Higher Cumulants in the Cluster Expansion in QCD

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Abstract

In this work an extension of the Gaussian model of the stochastic vacuum is presented. It consists of including higher cumulants than just the second one in the cluster expansion in QCD. The influence of nonabelian fourth cumulants on the potential of a static quark-antiquark pair is examined and the formation of flux tubes between a static $q\bar{q}$-pair is investigated. It is found that the fourth cumulants can contribute to chromomagnetic flux tubes. Furthermore the contribution of fourth cumulants to the total cross section of soft high energy hadron-hadron scattering is examined and it is found that the fourth cumulants do not change the general picture obtained in the Gaussian model.
I. INTRODUCTION

The linked cluster expansion \([1], [2]\) is a powerful tool in statistical mechanics to evaluate expectation values. Especially if the measure for forming the average is not known it allows a systematic expansion. In particle physics this method is much less widely used. Here the weight, given by the exponential function of the action is known at least formally and therefore there are two principle ways to evaluate expectation values:

The perturbative one where the action is split in a part quadratic in the fields, and the rest, which is expanded as a power series. The Gaussian integral due to the quadratic part can be performed and this yields the well known perturbation series \([3], [4]\).

Another way is to discretize the action in a gauge invariant way on the lattice \([5], [6]\) and perform the functional integration with numerical methods.

In the model of the stochastic vacuum (MSV) \([7], [8], [9]\) the linked cluster expansion of the long distance behaviour of QCD is the starting point. It turns out that if the stochastic variables are taken to be the gluon field strenghts a convergent cluster expansion yields automatically the salient feature of non-perturbative QCD namely confinement. This approach has been successfully applied to many processes and it turned out that the simplest form of the cluster expansion namely the one where all cumulants higher than the second one are neglected yields a surprisingly good description of many phenomena ranging from charmonium spectroscopy to high energy scattering (see the reviews \([10], [11], [12], [13]\) and the literature quoted there). A process with vanishing higher cumulants is just a Gaussian one where the functional integrals can be performed explicitly and all can be reduced to one correlator.

Though the Gaussian approximation is astonishingly successful, it is nevertheless worthwhile to investigate which principally new effects can occur if higher cumulants are taken into account. This is the aim of the present paper. If one wants to study the gauge invariant field contribution of a static quark-antiquark pair and high energy scattering of hadrons, one has to evaluate the vacuum expectation value of a product of traces of Wegner-Wilson loops. In that case the leading term is the expectation value of the product of four field strength tensors. A fourth cumulant can contribute directly to that expectation value and for that reason in this paper we concentrate on the fourth cumulant.

It is structured in the following way: In the second section we shortly repeat some basic
features of the linked cluster expansion and some important technical features of the model of the stochastic vacuum (MSV). In section III we discuss some possible choices of higher cumulants and the way how we calculate the effects of these higher cumulants. We present the results obtained in this way and discuss them.

II. LINKED CLUSTER EXPANSION AND THE MODEL OF THE STOCHASTIC VACUUM

The quantity we are mainly interested in, is the Wegner-Wilson loop in QCD [5], [6]. It is defined as the vacuum expectation value of the gauge invariant path ordered closed line integral over the exponential of the color-potential

\[ \langle W[C] \rangle = \langle P e^{-ig \oint_C A_\mu(x) dx^\mu} \rangle \]  

where \( A_\mu(z) = \sum A_\mu^a \lambda^a / 2 \) is the SU(3)-Lie-Algebra valued gluon potential and P denotes path ordering.

Using the non-Abelian Stokes theorem [14], [15], [16], we can express \( W[C] \) by a surface integral over the gluon field strength

\[ \langle W[C] \rangle = \langle P S \exp \left[ -\sum g^n / 2^n n! \int_S d\sigma_{\mu_1 \nu_1} (x_1) \ldots d\sigma_{\mu_n \nu_n} (x_n) \langle \langle F_{\mu_1 \nu_1} (x_1) \ldots F_{\mu_n \nu_n} (x_n) \rangle \rangle \right] \]  

where Lorentz and gauge invariance imply \( \langle \langle F_{\mu \nu} (x) \rangle \rangle = 0 \) and the series starts with \( n = 2 \), the “Gaussian” cluster.
FIG. 1: Transition of path ordering to surface ordering with the non-abelian Stokes theorem.

By expanding the exponentials and rearranging the terms one can show that the cumulants \( \langle \langle . \rangle \rangle \) can be expressed through the expectation values \( \langle . \rangle \) \[2]\.

\[
\langle \langle F_1 F_2 \rangle \rangle = \langle F_1 F_2 \rangle \quad \langle \langle F_1 F_2 F_3 \rangle \rangle = \langle F_1 F_2 F_3 \rangle
\] (4)

and for \( x_1 < x_2 \ldots < x_n \)

\[
\langle \langle F_1 F_2 F_3 F_4 \rangle \rangle = \langle F_1 F_2 F_3 F_4 \rangle - \langle F_1 F_2 \rangle \langle F_3 F_4 \rangle - \langle F_1 F_3 \rangle \langle F_2 F_4 \rangle - \langle F_1 F_4 \rangle \langle F_2 F_3 \rangle
\]

\ldots (5)

where \( F_i = F_{\mu_1 \nu_1}(x_i) \).

For a Gaussian process all cumulants higher than the second one vanish by definition and we obtain directly for such a process

\[
\langle W[C] \rangle = P_S \exp \left[ -\frac{g^2}{2^{2/2}} \int_S \int_S \sigma_{\mu_1 \nu_1}(x_1) \sigma_{\mu_2 \nu_2}(x_2) \langle \langle F_{\mu_1 \nu_1}(x_1) F_{\mu_2 \nu_2}(x_2) \rangle \rangle \right]
\] (6)

from which the area law of the W-loop can be derived easily \[3\], \[7\] if we assume that the correlator itself falls off with a characteristic correlation length \( a \).

In order to evaluate expectation values of two (or more) Wegner-Wilson loops the simple factorization of the matrices does not suffice. In all applications studied so far, the following factorization hypothesis has been made for the color components of the field

\[
\langle F_1^{a_1} F_2^{a_2} F_3^{a_3} F_4^{a_4} \rangle = \langle F_1^{a_1} F_2^{a_2} \rangle \langle F_3^{a_3} F_4^{a_4} \rangle
\]

\[ + \langle F_1^{a_1} F_3^{a_3} \rangle \langle F_2^{a_2} F_4^{a_4} \rangle + \langle F_1^{a_1} F_4^{a_4} \rangle \langle F_2^{a_2} F_3^{a_3} \rangle \]

(7)
with
\[ F_{\alpha i}^a = F_{\mu_i \nu_i}^a(x_i, w) \] (8)

\( F_{\mu_i \nu_i}(x_i, w) \) is the color-parallel transported field:
\[ F_{\mu_i \nu_i}(x_i, w) = \Phi^{-1}(x_i, w) F_{\mu \nu}(x_i) \Phi(x_i, w) \] (9)

with
\[ \Phi(x_i, w) = P \exp \left[ -ig \int_0^1 d\lambda A_\mu \left( w + \lambda(x_i - w) \right) (x_i^\mu - w^\mu) \right] \] (10)

being the color-parallel transporter from \( x \) to \( w \) in the adjoint representation. Transformation (9) is necessary in order to give the non-local correlators a gauge invariant meaning.

In the applications of the MSV to the calculations of more than one W-loop the following approximations are made: In the correlator \( \langle \langle F_{\mu_1 \nu_1}(x_1, w) F_{\mu_2 \nu_2}(x_2, w) \rangle \rangle \) the dependence on the reference point \( w \) is neglected which is certainly only justifiable for a “reasonable” choice of the point \( w \). A more technical assumption is the following: if the exponentials are expanded as power series in the correlator \( \langle \langle F_{\mu_1 \nu_1}(x_1, w) F_{\mu_2 \nu_2}(x_2, w) \rangle \rangle \) only the term leading in \( (a/L) \) is taken into account, where \( L \) is the linear dimension of the loop and \( a \) the correlation length of the correlator. For a discussion and possible resolution of this rather ugly assumption see [17], [13].

Under these assumptions the following problems necessitating the evaluation of two loops have been solved: The calculation of the gauge invariant color field density \( \sum_a E_k^a(x)^2 \), \( \sum_a B_k^a(x)^2 \) [17] and of total or diffractive cross sections in soft high energy reactions [18], [13], [19], [20].

In the next section we investigate in which way the very consistent results of the Gaussian model could be modified by the presence of higher cumulants. It is not astonishing that in that case the program cannot be performed as far as in the Gaussian approximation. We therefore concentrate on the leading effects of some higher cumulants.

III. HIGHER CUMULANTS

A. Choice of the cumulants

In order to relax the factorization hypothesis of the MSV which only takes into account cumulants of order two, we introduce higher cumulants. The cumulant expression for a
correlator of fourth order is given by:

\[
\langle F_{\mu_1\nu_1}^{a_1} F_{\mu_2\nu_2}^{a_2} F_{\mu_3\nu_3}^{a_3} F_{\mu_4\nu_4}^{a_4} \rangle = \left( \langle F_{\mu_1\nu_1}^{a_1} F_{\mu_2\nu_2}^{a_2} \rangle \langle F_{\mu_3\nu_3}^{a_3} F_{\mu_4\nu_4}^{a_4} \rangle + \langle F_{\mu_1\nu_1}^{a_1} F_{\mu_3\nu_3}^{a_3} \rangle \langle F_{\mu_2\nu_2}^{a_2} F_{\mu_4\nu_4}^{a_4} \rangle + \langle F_{\mu_1\nu_1}^{a_1} F_{\mu_4\nu_4}^{a_4} \rangle \langle F_{\mu_2\nu_2}^{a_2} F_{\mu_3\nu_3}^{a_3} \rangle + \langle F_{\mu_1\nu_1}^{a_1} F_{\mu_2\nu_2}^{a_2} F_{\mu_3\nu_3}^{a_3} F_{\mu_4\nu_4}^{a_4} \rangle \right), \tag{11}
\]

We are not striving to find the most general ansatz for the fourth cumulant since a full inclusion of all fourth cumulants compatible with the general requirements of Lorentz and gauge covariance would introduce many new unknown parameters into the model. Even for the local condensate there are six independent ones \[21\]. Any quantitative computation would lose its predictive power. Therefore we restrict ourselves to studying the influence of some possible cumulants in a qualitative way. Of course the cumulants are not independent of each other, but there are some relations between them due to the equations of motion \[21\], \[22\]. However these relations are not sufficient to determine the higher cumulants through the lower ones.

Two non-abelian example structures for a fourth cumulant, which satisfy the requirements of gauge and Lorentz invariance and which satisfy the normalisation condition:

\[
\langle \langle g^4 F_{\mu_1\nu_1}^{a_1} F_{\mu_2\nu_2}^{a_2} F_{\mu_3\nu_3}^{a_3} F_{\mu_4\nu_4}^{a_4} \rangle \rangle = \mathcal{N} \left( \langle g^2 FF \rangle \right)^2, \tag{12}
\]

are:

\[
\begin{align*}
\langle \langle g^4 F_{\mu_1\nu_1}^{a_1}(x_1, w) F_{\mu_2\nu_2}^{a_2}(x_2, w) F_{\mu_3\nu_3}^{a_3}(x_3, w) F_{\mu_4\nu_4}^{a_4}(x_4, w) \rangle \rangle = & \frac{\mathcal{N}}{1152} \left( \langle g^2 FF \rangle \right)^2 \\
& \left\{ f_{a_1a_2e} f_{a_3a_4e} (\epsilon_{\mu_1\nu_1\mu_3\nu_3} \epsilon_{\mu_2\nu_2\mu_4\nu_4} - \epsilon_{\mu_1\nu_1\mu_4\nu_4} \epsilon_{\mu_2\nu_2\mu_3\nu_3}) \\
& + f_{a_3a_1e} f_{a_2a_4e} (\epsilon_{\mu_1\nu_1\mu_4\nu_4} \epsilon_{\mu_2\nu_2\mu_3\nu_3} - \epsilon_{\mu_1\nu_1\mu_2\nu_2} \epsilon_{\mu_3\nu_3\mu_4\nu_4}) \\
& + f_{a_2a_3e} f_{a_1a_4e} (\epsilon_{\mu_1\nu_1\mu_2\nu_2} \epsilon_{\mu_3\nu_3\mu_4\nu_4} - \epsilon_{\mu_1\nu_1\mu_3\nu_3} \epsilon_{\mu_2\nu_2\mu_4\nu_4}) \right\} D(z_1, z_2, z_3, z_4, z_5, z_6) \tag{13}
\end{align*}
\]

and

\[
\begin{align*}
\langle \langle g^4 F_{\mu_1\nu_1}^{a_1}(x_1, w) F_{\mu_2\nu_2}^{a_2}(x_2, w) F_{\mu_3\nu_3}^{a_3}(x_3, w) F_{\mu_4\nu_4}^{a_4}(x_4, w) \rangle \rangle = & -\frac{\mathcal{N}}{9216} \left( \langle g^2 FF \rangle \right)^2 \\
& \left\{ f_{a_1a_2e} f_{a_3a_4e} (\epsilon_{\mu_1\nu_1\mu_3\nu_3} \epsilon_{\mu_2\nu_2\mu_4\nu_4} - \epsilon_{\mu_1\nu_1\mu_4\nu_4} \epsilon_{\mu_2\nu_2\mu_3\nu_3}) \\
& + f_{a_3a_1e} f_{a_2a_4e} (\epsilon_{\mu_1\nu_1\mu_4\nu_4} \epsilon_{\mu_2\nu_2\mu_3\nu_3} - \epsilon_{\mu_1\nu_1\mu_2\nu_2} \epsilon_{\mu_3\nu_3\mu_4\nu_4}) \\
& + f_{a_2a_3e} f_{a_1a_4e} (\epsilon_{\mu_1\nu_1\mu_2\nu_2} \epsilon_{\mu_3\nu_3\mu_4\nu_4} - \epsilon_{\mu_1\nu_1\mu_3\nu_3} \epsilon_{\mu_2\nu_2\mu_4\nu_4}) \right\} D(z_1, z_2, z_3, z_4, z_5, z_6) \tag{14}
\end{align*}
\]
\[
\left\{ \delta_{a_1 a_2} \delta_{a_3 a_4} \left( \delta_{\mu_1 \mu_2} \delta_{\nu_1 \nu_2} - \delta_{\mu_1 \nu_2} \delta_{\mu_2 \nu_1} \right) \left( \delta_{\mu_3 \mu_4} \delta_{\nu_3 \nu_4} - \delta_{\mu_3 \nu_4} \delta_{\mu_4 \nu_3} \right) \\
+ \delta_{a_1 a_3} \delta_{a_2 a_4} \left( \delta_{\mu_1 \mu_3} \delta_{\nu_1 \nu_3} - \delta_{\mu_1 \nu_3} \delta_{\mu_3 \nu_1} \right) \left( \delta_{\mu_2 \mu_4} \delta_{\nu_2 \nu_4} - \delta_{\mu_2 \nu_4} \delta_{\mu_4 \nu_2} \right) \\
+ \delta_{a_1 a_4} \delta_{a_2 a_3} \left( \delta_{\mu_1 \mu_4} \delta_{\nu_1 \nu_4} - \delta_{\mu_1 \nu_4} \delta_{\mu_4 \nu_1} \right) \left( \delta_{\mu_2 \mu_3} \delta_{\nu_2 \nu_3} - \delta_{\mu_2 \nu_3} \delta_{\mu_3 \nu_2} \right) \right\} \\
D(z_1, z_2, z_3, z_4, z_5, z_6).
\]

Here \(z_i\) are the differences between the coordinates \(z_1 = x_1 - x_2, z_2 = x_1 - x_3, \ldots, z_6 = x_3 - x_4\). The scalar function \(D\) falls off fast if any of the distances \(|z_i|\) becomes large, it is normalized \(D(0, 0, \ldots, 0) = 1\). We have extracted the dimensionful quantity \((g^2 FF)^2\), the number which determines the strength of the full correlator \([11]\). The first structure is similar to the four gluon vertex, \(f_{a_1 a_2 e}\) being the structure constants of SU(3). The correlation lengths \(a_i\) are defined by the following conditions:

\[
\int_0^\infty D(0, \ldots, z_i, \ldots, 0) \, dz_i = a_i \quad i = 1 \ldots 6.
\]

In the following we put \(a_i = a\) and all distances are taken to be in units of the correlation length \(a\). In a Euclidean space we make a simple Gaussian ansatz for \(D\):

\[
D(z_1, z_2, z_3, z_4, z_5, z_6) = e^{-(z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2)/\lambda^2},
\]

where \(\lambda = \frac{2}{\sqrt{\pi}} a\). As the distances \(z_4, z_5\) and \(z_6\) can be expressed as a function of \(z_1, z_2\) and \(z_3\), \(D\) is a function of the latter three independent variables only. Ansatz \([14]\) is not supposed to be a realistic choice but rather a simple manageable expression taking into account the cluster property.

### B. Single Wegner-Wilson loop: Potential

In order to obtain the static potential of a quark-antiquark pair, the expectation value of the Wegner-Wilson loop is calculated through the cumulant expansion \([9], [7]\):

\[
\langle \text{Tr} W[S] \rangle = \text{Tr} P_S \exp \left[ -\frac{g^2}{4 \cdot 2!} \int_S \int_S d\sigma_{\mu_1 \nu_1}(x_1) d\sigma_{\mu_2 \nu_2}(x_2) \langle \langle F_{\mu_1 \nu_1}(x_1, w) F_{\mu_2 \nu_2}(x_2, w) \rangle \rangle \right] \\
+ \frac{g^4}{2^4 \cdot 4!} \int_S \int_S \int_S \int_S d\sigma_{\mu_1 \nu_1}(x_1) d\sigma_{\mu_2 \nu_2}(x_2) d\sigma_{\mu_3 \nu_3}(x_3) d\sigma_{\mu_4 \nu_4}(x_4) \\
\cdot \langle \langle F_{\mu_1 \nu_1}(x_1, w) F_{\mu_2 \nu_2}(x_2, w) F_{\mu_3 \nu_3}(x_3, w) F_{\mu_4 \nu_4}(x_4, w) \rangle \rangle \\
- \frac{g^6}{2^6 \cdot 6!} \cdots.
\]

(17)
where $S$ is the integration surface enclosed by the loop and $P_S$ denotes surface ordering.

The first ansatz (13) gives the following contribution to the potential, after performing the trace and summing over all color indices:

$$V_4^{(1)} = \lim_{T \to \infty} \frac{1}{T} P_S \left( \frac{1}{2^4 \cdot 4!} \int_S \int_S \int_S \int_S d\sigma_{\mu_1 \nu_1}(x_1) d\sigma_{\mu_2 \nu_2}(x_2) d\sigma_{\mu_3 \nu_3}(x_3) d\sigma_{\mu_4 \nu_4}(x_4) \right)$$

$$\times \frac{1}{128} \left( -\epsilon_{\mu_1 \nu_1 \mu_4 \nu_4} \epsilon_{\mu_2 \nu_2 \mu_3 \nu_3} - \epsilon_{\mu_1 \nu_1 \mu_2 \nu_2} \epsilon_{\mu_3 \nu_3 \mu_4 \nu_4} + 2 \epsilon_{\mu_1 \nu_1 \mu_3 \nu_3} \epsilon_{\mu_2 \nu_2 \mu_4 \nu_4} \right)$$

$$\mathcal{N} \left( \langle g^2 FF \rangle \right)^2 D(z_1, z_2, z_3, z_4, z_5, z_6). \quad (18)$$

Assuming that the Wegner-Wilson loop lies in a plane with the reference point also on this plane, the integration surface $S$ is just the rectangle enclosed by the loop. It is easy to see that the $\epsilon$-tensors vanish on this plane and that the cumulant of the first ansatz (13) does not contribute to the potential of the $q\bar{q}$-pair.

Inserting ansatz (14) and summing over all color indices gives the following fourth cumulant contribution to the potential:

$$V_4^{(2)} = \lim_{T \to \infty} \frac{1}{T} P_S \left( \frac{1}{9216 \cdot 2^4 \cdot 4!} \int_S \int_S \int_S \int_S d\sigma_{\mu_1 \nu_1}(x_1) d\sigma_{\mu_2 \nu_2}(x_2) d\sigma_{\mu_3 \nu_3}(x_3) d\sigma_{\mu_4 \nu_4}(x_4) \right)$$

$$\times \left\{ 16(\delta_{\mu_1 \mu_2} \delta_{\nu_1 \nu_2} - \delta_{\mu_1 \nu_2} \delta_{\mu_2 \nu_1})(\delta_{\mu_3 \mu_4} \delta_{\nu_3 \nu_4} - \delta_{\mu_3 \nu_4} \delta_{\mu_4 \nu_3}) \\
-2(\delta_{\mu_1 \mu_3} \delta_{\nu_1 \nu_3} - \delta_{\mu_1 \nu_3} \delta_{\mu_3 \nu_1})(\delta_{\mu_2 \mu_4} \delta_{\nu_2 \nu_4} - \delta_{\mu_2 \nu_4} \delta_{\mu_4 \nu_2}) \\
+16(\delta_{\mu_1 \mu_4} \delta_{\nu_1 \nu_4} - \delta_{\mu_1 \nu_4} \delta_{\mu_4 \nu_1})(\delta_{\mu_2 \mu_3} \delta_{\nu_2 \nu_3} - \delta_{\mu_2 \nu_3} \delta_{\mu_3 \nu_2}) \right\} \mathcal{N} \left( \langle g^2 FF \rangle \right)^2 D(z_1, z_2, z_3, z_4, z_5, z_6). \quad (19)$$

After suitable parametrisation of the rectangular integration surface bounded by the loop, a non-vanishing contribution to the potential is obtained (23):
The integrations and the limit \( T \to \infty \) can be performed numerically and the result is shown in figure \( 2 \). The fourth cumulant contribution to the potential is plotted as a function of the quark-antiquark distance \( R \). For large \( R \) a linear rise is found, as in the Gaussian model. For short distances the potential behaves as \( R^4 \), in contrast to the Gaussian model, where the short distance behaviour is like \( R^2 \).

\[
\langle g^2 F F \rangle^2 \quad V^{(2)}_4
\]

\[
\langle g^2 F F \rangle^3 \quad V^{(3)}_2
\]

\[
\langle g^2 F F \rangle^4 \quad V^{(4)}_2
\]

FIG. 2: On the left the numerical result for the fourth cumulant contribution (14) to the potential \( V^{(2)}_4 \) of a static \( q\bar{q} \)-pair is shown. For large quark separations \( R \) a linear rise is obtained whereas for small separation a \( R^4 \) behaviour is found. On the right the results of the Gaussian model for the Abelian (ab) and non-Abelian (n.ab) case are shown.

C. The field of a static \( q\bar{q} \)-pair

We consider a static quark-antiquark pair in a color singlet state at fixed space points. It gives rise to a chromoelectric and chromomagnetic field whose components form the field strength tensor. In order to calculate the squared field strength we introduce a plaquette \( P_{\mu\nu}(x) \), representing a small Wegner-Wilson loop in the \( \mu\nu \)-plane with center point \( x \) and dimension \( R_P \) (fig. \( 3 \)). In the limit \( R_P \to 0 \) its expansion gives the following relationship with the gauge invariant field density [17]:

\[
P_{\mu\nu}(x) = N_C - \frac{1}{4} R_P^4 g^2 \sum_a F_{\mu\nu}^a F_{\mu\nu}^a + \mathcal{O}(R_P^8) \quad (21)
\]
where there is no summation over $\mu$ and $\nu$. This plaquette is evaluated in the presence of a static $q\bar{q}$-pair whose world lines and color connectors form in turn a large rectangular loop of width $R$ and length $T$ going to infinity. We thus consider the quantity:

$$f_{\mu\nu}(x) := \frac{4}{R_P^4 g^2} \frac{\langle \text{Tr} W[\partial S] \text{Tr} P_{\mu\nu}(x) \rangle - \langle \text{Tr} W[\partial S] \rangle \langle \text{Tr} P_{\mu\nu}(x) \rangle}{\langle \text{Tr} W[\partial S] \rangle} \tag{22}$$

and

$$\lim_{R_P \to 0} f_{\mu\nu}(x) = \begin{pmatrix}
0 & -B_z^2 & -B_y^2 & -E_x^2 \\
-B_z^2 & 0 & -B_x^2 & -E_y^2 \\
-B_y^2 & -B_x^2 & 0 & -E_z^2 \\
-E_x^2 & -E_y^2 & -E_z^2 & 0
\end{pmatrix}, \tag{23}$$

$\vec{E}$ and $\vec{B}$ being the chromoelectric and chromomagnetic field. The coordinates of the plaquette are $x$, $y$ and $z$: because of rotational symmetry in the $x_1$-$x_2$-plane, we choose $y = 0$. Assuming that the Wegner-Wilson loop lies in the $x_3$-$x_4$-plane, $x$ is the perpendicular distance to the quark axis, whereas $z$ represents the distance from the origin parallel to the quark axis see fig. 3. We proceed by studying $f_{\mu\nu}$ in the limit $R_P \to 0$. After the expansion

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{Configuration of the Wegner-Wilson loop $W$ and the plaquette $P$. The quark-antiquark axis is oriented in the $x_3$-direction. The plaquette distance from the origin parallel to the quark-antiquark axis is represented by $z$ and the distance perpendicular to the quark axis is $x$. In this figure the plaquette $P_{34}$ is shown.}
\end{figure}
of the exponentials in (22) we obtain:

\[
f(x) = \frac{4}{R_P^4 g^2} \frac{1}{\langle \text{Tr} W \rangle} \cdot \left( \sum_{n=1}^{\infty} \frac{(-i)^n}{2^n n!} \int \cdots \int d\sigma_{\mu_1 \nu_1} \cdots d\sigma_{\mu_n \nu_n} \, \text{Tr} [T^{a_1} \cdots T^{a_n}] \int \int d\sigma^P \, d\sigma^P \right) \]

\[
\frac{(-i)^2}{2^{2n} 2} \, \text{ordered} \int \int d\sigma^W \int \cdots \int d\sigma^W \, \text{Tr} [T^{a_1} \cdots T^{a_n}] \int \int d\sigma^P \, d\sigma^P \]

\[
- \frac{(-i)^n}{2^{2n} 2} \, \text{free} \int \int d\sigma^W \int \cdots \int d\sigma^W \, \text{Tr} [T^{a_1} \cdots T^{a_n}] \int \int d\sigma^P \, d\sigma^P \]

\[
\frac{(-i)^n}{2^{2n} 2} \, \text{free} \int \int d\sigma^W \int \cdots \int d\sigma^W \, \text{Tr} [T^{a_1} \cdots T^{a_n}] \int \int d\sigma^P \, d\sigma^P \]

\[
\left( g^{a_1} F_{\mu_1 \nu_1} \cdots F_{\mu_n \nu_n} g^2 F^{a_1}_{\mu \nu} F^{a_2}_{\rho \sigma} \right) \left( g^{a_2} F_{\mu_2 \nu_2} \cdots F_{\mu_n \nu_n} g^2 F^{a_1}_{\mu \nu} F^{a_2}_{\rho \sigma} \right)
\]

(24)

where the integrations over the surface bounded by the Wegner-Wilson loop are surface ordered. The matrix elements of \( f \) are given by the different orientations of the plaquette. In the Gaussian model it was possible to insert the factorizations and rewrite the summations in such a way as to perform the summations [17]. Here this full summation is not feasible. Nevertheless to get an idea about what kind of contribution a fourth cumulant could give to the fields, we confine ourselves to the lowest non-trivial term. It should give a qualitative indication of the effects of higher cumulants. After inserting the cumulant definition (11) we obtain the leading contribution from the fourth cumulant:

\[
\delta f := \frac{4}{R_P^4 g^2} \frac{1}{\langle \text{Tr} W \rangle} \cdot \left( \sum_{n=1}^{\infty} \frac{(-i)^n}{2^n n!} \int \cdots \int d\sigma_{\mu_1 \nu_1} \cdots d\sigma_{\mu_n \nu_n} \, \text{Tr} [T^{a_1} \cdots T^{a_n}] \int \int d\sigma^P \, d\sigma^P \right) \]

\[
\frac{(-i)^2}{2^{2n} 2} \, \text{ordered} \int \int d\sigma^W \int \cdots \int d\sigma^W \, \text{Tr} [T^{a_1} \cdots T^{a_n}] \int \int d\sigma^P \, d\sigma^P \]

\[
- \frac{(-i)^n}{2^{2n} 2} \, \text{free} \int \int d\sigma^W \int \cdots \int d\sigma^W \, \text{Tr} [T^{a_1} \cdots T^{a_n}] \int \int d\sigma^P \, d\sigma^P \]

\[
\frac{(-i)^n}{2^{2n} 2} \, \text{free} \int \int d\sigma^W \int \cdots \int d\sigma^W \, \text{Tr} [T^{a_1} \cdots T^{a_n}] \int \int d\sigma^P \, d\sigma^P \]

\[
\left( g^{a_1} F_{\mu_1 \nu_1} \cdots F_{\mu_n \nu_n} g^2 F^{a_1}_{\mu \nu} F^{a_2}_{\rho \sigma} \right) \left( g^{a_2} F_{\mu_2 \nu_2} \cdots F_{\mu_n \nu_n} g^2 F^{a_1}_{\mu \nu} F^{a_2}_{\rho \sigma} \right)
\]

(25)

Since we are mainly interested in the \( x, z \)-dependence of \( \delta f \) we omit the \( x \) and \( z \)-independent normalization \( \langle \text{Tr} W \rangle \).

To compute the contribution of each ansatz (13) or (14) the traces must be performed and a summation over all color indices must be carried out. After suitable parametrization of the plane surfaces bounded by the Wegner-Wilson loop and the plaquette, we obtain for
the contribution of the first ansatz (13) [23]:

$$\delta f_1 = \frac{-N C}{18432 g^2} \mathcal{N} \left( \left\langle g^2 F F \right\rangle \right)^2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xi(x, z, R),$$  

(26)

and for the second ansatz (14) [23]:

$$\delta f_2 = -\frac{1}{147456 g^2} \mathcal{N} \left( \left\langle g^2 F F \right\rangle \right)^2 \begin{pmatrix} 0 & 4 & 4 & 4 \\ 4 & 0 & 4 & 4 \\ 4 & 4 & 0 & 5 \\ 4 & 4 & 5 & 0 \end{pmatrix} \xi(x, z, R).$$  

(27)

where both contributions have the same $x, z$-dependence:

$$\xi(x, z, R) := \int_{-1}^{1} R^2 ds_W ds'_W \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau_W d\tau'_W$$

$$\cdot \exp \left[ -\frac{1}{\lambda^2} \left[ 4x^2 + 2(z - \frac{1}{2} R s_W)^2 + 2(z - \frac{1}{2} R s'_W)^2 \\
+ \frac{1}{4} R^2 (s'_W - s_W)^2 + 2(\tau_W^2 + \tau'_W^2) + (\tau'_W - \tau_W)^2 \right] \right].$$  

(28)

The first ansatz (13) only contributes to the chromomagnetic field in the direction parallel to the quark axis:

$$\delta B^2_z = \frac{N C}{18432 g^2} \mathcal{N} \left( \left\langle g^2 F F \right\rangle \right)^2 \xi(x, z, R),$$  

(29)

whereas the second ansatz (14) contributes to all components of both the chromoelectric and the chromomagnetic field:

$$\delta B^2_x = \delta B^2_y = \delta B^2_z = \frac{4}{147456 g^2} \mathcal{N} \left( \left\langle g^2 F F \right\rangle \right)^2 \xi(x, z, R)$$

$$\delta E^2_x = \delta E^2_y = \frac{4}{147456 g^2} \mathcal{N} \left( \left\langle g^2 F F \right\rangle \right)^2 \xi(x, z, R)$$

$$\delta E^2_z = \frac{5}{147456 g^2} \mathcal{N} \left( \left\langle g^2 F F \right\rangle \right)^2 \xi(x, z, R).$$  

(30)

The numerical evaluation of the $x, z$-dependence $\xi(x, z, R)$ of the fourth cumulant contributions to the fields is shown in figure 4 for the $q\bar{q}$-separations $R = 1$ and $R = 8$. The result is that, for increasing separation between the $q\bar{q}$-pair, a flux tube is formed. With (28) and (30) the following conclusions can be drawn:
FIG. 4: The function $\xi(x, z, R)$ in arbitrary units for the $q\bar{q}$-separations $R = 1$ and $R = 8$. The quarks have coordinates $x = 0$ and $z = \pm 0.5a$ for $R = 1$ and $x = 0$ and $z = \pm 4a$ for $R = 8$. One observes the formation of a flux tube, note the different $z$-axis scales of the two plots.

- The first ansatz (13) leads to the formation of a purely chromomagnetic flux tube for the $z$-component of the field, but to no area law of the Wegner-Wilson loop.

- The contribution (30) of the second ansatz (14) can be split into a deformation of the vacuum field density with no preferred field components and a flux tube for the color-electric field in the $z$-direction. Of course both contribute to the energy density of the system. In this case confinement also reflects itself in the area law.

We thus find for the first ansatz (13) a contribution to a chromomagnetic flux tube which was not present in the Gaussian approximation. The easiest interpretation of this behaviour is that the flux tube has been created by a pair of magnetic monopoles. At first sight there seems to be a contradiction between the formation of a flux tube and the absence of a contribution to the area law of the Wegner-Wilson loop. However in the Wegner-Wilson loop the exponent $gA_\mu dx^\mu$ only takes into account the interaction of a color-\emph{electric} charge. Therefore it is not astonishing that the area law of the Wegner-Wilson loop (1) is not a criterium for confinement of color-\emph{magnetic} monopoles.
D. High Energy Scattering

Soft hadron-hadron high energy scattering has been extensively studied using the Gaussian model [10], [11], [12], [13], [20]. We have investigated the effect of relaxing the Gaussian approximation in much the same way as for the static $q\bar{q}$-pair. Both fourth cumulants (13) and (14) can be continued to Minkowski space in exactly the same way as the second cumulant and in the final results only the separations in transverse space enter; this means that the Euclidean expressions for the correlators can be used. In fig. 5 we give the total dipole-dipole cross section as a function of the dipole size for two equal dipoles. The qualitative behaviour is the same as for the Gaussian model: at short distances the cross section is proportional to the fourth power of the size of the dipoles, for large sizes it approaches a quadratic dependence. We have also studied the dependence of the cross section on dipole size with one fixed-size dipole. Again we find the same behaviour as in the Gaussian model namely for small sizes a dependence on the square of dipole size and for large sizes an approach to a linear behaviour.

IV. SUMMARY AND OUTLOOK

We have found that the contributions of higher cumulants in the cluster expansion do not lead to a change of the qualitative behaviour of phenomenological results of the Gaussian model. There are however interesting new features in the field configurations of a static $q\bar{q}$-pair. The resulting flux tube of ansatz (14) consists of an increased isotropic vacuum density inside the region described by the function $\xi(x, z, R)$ and in addition of a color-electric flux tube. The ansatz (13) leads to the formation of a color-magnetic flux tube, very similar to the color-electric flux tube of the Gaussian model. The easiest interpretation of such a flux tube is that it has been generated by a pair of confined monopoles. Though a static bare quark carries only color-electric charge it is not impossible that it acquires a magnetic charge in the vacuum. For a color-magnetic monopole without color-electric charge the area law of the Wegner-Wilson loop as evaluated in the MSV is then no longer an indication of confinement. It would therefore be very interesting to investigate higher cumulants on the lattice since they might reveal new features of non-Abelian gauge theories.
FIG. 5: The first two figures give the fourth cumulant contributions to the total dipole-dipole scattering cross section for ansatz (13) and (14) respectively. The last figure gives the results of the Gaussian model for the non-Abelian (n.ab) and the Abelian (ab) case.

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