Light kaonic atoms: from ”corrected” to ”summed up” Deser formula

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Accuracy of ”corrected Deser” and ”summed up Deser” formulas was checked for the $K^-p$ and $K^-d$ systems. It was found that the last one is much more accurate and should be used for connection the 1$s$ level shift to the corresponding scattering length.

Interaction of antikaons with nucleons is the basic input for studying quasi-bound states in exotic nuclei containing antikaons. Experimental data, which can be used for construction of a model of $\bar{K}N$ interaction, are not too rich. Some of the data are old and not very accurate, such as $K^-p$ cross-sections. They can be reproduced by an antikaon-nucleon strong potential directly, in a model independent way. Reproduction of others cannot be used for parameter fitting directly, it needs some additional model-dependent assumptions. It is the case of $K^-p$ correlation functions recently measured by ALICE at CERN.

The most promising is kaonic hydrogen, which is an atom consisting of $K^-$ and proton. It’s 1$s$ level is shifted in respect to the pure Coulomb value due to strong interaction between the particles. In principle, the shift and the width of kaonic hydrogen is the data, which can be calculated directly using the strong plus Coulomb interactions. However, it is much easier to calculate $K^-p$ scattering length using only strong $\bar{K}N$ interaction, and then use some formula, which connects the scattering length to the 1$s$ level shift. Such formula was suggested by Deser for pion-nucleon system [1].

Later the authors of [2] suggested a formula, which is more accurate for the case when the meson is heavier, like an antikaon. This ”corrected Deser” formula is widely used nowadays in a form

$$\Delta E_{K^-p}^{\text{col}} - \frac{1}{2} \Gamma_s^{K^-p} = -2\alpha^3 \mu^2 a_{K^-p}[1 - 2\alpha \mu a_{K^-p}(\ln \alpha - 1)]$$

by experimentalists and by theorists, who are not able to calculate 1$s$ level of kaonic hydrogen directly. It was shown in [3, 4], that the accuracy of the formula is about 10% for the two-body $K^-p$ system and much worse for the three-body $K^-d$ system.

However, the series, which leads to the corrected Deser formula in [2] can be summed up. The result is a formula, which I will call ”summed up Deser” formula

$$\Delta E_{K^-p}^{\text{sum}} - \frac{1}{2} \Gamma_s^{K^-p} = -2\alpha^3 \mu^2 a_{K^-p}/[1 + 2\alpha \mu a_{K^-p}(\ln \alpha - 1)].$$

To the best of my knowledge, first it was wrote down in [5]. The authors of [5] introduce it in a footnote: ”In the case of kaonic atoms, higher-order Coulomb corrections may turn out to be not completely negligible numerically . . . This issue is, however, relatively easy to cure since the large contribution comes from an iteration of a particular diagram to all orders. Replacing the factor $1 - 2\mu a(\ln \alpha - 1)a_p$ by $(1 + 2\mu a(\ln \alpha - 1)a_p)^{-1}$ already captures the bulk of the effect. We shall not further elaborate on this issue.”

Such summed up formula should be more accurate than the first two terms of the series. The question is, how accurate is it, especially for the three-body system.

Knowledge of the accuracy of the approximate Deser-type formulas is useful not only for theorists, but for experimentalists as well. Quite a few experiments plan or performing measurements of antikaon-nucleon or -nucleus systems. In particular, SIDDHARTA-2 [6] or J-PARK E57 [7] experiments aim to measure 1$s$ level shift in kaonic deuterium.

Authors of [8, 9] mentioned in passing ”summed up” Deser formula as ”substantially improving the agreement” or ”reproducing the level shifts considerably better”. Accuracy of ”summed up” Deser formula was checked in [10], where for the two-body $K^-p$ system it was found to be much higher than for the ”corrected” Deser formula. The results for the three-body $K^-d$ system, however, are not so clear. The problem is that 1$s$ level shift and width of kaonic deuterium were evaluated in [10] from variational calculation with energy-independent $ar{K}N$ potential, while $K^-d$ scattering length, necessary for the approximate formulas, was calculated using the fixed-center approximation. Therefore, it is not absolutely clear, what is the accuracy of the approximate formulas themselves, and what are the errors introduced by the approximations.

In previous years we performed a series of calculations of different states and reactions in $\bar{K}N$ and $\bar{K}KN$ systems [11]. The calculations were performed using dynamically exact Faddeev-type AGS equations. As an input we used three different versions of the $\bar{K}N$ interaction, constructed by us. The potentials reproduce low-energy $K^-p$ experimental data including 1$s$ level shift and width of kaonic hydrogen. In contrast to most of the authors of $\bar{K}N$ interaction models, we calculated energy of the 1$s$ level and its width directly by solving Lippmann-Schwinger equation with strong plus Coulomb potentials. The scattering lengths, given by the strong antikaon-nucleon potential, were also calculated, so that we can compare the directly calculated characteristics of

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TABLE I. 1s level shift $\Delta E_{1s}^{K^-p}$ (eV) and width $\Gamma_{1s}^{K^-p}$ (eV) of kaonic hydrogen calculated using the "corrected Deser" and "summed up Deser" formulas together with the exact results.

|            | Corrected Deser | Summed up Deser | Exact |
|------------|-----------------|-----------------|-------|
| $\Delta E_{1s}^{K^-p}$ | $\Gamma_{1s}^{K^-p}$ | $\Delta E_{1s}^{K^-p}$ | $\Gamma_{1s}^{K^-p}$ | $\Delta E_{1s}^{K^-p}$ | $\Gamma_{1s}^{K^-p}$ |
| $V_{K_N}^{1,SIDD}$ | -328 | 579 | -318 | 593 | -313 | 597 |
| $V_{K_N}^{2,SIDD}$ | -322 | 589 | -312 | 603 | -308 | 602 |
| $V_{K_N}^{Chiral}$ | -326 | 544 | -318 | 559 | -313 | 561 |
| $V_{K_N}^{1,KEK}$ | -383 | 404 | -381 | 429 | -377 | 434 |
| $V_{K_N}^{2,KEK}$ | -381 | 482 | -376 | 509 | -373 | 514 |

kaonic hydrogen with those given by approximate formulas.

In addition, 1s shift caused by strong interaction in kaonic deuterium was evaluated using Faddeev-type three-body equations with directly included strong plus Coulomb interactions in [12]. Since $K^-d$ scattering length was calculated before as well, it is a unique possibility to check the approximate formulas, connecting scattering length with characteristics of kaonic atoms for two- and three-body systems, comparing them with the exact results.

Three our antikaon-nucleon potentials are: two phenomenological potentials having one- $V_{K_N}^{1,SIDD}$ or two-pole $V_{K_N}^{2,SIDD}$ structure of $\Lambda(1405)$ resonance [13], and a chiral potential $V_{K_N}^{Chiral}$ [14]. All three potentials reproduce the most recent and most accurate result of the 1s level shift of kaonic hydrogen measurement performed by SIDDHARTA collaboration [11].

The $K^-p$ scattering lengths of the three antikaon-nucleon potentials [13, 14], reproducing SIDDHARTA data are:

$$a_{K^-p}^{1,SIDD} = -0.76 + i 0.89 \text{ fm}$$  \hspace{1cm} (3)
$$a_{K^-p}^{2,SIDD} = -0.74 + i 0.90 \text{ fm}$$  \hspace{1cm} (4)
$$a_{K^-p}^{Chiral} = -0.77 + i 0.84 \text{ fm}.$$  \hspace{1cm} (5)

I also used our previously constructed phenomenological potentials from [16], which do not reproduce SIDDHARTA data, but can be useful for the checks of the approximate formulas. The $K^-p$ scattering lengths given by the one-pole $V_{K_N}^{1,KEK}$ and the two-pole $V_{K_N}^{2,KEK}$ potentials are:

$$a_{K^-p}^{1,KEK} = -1.00 + i 0.68 \text{ fm}$$  \hspace{1cm} (6)
$$a_{K^-p}^{2,KEK} = -0.96 + i 0.80 \text{ fm}.$$  \hspace{1cm} (7)

The 1s level shifts and widths of kaonic hydrogen calculated using the "corrected Deser" Eq. (11) and "summed up Deser" Eq. (2) formulas are presented in Table I together with the exact results. It is seen that the "summed up Deser" formula lead to much more accurate $\Delta E_{1s}^{K^-p}$ (eV) and $\Gamma_{1s}^{K^-p}$ values corresponding to the $K^-p$ scattering lengths Eqs. (11,13) and Eq. (14) than the "corrected Deser" formula. The accuracy of the "summed up Deser" formula varies within 0.8 – 1.6% for the shift and 0.2 – 1.1% for the width.

The three potentials reproducing SIDDHARTA experimental data: phenomenological $V_{K_N}^{1/2,SIDD}$ [13] and chiral potential $V_{K_N}^{Chiral}$ [14], were used in the three-body calculations of the $KNN$ system with spin 1. The $K^-d$ scattering lengths were evaluated by solving the dynamically exact Faddeev-type AGS equations with strong $KNN$ potentials only:

$$a_{K^-d}^{1,SIDD} = -1.49 + i 1.24 \text{ fm}$$  \hspace{1cm} (8)
$$a_{K^-d}^{2,SIDD} = -1.51 + i 1.25 \text{ fm}$$  \hspace{1cm} (9)
$$a_{K^-d}^{Chiral} = -1.59 + i 1.32 \text{ fm}.$$  \hspace{1cm} (10)

Dynamically exact calculation of the kaonic deuterium characteristics were performed using Faddeev-type equations with directly included Coulomb interaction in [12]. The same three strong antikaon-nucleon potentials were used there. The 1s level shifts and widths of the three-body kaonic atom corresponding to the scattering lengths of $K^-d$ system Eq. (8,9,10) are presented in Table I. We used the same Eq. (11) and Eq. (2) for the "corrected" and "summed up Deser" formulas as for the two-body $K^-p$ case with $\mu$ being the $K^-d$ reduced mass.

The results of one more method of kaonic deuterium characteristics calculations are denoted in Table I as "$K^-d$ optical potential". To obtain them we: calculated low-energy $K^-d$ scattering amplitudes using dynamically exact three-body AGS equations, constructed an optical $K^-d$ two-body potential, reproducing the three-body amplitudes, and evaluated the 1s level shift and width using two-body Lippmann-Schwinger equations with the optical $K^-d$ and Coulomb potentials [13].

It is seen from Table I that in this three-body case "summed up Deser" result is slightly worse in reproducing the exact results: 0.1 – 3.3% for the shift and 0.8 – 5.3% for the width, but is much better than the results of "corrected Deser" formula. As for the "$K^-d$ optical potential" method, it gives comparable accuracy for the 1s shift (0.8 – 2.4%), but is worse for the width (5.1 – 9.7%) than the "summed up Deser".

Keeping all the above in mind, I suggest to use "summed up" Deser formula Eq. (2) for evaluation 1s level shift and width of kaonic hydrogen and kaonic deuterium from $K^-p$ and $K^-d$ scattering length correspondingly. Its accuracy is better than 2% for the two-body $K^-p$ system and 6% for the three-body system.

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TABLE II. 1s level shift $\Delta E_{1s}^{K^-d}$ (eV) and width $\Gamma_{1s}^{K^-d}$ (eV) of kaonic deuterium calculated using the "corrected Deser", "summed up Deser" formulas and using $K^-d$ optical potentials together with the exact results.

|                  | Corrected                          | Summed up                         | Exact                  | $K^-d$ optical potential |
|------------------|------------------------------------|-----------------------------------|------------------------|-------------------------|
|                  | $\Delta E_{1s}^{K^-p}$ | $\Gamma_{1s}^{K^-p}$ | $\Delta E_{1s}^{K^-p}$ | $\Gamma_{1s}^{K^-p}$ | $\Delta E_{1s}^{K^-p}$ | $\Gamma_{1s}^{K^-p}$ |
| $V_{KN}^{1,\text{SIDD}}$ | -826       | 731                           | -792                   | 921                    | -767                    | 928                     | -785                   | 1018                   |
| $V_{KN}^{2,\text{SIDD}}$ | -835       | 727                           | -800                   | 923                    | -782                    | 938                     | -797                   | 1025                   |
| $V_{KN}^{\text{Chiral}}$ | -876       | 724                           | -836                   | 951                    | -835                    | 1004                    | -828                   | 1055                   |

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