Stable Envelopes for Slices of the Affine Grassmannian

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Abstract

The affine Grassmannian associated to a reductive group \( G \) is an affine analogue of the usual flag varieties. It is a rich source of Poisson varieties and their symplectic resolutions. These spaces are examples of conical symplectic resolutions dual to the Nakajima quiver varieties. We study the cohomological stable envelopes of D. Maulik and A. Okounkov [MO] in this family. We construct an explicit recursive relation for the stable envelopes in the \( G = \text{PSL}_2 \) case and compute the first-order correction in the general case. This allows us to write an exact formula for multiplication by a divisor.

1 Introduction

1.1 Overview

In their influential paper [MO] D. Maulik and A. Okounkov studied equivariant cohomology \( H^*_T (X^\vee) \) of Nakajima quiver varieties \( X^\vee[N2,N,N3] \). They showed that \( H^*_T (X^\vee) \) are modules over a quantum group called Yangian. The key tools in their constructions were the \( R \)-matrices that essentially came from a family special bases, called the stable envelopes. Their construction of stable envelopes works for quite general for a symplectic resolution with a torus action [see assumptions in Chapter 3 in [MO] or in Section 3].

From a perspective of gauge theories, Nakajima quiver varieties are Higgs branches of moduli spaces of vacua in a 3d supersymmetric field theory. The 3d mirror symmetry is known to exchange the Coulomb \( (X) \) and the Higgs branch \( (X^\vee) \) of vacua [IS,SW]. To understand the relations between the enumerative geometry of \( X \) and \( X^\vee \), we study the Coulomb side directly.

The key object to get Coulomb branches is the moduli space called the affine Grassmannian \( \text{Gr} \). It is associated to a complex connected reductive group \( G \), and one can think of it as an analogue of flag varieties for a corresponding Kac-Moody group. It is known to have deep connections with representation theory and Langlands duality [G, MV, MV2]. A big family of Coulomb branches [BFN, BFN2] is given by the transversal slices in affine Grassmannians. To give an idea of what these slices are, recall that the affine Grassmannian \( \text{Gr} \) has a cell structure similar to the Schubert cells in the ordinary flag variety. A transversal slice \( \text{Gr}^\lambda_\mu \) describes how one orbit is attached to another. Similarly to Nakajima quiver varieties, they are naturally algebraic Poisson varieties and, in some cases, admit a smooth symplectic resolution \( \text{Gr}^\lambda_\mu \to \text{Gr}^\lambda_\mu \), which is a notion of independent mathematical interest [K, K2, BK]. The goal of this work is to study the equivariant cohomology of these resolutions.

We study the stable basis \( \text{Stab}_\epsilon (p) \) introduced in [MO]. It is also given by the images of \( 1_p \in H^*_T (p) \) under a map

\[
\text{Stab}_\epsilon : H^*_T (X^\Lambda) \to H^*_T (X).
\]

Note that these are going in the "wrong" way, comparing to the natural restriction maps.

The main choice one has to make is the choice of attracting directions in a torus, which is denoted here by \( \mathcal{C} \) as a Weyl chamber. Informally, the classes \( \text{Stab}_\epsilon (p) \) are "corrected" versions of the (Poincaré dual) fundamental classes of the attracting varieties to \( p \). The notion of attracting variety clearly depends on a choice of \( \mathcal{C} \).
This basis is a reach source of enumero-geometric data, see [MO]. Even for purely computational convenience they are useful because they are non-localized classes of low degree, and this allows one to use the degree bounds effectively. However, this comes at a price. As opposed to the fixed point basis, the multiplication by a divisor is no longer diagonal. Fortunately, it’s not that far from it.

For a $T$-equivariant line bundle $T$ we have

$$c_1^T(\mathcal{L}) \cup \text{Stab}_\epsilon(p) = \iota_\ast c_1^T(\mathcal{L}) \cdot \text{Stab}_\epsilon(p) + h \sum_{q < p} c_{p,q}^\epsilon \text{Stab}_\epsilon(q)$$

for some $c_{p,q}^\epsilon \in \mathbb{Q}$. Moreover, these numbers $c_{p,q}^\epsilon$ can be easily recovered if one knows $\iota_\ast \text{Stab}_\epsilon(p)$ modulo $h^2$.

We find the restrictions $\iota_\ast \text{Stab}_\epsilon(p)$ in two steps

- Use the wall-crossing behaviour of $\iota_\ast \text{Stab}_\epsilon(p)$ modulo $h^2$ to reduce the computation to a similar computation on a wall in Lie $A$, which turns out to be a slice for $G = \text{PSL}_2$, the case of $A_1$-type. This is done in Theorem 3.19
- Find the restrictions for $A_1$-case via the action of Steinberg correspondences. This computation ends with Theorem 3.18 and has a flavor similar to the computation in C. Su’s thesis[S].

For special line bundles $\mathcal{E}_i$ spanning $\text{Pic}(X) \otimes \mathbb{Q}$ we get the following the main formula

$$c_1^T(\mathcal{E}_i) \cup \text{Stab}_\epsilon(p) = \text{Stab}_\epsilon \left[ H^i(p) + h \sum_{j < i} \Omega_{ij} - \epsilon, -\epsilon(p) - h \sum_{i < j} \Omega_{ij} - \epsilon, -\epsilon(p) \right]$$

in Theorem 3.26.

Since multiplications by $c_1^T(\mathcal{E}_i)$ act with simple spectrum, this formula uniquely determines the stable envelopes.

1.2 Structure

The paper is organized as follows. In Section 2 we recall the main facts about the slices in the affine Grassmannian, in Section 3 we prove the main results about the stable envelopes. For reader’s convenience, we keep the proofs of the technical facts until the Appendix A.

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2 Slices of the affine Grassmannian

In this section we recall some facts about slices of the affine Grassmannians.

2.1 Representation-theoretic notation

Let us present notation we use for common representation-theoretic objects. One major difference from standard notation is that we use non-checked notation for coobjects (coweights, coroots, etc) and checked for ordinary ones (weights, roots, etc). This is common in the literature on affine Grassmannian since it simplifies notation. And unexpected side effect is that weights in equivariant cohomology will have checks. We hope this won’t cause confusion.

- $G$ is a connected simple complex group unless stated otherwise,
• $A \subset G$ is a maximal torus,
• $B \supset A$ is a Borel subgroup,
• $\mathfrak{g}$ is the Lie algebra of $G$
• $X_*(A) = \text{Hom}(\mathbb{C}^\times, A)$ is the cocharacter lattice, it’s equal to the coweight lattice if $G$ is of adjoint type,
• $X_*(A)_+ \subset X_*(A)$ the submonoid of dominant cocharacters.
• $W = N(A)/A$ is the Weyl group of $G$,
• $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha$ is the root decomposition of $\mathfrak{g}$.
• $\rho^\vee$ is the halfsum of positive roots.
• $K(\bullet, \bullet)$ Weyl-invariant scalar product on cocharacter space $X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$, normalized in such a way that for the shortest coroot the length squared is 2 (equivalently, the length squared of the longest root is 2).

We identify all weights with Lie algebra weights (in particular we use additive notation for the weight of a tensor product of weight subspaces).

2.2 Classical constructions

2.2.1 Affine Grassmannian

Let $\mathcal{O} = \mathbb{C}[[t]]$ and $\mathcal{K} = \mathbb{C}((t))$. We will refer to $\mathcal{D} = \text{Spec} \mathcal{O}$ and $\mathcal{D}^\times = \text{Spec} \mathcal{K}$ as the formal disk and the formal punctured disk respectively.

Let $G$ be a connected reductive algebraic group over $\mathbb{C}$. The affine Grassmannian $\text{Gr}_G$ is the moduli space of

$$\left\{ (\mathcal{P}, \varphi) \left| \begin{array}{l} \mathcal{P} \text{ is a } G\text{-principle bundle over } \mathcal{D}, \\
\varphi: \mathcal{P}_0|_{\mathcal{D}^\times} \sim \mathcal{P}|_{\mathcal{D}^\times} \text{ is a trivialization of } \mathcal{P} \\
\text{over the punctured disk } \mathcal{D}^\times \end{array} \right. \right\}$$

where $\mathcal{P}_0$ is the trivial principal $G$-bundle over $\mathcal{D}$. It is representable by an ind-scheme.

One can give a less geometric definition of the $\mathbb{C}$-points of the affine Grassmannian

$$\text{Gr}_G(\mathbb{C}) = G(\mathcal{K})/G(\mathcal{O})$$

where we use notation $G(R)$ for $R$-points of the scheme $G$ given a $\mathbb{C}$-algebra $R$. We will use this point of view when we talk about points in $\text{Gr}_G$.

The see that these are exactly the $\mathbb{C}$-points of $\text{Gr}_G$ (i.e. the pairs $(\mathcal{P}, \varphi)$ as in (1)), note that on $\mathcal{D}$ any $G$-principle bundle is trivializable, so take one trivialization $\psi: \mathcal{P}_0 \rightarrow \mathcal{P}$. This gives a composition of isomorphisms

$$\mathcal{P}_0|_{\mathcal{D}^\times} \xrightarrow{\psi|_{\mathcal{D}^\times}} \mathcal{P}|_{\mathcal{D}^\times} \xrightarrow{\varphi^{-1}} \mathcal{P}_0|_{\mathcal{D}^\times}$$

which is a section of $\mathcal{P}_0|_{\mathcal{D}^\times}$, i.e. an element of $G(\mathcal{K})$. Change of the trivialization $\psi$ precomposes with a section of $\mathcal{P}_0$, in other words, precomposes with an element of $G(\mathcal{O})$. This gives a quotient by $G(\mathcal{O})$ from the left.

We restrict ourselves to the case of connected simple (possibly not simply-connected) $G$ in what follows without loosing much generality. Restating the results to allow arbitrary connected complex reductive group is straightforward.

From now on we fix a connected simple complex group $G$. Since it can’t cause confusion, we omit later $G$ in the notation $\text{Gr}_G$ and just write $\text{Gr}$. 3
The group \( G(K) \times \mathbb{C}^\times \) naturally acts on \( \text{Gr} \). In the coset formulation the \( G(K) \)-action is given by left multiplication and \( \mathbb{C}^\times \)-acts by scaling variable \( t \) in \( K \) with weight 1. In the moduli space description \( G(K) \) acts by changes of section \( g \cdot (P, \varphi) = (P, \varphi g^{-1}) \) (these two actions are the same action if one takes into account identification (2)), \( \mathbb{C}^\times \) scales \( D \) such that the coordinate \( t \) has weight 1. We will later denote this \( \mathbb{C}^\times \) as \( \mathbb{C}^\times \times \mathbb{h} \) when we want to emphasize that we use this algebraic group and its natural action.

We will need actions by subgroups of this group, namely \( A \subset G \subset G(O) \subset G(K) \). It will also be useful to consider extended torus \( T = A \times \mathbb{C}^\times \times \mathbb{h} \), where \( \mathbb{C}^\times \times \mathbb{h} \) part comes from the second term in the product \( G(K) \times \mathbb{C}^\times \times \mathbb{h} \).

The canonical projection \( T \to \mathbb{C}^\times \times \mathbb{h} \) gives a character of \( T \) which we call \( \mathbb{h} \) (this explains the subscript \( \mathbb{h} \) in the notation \( \mathbb{C}^\times \times \mathbb{h} \)). Then the weight of coordinate \( t \) on \( D \) is \( \mathbb{h} \) by the construction of the \( \mathbb{C}^\times \times \mathbb{h} \)-action.

**Remark.** We call the maximal torus of \( G \) by \( A \) to have notation similar to [MO], i.e. \( T \) is the maximal torus acting on the variety, and \( A \) is the subtorus of \( T \) preserving the symplectic form.

The partial flag variety \( G/P \) (where \( P \) is parabolic) has a well-known decomposition by orbits of \( B \)-action called Schubert cells. The affine Grassmannian has a similar feature.

One can explicitly construct fixed points of the \( A \)-action. Given a cocharacter \( \lambda : \mathbb{C}^\times \to T \), one can construct a map using natural inclusions
\[
\mathcal{D} \to \mathbb{C}^\times \to T \hookrightarrow G
\]
i.e. an element of \( G(K) \). Projecting it naturally to \( \text{Gr} \) we get an element \([\lambda] \in \text{Gr}\). These points are \( A \)-fixed and even the following stronger statements are true.

**Proposition 2.1.**
1. \( \text{Gr}^A = \bigsqcup_{\lambda \in X^*_+} \{[\lambda]\} \)
2. Moreover, \( \text{Gr}^T = \text{Gr}^A \)

Using these elements we define
\[
\text{Gr}^\lambda = G(O) \cdot [\lambda]
\]
as their orbits. These have the following properties

**Proposition 2.2.**
1. For any \( w \in W \) one has \( \text{Gr}^{w\lambda} = \text{Gr}^\lambda \).
2. \( \text{Gr} = \bigsqcup_{\lambda \in X^*_+} \text{Gr}^\lambda \)
3. \( \overline{\text{Gr}^\lambda} = \bigsqcup_{\mu \in X^*_+} \text{Gr}^\mu \)
4. As a \( G \)-variety, \( \text{Gr}^\lambda \) is isomorphic to the total space of a \( G \)-equivariant vector bundle over a partial flag variety \( G/P_\lambda^- \) with parabolic \( P_\lambda^- \) whose Lie algebra contains all roots \( \alpha^\vee \) such that \( \langle \alpha^\vee, \lambda \rangle \leq 0 \).
5. \( D \)-scaling \( \mathbb{C}^\times \times \mathbb{h} \) in \( T \) scales the fibers of this vector bundle over \( G/P_\lambda^- \) with no zero weights.
6. \( \text{Gr}^\lambda \) is the smooth part of \( \overline{\text{Gr}^\lambda} \). In particular, \( \overline{\text{Gr}^\lambda} \) is smooth iff \( \lambda \) is a minuscule coweight or zero.

**Corollary 2.3.** \( \text{Gr} \) admits a \( T \)-invariant cell structure.
Proof. Each $\mathbf{G}/\mathbf{P}_\lambda$ admits an $A$-invariant cell structure, namely Schubert cells. $A$-equivariant vector bundles over these cells give $A$-invariant cells for each of $\text{Gr}^\lambda$. Hence we have an $A$-invariant cell structure for $\text{Gr}$.

Moreover this cell structure is invariant under $D$-scaling $\mathbb{C}_h^\times$-action, because it just scales the fibers of the vector bundles. Finally, we get that this cell decomposition $T$-invariant.

**Corollary 2.4.** Orbits of diagonal $G(\mathcal{X})$-action on $\text{Gr} \times \text{Gr}$ are in bijection with dominant cocharacters.

**Proof.** First recall that for any group $H$ and a subgroup $K$ one has a bijection

$$
\left\{ \text{H-diagonal orbits of } H/K \times H/K \right\} \simeq K\backslash H/K \simeq \left\{ \text{K-orbits of } H/K \right\}
$$

Applying this for $H = G(\mathcal{X})$ and $K = G(\mathcal{O})$ and that the rightmost set is in bijection with dominant cocharacters by Proposition 2.2 we get the statement.

We write $L_1 \xrightarrow{\lambda} L_2$ if $(L_1, L_2) \in \text{Gr} \times \text{Gr}$ is in the $G(\mathcal{X})$-orbit indexed by a dominant $\mu \leq \lambda$.

**Remark.** Explicitly, one says that $L_1 \xrightarrow{\lambda} L_2$ iff picking representatives $L_1 = g_1 G(\mathcal{O})$, $L_2 = g_2 G(\mathcal{O})$ with $g_1, g_2 \in G(\mathcal{X})$ we have

$$
G(\mathcal{O}) g_1^{-1} g_2 G(\mathcal{O}) \subset G(\mathcal{O}) [\mu] \text{ for } \mu \leq \lambda
$$
or, equivalently,

$$
g_1^{-1} g_2 G(\mathcal{O}) \in G(\mathcal{O}) \cdot [\lambda]
$$

Note that this independent on the choice of the representatives of $L_1$, $L_2$.

### 2.2.2 Transversal slices

As one can see, elements $g(t)$ of $G(\mathcal{O}) \subset G$ can be characterized as the elements of $g(t) \in G(\mathcal{X})$ such that $\lim_{t \to 0} g(t) \in G \subset G(\mathcal{X})$.

We can use a similar way to define a subgroup transversal to $G(\mathcal{O})$. Let

$$
G_1 = \left\{ g(t) \in G \mid \lim_{t \to \infty} g(t) = e \right\}
$$

This is a subgroup of $G(\mathcal{X})$.

$G(\mathcal{O})$ and $G_1$ are transversal in the sense that

$$
T_e G(\mathcal{X}) = T_e G(\mathcal{O}) \oplus T_e G_1
$$

because

$$
T_e G(\mathcal{X}) = g((t)) \quad T_e G(\mathcal{O}) = g[t] \quad T_e G_1 = t^{-1} g[t^{-1}]
$$

One uses $G_1$ to analyze how $G(\mathcal{O})$-orbits in $\text{Gr}$ are attached to each other. The ind-varieties

$$
\text{Gr}_\mu = G_1 \cdot [\mu]
$$
is the slice transversal to $\text{Gr}^\mu$ at $[\mu]$. We will use use the convention that $\mu$ is dominant.

The main object of interest for us are the following varieties

$$
\text{Gr}^\lambda_\mu = \overline{\text{Gr}^\lambda} \cap \text{Gr}_\mu
$$

These are non-empty iff $\mu \leq \lambda$. 5
Proposition 2.5.

1. $\text{Gr}_\mu^\lambda$ is a $T$-invariant subscheme of $\text{Gr}$;
2. The only $A$-fixed (and $T$-fixed) point of $\text{Gr}_\mu^\lambda$ is $[\mu]$;
3. The $D$-scaling $\mathbb{C}_h^\times$-action contracts $\text{Gr}_\mu^\lambda$ to $[\mu]$.
4. $\text{Gr}_\mu^\lambda$ is a normal affine variety of dimension $\langle 2\rho^\vee, \lambda - \mu \rangle$.

2.3 Poisson structures and symplectic resolutions

2.3.1 Poisson structure

The affine Grassmannian has a natural Poisson structure. Let us describe it using the Manin triples introduced by V. Drinfeld [D, D2]. First recall that $G(\mathbb{K})$ is a Poisson ind-group. It is given by a Manin triple $(g[[t]], t^{-1}g[t^{-1}], g((t)))$, if we equip $g((t))$ with the standard $G$-invariant scalar product

$$(xt^n, yt^m) = K(x, y) \delta_{n+m+1,0},$$

where $\delta_{\cdot, \cdot}$ is the Kronecker delta. Then the subalgebras $g[[t]], t^{-1}g[t^{-1}]$ are isotropic and the pairing between them is non-degenerate. Thus $g((t))$ is a Lie bialgebra and $g[[t]]$ is a Lie subbialgebra.

We have that $G(\mathbb{K})$ is a Poisson ind-group and $G(G)$ is its Poisson subgroup. By theorem of Drinfeld [D2] the quotient $\text{Gr} = G(\mathbb{K})/G(G)$ is Poisson. The explicit construction shows that since the pairing (3) has weight $h$, we have the following important property

Proposition 2.6. The Poisson bivector on $\text{Gr}$ is a $T$-eigenvector in $H^0(\text{Gr}, \Lambda^2 \text{Gr})$ with weight $-h$.

Proof. The pairing is invariant with respect to $A$, and is of degree $-1$ in $t$, so the $T$ torus acts with weight $-h$. □

The slices $\text{Gr}_\mu^\lambda \subset \text{Gr}$ are Poisson subvarieties. Moreover, their smooth parts are symplectic leaves, however not all symplectic leaves have this form: they might be shifted by the $G$-action.

2.3.2 Resolutions of slices

We want to study the geometry of the symplectic resolutions of $\text{Gr}_\mu^\lambda$. Unfortunately, these spaces do not always possess a symplectic resolution. We’ll follow [KWWY] to construct resolution in special cases.

For construction of resolutions it is convenient to assume that the cocharacter lattice is as large as possible, that is, equal to the coweight lattice. This happens when $G$ is of adjoint type (the other extreme from being semi-simple). This assumption doesn’t change the space we are allowed to consider: If $\tilde{G} \to G$ is a finite cover, then there is a natural inclusion $\text{Gr}_G \to \text{Gr}_{\tilde{G}}$ with the image being a collection of connected components of $\text{Gr}_{\tilde{G}}$. Since the slices are connected, they belong only to one component and can be thought as defined for $\tilde{G}$. Last but not least the $T$-equivariant structures are compatible if one takes into account that the torus for $\tilde{G}$ is a finite cover over the torus for $G$.

From now on we assume that $G$ is a connected simple group of adjoint type (so the center is trivial) and make no distinction between cocharacters and coweights unless otherwise is stated.

Consider a sequence of dominant weights $\lambda = (\lambda_1, \ldots, \lambda_l)$ Define the (closed) convolution product:

$$\text{Gr}^\lambda = \left\{ (L_1, \ldots, L_l) \in \text{Gr}^\times | e \cdot G(\mathbb{O}) \xrightarrow{\lambda_1} L_1 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_{l-1}} L_{l-1} \xrightarrow{\lambda_l} L_l \right\}$$
There’s a map

\[ m_\lambda^\prime: \text{Gr}^\lambda \to \text{Gr}, \]
\[(L_1, \ldots, L_l) \mapsto L_l. \]

One can show that the image of \( m_\lambda^\prime \) is inside \( \text{Gr}^\lambda \), where \( \lambda = \sum_i \lambda_i \), so we can actually make a map

\[ m_\lambda: \text{Gr}^\lambda \to \text{Gr}^\lambda. \]

Then let us define

\[ \text{Gr}^\lambda_{\mu} = \sum_i^{-1} \left( \text{Gr}^\lambda_{\mu} \right). \]

These spaces and maps are useful to construct symplectic resolutions of singular affine \( \text{Gr}^\lambda_{\mu} \). Let all \( \lambda_i \) in \( \lambda \) be fundamental coweights (which means this is a maximal sequence which splits \( \lambda = \sum_i \lambda_i \) into nonzero dominant coweights). We will also need the corresponding irreducible highest weight representations \( V(\lambda_k) \) of the Langlands dual group \( G^\vee \). Then there’s the following result (see [KWWY], Theorem 2.9)

**Theorem 2.7.** The following are equivalent

1. \( \text{Gr}^\lambda_{\mu} \) possesses a symplectic resolution;
2. \( \text{Gr}^\lambda_{\mu} \) is smooth and thus \( m_\lambda \) gives a symplectic resolution of singularities of \( \text{Gr}^\lambda_{\mu} \).
3. There do not exist coweights \( \nu_1, \ldots, \nu_l \) such that \( \sum_i \nu_i = \mu \), for all \( k \), \( \nu_k \) is a weight of \( V(\lambda_k) \) and for some \( k \), \( \nu_k \) is not an extremal weight of \( V(\lambda_k) \), that is, not in the Weyl orbit of \( \lambda_k \).

**Corollary 2.8.** If all \( \lambda_i \) are minuscule, then all three statements of the Theorem 2.7 hold.

**Corollary 2.9.** If \( G = \text{PSL}_n \), then all fundamental coweights are minuscule and for all dominant coweights \( \mu \leq \lambda \) all three statements of the Theorem 2.7 hold.

**Corollary 2.10.** If \( G \) is not \( \text{PSL}_n \), then there exist dominant coweights \( \mu \leq \lambda \) such that all three statements of the Theorem 2.7 are not satisfied.

From this time on we assume that the pair of dominant coweights \( \mu \leq \lambda \) is such that the statements of the Theorem 2.7 hold.

There are important special cases of these symplectic resolutions

**Example 2.1.** Let \( G = \text{PSL}_2 \), then the cocharacter and coroot lattice are character (or, equivalently, weight) and root lattices of \( \text{SL}_2 \) respectively. For any integer \( n \geq 1 \) by setting \( \lambda = (n+1)\omega \) and \( \mu = (n-1)\omega \) we get that \( \text{Gr}^\lambda_{\mu} \) is \( A_n \)-singularity and the map \( \text{Gr}_{\mu}^{(\omega, \ldots, \omega)} \to \text{Gr}_{\mu}^{\lambda} \) is a symplectic resolution of singularity.

**Example 2.2.** Let \( \nu \) be a minuscule coweight of \( G \) and \( \iota \) is the Cartan involution and set \( \lambda = \nu + \iota \nu \), \( \mu = 0 \). Then we get a resolution \( \text{Gr}_{\mu}^{(\nu, \iota \nu)} \to \text{Gr}_{\mu}^{\nu + \iota \nu} \) is the Springer resolution. Here \( \text{Gr}_{\mu}^{(\nu, \iota \nu)} \cong T^*(G/P_\nu) \) (the cotangent bundle of a partial flag variety), where \( P_\nu \) is the maximal parabolic corresponding to \( \nu \) (i.e. the Lie algebra of \( P_\nu \) has all roots \( \alpha^\vee \) of \( G \) satisfying \( \langle \nu, \alpha^\vee \rangle \leq 0 \)). \( \text{Gr}_{\mu}^{\nu + \iota \nu} \) is the affinization.

**Example 2.3.** Let \( G = \text{PSL}_n \), \( \omega_1 \) be highest weight of the defining representation of \( \text{SL}_n \) (so it is a \( \text{PSL}_n \)-coweight). Set also \( \lambda = n\omega_1 \), \( \lambda = (\omega_1, \ldots, \omega_1) \) and \( \mu = 0 \). Then \( \text{Gr}_{\mu}^{(\omega_1, \ldots, \omega_1)} \cong T^*(G/B) \) is the cotangent bundle of full flag variety, \( \text{Gr}_{\mu}^{n\omega_1} \) is the nilpotent cone and \( \text{Gr}_{\mu}^{(\omega_1, \ldots, \omega_1)} \to \text{Gr}_{\mu}^{n\omega_1} \) is the Springer resolution.
2.4 Group action on resolutions

Recall that $G(\mathcal{X}) \rtimes \mathbb{C}_k^\times$ acts on $\mathbf{Gr}$, so it also acts diagonally on $\mathbf{Gr} \times l$ for any $l$.

Proposition 2.11.

1. For any sequence $\lambda$ the subscheme $\mathbf{Gr}^\lambda \subset \mathbf{Gr} \times l$ is $G(\mathcal{O}) \rtimes \mathbb{C}_k^\times$-invariant

2. The resolution of singularities $m_\lambda: \mathbf{Gr}^\lambda \rightarrow \mathbf{Gr}^\lambda$ is $\mathbf{T}$-equivariant.

Proof.

1. Let us first prove that $\mathbf{Gr}^\lambda$ is $G(\mathcal{O})$-invariant. For any $g \in G(\mathcal{O})$ we have

$$g \cdot h \mathcal{G}(\mathcal{O}) = (gh) \mathcal{G}(\mathcal{O}) = (\Ad_g^{-1} h) \mathcal{G}(\mathcal{O})$$

Then we have

$$h_1 \mathcal{G}(\mathcal{O}) \overset{\lambda}{\rightarrow} h_2 \mathcal{G}(\mathcal{O})$$

$$\Leftrightarrow \mathcal{G}(\mathcal{O}) h_1^{-1} h_2 \mathcal{G}(\mathcal{O}) \subset \mathcal{G}(\mathcal{O}) [\mu] \text{ for some } \mu \leq \lambda$$

$$\Leftrightarrow \mathcal{G}(\mathcal{O}) Ad_g^{-1} (h_1^{-1}) Ad_g (h_2) \mathcal{G}(\mathcal{O}) \subset \mathcal{G}(\mathcal{O}) [\mu] \text{ for some } \mu \leq \lambda$$

$$\Leftrightarrow g \cdot h_1 \mathcal{G}(\mathcal{O}) \overset{\lambda}{\rightarrow} g \cdot h_2 \mathcal{G}(\mathcal{O})$$

A similar computation holds for $C \times$.

2. Follows from the action being diagonal and the definition of $m_\lambda$.

Proposition 2.12. The fixed locus $X^\mathbf{T}$ is the following disjoint union of finitely many points

$$X^\mathbf{T} = \left\{ ([\mu_1], \ldots, [\mu_l]) \in (\mathbf{Gr} \times l) \mid \forall i \mu_i - \mu_{i-1} \text{ is a weight of } V(\lambda_i) \right\}.$$ 

Proof. Since the $\mathbf{T}$-action is diagonal, we have that $(L_1, \ldots, L_l) \in X^\mathbf{T}$ iff $\forall i L_i$ has form $[\mu_i]$ for some coweight $\mu_i$. Next, conditions for such points to be in $X$ are

a A condition to be in $\mathbf{Gr}^\lambda$. That is, $\forall i L_{i-1} \overset{\lambda}{\rightarrow} L_i$ which gives that $\mu_i - \mu_{i-1}$ is a weight of $V(\lambda_i)$ with convention $\mu_0 = 0$ to make $L_0 = [0] = e \cdot \mathcal{G}(\mathcal{O})$.

b A condition to be in $m^{-1}_\lambda(\mathbf{Gr}^\mu)$. That is, $L_l = m^{-1}_\lambda(L_1, \ldots, L_l) = [\mu]$, which is equivalent to $\mu_l = \mu$.

This gives the description of $X^\mathbf{T}$ in the proposition.

Remark. If $\lambda_i$ is minuscule, the constraint of $\nu_i$ to be a weight of $V(\lambda_i)$ just means that $\mu_i - \mu_{i-1} \in W\lambda_i$.

Remark. If all $\lambda_i$ are minuscule, the fixed points are in bijection with a basis of the weight $\mu$ subspace of the $G^\vee$-representation $V(\lambda_1) \otimes \cdots \otimes V(\lambda_l)$.

Notation. For a fixed point $p \in X^\mathbf{T}$ there are two related sequences of coweights. Let

$$p = ([0], [\mu_1], \ldots, [\mu_{l-1}], [\mu]).$$

Then for all $i$, $0 < i < l$, we define

$$\Sigma_i (p) = \mu_i,$$

and for coherent notation we set

$$\Sigma_0 (p) = 0,$$

and

$$\Sigma_l (p) = \mu.$$
This gives a sequence
\[ \Sigma(p) = (\Sigma_0(p), \ldots, \Sigma_l(p)) \, . \]
We also define for all \( i, 1 \leq i \leq l \)
\[ \Delta_i(p) = \Sigma_i(p) - \Sigma_{i-1}(p) \, . \]

One forms the second sequence from these coweights
\[ \Delta(p) = (\Delta_1(p), \ldots, \Delta_l(p)) \, . \]

Informally speaking, \( \Delta(p) \) is the sequence of increments, so \( \Sigma(p) \) is the sequence of their partial sums. This explains the notation.

One useful way to think about the fixed points of \( \text{Gr}^{\lambda}_{\mu} \) is that they are one-to-one certain piecewise-linear paths in the coweight lattice. Let \( P_p \) be the piecewise linear path connecting points
\[ 0 = \Sigma_0(p), \Sigma_1(p), \ldots, \Sigma_{l-1}(p), \Sigma_l(p) = \mu \]
in the coweight lattice.

The paths \( P_p \) are the ones that have \( l \) segments, start at the origin, and end at \( \mu \). The \( i \)th segment must be a weight of \( V(\lambda_i) \) (as said earlier, this is equivalent to being a Weyl reflection of \( \lambda_i \) if \( \lambda_i \) is minuscule). See Figure 1 for an example of a typical path in the case \( G = \text{PSL}_3 \) and \( \text{Gr}^{(\omega_1, \omega_2)}_{\omega_1 + \omega_2} \).

Using the paths \( P_p \) is useful for a combinatorial description of the tangent weights at the fixed points. We postpone the proof until the appendix.

**Theorem 2.13.** The only weights in \( T_p X \) are of form \( \alpha^\vee + n\hbar \) for a root \( \alpha^\vee \) and \( n \in \mathbb{Z} \). Moreover, the multiplicity of weight \( \alpha^\vee + n\hbar \) is the number of crossings of \( P_p \) with the hyperplane
\[ \langle \bullet, \alpha^\vee \rangle - \left( n + \frac{1}{2} \right) = 0 \]
in the direction of the halfspace containing 0.

An illustration of the statement of Theorem 2.13 is shown on Figure 2. The dashed lines and green halfphanes correspond to pairs of weights, the blue dots show the intersections under consideration. We used the standard identification of (short) roots \( \alpha_1^\vee \) and \( \alpha_2^\vee \) with vectors of length squared 2 via a properly normalized invariant inner product to visualize them.
3 Stable Envelopes

In this section we introduce the stable envelopes and derive how multiplication by a divisor acts on them.

From now on the base ring for all cohomology is the field $\mathbb{Q}$, unless stated otherwise. All the results generalize straightforwardly to the case of any field of characteristic 0.

3.1 Preliminaries on Equivariant Cohomology

Computations in equivariant cohomology heavily rely on the well-known localization theorem [AB].

**Theorem 3.1.** (Atiyah-Bott) The restriction map in the localized equivariant cohomology

$$H^*_T (X)_{loc} \to H^*_T (X^T)_{loc}$$

is an isomorphism.

Here and later we use notation $H^*_T (X)_{loc} := H^*_T (X) \otimes_{H^*_T (pt)} \text{Frac} H^*_T (pt)$.

If $X$ is proper then there is the fundamental class $[X] \in H^{2 \dim X}_2 (X)$. By this gives a pushforward to a point homomorphism

$$H^*_T (X) \to H^*_T (pt)$$

$$\gamma \mapsto \int_X \gamma$$

quite often called integration by analogy with de Rham cohomology. Then the localization theorem allows one to compute this using only restrictions to the fixed locus.

$$\int_X \gamma = \sum_{Z \subset X^T} \int_Z e^T (N_{Z/X})$$

where the sum is taken over all components $Z \subset X^T$, $e^T (\bullet)$ is the equivariant Euler class, $N_{Z/X}$ is the normal bundle to $Z$ in $X$.

If $X$ is not proper the fundamental class lies in Borel-Moore (equivariant) homology $[X] \in H^{T,BM}_{2 \dim X} (X)$. This pairs naturally with the equivariant cohomology with compact support $H^*_T (X)$ and gives a pushforward map

$$H^*_T (X) \to H^*_T (pt).$$
In general, we have the following natural maps

\[
\begin{array}{ccc}
H^*_{T,c}(X) & \longrightarrow & H^*_T(X) \\
\downarrow \tau & & \downarrow \tau \\
H^*_{T,c}(X^T) & \longrightarrow & H^*_T(X^T)
\end{array}
\] (5)

The vertical arrows are restrictions. \( X^T \) is a closed subvariety, so the inclusion \( \iota: X^A \to X \) is proper and the pullback is well-defined in the cohomology with compact support. After the localization the vertical arrows in (5) become isomorphisms.

Assume now that \( X^T \) is proper. Then the bottom arrow in (5) becomes an isomorphism because the compact support condition becomes vacuous. Then we get an isomorphism

\[
H^*_{T,c}(X) \xrightarrow{\sim} H^*_T(X)
\]

which gives the pushforward

\[
H^*_T(X)_{loc} \to H^*_T(pt)_{loc}
\]

We still denote it by \( \int_X \gamma \). The relation (4) still holds. Some authors use it to define \( \int_X \gamma \). As a side affect, a priori, the answer one gets by this definition is in \( H^*_T(pt)_{loc} = \text{Frac} H^*_T(pt) \) because of the division by Euler classes. However, we see that the image of non-localized compactly supported classes \( H^*_{T,c}(X) \) is inside \( H^*_T(pt) \).

This defines on \( H^*_T(X)_{loc} \) the structure of the Frobenius algebra over \( H^*_T(pt)_{loc} \). In particular, we often use the natural pairing

\[
\langle \gamma_1, \gamma_2 \rangle_X = \int_X \gamma_1 \cup \gamma_2.
\]

If \( X \) is clear we omit it in the notation.

We think of \( H^*_T(pt) \) as the ground ring. We usually write the multiplication in \( H^*_T(pt) \) by \( \cdot \) to simplify notation.

The multiplication in \( H^*_T(X) \) is still denoted by \( \cup \). We also refer to it as classical multiplication as opposed to its deformation, quantum multiplication.

### 3.2 Stable Envelopes

#### 3.2.1 Assumptions

Here we introduce the stable envelopes and some of their properties following [MO].

Let us list necessary assumptions for the existence of stable envelopes.

Let \( X \) be a non-singular algebraic variety and \( \omega \in H^0(\Omega^2_X) \) be a holomorphic symplectic form on \( X \). Let a pair of tori \( A \subset T \) act on \( X \) in such a way, that:

1. \( \omega \) is an eigenvector of \( T \).
2. There is a proper \( T \)-equivariant map \( \pi: X \to X_0 \) to an affine \( T \)-variety \( X_0 \).
3. \( X \) is a formal \( T \)-variety. [GKM]

We denote the \( T \)-weight of \( \omega \) as \( h \). Let \( A \subset T \) be a subtorus preserving \( \omega \).
3.2.2 Chambers

We study walls in $a_R$, the real vector space associated to the cocharacter lattice of $A$,

$$a_R = X_*(A) \otimes \mathbb{R}.$$

The walls $H_{\alpha^\vee}$ are given by linear equations $\alpha^\vee = 0$ and split this space into disconnected domains called chambers:

$$a_R \setminus \bigcup_{\alpha^\vee} H_{\alpha^\vee} = \bigsqcup \mathcal{C}_i.$$

For generic resolutions of slices the configuration is exactly the Weyl chambers from representation theory. Even though in certain examples we might miss walls corresponding to some roots, we keep calling them Weyl chambers.

3.2.3 Attracting manifolds

Let $\lambda$ be an $A$-cocharacter. Then the existence and value of the limit

$$\lim_{t \to 0} \lambda(t) \cdot x \in X^A$$

does not change if we vary $\lambda$ without crossing the walls. I.e. it depends only on a choice of chamber $\mathcal{C}$ and we denote it by $\lim_{\mathcal{C}} x$.

For any subvariety $Z \subset X$ let us denote

$$\text{Attr}_{\mathcal{C}}(Z) = \{ x \in X | \lim_{\mathcal{C}} x \in Z \}.$$  

We call this set the attractor of $Z$.

More generally, the attracting manifolds can be defined for a face of smaller dimension in the stratification of $a_R$ by walls. We will use it later.

3.2.4 Partial order

The attractors define a partial order on components of $X^A$.

We take the relation

$$\text{Attr}_{\mathcal{C}}(Z) \cap Z' \neq \emptyset \implies Z \geq_{\mathcal{C}} Z'$$

and take its transitive closure.

Now we can define the full attractor as

$$\text{Attr}_{\mathcal{C}}(Z) = \bigsqcup_{Z' \leq_{\mathcal{C}} Z} \text{Attr}_{\mathcal{C}}(Z).$$

This set is closed.

3.2.5 Support and degree in $A$

We say that a class $\gamma \in H_T^*(X)$ is supported on a closed $T$-invariant subset $Z$ if the pullback vanishes

$$i^* \gamma = 0$$

for the inclusion $i : X \setminus Z \hookrightarrow X$. Notation is the following

$$\text{supp} \gamma \subset Y.$$

Since $A$ acts on $X^A$ trivially,

$$H_T^*(X^A) = H_{T/A}^*(X^A) \otimes_{H_{T/A}^*(pt)} H_T^*(pt).$$

We have an increasing filtration of $H_{T/A}^*(pt)$-modules on $H_T^*(pt)$ induced by the degree filtration on $H_A^*(pt)$. This induces a filtration on $H_T^*(X^A)$. We call this degree $\text{deg}_{A}$, the degree in $A$. 12
3.2.6 Polarizations

Definition of stable envelopes depends on a certain sign choice provided by polarizations.

Let $C$ be a face in the stratification of $\mathfrak{a}_0$ by walls (possibly not of the highest dimension, so it’s not necessary a chamber). Given $Z \subset X^A$ a component of the fixed locus, the restriction of the tangent bundle $T_X$ of $X$ to $Z$ splits into attracting, constant and repelling parts:

$$T_X|_Z = N_{Z/X}^e \oplus T_Z \oplus N_{Z/X}^-.$$  

(repelling part is exactly the attraction part with respect to $-C$, the opposite face).

If we consider the normal bundle to $Z$ in $X$, we get a splitting

$$N_{Z/X}|_Z = N_{Z/X}^e \oplus N_{Z/X}^-.$$  

The symplectic form pairs $N_{Z/X}^e$ and $N_{Z/X}^-$, which gives an isomorphism of $T$-equivariant bundles

$$\left( N_{Z/X}^e \right) \cong h \otimes N_{Z/X}^-.$$  

Since the restriction of $h$ to $A$ is trivial,

$$\epsilon_A \left( N_{Z/X}^e \right) = \epsilon_A \left( N_{Z/X}^- \right) = (-1)^{\text{codim } Z/2} \left[ \epsilon_A \left( N_{Z/X}^- \right) \right]^2.$$  

A polarization on $Z$ is a choice of $\epsilon \in H^*_A (Z)$, such that

$$\epsilon^2 = (-1)^{\text{codim } Z/2} \epsilon_A \left( N_{Z/X}^- \right).$$  

One can easily see that

$$\epsilon = \epsilon_A \left( N_{Z/X}^- \right)$$

is a polarization. Moreover, any polarization is

$$\pm \epsilon_A \left( N_{Z/X}^- \right),$$

so a choice of polarization is just a choice of sign. This sign we denote later by

$$\sigma_{Z/X}^{\epsilon, \epsilon} = \frac{\epsilon_q}{\epsilon_A \left( N_{Z/X}^- \right)} \in \{ \pm 1 \}.$$  

The polarization on $Z$, dual to $\epsilon$ is defined as

$$\bar{\epsilon} = (-1)^{\text{codim } Z/2} \epsilon_A \left( N_{Z/X}^- \right).$$

A polarization on $X^A$ is a choice of polarization $\epsilon_Z$ for each component $Z \subset X^A$.

3.2.7 Definition of stable envelopes

Now we are ready to give the definition of stable envelopes.

Fix a chamber $C$ and a polarization $\epsilon$ on $X^A$.

Theorem 3.2. (Maulik-Okounkov) There exists a unique map of $H^*_T (pt)$-modules

$$\text{Stab}_\epsilon : H^*_T (X^A) \to H^*_T (X)$$

such that for any component $Z \subset X^A$ and $\gamma \in H^*_T (Z)$, the stable envelope $\Gamma = \text{Stab}_\epsilon (\gamma)$ satisfies

1. $\text{supp} \, \Gamma \subset \text{Attr}^T (Z),$

2. $\iota^*_Z \Gamma = \pm \epsilon_A \left( N_{Z/X}^- \right) \cup \gamma$, where the sign is fixed by the polarization: $\iota^*_Z \Gamma|_A = \epsilon_Z \cup \gamma|_A.$
3. \( \deg_A \iota_{z'}^* \Gamma < \frac{1}{2} \dim X \) for any \( Z' \ll \epsilon Z \).

In the spaces of our interest \( X^A \) is finite discrete, so \( H^*_T (X^A) \) has a basis \( 1_p \) of classes those only non-zero restriction is 1 at \( p \in X^A \). To simplify notation we write

\[
\text{Stab}_\epsilon^c (p)
\]

instead of

\[
\text{Stab}_\epsilon^c (1_p).
\]

The classes \( \text{Stab}_\epsilon^c (p) \) may be thought of as a refined version of (the Poincaré-dual class to) the fundamental cycle \( \pm \left[ \text{Attr}_\epsilon (p) \right] \). This explains the first two conditions in the Theorem 3.2. The last condition is required to ensure these classes behave well under deformations of the symplectic variety.

### 3.2.8 First properties

Let us recall some results about stable envelopes.

From the definition of stable envelopes the restriction matrix \( \iota_{z'}^* \text{Stab}_\epsilon^c (p) \) is triangular with respect to order \( \ll \epsilon \). The values on the diagonal are prescribed. In the class of symplectic resolutions one knows that the off-diagonal terms are "small" in the following sense.

**Theorem 3.3.** Let \( X \) be a symplectic resolution. Then for any components \( Z, Z' \subset X^A, Z' \ll \epsilon Z \), and \( \gamma \in H^*_T (Z) \), we have

\[
\iota_{z'}^* \text{Stab}_\epsilon^c (\gamma) \in \mathbb{h} H^*_T (Z').
\]

Then if one restricts \( \text{Stab}_\epsilon^c (p) \) to the torus \( A \) (sets \( \mathbb{h} = 0 \)), it has a non-zero coefficient only at \( p \).

Since the fixed point is a basis in the localized cohomology \( H^*_T (X)_{loc} \), stable envelopes also form a basis in the localized cohomology.

Another nice property is that the basis, dual to \( \{ \text{Stab}_\epsilon^c (p) \} \), is given by taking the stable envelopes for the opposite chamber and polarization, \( -\epsilon \) and \( \tau \).

**Theorem 3.4.** For \( p, q \in X^A \), we have

\[
\left\langle \text{Stab}_{-\epsilon}^r (q), \text{Stab}_\epsilon^c (p) \right\rangle = \delta_{q,p},
\]

where \( \delta_{p,q} \) is the Kronecker delta

\[
\delta_{p,q} = \begin{cases} 
1, & \text{if } p = q, \\
0, & \text{otherwise}.
\end{cases}
\]

**Remark.** Note, in particular, that the pairing lies in non-localized cohomology \( H^*_T (pt) \) since the class

\[
\text{Stab}_{-\epsilon}^r (q) \cup \text{Stab}_\epsilon^c (p)
\]

has compact support because of the support conditions on the stable envelopes and the choice of opposite chambers. That is, the support is \( \pi^{-1} (x) \subset X \), a proper subset since \( \pi \) is proper.

Similar arguments are crucial later for our computation.

### 3.2.9 Steinberg correspondences

Let us first introduce the notion of Lagrangian correspondence.

Recall that a subvariety \( L \subset M \) of an algebraic symplectic variety \( (M, \omega_M) \) is called Lagrangian if the restriction \( \iota_{z'}^* \omega_M \) of the symplectic form vanishes and \( \dim Z = \frac{1}{2} \dim M \).

Now let \( (X, \omega_X) \) and \( Y \) be two algebraic symplectic varieties. We equip \( X \times Y \) with a symplectic form \( p_X^* \omega_X - p_Y^* \omega_Y \) (here \( p_X \) and \( p_Y \) are natural projections of \( X \times Y \) to \( X \) and \( Y \)). Then we say that

\[
L \subset X \times Y
\]
is a Lagrangian correspondence if it’s Lagrangian with respect to the chosen symplectic structure.

If both $X$ and $Y$ are $T$-varieties and $L$ is $T$-invariant and proper over $X$, this defines a map

$$\Theta_L : H^*_T(Y) \xrightarrow{p^*_Y} H^*_T(L) \xrightarrow{p^*_X} H^*_T(X).$$

We say that a Lagrangian correspondence $L \subset X \times Y$ is Steinberg if there exist proper $T$-equivariant maps

$$\begin{array}{c}
Y \\
\downarrow \\
X \longrightarrow V
\end{array}$$

to a $T$-equivariant affine variety $V$ such that

$$L \subset X \times V.$$

Let $(M, \omega_M)$ be an algebraic symplectic variety with an $A$-action preserving $\omega_M$, and let $L \subset M$ be an $A$-invariant Lagrangian subvariety. Choose a polarization $\epsilon$ on $M$. Then we can define the Lagrangian residue $\text{Res}_Z L \in H^*_T(M^A)$ to be the unique class such that

$$\iota^*_M [L] = \epsilon \cup \text{Res}_Z L + \ldots$$

where dots stand for terms of smaller $A$-degree and $[L]$ is the pushforward of $1 \in H^*_T(L)$ along the inclusion $L \hookrightarrow M$ (i.e. the Poincaré dual class to the fundamental cycle of $L$).

Now let $M = X \times Y$ with algebraic symplectic varieties $X$ and $Y$. Then $M^A = X^A \times Y^A$. Fix polarizations $\epsilon_X$ and $\epsilon_Y$ on $X^A$ and $Y^A$ respectively. Then we define the polarization on $X^A \times Y^A$ to be $\epsilon_X \epsilon_Y$.

Let $\Theta$ be the map given by a Steinberg correspondence $L$, and $\Theta_A$ the map by $\text{Res}_{X^A \times Y^A} L$ (this depends on choice $\epsilon_X$ and $\epsilon_Y$ as above). Then we have the following theorem.

**Theorem 3.5.** The diagram

$$\begin{array}{ccc}
H^*_A(Y^A) & \xrightarrow{\Theta_A} & H^*_A(Y) \\
\downarrow & & \downarrow \\
H^*_A(X^A) & \xrightarrow{\Theta} & H^*_A(X)
\end{array}$$

commutes. The polarization above is $\epsilon_Y$, below it is $\epsilon_X$.

### 3.2.10 Torus restriction

Let $\mathcal{C}$ be a chamber and let $\mathcal{C}'$ be a face of $\mathcal{C}$ of some dimension. Let $A' \subset A$ be a connected subtorus, such that $X_*(A') \otimes \mathbb{R} = a'_R = \text{span} \mathcal{C}' \subset a_R$. Then there’s a projection of the cone $\mathcal{C}$ to $a_R / a'_R$ which we denote by $\mathcal{C}''$.

Moreover, fix polarizations $\epsilon$ and $\epsilon'$ on $X^A$ and $X^{A'}$ respectively. This induces a polarization $\epsilon''$ of $X^A$ as a subspace of $X^{A'}$ by the following. Let $p \subset X^A$ and $Z \subset X^{A'}$, such that $p \subset Z$. If we present $\epsilon_p$ as a product of $A$-weight

$$\epsilon_p = \prod \alpha^\vee,$$

then we can write

$$\iota^*_p \epsilon' = \pm \prod_{\alpha^\vee |_{A'} \neq 0} \alpha^\vee \bigg|_{A'}.$$

Then we set

$$\epsilon'' = \pm \prod_{\alpha^\vee |_{A'} = 0} \alpha^\vee \bigg|_{A'/A}.$$
with the same sign as for $I_p^* e_Z$. One can rewrite this in terms of signs:

$$
\sigma_{p/X} = \sigma_{Z/X} \sigma_{p/Z}
$$

using the fact that the polarization $e^{A/A'} (N_{p/X})$ is induced in this sense by $e^A (N_{p/X})$ and $e^{A'} (N_{Z/X})$.

With these choices we have the following statement.

**Theorem 3.6.** The diagram

$$
\begin{array}{ccc}
H^*_T (X^A) & \xrightarrow{\text{Stab}_Z} & H^*_T (X) \\
\downarrow \text{Stab}_{e^{A'}} & & \downarrow \text{Stab}_{e'} \\
H^*_T (X^{A'}) & \xrightarrow{\text{Stab}_{e_Z}} & H^*_T (X)
\end{array}
$$

commutes.

### 3.3 Stable Envelopes restrictions modulo $\hbar^2$: $A_1$ type

In this case we have enough recursive relations similar to the ones which appear in [S]. They arise from action by certain Steinberg correspondences. These correspondences appeared in the context of the convolution affine Grassmannian of type $A_1$ at least in [CK]. Even before that these correspondences were studied by H. Nakajima[N3], since the slices in type $A$ appear to have representation as Nakajima quiver varieties[MV3].

#### 3.3.1 Steinberg correspondences in type $A_1$

In type $A_1$ the connected group of the adjoint type is $G = \text{PSL}_2$, the fundamental coweight $\omega$ is minuscule, the simple root is $\alpha = 2\omega$. Let us denote the slice as $X_0 = \text{Gr}^{e_{2\omega}}_\omega$. One can think $\geq 0$ to have $k\omega$ dominant, but it won’t affect the computations. We do assume $l > 1$ and $|k| < l$, otherwise $X_0$ is just a point or an empty space. Then $X = \text{Gr}^{e_{2\omega}}_\omega$ is the resolution of $X_0$, where $\lambda$ is an $l$-tuple

$$
\lambda = (\omega, \ldots, \omega).
$$

First let us introduce the following subvarieties of $X$. Recall that by definition a point in $X$ is an $(l + 1)$-tuple $(L_0, L_1, \ldots, L_l)$ of points in $\text{Gr}$ (here we use the trick $L_0 = e \cdot G(0)$ as before). Then for every $i$, $1 \leq i < l$ define

$$
X^i = \{(L_0, L_1, \ldots, L_l) \in X | L_{i-1} = L_{i+1}\}.
$$

I.e. $X^i$ is the equalizer of the projections to $(i - 1)$th and $(i + 1)$th factor.

We will also need a slice $Y_0 = \text{Gr}^{e_{2\omega}}_{k\omega}$. In case $l > 2$, its resolution is $Y = \text{Gr}^{e_{2\omega}}_{k\omega}$, where $\nu$ is an $(l - 2)$-tuple

$$
\nu = (\omega, \ldots, \omega).
$$

In case $l = 2$, we have $Y_0 = Y = \{e \cdot G(0)\}$ is the point if $k = 0$, and $Y_0$, $Y$ are empty spaces if $k \neq 0$.

Then there are the following natural $T$-equivariant "forgetful" projections:

$$
\pi_i : X^i \rightarrow Y
$$

$$(L_0, \ldots, L_l) \mapsto (L_0, \ldots, L_{i-2}, L_{i-1} = L_{i+1}, L_{i+2}, \ldots, L_l).$$

In English, we forget $L_i$ and identify $L_{i-1}$ and $L_{i+1}$. It’s an easy check that for $l > 2$ the resulting $(l - 2)$-tuple satisfies the defining relation of $Y$ (or is the only point $e \cdot G(0)$ on the case $l = 2$).
Proposition 3.7. $X^i \to Y$ is a $\mathbb{P}^1$-bundle.

Proof. Since $X^i$ adds one more element of $\text{Gr}$ to a sequence from $Y$, it is natural to consider $X^i \to Y$ as a subbundle of the trivial fibration

$$
X^i \leftarrow Y \times \text{Gr}
$$

defined by the equations

$$L_{i-1} \xrightarrow{\omega} L_i$$

where $L_0, \ldots, L_{i-2}, L_{i-1} = L_{i+1}, L_{i+2}, \ldots, L_i$ are projections of the point in $Y$, and $L_i$ is an element in $\text{Gr}$ (the one we want to reconstruct). The fiber is $\mathbb{P}^1$ and this fibration is locally trivial.

Recall that $X$ and $Y$ have symplectic forms $\omega_X$ and $\omega_Y$ as described in Section 2.

Proposition 3.8. The natural pullbacks of the symplectic forms from $X$ and $Y$ to $X^i$ are equal:

$$i^*_X \omega_X = \pi^*_i \omega_Y$$

Then we get that $X^i \times_Y X^i$ is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over $Y$. In particular, $X^i \times_Y X^i$ is smooth.

We have the following commutative diagram

$$
\begin{array}{ccc}
Y & \xleftarrow{\pi_i} & X^i \\
\downarrow & & \downarrow \\
Y_0 & \xleftarrow{i} & X
\end{array}
$$

The vertical arrows are natural maps from the definition of resolutions, the inclusion $Y_0 \hookrightarrow X_0$ is induced by the inclusion of closed cells $\text{Gr}^{([l-2])}\omega \hookrightarrow \text{Gr}^{[l]}\omega$.

This gives that, first of all, there’s a natural inclusion $X^i \times_Y X^i \subset X^i \times_{X_0} X^i$, where the second fiber product is with respect to the dashed line on the diagram. This is because the dashed line factors through $\pi_i$. Secondly, we have a natural inclusion $X^i \times_{X_0} X^i \subset X \times_{X_0} X$ because the inclusion $X^i \hookrightarrow X$ commutes with the maps to $X_0$ as the diagram shows. Finally, because all morphisms are $T$-equivariant, the inclusions are also $T$-equivariant.

Let us denote $L^i = X^i \times_Y X^i$. The statements above allows us to write $L^i \subset X \times_{X_0} X$.

Example 3.1. In the case of the resolution of $A_n$-singularity, i.e. when $l = n + 1$, $k = l - 1$, we have $Y_0 = \text{Gr}^{([1])}\omega = pt$, so $Y = pt$. The subvariety $X^i \subset X$ is an irreducible component $E_i$ of the exceptional divisor $E_i$, $X^i \simeq \mathbb{P}^1$. Then $X^i \times_Y X^i$ is $\mathbb{P}^1 \times \mathbb{P}^1$, and the inclusion $X^i \times_Y X^i \subset X \times X$ is just $E_i \times E_i \subset X \times X$. Since $E$ maps to one point in $X_0$, we even have $X^i \times_Y X^i \subset X \times_{X_0} X$.

Proposition 3.9. $L^i \subset X \times_{X_0} X$ is a Steinberg correspondence.

Proof. First, we have $L^i \subset X \times_{X_0} X$ for a proper morphism $X \to X_0$ as shown before. So we only need to check if $L^i$ is Lagrangian.

Let $p_1, p_2: L^i \to X$ be the projections to the first and second factor in $X \times X$. Then we need to check

$$p^*_1 \omega_X - p^*_2 \omega_X = 0.$$

We also have the natural inclusion $L^i \subset X^i \times X^i$ and corresponding projections $\tilde{p}_1, \tilde{p}_2: L^i \to X^i$. These are related to $p_k$’s by

$$p_k = \iota_X \tilde{p}_k.$$

By definition $L^i = X^i \times_Y X^i$, so

$$\pi^i \tilde{p}_1 = \pi^i \tilde{p}_2.$$
Then from the Proposition 3.8 we can write
\[ p_1^* \omega_X - p_2^* \omega_X = (p_1^* - p_2^*) (p_1^* \omega_X) = (p_1^* - p_2^*) (p_1^* \omega_Y) = 0, \]
which finishes the proof.

Let us discuss what are the elements \((X^i)^A\) and what are the fibers of \((X^i)^A \rightarrow Y^A\).

**Proposition 3.10.**

1. A point \(p \in X^A\) is in \((X^i)^A\) if and only if
   \[ \Sigma_{i-1} (p) = \Sigma_{i+1} (p). \]

2. Every point \(q \in Y^A\) has exactly two preimages \(q_+, q_- \in (X^i)^A\) uniquely determined by the following conditions:
   \[ \Sigma_j (q_{\pm}) = \Sigma_j (q) \text{ for all } j < i, \]
   \[ \Sigma_j (q_{\pm}) = \Sigma_{j-2} (q) \text{ for all } j > i, \]
   \[ \Sigma_i (q_{\pm}) = \Sigma_{i-1} (q) \pm \omega. \]

**Proof.** The both parts follow directly from the definitions of \(X^i, X^i \rightarrow Y\) and the description \(X^A\).

The following statements are useful in localization computations.

**Proposition 3.11.** Let \(q_+, q_- \in (X^i)^A\) and \(q \in Y^A\) be as in Proposition 3.10. Then we have the following formulas

1. \[ \iota_{q_{\pm}}^* e^T (N_{X^i/X}) = \mp (\alpha^\vee + \langle \alpha^\vee, \Sigma_{q_{\pm}} (i) \rangle) h \]

2. \[ \frac{e^T (N_{q_{\pm}/X})}{e^T (N_{q/Y})} = \pm (\alpha^\vee + \langle \alpha^\vee, \Sigma_q (i - 1) \rangle) h \]

**Proof.**

1. Let us check what effect on the tangent space the condition \(L_{i-1} = L_{i+1}\) has. It gives an extra relation that the restrictions to \((i-1)\)th and \((i+1)\)th components must be equal. However, due to adjacency conditions with \(i\)th component, all but one weight vectors must be already equal in \(X\). The only new condition is on the part of weight
   \[ \mp (\alpha^\vee + \langle \alpha^\vee, \Sigma_{q_{\pm}} (i) \rangle h}. \]
   Instead of two independent components, we now have one (note that it can’t be that components from both sides must be 0). This gives the relation of the Euler classes.

2. By comparison of tangent spaces of \(X^i\) and \(Y\) (or even at the weight multiplicities) we see that \(X^i\) has one more tangent vector of weight
   \[ \pm (\alpha^\vee + \langle \alpha^\vee, \Sigma_q (i - 1) \rangle h}. \]

This proves the statement.
If \( p \in (X^i)^A \) we denote by \( s_i p \) the other point in \( (X^i)^A \) which has the same image under \( X^i \to Y \).

In notation of Proposition 3.10
\[
\begin{align*}
s_i q_+ &= q_-, \\
s_i q_- &= q_+.
\end{align*}
\]

If \( p \in (X \setminus X^i)^A \), then we define \( s_i p = p \).

One can easily see that \( s_i \) are involutions on \( X^A \).

The motivation for this notation is that this operation swaps \( \Delta_i (p) \) and \( \Delta_{i+1} (p) \) as an adjacent transposition. This leads to the following statement.

**Proposition 3.12.** Any two points in \( X^A \) can be related by a sequence of \( s_i \)'s.

**Proof.** Since for any \( p \in X^A \) we have
\[
\sum \Delta_i (p) = \Sigma_i (p) = k \omega
\]
then any two points in \( X^A \) differ only by permuting positions of \( +\omega \) and \( -\omega \) in \( \Delta (p) \). Any permutation can be written as a product of adjacent transpositions, which gives a relation by a sequence of \( s_i \)'s. \( \square \)

The correspondence \( L^i \) is \( T \)-invariant, so it defines a map which we denote by \( \Theta^i : H_T^*(X) \to H_T^*(X) \).

A polarization \( \epsilon \) on \( X^A \) induces a polarization on \( X^A \times X^A \) as in Theorem 3.5.

**Proposition 3.13.** The Lagrangian residue \( \text{Res}_{X^A \times X^A} L^i \) is supported on \( (L^i)^A \). The \( A \)-equivariant restrictions to \( (p, q) \in (L^i)^A \) is the following
\[
\iota^*_{(p, q)} \text{Res}_{X^A \times X^A} L^i = \begin{cases} -1, & \text{if } p = q, \\
\epsilon_p \epsilon_q & \text{if } p \neq q. \end{cases}
\]

**Proof.** By dimension count
\[
\iota^*_{(p, q)} \text{Res}_{X^A \times X^A} L^i \in H_0^A (pt) = \mathbb{Q}
\]
Restricting the definition of of the Lagrangian residues to the subtorus \( A \), we get
\[
\iota^*_{(p, q)} L^i \big|_A = \epsilon_p \epsilon_q \iota^*_{(p, q)} \left[ \text{Res}_{X^A \times X^A} L^i \right]
\]
So
\[
\iota^*_{(p, q)} \text{Res}_{X^A \times X^A} L^i = \frac{\epsilon_A (N_{(p,q)}/L^i)}{\epsilon_p \epsilon_q}
\]

One can compute
\[
\epsilon_A (N_{(p,q)}/L^i) = \begin{cases} (-1)^{\dim Y/2} (\alpha^\vee)^{\dim X}, & \text{if } p = q, \\
(-1)^{\dim Y/2} (\alpha^\vee)^{\dim X} \epsilon_p \epsilon_q, & \text{if } p \neq q, \end{cases}
\]
and
\[
\epsilon_p \epsilon_q = \epsilon_q \epsilon_p = (-1)^{\dim X/2} (\alpha^\vee)^{\dim X} \epsilon_p \epsilon_q
\]
This gives the statement of the Proposition. \( \square \)

We denote by
\[
\text{Res} \Theta^i : H_T^*(X^A) \to H_T^*(X^A)
\]
the map given by the Lagrangian residues \( \text{Res}_{X^A \times X^A} L^i \).
3.3.2 Recursion

The main ingredient for the recursion is interaction of $\Theta^i$ with the stable envelopes and the fixed points basis.

**Proposition 3.14.** The correspondence $\Theta^i$ has the following (left) action on $\text{Stab}_\mathcal{C}^\epsilon(p)$, $p \in X^A$:

$$\Theta^i \circ \text{Stab}_\mathcal{C}^\epsilon(p) = -\text{Stab}_\mathcal{C}^\epsilon(p) + \frac{\epsilon_p}{\epsilon_{s_ip}} \text{Stab}_\mathcal{C}^\epsilon(s_ip).$$

Moreover,

$$\frac{\epsilon_p}{\epsilon_{s_ip}} \in \{\pm 1\}.$$  

**Proof.** By the Theorem 3.5 we get

$$\Theta^i \circ \text{Stab}_\mathcal{C}^\epsilon(p) = \text{Stab}_\mathcal{C}^\epsilon\left(\text{Res} \Theta^i 1_p\right)$$

If $p \in (X \setminus X^i)^A$, both $\text{Res} \Theta^i 1_p = 0$ and

$$-\text{Stab}_\mathcal{C}^\epsilon(p) + \frac{\epsilon_p}{\epsilon_{s_ip}} \text{Stab}_\mathcal{C}^\epsilon(s_ip) = 0,$$

since $s_ip = 0$ by definition. This proves the Proposition for all $p \in (X \setminus X^i)^A$.

Now assume $p \in (X^i)^A$. From Proposition 3.13 and Proposition 3.10, we have

$$\text{Stab}_\mathcal{C}^\epsilon\left(\text{Res} \Theta^i 1_p\right) = \text{Stab}_\mathcal{C}^\epsilon\left(-1_p + \frac{\epsilon_p}{\epsilon_{s_ip}} 1_{s_ip}\right)$$

$$= -\text{Stab}_\mathcal{C}^\epsilon(p) + \frac{\epsilon_p}{\epsilon_{s_ip}} \text{Stab}_\mathcal{C}^\epsilon(s_ip).$$

This proves the formula for $p \in (X^i)^A$.

The remark about $\frac{\epsilon_p}{\epsilon_{s_ip}} \in \{\pm 1\}$ follows from the fact that $\epsilon_q = \pm (\alpha^\vee)^{\dim X/2}$ in type $A_1$ for any $q \in X^A$.

There’s a similar fact for the restrictions $\epsilon_q^*$.  

**Proposition 3.15.** The correspondence $\Theta^i$ has the following (right) action on $\epsilon_q^* q \in X^A$:

$$\epsilon_q^* \circ \Theta^i = \frac{\alpha^\vee + \langle \alpha^\vee, \Sigma_i(q) \rangle}{\alpha^\vee + \langle \alpha^\vee, \Sigma_{i-1}(q) \rangle} \frac{h}{\hbar} \left[-\epsilon_q^* + \epsilon_{s_ip}^* q\right]$$

**Proof.** This is a straightforward localization computation.  

In the recursive relations it’s more convenient to work with reduced stable envelopes $\overline{\text{Stab}}$, i.e. stable envelopes with a different normalization:

$$\overline{\text{Stab}_\mathcal{C}^\epsilon}(p) = \frac{1}{\epsilon_p} \text{Stab}_\mathcal{C}^\epsilon(p).$$

These classes no longer depend on the polarization. The only restriction which is not divisible by $\hbar$ is the diagonal element

$$\epsilon_p^* \overline{\text{Stab}_\mathcal{C}^\epsilon}(p) = 1 \mod \hbar.$$

As a side effect, these classes lie only in localized cohomology, but we won’t use these classes when we need non-localized statements. One can rewrite all recursions in terms of original Stab, just inserting appropriate polarizations.
The recursion relation appear from application of both Proposition 3.14 and Proposition 3.15 to the following expression
\[ t_q^\ast \circ \Theta^i \circ \text{Stab}_p^\ast(p). \]

This gives the following relations similar to the ones in [S].

**Proposition 3.16.** Let \( p, q \in X^A \), \( \mathcal{C} \) be a Weyl chamber. Then
\[
\iota_q^\ast \text{Stab}_p^\ast(s,p) = - \langle \alpha^\vee, \Delta_i(q) \rangle \hbar \frac{\alpha^\vee + \langle \alpha^\vee, \Sigma_i(q) \rangle}{\alpha^\vee + \langle \alpha^\vee, \Sigma_{i-1}(q) \rangle} \hbar t_q^\ast \text{Stab}_p^\ast(p) + \frac{\alpha^\vee + \langle \alpha^\vee, \Sigma_i(q) \rangle}{\alpha^\vee + \langle \alpha^\vee, \Sigma_{i-1}(q) \rangle} \hbar t_{s,q}^\ast \text{Stab}_p^\ast(p)
\]
and
\[
\iota_{s,q}^\ast \text{Stab}_p^\ast(p) = \frac{\alpha^\vee + \langle \alpha^\vee, \Sigma_i(q) \rangle}{\alpha^\vee + \langle \alpha^\vee, \Sigma_{i-1}(q) \rangle} \hbar \frac{\alpha^\vee + \langle \alpha^\vee, \Sigma_i(q) \rangle}{\alpha^\vee + \langle \alpha^\vee, \Sigma_{i-1}(q) \rangle} \hbar t_q^\ast \text{Stab}_p^\ast(s,p)
\]

**Proof.** From Proposition 3.15 and Proposition 3.14 we have two ways to express \( t_q^\ast \circ \Theta^i \circ \text{Stab}_p^\ast(p) \):
\[
t_q^\ast \circ \Theta^i \circ \text{Stab}_p^\ast(p) = \frac{\alpha^\vee + \langle \alpha^\vee, \Sigma_i(q) \rangle}{\alpha^\vee + \langle \alpha^\vee, \Sigma_{i-1}(q) \rangle} \hbar [- \iota_q^\ast \text{Stab}_p^\ast(p) + \iota_{s,q}^\ast \text{Stab}_p^\ast(p)]
\]
\[
t_q^\ast \circ \Theta^i \circ \text{Stab}_p^\ast(p) = - \iota_q^\ast \text{Stab}_p^\ast(s,p) + \frac{\epsilon_{s,q}}{\epsilon_{s,p}} \iota_q^\ast \text{Stab}_p^\ast(s,p).
\]

By algebraic manipulations with these expressions we get the relations in the Proposition. \( \square \)

**Remark.** One can use this relation to recursively compute the stable envelopes, starting either from \( p \) for which all restrictions are known (i.e. the minimal \( p \) with respect to \( \mathcal{C} \)), or starting from \( q \) restriction to which are all known (i.e. the maximal \( q \) with respect to \( \mathcal{C} \)).

**Remark.** These relations modulo \( \hbar \) just give
\[
\iota_{s,q}^\ast \text{Stab}_p^\ast(p) = \iota_q^\ast \text{Stab}_p^\ast(s,p) \mod \hbar.
\]

So if we know that only the diagonal restriction is \( 1 \mod \hbar \) and all other are zero (say, from \( \text{Stab}_p^\ast(p) \) for the minimal \( p \) with respect to \( \mathcal{C} \)), then all other stable envelopes have this property. We know this from the definition of the stable envelopes combined with the property of off-diagonal terms of a symplectic resolution. Here we derive it just from commutativity with Steinberg correspondences and knowing the stable envelope of the minimal point.

### 3.3.3 Formulas for stable envelopes restrictions: type \( A_1 \)

Since for our main application we are interested in the stable envelopes modulo \( \hbar^2 \), let us present the following version of 3.17.

**Proposition 3.17.** Let \( p, q \in X^A \), \( \mathcal{C} \) be a Weyl chamber, \( 1 \leq i < l \). Then
\[
\iota_q^\ast \text{Stab}_p^\ast(s,p) = \iota_{s,q}^\ast \text{Stab}_p^\ast(p) + \langle \alpha^\vee, \Delta_i(q) \rangle [\delta_{p,s,q} - \delta_{p,q}] \frac{\hbar}{\alpha^\vee} \mod \hbar^2.
\]

**Proof.** This is a straightforward computation using that
\[
\iota_q^\ast \text{Stab}_p^\ast(p) = \delta_{p,q} \mod \hbar.
\]
as discussed above. \( \square \)

Let us also introduce the following numerical quantity for \( p \in X^A \)
\[
|p|_\mathcal{C} = \frac{1}{2} \sum_i \langle \alpha^\vee, \Sigma_i(p) \rangle,
\]
where \( \alpha^\vee \) is the positive root with respect to \( \mathcal{C} \). Since all \( \Sigma_i(p) \) are coweights, the pairings are integers, so \( |p|_\mathcal{C} \in \mathbb{Z} \).
It is easy to check that $q<\varepsilon p \implies |q|_\varepsilon < |p|_\varepsilon$.
This implies that the minimal $|p|_\varepsilon$ is for the minimal point of $X^\Lambda$ with respect to $\varepsilon$, $p^{\varepsilon}_{\text{min}}$. I.e. $p^{\varepsilon}_{\text{min}}$ is the unique point of the following form:

$$\Delta \left( p^{\varepsilon}_{\text{min}} \right) = \left(-\omega_\varepsilon, \ldots, -\omega_\varepsilon, \omega_\varepsilon, \ldots, \omega_\varepsilon \right),$$

where $\omega_\varepsilon$ is the minimal strictly dominant coweight with respect to $\varepsilon$.

**Remark.** One can identify points $p \in X^\Lambda$ with partitions in a square $l^2 \times l^2$. Then $|p|_\varepsilon$ is the size of the partition for one chamber $\varepsilon = \varepsilon_+$, and the size of the complement for the opposite chamber, $\varepsilon = \varepsilon_-$ (both up to adding a constant). Moreover, the order $<_{\varepsilon_+}$ is the inclusion of partitions, $<_{\varepsilon_-}$ is the inclusion of complements.

Now we are ready to prove the main result in type $A_1$.

**Theorem 3.18.** Let $p, q \in X^\Lambda$ be two fixed points. Let $\alpha_\varepsilon$ be the positive coroot with respect to $\varepsilon$ and $\alpha_\varepsilon^\vee$ the corresponding root. Then

1. If $p = q$, then

$$\iota^*_q \text{Stab}^\varepsilon (p) = \varepsilon_p \left[ 1 + \left( |p|_\varepsilon + C \right) \frac{h}{\alpha_\varepsilon^\vee} \right] \mod h^2,$$

where the constant $C \in \mathbb{Z}$ does not depend on $p$.

2. If there are $i, j$, $0 \leq i < j \leq l$ such that

$$\Delta_i (q) = \Delta_i (p) - \alpha_\varepsilon,$$

$$\Delta_j (q) = \Delta_j (p) + \alpha_\varepsilon,$$

$$\Delta_k (q) = \Delta_k (p) \text{ for all } k, k \neq i, k \neq j,$$

then

$$\iota^*_q \text{Stab}^\varepsilon (p) = \frac{\varepsilon_p \alpha_\varepsilon^\vee}{\alpha_\varepsilon^\vee} \mod h^2.$$

3. Otherwise

$$\iota^*_q \text{Stab}^\varepsilon (p) = 0 \mod h^2,$$

**Proof.** We work with $\text{Stab}$, the statements for $\text{Stab}$ are recovered by a simple multiplication on polarization $\varepsilon$.

The proof is by induction on $|p|_\varepsilon$.

**Base case**

The minimal value of $|p|_\varepsilon$ is $|p^{\varepsilon}_{\text{min}}|_\varepsilon$ as mentioned above. Moreover, $p^{\varepsilon}_{\text{min}}$ is the unique point with this weight.

Since $p^{\varepsilon}_{\text{min}}$ is the minimal point with respect to $\varepsilon$, for all $q \neq p^{\varepsilon}_{\text{min}}$

$$\iota^*_q \text{Stab}^\varepsilon \left(p^{\varepsilon}_{\text{min}} \right) = 0.$$

This agrees with the statement of the theorem, because there is no $q$ satisfying (6) for $i < j$: one can only subtract $\alpha_\varepsilon$ from $\Delta_i (p^{\varepsilon}_{\text{min}}) = \omega_\varepsilon$ which are in the second half of the sequence $\Delta \left( p^{\varepsilon}_{\text{min}} \right)$, and $\alpha_\varepsilon$ can be added to $\Delta_j (p^{\varepsilon}_{\text{min}}) = -\omega_\varepsilon$ which are in the first half of $\Delta \left( p^{\varepsilon}_{\text{min}} \right)$.

The check of the diagonal is trivial: the weights of $N^{-\varepsilon}_{p/X}$ are of form

$$-(\alpha_\varepsilon^\vee + nh) = -\alpha_\varepsilon^\vee \left[ 1 + n \frac{h}{\alpha_\varepsilon^\vee} \right], \quad n \in \mathbb{Z}$$

So

$$\varepsilon^T \left(N^{-\varepsilon}_{p/X} \right) = \left(-\alpha_\varepsilon^\vee \right)^{\dim X/2} \left[ 1 + N \frac{h}{\alpha_\varepsilon^\vee} \right] \mod h^2, \quad N \in \mathbb{Z}.$$
This implies

\[ t^*_{p_{min}} \overline{\text{Stab}_\epsilon} (p_{min}^\epsilon) = 1 + N \frac{h}{\alpha^\epsilon}. \]

After a shift of \( N \) by an integer constant \( |p_{min}|_{\epsilon} \) we get the statement of the Theorem.

Thus the base case is proved.

**Inductive step**

Assume the statement is true for all \( p \in X^A \) such that \( |p|_\epsilon = N \), where \( N \geq |p_{min}|_{\epsilon} \). Let’s check the statement for a point \( p \in X^A \) with \( |p|_\epsilon = N + 1 \).

Since \( |p|_\epsilon = N + 1 > |p_{min}|_{\epsilon} \), we have \( p \neq p_{min}^\epsilon \). This means that in \( \Delta (p) \) we can find a \( \omega_\epsilon \) followed by \(-\omega_\epsilon\):

\[
\Delta_m (p) = \omega_\epsilon,
\]
\[
\Delta_{m+1} (p) = -\omega_\epsilon
\]

for some \( m \). Then \( s_{m}p <_{\epsilon} p \).

Moreover,

\[ |s_{m}p|_{\epsilon} = |p|_{\epsilon} - 1 = N, \]

because

\[ \Sigma (s_{m}p) = (\Sigma_0 (p), \ldots, \Sigma_{m-1} (p), \Sigma_{m} (p) - \alpha_\epsilon, \Sigma_{m+1} (p), \ldots, \Sigma_l (p)). \]

We have 3 cases of \( q \):

1. \( q = p \).

   By Proposition 3.17 we have

\[ t^*_{p} \overline{\text{Stab}_\epsilon} (p) = t^*_{s_{m}p} \overline{\text{Stab}_\epsilon} (s_{m}p) + \langle \alpha^\epsilon, \Delta_m (p) \rangle \frac{h}{\alpha^\epsilon} \mod h^2 \]
\[ = t^*_{s_{m}p} \overline{\text{Stab}_\epsilon} (s_{m}p) + \langle \alpha^\epsilon, \Delta_m (p) \rangle \frac{h}{\alpha^\epsilon} \mod h^2 \]
\[ = t^*_{s_{m}p} \overline{\text{Stab}_\epsilon} (s_{m}p) + \frac{h}{\alpha^\epsilon} \mod h^2 \]

We know

\[ t^*_{s_{m}p} \overline{\text{Stab}_\epsilon} (s_{m}p) = 1 + (|s_{m}p|_{\epsilon} + C) \frac{h}{\alpha^\epsilon} \mod h^2 \]
\[ = 1 + (|p|_{\epsilon} - 1 + C) \frac{h}{\alpha^\epsilon} \mod h^2 \]

by the induction assumption.

This proves this case.

2. \( q = s_{m}p \).

   In this case the Proposition 3.17 says

\[ t^*_{s_{m}p} \overline{\text{Stab}_\epsilon} (s_{m}p) = t^*_{p} \overline{\text{Stab}_\epsilon} (s_{m}p) - \langle \alpha^\epsilon, \Delta_m (s_{m}p) \rangle \frac{h}{\alpha^\epsilon} \mod h^2. \]

We have \( t^*_{p} \overline{\text{Stab}_\epsilon} (s_{m}p) = 0 \) since \( s_{m}p <_{\epsilon} p \). Also

\[ \Delta_m (s_{m}p) = \Delta_{m+1} (p) = -\omega_\epsilon, \]

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so
\[ \langle \alpha^\vee, \Delta_m(s_mp) \rangle = -1. \]

Finally,
\[ t_{s_mp}^* \overline{\text{Stab}}_\epsilon (p) = \frac{\hbar}{\alpha^\vee} \mod \hbar^2. \]

Note that such a pair of \( q, p \) satisfy the property (6) for \( i = m \) and \( j = m + 1 \). After restoring \( \epsilon \) from the normalization, we get the statement of the Theorem.

3. Both \( q \neq p \) and \( q \neq s_mp \).

The Proposition \ref{prop:stable-envelopes} gives in this case
\[ t_q^* \overline{\text{Stab}}_\epsilon (p) = t_{s_mp}^* \overline{\text{Stab}}_\epsilon (s_mp) \mod \hbar^2. \]

Note that if \( s_mp \) and \( s_mq \) are satisfy relations (6), then \( p \) and \( q \) are also satisfy these, possibly for different \( i, j \). The reason is that \( s_m \) only permutes components of \( \Delta(s_mp) \) and \( \Delta(s_mq) \) in the same way. The only think we have to check is that the order of \( \Delta_i(s_mp) \) and \( \Delta_j(s_mp) \) is not reversed. Since \( s_m \) permutes \( \Delta_m(s_mp) \) and \( \Delta_m(s_mp) \) is not reversed. But then \( q = s_mp \), which is prohibited in this case.

Applying the induction assumption
\[ t_q^* \overline{\text{Stab}}_\epsilon (p) = \frac{\hbar}{\alpha^\vee} \mod \hbar^2 \]

if \( p \) and \( q \) are related by (6), and
\[ t_q^* \overline{\text{Stab}}_\epsilon (p) = 0 \mod \hbar^2 \]

otherwise.

Combining the results from these three cases, we prove the inductive step.

\[ \square \]

Remark. The constant \( C \) in the formula for the diagonal can be found at any point \( p \) comparing with the result on weight multiplicities obtained in the previous section. For example, at the minimal or the maximal point.

In what follows we’re not interested in the diagonal restrictions, so we don’t write the computation of \( C \).

3.4 Stable Envelopes restrictions modulo \( \hbar^2 \): general case

3.4.1 Wall-crossing for stable envelopes

The first fact we need to reduce the general case to type \( A_1 \) is the wall-crossing phenomena for restrictions of stable envelopes modulo \( \hbar^2 \). It is a straightforward consequence of the torus restriction Theorem \ref{thm:torus-restriction} and a close relative of \( R \)-matrices in [MO].

Theorem 3.19. Let \( X, A, \epsilon \) be as in the definition of the stable envelopes and \( X^A \) discrete. Pick two Weyl chambers \( C_+ \) and \( C_- \) sharing a face \( C' \) (of any codimension). This gives associated subtorus \( A' \subset A \), i.e. such connected subgroup that \( a'_R = X_*(A') \otimes_\mathbb{Z} \mathbb{R} = \text{span} C' \subset a_R \). For any \( p,q \in X^A, p \neq q \) we have the following

1. If \( p \) and \( q \) are in different components of \( X^{A'} \) then
\[ t_q^* \overline{\text{Stab}}_{\epsilon_+} (p) = t_q^* \overline{\text{Stab}}_{\epsilon_-} (p) \mod \hbar^2 \]
2. If \( p \) and \( q \) are in the same component of \( Z \subset X^{A'} \) then

\[
i_q^*\text{Stab}_{\xi_+} (p) - i_q^*\text{Stab}_{\xi_-} (p) = \epsilon_q' \left[ i_{q/Z}^*\text{Stab}_{\xi''_+} (p) - i_{q/Z}^*\text{Stab}_{\xi''_-} (p) \right] \mod h^2 \tag{7}
\]

where \( i_{q/Z}^*\text{Stab}_{\xi''_+} (p) \) and \( i_{q/Z}^*\text{Stab}_{\xi''_-} (p) \) are the stable envelopes on \( X^{A'} \) as in the torus restriction Theorem 3.6. The polarization \( \epsilon'' \) on \( X^{A'} \) is induced by \( \epsilon' \) on \( X^{A'} \) and \( \epsilon \) on \( X^{A} \) as in Theorem 3.6. On the RHS we use notation \( i_{q/Z}^* \) to emphasize that we restrict to \( q \) from \( Z \), not from \( X \).

\textbf{Proof.} The torus restriction Theorem 3.6 gives us that the following diagram is commutative

\[
\begin{array}{ccc}
H_T^* (X^{A'}) & \xrightarrow{\text{Stab}_{\xi_+}} & H_T^* (X) \\
\text{Stab}_{\xi''_+} & & \text{Stab}_{\xi''_+} \\
H_T^* (X^{A'}) & \xleftarrow{\text{Stab}_{\xi_-}} & H_T^* (X) \\
\end{array}
\]

The induced polarizations \( \epsilon' \), \( \epsilon'' \) and the projected chamber \( \xi''_+ \) are the same as in Theorem 3.6. Then we have

\[
i_q^*\text{Stab}_{\xi_+} (p) = i_{q/Z}^*\text{Stab}_{\xi'} \circ \text{Stab}_{\xi''_+} (p).
\]

Similar holds if we replace \( \xi_+ \) by \( \xi_- \):

\[
i_q^*\text{Stab}_{\xi_-} (p) = i_{q/Z}^*\text{Stab}_{\xi'} \circ \text{Stab}_{\xi''_-} (p).
\]

This gives

\[
i_q^*\text{Stab}_{\xi_+} (p) - i_q^*\text{Stab}_{\xi_-} (p) = i_{q/Z}^*\text{Stab}_{\xi'} \left[ \text{Stab}_{\xi''_+} (p) - \text{Stab}_{\xi''_-} (p) \right].
\]

If the fixed point if \( p \) is in the component \( Z_p \subset X^{A'} \), then the class \( \text{Stab}_{\xi''_+} (p) - \text{Stab}_{\xi''_-} (p) \) is supported on \( Z_p \). One gets this from the definition of the stable envelopes and that \( z_p \) contains refined attractor to \( p \) via any evolution by \( A/A' \).

Denote the component of \( X^{A'} \) containing \( q \) as \( Z_q \).

Let us first consider the case when \( p \) and \( q \) are in different components of \( X^{A'} \), i.e. \( Z_p \neq Z_q \). Then

- The class

\[
\text{Stab}_{\xi''_+} (p) - \text{Stab}_{\xi''_-} (p)
\]

is divisible by \( h \) since the diagonal contributions cancel and the off-diagonal contributions are divisible by \( h \) because \( X \) is a symplectic resolution.

- For any class \( \gamma \in H_T^* (X^{A'}) \) supported on \( Z_p \) the restriction

\[
i_{Z_p}^* \text{Stab}_{\xi} (\gamma)
\]

is divisible by \( h \) because \( X \) is a symplectic resolution.

Then by \( H_T^* (pt) \)-linearity of the stable envelopes and restrictions we get that

\[
i_q^*\text{Stab}_{\xi_+} (p) - i_q^*\text{Stab}_{\xi_-} (p)
\]

is divisible by \( h^2 \) which gives the first statement of the theorem.

Now let \( p \) and \( q \) be in the same component \( Z = Z_p = Z_q \). Then by one of the defining properties of the stable envelopes we get

\[
\text{Stab}_{\xi_+} (p) - \text{Stab}_{\xi_-} (p) = \sigma_n^Z \epsilon_T \left( \frac{N - \epsilon'}{Z/X} \right) \left[ \text{Stab}_{\xi''_+} (p) - \text{Stab}_{\xi''_-} (p) \right].
\]
Restricting to \( q \) we get

\[
\iota_q^*\text{Stab}_\epsilon\big( p \big) - \iota_q^*\text{Stab}_\epsilon\big( p \big) = \sigma_{q/Z}^{\epsilon'} \epsilon^T \left( N_{q/X}^\epsilon \right) \left[ \iota_q^*\text{Stab}_\epsilon\big( p \big) - \iota_q^*\text{Stab}_\epsilon\big( p \big) \right] \\
= \epsilon_q^' \left[ \iota_q^*\text{Stab}_\epsilon\big( p \big) - \iota_q^*\text{Stab}_\epsilon\big( p \big) \right] \mod \hbar^2.
\]

\( \square \)

Remark. In the case when \( \mathcal{C} \) is codimension one (a "wall") we have \( \mathcal{C}'' = -\mathcal{C}'' \) and there’s only one non-zero contribution on the RHS of (7).

Remark. Informally, the Theorem 3.19 says that if one fixes \( p, q \in X^A \) \( (p \neq q) \) and a polarization \( \epsilon \) variation of the chamber \( \mathcal{C} \) doesn’t change \( \iota_q^*\text{Stab}_\epsilon\big( p \big) \) modulo \( \hbar^2 \), unless one crosses a wall \( \mathcal{C}' \) where \( p \) and \( q \) are in the same component of \( X^{\text{span}\mathcal{C}'} \).

### 3.4.2 Formulas for stable envelopes restrictions: general case

Let us return back to the case \( X = \text{Gr}^{\lambda}_{\mu} \). Now we can prove the following generalization of Theorem 3.18.

**Theorem 3.20.** Given fixed points \( p, q \in X^A \), \( p \neq q \), a Weyl chamber \( \mathcal{C} \) and a polarization \( \epsilon \) we have the following restriction formula

1. If there are \( i, j, 0 \leq i < j \leq l \) and a coroot \( \alpha \) positive with respect to \( \mathcal{C} \), such that

\[
\Delta_i (q) = \Delta_i (p) - \alpha, \\
\Delta_j (q) = \Delta_j (p) + \alpha, \\
\Delta_k (q) = \Delta_k (p) \text{ for all } k \neq i, k \neq j,
\]

then

\[
\iota_q^*\text{Stab}_\epsilon\big( p \big) = \omega_{p,q} \frac{\hbar}{\epsilon'} \epsilon_p \mod \hbar^2,
\]

where the coefficient \( \omega_{p,q} \) can be found from any polarization \( \epsilon' \) on \( X^{\ker \alpha} \):

\[
\omega_{p,q} = \frac{\epsilon_q^'}{\epsilon_p'}.
\]

This coefficient does not depend on the choice of \( \epsilon' \). In general, \( \omega_{p,q} \) is not in \( \mathbb{Q} \).

2. Otherwise

\[
\iota_q^*\text{Stab}_\epsilon\big( p \big) = 0 \mod \hbar^2.
\]

**Proof.** Let \( q < \epsilon p \), otherwise the restriction of the stable envelope \( \iota_q^*\text{Stab}_\epsilon\big( p \big) \) is zero by the definition. Then it implies that \( \iota_q^*\text{Stab}_\epsilon\big( p \big) = 0 \). We’ll use this to find \( \iota_q^*\text{Stab}_\epsilon\big( p \big) \) by wall-crossing.

One can connect \( -\mathcal{C} \) to \( \mathcal{C} \) by a sequence of adjacent chambers

\[
-\mathcal{C} = \mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \mathcal{C}_k = \mathcal{C}.
\]

Moreover, if \( H_1, \ldots, H_k \) are the hyperplanes which split the space into chambers, then one can choose the sequence to be such that each \( H_i \) is crossed exactly once (for example, make a choice by following a straight line connecting a generic point in \( -\mathcal{C} \) with a generic point in \( \mathcal{C} \)).

In our case all the hyperplanes have form \( H_\alpha = \ker \alpha^\vee \) for a root \( \alpha^\vee \) by Proposition A.20.

By using the Theorem 3.19 for the chain of chambers \( -\mathcal{C} = \mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \mathcal{C}_k = \mathcal{C} \) we see, that the only change module \( \hbar^2 \) appears when we cross the wall \( H_\alpha = \ker \alpha^\vee \) such that \( p \) and \( q \) are in the
same component of $X^\ker \alpha^\vee$. By Corollary A.23 we know that there at most one such wall. To have exactly one wall, $p$ and $q$ most satisfy conditions from Theorem A.22. Since otherwise
\[
i_q^* \text{Stab}_\epsilon(p) = i_q^* \text{Stab}_{-\epsilon}(p) \mod \hbar^2 = 0 \mod \hbar^2
\]
we assume that conditions in Theorem A.22 are satisfied for one root $\alpha^\vee$. Then by Theorem 3.19
\[
i_q^* \text{Stab}_{\epsilon'}(p) = i_q^* \text{Stab}_{\epsilon''}(p) - i_q^* \text{Stab}_{-\epsilon''}(p)
\]
\[
= \epsilon'_q \left[ i_{q/2}^* \text{Stab}_{\epsilon''}(p) - i_{q/2}^* \text{Stab}_{-\epsilon''}(p) \right] \mod \hbar^2
\]
\[
= \epsilon'_q i_{q/2}^* \text{Stab}_{\epsilon''}(p) \mod \hbar^2
\]
where $\mathcal{C}''$ is the projection of $\mathcal{C}$ to $a_\mathbb{R}/\ker \alpha^\vee$, $Z \subset X^\ker \alpha^\vee$ is the component containing both $p$ and $q$, $\epsilon'$ is any polarization on $Z$, $\epsilon''$ is induced by $\epsilon$ and $\epsilon'$. By identification of $Z$ with the resolution of type $A_1$ slice in Theorem A.21 and the result for $A_1$-type in Theorem 3.18 we get that
\[
i_q^* \text{Stab}_{\epsilon''}(p) = h \frac{\epsilon''_p}{\overline{\alpha^\vee}} \mod \hbar^2
\]
if $p$ and $q$ satisfy conditions (8) and is 0 modulo $\hbar^2$ otherwise.
This gives
\[
i_q^* \text{Stab}_{\epsilon}(p) = h \frac{\epsilon''_p}{\overline{\alpha^\vee}} \mod \hbar^2.
\]
Recall that by definition of the induced polarization we have
\[
\epsilon_p = \epsilon'_p \epsilon''_p.
\]
Then
\[
i_q^* \text{Stab}_{\epsilon}(p) = \epsilon'_q \frac{h}{\epsilon'_p} \epsilon_p \mod \hbar^2.
\]
The factor
\[
\frac{\epsilon'_q}{\epsilon'_p}
\]
does not depend on a choice of polarization $\epsilon'$ since all polarizations differ only by $\pm 1$ on $Z$ and the extra sign would cancel.
This proves the statement.

\[\square\]

Remark. To see that $\omega_{p,q}$ is not in $\mathbb{Q}$ it’s enough to consider the first type $A_2$ case, $T^*\mathbb{P}^2$, which is one of the examples with the Picard rank one.

3.5 Classical Multiplication

3.5.1 Stable envelopes and classical multiplication

Let us first discuss in general relation between classical multiplication of a stable envelope by a divisor and restrictions of stable envelopes.

Proposition 3.21. Let $X$, $T$, $A$, $\hbar$, $\epsilon$, $\mathcal{C}$ be as in the definition of the stable envelopes. Assume moreover that $X$ is a symplectic resolution with $X^A$ finite.

1. For any $\mathcal{L} \in \text{Pic}_T(X)$ and $p, q \in X^A$, $q<\epsilon p$ there are such $c_{p,q}^\mathcal{L} \in \mathbb{Q}$ that
\[
c^T_1(\mathcal{L}) \cup \text{Stab}_\epsilon(p) = i_p^* c_1^T(\mathcal{L}) \cdot \text{Stab}_\epsilon(p) + \hbar \sum_{q<\epsilon p} c_{p,q}^\mathcal{L} \text{Stab}_\epsilon(q).
\]

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2. The number \( c^\xi_{p,q} \in \mathbb{Q} \) is uniquely reconstructed from the restriction of the stable envelope \( \iota_q^*\text{Stab}_\xi(p) \) modulo \( \hbar^2 \):

\[
c^\xi_{p,q} = \frac{\iota_q^*\text{Stab}_\xi(p)}{\hbar} \quad \left| \frac{\iota_q^*c^\Lambda_1(L) - \iota_q^*c^\Lambda_2(L)}{c^\Lambda_1} \right.
\]

Proof.

1. Since \( \{ \text{Stab}^\xi_{-\xi}(q) \} \) is the dual basis to \( \{ \text{Stab}^\xi_{\xi}(q) \} \), we have

\[
c_1^T(L) \cup \text{Stab}^\xi_{\xi}(p) = \sum_{q \in X^\Lambda} \left( \text{Stab}^\xi_{-\xi}(q), c_1^T(L) \cup \text{Stab}^\xi_{\xi}(p) \right) \text{Stab}^\xi_{\xi}(q)
\]

By localization we can compute the coefficients

\[
\left( \text{Stab}^\xi_{-\xi}(q), c_1^T(L) \cup \text{Stab}^\xi_{\xi}(p) \right) = \sum_{x \in X^\Lambda} \frac{\iota_q^*\text{Stab}^\xi_{-\xi}(q) \cdot c_1^T(L) \cdot \iota_q^*\text{Stab}^\xi_{\xi}(p)}{c^T(N_{x/X})}
\]

(10)

First of all, by the properties of the stable envelopes the only nonzero terms are the ones where \( q \leq x \leq \ell_e p \). In particular, this coefficient is zero unless \( q \leq \ell_e p \).

If \( q = p \) there’s only one term in (10):

\[
\left( \text{Stab}^\xi_{-\xi}(q), c_1^T(L) \cup \text{Stab}^\xi_{\xi}(p) \right) = \frac{\iota_p^*\text{Stab}^\xi_{-\xi}(p) \cdot c_1^T(L) \cdot \iota_p^*\text{Stab}^\xi_{\xi}(p)}{c^T(N_{p/X})} = \iota_p^*c_1^T(L)
\]

If \( q \neq p \), we know that for every term in (10) at least one of the stable envelope restrictions is divisible by \( \hbar \) since \( X \) is a symplectic resolution, see Theorem 3.3.

On the other hand, by dimension count the coefficients we have

\[
\left( \text{Stab}^\xi_{-\xi}(q), c_1^T(L) \cup \text{Stab}^\xi_{\xi}(p) \right) \in \mathbb{H}^2_{X^\Lambda}(pt)
\]

The pairing lies in non-localized cohomology \( \mathbb{H}^*_{X^\Lambda}(pt) \) since the class

\[
\text{Stab}^\xi_{-\xi}(q) \cup c_1^T(L) \cup \text{Stab}^\xi_{\xi}(p)
\]

has compact support. This follows from the support conditions on the stable envelopes and the choice of opposite chambers: the support is \( m_{-\xi}^{-1}([\mu]) \subseteq \text{Gr}_{\Lambda}^\Lambda = X \), a proper subset since \( m_{-\xi} \) is proper.

Thus

\[
\left( \text{Stab}^\xi_{-\xi}(q), c_1^T(L) \cup \text{Stab}^\xi_{\xi}(p) \right) \in h \mathbb{H}^0_{X^\Lambda}(pt) = \mathbb{H}^0_{X^\Lambda}.
\]

Denoting

\[
c^\xi_{p,q} = \frac{1}{h}\left( \text{Stab}^\xi_{-\xi}(q), c_1^T(L) \cup \text{Stab}^\xi_{\xi}(p) \right)
\]

we get the formula (9).

2. Let us restrict the formula (9) to a point \( q \).

\[
\iota_q^*[c_1^T(L) \cup \text{Stab}^\xi_{\xi}(p)] = \iota_q^*c_1^T(L) \cdot \iota_q^*\text{Stab}^\xi_{\xi}(p) + h \sum_{p' \leq \ell_e p} c^\xi_{p',q}\iota_q^*\text{Stab}^\xi_{\xi}(p').
\]

The LHS equals

\[
\iota_q^*c_1^T(L) \cdot \iota_q^*\text{Stab}^\xi_{\xi}(p).
\]
So we have
\[ [i_q^*c_1^T(\mathcal{L}) - i_p^*c_1^T(\mathcal{L})] \cdot i_q^*\text{Stab}_\xi(p) = h \sum_{p' < p} c_{p, p'}^q i_q^*\text{Stab}_\xi(p'). \]

The terms \( i_q^*\text{Stab}_\xi(p') \) are divisible by \( h \) unless \( p' = q \). Then we have modulo \( h^2 \)
\[ [i_q^*c_1^T(\mathcal{L}) - i_p^*c_1^T(\mathcal{L})] \cdot i_q^*\text{Stab}_\xi(p) = h c_{p, q}^q i_q^*\text{Stab}_\xi(q) \mod h^2. \]

This implies
\[
\begin{align*}
    c_{p, q}^q &= \left[ \frac{[i_q^*c_1^T(\mathcal{L}) - i_p^*c_1^T(\mathcal{L})] \cdot i_q^*\text{Stab}_\xi(p)}{h i_q^*\text{Stab}_\xi(q)} \right]_A \\
    &= \left[ \frac{[i_q^*c_1^A(\mathcal{L}) - i_p^*c_1^A(\mathcal{L})] \cdot i_q^*\text{Stab}_\xi(p)}{h i_q^*} \right]_A
\end{align*}
\]
as desired.

\[ \square \]

### 3.5.2 Line bundles

The slices have a natural collection line bundles
\[ \mathcal{L}_0, \ldots, \mathcal{L}_l \]
coming from pulling back the \( \mathcal{O}(1) \) of the affine Grassmannian \( \text{Gr} \). We keep the detailed discussion for the appendix, let us just list the properties of line bundles \( \mathcal{L}_i \) we use here.

**Proposition 3.22.** The second cohomology \( H^2_\text{T}(X) \) is generated as a vector space by \( c_1^T(\mathcal{L}_i) \), \( 0 < i < l \), and constants \( H^2_\text{H}(\text{pt}) \).

So, to study classical multiplication by elements of \( H^2_\text{T}(X) \) it’s enough to study classical multiplication by \( c_1^T(\mathcal{L}_i) \).

We know the action of \( c_1^T(\mathcal{L}_i) \) in the fixed point basis. It’s diagonal with eigenvalues given by restrictions to the fixed points.

**Proposition 3.23.** The weight of a line bundle \( \mathcal{L}_i \) at a fixed point \( p \in X^A \) is
\[ K(\Sigma_i(p), \bullet) + \frac{h}{2} K(\Sigma_i(p), \Sigma_i(p)). \]

We see that from the restriction \( c_1^T(\mathcal{L}_i) \) at a fixed point \( p \) we can uniquely reconstruct \( \Sigma_i(p) \). So, if we know restrictions of all \( c_1^T(\mathcal{L}_i) \), \( 0 < i < l \) at a given fixed point, we know the fixed point. This gives the following useful statement.

**Corollary 3.24.** The operators of classical multiplication \( c_1^T(\mathcal{L}_i) \) act with simple spectrum on \( H^2_\text{T}(X) \) loc. The fixed point classes give an eigenbasis of this action.

### 3.5.3 Main result on classical multiplication

To formulate the main result on classical multiplication it’s convenient to introduce auxiliary operators.

First we define two families of diagonal operators
\[ H^1: H^*_A(X^A) \to H^*_A(X^A) \]
\[ 1_p \mapsto \left[ K(\Delta_i(p), \bullet) + \frac{h}{2} K(\Delta_i(p), \mu) \right] 1_p \]
\[ \Omega^{ij}_A: H^*_A(X^A) \to H^*_A(X^A) \]
\[ 1_p \mapsto K(\Delta_i(p), \Delta_j(p)) 1_p \]
Then we introduce a family of strictly triangular operators. Let $i, j$ be such that $0 \leq i, j \leq l$, $\alpha^\vee$ be a root and $\epsilon$ be a polarization. We define

$$\Omega_{ij,\alpha,\epsilon}^\cdot : H_\lambda^\cdot (X^A) \to H_\lambda^\cdot (X^A)$$

by the following property: for any $p \in X^A$

$$\Omega_{-\alpha,\epsilon}^j (1_p) = \sigma^\epsilon_{p,q} K (\alpha, \alpha) \frac{1}{2} 1_q$$

if there exists $q \in X^A$ satisfying conditions (8) for these $i, j$ and $\alpha$. The signs $\sigma^\epsilon_{p,q} \in \{ \pm 1 \}$ are the following product of signs

$$\sigma^\epsilon_{p,q} = \sigma^\epsilon_q \sigma^\epsilon_p \frac{e^A (N - \tilde{C}^\epsilon_{q/X})}{e^A (N - \tilde{C}^\epsilon_{p/X})} \epsilon_q,$$

and where $\tilde{C}$ is any Weyl chamber adjacent to the wall $\alpha^\vee = 0$. The sign does not depend on this choice which we will show later.

If there’s no such $q$, we set

$$\Omega_{ij,\epsilon}^\cdot (1_p) = 0.$$

**Remark.** By the conditions on $\Delta (p)$’s the existence for such $q \in X^A$ is equivalent to the following two conditions on $p$

$$\langle \Delta_i (p), \alpha^\vee \rangle = 1,$n
$$\langle \Delta_j (p), \alpha^\vee \rangle = -1.$$

Now for any Weyl chamber we define the following operator

$$\Omega_{ij,\epsilon}^\cdot = \frac{1}{2} \Omega_{ij}^0 + \sum_{\alpha > 0} \Omega_{ij,\epsilon}^{\alpha}.$$

**Remark.** The notation $\Omega_{ij,\epsilon}^\cdot$ suggests that these operators are related to Casimir operators in representation theory. We’ll make this statement more precise later. The minus sign for $-\alpha$ in the definition of $\Omega_{ij,\epsilon}^\cdot$ is introduced to make this comparison easier.

**Remark.** If one chooses polarization $\epsilon_p = e^A \left( N - \tilde{C}^\epsilon_{p/X} \right)$, then the signs in $\Omega_{ij,\epsilon}^\cdot$ with a simple $\alpha$ (with respect to $\mathfrak{C}$) are all +1. We will need it later to choose the right basis in representation-theoretic language.

Similar to stable envelopes, for a simpler notation we write

$$H_i^\cdot (p) \text{ and } \Omega_{ij,\epsilon}^\cdot (p)$$

instead of

$$H_i^\cdot (1_p) \text{ and } \Omega_{ij,\epsilon}^\cdot (1_p).$$

Now we are ready to state the main result on classical multiplication in the stable basis.

**Theorem 3.25.** The classical multiplication is given by the following formula

$$c^T_1 (\mathcal{L}_k) \cup \text{Stab}^\epsilon_\mathfrak{C} (p) = \text{Stab}^\epsilon_\mathfrak{C} \left[ \sum_{i \leq k} H_i^\cdot (p) - h \sum_{i \leq k < j} \Omega_{ij,\epsilon}^\cdot (p) \right].$$
Proof. From the first part of Proposition 3.21 we get that
\[ c^T_1 ( \mathcal{L}_k ) \cup \text{Stab}_\mathcal{E}^\flat ( p ) = c^T_\mu ( \mathcal{L}_k ) \cdot \text{Stab}_\mathcal{E}^\flat ( p ) + \frac{\hbar}{q < p} c^k_{p,q} \text{Stab}_\mathcal{E}^\flat ( q ) . \]

Let us first rewrite the first term. By Corollary 3.23 we know the restriction of the first Chern classes
\[ c^T_\mu ( \mathcal{L}_k ) = K ( \Sigma_k ( p ), \bullet ) + \frac{\hbar}{2} K ( \Sigma_k ( p ), \Sigma_k ( p ) ) \]
\[ = K ( \Sigma_k ( p ), \bullet ) + \frac{\hbar}{2} K ( \Sigma_k ( p ), \mu ) - \frac{\hbar}{2} K ( \Sigma_k ( p ), \mu - \Sigma_k ( p ) ) \]
\[ = \sum_{i \leq k} \left[ K ( \Delta_i ( p ), \bullet ) + \frac{\hbar}{2} K ( \Delta_i ( p ), \mu ) \right] - \frac{\hbar}{2} \sum_{i < k < j} K ( \Delta_i ( p ), \Delta_j ( p ) ) , \]
where we used \( \sum \Delta_i ( p ) = \mu \). Using definitions of \( \mathbf{H}^i \) and \( \Omega_{ij} \), we can write
\[ c^T_\mu ( \mathcal{L}_k ) \cdot \text{Stab}_\mathcal{E}^\flat ( p ) = \text{Stab}_\mathcal{E}^\flat \left[ \sum_{i \leq k} \mathbf{H}^i ( p ) - \frac{\hbar}{2} \sum_{i < k < j} \frac{1}{2} \Omega_{ij} ( p ) \right] \quad (11) \]

Now let us compute the off-diagonal terms. The coefficients \( c^k_{p,q} \) can be found from the second part of Proposition 3.21.
\[ c^k_{p,q} = \frac{\epsilon_q}{\epsilon_\mu} \left. \frac{c^T_\mu ( \mathcal{L}_k ) - c^A_\mu ( \mathcal{L} )}{c^A_\mu ( \mathcal{L} )} \right|_{c^A_\mu ( \mathcal{L} )} \]

By Theorem 3.20 these are zero unless \( p \) and \( q \) are related by (8) for some \( i < j \) and a coroot \( \alpha \) positive with respect to \( \mathcal{E} \). For such \( p, q \)
\[ c^k_{p,q} = \omega_{p,q} \frac{\epsilon_q}{\epsilon_\mu} \frac{K ( \Sigma_k ( q ), \bullet ) - K ( \Sigma_k ( p ), \bullet )}{c^A_\mu ( \mathcal{L} )} \]
where one can compute the multiplier \( \omega_{p,q} \) by choosing the polarization on \( X^\ker \alpha^\vee \) to be
\[ e^A ( N^{-\mathcal{E}_\mu} / X ) \]
for some Weyl chamber in \( \ker \alpha^\vee \) (the multiplier does not depend on this choice). This gives
\[ \omega_{p,q} = \frac{e^A ( N^{-\mathcal{E}_\mu} / X )}{e^A ( N^{-\mathcal{E}_\mu} / X )} \]

Let \( \sim \) be any Weyl chamber adjacent to the "wall" \( \mathcal{E} \). Then \( e^A ( N^{-\mathcal{E}_\mu} / X ) \) differs from \( e^A ( N^{-\mathcal{E}_\mu} / X ) \) by a factor which is a power of \( \pm \alpha^\vee \) with the sign chosen to make the weight negative with respect to \( \mathcal{E} \). We see that this affect numerator and denominator by the same factor, so the multiplier can be computed by
\[ \omega_{p,q} = \frac{e^A ( N^{-\mathcal{E}_\mu} / X )}{e^A ( N^{-\mathcal{E}_\mu} / X )} . \]
Since the \( \omega_{p,q} \) did not depend on the choice of \( \mathcal{E} \), it does not depend on the choice of \( \sim \) as long as it is adjacent to the hyperplane \( \alpha^\vee = 0 \).

From the relation (8) we know that
\[ \Sigma_k ( p ) = \begin{cases} \Sigma_k ( q ) + \alpha, & \text{if } i \leq k < j, \\ \Sigma_k ( q ), & \text{otherwise.} \end{cases} \]
This gives
\[ K (\Sigma_k (q), \bullet) - K (\Sigma_k (p), \bullet) = \begin{cases} \frac{-K (\alpha, \alpha)}{2} \alpha', & \text{if } i \leq k < j, \\ 0, & \text{otherwise}. \end{cases} \]

Combining this together, we get that \( c^k_{p,q} \) is not zero if \( p \) and \( q \) are related by (8) for \( i < j \) and \( i \leq k < j \). Then
\[ c^k_{p,q} = -\sigma_{p,q}^r \frac{K (\alpha, \alpha)}{2} \]
with the signs
\[ \sigma_{p,q}^r = \frac{\epsilon_p e^\Lambda (N_{q/X}^{-\xi})}{\epsilon_q e^\Lambda (N_{p/X}^{-\xi})} \]
equal to the ones introduced in the definition of \( \Omega_{i,j}^{\alpha,\epsilon} \).

Comparing with the definition of \( \Omega_{i,j}^{\alpha,\epsilon} \), we find that the off-diagonal contribution is
\[ \text{Stab}_{\epsilon_C} \left[ -\hbar \sum_{i \leq k < j} \sum_{\alpha > 0} \Omega_{i,j}^{\alpha,\epsilon} (p) \right]. \quad (12) \]

Adding (11) and (12) finishes the proof of the Theorem. \( \square \)

The formula becomes cleaner if we use line bundles \( \mathcal{E}_i \) from (21) instead of \( \mathcal{L}_i \). There’s a straightforward reformulation of Theorem 3.25.

**Theorem 3.26.** The classical multiplication is given by the following formula
\[ c^T (\mathcal{E}_i) \cup \text{Stab}_{\epsilon_C} (p) = \text{Stab}_{\epsilon_C} \left[ H^i (p) + \hbar \sum_{j < i} \Omega_{i,j}^{\epsilon,\epsilon} (p) - \hbar \sum_{i < j} \Omega_{i,j}^{\epsilon,\epsilon} (p) \right]. \]

Let us finish with the following remark.

**Remark.** Since the multiplication by \( c^T (\mathcal{L}_i) \) has simple spectrum by Corollary 3.24, knowing the exact form of the multiplication uniquely determines the transition matrix to the eigenbasis [up to a rescaling of the eigenbasis]. The fixed point basis is the eigenbasis, and the normalization is fixed by the polarization, so the restriction of the stable envelopes to the fixed points is uniquely reconstructed from the formulas in Theorem 3.25 or 3.26.

### A T-equivariant geometry

To perform computations in \( T \)-equivariant cohomology, it’s important to know the weights of the tangent spaces to the \( T \)-fixed locus and \( T \)-equivariant line bundles.

In this section we fix dominant cocharacters \( \mu \leq \lambda \), a split of \( \lambda \) into a sequence of fundamental coweights \( \lambda = (\lambda_1, \ldots, \lambda_l) \). We write \( X = \text{Gr}_\mu^{\lambda} \) to simplify notation.

#### A.1 Tangent weights at \( T \)-fixed points

Weights in the tangent spaces to fixed points are an important data in equivariant computations. In this section we describe this data for the resolution of slices of the Affine Grassmannian and related spaces.
A.1.1 Tangent spaces of the affine Grassmannian

First let us consider the tangent space at $T$-fixed point $[\lambda] \in \mathbf{Gr}$ and denote it as $T_\lambda$. The following proposition describes $T_\lambda$ explicitly as a $T$-representation.

Then we can generalize it to the following

**Proposition A.1.** There is an isomorphism of $T$-representations

$$T_\lambda \cong g((t))/\text{Ad}_{t^\lambda}(g[[t]]) \cong \text{Ad}_{t^\lambda}(t^{-1}g[t^{-1}]).$$

**Proof.** The left action gives a natural map:

$$p_\lambda: G(\mathbb{K}) \to \mathbf{Gr}, \quad g \mapsto g \cdot [\lambda]$$

Let us equip $G(\mathbb{K})$ with the $T$-action, where $A$ acts by conjugation and the loop-rotating $C^\times$ acts naturally (they commute in this case, so we can say that it gives a $T$-action).

The map $p_\lambda$ is

- $A$-equivariant because $A$ commutes with $t^\lambda$ and $A \subset G(\mathcal{O})$.
- $C^\times$-equivariant since for any $s \in C^\times$ we have
  $$s \cdot t^\lambda = t^\lambda s(s)$$
  and by definition $\lambda(s) \in A \subset G(\mathcal{O})$, so $(s \cdot g) \cdot [\lambda] = s \cdot (g \cdot [\lambda])$.

Thus the map $p_\lambda$ is $T$-equivariant.

The derivative at the identity is a surjection of vector spaces

$$dp_\lambda: g((t)) \to T_\lambda$$

and since $p_\lambda$ is $T$-equivariant and the identity of $G(\mathbb{K})$ is a $T$-fixed point, this is a map of $T$-representations.

The kernel of $p_\lambda$ is easy to find. A vector $\xi \in g((t))$ is in the kernel iff $\text{Ad}_{t^\lambda} \xi \in g[[t]]$ since being in $gOt$ is equivalent to being killed by $G(\mathcal{O})$ and $\text{Ad}_{t^\lambda}$ comes from commuting with $t^\lambda$. Finally,

$$\ker dp_\lambda = \text{Ad}_{t^\lambda}(g[[t]])$$

which gives the isomorphism in the proposition. $\square$

We will need the following lemma to explicitly describe the weight subspaces of $T_\lambda$.

**Lemma A.2.** Let $V$ be a $A(\mathbb{K}) \rtimes C^\times$-representation and denote by $V^\mu$ the $T$-weight subspace with a $T$-weight $\mu$. Then the action map $a_{t^\lambda}$ induces isomorphisms

$$a_{t^\lambda}: V^\mu \to V^\mu_{\mu^\vee + \langle \mu^\vee, \lambda \rangle h}.$$

**Proof.** A straightforward computation of the weight of $\text{Ad}_{t^\lambda}V^\mu$ gives $\mu^\vee + \langle \mu^\vee, \lambda \rangle h$, so we get the required map. Since $\text{Ad}_{t^\lambda}$ is invertible, this is an isomorphism. $\square$

**Corollary A.3.** For the tangent space $T_\lambda \subset g((t))$ the following is true

1. $T_\lambda$ is the direct sum of all $T$-weight subspaces of $g((t))$ satisfying condition
   $$k - \langle \omega^\vee, \lambda \rangle < 0$$
   on the $T$-weight $\omega^\vee + kh$, where $\omega^\vee$ is the $A$-part and $kh$ is the $C^\times$-part of this weight.

2. The only non-zero weight subspaces of $T_\lambda$ are
   - $(\dim g)$-dimensional subspaces of weight $-kh$ for $k \in \mathbb{Z}_{\geq 0}$;
   - $1$-dimensional subspaces of weight $\alpha^\vee + \langle \alpha^\vee, \lambda \rangle h - kh$ for $k \in \mathbb{Z}_{\geq 0}$ and each root $\alpha^\vee$ of $g$.

**Proof.** To prove part 1 first note that this it true when $\lambda = 0$ and then $T_\lambda = T_0 = t^{-1}g[t^{-1}]$.

For a general cocharacter $\lambda$ use $T_\lambda = \text{Ad}_{t^\lambda}T_0$. The image given by the isomorphisms described in Lemma A.2 is exactly the direct sum needed.

The part 2 is a simple corollary if one takes into account well-known dimensions of $T$-weight subspaces of $g((t))$. $\square$
A.1.2 Tangent spaces of the $G(\mathcal{O})$-orbits of the affine Grassmannian

We will also need the following fact about the closed cells $\text{Gr}^\lambda$

**Proposition A.4.** Given a dominant cocharacter $\lambda$ and any element $w \in W$ of the Weyl group, the closed cell $\text{Gr}^\lambda$ is smooth at the fixed point $[w\lambda]$. Let us denote the tangent space at this point as $T^\lambda_{w\lambda}$.

Then the action by $G(\mathcal{O})$ gives an isomorphism

$$T^\lambda_{w\lambda} \cong \mathfrak{g}[t] \cap \text{Ad}_{t\cdot \lambda}(t^{-1}\mathfrak{g}[t^{-1}])$$

of $T$-representations.

**Proof.** Recall that the smooth locus $\text{Gr}^\lambda \subset \text{Gr}_\lambda^\lambda$ is a $G$-subvariety by Proposition 2.2. Since the $W$-action on $A$-invariant point $[\lambda]$ can be made by taking representatives for elements $W$, we can get that all the points in the Weyl orbit are in the smooth locus. Hence for all $w \in W$ the fixed point $[w\lambda]$ is in the smooth locus.

The tangent space $T^\lambda_{w\lambda}$ by the definition of $\text{Gr}^\lambda$ is the subspace of $T^\lambda_w$ given by the infinitesimal action of $G(\mathcal{O})$, i.e. the intersection

$$\mathfrak{g}[t] \cap T^\lambda_w$$

taking into account that $T^\lambda_w$ was identified with a subspace of $\mathfrak{g}(())$ by the infinitesimal action of a larger group $G(\mathfrak{K})$.

Finally, use Proposition A.1 to get

$$\mathfrak{g}[t] \cap T^\lambda_w = \mathfrak{g}[t] \cap \text{Ad}_{t\cdot \lambda}(t^{-1}\mathfrak{g}[t^{-1}])$$



A.1.3 Tangent spaces of the resolutions of slices

Now let us return to $X = \text{Gr}^\mu_\Lambda$. We assume that $\mu$ and $\Lambda$ are such that they satisfy conditions in Theorem 2.7, so $X$ is smooth. We want to describe the tangent space of $X$ at a fixed point.

In what follows it will be more convenient to consider $X$ as a subscheme of $\text{Gr}^\Lambda \times (l+1)$, not $\text{Gr}^\Lambda \times l$. I.e. we will think of points in $X$ not as $l$-tuples $(L_1, \ldots, L_l)$, but as an $(l+1)$-tuple $(L_0, L_1, \ldots, L_l)$ with condition $L_0 = e \cdot G(\mathcal{O})$. Then the incidence condition defining $\text{Gr}^\Lambda$ is expressed in a uniform way

$$L_0 \xrightarrow{\lambda_1} L_1 \xrightarrow{\lambda_2} \ldots \xrightarrow{\lambda_{l-1}} L_{l-1} \xrightarrow{\lambda_l} L_l.$$ 

Consider a fixed point $p \in X^T$. By definition the $T$-action on $X$ is induced from the $T$ action on $\text{Gr}^\Lambda \times (l+1)$, so the inclusion $X \hookrightarrow \text{Gr}^\Lambda \times (l+1)$ is $T$-equivariant. So we have

$$T_p X \subset T_p \text{Gr}^\Lambda \times (l+1)$$

as a $T$-representation.

Then we have associated sequences of coweights

$$\Sigma (p) = (\Sigma_0 (p), \Sigma_1 (p), \ldots, \Sigma_l (p)),$$

$$\Delta (p) = (\Delta_1 (p), \ldots, \Delta_l (p)).$$

In what follows we do not vary $p$, so we drop $p$ in the notation:

$$\Sigma = (\Sigma_0, \Sigma_1, \ldots, \Sigma_l),$$

$$\Delta = (\Delta_1, \ldots, \Delta_l).$$

This gives an explicit description

$$T_p \text{Gr}^\Lambda \times (l+1) = T_{\Sigma_0} \oplus T_{\Sigma_1} \oplus \cdots \oplus T_{\Sigma_l}$$
and we have
\[ T_pX \subset T_{\Sigma_0} \oplus T_{\Sigma_1} \oplus \cdots \oplus T_{\Sigma_l} \]
as a \( T \)-representation. This allows us to write \( \xi \in T_pX \) as
\[ \xi = (\xi_0, \ldots, \xi_l). \]

By the definition \( X \) is a subscheme of \( \text{Gr}^{(l+1)} \) by three types of conditions:

1. **Left boundary condition:**
   \[ L_0 = e \cdot G(0); \] (13)

2. **Right boundary condition:**
   \[ L_l \in \text{Gr}_\mu; \] (14)

3. **Incidence condition:**
   \[ L_0 \xrightarrow{\lambda_1} L_1 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_{l-1}} L_{l-1} \xrightarrow{\lambda_l} L_l. \] (15)

Each gives linear relations on a tangent vector at \( p \). Taking them all into account, we find \( T_pX \).

**Notation.** To write the relations in the simplest form we will need the following natural projection in (\( T \)-weight) graded spaces. Given a direct sum
\[ V = \bigoplus_{i \in I} V_i \]
and a graded subspace \( V' \) where some graded components are missed, i.e. given \( J \subset I \) set
\[ V' = \bigoplus_{i \in J} V_i \]
Then we have a natural projection \( V \twoheadrightarrow V' \) which we denote as restriction \( v \mapsto v|_{V'} \).

In our case \( V \) will be one of the spaces \( T_\lambda \) are weight graded, i.e. \( I \) is the \( T \)-weight lattice. The subspaces \( V' \) will be intersections \( T_\lambda \cap T_\mu \) and one can check from the description in Corollary A.3 that these are of the required type.

**Proposition A.5.** The relations on \( \xi_0, \ldots, \xi_l \) from the conditions (13), (14) and (15) are

1. **The left boundary condition** (13) gives
   \[ \xi_0 = 0; \] (16)

2. **The right boundary condition** (14) gives
   \[ \xi_l \in T_\mu \cap t^{-1}g[t^{-1}]; \] (17)

3. **The incidence condition** gives
   \[ \xi_i|_{T_i} = \xi_{i+1}|_{T_i} \quad \text{for } 0 \leq i < l, \] (18)

   where \( T_i = T_{\Sigma_i} \cap T_{\Sigma_{i+1}} \).

**Proof.** The left boundary condition is the easiest, \( L_0 \) is constant \( \Rightarrow \xi_0 = 0 \).

In case of the right boundary condition \( L_l \in \text{Gr}_\mu \) implies that \( L_l \) changes only by the left multiplication by \( G \). Recall that we used he left multiplication by \( G \) to identify the tangent space with a subspace of \( g(t) \). Under this identification the infinitesimal \( G_1 \) action is the subspace \( t^{-1}g[t^{-1}] \) which gives the condition on \( \xi_l \).

To analyze the incidence condition \( L_i \xrightarrow{\lambda_{i+1}} L_{i+1} \) in a neighborhood of \( L_i = [\Sigma_i] \) and \( L_{i+1} = [\Sigma_{i+1}] \), we look at perturbation of \( L_i, L_{i+1} \) by \( g_i, g_{i+1} \in G(K) \):
\[
L_i = g_i \cdot [\Sigma_i] \\
L_{i+1} = g_{i+1} \cdot [\Sigma_{i+1}]
\]
Then the condition \( g_i \cdot [\Sigma_i] \overset{\lambda_{i+1}}{\longrightarrow} g_{i+1} \cdot [\Sigma_{i+1}] \) is equivalent to saying that

\[
(g_i t^{\lambda_i})^{-1} g_i [\Sigma_i] = \text{Ad}_{\tau^{-1}} \left( (g_i^{-1} g_{i+1}) t^{-\Sigma_i} [\Sigma_{i+1}] = \text{Ad}_{\tau^{-1}} \left( (g_i^{-1} g_{i+1}) \right) [\Delta_{i+1}] \right)
\]

is in the closure of the \( G(\mathbb{O}) \)-orbit of \([\Delta_{i+1}]\). In other words, it is in \( \text{Gr}_{\Delta_{i+1}} \) for which we know the tangent space at \([\Delta_{i+1}]\), by Proposition A.4 it is

\[
T_{\Delta_{i+1}} = g[[t]] \cap \text{Ad}_{\tau^{-1}} \left( (t^{-1} g[t^{-1}]) \right).
\]

Differentiating the expression (19) with respect to \( g_i \) and \( g_{i+1} \) at the origin, we get that

\[
\text{Ad}_{\tau^{-1}} \left( (\xi_i - \xi) \right) \in T_{\Delta_{i+1}} + \text{Ad}_{\tau^{-1}} \left( g[[t]] \right),
\]

where the first term is the stabilizer of \([\Delta_{i+1}]\) from Proposition A.1.

Equivalently,

\[
\xi_i - \xi \in \left( \text{Ad}_{\tau^{-1}} \left( g[[t]] \right) \cap \text{Ad}_{\tau^{-1}} \left( (t^{-1} g[t^{-1}]) \right) \right) + \text{Ad}_{\tau^{-1}} \left( g[[t]] \right).
\]

Note that the second term is contained exactly all weight subspaces of \( \text{Ad}_{\tau^{-1}} \left( g[[t]] \right) \) which are not in \( \text{Ad}_{\tau^{-1}} \left( g[[t]] \right) \).

Saying that this difference belongs to this space is equivalent to the statement that the restriction to the complement space \( T_i = T_{\Sigma_i} \cap T_{\Sigma_{i+1}} \) is zero:

\[
(\xi_{i+1} - \xi_i) |_{T_i} = 0
\]

(here we once again used that any weight subspace of \( g((t)) \) either belongs to

\[
\text{Ad}_{\tau^{-1}} \left( g[[t]] \right) + \text{Ad}_{\tau^{-1}} \left( g[[t]] \right)
\]

or intersects trivially with it). \( \square \)

One observation which simplifies the analysis in the following

**Proposition A.6.** The equations in Proposition A.5 restrict to each \( T \)-weight subspace independently.

**Proof.** This follows from \( T \)-equivariance of conditions (13), (14) and (15) which implies that their infinitesimal forms in Proposition A.5 respect \( T \)-weight grading. \( \square \)

This allows us to analyze the tangent space \( T_p X \) by restricting to \( T \)-weight subspaces of \( g((t)) \) (only these weights are present in \( T_p \text{Gr}^{\times(t+1)} \) and hence in \( T_p X \)). However it’s more convenient to restrict to \( A \)-weight subspaces, i.e. consider \( T \)-weights which differ by a multiple of \( h \) simultaneously.

More explicitly, we’re interested in intersections

\[
T_{\Sigma_i} \cap \mathfrak{h}((t)) \text{ and } T_{\Sigma_i} \cap \mathfrak{g}_{a^\vee}((t)).
\]

**Proposition A.7.** \( T_{\Sigma_i} \cap \mathfrak{h}((t)) = 0 \).

**Proof.** Since \( \text{Ad}_{[\lambda]} \) for any cocharacter \( \lambda \) acts trivially on \( \mathfrak{h}((t)) \), we have that

\[
T_{\lambda} \cap \mathfrak{h}((t)) = \text{Ad}_{[\lambda]} \left( t^{-1} \mathfrak{g}[t^{-1}] \cap \mathfrak{h}((t)) \right)
\]

\[
= \text{Ad}_{[\lambda]} \left( t^{-1} \mathfrak{g}[t^{-1}] \cap \mathfrak{h}((t)) \right)
\]

\[
= t^{-1} \mathfrak{g}[t^{-1}] \cap \mathfrak{h}((t)) = t^{-1} \mathfrak{h}[t^{-1}].
\]

Note that this doesn’t depend on \( \lambda \). This implies that if we restrict a tangent vector \( \xi = (\xi_0, \ldots, \xi_l) \in T_{\Sigma_i} \) to \( T_{\Sigma_i} \cap \mathfrak{h}((t)) \) we get

\[
(\xi_0 t^{-1} \mathfrak{h}[t^{-1}], \ldots, \xi_l t^{-1} \mathfrak{h}[t^{-1}]).
\]
This gives that the incidence condition restricted to $T_pX \cap \mathfrak{h}((t))$ is simple:

$$\xi_{t^{-1}}|_{t^{-1}g[t^{-1}]} = \xi_{t^{-1}}|_{t^{-1}g[t^{-1}]}.$$ 

Combined with the left boundary condition $\xi_0 = 0$ we have

$$\xi_i|_{t^{-1}g[t^{-1}]} = 0 \text{ for all } i.$$ 

This is equivalent to $T_pX \cap \mathfrak{h}((t)) = 0$.

Let us denote $T_\lambda \cap \mathfrak{g}_{\alpha^\vee}((t))$ as $T_\alpha^\vee T_\lambda$ and similarly $T_pX \cap \mathfrak{g}_{\alpha^\vee}((t))$ as $T_p^\alpha X$.

**Corollary A.8.** $T_pX = \bigoplus_{\alpha^\vee} T_p^\alpha X$.

Our goal is to describe $T_p^\alpha X$.

For this purpose we introduce a convenient graphical representation. Let us start with $T_\lambda$. Its $T$-weight subspaces are 1-dimensional with weights $\alpha^\vee + kh$, $k \in \mathbb{Z}$. Then we can represent nonzero weight subspaces of $T_\lambda$ as dots on $k$-axis. By Corollary A.3 we have that these are all integer points strictly below $\langle \alpha^\vee, \lambda \rangle$. This is shown on Figure 3a.

In a similar way we can represent $T_\Sigma_0^\vee \oplus \cdots \oplus T_\Sigma_l^\vee$. If we plot a nontrivial weight subspace of $T_\Sigma_i^\vee$ with weight $\alpha^\vee + kh$ as a point with coordinates $(i, k)$ in the plane, then we have all points with integer coordinates strictly below the graph $\Gamma_h^\alpha$ piecewise linear connecting points $\langle \alpha^\vee, \Sigma_i \rangle$ (and $0 \leq i \leq l$), as shown on Figure 3b. From the Proposition 2.12 a sequence $\Sigma$ corresponding to a fixed point in $X$ is has $\Sigma_0 = 0$ and $\Sigma_l = \mu$, so the graph starts at $(0, 0)$ and ends at $(l, \langle \alpha^\vee, \mu \rangle)$. Moreover if all $\lambda_i$ are minuscule, then all increments in this path are $\pm 1$ or 0 since $\Sigma_{i+1} - \Sigma_i = \Delta_{i+1}$ is a Weyl reflection of $\lambda_{i+1}$.

In this pictorial language one can represent the conditions (13)-(15).

First, both Left Condition and Right Condition say that restrictions to certain $T$-weight subspaces of $T_\Sigma_0^\vee$ or $T_\Sigma_i^\vee$ must be trivial, i.e. we just have to exclude these weight subspaces from the picture. We’ll denote these by crosses in our picture. The Figure 4a shows how the Left Condition changes the leftmost column, corresponding to $T_\Sigma_0^\vee = t^{-1}g[t^{-1}]$. On the Figure 4b we show two possible cases which can appear in the rightmost column, corresponding to $T_\Sigma_i^\vee = T_\mu^\vee$.

The Incidence condition says that restrictions to $T$-weight subspaces of $T_\Sigma_0^\vee$ and $T_\Sigma_{i+1}^\vee$ are equal if the weight subspace belongs to their intersection (i.e. is non-trivial in both of them). We will show this by connecting corresponding dots with a straight lines. This is shown on Figure 5 in all three cases which can occur in the case of minuscule $\lambda_i$'s.
Figure 4: Left and Right Conditions

(a) Left Condition

(b) Right Condition changes \( T_{\alpha}^{\vee} \) if \( \langle \alpha^\vee, \mu \rangle > 0 \) and doesn’t change if \( \langle \alpha^\vee, \mu \rangle \leq 0 \).

Figure 5: Incidence conditions on \( T_{\alpha}^{\vee} \) and \( T_{\alpha}^{\vee} \).

If the dots are connected by a line, they give one tangent vector, since these components must be equal. If the left or right end of a chain of lines end with a cross of Left or Right boundary condition, then the restrictions to each dot on this line must be zero. These lines are "irrelevant", they don’t give a tangent vector. To make the picture clearer we’ll denote these ones by dashed lines and won’t put dots on them. Typical pictures we get are shown on Figure 6. The connected components of red (non-dashed) lines give 1-dimensional subspaces of \( T_{\alpha}^{\vee} \) of weight \( \alpha^\vee + k \hbar \) where \( k \) is the height of this line on the plot.

Let us mention that this also gives an explicit basis in \( T_{\alpha}^{\vee} \). If one fixes a basis vector \( e_{\alpha^\vee} \in g_{\alpha^\vee} \) then every segment at height \( k \) in the constructed plot gives a basis vector which is \( e_{\alpha^\vee} t^k \) for every \( T_{\Sigma_i}^{\vee} \) if the vertical line corresponding to \( i \) intersects with the segment (possibly at endpoints) and is zero for other \( T_{\Sigma_i}^{\vee} \).

Note that the plots for \( \alpha^\vee \) and \( -\alpha^\vee \) are reflections of each other, we can use it to draw a plot for a pair of roots simultaneously. Under the graph we draw the same picture for \( \alpha^\vee \) as we used to, above it we draw a reflected picture for \( -\alpha^\vee \). For a visual convenience we draw it in blue. A typical picture is shown on Figure 7.

A.1.4 Classical examples

Let us show how this works in classical examples listed before and coincides with well-known weights of fixed points tangent spaces.

Example A.1. Recall \( A_n \)-singularity resolution is \( Gr_{(n+1)\omega}^{(\omega, \ldots, \omega)} \rightarrow Gr_{(n-1)\omega}^{(n-1)\omega} \) where \( G = \text{PSL}_2 \) and \( \omega \).
Figure 6: A typical picture of $T^\alpha_p X$ at a fixed point $p$.

Figure 7: A typical picture of $T^\alpha_p X \oplus T^{-\alpha}_p X$ at a fixed point $p$. 
is the fundamental coweight. There are \( n + 1 \) fixed points \( p_0, \ldots, p_n \):
\[
\begin{align*}
\Delta (p_0) &= (-\omega, \omega, \omega, \ldots, \omega), \\
\Delta (p_1) &= (\omega, -\omega, \omega, \ldots, \omega), \\
& \quad \vdots \\
\Delta (p_n) &= (\omega, \omega, \omega, \ldots, -\omega).
\end{align*}
\]

The tangent weights at the point \( p_i \) are \( \alpha^\vee + (i - 1)h \) and \( -(\alpha^\vee + ih) \). An example of such computation is shown on Figure 8.

This coincides with computation via toric geometry (the singularity is a toric variety and hence admits a toric resolution).

**Example A.2.** Let \( \lambda = \nu + \iota \nu \) for a minuscule \( \nu \). Then \( \text{Gr}^\nu_0 \cong T^* G / P^{-\nu}_\nu \) as \( T \)-varieties, where \( P^{-\nu}_\nu \) is a maximal parabolic corresponding to \(-\nu\). I.e. the roots \( \alpha^\vee \) of \( P^{-\nu}_\nu \) are exactly such that \( \langle \nu, \alpha^\vee \rangle \leq 0 \).

Recall that the fixed points of \( T^* G / P^{-\nu}_\nu \) are of form \( w P^{-\nu}_\nu \) and are classified by left cosets \( w W \) in the Weyl group \( W \) of \( G \), \( w \in W \) (here \( W \) in the Weyl group of the Levi factor of \( P^{-\nu}_\nu \)). Then the tangent weights to the zero section \( G / P^{-\nu}_\nu \) at \( w P^{-\nu}_\nu \) are roots \( \alpha^\vee \) such that \( \langle w \nu, \alpha^\vee \rangle > 0 \) and the weights of the cotangent fiber over this point are \(-\alpha^\vee - h\) for the same \( \alpha^\vee \). Under identification of the fixed loci
\[
(T^* G / P^{-\nu}_\nu)^T \sim (\text{Gr}^\nu_0)^T
\]
\[
w P \mapsto p_w P,
\]
where \( \Sigma (p_w P) = (0, w \nu, 0) \), one gets the same answer from the Figure 9. Here we used that \(-w \nu \in W \cdot \iota \nu \) to get that \( p_w P \in \text{Gr}^\nu_0 \).

**Example A.3.** Let \( G = \text{PSL}_n \) and \( \omega_1 \) be the highest weight of the defining representation of \( \text{SL}_n \) (hence a minuscule coweight of \( \text{PSL}_n \)). Then for an \( n \)-tuple \( \lambda = (\omega_1, \ldots, \omega_1) \), \( \text{Gr}^\lambda_0 \cong T^* G / B \) as \( T \)-varieties.

Denote the elements of the Weyl orbit of \( \omega_1 \) as \( \{e_i \mid 1 \leq i \leq n\} \) with \( e_1 = \omega_1 \) and \( e_i - e_{i+1} \) being simple roots. Then \( T \)-fixed points of \( T^* G / B \) are \( w B \) for \( w \in W \). In our case, this is a root system of type \( A \), so \( W = S_n \), \( w \) is a permutation. Moreover, \( w \in W \) acts on \( \{e_i\} \) by applying permutation to the index: \( e_i \mapsto e_{w(i)} \). This allows us to identify the fixed loci by following assignment
\[
(T^* G / B)^T \sim (\text{Gr}^\lambda_0)^T
\]
\[
w B \mapsto p_w,
\]
where
\[
\Delta (p_w) = (e_{w_0 w^{-1}(1)}, e_{w_0 w^{-1}(2)}, \ldots, e_{w_0 w^{-1}(t)})
\]
and \( w_0 \) is the longest element in the Weyl group (explicitly \( w_0(i) = n - i \) in our case). We used well-known fact that \( \sum_{i=1}^{n} e_i = 0 \) to get the sum of all components to be zero, which means that the point in the image is in \( \text{Gr}^\lambda_0 \), not only in \( \text{Gr}^\lambda \). The tangent weights at \( w B \) are \( w \alpha^\vee \) and \(-w \alpha^\vee - h\) for all negative roots \( \alpha^\vee \).

The Figure 10 shows the tangent weights at a fixed point \( p_w \) of \( \text{Gr}^\lambda_0 \) corresponding to a fixed point \( w B \), \( w \in W \) in \( T^* G / B \). If \( \alpha^\vee = e_i - e_j \) and \( w_0 w^{-1}(i) < w_0 w^{-1}(j) \) (equivalently, \( w_0 w^{-1} \alpha^\vee \) is positive), then the weights of form \( \pm \alpha^\vee + h \mathbb{Z} \) are \( \alpha^\vee \) and \(-\alpha^\vee + h \). From this we confirm that the weights are the same as described before, if one takes into account that
1. All roots are of form \( e_i - e_j \), \( i \neq j \),
2. \( e_i - e_j \) is negative iff \( i > j \),
3. \( \alpha^\vee \) is negative iff \( w_0 \alpha^\vee \) is positive.
(a) For $\Delta(p_0) = (-\omega, \omega, \omega, \omega, \omega)$ the tangent weights at $p_0$ are $\alpha^\vee - \hbar$ and $-\alpha^\vee$.

(b) For $\Delta(p_1) = (\omega, -\omega, \omega, \omega, \omega)$ the tangent weights at $p_1$ are $\alpha^\vee$ and $- (\alpha^\vee + \hbar)$.

(c) For $\Delta(p_2) = (\omega, \omega, -\omega, \omega, \omega)$ the tangent weights at $p_2$ are $\alpha^\vee + \hbar$ and $- (\alpha^\vee + 2\hbar)$.

(d) For $\Delta(p_3) = (\omega, \omega, \omega, -\omega, \omega)$ the tangent weights at $p_3$ are $\alpha^\vee + 2\hbar$ and $- (\alpha^\vee + 3\hbar)$.

(e) For $\Delta(p_4) = (\omega, \omega, \omega, \omega, -\omega)$ the tangent weights at $p_4$ are $\alpha^\vee + 3\hbar$ and $- (\alpha^\vee + 4\hbar)$.

Figure 8: Tangent spaces at the $T$-fixed points of the resolution of type $A_4$. 

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Figure 9: Tangent spaces at the $T$-fixed points of $T^*G/P_\nu$

Figure 10: Tangent spaces at the $T$-fixed points of $T^*G/B$

A.1.5 Combinatorial description of weight multiplicities

As previously, let $X = \text{Gr}_{\lambda/\mu}$ and $\lambda$ and $\mu$ are such that they satisfy conditions in Theorem 2.7, so $X$ is smooth. We want to describe weight multiplicities in the tangent space to $p \in X^T$ with a corresponding sequence of coweights

$$\Sigma = (\Sigma_0, \Sigma_1, \ldots, \Sigma_l)$$

where we assume usual conditions $\Sigma_0 = 0$ and $\Sigma_l = \mu$ and do not write $p$ explicitly in the notation for $\Sigma$.

As previously, for each root $\alpha^\vee$ we use a graph $\Gamma_\alpha^\vee$ piecewise linearly connecting points $(i, (\alpha^\vee, \Sigma_i))$ in $i,k$-plane. We are interested in the intersection of this graph with the line $k = n + \frac{1}{2}, n \in \mathbb{Z}$. Let $N_-(\alpha^\vee, n + \frac{1}{2})$ be the number of the such crossings where $\Gamma_\alpha^\vee$ goes down as $i$ decreases. Similarly, $N_+(\alpha^\vee, n + \frac{1}{2})$ is the number of crossings where $\Gamma_\alpha^\vee$ goes down. The Figure 11 shows how this intersections may look like. Blue dots contribute to $N_-(\alpha^\vee, n + \frac{1}{2})$ and green squares contribute to $N_+(\alpha^\vee, n + \frac{1}{2})$.

**Proposition A.9.** Let $\alpha^\vee$ be a root and $n \in \mathbb{Z}$. Then the multiplicity of weight $\alpha^\vee + nh$ in $T_pX$ is

1. $N_+(\alpha^\vee, n + \frac{1}{2}) - 1$ if $0 < n + \frac{1}{2} < \langle \alpha^\vee, \mu \rangle$,

2. $N_+(\alpha^\vee, n + \frac{1}{2})$ otherwise.

**Proof.** Every point which is counted by $N_+(\alpha^\vee, n + \frac{1}{2})$ gives rise to the left end of a segment corresponding to a non-zero tangent vector with tangent weight if we don’t take into account the
Right condition. And this condition vanishes the rightmost vector if and only if
\[ 0 \leq n < \langle \alpha^\vee, \mu \rangle \iff 0 < n + \frac{1}{2} < \langle \alpha^\vee, \mu \rangle. \]

This gives −1 in this case, as stated in the Proposition.

If one wants to use \( N_-(\alpha^\vee, n + \frac{1}{2}) \) in place of \( N_+(\alpha^\vee, n + \frac{1}{2}) \), there is an easy relation between them.

**Proposition A.10.** The numbers \( N_+(\alpha^\vee, n + \frac{1}{2}) \) and \( N_-(\alpha^\vee, n + \frac{1}{2}) \) are related in the following way

1. \( N_+(\alpha^\vee, n + \frac{1}{2}) = N_-(\alpha^\vee, n + \frac{1}{2}) + 1 \) if \( 0 < n + \frac{1}{2} < \langle \alpha^\vee, \mu \rangle \),
2. \( N_+(\alpha^\vee, n + \frac{1}{2}) = N_-(\alpha^\vee, n + \frac{1}{2}) - 1 \) if \( \langle \alpha^\vee, \mu \rangle < n + \frac{1}{2} < 0 \),
3. \( N_+(\alpha^\vee, n + \frac{1}{2}) = N_-(\alpha^\vee, n + \frac{1}{2}) \) otherwise.

**Proof.** This is a consequence of elementary topology.

These Propositions allow us to get the following similar description in terms of \( N_-(\alpha^\vee, n + \frac{1}{2}) \).

**Corollary A.11.** Let \( \alpha^\vee \) be a root and \( n \in \mathbb{Z} \). Then the multiplicity of weight \( \alpha^\vee + nh \) in \( T_pX \) is

1. \( N_-(\alpha^\vee, n + \frac{1}{2}) - 1 \) if \( \langle \alpha^\vee, \mu \rangle < n + \frac{1}{2} < 0 \),
2. \( N_-(\alpha^\vee, n + \frac{1}{2}) \) otherwise.
Proof. Straightforward case-by-case consideration. 

Remark. We made the shift of \( n \) by \( \frac{1}{2} \) to make it manifest that the the multiplicities of \( \alpha^\vee + n\hbar \) and \(-\alpha^\vee - (n + 1)\hbar \) are the same. Indeed,

\[
-(n + \frac{1}{2}) = -(n + 1) + \frac{1}{2}
\]

and \( \Gamma_p^{-\alpha^\vee} \) is a reflection of \( \Gamma_p^{\alpha^\vee} \), so in these cases we’re looking at mirror pictures of intersections. The roles of “going upwards” and “going downwards” are swapped, so

\[
N_+ \left( \alpha^\vee, n + \frac{1}{2} \right) = N_- \left( -\alpha^\vee, -(n + 1) + \frac{1}{2} \right)
\]

and the conditions

\[
0 < n + \frac{1}{2} < \langle \alpha^\vee, \mu \rangle \iff \langle -\alpha^\vee, \mu \rangle < -(n + 1) + \frac{1}{2} < 0
\]

are equivalent.

This is expected because of the presence of the symplectic structure of weight \( \hbar \), which gives a nondegenerate pairing between spaces of these weights.

Moreover, one can keep track which tangent vectors have a non-zero pairing by the symplectic form. It’s non-zero if the end of the segment corresponding to one vector is the beginning of a segment for another vector. It’s not hard to compute the explicit number (depending on normalization of the symplectic form), but we since we won’t need it later we don’t compute it here.

Finally, Theorem 2.13 follows from Corollary A.8, Proposition A.9, and Corollary A.11. It describes the multiplicities in a way which doesn’t refer to the graph of pairings and might be convenient for a reader who wants to know only about weight multiplicities, not how the tangent vectors behave.

The gap between the statement of Theorem 2.13 and the preceding statements is closed by the following two notes

- The intersections of graph \( \Gamma_p^{\alpha^\vee} \) with the line \( k = n + \frac{1}{2} \) are in bijection with the intersection of the hyperplane \( \langle \bullet, \alpha^\vee \rangle - \left( n + \frac{1}{2} \right) = 0 \) with the piecewise linear path \( P_p \) corresponding to the fixed point \( p \) (i.e. connecting the points in the coweight lattice \( 0 = \Sigma_0, \Sigma_1, \ldots, \Sigma_{l-1}, \Sigma_l = \mu \ ));

- To avoid the second case in both Proposition A.9 and Corollary A.11 we can take the number \( N_0 \left( \alpha^\vee, n + \frac{1}{2} \right) \) of intersections with the path going in the direction of \( k = 0 \). In the language of lattice paths it’ll be all intersections with hyperplanes where the path goes to the halfspace containing the origin (i.e. ”returns” back to the origin halfspace, since it starts at the origin).

So, finally, we proved Theorem 2.13.

A.2 T-equivariant line bundles

Let us first remind facts about line bundles on \( \text{Gr} \). The reader can find these results in [KNR].

Theorem A.12.

1. There is an abelian group isomorphism

\[
\text{Pic} \, \text{Gr} \simeq \mathbb{Z}
\]
2. $\text{Gr}$ is ind-projective with respect to one of the generators $\mathcal{O}(1) \in \text{Pic} \text{Gr}$, i.e. there is a stratification of $\text{Gr}$:

$$\text{Gr}_1 \subset \ldots \text{Gr}_i \subset \cdots \subset \text{Gr}, \quad i \in \mathbb{N}$$

such that all $\text{Gr}_i$ are projective with respect to $\mathcal{O}(1)$ and $\bigcup_i \text{Gr}_i = \text{Gr}$.

Proof. See [KNR] for proofs.

One approach to $\mathcal{O}(1)$ comes from the theory of Kac-Moody groups.

Let us first assume that $G$ is simply-connected. Then there is a non-trivial central extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \hat{G} \longrightarrow G(\mathbb{X}) \rtimes \mathbb{C}_h^\times \longrightarrow 1$$

where $\hat{G}$ is the Kac-Moody group for $G$. This is a non-trivial $\mathbb{C}^\times$-principal bundle. The associated line bundle on $G(\mathbb{X})$ trivializes on right $G(\mathcal{O})$-orbits and gives rise to the line bundle $\mathcal{O}(1)$ on $\text{Gr}$.

This $\mathbb{C}^\times$-bundle trivializes over orbits of right $G(\mathcal{O})$-action on $G(\mathbb{X})$. This gives a principle $\mathbb{C}^\times$-bundle on the quotient, $\text{Gr}$. The associated line bundle is $\mathcal{O}(1)$.

If $G$ is not simply-connected, $\text{Gr}$ has several isomorphic connected components, each isomorphic to the affine Grassmannian of the associated simply-connected group [MV, MV2]. This gives an independent $\mathcal{O}(1)$ on each component. When we work with the resolutions of slices each flag does not leave the connected component since the resolutions are connected. Thus we can ignore the fact that on different components of $\text{Gr}$ the power of $\mathcal{O}(1)$ can be chosen differently.

A representation-theoretic description gives in general only a power of $\mathcal{O}(1)$, but it naturally comes with a $G(\mathbb{X}) \rtimes \mathbb{C}_h^\times$-equivariant structure.

Let $V$ be a nontrivial finite dimensional complex representation of $G$. Then by extending scalars $V_{\mathbb{X}} = V \otimes_\mathbb{C} \mathbb{X}$ we get a representation of $G(\mathbb{X})$. Moreover, loop scaling $\mathbb{C}_h^\times$ (i.e. scaling $\ell$) extends this to the action of $G(\mathbb{X}) \rtimes \mathbb{C}_h^\times$.

Consider $V_0 = V \otimes_\mathbb{C} \mathcal{O}$. This is a $G(\mathcal{O})$-invariant subspace of $V_{\mathbb{X}}$. Then for $g \in G(\mathbb{X})$ the subspace $gV_0$ depends only on the image of $g$ in $G$. This gives a $G(\mathbb{X}) \rtimes \mathbb{C}_h^\times$-equivariant vector bundle $\mathcal{V}$ of infinite rank over $\text{Gr}$.

Informally, we would like to consider the determinant line bundle of $\mathcal{V}$. This is ill-defined because of the infinite rank. Then one can correct the definition by ”subtracting” an infinite part of $\mathcal{V}$ by ”subtracting” a constant bundle, let’s say the constant vector bundle $\mathcal{V}_0$ with fibers $V_0$. Unfortunately, $\mathcal{V}_0$ is not a subbundle of $\mathcal{V}$. However, one can make sense of the virtual bundle $\mathcal{V} - \mathcal{V}_0$ by

$$\mathcal{V} - \mathcal{V}_0 = \mathcal{V} / \mathcal{V} \cap \mathcal{V}_0 - \mathcal{V}_0 / \mathcal{V} \cap \mathcal{V}_0,$$

since both quotient on RHS are coherent sheaves.

Applying the determinant to this object, we get a line bundle

$$\mathcal{L}_\mathcal{V} = \frac{\det (\mathcal{V} / \mathcal{V} \cap \mathcal{V}_0)}{\det (\mathcal{V}_0 / \mathcal{V} \cap \mathcal{V}_0)}$$

To compare $\mathcal{L}_\mathcal{V}$ to $\mathcal{O}(1)$, we need the notion of a Dynkin index [D3]. The representation $V$ gives a scalar product on $g$ by $\text{Tr}_V (\xi \eta)$ for $\xi, \eta \in g$. Restricting to the real Cartan $a_\mathbb{R} = X_\ast (A) \otimes_\mathbb{Z} \mathbb{R} \subset a$ (here we identify a cocharacter with its derivative at $e$) we get a Weyl-invariant positive-definite scalar product $(\bullet, \bullet)_V$. For a simple $g$ there is only one such scalar product up to a multiple. Thus we can write

$$(\bullet, \bullet)_V = d_V K (\bullet, \bullet)$$

for $d_V \in \mathbb{R}$, $d_V > 0$. With normalization of $K (\bullet, \bullet)$ we use (i.e. the length squared for the shortest coroot is 2) the number $d_V$ is an integer. The number $d_V$ is called the Dynkin index of $V$. For example, if $V$ is the defining representation of $SL_n$, then $d_V = 1$.

**Proposition A.13.** $\mathcal{L}_\mathcal{V} = \mathcal{O}(d_V)$.

Proof. The proof can be found in [KNR].
We will see the traces of this fact in the computation of the weights at $T$-fixed points.

**Proposition A.14.** For the line bundle $L_V$ the $T$-weight of the fiber at a $T$-fixed point $[\lambda]$ is

$$(\lambda, \bullet)_V + \frac{\hbar}{2} (\lambda, \lambda)_V$$

**Proof.** For the determinant line bundle the weight of the fiber is the sum of weights of $T$-eigenspaces counted with multiplicity.

Let us denote the set of all weights of $V$ as $W_V$. For a $\mu^\vee \in W_V$ we also denote by $m_{\mu^\vee}$ the dimension of the subspace in $V$ of weight $\mu^\vee$.

Consider the fiber of $\mathcal{V}/\mathcal{V} \cap \mathcal{V}_0$ at $[\lambda]$. By Lemma A.2 we have that the weights of $T$-eigenspaces are of form $\mu^\vee + k\hbar$ with an integer $k$ and a weight $\mu^\vee \in W_V$ such that $0 < k \leq (\mu^\vee, \lambda)$. All multiplicities are 1. The sum is of all weights is

$$\sum_{\mu^\vee \in W_V, (\mu^\vee, \lambda) > 0} m_{\mu^\vee} \sum_{k=1}^{(\mu^\vee, \lambda)} \mu^\vee + k\hbar = \sum_{\mu^\vee \in W_V, (\mu^\vee, \lambda) > 0} m_{\mu^\vee} (\mu^\vee, \lambda) \left[\mu^\vee + \frac{\hbar}{2} (\mu^\vee, \lambda) + 1\right].$$

Similarly, the fiber of $\mathcal{V}_0/\mathcal{V} \cap \mathcal{V}_0$ has $T$-eigenspaces o multiplicity 1 and the weights are of form $\mu^\vee + k\hbar$ for an integer $k$ and a weight $\mu^\vee \in W_V$ such that $(\mu^\vee, \lambda) < k \leq 0$. The sum is

$$\sum_{\mu^\vee \in W_V, (\mu^\vee, \lambda) < 0} m_{\mu^\vee} \sum_{k=(\mu^\vee, \lambda) + 1}^{0} \mu^\vee + k\hbar = \sum_{\mu^\vee \in W_V, (\mu^\vee, \lambda) < 0} m_{\mu^\vee} (\mu^\vee, \lambda) \left[\mu^\vee + \frac{\hbar}{2} (\mu^\vee, \lambda) + 1\right]$$

Subtracting these contributions extending summation to include $\mu^\vee \in W_V$ with $(\mu^\vee, \lambda) = 0$, we find the weight of $L_V$

$$\sum_{\mu^\vee \in W_V} m_{\mu^\vee} (\mu^\vee, \lambda) \mu^\vee + \frac{\hbar}{2} \sum_{\mu^\vee \in W_V} m_{\mu^\vee} (\mu^\vee, \lambda)^2 + \frac{\hbar}{2} \sum_{\mu^\vee \in W_V} (m_{\mu^\vee} \mu^\vee, \lambda)$$

(20)

First let us write $\mu$ in the first term of (20) as $\langle \mu^\vee, \bullet \rangle$

$$\sum_{\mu^\vee \in W_V} m_{\mu^\vee} \langle \mu^\vee, \lambda \rangle \mu^\vee = \sum_{\mu^\vee \in W_V} m_{\mu^\vee} \langle \mu^\vee, \lambda \rangle \langle \mu^\vee, \bullet \rangle$$

Then, note that $\langle \mu^\vee, \lambda \rangle$ is the eigenvalue of $\lambda$ acting on the eigenspace of weight $\mu$. This allows us to rewrite (20) in terms of traces:

$$\text{Tr}_V (\lambda \bullet) + \frac{\hbar}{2} \text{Tr}_V (\lambda \lambda) + \frac{\hbar}{2} \text{Tr}_V (\lambda)$$

The last term vanishes, since $g$ is simple, that is $g = [g, g]$ and every element can be written as a sum of commutators. Rewriting the result using the scalar product, we get

$$(\lambda, \bullet)_V + \frac{\hbar}{2} (\lambda, \lambda)_V$$

as desired. $\square$

By Proposition A.13 we can define the equivariant structure on $O(1)$ as on $d_V$th root of $L_V$. In general, this is an equivariant structure with respect to a connected cover of $T$. One can make it an honest $T$-equivariant structure on each connected component of $\text{Gr}$ by tensoring with certain weight, but we work with the structure we defined because it has a simple formula for weights at the fixed points.
Corollary A.15. For the line bundle $O(1)$ the $T$-weight of the fiber at a $T$-fixed point $|\lambda|$ is

$$K(\lambda, \bullet) + \frac{\hbar}{2} K(\lambda, \lambda).$$

Proof. This follows immediately from $O(1)$ being $d_V$-th root of $L$ as an equivariant line bundle, and the Proposition A.14. \qed

In particular, Corollary A.15 shows that the equivariant structure does not depend on the choice of $V$: changing the equivariant structure would give a constant summand to all fiber weights.

The line bundle $O(1)$ gives a collection of line bundles on $Gr^\lambda_{\mu}$ via coordinate-wise pullbacks:

$$L_i = \pi_i^* O(1),$$

where

$$\pi_i: Gr^\lambda_{\mu} \to Gr, \quad 0 \leq i \leq l$$

$$(L_0, L_1, \ldots, L_l) \mapsto L_i.$$

Then using Corollary A.15 and the definition of $L_i$ we prove Proposition 3.23.

Given a weight $\chi^\vee$ we denote the corresponding trivial line bundle as $\chi^\vee$ (i.e. the line bundle where $T$ scales the fiber by $\chi^\vee$). The check here means that $\chi^\vee$ a weight; it has nothing to do with taking the dual line bundle.

Proposition A.16. The line bundles $\mathcal{L}_0$ and $\mathcal{L}_l$ are trivial. More precisely,

$$\mathcal{L}_0 = 0,$$

$$\mathcal{L}_l = K(\mu, \bullet) + \frac{\hbar}{2} K(\mu, \mu).$$

Proof. The bundles $\mathcal{L}_0$ and $\mathcal{L}_l$ are pulled back from a point and $Gr_\mu$ respectively. Both have trivial Picard group, so the bundles are trivial. Now the question is what the equvariant structure is, i.e. how the fiber is scaled. This can be done by computing the weight at any fixed point via Corollary 3.23. \qed

It is also convenient to introduce the following line bundles.

$$\mathcal{E}_i = \mathcal{L}_i / \mathcal{L}_{i-1} \text{ for } 1 \leq i \leq l.$$  \hspace{1cm} (21)

Since $c_T^1 (\mathcal{L}_i)$ generate $H^2_T(X)$, then $c_T^1 (\mathcal{E}_i)$ generate it as well.

A.2.1 Homology of resolutions

The tight connection between the geometry of the affine Grassmannian of a reductive group $G$ and the representation theory of the Langlands dual group $G^\vee$ is explicit in the geometric Satake correspondence. Here we present a quick overview of it, the reader can find more details in [G, MV, MV2].

The affine Grassmannian $Gr$ has a natural (Whitney) stratification $S$. This allows one to define $D^b_S(Gr, \mathbb{C})$ the bounded derived category of $S$-constructable $\mathbb{C}$-sheaves with respect to this stratification. There is a full subcategory of perverse sheaves $Perv_S(Gr, \mathbb{C})$. This category is abelian, and even is a heart of certain t-structure, see [BBD]. In what follows we don’t change the stratification $S$, so we omit it in the notation: $Perv(Gr, \mathbb{C})$.

The perverse sheaves were introduced by P. Deligne to give a sheaf-theoretic approach to the intersection cohomology, see the change form [GM] to [GM2]. For a closure of strata $S \in S$ we can assign a perverse sheaf $IC(S)$ supported on $S$ called the intersection cohomology sheaf. One can think of this sheaf as a correction of the constant sheaf on $S$ on the lower stratum to preserve the features like Poincaré duality in the singular case. If $S$ is smooth, then $IC(S)$ is just the constant sheaf, possibly up to cohomological shift.
For the affine Grassmannian the distinguished perverse sheaves are $\text{IC} \left( \overline{\text{Gr}}^\lambda \right)$ which we denote later as $\text{IC}_\lambda$. These are simple objects in $\text{Perv} \left( \text{Gr}, \mathbb{C} \right)$.

There is a standard sheaf cohomology functor

$$H^*: \text{Perv} \left( \text{Gr}, \mathbb{C} \right) \to \text{Vect}_\mathbb{C}$$

$$\mathcal{F} \mapsto \bigoplus_i H^i \left( \text{Gr}, \mathcal{F} \right)$$

to the abelian category $\text{Vect}_\mathbb{C}$ of finite dimensional complex vector spaces.

The sheaf cohomology of the intersection cohomology sheaf

$$\text{IH}^* \left( \overline{\mathcal{S}} \right) := H^* \left( \text{IC} \overline{\mathcal{S}} \right)$$

give the dual space to intersection homology $\text{IH}_* \left( \overline{\mathcal{S}} \right)$ of $\overline{\mathcal{S}}$.

In one of the simplest versions, the geometric Satake correspondence states that the category of complex perverse sheaves on $\text{Gr}$ is equivalent

$$\text{Perv} \left( \text{Gr}, \mathbb{C} \right) \xrightarrow{\sim} \text{Rep} \left( G^\vee, \mathbb{C} \right)$$

to the abelian category of finite dimensional complex representations of the Langlands dual group $G^\vee$. This equivalence moreover

- respects the fiber functors, that is, the triangle

$$\xymatrix{ \text{Perv} \left( \text{Gr}, \mathbb{C} \right) & \text{Rep} \left( G^\vee, \mathbb{C} \right) \ar[l]_{H^*} \ar[r]^F \ar[ld]_{\sim} & \text{Vect}_\mathbb{C} }$$

is commutative. The functor $F$ here is the forgetful functor, which assigns the underlying vector space of the representation.

- respects the tensor structure of these categories. On $\text{Rep} \left( G^\vee, \mathbb{C} \right)$ the tensor product is the usual tensor product $\otimes$ of representations. The tensor structure $*$ on $\text{Perv} \left( \text{Gr}, \mathbb{C} \right)$ is harder to describe and is called the convolution. It is defined via natural operations with the convolution Grassmannian.

These properties immediately imply the isomorphism

$$H^* \left( \text{Gr}, \mathcal{F}_1 \ast \mathcal{F}_2 \right) \simeq H^* \left( \text{Gr}, \mathcal{F}_1 \right) \otimes H^* \left( \text{Gr}, \mathcal{F}_2 \right)$$

as $\mathbb{C}$-vector spaces.

Under the geometric Satake equivalence simple objects must go to simple objects. As we said above, the simple objects are the intersection cohomology sheaves $\text{IC}_\lambda$ of the closed cells parametrized by dominant coweight $\lambda$. In $\text{Rep} \left( G^\vee, \mathbb{C} \right)$ the simple objects are irreducible representations $V_\lambda$ which are parametrized by dominant highest weight $\lambda$. A coweight of $G$ is a weight of $G^\vee$, so these parametrizations naturally identify. The equivalence says that these

$$\text{IC}_\lambda \leftrightarrow V_\lambda$$

correspond to each other.

By the construction of the convolution product we have the intersection cohomology sheaf of the convolution Grassmannian $\text{Gr}^{(\lambda_1, \ldots, \lambda_l)}$ pushforwards along the map

$$\text{Gr}^{(\lambda_1, \ldots, \lambda_l)} \to \text{Gr}$$

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to the convolution $\text{IC}_{\lambda_1} \ast \cdots \ast \text{IC}_{\lambda_l}$. Since the sheaf cohomology is the pushforward to the point, we get

$$
\text{IH}^* \left( \text{Gr}^{(\lambda_1, \ldots, \lambda_l)} \right) \cong H^* \left( \text{Gr}, \text{IC}_{\lambda_1} \ast \cdots \ast \text{IC}_{\lambda_l} \right)
\cong H^* \left( \text{Gr}, \text{IC}_{\lambda_1} \right) \otimes \cdots \otimes H^* \left( \text{Gr}, \text{IC}_{\lambda_l} \right)
= \text{IH}^* \left( \text{Gr}^{\lambda_1} \right) \otimes \cdots \otimes \text{IH}^* \left( \text{Gr}^{\lambda_l} \right)
\cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_l}.
$$

If all $\lambda_i$ are minuscule, then the intersection cohomology is the same as ordinary cohomology, so

$$
H^* \left( \text{Gr}^{(\lambda_1, \ldots, \lambda_l)} \right) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_l}.
$$

If we restrict to the intersection cohomology sheaf of $\text{Gr}^{(\lambda_1, \ldots, \lambda_l)} \subset \text{Gr}^{(\lambda_1, \ldots, \lambda_l)}$, then we get a weight subspace

$$
H^* \left( \text{Gr}^{(\lambda_1, \ldots, \lambda_l)} \right) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_l} [\mu]
$$

by the construction of the geometric Satake.

Functoriality gives compatibility of these isomorphisms

$$
\begin{array}{ccc}
H^* \left( \text{Gr}^{(\lambda_1, \ldots, \lambda_l)} \right) & \overset{i^*}{\longrightarrow} & V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_l} \\
\downarrow \pi & & \downarrow \pi \\
H^* \left( \text{Gr}^{(\lambda_1, \ldots, \lambda_l)} \right) & \overset{\sim}{\longrightarrow} & V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_l} [\mu]
\end{array}
$$  \hfill (22)

where $i^*$ is the pullback by the inclusion

$$
i : \text{Gr}^\lambda_\mu \to \text{Gr}^\lambda
$$

and $\pi$ is the natural projection to a direct summand.

Now let us say a couple of words on the relation of these constructions to the equivariant cohomology. The spaces $\text{Gr}^\lambda_\mu$ admit a $T$-invariant cell structure, namely the Bialynicki-Birula decomposition. Then $\text{Gr}^\lambda_\mu$ are $T$-equivariantly formal, which imply two things

- The equivariant cohomology $H^*_T \left( \text{Gr}^\lambda_\mu, \mathbb{C} \right)$ is a free module over $H^*_T \left( \text{pt}, \mathbb{C} \right)$.
- The ordinary cohomology can be obtained from the equivariant cohomology by "killing" equivariant variables:

$$
H^* \left( \text{Gr}^{(\lambda_1, \ldots, \lambda_l)}, \mathbb{C} \right) \cong \mathbb{C} \otimes_{H^*_T \left( \text{pt}, \mathbb{C} \right)} H^*_T \left( \text{Gr}^{(\lambda_1, \ldots, \lambda_l)}, \mathbb{C} \right)
$$

is an isomorphism of graded algebras.

This implies that

$$
H^*_T \left( \text{Gr}^{(\lambda_1, \ldots, \lambda_l)}, \mathbb{C} \right) \cong H^*_T \left( \text{pt}, \mathbb{C} \right) \otimes \mathbb{C} H^* \left( \text{Gr}^{(\lambda_1, \ldots, \lambda_l)}, \mathbb{C} \right)
$$

as $H^*_T \left( \text{pt}, \mathbb{C} \right)$-modules (not as algebras!). By applying the geometric Satake,

$$
H^*_T \left( \text{Gr}^{(\lambda_1, \ldots, \lambda_l)}, \mathbb{C} \right) \cong H^*_T \left( \text{pt}, \mathbb{C} \right) \otimes \mathbb{C} V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_l} [\mu] .
$$

We will see later a similar statement on the localized level, but we derive it only from the combinatorics of the $T$-fixed points, with no use of the geometric Satake.

There an analogue of the geometric Satake for the equivariant cohomology $H^*_T \left( \text{Gr}^\lambda_\mu \right)$, given by V. Ginzburg and S. Riche[GR]. However, we do not use it here.
A.2.2 Generation of the second cohomology

One can pose a questions if the collection \{L_i\} has "enough" line bundles. That is, do they form a basis in Pic (X) \otimes_Z Q. For symplectic resolutions this question is equivalent to asking if the first Chern classes c_1 (L_i) generate the second rational cohomology \H^2 (X, Q), as the following proposition shows.

**Proposition A.17.** For a symplectic resolution X the first Chern class is the map

\[
\text{Pic}(X) \xrightarrow{c_1} \H^2 (X, \Z)
\]

is an isomorphism.

*Proof.* The first Chern class map

\[
\text{Pic}(X) = \H^1 (X, \O_X) \xrightarrow{c_1} \H^2 (X, \Z)
\]

comes from the long exact sequence for the exponential short exact sequence of sheaves

\[
0 \longrightarrow \Z \longrightarrow \O \longrightarrow \O_X \longrightarrow 0.
\]

Since we have vanishing \H^i (X, \O) = 0 for \(i > 0\) (see e.g. Kaledin\[K2\], Lemma 2.1), the first Chern class gives an isomorphism. The vanishing comes from \O being the canonical sheaf and the vanishing theorem of Grauert and Riemenschneider. \(\square\)

**Proposition A.18.** The rational second cohomology \H^2 (X, Q) is generated as a vector space by \L_i, 0 < i < l.

*Proof.* The space \Gr^{\lambda_1, \ldots, \lambda_l} is a bundle over \Gr^{\lambda_1, \ldots, \lambda_l-1} with a fiber \Gr^{\lambda_l}. The cell \Gr^{\lambda_l} is the flag variety for a maximal parabolic (\lambda_l is minuscule), so its second cohomology is one-dimensional and is generated by the first Chern class of any ample line bundle. Restriction of \L_i is ample (and even can be identified with one of equivariant bundles), so it generates the second cohomology of the fiber. Both \Gr^{\lambda_1, \ldots, \lambda_l-1} and \Gr^{\lambda_l} are connected, so by the spectral sequence of the fibration the second cohomology is generated by the second cohomology of \Gr^{\lambda_1, \ldots, \lambda_l-1} and c_1 (\L_i). Applying induction on \(l\) and using \Gr^0 = pt as the base case, we find that the rational second cohomologies are generated by \(c_1 (i), 1 \leq i \leq l\).

By the compatibility (22) we know that the restriction

\[
\H^i \left( \Gr^\lambda, Q \right) \xrightarrow{i_*} \H^i \left( \Gr^\mu, Q \right)
\]

is a surjection. Since \H^i \left( \Gr^\lambda, Q \right) is generated as a linear space by \(c_1 (\L_i), 0 < i \leq l\) and the functoriality of the Chern classes, we get that \H^i \left( \Gr^\lambda, Q \right) is generated by the same line bundles \L_i (or more pedantically, by their restrictions which have the same name by slightly abusing the notation). Finally, \L_i is trivial, so we can drop \(i = l\) and take only \(0 < i < l\) to generate the second cohomology. \(\square\)

This proposition, the isomorphism \Pic(X) = \H^1 (X, \O_X) \xrightarrow{c_1} \H^2 (X, \Z), and \T-formality of X proves Proposition 3.22

*Remark.* The generation of \Pic(X) \otimes_Z Q by "tautological" bundles \L_i is an analogue of Kirwan surjectivity for Nakajima quiver varieties.

The line bundles \L_i are not independent. Let us give a couple of examples how this dependence might look like.

**Example A.4.** If \(\lambda = \mu\) the space is a point and all the bundles are trivial. As a more interesting generalization, one can say that if \(\mu\) is dominant, \(\lambda_i\) is minuscule fundamental coweight and \(K (\lambda - \mu, \lambda_i) = 0\), then \L_i and \L_{i-1} differ by a weight.

**Example A.5.** Let \(G\) be of type \(D_n\). Let \(\omega_n-1\) and \(\omega_n\) be the highest weights of the half-spin representations, and let \(\alpha_{n-1}, \alpha_n\) are corresponding coroots. Then if \(\lambda = n_1 \omega_{n-1} + n_2 \omega_n, \mu = \lambda - \omega_{n-1} - \omega_n\) we have that the product of the line bundles corresponding to components with \(\omega_n\) is trivial. Similar holds for \(\omega_{n-1}\).
A.3 Behavior on walls in $A$

For each point $a \in a_0$ one can define the fixed point locus as $X^a = X^{na}$, where $n \in \mathbb{Z}$ is any integer such that $na \in X_\ast(A)$. Then for a generic $a$ the fixed locus is $X^A$. The locus where $X^a$ is larger is a union of hyperplanes called walls. Let us remind how positions of walls are related to the tangent weights of the fixed locus.

Let us first prove the following auxiliary statement

**Lemma A.19.** Let $T$ be a torus and $\pi: X \to X_0$ be a $T$-equivariant proper morphism to an affine variety. Assume, moreover, $X_0$ has the unique $T$-fixed point $x \in X_0$ and there is a $T$-cocharacter $\lambda: \mathbb{C}^\times \to T$ contracting $X_0$ to $x$. Then for any $T$-invariant closed subvariety $Y \subset X$ has a $T$-fixed point.

**Proof.** The image $\pi(Y)$ is closed $T$-invariant. By the action of $\lambda$ on any point, we get that $x \in \pi(Y)$. So $Y$ intersects non-trivially with $\pi^{-1}(x)$. Then $Y \cap \pi^{-1}(x)$ is non-empty and proper, invariant $T$. Since $T$ is a torus, there is a $T$-fixed point in $Y \cap \pi^{-1}(x)$ (by a theorem due to Borel[B]), and hence in $Y$.

**Remark.** We could not apply the theorem from [B] immediately to $Y$ since $X$ is not proper, it only has a proper map to $X_0$. However, extra conditions on $X_0$ allowed us to reduce our case to the well-known result.

Recall that we have such a proper map

$$m_\lambda: \text{Gr}_T^\lambda \to \text{Gr}_T^\mu$$

to a $T$-equivariant affine space. By Proposition 2.5 $\text{Gr}_T^\lambda$ is contracted to the unique fixed point by $C^*_h \subset T$. So the statement holds for $X = \text{Gr}_T^\lambda$. Now we can apply the lemma for the fixed locus $X^a$, for some $a \in a_0$. It’s easy to see that it is a closed subspace. Moreover, it is $T$-invariant because the $T$-action commutes with anything in $A \subset T$, in particular, with (any power of) $a$.

Then there’s an immediate corollary of Lemma A.19 and computation of tangent weights.

**Proposition A.20.** Given $a \in a_0$, the fixed locus $X^a$ is greater than $X^A$ only if there is a (non-affine) root $\chi^\vee$ such that $a \in \text{ker} \chi^\vee$.

For the rest of this section all objects (like $\text{ker} \chi^\vee$) are associated to the torus $A$, not $T$.

We want to show that components of $X^{\text{ker} \chi^\vee}$ are the resolution of slices of type $A_1$. Let us fix $X = \text{Gr}_T^\lambda$ and $X_0 = \text{Gr}_T^0$ for the fixed $G$ of any type, and in what follows we write $\text{Gr}$ explicitly only meaning the $A_1$-type affine Grassmannian.

So far we always assumed that in the sequence $\lambda = (\lambda_1, \ldots, \lambda_l)$ all $\lambda_i$ are non-zero. In this section it’s convenient to drop this assumption for the slices of $A_1$-type. This doesn’t give anything new: the condition

$$L_{i-1} \overset{0}{\to} L_i$$

simply means

$$L_{i-1} = L_i.$$  

So, if one drops all $\lambda_i = 0$, an isomorphic slice is obtained.

For $A_1$-type the connected group of the adjoint type is $\text{PSL}_2$. We denote the fundamental coweight as $\omega$ and the positive root as $\alpha = 2\omega$.

Let $p \in X^A$ and $\chi^\vee$ be a root of $G$. Then we consider

$$Z = \text{Gr}_{m\omega}^{(k_1\omega, \ldots, k_l\omega)},$$

where

$$m = \langle \mu, \chi^\vee \rangle,$$

$$k_i = \begin{cases} 0 & \text{if } \langle \Sigma_{i-1} (p) - \Sigma_i (p), \chi^\vee \rangle = 0, \\ 1 & \text{otherwise.} \end{cases}$$
In $Z$ there is a unique point $p'$ satisfying

$$\Sigma_i (p') = \langle \Sigma_i (p), \chi^\vee \rangle \omega$$

for all $i$. It's an easy check that it's indeed in $Z$.

Then we can define an inclusion map

$$i: Z \hookrightarrow X$$

given by

$$\left( gt^{\Sigma_i (p')} \text{PSL}_2 (\Omega), \ldots, gt^{\Sigma_l (p')} \text{PSL}_2 (\Omega) \right)$$

$$\mapsto \left( t_{\chi^\vee}^{\Sigma_i (p)} G (\Omega), \ldots, t_{\chi^\vee}^{\Sigma_l (p)} G (\Omega) \right),$$

where $g_1, \ldots, g_l \in \text{SL}_2 (\mathbb{C})$. We use the same name $\iota_{\chi^\vee}$ for a base change $\text{SL}_2 (K) \to G (K)$ of the root subgroup inclusion $\iota_{\chi^\vee}: \text{SL}_2 \to G$. Here we used the fact that for any coweight $\nu$ any point in $\text{Gr}$ can be presented in a form $gt^\mu \text{PSL}_2 (\Omega)$ for some $g \in \text{SL}_2 (\mathbb{C})$.

Then we get that the image of $Z$ is an $A$-invariant subvariety containing $p'$. Cocharacters in $\ker \chi^\vee$ act trivially on the $\text{SL}_2$-subgroup corresponding to $\chi^\vee$, so $i (Z) \subset \ker \chi^\vee$. Since the multiplicity formulas for weights of form $\pm \chi^\vee + nh$ are the same for $Z$ and $X$, $Z$ is an open subspace of $X^{\ker \chi^\vee}$. Moreover, $Z$ is proper over $X_0$ as one can see from the following diagram

$$\begin{array}{ccc}
Z & \xrightarrow{i} & X \\
\downarrow & & \downarrow \\
\text{Gr}_{m \omega}^{\Sigma k_i \omega} & \xrightarrow{\iota} & X_0
\end{array}$$

where the bottom inclusion is defined similarly to $i$. From this we see that $Z$ is a closed, and hence a union of components of $X^{\ker \chi^\vee}$.

$Z$ has a proper birational morphism to a normal irreducible scheme $\text{Gr}_{m \omega}^{\Sigma k_i \omega}$, so $Z$ is connected. Thus $Z$ is a component of $X^{\ker \chi^\vee}$.

From Lemma A.19 every component of $X^{\ker \chi^\vee}$ is of this form. Keeping track of the $T$-action on components of $X^{\ker \chi^\vee}$ we can summarize our consideration in the following theorem.

**Theorem A.21.** If $\alpha^\vee$ is a root, then any connected component $Z \subset X^{\ker \chi^\vee}$ is isomorphic to a resolution of a $\text{PSL}_2$-slice:

$$\text{Gr}_{m \omega}^{(\omega, \ldots, \omega)} \xrightarrow{\iota} Z$$

Moreover, this morphism is $T$-equivariant if $A \subset T$ acts on $\text{Gr}_{m \omega}^{(\omega, \ldots, \omega)}$ via character $\alpha^\vee$.

**Theorem A.22.** Let $\chi^\vee$ be a nontrivial character of $A$ and $p, q \in X^A$. Then $p$ and $q$ are in the same component of $X^{\ker \chi^\vee}$ if and only if the following conditions are satisfied

1. $\chi^\vee$ is a multiple of a root $\alpha^\vee$,

2. For all $i$

$$\Sigma_i (p) - \Sigma_i (q) \in \mathbb{Z} \alpha$$

for the coroot $\alpha$ corresponding to root $\alpha^\vee$ and $\omega^\vee$.

**Proof.** Let $p, q$ be in the same connected component of $X^{\ker \alpha^\vee}$. The condition on $\chi^\vee$ being equal to a root follows from Proposition A.20.
Since \( p, q \) in a connected component of \( X^\ker \alpha \), for any equivariant line bundle \( L \) the \( A \)-weights of fibers over \( p \) and \( q \) must be equal modulo \( \alpha \). By Corollary 3.23 we have for all \( i, 0 < i < l \),
\[
K(\Sigma_i(p), \bullet) = K(\Sigma_i(q), \bullet) \mod \alpha \vee,
\]
which implies
\[
\Sigma_i(p) - \Sigma_i(q) \in \alpha \vee \mathbb{Q}.
\]
Now one uses that \( \Sigma_i(p) - \Sigma_i(q) \) is in the coroot lattice to derive
\[
\Sigma_i(p) - \Sigma_i(q) \in \alpha \vee \mathbb{Z}.
\]
If \( \chi \vee \) is a multiple of a root \( \alpha \) and \( p, q \in X^A \) satisfy (23). Then by construction of the isomorphism of the component \( Z \subset X^\ker \alpha \) containing \( p \), one can find a point which maps to under \( i \). Thus \( q \) is in \( Z \).

This immediately implies the following statement we in the main text.

**Corollary A.23.** Let \( \chi_{1}^{\vee} \) and \( \chi_{2}^{\vee} \) be \( \mathbb{Q} \)-linearly independent characters of \( A \). If \( p, q \in X^A \) are in the same components in both \( X^\ker \chi_{1} \) and \( X^\ker \chi_{2} \), then \( p = q \).

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