ABSTRACT
It has been shown recently that graph signals with small total variation can be accurately recovered from only few samples if the sampling set satisfies a certain condition, referred to as the network nullspace property. Based on this recovery condition, we propose a sampling strategy for smooth graph signals based on random walks. Numerical experiments demonstrate the effectiveness of this approach for graph signals obtained from a synthetic random graph model as well as a real-world dataset.

Outline. The problem setup is discussed in III, where we present the random walk sampling method and discuss its properties in the context of the assortative planted partition model. The results of illustrative numerical experiments are presented in Section IV. We finally conclude in Section V.

1. INTRODUCTION
Modern information processing systems are generating massive datasets which are partially labeled mixtures of different media (audio, video, text). Many successful approaches to such datasets are based on representing the data as networks or graphs. In particular, within (semi-)supervised machine learning, we represent the datasets by graph signals defined over an underlying graph, which reflects the similarity relations between individual data points. These graph signals often conform to a smoothness hypothesis, i.e., the signal values of close-by nodes are similar.

Two key problems related to processing these datasets are (i) how to sample them, i.e., which nodes provide the most information about the entire dataset, and (ii) how to recover the entire graph signal representation of the dataset from these samples. These problems have been studied in [3] which proposed a convex optimization method for recovering a graph signal from a small number of samples. Moreover, a sufficient condition for this recovery method to be accurate has been presented. This condition is a reformulation of the stable nullspace property of compressed sensing to the graph signal setting.

Contribution. Based on the intuition provided by the recently derived network nullspace property, we propose a sampling strategy based on random walks. The effectiveness of this approach is confirmed via numerical experiments based on synthetic graph signals obtained from a particular random graph model, i.e., the assortative planted partition model, and graph signals induced by a real-world dataset containing product rating information of an online retail shop.

Notation. Vectors and matrices are denoted by boldface lower-case and upper-case letters, respectively. The vector with all entries equal to one (zero) is denoted \( 1 \) \((0)\). The \( \ell_1 \) and \( \ell_2 \) norm of a vector \( x = (x_1, \ldots, x_N)^T \) are denoted by \( \|x\|_1 \) and \( \|x\|_2 \) respectively.

Index Terms— compressed sensing, big data, graph signal processing, total variation, complex networks

II. PROBLEM FORMULATION
We consider massive heterogeneous datasets with intrinsic network structure represented by a graph \( \mathcal{G} = (V, E) \). The graph \( \mathcal{G} \) consists of the nodes \( V = \{1, \ldots, N\} \), which are connected by undirected edges \( \{i, j\} \in E \). Each node \( i \in V \) represents an individual data point and an edge \( \{i, j\} \in E \) connects nodes representing similar data points. For a given node \( i \in V \), we define its neighbourhood as

\[
\mathcal{N}(i) := \{ j \in V : \{i, j\} \in E \}.
\] (1)

The degree \( d_i := |\mathcal{N}(i)| \) of node \( i \in V \) counts the number of its neighbours.

Within (semi-)supervised learning, we associate each data point \( i \in V \) with a label \( x[i] \in \mathbb{R} \). These labels induce a graph signal \( x[i] : V \rightarrow \mathbb{R} \) defined over the graph \( \mathcal{G} \) underlying the dataset.

We aim at recovering a smooth graph signal \( x \) based on observing its values \( x[i] \) for all nodes \( i \in V \) which belong to the sampling set

\[
\mathcal{M} := \{i_1, \ldots, i_M\} \subseteq V.
\] (2)

The size \( M := |\mathcal{M}| \) of the sampling set is typically much smaller than the overall dataset, i.e., \( M \ll N \). For a fixed sampling budget \( M \) it is important to choose the sampling set such that the information obtained is sufficient to recover the overall graph signal. By considering a particular recovery method, called sparse label propagation (SLP), [3] presents the network nullspace property as a sufficient condition on the sampling set such that SLP recovers the overall graph signal from the samples.
The SLP recovery method is based on a smoothness hypothesis, which requires signal values of nodes belonging to the same cluster to be similar. This smoothness hypothesis then suggests to search for the particular graph signal which is consistent with the observed signal samples, and moreover has minimum total variation (TV)

$$
\|x\|_{\text{TV}} := \sum_{\{i,j\} \in E} |x[j] - x[i]|,
$$

which quantifies signal smoothness. Thus the recovery problem amounts to the convex optimization problem

$$
\hat{x} \in \arg \min \|x\|_{\text{TV}} \quad \text{s.t.} \quad \hat{x}_M = x_M.
$$

The SLP algorithm is nothing but the the primal-dual optimization method of Pock and Chambolle applied to the problem (4).

Let us from now on assume that the true underlying graph signal $x$ is clustered, i.e.,

$$
x = \sum_{C \in \mathcal{F}} a_C t_C,
$$

with the cluster indicator signals

$$
t_C[i] = \begin{cases} 
1, & \text{if } i \in C \\
0, & \text{else.}
\end{cases}
$$

For a partition $\mathcal{F} = \{C_1, \ldots, C_{|\mathcal{F}|}\}$ consisting of disjoint clusters $C_i$ with small cut-sizes, we have that the TV $\|x\|_{\text{TV}}$ is relatively small. Thus, we expect recovery based on TV minimization (4) to be accurate for signals of the type (5). Indeed, a sufficient condition for the solution $\hat{x}$ of (4) to coincide with $x = \sum_{C \in \mathcal{F}} a_C t_C$ can be formulated as

**Lemma 1.** We observe a clustered signal $x$ of the form (5) on the sampling set $M \subseteq V$. If each boundary edge $\{i,j\}$ with $i \in C_a, j \in C_b$ is connected to two sampled nodes in each cluster, i.e.,

$$
|M \cap C_a \cap N(i)| \geq 2, \text{ and } |M \cap C_b \cap N(j)| \geq 2,
$$

then (4) has a unique solution which moreover coincides with the true graph signal $x$.

## III. RANDOM WALK SAMPLING

We now present a particular strategy (summarized in Algorithm 1 below) for choosing the sampling set $M$ of nodes at which the graph signal should be sampled to obtain the observations $\{x[i]\}_{i \in M}$. Our strategy is based on parallel random walks which are started at randomly selected seed nodes. The endpoints of these random walks, which are run for a fixed number $L$ of steps, constitute the sampling set $M$.

In Figure 1 we illustrate the construction of the sampling set via the random walks $P_j$. Each random walk $P_j$ forms a finite sequence $\{v_1 = r_j, \ldots, v_L = i_j\}$ of nodes that are visited in successive steps of the walk.

**Algorithm 1** Random Walk Sampling

**Input:** random walk length $L$, sample budget $M$

**Initialize:** Sampling set $M = \emptyset$

1. for $j = 1 : M$
2. randomly select a seed (start) node $i_1$
3. perform a length-$L$ random walk $P_j \leftarrow (i_1, \ldots, i_L)$
4. $M \leftarrow M \cup \{i_j\}$
5. end for

**Output:** $M$

![Clustered graph signal](image)

**Fig. 1.** Clustered graph signal $x$ defined over a graph composed of two clusters $C_1$ and $C_2$.

The sampling strategy of Algorithm 1 is appealing since it allows for efficient implementation as the random walks can be follows in parallel. Moreover, for a particular random graph model, the sampling set $M$ delivered by Algorithm 1 conforms with Lemma 1. According to Lemma 1 we have to select from each cluster $C_i$ a number sampled nodes which is proportional to its cut-size $|\partial C_i|$. Thus, we have to sample more densely in those clusters which have large cut-size. We now show that the sampling set $M$ obtained by Algorithm 1 follows this rationale for graph signals obtained from the stochastic block model (SBM).

For a given partition $\mathcal{F} = \{C_1, \ldots, C_{|\mathcal{F}|}\}$ of the graph $G$ in clusters $C_i$ of size $N_i := |C_i|$, the SBM is a generative stochastic model for the edge set $E$ of the graph $G$. In its simplest form, which is called the assortative planted partition model (APPM), the SBM is defined by two parameters $p$ and $q$ which specify the probability that two particular nodes $i, j$ of the graph are connected by an edge $\{i, j\}$. In particular, two nodes $i, j \in C_i$ out of the same cluster are connected by an edge with probability $p$, i.e., $P\{\{i, j\} \in E\} = p$ for $i, j \in C_a$. Two nodes $i \in C_a, j \in C_b$ from different clusters $C_a$ and $C_b$ are connected by an edge with probability $q$, i.e., $P\{\{i, j\} \in E\} = q$ for $i \in C_a$ and $j \in C_b$.

Elementary derivations yield the expected degree $d_r$ of any node $i \in C_r$ belonging to cluster $C_r$ as

$$
\tilde{d}_r = E\{d_i\} = p(N_r - 1) + q(N - N_r).
$$

On the other hand, by similarly elementary calculations, the
expected cut-size $C_r := |\partial C_r|$ satisfies

$$C_r = qN_r(N - N_r).$$

(9)

Now consider a particular random walk $\mathcal{P}_j$ which is run in Algorithm 1. For a fixed node $i \in \mathcal{V}$, let $p_l(i)$ denote the probability that the random walk visits node $i$ in the $l$th step. A fundamental result in the theory of random walks over graphs states [8, page 159]

$$\lim_{l \to \infty} p_l(i) = \frac{d_i}{2|E|}$$

(10)

Thus, by running the random walks in Algorithm 1 sufficiently long (choosing $L$ sufficiently large), the probability that the delivered sampling set $\mathcal{M}$ contains a node $i \in C_r$ from cluster $C_r$ satisfies

$$P\{i \in \mathcal{M}\} \approx \frac{p(N_r - 1) + q(N - N_r)}{2|E|}.$$ 

(11)

Comparing (11) with (9) reveals that the sampling set delivered by Algorithm 1 indeed conforms with Lemma 1 which requires clusters with larger cut-size to be sampled more densely.

IV. NUMERICAL RESULTS

We tested the effectiveness of the sampling method given by Algorithm 1 was verified by applying it to different graph signals and using sparse label propagation (SLP) as the recovery method for obtaining the original graph signal from the samples. The SLP algorithm, derived in [3], is restated in Algorithm 2 for convenience. In Algorithm 2, we make use of the clipping operator $\tilde{T} : \mathbb{R}^{|E|} \to \mathbb{R}^{|E|}$ for edge signals defined element-wise as $(\tilde{T}(\mathbf{x}))[e] = (1/\max\{|\mathbf{x}[e]|, 1\})|\mathbf{x}[e]|$.

Table 1 shows that SLP is able to optimally recover the graph signal $x$ with the delivery of a sampling set $\mathcal{M}$ via Algorithm 1.

Algorithm 2 Sparse Label Propagation

Input: data graph $G$, sampling set $\mathcal{M}$, signal samples $\{x[i]\}_{i \in \mathcal{M}}$.

Initialize: $k := 0$, $D :=$ incidence matrix of $G$ for some arbitrary orientation, $z^{(0)} := 0$, $x^{(0)} := 0$, $\hat{x}^{(0)} := 0$, $y^{(0)} := 0$, maximum node degree $d_{\text{max}} := \max_{i \in \mathcal{V}} d_i$.

1: repeat

2: $y^{(k+1)} := \tilde{T}(y^{(k)} + (1/2\sqrt{d_{\text{max}}})Dz^{(k)})$

3: $r := x^{(k)} - (1/2\sqrt{d_{\text{max}}})DTy^{(k+1)}$

4: $x^{(k+1)} := \left\{ \begin{array}{ll}
\hat{x}^{(k)} & \text{for } i \in \mathcal{M} \\
r[i] & \text{else}
\end{array} \right.$

5: $z^{(k+1)} := 2x^{(k+1)} - x^{(k)}$

6: $\hat{x}^{(k+1)} := \hat{x}^{(k)} + x^{(k+1)}$

7: $k := k + 1$

8: until stopping criterion is satisfied

Output: $\hat{x}^{(k)} := (1/k)x^{(k)}$

Fig. 2. An APPM instance with 60 nodes and three clusters. Node colours represent the signal values.

Our numerical experiments involved $10^4$ independent simulation runs. Each simulation run is based on randomly generating an instance (see Figure 2) of the APPM for fixed parameter values $p = 3/10$, $q = 5/100$ and partition consisting of four clusters with sizes $|C_1| = 10$, $|C_2| = 20$, $|C_3| = 30$, $|C_4| = 40$ (cf. Section III). We then generated a clustered graph signal $x$ of the form (5) by choosing the cluster values $a_C$ as independent random variables $a_C \sim U(0, 1)$ (cf. (5)).

For each realization of the APPM, we constructed a sampling set $\mathcal{M}$ using Algorithm 1 which was then used to obtain the signal samples $\{x[i]\}_{i \in \mathcal{M}}$ and subsequently recovering the entire graph signal $x$ via Algorithm 2. We measured the recovery accuracy obtained by Algorithm 2 via the normalized empirical mean squared error (NMSE) of the signal estimate $\hat{x}$, i.e.,

$$\tilde{\varepsilon} := \frac{\left\| \hat{x}^{(l)} - x^{(l)} \right\|^2_2}{\left\| x^{(l)} \right\|^2_2}.$$ 

(12)

Here, $\varepsilon$, $x^{(l)}$ and $\hat{x}^{(l)}$ denote the NMSE, the original and the recovered graph signal, respectively, obtained in the $l$th simulation run. Note that $\varepsilon$ is random and often we are interested in its empirical mean

$$\bar{\varepsilon} := (1/10^4) \sum_{l=1}^{10^4} \tilde{\varepsilon}^{(l)}.$$ 

(13)

We evaluated the quality of the sampling set provided by Algorithm 1 for varying sampling budgets $M$ and a fixed length $L = 10$ of the random walks $\mathcal{P}_j$. In Table 1 we report the mean and standard deviation of the NMSE of $x$ for different sampling budgets $M$. Besides the expected
A real-world dataset on the Amazon website. For each product, it provides information such as the product name, price, and number of sales. The dataset consists of a collection of products purchased on the Amazon website. For each product, it provides a list of other products that are frequently co-purchased with it, as well as an average user rating.

The SLP algorithm was then applied for recovering the graph signal. This resulted in a mean NMSE of 0.332 ± 0.013 over 10 runs. For comparison, we also tested three graph clustering algorithms (also referred to as community detection algorithms) for selecting the sampling set. This comprised of first finding the partitioning of the nodes using the clustering algorithms and then randomly sampling from each cluster, where the number of samples in clusters was uniformly distributed according to the cut-size. For finding the clusters, we used an algorithm by Blondel et al. (also known as Louvain) [1], an algorithm by Newman [2], and one by Ronhovde et al. [3]. Choosing the sampling set via these methods and applying SLP for recovering the graph signal resulted in a NMSE of 0.369, 0.478, and 0.364 for the Louvain, Newman, and Ronhovde methods respectively (the value for the Ronhovde method is the average over 5 different clusterings corresponding to 5 values of its gamma parameter equally spaced between 0.1 and 0.5). We conclude that in this case our random walk method performs similarly to more computationally demanding clustering algorithms for sampling the graph signal.

### V. CONCLUSIONS

We proposed a novel random walk strategy for sampling graph signals representing massive datasets with intrinsic network structure. This strategy conforms with the rationale, which is supported by the recently derived network nullspace property, to sample more densely in clusters with large cut-size. The proposed sampling method has been tested on synthetic graph signals generated via an APPM. Our numerical experiments demonstrated that combining our sampling strategy with the SLP recovery algorithm, it is possible to recover graph signals with small error from only few samples. The effectiveness of our sampling strategy has been also verified numerically for graph signals obtained from a real-world dataset containing product rating information of an online retail shop.

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