre part, si $i \geq s+1, k \geq i$, $\Gamma_i^{s+m \times k}$ peut être calculé à partir du nombre $\Gamma_i^{s'+m' \times k'}$ de matrices de rang $(s'+1)$ de la forme $[\hat{A}' \hat{B}']$ où $A'$ est une matrice $s' \times k'$ persymétrique et $B'$ une matrice $(s'+m') \times k'$ persymétrique, où $s'$, $m'$ et $k'$ dépendent de $i$, $s$, $m$ et $k$. La preuve de ce résultat est basée sur une formule (donnée dans [4]) du nombre de matrices de rang $i$ de la forme $[\hat{A}]$ où $A$ est persymétrique et $b_-$ une matrice ligne avec entrées dans $\mathbb{F}_2$. Nous montrons également que le nombre $R$ de représentations dans $\mathbb{F}_2[T]$ des équations polynomiales

$$\begin{cases}
YZ + Y_1Z_1 + \ldots + Y_{q-1}Z_{q-1} = 0 \\
YU + Y_1U_1 + \ldots + Y_{q-1}U_{q-1} = 0
\end{cases}$$

associées aux sommes exponentielles

$$\sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU)$$

est donné par une intégrale sur la boule unité de $K^2$ et est une combinaison linéaire de $\Gamma_i^{s+m \times k}$ pour $i \geq 0$. Nous pouvons alors calculer explicitement le nombre $R$. 

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Abstract. Let $\mathbb{K}^2$ be the 2-dimensional vectorspace over $\mathbb{K}$ where $\mathbb{K}$ denotes the field of Laurent Series $\mathbb{F}_2((T^{-1}))$. We compute in particular exponential sums, (in $\mathbb{K}^2$) of the form
\[
\sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU) \quad \text{where } (t, \eta) \text{ is in the unit interval of } \mathbb{K}^2.
\]
We show that they only depend on the rank of some associated double persymmetric matrices with entries in $\mathbb{F}_2$, that is matrices of the form $\begin{bmatrix} A & B \end{bmatrix}$ where $A$ is a $s \times k$ persymmetric matrix and $B$ a $(s+m) \times k$ persymmetric matrix. (A matrix $[\alpha_{i,j}]$ is persymmetric if $\alpha_{i,j} = \alpha_{r,s}$ for $i + j = r + s$). Besides, we establish several formulas concerning rank properties of partitions of double persymmetric matrices, which leads to a recurrent formula for the number $\Gamma_i^{s+m} \times k$ of rank $i$ matrices of the form $\begin{bmatrix} A & B \end{bmatrix}$. We deduce from the recurrent formula that if $0 \leq i \leq \min(s-1, k-1)$ then $\Gamma_i^{s+m} \times k$ depends only on $i$. On the other hand, if $i \geq s+1, k \geq i$, $\Gamma_i^{s+m} \times k$ can be computed from the number $\Gamma_j^{s'+m'} \times k'$ of rank $(s'+1)$ matrices of the form $\begin{bmatrix} A' & B' \end{bmatrix}$ where $A'$ is a $s' \times k'$ persymmetric matrix and $B'$ a $(s'+m') \times k'$ persymmetric matrix, where $s', m'$ and $k'$ depend on $i, s, m$ and $k$. The proof of this result is based on a formula (given in [4]) of the number of rank $i$ matrices of the form $\begin{bmatrix} A \end{bmatrix}$ where $A$ is persymmetric and $b_-$ a one-row matrix with entries in $\mathbb{F}_2$.

We also prove that the number $R$ of representations in $\mathbb{F}_2[T]$ of the polynomial equations
\[
\begin{cases}
YZ + Y_1Z_1 + \ldots + Y_{q-1}Z_{q-1} = 0 \\
YU + Y_1U_1 + \ldots + Y_{q-1}U_{q-1} = 0
\end{cases}
\]
associated to the exponential sums
\[
\sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU) \quad \text{is given by an integral over the unit interval of } \mathbb{K}^2,
\]
and is a linear combination of the $\Gamma_i^{s+m} \times k$ for $i \geq 0$. We can then compute explicitly the number $R$. 

EXPONENTIAL SUMS AND RANK OF DOUBLE PERSYMMETRIC MATRICES OVER \( \mathbb{F}_2 \)

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1. NOTATION

1.1. ANALYSIS ON $\mathbb{K}$. We denote by $\mathbb{F}_2((\frac{1}{T})) = \mathbb{K}$ the completion of the field $\mathbb{F}_2(T)$, the field of rational functions over the finite field $\mathbb{F}_2$, for the infinity valuation $v = v_\infty$ defined by $v\left(\frac{B}{A}\right) = \deg B - \deg A$ for each pair $(A, B)$ of non-zero polynomials. Then every element non-zero $t$ in $\mathbb{F}_2((\frac{1}{T}))$ can be expanded in a unique way in a convergent Laurent series $t = \sum_{j=-\infty}^{-v(t)} t_j T^j$ where $t_j \in \mathbb{F}_2$.

We associate to the infinity valuation $v = v_\infty$ the absolute value $|\cdot|_\infty$ defined by $|t|_\infty = |t| = 2^{-v(t)}$.

We denote $E$ the Character of the additive locally compact group $\mathbb{F}_2((\frac{1}{T}))$ defined by

$$E\left(\sum_{j=-\infty}^{-v(t)} t_j T^j\right) = \begin{cases} 1 & \text{if } t_{-1} = 0, \\ -1 & \text{if } t_{-1} = 1. \end{cases}$$

We denote $\mathbb{P}$ the valuation ideal in $\mathbb{K}$, also denoted the unit interval of $\mathbb{K}$, i.e. the open ball of radius 1 about 0 or, alternatively, the set of all Laurent series

$$\sum_{i \geq 1} \alpha_i T^{-i} \quad (\alpha_i \in \mathbb{F}_2)$$

and, for every rational integer $j$, we denote by $\mathbb{P}_j$ the ideal $\{t \in \mathbb{K} | v(t) > j\}$. The sets $\mathbb{P}_j$ are compact subgroups of the additive locally compact group $\mathbb{K}$.

All $t \in \mathbb{F}_2((\frac{1}{T}))$ may be written in a unique way as $t = [t] + \{t\}$, $[t] \in \mathbb{F}_2[T]$, $\{t\} \in \mathbb{P}(-\mathbb{P}_0)$.

We denote by $dt$ the Haar measure on $\mathbb{K}$ chosen so that

$$\int_{\mathbb{P}} dt = 1.$$

Definition 1.1. We introduce the following definitions in $\mathbb{K}$:

- Let $s$, $m$ and $k$ denote rational integers such that $s \geq 2$, $m \geq 0$ and $k \geq 1$.

- A matrix $D = [\alpha_{i,j}]$ is said to be persymmetric if $\alpha_{i,j} = \alpha_{r,s}$ whenever $i+j = r+s$.

- Set $t = \sum_{i \geq 1} \alpha_i T^{-i} \in \mathbb{P}$, we denote by $D_{s \times k}(t)$ the following $s \times k$ persymmetric matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\ \alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\ \alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{s+k-2} & \alpha_{s+k-1} \end{pmatrix}.$$
Let \( \mathbb{K} \times \mathbb{K} = \mathbb{K}^2 \) be the 2-dimensional vector space over \( \mathbb{K} \). Let \( (t, \eta) \in \mathbb{K}^2 \) and 
\[
| (t, \eta) | = \sup \{|t|, |\eta| \} = 2^{-\inf(v(t), v(\eta))} .
\]

It is easy to see that \( (t, \eta) \longrightarrow |(t, \eta)| \) is an ultrametric valuation on \( \mathbb{K}^2 \), that is, \( (t, \eta) \longrightarrow |(t, \eta)| \) is a norm and \(|(t, \eta) + (t', \eta')| \leq \max \{|(t, \eta)|, |(t', \eta')|\}\).

We denote by \( d(t, \eta) = dt d\eta \) the Haar measure on \( \mathbb{K}^2 \) chosen so that the measure on the unit interval of \( \mathbb{K}^2 \) is equal to one, thus
\[
\iint_{\mathbb{K} \times \mathbb{K}} d(t, \eta) = \int_{\mathbb{K}} dt \int_{\mathbb{K}} d\eta = 1 \cdot 1 = 1 .
\]

Let \( (t, \eta) = \left( \sum_{i=-\infty}^{-v(t)} t_i T^i, \sum_{i=-\infty}^{-v(\eta)} \eta_i T^i \right) \in \mathbb{K}^2 \), we denote \( \chi \) the Character on \( (\mathbb{K}^2, +) \) defined by
\[
\chi \left( \sum_{i=-\infty}^{-v(t)} t_i T^i, \sum_{i=-\infty}^{-v(\eta)} \eta_i T^i \right) = E \left( \sum_{i=-\infty}^{-v(t)} t_i T^i \right) \cdot E \left( \sum_{i=-\infty}^{-v(\eta)} \eta_i T^i \right) = \begin{cases} 1 & \text{if } t_{-1} + \eta_{-1} = 0, \\ -1 & \text{if } t_{-1} + \eta_{-1} = 1. \end{cases}
\]

**Definition 1.2.** We introduce the following definitions in the two-dimensional \( \mathbb{K} \)-vector-space.

- Let \( k, s \) and \( m \) denote rational integers such that \( k \geq 1, \ s \geq 2 \) and \( m \geq 0 \).

- We denote by \( \mathbb{P} / \mathbb{P}_i \times \mathbb{P} / \mathbb{P}_j \) a complete set of coset representatives of \( \mathbb{P}_i \times \mathbb{P}_j \) in \( \mathbb{P} \times \mathbb{P} \), for instance \( \mathbb{P} / \mathbb{P}_{s+k-1} \times \mathbb{P} / \mathbb{P}_{s+m+k-1} \) denotes a complete set of coset representatives of \( \mathbb{P}_{s+k-1} \times \mathbb{P}_{s+m+k-1} \) in \( \mathbb{P} \times \mathbb{P} \).

- Set \( (t, \eta) = \left( \sum_{i \geq 1} \alpha_i T^{-i}, \sum_{i \geq 1} \beta_i T^{-i} \right) \in \mathbb{P} \times \mathbb{P} \).

We denote by \( D_{s \times (k-j+1)}^j(t, \eta) \) any \((2s + m) \times k\) matrix, such that after
a rearrangement of the rows, if necessary, we can obtain the following double persymmetric matrix \[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{s+k-2} & \alpha_{s+k-1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{s+m+k-2} & \beta_{s+m+k-1}
\end{pmatrix}
\]

We recall that the rank of a matrix does not change under elementary row operations.

- Let \( j \) be a rational integer such that \( 1 \leq j \leq k-1 \), set \((t, \eta) = (\sum_{i \geq 1} \alpha_i T^{-i}, \sum_{i \geq 1} \beta_i T^{-i}) \in \mathbb{P} \times \mathbb{P} \).

We denote by \( D_{(s\times(k+j+1))}^{j} \) any \((2s+m)\times(k-j+1)\) matrix such that after a rearrangement of the rows, if necessary, we can obtain the following double persymmetric matrix \[
\begin{pmatrix}
\alpha_j & \alpha_{j+1} & \alpha_{j+2} & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_{j+1} & \alpha_{j+2} & \alpha_{j+3} & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{j+s-2} & \alpha_{j+s-1} & \alpha_{j+s} & \ldots & \alpha_{k+s-3} & \alpha_{k+s-2} \\
\alpha_{j+s-1} & \alpha_{j+s} & \alpha_{j+s+1} & \ldots & \alpha_{k+s-2} & \alpha_{k+s-1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\beta_j & \beta_{j+1} & \beta_{j+2} & \ldots & \beta_{k-1} & \beta_k \\
\beta_{j+1} & \beta_{j+2} & \beta_{j+3} & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{j+s-2} & \beta_{j+s-1} & \beta_{j+s} & \ldots & \beta_{k+s-3} & \beta_{k+s-2} \\
\beta_{j+s-1} & \beta_{j+s} & \beta_{j+s+1} & \ldots & \beta_{k+s-2} & \beta_{k+s-1} \\
\beta_{j+s} & \beta_{j+s+1} & \beta_{j+s+2} & \ldots & \beta_{k+s-1} & \beta_{k+s} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{j+s+m-3} & \beta_{j+s+m-2} & \beta_{j+s+m-1} & \ldots & \beta_{k+s+m-4} & \beta_{k+s+m-3} \\
\beta_{j+s+m-2} & \beta_{j+s+m-1} & \beta_{j+s+m} & \ldots & \beta_{k+s+m-3} & \beta_{k+s+m-2} \\
\beta_{j+s+m-1} & \beta_{j+s+m} & \beta_{j+s+m+1} & \ldots & \beta_{k+s+m-2} & \beta_{k+s+m-1}
\end{pmatrix}
\]
For instance we can denote by $D_j^{s+m} \times_k (t, \eta)$ the following matrix

$$
\begin{pmatrix}
\alpha_j & \alpha_{j+1} & \alpha_{j+2} & \ldots & \alpha_{k-1} & \alpha_k \\
\beta_j & \beta_{j+1} & \beta_{j+2} & \ldots & \beta_{k-1} & \beta_k \\
\alpha_{j+1} & \alpha_{j+2} & \alpha_{j+3} & \ldots & \alpha_k & \alpha_{k+1} \\
\beta_{j+1} & \beta_{j+2} & \beta_{j+3} & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{j+s-2} & \alpha_{j+s-1} & \alpha_{j+s} & \ldots & \alpha_{k+s-3} & \alpha_{k+s-2} \\
\beta_{j+s-2} & \beta_{j+s-1} & \beta_{j+s} & \ldots & \beta_{k+s-3} & \beta_{k+s-2} \\
\beta_{j+s-1} & \beta_{j+s} & \beta_{j+s+1} & \ldots & \beta_{k+s-2} & \beta_{k+s-1} \\
\beta_{j+s} & \beta_{j+s+1} & \beta_{j+s+2} & \ldots & \beta_{k+s-1} & \beta_{k+s} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{j+s+m-3} & \beta_{j+s+m-2} & \beta_{j+s+m-1} & \ldots & \beta_{k+s+m-4} & \beta_{k+s+m-3} \\
\beta_{j+s+m-2} & \beta_{j+s+m-1} & \beta_{j+s+m} & \ldots & \beta_{k+s+m-3} & \beta_{k+s+m-2} \\
\alpha_{j+s-1} & \alpha_{j+s} & \alpha_{j+s+1} & \ldots & \alpha_{k+s-2} & \alpha_{k+s-1} \\
\beta_{j+s+m-1} & \beta_{j+s+m} & \beta_{j+s+m+1} & \ldots & \beta_{k+s+m-2} & \beta_{k+s+m-1} 
\end{pmatrix}
$$

If $j = 1$ we denote $D_1^{s+m} \times_k (t, \eta)$ by $D_1^{s+m} \times_k (t, \eta)$.

- We denote by $D_1^{s+m} \times_k (t, \eta)$ the following $(2s + m - 1) \times_k$ matrix, where the submatrix formed by the first $(2s + m - 2)$ rows is equal to the matrix $D_1^{s+m-1} \times_k (t, \eta)$, and the last row form a $1 \times k$ persymmetric matrix of the form $(\alpha_s, \beta_s, \alpha_{s+1} + \beta_{s+m-1}, \ldots, \alpha_{s+k-1} + \beta_{s+k+m-1})$

$$
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
\alpha_s & \beta_{s+m} & \alpha_{s+1} + \beta_{s+m+1} & \alpha_{s+2} + \beta_{s+m+2} & \ldots & \alpha_{k+s-2} + \beta_{k+s+m-2} & \alpha_{k+s-1} + \beta_{k+s+m-1} 
\end{pmatrix}
$$

- We denote by ker $D$ the nullspace of the matrix $D$ and $r(D)$ the rank of the matrix $D$. 
To simplify the notations concerning the exponential sums used in the proofs, we introduce the following definitions.

Let $g_{s,k,m}(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by

$$(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU) \in \mathbb{Z}.$$

Let $g(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by

$$(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU) \in \mathbb{Z}.$$

Let $g_1(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by

$$(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\deg Y \leq k-1} \sum_{\deg Z = s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU) \in \mathbb{Z}.$$

Let $g_2(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by

$$(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\deg Y \leq k-1} \sum_{\deg Z = s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta YU) \in \mathbb{Z}.$$

Let $f_1(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by

$$(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\deg Y \leq k-1} \sum_{\deg Z = s-1} E(tYZ) \sum_{\deg U \leq s+m-2} E(\eta YU) \in \mathbb{Z}.$$

Let $f_2(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by

$$(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\deg Y \leq k-1} \sum_{\deg Z = s-2} E(tYZ) \sum_{\deg U = s+m-1} E(\eta YU) \in \mathbb{Z}.$$

Let $h(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by

$$(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\deg Y \leq k-1} \sum_{\deg Z = s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta YU) \in \mathbb{Z}.$$

Let $v(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by

$$(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\deg Y \leq k-1} \sum_{\deg Z = s-2} E(tYZ) \sum_{\deg U \leq s+m-2} E(\eta YU) \in \mathbb{Z}.$$
• Let $\psi(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by
\[
(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\text{deg}Y=k-1} \sum_{\text{deg}Z \leq s} E(tYZ) \sum_{\text{deg}U \leq s+m} E(\eta YU) \in \mathbb{Z}.
\]

• Let $\phi(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by
\[
(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\text{deg}Y=k-1} \sum_{\text{deg}Z \leq s} E(tYZ) \sum_{\text{deg}U \leq s+m} E(\eta YU) \in \mathbb{Z}.
\]

• Let $\phi_1(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by
\[
(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\text{deg}Y=k} \sum_{\text{deg}Z \leq s} E(tYZ) \sum_{\text{deg}U \leq s+m} E(\eta YU) \in \mathbb{Z}.
\]

• Let $\phi_2(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by
\[
(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\text{deg}Y \leq k-2} \sum_{\text{deg}Z \leq s} E(tYZ) \sum_{\text{deg}U \leq s+m} E(\eta YU) \in \mathbb{Z}.
\]

• Let $\theta_1(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by
\[
(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\text{deg}Y=k-1} \sum_{\text{deg}Z=s} E(tYZ) \sum_{\text{deg}U \leq s+m} E(\eta YU) \in \mathbb{Z}.
\]

• Let $\theta_2(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by
\[
(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\text{deg}Y \leq k-2} \sum_{\text{deg}Z=s} E(tYZ) \sum_{\text{deg}U \leq s+m} E(\eta YU) \in \mathbb{Z}.
\]

• Let $\theta_3(t, \eta)$ be the quadratic exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by
\[
(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\text{deg}Y \leq k-2} \sum_{\text{deg}Z=s} E(tYZ) \sum_{\text{deg}U \leq s+m} E(\eta YU) \in \mathbb{Z}.
\]

• Consider the following partition of the matrix $D^{[s+m] \times k}(t, \eta)$
Consider the following partition of the matrix
\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{s+m+k-2} & \beta_{s+m+k-1} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{s+k-2} & \alpha_{s+k-1}
\end{pmatrix}.
\]

We define
\[
\sigma_{i,j}^{s-1 \times m_{x,y}}
\]

to be the cardinality of the following set
\[
\left\{(t, \eta) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \mid r(D^{[s+1 \times k]}(t, \eta)) = r(D^{[s+m \times k]}(t, \eta)) = i\right\}.
\]

- Consider the following partition of the matrix \(D^{[s+1 \times k]}(t, \eta)\)
We define
\[
\sigma_{i,i}^{s-1 \atop \alpha_s + \beta_{s+m-1}} \times k
\]
to be the cardinality of the following set
\[
\left\{ (t, \eta) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-2} \mid r(D^{s-1 \atop \alpha_s + \beta_{s+m-1}} \times k)(t, \eta) = r(D^{s+m-1 \atop \alpha_s + \beta_{s+m-1}} \times k)(t, \eta) = i \right\}.
\]

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-2} & 0 \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{s+m+1} & \beta_{s+m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_s + \beta_{s+m-1} & \alpha_{s+1} + \beta_{s+m+1} & \alpha_{s+2} + \beta_{s+m+2} & \ldots & \alpha_{k+s-2} + \beta_{k+s+m-2} & \alpha_{k+s-1} + \beta_{k+s+m-1} \\
\end{pmatrix}
\]

We define
\[
\sigma_{i,i}^{s-1 \atop \alpha_s + \beta_{s+m-1}} \times k
\]
to be the cardinality of the following set
\[
\left\{ (t, \eta) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \mid r(D^{s-1 \atop \alpha_s + \beta_{s+m-1}} \times k)(t, \eta) = r(D^{s+m-1 \atop \alpha_s + \beta_{s+m-1}} \times k)(t, \eta) = i \right\}.
\]

\[
\begin{pmatrix}
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
\alpha_s + \beta_{s+m-1} & \alpha_{s+1} + \beta_{s+m+1} & \alpha_{s+2} + \beta_{s+m+2} & \ldots & \alpha_{k+s-2} + \beta_{k+s+m-2} & \alpha_{k+s-1} + \beta_{k+s+m-1} \\
\end{pmatrix}
\]

- We consider the following partition of the matrix
\[
D^{s-1 \atop \alpha_s + \beta_{s+m-1}} \times k
\]
• Consider the following partition of the matrix
Let \( \{t, \eta \} \in \mathbb{F} / \mathbb{P}_{k+s-1} \times \mathbb{P} / \mathbb{P}_{k+s+m-1} \) be the cardinality of the following set

\[
\left\{(t, \eta) \in \mathbb{F} / \mathbb{P}_{k+s-1} \times \mathbb{P} / \mathbb{P}_{k+s+m-1} \mid r(D^{s \rightarrow 1 \rightarrow m} \times k(t, \eta)) = r(D^{s \rightarrow 1 \rightarrow m} \times k(t, \eta)) = r(D^{s \rightarrow 1 \rightarrow m} \times k(t, \eta)) = i \right\}
\]

\[
= \left\{(t, \eta) \in \mathbb{F} / \mathbb{P}_{k+s-1} \times \mathbb{P} / \mathbb{P}_{k+s+m-1} \mid r(D^{s \rightarrow 1 \rightarrow m} \times k(t, \eta)) = r(D^{s \rightarrow 1 \rightarrow m} \times k(t, \eta)) = r(D^{s \rightarrow 1 \rightarrow m} \times k(t, \eta)) = i \right\}.
\]

- Let \( (j_1, j_2, j_3) \in \mathbb{N}^3 \) we define

\[
\left\{(t, \eta) \in \mathbb{F} / \mathbb{P}_{k+s-1} \times \mathbb{P} / \mathbb{P}_{k+s+m-1} \mid r(D^{s \rightarrow 1 \rightarrow m} \times k(t, \eta)) = j_1, \quad r(D^{s \rightarrow 1 \rightarrow m} \times k(t, \eta)) = j_2,
\quad r(D^{s \rightarrow 1 \rightarrow m} \times k(t, \eta)) = j_3 \right\}.
\]
Consider the following partition of the matrix $D_{s+m}^{k}(t, \eta)$

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{s+k-2} & \alpha_{s+k-1} \\
\end{pmatrix}
\]

Let $(j_1, j_2, j_3) \in \mathbb{N}^3$ we define

\[
\sigma_{j_1, j_2, j_3} = \begin{pmatrix}
\alpha_{s-1} & \alpha_{s+m-1} & \alpha_{s+m} \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} \\
\alpha_{s-1} & \alpha_{s+m-1} & \alpha_{s+m} \\
\end{pmatrix}^{k}
\]

to be the cardinality of the following set

\[
\left\{(t, \eta) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \mid r(D_{s+m-1}^{s-1}(t, \eta)) = j_1, \ r(D_{s+m}^{s-1}(t, \eta)) = j_2, \ r(D_{s+m}^{s-1}(t, \eta)) = j_3\right\}.
\]

Consider the following partition of the matrix $D_{s+m}^{k}(t, \eta)$
EXPONENTIAL SUMS AND RANK OF DOUBLE PERSYMMETRIC MATRICES OVER $\mathbb{F}_2$

Let $(j_1, j_2, j_3, j_4, j_5, j_6) \in \mathbb{N}^6$, we define

\[
\# \left( \begin{array}{c|c}
    j_1 & j_2 \\
    \hline
    j_3 & j_4 \\
    \hline
    j_5 & j_6 \\
  \end{array} \right) \pmod{\mathbb{F}_{k+1} \times \mathbb{F}_{k+m-1}}
\]

to be the cardinality of the following set

\[
\left\{ (t, \eta) \in \mathbb{P}/\mathbb{P}_{k+1} \times \mathbb{P}/\mathbb{P}_{k+m-1} \mid r(D^{s+1}_{s+m-1} \times (k-1)}(t, \eta)) = j_1, \quad \right.
\]

\[
r(D^{s+1}_{s+m-1} \times k}(t, \eta)) = j_2,
\]

\[
r(D^{s+1}_{s+m} \times (k-1)}(t, \eta)) = j_3, \quad \right.
\]

\[
r(D^{s+1}_{s+m} \times k}(t, \eta)) = j_4,
\]

\[
r(D^{s+1}_{s+m} \times (k-1)}(t, \eta)) = j_5, \quad \right.
\]

\[
r(D^{s+1}_{s+m} \times k}(t, \eta)) = j_6 \}.
\]

- Consider the following partition of the matrix $D^{s+1}_{s+m}(t, \eta)$

\[
\left( \begin{array}{cccc|c}
  \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
  \alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  \alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
  \beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
  \beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  \beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  \beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
  \beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{s+m+k-2} & \beta_{s+m+k-1} \\
  \alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{s+k-2} & \alpha_{s+k-1} \\
  \end{array} \right)
\]

Let $(j_1, j_2, j_3, j_4, j_5, j_6) \in \mathbb{N}^6$, we define
Consider the following partition of the matrix $D_{[s+m] \times k}(t, \eta)$

$$
\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-2} & \alpha_{s+k-1} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-2} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-2} & \beta_{s+m+k-1} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{s+k-2} & \alpha_{s+k-1}
\end{bmatrix}
$$

Let $(j_1, j_2, j_3, j_4) \in \mathbb{N}^4$, we define

$$
\# \left( \frac{[j_1 \quad j_2]}{j_3 \quad j_4} \right)_{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}}
$$

to be the cardinality of the following set

$$
\left\{ (t, \eta) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \mid r(D_{[s+m] \times (k-1)}(t, \eta)) = j_1, \quad r(D_{[s+m-1] \times k}(t, \eta)) = j_2, \right. \\
r(D_{[s+m] \times (k-1)}(t, \eta)) = j_3, \quad r(D_{[s+m-1] \times k}(t, \eta)) = j_4 \right\}.
$$

- Set $(t, \eta) = (\sum_{i \geq 1} \alpha_i T^{-i}, \sum_{i \geq 1} \beta_i T^{-i}) \in \mathbb{P} \times \mathbb{P}$.

Let $\Gamma_i$ denote the number of double persymmetric $(2s + m) \times k$ rank matrices of the form $\frac{D_{[s+m]}(t)}{r((s+m) \times k)}$, that is
\[ \Gamma \left[ \frac{r + m}{k} \right] = \text{Card} \left\{ (t, \eta) \in \mathbb{P}/\mathbb{P}_{k+1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \mid r(D^{[s+1] \times k}(t, \eta)) = i \right\}. \]

- Set for \( 0 \leq i \leq \inf(2s + m, k) \)

\[ \Delta \left[ \frac{r + m}{k} \right] = \sigma_{i,i,i} - 3 \cdot \sigma_{i-1,i-1,i-1} + 2 \cdot \sigma_{i-2,i-2,i-2}. \]

- Set for \( 0 \leq j \leq s + m, k \geq s + j + 1 \)

\[ \Omega_{j+1}(1, m + s - 1, k) = \Gamma \left[ \frac{1}{j+1} \right] - 4 \cdot \Gamma \left[ \frac{1}{j+1} \right] \times k. \]

- We define

\[
\begin{array}{c|c|c|c}
|k-1| \quad k \\
\hline
j \quad j+1 \\
\end{array}
\]

\[ \begin{array}{c|c|c|c}
\alpha_{s-m}
\end{array} \]

\[ \begin{array}{c|c|c|c}
\beta_{s+m-} \\
\end{array} \]

to be the following subset of \( \mathbb{P}^2 \)

\[ \left\{ (t, \eta) = (\sum_{i \geq 0} \alpha_{s-i} T^{-i}, \sum_{i \geq 1} \beta_{s-i} T^{-i}) \in \mathbb{P}^2 \mid r(D^{[s+1] \times (k-1)}(t, \eta)) = j, r(D^{[s+1] \times k}(t, \eta)) = j \right\} \]

- We define

\[
\begin{array}{c|c|c|c}
|k-1| \quad k \\
\hline
j \quad j+1 \\
\end{array}
\]

\[ \begin{array}{c|c|c|c}
\alpha_{s-m}
\end{array} \]

\[ \begin{array}{c|c|c|c}
\beta_{s+m-} \\
\end{array} \]

to be the following subset of \( \mathbb{P}^2 \)

\[ \left\{ (t, \eta) = (\sum_{i \geq 0} \alpha_{s-i} T^{-i}, \sum_{i \geq 0} \beta_{s-i} T^{-i}) \in \mathbb{P}^2 \mid r(D^{[s+1] \times (k-1)}(t, \eta)) = j, r(D^{[s+1] \times k}(t, \eta)) = j+1 \right\} \]

- Similar expressions are defined in a similar way.

- We denote by \( R_q(k, s, m) \) the number of solutions \((Y_1, Z_1, U_1, \ldots, Y_q, Z_q, U_q)\) of the polynomial equations

\[
\begin{align*}
Y_1 Z_1 + Y_2 Z_2 + \ldots + Y_q Z_q &= 0 \\
Y_1 U_1 + Y_2 U_2 + \ldots + Y_q U_q &= 0
\end{align*}
\]

satisfying the degree conditions

\[ \deg Y_i \leq k - 1, \quad \deg Z_i \leq s - 1, \quad \deg U_i \leq s + m - 1 \quad \text{for} \quad 1 \leq i \leq q. \]
2. INTRODUCTION (REFER TO SECTION 1 AND SECTION 3)

The rational function field $\mathbb{F}_2(T)$ is completed with respect to an appropriate valuation to a field $\mathbb{K}$ (i.e. the field of Laurent Series). The unit interval of $\mathbb{K}$, that is, the open ball of radius 1 about 0, is a compact additive group. We shall use the Haar integral on this group. Let $\mathbb{K}^2$ be the 2-dimensional vectorspace over $\mathbb{K}$.

The main result of this paper is to obtain a formula for the number $\Gamma_i^{[s \ s+m] \times k}$ of double persymmetric $(2s + m) \times k$ rank i matrices.

In particular we shall show that if $0 \leq i \leq \inf(2s + m, k - 1)$ the number of double persymmetric $(2s + m) \times k$ rank i matrices depends only on i and if $i \geq s + 1$, $k \geq i$ the number of rank i matrices can be obtained from the formula for the number of rank $s+1$ matrices. The computation of the $\Gamma_i^{[s \ s+m] \times k}$ is based on a formula (given in [4]) of the number of rank i matrices of the form $[A \ b]$ where A is persymmetric and b_ a one-row matrix with entries in $\mathbb{F}_2$. We observe that the $\Gamma_i^{[s \ s+m] \times k}$ where $0 \leq i \leq \inf(2s + m, k)$ are solutions to the system

\[
(2.1) \quad \begin{cases}
\sum_{i=0}^{\inf(2s+m,k)} \Gamma_i^{[s \ s+m] \times k} = 2^{2k+2s+m-2}, \\
\sum_{i=0}^{\inf(2s+m,k)} \Gamma_i^{[s \ s+m] \times k} \cdot 2^{-i} = 2^{k+2s+m-2} + 2^{2k-2} - 2^{k-2}.
\end{cases}
\]

The first equation is obvious.

The second equation is a consequence of the identity

\[
\int_{\mathbb{F} \times \mathbb{F}} g(t, \eta) dt d\eta = \text{Card} \{ (Y, Z, U), \ deg Y \leq k - 1, \ deg Z \leq s - 1, \ deg U \leq s + m - 1 \mid Y \cdot Z = Y \cdot U = 0 \}
= 2^{2s+m} + 2^k - 1.
\]

2.1. SKETCH OF THE PROOF.

- Consider the following $[s \ s+m] \times k$ double persymmetric matrix
We establish the following recurrent formula for the number \( \Gamma_i^{\left[ \frac{s}{s+m} \times k \right]} \) of \( \left[ \frac{s}{s+m} \times k \right] \) double persymmetric matrices of rank \( i \):

\[
\Gamma_i^{\left[ \frac{s}{s+m} \times k \right]} = 2 \cdot \Gamma_{i-1}^{\left[ \frac{s-1}{s-1+m+1} \times k \right]} + 4 \cdot \Gamma_{i-2}^{\left[ \frac{s}{s+(m-1)} \times k \right]} - 8 \cdot \Gamma_{i-2}^{\left[ \frac{s-1}{s-1+m+2} \times k \right]} + \Delta_i^{\left[ \frac{s}{s+m} \times k \right]}.
\]

We observe that the \( \Gamma_i^{\left[ \frac{s}{s+m} \times k \right]} \), for \( 0 \leq i \leq \inf(2s + m, k) \) are solutions to the system

\[
\sum_{i=0}^{\inf(2s+m,k)} \Gamma_i^{\left[ \frac{s}{s+m} \times k \right]} = 2^{2k+2s+m-2},
\]

\[
\sum_{i=0}^{\inf(2s+m,k)} \Gamma_i^{\left[ \frac{s}{s+m} \times k \right]} \cdot 2^{-i} = 2^{k+2s+m-2} + 2^{2k-2} - 2^{k-2}.
\]

We establish that the remainder in (1) satisfies the following equality

\[
\Delta_i^{\left[ \frac{s}{s+m} \times k \right]} = \Delta_i^{\left[ \frac{s}{s+m} \times (i+1) \right]} \quad \text{for all} \quad 0 \leq i \leq 2s + m, \ k \geq i + 1.
\]

We establish from (1) and (3) a recurrent formula for the difference

\[
\Gamma_i^{\left[ \frac{s}{s+m} \times (k+1) \right]} - \Gamma_i^{\left[ \frac{s}{s+m} \times k \right]} \quad \text{for} \ 2 \leq i \leq 2s + m, \ k \geq i + 1.
\]

In fact we prove that
From (4) applying (2) we deduce by induction on $i$

$\Gamma_{i}^{\frac{s}{s+m}} \times (k+1) = 4 \cdot \left[ \Gamma_{i}^{\frac{s}{s+(m-1)}} \times (k+1) - \Gamma_{i+1}^{\frac{s}{s+(m-1)}} \times k \right]$ if $1 \leq i \leq s - 1$, $k \geq i + 1$.

(5) $\Gamma_{s+j}^{\frac{s}{s+m}} \times (k+1) - \Gamma_{s+j}^{\frac{s}{s+m}} \times k = 4 \cdot \left[ \Gamma_{s+j}^{\frac{s}{s+(m-1)}} \times (k+1) - \Gamma_{s+j-1}^{\frac{s}{s+(m-1)}} \times k \right] + R(j, s, m, k)$ if $0 \leq j \leq \inf(s + m, k - s + 1)$

where $R(j, s, m, k)$ is equal to

$2^{s-1} \left[ \Gamma_{j+1}^{\frac{1+s}{s+(m-1)}} \times (k+1) - \Gamma_{j}^{\frac{1+s}{s+(m-1)}} \times k \right] - 2^{s-1} \left[ \Gamma_{j}^{\frac{1+s}{s+(m-2)}} \times (k+1) - \Gamma_{j}^{\frac{1+s}{s+(m-2)}} \times k \right]$.

- From a formula of $\Gamma_{i}^{\frac{1}{1+m}}$ obtained in [4] we compute the remainder $R(j, s, m, k)$ in (5).

In fact we prove

(6) $R(j, s, m, k) = \begin{cases} 2^{k+s-1} & \text{if } j = 0, \ k \geq s + 1, \\ -2^{k+s-1} & \text{if } j = 1, \ k \geq s + 2, \\ 0 & \text{if } 2 \leq j \leq s + m - 1, \ k \geq s + j + 1, \\ -3 \cdot 2^{k+2s+m-2} & \text{if } j = s + m, \ k \geq 2s + m + 1. \end{cases}$

- From (4) applying (2) we deduce by induction on $i$

$\Gamma_{i}^{\frac{s}{s+m}} \times k = \begin{cases} 2^{k+s-1} \cdot (i+1) & \text{if } 1 \leq i \leq s - 1, \ k \geq i + 1, \\ -2^{k+s-1} \cdot 2^{2s+2i+m-2} - 3 \cdot 2^{3i-4} + 2^{2i-3} & \text{if } 1 \leq i = k \leq s + 1. \end{cases}$

We have now computed $\Gamma_{i}^{\frac{s}{s+m}}$ for $1 \leq i \leq s - 1$, $k \geq i$.

- From (5) applying (6) we obtain

(8) $\Gamma_{s+j}^{\frac{s}{s+m}} \times (k+1) - \Gamma_{s+j}^{\frac{s}{s+m}} \times k = 4 \cdot \left[ \Gamma_{s+j}^{\frac{s}{s+(m-1)}} \times (k+1) - \Gamma_{s+j-1}^{\frac{s}{s+(m-1)}} \times k \right]$

$\begin{cases} 2^{k+s-1} & \text{if } j = 0, \ k \geq s + 1, \\ -2^{k+s-1} & \text{if } j = 1, \ k \geq s + 2, \\ 0 & \text{if } 2 \leq j \leq s + m - 1, \ k \geq s + j + 1, \\ -3 \cdot 2^{k+2s+m-2} & \text{if } j = s + m, \ k \geq 2s + m + 1. \end{cases}$

- Using successively the recurrent formula in (8) we deduce

In the case $m \in \{0,1\}$

$\Gamma_{s+j}^{\frac{s}{s+j}} \times (k+1) - \Gamma_{s+j}^{\frac{s}{s+j}} = \begin{cases} 3 \cdot 2^{k+s-1} & \text{if } j = 0, \ k > s, \\ 21 \cdot 2^{k+s+3j-4} & \text{if } 1 \leq j \leq s - 1, \ k > s + j, \\ 3 \cdot 2^{2k+2s-2} - 3 \cdot 2^{k+4s-4} & \text{if } j = s, \ k > 2s, \end{cases}$
\[
\begin{align*}
\Gamma_{s+j}^{s+1} \times (k+1) - \Gamma_{s+j}^{s+1} \times k &= \begin{cases} 
2^{k+s-1} & \text{if } j = 0, \ k > s, \\
11 \cdot 2^{k+s-1} & \text{if } j = 1, \ k > s + 1, \\
21 \cdot 2^{k+s+3j-5} & \text{if } 2 \leq j \leq s, \ k > s + j, \\
3 \cdot 2^{2k+2s-1} - 3 \cdot 2^{k+4s-2} & \text{if } j = s + 1, \ k > 2s + 1.
\end{cases}
\end{align*}
\]

In the case \( m \geq 2 \)

\[
\begin{align*}
\Gamma_{s+m+j}^{s+m} \times (k+1) - \Gamma_{s+m+j}^{s+m} \times k &= \begin{cases} 
2^{k+s-1} & \text{if } j = 0, \ k > s, \\
3 \cdot 2^{k+s+2j-3} & \text{if } 1 \leq j \leq m - 1, \ k > s + j, \\
11 \cdot 2^{k+s+2m-3} & \text{if } m = m, \ k > s + m,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\Gamma_{s+m+j}^{s+m} \times (k+1) - \Gamma_{s+m+j}^{s+m} \times k &= \begin{cases} 
21 \cdot 2^{k+s+2m+3j-4} & \text{if } 1 \leq j \leq s - 1, \ k > s + m + j, \\
3 \cdot 2^{2k+2s+m-2} - 3 \cdot 2^{k+4s+2m-4} & \text{if } j = s, \ k > 2s + m.
\end{cases}
\end{align*}
\]

• Consider the following array \( \left( \Gamma_{s+j}^{s+m} \times k \right)_{0 \leq j \leq s+m, \ k \geq s+j} \):

\[
\begin{align*}
\Gamma_{s+m}^{s} \times s & \quad \Gamma_{s+m}^{s} \times (s+1) & \quad \Gamma_{s+m}^{s} \times (s+2) & \cdots & \quad \Gamma_{s+m}^{s} \times k & \cdots & \quad k \geq s \\
\Gamma_{s+m}^{s+1} \times (s+1) & \quad \Gamma_{s+m}^{s+1} \times (s+2) & \cdots & \quad \Gamma_{s+m}^{s+1} \times k & \cdots & \quad k \geq s + 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Gamma_{s}^{s+j} \times (s+j) & \quad \Gamma_{s}^{s+j} \times (s+j+1) & \quad \Gamma_{s}^{s+j} \times (s+j+2) & \cdots & \quad \Gamma_{s}^{s+j} \times k & \cdots & \quad k \geq s + j \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Gamma_{s+m}^{s+m} \times (s+m) & \quad \Gamma_{s+m}^{s+m} \times (s+m+1) & \cdots & \quad \Gamma_{s+m}^{s+m} \times k & \cdots & \quad k \geq s + m \\
\Gamma_{s+m+1}^{s+m} \times (s+m+1) & \quad \Gamma_{s+m+1}^{s+m} \times (s+m+2) & \cdots & \quad \Gamma_{s+m+1}^{s+m} \times k & \cdots & \quad k \geq s + m + 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Gamma_{s+m+1+j}^{s+m+1+j} & \quad \Gamma_{s+m+1+j}^{s+m+2+j} & \cdots & \quad \Gamma_{s+m+1+j}^{s+m+3+j} & \cdots & \quad k \geq s + m + 1 + j \\
\Gamma_{2s+m-1}^{s+m} \times (2s+m-1) & \quad \Gamma_{2s+m-1}^{s+m} \times (2s+m) & \cdots & \quad \Gamma_{2s+m-1}^{s+m} \times k & \cdots & \quad k \geq 2s + m - 1 \\
\Gamma_{2s+m+1}^{s+m} \times (2s+m+1) & \quad \Gamma_{2s+m+1}^{s+m} \times (2s+m+2) & \cdots & \quad \Gamma_{2s+m+1}^{s+m} \times k & \cdots & \quad k \geq 2s + m.
\end{align*}
\]

• To compute the elements in the array \( \left( \Gamma_{s+j}^{s+m} \times k \right)_{0 \leq j \leq s+m, \ k \geq s+j} \), we proceed by induction on \( j \) as follows:

Let \( l \) be a rational integer such that \( 0 \leq l \leq s + m - 1 \).

Assume that we have computed the elements in the following subarray \( \left( \Gamma_{s+j}^{s+m} \times k \right)_{0 \leq j \leq l, \ k \geq s+j} \).
Recall that from (7) \[ \Gamma_i^{s+m} \times k \] are known for \( 0 \leq i \leq s - 1, \ k \geq i. \)

From the first equation in (2) with \( k = s + l + 1, \sum_{i=0}^{s+l+1} \Gamma_i^{s+m} \times (s+l+1) = 2^{s+2l+m} \)
we deduce \( \Gamma_i^{s+m} \times (s+l+1) \) since the terms \( \Gamma_i^{s+m} \times (s+l+1) \) for \( i \leq s + l \) are known.

From the equations in (2) with \( k = s + l + 2 \) we have
\[
\begin{align*}
\sum_{i=0}^{s+l+2} \Gamma_i^{s+m} \times (s+l+2) &= 2^{s+2l+m+2}, \\
\sum_{i=0}^{s+l+2} \Gamma_i^{s+m} \times (s+l+2) \cdot 2^{-i} &= 2^{3s+m+l} + 2^{2s+2l+2} - 2^{s+l}.
\end{align*}
\]

We then deduce \( \Gamma_i^{s+m} \times (s+1+2) \) and \( \Gamma_i^{s+m} \times (s+l+2) \) since the terms \( \Gamma_i^{s+m} \times (s+1+2) \) for \( i \leq s + l \) are known.

Then from (9) or (10) we compute \( \Gamma_i^{s+m} \times k \) knowing \( \Gamma_i^{s+m} \times (s+1+2) \) for all \( k \geq s + l + 2. \)

We have now computed any element in the \( l + 1 \)-th row and the first element in the \( l + 2 \)-th row in the array \( \left( \Gamma_i^{s+m} \times k \right) \).

- Using the procedure above we prove successively by induction on \( j \) the following reduction formulas
\[
\begin{align*}
\Gamma_i^{s+m} \times k &= 8^{j-1} \cdot \Gamma_i^{s+(m-(j-1))} \times (k-(j-1)) \quad \text{for} \quad 1 \leq j \leq m, \ k \geq s + j, \\
\Gamma_i^{s+m+1+j} &= 8^{2j+m} \cdot \Gamma_i^{s-j} \times (k-m-2j) \quad \text{for} \quad 0 \leq j \leq s - 1, \ k \geq s + m + 1 + j.
\end{align*}
\]

### 2.2. ORGANIZATION OF THE PAPER

We proceed as follows:

- **In Section 1** are introduced main notations and definitions.

- **In Section 3**, we state the main theorems in the two dimensional \( \mathbb{K} \)-vector space.

- **In Section 4**, we establish results on exponential sums, (in \( \mathbb{P} \times \mathbb{P} \)) of the form \( g(t, \eta), h(t, \eta) \) or similar sums and we show that they only depend on rank properties of some corresponding double persymmetric matrices with entries in \( \mathbb{F}_2. \)

Consider in particular the quadratic exponential sum in \( \mathbb{P} \times \mathbb{P} \) defined by
\[
(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto g(t, \eta) = \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU) \in \mathbb{Z}.
\]
We associate to the exponential sum $g(t, \eta)$ the following double persymmetric matrix 

$$\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \cdots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \cdots & \alpha_{s+k-2} & \alpha_{s+k-1} \\
\beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \cdots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \cdots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \cdots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \cdots & \beta_{s+m+k-2} & \beta_{s+m+k-1}
\end{pmatrix}.$$ 

We get

(2.2)

$$g(t, \eta) = \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU) = 2^{2s+m+k-r(D_{s+m} \times k(t, \eta))}.$$ 

The proof of (2.2) is based on the following two similar identities

$$\sum_{\deg Z \leq s-1} E(tYZ) = 2^s \text{ if and only if } Y \in \ker D_{s \times k}(t),$$

$$\sum_{\deg U \leq s+m-1} E(\eta YU) = 2^{s+m} \text{ if and only if } Y \in \ker D_{(s+m) \times k}(\eta).$$

Furthermore we establish similar results concerning similar exponential sums.

For instance we have

Let $(t, \eta) \in \mathbb{P} \times \mathbb{P}$ and set

$$g_2(t, \eta) = \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U = s-1+m} E(\eta YU).$$

Then

(2.3)

$$g_2(t, \eta) = \begin{cases} 2^{2s+m+k-1-r(D_{s+m} \times k(t, \eta))} & \text{if } r(D_{s+m} \times k(t, \eta)) = r(D_{s+(m-1)} \times k(t, \eta)), \\ 0 & \text{otherwise} \end{cases}.$$ 

In the end of the section we obtain by (2.2), observing that $g(t, \eta)$ is constant on cosets of $\mathbb{P}_{k+s-1} \times \mathbb{P}_{k+s+m-1}$
Consider the following two partitions of the matrix $s$

Obviously $\Gamma$ the two expressions of $\Gamma$

$$D_{\binom{s}{s+m}} \cdot 2^{-k+2} \sum_{i=0}^{\inf(2s+m,k)} \Gamma \binom{s}{s+m+k}(q-1) \cdot 2^{-q}.$$

In Section 5, we establish a recurrent formula for the number $\Gamma \binom{s}{s+m+k}$ of rank $i$ matrices of the form $\binom{A}{B}$, where $A$ is a $s \times k$ persymmetric matrix and $B$ a $(s + m) \times k$ persymmetric matrix with entries in $\mathbb{F}_2$.

Consider the following two partitions of the matrix $D_{\binom{s}{s+m}}(t, \eta)$

$$\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{k-1} & \alpha_k & \\
\alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_k & \\
\vdots & \ddots & \ddots & \cdots & \ddots & \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \cdots & \alpha_{s+k-3} & \alpha_{s+k-2} & \\
\beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{k-1} & \beta_k & \\
\beta_2 & \beta_3 & \beta_4 & \cdots & \beta_k & \\
\vdots & \ddots & \ddots & \cdots & \ddots & \\
\beta_{s+1} & \beta_{s+2} & \beta_{s+3} & \cdots & \beta_{s+k-1} & \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \cdots & \beta_{s+m+k-2} & \beta_{s+m+k-1} & \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \cdots & \beta_{s+m+k-2} & \beta_{s+m+k-1} & \\
\end{pmatrix}$$

and

$$\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{k-1} & \alpha_k & \\
\alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_k & \\
\vdots & \ddots & \ddots & \cdots & \ddots & \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \cdots & \alpha_{s+k-3} & \alpha_{s+k-2} & \\
\beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{k-1} & \beta_k & \\
\beta_2 & \beta_3 & \beta_4 & \cdots & \beta_k & \\
\vdots & \ddots & \ddots & \cdots & \ddots & \\
\beta_{s+1} & \beta_{s+2} & \beta_{s+3} & \cdots & \beta_{s+k-1} & \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \cdots & \beta_{s+m+k-2} & \beta_{s+m+k-1} & \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \cdots & \beta_{s+m+k-2} & \beta_{s+m+k-1} & \\
\end{pmatrix}$$

Obviously $\Gamma \binom{s}{s+m+k}$ can be represented in the following two ways

$$\Gamma \binom{s}{s+m+k} = \sigma_{t,i,i,i} + \sigma_{t-i,i,i} + \sigma_{t-1,i-1,i,i} + \sigma_{t-2,i-1,i,i},$$

$$\Gamma \binom{s}{s+m+k} = \sigma_{t,i,i,i} + \sigma_{t-i,i,i} + \sigma_{t-1,i-1,i,i} + \sigma_{t-2,i-1,i,i}.$$
Following identity (2.8), we obtain by elementary rank considerations the following identity

\[ (2.7) \quad \sigma_{i,i}^{s-1} \cdot k = 2 \cdot \sigma_{i,i}^{2s+m-1} \cdot k \quad \text{for } 0 \leq i \leq \inf(2s + m - 2, k). \]

Using the binomial formula, we can write

\[ (2.8) \quad g_1^q = (h + f_1)^q = h^q + f_1^q + \sum_{i=1}^{q-1} \binom{q}{i} h^i \cdot f_1^{q-i}, \]

\[ (2.9) \quad g_2^q = (h + f_2)^q = h^q + f_2^q + \sum_{i=1}^{q-1} \binom{q}{i} h^i \cdot f_2^{q-i}. \]

Integrating (2.8) on the unit interval of \( \mathbb{K}^2 \) we get by using results similar to (2.25)

\[ (2.10) \quad \sum_{i=1}^{\inf(2s+m-1,k)} \sigma_{i-1,i,i} \cdot k - 2 \cdot \sigma_{i-1,i-1,i} \cdot 2^{-q} = 0 \quad \text{for all } q \geq 2. \]

From (2.10) we obtain
Combining (2.5), (2.6), (2.7), (2.11) and (2.13) we obtain the following recurrent formula for the number of \((2s + m) \times k\) double persymmetric matrices of rank \(i\):

\[
\sigma_{i-1,i,i} = 2 \cdot \sigma_{i-1,i-1,i} \quad \text{for } 0 \leq i \leq \inf(2s + m - 1, k).
\]

Integrating (2.9) on the unit interval of \(\mathbb{K}^2\) we get in the same way:

\[
\inf(2s+m-1,k) \sum_{i=1} \sigma_{i-1,i,i} = 2 \cdot \sigma_{i-1,i-1,i} + \sigma_{i-1,i-1,i} \quad \text{for } 1 \leq i \leq \inf(2s+m-1, k).
\]

Combining (2.5), (2.6), (2.7), (2.11) and (2.13) we obtain the following recurrent formula for the number of \((2s + m) \times k\) double persymmetric matrices of rank \(i\):

\[
\Gamma_i \left[ \begin{array}{c} s \\sqrt{s+m} \end{array} \right] = 2 \cdot \Gamma_{i-1} \left[ \begin{array}{c} s-1 \\sqrt{s-1+m+1} \end{array} \right] + 4 \cdot \Gamma_{i-1} \left[ \begin{array}{c} s \\sqrt{s+(m-1)} \end{array} \right] - 8 \cdot \Gamma_{i-2} \left[ \begin{array}{c} s-1 \\sqrt{s-1+m} \end{array} \right] + \Delta_i \left[ \begin{array}{c} s \\sqrt{s+m} \end{array} \right] \quad \text{for } 0 \leq i \leq \inf(2s + m, k)
\]

where \(\Delta_i \left[ \begin{array}{c} s \\sqrt{s+m} \end{array} \right] \) is equal to

\[
\sigma_{i,i,i} - 3 \cdot \sigma_{i-1,i-1,i-1} + 2 \cdot \sigma_{i-2,i-2,i-2}.
\]

We observe that we have for instance
We deduce (2.16) from the fact that for all

\[ \inf(2s+m-2, k-1) \]

\[ \sum_{j=0}^{\inf(2s+m-2, k-1)} a(s, m, k, q) \cdot 2^{-q} \cdot \# \left( \begin{array}{l} j \\ j \\ j \end{array} \right) \cdot \phi(t, \eta) \cdot \theta_{3}^{-1}(t, \eta) dt d\eta \]

is equal to zero, that is

\[ 0 \]
We deduce (2.17) in a similar way from the fact that for all \( q \geq 2 \) the integral
\[
\int_{\mathcal{P} \times \mathcal{P}} \theta_1(t, \eta) \cdot \theta_2^{-1}(t, \eta) dt d\eta
\]
is equal to zero, that is
\[
\inf(2s + m - 2, k - 1) \sum_{j=0}^{\frac{s-1}{s-m-1}} b(s, m, k, q) \cdot 2^{-jq} \cdot \left[ \left( \begin{array}{c} j \\ j \end{array} \right) \left( \begin{array}{c} j \\ j \end{array} \right) \right]_{\mathcal{P}/\mathcal{P}_{k+1} \times \mathcal{P}/\mathcal{P}_{k+m-1}} + \left( \begin{array}{c} j \\ j \end{array} \right)_{\mathcal{P}/\mathcal{P}_{k+1} \times \mathcal{P}_{k+m-1}} \left( \begin{array}{c} j \\ j \end{array} \right)_{\mathcal{P}/\mathcal{P}_{k+1} \times \mathcal{P}_{k+m-1}} = 0.
\]

In Section 7, we study some rank properties of submatrices of double persymmetric matrices. Consider the following partition of the matrix \( D \) into \((t, \eta)\),
\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
\beta_k & \beta_{k+1} & \beta_{k+2} & \ldots & \beta_{k+m+k-1} & \beta_{k+m+k-2} \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{s+m+k-2} & \beta_{s+m+k-1}
\end{pmatrix}
\]

In fact we prove that for all \( j \in [0, \inf(2s + m - 3, k - 2)] \) we have
\[
(2.18) \quad \# \left( \begin{array}{c} j \\ j \end{array} \right)_{\mathcal{P}/\mathcal{P}_{k+1} \times \mathcal{P}_{k+m-1}} = 0.
\]

We prove (2.18) by contradiction.
Assume on the contrary that there exist \( j_0 \in [0, \inf(2s + m - 3, k - 2)] \) such that
\[
\# \left( \begin{array}{c} j_0 \\ j_0 \end{array} \right)_{\mathcal{P}/\mathcal{P}_{k+1} \times \mathcal{P}_{k+m-1}} > 0.
\]
We show that

\[
\# \left( \begin{array}{cc}
\frac{j_0}{j_0} & \frac{j_0 + 1}{j_0} \\
\frac{j_0}{j_0} & \frac{j_0 + 1}{j_0}
\end{array} \right)_{P/P_{k+j-1} \times P/P_{k+j+m-1}} > 0 \implies \# \left( \begin{array}{cc}
\frac{j_0 - 1}{j_0} & \frac{j_0}{j_0} \\
\frac{j_0 - 1}{j_0} & \frac{j_0}{j_0}
\end{array} \right)_{P_1/P_{k+j-1} \times P_1/P_{k+j+m-1}} > 0
\]

\implies \ldots \implies \# \left( \begin{array}{ccc}
0 & 1 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{array} \right)_{P_{j_0}/P_{k+j-1} \times P_{j_0}/P_{k+j+m-1}} > 0
\]

which obviously contradicts

\[
\# \left( \begin{array}{ccc}
0 & 1 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{array} \right)_{P_{j_0}/P_{k+j-1} \times P_{j_0}/P_{k+j+m-1}} = 0.
\]

These inequalities are a consequence of rank properties of submatrices of double persymmetric matrices.

In Section 8, we study the remainder $\Delta_j^{\left[e \atop e + m\right] \times k}$ in the recurrent formula and we prove in particular that

\[
(2.19) \quad \Delta_j^{\left[e \atop e + m\right] \times (k+1)} - \Delta_j^{\left[e \atop e + m\right] \times k} = 0 \quad \text{if } k \geq j + 1.
\]

Consider still the following partition of the matrix $D_{(t, \eta)}$

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_{k} \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{k} & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{s+m+k-2} & \beta_{s+m+k-1}
\end{pmatrix}.
\]

Let $j \leq \inf(k - 2, 2s + m - 3)$. Then by elementary rank considerations and using (2.18) we obtain...
From (2.20), (2.21) we deduce

\[
\sigma_{j,j,j} \times k = \# \left( \begin{array}{ccc} j & j & j \\ j & j & j \\ j & j & j \end{array} \right) \frac{\#}{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}} + \# \left( \begin{array}{ccc} j & j & j \\ j & j & j \\ j & j & j \end{array} \right) \frac{\#}{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}}
\]

\[
+ \# \left( \begin{array}{ccc} j & j & j \\ j & j & j \\ j & j & j \end{array} \right) \frac{\#}{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}} + \# \left( \begin{array}{ccc} j & j & j \\ j & j & j \\ j & j & j \end{array} \right) \frac{\#}{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}}
\]

\[
= \# \left( \begin{array}{ccc} j & j & j \\ j & j & j \\ j & j & j \end{array} \right) \frac{\#}{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}}
\]

Further by (2.16), (2.17) and (2.18) we have

\[
\sigma_{j,j,j} \times (k-1) = 4 \cdot \sigma_{j,j,j} = \# \left( \begin{array}{ccc} j & j & j \\ j & j & j \\ j & j & j \end{array} \right) \frac{\#}{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}} + \# \left( \begin{array}{ccc} j & j & j \\ j & j & j \\ j & j & j \end{array} \right) \frac{\#}{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}}
\]

\[
+ \# \left( \begin{array}{ccc} j & j & j \\ j & j & j \\ j & j & j \end{array} \right) \frac{\#}{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}} + \# \left( \begin{array}{ccc} j & j & j \\ j & j & j \\ j & j & j \end{array} \right) \frac{\#}{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}}
\]

\[
= 4 \cdot \# \left( \begin{array}{ccc} j & j & j \\ j & j & j \\ j & j & j \end{array} \right) \frac{\#}{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}}
\]

From (2.20), (2.21) we deduce

\[
\sigma_{j,j,j} \times k = \sigma_{j,j,j} \times (j+1)
\]

(2.22) \quad \text{if } j \leq \inf(k - 2, 2s + m - 3).

Using (2.16), (2.17) we obtain easily

\[
\sigma_{j,j,j} \times (j+1) = \# \left( \begin{array}{ccc} j & j & j \\ j & j & j \\ j & j & j \end{array} \right) \frac{\#}{\mathbb{P}/\mathbb{P}_{j+s} \times \mathbb{P}/\mathbb{P}_{j+s+m}}
\]

(2.23)
From (2.23), (2.24) we deduce Theorem 3.3.

Recalling that \( \Delta_i \) is equal to

\[
\sigma_{i,i,i}^{s,m} \times j - 3 \cdot \sigma_{i-1,i-1,i-1}^{s,m} + 2 \cdot \sigma_{i-2,i-2,i-2}^{s,m}
\]

we deduce easily Theorem 3.4.

We get (2.19) from Theorem 3.5.

In Section 9, we establish a recurrent formula for the difference

\[
\Gamma_{i}^{s+m} \times (k+1) - \Gamma_{i}^{s+m} \times k
\]

for \( 2 \leq i \leq 2s + m, \quad k \geq i + 1 \).

In fact we prove that

\[
(2.25) \quad \Gamma_{i}^{s+m} \times (k+1) - \Gamma_{i}^{s+m} \times k = 4 \cdot \left( \Gamma_{i-1}^{s+(m-1)} \times (k+1) - \Gamma_{i-1}^{s+(m-1)} \times k \right) \quad \text{if} \quad 2 \leq i \leq s - 1, \quad k \geq i + 1,
\]

\[
(2.26) \quad \Gamma_{s+j}^{s+m} \times (k+1) - \Gamma_{s+j}^{s+m} \times k = 4 \cdot \left( \Gamma_{s+j-1}^{s+(m-1)} \times (k+1) - \Gamma_{s+j-1}^{s+(m-1)} \times k \right) + 2^{s-1} \left( \Gamma_{j+1}^{1+(s+m-1)} \times (k+1) - \Gamma_{j+1}^{1+(s+m-1)} \times k \right)
\]

\[
- 2^{s+1} \left( \Gamma_{j}^{1+2(s+m-2)} \times (k+1) - \Gamma_{j}^{1+2(s+m-2)} \times k \right) \quad \text{if} \quad 0 \leq j \leq s + m, \quad k \geq s + j + 1.
\]

To prove (2.25) we proceed as follows

Set for \( 1 \leq i \leq \inf(2s + m, k) \)

\[
\Omega_i(s, m, k) = \Gamma_{i}^{s+m} \times k - 4 \cdot \Gamma_{i-1}^{s+(m-1)} \times k.
\]

Then, from the recurrent formula (2.14) we get
From (2.25), (2.26) and (2.30) we get for

\[ \Omega_i(s, m, k) = 2 \cdot \Omega_{i-1}(s-1, m+1, k) + \Delta_i \left[ s \atop s+m \right] \times k. \]

Applying successively (2.27) we obtain

\[ \sum_{j=0}^{i-1} 2^j \cdot \Omega_{i-j}(s-j, m+j, k) = \sum_{j=0}^{i-1} 2^{j+1} \cdot \Omega_{i-(j+1)}(s-(j+1), m+(j+1), k) + \sum_{j=0}^{i-2} 2^j \cdot \Delta_i \left[ s-j \atop s-j+(m+j) \right] \times k. \]

Observing that

\[ \Omega_i(s-(i-1), m+(i-1), k) = \Gamma_i \left[ s-(i+1) \atop s+m \right] \times k - 4 \cdot \Gamma_i \left[ s-i+1 \atop s+(m-1) \right] \times k = 9 - 4 = 5, \]

we get by (2.28) after some simplifications

\[ \sum_{j=0}^{i-2} 2^j \cdot \Delta_i \left[ s-j \atop s-j+(m+j) \right] \times k. \]

Applying (2.19) and (2.29) we obtain (2.25).

Applying successively (2.27) we obtain

\[ \sum_{j=0}^{i-2} 2^j \cdot \Omega_{i-j}(s-j, m+j, k) + \sum_{j=0}^{i-2} 2^j \cdot \Delta_i \left[ s-j \atop s-j+(m+j) \right] \times k. \]

In Section 10, we compute explicitly

\[ 2^{s-1} \left[ \Gamma_{j+1} \left[ 1+(s+m-1) \atop 1 \right] \times (k+1) - \Gamma_{j+1} \left[ 1+(s+m-1) \atop 1 \right] \times k \right] - 2^{s+1} \left[ \Gamma_j \left[ 1+s+m-2 \atop 1 \right] \times (k+1) - \Gamma_j \left[ 1+s+m-2 \atop 1 \right] \times k \right] \]

if \[ 0 \leq j \leq s+m, \quad k \geq s+j+1. \]

In fact we prove

\[ R(j, s, m, k) = \begin{cases} 2^{k+s-1} & \text{if } j = 0, \quad k \geq s+1, \\ -2^{k+s-1} & \text{if } j = 1, \quad k \geq s+2, \\ 0 & \text{if } 2 \leq j \leq s+m-1, \quad k \geq s+j+1, \\ -3 \cdot 2^{2k+2s+m-2} & \text{if } j = s+m, \quad k \geq 2s+m+1. \end{cases} \]

From (2.25), (2.26) and (2.30) we get for \( s \geq 2, m \geq 0 \)
We show precisely and we establish (2.30) by applying the formula for \( \Gamma \) (3.9), with \( m \).

From (2.33), (2.34) we deduce the following results

We prove (2.33) by induction on \( j \).

In Section 11, we establish a formula for \( \Gamma_i^{s+m} \) (obtained in [4], see Theorem 3.9, with \( m \rightarrow s + m - 1, m \rightarrow s + m - 2 \)).

We show precisely

and

From (2.33), (2.34) we deduce the following results

We prove (2.33) by induction on \( j \).
Set

\[(2.38) \quad (H_j) \quad \Gamma_j^{[s+m] \times k} = \Gamma_j^{[j] \times (j+1)} = \gamma_j \quad \text{if} \quad s \geq j+1, k \geq j+1, m \geq 0.\]

Applying (2.1) we get, using successively (2.31) for \(i = 1, 2, 3\)

\[(2.39) \quad \Gamma_i^{[s+m] \times k} = \Gamma_i^{[1] \times 2} = \gamma_1 \quad \text{if} \quad s \geq 2, k \geq 2, m \geq 0.\]

\[(2.40) \quad \Gamma_2^{[s+m] \times k} = \Gamma_2^{[2] \times 3} = \gamma_2 \quad \text{if} \quad s \geq 3, k \geq 3, m \geq 0.\]

\[(2.41) \quad \Gamma_3^{[s+m] \times k} = \Gamma_3^{[3] \times 4} = \gamma_3 \quad \text{if} \quad s \geq 4, k \geq 4, m \geq 0.\]

Thus \((H_j)\) [see (2.38)] holds for \(j \in \{1, 2, 3\}\).

Assume now that \((H_j)\) holds for \(j \in \{1, 2, 3, \ldots, i-1\}\), that is

\[(2.42) \quad \bigwedge_{j=1}^{j=i-1} (H_j) \quad \Gamma_j^{[s+m] \times k} = \Gamma_j^{[j] \times (j+1)} = \gamma_j \quad \text{for all} \quad 1 \leq j \leq i-1, k \geq j+1, \ s \geq j+1, \ m \geq 0.\]

From \(\bigwedge_{j=1}^{j=i-1} (H_j)\) and (2.31) it follows

\[\Gamma_i^{[s+m] \times (k+1)} - \Gamma_i^{[s+m] \times k} = 4 \cdot \left[ \Gamma_{i-1}^{[s+(m-1)] \times (k+1)} - \Gamma_{i-1}^{[s+(m-1)] \times k} \right]\]

is equal to \(\gamma_i\) for all \(s \geq i+1, k \geq i+1\).

Hence \(\Gamma_i^{[s+m] \times (i+1)} = \Gamma_i^{[s+m] \times (i+1)}\) for all \(s \geq i+1, k \geq i+1\).

To get (2.33) we need only to show that \(\Gamma_i^{[s+m] \times (i+1)}\) is equal to \(\Gamma_i^{[i] \times (i+1)}\) for all \(s \geq i+1, m \geq 0\).

To do so we consider the following matrix
Setting \( s = i, m = 0 \) in (2.43), (2.44) we have

\[
\begin{align*}
\Gamma_m j \in \{+1\} \\
\beta_{m+1} & \beta_{m+2} \beta_{m+3} \ldots \beta_{i+m} \beta_{i+m+1} \\
\beta_{s+m-1} & \beta_{s+m} \beta_{s+m+1} \ldots \beta_{s+m+i-2} \beta_{s+m+i-1} \\
\beta_{s+m} & \beta_{s+m+1} \beta_{s+m+2} \ldots \beta_{s+m+i-1} \beta_{s+m+i}
\end{align*}
\]

Using the equations (2.1) with \( k = i+1 \) we obtain

\[
\sum_{j=0}^{i+1} \Gamma_j \left[ \begin{array}{c} s+m \\ j \end{array} \right] ^{x(i+1)} = 2^{2i+2s+m+2}
\]

and

\[
\sum_{j=0}^{i+1} \Gamma_j \left[ \begin{array}{c} s+m \\ j \end{array} \right] ^{x(i+1)} \cdot 2^{-j} = 2^{i+2s+m-1} + 2^{2i} - 2^{i-1}.
\]

Setting \( s = i, m = 0 \) in (2.43), (2.44) we have

\[
\sum_{j=0}^{i+1} \Gamma_j \left[ \begin{array}{c} i \\ j \end{array} \right] ^{x(i+1)} = 2^{4i}
\]

and

\[
\sum_{j=0}^{i+1} \Gamma_j \left[ \begin{array}{c} i \\ j \end{array} \right] ^{x(i+1)} \cdot 2^{-j} = 2^{3i-1} + 2^{2i} - 2^{j-1}.
\]

From (2.43), (2.44), (2.45), (2.46) and \( \bigwedge_{j=1}^{j=i-1} (H_j) \) we deduce (2.34).

In Section 12, using successively the recurrent formula (2.32) for the difference

\[
\Gamma \left[ \begin{array}{c} s+j \\ k \end{array} \right] ^{x(k+1)} - \Gamma \left[ \begin{array}{c} s+j \\ s+j \end{array} \right] ^{x k},
\]

with \( 0 \leq j \leq s + m, \ k \geq s + j + 1 \) we deduce

In the case \( m \in \{0, 1\} \)

\[
\sum_{j=0}^{i+1} \Gamma_j \left[ \begin{array}{c} s+m \\ j \end{array} \right] ^{x(i+1)} \cdot 2^{-j} = 2^{3i-1} + 2^{2i} - 2^{j-1}.
\]

\[
\left\{ \begin{array}{ll}
3 \cdot 2^{k+s-1} & \text{if } j = 0, \ k > s, \\
21 \cdot 2^{k+s+3j-4} & \text{if } 1 \leq j \leq s - 1, \ k > s + j, \\
3 \cdot 2^{k+2s-2} - 3 \cdot 2^{k+4s-4} & \text{if } j = s, \ k > 2s,
\end{array} \right.
\]

\[
\left\{ \begin{array}{ll}
3 \cdot 2^{k+s-1} & \text{if } j = 0, \ k > s, \\
21 \cdot 2^{k+s+3j-4} & \text{if } 1 \leq j \leq s - 1, \ k > s + j, \\
3 \cdot 2^{k+2s-2} - 3 \cdot 2^{k+4s-4} & \text{if } j = s, \ k > 2s,
\end{array} \right.
\]
(2.32) we obtain

\[ \Gamma_{s+j}^{s+1} \times (k+1) - \Gamma_{s+j}^{s+1} \times k = \begin{cases} 
2^{k+s-1} & \text{if } j = 0, k > s, \\
11 \cdot 2^{k+s-1} & \text{if } j = 1, k > s + 1, \\
21 \cdot 2^{k+s+3j-5} & \text{if } 2 \leq j \leq s, k > s + j, \\
3 \cdot 2^{k+2s-1} - 3 \cdot 2^{k+4s-2} & \text{if } j = s + 1, k > 2s + 1.
\end{cases} \]

In the case \( m \geq 2 \)

(2.49) \[ \Gamma_{s+m}^{s+m} \times (k+1) - \Gamma_{s+m}^{s+m} \times k = \begin{cases} 
2^{k+s-1} & \text{if } j = 0, k > s, \\
3 \cdot 2^{k+s+2j-3} & \text{if } 1 \leq j \leq m - 1, k > s + j, \\
11 \cdot 2^{k+s+2m-3} & \text{if } j = m, k > s + m,
\end{cases} \]

(2.50) \[ \Gamma_{s+m+j}^{s+m} \times (k+1) - \Gamma_{s+m+j}^{s+m} \times k = \begin{cases} 
21 \cdot 2^{k+s+2m+3j-4} & \text{if } 1 \leq j \leq s - 1, k > s + m + j, \\
3 \cdot 2^{k+2s+m-2} - 3 \cdot 2^{k+4s+2m-4} & \text{if } j = s, k > 2s + m.
\end{cases} \]

For instance to compute \( \Gamma_{s+j}^{s} \times (k+1) - \Gamma_{s+j}^{s} \times k \) for \( 0 \leq j \leq s - 1, k \geq s + j + 1 \) we proceed as follows:

From (2.32) with \( m = 0, j = 0, k > s + 1 \) we get, using (2.33)

(2.51)

\[ \Gamma_{s}^{s} \times (k+1) - \Gamma_{s}^{s} \times k = 4 \cdot \left[ \Gamma_{s}^{s-1+1} \times (k+1) - \Gamma_{s-1+1}^{s-1} \times k \right] + 2^{k+s-1}, \]

(2.52)

\[ \Gamma_{s-1}^{s-1+1} \times (k+1) - \Gamma_{s-1}^{s-1+1} \times k = 4 \cdot \left[ \Gamma_{s-2}^{s-1} \times (k+1) - \Gamma_{s-2}^{s-1} \times k \right] + 2^{k+s-2} = 2^{k+s-2}. \]

From (2.51), (2.52) we obtain

(2.53)

\[ \Gamma_{s+1}^{s} \times (k+1) - \Gamma_{s+1}^{s} \times k = 4 \cdot 2^{k+s-2} + 2^{k+s-1} = 3 \cdot 2^{k+s-1}. \]

We get in a similar way, using (2.34) with \( s \to s - 1 \)

(2.54)

\[ \Gamma_{s+1}^{s} \times (k+1) - \Gamma_{s+1}^{s} \times k = 21 \cdot 2^{k+s-1}. \]

Let \( 2 \leq j \leq s - 1, k > s + j \).

By (2.32) we obtain

\[ \Gamma_{s+j}^{s} \times (k+1) - \Gamma_{s+j}^{s} \times k = 4 \cdot \left[ \Gamma_{s+j-1}^{s+1} \times (k+1) - \Gamma_{s+j-1}^{s+1} \times k \right] \]

if \( 2 \leq j \leq s - 1, \)
\[
\Gamma_{\frac{s}{s+1}}^{s-1 (j+1)} \times (k+1) - \Gamma_{\frac{s}{s+1}}^{s-1 (j+1)} \times k = 4 \cdot \Gamma_{\frac{s}{s+1}}^{s-1 (j+1)} \times (k+1) - \Gamma_{\frac{s}{s+1}}^{s-1 (j+1)} \times k \] if \(2 \leq j \leq s - 1 + 1 = s - 1.\)

From the above equations we get

(2.55) \[
\Gamma_{\frac{s}{s+j}}^{s} \times (k+1) - \Gamma_{\frac{s}{s+j}}^{s} \times k = 4^2 \cdot \Gamma_{\frac{s}{s+j}}^{s} \times (k+1) - \Gamma_{\frac{s}{s+j}}^{s} \times k \] if \(2 \leq j \leq s - 1.\)

Using successively (2.55) we get from (2.54) with \(s \rightarrow s - j + 1\)

(2.56) \[
\Gamma_{\frac{s}{s+j}}^{s} \times (k+1) - \Gamma_{\frac{s}{s+j}}^{s} \times k = 4^{2(j-1)} \cdot \Gamma_{\frac{s}{s+j}}^{s} \times (k+1) - \Gamma_{\frac{s}{s+j}}^{s} \times k \] if \(2 \leq j \leq s - 1.\)

**In Sections 13, 14, 15, 16 and 17** we compute \(\Gamma_{\frac{s}{s+j}}^{s} \times k\) for \(0 \leq j \leq s + m, \ k \geq s + j, \ m \geq 0.\)

To do so, consider the following array \(\left( \Gamma_{\frac{s}{s+j}}^{s} \times k \right)_{0 \leq j \leq s + m, \ k \geq s + j}:\)

(2.57)

\[
\begin{array}{ccccccc}
\Gamma_{\frac{s}{s+1}}^{s+m} & \Gamma_{\frac{s}{s+1}}^{s+m} & \Gamma_{\frac{s}{s+1}}^{s+m} & \Gamma_{\frac{s}{s+1}}^{s+m} & \ldots & \Gamma_{\frac{s}{s+1}}^{s+m} \\
\Gamma_{\frac{s}{s+j}}^{s+m} & \Gamma_{\frac{s}{s+j}}^{s+m} & \Gamma_{\frac{s}{s+j}}^{s+m} & \Gamma_{\frac{s}{s+j}}^{s+m} & \ldots & \Gamma_{\frac{s}{s+j}}^{s+m} \\
\Gamma_{\frac{s}{s+m}} & \Gamma_{\frac{s}{s+m}} & \Gamma_{\frac{s}{s+m}} & \Gamma_{\frac{s}{s+m}} & \ldots & \Gamma_{\frac{s}{s+m}} \\
\Gamma_{\frac{s}{s+m+1}} & \Gamma_{\frac{s}{s+m+1}} & \Gamma_{\frac{s}{s+m+1}} & \Gamma_{\frac{s}{s+m+1}} & \ldots & \Gamma_{\frac{s}{s+m+1}} \\
\Gamma_{\frac{s}{s+m+2}} & \Gamma_{\frac{s}{s+m+2}} & \Gamma_{\frac{s}{s+m+2}} & \Gamma_{\frac{s}{s+m+2}} & \ldots & \Gamma_{\frac{s}{s+m+2}} \\
\Gamma_{\frac{s}{s+m+3}} & \Gamma_{\frac{s}{s+m+3}} & \Gamma_{\frac{s}{s+m+3}} & \Gamma_{\frac{s}{s+m+3}} & \ldots & \Gamma_{\frac{s}{s+m+3}} \\
\Gamma_{\frac{s}{s+m+4}} & \Gamma_{\frac{s}{s+m+4}} & \Gamma_{\frac{s}{s+m+4}} & \Gamma_{\frac{s}{s+m+4}} & \ldots & \Gamma_{\frac{s}{s+m+4}} \\
\Gamma_{\frac{s}{s+m+5}} & \Gamma_{\frac{s}{s+m+5}} & \Gamma_{\frac{s}{s+m+5}} & \Gamma_{\frac{s}{s+m+5}} & \ldots & \Gamma_{\frac{s}{s+m+5}} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}
\]
To compute the elements in the array \( \Gamma_{i+1}^{s+m} \times k \), we proceed by induction on \( j \) as follows:

Let \( l \) be a rational integer such that \( 0 \leq l \leq s + m - 1 \).

Assume that we have computed the elements in the following subarray \( \Gamma_{s+m}^{s+m} \times k \).

Recall that from (2.33), (2.34) \( \Gamma_{i}^{s+m} \times k \) are known for \( 0 \leq i \leq s - 1, \ k \geq i \).

From the first equation in (2.1) with \( k = s + 1 \), \( \sum_{i=0}^{s+l+1} \Gamma_{i}^{s+m} \times (s+l+1) = 2^{s+2l+m} \)
we deduce \( \Gamma_{s+l+1}^{s+m} \times (s+l+1) \) since the terms \( \Gamma_{i}^{s+m} \times (s+l+1) \) for \( i \leq s + l \) are known.

From the equations in (2.1) with \( k = s + l + 2 \) we get

\[
\begin{align*}
\left\{ \begin{array}{l}
\sum_{i=0}^{s+l+2} \Gamma_{i}^{s+m} \times (s+l+2) = 2^{s+2l+m+2}, \\
\sum_{i=0}^{s+l+2} \Gamma_{i}^{s+m} \times (s+l+2) \cdot 2^{-i} = 2^{s+2l+m} + 2^{s+2l+2} - 2^{s+l+1}.
\end{array} \right.
\]

We deduce \( \Gamma_{s+l+1}^{s+m} \times (s+1+2) \) and \( \Gamma_{s+l+2}^{s+m} \times (s+l+2) \) since the terms \( \Gamma_{i}^{s+m} \times (s+l+2) \) for \( i \leq s + l \) are known.

Then, from (2.49) or (2.50), in the case \( m \geq 2 \), we compute \( \Gamma_{s+l+1}^{s+m} \) knowing \( \Gamma_{s+l+1}^{s+m} \times (s+1+2) \) for all \( k \geq s + l + 2 \).

We have now computed any element in the \( l + 1 \)-th row and the first element in the \( l + 2 \)-th row in the array \( \Gamma_{s+j}^{s+m} \times k \), 0 \leq j \leq s + m, \ k \geq s+j \).

From the first equation in (2.1) with \( k = s \), \( \sum_{i=0}^{s} \Gamma_{i}^{s+m} \times s = 2^{4s+m-2} \)
we deduce \( \Gamma_{s}^{s+m} \times s \) since the terms \( \Gamma_{i}^{s+m} \times s \) for \( i \leq s - 1 \) are known by (2.33), (2.34).

From the equations in (2.1) with \( k = s + 1 \) we obtain

\[
\begin{align*}
\left\{ \begin{array}{l}
\sum_{i=0}^{s+1} \Gamma_{i}^{s+m} \times (s+1) = 2^{4s+m}, \\
\sum_{i=0}^{s+1} \Gamma_{i}^{s+m} \times (s+1) \cdot 2^{-i} = 2^{4s+m-1} + 2^{s} - 2^{s-1}.
\end{array} \right.
\]

We deduce \( \Gamma_{s}^{s+m} \times (s+1) \) and \( \Gamma_{s+1}^{s+m} \times (s+1) \) since the terms \( \Gamma_{i}^{s+m} \times (s+1) \) for \( i \leq s - 1 \) are known.

Then, from (2.49) in the case \( m \geq 2 \) with \( j = 0 \) we compute \( \Gamma_{s}^{s+m} \times k \) for all \( k \geq s + 1 \).
We have now computed any element in the subarray $\left( \Gamma_{s+m}^{k} \right)_{s+j} \cdot k \geq s+j$ with $l = 0$.

- From the equations in (2.1) with $k = s+2$, we have

$$
\begin{align*}
\sum_{i=0}^{s+2} \Gamma_{s+m}^{i} \cdot (s+2) = 2^{4s+m+2}, \\
\sum_{i=0}^{s+2} \Gamma_{s+m}^{i} \cdot (s+2) \cdot 2^{-i} = 2^{3s+m} + 2^{2s+2} - 2^{s}.
\end{align*}
$$

We deduce $\Gamma_{s+1}^{s+m} \cdot (s+2)$ and $\Gamma_{s+2}^{s+m} \cdot (s+2)$ since the terms $\Gamma_{i}^{s+m} \cdot (s+2)$ for $i \leq s$ are known.

Then from (2.49) in the case $m \geq 2$ with $j = 1$ we compute $\Gamma_{s+1}^{s+m} \cdot k$ knowing $\Gamma_{s+1}^{s+m} \cdot (s+2)$ for all $k \geq s+2$.

We have now computed any element in the subarray $\left( \Gamma_{s+m}^{k} \right)_{s+j} \cdot k \geq s+j$ with $l = 1$.

- From the equations in (2.1) with $k = s+3$, we get

$$
\begin{align*}
\sum_{i=0}^{s+3} \Gamma_{s+m}^{i} \cdot (s+3) = 2^{4s+m+4}, \\
\sum_{i=0}^{s+3} \Gamma_{s+m}^{i} \cdot (s+3) \cdot 2^{-i} = 2^{3s+m+1} + 2^{2s+4} - 2^{s+1}.
\end{align*}
$$

We deduce $\Gamma_{s+2}^{s+m} \cdot (s+3)$ and $\Gamma_{s+3}^{s+m} \cdot (s+3)$ since the terms $\Gamma_{i}^{s+m} \cdot (s+3)$ for $i \leq s+1$ are known.

Then from (2.49) with $j = 2$ we compute $\Gamma_{s+2}^{s+m} \cdot k$ knowing $\Gamma_{s+2}^{s+m} \cdot (s+3)$ for all $k \geq s+3$.

We have now computed any element in the subarray $\left( \Gamma_{s+m}^{k} \right)_{s+j} \cdot k \geq s+j$ with $l = 2$.

Further we deduce that

$$
(2.58) \quad \Gamma_{s+2}^{s+m} \cdot k = 8 \cdot \Gamma_{s+1}^{s+m-1} \cdot (k-1) \quad \text{for } k \geq s+2, m \geq 2.
$$

- To compute the elements in the subarray $\left( \Gamma_{s+m}^{k} \right)_{2 \leq j \leq m, k \geq s+j}$ we proceed by induction on $j$.

To do so, let $l$ be a rational integer such that $2 \leq l \leq m$.

Assume $(H_{l-1}) \quad \Gamma_{s+j}^{s+m} \cdot k = 8^{j-1} \cdot \Gamma_{s+1}^{s+(m-s)+(j-1)} \cdot (k-(j-1)) \quad \text{for } 1 \leq j \leq l-1, k \geq s+j$.

$(H_{l-1})$ holds for $l = 2$ and for $l = 3$ by (2.58).
To establish \((H_1)\) we need only to show that

\[
(2.59) \quad \Gamma_{s+l}^{[s+m] \times k} = 8^{l-1} \cdot \Gamma_{s+l}^{[s+(m-(l-1)) \times (k-(l-1))} \quad \text{for} \quad k \geq s + l.
\]

In the case \(k > s + l\) we proceed as follows:

From the equations in (2.1) with \(k = s + l + 1\), we have

\[
\sum_{i=0}^{s+l+1} \Gamma_{s+l+1}^{[s+m] \times (s+l+1)} = 2^{4s+m+2l},
\]

\[
\sum_{i=0}^{s+l+1} \Gamma_{s+l+1}^{[s+m] \times (s+l+1)}, 2^{-i} = 2^{3s+m+l-1} + 2^{2s+2l} - 2^{s+l-1}.
\]

We deduce \(\Gamma_{s+l}^{[s+m] \times (s+l+1)}\) and \(\Gamma_{s+l+1}^{[s+m] \times (s+l+1)}\) since by \((H_{l-1})\) the terms \(\Gamma_{s+l}^{[s+m] \times (s+l+1)}\) for \(i \leq s + l - 1\) are known.

Then from (2.49) with \(j = l\) we compute \(\Gamma_{s+l}^{[s+m] \times k}\) for all \(k \geq s + l + 1\) and (2.59) follows.

The case \(k = s + l\) is obtained in a similar way.

- To compute the elements in the remaining subarray \( \Gamma_{s+m+1+j}^{[s+m] \times k} \) \(0 \leq j \leq s-1, k \geq s+m+1+j\)
  we proceed still by induction on \(j\).

Let \(j = 0\), a similar proof of (2.58) gives the following identity:

\[
(2.60) \quad \Gamma_{s+m+1}^{[s+m] \times k} = 8^m \cdot \Gamma_{s+1}^{[s] \times (k-m)} \quad \text{for} \quad k \geq s + m + 1.
\]

Let \(l\) be a rational integer such that \(1 \leq l \leq s - 1\).

Assume \((H_{l-1})\) \(\Gamma_{s+m+1+j}^{[s+m] \times k} = 8^{2j+m} \cdot \Gamma_{s-j+1}^{[s-j] \times (k-m-2j)}\) \(0 \leq j \leq l-1, k \geq s + m + 1 + j\).

From (2.60) \((H_{l-1})\) holds for \(l = 1\).

To establish \((H_l)\) we need only to show that

\[
(2.61) \quad \Gamma_{s+m+1+l}^{[s+m] \times k} = 8^{2l+m} \cdot \Gamma_{s-l+1}^{[s-l] \times (k-(l-1))} \quad \text{for} \quad k \geq s + m + 1 + l.
\]

In the case \(k > s + m + 1 + l\) we proceed as follows:

From the equations in (2.1) with \(k = s + m + l + 2\), we get

\[
\sum_{i=0}^{s+m+l+2} \Gamma_{s+m+l+2}^{[s+m] \times (s+m+l+2)} = 2^{4s+3m+2l+2},
\]

\[
\sum_{i=0}^{s+m+l+2} \Gamma_{s+m+l+2}^{[s+m] \times (s+m+l+2)}, 2^{-i} = 2^{3s+2m+l} + 2^{2s+2m+2l+2} - 2^{s+m+l}.
\]

We deduce \(\Gamma_{s+m+l+1}^{[s+m] \times (s+m+l+2)}\) and \(\Gamma_{s+m+l+2}^{[s+m] \times (s+m+l+2)}\) since by \((H_{l-1})\) the terms \(\Gamma_{s+m+l+1}^{[s+m] \times (s+m+l+2)}\) for \(i \leq s + m + l\) are known.
Then from (2.50) with $j = l$ knowing $\Gamma_{s+m}^{s\times (s+m+2)}$ we compute $\Gamma_{s+m}^{s\times k}$ for all $k \geq s + m + l + 1$ and (2.61) follows.

The case $k = s + m + l + 1$ is obtained in a similar way.

3. STATEMENT OF RESULTS

Theorem 3.1. Let $q$ be a rational integer $\geq 1$, then

\[(3.1)\]

$$g_{k,s,m}(t, \eta) = g(t, \eta) = \sum_{deg Y \leq k-1} \sum_{deg Z \leq s-1} E(tYZ) \sum_{deg U \leq s+m-1} E(\eta YU) = 2^{2s+m+k-r(D_{s+m}^{s\times k}(t, \eta))},$$

\[(3.2)\]

$$\int_{P \times P} g_{k,s,m}(t, \eta) dtd\eta = 2^{(2s+m+k)(q-1)} \cdot 2^{k+2} \cdot \inf(2s+m,k) \sum_{i=0}^{\inf(2s+m,k)} \Gamma_{i}^{s\times k} \cdot 2^{-qi}.$$ 

Theorem 3.2. Let $s \geq 2$, $m \geq 0$, $k \geq 1$ and $0 \leq i \leq \inf(2s+m,k)$. Then we have the following recurrent formula for the number $\Gamma_{i}^{s\times k}$ of rank $i$ matrices of the form $[A]_{P}$, such that $A$ is a $s \times k$ persymmetric matrix and $B$ a $(s + m) \times k$ persymmetric matrix with entries in $F_{2}$:

\[(3.3)\]

$$\Gamma_{i}^{s\times k} = 2 \cdot \Gamma_{i-1}^{s-1\times (m+1)} + 4 \cdot \Gamma_{i-1}^{s\times (m-1)} - 8 \cdot \Gamma_{i-2}^{s-1\times m} + \Delta_{i}^{s\times m},$$

where the remainder $\Delta_{i}^{s\times m}$ is equal to

\[(3.4)\]

$$\sigma_{i,i,i}^{s\times m} - 3 \cdot \sigma_{i-1,i-1,i-1}^{s\times m} + 2 \cdot \sigma_{i-2,i-2,i-2}^{s\times m}.$$ 

Recall that

$$\sigma_{i,i,i}^{s\times m} = \sum_{\beta_{s+m-1}^{s-1\times m-1}} \beta_{s+m-1}^{s-1\times m-1} \times k.$$ 

is equal to the cardinality of the following set

$$\left\{(t, \eta) \in P/P_{k+s-1} \times P/P_{k+s+m-1} | r(D_{s-1\times m}^{s\times k}(t, \eta)) = r(D_{s\times m-1}^{s+m\times k}(t, \eta)) = r(D_{s+m\times k}^{s+m\times k}(t, \eta)) = i \right\}. $$
Theorem 3.3. Let $s \geq 2$ and $m \geq 0$, we have in the following two cases:

The case $1 \leq k \leq 2s + m - 2$

\[
\sigma_{i,i,i}^{\left(\frac{s}{s+m}\right)\times k} = \begin{cases} 
1 & \text{if } i = 0, \ k \geq 1, \\
4 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times i} - \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (i+1)} & \text{if } 1 \leq i \leq k - 1, \\
4 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times k} & \text{if } i = k.
\end{cases}
\]

The case $k \geq 2s + m - 2$

\[
\sigma_{i,i,i}^{\left(\frac{s}{s+m}\right)\times k} = \begin{cases} 
1 & \text{if } i = 0, \\
4 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times i} - \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (i+1)} & \text{if } 1 \leq i \leq 2s + m - 3, \\
4 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (2s+m-2)} & \text{if } i = 2s + m - 2.
\end{cases}
\]

Theorem 3.4. The remainder $\Delta_i^{\left(\frac{s}{s+m}\right)\times k}$ in the recurrent formula is equal to

\[
\begin{align*}
&\begin{cases} 
1 & \text{if } i = 0, k \geq 1, \\
4 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times 1} - \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times 2} & \text{if } i = 1, k \geq 2, \\
4 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times 1} - 3 & \text{if } i = 1, k = 1, \\
7 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times 2} - 12 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times 1} - \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times 3} + 2 & \text{if } i = 2, k \geq 3, \\
7 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times 2} - 12 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times 1} + 2 & \text{if } i = 2, k = 2, \\
7 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times i} - 14 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (i-1)} + 8 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (i-2)} - \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (i+1)} & \text{if } 3 \leq i \leq 2s + m - 3, k \geq i + 1, \\
7 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times i} - 14 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (i-1)} + 8 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (i-2)} & \text{if } 3 \leq i \leq 2s + m - 3, k = i, \\
7 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (2s+m-2)} - 14 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (2s+m-3)} + 8 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (2s+m-4)} & \text{if } i = 2s + m - 2, k \geq i, \\
-14 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (2s+m-2)} + 8 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (2s+m-3)} & \text{if } i = 2s + m - 1, k \geq i, \\
8 \cdot \Gamma_i^{\left(\frac{s-1}{s-1+m}\right)\times (2s+m-2)} & \text{if } i = 2s + m, k \geq i.
\end{cases}
\end{align*}
\]

Theorem 3.5. We have

\[
\Delta_i^{\left(\frac{s}{s+m}\right)\times k} = \Delta_i^{\left(\frac{s}{s+m}\right)\times (i+1)} \quad \text{for } i \in [0, 2s + m - 3], k \geq i + 1,
\]

\[
\Delta_i^{\left(\frac{s}{s+m}\right)\times k} = \Delta_i^{\left(\frac{s}{s+m}\right)\times i} \quad \text{for } i \in \{2s + m - 2, 2s + m - 1, 2s + m\}, k \geq i.
\]

Theorem 3.6. We have for all $m \geq 0$
We have for
\[\Gamma_j^{s+m} \times (k+1) - \Gamma_j^{s+m} \times k = 0 \quad \text{if} \quad 0 \leq j \leq s - 1, \quad k > j.\]

We have in the cases \(m \in \{0, 1\}\)

\[\Gamma_{s+j}^{s} \times (k+1) - \Gamma_{s+j}^{s} \times k = \begin{cases} 
3 \cdot 2^k s - 1 & \text{if} \quad j = 0, \quad k > s, \\
21 \cdot 2^{k+s+3j-4} & \text{if} \quad 1 \leq j \leq s - 1, \quad k > s + j, \\
3 \cdot 2^{2k+2s-2} - 3 \cdot 2^{k+4s-4} & \text{if} \quad j = s, \quad k > 2s, 
\end{cases}\]

\[\Gamma_{s+j}^{s+1} \times (k+1) - \Gamma_{s+j}^{s+1} \times k = \begin{cases} 
2^k s - 1 & \text{if} \quad j = 0, \quad k > s, \\
11 \cdot 2^k s - 1 & \text{if} \quad j = 1, \quad k > s + 1, \\
21 \cdot 2^{k+s+3j-5} & \text{if} \quad 2 \leq j \leq s, \quad k > s + j, \\
3 \cdot 2^{2k+2s-1} - 3 \cdot 2^{k+4s-2} & \text{if} \quad j = s + 1, \quad k > 2s + 1. 
\end{cases}\]

In the case \(m \geq 2\)

\[\Gamma_{s+m+j}^{s+m} \times (k+1) - \Gamma_{s+m+j}^{s+m} \times k = \begin{cases} 
2^k s - 1 & \text{if} \quad j = 0, \quad k > s, \\
3 \cdot 2^k s + 2j - 3 & \text{if} \quad 1 \leq j \leq m - 1, \quad k > s + j, \\
11 \cdot 2^{k+s+2m-3} & \text{if} \quad j = m, \quad k > s + m, 
\end{cases}\]

\[\Gamma_{s+m+j}^{s+m} \times (k+1) - \Gamma_{s+m+j}^{s+m} \times k = \begin{cases} 
21 \cdot 2^{k+s+2m+3j-4} & \text{if} \quad 1 \leq j \leq s - 1, \quad k > s + m + j, \\
3 \cdot 2^{2k+2s+m-2} - 3 \cdot 2^{k+4s+2m-4} & \text{if} \quad j = s, \quad k > 2s + m. 
\end{cases}\]

**Theorem 3.7.** We have for \(m \geq 1\)

\[\Gamma_{s+j}^{s+m} \times k = 8^{j-1} \cdot \Gamma_{s+1}^{s+(m-(j-1))} \times (k-(j-1)) \quad \text{if} \quad 1 \leq j \leq m, \quad k \geq s + j,\]

\[\Gamma_{s+1}^{s+(m-(j-1))} \times (s+1) = 2^{4s+(m-(j-1))} - 3 \cdot 2^{3s-1} + 2^{2s-1} \quad \text{if} \quad 1 \leq j \leq m, \quad k = s + j,\]

\[\Gamma_{s+1}^{s+(m-(j-1))} \times (k-(j-1)) = 3 \cdot 2^{k-j+s} + 21 \cdot [2^3s-1 - 2^{2s-1}] \quad \text{if} \quad 1 \leq j \leq m - 1, \quad k > s + j,\]

\[\Gamma_{s+1}^{s+(m-(j-1))} \times (k-(m-1)) = 11 \cdot 2^{k-m+s} + 21 \cdot 2^{3s-1} - 11 \cdot 2^{2s-1} \quad \text{if} \quad j = m, \quad k > s + m.\]

**Theorem 3.8.** We have for \(m \geq 0\)

\[\Gamma_{s+j}^{s+m} \times k = 8^{j+m} \cdot \Gamma_{s-j}^{s-(m-2j)} \times (k-m-2j) \quad \text{if} \quad 0 \leq j \leq s - 1, \quad k \geq s + m + 1 + j,\]
Theorem 3.9. We have

\[
\Gamma_{s-j} \times (s-j+1)
\]

\[
= 2^{4s-4j} - 3 \cdot 2^{3s-3j} - 1 + 2^{2s-2j} - 1
\]

if \(0 \leq j \leq s-1, k = s + m + 1 + j,\)

\[
\Gamma_{s-j} \times (k-m-2j)
\]

\[
= 21 \cdot [2^{k-m-3j+1} + 2^{3s-3j-1} - 5 \cdot 2^{2s-2j-1}]
\]

if \(0 \leq j \leq s-2, k > s + m + 1 + j,\)

\[
\Gamma_{2} \times (k-m-2s+2)
\]

\[
= 2^{2(k-m)-4s+4} - 3 \cdot 2^{k-m-2s+2} + 2
\]

if \(j = s-1, k > 2s + m.\)

Theorem 3.10. We have

\[
\Gamma_{s} \times k
\]

\[
= \begin{cases} 
1 & \text{if } i = 0, k \geq 1, \vspace{1mm} \\
21 \cdot 2^{3i-4} - 3 \cdot 2^{2i-3} & \text{if } 1 \leq i \leq s-1, k > i, \vspace{1mm} \\
3 \cdot 2^{k+s-1} + 21 \cdot 2^{3s-4} - 27 \cdot 2^{2s-3} & \text{if } i = s, k > s, \vspace{1mm} \\
21 \cdot [2^{k-2s+3i-4} + 2^{3i-4} - 5 \cdot 2^{4i-2s-5}] & \text{if } s+1 \leq i \leq 2s-1, k > i, \vspace{1mm} \\
2^{k+2s-2} - 3 \cdot 2^{k+4s-4} + 2^{6s-5} & \text{if } i = 2s, k > 2s. 
\end{cases}
\]

Theorem 3.11. We have

\[
\Gamma_{s} \times i
\]

\[
= \begin{cases} 
2^{2s+2i-2} - 3 \cdot 2^{3i-4} + 2^{2i-3} & \text{if } 1 \leq i \leq s, \vspace{1mm} \\
2^{2s+2i-2} - 3 \cdot 2^{3i-4} + 2^{4i-2s-5} & \text{if } s+1 \leq i \leq 2s. 
\end{cases}
\]

Theorem 3.12. We have

\[
\Gamma_{s-1} \times k
\]

\[
= \begin{cases} 
1 & \text{if } i = 0, k \geq 1, \vspace{1mm} \\
21 \cdot 2^{3i-4} - 3 \cdot 2^{2i-3} & \text{if } 1 \leq i \leq s-1, k > i, \vspace{1mm} \\
2^{k+s-1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3} & \text{if } i = s, k > s, \vspace{1mm} \\
11 \cdot 2^{k+s-1} + 21 \cdot 2^{3s-1} - 53 \cdot 2^{2s-1} & \text{if } i = s+1, k > s+1, \vspace{1mm} \\
21 \cdot [2^{k-2s+3i-5} + 2^{3i-4} - 5 \cdot 2^{4i-2s-6}] & \text{if } s+2 \leq i \leq 2s, k > i, \vspace{1mm} \\
2^{k+2s-1} - 3 \cdot 2^{k+4s-2} + 2^{6s-2} & \text{if } i = 2s+1, k > 2s+1. \end{cases}
\]

Theorem 3.13. We have for \(m \geq 2\)

\[
\Gamma_{s+1} \times i
\]

\[
= \begin{cases} 
2^{2s+2i-1} - 3 \cdot 2^{3i-4} + 2^{2i-3} & \text{if } 1 \leq i \leq s+1, \vspace{1mm} \\
2^{2s+2i-2} - 3 \cdot 2^{3i-4} + 2^{4i-2s-6} & \text{if } s+2 \leq i \leq 2s+1. \end{cases}
\]
Example. Let \( q = 3, k = 4, s = 3, m = 2 \). Then

\[
\Gamma_i^{[s+m] \times k} = \begin{cases} 
1 & \text{if } i = 0, k \geq 1, \\
21 \cdot 2^{3i-4} - 3 \cdot 2^{2i-3} & \text{if } 1 \leq i \leq s - 1, k > i, \\
2^{k+s-1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3} & \text{if } i = s, k > s, \\
3 \cdot 2^{k+s+2i-3} + 21 \cdot [2^{3i-4} - 2^{3i-s-4}] & \text{if } s + 1 \leq i \leq s + m - 1, k > i, \\
11 \cdot 2^{k+s+2m-3} + 21 \cdot 2^{3s+3m-4} - 53 \cdot 2^{2s+3m-4} & \text{if } i = s + m, k > s + m, \\
2^{k+2s+m-2} - 3 \cdot 2^{k+4s+2m-4} + 2^{6s+3m-5} & \text{if } i = 2s + m, k > 2s + m.
\end{cases}
\]

Theorem 3.14. We have for \( m \geq 2 \)

\[
\Gamma_i^{[s] \times i} = \begin{cases} 
2^{2s+2i+m-2} - 3 \cdot 2^{3i-4} + 2^{2i-3} & \text{if } 1 \leq i \leq s + 1, \\
2^{2s+2i+m-2} - 3 \cdot 2^{3i-4} + 2^{3i-s-4} & \text{if } s + 2 \leq i \leq s + m + 1, \\
2^{2s+2i+m-2} - 3 \cdot 2^{3i-4} + 2^{4i-2s-m-5} & \text{if } s + m + 2 \leq i \leq 2s + m.
\end{cases}
\]

Theorem 3.15. We denote by \( R_q(k, s, m) \) the number of solutions \((Y_1, Z_1, U_1, \ldots, Y_q, Z_q, U_q)\) of the polynomial equations

\[
\begin{align*}
Y_1 Z_1 + Y_2 Z_2 + \ldots + Y_q Z_q &= 0, \\
Y_1 U_1 + Y_2 U_2 + \ldots + Y_q U_q &= 0,
\end{align*}
\]

satisfying the degree conditions

\[
deg Y_i \leq k - 1, \quad deg Z_i \leq s - 1, \quad deg U_i \leq s + m - 1 \quad \text{for} \quad 1 \leq i \leq q.
\]

Then

\[
R_q(k, s, m) = \int_{\mathbb{P} \times \mathbb{P}} g_{k, s, m}(t, \eta)dt d\eta
\]

\[
= 2^{(2s+m+k)(q-1)} \cdot 2^{-k+2} \sum_{i=0}^{\inf(2s+m,k)} \Gamma_i^{[s] \times k} \cdot 2^{-qi}.
\]

Example. Let \( q = 3, k = 4, s = 3, m = 2 \). Then

\[
\Gamma_i^{[3] \times 4} = \begin{cases} 
1 & \text{if } i = 0, \\
9 & \text{if } i = 1, \\
78 & \text{if } i = 2, \\
648 & \text{if } i = 3, \\
15648 & \text{if } i = 4.
\end{cases}
\]

Hence the number \( R_3(4, 3, 2) \) of solutions

\((Y_1, Z_1, U_1, Y_2, Z_2, U_2, Y_3, Z_3, U_3)\) of the polynomial equations

\[
\begin{align*}
Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3 &= 0, \\
Y_1 U_1 + Y_2 U_2 + Y_3 U_3 &= 0,
\end{align*}
\]

satisfying the degree conditions

\[
deg Y_i \leq 3, \quad deg Z_i \leq 2, \quad deg U_i \leq 4 \quad \text{for} \quad 1 \leq i \leq 3,
\]

is equal to
\[
\int_{\mathbb{P} \times \mathbb{P}} g^3_{4,3,2}(t, \eta) dt \, d\eta = 2^{22} \cdot \sum_{i=0}^{4} \Gamma_i^{[3+2]} \times 2^{-3i}
\]
\[
= 2^{22} \cdot [1 + 9 \cdot 2^{-3} + 78 \cdot 2^{-6} + 648 \cdot 2^{-9} + 15648 \cdot 2^{-12}]
\]
\[
= 35356672.
\]

Example. Let \( q = 4, k = 6, s = 5, m = 0 \). Then
\[
\Gamma_i^{[5 \times 6]} = \begin{cases}
1 & \text{if } i = 0, \\
9 & \text{if } i = 1, \\
78 & \text{if } i = 2, \\
648 & \text{if } i = 3, \\
5280 & \text{if } i = 4, \\
42624 & \text{if } i = 5, \\
999936 & \text{if } i = 6.
\end{cases}
\]

Hence the number \( R_4(6, 5, 0) \) of solutions
\[
(Y_1, Z_1, U_1, Y_2, Z_2, U_2, Y_3, Z_3, U_3, Y_4, Z_4, U_4)
\]
of the polynomial equations
\[
\begin{aligned}
Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3 + Y_4 Z_4 &= 0, \\
Y_1 U_1 + Y_2 U_2 + Y_3 U_3 + Y_4 U_4 &= 0,
\end{aligned}
\]
satisfying the degree conditions
\[
\deg Y_i \leq 5, \quad \deg Z_i \leq 4, \quad \deg U_i \leq 4 \quad \text{for } 1 \leq i \leq 4
\]
is equal to
\[
\int_{\mathbb{P} \times \mathbb{P}} g^4_{4,5,0}(t, \eta) dt \, d\eta
\]
\[
= 2^{44} \cdot \sum_{i=0}^{6} \Gamma_i^{[5 \times 6]} \times 2^{-4i}
\]
\[
= 2^{44} \cdot [1 + 9 \cdot 2^{-4} + 78 \cdot 2^{-8} + 648 \cdot 2^{-12} + 5280 \cdot 2^{-16} + 42624 \cdot 2^{-20} + 999936 \cdot 2^{-24}]
\]
\[
= 37014016 \cdot 2^{20}.
\]

Example. The fraction of square double persymmetric \([ \begin{smallmatrix} s \\ s \\ s+m \end{smallmatrix} \times (2s+m) \]) matrices which
are invertible is equal to \( \frac{\Gamma_i^{[s \times m]} \times (2s+m)}{\sum_{i=0}^{2s+m} \Gamma_i^{[s \times m]} \times (2s+m)} = \frac{3}{8} \).

4. EXPONENTIAL SUMS FORMULAS ON \( \mathbb{K}^2 \)

In this section we compute exponential quadratic sums in \( \mathbb{P}^2 \) and show that they only depend on rank properties of some associated double persymmetric matrices. The following propositions are proved in [2].

**Proposition 4.1.** The following holds:

- For every rational integer \( j \), the measure of \( \mathbb{P}_j \) is \( 2^{-j} \).
- For every \( A \in \mathbb{F}_2[T], \ E(A) = 1 \).
- For \( u \in \mathbb{K} \) \( \nu(u) \geq 2 \Rightarrow E(u) = 1 \).
Proposition 4.2. Let \( j \) be a rational integer and \( u \in \mathbb{K} \). Then
\[
\int_{F_{j}} E(ut) dt = \begin{cases} 2^{-j} & \text{if } \nu(u) > -j, \\
0 & \text{otherwise.}
\end{cases}
\]

Proposition 4.3. Let \( j \) be a rational integer and let \( u \in \mathbb{K} \). Then
\[
\sum_{degB \leq j} E(Bu) = \begin{cases} 2^{j+1} & \text{if } \nu(\{u\}) > j + 1, \\
0 & \text{otherwise.}
\end{cases}
\]

Lemma 4.4. Let \( Y \in \mathbb{F}_2[T] \) and \( a \) a rational integer \( \geq 1 \). Then
\[
\sum_{degZ \leq a-1} E(tYZ) = \begin{cases} 2^a & \text{if } \nu(\{tY\}) > a, \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. The Lemma follows from the Proposition 4.2 with \( u = tY \) and \( j = a - 1 \).

Lemma 4.5. Let \( t \in \mathbb{P} \) and \( Y \in \mathbb{F}_2[T] \), \( degY \leq b - 1 \). Then
\[
\nu(\{tY\}) > a \iff Y \in \ker D_{a \times b}(t).
\]

Proof. Let \( Y = \sum_{j=0}^{b-1} \gamma_j T^j \), \( \gamma_j \in \mathbb{F}_2 \). Then
\[
tY = (\sum_{i \geq 1} \alpha_i T^{-i})(\sum_{j=0}^{b-1} \gamma_j T^j) = \sum_{i \geq 1} \gamma_i T^{-(i-j)}
\]
and
\[
\{tY\} = \left( \sum_{i=1}^{b} \alpha_i \gamma_i \right) T^{-1} + \left( \sum_{i=2}^{b+1} \alpha_i \gamma_i \right) T^{-2} + \ldots + \left( \sum_{i=s}^{b+a-1} \alpha_i \gamma_i \right) T^{-a} + \ldots
\]
Therefore \( \nu(\{tY\}) \geq a \) if and only if
\[
\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \ldots & \alpha_b \\
\alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \ldots & \alpha_{b+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\
\alpha_a & \alpha_{a+1} & \alpha_{a+2} & \alpha_{a+3} & \alpha_{a+4} & \ldots & \alpha_{b+a-1}
\end{bmatrix}
\begin{bmatrix}
\gamma_0 \\
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 \\
\vdots \\
\gamma_{b-1}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

\[
\iff D_{a \times b}(t)
\begin{bmatrix}
\gamma_0 \\
\gamma_1 \\
\vdots \\
\gamma_{b-1}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\iff Y \in \ker D_{a \times b}(t).
\]

Lemma 4.6. Let \((t, \eta) \in \mathbb{P} \times \mathbb{P}\). Then
\[
(1.1) \quad g(t, \eta) = \sum_{degY \leq k-1} \sum_{degZ \leq s-1} E(tYZ) \sum_{degU \leq s+m-1} E(\eta YU) = 2^{2s+m+k-r(D_{r+m} \times (t, \eta))}.
\]
Proof. By lemma 4.3 and lemma 4.7, we obtain with \( a \to s \) and \( b \to k \)
\[
\sum_{\deg Z \leq s-1} E(tYZ) = 2^s \quad \text{if and only if} \quad Y \in \ker D_{s \times k}(t).
\]
Similarly with \( t \to \eta \), \( a \to s + m \) and \( b \to k \) we get
\[
\sum_{\deg U \leq s + m - 1} E(\eta YU) = 2^{s + m} \quad \text{if and only if} \quad Y \in \ker D_{(s + m) \times k}(\eta).
\]
We then deduce
\[
\sum_{\deg Z \leq s - 1} E(tYZ) \sum_{\deg U \leq s + m - 1} E(\eta YU) = 2^{s + m} \quad \text{if and only if} \quad Y \in \ker D_{[s + m] \times k}(t, \eta).
\]
and we obtain
\[
\sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s + m - 1} E(\eta YU) = 2^{s + m} \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-2} E(tYZ) \sum_{\deg U \leq s + m - 1} E(\eta YU) + 1 = 2^{s + m} 2^{k-r} D_{[s + m] \times k}(t, \eta).
\]

Lemma 4.7. Let \((t, \eta) \in \mathbb{P} \times \mathbb{P}\) and set
\[
g_1(t, \eta) = \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s + m - 1} E(\eta YU).
\]
Then (4.2)
\[
g_1(t, \eta) = \begin{cases} 2^{s + m + k - 1 - r} D_{[s + m] \times k}(t, \eta) & \text{if } r(D_{[s + m] \times k}(t, \eta)) = r(D_{[s + m] \times k}(t, \eta)) \text{,} \\ 0 & \text{otherwise} \end{cases}.
\]
Proof. We have
\[
g_1(t, \eta) = \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s + m - 1} E(\eta YU) - \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-2} E(tYZ) \sum_{\deg U \leq s + m - 1} E(\eta YU).
\]
By (4.1) we obtain
\[
g_1(t, \eta) = 2^{s + m + k - 1 - r} D_{[s + m] \times k}(t, \eta) - 2^{s + m + k - 1 - r} D_{[s + m] \times k}(t, \eta).
\]
Now either
\[
r(D_{[s + m] \times k}(t, \eta)) = r(D_{[s + m] \times k}(t, \eta))
\]
or
\[
r(D_{[s + m] \times k}(t, \eta)) = r(D_{[s + m] \times k}(t, \eta)) + 1.
\]
Lemma 4.7 follows. □

Lemma 4.8. Let \((t, \eta) \in \mathbb{P} \times \mathbb{P}\) and set
\[
g_2(t, \eta) = \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s - 1 + m} E(\eta YU),
\]
\[
f_2(t, \eta) = \sum_{\deg Y \leq k-1} \sum_{\deg Z = s-1} E(tYZ) \sum_{\deg U \leq s - 2 + m} E(\eta YU),
\]
Let Lemma 4.10, which proves Lemma 4.9. □

We have

\[ E(tYZ) \sum_{\deg Z = s-2} E(\eta YU). \]

Then

\[ (4.3) \]

\[ g_2(t, \eta) = \begin{cases} 
2^{2s+m+k-1} - r(D[[s+1+\nu]]_{t,\eta}) & \text{if } r(D[[s+1+\nu]]_{t,\eta}) = r(D[[s+(m-1)]_{t,\eta}) , \\
0 & \text{otherwise}, 
\end{cases} \]

\[ f_1(t, \eta) = \begin{cases} 
2^{2s+m+k-2} - r(D[[s+1-\nu]]_{t,\eta}) & \text{if } r(D[[s+1-\nu]]_{t,\eta}) = r(D[[s+(m-1)]_{t,\eta}) , \\
0 & \text{otherwise}, 
\end{cases} \]

\[ f_2(t, \eta) = \begin{cases} 
2^{2s+m+k-2} - r(D[[s-1+\nu]]_{t,\eta}) & \text{if } r(D[[s-1+\nu]]_{t,\eta}) = r(D[[s+(m-1)]_{t,\eta}) , \\
0 & \text{otherwise}. 
\end{cases} \]

Proof. Similarly to the proof of Lemma 4.7. □

Lemma 4.9. Set \((t, \eta) = (\sum_{j \geq 1} \alpha_j T^{-j}; \sum_{j \geq 1} \beta_j T^{-j}) \in \mathbb{P} \times \mathbb{P} \text{ and } Y = \sum_{j=1}^{k-1} \delta_j T^j \in F_2[T], \deg Y \leq k - 1. \]

Then

\[ (4.6) \]

\[ E(Y(tT^{s-1} + \eta T^{s+m-1})) = \begin{cases} 
1 & \text{if } \sum_{j=1}^{k-1} (\alpha_{s+j} + \beta_{s+m+j}) \delta_j = 0, \\
-1 & \text{if } \sum_{j=1}^{k-1} (\alpha_{s+j} + \beta_{s+m+j}) \delta_j = 1. 
\end{cases} \]

Proof. We have

\[ Y(tT^{s-1} + \eta T^{s+m-1}) = \sum_{j=0}^{k-1} \delta_j T^j \cdot \left[ \sum_{i \geq 1} \alpha_i T^{-i} \cdot T^{s-1} + \sum_{l \geq 1} \beta_l T^{-l} \cdot T^{s+m-1} \right] \]

\[ = \sum_{j=0}^{k-1} \left[ \sum_{i \geq 1} \alpha_i \delta_j T^{s-1-i+j} + \sum_{l \geq 1} \beta_l \delta_j T^{s+m-1-l+j} \right] \]

\[ = \ldots + \sum_{j=0}^{k-1} (\alpha_{s+j} + \beta_{s+m+j}) \delta_j \cdot T^{-1} + \ldots \]

Now

\[ \nu (\{Y(tT^{s-1} + \eta T^{s+m-1})\}) > 1 \iff \sum_{j=0}^{k-1} (\alpha_{s+j} + \beta_{s+m+j}) \delta_j = 0, \]

\[ \nu (\{Y(tT^{s-1} + \eta T^{s+m-1})\}) = 1 \iff \sum_{j=0}^{k-1} (\alpha_{s+j} + \beta_{s+m+j}) \delta_j = 1, \]

which proves Lemma 4.9. □

Lemma 4.10. Let \((t, \eta) \in \mathbb{P} \times \mathbb{P} \text{ and set} \)

\[ h(t, \eta) = \sum_{\deg Y \leq k-1} \sum_{\deg Z = s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta YU). \]
Then
\[(4.7)\]
\[
\begin{align*}
h(t, \eta) &= \begin{cases} 
2^{2s+m+k-2-r(D)} & \frac{s-1}{s-1+m} \times k (t, \eta) \quad \text{if } r(D) = r(D) \\
0 & \text{otherwise }
\end{cases}
\end{align*}
\]

Lemma 4.11. Let \((t, \eta) \in \mathbb{P} \times \mathbb{P}\) and set
\[
\begin{align*}
h(t, \eta) &= \sum_{\deg Y \leq k-1} \sum_{\deg Z = s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta U), \\
v(t, \eta) &= \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-2} E(tYZ) \sum_{\deg U \leq s+m-2} E(\eta U), \\
f_s(t, \eta) &= \sum_{\deg Y \leq k-1} \sum_{\deg Z = s-1} E(tYZ) \sum_{\deg U \leq s+m-2} E(\eta U),
\end{align*}
\]
Proof. We have by the proof of lemma 4.10 and equation 1.10:
\[
\begin{align*}
h(t, \eta) &= \sum_{\deg Y \leq k-1} \sum_{\deg Z = s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta U) \\
&= \sum_{\deg Y \leq k-1} E(Y(tT^{s-1} + \eta T^{s+m-1})) \sum_{\deg Z \leq s-2} E(tYZ) \sum_{\deg U \leq s+m-2} E(\eta U) \\
&= 2^{2s+m-2} \sum_{\deg Y \leq k-1} E(Y(tT^{s-1} + \eta T^{s+m-1}))
\end{align*}
\]
\[
\begin{align*}
&= 2^{2s+m-2} \left[ \sum_{\deg Y \leq k-1} 1 - \sum_{\deg Y \leq k-1} \sum_{Y \in \ker D (\frac{s-1}{s-1+m} \times k (t, \eta))} \sum_{j=1}^{k-1} (\alpha_{s+j} + \beta_{s+m+j}) \delta_j = 0 \right] \\
&= 2^{2s+m-2} \left[ \sum_{\deg Y \leq k-1} 1 - \sum_{\deg Y \leq k-1} \sum_{Y \in \ker D (\frac{s-1}{s-1+m} \times k (t, \eta))} \sum_{j=1}^{k-1} (\alpha_{s+j} + \beta_{s+m+j}) \delta_j = 1 \right] \\
&= 2^{2s+m-2} \left[ 2 \sum_{\deg Y \leq k-1} 1 - \sum_{\deg Y \leq k-1} \sum_{Y \in \ker D (\frac{s-1}{s-1+m} \times k (t, \eta))} \sum_{j=1}^{k-1} (\alpha_{s+j} + \beta_{s+m+j}) \delta_j = 1 \right] \\
&= 2^{2s+m-2} \left[ 2 \sum_{\deg Y \leq k-1} 1 - \sum_{\deg Y \leq k-1} \sum_{Y \in \ker D (\frac{s-1}{s-1+m} \times k (t, \eta))} \sum_{j=1}^{k-1} (\alpha_{s+j} + \beta_{s+m+j}) \delta_j = 0 \right]
\end{align*}
\]
which proves Lemma 4.11. \(\square\)

Lemma 4.11. Let \((t, \eta) \in \mathbb{P} \times \mathbb{P}\) and set
Then we have
\begin{equation}
\textstyle h^q(t, \eta) = h(t, \eta) \cdot v^{q-1}(t, \eta) \quad \text{for every rational integer } q \geq 2,
\end{equation}
\begin{equation}
\textstyle f^q_1(t, \eta) = f_1(t, \eta) \cdot v^{q-1}(t, \eta) \quad \text{for every rational integer } q \geq 2,
\end{equation}
\begin{equation}
\textstyle f^q_2(t, \eta) = f_2(t, \eta) \cdot v^{q-1}(t, \eta) \quad \text{for every rational integer } q \geq 2.
\end{equation}

**Proof.** We get obviously
\[ h^2(t, \eta) = \left[ 2^{2s+m-2} \sum_{\substack{\deg Y_1 \leq k-1 \atop Y_1 \in \ker D}} E(Y_1(tT^{s-1} + \eta T^{s+m-1})) \right] \cdot \left[ 2^{2s+m-2} \sum_{\substack{\deg Y_2 \leq k-1 \atop Y_2 \in \ker D}} E(Y_2(tT^{s-1} + \eta T^{s+m-1})) \right]. \]
We set
\[ \begin{cases} Y_1 + Y_2 = Y_3 & \deg Y_3 \leq k - 1, \\ Y_2 = Y_4 & \deg Y_4 \leq k - 1. \end{cases} \]
Then we obtain
\[ h^2(t, \eta) = 2^{2s+m-2} \sum_{\substack{\deg Y_3 \leq k-1 \atop Y_3 \in \ker D}} E(Y_3(tT^{s-1} + \eta T^{s+m-1})) = 2^{2s+m-2} \sum_{\substack{\deg Y_4 \leq k-1 \atop Y_4 \in \ker D}} E(Y_4(tT^{s-1} + \eta T^{s+m-1})) = h(t, \eta) \cdot v(t, \eta). \]

By recurrence on \( q \) we get
\[ h^q(t, \eta) = h(t, \eta) \cdot v^{q-1}(t, \eta). \]

The proof of (4.12) and (4.10) is similar to the proof of (4.10), that is
\[ f^2_1(t, \eta) = \left[ 2^{2s+m-2} \sum_{\substack{\deg Y_1 \leq k-1 \atop Y_1 \in \ker D}} E(Y_1(tT^{s-1})) \right] \cdot \left[ 2^{2s+m-2} \sum_{\substack{\deg Y_2 \leq k-1 \atop Y_2 \in \ker D}} E(Y_2(tT^{s-1})) \right]. \]
We set
\[ \begin{cases} Y_1 + Y_2 = Y_3 & \deg Y_3 \leq k - 1, \\ Y_2 = Y_4 & \deg Y_4 \leq k - 1. \end{cases} \]
Then we obtain
\[ f^2_1(t, \eta) = 2^{2s+m-2} \sum_{\substack{\deg Y_3 \leq k-1 \atop Y_3 \in \ker D}} E(Y_3(tT^{s-1})) = 2^{2s+m-2} \sum_{\substack{\deg Y_4 \leq k-1 \atop Y_4 \in \ker D}} E(Y_4(tT^{s-1})) = f_1(t, \eta) \cdot v(t, \eta). \]
Then we obtain
\[ f \]
We have
\[ W \]
We set
\[ \begin{aligned}
(4.13) \\
E(Y_3 \iota T^{s-1}) & \cdot [2^{2s+m-2} \sum_{\deg Y_4 \leq k-1} |\sum_{Y_4 \in \ker D}^k|] \\
= f_1(t, \eta) \cdot v(t, \eta).
\end{aligned} \]
By recurrence on \( q \) we get respectively
\[ f_1^q(t, \eta) = f_1(t, \eta) \cdot v^{q-1}(t, \eta), \]
\[ f_2^q(t, \eta) = f_2(t, \eta) \cdot v^{q-1}(t, \eta). \]

\[ \square \]

**Lemma 4.12.** Let \((t, \eta) \in \mathbb{P} \times \mathbb{P} \) and \( q \geq 2 \), then we have for \( 1 \leq i \leq q - 1 \)
\[ (4.11) \]
\[ h^i f_1^{q-i} = h^i f_2^{q-i} = \begin{cases} 
\nu^q & \text{if } f_1 f_2 \neq 0, \\
0 & \text{otherwise,}
\end{cases} \]
that is
\[ (4.12) \]
\[ h^i(t, \eta)f_1^{q-i}(t, \eta) = h^i(t, \eta)f_2^{q-i}(t, \eta) \]
is equal to
\[ (4.13) \]
\[ \begin{cases} 
2^{2s+m-k-2 - r(D)}(t, \eta)) & \text{if } r(D)\left[ \begin{array}{c}
\frac{s-1}{s-1+m} \\
\frac{s-1}{s-1+m}
\end{array} \right] (t, \eta)) = r(D)\left[ \begin{array}{c}
\frac{s-1}{s-1+m} \\
\frac{s-1}{s-1+m}
\end{array} \right] (t, \eta)) = r(D)\left[ \begin{array}{c}
\frac{s-1}{s-1+m} \\
\frac{s-1}{s-1+m}
\end{array} \right] (t, \eta), \\
0 & \text{otherwise.}
\end{cases} \]

**Proof.** We have
\[ f_1(t, \eta) h(t, \eta) = [2^{2s+m-2} \sum_{\deg Y_3 \leq k-1} |\sum_{Y_3 \in \ker D}^k| E(Y_3 \iota T^{s-1}) \cdot [2^{2s+m-2} \sum_{\deg Y_4 \leq k-1} |\sum_{Y_4 \in \ker D}^k| E(Y_2 \iota T^{s-1} + \eta T^{s+m-1})] \\
= 2^{2s+m-2} \sum_{\deg Y_3 \leq k-1} |\sum_{Y_3 \in \ker D}^k| \sum_{\deg Y_4 \leq k-1} |\sum_{Y_4 \in \ker D}^k| E(t(Y_1 + Y_2) T^{s-1} E(\eta Y_4 T^{s+m-1}).
\]
We set
\[ \begin{aligned}
Y_1 + Y_2 = Y_3 & \quad \deg Y_3 \leq k - 1, \\
Y_2 = Y_4 & \quad \deg Y_4 \leq k - 1.
\end{aligned} \]
Then we obtain
\[ f_1(t, \eta) h(t, \eta) = 2^{2s+m-2} \sum_{\deg Y_3 \leq k-1} |\sum_{Y_3 \in \ker D}^k| \sum_{\deg Y_4 \leq k-1} |\sum_{Y_4 \in \ker D}^k| E(tY_3 T^{s-1}) E(\eta Y_4 T^{s+m-1}) \\
= \left[ 2^{2s+m-2} \sum_{\deg Y_3 \leq k-1} |\sum_{Y_3 \in \ker D}^k| \sum_{\deg Y_4 \leq k-1} |\sum_{Y_4 \in \ker D}^k| E(Y_3 \iota T^{s-1}) \cdot [2^{2s+m-2} \sum_{\deg Y_4 \leq k-1} |\sum_{Y_4 \in \ker D}^k| E(\eta Y_4 T^{s+m-1})] \\
= f_1(t, \eta) \cdot f_2(t, \eta).
\]
In the same way we get
Then

\[ f_2(t, \eta) \cdot h(t, \eta) = f_1(t, \eta) \cdot f_2(t, \eta). \]

By respectively (4.18), (4.9) and (4.10) we deduce

\[ h = \begin{cases} v & \text{if } h \neq 0, \\ 0 & \text{otherwise,} \end{cases} \]

\[ f_1 = \begin{cases} v & \text{if } f_1 \neq 0, \\ 0 & \text{otherwise,} \end{cases} \]

\[ f_2 = \begin{cases} v & \text{if } f_2 \neq 0, \\ 0 & \text{otherwise.} \end{cases} \]

Now by (4.14), (4.15), (4.16) and (4.17) we have for \( 1 \leq i \leq q - 1 \)

\[ h^i f_1^{q-i} = h^i f_2^{q-i} = \begin{cases} v^i \cdot v^{q-i} = v^q & \text{if } h \cdot f_1 = h \cdot f_2 = f_1 \cdot f_2 \neq 0, \\ 0 & \text{otherwise.} \end{cases} \]

Finally from (4.4), (4.5) and (4.18) we deduce (4.13). \( \square \)

**Lemma 4.13.** We denote by \( R_q(k, s, m) \) the number of solutions

\( (Y_1, Z_1, U_1, \ldots, Y_q, Z_q, U_q) \)

of the polynomial equations

\[ \begin{cases} Y_1 Z_1 + Y_2 Z_2 + \ldots + Y_q Z_q = 0, \\ Y_1 U_1 + Y_2 U_2 + \ldots + Y_q U_q = 0, \end{cases} \]

satisfying the degree conditions

\[ \deg Y_i \leq k - 1, \quad \deg Z_i \leq s - 1, \quad \deg U_i \leq s + m - 1 \quad \text{for} \quad 1 \leq i \leq q. \]

Then

\[ R_q(k, s, m) = \int_{\mathbb{P} \times 
\sum_{i=0}^{\inf(2s+m,k)} \sum_{(t, \eta) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}} 2^{q(2s+m+k-i)} \int_{\mathbb{P}_{k+s-1}} dt \int_{\mathbb{P}_{k+s+m-1}} dtd\eta \]

\[ = \sum_{i=0}^{\inf(2s+m,k)} \sum_{(t, \eta) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}} 2^{q(2s+m+k-i)} \int_{\mathbb{P}_{k+s-1}} dt \int_{\mathbb{P}_{k+s+m-1}} dtd\eta \]

\[ = \sum_{i=0}^{\inf(2s+m,k)} \Gamma_i \left[ \frac{2^{q(2s+m+k-i)} \cdot 2^{q(2s+m+k)} \cdot 2^{-(k+s+1)} \cdot 2^{-(k+s+m-1)}}{2^{q(2s+m+k)} \cdot 2^{-(k+s+1)} \cdot 2^{-(k+s+m-1)}} \right]. \]

\( \square \)
5. A RECURRENT FORMULA FOR THE NUMBER $\Gamma_{i,m}^{[s+m]}\times k$ OF RANK $i$ MATRICES OF THE FORM $[\frac{1}{4}]$, WHERE A IS A $s \times k$ PERSYMMETRIC MATRIX AND B A $(s+m) \times k$ PERSYMMETRIC MATRIX WITH ENTRIES IN $\mathbb{F}_2$

In this section we establish a recurrent formula for the number of rank $i$ matrices of the form $[\frac{1}{4}]$, where A and B are persymmetric.

**Lemma 5.1.** We have

$$\int_{\mathbb{F}_2^2} g_1^q(t,\eta)d\sigma = \int_{\mathbb{F}_2^2} (h + f_1)^q(t,\eta)d\sigma$$

**Proof.** By the binomial theorem we obtain

$$g_1^q = (h + f_1)^q = h^q + f_1^q + \sum_{i=1}^{q-1} \binom{q}{i} h^i f_1^{q-i}.$$

By integrating $g_1^q$ on the unit interval of $\mathbb{F}_2^2$ and using (4.15), (4.16) and (4.9) we get

$$\int_{\mathbb{F}_2^2} g_1^q(t,\eta)d\sigma = \int_{\mathbb{F}_2^2} (h + f_1)^q(t,\eta)d\sigma$$

**Lemma 5.2.** We have

$$\int_{\mathbb{F}_2^2} g_1^q(t,\eta)d\sigma = 2q(2s+m+k-1) \cdot 2^{-(2k+2s+m-2)} \cdot \sum_{i=0}^{\inf(2s+m-1,k)} \sigma_{i,i} \cdot 2^{-q_i}.$$

**Proof.** We have by (1.2) observing that $g_1(t,\eta)$ is constant on cosets of $\mathbb{F}_{k+s-1} \times \mathbb{F}_{k+s+m-1}$
Lemma 5.3. We have

\[(5.5) \int_{\mathbb{F}_2} f_1^q(t, \eta) dt d\eta = 2^{q(2s+m+k-2)} \cdot 2^{-2k+2s+m-3} \cdot \sum_{i=0}^{\text{inf}(2s+m-2,k)} \sigma_{i,i} \cdot 2^{-qi}.
\]

Proof. Similar to the proof of Lemma 5.2 using (4.14) and observing that \( f_1(t, \eta) \) is constant on cosets of \( \mathbb{F}_{k+s-1} \times \mathbb{F}_{k+s+m-2} \).

Lemma 5.4. We have

\[(5.6)
\int_{\{(t, \eta) \in \mathbb{F}_2^2 \mid f_1(t, \eta) \neq f_2(t, \eta) \neq 0\}} v^q(t, \eta) dt d\eta = 2^{q(2s+m+k-2)}\cdot 2^{-2k+2s+m-2} \cdot \sum_{i=0}^{\text{inf}(2s+m-2,k)} \sigma_{i,i,i} \cdot 2^{-qi}.
\]

Proof. We have by (4.12) and (4.13)

\[
\int_{\{(t, \eta) \in \mathbb{F}_2^2 \mid f_1(t, \eta) \neq f_2(t, \eta) \neq 0\}} v^q(t, \eta) dt d\eta = 2^{q(2s+m+k-2) - r(D^{\frac{s-1}{s+m-1}} k(t, \eta))} \int_{\mathbb{F}_{k+s-1}} dt \int_{\mathbb{F}_{k+s+m-1}} d\eta
\]

\[
\int_{\{(t, \eta) \in \mathbb{F}_2^2 \mid f_1(t, \eta) \neq f_2(t, \eta) \neq 0\}} v^q(t, \eta) dt d\eta = 2^{q(2s+m+k-2) - r(D^{\frac{s-1}{s+m-1}} k(t, \eta))} \int_{\mathbb{F}_{k+s-1}} dt \int_{\mathbb{F}_{k+s+m-1}} d\eta
\]

\[
= \sum_{i=0}^{\text{inf}(2s+m-2,k)} \sum_{(t, \eta) \in \mathbb{F}_2^{k+s-1} \times \mathbb{F}_2^{k+s+m-1}} 2^{q(2s+m+k-2-i)} \cdot 2^{-qi} \cdot 2^{q(2s+m+k-2) \cdot 2^{-2k+2s+m-2}} \cdot \sum_{i=0}^{\text{inf}(2s+m-2,k)} \sigma_{i,i,i} \cdot 2^{-qi}.
\]

Lemma 5.5. We have

\[
\sigma_{i,i} = 2 \cdot \sigma_{i,i}.
\]

Proof. We consider the following \((2s+m-1) \times k\) matrix denoted by

\[
D \left[ \begin{array}{c} \frac{s-1}{s+m-1} \\ \frac{s+1}{s+m-1} \end{array} \right] \times k
\]

(see Section 11)
We recall that the rank of a matrix does not change under elementary row operations.

On the above matrix, we add the \( s+m+j \)-th row to the \( j \)-th row for \( 0 \leq j \leq s-2 \) obtaining

\[
\begin{pmatrix}
\alpha_1 + \beta_{i+m} & \alpha_2 + \beta_{i+m+1} & \alpha_3 + \beta_{i+m+2} & \ldots & \alpha_k + \beta_{i+m+k-1} & \alpha_k \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{i+m} & \beta_{i+m+1} & \beta_{i+m+2} & \ldots & \beta_{i+m+k-1} & \beta_{i+m+k} \\
\end{pmatrix}
\]

In the above matrix we set \( \alpha_i + \beta_{i+m} = \mu_i \) for \( 1 \leq i \leq k+s-2 \),
\( \beta_j = \tau_j \) for \( 1 \leq j \leq k+s+m-2 \).

( Remark that the map \( \kappa : F_2^{2k+2s+m-4} \rightarrow F_2^{2k+2s+m-4} \) defined by
\( (\alpha_1, \alpha_2, \ldots, \alpha_{k+s-2}, \beta_1, \beta_2, \ldots, \beta_{k+s+m-2}) \mapsto (\mu_1, \mu_2, \ldots, \mu_{k+s-2}, \tau_1, \tau_2, \ldots, \tau_{k+s+m-2}) \)

is an isomorphism). We then obtain the below matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \ldots & \beta_{s+k-2} & \beta_{s+k-1} \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{s+k+m-2} & \beta_{s+k+m-1} \\
\end{pmatrix}
\]
Comparing the above matrix with the below matrix

\[
\begin{bmatrix}
\tau_1 \\
\tau_2 \\
\vdots \\
\tau_{s-1} \\
\tau_s \\
\tau_{s+1} \\
\vdots \\
\tau_{s+m} \\
\tau_{s+m-1}
\end{bmatrix}
= 
\begin{bmatrix}
\mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{k-1} & \mu_k \\
\mu_2 & \mu_3 & \mu_4 & \cdots & \mu_k & \mu_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mu_{s-1} & \mu_s & \mu_{s+1} & \cdots & \mu_{s+k-3} & \mu_{s+k-2} \\
\tau_1 & \tau_2 & \tau_3 & \cdots & \tau_{k-1} & \tau_k \\
\tau_2 & \tau_3 & \tau_4 & \cdots & \tau_k & \tau_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\tau_{m+1} & \tau_{m+2} & \tau_{m+3} & \cdots & \tau_{k+m-1} & \tau_{k+m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s+k-1} & \beta_{s+k+m-1} & \mu_k
\end{bmatrix}
\]

and observing that for all \(\mu_{s+k-1} \in \mathbb{F}_2\)

\[
\text{Card} \left\{ (\alpha_{s+k-1}, \beta_{s+k+m-1}) \in \mathbb{F}_2^2 \mid \alpha_{s+k-1} + \beta_{s+k+m-1} = \mu_{s+k-1} \right\} = 2
\]

we obtain the following equality

\[
\text{Card} \left\{ (t, \eta) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \mid r(D_{s-1+m}^{s-1+m} \times k)(t, \eta) = r(D_{s-1+m}^{s-1+m} \times k)(t, \eta) = i \right\}
\]

\[
= 2 \cdot \text{Card} \left\{ (t, \eta) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-2} \mid r(D_{s-1+m}^{s-1+m} \times k)(t, \eta) = r(D_{s-1+m}^{s-1+m} \times k)(t, \eta) = i \right\}
\]
Alternatively consider the following equivalences

$$\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2}
\end{pmatrix}
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2}
\end{pmatrix}
= \begin{pmatrix}
\alpha_1 + \beta_{m+1} & \alpha_2 + \beta_{m+2} & \alpha_3 + \beta_{m+3} & \ldots & \alpha_{k-1} + \beta_{m+k-1} & \alpha_k + \beta_{m+k} \\
\alpha_2 + \beta_{m+2} & \alpha_3 + \beta_{m+3} & \alpha_4 + \beta_{m+4} & \ldots & \alpha_k + \beta_{m+k} & \alpha_{k+1} + \beta_{m+k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} + \beta_{s-1+m} & \alpha_s + \beta_{s+m} & \alpha_{s+1} + \beta_{s+m+1} & \ldots & \alpha_{s+k-3} + \beta_{s+k+m-3} & \alpha_{s+k-2} + \beta_{s+k+m-2} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2}
\end{pmatrix}
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2}
\end{pmatrix}
\begin{pmatrix}
\alpha_1 + \beta_{m+1} & \alpha_2 + \beta_{m+2} & \alpha_3 + \beta_{m+3} & \ldots & \alpha_{k-1} + \beta_{m+k-1} & \alpha_k + \beta_{m+k} \\
\alpha_2 + \beta_{m+2} & \alpha_3 + \beta_{m+3} & \alpha_4 + \beta_{m+4} & \ldots & \alpha_k + \beta_{m+k} & \alpha_{k+1} + \beta_{m+k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} + \beta_{s-1+m} & \alpha_s + \beta_{s+m} & \alpha_{s+1} + \beta_{s+m+1} & \ldots & \alpha_{s+k-3} + \beta_{s+k+m-3} & \alpha_{s+k-2} + \beta_{s+k+m-2} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2}
\end{pmatrix}
We then get

$$\sigma_{i,j} = 2 \cdot \sigma_{i,i} \times k$$

Lemma 5.6. We have

$$(5.7)$$

$$\int_{\mathbb{P}_{k+2}} h^q(t, \eta) dt d\eta = 2q(2s+m+k-2) \cdot 2^{-(2k+2s+m-2)} \cdot \sum_{i=0}^{s-1} \sigma_{i,i} \times k$$

Proof. We have by (4.7) observing that $h(t, \eta)$ is constant on cosets of $\mathbb{P}_{k+2} \times \mathbb{P}_{k+s+m-1}$

$$\int_{\mathbb{P}_{k+2}} h^q(t, \eta) dt d\eta = \sum_{(t, \eta) \in / \mathbb{P}_{k+2} / \mathbb{P}_{k+s+m-1}} 2q(2s+m+k-2) \cdot r(D) [\frac{s-1}{s+1} \times k]$$

$$= \sum_{i=0}^{s-1} 2q(2s+m+k-2) \cdot \sigma_{i,i} \times k$$

Lemma 5.7. We have

$$(5.8)$$

$$\int_{\mathbb{P}_{k+2}} h^q(t, \eta) dt d\eta = \int_{\mathbb{P}_{k+2}} f^q(t, \eta) dt d\eta.$$

Proof. Immediately obtained by comparing (5.5) and (5.7) using Lemma 5.6.
Lemma 5.8. We have for $1 \leq i \leq \inf(2s + m - 1, k)$

\[
\sigma_{i-1, i, i} = 2 \cdot \sigma_{i-1, i-1, i} \tag{5.9}
\]

Proof. From (5.1), (5.4), (5.5), (5.7) and (5.8) we have

\[
\int_{g^2} g_i^q \int_{g^2} h_i^q + \int_{g^2} f_i^q = 2 \int_{g^2} f_i^q \cdot \int_{f_1 f_2 \neq 0} v^q
\]

\[
= 2 \cdot \int_{g^2} f_i^q + (2q - 2) \cdot \int_{f_1 f_2 \neq 0} v^q
\]

\[
\iff 2q(2s + m + k - 1) \cdot 2^{-(2k + 2s + m - 2)} \cdot \sum_{i=0}^{\inf(2s + m - 1, k)} \sigma_{i, i} \cdot 2^{-qi}
\]

\[
= 2 \cdot 2q(2s + m + k - 2) \cdot 2^{-(2k + 2s + m - 3)} \cdot \sum_{i=0}^{\inf(2s + m - 2, k)} \sigma_{i, i, i} \cdot 2^{-qi}
\]

\[
+ (2q - 2) \cdot 2q(2s + m + k - 2) \cdot 2^{-(2k + 2s + m - 2)} \cdot \sum_{i=0}^{\inf(2s + m - 2, k)} \sigma_{i, i, i} \cdot 2^{-qi}
\]

\[
\iff \sum_{i=0}^{\inf(2s + m - 1, k)} \sigma_{i, i} \cdot 2^{-qi}
\]

\[
= 2^{2q} \cdot \sum_{i=0}^{\inf(2s + m - 2, k)} \sigma_{i, i, i} \cdot 2^{-qi} + (2q - 2) \cdot 2^{-q} \sum_{i=0}^{\inf(2s + m - 2, k)} \sigma_{i, i, i} \cdot 2^{-qi}
\]

\[
\iff \sum_{i=0}^{\inf(2s + m - 1, k)} \sigma_{i, i} \cdot 2^{-qi}
\]

\[
= \sum_{i=0}^{\inf(2s + m - 2, k)} 4 \cdot \sigma_{i, i} \cdot 2^{-q(i+1)}
\]

\[
\iff \sum_{i=0}^{\inf(2s + m - 2, k)} \sigma_{i, i, i} \cdot 2^{-qi} - \sum_{i=0}^{\inf(2s + m - 2, k)} 2 \cdot \sigma_{i, i, i} \cdot 2^{-q(i+1)}
\]
Now we have obviously

\[\inf(2s+m-1,k) \quad \sum_{i=0}^{\sigma_{i,i}} \times k \cdot 2^{-qi}\]

\[\inf(2s+m-2,k+1) = \sum_{i=1}^{4 \times \sigma_{i-1,i-1}} \times k \cdot 2^{-qi}\]

\[\inf(2s+m-2,k) + \sigma_{i,i,i} \times k \cdot 2^{-qi} = \sum_{i=1}^{2 \times \sigma_{i-1,i-1,i-1} \times k \cdot 2^{-qi}} \]

The case \(k \leq 2s + m - 2\)

Observing that \(4 \times \sigma_{k,k} = 2 \times \sigma_{k,k,k}\), we deduce from (5.10)

(5.11)

\[\sum_{i=1}^{k} \sigma_{i,i} \times k - 4 \times \sigma_{i-1,i-1} \times k + 2 \times \sigma_{i-1,i-1,i-1} \times 2^{-qi} = 0 \text{ for all } q \geq 2\]

Now we have obviously

(5.12)

(5.13)

By (5.11), (5.12) and (5.13) we obtain for all \(q \geq 2\)

(5.14)

\[\sum_{i=1}^{k} \sigma_{i-1,i,i} + \sigma_{i,i,i} - 2 \times \sigma_{i-1,i-1,i-1} - 2 \times \sigma_{i-1,i-1,i-1} - \sigma_{i,i,i} + 2 \times \sigma_{i-1,i-1,i-1} \times 2^{-qi} = 0\]
Observing that \( \sigma \) from (5.17) we obtain that for all \( i \) and Lemma 5.8 is proved.

The case \( k \geq 2s + m - 1 \)

From (5.14) we deduce

\[
\sum_{i=0}^{2s+m-1} \sigma_{i,i} \cdot 2^{-qi} = \sum_{i=1}^{2s+m-1} 4 \cdot \sigma_{i-1,i-1} \cdot 2^{-qi} + \sum_{i=0}^{2s+m-2} \sigma_{i,i,i} \cdot 2^{-qi} - \sum_{i=1}^{2s+m-1} 2 \cdot \sigma_{i-1,i-1,i-1} \cdot 2^{-qi}.
\]

Observing that \( \sigma_{2s+m-1,2s+m-1,2s+m-1} = 0 \) we deduce from (5.16), (5.12) and (5.13) that for all \( q \geq 2 \)

\[
\sum_{i=1}^{2s+m-1} \left[ \sigma_{i-1,i,i} + \sigma_{i,i,i} - 2 \cdot \sigma_{i-1,i-1,i} - 2 \cdot \sigma_{i-1,i-1,i} - 2 \cdot \sigma_{i-1,i-1,i} + 2 \cdot \sigma_{i-1,i-1,i} \right] \cdot 2^{-qi} = 0.
\]

From (5.17) we obtain

\[
\sum_{i=1}^{2s+m-1} \sigma_{i-1,i,i} - 2 \cdot \sigma_{i-1,i-1,i} \cdot 2^{-qi} = 0 \text{ for all } q \geq 2
\]

and Lemma 5.8 is proved.
Lemma 5.9.

\begin{equation}
\int_{\mathbb{F}^2} g_2^q(t,\eta) d\eta = \int_{\{(t,\eta) \in \mathbb{F}^2 \mid h(t,\eta) \neq 0\}} v^q(t,\eta) d\eta + \int_{\{(t,\eta) \in \mathbb{F}^2 \mid f_2(t,\eta) \neq 0\}} v^q(t,\eta) d\eta \\
+ (2^q - 2) \cdot \int_{\{(t,\eta) \in \mathbb{F}^2 \mid f_3(t,\eta) - f_2(t,\eta) \neq 0\}} v^q(t,\eta) d\eta.
\end{equation}

Proof. Similar to the proof of Lemma 5.4.

Lemma 5.10. We have for \(1 \leq i \leq \inf(2s + m - 1, k)\)

\begin{equation}
\sigma_{i-1,i,i} + \sigma_{i-1,i-1,i} = \sigma_{i-1,i,i} + \sigma_{i-1,i-1,i}.
\end{equation}

Proof. From (5.11) and (5.19) we get for all \(q \geq 2\)

\begin{equation}
\int_{\mathbb{F}^2} g_2^q(t,\eta) d\eta - \int_{\mathbb{F}^2} g_2^q(t,\eta) d\eta = \int_{\{(t,\eta) \in \mathbb{F}^2 \mid f_1(t,\eta) \neq 0\}} v^q(t,\eta) d\eta - \int_{\{(t,\eta) \in \mathbb{F}^2 \mid f_1(t,\eta) \neq 0\}} v^q(t,\eta) d\eta.
\end{equation}

By (5.21) we get for all \(q \geq 2\) the following equivalences

\begin{align*}
&2^q(2s+m+k-1) \cdot 2^{-(2k+2s+m-2)} \cdot \sum_{i=0}^{\inf(2s+m-1,k)} \left[ \sigma_{i,i} - \sigma_{i,i} \right] \cdot 2^{-qi} \\
&= 2^q(2s+m+k-2) \cdot 2^{-(2k+2s+m-3)} \cdot \sum_{i=0}^{\inf(2s+m-2,k)} \left[ \sigma_{i,i} - \sigma_{i,i} \right] \cdot 2^{-qi}
\end{align*}

\[
\Leftrightarrow \sum_{i=0}^{\inf(2s+m-1,k)} \left[ \sigma_{i,i} - \sigma_{i,i} \right] \cdot 2^{-qi} = \sum_{i=0}^{\inf(2s+m-2,k)} \left[ \sigma_{i,i} - \sigma_{i,i} \right] \cdot 2^{-qi}
\]

\[
\Leftrightarrow \sum_{i=0}^{\inf(2s+m-1,k)} \left[ \sigma_{i-1,i,i} - \sigma_{i-1,i,i} \right] \cdot 2^{-qi} = \sum_{i=1}^{\inf(2s+m-2,k)+1} \left[ \sigma_{i-1,i,i} - \sigma_{i-1,i,i} \right] \cdot 2^{-qi}
\]
(5.22)
\[
\sum_{i=1}^{k} \left[ \sigma_{i-1,i,i} - \sigma_{i-1,i,i} - \sigma_{i-1,i-1,i} + \sigma_{i-1,i-1,i} \right] \cdot 2^{-qi} = 0.
\]

Now (5.22) holds for all \( q \geq 2 \) which deduce Lemma 5.10.

Lemma 5.11. We have
\[
\Gamma_i^s (x+m) = \sigma_i^s,i,i + \sigma_i^s-1,i,i + \sigma_i^s-1,i,i + \sigma_i^s-2,i,i.
\]

Proof. The proof is obvious.

Lemma 5.12. We have
\[
\sigma_{i-2,i-1,i} = \sigma_{i-2,i-1,i}.
\]

Proof. (5.24) follows from (5.23) and (5.20).

Lemma 5.13. The following holds:
\[
\sigma_{i-1,i,i} = 2 \cdot \sigma_{i-1,i-1,i}
\]

\[
\sigma_{i-1,i,i} + \sigma_{i-1,i-1,i} = \sigma_{i-1,i-1,i} + \sigma_{i-1,i,i}
\]

\[
\sigma_{i-2,i-1,i} = \sigma_{i-2,i-1,i}
\]
Theorem follows from Lemmas 5.8, 5.10, 5.11 and 5.12.

\[ \sigma_{i,i,i} = \sigma_{i,i,i} \]

\[ \Gamma_i^{[s+m]} x_k = \sigma_{i,i,i} + \sigma_{i-1,i,i} + \sigma_{i-1,i,i} + \sigma_{i-2,i,i} \]

\[ \Gamma_i^{[s+m]} x_k = \sigma_{i,i,i} + \sigma_{i-1,i,i} + \sigma_{i-1,i,i} + \sigma_{i-2,i,i} \]

Proof. The theorem follows from Lemmas 5.8, 5.10, 5.11 and 5.12.

\[ (5.25) \]

Let \( s \geq 2, m \geq 0, k \geq 1 \) and \( 0 \leq i \leq \inf (2s + m, k) \). Then we have the following recurrent formula for the number of rank \( i \) matrices of the form \([\frac{1}{n}]\), such that \( A \) is a \( s \times k \) persymmetric matrix and \( B \) a \((s + m) \times k\) persymmetric matrix with entries in \(\mathbb{F}_2\)

\[ (5.26) \]

\[ \sigma_{i,i,i} = 3 \cdot \sigma_{i-1,i-1,i-1} + 2 \cdot \sigma_{i-2,i-2,i-2} \]

Proof. Consider the following two matrices

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\alpha_x & \alpha_{x+1} & \alpha_{x+2} & \ldots & \alpha_{x+k-2} & \alpha_{x+k-1} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_k \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\beta_{m+2} & \beta_{m+3} & \beta_{m+4} & \ldots & \beta_{k+m} & \beta_{k+m+1} \\
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\alpha_x & \alpha_{x+1} & \alpha_{x+2} & \ldots & \alpha_{x+k-2} & \alpha_{x+k-1} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_k \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\beta_{m+2} & \beta_{m+3} & \beta_{m+4} & \ldots & \beta_{k+m} & \beta_{k+m+1} \\
\end{pmatrix}
\]

We have by Lemma 5.13
6. RANK PROPERTIES OF A PARTITION OF DOUBLE PERSYMMETRIC MATRICES

Consider the following partition of the matrix

\[
D = \begin{bmatrix}
\alpha_{x} & \beta_{x+m-} \\
\beta_{x+m-} & \alpha_{x}
\end{bmatrix} \times k
\]
By integrating some appropriate exponential sums on the unit interval of $\mathbb{R}^2$ with integral equal to zero, we deduce the following rank formulas for all $j \in [0, \inf(2s + m - 2, k - 1)]$:

$$
\psi(t, \eta) = \sum_{\deg Y = k-1} \sum_{\deg Z \leq s-1} \left( \sum_{\deg U \leq s-1+m} E(tYZ) \right) \sum_{\deg U \leq s-1+m} E(\eta YU).
$$

(6.1)

**Lemma 6.1.** Let $(t, \eta) \in \mathbb{P} \times \mathbb{P}$ and set

$$
\psi(t, \eta) = \begin{cases} 
2^{s+m+k-1-r(D[s^m] \times k)}(t, \eta) & \text{if } r(D[s^m] \times k)(t, \eta) = r(D[s^m](k-1))(t, \eta), \\
0 & \text{otherwise}.
\end{cases}
$$

Proof. We have

$$
\psi(t, \eta) = \sum_{\deg Y = k-1} \sum_{\deg Z \leq s-1} \left( \sum_{\deg U \leq s-1+m} E(tYZ) \right) \sum_{\deg U \leq s-1+m} E(\eta YU)
$$

$$
= \sum_{\deg Y = k-1} \sum_{\deg Z \leq s-1} \sum_{\deg U \leq s-1+m} E(tYZ) \sum_{\deg U \leq s-1+m} E(\eta YU) - \sum_{\deg Y = k-2} \sum_{\deg Z \leq s-1} \sum_{\deg U \leq s-1+m} E(tYZ) \sum_{\deg U \leq s-1+m} E(\eta YU)
$$

$$
= 2^{2s+m} \sum_{\deg Y = k-1} \sum_{\deg Z \leq s-1} \sum_{\deg U \leq s-1+m} 1 - 2^{2s+m} \sum_{\deg Y = k-1} \sum_{\deg Z \leq s-1} \sum_{\deg U \leq s-1+m} 1
$$

$$
= 2^{2s+m} \cdot 2^{k-r(D[s^m] \times k)}(t, \eta) - 2^{s+m} \cdot 2^{k-1-r(D[s^m](k-1))(t, \eta)}.
$$

□

**Lemma 6.2.** We have

$$
\phi^2(t, \eta) = \phi_1(t, \eta) \cdot \phi_2(t, \eta),
$$

where
We have the following equivalences

**Lemma 6.3.** We have the following equivalences

\[
\phi(t, \eta) = \sum_{\deg Y = k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta YU),
\]

\[
\phi_1(t, \eta) = \sum_{\deg Y = k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-2} E(\eta YU)
\]

and

\[
\phi_2(t, \eta) = \sum_{\deg Y \leq k-2} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta YU).
\]

**Proof.** We obtain

\[
\phi^2(t, \eta) = \left( 2^{s+m-1} \sum_{\deg Y_1 = k-1} E(\eta Y_1 T^{s+m-1}) \right) \left( 2^{s+m-1} \sum_{\deg Y_2 = k-1} E(\eta Y_2 T^{s+m-1}) \right) = \phi_1(t, \eta) \cdot \phi_2(t, \eta).
\]

We set

\[
\begin{cases}
Y_1 + Y_2 = Y_3 & \deg Y_3 \leq k-2, \\
Y_1 = Y_4 & \deg Y_4 = k-1.
\end{cases}
\]

Then we get obviously

\[
\begin{cases}
Y_1 \in \ker D^{s+m-1} \times (t, \eta) & \deg Y_1 = k-1, \\
Y_2 \in \ker D^{s+m-1} \times (t, \eta) & \deg Y_2 = k-1,
\end{cases}
\]

\[
\iff \begin{cases}
Y_3 \in \ker D^{s+m-1} \times (k-1) & \deg Y_3 \leq k-2, \\
Y_4 \in \ker D^{s+m-1} \times (t, \eta) & \deg Y_4 = k-1.
\end{cases}
\]

And we obtain

\[
\phi^2(t, \eta) = 2^{(2s+m-1)2} \sum_{\deg Y_4 = k-1} \sum_{\deg Y_3 \leq k-2} E(\eta Y_3 T^{s+m-1})
\]

\[
= \left[ 2^{2s+m-1} \sum_{\deg Y_4 = k-1} \right] \cdot \left[ 2^{2s+m-1} \sum_{\deg Y_3 \leq k-2} \right] = \phi_1(t, \eta) \cdot \phi_2(t, \eta).
\]

\[
\Box
\]

**Lemma 6.3.** We have the following equivalences

\[
\phi(t, \eta) \neq 0 \iff \phi^2(t, \eta) \neq 0
\]

\[
\iff \phi_1(t, \eta) \cdot \phi_2(t, \eta) \neq 0
\]

\[
\iff r(D^{s+m-1} \times (k-1)) = r(D^{s+m-1} \times (t, \eta)) = r(D^{s+m} \times (k-1)) = r(D^{s+m} \times (t, \eta)).
\]
**Proof.** By Lemma 6.1 with \( m \to m - 1 \) we have

(6.3)
\[
\phi_1(t, \eta) = \begin{cases} 
2^{2s+m+k-2-r(D^{[s+(m-1)] \times k}(t, \eta))} & \text{if } r(D^{[s+(m-1)] \times k}(t, \eta)) = r(D^{[s+(m-1)] \times (k-1)}(t, \eta)), \\
0 & \text{otherwise}
\end{cases}
\]

and by (4.5) with \( k \to k - 1, \ s \to s + 1 \) and \( m \to m - 1 \) we have

(6.4)
\[
\phi_2(t, \eta) = \begin{cases} 
2^{2s+m+k-2-r(D^{[s+m]} \times (k-1)}(t, \eta)) & \text{if } r(D^{[s+m]} \times (k-1)}(t, \eta)) = r(D^{[s+(m-1)] \times (k-1)}(t, \eta)), \\
0 & \text{otherwise}
\end{cases}
\]

Lemma 6.3 follows then from (6.3) and (6.4).

\[\square\]

**Lemma 6.4.** Let \((t, \eta) \in \mathbb{P} \times \mathbb{P}\), then

\[
\phi(t, \eta) = \sum_{\deg Y = k - 1} \sum_{\deg Z \leq s - 1} E(tYZ) \sum_{\deg U = s + m - 1} E(\eta YU)
\]

is given by

\[
\begin{align*}
2^{2s+m+k-j-2} & \quad \text{if } (t, \eta) \in [k-1, k] \times D^{[s+m-1]} \times \cdot, \\
& \begin{cases} 
\ \ j & \ j & \alpha_{s}, \\
\ \ j & \ j & \beta_{s+m}
\end{cases} \\
-2^{2s+m+k-j-2} & \quad \text{if } (t, \eta) \in [k-1, k] \times D^{[s+m-1]} \times \cdot, \\
& \begin{cases} 
\ \ j & \ j+1 & \beta_{s+m}
\end{cases}
\end{align*}
\]

otherwise.

**Proof.** We consider the following two cases in which by Lemma 6.3 \( \phi(t, \eta) \) is different from zero.

First case:
\[
r(D^{[s+m-1]} \times (k-1)}(t, \eta)) = r(D^{[s+m-1]} \times k)}(t, \eta)) = r(D^{[s+m]} \times (k-1)}(t, \eta)) = r(D^{[s+m]} \times (k-1)}(t, \eta)) = j.
\]

Second case:
\[
r(D^{[s+m-1]} \times (k-1)}(t, \eta)) = r(D^{[s+m-1]} \times k)}(t, \eta)) = r(D^{[s+m]} \times (k-1)}(t, \eta)) = j \quad \text{and} \quad r(D^{[s+m]} \times k)}(t, \eta)) = j+1.
\]

We obtain using (4.3):

In the first case
\[
\phi(t, \eta) = \sum_{\deg Y = k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta YU)
\]
\[
= \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta YU) - \sum_{\deg Y \leq k-2} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta YU)
\]
\[
= 2^{2s+m+k-1-j} - 2^{2s+m+k-2-j} = 2^{2s+m+k-2-j}.
\]

In the second case
\[
\phi(t, \eta) = \sum_{\deg Y = k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta YU)
\]
\[
= \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta YU) - \sum_{\deg Y \leq k-2} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta YU)
\]
\[
= 0 - 2^{2s+m+k-2-j} = -2^{2s+m+k-2-j}.
\]

And otherwise \( \phi(t, \eta) \) is equal to zero.

**Lemma 6.5.** We have
\[
\int_{\mathbb{P} \times \mathbb{P}^2} \phi^{2q+1}(t, \eta) dt d\eta = 0.
\]

**Proof.** The integral \( \int_{\mathbb{P} \times \mathbb{P}^2} \phi^{2q+1}(t, \eta) dt d\eta \) is equal to the number of solutions \((Y_1, Z_1, U_1, Z_2, U_2, \ldots, Y_{2q+1}, Z_{2q+1}, U_{2q+1})\) of the polynomial equations
\[
\begin{aligned}
\{ & Y_1 Z_1 + Y_2 Z_2 + \ldots + Y_{2q+1} Z_{2q+1} = 0, \\
& Y_1 U_1 + Y_2 U_2 + \ldots + Y_{2q+1} U_{2q+1} = 0,
\end{aligned}
\]
satisfying the degree conditions
\[
\deg Y_i = k-1, \quad \deg Z_i \leq s-1 \quad \deg U_i = s + m - 1 \text{ for } 1 \leq i \leq 2q + 1.
\]
Now \( 2q+1 \) is odd so \( \sum_{i=1}^{2q+1} Y_i U_i \) is equal to \( k+s+m-2 \). The Lemma follows.

**Lemma 6.6.** Let \( j \in \mathbb{N} \) such that \( 0 \leq j \leq \inf(2s + m - 1, k - 1) \), then
\[
\# \left( \frac{j}{j} \right)_{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}} = \# \left( \frac{j}{j} \right)_{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}}.
\]

**Proof.** We define
\[
\mathbb{A} = \{(t, \eta) \in \mathbb{P}^2 \mid r(D^{[s+m-1] \times (k-1)}(t, \eta)) = r(D^{[s+m-1] \times k}(t, \eta))
\]
\[
= r(D^{[s+m] \times (k-1)}(t, \eta)) = r(D^{[s+m] \times k}(t, \eta))\}
\]
and
\[
\mathbb{B} = \{(t, \eta) \in \mathbb{P}^2 \mid r(D^{[s+m-1] \times (k-1)}(t, \eta)) = r(D^{[s+m-1] \times k}(t, \eta)) = r(D^{[s+m] \times (k-1)}(t, \eta)),
\]
r\[
= r(D^{[s+m-1] \times (k-1)}(t, \eta)) + 1\}.
\]
By lemma 6.4 we have by observing that \( \phi(t, \eta) \) is constant on cosets of \( \mathbb{P}_{k+s-1} \times \mathbb{P}_{k+s+m-1} \)

\[
\int_{\mathbb{P} \times \mathbb{P}} \phi^{2q+1}(t, \eta) d\tau \eta = \int_{\mathbb{A}} \phi^{2q+1}(t, \eta) d\tau \eta + \int_{\mathbb{B}} \phi^{2q+1}(t, \eta) d\tau \eta
\]

\[
= \sum_{(t, \eta) \in A \cap \left( \mathbb{P} / \mathbb{P}_{k+s-1} \times \mathbb{P} / \mathbb{P}_{k+s+m-1} \right)} 2^{(2s+m+k-2-j)(2q+1)} \cdot \# \left( \frac{j}{j+1} \right) \left( \frac{\theta}{\eta} \right) \int_{\mathbb{P} / \mathbb{P}_{k+s-1} \times \mathbb{P} / \mathbb{P}_{k+s+m-1}} dt \int_{\mathbb{P} / \mathbb{P}_{k+s+m-1}} d\eta
\]

\[
+ \sum_{(t, \eta) \in B \cap \left( \mathbb{P} / \mathbb{P}_{k+s-1} \times \mathbb{P} / \mathbb{P}_{k+s+m-1} \right)} 2^{(2s+m+k-2-j)(2q+1)} \cdot \# \left( \frac{j}{j+1} \right) \left( \frac{\theta}{\eta} \right) \int_{\mathbb{P} / \mathbb{P}_{k+s-1} \times \mathbb{P} / \mathbb{P}_{k+s+m-1}} dt \int_{\mathbb{P} / \mathbb{P}_{k+s+m-1}} d\eta.
\]

Now by Lemma 6.5 we get for all \( q \in \mathbb{N} \)

\[
\inf_{2s+m-1,k-1} \sum_{j=0}^{2(2s+m+k-2-j)(2q+1)} \cdot \# \left( \frac{j}{j+1} \right) \left( \frac{\theta}{\eta} \right) \int_{\mathbb{P} / \mathbb{P}_{k+s-1} \times \mathbb{P} / \mathbb{P}_{k+s+m-1}} dt \int_{\mathbb{P} / \mathbb{P}_{k+s+m-1}} d\eta = 0,
\]

which proves Lemma 6.6.

Lemma 6.7. Let \( (t, \eta) \in \mathbb{P}^2 \), \( q \) be a rational integer \( \geq 2 \) and

\[
\theta_1(t, \eta) = \sum_{\deg Y = k-1} \sum_{\deg Z = s-1} E(tYZ) \sum_{\deg U \leq s+m-2} E(\eta YU),
\]

\[
\theta_2(t, \eta) = \sum_{\deg Y \leq k-2} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U = s+m-1} E(\eta YU).
\]
Then $\theta_1(t, \eta)$ is given by

$$
\begin{cases}
2^{2s+m+k-j-3} & \text{if } (t, \eta) \in \begin{bmatrix} k-1 & k \\ j & j \\ \cdot & \cdot \end{bmatrix} D_{s+m-1}^{s+1} \\
-2^{2s+m+k-j-3} & \text{if } (t, \eta) \in \begin{bmatrix} k-1 & k \\ j & j+1 \\ \cdot & \cdot \end{bmatrix} D_{s+m-1}^{s+1}
\end{cases}
$$

(6.5)

and $\theta_2(t, \eta)$ is given by

$$
\begin{cases}
2^{2s+m+k-2-j} & \text{if } (t, \eta) \in \begin{bmatrix} k-1 & k \\ \cdot & \cdot \\ j & \cdot \\ j & \cdot \end{bmatrix} D_{s+m-1}^{s+1} \\
0 & \text{otherwise,}
\end{cases}
$$

(6.6)

Proof. The proofs of (6.5), (6.6) are respectively similar to the proofs of Lemmas 6.4, 4.7.

Lemma 6.8. Let $(t, \eta) \in \mathbb{P}_2$ and $q$ be a rational integer $\geq 2$, then $\theta_1(t, \eta) \cdot \theta_2^{q-1}(t, \eta)$ is equal to

$$
\begin{cases}
2^{q-1} \cdot 2^{2s+m+k-3-j}q & \text{if } (t, \eta) \in \begin{bmatrix} k-1 & k \\ j & j \\ j & \cdot \end{bmatrix} D_{s+m-1}^{s+1} \\
-2^{q-1} \cdot 2^{2s+m+k-3-j}q & \text{if } (t, \eta) \in \begin{bmatrix} k-1 & k \\ j & j+1 \\ j & j+1 \end{bmatrix} D_{s+m-1}^{s+1}
\end{cases}
$$

(6.7)

Proof. We consider the following partition of the matrix

$$
D_{s+m-1}^{s+1} \times k
$$
\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{s+m+k-2} & \beta_{s+m+k-1}
\end{pmatrix}
\]

Obviously by (6.6) we have

\[(6.8)\]

\[
\theta_2^{q-1}(t, \eta) = \begin{cases} 
2^{(2s+m+k-2-j)(q-1)} & \text{if } r(D^{s-1+m \times (k-1)}(t, \eta)) = r(D^{s+m \times (k-1)}(t, \eta)) = j, \\
0 & \text{otherwise}.
\end{cases}
\]

Then by considering the above matrix, using (6.5), (6.8) and elementary rank properties we get

\[
\theta_1(t, \eta) \cdot \theta_2^{q-1}(t, \eta) \text{ is equal to } 2^{q-1} \cdot 2^{(2s+m+k-3-j)q} \text{ if and only if}
\]

\[
r(D^{s-1+m \times (k-1)}(t, \eta)) = r(D^{s+m-1 \times (k-1)}(t, \eta)) = r(D^{s+m \times (k-1)}(t, \eta)) = j
\]

and

\[
r(D^{s+m \times k}(t, \eta)) = j \text{ or } j + 1
\]

\[
\theta_1(t, \eta) \cdot \theta_2^{q-1}(t, \eta) \text{ is equal to } -2^{q-1} \cdot 2^{(2s+m+k-3-j)q} \text{ if and only if}
\]

\[
r(D^{s-1+m \times (k-1)}(t, \eta)) = r(D^{s+m-1 \times (k-1)}(t, \eta)) = r(D^{s+m \times (k-1)}(t, \eta)) = r(D^{s+m-1 \times k}(t, \eta)) = j
\]

and

\[
r(D^{s+m \times k}(t, \eta)) = j + 1
\]
And in all the other cases \( \theta_1(t, \eta) \cdot \theta_2^{q-1}(t, \eta) \) is equal to zero, which proves Lemma 6.8.

**Lemma 6.9.** We have

\[
\int_{\mathbb{P} \times \mathbb{P}} \theta_1(t, \eta) \cdot \theta_2^{q-1}(t, \eta) dt d\eta = 0.
\]

**Proof.** The integral \( \int_{\mathbb{P} \times \mathbb{P}} \theta_1(t, \eta) \cdot \theta_2^{q-1}(t, \eta) dt d\eta \) is equal to the number of solutions \((Y, Z, U, Y_1, Z_1, U_1, \ldots, Y_{q-1}, Z_{q-1}, U_{q-1})\) of the polynomial equations

\[
\begin{aligned}
&YZ + Y_1Z_1 + \ldots + Y_{q-1}Z_{q-1} = 0, \\
&YU + Y_1U_1 + \ldots + Y_{q-1}U_{q-1} = 0,
\end{aligned}
\]

satisfying the degree conditions

\[
\deg Y = k - 1, \quad \deg Z = s - 1 \quad \deg U \leq s + m - 2
\]

\[
\deg Y_i \leq k - 2, \quad \deg Z_i \leq s - 1 \quad \deg U_i = s + m - 1 \quad \text{for } 1 \leq i \leq q - 1.
\]

By degree considerations we have that \( \deg [YZ + Y_1Z_1 + \ldots + Y_{q-1}Z_{q-1}] \) is equal to \( s + k - 2 \).

Lemma 6.9 follows. \( \Box \)

**Lemma 6.10.** We have for all \( j \in \mathbb{Z} \) such that \( 0 \leq j \leq \inf(2s + m - 2, k - 1) \),

\[
\begin{aligned}
\# \left( \substack{ j_1 \\ j_3 \\ j_5 } \right) &+ \# \left( \substack{ j_1 \\ j_4 \\ j_5 } \right) - \# \left( \substack{ j_1 \\ j_2 \\ j_5 } \right)
\end{aligned}
\]

\[
\begin{aligned}
\&= 0.
\end{aligned}
\]

Recall that

\[
\begin{aligned}
\# \left( \substack{ j_1 \\ j_3 \\ j_5 } \right)
\end{aligned}
\]

denotes the cardinality of the following set

\[
\begin{aligned}
&\{ (t, \eta) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \mid r(D)_{s+m-1}^{(k-1)}(t, \eta) = j_1, \quad r(D)_{s+m-1}^{k}(t, \eta) = j_2, \\
r(D)_{s+m-1}^s(t, \eta) = j_3, \quad r(D)_{s+m-1}^{k}(t, \eta) = j_4, \\
r(D)_{s+m}^{s}(t, \eta) = j_5, \quad r(D)_{s+m}^{k}(t, \eta) = j_6 \}
\end{aligned}
\]

for \((j_1, j_2, j_3, j_4, j_5, j_6) \in \mathbb{N}^6\).

**Proof.** We define

\[
\begin{aligned}
\mathcal{A} = \{ (t, \eta) \in \mathbb{P}^2 \mid r(D)_{s+m-1}^{(k-1)}(t, \eta) = r(D)_{s+m-1}^{k}(t, \eta)
\end{aligned}
\]

\[
\begin{aligned}
= r(D)_{s+m}^{s}(t, \eta) = r(D)_{s+m}^{k}(t, \eta)
\end{aligned}
\]

\[
\begin{aligned}
= r(D)_{s+m}^{s}(t, \eta) = r(D)_{s+m}^{k}(t, \eta)
\end{aligned}
\]
Lemma 6.11. Let \((t, \eta) \in \mathbb{P} \times \mathbb{P}\) and \(q\) be a rational integer \(\geq 2\).

Set
\[
\phi_3(t, \eta) = \sum_{\deg Y \leq k-2} \sum_{\deg Z = s-1} E(tYZ) \sum_{\deg U \leq s+m-2} E(\eta YU).
\]
Then we have
\((6.9)\)
\[
\phi^\alpha_3(t, \eta) = \begin{cases} 
2(2s+m+k-3-j)(q-1) & \text{if } (t, \eta) \in \{ k-1 & k \\ j & j \}
\end{cases} 
\]
\(\begin{array}{c}
\cdot \\
j & j
\end{array} \quad \alpha_s - \\
\cdot \\
j & j
\end{cases} 
\beta_{s+m-}
\]
otherwise.

\textbf{Proof.} Lemma 6.11 follows from (4.4) with \(k \to k - 1\).
\(\Box\)

\textbf{Lemma 6.12.} Let \((t, \eta) \in \mathbb{P}^2\), and \(q\) be a rational integer \(\geq 2\), then \(\phi(t, \eta) \cdot \theta^\alpha_3(t, \eta)\) is equal to
\[
\begin{cases} 
2 \cdot 2(2s+m+k-3-j)q & \text{if } (t, \eta) \in \{ k-1 & k \\ j & j \}
\end{cases} 
\]
\(\begin{array}{c}
\cdot \\
j & j
\end{array} \quad \alpha_s - \\
\cdot \\
j & j
\end{cases} 
\beta_{s+m-}
\]
\((6.10)\)
\[
-2 \cdot 2(2s+m+k-3-j)q & \text{if } (t, \eta) \in \{ j & j \\
j & j \}
\end{array} \quad \alpha_s - \\
\cdot \\
j & j
\end{cases} 
\beta_{s+m-}
\]
otherwise.

\textbf{Proof.} Recall that by Lemma 6.4 \(\phi(t, \eta)\) is equal to
\[
\begin{cases} 
2^{2s+m+k-j-2} & \text{if } (t, \eta) \in \{ k-1 & k \\ \cdot & \cdot \}
\end{cases} 
\]
\(\begin{array}{c}
\cdot \\
j & j
\end{array} \quad \alpha_s - \\
\cdot \\
j & j
\end{cases} 
\beta_{s+m-}
\]
\[
-2^{2s+m+k-j-2} & \text{if } (t, \eta) \in \{ j & j \\
j & j \}
\end{array} \quad \alpha_s - \\
\cdot \\
j & j
\end{cases} 
\beta_{s+m-}
\]
otherwise.

Then by (6.9) we obtain by elementary rank properties (6.10) and Lemma 6.12 is proved.
\(\Box\)

\textbf{Lemma 6.13.} We have
\[
\int_{\mathbb{P} \times \mathbb{P}} \phi(t, \eta) \cdot \theta^\alpha_3(t, \eta)dt \, d\eta = 0.
\]
Lemma 6.14. We have for all

Lemma 6.13 follows.

By degree considerations we have that \( \deg Y \leq k - 1, \quad \deg Z \leq s - 1, \quad \deg U = s + m - 1, \)
\( \deg Y_i \leq k - 2, \quad \deg Z_i = s - 1, \quad \deg U_i \leq s + m - 2 \) for \( 1 \leq i \leq q - 1. \)

By degree considerations we have that \( \deg [YU + Y_1U_1 + \ldots + Y_{q-1}U_{q-1}] \) is equal to \( k+s+m-2. \)

Lemma 6.13 follows.

\[ \square \]

Lemma 6.14. We have for all \( j \in \mathbb{Z} \) such that \( 0 \leq j \leq \inf(2s + m - 2, k - 1). \)

\[ \# \left( \begin{array}{c|c|c} j & j & j \\ \hline j & j & j \\ \hline j & j & j \end{array} \right) \frac{q}{j} _{p/p_{k+s-1} \times p/p_{k+s+m-1}} - \# \left( \begin{array}{c|c|c} j & j & j \\ \hline j & j & j \\ \hline j & j & j \end{array} \right) \frac{q}{j} _{p/p_{k+s-1} \times p/p_{k+s+m-1}} = 0 \]

Recall that

\[ \# \left( \begin{array}{c} j_1 \ \ j_2 \ \ j_3 \ \ j_4 \\ \hline j_5 \ \ j_6 \end{array} \right) \frac{q}{j} _{p/p_{k+s-1} \times p/p_{k+s+m-1}} \]

denotes the cardinality of the following set

\[ \left\{ (t, \eta) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \mid r(D^{s+1_{s+m-1} \times (k-1)}(t, \eta)) = j_1, \quad r(D^{s+1_{s+m-1} \times k}(t, \eta)) = j_2, \right. \]
\[ \left. r(D^{s+1_{s+m-1} \times (k-1)}(t, \eta)) = j_3, \quad r(D^{s+1_{s+m-1} \times k}(t, \eta)) = j_4, \right. \]
\[ \left. r(D^{s+1_{s+m-1} \times (k-1)}(t, \eta)) = j_5, \quad r(D^{s+1_{s+m-1} \times k}(t, \eta)) = j_6 \right\} \]

for \( (j_1, j_2, j_3, j_4, j_5, j_6) \in \mathbb{N^6}. \)

Proof. We define

\[ A = \left\{ (t, \eta) \in \mathbb{P}^2 \mid r(D^{s+1_{s+m-1} \times (k-1)}(t, \eta)) = r(D^{s+1_{s+m-1} \times k}(t, \eta)) = r(D^{s+1_{s+m-1} \times (k-1)}(t, \eta)) \right\} \]
\[ = r(D^{s+1_{s+m-1} \times k}(t, \eta)) \]

and

\[ B = \left\{ (t, \eta) \in \mathbb{P}^2 \mid r(D^{s+1_{s+m-1} \times (k-1)}(t, \eta)) = r(D^{s+1_{s+m-1} \times k}(t, \eta)) \right\} \]
\[ = r(D^{s+1_{s+m-1} \times k}(t, \eta)) \]

and

\[ r(D^{s+1_{s+m-1} \times (k-1)}(t, \eta)) + 1 \].
By (6.10) we have, observing that \( \phi(t, \eta) \cdot \theta_3^{q-1}(t, \eta) \) is constant on cosets of \( \mathbb{P}_{k+s-1} \times \mathbb{P}_{k+s+m-1} \)

\[
\int_{\mathbb{P} \times \mathbb{P}} \phi(t, \eta) \cdot \theta_3^{q-1}(t, \eta) dt \eta = \int_{\mathbb{A}} \phi(t, \eta) \cdot \theta_3^{q-1}(t, \eta) dt \eta + \int_{\mathbb{B}} \phi(t, \eta) \cdot \theta_3^{q-1}(t, \eta) dt \eta
\]

\[
= \sum_{(t, \eta) \in \mathbb{A} \cap (\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1})} 2 \cdot 2^{(2s+m+k-3-r)(D_{\frac{s-1}{s+m-1}} \times (k-1))} (t, \eta) q \int_{\mathbb{P}_{s+k-1}} dt \int_{\mathbb{P}_{s+k+m-1}} d\eta
\]

\[
+ \sum_{(t, \eta) \in \mathbb{B} \cap (\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1})} -2 \cdot 2^{(2s+m+k-3-r)(D_{\frac{s-1}{s+m-1}} \times (k-1))} (t, \eta) q \int_{\mathbb{P}_{s+k-1}} dt \int_{\mathbb{P}_{s+k+m-1}} d\eta
\]

\[
= \inf(2s+m-2,k-1) \sum_{j=0}^{\inf(2s+m-2,k-1)} 2 \cdot 2^{(2s+m+k-3-j)q} \cdot \left( \frac{\beta s - \alpha}{\beta s + \alpha} \right)^{\frac{q}{p}} \cdot 2^{-(2k+2s+m-2)}
\]

\[
- \sum_{j=0}^{\inf(2s+m-2,k-1)} 2 \cdot 2^{(2s+m+k-3-j)q} \cdot \left( \frac{\beta s - \alpha}{\beta s + \alpha} \right)^{\frac{q}{p}} \cdot 2^{-(2k+2s+m-2)}.
\]

Now by Lemma 6.13 we get for all \( q \geq 2 \)

\[
\sum_{j=0}^{\inf(2s+m-2,k-1)} 2^{-jq} \cdot \left[ \left( \begin{array}{c} j \\ j \end{array} \right) \right]^{\frac{q}{p}} \cdot \left( \begin{array}{c} j \\ j \end{array} \right) \left( \begin{array}{c} j + 1 \\ j \end{array} \right)^{\frac{q}{p}} \left( \begin{array}{c} j + 1 \\ j \end{array} \right) = 0.
\]

\[
\Box
\]

7. RANK PROPERTIES OF SUBMATRICES OF DOUBLE PERSYMMETRIC MATRICES

Consider the following partition of the matrix \( D \):

\[
\begin{pmatrix}
\frac{s-1}{s+m-1} \\
\alpha_{s-1} \\
\beta_{s+m-1}
\end{pmatrix} \times k
\]

\( (t, \eta) \)
For all $i$ such that

$$\text{Lemma 7.1.}$$

By studying rank properties of submatrices of the above double persymmetric matrix, we deduce by contradiction the following rank formula for all $i \in [0, \inf(2s + m - 3, k - 2)]$

$$\# \left( \begin{array}{c|c|c|c|c} i & i & i & \cdots & i \hline i & i & i & \cdots & i \hline i & i & i & \cdots & i \hline \end{array} \right)_{F/P^{k+s-1} \times P/P_{k+s+m-1}^{k+s+m}} = 0.$$ 

Recall that

$$\# \left( \begin{array}{c|c|c|c|c} i & i & i & \cdots & i \hline i & i & i & \cdots & i \hline i & i & i & \cdots & i \hline \end{array} \right)_{F/P^{k+s-1} \times P/P_{k+s+m-1}^{k+s+m}}$$

denotes the cardinality of the following set

$$\{ (t, \eta) \in P/P_{k+s-1} \times P/P_{k+s+m-1} \mid r(D^{[s-m-1] \times (k-1)}(t, \eta)) = i, \ r(D^{[s-m-1] \times k}(t, \eta)) = i + 1, \ r(D^{[s-m+1] \times (k-1)}(t, \eta)) = i, \ r(D^{[s-m+1] \times k}(t, \eta)) = i + 1 \}.$$ 

**Lemma 7.1.** For all $i$ such that $0 \leq i \leq \inf(2s + m - 3, k - 2)$ we have

$$\# \left( \begin{array}{c|c|c|c|c} i & i & i & \cdots & i \hline i & i & i & \cdots & i \hline i & i & i & \cdots & i \hline \end{array} \right)_{F/P^{k+s-1} \times P/P_{k+s+m-1}^{k+s+m}} = 0.$$ 

**Proof.** Set $(t, \eta) = (\sum_{i \geq 1} \alpha_i T^{-i}, \sum_{i \geq 1} \beta_i T^{-i}) \in P \times P$ and let $(s, k, m, j) \in \mathbb{N}^* \times \mathbb{N}^* \times \mathbb{N} \times \mathbb{N}^*$, where $1 \leq j \leq k - 1$. 


We denote by \( D_j^{\frac{s}{s+m}} (t, \eta) \) the following \((2s + m) \times (k - j + 1)\) matrix

\[
\begin{pmatrix}
\alpha_j & \alpha_{j+1} & \alpha_{j+2} & \ldots & \alpha_{k-1} & \alpha_k \\
\beta_j & \beta_{j+1} & \beta_{j+2} & \ldots & \beta_{k-1} & \beta_k \\
\alpha_{j+1} & \alpha_{j+2} & \alpha_{j+3} & \ldots & \alpha_k & \alpha_{k+1} \\
\beta_{j+1} & \beta_{j+2} & \beta_{j+3} & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{j+s-2} & \alpha_{j+s-1} & \alpha_{j+s} & \ldots & \alpha_{k+s-3} & \alpha_{k+s-2} \\
\beta_{j+s-2} & \beta_{j+s-1} & \beta_{j+s} & \ldots & \beta_{k+s-3} & \beta_{k+s-2} \\
\beta_{j+s-1} & \beta_{j+s} & \beta_{j+s+1} & \ldots & \beta_{k+s-2} & \beta_{k+s-1} \\
\beta_{j+s} & \beta_{j+s+1} & \beta_{j+s+2} & \ldots & \beta_{k+s-1} & \beta_{k+s} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{j+s+m-3} & \beta_{j+s+m-2} & \beta_{j+s+m-1} & \ldots & \beta_{k+s+m-4} & \beta_{k+s+m-3} \\
\beta_{j+s+m-2} & \beta_{j+s+m-1} & \beta_{j+s+m} & \ldots & \beta_{k+s+m-3} & \beta_{k+s+m-2} \\
\alpha_{j+s-1} & \alpha_{j+s} & \alpha_{j+s+1} & \ldots & \alpha_{k+s-2} & \alpha_{k+s-1} \\
\beta_{j+s+m-1} & \beta_{j+s+m} & \beta_{j+s+m+1} & \ldots & \beta_{k+s+m-2} & \beta_{k+s+m-1}
\end{pmatrix}
\]

Remark that after a rearrangement of the rows in the above matrix we obtain the following double persymmetric matrix where the first \(s\) rows form a \(s \times (k - j + 1)\) persymmetric matrix and the last \(s + m\) rows form a \((s + m) \times (k - j + 1)\) persymmetric matrix with entries in \(\mathbb{F}_2\)

\[
\begin{pmatrix}
\alpha_j & \alpha_{j+1} & \alpha_{j+2} & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_{j+1} & \alpha_{j+2} & \alpha_{j+3} & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{j+s-2} & \alpha_{j+s-1} & \alpha_{j+s} & \ldots & \alpha_{k+s-3} & \alpha_{k+s-2} \\
\alpha_{j+s-1} & \alpha_{j+s} & \alpha_{j+s+1} & \ldots & \alpha_{k+s-2} & \alpha_{k+s-1} \\
\beta_j & \beta_{j+1} & \beta_{j+2} & \ldots & \beta_{k-1} & \beta_k \\
\beta_{j+1} & \beta_{j+2} & \beta_{j+3} & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{j+s-2} & \beta_{j+s-1} & \beta_{j+s} & \ldots & \beta_{k+s-3} & \beta_{k+s-2} \\
\beta_{j+s-1} & \beta_{j+s} & \beta_{j+s+1} & \ldots & \beta_{k+s-2} & \beta_{k+s-1} \\
\beta_{j+s} & \beta_{j+s+1} & \beta_{j+s+2} & \ldots & \beta_{k+s-1} & \beta_{k+s} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{j+s+m-3} & \beta_{j+s+m-2} & \beta_{j+s+m-1} & \ldots & \beta_{k+s+m-4} & \beta_{k+s+m-3} \\
\beta_{j+s+m-2} & \beta_{j+s+m-1} & \beta_{j+s+m} & \ldots & \beta_{k+s+m-3} & \beta_{k+s+m-2} \\
\beta_{j+s+m-1} & \beta_{j+s+m} & \beta_{j+s+m+1} & \ldots & \beta_{k+s+m-2} & \beta_{k+s+m-1}
\end{pmatrix}
\]

Proof by contradiction.
Assume on the contrary that there exists \(i_0 \in [0, \inf(2s + m - 3, k - 2)]\) such that

\[
(7.1) \quad \# \left( \begin{array}{c|c}
\frac{i_0}{i_0} & \frac{i_0 + 1}{i_0 + 1} \\
\frac{1}{i_0} & \frac{i_0 + 1}{i_0 + 1} \\
\end{array} \right) \geq 0.
\]
We are going to show that

\[
\# \left( \frac{i_0}{i_0} \begin{pmatrix} i_0 + 1 \\ i_0 \\ i_0 + 1 \end{pmatrix} \right) > 0 \implies \# \left( \frac{i_0 - 1}{i_0} \begin{pmatrix} i_0 + 1 \\ i_0 - 1 \\ i_0 \end{pmatrix} \right) > 0
\]

\[
\implies \ldots \implies \# \left( \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\
0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right) > 0,
\]

which obviously contradicts

\[
\# \left( \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\
0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right) = 0.
\]

By (7.1) there exists \((t_0, \eta_0) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}\) such that

\[
\begin{align*}
r(D_{\frac{s+m-1}{s+m-1}}(t_0, \eta_0)) &= r(D_{\frac{s+m-1}{s+m-1}}(t_0, \eta_0)) = r(D_{\frac{s+m}{s+m}}(t_0, \eta_0)) = i_0, \\
r(D_{\frac{s+m-1}{s+m-1}}(t_0, \eta_0)) &= r(D_{\frac{s+m-1}{s+m-1}}(t_0, \eta_0)) = r(D_{\frac{s+m}{s+m}}(t_0, \eta_0)) = i_0 + 1.
\end{align*}
\]

Set \((t_0, \eta_0) = (\sum_{i \geq 1} \alpha_i T^{-i}, \sum_{i \geq 1} \beta_i T^{-i})\), and consider the following partition of the matrix \(D_{\frac{s+m}{s+m}}(t_0, \eta_0)\)

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_s-1 & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{k-s-2} & \alpha_{k-s} \\
\beta_s-1 & \beta_s & \beta_{s+1} & \ldots & \beta_{k-s-3} & \beta_{k-s-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-2} & \beta_{s+m-1} & \beta_{s+m} & \ldots & \beta_{k+s+m-4} & \beta_{k+s+m-3} \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{k+s+m-3} & \beta_{k+s+m-2} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{k+s-2} & \alpha_{k+s} \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{k+s+m-2} & \beta_{k+s+m-1}
\end{pmatrix}
\]

By (7.1) we have \(r(D_{\frac{s+m-1}{s+m-1}}(t_0, \eta_0)) \leq i_0\) and \(r(D_{\frac{s+m-1}{s+m-1}}(t_0, \eta_0)) = i_0 + 1\).

It follows that \(r(D_{\frac{s+m-1}{s+m-1}}(t_0, \eta_0)) = i_0).
Let $a_1, a_2, \ldots, a_k$ denote the columns of $D^{\frac{s-1}{s+m-1} \times k}(t_0, \eta_0)$ that is,

$$\begin{align*}
a_j &= \begin{pmatrix}
\alpha_j \\
\beta_j \\
\alpha_{j+1} \\
\beta_{j+1} \\
\vdots \\
\alpha_{j+s-2} \\
\beta_{j+s-2} \\
\beta_{j+s-1} \\
\beta_{j+s} \\
\vdots \\
\beta_{j+s+m-3} \\
\beta_{j+s+m-2}
\end{pmatrix}.
\end{align*}$$

Since $r(D_2^{\frac{s-1}{s+m-1} \times (k-1)}(t_0, \eta_0)) = i_0$ and $r(D_2^{\frac{s-1}{s+m-1} \times k}(t_0, \eta_0)) = i_0 + 1$,

we have $a_1 \notin \text{span}\{a_2, a_3, \ldots, a_k\}$, therefore $r(D_2^{\frac{s-1}{s+m-1} \times (k-2)}(t_0, \eta_0)) = i_0 - 1$.

We have then

(7.2) $r(D_2^{\frac{s-1}{s+m-1} \times (k-2)}(t_0, \eta_0)) = i_0 - 1,$

(7.3) $r(D^{\frac{s-1}{s+m-1} \times (k-1)}(t_0, \eta_0)) = i_0,$

(7.4) $r(D^{\frac{s^s}{s+m-1} \times (k-1)}(t_0, \eta_0)) = i_0,$

(7.5) $r(D^{\frac{s^s}{s+m-1} \times (k-1)}(t_0, \eta_0)) = i_0$ and, see above,

(7.6) $r(D_2^{\frac{s-1}{s+m-1} \times (k-1)}(t_0, \eta_0)) = i_0.$
Now we consider the matrix obtained from the matrix $D\left[ s+m \right] ^{x k} (t_0, \eta_0)$ by delating the first column and replacing the last column by the first one:

$$
\begin{pmatrix}
\alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_1 \\
\beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_1 \\
\alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_2 \\
\beta_3 & \beta_4 & \ldots & \beta_k & \beta_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_s & \alpha_{s+1} & \ldots & \alpha_{k+s-3} & \alpha_{s-1} \\
\beta_s & \beta_{s+1} & \ldots & \beta_{k+s-3} & \beta_{s-1} \\
\beta_{s+1} & \beta_{s+2} & \ldots & \beta_{k+s-2} & \beta_s \\
\beta_{s+2} & \beta_{s+3} & \ldots & \beta_{k+s-1} & \beta_{s+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \ldots & \beta_{k+s+m-4} & \beta_{s+m-2} \\
\beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{k+s+m-3} & \beta_{s+m-1}
\end{pmatrix}
$$

From (7.2), (7.3), (7.4) and (7.5) we obtain by elementary rank considerations

$$
(7.7) \quad r(D_2^{[s+m]} \times (k-2) (t_0, \eta_0)) = i_0 - 1,
$$

$$
(7.8) \quad r(D_2^{[s+m]} \times (k-1) (t_0, \eta_0)) = i_0 - 1.
$$

Consider now the matrix $D_2^{[s+m]} \times (t_0, \eta_0)$ obtained by the matrix $D^{[s+m]} \times (t_0, \eta_0)$ by delating the first column, that is

$$
\begin{pmatrix}
\alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_s & \alpha_{s+1} & \ldots & \alpha_{k+s-3} & \alpha_{k+s-2} \\
\beta_s & \beta_{s+1} & \ldots & \beta_{k+s-3} & \beta_{k+s-2} \\
\beta_{s+1} & \beta_{s+2} & \ldots & \beta_{k+s-2} & \beta_{k+s-1} \\
\beta_{s+2} & \beta_{s+3} & \ldots & \beta_{k+s-1} & \beta_{k+s} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \ldots & \beta_{k+s+m-4} & \beta_{k+s+m-3} \\
\beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{k+s+m-3} & \beta_{k+s+m-2}
\end{pmatrix}
$$
From (7.2), (7.6), (7.7) and (7.8), we obtain
\[ r(D_2^{s+m-1} \times (k-1), (t_0, \eta_0)) = r(D_2^{s+m} \times (k-1), (t_0, \eta_0)) = i_0. \]

We get
\[ \# \left( \begin{array}{c|c} & \frac{i_0 - 1}{i_0} \frac{i_0}{i_0} \\ \hline \frac{i_0 - 1}{i_0} & \frac{i_0}{i_0} \end{array} \right) > 0. \]

Recall that
\[ \# \left( \begin{array}{c|c} & \frac{i_0 - 1}{i_0} \frac{i_0}{i_0} \\ \hline \frac{i_0 - 1}{i_0} & \frac{i_0}{i_0} \end{array} \right) > 0. \]

denotes the cardinality of the following set
\[ \{ (t, \eta) \in \mathbb{P}_1/\mathbb{P}_{k+s-1} \times \mathbb{P}_1/\mathbb{P}_{k+s+m-1} | r(D_2^{s+m-1} \times (k-2), (t, \eta)) = i_0 - 1, \quad r(D_2^{s+m} \times (k-1), (t, \eta)) = i_0, \]
\[ r(D_2^{s+m-1} \times (k-2), (t, \eta)) = i_0 - 1, \quad r(D_2^{s+m} \times (k-1), (t, \eta)) = i_0, \]
\[ r(D_2^{s+m} \times (k-2), (t, \eta)) = i_0 - 1, \quad r(D_2^{s+m} \times (k-1), (t, \eta)) = i_0 \}. \]

We have now proved that
\[ \# \left( \begin{array}{c|c} & \frac{i_0}{i_0} \frac{i_0 + 1}{i_0} \\ \hline \frac{i_0}{i_0} & \frac{i_0 + 1}{i_0} \end{array} \right) > 0 \quad \Rightarrow \quad \# \left( \begin{array}{c|c} & \frac{i_0 - 1}{i_0} \frac{i_0}{i_0} \\ \hline \frac{i_0 - 1}{i_0} & \frac{i_0}{i_0} \end{array} \right) > 0. \]

We repeat this procedure and obtain after finitely many steps
\[ (7.9) \quad \# \left( \begin{array}{c|c} & 1 \\ \hline 0 & 1 \end{array} \right) > 0. \]
From

\[
D_{\alpha_0+1} \left[ (s+m) \times (k-i_0) \right] (t_0, \eta_0) = \begin{pmatrix}
\alpha_{i_0+1} & \alpha_{i_0+2} & \alpha_{i_0+3} & \ldots & \alpha_{k-1} & \alpha_k \\
\beta_{i_0+1} & \beta_{i_0+2} & \beta_{i_0+3} & \ldots & \beta_{k-1} & \beta_k \\
\alpha_{i_0+2} & \alpha_{i_0+3} & \alpha_{i_0+4} & \ldots & \alpha_k & \alpha_{k+1} \\
\beta_{i_0+2} & \beta_{i_0+3} & \beta_{i_0+4} & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{i_0+s-1} & \alpha_{i_0+s} & \alpha_{i_0+s+1} & \ldots & \alpha_{k+s-3} & \alpha_{k+s-2} \\
\beta_{i_0+s} & \beta_{i_0+s+1} & \beta_{i_0+s+2} & \ldots & \beta_{k+s-3} & \beta_{k+s-2} \\
\alpha_{i_0+s+1} & \alpha_{i_0+s+2} & \alpha_{i_0+s+3} & \ldots & \alpha_{k+s-2} & \alpha_{k+s-1} \\
\beta_{i_0+s+2} & \beta_{i_0+s+3} & \beta_{i_0+s+4} & \ldots & \beta_{k+s-1} & \beta_{k+s} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{i_0+s+m-2} & \beta_{i_0+s+m-1} & \beta_{i_0+s+m} & \ldots & \beta_{k+s+m-4} & \beta_{k+s+m-3} \\
\beta_{i_0+s+m-1} & \beta_{i_0+s+m} & \beta_{i_0+s+m+1} & \ldots & \beta_{k+s+m-3} & \beta_{k+s+m+1} \\
\alpha_{i_0+s} & \alpha_{i_0+s+1} & \alpha_{i_0+s+2} & \ldots & \alpha_{k+s-1} & \alpha_{k+s} \\
\beta_{i_0+s+m} & \beta_{i_0+s+m+1} & \beta_{i_0+s+m+2} & \ldots & \beta_{k+s+m-2} & \beta_{k+s+m-1}
\end{pmatrix}.
\]

we obviously get

\[
\# \left( \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \right)_{\overline{\mathbb{F}}} = 0
\]

which clearly contradicts \((7.9)\).

Recall that

\[
\# \left( \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \right)_{\overline{\mathbb{F}}} = 0
\]

denotes the cardinality of the following set

\[
\left\{ (t, \eta) \in \mathbb{P}_{i_0}/\mathbb{P}_{k+s-1} \times \mathbb{P}_{i_0}/\mathbb{P}_{k+s+m-1} \mid r(D)_{s+m-1} \times (k-i_0) \right] (t, \eta) = 0, \quad r(D)_{s+m-1} \times (k-i_0) \right] (t, \eta) = 1,
\]

\[
r(D)_{s+m-1} \times (k-i_0) \right] (t, \eta) = 0, \quad r(D)_{s+m-1} \times (k-i_0) \right] (t, \eta) = 1,
\]

\[
r(D)_{s+m} \times (k-i_0) \right] (t, \eta) = 0, \quad r(D)_{s+m} \times (k-i_0) \right] (t, \eta) = 1.\]

\(\square\)

**Lemma 7.2.** We have

\[
(7.10) \quad \# \left( \begin{pmatrix} i & i+1 \\ i & i+1 \\ i \end{pmatrix} \right)_{\overline{\mathbb{F}}} = 0 \quad \text{if } 0 \leq i \leq \inf(2s+m-3, k-2),
\]
\[(7.11) \quad \# \left( \begin{array}{ccc} i & i & i \\ i & i & i \\ i & i & i \end{array} \right) \overset{\sigma}{\rightarrow} \# \left( \begin{array}{ccc} i & i & i \\ i & i & i \\ i & i & i \end{array} \right) \overset{\sigma}{\rightarrow} P^{k+s}_{k+s+m-1} P^{k+s}_{k+s+m-1} \quad \text{if } 0 \leq i \leq \inf(2s+m-2,k-1),\]

\[(7.12) \quad \# \left( \begin{array}{ccc} i & i & i \\ i & i & i \\ i & i & i \end{array} \right) \overset{\sigma}{\rightarrow} 2 \cdot \# \left( \begin{array}{ccc} i & i & i \\ i & i & i \\ i & i & i \end{array} \right) \overset{\sigma}{\rightarrow} P^{k+s}_{k+s+m-1} P^{k+s}_{k+s+m-1} \quad \text{if } 0 \leq i \leq \inf(2s+m-2,k-1).\]

**Proof.** The Lemma follows respectively from Lemmas 7.1, 6.14 and 6.10. 

8. **STUDY OF THE REMAINDER \( \Delta^s_{s+m} \times k \) IN THE RECURRENT FORMULA**

From the rank formulas established in sections 6 and 7, we deduce by elementary rank considerations the following formula for \( 1 \leq i \leq 2s + m, \ k \geq i+1 \)

\[
\Delta^s_{s+m} \times k = \sum_{j=i-2}^{i+1} a_j \cdot \Gamma^s_{s-1+m} \times j
\]

where the \( a_j \in \mathbb{Z} \) are explicitly determined.

We get

\[
\Delta^s_{s+m} \times k = \Delta^s_{s+m} \times (i+1) \quad \text{whenever } k \geq i + 1.
\]

Let \( (j_1,j_2,j_3) \in \mathbb{N}^3 \) recall that we define

\[
\sigma_{j_1,j_2,j_3}^s \times k
\]

to be the cardinality of the following set

\[
\left\{ (t,\eta) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \mid r(D^s_{s+m-1} \times k)(t,\eta) = j_1, \quad r(D^{s+m-1}_{s+m-1} \times k)(t,\eta) = j_2, \quad r(D^s_{s+m} \times k)(t,\eta) = j_3 \right\}.
\]
Lemma 8.1. For $1 \leq i \leq 2s + m - 3, \quad k \geq i + 1$ we have

\[
\sigma_{i,i,i}^{(s-1)} \alpha_s \begin{bmatrix} \alpha_{i+1} & \alpha_{i+2} & \cdots & \alpha_{i+k-1} \\ \beta_{i+k+1} & \beta_{i+k+2} & \cdots & \beta_{i+k+m-2} \\ \beta_{i+k+m+1} & \beta_{i+k+m+2} & \cdots & \beta_{i+k+m+k-2} \end{bmatrix} \times k = \sigma_{i,i,i}^{(s-1)} \beta_{i+k+m} \begin{bmatrix} \alpha_{i+1} & \alpha_{i+2} & \cdots & \alpha_{i+k-1} \\ \beta_{i+k+1} & \beta_{i+k+2} & \cdots & \beta_{i+k+m-2} \\ \beta_{i+k+m+1} & \beta_{i+k+m+2} & \cdots & \beta_{i+k+m+k-2} \end{bmatrix} \times (i+1)
\]

(8.1)

Proof. The formula (8.1) is obvious for $k = i+1$.

We consider the following partition of the matrix

\[
D = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k & \alpha_{k+1} \\ \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_k & \beta_{k+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{s-1} & \alpha_s & \alpha_{s+1} & \cdots & \alpha_{s+k-2} & \alpha_{s+k-1} \end{bmatrix} \times (t, \eta)
\]

Let $k \geq i + 2$. Using Lemma 7.1 we obtain by elementary rank considerations

(8.2)

\[
\sigma_{i,i,i}^{(s-1)} = \# \begin{bmatrix} \frac{i}{i} \frac{i}{i} \frac{i}{i} \frac{i}{i} \end{bmatrix} \frac{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}}{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}} + \# \begin{bmatrix} \frac{i - 1}{i} \frac{i}{i} \frac{i}{i} \frac{i}{i} \end{bmatrix} \frac{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}}{\mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}}
\]

Further, by Lemma 7.2 we have
Hence by (8.4) we obtain successively

\[
\begin{align*}
4 \cdot \sigma_{i,i,i} & = \# \left( \begin{array}{ccc} i & i & i \\ i & i & i \\ i & i & i \\ \end{array} \right) \ 4 \cdot \sigma_{i,i,i} + \# \left( \begin{array}{ccc} i & i & i \\ i & i & i \\ i & i & i \\ \end{array} \right) \ 
+ \# \left( \begin{array}{ccc} i & i & i \\ i & i & i \\ i & i & i \\ \end{array} \right) \\
& = 4 \cdot \# \left( \begin{array}{ccc} i & i & i \\ i & i & i \\ i & i & i \\ \end{array} \right)
\end{align*}
\]

By (8.2) we deduce

\[
\begin{align*}
\sigma_{i,i,i} \times (k-1) & = \sigma_{i,i,i} \times k
\end{align*}
\]

(8.4)

Hence by (8.4) we obtain successively

\[
\begin{align*}
& \sigma_{i,i,i} \times (k-1) \times (k-1) = \sigma_{i,i,i} \times (k-2) \times (k-2) = \sigma_{i,i,i} \times (i+1) \\
& \quad = \sigma_{i,i,i} \times (i+1)
\end{align*}
\]

\[\square\]

**Lemma 8.2.** For all \( i \) such that \( 1 \leq i \leq 2s + m - 3 \) we have

\[
\begin{align*}
\sigma_{i,i,i} & = 4 \cdot \Gamma_i \left[ \begin{array}{c} s-1 \\ s + m - 1 \end{array} \right] \times (i+1) \\
& = 4 \cdot \Gamma_i \left[ \begin{array}{c} s-1 \\ s + m - 1 \end{array} \right] \times (i+1) - \Gamma_i \left[ \begin{array}{c} s-1 \\ s + m - 1 \end{array} \right] \times (i+1)
\end{align*}
\]

(8.5)

**Proof.** Consider the following partition of the matrix

\[
D \left( \begin{array}{c} s-1 \\ s + m - 1 \end{array} \right) \times (i+1)
\]

\((t, \eta)\)
Further, by Lemma 7.2 we have

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_i & a_{i+1} \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{i+1} & \alpha_{i+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{i+s-2} & \alpha_{i+s-1} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_i & \beta_{i+1} \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{i+1} & \beta_{i+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+i-2} & \beta_{s+m+i-1} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{s+i-1} & \alpha_{s+i} \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{s+m+i-1} & \beta_{s+m+i}
\end{pmatrix}
\]

By elementary rank considerations and using Lemma 7.1 we obtain

\[
(8.6) \quad \sigma_{i,i,i}^{s-1 \times (i+1)} \times (i+1) = \# \left( \frac{i}{i} \frac{i}{i} \frac{i}{i} \right)_{P/P_{i+s} \times P/P_{i+s+m}} + \# \left( \frac{i-1}{i} \frac{i}{i} \frac{i}{i} \right)_{P/P_{i+s} \times P/P_{i+s+m}}
\]

\[
+ \# \left( \frac{i-1}{i} \frac{i}{i} \frac{i}{i} \right)_{P/P_{i+s} \times P/P_{i+s+m}} + \# \left( \frac{i-1}{i} \frac{i}{i} \frac{i}{i} \right)_{P/P_{i+s} \times P/P_{i+s+m}}
\]

\[
= \# \left( \frac{i}{i} \frac{i}{i} \frac{i}{i} \right)_{P/P_{i+s} \times P/P_{i+s+m}}.
\]

Further, by Lemma 7.2 we have

\[
(8.7) \quad 4 \cdot \sigma_{i,i,i}^{s-i \times i} = 4 \cdot 4 \cdot \Gamma_{i}^{s-1 \times (1+m)} \times i
\]

\[
= \# \left( \frac{i}{i} \frac{i}{i} \frac{i}{i} \right)_{P/P_{i+s} \times P/P_{i+s+m}} + \# \left( \frac{i}{i} \frac{i}{i} \frac{i}{i} \right)_{P/P_{i+s} \times P/P_{i+s+m}}
\]

\[
+ \# \left( \frac{i}{i} \frac{i}{i} \frac{i}{i} \right)_{P/P_{i+s} \times P/P_{i+s+m}} + \# \left( \frac{i}{i} \frac{i}{i} \frac{i}{i} \right)_{P/P_{i+s} \times P/P_{i+s+m}}
\]

\[
= 4 \cdot \# \left( \frac{i}{i} \frac{i}{i} \frac{i}{i} \right)_{P/P_{i+s} \times P/P_{i+s+m}} + \# \left( \frac{i}{i} \frac{i}{i} \frac{i}{i} \right)_{P/P_{i+s} \times P/P_{i+s+m}}
\]
= 4 \cdot \left( \begin{array}{ccc} i & i & i \\ 1 & i & 1 \\ i & 1 & i \end{array} \right) \frac{\Delta}{P/P_{i+1}} + 4 \cdot \Gamma_{i+1}^{s-1+m} \times (i+1).

Combining (8.6), (8.7) we deduce

\begin{equation}
(8.8) \quad 16 \cdot \Gamma_i^{s-1+m} \times (i) = 4 \cdot \sigma_{i,i,i} + 4 \cdot \Gamma_{i+1}^{s-1+m} \times (i+1)
\end{equation}

and (8.5) is established. \hfill \square

Lemma 8.3. Let $s \geq 2$ and $m \geq 0$, we have in the following two cases:

The case $1 \leq k \leq 2s + m - 2$

\begin{equation}
(8.9) \quad \sigma_{i,i,i} = \begin{cases} 1 & \text{if } i = 0, \quad k \geq 1, \\
4 \cdot \Gamma_i^{s-1+m} \times (i) - \Gamma_{i+1}^{s-1+m} \times (i+1) & \text{if } 1 \leq i \leq k - 1, \\
4 \cdot \Gamma_k^{s-1+m} \times (i) & \text{if } i = k.
\end{cases}
\end{equation}

The case $k \geq 2s + m - 2$

\begin{equation}
(8.10) \quad \sigma_{i,i,i} = \begin{cases} 1 & \text{if } i = 0, \\
4 \cdot \Gamma_i^{s-1+m} \times (i) - \Gamma_{i+1}^{s-1+m} \times (i+1) & \text{if } 1 \leq i \leq 2s + m - 3, \\
4 \cdot \Gamma_{2s+m-2}^{s-1+m} \times (2s+m-2) & \text{if } i = 2s + m - 2.
\end{cases}
\end{equation}

Proof. Combining (8.1) and (8.5) and recalling the definition of $\sigma_{i,i,i}$ we deduce easily Lemma 8.3. \hfill \square

Lemma 8.4. The remainder $\Delta_i^{s} \times (i)$ in the recurrent formula is equal to
Proof. From (8.9) and (8.10) and recalling that \( \Delta_{\text{8.11}} \)
we deduce easily Lemma 8.4. □

Proof. Follows immediately from Lemma 8.4.

In this section we deduce from the recurrent formula in Lemma 5.14 and from the fact
that the remainder \( \Delta_{i,s,m,k} \) is independent of \( k \) if \( k \geq i + 1 \), the following recurrent
formula

\[
\Gamma_i \left[ \frac{s}{s+m} \right] \times (k+1) - \Gamma_i \left[ \frac{s}{s+m} \right] \times k = 4 \cdot \Gamma_{i-1} \left[ \frac{s+(m-1)}{s+(m-1)} \right] \times (k+1) - \Gamma_{i-1} \left[ \frac{s+(m-1)}{s+(m-1)} \right] \times k
\]

\[+ R(i, s, m, k) \quad \text{if } 0 \leq i \leq \inf(2s + m, k - 1), \quad \text{where } R(i, s, m, k) \text{ is equal to} \]

\( \sum_{i=0}^{k-1} \left[ \frac{s}{s+m} \right] \times i \) for \( i \in \{2s + m - 2, 2s + m - 1, 2s + m\}, k \geq i + 1. \)
\[
\begin{align*}
2^{s-1} & \left[ \frac{1}{\Gamma_{j+1}^{1+(s+m-1)\times(k+1)} - \Gamma_{j+1}^{1+(s+m-1)\times k}} \right] \\
-2^{s+1} & \left[ \frac{1}{\Gamma_j^{1+(s+m-2)\times(k+1)} - \Gamma_j^{1+(s+m-2)\times k}} \right]
\end{align*}
\] if \( 1 \leq i \leq s - 1, \ k \geq i + 1, \)

\[
2^{s-1} \left[ \frac{1}{\Gamma_{j+1}^{1+(s+m-1)\times(k+1)} - \Gamma_{j+1}^{1+(s+m-1)\times k}} \right]
\] if \( i = s + j, \ 0 \leq j \leq \inf(s + m, k - s - 1). \)

If \( i = s + j \) we set \( R(i, s, m, k) = R(j, s, m) \) where \( 0 \leq j \leq \inf(s + m, k - s - 1). \)

**Lemma 9.1.** Let \( s \geq 2, m \geq 0 \) and \( k \geq 1 \), then \( \Gamma_i^{s+m} - 4 \cdot \Gamma_{i-1}^{s+(m-1)} \) is given by

\[
\begin{align*}
5 \cdot 2^{s-1} & + \sum_{j=0}^{i-2} 2^j \Delta_{i-j}^{s-j+(m+j)} \\
2^{s-1} & \left[ \frac{1}{\Gamma_{i-(s-1)}^{1+(s+m-1)\times k}} - 4 \cdot \Gamma_{i-s}^{1+(m-2)\times k} \right] + \sum_{j=0}^{i-2} 2^j \Delta_{i-j}^{s-j+(m+j)} \\
\end{align*}
\] if \( 2 \leq i \leq \inf(s - 1, k), \)

if \( s \leq i \leq \inf(k, 2s + m). \)

**Proof.** The first case \( 2 \leq i \leq \inf(s - 1, k) \)

Set for \( 1 \leq i \leq \inf(2s + m, k) \)

\[
\Omega_i(s, m, k) = \Gamma_i^{s+m} - 4 \cdot \Gamma_{i-1}^{s+(m-1)}
\]

so

\[
\Omega_{i-1}(s - 1, m + 1, k) = \Gamma_{i-1}^{s-1+(m+1)} - 4 \cdot \Gamma_{i-2}^{s-1+m}.
\]

We obtain then by Lemma 5.14

\[
\begin{align*}
\Gamma_i^{s+m} & = 2 \cdot \Gamma_{i-1}^{s-1+(m+1)} + 4 \cdot \Gamma_{i-1}^{s+(m-1)} - 8 \cdot \Gamma_{i-2}^{s-1+m} + \Delta_i^{s+m}, \\
\iff \Omega_i(s, m, k) & = 2 \cdot \Omega_{i-1}(s - 1, m + 1, k) + \Delta_i^{s+m}.
\end{align*}
\]

By (9.3) we obtain successively

\[
\Omega_i(s, m, k) = 2 \cdot \Omega_{i-1}(s - 1, m + 1, k) + \Delta_i^{s+m}
\]

\[
2 \cdot \Omega_{i-1}(s - 1, m + 1, k) = 2 \left( 2 \cdot \Omega_{i-2}(s - 2, m + 2, k) + \Delta_{i-2}^{s-1+(m+1)} \right)
\]
EXPONENTIAL SUMS AND RANK OF DOUBLE PERSYMMETRIC MATRICES OVER $\mathbb{F}_2$

\[
2^j \cdot \Omega_{i-j}(s-j, m+j, k) = 2^j \cdot \left(2 \cdot \Omega_{i-(j+1)}(s-(j+1), m+(j+1), k) + \Delta_{(s-j) \times (m+j)}\right)
\]

\[
2^{i-2} \cdot \Omega_{i-(i-2)}(s-(i-2), m+(i-2), k) = 2^{i-2} \cdot \left(2 \cdot \Omega_{i-(i-1)}(s-(i-1), m+(i-1), k) + \Delta_{(s-(i-2)) \times (m+(i-2))}\right)
\]

By summing the above equations we get

\[
\sum_{j=0}^{i-2} 2^j \cdot \Omega_{i-j}(s-j, m+j, k) = \sum_{j=0}^{i-2} 2^{j+1} \cdot \Omega_{i-(j+1)}(s-(j+1), m+(j+1), k) + \sum_{j=0}^{i-2} 2^j \cdot \Delta_{(s-j) \times (m+j)}\]
\[
= \sum_{j=1}^{i-1} 2^j \cdot \Omega_{i-j}(s-j, m+j, k) + \sum_{j=0}^{i-2} 2^j \cdot \Delta_{(s-j) \times (m+j)}\times k.
\]

By (9.4) we get after some simplifications

\[
\Omega_i(s, m, k) = 2^{i-1} \cdot \Omega_1(s-(i-1), m+(i-1), k) + \sum_{j=0}^{i-2} 2^j \cdot \Delta_{(s-j) \times (m+j)}\times k.
\]

By the definition of $\Omega_i(s, m, k)$ we have

\[
\Omega_1(s-(i-1), m+(i-1), k) = \Gamma_{\left[\frac{s-i+1}{s+m}\right] \times k} - 4 \cdot \Gamma_{\left[\frac{s-i+1}{s+m-1}\right] \times k}.
\]

Recall that $D_{(s-i+1) \times k}(t, \eta)$ denotes the following $(2s + m - i + 1) \times k$ matrix where the first $(s-i+1)$ rows form a $(s-i+1) \times k$ persymmetric matrix and the last $(s+m)$ rows form a $(s+m) \times k$ persymmetric matrix with entries in $\mathbb{F}_2$. 

By (9.5) we get after some simplifications
Then obviously we have

\[
\Gamma_1^{[s-i+1] \times k} = \text{Card} \left\{ (t, \eta) \in \mathbb{P}/\mathbb{P}_{k+s-i-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \mid r(D_{[s+m]}^{[s-i+1] \times k}(t, \eta)) = 1 \right\} = 3 + 3 + 3 = 9.
\]

And (9.1) follows in the first case.

The second case \(s \leq i \leq \inf(2s + m, k)\)

We proceed as in the first case.

Again by (9.3) we obtain successively

\[
\Omega_i(s, m, k) = 2 \cdot \Omega_{i-1}(s-1, m+1, k) + \Delta_i^{[s+m] \times k}
\]

\[
2 \cdot \Omega_{i-1}(s-1, m+1, k) = 2 \cdot \left( 2 \cdot \Omega_{i-2}(s-2, m+2, k) + \Delta_{i-1}^{[s-1+m+1] \times k} \right)
\]

\[
\vdots
\]

\[
2^j \cdot \Omega_{i-j}(s-j, m+j, k) = 2^j \cdot \left( 2 \cdot \Omega_{i-(j+1)}(s-(j+1), m+(j+1), k) + \Delta_{i-j}^{[s-j+m+j] \times k} \right)
\]

\[
\vdots
\]
Lemma 9.2 now follows from Theorem 9.1 using the above equations.

Proof.

(9.8)

\[ 2^s \cdot \Omega_{i-(s-2)}(s-(s-2), m + (s-2), k) = 2^{s-2} \cdot \left( 2 \cdot \Omega_{i-(s-1)}(s-(s-1), m + (s-1), k) + \Delta_{i-(s-2)}^{\left[ s-(s-2) \right]} \right) \]

By summing the above equations we obtain

(9.6)

\[
\sum_{j=0}^{s-2} 2^j \cdot \Omega_{i-j}(s-j, m+j, k) = \sum_{j=0}^{s-2} 2^{j+1} \cdot \Omega_{i-(j+1)}(s-(j+1), m+(j+1), k) + \sum_{j=0}^{s-2} 2^j \cdot \Delta_{i-j}^{\left[ s-j \right]} \times k
\]

By (9.6) we get after some simplifications

(9.7)

\[ \Omega_i(s, m, k) = 2^{s-1} \cdot \Omega_{i-(s-1)}(1, m + s - 1, k) + \sum_{j=0}^{s-2} 2^j \cdot \Delta_{i-j}^{\left[ s-j \right]} \times k \]

By the definition of \( \Omega_i(s, m, k) \) we have \( \Omega_{i-(s-1)}(1, m + s - 1, k) = \Gamma_i^{\left[ 1 + \frac{1}{(s-1)} \right]} \times k \), \( \Gamma_i^{\frac{1}{s}} \times k \). And (9.1) follows in the second case. \( \square \)

Lemma 9.2. We have

(9.8)

\[
\left\{ \begin{array}{l}
\Gamma_i^{\frac{s}{s+m}} \times (k+1) - \Gamma_i^{\frac{s}{s+m}} \times k = 4 \cdot \left[ \Gamma_i^{\left[ s \right]} \times (k+1) - \Gamma_i^{\left[ s \right]} \times k \right] \quad \text{if } 2 \leq i \leq s-1, \quad k \geq i+1,
\\
\Gamma_i^{\frac{s}{s+m}} \times (k+1) - \Gamma_i^{\frac{s}{s+m}} \times k = 4 \cdot \left[ \Gamma_i^{\left[ s \right]} \times (k+1) - \Gamma_i^{\left[ s \right]} \times k \right] \quad \text{if } s \leq i \leq 2s+m, \quad k \geq i+1
\end{array} \right.
\]

Proof. By (8.12) we obtain

\[
\sum_{j=0}^{i-2} 2^j \Delta_{i-j}^{\left[ s-j \right]} \times (k+1) = \sum_{j=0}^{i-2} 2^j \Delta_{i-j}^{\left[ s-j \right]} \times k \quad \text{for } 2 \leq i \leq s-1, \quad k \geq i+1
\]

and equally

\[
\sum_{j=0}^{s-2} 2^j \Delta_{i-j}^{\left[ s-j \right]} \times (k+1) = \sum_{j=0}^{s-2} 2^j \Delta_{i-j}^{\left[ s-j \right]} \times k \quad \text{for } 2 \leq s \leq i \leq 2s+m, \quad k \geq i+1
\]

Lemma 9.2 now follows from Theorem 9.1 using the above equations. \( \square \)
10. COMPUTATION OF THE REMAINDER $R(j,s,m,k)$ IN THE RECURRENT FORMULA FOR THE DIFFERENCE

$$\Gamma_i^{\left[ \frac{1+m}{s-j} \right] x^k} - \Gamma_i^{\left[ \frac{s+m}{s+j} \right] x^k} \quad \text{for } 0 \leq j \leq s + m, \, k > s + j$$

From a formula of $\Gamma_i^{\left[ \frac{1+m}{s-j} \right] x^k}$ obtained in [4] we compute in this section the remainder $R(j,s,m,k)$ in the recurrent formula for the difference $\Gamma_i^{\left[ \frac{s+m}{s+j} \right] x^k} - \Gamma_i^{\left[ \frac{s+m}{s+j} \right] x^k}$.

By definition we have for $s \leq i \leq \inf(2s + m, k)$

$$\Omega_{i-(s-1)}(1, m + s - 1, k) = \Gamma_i^{\left[ \frac{1+m}{s-j} \right] x^k} - 4 \cdot \Gamma_i^{\left[ \frac{1+m}{s-j} \right] x^k}.$$ 

Set $i - s = j$, then

$$\Omega_{j+1}(1, m + s - 1, k) = \Gamma_{j+1}^{\left[ \frac{1+m}{s-j} \right] x^k} - 4 \cdot \Gamma_{j+1}^{\left[ \frac{1+m}{s-j} \right] x^k}.$$ 

Lemma 10.1. We have by Theorem 3.8 [see [4]] with $m \rightarrow s + m - 1, m \rightarrow s + m - 2$

The case $s + m - 1 = 0, k \geq 2$

$$\Gamma_j^{\left[ \frac{1+(s+m-1)}{1} \right] x^k} = \Gamma_j^{\left[ \frac{1}{1} \right] x^k} = \begin{cases} 1 & \text{if } j = 0, \\ 3 \cdot (2^{k-1}) & \text{if } j = 1, \\ 2^{2k} - 3 \cdot 2^k + 2 & \text{if } j = 2. \end{cases}$$

The case $s + m - 1 = 1, k \geq 3$

$$\Gamma_j^{\left[ \frac{1+(s+m-1)}{1} \right] x^k} = \Gamma_j^{\left[ \frac{1}{1} \right] x^k} = \begin{cases} 1 & \text{if } j = 0, \\ 2^k + 5 & \text{if } j = 1, \\ 11 \cdot (2^{k-1}) & \text{if } j = 2, \\ 2^{2k+1} - 3 \cdot 2^{k+2} + 2^4 & \text{if } j = 3. \end{cases}$$

The case $3 \leq k \leq s + m$

$$\Gamma_{j+1}^{\left[ \frac{1+(s+m-1)}{1} \right] x^k} = \begin{cases} 2^k + 5 & \text{if } j = 0, \\ 3 \cdot 2^{k+2j-2} + 21 \cdot 2^{3j-2} & \text{if } 1 \leq j \leq k - 2, \\ 2^{2k+s+m-1} - 5 \cdot 2^{3k-5} & \text{if } j = k - 1. \end{cases}$$

The case $4 \leq s + m \leq k$

$$\Gamma_{j+1}^{\left[ \frac{1+(s+m-1)}{1} \right] x^k} = \begin{cases} 2^k + 5 & \text{if } j = 0, \\ 3 \cdot 2^{k+2j-2} + 21 \cdot 2^{3j-2} & \text{if } 1 \leq j \leq s + m - 2, \\ 11 \cdot 2^{k+2s+2m-4} - 2^{3s+3m-5} & \text{if } j = s + m - 1, \\ 2^{2k+s+m-1} - 3 \cdot 2^{k+2s+2m-2} + 2^{3s+3m-2} & \text{if } j = s + m. \end{cases}$$

The case $3 \leq k \leq s + m$

$$\Gamma_j^{\left[ \frac{1+(s+m-2)}{1} \right] x^k} = \begin{cases} 1 & \text{if } j = 0, \\ 2^k + 5 & \text{if } j = 1, \\ 3 \cdot 2^{k+2j-4} + 21 \cdot 2^{3j-5} & \text{if } 2 \leq j \leq k - 1, \\ 2^{2k+s+m-2} - 5 \cdot 2^{3k-5} & \text{if } j = k. \end{cases}$$
The case $4 \leq s + m \leq k$

$$\Gamma_j^{1+(s+m-2)} = \begin{cases} 
1 & \text{if } j = 0, \\
2^k + 5 & \text{if } j = 1, \\
3 \cdot 2^k + 2^{j-4} + 21 \cdot 2^{3j-5} & \text{if } 2 \leq j \leq s + m - 2, \\
11[2^k + s + 2m - 6 - 2^{3s-3m-8}] & \text{if } j = s + m - 1, \\
2^k + s + 2m - 2 - 3 \cdot 2^k + 2^{s+2m-4} + 2^{3s+3m-5} & \text{if } j = s + m. 
\end{cases}$$

Lemma 10.2. For $0 \leq j \leq s + m, k \geq s + j + 1$

$$2^{s-1} \left[ \Gamma_j^{1+(s+m-1)} \times (k+1) - \Gamma_j^{1+(s+m-1)} \times k \right] - 2^{s+1} \left[ \Gamma_j^{1+(s+m-2)} \times (k+1) - \Gamma_j^{1+(s+m-2)} \times k \right]$$

is equal to

$$\begin{cases} 
2^{k+s-1} & \text{if } j = 0, \quad k \geq s + 1, \\
-2^{k+s-1} & \text{if } j = 1, \quad k \geq s + 2, \\
0 & \text{if } 2 \leq j \leq s + m - 1, \quad k \geq s + j + 1, \\
-3 \cdot 2^{k+2s+m-2} & \text{if } j = s + m, \quad k \geq 2s + m + 1.
\end{cases}$$

Proof. From Lemma 10.1 we deduce

In the case $j = 0, k \geq s + 1$

$$2^{s-1} \left[ \Gamma_0^{1+(s+m-1)} \times (k+1) - \Gamma_0^{1+(s+m-1)} \times k \right] - 2^{s+1} \left[ \Gamma_0^{1+(s+m-2)} \times (k+1) - \Gamma_0^{1+(s+m-2)} \times k \right]$$

$$= 2^{s-1} \left[ (2^{k+1} + 5) - (2^k + 5) \right] - 2^{s+1} \left[ 1 - 1 \right] = 2^{k+s-1}.$$  

In the case $j = 1, k \geq s + 2$

$$2^{s-1} \left[ \Gamma_1^{1+(s+m-1)} \times (k+1) - \Gamma_1^{1+(s+m-1)} \times k \right] - 2^{s+1} \left[ \Gamma_1^{1+(s+m-2)} \times (k+1) - \Gamma_1^{1+(s+m-2)} \times k \right]$$

$$= 2^{s-1} \left[ (3 \cdot 2^{k+1} + 42) - (3 \cdot 2^k + 42) \right] - 2^{s+1} \left[ (2^{k+1} + 5) - (2^k + 5) \right]$$

$$= 3 \cdot 2^{k+s-1} - 2^{k+s+1} = -2^{k+s-1}.$$  

In the case $2 \leq j \leq m - 1, s + j + 1 \leq k \leq s + m$

$$2^{s-1} \left[ \Gamma_j^{1+(s+m-1)} \times (k+1) - \Gamma_j^{1+(s+m-1)} \times k \right] - 2^{s+1} \left[ \Gamma_j^{1+(s+m-2)} \times (k+1) - \Gamma_j^{1+(s+m-2)} \times k \right]$$

$$= 2^{s-1} \left[ (3 \cdot 2^{k+2j-1} + 21 \cdot 2^{3j-2}) - (3 \cdot 2^{k+2j-2} + 21 \cdot 2^{3j-2}) \right]$$

$$- 2^{s+1} \left[ (3 \cdot 2^{k+2j-3} + 21 \cdot 2^{3j-5}) - (3 \cdot 2^{k+2j-4} + 21 \cdot 2^{3j-5}) \right]$$

$$= 3 \cdot 2^{k+2j+s-3} - 3 \cdot 2^{k+2j+s-3} = 0.$$
In the case $2 \leq j \leq m - 1 \leq s + m - 2$, $k \geq s + m$

$$2^{s-1} \left[ \Gamma_{j+1} \left[ 1 + \frac{1}{s+m-1} \right] \times (k+1) - \Gamma_{j+1} \left[ 1 + \frac{1}{s+m-1} \right] \times k \right] - 2^{s+1} \left[ \Gamma_{j} \left[ 1 + \frac{1}{s+m-2} \right] \times (k+1) - \Gamma_{j} \left[ 1 + \frac{1}{s+m-2} \right] \times k \right]$$

$$= 2^{s-1} \left[ (3 \cdot 2^{k+2j-1} + 21 \cdot 2^{3j-2}) - (3 \cdot 2^{k+2j-2} + 21 \cdot 2^{3j-2}) \right]$$

$$- 2^{s+1} \left[ (3 \cdot 2^{k+2j-3} + 21 \cdot 2^{3j-5}) - (3 \cdot 2^{k+2j-4} + 21 \cdot 2^{3j-5}) \right]$$

$$= 3 \cdot 2^{k+2j+s-3} - 3 \cdot 2^{k+2j+s-3} = 0.$$

In the case $2 \leq m \leq s + m - 2$, $k \geq s + j + 1 \geq s + m + 1$

$$2^{s-1} \left[ \Gamma_{j+1} \left[ 1 + \frac{1}{s+m-1} \right] \times (k+1) - \Gamma_{j+1} \left[ 1 + \frac{1}{s+m-1} \right] \times k \right] - 2^{s+1} \left[ \Gamma_{j} \left[ 1 + \frac{1}{s+m-2} \right] \times (k+1) - \Gamma_{j} \left[ 1 + \frac{1}{s+m-2} \right] \times k \right]$$

$$= 2^{s-1} \left[ (3 \cdot 2^{k+2j-1} + 21 \cdot 2^{3j-2}) - (3 \cdot 2^{k+2j-2} + 21 \cdot 2^{3j-2}) \right]$$

$$- 2^{s+1} \left[ (3 \cdot 2^{k+2j-3} + 21 \cdot 2^{3j-5}) - (3 \cdot 2^{k+2j-4} + 21 \cdot 2^{3j-5}) \right]$$

$$= 3 \cdot 2^{k+2j+s-3} - 3 \cdot 2^{k+2j+s-3} = 0.$$

In the case $j = s + m - 1$, $k \geq s + j + 1 = 2s + m$

$$2^{s-1} \left[ \Gamma_{s+m} \left[ 1 + \frac{1}{s+m-1} \right] \times (k+1) - \Gamma_{s+m} \left[ 1 + \frac{1}{s+m-1} \right] \times k \right] - 2^{s+1} \left[ \Gamma_{s+m-1} \left[ 1 + \frac{1}{s+m-2} \right] \times (k+1) - \Gamma_{s+m-1} \left[ 1 + \frac{1}{s+m-2} \right] \times k \right]$$

$$= 2^{s-1} \left[ 11(2^{k+2s+2m-3} + 2^{3s+3m-5}) - 11(2^{k+2s+2m-4} + 2^{3s+3m-5}) \right]$$

$$- 2^{s+1} \left[ 11(2^{k+2s+2m-5} + 2^{3s+3m-8}) - 11(2^{k+2s+2m-6} + 2^{3s+3m-8}) \right]$$

$$= 11 \cdot 2^{k+3s+2m-5} - 11 \cdot 2^{k+3s+2m-5} = 0.$$

In the case $j = s + m$, $k \geq s + j + 1 = 2s + m + 1$

$$2^{s-1} \left[ \Gamma_{s+m+1} \left[ 1 + \frac{1}{s+m-1} \right] \times (k+1) - \Gamma_{s+m+1} \left[ 1 + \frac{1}{s+m-1} \right] \times k \right] - 2^{s+1} \left[ \Gamma_{s+m} \left[ 1 + \frac{1}{s+m-2} \right] \times (k+1) - \Gamma_{s+m} \left[ 1 + \frac{1}{s+m-2} \right] \times k \right]$$

$$= 2^{s-1} \left[ (2^{2k+s+m+1} - 3 \cdot 2^{k+2s+2m-1} + 2^{3s+3m-2}) - (2^{2k+s+m-1} - 3 \cdot 2^{k+2s+2m-2} + 2^{3s+3m-2}) \right]$$

$$- 2^{s+1} \left[ (2^{2k+s+m} - 3 \cdot 2^{k+2s+2m-3} + 2^{3s+3m-5}) - (2^{2k+s+m-2} - 3 \cdot 2^{k+2s+2m-4} + 2^{3s+3m-5}) \right]$$

$$= 2^{s-1} \left[ 3 \cdot 2^{2k+s+m-1} - 3 \cdot 2^{k+2s+2m-2} \right] - 2^{s+1} \left[ 3 \cdot 2^{2k+s+m-2} - 3 \cdot 2^{k+2s+2m-4} \right]$$

$$= -3 \cdot 2^{2k+2s+m-2}.$$

In all the others cases the proofs are similar. □

**Lemma 10.3.** We have for $s \geq 2$, $m \geq 0$ the following recurrent formula for the difference $\Gamma_i \left[ \frac{s}{s+m} \right] \times (k+1) - \Gamma_i \left[ \frac{s}{s+m} \right] \times k$ where $0 \leq i \leq 2s + m, k > i$. 
Proof. Lemma 10.3 follows from Lemma 9.2 and lemma 10.2.

\[ \begin{aligned}
\Gamma_{s+j}^{(k+1)}(s+m) &- \Gamma_{s+j}^{(k+1)}(s+m) = 4 \cdot \Gamma_{s+j}^{(k+1)}(s+m) - \Gamma_{s+j}^{(k+1)}(s+m) \\
\Gamma_{s+m}^{(k+1)}(s) - \Gamma_{s+m}^{(k+1)}(s) &= 4 \cdot \Gamma_{s+m}^{(k+1)}(s) - \Gamma_{s+m}^{(k+1)}(s)
\end{aligned} \]

if \( 1 \leq j \leq s-1, \ k \geq j+1, \)

11. COMPUATION OF \( \Gamma_{s+m}^{(k+1)}(s) \) FOR \( i \leq s-1, \ i \leq k-1, \ m \geq 0 \)

In this section we apply successively the recurrent formula (10.1) for the difference

\[ \Gamma_{i+1}^{(s+m)} - \Gamma_{i}^{(s+m)} \]

to compute \( \Gamma_{i}^{(s+m)} \) for \( i \leq s-1, \ i \leq k-1, \ m \geq 0. \)

Lemma 11.1. Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{s+k-2} & \alpha_{s+k-1} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{s+m+k-2} & \beta_{s+m+k-1}
\end{pmatrix}
\]

Let \( g_{k,s,m}(t, \eta) \) be the quadratic exponential sum in \( \mathbb{P} \times \mathbb{P} \) defined by

\[ (t, \eta) \in \mathbb{P} \times \mathbb{P} \rightarrow \sum_{\text{deg}Y \leq k-1} \sum_{\text{deg}Z \leq s-1} E(tYZ) \sum_{\text{deg}U \leq s+m-1} E(\eta YU) \in \mathbb{Z}. \]

We have

\[
\inf_{2(s+m,k)} \sum_{i=0}^{2s+m,k} \Gamma_{i}^{(s+m)} = 2^{2k+2s+m-2}
\]

and
(11.2) \[
\sum_{i=0}^{\inf(2s+m,k)} \Gamma_i \binom{s}{s+m}^k \cdot 2^{-i} = 2^{k+2s+m-2} + 2^{2k-2} - 2^{k-2}.
\]

Proof. The proof is by strong induction, that is:

By Lemma 11.0, we obtain, observing that \(g_{k,s,m}(t, \eta)\) is constant on cosets of \(P_{k+s-1} \times P_{k+s+m-1}\)

\[
\int_{P \times P} g_{k,s,m}(t, \eta) dtd\eta = \sum_{(t, \eta) \in P/P_{k+s-1} \times P/P_{k+s+m-1}} 2^{2s+m+k-r(D)} \int_{P_{k+s-1}} dt \int_{P_{k+s+m-1}} d\eta \sum_{i=0}^{\inf(2s+m,k)} \Gamma_i \binom{s}{s+m}^k \cdot 2^{2s+m+k-i} \int_{P_{k+s-1}} dt \int_{P_{k+s+m-1}} d\eta = 2^{-k+2} \sum_{i=0}^{\inf(2s+m,k)} \Gamma_i \binom{s}{s+m}^k \cdot 2^{-i}.
\]

On the other hand

\[
\int_{P \times P} g_{k,s,m}(t, \eta) dtd\eta = \text{Card} \{(Y, Z, U), \deg Y \leq k - 1, \deg Z \leq s - 1, \deg U \leq s + m - 1 | Y \cdot Z = Y \cdot U = 0\} = 2^{2s+m} + 2^k - 1.
\]

The above equations imply (11.2).

Lemma 11.2. We have for all \(j \geq 1\)

(11.3) \[
(H_j) \quad \Gamma_j \binom{s}{s+m}^k = \Gamma_j \binom{j}{j} \cdot 2^k \quad \text{for} \quad s \geq j + 1, \ k \geq j + 1, \ m \geq 0.
\]

Proof. The proof is by strong induction, that is:

If

- \((H_1)\) is true, and
- for all \(j \geq 1, \quad (H_1) \land (H_2) \land \ldots \land (H_j) \implies (H_{j+1})\)

then \((H_j)\) is true for all \(j \geq 1\).

\((H_1)\) is true

Indeed from (10.1) with \(j = 1\)

\[
\Gamma_1 \binom{s}{s+m}^{(k+1)} - \Gamma_1 \binom{s}{s+m}^k = 4 \left( \Gamma_0^s \binom{s}{s+(m-1)}^{(k+1)} - 2 \Gamma_0^s \binom{s}{s+(m-1)}^k \right) \quad \text{for all} \quad s \geq 2, \ k \geq 2, \ m \geq 0
\]

which implies that \((H_j)\) holds for \(j = 1\), that is

(11.4) \[
\Gamma_1 \binom{s}{s+m}^k = \Gamma_1 \binom{s}{s+m}^2 = \Gamma_1 \binom{1}{1} = \gamma_1 = 9 \quad \forall \, s \geq 2, \, \forall \, k \geq 2, \, m \geq 0.
\]

\((H_1) \Rightarrow (H_2)\)

From (10.1) with \(j = 2\)
\[ \Gamma_2^{s+m} \times (k+1) - \Gamma_2^{s+m} \times k = 4 \cdot \left[ \Gamma_1^{s+(m-1)} \times (k+1) - \Gamma_1^{s+(m-1)} \times k \right] \quad \text{for all } s \geq 3, \ k \geq 3, \ m \geq 0. \]

Hence by (11.4) we have

\[ \Gamma_2^{s+m} \times k = \Gamma_2^{s+m} \times 3 \quad \text{for all } s \geq 3, \ k \geq 3, \ m \geq 0. \]

Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_2 & \alpha_3 & \alpha_4 \\
\vdots & \vdots & \vdots \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} \\
\beta_1 & \beta_2 & \beta_3 \\
\beta_2 & \beta_3 & \beta_4 \\
\vdots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} \\
\vdots & \vdots & \vdots \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2}
\end{pmatrix}
\]

We have respectively by (11.1), (11.2) with \( k = 3 \)

\[ \sum_{i=0}^{3} \Gamma_1^{s+m} \times 3 = 2^{2s+m+4}, \]

\[ \sum_{i=0}^{3} \Gamma_1^{s+m} \times 3 \cdot 2^{-i} = 2^{2s+m+1} + 2^4 - 2. \]

From (11.6) and (11.7) using (11.4) we deduce

\[ \Gamma_2^{s+m} \times k = \Gamma_2^{s+m} \times 3 = 78 \quad \text{and} \quad \Gamma_3^{s+m} \times 3 = 2^{2s+m+4} - 88 \]

which implies, using (11.5), that \((H_j)\) holds for \( j = 2 \), that is

\[ \Gamma_2^{s+m} \times k = \Gamma_2^{s+m} \times 3 = 78 \quad \forall \ s \geq 3, \ \forall \ k \geq 3, \ \forall \ m \geq 0. \]

\((H_1) \land (H_2)\) implies \((H_3)\)

From (10.1) with \( j = 3 \)

\[ \Gamma_3^{s+m} \times (k+1) - \Gamma_3^{s+m} \times k = 4 \cdot \left[ \Gamma_2^{s+(m-1)} \times (k+1) - \Gamma_2^{s+(m-1)} \times k \right] \quad \text{for all } s \geq 4, \ k \geq 3, \ m \geq 0. \]
Hence by (11.8) we have

\[(11.9) \Gamma_{\frac{s}{s+m}}^{s}\times k = \Gamma_{\frac{s}{s+m}}^{s}\times 4 \text{ for all } s \geq 4, k \geq 4, m \geq 0.\]

Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\
\vdots & \vdots & \vdots & \vdots \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \alpha_{s+3} \\
\beta_1 & \beta_2 & \beta_3 & \beta_4 \\
\beta_2 & \beta_3 & \beta_4 & \beta_5 \\
\vdots & \vdots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \beta_{m+4} \\
\vdots & \vdots & \vdots & \vdots \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \beta_{s+m+3} \\
\end{pmatrix}
\]

We have respectively by (11.1), (11.2) with \(k = 4\)

\[(11.10) \sum_{i=0}^{4} \Gamma_{\frac{s}{s+m}}^{s}\times 4 = 2^{2s+m+6},\]

\[(11.11) \sum_{i=0}^{4} \Gamma_{\frac{s}{s+m}}^{s}\times 4 \cdot 2^{-i} = 2^{2s+m+2} + 2^{6} - 2^{2}.
\]

From (11.10) and (11.11) using (11.5) and (11.9) we deduce

\[
\Gamma_{\frac{s}{s+m}}^{s}\times 4 = 648 \quad \text{and} \quad \Gamma_{\frac{s}{s+m}}^{s}\times 4 = 2^{2s+m+6} - 736
\]

which implies, using (11.9), that \((H_j)\) holds for \(j = 3\), that is

\[(11.12) \Gamma_{\frac{s}{s+m}}^{s}\times k = \Gamma_{\frac{s}{s+m}}^{s}\times 4 = 648 = \gamma_3 \text{ for all } s \geq 4, k \geq 4, m \geq 0.\]

\((H_1) \land (H_2) \land \ldots \land (H_j) \implies (H_{j+1})\)

From (10.1) with \(j \to j+1\)

\[
\Gamma_{\frac{s}{s+m}}^{s}\times (k+1) - \Gamma_{\frac{s}{s+m}}^{s}\times k = 4 \cdot \left[ \Gamma_{\frac{s}{s+m}}^{s}\times (k+1) - \Gamma_{\frac{s}{s+m}}^{s}\times (k+1) \right] \text{ for all } s \geq j + 2, k \geq j + 2, m \geq 0.
\]

By \((H_j)\) we obtain

\[
\Gamma_{\frac{s}{s+m}}^{s}\times (k+1) - \Gamma_{\frac{s}{s+m}}^{s}\times k = \gamma_j - \gamma_j = 0.
\]

Indeed if \(m = 0\)
\[
\Gamma_j \left[ \begin{array}{c}
s + (m-1) \\
\end{array} \right] \times k = \Gamma_j \left[ \begin{array}{c}
s-1 \\
\end{array} \right] \times k = \Gamma_j \left[ \begin{array}{c}
j \\
\end{array} \right] \times (j+1) = \gamma_j
\]

since

\[s - 1 \leq j + 1, \quad k \geq j + 2 \geq j + 1, \quad m \to 1 \geq 0.\]

If \(m \geq 1\)

\[
\Gamma_j \left[ \begin{array}{c}
s + (m-1) \\
\end{array} \right] \times k = \Gamma_j \left[ \begin{array}{c}
j \\
\end{array} \right] \times (j+1) = \gamma_j
\]

since

\[s \geq j + 2 \geq j + 1, \quad k \geq j + 2 \geq j + 1, \quad m \to (m - 1) \geq 0.\]

Thus, we obtain

\[(11.13) \quad \Gamma_{j+1} \left[ \begin{array}{c}
s \\
\end{array} \right] \times k = \Gamma_{j+1} \left[ \begin{array}{c}
s+m \\
\end{array} \right] \times (j+2) \quad \text{for } s \geq j + 2, \quad k \geq j + 2, \quad m \geq 0.\]

Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{j+1} & \alpha_{j+2} \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{j+2} & \alpha_{j+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+j-1} & \alpha_{s+j} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{s+j} & \alpha_{s+j+1} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{j+1} & \beta_{j+2} \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{j+2} & \beta_{j+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \ldots & \beta_{j+m+1} & \beta_{j+m+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+j-1} & \beta_{s+m+j} \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{s+m+j} & \beta_{s+m+j+1}
\end{pmatrix}
\]

We have respectively by (11.1), (11.2) with \(k = j + 2\)

\[(11.14) \quad \sum_{i=0}^{j+2} \Gamma_i \left[ \begin{array}{c}
s \\
\end{array} \right] \times (j+2) = 2^{2j+2s+m+2}
\]

and

\[(11.15) \quad \sum_{i=0}^{j+2} \Gamma_i \left[ \begin{array}{c}
s \\
\end{array} \right] \times (j+2) \cdot 2^{-i} = 2^{j+2s+m} + 2^{j+2} - 2^{j}.
\]

Setting \(s = j+1, \ m = 0\) in (11.14), (11.15) we have
\begin{equation}
\sum_{i=0}^{j+2} \Gamma_{i}^{j+1} = 2^{4j+4}
\end{equation}

and

\begin{equation}
\sum_{i=0}^{j+2} \Gamma_{i}^{j+1} \cdot 2^{-i} = 2^{3j+2} + 2^{2j+2} - 2^{j}.
\end{equation}

From \((H_1) \land (H_2) \land \ldots \land (H_j)\), it follows that

\begin{equation}
\Gamma_{i}^{s} = \Gamma_{i}^{j+1} = \gamma_{i} \quad \text{for } 0 \leq i \leq j.
\end{equation}

We deduce from (11.14), (11.15)

\begin{equation}
\Gamma_{i}^{s+m} = \Gamma_{i}^{j+1} = \gamma_{i} \quad \text{for all } s \geq j + 2, \ k \geq j + 2, \ m \geq 0.
\end{equation}

\textbf{Lemma 11.3.} We have

\begin{equation}
\Gamma_{i}^{s+m} \cdot \Gamma_{i}^{j+1} = \Gamma_{i}^{s} \cdot \Gamma_{i}^{j+1} = \gamma_{j+1} \quad \text{for all } s \geq j + 2, \ k \geq j + 2, \ m \geq 0.
\end{equation}

\textbf{Proof.} We have respectively by (11.1), (11.2) with \(k = i+1, s = i\) and \(m = 0\), using (11.3)

\begin{equation}
\sum_{j=0}^{i+1} \Gamma_{j}^{i} \cdot \Gamma_{i+1}^{i+1} = \sum_{j=0}^{i} \gamma_{j} + \Gamma_{i+1}^{i+1} = 2^{4i}
\end{equation}
and

\[(11.23) \quad \sum_{j=0}^{i+1} \Gamma_j^{[j] \times (i+1)} \cdot 2^{-j} = \sum_{j=0}^{i} \gamma_j \cdot 2^{-j} + \Gamma_{i+1}^{[i] \times (i+1)} \cdot 2^{-(i+1)} = 2^{3i-1} + 2^{2i} - 2^{i-1}.\]

From (11.22) and (11.23) we get

\[2^{i+1} \cdot \left( \sum_{j=0}^{i} \gamma_j \cdot 2^{-j} + \Gamma_{i+1}^{[i] \times (i+1)} \cdot 2^{-(i+1)} \right) - \left( \sum_{j=0}^{i} \gamma_j + \Gamma_{i+1}^{[i] \times (i+1)} \right) = 2^{3i+1} - 2^{2i}\]

\[(11.24) \quad \iff \sum_{j=0}^{i} \gamma_j \cdot (2^{i+1-j} - 1) = 2^{3i+1} - 2^{2i}.\]

Hence by (11.24) we deduce

\[\sum_{j=0}^{i} \gamma_j = 3 \cdot 2^{3i-1} - 2^{2i-1}.\]

From (11.25) we get for \(i \geq 2\)

\[\sum_{j=0}^{i} \gamma_j - \sum_{j=0}^{i-1} \gamma_j = (3 \cdot 2^{3i-1} - 2^{2i-1}) - (3 \cdot 2^{3i-4} - 2^{2i-3})\]

\[\iff \Gamma_{i+1}^{[i] \times (i+1)} = \gamma_i = 21 \cdot 2^{3i-4} - 3 \cdot 2^{2i-3}.\]

\[\Box\]

**Lemma 11.4.** We have

\[(11.26) \quad \Gamma_k^{[s+m] \times k} = 2^{2k+2s+m-2} - 3 \cdot 2^{3k-4} + 2^{2k-3} \quad \text{for } 1 \leq k \leq s,\]

\[(11.27) \quad \sum_{j=0}^{s-1} \Gamma_j^{[s+m] \times k} = 3 \cdot 2^{3s-4} - 2^{2s-3} \quad \text{for } k \geq s,\]

\[(11.28) \quad \sum_{j=0}^{k-1} \Gamma_j^{[s+m] \times k} \cdot 2^{-j} = 7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3} \quad \text{for } k \geq s.\]
Proof. Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{s+k-2} & \alpha_{s+k-1} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{s+m+k-2} & \beta_{s+m+k-1}
\end{pmatrix}
\]

Obviously by (11.25) we have

\[
\begin{align*}
\Gamma_k \left[ \binom{s+m}{k} \right]^x &= 2^{2k+2s+m-2} - \sum_{j=0}^{k-1} \Gamma_j \left[ \binom{s+m}{k} \right]^x \\
&= 2^{2k+2s+m-2} - \sum_{j=0}^{k-1} \gamma_j \\
&= 2^{2k+2s+m-2} - (3 \cdot 2^{3(k-1)} - 2^{2(k-1)}) = 2^{2k+2s+m-2} - 3 \cdot 2^{3k-4} + 2^{2k-3}.
\end{align*}
\]

We deduce (11.27) from (11.25) with \( i = s - 1 \).

From (11.21) we obtain after some calculations

\[
\sum_{j=0}^{s-1} \Gamma_j \left[ \binom{s+m}{s+j} \right]^x \cdot 2^{-j} = 1 + \sum_{j=1}^{s-1} (21 \cdot 2^{2j-4} - 3 \cdot 2^{j-3}) = 7 \cdot 2^{s-4} - 3 \cdot 2^{s-3}.
\]

\[\blacksquare\]

**COMPUTATION OF**

\[
\Gamma_k \left[ \binom{s+m}{s+j} \right]^{(k+1)} - \Gamma_k \left[ \binom{s+m}{s+j} \right]^x \quad \text{for } 0 \leq j \leq s + m, \quad k \geq s + j + 1, \quad m \geq 0
\]

In this section we apply successively the recurrent formula (10.1) to compute explicitly the difference \( \Gamma_k \left[ \binom{s+m}{s+j} \right]^{(k+1)} - \Gamma_k \left[ \binom{s+m}{s+j} \right]^x \).

**Lemma 12.1.** We have

\[
(12.1) \quad \Gamma_k \left[ \binom{s}{s+j} \right]^{(k+1)} - \Gamma_k \left[ \binom{s}{s+j} \right]^x = \begin{cases} 
3 \cdot 2^{k+s-1} & \text{if } j = 0, \ k > s, \\
21 \cdot 2^{k+s+3j-4} & \text{if } 1 \leq j \leq s - 1, \ k > s + j, \\
3 \cdot 2^{2k+2s-2} - 3 \cdot 2^{k+4s-4} & \text{if } j = s, \ k > 2s.
\end{cases}
\]

**Proof.** From (10.1) we have the following formula
From (12.6), (12.7) we get

\[
\Gamma_{s+j+1}^{s+m} \times (k+1) - \Gamma_{s+j}^{s+m} \times k = 4 \cdot \left[ \Gamma_{s+j-1}^{s+(m-1)} \times (k+1) - \Gamma_{s+j-1}^{s+(m-1)} \times k \right]
\]

+ \begin{cases}
2^{k+s-1} & \text{if } j = 0, \ k \geq s + 1, \\
-2^{k+s-1} & \text{if } j = 1, \ k \geq s + 2, \\
0 & \text{if } 2 \leq j \leq s + m - 1, \ k \geq s + j + 1, \\
-3 \cdot 2^{2k+2s+m-2} & \text{if } j = s + m, \ k \geq 2s + m + 1.
\end{cases}

The case \( j = 0, k > s \)

Applying formula (12.2) we obtain using (11.21)

\[
(12.3) \quad \Gamma_{s}^{s} \times (k+1) - \Gamma_{s}^{s} \times k = 4 \cdot \left[ \Gamma_{s-1}^{s-1} \times (k+1) - \Gamma_{s-1}^{s-1} \times k \right] + 2^{k+s-1},
\]

\[
(12.4) \quad \Gamma_{s-1}^{s-1} \times (k+1) - \Gamma_{s-1}^{s-1} \times k = 4 \cdot \left[ \Gamma_{s-2}^{s-1} \times (k+1) - \Gamma_{s-2}^{s-1} \times k \right] + 2^{k+s-2} = 2^{k+s-2}.
\]

From (12.2), (12.4) we get

\[
(12.5) \quad \Gamma_{s}^{s} \times (k+1) - \Gamma_{s}^{s} \times k = 4 \cdot 2^{k+s-2} + 2^{k+s-1} = 3 \cdot 2^{k+s-1}.
\]

The case \( j = 1, k > s + 1 \)

We proceed as in the case \( j = 0 \) using (12.3) with \( s \rightarrow s - 1 \)

\[
(12.6) \quad \Gamma_{s+1}^{s} \times (k+1) - \Gamma_{s+1}^{s} \times k = 4 \cdot \left[ \Gamma_{s}^{s-1} \times (k+1) - \Gamma_{s}^{s-1} \times k \right] - 2^{k+s-1},
\]

\[
(12.7) \quad \Gamma_{s+1}^{s} \times (k+1) - \Gamma_{s+1}^{s} \times k = 4 \cdot \left[ \Gamma_{s}^{s-1} \times (k+1) - \Gamma_{s}^{s-1} \times k \right] - 2^{k+s-2}
\]

\[
= 4 \cdot 3 \cdot 2^{k+s-2} - 2^{k+s-2} = 11 \cdot 2^{k+s-2}.
\]

From (12.6), (12.7) we get
The case $j = 2$, $k > s + 2$

Proceeding as before, using (12.8) with $s \to s - 1$

\begin{align*}
(12.9) \quad \left[ \Gamma \left( \frac{s}{s+1} \right)^{(k+1)} - \Gamma \left( \frac{s}{s+1} \right)^{s} \right] = 4 \cdot 11 \cdot 2^{k+s-2} - 2^{k+s-1} = 21 \cdot 2^{k+s-1}.
\end{align*}

\begin{align*}
(12.10) \quad & \quad \left[ \Gamma \left( \frac{s-1}{s-1+1} \right)^{(k+1)} - \Gamma \left( \frac{s-1}{s-1+1} \right)^{s} \right] = 4 \cdot 21 \cdot 2^{k+s-2} = 21 \cdot 2^{k+s}.
\end{align*}

From (12.9), (12.10) we get

\begin{align*}
(12.11) \quad & \quad \left[ \Gamma \left( \frac{s}{s+2} \right)^{(k+1)} - \Gamma \left( \frac{s}{s+2} \right)^{s} \right] = 4 \cdot 21 \cdot 2^{k+s} = 21 \cdot 2^{k+s+2}.
\end{align*}

The case $2 \leq j \leq s - 1$, $k > s + j$

By (12.2) we obtain

\begin{align*}
& \quad \left[ \Gamma \left( \frac{s}{s+j} \right)^{(k+1)} - \Gamma \left( \frac{s}{s+j} \right)^{s} \right] = 4 \cdot \left[ \Gamma \left( \frac{s-1}{s+j-1} \right)^{(k+1)} - \Gamma \left( \frac{s-1}{s+j-1} \right)^{s-1} \right] \quad \text{if } 2 \leq j \leq s - 1,
\end{align*}

\begin{align*}
& \quad \left[ \Gamma \left( \frac{s-1}{s-1+j} \right)^{(k+1)} - \Gamma \left( \frac{s-1}{s-1+j} \right)^{s-1} \right] = 4 \cdot \left[ \Gamma \left( \frac{s-1}{s-1+j-1} \right)^{(k+1)} - \Gamma \left( \frac{s-1}{s-1+j-1} \right)^{s-1} \right] \quad \text{if } 2 \leq j \leq s - 1 + 1 - 1 = s - 1.
\end{align*}

From the above equations we get

\begin{align*}
(12.12) \quad \left[ \Gamma \left( \frac{s}{s+j} \right)^{(k+1)} - \Gamma \left( \frac{s}{s+j} \right)^{s} \right] = 4^2 \cdot \left[ \Gamma \left( \frac{s-1}{s-1+j-1} \right)^{(k+1)} - \Gamma \left( \frac{s-1}{s-1+j-1} \right)^{s-1} \right] \quad \text{if } 2 \leq j \leq s - 1.
\end{align*}

Using successively (12.12) we get
\[
\Gamma_{s+j}^\left[\begin{array}{c}s \\ s \end{array}\right] \times (k+1) - \Gamma_{s+j}^\left[\begin{array}{c}s \\ s \end{array}\right] \times k
= 4^2 \cdot \left[ \Gamma_{s-1+(j-1)}^\left[\begin{array}{c}s-1 \\ s-1 \end{array}\right] \times (k+1) - \Gamma_{s-1+(j-1)}^\left[\begin{array}{c}s-1 \\ s-1 \end{array}\right] \times k \right]
\text{ if } 2 \leq j \leq s - 1
\]
\[
= 4^4 \cdot \left[ \Gamma_{s-2+(j-2)}^\left[\begin{array}{c}s-2 \\ s-2 \end{array}\right] \times (k+1) - \Gamma_{s-2+(j-2)}^\left[\begin{array}{c}s-2 \\ s-2 \end{array}\right] \times k \right]
\text{ if } 2 \leq j - 1 \leq s - 1 - 1
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We deduce from the above equations

\[(12.14) \quad \sum_{l=0}^{s-2} 4^{2l} \cdot \left[ \Gamma_{2(s-l)}^{s-l} \times (k+1) - \Gamma_{2(s-l)}^{s-l} \times k \right] = \sum_{l=0}^{s-2} \left( 4^{2(l+1)} \cdot \left[ \Gamma_{2(s-l-1)}^{s-l-1} \times (k+1) - \Gamma_{2(s-l-1)}^{s-l-1} \times k \right] - 4^2 \cdot 2^{k+2(s-l)-2} \cdot 2^{4l} \right) = \sum_{l=1}^{s-1} \left( 4^{2l} \cdot \left[ \Gamma_{2(s-l)}^{s-l} \times (k+1) - \Gamma_{2(s-l)}^{s-l} \times k \right] - \sum_{l=1}^{s-1} 9 \cdot 2^{2k+2s+2l-4} \right) \]

We get from (12.14) after some simplifications

\[
\Gamma_{2s}^{s} \times (k+1) - \Gamma_{2s}^{s} \times k = 4^{2(s-1)} \cdot \left[ \Gamma_{2}^{1} \times (k+1) - \Gamma_{2}^{1} \times k \right] - 9 \cdot 2^{2k+2s-4} \cdot \sum_{l=1}^{s-1} 2^{2l} = 4^{2(s-1)} \left( 2^{2(k+1)} - 3 \cdot 2^{k+1} + 2 \right) - 3 \cdot 2^{2s-2} - 3 \cdot 2^{k+4s-4}.
\]

\textbf{Lemma 12.2.} We have

\[(12.15) \quad \Gamma_{s+j}^{s+1} \times (k+1) - \Gamma_{s+j}^{s+1} \times k = \begin{cases} 
2^{k+s-1} & \text{if } j = 0, k > s, \\
11 \cdot 2^{k+s-1} & \text{if } j = 1, k > s + 1, \\
21 \cdot 2^{k+s+3j-5} & \text{if } 2 \leq j \leq s, k > s + j, \\
3 \cdot 2^{2k+2s-1} - 3 \cdot 2^{k+4s-2} & \text{if } j = s + 1, k > 2s + 1.
\end{cases}
\]

\textbf{Proof.} We proceed as in the proof of Lemma 12.1.
The case \( j = 0, \ k > s \)

Applying formula (12.2) we obtain using (11.21)

(12.16) \[ \Gamma \left[ \frac{s+1}{s+1} \right]^{(k+1)} - \Gamma \left[ \frac{s+1}{s+1} \right]^k = 4 \cdot \left[ \Gamma \left[ \frac{s}{s-1} \right]^{(k+1)} - \Gamma \left[ \frac{s}{s-1} \right]^k \right] + 2^{k+s-1} = 2^{k+s-1}. \]

The case \( j = 1, \ k > s \)

From (12.2) we obtain using (12.1)

(12.17) \[ \Gamma \left[ \frac{s+1}{s+1} \right]^{(k+1)} - \Gamma \left[ \frac{s+1}{s+1} \right]^k = 4 \cdot \left[ \Gamma \left[ \frac{s}{s-1} \right]^{(k+1)} - \Gamma \left[ \frac{s}{s-1} \right]^k \right] - 2^{k+s-1} = 4 \cdot (3 \cdot 2^{k+s-1}) - 11 \cdot 2^{k+s-1}. \]

The case \( 2 \leq j \leq s, \ k > s + j \)

From (12.2) we obtain using (12.1) with \( j \rightarrow j - 1 \) (observing that \( 1 \leq j - 1 \leq s - 1 \))

(12.18) \[ \Gamma \left[ \frac{s+1}{s+j} \right]^{(k+1)} - \Gamma \left[ \frac{s+1}{s+j} \right]^k = 4 \cdot \left[ \Gamma \left[ \frac{s}{s+j-1} \right]^{(k+1)} - \Gamma \left[ \frac{s}{s+j-1} \right]^k \right] = 4 \cdot (21 \cdot 2^{k+s+3(j-1)-4}) = 21 \cdot 2^{k+s+3j-5}. \]

The case \( j = s+1 \)

(12.19) \[ \Gamma \left[ \frac{s+1}{2s+1} \right]^{(k+1)} - \Gamma \left[ \frac{s+1}{2s+1} \right]^k = 4 \cdot \left[ \Gamma \left[ \frac{s}{2s} \right]^{(k+1)} - \Gamma \left[ \frac{s}{2s} \right]^k \right] - 3 \cdot 2^{2k+2s-1} = 4 \cdot [3 \cdot 2^{2k+2s-2} - 3 \cdot 2^{k+4s-4}] - 3 \cdot 2^{2k+2s-1} = 3 \cdot 2^{2k+2s-1} - 3 \cdot 2^{k+4s-2}. \]

\[ \square \]

**Lemma 12.3.** We have in the case \( m \geq 2 \)

(12.20) \[ \Gamma \left[ \frac{s+m}{s+j} \right]^{(k+1)} - \Gamma \left[ \frac{s+m}{s+j} \right]^k = \begin{cases} 2^{k+s-1} & \text{if } j = 0, \ k > s, \\ 3 \cdot 2^{k+s+2j-3} & \text{if } 1 \leq j \leq m - 1, \ k > s + j, \\ 11 \cdot 2^{k+s+2m-3} & \text{if } j = m, \ k > s + m, \end{cases} \]

(12.21) \[ \Gamma \left[ \frac{s+m}{s+m+j} \right]^{(k+1)} - \Gamma \left[ \frac{s+m}{s+m+j} \right]^k = \begin{cases} 21 \cdot 2^{k+s+2m+3j-4} & \text{if } 1 \leq j \leq s - 1, \ k > s + m + j, \\ 3 \cdot 2^{2k+2s+m-2} - 3 \cdot 2^{k+4s+2m-4} & \text{if } j = s, \ k > 2s + m. \end{cases} \]

**Proof.** The case \( s+j \) with \( 0 \leq j \leq m, \ k > s + j \)

\( j = 0 \)
Applying formula (12.2) we obtain using (11.21), observing that \( m - 1 \geq 1 \geq 0 \)

\[
(12.22) \quad \Gamma_s^{s+m} \times (k+1) - \Gamma_s^{s+m} \times k = 4 \cdot \left[ \Gamma_s^{s+(m-1)} \times (k+1) - \Gamma_s^{s+(m-1)} \times k \right] + 2^{k+s-1} = 2^{k+s-1}.\]

\( j = 1 \)

Applying formula (12.2) we obtain using (11.21), observing that \( m - 2 \geq 0 \)

\[
(12.23) \quad \Gamma_s^{s+m} \times (k+1) - \Gamma_s^{s+m} \times k = 4 \cdot \left[ \Gamma_s^{s+(m-1)} \times (k+1) - \Gamma_s^{s+(m-1)} \times k \right] - 2^{k+s-1}
\]

\[
= 4 \cdot \left( 4 \cdot \left[ \Gamma_s^{s+(m-2)} \times (k+1) - \Gamma_s^{s+(m-2)} \times k \right] + 2^{k+s-1} \right) - 2^{k+s-1}
\]

\[
= 3 \cdot 2^{k+s-1}.
\]

\( 2 \leq j \leq m-1 \)

Assume \( 0 \leq l \leq j-2 \), then by (12.2) with \( j \to j-l \), \( m \to m-l \), we have

\[
(12.24) \quad \Gamma_{s+j-l}^{s+(m-l)} \times (k+1) - \Gamma_{s+j-l}^{s+(m-l)} \times k = 4 \cdot \left[ \Gamma_{s+j-l}^{s+(m-l-(l+1))} \times (k+1) - \Gamma_{s+j-l}^{s+(m-l-l+1)} \times k \right]
\]

for \( l = 0, 1, 2, \ldots, j-2 \).

From (12.24) we deduce

\[
(12.25) \quad \prod_{l=0}^{j-2} \left[ \Gamma_{s+j-l}^{s+(m-l)} \times (k+1) - \Gamma_{s+j-l}^{s+(m-l)} \times k \right] = \prod_{l=0}^{j-2} 4 \cdot \left[ \Gamma_{s+j-l}^{s+(m-l-(l+1))} \times (k+1) - \Gamma_{s+j-l}^{s+(m-l-l+1)} \times k \right]
\]

\[
= \prod_{l=1}^{j-1} 4 \cdot \left[ \Gamma_{s+j-l}^{s+(m-l)} \times (k+1) - \Gamma_{s+j-l}^{s+(m-l)} \times k \right].
\]

We then obtain by (12.25), (12.23), observing that \( m - (j-1) \geq 2 \)

\[
\Gamma_{s+j}^{s+m} \times (k+1) - \Gamma_{s+j}^{s+m} \times k = 4^{j-1} \cdot \left[ \Gamma_{s+1}^{s+(m-(j-1))} \times (k+1) - \Gamma_{s+1}^{s+(m-(j-1))} \times k \right]
\]

\[
= 2^{2j-2} \cdot 3 \cdot 2^{k+s-1} = 3 \cdot 2^{k+s+2j-3}.
\]

\( j = m \)
Assume $2 \leq l \leq m$ ($s + m - 1$) then by (12.2) with $j \to l$, $m \to l$, we get

\[(12.26)\]
\[
\Gamma_{s+l}^{s+1} \times (k+1) - \Gamma_{s+l}^{s+1} \times k = 4 \cdot \left[ \Gamma_{s+(l-1)}^{s+1} \times (k+1) - \Gamma_{s+(l-1)}^{s+1} \times k \right]
\]
for $l = 2, 3, \ldots, m$.

From (12.26) we deduce

\[(12.27)\]
\[
\prod_{l=2}^{m} \left[ \Gamma_{s+l}^{s+1} \times (k+1) - \Gamma_{s+l}^{s+1} \times k \right] = \prod_{l=2}^{m} 4 \cdot \left[ \Gamma_{s+(l-1)}^{s+1} \times (k+1) - \Gamma_{s+(l-1)}^{s+1} \times k \right]
\]
\[
= \prod_{l=1}^{m-1} 4 \cdot \left[ \Gamma_{s+l}^{s+1} \times (k+1) - \Gamma_{s+l}^{s+1} \times k \right].
\]

We then obtain by (12.27) and (12.15) with $j = 1$

\[
\Gamma_{s+m}^{s+m} \times (k+1) - \Gamma_{s+m}^{s+m} \times k = 4^{m-1} \cdot \left[ \Gamma_{s+1}^{s+1} \times (k+1) - \Gamma_{s+1}^{s+1} \times k \right]
\]
\[
= 2^{2m-2} \cdot 11 \cdot 2^{k+s-1} = 11 \cdot 2^{k+s+2m-3}.
\]

The case $s+m+j$ with $1 \leq j \leq s-1$

Assume $0 \leq l \leq m-1$, then by (12.2) with $j \to (m-l) + j$, $m \to m-l$,

(observing that $2 \leq m - l + j \leq s + (m-l) - 1$) we have

\[(12.28)\]
\[
\Gamma_{s+(m-l)+j}^{s+(m-l)} \times (k+1) - \Gamma_{s+(m-l)+j}^{s+(m-l)} \times k = 4 \cdot \left[ \Gamma_{s+(m-(l+1))+j}^{s+(m-(l+1))} \times (k+1) - \Gamma_{s+(m-(l+1))+j}^{s+(m-(l+1))} \times k \right]
\]
for $l = 0, 1, \ldots, m-1$.

From (12.28) we deduce

\[(12.29)\]
\[
\prod_{l=0}^{m-1} \left[ \Gamma_{s+(m-l)+j}^{s+(m-l)} \times (k+1) - \Gamma_{s+(m-l)+j}^{s+(m-l)} \times k \right] = \prod_{l=0}^{m-1} 4 \cdot \left[ \Gamma_{s+(m-(l+1))+j}^{s+(m-(l+1))} \times (k+1) - \Gamma_{s+(m-(l+1))+j}^{s+(m-(l+1))} \times k \right]
\]
\[
= \prod_{l=1}^{m} 4 \cdot \left[ \Gamma_{s+(m-l)+j}^{s+(m-l)} \times (k+1) - \Gamma_{s+(m-l)+j}^{s+(m-l)} \times k \right].
\]

We then obtain by (12.29) and (12.1)
Lemma 12.4. We have in the case \( m \in \{0, 1\} \)

\[
\Gamma \left[ s + m \right] \times (k+1) - \Gamma \left[ s + m + j \right] = 4^m \cdot \left[ \Gamma \left[ s \right] \times (k+1) - \Gamma \left[ s \right] \times k \right]
\]

\[= 2^{2m} \cdot 21 \cdot 2^{k+s+3j-4} = 21 \cdot 2^{k+s+2m+3j-4}.\]

The case \( s + m + j \) with \( j = s \)

Assume \( 0 \leq l \leq m - 1 \), then by (12.24) with \( j \to s + (m - l) \), \( m \to m - l \), we have

\[
(12.30)
\]

\[
\Gamma \left[ s + (m-l) \right] \times (k+1) - \Gamma \left[ s + (m-l) \right] \times k
\]

\[= 4 \cdot \left[ \Gamma \left[ s + (m-l) \right] \times (k+1) - \Gamma \left[ s + (m-l) \right] \times k \right] - 3 \cdot 2^{2k+2s+(m-l)-2} \quad \text{for } l = 0, 1, \ldots, m - 1.
\]

From (12.30) we deduce

\[
(12.31)
\]

\[
\sum_{l=0}^{m-1} 4^l \cdot \left[ \Gamma \left[ s + (m-l) \right] \times (k+1) - \Gamma \left[ s + (m-l) \right] \times k \right]
\]

\[= \sum_{l=0}^{m-1} 4^l \cdot \left( 4 \cdot \left[ \Gamma \left[ s + (m-l) \right] \times (k+1) - \Gamma \left[ s + (m-l) \right] \times k \right] - 3 \cdot 2^{2k+2s+(m-l)-2} \right)
\]

\[= \sum_{l=1}^{m} 4^l \cdot \left[ \Gamma \left[ s + (m-l) \right] \times (k+1) - \Gamma \left[ s + (m-l) \right] \times k \right] - \sum_{l=1}^{m} 3 \cdot 2^{2k+2s+m+l-3}.\]

We then obtain by (12.31) and (12.1) after some simplifications

\[
\Gamma \left[ s + m \right] \times (k+1) - \Gamma \left[ s + m \right] \times k
\]

\[= 2^{2m} \cdot (3 \cdot 2^{2k+2s-2} - 3 \cdot 2^{k+4s-4}) - (3 \cdot 2^{2k+2s+2m-2} - 3 \cdot 2^{2k+2s+m-2})
\]

\[= 3 \cdot 2^{2k+2s+m-2} - 3 \cdot 2^{k+4s+2m-4}.
\]

\[
\square
\]

Lemma 12.4. We have in the case \( m \in \{0, 1\} \)

\[
(12.32)
\]

\[
\Gamma \left[ s \right] \times (k+1) - \Gamma \left[ s \right] \times k
\]

\[= \begin{cases} 3 \cdot 2^{k+s-1} & \text{if } j = 0, \ k > s, \\ 21 \cdot 2^{k+s+3j-4} & \text{if } 1 \leq j \leq s - 1, \ k > s + j, \\ 3 \cdot 2^{2k+2s-2} - 3 \cdot 2^{k+4s-4} & \text{if } j = s, \ k > 2s, \end{cases}
\]
In the case \( m \geq 2 \), we get

\[
\Gamma_{s+m}^{(s+j)} - \Gamma_{s+m+j}^{(s+j)} = \begin{cases} 
2^{k+s-1} & \text{if } j = 0, \ k > s, \\
11 \cdot 2^{k+s-1} & \text{if } j = 1, \ k > s + 1, \\
21 \cdot 2^{k+s+3j-5} & \text{if } 2 \leq j \leq s, \ k > s + j, \\
3 \cdot 2^{k+2s-1} - 3 \cdot 2^{k+4s-2} & \text{if } j = s + 1, \ k > 2s + 1.
\end{cases}
\]

Proof. The assertions follow from Lemmas 12.1, 12.2 and 12.3.

\[\square\]

13. COMPUTATION OF \( \Gamma_{s+m}^{(s+j)} \) for \( j \in \{0, 1\} \), \( m \geq s + j \), \( m \geq 0 \)

From the equations (11.1) and (11.2) with \( k = s + 1 \) we deduce \( \Gamma_{s}^{(s+j)} \) and \( \Gamma_{s+1}^{(s+j)} \) since by (11.21) the terms \( \Gamma_{i}^{(s+j)} \) are known for \( i \leq s - 1 \). Then from the recurrent formula (12.34) with \( j = 0 \) we compute \( \Gamma_{s+m}^{(s+j)} \) for \( m \geq s + 2 \). The other results are obtained in a similar way.

Lemma 13.1. We have in the case \( m = 0 \)

\[
\Gamma_{s}^{s} = 2^{4s-2} - 3 \cdot 2^{3s-4} + 2^{2s-3},
\]

\[
\Gamma_{s}^{k} = 3 \cdot 2^{k+s-1} + 21 \cdot 2^{3s-4} - 27 \cdot 2^{2s-3} \quad \text{if } k > s,
\]

\[
\Gamma_{s+1}^{(s+1)} = 2^{4s} - 3 \cdot 2^{3s-1} + 2^{2s-1},
\]

\[
\Gamma_{s+1}^{k} = 21 \cdot [2^{k+s-1} + 2^{3s-1} - 5 \cdot 2^{2s-1}] \quad \text{if } k > s + 1.
\]

Proof. We get (13.1) from (11.26) with \( k = s, m = 0 \).

To prove (13.2) and (13.3) we proceed as follows:

Consider the matrix

\[
\begin{pmatrix}
2^{k+s-1} \\
11 \cdot 2^{k+s-1} \\
21 \cdot 2^{k+s+3j-5} \\
3 \cdot 2^{k+2s-1} - 3 \cdot 2^{k+4s-2}
\end{pmatrix}
\]
We have respectively by (11.1), (11.2), (11.27) and (11.28) with \( m = 0, k = s+1 \)

\[
\sum_{i=0}^{s+1} \Gamma^s_i \times (s+1) = 2^{4s},
\]

\[
\sum_{i=0}^{s+1} \Gamma^s_i \times (s+1) \cdot 2^{-i} = 2^{3s-1} + 2^{2s} - 2^{s-1},
\]

\[
\sum_{i=0}^{s-1} \Gamma^s_i \times (s+1) = 3 \cdot 2^{3s-4} - 2^{2s-3}
\]

and

\[
\sum_{i=0}^{s-1} \Gamma^s_i \times (s+1) \cdot 2^{-i} = 7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3}.
\]

From (13.5), (13.6), (13.7) and (13.8) we deduce after some calculations

\[
\Gamma^s_s \times (s+1) = 21 \cdot 2^{3s-4} - 3 \cdot 2^{2s-3},
\]

\[
\Gamma^s_{s+1} \times (s+1) = 2^{4s} - 3 \cdot 2^{3s-1} + 2^{2s-1}.
\]

By (12.32) with \( j = 0 \) we get

\[
\Gamma^s_s \times (k+1) - \Gamma^s_s \times k = 3 \cdot 2^{k+s-1} \text{ if } k > s.
\]

From (13.11), (13.8) we deduce

\[
\sum_{j=s+1}^{k} \left( \Gamma^s_s \times (j+1) - \Gamma^s_s \times j \right) = \sum_{j=s+1}^{k} 3 \cdot 2^{j+s-1}
\]
Consider the matrix
\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{s+1} & \alpha_{s+2} \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{s+2} & \alpha_{s+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{2s-1} & \alpha_{2s} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{2s} & \alpha_{2s+1} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{s+1} & \beta_{s+2} \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{s+2} & \beta_{s+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \ldots & \beta_{2s-1} & \beta_{2s} \\
\beta_s & \beta_{s+1} & \beta_{s+2} & \ldots & \beta_{2s} & \beta_{2s+1}
\end{pmatrix}
\]

We have respectively by (11.1), (11.2), (11.27) and (11.28) with \(m = 0, k = s+2\),

\[
\sum_{i=0}^{s+2} \Gamma_i^s \times (s+2) = 2^{4s+2},
\]

(13.13)

\[
\sum_{i=0}^{s+2} \Gamma_i^s \times (s+2) \cdot 2^{-i} = 2^{3s} + 2^{2s+2} - 2^s,
\]

(13.14)

\[
\sum_{i=0}^{s-1} \Gamma_i^s \times (s+1) = 3 \cdot 2^{3s-4} - 2^{2s-3}
\]

and

\[
\sum_{i=0}^{s-1} \Gamma_i^s \times (s+2) \cdot 2^{-i} = 7 \cdot 2^{3s-4} - 3 \cdot 2^{s-3}.
\]

(13.15)

From (13.13), (13.14), (13.15), (13.16) and (13.2) with \(k = s+2\) we obtain
By (13.19) we see that (13.22) holds for \( k = s+1 \), then (13.22) holds for \( 1 \leq s \leq k \).

In the case \( m = 1 \), we have

**Lemma 13.2.** In the case \( m = 1 \), we have

\[
\Gamma_{s+1}^{\times s} = 2^{4s+2} - 3 \cdot 2^{3s-4} - 2^{2s-3}.
\]
\[ \Gamma_{s+1}^{s+1} \times k = 2^{k+s-1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3} \quad \text{if } k > s, \quad (13.24) \]

\[ \Gamma_{s+1}^{s+1} \times (s+1) = 2^{4s+1} - 3 \cdot 2^{3s-1} + 2^{2s-1}, \quad (13.25) \]

\[ \Gamma_{s+1}^{s+1} \times k = 11 \cdot 2^{k+s-1} + 21 \cdot 2^{3s-1} - 53 \cdot 2^{2s-1} \quad \text{if } k > s + 1. \quad (13.26) \]

Proof. We proceed as in the proof of Lemma 13.1.

Proof of (13.23)
Follows from (11.26) with \( m = 1, k = s \).

Proof of (13.24), (13.25)
Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_s & \alpha_{s+1} \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{s+1} & \alpha_{s+2} \\
: & : & : & \vdots & : & : \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{2s-2} & \alpha_{2s-1} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{2s-1} & \alpha_{2s} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_s & \beta_{s+1} \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{s+1} & \beta_{s+2} \\
: & : & : & \vdots & : & : \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \ldots & \beta_{2s-2} & \beta_{2s-1} \\
\beta_s & \beta_{s+1} & \beta_{s+2} & \ldots & \beta_{2s-1} & \beta_{2s} \\
\beta_{s+1} & \beta_{s+2} & \beta_{s+3} & \ldots & \beta_{2s} & \beta_{2s+1}
\end{pmatrix}
\]

We have respectively by (11.1), (11.2), (11.27) and (11.28) with \( m = 1, k = s+1 \)

\[ \sum_{i=0}^{s+1} \Gamma_{i}^{s+1} \times (s+1) = 2^{4s+1}, \quad (13.27) \]

\[ \sum_{i=0}^{s+1} \Gamma_{i}^{s+1} \times (s+1) \cdot 2^{-i} = 2^{3s} + 2^{2s} - 2^{s-1}, \quad (13.28) \]

\[ \sum_{i=0}^{s-1} \Gamma_{i}^{s+1} \times (s+1) = 3 \cdot 2^{3s-4} - 2^{2s-3} \]

and

\[ \sum_{i=0}^{s-1} \Gamma_{i}^{s+1} \times (s+1) \cdot 2^{-i} = 7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3}. \quad (13.30) \]

From (13.27), (13.28), (13.29) and (13.30) we deduce after some calculations
(13.31) \[
\Gamma^{\left[\frac{s}{s+1}\right]}_{s+1} \times (s+1) = 21 \cdot 2^{3s-4} - 3 \cdot 2^{2s-3},
\]

(13.32) \[
\Gamma^{\left[\frac{s}{s+1}\right]}_{s+1} \times (s+1) = 2^{4s+1} - 3 \cdot 2^{3s-1} + 2^{2s-1}.
\]

By (13.33) with \( j = 0 \) we get

(13.33) \[
\Gamma^{\left[\frac{s}{s+1}\right]}_{s+1} \times (k+1) - \Gamma^{\left[\frac{s}{s+1}\right]}_{s+1} \times k = 2^{k+s-1} \quad \text{if} \quad k > s.
\]

From (13.33), (13.31) we deduce

\[
\sum_{j=s+1}^{k} \left( \Gamma^{\left[\frac{s}{s+1}\right]}_{s+1} \times (j+1) - \Gamma^{\left[\frac{s}{s+1}\right]}_{s+1} \times j \right) = \sum_{j=s+1}^{k} 2^{j+s-1}
\]

\[
\Leftrightarrow \sum_{j=s+2}^{k+1} \Gamma^{\left[\frac{s}{s+1}\right]}_{s+1} \times j - \sum_{j=s+1}^{k} \Gamma^{\left[\frac{s}{s+1}\right]}_{s+1} \times j = 2^{k+s} - 2^{2s}
\]

\[
\Leftrightarrow \Gamma^{\left[\frac{s}{s+1}\right]}_{s+1} \times (k+1) - \Gamma^{\left[\frac{s}{s+1}\right]}_{s+1} \times (s+1) = 2^{k+s} - 2^{2s}
\]

\[
\Leftrightarrow \Gamma^{\left[\frac{s}{s+1}\right]}_{s+1} \times (k+1) = 2^{k+s} - 2^{2s} + 2 \cdot 2^{3s-4} - 3 \cdot 2^{2s-3}
\]

(13.34) \[
\Leftrightarrow \Gamma^{\left[\frac{s}{s+1}\right]}_{s+1} \times (k+1) = 2^{k+s} + 2 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3} \quad \text{if} \quad k > s.
\]

By (13.31) we see that (13.34) holds for \( k = s \), then (13.34) holds for \( k \geq s \).

**Proof of (13.26)**

Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{s+1} & \alpha_{s+2} \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{s+2} & \alpha_{s+3} \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{2s-1} & \alpha_{2s} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{2s} & \alpha_{2s+1} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{s+1} & \beta_{s+2} \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{s+2} & \beta_{s+3} \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \ldots & \beta_{2s-1} & \beta_{2s} \\
\beta_s & \beta_{s+1} & \beta_{s+2} & \ldots & \beta_{2s} & \beta_{2s+1} \\
\beta_{s+1} & \beta_{s+2} & \beta_{s+3} & \ldots & \beta_{2s+1} & \beta_{2s+2}
\end{pmatrix}
\]

We have respectively by (11.1), (11.2), (11.27) and (11.28) with \( m = 1 \) and \( k = s+2 \)

(13.35) \[
\sum_{i=0}^{s+2} \Gamma^{\left[\frac{s}{s+1}\right]}_{s+1} \times (s+2) = 2^{4s+3},
\]
Hence by (13.39), (13.40) we deduce after some calculations

\[ \sum_{i=0}^{s-1} \Gamma_i \left[ {s+1 \atop s+1} \right] \times (s+2) \cdot 2^{-1} = 2^{3s+1} + 2^{2s+2} - 2^s, \]

(13.36)

\[ \sum_{i=0}^{s} \Gamma_i \left[ {s+1 \atop s+1} \right] \times (s+2) = 3 \cdot 2^{3s-4} - 2^{2s-3} \]

and

(13.37)

\[ \sum_{i=0}^{s-1} \Gamma_i \left[ {s+1 \atop s+1} \right] \times (s+2) \cdot 2^{-1} = 7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3}. \]

From (13.35), (13.36), (13.37), (13.38) and (13.24) with \( k = s+2 \) we obtain

\[ \Gamma \left[ {s \atop s+1} \right] \times (s+2) + \Gamma \left[ {s \atop s+2} \right] \times (s+2) \]

(13.39)

\[ = 2^{4s+3} - (3 \cdot 2^{3s-4} - 2^{2s-3}) - (2^{s+2+s-1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3}), \]

\[ \Gamma \left[ {s \atop s+1} \right] \times (s+2) \cdot 2^{-(s+1)} + \Gamma \left[ {s \atop s+2} \right] \times (s+2) \cdot 2^{-(s+2)} \]

(13.40)

\[ = 2^{3s+1} + 2^{2s+2} - 2^s - (7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3}) - (2^{s+2+s-1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3}) \cdot 2^{-s}. \]

Hence by (13.39), (13.40) we deduce after some calculations

(13.41)

\[ \Gamma \left[ {s \atop s+1} \right] \times (s+2) = 21 \cdot 2^{3s-1} - 9 \cdot 2^{2s-1}, \]

(13.42)

\[ \Gamma \left[ {s \atop s+2} \right] \times (s+2) = 2^{4s+3} - 3 \cdot 2^{3s+2} + 2^{2s+2}. \]

By (12.33) with \( j = 1 \) we get

(13.43)

\[ \Gamma \left[ {s \atop s+1} \right] \times (k+1) - \Gamma \left[ {s \atop s+1} \right] \times k = 11 \cdot 2^{k+s-1} \text{ if } k > s+1. \]

From (13.43), (13.41) we deduce

\[ \sum_{j=s+2}^{k} \left( \Gamma \left[ {s \atop s+1} \right] \times (j+1) - \Gamma \left[ {s \atop s+1} \right] \times j \right) = \sum_{j=s+2}^{k} 11 \cdot 2^{j+s-1} \]

\[ \Leftrightarrow \sum_{j=s+3}^{k+1} \Gamma \left[ {s \atop s+1} \right] \times j - \sum_{j=s+2}^{k} \Gamma \left[ {s \atop s+1} \right] \times j = 11 \cdot 2^{k+s} - 11 \cdot 2^{2s+1} \]

\[ \Leftrightarrow \Gamma \left[ {s \atop s+1} \right] \times (k+1) - \Gamma \left[ {s \atop s+1} \right] \times (s+2) = 11 \cdot 2^{k+s} - 11 \cdot 2^{2s+1} \]
\[
\begin{align*}
\Leftrightarrow & \quad \Gamma \left[ \alpha^s_{s+1} \right]^{(k+1)} = 11 \cdot 2^{k+s} - 11 \cdot 2^{2s+1} + 21 \cdot 2^{3s-1} - 9 \cdot 2^{2s-1} \\
\Leftrightarrow & \quad \Gamma \left[ \alpha^s_{s+1} \right]^{(k+1)} = 11 \cdot 2^{k+s} + 21 \cdot 2^{3s-1} - 53 \cdot 2^{2s-1} \quad \text{if } k > s + 1.
\end{align*}
\]

By (13.41) we see that (13.44) holds for \( k = s+1 \), then (13.44) holds for \( k \geq s + 1 \).

**Lemma 13.3.** *In the case \( m \geq 2 \), we have*

\[
\begin{align*}
\Gamma^s_{s+m} \times s & = 2^{4s+m-2} - 3 \cdot 2^{3s-4} + 2^{2s-3}, \\
\Gamma^s_{s+m} \times k & = 2^{k+s-1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3} \quad \text{if } k > s, \\
\Gamma^s_{s+1} \times (s+1) & = 2^{4s+m} - 3 \cdot 2^{3s-1} + 2^{2s-1}, \\
\Gamma^s_{s+1} \times k & = 3 \cdot 2^{k+s-1} + 21 \cdot [2^{3s-1} - 2^{2s-1}] \quad \text{if } k > s + 1, \\
\Gamma^s_{s+m} \times (s+2) & = 2^{4s+m+2} - 3 \cdot 2^{3s+2} + 2^{2s+2}.
\end{align*}
\]

**Proof.** We proceed as in the proof of Lemma 13.2

**Proof of (13.45)**

Follows from (11.26) with \( m \geq 2 \), \( k = s \).

**Proof of (13.46), (13.47)**

Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_s & \alpha_{s+1} \\
\alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{s+1} & \alpha_{s+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \cdots & \alpha_{2s-2} & \alpha_{2s-1} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \cdots & \alpha_{2s-1} & \alpha_{2s} \\
\beta_1 & \beta_2 & \beta_3 & \cdots & \beta_s & \beta_{s+1} \\
\beta_2 & \beta_3 & \beta_4 & \cdots & \beta_{s+1} & \beta_{s+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \cdots & \beta_{2s-2} & \beta_{2s-1} \\
\beta_s & \beta_{s+1} & \beta_{s+2} & \cdots & \beta_{2s-1} & \beta_{2s} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \cdots & \beta_{2s+m-1} & \beta_{2s+m}
\end{pmatrix}
\]

We have respectively by (11.1), (11.2), (11.27), and (11.28) with \( m \geq 2 \), \( k = s+1 \)
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(13.50) \[ \sum_{i=0}^{s+1} \Gamma_i \left[ \sigma_{s+m} \right]^{i+1} = 2^{4s+m}, \]

(13.51) \[ \sum_{i=0}^{s+1} \Gamma_i \left[ \sigma_{s+m} \right]^{i+1} \cdot 2^{-i} = 2^{3s+m-1} + 2^{2s} - 2^{s-1}, \]

(13.52) \[ \sum_{i=0}^{s-1} \Gamma_i \left[ \sigma_{s+m} \right]^{i+1} = 3 \cdot 2^{3s-4} - 2^{2s-3} \]

and

(13.53) \[ \sum_{i=0}^{s-1} \Gamma_i \left[ \sigma_{s+m} \right]^{i+1} \cdot 2^{-i} = 7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3}. \]

From (13.50), (13.51), (13.52) and (13.53) we deduce after some calculations

(13.54) \[ \Gamma_s \left[ \sigma_{s+m} \right]^{(s+1)} = 21 \cdot 2^{3s-4} - 3 \cdot 2^{2s-3}, \]

(13.55) \[ \Gamma_{s+1} \left[ \sigma_{s+m} \right]^{(s+1)} = 2^{4s+m} - 3 \cdot 2^{3s-1} + 2^{2s-1}. \]

By (12.34) with $j = 0$, $m \geq 2$ we get

(13.56) \[ \Gamma_s \left[ \sigma_{s+m} \right]^{(k+1)} - \Gamma_s \left[ \sigma_{s+m} \right]^{k} = 2^{k+s-1} \text{ if } k > s. \]

From (13.56), (13.57) we deduce

(13.57) \[ \sum_{j=s+1}^{k+1} \left( \Gamma_s \left[ \sigma_{s+m} \right]^{(j+1)} - \Gamma_s \left[ \sigma_{s+m} \right]^{j} \right) = \sum_{j=s+1}^{k} 2^{j+s-1} \]

\[ \Leftrightarrow \sum_{j=s+2}^{k+1} \Gamma_s \left[ \sigma_{s+m} \right]^{j} - \sum_{j=s+1}^{k} \Gamma_s \left[ \sigma_{s+m} \right]^{j} = 2^{k+s} - 2^{2s} \]

\[ \Leftrightarrow \Gamma_s \left[ \sigma_{s+m} \right]^{(k+1)} - \Gamma_s \left[ \sigma_{s+m} \right]^{(s+1)} = 2^{k+s} - 2^{2s} \]

\[ \Leftrightarrow \Gamma_s \left[ \sigma_{s+m} \right]^{(k+1)} = 2^{k+s} - 2^{2s} + 21 \cdot 2^{3s-4} - 3 \cdot 2^{2s-3} \]

\[ \Leftrightarrow \Gamma_s \left[ \sigma_{s+m} \right]^{(k+1)} = 2^{k+s} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3} \text{ if } k > s. \]

By (13.54) we see that (13.57) holds for $k = s$, then (13.57) holds for $k \geq s$.

Proof of (13.48), (13.49)

Consider the matrix
Hence by (13.62), (13.63) we deduce after some calculations

\[
\Gamma \sum_{i=0}^{s+2} \left[ s+m \right]_{x(s+2)} = 2^{4s+m+2},
\]

\[
\sum_{i=0}^{s+2} \Gamma \left[ s+m \right]_{x(s+2)} \cdot 2^{-i} = 2^{3s+m} + 2^{2s+2} - 2^s,
\]

\[
\sum_{i=0}^{s-1} \Gamma \left[ s+m \right]_{x(s+2)} = 3 \cdot 2^{3s-4} - 2^{2s-3}
\]

and

\[
\sum_{i=0}^{s-2} \Gamma \left[ s+m \right]_{x(s+2)} \cdot 2^{-i} = 7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3}.
\]

We have respectively by (11.1), (11.2), (11.27) and (11.28) with \( m \geq 2 \), \( k = s+2 \)

\[
(13.58)
\]

\[
(13.59)
\]

\[
(13.60)
\]

\[
(13.61)
\]

From (13.58), (13.59), (13.60), (13.61) and (13.62) with \( k = s+2 \) we obtain

\[
\Gamma \left[ s+m \right]_{x(s+2)} \cdot 2^{-s+1} + \Gamma \left[ s+m \right]_{x(s+2)}
\]

\[
= 2^{4s+m+2} - (3 \cdot 2^{3s-4} - 2^{2s-3}) - (2^{s+2} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3}),
\]

\[
\Gamma \left[ s+m \right]_{x(s+2)} \cdot 2^{-s+1} + \Gamma \left[ s+m \right]_{x(s+2)} \cdot 2^{-s+2}
\]

\[
= 2^{3s+m} + 2^{2s+2} - 2^s - (7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3}) - (2^{s+2} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3}) \cdot 2^{-s}.
\]

Hence by (13.62), (13.63) we deduce after some calculations
EXponential Sums and Rank of Double Persymmetric Matrices Over \( \mathbb{F}_2 \)

\[(13.64) \quad \Gamma_{s+1}^{s \cdot m} \times (s+2) = 21 \cdot 2^{3s-1} - 9 \cdot 2^{2s-1}, \]

\[(13.65) \quad \Gamma_{s+2}^{s \cdot m} \times (s+2) = 2^{4s+m+2} - 3 \cdot 2^{3s+2} + 2^{2s+2}. \]

By (12.34) with \( j = 1 \), \( m \geq 2 \) we get

\[(13.66) \quad \Gamma_{s+1}^{s \cdot m} \times (k+1) - \Gamma_{s+1}^{s \cdot m} \times k = 3 \cdot 2^{k+s-1} \quad \text{if} \quad k > s+1. \]

From (13.66), (13.64) we deduce

\[
\sum_{j=s+2}^{k} \left( \Gamma_{s+1}^{s \cdot m} \times (j+1) - \Gamma_{s+1}^{s \cdot m} \times j \right) = \sum_{j=s+2}^{k} 3 \cdot 2^{j+s-1} \\
\Leftrightarrow \sum_{j=s+3}^{k+1} \Gamma_{s+1}^{s \cdot m} \times j - \sum_{j=s+2}^{k} \Gamma_{s+1}^{s \cdot m} \times j = 3 \cdot 2^{k+s} - 3 \cdot 2^{2s+1} \\
\Leftrightarrow \Gamma_{s+1}^{s \cdot m} \times (k+1) - \Gamma_{s+1}^{s \cdot m} \times (s+2) = 3 \cdot 2^{k+s} - 3 \cdot 2^{2s+1} \\
\Leftrightarrow \Gamma_{s+1}^{s \cdot m} \times (k+1) = 3 \cdot 2^{k+s} - 3 \cdot 2^{2s+1} + 21 \cdot 2^{3s-1} - 9 \cdot 2^{2s-1} \\
\Leftrightarrow \Gamma_{s+1}^{s \cdot m} \times (k+1) = 3 \cdot 2^{k+s} + 21 \cdot 2^{3s-1} - 21 \cdot 2^{2s-1} \quad \text{if} \quad k > s+1. \]

By (13.64) we see that (13.67) holds for \( k = s+1 \), then (13.67) holds for \( k \geq s + 1 \).

\[\square\]

14. **Computation of** \( \Gamma_{s+j}^{s \cdot m} \times k \) **for** \( j \in \{2, 3\}, \ k \geq s + j, \ j \leq m \)

In this section we show the following reduction formulas to be needed in the induction proof in the next section

\[
\Gamma_{s+2}^{s \cdot m} \times k = 8 \cdot \Gamma_{s+1}^{s \cdot (m-1)} \times (k-1) \quad \text{if} \quad k \geq s + 2, \ m \geq 2, \\
\Gamma_{s+3}^{s \cdot m} \times k = 8 \cdot \Gamma_{s+1}^{s \cdot (m-2)} \times (k-2) \quad \text{if} \quad k \geq s + 3, \ m \geq 3. \\
\]

We recall that the right hand sides in the above equations have already been computed in section 13.

In fact we have
Lemma 14.1. We have

\[ \Gamma_{s+1}^{(s+m) \times (s+1)} \times (k-j-1)) = \begin{cases} 
2^{4s+1} - 3 \cdot 2^{3s-1} + 2^{2s-1} & \text{if } j = 2, k = s + 2, m = 2, \\
11 \cdot 2^{k+s-2} + 21 \cdot 2^{3s-1} - 53 \cdot 2^{2s-1} & \text{if } j = 2, k > s + 2, m = 2, \\
2^{4s+m-1} - 3 \cdot 2^{3s-1} + 2^{2s-1} & \text{if } j = 2, k = s + 2, m > 2, \\
3 \cdot 2^{k+s-2} + 21 \cdot [2^{3s-1} - 2^{2s-1}] & \text{if } j = 2, k > s + 2, m > 2, \\
2^{4s+1} - 3 \cdot 2^{3s-1} + 2^{2s-1} & \text{if } j = 3, k = s + 3, m = 3, \\
11 \cdot 2^{k+s-3} + 21 \cdot 2^{3s-1} - 53 \cdot 2^{2s-1} & \text{if } j = 3, k > s + 3, m = 3, \\
2^{4s+m-2} - 3 \cdot 2^{3s-1} + 2^{2s-1} & \text{if } j = 3, k = s + 3, m > 3, \\
3 \cdot 2^{k+s-3} + 21 \cdot [2^{3s-1} - 2^{2s-1}] & \text{if } j = 3, k > s + 3, m > 3. 
\end{cases} \]

Proof. Proof of (14.1)

By (13.65), (13.47) and (13.25) we get

\[ \Gamma_{s+2}^{(s+m) \times (s+2)} = 8 \cdot \Gamma_{s+1}^{(s+m-1) \times (s+1)} = 2^{4s+m+2} - 3 \cdot 2^{3s+2} + 2^{2s+2} \quad \text{if } m \geq 2, \]

(14.2)

\[ \Gamma_{s+2}^{(s+m) \times k} = 8 \cdot \Gamma_{s+1}^{(s+m-1) \times (k-1)} = 3 \cdot 2^{k+s+1} + 21 \cdot 2^{3s+2} - 21 \cdot 2^{2s+2} \quad \text{if } k > s + 2, \quad m \geq 3, \]

(14.3)

\[ \Gamma_{s+2}^{(s+m) \times k} = 8 \cdot \Gamma_{s+1}^{(s+m-1) \times (k-1)} = 11 \cdot 2^{k+s+1} + 21 \cdot 2^{3s+2} - 53 \cdot 2^{2s+2} \quad \text{if } k > s + 2. \]

Proof of (14.2)

Consider the matrix
Hence by (14.8), (14.9) we deduce after some calculations

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{s+2} & \alpha_{s+3} \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{s+3} & \alpha_{s+4} \\
& \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{2s} & \alpha_{2s+1} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{2s+1} & \alpha_{2s+2} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{s+2} & \beta_{s+3} \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{s+3} & \beta_{s+4} \\
& \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \ldots & \beta_{2s} & \beta_{2s+1} \\
\beta_s & \beta_{s+1} & \beta_{s+2} & \ldots & \beta_{2s+1} & \beta_{2s+2} \\
\beta_{s+1} & \beta_{s+2} & \beta_{s+3} & \ldots & \beta_{2s+2} & \beta_{2s+3} \\
& \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{2s+m+1} & \beta_{2s+m+2}
\end{pmatrix}
\]

We have respectively by (11.1), (11.2), (11.27) and (11.28) with \( m \geq 2, k = s+3 \)

\[
(14.4) \quad \sum_{i=0}^{s+3} \Gamma_{i}^{\left[\begin{smallmatrix} s+m \\ s \end{smallmatrix}\right] \times (s+3)} = 2^{4s+m+4},
\]

\[
(14.5) \quad \sum_{i=0}^{s+3} \Gamma_{i}^{\left[\begin{smallmatrix} s+m \\ s \end{smallmatrix}\right] \times (s+3)} \cdot 2^{-1} = 2^{3s+m+1} + 2^{2s+4} - 2^{s+1},
\]

\[
(14.6) \quad \sum_{i=0}^{s-1} \Gamma_{i}^{\left[\begin{smallmatrix} s+m \\ s \end{smallmatrix}\right] \times (s+3)} = 3 \cdot 2^{3s-4} - 2^{2s-3}
\]

and

\[
(14.7) \quad \sum_{i=0}^{s-1} \Gamma_{i}^{\left[\begin{smallmatrix} s+m \\ s \end{smallmatrix}\right] \times (s+3)} \cdot 2^{-1} = 7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3}.
\]

From (14.5), (14.6), (14.7), (13.46) and (13.48) with \( k = s+3 \) we obtain

\[
(14.8) \quad \Gamma_{s+2}^{\left[\begin{smallmatrix} s+m \\ s \end{smallmatrix}\right] \times (s+3)} + \Gamma_{s+3}^{\left[\begin{smallmatrix} s+m \\ s \end{smallmatrix}\right] \times (s+3)} = 2^{4s+m+4} - (3 \cdot 2^{3s-4} - 2^{2s-3}) - (2^{s+3} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3})
\]

\[= (3 \cdot 2^{s+3} + 21 \cdot 2^{3s-4} - 21 \cdot 2^{2s-1}),\]

\[
(14.9) \quad \Gamma_{s+2}^{\left[\begin{smallmatrix} s+m \\ s \end{smallmatrix}\right] \times (s+3)} \cdot 2^{-s+2} + \Gamma_{s+3}^{\left[\begin{smallmatrix} s+m \\ s \end{smallmatrix}\right] \times (s+3)} \cdot 2^{-s+3} = 2^{3s+m+1} + 2^{2s+4} - 2^{s+1} - (7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3})
\]

\[= (2^{s+3} + 21 \cdot 2^{3s-4} - 21 \cdot 2^{2s-3}) \cdot 2^{-s+1} - (3 \cdot 2^{s+3} + 21 \cdot 2^{3s-1} - 21 \cdot 2^{2s-1}) \cdot 2^{-s+2}.
\]

Hence by (14.8), (14.9) we deduce after some calculations
(14.10) \[ \Gamma_{s+2}^{\left[ \frac{s}{s+m} \right] \times (s+3)} = 21 \cdot 2^{3s+2} - 9 \cdot 2^{2s+2}, \]

(14.11) \[ \Gamma_{s+3}^{\left[ \frac{s}{s+m} \right] \times (s+3)} = 24s+m+4 - 3 \cdot 2^{3s+5} + 2^{2s+5}. \]

By \[12.34\] with \(j = 2\), \(m \geq 3\) we get

(14.12) \[ \Gamma_{s+2}^{\left[ \frac{s}{s+m} \right] \times (k+1)} - \Gamma_{s+2}^{\left[ \frac{s}{s+m} \right] \times k} = 3 \cdot 2^{k+s+1} \text{ if } k > s+2. \]

From (14.12), (14.10) we deduce

\[ \sum_{j=s+3}^{k} \left( \Gamma_{s+2}^{\left[ \frac{s}{s+m} \right] \times (j+1)} - \Gamma_{s+2}^{\left[ \frac{s}{s+m} \right] \times j} \right) = \sum_{j=s+3}^{k} 3 \cdot 2^{j+s+1} \]

\[ \Leftrightarrow \sum_{j=s+4}^{k+1} \Gamma_{s+2}^{\left[ \frac{s}{s+m} \right] \times j} - \sum_{j=s+3}^{k} \Gamma_{s+2}^{\left[ \frac{s}{s+m} \right] \times j} = 3 \cdot 2^{k+s+2} - 3 \cdot 2^{2s+4} \]

\[ \Leftrightarrow \Gamma_{s+2}^{\left[ \frac{s}{s+m} \right] \times (k+1)} - \Gamma_{s+2}^{\left[ \frac{s}{s+m} \right] \times (s+3)} = 3 \cdot 2^{k+s+2} - 3 \cdot 2^{2s+4} \]

\[ \Leftrightarrow \Gamma_{s+2}^{\left[ \frac{s}{s+m} \right] \times (k+1)} = 3 \cdot 2^{k+s+2} + 21 \cdot 2^{3s+2} - 21 \cdot 2^{2s+2} \text{ if } k > s+3. \]

By (14.10) we see that (14.13) holds for \(k = s+3\), then (14.13) holds for \(k \geq s+3\).

From (13.48) with \(m \to m-1\), \(k \to k-1\), \(m \geq 3\), \(k-1 > s+1\) and (14.13) we get

(14.14) \[ 8 \cdot \Gamma_{s+1}^{\left[ \frac{s}{s+m} \right] \times (k-1)} = 8 \cdot [3 \cdot 2^{k+s-2} + 21 \cdot (2^{3s-1} - 2^{2s-1})]. \]

(14.15) \[ \Gamma_{s+2}^{\left[ \frac{s}{s+m} \right] \times k} = 3 \cdot 2^{k+s+1} + 21 \cdot 2^{3s+2} - 21 \cdot 2^{2s+2}. \]

By (14.14), (14.15) we obtain for \(m \geq 3\), \(k \geq s+3\)

\[ \Gamma_{s+2}^{\left[ \frac{s}{s+m} \right] \times k} = 8 \cdot \Gamma_{s+1}^{\left[ \frac{s}{s+(m-1)} \right] \times (k-1)}. \]

Proof of (14.13)

By \[12.34\] with \(j = 2\), \(m = 2\) we get

(14.16) \[ \Gamma_{s+2}^{\left[ \frac{s}{s+2} \right] \times (k+1)} - \Gamma_{s+2}^{\left[ \frac{s}{s+2} \right] \times k} = 11 \cdot 2^{k+s+1} \text{ if } k > s+2. \]

From (14.16), (14.10) we deduce
Proof. We have

\[
\sum_{j=s+3}^{k} \left( \Gamma_{s+2}^{s+2} \times (j+1) - \Gamma_{s+2}^{s+2} \times j \right) = \sum_{j=s+3}^{k} 11 \cdot 2^{j+s+1}
\]

\[
\Leftrightarrow \sum_{j=s+4}^{k+1} \Gamma_{s+2}^{s+2} \times j - \sum_{j=s+3}^{k} \Gamma_{s+2}^{s+2} \times j = 11 \cdot 2^{k+s+2} - 11 \cdot 2^{2s+4}
\]

\[
\Leftrightarrow \Gamma_{s+2}^{s+2} \times (k+1) - \Gamma_{s+2}^{s+2} \times (s+3) = 11 \cdot 2^{k+s+2} - 11 \cdot 2^{2s+4}
\]

\[
\Rightarrow \Gamma_{s+2}^{s+2} = 11 \cdot 2^{k+s+2} - 11 \cdot 2^{2s+4} + 21 \cdot 2^{3s+2} - 9 \cdot 2^{2s+2}
\]

(14.17)

\[
\Rightarrow \Gamma_{s+2}^{s+2} = 11 \cdot 2^{k+s+2} + 21 \cdot 2^{3s+2} - 53 \cdot 2^{2s+2} \text{ if } k > s + 3.
\]

By (14.10) we see that (14.17) holds for \( k = s+3 \), then (14.17) holds for \( k \geq s + 3 \).

From (13.26) ( with \( k \to k - 1, \quad k - 1 > s + 1 \) ) and (14.17) we have

\[
\Gamma_{s+2}^{s+1} \times (k-1) = 8 \cdot \Gamma_{s+1}^{s+1} \times (s+1) = 8 \cdot [11 \cdot 2^{k+s-2} + 21 \cdot 2^{3s-1} - 53 \cdot 2^{2s-1}] \text{ if } k - 1 > s + 1.
\]

Lemma 14.2. We have

(14.18)

\[
\Gamma_{s+3}^{s+m} \times (s+3) = 8^2 \cdot \Gamma_{s+1}^{s+(m-2)} \times (s+1) = 8^2 [2^{4s+m-2} - 3 \cdot 2^{3s} + 2^{2s-1}] \quad \text{if } m \geq 3,
\]

(14.19)

\[
\Gamma_{s+3}^{s+m} \times k = 8^2 \cdot \Gamma_{s+1}^{s+(m-2)} \times (k-2) = 8^2 [3 \cdot 2^{k+s-3} + 21 \cdot 2^{3s-1} - 21 \cdot 2^{2s-1}] \quad \text{if } k > s + 3, \quad m \geq 4,
\]

(14.20)

\[
\Gamma_{s+3}^{s+1} \times (k-2) = 8^2 \cdot \Gamma_{s+1}^{s+1} \times (k-2) = 8^2 [11 \cdot 2^{k+s-3} + 21 \cdot 2^{3s-1} - 53 \cdot 2^{2s-1}] \quad \text{if } k > s + 3.
\]

Proof. Proof of (14.18)

By (14.11), (13.47) and (13.25) we get

\[
\Gamma_{s+3}^{s+m} \times (s+3) = 2^{4s+m+4} - 3 \cdot 2^{3s+5} + 2^{2s+5} \quad \text{if } m \geq 3,
\]
Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{s+3} & \alpha_{s+4} \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{s+4} & \alpha_{s+5} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{2s+1} & \alpha_{2s+2} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{2s+2} & \alpha_{2s+3} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{s+3} & \beta_{s+4} \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{s+4} & \beta_{s+5} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \ldots & \beta_{2s+1} & \beta_{2s+2} \\
\beta_s & \beta_{s+1} & \beta_{s+2} & \ldots & \beta_{2s+2} & \beta_{2s+3} \\
\beta_{s+1} & \beta_{s+2} & \beta_{s+3} & \ldots & \beta_{2s+3} & \beta_{2s+4} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{2s+m+2} & \beta_{2s+m+3}
\end{pmatrix}
\]

We have for \( m \geq 3, k = s+4 \)

\[(14.21) \quad \sum_{i=0}^{s+4} \Gamma_i \left[ \frac{s}{s+m} \right] x^{(s+4)} = 2^{4s+m+6}, \quad \text{by (11.1),} \]

\[(14.22) \quad \sum_{i=0}^{s+4} \Gamma_i \left[ \frac{s}{s+m} \right] x^{(s+4)} \cdot 2^{-i} = 2^{3s+m+2} + 2^{2s+6} - 2^{s+2} \quad \text{by (11.2),} \]

\[(14.23) \quad \sum_{i=0}^{s-1} \Gamma_i \left[ \frac{s}{s+m} \right] x^{(s+4)} = 3 \cdot 2^{3s-4} - 2^{2s-3} \quad \text{by (11.27),} \]

\[(14.24) \quad \sum_{i=0}^{s-1} \Gamma_i \left[ \frac{s}{s+m} \right] x^{(s+4)} \cdot 2^{-i} = 7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3} \quad \text{by (11.28),} \]

\[(14.25) \quad \Gamma_s \left[ \frac{s}{s+m} \right] x^{(s+4)} = 21 \cdot 2^{3s-4} + 53 \cdot 2^{2s-3} \quad \text{by (13.46) with } k = s+4, \]

\[(14.26) \quad \Gamma_{s+1} \left[ \frac{s}{s+m} \right] x^{(s+4)} = 21 \cdot 2^{3s-1} + 27 \cdot 2^{2s-1} \quad \text{by (13.48) with } k = s+4, \]

\[(14.27) \quad \Gamma_{s+2} \left[ \frac{s}{s+m} \right] x^{(s+4)} = 21 \cdot 2^{3s+2} + 3 \cdot 2^{2s+2} \quad \text{by (14.2) with } k = s+4. \]
From (14.21), . . . (14.27) we get

\[(14.28)\]

\[
\Gamma_{s}^{s+m} \times (s+4) + \Gamma_{s+4}^{s+m} \times (s+4)
\]

\[
= \sum_{i=0}^{s+4} \Gamma_{i}^{s+m} \times (s+4) - \sum_{i=0}^{s-1} \Gamma_{i}^{s+m} \times (s+4) - \Gamma_{s}^{s+m} \times (s+4) - \Gamma_{s+1}^{s+m} \times (s+4) - \Gamma_{s+2}^{s+m} \times (s+4)
\]

\[
= 2^{4s+m+6} - (3 \cdot 2^{3s-4} - 2^{2s-3} + 21 \cdot 2^{3s-4} + 53 \cdot 2^{2s-3} + 21 \cdot 2^{3s-1} + 27 \cdot 2^{2s-1} + 21 \cdot 2^{3s+2} + 3 \cdot 2^{2s+2}),
\]

\[(14.29)\]

\[
\Gamma_{s+3}^{s+m} \times (s+4) \cdot 2^{-(s+3)} + \Gamma_{s+4}^{s+m} \times (s+4) \cdot 2^{-(s+4)}
\]

\[
= \sum_{i=0}^{s+4} \Gamma_{i}^{s+m} \times (s+4) \cdot 2^{-i} - \sum_{i=0}^{s-1} \Gamma_{i}^{s+m} \times (s+4) \cdot 2^{-i} - \Gamma_{s}^{s+m} \times (s+4) \cdot 2^{-s} - \Gamma_{s+1}^{s+m} \times (s+4) \cdot 2^{-(s+1)}
\]

\[
- \Gamma_{s+2}^{s+m} \times (s+4) \cdot 2^{-(s+2)}
\]

\[
= 2^{4s+m+2} + 2^{2s+6} - 2^{s+2} - (7 \cdot 2^{3s-4} - 3 \cdot 2^{2s-3} + (21 \cdot 2^{3s-4} + 53 \cdot 2^{2s-3}) \cdot 2^{-s}
\]

\[
+ (21 \cdot 2^{3s-1} + 27 \cdot 2^{2s-1}), 2^{-(s+1)} + (21 \cdot 2^{3s+2} + 3 \cdot 2^{2s+2}) \cdot 2^{-(s+2)}
\]

\[
= 2^{4s+m+2} + 9 \cdot 2^{2s+2} - 5 \cdot 2^{s+2}.
\]

Hence by (14.28), (14.29) we deduce after some calculations

\[(14.30)\]

\[
\Gamma_{s+3}^{s+m} \times (s+4) = 21 \cdot 2^{3s+5} - 9 \cdot 2^{2s+5} \quad \text{if } m \geq 3,
\]

\[(14.31)\]

\[
\Gamma_{s+4}^{s+m} \times (s+4) = 2^{4s+m+6} - 3 \cdot 2^{3s+8} + 2^{2s+8} \quad \text{if } m \geq 3.
\]

By (12.34) with \( j = 3 \), \( m \geq 4 \) we get

\[(14.32)\]

\[
\Gamma_{s+3}^{s+m} \times (k+1) - \Gamma_{s+3}^{s+m} \times k = 3 \cdot 2^{k+s+3} \quad \text{if } k > s + 3.
\]

From (14.32), (14.30) we deduce

\[
\sum_{j=s+4}^{k} \left( \Gamma_{s+3}^{s+m} \times (j+1) - \Gamma_{s+3}^{s+m} \times j \right) = \sum_{j=s+4}^{k} 3 \cdot 2^{j+s+3}
\]

\[
\Leftrightarrow \sum_{j=s+5}^{k+1} \Gamma_{s+3}^{s+m} \times j - \sum_{j=s+4}^{k} \Gamma_{s+3}^{s+m} \times j = 3 \cdot 2^{k+s+4} - 3 \cdot 2^{2s+7}
\]

\[
\Leftrightarrow \Gamma_{s+3}^{s+m} \times (k+1) - \Gamma_{s+3}^{s+m} \times (s+4) = 3 \cdot 2^{k+s+4} - 3 \cdot 2^{2s+7}
\]

\[
\Leftrightarrow \Gamma_{s+3}^{s+m} \times (k+1) = 3 \cdot 2^{k+s+4} - 3 \cdot 2^{2s+7} + 21 \cdot 2^{3s+5} - 9 \cdot 2^{2s+5}
\]
From (14.36), (14.37) we deduce (14.37).

We have

\[ (14.33) \quad \Leftrightarrow \left[ s + \frac{x}{s+3} \right] \times (k+1) = 3 \cdot 2^{k+s+4} + 21 \cdot 2^{3s+5} - 21 \cdot 2^{2s+5} \quad \text{if } k > s + 4. \]

By (14.30) we see that (14.33) holds for \( k = s+3 \), then (14.33) holds for \( k \geq s + 3 \).

From (13.48) with \( m \to m-2, \quad k \to k-2, \quad m \geq 4, \quad k-2 > s+1 \) and (14.33) we get

\[ (14.34) \quad 8^2 \cdot \Gamma_{s+1}^{s+(m-2)} \times (k-2) = 8^2 \cdot [3 \cdot 2^{k+s-3} + 21 \cdot (2^{3s-1} - 2^{2s-1})], \]

\[ (14.35) \quad \Gamma_{s+3}^{s+m} \times k = 3 \cdot 2^{k+s+3} + 21 \cdot 2^{3s+5} - 21 \cdot 2^{2s+5}. \]

By (14.34), (14.35) we obtain for \( m \geq 4, \quad k \geq s + 4 \)

\[ \Gamma_{s+3}^{s+m} \times k = 8^2 \cdot \Gamma_{s+1}^{s+(m-2)} \times (k-2). \]

Proof of (14.20)

We have

\[ (14.36) \quad \Gamma_{s+3}^{s+3} \times (s+4) = 21 \cdot 2^{3s+5} - 9 \cdot 2^{2s+5} \quad \text{by (14.30) with } m = 3, \]

\[ (14.37) \quad \Gamma_{s+3}^{s+3} \times (k+1) - \Gamma_{s+3}^{s+3} \times k = 11 \cdot 2^{k+s+3} \quad \text{by (12.34) with } j = m = 3, \ k > s + 3. \]

From (14.36), (14.37) we deduce

\[
\sum_{j=s+4}^{k} \left( \Gamma_{s+3}^{s+3} \times (j+1) - \Gamma_{s+3}^{s+3} \times j \right) = \sum_{j=s+4}^{k} 11 \cdot 2^{j+s+3} \\
\Leftrightarrow \sum_{j=s+4}^{k+1} \Gamma_{s+3}^{s+3} \times j - \sum_{j=s+4}^{k} \Gamma_{s+3}^{s+3} \times j = 11 \cdot 2^{k+s+4} - 11 \cdot 2^{2s+7} \\
\Leftrightarrow \Gamma_{s+3}^{s+3} \times (k+1) - \Gamma_{s+3}^{s+3} \times (s+4) = 11 \cdot 2^{k+s+4} - 11 \cdot 2^{2s+7} \\
\Leftrightarrow \Gamma_{s+3}^{s+3} \times (k+1) = 11 \cdot 2^{k+s+4} - 11 \cdot 2^{2s+7} + 21 \cdot 2^{3s+5} - 21 \cdot 2^{2s+5} \\
(14.38) \quad \Leftrightarrow \Gamma_{s+3}^{s+3} \times (k+1) = 11 \cdot 2^{k+s+4} + 21 \cdot 2^{3s+5} - 53 \cdot 2^{2s+5} \quad \text{if } k > s + 4. \]

By (14.30) we see that (14.38) holds for \( k = s+3 \), then (14.38) holds for \( k \geq s + 3 \).

From (13.26) (with \( k \to k-2, \quad k-2 > s+1 \)) and (14.38) we have
In fact we have

Recall that the right hand side in the above equation has been computed in section 13.

We have

Lemma 15.1. We have

\[ \Gamma_{s+1}^{s} \times k = 8^{l-1} \cdot \Gamma_{s+1}^{s(m-l-1)} \times (k-l) \] if \( 1 \leq l \leq m, \ k \geq s + j. \]

In this section we prove by induction on \( j \) the following reduction formula

\[ \Gamma_{s+m}^{s} \times k = 8^{j-1} \cdot \Gamma_{s+1}^{s(m-s-j)} \times (k-(j-1)) \] if \( 1 \leq j \leq m, \ k \geq s + j. \]

Recall that the right hand side in the above equation has been computed in section 13.

In fact we have

\[ \Gamma_{s+1}^{s(m-s-j)} \times (k-(j-1)) = \begin{cases} 3 \cdot 2^{k-j-s} + 21 \cdot \left[ 2^{3s-1} - 2^{2s-1} \right] & \text{if } 1 \leq j \leq m - 1, \ k > s + j, \\ 2^{4s+(m-j-1)} - 3 \cdot 2^{3s-1} + 2^{2s-1} & \text{if } 1 \leq j \leq m, \ k = s + j, \\ 11 \cdot 2^{k-m+s} + 21 \cdot 2^{3s-1} - 53 \cdot 2^{2s-1} & \text{if } j = m, \ k > s + m. \end{cases} \]

Lemma 15.1. We have

\[ \Gamma_{s+l}^{s} \times (s+l) = 8^{l-1} \cdot \Gamma_{s+1}^{s(m-l-1)} \times (s+1) \] if \( 1 \leq l \leq m, \]

\[ \Gamma_{s+1}^{s(m-l-1)} \times (s+1) = 2^{4s+(m-l-1)} - 3 \cdot 2^{3s-1} + 2^{2s-1} \] if \( 1 \leq l \leq m, \]

\[ \Gamma_{s+1}^{s} \times k = 8^{l-1} \cdot \Gamma_{s+1}^{s(m-l-1)} \times (k-l) \] if \( 1 \leq l \leq m, \ k > s + l, \]

\[ \Gamma_{s+1}^{s(m-l-1)} \times (k-l) = 3 \cdot 2^{k-l+s} + 21 \cdot \left[ 2^{3s-1} - 2^{2s-1} \right] \] if \( 1 \leq l \leq m - 1, \ k > s + l, \]

\[ \Gamma_{s+1}^{s(m-l-1)} \times (k-(m-1)) = 11 \cdot 2^{k-m+s} + 21 \cdot 2^{3s-1} - 53 \cdot 2^{2s-1} \] if \( l = m, \ k > s + m. \]
Proof. Let \( l \) be a rational integer such that \( 2 \leq l \leq m - 1 \); we shall prove lemma 15.1 by strong induction on \( l \).

Assume
(15.6) \[
(H_{l-1}) \Gamma_{s+1}^{s+m} \times k = 8^{l-1} \cdot \Gamma_{s+1}^{s+(m-(j-1))} \times (k-(j-1)) \quad \text{for} \quad 1 \leq j \leq l-1 \quad k \geq s+j.
\]

We are going to show that for \( 2 \leq l \leq m - 1 \),
(15.6) \[
(H_{l-1}) \implies \Gamma_{s+l}^{s+m} \times k = 8^{l-1} \cdot \Gamma_{s+1}^{s+(m-(l-1))} \times (k-(l-1)) \quad \text{for} \quad k \geq s+l.
\]

By Lemmas 14.1, 14.2 \( (H_{l-1}) \) holds for \( l = 3,4 \).

The case \( k = s+l \)

We have

(15.7) \[
\Gamma_{j}^{s+m} \times (s+l) = 21 \cdot 2^{3j-4} - 3 \cdot 2^{2j-3} \quad \text{if} \quad 1 \leq j \leq s-1 \quad \text{by (11.21)}.
\]

(15.8) \[
\Gamma_{s}^{s+m} \times (s+l) = 2^{2s+l-1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3} \quad \text{by (13.46)},
\]

(15.9) \[
\Gamma_{s+1}^{s+m} \times k = 3 \cdot 2^{k+s-1} + 21[2^{3s-1} - 2^{2s-1}] \quad \text{if} \quad k > s+1 \quad \text{by (13.48)},
\]

(15.10) \[
\Gamma_{s+j}^{s+m} \times (s+l) = 8^{j-1} \cdot \Gamma_{s+1}^{s+(m-(j-1))} \times (s+l-(j-1))
\]
\[
= 3 \cdot 2^{2s+l+2j-3} + 21 \cdot [2^{3j+3s-4} - 2^{2s+3j-4}] \quad \text{if} \quad 1 \leq j \leq l-1
\]

by \( (H_{l-1}) \) and (13.48) with \( m \to m - (j-1) \geq 2 \), \( k \to s+l - (j-1) \), \( s+l - (j-1) > s+1 \).

Consider the matrix
From (15.13), (15.14) we get

\[
\begin{pmatrix}
α_1 & α_2 & α_3 & \cdots & α_{s+l-1} & α_{s+l} \\
α_2 & α_3 & α_4 & \cdots & α_{s+l} & α_{s+l+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
α_{s-1} & α_s & α_{s+1} & \cdots & α_{2s+l-3} & α_{2s+l-2} \\
α_s & α_{s+1} & α_{s+2} & \cdots & α_{2s+l-2} & α_{2s+l-1} \\
β_1 & β_2 & β_3 & \cdots & β_{s+l-1} & β_{s+l} \\
β_2 & β_3 & β_4 & \cdots & β_{s+l} & β_{s+l+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
β_{s-1} & β_s & β_{s+1} & \cdots & β_{2s+l-3} & β_{2s+l-2} \\
β_s & β_{s+1} & β_{s+2} & \cdots & β_{2s+l-2} & β_{2s+l-1} \\
β_{s+1} & β_{s+2} & β_{s+3} & \cdots & β_{2s+l-1} & β_{2s+l} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
β_{s+m} & β_{s+m+1} & β_{s+m+2} & \cdots & β_{2s+m+l-2} & β_{2s+m+l-1}
\end{pmatrix}.
\]

We have respectively by (11.1) and (11.27) with \(k = s+1\)

\[(15.11)\]
\[
\sum_{i=0}^{s+l} \Gamma_i^{s+m} \times (s+l) = 2^{4s+m+2l-2},
\]

\[(15.12)\]
\[
\sum_{i=0}^{s-1} \Gamma_i^{s+m} \times (s+l) = 3 \cdot 2^{3s-4} - 2^{2s-3}.
\]

From (15.8), (15.9), ..., (15.12) we get

\[(15.13)\]
\[
\Gamma_i^{s+m} \times (s+l) = 2^{4s+m+2l-2} - \sum_{j=0}^{s-1} \Gamma_j^{s+m} \times (s+l) - \Gamma_s^{s+m} \times (s+l) - \sum_{j=1}^{l-1} \Gamma_j^{s+m} \times (s+l)
\]
\[
= 2^{4s+m+2l-2} - (3 \cdot 2^{3s-4} - 2^{2s-3}) - (2^{2s+l-1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3})
\]
\[
- \sum_{j=1}^{l-1} (3 \cdot 2^{2s+l+2j-3} + 21 \cdot [2^{3j+3s-4} - 2^{2s+3j-4}])
\]
\[
= 2^{4s+m+2l-2} - 3 \cdot 2^{3s+3l-4} + 2^{2s+3l-4},
\]

\[(15.14)\]
\[
\Gamma_i^{s+(m-(l-1))} \times (s+1) = 2^{4s+(m-(l-1))} - 3 \cdot 2^{3s-1} + 2^{2s-1} \quad \text{by (13.47) since} \ m - (l - 1) \geq 2.
\]

From (15.13), (15.14) we get

\[(15.15)\]
\[
\Gamma_i^{s+l} \times (s+l) = 8^{l-1} \cdot \Gamma_i^{s+(m-(l-1))} \times (s+1) \quad \text{if} \ 1 \leq l \leq m-1, \ k = s+l.
\]

The case \(k > s+l\)

Consider the matrix
We have respectively by (11.1), (11.2), (11.27) and (11.28) with $m$ by (H) and (13.48) with $\Gamma$

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{s+l} & \alpha_{s+l+1} \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{s+l+1} & \alpha_{s+l+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{2s+l-2} & \alpha_{2s+l-1} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{2s+l-1} & \alpha_{2s+l} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{s+l} & \beta_{s+l+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \ldots & \beta_{2s+l-2} & \beta_{2s+l-1} \\
\beta_s & \beta_{s+1} & \beta_{s+2} & \ldots & \beta_{2s+l-1} & \beta_{2s+l} \\
\beta_{s+1} & \beta_{s+2} & \beta_{s+3} & \ldots & \beta_{2s+l} & \beta_{2s+l+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{2s+m+l-1} & \beta_{2s+m+l}
\end{pmatrix}
\]

We get

\[(15.16)\]

\[\Gamma_{s+j}^{s+m} = 2^{2s+l} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3} \quad \text{by (13.16)},\]

\[(15.17)\]

\[\Gamma_{s+j}^{s+m} = 8^{j-1} \cdot \Gamma_{s+j}^{s+(m-1)(j-1)} \times (s+l+1-(j-1))
\]

\[= 3 \cdot 2^{s+l+2j-2} + 21 \cdot [2^{3j+3s-4} - 2^{2s+3j-4}] \quad \text{if } 1 \leq j \leq l-1\]

by (H) and (13.38) with $m \rightarrow m - (j-1) \geq 2$, $k \rightarrow s+l+1-(j-1)$, $s+l+1-(j-1) > s+1$.

We have respectively by (11.1), (11.2), (11.27) and (11.28) with $m \geq 2$, $k = s+1+1$

\[(15.18)\]

\[\sum_{i=0}^{s+l+1} \Gamma_{s+j}^{s+m} = 2^{s+m+2l},\]

\[(15.19)\]

\[\sum_{i=0}^{s+l+1} \Gamma_{s+j}^{s+m} \cdot 2^{-i} = 2^{s+m+l-1} + 2^{s+2l} - 2^{s+l-1},\]

\[(15.20)\]

\[\sum_{i=0}^{s-1} \Gamma_{s+m}^{s+l+1} = 3 \cdot 2^{3s-4} - 2^{2s-3}\]

and

\[(15.21)\]

\[\sum_{i=0}^{s-1} \Gamma_{s+m}^{s+l+1} \cdot 2^{-i} = 7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3}.\]

From (15.16), ..., (15.21) we obtain
Hence by (15.22), (15.23) we deduce after some calculations

\[(15.24) \quad \Gamma_{s+l}^{s+m} \times (s+l+1) = 21 \cdot 2^{3s+3l-4} - 9 \cdot 2^{2s+3l-4},\]

\[(15.25) \quad \Gamma_{s+l+1}^{s+m} \times (s+l+1) = 2^{4s+m+2l} - 3 \cdot 2^{3s+3l-1} + 2^{2s+3l-1}.\]

By (12.34) with \(j = l, m \geq l + 1\) we get

\[(15.26) \quad \Gamma_{s+l}^{s+m} \times (k+1) - \Gamma_{s+l}^{s+m} \times k = 3 \cdot 2^{k+s+2l-3} \quad \text{if} \quad k > s + l.\]
By (15.24) we see that (15.27) holds for $k = s + l$, then (15.24) holds for $k \geq s + l$.

From (13.48) with $m \to m - (l - 1)$, $k \to k - (l - 1)$, $m \geq l + 1$, $k - (l - 1) > s + 1$ and (15.27) we get

\[(15.28)\]
\[
8^{l-1} \cdot \Gamma_{s+1}^{\lfloor s + (m - (l - 1)) \rfloor \times (k - (l - 1))} = 8^{l-1} \cdot [3 \cdot 2^{k - (l - 1) + s - 1} + 21 \cdot (2^{3s - 1} - 2^{s-1})],
\]

\[(15.29)\]
\[
\Gamma_{s+l}^{\lfloor s + m \rfloor \times k} = 3 \cdot 2^{k+s+2l-3} + 21 \cdot 2^{3s+3l-4} - 21 \cdot 2^{2s+3l-4}.
\]

By (15.28), (15.29) we obtain

\[(15.30)\]
\[
\Gamma_{s+l}^{\lfloor s + m \rfloor \times m} = 8^{l-1} \cdot \Gamma_{s+1}^{\lfloor s + (m - (l - 1)) \rfloor \times (k - (l - 1))} \quad \text{for} \quad l \leq m - 1, \quad k > s + l.
\]

Observe that in view of (15.30) the formulas (15.24) and (15.25) holds for $l = m$, it suffices to reproduce carefully the proof of the above formulas with $k \to s + m + 1$ instead of $k \to s + l + 1$.

In fact we have

\[(15.31)\]
\[
\Gamma_{s+m}^{\lfloor s + m \rfloor \times (s + m + 1)} = 21 \cdot 2^{3s+3m-4} - 9 \cdot 2^{2s+3m-4},
\]

\[(15.32)\]
\[
\Gamma_{s+m+1}^{\lfloor s + m \rfloor \times (s + m + 1)} = 2^{4s+3m} - 3 \cdot 2^{3s+3m-1} + 2^{2s+3m-1}.
\]

The case $k = s + m$, $l = m$

We have

\[(15.33)\]
\[
\Gamma_{s+m}^{\lfloor s + m \rfloor \times (s+m)} = 2^{4s+3m-2} - 3 \cdot 2^{3s+3m-4} + 2^{2s+3m-4} \quad \text{by} \quad (15.25) \quad \text{with} \quad l \to m - 1,
\]

\[(15.34)\]
\[
\Gamma_{s+1}^{\lfloor s + 1 \rfloor \times (s+1)} = 2^{4s+1} - 3 \cdot 2^{3s-1} + 2^{2s-1} \quad \text{by} \quad (13.25).
\]

From (15.33), (15.34) we obtain
The case \( k > s + m, \ l = m \)

We have

(15.35)
\[
\Gamma_{s+m}^{s+m} \times (s+m+1) = 21 \cdot 2^{3s+3m-4} - 9 \cdot 2^{2s+3m-4} \quad \text{by (15.31)},
\]

(15.36)
\[
\Gamma_{s+m}^{s+m+1} - \Gamma_{s+m}^{s+m} = 11 \cdot 2^{k+s+2m-3} \quad \text{by (12.34) with } j = m, \ k > s + m.
\]

From (15.35), (15.36) we deduce

\[
\sum_{j=s+m+1}^{k} \left( \Gamma_{s+m}^{s+m} \times (j+1) - \Gamma_{s+m}^{s+m} \times j \right) = \sum_{j=s+m+1}^{k} 11 \cdot 2^{j+s+2m-3}
\]

\[
\Rightarrow \sum_{j=s+m+2}^{k+1} \Gamma_{s+m}^{s+m} \times j - \sum_{j=s+m+1}^{k} \Gamma_{s+m}^{s+m} \times j = 11 \cdot 2^{k+s+2m-2} - 11 \cdot 2^{2s+3m-2}
\]

\[
\Rightarrow \Gamma_{s+m}^{s+m+1} - \Gamma_{s+m}^{s+m} = 11 \cdot 2^{k+s+2m-2} - 11 \cdot 2^{2s+3m-2}
\]

\[
\Rightarrow \Gamma_{s+m}^{s+m} \times (k+1) = 11 \cdot 2^{k+s+2m-2} + 21 \cdot 2^{3s+3m-4} - 9 \cdot 2^{2s+3m-4} \quad \text{if } k > s + m.
\]

By (15.31) we see that (15.37) holds for \( k = s + m, \) then (15.37) holds for \( k \geq s + m. \)

From (15.35) (with \( k \to k-(m-1), \ k-(m-1) > s + 1 \)) and (15.37) with \( k+1 \to k \) we have

\[
\Gamma_{s+m}^{s+m} \times k = 11 \cdot 2^{k+s+2m-3} + 21 \cdot 2^{3s+3m-4} - 53 \cdot 2^{2s+3m-4} \quad \text{if } k > s + m,
\]

\[
8^{m-1} \cdot \Gamma_{s+1}^{s+(m-1)} = 8^{m-1} \cdot [11 \cdot 2^{k-(m-1)+s-1} + 21 \cdot 2^{3s-1} - 53 \cdot 2^{2s-1}] \quad \text{if } k-(m-1) > s + 1.
\]

\[ \square \]

**Lemma 15.2.** We have

(15.38)
\[
\Gamma_{s+l}^{s+l} \times k = 8^{l-1} \cdot \Gamma_{s+1}^{s+(m-l-1)} \times (k-(l-1)) \quad \text{if } 1 \leq l \leq m, \ k \geq s + l,
\]
\[ \Gamma_{s+t} = 2^{4s+2l+2m-2} - 3 \cdot 2^{3s+3l-4} + 2^{2s+3l-4} \quad \text{if } 1 \leq l \leq m, \]  
(15.39)

\[ \Gamma_{s+t}^k = 3 \cdot 2^{k+2l+s-3} + 21 \cdot 2^{3s+3l-4} - 21 \cdot 2^{2s+3l-4} \quad \text{if } 1 \leq l \leq m-1, k > s+l, \]  
(15.40)

\[ \Gamma_{s+m}^k = 11 \cdot 2^{k+2m+s-3} + 21 \cdot 2^{3s+3m-4} - 53 \cdot 2^{2s+3m-4} \quad \text{if } k > s+m. \]  
(15.41)

**Proof.** Lemma 15.2 follows from Lemma 15.1.

16. **A REDUCTION FORMULA FOR** $\Gamma_{s+j+1}^{[s]}$ **IN THE CASE**

$1 \leq j \leq s-1, k \geq s+j+1$

In this section we prove the following reduction formula by induction on $j$

\[ \Gamma_{s+j+1}^{[s]} = 8^{2j} \cdot \Gamma_{s-j+1}^{[s-j]} \]  
(16.1)

We recall once more that the right hand side in the above equation has been computed in section 13.

In fact we have

\[ \Gamma_{s-j+1}^{[s-j]}(k-2j) = \begin{cases} 2^{4s-4j} - 3 \cdot 2^{4s-3j-1} + 2^{2s-2j-1} & \text{if } 0 \leq j \leq s-1, k = s+j+1, \\ 21 \cdot (2^{2s-3j-1} + 2^{3s-3j-1} - 5 \cdot 2^{2s-2j-1}) & \text{if } 0 \leq j \leq s-2, k > s+j+1, \\ 2^{2k-4s+4} - 3 \cdot 2^{k-2s+2} + 2 & \text{if } j = s-1, k > 2s. \end{cases} \]

**Lemma 16.1.** We have

\[ \Gamma_{s+2}^{[s]} = 8^2 \cdot \Gamma_{s-1}^{[s-1]}(k-2) \quad \text{if } k \geq s+2, \]  
(16.1)

\[ \Gamma_{s+2}^{[s]} = 2^{4s+2} - 3 \cdot 2^{3s+2} + 2^{2s+3} \quad \text{if } k = s+2, \]  
(16.2)

\[ \Gamma_{s+2}^{[s]} = 21 \cdot [2^{k+s+2} + 2^{3s+2} + 5 \cdot 2^{s+3}] \quad \text{if } k > s+2. \]  
(16.3)

**Proof.** Proof of (16.1) with $k = s+2$

From (13.20) and (13.3) (with $s \to s-1$) we get
\[
\Gamma_s \left[ \begin{matrix} s \\ s+2 \end{matrix} \right]^{(s+2)} = 2^{4s+2} - 3 \cdot 2^{3s+2} + 2^{2s+3},
\]

\[
S^2 \cdot \Gamma_s \left[ \begin{matrix} s-1 \\ s \end{matrix} \right]^{s} = 8 \cdot 2^{4s-4} - 3 \cdot 2^{3s-4} + 2^{2s-3}.
\]

Proof of (16.1) with \( k > s + 2 \)

Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{s+2} & \alpha_{s+3} \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{s+3} & \alpha_{s+4} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{2s} & \alpha_{2s+1} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{2s+1} & \alpha_{2s+2} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{s+2} & \beta_{s+3} \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{s+3} & \beta_{s+4} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \ldots & \beta_{2s} & \beta_{2s+1} \\
\beta_s & \beta_{s+1} & \beta_{s+2} & \ldots & \beta_{2s+1} & \beta_{2s+2}
\end{pmatrix}
\]

We have

(16.4) \( \sum_{i=0}^{s+3} \Gamma_i \left[ \begin{matrix} s \\ s+3 \end{matrix} \right]^{(s+3)} = 2^{4s+4} \) by (11.1) with \( m = 0, k = s+3 \),

(16.5) \( \sum_{i=0}^{s+3} \Gamma_i \left[ \begin{matrix} s \\ s+3 \end{matrix} \right]^{i(3-s)} = 2^{3s+1} + 2^{2s+4} - 2^{s+1} \) by (11.2) with \( m = 0, k = s+3 \),

(16.6) \( \sum_{i=0}^{s-1} \Gamma_i \left[ \begin{matrix} s \\ s+3 \end{matrix} \right]^{i(3-s)} = 3 \cdot 2^{3s-4} - 2^{2s-3} \) by (11.27) with \( m = 0, k = s+3 \),

(16.7) \( \sum_{i=0}^{s-1} \Gamma_i \left[ \begin{matrix} s \\ s+3 \end{matrix} \right]^{i(3-s)} \cdot 2^{-i} = 7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3} \) by (11.28) with \( m = 0, k = s+3 \),

(16.8) \( \Gamma_s \left[ \begin{matrix} s \\ s+1 \end{matrix} \right]^{(s+3)} = 21 \cdot 2^{3s-4} + 69 \cdot 2^{2s-3} \) by (13.2) with \( k = s+3 \),

(16.9) \( \Gamma_s \left[ \begin{matrix} s \\ s+1 \end{matrix} \right]^{(s+3)} = 21 \cdot [2^{3s-1} + 3 \cdot 2^{2s-2}] \) by (13.3) with \( k = s+3 \).
From (16.3), . . . , (16.9) with $k = s + 3$ we obtain

\[(16.10)\]

\[
\Gamma_{s+2}^{s} \times (s+3) + \Gamma_{s+3}^{s} \times (s+3) = 2^{4s+4} - (3 \cdot 2^{3s-4} - 2^{2s-3}) - (21 \cdot 2^{3s-4} + 69 \cdot 2^{2s-3}) - (21 \cdot [2^{3s-1} + 3 \cdot 2^{2s-1}]),
\]

\[(16.11)\]

\[
\Gamma_{s+2}^{s} \times (s+3) \cdot 2^{-(s+2)} + \Gamma_{s+3}^{s} \times (s+3) \cdot 2^{-(s+3)} = 2^{3s+1} + 2^{2s+4} - 2^{s+1} - (7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3}) - (21 \cdot 2^{3s-4} + 69 \cdot 2^{2s-3}) \cdot 2^{-s}
\]

\[\ldots\]

Hence by (16.10), (16.11) we deduce after some calculations

\[(16.12)\]

\[
\Gamma_{s+2}^{s} \times (s+3) = 21 \cdot 2^{3s+2} - 21 \cdot 2^{2s+3},
\]

\[(16.13)\]

\[
\Gamma_{s+3}^{s} \times (s+3) = 2^{4s+4} - 3 \cdot 2^{3s+5} + 2^{2s+7}.
\]

By (12.32) with $j = 2$, $s \geq 3$ we get

\[(16.14)\]

\[
\Gamma_{s+2}^{s} \times (k+1) - \Gamma_{s+2}^{s} \times k = 21 \cdot 2^{k+s+2} \quad \text{if} \quad k > s + 2.
\]

From (16.1), (16.12) we deduce

\[
\sum_{j=s+3}^{k} \left( \Gamma_{s+2}^{s} \times (j+1) - \Gamma_{s+2}^{s} \times j \right) = \sum_{j=s+3}^{k} 21 \cdot 2^{j+s+2}
\]

\[
\Leftrightarrow \sum_{j=s+4}^{k+1} \Gamma_{s+2}^{s} \times j - \sum_{j=s+3}^{k} \Gamma_{s+2}^{s} \times j = 21 \cdot 2^{k+s+3} - 21 \cdot 2^{2s+5}
\]

\[
\Leftrightarrow \Gamma_{s+2}^{s} \times (k+1) - \Gamma_{s+2}^{s} \times (s+3) = 21 \cdot 2^{k+s+3} - 21 \cdot 2^{2s+5}
\]

\[
\Leftrightarrow \Gamma_{s+2}^{s} \times (k+1) = 21 \cdot 2^{k+s+3} - 21 \cdot 2^{2s+5} + 21 \cdot (2^{3s+2} - 21 \cdot 2^{2s+3})
\]

\[(16.15)\]

\[
\Gamma_{s+2}^{s} \times (k+1) = 21 \cdot [2^{k+s+3} + 2^{3s+2} - 5 \cdot 2^{2s+3}] \quad \text{if} \quad k > s + 2.
\]

By (16.12) we see that (16.15) holds for $k = s + 2$, then (16.15) holds for $k \geq s + 2$.

From (13.3) (with $k \rightarrow k - 2$, $s \rightarrow s - 1$, $k - 2 > (s - 1) + 1$) and (16.15) we have
\[ \Gamma_{s+2}^{[s \times k]} = 21 \cdot [2^{k+s+2} + 2^{3s+2} - 5 \cdot 2^{2s+3}], \]
\[ 8^2 \cdot \Gamma_s^{[s \times (k-2)]} = 8^2 \cdot 21 \cdot [2^{k+s-4} + 2^{3s-4} - 5 \cdot 2^{2s-3}]. \]

Lemma 16.2. We have for \( s \geq 2 \)

\begin{align*}
\Gamma_{s+j+1}^{[s \times k]} &= 8^{2j} \cdot \Gamma_{s-j+1}^{[s-j \times (k-2j)]} & \text{if } 0 \leq j \leq s-1, \ k \geq s+j+1, \\
\Gamma_{s-j+1}^{[s-j \times (s-j+1)]} &= 2^{4s-4j} - 3 \cdot 2^{3s-3j-1} + 2^{2s-2j-1} & \text{if } 0 \leq j \leq s-1, \ k = s+j+1, \\
\Gamma_{s-j+1}^{[s-j \times (k-2j)]} &= 21 \cdot [2^{k-3j+s-1} + 2^{3s-3j-1} - 5 \cdot 2^{2s-2j-1}] & \text{if } 0 \leq j \leq s-2, \ k > s+j+1, \\
\Gamma_{s-j+1}^{[s-j \times (k-2j)]} &= \Gamma_{s-j+1}^{[s-j \times (k-2(s-1))]} = 2^{2k-4s+4} - 3 \cdot 2^{k-2s+2} + 2 & \text{if } j = s-1, \ k > 2s.
\end{align*}

Proof. We proceed as in section 15 by induction on \( j \).

Let \( l \) be a rational integer such that \( 2 \leq l \leq s-2 \).

Assume

\begin{equation}
(\text{H}_{l-1}) \quad \Gamma_{s+j+1}^{[s \times k]} = 8^{2j} \cdot \Gamma_{s-j+1}^{[s-j \times (k-2j)]} \quad \text{for } 0 \leq j \leq l-1, \ k \geq s+j+1.
\end{equation}

We are going to show that (H\(_l\)) holds, that is

\begin{equation}
(\text{H}_l) \quad \Gamma_{s+l+1}^{[s \times k]} = 8^{2l} \cdot \Gamma_{s-l+1}^{[s-l \times (k-2l)]} \quad \text{for } k \geq s+l+1.
\end{equation}

By Lemma 16.1 (H\(_{l-1}\)) holds for \( l = 2 \) (see (16.20)).

Proof of (16.21) with \( k = s+l+1 \)

Consider the matrix
We have

\[ \sum_{i=0}^{s+l+1} \Gamma_i^s \times (s+l+1) = 2^{4s+2l} \] by (11.21) with \( m = 0, k = s + l + 1, \)

\[ \sum_{i=0}^{s-1} \Gamma_i^s \times (s+l+1) = 3 \cdot 2^{3s-4} - 2^{2s-3} \] by (11.27) with \( m = 0, k = s + l + 1, \)

\[ \Gamma^s \times (s+l+1) = 3 \cdot 2^{2s+l} + 21 \cdot 2^{3s-4} - 27 \cdot 2^{2s-3} \] by (13.2) with \( k = s + l + 1, \)

\[ \Gamma^s \times (s+l+1) = 8^{2j} \cdot \Gamma_{s-j+1}^{s-j} \times (s+l+1-2j) \] by (16.20) for \( 0 \leq j \leq l - 1, k = s + l + 1 \)

\[ = 2^{2j} \cdot [21 \cdot (2^{(s+l+1-2j)}+s-j-1) + 2^{2(s-j)-1} - 5 \cdot 2^{2(s-j)-1}] \] by (13.1) with \( s \rightarrow s-j, k \rightarrow s+l+1-2j > s-j+1 \)

\[ = 21 \cdot [2^{2s+3j+l} + 2^{3s+3j-1} - 5 \cdot 2^{2s+4j-1}] \]

From (16.22), (16.23), (16.24) and (16.25) we get after some calculations

\[ \Gamma_{s+l+1}^s \times (s+l+1) = 2^{4s+2l} - (3 \cdot 2^{3s-4} - 2^{2s-3}) - (3 \cdot 2^{2s+l} + 21 \cdot 2^{3s-4} - 27 \cdot 2^{2s-3}) \]

\[ - \left( \sum_{j=0}^{l-1} 21 \cdot [2^{2s+3j+l} + 2^{3s+3j-1} - 5 \cdot 2^{2s+4j-1}] \right) \]

\[ = 2^{4s+2l} - 3 \cdot 2^{3s+3l-1} + 2^{2s+4l-1}. \]

By (13.3) with \( s \rightarrow s-l \) and (16.26) we obtain
Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{s+l+1} & \alpha_{s+l+2} \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{s+l+2} & \alpha_{s+l+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{2s-l-1} & \alpha_{2s+l} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{2s+l} & \alpha_{2s+l+1} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{s+l+1} & \beta_{s+l+2} \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{s+l+2} & \beta_{s+l+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \ldots & \beta_{2s-l-1} & \beta_{2s+l} \\
\beta_s & \beta_{s+1} & \beta_{s+2} & \ldots & \beta_{2s+l} & \beta_{2s+l+1}
\end{pmatrix}
\]

We have

(16.27)
\[
\sum_{i=0}^{s+l+2} \Gamma_i^s \times (s+l+2) = 2^{4s+2l+2}
\]
by (11.1) with \(m = 0, k = s + l + 2\),

(16.28)
\[
\sum_{i=0}^{s+l+2} \Gamma_i^s \times (s+l+2) \cdot 2^{-i} = 2^{3s+l} + 2^{2s+2l+2} - 2^{s+l}
\]
by (11.2) with \(m = 0, k = s + l + 2\),

(16.29)
\[
\sum_{i=0}^{s-1} \Gamma_i^s \times (s+l+2) = 3 \cdot 2^{3s-4} - 2^{2s-3}
\]
by (11.27) with \(m = 0, k = s + l + 2\),

(16.30)
\[
\sum_{i=0}^{s-1} \Gamma_i^s \times (s+l+2) \cdot 2^{-i} = 7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3}
\]
by (11.28) with \(m = 0, k = s + l + 2\),

(16.31)
\[
\Gamma_s^s \times (s+l+2) = 3 \cdot 2^{2s+l+1} + 21 \cdot 2^{3s-4} - 27 \cdot 2^{2s-3}
\]
by (13.2) with \(k = s + l + 2\),

(16.32)
\[
\Gamma_{s+j+1}^s \times (s+l+2) = 8^{2j} \cdot \Gamma_{s-j+1}^{s-j} \times (s+l+2-2j)
\]
by (16.20) for \(0 \leq j \leq l - 1, k = s + l + 2\).
\[2^{6j} \cdot [21 \cdot (2^{s+l+2-2j})^{(s-j)-1} + 2^{3(s-j)-1} - 5 \cdot 2^2(s-j)-1)]\] by (13.3) with \(s \to s-j, \ k \to s+l+2-2j > s-j+1\)
\[= 21 \cdot [2^{2s+3j+l+1} + 2^{3s+3j-1} - 5 \cdot 2^{2s+4j-1}]\]

From (16.2), \ldots, (16.32) with \(k = s+l+2\) we obtain

\[(16.33)\]
\[
\Gamma_{s+l+1}^{(s+l+2)} + \Gamma_{s+l+1}^{(s+l+2)} = 2^{4s+2l+2} - (3 \cdot 2^{3s-4} - 2^{3s-3}) - (3 \cdot 2^{2s+l+1} + 21 \cdot 2^{3s-4} - 27 \cdot 2^{2s-3}) - \sum_{j=0}^{l-1} 21 \cdot [2^{2s+3j+l+1} + 2^{3s+3j-1} - 5 \cdot 2^{2s+4j-1}]\]

\[(16.34)\]
\[
\Gamma_{s+l+1}^{(s+l+2)} = 2^{4s+2l+2} - 3 \cdot 2^{3s+3l+2} + 2^{2s+4l+3} - \sum_{j=0}^{l-1} 21 \cdot [2^{2s+3j+l+1} + 2^{3s+3j-1} - 5 \cdot 2^{2s+4j-1}] \cdot 2^{-(s+j+1)}\]

Hence by (16.33), (16.34) we deduce after some calculations

\[(16.35)\]
\[
\Gamma_{s+l+1}^{(s+l+2)} = 21 \cdot 2^{3s+3l-1} - 21 \cdot 2^{2s+4l-1},
\]

\[(16.36)\]
\[
\Gamma_{s+l+2}^{(s+l+2)} = 2^{4s+2l+2} - 3 \cdot 2^{3s+3l+2} + 2^{2s+4l+3}.
\]

By (12.32) with \(j = l+1, \ l+1 \leq s-1\) we get

\[(16.37)\]
\[
\Gamma_{s+l+1}^{(k+1)} - \Gamma_{s+l+1}^{(k)} = 21 \cdot 2^{k+s+3(l+1)-4} \quad \text{if} \quad k > s+l+1.
\]

From (16.37), (16.38) we deduce

\[
\sum_{j=s+l+2}^{k} \left( \Gamma_{s+l+1}^{(j+1)} - \Gamma_{s+l+1}^{(j)} \right) = \sum_{j=s+l+2}^{k} 21 \cdot 2^{j+s+3l-1}
\]
\[
\iff \sum_{j=s+l+3}^{k+1} \Gamma_{s+l+1}^{(j)} - \sum_{j=s+l+1}^{k} \Gamma_{s+l+1}^{(j)} = 21 \cdot 2^{k+s+3l} - 21 \cdot 2^{2s+4l+1}
\]
\[
\iff \Gamma_{s+l+1}^{(k+1)} - \Gamma_{s+l+1}^{(s+l+2)} = 21 \cdot 2^{k+s+3l} - 21 \cdot 2^{2s+4l+1}
\]
\[
\iff \Gamma_{s+l+1}^{(k+1)} = 21 \cdot 2^{k+s+3l} - 21 \cdot 2^{2s+4l+1} + 21 \cdot (2^{3s+3l-1} - 21 \cdot 2^{2s+4l-1})
\]
We have

\[ (16.38) \quad \Gamma_{s+l+1}^s \times (k+1) = 21 \left[ 2^{k+s+3l} + 2^{3s+3l-1} - 5 \cdot 2^{2s+4l-1} \right] \quad \text{if } k > s + l + 1. \]

By (16.35) we see that (16.38) holds for \( k = s + l + 1 \), then (16.38) holds for \( k \geq s + l + 1 \).

From (16.38) and (13.4) with \( s \rightarrow s - l, k \rightarrow k - 2l > s - l + 1 \) we get

\[ \Gamma_{s+l+1}^s = 8^{2l} \cdot \Gamma_{s-l+1}^{s-l} \times (k-2l) = 2^{6l} \cdot \left[ 21 \cdot (2^{k-2l+(s-l)-1} + 2^{3(s-l)-1} - 5 \cdot 2^{2(s-l)-1}) \right] \]

We have now established that

\[ (16.39) \quad \Gamma_{s+j+1}^s = 8^{2s-2} \cdot \Gamma_{2s}^1 \times (k-2(s-1)) \quad \text{if } 0 \leq j \leq s - 2, \ k \geq s + j + 1. \]

Proof of (16.16) with \( j = s - 1, \ k \geq 2s \)

We shall show that

\[ (16.40) \quad \Gamma_{2s}^s = 8^{2s-2} \cdot \Gamma_{2}^1 \times (k-2(s-1)) \quad \text{if } k \geq 2s. \]

If \( k = 2s \), (16.40) follows from (16.21) with \( l = s - 1 \) and \( k = s + (s-1)+1 = 2s \).

Let \( k > 2s \).

Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{2s} & \alpha_{2s+1} \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{2s+1} & \alpha_{2s+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{3s-2} & \alpha_{3s-1} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{3s-1} & \alpha_{3s} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{2s} & \beta_{2s+1} \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{2s+1} & \beta_{2s+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \ldots & \beta_{3s-2} & \beta_{3s-1} \\
\beta_s & \beta_{s+1} & \beta_{s+2} & \ldots & \beta_{3s-1} & \beta_{3s}
\end{pmatrix}.
\]

We have

\[ (16.41) \quad \sum_{i=0}^{2s} \Gamma_i^s \times (2s+1) = 2^{6s} \quad \text{by (11.1) with } m = 0, \ k = 2s + 1, \]

\[ (16.42) \quad \sum_{i=0}^{s-1} \Gamma_i^s \times (2s+1) = 3 \cdot 2^{3s-4} - 2^{2s-3} \quad \text{by (11.27) with } m = 0, \ k = 2s + 1, \]
(16.43) \[ \Gamma_s \times (2s+1) \quad \text{for } s = n \pm 1 \] \[ = 3 \cdot 2^{3s} + 21 \cdot 2^{3s-4} - 27 \cdot 2^{2s-3} \quad \text{by (13.2) with } k = 2s + 1, \]

(16.44) \[ \Gamma_s \times (2s+1) \quad \text{for } s = n \pm 1 \] \[ = 2^{6s} - \left( 3 \cdot 2^{3s-4} - 2^{2s-3} \right) - \left( 3 \cdot 2^{3s} + 21 \cdot 2^{3s-4} - 27 \cdot 2^{2s-3} \right) \] \[ = \sum_{j=0}^{s-2} 21 \cdot \left[ 2^{4s+3j} + 2^{3s+3j} - 5 \cdot 2^{2s+4j} - 1 \right] \] \[ = 21 \cdot 2^{6s-5}. \]

By (12.32) with \( j = s, \quad k > 2s \) we get

(16.45) \[ \Gamma_s \times (k+1) - \Gamma_s \times k = 3 \cdot 2^{2k+2s-2} - 3 \cdot 2^{k+4s-4} \quad \text{if } k > 2s. \]

From (16.46), (16.45) we deduce

\[ \sum_{j=2s+1}^{k+1} \Gamma_s \times j = \sum_{j=2s+1}^{k+1} \Gamma_s \times (k+1) = 2^{2k+2s} - 3 \cdot 2^{k+4s-3} - 5 \cdot 2^{6s-3} \]

\[ \sum_{j=2s+1}^{k+1} \Gamma_s \times j = 2^{2k+2s} - 3 \cdot 2^{k+4s-3} - 5 \cdot 2^{6s-3} + 21 \cdot 2^{6s-5} \]

(16.47) \[ \Gamma_s \times (k+1) = 2^{2k+2s} - 3 \cdot 2^{k+4s-3} + 2^{6s-5} \quad \text{if } k > 2s. \]

By (16.45) we see that (16.47) holds for \( k = 2s \), then (16.47) holds for \( k > 2s \). Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{k-2s+1} & \alpha_{k-2s+2} \\
\beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{k-2s+1} & \beta_{k-2s+2}
\end{pmatrix}
\]
We see easily that
\[
\Gamma_1 \left( \frac{1}{2} \right) \times (k-2(s-1)) = 3 \cdot (2^{k-2s+2} - 1).
\]

Hence
\[
(16.48) \quad \Gamma_2 \left( \frac{1}{2} \right) \times (k-2(s-1)) = 2^{2k-4s+4} - 3 \cdot (2^{k-2s+2} - 1) = 2^{2k-4s+4} - 3 \cdot 2^{k-2s+2} + 2.
\]

From \(16.38\) and \(16.47\) with \(k \geq 2s\) we get
\[
\Gamma_2 \left( \frac{s}{2s} \right) \times k = 2^{2k+2s-2} - 3 \cdot 2^{k+s-4} + 6s-5 = 8^{s-1} \cdot \Gamma_2 \left( \frac{1}{2} \right) \times (k-2(s-1)).
\]

\[
\square
\]

17. A REDUCTION FORMULA FOR \( \Gamma \left( \frac{s+m}{s+m+1+j} \right) \times k \) IN THE CASE

\[1 \leq j \leq s-1, \ k \geq s+m+1+j\]

In this section we prove the following reduction formula by induction on \(j\)
\[
\Gamma \left( \frac{s+m}{s+m+1+j} \right) \times k = 8^{j+m} \cdot \Gamma \left( \frac{s-j}{s-j+1} \right) \times (k-m-2j) \quad \text{if} \quad 0 \leq j \leq s-1, \ k \geq s+m+1+j.
\]

We recall again that the right hand side in the above equation has been computed in section \[10\]

In fact we have
\[
\Gamma \left( \frac{s-j}{s-j+1} \right) \times (k-m-2j) = \begin{cases} 2^{4s-4j} - 3 \cdot 2^{3s-3j-1} + 2^{2s-2j-1} & \text{if} \quad 0 \leq j \leq s-1, \ k = s+m+j+1, \\ 21 \cdot 2^{k-m+s-3j-1} + 2^{3s-3j-1} - 5 \cdot 2^{2s-2j-1} & \text{if} \quad 0 \leq j \leq s-2, \ k > s+m+j+1, \\ 2^{2k-2m-4s+4} - 3 \cdot 2^{k-m-2s+2} + 2 & \text{if} \quad j = s-1, \ k > 2s+m. 
\end{cases}
\]

Lemma 17.1. We have

\[
(17.1) \quad \Gamma \left( \frac{s+m}{s+m+1} \right) \times k = 8^m \cdot \Gamma \left( \frac{s}{s+1} \right) \times (k-m) \quad \text{if} \quad k \geq s+m+1,
\]

\[
(17.2) \quad \Gamma \left( \frac{s}{s+m+1} \right) \times (s+m) = 2^{4s+3m} - 3 \cdot 2^{3s+3m-1} + 2^{2s+3m-1} = 8^m \cdot \Gamma \left( \frac{s}{s+1} \right) \times (s+1) \quad \text{if} \quad k = s+m+1,
\]

\[
\Gamma \left( \frac{s}{s+1} \right) \times (s+1) = 2^{4s} - 3 \cdot 2^{3s-1} + 2^{2s-1},
\]

\[
(17.3) \quad \Gamma \left( \frac{s}{s+m+1} \right) \times k = 21 \cdot (2^{k+s+2m-1} + 2^{3s+3m-1} - 5 \cdot 2^{2s+3m-1}) = 8^m \cdot \Gamma \left( \frac{s}{s+1} \right) \times (k-m) \quad \text{if} \quad k > s+m+1,
\]

\[
\Gamma \left( \frac{s}{s+1} \right) \times (k-m) = 21 \cdot (2^{k-m+s-1} + 2^{3s-1} - 5 \cdot 2^{2s-1}).
\]
**Proof**. Proof of (17.2)

By (15.32) and (13.3) we get

\[
\Gamma \left[ \frac{s}{s+m} \right]_{x(s+m+1)} = 2^{4s+3m} - 3 \cdot 2^{3s+3m-1} + 2^{2s+3m-1} = 8^m \cdot \Gamma \left[ \frac{s}{s+1} \right]_{x(s+1)}.
\]

**Proof of (17.3)**

Consider the matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{s+m+1} & \alpha_{s+m+2} \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{s+m+2} & \alpha_{s+m+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{2s+m-1} & \alpha_{2s+m} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{2s+m} & \alpha_{2s+m+1} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{s+m+1} & \beta_{s+m+2} \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{s+m+2} & \beta_{s+m+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \ldots & \beta_{2s+m-1} & \beta_{2s+m} \\
\beta_s & \beta_{s+1} & \beta_{s+2} & \ldots & \beta_{2s+m} & \beta_{2s+m+1} \\
\beta_{s+1} & \beta_{s+2} & \beta_{s+3} & \ldots & \beta_{2s+m+1} & \beta_{2s+m+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{2s+2m} & \beta_{2s+2m+1}
\end{pmatrix}.
\]

We have

\[
(17.4) \quad \sum_{i=0}^{s+m+2} \Gamma \left[ \frac{s}{s+m} \right]_{x(s+m+2)} = 2^{4s+3m+2} \quad \text{by (11.1) with } k = s+m+2,
\]

\[
(17.5) \quad \sum_{i=0}^{s+m+2} \Gamma \left[ \frac{s}{s+m} \right]_{x(s+m+2)} \cdot 2^{-i} = 2^{3s+m} + 2^{2s+2m+2} - 2^{s+m} \quad \text{by (11.2) with } k = s+m+2,
\]

\[
(17.6) \quad \sum_{i=0}^{s-1} \Gamma \left[ \frac{s}{s+m} \right]_{x(s+m+2)} = 3 \cdot 2^{3s-4} - 2^{2s-3} \quad \text{by (11.27) with } k = s+m+2,
\]

\[
(17.7) \quad \sum_{i=0}^{s-1} \Gamma \left[ \frac{s}{s+m} \right]_{x(s+m+2)} \cdot 2^{-i} = 7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3} \quad \text{by (11.28) with } k = s+m+2,
\]

\[
(17.8) \quad \Gamma \left[ \frac{s}{s+m} \right]_{x(s+m+2)} = 2^{2s+m+1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3} \quad \text{by (13.46) with } k = s+m+2,
\]
From (17.4), (17.9) with $k = s + m + 2$ we obtain

\[
\Gamma_{s+m+1}^{s+m+2} \times (s+m+2) = 2^{s+m+2} - \frac{3}{2}(2^{2s-m} - 2^{2s-1}) - (2^{2s+m+1} + 2^{3s-4} - 11 \cdot 2^{2s-3}) - \sum_{j=1}^{m-1} [3 \cdot 2^{2s+m+2j-1} + 21 \cdot 2^{3s+3j-4} - 21 \cdot 2^{2s+3j-4}] - (21 \cdot 2^{3s+3m-4} + 35 \cdot 2^{2s+3m-4})
\]

\[
= 2^{4s+m+2} - 3 \cdot 2^{3s+3m-1} - 5 \cdot 2^{2s+3m-1},
\]

Hence by (17.11), (17.12) we deduce after some calculations

\[
\Gamma_{s+m+1}^{s+m+2} \times (s+m+2) = \frac{3}{2}(2^{2s-m} - 2^{2s-1}) - (2^{2s+m+1} + 2^{3s-4} - 11 \cdot 2^{2s-3}) - \sum_{j=1}^{m-1} [3 \cdot 2^{2s+m+2j-1} + 21 \cdot 2^{3s+3j-4} - 21 \cdot 2^{2s+3j-4}] - (21 \cdot 2^{3s+3m-4} + 35 \cdot 2^{2s+3m-4})
\]

\[
= 2^{4s+m+2} - 3 \cdot 2^{3s+3m-1} - 5 \cdot 2^{2s+3m-1}.
\]
\[
\sum_{j=s+m+2}^{k} \left( \Gamma_{s+m+1}^{s} \times (j+1) - \Gamma_{s+m}^{s+m} \times j \right) = \sum_{j=s+m+2}^{k} 21 \cdot 2^{j+s+2m-1}
\]
\[
\Leftrightarrow \sum_{j=s+m+3}^{k+1} \Gamma_{s+m+1}^{s} \times j - \sum_{j=s+m+1}^{k} \Gamma_{s+m}^{s+m} \times j = 21 \cdot 2^{k+s+2m} - 21 \cdot 2^{2s+3m+1}
\]
\[
\Leftrightarrow \Gamma_{s+m+1}^{s+s+m+2} = 21 \cdot 2^{k+s+2m} - 21 \cdot 2^{2s+3m+1}
\]
\[
\Leftrightarrow \Gamma_{s+m+1}^{s+m+1} \times (k+1) = 21 \cdot 2^{k+s+2m} - 21 \cdot 2^{2s+3m+1}
\]
\[
\Leftrightarrow \Gamma_{s+m+1}^{s+m+1} \times (k+1) = 21 \cdot 2^{k+s+2m} - 21 \cdot 2^{2s+3m+1} + 21 \cdot (2^{3s+3m-1} - 21 \cdot 2^{2s+3m-1})
\]
(17.16)
\[
\Leftrightarrow \Gamma_{s+m+1}^{s+m+1} \times (k+1) = 21 \cdot 2^{k+s+2m} + 2^{3s+3m-1} - 5 \cdot 2^{2s+3m-1} \text{ if } k > s + m + 1.
\]

By (17.13) we see that (17.16) holds for \( k = s + m + 1 \), then (17.16) holds for \( k \geq s + m + 1 \).

From (17.16) and (13.3) with \( k \to k - m > s + 1 \) we get
\[
\Gamma_{s+m+1+j}^{s+m} \times k = 8^{m} \cdot \Gamma_{s+m+1+j}^{s-m} \times (k-m)
\]
(17.17)
\[
\Gamma_{s+m+1+j}^{s+m} \times (s+m+1+j) = 8^{2j+m} \cdot \Gamma_{s+j+1}^{s-j} \times (k-m-2j)
\]
if \( 0 \leq j \leq s-1, \ k \geq s + m + 1 + j \),

(17.18)
\[
\Gamma_{s+m+1+j}^{s+m} \times (s+m+1+j) = 8^{2j+m} \cdot \Gamma_{s+j+1}^{s-j} \times (s-j+1)
\]
\[
= 8^{2j+m} \cdot (2^{4s-4j} - 3 \cdot 2^{3s-3j-1} + 2^{2s-2j-1})
\]
\[
= 2^{4s+3m+2j} - 3 \cdot 2^{3s+3m+3j-1} + 2^{2s+3m+4j-1}.
\]

(17.19)
\[
\Gamma_{s+m+1+j}^{s-m} \times k = 8^{2j+m} \cdot \Gamma_{s-j+1}^{s-j} \times (k-m-2j)
\]
\[
= 8^{2j+m} \cdot 21 \cdot [2^{k-m-3j+s-1} + 2^{3s-2j-1} - 5 \cdot 2^{2s-2j-1}]
\]
\[
= 21 \cdot 2^{k+2m+3j+s-1} + 2^{3s+3m+3j-1} - 5 \cdot 2^{2s+3m+4j-1}.
\]
if \( 0 \leq j \leq s-2, \ k > s + m + 1 + j \),

(17.20)
\[
\Gamma_{s+m}^{s-m} \times k = 8^{2s+m-2} \cdot \Gamma_{2}^{j} \times (k-m-2s+2)
\]
\[
= 8^{2s+m-2} \cdot [2^{2(k-m)-4s+4} - 3 \cdot 2^{k-m-2s+2} + 2]
\]
if \( j = s-1, \ k > 2s + m \).
We have
\[2^{2k+2s+m-2} - 3 \cdot 2^{k+2m+4s-4} + 2^{6s+3m-5}.

Proof. We will do the proof by induction on \(j\).
Let \(l\) be a rational integer such that \(1 \leq l \leq s - 2\).
Let \((H_{l-1})\) denote the following statement
\[
\text{(17.21)} \quad \Gamma_{s+m+1+j}^{s+m} \times^k \Gamma_{s-j}^{s-j+1} = 8^{2j+m} \cdot \Gamma_{s-j+1}^{s-j} \times^{(k-m-2)j}
\]
if \(0 \leq j \leq l - 1, \ k \geq s + m + 1 + j\).

By Lemma 17.1 \((H_{l-1})\) holds for \(l = 1\).
Assume that \((H_{l-1})\) holds.
We are going to show that \((H_l)\) holds, that is
\[
\text{(17.22)} \quad \Gamma_{s+m+1+l}^{s+m} \times^k \Gamma_{s-l}^{s-l+1} = 8^{2l+m} \cdot \Gamma_{s-l+1}^{s-l} \times^{(k-m-2l)}
\]
if \(k \geq s + m + l + 1\).

The case \(k = s + m + l + 1\)
Consider the matrix
\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{s+m+l} & \alpha_{s+m+l+1} \\
\alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{s+m+l+1} & \alpha_{s+m+l+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \cdots & \alpha_{2s+m+l-2} & \alpha_{2s+m+l-1} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \cdots & \alpha_{2s+m+l-1} & \alpha_{2s+m+l} \\
\beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{s+m+l} & \beta_{s+m+l+1} \\
\beta_2 & \beta_3 & \beta_4 & \cdots & \beta_{s+m+l+1} & \beta_{s+m+l+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \cdots & \beta_{2s+m+l-2} & \beta_{2s+m+l-1} \\
\beta_s & \beta_{s+1} & \beta_{s+2} & \cdots & \beta_{2s+m+l-1} & \beta_{2s+m+l} \\
\beta_{s+1} & \beta_{s+2} & \beta_{s+3} & \cdots & \beta_{2s+m+l} & \beta_{2s+m+l+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \cdots & \beta_{2s+2m+l-1} & \beta_{2s+2m+l}
\end{pmatrix}
\]

We have
\[
\text{(17.23)} \quad \sum_{i=0}^{s+m+l+1} \Gamma_{s+m}^{s+m} \times^{(s+m+l+1)} = 2^{4s+3m+2l}
\]
by (11.1) with \(k = s + m + l + 1\),
\[
\text{(17.24)} \quad \sum_{i=0}^{s-1} \Gamma_{s+m}^{s+m} \times^{(s+m+l+1)} = 3 \cdot 2^{3s-4} - 2^{2s-3}
\]
by (11.27) with \(k = s + m + l + 1\),
(17.25) 
\[ \Gamma_{s+m+1}^{s+m} \times (s+m+l+1) = 2^{2s+m+l} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3} \]  
by (13.40) with \( k = s + m + l + 1 \),

(17.26) 
\[ \Gamma_{s+j}^{s+m} \times (s+m+l+1) = 8^{j-1} \cdot \Gamma_{s+j+1}^{s+m} \times (s+m+l+1-(j-1)) = 3 \cdot 2^{2s+m+l+2j-2} + 21 \cdot 2^{3s+3j-4} - 21 \cdot 2^{2s+3j-4} \]  
by (13.40) for \( 1 \leq j \leq m - 1 \) and \( k = s + m + l + 1 \),

(17.27) 
\[ \Gamma_{s+m}^{s+m} \times (s+m+l+1) = 8^{j-1} \cdot \Gamma_{s+j}^{s+m} \times (s+m+l+1-(j-1)) = 11 \cdot 2^{2s+3m+l-2} + 21 \cdot 2^{3s+3m-4} - 53 \cdot 2^{2s+3m-4} \]  
by (15.11) for \( k = s + m + 2 \),

(17.28) 
\[ \Gamma_{s+j}^{s+m} \times (s+m+l+1) = 8^{j+m} \cdot \Gamma_{s+j+1}^{s-m-1} \times ((s+m+l+1)-(m-2j)) = 21 \cdot [2^{2s+3m+l+3j} + 2^{3s+3m+3j-1} - 5 \cdot 2^{2s+3m+4j-1}] \]  
by (H_{l-1}) (see (17.21)) if \( 0 \leq j \leq l - 1 \), \( k = s + m + l + 1 \),

From (17.23), ..., (17.28) with \( k = s + m + l + 1 \) we obtain after some calculations

(17.29) 
\[ \Gamma_{s+m+1}^{s+m} \times (s+m+l+1) = 2^{4s+3m+2l} - (3 \cdot 2^{3s-4} - 2^{2s-3}) - (2^{2s+m+l} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3}) \]  
\[ - \sum_{j=1}^{m-1} [3 \cdot 2^{2s+m+l+2j-2} + 21 \cdot 2^{3s+3j-4} - 21 \cdot 2^{2s+3j-4}] - (11 \cdot 2^{2s+3m+l-2} + 21 \cdot 2^{3s+3m-4} - 53 \cdot 2^{2s+3m-4}) \]  
\[ - \sum_{j=0}^{l-1} 21 \cdot [2^{2s+3m+l+3j} + 2^{3s+3m+3j-1} - 5 \cdot 2^{2s+3m+4j-1}] \]  
\[ = 2^{4s+3m+2l} - 3 \cdot 2^{3s+3m+3l-1} + 2^{2s+3m+4l-1} \]  

By (13.3) with \( s \to s - l \) and (17.29) we get

\[ \Gamma_{s+m+1}^{s+m} \times (s+m+l+1) = 8^{2l+m} \cdot [2^{4(s-l)} - 3 \cdot 2^{3(s-l)-1} + 2^{2(s-l)-1}] = 2^{6l+3m} \cdot \Gamma_{s-l+1}^{s-l} \times (s-l+1) \]

The case \( k > s + m + l + 1 \)
Consider the matrix
\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{s+m+l+1} & \alpha_{s+m+l+2} \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_{s+m+l+2} & \alpha_{s+m+l+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{2s+m+l-1} & \alpha_{2s+m+l} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{2s+m+l} & \alpha_{2s+m+l+1} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{s+m+l+1} & \beta_{s+m+l+2} \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{s+m+l+2} & \beta_{s+m+l+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s-1} & \beta_s & \beta_{s+1} & \ldots & \beta_{2s+m+l-1} & \beta_{2s+m+l} \\
\beta_s & \beta_{s+1} & \beta_{s+2} & \ldots & \beta_{2s+m+l} & \beta_{2s+m+l+1} \\
\beta_{s+1} & \beta_{s+2} & \beta_{s+3} & \ldots & \beta_{2s+m+l+1} & \beta_{2s+m+l+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \ldots & \beta_{2s+2m+l} & \beta_{2s+2m+l+1}
\end{pmatrix}
\]

We have

\[(17.30)\]
\[
\sum_{i=0}^{s+m+l+2} \Gamma_i \left[ \sum_{s+m}^{s+m+l+2} (s+m+l+2) \right] = 2^{4s+3m+2l+2} \quad \text{by (11.1) with } k = s + m + l + 2,
\]

\[(17.31)\]
\[
\sum_{i=0}^{s+m+l+2} \Gamma_i \left[ \sum_{s+m}^{s+m+l+2} (s+m+l+2) \right] \cdot 2^{-i} = 2^{3s+2m+l} + 2^{2s+2m+2l+2} - 2^{s+m+l} \quad \text{by (11.2) with } k = s + m + l + 2,
\]

\[(17.32)\]
\[
\sum_{i=0}^{s+m+l+2} \Gamma_i \left[ \sum_{s+m}^{s+m+l+2} (s+m+l+2) \right] \cdot 2^{-i} = 7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3} \quad \text{by (11.23) with } k = s + m + l + 2,
\]

\[(17.33)\]
\[
\sum_{i=0}^{s+m+l+2} \Gamma_i \left[ \sum_{s+m}^{s+m+l+2} (s+m+l+2) \right] = 3 \cdot 2^{3s-4} - 2^{2s-3} \quad \text{by (11.27) with } k = s + m + l + 2,
\]

\[(17.34)\]
\[
\Gamma_i \left[ \sum_{s+m}^{s+m+l+2} (s+m+l+2) \right] = 2^{2s+m+l+1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3} \quad \text{by (13.46) with } k = s + m + l + 2,
\]

\[(17.35)\]
\[
\Gamma_i \left[ \sum_{s+m}^{s+m+l+2} (s+m+l+2) \right] = 8^{j-1} \cdot \Gamma_i \left[ \sum_{s+(m-s-j-1)}^{s+(m+l+2-(j-1))} (s+m+l+2-(j-1)) \right] \quad \text{by (15.40) for } 1 \leq j \leq m - 1
\]

and
\[k = s + m + l + 2,
\]
\[
\Gamma\left[\frac{s+m}{s+m}\right] \times (s+m+l+2) = 11 \cdot 2^{(s+m+l+2)+2m-s-3} + 21 \cdot 2^{3s+3m-4} - 53 \cdot 2^{2s+3m-4} \quad \text{by (15.41) for } k = s + m + l + 2
\]
\[
= 11 \cdot 2^{2s+3m+l-1} + 21 \cdot 2^{3s+3m-4} - 53 \cdot 2^{2s+3m-4},
\]

\[
\Gamma\left[\frac{s}{s+m}\right] \times (s+m+l+2) = 8^{2j+m} \cdot \Gamma\left[\frac{s-j}{s+j+1}\right] \times ((s+m+l+2) - m - 2j)
\]
\[
= 8^{2j+m} \cdot 21 \cdot [2^{(s+m+l+2) - m - 3j + s - 1} + 2^{3s-3j-1} - 5 \cdot 2^{2s-2j-1}]
\]
\[
= 21 \cdot [2^{2s+3m+l+3j+1} + 2^{3s+3m+3j-1} - 5 \cdot 2^{2s+3m+4j-1}].
\]

From (17.30), . . . , (17.37) with \( k = s + m + l + 2 \) we obtain after some calculations

\[
\Gamma\left[\frac{s+m}{s+m+l+1}\right] \times (s+m+l+2) + \Gamma\left[\frac{s+m}{s+m+l+2}\right] = 2^{4s+3m+2l+2} - (3 \cdot 2^{2s-4} - 2^{2s-3}) - (2^{2s+m+l+1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3})
\]
\[
- \sum_{j=1}^{m-1} [3 \cdot 2^{2s+m+l+2j-1} + 21 \cdot 2^{3s+3j-4} - 21 \cdot 2^{2s+3j-4}] - (11 \cdot 2^{2s+3m+l-1} + 21 \cdot 2^{3s+3m-4} - 53 \cdot 2^{2s+3m-4})
\]
\[
- \sum_{j=0}^{l-1} 21 \cdot [2^{2s+3m+l+3j+1} + 2^{3s+3m+3j-1} - 5 \cdot 2^{2s+3m+4j-1}]
\]
\[
= 2^{4s+3m+2l+2} - 3 \cdot 2^{3s+3m+3l-1} - 5 \cdot 2^{2s+3m+4l-1},
\]

\[
\Gamma\left[\frac{s+m}{s+m+l+1}\right] \times (s+m+l+2) \cdot 2^{-(s+m+l+1)} + \Gamma\left[\frac{s+m}{s+m+l+2}\right] \cdot 2^{-(s+m+l+2)}
\]
\[
= 2^{3s+2m+l} + 2^{2s+2m+2l+2} - 2^{s+m+l} - (7 \cdot 2^{2s-4} - 3 \cdot 2^{s-3}) - (2^{2s+m+l+1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3}) \cdot 2^{-s}
\]
\[
- \sum_{j=1}^{m-1} [3 \cdot 2^{2s+m+l+2j-1} + 21 \cdot 2^{3s+3j-4} - 21 \cdot 2^{2s+3j-4}] \cdot 2^{-(s+j)}
\]
\[
- (11 \cdot 2^{2s+3m+l-1} + 21 \cdot 2^{3s+3m-4} - 53 \cdot 2^{2s+3m-4}) \cdot 2^{-(s+m)}
\]
\[
- \sum_{j=0}^{l-1} 21 \cdot [2^{2s+3m+l+3j+1} + 2^{3s+3m+3j-1} - 5 \cdot 2^{2s+3m+4j-1}] \cdot 2^{-(s+m+j)}
\]
\[
= 2^{3s+2m+l} + 9 \cdot 2^{2s+2m+2l-2} - 13 \cdot 2^{s+2m+3l-2}.
\]

Hence by (17.38), (17.39) we deduce

\[
\Gamma\left[\frac{s+m}{s+m+l+1}\right] \times (s+m+l+2) = 21 \cdot 2^{3s+3m+3l-1} - 21 \cdot 2^{2s+3m+4l-1},
\]
\[
\Gamma\left[\frac{s}{s+m+l+2}\right] \times (s+m+l+2) = 2^{4s+3m+2l+2} - 3 \cdot 2^{3s+3m+3l+2} + 2^{2s+3m+4l+3}.
\]
By (12.35) with \( j = l + 1 (\leq s - 1) \), \( k > s + m + l + 1 \) we get

\[
(17.42) \quad \Gamma_{s+m}^{s+m} \times (k+1)_{s+m+l+1} - \Gamma_{s+m}^{s+m} \times k_{s+m+l+1} = 21 \cdot 2^{k+s+2m+3l-1} \text{ if } k > s + m + l + 1.
\]

From (17.42), (17.40) we deduce

\[
\sum_{j=s+m+l+2}^{k} \left( \Gamma_{s+m}^{s+m} \times (j+1)_{s+m+l+1} - \Gamma_{s+m}^{s+m} \times j_{s+m+l+1} \right) = \sum_{j=s+m+l+2}^{k} 21 \cdot 2^{k+s+2m+3l-1}
\]

\[
\iff \sum_{j=s+m+l+3}^{k+1} \Gamma_{s+m}^{s+m} \times (j+1)_{s+m+l+1} - \sum_{j=s+m+l+2}^{k} \Gamma_{s+m}^{s+m} \times j_{s+m+l+1} = 21 \cdot 2^{k+s+2m+3l-1} - 21 \cdot 2^{2s+3m+4l+1}
\]

\[
\iff \Gamma_{s+m}^{s+m} \times (s+m+l+2) = 21 \cdot 2^{k+s+2m+3l-1} - 21 \cdot 2^{2s+3m+4l+1} + 21 \cdot (2^{3s+3m+3l-1} - 2^{2s+3m+4l-1})
\]

\[
(17.43)
\]

\[
\iff \Gamma_{s+m}^{s+m} \times (k+1) = 21[2^{k+s+2m+3l-1} + 2^{3s+3m+3l-1} - 5 \cdot 2^{2s+3m+4l-1}] \text{ if } k > s + m + l + 1.
\]

By (17.40) we see that (17.43) holds for \( k = s + m + l + 1 \), then (17.43) holds for \( k \geq s + m + l + 1 \).

From (17.43) and (13.4) with \( s \to s - l, k \to k - m - 2l (\geq s - l + 1) \) we get

\[
\Gamma_{s+m}^{s+m} \times k_{s+m+l+1} = 8^{2l+m} \cdot \Gamma_{s-l}^{s-l} \times (k-m-2l)_{s-l+1} = 2^{3l+6l} \cdot \left[ 21 \cdot (2^{(k-m-2l)+(s-l)-1} + 2^{3(s-l)-1} - 5 \cdot 2^{2(s-l)-1}) \right]
\]

\[
= 21[2^{k+s+2m+3l-1} + 2^{3s+3m+3l-1} - 5 \cdot 2^{2s+3m+4l-1}] \text{ if } k > s + m + l + 1.
\]

We have now established that

\[
\Gamma_{s+m}^{s+m} \times k_{s+m+1+j} = 8^{2j+m} \cdot \Gamma_{s-j}^{s-j} \times (k-m-2j)_{s-j+1} \text{ if } 0 \leq j \leq s - 2, \quad k \geq s + m + 1 + j.
\]

It remains to prove

\[
(17.44) \quad \Gamma_{2s+m}^{s+m} \times k_{2s+m} = 8^{2s+m-2} \cdot \Gamma_{2}^{1} \times (k-m-2(s-1)) \text{ if } k \geq 2s + m.
\]

The case \( k = 2s + m \)

(17.44) holds for \( l = s - 2 \), we then obtain by (16.48) with \( k \to 2s \)

\[
8^{2(s-1)+m} \cdot \Gamma_{2}^{1} \times 2 = 8^{2(s-1)+m} \cdot [2^{4} - 3 \cdot 2^{2} + 2] = 2^{6s+3m-6} \cdot 6 = 3 \cdot 2^{6s+3m-5} = \Gamma_{2s+m}^{s+m} \times (2s+m).
\]
The case $k > 2s + m$

We proceed as in the case $k > s + m + l + 1$ with $l = s - 1$.

From (17.38), with $l = s - 1$ we get

(17.45)\[\Gamma_{2s+m}^{s+m} \times (2s+m+1) = 24s+3m+2(s-1)+2 - 3 \cdot 2^{3s+3m+3(s-1)-1} - 5 \cdot 2^{2s+3m+4(s-1)-1} = 2^{6s+3m} - 3 \cdot 2^{6s+3m-4} - 5 \cdot 2^{6s+3m-5} = 21 \cdot 2^{6s+3m-5}.\]

By (12.35) with $j = s, k > 2s + m$ we get

(17.46)\[\Gamma_{2s+m}^{s+m} \times (k+1) - \Gamma_{2s+m}^{s+m} \times k = 3 \cdot 2^{2k+2s+m-2} - 3 \cdot 2^{k+4s+2m-4} \text{ if } k > 2s + m\]

From (17.46), (17.45) we deduce

\[
\sum_{j=2s+m+1}^{k} \left( \Gamma_{2s+m}^{s+m} \times (j+1) - \Gamma_{2s+m}^{s+m} \times j \right) = \sum_{j=2s+m+1}^{k} 3 \cdot 2^{2j+2s+m-2} - 3 \cdot 2^{j+4s+2m-4}
\]

\[
\Leftrightarrow \sum_{j=2s+m+2}^{k+1} \Gamma_{2s+m}^{s+m} \times j - \sum_{j=2s+m+1}^{k} \Gamma_{2s+m}^{s+m} \times j = 2^{2k+2s+m} - 3 \cdot 2^{k+4s+2m-3} - 5 \cdot 2^{6s+3m-3}
\]

\[
\Leftrightarrow \Gamma_{2s+m}^{s+m} \times (k+1) = 2^{2k+2s+m} - 3 \cdot 2^{k+4s+2m-3} - 5 \cdot 2^{6s+3m-3} + 21 \cdot 2^{6s+3m-5}
\]

(17.47)\[\Gamma_{2s+m}^{s+m} \times (k+1) = 2^{2k+2s+m} - 3 \cdot 2^{k+4s+2m-3} + 2^{6s+3m-5} \text{ if } k > 2s + m.
\]

By (17.45) we see that (17.47) holds for $k = 2s+m$, then (17.47) holds for $k \geq 2s + m$.

From (16.48) with $k \to k - m$, and (17.47) we obtain

\[
2^{2s+m-2} \cdot \left[ \Gamma_{2s+m}^{s+m} \times (k-2(s-1)) \right] = 2^{6s+3m-6} \cdot [2^{2(k-m)-4} + 3 \cdot 2^{k-m-2} + 2] = 2^{2k+m+2s-2} - 3 \cdot 2^{k+2m+4s-4} + 2^{6s+3m-5} = \Gamma_{2s+m}^{s+m} \times k.
\]

\[
\square
\]

18. PROOF OF THEOREMS 3.1, 3.2, . . . , 3.14, 3.15

18.1. PROOF OF THEOREM 3.1. Follows from (1.1) and (1.19).

18.2. PROOF OF THEOREM 3.2. Follows from Lemma 5.13.

18.3. PROOF OF THEOREM 3.3. Follows from Lemma 8.3.

18.4. PROOF OF THEOREM 3.4. Follows from Lemma 8.4.
18.5. PROOF OF THEOREM 3.5. Follows from Lemma 8.5.

18.6. PROOF OF THEOREM 3.6. We have
- (3.10) follows from (11.21).
- (3.11) follows from (12.1).
- (3.12) follows from (12.15).
- (3.13) follows from (12.20).
- (3.14) follows from (12.21).

18.7. PROOF OF THEOREM 3.7. Follows from Lemma 15.2.

18.8. PROOF OF THEOREM 3.8. We have
- (3.19) follows from (17.17).
- (3.20) follows from (16.17).
- (3.21) follows from (16.18) with $k \mapsto k - m$.
- (3.22) follows from (17.20).

18.9. PROOF OF THEOREM 3.9. We have in the following cases:

- From (13.24) we get $\Gamma_i^{[s]} \times k$ in the case $1 \leq i \leq s - 1$, $k > i$.
- From (13.2) we get $\Gamma_i^{[s]} \times k$ in the case $i = s$, $k > s$.
- From (16.10), (16.18) and (16.19) we deduce $\Gamma_i^{[s]} \times k$ in the case $s + 1 \leq i \leq 2s$, $k > i$.

18.10. PROOF OF THEOREM 3.10. We have in the following two cases:

- From (11.26) we get $\Gamma_i^{[s]} \times i$ in the case $1 \leq i \leq s$.
- From (16.16), (16.17) we deduce $\Gamma_i^{[s]} \times i$ in the case $s + 1 \leq i \leq 2s$.

18.11. PROOF OF THEOREM 3.11. We have in the following cases:

- From (13.24) we get $\Gamma_i^{[s + 1]} \times k$ in the case $1 \leq i \leq s - 1$, $k > i$.
- From (13.24) we get $\Gamma_i^{[s + 1]} \times k$ in the case $i = s$, $k > s$.
- From (13.26) we get $\Gamma_i^{[s + 1]} \times k$ in the case $i = s + 1$, $k > s + 1$.
- From (17.19) with $j = i - s - 2$ we get $\Gamma_i^{[s + 1]} \times k$ in the case $s + 2 \leq i \leq 2s$, $k > i$.
- From (17.20) with $m = 1$ we get $\Gamma_i^{[s + 1]} \times k$ in the case $i = 2s + 1$, $k > 2s + 1$.

18.12. PROOF OF THEOREM 3.12. We have in the following two cases:

- From (13.25) and (11.26) with $m = 1$ and $k \mapsto i$ we get $\Gamma_i^{[s]} \times i$ in the case $1 \leq i \leq s + 1$.
- From (17.18) with $m = 1$ and $j = i - s - 2$ we deduce $\Gamma_i^{[s + 1]} \times i$ in the case $s + 2 \leq i \leq 2s + 1$.

18.13. PROOF OF THEOREM 3.13. We have in the following cases:

- From (13.21) we get $\Gamma_i^{[s + m]} \times k$ in the case $1 \leq i \leq s - 1$, $k > i$.
- From (13.36) we get $\Gamma_i^{[s + m]} \times k$ in the case $i = s$, $k > s$. 
From (15.40) with \( l = i - s \) we get \( \Gamma_{\left\lfloor \frac{s}{s+m} \right\rfloor}^{\times k} \) in the case \( s + 1 \leq i \leq s + m - 1 \), \( k > i \).

From (15.41) we get \( \Gamma_{\left\lfloor \frac{s}{s+m} \right\rfloor}^{\times k} \) in the case \( i = s + m \), \( k > i \).

From (17.19) with \( j = i - s - m - 1 \) we get \( \Gamma_{\left\lfloor \frac{s}{s+m} \right\rfloor}^{\times k} \) in the case \( s + m + 1 \leq i \leq 2s + m - 1 \), \( k > i \).

From (17.20) we get \( \Gamma_{\left\lfloor \frac{s}{s+m} \right\rfloor}^{\times k} \) in the case \( i = 2s + m \), \( k > i \).

18.14. **PROOF OF THEOREM 3.14.** We have in the following three cases:

From (11.20) with \( k \to i \) and (13.47) we get \( \Gamma_{\left\lfloor \frac{s}{s+m} \right\rfloor}^{\times i} \) in the case \( 1 \leq i \leq s + 1 \).

From (15.39) with \( l \to i \) and (17.2) we get \( \Gamma_{\left\lfloor \frac{s}{s+m} \right\rfloor}^{\times i} \) in the case \( s + 2 \leq i \leq s + m + 1 \).

From (17.18) with \( j = i - s - m - 1 \) we get \( \Gamma_{\left\lfloor \frac{s}{s+m} \right\rfloor}^{\times i} \) in the case \( s + m + 2 \leq i \leq 2s + m \).

18.15. **PROOF OF THEOREM 3.15.** Follows from Lemma 4.13.

**References**

[1] Daykin David E, Distribution of Bordered Persymmetric Matrices in a finite field J. reine angew. Math, 203 (1960), 47-54.

[2] Hayes , D.R, The expression of a polynomial as a sum of three irreducibles Acta Arith., 11 (1966), 461-488.

[3] Landsberg, G Ueber eine Anzahlbestimmung und eine damit zusammenhangende Reihe, J. reine angew.Math, 111 (1893), 87-88.

[4] Cherly, J Exponential sums and rank of persymmetric matrices over \( F_2 \), arXiv: 0711.1306

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