Investigation of Pontryagin trace anomaly using Pauli-Villars regularization

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Abstract

In this paper, we investigate the Pontryagin trace anomaly for chiral fermions in a general curved background using Pauli-Villars regularization. We use both Feynman diagram method and Fujikawa’s method to calculate the parity-odd contribution (Pontryagin term). Our result indicates that the trace anomaly of energy-momentum tensor for chiral fermions has Pontryagin term

\[ P = \frac{i}{1536\pi^2} \epsilon^{\rho\sigma\kappa\lambda} R_{\rho\sigma} R^{\kappa\lambda} \]

which agrees with the work of Bonora et al [1].

keywords Trace anomaly, Pauli-Villars regularization, Chiral fermion

1 Introduction

Symmetries play an important role in our understanding of elementary particle physics. A symmetry of the classical action may be violated in the quantized version. This new feature of quantum theory was discovered in 1969 by Adler, Bell and Jackiw [2, 3] (Chiral anomaly) in the solving the problem of neutral pion decay \( \pi^0 \rightarrow \gamma\gamma \). Gravitation is taken as a gauge theory also suffers from anomalies [4]. The gauges include the general coordinate transformations (diffeomorphisms) and the conformal transformations (Weyl transformations). In this paper, we will study the trace or conformal anomaly.

We recall the definition of trace anomaly. The energy-momentum tensor in field theory is defined by

\[ T_{\mu\nu}(x) = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}}. \]  

Consider the conformal transformation

\[ g_{\mu\nu} \rightarrow e^{2\sigma(x)} g_{\mu\nu} \]

for an infinitesimal value of the parameter \( \sigma(x) \): \( g_{\mu\nu} \rightarrow (1 + 2\sigma(x))g_{\mu\nu} \), we have

\[ \delta S = \frac{1}{2} \int d^4x \sqrt{|g|} T_{\mu\nu} \delta g^{\mu\nu} = - \int d^4x \sigma(x) \sqrt{|g|} T^{\mu}_\mu. \]
The invariance of $S$ under conformal transformation requires that trace of energy-momentum tensor is $T_\mu^\mu = 0$. Generically, this classical traceless energy-momentum tensor is broken by quantum effects, that is

$$A = \langle T_\mu^\mu \rangle \neq 0.$$  

Here the $A$ is called trace or conformal anomaly \cite{521}.

In four-dimension, the most general form of the trace anomaly was found to be given by \cite{2224}

$$\langle T_\mu^\mu \rangle = cF + aG + bR^2 + b'\Box R + e\epsilon^{\alpha\beta\gamma\delta}R_{\alpha\beta\mu\nu}R_{\gamma\delta}^\mu\nu,$$

where $F = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R^2$ is the square of the Weyl tensor and $G = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta}R_{\alpha\beta} + R^2$ yields the Euler invariant. The coefficient $c, a, b$ and $b'$ are known at one-loop \cite{2530}. The last nontrivial term $\epsilon^{\alpha\beta\gamma\delta}R_{\alpha\beta\mu\nu}R_{\gamma\delta}^\mu\nu$ is the parity odd Pontryagin density which was first discussed by \cite{3132}. Later, the group of Bonora et al \cite{1} has claimed that the parity odd Pontryagin density (CP-odd term) in the trace anomaly of Weyl fermions exists. The coefficient of this term is purely imaginary signaling a violation of unitarity. These results were derived both using a standard perturbative Feynman diagrams computation around Minkowski spacetime in dimensional regularization and the heat kernel method. For a spin $\frac{1}{2}$ right-handed spinor, the Weyl anomaly is connected with the Seeley-DeWitt coefficients \cite{1203334}

$$\langle T_\mu^\mu \rangle = \frac{1}{180 \times 16\pi^2} \left( -\frac{9}{2} F + \frac{11}{4} G + i\frac{15}{8} \epsilon^{\alpha\beta\gamma\delta}R_{\alpha\beta\mu\nu}R_{\gamma\delta}^\mu\nu \right).$$

Thus the Pontryagin term is

$$P = \frac{i}{1536\pi^2} \epsilon^{\alpha\beta\gamma\delta}R_{\alpha\beta\mu\nu}R_{\gamma\delta}^\mu\nu.$$  

Our motivation for this work is that there are some different results for the Pontryagin term e.g. \cite{3537}. In this paper we are going to recalculate the trace anomaly for a chiral fermion using Pauli-Villars regularization to try to figure it out. The paper is organized as follows. In Section 2, we use Feynman diagram method to calculate the parity-odd contribution. We regulate the divergence by introducing a set of Pauli-Villars fields. In Section 3, we apply the Fujikawa’s method to evaluate the parity-odd term by Pauli-Villars regularization. We end with the conclusions. Some definitions and useful formulae are put in appendix.

## 2 Feynman diagram method

Following the works \cite{1293638}, we consider the free Dirac fermion theory in 4d. The action is

$$S = \int d^4 x \sqrt{|g|} \left[ i\bar{\psi}\gamma^\mu \left( \partial_\mu + \frac{1}{2} \omega_\mu \right) \psi - m\bar{\psi}\psi \right] = \int d^4 x \sqrt{|g|} \left[ i\bar{\psi}\gamma^\mu \nabla_\mu \psi - m\bar{\psi}\psi \right]$$  

where the $\gamma^\mu = e^\mu_a \gamma^a (\mu, \nu, \ldots \text{are} \text{world indices}, a, b, \ldots \text{are} \text{flat indices})$ and the $e^\mu_a$ is the inverse vierbein. The $\nabla_\mu = \partial_\mu + \frac{1}{2} \omega_\mu$ is the covariant derivative and $\omega_\mu$ is the spin connection:

$$\omega_\mu = e^b_{\mu} \Sigma_{ab},$$
where \( \Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b] \) are the Lorentz generators and the spin connection is
\[
\omega_{\sigma \mu} = e_{\nu a}(\partial_\mu e_\nu^a + e_\nu^a \Gamma_{\sigma \mu}^\nu).
\]

The Levi-Civita connection \( \Gamma_{\sigma \mu}^\nu \) is
\[
\Gamma_{\sigma \mu}^\nu = \frac{1}{2}g^{\rho \sigma}(\partial_\mu g_{\rho \nu} + \partial_\nu g_{\rho \mu} - \partial_\rho g_{\mu \nu}).
\]

We obtain the stress tensor by the following identities:
\[
\delta \sqrt{|g|} = \frac{1}{2} \sqrt{|g|} g^{\mu \nu} \delta g_{\mu \nu},
\]
\[
\delta \gamma^\mu = -\frac{1}{2} g^{\mu \sigma} \gamma^\nu \delta g_{\nu \rho},
\]
\[
\delta (\omega_{\mu \rho \sigma} \gamma^{\rho \sigma}) = \gamma^{\rho \sigma} \nabla_\sigma \delta g_{\rho \mu}.
\]

The action (8) is invariant under a local Lorentz transformation. From the action (8), we can compute the stress tensor
\[
\hat{T}_{\mu \nu}(x) = -\frac{i}{4} \left( \overrightarrow{\psi} \gamma_\mu \overleftarrow{\nabla}_\nu \psi + (\mu \leftrightarrow \nu) \right) + g_{\mu \nu} \left( \frac{i}{2} \overrightarrow{\psi} \gamma^\lambda \overleftarrow{\nabla}_\lambda \psi - m \overrightarrow{\psi} \overrightarrow{\psi} \right).
\]

According to this definition, the second term of \( \hat{T}_{\mu \nu}(x) \) in (13) drops out. Thus, we adopt the following stress tensor
\[
T_{\mu \nu}(x) = -\frac{i}{4} \left( \overrightarrow{\psi} \gamma_\mu \overleftarrow{\nabla}_\nu \psi + (\mu \leftrightarrow \nu) \right).
\]

Classically, the action (8) is invariant under general coordinate transformation. It follows that
\[
\partial_\mu T^{\mu \nu}(x) + \Gamma_{\mu \lambda}^{\mu} T^{\lambda \nu}(x) + \Gamma_{\nu \mu}^{\nu} T^{\mu \lambda} = 0.
\]

We set \( g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \), where \( h_{\mu \nu} \) is a small perturbation around flat background. Using the following expansions
\[
g^{\mu \nu} = \eta^{\mu \nu} - h^{\mu \nu} + (h^2)^{\mu \nu} + \ldots,
\]
\[
\sqrt{|g|} = 1 + \frac{1}{2} (tr h) + \frac{1}{8} (tr h)^2 - \frac{1}{4} h^{\mu \nu} h_{\mu \nu} + \ldots,
\]
\[
e_\alpha^\mu = \delta_\alpha^\mu - \frac{1}{2} h_\alpha^\mu + \frac{3}{8} (h^2)_\alpha^\mu + \ldots,
\]
\[
e_\alpha^\mu = \delta_\alpha^\mu + \frac{1}{2} h_\alpha^\mu - \frac{1}{8} (h^2)_\alpha^\mu + \ldots,
\]
\[
\omega_{\mu \rho \sigma} = \partial_\rho h_{\alpha \mu} + \frac{1}{4} h_\rho^\alpha \partial_\sigma h_{\alpha \mu} - \frac{1}{2} h_\rho^\alpha (\partial_\sigma h_{\alpha \mu} - \partial_\theta h_{\alpha \mu}) + \ldots.
\]
and the relation
\[ \{ \gamma^a, \Sigma^{bc} \} = i \epsilon^{abcd} \gamma^d \gamma^5, \] (18)
the action (8) is expanded as
\[
S = \int d^4x \left[ i \frac{2}{\hbar} (\delta^\mu_a - \frac{1}{2} h^\mu_a) \bar{\psi} \gamma^a \partial_\mu \psi - m \bar{\psi} \psi + \frac{1}{16} \epsilon^{\mu abc} \partial_\mu h_{a\lambda} h^5_b \bar{\psi} \gamma^c \gamma^5 \psi \right] + \ldots 
= S^{(0)} + S^{(1)} + S^{(2)} + \ldots. \] (19)
Where the \( S^{(k)} \) is the order \( k \) in the metric fluctuation \( h_{\mu\nu} \), those are
\[
S^{(0)} = \int d^4x \left[ \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi \right], \\
S^{(1)} = - \frac{i}{4} \int d^4x h^\mu_a \bar{\psi} \gamma^a \partial_\mu \psi, \\
S^{(2)} = \frac{1}{16} \int d^4x \epsilon^{\mu abc} \partial_\mu h_{a\lambda} h^5_b \bar{\psi} \gamma^c \gamma^5 \psi. \] (20)
The Feynman rules can be read off directly from above action. The fermion propagator, two-fermion-one-graviton vertex and two-fermion-two-graviton vertex (figure 1) are
\[
P : \frac{i}{\not{p} - m + i\epsilon}, \\
V_{ffg} : - \frac{i}{8} \left[ (p + p')_\mu \gamma^\nu + (p + p')_\nu \gamma^\mu \right], \\
V_{ffgg} : \frac{1}{64} t_{\mu \nu \mu' \nu' \kappa \lambda} (k - k')^\lambda \gamma^\kappa \gamma^5, \] (21)
where the coefficient \( t_{\mu \nu \mu' \nu' \kappa \lambda} \) is
\[
t_{\mu \nu \mu' \nu' \kappa \lambda} = \eta_{\mu \mu'} \epsilon_{\nu \nu' \kappa \lambda} + \eta_{\nu \nu'} \epsilon_{\mu \mu' \kappa \lambda} + \epsilon_{\mu \nu} \epsilon_{\nu \mu' \kappa \lambda} + \eta_{\mu \nu} \epsilon_{\mu \nu' \kappa \lambda}. \] (22)

Figure 1: The two-fermion-one-graviton vertex (\( V_{ffg} \)) and two-fermion-two-graviton vertex (\( V_{ffgg} \)).
To discuss the trace anomaly for chiral fermions, we define a chiral stress tensor
\[
T^{(R)}_{\mu\nu}(x) = -\frac{i}{4} \left( \overline{\psi} \gamma_\mu \nabla_\nu \psi + (\mu \leftrightarrow \nu) \right).
\] (23)

Using the equation of motion for \(\psi\), the trace of \(T^{(R)}_{\mu\nu}(x)\) become
\[
T^{(R)}_{\mu\mu}(x) = -m \overline{\psi} P R \psi.
\] (24)

Then the trace anomaly of chiral stress tensor is
\[
A = \lim_{m \to 0} \langle T^{(R)}_{\mu\mu}(x) + m \overline{\psi} P R \psi \rangle = \lim_{m \to 0} \langle T^{(R)}_{\mu\mu}(x) \rangle.
\] (25)

Where the modified chiral stress tensor \(\overline{T}^{(R)}_{\mu\nu}(x)\) is defined as
\[
\overline{T}^{(R)}_{\mu\nu}(x) = -\frac{i}{4} \left( \overline{\psi} \gamma_\mu \nabla_\nu \psi + (\mu \leftrightarrow \nu) \right) + \frac{m}{4} g_{\mu\nu} \overline{\psi} P R \psi.
\] (26)

We will directly use the expansion (17) together with Wick’s theorem to evaluate the trace anomaly (25). The modified chiral stress tensor (26) also admits an expansion
\[
T^{(R)}_{\mu\nu}(x) = T^{(R,0)}_{\mu\nu}(x) + T^{(R,1)}_{\mu\nu}(x) + \ldots,
\] (27)

where, to first order in \(h_{\mu\nu}\),
\[
T^{(R,0)}_{\mu\nu}(x) = -\frac{i}{4} \left( \overline{\psi} \gamma_\mu \nabla_\nu \psi + (\mu \leftrightarrow \nu) \right) + \frac{m}{4} g_{\mu\nu} \overline{\psi} P R \psi,
\]
\[
T^{(R,1)}_{\mu\nu}(x) = \frac{1}{2} \epsilon_{\rho\sigma} \overline{\psi} \partial_\rho h_{\mu\nu} \gamma_5 \psi + \frac{m}{4} h_{\mu\nu} \overline{\psi} P R \psi.
\] (28)

To use the intuitive Feynman diagram method, we introduce two new two-fermion-one-graviton vertices \(V_{ffg}\) and \(\tilde{V}_{ffg}\) (figure 2), which come from the \(T^{(R,0)}_{\mu\nu}(x)\) and \(T^{(R,1)}_{\mu\nu}(x)\) separately.

![Figure 2: Two new two-fermion-one-graviton vertex \(V_{ffg}\) and \(\tilde{V}_{ffg}\).](image)

From the expression (28), we obtain the two new vertices to be
\[
V_{ffg} = -\frac{i}{8} \left[ (p + p')_\mu \gamma_\nu + (p + p')_\nu \gamma_\mu - m \eta_{\mu\nu} \right] \frac{1 + \gamma^5}{2},
\]
\[
\tilde{V}_{ffg} = -\left[ \eta_{\sigma\nu} \epsilon_{\rho\mu\sigma\sigma} + \eta_{\mu\sigma} \epsilon_{\rho\mu\sigma\sigma} \right] (p' - p) g^a \gamma_5 \frac{1 + \gamma^5}{2} + \frac{m}{8} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) \frac{1 + \gamma^5}{2}.
\]
The expectation value of the modified chiral stress tensor trace $\mathcal{T}_{\sigma}^{(R)}(x)$ in the metric perturbation is

$$\langle \mathcal{T}_{\sigma}^{(R)}(x) \rangle = \langle \mathcal{T}_{\sigma}^{(R)}(x) e^{i(S^{(1)} + S^{(2)} + \cdots)} \rangle_0$$

$$= \langle \left( \mathcal{T}_{\sigma}^{(R,0)}(x) + \mathcal{T}_{\sigma}^{(R,1)}(x) + \cdots \right) \left( 1 + iS^{(1)} + (iS^{(2)} - \frac{1}{2}S^{(1)}S^{(1)}) + \cdots \right) \rangle_0$$

$$= i \langle \mathcal{T}_{\sigma}^{(R,0)}(x)S^{(1)} \rangle_0 + i \langle \mathcal{T}_{\sigma}^{(R,1)}(x)S^{(1)} \rangle_0 + i \langle \mathcal{T}_{\sigma}^{(R,0)}(x)S^{(2)} \rangle_0$$

$$\left. - \frac{1}{2} \langle \mathcal{T}_{\sigma}^{(R,0)}(x)S^{(1)}S^{(1)} \rangle_0 + \cdots \right)$$

(29)

where the $\langle \cdots \rangle_0$ denotes expectation value in the free theory with action $S^{(0)}$. We have omitted the tadpole diagrams which have vanish contributions. The $\langle \mathcal{T}_{\sigma}^{(R)}(x) \rangle$ can be separated into parity-odd term and parity-even term

$$\langle \mathcal{T}_{\sigma}^{(R)}(x) \rangle = \langle \mathcal{T}_{\sigma}^{(R)}(x) \rangle^{\text{odd}} + \langle \mathcal{T}_{\sigma}^{(R)}(x) \rangle^{\text{even}}.$$  

(30)

In this paper, we only consider the parity-odd terms which contain the Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$ factor.

We first consider the expectation value of $\mathcal{T}_{\sigma}^{(R)}(x)$ at $\mathcal{O}(h)$ order that is

$$\langle \mathcal{T}_{\sigma}^{(R)}(x) \rangle \big|_{\mathcal{O}(h)} = i \langle \mathcal{T}_{\sigma}^{(R,0)}(x)S^{(1)} \rangle_0 = \int d^4y \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} T_{\sigma\mu\nu}^\sigma(q,m) h_{\mu\nu}(y).$$

Using the Feynman rules (figure 3), the $T_{\sigma\mu\nu}(q,m)$ is

$$T_{\sigma\mu\nu}(q,m) = \frac{i}{32} \int \frac{d^4p}{(2\pi)^4} \text{tr} \left[ \frac{1}{p - m} \left( (2p - q)_{\mu} \gamma_{\nu} + (\mu \leftrightarrow \nu) \right) \frac{1}{p - \not{q} - m} (2p - \not{q} - 2m) \frac{1 + \gamma^5}{2} \right].$$

Figure 3: Feynman diagram corresponding to $i \langle \mathcal{T}_{\sigma}^{(R,0)}(x)S^{(1)} \rangle_0$ term.

After calculating the integral, there are no parity-odd term in $T_{\sigma\mu\nu}(q,m)$.

Then we calculate the expectation value of $\mathcal{T}_{\sigma}^{(R)}(x)$ at $\mathcal{O}(h^2)$ order. The expression is

$$\langle \mathcal{T}_{\sigma}^{(R)}(x) \rangle \big|_{\mathcal{O}(h^2)} = i \langle \mathcal{T}_{\sigma}^{(R,1)}(x)S^{(1)} \rangle_0 + i \langle \mathcal{T}_{\sigma}^{(R,0)}(x)S^{(2)} \rangle_0 - \frac{1}{2} \langle \mathcal{T}_{\sigma}^{(R,0)}(x)S^{(1)}S^{(1)} \rangle_0$$

$$= \int d^4y \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} T_{\sigma\mu\nu\rho\sigma}^{(1)}(q,m) h_{\mu\nu\rho\sigma}(y) h_{\mu\nu\rho\sigma}(x) + \int d^4y \int d^4z \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \times e^{-ik_1(x-y)} e^{-ik_2(z-x)} \left( T_{\sigma\mu\nu\rho\sigma}^{(2)}(k_1,k_2,m) + T_{\sigma\mu\nu\rho\sigma}^{(3)}(k_1,k_2,m) \right) h_{\mu\nu\rho\sigma}(y) h_{\mu\nu\rho\sigma}(z).$$  

(31)
Here the Feynman diagrams of $T^{(1)}_{\sigma\mu\nu\lambda}(q, m)$, $T^{(2)}_{\sigma\mu\nu\lambda}(k_1, k_2, m)$ and $T^{(3)}_{\sigma\mu\nu\lambda}(k_1, k_2, m)$ are the figure 4, figure 5 and figure 6, respectively.

\[
T^{(1)}_{\sigma\mu\nu\lambda}(q, m) = \frac{1}{8} \int \frac{d^4 p}{(2\pi)^4} \text{tr} \left[ \frac{1}{p - m} \left( (2p - q)\gamma_\mu + (\mu \leftrightarrow \nu) \right) \frac{1}{p - \bar{q} - m} \right] \times (-2\epsilon_{\mu'\nu'\alpha\beta}q^\alpha q^\beta + \frac{m}{4}\eta_{\mu'\nu'}) \frac{1 + \gamma_5}{2}. \tag{32}
\]

The result of (32) also has no Pontryagin term.

Using the Feynman rules (figure 4), the $T^{(1)}_{\sigma\mu\nu\lambda}(q, m)$ is

![Figure 4: Feynman diagram corresponding to $i\langle \mathcal{T}^{(R,1)}_{(R,1),\sigma\tau}(x) S^{(1)}\rangle_0$ term.](image)

The bubble diagram with two vertices is illustrated in figure 5. Its contribution to the trace anomaly comes from contracting the indices $\sigma$ and $\tau$ with $\eta^{\sigma\tau}$.

\[
T^{(1)}_{\sigma\mu\nu\lambda}(q, m) = \frac{1}{8} \int \frac{d^4 p}{(2\pi)^4} \text{tr} \left[ \frac{1}{p - m} \left( (2p - q)\gamma_\mu + (\mu \leftrightarrow \nu) \right) \frac{1}{p - \bar{q} - m} \right] \times (-2\epsilon_{\mu'\nu'\alpha\beta}q^\alpha q^\beta + \frac{m}{4}\eta_{\mu'\nu'}) \frac{1 + \gamma_5}{2}. \tag{32}
\]

The result of (32) also has no Pontryagin term.

There other two diagrams (the bubble graph (figure 5) and the triangle graph (figure 6)) have none zero contributions to the $\langle \mathcal{T}^{(R)}_{\sigma\tau}(x) S^{(1)}\rangle_0$. We set that the ingoing graviton has momentum $q$ and Lorentz labels $\sigma$, $\tau$ and two outgoing graviton are specified by $k_1$, $\mu$, $\nu$ and $k_2$, $\mu'$, $\nu'$, respectively. As the same as [1], we put the two external outgoing gravitons on-shell which means that the corresponding fields satisfy the EOM of gravity $R_{\mu\nu} = 0$. We choose the de Donder gauge

\[
\Gamma^\lambda_{\mu\nu}g^{\mu\nu} = 0. \tag{33}
\]

In momentum space this means that $k_1^2 = k_2^2 = 0$.

The bubble diagram with two vertices is illustrated in figure 5. Its contribution to the trace anomaly comes from contracting the indices $\sigma$ and $\tau$ with $\eta^{\sigma\tau}$.

![Figure 5: Bubble diagram with one $V_{ffg}$ and one $V_{ffg}$ corresponding to $i\langle \mathcal{T}^{(R,0)}(x) S^{(2)}\rangle_0$ term.](image)
The $T^{(2)}_{\sigma\mu\nu',\nu'}(k_1, k_2, m)$ is

$$T^{(2)}_{\sigma\mu\nu',\nu'}(k_1, k_2, m) = \frac{1}{512} \int \frac{d^4 p}{(2\pi)^4} t_{\mu\nu',\nu'}^\lambda p^\lambda (k_2 - k_1)$$

$$\text{tr} \left( \frac{1}{p - m} \gamma^\mu \gamma^5 \frac{1}{p - k_1 - k_2 - m} (2p - k_1 - k_2 - 2m)(1 + \gamma^5) \right).$$

We regulate the integral by introducing a set of Pauli-Villars fields \[39\] with masses $M_i$ and compute

$$T^{(2),\text{reg}}_{\sigma\mu\nu',\nu'}(k_1, k_2, m) = T^{(2)}_{\sigma\mu\nu',\nu'}(k_1, k_2, m) - \sum_i c_i T^{(2)}_{\sigma\mu\nu',\nu'}(k_1, k_2, M_i). \quad (34)$$

To remove the divergences, we impose the conditions

$$\sum_i c_i = 1, \quad \sum_i c_i M_i^2 = m^2. \quad (35)$$

Taking the limit $M_i \to \infty$, the (34) becomes

$$T^{(2),\text{reg}}_{\sigma\mu\nu',\nu'}(k_1, k_2, m) = d(k_1 \cdot k_2, m) k_1 \cdot k_2 t_{\mu\nu',\nu'}^\lambda k_1^\lambda k_2^\nu. \quad (36)$$

where the coefficient $d(k_1 \cdot k_2, m)$ is

$$d(k_1 \cdot k_2, m) = \frac{i}{1536\pi^2(k_1 \cdot k_2)^2} \left[ -9m^2(k_1 \cdot k_2)^2 - 3m^2 \right.$$

$$\times \sqrt{k_1 \cdot k_2(-2m^2 + k_1 \cdot k_2)} \log \left( \frac{m^2 - k_1 \cdot k_2 + \sqrt{k_1 \cdot k_2(-2m^2 + k_1 \cdot k_2)}}{m^2} \right) \left. \right]. \quad (37)$$

The triangle diagrams (figure 6) have three vertices, one $V_{ffg}$ and two $V_{fgg}$s. The calculation of the full triangle diagrams $T^{(3)}_{\sigma\mu\nu',\nu'}(k_1, k_2, m)$ are complicated. The evaluation of the Pontryagin term which is contained in the $\gamma^5$ sector is easier than the full triangle diagrams.

![Figure 6](image)

Figure 6: Triangle diagrams with one $V_{ffg}$ and two $V_{fgg}$s corresponding to $-\frac{1}{2}\left\langle T^{(R,0)}_{\sigma}(x)S(1)S(1) \right\rangle_0$ term.
The $\gamma^5$ sector of full triangle diagrams is

$$T^{(3)}_{\sigma_{\mu\nu}\nu'}(k_1, k_2, m) = \frac{i}{256} \int \frac{dp}{(2\pi)^2} \text{tr} \left[ \frac{1}{p - m} \left( (2p - k_1)_{\mu}\gamma_{\nu} + (\mu \leftrightarrow \nu') \right) \frac{1}{p - k_1 - m} \right] \times \left( (2p - 2k_1 - k_2)_{\mu}\gamma_{\nu'} + (\mu' \leftrightarrow \nu') \right) \frac{1}{p - k_1 - k_2 - m} (2p - k_1 - k_2 - 2m) \gamma^5$$

$$+ (k_1, \mu, \nu \leftrightarrow k_2, \mu', \nu').$$

As the same as the bubble diagram, we regulate the integral by introducing a set of Pauli-Villars fields $M_i$ with masses $M_i$ and compute

$$T^{(3), \text{reg}}_{\sigma_{\mu\nu}\nu'}(k_1, k_2, m) = T^{(3)}_{\sigma_{\mu\nu}\nu'}(k_1, k_2, m) - \sum_i c_i T^{(3)}_{\sigma_{\mu\nu}\nu'}(k_1, k_2, M_i).$$

Then we impose the conditions and take the limit $M_i \to \infty$, the becomes

$$T^{(3), \text{reg}}_{\sigma_{\mu\nu}\nu'}(k_1, k_2, m) =$$

$$\left( a(k_1 \cdot k_2, m) t^{(21)}_{\mu\nu'\nu'\kappa\lambda} + b(k_1 \cdot k_2, m) k_1 \cdot k_2 t_{\mu\nu'\nu'\kappa\lambda} + c(k_1 \cdot k_2, m) t^{(12)}_{\mu\nu'\nu'\kappa\lambda} \right) k_1^\kappa k_2^\lambda.$$

Where the coefficients $a(k_1 \cdot k_2, m)$, $b(k_1 \cdot k_2, m)$ and $c(k_1 \cdot k_2, m)$ are

$$a(k_1 \cdot k_2, m) = \frac{6m^2 k_1 \cdot k_2 + (k_1 \cdot k_2)^2 + 3m^4 \log^2 \left( \frac{m^2 - k_1 \cdot k_2 + \sqrt{k_1 \cdot k_2(-2m^2 + k_1 \cdot k_2)}}{m^2} \right)}{3072\pi^2(k_1 \cdot k_2)^2},$$

$$b(k_1 \cdot k_2, m) = \frac{1}{3072\pi^2(k_1 \cdot k_2)^2} \left[ (k_1 \cdot k_2)^2 \right.\left. - 6m^2 \sqrt{k_1 \cdot k_2(-2m^2 + k_1 \cdot k_2)} \log \left( \frac{m^2 - k_1 \cdot k_2 + \sqrt{k_1 \cdot k_2(-2m^2 + k_1 \cdot k_2)}}{m^2} \right) \right.\left. - 3m^4 \log^2 \left( \frac{m^2 - k_1 \cdot k_2 + \sqrt{k_1 \cdot k_2(-2m^2 + k_1 \cdot k_2)}}{m^2} \right) - 24m^2 k_1 \cdot k_2 \right],$$

$$c(k_1 \cdot k_2, m) = \frac{m^2}{2048\pi^2(k_1 \cdot k_2)^2} \left[ 2 \log \left( \frac{m^2 - k_1 \cdot k_2 + \sqrt{k_1 \cdot k_2(-2m^2 + k_1 \cdot k_2)}}{m^2} \right) \right.\left. + 4\sqrt{k_1 \cdot k_2(-2m^2 + k_1 \cdot k_2)} + m^2 \log \left( \frac{m^2 - k_1 \cdot k_2 + \sqrt{k_1 \cdot k_2(-2m^2 + k_1 \cdot k_2)}}{m^2} \right) \right.\left. + k_1 \cdot k_2 \left( 20 + \log^2 \left( \frac{m^2 - k_1 \cdot k_2 + \sqrt{k_1 \cdot k_2(-2m^2 + k_1 \cdot k_2)}}{m^2} \right) \right) \right],$$

and the quantity $t^{(21)}_{\mu\nu'\nu'\kappa\lambda}$ is defined to be

$$t^{(21)}_{\mu\nu'\nu'\kappa\lambda} = k_2\mu k_1\nu'\epsilon_{\nu'\nu'\kappa\lambda} + k_2\nu k_1\nu'\epsilon_{\mu'\nu'\kappa\lambda} + k_2\nu k_1\nu'\epsilon_{\mu'\nu'\kappa\lambda} + k_2\mu k_1\mu'\epsilon_{\mu'\mu'\kappa\lambda}. $$

(41)
Putting $T^{(2),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m)$ and $T^{(3),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m)$ together, we get

\[
T^{(2),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m) + T^{(3),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m) = \left( a(k_1 \cdot k_2, m) \ell^{(21)}_{\mu \nu \mu' \nu' \kappa \lambda} \\
+ (b(k_1 \cdot k_2, m) + d(k_1 \cdot k_2, m)) k_1 \cdot k_2 \ell^{(11)}_{\mu \nu \mu' \nu' \kappa \lambda} + c(k_1 \cdot k_2, m) \ell^{(12)}_{\mu \nu \mu' \nu' \kappa \lambda} \right) k_1^\kappa k_2^\lambda. \tag{42}
\]

From (37) and (40), we have the relation

\[
b(k_1 \cdot k_2, m) + d(k_1 \cdot k_2, m) = -a(k_1 \cdot k_2, m). \tag{43}
\]

This indicates that the $T^{(2),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m) + T^{(3),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m)$ satisfied the diffeomorphisms Ward identities

\[
k_1^\mu \left( T^{(2),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m) + T^{(3),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m) \right) = 0,
\]

\[
k_2^{\mu'} \left( T^{(2),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m) + T^{(3),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m) \right) = 0. \tag{44}
\]

Taking massless limit ($m \to 0$), the $T^{(2),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m) + T^{(3),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m)$ becomes

\[
T^{(2),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m) + T^{(3),\text{reg}}_{\sigma \mu \nu \mu' \nu'}(k_1, k_2, m) = -\frac{i}{3072 \pi^2} \left( k_1 \cdot k_2 \ell^{(21)}_{\mu \nu \mu' \nu' \kappa \lambda} - \ell^{(21)}_{\mu \nu \mu' \nu' \kappa \lambda} \right) k_1^\kappa k_2^\lambda.
\]

Putting all contributions together, we obtain

\[
\left< T^{(R)}_{\sigma}(x) \right>_{\text{odd}} = -\frac{i}{3072 \pi^2} \int d^4 y \int d^4 z \int \frac{d^4 k_1}{(2 \pi)^4} \int \frac{d^4 k_2}{(2 \pi)^4} e^{-ik_1 \cdot (x-y)} e^{-ik_2 \cdot (x-z)} \times \left( k_1 \cdot k_2 \ell^{(21)}_{\mu \nu \mu' \nu' \kappa \lambda} - \ell^{(21)}_{\mu \nu \mu' \nu' \kappa \lambda} \right) k_1^\kappa k_2^\lambda h_{\mu \nu}(y) h_{\mu' \nu'}(z) + \ldots
\]

\[
= -\frac{i}{768 \pi^2} \epsilon_{\nu \nu' \kappa \lambda} \left( \partial_{\mu} \partial_{\mu'} h_{\mu \nu} \partial_{\nu} \partial_{\nu'} h_{\mu' \nu'} - \partial_{\mu} h_{\mu \nu} \partial_{\nu} \partial_{\nu'} h_{\mu' \nu'} \right) + \ldots. \tag{45}
\]

The Pontryagin density has the following approximation in the metric fluctuation $h_{\mu \nu}$

\[
\epsilon_{\nu \nu' \kappa \lambda} R^{\nu \nu' \mu \nu}_{\rho \rho} R^{\rho \mu \kappa \lambda} = -2 \epsilon_{\nu \nu' \kappa \lambda} \left( \partial_{\mu} \partial_{\mu'} h_{\mu \nu} \partial_{\nu} \partial_{\nu'} h_{\mu' \nu'} - \partial_{\mu} h_{\mu \nu} \partial_{\nu} \partial_{\nu'} h_{\mu' \nu'} \right) + \ldots. \tag{46}
\]

Comparing the expression (46) with formula (45), we finally get the result

\[
\left< T^{(R)}_{\sigma}(x) \right>_{\text{odd}} = \frac{i}{1536 \pi^2} \epsilon_{\nu \nu' \kappa \lambda} R^{\nu \nu' \mu \nu}_{\rho \rho} R^{\rho \mu \kappa \lambda}. \tag{47}
\]

The coefficient of the parity-odd term is purely imaginary signaling a violation of unitarity. To solve this problem, the authors [1] had an argument that the neutrino can have mass. We have another idea on this problem. In our previous work [40], we connected the multi-valued function with bound state. Basing on this idea, we added the bound state contribution to get an anomaly free theory [41]. We can use this method to obtain the zero Pontryagin density by taking consider the bound state contribution. We note that the complex function $\log(x)$ is multi-valued, that is

\[
\log(x) = \text{Log}(x) + 2\pi n i, \quad n \in \mathbb{Z}. \tag{48}
\]
Where the argument of function $\text{Log}(x)$ is $\arg(\text{Log}(x)) \in (-\pi, \pi]$. Putting this new expression of $\log(x)$ into the (40), the function $a(k_1 \cdot k_2, m)$ becomes

$$a(k_1 \cdot k_2, m) = \frac{6m^2k_1 \cdot k_2 + (k_1 \cdot k_2)^2 + 3m^4 \left[ \log \left( \frac{m^2-k_1 \cdot k_2+\sqrt{k_1 \cdot k_2(-2m^2+k_1 \cdot k_2)}}{m^2} \right) + 2\pi ni \right]^2}{3072\pi^2(k_1 \cdot k_2)^2}.$$  

We find that the $a(k_1 \cdot k_2, m)$ becomes zero in the massless limit as long as

$$q^2 = (k_1 + k_2)^2 = 2k_1 \cdot k_2 = 4\sqrt{3}m^2, \quad n \in \{0, N\}.$$  

So the bubble (figure 5) and triangle diagrams (figure 6) represent a physical process that is a neutral bound state decay into two gravitons. This is the same as the pion decay $\pi^0 \to \gamma \gamma$.

3 Fujikawa’s method

Following the works of [35, 42], we use the method of Fujikawa [43, 44] to evaluate anomalies. We consider Lagrangian of a Dirac fermion $\psi$ in a curved spacetime, which is

$$\mathcal{L} = \sqrt{|g|} \left[ i\bar{\psi} \gamma^\mu \nabla_\mu \psi - m \bar{\psi} \gamma^5 \psi \right] = \sqrt{|g|} \left[ i\bar{\psi} \nabla P \psi - m \bar{\psi} P \psi \right],$$

(50)

where the $\nabla$ is $\nabla = \gamma^\mu \nabla_\mu = e^a_\mu \gamma^a \nabla_\mu$. The Lagrangian (50) which we will consider is different with the one in the paper [35]. In massless limit $m = 0$, the Lagrangian (50) has the form of a Weyl fermion

$$\mathcal{L} = \mathcal{L}_{m=0} = \sqrt{|g|} i\bar{\psi} \nabla P \psi = \sqrt{|g|} i\bar{\psi} R \nabla \psi R.$$  

(51)

The corresponding action $\widehat{S}$

$$\widehat{S} = \int d^4x \sqrt{|g|} i\bar{\psi} \nabla P \psi$$  

(52)

is invariant under conformal transformation (2). The other fields transform accordingly by the rules

$$\psi(x) \to \psi'(x) = e^{-\frac{3}{2}\sigma(x)} \psi(x),$$

$$\bar{\psi}(x) \to \bar{\psi}'(x) = e^{-\frac{3}{2}\sigma(x)} \bar{\psi}(x),$$

$$e^a_\mu \to e'^a_\mu = e^{\sigma(x)} e^a_\mu.$$  

(53)

To get a symmetric form of Lagrangian, we put the dynamical variables into a column vector $\phi$ as

$$\phi = \begin{pmatrix} \psi \\ \psi^c \end{pmatrix} = \begin{pmatrix} \psi \\ \psi_c \end{pmatrix}.$$  

(54)

The $\psi_c$ is the charge conjugated field

$$\psi_c = C^{-1} \psi^T.$$  

(55)
Then the Lagrangian \((50)\) has the following symmetric form
\[
\mathcal{L} = -\frac{1}{2} \phi^T T^\top O \mathcal{P} \phi + \frac{1}{2} m \phi^T \tilde{P} \phi. \tag{56}
\]

Here the matrix \(T, O, P\) and \(\tilde{P}\) are defined as
\[
T = \begin{pmatrix}
0 & \sqrt{|g|} C \\
\sqrt{|g|} C^T & 0
\end{pmatrix}, \quad O = \begin{pmatrix}
i \bar{\nabla} & 0 \\
0 & i \bar{\nabla}
\end{pmatrix},
\]
\[
P = \begin{pmatrix}
P_R & 0 \\
0 & P_L
\end{pmatrix}, \quad \tilde{P} = \begin{pmatrix}
P_R & 0 \\
0 & P_R
\end{pmatrix}. \tag{57}
\]

To regulate the divergences, we introduce the Pauli-Villars field \(\theta\) with mass \(M\). The Lagrangian of Pauli-Villars field is
\[
\mathcal{L}_{PV} = \sqrt{|g|} \left[ i \bar{\theta} \nabla P_R \theta - M \theta P_R \theta \right]. \tag{58}
\]

Collecting the PV fields by
\[
\chi = \begin{pmatrix}
\theta \\
\theta_c
\end{pmatrix}, \tag{59}
\]
the Lagrangian \((58)\) is rewritten in the form
\[
\mathcal{L}_{PV} = -\frac{1}{2} \chi^T T^\top O \mathcal{P} \chi + \frac{1}{2} M \chi^T \tilde{P} \chi. \tag{60}
\]

The matrix \(T, O, P\) and \(\tilde{P}\) are the same as \((57)\).

For simplifying our discussion, we denote \(\delta \phi = K \phi\) and \(\delta \chi = K \chi\) by the infinitesimal Weyl transformation on \(\phi\) and \(\chi\), respectively. From the conformal transformation \((53)\), we find that \(K = -\frac{3}{2} \sigma(x)\).

We now calculate the Pontryagin term. The regulated action has the form
\[
\tilde{S} = \int d^4 x \left( \mathcal{L} + \mathcal{L}_{PV} \right). \tag{61}
\]

The quantum theory is defined by the path integral
\[
e^{i\tilde{W}} = Z = \int \mathcal{D} \phi \mathcal{D} \chi e^{i\tilde{S}}. \tag{62}
\]

From this, we obtain
\[
\delta W = -i \frac{\delta Z}{Z} = \frac{\int \mathcal{D} \phi \mathcal{D} \chi \delta \tilde{S} e^{i\tilde{S}}}{\int \mathcal{D} \phi \mathcal{D} \chi e^{i\tilde{S}}} = \langle \delta \tilde{S} \rangle. \tag{63}
\]

According to the definition \((3)\), the trace anomaly is obtained by the computing the quantity
\[
\mathcal{A} = \int d^4 x \, \sigma(x) \sqrt{|g|} \langle T^\mu_{\mu} \rangle = -\langle \delta \tilde{S} \rangle = i \frac{\delta Z}{Z}. \tag{64}
\]
Where the $\mathcal{A}$ is connected with trace anomaly $A$ \cite{25} by the relation

$$A = \int d^4 x \sigma(x) \sqrt{|g|} A. \quad (65)$$

The $\delta Z$ is calculated directly

$$\begin{align*}
\delta Z &= \int D\phi D\chi e^{i\tilde{S}(\phi, \chi)} - \int D\phi D\chi e^{i\tilde{S}(\phi, \chi)} \\
&= \int D\phi D\chi e^{i\tilde{S}(\phi)} + i \int d^4 x \left[ \frac{1}{2} M\chi^T (TK + K^T T + \delta T) \tilde{P} \chi \right] - \int D\phi D\chi e^{i\tilde{S}(\phi, \chi)} \\
&= \int D\phi \left( \det \left[ M(TK + K^T T + \delta T) \tilde{P} \right] \right)^{-\frac{1}{2}} e^{i\tilde{S}(\phi)} - \int D\phi D\chi e^{i\tilde{S}(\phi, \chi)}. \quad (66)
\end{align*}$$

Here the jacobian of Pauli-Villars fields cancels the jacobian of the original fields $\phi$. The $P$ and $\tilde{P}$ are not invertible, so we cannot write the resulting determinant as a product of the determinant $\text{det} \left[ M(TK + K^T T + \delta T) \tilde{P} \right]^{-1}$ and others. As the authors \cite{15} claimed that the lack of an inverse for the chiral Weyl kinetic term has drastic consequences. So we factor the determinant as the following

$$\begin{align*}
\det \left[ M(TK + K^T T + \delta T) \tilde{P} \right]^{-1} &= \text{det} \left[ M(TK + K^T T + \delta T) \tilde{P} \tilde{P}^{-1} \right]^{-1} \\
&= 1 + \text{Tr} \left[ M(TK + K^T T + \delta T) \tilde{P} \tilde{P}^{-1} \right]. \quad (67)
\end{align*}$$

Then the $\delta Z$ becomes

$$\delta Z = \int D\phi D\chi \left( \text{det} \left[ M(TK + K^T T + \delta T) \tilde{P} \tilde{P}^{-1} \right] \right)^{-\frac{1}{2}} e^{i\tilde{S}(\phi, \chi)} - Z. \quad (68)$$

Where the new action $\tilde{S}'(\phi, \chi)$ is defined by

$$\tilde{S}'(\phi, \chi) = \int d^4 x \left( \tilde{\mathcal{L}} - \frac{1}{2} \chi^T T \tilde{O} \chi + \frac{1}{2} M\chi^T T \chi \right). \quad (69)$$

The determinant in expression \cite{68} may be written as

$$\begin{align*}
\text{det} \left[ M(TK + K^T T + \delta T) \tilde{P} \tilde{P}^{-1} \right] &= 1 + \text{Tr} \left[ M(TK + K^T T + \delta T) \tilde{P} \tilde{P}^{-1} \right]. \\
&= 1 + \text{Tr} \left[ M(TK + K^T T + \delta T) \tilde{P} \tilde{P}^{-1} \right]. \quad (70)
\end{align*}$$

Then the $\mathcal{A}$ can be expressed as

$$\begin{align*}
\mathcal{A} &= - \lim_{M \to \infty} \frac{i}{2} \text{Tr} \left[ M(TK + K^T T + \delta T) \tilde{P} \tilde{P}^{-1} \right] \\
&= -i \lim_{M \to \infty} \text{Tr} \left[ \left( K + \frac{1}{2} T^{-1} \delta T \right) \tilde{P} \left( 1 - \frac{\mathcal{O}}{M} \right)^{-1} \right]. \quad (71)
\end{align*}$$
Inserting the identity \(1 = \left(1 + \frac{Q}{M}\right)\left(1 + \frac{Q}{M}\right)^{-1}\) into (71), the \(\mathcal{A}\) becomes

\[
\mathcal{A} = -i \lim_{M \to \infty} \text{Tr} \left[ \left( K\tilde{P} + \frac{1}{2} T^{-1} \delta T \tilde{P} + \frac{\tilde{P} \delta \mathcal{O}}{M} + \frac{1}{2} T^{-1} \delta T \frac{\tilde{P} \mathcal{O}}{M} \right) \left( 1 - \left( \frac{\mathcal{O}}{M} \right)^2 \right)^{-1} \right].
\]

Using the background invariance of the kinetic term

\[\phi^T \left( T \mathcal{O} K + \frac{1}{2} \delta (T \mathcal{O}) \right) \phi = 0,\]  

(72)

the \(\mathcal{A}\) has the form

\[
\mathcal{A} = -i \lim_{M \to \infty} \text{Tr} \left[ \left( K\tilde{P} + \frac{1}{2} T^{-1} \delta T \tilde{P} - \frac{1}{2} \tilde{P} \delta \mathcal{O} \frac{1}{2} \right) \left( 1 - \left( \frac{\mathcal{O}}{M} \right)^2 \right)^{-1} \right].
\]

(73)

From the equation (72), we obtain

\[
\delta \mathcal{O} = i \begin{pmatrix} -\sigma \nabla & 3(\partial \sigma) \\ 0 & -\sigma \nabla + 3(\partial \sigma) \end{pmatrix}.
\]

(74)

Substituting (74) into the expression (73), we find

\[
\mathcal{A} = -i \lim_{M \to \infty} \text{Tr} \left[ \tilde{P} \tilde{Q} e^{\frac{(\mathcal{O})^2}{M^2}} \right],
\]

(75)

where the matrix \(\tilde{Q}\) is

\[
\tilde{Q} = \begin{pmatrix} \frac{\sigma}{2} + \frac{i}{2M} (\sigma \nabla - 3(\partial \sigma)) & 0 \\ 0 & \frac{\sigma}{2} + \frac{i}{2M} (\sigma \nabla - 3(\partial \sigma)) \end{pmatrix}.
\]

(76)

We only consider the Pontryagin term which is obtained from the \(\gamma^5\) sector in (75). As a non-vanishing Dirac trace with the \(\gamma^5\) requires at least four \(\gamma\)-matrices. The expression (75) can be simplified to

\[
\mathcal{A} = -i \lim_{M \to \infty} \text{Tr} \left[ \frac{\sigma}{2} \tilde{P} e^{\frac{(\mathcal{O})^2}{M^2}} \right] = -i \lim_{M \to \infty} \text{Tr} \left[ \sigma P_R e^{\frac{i(\mathcal{O})^2}{M^2}} \right].
\]

(77)

Where the \(\text{Tr} [\cdots]_4\) represents trace on the four dimensional Dirac matrices. The \(\mathcal{A}^{\text{odd}}\) which is associated with parity-odd anomaly term is

\[
\mathcal{A}^{\text{odd}} = - \frac{i}{2} \lim_{M \to \infty} \text{Tr} \left[ \gamma^5 \sigma e^{\frac{-i(\mathcal{O})^2}{M^2}} \right]_4 = - \frac{i}{2} a_4 (\gamma^5 \sigma, D).
\]

(78)

Where the \(D\) is \(D = (\nabla)^2\). We have used the heat kernel method (see appendix B or review paper [34]) to get the expression (78). From (90) in appendix B, we obtain
\[ A^{\text{odd}} = -\frac{i}{2(4\pi)^2} \frac{1}{12 \times 16} (-i) \int d^4x\sigma \sqrt{|g|} R_{\sigma\rho\mu\nu} R^{\mu\nu} i j \text{Tr} (\gamma^5 \gamma^\sigma \gamma^\rho \gamma^i \gamma^j) \]
\[ = \frac{i}{1536\pi^2} \int d^4x\sqrt{|g|} R_{\sigma\rho\mu\nu} R^{\mu\nu} i j \epsilon^{\sigma\rho\iota\jot} \] (79)

Where the extra \((-i)\) comes from Euclidean space back to Minkowski space-time. The parity-odd term of trace anomaly is related to the Kimura-Delbourgo-Salam anomaly \[46, 47\] which is the anomalous divergence of the axial current \(\langle i \bar{\psi} \gamma^\mu \gamma^5 \psi \rangle\) in a gravitational field, that is
\[ A^{\text{odd}} = \frac{1}{4} \nabla_\mu \langle i \bar{\psi} \gamma^\mu \gamma^5 \psi \rangle = \frac{i}{1536\pi^2} R_{\sigma\rho\mu\nu} R^{\mu\nu} i j \epsilon^{ij\sigma\rho} \] (80)

Then the parity-odd term of trace anomaly obtained by the Fujikawa’s method is the same as the one by Feynman diagram method \[47\]. The final result is the same as the work \[1\].

4 Conclusions and Discussions

In this paper, we have studied the Pontryagin trace anomaly for chiral fermions in a general curved background using Pauli-Villars regularization. To introduce massive PV fields, we have utilized a Dirac mass term. After taking the massless limit, we recovered the chiral field theory by inserting the chiral projector in proper place of Dirac field Lagrangian. We used both Feynman diagram method and Fujikawa’s method to calculate the parity-odd contribution. Our result indicates that the trace anomaly of energy-momentum tensor for chiral fermions has Pontryagin term
\[ P = \frac{i}{1536\pi^2} \epsilon_{\nu\sigma\rho\lambda} R^{\nu\sigma}_{\rho\mu} R^{\rho\mu\nu\lambda}. \] (81)

This agrees with the work of Bonora et al \[1\].

In our work, we have used the Pauli-Villars regularization instead of dimensional regularization. The reason is that how to treat the \(\gamma^5\) matrix is still open problem in dimensional regularization. In our next work, we will apply the other prescriptions of dimensional regularization to investigate the trace anomaly of chiral fermions in a curved background.

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A Conventions and notations

We use the flat metric \(\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)\) and work in units with \(\hbar = c = 1\). The chiral matrix \(\gamma^5\) is defined by
\[ \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \] (82)
It allows to define the left and right chiral projectors

\[ P_L = \frac{1 - \gamma^5}{2}, \quad P_R = \frac{1 + \gamma^5}{2}. \]  

(83)

A Dirac spinor \( \psi \) can be split into two Weyl spinors

\[ \psi = \psi_L + \psi_R = P_L \psi + P_R \psi. \]  

(84)

The charge conjugation matrix \( C \) is the matrix that satisfy

\[ C \gamma^\mu C^{-1} = - (\gamma^\mu)^T. \]  

(85)

We take \( C = i \gamma^0 \gamma^2 \) which has the properties

\[ C = -C^T = -C^{-1} = -C^\dagger = C^*. \]  

(86)

### B The heat kernel method

In this appendix, we collect some definitions and useful formulae from the review paper [34]. Let \( M \) be a smooth compact Riemannian manifold of dimension \( n \). The \( g_{\mu\nu} \) and \( \omega_\mu \) are metric tensor and spin connection of \( M \) respectively. The field strength of the connection \( \omega \) is

\[ \Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \omega_\mu \omega_\nu - \omega_\nu \omega_\mu. \]  

(87)

The Riemann curvature tensor is

\[ R^\mu_{\nu\rho\sigma} = \partial_\sigma \Gamma^\mu_{\nu\rho} - \partial_\rho \Gamma^\mu_{\nu\sigma} + \Gamma^\lambda_{\nu\rho} \Gamma^\mu_{\lambda\sigma} - \Gamma^\lambda_{\nu\sigma} \Gamma^\mu_{\lambda\rho}. \]  

(88)

Let \( D \) be self-adjoint operator and \( f \) be an auxiliary smooth function on \( M \). There is an asymptotic expansion as \( t \to 0 \)

\[ \text{Tr}_{L^2} \left( f e^{-tD} \right) \approx \sum_{k \geq 0} t^{(k-n)/2} a_k(f, D). \]  

(89)

The leading heat kernel coefficients \( a_4(f, D) \) is known as [33, 48]

\[ a_4(f, D) = \frac{1}{360} \frac{1}{(4\pi)^2} \int_M d^nx \sqrt{g} \text{Tr} \{ f(60E_{,k}^k + 60RE + 180E^2 + 12R_{,k}^k + 5R^2 - 2R_{ij} R^{ij} + 2R_{ijkl} R^{ijkl} + 30\Omega_{ij} \Omega^{ij}) \}. \]  

(90)

Where the ; denotes multiple covariant differentiation with respect to the Levi-Civita connection of \( M \). The \( R_{\mu\nu} := R^\rho_{\mu\rho\sigma} \) is the Ricci tensor and \( R := R^\mu_\mu \) is the scalar curvature. Let \( \nabla \) be the standard Dirac operator in curved space, that is

\[ \nabla = \gamma^\mu (\partial_\mu + \frac{1}{2} \omega_\mu), \]  

(91)

then the \( E \) and \( \Omega_{\mu\nu} \) associated with \( \nabla \) are

\[ E = -\frac{1}{4} R, \quad \Omega_{\mu\nu} = -\frac{1}{4} \gamma^\sigma \gamma^\rho R_{\sigma\rho\mu\nu}. \]  

(92)
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