Towards Automated Proof Strategy Generalisation

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Abstract

The ability to automatically generalise (interactive) proofs and use such generalisations to discharge related conjectures is a very hard problem which remains unsolved; this paper shows how we hope to make a start on solving this problem. We develop a notion of goal types to capture key properties of goals, which enables abstractions over the specific order and number of sub-goals arising when composing tactics. We show that the goal types form a lattice, and utilise this property in the techniques we develop to automatically generalise proof strategies in order to reuse it for proofs of related conjectures. We illustrate our approach with an example.

1 Introduction

When verifying large systems one often ends up applying the same proof strategy many times – albeit with small variations. An expert user/developer of a theorem proving system would often implement common proof patterns as a so-called tactics, and use this to automatically discharge “similar” conjectures. However, other users often need to manually prove each conjecture. Our ultimate goal is to automate the process of generalising a proof (possibly a few proofs) into a sufficiently generic proof strategy capable of proving “similar” conjectures. In this paper we make a small step towards this goal by developing a suitable representation with necessary strong formal properties, and give two generic methods which utilise this representation to generalise a proof.

Whilst the manual repetition of similar proofs have been observed across different formal methods, for example Event-B, B and VDM (see [5]), we will focus on a subset of separation logic [23], used to reason about pointer-based programs. In the subset, there are two binary operations $\ast$ and $\land$ and a predicate pure, with the following axioms:

\[(A \ast B) \ast C \iff A \ast (B \ast C) \quad (ax1)\]
\[\text{pure}(B) \rightarrow (A \land B) \ast C \iff (A \ast C) \land B \quad (ax2)\]

These axioms pertain specifically to separation logic, and allow pure/functional content to be expressed apart from shape content, used to describe resources. Now, consider the conjecture:

\[p : \text{pure}(e), h : c \ast ((f \ast (d \land b) \land e) \land e) \ast a \vdash ((c \ast f) \ast (d \land e)) \ast ((b \land e) \ast a) \quad (1)\]

which demonstrate the typical form of a goal resulting from proving properties about heap structures, which involve some resource content and some functional content. For example, one could view the $a,b,c,d,f$ as propositions about space on a heap, with $e$ containing some functional information – for example about order.

Figure 1 illustrates a proof of this conjecture in the Isabelle theorem prover. Next, consider the following “similar” conjecture:

\[p' : \text{pure}(d), h' : a \ast ((b \ast c) \land d) \ast e \vdash ((a \ast ((b \ast d) \ast c)) \ast e) \quad (2)\]

\[1\]However, note that we believe that our approach is still generic across different formal methods.
lemma assumes \( p: \text{pure}(e) \)
and \( h: \ (f * (d * b)) \land e) \land e) * a \)
shows \( g_1: \ ((c * f) * (d \land e)) * ((b \land e) * a) \)

apply (subst ax1) \( g_2: \ ((c * (f \land e) * (d \land e) * b) * a) \)
apply (subst ax1) \( g_3: \ (c * ((f \land e) * (d \land e)) * b) * a) \)
apply (subst ax1) \( g_4: \ (c * (f \land e)) * (d \land e) * b) * a) \)
apply (subst ax1) \( g_5: \ c * ((f \land e) * (d \land e) * b) * a) \)
apply (subst ax2) \( g_6: \text{pure}(e), g_7: \ (f * (d \land e) * b) * a) \)
apply (rule p) \( g_7: \ c * ((f * (d \land e) * b) \land e) \land e) * a) \)
apply (subst ax2) \( g_8: \text{pure}(e), g_9: \ (f * (d \land b) \land e) * a) \)
apply (rule p) \( g_9: \ (f * (d \land b) \land e) * a) \)
apply (rule h) \[ \]
done

Figure 1: Isabelle proof of (1). The shaded area illustrate the proof state, and the r.h.s shows the proof tree.

which again demonstrate the form of a typical proof. This conjecture can be proven by the following sequence of tactic applications:

apply (subst ax1); apply (subst ax1); apply (rule p'); apply (rule h')

Our goal is to be able to apply some form of analogous reasoning to use the proof shown in Figure 1 to automatically discharge (2). However, a naive reuse of this proof will not work since:

- There are different number of tactic applications in the two proofs.
- Naive generalisations such as “apply subst ax1 until it fails” will fail, since subst ax1 is still applicable for (2) after the first application. Continued application will cause the rest of the proof to fail.
- The “analogous” assumptions have different names, e.g. \( p \) and \( p' \), thus rule p will not work for (2).

Even if the proofs are not identical, they are still captured by the same proof strategy. In fact, the proofs can be described as simple version of the the mutation proof strategy developed to reason about functional properties in separation logic [19]. Here, we assume the existence of a hypothesis \( H \) (i.e. \( h \) or \( h' \)), with some desirable properties we will return to. The strategy can then be described as:

- continue applying \( ax1 \) while there are no symbols at the same position in the goal and \( H \);
- then apply \( ax2 \) while the goal does not match \( H \) and there is a fact \( P \) which can discharge the condition of \( ax2 \)
- finally, discharge the goal with \( H \).

Figure 2: Mutation
The rest of the paper will focus on how we can automatically discover such strategy from the proof shown in Figure 1. To achieve this a suitable proof strategy representation is required. Firstly, as we can see from the strategy, the representation needs to include properties about the sub-goals/proof states as well as information about tactics. Moreover, sub-goals arising from a tactic application are often treated differently (e.g. the condition arising from the use of ax2), thus some “flow information” is required. From this we argue that

A graph where the nodes contains the goals, and edges annotated with goal information working as channels for the goals, is a suitable representation to support the automatic generalisation of proof strategies from proofs.

Previously, a graph-based language to express proof strategy has been developed [10], and we will briefly summarise this in the next section. However, the annotation of goals on edges has not been developed, and developing this is a key challenge in achieving our ambitious goal. A key contribution of this paper is the development of such a goal type which serves as a specification of a particular goal and can be generalised across proofs. The example has shown that there is vast number of information required which the goal type need to capture:

- The conclusion to be proven, e.g. \(((c \ast f) \ast (d \land e)) \ast h) \ast a\) initially.
- The facts available, including local assumptions (such as \(p\) and \(q\)), and axioms/lemmas (e.g. \(ax1\) and \(ax2\)).
- Properties between facts and the conclusion (or other facts). For example, \(ax2\) is applied because the condition of it can be discharged by \(p\).
- Properties relating goals to tactics; e.g. after applying \(ax2\) one subgoal is discharged by \(p\) but not the other.

Moreover, other information could also be essential for a particular proof strategy, for example: definitions; fixed/shared variables; and variance between steps (e.g. for each step a “distance” between \(h\) and the goal is reduced).

We argue that a language need to be able to capture such properties, and we are not familiar with any proof language which can capture them in a natural way. The development of a goal type to capture this is a key contributions and the topic of §3. We then briefly show how this can be utilised when evaluating a conjecture over the strategy in §4. The second key contribution of this paper is the topic of §5. Here, we utilise the graph language and goal types to generalise a proof into a proof strategy, illustrated by re-discovering the mutation strategy. A key feature here is that we see the goal types as a lattice which can naturally be generalised. This is combined with graph transformations to find common generalisable sub-strategies and loops with termination conditions. We discuss related work and conclude in §6 and §7.

2 Background on the Proof Strategy Language

The graphical proof strategy language was introduced in [10] built upon the mathematical formalism of string diagrams [6]. A string diagram consists of boxes and wires, where the wires are used to connect the boxes. Both boxes and wires can contain data, and data on the edges provides a type-safe mechanism of composing two graphs. Crucially, string diagrams allow dangling edges. If such edge has no source, then this becomes and input for the graph, and dually, if an edge as no destination then it is the output of the graph.
In a proof strategy graph [10], the wires are labelled with goal types, which is developed in the next section. A box is either a tactic or a list of goals. Such goal boxes are used for evaluation by propagating them towards the outputs as shown in Figure 3.

Figure 3: Evaluation of goals [10]. Goal nodes, represented as circles, are evaluated by propagating them over tactics from inputs to the outputs. An edge works as a channel, and can contain many goals, however, a tactic works on one goal at a time.

There are two types of tactics. The first type is known as a graph tactic, which is simple a node holding one or more graphs which be unfolded. This is used to introduce hierarchies to enhance readability. A second usage, in the case it holds more then one child graphs, is to represent branching in the search space, as there multiple ways of unfolding such graph. Note however that, as explained in [10], graph tactics are evaluated in-place and are thus not unfolded first.

The other type of tactic is an atomic tactic. This corresponds to a tactic of the underlying theorem prover. Here, we here assume works on a proof state (containing named hypothesis, the open conjecture, fixed variables etc). When evaluated, such tactic turn a proof state (goal) into a list of new proof states (sub-goals). Since this may also involve search it is returns a set of such list of proof state, thus it has the type

\[ \text{proof state} \rightarrow \{\text{proof state}\} \]

Here, for a type \( \tau \), \([\tau]\) is the type of finite lists of \( \tau \) and \( \{\tau\} \) is the type of finite sets whose elements are of type \( \tau \).

For this paper, we assume two atomic tactics: subst \( \langle \text{arg} \rangle \) and rule \( \langle \text{arg} \rangle \), which performs a single substitution or resolution step, respectively. Here, \( \langle \text{arg} \rangle \) may be both a single rule or a set of rules (all of them are then attempted). It can also be a description of a set of rules, which we call a class and is introduced in the next section.

In order to apply an atomic tactic in the strategy language, it has to be typed with goal types, also introduced next. Let \( \alpha \) and the \( \beta_i \) represent goal type variables. A typed tactic is then a function of the form:

\[ \alpha \rightarrow \{[\beta_1] \times [\beta_2] \times \ldots \times [\beta_n]\} \]

This type has to be reflected in our representation of goal nodes, which we will return in §4 after we have developed our notion of goal types, which is the next topic.

### 3 Towards a Theory of Goal Types

#### 3.1 Classes

A goal type must be able to capture the intuition of the user, potentially using all the information listed in §1. This information is then used to guide the proof and send sub-goals to the correct tactic. To achieve this we firstly need to capture important properties of the conclusion of the conjecture. Next, it is important to note that, in general, most of the information available is not relevant, inclusion of it will
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act as noise (and increase the chance of “over-fitting” a strategy to a particular proof). Thus, we need to be able to separate the wheat from the chaff, and capture properties of the ‘relevant’ facts, where facts refer to both lemmas/axioms, and assumptions which are local to the conclusion. Henceforth we will term a fact or conclusion an element. There are a large set of such element properties, e.g.:

- a particular shape or sub-shape;
- the symbols used, or symbols at particular positions (e.g. top symbol);
- certain types of operators are available, e.g. \( [\text{I}] \) contains associative-commutative operators;
- the element contains variables we can apply induction to or (shared) meta-variables;
- certain rules are applicable;
- the element’s origin, e.g. it is from group theory or it is a property of certain operator.

This list is by no means complete, and here we will focus on two such properties:

- top symbol describes the top level symbol;
- has symbol describes the symbols it must contain.

Each such feature will have data associated:

\[
data := \text{int} \mid \text{term} \mid \text{position} \mid \text{boolean}
\]

where term refers to the term of the underlying logic, and a position refers to an index of a term tree. A class describes a family of elements where certain such features hold. A class, for example, could be a conclusion or a hypothesis, for which certain properties hold.

**Definition 1.** A class is a map

\[
class := \text{name} \xrightarrow{m} [[\text{data}]]
\]

such that for each name in the domain of a class, there is an associated predicate on an element, termed the matcher. There are two special cases where the predicates always succeeds or always fails on certain data, denoted by \( \top_f \) and \( \bot_f \) as described below. A class matches to a conclusion/fact if the predicate on each element holds.

The intuition behind the list of lists of data is that it represent a property in DNF form, e.g. \( [[a, b], [c]] \), which is equivalent to \((a \wedge b) \lor c\). For the conjecture in \( [1] \), \( \{(\text{top symbol} \mapsto [[\ast]])\},\{(\text{has symbol} \mapsto [[\text{pure}]]\}\} \) identifies the conclusion, while \( \{(\text{has symbol} \mapsto [[\text{pure}],[\ast]]\}\} \) identifies the first assumption, but not the second, and \( \{(\text{has symbol} \mapsto [[\text{pure}],[\ast]]\}\} \) captures both assumptions and the goal. We call this a semantic representation of the data.

We write the constant space of feature names as \( \mathcal{N} \) and, for a class \( C \), with \( n \in \mathcal{N} \), \( C(n) \) is the data associated with feature \( n \) for class \( C \). We define the semantic representation of the data for a particular feature in a class using the notation \( x^n \) for some data \( x \). By semantic representation, we mean that the structure of the list of data is mapped to a representation about which we can reason – for example, above where a list of lists of data represents a formula in DNF. It is then possible to reason about this data. For example, for the feature has symbol in the conjecture \( [1] \) we write for \( C(\text{has symbol})^4 \):

\[
[[a_1 \cdots a_m], \cdots, [b_1 \cdots b_n]]^4 = ([[a_1] \cap \cdots \cap [a_m]] \cup \cdots \cup ([b_1] \cap \cdots \cap [b_n]))
\]

where \( [a] \) denotes \( a \) as an atom.
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Classes form a bounded lattice \((C, \lor, \land, \top, \bot)\), on which we can define a meet and a join. We show how to compute the join \(\land\: \text{least upper bound}\), and meet \(\lor: \text{greatest lower bound}\) for two classes \(C_1\) and \(C_2\). We define the most general class as \(\top\) and the empty class as \(\bot\). We write the most general element of \(C(f)\) as \(\top_f\) and the least general to be \(\bot_f\).

**Definition 2.** \(C_1 \land C_2\) is the greatest lower bound of \(C_1\) and \(C_2\) if \(\forall n \in \mathcal{N}.(C_1 \land C_2)(n) = C_1(n) \land_n C_2(n)\), where \(\land_n\) computes the greatest lower bound for feature \(n\).

**Definition 3.** \(C_1 \lor C_2\) is the least upper bound of \(C_1\) and \(C_2\) if \(\forall n \in \mathcal{N}.(C_1 \lor C_2)(n) = C_1(n) \lor_n C_2(n)\), where \(\lor_n\) computes the least upper bound for feature \(n\).

For \(f = \text{top}\) or \(f = \text{has}\) we define \(\land_f\) and \(\lor_f\) as:

**Definition 4.** \(C_1(f) \land_f C_2(f) := C_1(f)^s \land C_2(f)^s\) and \(C_1(f) \lor_f C_2(f) := C_1(f)^s \lor C_2(f)^s\)

We further define \(\top_f\) and \(\bot_f\) to be \(\cup\) (the universal set) and \(\emptyset\) respectively. To show that classes form a partial order, we prove the following properties about meet and join:

**Theorem 1.** \(\land\) and \(\lor\) are commutative and associative operations.

**Proof.** It suffices to prove that \(\land_f\) and \(\lor_f\) commutative and associative for each \(f \in \mathcal{N}\). In our example we use Definition 4. This is provable since \(\cap\) and \(\cup\) are commutative, associative and idempotent operations in set theory.

**Theorem 2.** \(\land\) and \(\lor\) follow the absorption laws \(a \lor (a \land b) = a\), and \(a \land (a \lor b) = a\).

**Proof.** It suffices to prove that \(\land_f\) follow the absorption laws. This follows from the fact that \(\cap\) and \(\cup\) are set theoretic operations. It also follows that \(\land\) and \(\lor\) are idempotent; \(a \land a = a\), \(a \lor a = a\).

Since \(\bot\) is \(\emptyset\) and \(\top\) is \(\cup\), it is trivial to show that \(C \lor \bot = C\) and \(C \land \top = C\) for a class \(C\). Thus, a class form a bounded lattice.

Orthogonality is a key property to reduce non-determinism during evaluation of a strategy, whilst subtyping of classes is a key feature for our generalisation techniques discussed in [5].

**Definition 5.** \(C_1\) and \(C_2\) are orthogonal if \(\exists f \in \mathcal{N}. C_1(f) \land C_2(f) = \bot_f\). We write this as \(C_1 \bot C_2\). \(C_1\) is a subtype of \(C_2\), written \(C_1 \ll C_2\), if \(\forall f \in \mathcal{N}. (C_1(f) \land C_2(f)) = C_1(f)\).

As an example, consider a goal class with features \(\text{has}\) and \(\text{top}\):

\[
C_1 : \{ (\text{top} \mapsto [*]), (\text{has} \mapsto [*, \land, \lor, *]) \} \\
C_2 : \{ (\text{top} \mapsto [\land]), (\text{has} \mapsto [*, \land, \lor, *]) \} \\
C_3 : \{ (\text{top} \mapsto [*]), (\text{has} \mapsto [*, \land, \lor, *]) \}
\]

(4)

(5)

\(C_2 \land C_3\) as there is a feature \(\text{top}\) for which \(C_2(f) \land C_3(f)\), since by the semantics \([\land] \cap [*] = \emptyset\). In order to determine whether \(C_1\) is a subtype of \(C_3\) we must show that \((C_1(f) \land C_3(f)) = C_3(f)\) for all features. Using definition 4 we must prove for \(\text{has}\):

\[
(([*] \cap [\land]) \cup ([\lor] \cap [*])) \cap ([*] \cap [\land] \cap [\lor]) = ([*] \cap [\land] \cap [\lor])
\]

which is true and the same for \(\text{top}\) which in this case follows trivially.

6
3.2 Links

A class identifies a cluster of elements with certain common properties. However, certain types of properties are between elements – e.g. a conditional fact can only be applied if the condition can be discharged. Moreover, certain properties rely on information pertaining to previous nodes in the proof tree, e.g. a measure has to be reduced in a rewriting step to ensure termination. Such properties include:

- common symbols between two elements, or the position they are at;
- common shapes between two elements;
- embedding of one element into another;
- some form of difference between elements
- some sort of measure reduces/increases between elements

We call such properties links. Moreover, we abstract links to make them relations between classes rather than between elements. Links are given an existential meaning: a link between two classes entails that there exists elements in them such that a property holds. In addition, we introduce a parent function on links to refer to the parent node. The meaning of this will become clearer in the next section, where we discuss evaluation.

Definition 6. A link is a map

\[ \text{link} := \text{name} \times \text{class} \times \text{class} \rightarrow \text{[[data]]} \]

such that for each name \( n \) in the domain of a link, there is an associated predicate \( n : \text{[[data]]} \times \text{element} \times \text{element} \rightarrow \mathbb{B} \) called a matcher. A link matches to a conclusion/fact if the predicate on each element holds.

We write the constant space of link names as \( \mathcal{N}_L \) and for a link \( L \), with \( n \in \mathcal{N}_L \), \( L(n, C_1, C_2) \) is the data associated with feature \( n \), classes \( C_1 \) and \( C_2 \), for link \( L \).

We will only consider the link features is\_match and symb\_at\_pos for this exposition. The data of the former are booleans in DNF, and its matcher succeeds if the result of an exact match between the elements is the same as the semantic value of the data. The data of the latter is lists of position, where for example

\[ \{ (\text{symb\_at\_pos}, C_1, C_2) \mapsto \text{[[pos]]} \} \]

states that there exists elements of classes \( C_1 \) and \( C_2 \) where the symbol at position \( pos \) is the same. To state that there is no position where this is the case, we introduce an element \( \perp_f \) for each \( f \in \mathcal{N}_L \), as we did with classes. In general, there will be more complicated links, with more complicated output data values. Defining these is ongoing work.

In order to define orthogonality and subtyping we define the meet and join for each name in \( \mathcal{N}_L \).

Definition 7. \( L_1 \land L_2 \) is the greatest lower bound of \( L_1 \) and \( L_2 \) if \( \forall n \in \mathcal{N}_L. (L_1 \land L_2)(n) = L_1(n) \land_n L_2(n) \), where \( \land_n \) computes the greatest lower bound for link feature \( n \).

Definition 8. \( L_1 \lor L_2 \) is the least upper bound of \( L_1 \) and \( L_2 \) if \( \forall n \in \mathcal{N}_L. (L_1 \lor L_2)(n) = L_1(n) \lor_n L_2(n) \), where \( \lor_n \) computes the least upper bound for link feature \( n \).
As with classes, we introduce a semantic representation for the links using notation \( x^s \) for some data \( x \). Since the data is a list of lists of positions, we use the same semantics as in (3). The intuition is that we should be able to generalise the link class to account for the same symbol to exist at multiple positions within the hypothesis and conclusions. The proofs and definitions of the lattice theory follow similarly to those for classes.

We then define orthogonality and subtyping for links:

**Definition 9.** \( L_1 \) and \( L_2 \) are orthogonal if \( \exists f \in \mathcal{L} : L_1(f) \land L_2(f) = \bot \). We write this as \( L_1 \perp L_2 \). \( L_1 \) is a subtype of \( L_2 \), written \( L_1 < : L_2 \), if \( \forall f \in \mathcal{L} : L_1(f) \land L_2(f) = L_1(f) \).

### 3.3 Goal Types

A goal type is a description of the conclusion, the related facts, and the links between them:

**Definition 10.** A goal type is a record:

\[
\text{GoalType} := \{ \text{link} : \text{link}, \text{facts} : \{ \text{class} \}, \text{concl} : \text{class} \}
\]

where \( \text{concl} \) is the class describing the conclusion of a goal, \( \text{facts} \) is a set of classes of relevant facts, and \( \text{link} \) is a link relating classes of facts and concl.

Note that we keep a set of classes of facts to account for specifying the existence of multiple classes of hypotheses. For example, in the our example conjecture, hypothesis \( p \) forms a class \( P \) (with top symbol pure), while \( h \) forms a class \( H \) (with has symbols \( [\land, *] \)). Henceforth we assume that all members of facts are orthogonal – dealing with the general case which allows overlapping is future work. Orthogonality and subtyping of two goal types reduces to orthogonality of their respective classes. Due to the assumptions of orthogonality between the facts, they have an universal interpretation for \( \perp \) and an existential interpretation for \(<\>:

\[
G_1 \perp G_2 := G_1(\text{concl}) \perp G_2(\text{concl}) \lor G_1(\text{link}) \perp G_2(\text{link}) \lor \\\\forall f_1 \in G_1(\text{fact}), f_2 \in G_2(\text{fact}), f_1 \perp f_2
\]

\[
G_1 < : G_2 := G_1(\text{concl}) < : G_2(\text{concl}) \land G_1(\text{link}) < : G_2(\text{link}) \land \\\\exists f_1 \in G_1(\text{fact}), f_2 \in G_2(\text{fact}), f_1 < : f_2
\]

### 4 Lifting of Goals and Tactics

Here, we will briefly outline how evaluation is achieved with the goal type introduced. Firstly, recall from Figure 3 that a single evaluation step is achieved by a tactic by consuming the input goal node on the input and produce the resulting sub-goals on the correct output edges. Since a goal nodes contains list of goals, this can be captured by meta graphical rewrite-rule shown in Figure 4. The details are given in [10], but one evaluation step works as follows:

1. Match and partly instantiate the LHS of the meta-rule.
2. Evaluate the tactic function for the matched input and output types.
3. Finish instantiating the RHS with the lists \( g_i \) from the tactic.
4. Apply the fully instantiated rule(s).
where $\alpha$ and $\beta_i$ are goal type variables. We assume $t$ is an atomic tactic, but this is trivial to extend to graph tactics. Further note that there are additional rules to split a list into a sequence of singleton lists and delete empty list nodes. For more details we refer to [10].

In the second step of this algorithm, the underlying tactic has to be lifted from proof state $\rightarrow \{\text{proof state}\}$ to the form $\alpha \rightarrow \{[\beta_1] \times [\beta_2] \times \ldots \times [\beta_n]\}$.

First we need to introduce a goal. This can be seen as an instance of a goal type for a particular proof state:

**Definition 11.** A goal is a record:

$$
\text{goal} := \{ fmap : \text{class} \to \{\text{fact}\}, \text{ps} : \text{proof state}, \text{parent} : \{\text{goal}\} \}
$$

where parent is either a singleton or empty set – empty if this is the first goal. Type checking relies on the “typing predicates” associated with classes and links. A goal $g$ is of type $G$, iff

- The conclusion in $g(ps)$ matches $G(\text{concl})$.
- For each class $c \in G(\text{facts})$, $g(fmap)(c)$ is defined, not empty, and each $f \in g(fmap)(c)$ matches $c$.
- For each $(l, c_1, c_2) \mapsto d \in G(\text{links})$ there exists elements $e_1 \in g(fmap)(c_1)$ and $e_1 \in g(fmap)(c_1)$ such that the $l(d, e_1, e_2)$ holds. Moreover, for each $e_1 \in g(fmap)(c_1)$ there must be an $e_2 \in g(fmap)(c_2)$, such that $l(d, e_1, e_2)$ (and dually the other way around).

Now, to lift a tactic we need to: unbifgoal $g$ to project the underlying proof state; apply the tactic; and lift the resulting proof states to goals of a type in $\{\beta_1, \ldots, \beta_n\}$ (which becomes instantiated to specific goal types when matching the RHS in the first step). Then, for a list $L$ of proof states, let $lp(\beta_1, \ldots, \beta_n; L)$ be the set of all partitions of $L$ lifted into $n$ lists of goals $\{\text{map lift } L_1), \ldots, \text{map lift } L_n\}$, such that all of the goals in the $i$-th list have goal type $\beta_i$. Then, we define lifting as:

$$
\text{lift}(tac) = \lambda g. \begin{cases} 
lp(\beta_1, \ldots, \beta_n; tac(unlift(g))) & \text{if } g \text{ is of type } \alpha \\
\emptyset & \text{otherwise}
\end{cases}
$$

We are then left to define unlifting and lifting for a single goal node and a single goal type. Firstly, a naive unlifting of a goal simply projects the goal state. More elaborate unliftings are tactic dependent, and may e.g. add all facts from a particular fact class as active assumptions beforehand.

Lifting is a partial function, and an element of $lp$ is only defined if lifting of all elements succeeds. There are several (type-safe) ways to implement lifting. Here, we show a procedure which assumes that all relevant information is passed down the graph from the original goal node. Any fact “added” to a goal node is thus a fact generated by the tactic. However, one may “activate” existing facts explicitly in the tactic which will then be used by lifting. A new goal $g'$ is then lifted as follows, using the (new) proof state $ps'$, previous goal $g$, and goal type $G$ as follows:

1. Set fields $g'(\text{parent})$ to $g$, and $g'(\text{ps})$ to $ps'$, fail if the conclusion does not match $G(\text{concl})$.
2. For each $c \in G(\text{facts})$, set $g'(\text{facts})(c)$ to be all facts in the range of $g(\text{facts})$ and newly generated facts which matches $c$. If for any $c \in G(\text{facts})$, $g'(\text{facts})(c)$ is empty (or undefined) then fail.
3. Check all link features. For each $c \in G(\text{facts})$ which is used by a link feature, filter out any element $e \in g'(\text{facts})(c)$ not “captured” by a link related link match. Fail if there does not exist an element in the related classes which holds for any of the links or any $g'(\text{facts})(c)$ (for $c \in G(\text{facts})$) is empty after this filtering step.
5 Generalising Strategies

A proof is generalised into a strategy by first lifting the proof tree into a proof strategy graph, and then apply graph transformation techniques which utilises the goal type lattice to generalise goal types. Simple generalisation of tactics are also used. One important property when performing such generalisations, is that any valid proofs on a strategy should also be valid after, which we will provide informal justification for below. However, note that we do not deal with termination.

5.1 Deriving Goal Types from Proof States

In this section we will discuss how to generalise the proof shown in Figure 1 into the mutation strategy shown in Figure 2 utilising the lattice structure of goal types.

However, first we need to turn the proof tree of Figure 1 into a low-level proof strategy graph of the same shape. We utilise techniques described in [25] to get the initial proof tree. Now, since the shape is the same this reduces to (1) generalising proof states into goal types and (2) generalising the tactics.

(1) To generalise the proof state into goal type we have taken an approach which can be seen as a “locally maximum” derivation of goal type, where each assumption becomes a separate class, and make each class as specific as possible. Any link features that holds are also included. Consequently, the goal type will be as far down the lattice as possible whilst still being able to lift the goal state it is derived from. To illustrate, we will show how the proof state (1) is lifted to goal type $GT_1$. Let

$$H = \{ \text{has\_symbol} \mapsto [[*, \wedge]], \{ \text{top\_symbol} \mapsto [[*]] \} \}$$

$$P = \{ \text{has\_symbol} \mapsto [[\text{pure}]], \text{top\_symbol} \mapsto [[\text{pure}]] \}$$

$$G = \{ \text{has\_symbol} \mapsto [[*, \wedge]], \text{top\_symbol} \mapsto [[*]] \}$$

$$L = \{ (\text{symb\_at\_pos}, G, H) \mapsto [[\bot]], (\text{symb\_at\_pos}, G, P) \mapsto [[\bot]], (\text{symb\_at\_pos}, H, P) \mapsto [[\bot]] \}.$$
Then GTI becomes \( \{ \text{link} : L, \text{facts} : \{ H, P \}, \text{concl} : G \} \). Note that the last two link features are useless, and are therefore ignored henceforth. However, this shows that in the presence of larger goal states and/or more properties heuristics will be required to reduce the size of the goal types, and filter out such “useless information”. This is future work.

(2) Tactics are kept with the difference that if a local assumption is used (e.g. \( h \) or \( p \)) their respective class is used instead.

The resulting tree is shown left-most of Figure 5. For space reasons we have not included the goal types, but provided a name when referred to in the text. This is slightly more general than the original proof as it allows a very slight variation of the goals. However, it still e.g. relies on the exact number of application of each tactic.

### 5.2 Generalising Tactics

Next, we need to generalise tactics. A simple example of this is when sets of rules are used as arguments for the subst and rule tactics. Here, subst \( R_1 \) and subst \( R_2 \) can be generalised into subst \((R_1 \cup R_2)\). Another example turns a tactic into a graph tactic which nest both these tactics (and can be unfolded to either). A proviso for both is that their input and output goal types can be generalised. Both these generalisations only increases the search space and are thus proper generalisations.

Graph tactics can also be generalised by generalising the graph they nest into one. We return to this with an example below. We will use the notation \( gen(t_1, t_2) \) for the generalisation of the two given tactics.

### 5.3 Generalising Goal Types

In the context of goal types: generalisation refers to computing the most general goal type for two existing goal types; while weakening applies to only one goal type and makes the description of it more general. Crucial to both generalisation and weakening is that multiple possible generalised and weakened goal types exist.

We use the notion of a least upper bound for a goal type lattice, described in \[\ref{sec:generalisation}\] using the join operator \( \lor \), to define generalisation for goal types. For a class \( C \), we write:

**Definition 12.** \( C \) is a generalisation of \( C_1 \) and \( C_2 \), also written \( C = gen(C_1, C_2) \), if \( \forall f \in N. C(f) = C_1(f) \lor C_2(f) \).

As an example, consider the two classes shown in \[\ref{fig:example1}\] and \[\ref{fig:example2}\]. We can compute \( G = gen(C_1, C_2) \) by appealing to the set theoretic semantics and tranferring back to the class representation. For \( f_1 = \text{top}\_\text{symbol} \) and \( f_2 = \text{has}\_\text{symbol} \) we compute

\[
\begin{align*}
C(f_1) &= ([\land] \cup \{ [\ast] \}) \\
C(f_2) &= ((\{ [\ast] \} \cap \{ \land \}) \cup ([\lor] \cap \{ [\ast] \})) \\
&= ([\land] \cup ([\lor] \cap \{ [\ast] \}))
\end{align*}
\]

producing a generalised class:

\[
C = \{ (\text{top}\_\text{symbol} \mapsto \{ [\lor], [\ast] \}), (\text{has}\_\text{symbol} \mapsto \{ [\ast], [\land], [\land], [\lor] \}) \}
\]

The definition of generalisation for links extends similarly from its associated lattice theory described in \[\ref{sec:generalisation}\]. Recall that we assume orthogonality of fact classes. We define a function \( \text{gen}\_\text{map} \) over two sets of (fact) classes, which generalises pairwise each fact class. Here, for any two fact classes \( H_1 \) and \( H_2 \) in the generalised set of fact classes, where \( H_1 <: H_2 \) and \( H_1 \perp H_2 \) we only retain \( H_2 \), thus ensuring orthogonality. We can then define a function \( \text{gen} \) on goal types to be
5.4 (Re-)Discovering the Mutation Strategy

Armoured with the techniques for generalising the edges and nodes of a proof strategy, we now develop two techniques which allows us to generate our proof into the required mutation strategy.

Firstly, we need to abstract over the number of repeated sequential applications of the same tactic – i.e. we need to discover loops. When working in a standard LCF tactic language [9], the problem is to know: (a) on which goals (in the case of side conditions) the tactic should be repeated, and (b) when to stop. This was highlighted in [7], where a regular expression language, closely aligned with common LCF tacticals, was used to learn proof tactics, and hand-crafted heuristics were defined to state when to stop a loop (which by the way would fail for our example).

The advantage of our approach, is that we can utilise the goal types to identify termination conditions – reducing termination and goal focus to the same case, thus also handling the more general proof-by-cases paradigm. We illustrate our approach with what can be seen as an inductive representation of tactic looping, as shown by rules loop1 and loop2 of Figure 6. For loop 1, we can see that it is correct since $B \perp C$ ensures that a goal will exit the loop when it matches $C$. Moreover, the $B <: A$ pre-condition ensures that the tactic can handle the input type. For loop2, similar arguments holds for the generalised $\text{gen}(B, B')$ edge.

Consider the left most graph of Figure 5 which is the proof tree lifted to a graph. Here, the stippled box highlights the sub-graph which matches with the rules shown above. loop1 is applied first, followed by two applications of loop2. The classes are identical so we only discuss link classes, which have the following values:

\[
\begin{align*}
\text{GT}_1(\text{link}) &= \text{GT}_2(\text{link}) = \{(\text{symb\_at\_pos}, G, H) \mapsto [[]]\} \\
\text{GT}_3(\text{link}) &= \{(\text{symb\_at\_pos}, G, H) \mapsto [[1]]\}
\end{align*}
\]

where $\text{GT}_2$ denote the goal types in the intermediate stages of the repeated application of tactic subst ax1. Now, for the sequence to be detected as a loop, we must first discover

\[
\text{GT}_2' = \text{gen}(\text{gen}(\text{GT}_2, \text{GT}_2), \text{gen}(\text{GT}_3)) = \text{GT}_1
\]

and show $\text{GT}_2' <: \text{GT}_0$ and $\text{GT}_2' \perp \text{GT}_3$. These are both true since $\text{GT}_2'$ and $\text{GT}_1$ are equal, and $\text{GT}_2'$ and $\text{GT}_3$ are orthogonal due to the existence of $\perp$ in the data argument denoting an empty feature.

The next step (s2) of Figure 5 layers the highlighted sub-graphs into the graph tactics pax2a and pax2b. Such layering can be done for a (connected) sub-graph if the inputs and outputs of the sub-graphs are respectively orthogonal.
Next, we again apply rule loop1 to the pax2a and pax2b sequence. However, this requires us to generalise these two graph tactics, i.e. combining the two graphs they contain into one. Now, as shown in [6], in the category of string graphs, two graphs are composed by a push-out over a common boundary. We can combine two graph tactics in the same way by a push-out over the largest common sub-graph. This is shown on the right-most diagram of Figure 6, which becomes the last step (s3) of Figure 5.

This graph is in fact the mutation strategy of Figure 2, with the addition that we have given semantics to the edges. Now, the first feedback loop is identified by \{(symb.at.pos, concl, H) \mapsto [\perp]\}\, while the second feedback loop is identified by \{(is.match, concl, H) \mapsto [[false]]\}.

6 Related Work

We extend [10], which introduces the underlying strategy language, by developing a theory for goal types which we show form a lattice, and using this property to develop techniques for generalising strategies.

Our goal types can be seen as a lightweight implementation of pre/post-condition used in proof planning [4] – with the additional property that the language captures the flow of goals. It can be seen as further extending the marriage of procedural and declarative approaches to proof strategies [2,13,8], and addressing issues related to goal flow and goal focus highlighted in [1] – for a more detailed comparison we refer to [10].

The lattice based techniques developed for goal type generalisation is similar to antiunification [22] which generalises two terms into one (with substitutions back to the original terms). Whilst each feature is primitive, the goal type has several dimensions. More expressive class/link features, which is future work, may require higher-order anti-unification [18] – and such ideas may also be applicable to graph generalisations. Other work that may become relevant for our techniques are graph abstraction transformations used in algorithmic heap-based program verification techniques, such as [3], and for parallelisation of functional programs [12].

As already discussed, the problem when ignoring goal information, is that one cannot describe e.g. where to send a goal or when to terminate a loop, in a way sufficiently abstract to capture a large class of proofs. Instead, often crude, heuristics have to be used in the underlying tactic language. This is the case for [7], which uses a regular expression language (close to LCF tactics), originally developed in [14] to learn proof plans. [14] further claims that explanation based generalisation (EBG) [21] is applied to derive pre/post-conditions, but no details of this are provided. An EBG approach is also applied to generalise Isabelle proof terms into more generic theorems in [15]. This could provide an alternative starting point for us, however, one may argue that much of the user intent will be lost by working in the low-level proof term representation. Further, note that our work focuses on proof of conjectures which requires structure, meaning machine learning techniques – such as [24], which learns heuristics to select relevant axioms/rules for automated provers – are not sufficient. However, in [11], an approach to combine essentially our techniques, with more probabilistic techniques to cluster interactive proofs [17], was outlined.

We would also like to utilise work on proof and proof script refactoring [26]. This could be achieved either as a pre-processing step, or by porting these techniques to our graph based language. Finally, albeit for source code, [20] argues for the use of graphs to perform refactorings, which further justifies our graph based representation of proof strategies for the work presented here.

7 Conclusion and Future Work

In this paper we have reported on our initial results in creating a technique to generalise proof into high-level proof strategies which can be used to automatically discharged similar conjectures. This paper
has two contributions: (1) the introduction of goal type to describe properties of goals using a lattice structure to enable generalisations; (2) two generic techniques, based upon loop discovery to generalise a proof strategy. The techniques was motivated and illustrated by an example from separation logic. We are in the process of implementation in Isabelle combined with the Quantomatic graph rewriting engine [16]. Next plan to implement these methods in order to test them on more examples, using a larger set of properties to represent the goal types. In particular, we are interested in less syntactic properties, such as the origin of a goal, or if it is in a decidable sub-logic.

We also showed how the lattice structure corresponds to sub-typing, and we plan to incorporate sub-typing in the underlying theory of the language in order to utilise it when composing graphs. Further, we plan to develop more techniques for generalising graphs, which may include develop an underlying theory of graph generalisation, which will be less restrictive than rewriting.

Finally, we have already touched upon the need for heuristic guidance in this work, as there will be many ways of generalising. We are also planning to apply the techniques to extract strategies from a corpus of proofs. Here we believe we have a much better chance of finding and generalising common sub-strategies, and may also incorporate probabilistic techniques as a pre-filter [11]. Such work may help to indicate which class/link features are more common, and can be used to improve the generalisation heuristics discussed above. Further, we would like to remove the restriction that facts have to be orthogonal, and improve the sub-typing to handle this case.

We only briefly discussed the process of turning proofs into initial low-level proof strategy graphs. With partners on the AI4FM project (www.ai4fm.org) we are working on utilising their work on capturing the full proof process, where the user may (interactively) highlight the key features of a proof (step) [25]. This can further help the generalisation heuristics.

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