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Časopis pro pěstování matematiky, Vol. 114 (1989), No. 2, 181--186

Persistent URL: [http://dml.cz/dmlcz/108706](http://dml.cz/dmlcz/108706)

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NATURAL TRANSFORMATIONS OF HIGHER ORDER TANGENT BUNDLES AND JET SPACES

Ivan Kolář, Gabriela Vosmanská, Bíno

Dedicated to Professor Otakar Borůvka on the occasion of his ninetieth birthday

(Received May 22, 1987)

Summary. We deduce that all natural transformations of the functor of the r-th order tangent vectors into itself are the homotheties only. We also determine all natural transformations of the r-th order jet functor into itself.

Keywords: Natural transformation, r-th order tangent vector, r-jet.

AMS Classification: 58A20.

Using a general method developed in [5], we first deduce that all natural transformations of the r-th order tangent functor $T^r$ into itself are the homotheties only. From the general point of view it is worth pointing out that this property is related with the fact that $T^r$ does not preserve products, and to contrast it with a recent result by G. Kainz and P. Michor, [3], which describes all natural transformations of the product-preserving differential geometric functors in terms of the homomorphisms of the related Weil algebras. Then we prove in a similar way that for $r \geq 2$ the only natural transformations of the r-th jet functor $J^r$ into itself are the identity and the contraction, while in the first order case, in which we deal with vector bundles, we have the one-parameter family of all homotheties. The authors hope that this interesting fact on a certain rigidity of the higher order jet spaces will lead to a deeper understanding of some general features of the higher order differential geometry. — All manifolds and maps are assumed to be infinitely differentiable.

1. Let $T^rM = J^r(M, \mathbb{R})_0$ be the space of all r-jets of a manifold $M$ into $\mathbb{R}$ with target 0. Since $\mathbb{R}$ is a vector space, $T^rM$ has a canonical structure of a vector bundle over $M$. The dual vector bundle $T^*M := (T^rM)^*$ is called the r-th order tangent bundle of $M$, [8]. Given a map $f: M \to N$, the jet composition $V \mapsto V \circ j^r_x f$, $V \in T^r_{f(x)}N$, determines a linear map $T^r_{f(x)}N \to T^r_x M$. The dual map $T^r_x M \to T^r_{f(x)}N$ will be denoted by $T^r_x f$ and called the r-th order tangent map of $f$ at $x$. This defines a functor $T^r$ from the category $\mathcal{M}$ of all manifolds and maps into the category $\mathcal{V}'\mathcal{B}$ of vector bundles.

If $x^i$ are some local coordinates on $M$, then the induced fibre coordinates $u_i, u_{i_1 i_2}, ..., u_{i_1 ... i_r}$ (symmetric in all indices) on $T^rM$ correspond to the polynomial representant $u_i x^i + u_{i_1 i_2} x^{i_1} x^{i_2} + ... + u_{i_1 ... i_r} x^{i_1} ... x^{i_r}$ of any element $U \in T^rM$. 

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A linear functional on $T^*M$ with the fibre coordinates $X^1, X^{i_1 i_2}, \ldots, X^{i_1 \ldots i_r}$ (symmetric in all indices) has the form

$$X^i u_i + X^{i_1 i_2} u_{i_1 i_2} + \ldots + X^{i_1 \ldots i_r} u_{i_1 \ldots i_r}.$$  

Let $y^p$ be some local coordinates on $N$, let $Y^p, Y^{p_1 p_2}, \ldots, Y^{p_1 \ldots p_r}$ be the induced fibre coordinates on $T^*N$ and let $y^p = f^p(x^i)$ be the coordinate expression of a map $f: M \to N$. Evaluating the jet composition $V \circ j^e_x f, V \in T^*_f(x)N$, we deduce by (1) the following coordinate expression of $T^e f$, cf. [4],

$$V^p = \frac{\partial f^p}{\partial x^i} X^i + \frac{1}{2!} \frac{\partial^2 f^p}{\partial x^{i_1} \partial x^{i_2}} X^{i_1 i_2} + \ldots + \frac{1}{r!} \frac{\partial^r f^p}{\partial x^{i_1} \ldots \partial x^{i_r}} X^{i_1 \ldots i_r}.$$

where the dots in the middle row denote a polynomial expression, each term of which contains at least one partial derivative of $f^p$ of an order at least two.

Since $T^e$ is a functor with values in the category $\mathcal{Y} \mathcal{A}$, for every $k \in \mathbb{R}$ the homotheties

$$(k)_M: T^e M \to T^e M, \quad X \mapsto kX$$

represent natural transformations of $T^e$ into itself.

**Proposition 1.** All natural transformations $T^e \to T^e$ form the one-parameter family (3) with any $k \in \mathbb{R}$.

**Proof.** First, consider $T^e$ as a functor on the subcategory $\mathcal{M}_n \subset \mathcal{M}$ of all $n$-dimensional manifolds and their local diffeomorphisms. Since $T^e$ is an $r$-th order functor, its standard fibre $S = T^e_0 \mathbb{R}^n$ is a $G^e_n$-space, where $G^e_n$ means the group of all invertible $r$-jets of $\mathbb{R}^n$ into $\mathbb{R}^n$ with source and target 0. By (2), the action of an element $(a^i_j, a^i_{j_1 j_2}, \ldots, a^i_{j_1 \ldots j_r}) \in G^e_n$ on $(X^i, X^{i_1 i_2}, \ldots, X^{i_1 \ldots i_r}) \in S$ is

$$X^i = a^i_j X^j + a^i_{j_1 j_2} X^{j_1 j_2} + \ldots + a^i_{j_1 \ldots j_r} X^{j_1 \ldots j_r}.$$

where the dots in the middle row denote a polynomial expression, each term of which contains at least one of the quantities $a^i_{j_1 j_2}, \ldots, a^i_{j_1 \ldots j_r}$. In the sequel we shall write shortly $(X^1, X^{i_1 i_2}, \ldots, X^{i_1 \ldots i_r}) = (X_1, X_2, \ldots, X_n)$.

According to a general theory, cf. [2], [7], the natural transformations $T^e \to T^e$ are in bijection with $G^e_n$-equivariant maps $f: S \to S$. There is a canonical injection $i$:  

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GL(n, R) → G_r transforming every matrix into the r-jet at 0 from the corresponding linear transformation of R^n. The subgroup i(GL(n, R)) ⊂ G_r is characterized by \( a^j_{1...s} = 0, \ldots, a^j_{1...r} = 0 \). First consider the equivariance of \( f = (f_1, \ldots, f_r) \) with respect to the homotheties \( a^j_k = k\delta^j_k \). Using (4) we obtain

\[
\begin{align*}
    kf_1(X_1, \ldots, X_s, \ldots, X_r) &= f_1(kX_1, \ldots, k^rX_s, \ldots, k^rX_r) \\
    k^rf_2(X_1, \ldots, X_s, \ldots, X_r) &= f_2(kX_1, \ldots, k^rX_s, \ldots, k^rX_r) \\
    k^rf_s(X_1, \ldots, X_s, \ldots, X_r) &= f_s(kX_1, \ldots, k^rX_s, \ldots, k^rX_r)
\end{align*}
\]

To discuss (5), we need the following simple property of the globally defined smooth homogeneous functions, a proof of which can be found e.g. in [9].

**Lemma.** Let \( g(x^1, y^1, \ldots, z^1) \) be a smooth function defined on \( R^m \times R^n \times \ldots \times R^p \), and let \( a > 0, b > 0, \ldots, c > 0, d \) be real numbers such that

\[
k^2g(x^1, y^2, \ldots, z^s) = g(k^a x^1, k^b y^2, \ldots, k^c z^s)
\]

for every real \( k > 0 \). Then \( g \) is a sum of polynomials of degrees \( \xi \) in \( x^1 \), \( \eta \) in \( y^2, \ldots, \zeta \) in \( z^s \) satisfying

\[
a^\xi + b^\eta + \ldots + c^\zeta = d
\]

If there are no non-negative integers \( \xi, \eta, \ldots, \zeta \) with the property (7), then \( g \) is the zero function.

According to this lemma, \( f_1 \) is linear in \( X_1 \) and independent of \( X_2, \ldots, X_r \), while \( f_s = g_s(X_1) + h_s(X_1, \ldots, X_{s-1}) \), where \( g_s \) is linear in \( X_s \) and \( h_s \) is a certain polynomial in \( X_1, \ldots, X_{s-1}, 2 \leq s \leq r \). Considering the equivariance of \( f \) with respect to the whole subgroup \( i(GL(n, R)) \), we find that \( g_s \) is a \( GL(n, R) \)-equivariant map of the \( s \)-th symmetric tensor power \( S^n R^n \) into itself. By the classical theory of the invariant tensors, \( g_s = c_s X_s \) (or, explicitly, \( g^{i_1 \ldots i_s} = c_s X^{i_1 \ldots i_s} \)) with any \( c_s \in R \), cf. [1].

Further, consider the equivariance with respect to the kernel of the jet projection \( G^r_r \rightarrow G^s_s = GL(n, R) \), which is characterized by \( a^j_i = \delta^j_i \). Then the first line of (4) implies

\[
c^j_1 X_1^j + a^j_{1...j}(c^j_2 X^{j_1 j_2} + h^{j_1 j_2}(X_1)) + \ldots + a^j_{1...j}(c^j_r X^{j_1 ... j_r} + h^{j_1 ... j_r} (X_1, \ldots, X_{r-1})) = c^j_1 (X_1^j + a^j_{1...j} X^{j_1 j_2} + \ldots + a^j_{1...j} X^{j_1 ... j_r}).
\]

Setting \( a^j_{1...j} = 0 \) for all \( s > 2 \), we find \( c^j_2 = c^j_1 \) and \( h^{j_1 j_2}(X_1) = 0 \). By a recurrence procedure of this type we further deduce \( c^j_s = c^j_1 \) and \( h^{j_1 ... j_s}(X_1, \ldots, X_{s-1}) = 0 \) for all \( s = 3, \ldots, r \).

This implies that the restriction of every natural transformation \( T^r \rightarrow T^r \) to each subcategory \( M_n \subset M \) is a homothety with a coefficient \( k_n \). Taking into account the injection \( R^n \rightarrow R^{n+m}, (x_1, \ldots, x_n) \rightarrow (x_1, \ldots, x_n, 0, \ldots, 0) \), we find \( k_{n+m} = k_n \) for all \( m \) and \( n \). This completes the proof of Proposition 1.
2. Let \( f: M \to M \) be a local diffeomorphism and let \( g: N \to N \) be any map. Then there is an induced map \( J^r(f, g) \) from the space \( J^r(M, N) \) of all \( r \)-jets of \( M \) into \( J^r(M, N) \) given by
\[
J^r(f, g)(X) = (j^r_f g) \circ X \circ (j^r_{f^{-1}})
\]
where \( x = a X \) or \( y = b X \) is the source or the target of \( X \in J^r(M, N) \) and the inverse map \( f^{-1} \) is constructed locally, cf. [6]. This defines a functor \( J^r \) from the product category \( \mathcal{M}_m \times \mathcal{M}_n \) into the category of fibred manifolds (we consider \( J^r(M, N) \) as a fibred manifold over \( M \times N \)).

Denote by \( \hat{f}: M \to N \) the constant map of \( M \) into \( J^r(M, N) \). Obviously, the assignment \( X \mapsto j^r_{aX} \hat{f} X \) is a (trivial) natural transformation of \( J^r \) into itself called the contraction.

For \( r = 1 \), \( J^1(M, N) \) coincides with \( \text{Hom}(TM, TN) \) which is a vector bundle over \( M \times N \).

**Proposition 2.** For \( r \geq 2 \) the only natural transformations \( J^r \to J^r \) are the identity and the contraction. For \( r = 1 \) all natural transformations \( J^1 \to J^1 \) form the one-parameter family of homotheties \( A \mapsto kA, k \in \mathbb{R} \).

**Proof.** We shall consider the subcategory \( \mathcal{M}_m \times \mathcal{M}_n \subset \mathcal{M}_m \times \mathcal{M}_n \) only, since the remaining part of the proof is quite similar to the end of the proof of Proposition 1. The standard fibre \( S = J^r_0(\mathbb{R}^m, \mathbb{R}^n) \) is a \( G^r_m \times G^r_n \)-space, see [6]. The action of \( (A, B) \in G^r_m \times G^r_n \) on \( X \in S \) is given by the jet composition
\[
X = B \circ X \circ A^{-1}.
\]
Quite analogously to the classical case, the natural transformations \( J^r \to J^r \) are in bijection with the \( G^r_m \times G^r_n \)-equivariant maps \( f: S \to S \).

Write \( A^{-1} = (a^1_1, \ldots, a^r_1, \ldots, a^r_s), B = (b^1_q, \ldots, b^r_q), X = (X^1_1, \ldots, X^r_t) = (X_1, \ldots, X_r) \). First, consider the equivariance of \( f = (f_1, \ldots, f_r) \) with respect to the homotheties \( a^j_i = k^{-1} \delta^j_i \) in \( i(GL(m, \mathbb{R})) \). This gives the homogeneity conditions of type (5). Taking into account the homotheties \( b^r_q = k \delta^r_q \) in \( i(GL(n, \mathbb{R})) \), we further find
\[
\begin{align*}
(kf_1(X_1, \ldots, X_r) &= f_1(kX_1, \ldots, kX_r) \\
(11) \quad kf_r(X_1, \ldots, X_r) &= f_r(kX_1, \ldots, kX_r).
\end{align*}
\]
Applying our lemma to both (5) and (11), we deduce that \( f_s \) is linear in \( X_s \) and independent of the other coordinates, \( s = 1, \ldots, r \). Further, the equivariance with respect to the subgroup \( i(GL(m, \mathbb{R})) \times i(GL(n, \mathbb{R})) \subset G^r_m \times G^r_n \). This yields that \( f_s \) corresponds to a \( GL(m, \mathbb{R}) \times GL(n, \mathbb{R}) \)-equivariant map of \( \mathbb{R}^s \otimes S^r \mathbb{R}^{m^*} \) into itself. By Lemma 3 of [5], we have \( f_s = c_s X_s \) (or, explicitly, \( f^r_{t_1 \ldots t_s} = c_s X^r_{t_1 \ldots t_s} \)) with any \( c_s \in \mathbb{R} \).

For \( r = 1 \) we have deduced \( f^r_1 = c_1 X^r_1 \), which proves Proposition 2. For \( r = 2 \) consider the equivariance with respect to the kernel of the jet projection \( G^2_m \times
× \(G^2_n \rightarrow G^1_m \times G^1_m\). Taking into account the coordinate form of the jet composition, we find that the action of an element \((\delta^1_j, a^1_{j,k}), (\delta^2_q, b^2_{q,p})\) on \((X^p_i, X^q_j)\) is \(X^p_i = X^p_i\) and

\[
X^q_j = X^q_j + b^q_{qp}X^q_j + X^p_ka^k_{ij}.
\]

(12)

Then the equivariancy condition for \(f^p_{ij}\) reads

\[
c_2X^p_{ij} + c_1^2 b^p_{qp}X^q_j + c_1 X^p_k a^k_{ij} = c_2(X^p_{ij} + b^p_{qp}X^q_j + X^p_ka^k_{ij})
\]

(13)

This implies \(c_1 = c_2 = 0\) or \(c_1 = c_2 = 1\). Assume by induction that Proposition 2 holds for the order \(r - 1\). Consider the equivariancy with respect to the kernel of the jet projection \(G^r_m \times G^r_m \rightarrow G^{r-1}_m \times G^{r-1}_m\). The action of an element \((\delta^r_j, 0, \ldots, 0, a^r_{j_1, \ldots, j_r})\), \((\delta^q_p, 0, \ldots, 0, b^q_{q_p, \ldots, q_r})\) leaves \(X_1, \ldots, X_{r-1}\) unchanged and

\[
X^p_i = X^p_i + b^p_{q_i}X^q_i + \ldots + X^p_r a^p_{r_i}.
\]

(14)

Then the equivariancy condition for \(f^p_{i_1, \ldots, i_r}\) requires

\[
c_1X^p_{i_1, \ldots, i_r} + c_1^2 b^p_{q_1, \ldots, q_r}X^q_i + \ldots + X^p_{r}a^p_{i_1, \ldots, i_r} =
\]

\[
= c_1(X^p_{i_1, \ldots, i_r} + b^p_{q_1, \ldots, q_r}X^q_i + \ldots + X^p_{r}a^p_{i_1, \ldots, i_r}).
\]

(15)

This implies \(c_r = c_1 = 0\) or \(1\), QED.

References

[1] J. A. Dieudonné, J. B. Carrel: Invariant Theory. Old and New, Academic Press, New York—London 1971.

[2] J. Janyška: Geometrical properties of prolongation functors. Časopis pěst. mat. 110 (1985), 77—86.

[3] G. Kainz, P. Michor: Natural transformations in differential geometry. Czechoslovak Math. J. 37 (112) (1987), 584—607.

[4] T. Klein: Connections on higher order tangent bundles. Časopis pěst. mat. 106 (1981), 414—421.

[5] I. Kolář: Some natural operations in differential geometry. Proc. Conf. Differential Geometry and its Applications, Brno 1986, D. Reidel, 1987, 91—110.

[6] I. Kolář, G. Vosmanská: Natural operations with second order jets. Rendiconti del Circolo Matematico di Palermo, Serie II. numero 14—1987, 179—186.

[7] R. S. Palais, C. L. Terng: Natural bundles have finite order. Topology, 16 (1977), 271—277.

[8] F. W. Pohl: Differential geometry of higher order. Topology, 1 (1962), 169—211.

[9] G. Vosmanská: Natural transformations of jet spaces. Thesis (Czech), Brno 1987.

Souhrn

PŘIROZENÉ TRANSFORMACE TEČNÝCH VECROU VYŠŠÍHO ŘÁDU
A JETOVÝCH PROSTORŮ

IVAN KOLÁŘ, GABRIELA VOSMANSKÁ

Dokazuje se, že všechny přirozené transformace funkcí tečných vektorů r-ťeho řádu do sebe jsou pouze homotetie. Určují se rovněž všechny přirozené transformace funkcí jetů r-ťeho řádu do sebe.
Резюме

НАТУРАЛЬНЫЕ ПРЕОБРАЗОВАНИЯ РАССЛОЕНИЙ КАСАТЕЛЬНЫХ ВЕКТОРОВ ВЫСШЕГО ПОРЯДКА И ПРОСТРАНСТВ СТРУЙ

Ivan Kolář, Gabriela Vosmanská

Показывается, что гомотетии являются единственными естественными преобразованиями функтора касательных векторов высшего порядка в себя. Определяются также все естественные преобразования функтора струй любого порядка в себя.

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