ON THE NOTIONS OF EQUILIBRIA FOR TIME-INCONSISTENT STOPPING PROBLEMS IN CONTINUOUS TIME

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ABSTRACT. A new notion of equilibrium, which we call strong equilibrium, is introduced for time-inconsistent stopping problems in continuous time. Compared to the existing notions introduced in [4] and [3], which in this paper are called mild equilibrium and weak equilibrium respectively, a strong equilibrium captures the idea of subgame perfect Nash equilibrium more accurately. When the state process is a continuous-time Markov chain and the discount function is log sub-additive, we show that an optimal mild equilibrium is always a strong equilibrium. Moreover, we provide a new iteration method that can directly construct an optimal mild equilibrium and thus also prove its existence.

1. INTRODUCTION

On a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})\) consider an optimal stopping problem in continuous time

\[
\sup_{\tau \in \mathcal{T}} \mathbb{E}_x [\delta(\tau) X_\tau],
\]

where \(X = (X_t)_{t \in [0, \infty)}\) is a time-homogeneous Markov process taking values in some space \(X \subset \mathbb{R}\), \(\mathcal{T}\) is a set of stopping times, \(\delta\) is a discount function, and \(\mathbb{E}_x\) is the expectation given \(X_0 = x\). It is well known that when \(\delta\) is not exponential, the problem (1.1) may be time-inconsistent. That is, the optimal stopping strategy obtained today may not be optimal in the eyes of future selves. One way to approach such time inconsistency is consistent planning (see [10]), which is formulated as a subgame perfect Nash equilibrium: once an equilibrium strategy is enforced over the planning horizon, the current self has no incentive to deviate from it, given all future selves will follow the equilibrium strategy.

There are two general notions of equilibrium stopping strategies in continuous time in the literature. The first notion is proposed in [4] and further studied in [5, 7, 6], which we will call mild equilibrium in this paper. Following [4, Definition 3.3] and [7, Definition 2.2], we have the following definition of mild equilibrium.

**Definition 1.1.** A measurable set \(S \subset X\) is said to be a mild equilibrium, if

\[
\begin{align*}
x &\leq \mathbb{E}_x [\delta(\tau_S) X_{\tau_S}], & \forall x \notin S, \\
x &\geq \mathbb{E}_x [\delta(\tau^+_S) X_{\tau_S}], & \forall x \in S,
\end{align*}
\]

where

\[
\tau_S := \inf\{t \geq 0 : X_t \in S\}, \quad \text{and} \quad \tau^+_S := \inf\{t > 0 : X_t \in S\}.
\]
In the above $S$ is the stopping region, and the economic interpretation for Definition 1.1 is clear, there is no incentive to deviate. That is, in (1.2) when $x \notin S$, it is better to continue and get $E_x[\delta(\tau_S)X_{\tau_S}]$, rather than to stop and get $x$; on the surface a similar statement applies to (1.3). However, when the time of return for $X$ is 0 (i.e., $P(\tau_x = 0 | X_0 = x) = 1$), which is satisfied for continuous-time Markov chain and many one-dimension diffusion processes, $\tau_S = \tau_S^+$ and thus (1.3) trivially holds. In other words, when the time of return is 0, there is no actual deviation captured by (1.3) from stopping to continuing, and Definition 1.1 is equivalent to the following.

**Definition 1.2.** A measurable set $S \subset \mathbb{X}$ is said to be a mild equilibrium, if

$$x \leq E_x[\delta(\tau_S)X_{\tau_S}] =: J(x, S), \quad \forall x \notin S. \tag{1.5}$$

Consequently, with the time of return being 0 the notion of mild equilibrium cannot fully capture the economic meaning of equilibrium. It is easy to see that $S = \mathbb{X}$ is always a mild equilibrium, and it is not clear why always stopping immediately is a reasonable strategy.

As can be seen from [4, 5, 7, 6], there is often a continuum of mild equilibria in many natural models, which naturally leads to the question of equilibrium selection. In [7], optimal mild equilibrium in the sense of point-wise dominance is considered. In particular, from [7, Definition 2.3] we have the following definition.

**Definition 1.3.** A mild equilibrium $S^*$ is said to be optimal, if for any other mild equilibrium $S$,

$$x \vee J(x, S^*) \geq x \vee J(x, S) \iff J(x, S^*) \geq J(x, S), \quad \forall x \in \mathbb{X}. \tag{1.6}$$

Note that $x \vee J(x, S)$ represents the value associated with the stopping region/strategy $S$. In [7] the existence of optimal equilibrium is established. A discrete-time version is in [8].

The second notion of equilibrium, which we call weak equilibrium in this paper, is proposed in [3] and further investigated in [2]. Following [3], we have the definition of weak equilibrium (we adapt the definition slightly for our setting).

**Definition 1.4.** A measurable set $S \subset \mathbb{X}$ is said to be a weak equilibrium, if

$$\begin{cases} x \leq E_x[\delta(\tau_S)X_{\tau_S}], & \forall x \notin S, \\ \liminf_{\epsilon \downarrow 0} \frac{x - E_x[\delta(\tau_S^\epsilon)X_{\tau_S^\epsilon}]}{\epsilon} \geq 0, & \forall x \in S, \end{cases} \tag{1.6}$$

where

$$\tau_S^\epsilon = \inf\{t \geq \epsilon : X_t \in S\}. \tag{1.7}$$

Compared to (1.3), the first-order condition (1.6) does capture the deviation from stopping to continuing. However, similar to that for time-inconsistent control (see e.g., [1, Remark 3.5] and [7]), the first-order criterion does not correspond to the equilibrium concept perfectly: when the limit in (1.6) equals zero, it is possible that for all $\epsilon > 0$, $x < E_x[\delta(\tau_S^\epsilon)X_{\tau_S^\epsilon}]$, in which case there is an incentive to deviate.

To sum up, the economic interpretation of being “equilibrium” for mild and weak ones is inadequate. To resolve this issue, we introduce the following concept of strong equilibrium, which is inspired by [9].

**Definition 1.5.** A measurable set $S \subset \mathbb{X}$ is said to be a strong equilibrium, if

$$\begin{cases} x \leq E_x[\delta(\tau_S)X_{\tau_S}], & \forall x \notin S, \\ \exists h(x) > 0, \text{ s.t. } \forall \epsilon \in (0, h(x)), \ x \geq E_x[\delta(\tau_S^\epsilon)X_{\tau_S^\epsilon}], & \forall x \in S. \end{cases} \tag{1.8}$$
Compared to (1.3) and (1.6), condition (1.8) not only captures the deviation from stopping to continuing, but also more precisely indicates the disincentive of such deviation. Consequently, a strong equilibrium delivers better economic meaning as being an “equilibrium”.

In this paper, when $X$ is a Markov chain we show that an optimal mild equilibrium is a strong equilibrium (see Theorem 2.1). (Obviously, a strong equilibrium is also weak, and a weak equilibrium is also mild.) We also provide examples showing that a strong equilibrium may not be an optimal mild equilibrium, and a weak equilibrium may not be strong. Therefore, we thoroughly obtain the relation between mild, weak, strong, and optimal mild (and thus optimal weak, optimal strong) equilibria. Moreover, we provide a new iteration method which directly constructs an optimal mild equilibrium and thus also establish its existence (see Theorem 2.2). In [8, 7], an optimal equilibrium is constructed by the intersection of all (mild) equilibria. In principle, this requires us to first find all (mild) equilibria in order to get the optimal one, which may not be implementable in many cases. The new iteration method proposed in this paper is much easier and more efficient to implement. It would be interesting to see whether such results can be extended to diffusion models, which we will leave for future research.

The rest of the paper is organized as follows. Section 2 collects the main results of the paper. An optimal mild equilibrium is proved to be a strong equilibrium, and can be directly constructed via a new iteration method. Section 3 focuses on a concrete two-state model, which demonstrates the differences between these equilibria.

2. The Main Results

In this section, we apply the concepts in Section 1 to a continuous-time Markov chain and present our main results under this setting. Let $X = (X_t)_{t \geq 0}$ be a time-homogeneous continuous-time Markov chain. It has a finite or countably infinite state space $X \subset [0, \infty)$. Let $\lambda_x$ be the transition rate out of the state $x \in X$, and $q_{xy}$ be the transition rate from state $x$ to $y$ for $y \neq x$. Then we have that $\lambda_x = \sum_{y \neq x} q_{xy}$. The discount function $t \mapsto \delta(t)$ is assumed to be non-exponential and decreasing, with $\delta(0) = 1$ and $\lim_{t \to \infty} \delta(t) = 0$. Let the filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$ be generated by $X$. Furthermore, we make the following assumptions on $X$ and $\delta(\cdot)$.

Assumption 2.1. (i) $C := \sup X < \infty$ and $\lambda := \sup_{x \in X} \lambda_x < \infty$.

(ii) $X$ is irreducible, i.e., for any $x, y \in X$, $\inf\{t \geq 0 : X_t = y | X_0 = x\} < \infty$, a.s..

Assumption 2.2. (i) $\delta$ is log-subadditive, i.e.,

\[ \delta(s)\delta(t) \leq \delta(s + t), \quad \forall s, t > 0. \tag{2.1} \]

(ii) $t \mapsto \delta(t)$ is differentiable at $t = 0$, and $\delta'(0) < 0$.

Remark 2.1. Assumption 2.2 (i) is closely related to decreasing impatience in Behavioral Economics and Finance and commonly used when studying non-exponential discounting problems; see e.g., [8, 7, 4].

The following is the first main result of this paper, which shows that an optimal mild equilibrium is a strong equilibrium. The proof is provided in Section 2.1.

Theorem 2.1. Let Assumptions 2.1 and 2.2 hold. If $S$ is an optimal mild equilibrium, then it is a strong equilibrium.
Since all mild equilibria are strong equilibria, an optimal mild equilibrium will generate larger values than any strong equilibrium as well. With Theorem 2.1, we can conclude that any optimal mild equilibrium is a strong equilibrium and in fact is an optimal strong equilibrium.

The following is the second main result of this paper. It provides an iteration method which directly constructs an optimal mild equilibrium, and thus also establishes the existence of weak, strong, and optimal mild equilibria. The proof of this result is presented in Section 2.2.

**Theorem 2.2.** Let $S_0 := \emptyset$, and
$$ S_{n+1} := S_n \cup \left\{ x \in \mathcal{X} \setminus S_n : x > \sup_{S \subseteq \mathcal{X} \setminus \{x\}} J(x, S) \right\}. \tag{2.2} $$

Let
$$ S_\infty := \bigcup_{n=0}^{\infty} S_n. \tag{2.3} $$

If Assumptions 2.1 (i) and 2.2 (ii) hold, then $S_\infty$ is an optimal mild equilibrium. If in addition Assumption 2.2 (ii) holds, then $S_\infty$ is a strong equilibrium.

2.1. **Proof of Theorem 2.1.** Recall $\tau_S, \tau^c_S, J(\cdot, \cdot)$ defined in (1.4), (1.7), (1.5) respectively. We have the following characterization of (1.6) in Definition 1.2.

**Proposition 2.1.** Let Assumptions 2.1 and 2.2 (ii) hold. Then $S \subset \mathcal{X}$ is a weak equilibrium if and only if $S$ is a mild equilibrium and for all $x \in S$,
$$ x(\lambda_x - \delta'(0)) \geq \sum_{y \in \mathcal{X} \setminus \{x\}} y q_{xy} + \sum_{y \in \mathcal{S}} J(y, S) q_{xy}. $$

**Proof.** By definition, we only need to check condition (1.6) in Definition 1.2 is equivalent to the above inequality.

Denote $T_x := \inf\{ t \geq 0 : X_t \neq x, X_0 = x \}$ as the holding time at state $x$, which has exponential distribution with parameter $\lambda_x$. Then
$$ \mathbb{E}_x[\delta(\tau^c_S)X_{\tau^c_S}] = \mathbb{E}_x[\delta(\tau^c_S)X_{\tau^c_S}1_{\{T_x > \varepsilon\}}] + \sum_{y \in \mathcal{X} \setminus \{x\}} \mathbb{E}_x[\delta(\tau^c_S)X_{\tau^c_S}1_{\{T_x \leq \varepsilon, X_{T_x} = y, T_y + T_x > \varepsilon\}}] + O(\varepsilon^2) $$
$$ = \delta(\varepsilon)x e^{-\lambda_x \varepsilon} + \left\{ \sum_{y \in \mathcal{X} \setminus \{x\}} \delta(\varepsilon) y q_{yx} \lambda_x + \sum_{y \in \mathcal{S}} \mathbb{E}_y[\delta(\varepsilon + \tau_S)X_{\tau_S}] q_{xy} \lambda_x \right\} (\lambda_x \varepsilon + O(\varepsilon^2)) + O(\varepsilon^2). $$

Notice that $\delta(\varepsilon) = 1 + \delta'(0) \varepsilon + O(\varepsilon^2)$. Therefore we have
$$ \mathbb{E}_x[\delta(\tau^c_S)X_{\tau^c_S}] = x + \left\{ -x(\lambda_x - \delta'(0)) + \sum_{y \in \mathcal{X} \setminus \{x\}} y q_{xy} + \sum_{y \in \mathcal{S}} q_{xy} \mathbb{E}_y[\delta(\varepsilon + \tau_S)X_{\tau_S}] \right\} \varepsilon + o(\varepsilon). $$

Therefore, (1.6) is equivalent to
$$ x(\lambda_x - \delta'(0)) \geq \sum_{y \in \mathcal{X} \setminus \{x\}} y q_{xy} + \sum_{y \in \mathcal{S}} \mathbb{E}_y[\delta(\tau_S)X_{\tau_S}] q_{xy}. $$

**Corollary 2.1.** Let Assumptions 2.1 and 2.2 (ii) hold. If $S$ is a mild equilibrium and satisfies
$$ x(\lambda_x - \delta'(0)) > \sum_{y \in \mathcal{X} \setminus \{x\}} y q_{xy} + \sum_{y \in \mathcal{S}} \mathbb{E}_y[\delta(\tau_S)X_{\tau_S}] q_{xy}, $$
then it is a strong equilibrium.

For the rest of the paper, we will sometimes use the notation

\[ \rho(x, S) := \inf\{t \geq 0 : X_t^x \in S\} \]

in the place of \( \tau_S \) to emphasize the initial state \( X_0 = x \) (\( X^x \) here is the Markov chain starting at \( x \)).

**Lemma 2.1.** Let Assumption 2.2 (i) hold. For \( x \in S \), denote \( \hat{S} = S \setminus \{x\} \). If \( S \) is an optimal mild equilibrium, then for any \( y \notin S \),

\[ J(y, \hat{S}) - J(y, S) \geq E_y[\delta(\tau_S)1_{\{X_{\tau_S}=x\}}](J(x, \hat{S}) - x). \]

**Proof.** Since \( \hat{S} \subset S \), we have \( \rho(y, S) \leq \rho(y, \hat{S}) \). Then

\[
J(y, \hat{S}) - J(y, S) = E_y[\delta(\rho(y, \hat{S}))X_{\rho(y, \hat{S})}\{x_{\rho(y, \hat{S})}=x\}] + E_y[\delta(\rho(y, S))X_{\rho(y, S)}\{x_{\rho(y, S)}\in S\}] - E_y[\delta(\rho(y, S))X_{\rho(y, S)}] \\
= E_y[\delta(\rho(y, \hat{S}))X_{\rho(y, \hat{S})}\{x_{\rho(y, \hat{S})}=x\}] + E_y[\delta(\rho(y, S))X_{\rho(y, S)}\{x_{\rho(y, S)}\in S\}] - E_y[\delta(\rho(y, S))X_{\rho(y, S)}] \\
= E_y[\delta(\rho(y, \hat{S}))X_{\rho(y, \hat{S})}\{x_{\rho(y, \hat{S})}=x\}] - xE_y[\delta(\rho(y, S))1_{\{X_{\rho(y, S)}=x\}}] \\
\geq E_y[\delta(\rho(y, S))1_{\{X_{\rho(y, S)}=x\}}]\{\mathbb{E}[\delta(\rho(y, \hat{S}) - \rho(y, S))X_{\rho(x, \hat{S})}|F_\rho(y, S)]\} - xE_y[\delta(\rho(y, S))1_{\{X_{\rho(y, S)}=x\}}] \\
= E_y[\delta(\tau_S)1_{\{X_{\tau_S}=x\}}]\{\mathbb{E}[\delta(\tau_S)X_{\tau_S}] - x\},
\]

where we use (2.1) for the inequality above. \( \square \)

**Lemma 2.2.** Let Assumption 2.2 (i) hold. If \( S \) is an optimal mild equilibrium, then for any \( x \in S \) we have that

\[ x \geq J(x, \hat{S}), \quad \text{where} \quad \hat{S} = S \setminus \{x\}. \]

As a result, \( 0 \notin S \) and \( J(y, S) > 0 \) for all \( y \in \mathbb{X} \).

**Proof.** If \( \hat{S} \) is also a mild equilibrium, then

\[ x \leq J(x, \hat{S}) \leq J(x, S) = x, \]

and thus \( x = J(x, \hat{S}) \).

If \( \hat{S} \) is not a mild equilibrium, then there exists \( y \notin \hat{S} \) such that \( J(y, \hat{S}) < y \leq J(y, S) \). By Lemma 2.1

\[ 0 > J(y, \hat{S}) - J(y, S) \geq E_y[\delta(\tau_S)1_{\{X_{\tau_S}=x\}}](J(x, \hat{S}) - x), \]

which implies that

\[ x > J(x, \hat{S}). \quad (2.4) \]

Now suppose \( 0 \in S \). By the above result, we have \( 0 \geq J(0, S \setminus \{0\}) \). Since \( X_{\tau_S \setminus \{0\}} > 0 \), \( J(0, S \setminus \{0\}) > 0 \), which is a contraction. As a result, \( 0 \notin S \) and \( J(y, S) > 0 \) for all \( y \in \mathbb{X} \). \( \square \)

**Proof of Theorem 2.1** By Assumption 2.2, \( \delta(t) \geq e^{\delta(0)t} \) for all \( t \geq 0 \). Moreover, there exist \( t_0 > 0 \) such that for \( t > t_0 \), \( \delta(t) > e^{\delta(0)t} \) since \( \delta \) is non-exponential. As a result, for any \( x \in \mathbb{X} \),

\[ E_x[\delta(T_x)] = \int_0^\infty \delta(t)e^{-\lambda_xt}dt > \int_0^\infty \lambda_x e^{(\delta(0)-\lambda_x)t}dt = \frac{\lambda_x}{\lambda_x - \delta(0)}. \]

Denote \( c_x := \frac{\lambda_x}{\lambda_x - \delta(0)}. \)
If \( S = \{x\} \), then
\[
\sum_{y \neq x} J(y, S)q_{xy}x \leq x \sum_{x \neq y} E_y[\delta(T_y)]q_{xy}x < x(\lambda_x - \delta'(0)),
\]
which implies that \( S \) is a strong equilibrium.

For the rest of the proof, we assume \( S \) contains at least two points. Fix any \( x \in S \), we have
\[
J(x, \hat{S}) = \sum_{y \in S \setminus \{x\}} \frac{q_{xy}}{\lambda_x} E_x[\delta(\tau_S)X_{T_x} = y] + \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} E_x[\delta(\tau_S)X_{T_x} = y].
\]
Since for \( y \in S \setminus \{x\} \),
\[
E_x[\delta(\tau_S)X_{T_x} = y] = y E_x[\delta(\tau_S)X_{T_x} = y] = y E_x[\delta(T_x)X_{T_x} = y] = y E_x[\delta(T_x)],
\]
and for \( y \in S^c \),
\[
E_x[\delta(\tau_S)X_{T_x} = y] \geq E_x[\delta(T_x)\delta(\tau_S - T_x)X_{T_x} = y] = E_x[\delta(T_x)X_{T_x} = y] \cdot E_x[\delta(\tau_S - T_x)X_{T_x} = y] = E_x[\delta(T_x)] \cdot J(y, \hat{S}),
\]
we have that
\[
J(x, \hat{S}) \geq \left( \sum_{y \in S \setminus \{x\}} \frac{q_{xy}}{\lambda_x} y + \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} J(y, \hat{S}) \right) \cdot E_x[\delta(T_x)]. \tag{2.5}
\]
Denote
\[
I := \sum_{y \in S \setminus \{x\}} \frac{q_{xy}}{\lambda_x} y, \quad II := \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} J(x, S), \quad \hat{I} := \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} J(y, \hat{S}).
\]
By Lemma 2.2, \( y > 0 \) for all \( y \in \hat{S} \) and \( J(y, \hat{S}) > 0 \) for all \( y \notin \hat{S} \), thus \( I + \hat{I} > 0 \). This together with \( E_x[\delta(T_x)] > c_x \) implies that
\[
J(x, \hat{S}) > (I + \hat{I})c_x.
\]
Then
\[
x - J(x, \hat{S}) < x - (I + \hat{I})c_x
\]
\[
= x - (I + II)c_x + (II - \hat{I})c_x
\]
\[
= x - (I + II)c_x + c_x \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} (J(y, S) - J(y, \hat{S}))
\]
\[
\leq x - (I + II)c_x + c_x \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} (E_y[\delta(\tau_S)1_{\{X_{T_x} = y\}}]x - J(x, \hat{S})),
\]
where the last line follows from Lemma 2.1. Thus
\[
\left( 1 - c_x \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} (E_y[\delta(\tau_S)1_{\{X_{T_x} = y\}}]) \right)(x - J(x, \hat{S})) < x - (I + II)c_x. \tag{2.6}
\]
Notice that
\[
c_x \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} (E_y[\delta(\tau_S)1_{\{X_{T_x} = y\}}]) \leq c_x \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} \leq c_x < 1.
\]
Then by Lemma 2.2,
\[
x - (I + II)c_x > 0, \quad \forall x \in X,
\]
which implies \( S \) is a strong equilibrium.
2.2. Proof of Theorem 2.2. We start with the following lemma, which in particular indicates that a smaller mild equilibrium generates larger values.

**Lemma 2.3.** Let Assumption 2.1 (i) hold. If \( S \) is a mild equilibrium, then for any subset \( R \subset X \) with \( S \subset R \), we have
\[
J(x, S) \geq J(x, R), \quad \forall x \in X.
\]

**Proof.** Since \( S \subset R \), \( \rho(x, S) \geq \rho(x, R) \) for all \( x \in X \).
\[
J(x, S) = \mathbb{E}_x[\delta(\rho(x, S))X_{\rho(x, S)}] = \mathbb{E}_x[\mathbb{E}_x[\delta(\rho(x, S))X_{\rho(x, S)}|\mathcal{F}_{\rho(x, R)}]] \\
\geq \mathbb{E}_x[\delta(\rho(x, R))\mathbb{E}_x[\delta(\rho(x, S) - \rho(x, R))X_{\rho(x, S)}|\mathcal{F}_{\rho(x, R)}]] \\
= \mathbb{E}_x[\delta(\rho(x, R))\mathbb{E}_{X_{\rho(x, R)}}[\delta(\rho(X_{\rho(x, R)}, S))X_{\rho(x, S)}]] \\
\geq \mathbb{E}_x[\delta(\rho(x, R))X_{\rho(x, R)}] = J(x, R).
\]

The last inequality holds because \( S \) is a mild equilibrium and by definition,
\[
\mathbb{E}_{X_{\rho(x, R)}}[\delta(\rho(X_{\rho(x, R)}, S))X_{\rho(x, S)}] \geq X_{\rho(x, R)}.
\]

\( \square \)

**Corollary 2.2.** Let Assumption 2.2 (i) hold. If \( S \) is the smallest mild equilibrium, i.e. \( S \subset \overset{\sim}{S} \) for any mild equilibrium \( \overset{\sim}{S} \), then \( S \) is an optimal mild equilibrium.

Thanks to this corollary, in order to show \( S_\infty \) defined in (2.3) is an optimal mild equilibrium, it suffices to show that \( S_\infty \) is the smallest one.

Recall \( S_n \) defined in (2.2). We have the following lemma.

**Lemma 2.4.** For any mild equilibrium \( R \), we have that \( S_n \subset R \) for all \( n \in \mathbb{N} \).

**Proof.** We prove this lemma by induction. First \( S_0 \subset R \). Suppose \( S_n \subset R \) for \( n \geq 0 \). Since \( R \) is a mild equilibrium, for any \( x \notin R \),
\[
x \leq J(x, R) \leq \sup_{S:S_n \subset S \subset X \setminus \{x\}} J(x, S).
\]

Therefore \( x \notin S_{n+1} \). As a result, \( S_{n+1} \subset R \) for all \( n \in \mathbb{N} \).

\( \square \)

**Lemma 2.5.** Let Assumption 2.1 (i) hold. For \( y \notin S_\infty \), denote
\[
V_n := \sup_{S:S_n \subset S \subset X \setminus \{y\}} J(y, S), \quad V_\infty := \sup_{S:S_\infty \subset S \subset X \setminus \{y\}} J(y, S),
\]
then we have \( V_n \nearrow V_\infty \), \( n \to \infty \).

**Proof.** Since \( S_\infty = \bigcap_{n \geq 1} S_n \), we have \( \rho(y, S_\infty \setminus S_n) \to \infty, n \to \infty \). Then for any \( \varepsilon > 0 \), there exists \( N = N(\varepsilon, y) \) such that for \( n > N \), \( \mathbb{E}_y[\delta(\tau_{S_\infty \setminus S_n})] < \varepsilon \) since \( \lim_{t \to \infty} \delta(t) = 0 \).

For any \( R_n \) such that \( S_n \subset R_n \subset X \setminus \{y\} \), denote \( \overline{R_n} := R_n \cup S_\infty \), then we have,
\[
J(y, R_n) - J(y, \overline{R_n}) = \mathbb{E}_y[\delta(\tau_{R_n})X_{\tau_{R_n}} - \delta(\tau_{\overline{R_n}})X_{\tau_{\overline{R_n}}}1_{\{X_{\tau_{\overline{R_n}}} \in S_\infty \setminus R_n\}}] \\
\leq C \mathbb{E}_y[\delta(\tau_{R_n})1_{\{X_{\tau_{R_n}} \in S_\infty \setminus R_n\}}] \\
\leq C \mathbb{E}_y[\delta(\tau_{S_\infty \setminus R_n})1_{\{X_{\tau_{R_n}} \in S_\infty \setminus R_n\}}] \\
\leq C \varepsilon
\]
Since \( S_\infty \subset \overline{R_n} \subset \mathbb{X}\setminus\{y\} \), by definition, \( J(y, \overline{R_n}) \leq V_\infty \). Therefore we have that for any \( \varepsilon > 0 \), there exists \( N \) such that for any \( n \geq N \),

\[
V_n = \sup_{R_n, S_n \subset R_n \subset \mathbb{X}\setminus\{y\}} J(y, R_n) \leq V_\infty + C\varepsilon.
\]

Clearly \( S_n \subset S_{n+1} \) implies that \( V_n \) is non-increasing and \( V_n \geq V_\infty \) for all \( n \). This completes the proof that \( V_n \searrow V_\infty, n \to \infty \). \( \square \)

**Proof of Theorem 2.2.** By Corollary 2.2 and Lemma 2.4 to show that \( S_\infty \) is an optimal mild equilibrium, it suffices to show \( S_\infty \) is a mild equilibrium.

Suppose \( S_\infty \) is not a mild equilibrium. Then

\[
\alpha := \sup_{x \in \mathbb{X}} \{ x - J(x, S_\infty) \} > 0.
\]

For any \( \varepsilon > 0 \), there exists \( y \notin S_\infty \) such that \( y - J(y, S_\infty) \geq \alpha - \varepsilon \). Since \( y \notin S_n \) for all \( n \geq 0 \), we have

\[
y \leq \sup_{S: S_n \subset S \subset \mathbb{X}\setminus\{y\}} J(y, S), \quad \forall n \geq 0.
\]

By Lemma 2.5,

\[
y \leq \sup_{S: S_\infty \subset S \subset \mathbb{X}\setminus\{y\}} J(y, S).
\]

Thus, there exists subset \( R \) with \( S_\infty \subset R \subset \mathbb{X}\setminus\{y\} \) such that

\[
y \leq J(y, R) + \varepsilon.
\]

Then we have \( J(y, R) - J(y, S_\infty) \geq y - \varepsilon + \alpha - \varepsilon - y = \alpha - 2\varepsilon \). Since \( S_\infty \subset R \), \( \rho(y, S_\infty) \geq \rho(y, R) \).

It follows that

\[
J(y, R) - J(y, S_\infty) = \mathbb{E}_y[\delta(\rho(y, R))X_{\rho(y, R)}] - \mathbb{E}_y[\mathbb{E}_y[\delta(\rho(y, S_\infty))X_{\rho(y, S_\infty)}|F_{\rho(y, R)}]]
\]

\[
\leq \mathbb{E}_y[\delta(\rho(y, R))X_{\rho(y, R)}] - \mathbb{E}_y[\delta(\rho(y, R))\mathbb{E}_y[\delta(\rho(y, S_\infty) - \rho(y, R))X_{\rho(y, S_\infty)}|F_{\rho(y, R)}]]
\]

\[
= \mathbb{E}_y[\delta(\rho(y, R))(X_{\rho(y, R)} - \mathbb{E}X_{\rho(y, R)}[\delta(\rho(X_{\rho(y, R)}), S_\infty))X_{\rho(y, R), S_\infty})]]
\]

\[
\leq \mathbb{E}_y[\delta(\rho(y, R))\alpha]
\]

\[
\leq \mathbb{E}_y[\delta(T_y)]\alpha.
\]

By Assumption 2.2 (i), \( \lambda = \sup_{x \in \mathbb{X}} \lambda_x < \infty \) and since \( y \notin R \), we have \( 0 < \mathbb{E}_y[\delta(T_y)] < c < 1 \) where \( c = \int_0^\infty \delta(t)\lambda e^{-\lambda t} dt \). By choosing \( 0 < \varepsilon \leq \frac{\alpha(1-c)}{2} \), we obtain a contradiction.

Next let us prove \( S_\infty \) is a strong equilibrium. If \( X \) is irreducible, then \( S_\infty \) is a strong equilibrium by Theorem 2.1. In general, following the proof for Proposition 2.1, to show \( S_\infty \) is a strong equilibrium, it suffices to show that for any \( x \in S_\infty \) with \( \lambda_x > 0 \),

\[
x(\lambda_x - \delta'(0)) > \sum_{y \in S_\infty \setminus\{x\}} q_{xy} q_{xy} + \sum_{y \in S_\infty} \mathbb{E}_y[\delta(\tau_S)X_{\tau_S}]q_{xy}.
\]

Take \( x \in S_\infty \) with \( \lambda_x > 0 \). Following the argument for (2.5), we have that

\[
J(x, \hat{S}_\infty) \geq \left( \sum_{y \in S_\infty \setminus\{x\}} \frac{q_{xy}}{\lambda_x} y + \sum_{y \notin S_\infty} \frac{q_{xy}}{\lambda_x} J(y, \hat{S}_\infty) \right) \cdot \mathbb{E}_x[\delta(T_x)]
\]

where \( \hat{S}_\infty = S_\infty \setminus\{x\} \). Using an argument similar to that for (2.6), we have that

\[
(1 - c_x \sum_{y \notin S_\infty} \frac{q_{xy}}{\lambda_x} (\mathbb{E}_y[\delta(\tau_{S_\infty})1_{\{\tau_{S_\infty} = x\}}])) (x - J(x, \hat{S}_\infty)) \leq x - (I_\infty + I_\infty)c_x,
\]
where

\[ I_\infty := \sum_{y \in S_\infty \setminus \{x\}} \frac{q_{xy}}{\lambda_x} y \quad \text{and} \quad II_\infty := \sum_{y \in S_\infty \setminus \{x\}} \frac{q_{xy}}{\lambda_x} f(y, S_\infty). \]

Since \( S_\infty \) is the smallest mild equilibrium, \( \hat{S}_\infty \) is not a mild equilibrium. Then \( x > J(x, \hat{S}_\infty) \) by (2.4). Therefore,

\[ x - (I_\infty + II_\infty)c_x > 0, \]

which implies (2.7). \( \square \)

3. A Two-state Example

In this section, we will use an example to illustrate that a mild equilibrium may not be a weak equilibrium, a weak equilibrium may not be a strong equilibrium and a strong equilibrium may not be an optimal mild equilibrium.

Consider a two-state continuous-time Markov chain \( X_t \in \{a, b\} \) for \( t \geq 0 \). Assume \( a > 0, b > 0 \) and without loss of generality we assume \( a > b \). The generator is

\[ Q = \begin{bmatrix} -\lambda_a & \lambda_a \\ \lambda_b & -\lambda_b \end{bmatrix}, \]

where \( \lambda_a > 0 \) and \( \lambda_b > 0 \).

There are four subsets of \( \{a, b\} \). Clearly \( S = \emptyset \) and \( S = \{b\} \) cannot be mild equilibria and \( S = \{a, b\} \) is a mild equilibrium. Next, let’s check when \( S = \{a\} \) is a mild equilibrium.

By definition \( S = \{a\} \) is a mild equilibrium if and only if

\[ b \leq a\mathbb{E}_b[\delta(T_b)] = a \int_0^\infty \delta(t) \lambda_b e^{-\lambda_b t} dt. \]

Consider the following cases.

(i) If \( \frac{b}{a} = \int_0^\infty \delta(t) \lambda_b e^{-\lambda_b t} dt < 1 \), then both \( \{a\} \) and \( \{a, b\} \) are optimal mild equilibria and thus both are strong equilibria.

(ii) If \( \frac{b}{a} < \int_0^\infty \delta(t) \lambda_b e^{-\lambda_b t} dt < 1 \), then \( \{a\} \) is the only optimal mild equilibrium, which is also a strong equilibrium. But the mild equilibrium \( \{a, b\} \) may not be a weak equilibrium. For example, when \( \frac{b}{a} < \frac{\lambda_b}{\lambda_a - \delta'(0)} < 1 \), the second condition for weak equilibrium is violated at state \( b \), thus it is not a weak equilibrium.

(iii) \( \frac{\lambda_b - \delta'(0)}{\lambda_a} < 1 < \frac{a}{b} \) holds automatically since \( a > b \) and \( \delta'(0) < 0 \). If \( \frac{\lambda_b}{\lambda_a - \delta'(0)} < \frac{b}{a} < \int_0^\infty \delta(t) \lambda_b e^{-\lambda_b t} dt < 1 \), then \( \{a, b\} \) is not an optimal mild equilibrium, but it is a weak equilibrium and also a strong equilibrium.

(iv) If \( \frac{\lambda_b}{\lambda_a - \delta'(0)} = \frac{b}{a} < 1 \), \( \{a, b\} \) is a weak equilibrium, but it may not be a strong equilibrium, i.e. condition (1.8) on strong equilibrium may not hold at state \( b \). This can be shown by computing the related term of order \( \varepsilon^2 \).

Since

\[ \mathbb{P}(X_\varepsilon = a|X_0 = b) = \lambda_b \varepsilon - \frac{\lambda_b^2 + \lambda_a \lambda_b}{2} \varepsilon^2 + o(\varepsilon^2), \]

and

\[ \mathbb{P}(X_\varepsilon = b|X_0 = b) = 1 - \lambda_b \varepsilon + \frac{\lambda_b^2 + \lambda_a \lambda_b}{2} \varepsilon^2 + o(\varepsilon^2), \]

\[ c_x > 0, \]

which implies (2.7). \( \square \)
we have  \[ b - \mathbb{E}_b[\delta(\varepsilon)X_\varepsilon] \]
\[ = b - \delta(\varepsilon)[a\mathbb{P}(X_\varepsilon = a|X_0 = b) + b\mathbb{P}(X_\varepsilon = b|X_0 = b)] \]
\[ = b - (1 + \delta'(0)\varepsilon + \frac{\delta''(0)}{2}\varepsilon^2 + o(\varepsilon^2))[a(\lambda_\varepsilon - \frac{\lambda_b^2 + \lambda_a\lambda_b}{2}\varepsilon^2 + o(\varepsilon^2)) + b(1 - \lambda_\varepsilon + \frac{\lambda_b^2 + \lambda_a\lambda_b}{2}\varepsilon^2 + o(\varepsilon^2))] \]
\[ = (b\lambda_b - a\lambda_b - b\delta'(0))\varepsilon + [b(\lambda_b\delta'(0) - \frac{\lambda_b^2 + \lambda_a\lambda_b}{2} - \frac{\delta''(0)}{2}) - a(\delta'(0)\lambda_b - \frac{\lambda_b^2 + \lambda_a\lambda_b}{2})]\varepsilon^2 + o(\varepsilon^2) \]
Therefore when the first order term and the second order term respectively satisfy
\[ b(\lambda_b - \delta'(0)) - a\lambda_b = 0, \tag{3.1} \]
and
\[ b(\lambda_b\delta'(0) - \frac{\lambda_b^2 + \lambda_a\lambda_b}{2} - \frac{\delta''(0)}{2}) - a(\delta'(0)\lambda_b - \frac{\lambda_b^2 + \lambda_a\lambda_b}{2}) < 0, \tag{3.2} \]
\{a, b\} is a weak equilibrium but not a strong equilibrium. Using \eqref{3.1}, \eqref{3.2} can be simplified to
\[ \lambda_a + \lambda_b < \frac{\delta''(0) - 2(\delta'(0))^2}{-\delta'(0)}. \tag{3.3} \]

An interesting case is when \( \delta(t) = \frac{1}{1 + \beta t} \). Then \eqref{3.3} does not hold: \( \delta'(0) = -\beta \) and \( \delta''(0) = 2\beta^2 \).

In this case \( \frac{\delta''(0) - 2(\delta'(0))^2}{-\delta'(0)} = 0 \), which contradicts \( \lambda_a + \lambda_b > 0 \). That means if we have hyperbolic discount function, a weak equilibrium is always a strong equilibrium in the two-state setting.

But when \( \delta(t) = (1 + \beta t)^{-\frac{1}{\beta}} \), then it can easily be seen that \eqref{3.3} holds: \( \delta'(0) = -\frac{\beta}{\beta^2}, \delta''(0) = \frac{2}{\beta^2} \) implies that when \( 0 < \lambda_a + \lambda_b < \frac{\beta}{2} \) and \( \frac{b}{a} = \frac{2\lambda_b}{2\lambda_b + \beta}, \) \{a, b\} is a weak equilibrium but not a strong equilibrium. In this case, \{a, b\} is not an optimal mild equilibrium.

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