W = 0 Complex Structure Moduli Stabilization on CM-type K3 × K3 Orbifolds: Arithmetic, Geometry and Particle Physics

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Abstract: It is an important question in string compactification whether complex structure moduli stabilization inevitably ends up with a vacuum expectation value of the superpotential ⟨W⟩ of the order of the Planck scale cubed. Any thoughts on volume stabilization and inflation in string theory, as well as on phenomenology of supersymmetric Standard Models, will be affected by the answer to this question. In this work, we follow an idea for making ⟨W⟩ ≃ 0 where the internal manifold has a vacuum complex structure with arithmetic characterization, and address Calabi–Yau fourfold compactification of F-theory. The moduli space of K3 × K3 orbifolds contain infinitely many such vacua. Arithmetic conditions for a ⟨W⟩ = 0 flux are worked out, and then all the K3 moduli have supersymmetric mass. Possible gauge groups, matter representations and discrete symmetries are studied for the case of Z2-orbifolds.

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1. Introduction

In an F-theory compactification on a Calabi–Yau fourfold $Y$, the effective theory in 3+1-dimensions has the superpotential

$$ W = W_{\text{cpx str}} \propto \int_Y G \wedge \Omega_Y; \quad (1) $$

$G$ is a four-form flux in the M-theory formulation of F-theory taking value in $\mathbb{R}^4$,

$$ \frac{1}{2} c_2(TY) + H^4(Y; \mathbb{Z}), \quad (2) $$

and $\Omega_Y$ a holomorphic $(4,0)$-form on $Y$. $\Omega_Y$ varies relatively to $H^4(Y; \mathbb{Z})$ over the moduli space $\mathcal{M}_{\text{cpx str}}^{[Y]}$ of complex structure of $Y$, so the superpotential $W$ is regarded as a function (a section of an appropriate line bundle in fact) on $\mathcal{M}_{\text{cpx str}}^{[Y]}$. When a topological flux $G$ is fixed, the $F$-term condition determines the vacuum expectation value (vev) of the complex structure parameters $\langle z \rangle \in \mathcal{M}_{\text{cpx str}}^{[Y]}$, and consequently the vev of $W$ proportional to $\int_Y G \wedge \langle \Omega_Y \rangle$, where $\langle \Omega_Y \rangle := \Omega_Y(\langle z \rangle)$.

The vev of $W$ determined in this way are quite often of the order of $M_{\text{Pl}}^3$, where $M_{\text{Pl}}$ is the Planck scale in (3+1)-dimensions [2]. This means that the vacuum\(^1\) has AdS supersymmetry with the cosmological constant of the order of $-M_{\text{Pl}}^4$, and the gravitino mass is of the order of $M_{\text{Pl}}$. Once a topological flux $G$ is chosen, there is no chance of continuous tuning of compactification parameters (because the complex structure moduli fields are expected to have large masses). Certainly such a large negative cosmological

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\(^1\) In this article, we do not discuss stabilization of Kähler moduli. It makes sense to focus on stabilization of complex structure moduli and pose a question if it is possible to achieve $|\langle W \rangle_{\text{cpx str}}| \ll M_{\text{Pl}}^3$, when one ignores a possibility that $|\langle W \rangle| \ll M_{\text{Pl}}^3$ as a result of cancellation between $\langle W \rangle_{\text{cpx str}} \sim M_{\text{Pl}}^3$ and $\langle W \rangle_{\text{Kähler}} \sim M_{\text{Pl}}^2$ in a non-geometric/non-perturbative stabilization of Kähler moduli.
constant is not a good approximation to the vacuum we live in. Large gravitino mass and its anomaly mediation to gauginos are a fatal blow to the electroweak-ino dark matter scenario, and also to supersymmetric grand unification.

If there is a dynamics, mechanism, theoretical principle or anything else that renders the vev of the superpotential much smaller than $M_{Pl}^2$, therefore, it is worth investigating it further. In this article, we pick up an idea of [3,4], and elaborate more on it. The idea is to focus on the subset $M_{alg}^z \subset M_{cpx \ str}^z$ of the complex structure moduli space; it is the set of points $\langle z \rangle$ in $M_{cpx \ str}^z$ where all the Hodge components $H^{p,4-p}(Y_{\langle z \rangle}; \mathbb{C})$ (with $p = 0, 1, 2, 3, 4$) have basis elements in $H^4(Y_{\langle z \rangle}; \overline{\mathbb{Q}})$. Now that the period integrals form a finite dimensional vector space over $\overline{\mathbb{Q}}$, it is not unthinkable any more that their linear combination by an appropriately chosen set of flux quanta $G$ in (2) vanishes. As an attractive phenomenology idea for the small value of the cosmological constant in this local universe, therefore, three directions of further investigation will be motivated: (a) to look for a dynamics or theoretical principle that will favor a choice of $\langle z \rangle$ from the subset $M_{alg}^z$ than from $M_{cpx \ str}^z$, (b) to derive physics consequences of the idea other than the original input $\langle W \rangle \simeq 0$, and (c) to elaborate more—as string-phenomenology—on how/when a choice $\langle z \rangle \in M_{alg}^z$ renders $\langle W \rangle \simeq 0$.

In Ref. [5], the authors pursued the direction (c) for choices of $\langle z \rangle$ from an even smaller subset $M_{CM}^z \subset M_{alg}^z$. The subset $M_{CM}^z$ consists of choices of complex structure $\langle z \rangle$ where the compactification manifold $Y_{\langle z \rangle}$ is of CM-type, a notion generalizing the complex multiplication on elliptic curves. While we wait until Sect. 2.4 in this article to spell out what the CM-type means in math language (a more systematic review is found in [5,6]), there are two characterizations of the CM nature of $z \in M_{cpx \ str}^z$ that can be stated in string theory. First, in a Type IIB Calabi–Yau orientifold case, the CM nature of a Calabi–Yau threefold $M_z$ for compactification has been conjectured [7] to be a half of the necessary and sufficient condition for the $N = (2,2)$ worldsheet superconformal field theory to be described by a rational CFT. This observation may (or may not) shed a bit of light in the research direction (a) above. The other characterization of a CM-type complex structure $z$ is that the basis elements of $H^{p,4-p}(Y_z; \mathbb{C})$ are not just in $H^4(Y_z; \overline{\mathbb{Q}})$, but are subject to stronger control under the Galois group action. Due to this nature, it turns out [5] that the F-term (supersymmetry) conditions on the flux quanta for a given CM-type complex structure $z$ are highly degenerate as a consequence, and are satisfied by a space of flux quanta of higher dimension (relatively to the estimation for $\langle z \rangle \in M_{alg}^z$ in [4]).

Study in this article has two motivations. One is to continue on the research direction (c) for $\langle z \rangle \in M_{CM}^z$ for fourfolds $Y$ in the context of M-theory/F-theory compactification; the study of [5] was only in the context of Type IIB Calabi–Yau orientifolds, so $Y$ were perplexing. 2 Such a large negative cosmological constant is not an immediate consequence of $\langle W \rangle \sim M_{Pl}^2$, because of cancellation that takes place for a special kinds of Kähler potential. One has to make sure, though, that such a special form is maintained even after Kaluza–Klein / string / quantum / non-perturbative corrections are taken into account.

3 Here, the field $\overline{\mathbb{Q}} \subset \mathbb{C}$ consists of all the algebraic numbers.

4 The phenomenology idea of assuming $\langle z \rangle \in M_{CM}^z$ (with an extra assumption on Kähler moduli vev) therefore attributes the small value of the cosmological constant not to a symmetry of the field theory on the space-time $\mathbb{R}^{3,1}$ (or $\mathbb{R}^{3,1} \times M_z$), but to an immensely large symmetry (chiral algebra) of the conformal field theory on the worldsheet.

5 Neither did we study stabilization of the moduli of D7-brane configurations, nor particle-physics consequences of such compactifications in [5].
of the form \((E_\phi \times M)/\mathbb{Z}_2\) for some Calabi–Yau threefold \(M\) and an elliptic curve \(E_\phi\), both of CM type. The other is to extract consequences on particle physics / complex structure moduli stabilization in F-theory, which is in the research direction (b).

We do not attempt at making a progress in the direction (a) in this article. It is worth reminding ourselves, though, that a CM-type K3 surface has a defining equation with all the coefficients being algebraic numbers \([8,9]\). So, if a fourfold \(Y\) is a K3 x K3 orbifold of a pair of CM-type K3 surfaces, \(Y\) has defining equations with all the coefficients in \(\mathbb{Q}\). The \(L\)-function can be defined for each one of such arithmetic models of \(Y\). Recent articles \([10–13]\) suggest—under certain assumptions—that Calabi–Yau threefolds \(M_z\) for certain \(z \in M_{\text{alg}}\) have a simple rational Hodge substructure in \(H^3(M_z; \mathbb{Q})\) whose \(L\)-function is modular. It will be exciting, if such a research direction (and its extension to F-theory) manages to elevate such an arithmetic aspect into a necessary theoretical principle in string/M-theory in the future.

Here is what we do in this article. We work exclusively on fourfolds \(Y\) obtained in the form of K3 x K3 orbifolds.\(^6\) This is because CM-type complex structure is known to exist in a most systematic way for this class of fourfolds (brief review in Sect. 2.1).\(^7\) We deal with a simplest class of \(\mathbb{Z}_2\)-orbifolds of K3 x K3 in Sects. 2 and 4, while Sect. 3 deals with more general orbifolds of K3 x K3. M-theory compactification on such fourfolds down to \(\mathbb{R}^{2,1}\) is studied in Sects. 2 and 3; we work out the conditions for a non-trivial supersymmetric flux with \(\langle W \rangle = 0\) to exist, and also examine the mass terms, interactions and symmetries of the complex structure moduli fields. Section 4 is devoted to F-theory compactification down to \(\mathbb{R}^{3,1}\). Some attempts are made in finding fourfolds \(Y\) birational to a K3 x K3 orbifold so that \(Y\) have flat elliptic fibrations. Results of Sect. 2 are recycled (with a bit of care), and we see that the complex structure moduli of the pair of K3 surfaces can be given large masses by a flux satisfying \(DW = W = 0\). Sections 4.3 and 4.4 also derive constraints on possible choices of non-Abelian gauge group and matter curve configuration.\(^8\)

This study can be seen as an example that a phenomenological idea for small \(\langle W \rangle\) may have particle physics consequences apparently totally unrelated to the cosmological constant: discrete gauge symmetry (Sect. 3.2.4), approximate accidental symmetry in the effective theory (Sects. 2.4.3, 2.5 and 3.3) and constraints on choices of non-Abelian gauge groups and matter curve configuration.

### 2. Supersymmetric Flux Vacua on CM-type \((K3 \times K3)/\mathbb{Z}_2\) Orbifolds

#### 2.1. CM-type Calabi–Yau fourfolds and Borcea–Voisin orbifolds

In the case \(Y = E\) is an elliptic curve, a one-dimensional Calabi–Yau manifold, the complex structure of \(E\) is of CM-type, by definition, if \(E\) has complex multiplication (see \([6, \S 2.1, B.1.5, \text{and B.1.6}]\) or footnote 28 in this article, for example, for more background information).

The set of CM points \(\mathcal{M}_{\text{CM}}^{[E]}\) in the moduli space of complex structure of elliptic curves \(\mathcal{M}_{\text{cpx str}}^{[E]} \cong \mathcal{H}/\text{SL}(2; \mathbb{Z})\) is completely understood; CM points in the upper complex half

\(^6\) This class of fourfolds includes (modulo birational transformation) orbifolds of (an elliptic curve) x (a Borcea–Voisin Calabi–Yau threefold).

\(^7\) A more extensive review is found in \(\S 2.2\) and appendix B.1 of \([6]\).

\(^8\) Here is a cautionary remark: we presented in Sect. 4 only the F-theory geometry construction in which we have confidence; we have a sense of feeling that there will be more constructions for F-theory geometry with CM-type Hodge (sub)structure, even within the simplest class of \(\mathbb{Z}_2\)-orbifold of K3 x K3 (see footnotes 86, 87 and 92). So, it is too early to take those constraints as a final statement, or to take them out of the context.
plane $\mathcal{H}$ are the set of the roots of any quadratic polynomial equation of one variable with coefficients in $\mathbb{Q}$. They are labeled by the imaginary quadratic fields $K$; the CM points sharing the same imaginary quadratic field forms an orbit under the action of $\text{GL}(2; \mathbb{Q}) = \mathbb{G} \text{Sp}(2; \mathbb{Q})$. In the case $Y = X$ is a K3 surface (a two-dimensional Calabi–Yau manifold) with a transcendental lattice $T_X$, it is also known that any CM point in the moduli space $\mathcal{M}_{CM}^{\text{cpx str}}(X; T_X)$ is associated with a CM field $K$ of degree $[K : \mathbb{Q}] = \text{rank}(T_X)$; the CM points sharing the same CM field $K$ form orbits under the action of the group $\mathbb{G} \text{O}(T_X; \mathbb{Q})$ on $\mathcal{M}_{CM}^{\text{cpx str}}(X; T_X) \subset \mathcal{M}_{\text{cpx str}}^{\text{cpx str}}(X; T_X) = \text{Isom}(T_X) \setminus D(T_X)$; here, $D(T_X)$ is the period domain of the signature $(2, \text{rank}(T_X) - 2)$ lattice $T_X$ and $\text{Isom}(T_X)$ the group of integral isometries of $T_X$. In particular, we know that there are infinitely many CM points in the moduli space of complex structure of elliptic curves and K3 surfaces.

When it comes to the case $Y = M$ is a Calabi–Yau threefold, or a Calabi–Yau fourfold $Y$, however, much less is known. It is believed that the Calabi–Yau threefolds $M$ realized by rational CFT’s have complex structure of CM type [7], but they are nothing more than a small number of isolated points in the moduli space. Although the group $\text{Sp}(b_3(M))$ is a symmetry of some of the relations that the Hodge structure of a Calabi–Yau threefold $M$ satisfies, yet the action of the group takes a point in $\mathcal{M}_{\text{cpx str}}^{\text{cpx str}}(Y; T_Y)$ outside of $\mathcal{M}_{\text{cpx str}}^{\text{cpx str}}(Y; T_Y)$, in general; the latter observation also holds true in the case $Y$ is a Calabi–Yau fourfold, when the group $\text{Sp}(b_3(M))$ is replaced by the isometry group of the lattice $H^4(Y; \mathbb{Z})$. So, in particular, we do not have an argument in the case $Y$ is a threefold or a fourfold that infinitely many CM points $\mathcal{M}_{CM}^{[Y]}$ show up in the form of orbits of $\mathbb{G} \text{Sp}(b_3)$ or $\mathbb{G} \text{O}(b_4(Y))$. Indeed, the André–Oort conjecture hints that there are not so many CM points available in $\mathcal{M}_{\text{cpx str}}^{[Y]}$ in those cases. For more information, see [6, §2.2].

For a special class of topological types of Calabi–Yau threefolds $[Y = M]$ or of fourfolds $[Y]$, however, it is possible to identify systematically a set of points $(z)$ of $\mathcal{M}_{\text{cpx str}}^{[Y]}(M)$ where $H^3(Y(z); \mathbb{Q})$ or $H^4(Y(z); \mathbb{Q})$ has a CM-type rational Hodge substructure. An idea, originally in [16,17], is to take a product of a CM-type elliptic curve $E$ and a CM-type K3 surface, or of a pair of CM-type K3 surfaces, first, and then to take an orbifold that preserves the Calabi–Yau condition. Not all the topological types available for a Calabi–Yau three/four-fold will be realized in this construction. The moduli space $\mathcal{M}_{\text{cpx str}}^{[Y]}$ of a three/four-fold $Y$ constructed in that way contains an orbifold locus $\mathcal{M}_{\text{cpx str}}^{[Y]BV}$ where the orbifold singularity of $Y(z)$ is not deformed in complex structure, as long as the building block $E$ or K3 surfaces are of CM-type, and the vacuum choice $\langle z \rangle$ of the complex structure of $Y$ is in the orbifold locus $\mathcal{M}_{\text{cpx str}}^{[Y]BV}$, then $H^3(Y(z); \mathbb{Q})$ or $H^4(Y(z); \mathbb{Q})$ has a rational Hodge substructure of CM-type indeed.

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9 See §A.2 of [6] (and a passage before Sect. 2.3.3 in this article) for the definition of a CM field.

10 The groups $\mathbb{G} \text{Sp}$ and $\mathbb{G} \text{O}$ consist of linear transformations that preserve skew-symmetric and symmetric bilinear forms, respectively, up to overall scalar multiplications.

11 This argument still does not rule out infinitely many CM points; in fact infinitely many CM points are contained in the 101-dimensional moduli space of the quintic Calabi–Yau threefolds (e.g., see [6, footnote 18] for references). The Fermat sextic fourfold [14] is CM-type (e.g., [15]).

12 There is a review material on rational Hodge structure in Sect. 2.3 in this article.

13 It is a stronger condition for a rational Hodge structure on $H^4(Y; \mathbb{Q})$ to be of CM-type than for it to have a rational Hodge substructure that is of CM-type. See the discussion at the end of Sect. 2.2. Whether the Coleman–Oort conjecture is relevant in the current context (whether supersymmetric flux is available for moduli stabilization) should also be reconsidered along this line.

14 We do not talk about choice of Kähler moduli in this article. Whether the orbifold singularity is resolved or not, discussions in this article are valid.
The simplest class of Calabi–Yau fourfolds $Y$ as $K3 \times K3$ orbifolds is of the form $Y = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$. Both of the $K3$ surfaces $X^{(1)}$ and $X^{(2)}$ are assumed to have a non-symplectic automorphism of order two, $\sigma^{(1)}$ and $\sigma^{(2)}$, respectively; the holomorphic $(2,0)$-forms $\Omega^{(i)}_{X^{(i)}}$ and $\Omega^{(j)}_{X^{(j)}}$ get transformed as $\sigma^{(i)}_{\ast} \Omega^{(i)}_{X^{(i)}} = -\Omega^{(i)}_{X^{(i)}}$ for $i = 1, 2$. By choosing the generator $\sigma$ of the orbifold group $\mathbb{Z}_2$ to be $(\sigma^{(1)}, \sigma^{(2)})$, the orbifold $Y$ becomes Calabi–Yau because $(\Omega^{(1)}_{X^{(1)}} \wedge \Omega^{(2)}_{X^{(2)}})$ is invariant under the generator $\sigma$, yet $\Omega^{(i)}_{X^{(i)}}$’s are not. We call a subclass of those fourfolds—those where both $\sigma^{(1)}$ and $\sigma^{(2)}$ act purely non-symplectically (more explanations in the next paragraph and also in Sect. 3.2 (footnotes 65 and 66 in particular)—as Borcea–Voisin $K3 \times K3$ orbifolds in this article; more general orbifolds (a) where $\sigma^{(1)}$ and/or $\sigma^{(2)}$ are non-symplectic but not purely non-symplectic, or (b) where the orbifold group is not $\mathbb{Z}_2$) as generalized Borcea–Voisin $K3 \times K3$ orbifolds. Until the end of this Sect. 2, we deal with M-theory compactifications on a Borcea–Voisin fourfold.

Reference [18] provides a theory of topological classification of a pair $(X, \sigma)$ of a $K3$ surface $X$ and an automorphism $\sigma \in \text{Aut}(X)$ of order two ($\sigma^{2} = \text{Id}_X$) acting non-symplectically ($\sigma^{\ast} \Omega_X \neq \Omega_X$) on the holomorphic $(2, 0)$-form $\Omega_X$. To be more precise, it classifies $(S_0, T_0, \sigma)$ modulo isometry of $H^2(X; \mathbb{Z})$, where $S_0$ and $T_0$ are mutually orthogonal primitive sublattices of $H^2(X; \mathbb{Z})$ of signature $(1, r - 1)$ and $(2, 20 - r)$, respectively, and $\sigma$ an isometry of $H^2(X; \mathbb{Z})$ that acts trivially on $S_0$ and as $(-1) \times$ on $T_0$. This lattice-theory classification of $(S_0, T_0, \sigma)$ is regarded as that of non-symplectic automorphisms of order two, because one may choose $\mathbb{Z} \Omega_X$ from $D(T_0)/\text{Isom}(T_0)$. For such a complex structure, the transcendental lattice $T_X$ is contained within $T_0$, and $\sigma^{\ast} \Omega_X = -\Omega_X$; the Néron–Severi lattice $S_X$ contains $S_0$. The list of [18] consists of 75 choices of $(S_0, T_0, \sigma)$. So, we have 75 choices of $(S^{(i)}_0, T^{(i)}_0, \sigma^{(i)})$ for each one of $i = 1, 2$; for a given choice, a topological family of Borcea–Voisin orbifolds is available for M-theory compactification. In the rest of this section, supersymmetric flux configuration is studied for a vacuum complex structure in

$$\mathcal{M}_{\text{CM}}^{[X(T^{(1)}_0)]} \times \mathcal{M}_{\text{CM}}^{[X(T^{(2)}_0)]} \subset \mathcal{M}_{\text{cpx str}}^{[X(T^{(1)}_0)]} \times \mathcal{M}_{\text{cpx str}}^{[X(T^{(2)}_0)]} = \mathcal{M}_{\text{cpx str}}^{[Y]BV}.$$ (3)

Here is a remark before moving on. One may also construct a Calabi–Yau fourfold as an orbifold of two elliptic curves $E_\phi$, $E_\tau$, and a K3 surface $X^{(2)}$, instead of a pair of $K3$ surfaces:\footnote{In Sects. 2 and 3, we do not distinguish a pair of fourfolds that are mutually birational and have the same number of complex structure and Kähler deformations. That is enough for the purpose of analysing supersymmetric flux configuration and complex structure moduli effective field theory. For example, an orbifold $(E_\phi \times E_\tau \times X^{(2)})/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ has $C^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ singularity along a curve $Z_{(2)} \subset X^{(2)}$; the fourfold $(E_\phi \times (E_\tau \times X^{(2)})/\mathbb{Z}_2)$ in the first line and $((E_\phi \times E_\tau)/\mathbb{Z}_2 \times X^{(2)})/\mathbb{Z}_2$ in the second line are regarded as different resolutions of the $C^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ singularity (cf [2]). Two flops convert one to the other. For this reason, we do not even make a clear distinction between an orbifold with singularity and a non-singular manifold obtained as a crepant resolution of the orbifold in Sects. 2 and 3.}

$$Y = \left( E_\phi \times \left( E_\tau \times X^{(2)} \right) / \mathbb{Z}_2 \right) / \mathbb{Z}_2 =: \left( E_\phi \times M \right) / \mathbb{Z}_2,$$

$$= \left( (E_\phi \times E_\tau) / \mathbb{Z}_2 \times X^{(2)} \right) / \mathbb{Z}_2 =: \left( X^{(1)} \times X^{(2)} \right) / \mathbb{Z}_2.$$ (4)

This is for Type IIB Calabi–Yau orientifolding compactification, where the Calabi–Yau threefold is $M = (E_\tau \times X^{(2)})/\mathbb{Z}_2$. This construction is nothing more than a special case of the Borcea–Voisin $K3 \times K3$ orbifolds; we can see the combination $X^{(1)} = (E_\phi \times E_\tau)/\mathbb{Z}_2 =: \text{Km}(E_\phi \times E_\tau)$ as the K3 surface $X^{(1)}$; along with an involution $\sigma^{(1)}$
that multiplies \((-1)\) to \(E_{\tau}\), the pair \((X^{(1)}, \sigma_{(1)})\) becomes one of the 75 topological types classified by Nikulin (the one with \(T_0 = U[2]U[2]\)). For this reason, we do not lose generality at all by thinking only of K3 \(\times\) K3 orbifolds.

2.2. \(H^4((X^{(1)} \times X^{(2)})/\mathbb{Z}_2; \mathbb{Q})\) and complex structure deformations. In an M-theory compactification on a fourfold \(Y\) and its F-theory limit, a flux is in \(H^4(Y; \mathbb{Q})\), and the complex structure moduli in \(H^4(Y; \mathbb{C})\). Before starting analysis for conditions for supersymmetric flux vacua with \((W) = 0\) to exist, we need to remind ourselves of a bit of math of the cohomology groups of the Borcea–Voisin orbifold \(Y = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2\) with both \(X^{(1)}\) and \(X^{(2)}\) being a K3 surface.

The fourfold \(Y = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2\) would remain singular, if it stays precisely at the orbifold locus without complex structure deformation or Kähler parameter resolution. Because we do not assume anything about the vacuum value of the Kähler parameter, we do not need to think that \(Y\) is singular, and moreover, we can always take a limit from non-zero resolution to the orbifold limit, if we wish. So, topology of the fourfold \(Y\) is well-defined. In Sect. 4, we will use \(Y^{BV}\) for the non-singular fourfold after resolution, and \(Y_0\) the orbifold without deformation or resolution of the \(\mathbb{C}^2/\mathbb{Z}_2\) singularity.

To describe the topology of \(Y\), we need one more preparation. The non-symplectic automorphism \(\sigma_{(i)} : X^{(i)} \to X^{(i)}\) may have fixed points (for \(i = 1, 2\) individually), and the locus of fixed points are denoted by \(Z_{(i)}\) for \(i = 1, 2\). The set \(Z_{(i)}\) of fixed points in \(X^{(i)}\) consists of curves whose irreducible components are disjoint from one another, when \(\sigma_{(i)}\) acts non-symplectically and is order 2. Among the 75 choices of \((S_0, T_0, \sigma)\) in [18], this subset \(Z\) of fixed points is empty in just one choice, where \(S_0 = U[2]E_8[2]\). The subset \(Z\) consists of two disjoint elliptic curves in the choice with \(S_0 = U E_8[2]\). For all other 73 choices, the set \(Z\) consists of one curve \(C_{(g)}\) of genus \(g = (22 - r - a)/2\) in addition to \(k = (r - a)/2\) rational curves \(\mathbb{P}^1\) [18]:
back to the exceptional divisor of the resolved $Y$, and then is taken a wedge product with the Poincaré dual of the exceptional divisor ($=\text{mapped by the Gysin homomorphism}$).

In the family of fourfolds $[Y]$, the horizontal component of $H^4(Y; \mathbb{Q})$ is

$$H^4_H(Y; \mathbb{Q}) = \left(T_0^{(1)} \otimes T_0^{(2)}\right) \otimes \mathbb{Q} \oplus H^1(Z(1); \mathbb{Q}) \otimes H^1(Z(2); \mathbb{Q}),$$

(7)

where the first term is from $[H^4(X(1) \times X(2); \mathbb{Q})]^\sigma$, and the second term from $H^2(Z(4); \mathbb{Q})$. The vertical component is

$$H^4_Y(Y; \mathbb{Q}) = \left(S_0^{(1)} \otimes S_0^{(2)}\right) \otimes \mathbb{Q} \oplus H^4(X(1); \mathbb{Q}) \otimes H^0(X(2); \mathbb{Q})$$

$$\quad \oplus H^0(X(1); \mathbb{Q}) \otimes H^4(X(2); \mathbb{Q})$$

$$\quad \oplus H^2(Z(1); \mathbb{Q}) \otimes H^0(Z(2); \mathbb{Q}) \oplus H^0(Z(1); \mathbb{Q}) \otimes H^2(Z(2); \mathbb{Q}),$$

(8)

where the first line and the second line come from $[H^4(X(1) \times X(2); \mathbb{Q})]^\sigma$ and $H^2(Z(4))$, respectively. The entire 4th cohomology group $H^4(Y; \mathbb{Q})$ is covered by the direct sum of the horizontal component and the vertical component in the case of the family of $[Y]$ over $\mathcal{M}_{cpx \ str}^Y$. The holomorphic 4-form $\Omega_Y$ varies for $z \in \mathcal{M}_{cpx \ str}^Y$, but it does so only within $H^4_H(Y; \mathbb{Q}) \otimes \mathbb{Q} \mathbb{C}$. When the point $z$ is in the subvariety $\mathcal{M}_{cpx \ str}^{YBV} \subset \mathcal{M}_{cpx \ str}^Y$, $\Omega_Y$ remains within $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q} \mathbb{C}$.

At any point $z \in \mathcal{M}_{cpx \ str}^Y$, a $(z$-dependent) Hodge structure is introduced in the vector space $H^4_H(Y; \mathbb{Q})$; the vertical subspace $H^4_Y(Y; \mathbb{Q})$ contains only the 0 Hodge structure. For a vacuum complex structure $(z)$ within $\mathcal{M}_{cpx \ str}^{YBV}$, the vector subspace $H^1(Z(1); \mathbb{Q}) \otimes H^1(Z(2); \mathbb{Q})$ supports a rational Hodge substructure of level 2, and $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$ a rational Hodge substructure of level 4. Linear fluctuation $\delta z$ in the complex structure from $(z)$ are in the Hodge (3, 1) component of $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{C}$ (there are $(20 - r(1)) + (20 - r(2))$ such deformations) and also in the vector space $H^{1,0}(Z(1); \mathbb{C}) \otimes H^{1,0}(Z(2); \mathbb{C})$ (there are $g(1)g(2)$ of them); the former group of fluctuations are within $\mathcal{M}_{cpx \ str}^{YBV}$ and the latter group ventures out from $\mathcal{M}_{cpx \ str}^{YBV}$ into $\mathcal{M}_{cpx \ str}^Y$ by deforming the $\mathbb{C}^2/\mathbb{Z}_2$ singularity of $Y(z)$. At the quadratic order in the deformation of complex structure, $\Omega_{Y_z} \simeq \Omega_{Y(z)} + (\delta z)^a \psi_a + (\delta z)^a (\delta z)^b \psi_{ab}$ for $z = (z) + \delta z$, the quadrature of the $(40 - r(1) - r(2))$ complex structure deformations do not bring $\Omega_{Y_z}$ out of $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{C}$. The quadrature involving $g(1)g(2)$ complex structure deformations, however, may be in the entire $H^4_H(Y(z); \mathbb{C})$.

The observation above on the Hodge substructures on $H^4(Y(z); \mathbb{Q})$ and finitely perturbed $\Omega_{Y_z}$ on them indicates that a non-trivial flux is necessary at least in the $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$ component in order to generate mass terms of the $(40 - r(1) - r(2))$ moduli fields. The $g(1)g(2)$ moduli fields may also acquire mass terms from a flux in $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$, or they may not. We take it out of the scope of this article to study

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20 The notions of a rational Hodge substructure and its level will be introduced in Sect. 2.3.2.

21 Comments on the $r(1) = r(2) = 20$ case will be found later in this article.

22 Here is a heuristic argument. Think of a case that a Borcea–Voisin orbifold $Y = (X(1) \times X(2))/\mathbb{Z}_2$ has a mirror $Y^\varnothing = (X_0^{(1)} \times X_0^{(2)})/\mathbb{Z}_2$ using the mirror $X_0^{(1)}$ and $X_0^{(2)}$ of $X(1)$ and $X(2)$; suppose that $X_0^{(i)}$ can be
\( \Omega_Y \) at the quadratic order in \( \delta z \) including those \( g(1)_r \text{g}(2)_r \) moduli. So it is not a necessary condition—at this moment—for all the complex structure moduli stabilization that \( H^1(Z(1)_r; \mathbb{Q}) \otimes H^1(Z(2)_r; \mathbb{Q}) \) contains a level-0 rational Hodge substructure.\(^{23}\) In this article, therefore, we assume that the Hodge structure on \((T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q})\) is of CM-type (starting in Sect. 2.3), and study when and how supersymmetric flux is admitted in this component; we do not ask whether the Hodge structure on \( H^1(Z(1)_r; \mathbb{Q}) \otimes H^1(Z(2)_r; \mathbb{Q}) \) is CM-type, or has a level-0 Hodge substructure.

2.3. The conditions of supersymmetric fluxes in M/F-theory: cases with CM-type Calabi–Yau compactifications.

2.3.1. Two perspectives As is well-known, there are two different perspectives in describing the way a topological flux \( G \in H^4(Y; \mathbb{Q}) \) in a Calabi–Yau fourfold \( Y \) stabilizes the complex structure moduli of \( Y \). One is more physical, and the other more mathematical, as we repeat them shortly. Either way, the condition for supersymmetry is stated concisely by the F-term condition\(^ {24}\)

\[
DW = 0 : \quad G^{(1,3)} = 0
\]

and the additional condition for the Minkowski spacetime and \( m_{3/2} = 0 \) after compactification,

\[
W = 0 : \quad G^{(0,4)} = 0.
\]

In the more physical perspective, we think that a topological flux \( G \) is specified as a part of data of compactification first, and then the superpotential (1) gives rise to non-trivial scalar potential of the complex structure moduli fields of \( Y \); the expectation value of those fields adjust themselves in the early period of time in the universe to arrive at a potential minimum, where the resulting complex structure of \( Y \) is such that the Hodge \((1, 3)\) component of the topological \( G \in H^4(Y; \mathbb{Q}) \) must be absent when measured in that complex structure. For such a topological \( G \) and the complex structure of \( Y \) so determined, it is a non-trivial question whether the Hodge \((0, 4)\) component of \( G \) vanishes (the condition (10) is satisfied) or not.

Footnote 22 continued chosen from Nikulin’s list, where \( r^0_{(i)} = (20 - r_{(i)}) \) and \( a^0_{(i)} = a_{(i)} \) for \( i = 1, 2 \). The intersection ring of the mirror \( Y^c \) can be used to infer finite perturbation of the complex structure \( \Omega_Y \) of \( Y \). The fourfold \( Y^c \) has three groups of divisors, \( D^{(1)} \) that originate from the divisors of \( X^c_0^{(1)} \), \( D^{(2)} \) that originate from the divisors of \( X^c_0^{(2)} \), and the exceptional divisors \( D_\alpha \) associated with the \( \mathbb{C}^2/Z_2 \) orbifold singularity. The intersection numbers of the form \((D_\alpha)_2 \cdot D^{(1)} \), \( D^{(2)} \) are determined by the intersection numbers of the curve of the involution-fixed points in \( X^c_0^{(2)} \) with \( D^{(i)} \), and are non-zero when the curves of fixed points are non-empty. This observation hints that there is a good chance that a quadratic order perturbation of \( \Omega_Y \) deforming the \( \mathbb{C}^2/Z_2 \) singularity turns into a mass term in the presence of a flux in \((T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q})\).

There are logical gaps to fill, however. One is that the classical intersection ring in \( Y^c \) has an immediate information on the \( \Omega_Y \) in the large complex structure limit of \( Y \), not in the zero deformation limit (orbifold limit) of \( \Omega_Y \). The other is that a flux needs to be in \( W_{(20|02)} \) component within \((T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q})\) as we will see in Sect. 2.4.

\(^{23}\) This condition is equivalent to existence of an algebraic curve in \( Z(1)_r \times Z(2)_r \) other than a copy of \( Z(1)_r \times \text{pt} \) or \( \text{pt} \times Z(2)_r \).

\(^{24}\) Here, we use the superpotential (1) and the Kähler potential obtained by dimensional reduction. All kinds of corrections expected in an effective theory of four supersymmetry charges are not taken into account.
In the more mathematical perspective, on the other hand, we pose questions that are concerned about classification of flux vacua, forgetting about cosmological time evolution before the complex structure moduli fields come down the potential to their vacuum value. We pick up one point in the complex structure moduli space \( z \in \mathcal{M}_{\text{cpx str}}[Y] \), and ask if there is any topological flux \( G \in H^4(Y; \mathbb{Q}) \) whose \( H^{1,3}(Y; \mathbb{C}) \) component vanishes; here, \( Y_z = Y \) is the fourfold of the topological type \([Y]\) with the complex structure corresponding to the point \( z \in \mathcal{M}_{\text{cpx str}}[Y] \), emphasizing the \( z \)-dependence. The condition (10) can also be phrased in the same way. At a generic point \( z \in \mathcal{M}_{\text{cpx str}}[Y] \), only the trivial purely horizontal flux \( G = 0 \in H^4(Y; \mathbb{Q}) \) satisfy the conditions (9). Points in \( \mathcal{M}_{\text{cpx str}}[Y] \) where non-trivial fluxes \( G \in H^4(Y; \mathbb{Q}) \) satisfy the condition (9) form a special sub-locus of \( \mathcal{M}_{\text{cpx str}}[Y] \). This is a Noether–Lefschetz problem in a Calabi–Yau fourfold \([Y]\). In this article, we exploit the latter perspective.

2.3.2. Math supplementary notes: simple Hodge substructure, and CM-type Before doing anything else, we should have a definition of simple Hodge structure and a minimum account for what CM-type stands for.

(Definition) Let \( V_\mathbb{Q} \) be a vector space over \( \mathbb{Q} \). A decomposition of the vector space \( V_\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{C} \) over \( \mathbb{C} \) into the form of

\[
V_\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{p+q=n} V_\mathbb{C}^{p,q} \quad \left( V_\mathbb{C}^{p,q} = V_\mathbb{C}^{q,p} \right)
\]

of vector subspaces \( V_\mathbb{C}^{p,q} \) for non-negative integers \( p, q, \) and \( n \), is called a rational Hodge structure of weight \( n \). For a smooth compact Kähler manifold \( M \), the cohomology group \( H^n(M; \mathbb{Q}) \) has a rational Hodge structure of weight \( n \) given by the complex structure of the Kähler manifold \( M \), for example.

A rational Hodge structure on a vector space \( V_\mathbb{Q} \) is said to be simple, if there is no vector proper subspace \( W_\mathbb{Q} \subset V_\mathbb{Q} \) over \( \mathbb{Q} \) so that \( \bigoplus_{p+q} (V_\mathbb{C}^{p,q} \cap (W_\mathbb{Q} \otimes \mathbb{C})) \) reproduces \( (W_\mathbb{Q} \otimes \mathbb{C}) \). When such a proper subspace \( W_\mathbb{Q} \) exists, \( V_\mathbb{Q} \) [resp. \( W_\mathbb{Q} \)] is said to have [resp. to support] a rational Hodge substructure. An example of rational Hodge structure that is not simple is the Hodge structure on \( H^2(X; \mathbb{Q}) \) of an algebraic K3 surface \( X \); both \( S_X \otimes \mathbb{Q} \) and \( T_X \otimes \mathbb{Q} \) support a rational Hodge substructure of \( H^2(X; \mathbb{Q}) \). When a vector space \( V_\mathbb{Q} \) with a rational Hodge structure is decomposed into vector subspaces

\[25\] We treat fluxes in this article as elements in the \( \mathbb{Q} \)-coefficient cohomology, not in the \( \mathbb{Z} \)-coefficient, and the upper bound on the D3-brane charge is not imposed. At this level of analysis, fluxes in the purely vertical part and purely horizontal part can be regarded completely independent.

\[26\] cf see footnote 13.
over \( \mathbb{Q} \), \( V_{\mathbb{Q}} \cong \bigoplus_{a \in A} W_a \), and each \( W_a \) supports a rational Hodge substructure that is simple, we say that it is a **simple component decomposition of the rational Hodge structure**.

A simple component \( W_a \) in such a decomposition is said to be **level-\( \ell \)**, when \( \ell := \text{Max}(\{ p-q \mid V^{p,q} \cap (W_a \otimes \mathbb{C}) \neq \emptyset \}) \). For example, \( T_X \otimes \mathbb{Q} \) of an algebraic K3 surface \( X \) is a simple component of level-2, and \( S_X \otimes \mathbb{Q} \) contains only level-0 simple components. (the end of the Definition)

(Definition) **CM-type**\(^\text{27} \) is a special property—see the next paragraph—that a rational Hodge structure on a vector space over \( \mathbb{Q} \) may have. For a complex manifold \( M \), its complex structure introduces a rational Hodge structure on \( H^m(M; \mathbb{Q}) \) and one may ask whether the rational Hodge structure is of CM-type or not. A choice of complex structure of \( M \) [resp. a point in \( \mathcal{M}^{[M]}_{\text{cm, stf}} \)] is said to be **CM-type** [resp. a **CM point**] when that is the case.

Suppose that a vector space \( V_{\mathbb{Q}} \) over \( \mathbb{Q} \) is given a rational Hodge structure. It is of **CM-type** when the algebra of Hodge-structure-preserving \( \mathbb{Q} \)-linear maps from \( V_{\mathbb{Q}} \) to itself—\( \text{End}_{\text{Hdg}}(V_{\mathbb{Q}}) \)—is abelian, and has \( \text{dim}_\mathbb{Q}(\text{End}_{\text{Hdg}}(V_{\mathbb{Q}})) = \text{dim}_\mathbb{Q} V_{\mathbb{Q}} \). (the end of Definition)

This is a property of the Hodge structure on \( H^1(T^2; \mathbb{Q}) \) for elliptic curves \( T^2 \) with complex multiplication.\(^\text{28} \) CM type is a notion that generalizes the complex multiplication on elliptic curves to more general complex manifolds. When a CM-type rational Hodge structure on \( V_{\mathbb{Q}} \) is simple, then the algebra \( \text{End}_{\text{Hdg}}(V_{\mathbb{Q}}) \) is always a field with a special property; this class of fields is called a CM field; a brief review on the properties of CM fields is found, for example, in [6, §A.2].

2.3.3. A frequently used property The following is a textbook-level material in math, but is a powerful tool frequently used in this article. So, we include its statement in this article for the convenience of readers (a little more explanation is found in [6, §B.2]).

Let \( V_{\mathbb{Q}} \) be a vector space over \( \mathbb{Q} \), and \( F \) a number field of degree \( \text{[} F : \mathbb{Q} \text{]} = \text{dim}_\mathbb{Q} V_{\mathbb{Q}} \); suppose that \( F \) acts on \( V_{\mathbb{Q}} \) through \( \phi : F \hookrightarrow \text{End}_\mathbb{Q}(V_{\mathbb{Q}}) \). Let us consider an (arbitrary) isomorphism \( \iota : F \cong V_{\mathbb{Q}} \) as a vector space over \( \mathbb{Q} \). Then the action of \( \phi(F) \subset \text{End}_\mathbb{Q}(V_{\mathbb{Q}}) \) on \( V_{\mathbb{Q}} \otimes_\mathbb{Q} F_{\text{nc}} \) can be diagonalized simultaneously; to be more specific, \( V_{\mathbb{Q}} \otimes_\mathbb{Q} F_{\text{nc}} \) has a diagonalization basis\(^\text{29} \)

\[
V_{\mathbb{Q}} \otimes_\mathbb{Q} F_{\text{nc}} = \text{Span}_{F_{\text{nc}}} \{ v_a \mid a = 1, \ldots, \text{dim}_\mathbb{Q} V_{\mathbb{Q}} \}, \tag{12}
\]

there is a 1-to-1 correspondence between those \( \text{dim}_\mathbb{Q} V_{\mathbb{Q}} \) basis elements and the set of all the \( [ F : \mathbb{Q} ] \) embeddings \( \Phi^\text{full}_F : \text{Hom}_\mathbb{Q}(F, \overline{\mathbb{Q}}) \), and

\[
x \cdot v_a = v_a \rho_a(x), \quad \rho_a \in \Phi^\text{full}_F, \quad \forall x \in \phi(F). \tag{13}
\]

\(^\text{27} \) As the present authors have already included a pedagogical review on this in [6], a brief explanation in the following is kept to the minimum.

\(^\text{28} \) For an elliptic curve \( T^2 = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \) with \( \tau^2 + 1 = 0 \), for example, not necessarily one-to-one holomorphic maps such as \( (i) \times 1 : T^2 \to T^2 \) and \( (i + 2i) \times 1 : T^2 \to T^2 \) that multiply complex numbers are examples of complex multiplication operations. More generally, an elliptic curve \( T^2 = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \) has non-trivial complex multiplication operations if and only if there is a set of non-trivial integers \( (a, b, c) \) satisfying \( at^2 + b\tau + c = 0 \). Each complex multiplication operation induces a \( \mathbb{Q} \)-linear map \( H^1(T^2; \mathbb{Q}) \to H^1(T^2; \mathbb{Q}) \) that preserves the Hodge decomposition of \( H^1(T^2; \mathbb{Q}) \otimes \mathbb{C} \). The field \( \text{End}_{\text{Hdg}}(H^1(T^2; \mathbb{Q})) \) of an elliptic curve with complex multiplication is an imaginary quadratic field (the easiest class of CM fields) whose extension degree is equal to \( \text{dim}_\mathbb{Q}(H^1(T^2; \mathbb{Q})) = 2 \).

\(^\text{29} \) The superscript “nc” stands for the normal closure.
Moreover, when we express the eigenvectors \( v_a \) as \( F^{nc} \)-coefficient linear combinations of a \( \mathbb{Q} \)-basis \( \{ e_i \mid i = 1, \ldots, \dim \mathbb{Q} V_\mathbb{Q} \} \) of \( V_\mathbb{Q} \), \( v_a = \sum e_i c_a^i \), there exists a basis \( \{ y_i \mid i = 1, \ldots, [F : \mathbb{Q}] \} \) of the vector space \( F \) over \( \mathbb{Q} \) so that

\[
c_a^i = \rho_a(y_i).
\]

In the context of this article, we wish to use the property above for \( V_\mathbb{Q} \) as \( T_X \otimes \mathbb{Q} \) of a CM-type K3 surface \( X \), and also as one of simple components of \( H^4(Y; \mathbb{Q}) \) with a CM-type rational Hodge structure. The role of the field \( F \) above is played by the CM field \( \text{End}_{\text{Hdg}}(V_\mathbb{Q}) \). In that context, each one of the eigenvectors, say, \( v_a \), belongs to a definite Hodge \((p_a, q_a)\) component.

2.3.4. The conditions With the jargons prepared in Sect. 2.3.2, and the special property of CM-type Hodge structure explained in Sect. 2.3.3, we can now translate the conditions on supersymmetric fluxes as follows. First, let

\[
H^4(Y; \mathbb{Q}) \cong \bigoplus_{a \in A} \left( H^4(Y; \mathbb{Q})_a \right)
\]

be a simple component decomposition of the rational Hodge structure of \( H^4(Y; \mathbb{Q}) \) at \( z \in \mathcal{M}_{[Y]}^{\text{cpx str}} \). We already take the second (math) perspective in Sect. 2.3.1. A flux \( G \) in \( H^4(Y; \mathbb{Q}) \)—if there is any—can be decomposed into \( \sum a \in A G_a \) with \( G_a \in \left( H^4(Y; \mathbb{Q})_a \right) \). Accordingly. Whether \( G \) satisfies the \( DW = 0 \) condition or the \( DW = W = 0 \) condition for the complex structure \( z \) can be discussed separately for individual \( G_a \)’s.

Now, the property stated in Sect. 2.3.3 implies for a CM-type simple Hodge structure in \( \left( H^4(Y; \mathbb{Q})_a \right) \) that there is a basis \( \{ v_b^{(a)} \} \) in \( \left( H^4(Y; \mathbb{C})_a \right) \) over which the action of the CM field \( \text{End}_{\text{Hdg}}(\left( H^4(Y; \mathbb{Q})_a \right)) \) is diagonalized; \( b = 1, \ldots, \dim \mathbb{Q}(\left( H^4(Y; \mathbb{Q})_a \right)) \). An intriguing property of the coefficients \( c_b^i \)’s in (14) and hence that of the basis elements \( v_b^{(a)} \)’s is that the Galois group \( \text{Gal}(\text{End}_{\text{Hdg}}(\left( H^4(Y; \mathbb{Q})_a \right))/\mathbb{Q}) \) acts on them as permutation and transitively; \( \sigma \cdot \rho_b =: \rho_{\sigma(b)} \), and \( \sigma(v_b^{(a)}) = v_{\sigma(b)}^{(a)} \). When a rational element \( G_a \in \left( H^4(Y; \mathbb{Q})_a \right) \) is decomposed in this basis, \( G_a = \sum_b v_b^{(a)} g_b^{(a)} \) with coefficients \( g_b^{(a)} \)’s in \( \overline{\mathbb{Q}} \). \( G_a \) should be invariant under the Galois group action, so we obtain \( \sigma(g_b^{(a)}) = g_{\sigma(b)}^{(a)} \). In particular, if \( G_a \) is such that \( g_b^{(a)} \neq 0 \) for some \( b \), then \( g_b^{(a)} \neq 0 \) for all \( b = 1, \ldots, \dim \mathbb{Q}(\left( H^4(Y; \mathbb{Q})_a \right)) \).

Therefore, the condition that a topological flux \( G \in H^4(Y; \mathbb{Q}) \) does not have the (1, 3) Hodge component is translated as follows:

\[
\forall G_a \in \left( H^4(Y; \mathbb{Q})_a \right) \quad \text{if} \quad \left( \left( H^4(Y; \mathbb{Q}) \right)_a \otimes \mathbb{C} \right) \cap H^{1,3} = \phi,
\]

\[
G_a = 0 \quad \text{if} \quad \left( \left( H^4(Y; \mathbb{Q})_a \right) \otimes \mathbb{C} \right) \cap H^{1,3} \neq \phi. \tag{16}
\]

In particular, if all the simple components have non-empty Hodge (1, 3) components, then only the trivial flux \( G = 0 \) is consistent with the \( DW = 0 \) condition at \( z \in \mathcal{M}_{[Y]}^{\text{cpx str}} \). Similarly, the condition that the topological flux \( G \in H^4(Y; \mathbb{Q}) \) has neither the (1, 3)-component nor (0, 4) component is translated as follows:

\[
\forall G_a \in \left( H^4(Y; \mathbb{Q})_a \right) \quad \text{if} \quad \left( \left( H^4(Y; \mathbb{Q})_a \right) \otimes \mathbb{C} \right) \cap \left( H^{1,3} \oplus H^{0,4} \right) = \phi,
\]
\[ G_a = 0 \quad \text{if} \quad \left( \left( H^4(Y_\tau : \mathbb{Q}) \right) \otimes_{\mathbb{Q}} \mathbb{C} \right) \cap \left( H^{1,3} \oplus H^{0,4} \right) \neq \phi. \quad (17) \]

The \( DW = W = 0 \) condition, and hence this last condition is further translated as follows: \( G_a = 0 \) in all the simple components with the level \( \ell > 0 \).

### 2.4. Cases with a generic CM point in \( D(T_0) \)

In Sects. 2.4 and 2.5, we work out the conditions (16, 17) for existence of a non-trivial supersymmetric flux in the \( (T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q} \) component for a vacuum complex structure in (3). It is done by translating the conditions (16, 17) into arithmetic characterizations on the vacuum complex structure. Before doing anything, however,

In Sect. 2.4, we deal with the cases where complex structure of \( X^{(1)} \) and \( X^{(2)} \) are CM-type but otherwise generic in the period domains \( D(T_0^{(1)}) \) and \( D(T_0^{(2)}) \); this means that \( T_X^{(i)} = T_0^{(i)} \). Analysis in Sects. 2.4.1 and 2.4.2 reveals that the condition (16) for a \( DW = 0 \) flux is translated to (47), and the condition (17) for a \( DW = W = 0 \) flux to (46); busy readers might choose to skip the analysis and proceed to a recap in p. 28 at the end of Sect. 2.4.2. The effective field theory (including mass matrices and symmetries) of complex structure moduli fields is studied in Sect. 2.4.3.

#### 2.4.1. Tensor product of a pair of CM-type Hodge structures

For a complex structure in (3) generic enough to have \( T_0^{(i)} = T_X^{(i)} \), the rational Hodge structure on \( V_i := T_0^{(i)} \otimes \mathbb{Q} \) is simple and CM-type (by assumption) for both \( i = 1, 2 \); let \( K^{(i)} \) denote their CM fields. It is then known [16, Prop. 1.2] that the rational Hodge structure on \( V_1 \otimes V_2 = (T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q} \) is also of CM type. The rational Hodge structure on \( (T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q} \) is not necessarily simple, however.

In fact, the non-simple nature of a rational Hodge structure of \( H^4(Y; \mathbb{Q}) \) (or of \( H^3(M; \mathbb{Q}) \)) is an essential ingredient for \( \langle W \rangle = 0 \) [22]. In Ref. [5], for example, \( M = (E_\tau \times X^{(2)}) / \mathbb{Z}_2 \) with a CM elliptic curve \( E_\tau \) and a CM-type K3 surface \( X^{(2)} \) is used for a Type IIB orientifold; the authors found \( DW = W = 0 \) fluxes by exploiting a case where the rational Hodge structure is not simple on \( V_1 \otimes V_2 \) with \( V_1 = H^1(E; \mathbb{Q}) \) and \( V_2 = T_X^{(2)} \otimes \mathbb{Q} \). We will also do so on \( V_1 \otimes V_2 = (T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q} \) in this article.

It was not difficult to work out the simple component decomposition of \( (V_1 \otimes V_2) \) in [5], when \( V_1 = H^1(T^2; \mathbb{Q}) \) is of just two-dimensions, and we know that \( K^{(i)} \) is an imaginary quadratic field. For a general \( V_1 = T_X^{(1)} \otimes \mathbb{Q} \) and \( K^{(1)} \) of a CM-type K3 surface \( X^{(1)} \), however, we need to be equipped with an understanding on general structure of the simple component decomposition of \( V_1 \otimes V_2 \). That is what we do in Sect. 2.4.1 (by exploiting [23, §5]), and we will arrive at (19, 24, 26, 35).

**Step 1:** The algebras of endomorphisms \( K^{(1)} \subset \text{End}_{\text{Hdg}}(V_1) \) and \( K^{(2)} \subset \text{End}_{\text{Hdg}}(V_2) \) give rise to an algebra of Hodge endomorphisms of the vector space \( (V_1 \otimes_{\mathbb{Q}} V_2) \); \( K^{(1)} \otimes_{\mathbb{Q}} K^{(2)} \leftarrow \text{End}_{\text{Hdg}}(V_1 \otimes_{\mathbb{Q}} V_2) \). Similarly to the fact that the rational Hodge structure on \( (V_1 \otimes_{\mathbb{Q}} V_2) \) is not necessarily simple, the algebra \( (K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}) \) of endomorphisms of \( (V_1 \otimes_{\mathbb{Q}} V_2) \) is not necessarily a field. The first step is to look at the structure of the algebra \( K^{(1)} \otimes_{\mathbb{Q}} K^{(2)} \).

First, let us explain\(^{30}\) the decomposition (19). The field extension \( K^{(1)} \) over \( \mathbb{Q} \) is always expressed in the form of \( K^{(1)} = \mathbb{Q}(\alpha) \) for some \( \alpha \in K^{(1)} \). Let \( f_{\alpha/\mathbb{Q}} \in \mathbb{Q}[x] \)

---

\(^{30}\) Any introductory textbook on field theory (such as [24,25]) will be useful in following the discussions in Sects. 2.4.1 and 2.4.2.
be a minimal polynomial of $\alpha \in K^{(1)}$ over $\mathbb{Q}$, which means that $K^{(1)} \cong \mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(f_{\alpha}/\mathbb{Q})$, and

$$K^{(2)} \otimes_{\mathbb{Q}} K^{(1)} \cong K^{(2)} \otimes_{\mathbb{Q}} \mathbb{Q}[x]/(f_{\alpha}/\mathbb{Q}) = K^{(2)}[x]/(f_{\alpha}/\mathbb{Q}).$$  \hspace{1cm} (18)

Although the minimal polynomial $f_{\alpha}/\mathbb{Q}$ is irreducible in the ring $\mathbb{Q}[x]$, it may in principle be factorizable in the ring $K^{(2)}[x]$; let $f_{\alpha}/\mathbb{Q}(x) = \prod_{i=1}^{r} g_{i}(x)$ be an irreducible factorization, where $g_{i}(x) \in K^{(2)}[x]$. The Chinese remainder theorem is used to obtain

$$(K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}) \cong K^{(2)}[x]/(f_{\alpha}/\mathbb{Q}) \cong \bigoplus_{i=1}^{r} K^{(2)}[x]/(g_{i}) =: \bigoplus_{i=1}^{r} L_{i}.  \hspace{1cm} (19)$$

The algebra $K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}$ is decomposed into a direct sum of number fields $K^{(2)}[x]/(g_{i})$; each component is a degree $[L_{i} : K^{(2)}] = \deg(g_{i})$ extension field over $K^{(2)}$.

Second, let us spell out the relation between the sets of embeddings of $K^{(1)}$ and $K^{(2)}$,

$$\Phi^{\text{full}}_{K^{(1)}} := \text{Hom}_{\mathbb{Q}}(K^{(1)}, \overline{\mathbb{Q}}) \quad \text{and} \quad \Phi^{\text{full}}_{K^{(2)}} := \text{Hom}_{\mathbb{Q}}(K^{(2)}, \overline{\mathbb{Q}}),$$  \hspace{1cm} (20)

respectively, and those of the number fields $L_{i}$; remember that the set of embeddings of the CM fields play an important role in describing a Hodge structure of CM-type (Sect. 2.3.3). The set of embeddings $\Phi^{\text{full}}_{K^{(1)}} \times \Phi^{\text{full}}_{K^{(2)}}$ of the algebra $K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}$ is decomposed into

$$\Phi^{\text{full}}_{K^{(1)}} \times \Phi^{\text{full}}_{K^{(2)}} = \bigcup_{i=1}^{r} \Phi^{\text{full}}_{L_{i}},$$

$$\Phi^{\text{full}}_{L_{i}} = \left\{ (\rho^{(1)}, \rho^{(2)}) \mid \rho^{(1)}(\alpha) \text{ is a root of } (\rho^{(2)}(g_{i}))(x) = 0 \right\}; \hspace{1cm} (21)$$

obviously individual $\Phi^{\text{full}}_{L_{i}}$’s consist of $\deg(g_{i}) \times [K^{(2)} : \mathbb{Q}]$ distinct embeddings of the number field $L_{i}$ (so the notation $\Phi^{\text{full}}_{L_{i}}$ is appropriate), and the subsets $\Phi^{\text{full}}_{L_{i}}$ for $i = 1, \ldots, r$ are mutually exclusive in $\Phi^{\text{full}}_{K^{(1)}} \times \Phi^{\text{full}}_{K^{(2)}}$, because the polynomial $f_{\alpha}/\mathbb{Q}$ is separable. Now, both the algebra $K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}$ and its set of embeddings $\Phi^{\text{full}}_{K^{(1)}} \times \Phi^{\text{full}}_{K^{(2)}}$ have decompositions, (19) and (21), respectively. The two decompositions are compatible in fact, in that the embeddings in $\Phi^{\text{full}}_{L_{i}}$ are trivial on the other direct sum components, $L_{j}$’s with $j \neq i$, as follows. As a part of the Chinese remainder theorem, there exists $a_{i} \in K^{(2)}[x]/(g_{i})$ for $i = 1, \ldots, r$ so that

$$1 = \sum_{i} a_{i} f_{i}' \in K^{(2)}[x]/(f_{\alpha}/\mathbb{Q}), \quad f_{i}' := \prod_{j \neq i} g_{j}, \hspace{1cm} (22)$$

in line with\textsuperscript{31} the decomposition $K^{(2)}[x]/(f_{\alpha}/\mathbb{Q}) \cong \bigoplus_{i} K^{(2)}[x]/(g_{i})$. So, an element in $L_{j}$ can be regarded as a polynomial in $K^{(2)}[x]$ times $a_{i} f_{j}' \mod f_{\alpha}/\mathbb{Q}$, whose image by any embedding in $\Phi_{L_{i}}^{\text{full}}$ with $i \neq j$ vanishes because $f_{j}'$ contains the factor $g_{i}$.

It is useful to note that the Galois group $\text{Gal}(((K^{(1)} K^{(2)})^{\text{nc}}/\mathbb{Q})$ acts on the set $\Phi^{\text{full}}_{K^{(1)}} \times \Phi^{\text{full}}_{K^{(2)}}$; a Galois transformation $\sigma \in \text{Gal}(((K^{(1)} K^{(2)})^{\text{nc}}/\mathbb{Q})$ converts an embedding $\rho^{(1)} \otimes$

\textsuperscript{31} memo: The Chinese remainder theorem is valid for a PID $R$, and its ideals $P_{j}$ that are mutually prime. Let $P_{1} = (p_{1})_{R}$. Then the element $a_{j} \in R/P_{j}$ satisfies $a_{j} \cdot (\prod_{j \neq i} p_{j}) = 1 \in R/(p_{1})$. The homomorphism from $R/P$ to $\oplus_{i}(R/P_{j})$ is obtained by just dividing further; divide a residue mod $\prod_{j} p_{j}$ by $p_{j}$ to find its residue mod $p_{j}$. Under the homomorphism from $\oplus_{i}(R/P_{j})$ to $R/P$, on the other hand, $(y_{1}, \ldots, y_{r}) \in \oplus R/(p_{j})$ is assigned $\sum_{j} y_{j} a_{j}(\prod_{j \neq i} p_{j}) \in R/(\prod_{k} p_{k})$. 

\[ \rho^{(2)} \in \Phi_{K^{(1)}}^{\text{full}} \times \Phi_{K^{(2)}}^{\text{full}} \] to another embedding given by \( \sigma \cdot (\rho^{(1)} \otimes \rho^{(2)}) : K^{(1)} \otimes_{\mathbb{Q}} K^{(2)} \to \mathbb{Q} \to \mathbb{Q} \). The decomposition (21) can be regarded as the orbit decomposition under this group action. So, this observation further indicates that the decomposition (21) is independent of the choice of the primitive element \( \alpha \) of \( K^{(1)} \cong \mathbb{Q}(\alpha) \).

Instead of exploiting the structure of \( K^{(1)} \) as \( \mathbb{Q}(\sqrt{a}) \), we could have exploited the same structure of \( K^{(2)} ; \; K^{(2)} \) is regarded as \( K^{(2)} \cong \mathbb{Q}(\sqrt{a'}) \) for an appropriate choice of \( \alpha' \in K^{(2)} ; \) find its minimal polynomial over \( \mathbb{Q} \), and factorize the polynomial over \( K^{(1)} \) to find another decomposition of \( K^{(1)} \otimes_{\mathbb{Q}} K^{(2)} \) into a direct sum of number fields. So, yet another decomposition of the set \( \Phi_{K^{(1)}}^{\text{full}} \times \Phi_{K^{(2)}}^{\text{full}} \) also follows. This decomposition must be the orbit decomposition of the action of \( \text{Gal}((K^{(1)} K^{(2)})^\text{nc} / \mathbb{Q}) \) on \( \Phi_{K^{(1)}}^{\text{full}} \times \Phi_{K^{(2)}}^{\text{full}} \), where the same group acts on the same set precisely in the same way as before. So, the decomposition of the embeddings should be independent of whether we exploit \( K^{(1)} \cong \mathbb{Q}(\sqrt{a}) \) or \( K^{(2)} = \mathbb{Q}(\sqrt{a'}) \), and so is the decomposition of the algebra \( K^{(1)} \otimes_{\mathbb{Q}} K^{(2)} \cong \bigoplus_{i=1}^{r} L_i \). It also follows that \( [L_i : \mathbb{Q}] \) is divisible by both \( [K^{(2)} : \mathbb{Q}] \) and \( [K^{(1)} : \mathbb{Q}] \).

Step 2: Remember that one can find a non-canonical isomorphism \( i_1 : K^{(1)} \cong V_1 \) and \( i_2 : K^{(2)} \cong V_2 \) as vector spaces over \( \mathbb{Q} \). Then an isomorphism

\[ i_1 \otimes i_2 : (K^{(1)} \otimes_{\mathbb{Q}} K^{(2)}) \cong V_1 \otimes_{\mathbb{Q}} V_2 \] (23)

combined with (19) introduces a decomposition of the vector space

\[ V_1 \otimes_{\mathbb{Q}} V_2 \cong \bigoplus_{i=1}^{r} W_i \] (24)

Individual \( W_i \)'s in \( V_1 \otimes V_2 \) are vector subspaces over \( \mathbb{Q} \); the number field \( L_i \) acts on \( W_i \), \( [L_i : \mathbb{Q}] = \dim_{\mathbb{Q}} W_i \), and each one of the simultaneous eigenstates \( v_a \in W_i \otimes_{\mathbb{Q}} (K^{(1)} K^{(2)})^\text{nc} \) of the action of \( L_i \) is in a definite Hodge \((p,q)\) component (Sect. 2.3.3), so all the elements in \( L_i \) are in \( \text{End}_{Hdg}(W_i) \). So, the decomposition (24) over \( \mathbb{Q} \) is compatible with the rational Hodge substructure, and each \( W_i \) has a rational Hodge structure of CM type.

Step 3: Independently from the decomposition (24) of \( V_1 \otimes V_2 \) that follows from the structure (19), one may also think of a simple component decomposition of a not necessarily CM-type rational Hodge structure of \( V_{\mathbb{Q}} \):

\[ V_{\mathbb{Q}} \cong \bigoplus_{a \in A} V_a \] (25)

Combining this structure (25) and the structure theorem of semi-simple algebras, one can state—as we do in the following—the structure of the entire algebra \( \text{End}_{Hdg}(V_{\mathbb{Q}}) \); as a reminder, \( K^{(1)} \otimes_{\mathbb{Q}} K^{(2)} \) is a part of \( \text{End}_{Hdg}(V_{\mathbb{Q}}) \).

For any pair of simple components \( V_a \) and \( V_b \) in (25), any \( \phi \in \text{Hom}_{Hdg}(V_a, V_b) \) is either a zero map or an invertible Hodge morphism.\(^{32}\) One can think of grouping the simple components \( \{V_a \mid a \in A\} \) into Hodge-isomorphism classes based on whether the set \( \text{Hom}_{Hdg}(V_a, V_b) \) is empty or non-empty (i.e., a Hodge isomorphism exists). The set of Hodge isomorphism classes of the simple components in \( V_{\mathbb{Q}} \) is denoted by \( A \), and one can think of the decomposition

\[ V_{\mathbb{Q}} \cong \bigoplus_{a \in A} (\bigoplus_{a \in A; [a]=a} V_a) =: \bigoplus_{a \in A} V_a \] (26)

\(^{32}\) If \( \phi \) is not surjective, then \( V_b \) has a rational Hodge substructure, which contradicts against the assumption that the rational Hodge structure on \( V_b \) is simple. If \( \phi \) has a non-trivial kernel, then implies that \( V_a \) has a rational Hodge substructure, which is a contradiction once again. So, \( \phi \) must be an isomorphism between the vector spaces \( V_a \) and \( V_b \) over \( \mathbb{Q} \).
The algebra of Hodge endomorphisms of $V_Q$ has the structure
\[ \text{End}_{\text{Hdg}}(V_Q) \cong \bigoplus_{\alpha \in \mathcal{A}} M_{n_\alpha \times n_\alpha}(D_\alpha), \quad D_\alpha = [\alpha] = \text{End}_{\text{Hdg}}(V_\alpha), \] (27)
where $n_\alpha$ is the number of simple components ($V_\alpha$’s) that fall into a given Hodge-isomorphism class $\alpha$, $[\alpha] = \alpha$. $D_\alpha$ is a division algebra over $\mathbb{Q}$ (because all the non-zero element is invertible). Therefore, $\text{End}_{\text{Hdg}}(V_Q)$ is a semi-simple algebra over $\mathbb{Q}$, and the factor $M_{n_\alpha \times n_\alpha}(D_\alpha)$ for a Hodge-isomorphism class $\alpha \in \mathcal{A}$ is a simple algebra.\(^{33}\)

Now, we can invoke a few known facts about semi-simple algebras. One is that $\text{End}_{\text{Hdg}}(V_\alpha)$ for some $\alpha \in \mathcal{A}$ is represented on this copy of the vector space $V_\alpha$.\(^{\text{full}}\)

Second, we will see how $V_\alpha$ against the fact that all the representations in $\mathcal{A}$ are non-trivial.\(^{\text{full}}\)

\[ \text{dim}_\mathbb{Q} V_\alpha = \text{dim}_\mathbb{Q} D_\alpha. \] (28)

As another fact (\cite[Thm. II.4.11]{26} or \cite[Cor.2.2.3]{27}),

\[ \text{dim}_\mathbb{Q} D_\alpha = q_\alpha^2[K_\alpha : \mathbb{Q}] \] (29)

for some $q_\alpha \in \mathbb{N}_{>0}$, where $K_\alpha$ is the centre of the division algebra $D_\alpha$.

Step 4: The general structure of $\text{End}_{\text{Hdg}}(V_Q)$ in Step 3 is for a general rational Hodge structure not necessarily of CM type, whereas the CM-type nature of the Hodge structure on $V_1 \otimes V_2$ has been exploited in Steps 1 and 2. Let us see in the following (by following \cite[§5]{23}) how the decomposition (24) is related to (26) in Step 3, and how $K^{(1)} \otimes K^{(2)}$ with the structure (19) fits into the general structure (27) of $\text{End}_{\text{Hdg}}(V_Q)$ in Step 3, when $V_Q = V_1 \otimes \mathbb{Q} V_2$.

First observation is that one $\alpha \in \mathcal{A}$ is assigned to each label $i \in \{1, \cdots, r\}$ in the decomposition (19, 24); the corresponding $\alpha$ is denoted by $\alpha(i)$. To see this correspondence, think of

\[ L_i \hookrightarrow (K^{(1)} \otimes K^{(2)}) \hookrightarrow \text{End}_{\text{Hdg}}(V_Q) \to M_{n_\alpha \times n_\alpha}(D_\alpha) \] (30)

for a given $i \in \{1, \cdots, r\}$ and an arbitrary $\alpha \in \mathcal{A}$. The image of $L_i$ must be non-trivial at least for one $\alpha \in \mathcal{A}$; now we wish to see that that is the case for only one Hodge isomorphism class $\alpha$ in $\mathcal{A}$.

Suppose that the image of $L_i$ is non-zero for $\alpha_0 \in \mathcal{A}$. Then the vector space $V_{\alpha_0}$ contains a vector subspace isomorphic to $L_i$, and the algebra $L_i \hookrightarrow M_{n_{\alpha_0} \times n_{\alpha_0}}(D_{\alpha_0})$ is represented on this copy of the vector space $L_i$ as a full set of $\Phi_{L_i}^{(\alpha)}$. If there were distinct $\alpha_0, \alpha_0' \in \mathcal{A}$ where $L_i$ is embedded non-trivially, then the set of representations $\Phi_{L_i}^{\text{full}}$ would appear more than once in $V_{\alpha_0} \oplus V_{\alpha_0} \subset V_Q = (V_1 \otimes V_2)$; that contradicts the fact that all the representations in $\Phi_{K^{(1)}}^{\text{full}} \times \Phi_{K^{(2)}}^{\text{full}}$ appear just once on $V_Q$. We have thus established a claim that there is just one $\alpha \in \mathcal{A}$ where the image of $L_i$ in $M_{n_\alpha \times n_\alpha}(D_\alpha)$ is non-trivial.

Second, we will see how $L_i$ fits into the algebra $M_{n_\alpha \times n_\alpha}(D_\alpha)$ with $\alpha = \alpha(i)$ by exploiting the CM nature of the Hodge structure on $V_\alpha$. The following argument (built on Step 3) is almost\(^{34}\) a copy of the logic of §5 of \cite{23}.

For a given $\alpha \in \mathcal{A}$, now consider a set of the label $i$’s in $\{1, \cdots, r\}$ with $\alpha(i) = \alpha$. Due to the CM nature, the relation

\[ \sum_{i \text{ s.t. } (\alpha(i) = \alpha)} [L_i : \mathbb{Q}] = \text{dim}_\mathbb{Q} V_\alpha \] (31)

\(^{33}\) It is a simple algebra in the sense that it does not have a non-trivial two-sided ideal.

\(^{34}\) The original version \cite[§5]{23} is for $V_Q = H^1(A; \mathbb{Q})$ for an abelian variety $A$. cf \cite{28,29} for $V_Q = T_X$ of a K3 surface $X$. 
implies that
\[
\dim_{\mathbb{Q}} V_{\alpha} = n_{\alpha} \dim_{\mathbb{Q}} V_{\alpha ([a] = \alpha)} = n_{\alpha} q_{\alpha}^2 [K_{\alpha} : \mathbb{Q}].
\] (32)

On the other hand, the algebra
\[
L' = \left( \bigoplus_{i \text{ s.t.}}^{(\alpha(i) = \alpha)} L_i \right) \cdot (K_{\alpha} 1_{n_\alpha \times n_\alpha}) \subset M_{n_\alpha \times n_\alpha} (D_{\alpha})
\] (33)

remains to be a commutative subalgebra, and any commutative subalgebra of a central simple algebra \( M_{n_\alpha \times n_\alpha} (D_{\alpha}) \) is bounded in its dimension by
\[
\sum_{i \text{ s.t.}} [L_i : \mathbb{Q}] \leq \dim_{\mathbb{Q}} L' \leq n_{\alpha} \times q_{\alpha} \times [K_{\alpha} : \mathbb{Q}].
\] (34)

So, by combining (31, 32) against (34), we can see that \( q_{\alpha} = 1 \) (which means that \( D_{\alpha} = K_{\alpha} \)), and also that \( K_{\alpha} 1_{n_\alpha \times n_\alpha} \) is contained in \( \bigoplus_{i : (\alpha(i) = \alpha)} L_i \). The latter statement further indicates\(^{35}\) that those \( L_i \)'s can be regarded as an extension of \( K_{\alpha(i)} \). For the field \( L_i \) to be a non-trivial extension of \( K_{\alpha(i)} \), at least some of the endomorphisms in \( L_i \subset M_{n_\alpha \times n_\alpha} (K_{\alpha}) \) must mix multiple different simple components \( V_{\alpha} \) with \([a] = \alpha\).

To summarize,
\[
V_{\alpha} \cong \bigoplus_{i \text{ s.t.}}^{(\alpha(i) = \alpha)} W_i.
\] (35)

\( K_{\alpha} \) is the endomorphism field of the CM-type simple rational Hodge structure of \( V_{\alpha} \) (such that \([a] = \alpha\)), \( L_i \) is an extension of \( K_{\alpha(i)} \), and \( \bigoplus_{i \text{ s.t.}}^{(\alpha(i) = \alpha)} L_i \hookrightarrow M_{n_\alpha \times n_\alpha} (K_{\alpha}) = \operatorname{End}_{\text{Hdg}}(V_{\alpha}). \)

(Remark) The vector space \( W_i \otimes \overline{\mathbb{Q}} \) has a well-motivated basis; basis elements are in one-to-one with the embeddings \( \rho^{(1)} \otimes \rho^{(2)} \) in \( \Phi_{\text{full}}^{L_i} \). This is just a special case of Sect. 2.3.3 with \( F = L_i \) and \( V_{\overline{\mathbb{Q}}} = W_i \). Each one of the basis elements are also associated with a particular Hodge \((p,q)\) type, so each embedding \( \rho^{(1)} \otimes \rho^{(2)} \) of \( L_i \) has its corresponding Hodge type \((p,q)\). This correspondence will be exploited in the following analysis.

2.4.2. \( DW = 0 \) flux and \( DW = W = 0 \) flux, assuming \( T_X = T_0 \)

Toward the end of Sect. 2.3, we used the language of the simple Hodge component decomposition to write down the conditions for the presence of a non-trivial supersymmetric flux. Whether a non-trivial flux with those conditions exists or not can be studied for individual simple rational Hodge components. For simple Hodge components that are mutually Hodge-isomorphic, say, \( \Phi : V_{\alpha} \cong V_{\beta}, [a] = [b] = \alpha \in \mathcal{A}, \)
\[
h^{p,q}[V_{\alpha}] = h^{p,q}[V_{\beta}]
\] (36)

holds for all \((p,q)\). We can thus talk of the level of individual Hodge-isomorphism classes, \( \alpha \in \mathcal{A} \), and we can also study whether fluxes with \( DW = 0 \) and/or \( W = 0 \) exists for individual Hodge-isomorphism classes.

- In a \( V_{\alpha} \) of level 0, there are only Hodge \((2,2)\) components (by definition). Any rational flux here satisfies both of the \( DW = 0 \) and \( W = 0 \) conditions.

---

\(^{35}\) This is because \( K_{\alpha} 1 \ni 1 = \sum_i \epsilon_i \in \otimes_i L_i; K_{\alpha} 1 \cdot (0, \ldots, \epsilon_i, \ldots, 0) \subset L_i \subset \otimes_j L_j \) is a subfield of \( L_i \) and is isomorphic to \( K_{\alpha} \).
• In a $V_q$ of level 2, any non-zero rational flux breaks the $DW = 0$ condition (although $W = 0$ would be satisfied).
• There is just one level-4 simple rational Hodge component of $H^4(Y; \mathbb{Q})$ of a Calabi–Yau fourfold $Y$, so this simple component alone forms one Hodge–isomorphism class of the simple components in $H^4(Y; \mathbb{Q})$. This simple component admits a rational flux with $DW = 0$ if and only if $h^{3,1} = 0$ holds in this simple component. Let us say that a simple component is $(3,1)$-free if the component has $h^{3,1} = 0$. Even when this condition is satisfied, such a flux does not satisfy the $W = 0$ condition.

Let us continue to focus on a Borcea–Voisin orbifold $Y = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$ of a pair of CM-type K3 surfaces with $T_X^{(1)} = T_0^{(1)}$ and $T_X^{(2)} = T_0^{(2)}$. We have seen in Sect. 2.4.1 that the Hodge structure on $K$ has the decomposition (24), which is compatible with the Hodge–isomorphism-class decomposition (26), although (24) may be a finer classification than (26). Therefore, we can rephrase the criteria for the existence of non-trivial supersymmetric fluxes, which is stated above, by simply replacing Hodge–isomorphism classes of simple components by individual components $W_i$ in (24).

In the decomposition (24) of the Hodge structure on $V_1 \otimes \mathbb{Q} V_2$, the individual components $W_i$ are either level-4, level-2, or level-0. We will see, first, that there are at most two $W_i$’s that are not level-2 (so, a $DW = 0$ flux is possible only in those at most two $W_i$’s); this is the Step 1 below. In Step 2, we work out the conditions on the CM fields $K^{(1)}$ and $K^{(2)}$ for those one or two component(s) to be $(3,1)$ free, so that a $DW = 0$ flux is indeed available. A physics recap (Step 3) comes at the end of this Sect. 2.4.2.

Step 1: To show that there are at most two $W_i$’s, let us introduce some notations. We denote the extension degrees of $K^{(1)}$ and $K^{(2)}$ over $\mathbb{Q}$ by $n_1 := [K^{(1)} : \mathbb{Q}]$ and $n_2 := [K^{(2)} : \mathbb{Q}]$, respectively. The embeddings of $K^{(i)}$ with $i = 1, 2$ are denoted by $\text{Hom}(K^{(i)}, \mathbb{Q}) = \{\rho^{(i)}(20), \rho^{(i)}(02), \rho^{(i)}_3, \ldots, \rho^{(i)}_{n_i}\}$, where $\rho^{(i)}_{20}$ and $\rho^{(i)}_{02}$ correspond to the $(2,0)$ component and $(0,2)$ component of $H^2(X^{(i)})$, respectively, in the sense of a remark at the end of Sect. 2.4.1;

the action of $x \in K^{(i)}$ on the $(2,0)$-form $\Omega^{(i)}_X$ of $X^{(i)}$ is $x : \Omega^{(i)}_X \mapsto \rho^{(i)}_{20}(x) \cdot \Omega^{(i)}_X$ for any $x \in K^{(i)}$. Let us denote by $L_{(20)}$ the number field $L_{i}$ for which $\Phi^{\text{full}}_{L_i}$ contains $\rho^{(1)}_{20} \otimes \rho^{(2)}_{20}$, and by $L_{(02)}$ the number field $L_{j}$ for which $\Phi^{\text{full}}_{L_j}$ contains $\rho^{(1)}_{20} \otimes \rho^{(2)}_{02}$. For those $i$ and $j$, the vector spaces $W_i$ and $W_j$ are denoted by $W_{(20)}$ and $W_{(02)}$, respectively. Note that both $i \neq j$ and $i = j$ are possible. We claim that these (at most) two components have a chance to be different from level-2, and that all other $W_k$’s in (24) are level-2.

Obviously, $W_{(20)}$ is always the unique level-4 component. $\Phi^{\text{full}}_{L_{(20)}}$ contains $\rho^{(1)}_{20} \otimes \rho^{(2)}_{20}$.

To see that all other $W_k$’s except $W_{(02)}$ are level-2, note first that every Hodge $(3,1)$ component in $(V_1 \otimes V_2) \otimes \mathbb{C}$ corresponds to an embedding of the form

\[\rho^{(1)}_{20} \otimes \rho^{(2)}_b\] with $3 \leq b \leq n_2$ \hspace{1cm} (37)
\[\rho^{(1)}_a \otimes \rho^{(2)}_{20}\] with $3 \leq a \leq n_1$. \hspace{1cm} (38)

36 Because of (35).
because a $(3, 1)$-form in $(V_1 \otimes V_2) \otimes \mathbb{C}$ is always a product of a $(2, 0)$-form in $V_1 \otimes \mathbb{C}$ and a $(1, 1)$-form in $V_2 \otimes \mathbb{C}$, or vice versa. On the other hand, each set of embeddings $\Phi_{full}^L_k$ contains at least one element of the form $\rho_{\beta(20)}^{(1)} \otimes \rho_{\beta(2)}^{(2)}$ for some $\beta$ in $(20), (02), 3, \ldots, n_2$, because $\Phi_{full}$ forms an orbit under the Galois group action. So, $\Phi_{full}^L_k$ for $k \neq (20, 20), (20, 02)$ contains $\rho_{\beta(20)}^{(1)} \otimes \rho_{\beta(2)}^{(2)}$ with $\beta \in \{3, \ldots, n_2\}$, and the corresponding $W_k$ is of level 2. We conclude that a $DW = 0$ flux is not impossible only within $W_{(20, 20)}$ and $W_{(20, 02)}$.

Step 2: Now let us work out the conditions for non-trivial fluxes to exist in $W_{(20, 20)}$ and $W_{(20, 02)}$ in terms of the CM fields $K^{(1)}, K^{(2)}$, and their actions on $T^{(1)}_X$ and $T^{(2)}_X$. The analysis will take several pages, but the conclusion can be summarized quite simply; a non-trivial flux with $DW = 0$ exists if and only if equation (47) is satisfied. A stronger condition (46) is necessary and sufficient for a non-trivial $DW = W = 0$ flux.

We first study the level-4 component $W_{(20, 20)}$. Recall that a non-trivial flux in a level-4 component preserves the $DW = 0$ condition if and only if the component is $(3, 1)$-free, i.e. free of Hodge $(3, 1)$ components.\(^{37}\) We are thus interested in when the component is $(3, 1)$-free. Since we know all the elements in $\Phi_{full}^{K^{(1)}} \times \Phi_{full}^{K^{(2)}}$ that correspond to Hodge $(3, 1)$ components, (37) and (38), our task reduces to finding out whether or not $\Phi_{full}^{L_{(20, 20)}}$ contains such embeddings. This is equivalent to working out whether or not there exists an action of $\text{Gal}((K^{(1)} K^{(2)})_{nc}/\mathbb{Q})$ that maps $\rho_{\beta(20)}^{(1)} \otimes \rho_{\beta(20)}^{(2)}$ to one of such embeddings that correspond to $(3, 1)$ components, since $\Phi_{full}^{L_{(20, 20)}}$ is generated by $\text{Gal}((K^{(1)} K^{(2)})_{nc}/\mathbb{Q})$ acting on $\rho_{\beta(20)}^{(1)} \otimes \rho_{\beta(20)}^{(2)}$ (see p. 18). Such a map must be contained in $G^{(1)}_{(20)} := \text{Gal}((K^{(1)} K^{(2)})_{nc}/\rho_{\beta(20)}^{(1)}(K^{(1)}))$ or $G^{(2)}_{(20)} := \text{Gal}((K^{(1)} K^{(2)})_{nc}/\rho_{\beta(20)}^{(2)}(K^{(2)}))$, since either $\rho_{\beta(20)}^{(1)}$ or $\rho_{\beta(20)}^{(2)}$ must be held fixed by the map. Thus the component $W_{(20, 20)}$ is $(3, 1)$-free, if and only if the following two conditions are satisfied simultaneously:

(i) There is no element $\sigma^{(1)} \in G^{(1)}_{(20)}$ such that $\sigma^{(1)} \circ \left(\rho_{\beta(20)}^{(1)} \otimes \rho_{\beta(20)}^{(2)}\right) = \rho_{\beta(20)}^{(1)} \otimes \rho_{\beta(20)}^{(2)}$, for any $3 \leq b \leq n_2$.

(ii) There is no element $\sigma^{(2)} \in G^{(2)}_{(20)}$ such that $\sigma^{(2)} \circ \left(\rho_{\beta(20)}^{(1)} \otimes \rho_{\beta(20)}^{(2)}\right) = \rho_{\beta(20)}^{(1)} \otimes \rho_{\beta(20)}^{(2)}$, for any $3 \leq a \leq n_1$.

Let us work out in turn when each one of these conditions is satisfied. We first focus on the condition (i). We define $N_1$ to be the extension degree\(^{38}\) $N_1 := [L_{(20, 20)} : K^{(1)}]$. There are $N_1 - 1$ non-trivial actions in $G^{(1)}_{(20)}$, which map $\rho_{\beta(20)}^{(1)} \otimes \rho_{\beta(20)}^{(2)}$ to $\rho_{\beta(20)}^{(1)} \otimes \rho_{\beta(20)}^{(2)}$ for some $\beta = (02), 3, \ldots, n_2$. Except for the one element that maps to $\beta = (02)$, which may or may not exist, each one of them will violate the condition (i). We thus immediately conclude that the condition (i) is violated whenever $N_1 > 2$.

There are two ways to satisfy the condition (i) when $N_1 \leq 2$, i.e. $W_{(20, 20)}$ does not contain Hodge $(3, 1)$ components of the form (37):

\(^{37}\) Note that any non-trivial flux in a level-4 component violates the $W = 0$ condition.

\(^{38}\) Strictly speaking, $L_{(20, 20)}$ is defined to be an abstract extension field of $K^{(2)}$, $L_{(20, 20)} = K^{(2)}[x]/g(x)$ for some $g \in K^{(2)}[x]$ and the endomorphism field $K^{(1)}$ is not a subfield of it, so the extension degree $[L_{(20, 20)} : K^{(1)}]$ does not make sense. However, since we know that $L_{(20, 20)}$ is isomorphic to $K^{(1)}[x]/h(x)$ with some $h \in K^{(1)}[x]$, we abuse the notation and define $[L_{(20, 20)} : K^{(1)}] := [\psi(L_{(20, 20)}) : K^{(1)}]$ with an isomorphism $\psi : K^{(2)}[x]/g(x) \rightarrow K^{(1)}[x]/h(x)$.}
(i-1) When $N_1 = 1$, the condition (i) is always satisfied. This is because there are no non-trivial action in $G^{(1)}_{(20)}$. Note that this also means that $W_{(20)|20} \neq W_{(20)|02}$ in this case, because $\Phi^\text{full}_{L_{(20)20}}$ does not contain $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}$.

(i-2) When $N_1 = 2$ and $\Phi^\text{full}_{L_{(20)20}}$ contains $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}$, then the condition (i) is satisfied. This is because the only non-trivial action of $G^{(1)}_{(20)}$ maps $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}$ to $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}$. At the same time this means that $W_{(20)|20} = W_{(20)|02}$.

Note that, even when $N_1 = 2$, if $\Phi^\text{full}_{L_{(20)20}}$ does not contain $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}$, the condition (i) is violated; the only non-trivial element in $G^{(1)}_{(20)}$ will map $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}$ to an embedding of the form (37).

Similarly, defining $N_2 := [L_{(20)20} : K^{(2)}]$, one can argue that there are only two ways to satisfy the condition (ii), i.e. $W_{(20)|20}$ does not contain Hodge (3, 1) components of the form (38):

(ii-1) When $N_2 = 1$, the condition (ii) is satisfied, since there are no non-trivial action in $G^{(2)}_{(20)}$. This means that $W_{(20)|20} \neq W_{(20)|02}$ in this case, because $\Phi^\text{full}_{L_{(20)20}}$ does not contain $\rho^{(1)}_{(02)} \otimes \rho^{(2)}_{(20)}$, which is the complex conjugate of $\rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}$ and must be contained in $\Phi^\text{full}_{L_{(20)02}}$.

(ii-2) When $N_2 = 2$ and $\Phi^\text{full}_{L_{(20)20}}$ contains $\rho^{(1)}_{(02)} \otimes \rho^{(2)}_{(20)}$, then the condition (ii) is again satisfied. At the same time this means that $W_{(20)|20} = W_{(20)|02}$.

Now we are ready to see when the conditions (i) and (ii) are simultaneously satisfied. In order to satisfy both (i) and (ii), there seems to be four choices for not having a Hodge (3, 1) component in $W_{(20)|20}$, i.e. two choices for the condition (i) and another pair of choices for the condition (ii). However, two of them, (i-1)-(ii-2) and (i-2)-(ii-1) cannot happen; (i-1) or (ii-1) imply $W_{(20)|20} \neq W_{(20)|02}$, whereas (i-2) or (ii-2) imply $W_{(20)|20} = W_{(20)|02}$, thus contradiction.

In summary, there are two cases, (i-1)-(ii-1) and (i-2)-(ii-2), where the level-4 component $W_{(20)|20}$ is (3, 1)-free. For these two cases, let us leave the list of embeddings in $\Phi^\text{full}_{L_{(20)20}}$ for clarification, and also rephrase these conditions on the $\text{Gal}((K^{(1)}_{1} K^{(2)}_{2})_{nc} / \mathbb{Q})$ action in terms of $K^{(1)}$, $K^{(2)}$ and their actions.

• Firstly, let us choose (i-2) and (ii-2), i.e. $[L_{(20)20} : K^{(1)}] = [L_{(20)20} : K^{(2)}] = 2$ (so, $n_1 = n_2 = n$) and $W_{(20)|20} = W_{(20)|02}$, to satisfy the conditions (i) and (ii). The contents of $\Phi^\text{full}_{L_{(20)20}}$ are

\[
\Phi^\text{full}_{L_{(20)20}} = \left\{ \rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(20)}, \rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}, \rho^{(1)}_{(02)} \otimes \rho^{(2)}_{(20)}, \rho^{(1)}_{(02)} \otimes \rho^{(2)}_{(02)} \right\} \\
\cup \left\{ \rho^a_{(1)} \otimes \rho^b_{(2)} \mid 3 \leq a \leq n, 3 \leq b \leq n, a, b \text{ appearing twice} \right\}.
\]

(39)

There are $4 + 2 \times (n - 2) = 2n$ embeddings of $L_{(20)20} = L_{(20)02}$; $2n - 2$ embeddings among them correspond to Hodge (2, 2) components, and the other two are the (4,0) and (0,4) Hodge components; indeed there are no Hodge (3,1) or (1,3) components. This case turns out to happen if and only if

\[
\rho^{(1)}_{(20)}(K^{(1)}_{0}) = \rho^{(2)}_{(20)}(K^{(2)}_{0}) \subseteq \mathbb{Q} \quad \text{and} \quad \rho^{(1)}_{(20)}(K^{(1)}) \neq \rho^{(2)}_{(20)}(K^{(2)}),
\]

(40)
where \( K_0^{(1)} \) and \( K_0^{(2)} \) are the maximal totally real subfields of \( K^{(1)} \) and \( K^{(2)} \), respectively.39

- Alternatively, we can choose (i-1) and (ii-1) to satisfy the conditions (i) and (ii). In this case, \([L_{(20)(20)} : K^{(1)}] = [L_{(20)(20)} : K^{(2)}] = 1\) and \(W_{(20)(20)} \neq W_{(20)(02)}\). This happens if and only if

\[
\rho_{(20)}^{(1)}(K^{(1)}) = \rho_{(20)}^{(2)}(K^{(2)}) \subset \mathbb{Q}.
\]

The contents of \( \Phi_{\text{full}}^{(L_{(20)(20)})} \) are

\[
\Phi_{\text{full}}^{(L_{(20)(20)})} = \left\{ \rho_{(20)}^{(1)} \otimes \rho_{(20)}^{(2)}, \rho_{(02)}^{(1)} \otimes \rho_{(02)}^{(2)} \right\} \cup \left\{ \rho_{a}^{(1)} \otimes \rho_{b}^{(2)} \mid 3 \leq a \leq n_1, 3 \leq b \leq n_2, a, b \text{ appearing once} \right\},
\]

and there are \(2 + (n - 2) = n\) embeddings of \(L_{(20)(20)}\); two of them correspond to the Hodge \((4, 0)\) and \((0, 4)\) components, and the rest correspond to \((2, 2)\) components. There are no \((3, 1)\) or \((1, 3)\) components.

Let us move on to the \(W_{(20)(02)}\) component. This component is level-4 when \(W_{(20)(20)} = W_{(20)(02)}\), and is level-0 or level-2 otherwise. We are interested in how \(K^{(1)}, K^{(2)}\) and their actions on \(T_X^{(1)}\) and \(T_X^{(2)}\) controls whether this component is \((3, 1)\)-free, especially whether it is level-0, or not. Almost the same analysis as above can be carried out, and there turn out to be only two cases where the component becomes \((3, 1)\)-free:

- The first case is where \([L_{(20)(02)} : K^{(1)}] = [L_{(20)(02)} : K^{(2)}] = 2\) and \(W_{(20)(20)} = W_{(20)(02)}\) holds. The component is level-4, and this case has already been considered in the analysis of \(W_{(20)(20)}\) as the (i-2)-(ii-2) case.

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39 Let us formally state the claim and prove it here. The claim is that if and only if \([L_{(20)(20)} : K^{(2)}] = 2\) and \(\Phi_{\text{full}}^{(L_{(20)(20)})}\) contains both \(\rho_{(20)}^{(1)} \otimes \rho_{(20)}^{(2)}\) and \(\rho_{(02)}^{(1)} \otimes \rho_{(02)}^{(2)}\), then \(\rho_{(20)}^{(1)}(K_0^{(1)}) = \rho_{(20)}^{(2)}(K_0^{(2)})\) and \(\rho_{(20)}^{(1)}(K^{(1)}) \neq \rho_{(20)}^{(2)}(K^{(2)})\).

Recall that \(L_{(20)(20)} = K^{(2)}[x]/g(x)\), where \(g \in K^{(2)}[x]\) and is a degree-2 polynomial. The two roots of \(\rho_{(20)}^{(2)}(g(x))\), \(\alpha_+\) and \(\alpha_-\), must correspond to a simple generator \(\alpha\) of \(K^{(1)}\), i.e. \(K^{(1)} = \mathbb{Q}(\alpha)\), such that \(\alpha_+ = \rho_{(20)}^{(1)}(\alpha)\) and \(\alpha_- = \rho_{(02)}^{(1)}(\alpha)\). This means that \(\alpha_+\) and \(\alpha_-\) are complex conjugate to each other. Let us explicitly define \(g(x) = x^2 + a_1 x + a_0\) with \(a_1, a_0 \in \mathbb{K}^{(2)}\). Then from the explicit form of the roots, one can conclude that \(a_0, a_1 \in \mathbb{R}\). This implies that \(\mathbb{Q}(\alpha_+), \mathbb{Q}(\alpha_-)\) are degree-2 extensions of a totally real field \(\rho_{(20)}^{(2)}(K_0^{(2)})\), which must equal to \(\rho_{(1)}^{(1)}(K_0^{(1)})\). Note that \(\rho_{(20)}^{(2)}(K^{(2)}) \neq \rho_{(20)}^{(1)}(K^{(1)})\) because \(\alpha_+ \neq \rho_{(20)}^{(2)}(K^{(2)})\).

Conversely, let us assume \(\rho_{(20)}^{(2)}(K_0^{(2)}) = \rho_{(20)}^{(1)}(K_0^{(1)})\) and \(\rho_{(20)}^{(2)}(K^{(2)}) \neq \rho_{(20)}^{(1)}(K^{(1)})\). Denoting the totally real field \(\rho_{(20)}^{(2)}(K_0^{(2)}) = \rho_{(20)}^{(1)}(K_0^{(1)})\) by \(K_0\), the composite field \(\rho_{(20)}^{(1)}(K^{(1)})\) can be rewritten as \(\rho_{(20)}^{(1)}(K^{(1)}) = \rho_{(20)}^{(1)}(K_0^{(1)})\rho_{(20)}^{(2)}(K^{(2)})\), which is the Galois action that maps \(\eta^{(1)}\) to its complex conjugate and leaves everything else will map \(\rho_{(20)}^{(1)}(K_0^{(1)})\) to \(\rho_{(20)}^{(2)}(K^{(2)})\), so the latter is also contained in \(\Phi_{\text{full}}^{(L_{(20)(20)})}\). One can also see that \([\rho_{(20)}^{(1)}(K^{(1)})\rho_{(20)}^{(2)}(K^{(2)}) : \rho_{(20)}^{(2)}(K^{(2)})] = 2\), so \([L_{(20)(20)} : K^{(2)}] = 2\).
The component is also \((3, 1)\)-free when \( [L_{(20|02)} : K^{(1)}] = [L_{(20|02)} : K^{(2)}] = 1 \) holds. This is equivalent to
\[
\rho^{(1)}_{(20)}(K^{(1)}) = \rho^{(2)}_{(02)}(K^{(2)}) \subseteq \overline{\mathbb{Q}} \tag{43}
\]
and in this case, \( W_{(20|02)} \neq W_{(20|20)} \) holds, which means that the \( W_{(20|02)} \) component is level-0. The contents of \( \Phi_{L_{(20|02)}}^{\text{full}} \) are
\[
\Phi_{L_{(20|02)}}^{\text{full}} = \left\{ \rho^{(1)}_{(20)} \otimes \rho^{(2)}_{(02)}, \rho^{(1)}_{(02)} \otimes \rho^{(2)}_{(20)} \right\} \\
\cup \left\{ \rho^{(1)}_{a} \otimes \rho^{(2)}_{b} \mid 3 \leq a \leq n_1, \ 3 \leq b \leq n_2, \ a, b \text{ appearing once} \right\}, \tag{44}
\]
and all the embeddings are associated with Hodge \((2, 2)\) components, and thus the component \( W_{(20|02)} \) is indeed level-0. Note that \( \rho^{(1)}_{(20)}(K^{(1)}) = \rho^{(2)}_{(02)}(K^{(2)}) \) is equivalent to \( \rho^{(1)}_{(20)}(K^{(1)}) = \rho^{(2)}_{(20)}(K^{(2)}) \), since \( \rho^{(2)}_{(02)}(K^{(2)}) \) is the complex conjugate of \( \rho^{(2)}_{(20)}(K^{(2)}) \), which is \( \rho^{(2)}_{(20)}(K^{(2)}) \) itself because it is a CM field. This means that, when \( W_{(20|02)} \neq W_{(20|20)}, W_{(20|20)} \) is \((3,1)\)-free if and only if \( W_{(20|02)} \) is level-0.

The analysis of the Hodge structures of \( W_{(20|20)} \) and \( W_{(20|02)} \) in the last pages can be summarized in a very simple way:

A). When\(^{40}\)
\[
\rho^{(1)}_{(20)}(K^{(1)}_0) = \rho^{(2)}_{(20)}(K^{(2)}_0) \subset \overline{\mathbb{Q}} \quad \text{and} \quad \rho^{(1)}_{(20)}(K^{(1)}_0) \neq \rho^{(2)}_{(20)}(K^{(2)}), \tag{45}
\]
the component \( W_{(20|20)} = W_{(20|02)} \) in (24) is level-4 and \((3, 1)\)-free. All other \( W_i \) components in (24) are level-2.

B). When
\[
\rho^{(1)}_{(20)}(K^{(1)}_0) = \rho^{(2)}_{(20)}(K^{(2)}) \subset \overline{\mathbb{Q}}, \tag{46}
\]
then there are two components in (24) that are not level-2: \( W_{(20|20)} \) is level-4 and \((3, 1)\)-free, and \( W_{(20|02)} \) is level-0. All other components are level-2.

C). When neither (45) nor (46) is satisfied, none of the component \( W_i \) in (24) is \((3,1)\)-free.

Before getting into the recap of the physical consequences, it is worth while to take a slightly different view on cases A) and B): A non-trivial flux satisfying \( DW = 0 \) condition is possible only when the condition (45) or (46) is satisfied. Noticing that \( \rho^{(1)}_{(20)}(K^{(1)}) = \rho^{(2)}_{(20)}(K^{(2)}) \) implies \( \rho^{(1)}_{(20)}(K^{(1)}_0) = \rho^{(2)}_{(20)}(K^{(2)}_0) \), it is clear that such flux configurations exist if and only if
\[
\rho^{(1)}_{(20)}(K^{(1)}_0) = \rho^{(2)}_{(20)}(K^{(2)}_0) =: K_0 \tag{47}
\]
holds. Let us call the situation case A+B), since it combines the cases A) and B). Introducing some generators \( \eta^{(1)}, \eta^{(2)} \in \overline{\mathbb{Q}} \) such that \( K_0(\eta^{(1)}) = \rho^{(1)}_{(20)}(K^{(1)}) \) and \( K_0(\eta^{(2)}) = \rho^{(2)}_{(20)}(K^{(2)}) \), one can see that the case B) is a non-generic situation where

\(^{40}\) Let us remind ourselves that \( K_0^{(i)} \) is defined to be the maximal totally real subfield of \( K^{(i)} \) for \( i = 1, 2 \).
$K_0(\eta^{(1)})$ coincides with $K_0(\eta^{(2)})$, and the case A is the generic situation complementary to it. The condition (16) for a $DW = 0$ flux has been translated into an arithmetic characterization (47), and a stronger condition (17) for a $DW = W = 0$ flux into a stronger characterization (46).

Step 3 (a physics recap): Now let us discuss the physical consequences, although most of what follows is included in the discussion so far. As a first physical consequence, one can see that there is no topological flux satisfying the $DW = 0$ condition, if $n_1 := [K^{(1)} : \mathbb{Q}]$ is not equal to $n_2 := [K^{(2)} : \mathbb{Q}]$; as we have been assuming $T_X^{(1)} = T_0^{(1)}$ and $T_X^{(2)} = T_0^{(2)}$ in this Sect. 2.4.2, this condition is equivalent to rank$(T_0^{(1)}) = \text{rank}(T_0^{(2)})$. Furthermore, a non-trivial supersymmetric flux exists only in either one of these:

- In case A), with the condition (45), the component $W_{(20|20)} = W_{(20|02)}$ is level-4 and a $2 \times (n = n_1 = n_2)$-dimensional subspace of the $n^2$-dimensional vector space $V_1 \otimes \mathbb{Q} V_2$. Any flux in this component satisfies the $DW = 0$ condition but always violates the $W = 0$ condition.
- In case B), with the condition (46), $W_{(20|02)}$ is an $n$-dimensional subspace of $V_1 \otimes \mathbb{Q} V_2$ and is level-0. One has $n$-dimensional degrees of freedom to turn on the flux in this component without violating $DW = 0$ or $W = 0$ conditions. Another $n$-dimensional subspace $W_{(20|20)}$ is also free of $(3, 1)$-components, but since it contains the $(4, 0)$ component by definition, turning on any flux in this component violates the $W = 0$ condition. In summary, non-trivial flux vacua with $DW = 0$ and $W = 0$ is possible, if and only if (46) is satisfied.

As a reminder, we did not study arithmetic characterization for the $H^1(Z_{(1)}; \mathbb{Q}) \otimes H^1(Z_{(2)}; \mathbb{Q})$ component of (7) to support a $DW = 0$ flux ($W = 0$ is automatic).

The conclusion above is similar to, and also a generalization of the study by Aspinwall–Kallosh [30]. They chose the pair of K3 surfaces $X^{(1)}$ and $X^{(2)}$ to be attractive, that is, rank$(T_X^{(1)}) = \text{rank}(T_X^{(2)}) = 2$, and studied topological fluxes satisfying the $DW = 0$ condition as well as ones satisfying both of the $DW = 0$ and $W = 0$ conditions. Note that attractive K3 surfaces are always of CM-type with endomorphism fields being imaginary quadratic fields. The condition (47) follows immediately from their set-up because $K_0^{(1)} = K_0^{(2)} = \mathbb{Q}$ with $K^{(1)} = \mathbb{Q}(\sqrt{-d_1})$, $K^{(2)} = \mathbb{Q}(\sqrt{-d_2})$ in this case. The condition (47) for non-trivial fluxes with $DW = 0$ is regarded as a generalization of the rank$(T_X^{(1)}) = \text{rank}(T_X^{(2)}) = 2$ setup in [30]. For fluxes with $DW = W = 0$ to exist, [30] concluded that $K^{(1)} = \mathbb{Q}(\sqrt{-d_1})$ should be isomorphic to $K^{(2)} = \mathbb{Q}(\sqrt{-d_2})$; we have seen that this condition should be generalized to (46). See also footnote 56 in Sect. 2.5.

2.4.3. Complex structure moduli masses with $W = 0$ Now that we have worked out the conditions for non-trivial supersymmetric flux to exist in terms of arithmetic of the endomorphisms fields $K^{(1)}$ and $K^{(2)}$, let us move on to see whether such fluxes generate mass of complex structure moduli of M-theory compactification on $Y = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$. In this Sect. 2.4.3, we assume that the vacuum complex structure of the pair of

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41 References [30, 31, 92] considered compactification by $Y = X^{(1)} \times X^{(2)}$, but since they did not take an orbifold, their set-up is different from the one in this article. When it comes to the study of supersymmetric fluxes within $(T_X^{(1)} \otimes T_X^{(2)}) \otimes \mathbb{Q}$, however, cases in [30, 31] can be regarded as a special case of the study in this section. The scope of [92] is not limited to rank$(T_X^{(1)}) = 2$ or vacuum complex structure within $\mathcal{M}_CM^{(T_X^{(1)})} \times \mathcal{M}_CM^{(T_X^{(2)})}$, on the other hand.
K3 surfaces $X^{(1)}$ and $X^{(2)}$ are of CM-type, generic enough in $D(T_0^{(i)})$ so that $T_X^{(i)} = T_0^{(i)}$, and satisfy the condition (46); low-energy effective field theory (including the mass matrix) of the fluctuation fields around the vacuum complex structure is studied in the following.\footnote{We restrict our attention to the fields of complex structure deformation within $\mathcal{M}_{\text{cpx str}}^{[Y]BV}$, not to the full deformation in $\mathcal{M}_{\text{cpx str}}^{[Y]}$. Nothing is lost when $g_{(1)}g_{(2)} = 0$, because there is no complex structure moduli deforming away from the orbifold limit then. For cases $g_{(1)}g_{(2)} \neq 0$, we do not have something to add to what we have already written in Sect. 2.2.}

At the vacuum, a holomorphic (2,0) form $\Omega_X^{(i)}$ of the K3 surface $X^{(i)}$ can be chosen to be $\nu^{(i)}_{(20)}$ (by choosing the normalization of $\Omega_X^{(i)}$). Over the moduli space $\mathcal{M}_{\text{cpx str}}^{[X(T_0^{(i)})]}$ around the vacuum $\{z^{(i)}\}$, the 2-form $\Omega_X^{(i)}(z^{(i)})$ that is holomorphic and purely of Hodge (2,0)-type in the complex structure of $z^{(i)} \in \mathcal{M}_{\text{cpx str}}^{[X(T_0^{(i)})]}$, is parameterized by

$$\Omega_X^{(i)} = \nu^{(i)}_{(20)} + \nu^{(i)}_{(02)} - \frac{(\nu^{(i)}_{(20)}, \nu^{(i)}_{(02)})}{2C^{(i)}} M_X^{(i)} \nu^{(i)}_{(02)},$$

(48)

where $\nu^{(i)}$ collectively denotes\footnote{See below (53) for a component description of the moduli fluctuation fields.} the local coordinates of the moduli space $\mathcal{M}_{\text{cpx str}}^{[X(T_0^{(i)})]}$ around the vacuum $\{z^{(i)}\}$ (i.e., the moduli field fluctuations around the vacuum), and is regarded as an element of $[T_0^{(i)} \otimes \mathbb{C}]^{(1,1)}$—the (1,1) Hodge component with respect to $\{z^{(i)}\}; \nu^{(i)}_{(20)}$ and $\nu^{(i)}_{(02)}$ are also fixed against $T_X^{(i)} \otimes \mathbb{Q}$ and provide a fixed frame with which we describe deformation of complex structure of $X^{(i)}$; finally, $C^{(i)} = \nu^{(i)}_{(20)}, \nu^{(i)}_{(02)}$. The four-form $\Omega_Y = \Omega_X^{(i)} \wedge \Omega_X^{(2)}$ to be fed into the flux superpotential (1) is

$$\Omega_Y = \nu^{(1)}_{(20)} \nu^{(2)}_{(20)} + (\nu^{(1)}_{(20)}, \nu^{(1)}_{(02)}),$$

(49)

$$- \nu^{(1)}_{(20)} \nu^{(2)}_{(02)} (2C^{(i)})^{(2)} C^{(2)} M_X^{(i)} + C^{(1)} T_X^{(i)} + C^{(2)} T_X^{(i)} + C^{(1)} T_X^{(i)} + C^{(2)} T_X^{(i)} + C^{(3)} T_X^{(i)}.$$

(49)

Suppose that a non-trivial flux is in the $W_{(20)02}$ component; the condition (46) is implicit now. Then the contributions to $\Omega_Y$ in the first line of (49) do not yield any terms in the effective superpotential, so both of the $DW = 0$ and $W = 0$ conditions are satisfied at the vacuum, as designed. A flux in $W_{(20)02}$ gives rise to terms that are quadratic in the fluctuation of the $2(n_{1,2} - 2) = 2(20 - r_{i(1,2)})$ moduli fields—that would deform the complex structure of $Y$ (within $\mathcal{M}_{\text{cpx str}}^{[Y]BV})$ from the CM-type vacuum complex structure $\langle z \rangle$. Note also that a flux in $W_{(20)02}$ does not generate a cubic or quartic terms of those moduli fields $t^{(i=1,2)}$, but just yields the mass terms.

With a closer look, one finds that the mass matrix is Dirac type, and that the product of the mass eigenvalues is real. As a first step to see this, we write down the mass terms\footnote{We could use an integral basis of the lattices $T_0^{(i)}$ for a fixed frame, but the choice in the main text is obviously much more convenient for the discussion here.} of the mass eigenvalues is real. As a first step to see this, we write down the mass terms

$$\langle \rho^{(1)}_{(20)} \rangle^{-1} \circ \rho^{(2)}_{(02)} : K^{(2)} \to \overline{\mathbb{Q}} \to K^{(1)};$$

(50)
this isomorphism can be used to set up a 1-to-1 correspondence between the embeddings of $K^{(1)}$ and those of $K^{(2)}$;

$$\rho^{(2)}_{\beta(\alpha)} := \rho^{(1)}_{\alpha} \circ \left( (\rho^{(1)}_{\alpha})^{-1} \circ \rho^{(2)}_{\beta(\alpha)} \right), \quad \alpha \in \{(20), (02), 3, \cdots, n\}. \quad (51)$$

Then $\Phi^{\text{full}}_{L^{(20,02)}} = \{ \rho^{(1)}_{\alpha} \otimes \rho^{(2)}_{\beta(\alpha)} \mid \alpha = (20), (02), 3, \cdots, n \}$ in this notation. The fact that a flux must be in the $\mathbb{Q}$-coefficient cohomology, rather than in the $\mathbb{R}$ or $\mathbb{C}$-coefficient cohomology groups, is translated into the condition $n^{ij} \in \mathbb{Q}$ defined below, when we use $v^{(1)}_{\alpha} \otimes v^{(2)}_{\beta}$’s for a basis of the cohomology:

$$\int_G (v^{(1)}_{\alpha} \otimes v^{(2)}_{\beta}) =: \sum_{i,j} n^{ij} \rho^{(1)}_{\alpha} (y^{(1)}_i) \rho^{(2)}_{\beta(\alpha)} (y^{(2)}_j) =: G_{\alpha\beta(\alpha)}, \quad \alpha \in \{(20), (02), 3, \cdots, n\},$$

where $\{y^{(1)}_i \mid i = 1, \cdots, n\}$ and $\{y^{(2)}_j \mid j = 1, \cdots, n\}$ are the basis of $K^{(1)}$ and $K^{(2)}$, respectively, over $\mathbb{Q}$, introduced in Sect. 2.3.3. $G_{(20)(02)}$ must be an algebraic number within $\rho^{(1)}_{(2)}(K^{(1)}) = \rho^{(2)}_{(02)}(K^{(2)})$. Other $G_{\alpha\beta(\alpha)}$’s are Galois conjugate of $G_{(20)(02)}$:

$$\sigma_{\alpha}(G_{(20)(02)}) = G_{\alpha\beta(\alpha)}, \quad (52)$$

where $\sigma_{\alpha} \in \text{Gal}((\rho^{(1)}_{(20)}(K^{(1)}))^{\text{nc}}/\mathbb{Q})$ that brings $\rho^{(1)}_{(20)}$ to $\sigma_{\alpha} \cdot \rho^{(1)}_{(20)} = \rho^{(1)}_{\alpha}$ and $\rho^{(2)}_{(02)}$ to $\sigma_{\alpha} \cdot \rho^{(2)}_{(02)} = \rho^{(2)}_{\beta(\alpha)}$. So, the mass matrix is of the form

$$W \propto -\frac{G_{(20)(02)}}{2C^{(2)}} (t^{(2)}_1, t^{(2)}_2) T^*_\chi - \frac{(G_{(20)(02)})^{c.c.}}{2C^{(1)}} (t^{(1)}_1, t^{(1)}_2) T^*_\chi + \sum_{a=3}^n \sigma_a(G_{(20)(02)}) t^{(1)}_a t^{(2)}_a, \quad (53)$$

when we parametrize the moduli by $t^{(1)} = \sum_{a=3}^n t^{(1)}_a v^{(1)}_{\alpha}$ and $t^{(2)} = \sum_{a=3}^n t^{(2)}_a v^{(2)}_{\beta}$.

The Dirac structure of the mass matrix becomes manifest only after examining the mass terms $\propto (t^{(2)}_1, t^{(2)}_2)$ and $\propto (t^{(1)}_1, t^{(1)}_2)$ that are apparently Majorana. A key observation is that $(v^{(i)}_{(20)}, v^{(i)}_{\gamma}) = 0$ for any $\gamma \in \{(20), 3, \cdots, n\}$. Applying the Galois transformations, note that

$$\begin{align*}
(v^{(i)}_{\alpha}, v^{(i)}_{\alpha'}) & = \sigma_{\alpha}(C^{(i)}), \\
(v^{(i)}_{\gamma}, v^{(i)}_{\gamma'}) & = 0 \quad \text{for } \gamma \neq \overline{\alpha}.
\end{align*} \quad (54)$$

Using this property, the moduli effective superpotential is written in the following form:

$$\sum_{a' \in \{2, \cdots, n/2\}} \left( t^{(1)}_{a'}, t^{(2)}_{a'} \right) \begin{pmatrix}
-G_{(20)(02)}^{c.c.} & \sigma_a(C^{(1)}) \\
\sigma_{\overline{a}}(C^{(1)}) & -G_{(20)(02)}\sigma_{\overline{a}}(C^{(2)})
\end{pmatrix}
\begin{pmatrix}
t^{(1)}_{a'} \\
t^{(2)}_{a'}
\end{pmatrix}. \quad (55)$$

Note that the map $\Phi^{\text{full}}_{K^{(i)}} \ni \rho^{(i)}_a \mapsto \sigma_a \cdot \rho^{(i)}_a \in \Phi^{\text{full}}_{K^{(i)}}$ is one-to-one map, and that the basis vectors $v^{\gamma}$ have a component description (14) for a $\mathbb{Q}$-basis of $T^{(i)}_\gamma$.

Here is a little more set of notations. The $n$ embeddings $\Phi^{\text{full}}_{K^{(i)}}$ form $n/2$ pairs under the complex conjugations in $\mathbb{Q}$ (and also in the CM fields $K^{(i)}$); $\rho^{(i)}_a$ is paired with $\text{cc} \cdot \rho^{(i)}_{\overline{a}} = \rho^{(i)}_a \cdot \text{conj.}$, which is denoted by $\rho^{(i)}_{\overline{a}}$; the set $\Phi^{\text{full}}_{K^{(i)}}$ can be grouped into two $\{\rho^{(i)}_a \mid a' \in \{(20), 2, \cdots, n/2\}\}$ and $\{\rho^{(i)}_{\overline{a}} \mid a' \in \{(20), 2, \cdots, n/2\}\}$; a separation into two in this way is not unique. Note also that $\overline{\beta(\alpha)} = \beta(\overline{\alpha})$. 

\[\text{\footnotesize Notes:} \]
This mass matrix is obviously Dirac type, and is furthermore split into \((n/2 - 1)\) blocks of \(2 \times 2\) matrices.

The product of all the mass eigenvalues is in \(\mathbb{R}\). This is so even at the level of the individual \(2 \times 2\) mass matrices; the product is the determinant of the mass matrix above, which is

\[
(C^{(1)} C^{(2)})^{-1} \left( |G_{(20)(02)}|^2 \sigma_{d'} (C^{(1)} C^{(2)}) \right) \in \mathbb{R}, \tag{56}
\]

because \(C^{(1)}, C^{(2)} \in \mathbb{R} \cap \rho_{(20)}^{(1)}(K^{(1)})\).

To summarize, for a given vacuum complex structure in \(\mathcal{M}^{[X(T^{(1)}_0)]}_{CM} \times \mathcal{M}^{[X(T^{(2)}_0)]}_{CM}\) satisfying the condition (46), each choice of a flux from \(W_{(20)(02)} \simeq \mathbb{Q}^n\) is consistent with the \(DW = 0\) and \(W = 0\) conditions, and the \((n - 2)\) Dirac mass eigenvalues (all the values in \(\mathbb{Q}\)) can be computed systematically. As a reminder, \(n = [K^{(1)} : \mathbb{Q}] = [K^{(2)} : \mathbb{Q}] = \text{rank}(T^{(i)}_X) = 22 - r(1,2)\). The Dirac type mass matrix and the real nature of the product of the mass eigenvalues are a common (and unexpected)\(^{48}\) consequence the class of flux vacua under consideration.

The moduli stabilization discussed above appears similar to the one in \([30,31,92]^{49}\). Direct comparison with \([30,31,92]\) is easier in the cases we discuss in Sect. 2.5 (see footnote 56). When we take \(T^{(i)}_X = T^{(i)}_0\) (as in this Sect. 2.4) and set \(\text{rank}(T^{(i)}_X) = 2\) (as in \([30,31]\)), all the complex structure moduli fields of \(Y = (X^{(1)}_X \times X^{(2)}_X)\) whose mass discussed in \([30,31]\) are now projected out in the orbifold \(Y = (X^{(1)}_X \times X^{(2)}_X)/\mathbb{Z}_2\) here. The mass and stabilization of the \((40 - r(1) - r(2))\) moduli above is a special case of those in \([92]\) (in that the vacuum complex structure of \(X^{(i)}\) is assumed to be CM-type in this article).

The moduli mass from fluxes (with \(\langle W \rangle = 0\)) above is closer to the one in \([5]\). Discussion there correspond to a special case of the above result in this article; Ref. [5] was for \(X^{(1)} = \text{Km}(E_\phi \times E_\tau)\), the Type IIB orientifold set-up. Although [5] only argued that the complex structure moduli of Type IIB Calabi–Yau threefold \(M = (E_\tau \times X^{(2)}_X)/\mathbb{Z}_2\) and the axi-dilaton chiral multiplets are stabilized along with \(\langle W \rangle = 0\), the discussion above shows that the moduli fields of D7-brane positions are also stabilized along with \(\langle W \rangle = 0\).

Having studied the mass terms in (53, 55) let us now have a look at the whole low energy effective theory superpotential of the complex structure moduli fields \(t^{(1,2)}\) from the perspective of symmetry. The superpotential have \(U(1)^{\mathbb{Z}_2 - 1} \times U(1)_R\) symmetry. All the moduli fields \(t^{(i)}_a\) have +1 charge under the \(R\)-symmetry; there is also one \(U(1)\) symmetry for each one of \(a' \in [2, \cdots, n/2]\), where the chiral multiplets \(t^{(1)}_{a'}\) and \(t^{(2)}_{a'}\) have charge +1, and the chiral multiplets \(t^{(1)}_{a'}\) and \(t^{(2)}_{a'}\) charge \(-1\). This symmetry is a part of the symmetry of the Kähler potential, which is

\[
K = -\sum_{i=1}^{2} \ln (\Omega_{X^{(i)}} \Omega_{X^{(i)}}) \tag{57}
\]

\(^{47}\) For a generic choice of a flux \(G\) in \(W_{(20)(02)}\), this combination would not vanish, which means that all the \(2(n - 2)\) moduli fields have non-zero masses.

\(^{48}\) The vacuum complex structure of the pair of K3 surfaces being CM-type does not imply at all such things as period integrals being real, or the field of moduli having embedding into \(\mathbb{R}\). Here, we pay attention to the product of the mass eigenvalues as an exercise problem for potential applications to the strong CP problem.

\(^{49}\) The relation to [2] is discussed in Sect. 4.3.
\[
\begin{align*}
= \sum_{i=1}^{2} \ln \left( C^{(i)} + \sum_{a} \sigma_{a}(C^{(i)}) t_{a}^{(i)} (t_{a}^{(i)})^{\dagger} + \sum_{a', b'}^{2-n/2} \left( \sigma_{a'}(C^{(i)}) t_{a'}^{(i)} (t_{b'}^{(i)})^{\dagger} \right) \left( \sigma_{b'}(C^{(i)}) (t_{b'}^{(i)})^{(i)} \right)^{\dagger} \right). \tag{57}
\end{align*}
\]

To understand the origin and nature of the \( U(1)^{n/2-1} \) symmetry in the moduli effective field theory, it helps to reflect more upon the symmetry of the Kähler potential, the non-linear sigma model metric (57). The target space \( M_{\text{cpx str}}^{[Y]BV} \) is a homogeneous space with the symmetry group

\[
\mathbb{G}U \text{Isom}(T^{(1)}_{X}; \mathbb{C}) \times \mathbb{G}U \text{Isom}(T^{(2)}_{X}; \mathbb{C}),
\]

where

\[
\mathbb{G}U \text{Isom}(L; \mathbb{C}) := \left\{ g \in \text{Isom}(L \otimes \mathbb{C}) \mid (g^{\dagger} \bar{x}, g x)_{L} = 3c_{g}(x)(\bar{x}, x)_{L} \text{ for } \forall x \in L \otimes \mathbb{C} \right\}
\]

for a lattice \( L; c_{g} \) can be any constant independent of \( x \). For any (not necessarily CM) point in \( M_{\text{cpx str}}^{[Y]BV} \) chosen as a vacuum, the isotropy group—the symmetry group linearly realized on the fluctuations fields—is

\[
\text{SO}(n_{1} - 2) \times \text{SO}(n_{2} - 2);
\]

we have seen that \( n_{1} = n_{2} =: n \) under the condition (46). The \( U(1)^{n/2-1} \) symmetry of the full moduli effective theory (including the moduli mass terms due to fluxes) is the Cartan part of the diagonal \( \text{SO}(n - 2) \). They are global symmetries.\(^{50}\)

It is worth noting that the presence of the symmetry \( U(1)^{n/2-1} \) in the non-linear sigma model in \( \mathbb{R}^{2,1} \) in M-theory (\( \mathbb{R}^{3,1} \) in F-theory) is essentially due to the nature of the period domain of a K3 surface\(^{51}\) rather than that of a Calabi–Yau manifold of higher dimensions. As already briefly referred to\(^{52}\) in Sect. 2.1, one could think of the space of rational Hodge structures on \( H^{3}(M; \mathbb{Q}) \) for a family of Calabi–Yau threefolds \([M]\), which is a homogeneous space just like \( D(T_{0}^{i})'s \) are; the complex structure moduli space of \([M]\) is only a subspace of the homogeneous space, so the symmetry of the non-linear sigma model of the complex structure moduli of \([M]\) cannot be simply stated by just referring to the vector space \( H^{3}(M; \mathbb{Q}) \) and the skew-symmetric intersection form on it. The same discussion applies also to Calabi–Yau fourfolds.

Furthermore, those continuous symmetries—either \( U(1)^{n/2-1} \) or \( \text{SO}(n-2) \times \text{SO}(n-2) \)—of the non-linear sigma model in \( \mathbb{R}^{2,1} \) or \( \mathbb{R}^{3,1} \) cannot be attributed to a symmetry of the geometry \( X^{(1)} \times X^{(2)} \). A symmetry transformation on \( X^{(i)} \) would manifest itself as a symmetry action on \( H^{2}(X^{(i)}; \mathbb{Q}) \); a transformation on \( H^{2}(X^{(i)}; \mathbb{C}) \) that cannot be

\(^{50}\) A non-trivial discrete subgroup in \( U(1)^{n/2-1} \) may be gauged, in the sense that a discrete subgroup of the symmetry of a vacuum complex structure may be regarded as a part of the isotropy group of the form \( \text{Isom}(T^{(1)}_{X})_{Hdg \ Amp} \times \text{Isom}(T^{(2)}_{X})_{Hdg \ Amp} \), which induces automorphisms (unphysical difference) of \( X^{(1)} \times X^{(2)} \).

\(^{51}\) The complex structure moduli effective superpotential (53) is of very specific—purely quadratic—form also essentially due to this. There is an argument on the ground of genericity that \( \langle W \rangle = 0 \) must be associated with some discrete R-symmetry (more than \( \mathbb{Z}_{2} \)), and then the moduli superpotential must be of the form \( W \sim \sum_{i} X_{i} f_{i}(\phi) \), where \( X_{i} \) are chiral fields that transform the same way as \( W \) under the R-symmetry, and \( \phi \)'s other moduli fields that are neutral under the R-symmetry [32]. It then follows in this regime when all those fields have masses. The moduli superpotential on K3 x K3 orbifolds in this article is not within this genericity regime.

\(^{52}\) Interested readers are referred to [16,33].
derived from one on $H^2(X^{(i)}; \mathbb{Q})$ does not have an interpretation as an $X^{(i)} \to X^{(i)}$ map. Those continuous symmetries are not symmetries of the geometry $X^{(1)} \times X^{(2)}$, but are symmetries of their moduli spaces. They are accidental symmetry in the low-energy effective theory.

The continuous $U(1)^{\frac{n}{2}-1} \times U(1)_R$ symmetry in the moduli effective theory are likely not to be an exact symmetry apart from its possible non-trivial discrete subgroup (cf footnote 50). This expectation is from general arguments in quantum gravity; as for the $U(1)_R$ part, one may also argue this by computing triangle anomalies against the Standard Model gauge groups (e.g., [34]). The source of explicit breaking of the symmetry may be the anomalies with gauge fields, stringy non-perturbative effects, or just stringy perturbative corrections to the approximation $K = -\ln(\int_Y \Omega_Y \wedge \Omega_Y)$ and $W \propto \int_Y G \wedge \Omega_Y$. Better understanding on the source of explicit breaking will give us a better hint on a discrete exact symmetry in the effective theory containing all of moduli, the supersymmetric Standard Model and anything else. In the case the discrete exact symmetry is larger than the symmetry acceptable at TeV scale (such as a subgroup of $U(1)_R$ larger than $\mathbb{Z}_2$ R symmetry), the domain wall problem sets constraints on inflation and the thermal history after that. If the explicit breaking leaves only the $\mathbb{Z}_2$ subgroup of the $U(1)_R$ symmetry, then the source of the explicit breaking also determines the gravitino mass.

2.5. Cases with $T_X \subsetneq T_0$. Think of a case where the vacuum complex structure of $X^{(i)}$ is still of CM-type, but not generic enough to have $T_X^{(i)} = T_0^{(i)}$ for at least one of $i = 1, 2; T_X^{(i)} \subsetneq T_0^{(i)}$ and $S_0^{(i)} \subseteq S_X^{(i)}$, then. Put differently, the vacuum complex structure of $X^{(i)}$ is in a Noether–Lefschetz locus of $D(T_0^{(i)})$, where there must be an element of $H^2(K3; \mathbb{Z})$ (Poincaré dual of a 2-cycle) that becomes algebraic. Now, the rank of $T_X^{(i)}$ is even (because of its CM nature), but $T_0^{(i)}$ in Nikulin’s list may be of odd rank. We will see below that much the same story unfolds for a $DW = 0$ flux, and also for a $DW = W = 0$ flux; one difference, though, is that there is one more way (without the relation (46)) to stabilize moduli in $\mathcal{M}^{cpx str}$ by a $DW = W = 0$ flux, when $T_X^{(i)} \subsetneq T_0^{(i)}$ for both $i = 1, 2$.

Let $T_0^{(i)}$ be the negative definite lattice $[(T_X^{(i)})^\perp \subset T_0^{(i)}]$, so that

$$T_0^{(i)} \otimes \mathbb{Q} \cong (T_X^{(i)} \otimes \mathbb{Q}) \oplus (T_0^{(i)} \otimes \mathbb{Q}).$$  \hspace{1cm} (61)

53 There are tight constraints on how R-symmetry charge is assigned on the particles in supersymmetric Standard Models. On the other hand, we need to know how the moduli fields $\alpha^{(i)}$ couple to the Standard Model particles to find out how (or whether) the $U(1)^{n/2-1}$ symmetry can be extended to the whole low-energy effective theory.

54 Alternatively, one may focus on the common subset of $\text{Isom}(T_X^{(i)}) \times \text{Isom}(T_X^{(2)})$ and $U(1)^{n/2-1}$, which will be more mathematical study, to infer what the discrete gauged symmetry is.

55 Memo: The 2-cycle in question may or may not be norm (-2). If it is not norm (-2), then the limit vacuum complex structure is in a subvariety of $D(T_0^{(i)})$, and such a K3 surface $X^{(i)}$ is obtained by just tuning the complex structure. If it is norm (-2), then the limit vacuum complex structure is found only in the closure $D(T_0^{(i)})$, not within $D(T_0^{(i)})$, if that matters. Such a K3 surface $X^{(i)}$ is obtained by taking a limit in the complex structure so an $A_1$-singularity emerges, and then by resolving it.
The \((T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}\) component of \(H^4_H(Y; \mathbb{Q})\) is then expanded as follows:

\[
(T_0^{(1)} \otimes T_0^{(2)}) = (T_X^{(1)} \otimes T_X^{(2)}) \oplus (T_X^{(1)} \otimes \bar{T}_0^{(2)}) \oplus (\bar{T}_0^{(1)} \otimes T_X^{(2)}) \oplus (\bar{T}_0^{(1)} \otimes \bar{T}_0^{(2)}).
\]  

(62)

The two components \((T_X^{(1)} \otimes \bar{T}_0^{(2)})\) and \((\bar{T}_0^{(1)} \otimes T_X^{(2)})\) with rational Hodge substructure are always level-2, and the \((\bar{T}_0^{(1)} \otimes \bar{T}_0^{(2)})\) component always level-0, if it is non-empty. The component \(T_X^{(1)} \otimes T_X^{(2)}\) contains a level-4 component; whether the rational Hodge structure on this component is simple or not depends.

Suppose that the condition (46) is satisfied. Then any rational flux in \(\bar{T}_0^{(1)} \otimes \bar{T}_0^{(2)} \otimes \mathbb{Q}\) and the \(i = (20|02)\) component of \(T_X^{(1)} \otimes T_X^{(2)} \otimes \mathbb{Q}\) satisfies the \(DW = 0\) and \(W = 0\) conditions. When a flux is non-zero only in the \(W_{(20|02)}\) component within \(T_X^{(1)} \otimes T_X^{(2)}\), the moduli effective theory superpotential (53) remains as it is if it is interpreted as follows; the third term of (53) only involves the moduli fluctuation fields within \(D(T_X^{(1)}) \times D(T_X^{(2)}),\) while \((t^{(2)}, t^{(2)})\) and \((t^{(1)}, t^{(1)})\) in the first two terms of (53) are meant to include all the fluctuations fields in \(D(T_0^{(1)}) \times D(T_0^{(2)}).\) When a non-zero flux in \(\bar{T}_0^{(1)} \otimes \bar{T}_0^{(2)} \otimes \mathbb{Q},\) there is one more term in the effective superpotential, which is the Dirac-type mass term of the moduli fluctuation fields in \(N_{D(T_X^{(1)})\bar{D}(T_0^{(1)})}\) (normal directions) and \(N_{D(T_X^{(2)})\bar{D}(T_0^{(2)})}.\) Because the (stabilizing) mass terms for the fluctuations within \(D(T_X^{(1)}) \times D(T_X^{(2)})\) rely on the flux in \(W_{(20|02)},\) the condition (46) is necessary (apart from the caveat mentioned below). This moduli effective theory has an \(U(1)\) symmetry, where all the moduli fluctuation fields have \(+1\) \(R\)-charge; to see this, we almost have to repeat the argument in Sect. 2.4.3, and the fact that a flux in \(\bar{T}_0^{(1)} \otimes \bar{T}_0^{(2)}\) also generates only the mass term. There is no additional non-\(R\) \(U(1)\) symmetry where the moduli fluctuation fields in \(N_{D(T_X^{(1)})\bar{D}(T_0^{(1)})\text{ and }N_{D(T_X^{(2)})\bar{D}(T_0^{(2)})}\) are charged, however. This is because there is no Dirac-like structure for those moduli fields in the \((t^{(2)}, t^{(2)})\) and \((t^{(1)}, t^{(1)})\) in the first two terms in the superpotential (53).

One caveat in the argument above is the case there is no moduli fluctuation fields within \(D(T_X^{(1)}) \times D(T_X^{(2)}),\) which is when both \(X^{(1)}\) and \(X^{(2)}\) are attractive (\(\text{rank}(T_X^{(i)}) = 2, \text{rank}(S_X^{(j)}) = 20\)) K3 surfaces. Even when the condition (46) is not satisfied, a flux in \(\bar{T}_0^{(1)} \otimes \bar{T}_0^{(2)}\) provide a mass term for all the moduli fluctuation fields in \(D(T_0^{(1)}) \times D(T_0^{(2)})\) if the condition

\[
\text{rank}(\bar{T}_0^{(1)}) = \text{rank}(\bar{T}_0^{(2)})
\]

is satisfied. The mass matrix is Dirac type then. This is because the mass matrix from the flux here is always Dirac type.

3. General K3 x K3 Orbifolds

The Borcea–Voisin orbifold \((X^{(1)} \times X^{(2)})/\mathbb{Z}_2\) in the previous section is regarded as one way to construct a Calabi–Yau variety of higher dimensions by using K3 surfaces (and/or

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56 This mechanism is quite close to the one in [30,31,92]; in the terminology of [30,31], the flux in \((\bar{T}_0^{(1)} \otimes \bar{T}_0^{(2)}) \otimes \mathbb{Q}\) corresponds to a part of \(G_0\)-flux, and the one in \((T_X^{(1)} \otimes T_X^{(2)}) \otimes \mathbb{Q}\) to \(G_1\)-flux.
elliptic curves). So, there is an obvious generalization; think of any supersymmetry-preserving orbifold of a product of K3 surfaces (and/or elliptic curves); the orbifold group \( \Gamma \) is not necessarily \( \mathbb{Z}_2 \) [35]. In this Sect. 3, we bring known materials together from the literatures, to have a broad brush picture of possible variety in the construction, to identify open math problems for a complete classification, and to repeat the same study as in Sects. 2.4 and 2.5 for the cases with \( \Gamma \neq \mathbb{Z}_2 \).

### 3.1. K3 x K3 orbifold.

Consider an orbifold \( Y_0 = (X^{(1)} \times X^{(2)}) / \Gamma \), where \( X^{(1)} \) and \( X^{(2)} \) is a pair of K3 surfaces. The orbifold group \( \Gamma \) should be a subgroup of \( \text{Aut}(X^{(1)}) \times \text{Aut}(X^{(2)}) \), first of all. For the action of the orbifold group \( \Gamma \) to preserve supersymmetry, one more condition needs to be imposed. To state the condition, we prepare some notations.

Under the projection \( p_i : \text{Aut}(X^{(1)}) \times \text{Aut}(X^{(2)}) \to \text{Aut}(X^{(i)}) \), let \( G_i := p_i(\Gamma) \). Let \( \alpha_i' : \text{Aut}(X^{(i)}) \to \text{Isom}(T^{(i)}_{X})^{\text{Hdg Amp}} \) be the projection that fits into the exact sequence\(^{57}\)

\[
1 \to \text{Aut}_N(X^{(i)}) \to \text{Aut}(X^{(i)}) \to \text{Isom}(T^{(i)}_{X})^{\text{Hodge Amp}} \to 1. \tag{64}
\]

Because the elements of \( \text{Isom}(T^{(i)}_{X})^{\text{Hdg Amp}} \) acts on the holomorphic (2,0) form \( \Omega^2_{X^{(i)}} \) faithfully, \( \alpha_i'((\sigma)) \in \text{Isom}(T^{(i)}_{X})^{\text{Hdg Amp}} \) for \( \sigma \in \text{Aut}(X^{(i)}) \) may well be identified with the complex phase \( \alpha_i((\sigma)) \) in \( \sigma^* \Omega^2_{X^{(i)}} = \alpha_i((\sigma)) \Omega^2_{X^{(i)}} \). With those preparations, the supersymmetry condition is written as

\[
\forall \sigma \in \Gamma, \quad \alpha_1(p_1(\sigma)) \alpha_2(p_2(\sigma)) = 1 \in \mathbb{C}. \tag{65}
\]

We will discuss only the cases that the group \( \Gamma \) has a finite number of elements.\(^ {58}\)

An equivalent way to state the supersymmetry condition is that there is a group \( \Delta \), so that\(^ {59,60}\)

\[
\alpha_i'(G_i) \cong \Delta, \quad \Gamma \subset G_1 \times_{\Delta} G_2. \quad \tag{66}
\]

When we impose the Calabi–Yau condition \( h^{p,0}(Y_0) = 0 \) for \( p = 1, 2, 3 \) in addition to \( h^{4,0}(Y_0) = 1 \), the group \( \Delta \) needs to be something other than\(^ {61}\) the trivial group \( \{1\} \).

The two K3 surfaces \( X^{(1)} \) and \( X^{(2)} \) for an M-theory/F-theory compactification come with one Kähler form for each one of them. The orbifold group action by \( G_i \) for \( i = 1, 2 \) should preserve the Kähler form on \( X^{(i)} \) (so the orbifold defines a consistent theory).\(^ {62}\)

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\(^{57}\) In this section, we use the same notation as in [31,36] without spelling out their definitions. Reviews in [29,31,36,37] will also be useful.

\(^{58}\) It sounds like an orbifold \( (X^{(1)} \times X^{(2)}) / \Gamma \) with \( |\Gamma| = \infty \) would yield a pathological “Calabi–Yau fourfold”, although we are not absolutely sure if such possibilities should be completely ruled out.

\(^{59}\) The isomorphism \( \alpha_i'(G_i) \cong \alpha_2'(G_2) \) should be such that their representations \( \alpha_1 \) and \( \alpha_2 \) are complex conjugate.

\(^{60}\) Complete classification of \( (S^{(i)}, T^{(i)}, G_i; G_{s,i}, \Delta) \) for \( i = 1, 2 \) and \( \Gamma \subset G_1 \times_{\Delta} G_2 \) will be redundant for classification of variety in the generalized Borcea–Voisin fourfolds for compactification. For example, in a case \( \Gamma \) has a structure of \( \Gamma \cong \Gamma_0 \times G_1' \times G_2' \) with \( \Gamma_0 \subset \text{Aut}(X^{(1)}) \times_{\Delta} \text{Aut}(X^{(2)}) \) and \( G_i' \subset \text{Aut}_N(X^{(i)}) \) for \( i = 1, 2 \), one may replace a compactification by \( (X^{(1)} \times X^{(2)}) / \Gamma \) with a compactification by \( (X^{(1)}_{ct} \times X^{(2)}_{ct}) / \Gamma_0, \) where \( X^{(i)}_{ct} \) is a crepant resolution of \( X^{(i)}/G_i' \).

\(^{61}\) It can be shown ([38] Thm. 0.1 (a), Thm. 3.1 (a) and Cor 3.2) that a K3 surface with \( \Delta \neq \{1\} \) is always algebraic.

\(^{62}\) It is a version of this statement that reflects the underlying theoretical principles more directly: The metric on \( X^{(i=1,2)} \) is expressed as a positive definite 3-plane within \( H^2(X^{(i)}; \mathbb{R}) \), and all the elements \( g \) in
3.2. K3 surfaces with non-symplectic automorphisms

3.2.1. Discrete classification Just like we used Nikulin’s classification in the previous section, one can think of a similar classification problem whose answer can be used for this general form of the Borcea–Voisin orbifolds. Here is how we formulate the problem: how many different choices of \((S \oplus T, G)\) there are modulo \(Isom(\Pi_3,19)\), subject to the conditions

- \(G\) is a finite subgroup of \(Isom(\Pi_3,19)\), and \(S\) and \(T\) are mutually orthogonal primitive sublattices of \(\Pi_3,19 \cong H^2(K3; \mathbb{Z})\) such that \((S \oplus T) \otimes \mathbb{Q} \cong \Pi_3,19 \otimes \mathbb{Q}\),
- \(g(T) = T\) (and also \(g(S) = S\)) for any \(g \in G\),
- for any \(g \in G\) whose \(g|_T\) is non-trivial, \(g|_T\) is not identity on any vector subspace of \(T \otimes \mathbb{Q}\).
- \(S\) and \(T\) have signature \((1, r - 1)\) and \((2, 20 - r)\) (with \(1 \leq r \leq 20\)),
- for any \(g \in G\), the sublattice \(S^g := \{x \in S \mid g \cdot x = x\}\) contains 1 signature-positive direction.

The first three conditions characterize \(G\), \(T\), and \(S\) as a set of automorphism group, transcendental lattice and Néron–Severi lattice of a K3 surface. The last two conditions reflect the Calabi–Yau condition of \(Y_0(\Delta \neq \{1\})\) and the \(\Gamma\)-invariance of the Kähler parameter discussed in Sect. 3.1.

For one choice \((S \oplus T, G)\), we can determine two groups

\[G_s := \text{Ker}(G \to Isom(T)),\]
\[\Delta := \text{Im}(G \to Isom(T)).\] (67) (68)

So, the classification of \((S \oplus T, G)\) may well be regarded as classification of the data \((S \oplus T, G; G_s, \Delta)\). Furthermore, one may state the result of the classification by listing up all possible choices \(1 \to G_s \to G \to \Delta \to 1\) first, and then by listing up of all possible lattice pairs \(S \oplus T\) for \((G; G_s, \Delta)\). Nikulin’s classification implies that the case \(G_s \cong \{1\}\) and \(G \cong \Delta \cong \mathbb{Z}_2\) contains 75 different choices of \(S \oplus T\), as we have referred to in Sect. 2.

There are at most 41 different cases of the group \(\Delta\) including \(\Delta \cong \{1\}\) ([38, Cor 3.2] and [39]); all the possible groups \(\Delta\) are cyclic groups \(\mathbb{Z}_m\) for some \(m \in \mathbb{N}_{\geq 0}\) ([38, Thm. 3.1 (b,c)], [40, Lemma 2.1], [29, Cor 3.3.4]), and the list of 41 \(m\)'s (Table 1 of [39]) are reprinted explicitly in Table 1 here for convenience of the readers.

The rank \((22 - r)\) of the lattice \(T\) should be divisible by \(\varphi(m)\).

\(G_i\) of the orbifold group should preserve this 3-plane as a whole; \(g\) induces an SO(3) rotation on the 3-plane, which is regarded as a rotation around one axis. We require further that this rotation axis is common for all \(g \in G_i\), in which case the orbifold becomes Calabi–Yau.

The common axis of rotation within the 3-plane is identified with the Kähler form modulo normalization, and the other two directions orthogonal to the axis are identified with \(\Omega_X(x)\).

63 See [38, Thm. 3.1 (b)] for the third condition.
64 See [38, Thm. 0.1 (a) and 3.1 (a)] for the fourth condition, and [38, Lemma 4.2 (a)] for the fifth condition.
65 Nikulin’s list contain \(S \oplus T\) where \(G \cong \Delta \cong \mathbb{Z}_2\) acts trivially on \(S\). When the last condition is relaxed, then there may be more choices of \(S \oplus T\) for the case of \(G_s \cong \{1\}\) and \(G \cong \Delta \cong \mathbb{Z}_2\). When the latter condition is satisfied, we say that the action of \(G \cong \Delta\) is purely non-symplectic.
66 In the case of \(G_s \cong \{1\}\) and \(G \cong \Delta \cong \mathbb{Z}_p\) for a prime number \(p\), it is enough to list up all the \((S, T)'s\) where \(G \cong \Delta\) acts trivially on \(S\). Even when one finds \((S', T')\) where \(G \cong \Delta\) acts non-trivially on \(S'\), one may find \(S \subset S'\) where \(G \cong \Delta\) acts trivially; a K3 surface with its complex structure in \(D(T')\) may be regarded as a special case of a K3 surface with its complex structure in \(D([S^\perp])\).
the unique $K_3$ surface admitting $\Delta_1$ known to be $66$ [45].

There are at most 82 different choices of $G_s$ including $G_s \cong \{1\}$. They should be a subgroup of 11 different finite groups listed in [41] (two of them are $\mathbb{Z}_6$ and $\mathbb{S}_5$; see [41] for nine others). See [42] for the list of those all 82 finite groups. It is also known that if $G_s$ has an element $g$ of order $n > 1$, then it must be [48] that $n \leq 8$ first of all, and secondly, rank($S$) $\geq 9$ if $n = 2$, rank($S$) $\geq 13$ if $n = 3$, rank($S$) $\geq 15$ if $n = 4$, rank($S$) $\geq 17$ if $n = 5$, 6, and rank($S$) $\geq 19$ if $n = 7, 8$ [29, Cor 15.1.8].

Therefore, there can be at most $[82 \times 41]$ different choices of the finite groups $G_s$ and $\Delta$; the choice $G_s \cong \{1\}$ and $\Delta \cong \mathbb{Z}_m$ in Sect. 2 is one of this $[82 \times 41]$ choices. In fact, not all the 82 x 41 choices can be realized. A group $\Delta \cong \mathbb{Z}_m$ with larger $m$ requires that the lattice $T$ has a larger rank because $\varphi(m) / \text{rk}(T)$, whereas a larger group $G_s$ requires $S$ with a larger rank (and $T$ with a smaller rank). Using the data available in [42], the range of $m_{\varphi(m)}$ can be narrowed down for each choice of $G_s$; Table 2 is the summary (cf. also [45, 46]).

For $\Delta \cong \mathbb{Z}_m$ with $m = 66, 44, 33, 50, 25, 40, \text{ and } 60$, for which $G_s = \{1\}$ (and $G \cong \Delta$) is the only option, all the possible $S \oplus T$’s have been worked out by using lattice theory and a bit of geometry [39, Lemma (1.2)]. It turns out that there is just one choice of $S \oplus T$ for each one of $m = 66, 44, 33, 50, 25, 40$, and that $G \cong \Delta$ happens to act on $S$ trivially. There is no choice of $S \oplus T$ where $m = 60$ [47]. For $\Delta \cong \mathbb{Z}_m$ with $m = 17$ and 19, see [48].

For general ($G; G_s, \Delta$), complete classification of the choices of $S \oplus T$ is not available yet. For cases with $G_s = \{1\}$ (so $G \cong \Delta$), all the possibilities of $(S \oplus T)$ with $G \cong \{1\}$ worked out by [38, Thm 4.5]. If we are to demand that $|\text{Aut}(X)| < \infty$ (not just $|G| < \infty$), then just the three in the list, $G_s = \{1\}$, $\mathbb{Z}_2$ and $S_3$, are possible [43, 44].

An immediate consequence of this fact is that, if we are to choose a finite subgroup $G \subset \text{Aut}(X)$, then any element in $G$ has an order not larger than $8 \times 66$; in fact there is no such an element of order $8 \times 66$ because the unique $K_3$ surface admitting $\Delta \cong \mathbb{Z}_{66}$ does not have symplectic automorphisms. The true upper bound is known to be $66$ [45].

---

**Table 1.** The list of $m := |\Delta|$ and the corresponding $\varphi(m)$

| $G_s$ in [42] | $c$ | $\Delta \cong \mathbb{Z}_m$ |
|---------------|-----|----------------|
| $\{1\}$       |     | any 41 $m$'s  |
| $\#1$         | $(c = 8)$ | $26$ $m_{\varphi(m)}$'s with $\varphi(m) \leq 12$ |
| $\#2, 3$      | $(c = 12)$ | $18$ $m_{\varphi(m)}$'s with $\varphi(m) \leq 8$ |
| $\#4, 6, 9, 10, 21$ | $c = 14, 15$ | $13$ $m_{\varphi(m)}$'s with $\varphi(m) \leq 6$ |
| $17 G_s$'s     | $c = 16, 17$ | $9$ $m_{\varphi(m)}$'s with $\varphi(m) \leq 4$ |
| $56 G_s$'s     | $c = 18, 19$ | $m_{\varphi(m)}(\in \{1, 2, 3, 2, 4, 2, 6\}$ |

The 82 choices of $G_s$ are grouped into six by their value of $c$ listed in Table 2 of [42]. For those six groups of $G_s$'s, the possible range of $m_{\varphi(m)}$ is determined by the condition $\varphi(m) \leq 21 - c$, shown on the right.
\( \Delta \cong \mathbb{Z}_m \) acting trivially on \( S \) have been classified, however. For cases with an \( m \) that is divisible by two (or more) prime numbers (such as \( m = 6, 10, 15, \cdots \)), it turns out that both \( S \) and \( T \) have to be unimodular; see [49] for the list of \( S \oplus T \) for the \( m \)'s that are not in the form of \( m = p^k \) for a single prime number \( p \). For cases with \( m = p^k \), this is an immediate generalization of the classification of Nikulin [18]. See [50–52] for the \( m = 2^2 \) case (there are 12 \( S \oplus T \)), the \( m = 2^3 \) case (there are 3 \( S \oplus T \)), and the \( m = 2^n \) case (\( S = UD_4 \) unique), while there is no choice of \( S \oplus T \) for the case \( m = 2^5 \) [49,53]. For the cases with \( m = 3^k \), and \( m = 5, 7, 11, 13, 17, 19 \), see [54–56] and [57], respectively.

For the cases with \( G_s = \{1\} \) (so \( G \cong \Delta \cong \mathbb{Z}_m \)), one may think of the classification of \( (S \oplus T) \)'s where \( G \) may act on \( S \) non-trivially. Only partial results are known. Results for \( m = 17, 19, 40, 25, 50, 33, 44, 66, 60 \) have been quoted earlier already. The \( m = 2^5 \) case has just one choice of \( S \oplus T \), where \( G \cong \mathbb{Z}_2 \) acts on the rank-6 \( S = UD_4 \) (the same as the unique choice for the \( m = 2^4 \) case) non-trivially on a 2-dimensional subspace through a quotient \( \mathbb{Z}_3 \to \mathbb{Z}_4 \) (and trivially on a 4-dimensional subspace) [58,59]. For a similar study in the case of \( G_s = \{1\} \) and \( G \cong \Delta \cong \mathbb{Z}_m \) with \( m = 2^4, m = 2^5, \) and \( m = 2^2 \), see [52,60–62].

Just like there is only small number of choices of \( (S \oplus T) \) is available for a large \( \Delta \), it is also known that there are tight constraints on the possible choices of \( (S \oplus T) \) when \( G_s \) is large. See such references as [38,63–66, Thm. 4.7], and [67].

This Sect. 3.2.1 is a literature survey, relying mostly on [29] as a guide. We wished to learn what is known as well as what has not been known about how much the \( \mathbb{Z}_2 \) orbifold in Sect. 2 can be generalized.

### 3.2.2. Period domains for \( K3 \) surfaces with automorphisms

The period integrals (complex structure) of a \( K3 \) surface \( X \) should be in the period domain \( D(T) \) for one of the choice \( (S \oplus T) \), when \( X \) has an automorphism \( (G; G_s, \Delta) \), but the converse is not true. For a complex structure to be consistent with the automorphism group \( (G; G_s, \Delta) \) with \( \Delta \neq \{1\} \), \( G|_T = \Delta \cong \mathbb{Z}_m \) needs to be a Hodge isometry.

The subspace \( D(T) \) consistent with such a non-symplectic automorphism group is specified as follows. Note that the action of \( \Delta \cong \mathbb{Z}_m \) on \( T \otimes \mathbb{Q} \) is always of the form

\[
T \otimes \mathbb{Q} \cong (N_m)^\otimes \ell \quad \text{with} \quad \ell := \text{rk}(T)/\varphi(m)
\]

where \( \mathbb{Z}_m \) acts as \( \mathbb{Q} \)-valued matrices on a \( \varphi(m) \)-dimensional vector space \( N_m \) over \( \mathbb{Q} \); the generator \( [[\sigma]] \) of \( \mathbb{Z}_m \) has the set of eigenvalues \( \{\zeta^a_m \mid a \in [\mathbb{Z}_m]^\times\} \). So, \( T \otimes \mathbb{C} \) is divided into \( \varphi(m) \) distinct eigenspaces of \( \mathbb{Z}_m \), \( \oplus_{a \in [\mathbb{Z}_m]^\times} V_a \), where \( [[\sigma]]|_{V_a} = \zeta^a_m \). Individual \( V_a \)'s are of \( \ell \)-dimensions over \( \mathbb{C} \). So, the complex structure should be in

\[
D(V_a) := \mathbb{P}[V_a] \cap D(T)
\]

for some \( a \in [\mathbb{Z}_m]^\times \). The subvariety \( D(V_a) \) of \( D(T) \) is determined only by \( \Delta \cong \mathbb{Z}_m \), independent of the symplectic subgroup of the automorphism \( G_s \).

This extra condition on the complex structure moduli space was absent in the case of \( \mathbb{Z}_2 \) orbifold in the previous section, because \( \varphi(m = 2) = 1 \), and \( T \otimes \mathbb{Q} \cong V_{a=1} \). For the cases with \( m > 2 \), however, \( V_a \subsetneq T \otimes \mathbb{C} \), and \( D(V_a) \subsetneq D(T) \). In fact, there is just one pair of \( D(V_a) \) and \( D(V_{a'}) \) with \( a, a' \in [\mathbb{Z}_m]^\times \) and \( a' = -a \in \mathbb{Z}_m \); that is because the intersection matrix remains non-zero only between \( V_{b'}-V_{b} \) pairs with \( b' = -b \in \mathbb{Z}_m \) (remember that \( [[\sigma]] \) is an isometry of \( T \)), and the 2-dimensional positive signature
directions must be contained only in one of those pairs. In that non-empty pair $D(V_{a0})$ and $D(V_{-a0})$, the $\Omega^2 = 0$ condition is automatically satisfied in $\mathbb{P}(V_{a0})$ and $\mathbb{P}(V_{-a0})$, so $D(V_{a0})$ and $D(V_{-a0})$ are open subspace of $\mathbb{P}^{\ell-1}$ specified by the $(\Omega, \overline{\Omega}) > 0$ condition [57,68].

In the cases with $\varphi(m) = \text{rk}(T)$, so $\ell = 1$, the subvarieties $D(V_a)$ are of 0-dimensions, so they are isolated points. This is consistent with the fact that a CM-type K3 surface corresponds to an isolated point on the moduli space, as discussed later.

3.2.3. K3 surfaces of CM-type and with non-symplectic automorphisms Not all the points in the moduli space $D(V_{a0})$ of K3 surfaces with non-symplectic automorphisms correspond to K3 surfaces of CM-type. The subspace of $D(V_{a0})$ corresponding to CM-type K3 surfaces is characterized as in the discussion in the following. We focus on the cases with $\Delta \cong \mathbb{Z}_m$ for $m > 2$, but some parts of the discussion applies to the cases of involution, $m = 2$.

In the cases with $\ell = 1$ and $\varphi(m) = \text{rk}(T)$, the one point $D(V_{a0})$ corresponds to a CM-type K3 surface (cf [69]). This is because the algebra $\text{Span}_{\mathbb{Q}}\{[\sigma] \in \Delta\}$ is a part of the endomorphism algebra $\text{End}(T)_{\text{Hdg}}$, and already $\dim_{\mathbb{Q}}(\text{Span}_{\mathbb{Q}}\{[\sigma]\}) = \text{rk}(T)$.

The endomorphism field is isomorphic to $\mathbb{Q}(\zeta_m)$.

In the cases with $\ell := \text{rk}(T)/\varphi(m) > 1$, if a CM point is contained $D(V_{a0})$ outside of Noether–Lefschetz loci, then the CM field $K$ must be an extension of $\mathbb{Q}([\sigma]) \cong \mathbb{Q}(\zeta_m)$.

Beyond that, however, the authors have not been able to find a comprehensive and concise statement about how to find out all possible $K$’s for a given lattice $T$. For a given $T$ and $K$, CM points with the CM field $K$ are open subspace of $\text{Isom}(T \otimes \mathbb{Q}; K)^{[\sigma]}$; we do not know whether this action is transitive, or whether there is an action of a larger group.

3.2.4. Bonus symmetry By construction, the K3 surfaces $X^{(1)}$ and $X^{(2)}$ to be used in the orbifold construction have certain amount of automorphisms, $G_1$ and $G_2$, respectively. It happens to be the case for some $(S, T, G; G_s, \Delta)$, though, that a K3 surface $X$ with a generic complex structure in $D(T)$ has $\text{Aut}(X)$ larger than $G$.

For example, think of $(G; G_s, \Delta) = (\mathbb{Z}_m; [1], \mathbb{Z}_m)$ with an $m_{\varphi(m)}$ in Table 1 such that $\varphi(m)$ divides either one of 4, 12, 20. Then $(S, T)$ can be unimodular lattices of rank $(18, 4)$, $(10, 12)$ and $(2, 20)$. For a unimodular $T$, a K3 surface with a generic complex structure in $D(T)$ has a $\mathbb{Z}_2$ purely non-symplectic automorphism (generated by the combination of $(-1)$ multiplication on $T$ and id. on $S$) [49]. This automorphism is a part of the symmetry $\Delta \cong \mathbb{Z}_m$ that we imposed, if $m$ is even. For an odd $m$, namely $m = 5, 7, 11, 13, 25, 3, 21, 33, 9$, however, we have more automorphisms ($\mathbb{Z}_{2m} \subset \text{Aut}(X)$) than we imposed for orbifold construction ($G = \Delta = \mathbb{Z}_m$). See also [71], where similar

69 Those isolated points are further subject to identification by a certain finite index subgroup of $\text{Isom}(T)$. More is known in the literature about the identification of those isolated points (e.g., [49, §5]).

70 For a given CM field $K$, one can construct an even lattice $T$ of signature $(2, [K : \mathbb{Q}] - 2)$ with a simple rational Hodge structure of CM type by $K$. This is done [8] by choosing $\lambda \in K_0^*$ of certain kinds, introducing a $\mathbb{Q}$-bilinear form $q_{\lambda}$ on $K$ by using $\lambda$, and identifying a free rank-$[K : \mathbb{Q}]$ abelian subgroup of $K$ as $T$.

Conversely, for a given even lattice $T$ of signature $(2, [K : \mathbb{Q}] - 2)$ with a simple rational Hodge structure of CM-type by a CM field $K$, one can always find an appropriate $\lambda \in K_0^*$ and an embedding $T \hookrightarrow K$ so that $q_{\lambda}|_T = (-, -)_T$; this can be seen by exploiting the property (54). So, all the CM points in $D(T)$ and their CM field $K$ should satisfy the $K$-and-$T$ relation in the construction of [8].

It therefore follows, in particular, that $D(T)$ admits a CM point with the CM field $K$ only when $\text{discr}(T) \sim (-1)^{[K : \mathbb{Q}]/2}D_{K/\mathbb{Q}} \bmod (\mathbb{Q}^{	imes})^2$; here, $D_{K/\mathbb{Q}}$ is the discriminant of the field extension $K/\mathbb{Q}$ (see [70, §3] and references therein).
enhancement of automorphism groups are discussed in the case \((S, T)\) are not necessarily unimodular.

As another class of examples, we may think of a case of \((G; G_s, \Delta)\) with \(G_s \neq \{0\}, \mathbb{Z}_2, S_3\). It is then known that \(|\text{Aut}_s(X^{(1)})|=\infty\) (see footnote 67 and also [72]). So, there are more automorphisms than we impose in this class of examples.

Those bonus automorphisms are available for any point in the period domain \(D(T)\); this means that they act trivially on \(D(T)\). The automorphisms can be realized linearly in the effective theory (not a broken symmetry), because a choice of a complex structure in \(D(T)\) is not shifted by the automorphisms. This observation also indicates that the fluctuation fields of complex structure within \(D(T)\) are neutral under these symmetry transformation.\(^{71}\) Because those bonus automorphisms\(^{72}\) act on \(X^{(i)}\), and are non-trivial transformation on the orbifold geometry \(Y_0 = (X^{(1)} \times X^{(2)})/\Delta\), they are still non-trivial information on the effective field theory.\(^{73}\) When one considers F-theory applications (where the orbifold \(Y_0\) and its crepant resolutions are replaced by a birationally equivalent fourfold \(\tilde{Y}\) and some Kähler parameters are brought to zero (see Sect. 4)), one will be interested in working out how the symmetry acts\(^{74}\) on fluctuation fields other than the complex structure moduli in \(D(T)\). That is beyond the scope of this article, however.

### 3.3. Complex structure moduli masses with \(W = 0\)

For a general choice of the orbifold group \(\Gamma \subset G_1 \times_{\Delta} G_2\), we do not try to say what the cohomology \(H^4(Y; \mathbb{Q})\) is like (a statement analogous to (6) and/or (7)) for \(Y\), a minimal crepant resolution the orbifold \((X^{(1)} \times X^{(2)})/\Gamma\). In the cases \(\Gamma \cong G_1 \cong G_2 \cong \Delta = \mathbb{Z}_m\), the cohomology group \(H^4(Y; \mathbb{Q})\) contains [73]

\[
[(T^{(1)}_0 \otimes T^{(2)}_0) \otimes \mathbb{Q}]^{[\sigma]} \oplus (H^2(Z_{(1)} \times Z_{(2)}; \mathbb{Q}))^\oplus(m-1) \oplus (\cdots),
\]

where \(Z_{(1)}\) and \(Z_{(2)}\) are curve loci of points in \(X^{(1)}\) and \(X^{(2)}\), respectively, fixed under the group \(\Delta\). The last term stands for possible contributions from fixed loci \(Z_{(1)}\) (isolated pts), \(Z_{(2)}\), and \(Z_{(1)} \times (\text{isolated pts})\). The second term has a Hodge structure of level 2 (for a vacuum complex structure within \(\mathcal{M}_{\text{cpx\, st}}\)), and hosts the fluctuation fields of complex structure deforming the \(\mathbb{C}^2/\mathbb{Z}_m\) orbifold singularity. The \(H^2(Z_{(1)} \times Z_{(2)}; \mathbb{Q})\) component may contain a level-0 rational Hodge structure in some cases (see footnote 23 in Sect. 2.2). Possible contributions

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\(^{71}\) Discussion here focuses on automorphisms available for a generic point in \(D(T)\), once \((G; G_s, \Delta)\) and \((S, T)\) are given. For special loci in \(D(T)\), there can be larger group of automorphisms. We will see in Sect. 3.3 in the case of \(\Delta \cong \mathbb{Z}_m\) with \(m > 2\) that \(D(T)\) moduli are stabilized by a DW = \(W = 0\) flux only in the case the vacuum complex structure minimizes the rank of the transcendental lattice \(T_X\) to \(\varphi(m)\). So, the question of real interest is not necessarily about a generic point in \(D(T)\).

\(^{72}\) Besides the bonus automorphisms \((\text{Aut}(X^{(1)}) \times \text{Aut}(X^{(2)}) \backslash (G_1 \times G_2))\), there are also automorphisms \((G_1 \times G_2) \backslash \Gamma\) acting non-trivially on the orbifold \(Y_0 = (X^{(1)} \times X^{(2)})/\Gamma\) by construction. Note that \(\Gamma \subset G_1 \times_{\Delta} G_2 \subseteq G_1 \times G_2\) (because \(\Delta \neq \{1\}\)). Both of the bonus automorphisms and the by-construction automorphisms present themselves as symmetries of the low-energy effective theory.

\(^{73}\) Some of those bonus automorphisms (symmetries) may be broken by a non-trivial flux in \(H^4(Y; \mathbb{Q})\). It is the symmetry respected by the flux that matters in the low-energy effective theory and cosmology after inflation.

\(^{74}\) Choices of configuration of metric and other fields that become equivalent under automorphisms are regarded as one and the same point in the space of path integral. So, an automorphism may be regarded as a gauge symmetry. It makes sense to study non-trivial representations of those automorphisms (gauge symmetries) on field fluctuations instead of throwing away all the modes in non-trivial representations, because two particle excitation state can be gauge-symmetry neutral, while each particle is not.
are also level 0. So, \( W = DW = 0 \) flux are available in those level-0 components. Those fluxes may (or may not) give rise to the complex structure moduli fluctuation fields that move away from \( \mathcal{M}_{\text{cpx str}} \) into \( \mathcal{M}_{\text{cpx str}} \), but they do not generate a mass term or interaction term of moduli fluctuation fields within \( \mathcal{M}_{\text{cpx str}} \).

Let us now focus on supersymmetric fluxes available within the \( [(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}]^{[\sigma]} \) component; only such fluxes can stabilize (generate mass terms of) the \((\ell(1) - 1) + (\ell(2) - 1)\) moduli fluctuation fields\(^{75}\) in \( \mathcal{M}_{\text{cpx str}} \). In the case of \([[\sigma]] = \Delta \cong \mathbb{Z}_2\), we have nothing to modify\(^{76}\) in the discussions in Sects. 2.4 and 2.5. In a case of \( \Delta \cong \mathbb{Z}_m \) with \( m > 2 \), let us start our discussion with an assumption that \( T_X^{(i)} = T_0^{(i)} \), i.e., a generic CM complex structure available within \( D(V_{a0}) \times D(V_{-a0}) \). A few observations to be added to the discussions in Sects. 2.4.1 and 2.4.2 are the following.

First, only a proper subspace of \( (T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q} \) survives the orbifold projection, as we consider a case with \( m > 2 \) now. Second, the decomposition (24) of the vector space \( (T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q} \) is still useful; keeping in mind the fact that individual components \( W_i \) in (24) are in one-to-one with the orbits \( \Phi^\text{full}_{L_i} \) under the action of \( \text{Gal}((K^{(1)} K^{(2)})^\text{nc}/\mathbb{Q}) \), and also the fact that both \( K^{(1)} \) and \( K^{(2)} \) contain a subfield \( \mathbb{Q}[[\sigma]] \), one concludes that \([[\sigma]] \) acts on each one of \( W_i \)'s either trivially entirely or non-trivially entirely on that \( W_i \).

So, we have

\[
[[T_0^{(1)} \otimes T_0^{(2)} \otimes \mathbb{Q}]]^{[\sigma]} \cong \oplus_{i \in \{\text{neut}\}} \oplus_{\Delta \subseteq \mathbb{Z}_m} \oplus \cdots W_i, \tag{72}
\]

where only the subset of \( \{1, \cdots, r\} \) where \( W_i \) is neutral under \( \Delta \)—denoted by \( \text{neut} \)—is retained on the right hand side. Finally, the component with \( i = (20)20 \) is in the subset \( \{\text{neut}\} \), but the component \( i = (20)02 \) is not.\(^{77}\)

We can review the conclusions in Sect. 2.4.2 with those three observations in mind. Now (for \( m > 2 \)), the case-A in page 27 is not logically possible. Besides the case-C, where there is no flux with the \( DW = 0 \) condition available, the only possibility for a supersymmetric flux is the case-B in page 27, where all but one components \( W_i \) in (72) are level-2, and the remaining \( W_{(20)20} \) is simple and level-4. So, to conclude (when \( m > 2 \) and \( T_X^{(i)} = T_0^{(i)} \)), a \( DW = 0 \) flux is possible if and only if the condition (46) is satisfied; such a flux is in the level-4 \( W_{(20)20} \) component, so \( \langle W \rangle \neq 0 \). There is no chance for a \( DW = W = 0 \) flux in \( [[(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}]]^{[\sigma]} \) when \( m > 2 \) and \( T_X^{(i)} = T_0^{(i)} \), because the level-0 \( W_{(20)20} \) component does not survive the orbifold projection when \( m > 2 \).

A \( DW = W = 0 \) flux is possible within \( [[(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}]]^{[\sigma]} \) if and only if \( T_X^{(i)} \not\subset T_0^{(i)} \) for both \( i = 1, 2 \); it is not enough to have \( T_X^{(i)} \not\subset T_0^{(i)} \) for just one of \( i = 1, 2 \). To see this, remember that

\[
[[T_0^{(1)} \otimes T_0^{(2)}]]^{[\sigma]} \cong [[(T_X^{(1)} \otimes T_X^{(2)})]]^{[\sigma]} \oplus [(T_X^{(1)} \otimes T_X^{(2)})][[\sigma]] \oplus [((T_0^{(1)} \otimes T_0^{(2)})[[\sigma]] \oplus [(\overline{T_0^{(1)}} \otimes \overline{T_0^{(2)})]]^{[\sigma]}, \tag{73}
\]

\(^{75}\) \((\ell(1)) - 2\) instead of \((\ell(1)) - 1\) in the case of \( m = 2 \).

\(^{76}\) The cohomology group \( H^2(Y; \mathbb{Q}) \) outside of the \( [[(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}]]^{[\sigma]} \) should be modified, when \( \Gamma \) acts non-trivially on \( q_0^{(i)} := [T_0^{(i)}] \subset H^2(X^{(i)}; \mathbb{Z}) \), for \( i = 1 \) or 2.

\(^{77}\) Note that \( a_0 = 1 \) and \(\mathbb{Z}_m \) only when \( m = 2 \) and \( a_0 = 1 \).
the middle two components on the right-hand side are purely level-2, and the first component consists only of level-2 and level-4 Hodge components; the latter statement is obtained by repeating the argument above (for $T_0^{(1)} \otimes T_0^{(2)}$ with $T_X^{(i)} = T_0^{(i)}$ there). So, a $DW = W = 0$ flux can only be in the last component. Such a flux cannot generate a mass term for the moduli field fluctuations in $D(T_X^{(1)}) \cap D(V_{a0})$ and $D(T_X^{(2)}) \cap D(V_{-a0})$, however.

Therefore, the only possibility for a $DW = W = 0$ flux stabilizing all the complex structure moduli, if $m > 2$, is when $\text{rank}(T_X^{(1)}) = \text{rank}(T_X^{(2)}) = \varphi(m)$ so that there is no moduli within $[(T_X^{(1)} \otimes T_X^{(2)}) \otimes \mathbb{C}]^{[\varphi]}$. The condition (46) is satisfied automatically then $(K^{(1)} \cong K^{(2)} \cong \mathbb{Q}(\tau_m))$, but there is no $W_{(20)|02}$ component to support a $DW = W = 0$ flux when $m > 2$. The $(\ell^{(1)} - 1) + (\ell^{(2)} - 1)$ moduli field fluctuations have Dirac type mass terms from a flux in the $(T_0^{(1)} \otimes T_0^{(2)}) \otimes \mathbb{Q}$ component. So, for all those moduli fields to have masses, $\ell^{(1)} = \ell^{(2)}$ is also necessary, just like the condition (63) in Sect. 2.5. Just like in Sects. 2.4.3 and 2.5, this moduli effective field theory has an approximate $U(1) \times U(1)_R$ symmetry.

4. F-theory Applications and Particle Physics Aspects

In the earlier sections, we have discussed the supersymmetry conditions (9, 10) of fluxes on CM-type Borcea–Voisin Calabi–Yau fourfolds $Y = (X^{(1)} \times X^{(2)}/\mathbb{Z}_2$, and also stabilization of the complex structure moduli. The analysis in Sects. 2 and 3 can be read in the context of M-theory compactification on such fourfolds down to $(2 + 1)$-dimensions; the orbifold geometry $Y_0 = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$ is singular, but the study in Sects. 2 and 3 in that M-theory context should be read as that for a fourfold $Y^{BV}$ which is the minimal and crepant resolution of $Y_0$, with positive values of Kähler parameters for the exceptional cycles.

To think of an F-theory compactification down to $(3 + 1)$-dimensions, however, we need a Calabi–Yau fourfold $\tilde{Y}$ that has a flat elliptic fibration. When F-theory is compactified on $\tilde{Y}$ such that $\mathcal{M}_{cpx \ str}^{[\tilde{Y}]}$ is contained in $\mathcal{M}_{cpx \ str}^{[Y]BV}$, the analysis for presence of a non-trivial supersymmetric flux and stabilization of moduli in $\mathcal{M}_{cpx \ str}^{[\tilde{Y}]}$ is still valid.

In a large fraction of this section, we will be concerned about when and how one can find $\tilde{Y}$ birational to $Y^{BV}$. When a $\tilde{Y}$ is available, its geometry should determine gauge groups and possible matter representations in the effective theory on $(3 + 1)$-dimensions, motivated by $\langle W \rangle \simeq 0$. We will take steps to read out those implications.

4.1. Elliptic fibred K3 surface with a non-symplectic involution One of the technical problems that we face in the context of F-theory compactification is to find, for a given Calabi–Yau variety $Y$ for an M-theory compactification, a set of $(Y, B, \pi)$, where $\pi : Y \to B$ is a flat elliptic fibration and $B$ a base manifold. This is much easier in a

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78 One may also notice (when $m > 2$) that the third term in the expansion in (48) vanishes. So, in the expression for the Kähler potential (57), the third term in the second line vanishes.

79 For example, eq. (6) is justified for smooth manifolds $Y = Y^{BV}$.

80 This condition is for absence of an exotic particle spectrum on $\mathbb{R}^{3,1}$. So, this is a phenomenological constraint.

81 We also need to take the limit in the Kähler moduli so that the volume of the elliptic fibre vanishes, and to keep some part of the purely vertical part of $H^1(Y; \mathbb{Q})$ free from fluxes [74], in order to restore the $SO(3, 1)$ symmetry.
lower dimensional set-up; the classification problem has a long history in the case of \( \dim C Y = 2 \). Nearly a complete classification of \((Y = X(1), B = \mathbb{P}^1(1), \pi_{X(1)}^f)\) is available ([75,76] and references therein) for K3 surfaces \( X(1) \) associated with the 75 choices of \((S_0^{(1)}, T_0^{(1)}, \sigma(1))\) of Nikulin, as we will review briefly in this Sect. 4.1. Such an elliptic fibration \((X(1), \mathbb{P}^1(1), \pi_{X(1)}^f)\) is used in Sects. 4.3 and 4.4 to construct a fourfold \( \tilde{Y} \) birational to a Borcea–Voisin orbifold \( Y_0 = (X(1) \times X(2))/\mathbb{Z}_2 \) where there is a flat elliptic fibration morphism \( \pi : \tilde{Y} \to B_3 \).

We begin with recalling known facts about how we find elliptic fibration morphisms from an algebraic K3 surface to \( \mathbb{P}^1 \). There exists a genus-one curve fibration\(^{82}\) morphism from a K3 surface \( X(1) \) to \( \mathbb{P}^1 \) if and only if there exists a divisor class \([f] \in S_X^{(1)}\) with \([f]^2 = 0\) [77]. The corresponding fibration morphism is denoted by \( \pi_{X(1)}^f : X(1) \to \mathbb{P}^1(1) \). For a genus-one fibration morphism \( \pi_{X(1)}^f : X(1) \to \mathbb{P}^1(1) \) to be an elliptic fibration morphism,\(^{83}\) there must exist another divisor class \([s] \in S_X^{(1)}\) satisfying \((s, f) = +1\) and \((s, s) = -2\). The primitive sublattice generated by \([f]\) and \([s]\) within \( S_X^{(1)} \) is isomorphic to \( U \) then. To repeat, existence of an elliptic fibration is equivalent to existence of a factor \( U \) in \( S_X^{(1)} \).

In the context of F-theory applications, when we write \( S_X^{(1)} = U \oplus W \), the lattice \( W \) contains the information of non-Abelian 7-brane gauge groups, the number of \( U(1) \) gauge fields, and also the spectrum of charges under those gauge groups in 7+1-dimensions. So, a well-motivated classification of elliptic fibration morphisms of \( X(1) \) is equivalent to classifying\(^{84}\) primitive embeddings of \( U \) into \( S_X^{(1)} \) modulo isometry of the lattice \( S_X^{(1)} \). One and the same K3 surface \( X(1) \) (with a common \( S_X^{(1)} \) and \( T_X^{(1)} \)) may have multiple different types of elliptic fibration morphisms in this classification; one of the most famous examples is the case \( S_X = U \oplus E_8^{\oplus 2} \cong \Pi_{11,17} \cong U \oplus (D_{16}; \mathbb{Z}_2) \). An F-theory limit takes the volume of a fibre elliptic curve class \([f] \in U\) to zero, so different choices of \( U \subset S_X^{(1)} \) correspond to different F-theory limits.

In general, \( S_0^{(1)} \subset S_X^{(1)} \); when the complex structure of the K3 surface \( X(1) \) is not fully generic in the period domain of the lattice \( T_0^{(1)} \), \( S_X^{(1)} \) is strictly larger than \( S_0^{(1)} \) (and \( T_X^{(1)} \) is strictly smaller than \( T_0^{(1)} \)). Although it is enough to find a factor \( U \) within

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\(^{82}\) In a genus-one curve fibration, the fibre of a generic point in the base is a curve of genus one; a section \( s : \mathbb{P}^1 \to X(1) \) that covers the base just once does not necessarily exist. In an elliptic fibration, we require that such a section exists; the image of a section is often denoted by \( s \) (by abusing notation). This is to follow the terminology in F-theory community. Genus-one curve fibrations here correspond to elliptic fibrations in math literatures, and elliptic fibrations here to Jacobian elliptic fibrations there. In this article, we stick to the terminology of F-theory community.

\(^{83}\) In this article, we consider only F-theory compactifications down to (3+1)-dimensional space-time, by taking the limit of the Kähler moduli of the fibre elliptic curve. In this context, a fourfold \( Y_1 \) with a genus-one fibration over a threefold \( B \) yields the same effective theory on 3+1-dimensions as a fourfold \( \tilde{Y} \) with an elliptic fibration over \( B \), when \( \tilde{Y} \) is the Jacobian fibration of \( Y_1 \). For this reason, it is fine to restrict our attention only to fourfolds with elliptic fibration morphisms; it should be remembered however that \( \tilde{Y} \) cannot necessarily be made non-singular and Calabi–Yau even when \( Y_1 \) is [78,79]. So, one should be careful about what kind of singularity still leads to sensible physics when one deals exclusively with fourfolds with elliptic fibrations [80].

In this article, however, we do not try to explore that borderline, and restrict our attention only to F-theory compactifications that can be associated with non-singular Calabi–Yaus \( \tilde{Y} \) with elliptic fibration morphisms.

\(^{84}\) A review addressed to string theorists is found in [36].
$S_X^{(1)}$ in constructing an elliptic fibration $\pi_{X^{(1)}} : X^{(1)} \to \mathbb{P}^1_1$, we wish to use the elliptic fibration morphism to construct an elliptic fibration $\pi_Y : Y \to B_3$ with some threefold $B_3$ (which is to be constructed in the following). So, we need to be concerned how the elliptic fibration morphism $\pi_{X^{(1)}} : X^{(1)} \to \mathbb{P}^1_1$ behaves under the generator $\sigma$ of the $\mathbb{Z}_2$ orbifold. We stick to the simplest case where the $U$ sublattice is within $S_X^{(1)}$, which means that

$$\sigma_*^{(1)} : [f] \mapsto [f], \quad \sigma_*^{(1)} : [s] \mapsto [s]. \quad (74)$$

There are two types in the way the involution $\sigma_{(1)}$ acts on a K3 surface with elliptic fibration $(X^{(1)}, \mathbb{P}^1_1, \pi_{X^{(1)}})$ [76, Prop. 2.3]. It always maps the zero section $s$ to itself, but it may be either an identity $\sigma_{(1)}|_s = \text{id}_s$ (Type 1 (referred to as type b in [75])), or a non-trivial holomorphic involution (Type 2 (referred to as type a in [75])). An involution of Type 1 acts on individual fibre elliptic curves, while an involution of Type 2 exchanges two fibre curves (except the fibres over the two $\sigma_{(1)}|_s$-fixed points in the base $\mathbb{P}^1_1$).

Let us take a few examples from the 75 choices of $(S_0^{(1)}, T_0^{(1)}, \sigma_{(1)})$ of [18]. For $S_0^{(1)} = U[2] E_8[2]$, there is no primitive embedding of $U$ into $S_0^{(1)}$, so there is no elliptic fibration [76, Thm. 2.6.(i)]. For $S_0^{(1)} = U$, there is unique elliptic fibration, which is Type 1. Think of the case $S_0^{(1)} = U E_8[2]$, next. An obvious primitive embedding of $U$ into $S_0^{(1)}$ corresponds to an elliptic fibration of Type 2; this embedding is actually the only one available for this $S_0^{(1)}$ [76, Thm. 2.6.(ii)]. For the choice $(S_0^{(1)}, T_0^{(1)})$ for $T_0 = U[2]\mathbb{P}^2_1$, which is for $X^{(1)} = \text{Km} (E \times E')$ for mutually non-isogenous elliptic curves $E$ and $E'$, there are 11 different elliptic fibrations (modulo Isom$(S_0^{(1)})$) [82]; three out of the 11 elliptic fibrations $(\mathcal{J}_{1,2,3}$ in [82]) are Type 2, and the remaining eight $(\mathcal{J}_{4,...,11})$ are Type 1. In the study of [75, 76], it turns out that more than 60 choices out of the 75 in [18] admit at least one elliptic fibration; choices with larger [resp. smaller] $g_{(1)} = (22 - r_{(1)} - a_{(1)})/2$ tend to have less [resp. more] inequivalent primitive embeddings $U \leftrightarrow S_0^{(1)}$ and inequivalent elliptic fibrations consequently. A pair $(\pi_{X^{(1)}}, \sigma_{(1)})$ of Type 2 is rare relatively to one of Type 1, and is possible only for the choices with $g_{(1)} \leq 1$. For more information, see [75, 76] and references therein.

### 4.2. Borcea–Voisin manifold and Weierstrass model

For an F-theory compactification, we need a Calabi–Yau fourfold $Y$ that has an elliptic fibration $\pi : Y \to B_3$ and its section $\sigma : B_3 \to Y$. It is not obvious in F-theory (due to the lack of its theoretical formulation) which one of $Y$ and $Y'$ should be regarded as input data of compactification, when there is a birational pair of Calabi–Yau varieties $Y$ and $Y'$ with no difference in cycles of finite volume or the number of complex structure deformation parameters. This article deals with F-theory compactification on such an equivalence class of Calabi–Yau fourfolds that is represented by a non-singular model $\tilde{Y}$ with a flat elliptic fibration, $\tilde{Y} \to B_3$. Although the Borcea–Voisin manifold $Y_{BV}$—the minimal resolution of the Borcea–Voisin orbifold $Y_0 = (X^{(1)} \times X^{(2)})/\mathbb{Z}_2$—is non-singular, it is hard, or even seems to be impossible for some choices of $(S_0^{(1)}, T_0^{(1)}, [f])$, to find a flat elliptic fibration on

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85 Although we attempted to write down the equivalence relation explicitly above, the choice of the relations may have to be refined or corrected from the version written there.

86 Suppose that a singular fibre of $\pi_{X^{(1)}} : X^{(1)} \to \mathbb{P}^1_1$ contains both an irreducible component in $Z_{(1)}$ and also $\mathbb{P}^1$ not in $Z_{(1)}$—(**). Then the fibration $\text{Bl}_{\sigma} \text{−fixed}(X^{(1)} \times X^{(2)}) \to \mathbb{P}^1_1$ is not flat. Apart from the
\(Y^{BV}\). So, for F-theory applications, let us find \(\tilde{Y}\) that is birational to \(Y^{BV}\), along with a threefold \(B_3\) so that there is a flat elliptic fibration \(\tilde{Y} \to B_3\).

We will find such \(\tilde{Y}\) and \(B_3\) in Sects. 4.3 and 4.4 as a resolution of a Weierstrass model fourfold \(Y^W\); see (79) and (81). As a first step for that purpose, consider an orbifold\(^{87}\) \(Y^W_0 = (X^{(1)W} \times X^{(2)W})/\mathbb{Z}_2\). \(X^{(1)W}\) is the Weierstrass model of a non-singular K3 surface \(X^{(1)}\), which is obtained from \((X^{(1)}, \mathbb{P}^1, \pi_{X^{(1)}})\) discussed in Sect. 4.1 by collapsing \((-2)\)-curves in the singular fibres of \(\pi_{X^{(1)}}\) except those that intersect the section \(s\) of \(\pi_{X^{(1)}}\). The \(\mathbb{Z}_2\) quotient is by \((\sigma_{(1)W}, \sigma_{(2)})\), where \(\sigma_{(1)W}\) is described below.

A K3 surface \(X^{(1)}\) of interest in this article is in a family parameterized by the (CM points in the) period domain \(D(T_0^{(1)})\) characterized by the pair \((S_0^{(1)}, T_0^{(1)})\), where \(\sigma_{(1)}\) acts identically on \(S_0^{(1)}\) and by \(\{(−1)×1\}\) on \(T_0^{(1)}\), as a reminder. Its Weierstrass model \(X^{(1)W}\), however, is regarded as \(X^{(1)}\) with \(S_0^{(1)W} = U\) in the Type 1 case, and the period domain \(D(T_0^{(1)})\) as a special subspace in \(\overline{D(T_0^{(1)W})}\); \(T_0^{(1)W} = \mathbb{P}^2 \oplus E_8\) now. The involution \(\sigma_{(1)W}\) on \(X^{(1)W}\) is that of \(X^{(1)}\) with \(S_0^{(1)} = U\), which multiplies \((-1)\) to the \(y\) coordinate of the Weierstrass equation \(y^2 = x^3 + f(t)x + g(t)\).

In the Type 2 case, its Weierstrass model \(X^{(1)W}\) is regarded as \(X^{(1)}\) with \(S_0^{(1)W} = U \oplus E_8\), and the period domain \(D(T_0^{(1)})\) as a special subspace in \(\overline{D(T_0^{(1)W})}\); \(T_0^{(1)W} = \mathbb{P}^2 \oplus E_8\) now. The involution \(\sigma_{(1)W}\) on \(X^{(1)W}\) is that of \(X^{(1)}\) with \(S_0^{(1)} = U \oplus E_8\), which multiplies \((-1)\) to the inhomogeneous coordinate \(t\) of the base \(\mathbb{P}^1\), where the Weierstrass equation is \(y^2 = x^3 + f(t^2)x + g(t^2)\) \([75, 76]\).

The orbifold \(Y^W_0\) is now well-defined; we claim now that there is a regular map \(Y_0 \to Y^W_0\), and that this map is birational. To see that they are birational, note first that there is a field isomorphism \(\mathbb{C}(X^{(1)W}) \cong \mathbb{C}(X^{(2)})\) because of the birationality between \(X^{(1)}\) and \(X^{(1)W}\). The action of \((\sigma_{(1)W}, \sigma_{(2)})\) on the left and that of \((\sigma_{(1)}, \sigma_{(2)})\) on the right are compatible with this field isomorphism, so we have

\[
\mathbb{C}(Y^W_0) \cong \left[\mathbb{C}(X^{(1)W}) \mathbb{C}(X^{(2)})\right]^\mathbb{Z}_2 \cong \left[\mathbb{C}(X^{(1)}) \mathbb{C}(X^{(2)})\right]^\mathbb{Z}_2 \cong \mathbb{C}(Y_0).
\]

(75)

So, they are birational indeed. The regularity of the map \(Y_0 \to Y^W_0\) follows from

\[
(C[U_i]C[V])^\mathbb{Z}_2 \hookrightarrow \left(\mathbb{C}[X^{(1)}] \mathbb{C}[V]\right)^\mathbb{Z}_2
\]

(76)

\(^{87}\) choice of \(S_0^{(1)} = U \oplus E_8\), which has an elliptic fibration of Type 2, all other Type 2 elliptic fibrations available in K3 surfaces with a non-symplectic involutions fall into the category (**)\(^{88}\). Elliptic fibrations of Type 1 that stay out of the category (**) is when \(S_0^{(1)} = U \oplus W_0\), with \(W_0\) containing only \(A_1\)’s and the Mordell–Weil group, but no other \(ADE\)-type lattices. Such \(S_0^{(1)} = U \oplus W_0\) constitutes a small fraction of the tables in \([76]\).

\(^{88}\) It is likely that the constructions of \((\tilde{Y}, B_3, \pi)\) starting from here are not the most general ones with a moduli space containing \(D(T_0^{(1)}) \times D(T_0^{(2)})\). The authors are not yet ready to write down a broader class of constructions, however.
for open patches $V$ of $X^{(2)}$; here, $U_i$'s are open patches of $X^{(1)W}$ and $X^{(1)}_i$'s those of $X^{(1)}$ so that $X^{(1)}_i$'s are mapped to $U_i$'s under the regular map $X^{(1)} \to X^{(1)W}$.

Construction of $Y^{W0}_0$ from $Y^{BV}_0$ or from $Y^{W}_0$ is essentially the same for both Type 1 and Type 2. From this point on, however, we need to deal with the Type 1 and 2 cases separately in the construction of a Weierstrass-model fourfold $Y^W$ and a non-singular model $\tilde{Y}$ with a flat fibration.

4.3. Fibration and involution of Type 1

4.3.1. Construction of $\tilde{Y}$, and Gauge group and matter representations

In the case of a pair of fibration and involution of Type 1, a Weierstrass model $Y^W$ is obtained by once blowing up $Y^{W0}_0$ ($Y^{W'}_0 \to Y^{W0}_0$), and then blowing it down ($Y^{W'} \to Y^W$), as we elaborate a bit more in the following.

Construction of $Y^{W'}$ from $Y^W_0$ is as follows. The $\mathbb{Z}_2$-orbifold $Y^W_0$ has a two-dimensional locus of singularity that is $A_1$-type for each isolated component of $[Z(2)] \subset B^{(2)}$. The two transverse directions are the transverse direction of $[Z(2)] \subset B^{(2)}$ and also the elliptic fibre direction. For a generic point in $[Z(2)] \subset B^{(2)}$, the locus of $A_1$-singularity consists of two pieces of curves, one of which is a three-fold cover over $\mathbb{P}^1_{(1)}$ and the other a one-fold cover. The proper transform of $Y^W_0$ in a blow-up centred along the latter singular locus (the one-fold covering one) is denoted by $Y^{W'}_0$; one may also think of the blow-up along both of the singular loci, where the proper transform is denoted by $Y^{W''}$. See (79) and Fig. 1.

The Weierstrass model $Y^W$ is obtained from $Y^{W'}$ by collapsing the divisors over $[Z(2)] \times \mathbb{P}^1_{(1)}$ that are non-exceptional in the blow-up $Y^{W'} \to Y^W_0$ (see Fig. 1). This variety $Y^W$ has a projection $\pi : Y^W \to B_w := (\mathbb{P}^1_{(1)} \times B^{(2)})$, and is given by

$$y^2 = \tilde{x}^3 + V^2 f(t)\tilde{x} + V^3 g(t) \quad (77)$$

in one of its Affine patch.

The Affine coordinates ($\tilde{x}$, $\tilde{y}$, $t$, $V$, $u$) of $Y^W$ are related with the coordinates ($x$, $y$, $t$, $v$, $u$) of $X^{(1)W}$ and $X^{(2)}$ through

$$t = t, \quad u = u, \quad V = v^2, \quad \tilde{x} = xv^2, \quad \tilde{y} = yv^3. \quad (78)$$

Remember that the involution $\sigma^{(1)W}$ acts trivially on $t$, $u$, and $x$, and by $[-1 \times]$ on $y$ and $v$. Here is a summary (all the arrows between $Y$'s are regular and birational):
Fig. 1. Schematic picture of the singular fibre geometry for a generic point in \([Z(2)]\) in (a) \(Y^W_0\), (b) \(Y^{W'}\), (c) \(Y^{W''}\), and (d) \(Y^W\)

\[
\begin{array}{ccccccccc}
Y^{BV} & \rightarrow & Y^{W''} & \rightarrow & Y^W & \leftarrow & v^*(Y^W) & \leftarrow & v^*(Y^{W''}) & \rightarrow & Y \\
Y_0 & \rightarrow & Y^W_0 & \leftarrow & Y^{W'} & \rightarrow & Y^W & \leftarrow & v^*(Y^W) & \leftarrow & Y \\
B_w & \leftarrow & B_3 & \rightarrow & Y^{W''} & \leftarrow & Y^W & \leftarrow & v^*(Y^W) & \leftarrow & Y \\
\end{array}
\]

See the following discussions for \(B_3 = \text{Bl}_{pt_\times[Z(2)]}(B_w), v^*(Y^W), Y, \) and \(\tilde{Y}\).

So long as complex structure of \(X^{(1)}\) is that of a generic one in \(D(T^{(1)}_0) = D(I_{2,18})\), which means that \(S_X^{(1)} = S_0^{(1)} = U\), there is no difference between \(Y_0\) and \(Y^W_0\); \(Y^{W''}\) is nothing but \(Y^{BV}\); the projection \(\pi : Y^{W''} = Y^{BV} \rightarrow (\mathbb{P}^1(1) \times B(2))\) yields a flat elliptic fibration, so \(\tilde{Y} = Y^{W''}\) and \(B_3 = B_w\). The discriminant locus \(\Delta_{\text{discr}}\) of the elliptic fibration \(Y^W \rightarrow B_w\) is of the form

\[
\Delta_{\text{discr}} = \Delta_f + \Delta_b, \quad \Delta_f = (24\text{pts}) \times B(2), \quad \text{and} \quad \Delta_b = 6(\mathbb{P}^1(1) \times [Z(2)]). \quad (80)
\]

On a generic point in \(\Delta_b\), the singular fibre in \(\tilde{Y} = Y^{W''}\) is the \(I_6^*\)-type in the Kodaira classification [81], and the equation (77) is completely in the non-split type over \(\mathbb{P}^1_{(1)}\) (and also over \(\mathbb{P}^1_{(1)} \times [Z(2)]\)) [83]. So, the \(\mathcal{N} = 1\) supersymmetric effective theory on \(\mathbb{R}^{3,1}\) has one vector multiplet with the gauge group \(G_2\) for each one of the isolated components\(^{89}\) of \([Z(2)]\). The 7-branes \(\Delta_f\) do not yield a massless vector multiplet on the effective theory on \(\mathbb{R}^{3,1}\).

\(^{89}\) There are \(k_2 + 1\) isolated components for all the (75-2) choices of \((S_0^{(2)}, T_0^{(2)})\) from Nikulin’s list.
There may be massless $\mathcal{N} = 1$ chiral multiplets (matter fields) charged under those $G_2$ gauge groups, possibly in the adjoint representation, and also possibly in the 7-representation, because matter hypermultiplets in those two representations can be present in F-theory compactifications to 5+1-dimensions [83,84]. None of them must be charged under multiple $G$’s, because all the irreducible components of $[Z(2)] \subset B^{(2)}$ are disjoint from each other [18]. All those matter fields are in self-real representations, so there is no such things as a formula for the net chirality. Although the (20 $- r_{(1)}(1) + 20 - r_{(2)}(2) = 38 - r_{(2)}(2)$ complex structure moduli of $Y^{W^{\prime}} = Y^{BV}$ remain to be gauge-group neutral moduli chiral multiplets in the effective theory on $\mathbb{R}^{3,1}$, the $g_{(1)} g_{(2)} = 10 g_{(2)}$ complex structure moduli of $Y^{BV}$ are likely to be part of $G_2$-charged matter chiral multiplets; solid evidence for this statement can be provided by studying F-theory compactification on a threefold $M^{BV}$ as the crepant resolution of $(X^{(1)} \times E_7)/\mathbb{Z}_2$.

Now, let us turn to cases where $X^{(1)}$ and $(X^{(1)})^W$ are not isomorphic. In terms of the lattice, let $S_X^{(1)} = \mathbb{Z} \subset W$; $R$ denote the sublattice of $W$ generated by the norm (−2) elements of $W$; $(X^{(1)})$ and $(X^{(1)})^W$ are not mutually isomorphic if and only if $R$ is non-empty. $Y^{BV}$ and $Y^{W^{\prime}}$ are not mutually isomorphic either in such cases. A flat elliptic fibration ($\tilde{Y}, B_3, \pi$) is constructed as reviewed below by starting from $Y^W \rightarrow B_w$, or from $Y^{W^{\prime}} \rightarrow B_w$.

Suppose first that the lattice $R$ contains only $A_n$’s, not $D_n$’s or $E_\nu$’s. It is then known that we can take $B_w$ as $B_3$; $Y^{W^{\prime}}$ has singularity of $A_n$ type over the 7-brane $(n + 1)(pt \times B^{(2)} \subset \Delta_f$, so those codimension-2 singularity is resolved canonically; after a small resolution, a non-singular Calabi–Yau fourfold $\bar{Y}$ is obtained in this case [85,86].

The gauge group on the 7-brane $pt \times B^{(2)}$ becomes $\text{Sp}$-type in the effective theory on $\mathbb{R}^{3,1}$ (and a product of $G_2$’s is from $\mathbb{P}^1(1) \times [Z(2)]$ as before). The matter fields must be in the bifundamental representation of $G_2$ and $Sp$ [87], besides those in the adjoint representations, $G_2 \sim 7$, and the Sp rank-2 antisymmetric representation [84,86] (consistent with the Type IIB brane constructions). All the fourfolds $Y^{BV}$, $Y_0$, $Y^{W^{\prime}}$, $Y^{W^{\prime}}$ and $\bar{Y}$ are Calabi-Yau and are birational, and no cycles of finite volume are added or removed. The $(20 - r_{(1)} - \text{rk}(R)) + 20 - r_{(2)}$ complex structure moduli remain neutral chiral multiplets on $\mathbb{R}^{3,1}$; other complex structure moduli of $Y^{BV}$ will remain massless chiral multiplets on $\mathbb{R}^{3,1}$, but as a part of gauge-charged matter fields (they are the $g_{(1)} g_{(2)} = 10 g_{(2)}$ moduli deforming the $C^2/\mathbb{Z}_2$ singularity of $Y_0$ and the $r(\bar{R})$ moduli of $X^{(1)}$ that reduces $S_X^{(1)}$ back to $S_0^{(1)}$).

90 In the construction of $\tilde{Y}$ in the main text, we consider choosing a complex structure of $X^{(1)}$ from $D(T^{(1)}_0) = D(U_{2,18})$ in such a way that $S_X^{(1)}$ is enhanced from $S_0^{(1)} = U$ to $U \oplus W$ with $W$ containing $A_n$’s. When we consider a complex structure so that $S_0^{(1)} = U \oplus W_0$, $W_0 = W$, and $R \subset W_0$ contains only $A_1$’s (cf footnote 86), however, one may think of another construction of $(Y, B_3, \pi)$. That is to choose $Y^{BV}$ as the fourfold, and $B_3 \equiv \mathbb{P}^1(1) \times B^{(2)}$; this is a flat elliptic fibration [81, Prop. 3.1] in such a case. It is a question of interest whether $Y^{BV}$ is isomorphic to $\tilde{Y}$ in the main text and whether the matter spectra are the same or not.

91 This gauge group $G_2$ in the $\mathbb{R}^{3,1}$ effective theory is enhanced to $SO(7)$ or $SO(8)$, for example, when the cubic polynomial $x^3 + f(t)x + g(t)$ is factorized into a product of a pair of linear and quadratic polynomials, or of three linear polynomials.

92 It is desirable to carry out the Higgs cascade analysis [83,88] of all those kinds of constructions in Sects. 4.3 and 4.4, where F-theory prediction including matter multiplicity information is compared against symmetry breaking processes in the effective field theory on (3+1)-dimensions (or on (5+1)-dimensions). The authors consider that such a study will uncover much more aspects of F-theory compactification on K3 x K3 orbifolds (or on Borcea–Voisin orbifolds) than those presented in Sects. 4.3 and 4.4.
Suppose next that the lattice $R$ contains a factor $D_n$ or $E_6$, corresponding to a singular fibre of $I_{n-4}^*$ type or IV$^*$ in $X^{(1)}$ over $pt_*$ $\in \mathbb{P}^1$. The known prescription$^{93}$ is to set$^{94}$

$$B_3 = B|_{pt_\times \mathbb{P}^1} \times [Z_{(2)}] B_{w},$$

and think of $\nu^*(Y^W)$ with a Weierstrass fibration over $B_3$ for a moment; $\nu : B_3 \rightarrow B_w$ is the blow-up map. The fourfold $\nu^*(Y^W)$ has a parabolic singularity at $\{ \tilde{y} = \tilde{x} = 0 \}$ in the fibre of the exceptional locus $E$ of $B_3 = B|_{pt_\times \mathbb{P}^1} \times [Z_{(2)}] B_{w}$. The ambient space of $\nu^*(Y^W)$ is blown-up three times with the centre in the fibre of $E$, and now the proper transform $\nu^*(Y^W)$ has only $A_{n-5}$ singularity (assuming an even $n > 4$; none for $I_0^*$ or IV$^*$). The fourfold $\nu^*(Y^W)$ is not Calabi–Yau due to the non-trivial morphisms $\nu^*(Y^W) \rightarrow \nu^*(Y^W) \rightarrow Y^W$, but there is a morphism $\nu^*(Y^W) \rightarrow Y$ to a fourfold ramified along the canonical divisor of $\nu^*(Y^W)$, so $Y$ is a Calabi–Yau fourfold. There is also a projection morphism $Y \rightarrow B_3$ (see (79)). The fourfold $Y$ has $D_4$ singularity in the fibre of $\Delta_b$ (the proper transform of $\Delta_b$ under $\nu : B_3 \rightarrow B_w$).

In the case of $I_{n-4}^*$, the fourfold $Y$ also has $D_{4+n}$ singularity in the fibre of $pt_\times B^{(2)}$, there is also $A_{n-5}$ singularity (if $n > 4$) in the fibre of the exceptional divisor $E$ (statements in the rest of this paragraph is for an even $n$). Those singularities in $Y$ should be resolved canonically; further small resolution in the fibre of codimension-2 loci in $B_3$ yields $\tilde{Y}$ that has a flat elliptic fibration over $B_3$ $[85]$. The 7-brane $pt_\times B^{(2)}$ yields SO($2n$) gauge group in the effective theory on $\mathbb{R}^{3,1}$; the effective theory also has an $(\text{Sp}(n-4)/2)k_{2+1}$ gauge group (for an even $n > 4$).$^{95}$ A $I_0^*-I_{n-4}^*$ collision may yield chiral multiplets of 4D $\mathcal{N} = 1$ supersymmetry in the $G_2$–$\text{Sp}(n-4)/2$ bifundamental, and in the $\text{Sp}(n-4)/2$–SO$(2n)$ bifundamental representations (in the case $n = 4$ there is no matter fields associated particularly with the $I_0^*-I_0^*$ collision) $[86]$. Cases with an odd $n > 4$ are less trivial, but remain similar $[83,84]$.

In the case of IV$^*$, we have an $F_4$ vector multiplet on $\mathbb{R}^{3,1}$ from the brane $pt_\times B^{(2)}$ $[83,86]$. Chiral multiplets may arise from the $I_0^*-\text{IV}^*$ collision, which are in the fundamental representations of $G_2$ and $F_4$, but there is no matter in a mixed representation $[84,86]$. The types of matter representations available are the same for all $(k_2 + 1)$ singularity collisions along the $(k_2 + 1)$ disjoint components of $pt_\times [Z_{(2)}]$. Details of the massless spectrum may be different due to a choice of a four-form flux in the non-horizontal part of $H^4(\tilde{Y}; \mathbb{Q})$.

In the case the lattice $R \subset W$ contains a factor $E_7$, $B_3$ is obtained by blowing-up $B_w$ twice; $\tilde{Y}$ is also obtained in a similar procedure. The gauge group on $\mathbb{R}^{3,1}$ becomes $(G_2 \times \text{SU}(2))k_{2+1} \times E_7$. Matter chiral multiplets charged under $E_7$ and singlet under $(G_2 \times \text{SU}(2))k_{2+1}$ $[86]$. Such matter $E_7$–$\text{56}$ fields, however, are not associated with $I_0^*-\text{III}^*$ collision; the base $B_3$ has divisors $\mathbb{P}^1 \times [Z_{(2)}]$, two exceptional divisors in $B_3 \rightarrow B_w$, and $pt_\times B^{(2)}$, over which a generic fibre is of $I_0^*$-type, type-III, non-singular, and of type-III$^*$, respectively; there is no enhancement of singularity over the non-singular–III$^*$ collisions, so there is no $E_7$–$\text{56}$ matter arising from there. The $E_7$–$\text{56}$ matter field may arise from the intersection of $pt_\times B^{(2)}$ with other irreducible components of the discriminant locus, but there is no such intersection

$^{93}$ The prescription of Ref. $[86]$ is to replace $\nu^*(Y^W)$ by the Affine part of $Y$ and then to add the zero section by hand, without discussing birational map between them. In that prescription, the Calabi–Yau condition of $Y$ had to be tested independently from the Calabi–Yau nature of $Y^W$.

$^{94}$ We have in mind that the Kähler parameter is such that the exceptional divisor in the blow-up $B_3 \rightarrow B_w$ has a non-zero positive volume.

$^{95}$ $\text{Sp}(n) = \text{USp}(2n)$ in this notation.
in the K3 x K3 orbifolds we consider in this article. So, there is no $E_7$-charged matter field.

4.3.2. More consequences in physics In all those cases\(^{96}\) where $R$ involves $D_4$, $E_6$ or $E_7$, birational morphisms between the two Calabi–Yau’s $Y^{BV}$ and $Y$ in (79) can be constructed for any choice of moduli in $D([S_X^{(1)}]_\perp) \times D(T_0^{(2)})$. So, those deformation degrees of freedom and their corresponding cohomology groups (i.e., $T_X^{(1)} \otimes T_0^{(2)}$) will remain to be there for $Y$ and $\tilde{Y}$. The $g(1)g(2)$ complex structure moduli of $Y^{BV}$ (and the $H^2(Z_{(1)} \times Z_{(2)}; \mathbb{Q})$ component in $H^4(Y^{BV}; \mathbb{Q})$) may or may not be present in $\tilde{Y}$, but even when they are present, they will be part of $G_2$-charged matter fields. The $\operatorname{rk}(R)$ moduli fields necessary in enhancing the $D_n$, $E_6$ or $E_7$ singularity in $Y^W$ may either be part of gauge-charged matter fields, or be absent as massless degrees of freedom in the F-theory compactification.

If there were any chance of accommodating grand unification of the Standard Model in this Type 1 framework, a GUT gauge group such as $\operatorname{SU}(5)$, $\operatorname{SO}(10)$, $E_6$, and $E_7$ at the level of $(7+1)$-dimensions could only be from $\Delta_f$, because those gauge groups do not fit within $\operatorname{SO}(8)$. We have seen above that implementing $A_4 = \operatorname{SU}(5)$ or $E_6$ in $R \subset W$ of $X^{(1)}$ does not result in an $\operatorname{SU}(5)$ or $E_6$ gauge group on $(3+1)$-dimensions due to the monodromy at the $I_0^6$–$R$ collision.\(^{97}\) Even when we require $D_5$ or $E_7$ within $R \subset W$, there is no chance having a matter field in the spinor representation of $\operatorname{SO}(10)$, or in the $56$ representation of $E_7$, as stated earlier. Furthermore, there is no massless adjoint chiral multiplets of $E_7$ because $h^{0,1}(B^{(2)}) = h^{0,2}(B^{(2)}) = 0$.

The conditions for a $DW = W = 0$ flux (or a $DW = 0$ flux) and study of complex structure moduli stabilization in Sects. 2.4 and 2.5 can be recycled without modifications for F-theory, as we see below. We stick to the Type 1 case available for $S_0^{(1)} = U$ and $T_0^{(1)} = I_{2,18}$. For a CM-type vacuum complex structure such that $T_X^{(1)} = T_0^{(1)}$, then $T_X^{(2)}$ should\(^{98}\) also be of rank 20, so that $T_X^{(2)}$ has a CM point in $D(T_X^{(2)})$ with the CM field $K^{(2)}$ satisfying the condition (46) (or (45)). This means that $\operatorname{rank}(T_0^{(2)})$ is either 20 (when $\operatorname{rank}(S_0^{(2)}) = 2$) or 21 (when $\operatorname{rank}(S_0^{(2)}) = 1$); there are three such pairs $(S_0^{(2)}, T_0^{(2)})$ in Nikulin’s list $(S_0^{(2)} = (+2), U, U[2])$. For any one of the three choices of $(S_0^{(2)}, T_0^{(2)})$, all the 18+(19 or 18) complex structure fluctuation fields in $D(T_0^{(1)}) \times D(T_0^{(2)})$ are valid Calabi–Yau deformations of $\tilde{Y} = Y^{W''}$, not just of $Y^{BV}$. A $DW = W = 0$ flux provides large supersymmetric masses to all those complex structure moduli fluctuations.

For a CM-type vacuum complex structure with $T_X^{(1)} \subsetneq T_0^{(1)}$, for example, when $S_X^{(1)} \supset U \oplus R$, the CM field $K^{(1)}$ has a degree $[K^{(1)} : \mathbb{Q}] = \operatorname{rank}(T_X^{(1)}) < 20$. So, the necessary condition $\operatorname{rank}(T_X^{(1)}) = \operatorname{rank}(T_X^{(2)})$ for (46), which is also for a non-

\(^{96}\) One may think of a case a complex structure is tuned in $D(T_0^{(1)} = I_{2,18})$ so that $(S_X^{(1)}, T_X^{(1)})$ just happens to be identical to one of $(S_0, T_0)$ in Nikulin’s list. It is a question of interest whether there is an isomorphism between $Y^{BV}$ and $\tilde{Y}$ constructed as in the main text.

\(^{97}\) An exception is when $S_0^{(2)} = U[2]E_6[2]$ in the list of Nikulin, because $Z_{(2)}$ is empty and there is no $I_0^6$–$R$ collision; this scenario is still not suitable for GUT, however, because there is no massless matter chiral multiplets charged under $R$.

\(^{98}\) This rank($T_X^{(2)}$) = $[K^{(1)} : \mathbb{Q}]$ condition is only a necessary condition for an existence of such a CM point (cf footnote 70). At least in the case of $T_0^{(2)} = T_X^{(1)}$, we are sure that $D(T_0^{(2)})$ contains a CM point whose CM field is isomorphic to the CM field $K^{(1)}$ of a CM point in $D(T_X^{(1)})$. 


trivial $DW = W = 0$ flux, allows a choice of $(S_0^{(2)}, T_0^{(2)})$ from a broader subset of Nikulin’s list. The complex structure deformation fields in $D(T_X^{(1)}) \times D(T_0^{(2)})$ obtain large supersymmetric masses by a $DW = W = 0$ flux, which one can see by repeating the same discussion as in Sect. 2.5.

The complex structure moduli stabilization in [2] can be regarded as a special case of the general discussion above. Our interpretation is that the fourfolds for F-theory in [2] correspond to $(S_0^{(1)}, T_0^{(1)}) = (U, \Pi_{2,18})$ as stated above, $(S_0^{(2)}, T_0^{(2)})$ that of a Kummer surface $(r_2) = 18, a_2 = 4, k_2 = 7$ and $g_2(2) = 0$, $T_X^{(1)} = U[2]^{\oplus 2} \subseteq T_0^{(1)}$ and $T_X^{(2)} = T_0^{(2)} = U[2]^{\oplus 2}$. The discussion above further indicates that there should be a flux with the vev $\langle W \rangle = 0$, when we choose the vacuum complex structure of all the tori in $X^{(1)} \sim (T^2 \times T^2)/\mathbb{Z}_2$ and $X^{(2)} \sim (T^2 \times T^2)/\mathbb{Z}_2$ so that they all have complex multiplication, and the condition (46) is satisfied.

4.4. Fibration and involution of Type 2 In the Type 2 case, we start from the projection map $Y_W^0 \to B_{w0}$, which is in between singular varieties; $B_{w0} := (\mathbb{P}_1^{(1)} \times X^{(2)})/\mathbb{Z}_2$. Consider the canonical resolution of the $A_1$-singularity of $B_{w0}$, $v : B_w := \tilde{B}_{w0} \to B_{w0}$, and set $Y^W := v^*(Y_0^W)$. Now the projection $Y^W \to B_w$ is a Weierstrass model over a non-singular threefold $B_w$. The fourfold $Y^W$ satisfies the Calabi–Yau condition because $v : B_w \to B_{w0}$ is crepant.

$$
\begin{array}{cccccc}
\ y^{BV} & \ y_0^W & \ y^W & v^*(Y^W) & v^*(Y^W) & \ y \\
\ \downarrow & \ \downarrow & \ \downarrow & \ \downarrow & \ \downarrow & \ \downarrow \\
\ y_0 & \ y_0^W & \ y^W & v^*(Y^W) & v^*(Y^W) & \ y \\
\ \downarrow & \ \downarrow & \ \downarrow & \ \downarrow & \ \downarrow & \ \downarrow \\
\ b_{w0} = (\mathbb{P}_1^{(1)} \times X^{(2)})/\mathbb{Z}_2 & \ b_w = \tilde{b}_{w0} & \ b_w = \tilde{b}_{w0} & \ b_3 & \ b_3 & \ b_3
\end{array}
$$

(81)

See the following discussions for $v^*$, $Y$, and $\tilde{Y}$.

So long as complex structure of $X^{(1)}$ corresponds to a generic point in $D(T_0^{(1)}W) = D(U^{\oplus 2}E_8[2])$, which means that $S_X^{(1)} = S_0^{(1)} = U \oplus E_8[2]$, the Weierstrass model $Y^W$ is already non-singular; the projection $Y^W \to B_w$ is a flat elliptic fibration, so we can set $\tilde{Y} = Y^W$ and $B_3 = B_w$. The base threefold $B_3$ is a $\mathbb{P}^1$-fibration over $B^{(2)}$; the $\mathbb{P}^1$-fibre degenerates into three irreducible pieces ($\mathbb{P}^1 + 2\mathbb{P}^1 + \mathbb{P}^1$) over [$Z_{(2)}$] $\subset B^{(2)}$. Note that there is no difference between $Y_0$ and $Y_0^W$, and that $Y^{BV}$ and $Y^W$ are identical in this generic complex structure. The discriminant locus $\Delta_{\text{discr}}$ of the elliptic fibration $Y^W \to B_w$ consists of 12 isolated components, each one of which is a double cover over $B^{(2)}$ ramified over [$Z_{(2)}$]; each piece is isomorphic to the K3 surface $X^{(2)}$. Here, we assume on the ground of genericity that the 12 pairs of $I_1$ fibres of $X^{(1)}$ stay away from the two fixed points of $\mathbb{P}_1^{(1)}$ under the action of $\sigma^{(1)}$. There is no non-abelian gauge group in the effective theory on $\mathbb{P}^{3,1}$ then.

When the vacuum complex structure of $X^{(1)}$ is tuned so that some of the 12 pairs of $I_1$ fibre come on top of each other (but remain distant from the $\sigma^{(1)}$-fixed locus), $S_X^{(1)}$ may be different from $S_0^{(1)} = U \oplus E_8[2]$, and in particular, the sublattice $R$ of $W$ in $S_X^{(1)} =: U \oplus W$ may contain a pair of copies of an ADE-type root lattice. Because
the discriminant locus of the ADE-type fibre forms a single irreducible component, the effective theory on \( \mathbb{R}^{3,1} \) will have a gauge group of that ADE type, with one chiral multiplet in the adjoint representation (because \( h^{0,2}(X^{(2)}) = 1 \)). A non-trivial gauge flux on these 7-branes may reduce the ADE symmetry further down to a smaller non-abelian gauge group, but we cannot obtain a chiral spectrum on \( \mathbb{R}^{3,1} \) in this way (note that \( c_1(X^{(2)}) = 0 \)).

Consider instead an \( X^{(1)} \) that has a singular fibre at a fixed point of \( \sigma(1) \) in \( \mathbb{P}^1(1) \). Suppose that the singular fibre is \( I_{2n} \) [resp. \( IV^* \) or \( I_0^* \)], and all the other singular fibres of \( X^{(1)} \) are of \( I_1 \) type and are away from the \( \sigma(1) \)-fixed points. The discriminant \( \Delta_{\text{discr}} \) consists of three distinct groups of components. One of them consists of \((12 - n) \) [resp. 8 or 9] copies of \( X^{(2)} \) that do not yield a non-abelian gauge group on \( \mathbb{R}^{3,1} \). Another is a section of the \( \mathbb{P}^1 \)-fibration over \( B(2) \), which yields the SU\((2n)\) [resp. \( E_6 \), or SO\((7)\) (due to monodromy)] gauge group on \( \mathbb{R}^{3,1} \). The last group of 7-branes is the \((k_2 + 1) \) isolated pieces of the exceptional divisors associated with the \( \sigma(1) \)-fixed point in \( \mathbb{P}^1(1) \) where \( X^{(1)} \) has the singular fibre. Each one of those 7-branes yields a gauge group \( SU(n) \) [resp. \( SU(3) \) or \( SU(2) \) (monodromy is absent)] on \( \mathbb{R}^{3,1} \).

In the case of \( I_{2n} \) [resp. \( I_0^* \)], we can set \( B_3 = B_w \), and \( \tilde{Y} \) as the canonical resolution of \( Y^W \) for its codimension-2 singularities followed by a small resolution in the fibre of \( I_{2n} - I_n \) collision [resp. \( I_0^* \)-III collision]. The projection \( \tilde{Y} \to Y^W \to B_3 \) is flat [85]. In the case of \( IV^* \), we can use as \( B_3 \) the blow-up of \( B_w \) centred at the intersection of the \( E_6 \) (Kodaira type \( IV^* \)) 7-brane and the \( SU(3) \) (Kodaira type \( IV \)) 7-branes. \( Y^W \) is pulled backed to be \( \nu^*(Y^W) \) fibred over \( B_3 \); it will be possible to construct birational and regular maps \( \nu^*(Y^W) \to Y^W \) and \( \nu^*(Y^W) \to Y \) (as in Sect. 4.3), where \( Y \) is Calabi–Yau [86], and \( \tilde{Y} \) is obtained as a canonical resolution of the codimension-2 singularities of \( Y \).

If there is any chance of accommodating a GUT gauge group, one might first consider \( SU(5) \) as a part of \( SU(6) \). In this case, there may be 4D \( N=1 \) chiral multiplets in the \( SU(6) - SU(3) \) bifundamental representation localized at the \( I_6 - I_3 \) collision matter curves. But, there is no matter in the rank-2 anti-symmetric representation. The other possibility is \( E_6 \). But, there is no matter fields in the \( E_6 - 27 \) representation; to see this, note that there is no singularity enhancement at the intersection of the \( E_6 \) 7-brane with the exceptional divisor of \( \nu' : B_3 \to B_w \), and that there can be no singularity enhancement away from the orbifold loci. There may be \( E_6 \)-adjoint chiral multiplets from the \( E_6 \) 7-brane, but its irreducible decomposition to \( SU(5) \) subgroup cannot yield a reasonably successful phenomenology [89]. To summarize, it is not possible to implement GUT phenomenology in any one of the constructions considered in this Sect. 4.4.

There is not much to add particularly for the Type 2 case on the flux-induced supersymmetric mass terms of the complex structure moduli fields. The discussion at the end of Sect. 4.3 can be repeated with minimal changes,\(^{100}\) the only difference from the Type 1 case is that \((S_0^{(1)}, T_0^{(1)}) = (UE_8[2], U \oplus B E_8[2])\) rather than \((U, U \oplus B E_8)\).

For a K3 surface \( X^{(1)} \) that corresponds to \( S_0 = U \oplus E_8[2] \), there automatically exists two non-symplectic involutions. One acts on the base, and the other on the fibre. So,\(^{99}\) There is a rule on the Kodaira type of a singular fibre that can appear over the base point of \( \mathbb{P}^1(1) \) fixed by the Type 2 involution [75,76,91]. Those Kodaira types are consistent with the rule.

\(^{100}\) The \( C_2^2/\mathbb{Z}_2^2 \)-deforming moduli of \( Y^{BV} \) and corresponding deformations in \( \tilde{Y} \) are localized in the fibre of non-abelian 7-branes in the case of Type 1, but that is not the case generically for a Type 2 fibration and involution. So, there are gauge-neutral moduli fields whose stabilization / mass term is not discussed in this article (cf. discussion at the end of Sect. 2.2 and footnote 22).
their combination also yields a non-trivial symplectic subgroup of the automorphisms. This means that all the compactifications in a Type 2 case has an extra $\mathbb{Z}_2$ symplectic (=non-R) symmetry in the effective theory (unless the flux breaks it).

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A. Type IIB Orientifold Case

As a special case of the analysis of supersymmetric flux configurations for M/F-theory in Sect. 2, the case of Type IIB orientifold compactification on a Borcea–Voisin threefold $M = (E_\tau \times X^{(2)})/\mathbb{Z}_2$ is covered (see (4)); $X^{(1)} = \text{Km}(E_\phi \times E_\tau)$ corresponds to a choice of $(S_0^{(1)}, T_0^{(1)}, \sigma^{(1)})$ from [18] where $T_0^{(1)} = U[2]U[2], r(1) = 18, a(1) = 4$ and $g(1) = 0$. The conditions (45, 46) for the case of $X^{(1)} = \text{Km}(E_\phi \times E_\tau)$ should therefore be equivalent\(^{101}\) to the conditions worked out in [5]. The two sets of conditions do not look similar at first sight (as reviewed below), but we confirm in the following that they are equivalent indeed. So, this appendix is regarded as a supplementary note to [5]; consistency check in this appendix also gives confidence in the study in Sect. 2 in this article.

Let us start off by recalling the Type IIB conditions in [5] for a non-trivial supersymmetric flux. $K^{(2)}$ and $K_E$ are the endomorphism fields of the CM-type Hodge structure on $T_X^{(2)}$ and $H^1(E_\tau; \mathbb{Q})$, respectively. $n := \text{rank}(T_X^{(2)})$.

When the untwisted sector $T_X^{(2)} \otimes \mathbb{Q} H^1(E_\tau; \mathbb{Q})$ is itself a simple component of the rational Hodge structure,\(^{102}\) it is level-3 and $K^{(2)} \otimes \mathbb{Q} K_E$ is the endomorphism field. A non-trivial $DW = 0$ flux exists if and only if

$$\left( K^{(2)} \otimes \mathbb{Q} K_E \right)^r \cong \mathbb{Q}(\phi), \quad [\mathbb{Q}(\phi) : \mathbb{Q}] = 2. \quad (82)$$

The half set\(^{103}\)

$$\Phi = \left\{ \rho^{(2)}_{(20)} \otimes \rho^{(\tau)}_{(10)}, \rho^{(2)}_{a=1,\ldots,n-2} \otimes \rho^{(\tau)}_{(01)}, \rho^{(2)}_{(02)} \otimes \rho^{(\tau)}_{(10)} \right\} \quad (83)$$

\(^{101}\) In section 3.3 of [5], we worked out orientifold projection on the moduli of the threefold $M$, and found that the twisted sector moduli of complex structure of $M$ are projected out. In this article, the absence of such moduli is understood as absence of the $H^1(Z_{(1)}; \mathbb{Q}) \otimes H^1(Z_{(2)}; \mathbb{Q})$ component; it is of $g(1)g(2) = 0$ dimension.

\(^{102}\) Then there is no chance for a non-trivial flux with $W = 0$.

\(^{103}\) Recall that we always consider the reflex field in the sense of Weil intermediate Jacobian, i.e. the Jacobian $J_W(M)$ associated with $H^{0,3}(M) \oplus H^{2,1}(M)$. 

of all the \(2n\) embeddings \(K^{(2)} \otimes K_E \rightarrow \bar{\mathbb{Q}}\) is used in determining the reflex field.\(^{104}\)

When \(T_X^{(2)} \otimes \mathbb{Q} H^1(E_\tau; \mathbb{Q})\) is not a simple component, instead, \(K^{(2)}\) has a structure of \(K_0 \mathbb{Q}(\xi_S)\) for its totally real subfield \(K_0\) and an imaginary quadratic field \(\mathbb{Q}(\xi_S)\) isomorphic to \(K_E\), and \(T_X^{(2)} \otimes \mathbb{Q} H^1(E_\tau; \mathbb{Q})\) has a structure \(K^{(2)} \oplus K^{(2)}\) under the action of the algebra \(K^{(2)} \otimes \mathbb{Q} K_E\) (\(K_E\) acts through an isomorphisms \(\mathbb{Q}(\xi_S) \cong K_E\)). For a non-trivial \(DW = 0\) flux to exist, it is necessary and sufficient that

\[
(K^{(2)})^r \cong \mathbb{Q}(\phi) \cong \mathbb{Q}(\tau). \tag{84}
\]

A few more words are necessary for this condition to have a clear meaning. Let \(\theta_{a=1,\ldots,n/2}\) be the embeddings \(K_0 \rightarrow \bar{\mathbb{Q}}\), and \(\theta^+_a\) those of \(K^{(2)}\) so that their restriction on \(K_0\) are \(\theta_a\), and \(\theta^-_a(\xi_S)\) [resp. \(\theta^+_a(\xi_S)\)] is in the upper [resp. lower] complex half plane. The reflex field \((K^{(2)})^r\) in the condition (84) should be for the half set\(^{105}\)

\[
\left\{ \theta^+_a | a = 1, \ldots, n/2 \right\}. \tag{85}
\]

The case \(T_X^{(2)} \otimes H^1(E_\tau; \mathbb{Q})\) is simple: Now, we begin with making the condition (82) more explicit. To this end, a set of notations is introduced in order to capture the structure of the fields \(K^{(2)}\) and \(K_E\). As a general property of CM fields, \(K^{(2)}\) has a structure of \(K_0(\chi)\) where \(K_0\) is the totally real subfield of \(K^{(2)}\), and \(\chi\) an element of \(K^{(2)}\) with the following properties: \(\chi^2 \in K_0\), and the element \(Q := -\chi^2\) in \(K_0\) is mapped onto the real positive axis by all the \([K_0 : \mathbb{Q}] = n/2\) embeddings \(K_0 \rightarrow \bar{\mathbb{Q}}\). Similarly, \(K_E = \mathbb{Q}(\xi)\) for some \(\xi \in K_E\) such that \(p := -\xi^2 \in \mathbb{Q}_{>0}\). The vector space \(K^{(2)} \otimes \mathbb{Q} K_E\) is regarded 4-dimensional over \(K_0\) generated by \(\{1, \chi, \xi, \chi\xi\}\); the totally real subfield of \(K^{(2)} \otimes K_E\)—denoted by \(K_0^{\text{tot}}\)—is 2-dimensional over \(K_0\) generated by \(\{1, \chi\xi\}\).

The condition that the reflex field in (82) is an imaginary quadratic extension of \(\mathbb{Q}\) is equivalent to existence of \(\eta \in K^{(2)} \otimes K_E\) such that its images by the \(n\) embeddings in \(\Phi\) are all identical \(\eta \in \bar{\mathbb{Q}}\) which generates an imaginary quadratic field \(\mathbb{Q}(\eta)\). For

\[
\eta = A + B\chi + C\xi + D\chi\xi \in K^{(2)} \otimes K_E, \quad A, B, C, D \in K_0, \tag{86}
\]

the condition \(\eta^2 \in \mathbb{Q}\) is equivalent to

\[
AC = QBD, \quad AB = pCD, \quad AD + BC = 0, \quad (A^2 - QB^2 - pC^2 + pQD^2) \in \mathbb{Q}. \tag{87}
\]

This leaves five distinct possibilities: i) none of \(A, B, C, D\) is zero, ii-A) \(A \neq 0\), and \(B = C = D = 0\), ii-B) \(B \neq 0\) and three others are zero, ii-C) \(C \neq 0\) and three others are zero, and ii-D) \(D \neq 0\) and three others are zero.

In fact, only the possibility ii-C) is viable. The possibility i) runs into a contradiction: \((B/D)\) is a well-defined element of the totally real field \(K_0\) in this possibility, and yet one can derive that \((B/D)^2 = -p \in \mathbb{Q}_{<0}\). In the possibilities ii-A) and ii-D), the element \(\eta = A\) or \(\eta = D\chi\xi\) would not generate a totally imaginary extension over \(K_0^{\text{tot}}\). The

\(^{104}\) Note that we started out in F/M-theory analysis in Sect. 2 in this article by assuming that \(X^{(1)} = \text{Km}(E_\phi \times E_\tau)\) is of CM type (that both \(E_\tau\) and \(E_\phi\) are CM elliptic curves). In the analysis of [5], however, the CM nature of \(E_\phi\), namely \([\mathbb{Q}(\phi) : \mathbb{Q}] = 2\), follows from the CM nature of \(X^{(2)}\) and \(E_\tau\) and the supersymmetry conditions on a non-trivial flux.

\(^{105}\) It was not clearly stated in [5] which half set of the \(n\) embeddings of \(K^{(2)}\) should be used in determining the reflex field in (84).
posibility ii-B) cannot be consistent with the condition that the images of \( \eta = B_X \) under the \( n \) embeddings in \( \Phi \) should be all identical; \( \rho_{(2)}^{(20)}(B_X) = -\rho_{(02)}^{(20)}(B_X) \neq 0 \).

Let us focus on the remaining ii-C) possibility. The condition (87) implies that \( C^2 \in \mathbb{Q} \). There are two cases, \((*)1) C \neq 0 \in \mathbb{Q} \), and \((*)2) C \notin \mathbb{Q} \) whose square is a positive rational number \( r \in \mathbb{Q}_{>0} \) that is not a square.

In the case \((*)1)\), the condition that all the \( n \) images of \( \eta = C_t \) are identical is satisfied if and only if \( n = 2 \); if \( n > 2 \), then \( \rho_{(2)}^{(2)}(\eta) = C \rho_{(01)}^{(2)}(\tau) \) cannot be the same as the images \( \rho_{(02)}^{(2)}(C) \rho_{(10)}^{(2)}(\tau) \) and \( \rho_{(02)}^{(2)}(C) \rho_{(10)}^{(2)}(\tau) \). So, \( K^{(2)} = \mathbb{Q}(\sqrt{r}) \) must be some imaginary quadratic field, and the reflex field \( (K^{(2)} \otimes K_E)^\vee \) must be \( \mathbb{Q}(\sqrt{-p}) \cong K_E \). It follows that \( E_\phi \) also has the endomorphism field \( \mathbb{Q}(\sqrt{-p}) \), \( E_\tau \) and \( E_\phi \) are isogenous (and are both CM), and \( X^{(1)} = \text{Km}(E_\phi \times E_\tau) \) has a rank-20 Néron–Severi lattice. So, to conclude, the case \((*)1)\) solution to the condition (87) implies that \( T_X^{(1)} \subset T_0^{(1)} \), \( K^{(1)} \cong \mathbb{Q}(\phi) \cong K_E \cong \mathbb{Q}(\sqrt{-p}) \), \( K^{(2)} \) is an imaginary quadratic field (and is not isomorphic to \( \mathbb{Q}(\sqrt{-p}) \) as assumed before (82)), and the condition (45) is satisfied; both \( \rho_{(2)}^{(2)}(K_0^{(1)}) = \rho_{(20)}^{(2)}(K_0^{(2)}) = \mathbb{Q} \).

In the case \((*)2)\), the totally real field \( K_0 \) must be a real quadratic field. To see this, note that \( K_0 \) contains \( \mathbb{Q}(C) \cong \mathbb{Q}(\sqrt{r}) \), which means that \( n/2 \geq 2 \). The condition that all the \( n \) images of \( \eta = C_t \) should be the same now implies that \( \rho_{(2)}^{(2)}(C) = -\rho_{(20)}^{(2)}(C) \).

Because the \( n/2 \) embeddings of \( K_0 \) should yield the same number of two different embeddings of the subfield \( \mathbb{Q}(C) \), \( (n-2)/2 \) must be equal to \( 2/2 \); \( n = 4 \). Therefore, \( K^{(2)} \cong \mathbb{Q}(\sqrt{r}, C) \), its totally real subfield must be \( K_0^{(2)} \cong \mathbb{Q}(C) \), and \( (K^{(2)} \otimes K_E)^\vee \cong \mathbb{Q}(\sqrt{-pr}) \).

Now, the remaining condition in (82) is \( \mathbb{Q}(\phi) \cong \mathbb{Q}(\sqrt{-pr}) \). So, it turns out that \( K^{(1)} \cong \mathbb{Q}(\sqrt{-p}, \sqrt{-pr}) \), and \( K_0^{(1)} \cong \mathbb{Q}(\sqrt{r}) \). Thus, to summarize, the case \((*)2)\) solution to the condition (82) implies that \( T_X^{(1)} = T_0^{(1)} \), \( K_0^{(1)} \cong K_0^{(2)} \cong \mathbb{Q}(\sqrt{r}) \), and hence the condition (45), in particular.

**The case \( T_X^{(2)} \otimes_{\mathbb{Q}} H^1(E_\tau; \mathbb{Q}) \) is not simple:** Let us now turn to the case \( T_X^{(2)} \otimes_{\mathbb{Q}} H^1(E_\tau; \mathbb{Q}) \) is not itself a simple component of the rational Hodge structure. The condition that the reflex field \( (K^{(2)}\vee) \) with respect to the half set (85) should be imaginary and quadratic implies in fact that \( n/2 = 1 \). So, \( K^{(2)} \) also needs to be an imaginary quadratic field. To conclude, the condition (85) implies \( K^{(1)} = \mathbb{Q}(\phi) \cong K_E \), \( T_X^{(1)} \not\subset T_0^{(1)} \), and \( K^{(2)} \) is also isomorphic to \( \mathbb{Q}(\phi) \); the condition (46) is satisfied.

To wrap up, here is what we learned in this appendix, stated in a colloquial language. Although it is not apparent from the Type IIB conditions (82, 85) in [5], only small classes of CM fields \( K^{(2)} \) can satisfy either one of those conditions; the analysis in this appendix left \([K^{(2)} : \mathbb{Q}] = 2, 4 \) as the only possibilities, in particular. The M/F-theory condition (45, 46) in the main text of this article also imply \([K^{(2)} : \mathbb{Q}] = 2, 4 \), because the CM field \( K^{(1)} \) for \( X^{(1)} = \text{Km}(E_\tau \times E_\phi) \) can only be degree-4 or degree-2 extension over \( \mathbb{Q} \). So, both perspectives led us to the same result.

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