Riemann reciprocity in higher dimensions

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Abstract

The reciprocity law for abelian differentials of first and second kind is generalized to higher-dimensional varieties. It is shown that $H^1(V)$ of a polarized variety $V$ is encoded in the Laurent data along a curve germ in $V$, with the polarization form on $H^1(V)$ corresponding to the one-dimensional residue pairing. This associates an extended abelian variety to $V$; if $V$ is an abelian variety itself, our construction “extends” it, even when $V$ is not a Jacobian.

1 Introduction

The reciprocity law, or bilinear relation, for abelian differentials of the first and second kind is classically formulated as follows. Let $X$ be a compact Riemann surface of genus $g$. Suppose $\omega$ is a holomorphic one-form (an abelian differential of the first kind) and $\eta$ is a meromorphic one-form with exactly one pole, necessarily of order $\geq 2$, at some point $p \in X$ (an abelian differential of the second kind). Let $A_1, \ldots, A_g, B_1, \ldots, B_g$ (resp. $A'_1, \ldots, A'_g, B'_1, \ldots, B'_g$) denote the periods of $\omega$ (resp. $\eta$) over a standard symplectic basis for $H_1(X, \mathbb{Z})$. Then

\begin{equation}
\sum_{j=1}^{g} A_j B'_j - A'_j B_j = 2\pi i \text{res}_p (f \eta),
\end{equation}

where $f$ is a holomorphic function defined near $p$, with $df = \omega$. 

The left-hand side of (1) can be more easily recognized for what it is when written as a matrix product

\[
(A_1 \ldots B_g) \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \begin{pmatrix} A'_1 \\ \vdots \\ B'_g \end{pmatrix}.
\]

The forms \(\omega\) and \(\eta\) represent cohomology classes in \(H^1(X, \mathbb{C})\) determined by their vectors of periods. The polarization form \(Q\) is defined as the dual of the intersection form on \(H_1(X, \mathbb{Z})\), and so the matrix product above equals \(Q([\omega], [\eta])\).

As to the right-hand side of (1), let \(g\) be a meromorphic function defined near \(p\) with \(dg = \eta\). Writing \(<f, g>\) for the residue pairing \(\text{res}_p(fdg)\), one has the following version of (1):

\[
Q([\omega], [\eta]) = 2\pi i <f, g>.
\]

It is this statement that we wish to generalize in this paper for varieties of dimension greater than one. The pointed Riemann surface is replaced by a smooth complex projective variety \(V\) with an irreducible ample divisor \(D\) and a regular point \(p\) on \(D\) (we do not assume that \(D\) is smooth). We also choose a smooth curve germ \(\mathcal{X}\) on \(V\) through \(p\), transversal to \(D\). The Chern class of \(D\) defines a polarization form \(Q\) on \(H^1(V, \mathbb{C})\). The claim is that \(Q\) is again expressed via the one-dimensional residue pairing. Specifically, let \(\omega\) be a holomorphic one-form (a simple abelian differential of the first kind) on \(V\), and let \(\eta\) be a closed meromorphic one-form on \(V\) with poles only along \(D\) (a simple abelian differential of the second kind). Such forms again represent cohomology classes in \(H^1(V, \mathbb{C})\) (Proposition 2.1), and in Theorem 5.1 we establish a similar relation

\[
Q([\omega], [\eta]) = (-1)^{n-1}2\pi i <f, g> ,
\]

where now \(f\) and \(g\) are, respectively, holomorphic and meromorphic functions on \(\mathcal{X}\), whose differentials are the one-forms \(\omega\) and \(\eta\) pulled back to \(\mathcal{X}\):

\[
df = \omega|_{\mathcal{X}} , \quad dg = \eta|_{\mathcal{X}},
\]

and \(<f, g> = \text{res}_p(fdg)\) denotes the residue pairing on \(\mathcal{X}\) at \(p\).

Returning to the one-dimensional case, with additional notation and terminology a more complete statement is possible. Let \(H = \mathbb{C}((z))\) (the field programming code goes here.)
of formal Laurent power series, \( \mathcal{H}_+ = \mathbb{C}[[z]] \), and \( \mathcal{H}' = \mathbb{C}((z))/\mathbb{C} \). The pairing \(<f,g> = \text{res}_{z=0} f dg\) on \( \mathcal{H} \) is skew-symmetric; it is non-degenerate on \( \mathcal{H}' \). Choosing a formal local parameter \( u : \hat{\mathcal{O}}_{X,p} \xrightarrow{\sim} \mathcal{H}_+ \) on \( X \) at \( p \) defines maps

\[
\Gamma(X, \mathcal{O}_X(*p)) \rightarrow \mathcal{H}' \quad \text{and} \quad \Gamma(X, \Omega^1_X(*p)) \rightarrow \mathcal{H}dz ,
\]

both of which will also be denoted \( u \).

Put \( K_0 = u(\Gamma(X, \mathcal{O}_X(*p))) \) and \( \Omega = \{ f \in \mathcal{H}' \mid df \in u(\Gamma(X, \Omega^1_X(*p))) \} \). Then \( K_0 \subset \Omega \) are each other’s annihilators in \( \mathcal{H}' \), i.e. \( \Omega = K_0^\perp \) and \(<,>\) induces a perfect pairing on \( K_0^\perp/K_0 \). The reciprocity law now says that there is a symplectic isomorphism

\[
(H^1(X, \mathbb{C}), Q) \rightarrow (K_0^\perp/K_0, 2\pi i <,>) .
\]

Noting that \( (H^1(X, \mathbb{C}), Q) \) is a polarized Hodge structure, we may go further and identify all of its components in terms of the Laurent data. Thus \( H^{1,0} = F^1 H^1(X, \mathbb{C}) \) is simply \( K_0^\perp \cap \mathcal{H}_+ \). As to the integral structure, put \( K = \{ f \in \mathcal{H}' \mid e^f \in u(\Gamma(X - \{p\}, \mathcal{O}^*)) \} \). Then \( K_0 \subset K \subset K_0^\perp \) and \( \Lambda := K/K_0 \) is isomorphic to \( H^1(X, \mathbb{Z}) \). Finally, let \( U \subset K_0^\perp/K_0 \) be the image of \( H^{0,1} \) under (4) and, in turn, let \( Z \) denote the preimage of \( U \) under the projection \( K_0^\perp \rightarrow K_0^\perp/K_0 \). Then \( Z \) is a maximal isotropic subspace of \( \mathcal{H}' \).

These results, most of which may already be found in [SW], led Arbarello and De Concini [AD] to codify triples \((Z,K_0,\Lambda)\) with properties as above under the name of extended abelian varieties. They are also called extended Hodge structures of weight one in [K]. Evidently, to each such there corresponds a unique polarized Hodge structure of weight one, although going back there are infinitely many choices (see (6.5)). Thus one has a “de-extension” map with infinite fibers

\[
\{\text{extended abelian varieties}\} \rightarrow \{\text{abelian varieties}\} .
\]

In this paper it is shown that all of the above generalizes to higher-dimensional varieties. We again meet the subspaces \( K_0 \subset \Omega \) of \( \mathcal{H}' \) with \( \Omega = K_0^\perp \), and a reciprocity law analogous to (4) holds (Theorem 6.2): there is a symplectic isomorphism

\[
(H^1(V, \mathbb{C}), Q) \rightarrow (K_0^\perp/K_0, (-1)^{n-1}2\pi i <,>) .
\]
We also construct the remaining components of an extended abelian variety whose associated Hodge structure is that of $H^1(V, \mathbb{C})$ for any variety $V$ with a divisor $D$ as above (Theorem 3.3). In particular, this applies when $V$ is an abelian variety and $D$ is its theta divisor. Thus we obtain a method of “inverting” the “de-extension” map (3), or “extending” a Hodge structure of weight one, even if the latter did not come from geometry. This may be useful in approaching the Schottky problem, which was, in fact, our original motivation.

We leave with a question suggested by the present work: does this generalize for differential forms of higher degree, and is there a good notion of an extended Hodge structure of weight higher than one?

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2 $H^1$ of a polarized variety

We will be studying a smooth complex projective variety $V$ with an irreducible ample divisor $D$, not necessarily smooth. By Grothendieck’s Algebraic De Rham Theorem [Gr],

\[ H^k(V - D, \mathbb{C}) \cong \frac{\Gamma(\tilde{\Omega}^k_V(\ast D))}{d\Gamma(\Omega_{V}^{k-1}(\ast D))}, \]

where the tilde in $\tilde{\Omega}^k_V(\ast D)$ denotes the subsheaf of $d$-closed forms in $\Omega^k_V(\ast D)$.

Proposition 2.1 With the above assumptions,

\[ H^1(V, \mathbb{C}) \cong H^1(V - D, \mathbb{C}) \cong \frac{\Gamma(\tilde{\Omega}^1_V(\ast D))}{d\Gamma(\Omega_{V}^{0}(\ast D))}. \]

Proof. First, assume $D$ is smooth. Then there is a sequence of sheaf complexes $0 \to \Omega^\bullet_V \to \Omega^\bullet_V(\log D) \to \Omega^\bullet_D \to 0$ inducing

\[ 0 \to H^1(V) \to H^1(V - D) \overset{P.R.}{\to} H^0(D) \overset{\gamma}{\to} H^2(V) \to \ldots \]

Here $P.R.$ stands for the Poincaré Residue and $\gamma$ denotes the Gysin map, which is obtained by applying Poincaré duality on the source and the target.
of the map $H_{2n-2}(D) \rightarrow H_{2n-2}(V)$, where $n = \dim_{\mathbb{C}} V$. The latter map is injective: the image of a suitable generator of $H_{2n-2}(D)(\cong \mathbb{C})$ is the fundamental class of $D$ in $H_{2n-2}(V)$, which cannot be zero by the irreducibility of $D$. Hence $\gamma$ is injective too, which proves the assertion.

In the general case, when $D$ is not smooth, the above argument does not work. Instead, consider the spectral sequence

$$E_2^{p,q} = H^p(V, \mathcal{H}^q_{\mathcal{O}_V}(\ast D)) \Rightarrow H^*(V - D, \mathbb{C}) .$$

It yields the very same exact sequence

$$0 \rightarrow H^1(V) \rightarrow H^1(V - D) \xrightarrow{R} H^0(D) \xrightarrow{\nu} H^2(V) \rightarrow \ldots$$

where now $R$ is the cohomological residue map induced by the sheaf isomorphism $\mathcal{H}^1_{\mathcal{O}_V}(\ast D) \cong \mathbb{C}_D$, and $\nu$ is the map sending $1_D$ to (the Poincaré dual of) the fundamental class of $D$ in $H^2(V)$ (see [GH], p. 458). Again, the map $\nu$ must be injective, i.e. $H^1(V) \cong H^1(V - D)$ in all cases. $\blacksquare$

We note that the isomorphism of the above Proposition makes the Hodge filtration on $H^1(V, \mathbb{C})$ obvious:

$$F^1 H^1(V, \mathbb{C}) = \Gamma(V, \Omega^1_{\mathcal{O}_V}) .$$

There is also a corresponding isomorphism of quotients

$$(8) \quad \frac{\Gamma(V, \tilde{\Omega}^1_{\mathcal{O}_V}(\ast D))}{\Gamma(V, \Omega^1_{\mathcal{O}_V}) + d\Gamma(V, \mathcal{O}_V(\ast D))} \cong H^1(V, \mathcal{O}_V) .$$

For future use, we will need to define it explicitly. Let $\mathcal{U} = \{U_\alpha\}$ be an affine open cover of $V$ or an acyclic refinement of such; thus $\check{H}^1(\mathcal{U}, \mathcal{O}_V) \cong H^1(V, \mathcal{O}_V)$. The Čech cochain $\{g_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_V)\}$ giving the image of $\eta \in \Gamma(V, \tilde{\Omega}^1_{\mathcal{O}_V}(\ast D))$ in $H^1(V, \mathcal{O}_V)$ can be described as follows:

$$\{g_{\alpha\beta}\} = \tilde{\delta}\{\mu_\alpha\} ,$$

where $\mu_\alpha$ is a meromorphic function on $U_\alpha$ such that $\eta|_{U_\alpha} - d\mu_\alpha$ is a holomorphic one-form on $U_\alpha$. In particular, when $U_\alpha$ is simply connected, we may assume that $\eta|_{U_\alpha} = d\mu_\alpha$, and if $\eta$ is already holomorphic over $U_\alpha$, then we may take $\mu_\alpha = 0$. 
3 Polarization and residues

The Hodge structure on $H^1(V, \mathbb{C})$ is polarized by the divisor $D$. Concretely, the cup product

$$Q : H^1(V, \mathbb{C}) \otimes H^1(V, \mathbb{C}) \xrightarrow{\cup} H^{2n}(V, \mathbb{C}) \xrightarrow{\int} \mathbb{C}$$

(9)

gives an integrally-defined perfect pairing. Here $c[D] \in H^2(V, \mathbb{Z})$ stands for the first Chern class of $\mathcal{O}_V(D)$, and $\int$ denotes the topological trace, i.e. the map $H^{2n}_{DR}(V, \mathbb{C}) \xrightarrow{\sim} \mathbb{C}$ obtained by integrating $C^\infty$ complex-valued $2n$-forms over $V$.

Thinking of $c[D]$ as an element of $H^1(V, \Omega^1_V)$, we will generally prefer a more algebraic version

$$Q : H^0(V, \Omega^1_V) \otimes H^1(V, \mathcal{O}_V) \xrightarrow{\cup} H^n(V, \mathcal{O}_V^n) \xrightarrow{\int} H^{2n}(V, \mathbb{C})$$

(10)

Remark 3.1 No notational distinction is made between the classes $[\omega]$, $[\eta]$ in $H^1(V, \mathbb{C})$ and in $H^0(V, \Omega^1_V)$ or $H^1(V, \mathcal{O}_V)$. Likewise, both pairings above are denoted $Q$. This should cause no confusion, since the pairings are compatible with the natural maps $H^0(V, \Omega^1_V) \hookrightarrow H^1(V, \mathbb{C})$ and $H^1(V, \mathcal{O}_V) \twoheadrightarrow H^1(V, \mathcal{O}_V)$.

Even more algebraically, $\int = (2\pi i)^n tr$, where $tr$ is Grothendieck’s trace isomorphism (cf. [D], (2.2)). As the latter is the least explicit part in the definition of $Q$, let us elaborate on it.

First recall from [L] and [GH] a construction of the local trace

$$H^n_{\{p\}}(V, \Omega^0_V) \xrightarrow{tr_p} \mathbb{C}. $$

Let $U$ be a polycylindrical neighborhood of $p$ in $V$. Then

$$H^n_{\{p\}}(V, \Omega^0_V) \cong H^n_{\{p\}}(U, \Omega^0_U),$$

by excision. And $U$ being Stein (and $n \geq 1$) implies

$$H^n_{\{p\}}(U, \Omega^0_U) \cong H^{n-1}(U - \{p\}, \Omega^0_U).$$

These isomorphisms are composed with the residue morphism

$$\text{Res} : H^{n-1}(U - \{p\}, \Omega^0_U) \xrightarrow{\cong} \mathbb{C}.$$
To define $\text{Res}$, let $t_1, \ldots, t_n$ be a coordinate system on $U$ centered at $p$. Thus $D_j = \{t_j = 0\}$ are $n$ smooth hypersurfaces in $U$ intersecting normally at $p$. We assume that $D_1 = D \cap U$, although this will not be necessary until the next section. Put $U_j = U - D_j$, and let $D^+ = D_1 + \ldots + D_n$. Evidently, the $U_j$’s form an acyclic open cover of $U - \{p\}$. Then a class in $H^{n-1}(U - \{p\}, \Omega^n_U)$ is represented by a holomorphic form on $U_1 \cap \ldots \cap U_n = U - D^+$. In fact, we may assume that this form is a restriction to $U_1 \cap \ldots \cap U_n$ of a meromorphic form $\psi \in \Gamma(U, \Omega^n_U(*D^+))$. The form $\psi$ can be expanded in a Laurent series:

$$\psi = \sum a_{k_1, \ldots, k_n} t_1^{k_1} \ldots t_n^{k_n} dt_1 \wedge \ldots \wedge dt_n .$$

Finally,

$$\text{Res } \psi \overset{\text{def}}{=} a_{-1, \ldots, -1} .$$

It is a basic fact of residue theory that $\text{Res } \psi$ is independent of the parameter system $(t_1, \ldots, t_n)$.

We note one property of $\text{Res}$:

$$\text{Res} : \Gamma(U, \Omega^n_U(*D^+)) \longrightarrow \mathbb{C}$$

for future reference. Assume $\psi \in \Gamma(U, \Omega_U(*D^+))$ has only simple poles along $D_2, \ldots, D_n$, i.e. can be written as

$$\psi = \frac{h dt_1 \wedge \ldots \wedge dt_n}{t_1^m t_2 \cdots t_n} ,$$

with $h$ a holomorphic function. Then

$$\text{Res } \psi = \text{res}_0 \frac{\tilde{h}}{t_1^m} ,$$

where $\tilde{h}(t_1) = h(t_1, 0, \ldots, 0)$, and res$_0$ denotes the usual residue at 0 of a meromorphic one-form in one variable.

The reason for bringing in the local trace is the following commutative diagram $\square$:

$$H^n_{(p)}(V, \Omega^n_V) \xrightarrow{\cong} H^n(V, \Omega^n_V)$$

$$tr_p \gtrless \cong \quad \cong' tr$$

$\mathbb{C}$

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To tie in with the above description of \( tr_p \), it remains to explain the isomorphism

\[
H^n(V, \Omega^n_V) \to H^n_{\{p\}}(V, \Omega^n_V) \to H^{n-1}(U - \{p\}, \Omega^n_U)
\]

in Čech cohomology. Take an affine (and hence acyclic) open covering \( U = \{U_\alpha\} \) of \( V \), so that \( H^n(V, \Omega^n_V) \cong \check{H}^n(U, \Omega^n_U) \). We may assume that the neighborhood \( U \) of \( p \) selected above is entirely contained in some \( U_\alpha \). Then, throwing in \( U_i = U - D_i \) and \( U_0 := U \), we get an acyclic refinement \( \check{U} \) of \( U \). Hence \( \check{H}^n(\check{U}, \Omega^n_V) \) is still isomorphic to \( H^n(V, \Omega^n_V) \).

Now, an \((n-1)\)-cochain over \( U - \{p\} \) with coefficients in \( \Omega^n_U \) is a section of \( \Omega^n_U \) over \( U_0 \cap U_1 \cap \cdots \cap U_n \). Regarding it as a section of \( \Omega^n_V \) over \( U_0 \cap U_1 \cap \cdots \cap U_n \) defines an \( n \)-cochain with coefficients in \( \Omega^n_V \). This is the cochain map underlying the isomorphism

\[
H^{n-1}(U - \{p\}, \Omega^n_U) \cong H^n(V, \Omega^n_V)
\]

To define its inverse, just read this backwards: take a cocycle in \( \check{C}^n(\check{U}, \Omega^n_V) \), isolate its component in \( \Gamma(U_0 \cap U_1 \cap \cdots \cap U_n, \Omega^n_V) \), and view it as an element in \( \Gamma(U_1 \cap \cdots \cap U_n, \Omega^n_U) \).

### 4 The Laurent data

Let \( p \) be a regular point on \( D \), and \( \mathcal{X} \) a germ of a smooth curve through \( p \), normal to \( D \), cut out by \( t_2 = \ldots = t_n = 0 \). \( z := t_1|_{\mathcal{X}} \) defines a coordinate on \( \mathcal{X} \), identifying \( \mathcal{O}_{\mathcal{X}, p} \) with \( \mathbb{C}[[z]] \). We will use the notation \( \mathcal{H} = \mathbb{C}((z)) \), \( \mathcal{H}_+ = \mathbb{C}[[z]] \), and \( \mathcal{H}' = \mathcal{H}/\mathbb{C} \), and we will write \( u \) for the Laurent expansion in terms of \( z \) of the global meromorphic objects on \( V \) restricted to \( \mathcal{X} \). Thus we have two compositions, both denoted \( u \):

\[
\Gamma(V, \mathcal{O}_V(*D)) \to \mathcal{O}_{\mathcal{X}, p}(*p) \to \mathcal{H} \to \mathcal{H}'
\]

and

\[
\Gamma(V, \Omega_V(*D)) \to \Omega^1_{\mathcal{X}, p}(*p) \to \mathcal{H}dz.
\]

Now we introduce two important subspaces of \( \mathcal{H}' \) carrying information about \( V \): \( K_0 := u(\Gamma(V, \mathcal{O}_V(*D))) \) and

\[
\Omega := \{ f \in \mathcal{H}' \mid df \in u(\Gamma(V, \check{\Omega}^1_V(*D))) \}
\]
It is obvious that $K_0 \subset \Omega$ and that $u$ (followed by $d^{-1} : \mathcal{H}dz \cong \mathcal{H}'$) induces a surjection
\begin{equation}
H^1(V, C) \cong \frac{\Gamma(V, \tilde{\Omega}_V^1(\ast D))}{d\Gamma(V, \mathcal{O}_V(\ast D))} \longrightarrow \frac{\Omega}{K_0}.
\end{equation}
We also have surjective morphisms
\[ H^0(V, \Omega^1_V) = \Gamma(V, \tilde{\Omega}_V^1) \longrightarrow \Omega \cap \mathcal{H}' \]
and
\[ H^1(V, \mathcal{O}_V) \cong \frac{\Gamma(V, \Omega^1_V)}{\Gamma(V, \Omega^1_V) + d\Gamma(V, \mathcal{O}_V(\ast D))} \rightarrow \frac{\Omega}{\Omega \cap \mathcal{H}' + K_0}. \]

The space $\mathcal{H}$ carries a symplectic form
\[ <f, g> = \text{res}_0 (fdg), \]
which is non-degenerate on $\mathcal{H}'$. This form induces a symplectic pairing, also denoted $< , >$
\begin{equation}
(\Omega \cap \mathcal{H}') \times \frac{\Omega}{\Omega \cap \mathcal{H}'} \rightarrow \mathbb{C}.
\end{equation}
A comparison of this with the polarization form $Q$ (10) constitutes our main result.

5 The generalized reciprocity law

**Theorem 5.1** Let $\omega \in \Gamma(V, \Omega^1_V)$ and $\eta \in \Gamma(V, \tilde{\Omega}_V^1(\ast D))$ represent classes in $H^0(V, \Omega^1_V)$ and $H^1(V, \mathcal{O}_V) \cong \Gamma(V, \tilde{\Omega}_V^1(\ast D))/\Gamma(V, \Omega^1_V) + d\Gamma(V, \mathcal{O}_V(\ast D))$, respectively, and suppose $u(\omega) = df$ and $u(\eta) = dg$ for some $f \in \Omega \cap \mathcal{H}'$ and $g \in \Omega$. Then
\[ Q([\omega], [\eta]) = (-1)^{n-1} 2\pi i <f, g>. \]

**Proof.** We will work in the covering $\tilde{\mathcal{U}}$ as above. The class $[\omega]$ is represented by the cochain $\{\omega_\alpha = \omega|_{U_\alpha}\}$, whereas $[\eta] = \{[g_{\alpha, \beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_V)]\}$ has been described as the end of Section 4. We single out
\[ g_{01} = \mu_1|_{U_0 \cap U_1} - \mu_0|_{U_0 \cap U_1} : \]
since \( U_0 = U \) is contractible, we may assume \( d\mu_0 = \eta|_U \), and since \( \eta \) is holomorphic over \( U_1 \), we may take \( \mu_1 = 0 \). Thus \( g_{01} = -\mu_0|_{U_0 \cap U_1} \) is a holomorphic function on \( U_0 \cap U_1 \), with \( \eta|_{U_0 \cap U_1} = -dg_{01} \).

And \( c_{[D]} \in H^1(V, \Omega^1_V) \) is represented by

\[
(14) \quad \left\{ -\frac{1}{2\pi i} \frac{dh_{\alpha \beta}}{h_{\alpha \beta}} + d\ell \right\},
\]

where \( h_{\alpha \beta} = 0 \) is a local equation of \( D \) in \( U_\alpha \cap U_\beta \) and \( \ell \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_V^*) \) extends across \( D \) as an invertible holomorphic function, if \( D \) meets the closure of \( U_\alpha \cap U_\beta \).

As explained in Section 3,

\[
Q(\omega, [\eta]) = (2\pi i)^n \text{tr} \left( [\omega] \sim [\eta] \sim c^{n-1}_{[D]} \right),
\]

and the trace can be computed locally, by taking the residue of the component at \( U_0 \cap U_1 \cap \ldots \cap U_n \) of a Čech cochain representing \( [\omega] \sim [\eta] \sim c^{n-1}_{[D]} \) in \( H^n(V, \Omega^1_V) \). Thus it suffices to take the wedge product of the restrictions to \( U_0 \cap U_1 \cap \ldots \cap U_n \) of \( \omega_0, g_{01} \), and of the Čech components of \( c_{[D]} \) over \( U_0 \cap U_1, U_1 \cap U_2, \ldots, U_{n-1} \cap U_n \). However, in the case of the cochain \( \left[ c_{[D]} \right] \) we have no information on the singularities these components have along the \( D_j \)'s, except the one over \( U_1 \cap U_2 \), which is \( -\frac{1}{2\pi i} \frac{dt_1}{t_1} + d\ell \). This is unsuitable for computing the residue. Thus we need other cochains representing \( c_{[D]} \).

Now, replacing a divisor by a linearly equivalent one does not affect the Chern class. And just as \( v_1 t_1^N \in \mathcal{O}_{V,p} \) is a germ of a global meromorphic function on \( V \) for an appropriate \( N \) and some \( v_1 \in \Gamma(U, \mathcal{O}_V^*) \), we shall assume temporarily that the same is true of the other coordinate functions \( t_2, \ldots, t_n \). Then \( v_1(t_j/t_1)^N \) \( (j = 2, \ldots, n) \) also come from meromorphic functions on \( V \). Consequently, for each \( j = 2, \ldots, n \) there is a cochain representing \( c_{[N,D]} \) whose component on \( U_j \cap \ldots \cap U_n \) is \( -\frac{N}{2\pi i} \frac{dt_j}{t_j} + d\ell_j \), with \( \ell_j \) extending as an invertible holomorphic function on all of \( U \). Dividing by \( N \) gives new cochains representing \( c_{[D]} \). Using these cochains,

\[
Q(\omega, [\eta]) = (2\pi i)^n \text{Res} \omega_0 g_{01} \wedge \left( -\frac{1}{2\pi i} \frac{dt_2}{t_2} + d\ell_2 \right) \wedge \ldots \wedge \left( -\frac{1}{2\pi i} \frac{dt_n}{t_n} + d\ell_n \right)
\]

\[
(15) = (-1)^{n-1} 2\pi i \text{Res} \omega_0 g_{01} \wedge \frac{dt_2}{t_2} \wedge \ldots \wedge \frac{dt_n}{t_n}
\]

The last reduction is possible because \( d\ell_j \)'s do not contribute to the residue, having no poles or zeroes along any component of \( D^+ \).
At this point we recall that the residue is independent of the parameter system (see, e.g. [1]). Thus we may return to the one in which \( t_1 \) is a local equation of \( D \), while \( t_2, \ldots, t_n \) are arbitrary.

By (11) the multidimensional residue in (13) reduces to the one-dimensional
\[
(-1)^{n-1}2\pi i \text{res}_0 (g_0 \omega_0 |_{\mathcal{X}}) = (-1)^{n-1}2\pi i \ < -g, f > ,
\]
and we end up with
\[
Q([\omega], [\eta]) = (-1)^{n-1}2\pi i \ < f, g > .
\]
\[\square\]

**Corollary 5.2** \( < x, y > = 0 \) whenever \( x \in \Omega \cap H'_+ + K_0 \) and \( y \in K_0 \). Consequently, the pairing (13) induces
\[
(\Omega \cap H'_+) \times \frac{\Omega}{\Omega \cap H'_+ + K_0} \to \mathbb{C} ,
\]
also denoted \( <, > \), and the surjection
\[
H^0(V, \Omega^1_V) \times H^1(V, \mathcal{O}_V) \to (\Omega \cap H'_+) \times \frac{\Omega}{\Omega \cap H'_+ + K_0}
\]
transforms the pairing \( Q \) on the source into \( (-1)^{n-1}2\pi i \ <, > \) on the target.
\[\square\]

**Corollary 5.3** The surjection
\[
H^1(V, \mathbb{C}) \cong \frac{\Gamma(V, \hat{\Omega}^1_V (\ast D))}{d\Gamma(V, \mathcal{O}_V (\ast D))} \to \frac{\Omega}{K_0}
\]
is a symplectic isomorphism.

**Proof.** The map in question induces a graded surjection
\[
H^0(V, \Omega^1_V) \oplus H^1(V, \mathcal{O}_V) \cong \\
\cong \Gamma(V, \Omega^1_V) \oplus \frac{\Gamma(V, \hat{\Omega}^1_V (\ast D))}{\Gamma(V, \Omega^1_V) + d\Gamma(V, \mathcal{O}_V (\ast D))} \\
\to (\Omega \cap H'_+) \oplus \frac{\Omega}{\Omega \cap H'_+ + K_0}
\]
which transforms the polarization form $Q$ on the left into the residue pairing $(-1)^{n-1}2\pi i < , >$ on the right. However, $Q$ is non-degenerate, i.e. for any non-zero $x \in H^0(V, \Omega^1_V)$ there exists $y \in H^1(V, \mathcal{O}_V)$ with $Q(x, y) \neq 0$, and similarly for any non-zero $y \in H^1(V, \mathcal{O}_V)$. Therefore, the image of any such $x$ or $y$ in $\Omega \cap \mathcal{H}_+^0$ (resp. in $\Omega/\Omega \cap \mathcal{H}_+^0 + K_0$) cannot be 0. This shows that the graded map is an isomorphism — in fact, a symplectic one, — which implies the same for the original map.

6 Extended Hodge structure

In this last section we will show that the above constructions can be completed to a full extended Hodge structure of weight one (= an extended abelian variety).

We already have $K_0 \subset \mathcal{H}$. Now we use the isomorphism $H^1(V, \mathbb{C}) \cong \Omega/K_0$ to define $U$ as the image of $H^{0,1} \subset H^1(V, \mathbb{C})$ in $\Omega/K_0$. Put $Z$ = the preimage of $U$ in $\Omega$ under the projection $\Omega \rightarrow \Omega/K_0$. Evidently, $U$ is a complement of $\Omega \cap \mathcal{H}_+^0$ in $\Omega/K_0$, and $Z$ is a complement of $\mathcal{H}_+^0$ in $\mathcal{H}$. And since $U$ projects isomorphically onto $\Omega/\Omega \cap \mathcal{H}_+^0 + K_0$ under the projection

\[
\frac{\Omega}{K_0} \rightarrow \frac{\Omega \cap \mathcal{H}_+^0 + K_0}{K_0},
\]

$U$ is perfectly paired with $\Omega \cap \mathcal{H}_+^0$ under the pairing induced by $< , >$ on $\Omega/K_0$. From this we deduce that $Z$ is a maximal isotropic subspace of $\mathcal{H}$. Furthermore, $\Omega \subseteq K_0^\perp$, i.e. $Z \subset K_0^\perp$. Thus $Z$ is sandwiched between $K_0$ and $K_0^\perp$:

\[
K_0 \subset Z \subset K_0^\perp.
\]

**Lemma 6.1** The subspaces $K_0$ and $\Omega$ of $\mathcal{H}$ are annihilators of each other with respect to the symplectic form $< , >$.

**Proof.** By (5.2) $K_0 \subseteq \Omega^\perp$. And by (5.3) and by non-degeneracy of $Q$, $< , >$ induces a nondegenerate pairing on $\Omega/K_0$. However, $< , >$ also induces a nondegenerate pairing on $\Omega/\Omega^\perp$, which is a surjective image of $\Omega/K_0$. But any symplectic surjection of a vector space with a non-degenerate symplectic form must be an isomorphism, as the proof of (5.3) shows. Hence $\Omega^\perp = K_0$. 

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Now, $\Omega \subseteq K^\perp_0$, hence $K^\perp_0 \subseteq \Omega^\perp = K_0$. But $K_0 \subseteq K^\perp_0$; so $K_0 = K^\perp_0$. Then $<,>$ is nondegenerate on $K^\perp_0/K^\perp_0 = K^\perp_0/K_0$. In particular, its maximal isotropic subspaces must be of dimension $\frac{1}{2} \dim (K^\perp_0/K_0)$. However, $Z/K_0$ is already a maximal isotropic subspace in $K^\perp_0/K_0$, and it is only of dimension $g = \frac{1}{2} \dim (\Omega/K_0)$. Therefore, $K^\perp_0 = \Omega$.  

Combining this lemma with Corollary 5.3 yields

**Theorem 6.2** There is a symplectic isomorphism

$$(H^1(V, \mathbb{C}), \; Q) \longrightarrow (K^\perp_0/K_0, \; (-1)^{n-1}2\pi i <,>) .$$

We now introduce the remaining components of the extended abelian variety. Define $\Lambda \subset \Omega/K_0 = K^\perp_0/K_0$ as the image of the lattice $H^1(V, \mathbb{Z})$ under the isomorphism $H^1(V, \mathbb{C}) \sim \to K^\perp_0/K_0$, and let $K$ be the preimage of $\Lambda$ under the projection $K^\perp_0 \to K^\perp_0/K_0$.

With this notation we may summarize our results as follows.

**Theorem 6.3** The triple $(Z, K_0, \Lambda)$ associated to the pointed polarized variety $(V, D, p)$ is an extended abelian variety.

Indeed, the definition of an extended abelian variety in [AD] calls for $Z$ to be a maximal isotropic subspace of $\mathcal{H}'$ with $\mathcal{H}' \oplus Z = \mathcal{H}'$, $K_0$ must be a subspace of $Z$ and $\Lambda$ a lattice in $K^\perp_0/K_0$ such that

$$(\Lambda, K^\perp_0/K_0, 2\pi i <,>)$$

constitutes a polarized Hodge structure of weight one, with the Hodge decomposition induced by the direct sum decomposition of $\mathcal{H}'$: the $(1, 0)$-component of $K^\perp_0/K_0$ is $K^\perp_0 \cap \mathcal{H}'_+$, and the $(0, 1)$-component is $(K^\perp_0 \cap Z)/K_0$. These conditions have been established already.

**Remark 6.4** In [SW] and [AD] the polarization form $Q$ on the first cohomology of a Riemann surface corresponds to $\frac{1}{2\pi i} <, >$ instead of our $2\pi i <, >$. The discrepancy is due to a different convention adopted in these papers: they identify $\Lambda$ with $H^1(X, 2\pi i \mathbb{Z})$ rather than $H^1(X, \mathbb{Z})$, as we do.

**Remark 6.5** The construction of an extended abelian variety associated to
V obviously depends on the choice of the ample divisor D and the point p. It also depends on the coordinate system \((t_1, \ldots, t_n)\) at p, without which we would not be able to define the Laurent expansions and hence the map \(u\). However, all such choices are equally good for our purposes.

We end with one last observation. In the curve case we had \(K = \{ f \in \mathcal{H}' \mid e^{2\pi i f} \in u(\Gamma(V - \{p\}, \mathcal{O}_V)) \} = u(\Gamma(X, \mathcal{O}_X(*p)/\mathbb{Z}))\).

It turns out, \(K\) admits a similar identification in the multidimensional situation.

**Lemma 6.6** \(K = u(\Gamma(V, \mathcal{O}_V(*D)/\mathbb{Z}))\).

**Proof.** Let us write \(\tilde{K}\) for \(u(\Gamma(V, \mathcal{O}_V(*D)/\mathbb{Z}))\). Our point of departure in identifying \(K\) with \(\bar{K}\) is the exact sequence

\[
0 \to \mathbb{Z}_V \to \mathcal{O}_V(*D) \to \mathcal{O}_V(*D)/\mathbb{Z} \to 0.
\]

The corresponding cohomology sequence reads, in part,

\[
H^0(V, \mathcal{O}_V(*D)) \to H^0(V, \mathcal{O}_V(*D)/\mathbb{Z}) \to H^1(V, \mathbb{Z}) \to H^1(V, \mathcal{O}_V(*D))
\]

The last term is isomorphic to \(H^1(V - D, \mathcal{O}_V) = 0\). Therefore, applying \(u\) yields

\[
0 \to K_0 \to \tilde{K} \to \bar{K}/K_0 \to 0.
\]

Thus \(u\) identifies \(H^1(V, \mathbb{Z})\) with \(\tilde{K}/K_0\). It is easy to see that \(\tilde{K} \subset \Omega\), and we conclude that \(\bar{K} = K\).  \(\square\)

**Remark 6.7** Connecting \(\Gamma(V, \mathcal{O}_V(*D)/\mathbb{Z})\) with \(\Gamma(V - D, \mathcal{O}_V^*)\) by means of the exponential sequence on \(V - D\), we also get

\[
K = \{ f \in \mathcal{H}' \mid e^{2\pi i f} \in u(\Gamma(V - D, \mathcal{O}_V^*)) \}.
\]

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