Universality theorem for the iterated integrals of the logarithm of the Riemann zeta-function

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Abstract. In this paper, we prove the universality theorem for the iterated integrals of the logarithm of the Riemann zeta-function on a line parallel to the real axis.

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1 Introduction and main results

Let $\zeta(s)$ denote the Riemann zeta-function. In this paper, we consider the function $\tilde{\eta}_m(s)$ given by

$$\tilde{\eta}_m(s+it) = \int_{\sigma}^{\infty} \tilde{\eta}_{m-1}(\alpha+it) \, d\alpha \quad \text{and} \quad \tilde{\eta}_0(s+it) = \log \zeta(s+it) = \int_{\infty}^{\sigma} \frac{\zeta'}{\zeta}(\alpha+it) \, d\alpha$$

for $s = \sigma + it \in G$, where

$$G = \mathbb{C} \setminus \left( \bigcup_{\rho = \beta + i\gamma} \{ s = \sigma + i\gamma; \sigma \leq \beta \} \cup (-\infty, 1) \right).$$

Here $\rho = \beta + i\gamma$ denote the nontrivial zeros of the Riemann zeta-function $\zeta(s)$. This function $\tilde{\eta}_m(s)$ is first introduced by Inoue [10], who investigated various its properties. Recently, he and the author [7] showed that the set $\{ \tilde{\eta}_m(\sigma+it); t \in \mathbb{R} \}$ is dense in the complex plane for $m \geq 1$ and $1/2 \leq \sigma < 1$ by using the mean value theorem in [10, Thm. 5]. We note that Bohr [3] classically proved that the set $\{ \log \zeta(\sigma+it); t \in \mathbb{R} \}$ is dense in the complex plane for $1/2 < \sigma < 1$, and it remains to determine whether or not the set $\{ \log \zeta(1/2+it); t \in \mathbb{R} \}$ is dense in the complex plane.
Historically, Bohr’s result evolved into the Bohr–Jessen limit theorem [4,5,12], Voronin’s multidimensional denseness theorem [25], and Voronin’s universality theorem [26]. Our goal in this paper is to show the universality theorem for \( \tilde{\eta}_m(s) \) in the strip \( D = \{ s = \sigma + it; 1/2 < \sigma < 1 \} \). We refer to textbooks [13, 15, 18, 23] for the theory of the universality theorem and refer to a survey paper [20] for recent studies.

For other recent studies on \( \tilde{\eta}_m(s) \), we remark that Inoue [11] studied the extreme values and the large deviation estimates like [16]. Inoue, Mine, and the author [8] studied the probability density function, the discrepancy, and the large deviation estimates like [4, 5, 17].

Now, we state the main theorem of the present paper.

**Theorem 1.** Let \( m \) be a nonnegative integer, let \( K \) be a compact subset of \( D \) with connected complement, and let \( f \) be a continuous function on \( K \) holomorphic in the interior of \( K \). Then for any \( \varepsilon > 0 \), we have

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [T, 2T]: \sup_{s \in K} \left| \tilde{\eta}_m(s + i\tau) - f(s) \right| < \varepsilon \right\} > 0,
\]

where \( \text{meas} \) denotes the Lebesgue measure.

We remark that in the set lying on the left-hand side of (1.1), the set of all \( \tau \in [0, T] \) such that the function \( \tilde{\eta}_m(s + i\tau) \) is not holomorphic for some \( s \in K \) is excluded.

Using this theorem, we get the following corollaries.

**Corollary 1.** Let \( m \) be a nonnegative integer, let \( n \) be a positive integer, and let \( 1/2 < \sigma < 1 \). Then the set

\[
\left\{ \left( \tilde{\eta}_m(\sigma + it), \tilde{\eta}_m'(\sigma + it), \ldots, \tilde{\eta}_m^{(n-1)}(\sigma + it) \right) \in \mathbb{C}^n; \ t \in \mathbb{R} \right\}
\]

is dense in \( \mathbb{C}^n \).

**Corollary 2.** Let \( m \) be a nonnegative integer, let \( n \) be a positive integer, and let \( 1/2 < \sigma < 1 \). If \( F_0, F_1, \ldots, F_J : \mathbb{C}^n \to \mathbb{C} \) are continuous functions and satisfy

\[
\sum_{j=0}^{J} s^2 F_j \left( \tilde{\eta}_m(s), \tilde{\eta}_m'(s), \ldots, \tilde{\eta}_m^{(n-1)}(s) \right) \equiv 0
\]

for all \( s \in \mathbb{C} \), then \( F_0, F_1, \ldots, F_J \equiv 0 \).

These corollaries can be proved along the same line as in [13, Chap. VII, Sect. 2]. Therefore we omit the proofs.

## 2 Proof of Theorem 1

### 2.1 Preliminaries

We begin with giving some definitions and notations. Let \( m \) be a nonnegative integer, and let \( K \) be a compact subset of \( D \) with connected complement. Since without the assumption of the Riemann hypothesis we cannot find whether or not the function \( \tilde{\eta}_m(s + i\tau) \) is holomorphic for \( s \in K \), we need to use the shift \( \tau \) such that the function \( \tilde{\eta}_m(s + i\tau) \) is holomorphic for \( s \in K \). For such a reason, we first prepare the following notations:

- \( |K| := \max_{s \in K} \text{Im}(s) - \min_{s \in K} \text{Im}(s) \).
- \( \tau_0(K) = (\max_{s \in K} \text{Im}(s) + \min_{s \in K} \text{Im}(s))/2 \).
- \( \sigma_0(K) = (1/2 + \min_{s \in K} \text{Re}(s))/2 \).
• For any $\Delta > 0$, put

$$G_{\sigma_0(K), \Delta} = \mathbb{R} \setminus \left\{ \bigcup_{\rho = \beta + i\gamma} \left( \gamma - \tau_0(K) - \Delta, \gamma - \tau_0(K) + \Delta \right) \cup \left( -\tau_0(K) - \Delta, -\tau_0(K) + \Delta \right) \right\}.$$

• For any $T > 0$, let $I_K(T) = G_{\sigma_0(K), |K|+1} \cap [T, 2T]$.

Note that $\tau \in G_{\sigma_0(K), |K|+1}$ implies $K + i\tau \subset G$. Thus the function $\tilde{\eta}_m(s + i\tau)$ is holomorphic for $s \in K$ when $\tau \in G_{\sigma_0(K), |K|+1}$. We further note that $\text{meas}(I_K) \sim T$ as $T \to \infty$ by using the zero density estimate. Remark that assuming the Riemann hypothesis, we have $I_K(T) = [T, 2T]$.

Let $\mathbb{T}$ denote the unit circle on the complex plane and define $\Omega$ by $\Omega = \prod_p \mathbb{T}_p$, where the product runs over all prime numbers $p$, and $\mathbb{T}_p = \mathbb{T}$. Since $\mathbb{T}_p$ is a compact topological Abelian group, there exists a unique probability Haar measure $m_p$ on $(\mathbb{T}_p, \mathcal{B}(\mathbb{T}_p))$. Here $\mathcal{B}(S)$ denotes the Borel $\sigma$-field of the topological space $S$. Using the Kolmogorov extension theorem, we can construct the probability Haar measure $m = \otimes_p m_p$ on $(\Omega, \mathcal{B}(\Omega))$. For any $\omega = (\omega(p))_p \in \Omega$ and $n \in \mathbb{N}$ such that $n \geq 2$, let

$$\omega(n) = \prod_{j=1}^k \omega(p_j)^{r_j},$$

where $n = p_1^{r_1} \cdots p_k^{r_k}$ is the prime factorization of $n$.

Fix a nonnegative integer $m$ and let $T$ be a positive real number. We fix real numbers $\sigma_L = \sigma_L(K)$ and $\sigma_R = \sigma_R(K)$ satisfying

$$\sigma_0(K) < \sigma_L < \min_{s \in K} \Re(s) \quad \text{and} \quad \max_{s \in K} \Re(s) < \sigma_R < 1,$$

and denote by $\mathcal{R} = \mathcal{R}(K)$ the rectangle

$$\mathcal{R} = (\sigma_L, \sigma_R) \times i \left( \min_{s \in K} \Im(s) - \frac{1}{2}, \max_{s \in K} \Im(s) + \frac{1}{2} \right),$$

which includes $K$ (see Fig. 1). Note that $\tau \in G_{\sigma_0(K), |K|+1}$ implies $\mathcal{R} + i\tau \subset G$ by the definition of $\mathcal{R}$. Let $\mathcal{H}(\mathcal{R})$ denote the set of holomorphic functions on $\mathcal{R}$ equipped with the topology of uniform convergence on compact subsets. This topology is metrizable in the following way: We take a sequence of compact sets $\{K_j\}_{j=1}^\infty$ satisfying

• $\mathcal{R} = \bigcup_{j=1}^\infty K_j$,
• $K_j \subset K_{j+1}^\circ$ for any $j \in \mathbb{N}$, where $A^\circ$ is the interior of a set $A$,
• for any compact subset $K$ of $\mathcal{R}$, there exists $j \in \mathbb{N}$ such that $K \subset K_j$,

and define the metrics $d_j$, $j = 1, 2, \ldots$, on $\mathcal{H}(\mathcal{R})$ by $d_j(f, g) = \sup_{s \in K_j} |f(s) - g(s)|$ for $f, g \in \mathcal{H}(\mathcal{R})$. For the sequence $\{d_j\}_{j=1}^\infty$ of metrics, we put

$$d(f, g) = \sum_{j=1}^\infty \frac{\min\{d_j(f, g), 1\}}{2^j}$$

for $f, g \in \mathcal{H}(\mathcal{R})$. The metric $d$ induces the desired topology on $\mathcal{H}(\mathcal{R})$. 

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We define the probability measures $Q_T$ and $Q$ on $(\mathcal{H}(\mathcal{R}), \mathcal{B}(\mathcal{H}(\mathcal{R})))$ by

$$Q_T(A) = \frac{1}{\text{meas}(\mathcal{I}_K(T))} \text{meas}\{\tau \in \mathcal{I}_K(T): \tilde{\eta}_m(s + i\tau) \in A\},$$

$$Q(A) = m\{\omega \in \Omega: \tilde{\eta}_m(s, \omega) \in A\}$$

for $A \in \mathcal{B}(\mathcal{H}(\mathcal{R}))$, where $\mathcal{H}(\mathcal{R})$-valued random variables $\tilde{\eta}_m(s, \omega)$ are defined by

$$\tilde{\eta}_m(s, \omega) = \sum_{n=2}^{\infty} \frac{A(n)\omega(n)}{n^s\log n^{m+1}}.$$

Here $A$ denotes the von Mangoldt function given by

$$A(n) = \begin{cases} \log p & \text{if } n \text{ is a power of a prime number } p, \\ 0 & \text{otherwise.} \end{cases}$$

We will show the following two propositions.

**Proposition 1.** The probability measure $Q_T$ on $(\mathcal{H}(\mathcal{R}), \mathcal{B}(\mathcal{H}(\mathcal{R})))$ converges weakly to $Q$ as $T \to \infty$.

**Proposition 2.** The support of the probability measure $Q$ on $(\mathcal{H}(\mathcal{R}), \mathcal{B}(\mathcal{H}(\mathcal{R})))$ coincides with $\mathcal{H}(\mathcal{R})$.

Throughout the paper, we use the following conventions:

- For a probability space $(X, \mathcal{F}, \mu)$ and an integrable function $f$ on $X$, we denote $E^\mu[f] = \int_X f \, d\mu$.
- Implicit constants may depend on $m$.

### 2.2 Proof of Proposition 1

We will prove the proposition by using the modern method as in [15] and [14]. Although the basic strategy of the proof is based on the method of Bagchi [1], we note that we need not use the Birkhoff–Khinchin ergodic theorem.
To show the proposition, we use the smoothing technique in analytic number theory. We fix a real-valued smooth function \( \varphi(x) \) on \([0, \infty)\) with compact support satisfying \( \varphi(x) = 1 \) on \([0, 1]\) and \( 0 \leq \varphi(x) \leq 1 \) for all \( x \in \mathbb{R} \). Let \( \hat{\varphi}(s) \) denote the Mellin transform of \( \varphi(x) \), that is, \( \hat{\varphi}(s) = \int_0^\infty \varphi(x)x^{s-1}\,dx \) for \( \text{Re}(s) > 0 \). We recall the basic properties of the Mellin transform.

**Lemma 1.** We have:

(i) The Mellin transform \( \hat{\varphi}(s) \) can be meromorphically continued to \( \text{Re}(s) > -1 \) and has only one simple pole at \( s = 0 \) with residue \( \varphi(0) = 1 \).

(ii) Let \(-1 < A < B \) be real numbers. Then for any positive integer \( N > 0 \), there exists \( C(A, B; N) > 0 \) such that

\[
|\hat{\varphi}(s)| \leq C(A, B; N)(1 + |t|)^{-N}
\]

for \( s = \sigma + it \) with \( A \leq \sigma \leq B \) and \( |t| \geq 1 \).

(iii) For any \( c > 0 \) and \( x > 0 \), we have the Mellin inversion formula

\[
\varphi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{\varphi}(s)x^{-s}\,ds.
\]

**Proof.** The proof can be found in [14, App. A].

We use the following lemmas, which are slightly modified versions of Lemmas 2.1 and 2.2 in Granville and Soundararajan [9].

**Lemma 2.** Let \( y \geq 2 \) and \( |t| \geq y + 3 \) be real numbers. Let \( 1/2 \leq \sigma_0 < 1 \) and suppose that the rectangle \( \{z: \sigma_0 < \text{Re}(s) \leq 1, |\text{Im}(z) - t| \leq y + 2\} \) is free of zeros of \( \zeta(z) \). Then for \( \sigma_0 < \sigma \leq 1 \), we have

\[
\tilde{\eta}_m(\sigma + it) = \sum_{2 \leq n \leq y} \frac{A(n)}{n^{\sigma + it}(\log n)^{m+1}} + O_m\left(\frac{\log |t|}{(\sigma' - \sigma_0)^2} y^\sigma - \sigma\right),
\]

where \( \sigma' = \min\{\sigma_0 + 1/\log y, (\sigma + \sigma_0)/2\} \).

**Proof.** For the case \( m = 0 \), the statement has been proved in Lemma 2.1 in [9]. For the other cases, the proof can be done in the same way.

**Lemma 3.** Fix \( 1/2 < \sigma_0 < \sigma_1 \leq 1 \), and let \( T \) and \( y \) be large numbers with \( T \geq y + 3 \). Put

\[
\mathcal{L}(T; \sigma_0, y) = \left( \bigcup_{\rho = \beta + i\gamma, \beta > \sigma_0} \left( \gamma - (y + 3), \gamma + (y + 3) \right) \right) \\
\cup \left[ \frac{1}{2}T, \frac{1}{2}T + (y + 3) \right] \cup \left[ \frac{5}{2}T - (y + 3), \frac{5}{2}T \right].
\]

Then we have

\[
\tilde{\eta}_m(\sigma + it) = \sum_{2 \leq n \leq y} \frac{A(n)}{n^{\sigma + it}(\log n)^{m+1}} + O_m\left(y^{\sigma_0 - \sigma_1} (\log T)^{3}\right)
\]

for \( \sigma_1 \leq \sigma \leq 1 \) and \( t \in [(1/2)T, (5/2)T] \setminus \mathcal{L}(T; \sigma_0, y) \) and the estimate

\[
\text{meas}\left(\mathcal{L}(T; \sigma_0, y)\right) \ll T^{3/2 - \sigma_0}(\log T)^5.
\]
Proof. The first statement can be obtained by applying Lemma 2. The last assertion follows from the zero-density estimate \( N(\sigma_0, T) \ll T^{3/2-\sigma_0} (\log T)^3 \) (see, e.g., [24, Thm. 9.19 A]). \( \square \)

We will use Lemma 3 with \( \sigma_0 = \sigma_0(\mathcal{K}) \) and \( \sigma_1 = \sigma_L \) in the next lemma. Let

\[
\mathcal{X}_\mathcal{K}(T) = \mathcal{G}_{\sigma_0(\mathcal{K}), [\sigma_0(\mathcal{K})+Y(T)+d} \cap [T, 2T] \quad \text{and} \quad Y(T) = (\log T)^4/(\sigma_L-\sigma_0(\mathcal{K}))
\]

and define the function

\[
\tilde{\eta}_{m, X}(s+i\tau) := \sum_{n=2}^{\infty} \frac{A(n)\varphi(n/X)}{n^{s+i\tau}(\log n)^{m+1}} = \sum_{2 \leq n \leq X} \frac{A(n)}{n^{s+i\tau}(\log n)^{m+1}} + R_{m,X}(s+i\tau)
\]

for \( X \geq 2 \). Note that \( \mathcal{X}_\mathcal{K}(T) \subset \mathcal{I}_\mathcal{K}(T) \) and \( \text{meas}(\mathcal{X}_\mathcal{K}(T)) \sim T \) as \( T \to \infty \) by the zero-density estimate. Then we have the following lemma.

**Lemma 4.** For any compact subset \( C \) of \( \mathcal{R} \), we have

\[
\lim_{X \to \infty} \limsup_{T \to \infty} \frac{1}{\text{meas}(\mathcal{I}_\mathcal{K}(T))} \int_{\mathcal{X}_\mathcal{K}(T)} \sup_{s \in C} \left| \tilde{\eta}_m(s+i\tau) - \tilde{\eta}_{m,X}(s+i\tau) \right| d\tau = 0.
\]

Proof. Let \( T \) be a large number. We note that for any \( \tau \in \mathcal{X}_\mathcal{K}(T) \), the function \( \tilde{\eta}_m(s+i\tau) \) is holomorphic on \( \mathcal{R} \). By Cauchy’s formula,

\[
\tilde{\eta}_m(s+i\tau) - \tilde{\eta}_{m,X}(s+i\tau) = \frac{1}{2\pi i} \int_{\partial C} \frac{\tilde{\eta}_m(z+i\tau) - \tilde{\eta}_{m,X}(z+i\tau)}{z-s} \, dz
\]

for \( s \in C \) and \( \tau \in \mathcal{X}_\mathcal{K}(T) \), where \( \partial A \) denotes the boundary of a set \( A \). Hence we have

\[
\frac{1}{\text{meas}(\mathcal{I}_\mathcal{K}(T))} \int_{\mathcal{X}_\mathcal{K}(T)} \sup_{s \in C} \left| \tilde{\eta}_m(s+i\tau) - \tilde{\eta}_{m,X}(s+i\tau) \right| d\tau \ll \frac{1}{\text{dist}(C, \partial \mathcal{R}) \text{meas}(\mathcal{I}_\mathcal{K}(T))} \int_{\partial \mathcal{R}} |dz| \int_{\mathcal{X}_\mathcal{K}(T)} \left| \tilde{\eta}_m(z+i\tau) - \tilde{\eta}_{m,X}(z+i\tau) \right| d\tau
\]

\[
\ll \frac{\ell(\partial \mathcal{R})}{\text{dist}(C, \partial \mathcal{R}) \text{meas}(\mathcal{I}_\mathcal{K}(T))} \sup_{\sigma \in \partial \mathcal{R}} \int_{\mathcal{X}_\mathcal{K}(T)} \left| \tilde{\eta}_m(\sigma+it) - \tilde{\eta}_{m,X}(\sigma+it) \right| dt,
\]

where \( \text{dist}(C, \partial \mathcal{R}) \) denotes the distance between \( C \) and \( \partial \mathcal{R} \), and \( \ell(\partial \mathcal{R}) \) is the length of \( \partial \mathcal{R} \). We put

\[
\mathcal{X}_\mathcal{K}'(T) = \left[ \frac{1}{2} T, \frac{5}{2} T \right] \setminus \ell \left( T; \sigma_0(\mathcal{K}), \frac{Y(T)}{2} \right).
\]

Now we have that

\[
\int_{\mathcal{X}_\mathcal{K}'(T)} \left| \tilde{\eta}_m(\sigma+it) - \tilde{\eta}_{m,X}(\sigma+it) \right| dt \leq \int_{\mathcal{X}_\mathcal{K}'(T)} \left| \tilde{\eta}_m(\sigma+it) - \sum_{2 \leq n \leq Y(T)/2} \frac{A(n)}{n^{\sigma+it}(\log n)^{m+1}} \right| dt
\]

\[
+ \int_{\mathcal{X}_\mathcal{K}'(T)} \left| \sum_{2 \leq n \leq Y(T)/2} \frac{A(n)}{n^{\sigma+it}(\log n)^{m+1}} - \tilde{\eta}_{m,X}(\sigma+it) \right| dt.
\]
For the first integral, we have
\[
\int_{X'_K(T)} \left| \tilde{\eta}_m(\sigma + it) - \sum_{2 \leq n \leq Y(T)/2} \frac{A(n)}{n^{\sigma + it}(\log n)^{m+1}} \right| \, dt \ll \frac{\text{meas}(X'(T))}{\log T}
\]
by Lemma 3. For the second integral, we have
\[
\left( \int_{X'_K(T)} \left| \sum_{2 \leq n \leq Y(T)/2} \frac{A(n)}{n^{\sigma + it}(\log n)^{m+1}} - \tilde{\eta}_{m,K}(\sigma + it) \right| \, dt \right)^2 \leq \text{meas}(X'(T)) \int_{\tau / 2}^{5T/2} \left| \sum_{2 \leq n \leq Y(T)/2} \frac{A(n)}{n^{\sigma + it}(\log n)^{m+1}} - \tilde{\eta}_{m,K}(\sigma + it) \right|^2 \, dt
\]
by using the Cauchy inequality. The latter mean square value is estimated by
\[
\left( \frac{5/2}{T} \right)^2 \sum_{2 \leq n \leq Y(T)/2} \frac{A(n)}{n^{\sigma + it}(\log n)^{m+1}} - \tilde{\eta}_{m,K}(\sigma + it) \right|^2 \, dt \ll \left( \sum_{n \leq X} \frac{A(n)^2(1 - \varphi(n/X))^2}{n^{2\sigma_0(K)(\log n)^{2m+2}}} \right) T + E_1,
\]
where
\[
E_1 \ll \sum_{X < l < n \leq Y(T)/2} \frac{A(l)A(n)(1 - \varphi(l/X))(1 - \varphi(n/X))}{l^{\sigma_0}(n^{\sigma}(\log n)^{m+1}(\log n)^{m+1}(\log n/l)} \ll \sigma_0(K) Y(T)^{2 - 2\sigma_0(K)} \log(Y(T)).
\]
Combining the above estimates with the estimate \( \text{meas}(I_K(T)) \sim T \), we obtain
\[
\limsup_{T \to \infty} \frac{1}{\text{meas}(I_K(T))} \int_{X_K(T)} \sup_{s \in C} \left| \tilde{\eta}_m(s + it) - \tilde{\eta}_{m,K}(s + it) \right| \, dt \ll K, \sigma, C \left( \sum_{n > X} \frac{A(n)^2}{n^{2\sigma_0(K)(\log n)^{2m+2}}} \right)^{1/2},
\]
and the right-hand side of the above inequality tends to 0 as \( X \to \infty \) since \( 2 \sigma_0(K) > 1 \). This completes the proof.

Next, we define the \( \mathcal{H}(\mathcal{R}) \)-valued random variables
\[
\tilde{\eta}_{m,K}(s, \omega) = \sum_{n=2}^{\infty} \frac{A(n)\omega(n)\varphi(n/X)}{n^2(\log n)^{m+1}}.
\]
We will prove the following lemma.
Lemma 5. For any compact subset $C$ of $\mathcal{R}$, we have

$$\lim_{X \to \infty} \mathbb{E}^m \left[ \sup_{s \in C} |\tilde{\eta}_m(s, \omega) - \tilde{\eta}_m, X(s, \omega)| \right] = 0.$$ 

To prove this lemma, we prepare the following lemma.

Lemma 6. We have the following:

(i) The series $\sum_p \omega(p) p^{-s}(\log p)^{-m}$ is holomorphic on the domain $\text{Re}(s) > \sigma_0(\mathcal{K})$ for almost all $\omega \in \Omega$. Here the sum runs over all prime numbers.

(ii) The function $\tilde{\eta}_m(s, \omega)$ is holomorphic on the domain $\text{Re}(s) > \sigma_0(\mathcal{K})$ for almost all $\omega \in \Omega$. Moreover, we have the representation

$$\tilde{\eta}_m(s, \omega) = \sum_p \frac{\text{Li}_{m+1}(p^{-s}\omega(p))}{(\log p)^m}$$

on the same region for almost all $\omega \in \Omega$. Here $\text{Li}_J$ is the polylogarithm function defined by $\text{Li}_J(z) = \sum_{n=1}^\infty z^n / n^J$ for $|z| < 1$ and $J \in \mathbb{N}$.

(iii) For almost all $\omega \in \Omega$, there exists $C_1(\mathcal{K}, \sigma_L, \omega) > 0$ such that

$$|\tilde{\eta}_m(s, \omega)| \leq C_1(\mathcal{K}, \sigma_L, \omega)(|t| + 2)$$

for $\text{Re}(s) \geq (\sigma_0(\mathcal{K}) + \sigma_L)/2$.

(iv) There exists $C_2(\mathcal{K}, \sigma_L) > 0$ such that

$$\mathbb{E}^m[|\tilde{\eta}_m(s, \omega)|] \leq C_2(\mathcal{K}, \sigma_L)(|t| + 2)$$

for $\text{Re}(s) \geq (\sigma_0(\mathcal{K}) + \sigma_L)/2$.

Proof. We first show that the series $\sum_p \omega(p) p^{-\sigma_0(\mathcal{K})(\log p)^{-m}}$ converges for almost all $\omega \in \Omega$. We have

$$\mathbb{E}^m \left[ \frac{\omega(p)}{p^{\sigma_0(\mathcal{K})(\log p)^m}} \right] = 0 \quad \text{and} \quad \sum_p \mathbb{E}^m \left[ \frac{\omega(p)}{p^{\sigma_0(\mathcal{K})(\log p)^m}} \right]^2 \leq \sum_{n=1}^\infty \frac{1}{n^{2\sigma_0(\mathcal{K})}} < \infty.$$ 

Hence the statement follows by applying Kolmogorov’s theorem (see, e.g., [14, App. B]). Combining the basic property of the Dirichlet series (see, e.g., [21, Thm. 1.1]) with it, we have Lemma 6(i).

Next, we prove Lemma 6(ii). For almost all $\omega \in \Omega$ satisfying Lemma 6(i), we have

$$\tilde{\eta}_m(s, \omega) = \lim_{X \to \infty} \sum_{2 \leq n \leq X} \frac{A(n)\omega(n)}{n^s(\log n)^{m+1}}$$

$$= \lim_{X \to \infty} \left( \sum_{p \leq X} \frac{\omega(p)}{p^s(\log p)^m} + \sum_{p^k \leq X, k \geq 2} \frac{\omega(p)^k}{k^{m+1}p^{ks}(\log p)^m} \right)$$

$$= \sum_p \frac{\omega(p)}{p^s(\log p)^m} + \sum_{k=2}^\infty \sum_{p^k \leq X} \frac{\omega(p)^k}{k^{m+1}p^{ks}(\log p)^m}$$

(2.2)
for \( \text{Re}(s) > \sigma_0(K) \) since the latter sum converges absolutely. This gives the first statement of Lemma 6(ii). On the other hand, we have

\[
\sum_{p} \frac{\text{Li}_{m+1}(p^{-s} \omega(p))}{(\log p)^m} = \lim_{X \to \infty} \left( \sum_{p \leq X} \frac{\omega(p)}{p^s (\log p)^m} + \sum_{p \leq X} \sum_{k=2}^{\infty} \frac{\omega(p)^k}{k^{m+1} p^s (\log p)^m} \right)
\]

\[
= \sum_{p} \frac{\omega(p)}{p^s (\log p)^m} + \sum_{p} \sum_{k=2}^{\infty} \frac{\omega(p)^k}{k^{m+1} p^s (\log p)^m}
\]

for \( \text{Re}(s) > \sigma_0(K) \), because of the same reason. This gives the second statement of Lemma 6(ii).

Finally, we will show Lemma 6(iii) and (iv). Define the random variables

\[
S_{\xi}(\omega) = \sum_{2 \leq n \leq \xi} \frac{A(n) \omega(n)}{n^{\sigma_0(K) (\log n)^{m+1}}}
\]

and

\[
A_{\xi}(\omega) = \sum_{p \leq \xi} \frac{\omega(p)}{p^{\sigma_0(K)} (\log p)^m}
\]

for \( \xi \geq 2 \) and \( \omega \in \Omega \). By the almost all convergence of the series \( \sum_{p} \omega(p) p^{-\sigma_0(K)} (\log p)^{-m} \) we find that \( \{A_{\xi}(\omega)\}_{\xi \geq 2} \) is bounded almost surely. We can write

\[
S_{\xi}(\omega) = A_{\xi}(\omega) + \sum_{p^k \leq \xi, k \geq 2} \frac{\omega(p)^k}{k^{m+1} p^{\sigma_0(K)} (\log p)^m},
\]

and the latter term is bounded. Therefore, for almost all \( \omega \in \Omega \), we have

\[
|S_{\xi}(\omega)| \ll M(K, \omega)
\]

for \( \xi \geq 2 \) with some positive constant \( M(K, \omega) \). We fix \( \omega \in \Omega \) that satisfies inequality (2.3). Estimate (2.3) and partial summation yield

\[
\tilde{\eta}_m(s, \omega) = \int_{2-}^{\infty} \frac{dS_{\xi}(\omega)}{\xi^{\sigma_{\xi+1}-\sigma_{\xi}}(\log \xi)^{m+1}} \ll \xi^{\sigma_{\xi+1}-\sigma_{\xi}} |t| + 2
\]

(2.4)

for \( \text{Re}(s) \geq (\sigma_0(K) + \sigma_L)/2 \), which gives Lemma 6(iii). Using equation (2.4) and the Cauchy–Schwarz inequality, we get the estimate

\[
\mathbb{E}^m[|\tilde{\eta}_m(s, \omega)|] \ll (|t| + 2) \int_{2-}^{\infty} \frac{\mathbb{E}^m[|S_{\xi}(\omega)|] \, d\xi}{\xi^{\sigma_{\xi+1}-\sigma_{\xi}}(\log \xi)^{m+1}} \ll (|t| + 2) \int_{2-}^{\infty} \frac{(\mathbb{E}^m[|S_{\xi}(\omega)|^2])^{1/2} \, d\xi}{\xi^{\sigma_{\xi+1}-\sigma_{\xi}}} \]

for \( \text{Re}(s) \geq (\sigma_0(K) + \sigma_L)/2 \). Since

\[
\mathbb{E}^m[|S_{\xi}(\omega)|^2] = \sum_{2 \leq n \leq \xi} \frac{A(n)^2}{n^{2\sigma_0(K)(\log n)^{2m+1}}} \ll \sum_{n=2}^{\infty} \frac{A(n)^2}{n^{2\sigma_0(K)(\log n)^{2m+1}}} \ll K
\]

for \( \xi \geq 2 \), we have the conclusion.

Remark 1. Statement (i) of this lemma will be used in Subsection 2.3. In proving the universality theorem, Rademacher–Menshov theorem (see, e.g., [15, App. B]) is usually used. However, we need not use it this time because the second term of (2.2) is an absolutely convergent series.

Lith. Math. J., 62(3):315–332, 2022.
Proof of Lemma 5. Let $C$ be a compact subset of $\mathcal{R}$ and fix $\omega \in \Omega$ satisfying (ii) and (iii) of Lemma 6. By the Mellin inversion formula of $\varphi(x)$, stated in Lemma 1(iii), we have

$$
\tilde{\eta}_{m,X}(z, \omega) = \frac{1}{2\pi i} \sum_{n=2}^{\infty} \int_{c-i\infty}^{c+i\infty} \frac{A(n)\omega(n)\varphi(\xi)}{n^2 \log n} \left( \frac{n}{X} \right)^{-\xi} \frac{d\xi}{c-i\infty} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{\eta}_{m}(z + \xi, \omega)\varphi(\xi)X^\xi \, d\xi
$$

for $z \in \partial \mathcal{R}$ and $c > 1$, where the interchange of the sum and the integral is justified by Fubini’s theorem and Lemma 1(ii). We will replace the contour from $c-i\infty$ to $c+i\infty$ by that from $-\delta - i\infty$ to $-\delta + i\infty$ with $\delta = (\sigma_L - \sigma_0(K))/4$. By our choice of $\varphi(x)$ and Lemma 1(i) we have

$$
\tilde{\eta}_{m}(z, \omega) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \tilde{\eta}_{m}(z + \xi, \omega)\varphi(\xi)X^\xi \, d\xi,
$$

where such a replacement is justified by Lemma 1(ii) and Lemma 6(iii). Using Cauchy’s formula, we have

$$
\sup_{s \in C} |\tilde{\eta}_{m}(s, \omega) - \tilde{\eta}_{m,X}(s, \omega)| \leq \frac{1}{2\pi \text{dist}(C, \partial \mathcal{R})} \int_{\partial \mathcal{R}} |\tilde{\eta}_{m}(z, \omega) - \tilde{\eta}_{m,X}(z, \omega)| \, |dz|
$$

and

$$
\leq \frac{X^{-\delta}}{4\pi^2 \text{dist}(C, \partial \mathcal{R})} \int_{\partial \mathcal{R}} |dz| \left( \int_{-\infty}^{\infty} |\tilde{\eta}_{m}(-\delta + \text{Re}(z) + i(t + \text{Im}(z)), \omega)| \varphi(-\delta + it) \, dt \right).
$$

Taking the expectation, we obtain

$$
\mathbb{E}^{m} \left[ \sup_{s \in C} |\tilde{\eta}_{m}(s, \omega) - \tilde{\eta}_{m,X}(s, \omega)| \right] \leq \frac{\ell(\partial \mathcal{R})X^{-\delta}}{4\pi^2 \text{dist}(C, \partial \mathcal{R})} \sup_{z \in \partial \mathcal{R}} \left( \int_{-\infty}^{\infty} \mathbb{E}\left[ |\tilde{\eta}_{m}(-\delta + \text{Re}(z) + i(t + \text{Im}(z)), \omega)| \right] \varphi(-\delta + it) \, dt \right)
$$

$$
\ll_{\mathcal{K}, \mathcal{R}, C} X^{-\delta} \to 0
$$

as $X \to \infty$ by estimates (iv) of Lemma 6 and (ii) of Lemma 1. This completes the proof. \( \square \)

We give one more lemma to prove the proposition. Let $\mathcal{P}_0$ be a finite subset of the set of prime numbers. Define the probability measures $\mathbb{H}^{\mathcal{P}_0}_T$ on $(\prod_{p \in \mathcal{P}_0} T_p, \mathcal{B}(\prod_{p \in \mathcal{P}_0} T_p))$ by

$$
\mathbb{H}^{\mathcal{P}_0}_T(A) = \frac{1}{\text{meas}(\mathcal{I}_K(T))} \text{meas}\{\tau \in \mathcal{I}_K(T): (p^{j\tau})_{p \in \mathcal{P}_0} \in A\}
$$

for $A \in \mathcal{B}(\prod_{p \in \mathcal{P}_0} T_p)$ and denote $\mathbf{m}_{\mathcal{P}_0} = \otimes_{p \in \mathcal{P}_0} \mathbf{m}_p$. Then we have the following lemma.

**Lemma 7.** The probability measure $\mathbb{H}^{\mathcal{P}_0}_T$ converges weakly to $\mathbf{m}_{\mathcal{P}_0}$ as $T \to \infty$. 
Proof. It suffices to check that the Fourier transform of $H_{T}^{P_0}$ converges pointwise to that of $m_{P_0}$ as $T \to \infty$. Let $\mathcal{F}(\cdot;H_{T}^{P_0})$ denote the Fourier transform of $H_{T}^{P_0}$. For any $n = (n_p)_{p \in P_0} \in \mathbb{Z}^{P_0}$, we have

$$\mathcal{F}(n; H_{T}^{P_0}) = \int_{\prod_{p \in P_0} \mathbb{P}^n} \prod_{p \in P_0} x^{n_p} dH_{T}^{P_0}((x_p)_{p \in P_0}) = \frac{1}{\text{meas}(\mathcal{I}_K(T))} \int_{\mathcal{I}_K(T)} \prod_{p \in P_0} p^{n_p \tau} d\tau.$$  

If $n = 0$, then we have $\lim_{T \to \infty} \mathcal{F}(0; H_{T}^{P_0}) = 1$. If $n \neq 0$, then we have

$$\mathcal{F}(n; H_{T}^{P_0}) = \frac{1}{\text{meas}(\mathcal{I}_K(T))} \exp \left( \frac{\pi \sum_{p \in P_0} n_p \log p}{\text{meas}(\mathcal{I}_K(T))} \right) \left( T - \frac{\text{meas}(\mathcal{I}_K(T))}{\text{meas}(\mathcal{I}_K(T))} \right).$$

The above first term tends to 0 trivially, and so does the second term by the estimate $\text{meas}(\mathcal{I}_K(T)) \sim T$. This completes the proof.

Proof of Proposition 1. Let $P_T$ denote the probability measure on $(\mathcal{I}_K(T), \mathcal{B}(\mathcal{I}_K(T)))$ given by

$$P_T(E) = \frac{1}{\text{meas}(\mathcal{I}_K(T))} \text{meas}(E), \quad E \in \mathcal{B}(\mathcal{I}_K(T)).$$

Let $F : \mathcal{H}(\mathcal{R}) \to \mathbb{R}$ be a bounded Lipschitz continuous function. Then there exist $M(F), C(F) \geq 0$ such that

$$|F(f)| \leq M(F) \quad \text{and} \quad |F(f) - F(g)| \leq C(F)d(f, g)$$

for all $f, g \in \mathcal{H}(\mathcal{R})$. It suffices to show that $|E^{Q_T}[F] - E^{Q}[F]| \to 0$ as $T \to \infty$ by the property of weak convergence (see, e.g., [6, Thm. 3.9.1]). We find that

$$E^{Q_T}[F] - E^{Q}[F] = E^{P_T}[F(\tilde{\eta}_m(s + i\tau))] - E^{m}[F(\tilde{\eta}(s, \omega))]$$

$$\leq |E^{P_T}[F(\tilde{\eta}_m(s + i\tau))] - E^{P_T}[F(\tilde{\eta}_m(s + i\tau))]| + |E^{m}[F(\tilde{\eta}_m(s, \omega))] - E^{m}[F(\tilde{\eta}, s, \omega)]|$$

$$=: \Sigma_1(T, X) + \Sigma_2(T, X) + \Sigma_3(T, X).$$

First, we estimate $\Sigma_1(T, X)$. We have

$$\Sigma_1(T, X) \leq \frac{2M(F) \text{meas}(\mathcal{I}_K(T) \setminus \mathcal{X}_K(T))}{\text{meas}(\mathcal{I}_K(T))} + C(F)E^{P_T}[d(\tilde{\eta}_m(s + i\tau), \tilde{\eta}_m(s + i\tau)); \tau \in \mathcal{X}_K(T)],$$

where $E[X; A]$ stands for $E[X 1_A]$ for a random variable $X$ and the indicator function $1_A$ of a set $A$. The first term of the above inequality tends to 0 as $T \to \infty$, since

$$\mathcal{X}_K(T) \subset \mathcal{I}_K(T) \quad \text{and} \quad \lim_{T \to \infty} \frac{\text{meas}(\mathcal{X}_K(T))}{T} = \lim_{T \to \infty} \frac{\text{meas}(\mathcal{I}_K(T))}{T} = 1.$$
As for the second term, we have
\[ \mathbb{E}^{P(r)} \left[ d(\tilde{\eta}_m(s + i\tau), \tilde{\eta}_m,X(s + i\tau)); \tau \in \mathcal{X}_K(T) \right] \]
\[ \leq \sum_{j=1}^{\infty} \min \left\{ \mathbb{E}^{P(r)} \left[ d_j(\tilde{\eta}_m(s + i\tau), \tilde{\eta}_m,X(s + i\tau)); \tau \in \mathcal{X}_K(T) \right], 1 \right\}. \]

As we take \( \lim_{X \to \infty} \limsup_{T \to \infty} \), this term also tends to 0 by Lemma 4 and Lebesgue’s dominated convergence theorem.

Next, we estimate \( \Sigma_2(T, X) \). Put
\[ \mathcal{P}(\varphi, X) = \left\{ p : p \text{ divides } \prod_{n \in \mathbb{N} : \varphi(n/X) \neq 0} n \right\} \]
and define the continuous mapping \( \Phi_{m,X} : \prod_{p \in \mathcal{P}(\varphi, X)} \mathbb{T}_p \to \mathcal{H}(\mathbb{R}) \) by
\[ \Phi_{m,X}(x) = \sum_{n=2}^{\infty} \frac{A(n)\varphi(n/X)}{n^{s} \log n} \prod_{p \mid n} x_p^{\nu(p;n)}, \quad x = (x_p)_{p \in \mathcal{P}(\varphi, X)} \in \prod_{p \in \mathcal{P}(\varphi, X)} \mathbb{T}_p, \]
where \( \nu(p;n) \) is the exponent of \( p \) in the prime factorization of \( n \). Then we have
\[ \mathbb{E}^{P(r)} \left[ F(\tilde{\eta}_m,X(s + i\tau)) \right] = \mathbb{E}^{H_{T^{(\varphi,X)}} \circ \Phi_{m,X}^{-1}} [F] \to \mathbb{E}^{m \circ \Phi_{m,X}^{-1}} [F] = \mathbb{E}^{m} \left[ F(\tilde{\eta}_m,X(s, \omega)) \right] \]
as \( T \to \infty \) by Lemma 7 and the property of weak convergence (see, e.g., [2, Sect. 2, The Mapping Theorem]). Hence we obtain \( \lim_{T \to \infty} \Sigma_2(T, X) = 0 \).

Finally, we estimate \( \Sigma_3(T, X) \). We have
\[ \Sigma_3(T, X) \leq C(F) \sum_{j=1}^{\infty} \min \left\{ \mathbb{E}^{m} \left[ d_j(\tilde{\eta}_m(s, \omega), \tilde{\eta}_m,X(s, \omega)) \right], 1 \right\}, \]
and the right-hand side of this inequality tends to 0 as \( X \to \infty \) by Lemma 5 and Lebesgue’s dominated convergence theorem. This completes the proof. \( \square \)

### 2.3 Proof of Proposition 2

We will prove the proposition by using the standard method as in [18] and [23]. First, we recall the following lemmas.

**Lemma 8.** Let \( D \) be a simply connected region in the complex plane. Suppose that the sequence \( \{ f_n \}_{n=1}^{\infty} \) in \( \mathcal{H}(D) \) satisfies the following assumptions.

(i) If \( \mu \) is a complex Borel measure on \( (\mathbb{C}, \mathcal{B}(\mathbb{C})) \) with compact support contained in \( D \) such that \( \sum_{n=1}^{\infty} | \int_D f_n \, d\mu | < \infty \), then \( \int_D s^r \, d\mu(s) = 0 \) for all \( r = 0, 1, 2, \ldots \).

(ii) The series \( \sum_{n=1}^{\infty} f_n \) converges in \( \mathcal{H}(D) \).

(iii) For every compact set \( K \subset D \), \( \sum_{n=1}^{\infty} \max_{s \in K} | f_n(s) |^2 < \infty \).

Then the set of all convergent series \( \sum_{n=1}^{\infty} c(n) f_n, \ c(n) \in \mathbb{T}, \) is dense in \( \mathcal{H}(D) \).

**Proof.** See [18, Thm. 3.10] and [23, Thm. 5.7]. \( \square \)
Lemma 9. Let $\mu$ be a complex Borel measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in the half-plane $\sigma > \sigma_0$. Moreover, for $s \in \mathbb{C}$, define the function $f(s) = \int_{\mathbb{C}} \exp(sz) \, d\mu(z)$. Then $f$ is an entire function of exponential type. If $f$ does not vanish identically, then

$$\limsup_{r \to \infty} \frac{\log |f(r)|}{r} > \sigma_0.$$  

Proof. See [18, Lemma 4.10].

Lemma 10. Let $f$ be an entire function of exponential type, and let $\{\xi_m\}_{m=1}^{\infty}$ be a sequence of complex numbers. Moreover, assume that there are positive real constants $\lambda, \eta$, and $\omega$ such that

(i) $\limsup_{y \to \infty} \frac{\log |f(\pm iy)|}{y} \leq \lambda,$
(ii) $|\xi_m - \xi_n| \geq \omega |m - n|$ for all $m, n \in \mathbb{N},$
(iii) $\lim_{m \to \infty} \xi_m/m = \eta,$ and
(iv) $\lambda \eta < \pi.$

Then

$$\limsup_{m \to \infty} \frac{\log |f(\xi_m)|}{|\xi_m|} = \limsup_{r \to \infty} \frac{\log |f(r)|}{r}.$$  

Proof. This can be proved by using Bernstein’s theorem. For the proof of Bernstein’s theorem, see, for example, [19]. For the proof of this lemma by using Bernstein’s theorem, see, for example, [18, Thm. 4.12].

Lemma 11. Let $N$ be a positive integer. The set of all convergent series

$$\sum_{p>N} \frac{\omega(p)}{p^{s}(\log p)^m}, \quad \omega(p) \in \mathbb{T},$$

is dense in $\mathcal{H}(\mathcal{R})$.

Proof. By Lemma 6(i) we can take a convergent series $\sum_{p>N} \tilde{\omega}(p)/p^{s}(\log p)^m$ with some $\tilde{\omega}(p) \in \mathbb{T}$. We will check that the sequence $\{\tilde{\omega}(p)p^{-s}(\log p)^{-m}\}_{p>N}$ satisfies assumptions (i)–(iii) of Lemma 8 with $D$ replaced by $\mathcal{R}$. It is obvious that the sequence satisfies assumption (ii) of Lemma 8.

For any compact set $C \subset \mathcal{R}$, we have the estimate

$$\sum_{p>N} \max_{s \in C} |\tilde{\omega}(p)p^{-s}(\log p)^{-m}|^2 \leq \sum_{p} p^{-2\gamma_0(K)}(\log p)^{-2m} < \infty,$$

which gives assumption (iii) of Lemma 8.

We will see that the sequence $\{\omega(p)p^{-s}(\log p)^{-m}\}_{p>N}$ satisfies assumption (i) of Lemma 8 by using Lemmas 9 and 10. Let $\mu$ be a complex Borel measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in $\mathcal{R}$, which satisfies

$$\sum_{p>N} \left| \int_{\mathbb{C}} \tilde{\omega}(p)p^{-s}(\log p)^{-m} \, d\mu(s) \right| < \infty.$$  

(2.5)

Put $\rho(z) = \int_{\mathbb{C}} \exp(-sz) \, d\mu(s)$. We will show $\rho(z) \equiv 0$. By Eq. (2.5) we find that

$$\sum_{p} \frac{\rho(|\log p|)}{(\log p)^m} < \infty.$$  

(2.6)

Lith. Math. J., 62(3):315–332, 2022.
We take a large positive number \( M > 0 \) such that the support of \( \mu \) is contained in the region \( \{ s \in \mathcal{R}; |t| < M - 1 \} \). By the definition of \( \rho(z) \) we find that

\[
|\rho(\pm iy)| \leq \exp(My)|\mu|(\mathbb{C})
\]

for \( y > 0 \), where \( |\mu| \) stands for the total variation of \( \mu \). Hence we have

\[
\limsup_{y \to \infty} \frac{\log |\rho(\pm iy)|}{y} \leq M.
\]

(2.7)

Fix a positive number \( \eta \) satisfying

\[
M\eta < \pi
\]

(2.8)

and let \( A \) denote the set of all positive integers \( n \) such that there exists a positive number \( r \in ((n - 1/4)\eta, (n + 1/4)\eta) \) such that \( |\rho(r)| \leq \exp(-\sigma_R r) \). For any positive integer \( n \), put

\[
\alpha_n = \exp\left(\left(n - \frac{1}{4}\right)\eta\right) \quad \text{and} \quad \beta_n = \exp\left(\left(n + \frac{1}{4}\right)\eta\right).
\]

Note that \( |\rho(\log p)| > p^{-\sigma_p} \) for any \( n \notin A \) and \( \alpha_n < p \leq \beta_n \) by the definition of \( A \). Then we have

\[
\sum_p \frac{|\rho(\log p)|}{(\log p)^m} \geq \sum_{n=1}^{\infty} \sum_{\alpha_n < p \leq \beta_n} \frac{|\rho(\log p)|}{(\log p)^m} \geq \sum_{n \notin A} \sum_{\alpha_n < p \leq \beta_n} \frac{1}{p^{\sigma_p}(\log p)^m} \geq \sum_{n \notin A} \frac{\beta_n^{\sigma_p}(\log \beta_n)^m}{(\log p)^m} (\pi(\beta_n) - \pi(\alpha_n)).
\]

Using the prime number theorem \( \pi(x) = \int_2^x \frac{du}{\log u} + O(x \exp(-c\sqrt{\log x})) \) with some constant \( c > 0 \), we have

\[
\pi(\beta_n) - \pi(\alpha_n) = \int_{\alpha_n}^{\beta_n} \frac{du}{\log u} + O(\beta_n \exp(-c\sqrt{\log \alpha_n}))
\]

\[
\geq \frac{\beta_n(1 - e^{-\eta/2})}{\log \beta_n} + O(\beta_n \exp\left(-\frac{c}{2} \sqrt{\log \beta_n}\right)) \gg \eta \frac{\beta_n}{\log \beta_n}
\]

for sufficiently large \( n \in \mathbb{N} \). Hence we have

\[
\sum_p \frac{|\rho(\log p)|}{(\log p)^m} \gg \eta \sum_{n \notin A, n \geq n_0} \frac{\beta_n^{1-\sigma_p}(\log \beta_n)^{m+1}}{(\log p)^m} \gg \eta \sum_{n \notin A, n \geq n_0} \frac{\exp((1 - \sigma_R)\eta n)}{n^{m+1}} \gg \sum_{n \notin A, n \geq n_0} 1
\]

for some large constant \( n_0 \in \mathbb{N} \). Combining this estimate with inequality (2.6), we get that \( \sum_{n \notin A, n \geq n_0} 1 < \infty \). Therefore there exists a positive constant \( n_1 \in \mathbb{N} \) such that \( \{ n \in \mathbb{N}; n \geq n_1 \} \subset A \). By the definition of \( A \) we can take a sequence \( \{ \xi_n \}_{n \geq n_1} \) such that

\[
(n - \frac{1}{4})\eta < \xi_n \leq (n + \frac{1}{4})\eta \quad \text{and} \quad |\rho(\xi_n)| \leq \exp(-\sigma_R \xi_n).
\]
From this we have
\[
\lim_{n \to \infty} \frac{\xi_n}{n} = \eta \quad \text{and} \quad \limsup_{n \to \infty} \frac{\log |\rho(\xi_n)|}{\xi_n} \leq -\sigma_R \tag{2.9}
\]
and
\[
\xi_m - \xi_n \geq \left( m - \frac{1}{4} \right) \eta - \left( n + \frac{1}{4} \right) \eta = (m - n) \eta - \frac{\eta}{2} \geq \frac{\eta(m - n)}{2} \tag{2.10}
\]
for \( m > n \geq n_1 \). Applying Lemma 10 with (2.7), (2.8), (2.9), and (2.10), we have
\[
\limsup_{r \to \infty} \frac{\log |\rho(r)|}{r} \leq -\sigma_R. \tag{2.11}
\]
On the other hand, assuming that \( \rho(z) \) does not vanish identically, by Lemma 9 we have
\[
\limsup_{r \to \infty} \frac{\log |\rho(r)|}{r} > -\sigma_R,
\]
which contradicts inequality (2.11). Hence we obtain \( \rho(z) \equiv 0 \). Since \( \rho(z) \) is represented as
\[
\rho(z) = \sum_{r=0}^{\infty} \left( \frac{(-1)^r}{r!} \int_\mathbb{C} s^r d\mu(s) \right) z^r,
\]
we conclude that \( \int_\mathbb{C} s^r d\mu(s) = 0 \) for \( r = 0, 1, 2, \ldots \) by the uniqueness of the Taylor expansion, which gives assumption (i) of Lemma 8. This completes the proof. \( \square \)

Using this lemma, we show the following lemma.

**Lemma 12.** The set of all convergent series
\[
\sum_{p} \frac{\text{Li}_{m+1}(p^{-s}\omega(p))}{(\log p)^m}, \quad \omega(p) \in \mathbb{T},
\]
is dense in \( \mathcal{H}(\mathcal{R}) \).

**Proof.** Fix \( f \in \mathcal{H}(\mathcal{R}) \) and \( \varepsilon > 0 \). We put
\[
h_N(s, \omega_N) = \sum_{p > N} \left( \frac{\text{Li}_{m+1}(p^{-s}\omega(p))}{(\log p)^m} - \frac{\omega(p)}{p^s(\log p)^m} \right) = \sum_{p > N} \sum_{k \geq 2} \frac{\omega(p)^k}{k^{m+1} p^{ks} (\log p)^m}
\]
for \( s \in \mathcal{R} \) and \( \omega_N = (\omega(p))_{p > N} \in \prod_{p > N} \mathbb{T}_p \), which converges absolutely. Then we have
\[
\|h_N\|_\infty := \sup_{\omega_N \in \prod_{p > N} \mathbb{T}_p} \sup_{s \in \mathcal{R}} |h_N(s, \omega_N)| \leq \sum_{p > N} \sum_{k \geq 2} \frac{1}{k^{m+1} p^{ks} (\log p)^m} \to 0
\]
as \( N \to \infty \). Hence we can take a large \( N_0 \) such that \( \|h_{N_0}\|_\infty < \varepsilon/2 \). By Lemma 11 there exists a convergent series
\[
\sum_{p > N_0} c_0(p) p^{-s}(\log p)^{-m}
\]
for some $c_0(p) \in \mathbb{T}$, $p > N_0$, such that

$$d \left( f(s) - \sum_{p \leq N_0} \frac{\text{Li}_{m+1}(p^{-s})}{(\log p)^m}, \sum_{p > N_0} \frac{c_0(p)}{p^{s}(\log p)^m} \right) < \frac{\varepsilon}{2},$$

Putting

$$c(p) = \begin{cases} 1 & \text{if } p \leq N_0, \\ c_0(p) & \text{if } p > N_0, \end{cases}$$

we obtain

$$d \left( f(s), \sum_{p} \frac{\text{Li}_{m+1}(p^{-s}c(p))}{(\log p)^m} \right) \leq d \left( f(s) - \sum_{p \leq N_0} \frac{\text{Li}_{m+1}(p^{-s})}{(\log p)^m}, \sum_{p > N_0} \frac{c_0(p)}{p^{s}(\log p)^m} \right)$$

$$+ d \left( \sum_{p > N_0} \frac{\text{Li}_{m+1}(p^{-s}c(p))}{(\log p)^m}, \sum_{p > N_0} \frac{c(p)}{p^{s}(\log p)^m} \right)$$

$$\leq \frac{\varepsilon}{2} + \|h_{N_0}\|_\infty < \varepsilon.$$

This completes the proof. □

**Lemma 13.** Let $(X_n)_{n=1}^\infty$ be a sequence of $H(\mathbb{R})$-valued independent random variables such that the series $X = \sum_{n=1}^\infty X_n$ converges almost surely. Then the support of $X$ is the closure of the set of all convergent series of the form $\sum_{n=1}^\infty x_n$, where $x_n$ belongs to the support of the distribution of $X_n$ for all $n \geq 1$.

**Proof.** See [18, Thm. 7.10]. □

**Proof of Proposition 2.** By Lemma 6(ii) we find that the distribution of $\tilde{\eta}_m(s, \omega)$ equals that of $\sum_p \text{Li}_{m+1}(p^{-s}\omega(p))(\log p)^{-m}$. We will confirm that the support of the distribution of $\text{Li}_{m+1}(p^{-s}\omega(p)) \times (\log p)^{-m}$ equals the set of functions $\text{Li}_{m+1}(p^{-s}z)(\log p)^{-m}, z \in \mathbb{T}$. We note that we have the estimate

$$|\text{Li}_{m+1}(w_1) - \text{Li}_{m+1}(w_2)| \leq \frac{|w_1 - w_2|}{1 - \max\{|w_1|, |w_2|\}} \leq \frac{|w_1 - w_2|}{|w_1|} \leq |w_2|$$

for $|w_1|, |w_2| < 1$. First, we show the set of functions $\text{Li}_{m+1}(p^{-s}\omega(p))(\log p)^{-m}, z \in \mathbb{T}$, is included in the support of the distribution of $\text{Li}_{m+1}(p^{-s}\omega(p))(\log p)^{-m}$. We fix a function $\text{Li}_{m+1}(p^{-s}z)(\log p)^{-m}, z \in \mathbb{T}$. Then we have

$$\mathfrak{m} \left( d \left( \frac{\text{Li}_{m+1}(p^{-s}\omega(p))}{(\log p)^m}, \frac{\text{Li}_{m+1}(p^{-s}z)}{(\log p)^m} \right) < \varepsilon \right)$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} 1 \left\{ d \left( \frac{\text{Li}_{m+1}(p^{-s}e^{i\theta})}{(\log p)^m}, \frac{\text{Li}_{m+1}(p^{-s}z)}{(\log p)^m} \right) < \varepsilon \right\} d\theta$$

$$\geq \frac{1}{2\pi} \int_{0}^{2\pi} 1 \left\{ \frac{|e^{i\theta} - z|}{|p^{\sigma}z - 1|(\log p)^m} < \varepsilon \right\} d\theta > 0$$

by inequality (2.12), which gives the desired inclusion.
We will prove the converse inclusion. Let \( g \) be a holomorphic function in \( \mathcal{R} \) that is not in the set of functions \( \text{Li}_{m+1}(p^{-s}z)(\log p)^{-m}, \ z \in \mathbb{T} \). Since the mapping
\[
\mathbb{T} \ni z \mapsto d\left( \text{Li}_{m+1}(p^{-s}z)(\log p)^{-m}, g(s) \right) \in \mathbb{R}
\]
is continuous, we have
\[
\varepsilon_0 := \min_{z \in \mathbb{T}} d\left( \text{Li}_{m+1}(p^{-s}z)(\log p)^{-m}, g(s) \right) > 0.
\]
Hence we obtain
\[
\mathbf{m}\left( d\left( \text{Li}_{m+1}(p^{-s}(\log p)^{-m}, g(s) \right) < \varepsilon_0 \right) = 0.
\]
Therefore \( g \) does not belong to the support of the distribution of \( \text{Li}_{m+1}(p^{-s}(\log p)^{-m} \). Thus Lemmas 13 and 12 yield the conclusion. \( \square \)

2.4 Proof of Theorem 1

Let \( m \) be a nonnegative integer, let \( \mathcal{K} \) be a compact subset of \( \mathcal{D} \) with connected complement, and let \( \mathcal{R} \) be the rectangle in \( \mathbb{C} \) given by (2.1). Suppose that \( f \) is a continuous function on \( \mathcal{K} \) holomorphic in the interior of \( \mathcal{K} \). Fix \( \varepsilon > 0 \). By Mergelyan’s approximation theorem (see, e.g., [22, Thm. 20.5]), there exists a polynomial \( P \) such that \( \sup_{s \in \mathcal{K}} |P(s) - f(s)| < \varepsilon / 2 \). Let \( \Phi(P) \) be the set of all holomorphic functions \( g \) in \( \mathcal{H}(\mathcal{R}) \) such that \( \sup_{s \in \mathcal{K}} |g(s) - P(s)| < \varepsilon / 2 \). Then we find that the set \( \Phi(P) \) is open in \( \mathcal{H}(\mathcal{R}) \). By Propositions 1, 2 and the portmanteau theorem (see, e.g., [2, Thm. 2.1]) we have
\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [T, 2T]: \sup_{s \in \mathcal{K}} \left| \tilde{\eta}_m(s + i\tau) - P(s) \right| < \frac{\varepsilon}{2} \right\}
\geq \liminf_{T \to \infty} \frac{1}{\text{meas}(I_{\mathcal{K}}(T))} \text{meas}\left\{ \tau \in I_{\mathcal{K}}(T): \sup_{s \in \mathcal{K}} \left| \tilde{\eta}_m(s + i\tau) - P(s) \right| < \frac{\varepsilon}{2} \right\}
\geq \liminf_{T \to \infty} Q_T(\Phi(P)) \geq Q(\Phi(P)) > 0.
\]
This inequality and the inequality
\[
\sup_{s \in \mathcal{K}} \left| \tilde{\eta}_m(s + i\tau) - f(s) \right| \leq \sup_{s \in \mathcal{K}} \left| \tilde{\eta}_m(s + i\tau) - P(s) \right| + \sup_{s \in \mathcal{K}} \left| P(s) - f(s) \right|
\]
finish the proof. \( \square \)

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