Periodical solutions of Poisson-gradient dynamical systems with periodical potential

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Abstract

The main purpose of this paper is the study of the action that produces Poisson-gradient systems and their multiple periodical solutions. The Section 1 establishes the basic tools.

The section 2 underlines conditions in which the action \( \varphi(u) = \int_{T_0} \left[ \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F(t, u(t)) \right] dt^1 \wedge ... \wedge dt^p \), that produces the Poisson-gradient systems, is continuous, and some conditions in which the general action \( \varphi(u) = \int_{T_0} L \left( t, u(t), \frac{\partial u}{\partial t}(t) \right) dt^1 \wedge ... \wedge dt^p \) is continuously differentiable.

The Section 3 studies the multiple periodical solutions of a Poisson-gradient system in the case when the potential function \( F \) has a spatial periodicity.

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1 Introduction

In this paper we will note by \( W_{T}^{1,2} \) the Sobolev spaces of the \( u \in L^2 [T_0, R^n] \) functions, which have the weak derivative \( \frac{\partial u}{\partial t} \in L^2 [T_0, R^n] \), \( T_0 = [0, T^1] \times ... \times [0, T^p] \subset R^p \). The weak derivatives are defined using the space \( C_T^\infty \) of all
indefinitely differentiable multiple $T$-periodic function from $\mathbb{R}^p$ into $\mathbb{R}^n$. We consider $H^1_T$, the Hilbert space of the $W^{1,2}_T$. The geometry on $H^1_T$ is realized by the scalar product

$$\langle u, v \rangle = \int_{T_0} \left( \delta_{ij} u^i(t) v^j(t) + \delta_{ij} \delta^{\alpha\beta} \frac{\partial u^i}{\partial t^\alpha}(t) \frac{\partial v^j}{\partial t^\beta}(t) \right) dt^1 \wedge \ldots \wedge dt^p,$$

and the associated Euclidean norm $\| \cdot \|$. These are induced by the scalar product (Riemannian metric)

$$G = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \delta^{\alpha\beta} \delta_{ij} \end{pmatrix}$$

on $\mathbb{R}^{n+np}$ (see the jet space $J^1(T_0, \mathbb{R}^n)$).

Let $t = (t^1, \ldots, t^p)$ be a generic point in $\mathbb{R}^p$. Then the opposite faces of the parallelepiped $T_0$ can be described by the equations

$$S^+_i : t^i = 0, S^-_i : t^i = T^i$$

for each $i = 1, \ldots, p$. We denote

$$\| u \|_{L^2} = \int_{T_0} \delta_{ij} u^i(t) v^j(t) dt^1 \wedge \ldots \wedge dt^p,$$

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2} = \int_{T_0} \delta_{ij} \delta^{\alpha\beta} \frac{\partial u^i}{\partial t^\alpha}(t) \frac{\partial v^j}{\partial t^\beta}(t) dt^1 \wedge \ldots \wedge dt^p,$$

$$(u, v) = \delta_{ij} u^i v^j, \quad |u| = \sqrt{\delta_{ij} u^i u^j}.$$

We study the extremals of the action

$$\varphi(u) = \int_{T_0} \left[ \frac{1}{2} \left\| \frac{\partial u}{\partial t} \right\|^2 + F(t, u(t)) \right] dt^1 \wedge \ldots \wedge dt^p$$

on $H^1_T$ in the case when the potential $F$ function has spatial periodicity. So, we consider that there exist $P_1, \ldots, P_n \in \mathbb{R}$ so that $F(t, x + P_i e_i) = F(t, x)$, for any $t \in T_0$, $x \in \mathbb{R}^n$ and any $i \in \{1, \ldots, n\}$. The vectors $e_1, \ldots, e_n$ create a canonical base in the Euclidian space $\mathbb{R}^n$. The extremals of the action $\varphi$ are being determined with the minimizing sequences method. For the existence
of the minimizing bounded sequence we need to introduce the spatial periodicity condition of the potential function $F$. The function that realizes the minimum of the action $\varphi$ verifies the Euler-Lagrange equations, which in the case of the Lagrangian

$$L \left( t, u(t), \frac{\partial u}{\partial t} \right) = \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F(t, u(t))$$

defined on $H^1_T$, reduces to a Poisson-gradient PDEs, $\Delta u(t) = \nabla F(t, u(t))$,

$$u|_{s^-_i} = u|_{s^+_i}, \frac{\partial u}{\partial t}|_{s^-_i} = \frac{\partial u}{\partial t}|_{s^+_i}, i = 1, ..., p.$$

2 The action that produces Poisson-gradient systems

2.1 Multi-time Euler-Lagrange equations

We consider the multi-time variable $t = (t^1, ..., t^p) \in \mathbb{R}^p$, the functions $x^i : \mathbb{R}^p \to \mathbb{R}, (t^1, ..., t^p) \to x^i(t^1, ..., t^p), i = 1, ..., n,$ and we denote $x^i_\alpha = \frac{\partial x^i}{\partial t^\alpha}, \alpha = 1, ..., p$. The Lagrange function

$$L : \mathbb{R}^{p+n+np} \to \mathbb{R}, \left( t^\alpha, x^i, x^i_\alpha \right) \to L \left( t^\alpha, x^i, x^i_\alpha \right)$$

gives the Euler-Lagrange equations

$$\frac{\partial}{\partial t^\alpha} \frac{\partial L}{\partial x^i_\alpha} = \frac{\partial L}{\partial x^i}, i = 1, ..., n, \alpha = 1, ..., p$$

(second order PDEs system on the n-dimensional space).

The multi-time Lagrangian and Hamiltonian dynamics is based on the concept of multisymplecticity (polysymplecticity) [3], [5]-[12]. Our task is to develop some ideas from [1]-[2], [12]-[14] having in mind the single-time theory in [4].
2.2 Continuous action

We consider the Lagrange function $L : T_0 \times \mathbb{R}^n \times \mathbb{R}^{np} \to \mathbb{R}, (t^\alpha, u^i, u^i_\alpha) \to L(t^\alpha, u^i, u^i_\alpha)$, $u^i_\alpha = \frac{\partial u^i}{\partial t^\alpha}, \alpha = 1, \ldots, p, i = 1, \ldots, n, u^i : T_0 \to \mathbb{R}, (t^1, \ldots, t^p) \to u^i(t^1, \ldots, t^p),$ $L(t^\alpha, u^i, u^i_\alpha) = \frac{1}{2} \left| \frac{\partial u^i}{\partial t} \right|^2 + F(t, u(t)).$

In the following result we will establish the conditions in which the action $\varphi(u) = \int_{T_0} L(t, u(t), \frac{\partial u}{\partial t}(t)) dt^1 \wedge \ldots \wedge dt^p$ is continuous.

**Theorem 1.** Let $F : T_0 \times \mathbb{R}^n \to \mathbb{R}, (t, u) \to F(t, u)$ be a measurable function in $t$ for any $u \in \mathbb{R}^n$ and continuously differentiable in $u$ for any $t \in T_0$, $T_0 = [0, T^1] \times \ldots \times [0, T^p] \subset \mathbb{R}^p$. If exists $M \geq 0$ and $g \in C(T_0, \mathbb{R})$ such that $|\nabla_u F(t, u)| \leq M |u| + g(t)$ for any $t \in T_0$ and any $u \in \mathbb{R}$, then $\varphi(u) = \int_{T_0} \left[ \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F(t, u(t)) \right] dt^1 \wedge \ldots \wedge dt^p$ is continuous in $H^1_T$.

**Proof.** We consider the $(u_k)_{k \in \mathbb{N}}$ sequence which is convergent in $H^1_T$ and we will note by $u$ his limit. This leads us to the fact that

$$\|u_k - u\|^2 = \int_{T_0} \left[ |u_k(t) - u(t)|^2 + \left| \frac{\partial u_k}{\partial t}(t) - \frac{\partial u}{\partial t}(t) \right|^2 \right] dt^1 \wedge \ldots \wedge dt^p$$

$$= \|u_k - u\|^2_{L^2} + \left\| \frac{\partial u_k}{\partial t} - \frac{\partial u}{\partial t} \right\|^2_{L^2} \to 0$$

when $k \to \infty$. The convergence of $u_k$ to $u$ in $H^1_T$ is equivalent to the convergence of $u_k$ to $u$ and to the convergence of $\frac{\partial u_k}{\partial t}$ to $\frac{\partial u}{\partial t}$ in $L^2$. By consequence $\|u_k\|_{L^2}$ and $\left\| \frac{\partial u_k}{\partial t} \right\|_{L^2}$ are bounded in $\mathbb{R}$. In order to show the continuity of $\varphi$, we will prove that $|\varphi(u_k) - \varphi(u)| \to 0$ when $u_k \to u$ in $H^1_T$. In the evaluations that we will do for $|\varphi(u_k) - \varphi(u)|$ we will utilize the following
inequality $\|u\|^2 - \|v\|^2 \leq \|u\| \|u - v\| + \|v\| \|u - v\|$ which is true in any vectorial space with an scalar product. So, we obtain:

$$\left| \varphi (u_k) - \varphi (u) \right| \leq \frac{1}{2} \left( \int_{T_0} \| \frac{\partial u_k}{\partial t} \|_2^2 \ dt \right)^{1} \wedge \ldots \wedge dt^p - \int_{T_0} \left| \frac{\partial u}{\partial t} \right|_2^2 \ dt^1 \wedge \ldots \wedge dt^p$$

$$+ \int_{T_0} \left( \int_0^1 \left( \left( F(t, u_k(t)) - F(t, u(t)) \right) \ dt \right)^{1} \wedge \ldots \wedge dt^p \right)$$

$$\leq \frac{1}{2} \left[ \left\| \frac{\partial u_k}{\partial t} \right\|_2 \left\| \frac{\partial u_k}{\partial t} - \frac{\partial u}{\partial t} \right\|_2 + \left\| \frac{\partial u}{\partial t} \right\|_2 \left\| \frac{\partial u_k}{\partial t} - \frac{\partial u}{\partial t} \right\|_2 \right]$$

$$+ \int_{T_0} \left( \left( \int_0^1 \left( \left| \left( M |u_k(t) + s (u (t) - u_k (t)) | + g (t) \right) | u (t) - u_k (t) | \right) \right) \ dt \right)^{1} \wedge \ldots \wedge dt^p$$

$$\leq \frac{1}{2} \left[ \left\| \frac{\partial u_k}{\partial t} \right\|_2 \left\| \frac{\partial u_k}{\partial t} - \frac{\partial u}{\partial t} \right\|_2 + \left\| \frac{\partial u}{\partial t} \right\|_2 \left\| \frac{\partial u_k}{\partial t} - \frac{\partial u}{\partial t} \right\|_2 \right]$$

$$+ \int_{T_0} \left( \left( \int_0^1 \left( \left| \left( M |u_k (t) + (1 - s) (u_k (t) - u (t)) | + g (t) \right) u (t) - u_k (t) | \right) \right) \ dt \right)^{1} \wedge \ldots \wedge dt^p$$

Because the sequence $\left\| \frac{\partial u_k}{\partial t} \right\|_2$ is bounded, it exists $C_1$ such that $\left\| \frac{\partial u_k}{\partial t} \right\|_2 \leq C_1$ for any $k \in N$. In the following, we will note by $C_2 = \max_{t \in T_0} g (t)$ and we have

$$\left| \varphi (u_k) - \varphi (u) \right| \leq \frac{1}{2} C_1 \left\| \frac{\partial u_k}{\partial t} - \frac{\partial u}{\partial t} \right\|_2 + \frac{1}{2} \left\| \frac{\partial u}{\partial t} \right\|_2 \left\| \frac{\partial u_k}{\partial t} - \frac{\partial u}{\partial t} \right\|_2$$

$$+ M \left\| u \right\|_2 \left( \int_{T_0} \left| u (t) - u_k (t) \right|^2 \ dt \wedge \ldots \wedge dt^p \right) \frac{1}{2}$$
\[ + M \int_{T_0} |u_k(t) - u(t)|^2 \, dt^1 \wedge ... \wedge dt^p + C_2 \int_{T_0} |u(t) - u_k(t)| \, dt^1 \wedge ... \wedge dt^p \]
\[
\leq \frac{1}{2} \left( C_1 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2} \right) \left\| \frac{\partial u_k}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2} \\
+ \left( M \|u\|_{L^2} + C_2 (T^1 \cdot \cdot \cdot T^p)^{\frac{1}{2}} \right) \left( \int_{T_0} |u(t) - u_k(t)|^2 \, dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}} \\
+ M \int_{T_0} |u_k(t) - u(t)|^2 \, dt^1 \wedge ... \wedge dt^p = \frac{1}{2} \left( C_1 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2} \right) \left\| \frac{\partial u_k}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2} \\
+ \left( M \|u\|_{L^2} + C_2 (T^1 \cdot \cdot \cdot T^p)^{\frac{1}{2}} \right) \|u - u_k\|_{L^2} + M \|u_k - u\|^2_{L^2}.
\]

Because \( \left\| \frac{\partial u_k}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2} \rightarrow 0 \) and \( \|u_k - u\|_{L^2} \rightarrow 0 \), when \( k \rightarrow \infty \) it results that \( \varphi(u_k) \rightarrow \varphi(u) \); from here we obtain the continuity of \( \varphi \) in \( H^1_T \).

### 2.3 Continuously differentiable action

In order to obtain a more general result than the one found in the previous theorem, we define the action

\[ \varphi : W^{1,2}_T \rightarrow R, \varphi(u) = \int_{T_0} L \left( t, u(t), \frac{\partial u}{\partial t}(t) \right) \, dt^1 \wedge ... \wedge dt^p, \]

\[ T_0 = [0, T^1] \times ... \times [0, T^p] \subset R^p. \]

Concerning this action we have the following Theorem which extends the particular case \( p = 1 \) from [4].

**Theorem 2.** We consider \( L : T_0 \times R^n \times R^{np} \rightarrow R, (t, x, y) \rightarrow L(t, x, y), \)
a measurable function in \( t \) for any \( (x, y) \in R^n \times R^{np} \) and with the continuous partial derivatives in \( x \) and \( y \) for any \( t \in T_0 \). If here exist \( a \in C^1(R^+, R^+) \)
with the derivative $a'$ bounded from above, $b \in C(T_0, \mathbb{R}^n)$ such that for any $t \in T_0$ and any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^{np}$ to have

$$
\begin{align*}
|L(t, x, y)| & \leq a (|x| + |y|^2) b(t), \\
|\nabla_x L(t, x, y)| & \leq a |x| b(t), \\
|\nabla_y L(t, x, y)| & \leq a |y| b(t), \\
\end{align*}
$$

(1)

then, the functional $\varphi$ has continuous partial derivatives in $W^{1,2}_T$ and his gradient derives from the formula

$$
(\nabla \varphi (u), v) = \int_{T_0} \left[ \left( \nabla_x L \left( t, u(t) + \frac{\partial u}{\partial t} (t) \right), v(t) \right) \\
+ \left( \nabla_y L \left( t, u(t) + \frac{\partial u}{\partial t} (t) \right), \frac{\partial v}{\partial t} (t) \right) \right] dt^1 \wedge ... \wedge dt^p.
$$

(2)

**Proof.** It is enough to prove that $\varphi$ has the derivative $\varphi'(u) \in \left( W^{1,2}_T \right)^*$ given by the relation (2) and the function $\varphi' : W^{1,2}_T \to \left( W^{1,2}_T \right)^*$, $u \to \varphi'(u)$ is continuous. We consider $u, v \in W^{1,2}_T$, $t \in T_0$, $\lambda \in [-1, 1]$. We build the functions

$$
F (\lambda, t) = L \left( t, u(t) + \lambda v(t), \frac{\partial u}{\partial t} (t) + \lambda \frac{\partial v}{\partial t} (t) \right)
$$

and

$$
\Psi (\lambda) = \int_{T_0} F (\lambda, t) dt^1 \wedge ... \wedge dt^p.
$$

Because the derivative $a'$ is bounded from above, exist $M > 0$ such that

$$
\frac{a'(|u|) - a(0)}{|u|} = a'(c) \leq M. \text{ This means that } a(|u|) \leq M |u| + a(0).
$$

On the other side

$$
\frac{\partial F}{\partial \lambda} (\lambda, t) = \left( \nabla_x L \left( t, u(t) + \lambda v(t), \frac{\partial u}{\partial t} (t) + \lambda \frac{\partial v}{\partial t} (t) \right), v(t) \right)
$$

$$
+ \left( \nabla_y L \left( t, u(t) + \lambda v(t), \frac{\partial u}{\partial t} (t) + \lambda \frac{\partial v}{\partial t} (t) \right), \frac{\partial v}{\partial t} (t) \right) \leq a (|u(t) + \lambda v(t)|)
$$

$$
b(t) |v(t)| + a \left( \left| \frac{\partial u}{\partial t} (t) + \lambda \frac{\partial v}{\partial t} (t) \right| \right) b(t) \left| \frac{\partial v}{\partial t} (t) \right|.
$$
\[ \leq b_0 \left( M \left( |u(t)| + |v(t)| \right) + a(0) \right) |v(t)| \\
+ b_0 \left( M \left( \left| \frac{\partial u}{\partial t} (t) \right| + \left| \frac{\partial v}{\partial t} (t) \right| \right) + a(0) \right) \left| \frac{\partial v}{\partial t} (t) \right| , \]

where

\[ b_0 = \max_{t \in T_0} b(t). \]

Then, we have

\[ \left| \frac{\partial F}{\partial \lambda} (\lambda, t) \right| \leq d(t) \in L^1 (T_0, R^+). \]

Then Leibniz formula of differentiation under integral sign is applicable and

\[ \frac{\partial \Psi}{\partial \lambda} (0) = \int_{T_0} \frac{\partial F}{\partial \lambda} (0, t) \, dt^1 \wedge \ldots \wedge dt^p = \int_{T_0} \left[ \left( \nabla_x L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right), v(t) \right) \\
+ \left( \nabla_y L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right), \frac{\partial v}{\partial t} (t) \right) \right] dt^1 \wedge \ldots \wedge dt^p. \]

Moreover,

\[ \left| \nabla_x L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right) \right| \leq b_0 \left( M |u(t)| + |a(0)| \right) \in L^1 \left( T_0, R^+ \right) \]

and

\[ \left| \nabla_y L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right) \right| \leq b_0 \left( M \left| \frac{\partial u}{\partial t} (t) \right| + |a(0)| \right) \in L^2 \left( T_0, R^+ \right). \]

That is why

\[ \int_{T_0} \left[ \left( \nabla_x L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right), v(t) \right) \\
+ \left( \nabla_y L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right), \frac{\partial v}{\partial t} (t) \right) \right] dt^1 \wedge \ldots \wedge dt^p \]
\[ \leq \int_{T_0} \left| \nabla_x L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right) \right| |v(t)| \, dt^1 \wedge \ldots \wedge dt^p \]
\[ + \int_{T_0} \left| \nabla_y L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right) \right| \left| \frac{\partial v}{\partial t} (t) \right| \, dt^1 \wedge \ldots \wedge dt^p \]
\[
\leq b_0 \int_{T_0} (M |u(t)| + |a(0)|)|v(t)|\,dt^1 \wedge \ldots \wedge dt^p \\
+b_0 \int_{T_0} \left( M \left| \frac{\partial u}{\partial t}(t) \right| + |a(0)| \right) \left| \frac{\partial v}{\partial t}(t) \right|\,dt^1 \wedge \ldots \wedge dt^p
\]

By using the inequality Cauchy-Schwartz we find
\[
\left| \frac{\partial \Psi}{\partial \lambda}(0) \right| \leq b_0 \left( \int_{T_0} (M |u(t)| + |a(0)|)^2 \,dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}} \\
\cdot \left( \int_{T_0} |v(t)|^2 \,dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}} \\
+b_0 \left( \int_{T_0} \left( M \left| \frac{\partial u}{\partial t}(t) \right| + |a(0)| \right)^2 \,dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}} \left( \int_{T_0} \left| \frac{\partial v}{\partial t}(t) \right|^2 \,dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}} \\
\leq C_1 \left( \int_{T_0} |v(t)|^2 \,dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}} + C_2 \left( \int_{T_0} \left| \frac{\partial v}{\partial t}(t) \right|^2 \,dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}} \\
\leq \max \{ C_1, C_2 \} \, 2^{\frac{p}{2}} \left( \int_{T_0} \left( |v(t)|^2 + \left| \frac{\partial v}{\partial t}(t) \right|^2 \right) \,dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}} = C \|v\|.
\]

By consequence, the action \( \varphi \) has the derivative \( \varphi' \in (W^{1,2}_T)^* \) given by (2). The Krasnoselski theorem and the hypothesis (1) imply the fact that the application \( u \rightarrow \left( \nabla_xL \left( \cdot, u, \frac{\partial u}{\partial t} \right), \nabla_yL \left( \cdot, u, \frac{\partial u}{\partial t} \right) \right) \), from \( W^{1,2}_T \) to \( L^1 \times L^2 \), is continuous, so \( \varphi' \) is continuous from \( W^{1,2}_T \) to \( (W^{1,2}_T)^* \) and the proof is complete.

**Theorem 3.** If the action
\[
\varphi(u) = \int_{T_0} \left[ \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F(t, u(t)) \right] \,dt^1 \wedge \ldots \wedge dt^p
\]
is continuously differentiable on \( H^1_T \) and \( u \in H^1_T \) is a solution of the equation \( \varphi'(u) = 0 \) (critical point), then the function \( u \) has a weak Laplacian \( \Delta u \) (or the Jacobian matrix \( \frac{\partial u}{\partial t} \) has a weak divergence) and
\[
\Delta u = \nabla F(t, u(t))
\]
a.e. on $T_0$ and
\[ u \big|_{s_t^-} = u \big|_{s_t^+}, \quad \frac{\partial u}{\partial t} \big|_{s_t^-} = \frac{\partial u}{\partial t} \big|_{s_t^+}. \]

\textbf{Proof.} We build the function
\[ \Phi : [-1, 1] \to \mathbb{R}, \]
\[ \Phi (\lambda) = \varphi (u + \lambda v) = \int_{T_0} \left[ \frac{1}{2} \left| \frac{\partial}{\partial t} (u(t) + \lambda v(t)) \right|^2 + F(t, u(t) + \lambda v(t)) \right] \, dt^1 \wedge \ldots \wedge dt^p, \]
where $v \in C^\infty_T (T_0, \mathbb{R}^n)$. The point $\lambda = 0$ is a critical point of $\Phi$ if and only if the point $u$ is a critical point of $\varphi$. Consequently
\[ 0 = \langle \varphi' (u), v \rangle = \int_{T_0} \left[ \delta^\alpha\delta^\beta \delta_{ij} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} + \delta_{ij} \nabla^i F(t, u(t)) v^j(t) \right] \, dt^1 \wedge \ldots \wedge dt^p, \]
for all $v \in H^1_T$ and hence for all $v \in C^\infty_T$. The definition of the weak divergence,
\[ \int_{T_0} \delta^\alpha\delta^\beta \delta_{ij} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} \, dt^1 \wedge \ldots \wedge dt^p = - \int_{T_0} \delta^\alpha\delta^\beta \delta_{ij} \frac{\partial^2 u^i}{\partial t^\alpha \partial t^\beta} v^j \, dt^1 \wedge \ldots \wedge dt^p, \]
shows that the Jacobian matrix $\frac{\partial u}{\partial t}$ has weak divergence (the function $u$ has a weak Laplacian) and
\[ \Delta u (t) = \nabla F(t, u(t)) \]
a.e. on $T_0$. Also, the existence of weak derivatives $\frac{\partial u}{\partial t}$ and weak divergence $\Delta u$ implies that
\[ u \big|_{s_t^-} = u \big|_{s_t^+}, \quad \frac{\partial u}{\partial t} \big|_{s_t^-} = \frac{\partial u}{\partial t} \big|_{s_t^+}. \]
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**Theorem 4.** If \( F : T_0 \times \mathbb{R}^n \to \mathbb{R}, (t,x) \to F(t,x) \) functions have the properties:

1) \( F(t,x) \) is measurable in \( t \) for any \( x \in \mathbb{R}^n \) and continuously differentiable in \( x \) for any \( t \in T_0 \), and there exist the functions \( a \in C^1(\mathbb{R}^+, \mathbb{R}^+) \) with the derivative \( a' \) bounded from above and \( b \in C(T_0, \mathbb{R}^+) \) such that for any \( t \in T_0 \) and any \( u \in \mathbb{R}^n \) to have \( |F(t,u)| \leq a(|u|) b(t) \) and \( |\nabla_u F(t,u)| \leq a(|u|) b(t) \),

2) \( F(t,x) > 0 \), for any \( t \in T_0 \) and any \( x \in \mathbb{R}^n \),

3) For any \( i \in \{1, \ldots, n\} \) it exists \( P_i \in \mathbb{R} \) such that \( F(t,x + P_i e_i) = F(t,x) \), for any \( t \in T_0 \) and any \( x \in \mathbb{R}^n \),

4) The action \( \varphi_1(u) = \int_{T_0} F(t,u(t)) dt^1 \wedge \ldots \wedge dt^p \) is weakly lower semi-continuous on \( H_1^T \).

If \( \int_{T_0} F(t,u) dt^1 \wedge \ldots \wedge dt^p \to \infty \) when \( |u| \to \infty \), then, the problem \( \Delta u(t) = \nabla F(t,u(t)) \) with the boundary condition

\[
    u|_{s^-_i} = u|_{s^+_i}, \quad \frac{\partial u}{\partial t}|_{s^-_i} = \frac{\partial u}{\partial t}|_{s^+_i}
\]

has at least solution which minimizes the action

\[
    \varphi(u) = \int_{T_0} \left[ \frac{1}{2} \left| \frac{\partial u}{\partial t}(t) \right|^2 + F(t,u(t)) \right] dt^1 \wedge \ldots \wedge dt^p
\]

in \( H_1^T \).

**Proof.** From Theorem 2, the action \( \varphi \) is continuously differentiable. From the periodicity and the continuity of \( F \), it results that exists the function \( d \in L^1(T_0, \mathbb{R}) \) such that \( F(t,x) \geq d(t) \geq 0 \), for any \( t \in T_0 \) and any \( x \in \mathbb{R}^n \). By consequence \( \int_{T_0} F(t,u(t)) dt^1 \wedge \ldots \wedge dt^p = C_1 \geq 0 \). It results the inequality

\[
    \varphi(u) \geq \int_{T_0} \frac{1}{2} \left| \frac{\partial u}{\partial t}(t) \right|^2 dt^1 \wedge \ldots \wedge dt^p - C_1 \text{ for any } u \in H_1^T. \quad \text{As result} \quad \inf_{u \in H_1^T} \varphi(u) < \infty.
\]

Because \( \varphi(u) + C_1 \geq \int_{T_0} \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge \ldots \wedge dt^p \), for any \( u \in H_1^T \), we
have the same inequality and for any \( u = u_k \) where \( (u_k) \) it is a minimizing sequence for \( \varphi \) in \( H^1_T \). So, we obtain

\[
\int_{T_0}^T \frac{1}{2} \left| \frac{\partial u_k}{\partial t} \right|^2 dt^1 \wedge \ldots \wedge dt^p \leq C_2, \quad \text{for any } k \in \mathbb{N}.
\]  

(3)

We consider \( u_k = \overline{u}_k + \tilde{u}_k \), where \( \overline{u}_k = \frac{1}{T^1 \ldots T^p} \int_{T_0}^T u_k(t) dt^1 \wedge \ldots \wedge dt^p \). From the relation (3) and the Wirtinger inequality we have

\[
\| \tilde{u}_k \| \leq C_3, \quad k \in \mathbb{N}, \quad C_3 \geq 0.
\]  

(4)

On the other side, from the periodicity of \( F \) we find that \( \varphi (u + P_i e_i) = \varphi (u) \), for any \( i \in \{1, \ldots n\} \) and any \( u \in H^1_T \). If the sequence \( (u_k) \) is a minimizing one for the action \( \varphi \), then the sequence \( (u^*_k) \),

\[
u^*_k = \left( \overline{u}^1_k + \tilde{u}^1_k + k_1 P_1, \ldots, \overline{u}^n_k + \tilde{u}^n_k + k_n P_n \right)
\]
is also a minimizing sequence, for any \( k_1, \ldots, k_n \in \mathbb{Z} \). Obviously, we may choose \( k_i \in \mathbb{Z}, i = 1, \ldots, n \) such that

\[
0 \leq \overline{u}^i_k + k_i P_i \leq P_i, i = 1, \ldots, n.
\]

From the relations (3) and (4) it results that the sequence \( (u^*_k) \) is bounded, so the action \( \varphi \) has a minimizing bounded sequence. By eventually passing to a subsequence, we may consider that the \( (u^*_k) \) sequence is weakly convergent with the limit \( u \).

The Hilbert space \( H^1_T \) is reflexive. By consequence, the sequence \( (u^*_k) \) (or one of his subsequence) is weakly convergent in \( H^1_T \) with the limit \( u \). Because

\[
\varphi_2 (u) = \int_{T_0}^T \delta_{ij} \delta^{\alpha \beta} \frac{\partial u^i}{\partial t^\alpha} (t) \frac{\partial u^j}{\partial t^\beta} (t) dt^1 \wedge \ldots \wedge dt^p
\]
is convex it results that \( \varphi_2 \) is weakly lower semi-continuous, so

\[
\varphi (u) = \varphi_1 (u) + \varphi_2 (u)
\]
is weakly lower semi-continuous and \( \varphi (u) \leq \lim \varphi (u_k) \). This means that \( u \) is minimum point of \( \varphi \). From the Theorem 3 this means that \( u \) is solution of boundary value problem

\[
\Delta u (t) = \nabla F (t, u (t)) ,
\]  

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\[ u \mid_{S_i^-} = u \mid_{S_i^+}, \frac{\partial u}{\partial t} \mid_{S_i^-} = \frac{\partial u}{\partial t} \mid_{S_i^+}. \]

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