Research article

On some fractional integral inequalities for generalized strongly modified $h$-convex functions

Peiyu Yan$^{1,*}$, Qi Li$^{1}$, Yu Ming Chu$^{2,3,*}$, Sana Mukhtar$^{4}$, and Shumaila Waheed$^{4}$

$^1$ Basic Teaching Department, Shandong Huayu University of Technology, Dezhou, Shandong 253034, China
$^2$ Department of Mathematics, Huzhou University, Huzhou 313000, P. R. China
$^3$ Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha 410114, P. R. China
$^4$ Department of Mathematics, University of Okara, Okara, Pakistan

* Correspondence: Email: yan1982fang@163.com, chuyuming@zjhu.edu.cn.

Abstract: Convex functions play an important role in many areas of mathematics. They are especially important in the study of optimization problems where they are distinguished by a number of convenient properties. Many generalizations of convex functions exists in literature. The main aim of the article is to develop fractional integral inequalities for generalized strongly modified $h$-convex functions. Based on obtained fractional type integral inequalities we give some applications to the means. Our results are extension and generalization of many existing results.

Keywords: generalized convex functions; strongly convex functions; modified $h$-convex function

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1. Introduction

Linear functions are considered as simplest functions in linear spaces. The class of functions and sets that are just a step more complicated then linear ones namely convex functions and convex sets.

The subset $C$ of $\mathbb{R}^n$ is said to be convex if

$$px + qy \in C$$

$\forall x, y \in C, p \in (0, 1)$ and $q = 1 - p$. The function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if its epigraph is convex subset of $\mathbb{R}$. The convexity of sets and functions are the the objects of many studies during the past few decades. The convexity of a function and set make it so special because of its interesting
properties like convex function has global minima, it has non-empty relative interior and convex set is connected having feasible directions at any point.

Some of early contributions to convex analysis were made by Holder, Jensen and Minkowski. The importance of convex analysis is well known in optimization theory [2,3], inspite of many applications, many recent problems in economics and engineering the notion of convexity does not longer suffices. Hence it is always necessary to extend the notion of convexity to some general form to meet recent problems see [4–10], for further reading on fractional integral inequalities we refer [11–17]. Moreover, the new inequalities in analysis is always appreciable. The present paper is organized as follow: in the second section, we give some preliminary material. In the third section, we derive some fractional integral inequalities for generalized strongly modified \( h \)-convex function, whereas in the fourth section, we present applications of results to the mean. Finally, we conclude our results.

2. Preliminaries

We start from some preliminaries material and basic definitions.

**Definition 2.1.** [18] Let \( f : \varphi \rightarrow \mathbb{R} \) be an extended-real-valued function define on a convex set \( \varphi \subset \mathbb{R}^n \). Then the function \( f \) is convex on \( \varphi \) if

\[
f(tb_1 + (1-t)b_2) \leq tf(b_1) + (1-t)f(b_2),
\]

for all \( b_1, b_2 \in \varphi \) and \( t \in (0, 1) \).

**Definition 2.2.** [19] Choose the functions \( f, h : J \subset \mathbb{R} \rightarrow \mathbb{R} \) are non-negative. Then \( f \) is called \( h \)-convex function if

\[
f(tb_1 + (1-t)b_2) \leq h(t)f(b_1) + h(1-t)f(b_2),
\]

for all \( b_1, b_2 \in J \) and \( t \in [0, 1] \).

**Definition 2.3.** [20] Choose the functions \( f, h : J \subset \mathbb{R} \rightarrow \mathbb{R} \) are non-negative. Then \( f \) is called modified \( h \)-convex function if

\[
f(tb_1 + (1-t)b_2) \leq h(t)f(b_1) + (1-h(t))f(b_2),
\]

for all \( b_1, b_2 \in J \) and \( t \in [0, 1] \).

**Definition 2.4.** [21] Let \( \varphi \) be an interval in real line \( \mathbb{R} \). A function \( f : \varphi = [b_1, b_2] \rightarrow \mathbb{R} \) is said to be generalized convex with respect to an arbitrary bifunction \( \eta(b_1, b_2) : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{F} \) where \( E, F \in \mathbb{R} \) if

\[
f(tb_1 + (1-t)b_2) \leq f(b_2) + t\eta(f(b_1), f(b_2)),
\]

for all \( b_1, b_2 \in \varphi, t \in [0, 1] \).

**Definition 2.5.** A function \( f : \varphi = [b_1, b_2] \rightarrow \mathbb{R} \) is called \( \eta_h \) convex function if

\[
f(tb_1 + (1-t)b_2) \leq f(b_2) + h(t)\eta(f(b_1), f(b_2)),
\]

for all \( b_1, b_2 \in \varphi, t \in [0, 1] \) and \( h : J \rightarrow \mathbb{R} \) is a non-negative function.
Definition 2.6. [22] A function $f : \varphi = [b_1, b_2] \to \mathbb{R}$ is called strongly convex function with modulus $\mu$ on $\varphi$, where $\mu \geq 0$ if
\[
f(tb_1 + (1 - t)b_2) \leq tf(b_1) + (1 - t)f(b_2) - \mu(t(1 - t)(b_1 - b_2)^2),
\]for all $b_1, b_2 \in \varphi$ and $t \in [0, 1]$.

Definition 2.7. [23] A function $f : J \subset \mathbb{R} \to \mathbb{R}$ is said to be strongly $\eta$-convex function with respect to $\eta : E \times E \to \mathbb{F}$ where $E, F \in \mathbb{R}$ and modulus $\mu \geq 0$, if
\[
f(tb_1 + (1 - t)b_2) \leq f(b_2) + \eta(t f(b_1), f(b_2)) - \mu(t(1 - t)(b_1 - b_2)^2),
\]for all $b_1, b_2 \in J$ and $t \in [0, 1]$.

Definition 2.8. [24] Choose the functions $f, h : J \subset \mathbb{R} \to \mathbb{R}$ are non-negative. Then $f$ is called generalized strongly modified $h$-convex function if
\[
f(tb_1 + (1 - t)b_2) \leq f(b_2) + h(t)\eta(f(b_1), f(b_2)) - \mu(t(1 - t)(b_1 - b_2)^2),
\]for all $b_1, b_2 \in J$ and $t \in [0, 1]$.

Definition 2.9. [25] Let $0 < s \leq 1$. A function $f : J \subset \mathbb{R} \to \mathbb{R}$ is called $s$-$\phi$-convex with respect to bifunction $\phi : E \times E \to \mathbb{F}$ where $E, F \in \mathbb{R}$ (briefly $\phi$-convex) if
\[
f(tb_1 + (1 - t)b_2) \leq f(b_2) + t^s \phi(f(b_1), f(b_2)),
\]
The next remark provides the relations among the convexities.

Remark 1. 1. If $\eta(b_1, b_2) = b_1 - b_2$ then,(2.4) reduces to (2.1);
2. If $h(t) = t$ then, (2.4) reduces to (2.4);
3. If $h(t) = t$ and $\eta(b_1, b_2) = b_1 - b_2$ then,(2.5) reduces to (2.1);
4. If $\mu = 0$ and $\eta(b_1, b_2) = b_1 - b_2$ then,(2.5) reduces to (2.3);
5. If $\mu = 0$ and $\eta(b_1, b_2) = b_1 - b_2$ then,(2.8) reduces to (2.3);
6. If $\mu = 0, \eta(b_1, b_2) = b_1 - b_2$ and $h(t) = t$ then,(2.8) reduces to (2.1);
7. If $\mu = 0$ then,(2.8) reduces to (2.5);
8. If $h(t) = t$ then, (2.8) reduces to (2.7);
9. If $\mu = 0$ and $h(t) = t^s$ then, (2.8) reduces to (2.9).

Utilization of more complicated convex functions

Most of the modern problems in engineering and other applied sciences are non-convex in nature. So it is difficult to reach at favorite results by only the classical convexity. That’s why the convexity is generalized in many directions. To understand the generalization of convexity it may categorize as:

Some generalization are made to change the form of defining e.g. quasi convex [26], pseudo convex [27] and strongly convex [28].

Some generalizations are made by expanding the domain e.g. [29] and some generalization are made by changing the range set of convex functions e.g. [30]. So generalizations the convex is always appreciable.

The next lemmas are useful in proving the main results.
Lemma 2.10. [31] Let \( f : J \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( J \) such that \( f' \in L^1[b_1, b_2] \), where \( b_1, b_2 \in J \) with \( b_1 < b_2 \). If \( \alpha, \beta \in \mathbb{R} \), then

\[
\frac{\alpha f(b_1) + \beta f(b_2)}{2} + \frac{2 - \alpha - \beta}{2} f \left( \frac{b_1 + b_2}{2} \right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x)dx
= \frac{b_2 - b_1}{4} \int_0^1 \left( (1 - \alpha - t) f' \left( tb_1 + (1 - t) \frac{b_1 + b_2}{2} \right) \right) \left( (\beta - t) f' \left( \frac{b_1 + b_2}{2} + (1 - t) b_2 \right) \right) dt.
\]

(2.10)

Lemma 2.11. [31] For \( s > 0 \) and \( 0 \leq \varepsilon \leq 1 \), we have

\[
\begin{align*}
\int_0^1 |x - t|^s dt &= \frac{\varepsilon^{s+1} + (1 - \varepsilon)^{s+1}}{s+1}, \\
\int_0^1 t|x - t|^s dt &= \frac{\varepsilon^{s+2} + (s+1 + \varepsilon)(1-\varepsilon)^{s+1}}{s+1}, \\
\int_0^1 t^2|x - t|^s dt &= \frac{-2(\varepsilon - t)^{s+3} + (1 - \varepsilon)^{s+1}(s+1)(s+3) - 2(1 - \varepsilon)^{s+2}(s+3) + 2(t - \varepsilon)^{s+3}}{(s+1)(s+2)(s+3)}.
\end{align*}
\]

(2.11)

(2.12)

(2.13)

Lemma 2.12. [32] Let \( f : J \rightarrow \mathbb{R}, J \subseteq \mathbb{R} \) be a differentiable mapping on \( J \) with \( f'' \in L^1[b_1, b_2] \), where \( b_1, b_2 \in J, b_1 < b_2 \), then

\[
\frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x)dx - f \left( \frac{b_1 + b_2}{2} \right)
= \frac{(b_2 - b_1)^2}{16} \left[ \int_0^1 t^2 f''(t \frac{b_1 + b_2}{2} + (1 - t)b_1) dt + \int_0^1 (t - 1)^2 f''(t_2 + (1 - t) \frac{b_1 + b_2}{2}) dt \right].
\]

(2.14)

Lemma 2.13. [23] If \( f^n \) for ne\( \mathbb{N} \) exists and is integrable on \([b_1, b_2] \), then

\[
\frac{f(b_1) + f(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x)dx - \sum_{k=2}^{n-1} \frac{(k - 1)(b_2 - b_1)^k}{2(k + 1)!} f^{(k)}(b_1)
= \frac{(b_2 - b_1)^n}{2n!} \int_0^1 t^{n-1}(n - 2t) f^{(n)}(tb_1 + (1 - t)b_2) dt.
\]

(2.14)

Lemma 2.14. [25] Suppose that \( f : [b_1, b_2] \rightarrow \mathbb{R} \) is a differentiable function, \( g : [b_1, b_2] \rightarrow \mathbb{R}^+ \) is a continuous function and symmetric about \( \frac{b_1 + b_2}{2} \) and \( f' \) is an integrable function on \([b_1, b_2] \). Then

\[
\frac{f(b_1) + f(b_2)}{2} \int_{b_1}^{b_2} g(x)dx - \int_{b_1}^{b_2} f(x)g(x)dx
= \frac{b_2 - b_1}{4} \left\{ \int_0^1 \left( \int_{\frac{b_1 + b_2}{2}}^{\frac{b_1 + b_2}{2}} g(u)du \right) f' \left( \frac{1 + t}{2} b_1 + \frac{1 - t}{2} b_2 \right) dt \right. \\
+ \left. \int_0^1 \left( \int_{\frac{b_1 + b_2}{2}}^{\frac{b_1 + b_2}{2}} g(u)du \right) f' \left( \frac{1 - t}{2} b_1 + \frac{1 + t}{2} b_2 \right) dt \right\}. 
\]
3. Fractional integral inequalities

**Theorem 3.1.** Let \( f : J \to \mathbb{R} \), \( J \subseteq \mathbb{R} \) be a differentiable mapping on \( J \) with \( f' \in L^1([b_1, b_2]) \), where \( b_1, b_2 \in J, b_1 < b_2 \). If \( |f'(x)|^q \) for \( q \geq 1 \) and \( 0 \leq \alpha, \beta \leq 1 \), is generalized strongly modified \( h \)-convex function on \([b_1, b_2]\), then

\[
\left| \frac{\alpha f(b_1) + \beta f(b_2)}{2} + \frac{2 - \alpha - \beta}{2} f\left(\frac{b_1 + b_2}{2}\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x)dx \right| \\
\leq \left( \frac{b_2 - b_1}{8} \right)^2 \left\{ (1 - 2\alpha + 2\alpha^2)^{1-q} \left[ \frac{1}{2} \left( 1 - 2\alpha + 2\alpha^2 \right) |f'(b_2)|^q \right.ight.
\]
\[
+ \int_0^1 |1 - \alpha - t| \left( \frac{1 + t}{2} \right) \eta(|f'(b_1)|^q + |f'(b_2)|^q) dt \\
- \mu \left( b_1 - b_2 \right)^2 \left\{ -2\alpha^2 + 8\alpha^3 - 8\alpha + 5 \right\} \left( \frac{1 - 2\beta + 2\beta^2}{12} \right)^{1-q} \\
\times \left[ (1 - 2\beta + 2\beta^2) |f'(b_2)|^q + \int_0^1 |\beta - t| h\left( \frac{t}{2} \right) \eta(|f'(b_1)|^q, |f'(b_2)|^q) dt \\
- \frac{\mu}{4} (b_1 - b_2)^2 \left\{ -2\beta^2 + 8\beta^3 - 8\beta + 5 \right\} \right\} \right) \tag{3.1}
\]

**Proof.** The proof begins with \( f'(x) \in [b_1, b_2] \), then using Lemma (2.10), and power mean inequality we have for \( q > 1 \)

\[
\left| \frac{\alpha f(b_1) + \beta f(b_2)}{2} + \frac{2 - \alpha - \beta}{2} f\left(\frac{b_1 + b_2}{2}\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x)dx \right|
\leq \frac{b_2 - b_1}{4} \left\{ \int_0^1 |1 - \alpha - t| \left| f'\left( tb_1 + (1-t) \frac{b_1 + b_2}{2}\right) \right| dt + \int_0^1 |\beta - t| \left| f'\left( \frac{b_1 + b_2}{2} + (1-t) b_2 \right) \right| dt \right\}
\leq \frac{b_2 - b_1}{4} \left\{ \left( \int_0^1 |1 - \alpha - t| dt \right)^{1-q} \left[ \int_0^1 |1 - \alpha - t| \left( |f'(b_2)|^q + h\left( \frac{1 + t}{2} \right) \right) dt \right] \right. \\
\times \eta(|f'(b_1)|^q, |f'(b_2)|^q) - \mu \frac{1 + t}{2} \left( 1 - \frac{1 + t}{2} \right) (b_1 - b_2)^2 dt \right\} \right)^{1-q} + \left( \int_0^1 |\beta - t| dt \right)^{1-q} \\
\times \left[ \int_0^1 |\beta - t| \left( |f'(b_2)|^q + h\left( \frac{t}{2} \right) \eta(|f'(b_1)|^q, |f'(b_2)|^q) \right) - \mu \frac{t}{2} \left( 1 - \frac{t}{2} \right) (b_1 - b_2)^2 dt \right] \right\} \right)^{1-q} \tag{3.2}
\]

Using Lemma (2.11), we have

\[
\mu (b_1 - b_2)^2 \int_0^1 |1 - \alpha - t| \left( \frac{1 + t}{2} \right) \left( \frac{1 - t}{2} \right) dt = \frac{\mu}{4} (b_1 - b_2)^2 \left( -2\alpha^2 + 8\alpha^3 - 8\alpha + 5 \right) \tag{3.3}
\]

And

\[
\mu (b_1 - b_2)^2 \int_0^1 |\beta - t| \left( \frac{t}{2} \right) \left( \frac{1 - t}{2} \right) dt = \frac{\mu}{4} (b_1 - b_2)^2 \left( -2\beta^2 + 8\beta^3 - 8\beta + 5 \right) \tag{3.4}
\]
Substituting values from Eqs (3.3), (3.4) in inequality (3.2), we obtain

\[
\left| \frac{\alpha f(b_1) + \beta f(b_2)}{2} + 2 - \alpha - \beta \frac{f(b_1 + b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) dx \right| \\
\leq \left( \frac{b_2 - b_1}{8} \right)^2 \left\{ \left( 1 - 2\alpha + 2\alpha^2 \right) \frac{1}{2} \left( 1 - 2\alpha + 2\alpha^2 \right) |f'(b_2)|^q \right. \\
+ \left. \int_0^1 |1 - \alpha - t| \left( \left( 1 + \frac{1}{2} \right) \eta(|f'(b_1)|^q + |f'(b_2)|^q) \right) dt \right. \\
- \frac{\mu}{4} (b_1 - b_2)^2 \left( \frac{-2\alpha^4 + 8\alpha^3 - 8\alpha + 5}{12} \right) \left\} + \left( 1 - 2\beta + 2\beta^2 \right)^{1-rac{1}{q}} \\
\times \left\{ \left( 1 - 2\beta + 2\beta^2 \right) |f'(b_2)|^q + \int_0^1 \left| \beta - t \right| \left( \left( \frac{-2\beta^4 + 8\beta^3 - 8\beta + 5}{12} \right) \right) dt \right. \\
- \frac{\mu}{4} (b_1 - b_2)^2 \left( \frac{-2\beta^4 + 8\beta^3 - 8\beta + 5}{12} \right) \right\}.
\]

For \( q = 1 \), using Lemma (2.10) and Lemma (2.11), we have

\[
\left| \frac{\alpha f(b_1) + \beta f(b_2)}{2} + 2 - \alpha - \beta \frac{f(b_1 + b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) dx \right| \\
\leq \left( \frac{b_2 - b_1}{4} \right)^2 \left\{ \left( 1 - 2\alpha + 2\alpha^2 \right) |f'(b_2)| \\
+ \int_0^1 |1 - \alpha - t| \left( \left( 1 + \frac{1}{2} \right) \eta(|f'(b_1)|, |f'(b_2)|) \right) dt - \frac{\mu}{4} (b_1 - b_2)^2 \left\{ \left( \frac{-2\alpha^4 + 8\alpha^3 - 8\alpha + 5}{12} \right) \right\} \\
+ \frac{1}{2} \left( 1 - 2\beta + 2\beta^2 \right) |f'(b_2)| + \int_0^1 \left| \beta - t \right| \left( \left( \frac{-2\beta^4 + 8\beta^3 - 8\beta + 5}{12} \right) \right) dt \\
- \frac{\mu}{4} (b_1 - b_2)^2 \left( \frac{-2\beta^4 + 8\beta^3 - 8\beta + 5}{12} \right) \right\}.
\]

(3.5)

This completes the proof. \( \Box \)

**Remark 2.** If we take \( h(t) = t \) and \( \mu = 0 \) then inequality (3.1) reduces to inequality (13) in [33].

Taking \( \alpha = \beta \) in Theorem (3.1), we have following corollary.

**Corollary 1.** Let \( f : J \to \mathbb{R}, J \subseteq \mathbb{R} \) be a differentiable mapping on \( J \) with \( f' \in L^1[b_1, b_2] \), where \( b_1, b_2 \in J, b_1 < b_2 \). If \( |f'(x)|^q \) for \( q \geq 1 \) is generalized strongly modified h-convex function on \([b_1, b_2]\) and \( 0 \leq \alpha, \beta \leq 1 \), then
Remark 3. If we take \( h(t) = t \) and \( \mu = 0 \) then inequality (3.6) reduces to inequality (16) in [33].

By choosing \( \alpha = \beta = \frac{1}{2}, \frac{1}{3} \) in Theorem (3.1) respectively, we obtain following corollary.

Corollary 2. Let \( f : J \to \mathbb{R}, J \subseteq \mathbb{R} \) be a differentiable mapping on \( J \) with \( f' \in L^1[\alpha, \beta] \), where \( b_1, b_2 \in J, b_1 < b_2 \). If \( |f'(x)|^q \) for \( q \geq 1 \) is generalized strongly modified \( h \)-convex function on \([b_1, b_2]\) and \( 0 \leq \alpha, \beta, 1, \) then

\[
\left| \frac{\alpha}{2} \left[ f(b_1) + f(b_2) \right] + (1 - \alpha) f \left( \frac{b_1 + b_2}{2} \right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) dx \right|
\]

\[
\leq \left( \frac{b_2 - b_1}{4} \right) \left( \frac{1 - 2\alpha + 2\alpha^2}{2} \right)^{1 - \frac{1}{q}} \left\{ \left( \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f'(x) dx \right)^q + \int_0^1 |1 - \alpha - t| \right\}
\]

\[
\times \left( \frac{1 + t}{2} \right) \frac{1}{2} \eta (|f'(b_1)|^q, |f'(b_2)|^q) dt - \frac{\mu}{4} (b_1 - b_2)^2 \left( \frac{-2\alpha^4 + 8\alpha^3 - 8\alpha + 5}{12} \right)^{\frac{1}{2}} + \left( \frac{1 - 2\alpha + 2\alpha^2}{2} \right)
\]

\[
\times (|f'(b_2)|^q + \int_0^1 |\alpha - t| h \left( \frac{t}{2} \right) \eta (|f'(b_1)|^q, |f'(b_2)|^q) dt - \frac{\mu}{4} (b_1 - b_2)^2 \left( \frac{-2\alpha^4 + 8\alpha^3 - 8\alpha + 5}{12} \right)^{\frac{1}{2}} \right\}
\]

\[
= \left( \frac{b_2 - b_1}{8} \right) (2)^{1 - \frac{1}{q}} \left( \frac{1 - 2\alpha + 2\alpha^2}{2} \right)^{1 - \frac{1}{q}} \left\{ \left( \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f'(x) dx \right)^q + \int_0^1 |1 - \alpha - t| \right\}
\]

\[
h \left( \frac{1 + t}{2} \right) \frac{1}{2} \eta (|f'(b_1)|^q, |f'(b_2)|^q) dt - \frac{\mu}{4} (b_1 - b_2)^2 \left( \frac{-2\alpha^4 + 8\alpha^3 - 8\alpha + 5}{12} \right)^{\frac{1}{2}} + \left( \frac{1 - 2\alpha + 2\alpha^2}{2} \right)
\]

\[
\times (|f'(b_2)|^q + \int_0^1 |\alpha - t| h \left( \frac{t}{2} \right) \eta (|f'(b_1)|^q, |f'(b_2)|^q) dt - \frac{\mu}{4} (b_1 - b_2)^2 \left( \frac{-2\alpha^4 + 8\alpha^3 - 8\alpha + 5}{12} \right)^{\frac{1}{2}} \right\}.
\]

(3.6)
\[ \left| \frac{1}{6} \left[ f(b_1) + f(b_2) + 4f \left( \frac{b_1 + b_2}{2} \right) \right] - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x)dx \right| \leq \frac{5}{72} (b_2 - b_1) \left( \frac{18}{5} \right)^{\frac{3}{2}} \left\{ \left[ \frac{5}{18} f''(b_2)^q + \int_0^t \left\{ \left. \frac{2}{3} - \mu \left( \frac{1 + t}{2} \right)^q \right| f''(b_1) \right. \right| \right. \\
\left. \left. \left. \left| f''(b_2) \right| \right\} dt \right. \right. \\
\left. \left. \left. \left. - \frac{211}{324} \mu (b_1 - b_2)^2 \right\} \left\{ \frac{5}{18} f''(b_2)^q + \int_0^1 \left| \frac{1}{3} - \mu \left( \frac{t}{2} \right)^q \right| f''(b_1) \right. \right. \\
\left. \left. \left. \left| f''(b_2) \right| \right\} dt \right. \right. \\
\left. \left. \left. \left. - \frac{211}{324} \mu (b_1 - b_2)^2 \right\} \right\} \right\} \right\}, \] (3.8)

**Remark 4.** Setting \( q = 1 \) in Corollary (2), we have the following result.

**Corollary 3.** Let \( f : J \rightarrow \mathbb{R}, J \subseteq \mathbb{R} \) be a differentiable mapping on \( J \) with \( f' \in L^1[b_1, b_2] \), where \( b_1, b_2 \in I, b_1 < b_2 \). If \( |f''(x)|^q \) for \( q \geq 1 \) is generalized strongly modified \( h \)-convex function on \([b_1, b_2]\) and \( 0 \leq \alpha, \beta \leq 1 \), then

\[ \left| \frac{1}{2} \left[ f(b_1) + f(b_2) \right] + f \left( \frac{b_1 + b_2}{2} \right) \right| - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x)dx \leq \left( \frac{b_2 - b_1}{4} \right) \left\{ \frac{1}{2} |f''(b_2)| + \eta \left( |f''(b_1)|, |f''(b_2)| \right) \int_0^1 \left| \mu \left( \frac{1 + t}{2} \right)^q \right| dt \right\}, \] (3.9)

and

\[ \left| \frac{1}{6} \left[ f(b_1) + f(b_2) + 4f \left( \frac{b_1 + b_2}{2} \right) \right] - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x)dx \right| \leq \left( \frac{b_2 - b_1}{4} \right) \left\{ \frac{5}{9} |f''(b_2)| + \eta \left( |f''(b_1)|, |f''(b_2)| \right) \int_0^1 \left| \mu \left( \frac{1 + t}{2} \right)^q \right| dt \right\} \] (3.10)

**Remark 5.** If we take \( h(t) = t \) and \( \mu = 0 \) then inequalities (3.7)–(3.10) reduce to inequalities (17) and (18) in [33].

**Theorem 3.2.** Let \( f : J \subset [0, 1) \rightarrow \mathbb{R} \) be a differentiable mapping on \( J \) with \( f'' \in L^1[b_1, b_2] \), where \( b_1, b_2 \in J \) and \( b_1 < b_2 \). If \( |f''| \) is generarilized strongly modified \( h \)-convex on \([b_1, b_2]\), then

\[ \left| f \left( \frac{b_1 + b_2}{2} \right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x)dx \right| \leq \frac{(b_2 - b_1)^2}{16} \left\{ \left[ \frac{1}{3} f''(b_1) \right] + \int_0^1 \left| \frac{1}{2} h(t) \right| dt \right\}, \] (3.11)
Proof. From Lemma (2.12), we have
\[ \left| f \left( \frac{b_1 + b_2}{2} \right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) \, dx \right| \leq \frac{(b_2 - b_1)^2}{16} \left[ \int_0^1 t^2 \left| f'' \left( \frac{t + b_2}{2} \right) + (1 - t) b_1 \right| \, dt + \int_0^1 (t - 1)^2 \left| f'' \left( tb_2 + (1 - t) \frac{b_1 + b_2}{2} \right) \right| \, dt \right]. \]

Since \(|f''|\) is generalized strongly modified \(h\) convex function, so
\[ \left| f \left( \frac{b_1 + b_2}{2} \right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) \, dx \right| \leq \frac{(b_2 - b_1)^2}{16} \left[ \int_0^1 t^2 \left( |f''(b_1)| + h(t) \eta \left( \left| f'' \left( \frac{b_1 + b_2}{2} \right) \right|, |f''(b_1)| \right) \right) - \mu(t)(1 - t) \left( \frac{b_1 + b_2}{2} - b_1 \right)^2 \right] dt + \frac{1}{20} \left( \frac{b_1 + b_2}{2} - b_1 \right)^2 \int_0^1 (t - 1)^2 \, dt \times h(t) \eta \left( \left| f''(b_2) \right|, \left| f'' \left( \frac{b_1 + b_2}{2} \right) \right| \right) \, dt + \frac{1}{20} \left( \frac{b_1 + b_2}{2} - b_1 \right)^2 \int_0^1 (t - 1)^3 \, dt \right]. \] (3.12)

And
\[ \left| f \left( \frac{b_1 + b_2}{2} \right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) \, dx \right| \leq \frac{(b_2 - b_1)^2}{16} \left\{ \left[ \frac{1}{3} \left| f''(b_1) \right| + \int_0^1 t^2 h(t) \eta \left( \left| f'' \left( \frac{b_1 + b_2}{2} \right) \right|, |f''(b_1)| \right) \, dt - \frac{1}{20} \mu \left( \frac{b_1 + b_2}{2} - b_1 \right)^2 \right] \right\}. \]

This completes the proof. \( \square \)

Remark 6. If we take \(h(t) = t\) and \(\mu = 0\) then inequality (3.11) reduces to inequality (24) in [33].

**Theorem 3.3.** Let \( f : J \subset [0, 1) \rightarrow \mathbb{R} \) be a differentiable mapping on \( J \) with \( f'' \in L^1 \{ b_1, b_2 \} \), where \( b_1, b_2 \in J \) and \( b_1 < b_2 \). If \( |f''| \) for \( q \geq 2 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) is generalized strongly modified \(h\)-convex on \([b_1, b_2]\), then
\[
\left| f\left(\frac{b_1 + b_2}{2}\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) \, dx \right|
\leq \frac{(b_2 - b_1)^2}{16} \left( \frac{1}{3} |f''(b_1)|^q + \int_0^1 r^2 h(t) \eta \left( \left| f''\left(\frac{b_1 + b_2}{2}\right)\right|^q, \left| f''(b_1)\right|^q \right) \, dt \right)
+ \left( \frac{1}{3} |f''(b_1 + b_2/2)|^q + \int_0^1 (t-1)^2 h(t) \eta \left( \left| f''(b_2)\right|^q, \left| f''\left(\frac{b_1 + b_2}{2}\right)\right|^q \right) \, dt \right) - \frac{1}{20^q \mu \left(\frac{b_2 - b_1}{2}\right)^2}.
\]

(3.13)

**Proof.** Suppose that \( p \geq 1 \), using Lemma (2.12) and power mean inequality, we have

\[
\left| f\left(\frac{b_1 + b_2}{2}\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) \, dx \right|
\leq \frac{(b_2 - b_1)^2}{16} \left( \int_0^1 r^2 \left| f''\left(\frac{b_1 + b_2}{2} + (1-t)b_1\right)\right|^q \, dt \right)
+ \left( \int_0^1 (t-1)^2 \left| f''\left(\frac{b_1 + b_2}{2}\right)\right|^q \, dt \right)^\frac{1}{q}.
\]

Since \( |f''|^q \) is generalized strongly modified \( h \)-convex, then we have

\[
\int_0^1 r^2 \left| f''\left(\frac{b_1 + b_2}{2} + (1-t)b_1\right)\right|^q \, dt
\leq \int_0^1 r^2 f'' |b_1|^q \, dt + \int_0^1 r^2 h(t) \eta \left( \left| f''\left(\frac{b_1 + b_2}{2}\right)\right|^q, \left| f''(b_1)\right|^q \right) \, dt - \int_0^1 \mu (1-t)^2 t \left( \frac{b_1 + b_2}{2} - b_1 \right)^2 \, dt
\leq \frac{1}{3} f'' |b_1|^q + \int_0^1 r^2 h(t) \eta \left( \left| f''\left(\frac{b_1 + b_2}{2}\right)\right|^q, \left| f''(b_1)\right|^q \right) \, dt - \frac{1}{20^q \mu \left(\frac{b_2 - b_1}{2}\right)^2}.
\]

And

\[
\int_0^1 (t-1)^2 \left| f''\left(\frac{b_1 + b_2}{2}\right)\right|^q \, dt
\leq \int_0^1 (t-1)^2 f'' \left(\frac{b_1 + b_2}{2}\right) \, dt + \int_0^1 (t-1)^2 h(t) \eta \left( \left| f''(b_2)\right|^q, \left| f''\left(\frac{b_1 + b_2}{2}\right)\right|^q \right) \, dt
- \int_0^1 \mu (1-t)^2 t \left( b_2 - \frac{b_1 + b_2}{2} \right)^2 \, dt
\leq \frac{1}{3} f'' \left(\frac{b_1 + b_2}{2}\right) + \int_0^1 (t-1)^2 h(t) \eta \left( \left| f''(b_2)\right|^q, \left| f''\left(\frac{b_1 + b_2}{2}\right)\right|^q \right) \, dt - \frac{1}{20^q \mu \left(\frac{b_2 - b_1}{2}\right)^2}.
\]
After simplification, we have

\[
\left| \frac{f(b_1 + b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) dx \right| \\
\leq \frac{(b_2 - b_1)^2}{16} \left( \frac{1}{3} \right)^{\frac{1}{p}} \left\{ \left( \frac{1}{3} \right)^{\frac{1}{p}} |f''(b_1)|^q + \int_0^1 t^p h(t) \eta \left( t \left( bf''(b_1) \right)^q, f'' \left( \frac{b_1 + b_2}{2} \right)^q \right) dt - \frac{1}{20} \mu \left( b_2 - b_1 \right)^{2 \frac{1}{p}} \right\} \\
+ \left( \frac{1}{3} \right)^{\frac{1}{p}} f'' \left( \frac{b_1 + b_2}{2} \right)^q + \int_0^1 (t - 1)^2 h(t) \eta \left( f''(b_2)^q, f'' \left( \frac{b_1 + b_2}{2} \right)^q \right) dt - \frac{1}{20} \mu \left( b_2 - b_1 \right)^{2 \frac{1}{p}} \right\}.
\]

Which completes the proof. \( \square \)

**Remark 7.** If we take \( h(t) = t \) and \( \mu = 0 \), then inequality (3.13) reduces to inequality (25) in [33].

**Theorem 3.4.** Let \( f : \bar{J} \subset R \to R \) be a n-times differentiable generalized strongly modified h-convex, function on \( \bar{J} \) where \( b_1, b_2 \in \bar{J} \) with \( b_1 < b_2 \) and \( f \in L^1[b_1, b_2] \). If \( f'' \) is generalized strongly modified h-convex, function with \( \mu \geq 1 \), then for \( n \geq 2 \) and \( p \geq 1 \), we have

\[
\left| \frac{f(b_1) + f(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b_2 - b_1)^k}{2(k+1)!} f^{(k)}(b_1) \right| \\
\leq \frac{(b_2 - b_1)^n}{2n!} \left( \frac{n-1}{n+1} \right)^{1 - \frac{1}{p}} \left[ \frac{n-1}{n+1} \left| f^{(\alpha)}(b_2) \right|^p + \int_0^1 h(t) \left( \left| f^{(\alpha)}(b_1) \right|^p, \left| f^{(\alpha)}(b_2) \right|^p \right) dt - \mu \frac{(n-1)(n+3)(x-y)^2}{(n+1)(n+3)} \right].
\]

**Proof. Case-i:** Since it is known that \( f' \) is generalized strongly modified h-convex function, then using the property of modules, and Lemma (2.13), we have following inequality for \( p = 1 \)

\[
\left| \frac{f(b_1) + f(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b_2 - b_1)^k}{2(k+1)!} f^{(k)}(b_1) \right| \\
\leq \frac{(b_2 - b_1)^n}{2n!} \int_0^1 t^{n-1} (n-2t) \left| f^{(\alpha)}(tb_1 + (1-t)b_2) \right| dt.
\]

Using the definition of generalized strongly modified h-convex function, we have

\[
\left| \frac{f(b_1) + f(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b_2 - b_1)^k}{2(k+1)!} f^{(k)}(b_1) \right| \\
\leq \frac{(b_2 - b_1)^n}{2n!} \int_0^1 (n-2t) \left[ \left| f''(b_2) \right| + h(t) \eta \left( \left| f''(b_1) \right|, \left| f''(b_2) \right| \right) - \mu (x-y)^2 t(1-t) \right] \right| dt \\
\leq \frac{(b_2 - b_1)^n}{2n!} \left[ \left| f''(b_2) \right| \int_0^1 t^{n-1} (n-2t) dt + \eta \left( \left| f''(b_1) \right|, \left| f''(b_2) \right| \right) \int_0^1 h(t) t^{n-1} (n-2t) dt \\
- \mu (x-y)^2 \int_0^1 t^2 (1-t)(n-2t) dt \right].
\]
As
\[
\int_0^1 t^{n-1}(n-2t) dt = \frac{n-1}{n+1}, \tag{3.17}
\]
\[
\int_0^1 (1-t)(n-2t) dt = \frac{n-1}{(n+1)(n+3)}. \tag{3.18}
\]
Substituting (3.17) and (3.18) in (3.16), we have
\[
\left| \frac{f(b_1) + f(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) dx - \frac{\sum_{k=2}^{n-1} (k-1)(b_2 - b_1)^k}{2(k+1)!} f^{(k)}(b_1) \right|
\leq \frac{(b_2 - b_1)^n}{2n!} \left[ \int_0^1 t^{n-1}(n-2t) dt \right]^{\frac{1}{p} - \frac{1}{q}} \left[ \int_0^1 t^{n-1}(n-2t) dt \right]^{\frac{1}{q}} \left| f^{(n)}(b_2) \right|^{\frac{1}{p} - \frac{2}{q}} \left| f^{(n)}(b_1) \right| \left| (1-t)b_2 + t(b_2 - b_1) \right|^{\frac{1}{p}} dt\right]^{\frac{1}{q}}.
\]

**Case-ii** For $p > 1$ applying Holder inequality, we have
\[
\left| \frac{f(b_1) + f(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) dx - \frac{\sum_{k=2}^{n-1} (k-1)(b_2 - b_1)^k}{2(k+1)!} f^{(k)}(b_1) \right|
\leq \frac{(b_2 - b_1)^n}{2n!} \left[ \int_0^1 t^{n-1}(n-2t) dt \right]^{\frac{1}{p} - \frac{1}{q}} \left[ \int_0^1 t^{n-1}(n-2t) dt \right]^{\frac{1}{q}} \left| f^{(n)}(b_2) \right|^{\frac{1}{p} - \frac{2}{q}} \left| f^{(n)}(b_1) \right| \left| (1-t)b_2 + t(b_2 - b_1) \right|^{\frac{1}{p}} dt\right]^{\frac{1}{q}}.
\]

Using definition of generalized strongly modified h-convex function, we have
\[
\left| \frac{f(b_1) + f(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) dx - \frac{\sum_{k=2}^{n-1} (k-1)(b_2 - b_1)^k}{2(k+1)!} f^{(k)}(b_1) \right|
\leq \frac{(b_2 - b_1)^n}{2n!} \left[ \int_0^1 t^{n-1}(n-2t) dt \right]^{\frac{1}{p} - \frac{1}{q}} \left[ \int_0^1 t^{n-1}(n-2t) dt \right]^{\frac{1}{q}} \left| f^{(n)}(b_2) \right|^{\frac{1}{p} - \frac{2}{q}} \left| f^{(n)}(b_1) \right| \left| (1-t)b_2 + t(b_2 - b_1) \right|^{\frac{1}{p}} dt\right]^{\frac{1}{q}}.
\]

Which completes the proof. \(\square\)

**Remark 8.** If we take $h(t) = t$ then Theorem (3.13) reduces to Theorem (2.5) in [23].
Theorem 3.5. Suppose that $f : [b_1, b_2] \rightarrow \mathbb{R}$ is a differentiable function, $g : [b_1, b_2] \rightarrow \mathbb{R}^+$ is a continuous function and symmetric about $\frac{b_1+b_2}{2}$ and $|f'|$ is a generalized strongly modified $h$-convex function. Then

$$
\left| \frac{f(b_1) + f(b_2)}{2} \int_{b_1}^{b_2} g(x)dx - \int_{b_1}^{b_2} f(x)g(x)dx \right| \\
\leq \frac{b_2-b_1}{4} \left[ 2 |f'(b_2)| + K\eta(|f'(b_1)|, |f'(b_2)|) - \frac{\mu}{2}(1-t^2)(b_1-b_2)^2 \right] \int_0^1 \int_{\frac{b_1+tb_2}{2}}^{\frac{b_1+tb_2}{2}} g(u)du dt
$$

(3.19)

where $k = \max_{t\in[0,1]} |g(t)|$ and $g(t) = h\left(\frac{1-t}{2}\right) + h\left(\frac{1+t}{2}\right)$.

Proof. From Lemma (2.14) and the fact that $|f'|$ is a generalized strongly modified $h$-convex function, we have

$$
\left| \frac{f(b_1) + f(b_2)}{2} \int_{b_1}^{b_2} g(x)dx - \int_{b_1}^{b_2} f(x)g(x)dx \right| \\
\leq \frac{b_2-b_1}{4} \int_0^1 \int_{\frac{b_1+tb_2}{2}}^{\frac{b_1+tb_2}{2}} g(u) \left[ |f'(\frac{1+t}{2}b_1 + \frac{1-t}{2}b_2)| + |f'(\frac{1-t}{2}b_1 + \frac{1+t}{2}b_2)| \right] du dt
$$

$$
\leq \frac{b_2-b_1}{4} \int_0^1 \int_{\frac{b_1+tb_2}{2}}^{\frac{b_1+tb_2}{2}} g(u) \left[ 2 |f'(b_2)| + h\left(\frac{1+t}{2}\right) \eta(|f'(b_1)|, |f'(b_2)|) \\
-2\mu\left(1-t^2\right)(b_1-b_2)^2 + h\left(\frac{1-t}{2}\right) \eta(|f'(b_1)|, |f'(b_2)|) \right] du dt
$$

$$
= \frac{b_2-b_1}{4} \int_0^1 \int_{\frac{b_1+tb_2}{2}}^{\frac{b_1+tb_2}{2}} g(u) \left[ 2 |f'(b_2)| + h\left(\frac{1+t}{2}\right) + h\left(\frac{1-t}{2}\right) \eta(|f'(b_1)|, |f'(b_2)|) \\
-\frac{\mu}{2}(1-t^2)(b_1-b_2)^2 \right] du dt
$$

$$
= \frac{b_2-b_1}{4} \left[ 2 |f'(b_2)| + h\left(\frac{1+t}{2}\right) + h\left(\frac{1-t}{2}\right) \eta(|f'(b_1)|, |f'(b_2)|) - \frac{\mu}{2}(1-t^2)(b_1-b_2)^2 \right] \int_0^1 \int_{\frac{b_1+tb_2}{2}}^{\frac{b_1+tb_2}{2}} g(u)du dt
$$

where $k = \max_{t\in[0,1]} |g(t)|$ and $g(t) = h\left(\frac{1-t}{2}\right) + h\left(\frac{1+t}{2}\right)$. □

Remark 9. If we take $\mu = 0$, $h(t) = t^r$ then Theorem (3.15) reduces to Theorem (3) in [25].

Corollary 4. In theorem (3.15) if we choose $h(t) = t$ then $k = 1$ and $\mu = 0$, we have the inequality of the theorem (2) (Gordji, Dragomir and Delaver).

$$
\left| \frac{f(b_1) + f(b_2)}{2} \int_{b_1}^{b_2} g(x)dx - \int_{b_1}^{b_2} f(x)g(x)dx \right| \\
\leq \frac{b_2-b_1}{4} \left[ 2 |f'(b_2)| + \eta(|f'(b_1)|, |f'(b_2)|) \right] \int_0^1 \int_{\frac{b_1+tb_2}{2}}^{\frac{b_1+tb_2}{2}} g(u)du dt
$$

(3.20)
Corollary 5. In corollary (3.17) if we choose \( g = 1, \eta(x, y) = x - y \), we have the following inequality:

\[
\left| \frac{f(b_1) + f(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) dx \right| \leq \frac{b_2 - b_1}{8} [f'(b_1) + f'(b_2)].
\]

(3.21)

for convex functions that is equivalent to Theorem (1.2) in [1].

4. Application to means

For two positive numbers \( b_1 > 0 \) and \( b_2 > 0 \), define

\[
\begin{align*}
A(b_1, b_2) &= \frac{b_1 + b_2}{2}, \\
G(b_1, b_2) &= \sqrt{b_1 b_2}, \\
H(b_1, b_2) &= \frac{2b_1 b_2}{b_1 + b_2}, \\
L(b_1, b_2) &= \begin{cases} \frac{b_1^{s+1} - b_2^{s+1}}{(s+1)(b_2 - b_1)}, & b_1 \neq b_2 \\ b_1, & b_1 = b_2, \end{cases} \\
I(b_1, b_2) &= \begin{cases} \frac{\log b_2}{\log b_1}, & b_1 \neq b_2 \\ b_1, & b_1 = b_2 \end{cases} \\
H_{w,s}(b_1, b_2) &= \begin{cases} \left[ \frac{b_1^{s+w(b_1 b_2)^{s-1}} + b_2^{s+w(b_1 b_2)^{s-1}}}{w+2} \right]^{\frac{1}{s}}, & s \neq 0 \\ \sqrt{b_1 b_2}, & s = 0 \end{cases}
\end{align*}
\]

(4.1)

for \( 0 \leq w \leq \infty \). These means are respectively called the arithmetic, geometric, harmonic, generalized logarithmic, identric and Heronian means of two positive numbers \( b_1 \) and \( b_2 \).

Applying Theorem (3.1) to \( f(x) = x^s \) for \( s \neq 0 \) and \( x > 0 \) result in the following inequalities for means.

Theorem 4.1. Let \( b_1 > 0, b_2 > 0, q \geq 1, \) either \( s > 1 \) and \((s - 1)q \geq 1\) or \( s < 0 \). Then

\[
\begin{align*}
&\left| A(\alpha b_1^s, \beta b_2^s) + \frac{2 - \alpha - \beta}{2} A'(b_1, b_2) - L'(b_1, b_2) \right| \\
&\leq \left( \frac{b_2 - b_1}{8} \right) (2)^{\frac{1}{q}} \left( 1 - 2\alpha + 2\alpha^2 \right)^{1 - \frac{1}{q}} \left[ \frac{1}{2} (1 - 2\alpha + 2\alpha^2) |sb_2^s|^{\frac{1}{q}} \right. \\
&\quad + \int_0^1 [1 - \alpha - t] \left( h \left( \frac{1 + t}{2} \right) \eta (|sb_1^s|^{\frac{1}{q}}, |sb_2^s|^{\frac{1}{q}}) \right) dt \\
&\quad - \frac{\mu}{4} (b_1 - b_2)^2 \left( -2\alpha^4 + 8\alpha^3 - 8\alpha + 5 \right) \left[ \frac{1}{2} (1 - 2\beta + 2\beta^2) |sb_2^s|^{\frac{1}{q}} \right. \\
&\quad \times \left[ \frac{1}{2} (1 - 2\beta + 2\beta^2) |sb_2^s|^{\frac{1}{q}} \right] + \int_0^1 |\beta - t| h \left( \frac{t}{2} \right) \eta (|sb_1^s|^{\frac{1}{q}}, |sb_2^s|^{\frac{1}{q}}) dt \\
&\quad - \frac{\mu}{4} (b_1 - b_2)^2 \left( -2\beta^4 + 8\beta^3 - 8\beta + 5 \right) \left] \right|^{\frac{1}{s}} \right).
\end{align*}
\]

(4.2)
Theorem 4.3. For $b_1 > 0$, $b_2 > 0$, $b_1 \neq b_2$ and $q \geq 1$, we have

\[
\frac{\ln G^2(b_1, b_2)}{2} + \frac{2 - \alpha - \beta}{2} \ln A(b_1, b_2) - \ln I(b_1, b_2)
\leq \left(\frac{b_2 - b_1}{8}\right)^{\frac{1}{2}} \left\{ (1 - 2\alpha + 2\alpha^2)^{1 - \frac{1}{2}} \left[ \frac{1}{b_2} (1 - 2\alpha + 2\alpha^2) \right] \right\}^q
\]

\[
+ \int_0^1 |1 - \alpha - t| h \left( \frac{1 + t}{2} \right) \eta \left( \frac{1}{b_1}, \frac{1}{b_2} \right) dt
\]

\[
- \frac{\mu}{4} (b_1 - b_2)^2 \left( \frac{-2\alpha^4 + 8\alpha^3 - 8\alpha + 5}{12} \right)^{\frac{1}{2}} + (1 - 2\beta + 2\beta^2)^{1 - \frac{1}{q}}
\]

\[
\times \left[ \frac{1}{2} (1 - 2\beta + 2\beta^2) \left( \frac{1}{b_2} \right) + \int_0^1 |\beta - t| h(t) \eta \left( \frac{1}{b_1}, \frac{1}{b_2} \right) dt
\]

\[
- \frac{\mu}{4} (b_1 - b_2)^2 \left( \frac{-2\beta^4 + 8\beta^3 - 8\beta + 5}{12} \right)^{\frac{1}{2}} \right\}. \tag{4.3}
\]

Finally, we can establish an inequality for the Heronian mean as follows.

Theorem 4.4. For $b_2 > b_1 > 0$, $b_1 \neq b_2$ and $q \geq 4$, we have

\[
\left| H_{w,s}^*(b_1, b_2) \right| + H^{e,1}_{w, 0} \left( \frac{b_2}{b_1}, \frac{b_1}{b_2}, 1 \right) - H^{e,1}_{w, 0} \left( \frac{L(b_1, b_2)}{G^2(b_1, b_2)}, 1 \right)
\leq \frac{(b_2 - b_1) A(b_1, b_2)}{2G^2(b_1, b_2)} \left\{ \frac{1}{2} |s| \left( \frac{G^{2(s-1)}(b_1, 1)}{b_2} + \frac{w}{2} G^{s-\frac{1}{2}}(b_2, 1) \right) \right\}
\]

\[
+ \eta \left( \frac{|s|}{w + 2} \left( \frac{G^{2(s-1)}(b_1, 1)}{b_2} + \frac{w}{2} G^{s-\frac{1}{2}}(b_2, 1) \right), \frac{|s|}{w + 2} \left( \frac{G^{2(s-1)}(b_2, 1)}{b_1} + \frac{w}{2} G^{s-\frac{1}{2}}(b_2, 1) \right) \right)
\]

\[
\times \int_0^1 \left[ h \left( \frac{1 + t}{2} \right) + h(t) \right] dt - \left( \frac{5}{64} \right) \mu \left( \frac{(b_1 - b_2) A(b_1, b_2)}{G^2(b_1, b_2)} \right)^2 \right\}. \tag{4.4}
\]

Proof. Let $f(x) = \frac{x^s + \alpha x^{s+1}}{w + 2}$ for $x > 0$ and $s \notin (1, 4)$. Then

\[
f'(x) = \frac{s}{w + 2} \left( x^{s-1} + \frac{w}{2} x^{s+1} \right).
\]

By corollary (3.6) it follows that
\[
\begin{align*}
\left| \frac{1}{2} \left[ f \left( \frac{b_2}{b_1} \right) + f \left( \frac{b_1}{b_2} \right) \right] + f \left( \frac{b_1}{b_2} + \frac{b_2}{b_1} \right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(x) dx \right| \\
= \left| \frac{1}{2} \left\{ \left[ b_2^s w(b_1 b_2)^{\frac{s}{2}} + b_1^s \right] + \frac{b_1^s w(b_1 b_2)^{\frac{s}{2}} + b_2^s}{b_2^s (w + 2)} \right\} \right| \\
= \left| \frac{1}{w + 2} \left\{ \left( \frac{b_2}{b_1} \right)^{\frac{s}{2} + 1} - \left( \frac{b_1}{b_2} \right)^{\frac{s}{2} + 1} \right\} \right|
\end{align*}
\]

\[
= \frac{1}{H(b_1, b_2)} + \frac{1}{H^s_{w, s+1}(b_1, b_2)} + \frac{1}{H^s_{w, s+1}(b_1, b_2)} - \frac{L(b_1, b_2)}{G^s_{w, s+1}(b_1, b_2, 1)} \right|.
\] 

(4.5)

On the other hand, we have
\[
\frac{b_2 - b_1}{4 b_1 b_2} \left\{ \left[ f' \left( \frac{b_2}{b_1} \right) \right] + \eta \left\{ f' \left( \frac{b_1}{b_2} \right), f' \left( \frac{b_2}{b_1} \right) \right\} \right| \left[ h \left( \frac{1 + t}{2} \right) \right] \right| \left| dt - \frac{\mu}{2} \left( \frac{b_2 - b_1}{b_1 b_2} \right)^2 \left( \frac{5}{128} \right) \right|
\]
\[
= \frac{b_2 - b_1}{4 b_1 b_2} \left\{ \left[ f' \left( \frac{b_2}{b_1} \right) \right] + \eta \left\{ f' \left( \frac{b_1}{b_2} \right), f' \left( \frac{b_2}{b_1} \right) \right\} \right| \left[ h \left( \frac{1 + t}{2} \right) \right] \right| \left| dt - \frac{\mu}{2} \left( \frac{b_2 - b_1}{b_1 b_2} \right)^2 \left( \frac{5}{128} \right) \right|
\]
\[
= \frac{(b_2 - b_1)}{2G^s_{w, s+1}(b_1, b_2)} \left[ \left( \frac{1}{2} |s| \right) \left( \frac{1}{w + 2} \right) \left( G^{2(s-1)}(b_2, 1) \right) + \frac{w}{2} G^{s-\frac{1}{2}}(b_2, 1) \right]
\]
\[
+ \eta \left[ \left( \frac{1}{w + 2} \right) \left( G^{2(s-1)}(b_1, 1) \right) + \frac{w}{2} G^{s-\frac{1}{2}}(b_1, 1) \right] \left[ \frac{|s|}{w + 2} \left( G^{2(s-1)}(b_2, 1) \right) + \frac{w}{2} G^{s-\frac{1}{2}}(b_2, 1) \right] \right|.
\] 

(4.6)

Obviously (4.5) and (4.6) yield (4.4).

\[
5. \text{ Conclusions}
\]

Fractional differential and integral equations play increasingly important roles in the modeling of engineering and science problems. It has been established fact that, in many situations, these models provide more suitable results than analogous models with integer derivatives. Fractional integral inequality results when \(0 < q < 1\) can be developed when the nonlinear term is increasing and satisfies a one sided Lipschitz condition. Using the integral inequality result and the computation of the solution of the linear fractional equation of variable coefficients, Gronwall inequality results can
be established. In the present report, we developed the fractional integral inequalities for more broader class of convex functions named as generalized strongly modified $h$-convex functions, we also established some applications of derived inequalities to means. Our results extend and generalize many existing results, for example [1,23,33–35].

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Conflict of interest

The authors declare that no competing interests exist.

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