Argument shift method and sectional operators: applications to differential geometry

Alexey Bolsinov
School of Mathematics, Loughborough University, LE11 3TU, UK
Moscow State University, Moscow, Russia
February 9, 2016

What is this paper about?

This text does not contain any new results, it is just an attempt to present, in a systematic way, one construction which makes it possible to use some ideas and notions well-known in the theory of integrable systems on Lie algebras to a rather different area of mathematics related to the study of projectively equivalent Riemannian and pseudo-Riemannian metrics. The main observation can be formulated, yet without going into details, as follows:

The curvature tensors of projectively equivalent metrics coincide with the Hamiltonians of multi-dimensional rigid bodies.

Such a relationship seems to be quite interesting and may apparently have further applications in differential geometry. The wish to talk about this relation itself (rather than some new results) was one of motivations for this paper.

The other motivation was to draw reader’s attention to the argument shift method developed by A. S. Mischenko ana A. T. Fomenko [49] as a generalisation of S. V. Manakov’s construction [44]. This method is, in my opinion, a very simple, natural and universal construction which, due to its simplicity, naturality and universality, occurs in different areas of modern mathematics. There are only a few constructions in mathematics of this kind. In this paper, the argument shift method is only briefly mentioned but the main subject, the so-called sectional operators, is directly related to it.

We discuss some new results obtained in our three papers [9, 10, 14]. In this sense, the present work can be considered as a review, but I would like to shift the accent from the results to the way how using algebraic properties of sectional operators helps in solving geometric problems. That is why the exposition is essentially different from the above mentioned papers and details of the proofs, which are not directly related to our main subjects, are omitted. Two first sections are devoted to the definition and properties of sectional operators, in the following four, we discuss their applications in geometry. Also I would like to especially mention the note [8], in which we discussed the properties of sectional operators in a very general setting and which was conceptually very helpful for this paper. I am very grateful to all of my coauthors, Vladimir Matveev, Volodymyr Kiosak, Dragomir Tsonev, Stefan Roseman and Andrey Konyaev.
I wish to express special thanks to my teacher, Anatoly Timofeevich Fomenko, without whom this work would never appear.

1 Sectional operators on semisimple Lie algebras

We start with a brief overview on (one special type of) integrable Euler equations on semisimple Lie algebras (more details on this subject can be found in [6, 15, 26, 27, 47, 49, 51, 53, 60]).

Let \( g \) be a semisimple Lie algebra, \( R : g \rightarrow g \) an operator symmetric with respect to the Killing form \( \langle \cdot, \cdot \rangle \) on \( g \). The differential equation

\[
\dot{x} = [R(x), x], \quad x \in g,
\]

is Hamiltonian on \( g \) with respect to the standard Lie-Poisson structure and is called the Euler equation related to the Hamiltonian function \( H(x) = \frac{1}{2} \langle R(x), x \rangle \).

A classical, interesting and extremely difficult problem is to find those operators \( R : g \rightarrow g \) for which the system (1) is completely integrable.

One of such operators was discovered by S. Manakov in [44] and his idea then led to an elegant general construction developed by A. Mischenko and A. Fomenko [49], called the argument shift method and having many remarkable applications. In brief this construction for semisimple Lie algebras can be presented as follows.

Assume that \( R : g \rightarrow g \) satisfies the following identity

\[
[R(x), a] = [x, b], \quad x \in g,
\]

for some fixed \( a, b \in g, a \neq 0 \). Then the following statement holds

**Theorem 1 (A. Mischenko, A. Fomenko [49]).** Let \( R : g \rightarrow g \) be symmetric and satisfy (2). Then

- the system (1) admits the following Lax representation with a parameter:
  \[
  \frac{d}{dt} (x + \lambda a) = [R(x) + \lambda b, x + \lambda a];
  \]
- the functions \( f(x + \lambda a) \), where \( f : g \rightarrow \mathbb{R} \) is an invariant of the adjoint representation, are first integrals of (1) for any \( \lambda \in \mathbb{R} \) and, moreover, these integrals commute;
- if \( a \in g \) is regular, then (1) is completely integrable.

This construction has a very important particular case. If the Lie algebra \( g \) admits a \( \mathbb{Z}_2 \)-grading, i.e., a decomposition \( g = h + v \) (direct sum of subspaces) such that \([h, h] \subset h, [h, v] \subset v, [v, v] \subset h\), then we may consider \( R : h \rightarrow h \) satisfying (2) with \( a, b \in v \), and Theorem 1 still holds if we replace \( g \) by \( h \).

The most important example for applications (in particular, in the theory of integrable tops) is \( g = \text{sl}(n, \mathbb{R}), h = \text{so}(n, \mathbb{R}) \), with \( a \) and \( b \) symmetric matrices. This is the situation that was studied in the pioneering work by S. Manakov [44] leading to integrability of the Euler equations of \( n \)-dimensional rigid body dynamics.

From the algebraic point of view, the above construction still makes sense if we replace \( \text{so}(n) \) by \( \text{so}(p, q) \) and assume \( a, b \) to be symmetric operators with respect to the
corresponding indefinite form \( g \). Moreover, if we complexify our considerations we do not even notice any difference. However, to indicate the presence (but not influence) of the bilinear form \( g \), we shall denote the space of \( g \)-symmetric operators by \( \text{Sym}(g) \), and the Lie algebra of \( g \)-skew-symmetric operators by \( \text{so}(g) \).

**Definition 1.** We shall say that \( R : \text{so}(g) \to \text{so}(g) \) is a sectional operator associated with \( A, B \in \text{Sym}(g) \), if \( R \) is symmetric with respect to the Killing form and the following identity holds:

\[
[R(X), A] = [X, B], \quad \text{for all } X \in \text{so}(g).
\]  

We follow the terminology introduced by Fomenko and Trofimov in [26, 58] where they studied various generalisations of such operators (see also [49], [8]). Strictly speaking, the above definition is just a particular case of a more general construction. The term “sectional” was motivated by the following reason. The identities (2) and (3) suggest that one may represent \( R \) as \( \text{ad}^{-1}_A \text{ad}_B \), but in general we cannot do so because \( \text{ad}_A \) as a rule, is not invertible. That is why the operator \( R \) splits into different parts each of which acts independently on its own subspace (section). A similar partition of \( R \) into “sections” will be seen in the proof of Proposition 4 below.

**Remark 1.** In fact, there are two different types of sectional operators defined respectively by (2) and (3). In this paper we focus on those defined by (3) (Definition 1). The first type, in some sense more natural and fundamental, was introduced and studied by Mischenko and Fomenko in [49, 50]. Traditionally, the operators from this class are denoted by \( \varphi_{a,b,D} \), they possess many interesting properties and applications too, and we refer to [5, 7, 39, 40, 58] for examples and details.

In the next section we discuss basic properties of sectional operators in the sense of Definition 1.

## 2 Algebraic properties of sectional operators

The first property is well-known.

**Proposition 1.** Let \( R \) be a sectional operator associated with \( A, B \in \text{Sym}(g) \), i.e., satisfy (3) for all \( X \in \text{so}(g) \). Then \( A \) and \( B \) commute. Moreover, \( B \) lies in the center of the centralizer of \( A \). In particular, \( B \) can be written as \( B = p(A) \) for some polynomial \( p(\cdot) \).

**Proof.** Indeed, \( \langle [B, A], X \rangle = \langle A, [X, B] \rangle = \langle A, [R(X), A] \rangle = \langle [A, A], R(X) \rangle = 0 \) for any \( X \in \text{so}(g) \), so \( [A, B] = 0 \). Moreover, if instead of \( A \) we substitute any element \( \xi \) from its centralizer \( \mathfrak{z}_A = \{ Y \mid [Y, A] = 0 \} \), we obviously get the same conclusion \( [B, \xi] = 0 \), i.e., \( B \) lies in the center of the centralizer of \( A \). Here, by \( \langle , \rangle \) we understand the usual invariant form \( \langle X, Y \rangle = \text{tr} XY \).

The representation of \( B \) as a polynomial in \( A \) is a standard fact from matrix algebra: the centre of the centraliser of every square matrix \( A \) is generated by its powers \( A^k \), \( k = 0, 1, 2, \ldots \).

Given \( A \) and \( B = p(A) \), can \( R \) be reconstructed from the relation (3)? Let \( R_1 \) and \( R_2 \) be two operators satisfying (3). Then we have

\[
[R_1(X) - R_2(X), A] = 0,
\]
which means that the image of \( R_1 - R_2 \) belongs to the centraliser of \( A \) in \( \mathfrak{so}(g) \), i.e. the subalgebra

\[
\mathfrak{g}_A = \{ Y \in \mathfrak{so}(g) \mid [Y, A] = 0 \}.
\]

In other words, we see that \( R \) can be reconstructed from (3) up to an arbitrary operator with the image in \( \mathfrak{g}_A \).

Also we notice that \( \mathfrak{g}_A \) is an invariant subspace for \( R \). Indeed, if \( X \in \mathfrak{g}_A \), then \( X \) commute with \( B = p(A) \), therefore the right hand side of (3) vanishes and we have \([R(X), A] = 0\), i.e. \( R(X) \in \mathfrak{g}_A \).

From the algebraic viewpoint, these two properties mean that the induced operator

\[
\tilde{R} : \mathfrak{so}(g)/\mathfrak{g}_A \to \mathfrak{so}(g)/\mathfrak{g}_A
\]

is well defined and can be uniquely reconstructed from (3).

**Remark 2.** As an important particular case, assume that \( A \) is regular in the sense of the adjoint representation, i.e., the centraliser \( \mathfrak{z}_A \) of \( A \) has minimal possible dimension.

It is well known that in this case the centraliser of \( A \) is generated by its powers \( A^k \). Hence \( \mathfrak{g}_A = \mathfrak{z}_A \cap \mathfrak{so}(g) \) is trivial, as all the elements of \( \mathfrak{z}_A \) are \( g \)-symmetric, whereas \( \mathfrak{so}(g) \) consists of \( g \)-skew-symmetric matrices. Therefore the sectional operator \( R \) can be uniquely reconstructed from \( A \) and \( B \). Namely, \( R(X) = \text{ad}^{-1}_A \text{ad}_B(x) \), a well-known formula in the theory of integrable systems on Lie algebras [49].

It is interesting to notice that in the general case (i.e., for arbitrary \( A \)), there is another explicit formula for a partial “solution” of (3).

**Proposition 2.** Let \( B = p(A) \), then

\[
R_0(X) = \frac{d}{dt} \bigg|_{t=0} p(A + tX)
\]

is a sectional operator associated with \( A \) and \( B \). In particular, if \( A \) is regular, then \( R_0(X) = \text{ad}^{-1}_A \text{ad}_B(X) \) and it is a unique solution of (3).

*Proof.* Indeed, \([p(A + tX), A + tX] = 0\) and differentiating w.r.t. to \( t \) gives

\[
0 = \frac{d}{dt} \bigg|_{t=0} [p(A + tX), A + tX] = \left[ \frac{d}{dt} \bigg|_{t=0} p(A + tX), A \right] + [p(A), X],
\]

i.e., \([\frac{d}{dt} \bigg|_{t=0} p(A + tX), A] = [X, B] \), as required.

However, we also need to check that \( R_0(X) \in \mathfrak{so}(g) \), i.e., \( R_0(X)^* = -R_0(X) \), where \( * \) denotes “g–adjoint”:

\[
g(L^*u, v) = g(u, Lv), \quad u, v \in V.
\]

Since \( A^* = A, X^* = -X \), \( (p(A + tX))^* = p(A^* + tX^*) \) and ”\( \frac{d}{dt} \)” and ”\( * \)” commute, we have

\[
R_0(X)^* = \frac{d}{dt} \bigg|_{t=0} p(A + tX)^* = \frac{d}{dt} \bigg|_{t=0} p(A^* + tX^*) = \frac{d}{dt} \bigg|_{t=0} p(A - tX) = -\frac{d}{dt} \bigg|_{t=0} p(A + tX) = -R_0(X),
\]

as needed. Thus, \( R_0(X) \in \mathfrak{so}(g) \).
Finally, we check that \( R_0 \) is symmetric with respect to the Killing form. Since the Killing form on \( so(g) \) is proportional to a simpler invariant form \( \langle X, Y \rangle = \text{tr} XY \), we will use the latter for our verification. Without loss of generality we may assume that \( p(A) = A^k \) (the general case follows by linearity). Then

\[
R_0(X) = \frac{d}{dt} \bigg|_{t=0} (A + tX)^k = A^{k-1}X + A^{k-2}XA + \cdots + AXA^{k-2} + XA^{k-1},
\]

and we have

\[
\langle R_0(X), Y \rangle = \text{tr} \left( (A^{k-1}X + A^{k-2}XA + \cdots + AXA^{k-2} + XA^{k-1}) \cdot Y \right) = \text{tr} \left( X \cdot (YA^{k-1} + AYA^{k-2} + \cdots + A^{k-2}YA + A^{k-1}Y) \right) = \langle X, R_0(Y) \rangle,
\]

as required.

Another interesting property of sectional operators is that they satisfy the Bianchi identity. To see this, we use the following natural identification of \( \Lambda^2V \) and \( so(g) \):

\[
\Lambda^2V \leftrightarrow so(g), \quad v \wedge u = v \otimes g(u) - u \otimes g(v).
\] (6)

Here the bilinear form \( g \) is understood as an isomorphism \( g : V \to V^* \) between “vectors” and “covectors”. Taking into account this identification, we have the following

**Proposition 3.** \( R_0 \) defined by (5) satisfies the Bianchi identity, i.e.

\[
R_0(u \wedge v)w + R_0(v \wedge w)u + R_0(w \wedge u)v = 0 \quad \text{for all } u, v, w \in V.
\]

**Proof.** It is easy to see that our operator \( R_0 : \Lambda^2V \simeq so(g) \to so(g) \) can be written as \( R_0(X) = \sum_k C_k XD_k \), where \( C_k \) and \( D_k \) are some \( g \)-symmetric operators (in our case these operators are some powers of \( A \)). Thus, it is sufficient to check the Bianchi identity for operators of the form \( X \mapsto CXD \).

For \( X = u \wedge v \) we have

\[
C(u \wedge v)Dw = C(u \cdot g(v, Dw) - C(v \cdot g(u, Dw)
\]

Similarly, if we cyclically permute \( u, v \) and \( w \):

\[
C(v \wedge w)Du = C(v \cdot g(w, Du) - C(w \cdot g(v, Du)
\]

and

\[
C(w \wedge u)Dw = C(w \cdot g(u, Dw) - C(u \cdot g(w, Dw).\]

Summing these three expressions and taking into account that \( C \) and \( D \) are \( g \)-symmetric, we obtain zero, as required.

One more useful property is related to the case when \( B = p(A) = 0 \), for instance if \( p(\cdot) = p_{\text{min}}(\cdot) \) is the minimal polynomial for \( A \). This case seems to be meaningless, but the operator \( R_0(X) \) defined by (5) turns out to be non-trivial (as the derivative of \( p_{\text{min}}(\cdot) \) does not vanish!).

**Proposition 4.** Let \( p(A) = 0 \), then the image of \( R_0 \) defined by (5) is contained in \( g_A \), the centraliser of \( A \) in \( so(g) \):

\[
R_0(X) \in g_A = \{ Y \in so(g) \mid [Y, A] = 0 \}.
\]
Proof. We know from Proposition 2, that $R_0$ satisfies the relation $[R_0(X), A] = [X, p(A)]$. Since $p(A) = 0$, we get $R_0(X) \in \mathfrak{g}_A$. \qed

Remark 3. Does the image of $R_0(X) = \frac{d}{dt} p_{\min}(A + tX)|_{t=0}$ coincide with $\mathfrak{g}_A$? The answer depends on the structure of Jordan blocks related to each eigenvalue of $A$. Recall first of all that for a regular matrix $A$, the subalgebra $\mathfrak{g}_A$ is trivial, so the question becomes interesting only for singular $A$'s. A straightforward computation shows that for semisimple $A$ we have $\text{Im} R_0 = \mathfrak{g}_A$. This property still holds in a more general situation if in addition we assume that each eigenvalue $\lambda$ of $A$ admits at most two Jordan blocks. More precisely, the “bad” situation is when $\lambda$ admits more than 2 blocks of a non-maximal size. For example, if $\lambda$ has several $\lambda$-blocks of size $k$ and one $m < k$, then we still have $\text{Im} R_0 = \mathfrak{g}_A$.

As shown above, $R$ can be reconstructed from $A$ and $B$ modulo operators with images in $\mathfrak{g}_A$. It is natural to ask a converse question. Given a sectional operator $R : \text{so}(g) \to \text{so}(g)$, can we reconstruct $A$ and $B$?

Proposition 5. Assume that $R : \text{so}(g) \to \text{so}(g)$ is symmetric and satisfies at the same time two identities:

$$[R(X), A] = [X, B] \quad \text{and} \quad [R(X), A'] = [X, B'],$$

with $A, B, A', B' \in \text{Sym}(g)$. If $A$ and $A'$ are not proportional (modulo the identity matrix), then $B$ is proportional to $A$ and, therefore, $[R(X) - k \cdot X, A] = 0$ for some $k \in \mathbb{R}$. Moreover, if $A$ is regular, then $R = k \cdot \text{id}$.

Proof. Notice that adding scalar matrices to $A$ or $B$ does not change the equation, so we consider $A \mapsto A + c \cdot \text{Id}$ and $B \mapsto B + c \cdot \text{Id}$ as trivial transformations. Without loss of generality we may then assume all these operators $A, A', B, B'$ to be trace free. Moreover, we are allowed to complexify all the objects so that instead of $\text{so}(g)$ and $\text{Sym}(g)$ we may simply consider the spaces of symmetric and skew-symmetric complex matrices.

Let $y$ and $z$ be arbitrary symmetric matrices, then $[A', y], [A, z] \in \text{so}(g)$ and we have:

$$[R([A', y]), A] = [[A', y], B], \quad [R([A, z]), A'] = [[A, z], B'].$$

Since $R$ is symmetric with respect to the Killing form $\langle \cdot, \cdot \rangle$ we have

$$\langle [[A', y], B], z \rangle = \langle [R([A', y]), A], z \rangle = \langle R([A', y]), [A, z] \rangle = \langle [A', y], R([A, z]) \rangle = \langle y, [R([A, z]), A'] \rangle = \langle y, [[A, z], B'] \rangle = \langle [[B', y], A], z \rangle$$

Since $z$ is an arbitrary symmetric matrix, we conclude that

$$[[A', y], B] = [[B', y], A].$$

Similarly, $[[A, y], B'] = [[B, y], A']$. Using the Jacobi identity, it is not hard to see that $[B, A'] = [A, B']$. Rewriting (8) as

$$y(B'A - A'B) + (AB' - BA')y = B'yA + AyB' - ByA' - A'yB$$
and noticing that \([B, A'] = [A, B']\) implies \(B'A - A'B = AB' - BA'\), we get
\[
yT + Ty = B'yA + AyB' - ByA' - A'yB
\]
where \(T\) denotes \(AB' - BA'\).

This formula can be understood as a relation between two linear operators acting on the space of symmetric matrices \(y \in \text{Sym}(g)\). To get some consequences from this identity, we take a “kind of trace”. Recall that we consider \(A', A, B', B, y, T\) as usual symmetric (complex) matrices.

Instead of \(y\) we substitute the symmetric matrix of the form \(e_i v^\top + ve_i^\top\), where \(e_i\) and \(v\) are vector-columns (\(e_1, \ldots, e_n\) is the standard (orthonormal) basis), then apply the result to \(e_i\) and take the sum over \(i\). Here is the result:
\[
(e_i v^\top + ve_i^\top)Te_i + T(e_i v^\top + ve_i^\top)e_i = B'(e_i v^\top + ve_i^\top)Ae_i + ...
\]
\[
e_i(Tv, e_i) + v(Te_i, e_i) + Te_i(v, e_i) + Tv(e_i, e_i) = B'e_i(Av, e_i) + B'v(Ae_i, e_i) + ...
\]

Using obvious facts from Linear Algebra such as
\[
\sum_{i} (Te_i, e_i) = \text{tr } T, \quad \sum_{i} (e_i, e_i) = n, \quad \sum_{i} e_i(v, e_i) = v,
\]
we get
\[
Tv + \text{tr } T \cdot v + Tv + n \cdot Tv = B'A v + \text{tr } A \cdot B'v + ...
\]
Taking into account that \(A, A', B, B'\) are all trace free we have
\[
((n + 2)T + \text{tr } T \cdot \text{Id})v = (B'A + AB' - A'B - BA')v.
\]

Since \(v\) is arbitrary and \(T = B'A - A'B = AB' - BA'\), we finally get
\[
nT + \text{tr } T \cdot \text{Id} = 0,
\]
but this simply means that \(T = 0\). Hence we come to the identity of the form
\[
B'yA + AyB' = ByA' + A'yB. \tag{9}
\]

It remains to use the following simple statement: if \(A, B, A', B'\) are symmetric, \(A \neq 0\) and (9) holds for any symmetric \(y\), then either \(B = k \cdot A\), or \(A' = k \cdot A\) for some constant \(k \in \mathbb{R}\).

By our assumption, \(A\) and \(A'\) are not proportional, so we conclude that \(B = k \cdot A\) and therefore the identity \([R(X), A] = [X, B]\) becomes \([R(X) - k \cdot X, A] = 0\), as needed.

Now assume that \(A\) is regular. Then \(g_A = \{0\}\) (Remark 2) and \([R(X) - k \cdot X, A] = 0\) implies \(R(X) = k \cdot X\) for all \(X \in \text{so}(g)\), i.e., \(R = k \cdot \text{id}\), as was to be proved. \(\square\)

REMARK 4. A similar result for sectional operators of the first type (see Remark 1) was proved by A. Konyaev [39].

The next statement describes the eigenvalues of sectional operators. For regular and semisimple \(A\), this fact is well known, see [44, 49], and our observation is a natural generalisation of it.
Proposition 6. Let $R : \text{so}(g) \to \text{so}(g)$ be a sectional operator associated with $A$ and $B = p(A)$, where $p(\cdot)$ is a polynomial. Let $\lambda_1, \ldots, \lambda_s$ be the distinct eigenvalues of $A$. Then the numbers
$$\frac{p(\lambda_i) - p(\lambda_j)}{\lambda_i - \lambda_j}, \quad i \neq j,$$
are eigenvalues of $R$. Moreover, if $A$ possesses a non-trivial Jordan $\lambda_i$-block, then the number $p'(\lambda_i)$ is an eigenvalue of $R$ too (here $p'$ denotes the derivative of $p$).

Proof. Since $R$ is not always uniquely defined, we are not able to find all the eigenvalues of $R$ from $A$ and $B$. However, we can find some of them, namely those of the induced operator $\tilde{R}$, see (4). Clearly, the eigenvalues of $\tilde{R}$ form a part of the spectrum of $R$ and for our computations we may set $\tilde{R} = \tilde{R}_0$, where $\tilde{R}_0$ is explicitly defined by (5).

Using (5) we can easily describe a natural partition of $\text{so}(g)$ into invariant subspaces of each of which, as we shall see later, “carries” one single eigenvalue of $\tilde{R}_0$ only (some of them may accidentally coincide, but generically our invariant subspaces are exactly generalised eigenspaces of $\tilde{R}_0$).

For simplicity we shall assume that all the eigenvalues of $A : V \to V$ are real. The decomposition $V = \bigoplus_i V_{\lambda_i}$ into generalised eigenspaces of $A$ naturally induces the following decomposition of $\text{so}(g)$
$$\text{so}(g) = \bigoplus_{i \leq j} m_{ij}$$
where $m_{ij}$ (that can be understood as $V_{\lambda_i} \wedge V_{\lambda_j}$) is spanned by the operators of the form
$$v \wedge u = v \otimes g(u) - u \otimes g(v) \in \text{so}(g), \quad \text{with } v \in V_{\lambda_i}, \ u \in V_{\lambda_j},$$
where $g(u) \in V^*$ is the covector corresponding to $u \in V$ under the natural identification of $V$ and $V^*$ by means of $g$ (so that $g(v, u) = \langle v, g(u) \rangle$).

This decomposition becomes transparent in the matrix form if we use a basis adapted to the decomposition $V = \bigoplus_i V_{\lambda_i}$. Then
$$A = \begin{pmatrix} A_1 & A_2 & \cdots \\ A_2 & A_3 & \cdots \\ \vdots & \vdots & \ddots \\ \cdots & \cdots & \cdots & A_s \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_s \end{pmatrix}$$
and $\text{so}(g)$ can be written in a block form
$$\text{so}(g) = \left\{ \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1s} \\ M_{21} & M_{22} & \cdots & M_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ M_{s1} & M_{s2} & \cdots & M_{ss} \end{pmatrix} \right\},$$
where $M_{ii} \in \text{so}(g_i)$ (diagonal blocks), the blocks $M_{ij}, i < j$ (above the diagonal) are arbitrary and related to $M_{ij}'$ (below the diagonal) as $g_j M_{ij}' = -M_{ij}^\top g_i$. Then $m_{ii} = \text{so}(g_i) \subset \text{so}(g)$, i.e. consists of the diagonal block $M_{ii}$ while all the other blocks vanish and $m_{ij}$ consists of the pair of blocks $M_{ij}$ and $M_{ij}' (i < j)$, the others vanish.

The following facts can be easily verified and we omit details.
1. Each subspace $m_{ij}$ is $R_0$-invariant.

2. The restriction of $R_0$ onto $m_{ij}$ possesses a single eigenvalue, namely $\frac{p(\lambda_i) - p(\lambda_j)}{\lambda_i - \lambda_j}$.

3. The restriction of $R_0$ onto $m_{ii}$ possesses a single eigenvalue, namely $p'(\lambda_i)$.

The first is straightforward. The next is based on the following simple fact from matrix algebra. Let $B$ and $C$ be square matrices of sizes $k \times k$ and $m \times m$ respectively. Suppose that $\lambda$ and $\mu$ are eigenvalues of $B$ and $C$ respectively and $B$ and $C$ have no other eigenvalues. Then the eigenvalue of the operator $Y \mapsto CY - YB$ acting on $k \times m$-matrices $Y$, is unique and equal to $\lambda - \mu$. The third statement require some easy computation.

Thus, we know all the eigenvalues of the operator $R_0 : so(g) \to so(g)$. Recall that we are interested in the eigenvalues of the induced operator $\tilde{R}_0 : so(g)/g_A \to so(g)/g_A$. Under such a reduction, some of eigenvalues may disappear. As the last step of the proof, we are going to explain that all of them survive.

To that end, we use another fact from linear algebra (which explains which eigenvalues survive under reduction):

Let $\phi : V \to V$ be a linear operator with an invariant subspace $U \subset V$. Let $\lambda$ be an eigenvalue of $\phi$ and $V_{\lambda} \subset V$ be the generalised $\lambda$-eigenspace of $\phi$. Then $\lambda$ is an eigenvalue of the induced operator $\tilde{\phi} : V/U \to V/U$ if and only if $V_{\lambda} \not\subset U$.

Thus, in order to show that the above eigenvalues of $R_0$ survive under reduction, it is sufficient to check that $m_{ij}$ and $m_{ii}$ are not contained in $g_A$. To see this we need just to have a look at the structure of $g_A$. It can be easily checked that $g_A$ has the following block-diagonal matrix form (we use the same adapted basis as before):

$$g_A = \left\{ X = \begin{pmatrix} X_1 & & \\ & X_2 & \\ & & \ddots \end{pmatrix} , \ X_i \in g_{A_i} \right\}, \quad (10)$$

where $g_{A_i}$ is the centraliser of $A_i$ in $so(g_i)$, i.e. $g_{A_i} = \{ Y \in so(g_i) \mid YA_i = A_iY \}$. More detailed description of $g_A$ can be found in [10].

It is seen from this description that the subspace $m_{ij}$ lies "outside" $g_A$ and the intersection $m_{ij} \cap g_A$ is trivial so that $m_{ij} \not\subset g_A$.

For $m_{ii}$, the situation is different. According to its definition, $m_{ii}$ coincides with $so(g_i)$ and therefore $m_{ii}$ is contained in $g_A$ (see (10)) if and only if $m_{ii} = so(g_i) = g_{A_i}$, i.e. the matrix $A_i$ commutes with all the $g_i$-skew symmetric matrices $Y \in so(g_i)$. This happens, however, if and only if $A_i$ is a scalar matrix, i.e. $A_i = \lambda_i \cdot Id$. Otherwise, $g_{A_i}$ is strictly smaller than $so(g_i)$. According to our assumptions (see Proposition 6), $A$ possesses a non-trivial Jordan $\lambda_r$-block, i.e. $A_i$ is not scalar. Hence $m_{ii}$ is not contained in $g_A$ and therefore $\mu_i = p'(\lambda)$ is an eigenvalue of $R_0$ as needed. This completes the proof of Proposition 6.

**Remark 5.** The above results can, more or less automatically, be transferred to the case of operators $R : u(g, J) \to u(g, J)$ on the unitary Lie algebra (in formula (3) we take $X$ to be skew-hermitean and $A$ and $B$ hermitean). This case corresponds to the natural $\mathbb{Z}_2$-grading $gl(n, \mathbb{C}) = u(g, J) \oplus \text{Herm}(g, J)$.
Here is the summary of the above properties. Let $R : \text{so}(g) \to \text{so}(g)$ be a sectional operator associated with $A, B \in \text{Sym}(g)$. Then

- $A$ and $B$ commute, moreover $B = p(A)$ for some polynomial $p(\cdot)$.
- $R_0(X) = \frac{d}{dt}|_{t=0}p(A+tX)$ is a sectional operator associated with $A$ and $B = p(A)$. If $A$ is regular, then a sectional operator associated with $A$ and $B$ is unique and therefore $R = R_0$.
- $R_0$ satisfies the Bianchi identity.
- If $B = p(A) = 0$, e.g. if $p = p_{\text{min}}$ is the minimal polynomial of $A$, then the image of $R_0$ is contained in $g_A = \{ Y \in \text{so}(g) \mid [Y, A] = 0 \}$, the centraliser of $A$ in $\text{so}(g)$. Moreover, if each eigenvalue of $A$ possesses at most two Jordan blocks, then the image of $R_0$ coincides with $g_A$.
- Suppose that $R$ is, at the same time, a sectional operator for another pair $A', B' \in \text{Sym}(g)$. If $A \neq \lambda A' + \mu \text{Id}$, then $B$ is proportional to $A$ and, therefore, $[R(X) - k \cdot X, A] = 0$ for some $k \in \mathbb{R}$. Moreover, if $A$ is regular, then $R = k \cdot \text{Id}$.
- Let $\lambda_1, \ldots, \lambda_s$ be distinct eigenvalues of $A$ and $B = p(A)$. Then the numbers $\frac{p(\lambda_i) - p(\lambda_j)}{\lambda_i - \lambda_j}, i \neq j$, are eigenvalues of $R$. Moreover, if $A$ possesses a non-trivial Jordan $\lambda_i$-block, then the number $p'(\lambda_i)$ is an eigenvalue of $R$ too (here $p'$ denotes the derivative of $p$).

3 Projectively equivalent metrics: curvature tensor as a sectional operator

**Definition 2.** Two metrics $g$ and $\bar{g}$ on the same manifold $M$ are called projectively equivalent, if they have the same geodesics considered as unparametrized curves.

In the Riemannian case the local classification of projectively equivalent pairs $g$ and $\bar{g}$ was obtained by Levi-Civita in 1896 [25]. For pseudo-Riemannian metrics, this problem turned out to be much more difficult. For the most important cases, local forms for $g$ and $\bar{g}$ were obtained in [2, 34, 52], but the final solution has been obtained only recently [13, 12, 16, 17].

In analytic form, the projective equivalence condition for $g$ and $\bar{g}$ can be written in several equivalent ways. One of them is based on the $(1, 1)$–tensor $A = A(g, \bar{g})$ defined by

$$A^i_j := \frac{\det(\bar{g})}{\det(g)} \frac{1}{\bar{g}^{ik}g_{kj}}, \quad (11)$$

where $\bar{g}^{ik}$ is the contravariant inverse of $\bar{g}_{ik}$. Since the metric $\bar{g}$ can be uniquely reconstructed from $g$ and $A$, namely:

$$\bar{g}(\cdot, \cdot) = \frac{1}{\det(A)} g(A^{-1} \cdot, \cdot) \quad (12)$$

the condition that $\bar{g}$ is geodesically equivalent to $g$ can be written as a system of PDEs on the components of $A$. From the point of view of partial differential equations, $A$ is
more convenient than $\bar{g}$ as the corresponding system of partial differential equations on $A$ turns out to be linear [54]. In the index-free form, it can be written as follows (where $*$ means $g$–adjoint):

$$\nabla_u A = \frac{1}{2} (u \otimes d\text{tr} A + (u \otimes d\text{tr} A)^*).$$

(13)

**Definition 3.** We say that a (1,1)-tensor $A$ is *compatible* with $g$, if $A$ is $g$-symmetric, nondegenerate at every point and satisfies (13) at any point $x \in M$ and for all tangent vectors $u \in T_x M$.

A surprising relationship between sectional operators and geodesically equivalent metrics is explained by the following observation. Notice, first of all, that due to its algebraic symmetries (skew-symmetry with respect to $i, j$ and $k, l$ and symmetry with respect to permutation of pairs $(ij)$ and $(kl)$), the Riemann curvature tensor $R_{ijkl}$ can be naturally considered as a symmetric operator $R : \text{so}(g) \to \text{so}(g)$ (strictly speaking we need to raise indices $i$ and $k$ by means of $g$ to get the tensor of the form $R^a_{\quad i j}$).

Equivalently such an interpretation can be obtained by using identification (6) of $\Lambda^2 V$ and $\text{so}(g)$.

Thus, in the (pseudo)-Riemannian case, a curvature tensor can be understood as a linear map

$$R : \text{so}(g) \to \text{so}(g).$$

In this setting, by the way, the symmetry $R_{ijkl} = R_{klij}$ of the curvature tensor amounts to the fact that $R$ is self-adjoint w.r.t. the Killing form, and “constant curvature” means that $R = k \cdot \text{Id}$, $k = \text{const}$. So this point of view on curvature tensors is quite natural.

The following observation was made in [9].

**Theorem 2.** If $g$ and $\bar{g}$ are projectively equivalent, then the curvature tensor of $g$ considered as a linear map

$$R : \text{so}(g) \to \text{so}(g)$$

is a sectional operator, i.e., satisfies the identity

$$[R(X), A] = [X, B] \quad \text{for all } X \in \text{so}(g)$$

(14)

with $A$ defined by (11) and $B$ being the Hessian of $\frac{1}{2} \text{tr} A$, i.e. $B = \frac{1}{2} \nabla \text{grad} \text{tr} A$.

This result is, in fact, an algebraic interpretation of some equations on the components of curvature tensors of projectively equivalent metrics obtained in tensor form by A. Solodovnikov [57], see also [54].

**Proof.** Consider the compatibility condition for the PDE system (13). Namely, differentiate (13) by means of $\nabla_v$ and then compute $\nabla_v \nabla_u A - \nabla_u \nabla_v A - \nabla_{[v,u]} A$ in terms of $\text{tr} A$:

$$\nabla_v \nabla_u A - \nabla_u \nabla_v A - \nabla_{[v,u]} A = [v \otimes g(u) - u \otimes g(v), B].$$

It remains to notice that the left hand side of this identity is $[R(u \wedge v), A]$. Hence, taking into account that bi-vectors $v \wedge u = v \otimes g(u) - u \otimes g(v)$ generate $\Lambda^2 V \simeq \text{so}(g)$, we get (14) as required.

Hence we immediately obtain a strong obstruction to the existence of a projectively equivalent partner.
In order for $g$ to admit a projectively equivalent metric $\tilde{g}$ (which is not proportional to $g$, i.e., $\tilde{g} \neq \text{const} \cdot g$), the curvature tensor of $g$ must be a sectional operator for some $A \neq \text{const} \cdot \text{Id}$ and $B$.

**Remark 6.** Some other links between projectively equivalent metrics and integrable systems are discussed in [11, 45, 46].

### 4 Projectively equivalent metrics: Fubini theorem

Given a Riemannian metric $g$, how many geodesically equivalent metrics can $g$ admit? Typically, the answer is: just metrics of the form $\tilde{g} = \text{const} \cdot g$ (this can be seen, for example, from Corollary 1 which says that the algebraic structure of the curvature tensor of $g$ must be very special). Levi-Civita classification theorem gives a lot of non-trivial examples of projectively equivalent pairs $g$ and $\tilde{g}$ (more precisely, two-parameter families of such metrics). Can such a family be larger, for example, three-parametric? In the Riemannian case, the following classical result of Fubini [28, 29] clarifies the situation: if three essentially different metrics on an $(n \geq 3)$-dimensional manifold $M$ share the same unparametrized geodesics, and two of them (say, $g$ and $\tilde{g}$) are strictly nonproportional (i.e., the roots of the characteristic polynomial $\det(\tilde{g} - \lambda g)$ are all distinct) at least at one point, then they have constant sectional curvature.

Following [9], we will say that two metrics $g$ and $\tilde{g}$ are strictly nonproportional at a point $x \in M$, if the $g$-symmetric $(1,1)$-tensor $G = g^{-1}\tilde{g}$ (or equivalently, the tensor $A$ defined by (11)), is regular in the sense of Remark 2.

If one of the metrics is Riemannian, strict nonproportionality means that all eigenvalues of $G$ have multiplicity one and that was one of the key properties used by Fubini. In the pseudo-Riemannian case, this idea does not work as $G$ and $A$ may have non-trivial Jordan blocks. However, the conclusion of the Fubini theorem still holds for pseudo-Riemannian metrics.

**Theorem 3 ([9]).** Let $g$, $\tilde{g}$ and $\hat{g}$ be three geodesically equivalent metrics on a connected manifold $M^n$ of dimension $n \geq 3$. Suppose there exists a point at which $g$ and $\tilde{g}$ are strictly nonproportional, and a point at which $g$, $\tilde{g}$ and $\hat{g}$ are linearly independent. Then, the metrics $g$, $\tilde{g}$ and $\hat{g}$ have constant sectional curvature.

**Proof.** We simply use the uniqueness property for sectional operators (see Proposition 5). Assume that we have three geodesically equivalent metrics $g$, $\tilde{g}$, and $\hat{g}$ and choose a generic point $x \in M$. Then by Theorem 2, the Riemann curvature tensor $R$ of the metric $g$ at $x \in M$ satisfies at the same time two identities:

$$[R(X), A] = [X, B] \quad \text{and} \quad [R(X), A'] = [X, B'].$$

(15)

Here we assume that $A$ and $A'$ are not proportional modulo the identity matrix (otherwise we would have $\hat{g} = \lambda \tilde{g} + \mu g$ which is not the case) and $A$ is regular due to strict non-proportionality of $g$ and $\tilde{g}$. From now on, we may forget about the geometric meaning of $A, B, A', B'$ and start thinking of them as just certain $g$-symmetric operators. After this we simply apply Proposition 5 to conclude that $R = k(x) \cdot \text{id}$, i.e. the sectional curvature of $g$ is constant in all directions. The fact that this constant $k$ does not depend of a point $x$, follows from the well-known fact that if $\dim M \geq 3$ then $R = k(x) \cdot \text{id}$ implies that $k(x) = \text{const}$.  

12
This gives a proof in local setting, i.e. in a neighbourhood of a generic point where the above mentioned algebraic conditions on $A$ and $A'$ are satisfied. The fact that the set of such points is open and everywhere dense in $M$ is not obvious and needs additional arguments (see [9]).

5 New holonomy groups in pseudo-Riemannian geometry

In this section, we discuss the results and ideas developed in [10].

Let $M$ be a smooth manifold endowed with an affine symmetric connection $\nabla$. Recall that the holonomy group of $\nabla$ is a subgroup $\text{Hol}(\nabla) \subset \text{GL}(T_x M)$ that consists of the linear operators $A : T_x M \to T_x M$ being “parallel transport transformations” along closed loops $\gamma$ with $\gamma(0) = \gamma(1) = x$.

Holonomy groups were introduced by Élie Cartan in the twenties [22, 23] for the study of Riemannian symmetric spaces and since then the classification of holonomy groups has remained one of the classical problems in differential geometry. The fundamental results in this direction are due to Marcel Berger [4] who initiated the programme of classification of Riemannian and irreducible holonomy groups which was completed by D. V. Alekseevskii [1], R. Bryant [19, 20], D. Joyce [36, 37, 38], L. Schwachhöfer, S. Merkulov [48]. Very good historical surveys can be found in [21, 55].

The classification of Lorentzian holonomy groups has recently been obtained by T. Leistner [43] and A. Galaev [31]. However, in the general pseudo-Riemannian case, the complete description of holonomy groups is a very difficult problem which still remains open, and even particular examples are of interest (see [3, 18, 30, 32, 35]). We refer to [33] for more information on recent development in this field.

The following theorem describes a new series of holonomy groups on pseudo-Riemannian manifolds. As we shall see, the proof of this result essentially uses the concept and properties of sectional operators.

**Theorem 4 ([10]).** For every $g$-symmetric operator $A : V \to V$, the identity connected component $G^0_A$ of its centraliser in $\text{SO}(g)$

$$G_A = \{ X \in \text{SO}(g) \midXA = AX \}$$

is a holonomy group for a certain (pseudo)-Riemannian metric $g$.

Notice that in the Riemannian case this theorem becomes trivial: $A$ is diagonalisable and the connected component $G^0_A$ of its centraliser is isomorphic to the standard direct product $\text{SO}(k_1) \oplus \cdots \oplus \text{SO}(k_m) \subset \text{SO}(n)$, $\sum k_i \leq n$, which is, of course, a holonomy group. In the pseudo-Riemannian case, $A$ may have non-trivial Jordan blocks (moreover, any combination of Jordan blocks is allowed) and the structure of $G^0_A$ becomes more complicated.

**Proof.** We follow the traditional approach to the problem of description of holonomy groups based on the notion of a Berger (sub)algebra.

**Definition 4.** A map $R : \Lambda^2 V \to \text{gl}(V)$ is called a formal curvature tensor if it satisfies the Bianchi identity

$$R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \quad \text{for all } u, v, w \in V.$$  \hfill (16)
This definition simply means that $R$ as a tensor of type $(1, 3)$ satisfies all usual algebraic properties of curvature tensors:

$$R^m_{kij} = R^m_{kji} \quad \text{and} \quad R^m_{kij} + R^m_{ijk} + R^m_{jki} = 0.$$ 

**Definition 5.** Let $\mathfrak{h} \subset \mathfrak{gl}(V)$ be a Lie subalgebra. Consider the set of all formal curvature tensors $R : \Lambda^2 V \to \mathfrak{gl}(V)$ such that $\text{Im} \ R \subset \mathfrak{h}$:

$$\mathcal{R}(\mathfrak{h}) = \{ R : \Lambda^2 V \to \mathfrak{h} \mid R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0, \ u, v, w \in V \}.$$ 

We say that $\mathfrak{h}$ is a *Berger algebra* if it is generated as a vector space by the images of the formal curvature tensors $R \in \mathcal{R}(\mathfrak{h})$, i.e.,

$$\mathfrak{h} = \text{span}\{ R(u \wedge v) \mid R \in \mathcal{R}(\mathfrak{h}), \ u, v \in V \}.$$ 

Berger’s test (which is sometimes referred to as Berger’s criterion) is the following property of holonomy groups:

*Let $\nabla$ be a symmetric affine connection on $TM$. Then the Lie algebra $\mathfrak{hol}(\nabla)$ of its holonomy group $\text{Hol}(\nabla)$ is Berger.*

Usually the solution of the description problem for holonomy groups consists of two parts. First, one tries to describe all Lie subalgebras $\mathfrak{h} \subset \mathfrak{gl}(n, \mathbb{R})$ of a certain type satisfying Berger’s test (i.e., Berger algebras). This part is purely algebraic. The second (geometric) part is to find a suitable connection $\nabla$ for a given Berger algebra $\mathfrak{h}$ which realises $\mathfrak{h}$ as the holonomy Lie algebra, i.e., $\mathfrak{h} = \mathfrak{hol}(\nabla)$.

We follow the same scheme but will use, in addition, some ideas from projective differential geometry. As a particular case of projectively equivalent metrics $g$ and $\tilde{g}$ one can distinguish the following.

**Definition 6.** Two metrics $g$ and $\tilde{g}$ are said to be *affinely equivalent* if their geodesics coincide as parametrized curves.

It is not hard to see that this condition simply means that the Levi-Chivita connections $\nabla$ and $\tilde{\nabla}$ related to $g$ and $\tilde{g}$ are the same, i.e., $\tilde{\nabla} = \nabla$ or, equivalently,

$$\tilde{\nabla}g = 0.$$ 

If instead of $\tilde{g}$ we introduce a linear operator $A$ (i.e. tensor field of type $(1, 1)$) using the standard one-to-one correspondence $\tilde{g} \leftrightarrow A$ between symmetric bilinear forms and $g$-symmetric operators:

$$\tilde{g}(\xi, \eta) = g(A\xi, \eta),$$

then the classification of affinely equivalent pairs $g$ and $\tilde{g}$ is equivalent to the classification of pairs $g$ and $A$, where $A$ is covariantly constant w.r.t. the Levi-Civita connection $\nabla$ related to $g$.\footnote{The classification of such pairs has been recently obtained by C. Boubel [16].}

On the other hand, the existence of a covariantly constant $(1, 1)$-tensor field $A$ can be interpreted in terms of the holonomy group $\text{Hol}(\nabla)$:

*The connection $\nabla$ admits a covariantly constant $(1, 1)$-tensor field if and only if $\text{Hol}(\nabla)$ is a subgroup of the centralizer of $A$ in $\text{SO}(g)$:*

$$\text{Hol}(\nabla) \subset G_A = \{ X \in \text{SO}(g) \mid XAX^{-1} = A \}.$$
In this formula, by $A$ we understand the value of the desired $(1,1)$-tensor field at any fixed point $x_0 \in M$. Since $A$ is supposed to be covariantly constant, the choice of $x_0 \in M$ does not play any role.

It is natural to conjecture that for a generic metric $g$ satisfying $\nabla A = 0$, its holonomy group coincides with $G_A$ (or its identity component) exactly. That is just another interpretation of the statement of our theorem. In other words, we want to construct (local) examples of pseudo-Riemannian metrics that admit covariantly constant $(1,1)$-tensor fields with a given algebraic structure, and to check that their holonomy group is the largest possible, i.e. coincides with $G^0_A$. As usual, it will be more convenient to deal with the corresponding Lie algebra $\mathfrak{g}_A$.

If we formally apply Theorem 2 to affinely equivalent metrics $g$ and $\bar{g}$ (or equivalently to the pair $g, A$)\(^2\) and use the fact that $\text{tr } A = \text{const}$, we will see that $R$ satisfies a simpler equation

$$[R(X), A] = 0,$$

which, of course, directly follows from $\nabla A = 0$ and seems to make all the discussion above not relevant to our very particular situation. However, as we know from Proposition 4, formula (5) still defines a non-trivial operator, if $p(t)$ is a non-trivial polynomial satisfying $p(A) = 0$, for example, the minimal polynomial $p_{\text{min}}(\cdot)$ for $A$.

Thus, this discussion gives us a very good candidate for the role of a formal curvature tensor to verify the condition of Berger’s test. Indeed, consider the sectional operator (associated with the given $A$ and $B = 0$) defined by

$$R : \text{so}(g) \to \text{so}(g), \quad R(X) = \frac{d}{dt} p_{\text{min}}(A + tX)|_{t=0} \quad (17)$$

where

Using the natural identification (6) of $\Lambda^2 V$ with $\text{so}(g)$ and Proposition 3, we see immediately that this operator is a formal curvature tensor. According to Proposition 4, the image of this operator belongs to $\mathfrak{g}_A$ and, moreover, coincides with $\mathfrak{g}_A$ if $A$ satisfies certain algebraic conditions, in particular, if for each of eigenvalue of $A$ there are at most two Jordan blocks. In the context of Berger’s criterion, this means that under these additional assumptions on $A$, the algebra $\mathfrak{g}_A$ is Berger.

To prove this result for an arbitrary $A$, it is sufficient to use the $g$-orthogonal decomposition $V = \oplus V_\alpha$ of $V$ into invariant subspaces corresponding to the Jordan blocks $J_\lambda$’s of $A$. Such a decomposition always exists, see [41, 42], and it induces a natural partition of $\text{so}(g) = \oplus_{\alpha \leq \beta} \mathfrak{b}_{\alpha \beta}$ into invariant subspaces of $R$ (similar to the partition $\text{so}(g) = \oplus_{i \leq j} \mathfrak{m}_{ij}$ from Proposition 6 and, more precisely, a subpartition of it). After this one can continue working with each pair of Jordan blocks separately and construct an operator

$$R_{\alpha \beta} : \text{so}(g, V_\alpha \oplus V_\beta) \to \text{so}(g, V_\alpha \oplus V_\beta)$$

by using the same formula (17) with the minimal polynomial of the matrix $A|_{V_\alpha \oplus V_\beta}$ consisting just of these two Jordan blocks. This operator, $R_{\alpha \beta}$ then can be naturally extended to the whole algebra $\text{so}(g)$ by letting it to be zero on the natural complement of $\text{so}(g, V_\alpha \oplus V_\beta)$ in $\text{so}(g) = \text{so}(g, V)$.

\(^2\)The operator $A$ we use in this section is slightly different from that in Section 3, but the final conclusion will be the same.
Finally we set:

\[ R_{\text{formal}} = \sum_{\alpha \leq \beta} R_{\alpha\beta} : \text{so}(g) \to \text{so}(g), \tag{18} \]

The operator so obtained is just a block-wise modification of (17), the only difference is that now the minimal polynomial is appropriately chosen for each particular invariant subspace \( \nu_{ij} \).

**Proposition 7.** The operator \( R_{\text{formal}} \) defined by (18) is a formal curvature tensor whose image coincides with \( g_A \). In particular, the Lie algebra \( g_A \) is Berger.

The next step is a geometric realisation of this Berger algebra. In other words, for a given operator \( A : V \to V \), where \( V \) is identified with the tangent space of a manifold \( M \) at some fixed point \( x_0 \), we need to find a (pseudo)-Riemannian metric \( g \) on \( M \) and a \((1,1)\)-tensor field \( A(x) \) (with the initial condition \( A(x_0) = A \)) such that

1. \( \nabla A(x) = 0 \);
2. \( \text{hol}(\nabla) = g_A \).

Notice that the first condition guarantees that \( \text{hol}(\nabla) \subset g_A \). On the other hand, it is well known (Ambrose-Singer theorem) that the image of the curvature operator \( R_g(x_0) \) is contained in \( \text{hol}(\nabla) \). Thus, taking into account Proposition 7, the second condition can be replaced by

\[ 2'. R_g(x_0) \text{ coincides with the formal curvature tensor } R_{\text{formal}} \text{ (18)}. \]

Thus, our goal is to construct (at least one example of) \( A(x) \) and \( g(x) \) satisfying conditions 1 and 2'. To that end, we are going to use some special ansatz for \( A \) and \( g \). Namely we will assume that \( A(x) \) does not depend on \( x = (x_1, \ldots, x_k) \) at all (as was proved by A. P. Shirokov [56], such a coordinate system always exists if \( \nabla A = 0 \)), i.e.,

\[ A(x) = A = \text{const} \]

and \( g \) is quadratic in \( x \), more precisely,

\[ g_{ij}(x) = g^0_{ij} + \sum B_{ij,pq} x^p x^q \tag{19} \]

where \( B \) satisfies obvious symmetry relations, namely, \( B_{ij,pq} = B_{ji,pq} \) and \( B_{ij,pq} = B_{ij,qp} \).

Thus, our goal is to find \( B_{ij,pq} \). It will be more convenient for us to replace \( B_{ij,pq} \) with \( B^i_{j, \ pq} = g_0^{\alpha} g_0^{\beta} B_{\alpha j, \beta p} \) and consider this \( B \) as a linear map

\[ B : \text{gl}(V) \to \text{gl}(V) \text{ defined by } B(X)^i_j = B^i_{j, \ pq} X^p_j, \]

where \( V \) is understood as the tangent space at the origin \( x_0 = 0 \).

We want \( g \) defined by (19) to satisfy the following three conditions:

1. \( A \) is \( g \)-symmetric;
2. \( \nabla A = 0 \);
3. \( R_g(x_0) = R_{\text{formal}} \), where \( x_0 = 0 \) in our local coordinates.
It can be easily checked that in terms of $B$, these conditions can be rewritten respectively as

$$AB(X) = B(AX) \quad \text{for any} \quad X \in \text{gl}(V), \quad (20)$$

$$[B(X), A] + [B(X), A]^* = 0 \quad \text{for any} \quad X \in \text{gl}(V), \quad (21)$$

$$R_{\text{formal}}(X) = -B(X) + B(X)^*, \quad X \in \text{so}(g, V). \quad (22)$$

The last formula (22), in fact, shows that $B$ can be understood as the extension of $R_{\text{formal}}$ from $\text{so}(g, V)$ to $\text{gl}(V)$ (with factor $-\frac{1}{2}$). In our case, such a natural extension indeed exists and can be defined by the formal expression $B = -\frac{1}{2}R_{\text{formal}}(\otimes)$ which can be explained as follows. Assume for simplicity that $R_{\text{formal}}$ is defined by (17) with $p_{\text{min}}(t) = \sum_{m=0}^{\infty} a_m t^m$. Then $R_{\text{formal}}(X)$ can be written as

$$\frac{d}{dt} \Big|_{t=0} \left( \sum_{m=0}^{n} a_m (A + t \cdot X)^m \right) = \sum_{m=0}^{n} a_m \sum_{j=0}^{m-1} A^{m-1-j} X A^j,$$

If in this expression we formally substitute $\otimes$ instead of $X$ (and use the factor of $-\frac{1}{2}$), we obtain a desired tensor of type $(2, 2)$:

$$B = -\frac{1}{2} \cdot \sum_{m=0}^{n} a_m \sum_{j=0}^{m-1} A^{m-1-j} \otimes A^j. \quad (23)$$

Notice that $B(X)$ for $X \in \text{gl}(V)$ is obtained from this this expression by replacing back $\otimes$ with $X$. After this remark, the verification of (20), (21), (22) is straightforward\(^3\) and the realisation part is completed. However, in general, $R_{\text{formal}}$ is a combination of operators $R_{\alpha\beta}$ related to each pair of Jordan blocks of $A$. But this does not represent any serious difficulty because we can use the same idea and set $B = \sum_{\alpha\beta} B_{\alpha\beta}$ where $B_{\alpha\beta}$ are the tensors constructed from $R_{\alpha\beta}$. Since the equations (20), (21), (22) are linear in $B$ in the natural sense, the conclusion, we need, will obviously hold for the sum $B = \sum_{\alpha\beta} B_{\alpha\beta}$. Geometrically, $B_{\alpha\beta}$ defines a direct product metric $g_{\alpha\beta} \times g_{\text{flat}}$, where $g_{\alpha\beta}$ is the metric on the sum of the subspace $V_\alpha \oplus V_\beta$ corresponding to the chosen pair of Jordan blocks and $g_{\text{flat}}$ is the flat metric on the orthogonal complement to $V_\alpha \oplus V_\beta$ whose components are constant in our local coordinates. This completes the proof. \(\square\)

### 6 On the Yano-Obata conjecture for c-projective vector fields

In the paper [14] we use sectional operators for studying global properties of c-projectively equivalent metrics. I would like to briefly mention some of our observations here as they could possibly lead to further applications of sectional operators in geometry.

\(^3\)It is interesting to notice that (21) follows immediately from Proposition 4 as in its proof we did not use that fact that $X$ was skew-symmetric, the conclusion of Proposition 4 still holds for any $X \in \text{gl}(V)$. 17
Definition 7. A curve \( \gamma(t) \) on a Kähler manifold \((M, g, J)\) is called \(J\)-planar, if
\[
\nabla_{\gamma'} \dot{\gamma} = \lambda \dot{\gamma},
\]
where \( \lambda \in \mathbb{C} \) is a complex number (depending on \( t \)), or equivalently
\[
\nabla_{\gamma'} \dot{\gamma} = \alpha \dot{\gamma} + \beta J \dot{\gamma}
\]
where \( \alpha, \beta \in \mathbb{R} \), and \( J \) is the complex structure on \( M \).

Definition 8. Two Kähler metrics \( g \) and \( \hat{g} \) on a complex manifold \((M, J)\) are called \(c\)-projectively equivalent, if they have the same \(J\)-planar curves.

The properties of \(c\)-projectively equivalent metrics are in many ways similar to those of metrics that are projectively equivalent in usual sense (cf. Section 3). By analogy with (11), we can introduce a linear operator
\[
A = \left( \frac{\det \hat{g}}{\det g} \right)^{\frac{1}{2(n+1)}} \hat{g}^{-1} g,
\]
where \( n = \dim_{\mathbb{C}} M \). Equivalently, \( \hat{g} = (\det A)^{-\frac{1}{2}} g A^{-1} \). Notice that \( A \) is hermitean w.r.t. both \( g \) and \( \hat{g} \).

We will say that \( g \) and \( A \) are \(c\)-compatible, if \( A \) is hermitean and \( g \) and \( \hat{g} \) are \(c\)-projectively equivalent. The following result was proved in [24] (cf. the compatibility condition (13)).

Theorem 5. A Kähler metric \( g \) and a hermitean operator \( A \) are \(c\)-compatible if and only if
\[
\nabla_u A = \text{pr}_C(u \otimes d \text{tr} A),
\]
where \( \text{pr}_C \) denotes the orthogonal projection to the subspace of hermitean operators.

The explicit formula for \( \text{pr}_C \) is as follows: \( \text{pr}_C L = \frac{1}{4}(L + L^* + J L J + J L^* J) \).

As in the (pseudo)-Riemannian case discussed in Section 3, the curvature tensor of a Kähler metric \( g \) can be naturally considered as an operator
\[
R : u(g, J) \rightarrow u(g, J),
\]
where \( u(g, J) \) is the unitary Lie algebra associated with the metric \( g \) and complex structure \( J \). It is a remarkable fact that if \( g \) and \( A \) are \(c\)-compatible, then \( R \) satisfies the following relation
\[
[R(X), A] = [X, B] \quad \text{for all } X \in u(g, J),
\]
where \( B = \nabla \text{grad}(\text{tr}A) \). In other words, \( R \) is a sectional operator but in the sense of another Lie algebra, namely \( u(g, J) \) instead of \( \text{so}(g) \). After Theorem 2, this property does not look very surprising. Here we discuss in brief just one relatively small part of our paper [14] in order to explain how this property of \( R \) can be used in \(c\)-projective geometry.

The paper [14] concerns two problems: local description of \(c\)-projectively equivalent metrics and Yano-Obata conjecture which states that essential \(c\)-projective vector fields\(^4\) may exists on a compact Kähler manifold \( M \) only in one very special case, namely, if \( M = \mathbb{C}P^n \) with the standard Fubini–Study metric.

\(^4\)An essential \(c\)-projective vector field is defined as a vector field whose flow preserves \(J\)-planar curves but changes the connection.
The proof of the Yano-Obata conjecture is based on our local description of c-projectively equivalent metrics but the main issue is “how to pass” from local explicit formulas for $g_{ij}(x)$ (which become very special if $g$ admits an essential c-projective vector field) to global conclusions. The main difficulty is that $g_{ij}$ itself has no simple scalar invariants, like e.g. eigenvalues. However such invariants can be constructed from the curvature tensor. Indeed, if we think of $R$ as an operator defined on $\mathfrak{u}(\mathfrak{g}, J)$, then we can consider its eigenvalues as scalar functions on $M$. Since $M$ is compact, these functions must be bounded and we may try to check this condition by using our local formulas. The next problem, however, is computational: how to find explicitly the eigenvalues of such a complicated tensor as $R$? That is where the properties of sectional operators come into play. Proposition 6 (more precisely, its unitary analog proved in [14]) gives a very simple formula for the eigenvalues. Analysing these eigenvalues (explicitly found by means of Proposition 6) has been an important part of our proof.

As a conclusion, just a few words about further possible applications of sectional operators. As was pointed out in Section 1, sectional operators $R : \mathfrak{h} \to \mathfrak{h}$ can be naturally defined for any $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{v}$. The above discussion shows that in the case $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{h} = \mathfrak{u}(p, q)$, the corresponding sectional operators admit a very natural geometric interpretation. What happens for other $\mathbb{Z}_2$-grading? Do these operators relate to any interesting geometric structures?

In his recent paper [16], C. Boubel has obtained a classification of covariantly constant $(1, 1)$-tensor fields not only on pseudo-Riemannian, but also on Kähler and hyper-Kähler manifolds of arbitrary signature. Can we generalise formulas (17), (18) and (23) to construct, in a similar way, examples of Kähler and hyper-Kähler manifolds with holonomy algebras $\mathfrak{z}_A \cap \mathfrak{u}(p, q)$ and $\mathfrak{z}_A \cap \mathfrak{sp}(p, q)$?

References

[1] D. V. Alekseevskii, *Riemannian spaces with unusual holonomy groups* Funct. Anal. Appl. 2 (1968) 97–105.

[2] A. V. Aminova, *Projective transformations of pseudo-Riemannian manifolds* J. Math. Sci. (N. Y.) 113 (2003), no. 3, 367–470.

[3] L. Bérard Bergery and A. Ikemakhen, *On the holonomy of Lorentzian manifolds* Proc. of Symposia in Pure Mathematics 54, Part 2: 27–39, 1993.

[4] M. Berger, *Sur les groupes d’holonomie des variétés à connexion affine et des variétés Riemanniennes* Bull. Soc. Math. France 83 (1955) 279–330.

[5] A. M. Bloch, V. Brinzanescu, A. Iserles, J. E. Marsden, and T. S. Ratiu, *A class of integrable flows on the space of symmetric matrices* Commun. Math. Phys., 290, 2009, 399-435.

[6] O. I. Bogoyavlensky, *Integrable Euler equations on Lie algebras, arising in problems of mathematical physics* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 48 (1984), no. 5, 883–938.

[7] A. V. Bolsinov *Commutative families of functions related to consistent Poisson brackets* Acta Appl. Math., 24 (1991), 253–274.
A. V. Bolsinov, A. Yu. Konyaev *Algebraic and geometric properties of quadratic Hamiltonians determined by sectional operators* Mathematical Notes December 2011, Volume 90, Issue 5-6, pp 666-677.

A. V. Bolsinov, V. Kiosak and V. S. Matveev, *A Fubini theorem for pseudo-Riemannian geodesically equivalent metrics* J. London Math. Soc. (2) 80 (2009) 341–356.

A. Bolsinov, D. Tsonev *On a new class of holonomy groups in pseudo-Riemannian geometry* J. Diff. Geom. 97 (2014) 377–394.

A. V. Bolsinov, V. S. Matveev, *Geometrical interpretation of Benenti systems*, J. of Geometry and Physics, 44 (2003), 489–506.

A. V. Bolsinov, V. S. Matveev, *Splitting and gluing lemmas for geodesically equivalent pseudo-Riemannian metrics* Trans. Amer. Math. Soc. 363 (2011), no. 8, 4081–4107.

A. V. Bolsinov, V. S. Matveev, *Local normal forms for geodesically equivalent pseudo-Riemannian metrics*, arXiv:1301.2492v2 [math.DG], 2013.

A. V. Bolsinov, V. S. Matveev, S. Rosemann, *Local normal forms for c-projectively equivalent metrics and proof of the Yano–Obata conjecture in arbitrary signature. Proof of the projective Lichnerowicz conjecture for Lorentzian metrics*, arXiv:1510.00275v1 [math.DG], 2015.

A. V. Bolsinov, A. V. Borisov, *Compatible Poisson Brackets on Lie Algebras* Mat. Zametki, 72:1 (2002), 11–34, Mathematical Notes, 2002, 72:1, 10–30.

C. Boubel, *The algebra of the parallel endomorphisms of a germ of pseudo-Riemannian metric* arXiv:1207.6544v5 [math.DG], 2013.

C. Boubel, *On the algebra of parallel endomorphisms of a pseudo-Riemannian metric* J. Differential Geom. 99 (2015) 1, 77–123.

C. Boubel, *On the holonomy of Lorentzian metrics* Annales de la faculté des sciences de Toulouse, vol. 16, n. 3, pp. 427–475, 2007. Prépublication de l’ENS Lyon / ENS Lyon preprint no. 323 (2004).

R. Bryant, *A survey of Riemannian metrics with special holonomy groups* Proc. ICM Berkeley, Amer. Math. Soc., 505–514 (1987).

R. Bryant, *Metrics with exceptional holonomy* Ann. Math. 126 (1987), 525–576 .

R. Bryant, *Classical, exceptional, and exotic holonomies: A status report* Besse, A. L. (ed.), Actes de la table ronde de géométrie différentielle en l’honneur de Marcel Berger, Luminy, France, 12–18 juillet 1992. Soc. Math. France. Sémin. Congr. 1 (1996), 93–165.

É. Cartan, *Les groupes d’holonomie des espaces généralisés* Acta.Math. 48, 1–42 (1926) ou Oeuvres complètes, tome III, vol. 2, 997–1038.
[23] É. Cartan, *Sur une classe remarquable d’espaces de Riemann* Bull. Soc. Math. France 54, 214–264 (1926), 55, 114–134 (1927) ou Oeuvres complètes, tome I, vol. 2, 587–659.

[24] V. V. Domashev, J. Mikes, *On the theory of holomorphically projective mappings of Kählerian spaces* Mat. Zametki 23 (1978), no. 2, 297–303.

[25] T. Levi-Civita, *Sulle trasformazioni delle equazioni dinamiche* Ann. di Mat., serie 2ª, 24 (1896), 255–300.

[26] A. T. Fomenko, V. V. Trofimov, *Integrable systems on Lie Algebras and Symmetric Spaces* Gordon and Breach, 1988.

[27] A. T. Fomenko *Symplectic Geometry. Methods and Applications* Gordon and Breach, 1988. Second edition 1995.

[28] G. Fubini, *Sui gruppi transformazioni geodetiche* Mem. Acc. Torino 53 (1903), 261–313.

[29] G. Fubini, *Sulle coppie di varietà geodeticamente applicabili* Acc. Lincei 14 (1905), 678–683 (1° Sem.), 315–322 (2° Sem.).

[30] A. Galaev, *The space of curvature tensors for holonomy algebras of Lorentzian manifolds* Differential Geometry and its Applications, 22 (2005), no. 1, 1–18.

[31] A. Galaev, *Metrics that realize all Lorentzian holonomy algebras* Int. J. Geom. Methods Mod. Phys., Italy. ISSN 0219-8878, 2006, vol. (3) 2006, no. 5–6, p. 1025–1045.

[32] A. S. Galaev, *Classification of connected holonomy groups of pseudo-Kählerian manifolds of index 2* arXiv:math.DG/0405098v2, 2005.

[33] A. S. Galaev and T. Leistner, *Holonomy Groups of Lorentzian Manifolds: Classification, Examples, and Applications* in Recent Developments in Pseudo-Riemannian Geometry, ESI Lect. Math. Phys., Eur., pp. 53–96, Math. Soc., Zürich, 2008

[34] V. I. Golikov, *Geodesic mappings of gravitational fields of general type* Trudy Sem. Vektor. Tenzor. Anal., 12 (1963), 79–129.

[35] A. Ikemakhen, *Examples of indecomposable non-irreducible Lorentzian manifolds* Ann. Sci. Math. Québec 20 (1996), no. 1, 53–66.

[36] D. Joyce, *Compact Riemannian 7-manifolds with holonomy G2. I* Journal of Differential Geometry 43 (1996), 291–328.

[37] D. Joyce, *Compact Riemannian 7-manifolds with holonomy G2. II* Journal of Differential Geometry 43 (1996), 329–375.

[38] D. Joyce, *A new construction of compact 8-manifolds with holonomy Spin(7)* Journal of Differential Geometry 53 (1999), 89–130.
[39] A. Yu. Konyaev, Uniqueness of the reconstruction of the parameters of sectional operators on simple complex Lie algebras (Russian) Mat. Zametki 90 (2011), no. 3, 384–393; translation in Math. Notes 90 (2011), no. 3-4, 365–372

[40] A. Yu. Konyaev, The bifurcation diagram and discriminant of a spectral curve of integrable systems on Lie algebras (Russian) Mat. Sb. 201 (2010), no. 9, 27–60; translation in Sb. Math. 201 (2010), no. 9-10, 1273–1305

[41] P. Lancaster, L. Rodman, Canonical forms for hermitian matrix pairs under strict equivalence and congruence SIAM Review, 47(2005), 407–443.

[42] D.B. Leep, L. M. Schueller, Classification of pairs of symmetric and alternating bilinear forms Expositiones Mathematicae, 17(1999), no. 5, 395–414.

[43] T. Leistner, On the classification of Lorentzian holonomy groups J. Differential Geom. 76 (2007) no. 3, 423–484.

[44] S.V. Manakov, Note on the integration of Euler’s equation of the dynamics of an N-dimensional rigid body Funct. Anal. Appl. 11 (1976), 328-329.

[45] V.S. Matveev, P. J. Topalov, Trajectory equivalence and corresponding integrals, Regular and Chaotic Dynamics, 3 (1998), no. 2, 30–45.

[46] V.S. Matveev, P. J. Topalov, Geodesic equivalence via integrability Geometriae Dedicata 96 (2003), 91–115.

[47] J. E. Marsden, T. S. Ratiu: Introduction to Mechanics and Symmetry Springer Verlag, New York, 1999.

[48] S. Merkulov and L. Schwachhöfer, Classification of irreducible holonomies of torsion-free affine connections Ann. Math. 150 (1999) 77–149.

[49] A. S. Mishchenko, A. T. Fomenko, Euler equations on finite-dimensional Lie groups Izv. Acad. Nauk SSSR, Ser. matem. 42, no. 2, 396–415 (1978) (Russian); English translation: Math. USSR-Izv. 12 (1978), no.2, 371–389.

[50] A. S. Mishchenko and A. T. Fomenko, Integrability of Euler equations on semisimple Lie algebras Trudy Sem. Vektor. Tenzor. Anal. 19 (1979), 3–94; English transl. in Selecta Math. Soviet. 2:4 (1982), 207–291.

[51] A. M. Perelomov, Integrable systems of classical mechanics and Lie algebras. Vol. I. Birkhäuser Verlag, Basel, 1990.

[52] A. Z. Petrov, Geodesic mappings of Riemannian spaces of an indefinite metric (Russian) Uchen. Zap. Kazan. Univ., 109(1949), no. 3, 7–36.

[53] A. G. Reyman, M. A. Semenov-Tian-Shansky, Integrable systems. A group theoretic approach Institute of Computer Science, Moscow-Izhevsk, 2003 (Russian).

[54] N. S. Sinjukov, Geodesic mappings of Riemannian spaces (in Russian) “Nauka”, Moscow, 1979.
[55] L. Schwachhöfer, *Connections with irreducible holonomy representations* Adv. Math. 160 (2001), no.1, 1–80.

[56] A. P. Shirokov, *On a property of covariantly constant affinors* Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 461–464 (Russian).

[57] A. S. Solodovnikov, *Projective transformations of Riemannian spaces* Uspehi Mat. Nauk (N.S.) 11 (1956), no. 4(70), 45–116.

[58] V. V. Trofimov and A. T. Fomenko, *Dynamical systems on the orbits of linear representations of Lie groups and the complete integrability of certain hydrodynamical systems* Functional Analysis and Its Applications 17 (1983), 2–29.

[59] V. V. Trofimov and A. T. Fomenko, *Non-invariant symplectic group structures and Hamiltonian flows on symmetric spaces* Trudy Sem. Vektor. Tenzor. Anal. 21 (1983), 23–83; English transl. in Selecta Math. Soviet. 7:4 (1988), 356–414.

[60] V. V. Trofimov and A. T. Fomenko, *Algebra and geometry of integrable Hamiltonian differential equations* Faktorial, Moscow, 1995.