FUNDAMENTAL GROUPS OF SMALL COVERS REVISITED

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Abstract. We study the topology of small covers from their fundamental groups. We find a way to obtain explicit presentations of the fundamental group of a small cover. Then we use these presentations to study the relations between the fundamental groups of a small cover and its facial submanifolds. In particular, we can determine when a facial submanifold of a small cover is $\pi_1$-injective in terms of some purely combinatorial data on the underlying simple polytope. In addition, we find that any 3-dimensional small cover has an embedded non-simply-connected $\pi_1$-injective surface. Using this result and some results of Schoen and Yau, we characterize all the 3-dimensional small covers that admit Riemannian metrics with nonnegative scalar curvature.

1. Introduction

The notion of small cover is first introduced by Davis and Januskiewicz [7] as an analogue of a smooth projective toric variety in the category of closed manifolds with $\mathbb{Z}_2$-torus actions ($\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$). An $n$-dimensional small cover is a closed $n$-manifold $M$ with a locally standard ($\mathbb{Z}_2$)$^n$-action whose orbit space can be identified with a simple convex polytope $P$ in an Euclidean space. A polytope is called simple if every codimension-$k$ face is the intersection of exactly $k$ distinct facets (codimension-one faces) of the polytope. Recall that two convex polytopes are combinatorially equivalent if there exists a bijection between their posets of faces with respect to the inclusion. All polytopes considered in this paper are convex, so we omit the word “convex” for brevity. And in most cases we make no distinction between a convex polytope and its combinatorial equivalence class.

The ($\mathbb{Z}_2$)$^n$-action on the small cover $M$ determines a ($\mathbb{Z}_2$)$^n$-valued characteristic function $\lambda$ on the set of facets of $P$, which encodes all the information of the isotropy groups of the non-free orbits. Indeed for any facet $F$ of $P$, the rank-one subgroup $\langle \lambda(F) \rangle \subset (\mathbb{Z}_2)^n$ generated by $\lambda(F)$ is the isotropy group of the codimension-one submanifold $\pi^{-1}(F)$ of $M$ where $\pi : M \to P$ is the orbit map of...
the \((\mathbb{Z}_2)^n\)-action. The function \(\lambda\) is non-degenerate in the sense that the values of \(\lambda\) on any \(n\) facets that are incident to a vertex of \(P\) form a basis of \((\mathbb{Z}_2)^n\). Conversely, we can recover the manifold \(M\) by gluing \(2^n\) copies of \(P\) according to the function \(\lambda\). For any proper face \(f\) of \(P\), define

\[ G_f = \text{the subgroup of } (\mathbb{Z}_2)^n \text{ generated by the set } \{\lambda(F) \mid f \subset F\}. \tag{1} \]

And define \(G_P = \{0\} \subset (\mathbb{Z}_2)^n\). Then \(M\) is homeomorphic to the quotient space

\[ P \times (\mathbb{Z}_2)^n / \sim \]  

where \((p,g) \sim (p',g')\) if and only if \(p = p'\) and \(g^{-1}g' \in G_{f(p)}\), and \(f(p)\) is the unique face of \(P\) that contains \(p\) in its relative interior. Let \(\Theta\) be the quotient map in (2).

\[ \Theta : P \times (\mathbb{Z}_2)^n \to P \times (\mathbb{Z}_2)^n / \sim. \tag{3} \]

It is shown in [2] that many important topological invariants of \(M\) can be easily computed in terms of the combinatorial structure of \(P\) and the characteristic function \(\lambda\). In particular, we can determine the fundamental group \(\pi_1(M)\) of \(M\) as follows. Let \(W_P\) be a right-angled Coxeter group with one generator \(s_F\) and relations \(s_F^2 = 1\) for each facet \(F\) of \(P\), and \((s_F s_{F'})^2 = 1\) whenever \(F, F'\) are adjacent facets of \(P\). Note that if \(F \cap F' = \emptyset\), \(s_F s_{F'}\) has infinite order in \(W_P\) (see [2] Proposition 1.1.1)]. According to [7] Lemma 4.4, \(W_P\) is isomorphic to the fundamental group of the Borel construction \(M_{(\mathbb{Z}_2)^n} = E(\mathbb{Z}_2)^n \times (\mathbb{Z}_2)^n, M\) of \(M\). It is shown in [7] Corollary 4.5] that the homotopy exact sequence of the fibration \(M \to M_{(\mathbb{Z}_2)^n} \to B(\mathbb{Z}_2)^n\) gives a short exact sequence

\[ 1 \to \pi_1(M) \xrightarrow{\psi} W_P \xrightarrow{\phi} (\mathbb{Z}_2)^n \to 1, \tag{4} \]

where \(\phi(s_F) = \lambda(F)\) for any facet \(F\) of \(P\), and \(\psi\) is induced by the canonical map \(M \hookrightarrow M \times E(\mathbb{Z}_2)^n \to M_{(\mathbb{Z}_2)^n}\). Hence \(\pi_1(M)\) is isomorphic to the kernel of \(\phi\). It follows that a small cover \(M\) is never simply connected. Moreover, the sequence (4) actually splits, so \(W_P\) is isomorphic to a semidirect product of \(\pi_1(M)\) and \((\mathbb{Z}_2)^n\).

Let \(\mathcal{F}(P)\) denote the set of facets of \(P\). For any proper face \(f\) of \(P\), we have the following definitions.

- Define \(\mathcal{F}(f^\perp) = \{F \in \mathcal{F}(P) \mid \dim(f \cap F) = \dim(f) - 1\}\). In other words, \(\mathcal{F}(f^\perp)\) consists of those facets of \(P\) that intersect \(f\) transversely.

- We call \(M_f = \pi^{-1}(f)\) the facial submanifold of \(M\) corresponding to \(f\). It is easy to see that \(M_f\) is a small cover over the simple polytope \(f\), whose characteristic function, denoted by \(\lambda_f\), is determined by \(\lambda\) as follows. Let

\[ \rho_f : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n / G_f \cong (\mathbb{Z}_2)^{\dim(f)} \]
be the quotient homomorphism. Then we have
\[ \lambda_f(f \cap F) = \rho_f(\lambda(F)), \; \forall F \in \mathcal{F}(f^\perp). \] (5)

- Similarly to \( W_P \), we obtain two group homomorphisms for \( W_f \):
\[ \psi_f : \pi_1(M_f) \to W_f, \quad \phi_f : W_f \to (\mathbb{Z}_2)^{\dim(f)}. \]

- Let \( i_f : f \to P \) and \( j_f : M_f \to M \) be the inclusion maps. Then \( i_f \) induces a natural group homomorphism \((i_f)_* : W_f \to W_P\) which sends the generator \( s_{f \cap F} \) of \( W_f \) to the generator \( s_F \) of \( W_P \) for any facet \( F \in \mathcal{F}(f^\perp) \). It is easy to check that \((i_f)_*\) is well defined.

Then we have the following diagram.

\[
\begin{array}{cccccc}
1 & \to & \pi_1(M_f) & \xrightarrow{\psi_f} & W_f & \xrightarrow{\phi_f} (\mathbb{Z}_2)^{\dim(f)} & \to & 1 \\
\downarrow{(i_f)_*} & & \downarrow{??} & & \downarrow{(i_f)_*} & & \downarrow{\rho_f} \\
1 & \to & \pi_1(M) & \xrightarrow{\psi} & W_P & \xrightarrow{\phi} (\mathbb{Z}_2)^n & \to & 1
\end{array}
\]

One may expect that this diagram commutes, i.e. \((i_f)_* \circ \psi_f = \psi \circ (j_f)_*\). But this is not true in general. Indeed, \((i_f)_*\) may not map \( \ker(\phi_f) \) into \( \ker(\phi) \) (see Example 1.1). It seems to us that there is no canonical way to relate the two maps \((j_f)_*\) and \((i_f)_*\), essentially because there is no canonical way to embed \((\mathbb{Z}_2)^{\dim(f)} = (\mathbb{Z}_2)^n/G_f\) into \((\mathbb{Z}_2)^n\). So to understand \((j_f)_* : \pi_1(M_f) \to \pi_1(M)\), we need to study the fundamental groups of small covers by some other methods.

**Example 1.1.** In Figure 1 we have a facet \( F \) in a 3-dimensional simple polytope \( P \) along with a \((\mathbb{Z}_2)^3\)-coloring, where \( e_1, e_2, e_3 \) is a basis of \((\mathbb{Z}_2)^3\). Then the Coxeter group \( W_F \) is generated by \( \{s_{F \cap F_i}\}_{1 \leq i \leq 4} \) and \((i_F)_* : W_F \to W_P\) sends \( s_{F \cap F_i} \) to the generator \( s_{F_i} \) of \( W_P \). Then \((i_F)_*(s_{F \cap F_1}s_{F \cap F_3}^{-1}) = s_{F_1}s_{F_3}^{-1}\). But

- \( \phi_F(F \cap F_1) = \phi_F(F \cap F_3) \), then \( s_{F \cap F_1}s_{F \cap F_3}^{-1} \in \ker(\phi_F) \subset W_F; \)
- \( \phi(F_1) = e_1 + e_3, \; \phi(F_3) = e_3, \) then \( s_{F_1}s_{F_3}^{-1} \notin \ker(\phi) \subset W_P. \)

So \((i_F)_*\) does not map \( \ker(\phi_F) \) into \( \ker(\phi) \).
A motive to study \((j_f)_* : \pi_1(M_f) \rightarrow \pi_1(M)\) is to see under what condition it is injective. A submanifold \(\Sigma\) embedded in \(M\) is called \(\pi_1\)-injective if the inclusion of \(\Sigma\) into \(M\) induces a monomorphism in the fundamental group. The \(\pi_1\)-injective submanifolds play important roles in the studying of geometry and topology of low dimensional manifolds. For example the existence of \(\pi_1\)-injective immersed surfaces with nonpositive Euler characteristic in a compact oriented 3-manifold is an obstruction to the existence of Riemannian metric with positive scalar curvature on the manifold (see \([26]\)). In addition, if an irreducible orientable 3-manifold has a non-simply-connected \(\pi_1\)-injective embedded surface, then the 3-manifold is a Haken manifold. It is shown in \([31]\) that Haken 3-manifolds satisfy the Borel Conjecture, so they are determined up to homeomorphism by their fundamental groups. The main result of this paper is the following theorem on the \(\pi_1\)-injectivity of a facial submanifold in a small cover.

**Theorem 1.2** (Theorem 3.3). Let \(M\) be a small cover over a simple polytope \(P\) and \(f\) be a proper face of \(P\). Then the following two statements are equivalent.

(i) The facial submanifold \(M_f\) is \(\pi_1\)-injective in \(M\).

(ii) For any \(F, F' \in \mathcal{F}(f^\perp)\), we have \(f \cap F \cap F' \neq \emptyset\) whenever \(F \cap F' \neq \emptyset\).

This result is a bit surprising since it tells us that the \(\pi_1\)-injectivity of \(M_f\) in \(M\) depends only on the combinatorics of the facets of \(P\) adjacent to \(f\). However the fundamental group of \(M\) depends not only on the combinatorial structure of \(P\) but also on the characteristic function of \(M\) in general.

The main ingredient of our proof of Theorem 3.3 is to use explicit presentations of the fundamental groups of small covers. We are also aware that methods from metric geometry can be used to prove \((\text{ii}) \Rightarrow (\text{i})\) in Theorem 3.3 when \(M\) is an aspherical manifold (see Proposition 3.6 and Remark 3.7). By \([8, \text{Theorem 2.2.5}]\), a small cover \(M\) over a simple polytope \(P\) is aspherical if and only if \(P\) is flag (i.e. a collection of facets of \(P\) have common intersection whenever they pairwise intersect).

For any \(n\)-dimensional simple polytope \(P\), let \(P^*\) be the dual (or polar) polytope of \(P\) (see \([33, \text{§2}]\)). Then its boundary \(\partial P^*\) is a simplicial \((n-1)\)-sphere. Any codimension-\(k\) face \(f\) of \(P\) corresponds to a unique \((k-1)\)-simplex in \(\partial P^*\) denoted by \(\sigma_f\). If \(f\) is the intersection of facets \(F_1, \ldots, F_k\) of \(P\), the vertices of \(\sigma_f\) are \(\sigma_{F_1}, \ldots, \sigma_{F_k}\). Let \(\text{Lk}(\sigma_f, \partial P^*)\) be the link of \(\sigma_f\) in \(\partial P^*\). Then the vertices of \(\text{Lk}(\sigma_f, \partial P^*)\) are \(\{\sigma_F | F \in \mathcal{F}(f^\perp)\}\).

**Remark 1.3.** The condition \((\text{ii})\) in Theorem 3.3 is equivalent to saying that two vertices in \(\text{Lk}(\sigma_f, \partial P^*)\) are connected by an edge in \(\text{Lk}(\sigma_f, \partial P^*)\) if and only if they are connected by an edge in \(\partial P^*\). But in general this condition does not imply that \(\text{Lk}(\sigma_f, \partial P^*)\) is a full subcomplex of \(\partial P^*\) (see \([3, \text{p.26}]\)). For example when \(P\)
is the 3-simplex $\Delta^3$, any 2-face $f$ of $P$ satisfies the condition (ii) in Theorem 3.3 while $\text{Lk}(\sigma_f, \partial P^*)$ is not a full subcomplex of $\partial P^* = \partial \Delta^3$.

A 3-belt on a simple polytope $P$ consists of three facets $F_1, F_2, F_3$ of $P$ which pairwise intersect but have no common intersection.

**Corollary 1.4.** Let $M$ be a small cover over a simple polytope $P$. For a facet $F$ of $P$, the facial submanifold $M_F$ is $\pi_1$-injective in $M$ if and only if $F$ is not contained in any 3-belt on $P$.

In addition, we obtain the following description of aspherical small covers in terms of the $\pi_1$-injectivity of their facial submanifolds.

**Proposition 1.5** (Proposition 3.6). A small cover $M$ over a simple polytope $P$ is aspherical if and only if all the facial submanifolds of $M$ are $\pi_1$-injective in $M$.

**Remark 1.6.** For a small cover $M$, a facial submanifold $M_f$ is $\pi_1$-injective in $M$ does not imply that $M_{f'}$ is $\pi_1$-injective in $M$ for any $f' \subset f$. Conversely, that $M_{f'}$ is $\pi_1$-injective in $M$ for all $f' \subset f$ does not imply that $M_f$ is $\pi_1$-injective in $M$ either.

**Example 1.7.** Let $M$ be any small cover over the 3-dimensional simple polytope $P_1$ or $P_2$ shown in Figure 2. By Theorem 3.3 we can conclude the following.

- For the triangular facet $F$ of $P_1$, the facial submanifold $M_F$ is $\pi_1$-injective in $M$. But for any 1-face $f \subset F$, $M_f$ is not $\pi_1$-injective in $M$.

- For the hexagonal facet $F'$ of $P_2$ whose vertex set is $\{v_1, v_2, v_3, v_4, v_5, v_6\}$, the facial submanifold $M_{F'}$ is not $\pi_1$-injective in $M$. But for any face $f \subset F'$, $M_f$ is $\pi_1$-injective in $M$.

Using Corollary 1.4 we can show that any 3-dimensional small cover has a non-simply-connected embedded $\pi_1$-injective surface.
Proposition 1.8 (Proposition 4.1). For any small cover $M$ over a 3-dimensional simple polytope $P$, there always exists a facet $F$ of $P$ so that the facial submanifold $M_F$ is $\pi_1$-injective in $M$.

By Proposition 4.1 and some results in Schoen-Yau [26], we can characterize all the 3-dimensional small covers that admit Riemannian metrics with nonnegative scalar curvature.

Proposition 1.9 (Proposition 4.9). A small cover $M$ over a simple 3-polytope $P$ can hold a Riemannian metric with nonnegative scalar curvature if and only if $P$ is combinatorially equivalent to the cube $[0,1]^3$ or a polytope obtained from $\Delta^3$ by a sequence of vertex cuts. In particular, all the orientable 3-dimensional small covers that can hold Riemannian metrics with nonnegative scalar curvature are the two orientable real Bott manifolds in dimension 3 and the connected sum of $k$ copies of $\mathbb{R}P^3$ for any $k \geq 1$.

The paper is organized as follows. In section 2, we construct some explicit presentation of the fundamental group of a small cover $M$ over a simple polytope $P$ (see Proposition 2.1) by constructing a special cell decomposition of $M$. Using this presentation of $\pi_1(M)$ and the presentation of the right-angled Coxeter group $W_P$, we construct an explicit monomorphism from $\pi_1(M)$ into $W_P$. In section 3, we obtain a necessary and sufficient condition for a facial submanifold $M_f$ of $M$ to be $\pi_1$-injective (Theorem 3.3). Moreover, we determine the kernel of the homomorphism $(j_f)_* : \pi_1(M_f,v) \to \pi_1(M,v)$ in Theorem 3.4. In section 4, we show in Proposition 4.1 that any 3-dimensional small cover has a non-simply-connected embedded $\pi_1$-injective surface. Then we characterize all the 3-dimensional small covers and real moment-angle manifolds that admit Riemannian metrics with nonnegative scalar curvature.

2. Presentations of the fundamental groups of small covers

2.1. Cell decompositions of small covers.

To obtain a presentation of the fundamental group of a small cover $M$, we can use the 2-skeleton of a cell decomposition of $M$. In [7] two kinds of cell decompositions of small covers are constructed.

(I) The first construction is to use a special type of Morse functions on $P$. The corresponding cell decompositions of the small cover $M$ are “perfect” in the sense that their cellular chain complexes with $\mathbb{Z}_2$-coefficients have trivial boundary maps.

(II) The second construction is to use a standard cubical subdivision of $P$ and decompose the small cover $M$ as a union of big cubes via the gluing procedure in [2].
But it is not so convenient for us to write a presentation of \( \pi_1(M) \) from either of these cell decompositions. In the construction (I), the gradient flow of the Morse function on \( M \) may flow from one critical point to another one with equal or higher index. So to obtain the representatives of the generators of \( \pi_1(M) \), we need to make some modification of the flow lines which could be hard to handle. In the construction (II), there are more than one 0-cell in the decomposition. So in reality we need to shrink a maximal tree in the 1-skeleton to a point to obtain the representatives of the generators of \( \pi_1(M) \). But there is no natural choice of a maximal tree in the 1-skeleton of this cell decomposition. In the following, we will slightly modify the construction (II) and obtain a new cell decomposition of \( M \) which has only one 0-cell.

Recall how to obtain a cubical subdivision of an \( n \)-dimensional simple polytope \( P \) (see [Sec.1.2]). First of all, the set of faces of \( P \), partially ordered by inclusion, forms a poset denoted by \( \mathcal{P} \). Let \( FL(\mathcal{P}) \) be the order complex of \( \mathcal{P} \) which consists of all flags in \( \mathcal{P} \). For any flag \( \alpha = (f_1 < \cdots < f_k) \in FL(\mathcal{P}) \), let \( |\alpha| = k \).

For any \( f, f' \in \mathcal{P} \) with \( f \leq f' \), we have the following definitions.

- Denote by \([f, f']\) the subposet \( \mathcal{P}_{\geq f} \cap \mathcal{P}_{\leq f'} \) of \( \mathcal{P} \).
- Denote by \([f, f']\) the subposet \( \mathcal{P}_{\geq f} \cap \mathcal{P}_{< f'} \) of \( \mathcal{P} \).

The order complex of \([f, f']\) and \([f, f']\) are denoted by \( FL([f, f']) \) and \( FL([f, f']) \), respectively. Then \( FL([f, f']) \) is a subcomplex of \( FL([f, f']) \).

For any face \( f \) of \( P \), let \( b_f \) be the barycenter of \( f \) (any point in the relative interior of \( f \) would suffice). Then any flag \( \alpha = (f_1 < \cdots < f_k) \in FL(\mathcal{P}) \) determines a unique simplex \( \Delta_\alpha \) of dimension \( |\alpha|-1 \) which is the convex hull of \( b_{f_1}, \cdots, b_{f_k} \). It is clear that the collection of simplices \( \{\Delta_\alpha \mid \alpha \in FL(\mathcal{P})\} \) defines a triangulation of \( P \), denoted by \( T(P) \), is called the barycentric subdivision of \( P \).

For any \( f \leq f' \in \mathcal{P} \), define

\[
\square_{[f,f']} = \bigcup_{\alpha \in FL([f,f'])} \Delta_\alpha; \quad \Delta_{[f,f']} = \bigcup_{\alpha \in FL([f,f'])} \Delta_\alpha.
\]

(6)

Note that \( \Delta_{[f,f']} \) is simplicially isomorphic the barycentric subdivision of a simplex of dimension \( \dim(f') - \dim(f) - 1 \). The complex \( \square_{[f,f']} \) is the cone of \( \Delta_{[f,f']} \) with the point \( b_{f'} \), i.e. \( \square_{[f,f']} = \text{Cone}_{b_{f'}}(\Delta_{[f,f']}) \). In fact \( \square_{[f,f']} \) is simplicially isomorphic to the standard simplicial subdivision of the cube of dimension \( \dim(f') - \dim(f) \).

The standard cubical subdivision of \( P \), denoted by \( \mathcal{C}(P) \), is the subdivision of \( P \) into \( \{\square_{[f,f']} \mid f \leq f' \in \mathcal{P}\} \).

For a small cover \( M \) over an \( n \)-dimensional simple polytope \( P \), the cubical subdivision of \( P \) and the construction (2) of \( M \) determines a cubical decomposition of \( M \), denoted by \( \mathcal{C}^*(M) \). The cubes in \( \mathcal{C}^*(M) \) are called small cubes which are images of the cubes in the \( 2^n \) copies of \( P \) under the quotient map \( \Theta \) in (3). For
any face \( f \) of \( P \), we define a family of big cubes associated to \( f \) in \( M \) by:

\[
C_f^{(g)} = \bigcup_{h \in g + G_f} \Theta(\square_{[f,P]}, h), \ g \in (\mathbb{Z}_2)^n, \ f \in \mathcal{P}.
\]

The \( G_f \subset (\mathbb{Z}_2)^n \) is defined in (1). Let \( \mathcal{C}(M) \) denote the set of all big cubes in \( M \).

\[
\mathcal{C}(M) = \{C_f^{(g)} | f \in \mathcal{P}, g \in (\mathbb{Z}_2)^n\}.
\]

Notice that \( C_f^{(g)} = C_f^{(g')}, \) if \( g - g' \in G_f \). So there are exactly \( 2^{\text{dim}(f)} \) big cubes associated to the face \( f \) in \( \mathcal{C}(M) \). In particular, there is only one big cube in \( \mathcal{C}(M) \) associated to a vertex \( v \) of \( P \), denoted by \( C_v \). In addition, there are exactly \( 2^n \) 0-cubes in \( \mathcal{C}(M) \) which are \( \{C_P^{(g)} | g \in (\mathbb{Z}_2)^n\} \), where \( P \) is considered as a face of itself. The reader is referred to [8, Sec.1.2] or [3, Ch.4] for more details of this construction.

Since there are \( 2^n \) 0-cells in \( \mathcal{C}(M) \), it is not so convenient for us to write a presentation of \( \pi_1(M) \) from \( \mathcal{C}(M) \). But notice that for any vertex \( v \) of \( P \), all the 0-cells in \( \mathcal{C}(M) \) are contained in the big \( n \)-cube \( C_v \). In the rest of our paper, we identify \( v \) with the center of \( C_v \). If we shrink \( C_v \) to the point \( v \), all other big cubes in \( \mathcal{C}(M) \) will be deformed simultaneously to give us a new cell decomposition of \( M \), denoted by \( \mathcal{D}_v(M) \) (see Figure 3). Specifically, for any face \( f \) of \( P \) and any \( g \in (\mathbb{Z}_2)^n \), let \( D_f^{(g)} \) denote the open cell in \( \mathcal{D}_v(M) \) that comes from the interior of the big cube \( C_f^{(g)} \). So all the open cells in \( \mathcal{D}_v(M) \) are

\[
\mathcal{D}_v(M) = \{D_f^{(g)} | f \subset P, g \in (\mathbb{Z}_2)^n\}. \quad (7)
\]

It is clear that for any face \( f \) of \( P \), \( D_f^{(g)} = D_f^{(g')} \) for any \( g - g' \in G_f \), \( g, g' \in (\mathbb{Z}_2)^n \). Let \( \lambda \) be the characteristic function of \( M \). Then the 2-skeleton of \( M \) with respect to \( \mathcal{D}_v(M) \) consists of the following cells.

- The only 0-cell is \( v \).
- The open 1-cells are \( \{D_F^{(g)} | v \notin F \in \mathcal{F}(P), g \in (\mathbb{Z}_2)^n\} \), where \( D_F^{(g)} = D_F^{(g + \lambda(F))} \) for any \( g \in (\mathbb{Z}_2)^n \). Note that if a facet \( F \) contains \( v \), then \( C_F^{(g)} \subset C_v \) for all \( g \in (\mathbb{Z}_2)^n \), which implies \( D_F^{(g)} = v \).
- The open 2-cells are \( \{D_{F \cap F'}^{(g)} | F \cap F' \neq \emptyset, F, F' \in \mathcal{F}(P), g \in (\mathbb{Z}_2)^n\} \), where

\[
D_{F \cap F'}^{(g)} = D_{F \cap F'}^{(g + \lambda(F))} = D_{F \cap F'}^{(g + \lambda(F) + \lambda(F'))} = D_{F \cap F'}^{(g + \lambda(F) + \lambda(F'))} \quad \text{for any} \ g \in (\mathbb{Z}_2)^n.
\]

Since every 1-cell \( D_F^{(g)} \) in \( \mathcal{D}_v(M) \) is attached to \( v \), \( D_F^{(g)} \) along with an orientation determines a generator of the fundamental group of \( M \). In the polytope \( P \), we orient the line segment \( \overline{vb_F} \) by going from \( v \) to \( b_F \), denoted by \( \overleftarrow{vb_F} \).

- For any facet \( F \) not containing \( v \), the closure of \( D_F^{(g)} \) is the union of \( \Theta(\overline{vb_F}, g) \) and \( \Theta(\overline{vb_F}, g + \lambda(F)) \) as a set (see Figure 3). Considering the
Figure 3. Shrinking the big cube $C_v$ to the point $v$

orientation, let $\beta_{F,g}$ denote the following oriented closed path based at $v$:

$$\beta_{F,g} = \Theta(((vb_F, g)) \cdot \Theta((vb_F, g + \lambda(F))^{-1}, \ g \in (Z_2)^n).$$

Then $\beta_{F,g}$ and $\beta_{F,g+\lambda(F)}$ correspond to the same 1-cell $D^{(g)}_F$, but they have opposite orientations. So we have $\beta_{F,g+\lambda(F)} = \beta_{F,g}^{-1}$ for any $g \in (Z_2)^n$.

Figure 4. The cell $D^{(g)}_{F \cap F'}$

- For any facet $F$ containing $v$, since $D^{(g)}_F = v$, we let $\beta_{F,g}$ denote the trivial loop at the point $v$ for any $g \in (Z_2)^n$, and define $\beta_{F,g+\lambda(F)} = \beta_{F,g}^{-1}$.

We can consider $\{\beta_{F,g} | F \in \mathcal{F}(P), g \in (Z_2)^n\}$ as a set of generators for $\pi_1(M, v)$. Moreover, any 2-cell $D^{(g)}_{F \cap F'}$ in $\mathcal{D}_v(M)$ determines a relation

$$\beta_{F,g} \beta_{F',g+\lambda(F)} = \beta_{F',g} \beta_{F,g+\lambda(F')}, \ g \in (Z_2)^n.$$  

This can be easily seen from the picture of $D^{(g)}_{F \cap F'}$ in Figure 4.
For any facet $F$ of $P$, we define a transformation $\sigma_F : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n$ by:
$$\sigma_F(g) = g + \lambda(F), \ g \in (\mathbb{Z}_2)^n.$$ (8)

Then we obtain the following proposition from the above discussion.

**Proposition 2.1.** By the above notations, a presentation of $\pi_1(M, v)$ is given by
$$\langle \beta_{F,g} , F \in \mathcal{F}(P) , g \in (\mathbb{Z}_2)^n | \beta_{F,g} \beta_{F,g}^{-1} = 1, \forall g \in (\mathbb{Z}_2)^n; \beta_{F,g} \beta_{F',g} \beta_{F',g}^{-1} = \beta_{F',g} \beta_{F',g}^{-1}, F \cap F' \neq \emptyset, \forall g \in (\mathbb{Z}_2)^n; \beta_{F,g} = 1, v \in F, \forall g \in (\mathbb{Z}_2)^n \rangle.$$ (9)

Note that if $F \cap F' \neq \emptyset$ where $v \notin F$, $v \in F'$, the relation (10) can be simplified to $\beta_{F,g} = \beta_{F,g} \beta_{F',g}$ for any $g \in (\mathbb{Z}_2)^n$. So if $v \notin F$ and $F'_1, \ldots, F'_n$ are all the facets of $P$ which contain $v$ and which intersect $F$ at the same time, we have
$$\beta_{F,g} = \beta_{F,g'}, \forall g, g' \in \langle \lambda(F'_1), \ldots, \lambda(F'_n) \rangle \subset (\mathbb{Z}_2)^n.$$ (12)

If we remove the redundant generators $\{ \beta_{F,g} = 1, v \in F, \forall g \in (\mathbb{Z}_2)^n \}$ from the above presentation of $\pi_1(M, v)$, we obtain a simplified presentation of $\pi_1(M, v)$:
$$\langle \beta_{F,g} , v \notin F \in \mathcal{F}(P) , g \in (\mathbb{Z}_2)^n | \beta_{F,g} \beta_{F,g}^{-1} = 1, v \notin F, \forall g \in (\mathbb{Z}_2)^n; \beta_{F,g} \beta_{F',g} \beta_{F',g}^{-1} = \beta_{F',g} \beta_{F',g}^{-1}, v \notin F, v \notin F', F \cap F' \neq \emptyset, \forall g \in (\mathbb{Z}_2)^n; \beta_{F,g} = \beta_{F,g'}, v \notin F, v \in F', F \cap F' \neq \emptyset, \forall g \in (\mathbb{Z}_2)^n \rangle.$$ (13)

It is clear that the above presentation of the fundamental group of $M$ only depends on the choice of the vertex $v$ and the ordering of the facets of $P$.

**Remark 2.2.** For any small cover $M$ over a simple polytope $P$, the minimal number of generators of $\pi_1(M)$ is equal to $m - n$ where $n = \dim(M)$ and $m$ is the number of facets of $P$. Indeed, the number of generators of $\pi_1(M)$ cannot be less than the first $\mathbb{Z}_2$-Betti number $b_1(M; \mathbb{Z}_2) = m - n$, since there is an epimorphism
$$\pi_1(M) \to H_1(M) \to H_1(M; \mathbb{Z}_2).$$

On the other hand, we can obtain a cell decomposition of $M$ from the construction (I) with a single 0-cell and exactly $m - n$ 1-cells.

But the presentation of $\pi_1(M)$ with minimal number of generators is not so useful for our study of the $\pi_1$-injectivity problem of the facial submanifolds of $M$.

### 2.2. Universal covering spaces of small covers.

For any simple polytope $P$, define
$$\mathcal{L}_P = P \times W_P / \sim$$
where $(p, \omega) \sim (p', \omega')$ if and only if $p = p'$ and $\omega' \omega^{-1}$ belongs the the subgroup of $W_P$ that is generated by $\{ s_F \mid F$ is any facet of $P$ that contains $p \}$. Indeed, if
Proof. For any $\mathcal{L}_P$ free with orbit space homeomorphic to $\text{So}((\mathbb{Z}_2)^k)$. This implies

$$(p,\omega) \sim (p,\omega') \iff \omega' = s_{F_1}^{\varepsilon_1} \cdots s_{F_k}^{\varepsilon_k} \cdot \omega, \ v, \varepsilon_1,\ldots,\varepsilon_k \in \{0,1\}. \quad (14)$$

There is a canonical action of $W_P$ on $\mathcal{L}_P$ defined by:

$$\omega' \cdot [(p,\omega)] = [(p,\omega')], \ p \in P, \ v, \omega' \in W_P, \quad (15)$$

where $[(p,\omega)]$ is the equivalence class of $(p,\omega)$ in $\mathcal{L}_P$.

By [6, Corollary 10.2], $\mathcal{L}_P$ is a simply connected manifold. Moreover, $\mathcal{L}_P$ is aspherical if and only if $P$ is a flag polytope (see [17, Proposition 3.4]).

Let $M$ be a small cover over $P$ with characteristic function $\lambda$. We have a homomorphism $\phi : W_P \to (\mathbb{Z}_2)^n$ where $n = \dim(P)$ and $\ker(\phi) \cong \pi_1(M)$ (see (14)).

**Lemma 2.3.** Suppose $F_1, \ldots, F_k$ are facets of $P$ with $F_1 \cap \cdots \cap F_k \neq \emptyset$. Then $\phi(s_{F_1}^{\varepsilon_1} \cdots s_{F_k}^{\varepsilon_k}) \neq 0$ as long as $\varepsilon_1, \ldots, \varepsilon_k \in \{0,1\}$ are not all 0.

**Proof.** Without loss of generality, we can assume that $\lambda(F_1) = e_1, \ldots, \lambda(F_k) = e_k$ where $\{e_1, \ldots, e_n\}$ is a basis of $(\mathbb{Z}_2)^n$. So $\phi(s_{F_1}^{\varepsilon_1} \cdots s_{F_k}^{\varepsilon_k}) = \varepsilon_1 e_1 + \cdots + \varepsilon_k e_k \neq 0$ if $\varepsilon_1, \ldots, \varepsilon_k$ are not all 0. \hfill \Box

The following proposition is contained in [3, Lemma 2.2.4] (but with few details for the proof). We give a proof here since some construction in the proof will be useful for our discussion later.

**Proposition 2.4.** The action of $\pi_1(M) \cong \ker(\phi) \subset W_P$ on $\mathcal{L}_P$ through (15) is free with orbit space homeomorphic to $M$. So $\mathcal{L}_P$ is a universal covering of $M$.

**Proof.** For any $p \in P, \ v, \omega' \in W_P$, we see from (14) and Lemma 2.3 that

$$(p,\omega) \sim (p,\omega') \iff \omega' = 1 \text{ or } \phi(\omega') \neq 0.$$

So $(p,\omega) \sim (p,\omega')$ for any $\omega' \neq 1 \in \ker(\phi)$. This implies that the action of $\ker(\phi)$ on $\mathcal{L}_P$ is free.

Next, we show $\mathcal{L}_P/\ker(\phi)$ is homeomorphic to $M$. Choose a vertex $v$ of $P$ and let $F_1, \ldots, F_n$ be the $n$ facets of $P$ meeting $v$. Suppose all the facets of $P$ are $F_1, \ldots, F_n, F_{n+1}, \ldots, F_m$. We can assume that $\lambda(F_i) = e_i, 1 \leq i \leq n$, where $\{e_1, \ldots, e_n\}$ is a basis of $(\mathbb{Z}_2)^n$. For any $g \in (\mathbb{Z}_2)^n$, define

$$\gamma_g = s_{F_{i_1}} \cdots s_{F_{i_l}} \in W_P, \ g = e_{i_1} + \cdots + e_{i_l} \in (\mathbb{Z}_2)^n, 1 \leq i_1 < \cdots < i_l \leq n. \quad (16)$$

In particular, $\gamma_{\lambda(F_i)} = \gamma_{e_i} = s_{F_i}$ and $\gamma_0 = 1 \in W_P$. It is easy to see that

$$\gamma_g^2 = 1 \in W_P, \ \phi(\gamma_g) = g, \ \forall g \in (\mathbb{Z}_2)^n. \quad (17)$$

So we have a monomorphism $\gamma : (\mathbb{Z}_2)^n \to W_P$ by mapping any $g \in (\mathbb{Z}_2)^n$ to $\gamma_g$. The image of $\gamma$ is the subgroup of $W_P$ generated by $s_{F_1}, \ldots, s_{F_n}$. Note that the definition of $\gamma$ depends on the vertex $v$ we choose.
For any element \( \omega \in W_P \), it is clear that \( \gamma_{\phi(\omega)}^{-1} \in \ker(\phi) \). Define
\[
\xi_{\omega} = \gamma_{\phi(\omega)}^{-1} \in \ker(\phi) \subset W_P.
\]
The definition of \( \xi_{\omega} \) depends on the choice of the vertex \( v \) of \( P \). By (13), we have
\[
\xi_{\omega} \cdot [(p, \omega)] = [(p, \gamma_{\phi(\omega)})], \quad \forall p, \omega \in W_P,
\]
So each orbit of the \( \ker(\phi) \)-action on \( \mathcal{L}_P \) has a representative of the form \([(p, \gamma_g)]\) where \( g \in (\mathbb{Z}_2)^n \).

Moreover, for any point \( p \) in the interior of \( P \), \([(p, \gamma_g)] \in \mathcal{L}_P \) are in the different orbits of the \( \ker(\phi) \)-action if \( g \neq g' \). This is because for any \( \omega \in \ker(\phi) \), we have \( \omega \cdot [(p, \gamma_g)] = [(p, \omega \gamma_g)] \), so \( \phi(\omega \gamma_g) = \phi(\gamma_g) = g \neq g' = \phi(\gamma_{g'}) \).

By the above discussion, a fundamental domain of the \( \ker(\phi) \)-action on the simply connected manifold \( \mathcal{L}_P \) can be taken to be the space \( Q_v \) defined below.
\[
Q_v = P \times \gamma((\mathbb{Z}_2)^n) / \sim
\]
where \( (p, \gamma_g) \sim (p', \gamma_{g'}) \) if and only if \( p = p' \in F_1 \cup \cdots \cup F_n \) and and \( \gamma_g \gamma_g^{-1} \) belongs to the subgroup of \( \gamma((\mathbb{Z}_2)^n) \) spanned by \( \{s_{F_i} = \gamma_{\lambda(F_i)} : p \in F_i, 1 \leq i \leq n\} \).

It is easy to see that \( Q_v \) is the gluing of \( 2^n \) copies of \( P \) by the same rule as (2) along the facets \( F_1, \cdots, F_n \) while leaving other facets \( F_{n+1}, \cdots, F_m \) not glued. Furthermore, the gluing rule (14) for \( \mathcal{L}_P \) tells us that the orbit space \( \mathcal{L} / \ker(\phi) \) is homeomorphic to the quotient of \( Q_v \) by gluing all the copies of \( F_{n+1}, \cdots, F_m \) on its boundary via the rule in (2), which is exactly the small cover \( M \). So the proposition is proved.

2.3. Isomorphism from \( \pi_1(M) \) to \( \ker(\phi) \) defined by group presentations.

For any small cover \( M \) over a simple polytope \( P \), we use the presentation of \( \pi_1(M) \) in Proposition 2.1 to construct an explicit isomorphism from \( \pi_1(M) \) to \( \ker(\phi) \) in this section. This isomorphism will be useful for us to study the relations between the fundamental groups of \( M \) and its facial submanifolds.

Suppose all the facets of \( P \) are \( F_1, \cdots, F_n, F_{n+1}, \cdots, F_m \) where \( v = F_1 \cap \cdots \cap F_n \) is a vertex \( P \). Then we have
\[
s_{F_i} s_{F_{i'}} = s_{F_{i'}} s_{F_i}, \quad 1 \leq i, i' \leq n.
\]
We can assume that \( \lambda(F_i) = e_i, \quad 1 \leq i \leq n \) where \( e_1, \cdots, e_n \) is a basis of \( (\mathbb{Z}_2)^n \).

In the presentation of \( \pi_1(M, v) \) in Proposition 2.1, we let for brevity
\[
\beta_{j,g} = \beta_{F_{j,g}}, \quad n + 1 \leq j \leq m.
\]
In addition, let \( \sigma_i = \sigma_{F_i} : (\mathbb{Z}_2)^n \rightarrow (\mathbb{Z}_2)^n \) where
\[
\sigma_i(g) = g + \lambda(F_i), \quad g \in (\mathbb{Z}_2)^n, 1 \leq i \leq m.
\]
Then according to (13), a (simplified) presentation of $\pi_1(M, v)$ is given by:

$$\langle \beta_{j,g}, n + 1 \leq j \leq m, g \in (\mathbb{Z}_2)^n | \beta_{j,g}\beta_{j,\sigma(g)} = 1, n + 1 \leq j \leq m, \forall g \in (\mathbb{Z}_2)^n; \beta_{j,g}\beta_{j',\sigma(g)} = \beta_{j',g}\beta_{j,\sigma(g)}, \forall g \in (\mathbb{Z}_2)^n \rangle$$

where

$$F_j \cap F_{j'} \neq \emptyset, n + 1 \leq j < j' \leq m;$$

$$\beta_{i,g} = \beta_{i,\sigma(g)}, \forall g \in (\mathbb{Z}_2)^n$$

$$F_i \cap F_j \neq \emptyset, 1 \leq i \leq n, n + 1 \leq j \leq m \rangle$$  \hfill (21)

On the other hand, we define a collection of elements of $W_P$ as follows.

$$\xi_{i,g} := \gamma_g s_{F_i} \gamma_{\sigma(g)} = \gamma_g s_{F_i} \gamma_{g + \lambda(F_i)} \in W_P, 1 \leq i \leq m, \forall g \in (\mathbb{Z}_2)^n.$$  \hfill (22)

Note that $\xi_{i,g} = 1 \in W_P$ for any $1 \leq i \leq n$ and $g \in (\mathbb{Z}_2)^n$. There are many relations between $\{\xi_{i,g} | n + 1 \leq j \leq m, g \in (\mathbb{Z}_2)^n\}$ shown in the following lemma.

**Lemma 2.5.** For any $g \in (\mathbb{Z}_2)^n$, we have

(i) $\phi(\xi_{i,g}) = 0$, i.e. $\xi_{i,g} \in \ker(\phi)$, $n + 1 \leq j \leq m$;

(ii) $\gamma_g s_{F_j} = \xi_{j,g} \gamma_{\sigma_j(g)}$, $n + 1 \leq j \leq m$;

(iii) $\xi_{i,g} \xi_{j,g} = 1 \in W_P, n + 1 \leq j \leq m$;

(iv) $\xi_{i,g} \xi_{j',\sigma_j(g)} = \xi_{j',g} \xi_{j,\sigma_j(g)}$, if $F_j \cap F_{j'} \neq \emptyset$ with $n + 1 \leq j < j' \leq m$.

(v) $\xi_{i,g} = \xi_{j,\sigma_j(g)}$, if $F_i \cap F_j \neq \emptyset, 1 \leq i \leq n, n + 1 \leq j \leq m$.

**Proof.** It is easy to prove (i)(ii)(iii) from the definition of $\xi_{i,g}$. As for (iv), we have

$$\xi_{i,g} \xi_{j',\sigma_j(g)} = \gamma_g s_{F_j} \gamma_{\sigma_j(g)} \cdot \gamma_{\sigma_j(g)} s_{F_j} \gamma_{\sigma_j(g)} = \gamma_g s_{F_j} s_{F_j} \gamma_{\sigma_j(g)};$$

$$\xi_{j',\sigma_j(g)} = \gamma_g s_{F_j} \gamma_{\sigma_j(g)} \cdot \gamma_{\sigma_j(g)} s_{F_j} \gamma_{\sigma_j(g)} = \gamma_g s_{F_j} s_{F_j} \gamma_{\sigma_j(g)}.$$

Note $\sigma_j(\sigma_j(g)) = \sigma_j(\sigma_j(g)) = g + \lambda(F_j) + \lambda(F_j')$, and $F_j \cap F_{j'} \neq \emptyset$ implies that $s_{F_j} s_{F_j'} = s_{F_j} s_{F_j'} \in W_P$. So we obtain $\xi_{i,g} \xi_{j',\sigma_j(g)} = \xi_{j',g} \xi_{j,\sigma_j(g)}$.

As for (v), since when $F_i \cap F_j \neq \emptyset$, $s_{F_i}$ commutes with $s_{F_j}$ in $W$. Then

$$\xi_{i,g} \xi_{j,\sigma_j(g)} = \gamma_{\sigma_j(g)} s_{F_j} \gamma_{\sigma_j(g)} = \gamma_g s_{F_j} s_{F_j} \gamma_{\sigma_j(g)} = \gamma_g s_{F_j} \gamma_{\sigma_j(g)} = \xi_{j,g}.$$

So the lemma is proved. \hfill \square

Note that (iv) and (v) in Lemma 2.5 can be unified to the following form

$$\xi_{i,g} \xi_{j,\sigma_j(g)} = \xi_{j,\sigma_j(g)} \xi_{i,g}, F_i \cap F_j \neq \emptyset, 1 \leq i \leq n, n + 1 \leq j \leq m.$$  \hfill (23)

**Lemma 2.6.** Any element of $\ker(\phi)$ can be written as a product of $\xi_{i,g}$.

**Proof.** Any element $\xi$ of $W_P$ can be written in the form

$$\xi = \gamma_{g_1} s_{F_1} \gamma_{g_2} s_{F_2} \cdots \gamma_{g_k} s_{F_k} \gamma_{g_{k+1}}$$

where $g_i \in (\mathbb{Z}_2)^n$ and $n + 1 \leq j_i \leq m$ for all $i = 1, \cdots, k + 1$. Note that we allow $\gamma_{g_i}$ to be the identity of $W_P$ in the above expression of $\xi$. Then from the left to right,
we can replace the $\gamma_s F_j$ in the expression of $\xi$ via the relation in Lemma 2.5(ii) until there is no $s F_j$ in the end. So we can write $\xi = \xi_{j_1,h_1} \xi_{j_2,h_2} \cdots \xi_{j_k,h_k},$ for some $h_1, \cdots, h_{k+1} \in (\mathbb{Z}_2)^n$. Since $\xi_{j,g} \in \ker(\phi)$, $\phi(\xi) = 0$ if and only if $\phi(\gamma_{h_{k+1}}) = h_{k+1} = 0$, i.e. $\gamma_{h_{k+1}} = 1 \in W_P$. This proves the lemma. \hfill \Box

Lemma 2.6 tells us that $\{\xi_{j,g} \mid n+1 \leq j \leq m, g \in (\mathbb{Z}_2)^n\}$ form a set of generators for $\ker(\phi)$. To understand the relations between these generators, we adopt some ideas originated from Poincaré’s study on discrete group of motions. The content of the following paragraph is taken from [30, Part II, Ch.2].

Suppose $Q$ is a normal fundamental polyhedron of the action of a discrete group $\Gamma$ on a simply connected $n$-manifold $X$. The polyhedron $\gamma Q$, $\gamma \in \Gamma$, are called chambers. For any $(n-1)$-dimensional face $F$ of $Q$, we denote by $a_F$ the element of $\Gamma$ taking $Q$ to the polyhedron adjacent to $Q$ along the face $F$, and by $F^*$ the inverse image of the face of $F$ under the action of $a_F$. Clearly we have

$$a_F \cdot a_{F^*} = e$$

(24)

The action by $a_F$ is called an adjacency transformation. Relations between the adjacency transformations are of the form $a_{F_1} \cdots a_{F_k} = e$ which correspond to the cycles of chambers

$$Q_0 = Q, \quad Q_1 = a_{F_1} Q, \quad Q_2 = a_{F_1} a_{F_2} Q, \cdots, \quad Q_k = a_{F_1} \cdots a_{F_k} Q = Q,$$

where $Q_i$ and $Q_{i-1}$ are adjacent along the face $a_{F_1} \cdots a_{F_{i-1}} F_i$.

- Each $(n-1)$-dimensional face $F$ of $Q$ defines a cycle of chambers $(Q, a_F Q, Q)$, which corresponds to the relation (24), called a pairing relation.
- Each $(n-2)$-dimensional face determines a cycle consisting of all chambers containing this face in the order in which they are encountered while circling this face. The corresponding relation is called a Poincaré relation.

**Theorem 2.7** (see [30]). The group $\Gamma$ is generated by adjacency transformations $\{a_F\}$. Moreover, the pairing relations together with Poincaré relations form a complete set of relations of the adjacency transformations in $\Gamma$.

**Lemma 2.8.** The Lemma 2.6(iii)(iv)(v) give a complete set of relations for the generators $\{\xi_{j,g} \mid n+1 \leq j \leq m, g \in (\mathbb{Z}_2)^n\}$ in $\ker(\phi)$.

**Proof.** According to the proof of Proposition 2.4, we can take the fundamental domain of the action of $\ker(\phi)$ on $\mathcal{L}_P$ to be $Q_v$ (see (19)) where $v = F_1 \cap \cdots \cap F_n$. For any $g \in (\mathbb{Z}_2)^n$, let $F_{j,g} \subset \partial Q_v$ denote the image of the copy of $F_j$ in $(P, \gamma_g)$ for each $n+1 \leq j \leq m$. It is easy to see that $\xi_{j,g}$ maps $F_{j,(j-g)}$ to $F_{j,g}$. So the adjacency transformation defined by the facet $F_{j,g}$ of $Q_v$ is exactly $\xi_{j,g}$. Moreover, we have:
The pairing relations for \(\{\xi_{j,g}\}\) are exactly given in Lemma 2.5(iii) since by definition \(\xi_{j,g}\) maps \(F_{j,\sigma(g)}\) to \(F_{j,g}\) for any \(g \in (\mathbb{Z}_2)^n\).

The Poincaré relations for \(\{\xi_{j,g}\}\) are exactly given by Lemma 2.5(iv)(v) (or equivalent the relations in (23)). This is because for any \(g \in (\mathbb{Z}_2)^n\) there are exactly four chambers of \(L_P\) around the face \(F_{i,g} \cap F_{j,g} \subset Q_v\) and we will meet \(F_{i,g}, F_{j,\sigma(g)}; F_{i,\sigma_j(g)}, F_{j,g}\) in order when circling around \(F_{i,g} \cap F_{j,g}\) (see Figure 5).

Then our lemma follows immediately from Theorem 2.7.

Now define a homomorphism \(\Psi : \pi_1(M, v) \to W_P\) by

\[\Psi(\beta_{j,g}) = \xi_{j,g}, \quad n + 1 \leq j \leq m, \quad \forall g \in (\mathbb{Z}_2)^n.\]  

(25)

By the presentation of \(\pi_1(M, v)\) given in (21) and Lemma 2.8, \(\Psi\) is well defined and is an isomorphism from \(\pi_1(M, v)\) to \(\ker(\phi)\). So we obtain the following.

**Lemma 2.9.** For any \(n\)-dimensional small cover \(M\) over a simple polytope \(P\), we have a short exact sequence

\[1 \longrightarrow \pi_1(M, v) \xrightarrow{\Psi} W_P \xrightarrow{\phi} (\mathbb{Z}_2)^n \longrightarrow 1.\]

**Remark 2.10.** The relation between \(\Psi\) and \(\psi\) (see (11)) is not very clear to us. The homomorphism \(\Psi\) is defined via the presentation of \(\pi_1(M)\) while \(\psi\) is induced by the map \(M \hookrightarrow M \times E(\mathbb{Z}_2)^n \to M(\mathbb{Z}_2)^n\). But the isomorphism between \(\pi_1(M(\mathbb{Z}_2)^n)\) and \(W_P\) is not very explicit.

Note that the sequence \(1 \longrightarrow \pi_1(M, v) \xrightarrow{\Psi} W_P \xrightarrow{\phi} (\mathbb{Z}_2)^n \longrightarrow 1\) splits since \(\phi \circ \gamma = \text{id}(\mathbb{Z}_2)^n\) (see (17)). Then \(W_P \cong \pi_1(M, v) \times (\mathbb{Z}_2)^n\) where \((\mathbb{Z}_2)^n\) acts on \(\pi_1(M, v)\) by: \(g' : \xi_{j,g} = \gamma_{g'-1}^\ast \xi_{j,g} \gamma^g = \xi_{j,g+g'}\) for any \(g, g' \in (\mathbb{Z}_2)^n\) (see (22)). In other words, \((\mathbb{Z}_2)^n\) permutates the \(2^n\) generators associated to each facet \(F_j\) in the presentation (21) of \(\pi_1(M, v)\).
2.4. Torsion in the fundamental groups of small covers.

Given a finite simplicial complex \( K \) with vertex set \( \{1, \ldots, m\} \), there is a right-angled Coxeter group \( W_K \) with generators \( s_1, \ldots, s_m \) and relations \( s_i^2 = 1 \), \( 1 \leq i \leq m \) and \( (s_i s_j)^2 = 1 \) for each 1-simplex \( \{i, j\} \) in \( K \). It is clear that

\[
W_K = W_K^{(1)}
\]

where \( K^{(1)} \) is the 1-skeleton of \( K \). It is shown in [9, Sec 1.2] that there is a finite cubical complex \( \mathbb{R}\mathcal{Z}_K \) determined by \( K \) whose fundamental group is isomorphic to the commutator subgroup \( [W_K, W_K] \) of \( W_K \). We call \( \mathbb{R}\mathcal{Z}_K \) the real moment-angle complex of \( K \). In some literature, \( \mathbb{R}\mathcal{Z}_K \) is also denoted by \((D^1, S^0)^K\).

**Definition 2.11** (Real Moment-Angle Manifold). For an \( n \)-dimensional simple polytope \( P, \mathbb{R}\mathcal{Z}_{\partial P} \) is a closed connected \( n \)-manifold called the real moment-angle manifold of \( P \), also denoted by \( \mathbb{R}\mathcal{Z}_P \). The Coxeter group \( W_{\partial P} \) clearly coincides with \( W_P \) defined earlier. So \( \pi_1(\mathbb{R}\mathcal{Z}_P) \cong [W_P, W_P] \). Let the set of facets of \( P \) be \( \{F_1, \ldots, F_m\} \). We can also obtain \( \mathbb{R}\mathcal{Z}_P \) by gluing \( 2^m \) copies of \( P \) via a function \( \mu : \{F_1, \ldots, F_m\} \to (\mathbb{Z}_2)^m \) in the same way as (2) where \( \{\mu(F_i) = e_i, 1 \leq i \leq m\} \) forms a basis of \((\mathbb{Z}_2)^m \) (see [7, §4.1] or [17]). Let

\[
\Theta : P \times (\mathbb{Z}_2)^m \to \mathbb{R}\mathcal{Z}_P
\]

be the quotient map. There is a canonical \((\mathbb{Z}_2)^m\)-action on \( \mathbb{R}\mathcal{Z}_P \) defined by

\[
g' \cdot \Theta(x, g) = \Theta(x, g + g'), \ x \in P^n, g, g' \in (\mathbb{Z}_2)^m,
\]

whose orbit space is \( P \). Let \( \pi_P : \mathbb{R}\mathcal{Z}_P \to P \) be the projection. For any proper face \( f \) of \( P \), it is easy to see that \( \pi_P^{-1}(f) \) consists of \( 2^m + \dim(f) - n - l \) copies of \( \mathbb{R}\mathcal{Z}_f \) where \( l \) is the number of facets of \( f \). Note that \( \mathbb{R}\mathcal{Z}_P \) is always a closed connected orientable manifold.

If there is a small cover \( M \) over \( P \), then there exists a subgroup \( H \cong (\mathbb{Z}_2)^{m-n} \) of \((\mathbb{Z}_2)^m \) where \( H \) acts freely on \( \mathbb{R}\mathcal{Z}_P \) (through the canonical action) whose orbit space is \( M \). In other words, \( \mathbb{R}\mathcal{Z}_P \) is a regular \((\mathbb{Z}_2)^{m-n}\)-covering space of \( M \).

We call \( K \) a flag complex if any finite set of vertices of \( K \), which are pairwise connected by edges, spans a simplex in \( K \). Suggestively we think of a non-flag complex as having a minimal empty simplex of some dimension greater than 1, i.e., a subcomplex equivalent to the boundary of a \( k \)-simplex that does not actually span a \( k \)-simplex.

By [9, Proposition 1.2.3], \( \mathbb{R}\mathcal{Z}_K \) is aspherical if and only if \( K \) is a flag complex. So when \( K \) is a flag complex, \( \pi_1(\mathbb{R}\mathcal{Z}_K) \) is torsion-free. For any finite simplicial complex \( K \), the minimal flag simplicial complex that contains \( K \) is called the flagification of \( K \), denoted by \( \text{fla}(K) \). Note that \( K \) and \( \text{fla}(K) \) have the same
1-skeleton. So we have
\[ [W_K, W_K] = [W_K^{(1)}, W_K^{(1)}] = [W_{\text{fla}(K)}^{(1)}, W_{\text{fla}(K)}^{(1)}] = [W_{\text{fla}(K)}, W_{\text{fla}(K)}] \cong \pi_1(\mathbb{R}Z_{\text{fla}(K)}). \]

This implies that \([W_K, W_K]\) is torsion-free for any finite simplicial complex \(K\). Then since \(W_K/[W_K, W_K] \cong (\mathbb{Z}_2)^m\), it is easy to see that any torsion element of \(W_K\) must have order 2. Then by [4], we obtain the following.

**Proposition 2.12.** Any element of the fundamental group of a small cover either has infinite order or has order 2.

Note that right-angled Coxeter groups are special instances of a more general construction called *graph products of groups*. The reader is referred to [22] for the study of the commutator subgroup of a general graph product of groups.

3. \(\pi_1\)-injectivity of the facial submanifolds of small covers

Let \(M\) be a small cover over a simple polytope \(P\) with characteristic function \(\lambda\). Given a proper face \(f\) of \(P\), we choose a vertex \(v\) of \(P\) contained in \(f\). Let all the facets of \(P\) be \(F_1, \ldots, F_m\), where

- \(F_1 \cap \cdots \cap F_n = v\), \(\lambda(F_i) = e_i, 1 \leq i \leq n, e_1, \ldots, e_n\) is a basis of \((\mathbb{Z}_2)^n\);
- \(F_1 \cap \cdots \cap F_{n-k} = f\), \(\dim(f) = k\). So \(G_f = \langle e_1, \ldots, e_{n-k} \rangle \subset (\mathbb{Z}_2)^n\).
- \(\mathcal{F}(f^+) = \{F_{n-k+1}, \ldots, F_n, \ldots, F_{n+r}\}, r \leq m - n\). So the codimension-one faces of \(f\) are \{\(f \cap F_{n-k+1}, \ldots, f \cap F_n, \ldots, f \cap F_{n+r}\}\} where the faces incident to \(v\) are \(f \cap F_{n-k+1}, \ldots, f \cap F_n\).

The facial submanifold \(M_f\) is a small cover over the simple polytope \(f\) whose characteristic function \(\lambda_f\) is given by [5]. Note that \(\{\lambda_f(f \cap F_i) \mid n-k+1 \leq i \leq n\}\) is a basis of \((\mathbb{Z}_2)^{\dim(f)}\). Then we can identify \((\mathbb{Z}_2)^{\dim(f)} \cong (\mathbb{Z}_2)^n/G_f\) with the subgroup \(\langle e_{n-k+1}, \ldots, e_n \rangle \subset (\mathbb{Z}_2)^n\) via a monomorphism

\[
\iota : (\mathbb{Z}_2)^{\dim(f)} \longrightarrow (\mathbb{Z}_2)^n
\]

\[
\lambda_f(f \cap F_i) \longmapsto \lambda(F_i) = e_i, \ n-k+1 \leq i \leq n.
\]

We will always assume this identification in the rest of this section. Under this identification, we can write the function \(\lambda_f\) as

\[
\lambda_f : \{f \cap F_{n-k+1}, \ldots, f \cap F_n, \ldots, f \cap F_{n+r}\} \rightarrow (\mathbb{Z}_2)^{\dim(f)} = \langle e_{n-k+1}, \ldots, e_n \rangle \subset (\mathbb{Z}_2)^n
\]

where \(\lambda_f(f \cap F_i) = e_i, n-k+1 \leq i \leq n\). Notice that \(\lambda_f(f \cap F_j)\) and \(\lambda(F_j)\) are not necessarily equal when \(n+1 \leq j \leq n+r\). But we have

\[
\lambda_f(f \cap F_j) - \lambda(F_j) \in G_f, n+1 \leq j \leq n+r. \quad (26)
\]
In addition, parallelly to the transformation $\sigma_i : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n$ in (20), we define a set of transformations $\sigma^f_i : (\mathbb{Z}_2)^{\dim(f)} \to (\mathbb{Z}_2)^{\dim(f)}$ by

$$\sigma^f_i(g) = g + \lambda_f(f_i \cap F_i), \quad g \in (\mathbb{Z}_2)^{\dim(f)}, \quad n - k + 1 \leq i \leq n + r. \quad (27)$$

Then we have

$$\sigma^f_i(g) - \sigma_i(g) = \lambda_f(f_i \cap F_i) - \lambda(F_i) = \begin{cases} 0, & n - k + 1 \leq i \leq n; \\ \in G_f & n + 1 \leq i \leq n + r. \end{cases} \quad (28)$$

By the presentation of $\pi_1(M,v)$ in Proposition 2.1, we can similarly obtain a simplified presentation of $\pi_1(M_f,v)$. Note that the codimension-one faces of $f$ that are not incident to $v$ are $f \cap F_{n+1}, \ldots, f \cap F_{n+r}$. We let $\beta^f_{j,g}$ denote the closed path of $M_f$ based at $v$ corresponding to $f \cap F_j$ for each $n + 1 \leq j \leq n + r$. Then according to (13), we have a presentation of $\pi_1(M_f,v)$:

$$\pi_1(M_f,v) = \langle \beta^f_{j,g}, n + 1 \leq j \leq n + r, g \in (\mathbb{Z}_2)^{\dim(f)} \mid$$

$$\beta^f_{j,g} \beta^f_{j',g} = 1, n + 1 \leq j \leq n + r, \forall g \in (\mathbb{Z}_2)^{\dim(f)};$$

$$\beta^f_{j,g} \beta^f_{j',g} = \beta^f_{j',g} \beta^f_{j,g}, \forall g \in (\mathbb{Z}_2)^{\dim(f)} \text{ where } F_j \cap F_{j'} \cap f \neq \emptyset, n + 1 \leq j < j' \leq n + r;$$

$$\beta^f_{j,g} = \beta^f_{j,\sigma^f_i(g)}, \forall g \in (\mathbb{Z}_2)^{\dim(f)} \text{ where } n - k + 1 \leq i \leq n, n + 1 \leq j \leq n + r \rangle. \quad (29)$$

**Lemma 3.1.** The inclusion map $j_f : M_f \to M$ induces a homomorphism

$$(j_f)_* : \pi_1(M_f,v) \longrightarrow \pi_1(M,v) \quad (30)$$

$$\beta^f_{j,g} \longmapsto \beta_{j,g}, \forall g \in (\mathbb{Z}_2)^{\dim(f)}$$

**Proof.** By the definition of $\mathcal{D}_v(M)$, it is easy to see that the restriction of $\mathcal{D}_v(M)$ to $M_f$ coincides with $\mathcal{D}_v(M_f)$. Since $P$ is convex, we can deform the oriented
line segment $\overrightarrow{vb_{f\cap F}}$ to $\overrightarrow{vb_{F}}$ in $P$ with the endpoint $v$ fixed (see Figure 6). If we do this deformation in $\Theta((P, g))$ for all $g \in \langle (Z_2)^{\dim(f)} \rangle$ simultaneously in $M$, we obtain a homotopy from the closed path $\beta_{j,g}^f \subset M_f$ to the closed path $\beta_{j,g} \subset M$ which fixes the base point $v$ for any $g \in \langle (Z_2)^{\dim(f)} \rangle$. This proves the lemma. □

We have the following constructions for any proper face $f$ of $P$.

- The inclusion map $i_f : f \to P$ induces a homomorphism $(i_f)_* : W_f \to W_P$, which sends the generator $s_{f \cap F_i}$ of $W_f$ to the generator $s_{F_i}$ of $W_P$ for any facet $F_i \in \mathcal{F}(f^\perp)$.
- From the monomorphism $\gamma : (Z_2)^n \to W_P$ defined in (16), we obtain a monomorphism $\Psi : \pi_1(M, v) \to W_P$ in (25). For the face $f$, we similarly define a monomorphism $\gamma^f : (Z_2)^{\dim(f)} \to W_f$ by sending $\lambda_f(f \cap F_i) = e_i$ to $s_{f \cap F_i}$ for any $n - k + 1 \leq i \leq n$. Let $\gamma^f = \gamma^f(g)$ for any $g \in \langle (Z_2)^{\dim(f)} \rangle$. Then since we identify $(Z_2)^{\dim(f)}$ with $\langle (e_{n-k+1}, \ldots, e_n) \rangle \subset (Z_2)^n$, we have

$$(i_f)_*(\gamma^f) = \gamma_i(g) = \gamma_g, \forall g \in \langle (Z_2)^{\dim(f)} \rangle.$$  \hspace{1cm} (31)

So by Lemma 2.9, we have a monomorphism $\Psi_f : \pi_1(M_f, v) \to W_f$ where

$$\Psi_f(\beta^f_{j,g}) = \xi^f_{j,g} := \gamma^f_g s_{f \cap F_i} \gamma^f_{\sigma_j(g)}.$$  \hspace{1cm} (32)

- Let $H_f$ be the normal subgroup of $W_P$ generated by the following set

$$\{s_{F} \mid f \subset F\} = \{s_{F_1}, \ldots, s_{F_{n-k}}\}.$$

And let $\eta_f : W_P \to W_P/H_f$ be the projection. Then by definition, $\gamma$ maps $G_f \subset (Z_2)^n$ isomorphically onto $H_f \subset W_P$.

- Let $\overline{\Psi} = \eta_f \circ \Psi$ and $(\overline{(i_f)}_*) = \eta_f \circ (i_f)_*$ be the compositions of $\Psi$ and $(i_f)_*$ with $\eta_f$.

**Lemma 3.2.** The following diagram is commutative.

$$\begin{array}{ccc}
\pi_1(M_f, v) & \xrightarrow{\Psi_f} & W_f \\
\downarrow{(i_f)_*} & & \downarrow{(i_f)_*} \\
\pi_1(M, v) & \xrightarrow{\Psi} & W_P/H_f
\end{array}$$  \hspace{1cm} (33)

**Proof.** For any $n + 1 \leq j \leq n + r$ and any $g \in \langle (Z_2)^{\dim(f)} \rangle$,

$$\overline{\Psi} \circ (j_f)_*(\beta^f_{j,g}) \overset{\text{E1}}{=} \overline{\Psi}(\beta_{j,g}) = \eta_f(\Psi(\beta_{j,g})) \overset{\text{E3}}{=} \eta_f(\xi_{j,g}) = \eta_f(\gamma_g s_{f \cap F_i} \gamma^f_{\sigma_j(g)});$$

$$\overline{(i_f)_*} \circ \Psi_f(\beta^f_{j,g}) \overset{\text{E2}}{=} (\overline{(i_f)_*})(\xi^f_{j,g}) \overset{\text{E4}}{=} (\overline{(i_f)_*})(\gamma^f_g s_{f \cap F_i} \gamma^f_{\sigma_j(g)}) = \eta_f(\gamma_g s_{f \cap F_i} \gamma^f_{\sigma_j(g)}).$$
By [23], we have \( \sigma_f^j(g) - \sigma_j(g) \in G_f. \) So \( \gamma_{\sigma_f^j(g)} \in \gamma_{\sigma_j(g)} \cdot H_f. \) Then \( \eta_f(\gamma_{\sigma_f^j(g)}) = \eta_f(\gamma_{\sigma_j(g)}) \), which implies \((ij) \ast \Psi_f(\beta_{j,g}^f) = \Psi \circ (j_f)_*(\beta_{j,g}^f)\). Then since \( \pi_1(M_f) \) is generated by \( \{\beta_{j,g}^f\} \), the lemma is proved. \( \square \)

**Theorem 3.3.** Let \( M \) be a small cover over a simple polytope \( P \) and \( f \) be a proper face of \( P \). Then the following two statements are equivalent.

(i) The facial submanifold \( M_f \) is \( \pi_1 \)-injective in \( M \).

(ii) For any \( F, F' \in \mathcal{F}(f^\perp) \), we have \( f \cap F \cap F' \neq \emptyset \) whenever \( F \cap F' \neq \emptyset \).

**Proof.** (i) \( \Rightarrow \) (ii). Using the conventions at the beginning of this section, we choose a vertex \( v \in f \) and let \( f = F_1 \cap \cdots \cap F_{n-k}, \mathcal{F}(f^\perp) = \{F_{n-k+1}, \ldots, F_n, \ldots, F_{n+r}\} \)

Assume that there exist two facets \( F_j, F'_j \in \mathcal{F}(f^\perp), n + 1 \leq j < j' \leq n + r \) so that \( f \cap F_j \cap F'_j = \emptyset \) while \( F_j \cap F'_j \neq \emptyset \). By the definition of \( W_f \), the element \( s_{f \cap F_j} s_{f \cap F'_j} \in W_f \) is of infinite order since \( (f \cap F_j) \cap (f \cap F'_j) = \emptyset \). So we have

\[
(s_{f \cap F_j} s_{f \cap F'_j})^2 \neq 1 \in W_f; \quad (s_{F_j} s_{F'_j})^2 = 1 \in W_P.
\]

Moreover, by the definitions of \( \Psi \) and \( \Psi_f \) (see (32)), it is easy to check that

\[
x_{j,j'}^f := \beta_{j,0,j',\lambda_f(j \cap F_j)} \beta_{j,j,\lambda_f(j \cap F_j)} \beta_{j,j',\lambda_f(j \cap F_j)} \Psi \rightarrow (s_{f \cap F_j} s_{f \cap F'_j})^2 \neq 1.
\]

Then since \( \Psi \) and \( \Psi_f \) are both monomorphisms, we have

\[
x_{j,j'}^f \neq 1 \in \pi_1(M_f, v); \quad x_{j,j'} = 1 \in \pi_1(M, v).
\]

The following claim is the heart of our argument.

**Claim:** \( (j_f)_*(x_{j,j'}^f) = x_{j,j'} = 1 \in \pi_1(M_f, v) \), i.e. \( x_{j,j'}^f \in \ker((j_f)_*) \).

By Lemma 3.1 \( (j_f)_*(x_{j,j'}^f) = \beta_{j,0,j',\lambda_f(j \cap F_j)} \beta_{j,j,\lambda_f(j \cap F_j)} \beta_{j,j',\lambda_f(j \cap F_j)} \Psi \rightarrow (s_{f \cap F_j} s_{f \cap F'_j})^2 \neq 1 \), which looks different from \( x_{j,j'}^f \). But since \( v \in f = F_1 \cap \cdots \cap F_{n-k} \) and \( f \cap F_j \neq \emptyset, f \cap F'_j \neq \emptyset \), we have \( F_i \cap F_j \neq \emptyset \) and \( F_i \cap F'_j \neq \emptyset \) for all \( 1 \leq i \leq n - k \). Note that the subgroup of \( (\mathbb{Z}_2)^n \) generated by \( \lambda(F_1), \ldots, \lambda(F_{n-k}) \) is \( G_f \) (see (11)). Then according to (11), we have the following relations among the generators \( \{\beta_{j,g}\} \) and \( \{\beta_{j,g}^f\} \) in the presentation (21) of \( \pi_1(M, v) \):

\[
\beta_{j,g} = \beta_{j,h}, \quad \beta_{j,g}^f = \beta_{j,h}^f, \quad \text{whenever } g - h \in G_f, g, h \in (\mathbb{Z}_2)^n.
\]

Then since \( \lambda_f(f \cap F_j) - \lambda_f(F_j) \in G_f, \lambda_f(f \cap F'_j) - \lambda_f(F'_j) \in G_f, \) we have \( \beta_{j',\lambda_f(f \cap F_j)} = \beta_{j',\lambda_f(F_j)}; \quad \beta_{j',\lambda_f(f \cap F'_j)} = \beta_{j',\lambda_f(F'_j)}; \quad \beta_{j',\lambda_f(f \cap F_j)} = \beta_{j',\lambda_f(F_j)} \).

It follows that \( (j_f)_*(x_{j,j'}^f) = x_{j,j'} \). The claim is proved.

By the above claim, \( \ker((j_f)_*) \) is not trivial, i.e. \( (j_f)_* \) is not injective.
(ii) $\Rightarrow$ (i). By Lemma 3.2, we have $\overline{id} \circ \Psi_f = \overline{\Psi} \circ (j_f)_*$. We already know that $\Psi_f$ is injective from Lemma 2.9. So to prove the injectivity of $(j_f)_*$, it is sufficient to show that $(\overline{id})_*$ is injective. Note that a presentation of $W_P/H_f$ is given by the generators $\{\overline{s}_F\}_{F \in \mathcal{F}(P)}$ with relations

$$\{\overline{s}_F = 1 \mid F \subset P \} \cup \{ \overline{s}_F^2 = 1 \} \cup \{(\overline{s}_F \overline{s}_F')^2 = 1, \forall F \cap F' \neq \emptyset\},$$

where $\overline{s}_F = \eta_f(s_F)$ for any $F \in \mathcal{F}(P)$. Using this presentation of $W_P/H_f$, we can define a homomorphism $\zeta_f : W_P/H_f \to W_f$ by

$$\zeta_f(\overline{s}_F) = \begin{cases} s_{f \cap F}, & F \in \mathcal{F}(f^\perp); \\ 1, & F \notin \mathcal{F}(f^\perp). \end{cases} \quad (34)$$

We claim that $\zeta_f$ is well defined. Indeed, for any $F, F' \in \mathcal{F}(f^\perp)$ with $F \cap F' \neq \emptyset$, we have $(\overline{s}_F \overline{s}_F')^2 = 1$ in $W_P/H_f$. Meanwhile, from the condition $f \cap F \cap F' \neq \emptyset$, we have $(s_{f \cap F}s_{f \cap F'})^2 = 1$ in $W_f$. This implies that $\zeta_f$ is well defined.

It is clear that $(\overline{id})_*(\overline{s}_{f \cap F}) = \overline{s}_F$ for any $F \in \mathcal{F}(f^\perp)$. So $\zeta_f \circ (\overline{id})_* = \text{id}_{W_f}$, which implies that $(\overline{id})_*$ is injective. This finishes the proof.

By the proof of Theorem 3.3, we obtain the following description of the kernel of $(j_f)_* : \pi_1(M_f, v) \to \pi_1(M, v)$ for any proper face $f$ of $P$ and any vertex $v \in f$.

**Theorem 3.4.** Let $M$ be a small cover over a simple polytope $P$ and $f$ be a proper face of $P$. Let $j_f : M_f \to M$ be the inclusion map. Then for any vertex $v \in f$, the kernel of $(j_f)_* : \pi_1(M_f, v) \to \pi_1(M, v)$ is the normal subgroup of $\pi_1(M_f, v)$ generated by the inverse image of the following set under $\Psi_f : \pi_1(M_f, v) \to W_f$.

$$\Xi_f = \{(s_{f \cap F}s_{f \cap F'})^2 \in W_f \mid v \notin F, v \notin F', f \cap F \cap F' = \emptyset \text{ while } F \cap F' \neq \emptyset\}.$$

**Proof.** Let $I_f$ be the normal subgroup of $\pi_1(M_f, v)$ generated by the set $\Psi_f^{-1}(\Xi_f)$. Let $J_f$ be the normal subgroup of $W_f$ generated by the set $\Xi_f$.

- By definition, $\Psi_f$ maps $I_f$ isomorphically onto $J_f$.
- For any $(s_{f \cap F}s_{f \cap F'})^2 \in \Xi_f$, the proof of (i) $\Rightarrow$ (ii) in Theorem 3.3 tells us that $\Psi_f^{-1}((s_{f \cap F}s_{f \cap F'})^2) \in\ker((j_f)_*)$. So $I_f$ is contained in $\ker((j_f)_*)$.
- We clearly have $J_f \subset \ker((\overline{id})_*)$ and hence $J_f \subset \ker((\overline{id})_*)$.

Then we obtain the following commutative diagram from the diagram (33).

$$\begin{array}{ccc}
\pi_1(M, v) / I_f & \xrightarrow{\psi_f} & W_f / J_f \\
\downarrow (j_f)_* & & \downarrow (\overline{j}_f)_* \\
\pi_1(M, v) & \xrightarrow{\overline{\Psi}} & W_P / H_f
\end{array} \quad (35)$$
where \((j_f)_\ast\), \(\Psi_f\) and \((i_f)_\ast\) are the homomorphisms induced by \((j_f)_\ast\), \(\Psi_f\) and \((i_f)_\ast\) under the quotients by \(I_f\) and \(J_f\), respectively.

So to prove \(\ker((j_f)_\ast) = I_f\), it is equivalent to prove that \((j_f)_\ast\) is injective. Note that \(\Psi_f\) is still injective. So by the commutativity of the diagram (35), to prove \((j_f)_\ast\) is injective, it is sufficient to show that \((i_f)_\ast\) is injective.

Indeed, the map \(\zeta_f : W_P/H_f \rightarrow W_f\) defined in (34) naturally induces a map \(\zeta'_f : W_P/H_f \rightarrow W_f/J_f\). It is easy to check that \(\zeta'_f\) is well defined and \(\zeta'_f \circ (i_f)_\ast = \text{id}_{W_f/J_f}\). This implies that \((i_f)_\ast\) is injective. So the theorem is proved. \(\square\)

If a proper face \(f\) of \(P\) is a simplex with \(\dim(f) \geq 2\), then for any \(F, F' \in \mathcal{F}(f^\perp)\), we have \(f \cap F \cap F' = (f \cap F) \cap (f \cap F')\) is nonempty. So the condition in Theorem 3.3 always holds for such a face \(f\). Then we obtain the following.

**Corollary 3.5.** Let \(M\) be a small cover over a simple polytope \(P\). If a proper face \(f\) of \(P\) is a simplex and \(\dim(f) \geq 2\), then the facial submanifold \(M_f\) is \(\pi_1\)-injective in \(M\).

We can use Theorem 3.3 to a description of aspherical small covers in terms of the \(\pi_1\)-injectivity of their facial submanifolds.

**Proposition 3.6.** A small cover \(M\) over a simple polytope \(P\) is aspherical if and only if all the facial submanifolds of \(M\) are \(\pi_1\)-injective in \(M\).

**Proof.** If \(M\) is aspherical, then \(P\) is a flag polytope by [8, Theorem 2.2.5]. Since any proper face of a flag simple polytope clearly satisfies the condition (ii) in Theorem 3.3, it follows that all the facial submanifolds of \(M\) are \(\pi_1\)-injective in \(M\). This proves the “only if” part of the proposition.

Let \(K_P = \partial P^*\) where \(P^*\) is the dual polytope of \(P\). Then \(K_P\) is a simplicial sphere. To prove that \(M\) is aspherical, it is equivalent to prove that \(P\) is a flag polytope (i.e. \(K_P\) is a flag simplicial complex) by [8, Theorem 2.2.5]. Furthermore, to prove that \(K_P\) is flag, it is equivalent to prove that for any simplex \(\sigma\) of \(K_P\) (including the empty simplex), the link \(\text{Lk}(\sigma, K_P)\) of \(\sigma\) in \(K_P\) has no empty triangle (see [10, Lemma 3.4]). For any proper face \(f\) of \(P\), let \(\sigma_f\) be the simplex in \(K_P\) corresponding to \(f\). Note that \(f\) is also a simple polytope and \(K_f = \partial f^*\) is isomorphic to \(\text{Lk}(\sigma_f, K_P)\) as a simplicial complex.

If all the facial submanifolds of \(M\) are \(\pi_1\)-injective in \(M\), then \(P\) has no 3-belts by Corollary 1.4. This implies that \(K_P\) contains no empty triangle (note that \(K_P\) is the link of the empty simplex). Moreover, for any facet \(F\) of \(P\), all the facial submanifolds of \(M_F\) must be \(\pi_1\)-injective in \(M_F\). So by the induction on the dimension of \(M\), we can assume that \(F\) is a flag polytope, i.e. \(K_F \cong \text{Lk}(\sigma_F, K_P)\) is a flag complex. Note that \(\sigma_F\) is a vertex of \(K_P\). For any codimension-\(k\) face \(f\)
of $P$ with $k \geq 2$, there exist $k$ distinct facets of $P$ so that $f = F_1 \cap \cdots \cap F_k$. The vertex set of $\sigma_f$ is $\{\sigma_{F_1}, \cdots, \sigma_{F_k}\}$ and

$$\text{Lk}(\sigma_f, K_P) = \bigcap_{i=1}^{k} \text{Lk}(\sigma_{F_i}, K_P).$$

Assume that $\text{Lk}(\sigma_f, K_P)$ contains an empty triangle with vertices $v_1, v_2, v_3$ (i.e. $v_1, v_2, v_3$ are pairwise connected by an edge in $\text{Lk}(\sigma_f, K_P)$, but they do not span a 2-simplex in $\text{Lk}(\sigma_f, K_P)$). Then since $\text{Lk}(\sigma_{F_i}, K_P)$ is a flag complex, $v_1, v_2, v_3$ must span a 2-simplex in $\text{Lk}(\sigma_{F_i}, K_P)$, denoted by $\tau_i$ for any $1 \leq i \leq k$. But since any simplex in $K_P$ is determined uniquely by its vertex set, we must have $\tau_1 = \cdots = \tau_k$. This implies that the 2-simplex $\tau_1$ is actually contained in $\text{Lk}(\sigma_f, K_P)$. This contradicts our assumption that $v_1, v_2, v_3$ do not span a 2-simplex in $\text{Lk}(\sigma_f, K_P)$. So $\text{Lk}(\sigma_f, K_P)$ contains no empty triangle. Then we can conclude that $K_P$ is a flag complex by [10, Lemma 3.4], since for any nonempty simplex $\sigma$ in $K_P$, there exists a unique face $f$ with $\sigma = \sigma_f$. This proves the “if” part of the proposition. □

Remark 3.7. There is another way to prove the “only if” part of Proposition 3.6 using ideas from metric geometry. By [8, Theorem 2.2.5], the natural piecewise Euclidean cubical metric $d_{\boxtimes}$ on $M$ with respect to the small cubes decomposition $\mathcal{C}^s(M)$ of $M$ is nonpositively curved if and only if $M$ is aspherical. Moreover, by [8, Corollary 1.7.3] the inclusion of the facial submanifold $M_f$ into $(M, d_{\boxtimes})$ as a cubical subcomplex is a totally geodesic embedding. Then the “only if” part of Proposition 3.6 follows from these facts because any totally geodesic embedding into a non-positively curved metric space will induce a monomorphism in the fundamental group (see [8, Remark 1.7.4]). But since the existence of a metric of nonpositive curvature is essential for this argument, this approach does not work when $P$ is not flag.

In Kirby’s list [16] of problems in low-dimensional topology, the Problem 4.119 asks one to find examples of aspherical 4-manifolds with $\pi_1$-injective immersed 3-manifolds with infinite fundamental group and wonders whether such kind of examples are common. If a small cover $M$ over a simple 4-polytope $P$ is aspherical, $P$ must be a flag polytope by [8, Theorem 2.2.5]. Since any facet $F$ of $P$ is also a flag polytope, $M_F$ is a closed aspherical 3-manifold (hence with infinite fundamental group) and, $M_F$ is $\pi_1$-injective in $M$ by Proposition 3.6. So all 4-dimensional aspherical small covers satisfy the condition in the Problem 4.119.

4. 3-DIMENSIONAL SMALL COVERS AND NONNEGATIVE SCALAR CURVATURE

In this section, we will use the results obtained in Section 3 to study the scalar curvature of Riemannian metrics on 3-dimensional small covers and real moment-angle manifolds. The reader is referred to [23] for the basic notions of Riemannian
geometry. Since every closed 3-manifold has a unique smooth structure (Moise’s theorem), we do not need to address the smooth structures here. First of all, we show that any 3-dimensional small cover has a non-simply-connected embedded $\pi_1$-injective surface.

**Proposition 4.1.** For any small cover $M$ over a 3-dimensional simple polytope $P$, there always exists a facet $F$ of $P$ so that the facial submanifold $M_F$ is $\pi_1$-injective in $M$.

**Proof.** By Corollary 1.4, the proposition is equivalent to saying that there always exists a facet $F$ of $P$ which is not contained in any 3-belt on $P$. Otherwise assume that every facet of $P$ is contained in some 3-belt on $P$. For the sake of brevity, we call a subset $D \subset \partial P$ a regular disk of $P$ if $D$ is the union of some facets of $P$ and it is homeomorphic to the standard 2-disk.

Suppose $F_1, \ldots, F_m$ are all the facets of $P$ where $F_1, F_2, F_3$ form a 3-belt on $P$. Then the closure of $P \setminus (F_1 \cup F_2 \cup F_3)$ consists of two disjoint regular disks of $P$, denoted by $D_1$ and $D_1'$. By our assumption, any facet $F$ in $D_1$ is contained in some 3-belt on $P$. Clearly the 3-belt containing $F$ can not contain any facet in $D_1'$. This implies that there exists a 3-belt in the regular disk $\tilde{D}_1 = D_1 \cup F_1 \cup F_2 \cup F_3$ other than $(F_1, F_2, F_3)$. Note that the number of facets in $\tilde{D}_1$ is strictly less than $P$. Next, we choose a 3-belt $(F_i, F_i, F_{i3}) \neq (F_1, F_2, F_3)$ in $\tilde{D}_1$. Then the closure of $\tilde{D}_1 \setminus (F_i \cup F_i \cup F_{i3})$ consists of two connected components, at least one of which is a regular disk of $P$, denoted by $D_2$. By our assumption any facet $F$ in $D_2$ is contained in a 3-belt in $P$ different from $(F_i, F_i, F_{i3})$, which must lie in the regular disk $\tilde{D}_2 = D_2 \cup F_i \cup F_i \cup F_{i3}$. By repeating this argument, we can obtain an infinite sequence of regular disks $\{\tilde{D}_i\}_{i=1}^{\infty}$ on $P$ where $\tilde{D}_i \supseteq \tilde{D}_{i+1}$ for all $i \geq 1$ and each $\tilde{D}_i$ contains a 3-belt. But this is impossible since $P$ has only finitely many facets. The proposition is proved. □

**Remark 4.2.** For an even dimensional small cover $M$, it is possible that there are no codimension-one $\pi_1$-injective facial submanifolds in $M$. Indeed, for any positive even integer $n$, let $P$ be the product of $n/2$ copies of 2-simplices $\Delta^2$ and let $M$ be any small cover $P$. For any $1 \leq j \leq n/2$, let $f_1^j, f_2^j, f_3^j$ be the three edges of the $j$-th copy of $\Delta^2$ in $P$. Then all the facets of $P$ are

$$\{F_i^j = \Delta^2 \times \cdots \times f_i^j \times \cdots \times \Delta^2 | 1 \leq j \leq n/2, 1 \leq i \leq 3\}.$$ 

Clearly $(F_1^j, F_2^j, F_3^j)$ is a 3-belt on $P$ for any $1 \leq j \leq n/2$. So every facet of $P$ is contained in some 3-belt on $P$. Then by Corollary 1.4 $M$ has no codimension-one $\pi_1$-injective facial submanifolds. But this kind of examples seem quite rare. In addition, it is not clear to us whether there are such kind of small covers in any odd dimension greater than 3.
Corollary 4.3. For any 3-dimensional simple polytope $P$, the real moment-angle manifold $\mathbb{R}Z_P$ has an embedded $\pi_1$-injective surface which is homeomorphic to $\mathbb{R}Z_F$ for some facet $F$ of $P$.

Proof. Suppose $P$ has $m$ facets. By Definition 2.11 there is a locally standard $(\mathbb{Z}_2)^m$-action on $\mathbb{R}Z_P$ with orbit space $P$. By the Four Color Theorem, there exists a small cover $M$ over $P$ and then $\mathbb{R}Z_P$ is a regular $((\mathbb{Z}_2)^m-3)$-covering space of $M$. Let $\eta : \mathbb{R}Z_P \to M$ be the covering map. By Proposition 4.1, there exists a facet $F$ of $P$ so that the facial submanifold $M_F$ is $\pi_1$-injective in $M$. Obviously $\eta^{-1}(M_F) = \pi_P^{-1}(F)$ is a disjoint union of copies of $\mathbb{R}Z_F$, where $\pi_P : \mathbb{R}Z_P \to P$ is the canonical projection. Let $j_F : M_F \to M$ be the inclusion and choose a copy of $\mathbb{R}Z_F$ in $\eta^{-1}(M_F)$ and let $\tilde{j}_F : \mathbb{R}Z_F \to \mathbb{R}Z_P$ be the inclusion. So $\eta, j_F$ and $\tilde{j}_F$ induce a commutative diagram on the fundamental groups below.

\[
\begin{array}{ccc}
\pi_1(\mathbb{R}Z_F) & \xrightarrow{(\tilde{j}_F)_*} & \pi_1(\mathbb{R}Z_P) \\
\eta_* & & \eta_* \\
\pi_1(M_F) & \xrightarrow{(j_F)_*} & \pi_1(M)
\end{array}
\]

Then since $(j_F)_*$ and $\eta_*$ are all injective, so is $(\tilde{j}_F)_*$.

By Schoen-Yau [26, Theorem 5.2], the existence of $\pi_1$-injective closed surfaces with nonpositive Euler characteristic in a compact orientable 3-manifold $M$ is an obstruction to the existence of Riemannian metric with positive scalar curvature on $M$. Moreover, if $M$ admits a Riemannian metric with nonnegative scalar curvature and has an immersed $\pi_1$-injective orientable closed surface $S$ with positive genus, then $S$ must be a torus and $M$ is Riemannian flat. In addition, after works of Schoen, Yau, Gromov and Lawson, Perelman’s proof of Thurston’s the Geometrization Conjecture led to a complete classification of closed orientable 3-manifolds which admit Riemannian metrics with positive scalar curvature (see [19]). These manifolds are the connected sum of spherical 3-manifolds and copies of $S^2 \times S^1$. Combing these results with Proposition 4.1, we can describe all the 3-dimensional small covers and real moment-angle manifolds which can hold Riemannian metrics with positive or nonnegative scalar curvature below.

Let $P$ be an $n$-dimensional simple convex polytope in $\mathbb{R}^n$ and $v$ a vertex of $P$. Choose a plane $H$ in $\mathbb{R}^n$ such that $H$ separates $v$ from the other vertices of $P$. Let $H_{\geq}$ and $H_{\leq}$ be the two half spaces determined by $H$ and assume that $v$ belongs to $H_{\geq}$. Then $P \cap H_{\geq}$ is an $(n-1)$-simplex, and $P \cap H_{\leq}$ is a simple polytope, which we refer to as a vertex cut of $P$. For example, a vertex cut of the 3-simplex $\Delta^3$ is combinatorially equivalent to $\Delta^2 \times [0, 1]$ (the triangular prism). We use the notation $vc^1(P)$ for any simple polytope that is obtained...
from $P$ by a vertex cut when the choice of the vertex is irrelevant. Note that up to combinatorial equivalence we can recover $P$ by shrinking the $(n - 1)$-simplex $P \cap H \geq$ on $\text{vc}^1(P)$ to a point. We also use the notation $\text{vc}^k(P)$ for any simple polytope that is obtained from $P$ by iterating the vertex cut operation $k$ times for any $k \geq 0$ where $\text{vc}^0(P) = P$. Note that for any simple polytope $P$ and any $k \geq 1$, $\text{vc}^k(P)$ is not a flag polytope since it always has a simplicial facet.

Remark 4.4. The simplicial polytope dual to any $\text{vc}^k(\Delta^3)$ is known as a stacked 3-polytope. A stacked $n$-polytope is a polytope obtained from an $n$-simplex by repeatedly gluing another $n$-simplex onto one of its facets (see [20]). One reason for the significance of stacked polytopes is that, among all simplicial $n$-polytopes with a given number of vertices, the stacked polytopes have the fewest possible higher-dimensional faces.

Lemma 4.5. Let $P$ be a 3-dimensional simple polytope with $m$ facets. Then the real moment-angle manifold $\mathbb{R}Z_{\text{vc}^1(P)}$ is diffeomorphic to the connected sum of two copies of $\mathbb{R}Z_P$ with $2^{m-3} - 1$ copies of $S^2 \times S^1$.

Proof. We can identify the 0-sphere $S^0$ with $\mathbb{Z}_2$. Then by the definition of $\mathbb{R}Z_{\text{vc}^1(P)}$, $\mathbb{R}Z_{\text{vc}^1(P)}$ is produced from $\mathbb{R}Z_P \times S^0$ by an equivariant surgery cutting $2^{m-3}$ balls from each copy of $\mathbb{R}Z_P$ and then connecting the boundary components by $2^{m-3}$ tubes $S^2 \times S^1$ (see [8, §6.4] or [11, §2]).

So indeed $\mathbb{R}Z_{\text{vc}^1(P)}$ is independent of where the vertex cut on $P$ is made. In general, $\mathbb{R}Z_{\text{vc}^k(P)}$ is a connected sum of $2^k$ copies of $\mathbb{R}Z_P$ with many copies of $S^2 \times S^1$ for any $k \geq 0$.

Definition 4.6 (Invariant Metric). For a simple 3-polytope $P$ with $m$ facets, a Riemannian metric $g$ on $\mathbb{R}Z_P$ (or a small cover $M$ over $P$) is called an invariant metric on $\mathbb{R}Z_P$ (or $M$) if the canonical ($\mathbb{Z}_2^m$)-action (or canonical ($\mathbb{Z}_2^3$)-action) on $\mathbb{R}Z_P$ (or $M$) is isometric with respect to $g$. Any invariant metric $g$ on $\mathbb{R}Z_P$ can be projected to an invariant metric on $M$ which is locally isomorphic to $g$.

The following theorem tells us that any equivariant surgery of codimension $\geq 3$ can preserve the existence of invariant metric of positive scalar curvature on a manifold with respect to a compact Lie group action.

Theorem 4.7 (cf. [1] Theorem 11.1). Let $M$ and $N$ be $G$-manifolds where $G$ is a compact Lie group. Assume that $N$ admits an $G$-invariant metric of positive scalar curvature. If $M$ is obtained from $N$ by equivariant surgeries of codimension at least three, then $M$ admits a $G$-invariant metric of positive scalar curvature.

Proposition 4.8. Let $P$ be a 3-dimensional simple polytope. The real moment-angle manifold $\mathbb{R}Z_P$ admits a Riemannian metric with nonnegative scalar curvature if and only if $P$ is combinatorially equivalent to the cube $[0, 1]^3$ or a polytope
obtained from $\Delta^3$ by a sequence of vertex cuts. In this case $\mathbb{R}Z_P$ is either the 3-dimensional torus, the 3-sphere or a connected sum of $(k-1)2^k+1$ copies of $S^2 \times S^1$ for some $k \geq 1$, and the nonnegative scalar curvature metric on these $\mathbb{R}Z_P$ can be assume be to invariant.

Proof. Assume that $\mathbb{R}Z_P$ admits a Riemannian metric with nonnegative scalar curvature.

(i) Suppose $P$ has no triangular facets. By Corollary 4.3 there exists a facet $F \neq \Delta^2$ of $P$ so that $\mathbb{R}Z_P$ has an embedded $\pi_1$-injective surface homeomorphic to $\mathbb{R}Z_F$. Then since $\mathbb{R}Z_F$ is an orientable surface with positive genus, $\mathbb{R}Z_F$ must be a flat manifold by [20, Theorem 5.2]. Then by [17, Theorem 1.2], $P$ is combinatorially equivalent to the cube $[0, 1]^3$.

(ii) Suppose $P$ has some triangular facets. Then unless $P$ is a 3-simplex, we can shrink a triangular facet on $P$ to a point and obtain a new simple polytope with less facets than $P$ (see Figure 7). If $P$ is a 3-simplex, the shrinking of its triangular facet to a point is invalid. Assume that after doing all possible (valid) shrinking of the triangular facets on $P$, we obtain a simple polytope $Q$ in the end.

- If $Q$ is a 3-simplex, then up to combinatorial equivalence $P$ can be obtained from the 3-simplex by a sequence of vertex cuts.

- If $Q$ is not a 3-simplex, then $Q$ has no triangular facets. But we claim that this is impossible. Indeed, $\mathbb{R}Z_P$ is homeomorphic to the connected sum of copies of $\mathbb{R}Z_Q$ with copies $S^2 \times S^1$. But by Corollary 4.3, $\mathbb{R}Z_Q$ has an embedded $\pi_1$-injective surface $\Sigma$ homeomorphic to $\mathbb{R}Z_F$ for some facet $F$ of $Q$. Then since $F$ is not a triangle, $\Sigma \cong \mathbb{R}Z_F$ is a closed orientable surface with positive genus. It is clear that the surface $\Sigma$ in each copy of $\mathbb{R}Z_Q$ is also $\pi_1$-injective in $\mathbb{R}Z_P$. Then by [20, Theorem 5.2], $\mathbb{R}Z_P$ is a flat manifold and hence $P$ is combinatorially equivalent to a 3-cube by case (i). This contradicts our assumption that $P$ has some triangular facets.

By the above discussion, $P$ is either combinatorially equivalent to $[0, 1]^3$ or a polytope obtained from $\Delta^3$ by a sequence of vertex cuts.

Conversely, we know that $\mathbb{R}Z_{[0, 1]^3}$ is the 3-dimensional torus which admits an invariant flat Riemannian metric, $\mathbb{R}Z_{\Delta^3}$ is the 3-sphere which admits an invariant Riemannian metric with positive scalar (sectional) curvature. By Lemma 4.5, it is easy to see that $\mathbb{R}Z_{vck(\Delta^3)}$ is diffeomorphic to the connected sum of $(k-1)2^k+1$ copies of $S^2 \times S^1$. Moreover, by the proof of Lemma 4.5, $\mathbb{R}Z_{vck(\Delta^3)}$ is obtained from $\mathbb{R}Z_{\Delta^3} \times (S^0)^k$ by a sequence of equivariant surgeries of codimension 3. So from Theorem 4.7 we can conclude that $\mathbb{R}Z_{vck(\Delta^3)}$ admits an invariant Riemannian metric with positive scalar curvature. The proposition is proved. \hfill $\square$
It is shown in [17] that all \( n \)-dimensional small covers that admit flat Riemannian metrics are exactly the small covers over the \( n \)-cube \([0, 1]^n\). These manifolds are called real Bott manifolds in [15]. According to the discussion in [15, §7]), there are exactly four diffeomorphism types among real Bott manifolds in dimension 3, two of which are orientable (the type \( G_1 \) and \( G_2 \) in the list of [22, Theorem 3.5.5]). Moreover, Choi-Masuda-Oum [5] found an interesting combinatorial method to classify all real Bott manifolds up to affine diffeomorphism.

**Proposition 4.9.** A small cover \( M \) over a simple 3-polytope \( P \) can hold a Riemannian metric with nonnegative scalar curvature if and only if \( P \) is combinatorially equivalent to the cube \([0, 1]^3\) or a polytope obtained from \( \Delta^3 \) by a sequence of vertex cuts. In particular, all the orientable 3-dimensional small covers that can hold Riemannian metrics with nonnegative scalar curvature are the two orientable real Bott manifolds in dimension 3 and the connected sum of \( k \) copies of \( \mathbb{R}P^3 \) for any \( k \geq 1 \).

**Proof.** Clearly, if \( M \) admits a Riemannian metric of nonnegative scalar curvature, so is \( \mathbb{R}Z_P \). Conversely, if \( \mathbb{R}Z_P \) admits a Riemannian metric \( g \) of nonnegative scalar curvature, Proposition 4.8 tells us that we can choose \( g \) to be an invariant metric on \( \mathbb{R}Z_P \). Then \( g \) projects to a metric on \( M \) which also has nonnegative scalar curvature (see Definition 4.6). The first statement is proved.

If \( M \) is orientable, we can assume that the range of the characteristic function \( \lambda \) of \( M \) is in the subset \( \{e_1, e_2, e_3, e_1 + e_2 + e_3\} \) of \( (\mathbb{Z}_2)^3 \) where \( \{e_1, e_2, e_3\} \) is a basis of \( (\mathbb{Z}_2)^3 \) (see [21, Theorem 1.7]). Then up to a change of basis, the value of \( \lambda \) around a triangular facet is equivalent to the right picture in Figure 8. The surgery on the 3-dimensional small cover \( M \) corresponding to the vertex cut in Figure 8 is the connected sum of \( M \) with \( \mathbb{R}P^3 \) (see [18, §5]). Then since the small cover over \( \Delta^3 \) is \( \mathbb{R}P^3 \), the second statement is proved. \(\square\)

A closed Riemannian flat 3-manifold cannot admit any Riemannian metric with positive scalar curvature (otherwise the 3-torus would admit a metric of positive scalar curvature which is impossible). So we have the following corollary.
Corollary 4.10. A small cover or the real moment-angle manifold over a simple polytope $P$ admits a Riemannian metric with positive scalar curvature if and only if $P$ is combinatorially equivalent to a polytope obtained from $\Delta^3$ by a sequence of vertex cuts.

We summarize results in this section along with some results from [17, §5] in the Table 1 below, where we list all the 3-dimensional simple polytopes and their dual simplicial polytopes over which the small covers (or real moment-angle manifolds) can hold Riemannian metrics with various curvature conditions.

| 3-dim. Small Cover | Simple 3-Polytope | (Dual) Simplicial 3-Polytope |
|---------------------|-------------------|------------------------------|
| Sectional/Ricci $> 0$ | $\Delta^3$ | $\Delta^3$ |
| Scalar $> 0$ | $\text{vc}^k(\Delta^3)$, $k \geq 0$ | Stacked 3-polytopes |
| Sectional/Ricci $\geq 0$ | $[0,1]^3$, $\Delta^3$, $\Delta^2 \times [0,1]$ | Octahedron, $\Delta^3$, Triangular bipyramid |
| Scalar $\geq 0$ | $[0,1]^3$, $\text{vc}^k(\Delta^3)$, $k \geq 0$ | Octahedron, Stacked 3-polytopes |

Table 1. 3-dimensional small covers which admit Riemannian metrics with various curvature conditions

In dimension $\geq 4$, characterizing all small covers and real moment-angle manifolds that admit Riemannian metrics of positive (or nonnegative) scalar curvature should be much harder than dimension 3. Indeed, in dimension 4 there are obstructions to the existence of metrics of positive scalar curvature from Seiberg-Witten invariants. In dimension $\geq 5$, the existence of Riemannian metrics with positive scalar curvature is intimately related to the existence of spin structures (see [12]) and some differential topological obstructions (e.g. $\hat{A}$-genus) from index theory of the Dirac operator (see [13, 14, 28]). The reader is referred to [29, 24] for a survey of this subject. But translating these conditions on small covers into conditions on the underlying simple polytopes seems quite difficult. In addition, the minimal hypersurface method (see [27, 25]) should be useful for us to study this problem as well.
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