Two Function Families and Their Application to Hankel Transform of Heat Kernel

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Abstract

In this paper, we study an asymptotic expansion of the heat kernel for a Laplace operator on a smooth Riemannian manifold without a boundary at enough small values of the proper time. The Seeley–DeWitt coefficients of this decomposition satisfy a set of recurrence relations, which we use to construct two function families of a special kind. Using these functions, we find the expansion of a heat kernel for the inverse Laplace operator for an arbitrary dimension of space. We show that the new functions have some important properties. For example, we can consider the Laplace operator on the function set as a shift one. Also we describe various applications useful in theoretical physics and, in particular, we find the decomposition of Green functions in terms of new functions.

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1 Introduction

The heat kernel method first appeared in [11] and since then plays an important role in theoretical and mathematical physics. The range of applications of this approach is very wide and currently includes thousands of works. For numerous examples, we refer the reader to the works [2, 3], which detail history of the issue and various applications.

As is known, a closed formula for the heat equation solution in general case does not exist due to technical difficulties. In this connection, as a rule, the main work with the heat kernel is focused on studying its asymptotic expansion both near zero [4–7] and at infinity [8–10]. As an example, we can recall finding the asymptotics of an operator trace in the case of non-zero mass [4, 9, 11].

On manifolds without boundary, a natural ansatz for finding the asymptotic expansion at small values of the proper time is the series (see formula (3)) in which the functions $a_n(x, y)$ are called Seeley–DeWitt coefficients. The point is that the Seeley–DeWitt coefficients obey the system of recurrence differential equations [4]. This property is very remarkable and is used to find the trace parts of the coefficients [12], as well as in different proofs, for example, in the Atiyah–Patodi–Singer theorem [13].

In our work we use the recurrence relations in a slightly different context. It turns out that, based on the Seeley–DeWitt coefficients, one can define a family of functions of a special form, and due to the property described above the new functions are transformed into each other by the action of the Laplace operator. Thus we obtain a set of chains (see (29), (30), (36), and (70)), which are closely related to the construction of asymptotic decompositions and various other applications.

The main purpose of our work is to study the connection between the following four objects: the heat kernel, the Seeley–DeWitt coefficients, the family of functions, and a fundamental solution of the Laplace operator. We show the relation between asymptotic expansion of the heat kernel and a Green function in terms of the new family of functions.

We believe that our work will be useful in theoretical and mathematical physics. Examples include loop calculations [17], a recently proposed approach to studying the fermion number [18, 19], as well as application in anomaly theory, in which the same integrals arise as in Section 6. The field of use may also include a study of integer powers of the Laplace operator.

Let us shortly describe the content of the paper. In Section 2 we formulate the background, give basic information about the heat kernel method, and briefly introduce the Hankel transform within the framework of the work with an exponential operator (see (16) and (17)).

Then, in Section 3 we define a family of $\Psi$-functions, the main building blocks of which are Seeley–DeWitt coefficients, and prove the key Theorem 1. It turns out that the functions can be connected not only by the action of the Laplace operator, but also by a partial derivative with respect to a parameter. We also consider two special cases of $\Psi$-functions for integer and half-integer index values.

In Section 4 we formulate and prove Theorem 2 on the relation between the asymptotic expansion of the heat kernel and a special family of fundamental solutions for the odd-dimensional case. In other words, we found the Hankel transform of the heat kernel in terms of well-studied Seeley–DeWitt coefficients. We also provide some examples of limit cases, when a mass parameter goes to zero. We discuss degeneracy and special cases.

Section 5 is devoted to the similar study for the even-dimensional case and contains several parts. First, in Section 5.1 we give a derivation of the asymptotic expansion of the fundamental solution of the Laplace operator near the diagonal in terms of $\Psi$-functions. Then, we introduce an additional family of $\Phi$-functions in terms of which we are going the represent the Hankel transform of the heat kernel. Further, in Section 5.2 we study the Hankel transform and prove Theorems 3, 5, and 6. The presence of several theorems is a consequence of the fact that the heat kernel can be represented as two parts, each of which solves the heat equation. In Section 5.3 we discuss limits and special cases.

In Section 6 we discuss different applications, such as late-time asymptotic behavior, Green function representation, cut-off regularization, and integral calculation. Conclusion section consists of some comments and remarks.
2 Problem statement

Basic concepts of heat kernel method. Let $M$ be a $d$-dimensional compact Riemannian manifold without a boundary. Points of the manifold we denote by letters $x, y, z$. Then, let us consider an open convex set $U \subset M$, so we suppose that $x, y, z \in U$. This means that all further calculations are performed locally in $U$. The smooth metric tensor equals to $g^{\mu \nu}(x)$ locally, where $\mu, \nu \in \{1, \ldots, d\}$. Moreover, let $V$ be a Hermitian vector bundle over $M$, so by $B_{\mu}(x)$ we denote smooth components of a Yang–Mills connection 1-form.

Let us formulate a problem, the solution $A(x)$ of which is called the heat kernel:

$$
\begin{aligned}
&\left\{ \begin{array}{l}
(\partial_\tau + A(x) + m^2)K(x, y; \tau) = 0; \\
K(x, y; 0) = g^{-1/2}(x)\delta(x - y),
\end{array} \right.
\end{aligned}
\tag{1}
$$

where $m$ is a positive constant mass parameter and $A(x)$ is a Laplace operator with smooth coefficients. In local coordinates it has the following form

$$
A(x) = -g^{-1/2}(x)D_{\mu\nu}g^{1/2}(x)g^{\mu\nu}(x)D_{\mu\nu} - v(x).
\tag{2}
$$

Here $D_{\mu\nu} = \partial_{\mu\nu} + B_{\mu}(x)$ is the covariant derivative, $v(x)$ is a smooth potential, and $g(x)$ is the metric tensor determinant. We know from the general theory that the Laplace operator has a discrete spectrum and can have only one accumulation point at $+\infty$, see [20]. Moreover, we will further assume that all eigenvalues of $A + m^2$ are positive.

Let us fix the form of the solution for problem (1) by using the choice of an ansatz. For the small enough values of the proper time $\tau$ the heat kernel is given by the following series

$$
K(x, y, \tau) = (4\pi\tau)^{-d/2} \Delta^{1/2}(x, y)e^{\sigma(x, y)/2\tau - \tau m^2} \sum_{k=0}^{+\infty} \tau^k a_k(x, y),
\tag{3}
$$

where $a_k(x, y), k \geq 0$, are the Seeley–DeWitt coefficients, $\sigma(x, y)$ is the Synge’s world function, and $\Delta(x, y)$ is the Van-Vleck–Morette determinant, which is defined by the formula

$$
\Delta(x, y) = (g(x)g(y))^{-1/2} \det \left( -\frac{\partial^2 \sigma(x, y)}{\partial x \partial y} \right).
\tag{4}
$$

For convenience, we introduce some useful notations

$$
\sigma_{\mu}(x, y) = \partial_{\nu}\sigma(x, y) \quad \text{and} \quad \sigma^{\mu}(x, y) = g^{\mu\nu}(x)\sigma_\nu(x, y),
\tag{5}
$$

then the following relations hold

$$
g^{\mu\nu}(x)\sigma_{\mu}(x, y)\sigma_{\nu}(x, y) = 2\sigma(x, y)
\tag{6}
$$

and

$$
\sigma^{\nu}(x, y)\partial_{x^\nu}\Delta^{1/2}(x, y) = \frac{1}{2} \left( d - g^{-1/2}(x)\partial_{x^\nu}g^{1/2}(x)g^{\mu\nu}(x)\partial_{x^\nu}\sigma(x, y) \right) \Delta^{1/2}(x, y).
\tag{7}
$$

Using the latter properties, after substituting the expansion (3) for the heat kernel into the heat equation (1), we obtain the following recurrence relations for the coefficients

$$
\begin{aligned}
&\sigma^{\mu}(x, y)D_{\mu}a_0(x, y) = 0, \quad a_0(x, x) = 1; \\
&(k + 1 + \sigma^{\mu}D_{\mu})a_{k+1}(x, y) = -\Delta^{-1/2}(x, y)A(x)\Delta^{1/2}(x, y)a_k(x, y), \quad k \geq 0.
\end{aligned}
\tag{8}
$$
Hankel transform. Let \( \lambda \) and \( \phi_\lambda \) be an eigenvalue and an eigenfunction of the operator \( A + m^2 \), respectively. It means that \( (A + m^2)\phi_\lambda = \lambda \phi_\lambda \). Using the assumptions described above we can write \( \lambda > 0 \). In this case the heat kernel has the following form

\[
K(x, y; \tau) = \sum_\lambda e^{-\tau \lambda} \phi_\lambda(x) \phi_\lambda^*(y).
\]

We introduce a Green function for the operator \( A(x) \) by the equality

\[
(A(x) + m^2)G(x, y) = \delta(x, y), \quad \text{where } \delta(x, y) = g^{-1/2}(x)\delta(x - y).
\]

Then we consider a problem similar to (1), but with the inverse operator

\[
\partial_\tau N(x, y; \tau) + \int_M d^4 z G(x, z)g^{1/2}(z)N(z, y; \tau) = 0, \quad N(x, y; 0) = 1(x, y).
\]

However, the solution of such problem requires the global considerations, while we are going to work only locally in \( U \). In this connection, we turn to a solution of an alternative problem by acting the operator \( A(x) \) on the left hand side of relation (11). Then the problem reduces to the form

\[
(\partial_\tau (A(x) + m^2) + 1)N(x, y; \tau) = 0, \quad N(x, y; 0) = 1(x, y).
\]

It is quite easy to show that the formula holds

\[
N(x, y; \tau) = \sum_\lambda e^{-\tau/\lambda} \phi_\lambda(x) \phi_\lambda^*(y).
\]

Let us note, that due to the orthogonality of the eigenfunctions we can investigate operator transforms separately for each eigenvalue. Using the exponential integration formula

\[
\lambda^k = \int_{\mathbb{R}_+} ds \frac{s^{k-1}}{(k-1)!} e^{-s/\lambda}
\]

and two properties of the Bessel function \( J_1(s) \)

\[
-\frac{s}{2} J_1(s) = \sum_{k=1}^{+\infty} \frac{(-s^2/4)^k}{k!(k-1)!}, \quad \int_{\mathbb{R}_+} ds J_1(s) = 1,
\]

after the summation of (13) by \( k \) we obtain

\[
e^{-\lambda \tau} = 1 + \sum_{k=1}^{+\infty} \int_{\mathbb{R}_+} \frac{ds}{s} \frac{(-s^2/4)^k}{k!(k-1)!} e^{-s/\lambda} = - \int_{\mathbb{R}_+} \frac{ds}{s} \sqrt{s} J_1(2\sqrt{s})(e^{-s/\lambda} - 1).
\]

Then, using the set of formulae (9), (13), and (16), we get the following relation between straight and inverse heat kernels

\[
K(x, y; \tau) = - \int_{\mathbb{R}_+} ds \sqrt{s} J_1(2\sqrt{s})(N(x, y; s) - 1(x, y)).
\]

Such kind of relation is quite remarkable, because it gives the way to investigate Green function decompositions. The rest of the paper is devoted to the study of relation (17) in terms of a family of new functions. Especially we note that the transformation with the kernel \( J_1(s) \) is the Hankel transform. It is very well known [14] and sometimes it is used to find solutions for differential equations [15].

Also in the following we are going to omit the arguments \( x, y \in U \) of the functions mentioned above, except in cases where it is necessary. Therefore, we have the following reductions:

\[
A = A(x), \quad a_j = a_j(x, y), \quad \Delta = \Delta(x, y), \quad \sigma = \sigma(x, y), \quad \mathbf{1} = \mathbf{1}(x, y),
\]

\[
K(\tau) = K(x, y; \tau), \quad N(s) = N(x, y, s).
\]
3 Family of $\Psi$-functions

**Theorem 1.** Let $\alpha \in \mathbb{C}$, $\omega \in \mathbb{R}_+$, and $g^\alpha = \{g_k^\alpha\}_{k=0}^{+\infty}$ be a set of complex numbers, depending on the parameter $\alpha$. Then we define the following special function of two variables $x, y \in U$, corresponding to the operator $A$, by the formula

$$\Psi^\alpha_{\omega}[g^\alpha] = \Delta^{1/2} \sum_{k=0}^{+\infty} \frac{(-1)^k g_k^\alpha}{2k} \omega^{k-\alpha} a_k. \quad (20)$$

Let us also require that the following additional relations hold

$$g_k^\alpha = g_k^\alpha(k - \alpha), \quad g_k^{\alpha+1} = -2g_k^{\alpha-1}, \quad k \geq 1. \quad (21)$$

Then we have

$$-2\partial_x \Psi^\alpha_{\omega}[g^\alpha] = \Psi^\alpha_{\omega+1}[g^{\alpha+1}], \quad (22)$$

$$\Delta^k \sigma^\alpha D_{\sigma} \Delta^k \Psi^\alpha_{\omega}[g^\alpha] = -A\Psi^\alpha_{\omega-1}[g^{\alpha-1}] - \alpha \Psi^\alpha_{\omega}[g^\alpha] + \frac{\omega}{2} \Psi^\alpha_{\omega+1}[g^{\alpha+1}]. \quad (23)$$

If we additionally assume, that the inequality $\text{Re}(\alpha) < k + d/2 - 1$ is satisfied for all $k$, for which $g_k^\alpha \neq 0$, then we have

$$A \Psi^\alpha_{\omega}[g^\alpha] = (d/2 - \alpha - 1)\Psi^\alpha_{\omega+1}[g^{\alpha+1}], \quad (24)$$

where $\sigma$ is the Syngue’s world function.

**Proof.** The first relation (22) follows from the differentiation by the parameter $\sigma$ and applying formulae (21). The second one can be verified by using formulae (8), (21), and (22), and using the equality $\kappa \omega^{k-\alpha} = \omega^{-\alpha} \partial_\omega \omega^\alpha \omega^{k-\alpha}$.

Then, let us apply the operator $A$ to the product $\Delta^{1/2} \sigma^\beta a_n$ with $\beta > 1 - d/2$ and $n \geq 0$. We get

$$A \left( \Delta^{1/2} \sigma^\beta a_n \right) = -\sigma^\beta \Delta^{1/2} (n + 1 + \sigma^\alpha D_{\alpha}) a_{n+1} - \beta \sigma^{\beta-1} \Delta^{1/2} (d + 2(\beta - 1) + 2\sigma^\alpha D_{\alpha}) a_n, \quad (25)$$

where we have used the relations from (8). Substituting this equality and making a change of variables, after application of the relations from (21) we obtain the last statement of the theorem. \hfill $\square$

**The case $\alpha \in \mathbb{Z}$.** Let us consider an example of functions from definition (20), where the parameter $\alpha$ takes only integer values. It is quite obvious that the set of coefficients $g^\alpha = \{g_k^\alpha\}_{k=0}^{+\infty}$ is uniquely defined by the relations from (21) and one non-zero value.

**Lemma 3.1.** Let $g_0^\alpha = 1$, then, taking into account the equalities from (21), we have

$$g_k^\alpha = \frac{(-2)^p}{\Gamma(k-p+1)} \text{ for all } k \geq 0 \text{ and } p \in \mathbb{Z}. \quad (26)$$

**Definition 1.** Let $p \in \mathbb{Z}$, $\omega \in \mathbb{R}_+$ and $g^p = \{g_k^p\}_{k=0}^{+\infty}$, where $g_k^p$ is from Lemma 3.1, then we define the set of functions, corresponding to the Laplace operator $A$, by the equality

$$\Psi^p_{\omega}[g^p] = \Psi^p_{\omega}[g^\alpha], \quad p \in \mathbb{Z}. \quad (27)$$

As it is noted earlier, the Seeley–DeWitt coefficient $a_k$ is defined for $k \geq 0$. Let us expand this definition to the instance $k \in \mathbb{Z}$ by the equality $a_k = 0$ for $k < 0$. In this case, the relations (8) are satisfied for all integer values of the parameter $k$.

**Lemma 3.2.** Functions from Definition 1 have alternative representation in the form

$$\Psi^p_{\omega} = \Delta^{1/2} \sum_{k=0}^{+\infty} \frac{(-1)^k \omega^k}{k!2^{k+1}} a_{p+k}, \quad p \in \mathbb{Z}. \quad (28)$$
Proof. The statement follows from definition (20), Lemma 3.1 and the remark that $g^p_k = 0$ for $k < p$. \qed

It follows from the last lemma that the function $\Psi^\sigma_p$ does not contain negative powers of $\omega$ in its series representation. This means that we can consider any value of $p$ from the set $\mathbb{Z}$.

**Corollary 3.3.** Let $d$ be even, then from Theorem 7 we have $A \Psi^\sigma_{d/2-1} = 0$.

Two important conclusions can be drawn from Theorem 7. Firstly, in the case of odd dimensions none of the functions $\Psi^\sigma_p$ lies entirely in the kernel of the Laplace operator. This leads to a doubly infinite chain of functions on which the Laplace operator is a shift operator. Hence, we have

$$\ldots \xrightarrow{A} \frac{\Psi_{k-1}^\sigma}{\Gamma(d/2 - k + 1)} \xrightarrow{A} \frac{\Psi_k^\sigma}{\Gamma(d/2 - k)} \xrightarrow{A} \frac{\Psi_{k+1}^\sigma}{\Gamma(d/2 - k - 1)} \xrightarrow{A} \ldots,$$

where $k \in \mathbb{Z}$ and $d$ is odd.

Secondly, in the case of even dimensions, it is important to consider the Corollary 3.3, from which it follows that the chain is interrupted, and we get two separate pieces of doubly infinite sequence

$$\ldots \xrightarrow{A} \Psi^\sigma_{d/2-2} \xrightarrow{A} \Psi^\sigma_{d/2-1} \xrightarrow{A} 0 \quad (30)$$

$$\Psi^\sigma_{d/2} \xrightarrow{A} -\Psi^\sigma_{d/2+1} \xrightarrow{A} 2\Psi^\sigma_{d/2+2} \xrightarrow{A} \ldots \quad (31)$$

Formula (22) also allows the visual representation in the form of a sequence. However, in this case, the dimension does not matter and we get

$$\ldots \xrightarrow{-2\partial_x} \Psi^\omega_{k-1} \xrightarrow{-2\partial_y} \Psi^\omega_k \xrightarrow{-2\partial_y} \Psi^\omega_{k+1} \xrightarrow{-2\partial_y} \ldots, \quad (32)$$

where $k \in \mathbb{Z}$.

**The case $\alpha \in \mathbb{Z} + 1/2$.** Let us consider the second useful example and extend Definition 1.

**Lemma 3.4.** Let $g^0_{1/2} = \sqrt{2\pi}$, then, taking into account the equalities from (21), we have $g^p_k = (-1)^k \Gamma(p - k) 2^p$ for all $k \geq 0$ and $p \in \mathbb{Z} + 1/2$.

**Definition 2.** Let $p \in \mathbb{Z} + 1/2$, $\omega \in \mathbb{R}$, and $g^p = \{g^p_k\}_{k=0}^{\infty}$, where $g^p_k$ is from Lemma 3.4. Then we extend Definition 7 to the set of half-integer indices by the relation

$$\Psi^\sigma_p = \Psi^\sigma_p[g^p], \quad p \in \mathbb{Z} + 1/2. \quad (33)$$

Taking into account definition (20), we conclude, that negative powers of $\sigma$ may occur in the case of half-integer indices. Hence, according to Theorem 7 the relation (21) holds only for $\alpha < d/2 - 1$. In other cases, we need to use generalized functions with the support at the point $y$. Anyway, equality (24) holds for the points $x \in U \setminus \{y\}$.

**Lemma 3.5.** Let $d$ be odd, then, given (18), we obtain $A \Psi^\sigma_{d/2-1}/(4\pi)^{d/2} = 1$.

**Proof.** Due to the presence of the relation (24) we have to check the equality only for the singular part with $\sigma^{1-d/2}$. Then, from the formula

$$-g^{-1/2}(x)\partial_x g^{1/2}(x)g^{\mu\nu}(x)\partial_x \frac{\Delta^{1/2}(x, y)}{(4\pi)^{d/2}} \sqrt{\pi} \sigma^{1-d/2}(x, y) = g^{-1/2}(x)\delta(x - y) \quad (35)$$

the statement of the lemma follows. \qed

Thus, in the case of half-integer index values we can also construct a sequence on which the Laplace operator is a shift operator. Such a chain terminates on one side and has the following form

$$\ldots \xrightarrow{A} \frac{1}{2} \Psi^\sigma_{d/2-3} \xrightarrow{A} \Psi^\sigma_{d/2-2} \xrightarrow{A} \Psi^\sigma_{d/2-1} \xrightarrow{A} (4\pi)^{d/2} 1. \quad (36)$$

The sequence, on which the derivative by the parameter $\sigma$ is a shift operator, is similar to (32) with the only change that $k \in \mathbb{Z} \rightarrow k \in \mathbb{Z} + 1/2$.
4 Odd-dimensional case

4.1 The main result

Theorem 2. Let $K(\tau)$ be the heat kernel expansion \((\ref{3})\) on an odd-dimensional manifold for small enough values of $\tau$, $\Psi$-functions be from Definitions \((\ref{4})\) and \((\ref{5})\) and $\omega > 0$. Then, under the general conditions of Section \((\ref{2})\) we have

$$K(\tau)e^{-\frac{\omega^2}{\pi \tau}} = - \int_{\mathbb{R}^+} \frac{\tau}{s} J_1(2\sqrt{\tau s}) \left( \sum_{n=1}^{\infty} \frac{f_n(s)}{(4\pi)^{d/2}} \Psi_{d/2-n} + \sum_{n\in\mathbb{Z}} \frac{g_n(s)}{4\pi} \Psi_{(d-1)/2-n} \right),$$  \hspace{1cm} (37)

where

$$f_n(s) = \sum_{k=0}^{n} \frac{(-s)^k (-m^2)^{n-k}}{k!(n-k+1)!}, \quad g_n(s) = \frac{\pi(-1)^{n-1} m^{n-1} s}{\Gamma(n+1/2)} F_1(1/2 - n, 2; -s/m^2),$$ \hspace{1cm} (38)

and $F_1$ is the confluent hypergeometric function of the first kind.

Proof. We are going to start with the representation \((\ref{3})\) and some useful remarks. Let us note that we can investigate only one-dimensional case, because a transition to the $d$-dimensional one can be achieved by applying the operator $-\frac{1}{(2\pi)^{-1}} \partial_{\omega}$ several times. Namely, $(d-1)/2$ times. It follows from formula \((\ref{24})\) of Theorem \((\ref{1})\) and the equality

$$\frac{1}{(4\pi \tau)^{d/2}} e^{-\omega^2/2\tau} = \left( -\frac{1}{2\pi} \frac{\partial}{\partial \omega} \right) \frac{1}{(4\pi \tau)^{d/2}} e^{-\omega^2/2\tau}. \hspace{1cm} (39)$$

Hence, without loss of generality we consider only $d = 1$. The statement can be achieved in several steps. First of all let us rewrite the left hand side as

$$\frac{\Delta^{1/2}}{(4\pi \tau)^{1/2}} e^{-\omega^2/2\tau} e^{-m^2\tau} = \sum_{n=0}^{\infty} \tau^n a_n = \frac{\Delta^{1/2}}{(4\pi \tau)^{1/2}} e^{-\omega^2/2\tau} \sum_{n=0}^{\infty} a_n (-\partial_{m^2})^n e^{-m^2\tau}. \hspace{1cm} (40)$$

Then we note one valuable relation

$$\frac{1}{(4\pi \tau)^{1/2}} e^{-\frac{\omega^2}{2\tau}} e^{-m^2\tau} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\Delta^{1/2}}{2\pi} \sum_{n=0}^{\infty} a_n (-\partial_{m^2})^n \int_{\mathbb{R}} \frac{\Delta^{1/2}}{2\pi} \frac{1}{\rho^2 + m^2} e^{-s/(\rho^2 + m^2)} \tau,$$ \hspace{1cm} (41)

that follows from applying the Fourier transform.

Let us transform the exponential $\exp(-\rho^2/m^2 \tau)$ in formula \((\ref{11})\) by applying relation \((\ref{16})\) with the parameter $\lambda = \rho^2 + m^2$. As a result of which, we get

$$K(\tau)e^{-\frac{\omega^2}{\pi \tau}} = - \int_{\mathbb{R}^+} \frac{\tau}{s} J_1(2\sqrt{\tau s}) \left[ \Delta^{1/2} \sum_{n=0}^{\infty} a_n (-\partial_{m^2})^n \right] \int_{\mathbb{R}} \frac{d\rho}{\rho^2 + m^2} e^{-s/(\rho^2 + m^2)} - 1 \right]. \hspace{1cm} (42)$$

Thereupon, let us note that the exponential in the last formula can be expanded in the series, because $m > 0$ and all integrals converge. Therefore, using the following relations

$$e^{-s/(\rho^2 + m^2)} = 1 + \sum_{k=1}^{\infty} \frac{(-s)^k}{k!(k-1)!} (-\partial_{m^2})^{k-1} \frac{1}{\rho^2 + m^2}, \quad \frac{1}{2\pi} \int_{\mathbb{R}} \frac{d\rho}{\rho^2 + m^2} = \frac{1}{2m} e^{-\sqrt{2m^2}}, \hspace{1cm} (43)$$

the expression in the square brackets can be rewritten in the form

$$\Delta^{1/2} \sum_{k=1}^{\infty} \frac{(-s)^k}{k!(k-1)!} (-\partial_{m^2})^{k-1} \left[ \sum_{n=0}^{\infty} a_n (-\partial_{m^2})^n \frac{1}{2m} e^{-\sqrt{2m^2}} \right]. \hspace{1cm} (44)$$
Actually, we need to investigate only the term with \( k = 1 \) in formula (44), because other terms can be obtained by differentiating with respect to the parameter \( m \). We see that our task is a combinatorial one. In order to find a formula, we perform the following procedure. We set the parameter \( \omega \) equal to the Synge’s world function \( \sigma \). Then, our construction will satisfy an additional equation (see below). After that we find a formula for the special case. And then, knowing this, we set the parameter \( \sigma \) equal to \( \omega \) and check, that the obtained formula gives the initial expansion.

Let us note that we have obtained the series by integer degrees of the parameter \( s \). It equals to \( \check{N}(s) = N(s) - 1 \), see formulae (22) and (17). Hence, it satisfies the equation \( (\partial_s (A + m^2) + 1) \check{N}(s) = -1 \). If we look for a solution in the form of a series \( \check{N}(s) = \sum_{k=1}^{\infty} (-s)^k G_k / k! \), we will find a system of recurrence relations for \( G_k \). Luckily, we are interested only in the first one, that has the following form \( (A + m^2)G_1 = 1 \). For this reason, we can find the solution as a series by \( \Psi \)-functions with unknown coefficients. Let us use the fact and take an ansatz in the form

\[
\Delta^{1/2} \sum_{n=0}^{+\infty} a_n (-\partial_m)\sigma^n \frac{1}{2\sqrt{m^2}} e^{-\sqrt{2m^2} \sigma} = \sum_{n=1}^{+\infty} \frac{b_n}{4\pi} \Psi_{1/2-n}^\sigma + \sum_{n \in \mathbb{Z}} \frac{c_n}{\sqrt{4\pi}} \Psi_n^\sigma.
\]  

(45)

Then, applying the operator \( A + m^2 \), see equation (24), and equating the answer to \( 1 \) we get two recurrence relations

\[
b_{n+1} = -\frac{n^2}{n-1} b_n \text{ for } n \geq 1, \quad \text{and} \quad c_{n+1} = -\frac{n^2}{n+1/2} c_n \text{ for } n \in \mathbb{Z}.
\]  

(46)

It means we need to calculate only two numbers \( b_1 \) and \( c_0 \). Moreover, we can limit ourselves to calculating the coefficients near \( a_0 \). Therefore, using the decomposition \( \exp(-\sqrt{2m^2} \sigma)/2m = 1/2m - \sqrt{\sigma/2} + \ldots \), we get \( b_1 = 1 \) and \( c_0 = \sqrt{\pi/m^2} \). Hence, we obtain

\[
b_n = \frac{(-m^2)^{n-1}}{\Gamma(n)}, \quad c_{n+1} = \frac{-1}{{2\pi}^{1/2}} \frac{(m^2)^{n-1/2}}{\Gamma(n+1/2)}.
\]  

(47)

Then, using two equalities

\[
(-\partial_m)^{k-1} \sum_{n=1}^{+\infty} \frac{(-m^2)^{n-1}}{\Gamma(n)} \Psi_{1/2-n}^\sigma = \sum_{n=k}^{+\infty} \frac{(-m^2)^{n-k}}{\Gamma(n-k+1)} \Psi_{1/2-n}^\sigma,
\]  

(48)

\[
(-\partial_m)^{k-1} \sum_{n \in \mathbb{Z}} \frac{\pi(-1)^n (m^2)^{n-1/2}}{\Gamma(n+1/2)} \Psi_{n}^\sigma = \sum_{n=k}^{+\infty} \frac{\pi(-1)^{n-k+1} (m^2)^{n-k+1/2}}{\Gamma(n-k+3/2)} \Psi_{n}^\sigma,
\]  

(49)

we can rewrite formula (44) in the following form

\[
\frac{1}{\sqrt{4\pi}} \sum_{k=1}^{+\infty} \frac{(-s)^k}{k! (k-1)!} \left( \sum_{n=k}^{+\infty} \frac{(-m^2)^{n-k}}{\Gamma(n-k+1)} \Psi_{1/2-n}^\sigma + \sum_{n \in \mathbb{Z}} \frac{\pi(-1)^{n-k+1} (m^2)^{n-k+1/2}}{\Gamma(n-k+3/2)} \Psi_{n}^\sigma \right).
\]  

(50)

Further, changing the order of summation and using the relation

\[
\sum_{k=1}^{+\infty} \frac{(s/m^2)^k}{k! (k-1)! \Gamma(n-k+3/2)} = \frac{s/m^2}{\Gamma(n+1/2)} \text{J}_{1/2-n, 2; -s/m^2},
\]  

(51)

we obtain the statement of the theorem. Let us note it again, that if we put the parameter \( \sigma \) equal to \( \omega \) in the \( \Psi \)-functions and expand (50) in a series, we will get exactly (44).

\[\square\]

4.2 Examples

Let us consider some informative calculations. We put \( \omega = \sigma \). The key fact in the proof of Theorem 2 was the inequality \( m > 0 \), thanks to which we have used the series expansion (43). Now we are going to
study the limit transition \( m \to +0 \) for a special case. Indeed, the functions from (58) have the following behavior near zero

\[
\lim_{m \to +0} f_n(s) = \frac{(-s)^n}{n!(n-1)!} \quad \text{for} \quad n \geq 0, \quad \text{and} \quad \lim_{m \to +0} g_n(s) = \frac{\pi(1-n)s^{n-1/2}}{\Gamma(n+1/2)\Gamma(n+3/2)} \quad \text{for} \quad n \in \mathbb{Z}.
\]  

(52)

This leads to the appearance of negative powers of the variable \( s \) in the second equality for \( n < 1 \). Hence, the decomposition at zero is destroyed. This means that either we have to use a different ansatz or impose additional restrictions. Let us choose the second way and consider the case, when \( a_n = \delta_{n0} \). Also, we put \( \Delta^{1/2} = 1 \) for convenience.

Firstly, we consider one-dimensional situation. In the case we can compute the integral (41) for \( m = 0 \) explicitly. It is equal to

\[
\frac{1}{2\pi} \int_{\mathbb{R}} d\rho e^{i\rho\sqrt{2}s}(e^{-s/\rho^2} - 1) = Q_1(s, \sigma) + Q_2(s, \sigma),
\]

(53)

where

\[
Q_1(s, \sigma) = s\left(\frac{\sigma}{2}\right)^{\frac{1}{2}}\text{F}2\left(\frac{3}{2}, 2; \frac{3\sigma}{2}\right) = \frac{1}{(4\pi)^{1/2}} \sum_{k=1}^{+\infty} \frac{(-s)^k}{k!} \left(\frac{\sigma}{2}\right)^{k-1/2} \frac{\Gamma(1/2 - k)}{\Gamma(k)},
\]

(54)

and

\[
Q_2(s, \sigma) = -\left(\frac{s}{\pi}\right)^{\frac{1}{2}}\text{F}2\left(\frac{1}{2}, 3; \frac{3\sigma}{2}\right) = -\frac{\sqrt{\pi}}{2} \sum_{k=0}^{+\infty} \frac{(s\sigma/2)^k}{k!\Gamma(k + 1/2)\Gamma(k + 3/2)}.
\]

(55)

Then, using the last formulae it is easy to check that

\[
\sum_{k=1}^{+\infty} f_k(s)\Psi_m^{\sigma/2-k} \bigg|_{a_n = \delta_{an, m=0}} = Q_1(s, \sigma), \quad \sum_{k=1}^{+\infty} f_k(s)\Psi_m^{\sigma/2-k} \bigg|_{a_n = \delta_{an, m=0}} = Q_2(s, \sigma).
\]

(56)

The last functions and their sum are depicted on the Figure 1 for \( \sigma = 1/2 \). We see that \( Q_1(s, 1/2) \) increases to \(+\infty\), when \( s \to +\infty \). Therefore, we can not integrate it. We should investigate the sum \( Q_1 + Q_2 \), because it goes to zero. The last sum is depicted on the Figure 2 for different values of the parameter \( \sigma \). We see that the function has damped oscillations near the abscissa axis. Actually, the damping, when \( s \to +\infty \), can be obtained from formula (58) explicitly by using the integration by parts.

![Figure 1: The functions \( Q_1(s, \sigma) \), \( Q_2(s, \sigma) \), and their sum, for fixed \( \sigma = 1/2 \), where \( 1e7 = 10^7 \).](image1)

![Figure 2: The sum \( Q_1(s, \sigma) + Q_2(s, \sigma) \) for different values of the parameter \( \sigma = 1, 1/2, 1/4 \).](image2)
correct, since the oscillations become very large (see Figure 2). So, permutation of the limit $\sigma \to +0$ and the integration by $s$ is not correct.

The situation $d > 1$ can be achieved by differentiation with respect to the parameter $\sigma$. So we obtain for odd values of the parameter $d$ the following formula

$$Q(d, s, \sigma) = \left( -\frac{\partial_\sigma}{2\pi} \right)^{d+1} [Q_1(s, \sigma) + Q_2(s, \sigma)]$$

$$= \frac{1}{(4\pi)^{d/2}} \sum_{k=0}^{+\infty} \frac{(-s)^k}{k!} \frac{(\sigma/2)^k}{\Gamma(k)} \frac{(d/2 - k)}{\Gamma(d/2 - k)}$$

$$- \left( -\frac{1}{2\pi} \right)^{d+1} \frac{\sqrt{\pi} s}{2} \sum_{k=0}^{+\infty} \frac{(s/2)^k \sigma^{k-(d-1)/2}}{\Gamma(k - (d + 1)/2) \Gamma(k + 3/2)}.$$  \hspace{1cm} (57)

The results for different odd values of the parameter $d$ are depicted on the Figure 3.

5 Even-dimensional case

5.1 Family of $\Phi$-functions

Let us find the asymptotic expansion of a solution for problem $AG = 1$. We denote it by $G = G(x, y)$. For the odd dimensional case the asymptotic expansion of the solution at $x \sim y$ is dictated by Lemma 3.5, that is $G = \Psi^d_{d/2-1}/(4\pi)^{d/2}$. In the case of even $d$ the asymptotic expansion is more intricate and contains a logarithmic component. This is due to the fact that the chain (30) is interrupted.

Let $d$ be even, then it is known, see [16], that for $x \sim y$ the asymptotic expansion has the form

$$G = \frac{\Delta^{1/2}}{(4\pi)^{d/2}} \sum_{k=0}^{d/2 - 2} \Gamma(d/2 - k - 1) (\sigma/2)^{k+1-d/2} a_k - \frac{[\ln \sigma]}{(4\pi)^{d/2}} \Psi^d_{d/2-1} + Q,$$  \hspace{1cm} (60)

where $Q = Q(x, y)$ is a smooth part.

**Lemma 5.1.** Under the conditions described above the part $Q$ from (60) has the form

$$Q = \frac{\Delta^{1/2}}{(4\pi)^{d/2}} \sum_{k=0}^{+\infty} \frac{(-\sigma/2)^k H_{k+1}}{(k + 1)!} a_{d/2+k} + \frac{\Theta_1}{(4\pi)^{d/2}},$$  \hspace{1cm} (61)
where $H_k$ is the $k$-th harmonic number and $\Theta_1 = \Theta_1(x, y)$ is a solution of $A\Theta_1 = \Psi_{d/2}^\sigma$.

**Proof.** Let us substitute the ansatz (60) into the equality (10) and use the relations (7), (8), (24), and (62), to obtain

$$A_0[\ln \sigma] = \frac{1}{\sigma}[A_0\sigma] + \frac{2}{\sigma}, \text{ where } A_0(x) = -g^{-1/2}(x)\partial_x g^{1/2}(x)g^{\mu\nu}(x)\partial_\mu\nu.$$  \hspace{1cm} (62)

Then, after some reductions, we obtain the equality

$$AQ = \frac{\Delta^{1/2}}{(4\pi)^{d/2}} \sum_{k=0}^{+\infty} \frac{(-1)^k\sigma^k}{(k+1)!^2} (d/2 + 2k + 1 + \sigma^\mu D_\mu) a_{d/2+k}.$$  \hspace{1cm} (63)

The same result can be achieved after calculation of $AQ$ by using representation (61). This fact proves the validity of the lemma statement.

**Theorem 3.** Let $\omega \in \mathbb{R}_+$, $g^k = \{g^k_n\}_{n=0}^{+\infty}$, $k \in \mathbb{Z}$, are sets of complex numbers. Then we define a function of two variables $x, y \in U$, corresponding to the operator $A$, by the formula

$$\Phi_\omega^k[i^k] = -[\ln \omega] \Psi_\omega^k + \Delta^{1/2} \sum_{n=0}^{+\infty} (\omega/2)^{n-k} g^k_n a_n.$$  \hspace{1cm} (64)

We assume that the numbers $g^k_n$ are defined by the following equalities

$$g^k_n = \frac{(-1)^{n-k}H_{n-k}}{(n-k+1)!^2}, \text{ for } n \geq k;$$

$$g^k_n = \frac{(-1)^{n-k}H_{n-k}}{(n-k)!}, \text{ for } n < k,$$  \hspace{1cm} (65)

where in the first equality we have used the analytic continuation of the Gamma function and harmonic number. Let $d$ be even. Then the functions $\Phi_\omega^k[i^k] = \{\Phi^k_n\}_{n=0}^{+\infty}$ for $k \geq 1$ satisfy the relations

$$A(\Phi_{d/2-k-1}^\sigma[i^{d/2-k-1}] + H_{k-1} \Psi_{d/2-k-1}^\sigma) = k(\Phi_{d/2-k}^\sigma[i^{d/2-k}] + H_{k-1} \Psi_{d/2-k}^\sigma),$$  \hspace{1cm} (66)

$$-2\partial_\omega \Phi_{k}^\sigma[i^k] = \Phi_{k+1}^\sigma[i^{k+1}] \text{ for all } k \in \mathbb{Z}.\hspace{1cm} (67)$$

**Proof.** The proof of relation (65) for $k \geq 1$ can be achieved by explicit calculations and using equalities (7), (8), (24), (25), and (62). The second formula follows from Lemma 5.1. The last relation can be obtained by differentiation with taking into account the equality

$$(k-n)g^k_n + (-1)^{n-k}g^k_n = \Phi_{d/2-k}^\sigma[i^{d/2-k}],$$  \hspace{1cm} (68)

that follows from formula (26) and definition (61). \hfill \Box

**Definition 3.** Let $d$ be even, $\omega \in \mathbb{R}_+$, $k \in \mathbb{Z}$, and the set of coefficients $g^k = \{g^k_n\}_{n=0}^{+\infty}$ be from Theorem 3. Then we define a function, corresponding to the Laplace operator $A$, by formula

$$\Phi_\omega^k = \Phi_\omega^k[i^k].$$  \hspace{1cm} (69)

Therefore, we have obtained two new sequences:

$$\Phi_{d/2-k}^\sigma[i^{d/2-k}] \Rightarrow \frac{1}{2} A_1(\Phi_{d/2-k}^\sigma + \Psi_{d/2-k}^\sigma) \Rightarrow A_1(\Phi_{d/2-k}^\sigma + \Psi_{d/2-k}^\sigma) \Rightarrow A_1(4\pi)^{d/2} - \Psi_{d/2-k}^\sigma,$$  \hspace{1cm} (70)

$$\Phi_{d/2-k}^\sigma[i^{d/2-k}] \Rightarrow -2\partial_\omega \Phi_{d/2-k}^\sigma[i^{d/2-k}] \Rightarrow -2\partial_\omega \Phi_{d/2-k}^\sigma[i^{d/2-k}] \Rightarrow -2\partial_\omega \Phi_{d/2-k}^\sigma[i^{d/2-k}] \Rightarrow \ldots.$$  \hspace{1cm} (71)

**Corollary 5.2.** Let $d$ be even, $k \geq 1$, and $g^k = \{g^k_n\}_{n=0}^{+\infty}$ be a set of coefficients, such that the relations (65) hold. Then we have $\Phi_{d/2-k}^\sigma[i^{d/2-k}] = \Phi_{d/2-k}^\sigma + \sum_{n=0}^{k-1} \alpha_k^k \Psi_{d/2-k-n}^\sigma$, where $\alpha_k^k \in \mathbb{C}$.\hfill \Box
Proof. It is enough to consider the difference \( \Phi_{d/2-k}^{\sigma} \{ g^{d/2-k} \} - \Phi_{d/2-k}^{\sigma} \) and use two facts. Firstly, negative degrees of \( \sigma \) are fixed uniquely by formula (64). Secondly, the difference contains only positive degrees of \( \sigma \). Hence, the statement of the corollary follows from Theorem 1.

Corollary 5.3. Let \( d \) be even, \( \omega \in \mathbb{R}_+ \), and \( k \in \mathbb{Z} \), then we have

\[
\Delta^{\frac{d}{2}} \sigma^n D_{\mu} \Delta^{-\frac{d}{2}} \Phi_k^\omega = -A \Phi_{k-1}^\sigma - k \Phi_k^\sigma + \frac{\omega}{2} \Phi_{k+1}^\omega - \Psi_k^\omega.
\]  

(72)

Proof. It is enough to use formulae (8), (23), (67), and the property \( \hat{g}_n^{\sigma} = \hat{g}_n^{k+1} \) that follows from (64).

Corollary 5.4. Let \( d \) be even, \( G \) and \( \Theta_1 \) be from Lemma 5.7 then we have \( \Phi_{d/2-1}^\sigma + \Theta_1 = (4\pi)^{d/2}G \).

To conclude this section, let us give an example of an explicit construction of the \( \Theta_1 \).

Lemma 5.5. Let the group of functions \( \{ f_k = f_k(x, y) \}_{k=0}^{+\infty} \) satisfy the following system of equations

\[
f_0 = 0, \ a_{d/2+k} + (k + 1 + \sigma^n D_{\mu})f_{k+1} = -\Delta^{-1/2} A \Delta^{1/2} f_k, \ k \geq 0.
\]

(73)

Then the series \( V = \Delta^{1/2} \sum_{k=0}^{+\infty} (\sigma/2)^k f_k/k! \) solves the equation \( AV = \Phi_{d/2}^\sigma \).

Proof. The statement of the lemma can be verified by explicit calculations with use of the equations.

5.2 The main result

Lemma 5.6. Let \( d \) be even and \( \tau \) be small enough, then the series \( \Omega^\omega(\tau) = \sum_{n=0}^{+\infty} \tau^n \Psi_{d/2+n}^\omega \) for \( \omega = \sigma \) gives a solution for the system

\[
(\partial_\tau + A) \Omega^\sigma(\tau) = 0, \ \Omega(0) = \Psi_{d/2}^\sigma.
\]

(74)

Proof. The statement follows from equality (23).

Corollary 5.7. Let \( \omega \in \mathbb{R}_+ \) and \( \tau \) be small enough. Then, under the conditions of Lemma 5.6 we have

\[
\frac{\Omega^\omega(\tau)}{(4\pi)^{d/2}} = \frac{\Delta^{1/2}}{(4\pi)^{d/2}} \sum_{n=d/2}^{+\infty} a_n \tau^n \sum_{k=0}^{n-d/2} \frac{(-\omega/2\tau)^k}{k!}.
\]

(75)

Proof. The statement follows from combination of Lemmas 3.2 and 5.6.

Theorem 4. Let \( \omega \in \mathbb{R}_+ \), \( K^\omega(\tau) = K(\tau) e^{-\omega(\tau - \sigma)/2} - e^{-\tau\omega} \Omega^\omega(\tau)/(4\pi)^{d/2} \), see formulae (3) and (76), on the even-dimensional manifold for small enough values of \( \tau \), and \( \Phi \)-functions and \( \Psi \)-functions be from Definitions 1 and 3. Then, under the general conditions of the Section 2, we have

\[
\tilde{K}^\omega(\tau) = -\int_{\mathbb{R}_+} ds \sqrt{s} C(2\sqrt{s} \tau) \left( \sum_{n=1}^{+\infty} \frac{f_n(s)}{(4\pi)^{d/2}} \Phi_{d/2-n}^\omega + \sum_{n=1}^{+\infty} \frac{g_n(s)}{(4\pi)^{d/2}} \Psi_{d/2-n}^\omega \right),
\]

(76)

where

\[
f_n(s) = \sum_{k=1}^{n} \frac{(-s)^k (-m^2)^{n-k}}{k!(k-1)! \Gamma(n-k+1)}
\]

(77)

and

\[
g_n(s) = \sum_{k=1}^{n} \frac{(-s)^k (-m^2)^{n-k} (-2\gamma - \ln(m^2/2) + \cdots)}{k!(k-1)! \Gamma(n-k+1)} + \cdots
\]

(78)

where \( \gamma \) is the Euler–Mascheroni constant.
Proof. First of all we are going to simplify the problem. Namely, due to the presence of the equalities \(22\) and \(67\) we can investigate only two dimensional case. Indeed, the situations \(d > 2\) can be obtained by using differentiation by the parameter \(\omega\). Hence, without loss of generality we work only with \(d = 2\).

Let us start from the right hand side of \((76)\). Using formulae \(43\) and \((75)\), \(K^\omega(\tau)\) can be rewritten in the form
\[
\tilde{K}^\omega(\tau) = \frac{\Delta^{1/2}}{4\pi r} e^{-\tau m^2} \sum_{n=0}^{+\infty} \tau^n a_n \left( e^{-\omega/2\tau} - \sum_{k=0}^{n-1} \frac{1}{k!} \left( -\omega \right)^k \right). \tag{80}
\]

Then, we introduce an integration operator \(S(\omega)\) and its inverse one, such that \(S^{-1}(\omega)S(\omega) = 1\), by the formulae
\[
S(\omega) = -\frac{1}{2} \int d\omega \bigg|_{\omega=-\omega}, \quad S^{-1}(\omega) = -2 \frac{d}{d\omega}. \tag{81}
\]

In this case the formula for \(\tilde{K}^\omega(\tau)\) has the following form
\[
\tilde{K}^\omega(\tau) = \frac{\Delta^{1/2}}{4\pi r} e^{-\tau m^2} \sum_{n=0}^{+\infty} a_n \left( S^n(\omega) e^{-\omega/2\tau} \right), \tag{82}
\]
or, using relation \(16\) and two-dimensional Fourier transform for \((4\pi)^{-1} \exp(-\omega/2\tau - m^2\tau)\), as it was done in the one-dimensional case earlier, we get
\[
\tilde{K}^\omega(\tau) = -\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_{\mathbb{R}_0^+} d\rho d\delta \rho^2 e^{i\rho \sqrt{\omega} \cos(\theta)} \left( e^{-s/(\rho^2 + m^2)} - 1 \right) = 2 \int_{\mathbb{R}^+} d\rho \rho J_0(\sqrt{2\rho^2 \omega}) \left( e^{-s/(\rho^2 + m^2)} - 1 \right). \tag{84}
\]

Now we need to transform the expression in the square brackets to the \(\Psi(\Phi)\)-functions representation. For this purpose we can use the series expansion for the exponential, because all integrals converge due to \(m > 0\). Let us use the following two relations
\[
e^{-s/(\rho^2 + m^2)} - 1 = \sum_{k=1}^{+\infty} \frac{(-s)^k}{k!(k-1)!} \left( -\frac{\partial}{\partial m^2} \right)^{k-1} \frac{1}{\rho^2 + m^2}, \tag{85}
\]
and
\[
\int_{\mathbb{R}^+} d\rho \rho J_0(\sqrt{2m^2\omega}) = K_0(\sqrt{2m^2\omega}) = \sum_{j=0}^{+\infty} \frac{H_j - \gamma - \ln \sqrt{m^2\omega}/2}{j!} \left( \frac{m^2\omega}{2} \right)^j, \tag{86}
\]
where \(K_0\) is the modified Bessel function of second kind. Thus, we need to find new representation for the expression
\[
\sum_{k=1}^{+\infty} \frac{(-s)^k}{k!(k-1)!} \left( -\frac{\partial}{\partial m^2} \right)^{k-1} \left[ \sum_{n=0}^{+\infty} a_n S^n(\omega) \left( \sum_{j=0}^{+\infty} \frac{H_j - \gamma - \ln \sqrt{m^2\omega}/2}{j!} \left( \frac{m^2\omega}{2} \right)^j \right) \right]. \tag{87}
\]

Indeed, we need to work only with the coefficient \(k = 1\), because other can be obtained by differentiation by the parameter \(m^2\). Here we must give arguments similar to those that were in Theorem \(22\), so we give the link to the paragraph after formula \(43\) to avoid repetition. Taking into account the previous reasoning, analyzing the terms from the sum leads to the ansatz \(\sum_{n=1}^{+\infty} b_n (\Phi_{1-n}^q + H_{n-1} \Psi_{1-n}^q)/(4\pi) + \sum_{n=1}^{+\infty} c_n \Psi_{1-n}^q/(4\pi)\), that we should choose. Moreover, we can use the fact, that the coefficient for \(k = 1\) gives \(4\pi 1 - \Psi_1^q\) after applying the operator \(A + m^2\) to it. Hence, we obtain two sets of recurrence relations
\[
b_{n+1} = -\frac{m^2}{n} b_n \quad \text{and} \quad c_{n+1} = -\frac{m^2}{n} c_n \quad \text{for} \ n \geq 1. \tag{88}
\]
The last equalities mean, that we need to find only initial coefficients \( b_1 \) and \( c_1 \). They follow from formula \((28)\), definition \((63)\), and the coefficients from \((87)\) corresponding to the term \( n = j = 0 \). Hence, we get \( b_1 = 1 \) and \( c_1 = -2\gamma - \ln(m^2/2) \), from which the main formulae follow

\[
b_n = \frac{(-m^2)^{n-1}}{(n-1)!} \quad \text{and} \quad c_n = \frac{(-m^2)^{n-1}}{(n-1)!} [-2\gamma - \ln(m^2/2)] \quad \text{for} \quad n \geq 1. \tag{89}
\]

Substituting this into \((87)\) and differentiating by the parameter \( m^2 \), as it was done in the one-dimensional case, we obtain

\[
\frac{1}{4\pi} \sum_{k=1}^{+\infty} \frac{(-s)^k}{k!(k-1)!} \left( \sum_{n=k}^{+\infty} \frac{(-m^2)^{n-k}}{\Gamma(n-k+1)} \left[ \Phi^q_{1-n} - \Phi^q_{1-n}(2\gamma + \ln(m^2/2)) \right] - \sum_{n=1}^{+\infty} \frac{(-m^2)^{n-k} H_{n-k} \Phi^q_{1-n}}{\Gamma(n-k+1)} \right), \tag{90}
\]

where we have used the following relations

\[
\frac{\partial_m \Gamma(n + \epsilon + 1)}{\Gamma(n + \epsilon + 1)} = H_{n+\epsilon} = \gamma, \tag{91}
\]

\[
\frac{\partial_m^k \ln(t)}{\partial t} \bigg|_{t=0} = \frac{\partial_m}{\partial t} \bigg|_{t=0}^{\epsilon+\epsilon} = \frac{\partial_m}{\partial t} \bigg|_{t=0}^{\epsilon+\epsilon} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)} \ln(x) + H_n - H_{n-k}. \tag{92}
\]

Then, changing the summation order and using the following formulae

\[
\frac{H_{n-k}}{\Gamma(n-k + 1)} = -\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \frac{e^{-\gamma\epsilon}}{\Gamma(n-k+1+\epsilon)}, \tag{93}
\]

\[
\sum_{n=1}^{+\infty} \frac{t^k H_{n-k}}{k!(k-1)! \Gamma(n-k+1)} = -\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \frac{e^{-\gamma\epsilon}}{\Gamma(n+\epsilon+1)} \Phi(1 - n - \epsilon; 2; -t), \tag{94}
\]

we obtain the theorem statement. Let us note it again, that if we put \( \sigma = \omega \) and expand \((90)\) in a series, we will get exactly \((87)\). \(\blacksquare\)

**Theorem 5.** Let \( \omega \in \mathbb{R}_+^2, \Omega^\omega(\tau) = \sum_{n=0}^{+\infty} \tau^n \Phi^\omega_{d/2+n} \) be from Lemma \((5.6)\) and \( \tau \) be small enough. Then, under the general conditions of the Section \((3)\) we have

\[
e^{-\tau m^2} \Omega^\omega(\tau) = -\int_{\mathbb{R}_+} ds \sqrt{\frac{\tau}{s}} J_1(2\sqrt{ts}) \left( \sum_{n=0}^{+\infty} g_n(s) \Phi^\omega_{d/2+n} \right),
\]

where

\[
g_n(s) = (-\partial_{m^2})^n \left[ e^{-s/m^2} - 1 \right]. \tag{95}
\]

**Proof.** The statement of the theorem follows from the explicit expression for \( \Omega(\tau) \), relation

\[
\tau^n \exp(-\tau m^2) = (-\partial_{m^2})^n \exp(-\tau m^2) \quad \text{for} \quad n \in \mathbb{N}, \tag{97}
\]

and formula \((16)\) for \( \lambda = m^2 \). \(\blacksquare\)

**Theorem 6.** Let \( K(\tau) \) be the heat kernel expansion \((3)\) on the even-dimensional manifold for small enough values of \( \tau, \omega \in \mathbb{R}_+^2, \) and \( \Phi \)-functions and \( \Psi \)-functions be from Definitions \((4)\) and \((5)\). Let also the functions \( \{f_n(s), g_n(s)\}_{n=1}^{+\infty} \) be from Theorem \((4)\) and \( \{g_n(s)\}_{n=0}^{+\infty} \) be from Theorem \((5)\). Then, under the general conditions of the Section \((3)\) we have

\[
K(\tau) e^{-\frac{\tau m^2}{2}} = -\int_{\mathbb{R}_+} ds \sqrt{\frac{\tau}{s}} J_1(2\sqrt{ts}) \left( \sum_{n=1}^{+\infty} \frac{f_n(s)}{(4\pi)^{d/2}} \Phi^\omega_{d/2-n} + \sum_{n \in \mathbb{Z}} \frac{g_n(s)}{(4\pi)^{d/2}} \Psi^\omega_{d/2+n} \right). \tag{98}
\]

**Proof.** The statement of the theorem follows from sum of formulae \((76)\) and \((95)\). \(\blacksquare\)
5.3 Examples

Here we are going to continue example considerations for \( \omega = \sigma \). In the previous section we have derived the Hankel transform of heat kernel expansion (3) in the even-dimensional case, see Theorems 3, 5, and 8. An important detail in the proof was the positive mass \( m > 0 \). It is thanks to this fact that we were able to use the series expansion (3), because integral (8) converges.

However, the transition \( m \to +0 \) to the massless case is not an easy task, because the transform of \( \Omega(\tau) \), see Theorem 5, contains the exponential \( \exp(-s/m^2) \). Hence, permutation of the limit \( m \to +0 \) and integration by \( s \) is not quite correct. Therefore, to study the limiting case we need to either use different decomposition or consider a special type restriction. Let us choose the second way and investigate the case \( a_n = \delta_{n0} \) and \( \Delta^{1/2} = 1 \). Moreover, firstly we study the simplest situation \( d = 2 \).

The condition \( a_n = \delta_{n0} \) means, that \( \Omega(\tau) \) equals to zero. So, the functions \( \{g_n(s)\}_{n \geq 0} \) are zero ones. Then, we can find the asymptotic behavior for other functions

\[
\lim_{m \to +0} f_n(s) = \frac{(-s)^n}{n!(n-1)!}, \quad \lim_{m \to +0} g_{-n}(s) = \frac{(-s)^n}{n!(n-1)!}(\ln(2) - \ln(s) - 3\gamma + H_n + H_{n-1}), \quad n \geq 1,
\]

where we have used (7) and (8), and the relation

\[
_1 \text{F}_1(1 - n - \epsilon, 2, -s/m^2) = (s/m^2)^{n-1+\epsilon} \Gamma(n+1+\epsilon), \quad \text{for} \ m \to +0.
\]

Then, using the summation with \( \Phi \)- and \( \Psi \)- functions, we get

\[
\sum_{n=1}^{+\infty} \frac{f_n(s) \Phi_{-n}^2}{4\pi} \bigg|_{a_n = \delta_{n0}, m = 0} = \frac{1}{4\pi} \sum_{n=1}^{+\infty} \frac{s^n(\sigma - H_{n-1})}{(n-1)!(n-1)\ln!} (\sigma/2)^{n-1},
\]

\[
\sum_{n=1}^{+\infty} \frac{g_{-n}(s) \Psi_{-n}^2}{4\pi} \bigg|_{a_n = \delta_{n0}, m = 0} = \frac{1}{4\pi} \sum_{n=1}^{+\infty} \frac{s^n(\ln(s) - \ln(2) + 3\gamma - H_n - H_{n-1})}{(n-1)!(n-1)\ln!} (\sigma/2)^{n-1}.
\]

Let us note that the last sums converge, but they have bad behaviour at infinity, because they increase exponentially. However, their sum \( Q(2, s, \sigma) \) is a good function, because it goes to zero at infinity. The same situation has been studied in the odd-dimensional case. This is a rather remarkable property of series that the logarithmic growth of \( \ln(s) \) can be compensated by introducing a harmonic numbers. Also we should note, that the last result in the restricted case can be obtained by explicit calculation of (34), see Appendix.

The case for \( d > 2 \) can be obtained by differentiating \(-\partial_\sigma/2\pi\) with respect to the parameter \( d/2 - 1 \) times. The result function \( Q(d, s, \sigma) \), when \( d \) is even, equals to

\[
Q(d, s, \sigma) = (-\partial_\sigma/2\pi)^{d/2-1} Q(2, s, \sigma)
\]

\[
= \frac{(-1)^{d/2-1}}{(4\pi)^{d/2}} \sum_{n=1}^{+\infty} \frac{s^n[\ln(s) - H_{n-1} - H_n - H_{n-d/2} + \ln(\sigma) + 3\gamma - \ln(2)]}{(n-1)!n!\Gamma(n-d/2+1)} (\sigma/2)^{n-d/2},
\]

and depicted on the Figure 3 for different values of the parameter \( d \). The last formula complements the definition for odd-dimensional case (57). Especially, we note that the decreasing of \( Q(2, s, \sigma) \) for \( s \to +\infty \) can be shown explicitly integrating (8) by parts.

6 Applications

Late-time asymptotics. Let us give an example of applications of new function families to study the asymptotic behaviour. We are going to consider even-dimensional case. This is due to the fact that sequence (30) terminates. Hence, we can construct two different solutions of the problem (1) and split the heat kernel asymptotics (3) into two parts.
Lemma 6.1. Let $d$ be even, $\omega \in \mathbb{R}^+$, $K(\tau)$ be from (3), and $\Omega^\omega(\tau)$ be the solution from Lemma 5.6. Then for sufficiently large values of the proper time $\tau$ we have

$$K^\omega(\tau) = K(\tau)e^{-\frac{\pi \tau}{\omega}} - \frac{\Omega^\omega(\tau)}{(4\pi)^{d/2}} = \frac{1}{(4\pi)^{d/2}} \sum_{n=1}^{+\infty} \frac{1}{\pi^n} \Psi_{d/2-n}^\omega.$$

Proof. It is easy to verify, that for small values of $\tau$ we can use asymptotic expansions for $K(\tau)$ and $\Omega^\omega(\tau)$ from formula (3) and Lemma 5.6. Then, using series representation for exponential from (3), we are making sure that $K^\omega(\tau)$ contains only negative degrees of $\tau$. Therefore we can move to the large values. Hence, collecting coefficients at the same degrees of $\tau$, we obtain the statement of the lemma.

Green function expansion. We can extract several useful corollaries from the Theorems 2 and 6. Indeed, in both statements, on the right side of the equality, we have the Hankel transform of a function. So this part is not presented in this paper.

Lemma 6.2. Under the conditions of Theorems 2 and 6 the following equalities hold

$$G = \frac{1}{(4\pi)^{d/2}} \left( \sum_{n=1}^{+\infty} \frac{(-m^2)^{-n-1}}{\Gamma(n)} \Psi_{d/2-n}^\sigma + \sum_{n \in \mathbb{Z}} \pi (-1)^n (m^2)^{n-1/2} \Gamma(n + 1/2) \Psi_{(d-1)/2-n}^\sigma \right), \quad \text{when } d \text{ is odd, and} \quad (106)$$

$$G = \frac{1}{(4\pi)^{d/2}} \left( \sum_{n=1}^{+\infty} \frac{(-m^2)^{-n-1}}{\Gamma(n)} \left[ \Psi_{d/2-n}^\sigma - \Psi_{d/2-n}^\sigma (2\gamma + \ln(m^2/2) - H_{n-1}) \right] + \sum_{n=1}^{+\infty} \frac{\Gamma(n)}{m^{2n}} \Psi_{d/2-1+n}^\sigma \right), \quad (107)$$

when $d$ is even.

Proof. It is enough to use inverse Hankel transform to equality (17), apply the Taylor expansion to the left hand side and use the results from Theorems 2 and 6.

Note that the first few terms of the asymptotic expansion in the four-dimensional case were written out in the paper [4]. Let us draw the attention that the last expansions should be considered as the asymptotic ones for $x \sim y$. Convergence issues should be discussed separately. They depend on the smoothness of the Laplace operator coefficients and the mass value. Some arguments on this subject can be found in the articles [4, 9, 11].

Cut-off regularization. The representations for (\Phi-)\Psi-functions obtained above, see formula (20) and definition (33), can be useful in different investigations with a cut-off regularization. For example, in loop calculations, the main building block for diagrams is a Green function. As a rule, it is represented as an integral of $K(x,y;\tau)$ from 0 to $+\infty$ by the variable $\tau$. If we want to explore the region $x \sim y$, we will face a problem, because the integral at $x = y$ diverges near zero. It means that we should use a regularization.

One of the convenient types of regularization is the cut-off one. It can be achieve by introducing a special parameter $\Lambda$ by the following substitution:

$$\sigma \to \sigma_\Lambda = \begin{cases} \sigma, & \text{for } \sigma \geq 1/2\Lambda^2; \\ 1/2\Lambda^2, & \text{for } \sigma < 1/2\Lambda^2. \end{cases} \quad (108)$$

After such procedure in the region $\sigma < 1/2\Lambda^2$ we have the exponential $\exp(-\sigma/\tau)$ instead of $\exp(-\sigma^2/\tau)$. Hence, the integral converges for $x = y$ near $\tau = 0$. Moreover, it can be verified, that the regularized Green function goes to $G(x, y)$ when $\Lambda \to +\infty$ in the sense of generalized functions.
Let us give some examples of regularized fundamental solutions, described above:

$$G \rightarrow G_\Lambda = G\big|_{\sigma \rightarrow \sigma_\Lambda},$$  \quad (109)$$
where the substitution affects only the parameter $\sigma$ as an independent one. We emphasize that the regularization does not change dependence of the Seeley–DeWitt coefficients on the variables $x$ and $y$.

**Integral calculations.** We have seen above that sometimes we need to compute integrals by variable $\tau$. For example, when finding the Green function. However, this is not the only example. Similar integrals arise in quantum field theory and in the theory of anomalies. In this regard, it makes sense to study constructions of the following form

$$\int_{\mathbb{R}_+} d\tau \tau^{-k/2} K(x, y; \tau).$$  \quad (110)$$
where it is implied that the integral exists on the upper limit.

Let us consider the following procedure. We introduce Euclidean coordinates $\hat{x}$ and $\hat{y}$ from smooth convex domain $\hat{U} \subset \mathbb{R}^k$. Then we define the distance between them as $\hat{\sigma} = |\hat{x} - \hat{y}|/2$. Hence, a heat kernel $\hat{K}(\hat{x}, \hat{y}; \tau)$ for the ordinary Laplace operator has the following form $(4 \pi \tau)^{-k/2} \exp(\hat{\sigma}/2 \tau)$. Now we note that

$$\int_{\mathbb{R}_+} d\tau \tau^{-k/2} K(x, y; \tau) = \lim_{\hat{y} \to \hat{x}} (4 \pi)^{k/2} \int_{\mathbb{R}_+} d\tau \hat{K}(\hat{x}, \hat{y}; \tau) \hat{K}(x, y; \tau)$$  \quad (111)$$

$$= (4 \pi)^{k/2} G,$$  \quad (112)$$
where $G$ is the Green function from Lemma 6.2 for $(d+k)$-dimensional case, in which the Seeley–DeWitt coefficients, the Van-Vleck–Morette determinant, and Synge’s world function are from Green function for $d$-dimensional case.

The latter formula means that we can study integrals (110) by using the transition into a space of higher dimension. In the last formula we have used $d \to d + k$. This trick, with further use of the representations for $(\Phi)$-$\Psi$-functions, allows us to find divergences and simplify calculations, see the section with $\ln \det(A)$ from [17].

**Integral transforms of the heat kernel.** Actually, the result of Theorems 2 and 6 and Lemma 5.1 allows us to find some integral transformations for the heat kernel. Let us assume, that an integral transformation is on the $\mathbb{R}_+$, and that a kernel $C(\tau)$ of the integral transform has a convergent Taylor decomposition $C(\tau) = \sum_{n=0}^{+\infty} c_n \tau^n$. Let us also remind the fact that the heat kernel $K(\tau)$ and the Green function $G$ from Lemma 6.2 depend on the mass parameter $m$. So we can write the following chain of relations

$$\int_{\mathbb{R}_+} d\tau C(\tau) K(\tau) = \sum_{n=0}^{+\infty} c_n \int_{\mathbb{R}_+} d\tau \tau^n K(\tau) = \sum_{n=0}^{+\infty} c_n (\partial_m^n)^n G,$$  \quad (113)$$
where we changed the order of the sum and the integral under the assumption that we have convergence. Otherwise, we assume that we are working with asymptotic series. Let us also note that if the integral transform contains negative powers of the variable $\tau$, then we can use the derivatives with respect to the parameter $\sigma$, considering it as an independent one.

7 Conclusion

In our paper, we have defined two families of new functions $(\Psi$ and $\Phi)$, and also have used them to find the Hankel transform of the heat kernel [8]. We believe that such calculations are very useful and important because they lead to a set of non-trivial relations, that allow us to study the transformations of the heat kernel. For example, the usual integration leads to the Green function. Other applications are listed in the Section [8].
Bessel functions

Separately, we pay attention to the fact that we perform all calculations locally, when \( x, y \in U \subset M \). Moreover, we assume that \( x \sim y \). This is necessary in order to be able to uniquely construct the Synge’s function \( \sigma(x, y) \) and the Seeley–DeWitt coefficients \( a_n(x, y) \). This raises interesting questions. In what cases is it possible to abandon locality and conduct reasoning globally? What restrictions should be imposed on the manifold \( M \)?

Let us assume that we were able to continue the function definitions in \( U \) in such a way, that all the function properties would be preserved. This means that we have abandoned the condition \( x \sim y \). Consider the following integral

\[
\eta_{p,n} = \int_U d^d z \Psi_n^p(x, z) \sqrt{g(z)} \Psi_n^p(z, y). \quad (114)
\]

Of course we have no ability to calculate the integral explicitly, but we can give some discussions for special cases. Let \( d \) be odd and \( p = d/2 - k \), where \( k \in \mathbb{N} \), then if we apply the operator \( A \) several times we get the delta-function under the integral. It means that \( A^k \eta_{p,n} = \Psi_n^p \). Another example is the following. Let \( d \) be even and \( p = d/2 - k \), where \( k \in \mathbb{N} \), then we obtain \( A^k \eta_{p,n} = 0 \). The same reasoning are possible for \( \Phi \)-functions.

Here we make some obvious remarks on the spectrum of the operator \( A + m^2 \). In Section 2, we assumed that such an operator has exclusively positive eigenvalues. If we have a non-positive value of the spectral parameter, then the integral along the \( \mathbb{R}^+ \) will diverge. Therefore, we must add the projector on the positive component of the spectrum in all our calculations.

If we assume that the manifold \( M \) has a boundary, then the heat kernel must satisfy boundary conditions. Such an assumption changes the asymptotic expansion and it should have additional correction terms. This means that the case of manifolds with a boundary must be considered separately and is not within the scope of this paper.

Let us make some comment on dimension of physical quantities. In this paper, we have used such formulae as \( \ln(m^2) \) or \( \ln(\sigma) \). We meant that all the values we have are dimensionless. If we consider the case when \( [\sigma] = L^2 \) and \( [m] = L^{-1} \), where \( L \) has the dimension of length, this is not a problem, because all calculations contain logarithm sums. Hence, we have only dimensionless combinations of the quantities.

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8 Appendix

**Lemma 8.1.** The integral \([44]\) from Theorem 4 has the following series expansion near zero for \( m = 0 \)

\[
T(\omega, s) = 2 \int_{\mathbb{R}^+} dp \rho J_0(\rho \sqrt{2\omega}) \left( e^{-s/\rho^2} - 1 \right) \quad (115)
\]

\[
= \sum_{k=1}^{+\infty} \frac{s^k [\ln(s) - 2H_{k-1} - H_k + \ln(\omega) + 3\gamma - \ln 2]}{(k-1)!} (\omega/2)^{k-1}. \quad (116)
\]

**Proof.** First of all we show that the integral satisfies the following differential equation

\[
2\partial_\omega \partial_s T(\omega, s) = T(\omega, s). \quad \text{It can be achieved by explicit differentiation, integration by parts, and using the properties for Bessel functions}
\]

\[
\frac{d}{dx} J_0(x) = -J_1(x), \quad \frac{1}{x} J_1(x) + \frac{d}{dx} J_1(x) = J_0(x). \quad (117)
\]
According to the above mentioned, let us take an ansatz for (115) in the form

\[ T(\omega, s) = g_0(\omega) + \sum_{k=1}^{\infty} \left( s^k \ln(s) f_k(\omega) + s^k g_k(\omega) \right), \]  

(118)

where the coefficients \( f_k(\omega) \) and \( g_k(\omega) \) should be found. Let us apply the operator \( 2\partial_s s \partial_s^2 \) to (118). Then we get the recurrent relations

\[
\begin{align*}
2\partial_s f_1(\omega) &= g_0(\omega); \\
2k(k+1)\partial_s f_{k+1}(\omega) &= f_k(\omega) \text{ for } k \geq 1; \\
2(2k+1)\partial_s f_{k+1}(\omega) - 2k(k+1)\partial_s g_{k+1}(\omega) &= g_k(\omega) \text{ for } k \geq 1.
\end{align*}
\]

(119)

To solve them, we have to find the initial conditions, using the integral representation (115). One can use the last calculations and the form of ansatz (118), we get

\[
2 \int_0^1 d\rho \rho J_0(\rho \sqrt{2\omega}) \left( e^{-s/\rho^2 - 1} \right) = \int_0^1 d\rho \left( e^{-s/\rho - 1} \right) + o(1)
\]

(120)

\[ = s \ln(s) + s\omega - s - \sum_{k=2}^{\infty} \frac{(-s)^k}{k!(k-1)} + o(1), \]

(121)

and

\[
2 \int_1^{+\infty} d\rho \rho J_0(\rho \sqrt{2\omega}) \left( e^{-s/\rho^2 - 1} \pm \frac{s}{\rho^2} \right) = \sum_{k=2}^{+\infty} \frac{(-s)^k}{k!(k-1)} + s \ln(\omega) + s \left( 2\gamma - \ln(2) \right) + o(1).
\]

(122)

Using the last calculations and the form of ansatz (118), we get

\[ \sum_{k=1}^{+\infty} s^k \ln(s) f_k(0) + \sum_{k=2}^{+\infty} s^k g_k(0) = s \ln(s). \]

(123)

Therefore, \( f_1(0) = 1 \) and \( f_k(0) = g_k(0) = 0 \) for \( k \geq 2 \). Finally, we need to find the coefficient \( g_1(\omega) \), that corresponds to \( s \). It can be achieved by subtracting the logarithmic part \( s \ln s \) and differentiating by the parameter \( s \) with a further transition \( s \to 0 \). So we get

\[
\partial_s \bigg|_{s=0} \left[ 2 \int_{\mathbb{R}_+} d\rho \rho J_0(\rho \sqrt{2\omega}) \left( e^{-s/\rho^2 - 1} \right) - s \ln(s) \right] = -2 \int_1^{+\infty} \frac{d\rho}{\rho} J_0(\rho \sqrt{2\omega}) \left( e^{-s/\rho^2 - 1} \right) + \gamma - 1
\]

(124)

\[
= \ln(\omega) + 3\gamma - \ln(2) - 1.
\]

(125)

where in the second equality we have used the change of variable \( \rho \to \rho/\sqrt{2\omega} \). It means that \( g_1(\omega) = \ln(\omega) + 3\gamma - \ln(2) - 1 \). Solving the recurrent relations, we find

\[
f_k(\omega) = \frac{(\omega/2)^{k-1}}{k!(k-1)!}; \quad g_k(\omega) = \frac{\ln(\omega) - 2H_{k-1} - H_k + 3\gamma - \ln 2}{k!(k-1)!} (\omega/2)^{k-1};
\]

(127)

which leads to the statement of the lemma. \qed
References

[1] V. A. Fock, *Die Eigenzeit in der Klassischen- und in der Quanten-mechanik*, Sow. Phys., 12, 404–425 (1937)

[2] D. V. Vassilevich, *Heat kernel expansion: user’s manual*, Phys. Rep., 388, 279–360 (2003)

[3] A. V. Ivanov, N. V. Kharuk, *Heat kernel: Proper-time method, Fock–Schwinger gauge, path integral, and Wilson line*, Theoret. and Math. Phys., 205:2, 1456–1472 (2020)

[4] B. S. DeWitt, *Dynamical Theory of Groups and Fields*, Gordon and Breach, New York, 1–248 (1965)

[5] P. B. Gilkey, *The spectral geometry of a Riemannian manifold*, J. Differ. Geom., 10, 601–618 (1975)

[6] P. B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem*, CRC Press, Boca Raton, 1–536 (1994)

[7] P. B. Gilkey, *Asymptotic Formulae in Spectral Geometry*, CRC Press, Boca Raton, 1–312 (2004)

[8] A. O. Barvinsky, Y. V. Gusev, G. A. Vilkovisky, V. V. Zhytnikov, *Asymptotic behaviors of the heat kernel in covariant perturbation theory*, J. Math. Phys., 35(7), 3543–3559 (1994)

[9] A. O. Barvinsky, V. F. Mukhanov, *New nonlocal effective action*, Phys. Rev. D, 66(6), 065007 (2002)

[10] A. O. Barvinsky, Y. V. Gusev, V. F. Mukhanov, D. V. Nesterov, *Nonperturbative late time asymptotics for the heat kernel in gravity theory*, Phys. Rev. D, 68(10), 105003 (2003)

[11] A. O. Barvinsky, G. A. Vilkovisky, *The generalized Schwinger-DeWitt technique in gauge theories and quantum gravity*, Phys. Rept. 119:1, 1–74 (1985)

[12] A. V. Ivanov, *Diagram technique for the heat kernel of the covariant Laplace operator*, Theoret. and Math. Phys., 198:1, 100–117 (2019)

[13] A. V. Ivanov, D. V. Vassilevich, *Atiyah–Patodi–Singer index theorem for domain walls*, J. Phys. A: Math. Theor., 53, 305201 (2020)

[14] H. Bateman, A. Erdelyi, *Tables of Integral Transforms, Volume II*, McGraw-Hill Book Company, New York, 1–467 (1954)

[15] A. D. Poularikas, *The Transforms and Applications Handbook*, CRC Press, Boca Raton, 1–911 (2018)

[16] M. Lüscher, *Dimensional regularisation in the presence of large background fields*, Ann. Phys., 142, 359–92 (1982)

[17] A. V. Ivanov, N. V. Kharuk, *Two-loop cutoff renormalization of 4-D Yang–Mills effective action*, J. Phys. G: Nucl. Part. Phys., 48, 015002 (2020)

[18] A. Alonso-Izquierdo, R. Fresneda, J. M. Guilarte, D. Vassilevich, *Soliton fermionic number from the heat kernel expansion*, Eur. Phys. J. C, 79, 525 (2019)

[19] C. Almeida, A. Alonso-Izquierdo, R. Fresneda, J. M. Guilarte, D. Vassilevich, *Non-topological fractional fermion number in the Jackiw-Rossi model*, [arXiv:2103.06826](https://arxiv.org/abs/2103.06826) [hep-th] (2021)

[20] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Berlin, Springer, 1–363 (2004)