Group actions on Banach spaces and a geometric characterization of a-T-menability

Piotr W. Nowak

Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240 USA.

Abstract

We prove a geometric characterization of a-T-menability through proper, affine, isometric actions on the Banach spaces $L^p[0,1]$ for $1 < p < 2$. This answers a question of A. Valette.

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Let $X$ be a normed space. An affine, isometric action of a group $\Gamma$ on $X$ is defined as $\Psi(g)v = \pi(g)v + \gamma(g)$ for $v \in X$, $g \in \Gamma$, where $\pi$ is a unitary (i.e. linear isometric) representation of $\Gamma$ on $X$ and $\gamma : \Gamma \to X$ satisfies the cocycle identity with respect to $\pi$, i.e. $\gamma(gh) = \pi(g)\gamma(h) + \gamma(g)$. The action is proper if $\lim_{g \to \infty} \|\Psi(g)v\| = \infty$ for every $v \in X$. This is equivalent to $\lim_{g \to \infty} \|\gamma(g)\| = \infty$. One can express this idea in the language of coarse geometry by saying that every orbit map is a coarse embedding.

The following definition is due to Gromov.

Definition 1 ([Gr, 6.A.III]) A second countable, locally compact group is said to be a-T-menable (has the Haagerup approximation property) if it admits a proper, affine, isometric action on a separable Hilbert space $\mathcal{H}$.

A-T-menability was designed as a strong opposite of Kazhdan’s property (T). We recall here a geometric characterization of property (T) known as the Delorme-Guichardet Theorem, for a detailed account of the subject see [BHV].

Definition 2 A second countable, locally compact group $\Gamma$ has Kazhdan’s Property (T) if and only if every affine isometric action of $\Gamma$ on a Hilbert space has a fixed point.

Email address: piotr.nowak@vanderbilt.edu (Piotr W. Nowak).
As suggested in the definition, a-T-menability turned out to be equivalent to the Haagerup property (this was proved in [BCV]), which arose in the study of approximation properties of operator algebras and has application to harmonic analysis. There are many other characterizations of a-T-menability, in particular Gromov showed [Gr, 7.A] that it is equivalent to existence of a proper isometric action on the (either real or complex) infinite dimensional hyperbolic space.

Recently N. Brown and E. Guentner [BG] proved that every discrete group admits a proper, affine and isometric action on an \( \ell_2 \)-direct sum \( (\sum \ell_{p_n})_2 \), for some sequence \( \{p_n\} \) satisfying \( p_n \to \infty \). Since there are discrete groups which are not a-T-menable, i.e. groups which are Kazhdan (T), an existence of a proper, affine, isometric action on a reflexive Banach space does not in general imply a-T-menability. Also results of G. Yu show that property (T) groups may admit proper, affine, isometric actions on the spaces \( \ell_p \) for \( p > 2 \) [Yu]. We also refer the reader to the recent article [BFGM] for a thorough study of similar questions in the context of property (T).

What we are interested in is to find Banach spaces actions on which imply or characterize a-T-menability. The motivation comes from a question of A.Valette, who in [CCJJV, Section 7.4.2] asked whether there are geometric characterizations of a-T-menability other than through actions on infinite-dimensional hyperbolic spaces. We prove the following

**Theorem 3** For a second countable, locally compact group \( \Gamma \) the following conditions are equivalent:

1. \( \Gamma \) is a-T-menable

2. \( \Gamma \) admits a proper, affine, isometric action on the Banach space \( L_p[0,1] \) for some \( 1 < p < 2 \)

3. \( \Gamma \) admits a proper, affine, isometric action on the Banach space \( L_p[0,1] \) for all \( 1 < p < 2 \)

Note that the results in [BG,Yu] show that Theorem 3 cannot be extended to \( p > 2 \) or to the class of reflexive or uniformly convex Banach spaces.

We also want to mention a problem raised in [Gr, 6.D3] by Gromov: for a given group \( \Gamma \) find all such \( p \geq 1 \) for which \( \Gamma \) admits a proper, affine, isometric action on \( \ell_p \). Our methods give some partial information on possible answers to this question, namely Proposition 8 states that only a-T-menable groups may admit such actions on \( \ell_p \) for \( 0 < p < 2 \).

A-T-menability is an important property in studying the Baum-Connes Conjecture. N. Higson and G. Kasparov showed [HK] that every discrete a-T-
menable group satisfies the Baum-Connes Conjecture with arbitrary coefficients.

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1 Proofs

We will use the fact that a-T-menability can be characterized in terms of existence of certain conditionally negative definite functions, which we define now.

By a kernel on a set $X$ we mean a symmetric function $K : X \times X \to \mathbb{R}$.

**Definition 4** A kernel $K$ is said to be conditionally negative definite if

$$\sum K(x_i, x_j) c_i c_j \leq 0$$

for all $n \in \mathbb{N}$ and $x_1, ..., x_n \in X$, $c_1, ..., c_n \in \mathbb{R}$ such that $\sum c_i = 0$.

A function $\psi : \Gamma \to \mathbb{R}$ on a metric group $\Gamma$, satisfying $\psi(g) = \psi(g^{-1})$ is said to be conditionally negative definite if $K(g, h) = \psi(gh^{-1})$ is a conditionally negative definite kernel.

It is easy to check that if $(\mathcal{H}, \| \cdot \|)$ is a Hilbert space then the kernel $K(x, y) = \|x - y\|^2$ is conditionally negative definite.

The following characterization is due to M.E.B. Bekka, P.-A. Cherix and A. Valette.

**Theorem 5** ([BCV]) A second countable, locally compact group $\Gamma$ is a-T-menable if and only if there exists a continuous, conditionally negative definite function $\psi : \Gamma \to \mathbb{R}_+$ satisfying $\lim_{g \to \infty} \psi(g) = \infty$.

To prove Theorem 3 we also need the following lemmas concerning conditionally negative definite functions and kernels on $L_p$-spaces. These facts where proved by Schoenberg [Sch], for a further discussion see e.g. [BL] Chapter 8.
Lemma 6 Let $K$ be a conditionally negative definite kernel on $X$ and $K(x,y) \geq 0$ for all $x,y \in X$. Then the kernel $K^\alpha$ is conditionally negative definite for any $0 < \alpha < 1$.

Proof. Let $K$ be a conditionally negative definite kernel. Then for every $t \geq 0$ the kernel $1 - e^{-tK} \geq 0$ is also conditionally negative definite and we have
\[
\int_0^\infty (1 - e^{-tK}) \, d\mu(t) \geq 0
\]
for every positive measure $\mu$ on $[0, \infty)$. For every $x > 0$ and $0 < \alpha < 1$ the following formula holds
\[
x^\alpha = c_\alpha \int_0^\infty (1 - e^{-tx}) \, t^{-\alpha-1} \, dt,
\]
where $c_\alpha$ is some positive constant. Thus $K^\alpha$ is also a conditionally negative definite kernel for every $0 < \alpha < 1$. \qed

Lemma 7 The function $\|x\|^p$ is conditionally negative definite on $L_p(\mu)$ when $0 < p \leq 2$.

Proof. The kernel $|x - y|^2$ is conditionally negative definite on the real line (as a square of the metric on a Hilbert space). By Lemma 6 for any $0 < p \leq 2$ the kernel $|x - y|^p$ is also conditionally negative definite on $\mathbb{R}$, i.e.,
\[
\sum |x_i - x_j|^p c_i c_j \leq 0
\]
for every such $p$, all $x_1, \ldots, x_n \in \mathbb{R}$ and $c_1, \ldots, c_n \in \mathbb{R}$ such that $\sum c_i = 0$. Integrate the above inequality with respect to the measure $\mu$ to establish the proof. \qed

It follows from the lemmas that the norm on $L_p(\mu)$ is a conditionally negative definite function, provided $1 \leq p \leq 2$.

To state the next proposition we define a more general notion of a proper action, it is necessary when talking about the spaces $L_p(\mu)$ for $p < 1$ which are not normable metric vector spaces. Thus, if $X$ is just a metric space we call an isometric action of $\Gamma$ on $X$ proper if the set $\{ g \in \Gamma \mid gU \cap U \}$ is finite for any bounded set $U \subset X$. For normed spaces this is consistent with the definitions stated in the introduction.
**Proposition 8** If a second countable, locally compact group $\Gamma$ admits a proper, affine, isometric action on a space $L_p(\mu)$ for some $0 < p < 2$ then $\Gamma$ is a-T-menable.

**Proof.** Given a proper, affine, isometric $\Gamma$-action on $L_p[0, 1]$ consider the function $\psi : \Gamma \to \mathbb{R}$, $\psi(g) = \|\gamma(g)\|^p$, where $\gamma$ is the cocycle associated with the action. Since the $p$-th power of the norm on $L_p[0, 1]$ is a conditionally negative definite function by Lemma 7, $\psi$ is a conditionally negative function on $\Gamma$. The considered $\Gamma$-action is proper thus $\lim_{g \to \infty} \|\gamma(g)\|^p = \infty$ and by Theorem 5, $\Gamma$ is a-T-menable. \(\square\)

In particular only a-T-menable groups may admit proper, affine isometric actions on the spaces $\ell_p$ for $0 < p < 2$ (cf. [Yu]).

**Proof of Theorem 3** (1) $\Rightarrow$ (3). Let $G$ be a locally compact, second countable, a-T-menable group. Then by [CCJJV, Theorem 2.2.2] there exists a standard probability space $(X, \mu)$ and a measure preserving action of $G$ on $X$ such that

1. there exists a sequence of Borel sets $A_n \subseteq X$ such that $\mu(A_n) = \frac{1}{2}$ and $\sup_{g \in B(e, n)} \mu(A_n g \Delta A_n) \leq \frac{1}{2^n}$,
2. the action is strongly mixing, i.e. $\langle f, f \cdot g \rangle \to 0$ when $g \to \infty$ for every $f \in L_2(X, \mu)$ such that $\int f \, d\mu = 0$.

Choose the sequence $v_n(x) = 1_{A_n}(x) - \frac{1}{2} \in L_2(X, \mu)$. Then $\|v_n\|_2 = \frac{1}{2}$ and

$$\int_X v_n(x) \, d\mu = 0$$

so by strong mixing,

$$\|v_n - v_n \cdot g\|_2 \to \sqrt{2}\|v_n\|_2,$$

when $g \to \infty$. Also, for $g \in B(e, n)$ we have

$$\|v_n - v_n \cdot g\|_2 = \mu(A_n g \Delta A_n) \leq \frac{1}{2^n}$$

for all $g \in B(e, n)$.

Now given $p < 2$ define

$$w_n(x) = |v_n(x)|^{2/p} \text{sign}(v_n(x)) \in L_p(X, \mu).$$

In other words, $w_n$ is a image of $v_n$ under the Mazur map, which is a uniform homeomorphism between unit balls of $L_p$-spaces, see [BL, Ch. 9.1] for details.
and estimates. Moreover this map clearly commutes with the regular representation. By the uniform continuity of the Mazur map and its inverse there exist constants $C, \delta > 0$ (which depend only on $p$) such that the sequence $w_n$ satisfies

\[ \sup_{g \in B(e,n)} \| w_n \cdot s - w_n \|_p \leq C/2^n, \]

\[ \| w_n \cdot g - w_n \|_p \geq \delta \text{ for all } g \in G \setminus B(e, S_n) \text{ for some } S_n > 0, \]

which depends on $n$ only (the sequence $\{S_n\}$ can be chosen to be increasing).

This allows to construct a proper affine isometric action on $L_p(X, \mu)$ in a standard way. Define $b : G \to (\bigoplus_{n=1}^{\infty} L_p(X, \mu))_p$ ($p$ denotes the $L_p$-norm on the infinite direct sum)

$$ b(g) = \bigoplus_{n=1}^{\infty} \rho(g) w_n - w_n $$

where $\rho : G \to \text{Iso}(L_p(X, \mu))$ is the right regular representation of $G$ on $X$. Then $b$ is a cocycle for the representation $\bigoplus \rho$ by standard calculations (see e.g. [BCV]).

This way we obtain a proper isometric action on $(\bigoplus_{n=1}^{\infty} L_p(X, \mu))_p$ and the only thing left to notice is that by construction in the proof of [CCJIV, Theorem 2.2.2] the measure $\mu$ is non-atomic, thus by the isometric classification of $L_p$-spaces, $L_p(X, \mu)$ is isometric to $L_p[0, 1]$ and the $p$-sum of infinitely many of these spaces is again isometric to $L_p[0, 1]$. Thus $G$ admits a proper, affine, isometric action on $L_p[0, 1]$.

(3) $\Rightarrow$ (2). This is obvious.

(2) $\Rightarrow$ (1). This implication is proved in Proposition 8 above. $\square$

Note that the above methods cannot be applied to other Banach spaces. J. Bre-tagnolle, D. Daculha-Castelle and J.L. Krivine showed [BDCK] that the function $\|x\|^p$, $0 < p \leq 2$, is a conditionally negative definite kernel on a Banach space $X$ if and only if $X$ is isometric to a subspace of $L_p(\mu)$ for some measure $\mu$. Together with Lemma 6 this covers all powers $p \geq 1$.

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