Thermodynamics of Kondo model with electronic interactions

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On the basis of Bethe ansatz solution of one dimensional Kondo model with electronic interaction, the thermodynamics equilibrium of the system in finite temperature is studied in terms of the strategy of Yang and Yang. The string hypothesis in the spin rapidity is discussed extensively. The thermodynamics quantities, such as specific heat and magnetic susceptibility, are obtained.

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I. INTRODUCTION

It is known that the study of exact solutions is helpful for the understanding of non-perturbative effects in the strongly correlated electronic systems. The exact solution of the one dimensional Kondo model with linearized dispersion in the absence of electronic interaction was found in [2]. The model with quadratic dispersion was shown to be exactly solvable at some value of electron-impurity coupling [3]. The present model was solved exactly by Bethe ansatz [4]. It was shown that the study of exact solutions is helpful for the understanding of non-perturbative effects in the strongly correlated electronic systems. The exact solution of the one dimensional Kondo model with linearized dispersion in the absence of electronic interaction was found in [2]. The model with quadratic dispersion was shown to be exactly solvable at some value of electron-impurity coupling [3]. The present model was solved exactly by Bethe ansatz [4].

In present paper, we study the thermodynamics of Kondo model with electronic interactions. The general thermal equilibrium are discussed exactly on the basis of the known Bethe ansatz solutions of the model [4]. The specific heat and magnetic susceptibility are obtained analytically in general and given explicitly in strong-coupling limit. The specific heat and the magnetic susceptibility at low temperature are discussed extensively. The thermodynamics quantities, such as specific heat and magnetic susceptibility, are obtained.

II. THE MODEL AND ITS SPECTRUM

The model Hamiltonian of a correlated electronic system which we consider reads

\[
H_0 = \sum_k \varepsilon(k) C_{k\alpha}^* C_{k\alpha} + \sum_{k_1, k_2, k_3, k_4} u \delta(k_1 + k_2, k_3 + k_4) C_{k_1, k_2}^* C_{k_3, k_4}^* C_{k_4, k_1} C_{k_2, k_3},
\]

where \( C_{k\alpha} \) annihilates an electron with momentum \( k \) and spin component \( \alpha \), and \( \varepsilon(k) = k^2/2 \) (in units of \( \hbar \) and of the electron mass). The electrons are coupled by both spin and charge interactions to a localized impurity,

\[
H_I = J \Psi_0^* \cdot S_0 + V \Psi_0^* \cdot \vec{S},
\]

where the field \( \Psi_0 \) is the Fourier transform of \( C_{k\alpha} \), \( S_0 \) is the spin of the impurity and \( \vec{S} \) is the spin of the electrons in the band.

The present model was solved exactly by Bethe ansatz with periodic boundary conditions [4]. It was shown that the model is integrable when the correlation strength \( u \) is proportional to the \( J \), the strength of electron-impurity coupling via spin. The Bethe Ansatz equations for the spectrum are

\[
e^{-ik_b L} = e^{-i\Theta(k_b)} \prod_{\nu=1}^M \frac{\lambda_\nu - k_j + iu/2}{\lambda_\nu - k_j - iu/2},
\]

\[- \prod_{\nu=1}^M \frac{\lambda_\nu - \lambda_\mu + iu}{\lambda_\nu - \lambda_\mu - iu} = \frac{\lambda_\mu - iu/2}{\lambda_\mu + iu/2} \prod_{\ell=1}^N \frac{\lambda_\mu - k_\ell - iu/2}{\lambda_\mu - k_\ell + iu/2},
\]

(1)
where $\theta(k_j) = 2\tan^{-1}(k_j/u)$. In the approximation $k_j \sim k_l$ for any $j,l$, the $S$ matrix of electron-electron will be independent of $u$. Then the Yang-Baxter equation will give no restrictions on between $u$ and $J$. This makes it easy to understand the usual Kondo problem where the linear dispersion relation is adopted, whence the model is solvable at any value of $J$.

### III. STRING HYPOTHESIS

For the ground state (ie. at zero temperature), the $k$'s and $\lambda$'s are real roots of the Bethe ansatz equation (2). For the excited state (ie. at nonzero temperature), however, they can be complex roots (3). We will not take account of the complex roots in the charge sector $k$ for repulsive interaction since it does not happen at low temperature. The complex roots $\lambda$ in spin sector always for a “bound state” with several other $\lambda$’s, which arises from the consistency of both hand side of the Bethe ansatz equation (3). The complex roots with the same real part $\lambda_n^\alpha$ form a $n$-string,

$$\Lambda^m = \lambda^m_\beta + i\frac{u}{2}m + O(\exp(-\delta N)) \quad (\delta > 0)$$

$$m = -n + 1, -n + 3, \ldots, n - 3, n - 1.$$  

(2)

The set of roots $\{\lambda_\nu | \nu = 1, 2, \ldots, M\}$ is then partitioned into a set of $n$-strings $\{\Lambda^m_\beta | m = -n + 1, -n + 3, \ldots, n - 3, n - 1; \beta = 1, 2, \ldots, M_n\}$. Obviously,

$$M = \sum_{n=1}^{\infty} nM_n,$$

where $M_n$ denotes the number of $n$-strings.

Substituting those $n$-strings into the eq.(4), we can find that the product of the fractions for the roots within the same $n$-string reduce to $(\lambda^m_\beta - k_j + i\alpha(u/2))/((\lambda^m_\beta - k_j - i\alpha(u/2))$ because of the alternative elimination between denominator and numerator. Hence the Bethe ansatz equation (4) becomes

$$e^{-ik_jL} = e^{-i\theta(k_j)} \prod_{\beta m} \frac{\lambda^m_\beta - k_j + i\alpha(u/2)}{\lambda^m_\beta - k_j - i\alpha(u/2)},$$

and

$$-\prod_{\beta m = -m+1}^{m-1} \frac{\Lambda^m_\beta - \Lambda^m_\beta + i\alpha(u/2)}{\Lambda^m_\beta - \Lambda^m_\beta - i\alpha(u/2)} = \frac{\Lambda^m_\beta - k_j - i\alpha(u/2)}{\Lambda^m_\beta - k_j + i\alpha(u/2)}. $$

(4)

The product of those equations (1) for $p = -n + 1, -n + 3, \ldots, n - 3, n - 1$ gives rise to

$$(-1)^n \prod_{\beta m} \Lambda^m_\beta - \lambda^m_\beta + i\alpha(u/2) \times$$

$$\left[ \frac{\lambda^m_\beta - \lambda^m_\beta + i\alpha(u/2)}{\lambda^m_\beta - \lambda^m_\beta - i\alpha(u/2)} \right]^{2} \times$$

$$\left[ \frac{\lambda^m_\beta - \lambda^m_\beta + i\alpha(u/2)}{\lambda^m_\beta - \lambda^m_\beta - i\alpha(u/2)} \right]^{N}$$

(5)

Taking the logarithm of eq.(3) and eq.(5) we have

$$k_j = \frac{2\pi}{L} I_j + \frac{1}{L} \sum_{\beta m} \theta_\beta^m (\lambda^m_\beta - k_j),$$

$$\Theta_{\alpha/2}(\lambda^m_\alpha) + \sum_{l=1}^{N} \Theta_{\alpha/2}(\lambda^m_\alpha - k_l) = 2\pi J^m_{\alpha} - \sum_{\beta m} A_{\alpha/2} (k_\beta^m - \lambda^m_\alpha),$$

(6)

where $\theta_\beta^m = 2\tan^{-1}(\beta^m)$ and

$$A_{\alpha/2} = \begin{cases} 1, & \text{for } p = m + n, |m - n| (\neq 0), \\ 2, & \text{for } p = m + n - 2, n + m - 4, \ldots, |n - n| + 2, \\ 0, & \text{otherwise}. \end{cases}$$
The $I_j$ and $J_n^α$ are quantum numbers, the $I_j$ are integers or half-integers depending on whether $M$ is even or odd, the $J_n^α$ are integers or half-integers depending on whether $N - M_n - n + 1$ is even or odd.

IV. THE THERMODYNAMICS LIMIT

The transcendental equations (3) for the real parts of the complex roots are obviously difficult to solve. It will be convenient to consider the thermodynamics limit, i.e. $N \to \infty$, $L \to \infty$ but $D = N/L$ is fixed. Introducing the density distributions of roots and holes

$$\frac{1}{L} \frac{dI(k)}{dk} = \rho(k) + \rho^h(k),$$

$$\frac{1}{L} \frac{dJ_n(\lambda)}{d\lambda} = \sigma_n(\lambda) + \sigma_n^h(\lambda),$$

we obtain from (3) the following set of integral equations,

$$\rho(k) + \rho^h(k) = \frac{1}{2\pi} - \frac{1}{L}K_1(k) + \sum_{n=1}^\infty K_{n/2}(k|\lambda')\sigma(\lambda'),$$

$$\sigma_n(\lambda) + \sigma_n^h(\lambda) = \frac{1}{L}K_{n/2}(\lambda) + K_{n/2}(\lambda|k')\rho(k') - \sum_{m=1}^\infty A_{nmp}K_{p/2}(\lambda|\lambda')\sigma_m(\lambda'),$$

where $K_n(x) = \pi^{-1}nu/(n^2u^2 + x^2)$. We have adopted a notation convention $K_n(x|y)\rho(y) = \int K_n(x - y)\rho(y)dy$ etc. in the above.

In term of the density distributions, the energy and the concentration of electrons as well as the number of down spins are given by

$$E = L \int_{-\infty}^{\infty} dk \rho(k)k^2,$$

$$D = \frac{N}{L} = \int_{-\infty}^{\infty} dk \rho(k),$$

$$M = \frac{N}{L} = \sum_{n=1}^\infty n \int_{-\infty}^{\infty} d\lambda \sigma_n(\lambda).$$

Thus the magnetic moment of the system is,

$$\mathcal{M} = \frac{1}{2}(N - 2M) + S_{imp}^z,$$

$$= \frac{L}{2} \int_{-\infty}^{\infty} \rho(k)dk - L \sum_{n=1}^\infty n \int_{-\infty}^{\infty} \sigma_n(\lambda)d\lambda + S_{imp}^z,$$

where $S_{imp}^z$ stands for the spin of the impurity, and the $g$ factor is put to unit.

The ground state of the present model is a Fermi sea described by $\rho(k)$ with real rapidity $\lambda$, i.e. $\rho(k) = 0$ for $|k| > k_F$ and $\rho^h(k) = 0$ for $|k| < k_F$; $\sigma_1(\lambda) \neq 0$ but $\sigma_n(\lambda) = 0 (n > 1)$, which is the case at zero temperature. Away from zero temperature, the density distributions of roots and holes with respective to the momentum $k$ and the spin rapidity $\lambda$ should be determined by the principles of statistical physics. Next section we will discuss this issue extensively on the basis of the strategy of Yang and Yang.

V. THERMAL EQUILIBRIUM

For a given $\rho(k)$ and $\rho^h(k)$, the number of roots and that of holes in the neighborhood $dk$ are $L\rho dk$ and $L\rho^h dk$ respectively. Obviously, the total number of roots and holes in the neighborhood is $L(\rho + \rho^h)dk$. For a given $\sigma_n(\lambda)$ and $\sigma_n^h(\lambda)$, $L\sigma_n d\lambda$ and $L\sigma_n^h d\lambda$ give rise to the number of $n$-string and the number of the vacancies of $n$-strings (holes) in the neighborhood $d\lambda$, while $L(\sigma_n + \sigma_n^h) d\lambda$ gives rise to the total number of $n$-string and vacancies of $n$-string. Thus the total number of the possible choice of state in $dkd\lambda$ being consistent with given distribution functions in both charge and spin sectors is

$$\Xi(k, \lambda) = \frac{[L(\rho + \rho^h)dk]!}{[L\rho dk]![L\rho^h dk]!} \prod_n \frac{[L(\sigma_n + \sigma_n^h) d\lambda]!}{[L\sigma_n d\lambda]![L\sigma_n^h d\lambda]!}.$$  

As the total number of all possible state for given distribution functions is

$$\Xi = \prod_{k\lambda} \Xi(k, \lambda),$$

the total entropy $S$ will be obtained by taking logarithm of $\Xi$.  

3
where the Boltzmann constant is put to unit.
In the presence of the external magnetic field, we must add a Zeeman term to the original Hamiltonian. The Zeeman term commutes with the original Hamiltonian, the Bethe ansatz solution is still valid for present case. Therefore the energy of the system in the presence of external magnetic field will be

\[ E/L = \int (k^2 - H)\rho(k)dk + \sum_{n=1}^{\infty} 2nH \int \sigma_n(k)dk, \tag{12} \]

In order to obtain the thermal equilibrium at temperature \( T \), we should maximize the contribution to partition function from the state described by the density distributions of roots and holes. As maximizing the partition function is equivalent to minimizing the free energy \( F = E - TS - \mu N \). Here \( S \) and \( E \) are given by eq.(11) and eq.(12), \( \mu \) is the chemical potential for canonical ensembles. The \( \mu \) plays the role of the Lagrangian multiplier for the condition \( L \int \rho(k)dk = N = constant \) if one minimizing the Helmholtz free energy \( \Omega = E - TS \). This constraint implies that the ensemble has fixed number of particles. A constraint that the number of down spins are fixed was imposed in ref. \[8\] when discussing delta Fermi gas, whereas we will not impose no physics constraints in the following discussion.

Making use of the relations derived from eq.(11),

\[ \delta \rho^h(k) = -\delta \rho(k) + \sum_n K_{n/2}(k|\lambda)\delta \sigma_n(\lambda), \]

\[ \delta \sigma^h_n(\lambda) = -\delta \sigma_n(\lambda) + K_{n/2}(\lambda|k)\delta \rho(k) - \sum_{mp} A_{mp} K_{p/2}(\lambda|\lambda')\delta \sigma_m(\lambda'), \tag{13} \]

we obtain the following conditions from the minimum condition \( \delta F = 0 \), namely

\[ \epsilon(k) = -\mu + k^2 - H - T \sum_n K_{n/2}(k|\lambda) \ln(1 + e^{-\xi_n(\lambda)/T}), \tag{14} \]

\[ \xi_n(\lambda) = 2nH - T K_{n/2}(\lambda|k) \ln(1 + e^{-\epsilon(k)/T}) \]

\[ + T \sum_{mp} A_{mp} K_{p/2}(\lambda|\lambda') \ln(1 + e^{-\xi_n(\lambda')/T}), \tag{15} \]

where we have written

\[ \frac{\rho^h(k)}{\rho(k)} = \exp[\epsilon(k)/T], \]

\[ \frac{\sigma^h_n(\lambda)}{\sigma_n(\lambda)} = \exp[\xi_n(\lambda)/T]. \]

Principally, once \( \epsilon(k) \) and \( \xi(\lambda) \) are solved from eq.(14), the equilibrium distributions \( \rho(k) \) and \( \sigma_n(\lambda) \) at temperature \( T \) will be known from the following relations,
The free energy per unit length reads

\[
F/L = \int dk \rho(k) \left[ k^2 - \epsilon(k) - H - T(1 + e^{\epsilon(k)/T}) \ln(1 + e^{-\epsilon(k)/T}) \right]
+ \sum_n \int d\lambda \sigma_n(\lambda) \left[ 2nH - \xi_n(\lambda) - T(1 + e^{\xi_n(\lambda)/T}) \ln(1 + e^{-\xi_n(\lambda)/T}) \right].
\]  

Integrating eq. (15) over \( \lambda \) where the Boltzmann constant is put to unit. Consequently, the partition function is given by

\[
Z = \frac{1}{DK^2} \int (2\pi)^{L} dk \rho(k) \left[ k^2 - \epsilon(k) - H - T(1 + e^{\epsilon(k)/T}) \ln(1 + e^{-\epsilon(k)/T}) \right]
+ \sum_n \int d\lambda \sigma_n(\lambda) \left[ 2nH - \xi_n(\lambda) - T(1 + e^{\xi_n(\lambda)/T}) \ln(1 + e^{-\xi_n(\lambda)/T}) \right].
\]  

Integrating eq. (14) over \( k \) after multiplying it with \( D^{-1} \rho \), we get the chemical potential

\[
\mu = \frac{1}{D} \int (k^2 - \epsilon(k) - H) \rho(k) \left[ k^2 - \epsilon(k) - H - T(1 + e^{\epsilon(k)/T}) \ln(1 + e^{-\epsilon(k)/T}) \right] dk
- \frac{T}{D} \sum_n \int \int K_{n/2}(k - \lambda) \ln(1 + e^{-\xi_n(\lambda)/T}) \rho(k) d\lambda dk.
\]  

Integrating eq. (15) over \( \lambda \) and summing over \( n \) after multiplying it with \( D^{-1} \sigma_n \), we obtain the following relation

\[
\sum_n \int \xi_n(\lambda) \sigma_n(\lambda) d\lambda = \sum_n 2nH \int \sigma_n(\lambda) d\lambda
- T \sum_n \int \int K_{n/2}(\lambda - k) \ln(1 + e^{-\epsilon(k)/T}) \sigma_n(\lambda) d\lambda dk
+ T \sum_{mnq} A_{mnq} \int \int K_{q/2}(\lambda - \lambda') \ln(1 + e^{-\xi_n(\lambda')/T}) \sigma_n(\lambda) d\lambda' d\lambda.
\]  

Using the relations (18-19), we can write out the free energy in terms of \( \epsilon \) and \( \xi \) only,

\[
F = \mu N + T \int (K_1(k) - \frac{L}{2\pi}) \ln(1 + e^{-\epsilon(k)/T}) dk
- \sum_n T \int K_{n/2}(\lambda) \ln(1 + e^{-\xi_n(\lambda)/T}) d\lambda.
\]  

Consequently, the partition function is given by

\[
Z = e^{-F/T},
\]  

where the Boltzmann constant is put to unit.

The thermodynamics functions, partition function \( Z \), free energy \( F \), and thermal potential \( \Omega \) etc., are of importance, as knowing either of them, one is able to calculate all thermodynamics properties of the system in principle. However, it is difficult to obtain an analytic expressions of \( \epsilon(k) \) and \( \xi_n(\lambda) \) from the coupled non-linear integral equations (14) and (15). So we are not able to derive explicit results for thermodynamics quantities. Moreover, we will obtain some plausible results for some special cases in next section.

VI. THERMODYNAMIC QUANTITIES

In general, the free energy of our model should be calculated by formula (20), where the \( \epsilon(k) \) and \( \xi_n(\lambda) \) are determined by eq. (14-15). Then the other thermodynamic quantities are obtainable from thermodynamic relations.

In the Appendix, we show that if \( \mu \), \( \epsilon \) and \( \xi_n \) are implicit functions of some thermodynamic quantities (such as \( T \), \( L \)), the derivative of eq. (20) with respect to \( x \) is the same as the partial derivative of eq. (20) with respect to the explicit variable \( x \). It is easy to get the pressure of the system

\[
P = -\frac{\partial F}{\partial L} = \frac{T}{2\pi} \int \ln(1 + e^{-\epsilon(k)/T}) dk,
\]  

which is the same as Yang and Yang’s expression formally. However, the integral equation which \( \epsilon(k) \) obeys are different. In our present case the contributions from both impurity and the spin rapidity are involved.

In terms of \( \epsilon \) and \( \xi \), the entropy \( S = -(\partial F/\partial T) \) becomes

\[
S = -\int \ln(1 + e^{-\epsilon(k)/T}) \left[ \frac{L}{2\pi} + \frac{1}{\pi} u^2 + k^2 \right] dk
- \int \frac{e^{-\epsilon(k)/T}}{1 + e^{-\epsilon(k)/T}} \epsilon(k) \left[ \frac{L}{2\pi} + \frac{1}{\pi} u^2 + k^2 \right] dk
+ \sum_n \int \int \frac{1}{\pi} \frac{nu/2}{(nu/2)^2 + \lambda^2} \ln(1 + e^{-\xi_n(\lambda)/T}) d\lambda
+ \sum_n \int \int \frac{1}{\pi} \frac{nu/2}{(nu/2)^2 + \lambda^2} \frac{\xi_n(\lambda)/T}{1 + e^{\xi_n(\lambda)/T}} d\lambda.
\]  

The other thermodynamic quantities is formally obtainable, e.g.
\[ CV = T \frac{\partial S}{\partial T}, \quad M = - \frac{\partial F}{\partial H}, \quad \chi = \frac{\partial M}{\partial H}. \]  

(24)

For the sake of saving space, we did not write them out.

The free energy \([22]\) is conveniently partitioned as two parts. \( F = F_c + F_i \), here \( F_i \) stands for the contribution from the impurity,

\[ F_i = \frac{T}{\pi} \int \frac{u}{u^2 + k^2} \ln(1 + e^{-\epsilon(k)/T}) dk \]

\[ -\frac{T}{\pi} \sum_n \int \frac{nu/2}{(nu/2)^2 + \lambda^2} \ln(1 + e^{-\xi_n(\lambda)/T}) d\lambda, \]

and \( F_e \) for that from the electrons.

**A. Strong coupling limit**

For the strong coupling \( u >> 1 \) and non-vanishing external magnetic field, we keep the leading term in eq.\([23]\) and \([24]\),

\[ \epsilon(k) = k^2 - H - \mu, \]

\[ \xi_n(\lambda) = 2nH. \]  

(25)

The free energy related to impurity and electrons become

\[ F_i = T \int \frac{dk}{\pi} \frac{u}{u^2 + k^2} \ln[1 + e^{(\eta - k^2)/T}] \]

\[ -T \sum_n \int \frac{nu/2}{(nu/2)^2 + \lambda^2} \ln[1 + e^{-2nH/T}], \]

\[ F_e = \mu N - \frac{T}{2\pi} \int \ln[1 + e^{(\eta - k^2)/T}] dk, \]  

(26)

where \( \eta = \mu + H \). Since \( \tan^{-1}(k/u) \simeq k/u \) for \( u \gg 1 \), integrating eq.\([26]\) we have

\[ F_i = \frac{4}{\pi u} \int_{0}^{\eta} \frac{k^2}{1 + e^{(k^2 - \eta)/T}} dk - T \sum_n \ln[1 + e^{-2nH/T}], \]

\[ F_e = \mu N - \frac{2L}{\pi} \int_{0}^{\eta} \frac{k^2}{1 + e^{(k^2 - \eta)/T}} dk. \]  

(27)

Now we compute the common integration in eq.\([23]\),

\[ I = \int_{0}^{\eta/T} \frac{k^2}{1 + e^{(k^2 - \eta)/T}} dk, \]  

(28)

Changing variables of \( z = (k^2 - \eta)/T \) brings it to

\[ I = \frac{T}{2} \int_{-\eta/T}^{\eta/T} \frac{(zt + \eta)^{1/2}}{1 + e^z} dz. \]  

(29)

which is conveniently slit up into three terms,

\[ I = T \int_{-\eta/T}^{\eta/T} \frac{(zt + \eta)^{1/2}}{1 + e^z} dz + T \int_{0}^{\eta/T} \frac{(zt + \eta)^{1/2}}{1 + e^z} dz. \]  

(30)

The right end of the interval for integration in the second term can be regarded as \( \infty \) due to the contribution from large value of \( z \) is negligible. Integrating eq. \([31]\) after expanding the numerator as Taylor series, we obtain

\[ I = \frac{1}{3} \eta^{3/2} + \frac{T}{2} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n + 1/2)}{2\sqrt{\pi}} T^{2(n+1/2)} [1 - \left( \frac{1}{4} \right)^{n+1/2} \frac{\zeta(2n + 2)}{(\eta)^{n+1/2}}] \]  

(31)

where \( \zeta(2n + 2) \) is the Riemann Zeta function. Consequently, the free energy is obtained

\[ F_i = \frac{4}{\pi u} I - T \sum_{n=1}^{\infty} \ln[1 + e^{-2nH/T}], \]

\[ F_e = \mu N - \frac{2L}{\pi} I. \]  

(32)

**B. Thermodynamic quantities at low temperature**

In the low temperature approximation, eq.\([32]\) becomes

\[ F_i = \frac{4}{\pi u} I - T \sum_{n=1}^{\infty} e^{-2nH/T}. \]  

(33)

The magnetization:

The contributions of electrons and the impurity to the magnetization are obtained

\[ M_e = \frac{2L}{\pi} I_h, \]

\[ M_i = -\frac{4}{\pi u} I_h - 2 \sum_{n=1}^{\infty} n e^{-2nH/T}, \]  

(34)

where

\[ I_h = \frac{\partial I}{\partial H}, \quad I_{hh} = \frac{\partial^2 I}{\partial H^2}, \quad I_u = \frac{\partial^2 I}{\partial T^2}. \]  

(35)
effectively coupling between impurity and the conduction electrons leading to the formation of a local singlet, and the infra-red physics is dominated by a strong coupling fixed point.

The electron’s contributions to specific heat and magnetic susceptibility are

\[ C_e = \frac{\pi L}{6} (\mu + H)^{-1/2} T, \]
\[ \chi_e = \frac{L}{2\pi} (\mu + H)^{-1/2} + \frac{\pi LT^2}{16} (\mu + H)^{-5/2}. \] (40)

We have analyzed the thermodynamics of Kondo Model with electronic interactions, particularly discussed the case of strong coupling limit extensively. In that case we have shown the impurity’s contribution to the specific heat and magnetic susceptibility of the system is Fermi-liquid like and shown that in very low temperature the system has the property of spontaneous magnetization.

**APPENDIX**

Since \( \epsilon \) and \( \xi \) which solve eqs(14,13) evidently depend on \( \mu \), we should consider them as functionals of \( \mu \) which is usually functions of some thermodynamic variables \( x \) (such as \( T \) or \( L \)). The derivative of free energy (20) is given by

\[
\frac{\partial F}{\partial x} = \left( \frac{\partial F}{\partial x} \right)_{\mu, \epsilon, \xi} + N \frac{\partial \mu}{\partial x} + \int dk \frac{L}{2\pi} \frac{K_n/2(\epsilon)}{1 + e^{\epsilon(k)/T}} \frac{\partial \sigma_n}{\partial \mu} \frac{\partial \xi_n}{\partial \mu} \frac{\partial \xi_n}{\partial x}.
\] (41)

where \((\partial F/\partial x)_{\mu, \epsilon, \xi}\) stands for partial derivative of \( F \) with respect to the explicit variable \( x \) while \( \mu, \epsilon, \) and \( \xi \) are regarded as irrelevant to \( x \).

The derivative of (14) with respect to \( \mu \) is

\[
\frac{\partial \epsilon}{\partial \mu} = -1 + \sum_n \int d\lambda \frac{K_n/2(k - \lambda)}{1 + e^{\xi_n(\lambda)/T}} \frac{\partial \xi_n}{\partial \mu}.
\] (42)

Integrating (42) after multiplying both hand sides with \( \rho \), we obtain

\[
\int dk \frac{\partial \epsilon}{\partial \mu} \rho(k) = -\frac{N}{L} + \sum_n \int d\lambda \frac{\partial \xi_n}{\partial \mu} \frac{\partial \sigma_n}{\partial \mu} - \frac{1}{L} \sum_n \int d\lambda \frac{K_n/2(\lambda)}{1 + e^{\xi_n(\lambda)/T}} \frac{\partial \xi_n}{\partial \mu} \frac{\partial \xi_n}{\partial \mu} \frac{\partial \sigma_n}{\partial \mu} + \sum_{nmq} \int d\lambda d\lambda' \frac{A_{nmq}}{1 + e^{\xi_n(\lambda)/T}} K_q/2(\lambda - \lambda') \sigma_m(\lambda').
\] (43)
In deriving to the above equation, the second equation of eq. (16) has been used. We take derivative of eq.(15), then multiply both hand sides with $\sigma_n$ and integrate over $\lambda$. Summing over the subscript $n$ in what we obtained, we have

$$\sum_n \int d\lambda \frac{\partial \xi_n}{\partial \mu} \sigma_n(\lambda) = \sum_n \int dk d\lambda \frac{K_{n/2}(\lambda-k)\sigma_n(\lambda)}{1 + e^{\epsilon(k)/T}}$$

$$- \sum_{nm} \int d\lambda' \frac{\partial \xi_m}{\partial \mu} \frac{A_{nmq}}{1 + e^{\epsilon_m(\lambda')/T}} K_{q/2}(\lambda - \lambda')\sigma_n(\lambda).$$

(44)

With the help of the first equation of eq. (16), eq.(44) and eq.(43) give rise to

$$N + \int dk \frac{L/2\pi - K_1(k)}{1 + e^{\epsilon(k)/T}} \frac{\partial \epsilon}{\partial \mu} + \sum_n \int d\lambda \frac{K_{n/2}(\lambda)}{1 + e^{\epsilon_n(\lambda)/T}} \frac{\partial \xi_n}{\partial \mu} = 0.$$  

(45)

Thus the complete cancellation of the last three terms in eq. (11) concludes that

$$\frac{\partial F}{\partial x} = \left( \frac{\partial F}{\partial x} \right)_{\mu,\epsilon,\xi}. $$

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