SCHRÖDINGER TYPE OPERATORS WITH UNBOUNDED DIFFUSION 
AND POTENTIAL TERMS

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Abstract. We prove that the realization $A_p$ in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, of the Schrödinger type operator $A = (1 + |x|^\alpha)\Delta - |x|^{\beta}$ with domain $D(A_p) = \{ u \in W^{2,p}(\mathbb{R}^N) : Au \in L^p(\mathbb{R}^N) \}$ generates a strongly continuous analytic semigroup provided that $N > 2$, $\alpha > 2$, and $\beta > \alpha - 2$. Moreover this semigroup is consistent, irreducible, immediately compact and ultracontractive.

1. Introduction

In this paper we study the generation of analytic semigroups in $L^p$-spaces of Schrödinger type operators of the form

(1.1) \quad Au(x) = a(x)\Delta u(x) - V(x)u(x), \quad x \in \mathbb{R}^N,

where $a(x) = 1 + |x|^\alpha$ and $V(x) = |x|^\beta$ with $\alpha > 2$ and $\beta > \alpha - 2$. We investigate also spectral properties of such semigroups. In the case when $\alpha \in [0,2]$ and $\beta \geq 0$, generation results of analytic semigroups for suitable realizations $A_p$ of the operator $A$ in $L^p(\mathbb{R}^N)$ have been proved in [4].

For $\beta = 0$ and $\alpha > 2$, the generation results depend upon $N$ as it is proved in [7]. More specifically, if $N = 1, 2$ no realization of $A$ in $L^p(\mathbb{R}^N)$ generates a strongly continuous (resp. analytic) semigroup. The same happens if $N \geq 3$ and $p \leq N/(N-2)$. On the other hand, if $N \geq 3$ and $p > N/(N-2)$, then the maximal realization $A_p$ of the operator $A$ in $L^p(\mathbb{R}^N)$ generates a positive analytic semigroup, which is also contractive if $\alpha \geq (p-1)(N-2)$.

Generation results concerning the case where $\beta = 0$ and with drift terms of the form $|x|^\alpha - 2 x$ were obtained recently in [8].

Here we consider the case where $\alpha > 2$ and assume that $N > 2$. Let us denote by $A_p$ the realization of $A$ in $L^p(\mathbb{R}^N)$ endowed with its maximal domain

(1.2) \quad D_{p,max}(A) = \{ u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) : Au \in L^p(\mathbb{R}^N) \}.

In the main result of the paper we prove that, for any $1 < p < \infty$, the realization $A_p$ of $A$ in $L^p(\mathbb{R}^N)$, with domain

$D(A_p) = \{ u \in W^{2,p}(\mathbb{R}^N) : Au \in L^p(\mathbb{R}^N) \},$

generates a positive strongly continuous and analytic semigroup $(T_p(t))_{t \geq 0}$ for any $\beta > \alpha - 2$. This semigroup is also consistent, irreducible, immediately compact and ultracontractive.

The paper is structured as follows. In Section 2 we study the invariance of $C_0(\mathbb{R}^N)$ under the semigroup generated by $A$ in $C_b(\mathbb{R}^N)$ and show its compactness. In Section 3 we use
It is well-known that equations (2.1) admit at least a solution in $L^p(\mathbb{R}^N)$. Then, in Section 4 we prove the generation results.

**Notation.** For any $k \in \mathbb{N} \cup \{\infty\}$ we denote by $C^k_c(\mathbb{R}^N)$ the set of all functions $f : \mathbb{R}^N \to \mathbb{R}$ that are continuously differentiable in $\mathbb{R}^N$ up to $k$-th order and have compact support (say $\text{supp}(f)$). The space $C_b(\mathbb{R}^N)$ is the set of all bounded and continuous functions $f : \mathbb{R}^N \to \mathbb{R}$, and we denote by $\|f\|_\infty$ its sup-norm, i.e., $\|f\|_\infty = \sup_{x \in \mathbb{R}^N} |f(x)|$. We use also the space $C_0(\mathbb{R}^N) := \{f \in C_b(\mathbb{R}^N) : \lim_{|x| \to \infty} f(x) = 0\}$. If $f$ is smooth enough we set
\[
|\nabla f(x)|^2 = \sum_{i=1}^N |D_i f(x)|^2, \quad |D^2 f(x)|^2 = \sum_{i,j=1}^N |D_{ij} f(x)|^2.
\]

For any $x_0 \in \mathbb{R}^N$ and any $r > 0$ we denote by $B(x_0, r) \subset \mathbb{R}^N$ the open ball, centered at $x_0$ with radius $r$. We simply write $B(r)$ when $x_0 = 0$. The function $\chi_E$ denotes the characteristic function of the (measurable) set $E$, i.e., $\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = 0$ otherwise.

For any $p \in [1, \infty)$ and any positive measure $d\mu$, we simply write $L^p_\mu$ instead of $L^p(\mathbb{R}^N, d\mu)$. The Euclidean inner product in $L^2_\mu$ is denoted by $(\cdot, \cdot)_\mu$. In the particular case when $\mu$ is the Lebesgue measure, we keep the classical notation $L^p(\mathbb{R}^N)$ for any $p \in [1, \infty)$. Finally, by $x \cdot y$ we denote the Euclidean scalar product of the vectors $x, y \in \mathbb{R}^N$.

## 2. Generation of semigroups in $C_0(\mathbb{R}^N)$

In this section we recall some properties of the elliptic and parabolic problems associated with $A$ in $C_b(\mathbb{R}^N)$. We prove the existence of a Lyapunov function for $A$ in the case where $\alpha > 2$ and $\beta > \alpha - 2$. This implies the uniqueness of the solution semigroup $(T(t))_{t \geq 0}$ to the associated parabolic problem. Using a dominating argument, we show that $T(t)$ is compact and $T(t)C_0(\mathbb{R}^N) \subset C_0(\mathbb{R}^N)$.

First, we endow $A$ with its maximal domain in $C_b(\mathbb{R}^N)$
\[
D_{\text{max}}(A) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^N), \ 1 \leq p < \infty : Au \in C_b(\mathbb{R}^N)\}.
\]

Then, we consider for any $\lambda > 0$ and $f \in C_b(\mathbb{R}^N)$ the elliptic equation
\[
(2.1) \quad \lambda u - Au = f.
\]

It is well-known that equations (2.1) admit at least a solution in $D_{\text{max}}(A)$ (see [3] Theorem 2.1.1). A solution is obtained as follows.

Take the unique solution to the Dirichlet problem associated with $\lambda - A$ into the balls $B(0, n)$ for $n \in \mathbb{N}$. Using an Schauder interior estimates one can prove that the sequence of solutions so obtained converges to a solution $u$ of (2.1). It is also known that a solution to (2.1) is in general not unique. The solution $u$, which we obtained by approximation, is nonnegative whenever $f \geq 0$.

As regards the parabolic problem
\[
(2.2) \quad \begin{cases} u_t(t,x) = Au(t,x) & x \in \mathbb{R}^N, \ t > 0, \\
\ t(t, x) = f(x) & x \in \mathbb{R}^N, \end{cases}
\]
where \( f \in C_b(\mathbb{R}^N) \), it is well-known that one can associate a semigroup \((T(t))_{t \geq 0}\) of bounded operator in \( C_b(\mathbb{R}^N) \) such that \( u(t, x) = T(t)f(x) \) is a solution of (2.2) in the following sense:

\[
u \in C([0, +\infty) \times \mathbb{R}^N) \cap C^{1+\frac{\gamma}{2}+\sigma}_{loc}(0, +\infty) \times \mathbb{R}^N)\]

and \( u \) solves (2.2) for any \( f \in C_b(\mathbb{R}^N) \) and some \( \sigma \in (0, 1) \). Uniqueness of solutions to (2.2) in general is not guaranteed. Moreover the semigroup \((T(t))_{t \geq 0}\) is not strongly continuous in \( C_b(\mathbb{R}^N) \) and do not preserve in general the space \( C_b(\mathbb{R}^N) \). For more details we refer to Section 2.

Uniqueness is obtained if there exists a positive function \( \varphi(x) \in C^2(\mathbb{R}^N) \), called Lyapunov function, such that \( \lim_{|x| \to \infty} \varphi(x) = +\infty \) and \( A\varphi - \lambda \varphi \leq 0 \) for some \( \lambda > 0 \).

**Proposition 2.1.** Let \( N > 2, \alpha > 2 \) and \( \beta > \alpha - 2 \). Let \( \varphi = 1 + |x|^\gamma \) where \( \gamma > 2 \) then there existence constant \( C > 0 \) such that

\[ A\varphi \leq C\varphi \]

**Proof.** An easy computation gives

\[ A\varphi = \gamma(N + \gamma - 2)(1 + |x|^\alpha)|x|^\gamma - 2 - (1 + |x|^\gamma)|x|^\beta, \]

then since \( \beta > \alpha - 2 \) there existence \( C > 0 \) such that

\[ \gamma(N + \gamma - 2)(1 + |x|^\alpha)|x|^\gamma - 2 \leq (1 + |x|^\gamma)|x|^\beta + C(1 + |x|^\gamma). \]

Then we can assert that problem (2.2) admits an unique solution in \( C([0, \infty) \times \mathbb{R}^N) \cap C^{1,2}((0, \infty) \times \mathbb{R}^N) \) and problem (2.1) admits an unique solutions in \( D_{max}(A) \).

In order to investigate the compactness of the semigroup and the invariance of \( C_0(\mathbb{R}^N) \) we check the behaviour of \( T(t) \). We use the following result (see Theorem 5.1.11).

**Theorem 2.2.** Let us fix \( t > 0 \). Then \( T(t) \in C_0(\mathbb{R}^N) \) if and only if \( T(t) \) is compact and \( C_0(\mathbb{R}^N) \) is invariant for \( T(t) \).

Let \( A_0 \) be the operator defined by \( A_0 := a(x)\Delta \). By Example 7.3 or Proposition 2.2 (iii), we have that the minimal semigroup \((S(t))\) is generated by \((A_0, D_{max}(A_0)) \cap C_0(\mathbb{R}^N) \). Moreover the resolvent and the semigroup map \( C_0(\mathbb{R}^N) \) into \( C_0(\mathbb{R}^N) \) and are compact.

Set \( v(t, x) = S(t)f(x) \) and \( u(t, x) = T(t)f(x) \) for \( t > 0, x \in \mathbb{R}^N \) and \( 0 \leq f \in C_0(\mathbb{R}^N) \). Then the function \( w(t, x) = v(t, x) - u(t, x) \) solves

\[
\begin{align*}
    w(t, x) &= A_0w(t, x) + V(x)u(t, x), \quad t > 0, \\
    w(0, x) &= 0, \quad x \in \mathbb{R}^N.
\end{align*}
\]

So, applying Theorem 4.1.3, we have \( w \geq 0 \) and hence \( T(t) \leq S(t) \). Thus, \( T(t) \in C_0(\mathbb{R}^N) \), since \( S(t) \in C_0(\mathbb{R}^N) \) for any \( t > 0 \) (see Proposition 2.2 (iii)). Thus, \( T(t) \) is compact and \( C_0(\mathbb{R}^N) \) is invariant for \( T(t) \) (cf. Theorem 5.1.11)). Then we have proved the following proposition

**Proposition 2.3.** The semigroup \((T(t))\) is generated by \((A, D_{max}(A)) \cap C_0(\mathbb{R}^N) \), maps \( C_0(\mathbb{R}^N) \) into \( C_0(\mathbb{R}^N) \), and is compact.
3. Solvability of the elliptic problem in $L^p(\mathbb{R}^N)$

In this section we study the existence and uniqueness of the elliptic problem $\lambda u - A_p u = f$ for a given $f \in L^p(\mathbb{R}^N)$, $1 < p < \infty$ and $\lambda \geq 0$. Let us consider first the case $\lambda = 0$.

We note that the equation $(1 + |x|^\alpha)\Delta u - V u = f$ is equivalent to the equation

$$\Delta u - \frac{V}{1 + |x|^\alpha}u = \frac{f}{1 + |x|^\alpha} =: \tilde{f}.$$ 

Therefore we focus our attention to the $L^p$-realization $\tilde{A}_p$ of the Schrödinger operator

$$\tilde{A} = \Delta - \frac{V}{1 + |x|^\alpha} = \Delta - \tilde{V}.$$ 

Here $0 \leq \tilde{V} \in L^1_{\text{loc}}(\mathbb{R}^N)$. So, by standard results, it follows that $0 \in \rho(\tilde{A}_p)$ and

$$(-\tilde{A}_p)^{-1}\tilde{f}(x) = \int_{\mathbb{R}^N} G(x,y)\tilde{f}(y)dy,$$ 

where $G$ denotes the Green function of $\tilde{A}_p$ which is given by its heat kernel $\tilde{p}$

$$G(x,y) = \int_0^\infty \tilde{p}(t,x,y)dt.$$ 

Thus the function $u = \int_{\mathbb{R}^N} G(x,y)\frac{f(y)}{1 + |y|^\alpha}dy \in D_{\text{p,max}}(A)$ and solves $A_p u = f$ for every $f \in L^p(\mathbb{R}^N)$. So we have to study the operator

$$u(x) = Lf(x) := \int_{\mathbb{R}^N} G(x,y)\frac{f(y)}{1 + |y|^\alpha}dy.$$ 

To this purpose, we use the bounds of $G(x,y)$ obtained in [11] when the potential of $\tilde{A}_p$ belongs to the reverse Hölder class $B_q$ for some $q \geq N/2$.

We recall that a nonnegative locally $L^q$-integrable function $V$ on $\mathbb{R}^N$ is said to be in $B_q$, $1 < q < \infty$, if there exists $C > 0$ such that the reverse Hölder inequality

$$\left( \frac{1}{|B|} \int_B V^q(x)dx \right)^{1/q} \leq C \left( \frac{1}{|B|} \int_B V(x)dx \right)$$

holds for every ball $B$ in $\mathbb{R}^N$. A nonnegative function $V \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ is in $B_\infty$ if

$$\|V\|_{L^\infty(B)} \leq C \left( \frac{1}{|B|} \int_B V(x)dx \right)$$

for any ball $B$ in $\mathbb{R}^N$.

One can easily verify that

$$\tilde{V} \in \begin{cases} 
B_\infty & \text{if } \beta - \alpha \geq 0 \\
B_q & \text{if } \beta - \alpha > -\frac{N}{q} \\
B_{\frac{q}{2}} & \text{if } \beta - \alpha > -2 \\
B_{N} & \text{if } \beta - \alpha > -1 
\end{cases}$$
for some \( q > 1 \). So, it follows from \([11\), Theorem 2.7\] that, if \( \beta - \alpha > -2 \) then for any \( k > 0 \) there is some constant \( C_k > 0 \) such that for any \( x, y \in \mathbb{R}^N \)

\[
|G(x, y)| \leq C_k \frac{1}{(1 + m(x)|x - y|)^k} \cdot \frac{1}{|x - y|^{N-2}},
\]

where the function \( m \) is defined by

\[
\frac{1}{m(x)} := \sup_{r > 0} \left\{ r : \frac{1}{r^{N-2}} \int_{B(x, r)} \tilde{V}(y) dy \leq 1 \right\}, \quad x \in \mathbb{R}^N.
\]

Due to the importance of the auxiliary function \( m \) we give a lower bound.

**Lemma 3.1.** Let \( \alpha - 2 < \beta < \alpha \). There exists \( C = C(\alpha, \beta, N) \) such that

\[
m(x) \geq C (1 + |x|) \frac{2\alpha}{\alpha - \beta}.
\]

**Proof.** Fix \( x \in \mathbb{R}^N \), and set \( f_x(r) = \frac{1}{r^{N-2}} \int_{B(x, r)} \tilde{V}(y) dy, r > 0 \). Since \( \tilde{V} \in B_{N/2} \) implies \( V \in B_q \) for some \( q > \frac{N}{2} \), by \([11\, Lemma 1.2]\), we have

\[
\lim_{r \to 0} f_x(r) = 0 \quad \text{and} \quad \lim_{r \to \infty} f_x(r) = \infty.
\]

Thus, \( 0 < m(x) < \infty \).

In order to estimate \( \frac{1}{m(x)} \) we need to find \( r_0 = r_0(x) \) such that \( r \in [r_0, \infty[ \) implies \( f_x(r) \geq 1 \).

In this case we will have \( \frac{1}{m(x)} \leq r_0 \).

Since \( \tilde{V} \in B_{N/2} \), there exists a constant \( C_1 \) depending only \( \alpha, \beta, N \) such that

\[
\left( \frac{1}{|B|} \int_B \tilde{V}^{N/2}(y) dy \right)^{2/N} \leq C_1 \left( \frac{1}{|B|} \int_B \tilde{V}(y) dy \right)
\]

for any ball \( B \) in \( \mathbb{R}^N \). Then we have

\[
f_x(r) = \frac{\sigma_{N-2}^2}{|B(x, r)|} \int_{B(x, r)} \tilde{V}(y) dy \\
\geq \frac{\sigma_{N-2}^2}{C_1} \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} \tilde{V}(y)^{N/2} dy \right)^{2/N} \\
= \frac{\sigma_{N-2}^2}{C_1} \left( \int_{B(x, r)} \tilde{V}(y)^{N/2} dy \right)^{2/N}.
\]

Hence, if

\[
\int_{B(x, r)} \tilde{V}(y)^{N/2} dy > C_2 \geq 0,
\]

then \( f_x(r) \geq 1 \), where \( C_2 = C_2(\alpha, \beta, N) = \frac{C_{N/2}^{N/2}}{\sigma_{N-2}^{N-2}} \). Note that \( \tilde{V} \geq \tilde{V}^* \) in \( \mathbb{R}^N \setminus B(0, 1) \) with \( \tilde{V}^*(x) = \frac{1}{4} |x|^\beta - \alpha \). Hence,

\[
\int_{B(x, r)} \tilde{V}(y)^{N/2} dy \geq \int_{B(x, r) \setminus B(0, 1)} \tilde{V}(y)^{N/2} dy \geq \int_{B(x, r) \setminus B(0, 1)} \tilde{V}^*(y)^{N/2} dy
\]
\[
\int_{B(x,r)} \tilde{V}^*(y)^{N/2} dy \geq \int_{B(x,r) \cap B(0,1)} \tilde{V}^*(y)^{N/2} dy
\]

\[
\int_{B(x,r)} \tilde{V}^*(y)^{N/2} dy - \int_{B(0,1)} \tilde{V}^*(y)^{N/2} dy
\]

\[
= \int_{B(x,r)} (V^*)^N dy - \frac{2^{1-N/2} \sigma_{N-1}}{N(2-\alpha+\beta)}
\]

\[
(3.8)
\]

\[
\geq \sigma_{N-1} r_N \inf_{B(x,r)} (\tilde{V}^*)^N/C_3(\alpha, \beta, N)
\]

\[
(3.9)
\]

Let \( \eta = \frac{\alpha-\beta}{2} < 1 \) and \( \delta > 0 \) a parameter to be choose later, and set

\[
r_0 = \delta(1 + |x|)^\eta.
\]

By (3.8) condition (3.7) became

\[
\int_{B(x,r_0)} \tilde{V}^*(y)^{N/2} dy - C_2 \geq \sigma_{N-1} r_0^N \frac{\alpha-\beta}{N} - C_2 - C_3
\]

\[
= \sigma_{N-1} \frac{\delta^N (1 + |x|)^\eta \alpha-\beta}{(1 + |x| + \delta(1 + |x|)^\eta)^{\alpha-\beta}} - C_4
\]

\[
\geq \sigma_{N-1} \frac{\delta^N (1 + |x|)^\eta \alpha-\beta}{(1 + |x| + \delta(1 + |x|)^\eta)^{\alpha-\beta}} - C_4
\]

\[
\geq \sigma_{N-1} \frac{\delta^N (1 + |x|)^\eta \alpha-\beta}{((\delta + 1)(1 + |x|)^\eta)^{\alpha-\beta}} - C_4
\]

\[
= \sigma_{N-1} \frac{\delta}{(1 + \delta)^{\alpha-\beta}} - C_4.
\]

Since \( \frac{\alpha-\beta}{2} < 1 \) we can choose \( \delta > 0 \) such that \( \sigma_{N-1} \left( \frac{\delta}{(1 + \delta)^{\alpha-\beta}} \right)^N - C_4 \geq 0 \).

So, (3.7) is satisfied for \( r = r_0 \) and hence it is satisfied for any \( r > r_0 \). Thus, \( f_\infty(r) > 1 \) for \( r > r_0 \), and, hence, \( \frac{1}{m(x)} \leq r_0 = \delta(1 + |x|)^\eta \).

The same lower bound holds in the case \( \beta \geq \alpha \) as the following lemma shows.

**Lemma 3.2.** Let \( \beta \geq \alpha \). There exists \( C = C(\alpha, \beta, N) \) such that

\[
m(x) \geq C (1 + |x|)^{\frac{\beta-\alpha}{2}}.
\]

**Proof.** From [11] Lemma 1.4 (c)], there exist \( C_1 > 0 \) and \( 0 < \eta_0 < 1 \) such that

\[
m(x) \geq \frac{C_1 m(y)}{(1 + |x - y|m(y))^{\eta_0}}.
\]
In particular,
\[ m(x) \geq \frac{C_1 m(0)}{(1 + |x| m(0))^{\eta_0}}, \]
where \( \frac{1}{m(0)} = \sup_{r > 0} \{ r : f_0(r) \leq 1 \} \) with
\[ f_0(r) = \frac{1}{r^{N-2}} \int_{B(0,r)} \frac{|z|^\beta}{1 + |z|^\alpha} \, dz = \frac{\sigma_{N-1}}{r^{N-2}} \int_0^r \frac{\rho^{\beta + N-1}}{1 + \rho^\alpha} \, d\rho. \]
We have \( \frac{\sigma_{N-1}}{(\beta + N)(1 + r^\alpha)} r^{\beta+2} \leq f_0(r) \leq \frac{\sigma_{N-1}}{\beta + N} r^{\beta+2} \). Since \( \beta > 0 \) and \( \beta - \alpha + 2 > 0 \) it follows that \( \lim_{r \to 0} f_0(r) = 0 \) and \( \lim_{r \to \infty} f_0(r) = \infty \). Consequently,
\[ 0 < \sup_{r > 0} \{ r : f_0(r) \leq 1 \} < \infty \]
and, hence, \( m(0) = C_2 \) for some constant \( C_2 > 0 \). Then
\[ (3.11) \quad m(x) \geq \frac{C_1 C_2}{(1 + C_2 |x|)^{\eta_0}} \geq \frac{C_3}{(1 + |x|)^{\eta_0}} \]
for some constant \( C_3 > 0 \).

On the other hand, since \( \beta \geq \alpha \), we obtain by (3.3) that \( \tilde{V} \in B_{\infty} \). Then, by [11, Remark 2.9], we have
\[ (3.12) \quad m(x) \geq C_3 \tilde{V}^{1/2}(x) \geq C_6 |x|^{\frac{\beta}{2}} (1 + |x|)^{-\frac{\alpha}{2}}. \]
The thesis follows taking into account (3.11) and (3.12).

Applying the estimate (3.4) and the previous lemma we obtain the following upper bounds for the Green function \( G \).

**Lemma 3.3.** Let \( G(x, y) \) denotes the Green function of the Schrödinger operator \( \Delta - \frac{|x|^\beta}{1 + |x|^\alpha} \) and assume that \( \beta > \alpha - 2 \). Then,
\[ (3.13) \quad G(x, y) \leq C_k \frac{1}{1 + |x - y|^k} \frac{1}{(1 + |y|)^{\frac{\beta - \alpha}{2} - k}} \frac{1}{|x - y|^{N-2},} \quad x, y \in \mathbb{R}^N \]
for any \( k > 0 \) and some constant \( C_k > 0 \) depending on \( k \).

Using the above lemma we have the following estimate.

**Lemma 3.4.** Assume that \( \alpha > 2 \), \( N > 2 \) and \( \beta > \alpha - 2 \). Then there exists a positive constant \( C \) such that for every \( 0 \leq \gamma \leq \beta \) and \( f \in L^p(\mathbb{R}^N) \)
\[ (3.14) \quad \| |x|^\gamma L f \|_p \leq C \| f \|_p, \]
where \( L \) is defined in (3.2).

**Proof.** Let \( \Gamma(x, y) = \frac{G(x, y)}{1 + |y|} \), \( f \in L^p(\mathbb{R}^N) \) and
\[ u(x) = \int_{\mathbb{R}^N} \Gamma(x, y) f(y) \, dy. \]
We have to show that
\[ \| |x|^\gamma u \|_p \leq C \| f \|_p. \]
Let us consider the regions $E_1 := \{|x - y| \leq (1 + |y|)\}$ and $E_2 := \{|x - y| > (1 + |y|)\}$ and write

$$u(x) = \int_{E_1} \Gamma(x, y) f(y) dy + \int_{E_2} \Gamma(x, y) f(y) dy =: u_1(x) + u_2(x).$$

In $E_1$ we have

$$1 + |x| \leq 1 + |x - y| + |y| \leq 2.$$

So, by Lemma 8.2,

$$||x|^\gamma u_1(x)| \leq |x|^\gamma \int_{E_1} \Gamma(x, y)|f(y)|dy \leq \frac{1 + |x|^{\beta}}{1 + |y|^\alpha} \int_{E_1} |G(x, y)|f(y)dy \leq C(1 + |x|)^{\beta - \alpha} \int_{\mathbb{R}^N} G(x, y)|f(y)|dy \leq Cm^2(x)\tilde{u}(x),$$

where $\tilde{u}(x) = \int_{\mathbb{R}^N} G(x, y)|f(y)|dy$. By (3.3) we have $\tilde{V} \in B_{\frac{3}{2}}$. So applying Corollary 2.8, we obtain $||m^2\tilde{u}_p|| \leq C||f||_p$ and then $|||x|^\gamma u_1||_p \leq C||f||_p$.

In the region $E_2$, we have, by Hölder’s inequality,

$$||x|^\gamma u_2(x)| \leq |x|^\gamma \int_{E_2} \Gamma(x, y)|f(y)|dy = \int_{E_2} (|x|^\gamma \Gamma(x, y))^\frac{1}{p} (|x|^\gamma \Gamma(x, y))^{\frac{1}{p}} |f(y)|dy \leq \bigg( \int_{E_2} |x|^\gamma \Gamma(x, y)dy \bigg)^\frac{1}{p} \bigg( \int_{E_2} |x|^\gamma \Gamma(x, y)|f(y)|^p dy \bigg)^\frac{1}{p}. (3.15)$$

We propose to estimate first $\int_{E_2} |x|^\gamma \Gamma(x, y)dy$. In $E_2$ we have $1 + |x| \leq 1 + |y| + |x - y| \leq 2|x - y|$, then from (3.13) it follows that

$$|x|^\gamma \Gamma(x, y) \leq |x|^\gamma G(x, y) \leq \frac{1 + |x|^{\beta}}{|x - y|^k (1 + |y|)^{k - \beta N - 2} |x - y|^{N - 2}} \leq C \frac{1}{|x - y|^k (1 + |y|)^{k - \beta N - 2} |x - y|^{N - 2}}.$$  

For every $k > \beta - N + 2$, taking into account that $\frac{1}{|x - y|} < \frac{1}{1 + |y|}$, we get

$$|x|^\gamma \Gamma(x, y) \leq \frac{1}{(1 + |y|)^{k - \beta N - 2} |x - y|^{N - \beta}}.$$  

Since $\beta - \alpha + 2 > 0$ we can choose $k$ such that $\frac{k}{2}(\beta - \alpha + 2) + N - 2 - \beta > N$, then

$$\int_{E_2} |x|^\gamma \Gamma(x, y)dy \leq \int_{E_2} |x|^\gamma G(x, y)dy \leq C \int_{\mathbb{R}^N} \frac{1}{(1 + |y|)^{k - \beta N - 2} |x - y|^{N - \beta}} dy < C.$$

Moreover by the symmetry of $G$ we have

$$|x|^\gamma \Gamma(x, y) \leq |x|^\gamma G(x, y) \leq \frac{1 + |x|^{\beta}}{|x - y|^k (1 + |x|)^{k - \beta N - 2} |x - y|^{N - 2}}.$$
Taking into account that \( \frac{1}{|x-y|} \leq 2 \frac{1}{1+|x|} \), arguing as above we obtain

\[
\int_{E_2} |x|^\gamma \Gamma(x,y) dx \leq C.
\]

Hence (3.15) implies

\[
|||x|^\gamma u_2(x)|^p \leq C \int_{E_2} |x|^\gamma \Gamma(x,y) f(y)^p dy.
\]

Thus, by (3.17) and (3.16), we have

\[
\left\| \frac{1}{|x|^\alpha} \right\| \| | \frac{1}{|x|^\beta} \| \leq C \int_{E_2} |x|^\gamma \Gamma(x,y) f(y)^p dy dx
\]

\[
= C \int_{E_2} f(y)^p \left( \int_{E_2} |x|^\gamma \Gamma(x,y) dx \right) dy \leq C \| f \|_p^p.
\]

\[\Box\]

We are now ready to show the invertibility of \( A_p \) and \( D(A_p) \subset D(V) \).

**Proposition 3.5.** Assume that \( N > 2 \), \( \alpha > 2 \) and \( \beta > \alpha - 2 \). Then the operator \( A_p \) is closed and invertible. Moreover there exists \( C > 0 \) such that, for every \( 0 \leq \gamma \leq \beta \), we have

\[
|| | \cdot | \cdot | u \|_p \leq C \| A_p u \|_p, \quad \forall u \in D_{p,\max}(A).
\]

**Proof.** Let us first prove the injectivity of \( A_p \). Let \( u \in D_{p,\max}(A) \) such that \( A_p u = 0 \), in particular \( \tilde{A}_p u = 0 \). It follows that \( u \in D_{p,\max}(\tilde{A}) = D(\Delta) \cap D \left( \frac{|x|^{\beta}}{1+|x|} \right) \), see [9] (see [4, Theorem 2.5]). Then multiplying \( A_p u \) with \( u|u|^{p-2} \) and integrating over \( \mathbb{R}^N \) we obtain, by [6],

\[
0 = \int_{\mathbb{R}^N} u|u|^{p-2} \Delta u dx - \int_{\mathbb{R}^N} \frac{|x|^\beta}{1+|x|^{\alpha}} |u|^p dx
\]

\[
= -(p-1) \int_{\mathbb{R}^N} |u|^{p-2} |\nabla u|^2 dx - \int_{\mathbb{R}^N} \frac{|x|^\beta}{1+|x|^{\alpha}} |u|^p dx,
\]

from which we have \( u \equiv 0 \). So, by (3.12) we obtain the invertibility of \( A_p \).

By elliptic regularity one deduces that \( A_p \) is closed on \( D_{p,\max}(A) \). Finally, the estimate (3.18) follows from (3.14). \[\Box\]

The previous Theorem gives in particular the \( A_p \)-boundedness of the potential \( V \) and the following regularity result.

**Corollary 3.6.** Assume that \( N > 2 \), \( \alpha > 2 \) and \( \beta > \alpha - 2 \). Then

(i) there exists \( C > 0 \) such that for every \( u \in D_{p,\max}(A) \)

\[
\| V u \|_p \leq C \| A_p u \|_p;
\]
Theorem 3.7. Assume that 

\[ D_{p,\text{max}}(A) = \{ u \in W^{2,p}(\mathbb{R}^N) \mid Au \in L^p(\mathbb{R}^N) \} , \]

Proof. We have only to prove the inclusion \( D_{p,\text{max}}(A) \subset \{ u \in W^{2,p}(\mathbb{R}^N) \mid Au \in L^p(\mathbb{R}^N) \} \). Let \( u \in D_{p,\text{max}}(A) \). Then, by (i), \( Vu \in L^p(\mathbb{R}^N) \) and hence

\[ \Delta u = \frac{Au + Vu}{1 + |x|^\alpha} \in L^p(\mathbb{R}^N) . \]

So, the thesis follows from the Calderon-Zygmund inequality. \( \square \)

We can now state the main result of this section

**Theorem 3.7.** Assume that \( N > 2, \beta > \alpha - 2 \) and \( \alpha > 2 \). Then, \( [0, +\infty) \subset \rho(A_p) \) and \( (\lambda - A_p)^{-1} \) is a positive operator on \( L^p(\mathbb{R}^N) \) for any \( \lambda \geq 0 \). Moreover, if \( f \in L^p(\mathbb{R}^N) \cap C_0(\mathbb{R}^N) \), then \( (\lambda - A_p)^{-1}f = (\lambda - A)^{-1}f \).

Proof. Let us first prove that if \( 0 \leq \lambda \in \rho(A_p) \), then \( (\lambda - A_p)^{-1} \) is a positive operator on \( L^p(\mathbb{R}^N) \). To this purpose, take \( 0 \leq f \in L^p(\mathbb{R}^N) \) and set \( u = (\lambda - A_p)^{-1}f \). Then, by Corollary 3.6, \( u \in D(\tilde{A}_p) \) and

\[ -(\tilde{A}_p - \lambda q)u = qf =: \tilde{f} , \]

where \( q(x) = \frac{1}{1 + |x|^\alpha} \). Since \( \tilde{A}_p \) generates an exponentially stable and positive \( C_0 \)-semigroup \((\tilde{T}_p(t))_{t \geq 0}\) on \( L^p(\mathbb{R}^N) \) (see [1] Theorem 2.5), it follows that the semigroup \((e^{-t\lambda q\tilde{T}_p(t)})_{t \geq 0}\) generated by \( \tilde{A}_p - \lambda q \) is positive and exponentially stable. Hence, \( u = (\lambda q - \tilde{A}_p)^{-1}\tilde{f} \geq 0 \).

We show that \( E = [0, +\infty) \cap \rho(A_p) \) is an non-empty open and closed set in \([0, +\infty)\). By Proposition 3.5 we have \( 0 \in \rho(A_p) \) and hence \( E \neq \emptyset \). On the other hand, using the above positivity property and the resolvent equation we have \( (\lambda - A_p)^{-1} \leq (A_p)^{-1} = L \) for any \( \lambda \in E \) and therefore

\[ \| (\lambda - A_p)^{-1} \| \leq \| L \| , \]

it follows that the operator norm of \( (\lambda - A_p)^{-1} \) is bounded in \( E \) and consequently \( E \) is closed. Finally, since \( \rho(A_p) \) is an open set, it follows that \( E \) is open in \([0, +\infty)\). Thus, \( E = [0, +\infty) \).

Now in order to show the last statement we may assume \( f \in C_c^\infty \), the thesis will follow by density. Setting \( u := (\lambda - A_p)^{-1}f \), we obtain, by local elliptic regularity (cf. [2] Theorem 9.19)), that \( u \in C^{2+\sigma}_{\text{loc}}(\mathbb{R}^N) \) for some \( 0 < \sigma < 1 \). On the other hand, \( u \in W^{2,p}(\mathbb{R}^N) \), by Corollary 3.6. If \( p \geq \frac{N}{2} \), then by Sobolev’s inequality, \( u \in L^q(\mathbb{R}^N) \) for all \( q \in [p, +\infty) \). In particular, \( u \in L^q(\mathbb{R}^N) \) for some \( q > \frac{N}{2} \) and hence \( Au = -f + \lambda u \in L^q(\mathbb{R}^N) \). Moreover, since \( u \in C^{2+\sigma}_{\text{loc}}(\mathbb{R}^N) \), it follows that \( u \in W^{2,q}_{\text{loc}}(\mathbb{R}^N) \). So, \( u \in D_{q,\text{max}}(A) \subset W^{2,q}(\mathbb{R}^N) \subset C_b(\mathbb{R}^N) \), by Corollary 3.6 and Sobolev’s embedding theorem, since \( q > \frac{N}{2} \).

Let us now suppose that \( p < \frac{N}{2} \). Take the sequence \((r_n)\), defined by \( r_n = 1/p - 2n/N \) for any \( n \in \mathbb{N} \), and set \( q_n = 1/r_n \) for any \( n \in \mathbb{N} \). Let \( n_0 \) be the smallest integer such that \( r_{n_0} \leq 2/N \) noting that \( r_{n_0} > 0 \). Then, \( u \in D_{p,\text{max}}(A) \subset L^{q_1}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \), by the Sobolev embedding theorem. As above we obtain that \( u \in D_{q_1,\text{max}}(A) \subset L^{q_1}(\mathbb{R}^N) \). Iterating this
argument, we deduce that \( u \in D_{q_{\alpha,0},\max}(A) \). So we can conclude that \( u \in C_b(\mathbb{R}^N) \) arguing as in the previous case. Thus, \( Au = -f + \lambda u \in C_b(\mathbb{R}^N) \). Again, since \( u \in C^{2+\sigma}(\mathbb{R}^N) \), it follows that \( u \in W^2_{\text{loc}}(\mathbb{R}^N) \) for any \( q \in (1, +\infty) \). Hence, \( u \in D_{\max}(A) \). So, by the uniqueness of the solution of the elliptic problem, we have \( (\lambda - A_p)^{-1}f = (\lambda - A)^{-1}f \) for any \( f \in C^\infty_c(\mathbb{R}^N) \).

\[ \square \]

4. Generation of semigroups

In this section we show that \( A_p \) generates an analytic semigroup on \( L^p(\mathbb{R}^N) \), \( 1 < p < \infty \), provided that \( N > 2, \alpha > 2 \) and \( \beta > \alpha - 2 \).

We start by proving a weighted gradient estimate. To this purpose we need the following covering result from, see [11 Proposition 6.1], to prove a weighted gradient estimate.

**Proposition 4.1.** For every \( 0 \leq k < 1/2 \) there exists a natural number \( \zeta = \zeta(N,k) \) with the following property: Given \( F = \{ B(x, \rho(x)) \}_x \in \mathbb{R}^N \), where \( \rho : \mathbb{R}^N \to [0, \infty) \) is a Lipschitz continuous function with Lipschitz constant \( k \). Then there exists a countable subcovering \( \{ B(x_n, \rho(x_n)) \}_{n \in \mathbb{N}} \) of \( \mathbb{R}^N \) such that at most \( \zeta \) among the double balls \( \{ B(x_n, 2\rho(x_n)) \}_{n \in \mathbb{N}} \) overlap.

To prove the main result of this section we need the following weighted gradient estimate.

**Lemma 4.2.** Assume that \( N > 2, \alpha > 2 \) and \( \beta > \alpha - 2 \). Then there exists a constant \( C > 0 \) such that for every \( u \in D_{p,\max}(A) \) we have

\[
\|(1 + |x|^{\alpha-1})\nabla u\|_p \leq C(\|A_pu\|_p + \|u\|_p). \tag{4.1}
\]

**Proof.** Let \( u \in D_{p,\max}(A) \). We fix \( x_0 \in \mathbb{R}^n \) and choose \( \vartheta \in C^\infty_c(\mathbb{R}^N) \) such that \( 0 \leq \vartheta \leq 1 \), \( \vartheta(x) = 1 \) for \( x \in B(1) \) and \( \vartheta(x) = 0 \) for \( x \in \mathbb{R}^N \setminus B(2) \). Moreover, we set \( \vartheta_p(x) = \vartheta \left( \frac{x-x_0}{\rho} \right) \), where \( \rho = \frac{1}{4}(1 + |x_0|) \). We apply the well-known inequality

\[
\|\nabla v\|_{L^p(B(R))} \leq C\|v\|_{L^p(B(R))}^{1/2} \|\Delta v\|_{L^p(B(R))}^{1/2}, \quad v \in W^{2,p}(B(R)) \cap W^{1,p}_0(B(R)), \quad R > 0
\]

to the function \( \vartheta_p u \) and obtain for every \( \varepsilon > 0 \),

\[
\|(1 + |x|^{\alpha-1})\nabla u\|_{L^p(B(x_0,\rho))} \leq \|(1 + |x|^{\alpha-1})\nabla(\vartheta_p u)\|_{L^p(B(x_0,2\rho))} \leq C\|(1 + |x|^{\alpha})\Delta(\vartheta_p u)\|_{L^p(B(x_0,2\rho))}^{1/2} \|(1 + |x|^{\alpha-2})\vartheta_p u\|_{L^p(B(x_0,2\rho))}^{1/2}
\]

\[
\leq C \left( \varepsilon\|(1 + |x|^{\alpha})\Delta(\vartheta_p u)\|_{L^p(B(x_0,2\rho))} + \frac{1}{4\varepsilon}\|(1 + |x|^{\alpha-2})\vartheta_p u\|_{L^p(B(x_0,2\rho))} \right)
\]

\[
\leq C \left( \varepsilon\|(1 + |x|^{\alpha})\Delta u\|_{L^p(B(x_0,2\rho))} + \frac{2M}{\rho}\varepsilon\|(1 + |x|^{\alpha-2})u\|_{L^p(B(x_0,2\rho))} \right)
\]

\[
\leq C \left( \varepsilon\|(1 + |x|^{\alpha})\Delta u\|_{L^p(B(x_0,2\rho))} + \frac{M}{\rho^2}\varepsilon\|(1 + |x|^{\alpha-2})u\|_{L^p(B(x_0,2\rho))} \right)
\]

\[
\leq C \left( \varepsilon\|(1 + |x|^{\alpha})\Delta u\|_{L^p(B(x_0,2\rho))} + 8M\varepsilon\|(1 + |x|^{\alpha-2})u\|_{L^p(B(x_0,2\rho))} \right)
\]

\[
+ \left( 16M + \frac{1}{4\varepsilon} \right) \|(1 + |x|^{\alpha-2})u\|_{L^p(B(x_0,2\rho))}
\]
\[
\begin{align*}
&\leq C(M) \left( \varepsilon \|(1 + |x_0|)^{\alpha} \Delta u\|_{L^p(B(x_0, 2\rho))} + \varepsilon \|(1 + |x_0|)^{\alpha - 1} \nabla u\|_{L^p(B(x_0, 2\rho))} \right) \\
&\quad + \frac{1}{\varepsilon} \|(1 + |x_0|)^{\alpha - 2} u\|_{L^p(B(x_0, 2\rho))},
\end{align*}
\]

where \( M = \|\nabla \vartheta\|_{\infty} + \|\Delta \vartheta\|_{\infty} \). Since \( 2\rho = \frac{1}{2}(1 + |x_0|) \) we get

\[
\frac{1}{2}(1 + |x_0|) \leq 1 + |x| \leq \frac{3}{2}(1 + |x_0|), \quad x \in B(x_0, 2\rho).
\]

Thus

\[
\|(1 + |x|)^{\alpha - 1} \nabla u\|_{L^p(B(x_0, \rho))} \leq \left( \frac{3}{2} \right)^{\alpha - 1} \|(1 + |x_0|)^{\alpha - 1} \nabla u\|_{L^p(B(x_0, \rho))}
\]

\[
\leq C \left( \varepsilon \|(1 + |x_0|)^{\alpha} \Delta u\|_{L^p(B(x_0, 2\rho))} + \varepsilon \|(1 + |x_0|)^{\alpha - 1} \nabla u\|_{L^p(B(x_0, 2\rho))} \right) \\
\quad + \frac{1}{\varepsilon} \|(1 + |x_0|)^{\alpha - 2} u\|_{L^p(B(x_0, 2\rho))} \\
\leq C \left( 2^\alpha \varepsilon \|(1 + |x|)^{\alpha} \Delta u\|_{L^p(B(x_0, 2\rho))} + 2^{\alpha - 1} \varepsilon \|(1 + |x|)^{\alpha - 1} \nabla u\|_{L^p(B(x_0, 2\rho))} \right)
\]

\[(4.3)
\]

Let \( \{B(x_n, \rho(x_n))\} \) be a countable covering of \( \mathbb{R}^N \) as in Proposition 4.1 such that at most \( \zeta \) among the double balls \( \{B(x_n, 2\rho(x_n))\} \) overlap.

We write \((4.3)\) with \( x_0 \) replaced by \( x_n \) and sum over \( n \). To the limit as \( n \) tends to infinity, taking into account the covering result above, we get

we get

\[
\|(1 + |x|)^{\alpha - 1} \nabla u\|_p \leq C \left( \varepsilon \|(1 + |x|)^{\alpha} \Delta u\|_p + \varepsilon \|(1 + |x|)^{\alpha - 1} \nabla u\|_p \right)
\]

\[
\quad + \frac{1}{\varepsilon} \|(1 + |x|)^{\alpha - 2} u\|_p.
\]

Choosing \( \varepsilon \) such that \( \varepsilon C < 1/2 \) we have

\[
\frac{1}{2} \|(1 + |x|)^{\alpha - 1} \nabla u\|_p \leq \frac{1}{2} \|(1 + |x|)^{\alpha} \Delta u\|_p + \frac{C}{\varepsilon} \|(1 + |x|)^{\alpha - 2} u\|_p.
\]

It follows from Corollary 3.6 that \( \|x\|^{\alpha - 2} u\|_p \leq \|(1 + |x|^\beta) u\|_p \leq \|u\|_p + \|V u\|_p \leq \|u\|_p + C\|A_p u\|_p \) for every \( u \in D_{p,\max}(A) \) and some \( C > 0 \). Hence,

\[
\|(1 + |x|)^{\alpha - 1} \nabla u\|_p \leq C(\|A_p u\|_p + \|u\|_p)
\]

for all \( u \in D_{p,\max}(A) \). This ends the proof of the lemma.

The following lemma shows that \( C_c^\infty(\mathbb{R}^N) \) is a core for \( A_p \).

**Lemma 4.3.** Assume \( N > 2, \alpha > 2 \) and \( \beta > \alpha - 2 \). The space \( C_c^\infty(\mathbb{R}^N) \) is dense in \( D_{p,\max}(A) \) with respect to the graph norm.

**Proof.** Let us first observe that \( C_c^\infty(\mathbb{R}^N) \) is dense in \( W_c^{2,p}(\mathbb{R}^N) \) with respect to the operator norm. Let \( u \in W_c^{2,p}(\mathbb{R}^N) \) and consider \( u_n = \rho_n \ast u \), where \( \rho_n \) are standard mollifiers.
We have \( u_n \in C^\infty_c(\mathbb{R}^N) \), \( u_n \to u \) in \( L^p(\mathbb{R}^N) \) and \( D^2 u_n \to D^2 u \) in \( L^p(\mathbb{R}^N) \). Moreover, \( \text{supp} \, u_n \subseteq \text{supp} \, u + B(1) := K \) for any \( n \in \mathbb{N} \). Then

\[
\| A_p u - A u_n \|_p = \| A_p u - A u_n \|_{L^p(K)} \\
\leq \|(1 + |x|^\alpha)\Delta (u - u_n)\|_{L^p(K)} + \| |x|^\beta (u - u_n)\|_{L^p(K)} \\
\leq \|(1 + |x|^\alpha)\Delta (u - u_n)\|_{L^p(K)} + \| |x|^\beta \|_{L^\infty(K)} \| (u - u_n)\|_{L^p(K)} \to 0 \text{ as } n \to \infty.
\]

Now, let \( u \in D_{p,\text{max}}(A) \) and let \( \eta \) be a smooth function such that \( \eta = 1 \) in \( B(1) \), \( \eta = 0 \) in \( \mathbb{R}^N \setminus B(2) \), \( 0 \leq \eta \leq 1 \) and set \( \eta_n(x) = \eta \left( \frac{x}{n} \right) \). Then consider \( u_n = \eta_n u \in W^{2,p}_c(\mathbb{R}^N) \). First we have \( u_n \to u \) in \( L^p(\mathbb{R}^N) \) by dominated convergence. As regard \( A_p u_n \) we have

\[
A_p u_n(x) = (1 + |x|^\alpha)\Delta (\eta_n u)(x) - |x|^\beta \eta_n(x) u(x) \\
= \eta_n(x) A_p u(x) + 2(1 + |x|^\alpha)\nabla \eta_n(x) \nabla u(x) + (1 + |x|^\alpha)\Delta \eta_n(x) u(x) \\
= \eta_n(x) A_p u(x) + \frac{2}{n} (1 + |x|^\alpha) \nabla \eta \left( \frac{x}{n} \right) \nabla u(x) + \frac{1}{n^2} (1 + |x|^\alpha) \Delta \eta \left( \frac{x}{n} \right) u(x),
\]

and

\[
\eta_n A_p u \to A_p u \quad \text{in} \quad L^p(\mathbb{R}^N)
\]

by dominated convergence. As regards the last terms we consider that \( \nabla \eta(x/n) \) and \( \Delta \eta(x/n) \) can be different from zero only for \( n \leq |x| \leq 2n \), then we have

\[
\frac{1}{n} (1 + |x|^\alpha) \left| \nabla \eta \left( \frac{x}{n} \right) \right| \left| \nabla u \right| \leq C(1 + |x|^\alpha - 1) |\nabla u| \chi_{\{n \leq |x| \leq 2n\}},
\]

and

\[
\frac{1}{n^2} (1 + |x|^\alpha) \left| \Delta \eta \left( \frac{x}{n} \right) \right| \left| u \right| \leq C(1 + |x|^\alpha - 2) |u| \chi_{\{n \leq |x| \leq 2n\}}.
\]

The right hand side tends to 0 as \( n \to \infty \), since by Proposition 3 and Lemma 1 we have

\[
\| (1 + |x|^\alpha - 2) u \|_p \leq C(\| A_p u \|_p + |u|_p) \quad \text{and} \quad \| (1 + |x|^\alpha - 1) \nabla u \|_p \leq C(\| A_p u \|_p + |u|_p).
\]

Let us give now the main result of this section.

**Theorem 4.4.** Assume \( N > 2 \), \( \alpha > 2 \) and \( \beta > \alpha - 2 \). Then the operator \( A_p \) with domain \( D_{p,\text{max}}(A) \) generates an analytic semigroup in \( L^p(\mathbb{R}^N) \).

**Proof.** Let \( f \in L^p, \rho > 0 \). Consider the operator \( \tilde{A}_p := A_p - \omega \) where \( \omega \) is a constant which will be chosen later. It is known that the elliptic problem in \( L^p(B(\rho)) \)

\[
\begin{cases}
\lambda u - \tilde{A}_p u = f & \text{in } B(\rho) \\
u = 0 & \text{on } \partial B(\rho),
\end{cases}
\]

admits a unique solution \( u_\rho \) in \( W^{2,2}(B(\rho)) \cap W^{1,2}_0(B(\rho)) \) for \( \lambda > 0 \), (cf. [2] Theorem 9.15). Let us prove that that \( e^{i\theta \tilde{A}_p } \) is dissipative in \( B(\rho) \) for \( 0 \leq \theta \leq \theta_\alpha \) with suitable \( \theta_\alpha \in (0, \frac{\pi}{2}) \). To this purpose observe that

\[
\tilde{A}_p u_\rho = \text{div} \left( (1 + |x|^\alpha) \nabla u_\rho \right) - \alpha |x|^\alpha \frac{x}{|x|} \nabla u_\rho - |x|^\beta u_\rho - \omega u_\rho.
\]
Set \( u^* = \overline{u}_\rho |u_\rho|^{p-2} \) and recall that \( a(x) = 1 + |x|^\alpha \). Multiplying \( \widetilde{A}_\rho u_\rho \) by \( u^* \) and integrating over \( B(\rho) \), we obtain

\[
\int_{B(\rho)} \widetilde{A}_\rho u_\rho u^* \, dx = -\int_{B(\rho)} a(x) |u_\rho|^{p-4} |\text{Re}(\overline{u}_\rho \nabla u_\rho)|^2 \, dx
- \int_{B(\rho)} a(x) |u_\rho|^{p-4} |\text{Im}(\overline{u}_\rho \nabla u_\rho)|^2 \, dx - (p-2) \int_{B(\rho)} a(x) |u_\rho|^{p-4} \overline{u}_\rho \nabla u_\rho \text{Re}(\overline{u}_\rho \nabla u_\rho) \, dx
- \alpha \int_{B(\rho)} \overline{u}_\rho |u_\rho|^{p-2} |x|^{\alpha-1} \frac{x}{|x|} \nabla u_\rho \, dx - \int_{B(\rho)} (|x|^{\beta} + \omega) |u_\rho|^p \, dx.
\]

We note here that the integration by part in the singular case \( 1 < p < 2 \) is allowed thanks to [6]. By taking the real and imaginary part of the left and the right hand side, we have

\[
\text{Re} \left( \int_{B(\rho)} \widetilde{A}_\rho u_\rho u^* \, dx \right) = -(p-1) \int_{B(\rho)} a(x) |u_\rho|^{p-4} |\text{Re}(\overline{u}_\rho \nabla u_\rho)|^2 \, dx - \int_{B(\rho)} a(x) |u_\rho|^{p-4} |\text{Im}(\overline{u}_\rho \nabla u_\rho)|^2 \, dx
- \alpha \int_{B(\rho)} |u_\rho|^{p-2} |x|^{\alpha-1} \frac{x}{|x|} \nabla |u_\rho|^p \, dx - \int_{B(\rho)} \left( |x|^{\beta} + \omega \right) |u_\rho|^p \, dx
= -(p-1) \int_{B(\rho)} a(x) |u_\rho|^{p-4} |\text{Re}(\overline{u}_\rho \nabla u_\rho)|^2 \, dx - \int_{B(\rho)} a(x) |u_\rho|^{p-4} |\text{Im}(\overline{u}_\rho \nabla u_\rho)|^2 \, dx
- \alpha \int_{B(\rho)} |x|^{\alpha-2} |x|^\beta - |x|^{\beta - \omega} |u_\rho|^p \, dx
+ \int_{B(\rho)} \left( \frac{\alpha(N - 2 + \alpha)}{\rho} |x|^{\alpha-2} - \omega \right) |u_\rho|^p \, dx
\]

and

\[
\text{Im} \left( \int_{B(\rho)} \widetilde{A}_\rho u_\rho u^* \, dx \right) = -(p-2) \int_{B(\rho)} a(x) |u_\rho|^{p-4} \text{Im}(\overline{u}_\rho \nabla u_\rho) \text{Re}(\overline{u}_\rho \nabla u_\rho) \, dx
- \alpha \int_{B(\rho)} |u_\rho|^{p-2} |x|^{\alpha-1} \frac{x}{|x|} \text{Im}(\overline{u}_\rho \nabla u_\rho) \, dx.
\]

We can choose \( \tilde{c} > 0 \) and \( \omega > 0 \) (depending on \( \tilde{c} \)) such that

\[
\frac{\alpha(N - 2 + \alpha)}{\rho} |x|^{\alpha-2} - |x|^{\beta} - \omega \leq -\tilde{c} |x|^{\alpha-2}.
\]

So, we obtain

\[
-\text{Re} \left( \int_{B(\rho)} \widetilde{A}_\rho u_\rho u^* \, dx \right) \geq (p-1) \int_{B(\rho)} a(x) |u_\rho|^{p-4} |\text{Re}(\overline{u}_\rho \nabla u_\rho)|^2 \, dx
\]
Using a weak compactness and a diagonal argument, we can construct a sequence $(\rho_n)_{n\in\mathbb{N}}$. Moreover, there exists a constant $\theta$ such that the functions $(\rho_n)$ converge weakly in $W^{2,p}_{loc}$ to a function $u$ which satisfies

$$
\int_{B(\rho)} a(x)|u_\rho|^p dx - \int_{B(\rho)} \text{Im}(\overline{u}_\rho \nabla u_\rho)^2 dx + \int_{B(\rho)} |u_\rho|^p |x|^\alpha dx = (p-1)B^2 + C^2 + cD^2.
$$

Moreover,

$$
\left| \text{Im} \left( \int_{B(\rho)} \tilde{A}_p u_\rho u^* dx \right) \right| \leq |p-2| \left( \int_{B(\rho)} |u_\rho|^{p-4} a(x) |\text{Re}(\overline{\nabla} u_\rho)|^2 dx \right)^{\frac{1}{2}} \left( \int_{B(\rho)} |u_\rho|^{p-4} a(x) |\text{Im}(\overline{\nabla} u_\rho)|^2 dx \right)^{\frac{1}{2}} + \alpha \left( \int_{B(\rho)} |u_\rho|^{p-4} |x|^\alpha |\text{Im}(\overline{\nabla} u_\rho)|^2 dx \right)^{\frac{1}{2}} \left( \int_{B(\rho)} |u_\rho|^p |x|^\alpha dx \right)^{\frac{1}{2}}
$$

$$
= |p-2| BC + \alpha CD,
$$

where

$$
B^2 = \int_{B(\rho)} |u_\rho|^{p-4} a(x) |\text{Re}(\overline{\nabla} u_\rho)|^2 dx,
$$

$$
C^2 = \int_{B(\rho)} |u_\rho|^{p-4} a(x) |\text{Im}(\overline{\nabla} u_\rho)|^2 dx,
$$

$$
D^2 = \int_{B(\rho)} |u_\rho|^p |x|^\alpha dx.
$$

Let us observe that, choosing $\delta^2 = \frac{|p-2|^2}{4(p-1)} + \frac{\alpha^2}{4e}$ (which is independent of $\rho$), we obtain

$$
\left| \text{Im} \left( \int_{B(\rho)} \tilde{A}_p u_\rho u^* dx \right) \right| \leq \delta \left\{ -\text{Re} \left( \int_{B(\rho)} \tilde{A}_p u_\rho u^* dx \right) \right\}.
$$

If $\tan \theta_\alpha = \delta$, then $e^{\pm i\delta} \tilde{A}_p$ is dissipative in $B(\rho)$ for $0 \leq \theta \leq \theta_\alpha$. From [10] Theorem I.3.9] follows that the problem (4.4) has a unique solution $u_\rho$ for every $\lambda \in \Sigma_\theta$, $0 \leq \theta < \theta_\alpha$ where

$$
\Sigma_\theta = \{ \lambda \in \mathbb{C} \setminus \{0\} : |\text{Arg} \lambda| < \pi/2 + \theta \}.
$$

Moreover, there exists a constant $C_\theta$ which is independent of $\rho$, such that

$$
(4.5) \quad \|u_\rho\|_{L^p(B(\rho))} \leq \frac{C_\theta}{|\lambda|} \|f\|_{L^p}, \quad \lambda \in \Sigma_\theta.
$$

Let us now fix $\lambda \in \Sigma_\theta$, with $0 < \theta < \theta_\alpha$ and a radius $r > 0$. We apply the interior $L^p$ estimates (cf. [2] Theorem 9.11) to the functions $u_\rho$ with $\rho > r + 1$. So, by (4.5) we have

$$
\|u_\rho\|_{W^{2,p}(B(r))} \leq C_1 \left( \|\lambda u_\rho - \tilde{A}_p u_\rho\|_{L^p(B(r+1))} + \|u_\rho\|_{L^p(B(r+1))} \right) \leq C_2 \|f\|_{L^p}.
$$

Using a weak compactness and a diagonal argument, we can construct a sequence $(\rho_n) \to \infty$ such that the functions $(u_{\rho_n})$ converge weakly in $W^{2,p}_{loc}$ to a function $u$ which satisfies
\[ \lambda u - \widetilde{A}_p u = f \]

and

\[ \|u\|_p \leq \frac{C_\theta}{|\lambda|} \|f\|_p, \quad \lambda \in \Sigma_\theta. \]

Moreover, \( u \in D_{p,\text{max}}(A_p) \). We have now only to show that \( \lambda - \tilde{A}_p \) is invertible on \( D_{p,\text{max}}(A_p) \) for \( \lambda \in \Sigma_\theta \). Consider the set

\[ E = \{ r > 0 : \Sigma_\theta \cap C(r) \subset \rho(\tilde{A}_p) \} \]

where \( C(r) := \{ \lambda \in \mathbb{C} : |\lambda| < r \} \). Since, by Theorem 3.7, 0 is in the resolvent set of \( \tilde{A}_p \), then \( R = \sup E > 0 \). On the other hand, the norm of the resolvent is bounded by \( C_\theta/|\lambda| \) in \( C(R) \cap \Sigma_\theta \), consequently it cannot explode on the boundary of \( C(R) \), then \( R = \infty \) and this ends the proof of the theorem.

\[ \square \]

**Remark 4.5.** Since \( A_p \) generates an analytic semigroup \( T_p(\cdot) \) on \( L^p(\mathbb{R}^N) \) and the semigroups \( T_q(\cdot), q \in (1, \infty) \) are consistent, see Theorem 3.7, one can deduces (as in the proof of [4, Proposition 2.6]) using Corollary 3.6 that \( T_p(t)L^p(\mathbb{R}^N) \subset C^{1+\nu}(\mathbb{R}^N) \) for any \( t > 0, \nu \in (0,1) \) and any \( p \in (1, \infty) \).

We end this section by studying the spectrum of \( A_p \). We recall from Proposition 3.5 that

\[ \| |x|^{\beta} u \|_p \leq C\|A_p u\|_p, \quad \forall u \in D_{p,\text{max}}(A). \]

So, arguing as in [4] we obtain the following results

**Proposition 4.6.** Assume \( N > 2, \alpha > 2 \) and \( \beta > \alpha - 2 \) then

(i) the resolvent of \( A_p \) is compact in \( L^p \);

(ii) the spectrum of \( A_p \) consists of a sequence of negative real eigenvalues which accumulates at \( -\infty \). Moreover, \( \sigma(A_p) \) is independent of \( p \);

(iii) the semigroup \( T_p(\cdot) \) is irreducible, the eigenspace corresponding to the largest eigenvalue \( \lambda_0 \) of \( A \) is one-dimensional and is spanned by strictly positive functions \( \psi \), which is radial, belongs to \( C^{1+\nu}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \) for any \( \nu \in (0,1) \) and tends to 0 when \( |x| \to \infty \).

**References**

[1] G. Cupini, S. Fornaro, Maximal regularity in \( L^p \) for a class of elliptic operators with unbounded coefficients, *Diff. Int. Eqs.* 17 (2004), 259-296.

[2] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Second edition, Springer, Berlin, (1983).

[3] L. Lorenzi, M. Bertoldi, *Analytical Methods for Markov Semigroups*, Chapman & Hall/CRC, (2007).

[4] L. Lorenzi, A. Rhandi, On Schrödinger type operators with unbounded coefficients: generation and heat kernel estimates, (submitted). Available on ArXiv [http://arxiv.org/abs/1203.0734], 2012.

[5] G. Metafune, D. Pallara, M. Wacker, Feller Semigroups on \( \mathbb{R}^N \), *Semigroup Forum* 65 (2002), 159-205.

[6] G. Metafune, C. Spina, An integration by parts formula in Sobolev spaces, *Mediterranean Journal of Mathematics* 5 (2008), 359-371.

[7] G. Metafune, C. Spina, Elliptic operators with unbounded coefficients in \( L^p \) spaces, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 11* (2012), no. 2, 303-340.

[8] G. Metafune, C. Spina, C. Tacelli, Elliptic operators with unbounded diffusion and drift coefficients in \( L^p \) spaces, *Adv. Diff. Equat.* 19 (2014), no. 5-6, 473-526.
[9] N. Okazawa, An $L^p$ theory for Schr"{o}dinger operators with nonnegative potentials, *J. Math. Soc. Japan* 36 (1984), 675-688.

[10] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied mathematical sciences 44, Springer-Verlag, 1983.

[11] Z. Shen, $L_p$ estimates for Schr"{o}dinger operators with certain potentials, *Annales de l'institut Fourier* 45 (1995), 513-546.

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