Projective Points Over Matrices and Their Separability Properties

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Abstract. In this article we consider topological quotients of real and complex matrices by various subgroups and their connections to spacetime structures. These spaces are naturally interpreted as projective points. In particular, we look at quotients of nonzero matrices $M_2^*(F)$ by $GL_2(F), SL_2(F), O_2(F),$ and $SO_2(F)$ and prove various results about their topological separability properties. We discuss the interesting result that, as the group we quotient by gets smaller, the separability properties of the quotient improve.

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1. Introduction

There exists a substantial literature on the relationship between various algebras (real division algebras, Clifford algebras, etc.) and relativistic physics, [7–9]. Matrix algebras play a fundamental role, and in this article we extend the connection by considering topological quotients of matrix algebras. The resulting spaces, however, are non-Hausdorff. Such spaces are not as well studied as their Hausdorff counterparts, and for good reason: Hausdorff separability is prerequisite for many other properties, such as being locally Euclidean, that are essential in much of mathematics and its applications. But this is not the case in all of mathematics and its applications, and indeed we observe a growing interest in spaces which encode information by way of the lack of separability properties. Applications of these more “exotic”
spaces continue to appear in areas as diverse as category theory, algebra, operator algebra, algebraic geometry, logic, computer science, and mathematical physics (see, e.g., [1,4,10,11,13,19] for a sampling).

In a prior article [2], the first author followed up on results by Souček [16, 17] regarding the connection between a projective space built over a matrix ring and Penrose’s Twistor Theory [12,14,15]. It was subsequently recognized that a large class of such projective spaces over matrices could be constructed and analyzed in the same spirit, yielding generalizations of the fundamental twistor correspondence [3]. These matrix projective spaces organize and unify the Grassmannians of the underlying vector space in such a way that the incidence properties of subspaces are encoded into the (lack of) separation properties of the parent matrix projective space. In that paper [3], the nonzero matrices were quotiented by the subset of invertible matrices, as that was the context. (It is worth noting that other interesting studies of projective spaces over rings have been done, such as in [6,18], but our construction differs in an essential way that leads to nonhomogeneous spaces.) Inspired by these results, we found it interesting to consider the effect of quotienting by other natural subgroups, particularly those that have known significance and applications to physics and geometry. This article presents the results of a first foray into this territory for the case of real and complex 2 × 2 nonzero matrices quotiented by the most common Lie subgroups (the general and special linear, and the regular and special unitary/orthogonal groups). We are able to obtain a complete description of these spaces and their topologies, and we find an interesting (and reasonable) relationship between the separation properties and the size of the group we are quotienting by.

It will clarify the exposition to take each topological quotient in turn. In preparation for this, let \( \mathbb{F} \) denote the real or complex numbers, \( M_2(\mathbb{F}) \) the space of 2 × 2 matrices over \( \mathbb{F} \), and \( M_2^*(\mathbb{F}) \) the subspace of nonzero matrices. We will also be interested in the general linear group \( GL_2(\mathbb{F}) \), the special linear subgroup \( SL_2(\mathbb{F}) \), the orthogonal subgroup \( O_2(\mathbb{F}) \), and the special orthogonal subgroup \( SO_2(\mathbb{F}) \). In the case \( \mathbb{F} = \mathbb{C} \), of course, the orthogonal group is called the unitary group, and it and its special counterpart are denoted \( U(2) \) and \( SU(2) \), respectively.

Let \( D(\mathbb{F}) \) denote any one of \( GL_2(\mathbb{F}), SL_2(\mathbb{F}), O_2(\mathbb{F}), \) or \( SO_2(\mathbb{F}) \). The construction of the quotient spaces \( \mathbb{P}_{D}^0(\mathbb{F}) := M_2^*(\mathbb{F})/D(\mathbb{F}) \) largely follows the standard construction of a projective space of dimension \( k \) over a field: 1. Form the free \( \mathbb{F} \)-module of rank \( k + 1 \); 2. Throw out the zero element (and anything else as dictated by the context); 3. Define equivalence classes modulo scalar multiplication by some set of invertible elements. The projective space is then the set of equivalence classes endowed with the quotient topology. When dealing with scalars that come from a noncommutative ring, we must choose between right and left scalar multiplication. The difference is one of duality and will not concern us here (see [2]). In the present article, we will scalar multiply on the right, throw out only the zero element, consider a few different sets of scalars for our quotienting set, and set \( k = 0 \) (the projective point case).

For convenience, we recall the three weakest of the separation axioms.
Definition. We say that a point $P$ is $T_0$-separable from a point $Q$ if there exists a neighborhood of $P$ excluding $Q$. We say that a pair of points is $T_0$-separable if at least one of the points is $T_0$-separable from the other. A topological space is said to be $T_0$ (or Kolmogorov) if every pair of distinct points is $T_0$-separable.

Definition. We say that a pair of points is $T_1$-separable if each point has a neighborhood excluding the other. A topological space is said to be $T_1$ (or Fréchet) if every pair of distinct points is $T_1$-separable. We note that a space is $T_1$ if and only if every singleton set is closed.

Definition. We say that a pair of points is $T_2$-separable if each point has a neighborhood excluding the other such that the two neighborhoods are disjoint. A topological space is said to be $T_2$ (or Hausdorff) if every pair of distinct points is $T_2$-separable.

2. Construction and Analysis of the Case $D = GL$

The first space we wish to draw attention to is $P^0_{GL}(F)$. It is the instance with the largest quotienting set. The case that $F = \mathbb{C}$ was studied for other purposes in [2]. Here we will summarize what we need for the present discussion, appropriately generalized for either $\mathbb{C}$ or $\mathbb{R}$.

We are looking at equivalence classes of $M_*^2(F)$ modulo right multiplication by elements of $GL_2(F)$:

$$A \sim B \iff A = BA, \quad \Lambda \in GL_2(F).$$

(1)

A singular nonzero matrix, $A$, will have rank one, and thus the columns $\vec{a}_1, \vec{a}_2$ will be multiples:

$$(\lambda \vec{a} \quad \mu \vec{a}),$$

(2)

where not both $\lambda$ and $\mu$ are zero. By using elementary matrices in $GL_2(F)$ we can zero out the second column,

$$(\vec{a} \quad 0),$$

(3)

and then the remaining freedom consists of identifying nonzero multiples:

$$(\vec{a} \quad 0) \sim (\lambda \vec{a} \quad 0).$$

(4)

Thus, the equivalence classes deriving from the nonzero singular matrices are bijective with the projective line, $FP^1$. In particular, $\mathbb{C}P^1 \cong S^2$, and $\mathbb{R}P^1 \cong S^1$.

On the other hand, all invertible elements are identified by this relation to form a single equivalence class $\ast$. What is interesting is the way that this class interacts with the rest of the quotient.

The topology induced by the quotient mapping $q : M_*^2(F) \to P^0_{GL}(F)$ on the subspace of equivalence classes with singular representative is the standard topology on $FP^1$, while any neighborhood of a singular matrix will contain nonsingular matrices, so the projective space $P^0_{GL}(F)$ is homeomorphic to the disjoint union of a 2-sphere and a dense point when $F = \mathbb{C}$, and
the disjoint union of a circle and a dense point when $\mathbb{F} = \mathbb{R}$. See Fig. 1. We write:

$$P^0_{GL}(\mathbb{C}) = S^2 \sqcup \{\ast\}. \quad (5)$$

$$P^0_{GL}(\mathbb{R}) = S^1 \sqcup \{\ast\}. \quad (6)$$

The separation properties, closed points, and closures of non-closed points of these spaces are detailed in the following proposition.

**Proposition 1.** $P^0_{GL}(\mathbb{F})$ is $T_0$, but not $T_1$ (and hence non-Hausdorff).

1. $\ast$ is $T_0$-separable from any other point in $P^0_{GL}(\mathbb{F})$, but not $T_1$-separable.
2. Each pair of points $[A] \neq [B] \in \mathbb{FP}^1$ is $T_1$-separable, but not $T_2$-separable.
3. A singleton set containing a point $[A] \in \mathbb{FP}^1$ is closed, while the closure of the dense point is $P^0_{GL}(\mathbb{F})$.

**Proof.** Since $\ast$ is dense, the closure of the singleton $\{\ast\}$ is the whole space, and so $P^0_{GL}(\mathbb{F})$ is not $T_1$. We can see that $P^0_{GL}(\mathbb{F})$ is $T_0$ by looking at basic open sets in the parent space (cf. Fig. 2) and noting that the subspace topology on the singular classes is just the usual topology of $\mathbb{FP}^1$: Every open set of $M^*_n(\mathbb{F})$ will intersect $GL_2(\mathbb{F})$, so there is no way to separate any other point of $P^0_{GL}(\mathbb{F})$ from $\ast$. However, $\ast$ can be separated from any other point by the open set $\{\ast\}$. Any two distinct singular classes can be $T_1$-separated by using separating neighborhoods in the $\mathbb{FP}^1$ topology and adding the dense point, but any two such neighborhoods will overlap in the dense point. \qed

### 3. Construction and Analysis of the Case $D = SL$

We now consider the construction of $P^0_{SL}(\mathbb{F})$ using the same procedure above, but now we quotient by the smaller group $SL_2(\mathbb{F})$. It will be convenient once again to separate out the nonsingular and singular cases.

It turns out that in the construction of $P^0_{GL}(\mathbb{F})$, in the singular case, only unit determinant matrices are needed to get unique equivalence class representatives in bijection with $\mathbb{FP}^1$. In particular, for any column vector $\vec{a}$ and scalar $\beta$, $[\vec{a}, \beta \vec{a}] \sim [\beta \vec{a}, \vec{a}]$ via the composite transformation $\begin{pmatrix} 2\beta & 1 \\ -1 & 0 \end{pmatrix}$,
Figure 2. The curve represents the nonzero singular matrices $\text{Sing}(2, \mathbb{F})$. Any neighborhood of a singular matrix must contain nonsingular matrices, so that no equivalence class can be separated from the nonsingular class $\ast$. The quotient space is not $T_1$

$[\bar{a}, \beta \bar{a}] \sim [\bar{a}, 0]$ via the shear transformation $\begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix}$, and $[\bar{a}, 0] \sim [\alpha \bar{a}, 0]$ for any non-zero scalar $\alpha$ via the squeeze transformation $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. Thus, equivalence classes are represented by non-zero vectors in $\mathbb{F}^2$ up to scalar multiplication, resulting again in $\mathbb{F}P^1$, in $\mathbb{F}P^0_{SL}(\mathbb{F})$.

For the nonsingular case we have the following lemma, which follows directly from standard results in linear algebra:

**Lemma.** The equivalence classes of $\mathbb{F}P^0_{SL}(\mathbb{F})$ having nonsingular representatives are in bijection with the nonzero scalars, $\mathbb{F}^\ast$, each class represented by the determinant of any representative matrix.

So, the difference between this case and the case $D = GL$ is that the dense point in the $GL$ case “blows up” into a punctured plane ($\mathbb{F} = \mathbb{C}$) or punctured line ($\mathbb{F} = \mathbb{R}$) when we quotient only by $SL$ matrices. Or, in reverse, when we step up from a quotient of $SL$ matrices to a quotient of $GL$ matrices, all the points of $\mathbb{F}^\ast$ get identified. The topological interest lies, again, in the interaction between singular classes and the rest of the quotient space. First, we note that the set of nonsingular matrices is open in $M^*_2(\mathbb{F})$, so that in the quotient topology of $\mathbb{F}P^0_{SL}(\mathbb{F})$, $\mathbb{F}^\ast$ is open and $\mathbb{F}P^1$ is closed. The separability properties are detailed in the following proposition.

**Proposition 2.** $\mathbb{F}P^0_{SL}(\mathbb{F})$ is $T_1$, but not $T_2$ (*i.e.*, non-Hausdorff).

1. Any point $[A] \in \mathbb{F}^\ast$ is $T_2$-separable from any other point $[B] \in \mathbb{F}P^0_{SL}(\mathbb{F})$.
2. Each pair of points $[A] \neq [B] \in \mathbb{F}P^1$ is $T_1$-separable, but not $T_2$-separable.
3. Every singleton set is closed.
Proof. For statement 1, let $A$ be an arbitrary invertible matrix, and let $B$ be an arbitrary nonzero matrix with $\det(A) \neq \det(B)$. Consider the following diagram of maps.

$$
\begin{align*}
F & \xleftarrow{\det} M_2^*(F) \xrightarrow{q} \mathbb{P}^0_{SL}(F) \\
\end{align*}
$$

Choose $\varepsilon > 0$ so that the open balls $U_{\varepsilon}(\det(A))$ and $U_{\varepsilon}(\det(B))$ are disjoint in $F$. Then since $\det : M_2^*(F) \to F$ is continuous, the inverse images, $S$ and $T$, of these balls are disjoint open neighborhoods of $A$ and $B$ (respectively). But $S$ and $T$ are saturated relative to the quotient map $q$, and so $q(S)$ and $q(T)$ are disjoint open neighborhoods separating $[A]$ and $[B]$ in $\mathbb{P}^0_{SL}(F)$.

For statement 2, consider an element $[A] \in \mathbb{F}P^1$, and without loss of generality take $A$ to have the form $\left( \begin{array}{cc} a & 0 \\ b & 0 \end{array} \right)$, and fix $\theta \in [0, 2\pi)$ if $F = \mathbb{C}$ or $\theta \in \{0, \pi\}$ if $F = \mathbb{R}$. The sequence of invertible matrices $A_n = \left( \begin{array}{cc} a & 0 \\ b & e^{i\theta} \end{array} \right)$ will then converge to $A$. (Use the sequence $\left( \begin{array}{cc} a & e^{i\theta} \\ b & 0 \end{array} \right)$ if $a = 0$.) The image of this sequence under the quotient map $q$ produces a sequence of elements $[A_n] \in \mathbb{F}^*$ converging to $[A]$. As a result, we see that every open neighborhood of an element $[A] \in \mathbb{F}P^1$ will always contain a punctured open ball about the origin in $\mathbb{F}^*$. In particular, any two open neighborhoods of elements of $\mathbb{F}P^1$ will intersect in a region of the puncture of $\mathbb{F}^*$. We conclude that no two elements of $\mathbb{F}P^1$ are $T_2$-separable. See Fig. 3.

To see that any distinct $[A]$ and $[B]$ in $\mathbb{F}P^1$ are $T_1$-separable, use the fact that $M_2^*(F)$ is a normal space to separate the double-ray $q^{-1}([A])$ from the double-ray $q^{-1}([B])$ with a union $U$ of conical open neighborhoods. It follows that $q(U)$ is an open neighborhood of $[A]$ and $[B] \notin q(U)$. Similarly, $[B]$ can be separated from $[A]$, so the elements of $\mathbb{F}P^1$ are $T_1$-separable.

The last statement follows from the equivalent reformulation of a $T_1$ space.

Therefore, we may write

$$
\begin{align*}
\mathbb{P}^0_{SL}(\mathbb{C}) &= S^2 \sqcup \mathbb{C}^*. \\
\mathbb{P}^0_{SL}(\mathbb{R}) &= S^1 \sqcup \mathbb{R}^*. 
\end{align*}
$$

The topology is such that, individually, the punctured-$\mathbb{R}^i$ part carries the subspace topology inherited from the standard $\mathbb{R}^i$ topology, and the $S^i$ part a standard $S^i$ topology (for $i = 1$ or 2). However, as parts of the quotient, the $S^i$ is “infinitely close” to the puncture in $\mathbb{R}^i$. Neighborhoods of distinct elements in $S^i$ necessarily overlap in the region of the puncture (see Fig. 3).

4. Construction and Analysis of the Case $D = O$

Next, we consider the quotient by $O_2(F)$, commonly denoted $U(2)$ when $F = \mathbb{C}$, and $O(2)$ when $F = \mathbb{R}$. The analysis for this case is nicely facilitated by using left polar decompositions. Let $A \in M_2^*(F)$. Then $A$ can be factored as...
Figure 3. A visual heuristic of the quotient in the case $\mathbb{F} = \mathbb{C}$ and some representative neighborhoods

$A = P_A U_A$, where $U_A \in O_2(\mathbb{F})$ and $P_A = \sqrt{AA^*}$ is the unique positive semi-definite square root of the (positive semi-definite) Hermitian matrix $AA^*$. (Here, $A^*$ denotes the transpose if $\mathbb{F} = \mathbb{R}$, and the conjugate transpose if $\mathbb{F} = \mathbb{C}$.)

**Proposition 3.** For $A, B \in M_2^*(\mathbb{F})$, $A \sim B$ if and only if $AA^* = BB^*$, or equivalently, $P_A = P_B$.

**Proof.** If $A \sim B$, there exists $\Lambda \in O_2(\mathbb{F})$ such that $A = B \Lambda$. Then,

$$AA^* = (BA)(BA)^* = BB^*, \tag{9}$$

and so $P_A = P_B$. Conversely, let $A, B$, and their polar decompositions be given. If $P_A = P_B$, then we may define $\Lambda = U_B^{-1} U_A$. Then,

$$BA = P_B U_B U_B^{-1} U_A = P_A U_A = A. \tag{10}$$

□

Thus, over $\mathbb{C}$, the quotient is equivalent to the set of positive semi-definite Hermitian matrices, and over $\mathbb{R}$, to the positive semi-definite symmetric matrices. It is straightforward to check that these correspondences are, in fact, homeomorphisms.

We will write

$$\mathbb{P}_O^0(\mathbb{C}) \cong H_{\geq 0}^0 = H_{> 0} \sqcup (H_{\geq 0} - H_{> 0}), \tag{11}$$

$$\mathbb{P}_O^0(\mathbb{R}) \cong S_{\geq 0}^0 = S_{> 0} \sqcup (S_{\geq 0} - S_{> 0}), \tag{12}$$

where $H_{\geq 0}$ (resp., $S_{\geq 0}$) denotes the space of positive semi-definite Hermitian (resp., symmetric) matrices, $H_{> 0}$ (resp., $S_{> 0}$) denotes the space of positive-definite Hermitian (resp., symmetric) matrices, which corresponds to the subspace of equivalences classes of nonsingular matrices, and $(H_{\geq 0} - H_{> 0})$ (resp., $(S_{\geq 0} - S_{> 0})$) corresponds to the subspace of equivalence classes of singular matrices. To see this, note that $A = P_A U_A$ is singular if and only if $P_A$ is singular, since $U_A \in O_2(\mathbb{F})$ is invertible.
A classical map from physics, then, provides a nice visualization of this quotient in real (1+3)-dimensional Minkowski space $\mathbb{R}M^4$. Consider a vector $V^\alpha$ in $\mathbb{R}M^4$ with components $(V^0, V^1, V^2, V^3)$. Define the $\Psi$ map as follows:

$$\Psi(V^\alpha) \mapsto \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}.$$ 

This is a homeomorphism between real Minkowski vectors and Hermitian matrices [12]. Notice that

$$\det \Psi(V^\alpha) = (V^0)^2 - (V^1)^2 - (V^2)^2 - (V^3)^2 = \eta_{\alpha\beta}V^\alpha V^\beta,$$

where $\eta$ is the Lorentzian inner product. That is, the $\Psi$ map sends a real Minkowski vector to a Hermitian matrix with determinant equal to the squared length of the original vector.

Positive semi-definite $2 \times 2$ matrices have non-negative determinant and positive trace, which implies that the corresponding spacetime vectors must have non-negative length and strictly positive time component, $V^0 > 0$. So, for $F = \mathbb{C}$, positive semi-definite Hermitian matrices correspond to real future pointing timelike and lightlike vectors. We denote this space as $V = V_T \sqcup V_L$,

where

$$V_T = \{ V^\alpha \in \mathbb{R}M^4 \mid \eta_{\alpha\beta}V^\alpha V^\beta > 0 \},$$

and

$$V_L = \{ V^\alpha \in \mathbb{R}M^4 \mid \eta_{\alpha\beta}V^\alpha V^\beta = 0 \}.$$

As a subspace of $\mathbb{R}M^4$, the space $V$ consists of the future pointing light cone minus the origin, corresponding to singular classes in $P_0^O(F)$, along with its interior, corresponding to nonsingular classes. (See Fig. 4.)

Moreover, for $F = \mathbb{R}$, positive semi-definite symmetric matrices correspond to the subspace of $V$ with $V^2 = 0$, the same real future-pointing timelike and lightlike vectors in a real Minkowski space one dimension smaller, $\mathbb{R}M^3$. We will denote this as $V' = V'_T \sqcup V'_L$,

where

$$V'_T = \{ V^\alpha \in \mathbb{R}M^4 \mid \eta_{\alpha\beta}V^\alpha V^\beta > 0, V^2 = 0 \},$$

and

$$V'_L = \{ V^\alpha \in \mathbb{R}M^4 \mid \eta_{\alpha\beta}V^\alpha V^\beta = 0, V^2 = 0 \}.$$

Then we have an alternative formulation of the quotient:

$$P_0^O(\mathbb{C}) \cong V = V_T \sqcup V_L$$

$$P_0^O(\mathbb{R}) \cong V' = V'_T \sqcup V'_L$$

Topologically speaking, passing from the $O_2(F)$ quotient to the $GL_2(F)$ quotient amounts to collapsing the punctured light-cone to the compact space $FP^1$, and collapsing the entire interior of that cone to a single point. Or again,
the reverse perspective “unfolds” $\mathbb{F}P^1$ into the light-cone, and “unpacks” the dense point to an open, unbounded region of Minkowski space.

The separability properties of the quotient improve rather dramatically with this new subgroup.

**Proposition 4.** $\mathbb{P}^0_O(\mathbb{F})$ is metrizable. In particular,

1. Any two points are $T_2$-separable.
2. $\mathbb{P}^0_O(\mathbb{F})$ satisfies every separation axiom.

**Proof.** From the discussion above, $\mathbb{P}^0_O(\mathbb{F})$ is homeomorphic to a subspace of the metric space $\mathbb{R}M^4$. \qed

5. Construction and Analysis of the Case $D = SO$

Finally, we consider the quotient by $SO_2(\mathbb{F})$, commonly denoted $SU(2)$ over $\mathbb{C}$ and $SO(2)$ over $\mathbb{R}$. It will turn out to be quite closely related to the $O_2(\mathbb{F})$ quotient. Much like the $SL$ quotient refines the $GL$ quotient by the determinant, so too will the $SO$ quotient refine the $O$ quotient by the determinant. To this end, note that any $2 \times 2$ unitary matrix can be factored as $U = \Phi S$, where $S \in SO_2(\mathbb{F})$, and $\Phi$ is a matrix encoding the determinant content of $U$:

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & \det(U) \end{pmatrix}.$$  \hfill (15)
Together with the polar decomposition, we may then factor any $A \in M^*_2(\mathbb{F})$ as

$$A = P_A \Phi_A S_A,$$

where $P_A$ is Hermitian and positive semi-definite, $\Phi_A$ is as defined in (15), and $S_A \in SO_2(\mathbb{F})$.

**Proposition 5.** For $A, B \in M^*_2(\mathbb{F})$, $A \sim B$ if and only if $\det(A) = \det(B)$ and $AA^* = BB^*$. Equivalently, $A \sim B$ if and only if $\Phi_A = \Phi_B$ and $P_A = P_B$.

**Proof.** Let $A, B \in M^*_2(\mathbb{F})$ be given. If $A \sim B$, so that $A = BS$ for $S \in SO_2(\mathbb{F})$, then

$$\det(A) = \det(BS) = \det(B),$$

as $\det(S) = 1$. Moreover, we know from Proposition 3 and the fact that $S \in O_2(\mathbb{F})$ that $P_A = P_B$. We note that under these conditions we must have $\det(U_A) = \det(U_B)$, i.e., $\Phi_A = \Phi_B$.

Conversely, let $\det(A) = \det(B)$ and $P_A = P_B$. Defining $\Lambda = S_B^{-1}S_A \in SO_2(\mathbb{F})$, we have

$$BA = P_B \Phi_B S_BS_B^{-1}S_A = P_A \Phi_A S_A = A.$$  \hspace{1cm} (17)

Thus $A \sim B$.

This proposition allows us to identify equivalence classes as

$$[M] = \{A \in M^*_2(\mathbb{F}) \mid \det(A) = \det(M), AA^* = MM^*\}.$$  

Let us compare these classes to those of the $O_2(\mathbb{F})$ quotient. For each $[M]_O = \{A \in M^*_2(\mathbb{F}) \mid AA^* = MM^*\}$, if any matrix in $[M]_O$ has determinant zero, then every matrix in $[M]_O$ has determinant zero, so $[M]_{SO} = [M]_O$. Thus, the $SO_2(\mathbb{F})$ equivalence classes of singular matrices correspond to real future-pointing lightlike vectors in real Minkowski space, just as in the $O_2(\mathbb{F})$ quotient. On the other hand, if $[M]_O$ is an equivalence class of nonsingular matrices, then for all $A \in [M]_O$, $\det(AA^*) = \det(MM^*)$, which means that the determinants of $A$ and $M$ have the same magnitude, even if they are not identical. Let $S_\mathbb{F}$ denote the unit circle in $\mathbb{F}$, that is the collection of scalars whose magnitude is 1. Then for every nonsingular class $[M]_O$, the pre-image of the natural projection $\pi : \mathbb{P}^0_{SO}(\mathbb{F}) \to \mathbb{P}^0_O(\mathbb{F})$ is $[M]_{SO} \times S_\mathbb{F}$.

Thus,

$$\mathbb{P}^0_{SO}(\mathbb{C}) = \mathbb{V}_L \sqcup (\mathbb{V}_T \times S^1),$$ \hspace{1cm} (18)

and

$$\mathbb{P}^0_{SO}(\mathbb{R}) = \mathbb{V'}_L \sqcup (\mathbb{V'}_T \times \{-1, 1\}).$$  \hspace{1cm} (19)

As in the step from the $GL$ quotient to the $SL$ quotient, where the dense point “blew up” to $\mathbb{F}^*$, the step from $O_2$ to $SO_2$ “unfurls” each nonsingular class to an $S_\mathbb{F}$’s worth of points, encoding determinant information about the matrices in each class. Since the determinant represents the squared length of the corresponding spacetime vectors, we may regard the spheres as having radius equal to that shared squared length. We note that this interpretation is consistent with the limiting case of a light-like vector and the lack of sphere in that case (radius zero).

The separability properties, however, have little room to improve here:
Proposition 6. \( \mathbb{P}^0_{SO}(\mathbb{F}) \) is metrizable. In particular,

1. Any two points are \( T_2 \)-separable.
2. \( \mathbb{P}^0_{SO}(\mathbb{F}) \) satisfies every separation axiom.

6. Concluding Remarks

6.1. Discussion of Results

In this article we analyzed the case of real and complex \( 2 \times 2 \) nonzero matrices quotiented by the general and special linear groups, as well as the regular and special unitary/orthogonal groups. This was inspired by the insight from previous studies\([2,16]\) that such spaces can encode relevant geometrical and physical information in their topologies. In \([2]\), it is shown that the quotient \( \mathbb{P}^0_{GL}(\mathbb{C}) \) can be identified with projective 2-spinors at a spacetime point (the projective spinors forming a sphere). On the other hand, the projective line \( \mathbb{P}^1_{GL}(\mathbb{C}) \) is constructed by considering the free, rank 2 modules built over the ring of \( 2 \times 2 \) matrices \( M_2(\mathbb{C}) \) modulo the right scalar action by \( GL_2(\mathbb{C}) \):

\[
[(A, B)] \sim [(C, D)] \iff (A, B)\Lambda = (B\Lambda A) = (C, D),
\]

where \( A, B, C, D \in M_2(\mathbb{C}) \) and \( \Lambda \in GL_2(\mathbb{C}) \). It is shown in \([16]\) that \( \mathbb{P}^1_{GL}(\mathbb{C}) \) encodes information about Penrose’s twistor correspondence\([15]\). In particular, \( \mathbb{P}^1_{GL}(\mathbb{C}) \) splits into two pieces, one which is homeomorphic to projective twistor space \( \mathbb{CP}^3 \), and another which is homeomorphic to complexified, compactified complex Minkowski spacetime \( \mathbb{CM}^c \), and the non-Hausdorff topology of \( \mathbb{P}^1_{GL}(\mathbb{C}) \) facilitates the transfer of information between the two pieces according to the twistor correspondence. In the present article, the new spaces considered result in a non-Hausdorff combination of spheres and punctured lines in the \( D = SL \) case, a future-pointing light cone in the \( D = O \) case, and a product of spheres and a future-pointing light cone in the \( D = SO \) case.

Worth noting is that as the subgroup gets smaller, the separation properties improve (see Fig. 5). In the case of \( GL \) the result is a \( T_0 \) but not \( T_1 \) space (see Prop. 1), then in the case of \( SL \) the result is a \( T_1 \) but not \( T_2 \) space (see Prop. 2), and finally in the cases of both \( O \) and \( SO \), the results are both \( T_2 \) (in fact, metrizable, see Props. 4 and 6). The separation properties improve as the subgroup gets smaller since we are identifying fewer elements, and we are able to trace this very clearly geometrically.

Moreover, when one denominator is a subgroup of another, we can understand how the quotients morph into each other by considering matrix factorizations. First consider the case \( SL_2(\mathbb{F}) \subseteq GL_2(\mathbb{F}) \). We may factor any matrix \( A \) as

\[
A = M\Lambda,
\]

where \( M \) is a \( \mathbb{P}^0_{GL}(\mathbb{F}) \) representative for \( A \), and

\[
\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}).
\]

If \( A \) is nonsingular, we may factor further as

\[
A = M\Lambda = M\Omega\Sigma,
\]
Figure 5. A lattice diagram of the four subgroups considered, with some topological properties of the quotients indicated

where $\Sigma \in SL_2(\mathbb{F})$, and where $\Omega$ effectively encodes the determinant information of $\Lambda$. Specifically, for $\mathbb{F} = \mathbb{C}$,

$$\Omega = \text{diag}(\sqrt{\det \Lambda}, \sqrt{\det \Lambda})$$

(24)

and

$$\Sigma = \frac{1}{\sqrt{\det \Lambda}} \Lambda.$$  \hspace{1cm} (25)

For $\mathbb{F} = \mathbb{R}$,

$$\Omega = \text{diag}(\sqrt{|\det(\Lambda)|}, \text{sgn}(\det(\Lambda))\sqrt{|\det \Lambda|})$$

(26)

and

$$\Sigma = \frac{1}{\sqrt{|\det \Lambda|}} \begin{pmatrix} a & b \\ \text{sgn}(\det(\Lambda))c & \text{sgn}(\det(\Lambda))d \end{pmatrix}.$$ \hspace{1cm} (27)

(Here, sgn denotes the sign function.) Thus, if we can parameterize the set of all $M \pmod{GL}$, then we have a parameterization for the nonsingular part of $\mathbb{P}^0_{GL}(\mathbb{F})$, whereas if we can parameterize the set of all $M \pmod{GL}$ and the set of all $\Omega$, then we have a parameterization of the nonsingular part of $\mathbb{P}^0_{SL}(\mathbb{F})$. The difference between the $GL$ and $SL$ cases thus lies in the $\Omega$ term, which is clearly parameterized by $\mathbb{F} - \{0\}$. Of course, as we saw above, the nonsingular part in the $\mathbb{P}^0_{GL}(\mathbb{F})$ case is a single point, whereas the nonsingular part in the $\mathbb{P}^0_{SL}(\mathbb{F})$ case was shown to be $\mathbb{F} - \{0\}$, as suggested by the factorization above. Thus, we can imagine that as we increase the denominator from $SL$ to $GL$, the nonsingular part $\mathbb{F} - \{0\}$ gets identified into a point. On the other hand, for the singular case, as shown above, we can take $\Lambda \in SL_2$ in the factorization (21), so there is no need to factor further and introduce $\Omega$. Topologically, this means that no additional identifications take place among the singular classes when passing from the $SL$ quotient to a $GL$ quotient.

Consider the case $SO_2(\mathbb{F}) \subseteq O_2(\mathbb{F})$. As we saw, we may factor a nonsingular matrix $A$ as

$$A = P_A \Phi_A S_A,$$ \hspace{1cm} (28)

where $\Phi_A$ encodes the determinant information of the orthogonal part of $A$ and is parameterized by $S^1 (\mathbb{F} = \mathbb{C})$ or $S^0 (\mathbb{F} = \mathbb{R})$. Thus, for the nonsingular part of the quotient, the effect of passing from the $SO$ quotient to the $O$
quotient is to shrink each $S^1$ or $S^0$ to a point. Upon further identification, by increasing the denominator to $GL$, the interior of the forward light cone gets identified to a single point. For the singular parts, the $SO$ and $O$ quotients result in the forward light cone. Quotienting further, using the $GL$ squeeze matrices $K = \text{diag}(k, 1/k)$, $k \neq 0$, the light rays generating the cone boundary get identified by their generators (say, a $t = 1$ cross-section of the cone boundary), resulting in a sphere\[14\]. Since the cone interior has the cone as its boundary, it becomes intuitively clear in this process that the identified interior should become dense in the $GL$ quotient space.

The remaining case is $SO_2(\mathbb{F}) \subseteq SL_2(\mathbb{F})$. For the nonsingular classes, we can glue together elements $[A]_{SO}$ in the interior of the cone $V_T$, for example, by first multiplying by the $SL_2(\mathbb{F})$ matrix $\frac{1}{\sqrt{\det A}} \text{adj}(A)$, where $\text{adj}(A)$ denotes the adjugate of $A$. (The adjugate is the unique matrix satisfying $A \text{adj}(A) = \det(A)I$.) This results in a real scalar matrix, which as a spacetime vector, necessarily lies on the $t$-axis and has positive, real square-norm $\det A$. Now, in the $SO$ quotient, the nonsingular portion is the interior of a cone crossed with a sphere (Eqs. (18) and (19)). The information in the $S^i$ part of $[A]_{SO}$ combines with $\det(A)$ as a phase to represent a point on the punctured line in the $SL$ quotient. For the singular classes, once again the $SL$ squeeze matrix $K$ can be used to projectivize the future light cone into a sphere.

6.2. Open Questions

We’ve focused on the separation properties in this article, largely since our primary interest was in the connection to twistor spaces, as mentioned in [2,3,16,17]. Nonetheless, it would be interesting to determine what other topological properties these spaces have or do not have. For example, we know that the quotient by $GL$ is compact, but the others are not, and it is straightforward to show that all quotients studied here are path connected (see Fig. 5). It would also be interesting to determine the homotopy/homology structures.

It is, of course, possible to generalize these methods to $n \times n$ matrices. The cases of $\mathbb{P}^0_{GL(n)}(\mathbb{F})$ and $\mathbb{P}^0_{SL(n)}(\mathbb{F})$ come out quite nicely, as the piece corresponding to singular classes becomes

$$Gr_{\mathbb{F}}(1, n) \sqcup Gr_{\mathbb{F}}(2, n) \sqcup \cdots \sqcup Gr_{\mathbb{F}}(n - 1, n),$$

where $Gr$ refers to the Grassmanian. However, the topologies of the $O_n(\mathbb{F})$ and $SO_n(\mathbb{F})$ quotients become quite unwieldy, and provide ample space for further exploration. Further, it is interesting that $\mathbb{P}^0_O$ should find such a natural and classical connection to spacetime, but by such a different route than in [17], which relates to Penrose’s Twistor Theory. Of course, connections between Minkowski spacetime, its conformal structure, 2-spinors, Grassmannians, and projective spaces are fundamental to Twistor Theory. So, while we don’t yet see the full picture as to how the matrix projective spaces might be utilized, either computationally or conceptually, to explore twistor-like correspondences and their application to physics (e.g., Yang–Mills Theory [5] and Twistor-String Theory [20]), these results provide ample encouragement for further examination.
In this paper we restricted to the case of projective points over matrices, but in [3], projective spaces of higher dimension (lines, planes, ...) were studied. One might think that the point case is a special case of this more general setting. However, preliminary results show that all the higher dimension cases can be subsumed by an appropriately large projective point. We hope to publish these results in a future work.

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References
[1] Abramsky, S., Jung, A.: Domain theory. In: Handbook of Logic in Computer Science, vol. 3. Handb. Log. Comput. Sci., pp. 1–168. Oxford University Press, New York (1994)
[2] Agnew, A.F.: The twistor structure of the biquaternionic projective point. Adv. Appl. Clifford Algebras 13(2), 231–240 (2003). https://doi.org/10.1007/s00006-003-0009-6
[3] Agnew, A.F., Childress, S.P.: Matrix projective spaces and twistorlike incidence structures. J. Math. Phys. 50(12), 122503 (2009). https://doi.org/10.1063/1.3271042
[4] Allison, L.: A Practical Introduction to Denotational Semantics, Cambridge Computer Science Texts, vol. 23. Cambridge University Press, Cambridge (1986)
[5] Atiyah, M.F.: Geometry of Yang–Mills fields. In: Mathematical Problems in Theoretical Physics (Proc. Internat. Conf., Univ. Rome, Rome, 1977), Lecture Notes in Physics, vol. 80, pp. 216–221. Springer, Berlin (1978)
[6] Blunck, A., Havlicek, H.: The connected components of the projective line over a ring. Adv. Geom. 1(2), 107–117 (2001). https://doi.org/10.1515/advg.2001.008

[7] Dixon, G.M.: Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics, Mathematics and its Applications, vol. 290. Kluwer Academic Publishers Group, Dordrecht (1994)

[8] Doran, C., Lasenby, A.: Geometric Algebra for Physicists. Cambridge University Press, Cambridge (2003). https://doi.org/10.1017/CBO9780511807497

[9] Girard, P.R.: Quaternions, Clifford Algebras and Relativistic Physics. Birkhäuser Verlag, Basel (2007). Translated from the 2004 French original

[10] Hájíček, P.: Causality in non-Hausdorff space-times. Commun. Math. Phys. 21, 75–84 (1971). http://projecteuclid.org/euclid.cmp/1103857261

[11] Hochster, M.: Prime ideal structure in commutative rings. Trans. Am. Math. Soc. 142, 43–60 (1969). https://doi.org/10.2307/1995344

[12] Huggett, S.A., Tod, K.P.: An introduction to twistor theory, London Mathematical Society Student Texts, vol. 4, 2nd edn. Cambridge University Press, Cambridge (1994). https://doi.org/10.1017/CBO9780511624018

[13] Pedicchio, M.C., Tholen, W. (eds.): Categorical foundations, Encyclopedia of Mathematics and its Applications, vol. 97. Cambridge University Press, Cambridge (2004). Special topics in order, topology, algebra, and sheaf theory

[14] Penrose, R., Rindler, W.: Spinors and Space-time, vol. 1. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge (1984). https://doi.org/10.1017/CBO9780511564048. Two-spinor calculus and relativistic fields

[15] Penrose, R., Rindler, W.: Spinors and Space-time, vol. 2. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge (1986). https://doi.org/10.1017/CBO9780511524486. Spinor and twistor methods in space-time geometry

[16] Souček, V.: The conformal group action on $\mathbb{P}^1(\mathbb{C})$. Twistor Newsl. 13, 22–25 (1981)

[17] Souček, V.: Complex quaternions, their connection to twistor theory. Czech. J. Phys. B 32(6), 688–691 (1982). https://doi.org/10.1007/BF01596718

[18] Thas, J.A.: The $m$-dimensional projective space $S_m(M_n(GF(q)))$ over the total matrix algebra $M_n(GF(q))$ of the $n \times n$-matrices with elements in the Galois field $GF(q)$. Rend. Mat. 6(4), 459–532 (1971)

[19] Vickers, S.: Topology via Logic, Cambridge Tracts in Theoretical Computer Science, vol. 5. Cambridge University Press, Cambridge (1989)

[20] Witten, E.: Perturbative gauge theory as a string theory in twistor space. Commun. Math. Phys. 252(1–3), 189–258 (2004). https://doi.org/10.1007/s00220-004-1187-3

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