QCD Sum-Rule Invisibilty of the $\sigma$ Meson

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Abstract

QCD Laplace sum-rules for light-quark $I = 0,1$ scalar currents are used to investigate candidates for the lightest $q\bar{q}$ scalar mesons. The theoretical predictions for the sum-rules include instanton contributions which split the degeneracy between the $I = 0$ and $I = 1$ channels. The self-consistency of the theoretical predictions is verified through a Hölder inequality analysis, confirming the existence of an effective instanton contribution to the continuum. The sum-rule analysis indicates that the $f_0(980)$ and $a_0(1450)$ should be interpreted as the lightest $q\bar{q}$ scalar mesons. This apparent decoupling of the $f_0(400−1200)$ (or $\sigma$) and $a_0(980)$ from the quark scalar currents suggests a non-$q\bar{q}$ interpretation of these resonances.

1 Field-Theoretical Content of the Sum-Rule

The nature of the scalar mesons is a challenging problem in hadronic physics. In particular, a variety of interpretations exist for the lowest-lying isoscalar resonances [$f_0(400−1200)$, $f_0(980)$, $f_0(1370)$, $f_0(1500)$] and isovector resonances [$a_0(980)$, $a_0(1450)$] listed by the Particle Data Group (PDG). In particular, interpreting the $f_0(400−1200)$ (or $\sigma$) is particularly significant because of its possible interpretation as the $\sigma$ meson of chiral symmetry breaking. In this paper we will summarize and extend previous work which used QCD Laplace sum-rules to study the various possibilities for the lowest-lying, non-strange quark scalar mesons.

QCD sum-rules probe hadronic properties through correlation functions of appropriately chosen currents. In the $SU(2)$ flavour limit $m_u = m_d \equiv m$, the non-strange-quark $I = 0,1$ scalar mesons are studied via the scalar-current correlation function:

$$J_I(x) = \frac{m}{2} \left[ \bar{u}(x)u(x) + (-1)^I \bar{d}(x)d(x) \right] , \quad I = 0,1 \quad (1)$$
\[ \Pi_I(Q^2) = i \int d^4xe^{iq\cdot x} \langle O|TJ_I(x)J_I(0)|O \rangle \]  

(2)

Laplace sum-rules, which exponentially suppress the high-energy region, are obtained by applying the Borel transform operator \( \hat{B} \) to the appropriately-subtracted dispersion relation satisfied by (2) [3]:

\[ R_I^0(\tau) \equiv \frac{1}{\tau} \hat{B} \left[ \Pi_I(Q^2) \right] = \frac{1}{\pi} \int_0^{\infty} \text{Im} \Pi_I(t)e^{-\tau t} \, dt \]  

(3)

To leading order in the quark mass, the theoretical prediction for \( R_I^0 \) incorporates two-loop \( \overline{\text{MS}} \) scheme perturbative corrections [4], infinite correlation-length non-perturbative vacuum effects parametrized by the QCD condensates [3, 5], and finite-correlation length non-perturbative effects of instantons in the instanton liquid model [6, 7]:

\[ R_I^0(\tau) = \frac{3m^2}{16\pi^2\tau^2} \left( 1 + 4.821098 \frac{\alpha}{\pi} \right) + m^2 \left( \frac{3}{2} \langle m\bar{q}q \rangle + \frac{1}{16\pi} \langle \alpha G^2 \rangle + \pi \langle O_6 \rangle \right) \tau \]

\[ + (-1)^I \frac{m^2}{16\pi^2\tau^3} \left( \frac{\rho^2}{3} \right) \left[ K_0 \left( \frac{\rho^2}{2\tau} \right) + K_1 \left( \frac{\rho^2}{2\tau} \right) \right] \]  

(4)

where the quantity \( \rho = 1/(600 \text{MeV}) \) is the mean instanton size in the instanton liquid model [7]. The only theoretical source of isospin-breaking effects in (4) are instantons, which are known to have non-trivial contributions for only the scalar and pseudoscalar correlation functions.

We have used \( SU(2) \) symmetry for the dimension-four quark condensate contributions to (4) \( i.e. \langle \bar{u}u \rangle = \langle \bar{d}d \rangle \equiv \langle \bar{q}q \rangle \). The quantity \( \langle O_6 \rangle \) denotes the dimension-six quark condensates for which the vacuum saturation hypothesis [3] provides a reference value

\[ \langle O_6 \rangle = -f_{vs} \frac{88}{27} \alpha \langle \bar{q}q \rangle = -f_{vs}5.9 \times 10^{-4} \text{GeV}^6 \]  

(5)

where \( f_{vs} = 1 \) for exact vacuum saturation. Larger values of effective dimension-six operators found in [8] imply that \( f_{vs} \) could be as large as 2, suggesting a central value \( f_{vs} = 1.5 \). The quark condensate is determined by the GMOR (PCAC) relation, and the gluon condensate is given by [3]

\[ \langle \alpha G^2 \rangle = (0.045 \pm 0.014) \text{GeV}^4 \]  

(6)

Renormalization group improvement of (4) implies that \( \alpha \) and \( m \) are running quantities evaluated at the mass scale \( Q = \frac{1}{\sqrt{\tau}} \) in the \( \overline{\text{MS}} \) scheme. We use \( \Lambda_{\overline{\text{MS}}} \approx 300 \text{MeV} \) for three active flavours, consistent with current estimates of \( \alpha(M_\tau) \) and matching conditions through the charm threshold [1, 9].

Phenomenological analysis of the sum-rule (3) proceeds through the resonance plus continuum model [3]

\[ \text{Im} \Pi_I(t) = \text{Im} \Pi_I^{\text{res}} + \theta (t - s_0) \text{Im} \Pi_I^{QCD}(t) \]  

(7)

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where $\text{Im} \Pi_I^{\text{res}}$ denotes the resonance contributions, and $\text{Im} \Pi_I^{QCD}$ represents the theoretically-determined QCD continuum occurring above the continuum threshold $s_0$. Defining these continuum contributions as

$$c_I^0(\tau, s_0) = \frac{1}{\pi} \int_{s_0}^{\infty} \text{Im} \Pi_I^{QCD}(t)e^{-t\tau} \, dt$$  \hspace{1cm} (8)

leads to a revised sum-rule which isolates the theoretical and phenomenological (resonance) contributions:

$$S_I^0(\tau, s_0) \equiv R_I^0(\tau) - c_I^0(\tau, s_0) = \frac{1}{\pi} \int_{s_0}^{\infty} \text{Im} \Pi_I^{res}(t)e^{-t\tau} \, dt$$  \hspace{1cm} (9)

Traditionally, only the perturbative contributions are included in the continuum. However, the $Q^2$ analytic structure of the instanton contributions to $\Pi_I^{\text{inst}}(Q^2)$ implies the existence of an imaginary part $\text{Im} \Pi_I^{\text{inst}}(t)$ which leads to the following instanton continuum contribution \([10]\):

$$c_I^{0,\text{inst}}(\tau, s_0) = \frac{1}{\pi} \int_{s_0}^{\infty} \text{Im} \Pi_I^{\text{inst}}(t)e^{-t\tau} \, dt = (-1)^{I+1} \frac{3m^2}{8\pi} \int_{s_0}^{\infty} t J_1 \left( \rho_c \sqrt{t} \right) Y_1 \left( \rho_c \sqrt{t} \right) \, dt$$  \hspace{1cm} (10)

where $J_n(x)$ and $Y_n(x)$ denote Bessel functions. The instanton continuum contribution has been ignored in previous applications of instanton effects in sum-rules. It should be noted that this formulation of the instanton effects leads to improved IR behaviour when integrating over the instanton density because (10) approaches zero in the limit $\rho \to 0$.

### 2 Hölder Inequality Constraints

In the phenomenological analysis of QCD sum-rules, the behaviour of $S_0(\tau, s_0)$ as a function of Borel-parameter $\tau$ is used to extract the phenomenological resonance parameters through (9), raising the difficult question of the $\tau$ region where the theoretical prediction $S_0(\tau, s_0)$ is valid \([3]\). This question can be addressed via Hölder inequalities, which must be upheld if Laplace sum-rules are to be consistent with the physically-required positivity of $\text{Im} \Pi_I^{\text{res}}(t)$ within the integrand of (9) \([11]\):

$$\frac{S_I^0[\omega \tau + (1 - \omega)\delta \tau, s_0]}{(S_I^0[\tau, s_0])^\omega (S_I^0[\tau + \delta \tau, s_0])^{1-\omega}} \leq 1 \quad , \quad 0 \leq \omega \leq 1$$  \hspace{1cm} (11)

Provided that $\delta \tau$ is reasonably small ($\delta \tau \approx 0.1 \text{GeV}^{-2}$ appears to be sufficient \([11]\)), these inequalities are insensitive to the choice of $\delta \tau$, permitting a simple analysis of the inequality as a function of the Borel-parameter $\tau$.

The scalar-channel sum-rules satisfy the inequality in a fashion qualitatively similar to other channels \([11]\), supporting the self-consistency of the theoretical predictions. The instanton continuum \([10]\) is crucial to this agreement. Regions of validity in which the sum-rules satisfy the inequality \([11]\) are

$$0.3 \text{GeV}^{-2} \leq \tau \leq 1.7 \text{GeV}^{-2} \, , \, s_0 > 3 \text{GeV}^2 \quad (I = 0)$$  \hspace{1cm} (12)

$$0.3 \text{GeV}^{-2} \leq \tau \leq 1.1 \text{GeV}^{-2} \, , \, s_0 > 3 \text{GeV}^2 \quad (I = 1)$$  \hspace{1cm} (13)
3 Phenomenological Analysis

The sum-rule predictions of the properties of the lowest-lying $I = 0, 1$ quark scalar resonances can now be studied through (14). Since the resonances could have a substantial width, it is necessary to extend the narrow width approximation traditionally used in sum-rules. A flexible and numerically simple technique is to build up the resonance shape using $n$ unit-area square pulses [2, 12]

$$\frac{1}{\pi \text{Im}\Pi^{(n)}(t)} = \frac{2}{n\pi} \sum_{j=1}^{n} \sqrt{\frac{n-j+f}{j-f}} P_M \left[ t, \sqrt{\frac{n-j+f}{j-f}} \Gamma \right]$$

$$P_M(t, \Gamma) = \frac{1}{2M\Gamma} \left[ \Theta(t - M^2 + M\Gamma) - \Theta(t - M^2 - M\Gamma) \right]$$

A single square pulse models a broad nearly structureless contribution (such as a broad light $\sigma$) to $\text{Im}\Pi(t)$, while a Breit-Wigner resonance of a particle of mass $M$ and width $\Gamma$ can be expressed as a sum of several square pulses. The quantity $f$ can be fixed by normalizing the area of the $n$-pulse approximation to unity.

We begin the phenomenological analysis with the 4-pulse approximation (14) to $\text{Im}\Pi^{res}(t)$ so that (3) becomes

$$\frac{1}{\pi \text{Im}\Pi^{res}(t)} = F^2 M^4 \frac{1}{\pi \text{Im}\Pi^{(4)}(t)}, \quad S^I(\tau, s_0) = F^2 M^4 e^{-M^2 \tau} W_4(M, \Gamma, \tau)$$

$$W_4(M, \Gamma, \tau) = \frac{2}{4\pi} \sum_{j=1}^{4} \frac{1}{M\Gamma} \sinh \left[ M \sqrt{\frac{4-j+f}{j-f}} \Gamma \tau \right]$$

where $F$ is the strength with which the scalar current couples the vacuum to the resonance. The free parameters in this expression, the resonance-related quantities $F$, $M$, $\Gamma$ and the continuum-threshold $s_0$, can be extracted from a fit to the $\tau$ dependence of the theoretical expression $S^I(\tau, s_0)$. This is done by minimizing the $\chi^2$ defined by

$$\chi^2 = \frac{1}{N} \sum_{j=1}^{N} \frac{\left[ S^I(\tau_j, s_0) - F^2 M^4 e^{-M^2 \tau_j} W_4(M, \Gamma, \tau_j) \right]^2}{\epsilon(\tau_j)^2}$$

where the sum is over evenly spaced, discrete $\tau$ points in the ranges [12,13] consistent with the Hölder inequality. The weighting factor $\epsilon$ used for the evaluation of (18) is $\epsilon(\tau) = 0.2 S^I(\tau, s_0)$. This 20% uncertainty has the desired property of being dominated by the continuum at low $\tau$ and power-law corrections at large $\tau$. Other choices of the 0.2 prefactor in $\epsilon$ would simply rescale the $\chi^2$, so its choice has no effect on the values of the $\chi^2$-minimizing parameters.

In the $\chi^2$ minimization, the quark mass parameter $\hat{m}$ is now absorbed into the quantity $a = F^2 M^4 / \hat{m}^2$. The best-fit parameters are subjected to a Monte-Carlo simulation which includes the parameter ranges $1 \leq f_{\sigma} \leq 2$, a 15% variation in the instanton size $\rho$, and a simulation of continuum and OPE truncation uncertainties. This results in the 90% confidence level results for the best-fit parameters shown in Table [4]. Decreasing the number of pulses (to simulate a structureless resonance) does not alter the $\chi^2$, and
Table 1: Results of the Monte-Carlo simulation of 90% confidence-level uncertainties for the resonance parameters and continuum threshold for the $I = 0, 1$ channels.

| $I$ | $M$ (GeV)  | $s_0$ (GeV$^2$) | $a$ (GeV$^4$) | $\Gamma$ (GeV) |
|-----|------------|-----------------|--------------|--------------|
| 0   | 1.00 ± 0.09| 3.7 ± 0.4       | 0.08 ± 0.02  | 0.19 ± 0.14  |
| 1   | 1.55 ± 0.11| 5.0 ± 0.7       | 0.17 ± 0.04  | 0.22 ± 0.11  |

only leads to a rescaling of $\Gamma$. Two-resonance models recover the single-resonance results in Table 1, so there is no evidence of a hidden light resonance in either of the channels.

Thus we conclude that a QCD sum-rule analysis is consistent with the interpretation of the $f_0(980)$ and $a_0(1450)$ as the lightest non-strange quark scalar mesons. A light $\sigma$ meson [$f_0(400 - 1200)$] and the $a_0(980)$ appear to be decoupled from the quark scalar currents, suggesting a non-$q\bar{q}$ interpretation of these resonances.

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