Multiplicity and regularity of periodic solutions for a class of degenerate semilinear wave equations.

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Abstract
We prove the existence of infinitely many classical periodic solutions for a class of degenerate semilinear wave equations:

\[ u_{tt} - u_{xx} + |u|^{s-1}u = f(x, t), \]

for all \( s > 1 \). In particular we prove the existence of infinitely many classical solutions for the case \( s = 3 \) posed by Brézis in [Brezis83]. The proof relies on a new upper a priori estimate, for minimax values of, a perturbed from symmetry, strongly indefinite functional.\(^1\)

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1 Introduction

In this paper we construct infinitely many classical time-periodic solutions for the following semilinear degenerate wave equation with time-dependent forcing term $f$:

$$u_{tt} - u_{xx} + g(u) - f(t, x) = 0$$  \hspace{1cm} (1.1)

$$u(0, t) = u(π, t) = 0.$$

(1.2)

where $g(u) = |u|^{s-1}u$ and $F(x, t, u) = g(u) + f(x, t)$, where $f$ is of class $C^2$ and satisfies the Dirichlet boundary conditions.

Brézis problem [Brézis83]: It seems reasonable to conjecture that when $g(u) = u^3$ problem (1.1), (1.2) possesses a solution -even infinitely many solutions- for every $f$(or at least a dense set of $f$’s.)

Theorem 1.1. If $f \in C^2$ then there exists infinitely many classical solutions of (1.1), (1.2) for all $s > 1$.

Theorem 1.1 also prove the existence of classical solutions for a question of Bahri-Berestycki in [BB84] on the existence of infinitely many solutions of (1.1), (1.2) for the class of $g(u) = |u|^{s-1}u$.

The weak version of the conjecture of Brézis, the existence of weak solutions for a dense set of $f$’s has been shown to be true by Tanaka in [Tanaka86]. The problem (1.1), (1.2), for a given $f$, has been studied by Tanaka [Tanaka86], Bartsch-Ding-Lee [BDL99], for arbitrary $s > 1$, and Bolle-Ghoussoub-Tehrani [BGT2000], Ollivry Ollivry83 for the case $1 < s < 2$ however only weak solutions have been obtained. As already noticed in [Rabinowitz71] there are two classes of monotone functions for problem (1.1), (1.2), the strongly monotone $F$, $\frac{\partial F}{\partial u} \geq \alpha > 0$ which can be compared to the uniformly elliptic case and the degenerate monotone case which allows $\frac{\partial F}{\partial u} = 0$. These two classes of monotone functions have been extensively studied by Torelli [Torelli69], Rabinowitz [Rabinowitz71], Hall [Hall70], Hale [Hale69], in the small perturbative case, i.e. with a smallness assumption on $f$. No such a smallness assumption is assumed here and the result we prove is a global one.

The difficulty in proving the regularity of the weak solutions obtained by [Tanaka88], [BDL99], [BGT2000] lies in the strong monotonicity assumption which is required by the regularity approach of Brézis-Nirenberg, [BN78-2]. In [BN78-2] Brézis and Nirenberg show that an $L^\infty$ weak solution is smooth as long as $F$ is smooth and satisfies the strong monotonicity assumption $\frac{\partial F}{\partial u} \geq \epsilon > 0$ which fails here as $g(u)$ has a vanishing derivative. Note that in the highly degenerate case where $F$ vanishes in an interval, weak solutions in $L^\infty$ need not to be smooth, see [BN78-2] or [BN78-1] theorem I.8. Therefore, to find classical periodic solutions we will proceed differently. In [Rabinowitz78] Rabinowitz developed a regularity theory for this type of degeneracy where $\frac{\partial F}{\partial u} = 0$ is allowed but $g$ strictly monotone ($z_1 > z_2$ implies $g(z_1) > g(z_2)$) for equations of the type (1.1), (1.2) and with $f = 0$. The approach in [Rabinowitz78] consisted in seeking viscous approximative solutions, studying a modified equation analogue
of (1.4) with $f = 0$:

$$w_{tt}(\beta) - w_{xx}(\beta) = -|u|^{s-1}u(\beta) + \beta v_{tt}(\beta)$$

(1.3)

(Here $u(\beta) = v(\beta) + w(\beta)$ and $v(\beta)$ is the component of $u(\beta)$ in the direction of the infinite dimensional kernel of $\Box$, with the Dirichlet-periodic boundary conditions. The solution $u$ is split in such a way to tackle the problem stemming from the infinite dimensional kernel of $\Box$.) With the parameter $\beta$ and obtaining compactness via upper priori estimates independently of $\beta$ of the critical values of the modified problem (1.3), enabling him to send $\beta$ to 0 and then finding classical solutions. However the problem here contains the forcing term $f$ and the natural functional associated with the problem (1.1) is no longer even thus the minimax sets for finding critical values in [Rabinowitz78] do not apply for forced vibrations.

In the eighties and nineties a perturbation theory for this type of problems -perturbation from symmetry- was developed, by Bahri-Berestycki [BB81], Bahri-Lions [BahriLions88], Tanaka [Tanaka89], Struwe [Struwe90], Rabinowitz [Rabinowitz82] and Bolle [Bolle99]. The approaches consist in finding growth estimates on some minimax values, and if they grow fast enough, will imply the existence of critical values of the perturbed functional. Hence it is therefore natural to try to implement these approaches, to tackle the regularity issues stemming from the degenerate monotone semilinear term $g(u)$ and the infinite dimensional kernel of $\Box$ under Dirichlet boundary conditions, to the modified equation, seeking viscous approximative solutions:

$$w_{tt}(\beta) - w_{xx}(\beta) = -|u|^{s-1}u(\beta) + \beta v_{tt}(\beta) + f(t,x).$$

(1.4)

However the approaches by [BB81], [BahriLions88], [Bolle99], [Struwe90], [Rabinowitz82], do not provide an upper explicit upper estimates on the critical values, and this lead to serious difficulties to obtain compactness of $u(\beta)$, as $\beta \to 0$. For even functionals, the identity map is an admissible function in the set of maps considered for the minimax procedure. Information gleaned from the identity map in [Rabinowitz84] has lead to explicit a priori estimates and hence compactness for free vibrations. For forced vibration such an explicit map is lacking and to overcome these difficulties we construct a map in the minimax sets of Rabinowitz [Rabinowitz82], whose energy in $J_\beta$ is controlled independently of $\beta$. The additional estimate thus obtained lead to the needed compactness needed to pass to the limit as $\beta \to 0$.

Having constructed minimax values $c_n^m(\delta)$ with upper a priori estimates independently of, the Galerkin parameter $m$ and $\beta$, we need information on the growth of some minimax values $b_n^m$ to show that the $c_n^m(\delta)$ are critical values. To obtain the lower estimates of the growth of the $b_n^m$ we employ the functional $K$ introduced by Tanaka in [Tanaka88] and the Borsuk-Ulam lemma of Tanaka [Tanaka88], see lemma 2.3.

Another advantage of our approach is that it simplifies the weak solutions approach of Tanaka88. In Tanaka88 some technical lemmas are employed to get information on the index of the weak solution $u$, obtained by passing to
the limit in the Galerkin parameter \( m \), the index of the critical value of the approximate solution \( u^m_n \), obtained from the Galerkin scheme. Here the upper estimate on \( c^n_m(\delta) \) is also independent of \( m \) thus it allows to simplify the passage to the limit as \( m \to \infty \).

Once the compactness of the sequence \( u(\beta) \) is obtained, the regularity will follow by adapting the argument of [Rabinowitz78] to the problem considered here, in presence of a forcing term \( f(x,t) \).

Remark: Upper estimates for critical values via the approach of [Bolle99] and under Dirichlet boundary conditions are in [CDHL2004] by Castro, Ding and Hernandez-Linares, and Castro and Clapp [CastroClapp2006], for perturbation of a differential operator, the Laplacian, the noncooperative elliptic system:

\[
-\Delta u = |u|^{p-1}u + f_u(x,u,v) \quad (1.5)
\]
\[
\Delta v = |v|^{q-1}v + f_v(x,u,v) \quad (1.6)
\]
\[
v|_{\partial Q} = u|_{\partial Q} = 0. \quad (1.7)
\]

However the approaches in [CDHL2004], [CastroClapp2006] are incomplete as they rely on estimating \( \int_Q |\nabla \tau(u)|^2 dx \) for \( u \in H^1_0(Q) \) but the functional \( \tau : H^1_0(Q) \to \mathbb{R} \) is not Fréchet differentiable and the authors do not define what they mean by \( \nabla \tau(u)|u| \int_Q |\nabla \tau(u)|^2 dx \), for arbitrary \( u \in H^1_0(Q) \).

In Section 1: There is a functional \( I_\beta \) whose critical points correspond formally to solutions of (1.4). However as indicated by the approach of [Rabinowitz82], for technical reasons we will work with another functional \( J_\beta \). We prove Palais-Smale conditions at large energies independently of \( \beta \) for the functional \( J_\beta \) and show implications for the functional \( I_\beta \).

In Section 2: We construct the map \( H \) whose energy is bounded independently of \( \beta \). This is the main novelty of the paper which leads to the compactness needed to show the existence of classical solutions.

In Section 3 we adapt the arguments of [Rabinowitz78] and [Rabinowitz84] to end the proof. First we show that \( u(\beta) \) is a classical solution of the modified equation (1.4) then we obtain a \( C^0 \) estimate for \( w(\beta) \). This is followed by a \( C^0 \) on \( v(\beta) \), and the existence of a \( C^0 \)-solution \( u \) is proved. We then use the bootstrapping argument in [Rabinowitz78] to prove the existence of classical solutions. The multiplicity is deduced by noticing the lower estimates on the critical values \( c^n_m(\delta) \) go to infinity as \( n \to \infty \).

Functional \( I_\beta \):

We define the functional \( I_\beta \):

\[
I_\beta(u) = \int_Q \left[ \frac{1}{2}(u_t^2 - u_x^2 - \beta v_t^2) - \frac{1}{s+1}u^{s+1} \right] - f(x,t)u dx dt. \quad (1.8)
\]

We seek time-periodic solutions satisfying Dirichlet boundary conditions so we seek functions \( u \in \mathbb{R} \) with expansions of the form

\[
u(x,t) = \sum_{(j,k) \in \mathbb{N} \times \mathbb{Z}} \tilde{u}(j,k) \sin jxe^{ikt}
\]
and define the function space

$$||u||_{E^s} = \sum_{j \neq |k|} \frac{|Q|}{2} |k^2 - j^2|^s |\hat{u}(j, k)|^2 + \sum_{j = \pm k} |\hat{u}(j, k)|^2$$

where we denote by $E$ the space $E^s$ with $s = 1$. Define the functions spaces $E^+, E^-, N$ as follows:

$$N = \{ u \in E, \hat{u}(j, k) = 0 \text{ for } j \neq |k| \}$$

$$E^+ = \{ u \in E, \hat{u}(j, k) = 0 \text{ for } |k| \leq j \}$$

$$E^- = \{ u \in E, \hat{u}(j, k) = 0 \text{ for } |k| \geq j \},$$

where $w = w^+ + w^-$ where $w^+ \in E^+, w^- \in E^-$ and $v \in N$ and define the norm on $E \oplus N$

$$||u||_E^2 = ||w^+||_E^2 + ||w^-||_E^2 + \beta ||v||_{L^2}^2.$$ 

When $u$ is trigonometric polynomial, $I_\beta$ can also be represented as:

$$I_\beta(u) = \frac{1}{2} (||w^+||_E^2 - ||w^-||_E^2 - \beta ||v||_{L^2}^2) - \frac{1}{s + 1} ||u||_{L^{s+1}}^{s+1} - \int_Q fudxdt. \quad (1.9)$$

The spectrum of the linear operator $\partial_t^2 - \partial_x^2$ under Dirichlet boundary conditions in space and time-periodicity consists of

$$-k^2 + j^2$$

where the eigenfunctions are the $\sin jx \cos kt, \sin jx \sin kt$. The eigenfunctions here are ordered as in [Tanaka88] i.e

$$\ldots - \mu_3 \leq - \mu_2 \leq - \mu_1 < 0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots$$

where the $\mu_l$ are the eigenvalues of $\partial_t^2 - \partial_x^2$ and have multiplicity one. Rearranging the eigenvalues this way is possible because all the non-zero eigenspaces of $\partial_t^2 - \partial_x^2$ have finite multiplicity. The $\mu_l \to +\infty$ as $l \to +\infty$ and denote by $e_l$ the corresponding eigenfunctions, and we define the spaces

$$E^{+n} = \text{span}\{e_l, 1 \leq l \leq n\}.$$ 

For the Galerkin procedure we define the spaces

$$E^{-m} = \text{span}\{\sin jx \cos kt, \sin jx \sin kt, \ j + k \leq m, j < k\},$$

$$N^m = \text{span}\{\sin jx \cos jt, \sin jx \sin jt, \ j \leq m\}$$

which are employed in the minimax procedure.

We start by following the procedure of [Rabinowitz82] for perturbation problems by proving some properties of the functional $I_\beta$. The difference here is that additionally we show that the constants involved in all the proof are independent of $\beta$ to prepare for passing to the limit as $\beta \to 0$. 

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Lemma 1.1. Suppose that \( u \) is a critical point of \( I_\beta \). Then there is a constant \( a_6 \) depending on \( s, f \) but independent of \( \beta \) such that

\[
\int_Q |u|^{s+1} dx dt \leq a_6 \int_I \beta(u) + 1 dx dt \tag{1.10}
\]

\[
I_\beta(u) = I_\beta(u) - \frac{1}{2} I'(u) u
= \frac{s-1}{2(s+1)} \int_Q |u|^{s+1} dx dt - \frac{1}{2} \int_Q f u dx dt.
\]

Now, by applying Hausdorff-Young inequalities to \( \int_Q f u dx dt \), we deduce

\[
I_\beta(u) \geq \frac{s-1}{2(s+1)} \int_Q |u|^{s+1} dx dt - c_1(s) ||u||_{L^{s+1}}^s - \epsilon(s) ||u||_{L^{s+1}}^{s+1}, \tag{1.11}
\]

where \( \epsilon(s) \ll 1 \). \( c_1(s) \) are both independent of \( \beta \) hence

\[
I_\beta(u) \geq \frac{1}{4} \frac{s}{s+1} \int_Q |u|^{s+1} dx dt - c(f, s) \tag{1.12}
\]

We define the functional \( J_\beta \) which is amenable to minimax procedure. We start by defining a bump function \( \chi \in C^\infty(\mathbb{R}, \mathbb{R}) \):

\[
\begin{align*}
\chi(t) = 1, & \text{ if } t \leq 1 \\
\chi(t) = 0, & \text{ if } t > 2
\end{align*}
\tag{1.13}
\]

and \( -2 < \chi' < 0 \), for \( 1 < t < 2 \). Then define

\[
\mathcal{I}_\beta(u) = 2a_6(I^2(u) + 1)
\]

and

\[
\psi(u) = \chi(I^{-1}_\beta(u)) \int \frac{|u|^{s+1}}{s+1} dx dt
\]

\[
J_\beta(u) = \int Q \left[ \frac{1}{2}(u^2_t - u^2_x - \beta v^2_t) - \frac{1}{s+1} |u|^{s+1} - \psi(u) f(x,t) u \right] dx dt, \tag{1.14}
\]

which on \( E^+ \oplus E^- \oplus N^m \) can be rewritten as

\[
J_\beta(u) = \frac{1}{2} (||w^+||_{L^2}^2 - ||w^-||_{L^2}^2 - \beta ||v||_{L^2}^2) - \frac{1}{s+1} ||u||_{L^{s+1}}^{s+1} - \int Q \psi(u) f u dx dt. \tag{1.15}
\]

Lemma 1.2. If \( u \in \text{supp} \psi \) then there is a constant \( \alpha_3 \) independent of \( \beta \) such that

\[
|\int_Q f u dx dt| \leq \alpha_3 \mathcal{I}_\beta^{1/2}(u) + 1
\]
Proof: \[ |\int_Q fudxdt| \leq c(f,s)||u||_{L^{s+1}} \]
by Holder inequality, then if \( u \in \text{supp}\psi \), then
\[ I_\beta(u) \int_Q \frac{1}{s+1}|u|^{s+1}dxdt \leq 2 \]
hence
\[ |\int_Q fudxdt| \leq c(f,s)||u||_{L^{s+1}} \leq \alpha_3(I_\beta^{1/s} + 1) \]

**Lemma 1.3.** There is a constant \( \gamma_1 \) depending on \( f, s \) but independent of \( \beta \) such that
\[ |J_\beta(u) - J_\beta(-u)| \leq \gamma_1(|J_\beta(u)|^{1/s} + 1) \quad (1.16) \]

Proof:
\[ J_\beta(u) - J_\beta(-u) = \psi(u)\int_Q fudxdt + \psi(-u)\int_Q fudxdt \]
and by the previous lemma [1.2]:
\[ \psi(-u)\int_Q fudxdt \leq \alpha_3\psi(-u)\int_Q |I_\beta(u)|^{1/s} + 1dxdt \]
now
\[ J_\beta(u) = I_\beta(u) + \int_Q fudxdt - \int_Q \psi(u)fudxdt \]
thus
\[ |I_\beta(u)| \leq |J_\beta(u)| + 2\int_Q fudxdt \]
and
\[ \psi(-u)\int_Q fudxdt \leq \alpha_3\psi(-u)(|J_\beta(u)|^{1/s} + \int_Q fudxdt|^{1/s} + 1) \]
and the lemma follows.

**Lemma 1.4.** There are constants \( \alpha_0, M_0 > 0 \) depending on \( f, s \) independent of \( \beta \) such that whenever \( M \geq M_0 \), then \( J_\beta(u) \geq M \) and \( u \in \text{supp}\psi \) then \( I_\beta(u) \geq \alpha M_0 \)

Proof:
\[ I_\beta(u) \geq J_\beta(u) - 2\int_Q fudxdt \quad (1.17) \]
while if \( u \in \text{supp}\psi \) then

\[
|I_\beta(u)|^{\frac{1}{s+1}} + 1 \geq \frac{1}{\alpha_1} \int_Q |f| u dx dt
\]

or

\[
|I_\beta(u)|^{\frac{1}{s+1}} \geq \frac{1}{\alpha_1} \int_Q |f| u dx dt - C
\]  \hspace{1cm} (1.18)

and adding (1.17) and (1.18)

\[
I_\beta(u) + 2\alpha_1|I_\beta(u)|^{\frac{1}{s+1}} \geq J_\beta(u) - C \geq \frac{M}{2}
\]  \hspace{1cm} (1.19)

for \( M_0 \) large enough. If \( I_\beta(u) \leq 0 \), then by Young inequality

\[
\alpha_1|I_\beta(u)|^{\frac{1}{s+1}} \leq \frac{\alpha_1}{s+1} + \frac{1}{s+1}|I_\beta(u)|^{s+1}
\]  \hspace{1cm} (1.20)

while the inequality (1.19)

\[
\alpha_1|I_\beta(u)|^{\frac{1}{s+1}} \geq -I_\beta(u) + \frac{M}{2}
\]  \hspace{1cm} (1.21)

hence

\[
\frac{\alpha_1}{s+1} + \frac{1}{s+1}|I_\beta(u)|^{s+1} \geq -I_\beta(u) + \frac{M}{2} = |I_\beta(u)| + \frac{M}{2}
\]  \hspace{1cm} (1.22)

thus there is \( c(s) > 0 \) such that

\[
c(s)|I_\beta(u)| \leq -\frac{M}{4} < 0
\]  \hspace{1cm} (1.23)

and we have a contradiction.

**Lemma 1.5.** *Lemma 1.29 [Rabinowitz82]* In \( E^{+m} \oplus E^{-m} \oplus N^m \), there is a constant \( M_1 > 0 \) independent of \( \beta, m \) such that \( J_\beta(u) \geq M_1 \) and \( J_\beta'(u) = 0 \) implies that \( J_\beta(u) = I_\beta(u) \) and \( I_\beta'(u) = 0 \).

**Proof:**

We follow step by step the argument in [Rabinowitz82].

It suffices to show that

\[
I_\beta^{-1}(u) \int_Q \frac{1}{s+1} |u|^{s+1} dx dt \leq 1
\]  \hspace{1cm} (1.24)

\[
J_\beta'(u) = \int_Q w_t^2 - w_x^2 - \beta v_t^2 - |u|^{s+1} dx dt - \psi(u) \int_Q f u dx dt - \psi'(u) \int_Q f u dx dt
\]  \hspace{1cm} (1.25)
are

Let

\(\text{Smale condition is satisfied on} A\)

There is a constant

Lemma 1.6.

and

\(\beta \in \mathbb{E}\)

large energies independently of

\(\text{now we choose} \rho \) and

\(J'_{\beta}(u)u = (1 + T_1(u)) \int_Q u^{2} - w_{x}^{2} - \beta v_{t}^{2} dx dt - (1 + T_2(u)) \int_Q |u|^{s+1} dx dt - (\psi(u) + T_1(u)) \int_Q f u dx dt\)

which are exactly as in [Rabinowitz82]:

\(T_1(u) = \chi(I_{\beta}^{-1}(u)) \int_Q \frac{1}{s+1} |u|^{s+1} + a_4 dx dt) (2a_6)^2 \beta \int_Q |u|^{s+1} dx dt \int_Q f u dx dt\)

and

\(T_2(u) = \chi(I_{\beta}^{-1}(u)) \int_Q \frac{1}{s+1} |u|^{s+1} dx dt) (2a_6)^2 \beta \int_Q |u|^{s+1} dx dt + T_1(u)\)

\(\text{and the conclusion follows just as in [Rabinowitz82].}\)

We now show that the functional \(J_\beta\) satisfies the Palais-Smale condition at large energies independently of \(\beta \in E^+ \oplus E^- \oplus N^m\):

**Lemma 1.6.** There is a constant \(M_2\) independent of \(\beta\) such that the Palais-Smale condition is satisfied on \(A_{M_2} = \{u \in E^+ \oplus E^- \oplus N^m, \ J_\beta(u) \geq M_2\}\)

**Proof:**

Let \(u_i = w_i + v_i = w_i^+ + w_i^- + v_i\) a Palais-Smale sequence at large energies, there are \(M_2, K\) independent of \(\beta, m\) such that \(M_2 \leq J_\beta(u_i) \leq K\) and \(J'_{\beta}(u_i) \rightarrow 0\)

\[J_\beta(u_i) - \rho J'_{\beta}(u_i)(u_i) = \left( \frac{1}{2} - \rho(1 + T_1(u_i)) \right) \int_Q w_i^2 - w_i^{2} - \beta v_t^2 dx dt + |\rho(1 + T_2(u_i) - \frac{1}{s+1})| \int_Q |u_i|^{s+1} dx dt + (\rho(\psi(u_i) + T_1(u_i)) - \psi(u_i)) \int_Q f u_i dx dt\]

now we choose \(\rho = \frac{1}{2(1 + J_1(u_i))}\) then we have

\(\rho \rightarrow \frac{1}{2}\) independently of \(\beta\) as \(M_2 \rightarrow +\infty\)

\[J_\beta(u_i) - \rho J'_{\beta}(u_i)(u_i) = [\rho(1 + T_2(u_i) - \frac{1}{s+1})] \int_Q |u_i|^{s+1} dx dt + (\rho(\psi(u_i) + T_1(u_i)) - \psi(u_i)) \int_Q f u_i dx dt\]

\[\geq [\rho(1 + T_2(u_i) - \frac{1}{s+1}) - \frac{c(s)}{s+1}] \int_Q |u_i|^{s+1} dx dt - c(f, s)\]
where $\epsilon(s)$ can be chosen to be a small positive constant by applying Young inequality so that

$$[\rho(1+T_2(u_l) - \frac{1}{s+1}) - \frac{\epsilon(s)}{s+1}] > 0$$

(1.30)

and $c(f,s)$ is another constant depending on $f,s$, both being independent of $\beta$. Now recall that $J_\beta'(u_l) \to 0$ and $\rho \to \frac{1}{2}$

$$J_\beta(u_l) - \rho J_\beta'(u_l) u_l \leq K + \rho \|u_l\|_{E,\beta}$$

(1.31)

so we have the inequalities:

$$K + \rho \|u_l\|_{E,\beta} \geq J_\beta(u_l) - \rho J_\beta'(u_l) u_l \geq c_3(s) \|u_l\|_{L^{s+1}} - c_2(f,s)$$

(1.32)

thus

$$\int_Q |u_l|^{s+1} dx dt \leq c_4(f,s) \|u_l\|_{E,\beta} + K + c_2(f,s).$$

(1.33)

Now

$$J_\beta'(u_l)v_l = (1+T_1(u_l)) \int_Q \beta v_l^2 u_l v_l dx dt - (1+T_2(u_l)) \int_Q |u_l|^{s-1} u_l v_l dx dt - (\psi(u_l) + T_1(u_l)) \int_Q f v_l dx dt.$$

(1.34)

$u_l$ is a Palais-Smale sequence so there exists $\epsilon$ small such that

$$J_\beta'(u_l)v_l \leq \epsilon \|v_l\|_{E,\beta}$$

thus

$$(1+T_1(u_l))\beta \|v_l\|_{L^2}^2 \leq (1+T_2(u_l)) \int_Q |u_l|^{s-1} u_l v_l dx dt + (\psi(u_l) + T_1(u_l)) \int_Q f v_l dx dt + \epsilon \|v_l\|_{E,\beta}.$$

Now for $M_2$ large enough (independently of $\beta$) and we have

$$\frac{1}{2} \beta \|v_l\|_{L^2}^2 \leq (2 \int_Q |u_l|^s v_l dx dt + 2 \int_Q |f| v_l dx dt + \epsilon \|v_l\|_{E,\beta}$$

(1.35)

and applying Hölder inequality we deduce:

$$\frac{\beta}{2} \|v_l\|_{L^2}^2 \leq c \|u_l\|_{L^{s+1}} \|v_l\|_{L^{s+1}} + 2 \|v_l\|_{L^{s+1}} \|f\|_{L^{s+1}} + \epsilon \|v_l\|_{L^2}.$$

A similar computation gives

$$\|w_l^+\|_{E,\beta} \leq c \|u_l\|_{L^{s+1}} \|w_l^+\|_{L^{s+1}} + 2 \|w_l^+\|_{L^{s+1}} \|f\|_{L^{s+1}} + \epsilon \|w_l^+\|_E.$$  

(1.36)

We now estimate $\|v_l\|_{L^{s+1}} = v_l - w_l^+ - w_l^-$ hence

$$\|v_l\|_{L^{s+1}} \leq \|u_l\|_{L^{s+1}} + \|w_l^+\|_{L^{s+1}} + \|w_l^-\|_{L^{s+1}}$$

$$\leq c \|u_l\|_{E,\beta} + c \|w_l^-\|_E + c \|w_l^+\|_E$$

(1.37)

and

$$\|v_l\|_{E,\beta} + D(f,s).$$

(1.38)
where the constants $c, D(f,s)$ are independent of $\beta$ and \textbf{(1.33)} follows from \textbf{(1.37)} and the Sobolev inequality $||w_t||_{L^p} \leq c(p)||w_t||_E$ We can now deduce:

$$||u_t||^2_{E,\beta} \leq c(1 + ||u||^2_{\hat{L}^{s+1}})(||v||_{L^{s+1}} + ||w_{t}||_{L^{s+1}}) + c||u||_{E,\beta}$$

so $||u_t||_{E,\beta} < +\infty$ and Palais-Smale is satisfied.

\section{2 Estimates on minimax values independently of $\beta$}

\textbf{Lemma 2.1.} There is $R_n \to +\infty$ such that $J_{\beta}(u) \to -\infty$, uniformly as $||u||_{\beta,E} = R_n \to +\infty$ for $u \in E^+ \oplus \overline{E}^{-m} \oplus N^m$. As a result we can also assume that $R_{n+1} > 4R_n$, without loss of generality.

\textbf{Proof:}

Let

$$||u||^2_{E,\beta} = R_n^2 $$

then either $||w^+||^2_{E,\beta} \geq \frac{R_n^2}{3}$ or $||w^-||^2_{E,\beta} + \beta||v||^2 \geq \frac{2R_n^2}{3}$.  

\textbf{Case 1:} $||w^+||^2 \geq \frac{R_n^2}{3}$.

$$J_{\beta}(u) = \frac{1}{2}||w^+||^2_{E} - ||w^-||^2_{E} - \beta||v||^2_E - \frac{1}{s+1}||u||_{L^{s+1}}^{s+1} - \psi(u)\int_\mathcal{Q} fudxdt$$

$$\leq \frac{1}{2}||w^+||^2_{E} - \frac{1}{s+1}||u||_{L^{s+1}}^{s+1} - \psi(u)\int_\mathcal{Q} fudxdt$$

$$\leq \frac{1}{2}||w^+||^2_{E} - a(s)||u||_{L^{s+1}}^{s+1} - c(f)$$

by the Haussdorff-Young inequality and as $w^+ \in E^+ \oplus \overline{E}^{-m} \oplus N^m$ and $s > 1$ we also have:

$$\frac{1}{\nu_n}||w^+||_E \leq ||u||_{L^2} \leq ||u||_{L^{s+1}}$$

thus

$$J_{\beta}(u) \leq \frac{1}{2}||w^+||^2_{E} - \left(\frac{||w^+||_E}{\nu_n}\right)^{s+1} - c(f)$$

and for $R_n$ large enough $J_{\beta} \to -\infty$ uniformly.

\textbf{Case 2:} $||w^+||^2 < \frac{R_n^2}{3}$ hence $||w^-||^2 + \beta||v||^2 \geq \frac{2R_n^2}{3}$, thus:

$$J_{\beta} \leq -\frac{R_n^2}{3} - \frac{1}{s+1}||u||_{L^{s+1}}^{s+1} - \psi(u)\int_\mathcal{Q} fudxdt$$

$$\leq -\frac{R_n^2}{3} - a(s)||u||_{L^{s+1}}^{s+1} + c(f)$$

\textbf{(2.44)}
by Haussdorff-Young inequality and we can conclude again that $R_n$ large enough $J_{\beta} \to -\infty$ uniformly which ends the proof of the lemma. We now define the minimax sets and the minimax values which will lead to the existence of critical values:

Let $B(R, W)$ the closed ball, of radius $R$, in a subspace $W$ of $E \oplus N$:

$$B(R, W) = \{ u \in W \mid ||u||_{E, \beta} \leq R \}$$

$$D_n^m = \{ u \in E^{+n} \oplus E^{-m} \oplus N^m \text{ and } ||u||_{E, \beta} \leq R_n \}$$

$$\Gamma_n^m = \{ h : D_n^m \to E^{+m} \oplus E^{-m} \oplus N^m, \text{ } h \text{ continuous and odd }, h(u) = u, \text{ for } ||u||_{E, \beta} = R_n \}$$

$$\inf_{h \in \Gamma_n^m} \max_{u \in D_n^m} J_{\beta}(h(u))$$

$$U_n^m = \{ u_{n+1} = t e_{n+1} + u_n, \text{ } t \in [0, R_{n+1}], u_n \in B(R_{n+1}, E^{+n} \oplus E^{-m} \oplus N^m), ||u_{n+1}||_{E, \beta} \leq R_{n+1} \}$$

$$\Lambda_n^m = \begin{cases} H \in C(U_n^m, E^{+m} \oplus E^{-m} \oplus N^m), & H|_{D_n^m} \in \Gamma_n^m, \text{ and } H(u) = u \\
\text{ if } ||u||_{E, \beta} = R_{n+1}, \text{ or if } \text{ } u \in B(R_{n+1}, E^{+n} \oplus E^{-m} \oplus N^m) \setminus B(R_n, E^{+n} \oplus E^{-m} \oplus N^m) \\ \end{cases}$$

where the constants $R_n$ does not depend on $\beta$.

$$\Lambda_n^m(\delta) = \{ H \in \Lambda_n^m, J_{\beta}(H(u)) \leq b_n^m + \delta \text{ on } D_n^m \}$$

$$c_n^m = \inf_{H \in \Lambda_n^m} \max_{u \in U_n^m} J_{\beta}(H(u))$$

and

$$c_n^m(\delta) = \inf_{H \in \Lambda_n^m(\delta)} \max_{u \in U_n^m} J_{\beta}(H(u))$$

**Lemma 2.2.** \( \forall u \in D_n^m \cap E^{+n} \), there is a constant $C(n)$ independent of $\beta, m$ such that

$$J_{\beta}(u) \leq C(n) \tag{2.45}$$

**Proof:**

Let $u \in E^{+n}$

$$J_{\beta}(u) = \frac{1}{2} ||w^+||_E^2 - \frac{1}{2} ||w^-||_E^2 - \beta ||v_1||_2^2 - \int_1^s \frac{|u|^{s+1}}{s+1} dx dt - \psi(u) \int Q f u dx dt$$

$$\leq \frac{1}{2} ||w^+||_E^2 - \frac{1}{2} ||w^-||_E^2 - \beta ||v_1||_2^2 - \frac{1}{2} \int_1^s \frac{|u|^{s+1}}{s+1} dx dt + c(f, s) \tag{2.46}$$

$$\leq c(f, s) + \sup_{u \in E^{+n}} \frac{1}{2} ||w^+||_E^2 - \frac{1}{2} \int_1^s \frac{|u|^{s+1}}{s+1} dx dt$$

$$\leq c(f, s) + \sup_{u \in E^{+n}} \frac{1}{2} ||w^+||_E^2 - c(s, Q) ||u||_{L^2}^{s+1} \tag{2.47}$$

Now in $E^{+n}$

$$||u||_E^2 \leq \mu_n ||u||_{L^2}^2 \tag{2.48}$$
and on the other-hand
\[ \sup_{u \in E^{+n}} \frac{1}{2} ||w^+||_{L^2}^2 - c(s,Q)||u||_{L^2}^{*+1} > 0 \]
and is attained at say $\overline{w}$ hence we have
\[ c(s,Q)||\overline{w}||_{L^2}^{*+1} \leq \frac{1}{2} ||\overline{w}||_{L^2}^2 \leq \frac{1}{2} \mu_n ||\overline{w}||_{L^2}^2 \]
and we can conclude there is $C(n)$ depending on $n$ but independent of $\beta$ such that
\[ J_\beta(u) \leq C(n) \]

We now construct the map which leads to upper estimates independently of $\beta$ which is the main contribution of the paper.

**Theorem 2.1.** Let $0 < \delta < c_m^n - b_m^n$, then and there is a map $H \in \mathcal{L}_m^n(\delta)$ such that
\[ J_\beta(H(u,t)) \leq C(n+1) \]
in $U_m^n$ where $C(n+1)$ is independent of $\beta,m$.

Proof:
Let $h \in \Gamma^n$ a minimizing map for $b_m^n$, i.e.:
\[ J_\beta(h(u)) \leq b_m^n + \frac{\delta}{2} \]
on $D_m^n$.

The aim is to construct a function $H(u,t)$ which is the identity map when ||$u||_{E,\beta} = R_{n+1}$ and which coincides with a map $h(x)$ at $t = 0$, for which $J_\beta(H(u,t)) \leq c(n+1)$ a constant independent of the small parameter $\beta$.

Let and $u \in E^{+n}$ we have $J_\beta(h(u)) \leq b_m^n + \frac{\delta}{2}$ where $b_m^n$ is bounded independently of $m,\beta$. $h$ also satisfies $h(0) = 0$ which plays an important role in the proof. The idea is to deform $H(u,t)$ from $h$ at $t = 0$ to the zero map and then to the identity map while keeping $H(u,t) = u + te_{n+1}$, when $||u||_{E,\beta} = R_{n+1}$.

0 $\leq t \leq 1$: We construct a map $H(u,t)$ which vanishes somewhere between $R_n < ||u||_{E,\beta} < 3R_n$, which here we choose to vanish at $||u|| = 2R_n$:

We define the functions $H_1, H_2, H_3, H_4$ for all values of $t \geq 0$:

\[
\begin{cases}
0 \leq ||u||_{E,\beta} \leq R_n & H_1(u,t) = h(u) \\
R_n \leq ||u||_{E,\beta} \leq 2R_n & H_2(u,t) = (1-t)u + t(-u + \frac{2R_n}{||u||_{E,\beta}}u) \\
2R_n \leq ||u||_{E,\beta} \leq 3R_n & H_3(u,t) = (1-t)u + t(3u - \frac{6R_n u}{||u||_{E,\beta}} + \frac{1}{2}(3R_n^2 - 2) e_{n+1}) \\
3R_n \leq ||u||_{E,\beta} \leq R_{n+1} & H_4(u,t) = u + te_{n+1}
\end{cases}
\]

Then to define $H$ we restrict the domain of these functions to $0 \leq t \leq 1$:

\[
\begin{cases}
0 \leq ||u||_{E,\beta} \leq R_n & H(u,t) = H_1(u,t) \\
R_n \leq ||u||_{E,\beta} \leq 2R_n & H(u,t) = H_2(u,t) \\
2R_n \leq ||u||_{E,\beta} \leq 3R_n & H(u,t) = H_3(u,t) \\
3R_n \leq ||u||_{E,\beta} \leq R_{n+1} & H(u,t) = H_4(u,t)
\end{cases}
\]

then to define $H$ we restrict the domain of these functions to $0 \leq t \leq 1$:
To verify continuity we note that $H_1(u, 0) = h(u)$ and $H_2(u, 0) = H_3(u, 0) = H_4(u, 0) = u$

$H_1(||u||_{E, \beta} = R_n, t) = H_2(||u||_{E, \beta} = R_n, t) = u$

$H_2(||u||_{E, \beta} = 2R_n, t) = H_3(||u||_{E, \beta} = 2R_n, t) = (1 - t)u$

$H_3(||u||_{E, \beta} = 3R_n, t) = H_4(||u||_{E, \beta} = 3R_n, t) = u + te_{n+1}$, hence we can conclude that $H$ is continuous for $0 \leq t \leq 1$. To show that $J_\beta(H(u, t))$ is bounded independently of $\beta$ note that $H_1(u) = h(u)$ and that by hypothesis $J_\beta(h(u)) \leq b_0^n + \frac{J}{2}$. Also $H_2(u, t) \in E^{n}, H_3(u, t) \in E^{n+1}, H_4(u, t) \in E^{n+1}$ and by lemma 2.2 and 2.1 we conclude

$$J_\beta(H(u, t)) \leq c(n + 1).$$

(2.54)

$1 \leq t \leq 2$: We do not have any a priori estimates independently of $m$ on the dimension of the subspace in which $h(u)$ lies. We deform $h(u)$ to the 0-map thereby ensuring that the range of $H(u, 2)$ lies in $E^{n+1}$ at $t = 2$, where by lemmas 2.1, 2.2, 2.3, $J_\beta$ is bounded independently of $\beta$.

Define $h_1(u) = h(u)$ for $||u||_{E, \beta} \leq R_n$ and $h_1(u) = H_2(u, 1)$.

$$h_1 \in C([0, 1]; E)$$

$h_1$ this constructed is continuous as

$$H_1(||u||_{E, \beta} = R_n, 1) = H_2(||u||_{E, \beta} = R_n, 1) = u.$$  

(2.55)

Now,

$$h_1 : B(2R_n, E^{n+1}) \rightarrow E^{n+1} \oplus E^{-m} \oplus N^m$$

(2.56)

$H_1(u, t) = h_1((2 - t)x)$ is continuous. The map $(2 - t)u$

$$(2 - t)u : B(2R_n, E^{n+1}) \times [1, 2] \rightarrow B(2R_n, E^{n+1}) : (u, t) \rightarrow (2 - t)u$$

(2.57)

is composed with $h_1$ so the composition

$$H_5(u, t) = h_1 \circ (2 - t)u : E^{n+1} \times \mathbb{R} \rightarrow E^{n+1} \oplus E^{-m} \oplus N^m ; (u, t) \rightarrow h_1((2 - t)u)$$

(2.58)

is continuous. This way we have $H_5(u, 2) = h_1(0) = 0 \in E^{n+1},$ for $0 \leq ||u||_{E, \beta} \leq 2R_n$ which will be later continued as the 0-map for $2 \leq t \leq 3$ and $0 \leq ||u||_{E, \beta} \leq 2R_n$:

$$\begin{cases}
0 \leq ||u||_{E, \beta} \leq 2R_n & H_5(u, t) = h_1((2 - t)u) \\
2R_n \leq ||u||_{E, \beta} \leq 3R_n & H_6(u, t) = (3u - \frac{6R_n u}{||u||_{E, \beta}}) + te_{n+1} \\
3R_n \leq ||u||_{E, \beta} \leq R_{n+1} & H_7(u, t) = u + te_{n+1}
\end{cases}$$

$$\begin{cases}
0 \leq ||u||_{E, \beta} \leq 2R_n & H(u, t) = H_5(u, t) \text{ for } 1 \leq t \leq 2 \\
2R_n \leq ||u||_{E, \beta} \leq 3R_n & H(u, t) = H_6(u, t) \text{ for } 1 \leq t \leq 2 \\
3R_n \leq ||u||_{E, \beta} \leq 3R_n & H(u, t) = H_7(u, t) \text{ for } 1 \leq t \leq 2
\end{cases}$$

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\[ H_5(\|u\|_{E, \beta} = 2R_n, t) = H_6(\|u\|_{E, \beta} = 2R_n, t) = 0 \]

\[ H_1(u, 1) = H_5(u, 1) = u \text{ for } \|u\|_{E, \beta} \leq R_n \]

\[ H_2(u, 1) = H_5(u, 1) = h_1(u) \text{ for } R_n \leq \|u\|_{E, \beta} \leq 2R_n \]

by definition of \( h_1 \).

\[ H_3(u, 1) = H_6(u, 1) = \left(3u - \frac{6R_n u}{\|u\|_{E, \beta}} + \frac{\|u\|_{E, \beta}}{R_n} \right) + (\|u\|_{E, \beta} - 2)e_{n+1} \text{ for } 2R_n \leq \|u\|_{E, \beta} \leq 3R_n \]

We can now note that \( J_\beta(h_1(u)) \) is bounded independently of \( \beta \):

Case 1: \( 0 \leq \|u\| \leq R_n \) \( J_\beta(h_1(u)) = J_\beta(h(u)) \) and since \( h \) is a minimizing map, \( J_\beta(h(u)) \leq b_n^{\beta} + \delta \)

Case 2: \( R_n \leq \|u\| \leq 2R_n \), \( h_1(u) \in E^{+(n+1)} \) hence \( J_\beta(h_1(u)) \leq c(n+1) \)

\[ 2 \leq t \leq 3 \]

\[
\begin{align*}
0 &\leq \|u\|_{E, \beta} \leq 2R_n & G_1(u, t) &\equiv 0 \text{ for } 2 \leq t \leq 3 \\
2R_n &\leq \|u\|_{E, \beta} \leq 3R_n & G_2(u, t) &= \left(3u - \frac{6R_n u}{\|u\|_{E, \beta}} \right) + t\left(\|u\|_{E, \beta} - 2\right)e_{n+1} \text{ for } 2 \leq t \leq 3 \\
3R_n &\leq \|u\|_{E, \beta} \leq R_{n+1} & G_3(u, t) &= u + te_{n+1} \text{ for } 2 \leq t \leq 3 \\
3R_n &\leq \|u\|_{E, \beta} \leq 3R_n & G_4(u, t) &= (3 - t)G(u, t) + (t - 2)(u + te_{n+1}) \text{ for } 2 \leq t \leq 3
\end{align*}
\]

\[ G(u, t) \in E^{+(n+1)} \text{ and the identity map } Id(u, t) = u + te_{n+1} \in E^{+(n+1)} \]

where we know \( J_\beta \) is bounded independently of \( \beta \), so by a simple linear homotopy we can now ensure that \( H(u, t) = u + te_{n+1} \) at \( t = 3 \):

\[ H(u, t) = (3 - t)G(u, t) + (t - 2)(u + te_{n+1}), \quad 2 \leq t \leq 3 \quad (2.59) \]

then extend for all other values of \( t \), by \( H(u, t) = u + te_{n+1} \in E^{+(n+1)}, \quad 3 \leq t \leq R_{n+1} \).

Such an \( H \in A_n^m(\delta) \) with \( J_\beta(H(u, t)) \) bounded independently of \( \beta, m \)

which concludes the proof.

At this stage we know that by lemma 1.57 in [Rabinowitz82], \( c_n^m(\delta) \) is a critical value if \( c_n^m > b_n^m \). Now to show that there is a subsequence \( n_q \) such that this is the case we employ the comparison functional \( K \) introduced by Tanaka in lemma 2.2 in [Tanaka88]:

\[ K(w^+) = \frac{1}{2}||w^+||_E - \frac{a_0(s)}{s + 1}||w^+||_{L_{s+1}^{s+1}} \]

which satisfies the Palais-Smale condition. The functional \( K \) also satisfies the comparison property:

\[ J_\beta(w^+) \geq K(w^+) = a_1(f, s) \]
for any \( w^+ \in E^+ \), \( a_1(f,s) \) is a positive constant. We define the minimax sets:
\[
A_m^\sigma = \{ \sigma \in C(S^{m-n}, E^{+m}), \sigma(-x) = \sigma(x) \}
\]
where \( S^{m-n} \subset E^{+m} \) is the unit sphere in \( \mathbb{R}^{m-n+1} \), whose basis consists of eigenvectors \( \{e_n, ..., e_m\} \).

\[
x = \sum_{i=n}^{m} x_i e_i \quad \text{and} \quad \sum_{i=n}^{m} x_i^2 = 1
\]

and the minimax values
\[
\beta_m^n = \sup_{\sigma \in A_m^n} \min_{x \in S^{m-n}} K(\sigma(x))
\]

Properties of the minimax numbers \( \beta_m^n \) from [Tanaka88]: There exists sequences \( \nu(n), \tilde{\nu}(n) \)
\[
\nu(n) \leq \beta_m^n \leq \tilde{\nu}(n) \quad (2.61)
\]
such that \( \nu(n), \tilde{\nu}(n) \to \infty \) as \( n \to \infty \) (independently of \( m \)).

Borsuk-Ulam type theorem:

Lemma 2.3. [Tanaka88] Let \( a, b \in \mathbb{N} \). Suppose that \( h \in C(S^a, \mathbb{R}^{a+b}) \), and \( g \in C(\mathbb{R}^b, \mathbb{R}^{a+b}) \) are continuous mappings such that
\[
h(x) = h(-x) \quad \text{for all} \quad x \in S^a \quad (2.62)
\]
\[
g(-y) = -g(y) \quad \text{for all} \quad y \in \mathbb{R}^b \quad (2.63)
\]
and there is a \( r_0 \) such that \( g(y) = y \) for all \( r \geq r_0 \). Then \( h(S^a) \cap g(\mathbb{R}^b) \neq \emptyset \)

Lemma 2.4. [Tanaka88] Let \( \gamma \in \Gamma_n^m \) and \( \sigma \in A_n^m \), then
\[
[\gamma(D_m^n) \cup \{ u \in E^{+n} \oplus E^{-m} \oplus N^{-m}, ||u||_{\beta,E} \geq R_n \}] \cap \sigma(S^{m-n}) \neq \emptyset \quad (2.64)
\]

Proof: Apply the lemma above with \( a = m - n \) and \( b = \text{dimension}(E^{+n} \oplus E^{-m} \oplus N^{-m}) \). Then extend \( \gamma \) to all of \( E^{+n} \oplus E^{-m} \oplus N^{-m} \) by extending it by the identity map on \( \partial D_m^n \) and view \( \sigma(S^{m-n}) \) as embedded in \( E^{+m} \oplus E^{-m} \oplus N^{-m} \), then apply the preceding lemma 2.3.

Lemma 2.5. \( \forall n \in \mathbb{N} \),
\[
b_m^n \geq \beta_m^n - a_1 \quad (2.65)
\]
where \( a_1 \) is independent of \( n, m, \beta \).

Lemma 2.6. (Proposition 4.1 [Tanaka88]) Suppose that \( \beta_m^n < \beta_m^{n+1}, m > n+1 \), then there exists a \( u_m^n \in E^{+m} \) such that
\[
K(u_m^n) \leq \beta_m^n \quad (2.66)
\]
\[
K'(u_m^n) = 0 \quad (2.67)
\]
\[
\text{index} K''(u_m^n) \geq n \quad (2.68)
\]

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Lemma 2.7. (Proposition 5.1 [Tanaka88]) For any \( \varepsilon > 0 \), there is a constant \( C_\varepsilon > 0 \), such that for \( u \in E^+ \)

\[
\text{index}K''(u) \leq C_\varepsilon ||u||_{L^p_{(s-1)(1+\varepsilon)}}^{(s-1)(1+\varepsilon)}
\]  

(2.69)

Theorem 2.2. There is a subsequence \( n_q \) and \( c \) independent of \( \beta, m, n \) such that

\[
b_m^{n_q} > \frac{n_q^{s+1}}{}(2.70)
\]

Proof:

The inequality (2.61) implies that there is a subsequence \( n_q \) such that

\[
\beta_m^{n_q+1} > \beta_m^{n_q}.
\]

(2.71)

Then for \( \varepsilon > 0 \) small enough

\[
||u_m^{n_q}||_{s+1} \geq c||u_m^{n_q}||_{(s-1)(1+\varepsilon)}^{s+1} \geq c||u_m^{n_q}||_{(s-1)(1+\varepsilon)}^{s+1} \geq c_\varepsilon n_q \frac{n_q^{s+1}}{}.
\]

(2.72)

by combining (2.69) and (2.68). Now recalling lemma 2.5 and that for \( \varepsilon \) small enough, \( \beta_m^{n_q+1} > \beta_m^{n_q} \), the lemma follows.

To conclude we recall lemma 1.64 in [Rabinowitz82] which in our case implies that, for \( m \) large enough, independently of \( \beta \), if \( c_m^m = b_m^{n_q} \) for all \( n \geq n_1 \) then \( b_n \leq cn^{s+1} \). Then by lemma 1.57 in [Rabinowitz82], \( c_m^m(\delta) \) is a critical value of \( I_\beta \) in \( E^+ \oplus E^- \oplus N^m \).

3 Regularity

Theorem 3.1. Let \( f \) be \( C^2 \), for \( n \) large enough there is a classical solution \( u = v + w \) of the modified problem (1.4).

(3.74)
now taking $\phi = v^m_{ttt} \in N^m$ we have
\[
(\beta v^m_{tt}, v^m_{tt})_{L^2} = (|u^m_{nq}|^{s-1} u^m_{nq} + f, v^m_{tt})_{L^2}
\]
and
\[
\beta ||v^m_{tt}||^2_{L^2} = |||u^m_{nq}|^{s-1}||_{L^2} ||v^m_{tt}||_{L^2} + ||f||_{L^2} ||v^m_{tt}||_{L^2}
\]
\[
\beta ||v^m_{tt}||_{L^2} \leq c ||v^m_{tt}||_{L^2}
\]
now by the argument in the proof of the Palais-Smale property we also have
\[
||u^m||_E < c(n_q), \beta ||v^m||_{L^2} < c(n_q)
\]
(3.75)

hence
\[
||v^m_{tt}||_{L^2} \leq c(\beta, f)
\]
we now have
\[
w^m_{tt} - w^m_{xx} = \beta v^m_{tt} + |u^m_{nq}|^{s-1} u^m_{nq} + f^m(x, t) \in L^2
\]
hence $w^m \in H^1 \cap C^1$ by [Rabinowitz78] and [BCN80]. This now implies $w^m \in H^2, w^m \rightarrow w(\beta)$ pointwise and $w(\beta) \in H^1 \cap C^3$. Then if $\phi = v^m_{ttt}$ then
\[
(\beta v^m_{tt}, v^m_{ttt})_{L^2} = (|u^m_{nq}|^{s-1} u^m_{nq} + f, v^m_{ttt})_{L^2}
\]
and we deduce $||v^m_{ttt}||_{L^2} \leq c(\beta, f)$ hence $v^m_{ttt} \rightarrow v_{tt}(\beta) \in C^0$ hence $v(\beta)$ is $C^2$ and $w(\beta)$ is $C^1$ by applying [BCN80] to (1.3). We now have
\[
u^m_{nq} \rightarrow u(\beta) \in C^1 \text{ as } m \rightarrow \infty
\]
(3.76)
and since (3.71) holds for any $\phi \in E^{+m} \oplus E^{-m} \oplus N^m$ we can deduce
\[
I'(u(\beta))|_\phi = 0 \forall \phi \in E \oplus N,
\]
(3.77)
and $u(\beta)$ is a weak solution of (1.3). Now for any $\phi \in C^\infty \cap L^2(S^1)$ we have
\[
I'(u(\beta))[\phi(x+t) - \phi(x-t)] = \int_{Q} [-\beta(p''(x+t) - p''(-x+t) + |u(\beta)|^{s-1} u(\beta)) + f(x, t)]
\]
\[
[\phi(x+t) - \phi(-x+t)]dxdt
\]
Denoting $\psi(x, t) := [-\beta(p''(x+t) + |u(\beta)|^{s-1} u(x, t) + f(x, t)]$ and noting that the functions $\psi, \phi$ are periodic we deduce as in [Rabinowitz78] that
\[
\int_{0}^{2\pi} \int_{0}^{\pi} \psi(x, t)\phi(x+t)dxdt = \int_{0}^{\pi} \int_{0}^{2\pi} \psi(r, r-x)\phi(r)dxdr
\]
and
\[
\int_{0}^{\pi} \int_{0}^{2\pi} \psi(x, t)\phi(-x+t)dxdt = \int_{0}^{2\pi} \int_{0}^{2\pi} \psi(x, r+x)\phi(r)dxdr
\]
for all $\phi \in C^\infty \cap L^2(S^1)$ hence

$$
\int_0^\pi \psi(x, r + x) - \psi(x, r - x)dxdr = 0
$$

and we have

$$
2\pi \beta p''(r) = \int_0^\pi (\vert u(\beta)\vert^{s-1}u(\beta)(x, r-x) - \vert u(\beta)\vert^{s-1}u(\beta)(x, r+x)) + f(x, r-x) - f(x, r+x)dx
$$

(3.78)

so $p$ is $C^3$ since $u(\beta) \in C^1$. Since RHS of (1.4) is $C^1$ then by [BCN80] $w \in C^2$ and $u(\beta)$ is a classical solution of (1.4).

**Lemma 3.1.** There is a constant $c$ independent of $\beta, m$ such that

$$
\vert\vert w(\beta) \vert\vert_{C^0} \leq c
$$

(3.79)

**Proof:**

By (1.10), the bound on $c_0^m(\beta)$ independent of $\beta, m$ and (3.76), we deduce that

$$
\int \vert u(\beta)\vert^{s+1}dxdt \leq c(n_q) \text{ independent of } \beta.
$$

(3.80)

Then by (3.78) $\vert\vert \beta v_t \vert\vert_{L^1}$ is bounded independently of $\beta$, hence by Lovicarova’s formula [Lovicarova69] we conclude that there is a constant $c$

$$
\vert\vert w(\beta) \vert\vert_{C^0} \leq c(n_q)
$$

(3.81)

which is independent of $\beta$.

**Lemma 3.2.** There is a constant $c(n_q)$, independent of $\beta$ such that

$$
\vert\vert v(\beta) \vert\vert_{C^0} \leq c(n_q).
$$

(3.82)

**Proof:**

$\forall \phi \in N$,

$$
\int_0^\pi \int_0^{2\pi} (-\beta v_t(\beta) + (g(u(\beta)) + f(x, t))\phi dxdt = 0
$$

$$
\int_0^\pi \int_0^{2\pi} \beta v_t(\beta)\phi_t + (g(v(\beta) + w(\beta)) - g(w))\phi dxdt = -\int_0^\pi \int_0^\pi (f(x, t)) + g(w)\phi dxdt
$$

(3.83)

Define $q$:

$$
\begin{cases}
q(s) = 0, \text{ if } |s| \leq M. \\
q(s) = s + M \text{ if } s \geq M \text{ and } q(s) = s - M \text{ if } s \leq M.
\end{cases}
$$

(3.84)

Now define the function $\psi_K(z)$:

$$
\begin{cases}
\psi_K(z) = \max_{|\xi| \leq M_k} f_K(z + \xi) - f_K(\xi) \text{ if } z > 0. \\
\psi_K(z) = -\min_{|\xi| \leq M_k} (f_K(\xi) - f_K(z + \xi)) \text{ if } z < 0
\end{cases}
$$

(3.85)
$\psi_K$ is monotonically increasing and \( \lim_{z \to \pm \infty} \psi_K(z) = \pm \infty \). For \( z \geq 0 \), \( \mu(z) = \min(\psi(z), \psi(-z)) \). Define
\[
T_\delta = \{(x, t) \in [0, \pi] \times [0, 2\pi] \mid |v(\beta)| \geq \delta \}.
\]
By taking the test function \( \phi = q(v^+) - q(v^-) = v^+ - v^- \) and noting that \( q \) is strictly increasing we have the estimate following lemma 3.7 in [Rabinowitz78]:

\[
\int_{T_\delta} (g(v + w) - g(v))(q^+ - q^-)dxdt \geq \frac{M - \delta}{\|v\|_{C_0}} \mu(\delta) \int_{T_\delta} (|q^+| + |q^-|)dxdt \quad (3.86)
\]
hence:

\[
(\|g(w)\|_{C_0} + \|f\|_{C_0}) \int_T |q^+| + |q^-|dxdt \geq \frac{M - \delta}{\|v\|_{C_0}} \mu(\delta) \int_{T_\delta} (|q^+| + |q^-|)dxdt.
\]

(3.87)

Denoting \( \max(\|v^+\|_{C_0}, \|v^-\|_{C_0}) = \|v^\pm\|_{C_0} \) we have

\[
\mu \left( \frac{1}{2} \|v^\pm\|_{C_0} \right) \leq 4(\|f\|_{C_0} + \|g(w)\|_{C_0})
\]

(3.88)

and we can conclude that there is a constant \( c \) independent of \( \beta \) such that

\[
\|v(\beta)\|_{C_0} \leq c.
\]

(3.89)

**Lemma 3.3.** The family \( v(\beta) \) is equicontinuous.

**Proof:** \( u = v + w \). Define \( \tilde{v}(x, t) = v(x, t + h), \tilde{w}(x, t) = w(x, t + h) \) and \( \tilde{u} = \tilde{v} + \tilde{w} \). \( \tilde{f} = f(x, t + h), U = V + W \), where \( V = \tilde{v} - v, W = \tilde{w} - w \), \( q(V^+) = Q^+ \), \( q(V^-) = Q^- \)

\[
\int_T \beta V_\epsilon \phi_t dxdt + \int_T g(\tilde{v} + \tilde{w}) - g(u)dxdt = -\int_T g(\tilde{u}) - g(\tilde{v} + \tilde{w}) + \tilde{f} - f dxdt \quad (3.90)
\]

For \( \phi = q(V^+) - q(V^-) \) and \( V^+ = \tilde{v}^+ - v^+ \), we have

\[
\int_T [g(V + u) - g(u) + \tilde{f} - f](Q^+ - Q^-)dxdt \leq (\|f(\tilde{u}) - f(\tilde{v} + \tilde{w})\|_{C_0} + \|\tilde{f} - f\|_{C_0}) \int_T (|Q^+| + |Q^-|)dxdt \quad (3.91)
\]

and

\[
\int_T [g(V + u) - g(u)](Q^+ - Q^-)dxdt \geq \frac{\mu(\delta)(M - \delta)}{|V|_{C_0}} \int_T (|Q^+| + |Q^-|)dxdt. \quad (3.92)
\]

Since \( w(\beta) \in C^1 \) and \( f \in C^1 \) we deduce

\[
\|f(\tilde{u}) - f(\tilde{v} + \tilde{w})\|_{C_0} + \|\tilde{f} - f\|_{C_0} \leq c|\epsilon| \quad (3.93)
\]

where \( c \) is independent of \( \beta \), thus

\[
\mu(\frac{1}{2} |V^\pm|_{C_0}) \leq c|\epsilon| \quad (3.94)
\]

and the modulus of continuity of \( v(\beta) \) is independent of \( \beta \).
Theorem 3.2. The problem (1.1), (1.2) has an infinite number of weak solutions \( u = w + v \) where \( w \in C^1 \) and \( v \in C^0 \).

Proof:
\[
\|\beta v_t\|_{L^1} \to 0 \text{ as } \beta \to 0: \text{ Recalling the interpolation inequalities } [\text{Rabinowitz78}],[\text{Nirenberg59}] \text{ and (3.78)}:
\]
\[
\beta\|v_t\|_{L^1} \leq \beta\|v_t\|_{C^0}^{\frac{1}{2}}\|v(\beta)\|_{C^0}^{\frac{1}{2}} \to 0 \tag{3.95}
\]
and Lovicarova fundamental solution in [Lovicarova69] implies that \( w \in C^1 \).

Case 1:
If \( \exists \tau \) such that \( u(x, \tau - x) = \alpha \) for \( \forall x \in [0, \pi] \) then the boundary conditions imply \( \alpha = 0 \) and \( p(\tau - 2x) = p(\tau) + w(x, \tau - x) \), thus
\[
\|v\|_{C^1} \leq \|w\|_{C^1}. \tag{3.96}
\]

Case 2:
There is no \( \tau \) such that \( u(x, \tau - x) = 0 \), then there is \( \gamma > 0 \) such that
\[
\int_0^\pi s|u|^{s-1}(x, r-x)dx > \gamma, \forall r \in [0, 2\pi]. \tag{3.97}
\]
Differentiating (3.78) with refer to \( r \) and using the boundary conditions for \( u \) as in [Rabinowitz78] we obtain:
\[
-\pi \beta p''(r) + a(r)p'(r) = \int_0^\pi s|u|^{s-1}(x, r-x)[-\frac{1}{2}w_x(x, r-x) - w_r(x, r-x)] + \]
\[
s|u|^{s-1}(x, r+x)[-\frac{1}{2}w_x(x, r+x) + w_r(x, r+x)] + \]
\[
f_r(x, r+x) - f_r(x, r-x)dx, \tag{3.98}
\]
where \( a(r) = \int_0^\pi s|u|^{s-1}(\beta)(x, r-x) + s|u|^{s-1}(\beta)(x, r+x)dx. \)
Now by writing \( \phi(r) = p'(r) \) we have:
\[
-\pi \beta \phi''(r) + a(r)\phi(r) = h(r) \tag{3.99}
\]
where \( h \in C^0(S^1) \) and since \( f \in C^1 \) we deduce as in [Rabinowitz78] that \( \lim_{\beta \to 0} \phi(\beta) \) exists and is in \( H^1(S^1) \). Denoting this limit by \( \phi(0) \) we deduce that \( w \in C^1 \). This implies \( w \in C^2 \) and \( h \in C^1 \), as \( f \in C^2 \). Now (3.99) is valid a.e at \( \beta = 0 \) which implies \( \phi \in C^1 \) and \( u \in C^2 \) is a classical solution of (1.1), (1.2).

References
[BahriLions88] Bahri, A.; Lions, P.-L. Morse index of some min-max critical points. I. Application to multiplicity results. Comm. Pure Appl. Math. 41 (1988), no. 8, 10271037.
[BB81] Abbas Bahri and Henri Berestycki, A perturbation method in critical point theory and applications. Transactions of the American Mathematical Society, Vol. 267, No. 1. (Sep., 1981), pp. 1-32.

[BB84] Bahri, A.; Berestycki, H. Forced vibrations of superquadratic Hamiltonian systems. Acta Math. 152 (1984), no. 3-4, 143197.

[BDL99] Bartsch, T.; Ding, Y. H.; Lee, C. Periodic solutions of a wave equation with concave and convex nonlinearities. J. Differential Equations 153 (1999), no. 1, 121–141

[Bolle99] Bolle, Philippe. On the Bolza problem. J. Differential Equations 152 (1999), no. 2, 274–288

[BGT2000] P. Bolle, N. Ghoussoub and H. Tehrani, The multiplicity of solutions in non-homogeneous Boundary Value problems Manuscripta Mathematica, 101 (2000) 325-350,

[Brézis83] Brézis, Haïm. Periodic solutions of nonlinear vibrating strings and duality principles. Bull. Amer. Math. Soc. (N.S.) 8 (1983), no. 3, 409426.

[BCN80] Brézis, Haïm; Coron, Jean-Michel; Nirenberg, Louis. Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz. Comm. Pure Appl. Math. 33 (1980), no. 5, 667684

[BN78-1] Brézis, H., and Nirenberg, L., Characterizations of the ranges of some nonlinear operators and applications to boundary value problems. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)5(1978), no. 2, 225-326.

[BN78-2] Brézis, H. and Nirenberg, L., Forced vibrations for a nonlinear wave equation. Comm. Pure Appl. Math. 31 (1978), no. 1, 1–30. Communications on pure and applied mathematics

[CastroClapp2006] Castro, Alfonso; Clapp, Monica. Upper estimates for the energy of solutions of nonhomogeneous boundary value problems. Proc. Amer. Math. Soc. 134 (2006), no. 1, 167–175

[CDHL2004] Clapp, Monica; Ding, Yanheng; Hernandez-Linares, Sergio. Strongly indefinite functionals with perturbed symmetries and multiple solutions of nonsymmetric elliptic systems. Electron. J. Differential Equations 2004, No. 100, 18 pp.

[Chemin] [http://www.ann.jussieu.fr/MathModel/telechar.php?filename=polycopies/chemin.pdf](http://www.ann.jussieu.fr/MathModel/telechar.php?filename=polycopies/chemin.pdf)

[C83] Coron, J.M. Periodic solutions without assumption of analyticity. Mathematische Annalen 262,273-286 1983

[Hale66] Hale, Jack K. Periodic solutions of a class of hyperbolic equations containing a small parameter. Arch. Rational Mech. Anal. 23 1966 380398.
[Hall70] Hall, William S. On the existence of periodic solutions for the equations
\[ D_{tt}u + (-1)^p D_{x}^{2p}u = \varepsilon f(\cdot, \cdot, u) \]
J. Differential Equations 7 1970 509-526

[Lovicarova69] Hana Lovicarova Periodic solution of a weakly nonlinear wave
equation in one dimension Czech J Math 19 1969 No2 pages 324-342.

[MP75] A. Marino and G. Prodi Metodi perturbativi nella teori di Morse. Bol-
letino U.M.I. (4) 11, Suppl. fasc. 3 (1975),1-32.

[Nirenberg59] Nirenberg, L. On elliptic partial differential equations. Ann.
Scuola Norm. Sup. Pisa (3) 13 1959 115-162.

[Ollivry83] Ollivry, Jean-Pascal, Vibrations forcées pour une quation d’onde non
linaire. (French. English summary) [Forced vibrations for a nonlinear wave
equation] C. R. Acad. Sci. Paris Sr. I Math. 297 (1983), no. 1, 2932

[Rabinowitz71] Rabinowitz P. H., Time periodic solutions of nonlinear wave
equations, Manuscripta Mathematica.5, 165-194 (1971)

[Rabinowitz67] Rabinowitz, P. H., Periodic solutions of nonlinear hyperbolic
partial differential equations. Comm. Pure Appl. Math. 20 1967 145–205.

[Rabinowitz78] Rabinowitz, P., Free vibrations for a semilinear wave equation
31 (1978) no.1,31-68. Communications on pure and applied mathematics

[Rabinowitz82] Rabinowitz, P., Multiple critical points of perturbed symmetric
functionals Transactions of the American Mathematical Society, Vol 272,
No 2. (Aug.,1982),pp 753-769.

[Rabinowitz84] Paul Rabinowitz Large amplitude periodic solution of a semi-
linear wave equation CPAM 37 1984 189-206

[Struwe90] Struwe, Michael Variational methods. Applications to nonlinear
partial differential equations and Hamiltonian systems. Third edition.
Springer-Verlag, Berlin, 2000.

[Tanaka88] Tanaka, Kazunaga Infinitely many periodic solutions for the equa-
tion: \[ u_{tt} - u_{xx} \pm |u|^{p-1}u = f(x,t) \] II. Trans. Amer. Math. Soc. 307 (1988),
no. 2, 615-645.

[Tanaka86] Tanaka, Kazunaga Density of the range of a wave operator with
nonmonotone superlinear nonlinearity. Proc. Japan Acad. Ser. A Math.
Sci. 62 (1986), no. 4, 129132,

[Tanaka89] Tanaka, Kazunaga Morse indices at critical points related to the
symmetric mountain pass theorem and applications. Comm. Partial Dif-
ferential Equations 14 (1989), no. 1, 99128.

[Torelli69] Torelli, Giovanni Soluzioni periodiche dell’equazione non lineare \[ u_{tt} -
 u_{xx} + \varepsilon F(x, t, u) = 0. \] (Italian) Rend. Ist. Mat. Univ. Trieste 1 1969 123137.