Parallel-product decomposition of edge-transitive maps

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Abstract

The parallel product of two rooted maps was introduced by S. E. Wilson in 1994. The main question of this paper is whether for a given reflexible map $M$ one can decompose the map into a parallel product of two reflexible maps. This can be achieved if and only if the monodromy (or the automorphism) group of the map has at least two minimal normal subgroups. All reflexible maps up to 100 edges, which are not parallel-product decomposable, are calculated and presented. For this purpose, all degenerate and slightly-degenerate reflexible maps are classified.

Three different quotients of rooted maps are considered in the paper and a characterization of morphisms of rooted maps similar to the first isomorphism theorem for groups is presented. The monodromy quotient of a map is introduced, having the property that all the automorphisms project.

A theory of edge-transitive maps on non-orientable surfaces is developed. A concept of reduced regularity in the manner of Breda d’Azevedo is applied on edge-transitive maps. Using that, the concept of parallel-product decomposability is extended to edge-transitive maps, where a characterization in terms of minimal normal subgroups of the automorphism group is obtained. Additionally, using Petrie triality and the parallel-product decomposition, a new organization of edge-transitive maps is presented, providing a basis for future censuses.

Key words: rooted map, edge-transitive map, map quotients, monodromy quotient, parallel product, reflexible map, parallel-product decomposition.

1 Introduction

History and motivation. The history of edge-transitive maps, which also include regular (reflexible) and orientably regular maps, starts with ancient Greeks (the platonic solids, also some of the archimedean solids). In the 17th century, Kepler [20] worked on stellated polyhedra where some non-planar regular maps occured. In the 19th century, Heffter [18] considered orientably regular embeddings of complete graphs, while Klein [21] and Dyck [12] constructed some cubic regular maps on the surface of orientable genus 3, in the context of automorphic functions. In the beginning of the 20th century, regular maps were first used as geometrical representations of groups (Burnside [8]). More systematic study of regular maps continued with Brahana [3] and Coxeter and Moser [11], where regular maps were treated as geometrical, combinatorial and group theoretical objects. The basis for the modern treatment of general maps was set by Jones and Singerman [19] for orientable surfaces and by Bryant and Singerman [6] for non-orientable ones. The classic reference for maps became the book by Gross and Tucker [15]. In the last decade, research on maps of high symmetry has mainly focused on regular (and orientably regular) maps and Cayley maps. The recent paper by Richter, Sirán, Jajcay, Tucker and Watkins [30] provides a nice survey for Cayley maps.
For edge-transitive maps, Graver and Watkins [14] give the fundamental classification into 14 types according to the possession of some types of automorphisms. The existence of all of the types on infinitely many orientable surfaces was shown in the important work by Sirán, Tucker and Watkins [32].

The central problem of edge-transitive maps is construction and classification. The most common constructions of edge-transitive maps arise either from constructions of finite groups admitting one of 14 types of presentations [32] or as covers of smaller maps. Three natural approaches are used in the classification of edge-transitive maps, namely by the number of edges, by the underlying surface and by the underlying graph. The results of those classifications are several censuses [9, 37, 39].

It is known that all compact closed surfaces, other than the sphere, torus, projective plane and Klein bottle, necessarily contain a finite number of edge-transitive maps. The upper bound depends on the surface and it is easily obtained from Euler's formula or the Riemann-Hurwitz equation. All edge-transitive maps on the torus were classified by Sirán, Tucker and Watkins [32], the classification for the sphere was done by Grünbaum and Shephard [16], while a part of the classification for the Klein bottle was done by Potočnik and Wilson [29].

Before the age of fast computers, many authors (Brahana [3], Coxeter and Moser [11], Sherk [31], Garbe [13], Bergau and Garbe [2]) worked on the classification of regular and orientably regular maps and managed to classify all regular and orientably regular maps on surfaces of orientable genus up to 7 and non-orientable genus up to 8. In the 1970s, Wilson in his Ph.D. thesis [35] calculated most reflexible and chiral maps up to 100 edges [39] using a computer and running his Riemann surface algorithm [38]. The recent breakthrough in this field is due to Conder and Dobcsányi [10], who calculated all orientably regular maps on surfaces from genera 3 up to 15 and all the non-orientable reflexible maps on surfaces from non-orientable genera 2 up to 30 (Conder&Dobcsányi’s census [9]).

Since Wilson’s and Conder&Dobcsányi’s censuses present different information, a census was needed that would contain the information from both of them. Since chiral maps are not closed under the Petrie dual, the natural extension of the censuses, as observed by T. Pisanski, seemed to be edge-transitive maps. Such an extended census was the motivation for this paper. Since the census is so large, the author sought a shorter description of maps in terms of some kind of ”primitive” maps from which all other maps can be obtained using some set of operations. The algorithms for performing the operations needed to be of relatively low time complexity so the computations of ”non-primitive” maps remain simple. It turned out that the appropriate operation is the parallel product introduced by Wilson [34].

Overview of main results. Let us focus on some special class of edge-transitive maps – reflexible maps. The question is which reflexible maps are parallel-product decomposable, that is, a parallel product of two reflexible maps. The maps that are not parallel-product decomposable are of special interest as basic building blocks.

Usually, a map is represented by a set of flags and by three involutions, two of which commute, treated as permutations of the flags and intuitively giving instructions for gluing the flags together to form a surface. The group generated by these three involutions acts transitively on the set of flags and is called the monodromy group of the map. The automorphism group of a map is the group of permutations of the flags respecting the action of the monodromy group.

The main results of this article are the following group theoretical characterizations of parallel-product decomposability:

Theorem 7.1. A reflexible map is parallel-product decomposable if and only if the
monodromy group (and therefore also the automorphism group) contains at least two non-trivial minimal normal subgroups.

**Theorem 8.15.** An edge-transitive map $M$ is parallel-product decomposable if and only if $\text{Aut}(M)$ contains at least two minimal normal subgroups.

These two theorems are consequences of the main result of the paper:

**Theorem 5.1.** (Decomposition theorem) A map $M = (f, G, Z, \text{id})$ is parallel-product decomposable if and only if there are at least two normal non-transitive subgroups $H_1, H_2 \triangleleft G$, such that $H_1 \cap H_2 = \{1\}$ and $G \text{id}_{H_1} \cap G \text{id}_{H_2} = G \text{id}$.

**Paper layout.** The sections of this paper are organized as follows. Section 2 establishes the algebraic machinery necessary to discuss rooted maps on all surfaces in a manner similar to the article about Cayley maps by Richter, Širáň, Jajcay, Tucker and Watkins [30]. The algebraic machinery includes also rooted map morphisms and vertex-face-Petrie circuits triality.

Section 3 contains results about quotients of maps. Only one type of quotient of maps has appeared in the literature, namely a quotient here called an *automorphism quotient* defined by Malnič, Nedela and Škoviera [23], or regular covering map in [15]. Here we introduce two completely new quotients: a $K$-quotient and a monodromy quotient. A complete characterization of map morphisms in terms of quotients in the manner of the first isomorphism theorem on groups is given by Theorem 3.2. The *monodromy quotient*, which has the important property that all automorphisms project, is introduced at the end of the section.

Section 4 describes some interesting properties of the parallel product. After Wilson [34] introduced the parallel product, only a few authors considered it as an important operation on maps (see [5] for hypermaps). The most interesting result of this section is the construction of the smallest unique reflexible cover above any map. Similarly, the uniquely totally symmetric cover of a reflexible map is also constructed. Lifts of automorphisms in a parallel product of maps are studied.

Section 5 is devoted to the proof of Theorem 5.1.

Section 6 classifies all degenerate and slightly degenerate reflexible maps. These are basically the maps containing vertices of valence less than 3 or some kind of a degeneracy of the edges, such as loops or semi-edges. These degeneracies arise naturally in quotients and in the **triality** of duals and Petrie duals.

In Section 7 all parallel-product indecomposable reflexible maps up to 100 edges are presented. The most important theorem of this section is the decomposition theorem for reflexible maps, namely Theorem 7.1.

Section 8 extends the theory of edge-transitive maps introduced in [14, 32] to non-orientable surfaces and presents the organization of a census of edge-transitive maps using triality and the parallel product. By triality, the 14 types are reduced to the 6 basic types needed for the reconstruction of all edge-transitive maps. For these types, partial finite presentations of the corresponding automorphism groups are given, as well as a method to uniquely reconstruct the corresponding maps. Using the concept of **reduced regularity** introduced by Breda d’Azevedo [4], a theory of presentations for edge-transitive maps, is developed. Here we change the presentation of a map, and thus the monodromy group, so that both the new monodromy group and the automorphism group become regular. This approach enables us to use a characterization of parallel-product decomposability for edge-transitive maps, namely Theorem 8.15, and forms a basis for future work.
2 Definitions

In the present work we will denote by \( \text{Sym}_R(S) \) a symmetric group on \(|S|\) elements, i.e. the set of all the permutations on the elements of \( S \), such that the composition of the permutations is done from the left to the right. Also, \( \text{Sym}_R(S) \) naturally acts on the set \( S \) by the right action. Similarly, we will denote by \( \text{Sym}_L(S) \) the set of all the permutations of the elements of the set \( S \), but here we have the composition from the right to the left, as functions are usually composed. Also, \( \text{Sym}_L(S) \) naturally acts on the set \( S \) by the left action.

Let \( F = \langle t, l, r \mid t^2 = l^2 = r^2 = (tl)^2 = 1 \rangle \). A \textit{(finite) rooted map} \( M \) is a quadruple \( M = (f, G, Z, \text{id}) = (f_M, \text{Mon}(M), \text{Flags}(M), \text{id}_M) \), where \( Z \) is a finite set of \textit{flags}, \( G \leq \text{Sym}_R(Z) \) acts transitively and faithfully on \( Z \), \( f : F \to G \) is a group epimorphism and \( \text{id}_M \in Z \) a \textit{root flag}. Define \( T := f(t), L := f(l), R := f(r) \). The group \( G = \text{Mon}(M) \) is referred to as the \textit{monodromy group} of the rooted map \( M \). We will often denote an empty word from \( F \) or \( \text{Mon}(M) \) by \( \epsilon \), but sometimes also by \( 1 \). An identity mapping on a set \( S \) is often denoted by \( \text{Id} \).

Note that a monodromy group as an algebraic object is not just a group, but a group together with the labelled generators \((T, L \text{ and } R)\). For two groups \( G \) and \( K \) generated by \( k \) generators labelled with labels \( a_1, \ldots, a_k \), we will say that \( G \) and \( K \) are \textit{congruent} if there exists an isomorphism of \( G \) and \( K \), which respects the labelling, and therefore maps a generator of \( G \) labelled by \( a_i \) to the generator of \( K \) also labelled by \( a_i \), for \( i = 1, \ldots, k \). We will denote the congruence by \( G = K \) and the corresponding isomorphism will be called the \textit{congruence isomorphism}. When the groups we are working with are monodromy groups, the labels to be considered are \( T, L \) and \( R \).

If we ignore the choice of a root flag, we obtain maps named \textit{holey maps} used in [1]. The word "map" will refer to the word "rooted map" in this work. In general, a right action of a group \( G \) on a set \( Z \) will be denoted by \( (Z, G) \) and a left action by \( (G, Z) \). We will denote a stabilizer of an element \( z \in Z \) by \( G_z \).

The \textit{flag graph} \( \text{Co}(M) \) is the trivalent multigraph with a vertex set \( \text{Flags}(M) \), where each \( x \in \text{Flags}(M) \) is connected with flags \( x \cdot T, x \cdot L, x \cdot R \). If for \( x \in Z, W \in \{T, L, R\} \), it follows \( x \cdot W = x \), then a semi-edge emanates from \( x \). The involutions \( T, L \) and \( R \) naturally induce a 3-coloring of edges and semi-edges of \( \text{Co}(M) \).

Let \( M \) and \( N \) be two rooted maps. A morphism of the maps is a pair \( (\phi, \psi) \), where \( \psi : \text{Mon}(M) \to \text{Mon}(N) \) is an epimorphism of the groups, such that \( \psi \circ f_M = f_N \) and \( \phi : \text{Flags}(M) \to \text{Flags}(N) \) is an onto mapping, where \( \phi(\text{id}_M) = \text{id}_N \) and \( \phi(z \cdot g) = \phi(z) \cdot \psi(g) \) for every \( z \in \text{Flags}(M) \) and \( g \in \text{Mon}(M) \). A morphism of rooted maps is also called a \textit{covering projection}. In this case the map \( M \) is called a \textit{cover} of the map \( N \). Note that the notion of covering projection corresponds to the notion of covering projection of flag graphs described in [24]. Since such a projection can take an edge to a semi-edge, this kind of a projection is not a covering projection in the sense of topology, namely a local homeomorphism, but is more like the projection associated with an orbifold.

If both \( \phi \) and \( \psi \) are one-to-one then the pair \( (\phi, \psi) \) is an \textit{isomorphism} of the rooted maps. If we omit the condition \( \phi(\text{id}_M) = \text{id}_N \) then we get a \textit{generalized isomorphism} of rooted maps. Note that this is an isomorphism of holey maps.

An \textit{automorphism} of a rooted map \( M \) is a generalized isomorphism \( (\phi, \text{Id}) : M \to M \), where \( \text{Id} \) denotes the identity mapping of \( \text{Mon}(M) \). The group of all automorphisms is denoted by \( \text{Aut}(M) \). Since for any \( W \in \text{Mon}(M) \), \( x \in \text{Flags}(M) \), it follows \( \alpha(x \cdot W) = \alpha(x) \cdot W \), each automorphism is already defined by a mapping of a single flag. Thus \( \text{Aut}(M) \) acts semi-regularly on \( \text{Flags}(M) \). A map \( M \) is \textit{reflexive} if and only if \( \text{Aut}(M) \simeq \text{Mon}(M) \) and \( \text{Mon}(M) \) is also regular. In a slightly
general form this will be also proved in Proposition 8.12. Given \( W \in F \), we say that a rooted map \( M \) contains the automorphism \( \alpha_W \), if there is an automorphism of \( M \) taking the flag \( \mathbf{id} \) to the flag \( \mathbf{id} \cdot f(W) \). If two maps contain \( \alpha_W, W \in F \), we will say that the maps have \( \alpha_W \) in common.

The edges \( E(M) \) of a map \( M \) are the orbits of \( (T, L) \), where \( (T, L) \) denotes the subgroup of \( \text{Mon}(M) \) generated by \( T \) and \( L \). The vertices \( V(M) \) are the orbits of \( (T, R) \), the faces \( F(M) \) are the orbits of \( (L, R) \) and the Petrie circuits \( P(M) \) are the orbits of \( (TL, R) \). Let \( \text{Or}(M) := (RT, RL) \) denote the image in \( \text{Mon}(M) \) of the index two subgroup of \( F \) consisting of even length words. It is easy to see that the number of orbits of the action \((Z, \text{Or}(M))\) is 1 or 2. It is known, that in the case when \( T, L, R \) are not contained in any stabilizer of any flag (i.e. they are fixed-point-free), the map combinatorially represents a map on a compact closed surface. In this case, we say that the map is orientable if \( M \) has 2 orbits, and non-orientable otherwise. If \( TL \) has a fixed point then the map has a semi-edge. Note that if \( \text{Co}(M) \) does not have semi-edges, then orientability coincides with the graph \( \text{Co}(M) \) being bipartite.

The parallel product of two maps \( M = (f_1, G_1, Z_1, \mathbf{id}_1) \) and \( N = (f_2, G_2, Z_2, \mathbf{id}_2) \) is defined as \( M \parallel N := ((f_1, f_2), G_1 \times G_2) \). The Petrie dual is defined as \( \text{Petrie}(M) := (f, \mathbf{id}_1, \mathbf{id}_2) \). The monodromy group \( \text{Mon}(M \parallel N) \) is a subgroup of \( \text{Mon}(M) \times \text{Mon}(N) \) generated by \( (T_M, T_N), (L_M, L_N) \) and the flags are the subset of \( \text{Flags}(M) \times \text{Flags}(N) \) that is an orbit of \( \text{Mon}(M \parallel N) \) containing \( \mathbf{id}_1 \times \mathbf{id}_2 \). We will often denote a new root by \( \mathbf{id}_{1,2} \) or \( \mathbf{id}_{M,N} \). It is easy to see that the action of the new monodromy group is faithful. A pair \( (f_1, f_2) \) will be often denoted by \( f_{1,2} \) or \( f_{M,N} \) and similarly the set \( Z_1 \times Z_2 \) by \( Z_{1,2} \). Note that the parallel product is associative and commutative (up to isomorphism of the obtained maps). This was already noted by Wilson [34], where the product was introduced. A parallel product is said to be non-trivial if and only if it is not isomorphic to one of the factors.

Let \( du, pe \) be automorphisms of \( F \) defined by \( du : t \mapsto l, l \mapsto t, r \mapsto r \) and \( pe : t \mapsto l, l \mapsto t, r \mapsto r \). Then the dual of a map \( M = (f, G, Z, \mathbf{id}) \) is defined as \( \text{Du}(M) := (f \circ du, G, Z, \mathbf{id}) \). The Petrie dual is defined as \( \text{Pe}(M) := (f \circ pe, G, Z, \mathbf{id}) \). It should be noted that given a map \( M \), both \( \text{Du}(M) \) and \( \text{Pe}(M) \) have the same edges as \( M \), but \( \text{Du}(M) \) interchanges faces and vertices leaving Petrie circuits the same, while \( \text{Pe}(M) \) interchanges faces and Petrie circuits leaving vertices the same. Since \( (du, pe) \simeq S_3 \), as a subgroup of \( \text{Aut}(F) \), at most 6 non-isomorphic maps can be produced applying these two operations. Since all the maps obtained using the operations \( \text{Du} \) and \( \text{Pe} \) have the same automorphism group (only the roles of automorphisms are changed), we will often analyze only one representative of the class. The symmetry provided by \( (du, pe) \) will be called a triality and a class of maps obtained from a single map by applying the operations will be called a triality class.

Let \( p : \bar{X} \to X \) be a morphism of maps. Let \( \bar{f} \in \text{Aut}(\bar{X}) \). If there exists \( f \in \text{Aut}(X) \), such that \( p \circ \bar{f} = f \circ p \), then we say that \( \bar{f} \) projects (along \( p \)). On the other hand, if there is \( f \in \text{Aut}(X) \) and there exists a \( \bar{f} \in \text{Aut}(\bar{X}) \), such that \( p \circ \bar{f} = f \circ p \), we say that \( \bar{f} \) lifts (with \( p \)) and \( \bar{f} \in \text{Lifts}_p(f) \) is one of its lifts. Note that for \( W \in F \), if \( \alpha_W \in \text{Aut}(\bar{M}) \) projects, it projects to \( \alpha_W \).

If a root flag \( \mathbf{id} \) of a map \( M \) is changed to the flag \( \mathbf{id} \cdot W, W \in \text{Mon}(M) \), a re-rooted map is obtained. If \( W \in \{\epsilon, T, L, TL\} \), the obtained re-rooted map is said to be simply re-rooted. In general, re-rooted maps are not isomorphic as rooted maps, although they are isomorphic as holey maps.
3 Quotients of maps

In this section, for an arbitrary map \( M \), three different quotients are introduced, namely a \( K \)-quotient, for some subgroup \( K \leq \text{Mon}(M) \), a monodromy quotient and an automorphism quotient. It is shown that any image of a map by a map morphism is isomorphic to some \( K \)-quotient.

The topics discussed in this generalize the work of Malnič, Nedela and Škoviera [23]. They mainly worked with quotients of a regular map obtained through subgroups of the automorphism group, while in this paper we mainly work with quotients obtained through subgroups of the monodromy group.

Let \( (Z, G) \) be a transitive action. Then all the stabilizers are conjugate and their intersection is a normal subgroup \( H \triangleleft G \). Let \( \chi : G \to S_{|Z|} \) be the homomorphism of groups mapping \( g \in G \) to the corresponding permutation that acts on the elements of \( Z \) in the same manner as \( g \). It is easy to see that \( \ker \chi = H \). Since \( G/H \) is isomorphic to \( \chi(G) \), the isomorphism induces a map \( (Z, G/H) \) define by \( z \cdot Hg = z \cdot g \), for any \( z \in Z \) and \( g \in G \), where \( Hg \in G/H \). Since the action \( (Z, \chi(G)) \) is faithful, the action \( (Z, G/H) \) is also faithful. In this case \( H \) is called the kernel of the action \( (Z, G) \).

The following proposition defines a way of obtaining the first kind of quotient of a map.

**Proposition 3.1.** Let \( M = (f, G, Z, \text{id}) \) and let \( K \leq G \) be a subgroup, such that \( G_{\text{id}} \leq K \). Let \( H \) be the kernel of the action \( (G/K, G) \) and \( q : G \to G/H \) be the natural epimorphism. Then \( N = (q \circ f, G/H, G/K, K) \) is a map and there exists a map morphism \((p, q) : M \to N\).

**Proof.** Since the action \( (G/K, G/H) \) is transitive and faithful, \( N \) is a map. Note that since \( H \triangleleft K \), the action \( (G/K, G/H) \) is naturally defined by \( Ka \cdot Hb = KaHa^{-1}b = Kab \). Define \( p : Z \to G/K \) by \( p(\text{id} \cdot g) = Kg \) for any \( g \in G \). Let \( x \in Z \) and \( g, h \in G \) such that \( x = \text{id} \cdot g = \text{id} \cdot h \). Then \( gh^{-1} \in G_{\text{id}} \leq K \) and \( p(x) \) is well defined.

Let \( z \in Z \) and \( g \in G \) be arbitrary and \( h \in G \), such that \( z = \text{id} \cdot h \). Then \( p(z \cdot g) = p(\text{id} \cdot hg) = Khg \). Also, \( p(z) \cdot q(g) = p(\text{id} \cdot h) \cdot q(g) = Kh \cdot Hg = Khg \). As \( p(\text{id}) = K \) and \( p \) is obviously onto, \((p, q)\) is a map morphism.

Any map \( N \) obtained from \( M \) in the way shown in Proposition 3.1 is called a \( K \)-quotient and denoted by \( M/K \). The following theorem characterizes all the images of morphisms of a given map.

**Theorem 3.2.** Let \( M = (f, G, Z, \text{id}) \), \( N = (f_N, G_N, Z_N, \text{id}_N) \) and let \((\phi, \psi) : M \to N\) be a map morphism. Then \( N \) is isomorphic to \( M/K \), where \( K = \psi^{-1}((G_N)_{\text{id}_N}) \) and \((G_N)_{\text{id}_N} \) denotes the stabilizer of \( \text{id}_N \in Z_N \) of the action \( (Z_N, G_N) \). In particular, every image \( N \) of any map morphism from \( M \) is isomorphic to some \( K \)-quotient for \( G_{\text{id}} \leq K \leq G \).

**Proof.** By the definition of a map morphism, \( f_N = \psi \circ f \). Since \( \psi \) is onto, \( G_N \) is isomorphic to \( G/H \), where \( H = \ker \psi \). Let \( s : G_N \to G/H \) be that isomorphism and \( q : G \to G/H \) a natural epimorphism. Then \( q = s \circ \psi \).

Let \( K := \psi^{-1}((G_N)_{\text{id}_N}) \). Thus \( G_{\text{id}} \leq K \leq G \). The stabilizer of the coset \( K \) of the action \( (G/K, G) \) is exactly \( K \). The kernel of the action is thus the intersection of all the conjugates of \( K \):
\[
\bigcap_{a \in G} a^{-1} Ka = \bigcap_{a \in G} a^{-1} \psi^{-1}((G_N)_{id_N}) a = \bigcap_{a \in G} \psi^{-1}(a^{-1}(G_N)_{id_N} a) = \psi^{-1}\left(\bigcap_{a \in G_N} a^{-1}(G_N)_{id_N} a\right) = \psi^{-1}(\{1\}) = H.
\]

Note that the calculation above is true because \(\psi\) is faithful and thus the kernel of the action equals \(\bigcap_{a \in G_N} a^{-1}(G_N)_{id_N} a = \{1\}\). Thus \(M/K = (\psi \circ f, G/H, G/K, K, K)\).

For \(z \in Z_N\) and \(g \in G_N\) define a mapping \(r : Z_N \to G/K\) by \(r : id_N \cdot g \mapsto Ku\), where \(u\) is any element from \(\psi^{-1}(g)\). If \(u' \in \psi^{-1}(g)\) is any other such element, then \(u'u' \in \ker \psi \leq K\) and thus the definition is independent of the choice of \(u\). If \(id_N : g = id_N : h\), then \(gh^{-1} \in G_{id_N}\). Let \(u \in \psi^{-1}(g)\) and \(v \in \psi^{-1}(h)\). Then \(uv^{-1} \in \psi^{-1}(gh^{-1}) \leq K\) and \(Kr = K\). Hence the mapping \(r\) is well defined.

Let \(z \in Z_N\), \(g \in G_N\) be arbitrary and let \(h \in G_N\), such that \(id_N \cdot h = z\). Let \(u \in \psi^{-1}(g)\) and \(v \in \psi^{-1}(h)\). Then \(s(g) = Hu\) and \(s(h) = Hv\). Hence, \(r(z \cdot g) = r(id_N \cdot h) = K\). On the other hand, \(r(z) \cdot s(g) = r(id_N \cdot h) \cdot s(g) = Kv \cdot Hu = K\vspace{1mm}v\).

For \(x, y \in Z_N\), \(g, h \in G/H\) and \(u \in \psi^{-1}(g)\), \(v \in \psi^{-1}(h)\), let \(x = id_N \cdot g\) and \(y = id_N \cdot h\). Then \(r(x) = r(y)\) means \(uv^{-1} \in K\) implying that \(gh^{-1} \in (G_N)_{id_N}\) and \(x = y\). Therefore \(r\) is one-to-one and since it is always onto, it is a bijection. As \(r(id_N) = K\), the mapping \((r, s) : N \to M/K\) is a map isomorphism.

**Corollary 3.3.** A map \(M = (f, G, Z, id)\) is isomorphic to its \(G_{id}\)-quotient \((f, G, G/G_{id}, G_{id})\).

**Proof.** Take \((\phi, \psi) = (Id, Id) : M \to M\) and apply Theorem 3.2.

Another type of a quotient can be obtained in the following way.

**Proposition 3.4.** Let \(M = (f, G, Z, id)\) and \(H \triangleleft G\) be a normal subgroup. Let \(q : G \to G/H\) be the natural epimorphism. Let \(Z/H\) denote the set of orbits of the action \((Z, H)\) and let \(p : Z \to Z/H\) be defined as \(p : z \mapsto [z]\), where \([z]\) denotes the orbit containing the element \(z \in Z\). Then \(N = (q \circ f, G/H, Z/H, p(id))\) is a map isomorphic to the \(K\)-quotient \(M/G_{id}\).

**Proof.** Note that for a word \(W \in G/H\) there exists a word \(w \in G\), such that \(W = q(w)\). For an orbit \([x] \in Z/H\) we define an operation as \([x] \cdot W := [x \cdot w]\). For any other word \(v \in Mon(M)\), such that \(q(v) = W\), it follows \(HV = HW\). Since \(H\) is normal, \(x \cdot v = x \cdot hw = x \cdot w^{-1}hw = x \cdot wh'\), where \(h' \in H\). Therefore \([x \cdot w] = [x \cdot v]\). It is easy to verify that the operation indeed meets the conditions to be a right action. The action \((Z/H, G/H)\) is obviously transitive.

Let \(g \in G/H\), such that \(g\) stabilizes \(id\). There exists some \(u \in G\), such that \(q(u) = g\). Since \(id \cdot q = id \cdot u = [id]\), there exists some \(h \in H\), such that \(id \cdot u = id \cdot h\). Therefore, \(u^{-1} \in G_{id}\) and \(q(u^{-1}) \in HG_{id} = G_{id}H\). Note that since \(H\) is normal, \(G_{id}H\) is a subgroup of \(G\).

For \(g \in G_{id}H\), it follows \(g = sh\), for some \(s \in G_{id}\) and \(h \in H\). Then \([id] \cdot q(g) = [id] \cdot q(h) = [id] \cdot sh = [id]\). Hence, the stabilizer of \([id]\) is exactly \(q(G_{id}H)\).

Since the action \((Z, G)\) is faithful, the kernel of the action \(\bigcap_{x \in G} x^{-1}G_{id}x\) is trivial. Therefore

\[
\bigcap_{x \in G} x^{-1}G_{id}Hx = \left(\bigcap_{x \in G} x^{-1}G_{id}x\right) H = H
\]


since $H$ is normal. Thus $M/G_{id}H = (q \circ f, G/H, G/G_{id}H, G_{id}H)$. For the kernel of the action $(Z/H, G/H)$ it follows

$$\bigcap_{a \in G/H} a^{-1}q(G_{id}H)a = \bigcap_{x \in G} q(x)^{-1}q(G_{id}H)q(x) = q \left( \bigcap_{x \in G} x^{-1}G_{id}Hx \right) = q(H) = 1,$$

since $q$ is onto. Thus, $(Z/H, G/H)$ is faithful and $N$ is a map.

Let $r : Z/H \to G/G_{id}H$ be a mapping defined by $r : [id \cdot u] \mapsto G_{id}Hu$. Since $[id \cdot u] = [id \cdot v]$, for some $u, v \in G$, if and only if $uv^{-1} \in G_{id}H$, the mapping $r$ is well defined and one-to-one. Obviously, it is also onto. Let $Hu \in G/H$, for some $u \in G$, and let $v \in G$. Then $r([id \cdot v]) \cdot Hv = G_{id}HvHg = G_{id}Hvg$. On the other hand, $r([id \cdot v])Hg = r([id \cdot vg]) = G_{id}Hvg$. As $r([id]) = G_{id}H$, it follows that $(r, Id) : N \to M/G_{id}H$ is a map isomorphism.

The quotient defined in Proposition 3.4 is called the monodromy quotient induced by $H$. We denote the monodromy quotient of a map $M$ induced by a normal subgroup $H \triangleleft \text{Mon}(M)$ by $M \triangle H$. The corresponding projection is called the monodromy quotient projection.

The following proposition presents one of the most important properties of the monodromy quotient.

**Proposition 3.5.** Let $\bar{X} = (f, G, Z, [id])$ be a map, $H \triangleleft G$ a normal subgroup and $X = \bar{X} \triangle H$ be the monodromy quotient induced by $H$. Let $(p, q)$ be the monodromy quotient projection and $\bar{a} \in \text{Aut}(\bar{X})$. Then $\bar{a}$ projects. In particular, if for $W \in F$, the map $\bar{X}$ contains $\alpha_W$ then the map $X$ also contains $\alpha_W$.

**Proof.** Define $a([x]) := [\bar{a}(x)]$. Let $y \in [x]$. Then there exists $h \in H$, such that $y = x \cdot h$ and $\bar{a}(y) = \bar{a}(x \cdot h) = \bar{a}(x) \cdot h \in [\bar{a}(x)]$. Thus the mapping $a$ is well defined. For $W \in q(G)$, there exists $g \in G$, such that $q(g) = W$. Then $a([x]) \cdot W = a([x] \cdot g) = a([x \cdot g]) = [\bar{a}(x \cdot g)] = [\bar{a}(x)] \cdot g = [\bar{a}(x)] \cdot q(g) = a([x]) \cdot W$.

If $a([x]) = a([y])$, then $\bar{a}(x) = [\bar{a}(y)]$ and $\bar{a}(y) = \bar{a}(x) \cdot h = \bar{a}(y \cdot h)$ for some $h \in H$. Thus $x = y \cdot h$ and $[x] = [y]$, implying that $a$ is one-to-one. Obviously, it is also onto and thus $a \in \text{Aut}(X)$.

If for $W \in F$, $\bar{a} = \alpha_W \in \text{Aut}(\bar{X})$ then $\bar{a}([id \cdot X]) = [id \cdot f(W)]$. Therefore $a([id \cdot X]) = a([id \cdot X]) = [id \cdot f(W)] = [id \cdot X] \cdot q(f(W)) = [id \cdot X] \cdot (q \circ f)(W)$ meaning that $a = \alpha_W \in \text{Aut}(X)$.

An interesting observation made by Tucker [33] is that any map morphism $(\phi, \psi) : M \to N$ factors through a monodromy quotient of $M = (f, G, Z, [id])$ obtained using $H = \text{ker} \psi$. Let $(p, \psi) : M \to M \triangle H$ be the monodromy quotient projection. Then $(\phi, \psi) = (r, Id) \circ (p, \phi)$, where $r$ is uniquely defined by $\phi = r \circ p$, since $\phi$ and $p$ are onto. A reader can easily verify that $(r, Id) : M \triangle H \to N$ is indeed a map morphism.

When we are making a monodromy quotient of a map $M$, the new flags are orbits of a normal group $H \triangleleft \text{Mon}(M)$. The quotienting works, because the orbits are the blocks of imprimitivity of the action $(\text{Flags}(M), \text{Mon}(M))$. If we take any subgroup $K \leq \text{Aut}(M)$ then the orbits of that subgroup are also blocks of imprimitivity for the same action. This kind of quotients was discussed in [23]. We will call such a quotient an automorphism quotient.

Having in mind the results of this section we will often say that some map is a monodromy quotient of a map $M$ if it is isomorphic to some monodromy quotient of the map $M$. 


4 Parallel product and automorphisms

In this section some properties of the parallel product that include the lifts of automorphisms are discussed.

**Proposition 4.1.** If maps $M_i = (f_i, G_i, Z_i, \text{id})$, $i = 1, 2$, contain automorphisms $\alpha_w$ then the map $M_1 \parallel M_2$ also contains the automorphism $\alpha_w$.

*Proof.* A parallel product is obtained as:

$$M := M_1 \parallel M_2 = (f_{1,2}, G := f_{1,2}(F), Z := \text{Orb}_{\text{id}}^G(Z_{1,2}, \text{id}_{1,2}).$$

For a word $w \in F$, take $\alpha^i_w \in \text{Aut}(M_i)$ and let $\alpha = (\alpha^1_w, \alpha^2_w)$. Note that in this proof the superscripts are not the exponents but are used as indices. Let $z = (z_1, z_2) \in Z$ and $W = f_{1,2}(w) \in G$. Then:

$$\alpha(z \cdot W) = \alpha(z_1 \cdot f_1(w), z_2 \cdot f_2(w)) = (\alpha^1_w(z_1 \cdot f_1(w)), \alpha^2_w(z_2 \cdot f_2(w))) =$$

$$= (\alpha^1_w(z_1) \cdot f_1(w), \alpha^2_w(z_2) \cdot f_2(w)) = (\alpha^1_w(z_1), \alpha^2_w(z_2)) \cdot f_{1,2}(w) =$$

$$= \alpha(z) \cdot W.$$

As $\alpha^i_w$, $i = 1, 2$, are one-to-one, $\alpha$ is indeed an automorphism. Note that $\alpha = \alpha_w \in \text{Aut}(M)$. \hfill \Box

Since the parallel product is associative, the proposition can be generalized to a parallel product of a finite number of maps.

It was proven by Wilson [34], that if $h : M \to N$ is a morphism of rooted maps then $M \parallel N \simeq M$. This also yields $M \parallel M \simeq M$.

**Proposition 4.2.** A parallel product $M \parallel N$ is the unique minimal cover over $M$ and $N$. Any cover $C$ over $M$ and $N$ is a cover of $M \parallel N$.

*Proof.* Note that $(M \parallel N) \parallel C = (M \parallel C) \parallel (N \parallel C) = C \parallel C = C$. \hfill \Box

If we forget the word "rooted" in the Proposition 4.2 then the proposition is not true anymore. The example of that can be seen in Figure 3 later in Section 7.

Together with common automorphisms in two maps some other automorphisms can be present in a parallel product. In the case where factors are re-rooted maps, the following claim was noted in [34] and generalized here.

**Proposition 4.3.** Let $M_i = (f_i, G_i, Z_i, \text{id})$, $i = 1, \ldots, n$, be maps obtained by re-rooting a map $M$, $N = M_1 \parallel \cdots \parallel M_n$ be the parallel product and $\alpha$ a permutation of components in the Cartesian product $\prod^n_{i=1} Z_i$. If $\alpha$ maps the orbit of the action of the group $(f_i)_{i=1}^n$ acting on $\prod^n_{i=1} Z_i$ containing $(\text{id})_{i=1}^n$ to itself, then $\alpha \in \text{Aut}(M)$.

*Proof.* Let $\pi \in \text{Sym}_n$, such that $\alpha ((z_1, \ldots, z_n)) = (z_{\pi(1)}, \ldots, z_{\pi(n)})$. Note that $f = f_1 = \ldots = f_n$, since the maps $M_i$ are obtained by re-rooting of the same map. Let $g = (f)_{i=1}^n$. For any $W \in F$:

$$\alpha \left( (z_1, \ldots, z_n) \cdot g(W) \right) = \alpha \left( (z_1 \cdot f(W), \ldots, z_n \cdot f(W)) \right) =$$

$$= (z_{\pi(1)} \cdot f(W), \ldots, z_{\pi(n)} \cdot f(W)) =$$

$$= (z_{\pi(1)}, \ldots, z_{\pi(n)}) \cdot g(W) = \alpha \left( (z_1, \ldots, z_n) \right) \cdot g(W)$$

and the result follows. \hfill \Box

For a given map $M$ let $M^M$ denote the total parallel product of the map $M$ defined as the parallel product of all re-rooted maps obtained from the map $M$. 

Proposition 4.4. Let $M$ be an arbitrary rooted map.

1. If $M'$ and $M''$ are maps obtained from the map $M$ by re-rooting then $\text{Mon}(M) = \text{Mon}(M') = \text{Mon}(M'') = \text{Mon}(M' \parallel M'')$.

2. If $M'$ and $M''$ are maps obtained from $M$ by re-rooting, such that both of them have a root flag in the same orbit of $\text{Aut}(M)$, then $M'$ and $M''$ are isomorphic as rooted maps.

3. The total parallel product $M^M$ is a reflexible map. It is the smallest reflexible cover over the map $M$. Any reflexible cover of $M$ is also a cover of $M^M$.

Proof. Since $f = f_{M'} = f_{M''}$ and $f(F) \simeq (f, f)(F)$, (1) follows.

Let $\alpha \in \text{Aut}(M)$, such that $\alpha(id_{M'}) = id_{M''}$. Then $(\alpha, \text{Id})$ is an isomorphism of the rooted maps $M'$ and $M''$ and (2) follows.

Let $1, \ldots, n$, be the flags of the map $M$. Then $id_{M^M} = (1, \ldots, n)$. Using (1) $\text{Mon}(M^M) = \text{Mon}(M)$. Let $W \in \text{Mon}(M^M)$. Then $id_{M^M} \cdot W = id_{M^M}$ implies that $W$ viewed as an element of $\text{Mon}(M)$ stabilizes all the flags in $M$, thus it is contained in the kernel of the action of $\text{Mon}(M)$ on $\text{Flags}(M)$, which is trivial. Therefore $\text{Mon}(M^M)$ acts regularly on $\text{Flags}(M^M)$ and thus $M^M$ is reflexible.

Let $N$ be any reflexible cover over $M$ and $(p, q) : N \to M$ be the corresponding map morphism. Then $\text{Mon}(N) = q^{-1}(\text{Mon}(M))$. A reflexible map is completely determined by its monodromy group, since by Corollary 3.3 such a reflexible map $(f, G, Z, id)$ is isomorphic to the map $(f, G, G, 1)$, where $1 \in G$ denotes an identity element. In our case $N = M^M$, $\text{Mon}(N) = \text{Mon}(M)$ and $q$ is an identity mapping. Thus $M^M$ must be the unique minimal reflexible cover over $M$. It is also obvious that any reflexible cover over $M$ is also a cover of $M^M$. Therefore, (3) follows. \qed

From Proposition 4.4, the following corollary immediately follows.

Corollary 4.5. All re-rootings of a reflexible map are isomorphic. \qed

Therefore, when we are working with reflexible maps only, we can omit the roots, since any choice of root yields the same rooted map.

From the proof of Proposition 4.4 it can be seen that the minimal reflexible cover can be obtained in a much easier way then by calculating $M^M$. From $M = (f, G, Z, id)$ one just needs to construct $(f, G, G, 1)$ and this is already the minimal reflexible cover.

The following proposition is also very useful.

Proposition 4.6. Let $M$ and $N$ be rooted maps.

1. $\text{Du}(M \parallel N) = \text{Du}(M) \parallel \text{Du}(N)$.

2. $\text{Pe}(M \parallel N) = \text{Pe}(M) \parallel \text{Pe}(N)$. \qed

Proof. Let $M, N$ be $(f_i, G_i, Z_i, id_i)$, $i = 1, 2$, respectively. Then

$$\text{Du}(M \parallel N) = \text{Du} \left( (f_{1,2}, f_{1,2}(F), \text{Orb}^{f_{1,2}(F)}_{id_{1,2}}(Z_{1,2}), id_{1,2}) \right)$$

$$= (f_{1,2} \circ \text{Du}, (f_{1,2} \circ \text{Du})(F), \text{Orb}^{f_{1,2} \circ \text{Du}}_{id_{1,2}}((f_{1,2} \circ \text{Du})(F)) \circ F)$$

$$= (f_1 \circ \text{Du}, (f_1 \circ \text{Du})(F), 1, \text{id}_1) \parallel (f_2 \circ \text{Du}, (f_2 \circ \text{Du})(F), 2, \text{id}_2)$$

$$= \text{Du}(M) \parallel \text{Du}(N)$$.

The proof for the operation Pe is similar. \qed
For a given reflexible map $M$ we can construct a reflexible cover $N$, such that $Du(N) = Pe(N) = N$, i.e. a self-dual and a self-Petrie reflexible map. Such a map is called totally symmetric.

**Proposition 4.7.** Let $M$ be a reflexible map. Then the parallel product of all the maps obtained from $M$ by applying the compositions of the operations $Du$ and $Pe$ is totally symmetric and is unique minimal with these properties.

**Proof.** Denote by $S$ the set of all the non-isomorphic maps obtained from $M$ using the operations $Du$ and $Pe$ and denote the parallel product of all the maps in $S$ by $N$. Since the parallel product operation is commutative, any order of the factors in the parallel product of all the maps in $S$ always yields the map (isomorphic to) $N$. Let us prove that $N$ is self-dual. Since in the set $S$ there are all non-isomorphic maps obtained by the operations $Du$ and $Pe$ and the operations are involutions, performing $Du$ on all the elements of $S$ yields the same set. Therefore the parallel product yields a map isomorphic to $N$ and by Proposition 4.6 the map $N$ is self-dual. Similarly we show that $N$ is self-Petrie. If $N'$ is a cover of $M$ it follows that $Du(N')$ must be a cover of $Du(M)$ (and similarly for the operation $Pe$). Using that, a reader can easily verify the minimality and the uniqueness. \[\square\]

Some of those properties of the parallel product were already noted in [34] without a proof. This theory can be extended in several directions. A possible extensions include abstract polytopes [27]. Using the constructions in this section one can extend the results to abstract polytopes and get similar results to the ones by Hartley [17].

## 5 A parallel-product decomposition of a map

We consider factorizing a map $M$ as a parallel product. The factors are always the images of map morphisms. Our aim is to find criteria for splitting the map as a parallel product of two maps which are monodromy quotients. Monodromy quotients are of special interest, because all the automorphisms $\alpha_{M'} \in M$ project. In particular, a monodromy quotient of a reflexible map is reflexible.

A map $M$ is parallel-product decomposable if it is a non-trivial parallel product of two maps, such that the two maps are monodromy quotients of $M$.

**Theorem 5.1.** (Decomposition theorem) A map $M = (f, G, Z, \text{id})$ is parallel-product decomposable if and only if there are at least two normal non-transitive subgroups $H^1, H^2 \triangleleft G$, such that $H^1 \cap H^2 = \{1\}$ and $G_{\text{id}}H^1 \cap G_{\text{id}}H^2 = G_{\text{id}}$.

**Proof.** Let $M = M_1 \parallel M_2 = (f_{1,2}, G := f_{1,2}(F), Z := \text{Orb}_{G_{\text{id}}, 2}^G(Z_{1,2}), \text{id} := \text{id}_{1,2})$ be a non-trivial parallel product of maps, where $M_i = (f_i, G^i, Z_i, \text{id}_i)$, $i = 1, 2$. Note that the indices in the names of the groups are written as superscripts since subscripts are used for denoting stabilizers. The coordinate projections $(p_i, q_i) : Z \times G \rightarrow Z_i \times G^i$ are the covering projections of the maps $M \rightarrow M_i$. Denote the kernels of the epimorphisms $q_i$ by $H^i$. These are normal subgroups in $G$ and $H^1 \cap H^2 = \{(1, 1)\}$. Since the factors of the parallel product are monodromy quotients, they must be the monodromy quotients by these two normal subgroups. For monodromy quotients it is true: $q_i^{-1}(G_{\text{id}}^i) = G_{\text{id}}H^i$. But since $G_{\text{id}} = q_1^{-1}(G_{\text{id}}^1) \cap q_2^{-1}(G_{\text{id}}^2)$ it immediately follows: $G_{\text{id}}H^1 \cap G_{\text{id}}H^2 = G_{\text{id}}$. Thus if $M$ is a nontrivial parallel product of two maps that are the monodromy quotients of the product then it meets the conditions of the theorem.

Now, let $M = (f, G, Z, \text{id})$. By Corollary 3.3 we can assume that $M = (f, G, G/G_{\text{id}}^1, G_{\text{id}}^2)$. Let $H^1, H^2$ be the normal subgroups meeting the conditions of the
theorem. A trivial parallel-product decomposition would be obtained if one of the factors would be isomorphic to $M$ or to the trivial map. In the first case this would mean $H' \leq G_{\text{id}}$, but since the action $(G/G_{\text{id}}, G)$ is faithful this cannot happen. The second case is prevented by the non-transitivity condition.

By Proposition 3.4 the monodromy quotients of $M$ by $H^i$, $i = 1, 2$, are isomorphic to the maps $M_i := (f_i, G^i, Z_i, \text{id})$, where $G^i := G/H^i$, $f_i := q_i \circ f$, $q_i : G \rightarrow G/H^i$ is a natural epimorphism, $Z_i := G/G_{\text{id}}H^i$ and $\text{id} = G_{\text{id}}H^i$. Let $(p_i, q_i) : M \rightarrow M_i$ be the corresponding covering projections as in Proposition 3.4. It is easy to see that $p_i : G/G_{\text{id}} \rightarrow G/G_{\text{id}}H^i$ is defined by $p : G_{\text{id}}g \rightarrow G_{\text{id}}H^i g$, for any $g \in G$.

Let $M_1 \parallel M_2 = (f_{1,2}, K, X, \text{id})$, where $K = f_{1,2}(F)$, $\text{id} = (G_{\text{id}}H^1, G_{\text{id}}H^2)$ and $X$ is an orbit of the naturally induced action $(G_{\text{id}}H^1 \times G_{\text{id}}H^2, K)$ containing $\text{id}$. We will show that $M_1 \parallel M_2$ is isomorphic to $\overline{M}$ and therefore we have to find an isomorphism $\psi : K \rightarrow G$ and a bijection $\phi : X \rightarrow G/G_{\text{id}}$, such that $(\phi, \psi) : M_1 \parallel M_2 \rightarrow M$ is a map isomorphism.

Let $W \in K$. Then there exists $w_1 \in F$, such that $f_{1,2}(w_1) = W$. Let $\psi(W) := f(w_1)$. First we verify that $\psi$ is well defined. It is true that $f_{1,2}(w_1) = (q_1 \circ f(w_1), q_2 \circ f(w_1))$. If there is some other $w_2 \in F$, such that $f_{1,2}(w_2) = W$, we get $q_i \circ f(w_1) = q_i \circ f(w_2)$, $i \in \{1, 2\}$. This means $f(w_1)f^{-1}(w_2) \in H^1 \cap H^2 = \{1\}$ and it follows $f(w_1) = f(w_2)$. Hence, the mapping $\psi$ is well defined. Now we have to see that $\psi$ is a homomorphism of the groups. Let $g = (g_1, g_2), h = (h_1, h_2) \in K$. There are $w_1, w_2 \in F$, such that $g = f_{1,2}(w_1)$ and $h = f_{1,2}(w_2)$. Then $\psi(g) = f(w_1)$ and $\psi(h) = f(w_2)$. Since

$$f_{1,2}(w_1w_2) = (f_1(w_1w_2), f_2(w_1w_2)) = (f_1(w_1)f_1(w_2), f_2(w_1)f_2(w_2))$$

then $\psi(gh) = f(w_1w_2) = f(w_1)f(w_2) = \psi(g)\psi(h)$ and $\psi$ is a homomorphism.

Obviously, it is an epimorphism. Let $g \in \ker \psi$. There exists $w \in F$, such that $f_{1,2}(w) = g$ and $\psi(g) = f(w) = 1$. Thus $f_1(w) = f_2(w) = 1$ and since $g = (f_1(w), f_2(w))$, it follows that $g = 1$ and $\psi$ must be an isomorphism.

Let $z \in X$. Then $z = (G_{\text{id}}H^1 f_1(w), G_{\text{id}}H^2 f_2(w))$, for some $w \in F$. Define $\phi : X \rightarrow G/G_{\text{id}}$ by $\phi : z \mapsto G_{\text{id}} f(w)$. There may exist another $w' \in F$, such that $z = (G_{\text{id}}H^1 f_1(w'), G_{\text{id}}H^2 f_2(w'))$. Then

$$(G_{\text{id}}H^1 f_1(w)f_1^{-1}(w), G_{\text{id}}H^2 f_2(w')f_2^{-1}(w)) = (G_{\text{id}}H^1, G_{\text{id}}H^2 f(w')f_2^{-1}(w)) = (G_{\text{id}}H^1, G_{\text{id}}H^2).$$

Thus by the assumption of the theorem $f(w')f_2^{-1}(w) \in G_{\text{id}}H^1 \cap G_{\text{id}}H^2 = G_{\text{id}}$ and $\phi$ is well defined. Similarly we can see that $\phi$ is one-to-one. Since for any $w \in F$, it follows $(G_{\text{id}}H^1 f_1(w), G_{\text{id}}H^2 f_2(w)) \in X$, the mapping $\phi$ is onto.

Now we will verify that $(\phi, \psi)$ is an isomorphism of the maps $M_1 \parallel M_2$ and $M$. Obviously, $\phi(\text{id}) = G_{\text{id}}$ and $\psi \circ f_{1,2} = f$. Let $g \in K$. Then there exists $w_1 \in F$, such that $g = f_{1,2}(w_1)$ and $\psi(g) = f(w_1)$. Let $z = (G_{\text{id}}H^1 f_1(w), G_{\text{id}}H^2 f_2(w)) \in X$ for some $w_2 \in F$. Then

$$zg = (G_{\text{id}}H^1 f_1(w), G_{\text{id}}H^2 f_2(w)) \cdot f_{1,2}(w_1) = (G_{\text{id}}H^1 f_1(w_2w_1), G_{\text{id}}H^2 f_2(w_2w_1))$$

and $\phi(zg) = G_{\text{id}}f(w_2w_1) = G_{\text{id}}f(w_2)f(w_1)$. Also, $\phi(z)\psi(g) = G_{\text{id}}f(w_2)f(w_1)$. Therefore, a pair $(\phi, \psi)$ is an isomorphism.
6 Degeneracy of reflexible maps

In this section reflexible maps are classified into three families according to their degeneracy. The classification will be used in the following section, where all parallel-product indecomposable degenerate maps will be presented.

Let \( M \) be a reflexible map with a presentation of the monodromy group of the form

\[
\text{Mon}(M) = \langle T, L, R \mid W_1^{e_1} = W_2^{e_2} = \ldots = W_k^{e_k} = 1 \rangle, \quad e_i \geq 1, k \geq 7,
\]

such that \( W_1 = T, W_2 = L, W_3 = R, W_4 = LT, W_5 = RT, W_6 = RL, W_7 = TLR, \)

where \( e_1, e_2, e_3, e_4 \in \{1, 2\} \), and where \( W_i, i \geq 8 \) are words in \( \text{Mon}(M) \), such that the group is finite. Also, all \( e_i \) are the actual orders of the corresponding elements (words). The set of words \( \{W_1, \ldots, W_k\} \) is called a context. Any context contains at least the words \( W_1, \ldots, W_7 \). In the context chosen, a monodromy group can be denoted by a vector \( \text{Mon}(M) = (e_1, e_2, \ldots, e_k) \) or \( \text{Mon}(M) = (e_i)_{i=1}^k \). For a given map \( M \) the words in the context \( C \) are sufficient to define \( \text{Mon}(M) \), the context is said to be sufficient. A monodromy group \( \text{Mon}(M) \) can be easily obtained from the vector and the obtained reflexible map is \( M = (f, \text{Mon}(M), \text{Mon}(M), 1) \), where \( f \) is a homomorphism mapping \( t \mapsto W_1, l \mapsto W_2, r \mapsto W_3 \) and \( 1 \in \text{Mon}(M) \). Sometimes the notation is abused and the map is denoted directly by the corresponding vector. It is obvious that any monodromy group of a reflexible map can be written in the form described above, but some of the maps need larger contexts (i.e. more words \( W_i, i \geq 8 \)).

For two contexts \( C_1 \) and \( C_2 \) the common context is \( C_1 \cup C_2 \). Obviously, if some map is represented in a context \( C_1 \), it can be also easily represented in \( C_1 \cup C_2 \) by calculating the orders of the words in \( C_2 \setminus C_1 \) and adding those (redundant) relations.

A map \( M \) is slightly-degenerate if in any sufficient context \( C \) it follows \( e_i \geq 2 \), for all \( i = 1, \ldots, 7 \), and at least one of \( e_3, e_6, e_7 \) equals to 2. It is degenerate if at least one of \( e_i, i = 1, \ldots, 7 \), equals to 1. If a map is not degenerate or slightly-degenerate then it is non-degenerate. In this case \( e_i \geq 3, i = 5, 6, 7 \).

Note that in any sufficient context of a map \( M \) the words \( W_i, i = 1, \ldots, 7 \), are exactly the generators and the relations that determine the map’s properties, such as the degrees of the vertices, the co-degrees of the faces and the sizes of the Petrie circuits.

**Lemma 6.1.** Let \( M = (e_i)_{i=1}^k, N = (f_i)_{i=1}^k \) be two reflexible maps represented in the common context. Then \( M \parallel N = (\text{lcm}(e_i, f_i))_{i=1}^k \).

**Proof.** We can view both groups \( \text{Mon}(M) \) and \( \text{Mon}(N) \) as quotients of a free group \( F_0 := \langle T, L, R \rangle \). Let \( H_1 \) be the normal closure in \( F_0 \) of the set \( \{W_i^{e_i}\}_{i=1}^k \) and \( H_2 \) be the normal closure in \( F_0 \) of \( \{W_i^{f_i}\}_{i=1}^k \). Then \( \text{Mon}(M) = F_0 / H_1 \) and \( \text{Mon}(N) = F_0 / H_2 \). Let \( g \) be an element of the intersection \( H_1 \cap H_2 \). Then \( g \) can be expressed as a finite product of conjugates and powers of conjugates of \( W_i \). Since everything is happening in the free group, any exponent of \( W_i \) in the expression of \( g \) must be divisible by \( e_i \) and \( f_i \) and thus by \( \text{lcm}(e_i, f_i) \). Thus \( H_1 \cap H_2 \) is exactly the normal closure of \( \{W_i^{\text{lcm}(e_i, f_i)}\}_{i=1}^k \) and this set determines the relations of \( F_0 / (H_1 \cap H_2) \) in the finite presentation. Let \( f_i : F_0 \to F_0 / H_i, i = 1, 2 \), be the natural quotient projections. Let \( f := (f_1, f_2) \). Then \( \ker f = H_1 \cap H_2 \) and \( F_0 / (H_1 \cap H_2) \simeq f(F_0) \). But \( M = (f_1, F_0 / H_1, F_0 / H_1, H_1) \) and \( N = (f_2, F_0 / H_2, F_0 / H_2, H_2) \) and thus \( F_0 / (H_1 \cap H_2) = \text{Mon}(M \parallel N) \).

When analyzing the existence of some class of reflexible maps for which the context \( C = \{W_1, \ldots, W_7\} \) is sufficient, we can use triality. Note that the operations
Du and Pe permute the triple \((e_1, e_2, e_4)\) with the same permutation as the triple \((e_5, e_6, e_7)\). To describe the action of Du and Pe on the indices \(i = 1, \ldots, 7\) of \(e_i\), we can represent Du as a permutation \((1, 2)(5, 6)\) and Pe as \((2, 4)(6, 7)\).

**Proposition 6.2.** All degenerate reflexible maps are shown in Table 1.

| Name | \((T, L, R, TL, TR, LR, TLR)\) | \(|Mon(M)|\) |
|------|----------------|----------------|
| DM_1 | \(1, 1, 1, 1, 1, 1, 1\) | 1 |
| DM_2 | \(1, 1, 2, 1, 2, 2, 2\) | 2 |
| DM_3 | \(2, 1, 1, 2, 2, 1\) | 2 |
| DM_4 | \(1, 2, 1, 2, 1, 2\) | 2 |
| DM_5 | \(2, 2, 1, 2, 2, 1\) | 2 |
| DM_6(k), \(k > 0\) | \(2, 1, 2, 2, k, 2\) | 2k |
| DM_7(k), \(k > 0\) | \(1, 2, 2, 2, 2, k\) | 2k |
| DM_8(k), \(k > 0\) | \(2, 2, 2, 1, k, 2\) | 2k |
| DM_9 | \(2, 2, 1, 2, 2, 2\) | 4 |
| DM_{10} | \(2, 2, 2, 2, 1, 2\) | 4 |
| DM_{11} | \(2, 2, 2, 2, 2, 1\) | 4 |
| DM_{12} | \(2, 2, 2, 2, 2, 1\) | 4 |

Table 1: Degenerate reflexible maps.

*Proof.* First we prove that all the monodromy groups in Table 1 are uniquely determined by the context \(C = \{W_1, \ldots, W_7\}\). For all the maps in the table except \(DM_i(k), i = 6, 7, 8\), this is pretty obvious. By triality it is enough to check the group of \(DM_6(k)\). The relations here determine a dihedral group \(D_{2k}\) generated by \(T\) and \(TR\) that commute. One can easily see that any quotient of \(D_{2k}\) strictly decreases the orders of at least one of the (projected) generators.

Now we will make an analysis of what kind of degenerate maps can occur. Let \(e_1 = e_2 = 1\). Then \(e_4 = 1\). If \(e_3 = 1\) we get \(DM_1\). If \(e_3 = 2\) then it must be \(e_5 = e_6 = e_7 = 2\) (DM_2). Now, let \(e_1 = 1\) and \(e_2 = 2\). Since \(e_4 = 1\) implies \(e_2 = e_1\), it must be \(e_4 = 2\). If \(e_3 = 1\) then it must be \(e_5 = 1\), \(e_6 = e_7 = 2\) (DM_4 and by triality DM_5 and DM_6). If \(e_3 = 2\) then \(e_5 = 2\) and \(e_6 = e_7 = k \geq 1\) (DM_7(k) and by triality DM_6(k) and DM_6(k)). By triality, all the possibilities where one of \(e_1, e_2, e_4\) is 1 are exhausted. Assume \(e_1 = e_2 = e_4 = 2\). If \(e_3 = 1\) then \(e_5 = e_6 = e_7 = 2\) (DM_9). Let now \(e_3 = 2\). Since map has to be degenerate, one of \(e_5, e_6, e_7\) must be equal to 1. By triality we can assume \(e_5 = 1\). Then it must be \(e_6 = e_7 = 2\), otherwise the orders \(e_1, e_2\) collapse (DM_{10}, DM_{11}, DM_{12}). This exhausts all the possibilities for degenerate maps. \(\blacksquare\)

A similar analysis of degenerate maps was done in [22], but Širáň’s definition of degeneracy is different from ours. By Širáň, a reflexible map \(M\) is degenerate if one of the generators \(x = a_L, y = a_T, z = a_R \in Aut(M)\) equals to the identity. It is easy to see that Širáň’s degeneracy is equivalent to saying that one of \(e_1, e_2\) or \(e_3\) is equal to 1. Unfortunately, in [22] they forgot to include the map DM_5. They also use similar names for degenerate maps. Thus their maps DM_1, …, DM_7 correspond to ours DM_1, DM_2, DM_4, DM_3, DM_6, DM_7 and DM_9, respectively.

In Figure 1 all the flag graphs for degenerate maps are shown.

If a reflexible map is not degenerate then all the involutions \(T, L, R, TL\) are fixed-point-free. Such a map corresponds to a reflexible 2-cell embedding of some graph into a compact closed surface. Slightly-degenerate maps can be constructed
using the operations $Du$ and $Pe$ from a reflexible embedding of a cycle in some compact closed surface. The only possible such 2-cell embeddings are the embeddings of $k$-cycle in the sphere, denoted by $\varepsilon_k$, and in the projective plane with the $k$-cycle embedded as a non-contractible curve, denoted by $\delta_k$. Here the names are adopted from [36].

The monodromy group presentations of maps $\varepsilon_k$ and $\delta_k$ are shown in Table 2.

| Name   | Additional relations | Order |
|--------|----------------------|-------|
| $\varepsilon_k$, $k > 0$ even | $(LR)^k, (TLR)^k$ | $4k$  |
| $\varepsilon_k$, $k > 1$ odd  | $(LR)^k, (TLR)^{2k}$ | $4k$  |
| $\delta_k$, $k > 0$ even      | $T(LR)^k, T(TLR)^k$ | $4k$  |
| $\delta_k$, $k > 1$ odd       | $(LR)^{2k}, (TLR)^k$ | $4k$  |

Table 2: A monodromy group of each map in this table is obtained as $\langle T, L, R \mid T^2 = L^2 = R^2 = (TL)^2 = (RT)^2 = \ldots = 1 \rangle$, where instead of "\ldots" one should put the additional relations. All slightly-degenerate reflexible maps can be constructed from the maps in this table by using the operations $Du$ and $Pe$. Note that $\varepsilon_1 = DM_{11}$ and $\delta_1 = DM_{12}$ and thus degenerate and not included in Table 2.

7 Parallel-product decomposition of reflexible maps

For reflexible maps the decomposition theorem (Theorem 5.1) can be more specialized.

**Theorem 7.1.** A reflexible map $M$ is parallel-product decomposable if and only if $\text{Mon}(M)$ (and therefore also $\text{Aut}(M)$) contains at least two non-trivial minimal normal subgroups.
Proof. Since the monodromy group of a reflexible map is regular, the stabilizer is trivial. The conditions of Theorem 5.1 are reduced to the existence of two non-trivial normal subgroups $H^1$ and $H^2$, such that $H^1 \cap H^2 = \{1\}$. But in a finite group such subgroups exist if and only if two minimal non-trivial normal subgroups exist. Since for reflexible maps the monodromy group is isomorphic to the automorphism group, the result follows.

Example 7.2. To demonstrate how quotienting and the parallel-product decomposition work, see the examples in Figures 2 and 3. In both figures the map $M$ we are quotienting is a 4-cycle on the sphere. In Figure 2, $M$ and its quotients are represented by flag graphs. We note that the monodromy group $\text{Mon}(M)$ is isomorphic to the group $\mathbb{Z}_2 \times D_4$. This group has exactly 3 minimal normal subgroups. The flag graphs of each of the corresponding monodromy quotients are shown. All these maps are reflexible. By Theorem 7.1 a parallel product of any two yields the original map $M$.

\[
\text{Mon}(M) = \mathbb{Z}_2 \times D_4 = \langle T \rangle \times \langle L, R \rangle
\]

\[\begin{array}{c}
\text{DM}_4(4) \\
\langle T \rangle
\end{array} \quad \text{\begin{array}{c}
\delta_4 \\
\langle L R L R \rangle
\end{array}} \quad \text{\begin{array}{c}
\delta_4 \\
\langle T L R L R \rangle
\end{array}}\]

Figure 2: Monodromy quotients of $C_4$ on the sphere that yield a non-trivial parallel-product decomposition.

In Figure 3, a different quotient is obtained. The quotient arises as an automorphism quotient from the orbits of the automorphism that rotates the flags around the vertex in the lower left corner. The obtained map is not reflexible. One can easily see that in the quotient there are 2 orbits of the automorphism group on the flags, namely the orbit of the flags around the vertices of degree 1 and the orbit of the flags around the vertex of degree 2. If we re-root the maps in a way, such that the root flags are in the different orbits and make a parallel product of them, we obtain the smallest reflexible cover (by Proposition 4.4) which is again the map $M$. Note that the monodromy groups of $M$ and its quotient are isomorphic.

Thus a parallel-product indecomposable reflexible map is any reflexible map $M$, such that either $\text{Mon}(M)$ is a simple group or $\text{Mon}(M)$ has a unique minimal normal subgroup. The latter groups are called monolithic groups and the unique minimal normal subgroup is called a monolith. Since the operations $D_u$ and $P_e$ preserve a monodromy group, the operations are invariant for the parallel-product indecomposability.
Proposition 7.3. The map $DM_6(k)$ ($DM_7(k)$, $DM_8(k)$), $k > 2$ is parallel-product decomposable if and only if $k$ is not a prime power.

Proof. Number $k$ is not a prime power if and only if there exist $a, b > 1$, such that $gcd(a, b) = 1$ and $k = ab$. Using Lemma 6.1 and Table 1 it is easy to see that for any $a, b > 1$, $DM_6(a) \parallel DM_7(b) \simeq DM_6(lcm(a, b))$. Nontrivial factors of $DM_6(k)$ can be only degenerate maps with $L = 1$, so only: $DM_6(l)$, $l \geq 1$, $DM_2$ and $DM_3$. Since $DM_2$ and $DM_3$ are quotients of any $DM_6(l)$, $l > 2$, a parallel product with $DM_6(l)$ absorbs them. Also $DM_2 \parallel DM_3 \simeq DM_2 \parallel DM_6(1) \simeq DM_3 \parallel DM_6(1) \simeq DM_6(2)$. So if $k > 2$ and $DM_6(k)$ is parallel-product decomposable, then it must be a product of two factors of the form $DM_6(l)$. By Table 1 and Proposition 6.1 this is possible only when the conditions of the lemma are fulfilled. Using triality, the proofs for $DM_7(k)$ and $DM_8(k)$ immediately follow. $\square$

The monodromy groups of the maps $DM_i$, $i = 9, 10, 11, 12$, are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and thus by Theorem 7.1 the maps are parallel-product decomposable. The monodromy groups of $DM_i$, $i = 1, 2, 3, 4, 5$, are either trivial or isomorphic to $\mathbb{Z}_2$, implying that those maps are parallel-product indecomposable.

The following corollary immediately follows.

Corollary 7.4. All degenerate reflexible maps are parallel-product indecomposable except:

1. $DM_5(k)$, $DM_6(k)$ and $DM_7(k)$, for $k = 2$ and any $k > 2$ which is not a power of a prime,

2. $DM_9$, $DM_{10}$, $DM_{11}$ and $DM_{12}$.$\square$

Proposition 7.5. The only parallel-product indecomposable slightly-degenerate maps are the maps $\delta_k$, where $k = 2^n$, $n \geq 1$.

Proof. Since $Pe(\varepsilon_k) \simeq \delta_k$, for $k$ odd, we have to consider only the parallel-product decompositions of maps $\varepsilon_k$ for all $k > 1$ and $\delta_k$, for $k > 1$ even.

Take a context $C = \{T, L, R, TL, RT, RTL\}$. In this context $\varepsilon_k = (2, 2, 2, 2, 2, k)$, for $k > 0$ even, $\varepsilon_k = (2, 2, 2, 2, 2, k, 2k)$, for $k > 1$ odd, $DM_3 = (2, 1, 1, 2, 2, 1, 2)$ and $DM_7(k) = (1, 2, 2, 2, 2, k, k)$. By Proposition 6.1, it follows $\varepsilon_k \simeq DM_7(k) \parallel DM_3$, for any $k > 1$. $\square$
Now, let \( k > 0 \) and let \( l \geq 1 \) be any odd number. We will prove that \( \delta_{2k} \simeq \text{DM}_{7}(2^{k}l) \| \delta_{2n} \). This would mean that for any even \( u \) not equal to the power of 2, \( \delta_{u} \) is parallel-product decomposable. The monodromy groups for \( \delta_{2k} \) and \( \text{DM}_{7}(2^{k}l) \) are defined by relations:

\[
\text{Mon}(\delta_{2k}) : T^{2} = L^{2} = R^{2} = (TL)^{2} = (RT)^{2} = 1, (RL)^{2k} = (TLM)^{2k} = T,
\]

\[
\text{Mon}(\text{DM}_{7}(2^{k}l)) : T = L^{2} = R^{2} = (TL)^{2} = (RT)^{2} = (RL)^{2k+1} = (TLM)^{2k+1} = 1.
\]

Hence a monodromy group of the parallel product is defined by relations

\[
T^{2} = L^{2} = R^{2} = (TL)^{2} = (RT)^{2} = 1, (RL)^{2k+1} = (TLM)^{2k+1} = T
\]

and thus congruent to the monodromy group of a map \( \delta_{2k} \).

For a given map \( M \), denote by \( \epsilon_{5}(M) \), \( \epsilon_{6}(M) \) and \( \epsilon_{7}(M) \) the exponents of the words \( RT \), \( RL \), \( TLM \), respectively. For \( \delta_{2n} \) it follows \( \epsilon_{5} = 2 \), \( \epsilon_{6} = \epsilon_{7} = 2^{n+1} \).

Since these values are powers of 2 and \( \text{lcm}(2, 2^{n}) = \max(2, 2^{n}) \), at least one of \( \epsilon_{5}, \epsilon_{6}, \epsilon_{7} \) must be reached with the corresponding values \( e_{5}, e_{6}, e_{7} \) and \( e_{5}''e_{6}''e_{7}'' \) in two possible factors. Therefore, one of the factors should be one of \( \text{DM}_{7}(2^{n+1}) \), \( \delta_{2n} \) or \( \delta_{2n+1} \). In the case of \( \delta_{2n} \), we would not get a non-trivial product. In the case of \( \delta_{2n+1} \) the parallel product would be orientable, while \( \delta_{2n} \) is not orientable. Therefore, if we have a parallel-product decomposition, one of the factors must be \( \text{DM}_{7}(2^{n+1}) \). Then the other factor cannot be a degenerate map, because the context \( C \) is not sufficient to obtain the map \( \delta_{2n} \). Hence, one of the factors must be a map \( \delta_{l} \), for some \( l = 2^{n}, u < n \). But one can easily verify that in this case \( \text{DM}_{7}(2^{n+1}) \| \delta_{l} \simeq \delta_{2n+1} \). Thus \( \delta_{2n}, n \geq 1 \) is parallel-product indecomposable.

\[ \Box \]

Using computer programs LOWX [10] and MAGMA [7] all non-degenerate reflexible maps were calculated up to 100 edges. The results of the calculation match with Wilson’s census of rotary maps [39]. Among them, the ones with the monolithic monodromy group were selected and they are shown in Table 3.

**Theorem 7.6.** Up to triality, all parallel-product indecomposable non-degenerate reflexible maps up to 100 edges are presented in Table 3.

Using computer programs LOWX [10] and MAGMA [7] all non-degenerate reflexible maps were calculated up to 100 edges. The results of the calculation match with Wilson’s census of rotary maps [39]. Among them, the ones with the monolithic monodromy group were selected and they are shown in Table 3.

**Theorem 7.6.** Up to triality, all parallel-product indecomposable non-degenerate reflexible maps up to 100 edges are presented in Table 3.

| Name | \(|\text{Mon}| \) | \( \epsilon_{5} \) | \( \epsilon_{6} \) | \( \epsilon_{7} \) | Additional relations | Monolith |
|------|--------|--------|--------|--------|---------------------|--------|
| MN₁  | 24     | 3      | 3      | 4      | \((RTRL)^{2}, (LRT)^{2}(LR)^{2}\) | \( \mathbb{Z}_{2} \) |
| MN₂  | 32     | 4      | 8      | 8      | \((RTRL)^{2}, (LRT)^{2}(LR)^{2}\) | \( \mathbb{Z}_{2} \) |
| MN₃  | 60     | 3      | 5      | 5      | \((RTRL)^{2}, (LRT)^{2}(LR)^{2}\) | \( \mathbb{Z}_{2} \) |
| MN₄  | 64     | 4      | 4      | 4      | \((RTRL)^{2}, (LRT)^{2}(LR)^{2}\) | \( \mathbb{Z}_{2} \) |
| MN₅  | 64     | 4      | 8      | 8      | \((RTRL)^{2}, (LRT)^{2}(LR)^{2}\) | \( \mathbb{Z}_{2} \) |
| MN₆  | 64     | 4      | 16     | 16     | \((RTRL)^{2}, (LRT)^{2}(LR)^{2}\) | \( \mathbb{Z}_{2} \) |
| MN₇  | 72     | 4      | 4      | 6      | \((RTRL)^{2}, (LRT)^{2}(LR)^{2}\) | \( \mathbb{Z}_{2} \) |
| MN₈  | 96     | 3      | 8      | 12     | \((RTRL)^{2}, (LRT)^{2}(LR)^{2}\) | \( \mathbb{Z}_{2} \) |
| MN₉  | 96     | 6      | 8      | 12     | \((RTRL)^{2}, (LRT)^{2}(LR)^{2}\) | \( \mathbb{Z}_{2} \) |
| Name  | $c_5$ | $c_6$ | $c_7$ | Additional relations | Monolith |
|-------|-------|-------|-------|----------------------|---------|
| MN$_{10}$ | 108  | 3  | 6  | 6  | $L(RTRL)^2(RT)^2$ | $Z_3$ |
| MN$_{11}$ | 120  | 4  | 5  | 6  | $L(RT)^2 RL(RT)^3$, $T(LR)^3 TR(LR)^2$ | $A_5 \leq S_5$ |
| MN$_{12}$ | 120  | 6  | 6  | 6  | $L(RT)^2 RL(RT)^3$, $T(LR)^3 TR(LR)^2$ | $A_5 \leq S_5$ |
| MN$_{13}$ | 128  | 4  | 4  | 8  | $(RTL)^4$ | $Z_2$ |
| MN$_{14}$ | 128  | 4  | 16 | 16 | $(LRT)^2(RL)^2(RT)^2$, $(LR)^2 T(LR)^6 T$ | $Z_2$ |
| MN$_{15}$ | 128  | 4  | 32 | 32 | $(RTL)^2$, $(LRT)^2(LR)^4$ | $Z_2$ |
| MN$_{16}$ | 128  | 8  | 8  | 8  | $(LRT)^2(RL)^2(RT)^2$, $(LRT)^2(LR)^2(TR)^2$ | $Z_2$ |
| MN$_{17}$ | 128  | 8  | 16 | 16 | $(RTL)^2$, $(LRT)^2(LR)^4$ | $Z_2$ |
| MN$_{18}$ | 160  | 4  | 5  | 5  | $3$ | $Z_2$ |
| MN$_{19}$ | 192  | 3  | 6  | 8  | $2$ | $Z_2$ |
| MN$_{20}$ | 192  | 4  | 6  | 6  | $(T(LR)^2)^3$ | $Z_2$ |
| MN$_{21}$ | 192  | 6  | 6  | 8  | $LRTRLTLRL(RT)^2$ | $Z_2$ |
| MN$_{22}$ | 192  | 8  | 12 | 12 | $(RT)^2(LR)^2 TRLR$, $(LR)^3 TL(RL)^2 RT$ | $Z_2$ |
| MN$_{23}$ | 192  | 8  | 24 | 24 | $(RT(RL)^2)^3$, $(TLR)^3(LR)^3 T(LR)^4 TR$ | $Z_2$ |
| MN$_{24}$ | 200  | 4  | 4  | 10 | $(RTL)^5$ | $Z_2$ |
| MN$_{25}$ | 216  | 4  | 6  | 12 | $T(LR)^3$ | $Z_2$ |
| MN$_{26}$ | 216  | 6  | 12 | 12 | $L(RT)^2 RL(RT)^3$, $(TLRLR)^3 T(LR)^3$ | $Z_2$ |
| MN$_{27}$ | 256  | 4  | 4  | 8  | $2$ | $Z_2$ |
| MN$_{28}$ | 256  | 4  | 8  | 8  | $(LRTR)^2(LR)^2 TRLR$ | $Z_2$ |
| MN$_{29}$ | 256  | 4  | 16 | 16 | $(RTL)^4$, $(RT(RL)^2)^2, (LRT)^4(LR)^4$ | $Z_2$ |
| MN$_{30}$ | 256  | 4  | 32 | 32 | $(LRT)^3(RL)^2(RT)^2$, $(LR)^2 T(LR)^4 TR$ | $Z_2$ |
| MN$_{31}$ | 256  | 4  | 64 | 64 | $(RTL)^2$, $TLRT(LR)^3$ | $Z_2$ |
| MN$_{32}$ | 256  | 8  | 8  | 8  | $(LRT)^2(LR)^2(TR)^2$, $T(RTRLT)T(RTRL)^3$ | $Z_2$ |
| MN$_{33}$ | 256  | 8  | 16 | 16 | $(LTRLRT)^2$, $(RT)^3 RL(RT)^2(RL)^3$ | $Z_2$ |
| MN$_{34}$ | 256  | 8  | 16 | 16 | $(LRT)^2(RL)^2(RT)^2$, $(LRT)^3(RL)^2$ | $Z_2$ |
| MN$_{35}$ | 256  | 8  | 16 | 16 | $(LRT)^2(RL)^2 T(LR)^3 LTR$ | $Z_2$ |
| MN$_{36}$ | 256  | 8  | 16 | 16 | $(LRT)^2 T(LR)^3$ | $Z_2$ |
| MN$_{37}$ | 256  | 8  | 32 | 32 | $(RTL)^2$, $(LRT)^4(LR)^4$ | $Z_2$ |
| MN$_{38}$ | 300  | 3  | 6  | 10 | $2$ | $Z_2$ |
| MN$_{39}$ | 320  | 5  | 5  | 8  | $(LRTR)^2 T(LR)^2 TRLR$ | $Z_2$ |
| MN$_{40}$ | 320  | 5  | 8  | 10 | $(RT(RL)^2)^3$, $(TLR)^3 TR(LR)^2 TR$ | $Z_2$ |
| MN$_{41}$ | 320  | 8  | 10 | 10 | $(RT)^3(LR)^4 TL(TLR)^3 LR(TR)^3$ | $Z_2$ |
| MN$_{42}$ | 324  | 3  | 6  | 18 | $(LR)^2 T)^6$ | $Z_3$ |
| MN$_{43}$ | 324  | 6  | 6  | 9  | $(LTRL)^2 T(LR)^2 T, T(LR(TR)^2)^3$ | $Z_2$ |
| MN$_{44}$ | 324  | 6  | 9  | 18 | $(RT(RL)^2)^2$, $(LRT)^4 RL(RT)^2$ | $Z_3$ |
| MN$_{45}$ | 336  | 3  | 7  | 8  | $3$ | $Z_2$ |
| MN$_{46}$ | 336  | 3  | 8  | 8  | $(TLR)^2(LRT)^2(LR)^3 LT(RL)^2 R$ | $PSL(2,7)$ |
| MN$_{47}$ | 336  | 4  | 6  | 8  | $T(RTRL)^4$, $(RT(RL)^2)^3$ | $PSL(2,7)$ |
| MN$_{48}$ | 336  | 4  | 7  | 8  | $(RTL)^3$ | $PSL(2,7)$ |
| MN$_{49}$ | 336  | 6  | 6  | 8  | $(LTR)^3$, $(T(LR)^2)^3$ | $PSL(2,7)$ |
| MN$_{50}$ | 336  | 6  | 7  | 7  | $RTL(RT)^2 RL(RT)^2$ | $PSL(2,7)$ |
| MN$_{51}$ | 336  | 6  | 8  | 8  | $(RTL)^3$, $TL(RT)^2 LTRL(RT)^2$, $(T(LR)^2)^3$ | $PSL(2,7)$ |
| MN$_{52}$ | 384  | 4  | 6  | 24 | $(LRT)^3(RL)^2 TRL(RT)^2$ | $Z_2$ |
| MN$_{53}$ | 384  | 4  | 12 | 24 | $(LRT)^3 LTRLRL$ | $Z_2$ |
| MN$_{54}$ | 384  | 6  | 6  | 8  | $(RTL)^3$, $L(RT)^2(LR)^2 L(TR)^2 T(LR)^2 T$ | $Z_2$ |
| MN$_{55}$ | 384  | 6  | 6  | 8  | $(LRT)^3(RTRL)^2 R$ | $Z_2$ |
| MN$_{56}$ | 384  | 8  | 12 | 12 | $(T(LR)^2)^3$, $(LRT)^3 RL)^2$, $(RTL)^4, L(RT)^3(LR)^3 T$ | $Z_2$ |
| MN$_{57}$ | 384  | 8  | 12 | 12 | $(T(LR)^2 T(RL)^3 RTRLR, L(RT)^3(LR)^3 T$ | $Z_2$ |
For the maps MN₁ to MN₁₀ detailed descriptions are given in Table 4.

A genus symbol is a 6-tuple \([a, b, c, d, e, f] \) containing genera of maps \( M \), \( Du(M) \), \( Pe(M) \), \( Pe(Du(M)) \) and \( Du(Pe(Du(M))) \). If an entry \( x \) of a genus symbol is positive, then the corresponding map is orientable and its orientable genus is \( x \). If an entry \( x \) is negative then the corresponding map is non-orientable and its non-orientable genus is \(-x\). An isomorphism symbol is a 6-tuple \([[a, b, c, d, e, f]] \) that determines which among the maps from the sequence defined above are isomorphic. If two entries corresponding to two maps are equal then those maps are isomorphic. The hexagonal number is the number of different entries in an isomorphism symbol.

Table 4: Parallel-product indecomposable non-degenerate reflexible maps MN₁ to MN₁₀ in detail. If an underlying graph \( G \) has an edge multiplicity \( k > 1 \), the graph is denoted as \( G(k) \).

| Name  | Genus symb. | Hex. n. | Iso. symb. | Graph         |
|-------|-------------|--------|------------|---------------|
| MN₁   | [0, 0, −1, −1, −1, −1] | 3      | [[1, 1, 3, 5, 5]] | \( K_4 \)     |
| MN₂   | [2, 2, 2, 3, 2, 3] | 3      | [[1, 2, 1, 4, 2, 4]] | \( C_4(2) \) |
| MN₃   | [−1, −1, −1, −5, −1, −5] | 3      | [[1, 2, 1, 4, 2, 4]] | Petersen      |
| MN₄   | [1, 1, 1, 1, 1, 1] | 1      | [[1, 1, 1, 1, 1]]  | \( K_{4,4} \) |
| MN₅   | [3, 3, 5, 3, 5] | 3      | [[1, 2, 1, 4, 2, 4]] | \( K_{4,4} \) |
| MN₆   | [4, 4, 4, 7, 4, 7] | 3      | [[1, 2, 1, 4, 2, 4]] | \( C_8(2) \) |
| MN₇   | [1, 1, −5, −5, −5, −5] | 3      | [[1, 1, 3, 5, 5]]  | \( DK_{3,3,3} \) |
| MN₈   | [2, 2, 3, −16, 3, −16] | 6      | [[1, 2, 3, 4, 5, 6]] | Gen. Petersen \( G(8,3) \) |
| MN₉   | [6, 6, 7, −16, 7, −16] | 6      | [[1, 2, 3, 4, 5, 6]] | \( Q₃(2) \)    |
| MN₁₀  | [1, 1, 1, −11, 1, −11] | 3      | [[1, 2, 1, 4, 2, 4]] | Pappus         |

8 Edge-transitive maps

Automorphisms of edge-transitive maps can be studied by focusing on the situation around the edge with the root flag. In Figure 4, a set of automorphisms is defined according to how they map the root flag. In an edge-transitive map not all of those automorphisms are necessarily present. Let \( A \) be the set of all the named automorphisms in Figure 4. Note that those "named automorphisms" are not the real automorphisms, but more like the rules how the corresponding automorphisms should act, if they exist in an actual map. For a map \( M \), let \( A_M \) be a set of all automorphisms from \( A \) contained in \( M \). Actually, here we have in mind the set of the corresponding automorphisms of the map matching the rules defined by "named automorphisms" in \( A \). According to [14, 32], each edge-transitive map can be simply re-rooted, such that \( A_M \) is one of the fourteen edge-transitive types given in Table 5. Let \( A_T \) be a set of automorphisms ("rules") that a type \( T \) map should contain according to Table 5. There is a partial ordering relation \( \preceq \) on the set of
the types defined by $T \preceq T'$ ⇔ $A_T \subseteq A_{T'}$. A Hasse diagram for this ordering is shown in Figure 5. The rooting of an edge-transitive map in which the type can be read using Table 5 is called a canonical rooting.

An edge-transitive map can have at most two orbits of vertices, faces and Petrie-circuits. The degrees of the vertices in each of the orbits are denoted by $a_1, a_2$, the sizes of the faces by $b_1, b_2$, and the sizes of the Petrie circuits by $c_1, c_2$. By $(a_1, a_2; b_1, b_2; c_1, c_2)$ we denote the map symbol. If a map is vertex transitive then $a_1 = a_2 = a$ and we reduce the symbol to $(a; b_1, b_2; c_1, c_2)$. A similar rule extends to faces and Petrie circuits.

![Figure 4: Automorphisms "around" the edge $e$ with the root flag id.](image)

![Figure 5: The partial order of the types of edge-transitive maps.](image)

Let us consider a few properties of edge-transitive maps.

**Corollary 8.1.** Let $M$ be an edge-transitive map. The product of all 4 simply re-rooted maps is a reflexible map $N$ with $\text{Mon}(N) = \text{Mon}(M)$ and thus the smallest reflexible cover.

**Proof.** Note that from Proposition 4.4 it follows that $N \simeq M^M$. □

The following corollary follows immediately from Proposition 3.5.

**Corollary 8.2.** A monodromy quotient of an edge-transitive map of type $T$ is of type $T'$, such that $T \preceq T'$. □

The obvious corollary of Proposition 4.1 is the following.

**Corollary 8.3.** A parallel product of two canonically rooted edge-transitive maps of type $T$ is an edge-transitive map of type $T'$, such that $T \preceq T'$. □
Let \( \sigma \) be an automorphism group acting on a set \( V \). If \( \sigma \) is edge-transitive, then the orbits of \( V \) that determine the edges remain unchanged. The automorphism group \( \aut(Du(\sigma)) \) changes the role and becomes exactly \( \sigma \). Then \( \sigma^* = \sigma \) and \( \sigma^* = \sigma \).

Example 8.4. If we make a parallel product of two edge-transitive maps, the result need not be edge-transitive. By Proposition 4.1, only the lifts of common automorphisms are guaranteed. Even, for instance, if we make a parallel product of two simply re-rooted maps of type 4, where one map is rooted in \( \mathrm{id} \) and the other is rooted in \( \mathrm{id} \cdot T \) (relatively to the first map), the obtained parallel product in general may not be edge-transitive.

By Corollary 8.2 the following holds.

Corollary 8.5. If an edge-transitive map of type \( T \) is parallel-product decomposable then the factors are maps of type \( T' \), such that \( T' \succeq T \).

Consider now the impact of the operations \( Du \) and \( Pe \) on edge-transitive maps. For the purpose of an easier consideration, we denote \( 1 = 1^* = 1^P \) and \( 3 = 3^* = 3^P \).

Proposition 8.6. Let \( M \) be an edge-transitive map of type \( T \). Then \( Du(M) \) and \( Pe(\sigma) \) are also edge-transitive maps. Furthermore, if \( T \in \{1, 2, 2ex, 3, 4, 5\} \) then (by abusing the notation), the types of the map convert as follows:

\[
Du(T) = T^*, \quad Du(T^*) = T, \quad Du(T^P) = T^P, \\
Pe(T) = T, \quad Pe(T^*) = T^P, \quad Pe(T^P) = T^*.
\]

Proof. Performing the operation \( Du \) can be considered as a renaming (permuting) of the elements \( \{T, L, TL\} \). The orbits that determine the edges remain unchanged. The automorphism group \( \aut(Du(\sigma)) \) changes the role and becomes exactly \( \aut(Du(\sigma)) \), but the named automorphisms change their names according to the following. Let \( \sigma \in \aut(F) \) be an automorphism that defines the operation \( Du \). Then \( \sigma \in \aut(M) \), for \( W \in F \), acts like \( \sigma_{du(W)} \) in \( \aut(Du(\sigma)) \). For example, \( \sigma_{f_2} = \sigma_{trlt} \in \aut(M) \) acts like \( \sigma_{trlt} = \sigma_{f_2}^{-1} \in \aut(Du(\sigma)) \). A reader can easily verify, that for any type \( T \), the set \( T' \) is in a similar way transformed to a set \( \{\gamma_0, \ldots, \gamma_k\} \), where \( A_{T'} = \{\gamma_0, \ldots, \gamma_k\} \) and \( T' \) is exactly the transformation of the type \( T \) as claimed in the proposition. A proof for the operation \( Pe \) is similar.
Corollary 8.7. Each edge-transitive map can be obtained from some map of a type 1, 2, 2ex, 3, 4 or 5 by one of 6 possible compositions of the operations $D_u$ and $Pe$. □

As far as we are considering the analysis of edge-transitive maps through their automorphism (and also monodromy) groups, we can focus on the types 1, 2, 2ex, 3, 4, and 5. From now on we consider those types only.

From the classification in [14, 32] the partial presentations of automorphism groups of edge-transitive maps can be extracted. They are shown in Table 6. Note that the values of map symbols are used in presentations to denote partial presentations of maps having a prescribed map symbol. The relations, that are independent of a specific map symbol and therefore are present in any partial presentation of the corresponding type, are underlined in Table 6. The generators and those relations alone determine an universal automorphism group for the corresponding type. If a finite presentation of a group $G$ matches the partial presentation corresponding to a type $T$ (for some map symbol), we say that $G$ is of the type $T$. This means that $G$ is a finite quotient of the corresponding universal automorphism group.

| Type | A partial presentation for a given map symbol |
|------|-----------------------------------------------|
| 1    | $\langle \tau, \lambda, \theta_1 \mid \tau^2, \lambda^2, \theta_1^2, (\tau \lambda)^2, (\tau_1 \theta)^o, (\lambda \theta)_y, (\tau \lambda \theta)_z, \ldots \rangle$ |
| 2    | $\langle \tau, \theta_1, \theta_2 \mid \tau^2, \theta_1^2, \theta_2^2, (\tau_1 \theta_1)^o, (\tau_2 \theta_1)_y, (\tau_2 \theta_1)_z, \ldots \rangle$ |
| 2ex  | $\langle \tau, \sigma_f, \mid \tau_1^2, (\sigma_f)^z, \sigma_f^b, (\tau \sigma_f)^y, \ldots \rangle$ |
| 3    | $\langle \theta_1, \theta_2, \theta_3, \theta_4 \mid \theta_1^2, \theta_2^2, \theta_3^2, (\theta_1 \theta_2)^o, (\theta_2 \theta_3)^o, (\theta_2 \theta_4)^o, \ldots \rangle$ |
| 4    | $\langle \sigma_{x_1}, \sigma_{x_2}, \sigma_{x_3}, \mid \sigma_{x_1}^o, \sigma_{x_2}^o, \sigma_{x_3}^o, (\sigma_{x_1} \sigma_{x_2} \sigma_{x_3})^o, (\sigma_{x_1} \sigma_{x_2}^{-1} \sigma_{x_3})^o, \ldots \rangle$ |

Table 6: Partial presentations for automorphism groups of types 1, 2, 2ex, 3, 4, 5.

For a type $T$, a map $M$ is $T$-admissible if there is a subgroup $G \leq \text{Aut}(M)$, such that $G$ is generated by automorphisms $A_T$ and no $T' \geq T$ exists such that $G$ in $A_T \setminus A_T$ are contained in $G$. In this case, $G$ is called an $T$-admissible subgroup of $\text{Aut}(M)$. Note that $G$ is of type $T$.

To illustrate the situation here is an example.

Example 8.8. Take an orientable reflexible (type 1) map $M$. Then the orientation preserving subgroup $\text{Aut}^+(M)$ contains and is generated by $\{\varphi, \sigma_{x_1}, \sigma_{x_2}, \sigma_{f_1}, \sigma_{f_2}\}$. But this set is exactly $A_{2\text{ex}^R}$. The subgroup generated by the set does not contain any other named automorphisms of type $T \geq 2\text{ex}$. This is true because any other automorphism in Figure 4 is not orientation preserving and $M$ is orientable. Thus $M$ is $2\text{ex}^R$-admissible. Note that saying that a map is $2\text{ex}^R$-admissible in general means that a map is either of the type $2\text{ex}^R$ (chiral) or the type 1 (reflexible) and orientable. It is also equivalent to saying that the map is orientably regular.

It is obvious that every $T$-admissible automorphism subgroup of a map $M$ of type $T$ can be represented in a presentation matching the corresponding partial presentation of Table 6. The following proposition is about the construction of a map from a group in such a presentation.

Proposition 8.9. Any finite finitely presented group $K$ of type $T$ yields the unique $T$-admissible map $M$, such that $K$ is congruent to the $T$-admissible subgroup $G \leq \text{Aut}(M)$. The construction of $M$ is given in Table 7.
If such a rooted map $M$ existed then $\text{Aut}(M)$ would give rise to the unique labelling of the flags in the orbit containing the flag $\text{id} = \text{id}_M$ as follows. Let $G \leq \text{Aut}(M)$ be the $T$-admissible subgroup congruent to $K$. The orbits of $G \leq \text{Aut}(M)$ are blocks of imprimitivity for $\text{Mon}(M)$. Since $G$ is edge-transitive, there can be at most 4 orbits on the flags and a subgroup $Q = \langle \epsilon, T, L, TL \rangle$ of order (a most) 4 acts on the set of the orbits transitively. Since $Q$ is a small group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, one can easily verify that there is always a subgroup $S \leq Q$, such that $S$ acts regularly on the set of the orbits. Each flag $x$ can be uniquely labelled by a pair $(\alpha, w)$, $\alpha \in G$ and $w \in S$, such that $x = \alpha \cdot w$. To see that, let $x = \alpha_1(\text{id}) \cdot w_1 = \alpha_2(\text{id}) \cdot w_2$ for some $\alpha_1, \alpha_2 \in G$ and $w_1, w_2 \in S$. This would imply $\alpha_2^{-1}(\alpha_1(\text{id})) \cdot w_1 w_2^{-1} = \text{id}$. Since $\text{id}$ is in the same orbit as $\alpha_2^{-1}(\alpha_1(\text{id}))$ and $S$ acts regularly on the orbits, it first follows $w_1 = w_2$ and then by semi-regularity of $G$ it follows $\alpha_1 = \alpha_2$. Thus the labelling is unique and any edge-transitive map corresponds to the unique labelling $G \times S$.

The unique labelling alone already determines the map, since a label $(\alpha w, V)$ corresponds to the flag $\text{id} \cdot WV$. From this information it is straightforward to calculate the actions of the involutions $T$, $L$ and $R$ on the flags with the labels of the form $(\text{id}, w)$, $w \in S$. Since for $W \in \{ T, L, R \}$, $x = (\alpha, w)$, $\alpha \in \text{Aut}(M)$, it follows $x \cdot W = \alpha(\alpha^{-1}(x) \cdot W)$, the map is uniquely determined by the labelling.

Note that $S$ is determined by the type of the map. If the type is 1, 2, 2ex, 3, 4, 5, then, according to [14, 32], the corresponding sets $S$ are: $\{ \epsilon \}$, $\{ \epsilon, L \}$, $\{ \epsilon, T, L, TL \}$, $\{ \epsilon, T, L, TL \}$, $\{ \epsilon, T, L, TL \}$, respectively. From any finitely presented group $G$ corresponding to a type $T$, the unique labelling $G \times S$ and from that an $T$-admissible map are obtained. The construction for the types following the above description is presented in Table 7. Here $S$ is modelled by a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

For $\alpha, \beta \in \text{Aut}(M)$, $w \in S$, $x = (\alpha, w)$, it follows that $\beta(x) = \beta(\alpha \cdot w) = (\beta \circ \alpha, w)$. It is easy to verify that this is an action. This action of $G$ on the the labels is consistent with the action of $G$ on the flags. Also every named automorphism of

![Table 7: A construction of the corresponding T-admissible map from a partially presented group G of type T.](image)

**Proof.**
G maps the root flag exactly according to its name. Using this, a reader can verify that the maps obtained by the construction from Table 7 are indeed $T$-admissible. The conclusion of the proposition follows.

**Corollary 8.10.** Let $M$ be a $T$-admissible map and $G \leq \text{Aut}(M)$ the corresponding $T$-admissible subgroup. Then the flags of the map $M$ can be partitioned into the blocks of imprimitivity of $G$, such that $G$ acts regularly on the blocks.

**Proof.** Let $G \times S$ be the unique labelling from the proof of Proposition 8.9, where each flag $x \in \text{Flags}(M)$ can be uniquely labelled by $x = (\alpha, w)$, where $\alpha \in G$ and $w \in S$. Then $B_\alpha = \{ x = (\alpha, w) \mid w \in S \}$, $\alpha \in G$ determine the blocks. The action of $G$ on the labels is consistent with the action of $G$ on flags and is defined as $\beta \cdot B_\alpha = B_{\beta \alpha}$. Since $B_\alpha = B_\gamma$ if and only if $\alpha = \gamma$, the action is regular.

A similar approach in construction of maps from groups using finite presentations was used in [32], described in terms of an embedding of an associated Cayley graph in an orientable surface. One of the problems encountered in [32] was whether a finitely presented group matching a partial presentation for a type $T$ indeed induces a map of exactly the type $T$. They proved that if the group fulfills two conditions, it induces an associated Cayley map of an orientable edge-transitive map of exactly type $T$. The two conditions were essentially one forcing an orientability and one preventing other automorphisms in the obtained map that would imply a type $T' \succ T$. In Proposition 8.9 a generalized construction to obtain both orientable and non-orientable $T$-admissible maps from a finite finitely presented group of type $T$ is presented. Similar condition for limiting the group automorphisms as Condition 3.2 in [32] can be developed and by that extend some theorems from [32] on non-orientable maps.

The author of this work used the programs LOWX[9] and MAGMA[7] to calculate all possible presentations of automorphism groups of non-degenerate edge-transitive maps of types 1, 2, 2ex, 3, 4, 5 up to 100 edges. An edge-transitive map is non-degenerate if and only if all the values in a map symbol are greater or equal to 3. During the calculation all possible groups matching the partial presentations form Table 6 had to be calculated for the type 1 up to size 400, for the types 2 and 2ex up to size 200 and for the types 3, 4, 5 up to size 100. From those presentations one can by Proposition 8.9 construct all the corresponding $T$-admissible maps. All not $\text{Aut}(M)$-admissible maps were filtered out thus keeping the maps that are of the exact type as the presentation we started with.

The numbers of triality classes and the numbers of the maps obtained from them for edge-transitive types are shown in Table 8. For the type 1 (reflexible) and the type 2ex” (chiral) the numbers match with Wilson’s census of rotary maps [39].

| Type | Num. trial. class. | Num. all. maps |
|------|--------------------|----------------|
| 1    | 277                | 1223           |
| 2    | 3065               | 16044          |
| 2ex  | 66                 | 291            |
| 3    | 6033               | 30278          |
| 4    | 2980               | 11754          |
| 5    | 119                | 495            |

Table 8: Numbers of triality classes of non-degenerated edge-transitive maps and numbers of all maps that can be obtained from the classes using the operations $Du$ and $Pe$.  

□
Parallel-product decomposition can be applied to edge-transitive maps. The major obstacle to get a good characterization (like Theorem 7.1) for a parallel-product decomposability of an edge-transitive map of type $T$ is the non-regular action of the automorphism and the monodromy group. The problem can be solved by changing the presentation of the map, thus also changing the monodromy group.

An universal automorphism group for an edge-transitive map of type $T$ is any group $F = \langle \alpha_{w_1}, \ldots, \alpha_{w_k} | W_1 = \ldots = W_k = 1 \rangle$, where $\{\alpha_{w_1}, \ldots, \alpha_{w_k}\} \leq A_T$ is a set of named automorphisms and $\{W_1, \ldots, W_k\}$ is a set of relations, such that any automorphism group of any map $M$ of type $T$ is congruent to a quotient of $F$.

By Proposition 8.9, a $T$-admissible map $M$ is already determined by its $T$-admissible subgroup. Instead of using the construction in Table 7 one can work with different presentations of maps, not in terms of flags but in terms of merged flags, i.e. the blocks described in Corollary 8.10. But the question is, how should one define a new monodromy group, such that the automorphisms in the usual rooted map presentation would be also the automorphisms in the new presentation?

Consider the following example.

Example 8.11. Let $F = \langle \tau, \lambda, \rho | \tau^2 = \lambda^2 = \rho^2 = (\tau\lambda)^2 = 1 \rangle$ be an universal automorphism group for the type 1. Let $G$ be a quotient of $F$ that represents an automorphism group of a map $M$, and $f : F \to G$ the corresponding quotient projection. By Corollary 3.3, such a map can be represented as $M = (f, G, G, 1)$, where the projections of the generators $\tau, \lambda, \rho$ in the quotient are considered as $T, L, R$, respectively.

Now we illustrate the correspondence of the actions of the automorphism group and of the monodromy group. Let $N$ be any map of the type 1 (reflexible). Then $\tau(\text{id}) = \text{id} \cdot T$. Let $x \in \text{Flags}(N)$. By regularity there exists the unique $\alpha \in \text{Aut}(N)$, such that $x = \alpha(\text{id})$. Therefore $x \cdot T = \alpha(\text{id}) \cdot T = \alpha(\text{id} \cdot T) = \alpha(\tau(\text{id}))$. Thus if we label the flags of $N$ by the automorphisms, the right action of the monodromy group on the labels correspond to the action of $\text{Aut}(M)$ from the right, where $\tau, \lambda, \rho$ act like $T, L, R$, respectively.

The same concept can be used to define monodromy groups on maps with merged flags, such that the $T$-admissible subgroup of $\text{Aut}(M)$ acts regularly on the set of merged flags.

Note that this approach matches the concept of a reduced regularity introduced by A. Breda D’Azevedo [4] on hypermaps. Using the concept for defining new kinds of monodromy groups opens a new area of objects to be studied.

From now on, let $F_0 = \langle \tau, \lambda, \rho | \tau^2 = \lambda^2 = \rho^2 = (\tau\lambda)^2 = 1 \rangle$. Let $F = \langle \alpha_{w_1}, \ldots, \alpha_{w_k} | W_1 = \ldots = W_k = 1 \rangle$ be a universal automorphism group for a type $T$, where $\alpha_{w_i} \in A_T$, $w_r \in F_0$ and $W_s \in F$ relations. Let $G$ be a finite quotient of $F$ and $f : F \to G$ be the corresponding quotient projection. Define a generalized rooted map in the presentation $F$ as a quadruple $M = (f, G, G, 1)$, where $f : \text{Flags}(M) \to Z = \text{Flags}(M), \text{id})$, where $G$ acts transitively and faithfully from the right on some finite set $Z$ and $\text{id} \in Z$ is a root flag. The generators of the monodromy group are exactly $\{f(\alpha_{i})\}_{i = 1}^{k} \cup \{f(\alpha_{i}^{-1})\}_{i = 1}^{k}$, i.e. the images of the generators and their inverses in the presentation of $F$. Note that the monodromy group together with the chosen set of generators and their inverses determines the combinatorial and algebraic structure of a generalized rooted map. Therefore, a different choice of generators in general yields a completely different combinatorial and algebraic structure. This combinatorial structure is modelled as before by a corresponding colored graph $\text{Co}(M)$ that is the action graph of $\text{Mon}(M)$ determined by the chosen set of generators and their inverses. For the theory of action graphs see [25]. For generalized map presentations, morphisms, automorphisms, parallel product and quotients are defined in the same way as at the beginning of the paper. There we derived all the theory for the special presentation $F = F_0 = \langle \tau, \lambda, \rho | \tau^2 = \lambda^2 = \rho^2 = (\tau\lambda)^2 = 1 \rangle$, but instead of
the names of the generators $\tau$, $\lambda$, $\rho$, we used the names $t$, $l$, $r$, respectively. All the claims that did not include the structure of $F_0$ thus hold in general presentations of maps. Note that the only claims that actually used the structure of $F_0$ were the claims about the operations $Du$ and $Pe$. To differ this presentation from others, we will say that the map in this presentation is a map in an usual (rooted) map presentation.

Note that the concept of reduced regularity can be applied to any map $M$ with an usual presentation where merging of flags yields blocks of imprimitivity of $\text{Mon}(M)$, such that some subgroup of $\text{Aut}(M)$ acts regularly on the blocks. A new presentation may cause a loss of information meaning that there is no unique construction from a new presentation to the initial usually presented rooted map. In the edge-transitive case, Proposition 8.9 guarantees that the obtained reduced presentations are in one-to-one correspondence with the corresponding usual rooted map presentations.

Now let us prove a proposition that links monodromy groups and automorphism groups of generalized rooted maps in some presentation $F$. Note that regular generalized map means that $\text{Aut}(M)$ is regular on flags.

**Proposition 8.12.** Let $M$ be a generalized rooted map in presentation $F$. Then $|\text{Aut}(M)| \leq |\text{Flags}(M)| \leq |\text{Mon}(M)|$. There is equality if and only if the map is regular. In this case $G := \text{Mon}(M) \simeq \text{Aut}(M)$ and the generalized rooted map $M$ is isomorphic to $N = (f, G, G, 1)$.

**Proof.** Similarly like for an usual map presentation one can easily verify that $\text{Aut}(M)$ acts semiregularly on flags and $\text{Mon}(M)$ is transitive, thus $|\text{Aut}(M)| \leq |\text{Flags}(M)| \leq |\text{Mon}(M)|$.

Let $M$ be regular. To prove regularity of $\text{Mon}(M)$ it suffices to prove that the stabilizer $\text{Mon}(M)_{\text{id}}$ is trivial. Let $W \in \text{Mon}(M)$ and $\text{id} \cdot W = \text{id}$. Then for any $d \in \text{Flags}(M)$ there exists an automorphism $\alpha_d \in \text{Aut}(M)$, such that $\alpha_d(\text{id}) = d$. Thus

$$d \cdot W = \alpha_d(\text{id}) \cdot W = \alpha_d(\text{id} \cdot W) = \alpha_d(\text{id}) = d.$$ 

Therefore $W$ is contained in all the stabilizers and thus it is an element of the kernel of the action of $\text{Mon}(M)$ acting on $\text{Flags}(M)$. Since the action is faithful, it follows that $W = e$ and the action of $\text{Mon}(M)$ is regular.

On the other hand, if $\text{Mon}(M)$ is regular, let $d \in \text{Flags}(M)$. There is an unique element $W_d \in \text{Mon}(M)$, such that $d = \text{id} \cdot W_d$. Define $\alpha_d(x) = x \cdot W_d^{-1}W_dW_x$. By the regularity of $\text{Mon}(M)$, the mapping is well defined. Let $x \in \text{Flags}(M)$ and $W \in \text{Mon}(M)$. Then

$$\alpha_d(x \cdot W) = \alpha_d(\text{id} \cdot W_dW_x) = (\text{id} \cdot W_dW_x)(W_xW)^{-1}W_d(W_xW) =$$

$$= \text{id} \cdot W_dW_xW = (\text{id} \cdot W_dW_x) \cdot W = \alpha_d(x) \cdot W.$$ 

It is easy to see that $\alpha_d$ is one-to-one and thus onto. Thus, $\alpha_d \in \text{Aut}(M)$. Since for every $d \in \text{Mon}(M)$ it follows $\alpha_d(\text{id}) = d$, the group $\text{Aut}(M)$ is regular.

The mapping $\gamma : \text{Mon}(M) \rightarrow \text{Aut}(M)$, $\gamma : W_d \mapsto \alpha_d$ induces an isomorphism. Since $\text{id} \cdot W_dW_x = \alpha_d(\text{id}) \cdot W_x = \alpha_d(\text{id} \cdot W_x) = \alpha_d \circ \alpha_e(\text{id})$, the rest follows. \hfill $\square$

**Example 8.13.** To see an example of the use of Proposition 8.12, take a rooted map $M$ of the type $2ex^{2}$ (chiral) in an usual map presentation. From Table 6 we can see that such a map necessarily contains automorphisms $\sigma_{x_1}$ and $\varphi$. It is not hard to see that these two automorphisms generate $\text{Aut}(M)$. Let $F = (\sigma_{x_1}, \varphi \mid \varphi^2 = 1)$. Then $G := \text{Aut}(M)$ must be a quotient of $F$ with the quotient projection $f : F \rightarrow \text{Mon}(M)$. Hence $N = (f, G, G, 1)$ corresponds to the map $M$ but in the presentation $F$. Since the type of $M$ is $2ex^{2}$, the group $\text{Aut}(M)$ is not regular on $\text{Flags}(M)$. But in the new presentation $N$, the same automorphisms yield a regular
generalized rooted map. If we define $R := f(\sigma x_1)$ and $L := f(\varphi)$, we obtain a presentation that is often used when considering orientably regular maps. Note that the same procedure applies if $M$ is an orientable reflexible map. In this case the automorphisms in the new presentation are exactly the original automorphisms that preserve an orientation. Note also, that a corresponding flag graph is an action graph for generators $R, R^{-1}$ and $L$. This is the so called truncation of a map.

**Example 8.14.** For the type 2 take $F = \langle \tau, \theta_1, \theta_2 \mid \tau^2 = \theta_1^2 = \theta_2^2 = 1 \rangle$. Any finite quotient of $F$ determines a 2-admissable map as a regular generalized rooted map in the presentation $F$. If $f$ is the corresponding quotient projection, then the generators of the monodromy group are $f(\tau), f(\theta_1)$ and $f(\theta_2)$. But the monodromy group can be viewed as a monodromy group of some regular hypermap. Thus the study of 2-admissable maps is in a way equivalent to the study of regular hypermaps.

Since for an edge-transitive map the new monodromy group obtained using the concept of a reduced regularity is isomorphic (also congruent) to the automorphism group of the map, the final theorem immediately follows.

**Theorem 8.15.** An edge-transitive map $M$ is parallel-product decomposable if and only if $\text{Aut}(M)$ contains at least two minimal normal subgroups.

### 9 Conclusion

The main results of the paper are a survey and classification of the quotients of rooted maps, the decomposition theorem, its application to the classification of reflexible maps of at most 100 edges and its extension to edge-transitive maps. The necessary presentation theory using the concept of a reduced regularity [4] of edge-transitive maps is developed. The presentation theory can be extended beyond edge-transitive maps to introduce correspondences between different combinatorial objects of high symmetry. For instance, Example 8.14 shows that the classification of edge-transitive 2-admissible maps is about as hard as the classification of regular hypermaps. The study of several different objects of high symmetry (regular) is similar and depends only on the presentation of a universal automorphism group. For instance, a theory of highly symmetric abstract polytopes can be modelled in a similar way. Abstract polytopes have been studied extensively [27]. Much less is known about the chiral polytopes or other highly symmetric polytopes.

The decomposition theorem can be used with all such objects and the study of these can be reduced to the study of monolithic quotients of the corresponding universal automorphism group. Thus the importance of monolithic groups as monodromy groups of parallel-product indecomposable maps is emphasised.

Since the algorithm for constructing regular elementary abelian covers or regular maps is already developed [26], a next step could be to specialize that algorithm, so that the group of covering transformations would be a monolith in the monodromy group of the cover. Adding this operation to the set of operations $\{\text{Pe}, \text{Du}, \|\}$ would significantly reduce the set of parallel-product indecomposable maps. Another next step would be a study of monolithic groups with a non-abelian monolith. This seems to be a hard problem.

Some of the work in the big paper about Cayley maps [30] can also be extended to generalized rooted maps. The theory of this paper might also be useful in study of Cayley maps.

Similar approaches using a parallel product, a parallel-product decomposition and the introduced presentation theory can be used with any object with a semi-regular action of the automorphism group, where the object can be uniquely reconstructed from the group.
10 Acknowledgement

I would like to express my gratitude to my supervisors Tomaz Pisanski and Thomas W. Tucker and to Dragan Marušić for their guidance and support. I would also like to thank Dušanka Janezič and the National Chemical Institute in Ljubljana for letting me use their computer cluster Vrana. Also, the author acknowledges the extensive use of programs LOWX [9] and MAGMA[7].

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