1. Introduction

This paper is the second in a series of papers. Let $A$ be a smooth commutative (ordinary) algebra over a field of characteristic zero $k$. Let $C$ be a small $A$-linear stable $\infty$-category, and let $\mathcal{H}_\bullet(C/A)$ be the Hochschild homology of $C$ over $A$, which is defined as an $A$-module spectrum (equivalently, a differential graded (dg) $A$-module) endowed with an action of the circle $S^1$. In the first paper [8], we construct a lift of $\mathcal{H}_\bullet(C/A)$ to an $A \otimes_k S^1$-module spectrum endowed with an $S^1$-action which is compatible with the $S^1$-action on $A \otimes_k S^1$. Here $A \otimes_k S^1 \simeq A \otimes_{A_{S^1}} A$. Since $\text{Spec} A \otimes_k S^1$ may be regarded as the (derived) loop space of $S = \text{Spec} A$, such a lift can be thought of as a deformation/extension of $\mathcal{H}_\bullet(C/A)$ along $S = \text{Spec} A \to LS = \text{Spec} A \otimes_k S^1$ with respect to $S^1$-actions. Using a lift to $LS$ we construct a $D$-module structure on the periodic cyclic homology/complex $\mathcal{H}^\bullet(C/A)$. We provided two methods for constructing a lift.

(I) The first method uses the canonical extension of factorization homology to mapping stacks. When $C$ is an stable idempotent-complete $\infty$-category over a scheme $S$, there exists a canonically defined $S^1$-equivariant extension $\mathcal{H}_\bullet(C/S)_L$, of the relative Hochschild homology (chain complex) $\mathcal{H}_\bullet(C/S)$ to the (derived) loop space $LS$. This approach is simple and easy, and the resulting object has a nice functoriality. Moreover, it admits a vast generalization to the lifts of factorization homology of $E_n$-algebras.

(II) The second method uses the algebra of the pair $(\mathcal{H}_\bullet(C/A), \mathcal{H}_\bullet(C/A))$ of Hochschild cohomology and Hochschild homology (by which we mean a chain complex/spectra computing Hochschild cohomology and Hochschild homology). The algebraic structure may be thought of as a version of Cartan calculus and is defined as an algebra over a colored topological operad called Kontsevich-Soibelman operad. An advantage of this approach is a direct relation with the pair $(\mathcal{H}_\bullet(C/A), \mathcal{H}_\bullet(C/A))$. This relation is useful. For example, if we write $\mathcal{H}_\bullet(C) \in QC_1(LS)^{S_1}$ for the lift, then the pullback of $\mathcal{H}_\bullet(C)$ to $QC_1(S \times_k LS^{S_1})$ can be described in terms of the (dg) Lie module coming from the Lie derivation and the contraction map built in the pair $(\mathcal{H}^\bullet(C/A), \mathcal{H}_\bullet(C/A))$ together with the Kodaira-Spencer morphism for $C$. Here $(S \times_k LS)^{S_1}$ is the formal stack obtained from $S \times_k LS$ by the formal completion (see Section 2.2). This structure will be applied to the study of the resulting object in the future work.

Both have their own pleasant features so that it is desirable to compare them. The main result of this paper is a comparison of the two methods. We state the main result in a naive way (see Theorem 8.1 for the precise statement):

**Theorem 1.1.** Assume that $S$ is affine and smooth over $k$. The two lifts constructed by the two methods (I) and (II) coincide. Namely, $\mathcal{H}_\bullet(C)$ and $\mathcal{H}_\bullet(C/S)_L = \mathcal{H}_\bullet(C/k)$ coincide in the $\infty$-category $QC_1(LS)^{S_1}$ of $S^1$-equivariant Ind-coherent complexes on the derived loop space $LS$.

The remarkable feature of this comparison result is that it can be thought of as Koszul duality between two methods. The first method (I) mainly uses module objects over commutative algebras ($E_\infty$-algebras). On the other hand, in the second method (II), the main data is dg Lie algebra modules over dg Lie algebras. To compare two methods, it is necessary to relate modules appearing in (I) and dg Lie algebra modules in (II). One of the key relations is Koszul duality between modules over an augmented commutative algebra $B$ and modules over Koszul dual dg Lie algebra $D_\infty(B)$: results and machinery in [7] play an important role.
2. Preliminaries

Throughout this paper, $k$ is a field of characteristic zero and $A$ is a commutative smooth (ordinary) algebra over $k$.

2.1. Convention and Notation. We use the theory of $(\infty, 1)$-categories. The principal model of $(\infty, 1)$-categories in this paper is the theory of quasi-categories extensively developed in [10], [11]. Following [10], we call quasi-categories $\infty$-categories. We use the notation in [8]. In particular, we frequently use the following symbols:

- $\Delta^n$: the standard $n$-simplex
- $\mathcal{S}$: $\infty$-category of small spaces/$\infty$-groupoids. We denote by $\mathcal{S}$ the $\infty$-category of spaces in the enlarged universe (cf. [10, 1.2.16]).
- $\mathcal{C}^\infty$: the largest Kan subcomplex of an $\infty$-category $\mathcal{C}$. Namely, $\mathcal{C}^\infty$ is the largest $\infty$-groupoid contained in $\mathcal{C}$.
- $\mathcal{C}^{\text{op}}$: the opposite $\infty$-category of an $\infty$-category. We also use the superscript “op” to indicate the opposite category for ordinary categories and enriched categories.
- $\text{Map}_\mathcal{C}(C, C')$: the mapping space from an object $C \in \mathcal{C}$ to $C' \in \mathcal{C}$ where $\mathcal{C}$ is an $\infty$-category. We usually view it as an object in $\mathcal{S}$ (cf. [10, 1.2.2]).
- $\text{Cat}^\infty$: the $\infty$-category of small $\infty$-categories. $\text{Cat}^\infty$ is the $\infty$-cartegory of large $\infty$-categories.
- $\text{Fun}(A, B)$: the function complex for simplicial sets $A$ and $B$. If $A$ and $B$ are $\infty$-categories, we regard $\text{Fun}(A, B)$ as the functor category.
- $\text{CAlg}_R$: the $\infty$-category of $R$-module spectra where $R$ is a commutative algebra object (i.e. an $E_n$-algebra object) in the $\infty$-category $\text{Sp}$ of spectra. For a symmetric monoidal $\infty$-category $\mathcal{M}$ we write $\text{CAlg}(\mathcal{M})$ for the $\infty$-category of commutative algebra objects in $\mathcal{M}$.
- $\text{Mod}_B$: the (symmetric monoidal) $\infty$-category of $B$-module objects in $\text{Sp}$ for $B \in \text{CAlg}(\text{Sp})$. We also write $\text{QC}(B)$ or $\text{QC}(\text{Spec} B)$ for $\text{Mod}_B$. Namely, $\text{QC}(B) = \text{QC}(\text{Spec} B) = \text{Mod}_B$. We denote by $\text{Perf}_R$ the full subcategory of $\text{Mod}_B$, which consists of dualizable objects. For $B \in \text{CAlg}_R = \text{CAlg}(\text{Mod}_R)$, the forgetful functor induces an equivalence $\text{Mod}_B(\text{Mod}_R) \simeq \text{Mod}_B(\text{Sp})$. If $R \in \text{CAlg}_k$ is connective, we write $\text{CAlg}^\leq_0 R$ for the full subcategory of $\text{CAlg}_R$ spanned by connective objects with respect to the standard $t$-structure on $\text{Mod}_R$. Let $\text{CAlg}^{\leq, \bullet}_k$ denote the full subcategory of $\text{CAlg}^\leq_0$ spanned by connective objects almost of finite type over $k$. Let $\text{CAlg}^{\leq, \circ}_k$ denote the full subcategory of $\text{CAlg}^{\leq, \bullet}_k$ spanned by those objects $R$ such that $H_n(R) = 0$ for $n > 0$.
- $\text{QC}: \text{CAlg}_k \rightarrow \text{Cat}^\infty$: the functor which carries $B \in \text{CAlg}_k$ to $\text{QC}(B)$. This functor sends $B \rightarrow B'$ (corresponding to $f : \text{Spec} B' \rightarrow \text{Spec} B$) to the $*$-pullback functor $f^* : \text{QC}(\text{Spec} B') \rightarrow \text{QC}(\text{Spec} B)$.
- $\mathcal{Q}C : \text{CAlg}^{\leq, \circ}_k \rightarrow \text{Cat}^\infty$: the functor which carries $B \in \text{CAlg}^{\leq, \circ}_k$ to the $\infty$-category $\mathcal{Q}C(B) = \text{Ind}(\text{Coh}(B))$ of Ind-coherent sheaves/complexes over $\text{Spec} B$. This functor sends $B \rightarrow B'$ to the $!$-pullback functor $\mathcal{Q}C(B') \rightarrow \mathcal{Q}C(B')$ given by $M \mapsto M \otimes_B \omega_B$ for each $B$, where $\omega_B$ is the $!$-pullback $p'(k)$ of $k \in \mathcal{Q}C(k)$ along $p : \text{Spec} B \rightarrow \text{Spec} k$. See [2].
- $\mathcal{T} : \text{QC} |_{\text{CAlg}^{\leq, \circ}_k} \rightarrow \mathcal{Q}C$: the natural transformation which induces $\mathcal{T}_B : \text{QC}(B) \rightarrow \mathcal{Q}C(B)$ defined by $M \mapsto M \otimes_B \omega_B$ for each $B$, where $\omega_B$ is the $!$-pullback $p'(k)$ of $k \in \mathcal{Q}C(k)$ along $p : \text{Spec} B \rightarrow \text{Spec} k$. See [2].

2.2. Formal stacks. We put $S = \text{Spec} A \in \text{Aff}_k = (\text{CAlg}^{\leq}_k)^{\text{op}}$. As in [8] we use the theory of pointed formal stacks over $A$, which was developed in [5], [3, Vol.II]. The theory generalizes the theory of formal moduli problems developed in [12, X], which should be thought of as the theory of pointed formal stacks over $k$. We refer the reader to [5], [3], [12] for detail and to [8, Section 3.4] for a brief review.

Let $\text{Lie}_{\mathcal{A}}$ be the $\infty$-category of dg Lie algebras. The $\infty$-category $\text{Lie}_{\mathcal{A}}$ is obtained from the model category of dg Lie algebras (whose fibrations are termwise surjective maps) by inverting quasi-isomorphisms (another equivalent approach is to define it as the $\infty$-category obtained from algebras over the Lie operad $\text{Lie}$). Let $\text{Art}_{\mathcal{A}}^{\text{tsz}}$ be the full subcategory of $(\text{CAlg}^{\leq}_k)_{\mathcal{A}/A} := ((\text{CAlg}^{\leq}_k)_{\mathcal{A}})_{\mathcal{A}/A}$, which is spanned by trivial square zero extensions $A = A \oplus 0 \rightarrow A \oplus M \rightarrow A$ such that $M$ is a connective $A$-module of the form $\oplus_{1 \leq i \leq n} A^{[d_i]} [d_i]$ ($r_i \geq 0, d_i \geq 0$). We note that any object $R \rightarrow A$ of $\text{Art}_{\mathcal{A}}^{\text{tsz}}$ is a trivial...
square zero extension of the form $pr_1: A \oplus M \to A$ such that $M = \oplus_{1 \leq i \leq n} A^{\oplus r_i} [p_i]$ ($p_i \geq 0$). Thus $\text{Map}_{\text{CAlg}}(A \oplus M, A)$ is a contractible space since $A$ is an ordinary commutative reduced algebra over $k$. It follows that the composite functor $\text{Art}_{\text{A}^\times} \to (\text{CAlg}_{\text{sc}}^{0})_{A//A} \to (\text{CAlg}_{\text{sc}}^{0})_{A//A} \simeq \text{CAlg}_{\text{sc}}^{0}$ is fully faithful. We can also think of $\text{Art}_{\text{A}^\times}$ as a full subcategory of $\text{CAlg}_{\text{sc}}^{0}$.

By abuse of notation, we often write $R$ for an object $A \to R \to A$ of $\text{CAlg}_{\text{sc}}^{0}/A$. Similarly, we often omit the augmentations from the notation. Let $\text{TSZ}_{\text{A}}$ denote the opposite category of $\text{Art}_{\text{A}^\times}$. A pointed formal stack over $A$ is a functor $\text{Art}_{\text{A}^\times} \to S$ satisfying a certain “gluing condition” (cf. [5], [8, Section 3.4]). The $\infty$-category $\hat{\text{St}}_{\text{A}}$ of is a full subcategory of $\text{Fun}(\text{Art}_{\text{A}^\times}, S)$. The Yoneda embedding $\text{TSZ}_{\text{A}} \hookrightarrow \text{Fun}(\text{Art}_{\text{A}^\times}, S)$ factors through $\hat{\text{St}}_{\text{A}} \subset \text{Fun}(\text{Art}_{\text{A}^\times}, S)$, and we often regard $\text{TSZ}_{\text{A}}$ as a full subcategory of $\hat{\text{St}}_{\text{A}}$. Let $\text{Free}_{\text{Lie}} : \text{Mod}_A \to \text{Lie}_A$ be the free Lie algebra functor which is a left adjoint to the forgetful functor $\text{Lie}_A \to \text{Mod}_A$. Let $\text{Mod}_{\text{A}^\times} \subset \text{Mod}_A$ be the full subcategory that consists of objects of the form $\oplus_{1 \leq i \leq n} A^{\oplus r_i} [d_i]$ ($d_i \leq -1$). Let $\text{Lie}^f_{\text{A}}$ be the full subcategory of $\text{Lie}_A$, which is the essential image of the restriction of the free Lie algebra functor $\text{Mod}^f_{\text{A}} \to \text{Lie}_A$. Thanks to [5, 1.5.6], this adjoint pair induces an adjoint pair

$$F : \text{Lie}_A \cong \hat{\text{St}}_{\text{A}} : L.$$  

which are inverse to one another. This adjoint pair extends the pair of mutually inverse functors $Ch^\bullet : \text{Lie}^f_{\text{A}} \simeq \text{TSZ}_{\text{A}} : \text{D}_{\infty}$, which is the restriction of the Koszul duality adjoint pair $Ch^\bullet : \text{Lie}_A \simeq \text{(CAlg}_{\text{A}^\times}/\text{A})^{\text{op}} : \text{D}_{\infty}$ such that the left adjoint $Ch^\bullet$ is the Chevalley-Eilenberg cochain functor. For $L \in \text{Lie}_A$, we usually write $F_L$ for the associated formal stack $F(L) \in \hat{\text{St}}_{\text{A}}^\times$.

**Example 2.1.** Let $\hat{S} \times_k S$ be the pointed formal stack obtained from the formal completion of $S \times_k S^{\text{pr}_1}$ along the diagonal $\Delta : S \to S \times_k S$. The pointed formal stack $\hat{S} \times_k S$ is defined to be the functor $\text{Art}_{\text{A}^\times} \to S$ given by $[A \to R \to A] \mapsto \text{Map}_{\text{Art}_{\text{A}^\times}/S}(\text{Spec} R, S \times_k S) \simeq \text{Map}_{\text{Art}_{\text{A}^\times}/S}(\text{Spec} R, S)$. Let $LS$ denote the (derived) loop space defined by $LS = \text{Spec} A \times_k S^1$. It has the obvious $S^1$-action and the $S^1$-equivariant morphism $* : S \to LS$ determined by $A \times_{k} S^1 \to A \times_k * \simeq A$ induced by the contraction $S^1 \to *$, where the $S^1$-action on $*$ is trivial. Let $S \times_k LS$ be the pointed formal stack obtained from $S \times_k LS^{\text{pr}_1}$ along $\text{id} \times * : S \to S \times_k LS$ by the formal completion defined in the same way.

Let $\hat{\text{St}}_{\text{A}}$ denote the $\infty$-category of pointed formal moduli problems over $A$ defined in [3]. It can be considered as a full subcategory of $(\text{Fun}(\text{CAlg}_{\text{sc}}^{0}/S), \text{Spec}_A//\text{Spec}_A)$. We will dub an object of $\hat{\text{St}}_{\text{A}}$ as a pointed $!$-formal stack over $A$ (or $S$). The Yoneda embedding $\text{TSZ}_{\text{A}} \to (\text{Fun}(\text{CAlg}_{\text{sc}}^{0}/S), \text{Spec}_A//\text{Spec}_A)$ factors through $\hat{\text{St}}_{\text{A}}$. There exists a categorical equivalence $\hat{\text{St}}_{\text{A}} \simeq \text{Lie}_A$. See [3, Vol. II, Chap.5] (also [8, Section 7.1] for the quick review). According to [8, Construction 7.2, Remark 7.4], there exists a categorical equivalence $\Theta_A : \hat{\text{St}}_{\text{A}}^\times \simeq \hat{\text{St}}_{\text{A}}$ which commutes with $\text{TSZ}_{\text{A}} \to \hat{\text{St}}_{\text{A}}^\times$ and $\text{TSZ}_{\text{A}} \leftarrow \hat{\text{St}}_{\text{A}}$. We remark that this equivalence is due to the smoothness of $A$.

**Example 2.2.** Let $(S \times_k S)^{\hat{\text{g}}}$ denote $(S \times_k S) \times_{(S \times_k S), \text{Art}} S_{\text{Art}}$, that is determined by the diagonal $\Delta : S \to S \times_k S$ (see [3, Vol.II, Chap.4] or [8, the review after Remark 7.4]). By definition, $(S \times_k S)^{\hat{\text{g}}}$ is the functor $(\text{CAlg}_{\text{sc}}^{0}/S)_{A//A} \to S$ defined by $R \mapsto (S(R) \times S(R)) \times_{(S(R), \text{Spec}(S(R)), S(R))} S(R)$. We think of it as the pointed $!$-formal stack obtained from $S \times_k S$ by taking the formal completion along the diagonal $S \to S \times_k S$. Let $LS = \text{Spec} A \times_{k} S^1 = S \times_k S \times_k S$ be the free loop space of derived scheme $S$ over $k$. Let $(S \times_k LS)^{\hat{\text{g}}}$ denote $(S \times_k LS) \times_{(S \times_k LS), \text{Art}} S_{\text{Art}}$, determined by $\text{id} \times * : S \to S \times_k LS$. The equivalence $\Theta_A : \hat{\text{St}}_{\text{A}}^\times \simeq \hat{\text{St}}_{\text{A}}$ carries $S \times_k S$ and $S \times_k LS$ to $(S \times_k S)^{\hat{\text{g}}}$ and $(S \times_k LS)^{\hat{\text{g}}}$, respectively (see [8, Proposition 7.7]). Thus, we will regard $(S \times_k S)^{\hat{\text{g}}}$ and $(S \times_k LS)^{\hat{\text{g}}}$ as the images of $S \times_k S$ and $S \times_k LS$, respectively.

3. The absolute Hochschild homology and cyclic deformations

3.1. Let $\mathcal{C}$ be a small stable idempotent-complete $\infty$-category. Let $\text{St}$ denote the $\infty$-category which consists of small stable idempotent-complete $\infty$-categories where mapping spaces are spanned by exact
functors (see e.g. [1], [8, Section 2] for detail). There exists a closed symmetric monoidal structure on St such that the internal Hom/mapping object is given by Fun^ex(−, −). Here Fun^ex(−, −) indicates the full subcategory of Fun(−, −) spanned by exact functors. For R ∈ CAlg(Sp) we consider the symmetric monoidal stable ∞-category Perf_R as a commutative algebra object in the symmetric monoidal ∞-category St. We define St_R to be Mod_{Perf_R}(St) and refer to an object of St_R as an A-linear small stable (idempotent-complete) ∞-category. Namely, by an A-linear structure on C we mean a Perf_R-module structure on C. Moreover, St_R = Mod_{Perf_R}(St) inherits a symmetric monoidal structure from that on St.

3.2. Let H{*}(-/A) : St^∞_A → (Mod^S_A)^∞ be the symmetric monoidal functor which carries an A-linear stable small stable ∞-category D to the Hochschild homology A-module spectrum H{*}(D/A) (see [6, Section 6]). We define the symmetric monoidal functor H{*}(-/k) : St^∞_k → Mod^∞_k in the same way.

We apply the construction Mod(M) → CAlg(M) (see [11, Chapter 3, Theorem 4.5.3.1]) to H{*}(-/A) : St_A → Mod^S_A to obtain

\[
\begin{array}{ccc}
\text{Mod(St}_A) & \overset{\text{Mod}(H{*}(-/A))}{\longrightarrow} & \text{Mod(Mod}^S_A) \\
\downarrow & & \downarrow \\
\text{CAlg(St}_A) & \overset{\text{CAlg}(H{*}(-/A))}{\longrightarrow} & \text{CAlg(Mod}^S_A)
\end{array}
\]

where vertical functors coCartesian fibrations.

Let C be an object of St_A. We define Mod^alg*(St_A) to be the fiber product Mod(St_A) ×_{CAlg(St_A)} CAlg_A, which is determined by the base change along CAlg_A = CAlg(Mod_A) → CAlg(St^∞_A) which carries R to Perf^∞_R (it is obtained from the construction in [11, 4.8.5.21], see also [8, Section 2]). We set Mod^alg*(St_A)^+ := Mod^alg(St_A) ×_{CAlg_A} CAlg^+_A. Put Mod^alg*(St_A)_C := Mod^alg*(St_A)^+ ×_{St_A} {C}. Let Mod^alg*(St_A) ×_{CAlg_A} CAlg^+_A → St_A be the functor which carries C_R ∈ Mod_{Perf_R}(St_A) to C_R ⊗_{Perf_R} Perf_A (see [7, Section 5.1]). For the construction up to Mod(St_A) ≃ RMod(St_A) ×_{Alg_A(St_A)} CAlg(St_A)). Consider the coCartesian fiberation

\[\text{Mod}^\text{alg}*(\text{St}_A)_C := \{C\} ×_{\text{St}_A} \text{Mod}^\text{alg}*(\text{St}_A) ×_{\text{CAlg}_A} \text{CAlg}^+_A \rightarrow \text{CAlg}^+_A.\]

This coCartesian fiberation corresponds to the functor Def^E∞(C) : CAlg^+_A → Cat^∞ informally given by [R → A] → St_R ×_{St_A} {C}. If g : Cat^∞ → S denotes the functor defined by g(V) = V^≥ (that is obtained by taking the largest groupoid/Kan complex contained in V), the composition with g determines Def^E∞(C) = g ∘ Def^E∞(C) : CAlg^+_A → S. We write Def_C : Art^{tsz}_A → S for the composite Art^{tsz}_A ↪ CAlg^+_A ↪ Cat^∞ ↪ S. Let X : Art^{tsz}_A → S be a functor, that is, a pointed formal prestack. Since X is a colimit of (TSZ_A)_{/X} → TSZ_A × Fun(Art^{tsz}_A, S), there exists a canonical equivalence

\[\text{Map}_{\text{Fun(Art}^{tsz}_A, S)}(X, \text{Def}_C) \simeq \lim_{\text{Spec } R ∈ (\text{TSZ}_A)_{/X}} \text{Def}_C(R).\]

3.3. Let H be an A-module spectrum endowed with an S^1-action, that is, an object of Mod^S_A = Fun(BS^1, Mod_A). We briefly review the notion of cyclic deformations of H. Set the coCartesian fiberation

\[\text{Mod}(\text{Mod}^S_A)_H := \{H\} ×_{\text{Mod}_A} \text{Mod}(\text{Mod}^S_A) ×_{\text{CAlg}_A} (\text{CAlg}^S_A)_{/A} \rightarrow (\text{CAlg}^S_A)_{/A}.\]

Here CAlg^S_A ≃ Fun(BS^1, CAlg_A) ≃ CAlg(Mod^S_A), and (CAlg^S_A)_{/A} denotes the overcategory over the unit algebra A endowed with the trivial S^1-action. This corresponds to the functor (CAlg^S_A)_{/A} → Cat^∞ given by [B → A] → Mod_B(Mod^S_A) ×_{Mod_A} {C}. Let Art^{tsz}_A → (CAlg^S_A)_{/A} is the composite Art^{tsz}_A ↪ CAlg^+_A ⊗_{CAlg_A} (CAlg^S_A)_{/A} which is nothing but Art^{tsz}_A ⊗_{CAlg^S_A} Fun(BS^1, (CAlg^S_k)_{/A} ⊂ (CAlg^S_A)_{/A} in Section 6). Let Def^C(H) : Art^{tsz}_A → Cat^∞ be the functor corresponding to the base change Mod(\text{Mod}^S_A)_H ×_{(\text{CAlg}^S_A)_{/A}} Art^{tsz}_A → Art^{tsz}_A. As in the case of Def_C, we define Def^C(H) : Art^{tsz}_A →
\( \tilde{S} \) to be \( g \circ \text{Def}_H^\circ(H) : \text{Art}_A^{ts} \to \text{Cat}_\infty \to \tilde{S} \). We refer to \( \text{Def}_H^\circ(H)(R) \) as the \( \infty \)-groupoid/space of cyclic deformations of \( H \) to \( R \). Suppose that \( X : \text{Art}^{ts}_A \to S \) is a functor. There exists a canonical equivalence

\[
\text{Map}_{\text{Fun}(\text{Art}^{ts}_A, \tilde{S})}(X, \text{Def}_H^\circ(H)) \simeq \lim_{\text{Spec} R \in (TSZ_A)/X} \text{Mod}_{R \otimes_A S}^1(\text{Mod}_A^{S^1})^\simeq \times_{(\text{Mod}_A^{S^1})^\simeq} \{H\}.
\]

**Definition 3.1.** Let \( QC : \text{CAlg}_A \to \text{Cat}_\infty \) for the functor corresponding to the coCartesian fibration \( \text{Mod}(\text{Mod}_A) \to \text{CAlg}_A \) (cf. Section 2.1, [11, 4.5.3.1]). Then we define \( QC_H^\circ \) to be the composite functor

\[
\text{Art}_A^{ts} \xrightarrow{\text{forget}} \text{CAlg}_A \otimes^S \text{Fun}(BS^1, \text{CAlg}_A) \xrightarrow{QC} \text{Fun}(BS^1, \text{Cat}_\infty).
\]

We also define \( QC_H^\circ : \text{Fun}(\text{Art}_A^{ts}, S)^{op} \to \text{Fun}(BS^1, \text{Cat}_\infty) \) to be the right Kan extension of \( QC_H : \text{Art}_A^{ts} \to \text{Fun}(\text{Art}_A^{ts}, S)^{op} \) (we abuse notation by using the same symbol \( QC_H^\circ \)). Using this functor we regard \( \text{Def}_H^\circ(H)(R) \) as \( (QC_H^\circ(H)(R)^{S^1} \times QC_H^\circ(H)(R)^{S^1 \times QC_H^\circ(H)} \{H\}^\simeq \).

**3.4.** Take \( H \) to be \( \mathcal{H}_\bullet(C/A) \). The symmetric monoidal functor \( \mathcal{H}_\bullet(-/A) : \text{St}_A^{\otimes} \to (\text{Mod}_A^{S^1})^{\otimes} \) gives rise to the diagram

\[
\begin{array}{ccc}
\text{Mod}_{\text{alg}}(\text{St}_A)c & \to & \text{Mod}(\text{Mod}_A^{S^1})_H \\
\downarrow & & \downarrow \\
\text{CAlg}_A & \to & (\text{CAlg}_A^{S^1})_A \\
\end{array}
\]

By [7, Lemma 6.3], \( \text{Mod}(\text{St}_A) \to \text{Mod}(\text{Mod}_A^{S^1})_H \) preserves coCartesian morphisms. It follows that \( \text{Mod}_{\text{alg}}(\text{St}_A)c \to \text{Mod}(\text{Mod}_A^{S^1})_H \) preserves coCartesian morphisms.

Write \( h := \mathcal{H}_\bullet(-/A) : \text{CAlg}_A \simeq \text{CAlg}(\text{Alg}_1(\text{Mod}_A)) \to \text{CAlg}(\text{Mod}_A^{S^1}) \simeq \text{CAlg}_A^{S^1} \). Since \( h : \text{CAlg}_A \to \text{CAlg}_A^{S^1} \) is equivalent to the functor \( \otimes^S \text{Fun}(\text{Art}_A^{ts}, S^1) : \text{CAlg}_A \to \text{CAlg}_A^{S^1} \), given by the tensor by \( S^1 \) (see [7, Lemma 3.5]), it follows that if \( i : \text{CAlg}_A \to \text{CAlg}_A^{S^1} \) is the functor which sends each \( B \) to \( B \) with the trivial \( S^1 \)-action, there exists the natural transformation \( h \to i \) induced by the contraction \( S^1 \to * \) to the one-point space. For each \( B \in \text{CAlg}_A \), \( h(B) \to i(B) \) can be identified with \( B \otimes_A S^1 \to B \otimes_A * = B \). The natural transformation \( h \to i \) naturally extends to a natural transformation \( h^+ \to i^+ \) between functors \( \text{CAlg}_A \simeq \text{CAlg}(\text{Alg}_1(\text{Mod}_A)) \to \text{CAlg}(\text{Mod}_A^{S^1})_A \) where \( h^+ \) and \( i^+ \) are functors induced by \( h \) and \( i \) in the obvious way.

We write \( U_A \) for the full subcategory of \( (\text{Aff}_k(S)_{/S})_{S \times_k S} \) which consists of those objects \( \text{Spec} C \to S \times_k S \) such that \( \text{Spec} C \in TSZ_A \). Here \( S \times_k S \) in \( \text{Aff}_k(S_{/S}) \) indicates \( S \xrightarrow{\Delta} S \times_k S \xrightarrow{pr_1} S \) having the diagonal morphism followed by the first projection. Let \( U_A \) be the full subcategory of \( (\text{Aff}_k(S)_{/S})_{S \times_k S} \) which is obtained from \( U_A \) by adding the object \( \text{id} : S \times_k S \to S \times_k S \). We note that there exists an equivalence \( U_A \simeq U_A^p \), provided that \( \dim A > 0 \). Let \( \overline{r} : U_A^p \to \text{CAlg}_A^p \) be the forgetful functor. The natural transformation \( h^+ \to i^+ \) induces \( h^+ \circ \overline{r} \to i^+ \circ \overline{r} \) which is described as \( \tau : \Delta^1 \times U_A^p \to \text{CAlg}_A^{S^1}_A \).

**Proposition 3.2.** Let \( pr_2^*(C) \) denote the base change of \( C \) along \( pr_2 : S \times_k S \to S \), that is, \( C \otimes_{\text{Perf}_A} \text{Perf}_{(A \otimes_k A)} \). Then there exists an essentially unique functor \( \sigma \) filling the diagram

\[
\begin{array}{ccc}
\text{Mod}(\text{Mod}_A^{S^1})_H & \xrightarrow{\tau} & \text{CAlg}_A^{S^1}_A \\
\uparrow & & \downarrow \\
\Delta^1 \times U_A^p & \xrightarrow{\sigma} & \text{CAlg}_A^{S^1}_A \\
\end{array}
\]

such that

1. \( \sigma \) sends the initial object \((0, S \times_k S) \) of \( \Delta^1 \times U_A^p \) to \( \mathcal{H}_\bullet(pr_2^*(C)/A) \),
2. the functor \( \sigma : \Delta^1 \times U_A^p \to \text{Mod}(\text{Mod}_A^{S^1})_H \) sends any morphism to a coCartesian morphism.
We start with the following Lemma.

**Lemma 3.3.** Let $\pi : \mathcal{P} \to \mathcal{B}$ be a coCartesian fibration between $\infty$-categories. Let $f : I \to \mathcal{B}$ and $g : I \to \mathcal{B}$ be functors between $\infty$-categories. Suppose that we are given a natural transformation $\sigma : \Delta^1 \times I \to \mathcal{B}$ from $f$ to $g$. Let $u : I \to \mathcal{P}$ be a functor such that $f = \pi \circ u$. Then there exists an essentially unique $\pi : \Delta^1 \times I \to \mathcal{P}$ which extends $u : \{0\} \times I \to \mathcal{P}$ such that (i) $\sigma = \pi \circ \pi$, and (ii) for any object $x \in I$ the induced map $\Delta = \Delta^1 \times \{x\} \to \mathcal{P}$ determines a coCartesian morphism/edge in $\mathcal{P}$.

**Proof.** Consider $\pi^1 : \text{Fun}(I, \mathcal{P}) \to \text{Fun}(I, \mathcal{B})$ induced by $\sigma$. According to [10, 3.1.2.1 (1)] this functor is a coCartesian fibration. Passing to adjoints, we have $v : \Delta^0 \to \text{Fun}(I, \mathcal{P})$ and $\tau : \Delta^1 \to \text{Fun}(I, \mathcal{B})$ which correspond to $u$ and $\sigma$, respectively. Then there exists an essentially unique coCartesian morphism $\tau : \Delta^1 \to \text{Fun}(I, \mathcal{P})$ which lies over $\tau$ and extends $v$. Then according to [10, 3.1.2.1 (2)] $\pi : \Delta^1 \times I \to \mathcal{P}$ corresponding to $\tau$ satisfies the required property (ii). $\Box$

**Lemma 3.4.** Let $f : I \to \mathcal{B}$ be a functor between $\infty$-categories. Let $\pi : \mathcal{P} \to \mathcal{B}$ be a coCartesian fibration. Suppose that $s$ is an initial object of $I$. Let $\text{Fun}_R(I, \mathcal{P})$ denote the function complex over $\mathcal{B}$. Let $\text{Fun}_R^1(I, \mathcal{P})$ be the full subcategory of $\text{Fun}_R(I, \mathcal{P})$ spanned by $u : I \to \mathcal{P}$ such that $u(0)$ is a coCartesian morphism for any morphism $\alpha$ in $I$. The evaluation at $s$ induces an equivalence $\text{Fun}_R^1(I, \mathcal{P}) \simeq \pi^{-1}(f(s))$.

**Proof.** Let $\theta : \mathcal{B} \to \overline{\text{Cat}_\infty}$ be the functor corresponding to the coCartesian fibration $\pi$ through the straightening functor (cf. [10, 3.2]). According to [10, 3.3.3.2], $\text{Fun}_R^1(I, \mathcal{P})$ can be identified with a limit of the form $\pi^{-1}(f(s))$. Consider $\text{Fun}_R^1(I, \mathcal{P})$ for $s$ an initial object, $\{s\}^{\text{op}} \to I^{\text{op}}$ is cofinal. It follows that $\text{Fun}_R^1(I, \mathcal{P}) \to \pi^{-1}(f(s))$ is a categorical equivalence. $\Box$

**Proof of Proposition 3.2.** By Lemma 3.4, there exists an essentially unique $\pi' : U_A^{\text{op}} \to \text{Mod}^{\text{alg}}(\text{St}_A)_C$ over $\text{CA}_{\text{Alg}}^{\text{op}}_A$, which carries the initial object to $\text{pr}_2^0(C)$ lying over $A \otimes_k A$ and carries any morphism in $U_A^{\text{op}}$ to a coCartesian morphism. The composition with horizontal functors in the diagram 3.1 gives rise to $\sigma_0 : \{0\} \times U_A^{\text{op}} \to \text{Mod}(\text{Mod}^{\text{alg}}_A)_H$ such that $\pi_0 \circ \sigma_0 = \pi|_{\{0\} \times U_A^{\text{op}}} \circ \pi'$ and each morphism in $\{0\} \times U_A^{\text{op}}$ maps to a coCartesian morphism $\text{Mod}(\text{Mod}^{\text{alg}}_A)_H$ and the image of $(0, S \otimes_k S)$ is $\text{H}H_*\left(\text{pr}_2^0(C)/A\right)$. Applying Lemma 3.3 to the natural transformation $\pi$ and the diagram $\sigma_0$, we obtain $\sigma : \Delta^1 \times U_A^{\text{op}} \to \text{Mod}(\text{Mod}^{\text{alg}}_A)_H$ such that $\sigma$ lies over $\pi$, and for each $T \in U_A^{\text{op}}$, $\Delta^1 \times \{T\} \to \text{Mod}(\text{Mod}^{\text{alg}}_A)_H$ determines a coCartesian morphism. The functor $\sigma$ is unique in the sense that given the fixed image $\text{H}H_*\left(\text{pr}_2^0(C)/A\right)$ of $(0, S \otimes_k S)$, the $\infty$-category classifying $\pi$ having property (2) is the contractible space (see Lemma 3.4). To see this, it is enough to prove that the functor $\sigma$ sends any morphism to a coCartesian morphism. Let $f : \Delta^1 \to \Delta^1 \times U_A^{\text{op}}$ be a morphism in $\Delta^1 \times U_A^{\text{op}}$. If $f$ factors through $\{0\} \times U_A^{\text{op}}$, then the assertion is obvious since $\pi'$ maps any morphism to a coCartesian morphism, and $\text{Mod}^{\text{alg}}(\text{St}_A)_C \to \text{Mod}(\text{Mod}^{\text{alg}}_A)_H$ preserves coCartesian morphisms. Next, consider the case when the the source $s$ of $f$ is contained in $\{0\} \times U_A^{\text{op}}$ and the target $t$ is contained in $\{1\} \times U_A^{\text{op}}$. If $f_0$ denotes the morphism $s_0 \to t_0$ in $U_A^{\text{op}}$ determined by the composite $\Delta^1 \to \Delta^1 \times U_A^{\text{op}} \to U_A^{\text{op}}$, then $f$ is equivalent to the composite of $\{0\} \times f_0 : (s_0) \to (0, t_0)$ and $\Delta^1 = \Delta^1 \times \{t_0\} \to \Delta^1 \times U_A^{\text{op}}$. The functor $\sigma$ carries both morphisms to coCartesian morphisms so that $f$ maps to a coCartesian morphism. Finally, we consider the case when $f$ factors through $\{1\} \times U_A^{\text{op}}$. As in the previous case, we write $\{1\} \times f_0 : (1, s_0) \to (1, t_0)$ for $f$. Consider the morphism $g_{s_0}$ given by $\Delta^1 \times \{s_0\} \to \Delta^1 \times U_A^{\text{op}}$. By the previous case, the composite $f \circ g_{s_0}$ and $g_{s_0}$ map to coCartesian morphisms. It follows that $\pi$ maps to a coCartesian morphism. $\Box$

**Construction 3.5.** The limits of the restriction

$$
\{i\} \times U_A^{\text{op}} \to \text{CA}_{\text{Alg}}^{\text{op}}_A \xrightarrow{\text{QC}} \overline{\text{Cat}_\infty}
$$
induces \( \lim_{\Spec C \in U_A} QC(C \otimes_A S^1) \rightarrow \lim_{\Spec C \in U_A} QC(C) \). Using the extension to \( \Delta^1 \times U_A^{op} \) and taking \( S^1 \)-invariants, we obtain the commutative diagram

\[
\begin{array}{ccc}
QC((A \otimes_k A) \otimes_A S^1)^{S^1} & \rightarrow & \lim_{\Spec C \in U_A} QC(C \otimes_A S^1)^{S^1} \\
\downarrow & & \downarrow \\
QC(A \otimes_k A)^{S^1} & \rightarrow & \lim_{\Spec C \in U_A} QC(C)^{S^1}
\end{array}
\]

(3.2)

Since the forgetful functor \( \text{Mod} (\text{Mod} S^1) \rightarrow \text{Mod} (\text{Mod} S^1) \) preserves cocartesian morphisms, it follows from Proposition 3.2 and \( [10, 3.3.3.2] \) that \( \{0\} \times U_A^{op} \rightarrow \text{Mod}(\text{Mod} S^1)_H \rightarrow \text{Mod}(\text{Mod} S^1) \) induced by \( \sigma \), which is a section of \( \{0\} \times U_A^{op} \rightarrow \text{CAlg}_A \), determines an object of \( \lim_{\Spec C \in U_A} QC(C \otimes_A S^1)^{S^1} \). We denote the object by \( \mathcal{H} \).

**Corollary 3.6.** We regard \( \mathcal{H}_* (pr_2^*(C)/A) \) as an object of \( QC((A \otimes_k A) \otimes_A S^1)^{S^1} \) (cf. Construction 3.5). Then the image of \( \mathcal{H}_* (pr_2^*(C)/A) \) in \( \Spec C \in U_A, QC(C \otimes_A S^1)^{S^1} \) is naturally equivalent to \( \mathcal{H} \). Here Perf\(^{S^1}\) module structure on \( pr_2^*(C) \) in \( \mathcal{H}_* (pr_2^*(C)/A) \) defined to be the restriction of the Perf\(^{S^1}\) module \( pr_2^*(C) \) along \( \text{Perf}_A \rightarrow \text{Perf}^{S^1}_{A \otimes_k A} \) given by \( A \simeq A \otimes_k k \rightarrow A \otimes_k A \).

**Proof.** Apply Proposition 3.2.

We consider

\[
A \otimes_k S^1 \rightarrow (A \otimes_k A) \otimes_k S^1 \rightarrow (A \otimes_k A) \otimes_A S^1
\]

in \( \text{CAlg}_k^{S^1} \). By \( \otimes_k S^1 \) and \( \otimes_A S^1 \) we mean the tensor with \( S^1 \) in \( \text{CAlg}_k \) and \( \text{CAlg}_A \), respectively. The first arrow is induced by \( k \otimes_k A \rightarrow A \otimes_A A \) determined by \( k \rightarrow A \). The second arrow is the canonical morphism. The \( A \)-module structure of \( A \otimes_k A \) in \( (A \otimes_k A) \otimes_A S^1 \) is given by \( A \simeq A \otimes_k k \rightarrow A \otimes_A A \) determined by \( k \rightarrow A \). We also note that there exists a canonical equivalence \( (A \otimes_k A) \otimes_A S^1 \simeq (A \otimes_k A) \otimes_k S^1 \) in \( \text{CAlg}_A^{S^1} \).

**Proposition 3.7.**

1. There exists a canonical equivalence \( \mathcal{H}_* (C/k) \otimes_{A\otimes_k S^1} ((A \otimes_k A) \otimes_A S^1) \simeq \mathcal{H}_* (pr_2^*(C)/A) \) in \( \text{Mod}_1 (A \otimes_k A) \otimes_{A\otimes_k S^1} (\text{Mod}_A^{S^1}) \).

2. There exists a canonical equivalence \( \mathcal{H}_* (C/k) \otimes_{A\otimes_k S^1} A \simeq \mathcal{H}_* (C/k) \) in \( \text{Mod}_1 \).

**Proof.** We first prove (1). According to \( [7, \text{Lemma 6.3}] \) there exists a canonical equivalence

\[
\mathcal{H}_* (C/k) \otimes_{A\otimes_k S^1} ((A \otimes_k A) \otimes_k S^1) \simeq \mathcal{H}_* (pr_2^*(C)/k)
\]

in \( \text{Mod}_1 (A \otimes_k A) \otimes_{k} S^1 \). By \( [8, \text{Theorem 4.13}] \) there exists a canonical equivalence

\[
\mathcal{H}_* (pr_2^*(C)/k) \otimes_{((A \otimes_k A) \otimes_{A\otimes_k S^1})} ((A \otimes_k A) \otimes_{A\otimes_k S^1}) \simeq \mathcal{H}_* (pr_2^*(C)/A)
\]

in \( \text{Mod}_1 (A \otimes_k A) \otimes_{A\otimes_k S^1} (\text{Mod}_A^{S^1}) \). Combining two equivalences we obtain \( \mathcal{H}_* (C/k) \otimes_{A\otimes_k S^1} ((A \otimes_k A) \otimes_A S^1) \simeq \mathcal{H}_* (pr_2^*(C)/A) \).

The assertion (2) is the special case of \( [8, \text{Theorem 4.27}] \).

### 4. Modular Interpretation

Let \( C \) be an \( A \)-linear stable small \( \infty \)-category (cf. \([8, \text{Section 3}]\)).

**4.1.** We begin by introducing the purpose of Section 4. We let \( T_{A/k} [-1] \) denote the dg Lie algebra that corresponds to \( S \wedge k S \) via the categorical equivalence \( \text{Lie}_A \simeq \Sigma S^* \). The underlying complex is equivalent to the \((-1\)-shifted tangent module \( T_{A/k} \) of \( \text{Spec} A \) over \( k \). Let \( T_{A/k} [-1] \) be the dg Lie algebra obtained by cotensor by \( S^1 \in S \). The Hochschild homology \( \mathcal{H}_* (C/A) \) admits the action of the dg Lie algebra \( T_{A/k} [-1] \) which we call the canonical \( T_{A/k} [-1] \)-module (cf. \([8, \text{Definition 6.3}]\)). This action is constructed by using the algebraic structure of the Hochschild pair \( (\mathcal{H}_* (C/A), \mathcal{H}_* (C/A)) \) and the Kodaira-Spencer morphism for \( C \). See \([6] \) or \([7]\) for the convention on the Hochschild cohomology \( A \)-module spectrum \( \mathcal{H}_* (C/A) \in \text{Mod}_A \) and the Hochschild homology \( A \)-module spectrum \( \mathcal{H}_* (C/A) \).
4.2. According to [8, Lemma 7.17], there is a canonical equivalence

\[ R_C : \text{Rep}(\mathbb{T}_{A/k}[-1])^{\text{S}^1} \times_{\text{Mod}_{\text{Art}}^{\text{S}^1}} \{ \mathcal{H}\mathcal{H}_*^{\text{S}^1}(C/A) \} \]

\[
\simeq \text{lim}_{\text{Spec} C \in (TS_{A})/S \times S} \text{Rep}(\mathbb{D}_\infty(C \otimes_A S^1))^{\text{S}^1} \times_{\text{Mod}_{\text{Art}}^{\text{S}^1}} \{ \mathcal{H}\mathcal{H}_*^{\text{S}^1}(C/A) \}
\]

(see Section 2.2, [5] for \( \mathbb{D}_\infty \)). If we consider the image of the canonical \( \mathbb{T}_{A/k}[-1]^{\text{S}^1} \)-module \( \mathcal{H}\mathcal{H}_*^{\text{S}^1}(C/A) \) in the \( \infty \)-category on the right side, there is its presentation in terms of cyclic deformations. Let \( \mathcal{F}_*^{\text{C}} : \text{Art}_A^{\text{tsz}} \to \mathcal{S} \) be the functor informally defined by

\[ C \mapsto (\text{Rep}(\mathbb{D}_\infty(C \otimes_A S^1))^{\text{S}^1} \times_{\text{Mod}_{\text{Art}}^{\text{S}^1}} \{ \mathcal{H}\mathcal{H}_*^{\text{S}^1}(C/A) \})^\infty, \]

where \( \text{Rep}(\mathbb{D}_\infty(C \otimes_A S^1))^{\text{S}^1} \times_{\text{Mod}_{\text{Art}}^{\text{S}^1}} \{ \mathcal{H}\mathcal{H}_*^{\text{S}^1}(C/A) \} \) is the forgetful functor (see [7, Section 8.2 and Remark 8.14] and [8, Remark 6.6] for the detail).

Let \( \text{LMod} \circ \mathbb{D}_1 : \text{Alg}_{\text{tsz}}^+ (\text{Mod}_{\text{Art}}^+) \overset{\mathbb{D}_1}{\to} \text{Alg}_{\text{tsz}}^+ (\text{Mod}_{\text{Art}})^{\text{op}} \overset{\text{LMod}}{\to} \mathcal{C} \) denote the composite where the first functor is given by the \( \mathbb{E}_1 \)-Koszul duality functor \( \mathbb{D}_1 : \text{Alg}_{\text{tsz}}^+ (\text{Mod}_{\text{Art}}^+) \to \text{Alg}_{\text{tsz}}^+ (\text{Mod}_{\text{Art}})^{\text{op}} \) (see e.g. [7], [8]) followed by the forgetful functor \( \text{Alg}_{\text{tsz}}^+ (\text{Mod}_{\text{Art}}) \to \text{Alg}_1 (\text{Mod}_{\text{Art}}) \) (here we slightly abuse notation), and the second functor \( \text{LMod} \) indicates the functor corresponding to the Cartesian fibration \( \text{LMod}(\text{Mod}_{\text{Art}}) \to \text{Alg}_1 (\text{Mod}_{\text{Art}}) \) (see [7, Section 2] or [11, 4.2.1] for the notation). Note that \( \text{Rep} : (\text{Lie}_A)^{\text{op}} \to \mathcal{C} \) is the composite of \( \text{LMod} : \text{Alg}_1 (\text{Mod}_{\text{Art}})^{\text{op}} \to \mathcal{C} \) and the universal enveloping algebra functor \( U_1 : \text{Lie}_A \to \text{Alg}_1 (\text{Mod}_{\text{Art}}) \), and there exists \( U_1 \circ \mathbb{D}_\infty \simeq \mathbb{D}_1 \) between functors \( \text{Art}^{\text{tsz}}_{\text{Art}} \to \text{Alg}_{\text{tsz}}^+ (\text{Mod}_{\text{Art}}) \). Thus, there exists a canonical equivalence \( \text{LMod} \circ \mathbb{D}_1 |_{\text{Art}^{\text{tsz}}_{\text{Art}}} \simeq \text{Rep} \circ \mathbb{D}_\infty |_{\text{Art}^{\text{tsz}}_{\text{Art}}} : \text{Art}^{\text{tsz}}_{\text{Art}} \to \mathcal{C} \).

We briefly review the Koszul duality functor

\[ \text{QC}(C) = \text{Mod}_C (\text{Mod}_{\text{Art}}) \to \text{LMod}_{\mathbb{D}_1 (C)} (\text{Mod}_{\text{Art}}) \]

for \( C \in \text{CAlg}_{\text{tsz}}^+ \), which sends \( P \to \text{P} \otimes_C A \). Here we abuse notation by writing \( \text{QC} \) for \( \text{CAlg}_{\text{tsz}}^+ \). Note that \( \mathbb{D}_1 (C) \otimes_A C \to A \) exhibits \( A \) as a \( \mathbb{D}_1 (C) \)-bimodule. This integral kernel \( A \) determines a functor \( I_C : \text{Mod}_C (\text{Mod}_{\text{Art}}) \to \text{LMod}_{\mathbb{D}_1 (C)} (\text{Mod}_{\text{Art}}) \) given by \( P \mapsto P \otimes_C A \). If \( C \in \text{Art}^{\text{tsz}}_{\text{Art}} \), then \( I_C \) is fully faithful (see e.g. [5, 2.3.6]). By the construction in [7, Remark 5.9], \( I_C \) is functorial in \( C \in \text{CAlg}_{\text{tsz}}^+ \). That is, there is a natural transformation

\[ \mathcal{I} : \text{QC} \to \text{LMod} \circ \mathbb{D}_1 \]

between functors \( \text{CAlg}_{\text{tsz}}^+ \to \mathcal{C} \), such that the evaluation at each \( R \in \text{CAlg}_{\text{tsz}}^+ \) is equivalent to \( I_R \).

This natural transformation (its \( \text{S}^1 \)-equivariant version) determines \( \mathcal{J}^{\mathbb{D}^1}_{\mathcal{H}\mathcal{H}_*^{\text{S}^1}(C/A)} : \text{Def}^{\mathbb{D}^1}_{}(\mathcal{H}\mathcal{H}_*^{\text{S}^1}(C/A)) \to \)
Let us formulate Proposition 4.1. Let $\sigma_0 : \{0\} \times U_{op} \to \Delta^1 \times U_{op} \to \text{Mod}(\text{Mod}_{A}^{S^1})_{H}$ be the restriction of the functor $\sigma$ in Proposition 3.2. Since $\sigma_0$ carries any morphism to a coCartesian morphism, it gives rise to an object of

$$\lim_{f: \text{Spec } C \to S \times_k S \in U_A} \text{Def}^C(\mathcal{HH}_*(C/A))(C)$$

(cf. [10, 3.3.3.2]). We shall write $D^C_c$ for it. Let $\text{pr}_2^*(C)$ be the base change $C \otimes_{\text{Perf}_A} \text{Perf}_{A \otimes k A} \in \text{St}_{A \otimes k A}$ of $C \in \text{St}_A$ (cf. Proposition 3.2). For $f : \text{Spec } C \to S \times_k S$ in $U_A$, we write $C_f$ for $f^*(\text{pr}_2^*(C)) = \text{pr}_2^*(C) \otimes_{A \otimes k A} C$. Then $D^C_c$ is informally described as the homotopy coherent diagram of the collection

$$\{(\mathcal{HH}_*(C_f/A), \mathcal{HH}_*(C_f/A)) \otimes_{C \otimes A S^1} A \simeq \mathcal{HH}_*(C/A)\}_{f \in U_A}.$$

By definition, the image of $D^C_c$ in $\lim_{\text{Spec } C \to S \times_k S \in U_A} Q(C \otimes_{A S^1})$ is $\mathcal{H}$ (cf. Construction 3.5).

Passing to limits, the natural transformation $j^C_{\mathcal{HH}_*(C/A)} : \text{Def}^C(\mathcal{HH}_*(C/A)) \to F^C_{A \otimes \text{End}(\mathcal{HH}_*(C/A))}$ sends $D^C_c$ to an object of $\lim_{f: \text{Spec } C \to S \times_k S \in U_A} F^C_{A \otimes \text{End}(\mathcal{HH}_*(C/A))}(C)$ which we denote by $E^C_c$. We remark that “$\lim_{f: \text{Spec } C \to S \times_k S \in U_A}$” can be replaced with “$\lim_{f: \text{Spec } C \to \text{Spec } \text{mod}(T_{S \times_k S})}$” (cf. Lemma 5.1).

**Proposition 4.1.** The canonical $T_{A/k}[-1]^{S^1}$-module $\mathcal{HH}_*(C/A)$ corresponds to $E^C_c$ through the equivalence $R_{C}$. 

**Remark 4.2.** The underlying object of $D^C_c$ is $\mathcal{H}$, and $D^C_c$ maps to $E^C_c$. Roughly, Proposition 4.1 means that there is a recipe to obtain the canonical $T_{A/k}[-1]^{S^1}$-module $\mathcal{HH}_*(C/A)$ from cyclic deformations of $\mathcal{HH}_*(C/A)$ arising from deformations of $C$.

**Proof.** According to [7, Lemma 8.13], for $M \in \text{Lie}_A$ there exists a canonical equivalence

$$\text{Map}_{\text{Fun}(\text{Art}_{A}^{op}, S)}(\mathcal{F}_M, \mathcal{F}^{C \otimes \text{End}(\mathcal{HH}_*(C/A))}) \simeq \text{Rep}(M^{S^1})(\text{Mod}_{A}^{S^1}) \simeq \{\mathcal{HH}_*(C/A)\}$$

which is functorial in $M \in \text{Lie}_A$. Here $\mathcal{F}_M$ is a pointed formal stack associated to $M$. Through this equivalence for $M = T_{A/k}[-1]$ and $R_{C},$ $E^C_c$ is classified, as the object (on the right side), by the following composite of $K_S C$ and maps in Theorem 1.2 in [7]:

$$u : \mathcal{F}_{T_{A/k}[-1]} \simeq \widehat{S \times_k S/\mathcal{K}S^1} \text{Def}_{C} M^C_c \xrightarrow{J^C_{\mathcal{HH}_*(C/A)}} \text{Def}^C(\mathcal{HH}_*(C/A)) \xrightarrow{\text{J}^C} F^C_{A \otimes \text{End}(\mathcal{HH}_*(C/A))}$$

(this is an obvious consequence of the definition of the above sequence in [7]). For $X \in \text{St}_A$, there exists a canonical equivalence

$$\text{Map}_{\text{Fun}(\text{Art}_{A}^{op}, S)}(X, \text{Def}^C(\mathcal{HH}_*(C/A))) \simeq \text{Rep}(X^{S^1})(\text{Mod}_{A}^{S^1}) \simeq \{\mathcal{HH}_*(C/A)\}$$

which is functorial in $X \in \text{St}_A$ (see Section 3.3). In a similar vein, $D^C_c$ is the object of

$$\text{Map}_{\text{Fun}(\text{Art}_{A}^{op}, S)}(\widehat{S \times_k S/\mathcal{K}S^1} \text{Def}_{C} M^C_c \text{Def}^C(\mathcal{HH}_*(C/A)) \xrightarrow{\text{J}} \text{Def}^C(\mathcal{HH}_*(C/A)))(C)$$

that corresponds to $\widehat{S \times_k S/\mathcal{K}S^1} \text{Def}_{C} M^C_c \text{Def}^C(\mathcal{HH}_*(C/A))$ through the equivalence (this is an obvious consequence of the definitions of $K_S C$ and $M^C_c$ in [7]).

Now our claim follows from [7, Theorem 1.2], which says that $u$ is equivalent to

$$u : \mathcal{F}_{T_{A/k}[-1]} \simeq \widehat{S \times_k S/\mathcal{K}S^1} \text{Def}_{C} J^C \rightarrow \mathcal{F}^{C \otimes \text{End}(\mathcal{HH}_*(C/A))}.$$
This sequence appears in [7, Theorem 1.2], and the final arrow is defined in [7, Construction 8.6] (in loc. cit., we denote it by \( \mathbb{T}_{\mathcal{C}}^E : \mathcal{F}_{A \otimes \mathcal{H} \ast (A/C)} \to h_{A \otimes \text{End}(\mathcal{H} \ast (C/A))} \circ h \circ \mathbb{D}_2 \simeq h_{A \otimes \text{End}(\mathcal{H} \ast (C/A))} \circ \mathbb{D}_1 \circ h \)). The final arrow corresponds to an object of

\[
\text{Rep}(G^S \otimes \text{(Mod}_A^S) \times_{\text{Mod}_A^S} \{\mathcal{H} \ast (C/A)\})
\]
determined by \( \xi^S : \mathcal{G}^A \to \text{End}^L(\mathcal{H} \ast (C/A)) \). By definition ([8, Section 5]), the Kodaira-Spencer morphism \( T_{A/k} [-1] \to T_{\mathcal{C}} \) corresponds to the composite \( T_{A/k} [-1] \simeq S \times_k S \to \text{Def}_C \to T_{\mathcal{C}} \) via \( \text{Lie}_A \simeq \mathcal{S}^* \). We deduce that \( v \) is classified by an object of \( \text{Rep}(T_{A/k} [-1] \otimes (\text{Mod}_A^S) \times_{\text{Mod}_A^S} \{\mathcal{H} \ast (C/A)\}) \) determined by the composite \( T_{A/k} [-1] \simeq \mathcal{G}^A \to \mathcal{G}^A \to \text{End}^L(\mathcal{H} \ast (C/A)) \), that is, the canonical action of \( T_{A/k} [-1] \) on \( \mathcal{H} \ast (C/A) \).

**5. Quasicoherent complexes between Ind-coherent complexes and Lie algebra modules**

In this section, for the reader’s convenience, we review the diagram [8, Section 7.3], which involves the \( \infty \)-category of modules over a dg Lie algebra and the \( \infty \)-category of Ind-coherent complexes over a formal completion, see Proposition 5.2.

**5.1.** By abuse of notation we continue to write \( \text{QC} : \text{CAlg}_A^+ \to \text{Cat}_\infty \) for the composite functor \( \text{CAlg}_A^+ \to \text{CAlg}_A \to \text{Cat}_\infty \) where the second functor corresponds to the coCartesian fibration \( \text{Mod}(\text{Mod}_A) \to \text{CAlg}_A \). Namely, \( \text{QC} \) carries \( C \in \text{CAlg}_A \) to \( \text{Mod}_C \), and we write \( \text{QC}(C) \) for \( \text{Mod}_C \). Let \( \text{QC}(\text{Art}^{tsz}_A) : \text{Art}^{tsz}_A \to \text{Cat}_\infty \) be the functor given on objects by \( C \to \text{QC}(\text{Spec} C) = \text{Ind}(\text{Coh}(C)) \). Here \( \text{Coh}(C) \) is the full subcategory of \( \text{QC}(C) \) spanned by those objects which are bounded with coherent cohomology (with respect to the standard \( t \)-structure). For \( f : \text{Spec} C' \to \text{Spec} C \) in \( \text{TSZ}_A \), it carries \( f \) to the pushforward functor \( f_! : \text{QC}(\text{Spec} C) \to \text{QC}(\text{Spec} C') \) which is the right adjoint to the proper pushforward functor \( f_! \text{Ind}(\text{Coh}(\text{Spec} C')) : \text{QC}(\text{Spec} C') \to \text{Ind}(\text{Coh}(\text{Spec} C')) \). This functor is the restriction of the functor \( \xi^s : \mathcal{S}^* \to \text{Cat}_\infty \) constructed in [2], [3] (see also [8]).

We consider three functors from \( \text{Art}^{tsz}_A \to \text{Cat}_\infty : \text{QC}(\text{Art}^{tsz}_A), \text{QC}(\text{Art}^{tsz}_A), \text{Mod} \circ \text{D}_1(\text{Art}^{tsz}_A) \). Taking right Kan extensions of \( \text{QC}(\text{Art}^{tsz}_A), \text{QC}(\text{Art}^{tsz}_A), \text{Mod} \circ \text{D}_1(\text{Art}^{tsz}_A) \) along \( \text{Art}^{tsz}_A = (\text{TSZ}_A)^{op} \to (\mathcal{S}^* A)^{op} \), we define three functors

\[
\text{QC}'_H, \text{QC}'_I, \text{Rep}'_H : (\mathcal{S}^* A)^{op} \to \text{Cat}_\infty.
\]

We let \( \Upsilon' : \text{QC}(\text{Art}^{tsz}_A) \to \text{QC}(\text{Art}^{tsz}_A) \) denote the natural transformation between functors \( \text{Art}^{tsz}_A \to \text{Cat}_\infty \), which is induced by \( \Upsilon \) (cf. Section 2.1). Recall \( Z' : \text{QC}(\text{Art}^{tsz}_A) \to \text{Mod} \circ \text{D}_1(\text{Art}^{tsz}_A) \) from Section 4.2. Let

\[
\text{QC}'_I \xrightarrow{\Upsilon'} \text{QC}'_H \xrightarrow{\Upsilon} \text{Rep}'_H
\]

be the diagram obtained from \( \text{QC}(\text{Art}^{tsz}_A) \xrightarrow{\Upsilon} \text{QC}(\text{Art}^{tsz}_A) \xrightarrow{\Upsilon'} \text{QC}(\text{Art}^{tsz}_A) \xrightarrow{\Upsilon} \text{Mod} \circ \text{D}_1(\text{Art}^{tsz}_A) \) by taking the right Kan extensions. For \( W \in \mathcal{S}^* A \), \( \text{QC}'_H(W) \to \text{Rep}'_H(W) \) is naturally equivalent to

\[
\lim_{\text{Spec} C \in (\text{TSZ}_A)^{op}} \text{QC}(\text{Spec} C) \to \lim_{\text{Spec} C \in (\text{TSZ}_A)^{op}} \text{QC}(\text{Spec} C) \to \lim_{\text{Spec} C \in (\text{TSZ}_A)^{op}} \text{Mod}(\text{D}_1(\text{C})).
\]

Both \( \text{QC}'_H(W) \to \text{QC}'_I(W) \) and \( \text{QC}'_H(W) \to \text{Rep}'_H(W) \) are fully faithful.

For \( \text{Spec} B \in (\text{Aff}_k)^{S/S} \) we let \( \text{Spec} B \) denote the functor \( \text{Art}^{tsz}_A \to \text{S} \) defined as the restriction of the functor \( \text{CAlg}_A^{(S)}_A \to \text{S} \) correponded by \( B \) (cf. [5, Definition 2.2.7], [8, Section 3], Section 2.2). According to [5, 2.2.8], \( \text{Spec} B \) lies in \( \mathcal{S}^* A \subset \text{Fun}(\text{Art}^{tsz}_A, S) \). Let \( \text{comp} : (\text{Aff}_k)^{S/S} \to \mathcal{S}^* A \) be the formal completion functor given by the assignment \( \text{Spec} B \to \text{Spec} B \). The composite \( (\text{Aff}_k)^{S/S} \to \mathcal{S}^* A \simeq \text{Lie}_A \) is naturally equivalent to the functor \( (\text{Aff}_k)^{S/S} \subset (\text{CAlg}_A^{(S)}_A)^{\text{op}} \to \text{Lie}_A \). By using the definition of the formal completion, we easily see:
Lemma 5.1. Let \( Y \) be an object of \((\text{Aff}_{k})_{S//S}\). The functor \( \text{comp} \) induces an equivalence of \(\infty\)-categories \((\text{Aff}_{k})_{S//S} \times (\text{Aff}_{k})_{S//S} \xrightarrow{\sim} (\text{TSZ}_{A})_{\bar{Y}}\).

Given \( M \in S \), we let \( \otimes_{A} M : (\text{CAlg}_{S/A})^{+} : (\text{CAlg}_{S/A})^{+} \rightarrow (\text{CAlg}_{S/A})^{+} \) denote the functor given by tensor by \( M \) in \((\text{CAlg}_{S/A})^{+}\). Consider the composite

\[
\xi_{M} : (\text{CAlg}_{S/A})^{+} \otimes_{A} M \rightarrow \text{comp}_{\text{op}}(\twisted{\hat{S}}_{A}^{\ast})_{\text{op}} \xrightarrow{\Theta_{\text{op}}} (\twisted{\hat{S}}_{A}^{\ast})_{\text{op}}.
\]

We write \( (\text{Spec} C)^{\wedge} \) for the image of \( \hat{S}_{A}^{\ast} \) under the equivalence \( \Theta_{A} : \hat{S}_{A}^{\ast} \xrightarrow{\sim} \hat{S}_{A}^{\ast} \). The composite \( \xi_{M} \) carries \( B \) to \((\text{Spec} B \otimes_{A} M)^{\wedge}\).

Let \( X \) be a pointed formal stack over \( A \), that is, an object of \( \hat{S}_{A}^{\ast} \). We set \((\text{TSZ}_{A})_{X} = \text{TSZ}_{A} \times \hat{S}_{A}^{\ast} \) and consider the composite

\[
\eta_{M,G} : ((\text{TSZ}_{A})_{X})^{\text{op}} \rightarrow \text{forget}(\text{CAlg}_{S/A})^{+} \xrightarrow{\xi_{M}} (\hat{S}_{A}^{\ast})^{\text{op}} \xrightarrow{G} \hat{C}_{\infty}
\]

where the third functor \( G \) is either \( \text{QC}'_{H} \), \( \text{QC}'_{s} \) or \( \text{Rep}'_{H} \). Define

\[
\text{QC}'_{H}((\text{Spec} C \otimes_{A} M)^{\wedge}) \xrightarrow{\text{lim}_{\text{Spec} C \in (\text{TSZ}_{A})_{X}}} \text{QC}'_{s}((\text{Spec} C \otimes_{A} M)^{\wedge})
\]

These two functors are obviously fully faithful.

Next we consider the cases when \( M = S \) and \( M = s \). We set \( \text{QC}'_{H}(X) = \text{QC}'(X) \), \( \text{QC}'_{s}(X) = \text{QC}'_{s}(X) \), \( \text{Rep}'_{H}(X) = \text{Rep}'_{s}(X) \), \( \text{QC}'_{s}(X) = \text{QC}'(X) \), \( \text{QC}'_{s}(X) = \text{QC}'_{s}(X) \), \( \text{Rep}'_{H}(X) = \text{Rep}'_{s}(X) \). The construction of \( D_{M,X} \) is functorial with respect to \( M \): the assignment \( M \mapsto D_{M,X} \) can be promoted to \( S \rightarrow \text{Fun}(\Delta \cup_{[0]} \Delta, \hat{C}_{\infty}) \). In particular, the \( S^{1}\)-equivariant map \( S^{1} \rightarrow s \) induces a morphism \( D_{S^{1},X} \rightarrow D_{s,X} \) in \( \text{Fun}(B \times (\Delta \cup_{[0]} \Delta^{1}), \hat{C}_{\infty}) \) (see [8, Remark 7.11, Construction 7.12] for the formulation). Furthermore, we focus on the case when \( X = S \times_{k} \hat{S} \). Combined with these observations, the following is proved in [8, Proposition 7.13, Proposition 7.14, Construction 7.15]:

Proposition 5.2 ([8]). The followings hold:

1. There exists the diagram

\[
\text{QC}_{1}((S \times_{k} \hat{S})_{S}^{\wedge}) \cong \text{QC}_{1}^{\wedge}(S \times_{k} \hat{S}) \xrightarrow{\beta} \text{QC}_{H}^{\wedge}(S \times_{k} \hat{S}) \xrightarrow{\beta} \text{Rep}_{H}^{\wedge}(S \times_{k} \hat{S}) \cong \text{Rep}(\hat{T}_{A/k}[-1]^{S^{1}})(\text{Mod}_{A})
\]

\[
\text{QC}_{1}((S \times_{k} \hat{S})_{S}^{\wedge}) \cong \text{QC}_{s}^{\wedge}(S \times_{k} \hat{S}) \xrightarrow{\beta} \text{QC}_{s}^{\wedge}(S \times_{k} \hat{S}) \xrightarrow{\beta} \text{Rep}_{s}^{\wedge}(S \times_{k} \hat{S}) \cong \text{Rep}(\hat{T}_{A/k}[-1]^{S^{1}})(\text{Mod}_{A})
\]

in \( \text{Fun}(B \times \hat{S}, \hat{C}_{\infty}) \). This diagram up to equivalences is \( D_{S^{1},X} \rightarrow D_{s,X} \). The vertical functor on the right side is determined by the restriction along the diagonal morphism \( T_{A/k}[-1] \rightarrow T_{A/k}[-1]^{S^{1}} \). The vertical functor on the left side is the \( l \)-pullback functor along \( (\text{id}_{S} \times \iota)_{S} : (S \times_{k} \hat{S})_{S}^{\wedge} \rightarrow (S \times_{k} \hat{S})_{S}^{\wedge} \). Every horizontal functors is fully faithful.
(2) Taking $S^1$-invariants, we obtain

\[
QC_1((S \times_k LS)^1_S) \xrightarrow{(\beta^1_S)} QC^\wedge_H((S \times_k S)^1_S) \xrightarrow{(\beta^1_H)} \text{Rep}(T_{A/k}[1])^{S^1}(\text{Mod}^{S^1}_A)
\]

\[
QC_1((S \times_k S)^1_S) \xrightarrow{\text{Res}} QC^\wedge_H((S \times_k S)^1_S) \xrightarrow{\text{Res}} \text{Rep}(T_{A/k}[1])^{S^1}(\text{Mod}^{S^1}_A)
\]

where horizontal functors are fully faithful functors.

6. Formal completion

Let $W$ be a pointed formal stack over $A$. By definition (see Definition 3.1), $QC_H(W)$ is equivalent to $\lim_{\text{Spec } C \in (\text{TSZ}_A)/W} QC(C \otimes_A S^1)$. If we remember the theory of formal schemes, an object of $QC(C \otimes_A S^1) = \text{Mod}_{C \otimes_A S^1}(\text{Mod}_A)$ can not be thought of as being formally complete along $\text{Spec } A \to \text{Spec } C \otimes_A S^1$. In this section, we construct a sort of formal completions. We start with a general situation.

Let us consider the sequence

\[
\text{Art}^\text{tsz}_A \leq (\text{Aff}_k)^{\text{op}} \xrightarrow{\text{comp}} (\hat{\text{St}}_A)^{\text{op}} \simeq (\text{St}_A)^{\text{op}}
\]

where the first functor is the evident inclusion. Then this sequence induces the adjoint pairs

\[
\text{Fun}(\text{St}_A^{\text{op}}, \text{Cat}_\infty) \xrightarrow{\text{res}_2} \text{Fun}(\text{CAlg}^{\leq}_k, \text{Cat}_\infty) \xrightarrow{\text{res}_1} \text{Fun}(\text{Art}^\text{tsz}_A, \text{Cat}_\infty).
\]

The left adjoint functors are given by restrictions. The right adjoint $R_1$ and $R_2$ are given by right Kan extensions along $\text{Art}^\text{tsz}_A \to (\text{CAlg}^{\leq}_k)_{A//A}$ and $(\text{CAlg}^{\leq}_k)_{A//A} \to (\hat{\text{St}}_A)^{\text{op}}$, respectively.

Construction 6.1. Let $F : (\text{CAlg}^{\leq}_k)_{A//A} \to \text{Cat}_\infty$ be a functor. The typical example is $QC : (\text{CAlg}^{\leq}_k)_{A//A} \to \text{Cat}_\infty$ which carries $[A \to B \to A]$ to $\text{Mod}_B$. Let $p : F \to R_1 \circ \text{res}_1(F)$ be the unit map determined by the adjoint pair (res$_1, R_1$). Let $q : \text{res}_2 \circ R_2(R_1 \circ \text{res}_1(F)) \to R_1 \circ \text{res}_1(F)$ be the counit map determined by the adjoint pair (res$_2, R_2$). We obtain the diagram

\[
F \xrightarrow{p} R_1 \circ \text{res}_1(F) \xleftarrow{q} \text{res}_2 \circ R_2(R_1 \circ \text{res}_1(F)).
\]

Observe that $q$ is an equivalence. Note that $R_2(R_1 \circ \text{res}_1(F))$ is a right Kan extension of $F|_{\text{Art}^\text{tsz}_A} = \text{res}_1(F)$ along the fully faithful embedding $\text{Art}^\text{tsz}_A \to (\hat{\text{St}}_A)^{\text{op}}$. Thus, for $B \in (\text{CAlg}^{\leq}_k)_{A//A}$, $\text{res}_2 \circ R_2(R_1 \circ \text{res}_1(F))(B)$ is described as the limit $\lim_{\text{Spec } C \in (\text{TSZ}_A)/(\text{Spec } B)^{\wedge}} F(C)$. Similarly, $(R_1 \circ \text{res}_1(F))(B)$ is $\lim_{\text{Spec } C \in (\text{TSZ}_A)/(\text{Spec } B)^{\wedge}} F(C)$, where $\text{TSZ}_A/(\text{Spec } B) = (\text{TSZ}_A \times (\text{Aff}_k)^{\text{op}})/(\text{Aff}_k^{\text{op}})/(\text{Spec } B)$. Then $q$ is given by $\lim_{\text{Spec } C \in (\text{TSZ}_A)/(\text{Spec } B)^{\wedge}} F(C) \to \lim_{\text{Spec } C \in (\text{TSZ}_A)/(\text{Spec } B)^{\wedge}} F(C)$ induced by the equivalence $(\text{TSZ}_A)/(\text{Spec } B) \simeq (\text{TSZ}_A)/(\text{Spec } B)^{\wedge}$ which is determined by the fully faithful functor $(\text{TSZ}_A)/(\text{Spec } B) \to (\hat{\text{St}}_A)^{\text{op}}/(\text{Spec } B)^{\wedge}$, where the first functor is induced by comp (the fully faithfulness follows immediately from the definition of $\text{Spec } B$, see Lemma 5.1). It follows that $q$ is an equivalence. Set $\mathcal{R} = R_2 \circ R_1$. Using the inverse $q^{-1}$ of $q$, we obtain

\[
\text{comp}_F : F \to R_1 \circ \text{res}_1(F) \simeq \text{res}_2 \circ R_2(R_1 \circ \text{res}_1(F)) = \text{res}_2 \circ \mathcal{R} \circ \text{res}_1(F).
\]

Remark 6.2. In Construction 6.1, for $[A \to B \to A] \in (\text{CAlg}^{\leq}_k)_{A//A}$, objects of $R_1 \circ \text{res}_1(F)(B) \simeq \text{res}_2 \circ \mathcal{R} \circ \text{res}_1(F)(B)$ should be thought of as “objects on the formal neighborhood of $Spec A \to Spec B$”. The natural transformation $\text{comp}_F : F \to R_1 \circ \text{res}_1(F)$ sends an object of $F(B)$ to “its formal completion along Spec $A \to Spec B$”.

Example 6.3. Let $QC$ denote the composite functor $(\text{CAlg}^{\leq}_k)_{A//A} \to (\text{CAlg}^{\leq}_k)_{A//A} \to \text{CAlg}_A \to \text{Cat}_\infty$ induced by the functor $QC : \text{CAlg}_A \to \text{Cat}_\infty$. Applying Construction 6.1 to $F = QC$ we define $\text{comp}_{QC} : QC \to \text{res}_2 \circ \mathcal{R} \circ \text{res}_1(QC) = QC^\wedge_H(\text{CAlg}^{\leq}_k)_{A//A}$. Here we note that by definition $\mathcal{R} \circ \text{res}_1(QC) = QC_H^\wedge$. 
Example 6.5. Let $LMod \circ D_1$ denote the composite functor $\left( \text{CAlg}^{\leq 0}_{A//A} \right)_{A//A} \xrightarrow{\text{forget}} \text{Alg}_1^+(\text{Mod}_A) \xrightarrow{D_1} \text{Alg}_1^+(\text{Mod}_A)^{op} \xrightarrow{LMod} \hat{\text{CAlg}}_{\infty}$, that is, the functor which carries $B$ to $LMod_{D_1}(B)(\text{Mod}_A)$. Applying Construction 6.1 to $F = LMod \circ D_1$ we define $\text{comp}_{LMod \circ D_1} : LMod \circ D_1 \to \text{res}_2 \circ \mathcal{R} \circ \text{res}_1(LMod \circ D_1) = \text{Rep}_H^{|(\text{CAlg}^{\leq 0}_{A//A})_{A//A}}$. By definition, $\mathcal{R} \circ \text{res}_1(LMod \circ D_1) = \text{Rep}_H$.

Example 6.6. Let $\mathcal{R}$ denote the composite functor $LMod \circ U_1 \circ D_\infty : (\text{CAlg}^{\leq 0}_{k})_{A//A} \to \hat{\text{Cat}}_{\infty}$, that is, the functor which carries $B$ to $\text{Rep}(D_\infty(B))(\text{Mod}_A)$. Applying Construction 6.1 to $F = \text{Rep} \circ D_\infty$ we define $\text{comp}_{\text{Rep} \circ D_\infty} : \text{Rep} \circ D_\infty \to \text{res}_2 \circ \mathcal{R} \circ \text{res}_1(\text{Rep} \circ D_\infty)$. It is possible to prove that $\text{comp}_{\text{Rep} \circ D_\infty}$ is an equivalence (cf. [8, Lemma 7.17]).

Let $I : \text{QC} \to LMod \circ D_1$ be the natural transformation (see Section 4.2). This functor induces the natural transformation $I' : \text{QC}_H \to \text{Rep}_H$ (see Section 5). Applying Construction 6.1 to $\text{QC} \to LMod \circ D_1$, we have the commutative diagram

\[
\begin{array}{ccc}
\text{QC} & \xrightarrow{I} & LMod \circ D_1 \\
\downarrow \text{comp}_{\text{QC}} & & \downarrow \text{comp}_{LMod \circ D_1} \\
\text{QC}_H |_{(\text{CAlg}^{\leq 0}_{A//A})_{A//A}} & \xrightarrow{I'} & \text{Rep}_H |_{(\text{CAlg}^{\leq 0}_{A//A})_{A//A}}
\end{array}
\]

in $\text{Fun}((\text{CAlg}^{\leq 0}_{A//A}, \hat{\text{Cat}}_{\infty})$. If we put $T : \text{Art}_A^{\text{zar}} \to (\text{CAlg}^{\leq 0}_{k})_{A//A} \xrightarrow{S^1} \text{Fun}(BS^1, (\text{CAlg}^{\leq 0}_{A//A})_{A//A})$ where the second functor is induced by the tensor by $S^1$ in $(\text{CAlg}^{\leq 0}_{A//A})_{A//A}$, the composition with $T$ gives rise to the commutative diagram

\[
\begin{array}{ccc}
\text{QC} \circ T & \to & LMod \circ D_1 \circ T \\
\downarrow \text{comp}_{\text{QC} \circ T} & & \downarrow \text{comp}_{LMod \circ D_1 \circ T} \\
\text{QC}_H |_{(\text{CAlg}^{\leq 0}_{A//A})_{A//A}} \circ T & \to & \text{Rep}_H |_{(\text{CAlg}^{\leq 0}_{A//A})_{A//A}} \circ T
\end{array}
\]

in $\text{Fun}(\text{Art}_A^{\text{zar}}, \text{Fun}(BS^1, \hat{\text{Cat}}_{\infty}))$.

Proposition 6.6. The natural transformation $\text{comp}_{LMod \circ D_1} \circ T : LMod \circ D_1 \circ T \to \text{Rep}_H |_{(\text{CAlg}^{\leq 0}_{A//A})_{A//A}} \circ T$ is an equivalence.

Proof. This is essentially proved in [8]: we here review it. By definition, for each $C \in \text{Art}_A^{\text{zar}}$, the induced functor $LMod \circ D_1 \circ T(C) \to \text{Rep}_H |_{(\text{CAlg}^{\leq 0}_{A//A})_{A//A}} \circ T(C)$ in $\hat{\text{Cat}}_{\infty}$ can naturally be identified with

\[
\text{Rep}(D_\infty(C \otimes A S^1)) \simeq \lim_{\text{Spec } R \in (\text{TS}_A)/(\text{Spec } C \otimes A S^1)} \text{LMod}_{D_1}(C \otimes A S^1)(\text{Mod}_A) \to \text{LMod}_{D_1}(\text{Mod}_A) \simeq \text{lim}_{\text{Spec } R \in (\text{TS}_A)/(\text{Spec } C \otimes A S^1)} \text{Rep}(D_{\infty}(R))(\text{Mod}_A).
\]

where the equivalences in the sequence come from [7, Proposition 3.3, Proposition 7.1]. This is an equivalence by [8, Lemma 7.17]. This completes the proof. 

Taking into account the inverse of $\text{comp}_{LMod \circ D_1} \circ T$, we see:

Corollary 6.7. The natural transformation $\text{QC} \circ T \to LMod \circ D_1 \circ T$ factors as

\[
\text{QC} \circ T \to \text{QC}_H |_{(\text{CAlg}^{\leq 0}_{A//A})_{A//A}} \circ T \to \text{Rep}_H |_{(\text{CAlg}^{\leq 0}_{A//A})_{A//A}} \circ T \simeq LMod \circ D_1 \circ T.
\]

Corollary 6.8. Let $\mathcal{V}_L \in \text{Rep}(\mathcal{T}_A[k[-1]^S])$ denote the canonical $T_A[k[-1]^S]$-module $\mathcal{H}_4(C/A)$ (cf. [8, Definition 6.3, Construction 6.10, Section 4]). Then $\mathcal{V}_L$ lies in the essential image of the fully faithful functor $(\beta')^{S^1} : \text{QC}_H |_{(S \times_k S)^{S^1}} \to \text{Rep}_H |_{(S \times_k S)^{S^1}} \simeq \text{Rep}(T_A[k[-1]^S]) |_{(\text{Mod}_A^{S^1})}$.
Proof. According to Proposition 4.1, $V'_1$ is the image of $D_C^\vee$ under

\[
\lim_{\text{Spec } C \in (TSZ_A)/S \times S} \text{QC}(C \otimes_A S^1)^{S^1} \times_{\text{Mod}_A^{S^1}} \{ \mathcal{H} \mathcal{H}_\bullet(C/A) \} \xrightarrow{\text{forget}} \lim_{\text{Spec } C \in (TSZ_A)/S \times S} \text{QC}(C \otimes_A S^1)^{S^1} \\
\rightarrow \lim_{\text{Spec } C \in (TSZ_A)/S \times S} \text{LMod}_{\mathcal{D}_1(C \otimes_A S^1)}(\text{Mod}_A)^{S^1} \\
\simeq \lim_{\text{Spec } C \in (TSZ_A)/S \times S} \text{Rep}(\mathcal{D}_\infty(C \otimes_A S^1))(\text{Mod}_A)^{S^1} \\
\simeq \text{Rep}_{\mathcal{H}_1}^\wedge(S \times_k S)^{S^1} \\
\simeq \text{Rep}(T_{A/k}[−1])^{S^1}(\text{Mod}_A)^{S^1}
\]

where the second functor is induced by $I : \text{QC} \to \text{LMod} \circ \mathcal{D}_1$ (cf. Section 4.2), and the first equivalence and the second equivalence come from [7, Proposition 7.1] and [8, Lemma 7.18]. According to Corollary 6.7, the second arrow factors as

\[
\lim_{\text{Spec } C \in (TSZ_A)/S \times S} \text{QC}(C \otimes_A S^1)^{S^1} \rightarrow \lim_{\text{Spec } C \in (TSZ_A)/S \times S} \text{QC}_H^\vee((\text{Spec } C \otimes_A S^1)^{S^1}) = \text{QC}_H^\vee(S \times_k S)^{S^1} \\
\rightarrow \lim_{\text{Spec } C \in (TSZ_A)/S \times S} \text{LMod}_{\mathcal{D}_1(C \otimes_A S^1)}(\text{Mod}_A)^{S^1}.
\]

Thus, our claim follows. \qed

For later use, we give the following definition.

**Definition 6.9.** For $B \in (\text{CAlg}_{g_k}^{\leq 0})_{A/\bar{A}}$ we define

\[
\text{comp}_B^\vee : \text{QC}^\vee_H((\text{Spec } B) \rightarrow \text{QC}^\vee_H((\text{Spec } B)
\]

to be the morphism in $\text{Fun}(B S^1, \mathcal{C} \mathcal{A} \mathcal{T}_{\infty})$ which is obtained from $\text{comp}_{QC \circ T} : \text{QC} \circ T \to \text{QC}_H^\vee((\text{CAlg}_{g_k}^{\leq 0})_{A/\bar{A}} \circ T)$ by passing to right Kan extensions along $\text{Art}_{\bar{A}}$ to $\text{CAlg}_{g_k}^{\leq 0})_{A/\bar{A}}$. Taking $S^1$-invariants, we define the induced functor $(\text{comp}_B^\vee)^{S^1} : \text{QC}^\vee_H((\text{Spec } B) \rightarrow \text{QC}^\vee_H((\text{Spec } B)^{S^1}$.

7. **Revisiting Construction in Part I**

In this section, we revisit the construction of an object of

\[
\text{QC}_1(S)^{S^1} \times_{\text{QC}_1((S \times_k S)^{S^1})} \text{QC}_1((S \times_k LS)^{S^1})
\]

in [8, Section 5.6,7] in view of the results of this paper.

In *loc.cit.*, we denote the constructed object by $V_1$. Consider the diagram

\[
\begin{array}{ccc}
(S \times_k S)^{S^1} & \rightarrow & (S \times_k LS)^{S^1} \\
\downarrow \text{pr}_2 & & \downarrow \\
S & \rightarrow & LS.
\end{array}
\]

(7.1)

The vertical morphisms are second projections. The horizontal morphisms are determined by the morphism $S \to LS$ given by constant loops. The !-pullback functors along morphisms in the diagram 7.1 induces a categorical equivalence

\[
\text{QC}_1(LS)^{S^1} \Rightarrow \text{QC}_1(S)^{S^1} \times_{\text{QC}_1((S \times_k S)^{S^1})} \text{QC}_1((S \times_k LS)^{S^1}).
\]

Through this equivalence, $V_1$ defines an object $\mathcal{H}_\bullet(C)$ of $\text{QC}_1(LS)^{S^1}$.

We will summarize the construction by using results of this paper and highlighting several points relevant to the next section (cf. Theorem 8.1).

(Step 1) We first consider the image $V'_1$ of $V_1$ under the projection to $\text{QC}_1((S \times_k LS)^{S^1})$. We consider $\mathcal{H}_\bullet(C/A)$ to be the canonical $T_{S^1}[-1]$-module, which is an object of $\text{Rep}(T_{S^1}[-1])^{S^1}(\text{Mod}_A^S)$ (see Section 4, [8, Definition 6.3]).
Let \( \hat{H} \) be the object of \( \text{lim}_{\text{Spec} C \in U_A} QC(C \otimes_A S^1)S^1 \), defined in Construction 3.5 (see also Section 4.3). By Proposition 4.1, \( \mathcal{H}^*_{\bullet}(C/A) \) in \( \text{Rep} (\mathbb{T}_{S[-1]}S^1)(\text{Mod}_A^S) \) is naturally equivalent to the image of \( \hat{H} \) under the composite

\[
QC_{\hat{H}}^C(S \times_k S)S^1 \simeq \lim_{\text{Spec} C \in U_A} QC(C \otimes_A S^1)S^1 \xrightarrow{(\beta)_{S^1}} \lim_{\text{Spec} C \in U_A} QC(C \otimes_A S^1)S^1 \xrightarrow{(\beta)_{S^1}} \text{Rep}(\mathbb{T}_{S[-1]}S^1)(\text{Mod}_A^S)
\]

(see Proposition 5.2 and Definition 6.9). In particular, \( \mathcal{H}^*_{\bullet}(C/A) \) lies in the essential image of the fully faithful functor \( (\beta)_{S^1} : \lim_{\text{Spec} C \in U_A} QC'(C \otimes_A S^1)S^1 \hookrightarrow \text{Rep}(\mathbb{T}_{S[-1]}S^1)(\text{Mod}_A^S) \). Let \( \tilde{H} \) be the image of \( \hat{H} \) in \( QC^C_{\hat{H}}(S \times_k S)S^1 = \lim_{\text{Spec} C \in U_A} QC(C \otimes_A S^1)S^1 \), which can be identified with \( \mathcal{H}^*_{\bullet}(C/A) \) endowed the canonical \( \mathbb{T}_{S[-1]}S^1 \)-action in \( \text{Rep}(\mathbb{T}_{S[-1]}S^1)(\text{Mod}_A^S) \). Using the diagram

\[
QC_1((S \times_k LS)_S^1) \xrightarrow{\beta_{S^1}} QC^C_{\hat{H}}(S \times_k S)S^1 \xrightarrow{(\beta)_{S^1}} \text{Rep}(\mathbb{T}_{S[-1]}S^1)(\text{Mod}_A^S)
\]

(see Proposition 5.2 (2)), we define \( \mathbb{V}'_1 \) to be the image of \( \hat{H} \) in \( QC_1((S \times_k LS)_S^1) \).

(Step 2) By Proposition 5.2 (2), there exists the commutative diagram

\[
\begin{array}{ccc}
\lim_{\text{Spec} C \in U_A} QC_H(C \otimes_A S^1)S^1 & \xrightarrow{\beta_{S^1}} & \lim_{\text{Spec} C \in U_A} QC_1((\text{Spec } C \otimes_A S^1)^\wedge)S^1 \\
\downarrow & & \downarrow \\
\lim_{\text{Spec} C \in U_A} QC(C)S^1 & \xrightarrow{\simeq} & \lim_{\text{Spec} C \in U_A} QC_1(\text{Spec } C)S^1 \\
\end{array}
\]

in \( \text{Cat}_{\infty} \). Notice that there exists an \( S^1 \)-equivariant canonical equivalence

\[
\text{Spec}(A \otimes_k A) \otimes_A S^1 \simeq \text{Spec } A \times_k \text{Spec}(A \otimes_k S^1) = S \times_k LS
\]

over \( S = \text{Spec } A \), where the structure morphism from the right side is the first projection. Here the \( A \)-module structure of \( A \otimes_k A \) in \( (A \otimes_k A) \otimes_A S^1 \) is defined by \( A \simeq A \otimes_k k \rightarrow A \otimes_k A \) (that corresponds to the first projection \( S \times_k A \rightarrow S \)). It follows that \( (\text{Spec } A \otimes_k A) \otimes_A S^1)^\wedge \simeq (S \times_k LS)_S^\wedge \). The upper equivalence in the diagram 7.2 is the canonical functor

\[
QC_1((\text{Spec } A \otimes_k A) \otimes_A S^1)^\wedge S^1 \rightarrow \lim_{\text{Spec } C \in U_A} QC_1((\text{Spec } C \otimes_A S^1)^\wedge)S^1
\]

which is induced by the \(!\)-pullback functoriality over \( U_A \). The lower equivalence is the canonical functor \( QC_1((\text{Spec } A \otimes_k A)^\wedge)S^1 \rightarrow \lim_{\text{Spec } C \in U_A} QC_1((\text{Spec } C)^\wedge)S^1 \) defined in the same way. We conclude from the diagram 7.2 that the image of \( \mathbb{V}'_1 \) in \( QC_1((S \times_k LS)_S^1) \) can be identified with the image of \( \hat{H} \) under \( \lim_{\text{Spec } C \in U_A} QC_H(C \otimes_A S^1)S^1 \rightarrow \lim_{\text{Spec } C \in U_A} QC_H(C)S^1 \simeq \lim_{\text{Spec } C \in U_A} QC(C)S^1 \rightarrow QC_1((S \times_k S)_S^1) \).

(Step 3) We consider the image \( \mathbb{V}''_1 \) of \( \mathbb{V}'_1 \) in \( QC_1((S \times_k S)_S^1)S^1 \). There exists an equivalence \( \mathbb{V}''_1 \simeq \mathbb{V}''(\mathcal{H}^*_{\bullet}(C/A)) \) in \( QC_1((S \times_k S)_S^1)S^1 \) where \( \mathbb{V}''_1 \) is the \(!\)-pullback along the composite of the second projection and the canonical morphism \( (S \times_k S)_S^1 \rightarrow S \times_k S \) (see [8, Lemma 6.7]). For later use (see the proof of Theorem 8.1), we review the equivalence \( \mathbb{V}''_1 \simeq \mathbb{V}''(\mathcal{H}^*_{\bullet}(C/A)) \) in detail.

As observed in (Step 2), taking into account the commutative diagram 7.2, we see that \( \mathbb{V}''_1 \) is naturally equivalent to the image of \( \hat{H} \) under the composite

\[
\xi : QC^C_{\hat{H}}(S \times_k S) = \lim_{\text{Spec } C \in U_A} QC(C \otimes A S^1)S^1 \xrightarrow{\beta} QC^C_H(S \times_k S)S^1 \rightarrow QC_1((S \times_k S)_S^1)S^1
\]
From the diagram 3.2 and the natural transformation $T : QC \to QC_{I}$ between functors $\text{CAlg}_{k}^{\leq 0, \Diamond} \to \widehat{\text{Cat}}_{\infty}$ we obtain the diagram

$$\begin{array}{c}
\text{QC}(A \otimes_k A) \otimes A S^1 \ar[d] \ar[r] & \text{QC}(A \otimes_k A) \otimes A S^1 \ar[d] \ar[r] & \text{QC}_{I}(A \otimes_k A) S^1 \\
\text{lim}_{\text{Spec} C \in U_A} \text{QC}(C \otimes A S^1) \otimes A S^1 \ar[d] & \text{lim}_{\text{Spec} C \in U_A} \text{QC}(C) \otimes A S^1 \ar[d] & \text{lim}_{\text{Spec} C \in U_A} \text{QC}_{I}(C) S^1 \\
\end{array}$$

(7.3)

in $\widehat{\text{Cat}}_{\infty}$. The composite of the lower functors is $\xi$ up to the lower right equivalence in the diagram 7.2. The right vertical functor can be identified with the $!$-pullback functor $\text{QC}_{I}(A \otimes_k A) S^1 = \text{QC}_{I}(S \otimes_k S) S^1 \to \text{QC}_{I}((S \otimes_k S) S^1)$ along the canonical morphism $\text{QC}_{I}(S \otimes_k S) S^1 \to S \otimes_k S$ up to the lower right equivalence in the diagram 7.2. Recall that $\sigma_0 : \text{QC}_{I}(S \otimes_k S) S^1 \to \text{QC}_{I}(S \otimes_k S) S^1$ along the canonical morphism $\text{QC}_{I}(S \otimes_k S) S^1 \to S \otimes_k S$ up to the lower right equivalence in the diagram 7.2. We will prove the following comparison result.

We will prove the following comparison result.

**Theorem 8.1.** The image of $\text{HH}_{I}(C/k) \in \text{QC}(LS) S^1$ under

$$\begin{array}{c}
\text{QC}(LS) S^1 \ar[r]^{T_{A \otimes S^1}} & \text{QC}_{I}(LS) S^1 \ar[r] & \text{QC}_{I}(LS) S^1 \\
\end{array}$$

is $\text{HH}_{I}(C/k) \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1$. We will write $\text{HH}_{I}(C/k) \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1$ to emphasize the “first” term. Similarly, $(A \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1) \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1$ can be identified with $(A \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1) \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1$ induced by $A \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1$ in the “first” term. Similarly, $(A \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1) \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1$ can be identified with $A \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1$ in the “second” term. Thus, by Proposition 3.7 (1)

$$\begin{align*}
\text{HH}_{I}(C/k) & \simeq (A \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1) \otimes_{A \otimes S^1} \text{HH}_{I}(C/k) \\
& \simeq (A \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1) \otimes_{A \otimes S^1} \text{HH}_{I}(C/k) \\
& \simeq A \otimes_{A \otimes S^1} \text{HH}_{I}(C/k) = p_{LS}(\text{HH}_{I}(C/k)).
\end{align*}$$

It follows that

$$\begin{align*}
\kappa : (A \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1) \otimes_{A \otimes S^1} \text{HH}_{I}(C/k) & \simeq (A \otimes_{A \otimes S^1} \text{QC}_{I}(LS) S^1) \otimes_{A \otimes S^1} \text{HH}_{I}(C/k)) \simeq \text{pr}_{2}^{I}(\text{HH}_{I}(C/k))
\end{align*}$$

where the final equivalence comes from $A \otimes_{A \otimes S^1} \text{HH}_{I}(C/k) \simeq \text{HH}_{I}(C/k)$ (see Proposition 3.7 (2)), and the first equivalence comes from base changes along the commutative diagram in $\text{CAlg}_{k}$

$$\begin{array}{c}
k \otimes_{k} (A \otimes_{k} S^1) \ar[r]^{(k \to A) \otimes \text{id}} & A \otimes_{k} (A \otimes_{k} S^1) \\
\end{array}$$

(7.4)

and the equivalence $p_{LS}^{I}(\text{HH}_{I}(C/k)) \simeq \text{HH}_{I}(\text{pr}_{2}^{I}(C/A))$.

(Step 4) $\text{QC}_{I}((S \times_{k} LS) S^1) \otimes_{A \otimes S^1} \text{HH}_{I}(C/A) \simeq \text{QC}_{I}((S \times_{k} LS) S^1) \otimes_{A \otimes S^1} \text{HH}_{I}(C/A)$, and the equivalence $\text{QC}_{I}((S \times_{k} LS) S^1) \otimes_{A \otimes S^1} \text{HH}_{I}(C/A)$ induced by $\kappa$, determine an object $\text{QC}_{I}((S \times_{k} LS) S^1) \otimes_{A \otimes S^1} \text{HH}_{I}(C/A)$.

**8. Comparison Result**
is equivalent to $V_1$. Namely, $\Upsilon_{A \otimes_k S^1}(\mathcal{H}_\bullet(C/k)) \simeq \mathcal{H}_\bullet(C)$.

Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
LS & \xleftarrow{p_{LS}} & S \times_k LS \\
\downarrow & & \downarrow \\
S & \xleftarrow{pr_2} & S \times_k S
\end{array}
\]

(8.1)

where $pr_2$ and $p_{LS}$ are second projections, and other horizontal morphisms are canonical morphisms. Note that the left square corresponds to the diagram 7.4. When combined with the diagrams 7.2 and 7.3, the functors $\text{QC} : \text{CAlg}_{k}^{\leq 0} \to \text{Cat}$ and $\text{QC}_1 : \text{Fun}(\text{CAlg}_{k}^{\leq 0}, S)^{op} \to \text{Cat}$ induce the commutative diagram in $\text{Cat}$:

\[
\begin{array}{ccc}
\text{QC}(S)^{S^1} & \xrightarrow{\text{id}} & \text{QC}(S \times_k S)^{S^1} \\
\downarrow & & \downarrow \\
\text{QC}(S)^{S^1} & \xrightarrow{\lim_{\text{Spec} C \in U_A}} & \text{QC}(\text{Spec} C)^{S^1} \\
\downarrow & & \downarrow \\
\text{QC}_1(S)^{S^1} & \xrightarrow{\Upsilon_A} & \text{QC}_1((\text{Spec} C)^{S^1}) \\
\downarrow & & \downarrow \\
\text{QC}_1((S \times_k S)^{S^1}) & \xrightarrow{\lim_{\text{Spec} C \in U_A}} & \text{QC}_1((\text{Spec} C \otimes_A S^{1})^\wedge)^{S^1} \\
\downarrow & & \downarrow \\
\text{QC}_1((S \times_k LS)^{S^1}) & \xrightarrow{\text{iso}} & \text{QC}_1((S \times_k LS)^{S^1}) \\
\end{array}
\]

(8.2)

The composition of vertical functors (taking inverses of equivalences) determines the commutative diagram

\[
\begin{array}{ccc}
\text{QC}(S)^{S^1} & \xrightarrow{pr_2^*} & \text{QC}(S \times_k S)^{S^1} \\
\downarrow & & \downarrow \\
\text{QC}_1((S \times_k S)^{S^1}) & \xrightarrow{pr_2^*} & \text{QC}_1((S \times_k LS)^{S^1}) \\
\downarrow & & \downarrow \\
\text{QC}_1((S \times_k LS)^{S^1}) & \xrightarrow{pr_2^*} & \text{QC}_1((S \times_k LS)^{S^1}) \\
\end{array}
\]

(8.3)

which is naturally equivalent to the diagram obtained from the diagram 8.1 by $*$-pullback functors, $!$-pullback functors, and $\Upsilon$. The middle vertical functor is the composite $\text{QC}(S \times_k S)^{S^1} \xrightarrow{\Upsilon_A \otimes_k A} \text{QC}_1((S \times_k S)^{S^1})$, where the second functor is the $!$-pullback functor along the canonical morphism. The right vertical functor is defined in a similar way. This diagram 8.3 induces

$$
\text{QC}(LS)^{S^1} \xrightarrow{\omega} \text{QC}(S)^{S^1} \times_{\text{QC}(S \times_k S)^{S^1}} \text{QC}(S \times_k LS)^{S^1} \xrightarrow{\omega_1} \text{QC}_1(S)^{S^1} \times_{\text{QC}_1((S \times_k S)^{S^1})} \text{QC}_1((S \times_k LS)^{S^1}).
$$

Here, for ease of notation we write $FP$ and $FP_1$ for the fiber product in the middle and the fiber product on the right side, respectively. It will suffice to prove that there exists an equivalence between $V_1$ and the image of $\mathcal{H}_\bullet(C/k) \in \text{QC}(LS)^{S^1}$ in $FP$. By definition, the image of $\mathcal{H}_\bullet(C/k)$ in $FP$ is the data consisting of the pair $(A \otimes_{(A \otimes_k S^1)} \mathcal{H}_\bullet(C/k), A \otimes_k \mathcal{H}_\bullet(C/k)) \in \text{QC}(S)^{S^1} \times \text{QC}(S \times_k LS)^{S^1}$ together with the canonical equivalence

$$
\omega : (A \otimes_k A) \otimes_{(A \otimes_k S^1)} (A \otimes_k \mathcal{H}_\bullet(C/k)) \simeq \text{pr}_2^*(A \otimes_{(A \otimes_k S^1)} \mathcal{H}_\bullet(C/k))
$$

in $\text{QC}(S \times_k S)^{S^1}$, which is obtained from the pullback functoriality over the diagram 7.4. Consider the object $V_1$ of $FP$, defined by the pair $(\mathcal{H}_\bullet(C/A), \mathcal{H}_\bullet(\text{pr}_2^*(C/A))) \in \text{QC}(S)^{S^1} \times \text{QC}(S \times_k LS)^{S^1}$ together with the equivalence $\kappa : (A \otimes_k A) \otimes_{(A \otimes_k S^1)} \mathcal{H}_\bullet(\text{pr}_2^*(C/A)) \simeq \text{pr}_2^*(\mathcal{H}_\bullet(C/A))$ in (Step 3) in the previous section. From the construction in (Step 3), the image of $V_1$ in $FP_1$ is naturally equivalent to $V_1$. Therefore, it is enough to construct an equivalence between $V_1$ and the image of $\mathcal{H}_\bullet(C/k)$ in
To this end, we will use equivalences $f : A \otimes_{(A \otimes S^1)} \mathcal{H}_\bullet(C/k) \xrightarrow{\sim} \mathcal{H}_\bullet(C/A)$ (see Proposition 3.7 (2)) and $g : p^*_{L,S}(\mathcal{H}(C/k)) \xrightarrow{\sim} \mathcal{H}_\bullet(p^*_{2}(C)/A)$ (see (Step 3)). It gives rise to the equivalence $f \times g : (A \otimes_{(A \otimes S^1)} \mathcal{H}_\bullet(C/k), p^*_{LS}(\mathcal{H}(C/k))) \xrightarrow{\sim} (\mathcal{H}_\bullet(C/A), \mathcal{H}_\bullet(p^*_{2}(C)/A))$.

Unfolding the definition, the inverse of $\kappa$ is the composite of equivalences $\omega \xleftarrow{(\text{id} \times \iota)} \mathcal{H}_\bullet(C/k) \xrightarrow{\text{pr}_2 f} \mathcal{H}_\bullet(\mathcal{H}(C/k)) \xrightarrow{\omega} \mathcal{H}_\bullet(C/A) \xrightarrow{\text{pr}_2 g} \mathcal{H}_\bullet(\text{pr}^*_2(C)/A)$.

Thus, we have an equivalence $\kappa \circ (\text{id} \times \iota)^* g \simeq \text{pr}_2 f \circ \omega$. This equivalence and $f \times g$ define an equivalence between $V^\dagger$ and the image of $\mathcal{H}_\bullet(C/k)$ in $F^P$. 

Let $\Omega^\circ(C)$ and $\Omega^\bullet(C)$ be two $\mathbb{Z}/2\mathbb{Z}$-periodic right crystals (D-modules), which are constructed from $\mathcal{H}_\bullet(C/k)$ and $\mathcal{H}_\circ(C)$, respectively, in [8, Section 8]. Both $\Omega^\circ(C)$ and $\Omega^\bullet(C)$ have the underlying $\mathbb{Z}/2\mathbb{Z}$-periodic complex $\mathcal{H}_\bullet(\mathcal{C}/A)$, that is, the periodic cyclic homology/complex. By the equivalence $\Upsilon(A \otimes S^1(\mathcal{H}_\bullet(C/k)) \simeq \mathcal{H}_\circ(C)$ in QC!$(LS)^{S^1}$ in Theorem 8.1, we see:

**Corollary 8.2.** There exists an equivalence $\Omega^\circ(C) \simeq \Omega^\bullet(C)$ of $\mathbb{Z}/2\mathbb{Z}$-periodic right crystals (D-modules).

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