UNIQUENESS FOR DISCRETE SCHRÖDINGER EVOLUTIONS

PHILIPPE JAMING, YURIY LYUBARSKII, EUGENIA MALINNIKOVA, AND KARL-MIKAIL PERFEKT

Abstract. We prove that if a solution of the discrete time-dependent Schrödinger equation with bounded real potential decays fast at two distinct times then the solution is trivial. For the free Schrödinger operator and for operators with compactly supported time-independent potentials a sharp analog of the Hardy uncertainty principle is obtained, using an argument based on the theory of entire functions. Logarithmic convexity of weighted norms is employed in the case of general real-valued time-dependent bounded potentials. In the latter case the result is not optimal.

1. Introduction

The aim of this paper is to show that a non-trivial solution of a semi-discrete Schrödinger equation with bounded real potential cannot have arbitrarily fast decay at two different times. For the free evolution (with no potential) the result we obtain is precise and it can be interpreted as a discrete version of the dynamical Hardy Uncertainty Principle.

The usual formulation of the Uncertainty Principle is that a function and its Fourier transform can not both be arbitrarily well localized. In Hardy’s Uncertainty Principle, the localization is measured in terms of speed of decay at infinity: if $f \in L^2(\mathbb{R})$ is such that $f$ and its Fourier transform $\hat{f}$ satisfy

$$|f(x)| \leq C \exp(-\pi |x|^2), \quad |\hat{f}(\xi)| \leq C \exp(-\pi |\xi|^2), \quad x, \xi \in \mathbb{R},$$

for some constant $C > 0$, then there is a constant $A$ such that $f(x) = A \exp(-\pi |x|^2)$.

It is known that Uncertainty Principles may also be given dynamical interpretations in terms of solutions of the free Schrödinger Equation $[10,13,14]$. Hardy’s Uncertainty Principle is equivalent to the following statement:

(*) if $u(t, x)$ is a solution of the free Schrödinger equation $\partial_t u = i \Delta u$ and $|u(0, x)| + |u(1, x)| \leq C \exp(-x^2/4)$, then $u(0, x) = A \exp(-(1 + i)x^2/4)$.

The point is that the free Schrödinger Equation can be explicitly solved via the...
Fourier transform, from which the two formulations of the Hardy Uncertainty Principle are seen to be equivalent.

In a remarkable series of papers, L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega [9–12], and also in collaboration with Cowling [7], have extended the uniqueness statement (*) to solutions of Schrödinger equations with potentials, as well as to solutions of a wider class of partial differential equations that even includes some non-linear equations. Further results for covariant Schrödinger evolutions were obtained in [24]. Concerning the discrete setting, we note that a discrete dynamical interpretation of the Heisenberg uncertainty principle was given in [13].

In the present work we obtain uniqueness results for solutions of the discrete time-dependent Schrödinger equation

$$\partial_t u = i(\Delta_d u + Vu),$$

where $u : \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{C}$ and the potential $V = V(t,n)$ is a real-valued bounded function. $\Delta_d$ is the discrete Laplacian; for a complex valued function $f : \mathbb{Z} \to \mathbb{C}$,

$$\Delta_d f(n) := f(n+1) + f(n-1) - 2f(n).$$

We say that $u = u(t,n)$ is a strong solution of (1) if $u \in C^1([0,\infty), \ell^2)$. In the case that $V = V(n)$ is time-independent, $H = \Delta_d + V$ is a bounded self-adjoint operator in $\ell^2$, and for any $u(0,\cdot) \in \ell^2$ there exists a unique strong solution of (1) with $u(0,\cdot)$ as the initial value: $u(t,\cdot) = e^{itH}u(0,\cdot)$. If the potential $V$ is time-dependent but real, the $\ell^2$-norm of a solution is preserved.

For the free evolution we obtain what can be considered the discrete analog of the dynamical interpretation of Hardy’s Uncertainty Principle, which we then extend to equations with bounded real potentials. The results bear similarities to the continuous case, but there are at the same time fundamental differences. For instance, in the setting of free evolution, in both the continuous and the discrete case the optimal decay is given by the heat kernel at time 1. However, this means that the critical decay is different for the two situations. For the continuous case the standard heat kernel is $k(1,0,x) = (4\pi)^{-1/2} \exp(-x^2/4)$, while for the discrete case the heat kernel is $K(1,0,n) = e^{-\frac{1}{n}I_n(1)} \approx e^{-\frac{1}{n}(n!2^n)^{-1}}$, where $I_n$ are the modified Bessel functions, $I_n(z) = (-i)^n J_n(iz)$. Computations of the discrete heat kernel for the lattice and asymptotic connecting the two cases can be found in [5,6]. Discrete heat kernels also appeared as weights for convexity results for discrete harmonic functions in recent work by G. Lippner and D. Mangoubi [16].

To finish the introduction we now describe our main results in greater detail. First, we prove that if $u(t,n)$ solves the free equation

$$\partial_t u = i\Delta_d u$$

and satisfies

$$|u(0,n)| + |u(1,n)| \leq C\frac{1}{\sqrt{|n|}} \left(\frac{e}{2|n|}\right)^{|n|}, \quad n \in \mathbb{Z} \setminus \{0\},$$

then $u(t,n) = Ai^{-n}e^{-2it}J_n(1 - 2t)$, where $J_n$ is the Bessel function. This result is sharp: $|J_n(-1)|$ and $|J_n(1)|$ have precisely the growth of the right hand side in (2) as $|n| \to \infty$, see Proposition 2.4.
Then we investigate the discrete equation (1) with bounded potential in essentially two different ways using techniques of complex and real analysis. First, applying the theory of entire functions, we establish that if $V(n)$ is a compactly supported potential and $u$ is a strong solution of (1) satisfying the one-sided estimates
\[ |u(t, n)| \leq C \left( \frac{e}{(2 + \epsilon)n} \right)^n, \quad n > 0, \ t \in \{0, 1\}, \]
for some $\epsilon > 0$, then $u \equiv 0$. In the continuous setting, one-sided Hardy uncertainty principles have previously appeared in works of Nazarov [18] and Demange [8]. The corresponding results for continuous Schrödinger evolutions can be found in the recent survey [12].

In the second part of the paper, we use the real-variable approach following [10]. The main idea is to construct a weight function $\psi(t, n)$ which provides the logarithmic convexity of the weighted $\ell^2$ norms $\|\psi(t, \cdot)u(t, \cdot)\|_{\ell^2(\mathbb{Z})}$, where $u(t, n)$ is a strong solution of (1). This line of reasoning has its roots in celebrated results of T. Carleman and S. Agmon; the technique of Carleman estimates goes back to [3] and convexity principles for elliptic operators were described in [1]. The method allows us to consider general bounded potentials $V$, at the cost of having to assume stronger decay of $u(0, n)$ and $u(1, n)$ in both directions $n \to \pm \infty$. The main result, Theorem 4.2, says that if
\[ \| (1 + |n|)^{2 + |n|} u(0, n) \|_{\ell^2(\mathbb{Z})}, \| (1 + |n|)^{2 + |n|} u(1, n) \|_{\ell^2(\mathbb{Z})} < \infty \]
for some $\gamma > (3 + \sqrt{3})/2$, then $u \equiv 0$. We don’t expect this result to be sharp, but it does provide a universal decay condition which implies uniqueness of solutions of Schrödinger equations with general bounded potentials.

The paper is organized as follows: in Section 2 we discuss preliminaries of entire functions and use them to obtain our first results. Section 3 contains a precursory energy estimate for solutions of (1), which we need in order to justify the validity of many of our computations. Section 4 splits into several subsections discussing and proving the logarithmic convexity results we require, and in the final subsection the main result is proven.

Note that we will use the symbol $C$ to denote various constants in what follows. Unless otherwise indicated, its value might change from line to line. Also all $\| \cdot \|_2$-norms are to be understood as the $\ell^2$-norm in the variable $n$.

2. A uniqueness result for Schrödinger operators with compactly supported potentials

In this section, we use methods from complex analysis. For the reader’s convenience, we begin by briefly outlining some definitions and facts on entire functions of exponential type that we need. Details can be found in [15] (see in particular Lectures 8 and 9). Recall that an entire function $f$ is said to be of exponential type if for some $k > 0$
\[ |f(z)| \leq C \exp(k|z|). \]
In this case the type of an entire function \( f \) is defined by
\[
\sigma = \limsup_{r \to \infty} \log \max_{\phi \in [0, 2\pi]} \{ |f(re^{i\phi})|; \phi \in [0, 2\pi] \} < \infty.
\]
In particular, an entire function \( f \) is of zero exponential type if for any \( k > 0 \) there exists \( C = C(k) \) such that (3) holds.

Let \( f(z) \) be an entire function of exponential type, \( f(z) = \sum_{n=0}^{\infty} c_n z^n \). Then the type of \( f \) can be expressed in terms of the Taylor coefficients in the following way
\[
\limsup_{n \to \infty} n |c_n|^{1/n} = e\sigma.
\]

The growth of a function \( f \) of exponential type along different directions is described by the indicator function
\[
h_f(\varphi) = \limsup_{r \to \infty} \log |f(re^{i\varphi})|.
\]
This function is the support function of some convex compact set \( I_f \subset \mathbb{C} \) which is called the indicator diagram of \( f \). In particular
\[
h_f(\varphi) + h_f(\pi - \varphi) \geq 0.
\]
For example the indicator function of \( e^{az} \) for \( a \in \mathbb{C} \) is \( h(\varphi) = \Re(\alpha e^{i\varphi}) \) and its indicator diagram consists of a single point, \( \bar{a} \).

Clearly, \( h_{fg}(\varphi) \leq h_f(\varphi) + h_g(\varphi) \), implying that
\[
I_{fg} \subset I_f + I_g := \{ z = z_1 + z_2 : z_1 \in I_f, z_2 \in I_g \}.
\]
Recall that the Bessel functions \( J_n \) satisfy
\[
\exp(x(z - z^{-1})/2) = \sum_{n=-\infty}^{\infty} J_n(x)z^n, \quad z \neq 0.
\]
Moreover, for fixed \( x \),
\[
|J_n(x)| \sim \frac{1}{\sqrt{|n|}} \left( \frac{e}{2|n|} \right)^{|n|} \quad \text{as } |n| \to \infty.
\]

Our first observation is the following discrete analog of the classical Hardy uncertainty principle.

**Proposition 2.1.** Let \( u \in C^1([0, 1], \ell^2) \) satisfy the discrete free Schrödinger equation \( \partial_t u = i\Delta_d u \), and suppose that
\[
|u(0, n)|, |u(1, n)| \leq C \frac{1}{\sqrt{|n|}} \left( \frac{e}{2|n|} \right)^{|n|}, \quad n \in \mathbb{Z} \setminus \{0\}
\]
for some \( C > 0 \). Then \( u(t, n) = Ai^{-n}e^{-2it}J_n(1 - 2t) \) for all \( n \in \mathbb{Z} \) and \( 0 \leq t \leq 1 \), for some constant \( A \).
Proof. Consider the discrete Fourier transforms of $u(t, \cdot)$,

$$
\Phi(t, \theta) = \sum_{k=-\infty}^{\infty} u(t, k) \theta^k \in L^2(\mathbb{T}).
$$

We have $\partial_t \Phi(t, \theta) = i(\theta + \theta^{-1} - 2) \Phi(t, \theta)$. Thus

$$
\Phi(t, \theta) = e^{i(\theta + \theta^{-1} - 2)t} \Phi(0, \theta),
$$

and in particular

$$
\Phi(1, \theta) = e^{i(\theta + \theta^{-1} - 2)} \Phi(0, \theta).
$$

It follows from (7) that the functions $\theta \mapsto \Phi(s, \theta)$, for $s = 0$ and $s = 1$, admit holomorphic extensions to $\mathbb{C} \setminus \{0\}$:

$$
\Phi(s, \theta) = \sum_{k<0} u(k, s) \theta^k + \sum_{k \geq 0} u(k, s) \theta^k = \Phi^-(s, \theta) + \Phi^+(s, \theta), \ s \in \{0, 1\}.
$$

Furthermore, (7) implies that $\Phi^+(s, \theta)$ and $\Phi^-(s, 1/\theta)$, $s = 0$ and $s = 1$, are entire functions of exponential type whose indicator diagrams $I^+_s$ and $I^-_s$, respectively, are contained in the disk of radius $1/2$ centered at zero. Actually one can say more:

$$
|\Phi^+(s, \theta)|, |\Phi^-(s, 1/\theta)| \leq Ce^{\theta|\theta|/2}, \ s \in \{0, 1\}.
$$

This follows from the fact that the right-hand side of (7) is asymptotically equivalent to the coefficients in the Taylor expansion of $\exp(z/2)$. On the other hand it follows from (8) that $I^+_0 \subset I^-_0 + i$. Thus $I^+_0 = \{-i/2\}$ and $I^-_1 = \{i/2\}$.

Now let

$$
g(z) = g^+(z) + g^-(z) = e^{i(z+z^{-1})/2} \Phi(0, z) = e^{-i(z+z^{-1})/2} \Phi(1, z),
$$

where, as before, $g^\pm$ are the parts of the Laurent series of $g$ with respectively non-negative and negative powers. It follows that the indicator diagrams $I^\pm$ of the entire functions $g^+(z)$ and $g^-(1/z)$ coincide with $\{0\}$, so $g^+(z)$ and $g^-(1/z)$ are entire functions of type zero.

The relations (10) and (11) now yield that $g^+(iy)$ and $g^-(1/iy)$ are bounded for $y \in \mathbb{R} \setminus \{0\}$ and by the Phragmen-Lindelöf principle (see Lecture 6) $g^+, g^-$, and hence $g$, are constants. Finally $\Phi_0(z) = A \exp(-i(z + z^{-1})/2)$, yielding the required expression for $u(t, n)$.

Corollary 2.2. Let $u$ be as in Proposition 2.1 if in addition

$$
|u(0, n)| \left(\frac{2|n|}{e}\right)^{|n|} \sqrt{|n|} = o(1)
$$

as $n \to +\infty$ or $n \to -\infty$ then $u \equiv 0$.

Assuming only slightly stronger decay, one can apply similar techniques in order to obtain a uniqueness result for solutions of discrete Schrödinger equations with compactly supported time-independent potentials. In this case it suffices to demand that the solution decays just in one direction.
Theorem 2.3. Let \( u(t,n), \quad t > 0, \quad n \in \mathbb{Z} \) be a solution of (1), where the potential \( V \) does not depend on time and also \( V(n) \neq 0 \) just for a finite number of \( n \)'s. If, for some \( \varepsilon > 0 \),

\[
|u(t,n)| \leq C \left( \frac{e}{(2 + \varepsilon)n} \right)^n, \quad n > 0, \quad t \in \{0,1\},
\]

then \( u \equiv 0 \).

Proof. We may assume that \( V_n = 0 \) for \( n > N \) and for \( n < 0 \). Consider the bounded operator \( H = \Delta_d + V : \ell^2 \to \ell^2 \). The solution \( u(t,n) \) is then defined by

\[
u(\cdot,t) = e^{iHt}u(\cdot,0)
\]

and hence belongs to \( \ell^2 \) for all \( t > 0 \).

The absolutely continuous spectrum of \( H : \ell^2 \to \ell^2 \) is the segment \([0,4]\), each point with multiplicity 2. The continuous spectrum is parametrized naturally by the unit circle \( \mathbb{T} \):

\[
\lambda \in [0,4] \Rightarrow \lambda = 2 - \theta - \theta^{-1}, \quad \text{for some} \quad \theta \in \mathbb{T}.
\]

For each \( \theta \in \mathbb{T} \) set \( \lambda(\theta) = 2 - \theta - \theta^{-1} \) and denote by \( e^\mp(\theta) = e^\mp(\theta,n) \) the corresponding Jost solutions of the spectral problem

\[
Hx = \lambda(\theta)x,
\]

i.e. the solutions of (13) satisfying

\[
e^+(\theta,n) = \theta^n, \quad \text{for} \quad n > N, \quad \text{and} \quad e^-(\theta,n) = \theta^n \quad \text{for} \quad n < 0.
\]

We refer the reader to [19] and [17] for the precise definition and detailed discussion of Jost solutions.

Except for \( \theta = \pm 1 \), each of the pairs \( \{e^+(\theta),e^-(\theta)^{-1}\} \), \( \{e^-(\theta),e^-(\theta)^{-1}\} \) is a fundamental system of solutions of (13). Hence we have the representations

\[
e^+(\theta) = a^-(\theta)e^-(\theta) + b^-(\theta)e^-(\theta^{-1})
\]

\[
e^-(\theta) = a^+(\theta)e^+(\theta) + b^+(\theta)e^+(\theta^{-1})
\]

It is known, see e.g. [19], that \( a^\pm \) and \( b^\pm \) are rational functions of \( \theta \), with no poles on \( \mathbb{T} \), and also for \( 0 \leq n \leq N \) the functions \( e^\pm(\theta,n) \) are linear combinations of \( \theta^j \), \( j \in \{-N,-N+1,\ldots,2N\} \). In particular,

\[
\lim_{|\theta| \to +\infty} \frac{\log |a^+(\theta)|}{|\theta|} = \lim_{|\theta| \to +\infty} \frac{\log |b^+(\theta)|}{|\theta|} = 0
\]

and

\[
\lim_{|\theta| \to +\infty} \frac{\log |e^\pm(\theta,n)|}{|\theta|} = 0 \quad \text{for} \quad n = 0, \ldots, N.
\]
Consider the function
\[ \Phi(\theta, t) = \sum_{n=-\infty}^{\infty} u(t, n) e^{-\theta(n, \theta)} = \sum_{n=-\infty}^{-1} u(t, n) e^{-\theta(n, \theta)} + a^+(\theta) \sum_{n=0}^{\infty} u(t, n) e^{\theta(n, \theta)} + b^+(\theta) \sum_{n=0}^{\infty} u(t, n) e^{\theta(n, \theta)} + a^+(\theta) \sum_{n>N}^{\infty} u(t, n) e^{\theta(n, \theta)} + a^+(\theta) \sum_{n>N}^{\infty} u(t, n) e^{\theta(n, \theta)}.
\]

For all \( t \geq 0 \) these functions are in \( L^2(T) \). In addition the first and the third series in the right-hand side converge for \( |\theta| > 1 \) while the second one converges for \( |\theta| < 1 \). For \( t = 0 \) and \( t = 1 \) the second term also converges for \( |\theta| > 1 \), by the hypothesis \( (12) \), thus the functions \( \Phi(\theta, 0) \) and \( \Phi(\theta, 1) \) are holomorphic in \( \mathbb{C} \setminus \{0\} \) except perhaps at the poles of the functions \( a^+ \) and \( b^+ \). Actually, by the basic energy estimate in the next section one can extend this convergence property to \( \Phi(\theta, t) \) for all \( t \in [0, 1] \), see Corollary \( 3.2 \).

We have
\[ -i \frac{\partial \Phi(\theta, t)}{\partial t} = \sum_{n=-\infty}^{\infty} (Hu)(t, n) e^{-\theta(n, \theta)} = \sum_{n=-\infty}^{\infty} u(t, n) (He^-)(n, \theta) = (2 - \theta - \theta^{-1}) \Phi(\theta, t). \]

Hence \( \Phi(\theta, t) = e^{it(2 - \theta - \theta^{-1})} \Phi(\theta, 0) \), and in particular
\[ (16) \quad \Phi(\theta, 1) = e^{it(2 - \theta - \theta^{-1})} \Phi(\theta, 0), \]

a relation which extends to the whole complex plane outside of \( \theta = 0 \).

To derive a contradiction to \( \Phi \neq 0 \), we write \( \Phi(\theta, t) \) as
\[ \Phi(\theta, t) = \left( \sum_{n<0} u(t, n) \theta^n + \sum_{n=0}^{N} u(t, n)e^{-\theta(n, \theta)} + b^+(\theta) \sum_{n>N} u(t, n) \theta^{-n} \right) + a^+(\theta) \left( \sum_{n>N} u(t, n) \theta^n \right) =: A(\theta, t) + a^+(\theta) B(\theta, t). \]

Since \( u(t, \cdot) \in \ell^2 \), we clearly have
\[ \lim_{\theta \to \infty} \frac{\log |A(\theta, t)|}{|\theta|} = \lim_{\theta \to \infty} \frac{\log |a^+(\theta, t)|}{|\theta|} = 0, \]

while \( (12) \) yields that \( B(\theta, t), t = 0 \) and \( t = 1 \), are entire functions of exponential type at most \( (2 + \varepsilon)^{-1} \). Hence, for each \( \alpha \in [0, 2\pi] \) we have
\[ \limsup_{r \to \infty} \frac{\log |B(re^{i\alpha}, t)|}{r} \leq \frac{1}{2 + \varepsilon}, \quad t \in \{0, 1\}, \]

and therefore
\[ (17) \quad \limsup_{r \to \infty} \frac{\log |\Phi(re^{i\alpha}, t)|}{r} \leq \frac{1}{2 + \varepsilon}, \quad t \in \{0, 1\}. \]
By (6), we also have
\[ \limsup_{r \to \infty} \frac{\log |\Phi(re^{i\alpha}, t)|}{r} \geq -\frac{1}{2 + \varepsilon}, \quad t \in \{0, 1\}. \]
This leads to a contradiction unless \( \Phi \equiv 0 \), since according (16) we have
\[ \limsup_{y \to +\infty} \frac{\log |\Phi(iy, 1)|}{y} = 1 + \limsup_{y \to +\infty} \frac{\log |\Phi(iy, 0)|}{y} > 1 - \frac{1}{2 + \varepsilon} > \frac{1}{2 + \varepsilon}. \]
\[ \square \]

It would be of interest to extend this result to the case of potentials with fast decay, not necessarily compactly supported.

3. First energy estimate

In the remainder of the paper we will follow the ideas of [10], to prove that a solution of the discrete Schrödinger equation which decays sufficiently fast along both half-axes at two different moments of time is trivial.

We begin with an energy estimate for solutions of a non-homogeneous initial value problem and show that if the initial data is well-concentrated, the energy cannot spread out too fast.

Given \( \alpha > 0 \) and \( t \geq 0 \) denote
\[ \psi_\alpha(t) = \{\psi_\alpha(t, n)\}_{n \in \mathbb{Z}} = \{(1 + |n|)^{\alpha|n|/(1+t)}\}_{n \in \mathbb{Z}} \]

**Proposition 3.1.** Let \( V = V_1 + iV_2 \), with \( V_1, V_2 : [0, T] \times \mathbb{Z} \to \mathbb{R} \) and \( V_2 \) bounded and \( F : [0, T] \times \mathbb{Z} \to \mathbb{C} \) bounded. Let \( u : [0, T] \times \mathbb{Z} \to \mathbb{C} \), \( u \in C^1([0, T], \ell^2(\mathbb{Z})) \), satisfy
\[ \partial_t u(t, n) = i(\Delta u(t, n) + V(t, n)u + F(t, n)). \]
Assume that \( \{\psi_\alpha(0, n)u(0, n)\} \in \ell^2(\mathbb{Z}) \) for some \( \alpha \in (0, 1] \). Then, for \( T > 0 \),
\[ \|\psi_\alpha(T, n)u(T, n)\|_2^2 \leq e^{CT} \left(\|\psi_\alpha(0, n)u(0, n)\|_2^2 + \int_0^T \|\psi_\alpha(s, n)F(s, n)\|_2^2 ds\right). \]

**Proof.** Consider \( f(t, n) = \psi_\alpha(t, n)u(t, n) \) and let \( H(t) = \|f(t, n)\|_2^2 \). We fix \( \alpha \) till the end of the proof and write \( \psi = \psi_\alpha \).

We will perform several formal computations, assuming that \( H(t) \) is finite for all \( t \in [0, T] \), and then justify these computations at the end of the proof.

Define
\[ \kappa(n, t) = \log \psi(t, n) = \frac{\alpha}{1 + t}|n| \log(1 + |n|). \]

Then
\[ \partial_t f = i\Delta(\psi^{-1} f) + iV f + \partial_t \kappa \psi + i\psi F, \]
which we rewrite as \( \partial_t f = Sf + Af + iVf + i\psi F \), where \( S \) and \( A \) are symmetric and anti-symmetric operators, respectively. Explicitly

\[
Sf = \frac{i}{2} \left( \psi\Delta(\psi^{-1}f) - \psi^{-1}\Delta(\psi f) \right) + \partial_t \kappa f,
\]

\[
Af = \frac{i}{2} \left( \psi\Delta(\psi^{-1}f) + \psi^{-1}\Delta(\psi f) \right).
\]

Denote

\[
a_n = \frac{\psi_{n+1}}{\psi_n} - \frac{\psi_n}{\psi_{n+1}}, \quad b_n = \frac{\psi_{n+1}}{\psi_n} + \frac{\psi_n}{\psi_{n+1}}.
\]

We then rewrite

\[
(Sf)_n = -\frac{i}{2} (a_n f_{n+1} - a_{n-1} f_{n-1}) + (\partial_t \kappa)_n f_n,
\]

\[
(Af)_n = \frac{i}{2} (b_n f_{n+1} + b_{n-1} f_{n-1}) - 2if_n,
\]

In what follows we will use the notation \( a_n = a(t, n) \), and similarly for \( \psi_n \), et cetera.

We want to control the growth of \( H(t) \). Clearly, \( \partial_t H(t) = 2\Re\langle \partial_t f, f \rangle \) and thus

\[
\partial_t H(t) = 2\langle Sf, f \rangle - 2\Im\langle Vf, f \rangle - 2\langle \psi F, f \rangle.
\]

This implies

\[
\partial_t H(t) \leq 2\|\psi F\|_2 \|f\|_2 + 2\|V_2\|_\infty \|f\|_2 + \sum_n \left(2\partial_t \kappa_n + |a_n| + |a_{n-1}| \right) |f_n|^2.
\]

Our aim is to prove that for all \( n \in \mathbb{Z} \)

\[
2\partial_t \kappa_n + |a_n| + |a_{n-1}| \leq 2C,
\]

where \( C \) is a constant. We have

\[
\partial_t \kappa_n = -\frac{\alpha}{(1+t)^2} |n| \log(|n| + 1).
\]

Further, \( |a_n| \leq e^\alpha(|n| + 1)^\alpha \). Hence, if \( \alpha \leq 1 \) we obtain (23), for \( t \in [0, 1] \).

Therefore \( \partial_t \|f\|_2 \leq C\|f\|_2 + \|\psi F\|_2 \) and (19) follows.

In order to justify these computations we truncate the weight function \( \psi \) to an interval \([-N, N]\):

\[
\psi_N(n, t) = \begin{cases} (|n| + 1)^{(1+t)^{-1}}a|n|, & |n| \leq N, \\ (|N| + 1)^{(1+t)^{-1}}a|N|, & |n| > N. \end{cases}
\]

Since the solution \( u \) is in \( l^2 \), the relevant norms weighted by \( \psi_N \) are guaranteed to be finite and by running the above argument we obtain (23) and (19) for the weight \( \psi_N \), this time rigorously. The desired inequality follows by passing to the limit as \( N \to \infty \). \( \square \)
Corollary 3.2. Let $u : [0, 1] \times \mathbb{Z} \to \mathbb{C}$ be a strong solution of the Schrödinger equation
\[ \partial_t u(t, n) = i(\Delta u(t, n) + V(t, n)u), \]
where $V = V_1 + iV_2$ is as above. Further suppose that
\[ \sum_{n>0} n^{2\alpha n} |u(0, n)|^2 < \infty \]
for some $\alpha \leq 1$. Then for each $t \in [0, 1]$ we have
\[ \sum_{n>0} n^{\alpha n} |u(t, n)|^2 < \infty. \]

Proof. Define $\tilde{u}(t, n) = 0$ for $n < 0$ and $\tilde{u}(t, n) = u(t, n)$ for $n \geq 0$. Then $\tilde{u}$ satisfies $(18)$ with $F(t, n)$ bounded and vanishing for $n \notin \{-1, 0\}$. If we apply Proposition 3.1 to $\tilde{u}$ we obtain the required estimate $\square$

4. Logarithmic convexity of weighted $\ell^2$-norms

4.1. Preliminary discussion. From now on we fix $\gamma_0 > 0$ and suppose that $V : [0, T] \times \mathbb{Z} \to \mathbb{R}$ is bounded. Further, we assume that $u$ is a strong solution of
\[ \partial_t u = i(\Delta u + Vu) \]
such that $\|(1 + |n|)^{\gamma_0(1+|n|)} u(0, n)\|_2$ and $\|(1 + |n|)^{\gamma_0(1+|n|)} u(1, n)\|_2$ are finite.

Following the ideas of [10], we are looking for a weight
\[ \psi(t, n) = \exp(\kappa(t, n)) \]
(24)
to give us a logarithmically convex function $e^{-Ct(1-t)} H(t)$, where
\[ H(t) = \|\psi(t, n) u(t, n)\|_2 \]
and $C$ depends on $V$ and $\psi$.

We will first use such a convexity argument to show that for any $0 < \gamma < \gamma_0$ and any $t \in [0, 1]$,
\[ \|(1 + |n|)^{\gamma(1+|n|)} u(t, n)\|_2 < \infty. \]

This also implies that
\[ \|(C_0 + |n| + R_0 t(1-t))^\gamma (C_0 + |n| + R_0 t(1-t)) u(t, n)\|_2 < +\infty \]
(26)
for any $C_0, R_0 > 0$ and $t \in [0, 1]$, and we then set out to prove the logarithmic convexity in $t$ of this latter norm.

In both steps we consider weights of the form (24), with
\[ \kappa(t, n) = \gamma(|n| + R(t)) \ln^b(|n| + R(t)) \]
where either $1/2 < b < 1$ and $R(t) = 1$, or $b = 1$ and $R(t) = C_0 + R_0 t(1-t)$. As before we set $f(t, n) = \psi(t, n) u(t, n)$.

We will first assume that $b < 1$, prove estimates independent of $b$, and let $b \to 1$ to establish (25). This will allow us to justify the computations involved in the second step, when $b = 1$ and we prove the convexity of (26).
4.2. Formal computations. We collect here a number of formal identities which we need in the sequel. The first identities are the same as in the continuous case, found in for example [12], others are specific to the discrete case. We use the notation established in the proof of Proposition 3.1.

We already know that \( \partial_t H(t) = 2\langle Sf, f \rangle \), since \( V \) is real-valued, and thus

\[
\partial_t^2 H(t) = 2\langle S_t f, f \rangle + 4\Re\langle Sf, f_t \rangle \\
= 2\langle S_t f, f \rangle + 4\|Sf\|^2 + 2\langle [S, A]f, f \rangle + 4\Re\langle Sf + iVf, f \rangle \\
= 2\langle S_t f, f \rangle + 2\langle [S, A]f, f \rangle + 4\Re\langle Sf + iVf, Sf \rangle \\
= 2\langle S_t f, f \rangle + 2\langle [S, A]f, f \rangle + 2\|Sf + iVf\|^2 - \|Vf\|^2.
\]

Therefore we obtain that

\[
\|f\|^2 \partial_t^2 (\log H(t)) = \frac{\|2Sf + iVf\|^2\|f\|^2}{\|f\|^2} - 4\|\langle Sf, f \rangle\|^2 \\
+ 2(\langle S_t f, f \rangle + \langle [S, A]f, f \rangle) - \|Vf\|^2 \\
= \frac{\|2Sf + iVf\|^2\|f\|^2 - |\Re\langle 2Sf + iVf, f \rangle|^2}{\|f\|^2} \\
+ 2(\langle S_t f, f \rangle + \langle [S, A]f, f \rangle) - \|Vf\|^2 \\
\geq 2(\langle S_t f, f \rangle + \langle [S, A]f, f \rangle) - \|Vf\|^2.
\]

We reiterate that our aim is to show that

\[
(27) \quad \partial_t^2 \log H(t) \geq -2C
\]

for some \( C \geq 0 \), which implies the log-convexity of \( e^{-Ct(1-t)}H(t) \). The last term in the right-hand side above is clearly bounded below by \(-C\|f\|^2\) since \( V \) is bounded. It suffices to establish an estimate of the first two terms of the form

\[
(28) \quad \langle S_t f, f \rangle + \langle [S, A]f, f \rangle \geq -C\|f\|^2.
\]

We refer now to [22], (21). It follows that

\[
(2S_t f + 2[S, A]f)_n = -i/2(a'_n f_{n+1} - a''_{n-1} f_{n-1}) + \kappa''_n f_n,
\]

and finally

\[
(2S_t f + 2[S, A]f)_n = \nu_{n+1} f_{n+2} + \lambda_n f_{n+1} + \mu_n f_n + \lambda'_{n-1} f_{n-1} + \nu_{n-1} f_{n-2},
\]

where

\[
\nu_{n+1} = \frac{1}{2}(a_n b_{n+1} - a_{n+1} b_n), \\
\lambda_n = -ib_n(\kappa'_{n+1} - \kappa'_n) - ia''_n, \\
\mu_n = a_n b_n - a_{n-1} b_{n-1} + 2\kappa''_n,
\]

and, as before, the coefficients \( a_n, b_n \) are defined in (20).

Clearly \( \psi''_n = \kappa''_n \psi_n \), implying that \( a'_n = (\kappa'_{n+1} - \kappa'_n)b_n \) and

\[
\lambda_n = -2ib_n(\kappa'_{n+1} - \kappa'_n).
\]
4.3. Estimates with an auxiliary weight.

**Proposition 4.1.** Let $\gamma > 0$. Assume that $u$ is a strong solution of

$$\partial_t u = i(\Delta u + Vu)$$

where the potential $V$ is a bounded real-valued function. Let also

$$\| (1 + |n|)^{\gamma(1+|n|)} u(t,n) \|_2 < +\infty, \quad t \in \{0; 1\}.$$ 

Then, for all $t \in [0, 1]$, $\| (1 + |n|)^{\gamma(1+|n|)} u(t,n) \|_2 < +\infty$.

**Proof.** Consider the weight function

$$\psi(n) = e^{\kappa_b(n)}, \quad \kappa_b(n) = \gamma(1 + |n|) \ln^b(1 + |n|),$$

where $1/2 < b < 1$. Note that the hypotheses (29) combined with Proposition 3.1 show that $H_b(t) = \| \exp(\kappa_b(n))u(t,n) \|_2$ is finite for all $t$, allowing us to justify the computations of the preceding section for this choice of weight. We will show that $H(t) = H_b(t)$ satisfies (27) with some $C$ independent of $b$, whence

$$\| \exp(\kappa_b(n))u(t,n) \|_2^2 \leq e^{\frac{\gamma^2}{2}(1-t)} H_b(0)^{1-t} H_b(1)^t \leq e^{\frac{\gamma^2}{2}(1-t)} \| (1 + |n|)^{\gamma(1+|n|)} u(0,n) \|_2^{2(1-t)} \| (1 + |n|)^{\gamma(1+|n|)} u(1,n) \|_2^{2t}.$$ 

Letting $b \to 1$ and applying the monotone convergence theorem then concludes the proof.

We refer to computations in the previous section. In the current setting $S_t = 0$ and $\lambda_n = 0$ so relation (25) reduces to

$$\langle 2[\mathcal{S}, \mathcal{A}]f, f \rangle \geq -C\|f\|^2.$$ 

We have

$$\langle 2[\mathcal{S}, \mathcal{A}]f, f \rangle = \sum_n \mu_n |f_n|^2 + 2\Re \sum_n \nu_{n+1} f_{n+2} \overline{f_n},$$

where

$$\mu_n = a_n b_n - a_{n-1} b_{n-1} = \frac{\psi_{n+1}^2}{\psi_n^2} - \frac{\psi_n^2}{\psi_{n-1}^2} - \frac{\psi_n^2}{\psi_{n+1}^2} + \frac{\psi_{n-1}^2}{\psi_n^2},$$

and

$$\nu_{n+1} = \frac{1}{2} (a_n b_{n+1} - a_{n+1} b_n) = -\frac{\psi_n \psi_{n+2}}{\psi_{n+1}^2} + \frac{\psi_{n+1}^2}{\psi_{n+2} \psi_n},$$

where the coefficients $a_n$ and $b_n$ are defined in (20). By appealing to the second derivative of $x \mapsto (1+x) \ln^b(1+x)$ it is easy to verify that $\kappa_b(n+2) + \kappa_b(n) - 2\kappa_b(n+1)$ is always non-negative and uniformly bounded from above. Thus $\nu_{n+1}$ is uniformly bounded and $\mu_n \geq 0$. This implies (30). $\square$
4.4. Convexity estimate. In this subsection we consider the weight function given by
\[ \psi(t, n) = e^{\kappa(t, n)}, \quad \text{where} \quad \kappa(t, n) = \gamma(|n| + R(t)) \ln(|n| + R(t)), \]
and \( R(t) = C_0 + R_0 t(1 - t), \) \( R_0 > 0, \) \( C_0 \) being large enough. As before we define \( H(t) = \|u(t, n)\psi(t, n)\|_2^2. \)

**Lemma 1.** For every \( \gamma > (3 + \sqrt{3})/2 \) there exists \( C(\gamma) \) such that for \( C_0 > C(\gamma) \) and \( R(t) = C_0 + R_0 t(1 - t) \) we have
\[ \partial_t^2 \log H(t) \geq -\frac{4\gamma}{2\gamma - 3} R_0 \log R_0 - C_1 R_0 - C_2, \]
where \( C_1 \) and \( C_2 \) depend on \( \gamma \) and \( \|V\|_{\infty} \) only.

**Proof.** For \( n \geq 0 \) we have
\[ \frac{\psi(t, n + 1)}{\psi(t, n)} = (n + 1 + R(t))^\gamma \left( 1 + \frac{1}{n + R(t)} \right)^{\gamma(n + R(t))}, \]
and \( \psi_n = \psi_{-n} \) for \( n < 0. \) Hence \( a_n = -a_{-n-1} \) and \( b_n = b_{-n+1} \) for \( n < 0, \) which in turn implies that \( \mu_n = \mu_{-n} \) and \( \lambda_n = -\lambda_{-n-1} \) when \( n < 0. \) We have also
\[ \mu_0 = 2a_0 b_0 + 2\kappa''_n. \]
As before, we get
\[ |\nu_{n+1}| = \frac{\psi_{n+1}^2}{\psi_n \psi_{n+2}} - \frac{\psi_n \psi_{n+1}^2}{\psi_{n+1}^2} \leq C_3, \]
where \( C_3 \) depends on \( \gamma \) only.

Let \( \phi(M) = \gamma M \ln M \) and \( M = M(t, n) = |n| + R(t). \) In this notation we have for \( n \neq 0 \)
\[ \mu_n \geq \exp(2\phi(M + 1) - 2\phi(M)) - \exp(2\phi(M) - 2\phi(M - 1)) - C_4 + 2\kappa''_n, \]
where \( C_4 \) is a constant that depends only on \( \gamma. \) The derivatives of \( \kappa_n \) are
\[ \kappa'_n(t) = R'(t) \phi'(|n| + R(t)), \]
\[ \kappa''_n(t) = -2R_0 \phi'(|n| + R(t)) + (R'(t))^2 \phi''(|n| + R(t)). \]
Then, by the Taylor expansions, we obtain that, for each \( \epsilon > 0 \) and \( C_0 = C_0(\epsilon) \) large enough,
\[ \mu_n \geq 2\gamma e^{2\gamma} M^{2\gamma - 1} + \gamma e^{2\gamma} \left( \frac{(\gamma - 1)^2}{3} - \epsilon \right) M^{2\gamma - 3} + 2A^2 \gamma M^{-1} - 4R_0 \gamma (1 + \ln M) - C_4, \]
where \( A = |R'(t)| \) and \( n \neq 0. \) Further,
\[ \mu_0 \geq (2 - \epsilon) M^{2\gamma} e^{2\gamma} + 2A^2 \gamma M^{-1} - 4R_0 \gamma (1 + \ln M) - C_4. \]
We introduce the notation
\[ \sigma_n = 2\gamma e^{2\gamma} M^{2\gamma - 1} + \gamma e^{2\gamma} \left( \frac{(\gamma - 1)^2}{3} - 2\epsilon \right) M^{2\gamma - 3} + 2A^2 \gamma M^{-1}, \]
and
\[ \rho_n = \epsilon \gamma e^{2\gamma} M^{2\gamma - 3} - 4R_0 \gamma (1 + \ln M) \]
so that \( \mu_n \geq \sigma_n + \rho_n - C_4 \) for all \( n \). Note that by the inequality of arithmetic and geometric means we have
\[ \sigma_n^2 \geq 8A^2 \gamma e^{2\gamma} \left( 2M^{2\gamma - 2} + \left( \frac{(\gamma - 1)^2}{3} - 2\epsilon \right) M^{2\gamma - 4} \right). \]

For \( n \geq 0 \) we have also
\[ |\kappa'_{n+1} - \kappa'_n| = |R'(t)| (\phi'(M + 1) - \phi'(M)) = A\gamma \ln(1 + M^{-1}). \]
Hence, for sufficiently large \( C_0 \),
\[ |\lambda_n| = 2|(\kappa'_{n+1} - \kappa'_n)||b_n| \leq 2A\gamma e^{\gamma} M^{\gamma - 1} + A\gamma e^{\gamma} (\gamma - 1) M^{\gamma - 2} + A\gamma e^{\gamma} \left( \frac{3\gamma^2 - 10\gamma + 8}{12} + \epsilon \right) M^{\gamma - 3}, n \geq 0. \]

To estimate \( \partial^2_t (\log H(t)) \) we note that
\[ \langle S_t f + 2[S, A]f, f \rangle = \sum_n \mu_n |f_n|^2 + 2\Re \sum_n \nu_{n+1} f_{n+2} f_n + 2\Re \sum_n \rho_n f_{n+1} \overline{f_n} \geq \sum_n \sigma_n |f_n|^2 + 2\Re \sum_n \lambda_n f_{n+1} \overline{f_n} + \sum_n \rho_n |f_n|^2 - (C_3 + C_4) \sum_n |f_n|^2. \]
First, we consider the first two terms. If we show that for any \( x, y \geq 0 \)
\[ \sigma_n x^2 + \sigma_{n+1} y^2 \geq 4|\lambda_n|x y, \tag{31} \]
then the summation of these inequalities with \( x = f_n, y = f_{n+1} \) yields
\[ \sum_n \sigma_n |f_n|^2 + 2\Re \sum_n \lambda_n f_{n+1} \overline{f_n} \geq 0. \]
To show \eqref{31} we have to check that
\[ \sigma_n \sigma_{n+1} \geq 4|\lambda_n|^2, \quad n \geq 0. \tag{32} \]
Actually we show \eqref{31} only for \( n \geq 0 \). The relations for negative integers given in the beginning of the proof then imply the inequality for all \( n \).

Using the estimates above, we have
\[ \sigma_n^2 \sigma_{n+1}^2 \geq 64A^4 \gamma^4 e^{4\gamma} \left( 4M^{4\gamma - 4} + 8(\gamma - 1)M^{4\gamma - 5} \right) + 64A^4 \gamma^4 e^{4\gamma} \left( 4(\gamma - 1)(2\gamma - 3) + 4 \left( \frac{(\gamma - 1)^2}{3} - 2\epsilon \right) \right) M^{4\gamma - 6}. \]

While
\[ 16|\lambda_n|^4 \leq 64A^4 \gamma^4 e^{4\gamma} \left( 4M^{4\gamma - 4} + 8(\gamma - 1)M^{4\gamma - 5} \right) + 4A^4 \gamma^4 e^{4\gamma} \left( 6(\gamma - 1)^2 + 8 \left( \frac{3\gamma^2 - 10\gamma + 8}{12} + \epsilon \right) \right) M^{4\gamma - 6}. \]
Inequality (32) hence follows for sufficiently small $\epsilon$ when
\[
2(\gamma - 1)(2\gamma - 3) + \frac{2(\gamma - 1)^2}{3} > 3(\gamma - 1)^2 + \frac{3\gamma^2 - 10\gamma + 8}{3}.
\]
The last inequality is equivalent to $2\gamma^2 - 6\gamma + 3 > 0$, which holds for $\gamma > (3 + \sqrt{3})/2$.

Finally by minimizing in $M$ one obtains that, for $\gamma > 3/2$,
\[
\rho_n \geq \min_{M > 0} \{ \epsilon \gamma e^{2\gamma M^2-3} - 4R_0 \gamma (1 + \ln M) \} \geq -\frac{4\gamma}{2\gamma - 3} R_0 \ln R_0 - C_1 R_0,
\]
where $C_1$ depends on $\gamma$ and $\epsilon$. The conclusion of the lemma follows. \(\square\)

4.5. **Concluding arguments.** Using the weight $\psi(n, t, R_0)$ from the last section and Lemma 1, we obtain that
\[
H_{R_0}(t) \exp(-d(R_0, \gamma)t(1-t))
\]
is logarithmically convex, where
\[
d(R_0, \gamma) = \frac{2\gamma}{2\gamma - 3} R_0 \ln R_0 + \frac{C_1}{2} R_0 + \frac{C_2}{2}.
\]

Hence, for $t = 1/2$ we obtain
\[
H_{R_0}(1/2) \leq \exp \left( \frac{\gamma}{2(2\gamma - 3)} R_0 \ln R_0 + \frac{C_1}{8} R_0 + \frac{C_2}{8} \right) H_{R_0}(0)^{1/2} H_{R_0}(1)^{1/2}.
\]

But since $R(0) = R(1) = C_0$ we see that $H(0)$ and $H(1)$ do not depend on the choice of $R_0$. We obtain that
\[
|u(1/2, n)|^2 \exp(2\gamma(|n| + C_0 + R_0/4) \ln(|n| + C_0 + R_0/4))
\]
\[
\leq D \exp \left( \frac{\gamma}{2(2\gamma - 3)} R_0 \ln R_0 + \frac{C_1}{8} R_0 \right),
\]
where $D$ is a constant independent of $n$ and $R_0$. However, this last inequality is clearly impossible for large $R_0$ when $\gamma > 2$, unless $u(1/2, \cdot) \equiv 0$, which of course implies that $u \equiv 0$.

Our work of this section can be summarized as follows.

**Theorem 4.2.** Assume that $\gamma > (3 + \sqrt{3})/2$ and that $V(t, n)$ is a real-valued bounded function. If $u$ is a strong solution of
\[
\partial_t u = i(\Delta_d u + Vu)
\]
such that
\[
\left\|(1 + |n|)^{\gamma(1+|n|)} u(0, n) \right\|_L \leq \left\|(1 + |n|)^{\gamma(1+|n|)} u(1, n) \right\|_L < +\infty,
\]
then $u \equiv 0$.

Remark. This result is most likely not sharp. The authors expect that a milder decay condition (with $\gamma = 1 + \epsilon$) and even just one-sided decay should imply uniqueness as in the case of free Schrödinger evolution.
References

[1] S. Agmon Unicité et convexité dans les problèmes différentiels, Séminaire de Mathématique Supérieures, 13, (1966), Les Presses de l’Université de Montreal. Que., 1966.
[2] J. A. Barceló, L. Fanelli, S. Gutiérrez, A. Ruiz and M.C. Vilela, Hardy uncertainty principle and unique continuation properties of covariant Schrödinger flows, J. Funct. Anal. 264 (2013), no. 10, 2386–2425.
[3] T. Carleman, Sur un problème d’unicité pour les systèmes d’équations aux dérivées partielles à deux variables, Ark. Math. 26B (1939), 1–9.
[4] B. Cassano and L. Fanelli, Sharp Hardy uncertainty principle and Gaussian profiles of covariant Schrödinger evolutions, Trans. AMS, 367 (2015), no. 3, 2213–2233.
[5] F. Chang and S.-T. Yau A combinatorial trace formula, in Tsing Hua Lectures on Geometry and Analysis, International Press, Cambrige, AA, 1997, 107–116.
[6] F. Chang and S.-T. Yau Discrete Green’s Functions. J. Combin. Theory Ser. A 91 (2000), no. 1-2, 191–214.
[7] M. Cowling, L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega, The Hardy uncertainty principle revisited, Indiana Univ. Math. J. 59 (2010), no. 6, 2007–2025.
[8] B. Demange, Uncertainty principles associated to non-degenerate quadratic forms, Mém. Soc. Math. Fr. (N. S.), 119 (2009), 98 pp., ISBN: 92-8-2-85629-297-6.
[9] L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega, On uniqueness properties of solutions of Schrödinger equations, Comm. Partial Diff. Eq., 31 (2006), no.10-12, 1811–1823.
[10] L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega, Hardy’s uncertainty principle, convexity and Schrödinger evolutions, J. Eur. Math. Soc. (JEMS) 10 (2008), no. 4, 883–907.
[11] L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega. The sharp Hardy uncertainty principle for Schrödinger evolutions, Duke Math. J. 155 (2010), no. 1, 163–187.
[12] L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega. Uniqueness properties of solutions to Schrödinger equations, Bull. of Amer. Math. Soc., 49 (2012), 415-422.
[13] A. Fernández-Bertolin, Discrete uncertainty principles and virial identities, Appl. Comput. Harmon. Anal., to appear, doi:10.1016/j.acha.2015.02.004.
[14] Ph. Jaming, Uncertainty principles for orthonormal bases. Séminaire: Équations aux Dérivées Partielles. 2005–2006, Exp. No. XV, 16 pp., S’em in. Équ. Dériv. Partielles, École Polytech., Palaiseau, 2006.
[15] B. Ya. Levin, Lectures on entire functions, Translations of Mathematical Monographs, AMS, 1996
[16] G. Lippner and D. Mangoubi Harmonic functions on the lattice: Absolute monotonicity and propagation of smallness. Duke Math. J., to appear.
[17] V. Marchenko, Introduction to the theory of inverse problems of spectral analysis (Russian), Acta publishing house, Kharkov, 2005.
[18] F. L. Nazarov, Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type, St. Petersburg Math. J., 5 (1994), no. 4, 663–717.
[19] G. Teschl, Jacobi Operators and Completely Integrable Nonlinear Lattices, Mathematical Surveys and Monographs, Vol. 72, AMS, 2000.

Institut de Mathématiques de Bordeaux UMR 5251, Université de Bordeaux, cours de la Libération, F 33405 Talence cedex, France
E-mail address: philippe.jaming@math.u-bordeaux1.fr

Department of Mathematics, Norwegian University of Science and Technology 7491, Trondheim, Norway
E-mail address: yura@math.ntnu.no
E-mail address: eugenia@math.ntnu.no
E-mail address: karl-mikael.perfekt@math.ntnu.no