NEW ENERGY METHOD IN THE STUDY OF THE INSTABILITY
NEAR COUETTE FLOW

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Abstract. In this paper, we provide a new energy method to study the growth mechanism
of unstable shear flows. As applications, we prove two open questions about the instability
of shear flows. First, we obtain the optimal stability threshold of the Couette flow for
Navier-Stokes equations with small viscosity $\nu > 0$, when the perturbations are in critical
spaces. More precisely, we prove the instability for some perturbation of size $\nu^{\frac{1}{2}-\delta_0}$ with
any small $\delta_0 > 0$, which implies that $\nu^{\frac{1}{2}}$ is the sharp stability threshold. Second, we study
the instability of the linearized Euler equations around shear flows that are near Couette
flow. We prove the existence of a growing mode for the corresponding Rayleigh operator
and give a precise location of the eigenvalues. We think our method has a lot of possible
other applications.

1. Introduction

Hydrodynamic stability is an active field of fluid mechanics which deals with the stability
and the instability of fluid flows. Mathematically, this kind of problems can be reduced to
the study of the semi-group estimate related to the linearized operator. On one hand, a
classical way to obtain the semi-group estimates is to study the resolvent, eigenvalues, or
pseudospectrum. Also, additional arguments are required if the linearized operator is time-
dependent, such as freezing the coefficient in time. On the other hand, a well designed energy
method is usually helpful to obtain upper bound estimates of the semi-group. However, it
is difficult to use the energy method to capture the growth of the semi-group and obtain its
lower bound, especially if the growth is a transient growth. In this paper, we provide a new
energy method to study the lower bound of the semi-group generated by a time-dependent
linearized operator. It allows us to study the instability of shear flows for Navier-Stokes and
Euler equations.

1.1. Two dimensional Navier-Stokes and Euler equations. We consider the two di-
mensional incompressible Navier-Stokes and Euler equations in $\Omega = \mathbb{T} \times \mathbb{R}$:

\begin{equation}
\begin{aligned}
\partial_t U + U \cdot \nabla U + \nabla P - \nu \Delta U &= 0, \\
\nabla \cdot U &= 0,
\end{aligned}
\end{equation}

where $\nu \geq 0$ denotes the viscosity. We denote by $U = (U^{(1)}, U^{(2)})$ the velocity and $P$ the
pressure. Let $W = -\partial_y U^{(1)} + \partial_x U^{(2)}$ be the vorticity, which satisfies

$$\partial_t W + U \cdot \nabla W - \nu \Delta W = 0.$$ 

Let $b(t, y)$ solve the heat equation:

\begin{equation}
\begin{aligned}
\partial_t b(t, y) - \nu \partial_{yy} b(t, y) &= 0, \\
b(0, y) &= b_0(y).
\end{aligned}
\end{equation}

Then the shear flow $(b(t, y), 0)$ is a solution of (1.1) with vorticity $W = -\partial_y b$. For the Euler
case $(\nu = 0)$, the shear flows $(b(y), 0)$ are steady solutions to Euler equations. The special
case, \( b(t, y) = y \), the Couette flow \((y, 0)\) is a steady solution of (1.1) with \( W = -1 \) for both \( \nu > 0 \) and \( \nu = 0 \).

In this paper, we focus on the (in)stability of the shear flow \((b(t, y), 0)\). It is natural to introduce the perturbation. Let \( u = (u^{(1)}, u^{(2)}) = U - (b(t, y), 0) \) and \( \omega = W - (\partial_y b) \), then \( \omega \) satisfies
\[
\begin{align*}
\partial_t \omega + b(t, y) \partial_y \omega - \nu \Delta \omega - \partial_{yy} b(t, y) u^{(2)} &= -u \cdot \nabla \omega, \\
u &= -\nabla \cdot (-\Delta)^{-1} \omega = (\partial_y (-\Delta)^{-1} \omega, -\partial_x (-\Delta)^{-1} \omega), \\
\omega|_{t=0} &= \omega_{in}.
\end{align*}
\]

If \( b(t, y) = y \), the equation is simpler:
\[
\begin{align*}
\partial_t \omega + y \partial_y \omega - \nu \Delta \omega &= -u \cdot \nabla \omega, \\
u &= -\nabla \cdot (-\Delta)^{-1} \omega = (\partial_y (-\Delta)^{-1} \omega, -\partial_x (-\Delta)^{-1} \omega), \\
\omega|_{t=0} &= \omega_{in}.
\end{align*}
\]

We also introduce the linearized equation:
\[
\begin{align*}
\partial_t \omega + b(t, y) \partial_y \omega - \nu \Delta \omega - \partial_{yy} b(t, y) u^{(2)} &= 0, \\
u &= -\nabla \cdot (-\Delta)^{-1} \omega = (\partial_y (-\Delta)^{-1} \omega, -\partial_x (-\Delta)^{-1} \omega), \\
\omega|_{t=0} &= \omega_{in},
\end{align*}
\]
and if \( b(t, y) = y \)
\[
\begin{align*}
\partial_t \omega + y \partial_y \omega - \nu \Delta \omega &= 0, \\
\psi &= \Delta^{-1} \omega, \\
\omega|_{t=0} &= \omega_{in},
\end{align*}
\]

where \( \psi \) is the stream function.

1.2. Historical comments. The field of hydrodynamic stability started in the nineteenth century with Stokes, Helmholtz, Reynolds, Rayleigh, Kelvin, Orr, Sommerfeld and many others. The study of (in)stability of shear flows dates back to Rayleigh [57], Kelvin [42], and Sommerfeld [60]. Kelvin considered the linearized system (1.6). Indeed, if we denote by \( \hat{\omega}(t, k, \eta) \) the Fourier transform of \( \omega(t, x, y) \), then the solution of (1.6) can be written as
\[
\hat{\omega}(t, k, \eta) = \hat{\omega}_{in}(k, \eta + kt) \exp \left( -\nu \int_0^t |k|^2 + |\eta - ks + kt|^2 ds \right),
\]
\[
\hat{\psi}(t, k, \eta) = -\frac{\hat{\omega}_{in}(k, \eta + kt)}{k^2 + \eta^2} \exp \left( -\nu \int_0^t |k|^2 + |\eta - ks + kt|^2 ds \right),
\]
which gives that
\[
\begin{align*}
\|\partial_y P_{\#} \hat{\psi}\|_{L^2} + \langle t \rangle \|\partial_x P_{\#} \hat{\psi}\|_{L^2} &\leq C(t)^{-1} \|P_{\#} \hat{\omega}_{in}\|_{H^2}, \\
\|P_{\#} \hat{\omega}\|_{L^2} &\leq C \|P_{\#} \hat{\omega}_{in}\|_{L^2} e^{-\nu t^3},
\end{align*}
\]
where here we denote by \( P_{\#} f = f(x, y) - \frac{1}{2\pi} \int_{\mathbb{T}} f(x, y) dx \) the projection to the nonzero mode of \( f \). The first inequality in (1.8) is the inviscid damping and the second one is the enhanced dissipation. These two results are both related to the vorticity mixing effect.

Indeed, in [56], Orr observed an important phenomenon that the velocity tends to 0 as \( t \to \infty \). This phenomenon is called inviscid damping, which is the analogue in hydrodynamics of Landau damping found by Landau [13], which predicted the rapid decay of the electric field of the linearized Vlasov equation around homogeneous equilibrium. Mouhot and Villani [55] made a breakthrough and proved nonlinear Landau damping for the perturbation in Gevrey class (see also [10]). For the inviscid damping, the mechanism leading to the damping
is the vorticity mixing driven by shear flow or Orr mechanism \[50\]. See \([51, 58, 59]\) for similar phenomena in various systems. The nonlinear inviscid damping was first proved by Bedrossian and Masmoudi \[9\] for the perturbations in the Gevrey-\(m\) class \((1 \leq m < 2)\). We also refer to \([38, 36]\) and references therein for other related interesting results.

Due to the presence of the nonlocal term, the inviscid damping for general shear flows is a challenging problem even at the linear level. In the case of finite channel, Case \[15\] gave a formal proof of \(t^{-1}\) decay for the velocity. Lin and Zeng \[49\] gave the optimal linear decay estimates of the velocity for data in Sobolev spaces. Zillinger \[68\] proved the linear inviscid damping for a class of monotone shear flows which are close to Couette flow. Wei, Zhang and Zhao \[65\] proved the linear inviscid damping for general monotone shear flows. We also refer to \([40, 39]\) for a simplified proof and the linear inviscid damping in Gevrey class. For non-monotone flows such as the Poiseuille flow and the Kolmogorov flow, another phenomenon should be taken into consideration, which is the so-called vorticity depletion phenomenon, predicted by Bouchet and Morita \[14\] and later proved by Wei, Zhang and Zhao \[66, 67\]. See also \([5, 37, 50]\) for similar phenomena in vortex dynamics and MHD.

Very recently, Ionescu-Jia \[35\], and Masmoudi-Zhao \[53\] proved that the nonlinear inviscid damping holds for general linear stable monotone shear flows.

The instability of shear flow \((b(y), 0)\) is also well-studied for the Euler equation. Rayleigh \[57\] proved that if the shear flow \((b(y), 0)\) is linearly unstable, then \(b''(y)\) must change sign. Howard \[33\] proved the semicircle theorem which describe the possible location of eigenvalues. Lin \[47\] proved that if the shear flow \((b(y), 0)\) is in some class then this shear flow is unstable. We refer to \([13, 27, 29, 47, 61]\) for the instability results of different shear flows. All the results are obtained by studying the Rayleigh equations. For the asymptotic instability, Lin and Zeng \[49\] proved that nonlinear inviscid damping is not true for perturbations of the Couette flow in \(H^s\) \((s < \frac{3}{2})\). Deng and Masmoudi \[23\] proved some instability for initial perturbations in Gevrey-\(m\) class \((m > 2)\). We also refer to \([24, 25]\), where the instability of some toy models related to linearized Euler equations were studied.

The second phenomenon is enhanced dissipation. The decay rate is much faster than the diffusive decay rate of \(e^{-\nu t}\). The mechanism leading to the enhanced dissipation is also due to vorticity mixing. Generally speaking, the sheared velocity sends information to higher frequency, enhancing the effect of the diffusion. We also refer to \([1, 4, 20, 63, 32, 44]\).

For the nonlinear system, the Orr mechanism is known to interact with the nonlinear term due to the transient growth, creating a weakly nonlinear effect referred to as echoes. The basic mechanism is as follows: a mode which is near its critical time is creating most of the velocity field and at this time it can interact with a second wave to put energy in a mode which did not reach its critical time yet. When this third mode reaches its critical time, the result of the nonlinear interaction becomes very strong due to the amplification (the time delay explains the terminology ‘echo’). There are two ways to make sure the echo cascades are under control. One is to assume enough smallness of the initial perturbations such that the rapid growth of the enstrophy may not happen before the enhanced dissipative time-scale \(\nu^{-\frac{3}{2}}\). The other is to assume enough regularity (Gevrey class) of the initial perturbations such that one can pay enough regularity to control the growth caused by the echo cascade.

The following mathematical version of transition threshold problem was formulated by Bedrossian, Germain and Masmoudi \[7\]:

**Given a norm \(\|\cdot\|_X\), find a \(\beta = \beta(X)\) so that**

\[
\|\omega_{in}\|_X \leq \nu^{\beta} \Rightarrow \text{stability},
\]

\[
\|\omega_{in}\|_X \gg \nu^{\beta} \Rightarrow \text{instability}.
\]
One can reformulate this in terms of nonlinear enhanced dissipation and inviscid damping which yield asymptotic stability:

1. Given a norm \( \| \cdot \|_X (X \subset L^2) \), determine a \( \beta = \beta(X) \) so that if the initial vorticity satisfies \( \| \omega_{in} \|_X \ll \nu^\beta \), then for \( t > 0 \)

\[
\| \omega(x,y,t) \|_{L^2_{x,y}} \leq C \| \omega_{in} \|_X e^{-c \nu^{1/3} t} \quad \text{and} \quad \| u(x,y,t) \|_{L^2_{x,y}} \leq C \| \omega_{in} \|_X,
\]

hold for the Navier-Stokes equation (1.4).

2. Given \( \beta \), is there an optimal function space \( X \subset L^2 \) so that if the initial vorticity satisfies \( \| \omega_{in} \|_X \ll \nu^\beta \), then (1.9) holds for the Navier-Stokes equation (1.4)?

We summarize the results for Couette flow in different domains in the following tables:

**Table 1. 2D Couette flow**

| Space   | \( \beta \) | Boundary | Reference |
|---------|-------------|----------|-----------|
| \( H_x^\log L_y^2 \) | \( > \frac{1}{2} \) | No | [12, 52] |
| \( H_L \) | \( \geq \frac{1}{2} \) | No | [54] |
| \( G^2 \) | 0 | No | [11] |
| \( H^1 \) | \( \geq \frac{1}{2} \) | Non-slip | [17] |

**Table 2. 3D Couette flow**

| Space   | \( \beta \) | Boundary | Reference |
|---------|-------------|----------|-----------|
| \( H^L \) | \( \geq 1 \) | No | [6, 7, 8, 64] |
| \( H^1 \) | \( \geq 1 \) | Non-slip | [19] |

Let us also mention some other recent progress on the stability problem of different types of shear flows in different domains:

- 2D Kolmogorov flow: [48, 34, 67]
- 2D Poiseuille flow: [21, 22, 26]
- 2D monotone shear flow: [31]
- 3D Kolmogorov flow: [45]
- 3D pipe Poiseuille flow: [18]

Let us first introduce a corollary of the results in [52]:

**Corollary 1.1.** Let \( \nu > 0 \) be small enough, there exist \( \varepsilon_0 > 0 \) independent of \( \nu \), such that for every shear flow \((b(t,y),0)\) solving \( \partial_t b - \nu \partial_{yy} b = 0 \), with \( b(0,y) = b_{in}(y) \) satisfying

\[
\| b_{in}(y) - y \|_{L^\infty \cap H^1} \leq \varepsilon_0 \nu^{1/2},
\]

the linear and nonlinear enhanced dissipation holds for the shear flow \((b(t,y),0)\).

More precisely, for every \( \omega_{in}(x,y) \) the solution of the linear system (1.5) with initial data \( \omega_{in} \) satisfies:

\[
\| \omega(t) \|_{L^2_{x,y}} \leq C \| \omega_{in} \|_{H_x^\log L_y^2} e^{-c \nu^{1/3} t}.
\]

If the initial data satisfies \( \| \omega_{in} \|_{H^1_x L_y^2} \leq \varepsilon_0 \nu^{1/2} \), then the solution of the nonlinear system (1.3) with initial data \( \omega_{in} \) satisfies:

\[
\| \omega(t) \|_{L^2_{x,y}} \leq C \| \omega_{in} \|_{H_x^\log L_y^2} e^{-c \nu^{1/3} t}.
\]

Corollary 1.1 seems stronger than the results in [52], as the Couette flow is the special case with \( b_{in}(y) - y = 0 \). However, both results are equivalent. The main reason is that the difference between shear flow \((b(t,y),0)\) and Couette flow is of the same size as the perturbation \( \omega_{in} \). There is no essential difference between linearizing around Couette flow and around the shear flow \((b(t,y),0)\). However, if the difference between the shear flow \((b(t,y),0)\) and the Couette
flow is slightly larger, then the linearization around Couette flow is not accurate any more. Instead of studying (1.6), it is better to study (1.5), which can be regarded as a secondary linearization, see [3, 28, 46] for some linear instability results. Due to the dissipation effect, such linear growth is a transient growth, which could trigger nonlinear instability, and leads to the transition to turbulence [62, 41]. The secondary linearization gives a possible resolution to the Couette-Sommerfeld paradox: the linearized system around the Couette flow is stable for all Reynolds numbers [60], however experiments show that any small perturbation to the linear shear can lead to the transition from the shear flow to turbulence when the Reynolds number is large enough.

1.3. Main results. In this paper, we focus on the instability of shear flows. A usual way to study the instability is to prove the existence of growing mode, see [47]. Here, we introduce a new energy method to capture the growth.

1.3.1. Optimal instability of Couette flow in critical space. As the first application, we consider the viscous flow and study the optimality of the size of the initial perturbation. More precisely, we consider the case $X = H_x^1 L_y^2$ and show that $\nu^\frac{1}{2}$ is the stability threshold, i.e.,

$$||\omega_{in}||_{H_x^1 L_y^2} \gg \nu^\frac{1}{2} \Rightarrow \text{instability}.$$  

Our main result states as follows:

**Theorem 1.2.** Let $\nu > 0$ be small enough. For any small $\delta_0 > 0$, there exist $M > 0$ independent of $\nu$ and shear flows $(b_\nu(t, y), 0)$ such that $\partial_t b_\nu - \nu \partial_y b_\nu = 0$, with $b_\nu(0, y) = b_{in}(y)$ satisfying

$$\|b_{in}(y) - y\|_{L^\infty \cap H^1} \approx M \nu^{1-\delta_0}$$

and the linear and nonlinear enhanced dissipation (1.9) fail for the shear flow $(b_\nu(t, y), 0)$.

More precisely, there exist $\omega_{in}(x, y)$ with $\int_T \omega_{in}(x, y) dx = 0$ such that the solution of the linear system (1.5) with initial data $\omega_{in}$ satisfies:

for $t \in [0, T]$ with $T = \varepsilon_1 \nu^{-\frac{1}{3} + \delta_0 \ln(\nu^{-1})}$ and $\varepsilon_1 > 0$ small enough depending on $\delta_0$ and independent of $\nu$,

$$C \|\omega_{in}\|_{H_x^1 L_y^2} e^{C \nu^{\frac{1}{3} - \frac{\delta_0}{4}t}} \geq \|\omega_{\pm 1}(t)\|_{L_x^2 y} \geq c \|\omega_{in}\|_{H_x^1 L_y^2} e^{C \nu^{\frac{1}{3} - \frac{\delta_0}{4}t}}.$$  

If the initial data $\omega_{in}(x, y)$ satisfies $\int_T \omega_{in}(x, y) dx = 0$ and $\|\omega_{in}\|_{H_x^1 L_y^2} \approx \varepsilon_0 \nu^{\frac{1}{3} + \delta_1}$, then the solution of the nonlinear system, (1.3) with initial data $\omega_{in}$ satisfies:

for $t \in [0, T]$ with $T = \varepsilon_1 \nu^{-\frac{1}{3} + \delta_0 \ln(\nu^{-1})}$ and $\varepsilon_1 > 0$ small enough depending on $\delta_0, \delta_1$ and independent of $\nu$,

$$C \|\omega_{in}\|_{H_x^1 L_y^2} e^{C \nu^{\frac{1}{3} - \frac{\delta_0}{4}t}} \geq \|\omega_{\pm 1}(t)\|_{L_x^2 y} \geq c \|\omega_{in}\|_{H_x^1 L_y^2} e^{C \nu^{\frac{1}{3} - \frac{\delta_0}{4}t}}.$$  

Here $c > 0$ is a constant independent of $t, \nu, \varepsilon_1$ and $f_{\pm 1}(x, y) = \frac{1}{2\pi} \int_T f(x, y) e^{-i\pm x} dx e^{\pm i x}$ is the $\pm 1$ Fourier mode.

In particular at $t = T$, for both cases,

$$\|\omega_{\pm 1}(T)\|_{L_x^2 y} \geq \frac{c}{\varepsilon_1} \|\omega_{in}\|_{H_x^1 L_y^2}.$$  

**Remark 1.3.** We emphasize that the constants $c$ and $\varepsilon_1$ in (1.13) are independent of $\nu$. Since we consider the small viscosity problem, at time $T$, the amplification can be made as large as we want by taking the limit $\nu \to 0_+$. 

NEW ENERGY METHOD IN THE STUDY OF THE INSTABILITY NEAR COUETTE FLOW 5
Indeed, the \( L_ω \) modes. We track the evolution of \( g \) grows exponentially in time, and the enstrophy transfers from the zero mode to the nonzero modes. In Theorem 1.2 we find a flow that gradually deviates from a shear flow.

Remark 1.5. In Theorem 1.2 we find a flow that gradually deviates from a shear flow. Indeed, the \( L^2 \) norm of the nonzero mode of the total vorticity \( P_\neq W(t,x,y) = P_\neq ω(t,x,y) \) grows exponentially in time, and the enstrophy transfers from the zero mode to the nonzero modes. We track the evolution of \( ω(t,x,y) \) till \( T = ε_1 ν^{-\frac{3}{4} + \frac{1}{2} δ_0} ln(ν^{-1}) \). Due to the enstrophy conservation law for the Couette flow i.e. \( ||W(t) + 1||_{L^2} ≤ ||W(0) + 1||_{L^2} \), the exponential growth should stop at some finite time for the nonlinear problem. The growing time \( T \) in Theorem 1.2 is optimal in terms of \( ν \) up to a constant.

After time \( T \), there are two possibilities. One is that the nonzero modes decay back to 0, the flow first approaches to a shear flow and then to the Couette flow. The second one is that the nonzero modes do not decay immediately. One may expect that the laminar flow transits to turbulence or that the flow forms some cat’s eyes structure. In both cases, due to the dissipation effect, the flow will approach to the Couette flow as \( t \to +∞ \).

1.3.2. Instability of the inviscid flow. The new energy method was developed for the viscous problem but also works for the inviscid problem. We study the linearized Euler equation around the shear flow \( (b_0(y),0) \):

\[
∂_t ω + R_{M,γ} ω = 0,
\]

where

\[
b_0(y) = \int_0^y 1 + 2 √π M γ e^{-\frac{y'^2}{γ^2}} dy',
\]

and

\[
R_{M,γ} = b_0(y) ∂_x - b''_0(y) ∂_x (Δ)^{-1}
\]

is the Rayleigh operator. We have the following instability result.

**Theorem 1.6.** Let \( M_0 > 0 \) be big enough. For each \( M ≥ M_0 \) there exists \( 0 < γ_0 = γ_0(M) \) such that for \( 0 < γ ≤ γ_0 \), the Rayleigh operator \( R_{M,γ} \) has an unstable eigenvalue \( λ = λ_r + i λ_i \) such that \( -CMγ ≤ λ_r < -γ \) and \( |λ_i| ≤ 4γ √(ln(γ^{-1})) \). As a consequence, there exists \( ω_{in}(x,y) ∈ L^2_{x,y} \) such that for \( ∀ t > 0 \)

\[
∥e^{-tR_{M,γ}ω_{in}}∥_{L^2_{x,y}} ≥ C^{-1}e^{γt}∥ω_{in}∥_{L^2_{x,y}}.
\]

Here the constant \( C > 0 \) is independent of \( M \) and \( γ \).
Remark 1.7. The inviscid growing mode of Theorem 1.6 disappears for the Navier Stokes dynamic. Indeed, let $\mathcal{R}_{M,\gamma,\nu} = \mathcal{R}_{M,\gamma} - \nu \Delta = b_0(y) \partial_x - b_0'(y) \partial_x(\Delta)^{-1} - \nu \Delta$ with any small viscosity $\nu > 0$, then $\mathcal{R}_{M,\gamma,\nu}$ has no eigenvalues and we have

$$
(1.18) \quad \| P_k e^{-t \mathcal{R}_{M,\gamma,\nu}} \omega_{\text{in}} \|_{L^2_{x,y}} \leq C e^{-C k^2 \nu t^3 + CM(\gamma^2 + |k| \gamma^2) t} \| P_k \omega_{\text{in}} \|_{L^2_{x,y}},
$$

where $P_k f(x,y)$ is the projection to the $k$th mode of $f(x,y)$ in $x$ variable. The proof can be found in Appendix B.

It is an interesting project to study the long time behavior of solutions of the corresponding linear and nonlinear problems.

Remark 1.8. In [47], Lin proved the existence of (un)stable eigenvalues for the shear flow in the $K^+$ class (see [47] for the definition) with additional spectral assumption on the corresponding Schrödinger type operator. By our energy method, to obtain the existence of a growing mode, it is not necessary to check the spectral assumption.

Remark 1.9. For the nonlinear instability, we refer to [29]. We remark that such nonlinear instability also holds for the shear flow $(b_0(y), 0)$.

Notations: Through this paper, we will use the following notations. We use $C$ (or, $c$) to denote a positive big (or, small) enough constant which may be different from line to line. We also use $C_M$ (or, $c_M$) to emphasize that such constant depends on a variable $M$.

We use $f \lesssim g$ ($f \approx g$) to denote $f \leq C g$ ($C^{-1} g \leq C f$).

Given a function $f(t,y)$, we denote its derivation in $y$ by $f'(t,y) = \partial_y f(t,y)$, $f''(t,y) = \partial_y^2 f(t,y)$, and denote its Fourier transform in $x$ by

$$
\hat{f}(k,y) = \frac{1}{2\pi} \int_T f(x,y) e^{-ikx} dx,
$$

and denote its Fourier transform in $(x,y)$ by

$$
\hat{f}(k,\xi) = \frac{1}{4\pi^2} \int_T \int_R f(x,y) e^{-ikx} e^{-i\xi y} dy dx.
$$

We denote the projection to the $k$th mode of $f(x,y)$ by

$$
P_k f(x,y) = f_k(x,y) = \frac{1}{2\pi} \left( \int_T f(x',y) e^{-ikx'} dx' \right) e^{ikx},
$$

and denote the projection to the non-zero mode by

$$
P_{\neq} f(x,y) = f_{\neq}(x,y) = \sum_{k \in \mathbb{Z} \setminus 0} f_k(x,y).
$$

We also use $\hat{f}(k,\xi)$ to denote $\hat{f}(k,\xi)$ to emphasize it is the Fourier transform of the $k$ mode. For a function $f(t, x, y)$ we introduce the following function spaces which are of the same spirit as the Chemin-Lerner’s Besov space [16],

$$
\| f(t) \|_{F_{L^1} L^q_y} = \sum_{k \in \mathbb{Z}} \left( \int_R |\hat{f}_k(t, y)|^q dy \right)^{1/q},
$$
and
\[
\|f\|_{L^p([0,T],L^q_y)}^p = \sum_{k \in \mathbb{Z}} \left( \int_0^T \left( \int_\mathbb{R} |\tilde{f}_k(t,y)|^q dy \right)^{\frac{p}{q}} dt \right)^\frac{1}{p}.
\]

2. Main ideas and sketch of the proof

In this section, we present the main ideas of the new energy method and the proof of instability. We study the linearized system around a shear flow \((b_\nu(t,y),0)\) where
\[
b'_\nu(t,y) = 1 + 2\sqrt{\frac{\pi M \gamma^2}{4\nu t + \gamma^2}} e^{-\frac{y^2}{4\nu t + \gamma^2}}, \quad b_\nu(t,0) = 0.
\]

(2.1)

It is easy to check that \(b_\nu(t,y)\) solves (1.2) with initial data
\[
b_{in}(y) = \int_0^y 1 + 2\sqrt{\pi M \gamma e^{-\frac{y'^2}{4\nu t + \gamma^2}}} dy'.
\]

For the viscous problem \((\nu > 0)\), \(b_\nu(t,y)\) varies with time. And for the inviscid problem \((\nu = 0)\), we study the time-independent shear flow \((b_0(y),0)\) with
\[
b'_0(y) = 1 + 2\sqrt{\pi M \gamma} e^{-\frac{y'^2}{4\nu t + \gamma^2}}, \quad b_0(0) = 0.
\]

(2.2)

It holds for \(\nu \geq 0\) that
\[
\sup_{t \geq 0} \|b_\nu(t,y) - y\|_{L^\infty_y} = \pi M \gamma^2,
\]
\[
\|b_\nu(t,y) - y\|_{H^1_y} = \frac{(2\pi)^{\frac{3}{4}} M \gamma^2}{(4\nu t + \gamma^2)^{\frac{1}{4}}},
\]
which means that the shear flow \((b_\nu(t,y),0)\) is a perturbation of Couette flow.

We construct the shear flow based on Gaussian function as its time evolution through heat equation has a precise formula. We remark that in [49] the authors also chose Gaussian related functions to prove the Kelvin’s cat’s eyes structure near Couette flow for inviscid problem.

For the viscous problem, we consider the case \(\nu = \gamma^{\frac{3}{1-2\delta_0}}\) for any small \(\delta_0 > 0\), so that this shear flow \((b_\nu(t,y),0)\) is close to the Couette flow in the sense:
\[
\sup_{t \in [0,\gamma^2/\nu]} \|b_\nu(t,y) - y\|_{L^\infty_y \cap H^1_y} \approx M \nu^{\frac{1}{2} - \delta_0}.
\]

2.1. New energy method. Usually, the key step for studying (non)linear instability is finding the growing mode of the linearized operators. This approach is effective in dealing with time-independent linearized operators [30, 47]. However, it is hard to reduce the instability problem to the eigenvalue problem if the linearized operator is time-dependent, especially, for problems where the distribution of eigenvalue varies in time. For the viscous problem, the background shear flow \(b_\nu(t,y)\) varies with time, so the corresponding linearized operator is time-dependent.

In this paper, we develop a new energy estimate method to describe the growth of each frequency and obtain the exponential growth. We would like to remark again that the linearized operator for the viscous problem has no eigenvalues, even with the time independent coefficients, see Remark 1.7.
We first introduce the modified linearized equation for \( \nu \geq 0 \):
\[
\partial_t g + y \partial_z g - \partial_y^2 b_v(t, y) \partial_x(\Delta)^{-1} g - \nu \Delta g = 0,
\]
where we replace the transport term \( b_v(t, y) \partial_z g \) by \( y \partial_z g \) and keep the nonlocal term \(-\partial_y^2 b_v(t, y) \partial_x(\Delta)^{-1} g\), since \( b_v(t, y) - y \) is small. The control of the difference \((b_v(t, y) - y) \partial_z g \) (between (1.5) and (2.3)) will be carefully studied in Section 4.

It is natural to introduce the linear change of coordinate \( z = x - ty \) and let
\[
h(t, z, y) = h(t, x - ty, y) = g(t, x, y).
\]
Then \( h \) satisfies
\[
\partial_t h(t, z, y) + \frac{4\sqrt{\pi} M \gamma^2 y}{(4\nu t + \gamma^2)^{3/2}} e^{-\frac{y^2}{4\nu t + \gamma^2}} \partial_x(\Delta_L)^{-1} h(t, z, y) - \nu \Delta_L h(t, z, y) = 0,
\]
where \( \Delta_L = \partial_{zz} + (\partial_y - t \partial_x)^2 \). By taking Fourier transform in \((z, y)\), we get that
\[
\partial_t \hat{h}_k(t, \xi) + \nu (k^2 + (\xi - kt)^2) \hat{h}_k(t, \xi) \approx - \int_{\mathbb{R}} M \gamma^2 (\xi - \eta) e^{-\nu t} \eta^2 |\xi - \eta|^2 \langle \eta - kt \rangle^2 + k^2 d\eta = 0,
\]
An easy argument shows that (see Lemma \[3.3\])
\[
||\hat{h}_k(t, \xi)||_{L^\infty} \leq e^{-\nu k^2(t+\frac{3}{12})} e^{M\pi^2 t} ||\hat{h}_k(0, \xi)||_{L^\infty},
\]
from which, we can see that for the viscous problem \((\nu = \gamma^{\frac{1}{3}} \frac{1}{2\kappa_0})\), the possible growth only happens for low frequencies in \( z \) and \( t \lesssim \gamma^{\frac{1}{2}} \nu^{-\frac{1}{2}} = \gamma^{-1 - \frac{3k}{2\kappa_0}} \). This shows that the growth obtained for the viscous flow is a transient growth. However, for the inviscid problem, the growth could be sustained for all the time, and this allows us to prove the existence of unstable eigenvalues.

To capture the growth, we choose the initial data that has only \( \pm 1 \) modes,
\[
h_{in}(z, y) = 2 \cos(z) h_{in}(y), \quad \hat{h}_{in}(\xi) = \begin{cases} 1, & |\xi| \leq 2\varepsilon_1 \gamma^{-1} \ln(\gamma^{-1}), \\ 0, & |\xi| > 2\varepsilon_1 \gamma^{-1} \ln(\gamma^{-1}). \end{cases}
\]

The corresponding initial data of the linear system (2.3) is
\[
g_{in}(x, y) = 2 \cos(x) h_{in}(y).
\]
From (2.5) and the choice of \( h_{in} \), one can easily check that \( \hat{h}_k(t, \xi) \equiv 0 \) for \( k \neq \pm 1 \), \( \hat{h}_{-1}(t, -\xi) = \hat{h}_1(t, \xi) \) for all \( t \geq 0 \) and \( \xi \in \mathbb{R} \). Here the constant \( M > 0 \) is large enough determined in (3.17), and the constant \( \varepsilon_1 \) is small enough determined in (3.10).

For the viscous problem \((\nu = \gamma^{\frac{1}{3}} \frac{1}{2\kappa_0})\), we can see that \( b_v(t, y) \) is close to \( b_{in}(y) \) for \( t \in [0, \varepsilon_1 \nu^{-\frac{1}{2}} + \frac{3}{2} \ln(\nu^{-1})] \), and the dissipation effect is weak. Thus for both viscous and inviscid problems, we only need to catch the growth of \( \hat{h}_1(t, \xi) \) which satisfies
\[
\hat{h}_1(t, \xi) = \int_{\mathbb{R}} M \gamma^2 (\xi - \eta) e^{-\gamma^2 |\eta|^2} \frac{\hat{h}_1(t, \eta)}{\langle \eta - t \rangle^2 + 1} d\eta + \text{lower order terms},
\]
\[
\hat{h}_1(0, \xi) = \hat{h}_{in}(\xi).
\]
The kernel
\[
M \gamma^2 (\xi - \eta) e^{-\gamma^2 |\eta|^2} \frac{1}{\langle \eta - t \rangle^2 + 1} \lesssim \gamma
\]
Let us assume
\[ \dot{h}_1(t, \xi) \in [t + \frac{1}{2m} \gamma^{-1}, t + 3 \gamma^{-1}] \]
for some \( m \geq 2 \). The scenario we are most interested in is a low to high transition. On one hand, at time \( t \), those frequencies \( \eta \) which are close to \( t \) have a strong effect that excites those frequencies \( \xi \in [t + \gamma^{-1}, t + 3 \gamma^{-1}] \):
\[
\dot{h}_1(t, \xi) \in [t + \frac{1}{2m} \gamma^{-1}, t + 3 \gamma^{-1}] \geq \gamma \inf_{|\eta-t| \leq C} \hat{h}_1(t, \eta).
\]

On the other hand, those frequencies \( \xi \) which are close to \( t \) do not change too much.

For the formal argument it is enough to follow the growth of two frequencies. Formally, let us assume
\[
(2.8) \quad \inf_{|\eta-t| \leq C} \hat{h}_1(t, \eta) \geq \hat{h}_1(0, 0), \quad \text{for} \quad 0 \leq t \leq \frac{1}{2} \gamma^{-1}
\]
then for \( t \) close to 0, the two frequencies \( \xi = \gamma^{-1} \) and \( \xi = 2 \gamma^{-1} \) are forced by \( \hat{h}_1(0, 0) \). From \( t = 0 \) to \( t = \gamma^{-1} \) we ignore the effect when \( t \) and \( \xi \) are close. Thus we get
\[
\hat{h}_1(\gamma^{-1}, \gamma^{-1}) \geq \hat{h}_1(0, \gamma^{-1}) + \int_0^{\gamma^{-1}} \partial t \hat{h}_1(s, \gamma^{-1}) ds \geq \hat{h}_1(0, \gamma^{-1}) + \hat{h}_1(0, 0),
\]
and
\[
h_1(\gamma^{-1}, 2 \gamma^{-1}) \geq \hat{h}_1(0, 2 \gamma^{-1}) + \int_0^{\gamma^{-1}} \partial t \hat{h}_1(s, \gamma^{-1}) ds \geq \hat{h}_1(0, 2 \gamma^{-1}) + \hat{h}_1(0, 0).
\]

Let us use the following lower bound instead:
\[
\hat{h}_1(\gamma^{-1}, \gamma^{-1}) = \hat{h}_1(0, \gamma^{-1}) + \hat{h}_1(0, 0), \quad \hat{h}_1(\gamma^{-1}, 2 \gamma^{-1}) = \hat{h}_1(0, 2 \gamma^{-1}) + \hat{h}_1(0, 0).
\]

In general, from \( t = (j - 1) \gamma^{-1} \) to \( t = j \gamma^{-1} \) with \( j \geq 1 \),
\[
\hat{h}_1(j \gamma^{-1}, j \gamma^{-1}) = \hat{h}_1((j - 1) \gamma^{-1}, j \gamma^{-1}) + \hat{h}_1((j - 1) \gamma^{-1}, (j - 1) \gamma^{-1}),
\]
\[
\hat{h}_1((j - 1) \gamma^{-1}, (j + 1) \gamma^{-1}) = \hat{h}_1((j - 1) \gamma^{-1}, (j + 1) \gamma^{-1}) + \hat{h}_1((j - 1) \gamma^{-1}, (j - 1) \gamma^{-1}),
\]
\[
\hat{h}_1((j - 1) \gamma^{-1}, (j + m) \gamma^{-1}) = \hat{h}_1((j - 1) \gamma^{-1}, (j + m) \gamma^{-1}), \quad m \geq 2.
\]

Note that \( \hat{h}_1(0, \xi) = 1 \) for \( |\xi| \leq \epsilon_1 \gamma^{-1} \ln(\gamma^{-1}) \), so it is easy to check that
\[
\hat{h}_1((j - m) \gamma^{-1}) = 1, \quad j \geq 0, \quad m \geq 2,
\]
\[
\hat{h}_1((j - 1) \gamma^{-1}, (j - 2) \gamma^{-1}) = 1 + \hat{h}_1((j - 2) \gamma^{-1}, (j - 2) \gamma^{-1}),
\]
\[
\hat{h}_1((j - 1) \gamma^{-1}, (j + 1) \gamma^{-1}) = 1 + \hat{h}_1((j - 1) \gamma^{-1}, (j - 1) \gamma^{-1}) = 1 + \hat{h}_1((j - 2) \gamma^{-1}, (j - 1) \gamma^{-1}) + \hat{h}_1((j - 2) \gamma^{-1}, (j - 2) \gamma^{-1})
\]
\[
= \hat{h}_1((j - 2) \gamma^{-1}, (j - 1) \gamma^{-1}) + \hat{h}_1((j - 1) \gamma^{-1}, j \gamma^{-1}),
\]
and thus \( \hat{h}_1((j - 1) \gamma^{-1}, (j + 1) \gamma^{-1}) = b_j \) where \( \{b_j\} \) is the Fibonacci sequence, which grows exponentially. Indeed, the numerical evidence (see Figure 1) gives the above growth mechanism.

The formal assumption (2.8) is not very accurate. To estimate \( \inf_{|\eta-t| \leq C} \hat{h}_1(t, \eta) \), we will subdivide the time interval \([j \gamma^{-1}, (j + 1) \gamma^{-1}]\) into smaller intervals of size \( \gamma^{-\frac{2}{3}} \), see the definition of \( I_m \) and Lemma 3.5 in Section 3. Mathematically, we prove that
\[
\hat{h}_1((j - 1) \gamma^{-1}, \xi) \in [j \gamma^{-1}, (j + 1) \gamma^{-1}] \geq b_j, \quad \text{for} \quad 1 \leq j \leq \epsilon_1 \ln(\gamma^{-1})
\]
which describes the exponential growth of the positive parts for each graph showed in the picture. See Section 3 for more details.

Based on the linear instability proved by the new energy method, we solve two important questions, namely, the optimal instability of the Couette flow for the viscous problem and the existence of growing mode for the inviscid problem.

2.2. Nonlinear instability of the viscous problem. In Section 4, we study the nonlinear viscous system (1.3), we first get a priori estimates by using classical energy method via the ghost type weight [2] and bootstrap argument, then prove the solution of system (1.3) has exponential growth from the initial data

\[ \omega_{in}(x, y) = \frac{\varepsilon_0 \nu^{\frac{2}{3} + \delta_1 - \frac{1}{3} \delta_0}}{\sqrt{\varepsilon_1 \sqrt{-\ln \nu}}} g_{in}(x, y), \]

where \( g_{in} \) is given in (2.7), the constant \( \delta_1 \) is small enough determined in (4.21), and \( \varepsilon_0 \) is a positive small constant which is determined in the proof.

2.3. Inviscid unstable shear flow. In Section 5, we study the linear instability and study the existence and location of eigenvalues for the Rayleigh operator \( R = b(y) - b''(y)(\partial_y^2 - 1)^{-1} \) corresponding to mode 1. We prove the existence result by using a contradiction argument. We first show that \( R \) has no embedded eigenvalue (see Lemma 5.7). If \( R \) has no eigenvalues, we prove that for any \( w \in L^2 \) and \( t \geq 0 \), it holds that

\[ \| e^{itR}w \|_{L^2} \leq C\|w\|_{L^2} \]

where \( C \) is a constant independent of \( \gamma, t \).

However, by the new energy method introduced in Section 3, we prove that

\[ \| e^{itR}w \|_{L^2} \geq ce^{ct} \|w\|_{L^2} \]
Proposition 3.1. There exists $f$ where $g$ exists and $c$ are constants which satisfies that

Proposition 3.2. There exists $\nu$ of the linear system (2.3) for both viscous and inviscid case ($\nu = \gamma^{-\frac{3}{2\delta_0}}$ and $\nu = 0$) from the initial data (2.7). Now we introduce the semi-group

$$S_\nu(t) : L^2_{x,y} \rightarrow L^2_{x,y}, \text{ for } \nu \geq 0,$$

which satisfies that $g(t, x, y) = S_\nu(t)f(x, y)$ solves the linear system (2.3) with initial data $f(x, y)$. By taking the well chosen initial data, we prove the following propositions.

**Proposition 3.1.** There exists $M_0 > 0$, for any $M > M_0$, and any small $\delta_0 > 0$, there exist constants $c_0, c_1, \varepsilon_1, \gamma_0 > 0$ such that for $0 < \gamma \leq \gamma_0$, $\nu = \gamma^{-\frac{3}{2\delta_0}}$, and $0 \leq t \leq T = \varepsilon_1 \gamma^{-1}(\ln \nu^{-1})$, it holds that

$$\|P_{\pm 1} (S_\nu(T)g_{in})\|_{L^2_{x,y}} \geq c_0 e^{c_1 \gamma t} \|g_{in}\|_{L^2_{x,y}},$$

and in particular

$$\|P_{\pm 1} (S_\nu(T)g_{in})\|_{L^2_{x,y}} \geq c_0 \nu^{-c_1 \varepsilon_1} \|g_{in}\|_{L^2_{x,y}},$$

where $g_{in}(x, y)$ is given in (2.7).

**Proposition 3.2.** There exists $M_0 > 0$ and for any $M > M_0$, there exist constants $c_0, c_1, \varepsilon_1, \gamma_0 > 0$ such that for $0 < \gamma \leq \gamma_0$, and $0 \leq t \leq T = \varepsilon_1 \gamma^{-1}(\ln \gamma^{-1})$, it holds that

$$\|P_{\pm 1} (S_0(t)g_{in})\|_{L^2_{x,y}} \geq c_0 e^{c_1 \gamma t} \|g_{in}\|_{L^2_{x,y}},$$

and in particular

$$\|P_{\pm 1} (S_0(T)g_{in})\|_{L^2_{x,y}} \geq c_0 \gamma^{-c_1 \varepsilon_1} \|g_{in}\|_{L^2_{x,y}},$$

where $g_{in}(x, y)$ is given in (2.7).

Recall that $h(t, z, y)$ is the solution of (2.4) satisfying

$$h(t, z, y) = h(t, x - ty, y) = g(t, x, y) = S_\nu(t)g_{in}(x, y), \text{ for } \nu \geq 0.$$

Accordingly,

$$2\pi \|\hat{h}(t)\|_{L^2_{k,\xi}} = \|h(t)\|_{L^2_{x,y}} = \|g(t)\|_{L^2_{x,y}} = \|S_\nu(t)g_{in}\|_{L^2_{x,y}}, \text{ for } \nu \geq 0.$$

It is sufficient to prove Propositions 3.1 and 3.2 by giving the lower bound estimate of $\|\hat{h}(t)\|_{L^2_{k,\xi}}$. Next, we focus on the system (2.5).
3.1. Upper bound estimate in $L^\infty_\xi$. We first give the upper bound estimate of $\|\hat{h}_k(t, \xi)\|_{L^\infty_\xi}$.

**Lemma 3.3.** Let $\hat{h}_k(t, \xi)$ be the solution to (2.5). For any $\nu \geq 0$, we have the following upper bound estimate:

$$\|\hat{h}_k(t, \xi)\|_{L^\infty_\xi} \leq e^{-\nu k^2(t + \frac{\nu^3}{4})} e^{M\pi \gamma t} \|\hat{h}_k(0, \xi)\|_{L^\infty_\xi}. \quad (3.1)$$

**Proof.** From (2.5), we deduce that

$$\left| \partial_t \hat{h}_k(t, \xi) + \nu (k^2 + (\xi - kt)^2) \hat{h}_k(t, \xi) \right| \leq \int_{\mathbb{R}} M\gamma^2 |\xi - \eta| |e^{-\nu(\xi^2 + \xi^2/4)}| \frac{k \|\hat{h}_k(t, \xi)\|_{L^\infty_\xi}}{(\eta - kt)^2 + k^2} d\eta$$

$$\leq M \|\hat{h}_k(t, \xi)\|_{L^\infty_\xi} \int_{\mathbb{R}} k \sup_\eta \left| \gamma^2 |\xi - \eta| e^{-\nu(\xi^2 + \xi^2/4)} \right| \frac{d\eta}{(\eta - kt)^2 + k^2} \leq M\pi \gamma \|\hat{h}_k(t, \xi)\|_{L^\infty_\xi}. \quad (3.2)$$

Here we use the fact that

$$\sup_\eta \left| \gamma^2 |\xi - \eta| e^{-\nu(\xi^2 + \xi^2/4)} \right| \leq \frac{\gamma^2}{\sqrt{2\pi \nu + \gamma^2/4}} \leq \gamma. \quad (3.3)$$

Then by using Gronwall's inequality, we have

$$\|\hat{h}_k(t, \xi)\|_{L^\infty_\xi} \leq e^{-\nu k^2(t + \xi^2/4 + \xi^3/12)} \left( \|\hat{h}_k(0, \xi)\|_{L^\infty_\xi} + \int_0^t e^{\nu(k^2 + \xi^2 s - \xi^3 s/2 + k^2 s^3/4)} M\pi \gamma \|\hat{h}_k(s, \xi)\|_{L^\infty_\xi} ds \right) \quad (3.4)$$

$$= e^{-\nu k^2(t + \xi^2/4 + \xi^3/12)} \|\hat{h}_k(0, \xi)\|_{L^\infty_\xi} + \int_0^t e^{-\nu k^2(t - s)\left(\xi^2/2 - \frac{t + s}{4}\right)^2 + \frac{3(t - s)}{12}} + \frac{3(t - s)}{4} M\pi \gamma \|\hat{h}_k(s, \xi)\|_{L^\infty_\xi} ds.$$ 

It follows from the fact

$$e^{-\nu k^2 t \left(\frac{t}{2} - \frac{t}{4}\right)^2} \leq 1, \quad e^{-\nu k^2 t \left(\frac{t}{2} - \frac{t}{4}\right)^2 + \frac{3(t - s)}{4}} \leq 1 \quad (3.5)$$

that

$$\|\hat{h}_k(t, \xi)\|_{L^\infty_\xi} \leq e^{-\nu k^2(t + \frac{\nu^3}{4})/2} \|\hat{h}_k(0, \xi)\|_{L^\infty_\xi} + \int_0^t e^{\nu k^2(s + \frac{\nu^3}{4} - t + \frac{\nu^3}{4})} M\pi \gamma \|\hat{h}_k(s, \xi)\|_{L^\infty_\xi} ds. \quad (3.6)$$

By using Gronwall's inequality again, we have

$$\|\hat{h}_k(t, \xi)\|_{L^\infty_\xi} \leq e^{-\nu k^2(t + \frac{\nu^3}{4})} \|\hat{h}_k(0, \xi)\|_{L^\infty_\xi} + e^{-\nu k^2(t + \frac{\nu^3}{4})} \|\hat{h}_k(0, \xi)\|_{L^\infty_\xi} \int_0^t M\pi e^{M\pi \gamma (t - s)} ds$$

$$= e^{-\nu k^2(t + \frac{\nu^3}{4})} e^{M\pi \gamma t} \|\hat{h}_k(0, \xi)\|_{L^\infty_\xi}, \quad \Box$$

which is (3.3).
3.2. Lower bound estimate for the viscous problem. In this subsection, we use the
new energy method to study the lower bound for the viscous problem ($\nu = \gamma^{-\frac{1}{2(\alpha_0)}}$). Note
that the initial data $h_{in}(z, y)$ (see (2.6)) has only $\pm 1$ modes with $\hat{h}_{in,1}(\xi) = \hat{h}_{in,-1}(\xi)$. So the
solution to the linear system satisfies $\hat{h}_{k}(t, \xi) \equiv 0$ for $k \neq \pm 1$ and $\hat{h}_{-1}(t, -\xi) = \hat{h}_{1}(t, \xi)$ for all
t $\geq 0$ and $\xi \in \mathbb{R}$. Therefore, we focus on the $k = 1$ mode, and study the following system:

$$\frac{\partial \hat{h}(t, \xi)}{\partial t} = \int_{\mathbb{R}} M\gamma^2(\xi - \eta)e^{-\left(\nu t + \frac{\gamma^2}{4}\right)||\xi - \eta||^2} \frac{\hat{h}(t, \eta)}{(\eta - t)^2 + 1} d\eta - \nu(1 + (\xi - t)^2)\hat{h}(t, \xi),$$

$$\hat{h}(0, \xi) = \hat{h}_{in}(\xi) \geq 0.$$

For simplicity of notation, here we use $\hat{h}(t, \xi)$ instead of $\hat{h}_{1}(t, \xi)$.

Analyzing the evolution from the initial time $t = 0$, we have the following observations.

The kernel in (3.5) reaches its maximum $\sqrt{2M\gamma^2} / \sqrt{\pi} \approx \gamma$ at $\eta = t$, $\eta - t = \sqrt{\frac{\gamma}{4\nu t + \gamma^2}} \approx \gamma^{-1}$, and satisfies

$$M\gamma^2(\xi - \eta)e^{-\left(\nu t + \frac{\gamma^2}{4}\right)||\xi - \eta||^2} \frac{1}{(\eta - t)^2 + 1} \geq c_\gamma$$

for $\xi \in [t + c_\gamma^{-1}, t + C_\gamma^{-1}]$, $\eta \in [t - C, t + C]$. Roughly speaking, for fixed $t \geq 0$, we call $[t - C, t + C]$ the excitation region, $[t + c_\gamma^{-1}, t + C_\gamma^{-1}]$
the region of growth, and $[t + C_\gamma^{-1}, C_1 \gamma^{-1} \ln(\gamma^{-1})]$ the waiting region, see Figure 2. One can
see that, at $t = 0$, the value of the integral in (3.5) is bigger than $c_\gamma$ for $\xi$ in the region of
growth. Meanwhile, for such $\xi$, as $\nu = \gamma^{-\frac{1}{2(\alpha_0)}}$, the dissipation effect is very weak. As a result, $\hat{h}(0, \eta)$ with $\eta$ in the excitation region will excite $\hat{h}(0+, \xi)$ for $\xi$ in the region of growth. For $\xi$
in the waiting region, both the integral and the dissipation term are extremely small, which
leads to the fact that $\hat{h}(0+, \xi)$ will stay close to 1. The region of growth is far in front of
the excitation region, and this two regions move in the same direction as time changes. For
each fixed frequency $\eta > C_\gamma^{-1}$, it first belongs to the waiting region, then enters into the
region of growth, at last enters into the the excitation region. In a word, $\hat{h}(t, \eta)$ will first be
excited and then excites the one with higher frequency in the new region of growth, which
shows that the function $\hat{h}(t, t)$ is increasing with respect to $t$. At the end, a cascade effect
generated from the huge scale of the region of growth causes the exponential growth.

![Figure 2. excitation region, region of growth and waiting region](image-url)
To clarify the growth mechanism, we divide the time-frequency into small intervals, and study the evolution of \( \hat{h}(t, \xi) \) during each small time interval. Let

\[
\mathbb{R}^+ \times \mathbb{R} = \bigcup_{m \in \mathbb{Z}^+, n \in \mathbb{Z}} I_m \times I_n,
\]

where \( I_n = (\frac{n}{N} \gamma^{-1}, \frac{n+1}{N} \gamma^{-1}] \), and \( N = \lfloor \frac{1}{\gamma^{-1}} \rfloor \). Here we denote by \(|a| \) the biggest integer that is not greater than \( a \). We also define \( T_m = \frac{m}{N} \gamma^{-1} \) which are the end points for the time intervals. Now we give precise definitions of the excitation region, the region of growth, and the waiting region. For \( t \in I_m \), we divide into 4 cases depending on the frequency \( \xi \in \mathbb{R} = \mathcal{I}_m^r \cup \mathcal{I}_m^e \cup \mathcal{I}_m^g \cup \mathcal{I}_m^w, \) where

Case 1. \( \xi \in \mathcal{I}_m^r = \left( \bigcup_{n \leq -3} I_n \right) \cup \left( \bigcup_{n > 2 \xi_1 \ln(\gamma^{-1})N} I_n \right) \);

Case 2. \( \xi \in \mathcal{I}_m^e = \bigcup_{|n-m| \leq 2} I_n; \)

Case 3. \( \xi \in \mathcal{I}_m^g = \bigcup_{m+3 \leq n \leq m+2+4N} I_n; \)

Case 4. \( \xi \in \mathcal{I}_m^w = \bigcup_{m+3+4N \leq n \leq 2\xi_1 \ln(\gamma^{-1})N} I_n. \)

For Case 1, \( \xi \in \mathcal{I}_m^r \) is not in the excitation region nor the region of growth, we only need the upper bound estimate given in Remark 3.4. For Case 2, the interval \( \mathcal{I}_m^e \) contains the excitation region, we prove that \( \hat{h}(t, \xi) \) barely change during the time interval \( I_m \), see Lemma 3.5. For Case 3, the interval \( \mathcal{I}_m^g \) is basically the region of growth, \( \hat{h}(t, \xi) \) keeps growing during the time interval \( I_m \), see Lemma 3.6. For Case 4, \( \mathcal{I}_m^w \) is basically the waiting region, we prove that \( \hat{h}(t, \xi) \geq \frac{10}{N} \), see Lemma 3.7. The growth accumulated in Case 3 will eventually produce exponential growth.

**Remark 3.4.** It holds for \( t \geq 0 \) that

\[
\| \hat{h}(t) \|_{L_\xi^\infty} \leq e^{M\pi \gamma t} \| \hat{h}_0 \|_{L_\xi^\infty} = e^{M\pi \gamma t}. \tag{3.6}
\]

**Proof.** The estimate (3.6) follows directly from Lemma 3.3 by taking the well chosen initial data (2.6). \( \square \)

Now, let us focus on Case 2.

**Lemma 3.5.** Let \( \hat{h} \) be the solution of (3.5). For \( t \in I_m, \xi \in I_n, \) with \( m \leq \frac{3}{2} \xi_1 \ln(\gamma^{-1})N - 1 \) and \( |n-m| \leq 2 \), it holds that

\[
\hat{h}(T_m, \xi) - \frac{6M\pi}{N^2} e^{\frac{M\pi (m+1)}{N}} \leq \hat{h}(t, \xi) \leq \hat{h}(T_m, \xi) + \frac{6M\pi}{N^2} e^{\frac{M\pi (m+1)}{N}}.
\]

**Proof.** It follows from (3.5) that

\[
\partial_t \hat{h}(t, \xi) = \int_{\cup_{|\xi| \leq |n-m| \leq 1} I_i} M \gamma^2 (\xi - \eta) e^{-(\nu t + \frac{\gamma^2}{4})|\xi - \eta|^2} \frac{1}{(\eta-t)^2 + 1} \hat{h}(t, \eta) d\eta
\]

\[
+ \int_{\cup_{|\xi| > |n-m| > 1} I_i} M \gamma^2 (\xi - \eta) e^{-(\nu t + \frac{\gamma^2}{4})|\xi - \eta|^2} \frac{1}{(\eta-t)^2 + 1} \hat{h}(t, \eta) d\eta
\]

\[
- \nu (1 + (\xi - t)^2) \hat{h}(t, \xi) \overset{\text{def}}{=} A_m + B_m + C.
\]

As \( \xi \in I_n, |n-m| \leq 2, \) and \( \eta \in \cup_{|\xi| \leq |n-m| \leq 1} I_i, \) it holds that \( |\xi - \eta| \leq \frac{4}{N} \gamma^{-1} \). Then from (3.6), we have

\[
|A_m| \leq M \gamma^2 \frac{4}{N} \gamma^{-1} \int_{\mathbb{R}} \frac{1}{(\eta-t)^2 + 1} d\eta \| \hat{h}(t, \xi) \|_{L_\xi^\infty} \leq \frac{4M\pi \gamma}{N} e^{M\pi \gamma t}.
\]
Then by using (3.2), we have
\[
|\mathcal{B}_m| \leq M \gamma \int_{|\eta-t| \geq \frac{\gamma}{2}} \frac{1}{(\eta-t)^2 + 1} d\eta \| \hat{h}(t, \xi) \|_{L^\infty} \leq 2MN \gamma^2 e^{M \gamma^2 t},
\]
As \( t \in I_m \), and \( \xi \in I_n, |n - m| \leq 2 \), we have \( |\xi - t| \leq \frac{3}{N} \gamma^{-1} \), then we have
\[
|\mathcal{C}| \leq 10 \nu \gamma^{-2} \frac{N^2}{2} \| \hat{h}(t, \xi) \|_{L^\infty} \leq 10 \nu \gamma^{-2} \frac{N^2}{2} e^{M \gamma^2 t}.
\]
Recalling that \( N = [\gamma^{-1}] \), \( \nu = \gamma^{1-2m} \), and \( \gamma > 0 \) small enough, we have
\[
\| \partial_t \hat{h}(t, \xi) \|_{L^\infty} \leq |\mathcal{A}_m| + |\mathcal{B}_m| + |\mathcal{C}| \leq \frac{5}{4}|\mathcal{A}_m| \leq \frac{5M \pi \gamma}{N} e^{M \gamma^2 t}.
\]
Therefore, for \( \xi \in I_n, |n - m| \leq 2 \), it holds that
\[
\hat{h}(T_m, \xi) - \frac{6M \pi}{N^2} e^{\frac{M \pi (m+1)}{N}} \leq \hat{h}(t, \xi) \leq \hat{h}(T_m, \xi) + \frac{6M \pi}{N^2} e^{\frac{M \pi (m+1)}{N}}.
\]
Next, we show the main growth and study Case 3.

**Lemma 3.6.** Let \( \hat{h} \) be the solution of (3.5). There exists \( \varepsilon_1 > 0 \) such that, for \( t \in I_m = [T_m, T_{m+1}], \xi \in I_n \) with \( m \leq \frac{3}{2} \varepsilon_1 \ln(\gamma^{-1})N - 1 \), and \( m + 3 \leq n \leq m + 2 + 4N \), if it holds that
\[
\inf_{\tau \in I_m} \inf_{\eta \in \cup_{|i-m| \leq 1} I_i} \hat{h}(\tau, \eta) \geq \frac{9}{10},
\]
we have
\[
\hat{h}(t, \xi) \geq \hat{h}(T_m, \xi) + (t - T_m) \frac{4\gamma M (n - m - 2) \pi}{9N} e^{-\frac{(m-2)^2}{2N^2}} \inf_{\tau \in I_m} \inf_{\eta \in \cup_{|i-m| \leq 1} I_i} \hat{h}(\tau, \eta),
\]
and in particular at the end point,
\[
\hat{h}(T_{m+1}, \xi) \geq \hat{h}(T_m, \xi) + \frac{4M (n - m - 2) \pi}{9N^2} e^{-\frac{(m-2)^2}{2N^2}} \inf_{\tau \in I_m} \inf_{\eta \in \cup_{|i-m| \leq 1} I_i} \hat{h}(\tau, \eta).
\]

**Proof.** Similar to (3.7), we write
\[
\partial_t \hat{h}(t, \xi) = \mathcal{A}_m + \mathcal{B}_m + \mathcal{C},
\]
where \( \mathcal{A}_m \) describes the effect from the excitation region. For \( \xi \in I_n, n - m \geq 3 \), it holds that \( (\xi - \eta) \geq \frac{\gamma}{2} \frac{(n-2)^2}{N} \eta \) with \( \eta \in \cup_{|i-m| \leq 1} I_i \). Then we can see from the assumption (3.8) that
\[
\mathcal{A}_m \geq \frac{M \gamma (n - m - 2) \pi}{N} e^{-\frac{8\pi}{9} (m-2)^2} \frac{8\pi}{9} \frac{1}{(\eta-t)^2 + 1} d\eta \inf_{\tau \in I_m} \inf_{\eta \in \cup_{|i-m| \leq 1} I_i} \hat{h}(\tau, \eta)
\]
\[
\geq \frac{4M \gamma (n - m - 2) \pi}{5N} e^{-\frac{(m-2)^2}{2N^2}}.
\]
As \( t \in I_m, \xi \in I_n \) with \( m + 3 \leq n \leq m + 2 + 4N \) and \( m \leq \frac{3}{2} \varepsilon_1 \ln(\gamma^{-1})N - 1 \), it is clear that \( t \leq \frac{3}{2} \varepsilon_1 \gamma^{-1} \ln(\gamma^{-1}) \). One can easily check that
\[
\frac{1}{4} \mathcal{A}_m \geq \frac{M \gamma (n - m - 2) \pi}{5N} e^{-8} \geq \frac{M \gamma^2 (n - m - 2) \pi}{5} e^{-8},
\]
\[ |\mathcal{B}_m| \leq 2MN\gamma^2e^{M\pi rt} \leq 2M\gamma^{\frac{5}{2} - \frac{3}{4} \varepsilon_1 M\pi}, \]
\[ |\mathcal{C}| \leq \frac{\nu(n - m + 2)^2 \gamma^{-2}}{N^2} e^{M\pi rt} \leq \gamma^{\frac{5}{3} + 6\delta_0 - \frac{3}{2} \varepsilon_1 M\pi} (n - m + 2)^2 \leq 5\gamma^{\frac{4}{3} + 6\delta_0 - \frac{3}{2} \varepsilon_1 M\pi} (n - m + 2). \]

Then, by taking
\[ \varepsilon_1 = \frac{2\delta_0}{M\pi}, \]
we get
\[ |\mathcal{B}_m| \leq 2M\gamma^{\frac{5}{3} - 3\delta_0} \leq \frac{1}{4} A_m, \quad |\mathcal{C}| \leq 5\gamma^{\frac{4}{3} + 3\delta_0} (n - m + 2) \leq \frac{1}{4} A_m \]
for \( \delta_0 \leq \frac{1}{10} \) and \( \gamma \) small enough.

Therefore, we get that
\[ \partial_t \hat{h}(t, \xi) \geq \frac{1}{2} A_m \geq \frac{4M\gamma(n - m - 2)\pi}{9N} e^{(n - m - 2)^2} \inf_{\tau \in I_m} \inf_{\eta \in \cup_{|i - m| \leq 1} I_i} \hat{h}(\tau, \eta), \]
and
\[ \hat{h}(t, \xi) \geq \hat{h}(T_m, \xi) + (t - T_m) \frac{4\gamma M(n - m - 2)\pi}{9N} e^{(n - m - 2)^2} \inf_{\tau \in I_m} \inf_{\eta \in \cup_{|i - m| \leq 1} I_i} \hat{h}(\tau, \eta), \]
which give the lemma. \( \Box \)

For Case 4, we study the time evolution in the waiting region. The dissipation effect will be obvious, that is to say, \( |\mathcal{C}| \) will be comparable to \( A_m \). So we do not expect \( \hat{h}(t, \xi) \) to grow for \( \xi \in \mathcal{F}_m \). We give a lower bound estimate.

**Lemma 3.7.** Let \( \hat{h} \) be the solution of \( (3.3) \). For \( t \in I_m \) and \( \xi \in I_n \) with \( m \leq \frac{3}{2} \varepsilon_1 \ln(\gamma^{-1})N - 1 \), \( m + 3 + 4N \leq n \leq 2\varepsilon_1 \ln(\gamma^{-1})N \), and \( \varepsilon_1 \) given in Lemma 3.6, if it holds that
\[ \inf_{0 \leq m' \leq m} \inf_{\tau \in I_m} \inf_{\eta \in \cup_{|i - m| \leq 1} I_i} \hat{h}(\tau, \eta) \geq 0, \]
we have
\[ \hat{h}(t, \xi) \geq \frac{19}{20}. \]

**Proof.** Under the assumption \( (3.12) \), it holds that
\[ A_{m'} \geq 0 \] for \( 0 \leq m' \leq m \).

Then, by \( m + 3 + 4N \leq n \leq 2\varepsilon_1 \ln(\gamma^{-1})N \), we have for \( t \in \cup_{m' = 0}^{m'} I_{m'}, \xi \in I_n \) that
\[ \partial_t \hat{h}(t, \xi) \geq -4\varepsilon_1^2 \gamma^{1 + 6\delta_0} (\ln \gamma)^2 \hat{h}(t, \xi) - 2MN\gamma^2 e^{M\pi rt}. \]

Using Gronwall’s inequality, we deduce that
\[ \hat{h}(t, \xi) \geq e^{-4\varepsilon_1^2 \gamma^{1 + 6\delta_0} (\ln \gamma)^2 t} \left( 1 - \int_0^t 2MN\gamma^2 e^{(M\pi + 4\varepsilon_1^2 \gamma^{1 + 6\delta_0} (\ln \gamma)^2)s} ds \right). \]

With \( \gamma \) small enough, it follows that
\[ \inf_{t \in I_m} \hat{h}(t, \xi) \geq e^{-6\varepsilon_1^2 \gamma^{6\delta_0} (\ln \gamma^{-1})^3} \left( 1 - \frac{\gamma^{\frac{2}{3} - 3\delta_0}}{\pi} \right) \geq \frac{19}{20}. \] \( \Box \)

The assumptions \( (3.8) \) and \( (3.12) \) hold naturally for the linear system with our well chosen initial data, see the following lemma.
Lemma 3.8. With our well chosen initial data, assumption (3.8) in Lemma 3.6 and assumption (3.12) in Lemma 3.7 are automatically satisfied.

Proof. From Lemma 3.5 and the fact \( \gamma \) small enough, we have for \( m = 0, 1 \) that

\[
(3.14) \quad \inf_{\tau \leq \eta \leq \tau + \Delta} h(\tau, \eta) \geq \frac{19}{20} - j \frac{6M\pi}{N^2} \gamma^{-3\delta_0} \geq \frac{9}{10}, \quad j = 1, 2, 3, 4.
\]

Next, we use mathematical induction to complete the proof. We assume that (3.14) holds for \( 0 \leq m \leq K - 1 \) with \( K \leq \frac{3}{2} \varepsilon_1 \ln(\gamma^{-1})N - 1 \), then assumption (3.8) and assumption (3.12) holds for \( 0 \leq m \leq K - 1 \). We will show that (3.14) holds for \( m = K \). Actually, we only need to prove that

\[
\inf_{\tau \leq \eta \leq \tau + \Delta} h(\tau, \eta) \geq \frac{19}{20} - j \frac{6M\pi}{N^2} \gamma^{-3\delta_0}.
\]

If \( K \leq 4N \), then the frequency \( \xi \in I_{K+2} \) stay in the region of growth for \( t \in \cup_{m=0}^{K-1} I_m = [0, T_K] \), then we deduce from Lemma 3.6 that

\[
\hat{h}(T_K, \xi) \geq 1, \quad \text{for } \xi \in I_{K+2}, \quad K \leq 2\sqrt{\delta_0} \sqrt{-\ln \gamma N}.
\]

If \( 4N + 1 \leq K \leq \frac{3}{2} \varepsilon_1 \ln(\gamma^{-1})N - 1 \), the frequency \( \xi \in I_{K+2} \) will first belong to the waiting region for \( t \in \cup_{m=0}^{K-1} I_m = [0, T_{K-4N}] \). We get from Lemma 3.7 that

\[
\hat{h}(T_{K-4N}, \xi) \geq \frac{19}{20} \quad \text{for } \xi \in I_{K+2}, \quad 4N + 1 \leq K \leq \frac{3}{2} \varepsilon_1 \ln(\gamma^{-1})N - 1.
\]

Then such \( \xi \) belong to the region of growth for \( t \in \cup_{m=K-4N}^{K-1} I_m = [T_{K-4N}, T_K] \), and it holds from Lemma 3.6 that

\[
\hat{h}(T_K, \xi) \geq \frac{19}{20}, \quad \text{for } \xi \in I_{K+2}, \quad 4N + 1 \leq K \leq \frac{3}{2} \varepsilon_1 \ln(\gamma^{-1})N - 1.
\]

Thus, for all \( 1 \leq K \leq \frac{3}{2} \varepsilon_1 \ln(\gamma^{-1})N - 1 \), we get from Lemma 3.5 that

\[
\inf_{\tau \leq \eta \leq \tau + \Delta} h(\tau, \eta) \geq \frac{19}{20} - j \frac{6M\pi}{N^2} \gamma^{-3\delta_0},
\]

and then (3.14) holds for \( K \).

Hence, by the principle of mathematical induction, (3.14) holds for all \( m \leq \frac{3}{2} \varepsilon_1 \ln(\gamma^{-1})N - 1 \). The desired result follows immediately.

Lemma 3.8 together with Lemma 3.5, 3.7 give the evolution of \( \hat{h}(t, \xi) \) in the excitation region, region of growth, and waiting region respectively. In the proof of Lemma 3.8, we only use the fact that \( \hat{h}(t, \xi) \) is growing in the region of growth, but do not care the amount of the growth. Next, we give the proof of Proposition 3.1 and show how the growth accumulated in the region of growth eventually produces an exponential growth.

Proof of Proposition 3.1. We prove the following statement by mathematical induction:

Let \( \{a_j\} \) and \( \{b_j\} \) satisfy \( a_0 = b_0 = 1 \), and \( j \geq 1 \),

\[
a_j = a_{j-1} + b_{j-1}, \quad b_j = a_{j-1} + 1.
\]

Then, for \( 1 \leq j \leq \frac{5}{4} \varepsilon_1 \ln(\gamma^{-1}) + 2 \),

\[
\text{10} \varepsilon_j \geq a_{j-1}, \quad \hat{h}(j \gamma^{-1}, \xi)|_{\xi \in [(j+2)\gamma^{-1}, (j+3)\gamma^{-1}]} \geq 1,
\]

\[
\hat{h}(j \gamma^{-1}, \xi)|_{\xi \in [j \gamma^{-1}, (j+1)\gamma^{-1}]} \geq a_j, \quad \hat{h}(j \gamma^{-1}, \xi)|_{\xi \in [(j+1)\gamma^{-1}, (j+2)\gamma^{-1}]} \geq b_j.
\]
where 
\[
\mathcal{E}_j \overset{\text{def}}{=} \inf_{(j-1)N \leq m \leq jN-1} \left( \inf_{\tau \in I_m} \left( \inf_{\eta \in \cup_{|i-m| \leq 1} I_i} \hat{h}(\tau, \eta) \right) \right).
\]

Base case \((j = 1, 2)\): From Lemma 3.8 we can see that 
\[
\mathcal{E}_1 \geq \frac{9}{10}.
\]

Then from Lemma 3.6 we know for \(\xi \in I_n\) with \(N + 2 \leq n \leq 2 + 4N\) that 
\[
\hat{h}(\gamma^{-1}, \xi) \geq \hat{h}(0, \xi) + \sum_{m=0}^{N-1} \frac{4M(n-m-2)}{9N^2} e^{-\frac{(n-m-2)^2}{2N^2}} \mathcal{E}_1
\]
\[
\geq \hat{h}(0, \xi) + \sum_{m=0}^{N-1} \frac{4M(n-m-2)}{9N^2} e^{-\frac{(n-m-2)^2}{2N^2}} \mathcal{E}_1
\]
\[
\geq \hat{h}(0, \xi) + \frac{4M\pi}{9N^2} e^{-\frac{\sigma^2}{2N^2}} d\sigma \mathcal{E}_1
\]
\[
\geq \hat{h}(0, \xi) + \frac{4M\pi}{9} \left( e^{-\frac{(n-N)^2}{2N^2}} - e^{-\frac{\sigma^2}{2N^2}} \right) \mathcal{E}_1.
\]

Similarly, from Lemma 3.5 and Lemma 3.8 for \(\xi \in I_n\) with \(n = N, N + 1\), it holds that 
\[
\hat{h}(\gamma^{-1}, \xi) \geq \hat{h}(0, \xi) + \frac{4M\pi}{9} \left( 1 - e^{-\frac{(N-2)^2}{2N^2}} \right) \mathcal{E}_1 = \frac{12M\pi\gamma^{-3\delta_0}}{N^2}.
\]

Then, for \(N \leq n \leq 3N - 1\), it holds that 
\[
\hat{h}(\gamma^{-1}, \xi)|_{\xi \in I_n} \geq \hat{h}(0, \xi) + \frac{4M\pi}{9} \left( e^{-\frac{(N-N)^2}{2N^2}} - e^{-\frac{\sigma^2}{2N^2}} \right) \mathcal{E}_1 = 1 + \frac{4M\pi}{9} (e^{-2} - e^{-\frac{9}{2}}) \mathcal{E}_1.
\]

Let 
\[
M \geq \frac{5}{2\pi (e^{-2} - e^{-\frac{9}{2}})},
\]

then we have 
\[
\hat{h}(\gamma^{-1}, \xi) \geq 1 + \frac{10}{9} \mathcal{E}_1 \geq 2, \text{ for } \xi \in [\gamma^{-1}, 3\gamma^{-1}].
\]

Therefor, using the same argument to Lemma 3.8 we can see that 
\[
\mathcal{E}_2 \geq \hat{h}(\gamma^{-1}, \eta)|_{\eta \in [\gamma^{-1}, 2\gamma^{-1}]} - \frac{1}{10} \geq \frac{19}{10},
\]

which will excite \(\hat{h}(2\gamma^{-1}, \xi)|_{\xi \in [2\gamma^{-1}, 4\gamma^{-1}]}\) to 
\[
\hat{h}(2\gamma^{-1}, \xi)|_{\xi \in [2\gamma^{-1}, 3\gamma^{-1}]} \geq \hat{h}(\gamma^{-1}, \xi)|_{\xi \in [2\gamma^{-1}, 3\gamma^{-1}]} + \frac{10}{9} \mathcal{E}_2 \geq 2 + \frac{10}{9} \frac{19}{10} \geq 4,
\]

and 
\[
\hat{h}(2\gamma^{-1}, \xi)|_{\xi \in [3\gamma^{-1}, 4\gamma^{-1}]} \geq \hat{h}(\gamma^{-1}, \xi)|_{\xi \in [3\gamma^{-1}, 4\gamma^{-1}]} + \frac{10}{9} \mathcal{E}_2 \geq 1 + \frac{10}{9} \frac{19}{10} \geq 3.
\]

We also have 
\[
\hat{h}(2\gamma^{-1}, \xi)|_{\xi \in [4\gamma^{-1}, 5\gamma^{-1}]} \geq 1.
\]

Inductive step: Suppose that the statement (3.15) holds for \(1, 2, \ldots, j\), then let us prove that it holds for \(j + 1\):
We have for the same reason as (3.18) that
\[
\frac{10}{9} E_{j+1} \geq \frac{10}{9} \left( \hat{h}(j\gamma^{-1}, \eta) \mid_{\eta \in [j\gamma^{-1}, (j+1)\gamma^{-1}]} - \frac{1}{10} \right) \geq a_j + \frac{a_j - 1}{9} \geq a_j.
\]
We also have that
\[
4M\pi \frac{9}{9} \left( e^{-\frac{(4N-N)^2}{2N^2}} - e^{-\frac{(4N)^2}{2N^2}} \right) E_j \geq \frac{4M\pi \cdot 9a_j}{10} \left( e^{-\frac{9}{2}} - e^{-8} \right) \geq \frac{1}{20}.
\]
Therefore, by Lemma 3.7, we deduce for \( 2 \leq j \leq \frac{5}{4} \tilde{c}_1 \ln(\gamma^{-1}) + 1 \) that
\[
\hat{h}(j\gamma^{-1}, \xi)_{|\xi \in [(j+2)\gamma^{-1}]} \geq \frac{19}{20} + \frac{4M\pi}{9} \left( e^{-\frac{(4N-N)^2}{2N^2}} - e^{-\frac{(4N)^2}{2N^2}} \right) \geq \frac{19}{20} + \frac{1}{20} = 1.
\]

Similar to (3.16), we have from (3.17) that
\[
\hat{h}((j+1)\gamma^{-1}, \xi)_{|\xi \in [(j+1)\gamma^{-1}, (j+2)\gamma^{-1}]} \geq \hat{h}(j\gamma^{-1}, \xi)_{|\xi \in [(j+1)\gamma^{-1}, (j+2)\gamma^{-1}]} + \frac{10}{9} E_{j+1} \geq b_j + a_j = a_{j+1},
\]
\[
\hat{h}((j+1)\gamma^{-1}, \xi)_{|\xi \in [(j+2)\gamma^{-1}, (j+3)\gamma^{-1}]} \geq \hat{h}(j\gamma^{-1}, \xi)_{|\xi \in [(j+2)\gamma^{-1}, (j+3)\gamma^{-1}]} + \frac{10}{9} E_{j+1} \geq 1 + a_j = b_{j+1}.
\]
That is, the statement also holds for \( j+1 \).

Recalling the definition of the series \( \{a_j\} \) and \( \{b_j\} \), one can see that
\[
b_j = a_{j-1} + 1 = a_{j-2} + b_{j-2} + 1 = b_{j-1} + b_{j-2}.
\]
So \( \{b_j\} \) is Fibonacci sequence and \( \{a_j\} \) is the summation of Fibonacci sequence. Therefore we know that there exists constants \( \tilde{c}_0 \geq \frac{1}{2} \) and \( \tilde{c}_1 \geq \ln \frac{2}{
\]
As a conclusion, at each time \( t \in [j\gamma^{-1}, (j+1)\gamma^{-1}] \) with \( 1 \leq j \leq \frac{5}{4} \tilde{c}_1 \ln(\gamma^{-1}) + 1 \) we have
\[
\hat{h}(t, \xi) \geq \left\{ \begin{array}{ll}
\tilde{c}_0 e^{-\tilde{c}_1 t} e^{\tilde{c}_1 \gamma t}, & \xi \in [(j+1)\gamma^{-1}, (j+2)\gamma^{-1}]; \\
19/20, & \xi \in [(j+2)\gamma^{-1}, 2\varepsilon_1 \gamma^{-1} \ln(\gamma^{-1}) - 1].
\end{array} \right.
\]
From (2.6), we can see that
\[
\|b_{in}\|_{L^2_y} = 4\pi \sqrt{\varepsilon_1 \gamma^{-1}} \sqrt{-\ln(\gamma)}.
\]
Therefore, for \( t \in [0, \frac{5}{4} \varepsilon_1 \gamma^{-1} \ln(\gamma^{-1})] \), there exists \( c_0, c_1 > 0 \) such that
\[
\|\hat{h}(t, \xi)\|_{L^2_x} \geq \tilde{c}_0 e^{-\tilde{c}_1 t} e^{\tilde{c}_1 \gamma t} \gamma^{-\frac{1}{2}} + \frac{19}{20} |2\varepsilon_1 \ln(\gamma^{-1}) \gamma^{-1} - t - 4\gamma^{-1}| \frac{1}{2}
\geq \tilde{c}_0 e^{c_1 \gamma t} \sqrt{\varepsilon_1 \gamma^{-1}} \sqrt{-\ln(\gamma)} = \frac{1}{\sqrt{2\pi}} c_0 e^{c_1 \gamma t} \|b_{in}\|_{L^2_y}.
\]
As \( \nu = \gamma^{-\frac{3}{20}} \) , it is clear that
\[
T = \varepsilon_1 \gamma^{-1} (\ln \nu^{-1}) \leq \frac{5}{4} \varepsilon_1 \gamma^{-1} \ln(\gamma^{-1}).
\]
Recalling that \( \hat{h}(t, \xi) \) we studied here is the 1 mode of \( \hat{h}(t, k, \xi) \), one can do the same estimate for the \(-1\) mode, and get for \( t \in [0, T] \) that
\[
\|P_{\pm} S_{\nu}(t) g_{in}\|_{L^2_{x,y}} = 2\pi \|\hat{h}_{\pm 1}(t)\|_{L^2_x} \geq \sqrt{2} c_0 e^{c_1 \gamma t} \|b_{in}\|_{L^2_x L^2_y} = c_0 e^{c_1 \gamma t} \|g_{in}\|_{L^2_x L^2_y},
\]
which gives Proposition 3.1 \( \square \)
As can be seen from the above proof, the result of Proposition 3.1 is still valid for $0 \leq \nu < \gamma^{-\frac{3}{2}}w_0$.

3.3. **Lower bound estimate for the inviscid problem.** In this subsection, we give the application of the new energy method to the inviscid problem. Similar to the viscous problem, we focus on the $k = 1$ mode, and study the following system:

\[
\begin{aligned}
\partial_t \hat{h}(t, \xi) &= \int_{\mathbb{R}} M \gamma^2 (\xi - \eta) e^{-\frac{\xi^2}{2} |\xi - \eta|^2} \frac{1}{(\eta - t)^2 + 1} \hat{h}(t, \eta) d\eta, \\
\hat{h}(0, \xi) &= \hat{h}_n(\xi) \geq 0.
\end{aligned}
\]

We also divide the time-frequency into small intervals, and study the evolution of $\hat{h}(t, \xi)$ during each small time interval. For $t \in I_m$, we divide into 4 cases depending on the frequency $\xi \in \mathbb{R} = \mathcal{I}_m^r \cup \mathcal{I}_m^e \cup \mathcal{I}_m^g \cup \mathcal{I}_m^w$, where

**Case 1.** $\xi \in \mathcal{I}_m^r = \left( \bigcup_{n \leq m-3} I_n \right) \cup \left( \bigcup_{n > 2 \varepsilon_1 \ln(\gamma^{-1})N} I_n \right)$;

**Case 2.** $\xi \in \mathcal{I}_m^e = \bigcup_{|n - m| \leq 2} I_n$;

**Case 3.** $\xi \in \mathcal{I}_m^g = \bigcup_{m + 3 \leq n \leq m + 2 + 4N} I_n$;

**Case 4.** $\xi \in \mathcal{I}_m^w = \bigcup_{m + 3 + 4N \leq n \leq 2 \varepsilon_1 \ln(\gamma^{-1})N} I_n$.

For Case 1, $\xi \in I_n$ is not in the excitation region nor the region of growth, we have from Lemma 3.3 with $\nu = 0$ that

\[
\|\hat{h}(t)\|_{L^\infty_\xi} \leq e^{M \pi \gamma t} \|\hat{h}_n\|_{L^\infty_\xi} = e^{M \pi \gamma t}, \quad \text{for } t \geq 0.
\]

For Case 2, by using the same argument in Lemma 3.5 one can see that the self interaction in the excitation region is weak.

**Lemma 3.9.** Let $\hat{h}$ be the solution of (3.20). For $t \in I_m$, $\xi \in I_n$, with $m \leq \frac{3}{2} \varepsilon_1 \ln(\gamma^{-1})N - 1$ and $|n - m| \leq 2$, it holds that

\[
\hat{h}(T_m, \xi) - \frac{6 M \pi}{N^2} e^{\frac{M \pi (m-1)}{N}} \leq \hat{h}(t, \xi) \leq \hat{h}(T_m, \xi) + \frac{6 M \pi}{N^2} e^{\frac{M \pi (m-1)}{N}}.
\]

For Case 3, we treat the region of growth and give the growing rate.

**Lemma 3.10.** Let $\hat{h}$ be the solution of (3.20). There exists $\varepsilon_1 > 0$ such that, for $t \in I_m$, $\xi \in I_n$ with $m \leq \frac{3}{2} \varepsilon_1 \ln(\gamma^{-1})N - 1$, and $m + 3 \leq n \leq m + 2 + 4N$, if it holds that

\[
\inf_{\tau \in I_m} \inf_{\eta \in \bigcup_{|\xi - m| \leq 1} I_t} \hat{h}(\tau, \eta) \geq \frac{9}{10},
\]

we have

\[
\hat{h}(t, \xi) \geq \hat{h}(T_m, \xi) + (t - T_m) \frac{4 \gamma M (n - m - 2) \pi}{9 N} e^{-\frac{(n - m - 2)^2}{2 N^2}} \inf_{\tau \in I_m} \inf_{\eta \in \bigcup_{|\xi - m| \leq 1} I_t} \hat{h}(\tau, \eta).
\]

**Proof.** Similar to the viscous problem, we write

\[
\partial_t \hat{h}(t, \xi) = \int_{\mathbb{R}} M \gamma^2 (\xi - \eta) e^{-\frac{\xi^2}{2} |\xi - \eta|^2} \frac{1}{(\eta - t)^2 + 1} \hat{h}(t, \eta) d\eta
\]

\[
= \int_{\bigcup_{\xi \in Z, |\xi - m| \leq 1} I_t} M \gamma^2 (\xi - \eta) e^{-\frac{\gamma^2}{2} |\xi - \eta|^2} \frac{1}{(\eta - t)^2 + 1} \hat{h}(t, \eta) d\eta
\]

\[
+ \int_{\bigcup_{\xi \in Z, |\xi - m| > 1} I_t} M \gamma^2 (\xi - \eta) e^{-\frac{\gamma^2}{2} |\xi - \eta|^2} \frac{1}{(\eta - t)^2 + 1} \hat{h}(t, \eta) d\eta
\]

\[
= A_m + B_m,
\]
and have
\[ A_m \geq \frac{M\gamma(n-m-2)}{N} e^{-\alpha t} \frac{\varepsilon}{9} \inf_{\tau \in I_m} \inf_{\eta \in |\omega_{i-m}| \leq 1} \hat{h}(\tau, \eta), \]
\[ |B_m| \leq 2MN \gamma^2 e^{M\pi t} \leq 2MN \gamma^2 \frac{3M\pi}{4}. \]

By taking
\[ \varepsilon_1 = \frac{1}{9M\pi}, \]
one can easily check that \(|B_m| \leq \frac{1}{2}A_m\), which gives the result of this lemma.

For case 4, we have a similar estimate to the one in the viscous case.

**Lemma 3.11.** Let \( \hat{h} \) be the solution of (3.20). For \( t \in I_m \) and \( \xi \in I_n \) with \( m \leq \frac{3}{2} \varepsilon_1 \ln(\gamma^{-1})N-1 \), \( m + 3 + 4N \leq n \), \( |\omega| \leq 2\varepsilon_1 \ln(\gamma^{-1})N \), and \( \varepsilon_1 \) given in Lemma 3.10, if it holds that
\[ \inf_{0 \leq m' \leq m} \inf_{\tau \in I_m} \inf_{\eta \in |\omega_{i-m}| \leq 1} \hat{h}(\tau, \eta) \geq 0, \]
we have
\[ \hat{h}(t, \xi) \geq \frac{19}{20}. \]

**Lemma 3.12.** With our well chosen initial data, assumption (3.22) in Lemma 3.10 and assumption (3.25) in Lemma 3.11 are automatically satisfied.

**Proof of Proposition 3.2.** Proposition 3.2 can be proved by combining Lemma 3.9, 3.12 and following the proof of Proposition 3.1. We omit the details.

4. **Nonlinear Instability of the Viscous Flow**

In this section, we focus on the nonlinear system of the viscous flow and give the proof of Theorem 1.2. The nonlinear instability is produced from the linear instability which was derived by the new energy method in Section 3.

We rewrite (1.3) in the following form:
\[ \begin{cases} \partial_t \omega_{\Delta} + y \partial_x \omega_{\Delta} - \partial_y^2 b \partial_x (\Delta)^{-1} \omega_{\Delta} - \nu \Delta \omega_{\Delta} = -\mathcal{L} - \mathcal{N}^{(1)} - \mathcal{N}^{(2)} - \mathcal{N}^{(3)}, \\ \omega_{\Delta}|_{t=0} = \omega_0 \end{cases} \]
\[ \begin{cases} (u_{\Delta}^{(1)}, u_{\Delta}^{(2)}) = (\partial_y (\Delta)^{-1} \omega_{\Delta}, -\partial_x (\Delta)^{-1} \omega_{\Delta}), \\ \omega_{\Delta}|_{t=0} = \omega_0 = \frac{\varepsilon_1 \nu \pi^2 + \delta_b}{\sqrt{\nu \pi} \gamma} g_0, \end{cases} \]
where \( b(t, y) \) is given in (2.1), \( g_0 \) is given in (2.7) which has only \( \pm 1 \) modes, \( \varepsilon_1 = \frac{2\delta_b}{M\pi} \) given in (3.10), and \( \gamma = \nu \frac{1}{2} \frac{\varepsilon_1 \pi^2}{\delta_b} \).

Here
\[ \mathcal{L} = (b - y) \partial_y \omega_{\Delta}, \quad \mathcal{N}^{(1)} = (u_{\Delta}^{(1)} \partial_x \omega_{\Delta})_{\Delta} + (u_{\Delta}^{(2)} \partial_y \omega_{\Delta})_{\Delta}, \]
\[ \mathcal{N}^{(2)} = u_{\Delta}^{(1)} \partial_y \omega_{\Delta}, \quad \mathcal{N}^{(3)} = u_{\Delta}^{(2)} \partial_y \omega_0, \]
\( \omega_0(t, y) \) is the 0 mode of vorticity which satisfies
\[ \begin{cases} \partial_t \omega_0 - \nu \partial_y^2 \omega_0 = - (u_{\Delta}^{(1)} \partial_x \omega_{\Delta})_{0} - (u_{\Delta}^{(2)} \partial_y \omega_{\Delta})_{0}, \\ \omega_0|_{t=0} = 0, \end{cases} \]
and \( u^{(1)}_0 \) is the 0 mode of horizontal velocity which satisfies
\[
\begin{cases}
\partial_t u^{(1)}_0 - \nu \partial_y^2 u^{(1)}_0 = -(u^{(1)}_\neq \partial_x u^{(1)}_\neq )_0 - (u^{(2)}_\neq \partial_y u^{(1)}_\neq )_0 \\
u u^{(1)}_0 |_{t=0} = 0.
\end{cases}
\]

Then we have
\[
\omega_\neq(t, x, y) = S_\nu(t) \omega_{in}(x, y) - \int_0^t S_\nu(t-s)(\mathcal{L} + \mathcal{N}^{(1)} + \mathcal{N}^{(2)} + \mathcal{N}^{(3)})(s, x, y) ds,
\]
(4.2) \( \omega_\neq(t, x, y) = -\int_0^t e^{(t-s)\nu \partial_y^2} \left( (u^{(1)}_\neq \partial_x u^{(1)}_\neq )_0 + (u^{(2)}_\neq \partial_y u^{(1)}_\neq )_0 \right)(s, x, y) ds,
\]
(4.3) \[ u^{(1)}_0(t, y) = -\int_0^t e^{(t-s)\nu \partial_y^2} \left( (u^{(1)}_\neq \partial_x u^{(1)}_\neq )_0 + (u^{(2)}_\neq \partial_y u^{(1)}_\neq )_0 \right)(s, x, y) ds. \]

From (2.6) and (2.7), we have
\[
\|\omega_{in}\|_{F_{\mathcal{L}} L^2_y} = \frac{\varepsilon_0 \nu^{2+\delta_1} \sqrt{1 - \frac{1}{4} \delta_0}}{\sqrt{2\pi e\sqrt{1 - \ln \gamma}}} \|g_{in}\|_{L^2_{x,y}} = \frac{\varepsilon_0 \nu^{2+\delta_1} \sqrt{1 - \frac{1}{4} \delta_0}}{\sqrt{2\pi e\sqrt{1 - \ln \gamma}}} \|h_{in}\|_{L^2_y} = \frac{\sqrt{8\varepsilon_0 \nu^{2+\delta_1}}}{\sqrt{2\pi e\sqrt{1 - \ln \gamma}}} \|s\|_{L^2_{x,y}}.
\]

We first give upper bound estimates for the semi-group \( S_\nu(t) \), then prove the upper bound estimates for the solution of (4.1) by Duhamel’s principle and bootstrap argument. Compared to [52], here we modify the function spaces of the solution which simplifies the proof. Then we prove the lower bound estimates in Theorem [1,2] by using the linear lower bound estimate obtained in Proposition [3,4] together with the upper bound estimates.

4.1. Upper bound estimates of \( S_\nu(t) \). In this subsection, we give upper bound estimates of the solution to (2.3) with \( \gamma = \nu^{\frac{1}{4}} - \frac{1}{2} \delta_0 \) from general initial data.

**Proposition 4.1.** Given \( f(x, y) \in F_{\mathcal{L}} L^2_y(\mathbb{T} \times \mathbb{R}) \) such that \( \int_{\mathbb{R}} f(x, y) dx = 0 \). There exists constant \( C_0, C_1 > 0 \) such that for any \( T \geq 0 \),
\[
\|e^{-\frac{1}{2}C_0 M^2 \gamma_1} S_\nu(t) f\|_{L^\infty_2([0, T]; \mathcal{F}_{\mathcal{L}} L^2_{y})} \leq C_1 \|f\|_{\mathcal{F}_{\mathcal{L}} L^2_{y}};
\]
(4.6) \[ \|e^{-\frac{1}{2}C_0 M^2 \gamma_1} \nabla S_\nu(t) f\|_{L^2_2([0, T]; \mathcal{F}_{\mathcal{L}} L^2_{y})} \leq C_1 \nu^{-\frac{1}{2}} \|f\|_{\mathcal{F}_{\mathcal{L}} L^2_{y}}; \]
(4.7) \[ \|e^{-\frac{1}{2}C_0 M^2 \gamma_1} \partial_x S_\nu(t) f\|_{L^2_1([0, T]; \mathcal{F}_{\mathcal{L}} L^2_{y})} \leq C_1 \nu^{-\frac{1}{2}} \|f\|_{\mathcal{F}_{\mathcal{L}} L^2_{y}}; \]
(4.8) \[ \|e^{-\frac{1}{2}C_0 M^2 \gamma_1} \nabla (-\Delta)^{-1} S_\nu(t) f\|_{L^\infty_2([0, T]; \mathcal{F}_{\mathcal{L}} L^2_{y})} \leq C_1 \|f\|_{\mathcal{F}_{\mathcal{L}} L^2_{y}}; \]
(4.9) \[ \|e^{-\frac{1}{2}C_0 M^2 \gamma_1} \partial_x (-\Delta)^{-1} S_\nu(t) f\|_{L^2_1([0, T]; \mathcal{F}_{\mathcal{L}} L^2_{y})} \leq C_1 \|f\|_{\mathcal{F}_{\mathcal{L}} L^2_{y}}. \]
(4.10)

Here \( S_\nu(t) f(x, y) \) denotes the solution of (2.3) with initial data \( f(x, y) \) defined in Section 3.

**Proof.** By the change of coordinate, it suffices to study the system (2.5) with initial data \( \hat{h} \xi(0, \xi) = \hat{f}(\xi) \). We introduce the ghost type weight \( e^{\arctan(\frac{\xi}{\pi})} \) which satisfies
\[
e^{-\pi} \leq e^{\arctan(\frac{\xi}{\pi})} \leq e^\pi.
\]
This kind of weight is first used by Alinhac [2], and it is useful in studying stability problem in fluid flow, see [3,12].

By energy method, we have
\[
\partial_t \int_\mathbb{R} e^{2 \arctan(\frac{\xi}{\pi})} |\hat{h}(t, \xi)|^2 d\xi
\]
By Hölder’s and Young’s convolution inequality, we have

\[ III. \]

\[ = -2 \int_{\mathbb{R}} \frac{|k|^2}{(\xi - kt)^2 + k^2} e^{2\arctan\left(\frac{\xi}{k} - t\right)} |\hat{h}_k(t, \xi)|^2 d\xi \]

\[ + 2 \int_{\mathbb{R}} \int_{\mathbb{R}} M \gamma^2 (\xi - \eta)e^{-(\nu t + \frac{\eta^2}{2})|\xi-\eta|^2} \frac{k}{(\eta - kt)^2 + k^2} \hat{h}_k(t, \eta) d\eta \ e^{2\arctan\left(\frac{\xi}{k} - t\right)} \hat{h}_k(t, \xi) d\xi \]

\[ - 2 \int_{\mathbb{R}} \nu e^{2\arctan\left(\frac{\xi}{k} - t\right)} (k^2 + (\xi - kt)^2) |\hat{h}_k(t, \xi)|^2 d\xi \]

\[ \overset{\text{def}}{=} I + II + III. \]

which implies

\[ \partial_t \int_{\mathbb{R}} e^{2\arctan\left(\frac{\xi}{k} - t\right)} |\hat{h}_k(t, \xi)|^2 d\xi \]

\[ \leq -2 \int_{\mathbb{R}} \frac{|k|^2}{(\xi - kt)^2 + k^2} e^{2\arctan\left(\frac{\xi}{k} - t\right)} |\hat{h}_k(t, \xi)|^2 d\xi \]

\[ + CM\gamma^2 \left( \int_{\mathbb{R}} \frac{|k|^2}{(\xi - kt)^2 + k^2} |\hat{h}_k(t, \xi)|^2 d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \frac{1}{(\xi - kt)^2 + k^2} d\xi \right)^{\frac{1}{2}} \ ||e^{2\arctan\left(\frac{\xi}{k} - t\right)} \hat{h}_k(t, \xi)||_{L_x^2} \]

\[ \leq CM\gamma^2 \left( \int_{\mathbb{R}} \frac{|k|^2}{(\xi - kt)^2 + k^2} e^{2\arctan\left(\frac{\xi}{k} - t\right)} |\hat{h}_k(t, \xi)|^2 d\xi \right)^{\frac{1}{2}} \ ||e^{2\arctan\left(\frac{\xi}{k} - t\right)} \hat{h}_k(t, \xi)||_{L_x^2}, \]

From which we deduce that

\[ \partial_t \int_{\mathbb{R}} e^{-C_0 M^2 \gamma t} e^{2\arctan\left(\frac{\xi}{k} - t\right)} |\hat{h}_k(t, \xi)|^2 d\xi \]

\[ \leq - \int_{\mathbb{R}} e^{-C_0 M^2 \gamma t} e^{2\arctan\left(\frac{\xi}{k} - t\right)} \frac{|k|^2}{(\xi - kt)^2 + k^2} |\hat{h}_k(t, \xi)|^2 d\xi \]

\[ - 2 \int_{\mathbb{R}} \nu e^{-C_0 M^2 \gamma t} e^{2\arctan\left(\frac{\xi}{k} - t\right)} (k^2 + (\xi - kt)^2) |\hat{h}_k(t, \xi)|^2 d\xi. \]
It follows that
\[
\int_{\mathbb{R}} e^{-\frac{1}{2}C_0M^2\gamma t}e^{2\arctan \left( \frac{\xi}{k} \right) - t} |\hat{h}_k(t, \xi)|^2 d\xi \\
+ \int_{0}^{t} \int_{\mathbb{R}} e^{-C_0M^2\gamma s} \frac{|k|^2}{(\xi - ks)^2 + k^2} e^{2\arctan \left( \frac{\xi}{k} - s \right)} |\hat{h}_k(s, \xi)|^2 d\xi ds \\
+ 2 \int_{0}^{t} \int_{\mathbb{R}} v e^{-C_0M^2\gamma s} e^{2\arctan \left( \frac{\xi}{k} - s \right) (k^2 + (\xi - ks)^2)} |\hat{h}_k(s, \xi)|^2 d\xi ds \\
\leq \int_{\mathbb{R}} e^{2\arctan \left( \frac{\xi}{k} \right) - \nu} |\hat{f}(\xi)|^2 d\xi.
\]
(4.12)

Summing up the above inequality in \( k \), we get
\[
\|e^{-\frac{1}{2}C_0M^2\gamma t} \hat{h} \|_{L^\infty_{t}([0,T]; L^2_{\theta})} \leq C_1 \|f\|_{L^1_{t} L^2_{\theta}},
\]
and
\[
\|e^{-\frac{1}{2}C_0M^2\gamma t} (k, \xi - kt) \hat{h} \|_{L^2_{t}([0,T]; L^2_{\theta})} \leq C_1 \nu^{-\frac{1}{2}} \|f\|_{L^1_{t} L^2_{\theta}}.
\]

As
\[
\|\hat{h}_k(t)\|_{L^2_{\theta}} = \|(S_\nu(t)f)_k\|_{L^2_{\theta}}, \quad \|(k, \partial_y - ikt) \hat{h}_k(t)\|_{L^2_{\theta}} = \|(k, \partial_y)(S_\nu(t)f)_k\|_{L^2_{\theta}}
\]
the estimates (4.6) and (4.7) follows immediately.

Next, we prove (4.8). By using Cauchy inequality, we have
\[
\int_{\mathbb{R}} \nu^{\frac{1}{2}} |k| e^{-C_0M^2\gamma t} e^{2\arctan \left( \frac{\xi}{k} - t \right)} |\hat{h}_k(t, \xi)|^2 d\xi \\
\leq \int_{\mathbb{R}} e^{-C_0M^2\gamma t} \frac{|k|^2}{(\xi - kt)^2 + k^2} e^{2\arctan \left( \frac{\xi}{k} - t \right)} |\hat{h}_k(t, \xi)|^2 d\xi \\
+ \int_{\mathbb{R}} v e^{-C_0M^2\gamma t} e^{2\arctan \left( \frac{\xi}{k} - t \right) (k^2 + (\xi - kt)^2)} |\hat{h}_k(t, \xi)|^2 d\xi.
\]

Then, we deduce from (4.11) that
\[
\partial_t \int_{\mathbb{R}} e^{-C_0M^2\gamma t} e^{2\arctan \left( \frac{\xi}{k} - t \right)} |\hat{h}_k(t, \xi)|^2 d\xi \leq -\nu^{\frac{1}{2}} |k| \left( \int_{\mathbb{R}} e^{-C_0M^2\gamma t} e^{2\arctan \left( \frac{\xi}{k} - t \right)} |\hat{h}_k(t, \xi)|^2 d\xi \right)^{\frac{1}{2}}.
\]

Therefore,
\[
\partial_t \left( \int_{\mathbb{R}} e^{-C_0M^2\gamma t} e^{2\arctan \left( \frac{\xi}{k} - t \right)} |\hat{h}_k(t, \xi)|^2 d\xi \right)^{\frac{1}{2}} \leq -\nu^{\frac{1}{2}} |k| \left( \int_{\mathbb{R}} e^{2\arctan \left( \frac{\xi}{k} \right) - \nu} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.
\]

Then, it follows from Gronwall’s inequality that
\[
\left( \int_{\mathbb{R}} e^{-C_0M^2\gamma t} e^{2\arctan \left( \frac{\xi}{k} - t \right)} |\hat{h}_k(t, \xi)|^2 d\xi \right)^{\frac{1}{2}} \leq e^{-\nu^{\frac{1}{2}} |k| t} \left( \int_{\mathbb{R}} e^{2\arctan \left( \frac{\xi}{k} \right) - \nu} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}},
\]
and
\[
\int_{0}^{T} \left( \int_{\mathbb{R}} e^{-C_0M^2\gamma t} e^{2\arctan \left( \frac{\xi}{k} - t \right)} |\hat{h}_k(t, \xi)|^2 d\xi \right)^{\frac{1}{2}} dt \leq C\nu^{-\frac{1}{2}} |k|^{-1} \left( \int_{\mathbb{R}} e^{2\arctan \left( \frac{\xi}{k} \right) - \nu} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.
\]

As a result, we get
\[
\|e^{-\frac{1}{2}C_0M^2\gamma t} k |\hat{h}_k(s, \xi)| \|_{L^2_{t} L^2_{\theta}} \leq C\nu^{-\frac{1}{2}} \|\hat{f}(\xi)\|_{L^2_{\theta}},
\]
and (4.8) follows immediately.
Recalling inequality (4.12) and the fact that \( \tilde{f}_k(y) = 0 \) for \( k = 0 \), we have \( \tilde{h}_k(t, y) \equiv 0 \) and
\[
\| e^{-\frac{1}{2} C_0 M^2 \gamma t} \nabla (\Delta)^{-1} S_\nu(t) f \|_{L^\infty_T([0,T]; \mathcal{F}_1 L_k^1 L_\infty^\infty)}
\leq C \sum_{k \in \mathbb{Z}/0} \left| e^{-\frac{1}{2} C_0 M^2 \gamma t} \frac{|k|}{k^2 + (\xi - k t)^2} |\tilde{h}_k(t, \xi)| \right|_{L^\infty_T([0,T])}
\leq C \sum_{k \in \mathbb{Z}/0} \| e^{-\frac{1}{2} C_0 M^2 \gamma t} \tilde{h}_k(t, \xi) \|_{L^\infty_T([0,T]; L_k^2)}
\leq C \sum_{k \in \mathbb{Z}/0} \| \tilde{f}_k(\xi) \|_{L_k^2} = C \| f \|_{\mathcal{F}_1 L_k^1 L_\infty^\infty},
\]
which is (4.9).

At last, we turn to (4.10). By using (4.12), we deduce that
\[
\| e^{-\frac{1}{2} C_0 M^2 \gamma t} \partial_x (\Delta)^{-1} S_\nu(t) f \|_{L^2_T([0,T]; \mathcal{F}_1 L_k^1 L_\infty^\infty)}
\leq \sum_{k \in \mathbb{Z}/0} \left| e^{-\frac{1}{2} C_0 M^2 \gamma t} \frac{|k|}{k^2 + (\xi - k t)^2} |\tilde{h}_k(t, \xi)| \right|_{L^2_T([0,T]; L_k^1)}
\leq C \sum_{k \in \mathbb{Z}/0} \left( \int_0^T \int \frac{|k|^2}{k^2 + (\xi - k s)^2} e^{2 \arctan \left( \frac{\xi}{k} \right)} |\tilde{h}_k(s, \xi)|^2 \, d\xi \, ds \right)^\frac{1}{2}
\leq C \sum_{k \in \mathbb{Z}/0} \| \tilde{f}_k(\xi) \|_{L_k^2} = C \| f \|_{\mathcal{F}_1 L_k^1 L_\infty^\infty},
\]
which gives (4.10).

4.2. Upper bound estimates for the nonlinear equation. In this subsection, we use the bootstrap argument to prove the upper bound estimates for the solution of (4.1).

Suppose for some \( 0 \leq \bar{T} \leq T = \varepsilon_1 \gamma^{-1} \ln(\nu^{-1}) \), the following inequalities hold:
\[
(4.13) \quad \| e^{-\frac{1}{2} C_0 M^2 \gamma t} \omega \|_{L^\infty_T([0,T]; \mathcal{F}_1 L_k^1 L_\infty^\infty)} \leq 2 C_2 \| \omega \|_{\mathcal{F}_1 L_k^1 L_\infty^\infty};
(4.14) \quad \| e^{-\frac{1}{2} C_0 M^2 \gamma t} \partial_y \omega \|_{L^2_T([0,T]; \mathcal{F}_1 L_k^1 L_\infty^\infty)} \leq 2 C_2 \nu^{-\frac{1}{2}} \| \omega \|_{\mathcal{F}_1 L_k^1 L_\infty^\infty};
(4.15) \quad \| e^{-\frac{1}{2} C_0 M^2 \gamma t} \partial_y u \omega \|_{L^2_T([0,T]; \mathcal{F}_1 L_k^1 L_\infty^\infty)} \leq 2 C_2 \nu^{-\frac{1}{2}} \| \omega \|_{\mathcal{F}_1 L_k^1 L_\infty^\infty};
(4.16) \quad \| e^{-\frac{1}{2} C_0 M^2 \gamma t} u^{(1)} \|_{L^\infty_T([0,T]; \mathcal{F}_1 L_k^1 L_\infty^\infty)} \leq 2 C_2 \| \omega \|_{\mathcal{F}_1 L_k^1 L_\infty^\infty};
(4.17) \quad \| e^{-\frac{1}{2} C_0 M^2 \gamma t} u^{(2)} \|_{L^2_T([0,T]; \mathcal{F}_1 L_k^1 L_\infty^\infty)} \leq 2 C_2 \| \omega \|_{\mathcal{F}_1 L_k^1 L_\infty^\infty};
(4.18) \quad \| e^{-\frac{1}{2} C_0 M^2 \gamma t} u^{(1)} \|_{L^\infty_T([0,T]; \mathcal{F}_1 L_k^1 L_\infty^\infty)} \leq 2 C_2 \| \omega \|_{\mathcal{F}_1 L_k^1 L_\infty^\infty};
(4.19) \quad \| e^{-\frac{1}{2} C_0 M^2 \gamma t} \partial_y \omega \|_{L^2_T([0,T]; \mathcal{F}_1 L_k^1 L_\infty^\infty)} \leq 2 C_2 \nu^{-\frac{1}{2}} \| \omega \|_{\mathcal{F}_1 L_k^1 L_\infty^\infty},
\]
where \( C_2 > 0 \) is a constant which will be determined later.

**Proposition 4.2 (Bootstrap).** Let \( \nu > 0 \) be small enough, and \( \omega \) be the solution of (4.1) for some \( 0 < \bar{T} \leq \varepsilon_1 \gamma^{-1} \ln(\nu^{-1}) \) with \( \gamma = \nu^{-\frac{1}{2}} \varepsilon_0 \), the estimates (4.13)-(4.19) hold for some constant \( C_2 > 0 \) on \([0, \bar{T}]\). Then there exists constants \( \delta_1, \varepsilon_0 > 0 \) so that these same estimates hold with all the occurrences of \( 2 \) on the right-hand side replaced by \( 1 \).

We first give some auxiliary lemmas.
Lemma 4.3. Let
\[ f(t, x, y) \in \tilde{L}^{p_1}_t \left( [0, \widetilde{T}]; \mathcal{F}L^1_y L_y^q (\mathbb{T} \times \mathbb{R}) \right), \quad g(t, x, y) \in \tilde{L}^{p_2}_t \left( [0, \widetilde{T}]; \mathcal{F}L^1_y L_y^q (\mathbb{T} \times \mathbb{R}) \right), \]
then we have
\[ \|fg\|_{\tilde{L}^p_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)} \leq \|f\|_{\tilde{L}^{p_1}_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)} \|g\|_{\tilde{L}^{p_2}_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)}, \]
where \( p, p_1, p_2, q, q_1, q_2 > 0 \) are constants satisfying
\[ (4.20) \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}. \]
As
\[ \|f\|_{\tilde{L}^{p_1}_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)} + \|g\|_{\tilde{L}^{p_2}_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)} = \|fg\|_{\tilde{L}^p_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)}, \]
it follows from (4.20) that
\[ \|fg\|_{\tilde{L}^p_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)} \leq \|f\|_{\tilde{L}^{p_1}_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)} \|g\|_{\tilde{L}^{p_2}_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)}. \]
Proof. By using Hölder’s inequality and Young’s convolution inequality, we have
\[
\|fg\|_{\tilde{L}^p_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)} \\
= \sum_{k \in \mathbb{Z}} \left( \int_0^{\widetilde{T}} \left( \int_\mathbb{R} \left| \int_\mathbb{Z} \hat{f}_j \hat{g}_{k-j} (t, y) \right|^q dy \right) dt \right)^{\frac{1}{p}} \\
\leq \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \left( \int_0^{\widetilde{T}} \left( \int_\mathbb{R} \left| \int_\mathbb{Z} \hat{f}_j \hat{g}_{k-j} (t, y) \right|^q dy \right) dt \right)^{\frac{1}{p}} \\
\leq \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \left( \int_0^{\widetilde{T}} \left( \int_\mathbb{R} |f_j(t,y)|^q dy \right) dt \right)^{\frac{1}{p_1}} \left( \int_0^{\widetilde{T}} \left( \int_\mathbb{R} |g_{k-j}(t,y)|^{q_2} dy \right) dt \right)^{\frac{1}{p_2}} \\
= \|f\|_{\tilde{L}^{p_1}_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)} \|g\|_{\tilde{L}^{p_2}_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)},
\]
which gives the lemma. □

Lemma 4.4. Under the bootstrap assumptions (4.13)–(4.19), it holds that
\[ \|e^{-\frac{1}{2}C_0 M^2 \gamma t} \mathcal{L}\|_{\tilde{L}^1_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)} \leq CC_2 M^{-2} \nu^{-\frac{1}{2}} \|\omega_{\text{in}}\|_{\mathcal{F}L^1_y L^q_y}, \]
and
\[ \sum_{i=1,2,3} \|e^{-\frac{1}{2}C_0 M^2 \gamma t} \mathcal{A}(i)\|_{\tilde{L}^1_t([0,\widetilde{T}];\mathcal{F}L^1_y L^q_y)} \leq CC_2 e^{\frac{1}{2}C_0 M^2 \gamma \widetilde{T}} \varepsilon_0 \delta_1 \|\omega_{\text{in}}\|_{\mathcal{F}L^1_y L^q_y}. \]
We also have
\[
\sum_{i=1} \|e^{-\frac{1}{2}C_0 M^2 \gamma t} (u^{(1)} \frac{\partial_x u^{(1)}}{\partial x} + u^{(2)} \frac{\partial_y u^{(1)}}{\partial y})\|_{\tilde{L}^1_t([0,\widetilde{T}];L^q_y)} \\
+ \|e^{-\frac{1}{2}C_0 M^2 \gamma t} (u^{(1)} \frac{\partial_x \omega^{(1)}}{\partial x} + u^{(2)} \frac{\partial_y \omega^{(1)}}{\partial y})\|_{\tilde{L}^1_t([0,\widetilde{T}];L^q_y)}
\]
\[ \leq C \varepsilon_0 C^2 e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} \varepsilon_0 \gamma^1 \| \omega \|_{FL^{1,1}_y} \].

**Proof.** Recalling that \( \| b_\nu(t, y) - y \|_{L^\infty_y} = \pi M^2 \), we get from (4.14) that

\[ \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} \|_{L^1_y([0, \tilde{T}], FL^{1,1}_y)} \]

\[ \leq ||b_\nu(t, y) - y||_{L^\infty_y([0, \tilde{T}], L^\infty_y)} e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} \| \tilde{L}^1_y([0, \tilde{T}], FL^{1,1}_y) \]

\[ \leq C \varepsilon_0 C^2 e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} \varepsilon_0 \gamma^1 \| \omega \|_{FL^{1,1}_y}. \]

From (4.5), (4.14)-(4.19), and Lemma 4.3, we get that

\[ \sum_{i=1,2,3} \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} N^{(i)} \|_{L^1_y([0, \tilde{T}], FL^{1,1}_y)} \]

\[ \leq \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} (u^{(1)} \partial_x u^{(1)}) \|_{L^1_y([0, \tilde{T}], FL^{1,1}_y)} + \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} (u^{(2)} \partial_y u^{(1)}) \|_{L^1_y([0, \tilde{T}], FL^{1,1}_y)} \]

\[ + \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} (u^{(1)} \partial_x u^{(1)}) \|_{L^1_y([0, \tilde{T}], FL^{1,1}_y)} + \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} (u^{(2)} \partial_y u^{(1)}) \|_{L^1_y([0, \tilde{T}], FL^{1,1}_y)} \]

\[ \leq 16 C_0^2 e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} \| \omega \|_{FL^{1,1}_y}. \]

A similar argument shows that

\[ \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} (u^{(1)} \partial_x u^{(1)}) \|_{L^1_y([0, \tilde{T}], L^2_y)} + \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} (u^{(1)} \partial_x u^{(1)}) \|_{L^1_y([0, \tilde{T}], L^2_y)} \]

\[ + \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} (u^{(2)} \partial_y u^{(1)}) \|_{L^1_y([0, \tilde{T}], L^2_y)} + \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} (u^{(2)} \partial_y u^{(1)}) \|_{L^1_y([0, \tilde{T}], L^2_y)} \]

\[ \leq C \varepsilon_0 C^2 e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} \| \omega \|_{FL^{1,1}_y}. \]

Here we use the fact that

\[ \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} \partial_x u^{(1)} \|_{L^1_y([0, \tilde{T}], L^2_y)} + \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} \partial_y u^{(1)} \|_{L^2_y([0, \tilde{T}], L^2_y)} \]

\[ \leq \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} \partial_x u \|_{L^1_y([0, \tilde{T}], L^2_y)} + \| e^{-\frac{1}{2} C_0 M^2 \gamma \tilde{T}} \partial_y u \|_{L^2_y([0, \tilde{T}], L^2_y)}. \]

Now, we are able to prove Proposition 4.2.
Proof of Proposition 4.2. We first give the estimates for $u^{(1)}_0$ and $\omega_0$. By using (4.3), (4.4), (4.14) and (4.17), Lemma 4.4 and the properties of heat kernel, we get for $0 \leq t \leq \tilde{T}$ that

$$
\| e^{-\frac{1}{2}C_0M^2\gamma_t} u^{(1)}_0(t) \|_{L^2_y} + \| e^{-\frac{1}{2}C_0M^2\gamma_t} \omega_0(t) \|_{L^2_y} 
\leq \| e^{-\frac{1}{2}C_0M^2\gamma_t} \int_0^t e^{(s-t)\nu\partial_y^2} \left( \left( \frac{1}{2} \partial_y \omega_0 \right)_0 + \left( \frac{1}{2} \partial_y \omega_0 \right)_0 \right) (s, y) ds \|_{L^2_y} 
+ \| e^{-\frac{1}{2}C_0M^2\gamma_t} \int_0^t e^{(s-t)\nu\partial_y^2} \left( \left( \frac{1}{2} \partial_y \omega_0 \right)_0 + \left( \frac{1}{2} \partial_y \omega_0 \right)_0 \right) (s, y) ds \|_{L^2_y} 
\leq \| e^{-\frac{1}{2}C_0M^2\gamma_t} \left( \frac{1}{2} \partial_y \omega_0 \right)_0 \|_{L^1([0, \tilde{T}]; L^2_y)} + \| e^{-\frac{1}{2}C_0M^2\gamma_t} \left( \frac{1}{2} \partial_y \omega_0 \right)_0 \|_{L^1([0, \tilde{T}]; L^2_y)} 
+ \| e^{-\frac{1}{2}C_0M^2\gamma_t} \left( \frac{1}{2} \partial_y \omega_0 \right)_0 \|_{L^1([0, \tilde{T}]; L^2_y)} + \| e^{-\frac{1}{2}C_0M^2\gamma_t} \left( \frac{1}{2} \partial_y \omega_0 \right)_0 \|_{L^1([0, \tilde{T}]; L^2_y)} 
\leq C \| e^{-\frac{1}{2}C_0M^2\gamma_t} \left( \frac{1}{2} \partial_y \omega_0 \right)_0 \|_{L^1([0, \tilde{T}]; L^2_y)} \| \omega_{in} \|_{F_{L^1_y} L^2_y}.
$$

It follows from the Gagliardo-Nirenberg interpolation inequality that

$$
\| e^{-\frac{1}{2}C_0M^2\gamma_t} u^{(1)}_0(t) \|_{L^\infty([0, \tilde{T}]; L^2_y)} \leq C \| e^{-\frac{1}{2}C_0M^2\gamma_t} u^{(1)}_0(t) \|_{L^\infty([0, \tilde{T}]; L^2_y)} \| e^{-\frac{1}{2}C_0M^2\gamma_t} \omega_0 \|_{L^\infty([0, \tilde{T}]; L^2_y)} 
\leq C \| e^{-\frac{1}{2}C_0M^2\gamma_t} \left( \frac{1}{2} \partial_y \omega_0 \right)_0 \|_{L^1([0, \tilde{T}]; L^2_y)} \| \omega_{in} \|_{F_{L^1_y} L^2_y},
$$

Recalling the property of heat kernel that

$$
\| \partial_y e^{\nu \partial_y^2 f} \|_{L^2([0, \tilde{T}]; L^2_y)} \leq C \nu^{-\frac{1}{2}} \| f \|_{L^2_y},
$$

we have

$$
\| e^{-\frac{1}{2}C_0M^2\gamma_t} \partial_y \omega_0 \|_{L^2([0, \tilde{T}]; L^2_y)} 
\leq \| e^{-\frac{1}{2}C_0M^2\gamma_t} \int_0^t \partial_y e^{(s-t)\nu\partial_y^2} \left( \left( \frac{1}{2} \partial_y \omega_0 \right)_0 + \left( \frac{1}{2} \partial_y \omega_0 \right)_0 \right) (s, y) ds \|_{L^2([0, \tilde{T}]; L^2_y)} 
\leq \int_0^{\tilde{T}} e^{-\frac{1}{2}C_0M^2\gamma s} \left( \int_0^s \left\| \partial_y e^{(s-t)\nu\partial_y^2} \left( \left( \frac{1}{2} \partial_y \omega_0 \right)_0 + \left( \frac{1}{2} \partial_y \omega_0 \right)_0 \right) (s) \right\|_{L^2_y}^2 dt \right)^{\frac{1}{2}} ds 
\leq C \nu^{-\frac{1}{2}} \int_0^{\tilde{T}} e^{-\frac{1}{2}C_0M^2\gamma s} \left( \left( \frac{1}{2} \partial_y \omega_0 \right)_0 + \left( \frac{1}{2} \partial_y \omega_0 \right)_0 \right) (s) \|_{L^2_y} ds 
\leq CC_2 \| e^{\frac{1}{2}C_0M^2\gamma \tilde{T}} \omega_0 \|_{L^2_y} \| \omega_{in} \|_{F_{L^1_y} L^2_y}.
$$

From (4.2) and Proposition 4.1 we have

$$
\| e^{-\frac{1}{2}C_0M^2\gamma_t} \omega_{in} \|_{L^\infty([0, \tilde{T}]; F_{L^1_y} L^2_y)} 
\leq \| e^{-\frac{1}{2}C_0M^2\gamma_t} S_v(t) \omega_{in} \|_{L^\infty([0, \tilde{T}]; F_{L^1_y} L^2_y)} 
+ \| e^{-\frac{1}{2}C_0M^2\gamma_t} \int_0^t S_v(t-s) (L + \sum_{i=1,2,3} N^{(i)}(s)) ds \|_{L^\infty([0, \tilde{T}]; F_{L^1_y} L^2_y)} 
\leq C_1 \| \omega_{in} \|_{F_{L^1_y} L^2_y} + \| e^{-\frac{1}{2}C_0M^2\gamma_t} L \|_{L^1([0, \tilde{T}]; F_{L^1_y} L^2_y)} + \sum_{i=1,2,3} \| e^{-\frac{1}{2}C_0M^2\gamma_t} N^{(i)} \|_{L^1([0, \tilde{T}]; F_{L^1_y} L^2_y)}
$$

NEW ENERGY METHOD IN THE STUDY OF THE INSTABILITY NEAR COUETTE FLOW 29
We study the solution from (4.13) and Proposition 4.2. By Proposition 3.1, Proposition 4.1, Proposition 4.2, and 4.3.

By taking \( C \) and \( \varepsilon_0 \) big enough, and \( \varepsilon_0 \) small enough, we prove the Proposition 4.2. □

4.3. Lower bound estimate to the nonlinear equation. Now, we start to prove Theorem 1.2.

Proof of Theorem 1.2. We study the solution \( \omega(t, x, y) \) of nonlinear system (1.3) with initial data \( \omega_{in} \) given in (4.1) which satisfying \( \int_T \omega_{in}(x, y)dx = 0 \). The upper bound in (1.12) follows from (4.13) and Proposition 4.2. By Proposition 3.1, Proposition 4.1, Proposition 4.2, and Lemma 4.4, we have the lower bound estimate

\[
\|\omega_{\pm 1}(t)\|_{L^2_t L^2_y} \geq C_0 \varepsilon_t \|\omega_{in}\|_{L^2_t L^2_y} - C_0 \nu \varepsilon_t \|\omega_{in}\|_{L^2_t L^2_y} \geq \frac{1}{2} C_0 \varepsilon_t \|\omega_{in}\|_{H^1_t L^2_y}.
\]

The last inequality follows from the properties of our well chosen initial data that

\[
\|\omega_{in}\|_{H^1_t L^2_y} = \|\omega_{in}\|_{L^2_t L^2_y} = \frac{1}{\sqrt{\pi}} \|\omega_{in}\|_{F L^1_t L^2_y}.
\]

In particular, we have

\[
\|\omega(T)\|_{L^2_t L^2_y} \geq \frac{C_0}{2\nu \varepsilon_1} \|\omega_{in}\|_{H^1_t L^2_y} \text{ for } T = \varepsilon_1^{-1} \ln(\nu^{-1}).
\]
Lemma 5.2. There exists in Section 3.\textsuperscript{3} \( e \) in Lemma 5.7 by a classical ODE argument.

Finally, we get from Proposition 3.1 and Proposition 4.1 that Proposition 4.2, we have

\[
\| \int_0^t S_\nu(t-s) \mathcal{L}(s, x, y) ds \|_{L^\infty_{\nu}(\mathbb{R}; \mathbb{R}^2)} \leq C \nu^{\frac{1}{2}} \| \omega_{in} \|_{H^1_y L^2_x}.
\]

Finally, we get from Proposition 3.1 and Proposition 4.1 that

\[
Ce^{\frac{1}{2}C_0 \nu^{1/2}} \| \omega_{in} \|_{H^1_y L^2_x} \geq \| \omega(t) \|_{L^2_y L^2_x} \geq \frac{1}{2} c_0 e^{C_1 \nu} \| \omega_{in} \|_{H^1_y L^2_x}.
\]

\[\Box\]

5. Linear instability for the inviscid flow

In this section, we study the inviscid flow and prove Theorem 1.6. The proof is based on a combination of the new energy method and the classical ODE argument.

After taking Fourier transform of (2.2) in \( x \), we have

\[
\mathcal{R}_{M, \gamma} \omega(k, y) = \mathcal{F}_{x \rightarrow k} \left( b_0(y) \partial_x \omega - b_0''(y) \partial_{k} (\Delta)^{-1} \omega \right) (k, y)
= ik \left( b_0(y) - b_0''(y)(\partial_y^2 - k^2)^{-1} \right) \omega(k, y).
\]

In this section, we only focus on the \( k = 1 \) mode, and study the operator

\[
\mathcal{R} = b_0(y) - b_0''(y)(\partial_y^2 - 1)^{-1}.
\]

If \( \varsigma = \varsigma_r + i \varsigma_i \) is an eigenvalue of \( \mathcal{R} \), then there exists \( \omega_\varsigma(y) \in L^2_y \) such that

\[
\mathcal{R} \omega_\varsigma(y) = \varsigma \omega_\varsigma(y).
\]

Let \( \varsigma = i \lambda \), and \( \omega_\lambda(x, y) = e^{ix} \omega_\varsigma(y) \). Then, it holds that

\[
\mathcal{R}_{M, \gamma} \omega_\lambda = \lambda \omega_\lambda.
\]

Therefore, to prove Theorem 1.6, we prove the following proposition:

Proposition 5.1. Let \( M_0 > 0 \) be big enough. For each \( M \geq M_0 \) there exists \( 0 < \gamma_0 = \gamma_0(M) \) such that for \( 0 < \gamma \leq \gamma_0 \), the following properties hold for \( \mathcal{R} \):

1. \( \mathcal{R} \) has no embedded eigenvalue.
2. \( \mathcal{R} \) has eigenvalues, and the number of eigenvalues is finite.
3. If \( \varsigma = \varsigma_r + i \varsigma_i \) is an eigenvalue of \( \mathcal{R} \), then \( \gamma \leq |\varsigma| \leq CM \gamma \) and \( |\varsigma_r| < 4 \gamma \sqrt{\ln \left( \ln \left( \gamma^{-1} \right) \right)} \).

Proof of in Proposition 5.1.1. The first statement that \( \mathcal{R} \) has no embedded eigenvalue is proved in Lemma 5.7 by a classical ODE argument.

Let us focus on the main result (2) namely, the existence of eigenvalues. We have the following lower bound estimates for the semi-group \( e^{-it \mathcal{R}} \) by our new energy estimate presented in Section 3.

Lemma 5.2. There exists \( \omega_{in}(y) \in L^2_y \) such that for \( t \in [0, T] \) with \( T = \varepsilon_1 \gamma^{-1} \ln \left( \gamma^{-1} \right) \) and \( \varepsilon_1 > 0 \) small, it holds that

\[
\| e^{-it \mathcal{R}} \omega_{in} \|_{L^2_y} \geq C^{-1} e^{C^{-1} \gamma t} \| \omega_{in} \|_{L^2_y},
\]

where \( C \) is independent of \( \gamma \).
Proof. We study the following system
\[ \partial_t \omega(t, x, y) + y \partial_y \omega(t, x, y) - \partial_y^2 b_0(y) \partial_x (\Delta)^{-1} \omega(t, x, y) = -(b_0(y) - y) \partial_x \omega(t, x, y), \]
with initial data \( \omega_{in}(x, y) = 2 \cos(x) \omega_{in}(y) \). Note that
\begin{equation}
\| \omega(t) \|_{L^2_{x,y}} = \| \partial_x \omega(t) \|_{L^2_{x,y}} = \sqrt{2} \| e^{-Rt} \omega_{in} \|_{L^2_y} \quad \text{and} \quad \| \omega_{in} \|_{L^2_{x,y}} = \sqrt{2} \| \omega_{in} \|_{L^2_y}.
\end{equation}

By following the proof of Proposition 4.1, we have for \( t \leq T \)
\[ \| e^{-\frac{i}{2} C_0 \gamma t} S_0(t) \omega_{in} \|_{L^\infty_t L^2_{x,y}} \leq C \| \omega_{in} \|_{L^2_{x,y}}. \]

We also have
\[ \omega(t, x, y) = S_0(t) \omega_{in}(x, y) - \int_0^t S_0(t-s) (b_0(y) - y) \partial_x \omega(s, x, y) ds, \]
and then
\[ \| e^{-\frac{i}{2} C_0 \gamma t} S_0(t) \omega_{in} \|_{L^\infty_t L^2_{x,y}} \leq \| e^{-\frac{i}{2} C_0 \gamma t} S_0(t) \omega_{in} \|_{L^\infty_t L^2_{x,y}} + \int_0^t S_0(t-s) (b_0(y) - y) \partial_x \omega(s, x, y) ds \|_{L^\infty_t L^2_{x,y}} \]
\[ \leq C \| \omega_{in} \|_{L^2_{x,y}} + C \| b_0(y) - y \|_{L^\infty_y} T \| e^{-\frac{i}{2} C_0 \gamma t} \partial_x \omega \|_{L^\infty_t L^2_{x,y}} \]
\[ \leq C \| \omega_{in} \|_{L^2_{x,y}} + C M \gamma^2 T \| e^{-\frac{i}{2} C_0 \gamma t} \omega \|_{L^\infty_t L^2_{x,y}}, \]
which gives that for \( t \leq T \),
\[ \| e^{-\frac{i}{2} C_0 \gamma t} \omega \|_{L^\infty_t L^2_{x,y}} \leq C \| \omega_{in} \|_{L^2_{x,y}}. \]

From Proposition 3.2, we obtain that for well-chosen initial data, it holds that
\[ \| S_0(t) \omega_{in} \|_{L^2_{x,y}} \geq C^{-1} e^{C \gamma t} \| \omega_{in} \|_{L^2_{x,y}}, \quad \text{for } 0 \leq t \leq T. \]

Thus
\[ \| \omega(t) \|_{L^2_{x,y}} \geq \| S_0(t) \omega_{in} \|_{L^2_{x,y}} - \frac{i}{2} C_0 \gamma t \| e^{-\frac{i}{2} C_0 \gamma t} \int_0^t S_0(t-s) (b_0(y) - y) \partial_x \omega(s) ds \|_{L^\infty_t L^2_{x,y}} \]
\[ \geq C^{-1} e^{C \gamma t} \| \omega_{in} \|_{L^2_{x,y}} - C M \gamma^2 T e^{\frac{i}{2} C_0 \gamma t} \| \omega_{in} \|_{L^2_{x,y}} \geq C^{-1} e^{C \gamma t} \| \omega_{in} \|_{L^2_{x,y}}, \]
which together with (5.2) gives the corollary. \( \square \)

Normally, for self-adjoint operators the exponential growth given in the lower bound estimate of the semi-group \( e^{-itR} \) \[ (5.1) \] implies the existence of unstable eigenvalues of \( R \) for \( \gamma \) small enough. Here \( R \) is not self-adjoint. To prove the existence of an unstable eigenvalue, we will use a contradiction argument and an upper bound estimate of the semi-group \( e^{-itR} \). Indeed, under the assumption that \( R \) has no eigenvalues, we will prove that for \( \gamma \) small
\begin{equation}
\| e^{-itR} \omega_{in} \|_{L^2_y} \leq C \| \omega_{in} \|_{L^2_y}, \end{equation}
where \( C \) is a constant independent of \( \gamma \). By taking \( \gamma \) small, the upper bound \[ (5.3) \] and the lower bound estimates \[ (5.1) \] of the semi-group lead to a contradiction.

The estimate \[ (5.3) \] is obtained by the representation formula and the resolvent estimate. For the bounded domain case, namely, if \( \Omega \) is replaced by \( \mathbb{T} \times [0, 1] \), the upper bound estimate is well studied in \[ (65) \], for the unbounded domain case, the result \[ (5.3) \] is new. Here we also need the uniformity in \( \gamma \).
Let us admit \((5.3)\) which is given in Section 5.4 and finish the proof of the existence of eigenvalues. Suppose that \(\mathcal{R}\) has no eigenvalues, then by Lemma \(5.14\) we get the upper bound estimate \((5.3)\) which contradicts Lemma \(5.2\) by taking \(\gamma\) small enough.

The third statement follows from Lemma \(5.8\) and Lemma \(5.9\). Moreover, Corollary \(5.10\) gives the fact that \(\mathcal{R}\) has finite number of eigenvalues. The rest parts of this section are mainly to prove \((5.3)\).

\section{5.1 Homogeneous Rayleigh equation.}

Let \(\zeta = \zeta_r + i\zeta_i \in \mathbb{C}\) be an eigenvalue of \(\mathcal{R}\), then there exists \(0 \neq \psi \in H_y^2\) solution of the homogeneous Rayleigh equation

\begin{equation}
\psi''(y) - \psi(y) - \frac{b_0''(y)}{b_0(y) - \zeta} \psi(y) = 0. \tag{5.4}
\end{equation}

As there is singularity in the coefficients when \(\zeta \in \mathbb{R}\), we first construct a regular solution \(\phi(y, \zeta)\) (not necessary in \(H_y^2\) of \((5.4)\).

\textbf{Proposition 5.3.} There exit \(0 < \varepsilon_3 \leq 1\), \(0 < C_4 \leq 1\) which depend only on the upper and lower bounds of \(b_0'(y)\), such that for \(\zeta \in \mathbb{C}\) with \(0 \leq |\zeta_i| \leq \varepsilon_3\), \((5.4)\) has a regular solution

\[\phi(y, \zeta) = (b_0(y) - \zeta) \phi_1(y, \zeta_r) \phi_2(y, \zeta),\]

which satisfies \(\phi(y, \zeta) = 0\) and \(\phi'(y, \zeta) = b_0(y)\). Here \(y_c = b_0^{-1}(\zeta_r)\), \(\phi_1(y, \zeta_r)\) is a real function that solves

\begin{equation}
\begin{aligned}
\phi_1(y, \zeta_r) & = 1, \\
\phi_1'(y, \zeta_r) & = 0,
\end{aligned} \tag{5.5}
\end{equation}

and \(\phi_2(y, \zeta)\) solves

\begin{equation}
\begin{aligned}
\partial_y ((b_0(y) - \zeta_r)^2 \phi_1'(y, \zeta_r)^2 & - \phi_1'(y, \zeta_r)^2) - \phi_1'(y, \zeta_r) = 0, \\
\phi_2(y_c, \zeta) & = 1, \\
\phi_2'(y_c, \zeta) & = 0.
\end{aligned} \tag{5.6}
\end{equation}

It holds that

\begin{equation}
\phi_1(y, \zeta_r) \geq 1, \quad C_4 C_1 |y - y_c| \leq \phi_1(y, \zeta_r) \leq C_4 |y - y_c|, \quad \phi_1'(y, \zeta_r) \leq 1, \quad \phi_1'(y, \zeta_r) \geq C_4 \phi_1(y, \zeta_r) \quad \forall y \in \mathbb{R},
\end{equation}

\begin{equation}
|\phi_1(y, \zeta_r) - 1| \leq C |y - y_c|^2 \quad \forall y \in \mathbb{R}, \quad \phi_1'(y, \zeta_r) \geq C_4 \phi_1(y, \zeta_r) \quad \forall y \in \mathbb{R},
\end{equation}

\begin{equation}
\phi_1'(y, \zeta_c) \leq C_4 |y - y_c| \quad \forall y \in \mathbb{R}, \quad \phi_1'(y, \zeta_c) \geq C_4 \phi_1(y, \zeta_c) \quad \forall y \in \mathbb{R},
\end{equation}

\begin{equation}
\phi_1''(y, \zeta_c) \leq C_4 \phi_1(y, \zeta_c) \quad \forall y \in \mathbb{R},
\end{equation}

where \(C_4 > 0\) is a constant independent of \(\zeta_r\) and \(\gamma\).

It is easy to check that

\[\phi(y, \zeta) \quad \text{and} \quad \varphi^+(y, \zeta),\]

\[\phi(y, \zeta) \quad \text{and} \quad \varphi^-(y, \zeta),\]

where

\[\varphi^\pm(y, \zeta) = \phi(y, \zeta) \int_{y_c}^y \frac{1}{\phi^2(y', \zeta)} dy',\]

are two fundamental sets of solutions of \((5.4)\). Therefore, if \(\psi(y, \zeta) \in H_y^2\) is a solution of \((5.4)\), then \(\psi(y, \zeta)\) has the following form

\begin{equation}
\psi(y, \zeta) = a_1^- \phi(y, \zeta) + a_2^- \varphi^-(y, \zeta) = a_1^+ \phi(y, \zeta) + a_2^+ \varphi^+(y, \zeta), \tag{5.10}
\end{equation}
where \(a_1^\pm, a_2^\pm\) are constants. Note that by Proposition 5.3, \(\varphi^\pm\) are well defined for \(|\zeta| \leq \varepsilon_3\).

We give a criterion for whether \(\zeta\) is an eigenvalue of \(\mathcal{R}\).

**Lemma 5.4.** Let

\[
\mathcal{D}(\zeta) = \int_{-\infty}^{+\infty} \frac{1}{\phi^2(y', \zeta)} dy'.
\]

Then, a number \(\zeta \in \mathbb{C}\) with \(0 < |\zeta| \leq \varepsilon_3\) is an eigenvalue of \(\mathcal{R}\) if and only if \(\mathcal{D}(\zeta) = 0\).

The criterion function \(\mathcal{D}(\zeta)\) can be extended to \(\mathbb{R}\).

**Lemma 5.5.** It holds that

\[
\lim_{\zeta \to 0^\pm} \mathcal{D}(\zeta) = \mathcal{J}_1(\zeta) \mp i \mathcal{J}_2(\zeta),
\]

where

\[
\mathcal{J}_1(\zeta_r) = \frac{1}{b_0'(y_c)} \Pi_1(\zeta_r) + \Pi_2(\zeta_r), \quad \mathcal{J}_2(\zeta_r) = \pi \frac{b''_0(y_c)}{b_0^3(y_c)}.
\]

and

\[
\Pi_1(\zeta_r) = P.V. \int \frac{b_0'(y_c) - b_0'(y)}{(b_0(y) - \zeta_r)^2} dy, \quad \Pi_2(\zeta) = \int_{-\infty}^{+\infty} \frac{1}{(b_0(y) - \zeta_r)^2} \left( \frac{1}{\phi^2_1(y, \zeta_r)} - 1 \right) dy.
\]

Recall that \(\text{Ran } b_0(y) = \mathbb{R}\) is the continuous spectrum of \(\mathcal{R}\). If \(\zeta \in \mathbb{R}\) is an eigenvalue of \(\mathcal{R}\), we call \(\zeta\) an embedded eigenvalue.

**Lemma 5.6.** A number \(\zeta_r \in \mathbb{R}\) is an embedded eigenvalue of \(\mathcal{R}\) if and only if

\[
\mathcal{J}_1^2(\zeta_r) + \mathcal{J}_2^2(\zeta_r) = 0.
\]

The proofs of the above proposition and lemmas can be found in Appendix C.

5.2. \(\mathcal{R}\) has no embedded eigenvalues. In this subsection, we show that there is no embedded eigenvalue in \(\mathbb{R}\).

**Lemma 5.7.** Under the assumptions on \(M\) and \(\gamma\) in Proposition 5.3, the operator \(\mathcal{R}\) does not have any embedded eigenvalue.

**Proof.** By the definition of \(b_0(y), b_0'(y) = 0\) only at \(y = 0\), thus \(\mathcal{J}_2(y) \neq 0\) if \(y \neq 0\). By Lemma 5.6, to prove Lemma 5.7, it suffices to show that \(\mathcal{J}_1(0) \neq 0\).

From Lemma 5.5, we have \(\lim_{\zeta \to 0} \mathcal{D}(i\zeta) = \Pi_1(0) + \Pi_2(0)\). And it follows from (C.5) that

\[
\Pi_1(0) = -\mathcal{H}(\varphi^2_0(b_0^{-1}))(0) = -\int_{\mathbb{R}} \frac{1}{v} \frac{b''_0(b_0^{-1}(v))}{(b_0^{-1}(v))^3} dv
\]

\[
= -\int_{\mathbb{R}} \frac{1}{b_0(y)} \frac{b_0'(y)}{(b_0'(y))^2} dy = 4\sqrt{\pi} M \int_{\mathbb{R}} \frac{y}{b_0(y)} \frac{1}{(b_0(y))^2} dy \geq CM,
\]

where we use the change of coordinate \(y = b_0^{-1}(v)\). From (5.7), one can easily check that

\[
|\Pi_2(0)| = \left| \int_{-\infty}^{+\infty} \frac{1}{b_0^2(y)} \left( \frac{1}{\phi^2_1(y, 0)} - 1 \right) dy \right| \leq C.
\]

Finally, taking \(M\) big enough, we have that

\[
\mathcal{J}_1(0) \geq \Pi_1(0) - |\Pi_2(0)| \geq CM > 0,
\]

and there is no embedded eigenvalue of \(\mathcal{R}\). \(\square\)
5.3. Possible locations of the eigenvalues. In this subsection, we show that $\mathcal{R}$ has at most finite number of eigenvalues, moreover they are only in $E = \mathbb{C} \setminus (E_1 \cup E_2 \cup E_3)$, where

$$E_1 = \{ \epsilon \in \mathbb{C} | |\epsilon| \geq 4\gamma \sqrt{\ln \left( \ln(\gamma^{-1}) \right)}, 0 < |\epsilon| < 8\sqrt{\pi} M\gamma \},$$

$$E_2 = \{ \epsilon \in \mathbb{C} | |\epsilon| \geq 8\sqrt{\pi} M\gamma \}, \quad E_3 = \{ \epsilon \in \mathbb{C} | |\epsilon| \leq 4\gamma \sqrt{\ln \left( \ln(\gamma^{-1}) \right)}, 0 < |\epsilon| \leq \gamma \}.$$

We first show that $\mathcal{R}$ has no eigenvalues in $E_1 \cup E_2$ by an energetic argument of the Rayleigh type.

**Lemma 5.8.** The operator $\mathcal{R}$ has no eigenvalues in $E_1 \cup E_2$.

**Proof.** We first assume that $c \in E_1$ is an eigenvalue. Taking the inner product of (5.4) with $\psi$, we have

$$\|\psi\|^2_{H^2} = \|\psi\|^2_{L^2} + \|\psi\|^2_{L^2} = - \int_{\mathbb{R}} \frac{b''}{b_0 - c} |\psi|^2 dy.

We write

$$\int_{\mathbb{R}} \frac{b''}{b_0 - c} |\psi|^2 dy = \int_{\mathbb{R} \setminus D} \frac{b''}{b_0 - c} |\psi|^2 dy + \int_{D} \frac{b''}{b_0 - c} |\psi|^2 dy \overset{\text{def}}{=} I + II,$n

where $D = [\epsilon_r - \min(\frac{1}{2}|\epsilon_r|, \frac{1}{2}), \epsilon_r + \min(\frac{1}{2}|\epsilon_r|, \frac{1}{2})]$. It is easy to check that $\|\psi\|^2_{L^2} \leq \frac{1}{2} \|\psi\|^2_{H^2}$.

For $I$, by integration by parts, we have

$$II = \int_{D} \frac{b''}{b_0 - c} |\psi|^2 dy = \int_{D} \frac{b''}{b_0} |\psi|^2 dy \ln(b_0 - c) dy$$

$$= - \int_{D} \frac{b''}{b_0} \left( \frac{b''}{b_0} \right)^2 |\psi|^2 \ln(b_0 - c) + 2 \frac{b''}{b_0} \Re(\psi \psi') \ln(b_0 - c) + 2 \frac{b''}{b_0} |\psi|^2 \ln(b_0 - c) \right| \epsilon_r - \min(\frac{1}{2}|\epsilon_r|, \frac{1}{2}) \right. \left. \epsilon_r + \min(\frac{1}{2}|\epsilon_r|, \frac{1}{2}) \right)$$

Then we have

$$|II| \leq C \left( \|b''\|_{L^\infty(D)} + \|b''\|_{L^\infty(D)} \right) \|\psi\|^2_{L^2} \ln(b_0 - c) \|L^2(D)$$

$$+ C \|b''\|_{L^\infty(D)} \|\psi\|^2_{L^2} \|\psi\|^2_{L^2} \ln(b_0 - c) \|L^2(D)$$

$$+ C \|b''\|_{L^\infty(D)} \|\psi\|^2_{L^2} \left( \ln \left( b_0(\epsilon_r + \min(\frac{1}{2}|\epsilon_r|, \frac{1}{2}) - c) \right) + \ln \left( b_0(\epsilon_r - \min(\frac{1}{2}|\epsilon_r|, \frac{1}{2}) - c) \right) \right).$$

Recall the definition of $D$. We have for $p = 1, 2$ that

$$\int_{D} |\ln(b_0 - c)|^p dy \leq C \int_{|z| \leq \frac{1}{2} \min(\frac{1}{2}|\epsilon_r|, \frac{1}{2})} |\ln(z)|^p dz$$

$$\leq C \int_{|z| \leq \frac{1}{2} \min(\frac{1}{2}|\epsilon_r|, \frac{1}{2})} |y|^p e^y dy \leq C \int_{-\infty}^{|\ln(\frac{1}{2} \min(\frac{1}{2}|\epsilon_r|, \frac{1}{2}))|} |y|^p e^y dy.$$
Then we have
\[ \int_D |\ln(b_0 - c)|^p dy \leq C c_r |\ln(\frac{3}{4} c_r)|^p \] for \(|c_r| \leq 1\), \[ \int_D |\ln(b_0 - c)|^p dy \leq C |c_r| > 1. \]

For the case \(2\gamma \ln(\gamma) \geq |c_r| \geq 4\gamma \ln(\ln(\gamma))\), we have
\[ |b''_0(y)| \leq C M \sqrt{\ln(\ln(\gamma))} \left(\ln(\ln(\gamma))\right)^4, \quad |b'''_0(y)| \leq C M \frac{\ln(\ln(\ln(\gamma)))}{\gamma(\ln(\ln(\gamma)))^4}, \]
and
\[ |\ln(b_0(c_r \pm \min(\frac{1}{2}|c_r|, \frac{1}{2})) - c)| \leq C \ln(\gamma), \quad \|\ln(b_0 - c)\|_{L^p(D)}^p \leq C_\gamma(\ln(\ln(\ln(\gamma))))^{p+1} \] for \(p = 1, 2\).

For \(|c_r| \geq 2\gamma \ln(\ln(\gamma))\), we have
\[ |b''_0(y)| \leq C M^3 \gamma, \quad |b'''_0(y)| \leq C M^2, \]
and
\[ |\ln(b_0(c_r \pm \min(\frac{1}{2}|c_r|, \frac{1}{2})) - c)| \leq C \ln(\gamma), \quad \|\ln(b_0 - c)\|_{L^p(D)}^p \leq C \] for \(p = 1, 2\).

Combining the above estimates and the Gagliardo-Nirenberg interpolation inequality, we have
\[ |II| \leq \frac{1}{4} \|\psi\|_{H^1}^2, \]
and
\[ \|\psi\|_{H^1}^2 \leq \frac{1}{2} \|\psi\|_{H^1}^2, \]
which means that \(\psi \equiv 0\) and \(c \in E_1\) could not be an eigenvalue.

Next, we assume that \(c \in E_2\) is an eigenvalue. As \(|c_1| \geq 8\ln(M)\), we have
\[ \int_R \frac{b''_0}{b_0 - c} |\psi|^2 dy \leq \frac{1}{|c_1|} \int_R |b''_0| dy \|\psi\|_{L^p}^2 \leq \frac{2\sqrt{M}\ln\gamma\|\psi\|_{H^1}^2}{8\sqrt{M}\gamma} \leq \frac{1}{2} \|\psi\|_{H^1}^2, \]
which also implies \(\psi \equiv 0\) and \(c \in E_2\) cannot be an eigenvalue.

Next, we show that \(\mathcal{R}\) has no eigenvalues in \(E_3\). We use an ODE argument and study the Wronskian \(D(c)\).

**Lemma 5.9.** Under the assumptions on \(M\) and \(\gamma\) in Preposition 5.1, the operator \(\mathcal{R}\) has no eigenvalues in \(E_3\).

**Proof.** From Lemma 5.4, it suffices to show that \(D(c) \neq 0\) for \(c \in E_3\). We prove that, the real part of \(D(c) \neq 0\) for \(|c_r| \lesssim \frac{1}{4}\gamma\) and \(|c_1| \leq \gamma\), and the imaginary part of \(D(c) \neq 0\) for \(\frac{1}{4}\gamma \geq |c_r| \leq 4\ln(\ln(\ln(\gamma)))\) and \(|c_1| \leq \gamma\).

We write
\[ D(c) = \int_{-\infty}^{+\infty} \frac{1}{(b_0(y) - c)^2} dy + \int_{-\infty}^{+\infty} \frac{1}{(b_0(y) - c)^2} \left(\frac{1}{\phi_1^2(y, c)\phi_2^2(y, c)} - 1\right) dy = I + II. \]

For \(II\), we have
\[ \int_{-\infty}^{+\infty} \frac{1}{(b_0(y) - c)^2} \left(\frac{1}{\phi_1^2(y, c)\phi_2^2(y, c)} - 1\right) dy \]
From (5.7) and (5.8), we have

\[\| φ \|_2 \leq \int_{-\infty}^{\infty} \left( \frac{(b_0(y) - c_r)^2 - c_t^2 + 2i c_i (b_0(y) - c_r)}{(b_0(y) - c_r)^2 + c_t^2} \right) dy' \]

\[= \int_{-\infty}^{\infty} \left( (b_0(y) - c_r)^2 - c_t^2 \right) \frac{\left( \phi_{2,r}^2 + \phi_{2,i}^2 \right) (1 - \phi_1^2 (\phi_{2,r}^2 + \phi_{2,i}^2) - 2 \phi_{2,i}^2)}{\phi_1^2 (\phi_{2,r}^2 + \phi_{2,i}^2)^2} dy' \]

\[+ 2i \int_{-\infty}^{\infty} \frac{c_i (b_0(y) - c_r)}{(b_0(y) - c_r)^2 + c_t^2} \phi_{2,r}^2 \phi_{2,i}^2 \right) dy' \]

\[= II_r + i II_i. \]

From (5.7) and (5.8), we have

\[
|φ_1(y) - 1| \leq C|y - y_t|^2, \quad |φ_{2,r}(y) - 1| \leq C|y - y_t|^2, \quad \text{for } |y - y_t| \leq 1, \\
|φ_{2,i}(y) - 1| \leq C|c_i| \min(|y - y_t|, 1), \quad \text{for } y \in \mathbb{R}. \]

Then we have for \(|y - y_t| \leq 1\) that

\[
|1 - \phi_1^2 (\phi_{2,r}^2 + \phi_{2,i}^2)| \leq C|y - y_t|^2. \]

And then,

\[
|II_r| \leq C \int_{[y_t - 1,y_t + 1]} \frac{|y - y_t|^2 + |c_i||y - y_t|}{(b_0(y) - c_r)^2 + c_t^2} dy + C \int_{\mathbb{R}\backslash[y_t - 1,y_t + 1]} \frac{1}{|y - y_t|^2} dy \leq C, \]

and

\[
|II_i| \leq C \int_{[y_t - 1,y_t + 1]} \frac{\left|c_i\right||y - y_t|}{(b_0(y) - c_r)^2 + c_t^2} dy + C \int_{\mathbb{R}\backslash[y_t - 1,y_t + 1]} \frac{|c_i|}{|y - y_t|^2} dy \leq \left|c_i\right| \ln |c_i| \leq C γ \ln γ^{-1}, \]

where the constant \(C\) does not depend on \(M\).

Recall that \(b_0(y)\) is a strictly monotonic function. Let \(v = b_0(y)\), we have

\[
I = \int_{-\infty}^{\infty} \frac{1}{(b_0(y) - c_r)^2} dy = \int_{-\infty}^{\infty} \frac{\partial_v (b_0^{-1}) (v)}{(v - c)^2} dv = \int_{-\infty}^{\infty} \frac{\partial_v^2 (b_0^{-1}) (v)}{v - c} dv
\]

\[
= \int_{-\infty}^{\infty} \frac{(v - c_r) \partial_v^2 (b_0^{-1}) (v)}{(v - c_r)^2 + c_t^2} dv + \int_{-\infty}^{\infty} \frac{ic_i \partial_v^2 (b_0^{-1}) (v)}{(v - c_r)^2 + c_t^2} dv \overset{\text{def}}{=} I_r + i I_i.
\]

Direct calculations show that

\[
\partial_v (b_0^{-1}) (v) = \frac{1}{b_0'(b_0^{-1}(v))}, \quad \partial_v^2 (b_0^{-1}) (v) = - \frac{b_0''(b_0^{-1}(v))}{(b_0'(b_0^{-1}(v)))^3}.
\]
Then we have
\[
I_r = - \int_{-\infty}^{+\infty} \frac{(b_0(y) - c_r)}{(b_0(y) - c_r)^2 + c_i^2} \, dy = 4\sqrt{\pi} M \int_{-\infty}^{+\infty} \frac{1}{(b_0'(y))^2} \frac{(b_0(y) - c_r) e^{-y^2}}{(b_0(y) - c_r)^2 + c_i^2} \, dy
\]
\[
= 4\sqrt{\pi} M \left( \int_{\mathbb{R}\setminus [0,2y_c]} \ldots dy + \int_0^{y_c} \ldots dy + \int_{y_c}^{2y_c} \ldots dy \right)
\]
\[
\overset{\text{def}}{=}) 4\sqrt{\pi} M \left( K_1 + K_2 + K_3 \right).
\]
We remark that $\|b_0'(y) - 1\|_{L^\infty} \leq CM\gamma$, so we have $|y_c - c_r| \leq CM\gamma^2$, and $|y_c| \leq \frac{\sqrt{2}}{4}\gamma$. Then, it is clear that $K_1 > 0$, $K_2 < 0$, and $K_3 > 0$. As $\frac{y}{\gamma} e^{-\frac{y^2}{\gamma^2}}$ grows on $[0, \frac{\sqrt{2}}{4}\gamma]$, so
\[
\min_{y \in [y_c, 2y_c]} \frac{y}{\gamma} e^{-\frac{y^2}{\gamma^2}} \geq \max_{y \in [0, y_c]} \frac{y}{\gamma} e^{-\frac{y^2}{\gamma^2}}.
\]
Then we have $K_2 + K_3 \geq -CM\gamma$. Indeed, we have
\[
K_2 + K_3 \geq \int_0^{y_c} \frac{1}{(b_0'(y))^2} \frac{(b_0(y) - c_r)}{(b_0(y) - c_r)^2 + c_i^2} + \frac{1}{(b_0(2y_c - y) - c_r)} \, dy
\]
\[
= \int_0^{\max(y_c, 2\gamma)} \ldots dy + \int_{\max(y_c, 2\gamma)}^{y_c} \ldots dy \overset{\text{def}}{=} J_1 + J_2.
\]
For $0 \leq y_1 \leq y_2$, we have $1 \leq b_0(y_2) \leq b_0(y_1) \leq 1 + CM\gamma$, and then
\[
\frac{1}{(b_0(y))^2} \frac{(b_0(y) - c_r)}{(b_0(y) - c_r)^2 + c_i^2} + \frac{1}{(b_0(2y_c - y))} \frac{(b_0(2y_c - y) - c_r)}{(b_0(2y_c - y) - c_r)^2 + c_i^2}
\]
\[
\geq \frac{1}{(b_0(y_c))^2} \frac{(b_0(y) - c_r)}{(b_0(y) - c_r)^2 + c_i^2} + \frac{1}{(b_0(2y_c - y) - c_r)} \frac{(b_0(2y_c - y) - c_r)}{(b_0(2y_c - y) - c_r)^2 + c_i^2}.
\]
It holds that $b_0(y) - c_r = b_0(\tilde{y}_1)(y - y_c)$ and $b_0(2y_c - y) - c_r = b_0(\tilde{y}_2)(2y_c - y)$ for some $\tilde{y}_1 \in (y, y_c]$ and $\tilde{y}_2 \in [2y_c - y, y_c]$, we have $y_2 \geq \tilde{y}_2$ and $1 \leq b_0'(\tilde{y}_2) \leq b_0'(\tilde{y}_1)$. Therefore
\[
\frac{(b_0(y) - c_r)}{(b_0(y) - c_r)^2 + c_i^2} + \frac{(b_0(2y_c - y) - c_r)}{(b_0(2y_c - y) - c_r)^2 + c_i^2}
\]
\[
= \frac{(b_0(\tilde{y}_1) - b_0(\tilde{y}_2))(y - y_c) + (b_0(\tilde{y}_1) b_0(\tilde{y}_2)(y - y_c)^2 - c_i^2)}{((b_0(\tilde{y}_1))^2(y - y_c)^2 + c_i^2) ((b_0(\tilde{y}_1))^2(2y_c - y)^2 + c_i^2)} \overset{\text{def}}{=} Q.
\]
For $y_c - y \geq 2|c_i|$, we have $b_0'(\tilde{y}_1) b_0'(\tilde{y}_2)(y - y_c)^2 - c_i^2 \geq 0$, then $J_1 \geq 0$. It is easy to check that
\[
|Q| \leq CM\gamma \frac{|y - y_c|}{(y - y_c)^2 + c_i^2},
\]
and
\[
|J_2| \leq CM\gamma \int_0^{|c_i|} \frac{x}{x^2 + c_i^2} \, dx \leq CM\gamma.
\]
For $K_1$, we have

$$K_1 = \int_{\mathbb{R} \setminus [0,2y_c]} \frac{1}{2} \left( \frac{b_0(y) - c_r}{(b_0'(y))^2} \right) dy = \int_{\mathbb{R} \setminus [-\gamma,\gamma]} \frac{1}{2} \left( \frac{b_0(y) - c_r}{(b_0'(y))^2} \right) dy \geq C \int_{\mathbb{R} \setminus [-\gamma,\gamma]} \frac{1}{2} \left( \frac{b_0(y) - c_r}{(b_0'(y))^2} \right) dy + \gamma^2 \geq C \int_{\mathbb{R} \setminus [-\gamma,\gamma]} \frac{1}{2} \frac{y^2}{y^2} dy = C \int_{\mathbb{R} \setminus [-1,1]} e^{-y^2} dy = C > 0.$$  

Combing the above estimates, we have $I_r \geq CM$.

Next, we show that $|I_i| \geq CM \left( \frac{1}{\ln(\gamma^{-1})} \right)^2$ for $|c_r| \geq \frac{1}{4} \gamma$ and $|c_i| \leq \gamma$. Recall that

$$I_i = \int_{-\infty}^{+\infty} \frac{c_i \partial_2^2 (b_0^{-1})(0)}{(v - c_r)^2 + \gamma^2} dv = - \int_{-\infty}^{+\infty} \frac{c_i b_0''(y)}{(b_0(y) - c_r)^2 + \gamma^2} dy$$

$$= 4\sqrt{\pi} M \int_{-\infty}^{+\infty} \frac{c_i y^2 e^{-y^2}}{(b_0(y) - c_r)^2 + \gamma^2} dy.$$  

We have $c_i I_i > 0$ for $c_r > 0$ and $c_i I_i < 0$ for $c_r < 0$. Without loss of generally, we assume $c_r, c_i > 0$. Then we write

$$I_i = 4\sqrt{\pi} M \int_{0}^{+\infty} \frac{c_i y^2 e^{-y^2}}{(b_0(y) - c_r)^2 + \gamma^2} \left( \frac{1}{(b_0'(y))^2} \left( \frac{1}{(b_0(y) - c_r)} \right)^2 + \gamma^2 \right) dy.$$

It is clear for each $y \in (0, +\infty)$ we have

$$\left( \frac{1}{(b_0'(y))^2} \left( \frac{1}{(b_0(y) - c_r)} \right)^2 + \gamma^2 \right) > 0.$$  

Therefore, we have

$$|I_i| \geq CM \int_{y_c}^{y_c + c_i} \frac{c_i y^2 e^{-y^2}}{(y - y_c)^2 + \gamma^2} dy = CM \sqrt{\ln(\gamma^{-1})} \left( \frac{1}{\ln(\gamma^{-1})} \right)^4 \int_{y_c}^{y_c + c_i} \frac{c_i y^2 e^{-y^2}}{(y - y_c)^2 + \gamma^2} dy \geq CM \sqrt{\ln(\gamma^{-1})} \left( \frac{1}{\ln(\gamma^{-1})} \right)^4 .$$

As a conclusion, we have

$$|II_r| \leq C, \quad |II_i| \leq C \gamma \ln(\gamma^{-1}) \quad \text{for } c_r \in \mathbb{R}, \quad |c_i| \leq \gamma; \quad I_r \geq CM \quad \text{for } |c_r| \leq \frac{1}{4} \gamma, \quad |c_i| \leq \gamma, \quad |I_i| \geq CM \sqrt{\ln(\gamma^{-1})} \left( \frac{1}{\ln(\gamma^{-1})} \right)^4 \quad \text{for } \frac{1}{4} \gamma < |c_r| < 4\gamma \sqrt{\ln(\gamma^{-1})}, \quad |c_i| \leq \gamma.$$  

Then by taking $M$ big enough, we have $D(c) \neq 0$ for $c \in E_3$. Thus we proved Lemma 5.9. 

**Corollary 5.10.** Under the assumptions on $M$ and $\gamma$ in Preposition 5.1, $\mathcal{R}$ could have at most a finite number of eigenvalues.
Proof. By Remark C.1 we get that \( c \in E \) is an eigenvalue of \( \mathcal{R} \) if and only if \( \mathcal{D}(c) = 0 \). And \( \mathcal{D}(c) \) is analytic. As \( E \) is a bounded set, there could only be at most a finite number of zero points of \( \mathcal{D}(c) \).

\[ \square \]

Remark 5.11. The Cauchy’s argument principle gives that the number of eigenvalues can be estimated by studying the contour integral \( \frac{1}{2\pi i} \oint_{\partial E} \frac{\mathcal{D}'(c)}{\mathcal{D}(c)} dc \).

5.4. Upper bound estimate. Now, we prove (5.3) by assuming that \( \mathcal{R} \) has no eigenvalues. Note that \( \mathcal{R} \) has no embedded eigenvalues. We consider the following equation

\[
\begin{cases}
\partial_t w(t, y) + i \mathcal{R} w(t, y) = 0, \\
(\partial_y^2 - 1) \Psi(t, y) = w(t, y), \\
w|_{t=0}(y) = w_{in}(y) = (\partial_y^2 - 1) \Psi_{in}(y),
\end{cases}
\]

and study the upper bound of the semi-group \( e^{-it\mathcal{R}} \) by its representation formula:

\[
\Psi(t, y) = \lim_{\epsilon_i \to 0^+} \frac{1}{2\pi i} \int_{\epsilon_i} \left[ e^{-i(\epsilon_r - i\epsilon_i)t}(\epsilon_r - i\epsilon_i - \mathcal{L})^{-1} \Psi_{in} - e^{-i(\epsilon_r + i\epsilon_i)t}(\epsilon_r + i\epsilon_i - \mathcal{L})^{-1} \Psi_{in} \right] d\epsilon_r
\]

where \( \mathcal{L} = (\partial_y^2 - 1)^{-1} \mathcal{R}(\partial_y^2 - 1) \) and \( \sigma(\mathcal{L}) = \mathbb{R} \). Let \( (\epsilon - \mathcal{L})^{-1} \Psi_{in} = i \Phi(y, \epsilon) \), then \( \Phi(y, \epsilon) \) solves the inhomogeneous Rayleigh equation:

\[
\partial_y^2 \Phi(y, \epsilon) - \Phi(y, \epsilon) - \frac{b_0''}{b_0} \Phi(y, \epsilon) = i \frac{w_{in}(y)}{b_0(y) - \epsilon}.
\]

Recall that \( \phi(y, \epsilon) \) given in Proposition 5.3 solves the homogeneous Rayleigh equation (5.4), it is easy to check that

\[
\partial_y \left( \phi^2(y, \epsilon) \partial_y \left( \frac{\Phi(y, \epsilon)}{\phi(y, \epsilon)} \right) \right) = i w_{in}(y) \phi_1(y, \epsilon).
\]

Here we briefly write \( \phi_1(y, \epsilon) = \phi_1(y, \epsilon_r) \phi_2(y, \epsilon) \). One can see that \( \phi_1(y, \epsilon) = \phi_1(y, \epsilon_r) \) if \( \epsilon_i = 0 \), and \( \phi_1(y, \epsilon) \) is well defined for \( \epsilon \in \mathbb{C} \).

For \( \epsilon_i \neq 0 \), there is a unique the solution \( \Phi(y, \epsilon) \) of (5.12) which decays at infinity. Then \( \Phi(y, \epsilon) \) can be written as follows:

\[
\Phi(y, \epsilon) = i \phi(y, \epsilon) \int_{-\infty}^{y} \int_{-\infty}^{y'} \frac{w_{in}(y'') \phi_1(y'', \epsilon) dy''}{\phi^2(y', \epsilon)} dy' - i \mu(w_{in}, \epsilon) \phi(y, \epsilon) \int_{-\infty}^{y} \frac{1}{\phi^2(y', \epsilon)} dy'
\]

where \( \mu(w_{in}, \epsilon) = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{w_{in}(y'') \phi_1(y'', \epsilon) dy''}{\phi^2(y', \epsilon)} dy'}{\int_{-\infty}^{+\infty} \frac{1}{\phi^2(y', \epsilon)} dy'} \).

Indeed, we have the following lemma.
Lemma 5.12. Let \( w_{in} \in C^\infty_c(\mathbb{R}) \), then for any \( y \in \mathbb{R} \), it holds that \( \Phi(y, \epsilon) \in L^1_c \) and

\[
\lim_{\epsilon_i \to 0^\pm} \Phi(y, \epsilon_r + i\epsilon_i) = \begin{cases} 
\int_{-\infty}^{y} \frac{y' w_{in}(y') \phi(y', \epsilon_r) dy'}{\phi^2(y', \epsilon_r)} dy \\
- i \mu_{\pm}(w_{in}, \epsilon_r) \phi(y, \epsilon_r) \int_{-\infty}^{y} \frac{1}{\phi^2(y', \epsilon_r)} dy', \quad y < y_c, \\
\int_{+\infty}^{y} \frac{y' w_{in}(y') \phi(y', \epsilon_r) dy'}{\phi^2(y', \epsilon_r)} dy' \\
- i \mu_{\pm}(w_{in}, \epsilon_r) \phi(y, \epsilon_r) \int_{+\infty}^{y} \frac{1}{\phi^2(y', \epsilon_r)} dy', \quad y > y_c,
\end{cases}
\]

where

\[
\mu_{\pm}(\epsilon_r) = \frac{\mathcal{J}_3(w_{in}, \epsilon_r) \pm i \mathcal{J}_4(w_{in}, \epsilon_r)}{\mathcal{J}_1(\epsilon_r) \mp i \mathcal{J}_2(\epsilon_r)},
\]

with \( \mathcal{J}_1(\epsilon_r), \mathcal{J}_2(\epsilon_r) \) given in Lemma 5.5, and

\[
\mathcal{J}_3(w_{in}, \epsilon_r) = P.V. \int_{-\infty}^{y} \frac{y' w_{in}(y'') \phi(y'', \epsilon_r) dy''}{\phi^2(y'', \epsilon_r)} dy', \quad \mathcal{J}_4(w_{in}, \epsilon_r) = \pi \frac{w_{in}(y_c)}{(b_0(y_c))^2}.
\]

Proof. First, we show \( \lim_{\epsilon_i \to 0^\pm} \mu(w_{in}, \epsilon) = \mu_{\pm}(\epsilon_r) \). It follows from Lemma 5.5 and the definition of \( \mathcal{D}(\epsilon) \) that

\[
\lim_{\epsilon_i \to 0^\pm} \int_{-\infty}^{+\infty} \frac{1}{(b_0(y') - \epsilon)^2 \phi_1^2(y', \epsilon)} dy' = \mathcal{J}_1(\epsilon_r) \mp i \mathcal{J}_2(\epsilon_r).
\]

We write the numerator of \( \mu(w_{in}, \epsilon) \) as

\[
\int_{-\infty}^{+\infty} \frac{y' w_{in}(y'') \phi(y'', \epsilon)}{(b_0(y') - \epsilon)^2 \phi_1^2(y', \epsilon)} dy' = \int_{-\infty}^{+\infty} \frac{y' w_{in}(y'') (\phi(y'', \epsilon) - 1)}{(b_0(y') - \epsilon)^2 \phi_1^2(y', \epsilon)} dy' + \int_{-\infty}^{+\infty} \frac{y' w_{in}(y'') dy''}{(b_0(y') - \epsilon)^2 \phi_1^2(y', \epsilon)} dy' \\
+ \int_{-\infty}^{+\infty} \frac{y' w_{in}(y'') dy'' - w_{in}(y_c) b_0'(y_c)(b_0(y') - \epsilon_r)}{(b_0(y') - \epsilon)^2 \phi_1^2(y', \epsilon)} dy' + \frac{w_{in}(y_c)}{(b_0(y_c))^2} \int_{-\infty}^{+\infty} \frac{b_0'(y') (b_0(y') - \epsilon_r)}{(b_0(y') - \epsilon)^2} dy'.
\]

From (5.7), (5.8) and the Lebesgue dominated convergence theorem, we have

\[
\lim_{\epsilon_i \to 0^\pm} \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3
\]

\[
= \int_{-\infty}^{+\infty} \frac{y' w_{in}(y'') (\phi(y'', \epsilon) - 1)}{(b_0(y') - \epsilon)^2 \phi_1^2(y', \epsilon)} dy' + \int_{-\infty}^{+\infty} \frac{y' w_{in}(y'') dy''}{(b_0(y') - \epsilon)^2 \phi_1^2(y', \epsilon)} dy' \\
+ \int_{-\infty}^{+\infty} \frac{y' w_{in}(y'') dy'' - w_{in}(y_c) b_0'(y_c)(b_0(y') - \epsilon_r)}{(b_0(y') - \epsilon)^2 \phi_1^2(y', \epsilon)} dy'.
\]
Let $v = b_0(y')$, we have
\[ I_4 = \frac{w_m(y_c)}{(b_0(y_c))^2} \int_{-\infty}^{+\infty} \frac{b'_0(y')(b_0(y') - c_r)}{(b_0(y') - c)^2} dy' = \frac{w_m(y_c)}{(b_0(y_c))^2} \int_{-\infty}^{+\infty} \frac{v + i c_i}{v^2} dv, \]
and then
\[ \lim_{c_i \to 0 \pm} I_4 = \frac{w_m(y_c)}{(b_0(y_c))^2} \text{P.V.} \int \frac{b'_0(y')(b_0(y') - c_r)}{(b_0(y') - c)^2} dy' + \pi i \frac{w_m(y_c)}{(b_0(y_c))^2}. \]

It follows that
\[ \lim_{c_i \to 0 \pm} \int_{-\infty}^{+\infty} \int y_c \frac{w_m(y'')\phi_1(y'', c)}{(b_0(y') - c)^2 \phi_1^2(y', c)} dy' \]
\[ = \text{P.V.} \int \int y_c \frac{w_m(y'')\phi_1(y'', c)}{(b_0(y') - c)^2 \phi_1^2(y', c)} dy' + \pi i \frac{w_m(y_c)}{(b_0(y_c))^2}. \]

For any fixed $y \in \mathbb{R}$, due to the fact that $w_m$ has compact support, it holds that
\[ \left| \int y_c \frac{w_m(y'')\phi_1(y'', c)}{(b_0(y') - c)^2 \phi_1^2(y', c)} dy' \right| \leq C |y - y_c|. \]

Then the integrability of $\Phi(y, c)$ in $c_r$ can be obtained by the same argument as (C.4) in the proof of Lemma 5.4. Indeed we have
\[ (5.15) \quad \left| e^{\frac{C_4}{2}|y - y_c|} \Phi(y, c) \right| \leq C, \]
where $C$ is independent of $c_i$. \hfill $\square$

The following estimates hold for $J_1(c_r), J_2(c_r)$, and $J_3(\cdot, c_r)$.

**Lemma 5.13.** Under the assumptions on $M$ and $\gamma$ in Preposition 5.7, it holds that
\[ (5.16) \quad J_1^2(c_r) + J_2^2(c_r) \geq C_M, \quad \text{for } c_r \in \mathbb{R}, \]
\[ (5.17) \quad \|J_3(f, c_r)\|_{L^0_c} \leq C_M \|f\|_{L^0_c} \quad \text{for } f(y) \in C_c^\infty(\mathbb{R}), \]
where $C_M$ is independent of $\gamma$.

**Proof.** Recall that
\[ J_1(c_r) = \frac{1}{b_0'(y_c)} \Pi_1(c_r) + \Pi_2(c_r). \]
As $M\gamma$ is small enough, we have
\[ 1 \leq b_0'(y) \leq \frac{3}{2} \quad \text{for } \forall y \in \mathbb{R}. \]
Then, from (5.7), we have that
\[ -C_6 \leq \Pi_2(c_r) \leq -C_7 < 0, \]
where $C_6, C_7 > 0$ are two constants independent of $M$ and $\gamma$.

Similar to Lemma 5.9, we have $\frac{1}{b_0'(y_c)} \Pi_1(c_r) \geq CM$ for $|c_r| \leq \frac{1}{4} \gamma$. Thus by taking $M$ big enough, we have $J_1 \geq C$ for $|c_r| \leq \frac{1}{4} \gamma$.

Next, we write
\[ \Pi_1(c_r) = \text{P.V.} \int \frac{b_0'(y_c) - b_0'(y)}{(b_0(y) - c_r)^2} dy = \text{P.V.} \int \frac{(b_0'(y_c) - b_0'(b_0^{-1}(v))) \partial_v b_0^{-1}(v)}{(v - c_r)^2} dv. \]
\[
\begin{aligned}
&= - \text{P.V.} \int \frac{b'_0(y) - b'_0(b_0^{-1}(v))}{b'_0(b_0^{-1}(v))} \partial_v \frac{1}{v - c_r} dv = - b'_0(y) \text{P.V.} \int \frac{b''_0(b_0^{-1}(v))}{(b'_0(b_0^{-1}(v)))^3} \frac{1}{v - c_r} dv \\
&= - b'_0(y) \text{P.V.} \int \frac{b'_0(v)}{v - c_r} dv - b'_0(y) \text{P.V.} \int \left( \frac{b''_0(b_0^{-1}(v))}{(b'_0(b_0^{-1}(v)))^3} - b''_0(v) \right) \frac{1}{v - c_r} dv \\
&\overset{\text{def}}{=} \Pi_{1,1}(c_r) + \Pi_{1,2}(c_r).
\end{aligned}
\]

Taking \( V = \frac{v}{\gamma} \), we have

\[
\Pi_{1,1}(c_r) = - \text{P.V.} \int \frac{b'_0(v)}{v - c_r} dv = 4\sqrt{\pi} \text{MP.V.} \int \frac{v e^{-\frac{v^2}{\pi}}}{v - c_r} dv = 4\sqrt{\pi} \text{MP.V.} \int \frac{V e^{-V^2}}{V - \frac{c_r}{\gamma}} dV = -4\sqrt{\pi} \mathcal{M} \mathcal{H}(V e^{-V^2})(\frac{c_r}{\gamma}).
\]

It is clear that \( V e^{-V^2} \in H^1_0 \), then for \( \varepsilon > 0 \) there exists constants \( C_\varepsilon \) independent of \( \gamma \) that \( |\Pi_{1,1}(c_r)| \leq \varepsilon \) for \( |c_r| \geq C_\varepsilon \gamma \).

Recalling that \( \|b_0(y) - y\|_{L_\infty^\infty} \leq C \gamma^2 \), one can easily check that

\[
\left\| \frac{b''_0(b_0^{-1}(v))}{(b'_0(b_0^{-1}(v)))^3} - b''_0(v) \right\|_{L_\infty^1} \leq C \gamma^2
\]

and then

\[
|\Pi_{1,2}(c_r)| \leq \left\| b'_0(y) \text{P.V.} \int \left( \frac{b''_0(b_0^{-1}(v))}{(b'_0(b_0^{-1}(v)))^3} - b''_0(v) \right) \frac{1}{v - c_r} dv \right\|_{L_\infty^1} \leq C \gamma.
\]

Combining the estimates for \( \Pi_{1,1}(c_r) \) and \( \Pi_{1,2}(c_r) \), we can see for \( \gamma \) small enough, there exists constant \( C_8 \) independent of \( \gamma \) such that for \( |c_r| \geq C_8 \gamma \), \( \Pi_1(c_r) \leq \frac{C_7}{\gamma} \), and then \( |J_1(c_r)| \geq \frac{C_7}{\gamma} \).

For \( \frac{1}{4}\gamma \leq |c_r| \leq C_8 \gamma \), we have \( J_2(c_r) \geq C_M \), which gives \( (5.16) \).

Next, we turn to \( J_3 \). We write

\[
J_3(c_r) = \text{P.V.} \int_{\epsilon}^{\infty} \frac{f(b_0^{-1}(v')) \phi_1(b_0^{-1}(v'), c_r) \partial_v b_0^{-1}(v') dv' \partial_v b_0^{-1}(v)}{(v - c_r)^2} dv.
\]

\[
= \frac{1}{(b_0(y_c))^2} \text{P.V.} \int_{\epsilon}^{\infty} \frac{f(b_0^{-1}(v')) dv'}{(v - c_r)^2} \frac{1 + 3(v - c_r)^2}{(1 + (v - c_r)^2)^2}
\]

\[
+ \int_{-\infty}^{\epsilon} \frac{f(b_0^{-1}(v')) \phi_1(b_0^{-1}(v'), c_r) \partial_v b_0^{-1}(v') \partial_v b_0^{-1}(v)}{\phi_1^2(b_0^{-1}(v), c_r)} \frac{1 + 3(v - c_r)^2}{(b_0(y_c))^2(1 + (v - c_r)^2)^2} dv' \]

\[
\overset{\text{def}}{=} J_{3,1}(c_r) + J_{3,2}(c_r).
\]

As \( |v' - c_r| \leq |v - c_r| \), it follows from \( (5.7) \) that \( |\phi_1(b_0^{-1}(v'), c_r) - 1| \leq C|v' - c_r|^2 \), and \( \phi_1(b_0^{-1}(v'), c_r) \geq C e^{C_6|b_0^{-1}(v') - y_c|} \). Then one can easily check that

\[
\left| \phi_1(b_0^{-1}(v'), c_r) \partial_v b_0^{-1}(v') \partial_v b_0^{-1}(v) - \frac{1 + 3(v - c_r)^2}{\phi_1^2(b_0^{-1}(v), c_r)(b_0(y_c))^2(1 + (v - c_r)^2)^2} \right| \leq C \min(|v - c_r|, \frac{1}{|v - c_r|^2}).
\]
Therefore, we have
\[ \| J_{3,2} \|_{L^2_{\epsilon_r}} \leq C \left\| \int_{-\infty}^{+\infty} \frac{\min(|v - \epsilon_r|, \frac{1}{|v - \epsilon_r|^2})}{|v - \epsilon_r|} \int_{\epsilon_r}^{v} f(b_0^{-1}(v')) dv' dv \right\|_{L^2_{\epsilon_r}} \]
\[ \leq C \| f(b_0^{-1}(\cdot)) \|_{L^2_{\vartheta}} \leq C \| f \|_{L^2_{\vartheta}}. \]

For \( J_{3,1}(\epsilon_r) \), we have
\[ J_{3,1}(\epsilon_r) = -\frac{1}{(b_0'(y_c))^2} P.V. \int \int_{\epsilon_r}^{v} f(b_0^{-1}(v')) dv' \partial_v \left( \frac{1}{(v - \epsilon_r)(1 + (v - \epsilon_r)^2)} \right) dv \]
\[ = \frac{1}{(b_0'(y_c))^2} P.V. \int \frac{f(b_0^{-1}(v))}{(v - \epsilon_r)(1 + (v - \epsilon_r)^2)} dv \]
\[ = \frac{1}{(b_0'(y_c))^2} P.V. \int_{\epsilon_r-1}^{\epsilon_r+1} f(b_0^{-1}(v)) dv - \frac{1}{(b_0'(y_c))^2} \int_{\epsilon_r-1}^{\epsilon_r+1} f(b_0^{-1}(v))(v - \epsilon_r) dv \]
\[ + \frac{1}{(b_0'(y_c))^2} \int_{\mathbb{R} \setminus [\epsilon_r - 1, \epsilon_r + 1]} f(b_0^{-1}(v)) dv \]
\[ = \mathcal{H}(f(b_0^{-1}(\cdot)))(\epsilon_r) - \mathcal{H}(f(b_0^{-1}(\cdot)))(\epsilon_r) \]
\[ \quad - \int_{\mathbb{R}} f(b_0^{-1}(v + \epsilon_r)) \frac{\chi_{|v| \leq 1}}{v(1 + v^2)} dv - \int_{\mathbb{R}} f(b_0^{-1}(v + \epsilon_r)) \frac{\chi_{|v| \geq 1}}{v(1 + v^2)} dv \]
\[ \quad \cdot \frac{(b_0'(y_c))^2}{(b_0'(y_c))^2}, \]

where \( \mathcal{H}(f(b_0^{-1}(\cdot)))(\epsilon_r) = \int_{\mathbb{R} \setminus [-1,1]} f(b_0^{-1}(\epsilon_r - v)) dv \). By Young's convolution inequality and the properties of the (maximal) Hilbert operator, we have
\[ \| J_{3,1} \|_{L^2_{\epsilon_r}} \leq C \| f(b_0^{-1}(\cdot)) \|_{L^2_{\vartheta}} \leq C \| f \|_{L^2_{\vartheta}}. \]

The estimate (5.17) follows immediately. \( \square \)

Now we give the proof of the upper bound estimate (5.3).

**Lemma 5.14.** Under the assumptions on \( M \) and \( \gamma \) in Preposition 5.1, and the assumption that \( R \) has no eigenvalues, it holds that
\[ (5.18) \quad \| e^{-itR} w_{in} \|_{L^2_{\varrho}} \leq C \| w_{in} \|_{L^2_{\varrho}}, \]
where \( C \) is a constant independent of \( \gamma \).

**Proof.** Let \( \Psi(t, y) \) be the solution of (5.11). As there is no eigenvalue of \( R \), by (5.15) and Lemma 5.12, we have the following representation formula
\[ \Psi(t, y) = \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-it\epsilon_r} \lim_{\epsilon_r \to 0^+} \left( i\Phi(y, \epsilon_r - i\epsilon_i) - i\Phi(y, \epsilon_r + i\epsilon_i) \right) d\epsilon_r \]
\[ = -\frac{1}{\pi} \int_{-\infty}^{b_0(y)} e^{-it\epsilon_r} \phi(y, \epsilon_r) \frac{J_1 J_4 + J_2 J_3}{J_1^2 + J_2^2} \int_{\epsilon_r}^{y} \frac{1}{\phi^2(y', \epsilon_r)} dy' d\epsilon_r \]
\[ - \frac{1}{\pi} \int_{b_0(y)}^{+\infty} e^{-it\epsilon_r} \phi(y, \epsilon_r) \frac{J_1 J_4 + J_2 J_3}{J_1^2 + J_2^2} \int_{-\infty}^{y} \frac{1}{\phi^2(y', \epsilon_r)} dy' d\epsilon_r. \]
Instead of acting \((\partial_{yy} - 1)\) on the stream function \(\Psi(t, y)\), to estimate \(w(t, y)\), we use the duality argument to give the upper bound of \(\|w(t, y)\|_{L^2_y}\). We make inner product of \(w(t, y)\) with a test function \(g(y) \in C_c^\infty(\mathbb{R})\) such that \(\|g\|_{L^2_y} = 1\), and get

\[
\int_\mathbb{R} w(t, y)g(y)dy = \int_\mathbb{R} \Psi(t, y)(\partial^2_{yy} - 1)g(y)dy = \int_\mathbb{R} e^{-icr}K(c_r)dc_r,
\]

where

\[
K(c_r) = -\frac{1}{\pi} \frac{\mathcal{I}_1(c_r, \mathcal{I}_2(w_{in}, c_r)) + \mathcal{I}_2(c_r, \mathcal{I}_3(w_{in}, c_r))}{\mathcal{I}_1^2(c_r) + \mathcal{I}_2^2(c_r)} \int_{-\infty}^{+\infty} \phi(y, c_r)(\partial^2_{yy} - 1)g(y)
\cdot \left( \chi_{y>y_c} \int_{+\infty}^{y} \frac{1}{\phi^2(y', c_r)}dy' + \chi_{y<y_c} \int_{-\infty}^{y} \frac{1}{\phi^2(y', c_r)}dy' \right)
\]

\[
= \frac{1}{\pi} \frac{\mathcal{I}_1(c_r, \mathcal{I}_2(w_{in}, c_r)) + \mathcal{I}_2(c_r, \mathcal{I}_3(w_{in}, c_r))}{\mathcal{I}_1^2(c_r) + \mathcal{I}_2^2(c_r)} \int_{-\infty}^{+\infty} \frac{\int_{y_c}^{y} b''_0(y)\phi_1(y, c_r)g(y)dy + \phi(y', c_r)g'(y') - \phi'(y, c_r)g(y') + b'_0(y_c)g(y_c)}{\phi^2(y', c_r)}dy',
\]

and \(c_r = b(y_c)\). Here we have used the Fubini theorem in the second identity and integration by part twice in the third identity.

Hence by the definition of \(\mathcal{I}_3\) in \([5, 14]\), we have

\[
P.V. \int \frac{\int_{y_c}^{y} b''_0(y)\phi_1(y, c_r)g(y)dy}{\phi^2(y', c_r)}dy' = \mathcal{I}_3(b''_0g, c_r).
\]

We rewrite

\[
P.V. \int \frac{\phi(y', c_r)g'(y') - \phi'(y', c_r)g(y') + b'_0(y_c)g(y_c)}{\phi^2(y', c_r)}dy'
= P.V. \int \frac{g'(y')}{\phi(y', c_r)}dy' - P.V. \int \frac{g'(y')b'_0(y)\phi_1(y, c_r) - b'_0(y_c)g(y_c)}{\phi^2(y', c_r)}dy'
- P.V. \int \frac{g(y')(b_0(y') - c_r)\phi_1'(y', c_r)}{\phi^2(y', c_r)}dy' \overset{\text{def}}{=} \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3.
\]

For each term, we deduce that

\[
\mathcal{K}_1 = P.V. \int \frac{\partial_y g(y') - g(y_c)}{\phi(y', c_r)}dy' = P.V. \int \frac{(g(y') - g(y_c))\phi'(y', c_r)}{(b_0(y') - c_r)^2\phi_1^2(y', c_r)}dy'
= P.V. \int \frac{(g(y') - g(y_c))b_0(y')\phi_1(y', c_r)}{(b_0(y') - c_r)^2\phi_1^2(y', c_r)}dy'
+ P.V. \int \frac{(g(y') - g(y_c))(b_0(y') - c_r)\phi_1'(y', c_r)}{(b_0(y') - c_r)^2\phi_1^2(y', c_r)}dy' \overset{\text{def}}{=} \mathcal{K}_{1,1} + \mathcal{K}_{1,2},
\]

and

\[
\mathcal{K}_{1,1} + \mathcal{K}_{1,2} = -g(y_c)P.V. \int \frac{b_0'(y')\phi_1(y', c_r) - b'_0(y_c)}{(b_0(y') - c_r)^2\phi_1^2(y', c_r)}dy'
\]
\[= - g(y_c) \text{P.V.} \int_{-\infty}^{+\infty} \left( \frac{y' - c_0}{(b_0(y') - c_0)^2} \right) dy' - g(y_c) \int_{-\infty}^{+\infty} \frac{b_0'(y') - b_0'(y_c)}{b_0(y') - c_0} \left( \frac{1}{\phi_1(y', c_0)} - 1 \right) dy',\]

and

\[K_{1,2} + K_3 = - g(y_c) \int_{-\infty}^{+\infty} \left( \frac{b_0(y') - c_0}{(b_0(y') - c_0)^2} \right) dy' + b_0'(y_c) g(y_c) \int_{-\infty}^{+\infty} \frac{1}{(b_0(y') - c_0)^2} \left( \frac{1}{\phi_1^2(y', c_0)} - 1 \right) dy'.\]

Thus, we conclude that

\[K_1 + K_2 + K_3 = K_{1,1} + K_2 + K_{1,2} + K_3 = g(y_c) \text{P.V.} \int_{-\infty}^{+\infty} \left( \frac{b_0(y') - b_0'(y_c)}{b_0(y') - c_0} \right) dy' + b_0'(y_c) g(y_c) \int_{-\infty}^{+\infty} \frac{1}{(b_0(y') - c_0)^2} \left( \frac{1}{\phi_1^2(y', c_0)} - 1 \right) dy' = b_0'(y_c) g(y_c) J_1(c_0).\]

It follows from the above results that

\[K(c_0) = \frac{1}{\pi} \frac{J_1(c_0) J_4(w_{in}, c_0) + J_2(c_0) J_3(w_{in}, c_0)}{J_2^2(c_0) + J_2^2(c_0)} (J_3(b_0 g, c_0) + b_0'(y_c) g(y_c) J_1(c_0)).\]

Then by Lemma 5.13 and the fact that \(\|g\|_{L^2} = 1\), we have

\[\|w(t)\|_{L^2} \leq \|K(c_0)\|_{L^2} \leq C \left( \|w_{in}\|_{L^2} + \|J_3(w_{in}, c_0)\|_{L^2} \right) \left( \|b_0 g\|_{L^2} + \|J_3(b_0 g, c_0)\|_{L^2} \right) \leq C \|w_{in}\|_{L^2},\]

where \(C > 0\) is independent of \(\gamma\). \(\square\)

**Remark 5.15.** The estimate (5.18) holds if \(P_d R w_{in} = 0\) where \(P_d R\) is the spectral projection to the eigen-spaces which correspond to the discontinuous spectrum \(\sigma_d(R)\).

**Appendix A. Key Points for the Proof of Corollary 1.1**

Consistent with [52], in the appendix we use \(S_{\nu}(t, s) f\) to denote the solution of

(A.1)

\[\begin{cases}
\partial_t \omega + y \partial_x \omega - \nu \Delta \omega = 0, \\
\omega|_{t=s} = f(x, y),
\end{cases}\]

with \(\int_T f(x, y) dx = 0\).

We now consider the nonlinear equation

(A.2)

\[\begin{cases}
\partial_t \omega_\ne + y \partial_x \omega_\ne - \nu \Delta \omega_\ne = -\mathcal{L} - \mathcal{N}^{(1)} - \mathcal{N}^{(2)} - \mathcal{N}^{(3)}, \\
(u_\ne^{(1)}, u_\ne^{(2)}) = (\partial_y (-\Delta)^{-1} \omega_\ne, -\partial_x (-\Delta)^{-1} \omega_\ne), \\
\omega_\ne|_{t=0} = P_\ne w_{in},
\end{cases}\]

where \(b_\nu(t, y)\) is the solution of (1.2),

\[\mathcal{L} = (b_\nu - y) \partial_x \omega_\ne + \partial_y b_\nu \partial_x (-\Delta)^{-1} \omega_\ne, \quad \mathcal{N}^{(1)} = (u_\ne^{(1)} \partial_x \omega_\ne)_\ne + (u_\ne^{(2)} \partial_y \omega_\ne)_\ne,
\]
Similarly we introduce the linear change of coordinate in \([52]\).

\[ \omega_0(t, y) \] is the zero mode of vorticity which satisfies

\[ \omega(t, y) = \frac{\partial \omega_0}{\partial y} \]

\[ \omega_0|_{t=0} = \mathcal{P}_0 \omega_0. \]

and \( u^{(1)}_0 \) is the zero mode of horizontal velocity which satisfies

\[ \frac{\partial u^{(1)}_0}{\partial t} - \nu \frac{\partial^2 u^{(1)}_0}{\partial y^2} = - \left( u^{(1)}_0 \frac{\partial \omega^{(1)}_0}{\partial x} \right)_0 - \left( u^{(2)}_0 \frac{\partial \omega^{(1)}_0}{\partial y} \right)_0, \]

\[ u^{(1)}_0|_{t=0} = \mathcal{P}_0 u^{(1)}_0. \]

Here

\[ \|b_\nu(y) - y\|_{L^\infty_t H^1_y} \leq \varepsilon_0 \nu^{\frac{1}{2}}, \quad \|\omega_\nu\|_{H^2_t L^\infty_y} \leq \varepsilon_0 \nu^{\frac{1}{2}}. \]

In \([52]\), the authors take \( b_\nu(t, y) \equiv y \), and then \( \mathcal{L} \equiv 0 \). Therefore, to prove Corollary \[1.1\] it suffices to give an estimate for \( \mathcal{L} \) similar to the estimates in Lemma 3.3 of \([52]\):

\[ \| \ln(e + |D_x|) \mathcal{L}(t) \|_{L^1_t([0, T]; L^2_y)} \leq C \varepsilon_0 \| \ln(e + |D_x|) \omega_\nu \|_{L^2_t L^2_y}. \]

It follows from properties of heat kernel that

\[ \|b_\nu(t, y) - y\|_{L^\infty_t([0, T]; L^\infty_y \cap H^1_y)} + \nu^{\frac{1}{2}} \|\partial^2 y b_\nu\|_{L^2_t([0, T]; L^2_y)} \leq C \varepsilon_0 \nu^{\frac{1}{2}}. \]

Then we have

\[ \| \ln(e + |D_x|) \mathcal{L}(t) \|_{L^1_t([0, T]; L^2_y)} \leq \|b_\nu(t, y) - y\|_{L^\infty_t([0, T]; L^\infty_y)} \| \ln(e + |D_x|) \partial x \omega_\nu \|_{L^1_t([0, T]; L^2_y)} \]

\[ + \|\partial^2 y \omega_\nu\|_{L^2_t([0, T]; L^2_y)} \| \ln(e + |D_x|) \partial x (\Delta)^{-1} \omega_\nu \|_{L^2_t([0, T]; L^2_y)} \]

\[ \leq C(C_3 + C_5) \varepsilon_0 \| \ln(e + |D_x|) \omega_\nu \|_{L^2_t L^2_y}, \]

where \( C_3 \) and \( C_5 \) are given in Section 3 of \([52]\).

With this estimate, one can easily deduce the result of Corollary \[1.1\] by following the proof in \([52]\).

**Appendix B. Proof of Remark \[1.7\]**

In this section, we study the semi-group generalized by

\[ \mathcal{R}_{M, \gamma, \mathcal{R}} = b_0(y) \partial_x - b_0^\nu(y) \partial_x (\Delta)^{-1} - \nu \Delta. \]

We consider the following system:

\[ \partial_t \omega(t, x, y) + b_0(y) \partial_x \omega(t, x, y) - \nu \Delta \omega(t, x, y) - b_0^\nu(y) \partial_x \Delta^{-1} \omega(t, x, y) = 0. \]

Similarly we introduce the linear change of coordinate \( z = x - ty \) and \( h(t, z, y) = \omega(t, x, y) \). Then \( h(t, z, y) \) satisfies

\[ \partial_t h(t, z, y) + b_0(y) \partial_z h(t, z, y) - b_0^\nu(y) \partial_z \Delta^{-1} h(t, z, y) - \nu \Delta h(t, z, y) = 0. \]

By taking Fourier transform in \( z \), we get that

\[ \partial_t \hat{h}_k(t, \xi) + \nu(k^2 + (\xi - kt)^2) \hat{h}_k(t, \xi) + ik \mathcal{F}(z, y) \rightarrow (k, \xi) \left( (b_0(y) - y) h \right)(t, k, \xi) - \int_{\mathbb{R}} M \gamma^2(\xi - \eta)e^{-\frac{\xi^2}{\eta^2} + \frac{2}{\eta(\xi - \eta)^2} + \frac{k^2}{\eta - kt} + k^2} \, d\eta = 0. \]
Similar to Lemma 3.3, we deduce from the above equation that
\[
\partial_t \left( e^{\nu k^2 \left( t + \left( \frac{1}{2} - \frac{k^2}{2} \right)^2 + \frac{\nu^2}{12} \right)} \hat{h}_k(t, \xi) \right) \\
= -i k e^{\nu k^2 \left( t + \left( \frac{1}{2} - \frac{k^2}{2} \right)^2 + \frac{\nu^2}{12} \right)} \mathcal{F}(z, y) \to (k, \xi) \left( (b_0(y) - y) h \right) (t, k, \xi) \\
+ e^{\nu k^2 \left( t + \left( \frac{1}{2} - \frac{k^2}{2} \right)^2 + \frac{\nu^2}{12} \right)} \int_\mathbb{R} M \gamma^2 (\xi - \eta) e^{-\nu^2 |\xi - \eta|^2} \frac{k \hat{h}_k(t, \eta)}{(\eta - kt)^2 + k^2} d\eta = 0.
\]
It follows that
\[
\hat{h}_k(t, \xi) = e^{-\nu k^2 \left( t + \left( \frac{1}{2} - \frac{k^2}{2} \right)^2 + \frac{\nu^2}{12} \right)} \hat{h}_k(0, \xi) \\
- i k \int_0^t e^{-\nu k^2 \left( (t - s) + \left( \frac{1}{2} - \frac{k^2}{2} \right)^2 + \frac{\nu^2}{12} \right)} \mathcal{F}(z, y) \to (k, \xi) \left( (b_0(y) - y) h \right) (s, k, \xi) ds \\
+ \int_0^t e^{-\nu k^2 \left( (t - s) + \left( \frac{1}{2} - \frac{k^2}{2} \right)^2 + \frac{\nu^2}{12} \right)} \int_\mathbb{R} M \gamma^2 (\xi - \eta) e^{-\nu^2 |\xi - \eta|^2} \frac{k \hat{h}_k(s, \eta)}{(\eta - ks)^2 + k^2} d\eta ds.
\]
We deduce from Young’s convolution inequality that
\[
\left\| \int_\mathbb{R} M \gamma^2 (\xi - \eta) e^{-\nu^2 |\xi - \eta|^2} \frac{k \hat{h}_k(s, \eta)}{(\eta - ks)^2 + k^2} d\eta \right\|_{L^2_\xi} \leq CM \gamma^2 \| \hat{h}_k(s, \xi) \|_{L^2_\xi}.
\]
We also have
\[
\left\| \mathcal{F}(z, y) \to (k, \xi) \left( (b_0(y) - y) h \right) (s, k, \xi) \right\|_{L^2_\xi} \leq \left\| b_0(y) - y \right\|_{L^\infty_\xi} \| \hat{h}_k(s, \xi) \|_{L^2_\xi} \leq CM \gamma^2 \| \hat{h}_k(s, \xi) \|_{L^2_\xi}.
\]
Then, by (3.4) we have
\[
\| \hat{h}_k(t, \xi) \|_{L^2_\xi} \leq e^{-\nu k^2 \left( t + \frac{\nu^2}{12} \right)} \| \hat{h}_k(0, \xi) \|_{L^2_\xi} + CM (\gamma^2 + k \gamma^2) \int_0^t e^{\nu k^2 \left( s + \frac{\nu^2}{12} - t + \frac{\nu^2}{12} \right)} \| \hat{h}_k(s, \xi) \|_{L^2_\xi} ds,
\]
which together with Gronwall’s inequality gives (1.18). The cubic exponential decay implies that \( R_{M, \gamma, \nu} \) has no eigenvalues for any small viscosity \( \nu > 0 \).

**APPENDIX C. HOMOGENEOUS RAYLEIGH EQUATION**

Here we give the proofs of the proposition and lemmas given in Section 5.1. We modify the argument in [65] to adapt it to the unbounded domain case. We remark that the estimates in this section hold for general \( b_0(y) \) which satisfies:

(C.1) \( 0 < C^{-1} \leq |b_0(y)| \leq C \) for \( y \in \mathbb{R} \), \( \| b_0' \|_{L^\infty_0 \cap L^1_0} \leq C \).

To prove Proposition 5.3, we introduce function spaces \( X \) and \( Y \). For a function \( g(y, c_\tau) \) defined on \( \mathbb{R} \times \mathbb{R} \), we define
\[
\| g \|_X \overset{\text{def}}{=} \sup_{(y, c_\tau) \in \mathbb{R} \times \mathbb{R}} \left| \frac{g(y, c_\tau)}{\cosh(C(y - y_\tau))} \right|, \quad \| g \|_Y \overset{\text{def}}{=} \sum_{k=0}^2 \tilde{C}^{-k} \| \partial_{c_\tau}^k g \|_X.
\]
Here \( \tilde{C} > 0 \) is a constant which will be determined later.

**Proof of Proposition 5.3.** From (5.5), we write
\[
(C.2) \quad \phi_1(y, c_\tau) = 1 + \int_{y_\tau}^y \frac{1}{(b_0(y') - b_0(y_\tau))^2} \int_{y_\tau}^{y'} \phi_1(z, c_\tau) (b_0(z) - b_0(y_\tau))^2 dz dy'.
\]
Submitting $\phi(y, \epsilon) = (b_0(y) - \epsilon) \phi_1(y, \epsilon) \phi_2(y, \epsilon)$ into (5.4), we have that $\phi_2(y, \epsilon)$ satisfies

$$(b_0 - \epsilon) \phi_2'' + 2b_0' \phi_2' + 2(b_0 - \epsilon) \phi_1' \phi_2' + \frac{2i \epsilon b_0'}{b_0 - \epsilon} \phi_1 \phi_2 = 0,$$

which together with the boundary conditions $\phi_2(y, \epsilon) = 1$ and $\phi_2'(y, \epsilon) = 0$ gives (5.6). Then we write

\begin{equation}
\phi_2(y, \epsilon) = 1 - 2i \epsilon \int_y^{y_c} \frac{1}{(b_0(y') - b_0(y_c))^2} \int_y^{y_c} \frac{b_0'(z)(b_0(z) - \epsilon) \phi_1(z) \phi_1'(z) \phi_2(z) dz dy'}{b_0(z) - \epsilon}.
\end{equation}

Let $T_1$ be the integral operator that

$T_1 g(y, \epsilon_r) = T_0 \circ T_2, g(y, \epsilon_r) = \int_y^{y_c} \frac{1}{(b_0(y') - b_0(y_c))^2} \int_y^{y_c} g(z, \epsilon_r)(b_0(z) - b_0(y_c))^2 dz dy'$,

where

$T_0 g(y, \epsilon_r) = \int_y^{y_c} g(y', \epsilon_r) dy'$

and

$T_{k-1} g(y, \epsilon_r) = \int_y^{y_c} g(z, \epsilon_r)(b_0(z) - b_0(y_c))^k dz$.

By the definition of the operator $T$, we have $\phi_1 = 1 + T \phi_1$.

It is clear that

$$\|T_0 g\| \leq \sup_{(y, \epsilon_r) \in \mathbb{R} \times \mathbb{R}} \left| \int_y^{y_c} \frac{1}{\cosh(C(y - y_c))} \int_y^{y_c} g(y', \epsilon_r) \cosh(\tilde{C}(y' - y_c)) dy' \right| \leq C \|g\| \|y\| \leq \frac{1}{C} \|g\| \|y\|,$$

and

$$\|T_{0,2} g\| \leq \sup_{(y, \epsilon_r) \in \mathbb{R} \times \mathbb{R}} \left| \int_y^{y_c} \frac{(b_0(y') - b_0(y_c))^2 \cosh(\tilde{C}(y' - y_c))}{\cosh(\tilde{C}(y - y_c))} dy' \right| \leq \frac{1}{C} \|g\| \|y\| \leq \frac{1}{C} \|g\| \|y\|.$$

Here we use the fact that $|b_0(y') - b_0(y_c)| \leq |b_0(y) - b_0(y_c)|$ for $|y' - y_c| \leq |y - y_c|$.

It follows directly that

$$\|T_1 g\| \leq \frac{1}{C^2} \|g\| \|y\|.$$

Direct calculations show that

$$\partial_y T_1 g(y, \epsilon_r) = T_{0,2} g(y, \epsilon_r), \quad \partial_y^2 T_1 g(y, \epsilon_r) = -2b_0''(y) T_{2,3} g(y, \epsilon_r) + g(y, \epsilon_r).$$

We write

$$T_{2,3} g(y, \epsilon_r) = \frac{1}{(b_0(y) - b_0(y_c))^3} \int_y^{y_c} g(z, \epsilon_r)(b_0(z) - b_0(y_c))^2 dz$$

$$= \int_y^{y_c} \frac{(z - y_c)^2 (\int_0^1 b_0'(y_c + s(z - y_c)) ds)^2}{(y - y_c)^3 (\int_0^1 b_0'(y_c + s(y - y_c)) ds)^3} g(z, \epsilon_r) dz.$$
Therefore, we have
\[
\int_0^1 \left( \int_0^t y_c s(y - y_c) ds \right)^2 \frac{g(y_c + t(y - y_c), c) t^2 dt}{(y_0 - y_c)^2} = \int_0^1 \left( \int_0^t b(y_c + t(y - y_c), c) t^2 dt \right) g(y_c + t(y - y_c), c) t^2 dt \]
then we have
\[
\| T_{2.3} g \|_X \leq C \sup_{(y, c) \in \mathbb{R} \times \mathbb{R}} \left| \frac{1}{\cosh(C(y - y_c))} \int_0^1 \frac{g(y_c + t(y - y_c), c) t^2}{\cosh(Ct(y - y_c))} dy \right| \]
\[
\leq C \sup_{(y, c) \in \mathbb{R} \times \mathbb{R}} \left| \frac{1}{\cosh(C(y - y_c))} \int_0^1 \cosh(\tilde{C}t(y - y_c)) dy \right| \| g \|_X \leq C \| g \|_X,
\]
which gives that
\[
\frac{1}{C} \| \partial_y T_1 g \|_X \leq \frac{1}{C^2} \| g \|_X, \quad \frac{1}{C^2} \| \partial_y^2 T_1 g \|_X \leq \frac{C}{C^2} \| g \|_Y.
\]
It follows that
\[
\| T_1 g \|_Y \leq \frac{C}{C^2} \| g \|_Y.
\]
By taking $\tilde{C}$ big enough, we have that $I - T_1$ is invertible in the space $Y$. Thus,
\[
\phi_1(y, \epsilon_r) = (I - T_1)^{-1} 1,
\]
with the bound $\| \phi_1 \|_Y \leq C$. Then, from the expression (C.2), one can easily verify that $\phi_1(y, \epsilon_r) \geq 0$.

Let $F(y, \epsilon_r) = \frac{\phi_1'(y, \epsilon_r)}{\phi_1(y, \epsilon_r)}$. It is easy to check that
\[
F'(y, \epsilon_r) + F^2(y, \epsilon_r) + \frac{2b_0'(y)F(y, \epsilon_r)}{b_0(y) - \epsilon_r} = 1 = 0.
\]
It follows from $\phi_1(y_1, \epsilon_r) = 1$ and $\phi_1'(y_1, \epsilon_r) = 0$ that $F(y, \epsilon_r) = 0$. Then we can see that
\[
\lim_{y \to y_c} F'(y, \epsilon_r) = 1 - \lim_{y \to y_c} F^2(y, \epsilon_r) - \lim_{y \to y_c} \frac{2b_0'(y)F(y, \epsilon_r)}{b_0(y) - \epsilon_r}
\]
\[
= 1 - 2 \lim_{y \to y_c} F'(y, \epsilon_r).
\]
Therefore, we have $F'(y_c, \epsilon_r) = \frac{1}{3} > 0$. From (C.2) we have
\[
\phi_1'(y, \epsilon_r) = \frac{1}{(b_0(y) - b_0(y_1))^2} \int_{y_1}^y \phi_1(z, \epsilon_r) (b_0(z) - b_0(y_1))^2 dz,
\]
and $\phi_1'(y, \epsilon_r) > 0$ for $y > y_c$ and $\phi_1'(y, \epsilon_r) < 0$ for $y < y_c$. Therefore $\phi_1(y, \epsilon_r) \geq 0$ for $z \in [y, y_1]$ or $z \in [y_c, y]$. Then we have $\frac{F(y, \epsilon_r)}{b_0(y) - \epsilon_r} \geq 0$ for $\forall y \in \mathbb{R}$, and
\[
|F(y, \epsilon_r)| = \left| \frac{\phi_1'(y, \epsilon_r)}{\phi_1(y, \epsilon_r)} \right| = \left| \frac{1}{(b_0(y) - b_0(y_1))^2} \int_{y_1}^y \phi_1(z, \epsilon_r) (b_0(z) - b_0(y_1))^2 dz \right| \leq |y - y_c|.
\]
Next, we show that $|F(y, \epsilon_r)| \leq 1$. If $F$ attains its maximum (minimum) at $y_0$, we have $F'(y_0, \epsilon_r) = 0$ and
\[
F^2(y_0, \epsilon_r) = 1 - \frac{2b_0'(y_0)F(y_0, \epsilon_r)}{b_0(y_0) - \epsilon_r} \leq 1.
\]
Therefore $|F(y, \epsilon_r)| \leq 1$. If $F(y, \epsilon_r) > 1$ at $y_1$, we know that $y_1 > y_c$ as $F(y_1, \epsilon_r) > 0$. Then we have $F'(y_1, \epsilon_r) < 0$. Recall that $F'(y_c, \epsilon_r) = \frac{1}{3} > 0$, so there exits $y_2 \in (y_c, y_1)$ such that $F'(y_2, \epsilon_r) = 0$ and $F(y_2, \epsilon_r) \geq F(y_1, \epsilon_r) > 1$, which is impossible. If $F(y, \epsilon_r) < -1$ at $y_3$,
we know that \( y_3 < y_4 \) and \( F'(y, c_r) < 0 \), then \( F(y, c_r) \) decreases strictly on \([y_3, y_4]\), which contradicts \( F(y, c_r) = 0 \). As a conclusion, we have \( |F(y, c_r)| \leq \min(1, |y - y_4|) \). It follows that

\[
e^{-|y-y'|} \leq \frac{\phi_1(y', c_r)}{\phi_1(y, c_r)} \leq e^{|y-y'|}.
\]

Next, we show that for \( y \) such that \( |y - y_4| \geq 1 \), \( |F(y, c_r)| \geq C_4 > 0 \). With out lose of generally, we assume \( y \geq y_4 + 1 \), and have

\[
F(y, c_r) = \frac{1}{(b_0(y) - b_0(y_4))^2} \int_{y_4}^{y} \frac{\phi_1(z, c_r)(b_0(z) - b_0(y_4))^2}{\phi_1(y, c_r)} dz \\
\geq \frac{1}{(b_0(y) - b_0(y_4))^2} \int_{y_4}^{y} e^{-|y-z|}(b_0(z) - b_0(y_4))^2 dz \\
\geq \frac{(b_0(y) - \frac{1}{2}) - b_0(y_4))^2}{(b_0(y) - b_0(y_4))^2} \int_{y_4}^{y} e^{-|y-z|} dz \geq C.
\]

Here we use (C.1) that \( b_0(y) \) is a strictly monotonic function. Then we get the existence of \( \phi_1(y, c_r) \) and the estimates (5.7).

Similarly, we introduce operator \( T_2 \) and write (C.3) as

\[
\phi_2(y, c) = 1 - 2ic_i \int_{y_4}^{y} \frac{1}{(b_0(y_4) - c)^2} \frac{\phi_2(y')}{\phi_1(y, c_r)} dy' \\
= 1 + T_2\phi_2(y, c).
\]

Recalling that \( F(y, c_r) = \frac{\phi'_1(y, c_r)}{\phi_1(y, c_r)} \), we have

\[
\left| \int_{y_4}^{y} \frac{b_0'(z)(b_0(z) - c)}{b_0(z) - c} \phi_1(z, c_r)\phi'_1(z, c_r)\phi_2(z, c) dz \right| \\
\leq \int_{y_4}^{y} \frac{b_0'(z) \phi_1(z, c_r)\phi'_1(z, c_r)\phi_2(z, c) dz}{b_0(z) - c} + c_i \left| \int_{y_4}^{y} \frac{b_0'(z)F}{b_0(z) - c} \phi_2^2(z, c)\phi_2(z, c) dz \right|.
\]

As \( \phi'_1(y, c_r) > 0 \) for \( y > y_4 \), and \( \phi'_1(y, c_r) < 0 \) for \( y < y_4 \), we have

\[
\left| \int_{y_4}^{y} \frac{b_0'(z) \phi_1(z, c_r)\phi'_1(z, c_r)\phi_2(z, c) dz}{b_0(z) - c} \right| \\
\leq C\|\phi_2\|_{L^\infty} \int_{y_4}^{y} \frac{\phi_1(z, c_r)\phi'_1(z, c_r)\phi_2(z, c) dz}{C\|\phi_2\|_{L^\infty}(\phi_2^2(y, c_r) - 1)},
\]

and

\[
\left| \int_{y_4}^{y} \frac{b_0'(z)F}{b_0(z) - c} \phi_2^2(z, c)\phi_2(z, c) dz \right| \\
\leq C\|\phi_2\|_{L^\infty} \int_{y_4}^{y} \phi_2^2(z, c_r) dz \\
\leq \|\phi_2\|_{L^\infty} \int_{y_4}^{y+1} \phi_2^2(z, c_r) dz + C\|\phi_2\|_{L^\infty} \int_{y_4+1}^{y} \phi_1(z, c_r)\phi'_1(z, c_r) dz \\
\leq C\|\phi_2\|_{L^\infty}(\phi_1^2(y, c_r) + C).
\]

Here we use the fact that \( |F(y, c_r)| \leq \min(|y - y_4|, 1) \) and \( |F(y, c_r)| \geq C_4 > 0 \) for \( |y - y_4| \geq 1 \).
We have \((b_0(y') - c)^2 = (b_0(y') - c_r)^2 - c_r^2 + 2i(b_0(y') - c_r)c_i\). It follows that
\[
\int_{y_c} y^C \frac{\|\phi_\|_{L^\infty} (\phi_0^2(y', y) - 1) |(b_0(y') - c)^2 \phi_0^2(y')|}{(b_0(y') - c)^2 \phi_0^2(y')} dy' \leq C \|\phi_\|_{L^\infty},
\]
and
\[
\int_{y_c} y^C \frac{\|\phi_\|_{L^\infty} (\phi_0^2(y', y) + C) |(b_0(y') - c)^2 \phi_0^2(y', c_r)|}{(b_0(y') - c)^2 \phi_0^2(y', c_r)} dy' \leq C \|\phi_\|_{L^\infty}.
\]
Combining (C.3) and the above two inequalities, we have
\[
\|T_2\|_{L^\infty} \leq C c_i \|\phi_\|_{L^\infty}.
\]
For \(c_i\) small enough, we have that \(I - T_2\) is invertible in \(L^\infty_y\). Thus there exists
\[
\phi_2(y, c) = (I - T_2)^{-1} 1,
\]
such that
\[
\|\phi_2(y, c) - 1\|_{L^\infty} \leq C c_i.
\]
Taking derivative of (C.3), it holds that
\[
\phi'_2(y, c) = -2i c_i \frac{1}{(b_0(y) - c)^2 \phi_0^2(y)} \int_{y_c} y^C \frac{b_0'(z)(b_0(z) - c)}{b_0(z) - c} \phi_1(z, c_r) \phi'_1(z, c_r) \phi_2(z) \, dz.
\]
As \(|F(y, c, c_r)| \leq \min(1, |y - y_c|)\), we have
\[
\left| \int_{y_c} y^C \frac{b_0'(z)F}{b_0(z) - c} \phi_1^2(z, c) \phi_2(z) \, dz \right| \leq C \|\phi_\|_{L^\infty} \int_{y_c} y^C \phi_1^2(z, c) \, dz \leq C |y - y_c|, \text{ for } |y - y_c| \leq 1.
\]
Then by similar argument to \(|\phi_2(y) - 1|_{L^\infty}\), one can easily deduce that
\[
|\phi_2(y)| \leq C c_i,
\]
and \(|\phi_2'(y)|\) decay to 0 as \(|y| \to +\infty\).

We write from (5.6) that
\[
\phi_2'' = -\frac{2b_0\phi_0'}{b_0 - c} - 2 \frac{\phi_1^2}{\phi_0} \phi_2 - \frac{2ic b_0' \phi_1'}{(b_0 - c)(b_0 - c)} \phi_2,
\]
from which we can see that \(|\phi_2''|_{L^\infty} \leq C\). Then we get the existence of \(\phi_2(y, c)\) and the estimates (5.8).

The existence of \(\phi(y, c)\) and the estimate (5.9) follow immediately.

Then we give the prove of Lemma 5.4.

Proof of Lemma 5.4. From Proposition 5.3, since \(\phi(y, c) = (b_0(y) - c) \phi_1(y, c_r) \phi_2(y, c)\) is a solution to (5.4), then
\[
\varphi^-(y, c) = \phi(y, c) \int_{-\infty}^y \frac{1}{\phi_0^2(y', c)} dy'\]
is a solution to (5.4) which is independent of \(\phi(y, c)\). If \(c\) is an eigenvalue, and \(\psi(y, c) \in H^2_y\) is the corresponding eigenfunction, then by (5.10) and the fact that as \(y \to -\infty, \phi(y, c) \to +\infty\) and \(\varphi^-(y, c) \to 0\), then necessarily \(\lambda_1^{-1}\) in (5.10) is 0 and \(\lim_{y \to +\infty} \varphi^-(y, c) = 0\). Then it follows from the fact \(|\phi(y, c)| \geq C |y - y_c| e^{C_4 |y - y_c|}\) that \(D(c) = 0\).
Next we show that if \( D(c) = 0 \), then \( \varphi^- (y, c) \in H^1_y \). We claim that as \( y \to -\infty \), \( \varphi^- (y, c) \) and \( \varphi^- (y, c) \) decay to 0 exponentially. Indeed, by the fact that \( \phi_1 (y', c_r) \geq \phi_1 (y, c_r) \geq e^{C_4 (y-y)} \) for \( y' \leq y \leq y_c - 1 \), thus we have,

\[
\left| e^{\frac{C_4}{2} (y-y)} \varphi^- (y, c) \right| = \left| e^{\frac{C_4}{2} (y-y)} \phi (y, c) \int_{-\infty}^{y} \frac{1}{\phi^2 (y', c)} dy' \right|
\]

(C.4)

Similarly, we have for \( y' \leq y \leq y_c - 1 \)

\[
\left| e^{\frac{C_4}{2} (y-y)} \varphi^- (y, c) \right| = \left| e^{\frac{C_4}{2} (y-y)} \frac{\phi (y, c)}{\phi (y, c)} + e^{\frac{C_4}{2} (y-y)} F (y, c) \varphi^- (y, c) \right| \leq C.
\]

Here \( C \) is a constant independent of \( c_r \). As \( \varphi^+ (y, c) = \phi (y, c) \int_{y}^{+\infty} \frac{1}{\varphi (y', c)} dy' \) and \( \phi (y, c) \) are two independent solutions of (5.4), there exists two constants \( a_1 \) and \( a_2 \) such that

\[
\varphi^- (y, c) = \phi (y, c) \int_{-\infty}^{y} \frac{1}{\phi^2 (y', c)} dy' = a_1 \phi (y, c) + a_2 \phi (y, c) \int_{y}^{+\infty} \frac{1}{\phi^2 (y', c)} dy',
\]

then,

\[
0 = D(c) = \lim_{y \to +\infty} \int_{-\infty}^{y} \frac{1}{\phi^2 (y', c)} dy' = \lim_{y \to +\infty} \frac{\phi (y, c)}{y} = a_1 + \lim_{y \to +\infty} a_2 \int_{y}^{+\infty} \frac{1}{\phi^2 (y', c)} dy' = a_1.
\]

Thus we have \( \varphi^- (y, c) = a_2 \phi (y, c) \int_{y}^{+\infty} \frac{1}{\varphi (y', c)} dy' \). Similar to the case \( y \to -\infty \), one can prove that as \( y \to +\infty \), \( \varphi^- (y, c) \) and \( \varphi^- (y, c) \) decay to 0 exponentially. Therefore \( \varphi^- \in H^1_y \).

Then by the equation (5.4), \( \varphi^- \in H^2_y \), and \( c \) is an eigenvalue of the Rayleigh operator \( R \).

Remark C.1. Let \( \varphi^+ (y, c) = \phi (y, c) \int_{y}^{+\infty} \frac{1}{\varphi (y', c)} dy' \), then

\[
D(c) = \int_{R} \frac{1}{\phi^2 (y, c)} dy = \det \left( \begin{array}{cc} \varphi^- (y, c) & \varphi^+ (y, c) \\ \partial_y \varphi^- (y, c) & \partial_y \varphi^+ (y, c) \end{array} \right)
\]

is the Wronskian and \( D(c) \) is analytic.

Proof of Lemma C.2. We write

\[
D(c) = \int_{-\infty}^{+\infty} \frac{1}{(b_0 (y) - c)^2} dy + \int_{-\infty}^{+\infty} \frac{1}{(b_0 (y) - c)^2} \left( \frac{1}{\phi^2 (y, c_r) \phi^2 (y, c)} - 1 \right) dy \overset{\text{def}}{=} I + II.
\]

From (5.7) and the Lebesgue dominated convergence theorem, we get

\[
\lim_{c_r \to 0} II = \lim_{c_r \to 0} \int_{-\infty}^{+\infty} \frac{1}{(b_0 (y) - c_r - ic_c)^2} \left( \frac{1}{\phi^2 (y, c_r) \phi^2 (y, c + ic_c)} - 1 \right) dy = \int_{-\infty}^{+\infty} \frac{1}{(b_0 (y) - c_r)^2} \left( \frac{1}{\phi^2 (y, c_r)} - 1 \right) dy = \Pi_2 (c_r).
\]

Let \( v = b_0 (y) \), we have

\[
I = \int_{-\infty}^{+\infty} \frac{1}{(b_0 (y) - c)^2} dy = \int_{-\infty}^{+\infty} \partial_v (b_0^{-1} (v)) dv = \int_{-\infty}^{+\infty} \frac{\partial^2_v (b_0^{-1} (v))}{v - c} dv = \int_{-\infty}^{+\infty} \frac{i c \partial^2_v (b_0^{-1} (v))}{v - c} dv \overset{\text{def}}{=} I_r + i I_i.
\]
From the properties of the Hilbert transform and the Poisson kernel, we know that

\begin{equation}
\lim_{\epsilon \to 0^\pm} I = -\mathcal{H}(\partial^2 (b_0^{-1}))(\epsilon_r) \pm i\pi \partial^2 (b_0^{-1})(\epsilon_r)
\end{equation}

\begin{align*}
&= \frac{1}{b_0'(y_c)} \text{P.V.} \int_\mathbb{R} \frac{1}{v - \epsilon_r} \partial_v \left( \frac{b_0'(y_c) - b_0'(y)}{b_0'(y)} \right) dv \mp i\pi \frac{b_0''(y_c)}{(b_0'(y_c))^3} \\
&= \frac{1}{b_0'(y_c)} \text{P.V.} \int_\mathbb{R} \frac{b_0'(y_c) - b_0'(y)}{(b_0(y) - \epsilon_r)^2} dy \mp i\pi \frac{b_0''(y_c)}{(b_0'(y_c))^3} = \frac{1}{b_0'(y_c)} \Pi_1(\epsilon_r) \mp i\mathcal{J}_2(\epsilon_r),
\end{align*}

which gives the lemma. \( \square \)

**Proof of Lemma** [5.6] We first show that if \( \epsilon_r \in \mathbb{R} \) is an eigenvalue, then \( b_0''(y_c) = 0 \). Let \( \psi(y, \epsilon_r) \in H^1_y \) be the corresponding eigenfunction, namely, the solution of (5.4). If \( b_0''(y_c) \neq 0 \), we can see from (5.4) that \( \psi(y) = 0 \). Then by taking inner product with \( \psi \) on both sides of (5.4) and integration by parts, we have

\begin{align*}
\int_\mathbb{R} \left| \psi' - \frac{b_0' \psi}{b_0 - \epsilon_r} \right|^2 dy + \int_\mathbb{R} |\psi|^2 dy = 0,
\end{align*}

which implies that \( \psi \equiv 0 \). Thus, if \( \epsilon_r \) is an eigenvalue, \( b_0''(y_c) = 0 \) and \( \mathcal{J}_2(\epsilon_r) = 0 \). Therefore, we only need to study \( \mathcal{J}_1(\epsilon_r) \) under the assumption that \( b_0''(y_c) = 0 \).

For \( \epsilon_r \in \mathbb{R} \), \( \phi_2(y, \epsilon_r) \equiv 1 \) and \( \phi(y, \epsilon_r) = \phi_1(y, \epsilon_r)(b_0(y) - \epsilon_r) \) is a solution to (5.4), then

\begin{equation}
\varphi^-(y, \epsilon_r) = \phi(y, \epsilon_r) \int_\mathbb{R} 1 \frac{1}{\phi^2(y', \epsilon_r)} dy' \end{equation}

is another solution of (5.4).

We write

\begin{align*}
\varphi^-(y, \epsilon_r) &= \phi(y, \epsilon_r) \int_{-\infty}^y \frac{1}{\phi^2(y', \epsilon_r)} dy' \\
&= \phi(y, \epsilon_r) \int_{-\infty}^y \frac{1}{(b_0(y') - \epsilon_r)^2} dy' + \phi(y, \epsilon_r) \int_{-\infty}^y \frac{1}{(b_0(y') - \epsilon_r)^2} \left( \frac{1}{\phi^2(y', \epsilon_r)} - 1 \right) dy' \\
&\equiv I + II.
\end{align*}

As \( \phi_1(y, \epsilon_r) = 1 \) and \( \phi'_1(y, \epsilon_r) = 0 \), the integral function in \( II \) does not have singularity at \( y_c \), then \( II \) is well defined on \( \mathbb{R} \). We also deduce that

\begin{equation}
I = \frac{\phi(y, \epsilon_r)}{b_0'(y_c)} \int_{-\infty}^y \frac{b_0'(y_c) - b_0'(y)}{(b_0(y') - \epsilon_r)^2} dy' + \frac{\phi(y, \epsilon_r)}{b_0'(y_c)} \int_{-\infty}^y \frac{b_0''(y)}{(b_0(y') - \epsilon_r)^2} dy' \\
= \frac{\phi(y, \epsilon_r)}{b_0'(y_c)} \int_{-\infty}^y \frac{b_0'(y_c) - b_0'(y)}{(b_0(y') - \epsilon_r)^2} dy' - \phi(y, \epsilon_r) \frac{1}{b_0'(y_c)} \frac{1}{b_0(y) - \epsilon_r} \\
= \frac{\phi(y, \epsilon_r)}{b_0'(y_c)} \int_{-\infty}^y \frac{b_0'(y_c) - b_0'(y)}{(b_0(y') - \epsilon_r)^2} dy' - \phi(y, \epsilon_r) \frac{1}{b_0'(y_c)}.
\end{equation}

For the first term on the right hand side of (C.6), as \( b_0''(y_c) = 0 \), we have

\begin{align*}
\frac{\phi(y, \epsilon_r)}{b_0'(y_c)} \int_{-\infty}^y \frac{b_0'(y_c) - b_0'(y)}{(b_0(y') - \epsilon_r)^2} dy' &= \frac{\phi(y, \epsilon_r)}{b_0'(y_c)} \int_{-\infty}^y \frac{(y')^2}{(b_0(y) - \epsilon_r)^2} \left( \frac{b_0'(y) - b_0'(y')}{y' - y} - b_0''(y_c) \right) dy'.
\end{align*}
We can see that the integrand is not singular, so \( I \) is well defined at \( y_c \) and then on \( \mathbb{R} \). Then by using the same argument in Lemma 5.4 one can prove that \( \varphi \in H^1_y \) if and only if 
\[
\mathcal{J}_1(c_r) = 0.
\]
As a conclusion, \( c_r \in \mathbb{R} \) is an embedded eigenvalue of \( \mathcal{R} \) if and only if 
\[
\mathcal{J}_1^2(c_r) + \mathcal{J}_2^2(c_r) = 0.
\]

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References

[1] D. Albritton, R. Beekie, and M. Novack, Enhanced dissipation and Hörmander’s hypoellipticity, arXiv:2105.12308, (2021).
[2] S. Alinhac, The null condition for quasilinear wave equations in two space dimensions I, Invent. Math., 145 (2001), 597–618.
[3] B. J. Bayly, S. A. Orszag, and T. Herbert, Instability mechanisms in shear-flow transition, Annual review of fluid mechanics, 20 (1988), 359–391.
[4] J. Bedrossian and M. Coti Zelati, Enhanced dissipation, hypoellipticity, and anomalous small noise inviscid limits in shear flows, Arch. Ration. Mech. Anal., 224 (2017), 1161–1204.
[5] J. Bedrossian, M. Coti Zelati, and V. Vicol, Vortex axisymmetrization, inviscid damping, and vorticity depletion in the linearized 2D Euler equations, Ann. PDE, 5 (2019), Paper No. 4, 192.
[6] J. Bedrossian, P. Germain, and N. Masmoudi, Dynamics near the subcritical transition of the 3d Couette flow ii: Above threshold case, arXiv preprint arXiv:1506.03721, (2015).
[7] J. Bedrossian, P. Germain, and N. Masmoudi, On the stability threshold for the 3D Couette flow in Sobolev regularity, Ann. of Math. (2), 185 (2017), 541–608.
[8] J. Bedrossian, P. Germain, and N. Masmoudi, Dynamics near the subcritical transition of the 3D Couette flow I: Below threshold case, Mem. Amer. Math. Soc., 266 (2020), v+158.
[9] J. Bedrossian and N. Masmoudi, Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations, Publ. Math. Inst. Hautes Études Sci., 122 (2015), 195–300.
[10] J. Bedrossian, N. Masmoudi, and C. Mouhot, Landau damping: paraproducts and Gevrey regularity, Ann. PDE, 2 (2016), Art. 4, 71.
[11] J. Bedrossian, N. Masmoudi, and V. Vicol, Enhanced dissipation and inviscid damping in the inviscid limit of the Navier-Stokes equations near the two dimensional Couette flow, Arch. Ration. Mech. Anal., 219 (2016), 1087–1159.
[12] J. Bedrossian, V. Vicol, and F. Wang, The Sobolev stability threshold for 2D shear flows near Couette, J. Nonlinear Sci., 28 (2018), 2051–2075.
[13] L. Belyakov, S. Friedlander, and V. Yudovich, The unstable spectrum of oscillating shear flows, SIAM Journal on Applied Mathematics, 59 (1999), 1701–1715.
[14] F. Bouchet and H. Morita, Large time behavior and asymptotic stability of the 2D Euler and linearized Euler equations, Phys. D, 239 (2010), 948–966.
[15] K. M. Case, Stability of inviscid plane Couette flow, Phys. Fluids, 3 (1960), 143–148.
[16] J. Chemin and N. Lerner, Flow of non-Lipschitz vector-fields and Navier-Stokes equations, Journal of Differential Equations, 121 (1995), 314–328.
[17] Q. Chen, T. Li, D. Wei, and Z. Zhang, Transition threshold for the 2-D Couette flow in a finite channel, Arch. Ration. Mech. Anal., 238 (2020), 125–183.
[18] Q. Chen, D. Wei, and Z. Zhang, Linear stability of pipe Poiseuille flow at high Reynolds number regime, arXiv preprint arXiv:1910.14245, (2019).
[19] Q. Chen, D. Wei, and Z. Zhang, Transition threshold for the 2-D Couette flow in a finite channel, Arch. Ration. Mech. Anal., 238 (2020), 125–183.
[20] M. Coti Zelati, Stable mixing estimates in the infinite Péclet number limit, J. Funct. Anal., 279 (2020), 108562, 25.
[21] M. Coti Zelati, T. M. Elgindi, and K. Widmayer, Enhanced dissipation in the Navier-Stokes equations near the Poiseuille flow, Comm. Math. Phys., 378 (2020), 987–1010.
56 HUI LI, NADER MASMOUDI, AND WEIREN ZHAO

[22] A. Del Zotto, Enhanced dissipation and transition threshold for the Poiseuille flow in a periodic strip, arXiv preprint arXiv:2108.11602, (2021).
[23] Y. Deng and N. Masmoudi, Long time instability of the Couette flow in low gevrey spaces, arXiv preprint arXiv:1803.01246, (2018).
[24] Y. Deng and C. Zillinger, On the smallness condition in linear inviscid damping: monotonicity and resonance chains, Nonlinearity, 33 (2020), 6176–6194.
[25] Y. Deng and C. Zillinger, Echo chains as a linear mechanism: norm inflation, modified exponents and asymptotics, Arch. Ration. Mech. Anal., 242 (2021), 643–700.
[26] S. Ding and Z. Lin, Enhanced dissipation and transition threshold for the 2-d plane Poiseuille flow via resolvent estimate, arXiv preprint arXiv:2008.10057, (2020).
[27] S. Friedlander, W. Strauss, and M. Vishik, Nonlinear instability in an ideal fluid, in Annales de l’Institut Henri Poincaré C, Analyse non linéaire, vol. 14, Elsevier, 1997, 187–209.
[28] A. E. Gill, A mechanism for instability of plane Couette flow and of Poiseuille flow in a pipe, Journal of Fluid Mechanics, 21 (1965), 503–511.
[29] E. Grenier, On the nonlinear instability of Euler and Prandtl equations, Comm. Pure Appl. Math., 53 (2000), 1067–1091.
[30] E. Grenier, Y. Guo, and T. T. Nguyen, Spectral instability of general symmetric shear flows in a two-dimensional channel, Adv. Math., 292 (2016), 52–110.
[31] E. Grenier, T. T. Nguyen, F. Rousset, and A. Soffer, Linear inviscid damping and enhanced viscous dissipation of shear flows by using the conjugate operator method, J. Funct. Anal., 278 (2020), 108339, 27.
[32] S. He, Enhanced dissipation, hypoellipticity for passive scalar equations with fractional dissipation, J. Funct. Anal., 282 (2022), Paper No. 109319, 28.
[33] L. N. Howard, Note on a paper of John W. Miles, J. Fluid Mech., 10 (1961), 509–512.
[34] S. Ibrahim, Y. Maekawa, and N. Masmoudi, On pseudospectral bound for non-selfadjoint operators and its application to stability of Kolmogorov flows, Ann. PDE, 5 (2019), Paper No. 14, 84.
[35] A. Ionescu and H. Jia, Nonlinear inviscid damping near monotonic shear flows, arXiv preprint arXiv:2001.03087, (2020).
[36] A. Ionescu and H. Jia, Axi-symmetrization near point vortex solutions for the 2d euler equation, Communications on Pure and Applied Mathematics, (2021).
[37] A. Ionescu and H. Jia, Linear vortex symmetrization: the spectral density function, arXiv preprint arXiv:2109.12815, (2021).
[38] A. D. Ionescu and H. Jia, Inviscid damping near the Couette flow in a channel, Comm. Math. Phys., 374 (2020), 2015–2096.
[39] H. Jia, Linear inviscid damping in Gevrey spaces, Arch. Ration. Mech. Anal., 235 (2020), 1327–1355.
[40] H. Jia, Linear inviscid damping near monotone shear flows, SIAM J. Math. Anal., 52 (2020), 623–652.
[41] M. Karp and J. Cohen, On the secondary instabilities of transient growth in Couette flow, Journal of Fluid Mechanics, 813 (2017), 528–557.
[42] L. Kelvin, Stability of fluid motion: rectilinear motion of viscous fluid between two parallel plates, Phil. Mag, 24 (1887), 188–196.
[43] L. Landau, On the vibrations of the electronic plasma, Acad. Sci. USSR. J. Phys., 10 (1946), 25–34.
[44] H. Li and W. Zhao, Metastability for the dissipative quasi-geostrophic equation and the non-local enhancement, arXiv preprint arXiv:2107.10594, (2021).
[45] T. Li, D. Wei, and Z. Zhang, Pseudospectral bound and transition threshold for the 3D Kolmogorov flow, Comm. Pure Appl. Math., 73 (2020), 465–557.
[46] Y. C. Li and Z. Lin, A resolution of the Sommerfeld paradox, SIAM J. Math. Anal., 43 (2011), 1923–1954.
[47] Z. Lin, Instability of some ideal plane flows, SIAM J. Math. Anal., 35 (2003), 318–356.
[48] Z. Lin and M. Xu, Metastability of Kolmogorov flows and inviscid damping of shear flows, Arch. Ration. Mech. Anal., 231 (2019), 1811–1852.
[49] Z. Lin and C. Zeng, Inviscid dynamical structures near Couette flow, Arch. Ration. Mech. Anal., 200 (2011), 1075–1097.
[50] H. Liu, N. Masmoudi, C. Zhai, and W. Zhao, Linear damping and depletion in flowing plasma with strong sheared magnetic fields, J. Math. Pures Appl. (9), 158 (2022), 1–41.
[51] N. Masmoudi, B. Said-Houari, and W. Zhao, Stability of Couette flow for 2d Boussinesq system without thermal diffusivity, arXiv preprint arXiv:2010.01612, (2020).
[52] N. Masmoudi and W. Zhao, Enhanced dissipation for the 2D Couette flow in critical space, Comm. Partial Differential Equations, 45 (2020), 1682–1701.
[53] N. Masmoudi and W. Zhao, Nonlinear inviscid damping for a class of monotone shear flows in finite channel, arXiv preprint arXiv:2001.08564, (2020).
[54] N. Masmoudi and W. Zhao, Stability threshold of two-dimensional Couette flow in Sobolev spaces, Annales de l’Institut Henri Poincaré C, 39 (2022), 245–325.
[55] C. Mouhot and C. Villani, On Landau damping, Acta Math., 207 (2011), 29–201.
[56] W. M. Orr, The stability or instability of the steady motions of a perfect liquid and of a viscous liquid. part I: A perfect liquid, 27 (1907), 9–68.
[57] L. Rayleigh, On the Stability, or Instability, of certain Fluid Motions, Proc. Lond. Math. Soc., 11 (1879/80), 57–70.
[58] S. Ren and W. Zhao, Linear damping of Alfvén waves by phase mixing, SIAM J. Math. Anal., 49 (2017), 2101–2137.
[59] D. Ryutov, Landau damping: half a century with the great discovery, Plasma physics and controlled fusion, 41 (1999), A1.
[60] A. Sommerfeld, Ein beitrag zur hydrodynamischen erklarung der turbulenten flüssigkeitsbewegung, Atti del IV Congresso internazionale dei matematici, (1908), 116–124.
[61] M. Vishik and S. Friedlander, Nonlinear instability in two dimensional ideal fluids: the case of a dominant eigenvalue, Comm. Math. Phys., 243 (2003), 261–273.
[62] F. Waleffe, Transition in shear flows. Nonlinear normality versus non-normal linearity, Physics of Fluids, 7 (1995), 3060–3066.
[63] D. Wei, Diffusion and mixing in fluid flow via the resolvent estimate, Sci. China Math., 64 (2021), 507–518.
[64] D. Wei and Z. Zhang, Transition threshold for the 3d Couette flow in Sobolev space, Communications on Pure and Applied Mathematics, (2020).
[65] D. Wei, Z. Zhang, and W. Zhao, Linear inviscid damping for a class of monotone shear flow in Sobolev spaces, Comm. Pure Appl. Math., 71 (2018), 617–687.
[66] D. Wei, Z. Zhang, and W. Zhao, Linear inviscid damping and vorticity depletion for shear flows, Ann. PDE, 5 (2019), Paper No. 3, 101.
[67] D. Wei, Z. Zhang, and W. Zhao, Linear inviscid damping and enhanced dissipation for the Kolmogorov flow, Adv. Math., 362 (2020), 106963, 103.
[68] C. Zillinger, Linear inviscid damping for monotone shear flows, Trans. Amer. Math. Soc., 369 (2017), 8799–8855.

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