Some New Classes of Topological Spaces and Annihilator Ideals

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Dedicated to Professor William Wistar Comfort

Abstract. By a characterization of semiprime $SA$-rings by Birkenmeier, Ghirati and Taherifar in [4, Theorem 4.4], and by the topological characterization of $C(X)$ as a Baer-ring by Stone and Nakano in [11, Theorem 3.25], it is easy to see that $C(X)$ is an $SA$-ring (resp., $IN$-ring) if and only if $X$ is an extremally disconnected space. This result motivates the following questions: Question (1): What is $X$ if for any two ideals $I$ and $J$ of $C(X)$ which are generated by two subsets of idempotents, $Ann(I) + Ann(J) = Ann(I \cap J)$? Question (2): When does for any ideal $I$ of $C(X)$ exists a subset $S$ of idempotents such that $Ann(I) = Ann(S)$? Along the line of answering these questions we introduce two classes of topological spaces. We call $X$ an $EF$ (resp., $EZ$)-space if disjoint unions of clopen sets are completely separated (resp., every regular closed subset is the closure of a union of clopen subsets). Topological properties of $EF$ (resp., $EZ$)-spaces are investigated. As a consequence, a completely regular Hausdorff space $X$ is an $F_\alpha$-space in the sense of Comfort and Negrepontis for each infinite cardinal $\alpha$ if and only if $X$ is an $EF$ and $EZ$-space. Among other things, for a reduced ring $R$ (resp., $J(R) = 0$) we show that $Spec(R)$ (resp., $Max(R)$) is an $EZ$-space if and only if for every ideal $I$ of $R$ there exists a subset $S$ of idempotents of $R$ such that $Ann(I) = Ann(S)$.

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1 Preliminaries

A space $X$ is extremally disconnected (resp., basically disconnected) if the closure of every open subset is clopen in $X$ (resp., if the closure of any cozeroset is open). It is well known that $X$ is an extremally disconnected space if and only if any two disjoint open subsets of $X$ are completely separated if and only if every open subset of $X$ is $C^*$-embedded. In an extremally disconnected space all dense subsets are
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A topological space is said to be zero-dimensional if it is a non-empty $T_1$-space with a base consisting of clopen sets. Zero-dimensional spaces were defined by Sierpinski in [15]. All zero-dimensional spaces are completely regular. A zero-dimensional space need not be a normal space. The space $\beta T = W^* \times N^*$ where $T$ is the Tychonoff plank is an example of non-normal zero-dimensional space (see [7, Example 16.18]). A space $X$ is totally disconnected if and only if the components in $X$ are the singletons. Equivalently, $X$ is totally disconnected if and only if the only non-empty connected subsets of $X$ are the one-point sets. The following implications characterize the relationship among the notions defined above: $X$ is extremally disconnected and $T_3$ implies $X$ is zero-dimensional implies $X$ is totally disconnected. None of the implications can be reversed and counterexamples exist even in the class of metric spaces. For terminology and notations, the reader is referred to [6] and [7].

All rings are assumed to be commutative with identity. By a reduced ring, we mean a ring without nonzero nilpotent elements. For each subset $S$ of a ring $R$, $Ann(S) = \{ r \in R : rs = 0, \forall s \in S \}$. A ring $R$ is called a SA (resp., IN)-ring if for any two ideals $I, J$ of $R$, $Ann(I) + Ann(J) = Ann(K)$, for some ideal $K$ of $R$ (resp., $Ann(I) + Ann(J) = Ann(I \cap J)$) (see [4]). We denote the Jacobson radical of $R$ by $J(R)$. For terminology and notations, the reader is referred to [9].

Throughout this paper, we denote by $C(X)$, the ring of all real-valued continuous functions on a completely regular Hausdorff space $X$, and $C^*(X)$ is its subring of bounded functions. A completely regular Hausdorff space $X$ is an $F$-space if its cozerosets are $C^*$-embedded. Equivalently, $X$ is an $F$-space if finitely generated ideals of $C(X)$ are principal. For terminology and notations, the reader is referred to [7].

For the proof of the following lemma see [6, Corollary 3.6.5].

**Lemma 1.1.** If $A$ is a clopen subset of a topological space $X$, then $cl_{\beta X} A$ is a clopen subset of $\beta X$.

For the proof of the following theorems see [7, 1.17] and [7, 1.15].

**Theorem 1.2.** A subset $S$ of $X$ is $C^*$-embedded in $X$ if and only if any two completely separated sets in $S$ are completely separated in $X$.

**Theorem 1.3.** Two sets are completely separated if and only if they are contained in disjoint zero-sets. Moreover, completely separated sets have disjoint zero-set-neighborhoods.

## 2 $EF$-space

We call a topological space $X$ an $EF$-space if for any two collections $\mathcal{U}$ and $\mathcal{V}$ of clopen subsets of $X$ with $\bigcup\mathcal{U} \cap \bigcup\mathcal{V} = \emptyset$, we have $\bigcup\mathcal{U}$ and $\bigcup\mathcal{V}$ are completely separated. The class of $EF$-space contains the class of spaces which are sums of connected spaces, all spaces for which the closure of any union of clopen subsets is open (hence all extremally disconnected spaces) and all spaces which any union of clopen subsets
is $C^*$-embedded. If we take $X$ as the sum of $\mathbb{R}$ (with usual topology) and $\mathbb{N}$, then $X$ is an $EF$-space which is neither connected nor extremally disconnected. A zero-dimensional space need not be an $EF$-space. In fact, it is straightforward to check that if $X$ is a zero-dimensional space, then $X$ is an $EF$-space if and only if it is extremally disconnected. In this section, we prove that for any two ideals $I$ and $J$ of $C(X)$, which are generated by two subsets of idempotents, $\text{Ann}(I) + \text{Ann}(J) = \text{Ann}(I \cap J)$ (Question 1) if and only if $X$ is an $EF$-space if and only if $\beta X$ is an $EF$-space.

Lemma 2.1. Let $X$ be a normal space. The following statements are equivalent.

(a) $X$ is an $EF$-space.

(b) For any two collections $\mathcal{U}$ and $\mathcal{V}$ of clopen subsets of $X$ with $\bigcup \mathcal{U} \cap \bigcup \mathcal{V} = \emptyset$, we have $\text{cl}(\bigcup \mathcal{U}) \cap \text{cl}(\bigcup \mathcal{V}) = \emptyset$.

Proof. (a) ⇔ (b) This follows from the definition of $EF$-space and the fact that in a normal space disjoint closed subsets are completely separated.

Proposition 2.2. For a topological space $X$ consider the following statements:

(a) Any union of clopen subsets of $X$ is $C^*$-embedded.

(b) The closure of any union of clopen subsets of $X$ is an open subset.

(c) $X$ is an $EF$-space.

(d) Disjoint unions of clopen subsets have disjoint closures

(e) If $O_1$ and $O_2$ are disjoint open subsets with $O_1$ a union of clopen sets, then their closures are disjoint.

Then (a) ⇒ (c) ⇒ (d) and (b) ⇔ (e) ⇒ (c).

Proof. (a) ⇒ (c) Let $\mathcal{U}$ and $\mathcal{V}$ be two collections of clopen subsets and $\bigcup \mathcal{U} \cap \bigcup \mathcal{V} = \emptyset$. Define $f(x) = 1$ for $x \in \bigcup \mathcal{U}$ and $f(x) = 0$ for $x \in \bigcup \mathcal{V}$. Then $f \in C^*(\bigcup \mathcal{U} \cup \bigcup \mathcal{V})$. By hypothesis, there exists $g \in C^*(X)$ such that $g|_{\bigcup \mathcal{U} \cup \bigcup \mathcal{V}} = f$. Therefore $\bigcup \mathcal{U} \subseteq Z(1 - g)$ and $\bigcup \mathcal{V} \subseteq Z(g)$, i.e., they are completely separated.

(c) ⇒ (d) Trivial.

(b) ⇔ (e) This follows from [11 Proposition 3.29].

(e) ⇒ (c) Assume that $\mathcal{U}$ and $\mathcal{V}$ are two collections of clopen subsets and $\bigcup \mathcal{U} \cap \bigcup \mathcal{V} = \emptyset$. Then $\text{cl}(\bigcup \mathcal{U}) \cap \text{cl}(\bigcup \mathcal{V}) = \emptyset$. Because $\text{cl}(\bigcup \mathcal{U})$ and $\text{cl}(\bigcup \mathcal{V})$ are clopen subsets. Hence $\text{cl}(\bigcup \mathcal{U})$ and $\text{cl}(\bigcup \mathcal{V})$ are disjoint zero sets, so $\bigcup \mathcal{U}$ and $\bigcup \mathcal{V}$ are completely separated.

In the following example, we see that (a) does not imply (b) in general. This example was suggested to the author by A. Dow.

Example 2.3. Let $\mathbb{N}$ denote the integers and let $p$ be an ultra-filter on $\mathbb{N}$. Consider $X = \mathbb{N} \cup \{p\} \cup [0,1]$. Now take the quotient space of $X$ in which the point $p$ is identified with $0$ in $[0,1]$. Then, we can see that any union of clopen subsets is $C^*$-embedded. On the other hand $\mathbb{N}$ is the union of clopen sets but its closure includes $0$ from $[0,1]$ and so is not clopen.
**Proposition 2.4.** If the closure of any union of clopen subsets is open and any union of clopen subsets is $C^*$-embedded in its closure, then any union of clopen subsets of $X$ is $C^*$-embedded in $X$.

**Proof.** Let $U$ be a collection of clopen subsets and $A$ and $B$ be completely separated in $\bigcup U$. By Theorems 1.2 and 1.3 it is enough to prove that they are contained in disjoint zero-sets in $X$. By hypothesis, there are $g_1, g_2 \in C^*(cl(\bigcup U))$ such that $A \subseteq Z(g_1), B \subseteq Z(g_2)$ and $Z(g_1) \cap Z(g_2) = \emptyset$. Again by hypothesis, there exists an idempotent element $f \in C^*(X)$ such that $cl(\bigcup U) = coz(f)$. Now we define $h_1(x) = 0$, for all $x \notin coz(f)$ and $h_1(x) = (g_1 f)(x)$, for all $x \in coz(f)$. Also, define $h_2(x) = 1$, for all $x \notin coz(f)$ and $h_2(x) = (g_2 f)(x)$, for all $x \in coz(f)$. Then we have $h_1, h_2 \in C^*(X), A \subseteq Z(g_1) \subseteq Z(h_1), B \subseteq Z(g_2) \subseteq Z(h_2)$ and $Z(h_1) \cap Z(h_2) = \emptyset$. $\Box$

In [5], Comfort and Negrepontis restrict their attention to the class of zero-dimensional space. For a space $X$ and open subset $G$ of $X$, $G$ is of type $< \alpha$ if $G$ is the union of $< \alpha$ closed-and-open subsets of $X$. Then, in the sense of [5], a space $X$ is an $F_{\alpha}$-space if every open subset of $X$ of type $< \alpha$ is $C^*$-embedded in $X$ [5 pp. 350, 343]. So, by Proposition 2.2 we have the following result.

**Corollary 2.5.** If $X$ is an $F_{\alpha}$-space in the sense of [5] for each infinite cardinal $\alpha$, then $X$ is an EF-space.

**Example 2.6.** A closed subset of an EF-space need not be an EF-space. To see this, let $X = \beta \mathbb{N} \setminus \mathbb{N}$ as a closed subspace of the EF-space $\beta \mathbb{N}$. Then by [7, 6S], the sets $A' = cl_{\beta \mathbb{N}} A \setminus \mathbb{N}$ form a base for the open sets in $\beta \mathbb{N} \setminus \mathbb{N}$. So $X$ is a zero-dimensional space. On the other hand by [7, 6W, 3], $X$ is not extremally disconnected. Hence, $X$ is not an EF-space.

The following example shows that a $P$-space and hence a basically disconnected space need not be an EF-space.

**Example 2.7.** [7, 4. N] Let $X$ be an uncountable space in which all points are isolated points except for a distinguished point $s$. A neighborhood of $s$ is any set containing $s$ whose complement is countable, so any set containing $s$ is closed. It is easily seen that $X$ is a $P$-space. So $X$ is basically disconnected. Now, consider two disjoint uncountable subsets $U, V \subseteq X \setminus \{s\}$. Then we have $s \in cl U \cap cl V$. So $X$ is not an EF-space.

**Theorem 2.8.** A space $X$ is an EF-space if and only if $\beta X$ is an EF-space.

**Proof.** Let $\{A_\alpha : \alpha \in S\}$ and $\{A_\gamma : \gamma \in K\}$ be two collections of clopen subsets of $\beta X$ and

$$(\bigcup_{\alpha \in S} A_\alpha) \cap (\bigcup_{\gamma \in K} A_\gamma) = \emptyset.$$ 

Then we have

$$(\bigcup_{\alpha \in S} A_\alpha \cap X) \cap (\bigcup_{\gamma \in K} A_\gamma \cap X) = \emptyset.$$ 

By hypothesis, there are disjoint zero-sets $Z_1$ and $Z_2$ in $Z[X]$ such that;
\[(\bigcup_{\alpha \in S} A_\alpha) \cap X \subseteq Z_1 \text{ and } (\bigcup_{\gamma \in K} A_\gamma) \cap X \subseteq Z_2.\]

Therefore we have,
\[
\bigcup_{\alpha \in S} A_\alpha \subseteq \operatorname{cl}_{\beta X}(\bigcup_{\alpha \in S} A_\alpha) = \operatorname{cl}_{\beta X}((\bigcup_{\alpha \in S} A_\alpha) \cap X) \subseteq \operatorname{cl}_{\beta X} Z_1,
\]
\[
\bigcup_{\gamma \in K} A_\gamma \subseteq \operatorname{cl}_{\beta X}(\bigcup_{\gamma \in K} A_\gamma) = \operatorname{cl}_{\beta X}((\bigcup_{\gamma \in K} A_\gamma) \cap X) \subseteq \operatorname{cl}_{\beta X} Z_2.
\]

On the other hand \(Z_1 \cap Z_2 = \emptyset\) implies that \(\operatorname{cl}_{\beta X} Z_1 \cap \operatorname{cl}_{\beta X} Z_2 = \emptyset\). By normality of \(\beta X\), \(\operatorname{cl}_{\beta X} Z_1\) and \(\operatorname{cl}_{\beta X} Z_2\) are completely separated, i.e., \(\bigcup_{\alpha \in S} A_\alpha\) and \(\bigcup_{\gamma \in K} A_\gamma\) are completely separated.

Conversely, assume that \(\{A_\alpha : \alpha \in S\}, \{A_\gamma : \gamma \in K\}\) are two collections of clopen subsets of \(X\) and
\[
(\bigcup_{\alpha \in S} A_\alpha) \cap (\bigcup_{\gamma \in K} A_\gamma) = \emptyset.
\]

By Lemma 1.1 for each \(\alpha \in S\) and \(\gamma \in K\), \(\operatorname{cl}_{\beta X} A_\alpha\) and \(\operatorname{cl}_{\beta X} A_\gamma\) are clopen subsets of \(\beta X\), so
\[
(\bigcup_{\alpha \in S} \operatorname{cl}_{\beta X} A_\alpha) \cap (\bigcup_{\gamma \in K} \operatorname{cl}_{\beta X} A_\gamma) = \emptyset.
\]

This together with our hypothesis implies that \(\bigcup_{\alpha \in S} \operatorname{cl}_{\beta X} A_\alpha\) and \(\bigcup_{\gamma \in K} \operatorname{cl}_{\beta X} A_\gamma\) are completely separated in \(\beta X\). Hence \(\bigcup_{\alpha \in S} A_\alpha\) and \(\bigcup_{\gamma \in K} A_\gamma\) are completely separated in \(X\). \(\square\)

In the next theorem we will answer Question 1. This result is an algebraic characterization of a completely regular Hausdorff \(EF\)-space. First we need the following lemma.

**Lemma 2.9.** Let \(R\) be a reduced ring. The following statements are equivalent.

(a) For any two orthogonal ideals \(I\) and \(J\) of \(R\), \(\text{Ann}(I) + \text{Ann}(J) = R\).

(b) For any two ideals \(I\) and \(J\) of \(R\), \(\text{Ann}(I) + \text{Ann}(J) = \text{Ann}(I \cap J)\).

**Proof.** (a) \(\Rightarrow\) (b). We always have \(\text{Ann}(I) + \text{Ann}(J) \subseteq \text{Ann}(I \cap J)\). Now suppose that \(x \in \text{Ann}(I \cap J) = \text{Ann}(IJ)\). Then \(xIJ = 0\). So by (a) we have,
\[
\text{Ann}(xI) + \text{Ann}(J) = R.
\]

This shows that \(1 = a + b\), where \(a \in \text{Ann}(xI)\) and \(b \in \text{Ann}(J)\). Therefore \(x = xa + xb\), where \(xa \in \text{Ann}(I)\) and \(xb \in \text{Ann}(J)\) that is; \(\text{Ann}(I \cap J) \subseteq \text{Ann}(I) + \text{Ann}(J)\).

(b) \(\Rightarrow\) (a). If \(IJ = 0\), then by hypothesis, we have
\[
\text{Ann}(I) + \text{Ann}(J) = \text{Ann}(I \cap J) = \text{Ann}(IJ) = R.
\]

\(\square\)

Note that, for any subset \(S\) of \(C(X)\), \(\bigcup \text{COZ}(S) = \bigcup_{f \in S} \text{coz}(f)\).

**Theorem 2.10.** Let \(X\) be a completely regular Hausdorff space. The following statements are equivalent.

(a) The space \(X\) is an \(EF\)-space.
(b) For any two ideals $I$ and $J$ of $C(X)$ which are generated by two subsets of idempotents of $C(X)$, $Ann(I) + Ann(J) = Ann(I \cap J)$.

(c) For any two ideals $I$ and $J$ of $C^*(X)$ which are generated by two subsets of idempotent elements of $C^*(X)$, $Ann(I) + Ann(J) = Ann(I \cap J)$.

**Proof.** $(a) \Rightarrow (b)$ Assume that $I$ and $J$ are ideals of $C(X)$ generated by two subsets $S_1$ and $S_2$ of idempotents. By Lemma 2.9 we can let $IJ = 0$ and it is enough to prove that $Ann(I) + Ann(J) = C(X)$. Now $IJ = 0$ implies that 

\[ (\bigcup COZ(I)) \cap (\bigcup COZ(J)) = (\bigcup COZ(S_1)) \cap (\bigcup COZ(S_2)) = \emptyset. \]

So there are disjoint zero-sets $Z(f_1)$ and $Z(f_2)$ such that:

\[ \bigcup COZ(I) = \bigcup COZ(S_1) \subseteq Z(f_1) \] and \[ \bigcup COZ(J) = \bigcup COZ(S_2) \subseteq Z(f_2). \]

Therefore $f_1 \in Ann(I)$, $f_2 \in Ann(J)$ and $Z(f_1^2 + f_2^2) = \emptyset$. Hence $f_1^2 + f_2^2$ is a unit element in $Ann(I) + Ann(J)$, i.e., $Ann(I) + Ann(J) = C(X)$.

$(b) \Rightarrow (a)$ Suppose that $(\bigcup_{\alpha \in S} A_\alpha) \cap (\bigcup_{\beta \in K} A_\beta) = \emptyset$, where for each $\alpha \in S$, $\beta \in K$, $A_\alpha$ and $A_\beta$ are clopen subsets of $X$. It is easily seen that for each $\alpha \in S$ and $\beta \in K$, there are idempotent elements $e_\alpha$ and $e_\beta$ in $C(X)$ such that $A_\alpha = coz(e_\alpha)$ and $A_\beta = coz(e_\beta)$. Hence

\[ (\bigcup_{\alpha \in S} coz(e_\alpha)) \cap (\bigcup_{\beta \in K} coz(e_\beta)) = \emptyset. \]

Now assume that $I = < \{ e_\alpha : \alpha \in S \}$ and $J = < \{ e_\beta : \beta \in K \}$. Then we can see that $IJ = 0$. By hypothesis and Lemma 2.9, $Ann(I) + Ann(J) = C(X)$. Hence there are $f \in Ann(I)$ and $g \in Ann(J)$ such that $1 = f + g$. Therefore we have,

\[ \bigcup_{\alpha \in S} A_\alpha = \bigcup_{\alpha \in S} coz(e_\alpha) = \bigcup COZ(I) \subseteq Z(f), \]

and

\[ \bigcup_{\beta \in K} A_\beta = \bigcup_{\beta \in K} coz(e_\beta) = \bigcup COZ(J) \subseteq Z(g). \]

On the other hand $Z(f) \cap Z(g) = \emptyset$. Hence $X$ is an $EF$-space.

$(c) \Leftrightarrow (a)$ This is a consequence of $(a) \Leftrightarrow (b)$, Theorem 2.8 and the fact that $C^*(X)$ is isomorphic to $C(\beta X)$. $\square$

Recall that, for a subset $A$ of $X$, we have $M_A = \{ f \in C(X) : A \subseteq Z(f) \}$.

**Lemma 2.11.** $(a)$ For every subset $S$ of $C(X)$ we have, $Ann(S) = M(\bigcup COZ[S])$.

$(b)$ For subsets $A, B$ of $X$ we have, $cl_X A = cl_X B$ if and only if $M_A = M_B$.

**Proof.** $(a)$ Let $f \in Ann(S)$. Then $fg = 0$, for all $g \in S$. This implies that $\bigcup Coz[S] \subseteq Z(f)$, i.e., $f \in M(\bigcup Coz[S])$. Now $f \in M(\bigcup Coz[S])$, implies that, $Coz(g) \subseteq \bigcup Coz[S] \subseteq Z(f)$ for each $g \in S$, so $f \in Ann(S)$.

$(b)$ If $cl_X A = cl_X B$, then it is easily seen that $M_A = M_{cl_X A} = M_{cl_X B} = M_B$. Now suppose that $M_A = M_B$ and $x \in cl_X A$. Then $M_A = M_{cl_X A} \subseteq M_x$. If $x \notin cl_X B$, then by completely regularity of $X$, there exists $f \in C(X)$ such that $x \notin Z(f)$ and $B \subseteq Z(f)$, i.e., $f \in M_B \setminus M_x$, a contradiction. Hence $cl_X A \subseteq cl_X B$. Similarly, we
can prove that $\text{cl}_X B \subseteq \text{cl}_X A$. □

From [10], a commutative ring $R$ with identity is Baer if the annihilator of every nonempty subset of $R$ is generated by an idempotent. The next result is proved in [1]. Now we give a new proof using Lemma 2.11.

**Theorem 2.12.** [1, Theorem 3.5] $C(X)$ is a Baer ring if and only if $X$ is an extremally disconnected space.

**Proof.** Let $A$ be an open subset of $X$. Then by completely regularity of $X$, there exists a subset $S$ of $C(X)$ such that $A = \bigcup \text{COZ}[S]$. By hypothesis, there is an idempotent $e \in C(X)$ such that $\text{Ann}(S) = \text{Ann}(e)$. By Lemma 2.11, $M(\bigcup \text{COZ}[S]) = M(\text{coz}(e))$. Hence

$$\text{cl}A = \text{cl}(\bigcup \text{COZ}[S]) = \text{cl}(\text{coz}(e)) = \text{coz}(e)$$

is open.

Conversely, suppose that $S$ is a subset of $C(X)$. Then by hypothesis, $\text{cl}(\bigcup \text{COZ}[S])$ is open. So there exists an idempotent $e \in C(X)$ such that $\text{cl}(\bigcup \text{COZ}[S]) = \text{coz}(e)$. Again by Lemma 2.11, we have, $\text{Ann}(S) = M(\bigcup \text{COZ}[S]) = M(\text{coz}(e)) = \text{Ann}(e)$. □

Recall that a commutative ring $R$ is an $SA$-ring (resp., $IN$-ring) if for any two ideals $I$ and $J$ of $R$, $\text{Ann}(I) + \text{Ann}(J) = \text{Ann}(I \cap J)$, (see [4]).

**Corollary 2.13.** The following statements are equivalent.

(a) $X$ is an extremally disconnected space.

(b) $C(X)$ is an IN-ring.

(c) $C(X)$ is an SA-ring.

(d) The space of prime ideals of $C(X)$ is an extremely disconnected space.

**Proof.** This is a consequence of [4, Corollary, 4.5], and Theorem 2.12. □

In [16], Swardson introduced an $\alpha$-open subset as a set of the form $A = \bigcup U$, where $U$ is a collection of cozero-sets of $X$ with $|U| < \alpha$. She also defined an $F_\alpha$-space $X$ to be a Tychonoff space in which every $\alpha$-open subset of $X$ is $C^*$-embedded in $X$. She proved that a space $X$ is an $F_\alpha$-space if and only if any two disjoint $\alpha$-open subsets of $X$ are completely separated in $X$ (see [16, Theorem 2.3]).

**Theorem 2.14.** A Tychonoff space $X$ is an $F_\alpha$-space in the sense of [16] if and only if for any two $\alpha$-generated ideals $I$ and $J$ of $C(X)$, $\text{Ann}(I) + \text{Ann}(J) = \text{Ann}(I \cap J)$.

**Proof.** By [16, Theorem 2.3], the proof is similar to that of Theorem 2.10. □

Recall that, we mean of $\omega_1$-generated ideal is a countably generated ideal. We also note that if $I = (f_1, \ldots, f_n, \ldots)$ is a countably generated ideal in $C(X)$, then $\text{ann}(I) = \text{ann}(f)$, where $f = \sum_{i=1}^{\infty} \frac{|f_n|}{2^{i}(f_n + 1)}$.

**Corollary 2.15.** A topological space $X$ is an $F$-space if and only if for any two $f, g \in C(X)$, $\text{Ann}(f) + \text{Ann}(g) = \text{Ann}(fg)$. 
Proposition 3.1.  (a) Every extremally disconnected space is an EZ-space.

(b) If $X$ has a clopen $\pi$-base, then $X$ is an EZ-space.

(c) Every $T_1$-space with a dense set of isolated points is an EZ-space.

(d) Every open subset of an EZ-space is an EZ-space.

Proof.  (a) is obvious.

(b) Let $B$ be an open subset of $X$. Set $O$ as the collection of clopen subsets of $X$ contained in $cl_XB$. We claim that $cl_X(\bigcup O) = cl_XB$. Clearly $cl_X(\bigcup O) \subseteq cl_XB$. Suppose there is an $x \in cl_XB \setminus cl(\bigcup O)$. Since $cl_XB$ is regular closed set

Corollary 2.16.  [16] Proposition, 2.2] A completely regular Hausdorff space $X$ is an extremally disconnected space if and only if $X$ is an $F_\alpha$-space in the sense of [16] for each infinite cardinal $\alpha$.

Proof. This is a consequence of Corollary [2.13] and Theorem [2.14]
\( x \in \text{cl}_X \text{int}_X \text{cl}_X B, \) so that \( \text{int}_X \text{cl}_X B \cap (X \setminus \text{cl}_X (\bigcup \mathcal{O})) \) is nonempty. By hypothesis, there is a nonempty clopen subset in \( \text{int}_X \text{cl}_X B \cap (X \setminus \text{cl}_X (\bigcup \mathcal{O})) \). However, such a clopen set is contained in \( \text{cl}_X B \) and \( (X \setminus \text{cl}_X (\bigcup \mathcal{O})) \), a contradiction.

(c) A \( T_1 \)-space with a dense set of isolated points has a clopen \( \pi \)-base and hence is an \( EZ \)-space.

(d) Assume that \( Y \) is an open subset of \( X \). Then by hypothesis, for any open subset \( A \) of \( Y \), there exists a collection \( \{A_\alpha : \alpha \in S\} \) of clopen subsets of \( X \) such that \( \text{cl}_X (A) = \text{cl}_X (\bigcup_{\alpha \in S} A_\alpha) \). Now \( Y \) is open in \( X \), so

\[
\text{cl}_Y (A) = \text{cl}_X (A) \cap Y = \text{cl}_X (\bigcup_{\alpha \in S} A_\alpha) \cap Y = \text{cl}_Y (\bigcup_{\alpha \in S} A_\alpha \cap Y).
\]

\( \square \)

**Lemma 3.2.** Let \( X \) be a regular space. Then \( X \) is an \( EZ \)-space if and only if \( X \) has a clopen \( \pi \)-base.

**Proof.** By Proposition 3.1, the necessity is evident. Suppose \( X \) is an \( EZ \)-space and let \( O \) be any nonempty open subset of \( X \). Choose \( x \in O \) and by regularity choose an open neighborhood of \( x \), say \( T \), such that \( x \in T \subseteq \text{cl} T \subseteq O \). By hypothesis, \( \text{cl} T = \text{cl} U \) where \( U \) is a union of clopen subsets. Then any nonempty clopen subset of \( U \) is also a nonempty clopen subset of \( O \). Therefore, \( X \) has a clopen \( \pi \)-base. \( \square \)

By Lemma 3.2, [13] Proposition 18] and [13] Corollary 19], we have \( X \) has clopen \( \pi \)-base if and only if \( \beta X \) has a clopen \( \pi \)-base.

The following example shows that the regularity hypothesis, in Lemma 3.2, is not superfluous and hence an \( EZ \)-space need not have a clopen \( \pi \)-base.

**Example 3.3.** Let \( X = \mathbb{R} \) with the topology \( \mathcal{T} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\} \). Then \( X \) is a non-regular space and any nonempty open subset is dense, so \( X \) is an \( EZ \)-space. On the other hand \( X \) does not have a clopen \( \pi \)-base.

Example 3.3 motivates the following question: Does there exist a Hausdorff \( EZ \)-space with no clopen \( \pi \)-base?

Recall that from [3], a \( DC \)-space is a Tychonoff space \( X \) such that for each \( f \in C(X) \) there exists a family of open subsets \( \{U_i : i \in I\} \), the union of which is dense in \( X \), such that \( f \), restricted to each \( U_i \), is constant. By [2] Lemma 2.5], and Proposition 3.1 any separable \( DC \)-space is an \( EZ \)-space.

**Proposition 3.4.** Let \( X = \bigoplus_{\alpha \in S} X_\alpha \). Then \( X \) is an \( EZ \)-space if and only if each \( X_\alpha \) is an \( EZ \)-space.

**Proof.** (\( \Rightarrow \)) Each \( X_\alpha \) is open in \( X \). By Proposition 3.1 (d), each \( X_\alpha \) is an \( EZ \)-space.

(\( \Leftarrow \)) Suppose that \( A \) is an open subset in \( X \). Then \( A \cap X_\alpha \) is open in \( X_\alpha \). Therefore, for each \( \alpha \in S \), there exists a collection \( \{A_\alpha^\beta : \beta \in S_\alpha\} \) of clopen subsets of \( X_\alpha \) such that \( \text{cl}_{X_\alpha} (A \cap X_\alpha) = \text{cl}_{X_\alpha} (\bigcup_{\beta \in S_\alpha} A_\alpha^\beta) \). Now it is easy to see that \( \text{cl}_X A = \text{cl}_X \bigcup_{\alpha \in S} (\bigcup_{\beta \in S_\alpha} A_\alpha^\beta) \), i.e., \( X \) is an \( EZ \)-space. \( \square \)

**Proposition 3.5.** Every dense subset of an \( EZ \)-space is an \( EZ \)-space.
Proof. Let $Y$ be a dense subset of a topological space $X$ and $A$ be an open subset of $Y$. Then there exists an open subset $G$ of $X$ such that $A = G \cap Y$. By hypothesis, there exists a collection $\{A_\alpha : \alpha \in S\}$ of clopen subsets of $X$ such that $\text{cl}_X G = \text{cl}_X (\bigcup_{\alpha \in S} A_\alpha)$. On the other hand $\text{cl}_X A = \text{cl}_X (G \cap Y) = \text{cl}_X G$. So we have, $\text{cl}_Y A = \text{cl}_X A \cap Y = \text{cl}_X (\bigcup_{\alpha \in S} A_\alpha \cap Y) \cap Y = \text{cl}_Y (\bigcup_{\alpha \in S} A_\alpha \cap Y)$. □

A completely regular Hausdorff $EZ$-space need not even be a totally disconnected space. The following example, (i.e., (a)) was presented by Brian Scott for another purpose. Also, by Proposition [3.5] we have another example of an $EZ$-space which is not a zero-dimensional space.

Example 3.6. (a) For $n \in \mathbb{Z}^+$ let

$$D_n = \{(\frac{2k+1}{2^n}, \frac{1}{2^n}) : k = 0, \ldots, 2^n-1, 1\},$$

and let $D = \bigcup_{n \in \mathbb{Z}^+} D_n$. Now suppose that $X = D \cup \{(a,0) : a \in [0,1]\}$ as a subspace of $\mathbb{R}^2$ with the usual topology. Then $D$ is a countable dense subset of isolated points in $X$. By Proposition [3.11] (c), $X$ is an $EZ$-space. On the other hand $X$ is not totally disconnected and hence $X$ is not zero-dimensional.

(b) By Dowker’s Example [5, Example, 6.2.20], we have a zero-dimensional space $Y$ for which $\beta Y$ is not zero-dimensional. On the other hand, by Proposition [3.5] $\beta Y$ is an $EZ$-space.

A closed subset of an $EZ$-space need not be an $EZ$-space. Because $Y = [0,1]$ as a closed subset of $X$ in the above example (i.e., (a)), is not an $EZ$-space.

In the following theorem we answer Question 2; it is an algebraic characterization of a completely regular Hausdorff $EZ$-space (i.e., a space with a clopen $\pi$-base).

Theorem 3.7. Let $X$ be a completely regular Hausdorff space. The following statements are equivalent.

(a) $X$ is an $EZ$-space (i.e., $X$ has a clopen $\pi$-base).

(b) For every ideal $I$ of $C(X)$, there exists a subset $S$ of idempotent elements of $C(X)$ such that $\text{Ann}(I) = \text{Ann}(S)$.

(c) For every $f \in C(X)$, there exists a subset $S$ of idempotent elements of $C(X)$ such that $\text{Ann}(f) = \text{Ann}(S)$.

(d) For every cozero-set $H$ of $X$ there exists a collection $\{H_\alpha : \alpha \in S\}$ of clopen subsets of $X$ such that $\text{cl}_X \bigcap \bigcup_{\alpha \in S} H_\alpha$.

Proof. (a) ⇒ (b) For an ideal $I$ of $C(X)$ consider, $A = \bigcup COZ[I]$. Then by hypothesis, there exists a collection $\{A_\alpha : \alpha \in S\}$ of clopen subsets of $X$ such that $\text{cl}_X A = \text{cl}_X (\bigcup_{\alpha \in S} A_\alpha)$. It is easily seen that for each $\alpha \in S$, there exists an idempotent $e_\alpha$ such that $A_\alpha = \text{coz}(e_\alpha)$. Now suppose that $S = \{e_\alpha : \alpha \in S\}$. Then by Lemma [2.11] we have,

$$\text{Ann}(I) = M_{\bigcup COZ[I]} = M_{\bigcup COZ[S]} = \text{Ann}(S).$$
(b) ⇒ (a) Let $A$ be an open subset of $X$. We know that in a completely regular space $X$, $COZ[X]$ is a base for open subsets. So there exists a subset $K$ of $C(X)$ such that $A = \bigcup COZ[K]$. Now suppose that $I$ is the ideal generated by $K$ in $C(X)$. Then by hypothesis, there exists a subset $S$ of idempotent elements of $C(X)$ such that $M_A = Ann(I) = Ann(S) = M_{\bigcup COZ[S]}$. Therefore by Lemma 2.11, we have $cl_X(A) = cl_X(\bigcup COZ[S])$.

(b) ⇒ (c) For any $f \in C(X)$, we have $Ann(f) = Ann(\langle f \rangle)$. By hypothesis, there exists a subset $S$ of idempotent elements such that $Ann(\langle f \rangle) = Ann(S)$. Hence $Ann(f) = Ann(S)$.

(c) ⇒ (b) Let $I$ be an ideal of $C(X)$. Then $Ann(I) = \cap_{f \in I} Ann(f)$. By hypothesis, for each $f \in I$ there exists a subset $S_f$ of idempotent such that $Ann(f) = Ann(S_f)$. Therefore

$$Ann(I) = \bigcap_{f \in I} Ann(S_f) = Ann(\bigcup_{f \in I} S_f).$$

(c) ⇔ (d) This is similar to that of (a) ⇔ (b) step by step. □

In [11], M. Knox and W. Wm. McGovern define $X$ to be a qsz-space if for any $f \in C(X)^+$ there exists a countable sequence $K_n$ of clopen subsets $X$ such that $cl_X\text{coz}(f) = cl_X\bigcup_{n \in \mathbb{N}} K_n$. So, by the above theorem, we have the following corollary.

**Corollary 3.8.** [11] Proposition 3.9] If $X$ is a qsz-space, then $X$ is an EZ-space.

Recall that, a subspace $Y$ of a space $X$ is $z$-embedded in $X$ if for every zero-set $Z$ in $Y$ there is a zero-set $H$ in $X$ such that $Z = H \cap Y$, equivalently, for every cozeroset of $Y$ there is a cozeroset of $X$ which traces to it. For example, a $C^*$-embedded subspace is clearly $z$-embedded (see [12]).

**Corollary 3.9.** If $X$ is a completely regular Hausdorff EZ-space, then every $z$-embedded subspace is an EZ-space.

**Proof.** Let $Y$ be a $z$-embedded subspace of an EZ-space $X$ and $H$ a cozero-set in $Y$. By hypothesis and Theorem 3.7, there exists a cozero-set $C$ in $X$ and a collection $\{A_\alpha : \alpha \in S\}$ of clopen subsets of $X$ such that $H = C \cap Y$ and $cl_X C = cl_X(\bigcup_{\alpha \in S} A_\alpha)$. Therefore

$$cl_Y H = cl_Y(C \cap Y) = cl_X C \cap Y = cl_Y(\bigcup_{\alpha \in S} A_\alpha \cap Y).$$

□

**Proposition 3.10.** Let $X$ be an EZ-space. Then the following statements are equivalent.

(a) $X$ is an EF-space.

(b) If $\mathcal{U}$ and $\mathcal{V}$ are two collections of clopen subsets of $X$ with $\bigcup \mathcal{U} \cap \bigcup \mathcal{V} = \emptyset$, then $cl(\bigcup \mathcal{U}) \cap cl(\bigcup \mathcal{V}) = \emptyset$.

(c) The closure of any union of clopen subsets of $X$ is an open subset.
(d) $X$ is an extremally disconnected space.

(e) Any union of clopen subsets of $X$ is $C^*$-embedded.

**Proof.** By Proposition 2.2 and definitions, it is clear that (d) $\Rightarrow$ (e) $\Rightarrow$ (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c) Let $U$ be a collection of clopen subsets and let $B$ be an open subset of $X$ which is disjoint from any element of $U$. By [11, Proposition 3.29], it is enough to prove that $clB \cap cl(\bigcup U) = \emptyset$. By hypothesis, there exists a collection $O$ of clopen subsets such that $clB = cl(\bigcup O)$. We have $B \cap (\bigcup U) = \emptyset$, so

$$clB \cap (\bigcup U) = cl(\bigcup O) \cap \bigcup U = \emptyset.$$  

This shows that $\bigcup O \cap \bigcup U = \emptyset$. Now hypothesis implies that,

$$clB \cap cl(\bigcup U) = cl(\bigcup O) \cap cl(\bigcup U) = \emptyset.$$  

(c) $\Rightarrow$ (d) By hypothesis, for any open subset $B$ there exists a collection $O$ of clopen subsets such that $clB = cl(\bigcup O)$. Now, by hypothesis, $clB$ is open, i.e., $X$ is extremally disconnected.

It is well known that if $X$ is zero-dimensional and an $F_\alpha$-space for each infinite cardinal $\alpha$ in the sense of [16], then $X$ is an $F_\alpha$-space for each infinite cardinal $\alpha$, in the sense of [5]. In the next result we see that the two concepts of $F_\alpha$-space coincide for each infinite cardinal $\alpha$.

**Corollary 3.11.** Let $X$ be a completely regular Hausdorff space. The following statements are equivalent.

(a) $X$ is an extremally disconnected space.

(b) $X$ is $EF$ and $EZ$-space.

(c) $X$ is an $F_\alpha$-space in the sense of [16] for each infinite cardinal $\alpha$.

(d) $X$ is an $F_\alpha$-space in the sense of [5] for each infinite cardinal $\alpha$.

(e) The closure of any union of clopen subsets is open and $X$ is an $EZ$-space.

**Proof.** (a) $\iff$ (c) Follows from Corollary 2.16

(a) $\Rightarrow$ (b) By Propositions 2.2 and 3.1, any extremely disconnected space is $EF$ and $EZ$.

(b) $\Rightarrow$ (a) Let $X$ be $EF$ and $EZ$-space. Then by Proposition 3.10, $X$ is an extremally disconnected space.

(a) $\Rightarrow$ (d) Every extremally disconnected $T_3$-space is zero-dimensional. On the other hand any open subset of an extremally disconnected space is $C^*$-embedded so any union of clopen subsets is $C^*$-embedded. Therefore $X$ is an $F_\alpha$-space in the sense of [5] for each infinite cardinal $\alpha$.

(d) $\Rightarrow$ (a) If $X$ is an $F_\alpha$-space in the sense of [5] for each infinite cardinal $\alpha$, then by Corollary 2.5, $X$ is an $EF$-space. On the other hand $X$ is zero-dimensional so is an $EZ$-space. Hence by Proposition 3.10, $X$ is an extremally disconnected space.

(a) $\Rightarrow$ (e) This is obvious.

(e) $\Rightarrow$ (a) Let $A$ be an open subset. Then by hypothesis, there exists a collection $\{A_\alpha : \alpha \in S\}$ of clopen subsets such that $clA = cl(\bigcup_{\alpha \in S} A_\alpha)$. Again by hypothesis, $clA$ is open, i.e., $X$ is an extremally disconnected space. □
4 \textit{Spec}(R) \textit{as an EZ-space}

In this section, for a reduced ring \(R\), we prove that \(\text{Spec}(R)\) is an EZ-space if and only if for every ideal \(I\) of \(R\) there exists a subset \(S\) of idempotents of \(R\) such that \(\text{Ann}(I) = \text{Ann}(S)\) (a general case of Question 2) if and only if for any \(a \in R\), there exists a subset \(S\) of idempotent elements of \(R\) such that \(\text{Ann}(a) = \text{Ann}(S)\).

Also, for a ring \(R\) satisfying \(J(R) = 0\), we show that \(\text{Max}(R)\) is an EZ-space if and only if for every ideal \(I\) of \(R\) there exists a subset \(S\) of idempotents of \(R\) such that \(\text{Ann}(I) = \text{Ann}(S)\).

For \(a \in R\), let \(\text{supp}(a) = \{P \in \text{Spec}(R) : a \notin P\}\). It is easy to see that for any \(R\), \(\{\text{supp}(a) : a \in R\}\) forms a basis of open sets for \(\text{Spec}(R)\) (i.e., the space of prime ideals of \(R\)). This topology is called the Zariski topology. We use \(V(I)(V(a))\) to denote the set of \(P \in \text{Spec}(R)\), such that \(I \subseteq P(a \in P)\). Note that \(V(I) = \bigcap_{a \in I} V(a)\) and \(V(a) = \text{Spec}(R) \smallsetminus \text{supp}(a)\) (see [9]).

For an open subset \(A\) of \(\text{Spec}(R)\), let \(O_{A} := \{a \in R : A \subseteq V(a)\}\). Since for any \(a, b \in R\), \(V(a) \cap V(b) \subseteq V(a - b)\) and for each \(r \in R\), \(a \in O_{A}\), we have \(V(a) \subseteq V(ra)\), thus \(O_{A}\) is an ideal of \(R\). It is easy to see that \(O_{A} = \bigcap_{P \in A} P\) and \(V(O_{A}) = \text{cl}A\), where \(\text{cl}A\) is the cluster points of \(A\) in \(\text{Spec}(R)\).

An ideal \(I\) of a commutative ring \(R\) is said to be an annihilator ideal provided that \(\text{Ann}(\text{Ann}(I)) = I\), equivalently, if \(\text{Ann}(I) \subseteq \text{Ann}(x)\), and \(x \in R\), then \(x \in I\).

We need the following lemma which consists of some well-known results.

\textbf{Lemma 4.1.} Let \(R\) be a reduced ring.

(a) For ideals \(I, J\) of \(R\), \(\text{Ann}(I) \subseteq \text{Ann}(J)\) if and only if \(\text{int}V(I) \subseteq \text{int}V(J)\).

(b) For an open subset \(A\) of \(\text{Spec}(R)\), \(O_{A}\) is an annihilator ideal.

(c) If \(I\) is an annihilator ideal of \(R\), then there exists an open subset \(A\) of \(\text{Spec}(R)\) such that \(I = O_{A}\).

(d) For open subsets \(A, B\) of \(\text{Spec}(R)\), \(O_{A} = O_{B}\) if and only if \(\text{cl}A = \text{cl}B\).

(e) \(A \subseteq \text{spec}(R)\) is a clopen subset if and only if there is an idempotent \(e \in R\) such that \(A = V(e) = \text{supp}(1 - e)\).

(f) For any ideal \(I\) of \(R\), \(\text{Ann}(I) = O_{\bigcup \text{supp}(I)}\).

\textbf{Proof.} (a) let \(I\) and \(J\) be two ideals of \(R\) and \(P \in \text{int}V(I)\). Then there is an \(a \in R\) such that \(P \in \text{supp}(a) \subseteq V(I)\). Hence \(\text{supp}(Ia) = \text{supp}(I) \cap \text{supp}(a) = \emptyset\), thus \(Ia = 0\). This implies that \(a \in \text{Ann}(I) \subseteq \text{Ann}(J)\), so \(Ja = 0\). Therefore \(P \in \text{supp}(a) \subseteq \text{int}V(J)\). Conversely, let \(x \in \text{Ann}(I)\). Then \(Ix = 0\). so \(\text{supp}(x) \subseteq \text{int}V(I) \subseteq \text{int}V(J)\). This shows that \(\text{supp}(Jx) = \text{supp}(x) \cap \text{supp}(J) = \emptyset\). Hence \(Jx = 0\), i.e., \(x \in \text{Ann}(J)\).

(b) Let \(\text{Ann}(O_{A}) \subseteq \text{Ann}(x)\). By (a),

\[A \subseteq \text{int}clA = \text{int}V(O_{A}) \subseteq \text{int}V(x) \subseteq V(x)\].
So $x \in O_A$.

e) Suppose that $I$ is an annihilator ideal and $A = intV(I)$. We claim that $I = O_A$. If $a \in I$, then $intV(I) \subseteq V(a)$, i.e., $a \in O_A$. Now let $a \in O_A$. Then $A = intV(I) \subseteq intV(a)$, so $Ann(I) \subseteq Ann(a)$. On the other hand $I$ is an annihilator, hence $a \in I$.

d) We have $O_A = O_B$. This implies that $V(O_B) = V(O_A)$, i.e., $cl(B) = cl(A)$. Conversely, $clA = clB$ implies that $O_A = O_{clA} = O_{clB} = O_B$.

e) Let $A$ be a clopen subset, $I = O_A$ and $J = O_A^c$. Then $A = clA = V(O_A) = V(I)$ and $A^c = V(O_A^c) = V(J)$. Hence $V(I) \cup V(J) = \emptyset$, so there are $a \in I$ and $b \in J$ such that $1 = a + b$. But $V(a) \cup V(b) = Spec(R)$, thus we have $ab = 0$, this implies that $a = a^2$ and $V(I) = V(a)$. The converse is evident.

(f) If $r \in Ann(I)$, then $ra = 0$, for all $a \in I$, so $\bigcup supp(I) \subseteq V(r)$, this shows that $r \in O_{\bigcup supp(I)}$. Now $r \in O_{\bigcup supp(I)}$ implies that $supp(a) \subseteq V(r)$, for all $a \in I$, so $supp(a) \cap supp(r) = supp(ra) = \emptyset$, i.e., $ra = 0$. Hence $r \in Ann(I)$. 

**Theorem 4.2.** Let $R$ be a reduced ring. The following statements are equivalent.

(a) The space of prime ideals, $Spec(R)$, is an EZ-space.

(b) For every ideal $I$ of $R$, there exists a subset $E$ of idempotents of $R$ such that $Ann(I) = Ann(E)$.

(c) For every $a \in R$, there exists a subset $S$ of idempotents of $R$ such that $Ann(a) = Ann(S)$.

(d) For any $a \in R$, there exists a clopen subset $S$ of $Spec(R)$ such that $cl(supp(a)) = cl(\bigcup supp(S))$.

**Proof.** (a) $\Rightarrow$ (b) Let $I$ be an ideal of $R$. Then we have $\bigcup supp(I)$ is an open subset of $Spec(R)$. By hypothesis, there exists a collection $\{A_\alpha : \alpha \in S\}$ of clopen subsets such that $cl(\bigcup supp(I)) = cl(\bigcup_{\alpha \in S} A_\alpha)$. By Lemma 4.1 for each $\alpha \in S$ there exists an idempotent $e_\alpha$ such that $A_\alpha = supp(e_\alpha)$. Therefore,

$$cl(\bigcup supp(I)) = cl(\bigcup_{\alpha \in S} supp(e_\alpha)).$$

Again by Lemma 4.1 we have $Ann(I) = Ann(E)$ where $E = \{e_\alpha : \alpha \in S\}$.

(b) $\Rightarrow$ (a) Let $A$ be an open subset of $Spec(R)$. Then there exists a subset $K$ of $R$ such that $A = \bigcup supp[K]$. Now suppose that $I$ be the ideal generated by $K$ in $R$. Then by hypothesis and Lemma 4.1, there exists a subset $E$ of idempotents of $R$ such that $O_A = Ann(I) = Ann(E) = O_{\bigcup supp[E]}$. Therefore, Lemma 4.1 implies that $cl(A) = cl(\bigcup supp[E])$.

(b) $\Rightarrow$ (c) This is evident.

(c) $\Rightarrow$ (b) For an ideal $I$ of $R$ we have $Ann(I) = \bigcap_{a \in I} Ann(a)$. By hypothesis, for each $a \in R$ there exists a subset $S_a$ of idempotents such that $Ann(a) = Ann(S_a)$. Hence $Ann(I) = \bigcap_{a \in I} Ann(S_a) = Ann(\bigcup_{a \in I} S_a)$.

(c) $\Leftrightarrow$ (d) By Lemma 4.1 $Ann(a) = Ann(S)$ for some subset $S$ of $R$ if and only if $O_{supp(a)} = O_{\bigcup supp(S)}$ if and only if $cl(supp(a)) = cl(\bigcup supp(S))$. 

We denote by $Max(R)$ the space of maximal ideals of $R$. For $a \in R$, let $D(a) =$
\{M \in \text{Max}(R) : a \notin M\}. It is easy to see that for any \(R\), \(\{D(a) : a \in R\}\) forms a basis of open sets on \(\text{Max}(R)\). This topology is called the Zariski topology. We use \(M(I)(M(a))\) to denote the set of \(M \in \text{Max}(R)\), where \(I \subseteq M(a) \in M\). Note that \(M(I) = \bigcap_{a \in I} M(a)\) and \(M(a) = \text{Max}(R) \setminus D(a)\) (see [10]).

For an open subset \(A\) of \(\text{Max}(R)\), suppose that \(M_A := \{a \in R : A \subseteq M(a)\}\). Then we can see that \(M_A\) is an ideal of \(R\), \(M_A = \bigcap_{M \in A} M\) and \(M(M_A) = \text{cl}A\), where \(\text{cl}A\) is the cluster points of \(A\) in \(\text{Max}(R)\).

**Lemma 4.3.** Let \(R\) be a ring satisfying \(J(R) = 0\).

(a) For ideals \(I, J\) of \(R\), \(\text{Ann}(I) \subseteq \text{Ann}(J)\) if and only if \(\text{int}M(I) \subseteq \text{int}M(J)\).

(b) For an open subset \(A\) of \(\text{Max}(R)\), \(M_A\) is an annihilator ideal.

(c) If \(I\) is an annihilator ideal of \(R\), then there exists an open subset \(A\) of \(\text{Max}(R)\) such that \(I = M_A\).

(d) For open subsets \(A, B\) of \(\text{Max}(R)\), \(M_A = M_B\) if and only if \(\text{cl}B = \text{cl}A\).

(e) \(A \subseteq \text{Max}(R)\) is a clopen subset if and only if there is an idempotent \(e \in R\) such that \(A = M(e) = D(1 - e)\).

(f) For any ideal \(I\) of \(R\), \(\text{Ann}(I) = M(\bigcup_{I \in I})\).

**Proof.** The proof is similar to that of Lemma [11].

**Theorem 4.4.** Let \(R\) be a ring satisfying \(J(R) = 0\). Then \(\text{Max}(R)\) is an \(EZ\)-space if and only if for any ideal \(I\) of \(R\) there exists a subset \(E\) of idempotents such that \(\text{Ann}(I) = \text{Ann}(E)\).

**Proof.** The proof is similar to that of Theorem [4.2] (a) \(\iff\) (b). Recall that a ring \(R\) is called *potent* if idempotents can be lifted (mod \(J(R)\)) and every ideal \(I\) of \(R\) which is not contained in \(J(R)\) contains a non-zero idempotent [14].

**Corollary 4.5.** Let \(R\) be a ring satisfying \(J(R) = 0\). If \(R\) is potent, then for any ideal \(I\) of \(R\) there exists a subset \(E\) of idempotents such that \(\text{Ann}(I) = \text{Ann}(E)\).

**Proof.** This is a consequence of [18] Proposition 4.4] and Theorem [14].

**Remark 4.6.** The converse of Corollary 4.5 need not be true. For example, let \(R = \mathbb{Z}\). Then for any nonzero ideal \(I\) of \(R\) we have \(\text{Ann}(I) = \text{Ann}(E)\), where \(E = \{1\}\) and \(J(R) = 0\). But \(R\) is not a potent ring.

**Corollary 4.7.** Let \(X\) be a completely regular Hausdorff space. The following statements are equivalent.

(a) \(X\) is an \(EZ\)-space.

(b) \(\text{Spec}(C(X))\) is an \(EZ\)-space.

(c) \(\text{Max}(C(X))\) is an \(EZ\)-space.

**Proof.** This is a consequence of Theorems [4.7] 4.2 and 4.3.
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