Convex polygons and separation of convex sets

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Abstract

We prove that for any collection $F$ of $n \geq 2$ pairwise disjoint compact convex sets in the plane there is a pair of sets $A$ and $B$ in $F$ such that any line that separates $A$ from $B$ separates either $A$ or $B$ from a subcollection of $F$ with at least $n/18$ sets.

Keywords.- Convex polygon. Plane Compact Convex Set. Separating line.

1 Introduction

H. Tverberg\textsuperscript{5} proved that for each positive integer $k$, there is a minimum integer $f(k)$ such that for every collection $F$ of $f(k)$ or more plane compact convex sets with pairwise disjoint interiors, there is a line that separates one set in $F$ from a subcollection of $F$ with at least $k$ sets. R. Hope and M. Katchalski\textsuperscript{3} showed that $3k + 1 \leq f(k) \leq 12k - 1$.

Later E. Rivera-Campo and J. Töröcsik\textsuperscript{4} proved that in any collection $F$ of $n \geq 5$ pairwise disjoint compact convex sets in the plane, there is a pair of sets $A$ and $B$ such that any line that separates $A$ from $B$ separates either $A$ or $B$ from at least $\frac{n+28}{30}$ sets in $F$. In this paper we give a higher lower bound of $\frac{n}{18}$ sets in $F$ for any collection $F$ of $n \geq 2$ sets.

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2 Preliminary results

H. Edelsbrunner et al [1] proved the following theorem, see L. Fejes-Toth [2] for a related result.

**Theorem 1.** Any collection of \( n \geq 3 \) compact, convex and pairwise disjoint sets in the plane may be covered with non-overlapping convex polygons with a total of not more than \( 6n - 9 \) sides. Furthermore no more than \( 3n - 6 \) distinct slopes are required.

We adapt part of the proof given in [1] to obtain the following result.

**Lemma 1.** Let \( \mathcal{P} = \{P_1, P_2, \ldots, P_n\} \) be a collection of \( n \geq 3 \) pairwise disjoint convex polygons in the plane. There exists a collection \( \mathcal{R} = \{R_1, R_2, \ldots, R_n\} \) of pairwise non-overlapping convex polygons such that:

1. For \( i = 1, 2, \ldots, n \), \( P_i \) is contained in \( R_i \).
2. For \( i = 1, 2, \ldots, n \), each side of \( R_i \) supports a side of \( P_i \).
3. The total number of sides among polygons \( R_1, R_2, \ldots, R_n \) is at most \( 9n - 9 \).

**Proof.** A side \( s \) of a polygon \( P_i \in \mathcal{P} \) is reducible with respect to \( \mathcal{P} \) if the triangle \( t_s \) (not containing \( P_i \)), bounded by \( s \) and the lines supporting the sides of \( P_i \) incident to \( s \), does not intersect the interior of any another polygon \( P_j \). Equivalently, a side \( s \) of a polygon \( P_i \in \mathcal{P} \) is reducible with respect to \( \mathcal{P} \) if it vanishes before reaching the interior of a polygon \( P_j \) when it is translated in the direction orthogonal to \( s \) and away from \( P_i \) (see Fig. 1).

![Figure 1: Polygon \( P \) with a reducible side \( s \).](image-url)
We modify $P$ by growing, one at a time, polygons in $P$ as follows: if some polygon $P_i \in P$ has a reducible side $s$ with respect to $P$ substitute $P_i$ in $P$ with polygon $P_i \cup t_s$. Repeat this until no polygon in $P$ has a reducible side. Denote by $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$ the family of polygons thus obtained.

It is clear that $\mathcal{R}$ satisfies conditions 1) and 2), we claim it also satisfies condition 3. To prove this consider a third family $Q = \{Q_1, Q_2, \ldots, Q_n\}$ of convex polygons obtained from $\mathcal{R}$ by further growing polygons $R_1, R_2, \ldots, R_n$ in the following way.

A side $s$ of a polygon $R_i$ in $\mathcal{R}$ is free with respect to $\mathcal{R}$ if the open polygonal region, not containing $R_i$, which is determined by $s$ and the lines supporting the two sides of $R_i$ adjacent to $s$ contains no points of polygons in $\mathcal{R}$.

Let $C$ be a circle containing all polygons in $\mathcal{R}$. Starting with the non-free sides of polygons $R_i \in \mathcal{R}$, translate one at a time a side $s$ of a polygon $R_i \in \mathcal{R}$ (also in the direction orthogonal to $s$ and away from $R_i$) as far as possible without intersecting the interior of another (possibly already grown) polygon $R_j$ or the exterior of $C$ (see Fig. 2).

![Figure 2: Side $s$ of $R_i$ reaches a polygon $R_j$ or reaches circle $C$.]

Stop when no side can be translated as indicated and let $Q = \{Q_1, Q_2, \ldots, Q_n\}$ where, for $i = 1, 2, \ldots, n$, $Q_i$ is the polygon obtained from $R_i$ in this manner. We claim that if $C$ is chosen large enough, then each polygon in $Q$
contains at most 3 free sides and all non free sides of a polygon \( Q_i \) in \( Q \) are in contact with a polygon \( Q_j \) with \( i \neq j \).

By the choice of \( R \) no side of a polygon \( R_j \) vanishes, therefore for \( i = 1, 2, \ldots, n \) the number of sides of \( R_i \) is equal to the number of sides of \( Q_i \).

We define a plane graph \( G \) with one vertex \( v_i \) placed in the interior of each polygon \( Q_i \in Q \) and a vertex \( v_{n+1} \) placed outside circle \( C \). The edge set of \( G \) consists of 6 types of edges as follows:

a) If a side of a polygon \( Q_i \) and a side of a polygon \( Q_j \) have a segment \( l \) in common choose an arbitrary point \( p \) in \( l \) and draw an edge \( v_i v_j \) through \( p \) as in Fig. 3 (left).

b) If a vertex \( p \) of a polygon \( Q_i \) lies in the interior of a side of a polygon \( Q_j \) and \( p \) is not a vertex of a polygon in \( Q \) other than \( Q_i \), then draw an edge \( v_i v_j \) through \( p \) as in Fig. 3 (center).

c) If two or more polygons \( Q_{i_1}, Q_{i_2}, \ldots, Q_{i_r} \) share a vertex \( p \) which lies in the interior of a side of a polygon \( Q_j \) and \( p \) is not a point of any polygon \( Q_k \) other than \( Q_{i_1}, Q_{i_2}, \ldots, Q_{i_r} \), then there is a neighbourhood \( N \) of \( p \) containing no points of polygons \( Q_k \) with \( k \neq j, i_1, i_2, \ldots, i_r \). In this case draw a cycle with edges \( v_j v_{i_1}, v_{i_1} v_{i_2}, v_{i_2} v_{i_3}, \ldots, v_{i_{r-1}} v_i, v_i v_j \) as in Fig. 3 (right).

d) If three or more polygons \( Q_{i_1}, Q_{i_2}, \ldots, Q_{i_r} \) share a vertex \( p \) and \( p \) is not a point of any polygon \( Q_k \) other than \( Q_{i_1}, Q_{i_2}, \ldots, Q_{i_r} \), then there is a neighbourhood \( N \) of \( p \) containing no points of any polygon \( Q_k \) with \( k \neq i_1, i_2, \ldots, i_r \). In this case draw a cycle with edges \( v_{i_1} v_{i_2}, v_{i_2} v_{i_3}, \ldots, v_{i_{r-1}} v_i, v_i v_1 \) as in Fig. 3 (left).

e) If two polygons \( Q_i \) and \( Q_j \) share a vertex \( p \) and \( p \) is not a point of any polygon in \( R \) other than \( Q_i \) and \( Q_j \), then draw an edge \( v_i v_j \) through \( p \) as in Fig. 4 (right).

f) For each free side \( s \) of a polygon \( Q_i \) add an edge \( v_i v_{n+1} \) drawn through an interior of point \( p \) in \( s \).
Notice that edges of type f) are the only possible multiple edges of $G$. Therefore $G$ is a plane multigraph with $n + 1$ vertices and $m \leq (3(n + 1) - 6) + (x_2 + 2x_3) = (3n - 3) + (x_2 + 2x_3)$ edges, where for $i = 2, 3$, $x_i$ denotes the number of polygons in $Q$ with $i$ free sides.

We say that an edge $e = v_i v_j$ of $G$ intersects a side $s$ of a polygon in $Q$. 

Figure 4: Edges type d) (left) and edge type e) (right).

Figure 5: Graph $G$ with $n = 18$, $x_1 = 6$, $x_2 = 2$, $m_e = 1$ and $m_f = 10$. 

We say that an edge $e = v_i v_j$ of $G$ intersects a side $s$ of a polygon in $Q$. 

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if $e$ is drawn through either an interior point of $s$ or through a vertex of $s$. Each non free side among polygons in $Q$ is intersected by at least one edge of $G$ of type a), b), c), d) or e) and each free side is intersected by an edge of type f). We claim that if an edge $v_i v_j$ of $G$ is of type e) then two of the 4 sides intersected by $v_i v_j$ are intersected by at least another edge of $G$.

Proof of Claim. Let $v_i v_j$ be an edge of $G$ of type e). Then either two sides $s, t$ of $Q_i$ (of $Q_j$) or one side $s$ of $Q_i$ and one side $t$ of $Q_j$ are such that the lines supporting $s$ and $t$ separate $Q_i$ and $Q_j$ (see Fig. 6).

![Figure 6: Lines supporting $s$ and $t$ separate $Q_i$ and $Q_j.$](image)

Without loss of generality we assume the line supporting a side $s$ of $Q_i$ separates $Q_i$ and $Q_j$. Let $b$ be the other side of $Q_i$ incident in the common vertex $p$ of $Q_i$ and $Q_j$. Then side $b$ must contain a point of a polygon $Q_k$ other than $Q_i$ and $Q_j$, otherwise $b$ could be translated away from $Q_i$ in the direction orthogonal to $b$ which is not possible by the properties of $Q$ (see Fig. 7). This means that side $b$ is intersected by an edge of $G$ other than edge $v_i v_j$.

![Figure 7: Side $b$ could be translated away from $Q_i.$](image)

Analogously a side $c \neq b$, incident with $t$, is intersected by an edge of $G$ other than $v_i v_j$. □

Let $T$ denote the total number of sides among polygons in $Q$. Also let $m$, $m_e$ and $m_f$ denote the number of edges of $G$, the number of edges of $G$ of
type e) and the number of edges of $G$ of type f), respectively. Since polygons in $Q$ contain at most 3 free edges, $m_f = x_1 + 2x_2 + 3x_3$, where for $i = 1, 2, 3$ $x_i$ is the number of polygons in $Q$ containing $i$ free edges.

In order to bound the number of sides among polygons in $Q$ we add the number of sides intersected by edges of $G$ and subtract the number of sides intersected by at least two edges of $G$ (2 sides for each edge of type e)). Considering that edges of types a), b), c), and d), intersect at most 3 sides; edges of type e) intersect 4 sides and edges of type f) intersect one side, we have:

$$T \leq (3(m - (m_e + m_f)) + 4m_e + m_f) - 2m_e$$
$$= 3m - m_e - 2m_f$$
$$\leq 3m - 2m_f$$
$$\leq 3((3n - 3) + (x_2 + 2x_3)) - 2(x_1 + 2x_2 + 3x_3)$$
$$= 9n - 9 - 2x_1 - x_2$$
$$\leq 9n - 9.$$

Consider the following family of plane geometric graphs $G_t$, $t = 1, 2, \ldots$ given by Edelsbrunner et al. [1] where it is used to show that the bounds in the number of sides and slopes on Theorem 1 are tight.

$G_1$ is the graph with 7 vertices shown on Fig. 8 (left); for $t \geq 1$, $G_{t+1}$ is the graph obtained from $G_t$ as in Fig. 8 (right). For $k \geq 1$ the graph $G_k$ is a plane geometric graph with $n = 3k$ internal faces. The three inner-most faces are quadrilaterals while all other faces are hexagons.

Figure 8: Left: Graph $G_1$. Right: Graph $G_{t+1}$ obtained from graph $G_t$ (placed upside down inside the dotted triangle).
Place an hexagon inside each of the three inner-most faces of $G_k$ as in Fig. 9 (left), an octagon inside each outer-most internal face of $G_k$ as in Fig. 9 (right) and a nonagon inside each other internal face of $G_k$ as in Fig. 10.

Figure 9: Hexagon inside inner-most face of $G_k$. Octagon inside outer-most internal face of $G_k$.

Figure 10: Nonagon inside internal face of $G_k$.

Since each face of $G_k$ has an even number of sides, the placement of the polygons described above may be done in such a way that for each internal edge $e$ of $G_k$ a side $s_e$ of the polygon $R_i$ lying on one side of $e$ is parallel (and closed enough) to $e$ while a vertex $v_e$ of the polygon $R_j$ lying on the other side of $e$ lies on $e$ as in Fig. 11.

Let $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$ be the set of polygons described above. The total number of sides among polygons in $\mathcal{R}$ is

$$m = 3(6) + 3(8) + (n - 6)(9) = 9n - 12$$

Each side among polygons in $\mathcal{R}$ is relevant with respect to $\mathcal{R}$, therefore our bound on the number of sides in Lemma 1 is almost tight.
3 Main Results

In this section we present our main results.

Lemma 2. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be a collection of $n \geq 3$ pairwise disjoint convex polygons in the plane. There is a side $s$ of a polygon $P_i$ such that the line supporting $s$ separates $P_i$ from a subcollection $\mathcal{F}$ of $\mathcal{P}$ with at least $\frac{n}{18}$ polygons.

Proof. Let $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$ be a collection of pairwise disjoint convex polygons satisfying conditions 1), 2) and 3) in Lemma 1 and let $\mathcal{L} = \{l_1, l_2, \ldots, l_m\}$ be the (multi)set of lines supporting the sides of each polygon in $\mathcal{R}$. We include $c$ copies of a line $l$ if $l$ supports sides of $c$ polygons in $\mathcal{R}$. Therefore we can associate a unique polygon $R_i(k)$ to each line $l_k \in \mathcal{L}$ such that a side of $R_i(k)$ is supported by $l_k$. For $k = 1, 2, \ldots, m$ let $H_k^-$ be the closed halfplane determined by $l_k$ that does not contain $R_i(k)$.

Define a bipartite graph $F$ with a vertex $v_j$ for each polygon $R_j \in \mathcal{R}$ and a vertex $w_k$ for each line $l_k \in \mathcal{L}$. Graph $F$ has an edge $v_j w_k$ if polygon $R_j$ is contained in $H_k^-$. For each pair of polygons $R_i, R_j$ there is at least one side $s$ of one of them such that the line supporting $s$ separates $R_i$ from $R_j$. Therefore graph $F$ has at least one edge for each pair $\{i, j\}$ with $1 \leq i < j \leq n$. This implies that there is a vertex $w_k$ whose degree in $F$ is at least $\left(\begin{array}{c} n \end{array}\right)/m \geq \left(\begin{array}{c} n \end{array}\right)/(9n - 9) = n/18$. 

Figure 11: Polygons on both sides of an internal edge of $G_k$. 


This means that the line \( l_k \) separates polygon \( R_{i(k)} \) from at least \( n/18 \) polygons in \( \mathcal{R} \). By Property 2) in Lemma 1, line \( l_k \) supports a side of polygon \( P_{i(k)} \).

\[ \square \]

**Theorem 2.** In any collection \( \mathcal{C} \) of \( n \geq 2 \) pairwise disjoint compact convex sets in the plane, there is a pair of sets \( A \) and \( B \) such that any line that separates \( A \) from \( B \) separates either \( A \) or \( B \) from a subcollection \( \mathcal{F} \) of \( \mathcal{C} \) with at least \( n/18 \) sets.

**Proof.** Let \( n \geq 2 \) be an integer and \( \mathcal{C} = \{C_1, C_2, \ldots, C_n\} \) be a collection of pairwise disjoint compact convex sets in the plane and let \( T \) be a triangle containing all sets in \( \mathcal{C} \) in its interior.

For any line \( l \) let \( t_l^- \) and \( t_l^+ \) be the number of sets in \( \mathcal{C} \) lying to the left and to the right of \( l \), respectively; for a horizontal line \( l \), \( t_l^- \) and \( t_l^+ \) are the number of sets in \( \mathcal{C} \) lying above and below \( l \), respectively. For \( 1 \leq i < j \leq n \) let \( l_{ij} \) be a line separating \( C_i \) from \( C_j \) for which \( \max\{t_{ij}^-, t_{ij}^+\} \) is as small as possible.

Each line \( l_{ij} \) determines two closed halfplanes \( H_{ij} \) and \( H_{ji} \) containing \( C_i \) and \( C_j \), respectively. For \( i = 1, 2, \ldots, n \), let \( P_i = T \cap (\bigcap_{j \neq i} H_{ij}) \). Then for \( i = 1, 2, \ldots, n \), \( P_i \) is a convex polygon that contains set \( C_i \) such that each side is supported by a line \( l_{ij} \) or by a side of \( T \).

Let \( \mathcal{P} = \{P_1, P_2, \ldots, P_n\} \). By Lemma 2 there is a side \( s \) of a polygon \( P_i \) such that the line \( l(s) \) supporting \( s \) separates \( P_i \) from a subcollection of \( \mathcal{P} \) with at least \( n/18 \) polygons.

Since no side of \( T \) separates polygons in \( \mathcal{P} \), \( l(s) \) is of the form \( l_{ij} \). By the choice of \( l_{ij} \) each line that separates sets \( C_i \) from \( C_j \) separates either \( C_i \) or \( C_j \) from at least \( n/18 \) sets in \( \mathcal{C} \).

\[ \square \]

**References**

[1] Herbert Edelsbrunner, Arch D. Robison and Xiao Jun Shen. Covering convex sets with non-overlapping polygons, *Discrete Mathematics* 81: 153 – 164, 1990.

[2] László Fejes Tóth. Illumination of convex discs, *Acta Mathematica Academiae Scientiarum Hungaricae* 29(3-4): 355 – 360, 1997.
[3] Rafael Hope and Meir Katchalski. Separating plane convex sets, *Mathematica Scandinavica* 66 (1): 44–46, 1990.

[4] Eduardo Rivera-Campo and Jeno Töröcsik. On separation of plane convex sets, *European Journal of Combinatorics* 14: 113 – 116, 1993.

[5] Helge Tverberg. A separation property of plane convex sets, *Mathematica Scandinavica* 45: 255–260, 1979.