KdV soliton interactions: a tropical view

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Abstract. Via a “tropical limit” (Maslov dequantization), Korteweg-deVries (KdV) solitons correspond to piecewise linear graphs in two-dimensional space-time. We explore this limit.

1. Introduction

The “tropical limit” (Maslov dequantization) of soliton solutions of the (scalar, real) KdV equation

\[ 4 u_t = u_{xxx} + 6 u u_x, \]

where a subscript indicates a partial derivative with respect to an independent variable, describes them as piecewise linear graphs in two-dimensional space-time. The corresponding exploration in this work is based on techniques developed in [1, 2], but our presentation will be fairly self-contained.

The most striking property of KdV solitons is surely the well-known fact that, after an interaction, they regain their “identity” (amplitude, width and speed) and only experience a shift in space. But what really happens during the interaction of solitons? After the groundbreaking work of Kruskal and Zabusky [3], this question has been addressed again and again [4–21]. Some authors argued that solitons always pass through one another (see, e.g., [12, 15]). Others suggested that solitons exchange their identities during the interaction (see, e.g., [5, 6]). Based on a certain decomposition of the 2-soliton solution, some proposed an intermediate wave that transfers energy between the two solitons [16, 20, 21]. There have been several attempts to “individualize” the a priori asymptotically (i.e., for \( t \ll 0 \)) defined solitons also during an interaction. Solitons do not really behave like classical particles, however. As localized disturbances of a continuous medium, they possess a wavelike nature. Instead of speaking of the same incoming \((t \ll 0)\) and outgoing \((t \gg 0)\) soliton, it is more adequate to speak of “instances” of a certain soliton state. Once solitons start to interact, they lose their individuality. This is also what the tropical limit shows: two KdV solitons (which can be regarded as parallel KP line solitons, see [1], Example 4.5) interact by exchanging a “virtual soliton”, see Section 3.4. The present work explores more generally KdV soliton interactions in the tropical limit. The analogy with a quantum scattering theory (also see [18–20]) is striking. But here a kind of second quantization is not necessary since the KdV equation (and any other evolution equation possessing solitons) is already a many-“particle” model. Solitons are the asymptotically free particles, and their interaction can be understood as an exchange of virtual particles. We should stress, however, that we are not attempting to interpret KdV as a quantum theory.
Remark 2.2. After having arranged (2.1) by a shift of the KdV hierarchy variables, an \( M \)-soliton solution is invariant under reflection of all independent variables, i.e., \( t^{(k)} \mapsto -t^{(k)}, k = 1, \ldots, N \).
Example 2.3. For the 2-soliton solution, setting \( N = 2 \), we obtain
\[
u(x, t) = 8(p_2^2 - p_1^2) \frac{p_1^2 \sinh^2 \theta_2 + p_2^2 \cosh^2 \theta_1}{(p_2 - p_1) \cosh(\theta_1 + \theta_2) + (p_2 + p_1) \cosh(\theta_1 - \theta_2)} .
\]
Since we set \( c_1 = c_2 = 0 \), we have \( u(x, t) = u(-x, -t) \), so that the solution is adapted to an obvious symmetry of the KdV equation. In particular, \( u(x, 0) \) is symmetric about \( x = 0 \) (also see Theorem 1 in [21], where the expressions for the symmetry event are more complicated due to a different form of the KdV equation and a different parametrization of the 2-soliton solution).

We find \( u_{xx}(0, 0) = -4(p_2 - p_1)(p_2 + p_1)(p_2^2 - 3p_1^2) \). Hence \( u(x, 0) \) has a minimum at \( x = 0 \) if \( p_2/p_1 < \sqrt{3} \) and a maximum if \( p_2/p_1 > \sqrt{3} \) [4,10,21], and we find \( u(0, 0) = 2(p_2^2 - p_1^2) \).

3. Tropical limit of KdV solitons

Let \( U_B \) be the region in \( \mathbb{R}^2 \) where the phase \( \Theta_B \) dominates all others, i.e.,
\[
U_B = \{ (x, t) \in \mathbb{R}^2 \mid \max\{\Theta_A(x, t) \mid A \in \{ -1, 1 \}^M \} = \Theta_B(x, t) \} ,
\]
considered at fixed values of the higher KdV hierarchy variables. \( U_B \) is connected, it may be empty for some \( B \). In a non-empty set \( U_B \) and sufficiently far away from the boundary, the approximation
\[
\log \tau \simeq \max\{\Theta_A \mid A \in \{ -1, 1 \}^M \}
\]
is valid, so that \( u \) vanishes (since \( \Theta_B \) is linear in \( x \)). As a consequence, \( u \) is localized along the boundary lines of non-empty dominating phase regions. The tropical limit of the KdV soliton solution is the piecewise linear planar graph consisting of such boundary lines, and the amplitude \( u \) on these lines. The structure of this graph is determined by the intersections of the dominating phase regions. In the following, we write \( U_{AB} := U_A \cap U_B \).

Remark 3.1. A precise formulation of what we call the tropical limit of KdV solitons is obtained by using the Maslov dequantization formula (see, e.g., [23])
\[
\lim_{\epsilon \to 0} \epsilon \sum_{A \in \{ -1, 1 \}^M} e^{\Theta_A/\epsilon} = \max\{\Theta_A \mid A \in \{ -1, 1 \}^M \} ,
\]
which replaces the operation of addition (of exponentials) by the maximum function (applied to the phases). This is a familiar step in “tropicalization”. It is usually accompanied by also replacing multiplication by addition (the “tropical product”).

Remark 3.2. If the logarithmic terms \( \delta_A \) were negligible, then we would have \( \log \tau \simeq \max_{A \in \{ -1, 1 \}^M} \{ \sum_{j=1}^M \alpha_j \theta_j \} = \sum_{j=1}^M |\theta_j| \), and the tropical limit of the KdV soliton would be given by the superimposition of the space-time lines corresponding to the constituent single solitons. (3.1) shows that this simplified limit corresponds to introducing “slow variables” via \( x \mapsto x/\epsilon \) and \( t \mapsto t/\epsilon \) in a soliton solution. This simply maps the latter to the corresponding solution of the \( \epsilon \)-dependent KdV equation \( u_t - (3/2) u u_x = \epsilon^2 u_{xxx} \), which formally approaches its “dispersionless” (or “quasiclassical”) limit, the inviscid Burgers (or Hopf, or Riemann) equation \( u_t - (3/2) u u_x = 0 \), as \( \epsilon \to 0 \). The \( \epsilon \)-dependent soliton solution does not tend to a solution of the inviscid Burgers equation, however. In the limit, KdV solitons “disappear” in the sense that their support becomes a set of measure zero in space-time, while \( u \) retains a finite value. The associated initial data become infinitely steep, so there is no corresponding local solution of the inviscid Burgers equation via the method of characteristics. It is also not adequate to think of the (simplified) tropical limit as a kind of non-smooth solution of the inviscid Burgers equation (e.g., similar to those obtained via “front tracking” [24]). This is so because irrespective how...
small $\epsilon$ is, the corresponding $\epsilon$-dependent solution necessarily retains dispersion since it remains solitonic. We have to conclude that the tropical limit cannot be regarded as a dispersionless limit.

Introducing

$$p_A := \sum_{j=1}^{M} \alpha_j p_j, \quad q_A := \sum_{j=1}^{M} \alpha_j p_j^3, \quad c_A := \delta_A + \sum_{k=3}^{N} \sum_{j=1}^{M} \alpha_j p_j^{2k-1} t^{(k)},$$

where $A = (\alpha_1, \ldots, \alpha_M)$, we have

$$\Theta_A - \Theta_B = (p_A - p_B) x + (q_A - q_B) t + c_A - c_B.$$  \hspace{1cm} (3.2)

There is a line at which two phases $\Theta_A$ and $\Theta_B$ “meet”, i.e., where $\Theta_A = \Theta_B$. It is given by

$$x = x_{AB}(t) := \frac{-q_A - q_B}{p_A - p_B} t - \frac{c_A - c_B}{p_A - p_B},$$  \hspace{1cm} (3.3)

assuming that $p_A \neq p_B$, and (3.2) can be written as

$$\Theta_A - \Theta_B = (p_A - p_B)(x - x_{AB}(t)).$$  \hspace{1cm} (3.4)

If $(x, t) \notin U_B$, there is a phase $\Theta_A$ such that $\Theta_A(x, t) > \Theta_B(x, t)$. In this case we say that the phase $\Theta_B$ is non-visible at the event $(x, t)$. An immediate consequence of the last identity is

$$p_A > p_B \implies \Theta_B(x, t) \text{ is non-visible for } x > x_{AB}(t), \quad \Theta_A(x, t) \text{ is non-visible for } x < x_{AB}(t).$$

Near an event on the boundary line (3.3) that is not a higher order coincidence of phases, we have $\tau \simeq e^{\Theta_A} + e^{\Theta_B}$ and thus $u \simeq 2(p_A - p_B)^2 e^{\Theta_A + \Theta_B} / (e^{\Theta_A} + e^{\Theta_B})^2$. Using $\Theta_A = \Theta_B$, this becomes $u \simeq \frac{1}{2} (p_A - p_B)^2$ (also see Appendix D in [1]).

Remark 3.3. Since the $p$’s are positive and distinct, for $M = 2$ we have $p_A \neq p_B$ if $A \neq B$. For $M > 2$, the latter condition holds if $p_1, \ldots, p_M$ are linearly independent over $\{\pm 1\}$.

3.1. $x$-asymptotics

For fixed $t$ and sufficiently large $x$ (we write $x \gg 0$), (3.4) implies $\Theta_{(A,...,1)} > \Theta_A$ for all $A \in \{-1, 1\}^M$, $A \neq (1, \ldots, 1)$. Furthermore, for $x \ll 0$ we have $\Theta_{(-1,...,-1)} > \Theta_A$ for all $A \in \{-1, 1\}^M$, $A \neq (-1, \ldots, -1)$. Hence $U_{(1,\ldots,1)}$ and $U_{(-1,\ldots,-1)}$ are dominating phase regions for $x \gg 0$, respectively $x \ll 0$.

3.2. Triple phase coincidences

At a triple phase coincidence three different phases satisfy $\Theta_A = \Theta_B = \Theta_C$. If

$$p_{ABC} := p_A(q_B - q_C) + p_B(q_C - q_A) + p_C(q_A - q_B) \neq 0,$$

then a corresponding event occurs at the time

$$t_{ABC} := -p_{ABC}^{-1} \left(p_A(c_B - c_C) + p_B(c_C - c_A) + p_C(c_A - c_B)\right).$$

Its $x$-coordinate is given by $x_{ABC} = x_{AB}(t_{ABC})$. We find that

$$x_{AC}(t) - x_{AB}(t) = -\frac{p_{ABC}}{(p_B - p_A)(p_C - p_A)} (t - t_{ABC}),$$
and
\[
\Theta_A - \Theta_C = (p_A - p_C)(x - x_{AB}(t)) - \frac{p_{ABC}}{p_B - p_A}(t - t_{ABC}) ,
\]
so that
\[
\Theta_A - \Theta_C = \frac{p_{ABC}}{p_A - p_B}(t - t_{ABC}) \quad \text{on the line } x = x_{AB}(t) .
\]
(3.5)

As a consequence, the half-line
\[
\{ x = x_{AB}(t) \mid t \geq t_{ABC} \} \text{ is non-visible if } \frac{p_{ABC}}{p_A - p_B} \lesssim 0 .
\]

At a triple phase event, \( \tau \simeq e^{\Theta_A} + e^{\Theta_B} + e^{\Theta_C} \), hence \( u \simeq \frac{1}{9}(p_A^2 + p_B^2 + p_C^2 - p_{AB} - p_{AC} - p_{BC}) \).

3.3. Appearances of single solitons and \( t \)-asymptotics

For \( A = (\alpha_1, \ldots, \alpha_M) \), let \( A_{(k)} := (\alpha_1, \ldots, \alpha_{k-1}, -\alpha_k, \alpha_{k+1}, \ldots, \alpha_M) \). On \( U_{AA_{(k)}} \) we obtain the single soliton expression \( \tau \simeq e^{-k} + e^k \), up to a factor that drops out in the expression for \( u \). According to (3.3), the boundary line between the two phases \( \Theta_A \) and \( \Theta_{A_{(k)}} \) is given by
\[
x = x_{AA_{(k)}} = -\frac{1}{2\alpha_k p_k} (\delta_A - \delta_{A_{(k)}}) - p_k^2 t - \sum_{j=3}^{N} p_{k_j}^{2j-2} t^{(j)} .
\]
Hence, a soliton (i.e., a visible part of such a line), moves from right to left along the \( x \)-axis. Its “height” is \( u \simeq 2p_k^2 \). Furthermore,
\[
x_{AA_{(k)}} - x_{BB_{(k)}} = -\frac{1}{2\alpha_k p_k} (\delta_A - \delta_{A_{(k)}}) + \frac{1}{2\beta_k p_k} (\delta_B - \delta_{B_{(k)}})
\]
(3.6)
is constant. For \( B \neq A, A_{(k)} \), the lines given by \( x = x_{AA_{(k)}} \) and \( x = x_{BB_{(k)}} \) are thus parallel.

There are \( 2^{M-1} \) such lines (for fixed \( k \)), since this is the number of different pairs \( A, A_{(k)} \). It is natural to interpret the visible segments as appearances of the \( k \)th soliton.

Now we show that the \( k \)th soliton is visible for large \( |t| \). We consider (3.5) on the line \( x = x_{AA_{(k)}}(t) \), with
\[
\frac{p_{AA_{(k)}}}{p_A - p_{A_{(k)}}} = p_k^2 (p_C - p_A) - (q_C - q_A) = \sum_{j \neq k} (\gamma_j - \alpha_j) p_j (p_k^2 - p_j^2) ,
\]
\[
= \sum_{j < k} (\gamma_j - \alpha_j) p_j (p_k^2 - p_j^2) + \sum_{j > k} (\alpha_j - \gamma_j) p_j (p_k^2 - p_j^2) .
\]
This is different from zero if \( C = (\gamma_1, \ldots, \gamma_M) \) is different from \( A \) and \( A_{(k)} \). (3.5) implies
\[
A = (1, \ldots, 1, \alpha_k, -1, \ldots, -1) \quad \Rightarrow \quad p_{AA_{(k)}}/|p_A - p_{A_{(k)}}| < 0 \quad \forall C \in \{-1,1\}^M
\]
\[
\Rightarrow \quad \{ x = x_{AA_{(k)}}(t) \mid t \ll 0 \} \quad \text{is visible}
\]
\[
A = (-1, \ldots, -1, \alpha_k, 1, \ldots, 1) \quad \Rightarrow \quad p_{AA_{(k)}}/|p_A - p_{A_{(k)}}| > 0 \quad \forall C \in \{-1,1\}^M
\]
\[
\Rightarrow \quad \{ x = x_{AA_{(k)}}(t) \mid t \gg 0 \} \quad \text{is visible}
\]
(using $0 < p_1 < p_2 < \cdots < p_M$). Moreover, for any other $A \in \{-1,1\}^M$ there is a $C \in \{-1,1\}^M$ such that, according to (3.5), $\Theta_C > \Theta_A$ for $t \ll 0$, respectively $t \gg 0$, so that no further boundary lines are visible for $|t| \gg 0$. (3.6) implies that

$$x(-1, \ldots, 1) < x_0 < \cdots < x(1, \ldots, 1)$$

for $t \ll 0$,

while we have

$$x(-1, \ldots, 1) < x_0 < \cdots < x(1, \ldots, 1)$$

for $t \gg 0$.

We conclude that, as time proceeds from $-\infty$ to $+\infty$, the solitons reappear in reversed order. Recalling the results in Section 3.1, we thus proved the asymptotic structure displayed in Fig. 1.

In particular, this implies that a phase $\Theta_A$, where $A$ is not of the form $(-1, \ldots, -1)$, $(1, \ldots, 1)$, $(1, 1, \ldots, 1, -1, -1, \ldots, -1)$ or $(-1, \ldots, -1, 1, \ldots, 1)$, can only be visible in a bounded region of the $xt$-plane. Moreover, passing in the asymptotic region $|t| \gg 0$ from left to right along the $x$-axis, from one phase $\Theta_A$ to the next, say $\Theta_A(1)$, requires $\alpha_k = -1$. Other transitions can therefore only occur in case of virtual solitons (which are bounded in space-time).

Figure 1. Asymptotic structure of an $M$-soliton KdV solution in the $xt$-plane. Time flows from bottom to top. A number $k$ refers to the $k$th soliton.

3.4. Two solitons

We achieve some simplification of expressions in the following by using the notation $\bar{1} := -1$, and writing, e.g., $\bar{1} \ldots \bar{1} \ldots \bar{1}$ instead of $(-1, \ldots, -1, 1, \ldots, 1)$. The tropical approximation of the 2-soliton solution is given by $\log \tau \approx \max\{\Theta_{\bar{1}1}, \Theta_{1\bar{1}}, \Theta_{11}\}$ with

$$\Theta_{\bar{1}1} = -\theta_1 - \theta_2 + \log(p_2 - p_1), \quad \Theta_{1\bar{1}} = \theta_1 - \theta_2 + \log(p_1 + p_2),$$

$$\Theta_{11} = -\theta_1 + \theta_2 + \log(p_1 + p_2), \quad \Theta_{11} = \theta_1 + \theta_2 + \log(p_2 - p_1),$$

and $\theta_j = p_j (x + p_j^2 t), 0 < p_1 < p_2$. Here we set $N = 2$. We recall that $\Theta_{\bar{1}1}$ dominates (all other phases) for $x \gg 0$, whereas $\Theta_{1\bar{1}}$ dominates for $x \ll 0$. There are six boundary lines. Those corresponding to asymptotic solitons are

1st soliton: $x = x_{11,11}(t) = -p_1^2 t - \frac{\ell}{2p_1}$ for $t \ll 0$ branch,

$x = x_{11,11}(t) = -p_1^2 t + \frac{\ell}{2p_1}$ for $t \gg 0$ branch,

shift: $x_{11,11}(t) - x_{1\bar{1}\bar{1}}(t) = \frac{\ell}{2p_1}$ where $\ell := \log \frac{p_2 + p_1}{p_2 - p_1} > 0$

2nd soliton: $x = x_{1\bar{1}\bar{1}}(t) = -p_1^2 t + \frac{\ell}{2p_2}$ for $t \ll 0$ branch,

$x = x_{1\bar{1}\bar{1}}(t) = -p_1^2 t - \frac{\ell}{2p_2}$ for $t \gg 0$ branch,

shift: $x_{\bar{1}1,11}(t) - x_{1\bar{1}\bar{1}}(t) = \frac{\ell}{2p_1}$. 


The constant shifts exactly correspond to the well-known (asymptotically determined) phase shifts, which are the only witnesses of an interaction of KdV solitons. Further boundary lines:

\[ x = x_{\overline{11},11}(t) = -\frac{1}{p_1 + p_2} (p_1^2 + p_2^2) t = -(p_1^2 - p_1 p_2 + p_2^2) t, \]
\[ x = x_{11,\overline{11}}(t) = -(p_1^2 + p_1 p_2 + p_2^2) t. \]

Triple phase coincidences occur at the times

\[ t_{\overline{11},11,11} = \frac{\ell}{2 p_1 p_2 (p_2 + p_1)}, \quad t_{11,\overline{11},11} = -\frac{\ell}{2 p_1 p_2 (p_2 - p_1)}, \]
\[ t_{11,\overline{11},11} = -\frac{\ell}{2 p_1 p_2 (p_2 - p_1)}, \quad t_{\overline{11},11,11} = \frac{\ell}{2 p_1 p_2 (p_2 - p_1)}. \]

These times are ordered as follows: \( t_{\overline{11},11,11} < t_{11,\overline{11},11} < 0 < t_{11,\overline{11},11} < t_{\overline{11},11,11}. \) Since

\[ \Theta_{11} - \Theta_{\overline{11}} = 2 \theta_2 - \ell = -2 \ell < 0 \quad \text{on} \quad x = x_{\overline{11},11}(t), \]
\[ \Theta_{11} - \Theta_{11} = 2 \theta_1 - \ell = -2 \ell < 0 \quad \text{on} \quad x = x_{11,\overline{11}}(t), \]

the triple phase events at \( t_{\overline{11},11,11} \) and \( t_{11,\overline{11},11} \) are non-visible. At the remaining two triple phase events \( (x_{11,\overline{11},11}, t_{\overline{11},11,11}) \) and \( (x_{11,\overline{11},11}, t_{11,\overline{11},11}) \), we have \( \Theta_{11} - \Theta_{\overline{11}} = 2 \ell > 0 \), respectively \( \Theta_{\overline{11}} - \Theta_{11} = 2 \ell > 0 \), so that these events are visible. Moreover, we have

\[ x_{\overline{11},11,11} - x_{11,\overline{11},11} = \frac{p_1^2 + p_1 p_2 + p_2^2}{p_1 p_2 (p_1 + p_2)} \ell > 0. \]

Furthermore,

\[ \Theta_{11} - \Theta_{\overline{11}} = 2 p_1 p_2 (p_2 - p_1) t - \ell \]
\[ \Theta_{\overline{11}} - \Theta_{11} = -2 p_1 p_2 (p_2 - p_1) t - \ell \quad \text{on} \quad x = x_{11,\overline{11}}(t), \]

implies \( \Theta_{11} < \Theta_{\overline{11}} \) for \( t < 0 \) and \( \Theta_{11} < \Theta_{\overline{11}} \) for \( t > 0 \) on the boundary line \( x = x_{11,\overline{11}}(t) \), so that the whole line is non-visible. We thus arrive at the situation described in Fig. 2. Two solitons interact by exchanging a “virtual soliton”.

### 3.5. Three solitons

Setting \( N = 3 \), we now have \( \theta_j = p_j (x + p_j^2 t + p_j^4 s) \), \( j = 1,2,3 \), with \( s := t^{(3)} \). Recall that \( 0 < p_1 < p_2 < p_3 \). The tropical approximation of the 3-soliton solution is given by log \( r \simeq \max\{ \Theta_0, \Theta_1, \ldots, \Theta_7 \} \), where \( \Theta_0 := \Theta_{\overline{111}}, \Theta_1 := \Theta_{1\overline{1}1}, \Theta_2 := \Theta_{11\overline{1}}, \Theta_3 := \Theta_{111}, \Theta_4 := \Theta_{\overline{1}11}, \Theta_5 := \Theta_{1\overline{1}1}, \Theta_6 := \Theta_{11\overline{1}}, \Theta_7 := \Theta_{111} \). Only the \( \Theta_2 \)- and \( \Theta_3 \)-regions are bounded in the \( xt \)-plane. In the following, we present results derived with the help of computer algebra. It turns out that there are no visible coincidences of more than four of these phases.

Among the \( \binom{3}{3} = 70 \) a priori possible coincidences of four phases, the following can be ruled out: \{0,1,2,3\}, \{0,1,4,5\}, \{0,1,6,7\}, \{0,2,4,6\}, \{0,2,5,7\}, \{0,3,4,7\}, \{1,2,5,6\}, \{1,3,4,6\}, \{1,3,5,7\}, \{2,3,4,5\}, \{2,3,6,7\}, \{4,5,6,7\}, where a number \( i \) represents the phase \( \Theta_i \). Of the remaining 58 4-phase coincidences, most are non-visible (for all values of the parameters). They can only occur if \( s = s_{ijkl} \), where \( s_{ijkl} \) is completely determined in terms of the parameters \( p_m \):

\[
s_{ijkl} = \frac{a_{ijkl}}{2 p_2 (p_2^2 - p_1^2) (p_3^2 - p_2^2)} + \frac{b_{ijkl}}{2 p_1 (p_2^2 - p_1^2) (p_3^2 - p_2^2)} + \frac{c_{ijkl}}{2 p_3 (p_3^2 - p_2^2) (p_3^2 - p_1^2)},
\]
Figure 2. To the left is a space-time plot of the tropical limit graph of a 2-soliton KdV solution \((p_1 = 1, p_2 = 1.5)\). The \(x\)-axis is horizontal, time proceeds from bottom to top. The colored regions are dominating phase regions. At \(t = t_{111111}\) soliton 2 splits into an instance of soliton 1 and a “virtual soliton” \(\bar{1}\), formally assigning to it the interpretation of a bound state composed of an antisoliton \(\bar{1}\) and a soliton 2. At \(t = t_{111111}\) it merges with soliton 1 to create a new instance of soliton 2. The second plot shows all boundary lines between pairs of phases. The whole green line, passing through the middle point, is non-visible. The third plot displays the tropical limit superimposed on a contour plot of the KdV solution. The last is a plot of the KdV solution \(u(x,t)\) over the \(xt\)-plane.

where

\[
\begin{align*}
    a_{0124} &= \ell_{12} + \ell_{23} \\
    a_{1235} &= \ell_{23} - \ell_{12} \\
    a_{1245} &= -2\ell_{13} + \ell_{12} + \ell_{23}
\end{align*}
\]

\[
\begin{align*}
    b_{0124} &= -\ell_{12} - \ell_{13} \\
    b_{1235} &= \ell_{12} - \ell_{13} \\
    b_{1245} &= -\ell_{13} - \ell_{12}
\end{align*}
\]

\[
\begin{align*}
    c_{0124} &= -\ell_{13} - \ell_{23} \\
    c_{1235} &= -\ell_{13} - \ell_{23} \\
    c_{1245} &= -\ell_{13} - \ell_{23}
\end{align*}
\]

with \(\ell_{ij} := \log[(p_j + p_i)/(p_j - p_i)]\), and we find

\[
\begin{align*}
    s_{2356} &= -s_{1245} \\
    s_{2456} &= -s_{1235} \\
    s_{3567} &= -s_{0124}
\end{align*}
\]

A 4-phase coincidence described by \(\{i, j, k, l\}\) can only be visible if for each subset of three indices the corresponding 3-phase coincidence is visible. Table 1 lists all (under the stated conditions) visible 3-phase coincidences. Fig. 3 provides some more information about the 3-soliton case and Fig. 4 shows an example.

Except for some relations implied by Table 1, like \(s_{0124} < s_{1245}\), the order of the values \(s_{ijkl}\) depends in a more complicated way on the parameters \(p_j\). In particular, we can draw the following conclusions.

- Only the 4-phase coincidences corresponding to \(\{0,1,2,4\}, \{1,2,3,5\}, \{1,2,4,5\}, \{2,3,5,6\}, \{2,4,5,6\}, \{3,5,6,7\}\) are visible, for certain values of \(s\).
- \(\Theta_2\) is non-visible if either \(s > s_{215}\) and \(p_3 < \sqrt{p_1^3 + p_1 p_2 + p_2^3}\), or \(s > s_{1235}\) and \(p_3 > \sqrt{p_1^3 + p_1 p_2 + p_2^3}\) (since then all 3-phase coincidences with \(t_{ijk}, 2 \in \{i,j,k\}\), are non-visible). In particular, \(\Theta_2\) is non-visible if \(s \gg 0\).
- \(\Theta_5\) is non-visible if either \(s < s_{2356}\) and \(p_3 < \sqrt{p_1^3 + p_1 p_2 + p_2^3}\), or \(s < s_{2456}\) and \(p_3 > \sqrt{p_1^3 + p_1 p_2 + p_2^3}\). In particular, \(\Theta_5\) is non-visible if \(s \ll 0\).

Although some general results can be obtained about arbitrarily high order phase coincidences, there is hardly a chance to achieve a classification, for arbitrary \(M\), of all possible evolutions of \(M\)-soliton KdV solutions or, equivalently, parallel \(M\)-soliton KP-II solutions, in a similar way as for the case of tree-shaped KP line soliton solutions [1, 2]. The simple
Table 1. Results about 3-phase coincidences for 3-soliton solutions.

| 3-phase coincidence time | visibility condition |
|--------------------------|----------------------|
| $t_{012}, t_{024}$      | $s < s_{0124}$       |
| $t_{014}$                | $s_{0124} < s$       |
| $t_{124}$                | $s_{1245} < s$       |
| $t_{145}$                | $s_{2456} < s < s_{1245}$ |
| $t_{245}$                | $s_{1235} < s < s_{1235}$ if $p_3 < \sqrt{p_1^2 + p_1 p_2 + p_2^2}$ |
| $t_{125}$                | $s < s_{1235}$       |
| $t_{135}$                | $s_{1235} < s$       |
| $t_{235}$                | $s_{2356} < s < s_{1235}$ |
| $t_{236}$                | $s < s_{2356}$       |
| $t_{256}$                | $s_{2356} < s < s_{2456}$ if $p_3 < \sqrt{p_1^2 + p_1 p_2 + p_2^2}$ |
| $t_{356}$                | $s_{2456} < s < s_{2356}$ if $p_3 > \sqrt{p_1^2 + p_1 p_2 + p_2^2}$ |
| $t_{357}, t_{367}, s_{567}$ | $s_{2356} < s < s_{3567}$ |
| $t_{456}$                | $s_{3567} < s$       |
| $t_{246}$                | $s < s_{2456}$       |

Figure 3. According to Remark 2.2, $u(x, t, s)$ has the property $u(x, 0, 0) = u(-x, 0, 0)$. Hence $u(x, 0, 0)$ has an extremum at $x = 0$, determined by the sign of $\frac{1}{4} u_{xx}(0, 0, 0) = - (p_1^4 + 3 p_2^2 + p_3^3) + 4 (p_1^2 p_2^2 - p_1^2 p_3^2 + p_2^2 p_3^2)$. The plot displays for $p_3/p_2$ (vertical axis) versus $p_2/p_1$ (horizontal axis) the boundary between the regions where $u(x, 0, 0)$ has a minimum (light region), respectively a maximum (dark region).

Figure 4. The first plot displays the tropical limit of a 3-soliton KdV solution for $s = 0$ (and $p_1 = 0.5$, $p_2 = 0.7$, $p_3 = 0.9$). Here all 24 phases are visible. The three phase regions extending to the bottom ($t \ll 0$) are given, from left to right, by $\Theta_{III}, \Theta_{I11}, \Theta_{1II}$, those extending to the top ($t \gg 0$) by $\Theta_{III}, \Theta_{111}, \Theta_{1II}$. In the middle we have two bounded phase regions where $\Theta_{1II}$, respectively $\Theta_{111}$, dominate. The second picture shows the superimposition of the tropical limit on a contour plot of the KdV solution. The third is a plot of the KdV solution $u(x, t, s)$ at $s = 0$. 

combinatorics (higher Tamari orders), underlying the latter case, has no counterpart in case of more general KP line soliton solutions (also see, e.g., [25] for an analysis of the general KP case).

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