Some characterizations of special curves in the Euclidean space $E^4$

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Abstract. In this work, first, we give some characterizations of helices and ccr curves in the Euclidean 4-space. Thereafter, relations among Frenet-Serret invariants of Bertrand curve of a helix are presented. Moreover, in the same space, some new characterizations of involute of a helix are presented.

1 Introduction

In the local differential geometry, we think of curves as a geometric set of points, or locus. Intuitively, we are thinking of a curve as the path traced out by a particle moving in $E^4$. So, investigating position vectors of the curves is a classical aim to determine behavior of the particle (curve).

Natural scientists have long held a fascination, sometimes bordering on mystical obsession for helical structures in nature. Helices arise in nanosprings, carbon nanotubes, $\alpha$–helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is

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structure of DNA [3]. This fact was published for the first time by Watson and Crick in 1953 [25]. They constructed a molecular model of DNA in which there were two complementary, antiparallel (side-by-side in opposite directions) strands of the bases guanine, adenine, thymine and cytosine, covalently linked through phosphodiesterase bonds. Each strand forms a helix and two helices are held together through hydrogen bonds, ionic forces, hydrophobic interactions and van der Waals forces forming a double helix, lipid bilayers, bacterial flagella in Salmonella and E. coli, aerial hyphae in actinomycete, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells (helico-spiral structures) [4, 5].

Helix is one of the most fascinating curves in science and nature. Also we can see the helix curve or helical structures in fractal geometry, for instance hyperhelices [23]. In the field of computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or the design of highways, etc. [26]. From the view of differential geometry, a helix is a geometric curve with non-vanishing constant curvature $\kappa$ and non-vanishing constant torsion $\tau$ [2]. The helix may be called a circular helix or W-curve [12, 17].

It is known that straight line ($\kappa(s) = 0$) and circle ($\tau(s) = 0$) are degenerate-helix examples [13]. In fact, circular helix is the simplest three-dimensional spirals. One of the most interesting spiral examples are k-Fibonacci spirals. These curves appear naturally from studying the k-Fibonacci numbers $\{F_k,n\}_{n=0}^\infty$ and the related hyperbolic k-Fibonacci function. Fibonacci numbers and the related Golden Mean or Golden section appear very often in theoretical physics and physics of the high energy particles [7, 8]. Three-dimensional k-Fibonacci spirals was studied from a geometric point of view in [9].

Indeed, in Euclidean 3-space $E^3$, a helix is a special case of the general helix. A curve of constant slope or general helix in Euclidean 3-space is defined by the property that the tangent makes a constant angle with a fixed straight line called the axis of the general helix. A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [22] for details) says that: A necessary and sufficient condition that a curve be a general helix is that the ratio $\kappa/\tau$ is constant along the curve, where $\kappa$ and $\tau$ denote the curvature and the torsion, respectively.

The notation of a generalized helix in $E^3$ can be generalized to higher dimensions in the same definition is proposed but in $E^n$, i.e., a generalized helix as a curve $\psi : R \to E^n$ such that its tangent vector forms a constant angle with a given direction $U$ in $E^n$ [20].

Two curves which, at any point, have a common principal normal vector
are called Bertrand curves. The notion of Bertrand curves was discovered by J. Bertrand in 1850. Bertrand curves have been investigated in $\mathbb{E}^n$ and many characterizations are given in [10]. Thereafter, by theory of relativity, investigators extend some of classical differential geometry topics to Lorentzian manifolds. For instance, one can see, Bertrand curves in $\mathbb{E}^n_1$ [6], in $\mathbb{E}^3_1$ for null curves [1], and in $\mathbb{E}^4_1$ for space-like curves [27]. In the fourth section of this paper, we follow the same procedure as in [27].

In this work, first, we aim to give some new characterizations of helices and ccr curves in terms of recently obtained theorems. Thereafter, we investigate relations among Frenet-Serret invariants of Bertrand curve couples, when one of is helix, in the Euclidean 4-space. Moreover, we observe that Bertrand curve of a helix is also a helix; and cannot be a spherical curve, a general helix and a 3-type slant helix, respectively. We also express some characterizations of involute of a helix. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

2 Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space $\mathbb{E}^4$ are briefly presented (A more complete elementary treatment can be found in [11]).

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be an arbitrary curve in the Euclidean space $\mathbb{E}^4$. Recall that the curve $\alpha$ is said to be of unit speed (or parameterized by arclength function $s$) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle \ldots \rangle$ is the standard scalar (inner) product of $\mathbb{E}^4$ given by

$$\langle \xi, \zeta \rangle = \xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3 + \xi_4\zeta_4,$$

for each $\xi = (\xi_1, \xi_2, \xi_3, \xi_4), \zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in \mathbb{E}^4$. In particular, the norm of a vector $\xi \in \mathbb{E}^4$ is given by

$$\|\xi\| = \sqrt{\langle \xi, \xi \rangle}.$$

Let $\{T(s), N(s), B(s), E(s)\}$ be the moving frame along the unit speed curve $\alpha$. Then the Frenet-Serret formulas are given by [10, 21]

$$\begin{bmatrix}
T' \\
N' \\
B' \\
E'
\end{bmatrix} =
\begin{bmatrix}
0 & \kappa & 0 & 0 \\
-\kappa & 0 & \tau & 0 \\
0 & -\tau & 0 & \sigma \\
0 & 0 & -\sigma & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B \\
E
\end{bmatrix}.$$  (1)
Here $T, N, B$ and $E$ are called, respectively, the tangent, the normal, the binormal and the trinormal vector fields of the curve and the functions $\kappa(s), \tau(s)$ and $\sigma(s)$ are called, respectively, the first, the second and the third curvature of a curve in $E^4$. Also, the functions $H_1 = \frac{\tau}{\kappa}$ and $H_2 = \frac{\kappa}{\sigma}$ are called Harmonic Curvatures of the curves in $E^4$, where $\kappa \neq 0, \tau \neq 0$ and $\sigma \neq 0$. Let $\alpha : I \subset \mathbb{R} \rightarrow E^4$ be a regular curve. If tangent vector field $T$ of $\alpha$ forms a constant angle with unit vector $U$, this curve is called an inclined curve or a general helix in $E^4$. Recall that, a curve $\psi = \psi(s)$ is called a 3-type slant helix if the trinormal lines of $\alpha$ make a constant angle with a fixed direction in $E^4$ [24]. Recall that if a regular curve has constant Frenet curvatures ratios, (i.e., $\frac{\tau}{\kappa}$ and $\frac{\sigma}{\tau}$ are constants), then it is called a ccr-curve [16]. It is worth noting that: the W-curve, in Euclidean 4-space $E^4$, is a special case of a ccr-curve.

Let $\alpha(s)$ and $\alpha^*(s)$ be regular curves in $E^4$. $\alpha(s)$ and $\alpha^*(s)$ are called Bertrand Curves if for each $s_0$, the principal normal vector to $\alpha$ at $s = s_0$ is the same as the principal normal vector to $\alpha^*(s)$ at $s = s_0$. We say that $\alpha^*(s)$ is a Bertrand mate for $\alpha(s)$ if $\alpha(s)$ and $\alpha^*(s)$ are Bertrand Curves.

In [14] Magden defined in the same space, a vector product and gave a method to establish the Frenet-Serret frame for an arbitrary curve by the following definition and theorem:

**Definition 1** Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ be vectors in $E^4$. The vector product in $E^4$ is defined by the determinant

$$a \wedge b \wedge c = \begin{vmatrix}
  e_1 & e_2 & e_3 & e_4 \\
  a_1 & a_2 & a_3 & a_4 \\
  b_1 & b_2 & b_3 & b_4 \\
  c_1 & c_2 & c_3 & c_4
\end{vmatrix},$$

(2)

where $e_1, e_2, e_3$ and $e_4$ are mutually orthogonal vectors (coordinate direction vectors) satisfying equations

$$e_1 \wedge e_2 \wedge e_3 = e_4, \quad e_2 \wedge e_3 \wedge e_4 = e_1, \quad e_3 \wedge e_4 \wedge e_1 = e_2, \quad e_4 \wedge e_1 \wedge e_2 = e_3.$$

**Theorem 1** Let $\alpha = \alpha(t)$ be an arbitrary regular curve in the Euclidean space $E^4$ with above Frenet-Serret equations. The Frenet apparatus of $\alpha$ can be written as follows:

$$T = \frac{\alpha'}{||\alpha'||},$$

$$N = \frac{||\alpha'||^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha'}{||\alpha'||^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha'||}.$$
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\[ B = \mu E \wedge T \wedge N, \]
\[ E = \mu \frac{T \wedge N \wedge \alpha'''}{||T \wedge N \wedge \alpha'''||}, \]
\[ \kappa = \frac{||\alpha'||^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha'||}{||\alpha'||^4}, \]
\[ \tau = \frac{||T \wedge N \wedge \alpha''|| ||\alpha'||}{||\alpha'||^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha'||}, \]
\[ \sigma = \frac{\langle \alpha^{(IV)}, E \rangle}{||T \wedge N \wedge \alpha'''|| ||\alpha'||}. \]

and

where \( \mu \) is taken \(-1\) or \(+1\) to make \(+1\) the determinant of \([T, N, B, E]\) matrix.

3 Some new results of helices and ccr curves

In this section we state some related theorems and some important results about helices and ccr curves:

**Theorem 2** Let \( \alpha = \alpha(s) \) be a regular curve in \( E^4 \) parameterized by arclength with curvatures \( \kappa, \tau \) and \( \sigma \). Then \( \alpha = \alpha(s) \) lies on the hypersphere of center \( m \) and radius \( r \in \mathbb{R}^+ \) in \( E^4 \) if and only if

\[ \rho^2 + \left( \frac{1}{\tau} \frac{d\rho}{ds} \right)^2 + \frac{1}{\sigma^2} \left[ \rho \tau + \frac{d}{ds} \left( \frac{1}{\tau} \frac{d\rho}{ds} \right) \right]^2 = r^2, \]

where \( \rho = \frac{1}{\kappa} \) [16].

**Theorem 3** Let \( \alpha = \alpha(s) \) be a regular curve in \( E^4 \) parameterized by arclength with curvatures \( \kappa, \tau \) and \( \sigma \). Then \( \alpha \) is a generalized helix if and only if

\[ H_2 + \sigma H_1 = 0, \]

where \( H_1 = \frac{\kappa}{\tau} \) and \( H_2 = \frac{1}{\sigma} H_1' \) are the Harmonic Curvatures of \( \alpha \) [15].

**Theorem 4** Let \( \alpha = \alpha(s) \) be a regular curve in \( E^4 \) parameterized by arclength with curvatures \( \kappa, \tau \) and \( \sigma \). Then \( \alpha \) is a type 3-slant helix (its second binormal vector \( E \) makes a constant angle with a fixed direction \( U \)) if and only if

\[ \tilde{H}_2 + \sigma \tilde{H}_1 = 0, \]

where \( \tilde{H}_1 = \frac{\sigma}{\tau} \) and \( \tilde{H}_2 = \frac{1}{\kappa} \tilde{H}_1' \) are the Anti-Harmonic Curvatures of \( \alpha \) [18].
With the aid of the above theorems, one can easily obtain the following important results:

**Theorem 5** Let \( \alpha = \alpha(s) \) be a helix in \( E^4 \) with non-zero curvatures.
1. \( \alpha \) can not be a generalized helix
2. \( \alpha \) can not be a \( \beta \)-type slant helix
3. If \( \alpha \) lies on the hypersphere \( S^3 \), then, the sphere's radius is equal to \( \sqrt{\tau^2 + \sigma^2} \).

**Theorem 6** Let \( \alpha = \alpha(s) \) be a ccr curve in \( E^4 \) with non-zero curvatures \( \kappa(s) \), \( \tau(s) = a \kappa(s) \) and \( \sigma(s) = b \kappa(s) \). Then
1. \( \alpha \) can not be a generalized helix
2. \( \alpha \) can not be a \( \beta \)-type slant helix
3. If \( \alpha \) lies on the hypersphere \( S^3 \), then, if and only if, the following equation is satisfied:

\[
f^2 + \frac{\tau^2}{4a^2} + \frac{f}{4a^2b^2}(2a^2 + f'')^2 = r^2, \tag{6}
\]

where the function \( f = f(s) = \rho^2(s) = \frac{1}{\kappa^2(s)} \).

### 4 Bertrand curve of a helix

In this section we investigate relations among Frenet-Serret invariants of Bertrand curve of a helix in the space \( E^4 \).

**Theorem 7** Let \( \delta = \delta(s) \) be a helix in \( E^4 \). Moreover, \( \xi \) be Bertrand mate of \( \delta \). Frenet-Serret apparatus of \( \xi \), \( \{T_\xi, N_\xi, B_\xi, E_\xi, \kappa_\xi, \tau_\xi, \sigma_\xi\} \), can be formed by Frenet apparatus of \( \delta \) \( \{T, N, B, E, \kappa, \tau, \sigma\} \).

**Proof.** Let us consider a helix (W-curve, i.e.) \( \delta = \delta(s) \). We may express

\[
\xi = \delta + \lambda N. \tag{7}
\]

We know that \( \lambda = c = \text{constant} \) (cf. [11]). By this way, we can write that

\[
\frac{d\xi}{ds_E} \frac{ds_E}{ds} = T_\xi \frac{ds_E}{ds} = (1 - \lambda \kappa) T + \lambda \tau B.
\]

So, one can have

\[
T_\xi = \frac{(1 - \lambda \kappa) T + \lambda \tau B}{\sqrt{(1 - \lambda \kappa)^2 + (\lambda \tau)^2}}, \tag{8}
\]
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\[ \frac{d\xi}{ds} = \left\| \xi' \right\| = \sqrt{(1 - \lambda \kappa)^2 + (\lambda \tau)^2}. \] \hspace{1cm} (9)

In order to determine relations, we differentiate:

\[ \xi'' = \left[ \kappa - \lambda (\kappa^2 + \tau^2) \right] N + (\lambda \tau \sigma) E, \]
\[ \xi''' = \kappa \left[ \lambda (\kappa^2 + \tau^2) - \kappa \right] T + \tau (\kappa - \lambda (\kappa^2 + \tau^2 + \sigma^2)) B, \]
\[ \xi^{(IV)} = l_1 N + l_2 E \] \hspace{1cm} (10)

where

\[ l_1 = \kappa^3 (\lambda \kappa - 1) + \lambda \tau^2 (2 \kappa^2 + \tau^2 + \sigma^2), \]
and

\[ l_2 = \tau \sigma (\kappa - \lambda (\kappa^2 + \tau^2 + \sigma^2)). \]

Using the above equations, we can form

\[ \left\| \xi' \right\|^2 \xi'' - \langle \xi', \xi'' \rangle \xi' = K^2 \left[ (\kappa - \lambda (\kappa^2 + \tau^2)) N + (\lambda \tau \sigma) E \right], \]

where

\[ K = \sqrt{(1 - \lambda \kappa)^2 + (\lambda \tau)^2}. \]

Therefore, we obtain the principal normal and the first curvature, respectively,

\[ N_\xi = \frac{1}{L} \left[ (\kappa - \lambda (\kappa^2 + \tau^2)) N + (\lambda \tau \sigma) E \right], \] \hspace{1cm} (11)

and

\[ \kappa_\xi = \frac{L}{K^2}, \] \hspace{1cm} (12)

where

\[ L = \sqrt{[\kappa - \lambda (\kappa^2 + \tau^2)]^2 + (\lambda \tau \sigma)^2}. \]

Now, we can compute the vector form \( T_\xi \wedge N_\xi \wedge \xi''' \) as the following:

\[ T_\xi \wedge N_\xi \wedge \xi''' = \frac{1}{MK} \begin{vmatrix} T & N & B & E \\ 1 - \lambda \kappa & 0 & \lambda \tau & 0 \\ 0 & \kappa - \lambda (\kappa^2 + \tau^2) & 0 & \lambda \tau \sigma \\ l_1 & 0 & l_2 & 0 \end{vmatrix} = \frac{M}{K} \left[ \lambda \tau \sigma N - (\kappa - \lambda (\kappa^2 + \tau^2)) E \right], \]

where

\[ M = \tau \left[ \lambda (\kappa^2 + \tau^2 + \sigma^2) - \kappa (1 + \lambda^2 \sigma^2) \right]. \]
Since, we have
\[ E_\xi = -\frac{1}{L} \left[ \lambda \tau \sigma N - [\kappa - \lambda(\kappa^2 + \tau^2)]E \right]. \quad (13) \]

By this way, we have the third curvature as follows:
\[ \tau_\xi = \frac{M}{K^2 L}. \quad (14) \]

Besides, considering last equation of Theorem 1, one can calculate
\[ \sigma_\xi = \frac{\kappa \sigma}{L}. \quad (15) \]

Now, to determine the third vector field of Frenet frame, we write
\[ E_\xi \wedge T_\xi \wedge N_\xi = -\frac{1}{K L^2} \begin{vmatrix} T & N & B & E \\ 0 & \lambda \tau \sigma & 0 & \lambda(\kappa^2 + \tau^2) - \kappa \\ 1 - \lambda \kappa & 0 & \lambda \tau & 0 \\ 0 & \kappa - \lambda(\kappa^2 + \tau^2) & 0 & \lambda \tau \sigma \end{vmatrix}. \]

So we obtain:
\[ B_\xi = -\frac{1}{K} [\lambda \tau T + (1 - \lambda \kappa)B]. \quad (16) \]

It is worth to note that \( \mu = 1 \).

Considering obtained equations, we get:

**Theorem 8** Let \( \delta = \delta(s) \) be a helix in \( E^4 \). Moreover, let \( \xi \) be a Bertrand mate of \( \delta \). Then
1. \( \xi \) is also a helix.
2. \( \xi \) can not be a generalized helix.
3. \( \xi \) can not be a 3-type slant helix.
4. If \( \xi \) lies on the hypersphere \( S^3 \), then, the sphere’s radius is equal to
\[ \frac{\sqrt{\tau_\xi^2 + \sigma_\xi^2}}{\kappa \sigma} = \frac{\sqrt{\tau^2 + (1 - \lambda \kappa)^2 \sigma^2}}{\kappa \sigma}. \]

5 **Involute-evolute curve of a helix**

In this section, first we correct the computations in the paper [19] and then we obtain new results:
Theorem 9 Let $\xi = \xi(s)$ be involute of $\delta$. Let $\delta$ be a helix in $E^4$. The Frenet apparatus of $\xi$, $\{T_\xi, N_\xi, B_\xi, E_\xi, \kappa_\xi, \tau_\xi, \sigma_\xi\}$, can be formed by Frenet apparatus of $\delta \{T, N, B, E, \kappa, \tau, \sigma\}$ and take the following form.

\[
T_\xi = N, \quad N_\xi = \frac{-\kappa T + \tau B}{\sqrt{\kappa^2 + \tau^2}}, \quad B_\xi = -E, \quad E_\xi = \frac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}},
\]

(17)

and

\[
\kappa_\xi = \frac{\sqrt{\kappa^2 + \tau^2}}{\kappa |c - s|}, \quad \tau_\xi = \frac{\tau \sigma}{\kappa \sqrt{\kappa^2 + \tau^2} |c - s|}, \quad \sigma_\xi = -\frac{\sigma}{\sqrt{\kappa^2 + \tau^2} |c - s|},
\]

(18)

where

\[
\frac{ds_\xi}{ds} = \kappa |c - s|.
\]

(19)

**Proof.** The proof of the above theorem is similar as the proof of the previous theorem.

Theorem 10 Let $\xi$ and $\delta$ be unit speed regular curves in $E^4$. $\xi$ be involute of $\delta$. Then

1. $\xi$ cannot be a helix.
2. $\xi$ is a ccr-curve.
3. $\xi$ cannot be a generalized helix.
4. $\xi$ cannot be a 3-type slant helix.
5. $\xi$ cannot be lies on the hypersphere $S^3$.

**Proof.** The proof of points 1, 2, 3 and 4 are obviously. In the following we will proof the point 5:

Integrating the equation (19), we have

\[
|c - s| = \sqrt{\frac{2s_\xi}{\kappa}},
\]

which leads to

\[
\kappa_\xi = \frac{A_1}{\sqrt{s_\xi}}, \quad \tau_\xi = \frac{A_2}{\sqrt{s_\xi}}, \quad \sigma_\xi = \frac{A_3}{\sqrt{s_\xi}},
\]

(20)

where

\[
A_1 = \sqrt{\frac{\kappa^2 + \tau^2}{2\kappa}}, \quad A_2 = -\frac{\tau \sigma}{2\kappa (\kappa^2 + \tau^2)}, \quad A_3 = -\frac{\sigma \sqrt{\kappa}}{\sqrt{2(\kappa^2 + \tau^2)}}.
\]
Then if the evolute \( \xi \) lies in the hypersphere the equation (6) must be satisfied. Substituting \( f = \frac{s_\xi}{A_1^2} \), \( \kappa_\xi = \frac{A_2}{\sqrt{s_\xi}} \), \( B_1 = \frac{r_\xi}{\kappa_\xi} \), and \( B_2 = \frac{s_\xi}{\kappa_\xi} \) in the equation (6), we have

\[
\frac{s_\xi (B_1^2 + B_2^2)}{A_1^2 B_1^2} + \frac{1}{4A_1^2 B_1^2} = r^2,
\]

which is contradiction because the radius \( r \) of the sphere must be constant and the coefficient of \( s_\xi \) can not be equal zero. The proof is completed. ■

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