Empirical Explorations in Training Networks with Discrete Activations

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Abstract

We present extensive experiments training and testing hidden units in deep networks that emit only a predefined, static, number of discretized values. These units provide benefits in real-world deployment in systems in which memory and/or computation may be limited. Additionally, they are particularly well suited for use in large recurrent network models that require the maintenance of large amounts of internal state in memory. Surprisingly, we find that despite reducing the number of values that can be represented in the output activations from $2^{32} - 2^{64}$ to between 64 and 256, there is little to no degradation in network performance across a variety of different settings. We investigate simple classification and regression tasks, as well as memorization and compression problems. We compare the results with more standard activations, such as tanh and relu. Unlike previous discretization studies which often concentrate only on binary units, we examine the effects of varying the number of allowed activation levels. Compared to existing approaches for discretization, the approach presented here is both conceptually and programatically simple, has no stochastic component, and allows the training, testing, and usage phases to be treated in exactly the same manner.

1 Introduction and Related Work

Almost all popular neural network training algorithms rely on gradient-based learning. For reliable computation of the gradients, it is useful for the hidden unit activations to be continuous and smooth. If the activation has large plateaus or discontinuities, gradient-based learning becomes difficult or even impossible. This is a large part of what motivated the move from neural networks based on synthetic discrete neurons, with hard thresholds, to the use of units with a sigmoid non-linearity, as well as the back-propagation algorithm to compute the gradients [1][2]. Interestingly, despite the use of the sigmoid non-linearity to smooth the derivatives, even early on in neural network research it was observed that often, after training, the units clustered their activations around the extrema [3] – thereby potentially under-utilizing their full representational capacity.

Recently, there has been renewed interest in using discrete outputs for the activation of the hidden units and weights of a trained network. Though perhaps closer in some regards to biologically plausible spiking neurons, much of the research in discretization of outputs and weights has stemmed from pragmatic concerns. These units provide benefits in the deployment of systems in which memory and/or computation may be limited, such as cell-phones and specialized hardware designed for the forward propagation of large networks. Additionally, they are particularly well suited for use in large recurrent network models that require the maintenance of large amounts of internal state in memory.
To date, most commonly, research has focused on “binarizing” the networks – both the weights and activations (larger alphabets are less often explored, though work has been done in that direction; see [3]). The focus of this paper is on the discretization of the outputs of the hidden units. Unlike previous studies, we will empirically examine what happens when the units are allowed to output between 2 to 256 discrete values. The trade-offs between higher cardinality alphabets and the number of employed hidden units will be examined in the experiments.

In the last decade, the use of rectified linear units, as well as a number of other non-linearities, has shown that it is possible to train networks that do not strictly adhere to the smoothness constraints often considered as necessary [5–7] – even with only few or no changes to the learning algorithms. When purely discrete outputs are desired, however, such as with binary units, a number of additional steps are normally taken [8, 2, 9–12] or evolutionary strategies used [13]. At a high level, many of the methods employ a stochastic binary unit and inject noise during the forward pass to sample the units and the associated effect on the network’s outputs. With this estimation, it is possible to calculate a gradient and pass it through the network. One interesting benefit of this method is its use in generative networks. In generative networks, if the units are employed stochastically in the forward propagation phase, they can be beneficial for generating multiple responses to a single input. For example, see [8] (in particular the task of generating the bottom half of hand written digits, given only the top-half). In the same work, [8] extend [11] to show that learning with stochastic units may not even be necessary if they are used within a larger deterministic network. Other existing binarization methods (e.g. [10]) liken the process to dropout [14] and its regularization effects. Instead of randomly setting activations to zero when computing gradients, they binarize both the activations and the weights.

2 Approach & Intuition

In this section, we describe the SUDO unit: Sigmoid-Underlying, Discrete-Output Units. In its simplest implementation, the SUDO unit is instantiated with a pre-defined set of output discretization levels, \( L \), that output a value between between a bounded range – e.g. either 0 and 1 or -1 and +1. See Figure 1. When the output is scaled between -1 and +1, the discretized output is computed as follows (shown in expanded form for clarity):

\[
\text{function SUDO\_Activation (input, levels)}:
\begin{align*}
\text{underlying} & \leftarrow \tanh(\text{input}) \\
\text{activation\_step} & \leftarrow 2/(\text{levels} - 1) \\
\text{plateauRange} & \leftarrow 2/\text{levels} \\
\text{output} & \leftarrow \left(\left\lceil \left(\text{underlying} + 1.0\right)/\text{plateauRange} \right\rceil - 1.0\right) \times \text{activation\_step} \\
\text{return} & \left(-1.0 + \text{output}\right)
\end{align*}
\]

- **On a practical note:** to replicate the results in this study or to use SUDO units within standard neural network packages, a potential unexpected behavior should be avoided. If the \( \tanh \) function is approximated as -1.0 for very small large negative values of \( \text{input} \) (the lower extrema) this function, as shown, yields an extra discretization level. This has been noted as a subtle problem in a popular language/package. There are many simple “tweaks” that can be used to avoid this issue that do not change the effectiveness of the procedure – such as clipping the outputs or multiplying the \( \tanh(\text{input}) \) by a number smaller than, but close to, 1.0 (e.g. 0.9999).

While there have been numerous studies that have examined the effects of binarizing activations, as described in the previous section, there have been fewer that have empirically examined
Figure 1: SUDO units, shown with 2, 4, 9, 64 and 256 levels. In the regions of largest slope in the underlying sigmoid/tanh function, the discretization levels change the fastest. There is no requirement to constrain the number of discretization levels to a power of 2 (as shown with $L = 9$), though in practice that may be preferred for memory efficiency.
the effects of increasing the number of discretization levels, \( L \). As we will demonstrate, as \( L \) is increased, many of the difficulties of training discretized units are mitigated, and simple mechanisms perform well. This allows the use of all the currently popular training algorithms with no modification.

Of course, naively backpropagating errors with SUDO-units will quickly run into problems as the activations are not well suited for calculating derivatives: they are both discontinuous and are largely characterized only by piece-wise constant functions. In contrast, standard sigmoid and tanh activations do not suffer from this problem. Relu units do partially share the same difficulties; however, because they are unbounded when positive, coupled with the fact that there is only a single discontinuity, gradient based methods continue to work.

In order to use gradient based methods with SUDO units, we need to define a suitable derivative. We simply use the derivative of the underlying function that we are discretizing. For the case shown above, the derivative is simply the derivative of \( \tanh(x) \) which is \( 1.0 - \tanh^2(x) \). In the forward pass of the network, the output of each unit is the discretized output. In the backward pass, we simply ignore the discretization and use the derivatives from the underlying function.

Why could this work in training? There are two important facets of this activation function to consider. First, if we had tried to use the discretized outputs in the backpropagation phase, the plateaus would not have given usable derivatives. By ignoring the discretizations, the weights of the network will still move in the wanted directions. The difference is, however, that unlike with standard tanh units, any single move may not actually change the unit’s output. In an extreme case, it is possible that with a low enough learning rate, the entire network’s output may not change despite all the weight changes made. This extreme case may cause a slow down in training, but, most importantly, it will not end training. Instead, in the next backprop phase, the weights will again be directed to move, and of those that move in the same direction, some will cross a discretization threshold. This changes the unit’s, and eventually, the network’s, output.

Second, we need to carefully examine Figure 1. Note that the plateaus are not evenly sized. This is most easily noticed in the middle two plots. Note that where the magnitude of the derivative for the underlying tanh function is maximum is where the plateau is the smallest size. This can be beneficial in practice; the unit’s output will change most rapidly where the derivative of tanh changes most rapidly, thereby keeping the expected movement closer to the real movement through the discretization.

To further our intuition of how these units perform in practice, we present three figures showing how SUDO, tanh and relu perform in tiny networks trained to fit a one dimensional parabola. The first, Figure 2, shows how well the parabola is fit as training progresses with a variety of activations. For all of the networks, there is a single linear output unit and two hidden units. The network is trained with Stochastic Gradient Descent with momentum (SGD+Momentum).

Perhaps the most telling results are the training curves with SUDO with \( L = 2 \) (SUDO-2). The resulting fit to the parabola matches closely with intuition; the different levels of discretization (almost) symmetrically reduce the error in a straightforward manner. As \( L \) is increased (moving to the right in the figure), the performance matches the networks trained with tanh and relu activations. Figures 3 and 4 show the same, but with networks that have 4 and 10 hidden units respectively. Again, the performance follows close to intuition.
Figure 2: Training to fit a parabola with 2 hidden units. Area is the error between the actual and predicted. Shown with hidden unit activations of (From Left to Right) Tanh, Relu, SUDO-2, SUDO-4, SUDO-8, SUDO-128 and SUDO-256. From Top to Bottom: Progress through epochs. This provides an intuitive demonstration for how the discretized units change the network’s performance. In the most clear example, with SUDO-2, we see that the effect of the binary discretization levels in approximating the parabola. The network has found a reasonable, symmetric, approximation, but cannot overcome the discretization artifacts – with only the 2 hidden units.
Figure 3: Training to fit a parabola with 4 hidden units. Area is the error between the actual and predicted. Shown with hidden unit activations of (From Left to Right) Tanh, Relu, SUDO-2, SUDO-4, SUDO-8, SUDO-128 and SUDO-256. From Top to Bottom: Progress through epochs.
Figure 4: Training to fit a parabola with 10 hidden units. Area is the error between the actual and predicted. Shown with hidden unit activations of (From Left to Right) Tanh, Relu, SUDO-2, SUDO-4, SUDO-8, SUDO-128 and SUDO-256. From Top to Bottom: Progress through epochs.
3 Experiments

In this section, we present a set of five experiments. While modest in scope in comparison to the large-scale vision tasks addressed by recent deep-learning research, they serve to elucidate important facets of the performance of the SUDO units. We explore a very large number of architectures, network models, and hyper-parameters to ensure that we are using the SUDO units effectively. We also allow the exact same amount of tuning to the baseline models for fairness.

3.1 Simple Binary Classification: Checkerboard

The goal of this problem is to correctly classify points on plane, based on their real-valued \(x, y\) coordinates, as belonging to either a 'black' or 'red' class. The target classification follows a checkerboard pattern, as shown in Figure 5. This is a real-valued version of the traditional X-OR-based problem used to analyze and study early neural network training \([15, 16]\).

Because we did not know whether the SUDO units would work well in a single layer or multiple, or even how many units should be used per layer, a very large variety of experiments are performed. Three basic architectures are used. The first has 2 input units, a single hidden layer with \(H\) units of type \(A\) and a single, tanh, output unit. We tested \(H\in\{5, 10, 20, 50, 100\} \times A\in\{\text{tanh, relu, SUDO}\{2, 4, 8, 16, 32, 64, 128, 256\}\}\. The results are shown in Table 1. Because this is a new training regime, three learning rates (0.001, 0.0001, 0.00001) were attempted with each activation/architecture combination and the best learning rate selected for each experiment. Each experiment is replicated three times. The results in the table show the average performance for the best setting, chosen individually for each experiment. Note that in this problem, as well as all the others explored in this paper, the relu and tanh units were given the exact same parameter tuning setup. The accuracies are measured on a set of 250,000 uniformly spaced points.

With a single hidden layer, the relu network performs the best when given a large number of hidden units. For the majority of settings of \(L\), the SUDO units perform more similarly to Figure 5: The 5000 samples chosen for training the binary classification checkerboard problem. This is a real-valued version of the classic X-OR problem for training neural networks.
the tanh units; this is a general trend that will be observed in most of the problems explored. Training with the larger networks did not provide a substantial benefit to either tanh or SUDO units. Next, lets examine what happens when the number of hidden layers is increased.

In the second architecture, a similar network was created to the first, but with 2 identically sized hidden layers. The same variants of $H$ and $A$ were explored for this architecture. All the layers are fully connected to the previous layer, with no skip connections between layers. The results are shown in Table 2. Here, the results have dramatically changed. First, note that tanh and SUDO-256 match or outperform relu units. Further, despite the fact that SUDO-256 can only output 256 unique values, tanh does not perform better. In fact, reducing $L$ to between 32 and 64 often rivals the best performances.

What about simple binary outputs or other lower cardinality units? Looking at SUDO-2 (and SUDO-4), we see that although they do not match the performance of the other activations when given a small number of hidden units per layer, as the number of hidden units increases, the simplest binary activations also perform well.

In the third architecture, we examine the performance of the SUDO units in a deeper network. Here, the network has 4 hidden layers, each with the same number of hidden units as in previous experiments. The results are consistent with those found earlier: even a low number of discrete levels, $L \geq 64$, matches the best performance. The SUDO units are able to keep up, and even surpass, the full resolution tanh and relu units.

Beyond the quantitative error measurements, it is illuminating to examine the decision surfaces created by the trained networks, see Figure 6. With a single hidden layer, the relu layers perform better than the tanh or SUDO units; samples with a single hidden layer are shown in the first row of Figure 6.

The second row shows that when multiple hidden layers are employed, the SUDO units (across all of the L discretization levels above 32) perform similarly to relu and tanh.

In the final two row of Figure 6, results with 4 hidden layers are shown. With 200 units, the results look very close to perfect. In the last row, with only 5 hidden units, relu appears significantly worse than with 200 hidden units and significantly worse than SUDO-256.

In the above description, we have largely concentrated on the SUDO units with $L = 256$. How do SUDO units with $L = 2$ carve the decision surface? In Figure 7, results with 10, 50 and 200 units are shown. The SUDO units with $L = 2$ perform well, with 200 units, the results look very close to perfect. In the last row, with only 5 hidden units, relu appears significantly worse than with 200 hidden units and significantly worse than SUDO-256.

An interesting note to this problem is that with a single hidden layer, SUDO-8/16/32 performed well while other $L$ settings did not. This is likely due to the number of decision boundaries that need to be placed to solve this problem accurately (see Figure 6(d,e)). We did not find this to be a trend in other problems.
Table 2: Checkerboard Accuracies with 2 Hidden Layers

| Activation | Hidden Units Per Layer | 5   | 10  | 50  | 100 | 200 |
|------------|------------------------|-----|-----|-----|-----|-----|
| tanh       | 81.3%                  | 97.5% | 98.1% | 97.8% | 97.0% |
| relu       | 71.9%                  | 91.1% | 98.0% | 97.7% | 97.8% |
| sudo-2     | 54.4%                  | 71.1% | 97.4% | 98.4% | 97.6% |
| sudo-4     | 70.8%                  | 95.8% | 98.7% | 97.0% | 96.6% |
| sudo-8     | 77.8%                  | 94.9% | 97.8% | 96.9% | 97.0% |
| sudo-16    | 81.8%                  | 96.4% | 97.6% | 97.6% | 96.8% |
| sudo-32    | 81.0%                  | 96.9% | 98.0% | 97.7% | 97.0% |
| sudo-64    | 81.7%                  | 97.7% | 98.0% | 98.0% | 97.1% |
| sudo-128   | 83.2%                  | 97.2% | 98.2% | 97.6% | 97.5% |
| sudo-256   | 84.1%                  | 96.8% | 98.0% | 97.9% | 96.7% |

Table 3: Checkerboard Accuracies with 4 Hidden Layers

| Activation | Hidden Units Per Layer | 5   | 10  | 50  | 100 | 200 |
|------------|------------------------|-----|-----|-----|-----|-----|
| tanh       | 92.8%                  | 98.1% | 98.4% | 98.4% | 98.3% |
| relu       | 81.8%                  | 97.1% | 97.9% | 97.8% | 98.1% |
| sudo-2     | 53.8%                  | 67.9% | 89.8% | 96.1% | 96.8% |
| sudo-4     | 66.3%                  | 91.5% | 98.4% | 97.2% | 97.3% |
| sudo-8     | 80.7%                  | 97.4% | 98.2% | 98.0% | 98.0% |
| sudo-16    | 88.4%                  | 98.0% | 98.3% | 98.1% | 98.0% |
| sudo-32    | 91.1%                  | 98.1% | 98.2% | 97.7% | 97.8% |
| sudo-64    | 92.1%                  | 98.5% | 98.4% | 98.4% | 97.9% |
| sudo-128   | 91.0%                  | 97.7% | 98.3% | 98.4% | 98.3% |
| sudo-256   | 93.5%                  | 98.3% | 98.4% | 98.4% | 98.3% |

200 units are shown. With 200 units, even the binary units perform well. In contrast, with only 10 units, the boundaries are less aligned with the underlying distribution.
Figure 6: Decision surfaces for the Checkerboard problem. (a-c): networks with a single hidden layer with 200 units. Relu, tanh, and SUDO-256 activation units shown. (g-f) Networks with 2 hidden layers. (g-l) Networks with 4 hidden layers, with 200 and 5 hidden units per layer.

Figure 7: Decision surfaces for networks with 4 hidden layers and binary activations in the hidden units (SUDO-2). Shown (a-c) with varying numbers of hidden units 10,50 and 200.
3.2 Simple Regression

An important unanswered question remains after examining only classification problems, such as the Checkerboard problem described in the previous section. Does the discretized nature of the activation provide an advantage over continuous activations by easily creating sharp decision boundaries? And, if that advantage is present, then will the SUDO units fare worse when asked to smoothly approximate response surfaces? In this experiment, we examine how well the SUDO units can be used to approximate smooth curves. The goal is to understand whether it is possible for the networks with discrete outputs to represent a regression function as accurately as is represented with continuous and relu units.

Consider the function $z = \sin(x \times 10.0) \times \cos(y \times 5.0)$, in the range of $x = [-1, 1]$ and $y = [-1, 1]$. (See Figure 8). How does the discretized output affect the ability of the network to approximating this surface?

Two basic network architectures were employed for these experiments. The first had 2 hidden layers and the second had 4. For each architecture, the number of hidden units ($H$) in each layer was varied $H \in \{10, 20, 50\}$. Three learning rates were attempted for each experiment (0.001, 0.0001, 0.00001). The best learning rate was used for each setting (e.g. each cell of the table). Each trial was replicated 5 times with random weights. The results are shown in Tables 4 & 5.

Compared with the previous experiments, here, the setting of $L$ has an exaggerated role in the performance of the approximation — especially with networks with 10 and 20 hidden units. Small values of $L$ perform significantly worse than larger $L$ and tanh and relu units. Intuitively, this makes sense as with fewer hidden units, there are fewer ways of combining the limited number of output values to approximate the real values.

Another important method of looking at these results is where the network’s memory “bits” are allocated. For example, if we allow 160 bits of state to be kept, how should they be allocated, given Table 4? One possibility is to create a network with 2 hidden layers, 20 hidden units per layer and $L = 16$. From the table, we see that this would yield an error of 0.60. Alternatively, if we use only 10 hidden units in the same architecture, we could double the bits per hidden unit. However, that would yield a larger error. The number of bits allocated for the representation is

Figure 8: The surface to be approximated: $z = \sin(x \times 10.0) \times \cos(y \times 5.0)$. 

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Table 4: Error in approximating a real-value function. Networks have 2 hidden layers.

| Activation | Hidden Units Per Layer |
|------------|------------------------|
|            | 10  | 20  | 50  |
| tanh       | 0.54| 0.23| 0.11|
| relu       | 0.92| 0.36| 0.11|
| sudo-2     | 13.72| 7.70| 3.39|
| sudo-4     | 6.69 | 2.04| 0.78|
| sudo-8     | 3.67 | 0.96| 0.38|
| sudo-16    | 2.51 | 0.60| 0.21|
| sudo-32    | 1.40 | 0.48| 0.13|
| sudo-64    | 1.12 | 0.37| 0.10|
| sudo-128   | 0.85 | 0.32| 0.14|
| sudo-256   | 0.71 | 0.25| 0.12|

Table 5: Error in approximating a real-value function. Networks have 4 hidden layers.

| Activation | Hidden Units Per Layer |
|------------|------------------------|
|            | 10  | 20  | 50  |
| tanh       | 0.12| 0.02| 0.01|
| relu       | 0.33| 0.09| 0.02|
| sudo-2     | 15.84| 7.85| 4.09|
| sudo-4     | 8.20 | 2.39| 0.69|
| sudo-8     | 2.77 | 0.82| 0.32|
| sudo-16    | 1.20 | 0.41| 0.16|
| sudo-32    | 0.87 | 0.30| 0.08|
| sudo-64    | 0.35 | 0.16| 0.08|
| sudo-128   | 0.21 | 0.07| 0.04|
| sudo-256   | 0.19 | 0.04| 0.02|

Not sufficient to predict the performance. How the bits are distributed (the architecture and discretization) must be considered.

In Table 5, the experiment is repeated with 4 hidden layers. Again, for each cell, the best learning rate is determined independently. Beyond the improved performance across all of the experiments, notice that the SUDO units ($L = 256$) are able to perform similarly to relu and tanh units. As before, the more bits that we allocate to increasing $L$ (while keeping everything else constant) improves performance. Even with 50 units, note that the performance of simple binary units (SUDO-2), as well as other low $L$ settings, remains relatively poor.
3.3 Effective Network Capacity / Memorization

Though the use of networks as simple associative memory devices [17–20] has largely fallen out of research favor, comparing how much a network can memorize with different activation functions may yield insight into the relative size of networks needed. Here, we examine how accurately a network can reconstruct an image given only each pixel’s \([x, y]\) coordinate. Unlike the other tasks presented here, as well as those most commonly explored in the research literature, there is no explicit notion of generalization. The network is trained on a single grayscale image and then queried with an arbitrary pixel’s coordinates to retrieve its intensity.

In these experiments, the original image is a grayscale intensity image of size 150 \(\times\) 150. The image has been post-processed with pseudo-HDR to ensure variation in intensity values. It is shown in Figure 10(a).

The results are not indicative of a theoretical measure of network capacity. They are, however, indicative of the practical accessible capacity given a particular learning approach. The learning approach (in this case ADAM [21] and/or SGD+Momentum) has a very large role in determining how the network represents the information – both the form and efficiency. Different learning approaches may be able to better utilize a network’s capacity. Nonetheless, we propose that this is an interesting experiment as (1) the tested optimization procedures are the most commonly used. And, (2), the same learning procedure is used across all of the trials with all of the hidden units, yielding relative results valid for comparison.

Table 6: SSE memorizing image. Networks have 1 hidden layer. Last Activation is tanh.

| Activation | Hidden Units Per Layer |
|------------|------------------------|
|            | 50 | 100 | 200  |
| tanh       | 1509.18 | 1535.56 | 1547.50 |
| relu       | 1436.83 | 1386.76 | 1370.26 |
| sudo-2     | 1702.52 | 1715.54 | 1750.73 |
| sudo-4     | 1648.93 | 1644.20 | 1649.05 |
| sudo-8     | 1608.58 | 1634.98 | 1597.16 |
| sudo-16    | 1541.39 | 1535.25 | 1533.82 |
| sudo-32    | 1498.50 | 1511.71 | 1546.28 |
| sudo-64    | 1529.45 | 1530.31 | 1563.23 |
| sudo-128   | 1520.51 | 1541.00 | 1544.91 |
| sudo-256   | 1530.65 | 1529.95 | 1568.84 |

When using a single hidden layer, the discrete activations can achieve similar performance to the tanh units across all sizes of hidden layers. However, discretization levels between 32-256 are required. The relu units have a clear advantage over both. Next, in Table 7 and Table 8, we examine the performance with 2 and 4 hidden layers. One of the peculiarities of these results is that for tanh, a larger network did not consistently yield improved results. Since the performance of SUDO units often approximates tanh, this trend also extended to many SUDO settings. This is likely due to the settings of the training hyper-parameters. Nonetheless, we did not change them for these experiments to keep them consistent across the paper.

In Tables 7 & 8, the performances of tanh and SUDO-256 are similar: relu continues to outperform both. This is the largest difference in performance between tanh/SUDO and relu witnessed in this paper. To see if the disparity continues, two extra experiments are constructed.

\(^2\)Note that this task is not the same as autoencoding images (autoencoding will be explored in Section 3.4). For autoencoding, the entire image is presented at once, and usually many images can be reconstructed by the network. In contrast, for these experiments, the network is only given the coordinates of the pixel and must recall the pixel’s correct intensity value for the single training image.
Table 7: SSE memorizing image. Networks have 2 hidden layers. Last Activation is tanh.

| Activation | Hidden Units Per Layer |
|------------|-----------------------|
|            | 50        | 100        | 200        |
| tanh       | 1285.85   | 1289.60    | 1319.10    |
| relu       | 1201.02   | 1152.29    | 1145.68    |
| sudo-2     | 1505.62   | 1454.44    | 1370.68    |
| sudo-4     | 1446.20   | 1448.04    | 1462.08    |
| sudo-8     | 1367.80   | 1424.81    | 1458.70    |
| sudo-16    | 1265.13   | 1323.53    | 1350.90    |
| sudo-32    | 1266.70   | 1292.24    | 1372.25    |
| sudo-64    | 1267.70   | 1298.24    | 1372.25    |
| sudo-128   | 1233.91   | 1298.23    | 1301.24    |

Table 8: SSE memorizing image. Networks have 4 hidden layers. Last Activation is tanh.

| Activation | Hidden Units Per Layer |
|------------|-----------------------|
|            | 50        | 100        | 200        |
| tanh       | 878.56    | 895.68     | 935.90     |
| relu       | 937.17    | 871.23     | 852.40     |
| sudo-2     | 1479.91   | 1418.88    | 1372.66    |
| sudo-4     | 1292.34   | 1263.84    | 1334.85    |
| sudo-8     | 994.39    | 1168.80    | 1244.86    |
| sudo-16    | 916.12    | 988.33     | 1147.57    |
| sudo-32    | 917.15    | 923.70     | 1071.47    |
| sudo-64    | 889.95    | 911.23     | 996.22     |
| sudo-128   | 888.76    | 872.15     | 986.37     |
| sudo-256   | 918.70    | 884.72     | 964.91     |

First, does the advantage continue with deeper architectures? Table 9 shows the results with a network of depth 10. We see that relu units again outperform both tanh and SUDO results. The increased depth does not equalize the performance of the tanh or the SUDO units with the relu activations. How do we regain the lost performance in comparison to relu? We can “rectify” the SUDO units as well; see the appendix for in-depth details and experiments.

Second, we look at the distribution of activations for the SUDO units (across all of the SUDO hidden units in the networks trained with 4 hidden layers and 50 hidden units per layer). This is compared to the activation of the tanh units when discretized into 8 equal sized bins. We see a common trend: in the discretized and non-discretized units, the extrema get the most activations. The histograms are shown in Figure 9.

To see how well each network does in the task of reconstruction, Figure 10 shows the reconstruction of the original image achieved by the networks.
Table 9: SSE memorizing image. Networks have 10 hidden layers. Last Activation is tanh.

| Activation | Hidden Units Per Layer |
|------------|-----------------------|
|            | 50        | 100        | 200        |
| tanh       | 885.43    | 939.55     | 857.40     |
| relu       | 769.27    | 745.33     | 662.66     |
| sudo-2     | 1625.15   | 1573.95    | 1545.70    |
| sudo-4     | 1412.62   | 1373.34    | 1398.23    |
| sudo-8     | 1165.48   | 1116.34    | 1107.36    |
| sudo-16    | 1024.29   | 965.74     | 968.67     |
| sudo-32    | 927.06    | 933.10     | 930.50     |
| sudo-64    | 929.23    | 856.97     | 881.00     |
| sudo-128   | 909.19    | 892.16     | 887.31     |
| sudo-256   | 903.90    | 889.50     | 874.53     |

Figure 9: Distributions of hidden unit output activations, post-training. Histogram shows the results across all layers of the 4-hidden layers memorization networks, trained with 50 hidden units. SUDO-4,8,16,64 and tanh (discretized to 8 equal sized bins) shown.
Figure 10: Performance on the memorization task. Original image of pier and water, composed of $150 \times 150$ pixels, shown on top. Networks have 4 hidden layers. SUDO-2 (d) yields an unrecognizable image. With 4 discretization levels (e) the dock becomes recognizable. By 256, the reconstruction has vastly improved.
3.4 Autoencoding

In this section, we examine the standard task of autoencoding images. For these tests, networks were trained on ImageNet images, scaled to 32x32, and tested on an independent test set, also from ImageNet [22].

For these experiments, two very different network architectures were used: one with a series of convolutions and the other fully connected. As described below, the “bottleneck” of the convolution is much larger in the conv-nets than in the fully connected nets and thereby the convolution-net outperformed the fully connected network for all comparable settings. No attempt was made to balance the two architecture’s bottleneck sizes as, for this study, the sole important comparison is between the hidden unit activations of the same architecture.

Table 10 shows the results with the convolution architecture. The first set of experiments were conducted with 4 layers of 3x3 convolutions with stride 2x2. The layers had 50, 50, 40, and 20 filters per layer. This was followed by 4 layers in the decode step of 2d-transpose (deconvolution layers) of depth 40, 50, 50, 20 and finally a 3 (RGB) layer for recreating the image. The final activations of the outputs were a tanh.

Table 10: Autoencoding Networks  
Table 11: Autoencoding Networks, Fully Connected - Relative SSE Errors

| Activation | Network Scale | Network Scale |
|------------|---------------|---------------|
|            | 1 | ×2 | ×4 | ×8 |
| tanh       | 1.00 | 0.68 | 0.35 | 0.06 |
| relu       | 1.02 | 0.69 | 0.37 | 0.09 |
| sudo-2     | 1.92 | 1.58 | 1.16 | 0.87 |
| sudo-4     | 1.41 | 1.01 | 0.68 | 0.35 |
| sudo-8     | 1.13 | 0.79 | 0.46 | 0.17 |
| sudo-16    | 1.05 | 0.71 | 0.38 | 0.11 |
| sudo-32    | 1.01 | 0.68 | 0.34 | 0.07 |
| sudo-64    | 0.99 | 0.66 | 0.34 | 0.07 |
| sudo-128   | 0.98 | 0.65 | 0.33 | 0.07 |
| sudo-256   | 0.98 | 0.65 | 0.33 | 0.06 |

For ease of reading this table, we set the performance of the conv-net with a tanh activation function as the baseline. Accordingly, the first column of Table 10 shows the relative results of all of the hidden activation types explored. The lower the number, the lower the SSE and therefore the better the performance. The subsequent three columns of Table 11 (×2, ×4, ×8) show the performance achieved by increasing the number of filters per layer (for all layers) by a factor of ×2, ×4, ×8. As expected, there is a sharp decrease in the error across all of the hidden unit activations as the number of filters per layer increases. Importantly, note that the good performance, relative to the tanh and relu units is achieved by SUDO-64, and continues as the discretization levels are increased for every network size.

We next repeat the same set of experiments using networks with fully connected layers. For these experiments, the fully connected architecture had 4 layers for the encoding; these consisted of 50, 50, 40 and 20 hidden units. Next, 3 layers for decoding had 40, 50, 50 hidden units. This was followed by a reconstruction layer of size 32x32x3 (RGB) with tanh units. Table 11 shows the results, again relative to the score of the tanh, scale 1, with conv-nets. As in Table 10, the columns of Table 11 (×2, ×4, ×8) show the performance relative to the number of hidden units per layer. Like the results with the conv-nets, larger networks have improved performance. Also, again, we see little difference in performance between the SUDO activations and the better of tanh and relu as \(L\) is increased.
3.5 MNIST Digit Classification

As a final test, we examine the performance of SUDO units on the standard MNIST digit classification task [23]. For this test, we use fully connected networks and vary the number of hidden units per layer and the number of hidden layers. Table 12 shows the results with a network employing a single hidden layer. Table 12 provides a similar set of results using a network with 4 hidden layers. Each entry shown in the tables is the average of 5 networks trained with randomly initialized starting weights [3]. For this task, very few discretization levels provide competitive performance when the number of hidden units is > 3. For example, SUDO-8 and SUDO-16 often perform as well as tanh and surpass relu in performance throughout the range of number of hidden units used in the 4 hidden layer architecture.

Table 12: MNIST Accuracy Results. Networks have 1 hidden layer.

| Activation | 2 | 3 | 4 | 10 | 50 | 100 |
|------------|---|---|---|----|----|-----|
| tanh       | 64.7% | 77.6% | 85.7% | 93.6% | 97.1% | 97.7% |
| relu       | 69.7% | 81.2% | 86.8% | 94.1% | 97.2% | 97.8% |
| sudo-2     | 38.9% | 67.8% | 82.2% | 90.8% | 96.1% | 97.2% |
| sudo-4     | 56.9% | 76.6% | 84.0% | 92.9% | 96.8% | 97.7% |
| sudo-8     | 63.0% | 76.9% | 85.6% | 93.4% | 96.9% | 97.7% |
| sudo-16    | 61.9% | 79.6% | 85.2% | 93.9% | 96.9% | 97.7% |
| sudo-32    | 64.5% | 79.4% | 86.1% | 93.6% | 97.1% | 97.7% |
| sudo-64    | 64.1% | 78.6% | 86.6% | 93.4% | 97.1% | 97.8% |
| sudo-128   | 61.5% | 80.7% | 86.4% | 93.5% | 97.3% | 97.7% |
| sudo-256   | 63.0% | 78.1% | 85.6% | 93.6% | 97.2% | 97.8% |

Table 13: MNIST Accuracy Results. Networks have 4 hidden layers.

| Activation | 2 | 3 | 4 | 10 | 50 | 100 |
|------------|---|---|---|----|----|-----|
| tanh       | 55.7% | 77.4% | 86.1% | 94.1% | 97.2% | 97.9% |
| relu       | 33.6% | 66.7% | 81.5% | 93.7% | 97.2% | 97.8% |
| sudo-2     | 25.6% | 36.0% | 41.4% | 88.2% | 95.8% | 97.2% |
| sudo-4     | 46.0% | 66.1% | 79.9% | 92.5% | 96.9% | 97.8% |
| sudo-8     | 52.6% | 71.7% | 83.0% | 93.4% | 97.2% | 97.9% |
| sudo-16    | 56.9% | 70.5% | 84.1% | 93.9% | 97.3% | 98.0% |
| sudo-32    | 64.5% | 74.2% | 84.1% | 93.9% | 97.2% | 97.8% |
| sudo-64    | 57.3% | 75.6% | 85.4% | 94.2% | 97.3% | 98.0% |
| sudo-128   | 61.1% | 75.8% | 84.3% | 93.7% | 97.3% | 97.9% |
| sudo-256   | 62.9% | 75.4% | 85.0% | 94.0% | 97.2% | 98.1% |

[3] Though alternate architectures with convolutions are known to provide state of the art results, for clarity with the effects of number of hidden units, we use only fully connected layers. Again, as before, we are most interested in seeing the relative performance of the SUDO units to the tanh and relu activations.
4 Discussion & Future Work

The most salient finding in this work is that reducing the number of outputs allowed in a network’s hidden units from $2^{32}$ (a typical floating point representation) to between only 64-256 unique outputs does not have a noticeable impact on performance. This is valuable in many modern scenarios where computation needs to be limited (e.g. mobile devices) or where memory is at a premium, such as pixel recurrent neural networks.

A second interesting trend that was noted is the role of network depth. With lower values of $L$, shallow networks exhibited severely degraded performance. However, as the depth of the network was increased (even to modest depth of 2 or 4), networks with small $L$ exhibited large improvements in performance. For example, in the memorization task (Section 3.3), doubling the number of units in a single hidden layer had far less performance improvement compared to adding another layer with the same number of units. This is similar to the performance improvements seen with more common activations.

Though there has been a significant previous work in “binarizing” networks (e.g. setting $L = 2$), the difficulties that were experienced in training and using these networks is greatly dissipated when the discretization level is increased. With large $L$, no modifications to the training algorithms is required. In this study, standard TensorFlow with both ADAM and SGD+Momentum were used, with the exact same settings for training, validation and testing.

In this study, we used the sigmoid or tanh as the underlying activation function to be discretized. In the cases in which the SUDO units did not perform as well as relu, generally, tanh did not either. The most straightforward method to address this is to change the underlying function. In the appendix, we examine what happens when the SUDO units are replaced with rectified-SUDO units. Analogously to relu units, these units emit a 0 for large parts of the activation. All of the experiments in this paper are repeated with rectified-SUDO units. The performance is much closer to standard relu units.

The simplicity of the approach opens a number of new potential avenues for research. For example, future work should be conducted with units with different discretization levels ($L$) within the same network. As was witnessed when examining the real-valued checkerboard problems, some of the middle discretization settings performed better than those both smaller and larger. A simple method of capturing this benefit is to use different $L$’s for tackling the same problem, within the same network.

An alternative approach to a priori guessing the right $L$ is to make this a learnable parameter. A similar approach was taken in [24], in which a piece-wise linear activation function was learned for each neuron with promising results when compared to static rectified linear units.

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Appendix A Rectified SUDO Units

The results reported in this paper have shown that the performance of the SUDO units closely tracks the performance of tanh units. However, in a few of the problems described in Section 3 the rectified linear units were able to outperform tanh (and therefore SUDO). The most stark case was found in the memorization task, Section 3.3, as well as a few instantiations of the the Checkerboard binary classification task, Section 3.1.

Here, we briefly describe a simple method to more closely approximate the performance of the relu units by “rectifying” the SUDO units. The resulting activation of the R-SUDO units is as follows:

function RECTIFIED_SUDO_Activation(input, levels):
    underlying ← tanh(input)
    if underlying < 0: then
        return (0);
    else
        activation_step ← 2/(levels - 1)
        plateauRange ← 2/levels
        output ← (⌊(underlying + 1.0)/plateauRange⌋ − 1.0) * activation_step
        return (−1.0 + output)
    end if

Several points should be noted about this simple method of rectifying the SUDO units.

• As with relu units, these units propagate no derivatives when underlying < 0.

• In this naive implementation, note that the number of activations for b bits is reduced from $2^b$ to $(2^b - 1 + 1)$. This is wasteful in terms of bits. Nonetheless, for ease of comparison to the rest of the results, and so that the tables in the appendix and the main body of the paper can be compared cell by cell, we keep this representation. In practice, to achieve $2^b$ discretization levels, they should all be placed in the $\geq 0$ range.

• Like the relu-6 variant, this activation has a sharp non-linearity and a maximum output activation [25]. The use of an underlying linear activation, such as with relu-6, can easily be substituted here.

In the remainder of this section, we recreate all of the experiments with R-SUDO units and provide all the results for completeness. In particular, examine the results in Section A.3. The R-SUDO results now match relu closely. Additionally, where tanh outperforms relu, SUDO tends to outperform R-SUDO.

When examining the results, note that the experiments with tanh and relu are rerun. Though they should be similar to those observed earlier, due to random weight initializations, differences may be present. Any differences give an indication of the typical variance experienced in these experiments. In both the main body and in this appendix, each table cell is the average of multiple runs, as described in Section 3.
## A.1 Checkerboard with R-SUDO units

Table 14: Checkerboard Accuracies with 1 Hidden Layer. Compare to Table 1.

| Activation | 5      | 10     | 50     | 100    | 200    |
|------------|--------|--------|--------|--------|--------|
| tanh       | 59.8%  | 80.5%  | 67.7%  | 55.0%  | 53.3%  |
| relu       | 58.0%  | 66.9%  | 86.9%  | 87.9%  | 92.9%  |
| r-sudo-2   | 59.1%  | 64.8%  | 80.3%  | 80.2%  | 82.1%  |
| r-sudo-4   | 57.7%  | 66.0%  | 85.8%  | 87.3%  | 89.1%  |
| r-sudo-8   | 61.3%  | 72.1%  | 92.7%  | 93.6%  | 94.3%  |
| r-sudo-16  | 64.5%  | 77.9%  | 94.5%  | 95.8%  | 96.0%  |
| r-sudo-32  | 62.5%  | 77.9%  | 94.7%  | 96.1%  | 96.8%  |
| r-sudo-64  | 66.9%  | 77.7%  | 94.5%  | 96.0%  | 96.9%  |
| r-sudo-128 | 65.2%  | 75.9%  | 94.9%  | 96.3%  | 97.0%  |
| r-sudo-256 | 65.6%  | 76.9%  | 95.3%  | 96.3%  | 96.9%  |

Table 15: Checkerboard Accuracies with 2 Hidden Layers. Compare to Table 2.

| Activation | 5      | 10     | 50     | 100    | 200    |
|------------|--------|--------|--------|--------|--------|
| tanh       | 82.3%  | 97.2%  | 98.2%  | 98.1%  | 97.6%  |
| relu       | 73.9%  | 91.8%  | 97.2%  | 97.9%  | 97.6%  |
| r-sudo-2   | 52.9%  | 52.6%  | 54.5%  | 57.9%  | 58.3%  |
| r-sudo-4   | 55.9%  | 75.6%  | 95.6%  | 95.5%  | 94.7%  |
| r-sudo-8   | 65.4%  | 83.2%  | 98.4%  | 97.5%  | 98.4%  |
| r-sudo-16  | 72.4%  | 89.2%  | 98.0%  | 97.6%  | 97.5%  |
| r-sudo-32  | 72.6%  | 91.9%  | 98.0%  | 97.7%  | 97.7%  |
| r-sudo-64  | 76.5%  | 92.7%  | 98.6%  | 98.0%  | 98.0%  |
| r-sudo-128 | 78.0%  | 93.4%  | 98.2%  | 98.0%  | 98.1%  |
| r-sudo-256 | 76.4%  | 94.7%  | 98.0%  | 98.1%  | 98.2%  |

Table 16: Checkerboard Accuracies with 4 Hidden Layers. Compare to Table 3.

| Activation | 5      | 10     | 50     | 100    | 200    |
|------------|--------|--------|--------|--------|--------|
| tanh       | 95.0%  | 97.7%  | 98.4%  | 98.5%  | 98.1%  |
| relu       | 84.8%  | 96.7%  | 97.9%  | 97.8%  | 97.9%  |
| r-sudo-2   | 52.1%  | 52.6%  | 52.0%  | 52.6%  | 55.0%  |
| r-sudo-4   | 53.1%  | 57.2%  | 96.6%  | 97.7%  | 98.4%  |
| r-sudo-8   | 61.4%  | 93.1%  | 97.7%  | 96.7%  | 97.7%  |
| r-sudo-16  | 72.2%  | 93.0%  | 98.3%  | 98.1%  | 98.1%  |
| r-sudo-32  | 80.2%  | 93.8%  | 98.2%  | 97.9%  | 97.7%  |
| r-sudo-64  | 80.2%  | 95.4%  | 98.1%  | 98.0%  | 97.8%  |
| r-sudo-128 | 84.7%  | 95.2%  | 98.2%  | 98.1%  | 98.0%  |
| r-sudo-256 | 85.1%  | 95.8%  | 98.1%  | 98.1%  | 98.0%  |
A.2 Simple Regression with R-SUDO units

Table 17: Error in approximating a real-value function. Networks have 2 hidden layers. Compare to Table 1.

| Activation | Hidden Units Per Layer |
|------------|------------------------|
|            | 10 | 20 | 50 |
| tanh       | 0.55 | 0.24 | 0.11 |
| relu       | 3.13 | 0.36 | 0.13 |
| r-sudo-2   | 21.45 | 21.47 | 15.53 |
| r-sudo-4   | 14.20 | 6.85 | 2.08 |
| r-sudo-8   | 7.64 | 1.89 | 0.70 |
| r-sudo-16  | 3.94 | 0.90 | 0.29 |
| r-sudo-32  | 2.60 | 0.53 | 0.14 |
| r-sudo-64  | 1.91 | 0.39 | 0.09 |
| r-sudo-128 | 1.78 | 0.31 | 0.07 |
| r-sudo-256 | 1.31 | 0.26 | 0.06 |

Table 18: Error in approximating a real-value function. Networks have 4 hidden layers. Compare to Table 5.

| Activation | Hidden Units Per Layer |
|------------|------------------------|
|            | 10 | 20 | 50 |
| tanh       | 0.13 | 0.02 | 0.01 |
| relu       | 0.47 | 0.11 | 0.02 |
| r-sudo-2   | 21.40 | 21.74 | 19.97 |
| r-sudo-4   | 16.26 | 10.75 | 3.22 |
| r-sudo-8   | 9.16 | 2.19 | 0.55 |
| r-sudo-16  | 3.26 | 0.78 | 0.26 |
| r-sudo-32  | 1.90 | 0.36 | 0.10 |
| r-sudo-64  | 0.81 | 0.20 | 0.05 |
| r-sudo-128 | 0.44 | 0.10 | 0.03 |
| r-sudo-256 | 0.41 | 0.07 | 0.02 |
### A.3 Memorization with R-SUDO units

Table 19: SSE memorizing image. Networks have 2 hidden layer. Last Activation is tanh. Compare to Table 7.

| Hidden Units Per Layer | 50    | 100   | 200   |
|------------------------|-------|-------|-------|
| tanh                   | 1276.81 | 1318.71 | 1350.49 |
| relu                   | 1187.68 | 1168.15 | 1129.43 |
| r-sudo-2               | 1603.53 | 1521.72 | 1443.99 |
| r-sudo-4               | 1372.24 | 1314.86 | 1299.39 |
| r-sudo-8               | 1231.86 | 1184.61 | 1175.17 |
| r-sudo-16              | 1171.52 | 1129.30 | 1105.97 |
| r-sudo-32              | 1144.96 | 1117.43 | 1095.69 |
| r-sudo-64              | 1133.57 | 1100.98 | 1094.33 |
| r-sudo-128             | 1113.24 | 1101.18 | 1089.25 |
| r-sudo-256             | 1118.84 | 1115.16 | 1079.66 |

Table 20: SSE memorizing image. Networks have 4 hidden layers. Last Activation is tanh. Compare to Table 8.

| Hidden Units Per Layer | 50    | 100   | 200   |
|------------------------|-------|-------|-------|
| tanh                   | 992.67 | 978.79 | 995.11 |
| relu                   | 915.97 | 879.11 | 846.82 |
| r-sudo-2               | 1661.37 | 1633.06 | 2442.41 |
| r-sudo-4               | 1364.00 | 1282.73 | 1215.14 |
| r-sudo-8               | 1185.38 | 1109.68 | 1070.57 |
| r-sudo-16              | 1054.54 | 995.39  | 930.72  |
| r-sudo-32              | 959.88  | 903.00  | 854.91  |
| r-sudo-64              | 924.95  | 871.47  | 812.99  |
| r-sudo-128             | 891.32  | 848.97  | 801.34  |
| r-sudo-256             | 897.88  | 815.65  | 802.37  |

Table 21: SSE memorizing image. Networks have 10 hidden layers. Last Activation is tanh. Compare to Table 9.

| Hidden Units Per Layer | 50    | 100   | 200   |
|------------------------|-------|-------|-------|
| tanh                   | 898.96 | 909.04 | 875.31 |
| relu                   | 751.49 | 701.92 | 681.25 |
| r-sudo-2               | 1771.92 | 1775.78 | 1758.56 |
| r-sudo-4               | 1552.20 | 1505.91 | 1432.91 |
| r-sudo-8               | 1287.86 | 1199.02 | 1145.87 |
| r-sudo-16              | 1097.75 | 1001.96 | 950.51  |
| r-sudo-32              | 920.75  | 868.79  | 800.57  |
| r-sudo-64              | 838.35  | 779.91  | 729.90  |
| r-sudo-128             | 795.00  | 736.38  | 740.50  |
| r-sudo-256             | 811.02  | 674.58  | 680.06  |
A.4 Autoencoding with R-SUDO units

In Section 3.4, these errors were presented relative to the average performance of the 5 trials with the tanh units in a convolution network. That same baseline is used here to make the comparison straightforward.

Table 22: Autoencoding Networks Convolution-Based - Relative SSE Errors. Compare to Table 10.

| Network Scale | tanh | relu | r-sudo-2 | r-sudo-4 | r-sudo-8 | r-sudo-16 | r-sudo-32 | r-sudo-64 | r-sudo-128 | r-sudo-256 |
|---------------|------|------|----------|----------|----------|-----------|-----------|-----------|------------|------------|
| 1             | 1.00 | 1.02 | 5.08     | 1.96     | 1.30     | 1.14      | 1.05      | 1.04      | 1.01       | 1.05       |
| x2            | 0.68 | 0.68 | 5.11     | 1.46     | 0.97     | 0.81      | 0.73      | 0.69      | 0.67       | 0.68       |
| x4            | 0.36 | 0.39 | 4.79     | 1.05     | 0.66     | 0.48      | 0.41      | 0.38      | 0.36       | 0.35       |
| x8            | 0.06 | 0.08 | 2.53     | 0.71     | 0.35     | 0.21      | 0.14      | 0.11      | 0.10       | 0.10       |

Table 23: Autoencoding Networks, Fully Connected - Relative SSE Errors. Compare to Table 11.

| Network Scale | tanh | relu | r-sudo-2 | r-sudo-4 | r-sudo-8 | r-sudo-16 | r-sudo-32 | r-sudo-64 | r-sudo-128 | r-sudo-256 |
|---------------|------|------|----------|----------|----------|-----------|-----------|-----------|------------|------------|
| 1             | 2.30 | 2.39 | 4.13     | 3.54     | 3.16     | 2.97      | 2.91      | 3.05      | 2.70       | 2.73       |
| x2            | 2.00 | 1.99 | 3.82     | 3.09     | 2.74     | 2.62      | 2.58      | 2.51      | 2.40       | 2.44       |
| x4            | 1.64 | 1.85 | 3.64     | 2.77     | 2.55     | 2.38      | 2.27      | 2.12      | 2.16       | 2.14       |
| x8            | 1.30 | 1.47 | 3.53     | 2.61     | 2.25     | 2.05      | 1.96      | 1.90      | 1.83       | 1.82       |
### A.5 MNIST with R-SUDO units

Table 24: MNIST Accuracy Results. Networks have 1 hidden layer. Compare to Table 12.

| Activation | 2   | 3   | 4   | 10  | 50  | 100 |
|------------|-----|-----|-----|-----|-----|-----|
| tanh       | 63.9% | 78.8% | 85.9% | 93.6% | 97.1% | 97.7% |
| relu       | 66.1% | 80.9% | 86.5% | 93.8% | 97.2% | 97.9% |
| r-sudo-2   | 27.6% | 36.1% | 42.3% | 76.7% | 96.3% | 97.2% |
| r-sudo-4   | 43.5% | 64.2% | 80.8% | 91.8% | 96.4% | 97.3% |
| r-sudo-8   | 48.2% | 72.8% | 82.1% | 92.3% | 96.6% | 97.5% |
| r-sudo-16  | 52.1% | 73.8% | 84.0% | 92.4% | 96.6% | 97.3% |
| r-sudo-32  | 55.1% | 72.4% | 83.0% | 92.7% | 96.6% | 97.4% |
| r-sudo-64  | 53.6% | 74.4% | 84.9% | 92.7% | 96.6% | 97.4% |
| r-sudo-128 | 51.3% | 74.6% | 84.0% | 92.6% | 96.7% | 97.5% |
| r-sudo-256 | 52.1% | 72.3% | 83.3% | 92.7% | 96.6% | 97.4% |

Table 25: MNIST Accuracy Results. Networks have 4 hidden layers. Compare to Table 13.

| Activation | 2   | 3   | 4   | 10  | 50  | 100 |
|------------|-----|-----|-----|-----|-----|-----|
| tanh       | 63.8% | 73.5% | 83.8% | 93.9% | 97.3% | 97.9% |
| relu       | 30.3% | 54.1% | 81.2% | 93.7% | 97.2% | 97.8% |
| r-sudo-2   | 15.0% | 17.8% | 25.0% | 47.2% | 91.6% | 96.4% |
| r-sudo-4   | 18.7% | 17.3% | 33.0% | 90.4% | 96.6% | 97.6% |
| r-sudo-8   | 14.1% | 28.6% | 58.2% | 92.1% | 96.8% | 97.7% |
| r-sudo-16  | 25.2% | 46.8% | 56.4% | 92.2% | 96.9% | 97.5% |
| r-sudo-32  | 25.4% | 58.4% | 54.6% | 92.7% | 97.0% | 97.7% |
| r-sudo-64  | 29.9% | 43.3% | 71.5% | 92.5% | 96.9% | 97.5% |
| r-sudo-128 | 33.2% | 56.9% | 63.0% | 92.7% | 96.9% | 97.7% |
| r-sudo-256 | 31.1% | 41.0% | 63.9% | 92.5% | 96.8% | 97.8% |