NEW APPROACH TO OCTONIONS AND CAYLEY ALGEBRAS

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ABSTRACT We announce a new approach to the octonions as quasias-
associative algebras. We strip out the categorical and quasi-quantum group
considerations in our longer paper and present here (without proof) some of
the more algebraic conclusions.

1 INTRODUCTION

Usually one recognises the nonassociativity of the octonions by saying that
they are instead alternative algebras. While this is true, the property of
being alternative has a much weaker character than associativity and, as a
result, many standard ideas and constructions for associative algebras do not
go through in the alternative case. In our paper \cite{1}, to which this note is a
short introduction, we have introduced a full solution to this problem based
on modern ideas from category theory and Drinfeld’s theory of quasiquantum
groups.

Without going into any details (see \cite{1}), the new formulation is that the
octonions and other Cayley algebras live naturally as objects in a monoidal

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category. For any three objects $V, W, Z$ in such a category there is an associator isomorphism $\Phi_{V, W, Z} : (V \otimes W) \otimes Z \to V \otimes (W \otimes Z)$ which performs the rebracketting. Mac Lane’s pentagon condition on $\Phi$ ensures that we can do all constructions as if there are no brackets (i.e. as if $\otimes$ is strictly associative). After writing any desired constructions as the composition of various maps, we simply insert $\Phi$ as needed for the compositions to make sense, and all different ways to do this will give the same result (this is Mac Lane’s coherence theorem). So working in such a category is no harder than usual associative linear algebra. For example, an algebra $A$ in such a category means

\[ \bullet \circ (\bullet \otimes \text{id}) = \bullet \circ (\text{id} \otimes \bullet) \circ \Phi_{A, A, A} \]

for the product $\bullet$, where $\Phi$ is inserted for the bracketting to make sense. So, recognizing the octonions as such a quasiassociative algebra (or quasialgebra for short) makes them as good as associative in the precise sense explained above.

In [1] we introduce and study a class of such quasialgebras that contain composition algebras and more general algebras obtained by a generalised Cayley-Dickson process. All algebras are considered over a field $k$ of characteristic different from 2. The required class of quasialgebras arises naturally by a certain ‘Drinfeld twisting’ or deformation of classical group algebras, as follows.

## 2 QUASIALGEBRAS $k_FG$

First of all, we know that if we consider the set of complex numbers, the quaternions or the octonions, all these algebras have something in common: If we choose a suitable basis and remove the $\pm$ signs from the multiplication tables of these algebras, we have the tables of the additive groups $G = \mathbb{Z}_2$ (for complex numbers), $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ (for quaternions) and $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (for octonions). We view the signs in the multiplication tables as an invertible 2-cochain $F : G \times G \to k$ (a nowhere-zero function which is 1 when either argument is the group identity $e \in G$). Writing $F(x, y) = (-1)^{f(x, y)}$, one has explicitly [1].

\[ G = \mathbb{Z}_2, \quad f(x, y) = xy, \quad x, y \in \mathbb{Z}_2 \quad \text{(Complex numbers)}, \]

\[ G = (\mathbb{Z}_2)^2, \quad f(\bar{x}, \bar{y}) = x_1 y_1 + (x_1 + x_2) y_2 \quad \text{(Quaternions)}, \]

where $\bar{x} = (x_1, x_2) \in G$ is a vector notation and the components $x_1, x_2$ are viewed in the field $\mathbb{Z}_2$.

\[ G = (\mathbb{Z}_2)^3, \quad f(\bar{x}, \bar{y}) = \sum_{i \leq j} x_i y_j + y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3 \quad \text{(Octonions)}, \]
where $\vec{x} = (x_1, x_2, x_3) \in G$ is a vector notation. Similarly for higher Cayley algebras where $G = (\mathbb{Z}_2)^n$.

From the group $G$ and the cochain $F$ we recover the complex, quaternion and octonion algebras as the ‘deformation’ $k_F G$ of the group algebra of the appropriate $G$. This is the vector space with basis labeled by $G$ and the product

$$x \cdot y = F(x, y)xy$$

for $x, y \in G$, where $xy$ is the group product in $G$. So this is a kind of deformation of the usual group algebra of $G$. In quantum groups we do the deformation by introducing a parameter $q$ such that when $q$ tends to 1 we have the original algebras. Here we do the deformation by introducing a cochain $F$.

**PROPOSITION 1** [1] Let $G$ be a group and $F$ any invertible 2-cochain. Then $k_F G$ is a $G$-graded quasialgebra with associator $\Phi$ determined by the coboundary $\phi$ of $F$. Explicitly, it is quasiassociative in the sense

$$(x \cdot y) \cdot z = \phi(x, y, z)x \cdot (y \cdot z)$$

for all $x, y, z \in G$, and

$$\phi(x, y, z) = \frac{F(x, y)F(xy, z)}{F(y, z)F(x, yz)}.$$

To explain the setting here, for $G$ a group and $\phi : G \times G \times G \to k$ an invertible group 3-cocycle, the category of $G$-graded vector spaces becomes monoidal with the associator $\Phi$ determined by the 3-cocycle $\phi$ and the grading. A quasialgebra with this form of $\Phi$ is called a $G$-graded quasialgebra. It consists of an algebra $A$, a $G$-grading which respects the product and unit (so the degree of $1 \in A$ is $e \in G$, the group identity), and the quasiassociativity law

$$(a \cdot b) \cdot c = \phi(|a|, |b|, |c|)a \cdot (b \cdot c)$$

for all elements $a, b, c \in A$ of degree $|a|, |b|, |c|$. In our case, $k_F G$ is such a $G$-graded quasialgebra with $\phi$ built from $F$ and with $|x| = x$ for $x \in G$.

For the complex number and the quaternion algebras, $F$ is closed, i.e. $\phi$ is trivial and the algebras happen to be strictly associative. This is because $f$ in these cases is quadratic. As soon as we introduce a cubic or higher ‘interaction’ term in $f$, as in the case of the octonions, $\phi$ typically becomes nontrivial and the algebra $k_F G$ nonassociative. In the case of the Octonions it is

$$\phi(\vec{x}, \vec{y}, \vec{z}) = (-1)^{(|\vec{x} \times \vec{y}|) \cdot \vec{z}}$$

(the vector cross product and vector dot product in the exponent, i.e. the determinant $|\vec{x} \cdot \vec{y} \cdot \vec{z}|$). On the other hand, if we simply drop the cubic or
higher terms in the above family, we clearly obtain the corresponding Clifford algebra with negative signature $3$ (the relations are immediate and the dimensions match) i.e. these are obtained as $k_F G$ with

$$G = (\mathbb{Z}_2)^n, \quad f(x, y) = \sum_{i \leq j} x_i y_j \quad \text{(Clifford algebras)},$$

as the associative version of the octonion or Cayley algebra, which is another way to see the close relationship between these and Clifford algebras. The positive signature algebras are obtained similarly with $i < j$ in $f$.

Also, we are mainly interested in $G$ Abelian and specialise to this case from now on. For $\phi$ of the coboundary form, the category of $G$-graded spaces is symmetric, i.e. for any two objects $V, W$ there is a generalised transposition isomorphism $\Psi_{V,W} : V \otimes W \to W \otimes V$. A quasialgebra $A$ is quasicommutative if $\bullet = \bullet \circ \Psi_{A,A}$. This is the case for all $k_F G$ with $\Psi$ determined by a function $\mathcal{R}(x, y) = F(x, y)/F(y, x)$. Explicitly,

$$x \bullet y = \mathcal{R}(x, y)y \bullet x,$$

for all $x, y \in G$. For complex numbers, quaternions and octonions, etc., the function $\mathcal{R}$ has the simple form

$$\mathcal{R}(x, y) = \begin{cases} 1 & \text{if } x = e \text{ or } y = e \text{ or } x = y \\ -1 & \text{otherwise.} \end{cases}$$

We call this important case altercommutative. By contrast, for the Clifford algebras one has

$$\mathcal{R}(\vec{x}, \vec{y}) = (-1)^{\sum_{i \neq j} x_i y_j}$$

which is not in general altercommutative (for $n > 2$).

### 3 ALGEBRAIC PROPERTIES OF $k_F G$

We have just seen that the functions $\phi, \mathcal{R}$ built from $F$ allow for the categorical setting of the algebras $k_F G$ as quasiassociative and quasicommutative. In particular, the algebra is associative iff $\phi = 1$ and commutative iff $\mathcal{R} = 1$. We now summarise less obvious results expressing the more conventional algebraic properties of $k_F G$ in terms of these functions $\phi, \mathcal{R}$. We refer to [1] for proofs and details.

**PROPOSITION 2** [1] $k_F G$ is an alternative algebra iff

$$\phi^{-1}(y, x, z) + \mathcal{R}(x, y)\phi^{-1}(x, y, z) = 1 + \mathcal{R}(x, y)$$

\[\text{We would like to thank Tony Smith for asking us to clarify this point.}\]
\[ \phi(x, y, z) + R(z, y)\phi(x, z, y) = 1 + R(z, y) \]
for all \( x, y, z \in G \). In this case,
\[ \phi(x, x, y) = \phi(x, y, y) = \phi(x, y, x) = 1 \]
for all \( x, y \in G \).

Next we consider involutions. Since we have a special basis of \( k_F G \) it is natural to consider involutions diagonal in this basis.

**Proposition 3** [1] \( k_F G \) admits an involution which is diagonal in the basis \( G \) iff \( R(x, y) = s(x)s(y) \) for some 1-cochain \( s : G \to k \) (a nowhere-zero function with \( s(e) = 1 \)) obeying \( s^2 = 1 \). In this case, one has \( R(x, y) = R(y, x) \) and \( \phi(x, y, z) = \phi(z, y, x)^{-1} \) for all \( x, y, z \in G \).

The corresponding involution here is \( \sigma(x) = s(x)x \) for all \( x \in G \). Let \( A \) be a finite dimensional algebra with identity element 1 and let \( \sigma \) be an involution in \( A \). We say that \( \sigma \) is a strong involution if \( a + \sigma(a), a \bullet \sigma(a) \in k1 \) for all \( a \in A \).

**Proposition 4** [1] \( k_F G \) admits a diagonal strong involution \( \sigma \) iff
i) \( G \cong (\mathbb{Z}_2)^n \) for some \( n \),
ii) \( \sigma(e) = e, \sigma(x) = -x \) for all \( x \neq e \),
iii) \( k_F G \) is altercommutative.

Given \( k_F G \) we have a natural function \( s(x) = F(x, x) \) and consider now the possibility of defining a strong involution using this. For all statements of simplicity in the following we assume \( |G| > 2 \).

**Proposition 5** [1] If \( \sigma(x) = F(x, x)x \) for all \( x \in G \) defines a strong involution, then the algebra \( k_F G \) is simple and the following are equivalent
i) \( k_F G \) is an alternative algebra,
ii) \( k_F G \) is a composition algebra.

Finally, it is possible to characterize a natural class of \( k_F G \) algebras that are composition algebras,

**Proposition 6** [1] Let \( k_F G \) admit a strong diagonal involution \( \sigma(x) = s(x)x \). Then \( q(x) = x \bullet \sigma(x) \) makes \( k_F G \) a composition algebra iff
i) \( s(xy)F(x, y)^2F(xy, xy) = s(x)s(y)F(x, x)F(y, y) \), for all \( x, y \in G \).
ii) \( F(x, xz)F(y, yz)F(z, z)s(z) + F(x, yz)F(y, xz)F(xyz, xyz)s(xyz) = 0 \),
for all \( x, y \in G \) with \( x \neq y \).
An important corollary of the last result is:

COROLLARY 7 If $G \cong (\mathbb{Z}_2)^n$ then the Euclidean norm quadratic function defined by $q(x) = 1$ for all $x \in G$ makes $k_F G$ a composition algebra iff

i) $F^2(x, y) = 1$ for all $x, y \in G$

ii) $F(x, xz)F(y, yz) + F(x, yz)F(y, xz) = 0$, for all $x, y, z \in G$ with $x \neq y$.

In this case $\sigma(x) = F(x, x)x$ for all $x \in G$ is a strong involution and $k_F G$ is simple and alternative.

4 CAYLEY-DICKSON PROCESS FOR $k_F G$

We have a generalisation of the Cayley-Dickson process as follows. Again, details are in [1].

DEFINITION 8 Let $G$ be an Abelian group $F$ a 2-cochain on it. For any 1-cochain $s : G \to k$ and $\alpha \neq 0$ we define $G = G \times \mathbb{Z}_2$ and on it the 2-cochain $F$ and 1-cochain $\bar{s}$,

$$F(x, y) = F(x, y), \quad F(x, vy) = s(x)F(x, y), \quad F(vx, y) = F(y, x),$$

$$\bar{F}(vx, vy) = \alpha s(x)F(y, x), \quad \bar{s}(x) = s(x), \quad \bar{s}(vx) = -1$$

for all $x, y \in G$. Here $x \equiv (x, e)$ and $vx \equiv (x, \nu)$ denote elements of $G$, where $\mathbb{Z}_2 = \{e, \nu\}$ with product $\nu^2 = e$.

We say that $k_F \bar{G}$ is the generalised Cayley-Dickson extension of $k_F G$ associated to $s, \alpha$. The motivation is that if $\sigma(x) = s(x)x$ is a strong involution, then $k_F \bar{G}$ is the usual Cayley-Dickson extension of $k_F G$ associated to $\sigma, \alpha$. Note that since all unital composition algebras over $k$ are obtained by repeated Cayley-Dickson extension[6], they are all of the form of a quasialgebra $k_F G$ in the last proposition of the preceding section with $G$ a power of $\mathbb{Z}_2$.

The natural application for our generalised Cayley-Dickson process is when $k_F G$ admits a diagonal involution $\sigma(x) = s(x)x$ (but not necessarily strong). In this case we have:

PROPOSITION 9 The 3-cocycle $\tilde{\phi}$ of $k_F \bar{G}$ is given by

$$\tilde{\phi}(x, y, z) = \phi(x, y, z), \quad \tilde{\phi}(vx, y, z) = R(y, z)\phi(x, y, z),$$

$$\tilde{\phi}(x, vy, z) = R(y, z)R(xy, z)\phi(x, y, z), \quad \tilde{\phi}(x, y, vz) = R(x, y)\phi(x, y, z),$$

$$\tilde{\phi}(vx, vy, z) = R(xy, z)\phi(x, y, z), \quad \tilde{\phi}(vx, y, vz) = R(y, z)R(x, y)\phi(x, y, z),$$

$$\tilde{\phi}(x, vy, vz) = R(xy, z)\phi(x, y, z), \quad \tilde{\phi}(vx, vy, vz) = R(xy, z)R(x, y)\phi(x, y, z),$$

for $x, y, z \in G$. 
Using this calculation, one may show under the same assumptions:

**PROPOSITION 10**

i) $k_F \bar{G}$ is associative iff $k_F G$ is associative and commutative.

ii) If $k_F G$ has trivial centre then $k_F \bar{G}$ is alternative iff $k_F G$ is associative and $s(x) = -1$ for all $x \in G$ and $x \neq e$.

iii) If $s$ defines a strong diagonal involution then $k_F \bar{G}$ is alternative iff $k_F G$ is associative.

This extends to more general $k_F G$ some well-known considerations for octonions and higher Cayley algebras.

5 CONCLUDING REMARKS

We conclude with some remarks about other classes of quasialgebras. In fact, the input data for our $k_F G$ construction is clearly very general. If we denote the elements of the finite group $G$ by $\{x_1 = e, x_2, \ldots, x_n\}$ then we can represent the cochain $F$ by an $n \times n$ matrix with entries $F_{ij} = F(x_i, x_j)$.

It has 1 in the first row and column and all entries non-zero. Conversely, any such matrix will do for a cochain and yield a quasialgebra. Therefore it is a wide-open question what other groups and cochains might be interesting; here we list just a few natural classes.

First of all, motivated by composition algebras for the Euclidean norm (isomorphic to complex, quaternions or octonions), where the cochain is represented by certain normalized Hadamard matrices, a natural more general class of examples is $k_F G$ where $F$ is a general normalized Hadamard matrix. Some results for (and low-dimensional examples of) this kind of $k_F G$ are given in [1]. Hadamard matrices have even dimension but odd dimensional examples of $k_F G$ can be obtained by taking $F$ with first column and row 1 and the rest an Hadamard matrix.

Another general choice of $F$, overlapping with the extended Hadamard case, is (for $G$ any group),

$$F(x, y) = \begin{cases} 
1 & \text{if } x = e \text{ or } y = e \text{ or } x \neq y \\
-1 & \text{otherwise}
\end{cases}$$

where one may show that $k_F G$ is simple, commutative and in general nonassociative.

Other interesting examples of $k_F G$ quasialgebras come from the theory of finite fields, and include the generalisations of octonions based on Galois sequences in [5].

Finally, we would also like to recall that our approach answers such questions as what means a representation of the octonions. Following the method
explained in the introduction, a representation of the quasialgebra $k_FG$ means a $G$-graded vector space $V$ and a degree-preserving action $\triangleright$ obeying

$$(x \bullet y) \triangleright v = \phi(x, y, |v|)x \triangleright (y \triangleright v)$$

for all $v \in V$. This is explained in [1], where it is also shown that a representation is equivalent to an algebra map from $k_FG$ to a certain quasialgebra $\text{End}_\phi(V)$ of quasimatrices associated to any $V$.

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