Schur functions in noncommuting variables

Steph van Willigenburg
University of British Columbia

CALICO
15 October 2022
**Integer partitions**

An integer partition $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ of $n$ is a list of positive integers whose sum is $n$: $3221 \vdash 8$.

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell = 1^{r_1} 2^{r_2} \cdots n^{r_n}$. Then

$$\lambda! = \lambda_1! \lambda_2! \cdots \lambda_\ell!$$

and

$$\lambda^! = r_1! r_2! \cdots r_n!$$

**Example**

If $\lambda = 3221 = 1^1 2^2 3^1 4^0 5^0 6^0 7^0 8^0 \vdash 8$, then

$$\lambda! = 3! 2! 2! 1! = 6 \times 2 \times 2 \times 1 = 24$$

$$\lambda^! = 1! 2! 1! 0! 0! 0! 0! 0! 0! 0! = 2.$$
Integer partitions

An integer partition \( \lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0 \) of \( n \) is a list of positive integers whose sum is \( n \): \( 3221 \vdash 8 \).

Let \( \lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell = 1^{r_1}2^{r_2} \cdots n^{r_n} \). Then

\[
\lambda! = \lambda_1!\lambda_2! \cdots \lambda_\ell!
\]

and

\[
\lambda^! = r_1!r_2! \cdots r_n!
\]

Example

If \( \lambda = 3221 = 1^12^23^14^05^06^07^08^0 \vdash 8 \), then

\[
\lambda! = 3!2!2!1! = 6 \times 2 \times 2 \times 1 = 24
\]

\[
\lambda^! = 1!2!1!0!0!0!0!0!0! = 2.
\]
**Integer partitions**

An integer partition \( \lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0 \) of \( n \) is a list of positive integers whose sum is \( n \): \( 3221 \vdash 8 \).

Let \( \lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell = 1^{r_1}2^{r_2} \cdots n^{r_n} \). Then

\[
\lambda! = \lambda_1! \lambda_2! \cdots \lambda_\ell!
\]

and

\[
\lambda_i! = r_1! r_2! \cdots r_n!
\]

**Example**

If \( \lambda = 3221 = 1^1 2^2 3^1 4^0 5^0 6^0 7^0 8^0 \vdash 8 \), then

\[
\lambda! = 3! 2! 2! 1! = 6 \times 2 \times 2 \times 1 = 24
\]

\[
\lambda_i! = 1! 2! 1! 0! 0! 0! 0! 0! = 2.
\]
**Set partitions**

A set partition $\pi$ of $[n] = \{1, 2, \ldots, n\}$ is a partitioning of $[n]$ into disjoint sets $B_1, B_2, \ldots, B_\ell$ called blocks so

- $B_i \neq \emptyset$
- $B_1 \cup B_2 \cup \cdots \cup B_\ell = [n]$. 

$$\pi = B_1/B_2/\cdots/B_\ell \vdash [n]$$

**Example**

$\{1, 3, 4\}, \{2, 5\}, \{6\}, \{7, 8\}$

is a set partition of $[8]$, or

$$\pi = 134/25/6/78 \vdash [8].$$
**PARTITIONS AND PERMUTATIONS**

If \( \pi = B_1/B_2/\cdots/B_\ell \vdash [n] \), then

\[
\lambda(\pi) = |B_1||B_2| \cdots |B_\ell|
\]

with sizes weakly decreasing. If \( \delta \in S_n \), then

\[
\delta(\pi) = \delta(B_1)/\delta(B_2)/\cdots/\delta(B_\ell).
\]

**Example**

If \( \pi = 134/25/6/78 \vdash [8] \), then

\[
\lambda(134/25/6/78) = 3221
\]

and \( \delta = 14325876 \in S_8 \)

\[
\delta(134/25/6/78) = \delta(1)\delta(3)\delta(4)/\delta(2)\delta(5)/\delta(6)/\delta(7)\delta(8) = 132/45/8/76 = 123/45/67/8.
\]
**Partitions and permutations**

If \( \pi = B_1/B_2/\cdots/B_\ell \vdash [n] \), then

\[
\lambda(\pi) = |B_1||B_2|\cdots|B_\ell|
\]

with sizes weakly decreasing. If \( \delta \in S_n \), then

\[
\delta(\pi) = \delta(B_1)/\delta(B_2)/\cdots/\delta(B_\ell).
\]

**Example**

If \( \pi = 134/25/6/78 \vdash [8] \), then

\[
\lambda(134/25/6/78) = 3221
\]

and \( \delta = 14325876 \in S_8 \)

\[
\delta(134/25/6/78) = \delta(1)\delta(3)\delta(4)/\delta(2)\delta(5)/\delta(6)/\delta(7)\delta(8) = 132/45/8/76 = 123/45/67/8.
\]
**Partitions and Permutations**

If \( \pi = B_1/B_2/\cdots/B_\ell \vdash [n] \), then

\[
\lambda(\pi) = |B_1||B_2|\cdots|B_\ell|
\]

with sizes weakly decreasing. If \( \delta \in \mathfrak{S}_n \), then

\[
\delta(\pi) = \delta(B_1)/\delta(B_2)/\cdots/\delta(B_\ell).
\]

**Example**

If \( \pi = 134/25/6/78 \vdash [8] \), then

\[
\lambda(134/25/6/78) = 3221
\]

and \( \delta = 14325876 \in \mathfrak{S}_8 \)

\[
\delta(134/25/6/78) = \delta(1)\delta(3)\delta(4)/\delta(2)\delta(5)/\delta(6)/\delta(7)\delta(8)
\]

\[
= 132/45/8/76 = 123/45/67/8.
\]
PARTITIONS AND PERMUTATIONS

If \( \pi = B_1/B_2/\cdots/B_\ell \vdash [n] \), then

\[
\lambda(\pi) = |B_1||B_2|\cdots|B_\ell|
\]

with sizes weakly decreasing. If \( \delta \in S_n \), then

\[
\delta(\pi) = \delta(B_1)/\delta(B_2)/\cdots/\delta(B_\ell).
\]

**Example**

If \( \pi = 134/25/6/78 \vdash [8] \), then

\[
\lambda(134/25/6/78) = 3221
\]

and \( \delta = 14325876 \in S_8 \)

\[
\delta(134/25/6/78) = \delta(1)\delta(3)\delta(4)/\delta(2)\delta(5)/\delta(6)/\delta(7)\delta(8)
\]

\[
= 132/45/8/76 = 123/45/67/8.
\]
PARTITIONS AND PERMUTATIONS

If $\pi = B_1/B_2/\cdots/B_\ell \vdash [n]$, then

$$\lambda(\pi) = |B_1||B_2|\cdots|B_\ell|$$

with sizes weakly decreasing. If $\delta \in \mathfrak{S}_n$, then

$$\delta(\pi) = \delta(B_1)/\delta(B_2)/\cdots/\delta(B_\ell).$$

**Example**

If $\pi = 134/25/6/78 \vdash [8]$, then

$$\lambda(134/25/6/78) = 3221$$

and $\delta = 14325876 \in \mathfrak{S}_8$

$$\delta(134/25/6/78) = \delta(1)\delta(3)\delta(4)/\delta(2)\delta(5)/\delta(6)/\delta(7)\delta(8)$$

$$= 132/45/8/76 = 123/45/67/8.$$
PARTITIONS AND PERMUTATIONS

If \( \pi = B_1/B_2/\cdots/B_\ell \vdash [n] \), then

\[
\lambda(\pi) = |B_1||B_2|\cdots|B_\ell|
\]

with sizes weakly decreasing. If \( \delta \in S_n \), then

\[
\delta(\pi) = \delta(B_1)/\delta(B_2)/\cdots/\delta(B_\ell).
\]

**Example**

If \( \pi = 134/25/6/78 \vdash [8] \), then

\[
\lambda(134/25/6/78) = 3221
\]

and \( \delta = 14325876 \in S_8 \)

\[
\delta(134/25/6/78) = \delta(1)\delta(3)\delta(4)/\delta(2)\delta(5)/\delta(6)/\delta(7)\delta(8)
\]

\[
= 132/45/8/76 = 123/45/67/8.
\]
PARTITIONS AND PERMUTATIONS

If $\pi = B_1/B_2/\cdots/B_\ell \vdash [n]$, then

$$\lambda(\pi) = |B_1||B_2|\cdots|B_\ell|$$

with sizes weakly decreasing. If $\delta \in \mathfrak{S}_n$, then

$$\delta(\pi) = \delta(B_1)/\delta(B_2)/\cdots/\delta(B_\ell).$$

**Example**

If $\pi = 134/25/6/78 \vdash [8]$, then

$$\lambda(134/25/6/78) = 3221$$

and $\delta = 14325876 \in \mathfrak{S}_8$

$$\delta(134/25/6/78) = \delta(1)\delta(3)\delta(4)/\delta(2)\delta(5)/\delta(6)/\delta(7)\delta(8)$$

$$= 132/45/8/76 = 123/45/67/8.$$
Slash product

If

\[ S + n = \{s + n : s \in S\} \]

then for \( \pi \vdash [n] \) and \( \sigma = B_1/B_2/\cdots/B_\ell \vdash [m] \) the slash product is

\[ \pi \mid \sigma = \pi/(B_1 + n)/(B_2 + n)/\cdots/(B_\ell + n) \vdash [n + m]. \]

**Example**

If \( \pi = 134/25 \vdash [5] \) and \( \sigma = 1/23 \vdash [3] \) then

\[ \pi \mid \sigma = 134/25/6/78 \vdash [8]. \]
NCsym (Rosas-Sagan 2004)

NCsym is the algebra of symmetric functions in noncommuting variables $x_1, x_2, x_3, \ldots$

$$\text{NCsym} = \text{NCsym}^0 \oplus \text{NCsym}^1 \oplus \cdots \subset \mathbb{Q} \ll x_1, x_2, x_3, \ldots \gg$$

where $\text{NCsym}^0 = \text{span}\{1\}$ and for $n > 0$

$$\text{NCsym}^n = \text{span}\{m_\pi : \pi \vdash [n]\}$$
$$= \text{span}\{p_\pi : \pi \vdash [n]\}$$
$$= \text{span}\{e_\pi : \pi \vdash [n]\}$$
$$= \text{span}\{h_\pi : \pi \vdash [n]\}.$$

Note: The $e_\pi$ defined by Wolf in 1936.
Monomial function in NCSym

The monomial symmetric function in NCSym for $\pi \vdash [n]$ is

$$m_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

summed over all tuples $(i_1, i_2, \ldots, i_n)$ with

$$i_j = i_k$$

if and only if $j$ and $k$ are in the same block of $\pi$.

Example

$$m_{13/2} = x_1 x_2 x_1 + x_2 x_1 x_2 + x_1 x_3 x_1 + x_2 x_3 x_2 + \cdots$$
**Power sum function in NCSym**

The power sum symmetric function in NCSym for $\pi \vdash [n]$ is

$$p_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

summed over all tuples $(i_1, i_2, \ldots, i_n)$ with

$$i_j = i_k$$

if $j$ and $k$ are in the same block of $\pi$.

**Example**

$$p_{13/2} = x_1 x_2 x_1 + x_2 x_1 x_2 + \cdots + x_1^3 + x_2^3 + \cdots$$
Elementary function in NCSym

The elementary symmetric function in NCSym for $\pi \vdash [n]$ is

$$e_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

summed over all tuples $(i_1, i_2, \ldots, i_n)$ with

$$i_j \neq i_k$$

if $j$ and $k$ are in the same block of $\pi$.

Example

$$e_{13/2} = x_1 x_1 x_2 + x_1 x_2 x_2 + x_2 x_2 x_1 + x_2 x_1 x_1 + \cdots + x_1 x_2 x_3 + \cdots$$
**Complete homogeneous function in NCSym**

The complete homogeneous symmetric function in NCSym for $\pi \vdash [n]$ is

$$h_\pi = \sum_{\sigma} \lambda(\sigma \land \pi)! m_\sigma$$

where $\sigma \land \pi$ is the maximal set partition where every block is a subblock of $\sigma$ and $\pi$.

**Example**

$$h_{13/2} = 2m_{123} + m_{12/3} + m_{1/23} + 2m_{13/2} + m_{1/2/3}$$

$$123 \land 13/2 = 13/2 \land 13/2 = 13/2 \quad \lambda(13/2)! = 2$$

$$12/3 \land 13/2 = 1/23 \land 13/2 = 1/2/3 \land 13/2 = 1/2/3 \quad \lambda(1/2/3)! = 1$$
**Permutations and products**

**Fact:** If $\pi \vdash [n]$, $\delta \in S_n$ and $\delta \circ m_\pi = m_{\delta(\pi)}$, then for $b = p, e, h$

$$\delta \circ b_\pi = b_{\delta(\pi)}.$$ 

**Fact:** If $\pi, \sigma$ are set partitions, then for $b = p, e, h$

$$b_\pi b_\sigma = b_{\pi|\sigma}.$$ 

**Example**

| Example 128 | $132 \circ m_{12/3} = m_{13/2}$ | $p_{13/2}p_1 = p_{13/2|1} = p_{13/2/4}$ |
What’s in a name?

Let the variables commute:

$$\rho : \text{NCSym} \rightarrow \text{Sym}$$

**Theorem (Rosas-Sagan 2004)**

Let $\pi$ be a set partition.

$$\rho(m_{\pi}) = \lambda(\pi)! m_{\lambda(\pi)} \quad \rho(p_{\pi}) = p_{\lambda(\pi)}$$

$$\rho(e_{\pi}) = \lambda(\pi)! e_{\lambda(\pi)} \quad \rho(h_{\pi}) = \lambda(\pi)! h_{\lambda(\pi)}$$

**Note:** The images are classical monomial, power sum, elementary and complete homogeneous symmetric functions.

**Question:** Where are the Schur functions?
Partitions and diagrams

An integer partition \( \lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0 \) of \( n \) is a list of positive integers whose sum is \( n \): \( 3221 \vdash 8 \).

The diagram \( \lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0 \) is the array of boxes with \( \lambda_i \) boxes in row \( i \) from the top.
A semistandard Young tableau (SSYT) $T$ of shape $\lambda$ is a filling with $1, 2, 3, \ldots$ so rows weakly increase and columns increase.

\[
\begin{array}{cc}
1 & 1 \\
2 &
\end{array}
\]

Given an SSYT $T$ we have for commuting variables

\[
x^T = x_1^{\#_1s} x_2^{\#_2s} x_3^{\#_3s} \ldots .
\]

\[
x_1^2 x_2
\]
Schur functions

The Schur function in Sym is

$$s_\lambda = \sum_{T \text{ SSYT of shape } \lambda} x^T.$$ 

**Example**

$$s_{21} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3 + \cdots$$

\[
\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
2 & 2 & 3 & 3 & 3 & 3 & 3 & 2 & & & & \\
\end{array}
\]
A dotted Young tableau (DYT) $\hat{T}$ of shape $\lambda$ is a filling with $1, 2, 3, \ldots$ so rows weakly increase and columns increase, and $1, 2, \ldots, n$ dots appear exactly once.

Given a DYT $\hat{T}$ we have for noncommuting variables

\[ x^\hat{T} = x_i \text{ in position } j \text{ iff } i \text{ has } j \text{ dots above it}. \]

\[ x_1 x_2 x_1 \]
Rosas-Sagan Schur functions

The Rosas-Sagan Schur function in NCSym is

\[ S^{RS}_\lambda = \sum_{\dot{T} \text{ DYT of shape } \lambda} x^{\dot{T}}. \]

**Example**

\[ S^{RS}_{21} = 2x_1x_1x_2 + 2x_1x_2x_1 + 2x_2x_1x_1 + \cdots \]

\[
\begin{array}{cccccccc}
\text{i} & \text{i} & \text{i} & \text{i} & \text{i} & \text{i} & \text{i} & \text{i} \\
\text{2} & \text{2} & \text{2} & \text{2} & \text{2} & \text{2} & \text{2} & \text{2}
\end{array}
\]
Rosas-Sagan Schur functions

**Theorem (Rosas-Sagan 2004)**

Let $\lambda \vdash n$.

- The $S^{RS}_\lambda$ are linearly independent.
- We have $\rho(S^{RS}_\lambda) = n!s_\lambda$.

**Note:** However they are not a basis for NCSym because we only have one for each integer partition, not set partition.
**Theorem (Rosas-Sagan 2004)**

Let $\lambda \vdash n$.

- The $S_{\lambda}^{RS}$ are linearly independent.
- We have $\rho(S_{\lambda}^{RS}) = n! s_{\lambda}$.

**Note:** However they are not a basis for NCSym because we only have one for each integer partition, not set partition.

Rosas-Sagan 2004:

Is there a way to define functions ... for set partitions $\pi \vdash [n]$ having properties analogous to the ordinary Schur functions $s_{\lambda}$?
Rosas-Sagan Schur functions

**Theorem (Rosas-Sagan 2004)**

Let $\lambda \vdash n$.
- The $S^RS_\lambda$ are linearly independent.
- We have $\rho(S^RS_\lambda) = n!s_\lambda$.

**Note:** However they are not a basis for NCSym because we only have one for each integer partition, not set partition.

Rosas-Sagan 2004:

Is there a way to define functions ... for set partitions $\pi \vdash [n]$ having properties analogous to the ordinary Schur functions $s_\lambda$?

Yes there is!
The three musketeers

ALEXANDRE DUMAS

The Three Musketeers
ALEXANDRE DUMAS

The Three Musketeers

ALINIAEIFARD-LI-vW 2022
Schur functions revisited

**Definition**

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the Jacobi-Trudi identity

\[ s_\lambda = \det (h_{\lambda_i - i + j})_{1 \leq i, j \leq \ell} \]

where $h_0 = 1$ and $h_{-ve} = 0$.

**Example**

\[ s_{21} = \det \begin{pmatrix} h_2 & h_3 \\ h_0 & h_1 \end{pmatrix} = h_2 h_1 - h_3 h_0 = h_{21} - h_3 \]
**Schur functions revisited**

**Definition**

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the Jacobi-Trudi identity

$$s_\lambda = \det (h_{\lambda_i-i+j})_{1 \leq i, j \leq \ell}$$

where $h_0 = 1$ and $h_{-ve} = 0$.

**Example**

$$s_{21} = \det \begin{pmatrix} h_2 & h_3 \\ h_0 & h_1 \end{pmatrix} = h_2 h_1 - h_3 h_0 = h_{21} - h_3$$
**Definition**

Let \( \lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell \). Then the Jacobi-Trudi identity

\[
s_\lambda = \det (h_{\lambda_i - i + j})_{1 \leq i, j \leq \ell}
\]

where \( h_0 = 1 \) and \( h_{-ve} = 0 \).

**Example**

\[
s_{21} = \det \begin{pmatrix} h_2 & h_3 \\ h_0 & h_1 \end{pmatrix} = h_2 h_1 - h_3 h_0 = h_{21} - h_3
\]
**Definition**

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the Jacobi-Trudi identity

$$s_\lambda = \det (h_{\lambda_i-i+j})_{1 \leq i,j \leq \ell}$$

where $h_0 = 1$ and $h_{-ve} = 0$.

**Example**

$$s_{21} = \det \begin{pmatrix} h_2 & h_3 \\ h_0 & h_1 \end{pmatrix} = h_2 h_1 - h_3 h_0 = h_{21} - h_3$$
The noncommutative Leibniz formula for

\( A = (a_{ij})_{1 \leq i,j \leq n} \) with noncommuting entries \( a_{ij} \) is

\[
\det(A) = \sum_{\varepsilon \in S_n} \text{sgn}(\varepsilon) a_{1\varepsilon(1)} a_{2\varepsilon(2)} \cdots a_{n\varepsilon(n)}
\]

- product of the entries is taken top row to the bottom row
- \( \text{sgn}(\varepsilon) \) is the sign of permutation \( \varepsilon \).
“... AND ONE FOR ALL,” — DUMAS, T3M

**Definition**

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the **source Schur function**

$$s[\lambda] = \det \left( \frac{1}{(\lambda_i - i + j)!} h_{[\lambda_i - i + j]} \right)_{1 \leq i, j \leq \ell}$$

where $h[0] = h_\emptyset = 1$ and $h_{-\text{ve}} = 0$.

**Example**

$$s_{[21]} = \det \left( \begin{array}{cc}
\frac{1}{2!} h_{12} & \frac{1}{3!} h_{123} \\
\frac{1}{0!} h_\emptyset & \frac{1}{1!} h_1
\end{array} \right) = \frac{1}{2!} h_{12} \frac{1}{1!} h_1 - \frac{1}{3!} h_{123} \frac{1}{0!} h_\emptyset$$

$$= \frac{1}{2} h_{12}|1 - \frac{1}{6} h_{123} = \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}.$$
“... AND ONE FOR ALL,” — DUMAS, T3M

**Definition**

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the source Schur function

$$s[\lambda] = \det \left( \frac{1}{(\lambda_i - i + j)!} h[\lambda_i - i + j] \right)_{1 \leq i, j \leq \ell}$$

where $h[0] = h_\emptyset = 1$ and $h_{-ve} = 0$.

**Example**

$$s_{[21]} = \det \left( \begin{array}{cc} \frac{1}{2!} h_{12} & \frac{1}{3!} h_{123} \\ \frac{1}{0!} h_\emptyset & \frac{1}{1!} h_1 \end{array} \right) = \frac{1}{2!} h_{12} \frac{1}{1!} h_1 - \frac{1}{3!} h_{123} \frac{1}{0!} h_\emptyset$$

$$= \frac{1}{2} h_{12|1} - \frac{1}{6} h_{123} = \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}.$$
"... AND ONE FOR ALL," — DUMAS, T3M

**Definition**

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the source Schur function

$$ s[\lambda] = \det \left( \frac{1}{(\lambda_i - i + j)!} h[\lambda_i - i + j] \right)_{1 \leq i, j \leq \ell} $$

where $h[0] = h_\emptyset = 1$ and $h_{-ve} = 0$.

**Example**

$$ s[21] = \det \left( \frac{1}{2!} h_{12} \frac{1}{1!} h_1 \frac{1}{3!} h_{123} \right) = \frac{1}{2} h_{12} \frac{1}{1!} h_1 - \frac{1}{3} h_{123} \frac{1}{0!} h_\emptyset $$

$$ = \frac{1}{2} h_{12|1} - \frac{1}{6} h_{123} = \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}. $$
“... and one for all,” — Dumas, T3M

**Definition**

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the source Schur function

$$s[\lambda] = \det \left( \frac{1}{(\lambda_i - i + j)!} h[\lambda_i - i + j] \right)_{1 \leq i, j \leq \ell}$$

where $h[0] = h_\emptyset = 1$ and $h_{-ve} = 0$.

**Example**

$$s[21] = \det \left( \frac{1}{2!} h_{12} \frac{1}{0!} h_\emptyset \frac{1}{3!} h_{123} \frac{1}{1!} h_1 \right) = \frac{1}{2!} h_{12} \frac{1}{1!} h_1 - \frac{1}{3!} h_{123} \frac{1}{0!} h_\emptyset$$

$$= \frac{1}{2} h_{12} |1| - \frac{1}{6} h_{123} = \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}.$$
“... AND ONE FOR ALL,” — DUMAS, T3M

We now create a set partition $\pi \vdash [n]$ from an integer partition $\lambda \vdash n$ where

$$\lambda(\pi) = \lambda$$

using tableaux $T_\pi$ of shape $\lambda$ such that

- every element $1, 2, \ldots, n$ appears exactly once
- rows increase left to right
- first column of rows of same length increase top to bottom.

rows $\leftrightarrow$ blocks

**Example**

$$T_\pi = \begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 \\
7 & 8 \\
6
\end{array} \quad \text{↔} \quad \pi = 134/25/78/6$$
We now create a permutation $\delta_\pi \in \mathfrak{S}_n$ using $T_\pi$.

read by row $\longleftrightarrow$ one line notation

**Example**

$T_\pi = \begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 & \\
7 & 8 & \\
6 & \\
\end{array}$ $\longleftrightarrow$ $\delta_\pi = 13425786$
We now create a permutation $\delta_{\pi} \in \mathfrak{S}_n$ using $T_{\pi}$.

read by row $\longleftrightarrow$ one line notation

**Example**

$$\pi = 134/25/78/6 \longleftrightarrow T_{\pi} = \begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 & \\
7 & 8 & \\
6 & \\
\end{array} \longleftrightarrow \delta_{\pi} = 13425786$$
We now create a permutation $\delta_\pi \in \mathfrak{S}_n$ using $T_\pi$.

read by row $\longleftrightarrow$ one line notation

**Example**

$$\pi = 134/25/78/6 \quad \longleftrightarrow \quad T_\pi = \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 \\ 7 & 8 \\ 6 \end{array} \quad \longleftrightarrow \quad \delta_\pi = 13425786$$
"... and one for all," – Dumas, T3M

**Definition**

Let $\pi = B_1 / B_2 / \cdots / B_\ell$. Then the Schur function in NCSym is

$$s_\pi = \delta_\pi \circ s_{\lambda(\pi)} = \delta_\pi \circ \det \left( \frac{1}{(\lambda_i - i + j)!} h_{\lambda_i - i + j} \right)_{1 \leq i,j \leq \ell}.$$

**Example**

If $\pi = 13/2$ then $T_\pi = \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix}$ and $\delta_\pi = 132$.

$$s_{13/2} = 132 \circ s_{[21]} = 132 \circ \left( \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123} \right) = \frac{1}{2} h_{13/2} - \frac{1}{6} h_{123}$$
A refined basis

Theorem (Aliniaeifard-Li-vW 2022)

\[ \text{NCSym}^n = \text{span}\{s_\pi : \pi \vdash [n]\} \quad \rho(s_\pi) = s_{\lambda(\pi)} \]

\[ S^R_{\lambda} = \sum_{\delta \in \mathfrak{S}_n} \delta \circ s_{[\lambda]} \]

Note: \( \delta \circ s_\pi \neq s_{\delta(\pi)} \) in general.
A refned basis

**Theorem (Aliniaeifard-Li-vW 2022)**

\[
\text{NCSym}^n = \text{span}\{s_\pi : \pi \vdash [n]\} \quad \rho(s_\pi) = s_{\lambda(\pi)}
\]

\[
S^{RS}_\lambda = \sum_{\delta \in \mathcal{S}_n} \delta \circ s_{[\lambda]}
\]

**Note:** \(\delta \circ s_\pi \neq s_{\delta(\pi)}\) in general.
A refined basis

Theorem (Aliniaeifard-Li-vW 2022)

\[ \text{NCSym}^n = \text{span}\{s_\pi : \pi \vdash [n]\} \quad \rho(s_\pi) = s_{\lambda(\pi)} \]

\[ S^R_{\lambda} = \sum_{\delta \in S_n} \delta \circ s_{[\lambda]} \]

Note: \( \delta \circ s_\pi \neq s_{\delta(\pi)} \) in general.

Theorem (Aliniaeifard-Li-vW 2022)

For \( n \geq 5 \) we have \( n! \) different bases:

\[ \text{NCSym}^n = \text{span}\{\delta \circ s_\pi : \pi \vdash [n]\} \quad \rho(\delta \circ s_\pi) = s_{\lambda(\pi)} \]
Young Tableaux

A Young tableau (YT) $t$ of shape $\lambda$ is a filling with 1, 2, 3, $\ldots$ so each number appears exactly once.

We now create a permutation $\delta_t \in \mathfrak{S}_n$ using $t$.

read by row $\iff$ one line notation

Then

$$s_t = \delta_t \circ s_{[\lambda]}.$$  

**Example**

$$t = \begin{array}{c} 2 & 1 \\ 3 \end{array} \quad \delta_t = 213$$

$$s_t = 213 \circ s_{[21]} = 213 \circ \left(\frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}\right) = \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}$$
The row-equivalence class \([t]\) of \(t\) consists of \(\tilde{t}\)

- same shape as \(t\)
- same set of row entries as \(t\).

**Example**

\[
t = \begin{array}{cc}
2 & 1 \\
3 & \\
\end{array}
\quad [t] = \left\{ \begin{array}{cc}
2 & 1 \\
3 & \\
\end{array}, \begin{array}{cc}
1 & 2 \\
3 & \\
\end{array} \right\}
\]

\[
\delta_t = 213 \quad \delta_{\tilde{t}} = 123
\]
Young tableaux

The row-equivalence class \([t]\) of \(t\) consists of \(\tilde{t}\):

- same shape as \(t\)
- same set of row entries as \(t\).

**Example**

\[
\begin{align*}
t &= \begin{array}{cc}
2 & 1 \\
3 &
\end{array} \\
[t] &= \left\{ \begin{array}{cc}
2 & 1 \\
3 &
\end{array}, \begin{array}{cc}
1 & 2 \\
3 &
\end{array} \right\}
end{align*}
\]

\(\delta_t = 213\) \(\delta_{\tilde{t}} = 123\)
A tabloid basis

The tabloid Schur function in NCSym is

$$s[t] = \sum_{\tilde{t}} s_{\tilde{t}}$$

where $\tilde{t}$ and $t$ are row equivalent.

**Example**

$$s\begin{bmatrix} 2 & 1 \\ 3 \end{bmatrix} = 213 \circ s_{[21]} + 123 \circ s_{[21]}$$

$$= \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123} + \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123} = h_{12/3} - \frac{1}{3} h_{123}$$
A tabloid basis

The tabloid Schur function in $\text{NCSym}$ is

$$s_{[t]} = \sum_{\tilde{t}} s_{\tilde{t}}$$

where $\tilde{t}$ and $t$ are row equivalent.

**Theorem (Aliniaeifard-Li-vW 2022)**

We have basis

$$\text{NCSym}^n = \text{span}\{s_{[t]} : t \leftrightarrow \pi, \pi \vdash [n]\}.$$  

If $t$ has shape $\lambda$ then

$$\rho(s_{[t]}) = \lambda!s_\lambda.$$
**Specht modules**

Given Young tableau $t$

$$e_t = \left( \sum_\delta \text{sgn}(\delta) \delta \right) \circ [t]$$

where the sums are over all column-stabilizers of $t$ that permute elements within each column.

Given partition $\lambda$ we have Specht module; submodule of NCSym.

$$S^\lambda = \text{span}\{ e_t : t \text{ has shape } \lambda \}$$

**Theorem (Aliniaeifard-Li-vW 2022)**

$$S^\lambda \supseteq S^\lambda$$

$$e_t \mapsto e_t$$
**Skew diagrams**

For $\lambda \vdash n, \mu \vdash m$ the skew diagram $\lambda/\mu$ is the array of $n - m$ boxes contained in $\lambda$ but **not** in $\mu$.

A skew shape $\lambda/\mu$ is a **ribbon** if it is connected with no $2 \times 2$ square.

A ribbon can be denoted by row lengths $\alpha$: 5332/221
For $\lambda \vdash n, \mu \vdash m$ the skew diagram $\lambda/\mu$ is the array of $n - m$ boxes contained in $\lambda$ but not in $\mu$.

A skew shape $\lambda/\mu$ is a ribbon if it is connected with no $2 \times 2$ square.

A ribbon can be denoted by row lengths $\alpha$: 

```
5332/221
```
For $\lambda \vdash n$, $\mu \vdash m$ the skew diagram $\lambda/\mu$ is the array of $n - m$ boxes contained in $\lambda$ but not in $\mu$.

A skew shape $\lambda/\mu$ is a ribbon if it is connected with no $2 \times 2$ square.

A ribbon can be denoted by row lengths $\alpha$: 
For $\lambda \vdash n, \mu \vdash m$ the skew diagram $\lambda/\mu$ is the array of $n - m$ boxes contained in $\lambda$ but not in $\mu$.

A skew shape $\lambda/\mu$ is a ribbon if it is connected with no $2 \times 2$ square.

A ribbon can be denoted by row lengths $\alpha$: 3122.
Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the skew Schur function in Sym is

$$s_{\lambda/\mu} = \det \left( h_{\lambda_i - \mu_j - i + j} \right)_{1 \leq i, j \leq \ell}$$

$$= \sum_{T \text{ SSYT of shape } \lambda/\mu} x^T$$

$$= \sum_\nu c_{\mu \nu}^\lambda s_\nu$$

**skew Schur functions**
skew Schur functions

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the skew Schur function in $\text{Sym}$ is

$$s_{\lambda/\mu} = \det \left( h_{\lambda_i - \mu_j - i + j} \right)_{1 \leq i, j \leq \ell}$$

$$= \sum_{T \text{ SSYT of shape } \lambda/\mu} x^T$$

$$= \sum_{\nu} c_{\mu\nu}^\lambda s_{\nu}$$

and the $c_{\mu\nu}^\lambda$ are the Littlewood-Richardson coefficients.
skew Schur functions

Let \( \lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell \). Then the skew Schur function in Sym is

\[
s_{\lambda/\mu} = \det \left( h_{\lambda_i - \mu_j - i + j} \right)_{1 \leq i, j \leq \ell}
= \sum_{T \text{ SSYT of shape } \lambda/\mu} x^T
= \sum_{\nu} c_{\lambda \mu \nu} s_{\nu}
\]

and the \( c_{\lambda \mu \nu} \) are the Littlewood-Richardson coefficients.

Let \( \lambda/\mu \) be a ribbon \( \alpha \). Then the ribbon Schur function in Sym is

\[
r_\alpha = s_{\lambda/\mu}.
\]
**Skew Schur functions in noncommuting variables**

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the source skew Schur function

$$s_{[\lambda/\mu]} = \det \left( \frac{1}{(\lambda_i - \mu_j - i + j)!} h[\lambda_i - \mu_j - i + j] \right)_{1 \leq i, j \leq \ell}$$

and the skew Schur function in NCSym is

$$s_{(\delta, \lambda/\mu)} = \delta \circ s_{[\lambda/\mu]} = \delta \circ \det \left( \frac{1}{(\lambda_i - \mu_j - i + j)!} h[\lambda_i - \mu_j - i + j] \right)_{1 \leq i, j \leq \ell}.$$

**Theorem (Aliniaieifard-Li-vW 2022)**

$$\rho(s_{(\delta, \lambda/\mu)}) = s_{\lambda/\mu}$$
Skew Schur functions in noncommuting variables

\[ s_{[22/1]} = \det \begin{pmatrix} \frac{1}{1!} h_1 & \frac{1}{3!} h_{123} \\ 0! & h_\varnothing \end{pmatrix} = \frac{1}{1!} h_1 \frac{1}{2!} h_{12} - \frac{1}{3!} h_{123} \frac{1}{0!} h_\varnothing \]

\[ = \frac{1}{2} h_{1|12} - \frac{1}{6} h_{123} = \frac{1}{2} h_{1/23} - \frac{1}{6} h_{123} \]

\[ s_{(132,22/1)} = 132^o s_{[22/1]} = 132^o \left( \frac{1}{2} h_{1/23} - \frac{1}{6} h_{123} \right) = \frac{1}{2} h_{1/23} - \frac{1}{6} h_{123} \]
The Rosas-Sagan skew Schur function in NCSym is

\[ S_{\lambda/\mu}^{RS} = \sum_{\dot{T} \text{ DYT of shape } \lambda/\mu} x^\dot{T}. \]

**Theorem (Aliniaeifard-Li-vW 2022)**

Let \( \lambda \vdash n \) and \( \mu \vdash m \).

\[ \rho(S_{\lambda/\mu}^{RS}) = (n - m)! s_{\lambda/\mu} \]

\[ S_{\lambda/\mu}^{RS} = \sum_{\delta \in \mathfrak{S}_{n-m}} s_{(\delta, \lambda/\mu)} \]

\[ = \sum_{\nu \vdash (n-m)} c_{\mu\nu}^{\lambda} S_{\nu}^{RS} \]
Rosas-Sagan skew Schur functions

The Rosas-Sagan skew Schur function in NCSym is

$$S_{\lambda/\mu}^{RS} = \sum_{\hat{T} \text{ DYT of shape } \lambda/\mu} x^{\hat{T}}.$$ 

Theorem (Aliniaeifard-Li-vW 2022)

Let $\lambda \vdash n$ and $\mu \vdash m$.

$$\rho(S_{\lambda/\mu}^{RS}) = (n - m)! s_{\lambda/\mu}$$

$$S_{\lambda/\mu}^{RS} = \sum_{\delta \in \mathcal{S}_{(n-m)}} s_{(\delta, \lambda/\mu)}$$

$$= \sum_{\nu \vdash (n-m)} c_{\mu \nu}^{\lambda} S_{\nu}^{RS}$$

and the $c_{\mu \nu}^{\lambda}$ are the Littlewood-Richardson coefficients.
Noncommutative symmetric functions

Take the noncommutative symmetric functions (Gelfand-Krob-Lascoux-Leclerc-Thibon) with map $\mathcal{J}$ (Bergeron-Reutenauer-Rosas-Zabrocki).

$$\mathcal{J} : \sum_{j_1 \leq j_2 \leq \cdots \leq j_n} x_{j_1} x_{j_2} \cdots x_{j_n} \mapsto \frac{1}{n!} h_{[n]} \in \text{NCSym}$$

**Theorem (Aliniaeifard-Li-vW 2022)**

For the immaculate function $\mathcal{G}_\lambda$ of Berg-Bergeron-Saliola-Serrano-Zabrocki

$$\mathcal{J}(\mathcal{G}_\lambda) = s_{[\lambda]}.$$ 

For the noncommutative ribbon Schur function $r_\alpha$ of Gelfand et al.

$$\mathcal{J}(r_\alpha) = r_{[\alpha]}$$

the ribbon source Schur function: $r_{[\alpha]} = s_{[\lambda/\mu]}$. 


**Further avenues**

- Find **product** rules

\[ s_\pi s_\sigma = \sum_\tau c_{\pi\sigma}^{\tau} s_\tau \]

\[ S_\lambda^{RS} S_\mu^{RS} = \sum_\nu c_{\lambda\mu}^{\nu} S_\nu^{RS}. \]

- Find a **coproduct** rule

\[ \Delta(s_\pi) = \sum_{\sigma, \tau} d_{\sigma\tau}^{\pi} s_\sigma \otimes s_\tau. \]

- **Generalize** \( s_\pi \), for example to MacMahon symmetric functions.
- **Find the dual basis** to Schur functions in NCSym.
- **Relationship** to \( x \)-basis of Bergeron-Hohlweg-Rosas-Zabrocki.
“NEVER FEAR QUARRELS, BUT SEEK HAZARDOUS ADVENTURES”
– Alexandre Dumas, The Three Musketeers

Schur functions in noncommuting variables
Farid Aliniaeifard, Shu Xiao Li, SvW, Adv. Math. 406 37pp (2022)
“NEVER FEAR QUARRELS, BUT SEEK HAZARDOUS ADVENTURES”
– ALEXANDRE DUMAS, THE THREE MUSKETEERS

Schur functions in noncommuting variables
Farid Aliniaefard, Shu Xiao Li, SvW, Adv. Math. 406 37pp (2022)

Thank you very much,
and a big thank you to the organizers for a wonderful conference!