THE SINGULARITY CATEGORY OF AN ALGEBRA WITH
RADICAL SQUARE ZERO

ALES M.BOHADA

Abstract. This paper studies the singularity category of a locally bounded
$k$—linear category $\mathcal{C}$ with radical square zero. Following the work of Bautista
and Liu [4], we give a complete description of $D_{sg}(\mathcal{C})$. Examples are provided
to show how one can easily compute the generators of $D_{sg}(\mathcal{C})$ from the quiver
of $\mathcal{C}$.

Introduction

Representation theory of infinite quivers was initiated by Bautista-Liu-Paquette
[2,3,4,10,11], were they studied the Auslander-Reiten theory of certain subcategories
of the category of locally finite dimensional representations. Recently, Bautista and
Liu [4] gave a complete description of the bounded derived category of a connected
locally bounded $k$—category, using the notion of Galois covering which was general-
ized by them to derived categories from the well known Galois covering introduced
by Bongartz and Gabriel[5]. Our aim in this paper is to continue in the same path
by investigating the singularity category of a connected elementary locally bounded
$k$—category $\mathcal{C}$, where $k$ is a commutative field. By Gabriel’s theorem, we may iden-
tify $\mathcal{C}$ with the category $kQ/(kQ^+)^2$, where $Q$ is a connected locally finite quiver.
In [6](see also [12]) Chen proved that the singularity category of an artin algebra
with radical square zero is triangle equivalent to the category of finitely generated
projective modules over certain algebra which is a triangulated abelian category,
something that we will also get in case $Q$ is gradable quiver. Our approach is
completely different and works for finite and infinite dimensional algebras.

Since the concept of Galois covering was very fruitful so far, we will be attempting
the same thing for $D_{sg}(\mathcal{C})$, that is, we are looking for the best choice of a trian-
gulated category $\mathcal{I}$ to construct a Galois covering functor $\mathcal{I} \to D_{sg}(\mathcal{C})$. Note
that even if one is capable to find such category, it will only be interesting if we
can derive easily properties from $\mathcal{I}$ which will allowe us to understand completely
$D_{sg}(\mathcal{C})$.

Inspirited by the work [4], we will be able to give a complete description of
$D_{sg}(\mathcal{C})$ by proving the following theorem.

Theorem . Let $Q$ be a locally finite quiver. Then,
(1) There exists a Galois covering functor $D^b(rep^{-}(\hat{Q}^{\text{op}})[\Sigma^{-1}]) \to D_{sg}(\mathcal{C})$.
(2) $D^b(rep^{-}(\hat{Q}^{\text{op}})[\Sigma^{-1}]) \cong D_{sg}(\mathcal{C})$ if $Q$ is gradable.

2010 Mathematics Subject Classification. 16E35, 16G20, 16G70, 18E30, 18E35.
Key words and phrases. Representations of a quiver; modules over a linear category; triangu-
lated categories; derived categories; singularity categories; Galois covering.
The opposite quiver $Q^{\text{op}}$ of $Q$ is gradable and its grading is given by $(Q^{\text{op}})^n = Q^{-n}$, $\forall n \in \mathbb{Z}$, for more details, see for instance [3, section 7].

A morphism of quivers $\psi : Q \to Q'$ consists of two maps $\psi_0 : Q_0 \to Q_0$ and $\psi_1 : Q_1 \to Q'_1$, such that $\psi_1(Q_1(x,y)) \subseteq Q'_1(\psi_0(x),\psi_0(y))$, for all $x,y \in Q_0$. An isomorphism of quivers $\phi$ is a morphism of quivers such that $\phi_0$ and $\phi_1$ are two bijections. Let $G$ be any group, an action on $Q$ by $G$ is an automorphism of quiver.

1. Preliminaries and Background

This section will be devoted to notions which will play a crucial role, see [2,3,4] for more details.

1.1. Quivers. A quiver $Q$ is just a directed graph, more precisely it consists of two sets $Q_0$, $Q_1$ and two maps $s,t : Q_1 \to Q_0$. The elements of $Q_0$ are called the vertices, and those of $Q_1$ are called the arrows. The maps $s$ and $t$ assign a source $s(\alpha)$ and a target $t(\alpha)$ to every arrow $\alpha \in Q_1$. To an arrow $\alpha$ we associate its formal inverse $\alpha^{-1}$ with $s(\alpha^{-1}) = y$ and $t(\alpha^{-1}) = x$. In this way one can define a walk as a sequence $c_1 \ldots c_2 c_1$, where $c_j$ is a trivial path, arrow or inverse of arrow such that $s(c_{j+1}) = t(c_j)$. In case $c_j$ is an arrow for all $j$, then we call the sequence $c_n \ldots c_2 c_1$ a path of length $n$. We denote by $Q_k(x,y)$ the set of all paths of length $k$, and by $x^+$ (resp.$x^-$) the set of arrows $\alpha$ with $s(\alpha) = x$ (resp.the set of arrows $\gamma$ with $t(\gamma) = x$. Two important classes of quivers are locally finite quivers and strongly locally finite quivers, the first one say exactly that for all $x \in Q_0$, $x^+$ and $x^-$ are finite, and the second one as indicated by the name is locally finite and for all $k \in \mathbb{N}$ and all $x,y \in Q_0$, $Q_k(x,y)$ is finite.

Let $Q = (Q_0,Q_1)$ be a connected locally finite quiver and $w$ a walk in $Q$. The degree $\delta(w)$ of $w$ is defined as follow:

$$\delta(w) = \begin{cases} 
1 & \text{if } w \text{ is an arrow} \\
0 & \text{if } w \text{ is a trivial path} \\
-1 & \text{if } w \text{ is the inverse of an arrow}
\end{cases}$$

Thus this can be extended to any walk $w$ in $Q$ by the formula $\delta(u_1u_2) = \delta(u_1)\delta(u_2)$, hence if $p$ is a path in $Q$, then its degree is exactly its length. Another important class of quivers are those with the property that each closed walk has degree zero, this later will be called gradable quiver. Let $Q$ be a gradable quiver, then $Q_0 = \cup_{n \in \mathbb{Z}} Q^n$ such that $Q^m \cap Q^l = \emptyset$, if $m \neq l$. In this way a path from $x$ to $y$ with $x \in Q^m$ and $y \in Q^l$ is of length $l-m$. Moreover, a gradable locally finite quiver is strongly locally finite. The opposite quiver $Q^{\text{op}}$ of $Q$ is gradable and its grading is given by $(Q^{\text{op}})^n = Q^{-n}$, $\forall n \in \mathbb{Z}$, for more details, see for instance [3, section 7].

A morphism of quivers $\psi : Q \to Q'$ consists of two maps $\psi_0 : Q_0 \to Q_0$ and $\psi_1 : Q_1 \to Q'_1$, such that $\psi_1(Q_1(x,y)) \subseteq Q'_1(\psi_0(x),\psi_0(y))$, for all $x,y \in Q_0$. An isomorphism of quivers $\phi$ is a morphism of quivers such that $\phi_0$ and $\phi_1$ are two bijections. Let $G$ be any group, an action on $Q$ by $G$ is an automorphism of quiver.
Indeed, it is an action on both sets $Q_0$ and $Q_1$. In case the following property is verified, $g.x = x$ implies that $g$ is the identity of $G$, then we say that the action on $Q$ is free. Now a Galois $G$-covering is a morphism of quivers $\phi : Q \to Q'$ which satisfies the following conditions:

1. $\phi_0$ is surjective,
2. $\phi \circ g = \phi$, for all $g \in G$,
3. $x, y \in Q_0 \Rightarrow y = gx$ for some $g \in G$,
4. $x \in Q_0 \Rightarrow \phi_1$ induces two bijections $x^+ \to \phi_0(x)^+$ and $x^- \to \phi_0(x)^-$.

1.2. Path algebras and path categories. The path algebra $kQ$ of a quiver $Q$ over a field $k$ is the vector space with basis consisting of all paths of $Q$ and with the multiplication given by, if $p$ and $q$ are two paths in $Q$ then we write $pq$ is $s(p) = t(q)$ and zero otherwise. An admissible relation is an element of $kQ$ of the form $\sum_{i=1}^{r} \alpha_ip_i$, where $\alpha_i \in k$ and $p_i$ are paths in $Q$ with common source and a common target. An admissible ideal of $kQ$ is a two-sided ideal generated by non-zero admissible relations. In case $I$ is an admissible ideal, the algebra $kQ/I$ is called the bound quiver algebra.

Note that if the quiver $Q$ has infinitely many vertices, then $kQ/I$ has no identity element, thus the notion of the path category.

The path category is the category with objects the vertices of $Q$ i.e. elements of $Q_0$, and morphisms spaces are the vector spaces $kQ(x,y)$ with basis all the paths from $x$ to $y$. We will call a family of vector spaces $(I(x,y))_{(x,y) \in Q_0^2}$ an admissible ideal, if:

1. A two-sided ideal, that is, if for all $p \in I(x,y)$ and all $(q,r) \in I(y,z) \times I(w,x)$, $qp \in I(x,y)$ and $pr \in I(w,x)$,
2. Admissible, if $I(x,y) \subseteq kQ^{+2}(x,y)$ ($kQ^{+2}$ is the ideal generated by paths of length 2) and for any $x \in Q_0$, there exists $n \geq 2$ such that $kQ^{+n}(x,-) \subseteq I(x,-)$ and $kQ^{+n}(-,x) \subseteq I(-,x)$. In this way, one can define the quotient category or the bound path category $kQ/I$ with the same objects as $kQ$ and with morphisms spaces $kQ/I(x,y) = kQ(x,y)/I(x,y)$.

The small category $kQ/I$ will be called locally bounded if it verify the conditions below:

1. $x \neq y \Rightarrow x \not\cong y$, for all $x,y \in Q_0$,
2. $kQ/I(x,x)$ is local algebra for all $x \in Q_0$,
3. $\oplus_{y \in Q_0} kQ/I(x,y) \oplus kQ/I(y,x)$ is of finite dimension, for all $x,y \in Q_0$.

One can prove that the bound path category $kQ/I$ is locally bounded if and only if $Q$ is locally finite quiver and $I$ is an admissible ideal [5]. Thus, an elementary locally bounded category with radical square zero can be seen as the path category $kQ/(kQ^+)^2$, where $Q$ is a connected locally finite quiver.

From now one, $\mathcal{C}$ stand for an elementary locally bounded category.

1.3. Representations and modules. A left finitely generated module $M$ over $\mathcal{C}$ is a $k$–linear covariant functor $\mathcal{C} \to k$–mod, where $k$–mod is the category of finite dimensional $k$–vector spaces. The support of $M$ is the set $\text{supp}(M) = \{x \in Q_0/M(x) \neq 0\}$. The module $M$ will be called finitely generated if its support
is finite. The category of left finitely generated modules will be denoted by \( \mathcal{C}-\text{mod} \) which is obviously abelian subcategory of \( \mathcal{C}-\text{Mod} \) since \( \mathcal{C} \) is locally bounded. Let \( \mathcal{C}-\text{proj} \) be the additive subcategory of \( \mathcal{C}-\text{mod} \) generated by finitely generated projective modules of the form \( \mathcal{C}(x, -) \). By [3, Lemma 6.1], any finitely generated \( \mathcal{C} \)-module \( M \) has a projective cover \( \oplus_{x \in \text{supp}(M)} \mathcal{C}(x, -) \) in \( \mathcal{C}-\text{proj} \), thus \( \mathcal{C} \) has enough projectives. A representation \( N \) of \( Q \) is an object in \( \text{rep}(Q) \) and its consists of a family of finite dimensional \( k \)-vector spaces \( N(x), x \in Q_0 \), and a family of \( k \)-linear maps \( N(\beta) : N(x) \rightarrow N(y) \), where \( \beta \in Q_1 \). The support of \( N \) will not be just a set, but a full subquiver of \( Q \) generated by vertices for which \( M(x) \neq 0 \). We shall denote by \( \text{rep}^h(Q) \) the category of representations with finite support. An important subrepresentation of \( N \) is \( \text{soc}N \) called the socle of \( N \), it is defined by vector spaces \( (\text{soc}N)(x) \) which are the intersection of all kernels of maps \( N(\alpha) \), with \( \alpha \in x^+ \). Another important representations of \( Q \) are simple representations \( S_a \), injective representations \( I_a \), and finitely co-presented. The first ones are defined by \( S_a(a) = \mathbb{k} \) and zero for all vertices different from \( a \), and the second ones which will play an important role are defined by \( I_a(x), x \in Q_0 \) is the vector space with base \( Q(x, a) \) and linear maps \( I_a(\alpha) : I_a(x) \rightarrow I_a(y) \) sending a path of the form \( p\alpha \) to \( \alpha \), their category will be denoted by \( \text{inj}(Q) \) which is additive. Note that if \( Q \) is strongly locally finite, then the vector spaces \( I_a(x) \) are of finite dimension something which is true in case \( Q \) is gradable locally finite. The last ones are those \( M \) with injective co-resolution \( 0 \rightarrow M \rightarrow I \rightarrow J \rightarrow 0 \), where \( I, J \) are in \( \text{inj}(Q) \). Their category \( \text{rep}^-(Q) \) is abelian and clearly hereditary.

1.4. Localization of abelian categories and the singularity category. The localization of abelian categories is similar to the localization of triangulated categories, see [9]. Let \( \mathcal{A} \) be an abelian category, and \( \mathcal{B} \) a full subcategory of \( \mathcal{A} \). We are interested by categories of the form \( \mathcal{A}(\Sigma^{-1}) \), where \( \Sigma \) is the set of morphisms \( s \) of \( \mathcal{A} \) such that \( \text{Kers} \) and \( \text{Cokers} \) are in \( \mathcal{B} \) [9, Examples 2.5.d]. Since we want \( \mathcal{A}(\Sigma^{-1}) \) to be abelian, we will assume that \( \mathcal{B} \) is a thick subcategory, that is, it is closed under extensions, subobjects and quotient objects. More precisely, if \( 0 \rightarrow B \rightarrow C \rightarrow D \rightarrow 0 \) is a short exact sequence in \( \mathcal{A} \) with \( B \) and \( D \) are in \( \mathcal{B} \) if and only if \( C \) is also in \( \mathcal{B} \). Under this assumptions, \( \mathcal{A}(\Sigma^{-1}) \) is an abelian category, and the projection functor \( \mathcal{A} \rightarrow \mathcal{A}(\Sigma^{-1}) \) is exact. More precisely, the objects in \( \mathcal{A}(\Sigma^{-1}) \) are the objects in \( \mathcal{A} \) and \( \pi \) is the identity on objects. Morphisms are equivalence classes of roofs \( \xrightarrow{f} \mathcal{A} \). Where \( f \) is some morphism in \( \mathcal{A} \) and \( s \in \Sigma \). To be more precise, two roofs \( X \rightarrow Z \leftarrow Y, X \rightarrow Z' \leftarrow Y \) are equivalent, if there exists a third roof \( X \rightarrow Z'' \leftarrow Y \), which make the following diagram commute.

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & Y \\
\downarrow & \swarrow & \searrow \\
X & \xrightarrow{1} & Z'' & \xleftarrow{1} & Y
\end{array}
\]
The singularity category $D_{sg}(\mathcal{C})$ of $\mathcal{C}$ is the Verdier quotient $D^b(\mathcal{C})/K^b(\mathcal{C} - \text{proj})$, where $D^b(\mathcal{C})$ is the bounded derived category of complexes of finitely generated $\mathcal{C}$-modules and $K^b(\mathcal{C} - \text{proj})$ is the bounded homotopy category of complexes of finitely generated projective $\mathcal{C}$-modules. An important fact about the singularity category of $\mathcal{C}$ is that, $D_{sg}(\mathcal{C}) = 0$ if and only if $\mathcal{C}$ has finite global dimension, and this is the case, if $Q$ has no right infinite path. When we say that our quiver has an infinite path, this means that the quiver has an infinite path of type $\mathbb{A}_\infty$ or an oriented cycle.

1.5. Galois Covering Functor. All categories are assumed to be $k$–linear, that is, skeletally small in which morphism sets are $k$–vector spaces and the compositions of morphisms are $k$–bilinear. An action of a group $G$ on a $k$–linear category $\mathcal{A}$ is an automorphism of $\mathcal{A}$.

Actions on linear categories. Let $\mathcal{A}$ be a $k$-category with an action of a group $G$. The $G$-action is:

1. free if $g \cdot X \not= X$, for any object $X$ of $\mathcal{A}$ and any non-identity element $g$ of $G$;
2. locally bounded if, for any objects $X,Y \in \mathcal{A}$, $\text{Hom}_\mathcal{A}(X,g \cdot Y) = 0$ for all but finitely many $g \in G$;

A $k$-linear functor $k$-categories $E : \mathcal{A} \rightarrow \mathcal{B}$ is called $G$-stable if it has a $G$-stabilizer $\delta = (\delta_g)_{g \in G}$, where $\delta_g : E \circ g \rightarrow E$ are functorial isomorphisms such that $\delta_{h \cdot g,X} \circ \delta_{g,Y} = \delta_{g,h \cdot Y}$, for $g,h \in G$ and $X \in \mathcal{A}$. In case, $E$ induces a $k$-linear isomorphism $\oplus_{g \in G} \text{Hom}_\mathcal{A}(X,g \cdot Y) \rightarrow \text{Hom}_\mathcal{B}(E(X),E(Y)) : (u_g)_{g \in G} \mapsto \sum_{g \in G} \delta_g.Y \circ E(u_g)$, for each pair $X,Y$ of objects of $\mathcal{A}$. Then it will be called a Galois $G$-precovering. Recall that a $G$-stabilizer $\delta = (\delta_g)_{g \in G}$ is called trivial if $\delta_{g,X} = 1_X$ for all $g \in G$ and $X \in \mathcal{A}$. Then it will be called a Galois $G$-precovering.

Definition. Let $\mathcal{A}$ be a $k$-category with a free and locally bounded action of a group $G$. A Galois $G$-precovering $E : \mathcal{A} \rightarrow \mathcal{B}$ is called a Galois $G$-covering if it is dense.

2. Main Results

Proof of the theorem. Before we give the proof of our theorem, we shall need the following two lemmas which will prove extremely useful in what follows. Compare lemma 2.1 with [4, Proposition 3.5 (1)]. We will always assume that representations of $Q^{op}$ are locally finite support, that is, $(Q^{op})^n \cap \text{supp}(M)$ is finite for all $n \in \mathbb{Z}$.

2.1. Lemma. Let $Q$ be a gradable locally finite quiver and $M$ a representation in $\text{rep}(Q^{op})$. The following are equivalent:

1. There exist a short exact sequence $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$, where $N \in \text{rep}^b(Q^{op})$ and $L \in \text{inj}(Q^{op})$.
2. $M$ is finitely copresented.

Proof. Since $Q^{op}$ is gradable, we will make use of the functor $F$ defined in [4], that is, we prove the following: 1) $\Leftrightarrow F(M)$ has bounded cohomology $\Leftrightarrow$ 2). More precisely, the complex $F(M)^n$ over proj-$kQ/I$ is defined by its components, for each $n \in \mathbb{Z}$

$$F(M)^n = \oplus_{x \in Q^{op}} \mathcal{C}(x,-) \otimes M(x) \in \text{proj} - kQ/I,$$
and \(d^n_{F(M)} : \oplus_{x \in Q^{-n}} \mathcal{C}(x, -) \otimes M(x) \to \oplus_{y \in Q^{-n-1}} \mathcal{C}(y, -) \otimes M(y)\) to be the \(A\)-linear morphism given by the matrix \((d^n_{F(M)}(y, x))_{(y, x) \in Q^{-n-1} \times Q^{-n}}\), where

\[
d^n_{F(M)}(y, x) = \sum_{\alpha \in Q_1(y, x)} \mathcal{C}(\alpha, -) \otimes M(\alpha^o) : \mathcal{C}(x, -) \otimes M(x) \to \mathcal{C}(y, -) \otimes M(y).
\]

Assume that \(F(M)\) has bounded cohomology. Let us first show that the \(M\) has an injective hull in \(\text{inj}(Q^{op})\). Since \(F(M)\) is bounded above the support of \(M\) contains no right infinite path, thus by [2, Lemma 1.1], \(\text{soc}(M)\) is essential. It remains to prove that \(\text{soc}(M)\) is finitely supported. We claim that \(\forall x \in \bigcup_{n \leq l} Q^{-n}\) \(\text{soc}(M(x)) = 0\), where \(l\) is the biggest integer such that \(H^i(F(M)) \neq 0\). Indeed if this is not the case then, \(\text{soc}(M(x)) \neq 0\) for some \(x \in \bigcup_{n \leq l} Q^{-n}\), this means that for all arrows \(y \xmapsto{\alpha} x\) there exists a vector \(v \in M(x)\) such that \(\mathcal{C}(\alpha^o)(v) = 0\). Now take the tensor element \(\epsilon \otimes v \in \mathcal{C}(x, -) \otimes M(x)\). It is easy to see that \(d^n(\epsilon \otimes v) = d^n(y, x)(\epsilon \otimes v) = \sum_{\alpha \in Q_1(y, x)} \alpha \otimes M(\alpha^o)(v) = 0\), (here we assumed that \(x \in Q^{-m}\)). This contradicts the fact that for all \(n\) the image of \(d^n\) is in the radical, therefore \(\text{soc}(M)\) is finitely supported, and hence, \(M\) is a finitely cogenerated representation. \(F(M)\) has bounded cohomology \(\Rightarrow 1\). Let \(J\) be an injective hull of \(M\) and \(M'\) the subrepresentation of \(J\) which is isomorphic to \(M\). For the sake of simplicity we will see \(J\) like \(I_a\) for some \(a \in Q_0\). Now let \(x \in \bigcup_{k \leq l} Q^{-k}\) and \(p\) a path in \(M'(x)\). Since there is no relations in the quiver \(Q^{op}, q \beta^0 \neq 0\), this gives \(\alpha^0 \beta^0 \neq 0\) \((q = q^o \alpha^0, \alpha^0 : x \to z)\). Thus \(\beta \alpha = 0\). Now it’s easy to see that \(d^n(x, y) = \sum_{\alpha \in Q_1(y, x)} \mathcal{C}(\alpha, -) \otimes M'(\alpha^o) (\beta \otimes \epsilon) = \beta \alpha \otimes q^o = 0\). Since \(H^i(F(M')) = 0\), we have also \(H^i(F(M)) = 0\), then \(\mathcal{C}(\beta, -) \otimes M'(\beta^0)(\epsilon q^o \otimes q^\beta^0) = \beta \otimes q^\beta^0\). Therefore \(q \beta^0 \in M'(y)\). Let \(Y = Q^{-1} \cap \text{supp}(M')\), clearly \(Y = \{a^1_1, ..., a^1_p\}\) is finite and \(M'(a^1_i)\) is the vector space with basis all the paths that start at \(a^1_i\) and end at \(a\). We define a new representations \(W_i\) by the vector spaces \(W_i(a) = 0\) if \(a \notin \bigcup_{k \leq l} Q^{-k}\) \(W_i(a^1_i) = k \otimes M'(a^1_i)\) for any \(i \in \{1, ..., p\}\), \(W_i(a^1_{j+1}) = < a^1_{j+1} \xrightarrow{\alpha^{o}_{j+1}} a^1_j\) such that there exists a path \(p\) from \(a^1_j\) to \(a^1_i\) and \(0 \neq \alpha^o p \in M'(a^1_{j-1}) \otimes M'(a^1_j)\),..., \(W_i(a^1_{1}) = < \text{paths from } a^1_1 \text{ to } a^1_i \rangle \otimes M'(a^1_i)\), moreover linear maps are just the restrictions of linear maps coming from the representation \(M'\). Finally we put \(L = \bigoplus_{i=1}^p W_i \odot M'(a^1_i)\). The representation \(N\) is defined by \(N(x) = M'(x), \forall x \in \text{supp}(M') \setminus \bigcup_{k \leq l} Q^{-k}\) and the linear applications \(N(\alpha) = M'(\alpha)\) with \(x \xrightarrow{\alpha} y, x, y \in \text{supp}(M') \setminus \bigcup_{k \leq l} Q^{-k}\). It’s easy to see that \(N\) is a subrepresentation of \(M'\), so, we only need to show that the quotient \(M'/N\) is isomorphic to \(L = \bigoplus_{i=1}^p W_i \odot M'(a^1_i) = \bigoplus_{i=1}^p I_{a^1_i} \otimes M'(a^1_i)\). This later is immediate since for all \(x \in Q^{-n}\) such that \(l < n\) we have \(M'/N(x) = 0\), thus the identity does the job. Therefore, the existence of such short exact sequence \(0 \to N \to M' \to L \to 0\) is clear now.

(1) \(\Rightarrow F(M)\) has bounded cohomology. Applying the functor \(F\) to \(0 \to N \to M \to L \to 0\) where \(\text{supp}(N)\) is finite and \(L \in \text{inj}(Q^{op})\) give us the long exact sequence of cohomology, which implies that \(H^n(F(M))\) is not zero for only finitely many \(n \in \mathbb{Z}_n\), since \(F(L)\) has bounded cohomology by what is said above.

(2) \(\Rightarrow F(M)\) has bounded cohomology. Assume (2), then there exists an exact sequence \(0 \to M \to I \to J \to 0\). By what we proved above, \(F(I)\) and \(F(J)\) has bounded cohomology, thus \(F(M)\) has bounded cohomology, and we are done. \(\Box\)
2.2. Lemma. Let $Q$ be a gradable locally finite quiver. Then, $\text{rep}^-(Q^{op})[\Sigma^{-1}]$ is an abelian semisimple category, where $\Sigma = \{s/\text{Ker}(s), \text{Coker}(s) \in \text{rep}^b(Q^{op})\}$.

Proof. Let us first observe that $\text{rep}^b(Q^{op})$ is an abelian subcategory which is closed under extensions, subobjects and quotient objects, thus $\text{rep}^-(Q^{op})[\Sigma^{-1}]$ is abelian. Now, we need to prove that every short exact sequence in $\text{rep}^-(Q^{op})[\Sigma^{-1}]$ split. Firstly, by lemma 1, for any finitely copresented representation $M$, there exist a short exact sequence $0 \to N \to M \to L \to 0$, where $N$ is a representation with finite support and $L$ is isomorphic to a finite direct sum of objects in $\text{inj}(Q^{op})$. Now let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in $\text{rep}^-(Q^{op})[\Sigma^{-1}]$. As mentioned above, we may identify $A, B$ and $C$ with some objects in $\text{inj}(Q^{op})$.

Consider the following roof $A \xrightarrow{a} Z \xrightarrow{\beta} B$ which represents the morphism $f$, where $\text{Ker}\beta, \text{Coker}\beta$ and $\text{Ker}\alpha$ are in $\text{rep}^b(Q^{op})(\text{Ker}\alpha)$ is in $\text{rep}^b(Q^{op})$ because $f$ is a monomorphism in $\text{rep}^-(Q^{op})[\Sigma^{-1}]$ and $A$ is an injective object. Since $\text{rep}^-(Q^{op})$ is hereditary and $A$ is injective, the image of $\alpha$ is also injective. Take the short exact sequence $0 \to \text{Im}\alpha \xrightarrow{i} Z \xrightarrow{p} \text{Coker}\alpha \to 0$ in $\text{rep}^-(Q^{op})$, which is clearly split, then we have a diagram of short exact sequences in $\text{rep}^-(Q^{op})[\Sigma^{-1}]$.

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Im}\alpha & \xrightarrow{i} & Z & \xrightarrow{p} \text{coker}\alpha & \rightarrow & 0 \\
& & \downarrow{a} & \downarrow{\beta^{-1}} & \downarrow{\gamma} & & \\
0 & \rightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \rightarrow & 0
\end{array}
\]

Where $a$ is the inverse of $A \to \text{Im}\alpha$ in $\text{rep}^-(Q^{op})[\Sigma^{-1}]$, since $\text{Ker}\alpha$ is an object in $\text{rep}^b(Q^{op})$. Now the commutativity of the left square come from the fact that the morphism $\alpha$ is the composition of $\text{Im}\alpha \xrightarrow{i} Z$ and the projection $A \to \text{Im}\alpha$, that is, $f = \beta^{-1}\alpha = \beta^{-1}ia^{-1}$. This shows that the diagram commute and the two exact sequences are isomorphic. By the exactness of the projection functor, the short exact sequence $0 \xrightarrow{1} A \xrightarrow{f} B \xrightarrow{g} C \to 0$ split in $\text{rep}^-(Q^{op})[\Sigma^{-1}]$, and the lemma follows. \hfill \Box

One may ask whether the projection functor $\text{rep}^-(Q^{op}) \xrightarrow{P} \text{rep}^-(Q^{op})[\Sigma^{-1}]$ has a right adjoint functor or not, see [9, Proposition.4.1].

2.3. Definition (9, definition 4.1). We say that an object $M$ in $\text{rep}^-(Q^{op})$ is left-closed for $\Sigma$ if for any morphism $s \in \Sigma$, $\text{Hom}(s, M)$ is an isomorphism.

Now we give a negative answer to the question above.

2.4. Corollary. The projection functor $\text{rep}^-(Q^{op}) \xrightarrow{P} \text{rep}^-(Q^{op})[\Sigma^{-1}]$ has no right adjoint.

Proof. Assume that the projection functor has a right adjoint. First, let $M$ be a representation in $\text{rep}^-(Q^{op})$ which is not finitely supported. By the previous lemma, we can find a short exact sequence $0 \to N \to M \to L \to 0$, where $N \in \text{rep}^b(Q^{op})$ and $L \in \text{rep}^-(Q^{op})$. Applying the functor $\text{Hom}(\cdot, M)$ to the above sequence, we get $\text{Hom}(f, M) = 0$, which implies $f = 0$. This contradicts our assumption. Therefore infinitely supported representations in $\text{rep}^-(Q^{op})$ are not left-closed $\Sigma$. Now if $M$ is any representation in $\text{rep}^-(Q^{op})$ then by [9, proposition 4.1], there
Proof. Would be a representation $N$ which is left-closed for $\Sigma$ and a morphism $s : M \rightarrow N$ such that $P_2(s)$ is invertible. But the only choice for $N$ is an object in $\text{rep}^b(Q^{\text{op}})$, this yields, $M \equiv 0$. This is impossible in case $M$ is not finitely supported. We conclude that the projection functor has no right adjoint.

We still need some preparatory notions. The grading period $r_Q$ of a locally finite quiver $Q$ is the minimum of positive degrees of all closed walks. Clearly, $r_Q = 0$ if $Q$ is gradable. Now take a gradable connected component $Q$ of the quiver $Q^2$ which is also gradable [3, Lemma 7.2]. The translation group $G$ is the group generated by the translation $\rho$, that is, an automorphism of $Q$ sending $(x, n)$ to $(x, n + r_Q)$. Obviously, $G$ is torsion free group. By [3, Theorem 7.5(2)], there exists a Galois $G$–covering $\pi : ˜Q \rightarrow Q$ such that $\pi(x, n) = x$, called the minimal gradable covering of $Q$. This later induces a Galois $G$–covering $D^b(\mathcal{E}) \rightarrow D^b(\mathcal{E})$. The $G$–action on $\mathcal{E}$ gives a $G$–action on $D^b(\mathcal{E})$, and thus a $G$–action on $D_{sg}(\mathcal{E})$. Note that $\mathcal{E} = k\tilde{Q}^{\text{op}}/(k\tilde{Q}^+)^2$ is locally bounded since $\tilde{Q}$ is locally finite. Now, we need to define another group action on our category $D^b(\text{rep}^{-}(\tilde{Q}^{\text{op}})[\Sigma^{-1}])$. With the same way as in [4], let $G$ be the group generated by the automorphism $\vartheta = \rho_i^{-r_Q}$. Obviously, $G$ is also torsion free group and it acts on $D^b(\text{rep}^{-}(\tilde{Q}^{\text{op}})[\Sigma^{-1}])$ in a natural way. In order to study the relation between our functors and the two actions, we need the twist functor defined in [4, Lemma 4.3], that is, an automorphism $t$ that take a complex $X$ to another complex $t(X)$, where $t(X)^n = x^n$ and $t(X)^n \xrightarrow{-d^X_n} t(X)^{n+1}$, with $n \in \mathbb{Z}$. The advantage of using $t$ lies in the fact that the equivalence (that we will prove) $G : D^b(\text{rep}^{-}(\tilde{Q}^{\text{op}})[\Sigma^{-1}]) \rightarrow D_{sg}(\mathcal{E})$ will satisfies, $G(\partial^k X) = t^{rkX}G(X)$, for all $k \in \mathbb{Z}$ and for all $X \in D^b(\text{rep}^{-}(\tilde{Q}^{\text{op}})[\Sigma^{-1}])$.

**Theorem 4, Bautista-Liu.** Let $Q$ be a locally finite quiver. Then,

1. There exists a Galois covering functor $D^b(\text{rep}^{-}(\tilde{Q}^{\text{op}})) \rightarrow D^b(\mathcal{E})$, where $t(X)^n = x^n$ and $t(X)^n \xrightarrow{-d^X_n} t(X)^{n+1}$, with $n \in \mathbb{Z}$. The advantage of using $t$ lies in the fact that the equivalence (that we will prove) $G : D^b(\text{rep}^{-}(\tilde{Q}^{\text{op}})[\Sigma^{-1}]) \rightarrow D_{sg}(\mathcal{E})$ will satisfies, $G(\partial^k X) = t^{rkX}G(X)$, for all $k \in \mathbb{Z}$ and for all $X \in D^b(\text{rep}^{-}(\tilde{Q}^{\text{op}})[\Sigma^{-1}])$.

2. $D^b(\text{rep}^{-}(\tilde{Q}^{\text{op}}))[\Sigma^{-1}] \cong D_{sg}(\mathcal{E})$, if $Q$ is gradable.

3. $D^b(\mathcal{E})$ has almost split triangles if gldim $\mathcal{E}$ is finite.

Now, we are ready to prove our theorem.

**2.5. Theorem.** Let $Q$ be a locally finite quiver. Then,

1. There exists a Galois covering functor $D^b(\text{rep}^{-}(\tilde{Q}^{\text{op}})[\Sigma^{-1}]) \rightarrow D_{sg}(\mathcal{E})$.

2. $D^b(\text{rep}^{-}(\tilde{Q}^{\text{op}})[\Sigma^{-1}]) \cong D_{sg}(\mathcal{E})$, if $Q$ is gradable.

3. $D_{sg}(\mathcal{E})$ has no almost split triangles.

4. Any complex in $D_{sg}(\mathcal{E})$ is a finite direct sum of stalk complexes.

5. Any stalk complex of the form $S_a$ is isomorphic to a finite direct sum of stalk complexes of the form $\oplus S_a[t_i] \otimes V_i$, with $a_i \in \{t(\alpha)/\alpha \in a^+\}$, $t_i \in \mathbb{Z}$ and $V_i \in \text{mod} \rightarrow k$.

Note that in Bautista-Liu’s theorem (4) and (5) are not true in $D^b(\mathcal{E})$. We point out that our theorem still true in case we are working with infinite dimensional modules.

**Proof.** We first prove (1). Claim 1. $D^b(\text{rep}^{-}(\tilde{Q}^{\text{op}})[\Sigma^{-1}])$ is equivalent to $D_{sg}(\mathcal{E})$. Let $D^b_{\text{rep}^{-}(\tilde{Q}^{\text{op}})}(\text{rep}^{-}(\tilde{Q}^{\text{op}}))$ be the bounded derived category of $\text{rep}^{-}(\tilde{Q}^{\text{op}})$ with cohomology groups in $\text{rep}^b(\tilde{Q}^{\text{op}})$. It is well known that $D^b_{\text{rep}^{-}(\tilde{Q}^{\text{op}})}(\text{rep}^{-}(\tilde{Q}^{\text{op}}))$ is
a thick subtriangulated category of $D^b(rep^-(\hat{Q}^{op}))$. Let $i$ the canonical functor $D^b(rep^b(\hat{Q}^{op})) \to D^b(rep^b(\hat{Q}^{op}))(rep^-(\hat{Q}^{op}))$ and $j$ the functor from $D^b(rep^b(\hat{Q}^{op}))(rep^-(\hat{Q}^{op}))$ to $D^b(rep^b(\hat{Q}^{op}))(rep^-(\hat{Q}^{op}))$, which take a complex $X$ in $D^b(rep^b(\hat{Q}^{op}))(rep^-(\hat{Q}^{op}))$ to its cohomology complex $\oplus_i H^i(X)[-i]$. Now since $rep^-(\hat{Q}^{op})$ is hereditary, it is not so hard to see that these two functors are quasi-inverse, thus the natural functor $D^b(rep^b(\hat{Q}^{op})) \to D^b(rep^b(\hat{Q}^{op}))(rep^-(\hat{Q}^{op}))$ is an equivalence and $D^b(rep^b(\hat{Q}^{op}))$ is a thick triangulated subcategory of $D^b(rep^-(\hat{Q}^{op}))$. Now the functor $F$ is the composition of the equivalence $D^b(rep^-(\hat{Q}^{op})) \cong D^b(\hat{E})$ [4] with the projection functor $D^b(\hat{E}) \to D_{sg}(\hat{E})$. Since the equivalence between $D^b(rep^-(\hat{Q}^{op}))$ and $D^b(\hat{E})$ is given by the sum total complex, it sends objects of $D^b(rep^b(\hat{Q}^{op}))$ to zero object of $D_{sg}(\hat{E})$ and vice versa, thus the existence of a functor $G$ such that $F \cong G\pi$, that is, the following diagram of triangulated categories commute.

\[
\begin{array}{ccc}
D^b(rep^-(\hat{Q}^{op})) & \xrightarrow{\pi} & D^b(rep^-(\hat{Q}^{op}))/D^b(rep^b(\hat{Q}^{op})) \\
\downarrow{F} & & \downarrow{G} \\
D_{sg}(\hat{E}) & & \\
\end{array}
\]

Note that the kernel of $F$ is exactly $D^b(rep^b(\hat{Q}^{op}))$, thus, we only need to prove that $G$ is full dense. Let $f = \pi(\alpha)\pi(\beta)^{-1}$ be a morphism in $D_{sg}(\hat{E})$ with $C(\beta) \in K^b(proj - \hat{E})$. By the equivalence [4, Theorem 3.9] we can get two morphisms $\alpha',\beta'$ such that $C(\beta') \in D^b(rep^b(\hat{Q}^{op}))$. Now put $f' = \pi(\alpha')\pi(\beta')^{-1}$, then $G(f') = f$. The density now is clear and by theorem [10, Theorem 3.2] we may identify $D^b(rep^-(\hat{Q}^{op}))/D^b(rep^b(\hat{Q}^{op}))$ with $D^b(rep^-(\hat{Q}^{op})[\Sigma^{-1}])$.

Claim 2. The minimal gradable covering $\pi : \hat{Q} \to Q$ induce a Galois $G$-covering $D_{sg}(\hat{E}) \xrightarrow{\pi_S} D_{sg}(\hat{E})$.

By the equivalence $D^b(rep^-(\hat{Q}^{op})[\Sigma^{-1}]) \cong D_{sg}(\hat{E})$, for any two complexes $X,Y \in D^b(rep^-(\hat{Q}^{op})[\Sigma^{-1}])$ we have an isomorphism

\[G_{X,Y} : \oplus_{k \in \mathbb{Z}} \text{Hom}_{D^b(rep^-(\hat{Q}^{op})[\Sigma^{-1}])}(X,\mathcal{O}Y) \to \oplus_{k \in \mathbb{Z}} \text{Hom}_{D_{sg}(\hat{E})}(G(X),G(\mathcal{O}Y))(\ast),\]

With the same argument as in [4, Lemma 4.9], and since $G(\mathcal{O}Y) = t^{k+r}\mathcal{O}(p^*G(Y))$ we get an isomorphism

\[\oplus_{k \in \mathbb{Z}} \text{Hom}_{D_{sg}(\hat{E})}(G(X),G(\mathcal{O}Y)) \to \oplus_{p \in \mathbb{Z}} \text{Hom}_{D_{sg}(\hat{E})}(G(X),\mathcal{O}G(Y))(\ast\ast)\]

Now, by what we have said above, and the fact that $rep^-(\hat{Q}^{op})[\Sigma^{-1}]$ is semisimple, the $G$–action on $D_{sg}(\hat{E})$ is locally bounded. Moreover, the group is torsion free, therefore the $G$–action is free. Now, consider the following commutative square.

\[
\begin{array}{ccc}
D^b(\hat{E}) & \xrightarrow{\pi} & D_{sg}(\hat{E}) \\
\downarrow{\pi_D} & & \downarrow{\pi_S} \\
D^b(\hat{E}) & \xrightarrow{\pi} & D_{sg}(\hat{E}) \\
\end{array}
\]

Where $\pi$ is the projection functor and $\pi_D$ is a Galois $G$-covering which induces a dense triangulated functor $\pi_S$. We need to check that $\pi_S$ is $G$–stable, but this
Therefore $\pi^S$ is a Galois $G$-covering.

Claim 3. The composition $\pi^SG$ is a Galois $\mathfrak{S}$-covering. First observe that the nature of the $\mathfrak{S}$-action on $D^b(\text{rep}^-(\tilde{Q}^{op})[\Sigma^{-1}])$ and the fact that $\text{rep}^-(\tilde{Q}^{op})[\Sigma^{-1}]$ is semisimple force it to be locally bounded, and thus free. Consider the commutative diagram below, where the left triangle and the right square are commutatives and

Let $\vartheta \in \mathfrak{S}$.

\[
\begin{array}{ccc}
D^b(\text{rep}^-(\tilde{Q}^{op})) & \cong & D^b(\mathfrak{E}) \\
\downarrow & & \downarrow \pi \\
D^b(\text{rep}^-(\tilde{Q}^{op})[\Sigma^{-1}]) & \cong & D^b(\mathfrak{E})
\end{array}
\]

We have that $\pi^SG\vartheta \cong \pi^SF\vartheta \cong \pi^SG$, with a $\mathfrak{S}$-stabilizer $\pi(\vartheta) = (\pi(\vartheta_\mathfrak{S}))_{\vartheta \in \mathfrak{S}}$, where $\delta = (\delta_\vartheta)_{\vartheta \in \mathfrak{S}}$ is a $\mathfrak{S}$-stabilizer of the Galois $\mathfrak{S}$-covering $D^b(\text{rep}^-(\tilde{Q}^{op}) \to D^b(\mathfrak{E})$.

Now the three group isomorphisms $(*)$, $(**)$, and $(***)$, gives the isomorphism

\[
\oplus_{k \in \mathbb{Z}} \text{Hom}_{D_{\mathfrak{S}}(\mathfrak{E})}(\pi^S(X), \pi^S(Y)),
\]

The density is immediate and thus we get our claim.

(2) If $Q$ is gradable then $r_Q = 0$ and the group $\mathfrak{S}$ is trivial. By the group isomorphism above, the functor $\pi^SG$ is full dense and by [3, Lemma 2.6] it is faithful. Therefore we obtain what we want.

(3) This follows easily from the fact that there is no almost split triangles in $D^b(\text{rep}^-(\tilde{Q}^{op})[\Sigma^{-1}])$, (1) and [3, Theorem 3.7].

(4) Let $Z \in D_{\mathfrak{S}}(\mathfrak{E})$. The Galois $\mathfrak{S}$-covering $\pi^SG$ ensures the existence of an object $Z' \in D^b(\text{rep}^-(\tilde{Q}^{op})[\Sigma^{-1}])$ such that $\pi^SG(Z') \cong Z$. By lemma 2.1, this later is isomorphic to a finite direct sum $\oplus_i M_i[s_i]$, where $M_i \in \text{rep}^-(\tilde{Q}^{op})[\Sigma^{-1}]$, and $s_i \in \mathbb{Z}$. By lemma 2.2, $M_i \cong \oplus^n_i I_{a_i} \otimes V_i$, where $V_i \in \text{mod-}k$ and $a_i \in \mathbb{Q}_0$. By the left triangle and [4, Lemma 3.4], $G\pi(Z') \cong F(Z') \cong \oplus_{i=1}^n (S_{a_i} [t_i] \otimes V_i)[s_i]$. Now, since $\pi^S$ preserve simples, we get $\pi^SG\pi(Z') \cong \oplus_{i=1}^n (S_{\pi^S(a_i)}[t_i] \otimes V_i)[s_i]$.

(5) Let $S_a$ be a stalk complex in $D_{\mathfrak{S}}$ with $S_a$ a simple $\mathfrak{E}$-module. By the minimal gradable covering $\pi : \tilde{Q} \to Q$, there exists $b \in \tilde{Q}_0$ such that $\pi(b) = a$. Now let $\alpha^+ = \{ \alpha_i \to a_i \}$ which is finite since $Q$ is locally finite. Then $b^+ = \{ \beta_i \to b_i \}$ is also finite and $(b^+)^- = b^+$ in $\tilde{Q}^{op}$. By lemma 2.1 and lemma 2.2, the injective module $I_b$ is isomorphic to $\oplus I_b \otimes V_i$ and by the Galois covering proved in (1) the image of $I_b$ which is $S_a[t]$ is isomorphic to $\oplus S_a[t_i] \otimes V_i$. This proves the theorem.

\[ \square \]
2.1. **Examples.** Here we show that from the quiver below we can compute easily the generators of $D_{sg}(\mathcal{C})$. Consider the following quiver $Q$ which is clearly not gradable.

![Diagram of a quiver](image)

By the minimal gradable covering $\pi$ of $Q$ we can find three points in $\tilde{Q}$ such that $\pi(a_1) = 1$, $\pi(a_2) = 2$ and $\pi(a_3) = 3$. Let $a_3 \xrightarrow{\alpha} a_1$ and $a_3 \xrightarrow{\beta} a_2$, then we have two arrows in $\tilde{Q}^{op}$ $a_1 \xrightarrow{\alpha^*} a_3$ and $a_2 \xrightarrow{\beta^*} a_3$. Thus we get an exact short sequence in $\text{rep}^{-}(\tilde{Q}^{op})$, $0 \to S_{a_3} \to I_{a_3} \to I_{a_1} \otimes V_1 \oplus I_{a_2} \otimes V_2 \to 0$. Moreover, $S_{a_3}$ is finitely supported implies $I_{a_3} \cong I_{a_1} \otimes V_1 \oplus I_{a_2} \otimes V_2$ in $\text{rep}^{-}(\tilde{Q}^{op})$. This fact with the above theorem yields $S_3 \cong S_1[t_1] \oplus V_1 \oplus S_2[t_2] \otimes V_2$.

With the same argument we show that there exists a short exact sequences $0 \to S_{a_4} \to I_{a_4} \to I_{a_3} \otimes V_3 \oplus I_{a_5} \otimes V_5 \to 0$, thus $S_4 \cong S_3[t_3] \otimes V_3 \oplus S_5[t_5] \otimes V_5$. $S_7 \cong S_5[t_5] \otimes V_3 \oplus S_6[t_6] \otimes V_6$, $S_6 \cong S_4[t_4] \otimes V_4'$, $S_8 \cong S_7[t_7] \otimes V_7 \oplus S_5[t_5] \otimes V_9$, $S_11 \cong S_9[t_9] \otimes V_9 \oplus S_{10}[t_{10}] \otimes V_9 \oplus S_{12}[t_{12}]$, $S_{10} \cong S_8[t_8] \otimes V_8$, and clearly, $S_{13} \cong S_{12}[t_{12}]$, $S_{14} \cong S_{13}[t_{13}] \cong S_{12}[t_{12}]$. We conclude that any complex in $D_{sg}(\mathcal{C})$ is of the form $\oplus_i (S_i \otimes V_i)[t_i]$ where $S_i \in \{S_1, S_2, S_5, S_9, S_{12}\}$, $s_i, t_i \in \mathbb{Z}$ and $V_i \in \text{mod } k$.

Now, if we take the gradable quiver $A_{\infty}$

![Diagram of a quiver](image)

Clearly, objects in $D_{sg}(\mathcal{C})$ are of the form $\oplus_i S_a[t_i]$, for some $a \in (A_{\infty})_0$, $t_i \in \mathbb{Z}$.

We finish by a finite quiver with an oriented cycle. Consider the following quiver

![Diagram of a quiver](image)

Then by the minimal gradable covering, the quiver $\tilde{Q}$ has an infinite path, hence by the previous example the objects in $D_{sg}(\mathcal{C})$ are $\oplus_i S_1[t_i]$.

**Acknowledgement** The author wishes to express his thanks to Prof Shuping Liu for helpful discussions and to Institut des sciences mathématiques for the financial support.

**References**

[1] H. Asashiba, R. Hafezi, R. Vahed, Gorenstein versions of covering techniques for linear categories and their applications, J. Algebra 507(2018), 320-361.

[2] Bautista, R., Liu, S., Paquette, C, Representation theory of strongly locally finite quivers. Proc. Lond. Math. Soc. (2012).

[3] R. Bautista, S. Liu, covering theory for linear categories with application to derived categories, J. Algebra 406(2014), 173-225.

[4] R. Bautista, S. Liu, The bounded derived categories of an algebra with radical squared zero, J. Algebra 482(2017), 309-345.
[5] K. Bongartz, P. Gabriel, covering spaces in representation theory, Invent. Math. 65(1982) 331-378.
[6] X. W. Chen, The singularity category of an algebra with radical square zero, Doc. Math. 16(2011), 921-936.
[7] P. Deligne, Théorème de Lefschetz et critères de dégénérescence de suites spectrales, Publ. Math. IHES 35(1969), 107-126.
[8] P. Deligne, Décomposition dans la catégorie Dérivée, Motives (Seattle, WA, 1991), 115-128, Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.
[9] P. Gabriel, M. Zisman. Calculus of fractions and homotopy theory. Springer Ergebnisse 35(1967).
[10] J. I. Miyachi, Localization of triangulated categories and derived categories. J. Algebra 141, 463-483 (1991).
[10] C. Paquette, On the Auslander-Reiten quiver of the representations of an infinite quiver. Alg. Rep. Theory 16(6), 1685-1715 (2013).
[11] C. Paquette, Irreducible Morphisms and Locally finite dimensional representations. Alg. Rep Theory 19(5), 1239-1255 (2016).
[12] S. P. Smith, Equivalence of categories involving graded modules over path algebras of quivers, Adv. Math. 230(2012), 1780-1810.

BOUHADA, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE SHERBROOKE, SHERBROOKE, QUÉBEC, CANADA.

E-mail address: mohammed.bouhada@usherbrooke.ca