CONVERGENCE RATES FOR LOOP-ERASED RANDOM WALK AND OTHER LOEWNER CURVES

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Abstract. We estimate convergence rates for curves generated by the Loewner equation under the basic assumption that a convergence rate for the driving terms is known. An important tool is the “tip structure modulus”, a geometric measure of regularity for Loewner curves in the capacity parameterization which is analogous to Warschawski’s structure modulus, and is closely related to annuli crossings. The main application we have in mind is that of a random discrete-model curve approaching a Schramm-Loewner evolution (SLE) curve in the lattice size scaling limit. We carry out the approach in the case of loop-erased random walk in a simply connected domain. Under some mild assumptions of boundary regularity we obtain an explicit power-law rate for the convergence of the loop-erased random walk path towards the radial SLE$_2$ path in the supremum norm, the curves being parameterized by capacity. On the deterministic side we show that the tip structure modulus gives a sufficient geometric condition for a Loewner curve to be Hölder continuous in the capacity parameterization, assuming its driving term is Hölder continuous. We also briefly discuss the case when the curves are a-priori known to be Hölder continuous in the capacity parameterization and we obtain a power-law rate depending only on the regularity of the curves.

1. Introduction, Motivation, and Results

1.1. Introduction. The Loewner equation is a partial differential equation that produces a Loewner chain, a family of conformal mappings from a reference domain onto a continuously decreasing sequence of simply connected domains. The evolution is controlled by a real valued function called driving term which acts as a parameter. Under smoothness assumptions on the driving term the Loewner equation can be used to generate a growing continuous curve, by which we mean a continuous function from some interval into the reference domain. Conversely, starting from a suitable curve one can reverse the procedure to recover the driving term and so there is a correspondence between Loewner curves and their driving terms. In recent years Loewner’s equation has been successfully applied to study conformally invariant lattice-size scaling limits of certain discrete models from statistical physics. By taking a scaled Brownian motion as the driving term one
obtains the one-parameter family of random fractal Schramm-Loewner evolution (SLE) curves which are, essentially, the only possible conformally invariant scaling limits of cluster interfaces with a certain Markovian property; see [22]. Convergence to SLE has been proved in several cases; see, e.g., [23] and the references therein. The use of the Loewner equation and SLE techniques in this context has made it possible to give precise meaning to the (passage to the) scaling limit itself, but also to prove conformal invariance, and to give rigorous proofs of various predictions made by physicists. The latter is to large extent due to the fact that the SLE processes are amenable to computation via stochastic calculus.

In this paper we will be interested in quantifying the relationship between (random) rough Loewner curves with driving terms that are close in the supremum norm. To explain our interest, let us first consider a non-random setting. One can view the Loewner equation as a highly non-linear function from a space of driving terms to a suitable metric space of (parameterized) curves and it is natural to ask about its continuity properties, if any. This point of view is closely related to work by Lind, Marshall, and Rohde; see [16] and [11]. For example, Theorem 4.1 of [11] proves that curves driven by Hölder-1/2 driving terms with small semi-norm converge as curves if the driving terms converge. So the “Loewner function” is continuous when restricted to this collection of driving terms and our results can be used to show that it is Hölder continuous with an explicit exponent depending only on the semi-norm when sufficiently small. One can also ask similar questions, restricting attention to driving terms generating curves with some given regularity.

Our principal motivation, however, comes from the observation that although several discrete-model curves are known to converge (as curves up to reparameterization) to SLE curves, next to nothing appears to be known about the speeds of their convergence. (See the paper [4] by Beneš, Kozdron, and the author for a quantitative result of convergence of loop-erased random walk with respect to Hausdorff distance when the curves are viewed as compact sets.)

Good control over convergence rates would allow SLE techniques to be used on mesoscopic scales, that is, scales of order $\varepsilon^p$ with $p \in (0,1)$ where $\varepsilon$ is the lattice spacing. This is likely to be helpful for obtaining fine properties of corresponding discrete models; this question was raised by Schramm in connection with sharp estimation of critical exponents [23]. We may compare with a related model. So-called strong approximation results such as the KMT approximation or the Skorokhod embedding [13] allow for the coupling of simple random walk and Brownian motion in such a way that the paths are close with high probability, with error terms expressed explicitly in terms of the lattice spacing. This gives a natural way to use techniques for Brownian motion to deduce fine properties of simple random walk which can depend on behavior on mesoscopic scales. This approach has been used
by, e.g., Lawler, Lawler and Puckette, and Beneš; see [14] and [3] and the references therein. It thus seems reasonable that approximation results with explicit error terms for discrete models converging to SLE would be quite useful. Presently, all known proofs of convergence to SLE goes via convergence of the driving terms in one way or another, so it seems natural to take a convergence rate for the driving terms as a starting point. In particular since the work in [4] essentially reduces finding a convergence rate for the driving terms to finding a convergence rate for the so-called martingale observable in rough domains. We will show that a power-law convergence rate to an SLE curve can be derived from a power-law convergence rate for the driving terms provided some additional quantitative geometric information, related to crossing events, is available for the discrete curves, along with an estimate on the growth of the derivative of the SLE map. The approach is quite general and we believe it can be applied to several models as soon as the aforementioned information is available, though we carry out the specific probabilistic estimates only in the case of loop-erased random walk.

1.2. Overview, Results, and Related Work. Let us briefly sketch the setup and main ideas. We will do it in the half plane setting, but we will later work mostly in a radial setup. See Section 2 for precise definitions. Let \( W, W_n : [0, T] \to \mathbb{R} \) be continuous functions with
\[
\sup_{t \in [0, T]} |W(t) - W_n(t)| \leq \varepsilon,
\]
where \( \varepsilon > 0 \) is to be thought of as small but fixed. Let \( f(t, z) : \mathbb{H} \to H(t) \) and \( f_n(t, z) : \mathbb{H} \to H_n(t) \) be the solutions to the chordal Loewner equations driven by \( W \) and \( W_n \), respectively, and assume that the Loewner chains are generated by the curves \( \gamma \) and \( \gamma_n \) parameterized by capacity so that for each \( t \), \( H(t) \) and \( H_n(t) \) are the unbounded components of \( \mathbb{H} \setminus \gamma[0, t] \) and \( \mathbb{H} \setminus \gamma_n[0, t] \), respectively. (We can think of \( \gamma_n \) as the conformal image of a discrete-model curve on a lattice approximation of a smooth domain \( D \), where the mesh of the lattice is \( n^{-1} \), and the driving term of \( \gamma_n \) is coupled with a scaled Brownian motion \( W \) driving the chordal SLE curve \( \gamma \) so that the driving terms are at distance at most \( \varepsilon = n^{-q} \) for some \( q < 1 \).) Let \( y > 0 \); we will later choose \( y = y(\varepsilon) \). Let \( t \in [0, T] \). We can write
\[
|\gamma(t) - \gamma_n(t)| \leq |\gamma(t) - f(t, W(t) + iy)| + |f(t, W(t) + iy) - f(t, W_n(t) + iy)| + |f(t, W_n(t) + iy) - f_n(t, W_n(t) + iy)| + |f_n(t, W_n(t) + iy) - \gamma_n(t)| =: A_1 + A_2 + A_3 + A_4.
\]
We wish to estimate the \( A_j \) in terms of \( \varepsilon \). Suppose that there are \( \beta < 1 \) and \( c < \infty \) such that
\[
d|f'(t, W(t) + id)| \leq cd^{1-\beta} \quad \text{for all} \quad d \leq y.
\quad (1.1)
If this estimate holds, by integrating, \( A_1 \leq cy^{1-\beta} \). (Constants may change from line to line, and are assumed to depend only on the parameters and not on \( \varepsilon, y, \) etc.) By the distortion theorem the same bound holds for \( A_2 \) if \( y \geq \varepsilon \). The third term, \( A_3 \), represents the distance between two solutions to the Loewner equation, with driving terms at supremum distance at most \( \varepsilon \), evaluated at the same point. In Section 2.3 we will use the reverse-time Loewner flow to estimate quantities like this. In particular, we will see that if \( \text{Im} \ z = y \), then

\[
|f(t, z) - f_n(t, z)| \leq c \varepsilon y^{-1},
\]

with \( c \) depending only on \( T \). Hence \( A_3 \leq c \varepsilon y^{-1} \) and Cauchy’s integral formula implies that

\[
|y|f'(t, z)| - y|f'_n(t, z)|| = c \varepsilon y^{-1}.
\]

From this it follows, using Koebe’s estimate and (1.1), that if

\[
\Delta_n(t, y) := \text{dist} [f_n(t, W_n(t) + iy), \partial H_n(t)],
\]

then

\[
\Delta_n(t, y) \leq cy|f'_n(t, W_n(t) + iy)| \leq cy^{1-\beta} + c \varepsilon y^{-1};
\]

(1.2) see Proposition 2.4. (Note that we have made no explicit assumption on the behavior of \( |f'_n| \).) Now choose \( y(\varepsilon) = \varepsilon^p \), for some \( p \in (0, 1) \). Then,

\[
A_1 + A_2 + A_3 \leq c \varepsilon^p (1-\beta) + c \varepsilon^{1-p}
\]

and it remains to bound \( A_4 \). Clearly, \( A_4 \geq \Delta_n(t, \varepsilon^p) \) but we would like an upper bound in terms of \( \Delta_n(t, \varepsilon^p) \). To proceed, some additional information about the boundary behavior of \( f_n \) is necessary.

For this, we will use the so-called “tip structure modulus”, a geometric gauge of the regularity of a Loewner curve in the capacity parameterization that is, for our problem, the analog of Warschawski’s [26] measure with a similar name. Let \( \delta > 0 \) and consider \( S_{t, \delta} \), the set of all crosscuts of \( H_n(t) \) of diameter at most \( \delta \) that separate the tip, \( \gamma_n(t) \), from \( \infty \) in \( H_n(t) \). Each crosscut \( C \in S_{t, \delta} \) separates from \( \infty \) in \( H_n(t) \) a “maximal” piece \( \gamma_C \) of \( \gamma_n(t) \) containing \( \gamma_n(t) \) as a prime-end. We then define the tip structure modulus of \( \gamma_n(t), t \in [0, T] \), by

\[
\eta_{\text{tip}}(\delta) = \sup_{t \in [0, T]} \sup_{C \in S_{t, \delta}} \text{diam} \ \gamma_C.
\]

(See Section 3 for a precise definition.) Roughly speaking, \( \eta_{\text{tip}}(\delta) \) is the maximal distance the curve travels into a “bottle” with “bottleneck” opening smaller than \( \delta \) viewed from the point towards which the curve is growing. (Similar conditions have been used before; see below.) In Proposition 3.2 we show that

\[
|f_n(t, W_n(t) + iy) - \gamma_n(t)| \leq \eta_{\text{tip}}(c \Delta_n(t, y)),
\]

(1.3)
where $\eta_{\text{tip}}$ is the tip structure modulus for $\gamma_n$. Consequently, if we have a power-law bound on the tip structure modulus evaluated at $c \Delta_n(t, \varepsilon^p)$, that is, if

$$\eta_{\text{tip}}(c \Delta_n(t, \varepsilon^p)) \leq c' \left( \Delta_n(t, \varepsilon^p) \right)^r,$$

for some $r > 0$, then by (1.2)

$$A_4 \leq c \varepsilon^{p(1-\beta)r} + c \varepsilon^{(1-p)r}.$$

We stress that the estimate on $\eta_{\text{tip}}$ is only required to hold on the scale of $\Delta_n(t, \varepsilon^p)$ and note that the failure of the existence of a given bound on $\eta_{\text{tip}}$ implies certain crossing events for the curve. If the estimates hold uniformly in $t \in [0, T]$, then we have obtained a power-law bound in terms of $\varepsilon$ on $\sup_{t \in [0, T]} |\gamma(t) - \gamma_n(t)|$ and we can finally optimize over exponents.

To implement these ideas in a particular setting we need to show that the assumptions we used are satisfied uniformly in $t \in [0, T]$, with high probability in terms of $\varepsilon$. If a convergence rate for the driving terms (or martingale observable in rough domains) is known, then we believe it is possible to derive the remaining required information from existing results in the literature on discrete models without too much effort, and we derive the needed SLE derivative estimates, from estimates in [6], in this paper. Indeed, as already mentioned, the event that the geometric condition fails implies annuli crossing events that are reasonably well-understood for the models known to converge to SLE.

The organization of the paper is as follows. In Section 2.3 we discuss some preliminaries and prove the quantitative comparison estimates for solutions to the Loewner equation. These estimates might be of some independent interest; see the forthcoming [8]. We also consider a case when the curves are \textit{à-priori} known to be Hölder continuous in the capacity parameterization; we then get a power-law convergence rate depending only on the regularity of the curves. See Corollaries 2.6 and 2.7.

In Section 3 we define the tip structure modulus and prove the estimates implying (1.3). Then in Theorem 3.5 we show that if a Loewner curve $\gamma$ has the property that there is $M < \infty$ such that $\eta_{\text{tip}}(\delta) \leq M \delta$, $\delta < \delta_0$, and the driving term is Hölder continuous, then $\gamma$ is also Hölder continuous in the capacity parameterization with exponent depending only on $M$ and the exponent for the driving term. This is an analog of the fact that John domains are Hölder domains [20] for Loewner curves parameterized by capacity.

In Section 4 we apply the above ideas to obtain a power-law estimate on the convergence rate to radial SLE$_2$ for the loop-erased random walk (LERW) path. Let us informally state one version of the result; see Theorem 4.3 for a precise statement. Let $D_n$ be an $n^{-1}Z^2$ grid domain approximation of a fixed simply connected Jordan domain $D \ni 0$ with $C^{1+\alpha}$ boundary and inner radius from 0 equal to 1. (The proof works for the larger
class of quasidisks \[20\], but we then get a slower convergence rate which de-
pends on the constant in the Ahlfors three-point condition for \(D\). Let \(\gamma_n\) be the time-reversal of LERW on \(D_n\) from 0 to \(\partial D_n\) and let \(\tilde{\gamma}_n\) be its image in \(\mathbb{D}\) under the conformal map \(\psi_n : D_n \to \mathbb{D}\) with the usual normalization. Let \(\tilde{\gamma}\) be the radial SLE\(_2\) path in \(\mathbb{D}\) started uniformly on \(\partial \mathbb{D}\).

**Theorem.** For each \(n\) sufficiently large, there is a coupling of \(\tilde{\gamma}_n\) with \(\tilde{\gamma}\) such that
\[
P\left\{ \sup_{t \in [0,\sigma]} |\tilde{\gamma}_n(t) - \tilde{\gamma}(t)| > \varepsilon_n^{1/41} \right\} < \varepsilon_n^{1/41},
\]
where both curves are parameterized by capacity, \(\varepsilon_n = n^{-1/24}\) is the conver-
geance rate of the driving terms from \[4\], and \(\sigma\) is a stopping time. The same
estimate holds for the pre-images of the curves in \(D_n\).

(The stopping time \(\sigma = \sigma(\varepsilon, T)\) can be taken as the minimum of some
fixed \(T < \infty\) and the first time such that the forward SLE\(_2\) flow of \(\tilde{\gamma}(0)\)
is smaller than some given \(\varepsilon > 0\). Note that in this case \(\lim_{\varepsilon \to 0} \sigma(\varepsilon, T) = T\)
almost surely, see Appendix A.) This quantifies the convergence result \[15,\]
Theorem 3.9] of Lawler, Schramm, and Werner. As indicated, the proof
considers the couplings of \[4\] in which if \(s < 1/24\), then with probability
at least \(1 - n^{-s}\) the estimate \(\sup_{t \in [0,T]} |W_n(t) - W(t)| < n^{-s}\) holds for the
driving term \(W_n\) of the LERW on \(D_n\) and \(W\), a Brownian motion with
speed 2 on \(\partial \mathbb{D}\). Using the Brownian motion as driving term in the Loewner
equation we have a coupling of the LERW image and SLE\(_2\) for each \(n\),
with their driving terms close. We then show that the above reasoning can
be carried out on an event with large probability in terms of \(n\) to prove
Theorem 4.3. Some work is required to establish the needed geometric
condition for the LERW path; see Proposition 4.5.

In Appendix A we derive an estimate on the probability (in terms of \(y\))
that a bound of the type \[1.1\] holds for radial SLE from a corresponding
estimate for chordal SLE from \[6\]. This is where the stopping time \(\sigma = \sigma(\varepsilon, T)\)
is needed; it is related to the “disconnection time” when the radial
and chordal SLE\(_\kappa\) processes become singular with respect to each other.

Finally, in Appendix B we discuss a convergence rate result for a sequence
of grid-domain approximations of a quasidisk which allows us to directly
“transfer” the geometric condition to \(\mathbb{D}\).

Besides classical articles by Ahlfors, Warschawski, Becker, Pommenenke,
and others, which develop (Euclidean) geometric conditions for regularity
estimates on Riemann maps; see, e.g., \[26, 2, 17, 18, 25\] and the references
therein, there are close connections between the results and methods of this
paper and more recent work. Let us highlight some. We mentioned the work
by Lind, Marshall, and Rohde \[11\] and by Marshall and Rohde \[16\]; see also
Wong’s paper \[27\]. The paper by Aizenman and Burchard \[1\] character-
izes tightness for probability measures on a space of (discrete-model) curves
modulo reparameterization in terms of estimates on probabilities of annuli crossing events. The event that the geometric condition fails is contained in a union of crossing events of this type and this is what allows for estimation of probabilities. Kemppainen and Smirnov consider related questions and use similar conditions in [9] and a quantity somewhat similar to the tip structure modulus has been used by Lind and Rohde in [10].

1.3. Acknowledgements. Support from the Simons Foundation, Institut Mittag-Leffler, and the AXA Research Fund is gratefully acknowledged. I wish to thank Dmitry Belyaev, Don Marshall, and Steffen Rohde for interesting, helpful, and inspiring conversations on the topics of this paper, and Julien Dubédat and Alan Sola for their useful comments on the manuscript.

2. Preliminaries and the Deterministic Loewner Equation

2.1. Preliminaries. We start by setting some notation. We will write $D = \{z \in \mathbb{C} : |z| < 1\}$ for the unit disk in the complex plane. This is the basic reference domain, although we will occasionally also consider the upper-half plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$. Let $D \ni 0$ be a simply connected domain. By the Riemann mapping theorem there exists a unique conformal map $\psi : D \to D$ with $\psi(0) = 0$ and $\psi'(0) > 0$. If we do not state otherwise we will always assume that uniformizing conformal maps like $\psi$ are normalized in this way.

A crosscut $\mathcal{C}$ of a simply connected domain $D$ is an open Jordan arc in $D$ such that $\overline{\mathcal{C}} = \mathcal{C} \cup \{\zeta, \eta\}$ with $\zeta, \eta \in \partial D$. A crosscut partitions $D$ into exactly two disjoint components; see Chapter 2 of [20].

A (parameterized) curve $\gamma$ is a continuous function $\gamma(t) : I \to \mathbb{C}$ defined on some interval $I$ which we will usually assume to be $[0, T]$ for some fixed $T > 0$. Given two curves $\gamma_1, \gamma_2$ defined on the same interval, we measure their distance by the supremum norm $\sup_{t \in [0, T]} |\gamma_1(t) - \gamma_2(t)|$.

Let $\gamma : [0, T] \to \overline{\mathbb{D}}$ be a curve with $\gamma(0) \in \partial \mathbb{D}, 0 \notin \gamma[0, T]$, and for $t \in [0, T]$, let $D_t$ be the connected component of 0 of $\mathbb{D} \setminus \gamma[0, t]$. We say that $\gamma$ is parameterized by capacity if the normalized conformal maps $g_t : D_t \to \mathbb{D}$ satisfy $g_t'(0) = e^t$ for $t \in [0, T]$. A reparameterization (which is increasing) of a curve $\gamma$ is a new curve $\tilde{\gamma}$ obtained by $\tilde{\gamma}(t) = \gamma \circ \alpha(t)$, where $\alpha(t) : [0, T] \to [0, T]$ is a strictly increasing and continuous function. We will often, when no confusion is possible, treat a curve and its reparameterizations as the same. A (\(\mathbb{D}\)-) Loewner curve is a curve $\gamma$ in $\overline{\mathbb{D}}$ as above, parameterized by capacity, for which the following continuity condition holds: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $s, t \in [0, T]$ with $0 < t - s < \delta$ there is a crosscut $\mathcal{C}$ with $\text{diam } \mathcal{C} < \varepsilon$ that separates $K_t \setminus K_s$ from 0 in $D_t$. 


where $K_t = \mathbb{D} \setminus D_t$. Intuitively, a $\mathbb{D}$-Loewner curve is a continuous curve with no transversal self-crossings whose tip $\gamma(t)$ is always “visible” from 0. For example, if $\gamma$ is has no double points and is contained in $\mathbb{D}$ for $t \in (0, T]$, then it is a Loewner curve. By Theorem 1 of [19], the $\mathbb{D}$-Loewner curves are exactly the curves that can be described using the radial Loewner equation driven by a continuous driving term, as discussed in the next section. We will also consider (chordal) Loewner curves in $\mathbb{H}$ which are defined in a similar manner; we refer to Chapter 4 of [12] for more information. We just note that in this case it is convenient to parameterize $\gamma$ by the so-called half-plane capacity, that is, so that the conformal maps $g_t: \mathbb{H} \rightarrow \mathbb{H}$, where $\mathbb{H}_t$ is the unbounded connected component of $\mathbb{H} \setminus \gamma[0, t]$, satisfy $g_t(z) = z + 2t/z + o(1/|z|)$ at $\infty$. (That is, in this case the normalization is at a boundary point, and the tip of the curve is to be “visible” from this point at all times.)

We will often write “constants” depending on parameters as $c = c(a, b)$, etc. It is then to be understood that $c$ depends only on these parameters.

2.2. Loewner Equations. We will be interested in two versions of Loewner’s differential equation. We define radial and chordal Loewner vector fields by

$$\Phi_{\mathbb{D}}(z, \zeta) = -z\zeta + z, \quad \Phi_{\mathbb{H}}(z, \xi) = -2z - \xi.$$ 

The radial and chordal Loewner equations are then given by

$$\partial_t f(t, z) = \partial_z f(t, z) \Phi_X(z, W(t)), \quad f_0(z) = z, \quad z \in X,$$

(2.1)

$X = \mathbb{D}$ and $X = \mathbb{H}$ respectively. (We will sometimes refer to these equations the $\mathbb{D}$- and $\mathbb{H}$-Loewner PDEs and their solutions as $\mathbb{D}$- and $\mathbb{H}$-Loewner chains, etc.) Here, $W : [0, \infty) \rightarrow \partial X$ is a (continuous) function called the driving term. (In the radial case, we will sometimes write the driving term as $W(t) = e^{i\xi(t)}$ for a real valued function $\xi$ which, when no confusion is possible, is for brevity also referred to as the driving term.)

Let us discuss a few properties in the radial case. (Similar results hold for the chordal version.) For each $t_0 \geq 0$ the solution $f(t_0, \cdot) : \mathbb{D} \rightarrow D_{t_0}$ is a conformal map onto a simply connected domain $D_{t_0} \subset \mathbb{D}$. The family $(f(t, z))_{t \geq 0}$ of conformal mappings is called a Loewner chain. A Loewner pair $(f, W)$ consists of a function $f(t, z)$ and a real-valued (continuous) function $W(t)$, $t \geq 0$, such that $f$ is the solution to the Loewner equation with $W$ as driving term. Under some rather mild regularity assumptions on $W$ (e.g., that $W$ is Hölder-$\left(1/2+\varepsilon\right)$ for some $\varepsilon > 0$) there exists a curve $\gamma(t)$ such that $D_t$ is the component of the origin of $\mathbb{D} \setminus \gamma[0, t]$ and in this case we say that the Loewner chain is generated by the (Loewner) curve $\gamma$. Conversely, given a Loewner curve, one can associate via the Loewner equation a unique driving term such that the Loewner chain $(f_t)$ in the Loewner pair $(f, W)$ is generated by $\gamma$. In fact, the driving term is the
preimage in \( \partial \mathbb{D} \) of the tip of the growing curve. In terms of the inverse relationship we have

\[
\gamma(t) = \lim_{d \to 0^+} f(t, (1 - d)W(t)).
\] (2.2)

A sufficient condition for \((f, W)\) to be generated by a curve \(\gamma\) is that the limit \(2.2\) exists for all \(t \geq 0\) and that \(t \mapsto \gamma(t)\) is continuous; see Theorem 4.1 of [21]. The parameterization of \(\gamma\) given by \(2.2\) is the capacity parameterization.

We will sometimes use the notation \(f_t(z) = f(t, z), f' = \partial_z f, \) and \(\dot{f} = \partial_t f.\)

**Lemma 2.1.** There exists a constant \(c_0 < \infty\) such that the following holds. Let \(X \in \{\mathbb{D}, \mathbb{H}\}\). Suppose that \(f_t\) satisfies the X-Loewner PDE and that \(\text{dist}(z, \partial X) = d\). Then for \(s \geq 0\)

\[
e^{-\cos/d^2} |f'_t(z)| \leq |f'_{t+s}(z)| \leq e^{\cos/d^2} |f'_t(z)|
\] (2.3)

and

\[
|f_{t+s}(z) - f_t(z)| \leq c_0d |f'_t(z)|(e^{\cos/d^2} - 1).
\] (2.4)

**Proof.** See [6] for the proof in the chordal case. The radial case is proved in the same way.

For Hölder continuous driving terms the existence of the curve and its regularity in the capacity parameterization is completely determined by the local behavior at the tip, that is, the growth of the derivative of the conformal map close to the pre-image of the tip. However, there are Hölder-1/2 driving terms that do not generate curves; see [11]. The following result is a version of Proposition 3.9 of [6], but allows for a less regular driving term.

**Proposition 2.2.** Let \((f, W)\) be a \(\mathbb{D}\)-Loewner pair and assume that \(W(t) = e^{i\xi(t)}\) and \(\xi(t)\) is Hölder-\(\alpha\) on \([0, T]\) for some \(\alpha \leq 1/2\). If there exist constants \(c < \infty, d_0 > 0, \) and \(\beta < 1\) such that

\[
\sup_{t \in [0, T]} d|f'_t((1 - d)W(t))| \leq cd^{1-\beta}, \quad \forall d \leq d_0,
\] (2.5)

then \((f, W)\) is generated by a curve \(\gamma\) which for every \(\varepsilon > 0\) is Hölder-\(\alpha(1 - \beta)\) continuous in the capacity parameterization on \([\varepsilon, T]\) and Hölder-min\{\(\alpha, \alpha(1 - \beta)\}\) continuous on \([0, T]\). The analogous statement holds for \(\mathbb{H}\)-Loewner pairs.

**Proof.** The bound on the derivative implies that the limit

\[
\gamma(t) = \lim_{d \to 0^+} f_t((1 - d)W(t))
\]
exists for every \( t \in [0, T] \) and since the convergence is uniform \( \gamma(t) \) is a continuous function. Let \( s > 0 \) and set \( d = s^\alpha \). If \( t, t+s \in [0, T] \), we have
\[
|\gamma(t+s) - \gamma(t)| \leq |\gamma(t+s) - f_{t+s}((1-d)W(t+s))| + |f_{t+s}((1-d)W(t+s)) - f_{t+s}((1-d)W(t))| + |f_{t+s}((1-d)W(t)) - f_t((1-d)W(t))| + |f_t((1-d)W(t)) - \gamma(t)|.
\]
If \( t > 0 \), then the estimate (2.5) implies that the first and last terms are bounded by a constant times \( d^{1-\beta} = s^{\alpha(1-\beta)} \). By assumption \( |\xi(t+s) - \xi(t)| \leq cs^\alpha = cd \), so the distortion theorem implies that
\[
|f_{t+s}((1-d)W(t+s)) - f_{t+s}((1-d)W(t))| \leq cd^{1-\beta}.
\]
Finally, since \( s = d^{1/\alpha} \) and \( \alpha \leq 1/2 \), (2.4) implies
\[
|f_{t+s}(W(t)) - f_t((1-d)W(t))| \leq cd^{1-\beta}.
\]
Since \( d|f'_0((1-d)W(0))| = d \) and so cannot decay faster than linearly, we get the stated exponent on \([0, T]\). \( \square \)

2.3. An Estimate for the Reverse-Time Loewner Equation. We want to compare solutions to the Loewner equation corresponding to driving terms which are close in the supremum norm. We will use the reverse-time Loewner equation
\[
\frac{\partial}{\partial t} h_j(t, z) = \Phi_X(h_j, U_j(t)), \quad h_j(0, z) = z,
\]
where \( X \) equals \( \mathbb{D} \) and \( \mathbb{H} \) in the radial and chordal case, respectively. We say that \( U_j \) is the driving term for (2.6). If we take \( U_j(t) = W_j(t_0 - t) \) we have the well-known identity
\[
h_j(t_0, z; t_0) = f_j(t_0, z), \quad z \in X, \quad j = 1, 2,
\]
where \( f_j(t, z) \) solves the Loewner PDE (2.1) with \( W_j(t) \) as driving term. Note that these equalities only hold at the special time \( t = t_0 \). The families of conformal mappings \((h_j(\cdot, z))\) and \((f_j(\cdot, z))\) are in general different. Note also that solutions \( t \mapsto h(t, z) \) to (2.6) flow away from \( \partial X \) as \( t \) increases when \( z \in X \). This implies that if \( z \in X \) is fixed then the solution \( t \mapsto h(t, z) \) exists for all \( t \geq 0 \).

Now assume that \( \varepsilon \) and \( \nu \) are given positive numbers. Let \( z_1, z_2 \in X \) be given and suppose that
\[
\sup_{t \in [0, T]} |W_1(t) - W_2(t)| \leq \varepsilon, \quad |z_1 - z_2| \leq \nu \varepsilon.
\]
Set
\[
H(t) = h_1(t, z_1) - h_2(t, z_2),
\]
where the $h_j$ are assumed to solve the reverse-time Loewner equations (2.6) driven by
\[ \tilde{W}_j(t) := W_j(t_0 - t), \quad j = 1, 2. \]
Then $H(t_0) = f_1(t_0, z_1) - f_2(t_0, z_2)$. We differentiate with respect to $t$ and use (2.6) to obtain the linear differential equation
\[ \dot{H}(t) - H(t)\psi_X(t) = (\tilde{W}_2(t) - \tilde{W}_1(t))\xi_X(t), \]
where
\[ \psi_d(t) = \frac{h_1 h_2 - \tilde{W}_1\tilde{W}_2 - \frac{1}{2}(h_1 + h_2)(\tilde{W}_1 + \tilde{W}_2)}{(h_1 - \tilde{W}_1)(h_2 - \tilde{W}_2)}, \]
\[ \xi_d(t) = \frac{h_1^2 + h_2^2}{2(h_1 - \tilde{W}_1)(h_2 - \tilde{W}_2)} \]
and
\[ \psi_H(t) = \frac{2}{(h_1 - \tilde{W}_1)(h_2 - \tilde{W}_2)}, \]
\[ \xi_H(t) = \psi_H(t). \]
Here we have suppressed the dependence on $t$ in the right-hand sides. We can integrate the differential equation and with $u(t) = \exp\left\{- \int_0^t \psi_X(s) \, ds \right\}$ we find
\[ H(t) = u(t)^{-1} \left( H(0) + \int_0^t (\tilde{W}_2 - \tilde{W}_1) u \xi_X \, ds \right). \]
Hence, for $0 \leq t \leq t_0$,
\[ |H(t)| \leq |H(0)| e^{\int_0^t \Re \psi_X(s) \, ds} + \int_0^t |\tilde{W}_2 - \tilde{W}_1| e^{\int_s^t \Re \psi_X(r) \, dr} |\xi_X| \, ds. \quad (2.7) \]
Consequently, since
\[ \sup_{t \in [0, t_0]} |\tilde{W}_1(t) - \tilde{W}_2(t)| \leq \varepsilon, \quad |H(0)| = |z_1 - z_2| \leq \nu \varepsilon, \]
recalling that $|f_1(t_0, z_1) - f_2(t_0, z_2)| = |H(t_0)|$, we get the estimate
\[ |f_1(t_0, z) - f_2(t_0, w)| \leq \varepsilon \left( \nu e^{\int_0^{t_0} \Re \psi_X(s) \, ds} + \int_0^{t_0} e^{\int_s^{t_0} \Re \psi_X(r) \, dr} |\xi_X| \, ds \right). \quad (2.8) \]
The right-hand side in (2.8) can be estimated in different ways depending on what data is available. We would like an estimate that depends only on $\varepsilon$ and $d = \text{dist}(\{z_1, z_2\}, \partial X)$. Estimating naively, using only the fact that points flow away from $\partial X$ under the reverse flow, gives a bound of order $\varepsilon e^{O(d^{-2})}$. (This kind of estimate was used in [4].) We shall see that we can do much better.
2.3.1. The Chordal Case. To give some intuition, let us first briefly discuss the easier chordal case which will be treated in greater detail in [8]. Assume \( \nu = 1 \) for simplicity. Write \( z_j(t) = h_j(t, z_j) - \bar{W}_j(t) \). We can apply the Cauchy-Schwarz inequality to get
\[
\int_0^t \text{Re} \psi_X(t) \, dt \leq \int_0^t \frac{2}{|z_1(t)|^2} \, dt \leq \left( \int_0^t \frac{2}{|z_1(t)|^2} \, dt \right)^{1/2} \left( \int_0^t \frac{2}{|z_2(t)|^2} \, dt \right)^{1/2}.
\]
Since \( \partial_t \log |z_j(t)| = 2/|z_j(t)|^2 \) this can now be used to show that the right-hand side of (2.8) is bounded by \( \varepsilon d^{-1} \) times a constant depending only on \( T \), if \( \log |z_j(0)| \geq \bar{a}, j = 1, 2 \). (Note that there is no logarithmic correction.)

Remark. The estimate \( \varepsilon d^{-1} \) is essentially sharp if no further assumptions are made. Indeed, consider a driving term \( W_1(t) \) generating a Loewner chain such that for some fixed \( p < 1 \) very close to 1, \( t_0 > 0 \), there is a constant \( c > 0 \) such that \( |f_j'(t_0, W_1(t_0) + id)| \geq c d^{-p} \) as \( d \to 0 \). (As shown in [11] one can take \( W_2(t) = W_1(t) + \varepsilon \), then \( f_2(t, z) = f_1(t, z - \varepsilon) + \varepsilon \). Hence, for \( \varepsilon \leq d/2 \), by Koebe’s distortion theorem,
\[
|f_2(t_0, W_1(t_0) + id) - f_1(t_0, W_1(t_0) + id)| \geq |f_2(t_0, W_1(t_0) + id - \varepsilon) - f_1(t_0, W_1(t_0) + id)| \geq c d^{-p}.
\]
A similar example can be constructed for the radial case.

If more information is available we can do a bit better. The reader may check that \( \partial_t \text{Re} \log h_j'(t, z) = \text{Re}(2/|z_j(t)|^2) \). From this one can see that if a non-trivial power-law bound on the growth of the derivative at time \( t_0 \) holds, that is, if there exists \( c < \infty \) such that
\[
|f_j'(t_0, W_j(t_0) + id)| \leq c d^{-\beta_j}, \quad d \to 0+,
\]
then one gets a bound in (2.8) of order at most \( \varepsilon d^{-\frac{3}{2}} \) log \( d^{-1} \). This refinement is joint work with Rohde and Wong and details will appear in [8].

2.3.2. The Radial Case. We now move to the radial case \( X = \mathbb{D} \). In order to bound (2.8) we need to estimate \( \int_{t_0}^t \text{Re} \psi_X(s) \, ds \). The idea is to prove that for a constant \( q \) slightly larger than 1,
\[
\text{Re} \psi_\mathbb{D}(t) \leq q \sqrt{1 + |z_1(t)|} \sqrt{1 + |z_2(t)|}.
\]
where for \( t \in [0, t_0] \), we define
\[
z_j(t) = h_j(t, z_j) - \bar{W}_j(t).
\]
Note that $|z_j(0)| = |z_j|$. Once we have this estimate we can apply the Cauchy-Schwarz inequality to the corresponding bound on $\int_s^0 \Re \psi_{D}(s) \, ds$ to decouple the two flows and then compare with

$$\frac{1 + |z_j(t)|}{|1 - z_j(t)|^2} = \partial_t \log (1 - |z_j(t)|).$$

(2.10)

This last identity follows from the reverse-time Loewner equation (2.6). This will give a bound in (2.8) of order $\varepsilon d^{-q}$, where $q$ can be taken arbitrarily close to 1. (Arguing naively as in the chordal case gives a bound of order $\varepsilon d^{-4}$, but we shall actually make use of this bound below.) This is essentially optimal in this general setting as we saw above.

**Proposition 2.3.** Let $t_0 \in [0,T]$ be fixed and for $j = 1, 2$, let $(f_j, W_j)$ be $D$-Loewner pairs. For any $\rho > 1$ there exists $\varepsilon' = \varepsilon'(\rho) > 0$ and $c = c(\rho, T) < \infty$ such that the following holds. Suppose that $\varepsilon, d > 0$ are such that $|z_1 - z_2| \leq \varepsilon$, $|z_1|, |z_2| \geq 1 - d$, and

$$\sup_{t \in [0,t_0]} |W_1(t) - W_2(t)| \leq \varepsilon.$$

Then there exist $\varepsilon_0 = \varepsilon_0(\rho) > 0$ and $d_0 = d_0(\rho) > 0$ such that if $0 < \varepsilon < \varepsilon_0$ and $(4\varepsilon/\varepsilon')^{1/\rho} \leq d \leq d_0$, then

$$|f_1(t_0, z_1) - f_2(t_0, z_2)| \leq c_\varepsilon \varepsilon d^{-\rho}.$$

(2.11)

**Proof.** By factoring out $\bar{W}_1 \bar{W}_2$ we can write

$$\Re \psi_D(t) = \Re \left( \frac{z_1(t)z_2(t) - 1 - (z_1(t) + z_2(t)) + O(\varepsilon)}{(1 - z_1(t))(1 - z_2(t))} \right)$$

$$= \Re \left\{ \frac{(z_1(t)z_2(t) - 1 - (z_1(t) + z_2(t)) + O(\varepsilon))(1 - \overline{z_1(t)})(1 - \overline{z_2(t)})}{|1 - z_1(t)|^2|1 - z_2(t)|^2} \right\}.$$

(2.12)

This uses that $\bar{W}_1(t)\bar{W}_2(t) = 1 + O(\varepsilon)$ in the sense that $|\bar{W}_1(t)\bar{W}_2(t) - 1| \leq c\varepsilon$ for a universal constant $c$. For $z, w \in \overline{D}$ we now consider the function

$$R(z, w) = \Re \left\{ (zw - 1 - (z + w))(1 - \overline{z})(1 - \overline{w}) \right\}.\frac{1}{|1 - z||1 - w|\sqrt{(1 + |z|)(1 + |w|)}}.$$

The function $R$ is bounded and continuous on the closed bi-disk $\overline{D} \times \overline{D}$. We claim that $\sup_{z, w \in \partial D} R(z, w) \leq 1$. A computation shows that $R$ simplifies if $|z| = |w| = 1$ so that

$$R(z, w) = \frac{(1 - \Re z)(1 - \Re w) + \Im z \Im w}{2\sqrt{(1 - \Re z)(1 - \Re w)}}.$$

By changing coordinates $z = e^{i\theta}$ and $w = e^{i\mu}$, with $\theta, \mu \in [0, 2\pi]$, in the last expression we find

$$\left( R(e^{i\theta}, e^{i\mu}) \right)^2 = \cos^2 \left( \frac{\theta - \mu}{2} \right) \leq 1.$$
Let $\delta > 0$ be fixed and small. By the last expression and the continuity of $R$, there exists $\varepsilon'(\delta) > 0$ such that if $1 - \varepsilon' \leq |z|, |w| \leq 1$ then $R(z, w) \leq 1 + \delta/2$. (We think of $\varepsilon'$ as small but macroscopic compared to $\varepsilon$.) Hence, returning to the flows, by (2.12) and the bound on $R$, if $\varepsilon$ is sufficiently small compared to $\delta$, we have the estimate

$$Re\psi_D(t) = Re\left(\frac{z_1(t)z_2(t) - 1 - (z_1(t) + z_2(t)) + O(\varepsilon)}{(1-z_1(t))(1-z_2(t))}\right) \leq (1 + \delta)\frac{\sqrt{1+|z_1(t)|}}{|1-z_1(t)|} \cdot \frac{\sqrt{1+|z_2(t)|}}{|1-z_2(t)|}, \quad 0 \leq t \leq \tau,$$

(2.13)

where

$$\tau = \inf\{t \geq 0 : \min\{|z_1(t)|, |z_2(t)|\} = 1 - \varepsilon'\}.$$

Also, note the easy bound

$$Re\psi_D(t) \leq |\psi_D(t)| \leq 4\frac{\sqrt{1+|z_1(t)|}}{|1-z_1(t)|} \cdot \frac{\sqrt{1+|z_2(t)|}}{|1-z_2(t)|}, \quad 0 \leq t \leq t_0.$$  

(2.14)

We now split the integral

$$\int_0^{t_0} Re\psi_D(s) \, ds = \int_0^\tau Re\psi_D(s) \, ds + \int_\tau^{t_0} Re\psi_D(s) \, ds.$$  

We estimate the first integral using (2.13) and the Cauchy-Schwarz inequality

$$\int_0^\tau Re\psi_D(s) \, ds \leq (1 + \delta) \left(\int_0^\tau \frac{1+|z_1(s)|}{|1-z_1(s)|^2} \, ds\right)^{1/2} \left(\int_0^\tau \frac{1+|z_2(s)|}{|1-z_2(s)|^2} \, ds\right)^{1/2}.$$  

Using (2.10) and the fact that $|z_2(s)| \geq 0$ we see that

$$\int_0^\tau Re\psi_D(s) \, ds \leq (1 + \delta) \left(\log\frac{1}{|z_1(s)|}\right)^{1/2} \left(\log\frac{1}{|z_2(s)|}\right)^{1/2}.$$  

(2.15)

Thus, if, say, $|z_1| = 1 - d \geq |z_2| \geq 1 - \varepsilon'$ we obtain for any $\delta' > \delta$

$$|z_1(\tau) - z_2(\tau)| \leq \varepsilon \left(\nu e\int_0^\tau Re\psi_D(s) \, ds + \int_0^\tau e\int_\tau^\tau Re\psi_D(r) \, dr \, |\xi_D| \, ds\right) \leq \varepsilon d^{-(1+\delta')} (\nu + 1),$$

where we also used that

$$|\xi_D(t)| \leq \frac{\sqrt{1+|z_1(t)|}}{|1-z_1(t)|} \cdot \frac{\sqrt{1+|z_2(t)|}}{|1-z_2(t)|}$$

and tacitly assumed that $d$ is sufficiently small so that the resulting logarithmic correction is dominated by $d^{\delta - \delta'}$. Consequently we see that if $\varepsilon$ is sufficiently small so that we can choose

$$d \geq \left(\frac{2\varepsilon}{\varepsilon'} (\nu + 1)\right)^{1/(1+\delta')}$$

,
then
\[ \max\{|z_1(\tau)|, |z_2(\tau)|\} \leq 1 - \varepsilon' + \varepsilon d^{-(1+\delta')} (\nu + 1) \leq 1 - \frac{\varepsilon'}{2}. \]

Hence, by (2.14), the Cauchy-Schwarz inequality, and (2.10),
\[ \int_{\tau_0}^{\tau} \Re \psi_D(s) \, ds \leq 4 \left| \log \left( \frac{2}{\varepsilon'} \right) \right|. \]
Combining this with (2.15), the proof is complete. \( \boxdot \)

Remark. We believe that the function \( R(z, w) \) used in the last proof is bounded by 1 on the whole bi-disk, and with some work one should be able to verify this. (However, this is not true for \( |R(z, w)| \).) This would allow for taking \( \rho = 1 \) in (2.11). This would not improve the resulting convergence rate, so we will not pursue this here.

Suppose now that for \( j = 1, 2 \), \( f_j \) satisfies the derivative estimate (2.9) with \( \beta = \beta_j \) and \( c = c_j \). (In the radial case we consider the radial version of (2.9) and take \( \beta_j = 1 \); indeed, it is a general fact about (normalized) conformal maps that (2.9) always holds with \( \beta = 1 \) for some constant universal constant \( c < \infty \).) Set
\[ \rho_0 = \rho_0(\beta_1, \beta_2) = \begin{cases} 1 & \text{if } X = \mathbb{D}; \\ \frac{1}{2} \sqrt{(1+\beta_1)(1+\beta_2)} & \text{if } X = \mathbb{H}. \end{cases} \] (2.16)
Suppose \( \rho > \rho_0 \) and \( p \in (0, 1/\rho) \). Let \( \varepsilon > 0 \) and define
\[ d_* = \varepsilon^p. \] (2.17)
We have proved that for any \( z \) and \( w \) with \( |z - w| \leq \varepsilon \) at distance at least \( d_* \) from the boundary, if the driving terms satisfy \( \sup |W_1(t) - W_2(t)| \leq \varepsilon \), then there are \( c = c(\rho, p) < \infty \) and \( \varepsilon_0 = \varepsilon_0(\rho) > 0 \) such that if \( \varepsilon < \varepsilon_0 \), then
\[ |f_1(t_0, z) - f_2(t_0, w)| \leq ce^{1-\rho p}. \]
By estimating using Cauchy’s integral formula, we also get a bound relating the derivatives: Write \( f_j(z) = f_j(z, t_0) \). Then with \( d = \text{dist}(z, \partial X) \),
\[ |f'_1(z) - f'_2(z)| = \frac{1}{2\pi} \left| \int_{|\zeta - z| = r} \frac{f_1(\zeta) - f_2(\zeta)}{(z - \zeta)^2} \, d\zeta \right| \leq c\varepsilon d^{-\rho} r^{-1}, \]
where \( r \leq d/2 \). Taking \( d = 2r = \varepsilon^p \) this estimate combined with the reverse triangle inequality shows that there is a constant \( c = c(\rho, p, T) < \infty \) (recall that \( t_0 \leq T \)) such that
\[ \sup_{z: \text{dist}(z, \partial X) \geq \varepsilon^p} \| |f'_1(z)| - |f'_2(z)| \| \leq c \varepsilon^{1-(1+\rho)p}. \]
We have proved the radial part of the following result. (The chordal case is joint work with Rohde and Wong; see \[ \S \] for its complete proof.)
Proposition 2.4. Let $X \in \{ \mathbb{D}, \mathbb{H} \}$ and $T > 0$. Let $(f_j, W_j), j = 1, 2,$ be $X$-Loewner pairs so that $f_j$ solve \[(2.1)\] with $W_j$ as driving terms and assume that $f_j$ satisfy \[(2.5)\] with $\beta = \beta_j$ and $c = c_j < \infty$. Suppose $\rho > \rho_0$, where $\rho_0$ is defined by \[(2.16)\]. Assume that for $\varepsilon > 0$

$$\sup_{t \in [0,T]} |W_1(t) - W_2(t)| \leq \varepsilon, \quad |z - w| \leq \varepsilon$$

and for $p \in (0,1/\rho)$ define

$$d_* = \varepsilon^p. \quad (2.18)$$

There exist $c = c(T, \rho, p, c_1, c_2) < \infty$ and $\varepsilon_0 = \varepsilon_0(\rho, p) > 0$ such that if

$$\min \{ \text{dist}(z, \partial X), \text{dist}(w, \partial X) \} \geq d_*$$

and $\varepsilon < \varepsilon_0$, then

$$\sup_{t \in [0,T]} |f_1(t, z) - f_2(t, w)| + \sup_{t \in [0,T]} |d_*|f_1'(t, z)| - |d_*|f_2'(t, z)| | \leq \varepsilon^{1-\rho p}. \quad (2.19)$$

One way to interpret the last proposition is that information about the derivative of one of the conformal mappings transfers to the other via the Loewner equation if they are evaluated sufficiently far away from the boundary. The proper scale (or resolution) is determined by the distance between the driving terms. Note that we make no assumptions about the regularity of the driving terms; the above results are consequences of the structure of the Loewner equation alone.

2.4. Supremum Distance Between Loewner Curves. We will now consider two Loewner curves, $\gamma_j : [0,T] \rightarrow X, j = 1, 2,$ generating the $X$-Loewner pairs $(f_j, W_j)$ and suppose that

$$\sup_{t \in [0,T]} |W_1(t) - W_2(t)| \leq \varepsilon. \quad (2.19)$$

We are interested in estimating the supremum distance $\sup_{t \in [0,T]} |\gamma_1(t) - \gamma_2(t)|$ when the curves are parameterizes by capacity, in terms $\varepsilon$. We have the following estimate.

Proposition 2.5. Let $X \in \{ \mathbb{D}, \mathbb{H} \}$. For $j = 1, 2$, let $(f_j, W_j)$ be $X$-Loewner pairs generated by the curves $\gamma_j$ and suppose that $f_j$ satisfy \[(2.5)\] with $\beta = \beta_j$ and $c = c_j$. Suppose that $\rho > \rho_0$, where $\rho_0$ is given by \[(2.16)\]. Suppose that $\varepsilon > 0$ is such that

$$\sup_{t \in [0,T]} |W_1(t) - W_2(t)| \leq \varepsilon.$$
Let \( p \in (1,1/\rho) \) and set \( d = \varepsilon^p \). There exist \( c = c(T, \rho, p) < \infty \) and \( \varepsilon_0 = \varepsilon_0(\rho, p) > 0 \) such that if \( \varepsilon < \varepsilon_0 \), then

\[
\sup_{t \in [0,T]} |\gamma_1(t) - \gamma_2(t)| 
\leq c \varepsilon^{1-\rho p} + c \sup_{t \in [0,T]} (|\gamma_1(t) - f_1(t, (1-d)W_1(t))| \\
+ |\gamma_2(t) - f_2(t, (1-d)W_2(t))|), \tag{2.20}
\]

with \( f_j(t, (1-d)W_j(t)) \) replaced by \( f_j(t, W_j(t) + \text{id}) \) in the chordal case.

**Proof.** We will do the radial case. Write

\[
|\gamma_1(t) - \gamma_2(t)| \leq |\gamma_1(t) - f_1(t, (1-d)W_1(t))| \\
+ |f_1(t, (1-d)W_1(t)) - f_1(t, (1-d)W_2(t))| \\
+ |f_1(t, (1-d)W_2(t)) - f_2(t, (1-d)W_2(t))| \\
+ |f_2(t, (1-d)W_2(t)) - \gamma_2(t)|.
\]

Denote by \( b_1, \ldots, b_4 \) the four terms on the right-hand side in the last inequality in the order in which they appear. By the distortion theorem, since \( d \geq \varepsilon \) we have that

\[
b_2 \leq c \text{dist}(f_1(t, (1-d)W_1(t)), \partial f(t, \mathbb{D})) \leq cb_1.
\]

Finally, by Proposition 2.4, \( b_1 \leq c \varepsilon^{1-\rho p} \). \( \square \)

**Corollary 2.6.** For \( j = 1, 2 \), let \((f_j, W_j)\) be \( \mathbb{H} \)-Loewner pairs generated by the curves \( \gamma_j \) and assume that (2.19) holds. Suppose that there exist \( d_0 > 0 \), \( c < \infty \), and \( \beta < 1 \) such that \( f_j \) satisfy the estimate (2.5). Then for every

\[
r < 2 \frac{1-\beta}{3-\beta},
\]

there exist \( c = c(r, T) < \infty \) and \( \varepsilon_0 = \varepsilon_0(r, T) > 0 \) such that if \( \varepsilon < \varepsilon_0 \), then

\[
\sup_{t \in [0,T]} |\gamma_1(t) - \gamma_2(t)| \leq c \varepsilon^r.
\]

**Proof.** Under our assumptions \( \rho_0 = (1+\beta)/2 \). Let \( \rho > \rho_0 \) and \( 0 < p < 1/\rho \). We set \( d = \varepsilon^p \), apply Proposition 2.5 and integrate the bound on the derivatives to see that for \( \varepsilon > 0 \) sufficiently small,

\[
\sup_{t \in [0,T]} |\gamma_1(t) - \gamma_2(t)| \leq c \left( \varepsilon^{1-\rho p} + \varepsilon^{p(1-\beta)} \right).
\]

We optimize over exponents to find the stated bound for \( r \). \( \square \)

The proof of the next corollary is an analog for Loewner curves of the well-known fact that the Riemann map onto a Hölder domain satisfies a power-law bound on the growth of the derivative.
Corollary 2.7. For \( j = 1, 2 \), let \((f_j, W_j)\) be \(\mathbb{H}\)-Loewner pairs generated by the curves \(\gamma_j\) and assume that \((2.19)\) holds. Suppose that both curves are Hölder-\(\alpha\) continuous in the capacity parameterization, where \(\alpha > 0\). Then for every

\[
 r < \frac{2\alpha}{1 + \alpha},
\]

there exist \(c = c(r, T) < \infty\) and \(\varepsilon_0 = \varepsilon_0(r, T) > 0\) such that if \(\varepsilon < \varepsilon_0\), then

\[
\sup_{t \in [0, T]} |\gamma_1(t) - \gamma_2(t)| \leq c \varepsilon^r.
\]

Proof. We will prove a bound on the growth of the derivative and then apply the previous corollary. It is enough to consider \(f(t, z) := f_1(t, z)\) since we made the same assumptions on both Loewner chains. Write \(\gamma = \gamma_1\) and \(W = W_1\) and for \(t, t + s \in [0, T]\), let

\[
\tilde{\gamma} = f^{-1}(t, \gamma[t, t + s]).
\]

Then \(\tilde{\gamma}\) is a curve in \(\mathbb{H}\) “rooted” at \(W(t)\). Set \(d = \text{diam } \tilde{\gamma}\). Let \(z \in \tilde{\gamma}\) be a point such that \(|z - W(t)| = d/2\) and let \(\Gamma\) be the hyperbolic geodesic in \(\mathbb{H}\) connecting \(W(t)\) with \(z\). Then \(\Gamma\) contains a point \(w\) with \(\text{Im } w \geq d/4\). Note that by the distortion theorem, \(|f'(t, \gamma)| \approx |f'(t, W(t) + id)|\) so that Koebe’s 1/4 theorem implies that there is a universal constant \(c > 0\) such that

\[
\text{diam } f(t, \Gamma) \leq c \text{diam } \gamma[t, t + s] \leq c' s^\alpha.
\]

Hence, using \((2.21)\), there is a constant \(c < \infty\) such that

\[
d |f'(t, W(t) + id)| \leq c s^\alpha \leq c' d^{2\alpha},
\]

where the last inequality follows since \(\text{hcap } \tilde{\gamma} = 2s\) so that there is a universal constant \(c < \infty\) such that \(s \leq c d^2\). The diameter \(d\) depended on \(s\), but every \(d\) sufficiently small can be written like this since \(s \mapsto d\) is an increasing continuous function.

Remark. If \(\gamma(t)\) is Hölder-\(\alpha\) continuous in the capacity parameterization, then its driving term is at least Hölder-\(\alpha/2\): Using the notion of the proof of Corollary 2.7, we note that by the Beurling estimate, \(\text{diam } \tilde{\gamma} \leq c s^{\alpha/2}\) and by Lemma 2.1 of [15], we have \(|W(t + s) - W(t)| \leq c \text{diam } \tilde{\gamma} \leq c' s^{\alpha/2}\).
3. Geometric Conditions

This section develops a geometric condition that we will use in place of a bound on the growth of the derivative of the conformal map in order to measure the regularity of a Loewner curve locally at the tip. As pointed out in the introduction, several similar conditions have appeared in the literature. We will work in the radial setting, but the results hold also in the chordal setting with minor modifications in their statements and proofs.

Let $D \ni 0$ be a simply connected domain. Let $\psi : D \to \mathbb{D}$ be the uniformizing conformal map. We consider a radial Loewner curve $\gamma : [0, T] \to D$. That is, the conformal image of $\gamma$ in $\mathbb{D}$ using the conformal map $\psi$ is a $\mathbb{D}$-Loewner curve. In this section we write $D_t$ for the connected component of $D \setminus \gamma[0, t]$ containing the origin. For $s, t \in [0, T]$ with $s \leq t$ we let $\gamma_{s,t}$ denote the curve determined by $\gamma(r), r \in [s, t]$. For a crosscut $C$ of $D_t$ we write $J_C$ for the component of $D_t \setminus C$ of smaller diameter.

3.1. Tip Structure Modulus. For each $0 \leq t \leq T$ and $\delta > 0$, let $S_t, \delta$ be the collection of crosscuts of $D_t$ of diameter at most $\delta$ that separate $\gamma(t)$ from the origin in $D_t$. For a crosscut $C \in S_t, \delta$, define

$s_C = \inf\{s > 0 : \gamma[t - s, t] \cap C \neq \emptyset\}, \gamma_C = (\gamma(r), r \in [t - s_C, t]).$

(We set $s_C = t$ if $\gamma$ never intersects $C$.) We then define the tip structure modulus of $(\gamma(t), t \in [0, T])$ in $D$, $\eta_{\text{tip}}(\delta)$, by

$$\eta_{\text{tip}}(\delta) = \sup_{t \in [0, T]} \sup_{C \in S_t, \delta} \text{diam } \gamma_C. \quad (3.1)$$

Remark. In the chordal setting we consider instead crosscuts separating $\gamma(t)$ from $\infty$ in $H_t$ in the definition of the structure modulus. The remaining construction is the same.

It is useful to introduce some more terminology. We will say that the curve $\gamma$ has a $(\delta, \eta)$-bottleneck in $D$ if there exists $t \in [0, T]$ and $\zeta \in \partial D_t$ such that $\gamma(t)$ and $\zeta$ can be connected by a crosscut $C_t$ of $D_t$ and $\text{diam } J_{C_t} \geq \eta$ while $\text{diam } C_t \leq \delta$. This definition is similar to the one for “quasi-loops” given by Schramm in [22]. (But our definition is analogous to the John condition rather than Ahlfors’ three-point condition.) We say that the bottleneck is at $z_0$ if the points $\zeta$ and $\gamma(t)$ in the previous definition are contained in the disk $B(z_0, \eta/4)$.

Similarly, we will say that the curve $\gamma$ has a nested $(\delta, \eta)$-bottleneck in $D$ if there exist $t \in [0, T]$ and $C \in S_t, \delta$ with

$\text{diam } \gamma_C \geq \eta.$

That $\gamma(t), t \in [0, T]$ has no nested $(\delta, \eta)$-bottleneck in $D$ is clearly equivalent to having the inequality $\eta_{\text{tip}}(\delta) \leq \eta$ for the same curve.
Figure 1. A nested $(\delta, \eta)$-bottleneck with $\text{diam} C = \delta$ and $\text{diam} \gamma C \geq \eta$, where $\gamma C = \gamma[s, t]$. A 6-crossing event of a $(\delta, \eta)$-annulus for the whole curve.

Remark. The definition of nested bottleneck is independent of the particular chosen parameterization of the curve in the sense that any increasing reparameterization would do in the definition. The definition is not, however, symmetric with respect to reversibility of the curve.

The term “structure modulus” is borrowed from Warschawski [26] who used it in the following sense: the “structure modulus of the boundary of $D$” is defined by the function

$$\eta_W(\delta) = \sup_C \text{diam}\, J_C,$$

where the supremum is over all crosscuts (of $D$) of diameter at most $\delta$. Intuitively, the decay rate of $\eta_W$ places a restriction on bottlenecks/outward-pointing cusps in the boundary and this gives estimates on the regularity of the Riemann mapping from $\mathbb{D}$. For example, $D$ is a John domain if and only if $\eta_W(\delta) \leq A\delta$ for some constant $A < \infty$. One can use this to show (see [26]) that if $h < 2/(A^2\pi^2)$, then the Riemann map from $\mathbb{D}$ is Hölder-$h$ on the closed unit disk. The tip structure modulus is the natural analogue to $\eta_W$. 
for Loewner curves. For example, we will see that if tip structure modulus
decays linearly then the Loewner curve is Hölder continuous in the capacity
parameterization; see Theorem 3.5 below. Moreover, and importantly, the
tip structure modulus is related to annuli crossing events (see Figure 1),
the probabilities of which are often known how to control for discrete-model
curves; the connection between annuli crossings and regularity of curves is
well-known; see, e.g., [1].

3.2. Distance to the Tip. Let \((f,W)\) be a \(D\)-Loewner pair and assume it
is generated by a curve \(\gamma\). We use the notation

\[ \Delta_t(d) = \text{dist}(f_t((1-d)W_t), D_t), \]

where \(W_t = e^{i\xi_t}\) is the driving term for \(f\). Note that Koebe’s distortion
theorem implies that

\[ \Delta_t(d) \asymp d|f_t'((1-d)W_t)|. \]

Recall also that for each \(t\), the tip of the curve is given by taking the radial
limit

\[ \gamma(t) = \lim_{d \to 0^+} f_t((1-d)W_t). \]

We saw in Section 2.4 that we need to obtain uniform (in \(t\)) bounds on
\(|\gamma(t) - f_t((1-d)W_t)|\).

A lower bound on this quantity is clearly given by \(\Delta_t(d)\) and if we have a
bound for \(\eta_{\text{tip}}(\delta)\) in terms of \(\delta\), then we can also give an estimate from above
in terms of \(\Delta_t(d)\). Essentially, the bound depends only on \(\eta_{\text{tip}}\). We need the
following lemma.

**Lemma 3.1.** Let \(T < \infty\) be given. There exist constants \(0 < \rho_0, c_1 < \infty\)
with \(\rho_0\) universal and \(c_1 = c_1(T)\) such that the following holds. Let \(\gamma\) be a
curve in \(D\) generated by the Loewner pair \((f,W)\). If \(\Delta_t(d) < c_1\) then for
each \(t \in [0,T]\) there is a crosscut \(C = C_t\) of \(D_t\) that separates \(f_t((1-d)W_t)\)
and \(\gamma(t)\) from \(0\) in \(D_t\) while

\[ \text{diam} C \leq \rho_0 \Delta_t(d). \]

**Proof.** Let \(t \in [0,T]\) and set

\[ z_d = f_t((1-d)W_t). \]

We will write

\[ \Delta = \Delta_t(d) = \text{dist}(z_d, \partial D_t). \]

For \(\rho > 1\), consider \((\partial B(z_d, \rho \Delta)) \cap D_t\). The components of this set form
crosscuts of \(D_t\) and we let \(C_0\) be the subset of those crosscuts that separate
\(z_d\) from \(0\) in \(D_t\). (Since the inner radius of \(D_t\) from \(0\) is bounded below by
e\(-T/4\), \(C_0\) is non-empty whenever \(\rho \Delta\) is smaller than, say, \(e^{-T}/16\).) Let
\(C_{\rho}\) be the unique crosscut in \(C_0\) with the property that it separates every
other member in \(C_0\) from \(0\) in \(D_t\). Let \(O_{\rho}\) be the component of \(D_t \setminus C_{\rho}\)
that contains \(z_d\) and let \(E_{\rho}\) be the part of \(\partial O_{\rho}\) that is not contained in \(C_{\rho}\).
\[ z_d = f_t((1 - d)W_t) \]

Figure 2. Sketch for the proof of Lemma 3.1. The crosscut \( g_t(\mathcal{C}) \) separates \((1 - d)W_t\) and \( g_t(\mathcal{E}) \subset \partial \mathbb{D} \) from 0 in \( \mathbb{D} \). The harmonic measure of \( g_t(\mathcal{E}) \) from \((1 - d)W_t\) is at least 1/2. Hence \( W_t \in g_t(\mathcal{E}) \).

By Beurling’s projection theorem and the maximum principle there exists a universal \( \rho_0 < \infty \) such that for each \( \rho > \rho_0 \) there is constant \( c_0 = c_0(\rho, T) \) such that for all \( \Delta < c_0 \)

\[ \omega(z_d, \mathcal{E}_\rho, \mathcal{O}_\rho) > 1/2. \tag{3.2} \]

Let \( \mathcal{O} := \mathcal{O}_{2\rho_0}, \mathcal{C} := \mathcal{C}_{2\rho_0}, \text{ and } \mathcal{E} := \mathcal{E}_{2\rho_0} \). It follows from Beurling’s projection theorem that there is a constant \( c_1 < \infty \) depending only on \( T \) such that if \( \Delta < c_1 \), then the diameter of the pre-image of \( \mathcal{C} \) in \( \mathbb{D} \) is at most 1/2 and \( \text{(3.2)} \) holds with \( \rho \) replaced by \( 2\rho_0 \). We shall assume that \( \Delta < c_1 \) in the sequel. We claim that the pre-image of \( \mathcal{E} \) in \( \partial \mathbb{D} \) is an arc containing the point \( W_t \). Indeed, if \( g_t = f_t^{-1} \) then \( g_t(\mathcal{C}) \) is a crosscut of \( \mathbb{D} \) separating \( g_t(\mathcal{E}) \) and \((1 - d)W_t\) from 0. By conformal invariance, the harmonic measure of \( g_t(\mathcal{E}) \) from \((1 - d)W_t\) is strictly bigger than 1/2. Therefore, if \( W_t = e^{i\xi_t} \notin g_t(\mathcal{E}) \), then the arc \( g_t(\mathcal{E}) \) must contain a point \( e^{i\theta} \) with \( \theta \in [\xi_t + \pi, \xi_t + 2\pi] \). Since \( g_t(\mathcal{C}) \) separates \((1 - d)W_t \) and \( e^{i\theta} \) from 0, this would imply that \( \text{diam } g_t(\mathcal{C}) > 1/2 \) and this is a contradiction. We conclude that \( W_t \in g_t(\mathcal{E}) \) and consequently \( \gamma(t) \) is on the boundary of \( \mathcal{O} \) and so \( \mathcal{C} \) separates \( z_d \) and \( \gamma(t) \) from 0 in \( D_t \).
Proposition 3.2. Let $T < \infty$ be given. There exist constants $0 < c_1, c_2 < \infty$ depending only on $T$ such that the following holds. Let $\gamma$ be a curve in $\mathbb{D}$ generating the Loewner pair $(f, W)$. Suppose there exists a non-decreasing function $\eta(\delta)$ such that for each $\delta > 0$, the tip structure modulus for $(\gamma(t), t \in [0, T])$ in $\mathbb{D}$ satisfies $\eta_{\text{tip}}(\delta) \leq \eta(\delta)$. Then if $\Delta_t(d) < c_1$ for all $t \in [0, T]$,

$$|\gamma(t) - f_t((1 - d)W_t)| \leq \eta(c_2 \Delta_t(d)).$$

(3.3)

Proof. We use the notation from the proof of Lemma 3.1. Set

$$\delta_0 = 4\rho_0 \Delta.$$

Then by Lemma 3.1 there is a crosscut $C$ separating $z_d$ and $\gamma(t)$ from 0 in $D_t$ while $\text{diam} \, C \leq \delta_0$. If $\gamma(t) \in B(z_d, \eta(\delta_0))$ we are done. If not, there exists a subarc of $\partial B(z_d, \eta(\delta_0)) \cap \mathcal{O}$ that separates $\gamma(t)$ from $z_d$ in $\mathcal{O}$. Let $\gamma_C \subset \mathcal{O}$ be the curve determined by tracing $\gamma$ backwards from $\gamma(t)$ until $C$ is first hit. Then $C$ contains a crosscut of $D_t$ separating $\gamma_C$ from 0 in $D_t$. Moreover, $\text{diam} \, \gamma_C \geq \eta(\delta_0)$ by assumption and since $\text{diam} \, C \leq \delta_0$ this contradicts the assumption on $\eta_{\text{tip}}$. \hfill \Box

One can also estimate the distance to the tip directly in terms of $d$, the distance to the boundary in $\mathbb{D}$.

Proposition 3.3. Let $\gamma$ be a curve in $\mathbb{D}$ generating the Loewner pair $(f, W)$. Suppose there exists a non-decreasing function $\eta(\delta)$ such that for each $\delta > 0$, the tip structure modulus for $(\gamma(s), s \in [0, t])$ in $\mathbb{D}$ satisfies $\eta_{\text{tip}}(\delta) \leq \eta(\delta)$. Then if $d < 1/2$

$$|\gamma(t) - f_t((1 - d)W_t)| \leq \eta \left((2\pi A/\log 1/d)^{1/2}\right),$$

(3.4)

where $A$ may be chosen as $\min\{\pi(\text{diam} \, \gamma_{0,T})^2, \pi\}$.

Proof. The needed estimate is a classical result due to J. Wolff. We will give a short proof using extremal length. Consider $A = A(r, R) \cap \mathbb{D}$ centered around $W_t$, the pre-image of $\gamma(t)$ in $\partial \mathbb{D}$. Let $E$ and $F$ be the two boundary components of $A$ which are contained in $\partial \mathbb{D}$. By comparing with a half-annulus and mapping to a rectangle, using also the comparison principle for extremal length, we see that the extremal distance between $E$ and $F$ in $A$ is at most $\pi/\log(R/r)$. Hence, by conformal invariance and the definition of extremal length,

$$\frac{\pi}{\log(R/r)} \geq \frac{L^2}{A},$$

where $L$ is the euclidean length of the curve-family connecting $f(E)$ with $f(F)$ in $f(A)$ and $A$ is the euclidean area of $f(A)$. The number $A$ is clearly bounded above by the minimum of $\pi(\text{diam} \, \gamma_{0,T})^2$ and $\pi$. Consequently, by taking $r = d$ and $R = \sqrt{d}$ we see that there exists a crosscut $C'$ of $D_t$ separating $\gamma(t)$ and $z_d = f_t((1 - d)W_t)$ from 0 and the diameter of $C'$ is at
most \((2\pi A/(\log 1/d))^{1/2}\). We can now argue exactly as in the end of the proof of Proposition 3.2 with \(C\) replaced by \(C'\).

3.3. Hölder Regularity. We shall now see that the John-type condition \(\eta_{\text{tip}}(\delta) \leq A\delta\) forces a curve driven by a Hölder continuous function to be Hölder continuous in the capacity parameterization, with exponent depending only on \(A\) and the exponent for the driving term. We will derive a bound on the growth of the derivative as in (2.5) from the bound on \(\eta_{\text{tip}}\). Hölder regularity then follows from Proposition 2.2. The proof uses the length-area principle. The situation is different from the classical one; see, e.g., [26] or [20], in that our assumptions do not prevent large bottlenecks to form.

**Theorem 3.5.** Suppose that the radial Loewner pair \((f, e^{i\xi})\) is generated by a curve \(\gamma\). Assume that \(\xi\) is Hölder continuous and that there exist \(A < \infty\) and \(\delta_0 > 0\) such that the tip structure modulus for \((\gamma(t), t \in [0, T])\) in \(D\)
satisfies $\eta_{\text{tip}}(\delta) \leq A\delta$, $\delta < \delta_0$. Then $\gamma$ is Hölder continuous on $[0, T]$ with a Hölder exponent depending only on $A$ and the Hölder exponent for $\xi$.

**Remark.** A bound on the tip structure modulus alone cannot imply Hölder regularity of the path in the capacity parametrization; it is necessary to have some regularity of the driving term. Indeed, consider the chordal setting and take $\gamma$ to be the graph of $e^{-1/x}$, $x \in [0, 1]$. For this curve the tip structure modulus clearly decays linearly, uniformly in $t$. On the other hand, parameterize by half-plane capacity and note that there is a universal constant $c$ such that
\[ 2t = h_{\text{cap}}(\gamma[0, t]) \leq c \text{ height } \gamma[0, t] \cdot \text{diam } \gamma[0, t]. \]
(This follows, e.g., from a harmonic measure estimate.) Hence
\[ t/2 \leq ce^{-1/\gamma(t)}\gamma(t), \]
which shows that $\gamma$ is not Hölder continuous at $t = 0$. (By precomposing with slit map $\sqrt{z^2 - 4T}$ a similar example can be constructed with the “singularity” occurring at an arbitrary $T > 0$.) Moreover, if $W$ is the driving term for $\gamma$, then
\[ \text{diam } \gamma[0, t] \asymp \sqrt{t} + \sup_{s \in [0, t]} |W(s)|, \]
so $W$ is also not Hölder continuous. (In fact, a similar argument shows that if the driving term is Hölder-$\alpha$, $\alpha \leq 1/2$, at $t = 0$, then so is the curve.)

It is possible to take this example as a starting point to formulate a geometric condition that implies Hölder continuity for the driving term. We shall not, however, pursue this further here.

Before giving the proof of Theorem 3.5 we need a simple lemma.

**Lemma 3.6.** Let $f : \mathbb{D} \to D$ be a conformal map with $f(0) = 0$, $f'(0) > 0$. Define the Stolz cone
\[ S_r = \{1 - \rho e^{i\theta} : 0 \leq \rho \leq r, -\pi/4 \leq \theta \leq \pi/4\}, \]
set $z_r = f(1 - r)$. There is a universal constant $c < \infty$ such that
\[ \text{diam } f(S_r) \leq c \text{ diam } f(\sigma_r), \]
where $\sigma_r = [1 - r, 1]$ is the line segment connecting $1 - r$ and $1$.

**Proof.** Let $u = 1 - \rho e^{i\theta}$ be an arbitrary point in $S_r$. By Koebe’s distortion theorem there is a universal constant $c$ such that
\[ |f(u) - f(1 - \rho)| \leq c\rho |f'(1 - \rho)|. \]
Hence by Koebe’s estimate there is a universal constant $c'$ such that
\[ |f(u) - f(1 - \rho)| \leq c' \text{ dist}(f(1 - \rho), \partial D) \leq c' \text{ diam } f(\sigma_r), \]
and this concludes the proof. \qed
Proof of Theorem 3.5. Let \( t \in [0, T] \) and write \( W_t = e^{i\xi t} \). Without loss of generality we may assume that \( t > 0 \) and that \( W_t = 1 \). We suppress the dependence on \( t \) and write \( f \) for \( f_t \) and \( D \) for \( D_t \) etc. throughout the proof. Set \( z_r = f(1 - r) \) and \( \Delta_r = \text{dist}(z_r, \partial D) \). By Proposition 3.3 there is an \( r_0 \) depending only on \( A \) and \( \delta_0 \) such that \( \Delta_r \leq \delta_0 \) for all \( r \leq r_0 \). By taking \( r_0 \) smaller if necessary, depending only on \( T \), we can guarantee that the assumptions of Lemma 3.1 are satisfied so that there will exist a universal \( \rho_0 < \infty \) and a crosscut \( C \) contained in \( \partial B(z_r, \rho_0 \Delta_r) \) that separates \( z_r \) and \( \gamma(t) \) from 0 in \( D \). Let \( \sigma_r = [1 - r, 1] \). We claim that the hyperbolic geodesic \( f(\sigma_r) \) connecting \( z_r \) with \( \gamma(t) \) in \( D \) satisfies

\[
\text{diam} \ f(\sigma_r) \leq c\rho_0 A \Delta_r, \tag{3.5}
\]

where \( c \) is a universal constant. To prove this, note that since \( C \) separates \( \gamma(t) \) and \( z_r \) from 0, the hyperbolic geodesic \( f(\sigma_1) \supset f(\sigma_r) \) which connects \( \gamma(t) \) and 0 must intersect \( C \). (Since \( \gamma \) is a Loewner curve, \( \gamma(t) \) is always on the boundary of the simply connected domain \( D_t \ni 0 \).) Moreover, by the bound on the structure modulus, there is a curve \( \Gamma \) connecting \( \gamma(t) \) with \( C \) in \( D_t \) and

\[
\text{diam} \ \Gamma \leq 2A \text{diam} \ C \leq 4\rho_0 A \Delta_r.
\]

The Gehring-Hayman theorem; see, e.g., Chapter 4 of [20], now implies that there is a universal constant \( c_0 \) such that

\[
\text{diam} \ f(\sigma_r) \leq \text{diam} \ f(\sigma_1) \leq c(\text{diam} \ \Gamma + \text{diam} \ C)
\]

and this gives (3.5).

Using Lemma 3.6, the reminder of the proof now proceeds by a length-area type argument in a form used in Chapter 5 of [20]. Define

\[
\varphi(r) = \int_0^r |f'(1 - r)|^2 r \, dr.
\]

Then by Koebe’s distortion theorem there is a universal constant \( c_0 \) such that

\[
 r^2 |f'(1 - r)|^2 \leq c_0 \int_{r/2}^r r |f'(1 - r)|^2 dr \leq c_0 \varphi(r). \tag{3.6}
\]

This theorem also implies that there is a constant \( c_1 \) depending only on \( c_0 \) such that

\[
 \varphi(r) \leq c_1 \int_0^r \int_{-\pi/4}^{\pi/4} |f'(1 - re^{i\theta})|^2 r \, dr \, d\theta = c_1 \text{area} \ f(S_r),
\]

where \( S_r \) is the Stolz cone defined in the statement of Lemma 3.6. Now, by (3.5) and Lemma 3.6 we have that

\[
\text{area} \ f(S_r) \leq \frac{\pi^2}{4} (\text{diam} \ f(S_r))^2 \leq c_2 \Delta_r^2.
\]

Hence

\[
\varphi(r) \leq c_1 \text{area} \ f(S_r) \leq c_3 r^2 |f'(1 - r)|^2.
\]
Consequently, since $\varphi'(r) = r|f'(1-r)|^2$, we have for $r_0 > r$ and a constant $c_4$ depending only on $A$

$$\log \left( \frac{\varphi(r_0)}{\varphi(r)} \right) = \int_r^{r_0} \frac{\varphi'(r)}{\varphi(r)} \, dr \geq c_4^{-1} \log \left( \frac{r_0}{r} \right).$$

Taking exponentials, using (3.6), gives for $0 < r \leq r_0$

$$r^2 |f'(1-r)|^2 \leq c_5 r^{1-c_4},$$

where $c_5$ depends only on $r_0$. Hence if $\beta = 1 - 1/(2c_4) < 1$ we see that

$$r|f'(1-r)| \leq c_6 r^{1-\beta},$$

and by Proposition 2.2 this implies Hölder regularity with every exponent smaller than the minimum of $1/2$ and $(1 - \beta)h$, where $h$ is the exponent for $W$. □

4. Loop-Erased Random Walk and SLE

This section proves a convergence rate result for loop-erased random walk using the setup detailed in the previous sections.

4.1. Definitions. The radial Schramm-Loewner evolution, radial SLE$_\kappa$, is defined by taking $W(t) = e^{\sqrt{\kappa}B(t)}$ as driving term for the radial Loewner equation, where $B$ is standard Brownian motion. It is a fact that this Loewner chain is almost surely generated by a curve – the SLE$_\kappa$ path. This is a random fractal curve which is simple when $0 \leq \kappa \leq 4$, has double points when $4 < \kappa$ and is space filling when $\kappa \geq 8$. See [21] for proofs of these facts. In Appendix A we discuss a derivative estimate for radial SLE$_\kappa$ that we will state and use in this section when $\kappa = 2$. For technical reasons we use a stopping time $\sigma$ for the radial SLE path $\tilde{\gamma}$ further discussed in Appendix A.

Fix a small constant $\varepsilon > 0$. We then define

$$\sigma = \sigma(\varepsilon, T) = \inf \{ t \geq 0 : |g_t(-1) - W(t)| \leq \varepsilon \} \wedge T,$$

where $g_t = f_t^{-1}$ is the forward Loewner SLE$_2$ flow and $W(t)$ is the driving term for $g_t$.

**Proposition 4.1.** Let $\varepsilon > 0$ and $T < \infty$ be fixed and let $(f_s)$, $0 \leq s \leq \sigma$, be the stopped radial SLE$_2$ Loewner chain with $\sigma = \sigma(\varepsilon, T)$ defined by (4.1). For every $\beta \in (2(\sqrt{10} - 1)/9, 1)$ there exists a constant $c = c(\beta, \varepsilon, T) < \infty$ such for all $d_* < 1$

$$\mathbb{P} \left\{ \forall d \leq d_*, \sup_{s \in [0, \sigma]} d|f_s'(1-d)W(s)| \leq d^{1-\beta} \right\} \geq 1 - cd_*^{q(\beta)}.$$

where

$$q(\beta) = -1 + 2\beta + \frac{\beta^2}{4(1+\beta)}.$$
Proof. See Appendix A.

Let $D \ni 0$ be a simply connected domain and assume that the inner radius of $D$ with respect to 0 equals 1. We shall consider a particular discretization of $D$. A grid-domain with respect to $n^{-1}\mathbb{Z}^2$ is a simply connected domain whose boundary is a subset of the edge set of the graph $n^{-1}\mathbb{Z}^2$. We define $D_n = D_n(D)$, the $n^{-1}\mathbb{Z}^2$ grid-domain approximation of $D$, as the component of 0 of $\mathbb{C}$ minus those closed $n^{-1}\mathbb{Z}^2$ lattice faces that intersect $\partial D$. Then clearly $D_n$ is a grid-domain contained in $D$. Let $\psi_n : D_n \rightarrow \mathbb{D}$ be the normalized conformal map. We will assume, for simplicity, that $D$ is a Jordan domain with $C^{1+\alpha}$ boundary, where $\alpha > 0$.

Suppose $S = S(j), j = 0, 1, \ldots$, is a finite nearest-neighbor walk on (the vertices of $n^{-1}\mathbb{Z}^2$ contained in) $D_n$. We define the loop-erasure $\mathcal{L}(S) \subset S$ in the following way. If $S$ is already self-avoiding, set $\mathcal{L}(S) = S$. Otherwise, let $s_0 = \max\{ j : S(j) = S(0) \}$, and for $i > 0$, let $s_i = \max\{ j : S(j) = S(s_{i-1} + 1) \}$. If we let $n = \min\{ i : s_i = m \}$, then $\mathcal{L}(S) = \{ S(s_0), S(s_1), \ldots, S(s_n) \}$. Notice that $\mathcal{L}(S)(0) = S(0)$ and $\mathcal{L}(S)(s_n) = S(m)$, that is, the loop-erased walk has the same end points as the original walk $S$. Loop-erased random walk (LERW) from 0 to $\partial D_n$ in $D_n$ is the random self-avoiding walk $\gamma_n$ obtained by taking $S$ to be a simple random walk on $n^{-1}\mathbb{Z}^2$ started from 0 and stopped when reaching $\partial D_n$, and then setting $\gamma_n = \mathcal{L}(S)$. For a nearest-neighbor walk $S$, let $S^R$ be the time-reversed walk. It is known that LERW has the following symmetry with respect to time-reversal: The distribution of $(\mathcal{L}(S))^R$ is equal to that of $\mathcal{L}(S^R)$. Sometimes it is more convenient to consider $\mathcal{L}(S^R)$, and when we do we will call it the time-reversal of LERW and usually assume that the path is traced from the boundary towards 0.

4.2. Convergence Rate for the LERW Path. Lawler, Schramm, and Werner proved in [15] that, as $n \rightarrow \infty$, the image of the time-reversal of LERW path in $\mathbb{D}$, $\psi_n(\mathcal{L}(S^R))$, traced from $\partial D$ towards 0, converges weakly with respect to a natural metric on curves modulo increasing reparameterization towards the radial SLE$_2$ path started uniformly on $\partial D$. (See Theorem 3.9 of [15] for a precise statement.) The goal of this section is to prove Theorem [4.3], which is can be viewed as a quantitative version of Theorem 3.9 of [15].

Let $D$ be a simply connected $C^{1+\alpha}$ domain with grid domain approximation $D_n = D_n(D)$ as above. Let $\gamma_n$ be the time-reversal of LERW on $n^{-1}\mathbb{Z}^2$ from 0 to $\partial D_n$ and let $\bar{\gamma}_n = \psi_n(\gamma_n)$ be its image in $\mathbb{D}$ traced from the boundary and parameterized by capacity. (Since $\gamma_n$ is a simple curve that intersects $\partial D_n$ at only one point it follows that $\bar{\gamma}_n$ is a $\mathbb{D}$-Loewner curve for each $n$.) Let $W_n(t)$ be the Loewner driving term for $\bar{\gamma}_n$. Fix $s \in (0, 1/24)$, and define

$$
\varepsilon_n = n^{-s}.
$$
Theorem 4.2. For every $T > 0$ there exists $n_0 = n_0(T, s) < \infty$ such that the following holds. For each $n \geq n_0$ there is a coupling of $\gamma_n$ with Brownian motion $B(t)$, $t \geq 0$, where $e^{iB(0)}$ is uniformly distributed on the unit circle, with the property that
\begin{equation}
\mathbb{P}\left\{ \sup_{t \in [0,T]} |W_n(t) - W(t)| > \varepsilon_n \right\} < \varepsilon_n,
\end{equation}
where $W(t) = e^{iB(2t)}$.

Theorem 4.3. There exists $n_1 = n_1(\varepsilon, T, s) < \infty$ such that if $n \geq n_1$, then in the coupling of Theorem 4.2, if $\tilde{\gamma}$ denotes the radial SLE$_2$ path in $\mathbb{D}$ driven by $W$,
\begin{equation}
\mathbb{P}\left\{ \sup_{t \in [0,\sigma]} |\gamma_n(t) - \tilde{\gamma}(t)| > \varepsilon_n^m \right\} < \varepsilon_n^m,
\end{equation}
where both curves are parameterized by capacity,
\begin{equation*}
m = 1/41,
\end{equation*}
and $\sigma = \sigma(\varepsilon, T)$ is the stopping time defined by (4.1).

Remark. The proof of Theorem 4.3 (with minor modifications) would also work under the weaker assumption that $D$ is a quasidisk. (The class of quasidisks includes, e.g., the von Koch snowflake.) In this case the rate would depend on the constant in the Ahlfors three-point condition satisfied by $\partial D$; see Appendix B. We may also note that the conclusion (and proof) of Theorem 4.3 holds true in any coupling like the one of Theorem 4.2, with the proviso that $\varepsilon_n$ decays slower than $n^{-1/2}$.

Remark. By Lemma 4.7 the preimages of the curves (parameterized by capacity) in $D_n$ satisfy a similar estimate as in (4.3), namely,
\begin{equation*}
\mathbb{P}\left\{ \sup_{t \in [0,\sigma]} |\gamma_n(t) - \psi_n^{-1}(\tilde{\gamma}(t))| > \varepsilon_n^m \right\} < \varepsilon_n^m, \quad m = 1/41.
\end{equation*}

Remark. The coupling(s) of $W_n = e^{i\theta_n}$ and $W = e^{iB}$ in Theorem 4.2 are via Shorokhod embedding of $\theta_n$ into $B$.

In order to apply the work from previous sections we need to verify that the assumptions of these results hold with large probability. In Section 4.3 we will first estimate the probability of the event of the existence of a certain power-law bound for the tip structure modulus for the LERW path in $D_n$. We show in Section B that if $\partial D$ is sufficiently smooth ($C^{1+\alpha}$), then the image of the LERW path in $\mathbb{D}$ enjoys the same tip structure modulus up to constants. This uses a convergence rate result for grid domain approximations of quasidisks that we show follows from a result of Warshawski’s. In Appendix A we derive the needed estimate on the derivative of the SLE$_2$ conformal maps. These result are combined to prove Theorem 4.3 in Section 4.5.
4.3. Tip Structure Modulus for LERW in a Grid Domain. An important tool to get quantitative estimates for LERW is the Beurling estimate for simple random walk; see, e.g., [13]. There are many ways to formulate this result and we state only one version here.

**Lemma 4.4.** There exists a constant \(c < \infty\) such that the following holds.

Let \(A \subset \mathbb{Z}^2\) be an infinite connected set. Let \(S\) be simple random walk on \(\mathbb{Z}^2\) started from \(z\) and stopped at the time \(\tau_A\) at which \(S\) hits \(A\). Then for \(r > 1\)

\[
P\{|S(\tau_A) - z| \geq r \text{dist}(z, A)|\} \leq cr^{-1/2}.
\]

We can now formulate the main estimate.

**Proposition 4.5.** Let \(D_n\) be a grid domain with respect to \(n^{-1}\mathbb{Z}^2\) and assume that \(\text{inrad}(D_n) = 1\) and \(\text{diam} D_n \leq R < \infty\), where \(R\) is given. Let \(\gamma_n\) be the time-reversal of loop-erased random walk from 0 to \(\partial D_n\). Let \(\eta^{(n)}_\text{tip}\) be the tip structure modulus for \(\gamma_n\) in \(D_n\). There exists a universal constant \(c_0 < \infty\) and a constant \(c = c(R, r) < \infty\) such that if \(n\) is sufficiently large and \(\delta > c_0/n\), then

\[
P\{\eta^{(n)}(\delta) \leq \delta^r\} \geq 1 - c\delta^{1/5 - 11r/5} |\log \delta|.
\]

**Remark.** When we apply Proposition 4.5 we will choose \(\delta = \delta(n) \in \omega(n^{-1})\) (in the sense of Landau notation) so that \(\delta > c_0/n\) is automatically satisfied for \(n\) sufficiently large.

**Remark.** The Beurling estimate implies that there is a constant \(c < \infty\) such that

\[
P\{\text{diam } \gamma_n > R\} \leq cR^{-1/2}
\]

for large \(R\). This means that one can formulate and prove Proposition 4.5 with an estimate which is independent of the diameter of \(D_n\). (We still need to normalize in some way the inner radius with respect to the origin.)

4.4. Proof of Proposition 4.5. This section is devoted to the proof of Proposition 4.5 and towards the end the statement of a consequence that we will prove in Appendix 13. Although the result was formulated for the time-reversal of LERW, in the proof we shall consider the LERW generated by erasing the loops of simple random walk from 0 to \(\partial D_n\) (without the time-reversal). By time-reversal symmetry, this is sufficient.

The strategy of the proof is based on that of the proof of Lemma 3.4 in [22], but see also the related Lemma 3.12 of [15]. Let \(w\) be a fixed point in \(D_n\). Let \(A = A(w; \delta, \eta) = \{z : \delta < |z - w| < \eta\}\) be the \((\delta, \eta)\)-annulus about \(w\). Let \(\gamma\) be a curve in \(D_n\). We say that \(\gamma\) has a \(k\)-crossing of \(A\) if the number of components of \(\gamma \cap A\) that connects to two boundary components of \(A\) is at least \(k\). We equivalently say that \(\gamma\) has a \((\delta, \eta)\) \(k\)-crossing at \(w\), and that \(\gamma\) has a \((\delta, \eta)\) \(k\)-crossing if there exists a \(w\) such that \(\gamma\) has a \((\delta, \eta)\) \(k\)-crossing at \(w\). Recall that \(\eta(\delta)\) is a bound for the tip structure modulus
for \( \gamma \) in \( D_n \) if and only if for all \( t \), \( \gamma[0,t] \) has no nested \((\delta, \eta(\delta))\)-bottleneck in \( D_n \). Now let \( \gamma_n \) be the LERW path in \( D_n \) traced from \( \partial D_n \) towards \( 0 \). Consider the event that there is a nested \((\delta,2\eta)\)-bottleneck in \( \gamma_n \). This event is contained in the union of the following two events:

\[
E_5 = \{ \text{There is a } w \in D_n \text{ such that } \gamma_n \text{ has a } (\delta, \eta) \text{-crossing at } w \} \\
E_B = \{ \text{The random walk generating } \gamma_n \text{ comes within distance } \delta \text{ from } \partial D_n \text{ and then travels more than distance } \eta \text{ before hitting } \partial D_n \}
\]

To see this, suppose that a nested \((\delta, 2\eta)\)-bottleneck occurs with the crosscut \( C \) contained in the disk \( B(w, \delta) \), \( w \in \overline{D_n} \). If \(|w| > \eta\) while \( \text{dist}(w, \partial D_n) > \eta \), then a 6-crossing of \( A(w; \delta, \eta) \) must occur since \( \gamma_n \) connects \( \partial D_n \) with \( 0 \). However, if \(|w| \leq \eta\) or \( \delta < \text{dist}(w, \partial D_n) \leq \eta \) then only a 5-crossing of \( A(w; \delta, \eta) \) is implied. These events are contained in \( E_5 \). Close to the boundary, that is, if \( \text{dist}(w, \partial D_n) \leq \delta \), a nested bottleneck can form, roughly speaking, with the “help” of the boundary. Depending on \( \partial D_n \) this can happen in several ways; for example, the curve might enter a thin channel, pass a thin bottleneck, or a “bad” 4-crossing of \( A(w; \delta, \eta) \) might occur. In any configuration like this, the event \( E_B \) occurs.

We will estimate the probabilities of the two events, starting with the last. In this case the Beurling estimate immediately implies that there is a
constant $c < \infty$ such that
\[ \mathbb{P}\{E_B\} \leq c \left( \frac{\delta}{\eta} \right)^{1/2}. \]  
(4.5)

We proceed to bound $\mathbb{P}\{E_5\}$. Fix a point $w \in D_n$. Set
\[ d_0 = \text{dist}(w, \partial D_n) > 0 \]
and define
\[ B_1 = B(w, \eta/4), \quad B_2 = B(w, \eta/2). \]
For a curve $\gamma \subset D_n$, we let $Q'(\gamma; w, \delta, \eta)$ denote the event that $\gamma$ has a 3-crossing of a $(\delta, \eta)$-annulus whose smaller boundary component is contained in $B_1$. Similarly, let $Q''(\gamma; w, \delta, \eta)$ denote the event that $\gamma$ has a 5-crossing of a $(\delta, \eta)$-annulus whose smaller boundary component is contained in $B_1$. Clearly, the latter event is contained in the former. We will first estimate the probability of
\[ Q'' := Q''(\gamma_n; w, \delta, \eta). \]
Let $S(t) = S_n(t)$, $t = 0, 1, \ldots, \tau$ be the simple random walk generating $\gamma_n$; it is started from 0 and stopped at
\[ \tau = \min \{ t \geq 0 : S(t) \in \partial D_n \}. \]
when $\partial D_n$ is hit. Define
\[ s_1 = \min \{ t \geq 0 : S(t) \in B_1 \}, \quad t_1 = \min \{ t > s_1 : S(t) \notin B_2 \}, \]
and recursively for $j = 2, 3, \ldots$
\[ s_j = \min \{ t > t_{j-1} : S(t) \in B_1 \}, \quad t_j = \min \{ t > s_j : S(t) \notin B_2 \}. \]
Note that we have $s_1 = 0$ if $|w| \leq \eta/4$ and $s_1 > 0$ otherwise. We will write
\[ Q''_j := Q''(\mathcal{L}\{S[0, t_j]\}; w, \delta, \eta), \quad Q'_j := Q'(\mathcal{L}\{S[0, t_j]\}; w, \delta, \eta). \]
Clearly, $Q''_j \subset Q'_j$, but it does not necessarily hold that $Q''_{j+1} \subset Q'_j$ or $Q'_{j+1} \subset Q''_j$ because part of the curve forming a crossing may be erased. Note that for $m \geq 1$
\[ \mathbb{P}\{Q''\} \leq \mathbb{P}\{\tau > t_{m+1}\} + \mathbb{P}\left( \bigcup_{j=1}^m Q''_j \right). \]
We estimate $\mathbb{P}\{\tau > t_{m+1}\}$ in Lemma 4.6 below.

We have
\[ \mathbb{P}\left( \bigcup_{j=1}^m Q''_j \right) \leq \sum_{j=1}^m \mathbb{P}\left( Q''_j, \neg Q''_{j-1} \right). \]
To get the last estimate we split the event on the left-hand side according to the first time a 5-crossing occurs; here and in the sequel, for an event $A$ the symbol “$\neg A$” means the complement of $A$. To bound $\mathbb{P}(Q''_j, \neg Q''_{j-1})$ let us first discuss the analogous quantity for a 3-crossing. In the proof of Lemma 3.4 of [22] the estimate
\[ \mathbb{P}\left( Q'_j | \neg Q'_{j-1}, S[0, t_{j-1}] \right) \leq c(j - 1) \left( \frac{\delta}{\eta} \right)^{1/2} \]
was (essentially) given and as a consequence the estimate
\[ P\left(Q_j, -Q'_{j-1}\right) \leq P\left(Q'_j \mid -Q'_{j-1}\right) \leq c(j-1) \left(\frac{\delta}{\eta}\right)^{1/2} \] (4.6)
follows. The exponent in the right-hand side of (4.6) was not specified in [22]. However, it follows from the Beurling estimate that one can take the exponent to be 1/2. Indeed, the proof of (4.6) is as follows: Let \( \{C_k\}_k \) be the components of \( L\{S[0, s_j]\} \cap B_2 \) intersecting \( B_1 \) but not containing \( S(s_j) \). There are at most \( j-1 \) such components. Conditionally on \( S[0, t_{j-1}] \), if \( L\{S[0, t_j]\} \) is to contain a 3-crossing which was not there in \( L\{S[0, t_{j-1}]\} \), then \( S[s_j, t_j] \) has to come within distance \( \delta \) of \( C_k \cap B_1 \) for some \( k \) and then exit \( B_2 \) without hitting that same \( C_k \). (It may hit other components.) For each component, we can use the strong Markov property and the Beurling estimate to see that this conditional probability is bounded by \( c(\delta/\eta)^{1/2} \) and there are at most \( j-1 \) such components. We can then sum to get
\[ P\left(Q'_j \right) \leq \sum_{k=1}^{j} P\left(Q'_k, -Q'_{k-1}\right) \leq cj^2 \left(\frac{\delta}{\eta}\right)^{1/2} \]
Returning to \( P\left(Q'_j, -Q''_{j-1}\right) \) we have, noticing that \( (Q'_j \cap -Q''_{j-1}) \subset Q'_{j-1} \),
\[ P\left(Q''_j, -Q''_{j-1}\right) = P\left(Q'_j, -Q''_{j-1} \mid Q'_j\right) P\left(Q'_j\right) \leq cP\left(Q''_j, -Q''_{j-1} \mid Q'_{j-1}\right) j^2 \left(\frac{\delta}{\eta}\right)^{1/2} \]
We continue to write
\[ P\left(Q''_j, -Q''_{j-1} \mid Q'_{j-1}\right) \leq P\left(Q''_j \mid -Q''_{j-1}, Q'_{j-1}\right). \]
We can estimate the last expression by using that
\[ P\left(Q''_j \mid -Q''_{j-1}, Q'_{j-1}, S[0, t_{j-1}]\right) \leq c(j-1) \left(\frac{\delta}{\eta}\right)^{1/2}. \]
Indeed, this estimate is proved in exactly the same way as (4.6) using the Beurling estimate.

Combining our bounds we get
\[ P\left(\cup_{j=1}^{m} Q''_j\right) \leq cm^4 \frac{\delta}{\eta}. \] (4.7)
We now take \( \nu > 0 \) and let \( m = \lfloor \delta^{-\nu} \rfloor \). We then use Lemma [4.6] (here we write the estimate for \( d_0 > \eta/4 \); in the case \( d_0 \leq \eta/4 \) we use the second bound of Lemma [4.6]) to get
\[ P(Q'') \leq \left(1 - \frac{c_3}{|\log(16d_0/\eta)|}\right)^{\lfloor \delta^{-\nu} \rfloor} + c \frac{\delta^{1-4\nu}}{\eta} \leq c \delta^\nu |\log(16d_0/\eta)| + c \frac{\delta^{1-4\nu}}{\eta}. \] (4.8)
This bound is for a fixed \( w \). To conclude, note that there is a universal \( c < \infty \) such that we can (deterministically) cover \( D_n \) using at most \( cR^2\eta^{-2} \) overlapping disks \( B(w_k, \eta/4) \) in such a way for every \( w \) such that \( \gamma_n \) has a 5-crossing of \( A(w; \delta, \eta) \), the smaller boundary component of \( A(w; \delta, \eta) \) is contained in \( B(w_k, \eta/4) \) for some \( k \). Consequently, for \( c = c(R) < \infty \),

\[
P(\mathcal{E}_5) \leq c\eta^{-2}\delta^\nu |\log(16d_0/\eta)| + c\eta^{-3}\delta^{1-4\nu}. \tag{4.9}
\]

For any \( r \in (0, 1/11) \), if \( \eta = \delta^r \), we can take \( \nu = (1-r)/5 \) in \( 4.9 \) which makes both terms in the bound of the same (“polynomial”) order so that the right-hand side of \( 4.9 \) decays like \( \delta^{1/5-11r/5} \) with a logarithmic correction. Since this term is always larger than the one coming from \( \mathcal{E}_B \), this concludes the proof of Proposition 4.5\footnote{assuming Lemma 4.6}.

**Lemma 4.6.** There exist constants \( 0 < c_1, c_2 < 1 \) such that

\[
P\{\tau > t_{m+1}\} \leq \begin{cases} 
(1 - \frac{c_1}{|\log(16d_0\eta^{-1})|})^m & \text{if } d_0 > \eta/4; \\
(1 - c_2)^m & \text{if } d_0 \leq \eta/4.
\end{cases}
\]

**Proof.** We first assume that \( d_0 > \eta/4 \). Using, e.g., Proposition 6.4.1 of [13] we see that the probability that a simple random walk started just outside of \( B_2 \) exits \( B(z_0, 8d_0) \) before hitting \( B_1 \) is bounded below by

\[
\frac{|\log 2| - O((\eta\eta)^{-1})}{|\log(16d_0\eta^{-1})|} \geq \frac{|\log 2|}{2|\log(16d_0\eta^{-1})|}
\]

if \( \eta \eta > c_1 \), where \( c_1 < \infty \) is a universal constant. (This uses also that \( d_0 > \eta/4 \).) This estimate is a discrete version of the expression for the harmonic measure of one of the boundary components in an annulus. Moreover, there is a universal constant \( c > 0 \) such that the probability that simple random walk from \( \partial B(z_0, 8d_0) \) separates \( B(z_0, d_0) \) from \( \infty \) before hitting \( B(z_0, d_0) \) is bounded below by \( c \). (Recall that our assumptions imply that \( d_0 > c'/n \), where we can assume that \( c' \) is large.) Consequently, by the strong Markov property the probability that simple random walk started from \( \partial B_2 \) exits \( D_n \) before hitting \( B_1 \) is bounded below by \( c_1/|\log(16d_0\eta^{-1})| \). By iterating this argument using the strong Markov property,

\[
P\{\tau > t_{m+1}\} \leq \left(1 - \frac{c_1}{|\log(16d_0\eta^{-1})|}\right)^m. \tag{4.10}
\]

When \( d_0 \leq \eta/4 \) the Beurling estimate and the Markov property directly show that the right-hand side of \( 4.10 \) can be replaced by \( (1 - c_2)^m \), where \( c_2 > 0 \) is a universal constant. \( \square \)

If the boundary of the domain \( D \) that is being approximated is sufficiently regular, then the structure modulus on a sufficiently large mesoscopic scale for the image curve in \( \mathbb{D} \) is essentially the same as the one in \( D_n \). The next lemma, proved in Appendix [13] makes this precise.
Lemma 4.7. Suppose $D \ni 0$ is a simply connected domain Jordan domain with $C^{1+\alpha}$ boundary, where $\alpha > 0$. Let $D_n$ be the $n^{-1/2}$ grid domain approximation of $D$ and let $\gamma_n$ be a Loewner curve in $D_n$ connecting $\partial D_n$ with 0. There is a constant $C$ depending only on $\alpha$ and the diameter of $D$ such that the following holds. Set $0 < r < 1/2$ and $d_n = n^{-r}$ and let $\eta^{(n)}_{t_{\text{tip}}}(\delta; D_n)$ be the tip structure modulus for $\gamma_n$ in $D_n$. Then for all $n$ sufficiently large the tip structure modulus $\eta^{(n)}_{t_{\text{tip}}}(\delta; \mathbb{D})$ for $\psi_n(\gamma_n)$ in $\mathbb{D}$ satisfies

$$\eta^{(n)}_{t_{\text{tip}}}(c^{-1}d_n; \mathbb{D}) \leq c\eta^{(n)}_{t_{\text{tip}}}(d_n; D_n).$$

4.5. Proof of Theorem 4.3 We write $\gamma$ for the radial SLE$_2$ path in $\mathbb{D}$ corresponding to the Brownian motion in (4.2). We thus have a coupling of the radial SLE$_2$ and the image of the LERW path $\tilde{\gamma}_n$ and we will estimate the distance between these curves in this coupling. Take $s \in (0, 1/24)$ and $n > n_0$ where $n_0$ is as in Theorem 4.2 fix $\rho > 1$ and for $p \in (0, 1/\rho)$, let

$$\varepsilon_n = n^{-s}, \quad d_n = (\varepsilon_n)^p.$$  

For each $n \geq n_0$, we shall define three events each of which occurs with large probability in our coupling. On the intersection of these events we can apply our estimates from Sections 2 and 3.

(a) Let $A_n = A_n(s)$ be the event that the estimate

$$\sup_{t \in [0, T]} |W_n(t) - W(t)| \leq \varepsilon_n$$

holds. By Theorem 4.2 we know that there exists $n_0 < \infty$ such that if $n \geq n_0$ then

$$\mathbb{P}(A_n) \geq 1 - \varepsilon_n.$$  

(b) For $\beta \in (2(\sqrt{10} - 1)/9, 1)$, let $B_n = B_n(s, r, \beta, \varepsilon, T, c_B)$ be the event the radial SLE$_2$ Loewner chain $(f_t)$ driven by $W(t)$ satisfies the estimate

$$\sup_{t \in [0, \sigma]} d|f'(t, (1 - d)W(t))| \leq c_B d^{1-\beta}, \quad \forall d \leq d_n.$$  

(Recall that $\varepsilon, T$ were used in the definition of the stopping-time $\sigma \leq T$.) Then by Proposition 4.1 there exist $c_B < \infty$, independent of $n$, and $n_1 < \infty$ such that if $n \geq n_1$ then

$$\mathbb{P}(B_n) \geq 1 - c_B' d_n^{q_2(\beta)},$$  

where

$$q_2(\beta) = -1 + 2\beta + \frac{\beta^2}{4(1 + \beta)}.$$  

(c) For $r \in (0, 1/11)$, let $C_n = C_n(s, r, p, c_C, \alpha, \text{diam } D)$ be the event that the tip structure modulus for $\tilde{\gamma}_n$ in $\mathbb{D}$, $\eta^{(n)}_{t_{\text{tip}}}$, satisfies

$$\eta^{(n)}_{t_{\text{tip}}}(d_n) \leq c_C d_n^r.$$
We know from Proposition 4.5 and Lemma 4.7 that there exist $c_C, c'_C < \infty$, independent of $n$, and $n_2 < \infty$ such that if $n \geq n_2$ then
\[ \mathbb{P}(C_n) \geq 1 - c'_C d_n^{1/5-11r/5} |\log d_n|. \]
Consequently, there exist $c_B, c_C < \infty$ and $c < \infty$, all independent of $n$ (but depending on $s, r, p, \varepsilon, T, \beta, \alpha, \text{diam } D$), such that for all $n$ sufficiently large,
\[ \mathbb{P}(A_n \cap B_n \cap C_n) \geq 1 - c (\varepsilon_n + d_n^{q_2(\beta)} + d_n^{1/5-11r/5} |\log d_n|), \quad (4.11) \]
and on the event $A_n \cap B_n \cap C_n$ we can apply Lemma 3.4 with constants $c = c_C, c' = c_B$ independent of $n$ to see that there exists $c$ independent of $n$ (but depending on the above parameters) such that for all $n$ sufficiently large,
\[ \sup_{t \in [0, \sigma]} |\tilde{\gamma}_n(t) - \tilde{\gamma}(t)| \leq c (d_n^{p_1(1-\beta)} + \varepsilon_n^{1-(\rho p)r}). \quad (4.12) \]
We now wish to optimize over the parameters in the exponents. Since $d_n = \varepsilon_n^{p}$ we see that $d_n^{p(1-\beta)}$ dominates in (4.12) when $p \in (0, 1/(1 + \rho - \beta)]$ and $\varepsilon_n^{r(1-\rho p)}$ whenever $p \in [1/(1 + \rho - \beta), 1]$. Suppose $p \in (0, 1/(1 + \rho - \beta)]$.

Set
\[ \mu(\beta, r) = \min \left\{ r(1-\beta), -1 + 2\beta + \frac{\beta^2}{4(1+\beta)}, \frac{1}{5} - \frac{11r}{5} \right\}. \]
The optimal rate is given by optimizing $\mu$ over $\beta, r$ and then choosing $p$ very close to $1/(1 + \rho - \beta)$. (No improvement is obtained by considering $p \in [1/(1 + \rho - \beta), 1]$.) Let $\beta_* \in (2(\sqrt{10} - 1)/9, 1)$ solve
\[ 45\beta^3 - 128\beta^2 - 84\beta + 68 = 0. \]
(One can check that $\beta_* = 0.497 \ldots$) Then if $r_* = 1/(16 - \beta_*) \in (0, 1/11)$
\[ \mu(r_*, \beta_*) = \max \left\{ \mu(\beta, r) : \frac{2(\sqrt{10} - 1)}{9} < \beta < 1, 0 < r < \frac{1}{11} \right\} = 0.037 \ldots \]
Consequently, for every
\[ m < m_* = \frac{\mu(r_*, \beta_*)}{2 - \beta_*}, \]
we obtain bounds in (4.11) and (4.12) of order $\varepsilon_n^m$ for all $n$ sufficiently large. Since $1/41 < m_* = 0.024 \ldots$, this concludes the proof. \( \square \)

**Appendix A. Derivative Estimate for Radial SLE**

This section proves a derivative estimate for both chordal and radial SLE. The radial case was needed in Section 4 in the case $\kappa = 2$. The chordal case is a direct consequence of an estimate from [3], but the radial case requires a little bit of work. In this case, our goal is to estimate explicitly in terms of $d_*$ and $\beta$ the probability of the event that when $(f(t, z))$ is the radial SLE$_\kappa$ Loewner chain, the estimate $d |f'(t, (1-d)W(t))| \leq c d^{1-\beta}$ for all $d \leq d_*$ holds
uniformly in $t \in [0, T]$. This follows without much trouble from a moment estimate for the chordal reverse flow in [3]. The only issue is the change of coordinates from radial to chordal SLE which we begin by discussing. See also Section 7 of [4] where a similar but non-equivalent situation is dealt with. We will use ideas from [24].

A.1. Change of Coordinates. Let $(f_s, W_s)$ be a radial Loewner pair generated by the curve $\gamma(s)$ with $W_s$ continuous. Recall that $f_s : \mathbb{D} \to \mathbb{D} \setminus D_s$ and that $K_s$ is the hull generated by $\gamma[0, s]$. Let $g_s = f_s^{-1}$ and set $z_s = g_s(-1) W_s$. We will need to keep track of the “disconnection time” $\sigma'$ when $K_s$ first disconnects $-1$ from $0$ in $\mathbb{D}$, in other words, the first time that $z_s$ hits $1$. Fix $\varepsilon > 0$ small and $T < \infty$, and define

$$\sigma = \sigma(\varepsilon, T) = \inf \{ s \geq 0 : |1 - z_s| \leq \varepsilon \} \land T.$$ 

Clearly, $\sigma < \sigma'$. 

**Lemma A.1.** There exists a constant $c = c(\varepsilon, T) > 0$ such that

$$\inf_{s \in [0, T]} |g'_s\vee\sigma(-1)| \geq c.$$

**Proof.** The Loewner equation implies that with $z_s$ as above,

$$|g'_s(-1)| = \exp \left\{ \int_0^s \Re \frac{2}{(1 - z_s)^2} - 1 \, ds \right\}.$$

This shows that $|g'_s(-1)|$ is strictly decreasing in $s$ and that $|g'_{T\vee\sigma}(-1)| \geq c = c(\varepsilon, T) > 0$. □

**Remark.** Note that if $g_s$ is the radial SLE$_\kappa$ forward flow, and if $\theta_s := -i \log z_s = -i \log g_s(-1) - \sqrt{\kappa} B_s$, $\theta_0 = \pi$, then by Itô’s formula,

$$d\theta_s = \cot(\theta_s/2) \, ds - \sqrt{\kappa} dB_s.$$

If $\kappa < 4$, then it follows from [12, Lemma 1.27] that almost surely $\theta_s$ does not hit $\{0, 2\pi\}$ in finite time. Hence for each $T < \infty$, if $\kappa < 4$, then almost surely,

$$\lim_{\varepsilon \to 0} \sigma(\varepsilon, T) = T.$$

Consider now the Mobius transformation

$$\varphi : \mathbb{H} \to \mathbb{D}, \quad \varphi(z) = \frac{i - z}{i + z}.$$ 

Clearly $\varphi^{-1} \circ \gamma$ is a curve in $\mathbb{H}$ (for sufficiently small $s$) and for $s \geq 0$ we define

$$t(s) := \text{hcap}(\varphi^{-1}(\gamma[0, s]))/2.$$ 

For each $s \in [0, \sigma]$ let $F_t(s) : \mathbb{H} \to H_t(s) := \varphi^{-1}(D_s)$ be the conformal mapping satisfying the hydrodynamical normalization $F_t(s)(z) = z - 2t(s)/z +$
We have proved the following result.

In fact, by expanding Koebe’s distortion theorem to see that there exists a constant

\[ \Delta_s(z) : \mathbb{D} \to \mathbb{H}, \quad \Delta_s(z) = \frac{z \mu_{t(s)} - \lambda_s \mu_{t(s)}}{z - \lambda_s}, \]

(A.2)

where the reader may verify that if

\[ G_{t(s)}(z) = F_{t(s)}^{-1}(z), \quad g_s(z) = f_s^{-1}(z), \]

then

\[ \mu_{t(s)} = G_{t(s)}(i), \quad \lambda_s = g_s(-1). \]

In fact, by expanding \( G \) at infinity via (A.1),

\[ \operatorname{Im} \mu_{t(s)} = -\frac{g_s'(-1)}{g_s(-1)} = |g_s'(-1)|. \]

This uses that

\[ \operatorname{Re} \left( 1 - \frac{g_s''(-1)}{g_s'(-1)} \right) = -\frac{g_s'(-1)}{g_s(-1)}, \]

which holds because the left-hand side equals \( \partial_\theta [\partial_\phi g_s(e^{i\theta})] \) at \( \theta = \pi \), and \( g_s \) maps the circle to the circle locally at \(-1\) so that the change of the tangent is equal to the change of the argument which is the right-hand side.

By Lemma [A.1] and (A.3) there exists \( c_1 = c_1(\varepsilon, T) > 0 \) such that

\[ \operatorname{Im} \mu_{t(s)} \geq c_1, \quad s \in [0, \sigma]. \]

(A.4)

Consider the family \( (F_t), t \in [0, \tau] \) with the half-plane capacity parameterization. It satisfies the chordal Loewner PDE in \( t \) and we let \( U_t = \Delta_{s(t)}(W_{s(t)}) \) be the corresponding chordal driving term. Set

\[ \tau := t(\sigma). \]

The estimate (A.4) then implies that there is a constant \( T' = T'(\varepsilon, T) < \infty \) such that \( \tau \leq T' \). Indeed, in Theorem 3 of [24] it is shown that \( s'(t) = 4(\operatorname{Im} \mu_{t(s)})^2/|\mu_{t(s)} - U_t|^4 \) which is bounded away from 0 on \([0, \tau]\). Using (A.4) and that \( |W_s - \lambda_s| \geq \varepsilon \) for \( s \in [0, \sigma] \), we see that there exist constants \( 0 < c < \infty \) and \( d_0 > 0 \) depending only on \( \varepsilon \) and \( T \) such that for all \( d \leq d_0 \), uniformly in \( s \in [0, \sigma] \),

\[ \left| \operatorname{Re} \left( \Delta_s((1 - d)W_s) \right) - U_{t(s)} \right| \leq c d, \quad c^{-1}d \leq \operatorname{Im} \left( \Delta_s((1 - d)W_s) \right) \leq cd. \]

In other words, the hyperbolic distance between \( \Delta_s((1 - d)W_s) \) and \( U_{t(s)} + id \) is bounded by a constant depending only on \( \varepsilon \) and \( T \). Therefore we can use Koebe’s distortion theorem to see that there exists a constant \( c = c(\varepsilon, T) < \infty \) such that for all \( s \in [0, \sigma] \)

\[ |f'_{t(s)}((1 - d)W_s)| \leq |F'_{t(s)}(\Delta_s((1 - d)W_s))| \leq c |F'_{t(s)}(U_{t(s)} + id)|. \]

We have proved the following result.
Proposition A.2. Let $T < \infty$ and $\varepsilon > 0$ be given. Suppose that $(f_s, W_s)$ is a radial Loewner pair generated by the curve $\gamma(s)$. Define $\sigma = \sigma(\varepsilon, T)$ by (A.1). Let $(F_t, U_t)$ be the chordal Loewner pair generated by the curve $s \mapsto \varphi^{-1}(\gamma(s))$, $s \in [0, \sigma]$ reparameterized by half-plane capacity and let $\tau = t(\sigma)$. There exists $c = c(\varepsilon, T) < \infty$ and $d_0 = d_0(\varepsilon, T) > 0$ such that for all $d \leq d_0$,
\[
\sup_{s \in [0, \sigma]} |f'_s((1 - d)W_s)| \leq c \sup_{t \in [0, \tau]} |F'_t(U_t + id)|.
\]

Now assume that $(f_s)$ is the radial SLE$_\kappa$ Loewner chain. Then $\sigma$ is a stopping time for $(f_s)$ and $\tau$ is a stopping time for $(F_t)$. The law of the chordal driving term $U_t$ stopped at $\tau$ is absolutely continuous with respect to the law of standard linear Brownian motion with speed $\kappa$, as shown in [24]. Moreover, by (A.4) the Girsanov density is uniformly bounded above by a constant depending only on $\kappa$ and $\varepsilon, T$ used in the definition of $\sigma$. Since $(F_t)$ is absolutely continuous with respect to a chordal SLE$_\kappa$ Loewner chain and since the Girsanov density is uniformly bounded (for fixed $\kappa, \varepsilon, T$) using Proposition A.2 we can estimate the behavior of $\sup_{s \in [0, \sigma]} |f'_s((1 - d)W_s)|$ using standard chordal SLE.

A.2. Derivative Estimate for Chordal SLE. We now derive the needed estimate on the growth of the derivative in chordal coordinates. The estimate is essentially a direct consequence of work in [1] and we will describe the modifications here. Let $(F_t)$, $t \geq 0$ be the standard chordal SLE Loewner chain mapping $\mathbb{H}$ onto the unbounded connected component of $\mathbb{H} \setminus \gamma[0, t]$. We write $\hat{F}_t(z) = F_t(z + U_t)$, where $U$ is the chordal driving term for $(F_t)$. Recall that the chordal reverse SLE$_\kappa$ flow is the family of conformal mappings solving
\[
\hat{h}_t = -\frac{2}{h_t - \sqrt{\kappa}B_t}, \quad h_0(z) = z,
\]
where $B$ is standard Brownian motion. For fixed $t_0 > 0$, $|h'_{t_0}(z)|$ is equal to $|\hat{F}'_{t_0}(z)|$ in distribution. Hence (first) moment estimates for $|\hat{F}'_{t_0}|$ are reduced to corresponding estimates for $|h'_{t_0}|$ and these are often more easily obtained.

Note that scaling implies that for fixed $y > 0$, $|h'_t(iy)| \overset{d}{=} |h'_{ty-2}(i)|$. Define
\[
\zeta(\lambda) = \lambda + \frac{\sqrt{(4 + \kappa)^2 - 8\lambda\kappa} - (4 + \kappa)}{4}.
\]
We will assume that
\[
\lambda < \lambda_c = 1 + \frac{2}{\kappa} + \frac{3\kappa}{32}.
\]
In this range we quote the following estimate from [6]. See also [7] and the references therein.

Lemma A.3. Let $h_t$ be the chordal reverse SLE$_\kappa$ flow, $\kappa > 0$. There exists a constant $c < \infty$ such that for $\lambda < \lambda_c$,
\[
\mathbb{E}[|h'_t(i)|^\lambda] \leq ct^{\zeta(\lambda)/2}, \quad t \geq 1.
\] (A.5)
This result now implies the needed estimate which is a version of Proposition 4.2 of [6] with a decay rate; we will sketch the proof and refer the reader to [6] for more details. Let \( \kappa > 0 \) and define the function

\[
\rho(\beta) = \beta + \frac{2(1 + \beta)}{\kappa} + \frac{\beta^2 \kappa}{8(1 + \beta)}
\]

and

\[
q(\beta) = \min\{\lambda_c \beta, \rho(\beta) - 2\}, \quad \beta_+ < \beta < 1,
\]

where

\[
\beta_+ = \max\left\{0, \frac{4(\kappa \sqrt{8 + \kappa} - (4 - \kappa))}{(4 + \kappa)^2}\right\}.
\]

Note that \( q(\beta) > 0 \) for \( \beta \) in the above range.

**Proposition A.4.** Let \( T < \infty \) be fixed and let \((F_t)\) be the chordal \( \text{SLE}_\kappa \) Loewner chain, \( \kappa \in (0, 8) \). There exists a constant \( c < \infty \) depending only on \( T \) and \( \kappa \) such that for every \( \beta \in (\beta_+, 1) \) and \( y_* < 1 \) we have that

\[
P\left\{ \forall y \leq y_*, \sup_{t \in [0,T]} |\hat{F}'_t(iy)| \leq cy^{1-\beta}\right\} \geq 1 - cy_*^{q(\beta)}.
\]

**Proof.** (Sketch.) By the distortion theorem, scaling, and the fact that Brownian motion is almost surely weakly Hölder-(1/2), it is enough (see [6]) to show that for \( \beta_+ < \beta < 1 \)

\[
\sum_{n=N_*}^{\infty} \sum_{j=1}^{2^{2n}} \mathbb{P}(|\hat{F}'_{j2^{-2n}}(i2^{-n})| > 2^{3n}) \leq c2^{-N_*q(\beta)},
\]

where \( N_* = \lfloor \log y_*^{-1} \rfloor \). We have for \( 0 < \lambda < \lambda_c \) using scaling, Chebyshev’s inequality, and Lemma A.3

\[
\sum_{n=N_*}^{\infty} \sum_{j=1}^{2^{2n}} \mathbb{P}(|\hat{F}'_{j2^{-2n}}(i2^{-n})| > 2^{3n}) \leq \sum_{n=N_*}^{\infty} \sum_{j=1}^{2^{2n}} 2^{-n\lambda \beta} \mathbb{E}[|\hat{F}'_{j2^{-2n}}(i2^{-n})|^\lambda]
\]

\[
\leq c \sum_{n=N_*}^{\infty} \sum_{j=1}^{2^{2n}} 2^{-n\lambda \beta} \mathbb{E}[|h'_j(i)|^\lambda]
\]

\[
\leq c \sum_{n=N_*}^{\infty} \sum_{j=1}^{2^{2n}} 2^{-n\lambda \beta} j^{-\zeta/2}
\]

\[
\leq c \sum_{n=N_*}^{\infty} \sum_{j=1}^{2^{2n}} 2^{-n\lambda \beta}(1 + 2^{n(2-\zeta)})
\]

\[
\leq c(2^{-N_*\lambda \beta} + 2^{-N_* (\lambda \beta + \zeta - 2)}).
\]

Recall that \( \lambda \in (0, \lambda_c) \). Note that \( \zeta = 2 < 0 \) if and only if \( \kappa > 1 \), so for these \( \kappa \) the smaller exponent is \( \lambda \beta + \zeta - 2 \). In this range, we find \( q(\beta) \) by maximizing over \( 0 < \lambda < \lambda_c \) for \( \beta \) fixed so that \( q(\beta) = \max_\lambda \lambda \beta + \zeta(\lambda) - 2 \). The lower
bound $\beta_+$ is the smallest $\beta > 0$ such that $\beta > \beta_+$ implies $q(\beta) > 0$. When $\kappa \leq 1$, $\lambda \beta$ is the smaller exponent and we must restrict attention to $\beta > 0$. We pick the largest $\lambda = \lambda_c$. □

From this and the work in the previous subsection we immediately obtain the following proposition. Recall that the stopping time $\sigma$ was defined in (A.1).

**Proposition A.5.** Let $\kappa \in (0, 8)$. Let $\varepsilon > 0$ be fixed and let $(f_s)$, $0 \leq s \leq \sigma$, be the radial $\text{SLE}_\kappa$ Loewner chain stopped at $\sigma$ as defined by (A.1). For every $\beta \in (\beta_+, 1)$ there exists a constant $c = c(\beta, \kappa, \varepsilon, T) < \infty$ such that for $d_* < 1$,

$$P \left\{ \forall d \leq d_*, \sup_{\kappa \in [0, \sigma]} d|f'_s((1-d)W_s)| \leq cd^{1-\beta} \right\} \geq 1 - cd^*_{1-\beta}.$$

We note that when $\kappa = 2$

$$q(\beta) = -1 + 2\beta + \frac{\beta^2}{4(1+\beta)}, \quad \beta_+ = \frac{2(\sqrt{10} - 1)}{9}.$$

**Appendix B. Grid Domains and Transfer of Information to $\mathbb{D}$**

When mapping conformally a curve into a reference domain, bounds on the tip structure modulus for the curve are not automatically preserved. In this section we will consider the general case without reference to a specific discrete model. It seems that this general setting requires information about boundary regularity of the approximated domain (as opposed to information about the behavior of the discrete curve). In particular, we shall need uniform control of the distortion of annuli on the scales of the structure modulus.

**B.1. Grid Domains.** Recall the definition of a grid domain that was given in Section 4. Let $D \ni 0$ be simply connected, and assume that the inner radius with respect to 0 equals 1. Let $D_n = D_n(D)$ be the $n^{-1}2^2$ grid-domain approximation of $D$. Notice that every point on $\partial D_n$ is within distance $\sqrt{2}/n$ of a point on $\partial D$, so that the inner Hausdorff distance between $\partial D_n$ and $\partial D$ is at most $\sqrt{2}/n$. Let $\psi : D \to \mathbb{D}$ be the conformal map normalized by $\psi(0) = 0$ and $\psi'(0) > 0$. Similarly, for $n = 1, 2, \ldots$, let $\psi_n : D_n \to \mathbb{D}$ be conformal maps with the same normalization. The sequence of domains $D_n$ converge to $D$ in the Carathéodory sense, and so the $\psi_n$ converge to $\psi$ uniformly on compacts. Our goal will be to find a convergence rate for

$$\sup_{z \in D_n} |\psi_n(z) - \psi(z)|.$$

For this to be achievable we need some information about the regularity of the boundary of $D$. We will here consider the class of quasidisks, although it will be clear that similar methods can be used to handle other classes.
of domains where Euclidean geometric estimates on the behavior of the conformal mapping on the boundary are available.

B.2. Discrete Approximation of a Quasidisk. A quasicircle is the image of the unit circle under a quasiconformal mapping. A quasidisk is a (bounded) domain bounded by a quasicircle. See [20] for definitions and an overview from a conformal mapping point of view. A quasicircle is not necessarily rectifiable as the example of the von Koch snowflake shows.

We find it convenient to use an equivalent but more geometric definition, namely Ahlfors’ three-point condition: The closed Jordan curve \( \partial D \) is a quasicircle if and only if there exists a constant \( A < \infty \) such that for any two points \( x, y \in \partial D \) it holds that

\[
\text{diam} J(x, y) \leq A|x - y|,
\]

where \( J(x, y) \subset \partial D \) is the arc of smaller diameter connecting \( x \) with \( y \). One can consider the smallest such \( A \) as a measure of regularity. This regularity implies some uniform regularity for the grid-domain approximation \( D_n \) and this allows us to estimate the convergence rate of \( \psi_n \) using a result from [26]. See also Section 5 of [16] where similar questions are discussed.

Lemma B.1. Let \( D \) be a quasidisk satisfying (B.1) and let \( D_n \) be the \( n^{-1} \mathbb{Z}^2 \) grid domain approximation of \( D \). Let \( \psi, \psi_n \) be the normalized conformal maps from \( D \) and \( D_n \), respectively, onto \( \mathbb{D} \). Then there exists a constant \( c < \infty \) depending only on \( A \) and the diameter of \( D \) such that

\[
\sup_{z \in D_n} |\psi_n(z) - \psi(z)| \leq c \frac{\log n}{\sqrt{n}}.
\] (B.2)

Proof. We will first show that \( D_n \) satisfies (B.1) uniformly in \( n \) with a constant \( A' \) depending only on \( A \). Let \( x, y \in \partial D_n \). First we consider the case when \( |x - y| < 1/n \). Then since \( \partial D_n \) is a Jordan curve which is a subset of the edge set of \( n^{-1} \mathbb{Z}^2 \), we have that \( \text{diam} J(x, y) \leq \sqrt{2} |x - y| \). Now assume that \( |x - y| \geq 1/n \). Let \( \xi \) and \( \eta \) be points on \( \partial D \) closest to \( x \) and \( y \), respectively. Clearly, \( |x - \xi| \) and \( |y - \eta| \) are both at most \( \sqrt{2}/n \). Let \( \alpha, \beta \) be the two line segments connecting \( x \) with \( \xi \) and \( y \) with \( \eta \). First assume that the curve \( \Gamma = J(x, y) \cup \alpha \cup \beta \) separates \( J(\xi, \eta) \) from 0 in \( D \). Let \( Q_j, j = 1, \ldots, N \) be those lattice squares whose faces are outside of \( D_n \) but whose boundaries touch \( J(x, y) \). By the construction of \( D_n \) and the Jordan curve theorem, since \( \Gamma \) separates 0 from \( J(\xi, \eta) \), each \( Q_j \) is intersected by \( \alpha \cup \beta \cup J(\xi, \eta) \). Consequently,

\[
\text{diam} \Gamma \leq \text{diam} J(\xi, \eta) + 2\sqrt{2}/n \leq A|\xi - \eta| + 2\sqrt{2}/n.
\]

Hence,

\[
\text{diam} J(x, y) \leq \text{diam} \Gamma \leq A|x - y| + (2A + 2)\sqrt{2}/n.
\]

Now, if \( \Gamma \) does not separate \( J(\xi, \eta) \) from 0 in \( D \), then since \( \Gamma \) is a crosscut of \( D \), \( (\partial D_n \setminus J(x, y)) \cup \alpha \cup \beta \) does separate \( J(\xi, \eta) \) from 0 in \( D \). Thus, in
this case we can do the same argument as in the previous paragraph showing that \(\text{diam}(\partial D_n \setminus J(x, y)) \leq \text{diam} J(\xi, \eta) + 2\sqrt{2}/n\). But by definition, \(\text{diam} J(x, y) \leq \text{diam}(\partial D_n \setminus J(x, y))\).

Using also the estimate we obtained in the case when \(|x - y| < 1/n\) we conclude that,

\[\text{diam} J(x, y) \leq (3A + 2\sqrt{2})|x - y|.
\tag{B.3}
\]

By (B.3) there is a constant \(c\) depending only on \(A\) and the diameter of \(D\) such the Warschawshi structure moduli \(\eta_W^{(n)}\) of \(\partial D_n\) satisfy

\[\eta_W^{(n)}(\delta) \leq c\delta, \quad \delta \leq 1.
\]

Consequently, since \(D_n \subset D\) and each point on \(\partial D_n\) is within distance \(\sqrt{2}/n\) of a point on \(\partial D\), part (a) of Theorem VII in [26] implies (B.2). \(\square\)

Remark. The same proof works when \(D\) is a John domain.

For simplicity we will now assume that \(\partial D\) is \(C^{1+\alpha}\) for some \(\alpha > 0\), that is, we assume that there is a parameterization of \(\partial D\) which has a Hölder-\(\alpha\) derivative. By Kellogg’s theorem; see, e.g., [3], this assumption implies that the conformal map \(\psi : D \to \mathbb{D}\) (and \(\psi^{-1}\)) is in \(C^{1+\alpha}(\overline{D})\). (So we can take the conformal parameterization of \(\partial D\).) In particular \(\psi\) is bilipschitz on \(\overline{D}\), that is, there is a constant \(c < \infty\) depending only on \(\alpha\) and the diameter of \(D\) such that

\[c^{-1}|z - w| \leq |\psi(z) - \psi(w)| \leq c|z - w|, \quad z, w \in \overline{D}.
\tag{B.4}
\]

Similar uniform estimates, but of Hölder type, and corresponding versions of Lemma 4.7 (stated again below) hold if \(D\) is assumed to be a quasidisk. Indeed, the uniformizing conformal map and its inverse are then Hölder continuous on a neighborhood of \(\partial D\) with an exponent depending only on \(A\); see [20]. From (B.4) we immediately get the required control over distortion of annuli up to constants on sufficiently large scales. We can now prove Lemma 4.7 which we state again:

**Lemma B.2.** Suppose \(D \ni 0\) is a simply connected domain Jordan domain with \(C^{1+\alpha}\) boundary. Let \(D_n\) be the \(n^{-1/2}\) grid domain approximation of \(D\) and let \(\gamma_n\) be a Loewner curve in \(D_n\) connecting \(\partial D_n\) with 0. There is a constant \(c\) depending only on \(\alpha\) and the diameter of \(D\) such that the following holds. Set \(0 < r < 1/2\) and \(d_n = n^{-r}\) and let \(\eta^{(n)}(d_n; D_n)\) be the tip structure modulus for \(\gamma_n\) in \(D_n\). Then for all \(n\) sufficiently large the tip structure modulus for \(\psi_n(\gamma_n)\) in \(\mathbb{D}\) satisfies

\[\eta^{(n)}(c^{-1}d_n; \mathbb{D}) \leq c\eta^{(n)}(d_n; D_n)\].

**Proof.** Let \(\eta_n = \eta^{(n)}(d_n; D_n)\). We can assume that \(\eta_n \geq 2d_n\). It is enough to verify that there exists a constant \(c\) independent of \(n\) such that for all
annuli $\mathcal{A}(z) = \{ w : d_n \leq |w - z| \leq \eta_n \}, z \in D_n$ we have

$$\psi_n (\mathcal{A}(z) \cap D_n) \subset \{ w : c^{-1} d_n \leq |w - \psi_n(z)| \leq c \eta_n \} \cap \mathbb{D}.$$ 

But this follows immediately from Lemma B.1 with the assumption that $d_n = o(n^{-1/2})$ and (B.4). □

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