Dissipative analysis of linear coupled differential-difference systems with distributed delays

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Abstract

This paper presents a new dissipative analysis method for linear coupled differential-difference systems (CDDS) with general distributed delays in both state and output equation. More precisely, the distributed delay terms under consideration can contain any \(L_2\) functions which are approximated via a broader class of functions in comparison with an existing approach which is based on the approximation via Legendre polynomials. By using this broader class of functions, we propose Krasovskii functionals with more general structures as compared to the existing approach based on polynomials, and also a generalized integral inequality. Based on our generalized approximation technique with the proposed integral inequality, sufficient conditions for the stability of CDDS with dissipativity properties are derived in terms of linear matrix inequalities. Several numerical examples are presented to show the effectiveness of our proposed methodologies.

Keywords: Dissipative analysis; Distributed Delay; Integral Inequality; Coupled differential-difference systems; Approximations.

1. Introduction

Coupled differential-difference systems (CDDS), which are mathematically related to time delay systems Briat (2014), can characterize a broad class of models concerning delay or propagation effects Răsvan (2006). It includes standard or neutral time-delay systems or some singular delay systems as well Gu (2006). For more information on the topic of CDDS, see Gu & Liu (2009) and the references therein.

Over the past decades, series of significant results on the stability of CDDS Pepe (2005); Pepe et al. (2008) have been proposed based on the idea of constructing Krasovskii functionals. In particular, the idea of complete Krasovskii functional of linear time-delay systems Briat (2014) has been extended in to formulate a complete functional for a linear CDDS Gu & Liu (2009), which may be constructed numerically Li (2012) via semidefinite programming. To the best of our knowledge, however, no results have been proposed in the reviewed publications on linear CDDS concerning non-trivial (non-constant) distributed delays. Generally speaking, analyzing distributed delay may require much more effort due to the complexities induced by different types of distributed delay kernels. For the latest available approaches on distributed delays in time domain, see Münz et al. (2009); Fridman & Tsodik (2009); Goebel et al. (2011); Seuret et al. (2015b); Feng & Nguang (2016b).

In Seuret et al. (2015b), an approximation scheme is proposed to deal with \(L_2\) continuous distributed delay terms based on the application of Legendre polynomials. Although only at most two distributed delay kernels are considered, the stability conditions derived in Seuret et al. (2015b) are highly competent and exhibit a hierarchical feasibility enhancement with respect to the degree of the approximating Legendre...
polynomials. In this paper, we propose a new approach generalizing the results in Seuret et al. (2015b). Unlike the approximation scheme in Seuret et al. (2015b), our approximation is based on a more general class of orthogonal functions (this including the case of Legendre polynomials) or even non-orthogonal functions. The proposed methodology provides a unified solution which can handle the situations that multiple distributed matrix kernels are approximated individually over different integration intervals with general matrix structures. This is especially useful when the distributed delays among states and outputs are given over different delay intervals. Furthermore, unified measures are formulated with a matrix framework, which can characterize the corresponding approximation errors associated with targeted functions.

In this paper, we propose solutions on the dissipative analysis of a linear CDDS, which has distributed delays at both the states and output equation. Specifically, the distributed delay kernels considered here can be any \( \mathbb{L}_2 \) function, which can be approximated by a class of functions. Our CDDS model incorporates many delay models, such as the models in Münz et al. (2009); Gu & Liu (2009); Li (2012); Seuret et al. (2015b); Feng & Nguang (2016b), as special cases. Meanwhile, analysis of the behavior of the approximation errors is presented by using a matrix framework. It shows that, when orthogonal functions \( \{\varphi_i(\cdot)\}_{i=1}^\infty \) are chosen as the approximator, the resulting approximation errors converge to zeros in a \( \mathbb{L}_2 \) sense as \( d \to \infty \). Furthermore, a quadratic supply function is considered in this paper for the dissipative analysis. To incorporate the approximation errors into the resulting dissipative stability conditions, a general integral inequality is derived which introduces error related terms into its lower bound. By constructing a Krasovskii functional with the assistance of this inequality, sufficient conditions for the dissipative asymptotic (exponential) stability conditions can be derived in terms of linear matrix inequalities. The dissipative stability conditions are further proved to have a hierarchical feasibility enlargement under the framework of orthogonal functions, which is similar to the result in Seuret & Gouaisbaut (2015). Finally, Several numerical examples are given to demonstrate the effectiveness and capacity of the proposed methodologies.

**Notation**

The notations in this paper follow standard rules, though certain new symbols are introduced for the sake of having a compact presentation: \( \varrho(X) \) stands for the spectra radius of \( X \); notations \( \|X\|_q = (\sum_{i=1}^n |x_i|^q)^{\frac{1}{q}} \) and \( \|f(\cdot)\|_p = \left( \int_X |f(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \) and \( \|f(\cdot)\|_p = \left( \int_X \|f(\tau)\|_2^p \, d\tau \right)^{\frac{1}{p}} \) are the norms associated with \( \mathbb{R}^n \) and the Lebesgue functions space \( \mathbb{L}_q(X; \mathbb{R}) \) and \( \mathbb{L}_p(X; \mathbb{R}^m) \), respectively. In addition, \( \mathbb{L}_p(X; \mathbb{R}^m) \) contains locally integrable Lebesgue measurable functions with reference to \( \mathbb{L}_p(X; \mathbb{R}^m) \). Moreover, the set \( \mathbb{L}_2(X; \mathbb{Y}) \) contains of all functions which are Lebesgue integrable from \( X \) onto \( \mathbb{Y} \). We use \( \mathbb{H}^1(X; \mathbb{R}^m) \) to indicate the Sobolev space with the norm \( \sqrt{\|f(\cdot)\|_2^2 + \|f(\cdot)\|_2^2} \), which contains \( \mathbb{L}_2 \) functions whose weak derivatives belong to \( \mathbb{L}_2(X; \mathbb{R}^m) \). \( \text{Sy}(X) := X + X^\top \) is the sum of a matrix with its transpose. The standard gamma function is denoted by \( \gamma(\cdot) \). A column vector containing a sequence of objects is defined as \( \text{Col}_{i=1}^n x_i := \left[ \text{Row}_{i=1}^n x_i^\top \right]^\top = [x_1^\top \cdots x_n^\top \cdots x_n^\top]^\top \). In addition, we define \( \text{Col}_{i=1}^n x_i := \emptyset \) when \( n < 1 \), where \( \emptyset \) is an empty matrix with an appropriate column dimension based on specific contexts. \( * \) is applied to denote \( [s]YX = X^\top YX \) or \( X^\top Y[s] = X^\top YX \). \( O_{n \times n} \) denotes a \( n \times n \) zero matrix with the abbreviation form \( 0_n \), whereas \( 0_n \) denotes a \( n \times 1 \) column vector. The diagonal sum of two matrices and \( n \) matrices are defined as \( X \oplus Y = \text{Diag}(X, Y) \), \( \bigoplus_{i=1}^n X_i = \text{Diag}^n_{i=1}(X_i) \), respectively. \( \otimes \) denotes the Kronecker product. Furthermore, we assume the order of operations concerning matrices to be matrix (scalars) multiplications \( > \otimes > \oplus > + \). Finally, the notion of empty matrices is applied in this article to facilitate our derivation, whose rules of operations are in line with the definition in Matlab environment.

2. Preliminaries and Problem Formulations

In this section, we will provide definitions and lemmas, and also we will formulate our problem.
Definition 1 (Jacobi Polynomials). Given $\alpha > -1, \beta > -1$ and $r_2 > r_1 > 0$, the Jacobi polynomials defined over $[-r_2, -r_1]$ with the weighting function $(-r_1 - \tau)^\alpha (r_2 + \tau)^\beta$ are denoted as

$$J_d^{\alpha, \beta}(\tau)_{-r_2}^{-r_1} := \frac{\gamma(d + 1 + \alpha)}{d! \gamma(d + 1 + \alpha + \beta)} \sum_{k=0}^{d} \binom{d}{k} \frac{\gamma(d + k + 1 + \alpha + \beta)}{\gamma(k + 1 + \alpha)} \left(\frac{\tau + r_1}{r_2 - r_1}\right)^k, \quad \forall d \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

which satisfies

$$\forall d \in \mathbb{N}_0, \quad \int_{-r_2}^{-r_1} (-r_1 - \tau)^\alpha (r_2 + \tau)^\beta J_d^{\alpha, \beta}(\tau)_{-r_2}^{-r_1} [\tau] \, d\tau = \sum_{k=0}^{d} \frac{(r_2 - r_1)^{\alpha + 1} \gamma(k + 1 + \alpha) \gamma(k + 1 + \beta)}{k! (2k + \alpha + \beta + 1) \gamma(k + \alpha + \beta + 1)},$$

with $\mathbb{R}^{d+1} \ni J_d^{\alpha, \beta}(\tau)_{-r_2}^{-r_1} := \text{Col}_{d=-1}^{d} J_d^{\alpha, \beta}(\tau)_{-r_2}^{-r_1}, \forall \tau \in [-r_2, -r_1]$.

Let us define the notations $\mathbb{R}^{d+1} \ni \ell_\delta(\tau) = \int_0^{\delta} (\tau - r_1)^0 \, dr, \quad \text{and} \quad \mathbb{R}^{d+1} \ni \ell_\delta(\tau) = \int_0^{\delta} (-r_1 - \tau)^0 \, dr$ with $d; \delta \in \mathbb{N}_0$, which are in line with the Legendre polynomials Senet et al. (2015b) over $[-r_2, 0]$ and $[-r_2, -r_1]$, respectively. Note that it is easy to conclude $\forall d; \delta \in \mathbb{N}_0, \exists! \lambda_{d+1} \in \mathbb{R}^{(d+1) \times (d+2)}, \exists! \lambda_{d+1} \in \mathbb{R}^{(d+1) \times (d+2)}$ and $\exists! \ell_d \in \mathbb{R}^{(d+2) \times (d+1)}$, $\exists! \ell_d \in \mathbb{R}^{(d+2) \times (d+1)}$ such that

$$\frac{d}{d\tau} \ell_{d+1}(\tau) = \lambda_{d+1} \ell_d(\tau) \quad \text{and} \quad \frac{d}{d\tau} \ell_{d+1}(\tau) = \lambda_{d+1} \ell_{d+1}(\tau) = \lambda_d \ell_d(\tau).$$

Furthermore, we have $\forall d; \delta \in \mathbb{N}_0, \exists! \ell_{d+1} \in \mathbb{R}^{(d+1) \times (d+1)}$ and $\exists! \ell_{d+1} \in \mathbb{R}^{(d+1) \times (d+1)}$ such that

$$(r_1 + \tau) \frac{d}{d\tau} \int_0^{r_1} (\tau - r_1)^0 \, dr = \Omega_{d+1} \ell_d(\tau) \quad \text{and} \quad (r_1 + \tau) \frac{d}{d\tau} \int_0^{r_1} (-r_1 - \tau)^0 \, dr = \Omega_{d+1} \ell_{d+1}(\tau)$$

which is a property of the Jacobi polynomials with $\alpha = 0$ and $\beta = 1$. Finally, it is easy to see that $\forall d \in \mathbb{N}_0, \ell_d(0) = \ell_d(-r_1) = \int_0^{r_1} (0 - r_1)^0 \, dr = \int_0^{r_1} (-r_1 - r_1)^0 \, dr = 1_{d+1}^{d+1} := \text{Col}_{d=0}^{d+1} 1$ and $\ell_d(-r_1) = \ell_d(-r_2) = 1_{d+1} := \text{Col}_{d=0}^{d+1} (-1)^d$, where the notations $1_{d+1}$ and $1_{d+1}$ are applied in the following portions of this paper.

The following linear CDDS

$$\begin{align*}
x(t) &= A_1 x(t) + A_2 \xi(t - r_1) + A_3 \xi(t - r_2) + \int_{-r_1}^{0} A_4 \left( \text{Col} \left[ \ell_d(\tau), \ell_\delta(\tau), \phi(\tau) \right] \otimes \mu \right) \xi(t + \tau) \, d\tau \\
&\quad + \int_{-r_2}^{-r_1} A_5 \left( \text{Col} \left[ \ell_d(\tau), \ell_\delta(\tau), \phi(\tau) \right] \otimes \mu \right) \xi(t + \tau) \, d\tau + D_1 w(t), \\
x(t) &= A_6 x(t) + A_7 \xi(t - r_1) + A_8 \xi(t - r_2) \\
z(t) &= C_1 x(t) + C_2 \xi(t - r_1) + C_3 \xi(t - r_2) + \int_{-r_1}^{0} C_4 \left( \text{Col} \left[ \ell_d(\tau), \ell_\delta(\tau), \phi(\tau) \right] \otimes \mu \right) \xi(t + \tau) \, d\tau \\
&\quad + \int_{-r_2}^{-r_1} C_5 \left( \text{Col} \left[ \ell_d(\tau), \ell_\delta(\tau), \phi(\tau) \right] \otimes \mu \right) \xi(t + \tau) \, d\tau + C_6 \xi(t - r_1) + C_7 \xi(t - r_2) + D_2 w(t),
\end{align*}$$

with $r_2 > r_1 > 0$ and $d_1; \delta_1 \in \mathbb{N}_0$ is considered to be analyzed in this paper. Specifically, the initial condition of (5) is assumed to be $x(0) = \psi \in \mathbb{R}^n$ and $\xi(\theta) = \phi(\theta), \forall \theta \in [-r_2, 0)$, where $\psi(\cdot) \in \mathbb{H}^1([-r_2, 0); \mathbb{R}^n)$. Furthermore, $x(t) \in \mathbb{R}^n$ and $\xi(t) \in \mathbb{R}^n$ and $w(\cdot) \in \mathbb{L}_2 (\mathbb{R}^m; \mathbb{R}^m)$, $z(t) \in \mathbb{R}^m$ are the solution of (5), input signals, disturbances, and the regulated output, respectively, with $n; \mu \in \mathbb{N}$ and $m; q \in \mathbb{N}_0$. The state space real valued matrices of the non-distributed delay terms in (5) are $A_1; A_2; A_3; A_4; A_7; A_8$ and $C_1; C_2; C_3$ and $D_1; D_2$ with appropriate dimensions. Meanwhile, we define $f(\cdot) := \text{Col}_{k=0}^{k} f_k(\cdot) \in \mathbb{C}^1([-r_1, 0]; \mathbb{R}^{d_1})$ and $\tilde{f}(\cdot) := \text{Col}_{k=1}^{k} \tilde{f}_k(\cdot) \in \mathbb{C}^1([-r_2, -r_1]; \mathbb{R}^{d_2})$ with $d_2; \delta_2 \in \mathbb{N}_0$, which satisfy the conditions

$$\exists! \dot{M} \in \mathbb{R}^{d_2 \times d_2}, \exists! \dot{M} \in \mathbb{R}^{d_2 \times d_2} : \frac{df(\tau)}{d\tau} = \dot{M} f(\tau) \quad \text{and} \quad \frac{d\tilde{f}(\tau)}{d\tau} = \dot{M} \tilde{f}(\tau).$$
Furthermore, \( \varphi_1(\cdot) \in \mathbb{L}_2([-r_1, 0] ; \mathbb{R}^{\nu_1}) \) and \( \varphi_2(\cdot) \in \mathbb{L}_2([-r_2, -r_1] ; \mathbb{R}^{\nu_2}) \) with \( \nu_1, \nu_2 \in \mathbb{N}_0 \) are vector valued functions satisfying

\[
\int_{-r_1}^0 [\ell_{d_1}(\tau) \ f^T(\tau) \ \varphi_1^T(\tau)] \, d\tau > 0, \quad \int_{-r_2}^{-r_1} [\ell_{d_2}(\tau) \ f^T(\tau) \ \varphi_2^T(\tau)] \, d\tau > 0
\]  

(7)

which indicates the linear independence among the functions in \( \text{Col} [\ell_{d_1}(\tau), f(\tau), \varphi_1(\tau)] \) and \( \text{Col} [\ell_{d_2}(\tau), f(\tau), \varphi_2(\tau)] \), respectively
\(^1\). The terms \( A_4 \in \mathbb{R}^{n \times (\nu_1 + d)\mu} \), \( C_4 \in \mathbb{R}^{m \times (\nu_1 + d)\mu} \) and \( A_5 \in \mathbb{R}^{n \times (\nu_2 + d)\mu} \), \( C_5 \in \mathbb{R}^{m \times (\nu_2 + d)\mu} \), with \( d = d_1 + d_2 + 1 \) and \( \delta = \delta_1 + \delta_2 + 1 \), are given matrices which characterize the distributed delay terms in (5). Finally, \( A_7 \) and \( A_8 \) satisfy

\[
\sup \left\{ \varrho \left( A_7 e^{i\theta_1} + A_8 e^{i\theta_2} \right) : \theta_1; \theta_2 \in [0, 2\pi] \right\} < 1,
\]  

(8)

which ensures input to state stability for the associated difference equation in Gu (2010).

**Remark 1.** Any distributed delay term can be decomposed into the distributed delay representation in (5) without having constraints with respect to matrix dimensions or the numbers of different functions contained therein. In addition, we emphasize here that \( \varphi_1(\cdot) \) and \( \varphi_2(\cdot) \) can become a \( 0 \times 1 \) empty vector if \( \nu_1 = \nu_2 = 0 \).

In order to tackle the functions \( \varphi_1(\cdot) \in \mathbb{L}_2([-r_1, 0] ; \mathbb{R}^{\nu_1}) \) and \( \varphi_2(\cdot) \in \mathbb{L}_2([-r_2, -r_1] ; \mathbb{R}^{\nu_2}) \), which might not satisfy (6), in an efficient and unified way, the functions \( \ell_{d_1}(\cdot), f(\cdot) \) and \( \ell_{d_2}(\cdot), f(\cdot) \) are applied to approximate \( \varphi_1(\cdot) \) and \( \varphi_2(\cdot) \), respectively. Specifically, the approximations are denoted by the decomposition:

\[
\varphi_1(\tau) = \hat{\Gamma}_d \text{Col} (\ell_{d_1}(\tau), f(\tau)) + \hat{\varepsilon}_d(\tau), \quad \varphi_2(\tau) = \hat{\Gamma}_s \text{Col} (\ell_{d_2}(\tau), f(\tau)) + \hat{\varepsilon}_s(\tau).
\]  

(9)

where \( \hat{\Gamma}_d \) and \( \hat{\Gamma}_s \) are given coefficient with \( d = d_1 + d_2 + 1 \) and \( \delta = \delta_1 + \delta_2 + 1 \). Furthermore, \( \hat{\varepsilon}_d(\tau) := \varphi_1(\tau) - \hat{\Gamma}_d \text{Col} (\ell_{d_1}(\tau), f(\tau)) \) and \( \hat{\varepsilon}_s(\tau) := \varphi_2(\tau) - \hat{\Gamma}_s \text{Col} (\ell_{d_2}(\tau), f(\tau)) \) define the error of approximations. In addition, we define matrices

\[
S^{\nu_1 \times \nu_1} \ni \hat{E}_d := \int_{-r_1}^0 \hat{\varepsilon}_d(\tau) \hat{\varepsilon}_d^T(\tau) \, d\tau, \quad S^{\nu_2 \times \nu_2} \ni \hat{E}_s := \int_{-r_2}^{-r_1} \hat{\varepsilon}_s(\tau) \hat{\varepsilon}_s^T(\tau) \, d\tau
\]  

(10)

to measure the error residues of (9). Inspired by the principle of orthogonal approximation in Hilbert space Muscat (2014), one option for the \( \hat{\Gamma}_d \) and \( \hat{\Gamma}_s \) in (9) is

\[
\mathbb{R}^{\nu \times d} \ni \hat{\Gamma}_d := \int_{-r_1}^0 \varphi_1(\tau) [\ell_{d_1}(\tau) \ f^T(\tau)] \, d\tau \hat{F}_d, \quad \hat{\Gamma}_d^{-1} = \int_{-r_1}^0 [\ell_{d_1}(\tau) \ f^T(\tau)] \, d\tau, \\
\mathbb{R}^{\nu \times d} \ni \hat{\Gamma}_s := \int_{-r_2}^{-r_1} \varphi_2(\tau) [\ell_{d_2}(\tau) \ f^T(\tau)] \, d\tau \hat{F}_s, \quad \hat{\Gamma}_s^{-1} = \int_{-r_2}^{-r_1} [\ell_{d_2}(\tau) \ f^T(\tau)] \, d\tau.
\]  

(11)

**Remark 2.** (11) might be interpreted as a vector form of the standard approximations (Least Squares) in Hilbert space Muscat (2014). Specifically, let \( \{f_i(\cdot)\}_{i=1}^\infty \subset \mathbb{L}_2([-r_1, 0] ; \mathbb{R}) \) and \( \{f_i(\cdot)\}_{i=1}^\infty \subset \mathbb{L}_2([-r_2, -r_1] ; \mathbb{R}) \) to be the orthogonal bases (not necessary normal here) of the corresponding Hilbert space and satisfying (6) (Trigonometric series, for instance). Then one can conclude that by using (11) with \( d_1 = \delta_1 = -1 \), each entry of the error vectors \( \hat{\varepsilon}_d(\cdot) \) and \( \hat{\varepsilon}_s(\cdot) \) in (10) converges to zero in a \( \mathbb{L}_2 \) sense, respectively, as \( d \to \infty \) and \( \delta \to \infty \). Similarly, the aforementioned property also holds with \( d_2 = \delta_2 = 0 \) since \( \{\ell_i(\cdot)\}_{i=1}^\infty \subset \mathbb{L}_2([-r_1, 0] ; \mathbb{R}) \) and \( \{\ell_i(\cdot)\}_{i=1}^\infty \subset \mathbb{L}_2([-r_2, -r_1] ; \mathbb{R}) \) are the orthogonal bases of the corresponding Hilbert space. With \( d_2 = \delta_2 = 0 \) in (5), (9)–(11) generalizes the polynomials approximation scheme proposed in Seuret et al. (2015b) via a matrix framework.

The following property of Kronecker products will be used throughout the paper.

\(^1\)See Theorem 7.2.10 in Horn & Johnson (2012) for the motivation of the criteria in (7).
Lemma 1. \( \forall X \in \mathbb{R}^{n \times m}, \forall Y \in \mathbb{R}^{m \times p}, \forall Z \in \mathbb{R}^{q \times r}, \)
\[
(X \otimes I_q)(Y \otimes Z) = (XY) \otimes (I_qZ) = (XY) \otimes (ZI_r) = (X \otimes Z)(Y \otimes I_r).
\]
Moreover, \( \forall X \in \mathbb{R}^{n \times m}, \)
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \otimes X = \begin{bmatrix} A \otimes X & B \otimes X \\ C \otimes X & D \otimes X \end{bmatrix}
\]
for any \( A, B, C, D \) with appropriate dimensions.

Now we derive the following generalized new integral inequality which will be employed in deriving our main results.

Lemma 2. Given a set \( K \subseteq \mathbb{R} \cup \{-\infty, +\infty\} \) and \( U \in \mathbb{S}_m^0, n \in \mathbb{N} \) and a functions space
\[
\mathbb{L}_{\mathbb{R};0}(K) := \left\{ g(\cdot) \in \mathbb{L}_f(K; \mathbb{R}^d) : \int_K \varpi(\tau)g(\tau)d\tau < +\infty \right\},
\]
with a weighted function \( \varpi(\cdot) \in \mathbb{L}_{\mathbb{R};0}(K) \). Let \( f(\cdot) := \mathbb{C}o[\sum_{i=1}^n f_i(\cdot) \in \mathbb{L}_{\mathbb{R};0}(K; \mathbb{R}^d) \) and \( \varphi(\cdot) := \mathbb{C}o[\varphi(\cdot) \in \mathbb{L}_{\mathbb{R};0}(K; \mathbb{R}^n) \) with \( d \in \mathbb{N} \) and \( n \in \mathbb{N}_0 \), in which the functions in \( \mathbb{C}o[f(\cdot), \varphi(\cdot)] \) are linearly independent, then the inequality
\[
\int_K \omega(\tau)Ux(\tau)d\tau \geq [\star] (F_d \otimes U) \left[ \int_K \omega(\tau)F(\tau)x(\tau)d\tau \right] + [\star] (F_{d-1}^{-1} \otimes U) \left[ \int_K \omega(\tau)E(\tau)x(\tau)d\tau \right]
\]
holds \( \forall x(\cdot) \in \mathbb{L}_{\mathbb{R};0}(K) \). where \( \mathbb{S}_m^d \ni F_{d-1}^{-1} := \left[ \int_K \omega(\tau)f(\tau)f(\tau)d\tau \right] \) and \( \mathbb{R}^{n \times n} \ni F(\tau) := f(\tau) \otimes I_n \). In addition, \( \mathbb{R}^{n \times n} \ni E(\tau) := e(\tau) \otimes I_n \) with \( \mathbb{R}^n \ni e(\tau) := \varphi(\tau) - \Gamma f(\tau) \) and \( \mathbb{R}^{n \times d} \ni \Gamma := \left( \int_K \omega(\tau)e(\tau)\Gamma(\tau)d\tau \right)F_d \) and \( \mathbb{S}_m^d \ni E_{d} := \int_K \omega(\tau)e(\tau)e(\tau)d\tau \).

Proof. See Appendix A for details.

Remark 3. In (14), one can interpret it as using \( f(\cdot) \) to approximate \( \varphi(\cdot) \). For instance, the distributed kernels \( \varphi_1(\cdot) \) and \( \varphi_2(\cdot) \) in (9) correspond to the position of \( \varphi(\cdot) \). By using this interpretation, \( E_d \) in (14) can be considered as a measure of approximation error similar to the ones in (9). Furthermore, if \( f(\cdot) \) contains only orthogonal functions, then the behavior of \( E_d \) can be quantitatively characterized, which will be shown in Corollary 1 later. It can be seen that (14) reduces to Lemma 1 in Seuret et al. (2015b) if \( f(\cdot) \) contains only Legendre polynomials, then is \( \kappa = 0 \) and \( \{f_i(\cdot)\}_{i=0}^n \) to be Legendre polynomials. Moreover, by utilizing the Cauchy formula for repeated integrations (see (5),(6) and (25),(26) in Gyurkovics & Takács (2016)), the results in Park et al. (2015); Gyurkovics & Takács (2016) and Chen et al. (2016) are covered by (14) as special cases with \( \kappa = 0 \) and appropriate \( f(\cdot) \). Meanwhile, if \( \kappa = 0 \) and \( f(\cdot) \) contains only orthogonal functions, then (14) reduces to the inequalities in Feng & Nguang (2016a) with a reverse order of Kronecker product. In addition, with \( \kappa = 0 \) and \( K = [0 + \infty], \) (9) in Liu et al. (2016) is the special case of (14) with appropriate \( \varphi(\cdot) \) and \( f(\cdot) \). By letting \( \kappa = 0 \) and \( \varphi(\tau) = 1, \) (14) reduces to the result of Lemma 4 in Feng & Nguang (2016b). Finally, it is worthy to note that a summation inequality
\[
\sum_{k \in J} \omega(k)[\star]Ux(k) \geq [\star] (F_d \otimes U) \sum_{k \in J} \omega(k)F(k)x(k), \quad F^{-1} = \sum_{k \in J} \omega(k)f(k)f(\tau) \quad F \subseteq \mathbb{Z}, \#J \geq 2
\]
can be easily obtained based on the derivation of (14), which encompasses the results in Park et al. (2016); Seuret et al. (2015a) as special cases. Note that for a discrete system with finite length of delays indicating finite dimensions, (15) may produce a perfect bound with no conservatism at a finite \( d \).
An interesting result on the matrix $E_d$ in Lemma 2 is given in the following corollary.

**Corollary 1.** Given all the prerequisites in Lemma 2 and assume that $f(\cdot) = \text{Col}^d_{i=1} f_i(\cdot)$ where $\{f_i(\cdot)\}_{i=1}^\infty$ contains only orthogonal functions, we have

$$\forall d \in \mathbb{N}_0, \ 0 = E_d - f_{d+1}^{-1} \gamma_{d+1} r_{d+1}^T \preceq E_d,$$

(16)

where $\mathbb{R}^\kappa \supseteq \gamma_{d+1} := (\int_K \varpi(\tau) \varphi(\tau)f_{d+1}(\tau)d\tau)$ and $f_{d+1}(\cdot) \in \mathbb{L}_2([\omega_1; \omega_2])$. Furthermore, $f_{d+1}^{-1} = \int_K \varpi(\tau)f_{d+1}^2(\tau)d\tau$ is the $d+1$th diagonal element of $E_{d+1}$ defined in Lemma 2.

**Proof.** See Appendix B for details.

**Remark 4.** Let $d_1 = \delta_1 = -1$ and $\{f_i(\cdot)\}_{i=1}^\infty \subset \mathbb{L}_2([-\infty, 0]; \mathbb{R})$ and $\{f_i(\cdot)\}_{i=1}^\infty \subset \mathbb{L}_2([-\infty, -1]; \mathbb{R})$ to be the orthogonal bases of the corresponding Hilbert space and satisfying (6), then (10) follow the property in Corollary 1 with (11) and $\hat{f}(\cdot) = \text{Col}^{d_1}_{i=1} f_i(\cdot)$ and $\hat{f}(\cdot) = \text{Col}^{d_2}_{i=1} f_i(\cdot)$ in (9). Since $\ell_{d_i}(\cdot)$ and $\ell_{d_i}(\cdot)$ are orthogonal functions, thus Corollary 1 holds for (10) with (9) and $d_2 = d_2 = 0$. Given (7) and (16), it is easy to see that $\hat{E}_{\hat{d}^{-1}}$ and $\hat{E}_{\hat{d}^{-1}}$ are well defined, which allows one to apply (14). On the other hand, the corresponding error matrices become an empty matrix with appropriate dimensions if $\nu_1 = 0$ or $\nu_2 = 0$.

The system (5) can be re-expressed as

$$\dot{x}(t) = A \theta(t), \ \xi(t) = [\Omega_{\mu \times (2 \mu + q)} \equiv \Omega_{\mu \times \mu}] \theta(t), \ z(t) = \Sigma \theta(t)$$

(17)

where

$$A := \begin{bmatrix} D_1 & A_1 & A_2 & A_3 \end{bmatrix}, \ \Xi = \begin{bmatrix} A_6 & A_7 & A_8 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} C_6 & C_7 & C_8 \end{bmatrix},$$

$$\theta(t) := \text{Col} \left[ \begin{bmatrix} \xi(t-r_1) \xi(t-r_2) \end{bmatrix}, \ \begin{bmatrix} \omega(t) \xi(t-r_1) \xi(t-r_2) \end{bmatrix} \right] \in \mathbb{R}^\kappa,$$

(18) (19) (20) (21)

with $\kappa := q + n + 4 \mu + p + \nu \mu$ and $g = q_1 + q_2$, $\nu = \nu_1 + \nu_2$ and $q_1 := \mu d = \mu (d_1 + d_2 + 1)$ and $q_2 := \mu \delta = \mu (\delta_1 + \delta_2 + 1)$, and $\mathbb{R}^{\mu \times \mu} \supseteq E_d(\tau) := \text{Col} \left[ \ell_{d_1}(\tau), \ f(\tau) \right] \otimes I_d$ and $\mathbb{R}^{\mu \times \mu} \supseteq F_d(\tau) := \text{Col} \left[ \ell_{d_1}(\tau), \ f(\tau) \right] \otimes I_d$.

The error matrices are defined as $\hat{E}_d(\tau) := \hat{E}_d^{-1}(\tau) \hat{E}_d(\tau)$ and $\hat{E}_d(\tau) := \hat{E}_d^{-1}(\tau) \hat{E}_d(\tau)$ with the approximation coefficients defined in (11). Note that the distributed delay terms in (17) are derived based on the identities

$$\left( \text{Col} \left[ \ell_{d_1}(\tau), \ f(\tau), \varphi_1(\tau) \right] \otimes I_d \right) \xi(t+\tau) = \left[ \begin{bmatrix} f_{d_1}^T \Gamma_d \end{bmatrix} \otimes I_d \right] \hat{E}_d(\tau) \xi(t+\tau) + \left[ \begin{bmatrix} O_{d_1 \times \mu} \end{bmatrix} \otimes I_d \right] \hat{E}_d(\tau) \xi(t+\tau),$$

$$\left( \text{Col} \left[ \ell_{d_2}(\tau), f(\tau), \varphi_2(\tau) \right] \otimes I_d \right) \xi(t+\tau) = \left[ \begin{bmatrix} f_{d_2}^T \Gamma_d \end{bmatrix} \otimes I_d \right] \hat{E}_d(\tau) \xi(t+\tau) + \left[ \begin{bmatrix} O_{d_2 \times \mu} \end{bmatrix} \otimes I_d \right] \hat{E}_d(\tau) \xi(t+\tau),$$

which themselves are obtained via the property in (12).

3. Main Results on Dissipative Analysis

The results of the dissipative analysis presented in this section are based on the construction of a Krasovskii functional:

$$v(x(t), \xi(t+\tau)) = \eta^T(t) P \eta(t) + \int_{-r_1}^0 [Q_1 + (\tau + r_1) R_1] \xi(t+\tau) d\tau + \int_{-r_2}^{\tau} [Q_2 + (\tau + r_2) R_2] \xi(t+\tau) d\tau + \int_{-r_1}^0 \xi^T(t+\tau) [S_1 + (\tau + r_1) U_1] \xi(t+\tau) d\tau + \int_{-r_2}^{\tau} \xi^T(t+\tau) [S_2 + (\tau + r_2) U_2] \xi(t+\tau) d\tau,$$

(22)
where $P \in \mathbb{S}^l$ and $Q_1; Q_2; R_1; R_2; S_1; S_2; U_1; U_2 \in \mathbb{S}^\mu$ and

$$\eta(t) := \text{Col} \left[ x(t), \xi(t-r_1), \xi(t-r_2), \int_{-r_1}^{0} \dot{F}_d(\tau)\xi(t+\tau)d\tau, \int_{-r_2}^{-r_1} \dot{F}_d(\tau)\xi(t+\tau)d\tau \right] \in \mathbb{R}^l. \quad (23)$$

with $l := n + 2\mu + \varrho$, $\varrho := g_1 + g_2$ and $\dot{F}_d(\tau)$ and $\dot{F}_s(\tau)$ defined in (21).

**Remark 5.** If one wants to increase the values of $d$ and $\delta$ in (22) to incorporate more functions in the distributed delay terms in (23), then extra zeros need to be introduced to the coefficient matrices $A_4, A_5$ and $C_4, C_5$ in (5) in order to make (22) consistent with (17). In conclusion, there are no upper bound on the values of $d$ and $\delta$. Finally, (22) generalizes the Krasovskii functional in Seuret et al. (2015b) which only consider Legendre polynomials for the integral terms in (23).

The following lemma provides sufficient conditions for the stability of CDDS. It can be interpreted as a particular case of Theorem 3 in Gu & Liu (2009) with certain modifications.

**Lemma 3.** Given $r > 0$, consider a coupled differential-difference system

$$\dot{x}(t) = f(x(t), \xi(t+\cdot)), \quad y(t) = g(x(t), \xi(t+\cdot)), \quad f(0_\nu, 0_\nu(\cdot)) = 0_n, \quad g(0_\nu, 0_\nu(\cdot)) = 0_\nu(\cdot) \quad (24)$$

satisfying the prerequisites in the Theorem 3 of Gu & Liu (2009), where $\xi(t+\cdot) \in H^1([-r,0] \setminus \mathbb{R}^\mu$) and $\xi(t) = g(x(t), \xi(t+\cdot))$ is uniformly input to state stable. Then the origin of (24) is globally uniformly asymptotically stable, if there exists a differentiable functional $v(\xi_1, \xi_2) : \mathbb{R}^n \times H^1([-r,0] \setminus \mathbb{R}^\mu \rightarrow \mathbb{T} := \{x \in \mathbb{R} : x \geq 0\}$ with $v(0_\nu, 0_\nu(\cdot)) = 0$, such that the following conditions are satisfied:

$$\exists \epsilon_1, \epsilon_2 > 0, \forall \psi \in \mathbb{R}^n, \forall \phi(\cdot) \in H^1([-r,0] \setminus \mathbb{R}^\mu$, $\epsilon_1 \|\psi\|^2 \leq v(\psi, \phi(\cdot)) \leq \epsilon_2 \left[ \|\psi\|^2 \vee \|\phi(\cdot)\|^2 + \|\phi(\cdot)\|^2 \right]^2, \quad (25)$$

$$\exists \epsilon_3 > 0, \forall \psi \in \mathbb{R}^n, \forall \phi(\cdot) \in H^1([-r,0] \setminus \mathbb{R}^\mu$, $\dot{v}(\psi, \phi(\cdot)) \leq -\epsilon_3 \|\phi(\cdot)\|^2 \quad (26)$$

where

$$\dot{v}(\psi, \phi(\cdot)) := \frac{d}{dt}v(x(t), \xi(t+\cdot)) \bigg|_{x=\psi, \xi(t+\cdot)=\phi(\cdot)}, \quad \frac{d}{dt}f(x) = \limsup_{\eta \downarrow 0} \frac{f(x + \eta) - f(x)}{\eta} \quad (27)$$

with $\dot{x}(t)$ and $\xi(t+\cdot)$ satisfying (24).

**Definition 2 (Dissipativity).** Given $r > 0$, a coupled differential-difference system

$$\dot{x}(t) = f(x(t), \xi(t+\cdot), w(t)), \quad \xi(t) = g(x(t), \xi(t+\cdot)), \quad z(t) = h(x(t), \xi(t+\cdot), w(t)), \quad (28)$$

is dissipative with respect to the supply rate function $s(z(t), w(t))$, if there exists a differentiable functional $v(\xi_1, \xi_2) : \mathbb{R}^n \times H^1([-r,0] \setminus \mathbb{R}^\mu \rightarrow \mathbb{R}$ such that

$$\forall t \in \mathbb{T}, \dot{v}(x(t), \xi(t+\cdot)) - s(z(t), w(t)) \leq 0, \quad (29)$$

with $x(t), \xi(t+\cdot)$ and $z(t)$ satisfying (28). Under the assumption that $v(\xi_1, \xi_2) : \mathbb{R}^n \times H^1([-r,0] \setminus \mathbb{R}^\mu \rightarrow \mathbb{R}$ is differentiable, (29) is equivalent to the standard definition of dissipativity. (See Briat (2014) for the definition of dissipativity without delays)

To incorporate performance objectives (dissipativity) into the analysis of (17), a quadratic form

$$s(z(t), w(t)) = \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \quad (30)$$

with $J = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \in \mathbb{S}^{m+q}$, $J_1 \leq 0$ is applied in this paper which is taken from Scherer et al. (1997). For specific optimization objectives included by (30), see the details in Scherer et al. (1997).

The main dissipative analysis result for (5) is derived in terms of LMIs as given in the following theorem.
Theorem 1. Given \( d_1; d_2; \delta_1; \delta_2 \in \mathbb{N} \) and \( \nu_1; \nu_2 \in \mathbb{N} \) with \( r_2 > r_1 > 0 \) and the matrix parameters in (30) with \( J_1 \approx 0 \), and the matrices \( \tilde{F}_d, \hat{F}_d, \tilde{E}_d, \tilde{E}_g \) in (9)–(10) with \( \tilde{F}_d, \hat{F}_d \) in (11), then (17) is globally asymptotically (exponentially) stable and dissipative with respect to (30) if there exist \( P \in \mathbb{S}^d \) and \( Q_1; Q_2; R_1; R_2; S_1; S_2; U_1; U_2 \in \mathbb{S}^\mu \) such that the inequalities

\[
P := P + \left( O_{\nu_2 + 2\mu} \otimes [\tilde{F}_d \otimes Q_1] \oplus [\hat{F}_d \otimes Q_2] \right) + \Pi^T \left( G_1 \tilde{F}_d + G_1 - G_2 \hat{F}_d + G_2 \right) \left( \Pi \otimes S_2 \right) \Pi + W^T \left( r_1^{-2} \left[ H^T_1 \left( D_{d_1} + I_{d_1 + 1} \right) H_1 \otimes U_1 \right] + r_2^{-2} \left[ H^T_2 \left( D_{\delta_1} + I_{\delta_1 + 1} \right) H_2 \otimes U_2 \right] \right) W > 0,
\]

\[
Q_1 \geq 0, \quad Q_2 \geq 0, \quad R_1 \geq 0, \quad R_2 \geq 0, \quad S_1 \geq 0, \quad S_2 \geq 0, \quad U_1 \geq 0, \quad U_2 \geq 0,
\]

\[
\Phi = \begin{bmatrix} I_{d_1 + 1}^{-1} & O_{\mu \times \mu} & \sum \end{bmatrix} \begin{bmatrix} -S_1 - r_1 U_1 & (S_1 + r_1 U_1) (A \Phi + Y) \end{bmatrix} < 0
\]

hold, where \( \tilde{F}_{d+1} = J_1^{0} \left[ \hat{F}_{d+1}^{T}(\tau) \right] \hat{F}_{d+1}(\tau) d\tau \right) + \hat{F}_{d+1}^{T}(\tau) \hat{F}_{d+1}(\tau) d\tau \) and \( D_{d_1} = \bigotimes_{k=0}^{d_1} (2k+1) \) and \( \Pi = \mathrm{Col} \left[ \Xi, \left[ O_{(2\mu+\nu) \times \nu} \right] \right] \) and

\[
W = \mathrm{Col} \left[ \Xi, \left[ O_{(2\mu+\nu) \times \nu} \right] \right] \left( I_{d_1 + 1} \times O_{(d_1 + \mu) \times \mu} \right) + \left( I_{d_1 + 1} \times O_{(d_1 + \mu) \times \mu} \right) + \left( I_{d_1 + 1} \times O_{(d_1 + \mu) \times \mu} \right)
\]

and \( Y = [A_1 \ A_2 \ O_{\mu \times (2\nu-\mu)}] \), and \( A \) and \( \Sigma \) have been defined in (18)–(20). Furthermore,

\[
\Phi := S_{y} (\Theta_2 P \Theta_1 + [O_{\mu \times (\nu-\nu)} \bigotimes J_1]) \left( O_{\nu+2\mu} \otimes [\Pi^T ([\tilde{F}_d + G_1 \otimes U_1] + [\hat{F}_d + G_2 \otimes U_2]) \Pi] \otimes O_{\mu \nu} \right)
\]

\[
- \left( S_1 - S_2 - r_3 U_1 \right) \otimes S_2 \otimes J_1 \otimes O_{\nu} \otimes \left[ Q_1 - Q_2 - r_3 U_2 \right] \otimes \left[ \hat{F}_d \otimes R_1 \right] \oplus \left[ \hat{F}_d \otimes R_2 \right]
\]

\[
+ \left( \hat{E}_d \otimes R_1 \right) \oplus \left( \hat{E}_d \otimes R_2 \right)
\]

with \( 1_{d+1} := \mathrm{Col}_{i=0}^{d_1} 1 \) and \( \hat{1}_{d+1} := \mathrm{Col}_{i=0}^{d_1} (-1)^i \) and the parameters defined in (3), (4) and (6).

Proof. Firstly, we prove that the existence of the feasible solutions of (32) and (33) infers both (26) and (29). Subsequently, we show that the existence of the feasible solutions of (31) and (32) infers (25). The existence of the upper bound of \( v(x(t), \xi(t+\cdot)) \) can be independently proved without considering the inequalities (31)–(33).

Differentiate \( v(x(t), \xi(t+\cdot)) \) alongside the trajectory of (17) and consider (30), it produces

\[
v(x(t), \xi(t+\cdot)) - s(z(t), w(t)) = \Theta^T \left( \Theta_2 P \Theta_1 \right) v(t) + \xi^T \left( t + r_1 \right) \left( Q_1 + r_1 U_1 \right) \xi(t)
\]

\[
+ \xi^T \left( t - r_1 \right) \left( Q_2 + r_3 U_2 - Q_1 \right) \xi(t - r_1) - [s] Q_2 \xi(t - r_2) + [s] \left( S_1 + r_1 U_1 \right) \xi(t) + [s] \left( S_2 + r_3 U_2 - S_1 \right) \xi(t - r_1)
\]

\[
= \mathrm{Sy} \left( \Theta_2 P \Theta_1 \right) v(t) + \xi^T \left( t + r_1 \right) \left( Q_1 + r_1 U_1 \right) \xi(t)
\]

\[
+ \xi^T \left( t - r_1 \right) \left( Q_2 + r_3 U_2 - Q_1 \right) \xi(t - r_1) - [s] Q_2 \xi(t - r_2) + [s] \left( S_1 + r_1 U_1 \right) \xi(t) + [s] \left( S_2 + r_3 U_2 - S_1 \right) \xi(t - r_1)
\]

\[
= \mathrm{Sy} \left( \Theta_2 P \Theta_1 \right) v(t) + \xi^T \left( t + r_1 \right) \left( Q_1 + r_1 U_1 \right) \xi(t)
\]

\[
+ \xi^T \left( t - r_1 \right) \left( Q_2 + r_3 U_2 - Q_1 \right) \xi(t - r_1) - [s] Q_2 \xi(t - r_2) + [s] \left( S_1 + r_1 U_1 \right) \xi(t) + [s] \left( S_2 + r_3 U_2 - S_1 \right) \xi(t - r_1)
\]
\[-[s]S_2\dot{\xi}(t - r_2) - \int_{-r_1}^{0}[s]R_1x(t + \tau)d\tau - \int_{-r_2}^{-r_1}[s]R_2x(t + \tau)d\tau - \int_{-r_2}^{0}[s]U_1\dot{\xi}(t + \tau)d\tau - \int_{-r_2}^{-r_1}[s]U_2\dot{\xi}(t + \tau)d\tau + w^T(t)J_3w(t) + \vartheta^T(t)\left[\Sigma^TJ_1\Sigma + SY\left([O_{N_2} \times (x_{-m})], \Sigma^TJ_2\right)\right]\vartheta(t), \forall t \in \mathbb{R}, \]  
where \(\vartheta(t)\) and \(\Theta_1, \Theta_2\) have been defined in (21) and (41), respectively, and the matrices \(H_1\) and \(H_2\) in (41) are obtained via the relations

\[
\int_{-r_1}^{0} \dot{F}_{d}(\tau)\dot{\xi}(t + \tau)d\tau = \dot{F}_{d}(0)\xi(t) - \dot{F}_{d}(-r_1)\xi(t - r_1) - \left[\left(\tilde{A}_{d_1} \oplus \tilde{M}\right) \otimes I_{p}\right] \int_{-r_1}^{0} \dot{F}_{d}(\tau)\dot{\xi}(t + \tau)d\tau = \left[O_{q_x \times (q + 2\mu)} \left(\begin{array}{c} H_1 \otimes I_{p} \end{array}\right) \Pi O_{q_x \times \mu_2}\right] \vartheta(t), \]  

\[
\int_{-r_2}^{-r_1} \dot{F}_{d}(\tau)\dot{\xi}(t + \tau)d\tau = \dot{F}_{d}(-r_1)\xi(t - r_1) - \dot{F}_{d}(-r_2)\dot{\xi}(t - r_2) - \left[\left(\tilde{A}_{d_1} \oplus \tilde{M}\right) \otimes I_{p}\right] \int_{-r_2}^{-r_1} \dot{F}_{d}(\tau)\dot{\xi}(t + \tau)d\tau = \left[O_{q_x \times (q + 2\mu)} \left(\begin{array}{c} H_2 \otimes I_{p} \end{array}\right) \Pi O_{q_x \times \mu_2}\right] \vartheta(t), \]  

\[
\int_{-r_1}^{0} \dot{F}_{d}(\tau)\dot{\xi}(t + \tau)d\tau \geq \int_{-r_1}^{0} \dot{F}_{d}(\tau)\dot{\xi}(t + \tau)d\tau + \left[\left(\tilde{A}_{d_1} \oplus \tilde{M}\right) \otimes I_{p}\right] \int_{-r_1}^{0} \dot{F}_{d}(\tau)\dot{\xi}(t + \tau)d\tau, \]  

\[
\int_{-r_2}^{-r_1} \dot{F}_{d}(\tau)\dot{\xi}(t + \tau)d\tau \geq \int_{-r_2}^{-r_1} \dot{F}_{d}(\tau)\dot{\xi}(t + \tau)d\tau + \left[\left(\tilde{A}_{d_1} \oplus \tilde{M}\right) \otimes I_{p}\right] \int_{-r_2}^{-r_1} \dot{F}_{d}(\tau)\dot{\xi}(t + \tau)d\tau, \]  

\[
\int_{-r_1}^{0} [s]U_1\dot{\xi}(t + \tau)d\tau \geq \int_{-r_1}^{0} [s]U_1\dot{\xi}(t + \tau)d\tau + \left[\left(\tilde{A}_{d_1} \oplus \tilde{M}\right) \otimes I_{p}\right] \int_{-r_1}^{0} [s]U_1\dot{\xi}(t + \tau)d\tau = \int_{-r_1}^{0} \dot{F}_{d}(\tau)\dot{\xi}(t + \tau)d\tau, \]  

\[
\int_{-r_2}^{-r_1} [s]U_2\dot{\xi}(t + \tau)d\tau \geq \int_{-r_2}^{-r_1} [s]U_2\dot{\xi}(t + \tau)d\tau + \left[\left(\tilde{A}_{d_1} \oplus \tilde{M}\right) \otimes I_{p}\right] \int_{-r_2}^{-r_1} [s]U_2\dot{\xi}(t + \tau)d\tau = \int_{-r_2}^{-r_1} \dot{F}_{d}(\tau)\dot{\xi}(t + \tau)d\tau, \]  

\[
\int_{-r_1}^{0} \dot{F}_{d_1}(\tau)\dot{\xi}(t + \tau)d\tau = \dot{F}_{d_1}(0)\xi(t) - \dot{F}_{d_1}(-r_1)\dot{\xi}(t - r_1) - \left[\left(\tilde{A}_{d_1} \oplus \tilde{M}\right) \otimes I_{p}\right] \int_{-r_1}^{0} \dot{F}_{d_1}(\tau)\dot{\xi}(t + \tau)d\tau = \left[O_{q_x \times (q + 2\mu)} \left(\begin{array}{c} G_1 \otimes I_{p} \end{array}\right) \Pi O_{q_x \times \mu_2}\right] \vartheta(t), \]  

\[
\int_{-r_2}^{-r_1} \dot{F}_{d_1}(\tau)\dot{\xi}(t + \tau)d\tau = \dot{F}_{d_1}(-r_1)\dot{\xi}(t - r_1) - \dot{F}_{d_1}(-r_2)\dot{\xi}(t - r_2) - \left[\left(\tilde{A}_{d_1} \oplus \tilde{M}\right) \otimes I_{p}\right] \int_{-r_2}^{-r_1} \dot{F}_{d_1}(\tau)\dot{\xi}(t + \tau)d\tau = \left[O_{q_x \times (q + 2\mu)} \left(\begin{array}{c} G_2 \otimes I_{p} \end{array}\right) \Pi O_{q_x \times \mu_2}\right] \vartheta(t), \]  

\[
\int_{-r_1}^{0} \dot{F}_{d_2}(\tau)\dot{\xi}(t + \tau)d\tau = \dot{F}_{d_2}(0)\xi(t) - \dot{F}_{d_2}(-r_1)\dot{\xi}(t - r_1) - \left[\left(\tilde{A}_{d_2} \oplus \tilde{M}\right) \otimes I_{p}\right] \int_{-r_1}^{0} \dot{F}_{d_2}(\tau)\dot{\xi}(t + \tau)d\tau = \left[O_{q_x \times (q + 2\mu)} \left(\begin{array}{c} H_1 \otimes I_{p} \end{array}\right) \Pi O_{q_x \times \mu_2}\right] \vartheta(t). \]  

Where \(G_1\) and \(G_2\) are given in (36) and (37) which are derived by the relations

\[
\dot{w}(x(t), \dot{\xi}(t + \cdot)) - s(x(t), w(t)) \leq \vartheta^T(t)\left[\Phi + [s] (S_1 + r_1U_1) (A_0A + Y) - \Sigma^TJ_1\Sigma\right]\vartheta(t), \forall t \in \mathbb{T}, \]  

\[
\exists \varepsilon_3 < 0 : \forall t \in \mathbb{R}, \ \dot{v}(x(t), \dot{\xi}(t + \cdot)) - s(x(t), w(t)) \leq \varepsilon_3\|x(t)\|^2 \leq 0. \]  

Moreover, considering the matrix structure of \(\Phi + (A_0A + Y)^T (S_1 + r_1U_1) (A_0A + Y) - \Sigma^TJ_1\Sigma\) together with the properties of negative definite matrices, we can conclude that the proposition

\[
\exists \varepsilon_3 < 0 : \forall t \in \mathbb{R}, \ \dot{v}(x(t), \dot{\xi}(t + \cdot)) \leq \varepsilon_3\|x(t)\|^2 \]  

holds provided that \(\Phi + [s] (S_1 + r_1U_1) (A_0A + Y) - \Sigma^TJ_1\Sigma < 0\) and (32) hold. As a result, it is obvious that (32) with \(\Phi + [s] (S_1 + r_1U_1) (A_0A + Y) - \Sigma^TJ_1\Sigma < 0\) infers (26) considering (27) with (51) and (29).
Finally, applying Schur complement to \( \Phi + [s](S_1 + r_1 U_1)(A_d A + Y) - \Sigma^T J_1 \Sigma < 0 \) with (32) and \( J_1 < 0 \) gives (33). Hence we have proved that (32) with (33) infers both (26) and (29).

Now we start to prove that (31) with (32) infers (25). Let \( \|\phi(t)\|_2^2 := \int_{r_2}^0 \Phi^T(\tau)\phi(\tau)d\tau \). Given the structure of (22), it follows that \( \exists \lambda; \eta > 0 : \)

\[
v(x(t), \xi(t + \cdot)) \leq \eta^T(t)\lambda\eta(t) + \lambda \int_{r_2}^0 [\xi^T(t + \tau) \hat{F}_d^T(\tau)\xi(\tau + \tau)]d\tau \leq \lambda \|x(t)\|_2^2 + 3\lambda \|\xi(t + \cdot)\|_2^2 + \lambda \|\xi(t + \cdot)\|_2^2 + \lambda \|\xi(t + \cdot)\|_2^2 + \lambda \|\xi(t + \cdot)\|_2^2 + \lambda \|\xi(t + \cdot)\|_2^2 + \lambda \|\xi(t + \cdot)\|_2^2 + \lambda \|\xi(t + \cdot)\|_2^2 + \lambda \|\xi(t + \cdot)\|_2^2 + \lambda \|\xi(t + \cdot)\|_2^2
\]

which is derived via the property of quadratic forms: \( \forall X \in S^n, \exists \lambda > 0 : \forall x \in \mathbb{R}^n \setminus \{0\}, x^T(\lambda I_n - X)x > 0 \) together with (14) with \( \kappa = 0 \). Then (52) shows that it is possible to found an upper bound for (22) which satisfies (27).

To construct a lower bound for \( v(x(t), \xi(t + \cdot)) \), apply (14) to (22) with \( \kappa = 0 \) and appropriate \( f(\cdot) matching \) the \( \hat{F}_d(\cdot), \hat{F}_s(\cdot) \) terms in (22). Then the inequalities

\[
\int_{r_2}^0 \xi^T(t + \tau) Q_1 \xi(t + \tau)d\tau \geq [s] \left( \hat{F}_d \otimes Q_1 \right) \int_{r_2}^0 \hat{F}_d(\tau)\xi(t + \tau)d\tau, \\
\int_{r_2}^0 \xi^T(t + \tau) Q_2 \xi(t + \tau)d\tau \geq [s] \left( \hat{F}_s \otimes Q_2 \right) \int_{r_2}^0 \hat{F}_s(\tau)\xi(t + \tau)d\tau,
\]

and

\[
\int_{r_2}^0 [s]S_1 \hat{x}(t + \tau)d\tau \geq [s] \left( \hat{F}_d \otimes S_1 \right) \int_{r_2}^0 \hat{F}_d(\tau)\hat{x}(t + \tau)d\tau = \eta^T(t) \Pi(t) \left( G_1^T \hat{F}_d G_1 \otimes S_1 \right) \Pi(t),
\]

\[
\int_{r_2}^0 [s]S_2 \hat{x}(t + \tau)d\tau \geq [s] \left( \hat{F}_s \otimes S_2 \right) \int_{r_2}^0 \hat{F}_s(\tau)\hat{x}(t + \tau)d\tau = \eta^T(t) \Pi(t) \left( G_2^T \hat{F}_s G_2 \otimes S_2 \right) \Pi(t),
\]

can be derived via (47) and (48), provided that (32) hold. Furthermore, apply (14) to the integrals \( \int_{r_2}^0 (r_1 + \tau)\hat{F}_d(\tau)\hat{x}(t + \tau)d\tau \) and \( \int_{r_2}^0 (r_2 + \tau)\hat{F}_s(\tau)\hat{x}(t + \tau)d\tau \) in (22) with \( \kappa = 0 \) and appropriate \( f(\cdot) \) considering (1) with \( \alpha = \beta = 1 \). Then it yields

\[
\int_{r_1}^0 (r_1 + \tau)\hat{x}^T(t + \tau)U_1 \hat{x}(t + \tau)d\tau \geq r_1^2[s]\left((\hat{D}_{d_1} + I_{d_1 + 1}) \otimes U_1 \right) \int_{r_1}^0 (r_1 + \tau)\hat{x}^T(t + \tau)\hat{x}(t + \tau)d\tau \geq r_1^2[s]\left((\hat{D}_{d_1} + I_{d_1 + 1}) \otimes U_1 \right) \int_{r_2}^0 (r_2 + \tau)\hat{x}^T(t + \tau)\hat{x}(t + \tau)d\tau
\]

\[
= r_1^2[s]\eta^T(t)W^T(H_3^T (\hat{D}_{d_1} + I_{d_1 + 1}) H_4 \otimes U_2) W(t),
\]

where the matrices \( W \) in (34) and \( H_3; H_4 \) in (40) are obtained by the property in (4) with the relations

\[
\int_{r_1}^0 (r_1 + \tau) \hat{F}_d(\tau) \otimes I_{d_1} d\tau = \left( \hat{D}_{d_1} + I_{d_1 + 1} \right) \int_{r_1}^0 (r_1 + \tau) \hat{F}_d(\tau) \otimes I_{d_1} d\tau = \left( \hat{D}_{d_1} + I_{d_1 + 1} \right) \int_{r_2}^0 \hat{F}_d(\tau) \otimes I_{d_1} d\tau
\]

\[
= \left( \hat{D}_{d_1} + I_{d_1 + 1} \right) \int_{r_2}^0 (r_2 + \tau) \hat{x}(t + \tau) d\tau = \left( \hat{D}_{d_1} + I_{d_1 + 1} \right) \int_{r_2}^0 (r_2 + \tau) \hat{x}(t + \tau) d\tau = \left( \hat{D}_{d_1} + I_{d_1 + 1} \right) \int_{r_2}^0 (r_2 + \tau) \hat{x}(t + \tau) d\tau
\]

By applying (53)–(55) to (22) with (32), we can conclude that (25) is satisfied if (31) and (32) hold considering (27). This shows that feasible solutions of (31)–(33) infers the existence of the functional in (22) satisfying all the criteria in (25) and (26) considering (27), and (29). This finishes the proof. □
where only orthogonal functions defined over the corresponding intervals. (See the Definition in with tuning factors
without performance requirements are naturally incorporated by Theorem 1. Moreover, if \( d_2 = \delta_2 = 0 \) in (5), Theorem 1 with (11) reduces to the two delay channel version of the stability results in Seuret et al. (2015b). (Note that the method in Seuret et al. (2015b) only consider a single delay channel)

**Remark 6.** For \( J_1 \leq 0 \) with \( J_1 \neq 0 \), the corresponding stability conditions can be easily derived based on the factorization presented in Scherer et al. (1997). By allowing \( m, q \) to be zero, the scenario of stability analysis without performance requirements are naturally incorporated by Theorem 1. Moreover, if \( d_2 = \delta_2 = 0 \) in (5), Theorem 1 with (11) reduces to the two delay channel version of the stability results in Seuret et al. (2015b). (Note that the method in Seuret et al. (2015b) only consider a single delay channel)

**Remark 7.** Note that the position of the error matrix \( \hat{E}_d \) and \( \hat{E}_d \) in (33) may cause numerical problem if the eigenvalues of \( \hat{E}_d \) and \( \hat{E}_d \) are too small. To circumvent this potential problem, we can apply congruence transformations to \( \Phi \) so that instead of using \( \Phi < 0 \), we use

\[
* \Phi \left[ I_{m+q+n+5} \otimes (\eta_1 E_d^{-1} \otimes I_{\mu}) \oplus (\eta_2 E_d^{-1} \otimes I_{\mu}) \right] < 0,
\]

with tuning factors \( \eta_1, \eta_2 \in \mathbb{R} \). After the transformation, the diagonal elements of the transformed matrix in (56) are no longer associated with the error terms appear at off-diagonal elements.

Similar to the results given in Seuret & Gouaisbaut (2015) and Gyurkovics & Takács (2016), the following theorem is derived to show that Theorem 1 exhibits a hierarchical feasibility enhancement under the framework of orthogonal functions.

**Theorem 2.** Let either \( d_1 = \delta_1 = -1 \) or \( d_2 = \delta_2 = 0 \) with the assumption that \( \hat{f}(\cdot) := \text{Col}_{i=1}^{d_2} \hat{f}_i(\cdot) \in \mathbb{C}^1([-r_1, 0] \times \mathbb{R}^{d_2}) \) and \( \hat{f}(\cdot) := \text{Col}_{i=1}^{d_2} \hat{f}_i(\cdot) \in \mathbb{C}^1([-r_2, -r_1] \times \mathbb{R}^{d_2}) \) in which \( \{\hat{f}_i(\cdot)\}_{i=1}^{\infty} \) and \( \{\hat{f}_i(\cdot)\}_{i=1}^{\infty} \) contain only orthogonal functions defined over the corresponding intervals. (See the Definition in Feng & Nguang (2016a) ) Given the same prerequisites in Theorem 1 with the same matrix variables, we have

\[
\forall \lambda := (d + \delta) \in \mathbb{N}_0, \quad \mathcal{F}_\lambda \subseteq \mathcal{F}_{\lambda+1}
\]

where \( \mathcal{F}_\lambda := \left\{ r_1 > 0; \ r_2 > r_1 : \ (31)-(33) \ hold \right\} \).

**Proof.** Refer to Appendix C.

4. Numerical Simulations

In this section, numerical examples are presented to demonstrate the effectiveness of our proposed methods. All examples are tested in Matlab environment using Yalmip Löfberg (2004) with SDPT3 Toh et al. (2012) as the numerical solver. Note that we do not prescribe fixed positive eigenvalue margins for PSD (positive semi-definite) variables, rather the validity of a feasible result (minimization programs follow the same principle) is confirmed by verifying that all eigenvalues of the resulting PSD variables are strictly positive.

4.1. Stability analysis of a distributed delay system

Consider the following distributed delay system

\[
\dot{x}(t) = 0.33x(t) - 5 \int_{-\tau}^{0} \sin(\cos(12\tau))x(t + \tau) d\tau = 0.33x(t) - [0_d \ 5] \int_{-\tau}^{0} \text{Col} \left( \ell_{d_1}(\tau), \hat{f}(\tau), \varphi_1(\tau) \right) x(t + \tau) d\tau.
\]

where \( \ell_{d_1}(\tau) = j_{d_1}^{0,0}(\tau) \) with \( r_1 = r \) in line with Definition 1 and \( \varphi_1(\tau) = \sin(\cos(12\tau)) \). Furthermore, we let \( \hat{f}(\tau) = \text{Col} \left[ \text{Col}_{i=1}^{d_2} \sin 12i\tau, \text{Col}_{i=1}^{d_2} \cos 12i\tau \right] \) in (58), this gives \( \hat{M} = \begin{bmatrix} 0 & \Phi_{d_2}^{21} & \Phi_{d_2}^{18} \\ \Phi_{d_2}^{21} & 0 & \Phi_{d_2}^{18} \\ \Phi_{d_2}^{21} & \Phi_{d_2}^{18} & 0 \end{bmatrix} \) satisfying the first relation in (6).

Since without the distributed delay term, the system is unstable, thus the method in Münz et al. (2008) cannot be applied. Furthermore, since the function \( \varphi_1(\tau) = \sin(\cos(12\tau)) \) does not satisfy (6), the methods
in Seuret & Johansson (2009); Feng & Nguang (2016b) are not able to handle (58). Now apply the spectrum methods in Breda et al. (2005) to (58) with \( M = 200 \). The resulting information of the spectrum of (58) shows that the system is stable in the following intervals: \([0.093, 0.169]\), \([0.617, 0.692]\), \([1.14, 1.216]\), \([1.664, 1.739]\), \([2.188, 2.263]\) and \([2.711, 2.787]\).

Now apply a single delay version of Theorem 1 to (58), which is derived from the Krasovskii functional

\[
v(x(t), \xi(t + \cdot)) = \eta^T(t) P \eta(t) + \int_0^T \xi^T(t + \tau) [Q + (\tau + r) R] \xi(t + \tau) d\tau
\]

(59)
as a simplified version of (22), where \( P \in \mathbb{S}^{n+(d_1+1)\mu}, Q; R \in \mathbb{S}^\mu \) and \( \eta(t) := \text{Col} \left[ x(t), \int_0^T F_d(\tau) \xi(t + \tau) d\tau \right] \) with \( F_d(\tau) = \text{Col} \left( \ell_d(\tau), f(\tau) \right) \otimes I_\mu \). Furthermore, the corresponding \( \vartheta(t) \) in (21) and (49) is defined as \( \vartheta(t) := \text{Col} \left[ x(t), \xi(t - r), \int_0^T F_d(\tau) \xi(t + \tau) d\tau \right] \). The stability results are summarized in the following tables. The values of \( N \) and \( d_1, d_2 \) are presented when the boundaries of the related delay intervals can be detected by the corresponding methods. Finally, NoV denotes the number of variables of the corresponding stability conditions.

| Seuret et al. (2015b) | 0.093, 0.169 | 0.617, 0.692 |
|-----------------------|----------------|----------------|
| d_1 = 3, d_2 = 0 (NoV: 17) | d_1 = 4, d_2 = 3 (NoV: 38) |

Table 1: Stability Testing Results.

| Seuret et al. (2015b) | 0.093, 0.169 | 0.617, 0.692 |
|-----------------------|----------------|----------------|
| d_1 = 0, d_2 = 5 (NoV: 80) | d_1 = 0, d_2 = 5 (NoV: 80) |

Table 2: Stability Testing Results (NDV stands for the number of decision variables).

Note that with \( N = 25 \), the stable delay intervals \([1.664, 1.739]\), \([2.188, 2.263]\) and \([2.711, 2.787]\) still cannot be detected by the method in Seuret et al. (2015b); see Table 2. For the higher values for \( N > 25 \), our simulation shows that the computational time becomes too long to accurately obtain the values of the approximation coefficient term and the error term via the function \text{vpaintegral} in Matlab. On the other hand, the function \text{integral} in Matlab is not an alternative option due to its limited capacity of numerical accuracy when it comes to larger values of \( N \).

From the results given in Tables 1 and 2, one can clearly observe the advantage of our proposed method over the scheme in Seuret et al. (2015b).

4.2. Dissipative analysis with distributed delays

Consider a system of the form (5) with \( d_1 = \delta_1 = -1 \) and constant delays \( r_1 = 2, r_2 = 4.05 \) and the state space matrices

\[
A_1 = \begin{bmatrix} 0.01 & 0 \\ 0 & -3 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0.1 \\ 0.2 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix}, A_6 = I_2, A_7 = A_8 = O_2, D_1 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix},
\]

\[
A_4 \left( \begin{bmatrix} f(\tau) \\ \varphi_1(\tau) \end{bmatrix} \otimes I_2 \right) = \begin{bmatrix} 3 \sin(18\tau) & -0.3e^{\cos(18\tau)} \\ 0 & 3\sin(18\tau) \end{bmatrix}, A_5 \left( \begin{bmatrix} f(\tau) \\ \varphi_1(\tau) \end{bmatrix} \otimes I_2 \right) = \begin{bmatrix} -10\cos(18\tau) & 0 \\ 0.5e^{\sin(18\tau)} & -10\cos(18\tau) \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} -0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix}, C_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, C_3 = \begin{bmatrix} 0 & 0.1 \\ -0.1 & 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0.12 \\ 0.1 \end{bmatrix},
\]

\[
C_4 \left( \begin{bmatrix} f(\tau) \\ \varphi_1(\tau) \end{bmatrix} \otimes I_\mu \right) = 0.1 \otimes 0, C_7 \left( \begin{bmatrix} f(\tau) \\ \varphi_2(\tau) \end{bmatrix} \otimes I_\mu \right) = 0.2 \otimes 0.1, C_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, C_7 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix}
\]

(60)
with \( \varphi_1(\tau) = \varphi_2(\tau) = \text{Col} \left[ e^{\sin(18\tau)}, e^{\cos(18\tau)} \right] \). We find out that the system with (60) is stable by applying the toolbox in Breda et al. (2014). The dissipative objective for the system is the minimization of \( \mathcal{L}_2 \) gain \( \gamma \) corresponding to the parameters \( J_1 = -\gamma^{-1}I_2, J_2 = 0_2 \) and \( J_3 = \gamma \) in (30).

Even one assumes the method in Münn et al. (2009) can be extended to handle systems with multiple delay channels, it still cannot be applied here given that \( A_1 \) is unstable. In addition, since \( \varphi_1(\tau) = \varphi_2(\tau) \) here do not satisfy (6), thus the problem of dissipative analysis may not be solved by a simple extension of the corresponding dissipative conditions in Feng & Nguang (2016b) towards CDDS as in Feng (2017), even one assumes that the version of the dissipative conditions with two delay channels is derivable.

Let \( \dot{f}(\tau) = g_{d_2}(\tau) = \text{Col} \left[ 1, \text{Col}_{i=1}^{d_2} \sin 18i\tau, \text{Col}_{i=1}^{d_2} \cos 18i\tau \right] \) and \( \dot{f}(\tau) = g_{d_2}(\tau) \) in (60), these correspond to

\[
\dot{M} = 0 \oplus \begin{bmatrix} O_{d_2} & \bigoplus_{i=1}^{d_2} 18i & O_{d_2} \end{bmatrix}, \quad \ddot{M} = 0 \oplus \begin{bmatrix} O_{d_2} & \bigoplus_{i=1}^{d_2} 18i & O_{d_2} \end{bmatrix}
\]

(61) in line with (6). By using \( \dot{f}(\tau), \ddot{f}(\tau) \) and \( \varphi_1(\tau), \varphi_2(\tau) \) with (12) and (9), we obtain

\[
A_4 = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & -0.3 \end{bmatrix}, \quad A_5 = \begin{bmatrix} O_{2 \times 2d_2+2} & -10 & 0 & 0 & 0 \end{bmatrix}, \quad C_4 = [0.1 \oplus 0 \quad O_{2 \times 4d_2+4}], \quad C_5 = [0.2 \oplus 0.1 \quad O_{2 \times 4d_2+4}]
\]

(62)

which corresponds to the distributed delay terms in (60).

Now apply Theorem 1 with (11) to (60) and solve the minimization program with the objective \( \min \gamma \). The matrices \( \hat{\Gamma}_d, \hat{\Gamma}_s, \hat{E}_d, \hat{E}_s \) are calculated via the function vpaintegral in Matlab, which yields high-numerical precision results. Letting \( d_2 = \delta_2 = 1 \) yields a feasible result with \( \min \gamma = 0.64655 \). It requires 196 decision variables. For \( d_2 = \delta_2 = 2 \), we obtain feasible solutions with \( \min \gamma = 0.32346 \) with 376 variables. Finally, with \( d_2 = \delta_2 = 10 \) the program produces feasible solution with \( \min \gamma = 0.31264 \) with 4120 variables. It is worthy to mention that even with large values \( d_2 = \delta_2 = 10 \), the time required by vpaintegral to calculate the coefficients \( \hat{\Gamma}_d, \hat{\Gamma}_s, \hat{E}_d, \hat{E}_s \) is about 60.4 seconds.

On the other hand, with \( d_2 = \delta_2 = 0 \), \( \varphi_1(\tau) = \varphi_2(\tau) = \text{Col} \left[ \sin(18\tau), \cos(18\tau), e^{\sin(18\tau)}, e^{\cos(18\tau)} \right] \), and non-zeros value of \( d_1 \) and \( \delta_1 \) in (11), which corresponds to the Legendre polynomials approximation scheme with \( \hat{\Gamma}_d = r_1^{-1}D_d, \) and \( \hat{\Gamma}_s = r_5^{-1}D_s, \). The characteristics of the functions in \( \varphi_1(\tau) = \varphi_2(\tau) \) indicate that they might be very difficult to be approximated by polynomial functions. Let \( d_1 = \delta_1 = 15 \) with the corresponding \( A_4, A_5 \) and \( C_4, C_5, C_6 \). In this case, Theorem 1 yields no feasible solutions.

The aforementioned examples have demonstrated the core contribution of the proposed methodology in this paper. Namely, the incorporation of \( f(\cdot) \) and \( \ddot{f}(\cdot) \) in (5) and (22) can provide less conservative result for Theorem 1 compared to only applying polynomial functions.

5. Conclusion

In this paper, the dissipative analysis method for a linear CDDS incorporating distributed delays in state and output equations has been presented and given in terms of LMs. The proposed approach can handle distributed delay with \( \mathcal{L}_2 \) functions kernel and simultaneously incorporating approximation errors in the dissipative analysis conditions. In comparison to existing results in Seuret et al. (2015b) which depends on polynomials functions only to achieve approximations, the proposed method allows one to apply a broader class of elementary functions to approximate the distributed delay terms. As a result, our proposed methods can produce less conservative results in terms of the dissipative analysis conditions derived via constructing a Krasovskii functional with more general structures. Numerical examples have been presented to demonstrate the effectiveness of the proposed methodologies. A potential future direction is to investigate if the hierarchy dissipative results in Theorem 2 here can be derived without the original orthogonality constraints.
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Appendix A. Proof of Lemma 2

Proof. The proof of Lemma 2 is inspired by the results in Seuret et al. (2015b); Feng & Nguang (2016b). To begin with, note that the matrix \( F^{-1} \) is well defined based on the Theorem 7.2.10 in Horn & Johnson (2012) given that \( (L_{\mathbb{R}^d})^2(K; \mathbb{R}^d), J_{\infty} \varpi_{\tau}(r) \varphi_{\tau}(r) \) is an inner product space. Moreover, one can conclude again based on the aforementioned property that

\[
E_d := \int_K \varpi(\tau) e(\tau) e^T(\tau) d\tau = \left[-\Gamma I_\kappa\right] \int_K \varpi(\tau) \left[ f(\tau) \right] \left[ f^T(\tau) \varphi^T(\tau) \right] d\tau \geq 0, \quad (A.1)
\]

given rank \( [-\Gamma I_\kappa] = \kappa \) and the fact that all functions in \( \text{Col}(f(\cdot), \phi(\cdot)) \) are linearly independent. Consequently, \( E_d^{-1} \) is well defined.

Let \( y(\tau) := x(\tau) - F^T(\tau)(F_d \otimes I_n) \int_K \varpi(\theta) F(\theta) x(\theta) d\theta - E^T(\tau) (E_d^{-1} \otimes I_n) \int_K \varpi(\theta) E(\theta) x(\theta) d\theta \), in which the expressions of \( F(\cdot), E(\cdot) \) have been defined in Lemma 2. Note that the integrations in the expression of \( y(\cdot) \) are well defined since \( x(\cdot) \in L_{\mathbb{R}^d}(K; \mathbb{R}^n), \varpi(\cdot) \in L_j(K; \mathbb{T}) \) and \( f(\cdot) \in L_{\mathbb{R}^d}(K; \mathbb{R}^n), \varphi(\cdot) \in L_{\mathbb{R}^d}(K; \mathbb{R}^n) \). As a result, we have \( y(\cdot) \in L_{\mathbb{R}^d}(K; \mathbb{R}^n) \). Now we can derive

\[
j_{\infty} \varpi(\tau) e(\tau) f^T(\tau) d\tau = j_{\infty} \varpi(\tau) [\varphi(\tau) - \Gamma f(\tau)] f^T(\tau) d\tau = j_{\infty} \varpi(\tau) \varphi(\tau) f^T(\tau) d\tau - j_{\infty} \varpi(\tau) f(\tau)[*] d\tau
\]

\[
= j_{\infty} \varpi(\tau) \varphi(\tau) f^T(\tau) d\tau - (j_{\infty} \varpi(\tau) \varphi(\tau) f^T(\tau) d\tau) F_d \int_{\infty} \varpi(\tau) f(\tau) f^T(\tau) d\tau
\]

\[
= j_{\infty} \varpi(\tau) \varphi(\tau) f^T(\tau) d\tau - (j_{\infty} \varpi(\tau) \varphi(\tau) f^T(\tau) d\tau) F_d F_d^{-1} = 0_{\kappa \times d} \quad (A.2)
\]

based on the relations between \( e(\cdot), \varphi(\cdot) \) and \( f(\cdot) \).

Substituting the expression of \( y(\cdot) \) into \( j_{\infty} \varpi(\tau) y^T(\tau) U y(\tau) d\tau \), which is well defined given \( y(\cdot) \in L_{\mathbb{R}^d}(K; \mathbb{R}^n) \), and considering (A.2) yields

\[
j_{\infty} \varpi(\tau) y^T(\tau) U y(\tau) d\tau = j_{\infty} \varpi(\tau) x^T(\tau) U x(\tau) d\tau - 2 j_{\infty} \varpi(\tau) x^T(\tau) U F^T(\tau) d\tau (F_d \otimes I_n) \zeta
\]

\[
+ \zeta^T j_{\infty} \varpi(\tau) (F_d \otimes I_n) T^T(\tau) F_d^T(\tau) (F_d \otimes I_n) d\zeta - 2 j_{\infty} \varpi(\tau) x^T(\tau) U E^T(\tau) d\tau (E_d^{-1} \otimes I_n) \omega
\]

\[
+ \omega^T j_{\infty} \varpi(\tau) (E_d^{-1} \otimes I_n) E(\tau) U E^T(\tau) (E_d^{-1} \otimes I_n) d\omega
\]

where \( \zeta := j_{\infty} \varpi(\theta) F(\theta) x(\theta) d\theta \) and \( \omega := j_{\infty} \varpi(\theta) E(\theta) x(\theta) d\theta \). Now apply (12) to the term \( F(\tau) U \) and \( E(\tau) U \) and consider \( F(\tau) = f(\tau) \otimes I_n \) and \( E(\tau) = e(\tau) \otimes I_n \). Then we have

\[
F(\tau) U = (f(\tau) \otimes I_n) U = (I_d \otimes U)(f(\tau) \otimes I_n) = (I_d \otimes U) F(\tau),
\]

\[
E(\tau) U = (e(\tau) \otimes I_n) U = (I_n \otimes U)(e(\tau) \otimes I_n) = (I_n \otimes U) E(\tau).
\]

Moreover, since \( U = U^T \), it also infers that

\[
UF^T(\tau) = (F(\tau) U)^T = F^T(\tau)(I_d \otimes U), \quad UE^T(\tau) = (E(\tau) U)^T = E^T(\tau)(I_n \otimes U), \quad (A.5)
\]

given \( (X \otimes Y)^T = X^T \otimes Y^T \). Now apply (A.5) to some of the terms in (A.3). It follows that

\[
j_{\infty} \varpi(\tau) x^T(\tau) U F^T(\tau) d\tau (F_d \otimes I_n) \zeta = \zeta^T (I_d \otimes U)(F_d \otimes I_n) \zeta = \zeta^T (F_d \otimes U) \zeta,
\]

\[
j_{\infty} \varpi(\tau) x^T(\tau) U E^T(\tau) d\tau (E_d^{-1} \otimes I_n) \omega = \omega^T (I_d \otimes U)(E_d^{-1} \otimes I_n) \omega = \omega^T (E_d^{-1} \otimes U) \omega.
\]

\[\text{14}\]
Furthermore, by (A.4) and the fact that $F_d = F_d^T$, and $E_d = E_d^T$, we have

$$
\int K (F_d \otimes I_n)^T \varpi(\tau) F(\tau) U F^T(\tau) (F_d \otimes I_n) d\tau = \left( (F_d \otimes U) \int K \varpi(\tau) F(\tau) F^T(\tau) d\tau (F_d \otimes I_n) \right),
\int K (E_d^{-1} \otimes I_n)^T \varpi(\tau) E(\tau) U E^T(\tau) (E_d^{-1} \otimes I_n) d\tau = \left( (E_d^{-1} \otimes U) \int K \varpi(\tau) E(\tau) E^T(\tau) d\tau (E_d^{-1} \otimes I_n) \right). \tag{A.7}
$$

Now, utilizing the definitions $F(\tau) = f(\tau) \otimes I_n$ and $E(\tau) = e(\tau) \otimes I_n$ produces

$$
\int K \varpi(\tau) (f(\tau) \otimes I_n) (f^T(\tau) \otimes I_n) (F_d \otimes I_n) d\tau = \left[ (F_d^{-1} \otimes I_n) (F_d \otimes I_n) = I_{ndt} \right],
\int K \varpi(\tau) (e(\tau) \otimes I_n) (e^T(\tau) \otimes I_n) (E_d^{-1} \otimes I_n) d\tau = \left[ (E_d^{-1} \otimes I_n) (E_d^{-1} \otimes I_n) = I_{nx} \right]. \tag{A.8}
$$

Consequently, (A.7) can be further simplified into

$$
\int K (F_d \otimes I_n)^T \varpi(\tau) F(\tau) U F^T(\tau) (F_d \otimes I_n) d\tau = (F_d \otimes U) \int K \varpi(\tau) F(\tau) F^T(\tau) d\tau (F_d \otimes I_n) = F_d \otimes U,
\int K (E_d^{-1} \otimes I_n)^T \varpi(\tau) E(\tau) U E^T(\tau) (E_d^{-1} \otimes I_n) d\tau = (E_d^{-1} \otimes U) \int K \varpi(\tau) E(\tau) E^T(\tau) d\tau (E_d^{-1} \otimes I_n) = E_d^{-1} \otimes U \tag{A.10}
$$

by using (A.8) and (A.9). Substituting (A.10) into (A.3) and also considering the relations in (A.6) yields

$$
\int K \varpi(\tau) y^T(\tau) U y(\tau) d\tau = \int K \varpi(\tau) x^T(\tau) U x(\tau) d\tau - [s] (F_d \otimes U) \left[ \int K \varpi(\tau) F(\tau) x(\tau) d\tau \right] - [s] (E_d^{-1} \otimes U) \left[ \int K \varpi(\tau) E(\tau) x(\tau) d\tau \right]. \tag{A.11}
$$

Given $U \succeq 0$, it gives (14). This finishes the proof. $\square$

**Appendix B. Proof of Corollary 1**

**Proof.** Note that only the dimension of $f(\cdot)$ is related to $d$, whereas $\kappa$ as the dimension of $\varphi(\cdot)$ is independent from $d$. It is obvious to see that given $f(\cdot)$ containing only orthogonal functions, we have $F_{d+1} = F_d \oplus f_{d+1}$ (See the Definition 1 in Feng & Nguang (2016a)). By using this property, it follows that

$$
\forall d \in N_0, \quad e_{d+1} = \varphi(\tau) - \left[ \int K \varpi(\tau) \varphi(\tau) \left[ f^T(\tau) \quad f_{d+1}(\tau) \right] d\tau \right] (F_d \oplus f_{d+1}) \text{Col} (f(\tau), f_{d+1}(\tau))
= \varphi(\tau) - \left[ \int_K \varpi(\tau) \gamma_{d+1} \text{Col}(f(\tau), f_{d+1}(\tau)) \right], \quad \tag{B.1}
$$

where $\gamma_{d+1} = \int K \varpi(\tau) \varphi(\tau) f_{d+1}(\tau) d\tau$ the index $d$ is added to the symbols $\Gamma$ and $e(\tau)$ in Lemma 2 without causing inconsistencies. Considering (B.1) and (A.1), we have

$$
0 < E_{d+1} = \int K \varpi(\tau) e_{d+1}(\tau) e_{d+1}(\tau) d\tau = E_d - S y (\gamma_{d+1} \int K \varpi(\tau) e_{d+1}(\tau) f_{d+1}(\tau) d\tau) + \left( \int K \varpi(\tau) f_{d+1}(\tau) d\tau \right) \gamma_{d+1} \gamma_{d+1}; \quad \tag{B.2}
$$

By considering (A.2) and the fact that $\int K \varpi(\tau) f_{d+1}(\tau) f(\tau) d\tau = 0_d$ due to the orthogonality among \{f_i(\cdot)\}_{i=1}^{\infty}, we have

$$
O_{\kappa \times (d+1)} = \int K \varpi(\tau) e_{d+1}(\tau) \left[ f^T(\tau) \quad f_{d+1}(\tau) \right] d\tau = \int K \varpi(\tau) (e_{d}(\tau) - \gamma_{d+1} f_{d+1}(\tau)) \left[ f^T(\tau) \quad f_{d+1}(\tau) \right] d\tau
= \int K \varpi(\tau) (e_{d}(\tau) f^T(\tau) \quad f_{d+1}(\tau) e_{d}(\tau) \right) d\tau - \gamma_{d+1} \int K \varpi(\tau) \left[ f_{d+1}(\tau) f^T(\tau) \quad f_{d+1}(\tau) \right] d\tau
= [O_{\kappa \times d} \int K \varpi(\tau) f_{d+1}(\tau) e_{d}(\tau) d\tau] - [O_{\kappa \times d} \int K \varpi(\tau) f_{d+1}(\tau) d\tau \gamma_{d+1}]. \tag{B.3}
$$

As a result, we can conclude $\int K f_{d+1}(\tau) e_{d}(\tau) d\tau = \int K \varpi(\tau) f_{d+1}^2(\tau) d\tau \gamma_{d+1}$. Substituting this equality into (B.2) yields (16) given $\int K \varpi(\tau) f_{d+1}^2(\tau) d\tau > 0$ and $\gamma_{d+1} \gamma_{d+1} \succeq 0$. This finishes the proof. $\square$
Appendix C. Proof of Theorem 2

Proof. The proof here is inspired by the strategy in proving Theorem 8 in Seuret et al. (2015b). Recognize that the matrices in (31) and (33) are indexed by the degree value \( d; \delta \), whereas the inequalities in (32) are independent from them. Note that the index \( \lambda; \delta \) might be automatically attached to the related variables throughout the entire proof.

Without losing generality, we assume \( d_2 = \delta_2 = 0 \) and only consider the situation with a fixed \( d_1 \). With \( d_2 = \delta_2 = 0 \), we have \( d = d_1 + 1 \) and \( \delta = \delta_1 + 1 \) with \( \hat{F}_d = r_1^{-1}D_{d_1} \) and \( \hat{F}_\delta = r_3^{-1}D_{\delta_1} \). Let \( \lambda = (d + \delta) \in \mathbb{N}_0 \) and \( \text{Col}(r_1, r_2) \in \mathcal{F}_\lambda \) with \( \mathcal{F}_\lambda \neq \emptyset \). For a fixed \( d_1 \), we assume that

\[
P_{\lambda+1} := P_{\lambda} \oplus O_n, \quad \eta_{\lambda+1}(t) := \text{Col} \left[ \eta_\lambda(t), \int_{-r_2}^{-r_1} \hat{t}_{\delta_1+1}(\tau)\xi(t+\tau)d\tau \right]. \tag{C.1}
\]

which gives \( \Pi_{\delta_1+1} = W_{\delta_1+1} = \Pi_{\delta_1} \oplus I_\mu \). In addition, it is also true in this case that \( \exists g; h \in \mathbb{R}^{6+d_1+\delta_1} \)

\[
[s]D_{\delta_1+2}G_{d_2}^\delta + [s]D_{\delta_1+1} \otimes (2\delta_1 + 5) \begin{bmatrix} r_3^\delta & 0_{\delta_1+2} \end{bmatrix} = \left( \begin{bmatrix} [s]D_{\delta_1+1} + 0 \end{bmatrix} + (2\delta_1 + 5)gg^\top \right). \tag{C.2}
\]

Now considering the relations in (C.1)–(C.3) with (13), we have

\[
P_{\lambda+1} = \left( P_{\lambda} \oplus r_3^{-1}(2\delta_1 + 3)Q_2 \right) + r_3^{-1}(2\delta_1 + 5)gg^\top \otimes S_2 + r_3^{-2}(2\delta_1 + 4)hh^\top \otimes U_2. \tag{C.4}
\]

Given \( gg^\top \succeq 0 \), \( hh^\top \succeq 0 \) and (32), we can conclude that feasible solution of \( P_{\lambda} > 0 \) under (32) infer the existence of feasible solutions of \( P_{\lambda+1} > 0 \) with \( \delta_1 \leftarrow \delta_1 + 1 \).

Now we start to prove the hierarchical results with respect to \( \Phi_{\lambda} < 0 \) in (33). Note that the previous inequality can be written as

\[
\Phi_{\lambda} = \begin{bmatrix} \Phi & T(\hat{E}_\delta \otimes I_\mu) \\ * & -\hat{E}_\delta \otimes R_2 \end{bmatrix} < 0 \tag{C.5}
\]

where \( \Phi_{\lambda} \) can be obtained easily based on the structure of \( \Phi_{\lambda} \), and

\[
T := \begin{bmatrix} \text{Col} \begin{bmatrix} C_7, \ (S_1 + r_1U_1)A_6A_5, \ O_{2\mu\times(v_2+\delta)\mu}; J_2^\top C_7 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} O_{(m+\mu)\times(v_2+\delta)\mu} \end{bmatrix} \begin{bmatrix} O_{q\times v_2} & I_{v_2} \end{bmatrix}, \tag{C.6}
\]

with \( \Upsilon := \left( [O_{l\times(2\mu+q)} \ I_{l\times\mu_1}] \right)^\top P_{\lambda} \text{Col} \left[ I_{n_1}, O_{(2\mu+q)\times n_1} \right]. \)

Use Schur complement to \( -\hat{E}_\delta \otimes R_2 \) considering the fact that \( \hat{E}_\delta \) is invertible. Then it yields

\[
\Phi - T \left( \begin{bmatrix} \hat{E}_\delta + 1 + 3^{-\delta_1+1} \gamma_{\delta+1} \otimes R_2 \end{bmatrix} \right) \succeq 0 \tag{C.7}
\]

based on the relations in (16). Apply Schur complement again two time separately to (C.7) concludes that (C.7) holds if and only if

\[
\Psi_{\lambda+1} := \begin{bmatrix} \Phi & T(\hat{E}_{\delta+1} \otimes I_\mu) & T(\hat{E}_{\delta+1} \otimes I_\mu) \\ * & -r_3^{-1}(2\delta_1 + 3)R_2 & O_{\mu\times\mu_2} \\ * & * & -\hat{E}_{\delta+1} \otimes R_2 \end{bmatrix} < 0 \tag{C.8}
\]

Now note that without considering the new term related to \( U_2 \) at \( \delta_1 + 1 \), \( \Psi_{\lambda+1} \) contains all the elements to constitute \( \Phi_{\lambda+1} \) via congruence transformations under the assumption (C.1). Considering (C.3), we can conclude that there exist nonsingular matrices \( Z_1 \) and \( Z_2 \) such that

\[
\Phi_{\lambda+1} = [s] \left( Z_1^\top \Psi_{\lambda+1} Z_1 - \left( O_{m+q+3\mu} \oplus [r_3^{-1}(2\delta_1 + 5)gg^\top \otimes U_2] + O_{\mu\mu} \right) \right) Z_2 \tag{C.9}
\]
under the structure of $P_{\lambda+1}$ in (C.1), where $Z_1$ and $Z_2$ can be uniquely constructed according to the structures of $\Phi_{\lambda+1}$ and $\Psi_{\lambda+1}$. Since it is true that $\Phi_{\lambda} \preceq 0$ is equivalent to $\Psi_{\lambda+1} \preceq 0$ in (C.8) with $P_{\lambda+1}$ in (C.1), it shows $[\varepsilon] \Phi_{\lambda+1} Z_1 \preceq 0$ given $Z_1$ is nonsingular. Furthermore, because of the fact that $g g^\top \succeq 0$ and (32) and $Z_2$ is invertible, one can derive $\Phi_{\lambda+1} \preceq 0$ by (C.9). As a result, we have proved that given (32), the feasible solutions of $\Phi_{\lambda} \preceq 0$ with (C.1) infers the existence of the feasible solutions of $\Phi_{\lambda+1} \preceq 0$ with $\delta_1 \leftarrow \delta_1 + 1$.

It is important to point out that (57) is also valid considering $d_2 = \delta_2 = 0$ with the substitution $d_1 \leftarrow d_1 + 1$ with a fixed $\delta_1$, which can be easily proved considering what we have presented in Appendix C together with the application of congruence transformations. Finally, since the proof procedures in Appendix C are based on the orthogonality of $\ell_{\delta_1} (\cdot)$ and $\delta_{\delta_1} (\cdot)$, hence the corresponding proof for (57) with $d_1 = \delta_1 = 0$ and assuming $f (\cdot)$ and $f (\cdot)$ to contain orthogonal functions as stated in Theorem 2, can be easily constructed. This finishes the proof of this Theorem.

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