Physical constraints on Higgs masses and quartic couplings in the two Higgs doublet model

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Abstract. In the framework of the two Higgs doublet model we graphically illustrate the energy dependence of the Higgs masses and the quartic couplings. Our results are based on the theoretical constraints on the quartic Higgs couplings through the stability and triviality principles that are obtained, using the Lagrange multipliers method and the numerical solutions of the Renormalization Group Equations that give the energy dependence of the parameters of the Lagrangian. From these solutions we can also establish the region of validity of the model in terms of the initial conditions of the parameters.

1. Introduction
For over three decade and until now the standard model (SM) of particle physics [1 - 3] has provided us with an excellent theoretical framework consistent with the experiments [4]. However it has some limitations since it does not explain the hierarchy of the masses, the masses of the neutrinos and has 19 free parameters which are adjusted by the experiment and the detection of the Higgs particle [5] is an open problem for the experimental verification of the SM.

Although there are these limitations, the work of theorists has continued to find solution to these problems. A non-super symmetric extension of the SM that arises in a natural way is known in the literature as the two Higgs doublet model (2HDM), which incorporate an extra scalar field doublet.

In this model there are different ways of coupling the scalar field with the up-type quarks and down-type quarks [6 - 9], depending on the type of symmetry that is imposed on the scalar field. Here we consider the 2HDM type-II model, where the first scalar field is coupled only to the down-type quarks and second field is coupled to the up-type quarks.

2. The Extension of the Standard Model
The Higgs sector of the 2HDM consists of two scalar doublets $\Phi_1$ y $\Phi_2$, both having hypercharge $Y = 1$, which written in terms eight real scalars become:

$$\Phi_1 = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \phi_5 + i\phi_6 \\ \phi_7 + i\phi_8 \end{pmatrix}. \quad (1)$$
The most general Higgs potential which can be constructed with these two scalar doublets, that is renormalizable and compatible with the SM gauge symmetry contains 14 parameters. This number of parameters is reduced by imposing the $Z_2$ discrete symmetry \([10 - 12]\):

$$
\Phi_1 \rightarrow \Phi_1, \Phi_2 \rightarrow -\Phi_2 \quad \text{or} \quad \Phi_1 \rightarrow -\Phi_1, \Phi_2 \rightarrow \Phi_2 .
$$

(2)

This $Z_2$ symmetry choice automatically excludes the violation of CP symmetry, which simplifies the analysis.

The gauge invariant and renormalizable Higgs potential can be written as:

$$
V(\Phi_1, \Phi_2) = \mu_1^2 \Phi_1^* \Phi_1 + \mu_2^2 \Phi_2^* \Phi_2 + \lambda_1 (\Phi_1^* \Phi_1)^2 + \lambda_2 (\Phi_2^* \Phi_2)^2 + \lambda_3 (\Phi_1^* \Phi_2)^2 + \lambda_4 (\Phi_2^* \Phi_1)^2 + \frac{1}{2} \lambda_5 [(\Phi_1^* \Phi_2)^2 + (\Phi_2^* \Phi_1)^2].
$$

(3)

The neutral scalar fields of both doublets acquire non-zero vacuum expectation values (VEV) $v_i$, just as in the standard model, and $v_1^2 + v_2^2 = v^2 = (246 \text{ GeV})^2$. After the electroweak symmetry breaking, in the potential, the resulting physical particles in the Higgs sector are: two CP-even neutral Higgs scalars ($H^0, h^0$), one CP-odd neutral Higgs scalar ($A^0$), two charged Higgs bosons ($H^\pm, H^0$), and three Goldstone bosons ($G^+, G^0$) that contribute to the mass generation of the gauge vector bosons $W^\pm$ and $Z^0$ [13].

2.1. Scalars masses

The masses of the massive Higgs scalars, in terms of the quartic couplings and VEV of the potential, satisfy the following relations:

$$
M_{H^0, h^0}^2 = \lambda_1 v_1^2 + \lambda_2 v_2^2 \mp [\lambda_3 v_1^2 - \lambda_4 v_2^2]^2 + (2 \lambda v_1 v_2)^2, \quad (4)
$$

$$
M_{A^0}^2 = - \frac{4M_W^2}{g^2} \lambda_5, \quad (5)
$$

$$
M_{H^\pm}^2 = - \frac{2M_W^2}{g^2} (\lambda_4 + \lambda_5), \quad (6)
$$

where $\Lambda = \frac{1}{2}(\lambda_3 + \lambda_4 + \lambda_5)$.

2.1.1. Bounds for the $\lambda$ parameters. The minimum of the potential must be stable, and it is required to have positive masses and furthermore the potential must be bounded from below. We obtain the following boundary values for the quartic couplings from the positive values of the above expressions for the masses:

$$
\lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_4 + \lambda_5 < 0, \quad \lambda_5 < 0, \quad \lambda_4 < |\lambda_5|, \quad \lambda_4 \lambda_5 > \frac{1}{4}(\lambda_3 + \lambda_4 + \lambda_5)^2. \quad (7)
$$

3. Lagrange Multipliers Method

A rigorous way to obtain the quartic couplings restrictions is through the use of the Lagrange multipliers and the implementation of the vacuum stability conditions. The problem we consider involves constraints with equality and inequality relations, to be solved through the Karush-Kuhn-Tucker method.

Equation (3) can be written in terms of four gauge invariant combinations $x_i$ of the scalar doublets and the parameters $b_i$, defined as:

$$
x_1 = \Phi_1^* \Phi_1, \quad x_2 = \Phi_2^* \Phi_2, \quad x_3 = \text{Re}(\Phi_1^* \Phi_2), \quad x_4 = \text{Im}(\Phi_1^* \Phi_2). \quad (8)
$$
$$b_{11} = \lambda_1, \quad b_{22} = \lambda_2, \quad b_{33} = \lambda_3, \quad b_{44} = (\lambda_4 - \lambda_3).$$

(9)

Then the Higgs potential becomes:

$$V = \mu_1^2 x_1^2 + \mu_2^2 x_2^2 + b_{11} x_1^4 + b_{22} x_2^4 + b_{33} x_3^4 + b_{44} x_4^2.$$  

(10)

As one can see the Higgs potential consists of a quadratic part and a quartic part. We will focus our attention on the quartic part of the potential and construct a Lagrangian function to be analyzed.

3.1. Lagrange function

The Lagrange function of our problem has the form:

$$L(x) = F(x) + \sum_{i=1}^{N} \Lambda_i g_i(x) + \sum_{j=1}^{M} \Gamma_j h_j(x).$$  

(11)

Where $F(x)$ is the function to be minimized (quartic potential), $\Lambda_i$ and $\Gamma_j$ are the Lagrange multipliers, $g_i(x)$ are equality constraints and $h_j$ are inequality constraints. We build the constraints by using the Schwarz inequality and the vacuum expectation values restrictions, as follows:

$$h(x) = x_i x_j - x_i^2 - x_j^2 \geq 0, \quad g(x) = x_i + x_j - v^2 = 0.$$  

(12)

Then the Lagrange function can be written as:

$$L(x) = b_{11} x_1^4 + b_{22} x_2^4 + b_{33} x_3^4 + b_{44} x_4^2 + \Lambda_1 (x_1 + x_2 - v^2) + \Lambda_2 (x_1 x_2 - x_3^2 - x_4^2).$$  

(13)

The vacuum stability condition implies that:

$$L(x_{\text{min}}) > 0.$$  

(14)

3.1.1. Conditions of Karush-Kuhn-Tucker. The Karush-Kuhn-Tucker conditions arise by imposing the following conditions:

$$\frac{\partial L}{\partial x_i} = 2b_{11} x_i + b_{12} x_2 + \Lambda_1 x_2 + \Lambda_2 = 0,$$  

(15)

$$\frac{\partial L}{\partial x_2} = 2b_{22} x_2 + b_{12} x_1 + \Lambda_2 x_1 + \Lambda_1 = 0,$$  

(16)

$$\frac{\partial L}{\partial x_3} = 2b_{33} x_3 - 2\Lambda_2 x_2 = 0,$$  

(17)

$$\frac{\partial L}{\partial x_4} = 2b_{44} x_4 - 2\Lambda_2 x_2 = 0,$$  

(18)

$$\frac{\partial L}{\partial \Lambda_1} = x_i + x_j - v^2 = 0,$$  

(19)

$$\frac{\partial L}{\partial \Lambda_2} = x_i x_j - x_i^2 - x_j^2 = 0.$$  

(20)

Now we explore the various solutions. There are four possible cases:

$$x_3 \neq 0, x_4 = 0 \Rightarrow \Lambda_2 = b_{33},$$  

(21)

$$x_4 \neq 0, x_3 = 0 \Rightarrow \Lambda_2 = b_{44},$$  

(22)

$$x_i \neq 0, x_4 = 0 \Rightarrow \Lambda_2 = 0, \quad x_i = 0, x_4 \neq 0 \Rightarrow \Lambda_2 = 0,$$  

(23)

$$x_3 \neq 0, x_4 \neq 0 \Rightarrow \Lambda_2 = b_{33} = b_{44}.$$  

(24)
Solving the system of equations (15) – (20) for the variables $x_1$, $x_2$, and $\Lambda_1$, we obtain:

$$x_1 = \frac{(b_{12} + b_{33}) - 2b_{22}}{4b_1b_{22} - (b_{12} + b_{33})^2} \Lambda_1 > 0$$

(25)

$$x_2 = \frac{(b_{12} + b_{33}) - 2b_{11}}{4b_1b_{22} - (b_{12} + b_{33})^2} \Lambda_1 > 0$$

(26)

$$\Lambda_1 = \frac{4b_1b_{22} - (b_{12} + b_{33})^2}{2(b_{12} + b_{33} - b_{11} - b_{22})} v^2$$

(27)

Equations (25) – (27) imply that:

$$b_{11} + b_{22} - b_{33} - b_{12} > 0, \quad 4b_1b_{22} - (b_{12} + b_{33})^2 > 0$$

(28)

$$b_{12} + b_{33} - 2b_{11} < 0, \quad b_{12} + b_{33} - 2b_{22} < 0$$

(29)

Then the constraints that satisfy the conditions of positivity of the masses and stability for quartic couplings are:

$$\lambda_1 > 0, \quad \lambda_2 > 0$$

(30)

$$\lambda_3 < 0, \quad \lambda_4 + \lambda_5 < 0$$

(31)

$$\lambda_3 + \lambda_4 + |\lambda_5| < \lambda_1 + \lambda_2$$

(32)

$$-2\sqrt{\lambda_3\lambda_5} < \lambda_3 + \lambda_4 + |\lambda_5|$$

(33)

3.1.2. Extreme case. In this case the Higgs quartic potential has its lowest possible value, i.e., $V = 0$, then:

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + (\lambda_1 + \lambda_2 + \lambda_3) x_1 x_2 = (\sqrt{\lambda_4} x_1 - \sqrt{\lambda_5} x_2)^2 + (\lambda_1 + \lambda_2 + \lambda_3 + 2\sqrt{\lambda_4\lambda_5}) x_1 x_2 \geq 0$$

(34)

And we get:

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + (\lambda_1 + \lambda_2 + \lambda_3) x_1 x_2 = \left(\frac{x_1}{\sqrt{\lambda_1}}\right)^2 + \left(\frac{x_2}{\sqrt{\lambda_2}}\right)^2 + \left((\lambda_1 + \lambda_2 + \lambda_3) + 2\sqrt{\lambda_4\lambda_5}\right) x_1 x_2 \geq 0$$

(35)

The Higgs masses become:

$$M_{h^0} = \begin{cases} 4\lambda_1\lambda_2 v^2, & \lambda_1 \neq \lambda_2, \quad v_1 \neq v_2, \\ 2\lambda_1 v^2, & \lambda_1 = \lambda_2 = \lambda, \quad v_1 = v_2 = \frac{v}{\sqrt{2}}. \end{cases}$$

(36)

$$M_{h^+} = 0, \quad M_{h^-} = \left(\frac{1}{2}|\lambda_4 + \lambda_5|\right)^2 v^2, \quad M_{h^0} = \left(|\lambda_5|\right)^2 v^2.$$  

(37)

4. Triviality constraints

We explore the asymptotic behavior of the parameters in the model, and their relations, through the Renormalization Group Equations (RGE) [8]. The RGE are important in studying the physics of the standard model and its extensions. The RGE determine the dependence of the coupling constants and other parameters of the Lagrangian on $t$, defined as $t = \ln (E/m_b)$, where $E$ is the renormalization point.
energy. To numerically evaluate the energy dependence of the \( \lambda_i \) quartic couplings, we consider the RGE of the parameters, i.e., the gauge couplings, and the Yukawa couplings of the top and the down quark sectors.

\[
\frac{dg_k}{dt} = \frac{1}{(4\pi)^2} b_k g_k^3, (k=1,2,3),
\]

\[
\frac{dg_i}{dt} = \frac{1}{(4\pi)^2} \left[ \frac{9}{2} g_i^2 + \frac{1}{2} g_b^2 - \frac{17}{20} g_i^4 + \frac{9}{4} g_i^2 g_b^2 + 8 g_i^2 \right] g_i,
\]

\[
\frac{dg_b}{dt} = \frac{1}{(4\pi)^2} \left[ \frac{9}{2} g_b^2 + \frac{1}{2} g_i^2 - \frac{1}{4} g_i^4 + \frac{9}{4} g_i^2 g_b^2 + 8 g_i^2 \right] g_b,
\]

\[
\frac{d\lambda_1}{dt} = \frac{1}{(4\pi)^2} \left[ 24 \lambda_1^2 - 3 \lambda_1 (3 g_i^2 + g_b^2 - 4 g_i^2) + 2 \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2 \lambda_1 \lambda_2 + \frac{3}{8} (g_i^2 + g_b^2)^2 + \frac{3}{4} g_i^4 - 6 g_i^4 \right],
\]

\[
\frac{d\lambda_2}{dt} = \frac{1}{(4\pi)^2} \left[ 24 \lambda_2^2 - 3 \lambda_2 (3 g_i^2 + g_b^2 - 4 g_i^2) + 2 \lambda_2^2 + \lambda_1^2 + \lambda_3^2 - 2 \lambda_1 \lambda_2 + \frac{3}{8} (g_i^2 + g_b^2)^2 + \frac{3}{4} g_i^4 - 6 g_i^4 \right],
\]

\[
\frac{d\lambda_3}{dt} = \frac{1}{(4\pi)^2} \left[ 4 \lambda_3^2 + 4 (3 \lambda_1 + \lambda_2) (\lambda_1 - \lambda_2) - 3 \lambda_1 (3 g_i^2 + g_b^2 - 2 (g_i^2 + g_b^2)) + 2 \lambda_1^2 + 2 \lambda_2^2
\]

\[+ \frac{3}{4} (g_i^2 + g_b^2)^2 + \frac{3}{2} g_i^4 - 12 g_i^2 g_b^2 \right],
\]

\[
\frac{d\lambda_4}{dt} = \frac{1}{(4\pi)^2} \left[ 4 \lambda_4^2 + 4 \lambda_1 (\lambda_1 + \lambda_2 + 3 \lambda_3) - 3 \lambda_4 (3 g_i^2 + g_b^2 - 2 (g_i^2 + g_b^2)) + 8 \lambda_2^2 + 3 g_i^2 g_b^2 - 12 g_i^2 g_b^2 \right],
\]

\[
\frac{d\lambda_5}{dt} = \frac{1}{(4\pi)^2} \left[ 4(\lambda_1 + \lambda_2 + 2 \lambda_3 + 3 \lambda_4) - 3 (3 g_i^2 + g_b^2 - 2 (g_i^2 + g_b^2)) \right].
\]

The range of values, for the energy in GeV and the variable \( t \) are \( \left( E_0 = M_0, E_w = 1.234 \times 10^{13} \right) \), and \( (t_0 = 0, t_w = 25) \), where \( M_0 \) is the mass of the quark top and \( E_w \) corresponds to the electroweak unification energy, where \( g_t \left( E_0 \right) = g_t \left( E_w \right) \). Now we explore the energy bounds for the 2HDM, through the running of the quartic couplings which determine the mass values of the Higgs. For our analysis, we set the masses \( M_{H^+} = 609 \text{ GeV} \) and \( M_t = 621.9 \text{ GeV} \) given in [14].

The figure 1 and figure 2 show that the range of validity of the 2HDM is short, here \( M_t < E < 292 \text{ GeV} \), i.e., \( 0 < t < 0.52 \).

**Figure 1.** The energy dependence of the quartic couplings.

**Figure 2.** The energy dependence of Higgs masses.
In the case of figure 3 and figure 4 we see that the 2HDM is valid in the whole range of energies then $M_t < E < E_{\mu}$.

Figure 3. The energy dependence of the quartic couplings.

Figure 4. The energy dependence of Higgs masses.

5. Conclusions
We examined the mass content of the 2HDM extension of the SM, considering real vacuum expectation values of both Higgs fields in the case with CP symmetry. We have obtained through the mass formulas and through the Lagrange multipliers method, a set of constraints to be satisfied by the quartic couplings in the Higgs sector of the model. With the numerical solutions to the RGE, as shown in the figures, we conclude that the range of validity of the 2HDM depends on the initial values of the quartic parameters.

For small Higgs masses and small quartic couplings, the range of validity of the model goes up to the electro-weak unification scale and further.

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