ON A GENERALIZED CRANK FOR \( k \)-COLORED PARTITIONS

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Abstract. A generalized crank (\( k \)-crank) for \( k \)-colored partitions is introduced. Following the work of Andrews-Lewis and Ji-Zhao, we derive two results for this newly defined \( k \)-crank. Namely, we first obtain some inequalities between the \( k \)-crank counts \( M_k(r, m, n) \) for \( m = 2, 3 \) and \( 4 \), then we prove the positivity of symmetrized even \( k \)-crank moments weighted by the parity for \( k = 2 \) and \( 3 \). We conclude with several remarks on furthering the study initiated here.

1. Introduction

A partition \([2]\) \( \pi \) of a positive integer \( n \) is a finite weakly decreasing sequence of positive integers \( \pi_1 \geq \pi_2 \geq \cdots \geq \pi_r > 0 \) such that \( \sum_{i=1}^{r} \pi_i = n \). The \( \pi_i \) are called the parts of the partition. In 1944, Dyson [13] defined the rank of a partition as the largest part minus the number of parts and then observed that the rank appears to give combinatorial interpretations for Ramanujan’s two celebrated partition congruences modulo 5 and 7. Unfortunately, Dyson’s rank fails to do the same thing for Ramanujan’s third partition congruence modulo 11. Nevertheless, Dyson [13] postulated the existence of another partition statistic, which he coined as the “crank”, that would similarly explain Ramanujan’s third partition congruence. In 1988, Andrews and Garvan [5] finally captured Dyson’s elusive crank of partitions, motivated by the crank of certain vector partitions, which was first studied by Garvan [16]. In an earlier paper [15] we considered, in the same vein, two families \( \mathfrak{p} \) and \( \mathfrak{r}^* \) of multiranks for multipartitions, multi-overpartitions and multi-pods that lead to similar combinatorial interpretations for congruence properties enjoyed by these types of partitions. We introduce here yet a third family of generalized cranks for multipartitions, which apparently do not possess any significance in explaining congruences but, as we are going to demonstrate, have in common several neat distributional properties with the Andrews-Garvan-Dyson crank.

Our starting point is a recent paper by Bringmann and Dousse [10], in which they settled a longstanding conjecture by Dyson [14, p. 172] concerning the limiting shape of the crank generating function. More precisely, they considered a family of inverse theta functions.

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defined for \( k \in \mathbb{N} \) by

\[
C_k(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M_k(m, n) z^m q^n := \frac{(q; q)^{2-k}_\infty}{(zq; q)_\infty (z^{-1}; q)_\infty}.
\]  

(1.1)

Here and in what follows, we adopt the following customary notations \([2]\) in partitions and \(q\)-series:

\[
(a; q)_j := \prod_{i=0}^{j-1} (1 - aq^i), \quad j \in \mathbb{N}_0 \cup \{\infty\}.
\]

Since \( P(q) := (q; q)_\infty^{-1} \) is the generating function for the ordinary partitions, the product side of (1.1) suggests the following definition.

**Definition 1.1.** For \( k \geq 2 \), the generalized crank (abbreviated as \( k \)-crank in what follows) of a \( k \)-colored partition \( \vec{\pi} = (\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}) \) is defined as

\[
k\text{-crank}(\vec{\pi}) = \ell(\pi^{(1)}) - \ell(\pi^{(2)}),
\]

(1.2)

where \( \ell(\pi^{(i)}) \) denotes the number of parts in \( \pi^{(i)} \).

**Remark 1.2.** We note that the special case \( k = 2 \) yields the Hammond-Lewis birank \([19]\) of \( 2 \)-colored partitions, and the case \( k = 3 \) corresponds to the authors’ multirank \( r^* \) for \( 3 \)-colored partitions \([15]\). For \( k > 3 \), the \( k \)-crank defined by (1.2) appears to be new.

In view of (1.1), Definition 1.1 and the generating function for crank (1.4), it is clear that for \( k \geq 2 \), \( M_k(m, n) \) (resp. \( M_1(m, n) \)) enumerates the number of \( k \)-colored partitions (resp. ordinary partitions) of \( n \) with \( k \)-crank (resp. crank) equals \( m \). By convention, we simply write \( M(m, n) \) for \( M_1(m, n) \). Furthermore, we denote \( M_k(r, m, n) \) (resp. \( M(r, m, n) \)) by the number of \( k \)-colored partitions (resp. ordinary partitions) of \( n \) with \( k \)-crank (resp. crank) congruent to \( r \) modulo \( m \).

Given a \( k \)-colored partition \( \vec{\pi} = (\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}) \) with crank equals, say \( m \), we can pair with it a \( k \)-colored partition with crank equals \(-m\), simply by swapping \( \pi^{(1)} \) and \( \pi^{(2)} \). This observation immediately establishes the symmetry for \( k \)-crank:

\[
M_k(m, n) = M_k(-m, n).
\]

(1.3)

This parallels the symmetry for the crank of ordinary partitions, which is not so obvious due to its asymmetric definition (see \([9]\) for a direct combinatorial proof).

The generating function for \( M(m, n) \) was given in \([5, 16]\):

\[
C_1(z, q) = C(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m, n) z^m q^n = \frac{(q; q)_\infty}{(z; q)_\infty (z^{-1}; q)_\infty}.
\]

(1.4)

Putting \( z = -1 \) in (1.4) gives

\[
\sum_{n=0}^{\infty} (M(0, 2, n) - M(1, 2, n)) q^n = \frac{(q; q)_\infty}{(-q; q)_\infty^2}.
\]
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whose coefficient $M_0(2, n) - M_1(2, n)$ alternates in sign, a fact that was first observed by Andrews and Lewis.

**Theorem 1.3** (Theorem 1 in [6]). For all $n \geq 0$,

\[
M_0(2, 2n) > M_1(2, 2n), \quad M_1(2, 2n + 1) > M_0(2, 2n + 1).
\]

Following the work of Andrews and Lewis, we study the $k$-crank of $k$-colored partitions modulo 2, 3, 4 and obtain comparable results. The following is the $m = 2$ case.

**Theorem 1.4.** For $n \geq 0$,

\[
\begin{align*}
M_k(0, 2, 2n) &> M_k(1, 2, 2n), \text{ for } k = 2, 3, 4, \\
M_k(1, 2, 2n + 1) &> M_k(0, 2, 2n + 1), \text{ for } k = 2, 3, \\
M_4(0, 2, 2n + 1) &= M_4(1, 2, 2n + 1), \\
M_k(0, 2, n) &> M_k(1, 2, n), \text{ for } k \geq 5.
\end{align*}
\]

The rest of the paper is organized as follows. In Section 2, we establish some inequalities between $k$-cranks of $k$-colored partitions modulo 2, 3, and 4. Next in Section 3 we generalize (1.5) and (1.6) further by introducing the symmetrized $k$-crank moments. We conclude in the last section with some remarks and one conjecture on the unimodality of $M_k(m, n)$.

2. $k$-CRANK MODULO 2, 3, AND 4

2.1. The case $m = 2$.

**Proof of Theorem 1.4.** Taking $k = 2, z = -1$ in (1.1) gives

\[
\sum_{n=0}^{\infty} (M_2(0, 2, n) - M_2(1, 2, n)) q^n = \frac{1}{(-q; q^2)_{\infty}} := f(q),
\]

say, then we have to show that the coefficient of $q^n$ in (2.1) is positive/negative according to whether $n$ is even or odd. In other words, we need to prove that the coefficients of $f(-q)$ are all positive. Since

\[
f(-q) = \frac{1}{(q; q^2)_{\infty}} = \frac{(-q; q^2)_{\infty}}{(-q^2; q^2)_{\infty}} = (-q; q^2)_{\infty}^2,
\]

then the coefficients of $f(-q)$ are all positive. Actually $f(-q)$ is the generating function of pairs of partitions into distinct odd parts. This gives us the $k = 2$ case of (1.5) and (1.6).

Similarly, taking $k = 3, z = -1$ in (1.1) leads to

\[
\sum_{n=0}^{\infty} (M_3(0, 2, n) - M_3(1, 2, n)) q^n = \frac{1}{(-q; q^2)_{\infty}(q; q)_{\infty}} = \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}}.
\]

which analogously gives us the $k = 3$ case of (1.5) and (1.6).
For \( k \geq 4 \), we have
\[
\sum_{n=0}^{\infty} (M_k(0, 2, n) - M_k(1, 2, n)) q^n = \frac{1}{(-q; q)^2_{\infty}} \frac{1}{(q^2; q^2)^{k-2}_{\infty}} = \frac{1}{(q^2; q^2)^{k-4}_{\infty}}.
\]

If \( k = 4 \), we get \((1.7)\) and the \( k = 4 \) case of \((1.5)\), since the power of \( q \) in \((2.2)\) must be even.
When \( k \geq 5 \), \((1.8)\) is obvious since \((2.2)\) contains the factor \( 1/(q; q)^{\infty}_{\infty} \). This completes the proof.

\[\Box\]

2.2. The case \( m = 3 \). In the same paper [6], Andrews and Lewis proposed the following conjecture, which was first solved by Kane [21] and reproved in a more systematic setting by Kim [22] later.

**Theorem 2.1.** For \( n \geq 0 \),
\[
M(0, 3, 3n) > M(1, 3, 3n),
M(0, 3, 3n + 1) < M(1, 3, 3n + 1),
M(0, 3, 3n + 2) < M(1, 3, 3n + 2), \quad \text{for } n \neq 1, 4, 5,
M(0, 3, 3n + 2) = M(1, 3, 3n + 2), \quad \text{for } n = 4, 5.
\]

In contrast, we have the following result when considering \( k \)-crank modulo 3.

**Theorem 2.2.** For all \( n \geq 0 \),
\[
(2.3) \quad M_2(0, 3, 3n) > M_2(1, 3, 3n),
(2.4) \quad M_2(0, 3, 3n + 1) < M_2(1, 3, 3n + 1),
(2.5) \quad M_2(0, 3, 3n + 2) < M_2(1, 3, 3n + 2),
(2.6) \quad M_3(0, 3, 3n) > M_3(1, 3, 3n),
(2.7) \quad M_3(0, 3, 3n + 1) = M_3(1, 3, 3n + 1), M_3(0, 3, 3n + 2) = M_3(1, 3, 3n + 2),
(2.8) \quad M_k(0, 3, n) > M_k(1, 3, n), \quad \text{for } k \geq 4.
\]

**Proof.** We first note that assuming \( k = 2, z = e^{2\pi i/3} \) in \((1.1)\) gives
\[
(2.9) \quad \sum_{n=0}^{\infty} (M_2(0, 3, n) - M_2(1, 3, n)) q^n = \frac{(q; q)_{\infty}}{(q^3; q^3)_{\infty}} = (q; q^3)_{\infty} (q^2; q^3)_{\infty}.
\]

To prove \((2.3), (2.4)\) and \((2.5)\), we only need to show that the signs of coefficients of \( q^{3n} \), \( q^{3n+1} \) and \( q^{3n+2} \) in \((2.9)\) are “+ − −”. To that end, we consider the following 3-dissection:
\[
\frac{(q; q)_{\infty}}{(q^3; q^3)_{\infty}} = \frac{J_1}{J_3} = \frac{J_{12,27}}{J_3} - q \frac{J_{6,27}}{J_3} - q^2 \frac{J_{3,27}}{J_3},
\]
where \( J_s := (q^s; q^s)_{\infty} \) and \( J_{s,t} := (q^s; q^t)_{\infty} (q^{1-s}; q^t)_{\infty} (q^d; q^t)_{\infty} \) for \( 1 \leq s < t \). This is a simple consequence of the Jacobi Triple Product identity, see for example [8, p. 48, Entry 31] for
a proof. Next for \( k \geq 3 \), we also set \( z = e^{2\pi i/3} \) in (1.1) to have

\[
\sum_{n=0}^{\infty} (M_k(0, 3, n) - M_k(1, 3, n))q^n = \frac{1}{(q^2; q^3)_{\infty}(q; q)_{k-3}}.
\]

If \( k = 3 \), we have (2.6) and (2.7) since the power of \( q \) in (2.10) is always divisible by 3. When \( k \geq 4 \), note that (2.10) contains the factor \( 1/(q; q)_{\infty} \), thus we have (2.8).

**Remark 2.3.** Two remarks on Theorem 2.2 are in order. First note that (2.3)–(2.5) can be viewed as the infinite version of the so-called “First Borwein conjecture” [4, (1.1)], and can be deduced from Theorem 2.1 in [4]. Secondly, Chan and Mao obtained an improvement [12, Corollary 1.8] on Theorem 2.1. It is then natural to ask if there exists an analogous improvement on Theorem 2.2.

### 2.3. The case \( m = 4 \)

Andrews and Lewis obtained the following inequalities of crank modulo 4.

**Theorem 2.4** (Theorem 3 in [6]). For \( n > 0 \),

\[
M(0, 4, 2n) > M(1, 4, 2n), \text{ for } n \neq 1,
\]

\[
M(0, 4, 2n - 1) < M(1, 4, 2n - 1), \text{ for } n \neq 2,
\]

\[
M(2, 4, 2n) > M(1, 4, 2n),
\]

\[
M(2, 4, 2n - 1) < M(1, 4, 2n - 1).
\]

Similarly, the numbers \( M_k(r, 4, n) \) satisfy the following relations.

**Theorem 2.5.** For \( n > 0 \),

\[
M_2(0, 4, 4n) > M_2(2, 4, 4n) > M_2(1, 4, 4n), \text{ for } n \neq 1,
\]

\[
M_2(2, 4, 4n + 2) > M_2(0, 4, 4n + 2) > M_2(1, 4, 4n + 2),
\]

\[
M_2(1, 4, 2n + 1) > M_2(0, 4, 2n + 1) = M_2(2, 4, 2n + 1),
\]

\[
M_3(0, 4, 2n) > M_3(1, 4, 2n) = M_3(2, 4, 2n),
\]

\[
M_3(0, 4, 2n + 1) = M_3(1, 4, 2n + 1) > M_3(2, 4, 2n + 1),
\]

\[
M_k(0, 4, n) > M_k(1, 4, n) > M_k(2, 4, n), \text{ for } k \geq 4.
\]

**Proof.** Firstly, setting \( k = 2, z = i \) in (1.1) gives

\[
\sum_{n=0}^{\infty} (M_2(0, 4, n) - M_2(2, 4, n))q^n = \frac{1}{(-q^2; q^2)_{\infty}} = (q^2; q^4)_{\infty},
\]

which does not have any odd powers of \( q \) in the expansion, and the coefficients of \( q^{4n} \) are all positive except for \( n = 1 \), while the coefficients of \( q^{4n+2} \) are all negative.
Recall from (2.1),
\[
\sum_{n=0}^{\infty} (M_2(0, 4, n) + M_2(2, 4, n) - 2M_2(1, 4, n)) q^n
= \sum_{n=0}^{\infty} (M_2(0, 2, n) - M_2(1, 2, n)) q^n = \frac{1}{(-q; q)_{\infty}^2},
\]
combining this with (2.14) we obtain
\[
\sum_{n=0}^{\infty} (M_2(0, 4, n) - M_2(1, 4, n)) q^n = \frac{1}{2} \left\{ \frac{1}{(-q; q)_{\infty}^2} + \frac{1}{(-q^2; q^2)_{\infty}} \right\} =: \alpha(q),
\]
say. Now
\[
\alpha(-q) = \frac{1}{2} \left\{ \frac{1}{(q; q)_{\infty}^2} + \frac{1}{(-q^2; q^2)_{\infty}} \right\} = \frac{1}{2} \left\{ (-q; q)_{\infty}^2 + (q^2; q^4)_{\infty} \right\}
= \frac{1}{2} (-q; q^2)_{\infty} \left\{ (-q; q^2)_{\infty} + (q; q^2)_{\infty} \right\} = (-q; q^2)_{\infty} \sum_{n=0}^{\infty} \mathcal{OD}_0(n) q^n,
\]
where $\mathcal{OD}_0(n)$ is the number of partitions of $n$ into an even number of distinct odd parts. Consequently the coefficients of $q^n$ in $\alpha(-q)$ are positive except for $n = 2$.

In just the same way, we see that
\[
\sum_{n=0}^{\infty} (M_2(2, 4, n) - M_2(1, 4, n)) q^n = \frac{1}{2} \left\{ \frac{1}{(-q; q^2)_{\infty}^2} - \frac{1}{(-q^2; q^2)_{\infty}} \right\} := \beta(q),
\]
say, then
\[
\beta(-q) = \frac{1}{2} (-q; q^2)_{\infty} \left\{ (-q; q^2)_{\infty} - (q; q^2)_{\infty} \right\} = (-q; q^2)_{\infty} \sum_{n=0}^{\infty} \mathcal{OD}_1(n) q^n,
\]
where $\mathcal{OD}_1(n)$ is the number of partitions of $n$ into an odd number of distinct odd parts. Therefore the coefficients of $q^n$ in $\beta(-q)$ are positive for $n > 0$. Thus we have proved (2.11)–(2.13). The arguments for cases with $k \geq 3$ are similar so we choose to omit them. \qed

3. 2-CRANK AND 3-CRANK MOMENTS: SYMMETRIZED AND WEIGHTED

In 2003, Atkin and Garvan [7] demonstrated the importance of the moments of ranks and cranks in the study of further partition congruences. Later, Andrews [3] considered a combinatorial interpretation of the moments of rank by introducing a symmetrized rank moment. This in turn motivated Garvan [18] to consider the symmetrized crank moments in his study of the higher order spt-functions. To be more precise, the $j^\text{th}$ symmetrized crank moment as defined by Garvan [18] is
\[
\mu_j(n) = \sum_{m=-n}^{n} \binom{m + \frac{|n|}{2}}{j} M(m, n),
\]
where \( \lfloor x \rfloor := \max\{m \in \mathbb{Z} : m \leq x\} \) is the usual floor function. From the symmetry \( M(m, n) = M(-m, n) \), it is clear that \( \mu_{2j+1}(n) = 0 \).

In our setting, a natural analog for \( k \)-crank is given by

\[
\mu_{j,k}(n) = \sum_{m=-n}^{n} \left( m + \left\lfloor \frac{j-1}{2} \right\rfloor \right) M_k(m, n),
\]

which is also meaningful for even \( j \) only due to the symmetry \((1.3)\) enjoyed by the \( k \)-cranks.

Recently, Ji and Zhao \([20]\) considered the 2\( j \)th crank moment weighted by the parity of \( k \)-cranks, i.e.,

\[
\mu_{2j}(-1, n) := \sum_{m=-n}^{n} \left( m + j - 1 \right) (-1)^m M(m, n).
\]

They showed the following positivity property of \((-1)^n \mu_{2\cdot j}(-1, n)\), which encompasses Theorem 1.3 as the \( j = 0 \) case.

**Theorem 3.1** (Theorem 1.2 in \([20]\)). For \( n \geq j \geq 0 \), \((-1)^n \mu_{2\cdot j}(-1, n) > 0\).

Motivated by the work of Ji and Zhao, we consider the 2\( j \)th symmetrized moments of \( k \)-colored partitions weighted by the parity of \( k \)-cranks, defined as

\[
(3.1) \quad \mu_{2j,k}(-1, n) := \sum_{m=-n}^{n} \left( m + j - 1 \right) (-1)^m M_k(m, n), \quad \text{for} \ k \geq 2.
\]

When \( j = 0 \), \( k = 2, 3 \), (3.1) reduces to

\[
\mu_{0,2}(-1, n) = M_2(0, 2, n) - M_2(1, 2, n),
\]

\[
\mu_{0,3}(-1, n) = M_3(0, 2, n) - M_3(1, 2, n).
\]

So the following result parallels Theorem 3.1 and includes (1.5) and (1.6) as the \( j = 0 \) case.

**Theorem 3.2.** For \( n \geq j \geq 0 \), we have

\[
(3.2) \quad (-1)^n \mu_{2\cdot 2}(-1, n) > 0,
\]

\[
(3.3) \quad (-1)^n \mu_{2\cdot 3}(-1, n) > 0.
\]

With the help of Andrew’s \( j \)-fold generalization of \( q \)-Whipple’s theorem \([1, \text{p.199, Theorem 4}]\), we can derive the following explicit generating functions for \( \mu_{2j,k}(-1, n) \).

\[
(3.4) \quad \sum_{n=0}^{\infty} \mu_{2j,k}(-1, n) q^n = \frac{1}{(q; q)_{\infty}^k (-q; q)_{\infty}^2} \sum_{n_j \geq n_{j-1} \geq \ldots \geq n_1 \geq 1} \frac{(-1)^i q^{n_1 + n_2 + \ldots + n_j}}{(1 + q^{n_1})^2 (1 + q^{n_2})^2 \ldots (1 + q^{n_j})^2}.
\]
Furthermore, the above generating functions (3.4) is equivalent to the following form, which is the key identity for the proof of Theorem 3.2.

\[(3.5) \quad \sum_{n=0}^{\infty} \mu_{2j,k}(-1, n)q^n = \frac{1}{(q; q)_\infty^2(q; q)_{\infty}^2_{m_j > m_{j-1} > \cdots > m_1 \geq 1}} \sum (-1)^m m_1(m_2 - m_1) \cdots (m_j - m_{j-1}) q^{m_j}. \]

The proofs of (3.4) and (3.5) are similar to those of the corresponding results in [20], thus we omit the details here.

Now we are ready to derive (3.2) and (3.3).

**Proof of Theorem 3.2.** Replacing \( q \) by \(-q\) in (3.5) and putting \( k = 2 \), we see that

\[
\sum_{n=0}^{\infty} (-1)^n \mu_{2j,2}(-1, n)q^n = \frac{1}{(q; q)_\infty^2(q; q)_{\infty}^2_{m_j > m_{j-1} > \cdots > m_1 \geq 1}} \sum \frac{m_1(m_2 - m_1) \cdots (m_j - m_{j-1}) q^{m_j}}{(1 - (-q)^{m_1})(1 - (-q)^{m_2}) \cdots (1 - (-q)^{m_j})}.
\]

Given \( m_j > m_{j-1} > \cdots > m_1 \geq 1 \), define

\[(3.6) \quad \sum_{m=0}^{\infty} g_{m_1, m_2, \ldots, m_j}(m) q^m := \frac{(-q; q^2)_{\infty}}{(1 - (-q)^{m_1})(1 - (-q)^{m_2}) \cdots (1 - (-q)^{m_j})}. \]

For each \( m_i \), we discuss by two cases according to the parity: (i) \( m_i \) is odd, then such factors \((1 - (-q)^{m_i})\) in the denominator will all be cancelled by \((-q; q^2)_{\infty}\) since the \( m_i \) are distinct; (ii) \( m_i \) is even, then the coefficients of \( 1/(1 - (-q)^{m_i}) \) are all nonnegative. Thus we arrive at \( g_{m_1, m_2, \ldots, m_k}(m) \geq 0 \) and \( g_{m_1, m_2, \ldots, m_k}(0) = 1 \), together with the factor \( q^{m_j}, m_j \geq j \), we can deduce (3.2).

Similarly, replacing \( q \) by \(-q\) in (3.5) and taking \( k = 3 \), we find that

\[
\sum_{n=0}^{\infty} (-1)^n \mu_{2j,3}(-1, n)q^n = \frac{1}{(-q; q)_{\infty}(q; q)_{\infty}^2_{m_j > m_{j-1} > \cdots > m_1 \geq 1}} \sum \frac{m_1(m_2 - m_1) \cdots (m_j - m_{j-1}) q^{m_j}}{(1 - (-q)^{m_1})(1 - (-q)^{m_2}) \cdots (1 - (-q)^{m_j})}.
\]

The rest of the proof is similar by analysing (3.6) and is omitted. \( \square \)
4. Final Remarks

We conclude with several questions that merit further investigation.

1) Atkin and Garvan [7] defined the \( j \)th moment of the crank by

\[
M_j(n) = \sum_{m=-n}^{n} m^j M(m, n),
\]

then they reproved the following beautiful identity

\[
\sum_{m=-n}^{n} m^2 M(m, n) = 2np(n),
\]

due to Dyson [14], who gave a combinatorial proof.

Following the same line as the proof of Atkin and Garvan and by applying the chain rule for taking derivative, we also obtain an analogue of \((4.1)\) for \(k\)-colored partitions. For all positive integer \(k\), we have

\[
\sum_{m=-n}^{n} m^2 M_k(m, n) = \frac{2}{k} np_k(n),
\]

where \(p_k(n)\) denotes the number of \(k\)-colored partitions of \(n\). This is equivalent to saying the mean-square \(k\)-crank of \(k\)-colored partitions of \(n\) is exactly \(2n/k\). It should be clear that the right hand side of \((4.2)\) is always an integer, and it would be appealing to find a combinatorial proof for this identity.

2) Lewis [23] showed that

\[
N(0, 2, 2n) < N(1, 2, 2n), \quad \text{if } n \neq 1,
\]
\[
N(1, 2, 2n + 1) < N(0, 2, 2n + 1), \quad \text{if } n \neq 0,
\]

where \(N(r, m, n)\) denotes the number of partitions of \(n\) with rank congruent to \(r\) modulo \(m\). His proof is combinatorial (bijective) in nature and consists of the construction of maps

\[
\{\text{partitions of } 2n \text{ of even rank}\} \rightarrow \{\text{partitions of } 2n \text{ of odd rank}\}, \quad \text{and}
\]
\[
\{\text{partitions of } 2n + 1 \text{ of odd rank}\} \rightarrow \{\text{partitions of } 2n + 1 \text{ of even rank}\},
\]

that are injective, but not surjective. One naturally wonders if analogous combinatorial analysis can be applied to prove Theorem 1.4, or in general any one of the inequalities presented in Section 2.

3) When \(k \geq 4\), \((-1)^n \mu_{2j,k}(-1, n)\) do not possess the positivity property as \((-1)^n \mu_{2j,2}(-1, n)\) and \((-1)^n \mu_{2j,3}(-1, n)\). However, they may have other interesting properties. On the other hand, there are many more rank and crank identities. For example,

\[
M(0, 8, 4n + 1) + M(1, 8, 4n + 1) = M(3, 8, 4n + 1) + M(4, 8, 4n + 1),
\]
Table 1. A Table of values of $M_2(m, n)$

| $n \setminus m$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 |
|-----------------|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0               | 1  |    |    |    |    |    |    |    |    |    |    |    |    |
| 1               | 0  | 1  |    |    |    |    |    |    |    |    |    |    |    |
| 2               | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |
| 3               | 2  | 2  | 1  | 1  |    |    |    |    |    |    |    |    |    |
| 4               | 4  | 3  | 3  | 1  | 1  |    |    |    |    |    |    |    |    |
| 5               | 6  | 6  | 4  | 3  | 1  | 1  |    |    |    |    |    |    |    |
| 6               | 11 | 9  | 8  | 5  | 3  | 1  | 1  |    |    |    |    |    |    |
| 7               | 16 | 16 | 12 | 5  | 3  | 1  | 1  |    |    |    |    |    |    |
| 8               | 27 | 24 | 21 | 14 | 10 | 5  | 3  | 1  | 1  |    |    |    |    |
| 9               | 40 | 39 | 31 | 25 | 15 | 10 | 5  | 3  | 1  | 1  |    |    |    |
| 10              | 63 | 59 | 51 | 37 | 27 | 15 | 10 | 5  | 3  | 1  | 1  |    |    |
| 11              | 92 | 90 | 75 | 60 | 41 | 28 | 16 | 10 | 5  | 3  | 1  | 1  |    |
| 12              | 141| 131| 116| 90 | 67 | 43 | 29 | 16 | 10 | 5  | 3  | 1  | 1  |

and others for moduli 5, 7, 8, 9, 10 and 11. The readers are referred to [16,17,24] and the references therein for more details. It would be interesting to find similar identities for $k$-crank.

4) A sequence of numbers $a_1, a_2, \cdots, a_n$ is unimodal if it never increases after the first time it decreases, i.e., if for some index $j$ we have $a_1 \leq a_2 \leq \cdots a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots \geq a_n$. The numerical evidence (see Table 1) suggests the following conjecture.

**Conjecture 4.1.** For $n \geq 0$ and $k \geq 2$, the sequence $\{M_k(m, n)\}_{n=-n}^n$ is unimodal except for $n = 1, k = 2$.

Thanks to the symmetry (1.3), we only need to prove

(4.3) \[ M_k(m, n) \geq M_k(m + 1, n) \text{ for } 0 \leq m \leq n. \]

Note that for $k \geq 3$, and fix $0 \leq m \leq n$, we have

\[ M_k(m, n) = \sum_{j=m}^{n} M_{k-1}(m, j)p(n - j), \]

so it will suffice to prove this unimodality for $k = 2$. Moreover, using two easily described maps we can prove that for $0 \leq m \leq n$,

\[ M_2(m, n) \leq M_2(m, n + 1), \]
\[ M_2(i, i + m) < M_2(i + 1, i + m + 1), \text{ for } 0 \leq i \leq m - 1, \]
\[ M_2(m, 2m) = M_2(m + i, 2m + i), \text{ for } i \geq 0. \]

Consequently we establish “half” of (4.3), i.e.,

\[ M_k(m, n) \geq M_k(m + 1, n) \text{ for } \lfloor n/2 \rfloor \leq m \leq n. \]
Unlike other properties shared by both crank and $k$-crank, this unimodality is not true for crank. For example,

$$M(n, n) = M(n - 2, n) = 1 \text{ and } M(n - 1, n) = 0, \text{ for all } n \geq 4.$$ 

Lastly, we note that the asymptotic formula obtained by Bringmann and Manschot [11, Corollary 1.3] makes it plausible to give a computer-aided proof of Conjecture 4.1, but it would still be interesting to seek for a combinatorial proof.

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