The Exact Foldy-Wouthuysen Transformation for a Dirac Theory Revisited

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Abstract

The Exact Foldy-Wouthuysen transformation (EFWT) method is generalized here. In principle, it is not possible to construct the EFWT to any Hamiltonian. The transformation conditions are the same but the involution operator has a new form. We took a particular example and constructed explicitly the new involution operator that allows one to perform the transformation. We treat the case of the Hamiltonian with 160 possible CPT-Lorentz breaking terms, using this new technique. The transformation was performed and physics analysis of the equations of motion is shown.

Keywords: Dirac equation, CPT and Lorentz violating terms, Exact Foldy-Wouthuysen transformation.

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1 Introduction

The study of the possible candidates to break CPT-Lorentz symmetry is very important nowadays [1]. There are a large study been developed during the last ten years that shows the possible experiments that could give the more prominent physical effect to measure one of these fields [2]. Until now, non of them was directly observed. The most prominent theoretical approaches that consider these cases are based on indirect physical effects, as it is shown in [3, 4]. In other words, the search for this manifestations starts with an action that considers at least two independent fields as one can see on the recent papers [5]. For the non-relativistic scenario the the results are well established in [6, 7, 8, 9], for torsion field, for example. It is very interesting to see [10] that torsion field could be generated from the symmetry breaking. Some recent theoretical studies have been developed with the same phenomenological background [11, 12, 13, 14].

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Another possible phenomenological approach to this problem can be constructed step by step by searching for new terms in the Hamiltonian that describes this situation. Thinking this way, it makes sense the appearance of some terms in the equations of motion that could give a mix between an external known field with sufficient enough big amplitude to compensate the fact that the CPT-Lorentz terms have small amplitudes.

The idea is the same shown in [15], where the strong magnetic field could, in principle, change the trajectory of the Dirac particle that interacts with gravitational waves. It is important to take into account the corrections, made with canonical FW, to these results that were shown in [16]. In [17], the massive linearized gravity was studied and some possible experiments that could measure indirect effects of gravitational waves on Dirac fermions were indicated. However, solving the Dirac equation for the general case is not a simple procedure [18]. It is well known in literature that working with the EFWT is a more prominent approach to interpret a Dirac Hamiltonian than the canonical transformation [19]. But this is true not only for the fact that it can give us new terms, but it is a faster and more economic (in terms of algebraic calculation) procedure [15, 20, 21, 22]. One can see this transformation as a generalization of the usual FWT.

Let us perform a comparison on the two procedures. It is possible to see that in the usual FWT the multiplication on each step (on each order on $1/m$) by the term that makes the Hamiltonian even, generates a maximum of $1 + 2n$ even terms, where $n$ represents the number of terms of the previous Hamiltonian (see, for example, pages 48-51 in [23]). The maximum number of terms in the $n$-th Hamiltonian is straightforward obtained by the fact that this is an expansion in power series of an operator. The factor 2 on $1 + 2n$ expression is obtained in case when it does not commute with all original terms.

On the other hand, the EFWT impose the multiplication of all terms of the Hamiltonian by themselves. Analogous arguments give us the maximum of $1 + 2n^2$ on the expanded Hamiltonian. If the parameter of expansion here is also taken to be $1/m$, one can see that the possibility of having new terms in comparison with the usual method is greater. In many particular known cases [21, 24, 25], the anti-commutators on both cases are such that the results are the same! But it is not the general case. This was explicitly shown on [19]. In this paper we show another case where it happens.

In [26] the author performs in a very didactic way the formal comparison between the two methods. He also described which is the most efficient method for each possible applications. The explicit calculations are performed in the series of three works where the generality for the exact procedure becomes evident [27, 28, 29].

On [30, 31, 32], the authors worked on a series of papers in which the EFWT conditions were not satisfied. In these articles, the study of the CPT-Lorentz violating terms was used as a background to this transformation. It is possible to see in [33] the diagonalized Hamiltonian for all the possible terms that allows this procedure.

Using the result of [32], we developed an algorithm to construct a generalized involution operator for the EFWT. We show a method to construct the explicit form of the operator that allows the Hamiltonian to be diagonalized. In some sense, the logic here is inverse: we do not test if it is possible to perform the EFWT but we search for the operator that gives us this possibility.

By showing the explicit analytic form of this operator, the EFWT usual algorithm can be
applied to the initial Hamiltonian. We constructed the general operator and the complete case of CPT-Lorentz interacting with Dirac field [13] is studied here using the EFWT technique. We also compare the result with the usual transformation and two new terms show up.

2 The complete Hamiltonian for a Dirac Theory with CPT-Lorentz invariance violation

In Ref. [22], the authors present a table that specifies the 80 cases of CPT and Lorentz violating terms in the modified Dirac equation. A complete study of the EFWT, taking into account these 80 cases, is presented in Ref. [33].

However, it is worth mentioning that, in Refs [22, 33], a sort of terms were not considered. In order to perform the EFWT study of the complete set of cases, it is necessary that the Hamiltonian admits the involution operator [19, 21, 24, 25, 33]. In this work, we present a new table corresponding to all the CPT-Lorentz breaking terms. The main point is the search for an involution operator $J$, which satisfies the anti-commutation relation,

$$JH + HJ = 0, \quad (1)$$

for the complete set of terms, presented in the table.

Table 1: Interaction coefficients

| $m$ | $P^\nu e^\nu$ | $a_1$ | $P^\nu_{ij}$ | $b_0$ | $P^\nu_{i} g^{ij}_{ij}$ | $m_5$ | $P^\nu_{i\alpha}$ | $a_0$ | $P^\nu_{i}\sigma^{0\alpha}$ | $H^{0\mu}_{i\sigma}$ |
|-----|--------------|--------|---------------|------|-------------------------|------|-------------------|------|----------------------|-----------------|
| $\gamma^0$ | 1 | $\gamma^0_{ij}$ | $-\gamma^0_{\alpha\beta}$ | $\frac{1}{2}\sigma^{ij}$ | $i\gamma_5$ | $\gamma_5\gamma_1$ | $\gamma_0$ | $\frac{1}{2}\sigma^{0\mu}$ |
| $e^{00}$ | $-\gamma^0_{0}$ | $-\alpha^0_{ij}$ | $-\gamma^5_{\alpha\beta}$ | $-\frac{1}{2}\gamma^0_{ij}$ | $-i\gamma_0\gamma_5$ | $\gamma_5\alpha_1$ | $-1$ | $-\frac{1}{2}\sigma^{0\mu}$ |
| $d^{00}$ | $-i\gamma^5_{0}$ | $i\gamma^5_{\alpha\beta}$ | $-i\gamma^5_{ij}$ | $\frac{1}{2}\gamma^5_{ij}$ | $1$ | $-i\gamma_5$ | $\gamma_5\alpha_1$ | $\frac{1}{2}\gamma^5_{ij}$ |
| $g^{00}$ | $-\frac{1}{2}\alpha^0_{ij}$ | $-\frac{1}{2}\alpha^0_{ij}$ | $-i\gamma^5_{ij}$ | $\frac{1}{2}\alpha^0_{ij}$ | $\frac{1}{2}\alpha^0_{i\alpha}$ | $-\frac{1}{2}\alpha^0_{i\alpha}$ | $-\frac{1}{2}\alpha^0_{i\alpha}$ |
| $d^{00}$ | $-\gamma^0_{0}$ | $\gamma^5_{ij}$ | $-1$ | $\frac{1}{2}\gamma_5\gamma_{0\alpha\beta}$ | $-i\gamma_0$ | $-\alpha_1$ | $-\gamma_5$ | $-\frac{1}{2}\sigma^{0\mu}_{\alpha\beta}$ |
| $e^{00}$ | $-1$ | $-\gamma_1$ | $-\gamma_5_{ij}$ | $-\frac{1}{2}\gamma_5_{ij}$ | $-i\gamma_5$ | $-\gamma_5_{ij}$ | $-\gamma_0$ | $-\frac{1}{2}\sigma^{0\mu}_{ij}$ |
| $e^{00}$ | $\gamma^0_{ij}$ | $\gamma^5_{ij}$ | $\frac{1}{2}\gamma^5_{ij}$ | $\frac{1}{2}\gamma^5_{ij}$ | $i\gamma^5_{ij}$ | $-i\gamma^5_{ij}$ | $-\alpha_1$ | $\frac{1}{2}\gamma^5_{ij}$ |
| $g^{00}$ | $-\frac{1}{2}\sigma^{ik}_{ij}$ | $-\frac{1}{2}\sigma^{ik}_{ij}$ | $\frac{1}{2}\sigma^{ik}_{0\gamma_5}$ | $-\frac{1}{2}\sigma^{ik}_{0\gamma_5}$ | $-\frac{1}{2}\sigma^{ik}_{0\gamma_5}$ | $-\frac{1}{2}\sigma^{ik}_{0\gamma_5}$ | $-\frac{1}{2}\sigma^{ik}_{0\gamma_5}$ |

The quantities $a_\mu$, $b_\mu$, $m_5$, $e^{\mu}$, $d^{\mu\nu}$, $e^\mu$, $f^\mu$, $g^{\mu\nu\lambda}$, and $H_{\mu\nu}$ represent the CPT-Lorentz violating parameters [34, 35, 36]. We adopt notations as described in [23] for Dirac matrices and the useful notations for $P_i$, used in [22].

The terms highlighted in boldface, correspond to the empty spaces, in the table presented in [22]. These terms do not obey the anti-commutation relation (1), if one takes into account the following form of the involution operator

$$J = i\gamma^5\gamma^0, \quad (2)$$
The set of terms that obey relation (1), considering the involution operator (2), are presented in Ref [22]. From now on, we shall call the quantities in boldface as new terms and the quantities that are not in boldface, old terms.

In order to understand how the Hamiltonian can be obtained, directly from the table, let us present a simple example. The rule is based on the product of the line terms by the terms in the rows. We shall consider, for instance, the first line times the first row: \( \gamma^0 \times 1 \times m = \gamma^0 m \).

We got, in this case, the free Dirac equation term, which is the most trivial one.

Let us consider another example. The product of the sixth line by the first row. The terms inside the table must also taken into account. Such a multiplying gives two terms

\[
d^{00} \times (-\gamma^0 \gamma^5) \times m = -md^{00} \gamma^0 \gamma^5 \quad \text{and} \quad d^{00} \times (-\gamma^0 \gamma^5) \times P^* \nu e^\nu = -md^{00} \gamma^0 \gamma^5 P^* \nu e^\nu.
\]

Observe that both of them break \( C, P, PT \) and \( CT \) [2, 36]. It is remarkable to say that the study of this kind of terms, with EFWT considerations, depend on the correct choice of the involution operator, such that relation (1) is contemplated.

The general form of the involution operator [21, 37] has the following structure

\[
\hat{J} = M \times \hat{F},
\]

where \( M \) and \( \hat{F} \) are operators. They act on the matrices and functions space, respectively. In particular, the choice \( M = i\gamma^5 \gamma^0 \) and \( \hat{F} = \hat{1} \) corresponds to the usual operator used in previous works [22, 33]. However, as already mentioned above, the new terms in the table do not satisfy the anti-commutation relation (1), for such a choice. The main point here is the following: the choice of an appropriated involution operator, for a specific term of the table, involves the knowledge of exactly what symmetry is being broken (for each term of the table).

An interesting case is the vectorial part of the torsion field, \( b_l \). As one can check [2, 36], this term breaks \( T, CT, PT \) and \( CPT \). In the Hamiltonian, the torsion field is founded by the product of line 0 by row 6. It gives \( b_l \gamma^0 \gamma^5 \gamma^l \). It was showed that \( M = i\gamma^5 \gamma^0 \) and \( \hat{F} = \hat{T} \) represents a specific choice for the involution operator, such that the anti-commutation relation is obeyed [33]. However, it is not the only possible choice. In particular \( \hat{F} = CT, \hat{F} = PT \) and \( \hat{F} = CPT \), would work equally.

In order to perform the EFWT for all the terms in the table above, we present in the next section, a proposal of a new involution operator which anti-commutes with all the terms of the table.

### 3 The involution operator, an appropriated choice

We begin with an appropriated representation of the Hamiltonian with CPT-Lorentz breaking terms.

\[
H = \phi^A_1 H_{AB} \phi^B_2.
\]

Throughout this paper, the quantities with Latin indexes \( A \) and \( B \) are only associated with possible positions in the table. We would like to emphasize that these indexes are not space-time indexes. The possible values for \( A \) are running horizontally in the table, from 0 to 8. In the
case of $B$, the possible values are running vertically, from 0 to 9. In addition, the quantities $\phi_1^A$ and $\phi_2^B$ are the fields that appear in the top row and in the left column of the table, respectively, and the quantity $H_{AB}$ represents the terms contained in the cells of the table.

As shall be better understood in the next section, the EFWT works if, and only if, one can write the Hamiltonian on the form of Eq. (5). It may seem cumbersome, at a first sight, but it is not. Let us consider an example. The choice $A = 0$ and $B = 6$ (first column and the seventh row, respectively) leads us to $\phi_0^1 = m + P_*^e \nu$, $\phi_2^6 = d_{00}$ and $H_{0,6} = -\gamma^0 \gamma^5$. It gives exactly the two terms described in Eq. (3).

The quantity $\theta_{IK}$ is defined in order to assume the values 0 or 1. If $I \times K > 0$, $\theta_{IK} = 0$. On the other hand, if $I \times K < 0$, $\theta_{IK} = 1$. Actually, the product between $I$ and $K$ tells us if we are dealing with the new or old terms of the table. The quantities $O_{AB}$ also assume the 0 or 1 values. They are determined by the previous knowledge of which symmetry is being broken.

Let us consider an example, by setting $A = 6$ and $B = 0$. Then, $\phi_0^6 = b_l + P_\nu^* d^n$, $\phi_2^0 = \gamma^0$ and $H_{6,0} = \gamma^5 \gamma^l$. According to Eq. (5), the Hamiltonian for this case, is given by

$$H = \gamma^0 \gamma^5 \gamma^l (b_l + P_\nu^* d^n).$$

The next step is the choice of the $O_{AB}$ quantities. In Refs. [2, 36], there is a table with the properties of operators for Lorentz violation in QED. According to this table, one can consider that $O''_{AB} = 0$ and $O'_{AB} = O''_{AB} = 1$. Observe that from the Eq. (8), $I = 1$, $K = -6$ and $I \times K = -6$, for this reason we have $\theta_{6,0} = 1$. With these considerations, the corresponding involution operator is

$$\hat{J} = i \gamma^5 \gamma^0 \theta_{IK} PT. \quad (10)$$

As one can check, the anti-commutation relation is obeyed, when the the quantities $\mathcal{H}$ and $\hat{J}$ are described by the relations (9) and (10), respectively.

One can see that for the old terms of table, the product between $I$ and $K$ is always positive and the quantity $\theta$ in Eq. (6) is equal to zero. Consequently, in what concerns the old part of the table, we shall have, as expected, $M = i \gamma^5 \gamma^0$ and $\hat{F} = \hat{1}$.

4 Exact transformation with CPT

We present in this section, the EFWT of the Hamiltonian for a free spin-1/2 Dirac fermion $\Psi$ of mass $m$ in the standard-model extension [3, 18]. Let us begin with the following Hamiltonian

$$\mathcal{H} = \gamma^0 \gamma^5 \gamma^l (b_l + P_\nu^* d^n).$$
\[ \mathcal{H} = m(\gamma^0 - \gamma^0 c_0 - e_0 - d_{j0} \gamma^5 \gamma^j + \frac{1}{2} g_{ik0} \sigma^{ik}) + P^k(-\alpha_k + 2d_{0k} \gamma^5 - c_{j0} \alpha^j + c_{00} \alpha_k + i f_k \gamma^5 \gamma^0 - 2g_{0uj} \gamma^0 \sigma^{0j} - 2c_{0k} + d_{j0} \gamma^5 \alpha_j) - d_{00} \gamma^5 \alpha_k - e_k \gamma^0 + \frac{1}{2} \gamma^0 \sigma^{ij} g_{ijk} - ig_{000} \gamma_k \alpha^i) + a_j \alpha^j - b_0 \gamma^5 + iH_{0j} \gamma^j - b_j \gamma^5 \alpha^j - \frac{1}{2} \gamma^0 \sigma^{ij} H_{ij}. \] (11)

This Hamiltonian can be constructed directly from the table presented in the last section. However, it is not the most complete Hamiltonian that one can extract from the table. The main point of this work is the development of the operator described in \([61]\). As it is been used for the first time, it is worthwhile to deal with a Hamiltonian which the diagonalized result we could know at least the qualitative result. On the other hand, it would be very interesting from the physical point of view if the new EFWT generates unexpected terms in comparison with the usual transformation for the same action. We decide to pick just the terms represented in Eq. (11), because in \([18]\) the authors perform the usual FWT, taking into account this Hamiltonian. Performing the transformation for it we could validate our algorithm and also search for physical quantities mixed in a new form. The transformed Hamiltonian (with usual FWT) is the following:

\[ \tilde{H}^\text{tr} = \beta m + \frac{1}{2m} \left\{ (1 + \tilde{A}) \left[ (\delta_{ij} + \tilde{B}_{ij}) \tilde{P}^i + \tilde{C}_j \right]^2 + \tilde{D} \right\}, \] (12)

where

\[ \tilde{A} = -2c_{00} \gamma^0 \]
\[ \tilde{B}_{ij} = \frac{1}{2} \left[ 4(d_{0i} + d_{0j}) \gamma^5 \gamma^j - 4c_{ij} \gamma^0 + 4e^j_{mj} \left( g_{0i} + g_{0j} \right) \gamma^5 \gamma^0 \gamma^m \right] \]
\[ \tilde{C}_j = \frac{1}{2} \left[ -4m(c_{0j} + c_{j0}) + 4md_{ij} \gamma^5 \gamma^0 \gamma^i - 4md_{00} \gamma^5 \gamma^0 \gamma^j - 4me_j \gamma^0 + 2me^k_l m g_{ij} \gamma^5 \gamma^m \right. \]
\[ \left. - 4me^j_l g_{000} \gamma^5 \gamma^l + 4a_j \gamma^0 - 4b_{00} \gamma^5 \gamma^j + 4e^{jk} \gamma^0 \gamma^j - 4m^k_l, g_{ij} \gamma^5 \gamma^0 \gamma^l \right] \]
\[ \tilde{D} = -2m^2 c_{00} \gamma^0 - 2m^2 e_0 - 2m^2 d_{j0} \gamma^5 \gamma^j - m^2 e^{ik} g_{ik0} \gamma^5 \gamma^0 \gamma^l + 2ma_0 - 2mb_j \gamma^5 \gamma^0 \gamma^j + me^{ij} H_{ij} \gamma^5 \gamma^l \] (13)

Besides EWFT is more economic in algebra, it presents more detailed information with respect to the non-relativistic approximation \([38\, 39\, 40\, 41]\).

As a first step to perform the EFWT, we calculate the squared Hamiltonian \(\mathcal{H}^2\). In order to simplify the algebra, we shall write this quantity as

\[ \mathcal{H}^2 = m^2 \left( 1 + \frac{\tilde{H}^2}{m^2} \right), \] (14)

where \(\tilde{H}^2\) is given by

\[ \tilde{H}^2 = (1 + \tilde{A})[(\delta_{ij} + \tilde{B}_{ij}) \tilde{P}^i + \tilde{C}_j]^2 + \tilde{D}. \] (15)

\[ ^4 \text{Now and so on, we denote transformed quantities by using the "tr" index.} \]
The quantities $\bar{A}$, $\bar{B}_{ij}$, $\bar{C}_j$ and $\bar{D}$ are written in the form

$$
\bar{A} = -2c_{00} - d_{00} \gamma^0 + 2i g_{i00} \gamma^0 \alpha^i,
$$

$$
\bar{B}_{ij} = \frac{1}{2} [ -8 d_{0i} \gamma^0 \alpha_j - 4c_{ij} + 8 g_{0i0} \gamma^0 \epsilon^{lm} \Sigma_m + 8 c_{0i} \alpha_j + 4d_{ij} \gamma^5 + 4g_{mli} \epsilon^{lmj} \gamma^0 \gamma^5 
+ 4ig_{idl} \gamma^0 \gamma^5 \Sigma^l + 4ig_{000} \gamma^0 \alpha_j ],
$$

$$
\bar{C}_j = \frac{1}{2} [ -8m \gamma^0 c_{0j} + 4md_{ij} \gamma^0 \alpha^i - 4md_{0j} \gamma^0 \gamma^5 \alpha_j - 4me_{0j} + 4i g_{ijkl} \gamma^0 \epsilon^{jkl} \Sigma_l - 2mg_{00} \epsilon^{ij} \alpha^5 + 4\alpha_j + 4b_0 \gamma^5 \alpha_j 
- 4H_{0k} \gamma^0 \epsilon^{jkl} \Sigma_l - 4b_j \gamma^5 - 4H_{kl} \epsilon^{jkl} \gamma^0 \gamma^5 + 4i H_{ij} \gamma^0 \gamma^5 \Sigma_l ] ,
$$

$$
\bar{D} = -2m^2 c_{00} - 2m^2 \gamma^0 e_0 + 2m^2 d_{0j} \gamma^0 \alpha_j + 2m^2 \gamma^0 \alpha^i + m^2 \gamma^0 \sigma^{jk} g_{ik0} + 2m\gamma^0 a_0 - 2m \gamma^0 \gamma^5 \alpha^i b_j 
- m \alpha^j H_{ij} + \left( 1 - 2c_{00} + 2d_{00} \gamma^5 - 2i g_{i000} \gamma^0 \alpha^i \right) \frac{i \hbar e}{m c} \Sigma_k B^k .
$$

(16)

There are, in the last equation, even and odd terms. In the FW context, even and odd operators are written as

$$
M_{(EVEN)} = \frac{1}{2} (M + \gamma^0 M \gamma^0) \quad \text{and} \quad M_{(ODD)} = \frac{1}{2} (M - \gamma^0 M \gamma^0) .
$$

(17)

In the situation when there are many odd terms, one must take into account the following relation

$$
\mathcal{H}^{tr} = \tilde{J} \frac{1}{2} (\sqrt{\mathcal{H}^2 - \gamma^0 \sqrt{\mathcal{H}^2} \gamma^0}) + \gamma^0 \frac{1}{2} (\sqrt{\mathcal{H}^2 + \gamma^0 \sqrt{\mathcal{H}^2} \gamma^0}) ,
$$

(18)

where $\tilde{J}$ is given by Eq. [14]. The transformed Hamiltonian is denoted by $\mathcal{H}^{tr}$ which presents only even terms. For this reason, $\mathcal{H}^{tr}$ does not mix spinor components. Naturally, the calculation of $\sqrt{\mathcal{H}^2}$ should be performed and the result inserted in the Eq. [18]. Let us consider that $m^2 \gg \mathcal{H}^2$ in the Eq. [14], such that

$$
\sqrt{\mathcal{H}} = m \left( 1 + \frac{\mathcal{H}^2}{2m^2} \right) .
$$

(19)

After some algebra, the transformed Hamiltonian is given by

$$
\mathcal{H}^{tr} = \gamma^0 m + \frac{1}{2m} \left\{ (1 + A^{tr}) \left[ (\delta_{ij} + B^{tr}_{ij}) \bar{P}^i + C^{tr}_j \right] + D^{tr} \right\} ,
$$

(20)

where

$$
A^{tr} = -2\gamma^0 c_{00} - 2i \gamma^0 d_{00} + 2i g_{i00} \Sigma^i
$$

$$
B^{tr}_{ij} = \frac{1}{2} \left[ 8d_{0i} \gamma^0 \Sigma_j - 4\gamma^0 c_{ij} - 8g_{0i0} \epsilon^{lm} \Sigma_m - 8i c_{0i} \gamma^0 \Sigma_j + 4i \gamma^0 d_{ij} 
+ 4i g_{iml} \epsilon^{mj} - 4g_{ij} \Sigma^l + 4g_{000} \Sigma^i \right] ,
$$

$$
C^{tr}_j = \frac{1}{2} \left[ 8mc_{0j} + 4md_{ij} \Sigma^i - 4md_{0j} \Sigma_j - 4m \gamma^0 e_j + 2mg_{klj} \gamma^0 \epsilon^{klm} \Sigma_m - 4img_{j00} \gamma^0 
- 4mg_{000} \epsilon^{ij} \Sigma_l - 4ime_0 \gamma^0 \Sigma_j - 4md_{k0} \epsilon^{jkl} \Sigma_l - 2igm_{i0} \epsilon^{jkl} \gamma^0 + 4\gamma^0 a_j 
- 4b_0 \gamma^0 \Sigma_j + 4H_{0k} \epsilon^{jkl} \Sigma_l + 4ia_0 \gamma^0 \Sigma_j - 4ib_j \gamma^0 - 4\epsilon^{jkl} H_{kl} - 4H_{ij} \Sigma_l \right] ,
$$

$$
D^{tr} = -2m^2 \gamma^0 c_{00} + 2m^2 e_0 - 2m^2 d_{0j} \gamma^0 \Sigma_j - m^2 g_{ik0} \epsilon^{jkl} \Sigma_l + 2ma_0 - 2mb_j \Sigma_j 
- m \gamma^0 \epsilon^{ij} H_{ij} \Sigma_l + \gamma^0 \left( 1 + 2c_{00} + 2d_{00} - 2g_{i000} \gamma^0 \Sigma^i \right) \frac{i \hbar e}{m c} \Sigma_k B^k .
$$

(21)
We have considered Eq. (14) as the starting point, in order to obtain the transformed Hamiltonian (21). It is possible to see that there are nine new terms in (21) when compared to (13). The new terms are one in the quantity $A$ related to the coefficient $d_{00}$; two in $B_{ij}$ related to the coefficients $c_{0i}$ and $d_{ij}$; four in $C_{j}$ related to the coefficients $a_{0j}$, $b_{ij}$, $c_{0}$ and $g_{d0}$; and two terms in $D$ related to the coefficients $c_{00}$ and $d_{00}$ with the magnetic field. Nevertheless, the exact process has some advantages when compared to the usual one, as commented above. For instance, the new terms that appear in $D$, are relevant when the bound state of the theory is considered.

5 Bound State of the theory

The determination of which kind of experimental tests, like Penning trap, Clock comparison, torsion pendulum, among others (see [42, 43, 44, 45, 46], and references cited there in.), has a significant relevance in the scope of the standard model extension (SME) [3]. In order to determine the kind of experimental test that should be performed, considering the CPT-Lorentz violation terms, presented in the Dirac equation, one should derive the bound state of the theory. In Ref. [2], the authors present a table with a set of many possible bound states. It is expected that the bound associated with transformed Hamiltonian (21), could be found in such a table. Hence, with the knowledge of the bound, together its magnitude and the original Hamiltonian, one can determine the kind of appropriated experimental test should be performed (See [3, 47, 48, 49] for a theoretical framework about CPT-Lorentz breaking tests).

In this section, we derive the bound state of the Hamiltonian (21). Let us begin by taking into account the two components spinor

$$\psi = \left( \begin{array}{c} \phi \\ \chi \end{array} \right) \exp^{-imt}.$$  \hspace{1cm} (22)

From this point one can write, after some algebra, the Dirac equation in the Schrödinger form $i\partial_t \psi = \mathcal{H}\psi$. With these considerations, the Hamiltonian to $\phi$ is written as

$$\mathcal{H} = \frac{1}{2m}\{(1 + A)(\delta_{ij} + B_{ij})\bar{P}^i + C_{j}\}^2 + D\}, \hspace{1cm} (23)$$

where

$$A = -2c_{00} - 2id_{00} + 2g_{i00}\sigma^i,$$

$$B_{ij} = 4d_{0i}\sigma_j - 2c_{ij} - 4g_{0i}\epsilon^{jlm}\sigma_m - 4ic_{0i}\sigma_j + 2id_{ij} + 2g_{lmi}\epsilon^{ljm}j + 2g_{i00}\sigma^i,$$

$$C_{j} = 4mc_{0j} + 2md_{ij}\sigma^i - 2md_{00}\sigma_j - 2mc_j + mg_{klj}\epsilon^{klm}\sigma_m - 2img_{j00} - 2mg_{i00}\epsilon^{jlm}\sigma_m - 2mg_{0i}\epsilon^{jlm}\sigma_m - 2img_{j00} - 2a_{j} + 2g_{lmi}\epsilon^{ljm}j + 2g_{i00}\sigma^i,$$

$$D = -2m^2c_{00} + 2m^2c_0 - 2m^2d_{00}\sigma^j - m^2g_{i00}\epsilon^{ijkl}\sigma_l - 2ma_0 - 2mb_j\sigma^j + mc_{i}\epsilon^{ijkl}\sigma_l + [1 + 2c_{00} + 2id_{00} - 2g_{i00}\sigma^i]i\hbar e^{ijkl}\sigma_l B^k.$$

(24)

The bound state of the Hamiltonian (21) can be calculated by taking into account the Lorentz violating potential $V$, which corresponds to the term $D$, in the last equation. Actually, this
potential obeys the following relation

\[ V = -\tilde{\theta}_j \sigma^j, \]  

where \( \sigma \) represents the spin matrices. From this point, one can calculate the bound state of the theory:

\[ \tilde{\theta}_j = b_j + \frac{1}{2} \Gamma^{lmj} H_{lm} + m d_{j0} + \frac{1}{2} m e^{lmj} g_{lm0} - \left[ 1 + 2c_{00} + 2id_{00} - 2g_{000} \sigma^i \right] ihe \frac{1}{2mc^2} B_j. \]

As it was expected, this bound state is a specific combination of two parts related to the SME coefficients. The first part includes the coefficients \( b_j, H_{lm}, d_{j0}, \) and \( g_{lm0}, \) and the bound is based on atomic clock and other non-relativistic experiments that can involve maser/magnetometer (see, for example, table VII in [2]). In the second part, there is the presence of magnetic field which can be a remarkable and very important result from the experimental point of view. As it is known, the external fields in the Eq. (26), are very weak. However, the modulus of \( B \) may be sufficiently high, in order to compensate the weakness of the interactions \( c_{00}, d_{00}, \) and \( g_{000}. \) In another words, with a strong enough magnetic field, one can have indications, in principle, of the kind of motion generated by the external field commented above. It is, an indirect way of performing measurements of such a weak external fields.

6 Conclusions and discussions

The exact Foldy-Wouthuysen transformation was performed in the context of Dirac field interacting with many possible external fields associated with CPT-Lorentz violation.

The first result of the work is written in the form of a table, representing the Hamiltonian with the complete set CPT-Lorentz violating terms, in the Dirac equation. In such table, the terms highlighted in boldface do not anti-commute with the usual involution operator (2).

Another result of the work is the appropriated involution operator, given by Eq. (6), such that the anti-commutation relation with the Hamiltonian of the problem is achieved. Actually, Eq. (6), introduces a new possibility of performing EFWT. From now on, a large class of Hamiltonians admit the exact transformation, since involution operator (6) is used.

In section 4, the usual EFWT algorithm was applied to the initial Hamiltonian and the exact transformation was performed. As it was expected, the EWFT approach presents a transformed Hamiltonian (13) with additional terms, when compared to the Hamiltonian (21), where the usual FWT is used.

In the last section we derive the bound state of the theory, given by Eq. (26). It worth mentioning that the possibility of the weakness of CPT-Lorentz terms to be compensated by the presence of a strong magnetic field. Thus, one can understand the particle behavior due to the interactions with external field, it gives the possibility to measure the external fields in indirect way.
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