NEW EXPLICIT BOUNDS FOR MERTENS FUNCTION AND THE RECIPROCAL OF THE RIEMANN ZETA-FUNCTION

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ABSTRACT. In this paper, we establish new explicit bounds for the Mertens function $M(x)$. In particular, we compare $M(x)$ against a short-sum over the non-trivial zeros of the Riemann zeta-function $\zeta(s)$, whose difference we can bound using recent computations and explicit bounds for the reciprocal of $\zeta(s)$. Using this relationship, we are able to prove explicit versions of

$$M(x) \ll x \exp \left( -\eta_1 \sqrt{\log x} \right)$$

and

$$M(x) \ll x \exp \left( -\eta_2 (\log x)^{3/5} (\log \log x)^{-1/5} \right)$$

for some $\eta_i > 0$. Our bounds with the latter form are the first explicit results of their kind.

In the process of proving these, we establish another novel result, namely explicit bounds of the form

$$\frac{1}{\zeta(\sigma + it)} \ll (\log t)^{2/3} (\log \log t)^{1/4}.$$
Table 1. Computations for Theorems 1.1-1.2.

| \( \log x_0 \) | \( c_1(x_0) \) | \( c_2(x_0) \) | \( c_3(x_0) \) | \( c_4(x_0) \) |
|----------------|-------------|-------------|-------------|-------------|
| 363.11         | 0.4188      | 0.1148      | -           | -           |
| \( 1.0 \cdot 10^5 \) | 0.1154      | 0.3876      | 5.6144      | 0.0031      |
| \( 2.0 \cdot 10^5 \) | 0.1103      | 0.3968      | 5.5871      | 0.0086      |
| \( 3.0 \cdot 10^5 \) | 0.1080      | 0.4010      | 5.5719      | 0.0110      |
| \( 4.0 \cdot 10^5 \) | 0.1067      | 0.4036      | 5.5615      | 0.0125      |
| \( 5.0 \cdot 10^5 \) | 0.1058      | 0.4055      | 5.5535      | 0.0135      |
| \( 6.0 \cdot 10^5 \) | 0.1051      | 0.4069      | 5.5472      | 0.0142      |
| \( 7.0 \cdot 10^5 \) | 0.1046      | 0.4080      | 5.5418      | 0.0148      |
| \( 8.0 \cdot 10^5 \) | 0.1042      | 0.4088      | 5.5373      | 0.0153      |
| \( 9.0 \cdot 10^5 \) | 0.1038      | 0.4096      | 5.5333      | 0.0157      |
| \( 1.0 \cdot 10^6 \) | 0.1035      | 0.4102      | 5.5298      | 0.0160      |

**Theorem 1.2.** If \( x \geq x_0 \), then we can compute values \( c_3(x_0) \) and \( c_4(x_0) \) such that

\[
|M(x)| < c_3(x_0)x \exp\left(-c_4(x_0)(\log x)^{\frac{3}{2}}(\log \log x)^{-\frac{1}{2}}\right).
\]

For example, if \( \log x_0 = 100000 \), then \( c_3(x_0) = 5.61432 \) and \( c_4(x_0) = 0.00319 \) are admissible. Further computations are presented in Table 1.

We will prove Theorems 1.1-1.2 as well as give a complete comparison between Theorem 1.1 and Chalker’s results in Section 5. To prove Theorems 1.1-1.2, we combine analytic techniques with computational methods. That is, the truncated Perron formula tells us

\[
M(x) \approx \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s\zeta(s)} \, ds
\]

for some \( c > 1 \) and a point of truncation \( T \geq 1 \). So, we move the line of integration to the left and apply Cauchy’s residue theorem to compare the contour integral in (2) against a short-sum over the non-trivial zeros \( \rho = \beta + i\gamma \) of \( \zeta(s) \). The short-sum over zeros will be chosen such that we can explicitly bound it by appealing to a large database of zeros. The upper bound, which is presented in Lemma 4.3, will have size \( O(\sqrt{x}) \).

To bound the difference between the contour integral in (2) and the short-sum, we will require good bounds for the reciprocal of \( \zeta(s) \), which have the form

\[
\frac{1}{\zeta(\sigma + it)} \ll \log t \quad \text{and} \quad \frac{1}{\zeta(\sigma + it)} \ll (\log t)^{\frac{3}{2}}(\log \log t)^{\frac{1}{4}}
\]

within the classical and Korobov–Vinogradov zero-free regions respectively. Using the classical zero-free region, the second named author recently obtained the strongest explicit bounds of the first form in (3) in [12], by building upon previous work of Trudgian [26]. He also proves explicit bounds of the form \( 1/\zeta(\sigma + it) \ll (\log t)^{11/12} \), at the cost of a slightly worse constant. In Section 3, we go further and prove the first explicit bounds of the second form in (3); our result is presented in Corollary 3.2.
Applying our bounds for the difference between the contour integral in (2) and the short-
sum, which are presented in Lemmas 4.1-4.2 as well as our bound for the short-sum, we will
have determined an upper bound for $|M(x)|$ depending on $T$, $x$, and some other parameters
(which are well-explained later). Therefore, we complete the proof of each result by making
sensible choices for $T$ as a function of $x$.

When $x$ is small, explicit bounds of the form $M(x) \ll x/(\log x)^k$ with $k \in \{0,1\}$ are
typically used, so it is not an issue that our results only hold for large $x$. Of the published
results in the literature, Cohen, Dress, and El Marraki proved the current best explicit
bounds of the form $M(x) \ll x$ in [4] (see (42)), and Ramaré proved the best explicit bounds
of the form $M(x) \ll x/\log x$ in [18] (see (43)). When $x$ is larger, Theorems 1.1-1.2 will give
better bounds. In fact, if $x \geq 1$, then combining Hurst’s computations in [11] with the results
from [4, 18] and Theorems 1.1-1.2, we are able to prove the following corollary in Section 5.

Corollary 1.3. If $x \geq 1$, then

$$|M(x)| \leq \begin{cases} 4 & \text{if } 1 \leq x \leq 32, \\ 0.571\sqrt{x} & \text{if } 33 \leq x \leq 10^{16}, \\ \frac{x}{4345} - \frac{0.118x}{(\log x)^2} & \text{if } 10^{16} < x \leq e^{45.123}, \\ u(x)x \exp \left( -c_2(x)(\log x)^{\frac{3}{4}}(\log \log x)^{\frac{1}{4}} \right) & \text{if } e^{45.123} < x \leq e^{1772.504}, \\ 5.0951x \exp \left( -0.02196(\log x)^{\frac{3}{4}}(\log \log x)^{\frac{5}{4}} \right) & \text{if } e^{1772.504} < x \leq e^{e^{36.821}}, \\ 0.0055 + 20.8944 \exp \left( -\frac{0.4037}{\log \log x} \right) & \text{if } x > \exp(e^{36.821}), \\ \end{cases}$$

in which

$$u(x) := 0.09798 + \frac{0.0055 + 20.8944 \exp \left( -\frac{0.4037}{\log \log x} \right)}{\log \log x},$$
$$c_2(x) := \frac{1}{\sqrt{Z_1}} - \frac{\log \log x}{\sqrt{\log x}}, \quad \text{and} \quad Z_1 = 5.558691.$$

Theorems 1.1-1.2 and Corollary 1.3 will have several applications in the literature. For
example, by building upon work by Erdős and Szekeres [6], Bateman and Grosswald [1]
proved

$$L(x) = \frac{\zeta(3/2)}{\zeta(3)}x^{\frac{3}{4}} + \frac{\zeta(2/3)}{\zeta(2)}x^{\frac{1}{4}} + o(x^{\frac{1}{4}}),$$

in which $L(x)$ is the number of square-full integers up to $x$. Several authors have studied the
error in (5), including Suryanarayana and Sitaramachandra Rao [21], and the error in (5) is
closely related to bounds of the form $M(x) = o(x)$. So, by extension, our explicit results can
be used to establish explicit descriptions of the error in (5).

The remainder of this article is structured as follows. In Section 2 we introduce some
preliminary results and notation related to the Riemann zeta function, which will be useful
throughout. In Section 3 we state and prove explicit versions of the bounds in (3) that
we will need to apply; recall that our bound for the second form in (3) is the first explicit
result of its kind. Using these bounds, we will compare the contour integral in (2) against a
short-sum, and bound the short-sum, in Section 4. In Section 5, we use these observations to prove Theorems 1.1, 1.2 and Corollary 1.3.

2. Preliminary results

Unless otherwise stated, let \( s = \sigma + it \) be a complex number and \( \rho = \beta + i\gamma \) (for some \( 0 < \beta < 1 \)) be a non-trivial zero of the Riemann zeta function \( \zeta(s) \) throughout. For our purposes, it is also convenient to define the value \( H := 2e^{e^2} = 3.236.35598 \ldots \).

2.1. The Riemann hypothesis. The Riemann Hypothesis (RH) postulates that every non-trivial zero \( \rho \) satisfies \( \beta = 1/2 \). The Riemann height is a constant \( H > 0 \) such that the RH is known to be true for all \( |\gamma| \leq H \). Platt and Trudgian announced the latest Riemann height \( H := 3.000.175.332.800 \) in [16].

2.2. Zero-free regions of the Riemann zeta function. For our argument, it is important to avoid the zeros of \( \zeta(s) \) in the critical strip. To this end, we require zero-free regions. There are three commonly-seen forms of zero-free region:

\[
\sigma \geq 1 - \frac{1}{Z_1 \log t}, \quad \sigma \geq 1 - \frac{\log \log t}{Z_2 \log t^2}, \quad \text{and} \quad \sigma \geq 1 - \frac{1}{Z_3 (\log t)^{2/3} (\log \log t)^{1/3}}. \tag{6}
\]

In particular, for \( t \geq 3 \), we know that \( \zeta(s) \neq 0 \) in the regions defined in (6), with

\[
Z_1 = 5.558691, \quad Z_2 = 21.233, \quad \text{and} \quad Z_3 = 53.989. \tag{7}
\]

For more details about these zero-free regions, see [14], [27], and [2] respectively.

2.3. Truncated Perron formulae. Suppose that \( F(s) = \sum_{n \geq 1} a_n n^{-s} \) is a Dirichlet series, \( c > 0 \) is a real parameter larger than the abscissa of absolute convergence of \( F \), \( x \geq 1 \), and \( T \geq 1 \), then the truncated Perron formula tells us

\[
\left| \sum_{n \leq x} a_n - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s} \, ds \right| \leq E(x, T, c). \tag{8}
\]

The classical error is

\[
E(x, T, c) = \sum_{n=1}^{\infty} |a_n| \left( \frac{x}{n} \right)^c \min \left\{ 1, \frac{1}{T |\log \frac{x}{n}|} \right\}.
\]

Alternatively, Simonič uses [17] Thm. 7.1 in [20] Thm. 4] to see that \( E(x, T, c) \) may be re-written as

\[
E(x, T, c) = \frac{2F(c)x^c}{T} + 4e^c \left( \frac{e\phi(x)x\log T}{T} + \phi(ex) \right), \tag{9}
\]

where \( \phi(n) \) satisfies \( |a_n| \leq \phi(n) \).
2.4. Further results. One of the main objectives in this paper is to establish new bounds for the reciprocal of \( \zeta(s) \) in the Korobov–Vinogradov zero-free region. To this end, we need to import the following results. First, we have the following result due to Heath-Brown \[8\] Lem. 3.2, which will be used later to estimate the real part of the logarithmic derivative of \( \zeta(s) \). This lemma is related to well-known results from Jensen and Carleman (see \[22\] §3.71).

**Lemma 2.1** (Heath-Brown \[8\] Lem. 3.2]). Let \( f(z) \) be holomorphic for \( |z - a| \leq R \), and non-vanishing both at \( z = a \) and on the circle \( |z - a| = R \). Let \( \rho_j = a + r_j \exp(i\theta_j) \) be the zeros of \( f(z) \) in the disc, where \( \rho_j \) has multiplicity \( n_j \). Then

\[
\text{Re} \frac{f'}{f}(a) = - \sum_{\rho_j} n_j \left( \frac{1}{r_j} - \frac{r_j}{R^2} \right) \cos \theta_j + \frac{1}{\pi R} \int_0^{2\pi} (\cos \theta) \log |f(a + Re^{i\theta})| d\theta.
\]

Next, we have an explicit version of the Phragmén–Lindelöf principle, which is imported from \[24\] Lem. 3].

**Lemma 2.2** (Trudgian \[24\] Lem. 3]). Let \( a, b, Q \) be real numbers such that \( b > a \) and \( Q + a > 1 \). Let \( f(s) \) be a holomorphic function on the strip \( a \leq \text{Re} \ s \leq b \) such that

\[
|f(s)| < C \exp(\epsilon|t|)
\]

for some \( C > 0 \) and \( 0 < \epsilon < \frac{\pi}{b-a} \). Suppose further that there are \( A, B, \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0 \) such that \( \alpha_1 \geq \beta_1 \) and

\[
|f(s)| \leq \begin{cases} A|Q + s|^{\alpha_1} (\log |Q + s|)^{\alpha_2} & \text{for } \text{Re} \ s = a; \\ B|Q + s|^{\beta_1} (\log |Q + s|)^{\beta_2} & \text{for } \text{Re} \ s = b. \end{cases}
\]

Then, for \( a \leq \text{Re} \ s \leq b \), we have

\[
|f(s)| \leq \left( A|Q + s|^{\alpha_1} (\log |Q + s|)^{\alpha_2} \right)^{b - \text{Re} s \over b-a} \left( B|Q + s|^{\beta_1} (\log |Q + s|)^{\beta_2} \right)^{\text{Re} s - a \over b-a}.
\]

We also present a collection of upper bounds for \( |\zeta(s)| \), which will be useful for us later.

**Theorem 2.3** (Belotti \[2\] Thm. 1.1]). For every \(|t| \geq 3 \) and \( 1/2 \leq \sigma \leq 1 \), we have

\[
|\zeta(\sigma + it)| \leq 70.6995|t|^{4.43795(1-\sigma)^{3/2}} (\log |t|)^{2/3}.
\]

**Corollary 2.4.** For every \(|t| \geq 3 \), we have

\[
|\zeta(1 + it)| \leq 58.096(\log |t|)^{2/3}.
\]

**Proof.** We use same argument as in the first paragraph of \[25\] p. 2], which modifies Ford’s proof in \[7\]. The only difference is that we now apply said argument to \[2\] §5 (where Theorem 2.3 is proved), which itself is an improvement on Ford’s result. This proves the corollary. \(\square\)

**Lemma 2.5** (Ramaré \[19\] Lem. 5.4]). Let \( \gamma = 0.57721\ldots \) denote Euler’s constant. If \( \sigma > 1 \), then

\[
|\zeta(\sigma + it)| \leq \zeta(\sigma) \leq \frac{e^{\gamma(\sigma-1)}}{\sigma-1}.
\]

5
Table 2. Values for $R_1(Z_1,t_0)$ in Theorem 3.1, with corresponding $t_0$.

| $t_0$   | $R_1(Z_1,t_0)$ | $t_0$   | $R_1(Z_1,t_0)$ |
|---------|----------------|---------|----------------|
| 10      | 8.101          | $10^7$  | 2.711          |
| $10^2$  | 4.339          | $10^9$  | 2.518          |
| $10^3$  | 3.632          | $H$     | 2.307          |
| $10^4$  | 3.264          | $e^{40}$| 3.422          |
| $10^5$  | 3.021          |         | 2.134          |

3. The reciprocal of the Riemann zeta-function

Upper bounds for $1/|\zeta(s)|$ will be important later. A crucial estimate is by Simonić [20, Lem. 4], who shows that

$$\frac{1}{|\zeta(s)|} \leq \frac{4}{\sigma - \frac{1}{2}} \quad \text{for} \quad \frac{1}{2} < \sigma \leq \frac{3}{2} \quad \text{and} \quad 0 \leq t \leq 2e^{e^2} = H. \quad (10)$$

Further, by building upon work in [26], Leong proved the following result in [12, Cor. 4].

**Theorem 3.1** (Leong [12, Cor. 4]). Let $W_1 \geq Z_1$, which is defined in (7). If $t \geq t_0$ and $\zeta(\sigma + it) \neq 0$ in the region

$$\sigma \geq 1 - \frac{1}{W_1 \log t},$$

then

$$\left| \frac{1}{\zeta(\sigma + it)} \right| \leq R_1(W_1,t_0) \log t,$$

where $R_1(Z_1,H) = 3.422$ is admissible. Other values of $R_1(Z_1,t_0)$ with corresponding $t_0$ are presented in Table 2.

**Proof.** All values in Table 2 come from Leong’s [12, Cor. 4, Thm. 5], except for the two values of $R_1(Z_1,H)$ and $R_1(Z_1,e^{40})$. To compute the additional values, we follow the method of [12, Thm. 5] and obtain for $t \geq \exp(40)$ that $R_1(Z_1,e^{40}) = 2.134$, with rounded parameters $(d_1, \sigma_1, \eta) = (1.15809, 1.28233, 3.46431)$, according to the notation of [12, Thm. 5]. Similarly, for $H \leq t \leq 10^9$ we obtain $R_1(Z_1,H) = 3.422$, with rounded parameters $(d_1, \sigma_1, \eta) = (0.68994, 1.68609, 3.46265)$, according to the notation of [12, Thm. 4].

Using a similar argument and an asymptotically stronger zero-free region, namely the zero-free region of the form involving $Z_3$ in (6), we prove the following result. We remark that when using a zero-free region of the form involving $Z_3$ in (6), one usually obtains bounds of the form $1/\zeta(s) \ll (\log t)^{2/3}(\log \log t)^{1/3}$. Our methods reduce the exponent $1/3$ to $1/4$, although it is likely that this comes at the expense of a slightly worse constant. The novelty of this result notwithstanding, we do not expect this loss to have a meaningful impact on the final computations in Theorem 1.2, since we will only apply these bounds at large values of $t$. 
Corollary 3.2. Let \( W_3 \geq Z_3 \), which is defined in (7). If \( t \geq t_0 \geq H \) and \( \zeta(\sigma + it) \neq 0 \) in the region

\[
\sigma \geq 1 - \frac{1}{W_3(\log t)^{2/3}(\log \log t)^{1/3}},
\]

then

\[
\left| \frac{1}{\zeta(\sigma + it)} \right| \leq R_3(W_3, t_0)(\log t)^{2/3}(\log \log t)^{1/4},
\]

where \( R_3(Z_3, \hat{H}) = 40.943 \) is admissible. Other values of \( R_3(W_3, t_0) \) with corresponding \( t_0 \) and \( W_3 \) are presented in Table 3. In addition, we have for \( \sigma = 1 \) and \( t \geq \hat{H} \),

\[
\left| \frac{1}{\zeta(1 + it)} \right| \leq 35.05(\log t)^{2/3}(\log \log t)^{1/4}.
\]

3.1. Preliminaries. In order to prove Corollary 3.2, we require two preliminary ingredients, which rely on the results imported in Section 2.4.

Lemma 3.3. Let \( d > 0 \) be some real constant and suppose that \( \zeta(s) \neq 0 \) in the region

\[
\sigma \geq 1 - 1/Z_3(\log t)^{2/3}(\log \log t)^{1/3}.
\]

Let \( R > 0 \) be another parameter, chosen such that

\[
d(\log t)^{2/3}(\log \log t)^{1/3} \leq \frac{1}{Z_3(\log(t + R))^{2/3}(\log \log(t + R))^{1/3}}.
\]

Then, for \( t \geq 3 \), we have

\[- \text{Re} \frac{\zeta'}{\zeta}(\sigma + it) \leq \frac{4}{\pi R} \max_{0 \leq \theta \leq 2\pi} \left| \log |\zeta(\sigma + it + Re^{i\theta})| \right|,
\]

which holds uniformly for

\[
\sigma \geq 1 - \frac{d}{(\log t)^{2/3}(\log \log t)^{1/3}}.
\]

Proof. For \( \sigma_0 \) in the region stated in (12), we take \( f(s) = \zeta(s) \), \( a = s_0 = \sigma_0 + it_0 \), and \( \rho_k = s_0 + r_k \exp(i\theta_k) \) to be a zero of \( \zeta(s) \) in Lemma 2.1 (where we will make the substitution \( s_0 \to s \) at the end of the proof). This gives us

\[
\text{Re} \frac{\zeta'}{\zeta}(s_0) = - \sum_{\rho_k} n_k \left( \frac{1}{r_k} - \frac{r_k}{R^2} \right) \cos \theta_k + \frac{1}{\pi R} \int_{0}^{2\pi} (\cos \theta) \log |\zeta(s_0 + Re^{i\theta})| d\theta.
\]
Note that if a zero of \( \zeta(s) \) lies on the boundary \(|s - s_0| = R\), then since the zeros are separable, we can always find an arbitrarily smaller \( R \) where the boundary is free of zeros.

The rest of the proof is identical to that of [12, Lem. 2], where the reader is referred for full details. The only difference is the condition in (11). This is a result of noting that \((\log t)^{-2/3}(\log \log t)^{-1/3}\) is decreasing for \( t > e \) and requiring

\[
1 - \frac{1}{Z_3(\log(t + R)2/3(\log \log(t + R))^{1/3}} \leq 1 - \frac{d}{(\log t)^{2/3}(\log \log t)^{1/3}},
\]

which allows us to discard any contribution from the sum over zeros and gives rise to the condition in (11).

\[\Box\]

**Lemma 3.4.** Let \( \omega \geq 0 \) and \( t_0 \geq e^e \) be constants. Suppose \( t \geq t_0 \) and define

\[\sigma_t := 1 - \omega \left( \frac{\log \log t}{\log t} \right)^{2/3}, \quad \text{where} \quad \omega \leq \frac{e^{2/3}}{2}.\]  

(13)

For each \( t \geq t_0 \), if \( \sigma_t \leq \sigma \leq 1 \), then

\[|\zeta(\sigma + it)| \leq A_\omega(\log t)^{B_\omega},\]

in which \( B_\omega = \frac{2}{3} + 4.43795\omega^{3/2} \) and

\[A_\omega = \begin{cases} 
70.6995 & \text{if } \omega > 0, \\
58.096\sqrt{1+16t_0^{-2}} \left( 1 + \frac{\log \sqrt{1+16t_0^{-2}}}{\log t_0} \right) & \text{if } \omega = 0.
\end{cases}\]  

(14)

**Proof.** The proof is similar to that of [10, Lem. 7]. From Theorem 2.3, if \( 1/2 \leq \sigma \leq 1 \), then

\[|\zeta(\sigma + it)| \leq 70.6995 t^{4.43795(1-\sigma)^{3/2}} (\log t)^{2/3}.\]  

(15)

If \( \sigma_t \geq 1/2 \), then as a consequence of the assumption (13), considering that \((\log \log t/\log t)^{2/3}\) attains its maximum of \( e^{-2/3} \) at \( t = e^e \), we may use (15) for any \( \sigma \in [\sigma_t, 1] \). In view of this, since \((1-\sigma)^{3/2}\) is monotonically decreasing with \( \sigma \), if \( \sigma_t \leq \sigma \leq 1 \), then (15) gives

\[|\zeta(\sigma + it)| \leq 70.6995 \exp \left( 4.43795(1-\sigma_t)^{3/2} \log t + \frac{2}{3} \log \log t \right).\]

Or, written differently,

\[|\zeta(\sigma + it)| \leq A_\omega \exp (B_\omega \log \log t),\]

where \( A_\omega \) and \( B_\omega \) are defined as in (14). The desired result therefore follows for \( \sigma_t \leq \sigma \leq 1 \).

Note that if \( \omega = 0 \), then \( \sigma_t = 1 \), so this argument only yields the bound for choices of \( \omega > 0 \).

Next, we will prove the result holds in the range \( 1 \leq \sigma \leq 3 \) too, by applying Lemma 2.2 with the holomorphic function

\[f(s) = (s - 1)\zeta(s).\]

Note that this part of the argument will apply for any choice of \( \omega \geq 0 \). Assuming this choice, on the 1-line we have

\[|f(1 + it)| = |t| |\zeta(1 + it)| \leq 58.096 |2 + it| (\log |2 + it|)^{2/3}.\]

8
This inequality is verified numerically for $|t| < 3$, and is a consequence of Corollary 2.4 for $|t| \geq 3$. Also, on the line $s = 3 + it$, we have

$$|f(3 + it)| \leq 2 + |t| \zeta(3) < 58.096|4 + it|(|\log|4 + it||)^{2/3},$$

for all real $t$. So, Lemma 2.2 with $a = 1$, $b = 3$, $Q = 1$, $\alpha_1 = \beta_1 = 1$, $\alpha_2 = \beta_2 = 2/3$, and $A = B = 58.096$ tells us that

$$|f(s)| \leq 58.096|1 + s|(|\log|1 + s||)^{2/3},$$

for $1 \leq \sigma \leq 3$. Moreover, if $1 \leq \sigma \leq 3$ and $t \geq t_0 \geq e^e$, we have

$$\frac{|1 + s|}{|s - 1|} \leq \sqrt{16 + t^2} \leq \sqrt{1 + 16t_0^{-2}} \quad \text{and} \quad \log|1 + s| \leq \log(1 + t^2) \leq \log(1 + 16t_0^{-2}).$$

whereas for $t_0 \geq e^e$,

$$58.096\sqrt{1 + 16t_0^{-2}} \left(1 + \frac{\log(1 + 16t_0^{-2})}{\log t_0}\right) \leq 60.8301.$$

Thus for $1 \leq \sigma \leq 3$ and $t \geq t_0 \geq e^e$, the inequality (16) implies that

$$|\zeta(\sigma + it)| \leq 58.096\sqrt{1 + 16t_0^{-2}} \left(1 + \frac{\log(1 + 16t_0^{-2})}{\log t_0}\right) (\log t)^{2/3} < A_\omega(\log t)^{B_\omega},$$

where $A_\omega$ and $B_\omega$ are defined as in (14), as desired. \qed

### 3.2. Proof of Corollary 3.2

In order to prove Corollary 3.2, we first prove the more general Theorem 3.5.

**Theorem 3.5.** Let $\gamma$ be Euler’s constant and let there be positive parameters $d$, $d_1$, $\omega$, $W_3$, such that $W_3 = 1/d \geq Z_3$, and the conditions (13), (20), (23), and (27) are satisfied. Then in the region

$$\sigma \geq 1 - \frac{1}{W_3(\log t)^{2/3}(\log \log t)^{1/3}}, \quad t \geq t_0 \geq \tilde{H},$$

where $\zeta(\sigma + it) \neq 0$, we have

$$\left|\frac{1}{\zeta(\sigma + it)}\right| \leq R_3(\log t)^{2/3}(\log \log t)^{1/4},$$

where

$$R_3(d, d_1, \omega) := \left(\frac{a_0}{d_1^3}\right)^{1/4} \left(1 + \frac{\log 2}{\log t_0}\right)^{1/6} \exp\left(\frac{3\gamma d_1}{4(\log t_0)^{2/3}(\log \log t_0)^{1/3}}\right) + \frac{4(d_1 + d)}{\pi(\omega - d(\log \log t_0)^{1/3})} \left(B_\omega + \frac{B_\omega}{\log \log t_0} \log\left(1 + \frac{\log(1 + R_{\max})}{\log t_0}\right) + \frac{\log A_\omega}{\log \log t_0}\right),$$

in which $R_{\max}$ was defined in (22), $A_\omega$, $B_\omega$ were defined in Lemma 3.4, and $a_0 = A_\omega$ at $\omega = 0$.  

9
Proof. We will adapt the argument in [12] for our purposes. It revolves around the relationship
\[
\log \left| \frac{1}{\zeta(\sigma + it)} \right| = - \text{Re} \log \zeta(1 + \delta_1 + it) + \int_\sigma^{1+\delta_1} \text{Re} \left( \frac{\zeta'}{\zeta}(y + it) \right) \, dy
\]  
(17)
for some \( \delta_1 > 0 \). We will focus on the case when \( t \geq \hat{H} \). To begin, we bound the second term on the right-hand side of (17). To do this, suppose that \( d \) is a positive real number to be determined later, and \( s = \sigma + it \) such that \( t \geq t_0 \geq H \) and
\[
\sigma \geq 1 - \delta := 1 - \frac{d}{(\log t)^{2/3}(\log \log t)^{1/3}}.
\]  
(18)
Next, let \( R, d > 0 \) be chosen such that the condition in (11) is satisfied, by first supposing \( R \leq R_{\text{max}} \). Under these assumptions, (11) is satisfied if
\[
dZ_3 \leq \frac{(\log t)^{2/3}(\log \log t)^{1/3}}{(\log(t + R_{\text{max}})^{2/3}(\log \log(t + R_{\text{max}})))^{1/3}},
\]  
(19)
which in turn is satisfied if
\[
dZ_3 \leq \frac{(\log t_0)^{2/3}(\log \log t_0)^{1/3}}{(\log(t_0 + R_{\text{max}})^{2/3}(\log \log(t_0 + R_{\text{max}})))^{1/3}},
\]  
(20)
since the right-hand side of (19) is increasing to 1 as \( t \to \infty \). With this, Lemma 3.3 implies
\[
\left| \text{Re} \frac{\zeta'}{\zeta}(\sigma + it) \right| \leq \frac{4}{\pi R} \max_{0 \leq \theta \leq 2\pi} \log |\zeta(\sigma + it + R e^{i\theta})|.
\]  
(21)
for \( t \geq t_0 \) and \( \sigma \) as in (18). Next, suppose that \( \omega > 0 \) is another parameter to be determined later. We require that \( \omega, d, \) and \( R \) satisfy \( \sigma_t \leq 1 - \delta - R; \) recall that \( \sigma_t \) was defined in (13). Keeping this in mind, we write
\[
R = \frac{\omega(\log \log t)^{2/3} - d(\log \log t)^{-1/3}}{(\log t)^{2/3}},
\]  
where clearly,
\[
R \leq R_{\text{max}} := \omega \left( \frac{\log \log t_0}{\log t_0} \right)^{2/3}
\]  
(22)
and
\[
\frac{1}{R} \leq \frac{1}{\omega - d(\log \log t_0)^{-1}} \left( \frac{\log t}{\log \log t} \right)^{2/3}, \quad \text{in which} \quad \omega > \frac{d}{\log \log t_0}.
\]  
(23)
Therefore, by Lemma 3.4 (21) becomes
\[
\left| \text{Re} \frac{\zeta'}{\zeta}(\sigma + it) \right| \leq \frac{4}{\pi R} \log \left( A_\omega (\log(t + R_{\text{max}}))^{B_\omega} \right)
\]  
\[
\leq \frac{4}{\pi R} \left( B_\omega \log \log t + B_\omega \log \left( 1 + \frac{\log(1 + R_{\text{max}}/t_0)}{\log t_0} \right) + \log A_\omega \right)
\]  
\[
\leq \frac{4(\log t)^{2/3}(\log \log t)^{1/3}}{\pi(\omega - d(\log \log t_0)^{-1})} \left( B_\omega \right)
\]  
(24)
\[
\begin{align*}
&\quad + \frac{B_\omega}{\log \log t_0} \log \left(1 + \frac{\log(1 + R_{\max}/t_0)}{\log t_0}\right) + \frac{\log A_\omega}{\log \log t_0},
\end{align*}
\]
which holds in the region \(1 - \delta \leq \sigma \leq 3 - R_{\max}\); recall that \(A_\omega\) and \(B_\omega\) were defined in Lemma 3.4.

Next, we bound the first term on the right-hand side of (17), namely
\[
- \Re \log \zeta(1 + \delta_1 + it) = \log \left| \frac{1}{\zeta(1 + \delta_1 + it)} \right|.
\]
Since \(1 + \delta_1 > 1\), we will use the classical non-negativity argument involving the trigonometric polynomial \(2(1 + \cos \theta)^2 = 3 + 4 \cos \theta + \cos 2\theta\); see [23, §3.3] and [13] for more commentary on this approach. That is,
\[
\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it) \geq 1,
\]
which implies
\[
\left| \frac{1}{\zeta(1 + \delta_1 + it)} \right| \leq \left| \zeta(1 + \delta_1) \right|^{3/4} \left| \zeta(1 + \delta_1 + 2it) \right|^{1/4}. \tag{26}
\]
Next, let \(d_1 > 0\) be a parameter and choose
\[
\delta_1 := \frac{d_1}{(\log t)^{2/3}(\log \log t)^{1/3}}, \quad \text{in which} \quad d_1 < (2 - R_{\max})(\log t_0)^{2/3}(\log \log t_0)^{1/3}. \tag{27}
\]
The latter condition on \(d_1\) ensures that \(1 + \delta_1 \leq 3 - R_{\max}\). Thus, by Lemma 3.4 with \(\omega = 0\), we have
\[
\left| \zeta(1 + \delta_1 + 2it) \right|^{1/4} \leq a_0^{1/6} (\log(2t))^{1/6} \leq a_0^{1/6} \left(1 + \frac{\log 2}{\log t_0}\right)^{1/6} (\log t)^{1/6},
\]
in which \(a_0 = A_\omega\) with \(\omega = 0\). Lemma 2.5 also tells us
\[
\left| \zeta(1 + \delta_1) \right|^{3/4} \leq \left(\frac{\log t}{\log \log t}\right)^{1/4} \frac{3\gamma d_1/4}{d_1^{3/4}} \exp \left(\frac{3\gamma d_1/4}{(\log t_0)^{2/3}(\log \log t_0)^{1/3}}\right).
\]
Thus, (26) implies that if \(\sigma \geq 1 + \delta_1\), then
\[
\left| \frac{1}{\zeta(\sigma + it)} \right| \leq a_0^{1/6} \left(1 + \frac{\log 2}{\log t_0}\right)^{1/6} \exp \left(\frac{3\gamma d_1/4}{(\log t_0)^{2/3}(\log \log t_0)^{1/3}}\right) \frac{3\gamma d_1/4}{d_1^{3/4}} (\log t)^{2/3}(\log \log t_0)^{1/3}. \tag{28}
\]
So, if \(1 - \delta \leq \sigma \leq 1 + \delta_1\), then
\[
\left| \frac{1}{\zeta(\sigma + it)} \right| = \exp \left( - \Re \log \zeta(1 + \delta_1 + it) + \int_{\sigma}^{1+\delta_1} \Re \left( \frac{\zeta'}{\zeta}(y + it) \right) dy \right) \tag{by (17)}
\]
\[
\leq \left| \frac{1}{\zeta(1 + \delta_1 + it)} \right| \exp \left( \int_{\sigma}^{1+\delta_1} \Re \left( \frac{\zeta'}{\zeta}(y + it) \right) dy \right) \tag{by (25)}
\]
\[
\leq \left| \frac{1}{\zeta(1 + \delta_1 + it)} \right| \exp \left( (\delta_1 + \delta) \left| \Re \frac{\zeta'}{\zeta}(\sigma + it) \right| \right).\]
It follows from (24) and (28) that the right-hand side is bounded from above by
\[
\begin{align*}
(\log t)^{2/3} & \left(\frac{a_0 \log \log t}{d_1^3} \right)^{1/4} \left(1 + \frac{\log 2}{\log t_0}\right)^{1/6} \exp \left(\frac{3\gamma d_1}{4(\log t_0)^{2/3}(\log \log t_0)^{1/3}}\right) \\
& + \frac{4(d_1 + d)}{\pi(\omega - d(\log \log t_0)^{-1})} \left(B_\omega + \frac{B_\omega}{\log \log t_0} \log \left(1 + \frac{\log(1 + R_{\max}/t_0)}{\log t_0}\right) + \frac{\log a_0}{\log \log t_0}\right).
\end{align*}
\]
Furthermore, if \( \sigma \geq 1 + \delta_1 \), then we simply use (28), which is clearly less than (29). Therefore, we have proved the result for all \( \sigma \geq 1 - \delta \). Taking \( \delta \) as in [18] completes the proof. \( \square \)

**Proof of Corollary 3.2.** To obtain numerical values for \( R_3 \) in Theorem 3.5, we fix \( t_0 \) and \( W_3 \), then set \( d = W_3^{-1} \) and optimise over \( d_1 \) and \( \omega \), making sure the hypotheses and conditions of the theorem are fulfilled; once \( d_1 \) and \( \omega \) have been chosen, \( R_3(W_3, t_0) = R_3(d, d_1, \omega) \) in our statement of the result. Our optimised choices of \( d_1 \) and \( \omega \) are presented in Table 3. The special case of \( \sigma = 1 \) is obtained by instead omitting \( W_3 \) and setting \( d = 0 \). \( \square \)

### 4. Important Contour Integrals and the Short-Sum

In this section, we apply the results of Section 3 and prove three key lemmas, which will be important in the proofs of our main results. First, we relate the contour integral in (2) to a sum over zeros.

**Lemma 4.1.** If \( W \geq Z_1 \), \( x \geq 1 \), \( c = 1 + 1/\log x \), and \( T > H \), then
\[
\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s\zeta(s)} \, ds - \sum_{|\gamma| \leq H, \gamma \neq 0} \frac{1}{\gamma \sigma_1(t)} \right| \leq \frac{R_1(W, H)\gamma \log \log \log x}{\pi \log x} + \frac{41.155}{\sqrt{x}} + \left( \frac{R_1(W, H)((\log T)^2 - (\log H)^2)}{2\pi} \right) + 1.26 \cdot 10^{-5}\right) x^{\sigma_1(t)},
\]
where \( \sigma = \beta + i\gamma \) are non-trivial zeros of \( \zeta(s) \) and
\[
\sigma_1(t) = 1 - \frac{1}{W \log t}.
\]

**Proof.** Suppose that \( T > H \) is a parameter to be chosen. Define the contours \( C_i \) such that
- \( C_0 \) connects the nodes \( c - iT \) and \( c + iT \) via a straight line,
- \( C_1 \) connects the nodes \( c + iT \) and \( c + iT \) via a straight line,
- \( C_2 \) connects the nodes \( \sigma_1(T) + iT \) and \( \sigma_1(H) + iT \) along the path \( \sigma_1(t) + iT \),
- \( C_3 \) connects \( \sigma_1(T) + iT \) and \( \sigma_1(T) + iT \) via a straight line,
- \( C_4 \) connects the nodes \( -\frac{1}{2} + iH \) and \( -\frac{1}{2} + iH \) via a straight line,
- \( C_5 \) connects the nodes \( -\frac{1}{2} + iH \) and \( \sigma_1(H) - iT \) via a straight line,
- \( C_6 \) connects the nodes \( \sigma_1(T) - iT \) and \( \sigma_1(T) - iT \) along the path \( \sigma_1(t) - iT \),
- \( C_7 \) connects the nodes \( \sigma_1(T) - iT \) and \( c - iT \) via a straight line.
Note that the contours $C_2$ and $C_6$ are defined as such because we want to avoid the non-trivial zeros $\rho = \beta + i\gamma$ when $|\gamma| > H$. It follows from (6) with $Z_1 = 5.558691$ that the closed contour

$$C = \bigcup_{i=0}^{7} C_i$$

only contains the zeros of $\zeta(s)$ such that $|\gamma| \leq H$, hence Cauchy’s residue theorem tells us

$$I = \frac{1}{2\pi i} \int_C \frac{x^s}{s\zeta(s)} \, ds = \sum_{|\gamma| \leq H} \frac{x^{\frac{1}{2} + i\gamma}}{(\frac{1}{2} + i\gamma)\zeta'(\frac{1}{2} + i\gamma)}.$$

For convenience, we write

$$I_i = \frac{1}{2\pi i} \int_{C_i} \frac{x^s}{s\zeta(s)} \, ds,$$

hence

$$\frac{1}{2\pi i} \int_{e^{-iT}}^{e^{+iT}} \frac{x^s}{s\zeta(s)} \, ds = \sum_{|\gamma| \leq H} \frac{x^{\frac{1}{2} + i\gamma}}{(\frac{1}{2} + i\gamma)\zeta'(\frac{1}{2} + i\gamma)} - \sum_{i=1}^{7} I_i.$$

Use Theorem 3.1 to see

$$|I_1| + |I_7| \leq \frac{R_1(W, T)\log T}{\pi T} \int_{\sigma_1(T)}^{c} x^y \, dy = \frac{R_1(W, T)(ex - x^{\sigma_1(T)})\log T}{\pi T\log x} < \frac{R_1(W, T)ex\log T}{\pi T\log x}.$$

Use Theorem 3.1 again to see

$$|I_2| + |I_6| \leq \frac{R_1(W, H)x^{\sigma_1(T)}}{\pi} \int_{H}^{T} \frac{\log t}{t} \, dt = \frac{R_1(W, H)((\log T)^2 - (\log H)^2)x^{\sigma_1(T)}}{2\pi}.$$

Next, we are in the region $|t| \leq H$ and the RH has been verified in this region. Computations tell us

$$|I_3| + |I_5| < \frac{x^{\sigma_1(H)}}{\pi} \int_{-\frac{1}{2}}^{1} \frac{dy}{\sqrt{y^2 + H^2}\zeta(y + iH)} \leq 1.26 \cdot 10^{-5}\cdot x^{\sigma_1(H)}.$$

Finally, to bound $|I_4|$, we need to recall the fundamental equations

$$\zeta(s) = \xi(s)\zeta(1 - s), \quad \text{where} \quad \xi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1 - s),$$

and $z\Gamma(z) = \Gamma(z + 1)$. We also need to import the bounds $|\Gamma(\frac{1}{2} + it)| \leq \Gamma(\frac{1}{2}) < 0.28861$ and $|\sin z| \leq 6|z|/5$ from [15]. Apply these and (10) to see

$$|I_4| \leq \frac{x^{\frac{1}{2}}}{2\pi} \int_{-H}^{H} |\zeta(-\frac{1}{2} + it)|^{-1} \frac{dt}{\sqrt{\frac{1}{4} + t^2}} = \frac{x^{\frac{1}{2}}}{2\pi} \int_{-H}^{H} \frac{|\xi(\frac{3}{2} - it)| \, dt}{|\zeta(\frac{3}{2} - it)|\sqrt{\frac{1}{4} + t^2}}.$$
Define the contours

\[ \text{Proof.} \]

If \( x \geq 1, \ c = 1 + 1/\log x, \) and \( T > H = 2e^{e^2}, \) then

\[
\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s\zeta(s)} \, ds - \sum_{|\gamma| \leq H} \frac{1}{x^{1/2+i\gamma}} \right|
\leq \frac{R_3(W_3, T)(\log T)^{3/4}(\log \log T)^{1/4}e^x}{\pi T \log x}
\]

\[
+ \frac{R_3(W_3, T_0)x}{\pi} \int_{T_0}^{T} \frac{t^{\sigma_2(t)-1}(\log t)^{3/4}(\log \log t)^{1/4}}{t} \, dt
\]

\[
+ \frac{R_1(W_1, T_0)(\sigma_2(T_0) - \sigma_1(T_0)) \log T_0}{\pi T_0}
\]

\[
+ \frac{41.155}{\sqrt{x}} + \left( \frac{R_1(W_1, H)((\log T_0)^2 - (\log H)_{x=\sigma_1(T_0)})}{2\pi} + 1.26 \cdot 10^{-5} \right) x^{\sigma_1(T_0)},
\]

where \( \sigma = \beta + i\gamma \) are non-trivial zeros of \( \zeta(s) \),

\[
\sigma_1(t) = 1 - \frac{1}{W_1 \log t}, \quad \text{and} \quad \sigma_2(t) = 1 - \frac{1}{W_3(\log t)^{3/4}(\log \log t)^{1/4}}.
\]

\[ \text{Proof.} \]

Define the contours \( C_i \) such that

- \( C_0 \) connects the nodes \( c - iT \) and \( c + iT \) via a straight line,
- \( C_1 \) connects the nodes \( c + iT \) and \( \sigma_2(T) + iT \) via a straight line,
- \( C_2 \) connects the nodes \( \sigma_2(T) + iT \) and \( \sigma_2(T_0) + iT_0 \) along the path \( \sigma_2(t) + it \),
- \( C_3 \) connects the nodes \( \sigma_2(T_0) + iT_0 \) and \( \sigma_1(T_0) + iT_0 \) via a straight line,
- \( C_4 \) connects the nodes \( \sigma_1(T_0) + iT_0 \) and \( \sigma_1(H) + iH \) along the path \( \sigma_1(t) + it \),

Second, we prove another variant of Lemma 4.1.

Lemma 4.2. Suppose that \( W_1 \geq Z_1, \ W_3 \geq Z_3, \) and \( T_0 \) is chosen such that

\[
\frac{\log T_0}{\log \log T_0} \leq \left( \frac{W_3}{W_1} \right)^3.
\]

If \( x \geq 1, \ c = 1 + 1/\log x, \) and \( T > H = 2e^{e^2}, \) then

\[
\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s\zeta(s)} \, ds - \sum_{|\gamma| \leq H} \frac{1}{x^{1/2+i\gamma}} \right|
\leq \frac{R_3(W_3, T)(\log T)^{3/4}(\log \log T)^{1/4}e^x}{\pi T \log x}
\]

\[
+ \frac{R_3(W_3, T_0)x}{\pi} \int_{T_0}^{T} \frac{t^{\sigma_2(t)-1}(\log t)^{3/4}(\log \log t)^{1/4}}{t} \, dt
\]

\[
+ \frac{R_1(W_1, T_0)(\sigma_2(T_0) - \sigma_1(T_0)) \log T_0}{\pi T_0}
\]

\[
+ \frac{41.155}{\sqrt{x}} + \left( \frac{R_1(W_1, H)((\log T_0)^2 - (\log H)_{x=\sigma_1(T_0)})}{2\pi} + 1.26 \cdot 10^{-5} \right) x^{\sigma_1(T_0)},
\]

where \( \sigma = \beta + i\gamma \) are non-trivial zeros of \( \zeta(s) \),

\[
\sigma_1(t) = 1 - \frac{1}{W_1 \log t}, \quad \text{and} \quad \sigma_2(t) = 1 - \frac{1}{W_3(\log t)^{3/4}(\log \log t)^{1/4}}.
\]
• $C_5$ connects $\sigma_1(H) + iH$ and $-\frac{1}{2} + iH$ via a straight line,
• $C_6$ connects the nodes $-\frac{1}{2} + iH$ and $-\frac{1}{2} - iH$ via a straight line,
• $C_7$ connects the nodes $-\frac{1}{2} - iH$ and $\sigma_1(H) - iH$ via a straight line,
• $C_8$ connects the nodes $\sigma_1(H) - iH$ and $\sigma_1(T_0) - iT_0$ along the path $\sigma_1(t) - it$,
• $C_9$ connects the nodes $\sigma_1(T_0) - iT_0$ and $\sigma_2(T_0) - iT_0$ via a straight line,
• $C_{10}$ connects the nodes $\sigma_2(T_0) - iT_0$ and $\sigma_2(T) - iT$ along the path $\sigma_2(t) - it$,
• $C_{11}$ connects the nodes $\sigma_2(T) - iT$ and $c - iT$ via a straight line.

Again, the contours $C_2$, $C_4$, $C_8$ and $C_{10}$ are defined as such because we want to avoid the non-trivial zeros $\rho = \beta + i\gamma$ when $|\gamma| > H$. It follows from (6) with the values $Z_1 = 5.558691$ and $Z_3 = 53.989$ that the closed contour

$$C = \bigcup_{i=0}^{11} C_i$$

only contains the zeros of $\zeta(s)$ such that $|\gamma| \leq H$, hence Cauchy’s residue theorem tells us

$$I = \frac{1}{2\pi i} \int_C \frac{x^s}{s\zeta(s)} \, ds = \sum_{|\gamma| \leq H} \frac{x^{\frac{1}{2} + i\gamma}}{(\frac{1}{2} + i\gamma)\zeta'(\frac{1}{2} + i\gamma)}.$$

Again, we write

$$I_i = \frac{1}{2\pi i} \int_{C_i} \frac{x^s}{s\zeta(s)} \, ds,$$

hence

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s\zeta(s)} \, ds = \sum_{|\gamma| \leq H} \frac{x^{\frac{1}{2} + i\gamma}}{(\frac{1}{2} + i\gamma)\zeta'(\frac{1}{2} + i\gamma)} - \sum_{i=1}^{11} I_i.$$

Note that $T_0$ should be chosen to satisfy (30) in order to ensure $\sigma_1(T_0) \leq \sigma_2(T_0)$.

Use Corollary 3.2 to see

$$|I_1| + |I_{11}| \leq \frac{R_3(W_3, T)(\log T)^{\frac{3}{4}}(\log \log T)^{\frac{5}{4}}}{\pi T} \frac{\int_{\sigma_2(T)}^{c} x^y \, dy}{\int_{\sigma_1(T)}^{c} x^y \, dy} \leq \frac{R_3(W_3, T)(\log T)^{\frac{3}{4}}(\log \log T)^{\frac{5}{4}}}{\pi T \log x}.$$

Use Corollary 3.2 again to see

$$|I_2| + |I_{10}| \leq \frac{R_3(W_3, T_0)x}{\pi} \int_{\sigma_2(T_0)}^{c} \frac{x^{\sigma_2(t)-1}(\log t)^{\frac{3}{4}}(\log \log t)^{\frac{5}{4}}}{t} \, dt.$$

Use Theorem 3.1 to see

$$|I_3| + |I_0| \leq \frac{R_1(W_1, T_0)\log T_0}{\pi T_0} \int_{\sigma_1(T_0)}^{\sigma_2(T_0)} x^y \, dy \leq \frac{R_1(W_1, T_0)(\sigma_2(T_0) - \sigma_1(T_0))\log T_0}{\pi T_0} x^{\sigma_2(T_0)}.$$
Next, repeat the arguments in the proof of Lemma 4.1, which depend heavily on Theorem 3.1, to see
\[
\sum_{i=4}^{8} |I_i| \leq 41.155 \frac{\sqrt{x}}{x} + \left( \frac{R_1(W, H)((\log T_0)^2 - (\log H)^2)}{2\pi} + 1.26 \cdot 10^{-5} \right) x^{\sigma_1(T_0)}.
\]
To complete the proof, use the triangle inequality and these observations. □

Finally, we present an exact computation for the sum that appears in Lemmas 4.1-4.2.

**Lemma 4.3.** If \( H = 2e^{e^2} \), then
\[
\sum_{|\gamma| \leq H} \frac{\sqrt{x}}{\sqrt{\frac{1}{4} + \gamma^2 |\zeta'(\frac{1}{2} + i\gamma)|}} \leq 2.4\sqrt{x}.
\]

**Proof.** The result follows from an exact computation using a database of the zeros of the non-trivial zeros of \( \zeta(s) \) up to a large height. To run the computation, we used Odlyzko's tables of zeros, which are available at [this link](#). We also thank Dave Platt for independently verifying our computations. □

**Remark.** We fix \( H = 2e^{e^2} \) in this paper, because this is the largest ordinate that \( (10) \) will hold for. However, we have exact information about a lot more zeros, so it would be preferable to have a version of \( (10) \) that allows us to choose \( H \) larger.

5. **Bounds for Mertens’ function**

In this section, we prove the main results of this paper.

5.1. **Proof of Theorem 1.1.** Apply (8) with the error in (9), \( a_n = \mu(n) \), \( \phi(n) = 1 \), and \( F(\sigma) \leq \zeta(\sigma) \leq \sigma/(\sigma - 1) \) to obtain
\[
\left| M(x) - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s\zeta(s)} \, ds \right| \leq \frac{\nu_1(x,T)x \log x}{T},
\]
in which \( x \geq 1 \), \( T > 1 \), \( c = 1 + 1/\log x \), and
\[
\nu_1(x,T) = 2ec + \frac{4e^c}{\log x} \left( \frac{T}{x} + e \log T \right) = 2e + \frac{4e^c}{\log x} \left( \frac{e^{-\frac{1}{\log x}}}{2} + \frac{T}{x} + e \log T \right).
\]
Suppose that \( W \geq Z_1 \). Lemma 4.1, Lemma 4.3, and (31) imply
\[
|M(x)| \leq \frac{R_1(W,T)x \log T}{\pi T \log x} + \frac{41.155}{\sqrt{x}} + 2.4\sqrt{x} + \left( \frac{R_1(W,H)((\log T)^2 - (\log H)^2)}{2\pi} + 1.26 \cdot 10^{-5} \right) x^{\sigma_1(T)} + \frac{\nu_1(x,T)x \log x}{T}.
\]
So, let \( x \geq x_W > 1 \) and
\[
\log T_x = \sqrt{\frac{\log x}{W}}.
\]
It follows that

\[ x^{\sigma_1(T_x)} = x \exp \left( -\sqrt{\frac{\log x}{W}} \right) \]

and \( T_x \geq H = 2e^2 \) when \( \log x_{W} \geq W(e^2 + \log 2)^2 \). Recall that computations from Hurst \[11\] tell us that \( |M(x)| \leq 0.571\sqrt{x} \) for \( 33 \leq x \leq 10^{16} \). So, assert

\[ \log x_{W} = \max \{W(e^2 + \log 2)^2, 16 \log 10\} \quad \text{and} \quad T = T_x. \]

Insert these choices into (32) to see that for all \( W \) and \( T \) larger \( x \)

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Table 4. Computations for the constants in (34) and (35), which are valid for \( x \geq x_0 \geq x_W \approx \exp(363.10487) \) with \( W = Z_1 \).

| \( \log x_0 \) | \( c_1(x_0) \) | \( c_2(x_0) \) |
|----------------|-------------|-------------|
| 363.11         | 7.3 \cdot 10^1 | 0.008       |
| 489.15         | 1.5 \cdot 10^2 | 0.041       |
| 607.78         | 5.0 \cdot 10^2 | 0.086       |
| 864.36         | 3.5 \cdot 10^3 | 0.143       |
| 1474.63        | 1.3 \cdot 10^5 | 0.213       |
| 3364.98        | 6.0 \cdot 10^8 | 0.297       |
| 14305.32       | 1.7 \cdot 10^{29} | 0.368      |

Table 5. Chalker’s computations for \( c_i(x_0) \) in (35), from [3, Tab. 3.3].

Clearly, our results significantly refine every constant in Table 5. One of the main differences in our approach is that we choose \( W = Z_1 \), whereas the values in Table 5 come from choosing \( W \in \{6, 7, 8, 9, 10, 11, 12\} \). The remainder of our improvements come from (a) improved bounds on \( 1/\zeta(s) \), and (b) the updated contour argument that we used to prove Lemma 4.2.

5.2. **Proof of Theorem 1.2.** We now make a few modifications to the above argument in order to get the desired bounds of Theorem 1.2. Recall we have from (31) that

\[
\left| M(x) - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s\zeta(s)} \, ds \right| \leq \frac{\nu_1(x, T)|x\log x|}{T},
\]

in which \( x \geq 1, T > 1, c = 1 + 1/\log x \), and

\[
\nu_1(x, T) = 2ec + \frac{4e^c}{\log x} \left( \frac{T}{x} + e \log T \right).
\]
Combining this with Lemma 4.2 (assuming the assumptions therein) and Lemma 4.3, we have

\[
\frac{|M(x)|}{x} \leq \frac{\nu_1(x, T) \log x}{T} + \frac{1}{\sqrt{x}} \left( 2.4 + \frac{41.155}{x} \right) + \frac{R_3(W_3, T)(\log T)^{\frac{3}{2}}(\log \log T)^{\frac{1}{2}} e}{\pi T \log x} + \frac{R_3(W_3, T_0)x^{\sigma_2(T)-1}}{\pi T_0} \int_{T_0}^{T} \frac{(\log t)^{\frac{3}{2}}(\log \log t)^{\frac{1}{2}}}{t} dt + \frac{R_1(W_1, T_0) (\sigma_2(T_0) - \sigma_1(T_0)) \log T_0}{\pi T_0} x^{\sigma_2(T_0)-1} + \left( \frac{R_1(W_1, H)((\log T_0)^2 - (\log H)^2)}{2\pi} + 1.26 \cdot 10^{-5} \right) x^{\sigma_1(T_0)-1}.
\]

So, let

\[
\log T_x = (\log x)^{\frac{3}{2}}(\log \log x)^{-\frac{1}{2}}
\]

and note that \(T_x > H = 2e^{e^2}\) when \(x \geq x_1 = 2.12216 \cdot 10^{22}\).

**Completing the bound.** In order to move forward, we need to make some observations. Fix \(T = T_x, x_0 \geq x_1\), and write

\[
k_0(x) = \frac{(\log x)^{6/5}}{(\log x)^{3/5}} > 0,
\]

so that

\[
\log x = \exp \left( k_0(x) \frac{(\log x)^{\frac{3}{2}}}{(\log \log x)^{\frac{1}{2}}} \right).
\]

Note that for \(x \geq x_1, k_0(x) \leq 0.48763\). Thus, (37) implies

\[
\frac{\log x}{T_x} = \exp \left( -k_1(x) \frac{(\log x)^{\frac{3}{2}}}{(\log \log x)^{\frac{1}{2}}} \right),
\]

in which \(k_1(x) = 1 - k_0(x) > 0\). Note that \(k_1(x) \to 1^-\) as \(x \to \infty\), so this bound will be at its sharpest when \(k_0(x)\) is as small as possible. Further, fix a value of \(T_0\) satisfying (30) and \(T_{x_0} \geq T_0 \geq \tilde{H}\); the final inequality ensures we can compute \(R_3(W_3, T_0)\). We have

\[
x^{\sigma_2(T)-1} = \exp \left( -\frac{\log x}{W_3(\log T)^{\frac{3}{2}}(\log \log T)^{\frac{1}{2}}} \right) \leq \exp \left( -\frac{(5/3)^{\frac{1}{2}}}{W_3} \frac{(\log x)^{\frac{3}{2}}}{(\log \log x)^{\frac{1}{2}}} \right),
\]

since \(\log \log T \leq \frac{3}{5} \log \log x\). Therefore, (37) implies

\[
x^{\sigma_2(T)-1} \log x \leq \exp \left( -k_2(x, W_3) \frac{(\log x)^{\frac{3}{2}}}{(\log \log x)^{\frac{1}{2}}} \right),
\]

(39)
with

\[ k_2(x, W_3) = \frac{(5/3)^{\frac{4}{7}}}{W_3} - k_0(x). \]

Observe that \( k_2(x, W_3) > 0 \) only when \( x \) is sufficiently large. Additionally, since \( W_3 \geq Z_3 \), it is clear from the definitions that \( k_2(x, W_3) \leq k_1(x) \), so we also have that \( E_1(x) \leq E_2(x) \), and it is easily verified that \( E_1(x)/E_2(x) \) is decreasing for all \( x \geq x_1 \). Thus,

\[ E_1(x) \leq \frac{E_1(x_0)}{E_2(x_0)} E_2(x). \quad (40) \]

Finally, we can bound each of the terms in (36). Our choice of \( T \), (38), and (40) imply that the first term of (36) satisfies

\[ \frac{\nu_1(x, T) \log x}{T^x} \leq \nu_1(x_0, T_{x_0}) E_1(x) \leq \frac{E_1(x_0) \nu_1(x_0, T_{x_0})}{E_2(x_0)} E_2(x) \]

for all \( x \geq x_0 \). For the second term of (36), see that

\[ \frac{1}{\sqrt{x}} = \exp \left( -\frac{\log x}{2} \right) = \exp \left( \frac{k_1(x)(\log x)^{\frac{3}{2}}}{\log \log x} - \frac{\log x}{2} \right) E_1(x), \]

and the last exponent decreases on \( x \geq x_1 \). So, (40) and \( k_1(x) < 1 \) imply

\[ \frac{1}{\sqrt{x}} \left( 2.4 + \frac{41.155}{x} \right) \leq 2.4 + \frac{41.155}{x_0} \exp \left( \frac{(\log x_0)^{\frac{3}{2}}}{\log \log x_0} - \frac{\log x_0}{2} \right) \frac{E_1(x)}{E_2(x)}, \]

for all \( x \geq x_0 \geq x_1 \).

Next, (38), (40), and the estimate \( \log \log T \leq \frac{3}{5} \log \log x \) imply that the third term in (36) satisfies

\[
\frac{R_3(W_3, T)(\log T)^{\frac{3}{2}}(\log \log T)^{\frac{1}{2}} e}{\pi T \log x} = \frac{R_3(W_3, T)(\log T)^{\frac{3}{2}}(\log \log T)^{\frac{1}{2}} e}{\pi (\log x)^2} \left( \frac{\log x}{T} \right) \leq \frac{R_3(W_3, T_{x_0})(\log T)^{\frac{3}{2}}(\frac{3}{5} \log \log x)^{\frac{1}{2}} e}{\pi (\log x)^2} E_1(x) \leq \frac{R_3(W_3, T_{x_0})(3/5)^{\frac{1}{2}} e (\log x_0)^{\frac{5}{4}} e}{\pi (\log x_0)^{\frac{5}{2}}} E_1(x_0) E_2(x). \]

The estimate \( \log \log T \leq \frac{3}{5} \log \log x \) and (39) imply that the fourth term in (36) satisfies

\[
\frac{R_3(W_3, T_0)^{x \sigma_2(T)-1}}{\pi} \int_{T_0}^{T} \frac{(\log t)^{\frac{1}{2}}(\log \log t)^{\frac{1}{2}}}{t} dt \leq \frac{R_3(W_3, T_0)^{x \sigma_2(T)-1}(\log \log T)^{\frac{1}{2}}}{\pi} \int_{T_0}^{T} \frac{(\log t)^{\frac{1}{2}}}{t} dt
\]
so we also have

$$\leq \frac{3R_3(W_3, T_0)x^{\sigma_2(T)} - (\frac{3}{5}\log \log x)^{\frac{1}{2}}}{5\pi} (\log T)^{\frac{5}{3}}$$

$$\leq \frac{(3/5)^{\frac{3}{2}}R_3(W_3, T_0)}{\pi(\log \log x_0)^{\frac{1}{2}}} E_2(x).$$

Moreover, (39) implies

$$\log T_0 \leq \frac{(\log x)^{\frac{3}{2}}(\log \log x)^{\frac{1}{2}}}{W_1},$$

then we also have

$$x^{\sigma_1(T_0) - 1} \leq \exp \left( - \frac{(\log x)^{\frac{3}{2}}}{(\log \log x)^{\frac{1}{2}}} \right) \leq \exp \left( \frac{(k_2(x_0, W_3) - 1)(\log x)^{\frac{3}{2}}}{(\log \log x_0)^{\frac{1}{2}}} \right) E_2(x),$$

since $k_2(x_0, W_3) - 1 < 0$. So, if $T_0$ satisfies (41), then

$$\left( \frac{R_1(W_1, H)((\log T_0)^2 - (\log H)^2)}{2\pi} + 1.26 \cdot 10^{-5} \right) x^{\sigma_1(T_0) - 1}
\leq \left( \frac{R_1(W_1, H)((\log T_0)^2 - (\log H)^2)}{2\pi} + 1.26 \cdot 10^{-5} \right) \exp \left( \frac{(k_2(x_0, W_3) - 1)(\log x)^{\frac{3}{2}}}{(\log \log x_0)^{\frac{1}{2}}} \right) E_2(x).$$

Moreover, (39) implies

$$x^{\sigma_2(T_0) - 1} \leq \frac{x^{\sigma_2(T)}}{x \log x} \leq \frac{E_2(x)}{\log x},$$

so we also have

$$\frac{R_1(W_1, T_0)(\sigma_2(T_0) - \sigma_1(T_0)) \log T_0}{\pi T_0} x^{\sigma_2(T_0) - 1} \leq \frac{R_1(W_1, T_0)(\sigma_2(T_0) - \sigma_1(T_0)) \log T_0}{\pi T_0 \log x_0} E_2(x).$$

Combining all preceding observations in this subsection, we conclude that if $k_2(x, W_3) > 0$ and $T_0$ satisfies (30) and (41), then we have from (36) that for all $x \geq x_0 \geq x_1$,

$$|M(x)| \leq c_3(x_0) x \exp \left( -c_4(x_0) \frac{(\log x)^{\frac{3}{2}}}{(\log \log x)^{\frac{1}{2}}} \right),$$

in which $c_4(x) = k_2(x, W_3)$ and

$$c_3(x_0) = \left( \nu_1(x_0, T_{x_0}) + \left( 2.4 + \frac{41.155}{x_0} \right) \exp \left( \frac{(\log x_0)^{\frac{3}{2}}}{(\log \log x_0)^{\frac{1}{2}}} - \frac{\log x_0}{2} \right) \right) \frac{E_1(x_0)}{E_2(x_0)}$$

$$+ \frac{R_3(W_3, T_{x_0})(3/5)^{\frac{3}{2}}c (\log \log x_0)^{\frac{1}{2}}}{\pi(\log x_0)^{\frac{1}{2}}} E_1(x_0)$$

$$+ \frac{(3/5)^{\frac{3}{2}}R_3(W_3, T_{x_0})}{\pi(\log x_0)^{\frac{1}{2}}} + \frac{R_1(W_1, T_0)(\sigma_2(T_0) - \sigma_1(T_0)) \log T_0}{\pi T_0 \log x_0}$$

$$+ \left( \frac{R_1(W_1, H)((\log T_0)^2 - (\log H)^2)}{2\pi} + 1.26 \cdot 10^{-5} \right) e^{\frac{(k_2(x_0, W_3) - 1)(\log x_0)^{3/5}}{(\log \log x_0)^{1/5}}}. $$
Under these choices, we note that if \( \log x \) have that the condition (30) is satisfied, while the condition (41) is satisfied only when \( x \) range of and Ramaré [18] proved that we can use our results to obtain computations for the extended range \( \log x \). This logic could be applied to remove the condition (41) from our proof, which would mean choosing \( T \) therefore choosing \( c \) to do computations for a more restricted range of \( x \). For an extreme example, consider \( \log x_0 = 10^{10} \) and note that (41) tells us we must have \( T_0 \leq \exp(3\,368.76) \). Therefore, choosing \( T_0 \) as large as possible, we can take

\[
\max\{R_1(W_1, H), R_1(W_1, T_0)\} \leq 3.422, \quad \max\{R_3(W_3, T_0), R_3(W_3, T)\} \leq 40.944;
\]

these computations follow from Tables 2 and 3. So, this choice of \( T_0 \) satisfies all the conditions when \( \log x \geq 95\,191.34 \), and our computations assuming these choices for \( c_3(x_0) \) and \( c_4(x_0) \), which are the same functions in the statement of Theorem 1.2, are presented in Table 1.

**Remark.** If we make the same choices for \( W_1, W_3 \), but assert a larger value of \( T_0 \), then better computations for \( c_3(x_0) \) are available. The drawback of this is that we can only take a larger value of \( T_0 \) to do computations for a more restricted range of \( x \). For an extreme example, consider \( \log x_0 = 10^{10} \) and note that (41) tells us we must have \( T_0 \leq \exp(3\,368.76) \). Therefore, choosing \( T_0 \) as large as possible, we can take

\[
\max\{R_1(W_1, H), R_1(W_1, T_0)\} \leq 3.422 \quad \text{and} \quad \max\{R_3(W_3, T_0), R_3(W_3, T)\} \leq 35.142.
\]

It follows that \( c_3(x_0) = 4.54829 \) and \( c_4(x_0) = 0.02192 \) would be admissible constants when \( \log x_0 = 10^{10} \).

**Remark.** Recall from the proof of Lemma 4.2 that (30) ensured \( \sigma_1(T_0) \leq \sigma_2(T_0) \), so we could use (39) to see that

\[
x^{\sigma_1(T_0) - 1} \leq x^{\sigma_2(T_0) - 1} \leq \frac{x^{\sigma_2(T)} \log x}{x \log x} \leq \frac{E_2(x)}{\log x_0}.
\]

This logic could be applied to remove the condition (41) from our proof, which would mean that we can use our results to obtain computations for the extended range \( \log x \geq 72\,775.43 \), although this would diminish the outcomes we already have.

5.3. **Proof of Corollary 1.3.** A straightforward computation will show that \( |M(x)| \leq 4 \) for \( 1 \leq x \leq 32 \). In addition, Hurst [11] has verified computationally that if \( 33 \leq x \leq 10^{10} \), then \( |M(x)| \leq 0.571\sqrt{x} \). Next, Cohen, Dress, and El Marraki [4] proved

\[
|M(x)| < \frac{x}{4\,345} \quad \text{for} \quad x \geq 2 \, 160 \, 535, \tag{42}
\]

and Ramaré [18] proved

\[
|M(x)| < \frac{0.013 \, x}{\log x} - \frac{0.118 \, x}{(\log x)^2} \quad \text{for} \quad x \geq 1 \, 078 \, 853. \tag{43}
\]

To find the least \( \log x \) to three decimal places such that the right-hand side of (43) is smaller than the right-hand side of (42), set \( \log x = 36 \) and increment \( \log x \) by 0.001 until the right-hand side of (43) is smaller than the right-hand side of (42); the output is 45.123. Simple alterations can be made to speed up the outcome of this algorithm.

Next, Theorem 1.1 will not be valid until \( \log x \geq 363.11 \), so set \( \log x = 363.11 \) and increment \( \log x \) by 0.001 until the right-hand side of (43) is larger than the right-hand
side of (35) with $W = Z_1$ and $R_1(Z_1, T) \leq 2.134$; the output is $1772.504$. Repeat this process, making changes mutatis mutandis, to find where Theorem 1.2 becomes the better bound. Supposing we did our computations for $c_3(x)$ and $c_4(x)$ by setting $T_0 = \hat{H}$, $W_1 = Z_1$, $W_3 = Z_3$, $\max\{R_1(W_1, H), R_1(W_1, T_0)\} \leq 3.422$, and $\max\{R_3(W_3, T_0), R_3(W_3, T)\} \leq 40.944$, the outcome of our analysis is

$$|M(x)| \leq \begin{cases} 
4 & \text{if } 1 \leq x \leq 32, \\
0.571\sqrt{x} & \text{if } 33 \leq x \leq 10^{16}, \\
\frac{x^{\frac{4}{11}}}{\log x} - \frac{0.118x}{(\log x)^2} & \text{if } 10^{16} < x \leq e^{45.123}, \\
c_1(x)x\exp\left(-c_2(x)\sqrt{\log x}\right) & \text{if } e^{45.123} < x \leq e^{1772.504}, \\
c_3(x)\exp\left(-c_4(x)(\log x)^{\frac{3}{2}}(\log \log x)^{-\frac{1}{2}}\right) & \text{if } x > \exp(e^{36.821}). 
\end{cases}$$

(44)

In the corresponding ranges of $x$, $0.09797 < c_1(x) < 0.23427$, $0.24647 < c_2(x) < 0.42415$, $c_3(x) < 5.09591$, and $c_4(x) > 0.02196$. Furthermore, $c_1(x)$ decreases and computational experiments verify that

$$c_1(x) \leq u(x) := 0.09798 + \frac{0.0055 + 20.8944\exp\left(-\frac{0.4037}{\log \log x}\right)}{\log \log x}$$

for all $7.480 \leq \log \log x \leq 36.821$; note that $\log(1772.505) \approx 7.481$. Figure 1 demonstrates that $u(x)$ is a very sharp upper bound for $c_1(x)$ in this range. Combining these observations together, (4) follows naturally.

**Remark.** To find the function $u(x)$, we observed from the data that there must exist a decreasing function $\Delta$ such that $c_1(x) \leq 0.09798 + \Delta(\log \log x)/\log \log x$; the shape of this bound is chosen to be convenient. Next, we used curve-fitting software to describe the line of best fit through a sample of points $(y, \Delta(y))$, with $7.480 \leq y \leq 36.821$; almost all of the decaying behaviour is exhibited in the range $7.480 \leq y \leq 26.481$. The choice

$$\Delta(y) = \tau + 20.8944\exp\left(-\frac{0.4037}{y}\right)$$

for some $\tau > 0$ followed from this analysis. All that remained was to increment $\tau$ until a value was found such that the upper bound held for every $x$ under consideration.

**Remark.** Daval has released a preprint [5] which refines (42) into

$$|M(x)| < \frac{x}{160383} \quad \text{for } x \geq 8.4 \cdot 10^9.$$
This result will change the ranges (and shape) of (4). That is, (4) would become

\[
|M(x)| \leq \begin{cases} 
4 & \text{if } 1 \leq x \leq 32, \\
0.571 \sqrt{x} & \text{if } 33 \leq x \leq 10^{16}, \\
\frac{x}{160.383} & \text{if } 10^{16} < x \leq e^{1.806.498}, \\
c_1(x) x \exp \left( -c_2(x) \sqrt{\log x} \right) & \text{if } e^{1.806.498} < x \leq \exp(e^{36.821}), \\
\frac{x}{c_3(x) x \exp \left( -c_4(x) (\log x)^{3/2} (\log \log x)^{-1/2} \right)} & \text{if } x > \exp(e^{36.821}).
\end{cases}
\]

Since \( u(x) \) is already a sharp upper bound, it follows that the final result would become

\[
|M(x)| \leq \begin{cases} 
4 & \text{if } 1 \leq x \leq 32, \\
0.571 \sqrt{x} & \text{if } 33 \leq x \leq 10^{16}, \\
\frac{x}{160.383} & \text{if } 10^{16} < x \leq e^{1.806.498}, \\
u(x) x \exp \left( -c_2(x) \sqrt{\log x} \right) & \text{if } e^{1.806.498} < x \leq \exp(e^{36.821}), \\
5.09591 x \exp \left( -0.02196 (\log x)^{3/2} (\log \log x)^{-1/2} \right) & \text{if } x > \exp(e^{36.821}).
\end{cases}
\]

To be safe, we opted to not use Daval’s unpublished result, but it is interesting to see how Corollary \ref{cor:1.3} would improve once it is published.
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