Hidden Anderson Localization in Disorder-Free Ising-Kondo Lattice

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Anderson localization (AL) phenomena usually exits in systems with random potential. However, disorder-free quantum many-body systems with local conservation can also exhibit AL or even many-body localization transition. In this work, we show that the AL phase exists in a modified Kondo lattice without external random potential. The density of state, inverse participation ratio and temperature-dependent resistance are computed by classical Monte Carlo simulation, which uncovers the AL phase from previously studied Fermi liquid and Mott insulator regime. The occurrence of AL roots from quenched disorder formed by conservative localized moments. Interestingly, a many-body wavefunction is found, which captures elements in all three paramagnetic phases and is used to compute their quantum entanglement. In light of these findings, we expect the disorder-free AL phenomena can exit in generic translation-invariant quantum many-body systems.

I. INTRODUCTION

Localization phenomena due to random potential, namely the Anderson localization (AL), is at the heart position in modern condensed matter physics. When interplay with interaction, many-body localization (MBL) emergences and has intrigued intensive studies on disordered quantum many-body systems. Interestingly, if local conservation, e.g. $Z_2$ gauge symmetry, exits in Hamiltonian, AL/MBL can exit without external quenched disorder, thus certain translation-invariant quantum systems can exhibit AL or MBL. However, existing examples of disorder-free AL and MBL are still rare, in despite of general interests on relation to quantum thermalization, lattice gauge field, topological order and novel quantum liquid.

Recently, we have revisited a modified Kondo lattice model, namely the Ising-Kondo lattice (IKL), which is shown to reduce to fermions moving on static potential problem due to local conservation of localized moments, thus admits a solution by classical Monte Carlo (MC) simulation. On square lattice at half-filling, Fermi liquid (FL), Mott insulator (MI) and Néel antiferromagnetic insulator (NAI) are established. (See also Fig. 1.) When doping is introduced, spin-stripe physics emerges with competing magnetic ordered states, similar to $t-J$ model and $f$-electron materials. As emphasized by Antipov et. al. in the context of Falicov-Kimball model, since the weight of static potential satisfies Boltzmann distribution, and if temperature is high enough, the probability distribution of all configurations of static potential tends to be equal, thus may realize binary random potential distribution. When such intrinsic random potential is active, AL of fermions appears without external quenched disorder.

In this paper, we explore the possibility of AL in IKL model on square lattice. To simplify the discussion and meet with our previous work, here we focus on half-filling case though doping the half-filled system does not involve any technical difficulty. (An example on doped system is given in Appendix C.) By inspecting density of state, inverse participation ratio and temperature-dependent resistance, we will show that the AL phase emerges in intermediate coupling regime between metallic FL and insulating MI above antiferromagnetic critical temperature. (See Fig. 1.) The occurrence of AL results from quenched disorder, formed by the conservative localized moment at each site. Interestingly, we find a many-body wavefunction, which captures elements in all three paramagnetic phases and is used to compute their entanglement entropy. In light of these findings, we argue that the disorder-free AL phenomena could exit in more generic translation-invariant quantum many-body systems.

The remainder of this paper is organized as follows: In Sec. II, the IKL model model is introduced and its MC formalism is developed. In Sec. III, MC is performed and observable such as density of state, inverse participation ratio and temperature-dependent resistance are computed. Analysis on MC data shows the appearance of AL phase in intermediate coupling regime if thermal fluctuation destroys magnetic long-ranged order. A wave-
function is constructed and the entanglement entropy is evaluated. Sec. IV gives a summary and a brief discussion on AL in generic quantum many-body systems.

II. MODEL AND METHOD

A. IKL model

The IKL model on square lattice at half-filling is defined as follows:\textsuperscript{17}

$$\hat{H} = -t \sum_{\langle i,j \rangle, \sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + \frac{J}{2} \sum_{j\sigma} \hat{S}_j^z \hat{c}_{j\sigma}^\dagger \hat{c}_{j\sigma}$$

(1)

where itinerant electron interplays with localized f-electron moment via longitudinal Kondo exchange. Here, $\hat{c}_{j\sigma}$ is the creation operator of conduction electron and $\hat{S}_j^z$ denotes the localized moment of f-electron at site j. $t$ is the hopping integral between nearest-neighbor sites i and j and J is the longitudinal Kondo coupling, which is usually chosen to be antiferromagnetic ($J > 0$). In literature, this model (with x-axis anisotropy) is originally proposed to account for the anomalously small staggered magnetization and large specific heat jump at hidden order transition in URu$_2$Si$_2$.\textsuperscript{16,21} It can explain the easy-axis magnetic order and paramagnetic metal or bad metal behaviors in the global phase diagram of heavy fermion compounds.\textsuperscript{22-24}

In Ref.\textsuperscript{17}, we have observed that f-electron’s spin/localized moment at each site is conservative since $[\hat{S}_j^z, \hat{H}] = 0$. Therefore, taking the eigenstates of spin $\hat{S}_j^z$ as bases, the Hamiltonian Eq. 1 is automatically reduced to a free fermion moving on effective static potential $\{q_j\}$

$$\hat{H}(q) = -t \sum_{\langle i,j \rangle, \sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + \sum_{j\sigma} \frac{J\sigma}{4} q_j \hat{c}_{j\sigma}^\dagger \hat{c}_{j\sigma}$$

(2)

with q emphasizing its q dependence and $\hat{S}_j^z\{q_j\} = \frac{q_j}{2}\{q_j\}$, $q_j = \pm 1$. Now, the many-body eigenstate of original model Eq. 1 can be constructed via single-particle state of effective Hamiltonian Eq. 2 under given configuration of effective Ising spin $\{q_j\}$. So, Eq. 1 is solvable in the spirit of well-known Kitaev’s toric-code and honeycomb model.\textsuperscript{25,26} At finite-T, this model can be readily simulated by classical MC simulation.\textsuperscript{27} (We consider periodic $N_x = L \times L$ lattices with L up to 20.)

B. MC simulation

To be simulated by MC, we can write the partition function as

$$\mathcal{Z} = \text{Tr} e^{-\beta \hat{H}} = \text{Tr}_c \text{Tr}_S e^{-\beta \hat{H}} = \sum_{\{q_j\}} \text{Tr} e^{-\beta \hat{H}(q)}.$$

Here, the trace is split into c-fermion and $\hat{S}^z$, where the latter is transformed into the summation over all possible configuration $\{q_j\}$. For each single-particle Hamiltonian $\hat{H}(q)$, it can be easily diagonalized into

$$\hat{H}(q) = \sum_{n\sigma} E_{n\sigma} \hat{d}_{n\sigma}^\dagger \hat{d}_{n\sigma}$$

where $E_{n\sigma}$ is the single-particle energy level and $\hat{d}_{n\sigma}$ is the quasi-particle. The fermion $\hat{d}_{n\sigma}$ is related into $\hat{c}_{j\sigma}$ via

$$\hat{c}_{j\sigma} = |0\rangle \langle j| = \sum_n |0\rangle \langle j| |n\sigma\rangle \langle n\sigma| = \sum_n \hat{d}_{n\sigma} \phi_{n\sigma}^j,$$

with $\phi_{n\sigma}^j \equiv \langle j| |n\sigma\rangle$. Now, the trace over c-fermion can be obtained as

$$\text{Tr}_c e^{-\beta \hat{H}(q)} = \sum_{n\sigma} \langle n\sigma| e^{-\beta \sum_{m\sigma'} E_{m\sigma'} \hat{d}_{m\sigma'}^\dagger \hat{d}_{m\sigma'}} |n\sigma\rangle$$

$$= \prod_{n\sigma} (1 + e^{-\beta E_{n\sigma}}).$$

This is the familiar result for free fermion, however one should keep in mind that $E_{n\sigma}$ actually depends on the effective Ising spin configuration $\{q_j\}$, thus we write $E_{n\sigma}(q)$ to emphasize this fact. So, the partition function reads

$$\mathcal{Z} = \prod_{\{q_j\}, n\sigma} (1 + e^{-\beta E_{n\sigma}(q)}) = \sum_{\{q_j\}} e^{-\beta F(q)}$$

(3)

where we have defined an effective free energy

$$F(q) = -T \sum_{n\sigma} \text{ln}(1 + e^{-\beta E_{n\sigma}(q)}).$$

In this situation, we can explain $e^{-\beta F(q)}$ or $\rho(q) = \frac{1}{Z} e^{-\beta F(q)}$ as an effective Boltzmann weight for each configuration of $\{q_j\}$ and this can be used to perform Monte Carlo simulation just like the classic Ising model.

To calculate physical quantities, we consider generic operator $\hat{O}$, which can be split into part with only Ising spin $\{q_j\}$ and another part with fermions,

$$\hat{O} = \hat{O}^c + \hat{O}^q.$$

Then, its expectation value in the equilibrium ensemble reads

$$\langle \hat{O} \rangle = \langle \hat{O}^c \rangle + \langle \hat{O}^q \rangle = \frac{\text{Tr} \hat{O}^c e^{-\beta \hat{H}}}{\text{Tr} e^{-\beta \hat{H}}} + \frac{\text{Tr} \hat{O}^q e^{-\beta \hat{H}}}{\text{Tr} e^{-\beta \hat{H}}}$$

For $\hat{O}^q$, we have

$$\langle \hat{O}^q \rangle = \sum_{\{q_j\}} \langle \hat{O}^q(\{q_j\}) e^{\beta \sum_{q_j} \text{Tr} e^{-\beta \hat{H}(\{q_j\})}} \rangle$$

$$= \sum_{\{q_j\}} \langle \hat{O}^q(\{q_j\}) e^{-\beta F(q)} \rangle$$

$$= \sum_{\{q_j\}} \hat{O}^q(\{q_j\}) \rho(q).$$
In the Metropolis importance sampling algorithm, the above equation means we can use the simple average to estimate the expectation value like

$$\langle \hat{O} \rangle \simeq \frac{1}{N_m} \sum_{\{q_j\}} \hat{O}(q)$$

where $N_m$ is the number of sampling and the sum is over each configuration. $\hat{O}(q)$ is a number since we always work on the basis of $\{q_j\}$.

For $\hat{O}_c$,

$$\langle \hat{O}_c \rangle = \frac{\sum_{\{q_j\}} e^{\beta H} \sum_q \Omega(q) e^{-\beta \hat{H}(q)} \sum_{\{q_j\}} e^{-\beta F(q)}}{\sum_{\{q_j\}} e^{-\beta F(q)}}$$

and we can insert $\frac{e^{-\beta F(q)}}{e^{-\beta F(q)}}$ in the numerator, which leads to

$$\langle \hat{O}_c \rangle = \frac{\sum_{\{q_j\}} \Omega(q) e^{-\beta \hat{H}(q)} e^{-\beta F(q)}}{\Omega e^{-\beta \hat{H}(q)} \sum_{\{q_j\}} e^{-\beta F(q)}}$$

$$= \frac{\sum_{\{q_j\}} \Omega(q) e^{-\beta \hat{H}(q)} \rho(q)}{\Omega e^{-\beta \hat{H}(q)} \sum_{\{q_j\}} e^{-\beta F(q)}}$$

$$= \frac{\langle \Omega(q) \rangle \rho(q)}{\sum_{\{q_j\}} e^{-\beta F(q)}}.$$

This means

$$\langle \hat{O}_c \rangle \simeq \frac{1}{N_m} \sum_{\{q_j\}} \langle \Omega(q) \rangle,$$

where $\langle \Omega(q) \rangle = \frac{\sum_{\{q_j\}} e^{\beta H} \sum_q \Omega(q) e^{-\beta \hat{H}(q)} \sum_{\{q_j\}} e^{-\beta F(q)}}{\sum_{\{q_j\}} e^{-\beta F(q)}}$ is calculated based on the Hamiltonian $\hat{H}(q)$. More practically, such statement means if fermions are involved, one can just calculate with $\hat{H}(q)$. Then, average over all sampled configuration gives rise to desirable results.

### III. RESULT

In terms of MC, we have determined the finite temperature phase diagram in Fig. 1. In addition to well-established FL, MI and NAI in previous work, interestingly, we have found a AL phase in intermediate coupling regime at high $T$. There is no transition but crossover from FL to AL and AL to MI. Since the former three phases have been detailed studied\textsuperscript{17}, in this work, we focus on the AL phase.

To characterize the AL phase from FL or MI, we have used density of state (DOS), inverse participation ratio (IPR) and temperature-dependent resistance of conduction electrons\textsuperscript{3,8}. The DOS of $c$-fermion $N(\omega)$ is evaluated from

$$N(\omega) \simeq \frac{1}{N_m N_s} \sum_{\{q_j\}} \sum_{n} \delta(\omega - E_{n\sigma}(q)).$$

In FL, its DOS at Fermi energy ($N(0), \omega = 0$ is Fermi energy) is finite. For usual AL phase, it results from localization-delocalization transition from metallic FL states due to random potential, so its $N(0)$ is finite. As for MI, Mott gap driven by local magnetic fluctuation leads to vanishing $N(0)$\textsuperscript{17}.

The IPR measures tendency of localization and in our case, the energy/frequency-dependent IPR is used,

$$\text{IPR} \simeq \frac{1}{N_m N_s} \sum_{\{q_j\}} \sum_{n} \sum_{j} \delta(\omega - E_{n\sigma}(q))(\Phi_{n\sigma}^q)^4.$$

In a localized state, it has to saturate for large system size. In contrast, in a delocalized state, such as FL, it has size-dependence as $\text{IPR} \propto 1/V$, suggesting a well-defined inverse-volume behavior\textsuperscript{3,8}.

The temperature-dependent resistance is related to static conductance $\sigma_{dc}$ as $\rho = 1/\sigma_{dc}$, which reads

$$\sigma_{dc} = \frac{\pi t^2 e^2}{N_m} \sum_{\{q_j\}} \int d\omega \frac{\partial f_F(\omega)}{\partial \omega} \Phi^q(\omega).$$

The derivation of $\sigma_{dc}$ and the detailed form of $\Phi^q(\omega)$ can be found in Appendix B. Generally, localized phases show insulating behavior at low temperature while delocalized metallic phases have contrast tendency.

Now, from MC calculation of these quantities, e.g. Fig. 2, 3 and 4, we have found a AL phase in intermediate coupling, beside well-established FL and MI at high $T$ regime. In Fig. 2, AL has finite $N(0)$ though its strength is highly suppressed and looks like a pseudogap. The reason is that due to the preformed local antiferromagnetic order, the band gap begins to form at low $T$. When increasing temperature, excitation of localized moments appears and it acts like impurity scattering center in the well-formed antiferromagnetic background. Then, the conduction electron scatters from such impurity and contributes impurity bound state, which fills in the band gap\textsuperscript{28}. At high $T$, the long-ranged antiferromagnetic order melts but the band gap survives due to remaining local antiferromagnetic order. After considering impurity bound states, $N(0)$ in AL is finite and a pseudogap-like behavior appears.

In Fig. 3, we see IPR of FL satisfies the expected inverse-volume law while AL and MI have saturated IPR.
as if you were reading it naturally.
C. Entanglement entropy

The entanglement entropy $S_{EE}$ is used to characterize the universal quantum correlation in many-body state. For our model, we calculate $S_{EE}$ for each Slater determinant state $|\psi\rangle$ in given Ising configuration $\{g_i\}$ as follows:\n
We consider open boundary condition and use $|\psi\rangle$ to compute equal-time correlation function $g_{ij,\sigma}$ as $g_{ij,\sigma}^q = \langle\langle \hat{c}_i^{\dagger}\hat{c}_j|\psi\rangle|\psi\rangle$. Dividing our system into $A$ and $B$ parts, a reduced correlation function $g_{ij,\sigma}^q$ is constructed as

$$g_{ij,\sigma}^q = g_{ij,\sigma}^q \quad \text{for} \quad i, j \in A$$

and others are zero. Treating $g_{ij}^q$ as $N_A \times N_A$ matrix with non-zero eigenvalues $\{\xi_\alpha\}$ and $N_A$ being the number of sites in region $A$, the entanglement entropy between subsystem $A, B$ is

$$S_{EE} = -\sum_\alpha \xi_\alpha \ln \xi_\alpha + (1 - \xi_\alpha) \ln(1 - \xi_\alpha). \quad (11)$$

Using Eq. 11, we have computed $S_{EE}$ for many-body state Eq. 10, whose results are shown in Fig. 6. Firstly, $S_{EE}$ decreases monotonically from small-J to large-J regime, agreeing with the increased localization tendency. Another interesting feature is that when $J/t \approx 12$, $S_{EE}/L_c$ collapses into a single line, thus indicates a crossover from AL to MI at $T = \infty$. More close inspection on $S_{EE}$ shows a linear-dependence on $L_c$ in MI regime, which is the well-known area-law for $S_{EE}$. For FL regime, its $S_{EE}$ deviates from area-law with a logarithmic correction. A fitting in FL gives $S_{EE} \approx 0.275L_c \ln L_c + 5.05$ for $J/t = 2$. As for AL, it qualitatively obeys area-law though a small deviation exists due to inevitable mixing with delocalized states.

IV. CONCLUSION AND DISCUSSION

In conclusion, we have established a AL phase in a modified Kondo lattice model without external disorder potential. The presence of AL results from quenched disorder, formed by conservative localized moment at each site and is a stable phase even at infinite temperature. A many-body wavefunction is constructed to understand AL, FL and MI. Their entanglement entropy is computed and the area-law is violated in FL. In light of these findings, recall that interacting many-electron system can be rewritten as free electrons moving on fluctuated background field after Hubbard-Stratonovich transformation. If dynamics of the background field is frozen, it may act as random potential and AL or MBL phase can be observed in generic many-body Hamiltonian. Hopefully certain classic models, such as Hubbard and Kondo lattice, may support the presence of those localized states of matter. These intriguing possibilities will be left for our future work.

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Appendix A: Correlation function and spectral function

When we calculate fermion’s correlation function like $\langle\langle \hat{c}_i^{\dagger}\hat{c}_j\rangle\rangle$, we get

$$\langle\langle \hat{c}_i^{\dagger}\hat{c}_j\rangle\rangle = \frac{1}{N_m} \sum_{\{q_j\}} \langle\langle \hat{c}_i^{\dagger}\hat{c}_j\rangle\rangle \langle\langle \hat{c}_j\hat{c}_i\rangle\rangle.$$ 

Then, using the Wick theorem for these free fermions, we get

$$\langle\langle \hat{c}_i^{\dagger}\hat{c}_j\rangle\rangle = \langle\langle \hat{c}_i^{\dagger}\hat{c}_j\rangle\rangle + \langle\langle \hat{c}_i\hat{c}_j\rangle\rangle.$$ 

Next, for each one-body correlation function like $\langle\langle \hat{c}_i\hat{c}_j\rangle\rangle$, one can transform these objects into their quasiparticle basis,

$$g_{ij}^q \equiv \langle\langle \hat{c}_i\hat{c}_j\rangle\rangle = \sum_{m,n} \langle\langle \hat{d}_m^{\dagger}\hat{d}_n\rangle\rangle (\phi_m^*)^* \phi_n^* = \sum_{m,n} f_F(E_n(q)) \delta_{mn} (\phi_m^*)^* \phi_n^*,$$

Similarly, we have

$$\langle\langle \hat{c}_i\hat{c}_j\rangle\rangle = \sum_n (1 - f_F(E_n(q))) \phi_n^* \phi_n^* = \delta_{ij} - g_{ij}^q.$$ 

For calculating dynamic quantities like conductance, (imaginary) time-dependent correlation function such as $\langle\langle \hat{c}_i(\tau)\hat{c}_j\rangle\rangle$, $\langle\langle \hat{c}_i(\tau)\hat{c}_j\rangle\rangle$ has to be considered. It is easy to
Thus, assuming $\tau > 0$, one finds the following relations between time-dependent correlation functions and their Green’s function,

\[
G_{ij}^q(-\tau) = \langle \langle \hat{c}_i(\tau)\hat{c}_j(\tau) \rangle \rangle, \quad G_{ij}^q(\tau) = -\langle \langle \hat{c}_i(\tau)\hat{c}_j(\tau) \rangle \rangle,
\]

and

\[
G_{ij}^q(\omega_n) = \int_0^\beta d\tau e^{i\omega_n\tau} G_{ij}^q(\tau) = \sum_n \frac{\phi_n^i(\phi_n^j)^*}{i\omega_n - E_n(q)}.
\]

The corresponding retarded Green’s function is obtained via analytic continuity $i\omega_n \to \omega + i0^+$

\[
G_{ij}^q(\omega) = \sum_n \frac{\phi_n^i(\phi_n^j)^*}{\omega + i0^+ - E_n(q)}.
\]

An alternative expression for $A_{ij}^q(\omega)$ is

\[
A_{ij}^q(\omega) = -\frac{1}{\pi} \text{Im} G_{ij}^q(\omega) = \sum_n \phi_n^i(\phi_n^j)^* \delta(\omega - E_n(q)).
\]

The spectral function with momentum-dependence has essential importance to spectral experiments, which can be found as

\[
A^q(k,\omega) = \frac{1}{N_s} \sum_{ij} e^{-ik(R_i-R_j)} A_{ij}^q(\omega).
\]

**Appendix B: Static conductance and resistance**

The dc conductance is related to current-current correlation function as

\[
\sigma_{dc} = \lim_{\omega \to 0} \frac{\text{Im} \Lambda_{xx}(k = 0, \omega)}{\omega}
\]

and the static resistivity is $\rho = 1/\sigma_{dc}$. Here, the retarded current-current correlation function $\Lambda_{xx}(k = 0, \omega)$ can be obtained via its imaginary-time form

\[
\Lambda_{xx}(k, i\Omega_n) = \frac{1}{N_s} \sum_{ij} e^{ik(R_i-R_j)} \int d\tau e^{\Omega_n \tau} \langle \hat{J}_x(i,\tau)\hat{J}_x(j,0) \rangle.
\]

Here, $\hat{J}_x$ is $x$-axis component of current operator. Because our model is defined on a lattice, in terms of Peierls substitution, the external electromagnetic potential $A_x(i) \equiv A_{i,i+x}$ is introduced as

\[
-j = A_{ij} = -A_{ji}. \text{ Now, } \hat{J}_x(i) \text{ is derived as}
\]

\[
\hat{J}_x(i) = \lim_{\Delta A_x(i) \to 0} \frac{\delta \hat{H}}{\delta A_x(i)} = \text{ite} \sum_{\sigma} (\hat{c}_{i+x,\sigma}^\dagger \hat{c}_{i,\sigma} - \hat{c}_{i,\sigma}^\dagger \hat{c}_{i+x,\sigma}).
\]

Therefore,

\[
\langle \hat{J}_x(i,\tau)\hat{J}_x(j,0) \rangle = \frac{1}{N_m} \sum_{\{ij\}} \langle \langle \hat{J}_x(i,\tau)\hat{J}_x(j,0) \rangle \rangle
\]

and

\[
\frac{1}{(\text{ite})^2} \langle \langle \hat{J}_x(i,\tau)\hat{J}_x(j,0) \rangle \rangle = \sum_{ij,\sigma,\sigma'} g_{ij,\sigma,\sigma'}^q (g_{j+x,\sigma,\sigma'}^q - g_{j+x,\sigma,\sigma'}^q) + \delta_{\sigma,\sigma'} G_{ij,\sigma,\sigma'}^q \delta(\tau - \tau) G_{i+x,\sigma,\sigma'}^q
\]

\[
+ \delta_{\sigma,\sigma'} G_{ij,\sigma,\sigma'}^q \delta(\tau - \tau) G_{i,j+x,\sigma,\sigma'}^q + \delta_{\sigma,\sigma'} G_{ij,\sigma,\sigma'}^q \delta(\tau - \tau) G_{i,j,x,\sigma,\sigma'}^q
\]

\[
- \delta_{\sigma,\sigma'} G_{ij,\sigma,\sigma'}^q \delta(\tau - \tau) G_{i,j,x,\sigma,\sigma'}^q.
\]

Here, $g_{ij,\sigma,\sigma'}^q$ has no frequency-dependence and imaginary part, thus it cannot contribute to conductance and will
be neglected hereafter. Integrating over $\tau$ gives

$$
\int d\tau e^{i\Omega\tau} \frac{1}{(i\epsilon)^2} \langle \langle \hat{J}_x (i, \tau) \hat{J}_x (j, 0) \rangle \rangle
$$

$$= -T \sum_{\omega_n, \sigma} G^{q}_{j+\omega+i, \sigma}(\omega_n) G^{q}_{j+\omega+i, \sigma}(\omega_n + \Omega)$$

$$+ T \sum_{\omega_n, \sigma} G^{q}_{j, \sigma}(\omega_n) G^{q}_{j+\omega+i, \sigma}(\omega_n + \Omega)$$

$$+ T \sum_{\omega_n, \sigma} G^{q}_{j, \sigma}(\omega_n) G^{q}_{j+\omega+i, \sigma}(\omega_n + \Omega)$$

$$- T \sum_{\omega_n, \sigma} G^{q}_{j+\omega+i, \sigma}(\omega_n) G^{q}_{j, \sigma}(\omega_n + \Omega)$$

$$= \sum_{\sigma} \int d\omega_1 \int d\omega_2 \int d\omega_3 \left( f_F(\omega_1) - f_F(\omega_2) \right)$$

$$\times (i\epsilon)^2 \langle \langle \hat{J}_x (i, \tau) \hat{J}_x (j, 0) \rangle \rangle$$

$$= \langle \langle \hat{J}_x (i, \tau) \hat{J}_x (j, 0) \rangle \rangle$$

which leads to the retarded current-current correlation

$$\Lambda_{xx}(k, \omega + i0^+) = \frac{(i\epsilon)^2}{(2\pi)^2} \sum_{\omega_n, \omega_{n+\Omega}} \sum_{\{q\}} \sum_{ij, \sigma} \int d\omega_1 \int d\omega_2$$

$$\left( f_F(\omega_1) - f_F(\omega_2) \right)$$

$$\times (i\epsilon)^2 \langle \langle \hat{J}_x (i, \tau) \hat{J}_x (j, 0) \rangle \rangle$$

$$= \langle \langle \hat{J}_x (i, \tau) \hat{J}_x (j, 0) \rangle \rangle$$

Now, it is straightforward to get

$$\text{Im} \Lambda_{xx}(0, \omega) = \frac{\pi e^2}{N_s N_m} \sum_{\sigma} \sum_{\{q\}} \int d\omega_1 \left( f_F(\omega_1) - f_F(\omega_1 + \omega) \right)$$

$$\times \left[ -A^q_{j+\omega+i, \sigma}(\omega_1) A^q_{1+\omega+i, \sigma}(\omega_1 + \omega) + A^q_{j, \sigma}(\omega_1) A^q_{1+\omega+i, \sigma}(\omega_1 + \omega) + A^q_{j+i, \sigma}(\omega_1) A^q_{1+i, \sigma}(\omega_1 + \omega) + A^q_{j+i, \sigma}(\omega_1) A^q_{1+i, \sigma}(\omega_1 + \omega) \right].$$

Finally, the dc conductance is found to be

$$\sigma_{dc} = \frac{\pi e^2}{N_m} \sum_{\{q\}} \int d\omega \left( f_F(\omega) - f_F(\omega + \Omega) \right) \Phi^q(\omega)$$

with

$$\Phi^q(\omega) = \frac{1}{N_s} \sum_{\omega_n} \left[ -A^q_{j+i, \sigma}(\omega) A^q_{1+i, \sigma}(\omega) + A^q_{j+i, \sigma}(\omega) A^q_{1+i, \sigma}(\omega) + A^q_{j+i, \sigma}(\omega) A^q_{1+i, \sigma}(\omega) - A^q_{j+i, \sigma}(\omega) A^q_{1+i, \sigma}(\omega) \right].$$

Appendix C: Example for doped system at $T = \infty$

Here, we show the entanglement entropy $S_{EE}$ and IPR at Fermi energy IPR(0) for the doped system. We choose chemical potential $\mu$ as $\mu/t = -4, -3, -2, -1, 0$ and set $J/t = 8$. Because the low $T$ phase diagram of the doped system is rather complicated due to intertwined magnetic orders, instead, we focus on $T = \infty$ limit, where only paramagnetic phases survive.

Using Eq. 11, $S_{EE}$ and IPR(0) are shown in Fig. 7. We find that $S_{EE}$ for different $\mu$ has similar linear dependence on $L_c$, and IPR(0) in infinite system limit is finite. Thus, the AL phase is stable when deviating from half-filling, at least in $T = \infty$ limit.
12. M. Srednicki, Phys. Rev. E 50, 888 (1994).
13. M. Rigol, V. Dunjko and M. Olshanii, Nature (London) 452, 854 (2008).
14. J. B. Kogut, Rev. Mod. Phys. 51, 659 (1979).
15. A. Smith, J. Koolle, R. Moessner and D. L. Kovrizhin, Phys. Rev. Lett. 119, 176601 (2017).
16. A. E. Sikkema, W. J. L. Buyers, I. Affleck and J. Gan, Phys. Rev. B 54, 9322 (1996).
17. W.-W. Yang, J. Zhao, H.-G. Luo and Y. Zhong, Phys. Rev. B 100, 045148 (2019).
18. S. R. White and D. J. Scalapino, Phys. Rev. Lett. 80, 1272 (1998).
19. J. W. Lynn, S. Skanthakumar, Q. Huang, S. K. Sinha, Z. Hossain, L. C. Gupta, R. Nagarajan and C. Godart, Phys. Rev. B 55, 6584 (1997).
20. L. M. Falicov and J. C. Kimball, Phys. Rev. Lett. 22, 997 (1969).
21. J. A. Mydosh, P. M. Oppeneer, Rev. Mod. Phys. 83, 1301 (2011).
22. P. Coleman, Introduction to Many Body Physics, chapters 15 to 18 (Cambridge University Press, 2015).
23. Q. Si and S. Paschen, Phys. Stat. Solid. B 250, 425-438 (2013).
24. P. Coleman and A. H. Nevidomskyy, J. Low Temp. Phys. 161, 182 (2010).
25. A. Kitaev, Ann. Phys. 303, 2 (2003).
26. A. Kitaev, Ann. Phys. 321, 2 (2006).
27. M. M. Maska and K. Czajka, Phys. Rev. B 74, 035109 (2006).
28. M. Zonda, J. Okamoto and M. Thoss, arXiv:1907.04697.
29. M. Dzero, K. Sun, P. Coleman, and V. Galitski, Phys. Rev. B 85, 045130 (2012).
30. J. Eisert, M. Cramer and M. B. Plenio, Rev. Mod. Phys. 82, 277 (2010).
31. M. M. Wolf, Phys. Rev. Lett. 96, 010404 (2006).
32. D. Gioev and I. Klich, Phys. Rev. Lett. 96, 100503 (2006).
33. J. Hubbard, Phys. Rev. Lett. 3, 77 (1959).
34. R. L. Stratonovich, Sov. Phys.-Doklady, (2), (1958).