Abstract. We prove that for each positive integer $N$ the set of smooth, zero degree maps $\psi: S^2 \to S^2$ which have the following three properties:

(i) there is a unique minimizing harmonic map $u: B^3 \to S^2$ which agrees with $\psi$ on the boundary of the unit ball;

(ii) this map $u$ has at least $N$ singular points in $B^3$;

(iii) the Lavrentiev gap phenomenon holds for $\psi$, i.e., the infimum of the Dirichlet energies $E(w)$ of all smooth extensions $w: B^3 \to S^2$ of $\psi$ is strictly larger than the Dirichlet energy $\int_{B^3} |\nabla u|^2$ of the (irregular) minimizer $u$,

is dense in the set of all smooth zero degree maps $\phi: S^2 \to S^2$ endowed with the $H^{1/2}$-topology.

1. Introduction

In this note, we revisit a well-known topic, the study of singularities of maps $u: B^3 \to S^2$ which minimize the Dirichlet integral

$$E(u) = \int_{B^3} |\nabla u|^2 \, dx, \quad u \in W^{1,2}(B^3, S^2)$$

under a prescribed boundary condition $u|_{\partial B^3} = \varphi: S^2 \to S^2$. Here, $B^3$ stands for the open unit ball in $\mathbb{R}^3$, $S^2$ is the unit sphere, and

$$W^{1,2}(B^3, S^2) = \{ v = (v_1, v_2, v_3) \in W^{1,2}(B^3, \mathbb{R}^3) : |v(x)| = 1 \text{ for a.e. } x \in B^3 \}.$$

Moreover, for a map $\varphi$ in the fractional Sobolev space $H^{1/2}(S^2, S^2)$ we write

$$W^{1,2}_{\varphi}(B^3, S^2) = \{ v \in W^{1,2}(B^3, S^2) : v|_{\partial B^3} = \varphi \text{ in the trace sense} \}.$$

Minimizers of the Dirichlet integral (1.1) in $W^{1,2}_{\varphi}(B^3, S^2)$ satisfy the Euler–Lagrange system

$$-\Delta u = |\nabla u|^2 u \quad \text{in } B^3,$$

$$u|_{\partial B^3} = \varphi.$$

The main motivation behind the present work was to reach a deeper understanding of the mechanisms governing the onset of singularities of solutions, and the cardinality and structure of the set of minimizing solutions for a fixed boundary condition. We also wanted

Date: Version of June 4, 2014.
to know whether the \textit{Lavrentiev gap phenomenon}, cf. (1.3) below, occurs typically (in a precise topological meaning). Despite the work of numerous experts over the last three decades, this topic is still not fully understood. Our main result states, roughly speaking, that even in the case when there is no purely topological reason for the solution of (1.2) to be discontinuous, singularities of $u$ do occur under arbitrarily small (in the $H^{1/2}$ sense) perturbations of an \textit{arbitrary} smooth boundary data $\varphi$. A corollary of this and previously known results is that an arbitrarily small (in $H^{1/2}$) perturbation of the boundary data can lead to extreme nonuniqueness of solutions of (1.2).

Before giving formal statements of the results, let us sketch a broader perspective.

When $\deg \varphi \neq 0$, all solutions of (1.2) in $W^{1,2}(\mathbb{B}^3, S^2)$ obviously have singularities, as $\varphi$ has no continuous extension $u: \mathbb{B}^3 \to S^2$. By a celebrated classic theorem of Schoen and Uhlenbeck [11] the singular set of a \textit{minimizing} solution of (1.2) consists of isolated points. By another theorem of Almgren and Lieb [2], if the boundary condition $\varphi$ has square integrable derivatives on $S^2$, then the number of these points does not exceed a constant multiple of the \textit{boundary energy} $\int_{S^2} |\nabla_T \varphi|^2$. (Non-minimizing solutions can behave in a wild way: Rivi`ere [10] proves that for any non-constant boundary data $\varphi$ there exists an everywhere discontinuous solution of the harmonic map system (1.2).)

However, even when $\varphi: S^2 \to S^2$ satisfies $\deg \varphi = 0$ so that a priori there is no topological obstruction for a map $u \in W^{1,2}(\mathbb{B}^3, S^2)$ to be continuous — minimizers of $E$ in $W^{1,2}(\mathbb{B}^3, S^2)$ might be singular because this is energetically preferable. Hardt and Lin [8] give an example of a smooth zero degree boundary data $\tilde{\varphi}: S^2 \to S^2$ which is $H^{1/2}$-close to a constant map and has the following properties:

(a) Each minimizer $v$ of $E$ in $W^{1,2}(\mathbb{B}^3, S^2)$ has at least $N$ singular points (and $N$ can be prescribed a priori);

(b) The Lavrentiev gap phenomenon holds for $E$ in $W^{1,2}(\mathbb{B}^3, S^2)$. By this, we mean the following inequality:

$$\mu(\tilde{\varphi}) := \min_{W^{1,2}(\mathbb{B}^3, S^2)} E(u) < \overline{\mu}(\tilde{\varphi}) := \inf_{W^{1,2}(\mathbb{B}^3, S^2) \cap C^0(\mathbb{B}^3)} E(u).$$

An immediate consequence of (1.3) is that $W^{1,2}(\mathbb{B}^3, S^2) \cap C^0(\mathbb{B}^3)$ is not dense in $W^{1,2}(\mathbb{B}^3, S^2)$. Let us note that the boundary conditions constructed in [8] were small $H^{1/2}$ perturbations of a constant boundary data.

As Bethuel, Brezis and Coron have shown in their work on minimal connections [3], if the Lavrentiev gap phenomenon holds in $W^{1,2}(\mathbb{B}^3, S^2)$, then there are infinitely many different solutions of the harmonic map system (1.2) for this boundary condition $\varphi$, cf. [3, Theorem 6] (and also Pakzad [9, Theorem 1]). Other examples of unexpected and counterintuitive behavior of singularities of minimizing harmonic maps have been given by Almgren and Lieb in [2]. In particular, a minimizer $u$ of $E$ in $W^{1,2}(\mathbb{B}^3, S^2)$ can have a large number of singular points even if $\det \nabla_T \varphi \equiv 0$ on $S^2$ and $\varphi$ maps the whole sphere $S^2$ to a
smooth curve $\gamma$. The abstract of [2] ends with the phrase: “in particular, singularities in $u$ can be unstable under small perturbations of $\varphi$.”

Our main result ascertains that the message of the last sentence, *singularities can be unstable*, may be strengthened, i.e. replaced with a firm *singularities are unstable*, at least when one takes into account small perturbations of the boundary data in the topology of the trace space $H^{1/2}$. Here is the precise statement.

**Theorem 1.1.** Assume that $\varphi \in C^\infty(S^2, S^2)$ is an arbitrary smooth map with $\deg \varphi = 0$. Then, for each $\varepsilon > 0$ and each $N \in \mathbb{N}$ there exists a map $\tilde{\varphi} \in C^\infty(S^2, S^2)$ such that

(i) $\deg \tilde{\varphi} = 0$;
(ii) $\|\varphi - \tilde{\varphi}\|_{H^{1/2}} < \varepsilon$ and $\mathcal{H}^2(\{x \in S^2 : \varphi(x) \neq \tilde{\varphi}(x)\}) < \varepsilon$;
(iii) the Dirichlet integral $E$ has precisely one minimizer $\tilde{u} \in W^{1,2}(B^3, S^2)$; moreover, $\tilde{u}$ has at least $N$ point singularities in $B^3$.

Combining the above result with Bethuel, Brezis and Coron, [3, Theorems 5 and 6], one immediately obtains the following.

**Corollary 1.2.** Assume that $\varphi \in C^\infty(S^2, S^2)$ and $\deg \varphi = 0$. Let $\tilde{\varphi} \in C^\infty(S^2, S^2)$ be given by Theorem 1.1. Then the Lavrentiev gap phenomenon (1.3) holds for $\tilde{\varphi}$.

The overall plan of the proof of Theorem 1.1 is as follows. We first revisit Almgren’s and Lieb’s method of installing new singular points, see [2, Theorem 4.3]. Introducing appropriate modifications of their method, we select two antipodal points $\pm p \in S^2$ such that $\varphi$ maps them to the same point of $S^2$, $\varphi(p) = \varphi(-p)$ (the existence of such antipodal points is guaranteed by the assumption $\deg \varphi = 0$), and insert many tiny bubbles into $\varphi$ close to those two points to obtain the new boundary condition $\tilde{\varphi}$. (In a sense, this is an imitation of the main idea of Hardt and Lin, applied by them to a constant boundary condition $\varphi : S^2 \to \{*\}$.) This way, $\varphi$ is changed only in two little discs centered at $\pm p \in S^2$, so that the second statement in (ii) in Theorem 1.1 does hold.

To control the degree of $\tilde{\varphi}$ and to guarantee the uniqueness of minimizers of the Dirichlet integral in $W^{1,2}_\varphi(B^3, S^2)$, we employ the uniform boundary regularity of minimizing harmonic maps combined with the fact that harmonic maps are real analytic in the interior of the regular set. Finally, the distance from $\varphi$ to $\tilde{\varphi}$ is estimated by an application of the trace theorem.

The notation throughout the paper is standard. $B(x_0, r) = \{x \in \mathbb{R}^3 : |x - x_0| < r\}$ is the standard Euclidean open ball. We write

$$\partial E(\varphi) = \int_{S^2} |\nabla_T \varphi|^2 d\sigma$$

to denote the boundary energy of a map $\varphi : S^2 \to S^2$. For a map $u : B^3 \to S^2$ we set

$$J(x) \equiv J(u)(x) = \sqrt{\det (Du(x)Du(x)^T)}.$$

---

1 We have just changed Almgren’s and Lieb’s notation from $\varphi, \psi$ to our $u, \varphi$ respectively.
If the rank of $Du(x)$ is maximal, i.e. equal to 2, then $J(u)(x)$ measures how $Du(x)|_V$, where $V$ is the orthogonal complement of ker $Du(x)$, distorts the surface measure: for an arbitrary ball $B$ centered at $x$, the Jacobian $J(u)(x)$ is equal to the ratio of $H^2(Du(x)B)$ to $H^2(B \cap V)$.

2. Installing new singularities

We start with a theorem of Almgrem and Lieb, see [2, Theorem 4.3], which describes how to modify the boundary mapping so that its energy minimizer would have a singularity and the energy of the new minimizer would be almost the same as the energy of the initial one. This result will serve as a main tool in constructing $\tilde{\varphi}$ in the proof of Theorem 1.1.

Before giving the statement, we introduce the notation which will be useful in several places below.

**Definition 2.1.** For a fixed map $\psi : S^2 \to S^2$, which is smooth near $q \in S^2$ and fixed $\rho > 0$, we let $[\psi]_{q, \rho} : S^2 \to S^2$ denote any smooth boundary map which arises from $\psi$ by a small deformation in a neighborhood of $q$ so that the following four conditions are satisfied:

(a) $[\psi]_{q, \rho}(x) = \psi(x)$ whenever $|x - q| \geq \rho$;
(b) $[\psi]_{q, \rho}(x) \equiv \psi(q)$ if $|x - q| = \rho/2$;
(c) The restriction of $[\psi]_{q, \rho}$ to the annulus $\frac{\rho}{2} < |x - q| < \rho$ satisfies the Lipschitz condition with a Lipschitz constant $L$ which depends only on $\psi$ and not on $\rho$;
(d) $[\psi]_{q, \rho}$ is a diffeomorphism of the spherical cap $\{|x - q| < \rho/2\} \cap S^2$ onto the punctured sphere $S^2 \setminus \{\psi(q)\}$ such that the boundary Dirichlet integral energy of $[\psi]_{q, \rho}$ on this cap equals $8\pi + o(1)$ as $\rho \to 0$.

It is well known that such maps exist, e.g. a modification of the mapping obtained in [1, Appendix A.2]. If we identify the spherical cap from (d) with a disc we can map a concentric smaller disc to the whole sphere without a spherical cap consisting of points whose angular distance (in radians) from the point $\psi(p)$ are greater or equal $\pi - \frac{1}{j}$. To do this we use a properly rescaled and rotated inverse stereographic projection. It is a smooth conformal mapping and therefore its norm is equal twice the Hausdorff measure of the image (and therefore approaches $2 \cdot 4\pi$ as $j \to \infty$). The remaining annuli from the domain can be mapped into the punctured spherical cap left in the image without changing the Dirichlet integral too much.

We shall sometimes say that $[\psi]_{q, \rho}$ arises from $\psi$ by inserting a smooth bubble at $q$.

**Theorem 2.2.** Suppose $u : B^3 \to S^2$ is a minimizer which is unique for its boundary mapping $\psi : S^2 \to S^2$ and which has an interior singularity at $p \in B^3$. Assume $\psi$ has finite boundary Dirichlet integral energy and is smooth near $q \in S^2$ and let $\psi_j : S^2 \to S^2$ be any sequence of continuous boundary mappings such that $\psi_j = [\psi]_{q, 2/j}$ for all $j$ sufficiently large.

Finally, let $u_j$ be any minimizer in $B^3$ with boundary mapping $\psi_j$. Then, for all sufficiently large $j$, the mapping $u_j$ will have at least two interior singular points $q_j$ and $p_j$ such that $q_j \to q$ and $p_j \to p$ as $j \to \infty$. 
Since we had some trouble to follow the argument in [2] — in particular the lines 11–14 on page 521 — in full detail, we include here a more detailed variant of Almgren and Lieb’s proof, explaining the parts which were unclear for us.

**Proof.** The proof consists of five steps.

**Step 1.** We first show that $u_j \to u$ strongly in $H^1$. By [2] Theorem 1.1, $$E(u_j) < C \sqrt{\partial E(\psi_j)} < C \sqrt{\partial E(\psi)} + 8\pi + L,$$

so $\sup_j E(u_j) < \infty$ and $\sup_j \partial E(\psi_j) < \infty$. Therefore, by [2] Theorem 1.2 part (4)], after passing to subsequences $u_j$ and $\psi_j$ converge to some $u_0$ and $\psi_0$, respectively; the convergence (of each of these subsequences) is strong in $H^1$ and $u_0$ is a minimizer for its boundary mapping $\psi_0$. By the strong convergence we obtain the existence of another subsequence $f_k$ such that $\psi_{j_k}(x) \to \psi_0(x)$ for a.e. $x \in S^2$. However, by its very definition $\psi_j(x) \to \psi(x)$ for all $x \in S^2 \setminus \{q\}$, so that $\psi_0 = \psi$ a.e. and by the uniqueness of $u$ we obtain that $u_0 = u$.

**Step 2.** Now the existence of interior singular points $p_j$ of $u_j$ for sufficiently large $j$, as well as the convergence $p_j \to p$, follows from [2] Theorem 1.8 part (2)]. (In short, if all $u_j$ were regular in a small neighborhood of $p$, the scaled energy of $u$ over a small ball $B(p, \delta)$ would be small enough to guarantee the regularity of $u$ at $p$.)

**Step 3.** By the Boundary Regularity Theorem [12] and monotonicity formula (see e.g. [1] Corollary 1.7)], we may choose an $R > 0$ such that for each $r < R/2$ we have $\int_{B(q,2r)} |\nabla u(x)|^2 \, dx < 2\pi r$.

**Step 4.** The map $\psi : S^2 \to S^2$ is continuous near $q$, therefore for any $\varepsilon > 0$ we may find a $\delta > 0$ such that if $|x - q| < \delta$ then $|\psi(x) - \psi(q)| < \varepsilon$. Let us fix $\varepsilon > 0$ and assume that for a fixed small $r = \min(\delta, \frac{1}{2}R)$ independent of $j$ there is no singularity for each $u_j$ in the region $|x - q| < 2r$.

Combining the elementary inequality $2|J(x)| \leq |\nabla u(x)|^2$ and the co-area formula $$\int_{B^3} |J(u)(x)| \, dx = \int_{w \in S^2} \mathcal{H}^1(u^{-1}\{w\}) \, d\mathcal{H}^2(w),$$

see [4] Chapter 3], one obtains

$$\int_{B^3} |\nabla u(x)|^2 \, dx \geq 2 \int_{w \in S^2} \mathcal{H}^1(u^{-1}\{w\}) \, d\mathcal{H}^2(w). \tag{2.1}$$

For a fixed point $q \in S^2$, to shorten the notation, we write $S(q,a) = B(q,a) \cap S^2$ for the spherical cap formed by the intersection of the ball $B(q,a)$ and the unit sphere. $S(q,a,b) = (B(q,b) \setminus B(q,a)) \cap S^2$ is the intersection of the annulus $B(q,b) \setminus B(q,a)$ with the unit sphere. We also write $\mathcal{U}_t = \partial (B(q,t) \cap \mathbb{B}^3)$ for the boundary of the intersection of the unit ball and the ball centered at $q$ of radius $t$, and $\mathcal{U}_t^{-} = \partial B(q,t) \cap \mathbb{B}^3$ for the boundary of the ball centered at $q$ of radius $t$ intersected with the unit ball. Finally,
\( \mathcal{V}_\varepsilon = B(\psi(q), \varepsilon) \cap S^2 \) stands for the spherical cap established by the intersection of a ball centered at \( \psi(q) \) of radius \( \varepsilon \) and a unit sphere.

We will use (2.1) to estimate the energy of \( u_j \) for sufficiently large \( j \)'s in the region \( r < |x - q| < 2r \). We consider \( j > 2/r \), so that the strict inclusion \( S(q, 2/j) \subsetneq S_r \) holds. By assumption (d) we have \( \psi_j(S(q, 1/j)) = S^2 \setminus \{ \psi(q) \} \) and \( \psi_j \) is injective in this region i.e. for any \( y \in S^2 \setminus \{ \psi(q) \} \) the set \( \psi_j^{-1}(S(q, 1/j)) \) consists of only one point. By (a) and (c), we also have \( \psi_j(S(q, 1/j, 2/j)) \subsetneq \mathcal{V}_\varepsilon \) and \( \psi_j(S(q, 2/j, 2r)) \subset \mathcal{V}_\varepsilon \).

**Figure 1.** The domain of \( u_j \). The green, red, blue and orange area are: \( S(q, 1/j) \), \( S(q, 1/j, 2/j) \), \( S(q, 2/j, r) \) and \( U_t \setminus S_r \), respectively, while the gray area illustrates \( B^3 \cap B(q, 2r) \).

**Figure 2.** The image of \( u_j(U_t) \). Colors illustrate the images of corresponding areas in Figure 1.

Since, by the assumption above, \( u_j \) is continuous in \( \{|x - q| < 2r\} \), we have \( \deg (u_j|_{U_t}) = 0 \) for every \( t \leq 2r \) because the set \( U_t \) is topologically a sphere. Now, choose a number \( t \in (r, 2r) \), fix a point \( y \in S^2 \setminus \{ \psi(S(q, 1/j, r)) \} \) and consider the set \( (u_j|_{U_t})^{-1}(y) \) of all its preimages. We know that there exists a point \( a \in S(q, 1/j) \) such that \( \psi_j(a) = u_j(a) = y \); since the degree is 0 we deduce that there must be another point \( b \in U_t \) such that \( u_j(b) = y \) (with a reverse orientation than at \( a \)). This degree consideration shows that for each \( t \in (r, 2r) \) there exists a point \( x_t \in U_t \) such that \( u_j(x_t) = y \). A simple computation yields \( \mathcal{H}^2(S^2 \setminus \mathcal{V}_\varepsilon) = \pi \left( 3 + \left( 1 - \frac{\varepsilon}{2} \right)^2 \right) \).
Thus, for $\varepsilon$ small, by formula (2.1) we obtain
\[
\int_{\{r<|x-q|<2r\}} |\nabla u_j(x)|^2 \, dx \geq 2 \int_{S^2} \mathcal{H}^1(u_j^{-1}\{w\}) \, d\mathcal{H}^2 w \\
\geq 2 \cdot r \cdot \pi \left( 3 + \left( 1 - \frac{\varepsilon^2}{2} \right) \right) \\
> 7\pi r.
\]

Having in mind the inequality $\int_{B(q,2r)} |\nabla u|^2 \, dx < 2\pi r$ from Step 3, this is a contradiction to the strong convergence obtained in Step 1. Thus in the region $|x-q| < 2r$ for sufficiently large $j$'s each $u_j$ has a singularity $q_j$.

**Step 5.** Now it suffices to show that $q_j \to q$ as $j \to \infty$. Since $\varepsilon > 0$ was arbitrary, we may choose a sequence of $\varepsilon_j \searrow 0$ such that the corresponding $r_j \searrow 0$ and the region $B(q,2r_j)$ in which the singularity $q_j$ appears will shrink to $\{q\}$. □

**Remark 2.3.** The assertion of Theorem 2.2 holds true if we replace each $\psi_j$ by an approximation $\tilde{\psi}_j$ such that the modification in the region $|x-q| < \frac{1}{j}$ from Definition 2.1 (d) remains a diffeomorphism of the smaller disc to the whole sphere without a small cap centered at $\psi(q)$, such that for sufficiently large $j$'s this cap is contained in $\mathcal{V}_{\varepsilon}$ from Step 4. One may easily check that it does not affect the proof.

### 3. Construction of $\tilde{\varphi}$

The main idea is as follows: we will modify $\varphi$ on two antipodal sets (in fact, on two little antipodal discs in $S^2$) of small measures. The modified $\tilde{\varphi}$ will be arbitrarily close to $\varphi$ in the the trace space $H^{1/2}$ although its oscillations on these discs will be large in $C^0$. In the first step of the construction, we shall perturb the original mapping slightly, to make it constant on those two discs. Next, roughly speaking, we repeat the construction of Hardt and Lin in [8] in those regions to obtain our $\tilde{\varphi}$.

At the beginning of this section we recall without proofs a few known results which will be used in the proof of Theorem 1.1. In the second part we construct our boundary condition and we close the section with the proof of Theorem 1.1.

#### 3.1. Auxiliary propositions.

The following theorem is a restatement of boundary regularity criterion in [12]. This form, convenient for our purposes, is taken from [2, Theorem 1.10 (2)].

**Theorem 3.1.** There exists $\varepsilon > 0$ with the following properties. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is three times continuously differentiable with $f(0) = 0$, $\nabla f(0) = 0$, and each partial derivative of $f$ up to order 3 does not exceed $\varepsilon^2$ in absolute value. Suppose also that $\varphi_0 : \mathbb{R}^2 \to S^2$ is three times continuously differentiable and that each partial derivative of $\varphi_0$ up to order 3 does not exceed $\varepsilon^2$ in absolute value. Finally suppose that $u^*$ is a minimizer in the region
\[
\{(x,y,z) : x^2 + y^2 \leq 1 \text{ and } f(x,y) \leq z \leq 1\}
\]
and the boundary mapping $\varphi^*$ for $u^*$ satisfies the condition that

$$\varphi^*(x, y, f(x, y)) = \varphi_0(x, y) \text{ whenever } x^2 + y^2 < 1.$$  

Then there is a two times continuously differentiable mapping $u_0 : \mathbb{R}^3 \to S^2$ such that each partial derivative of $u_0$ up to order 2 does not exceed $\varepsilon$ in absolute value and $u^*$ coincides with $u_0$ in the region

$$\left\{ (x, y, z) : x^2 + y^2 \leq \frac{1}{2} \text{ and } f(x, y) \leq z \leq \varepsilon \right\}.$$  

The next theorem was discovered by Almgren and Lieb; a precise statement can be found in [2, Theorem 4.1 (1)]. It asserts that the boundary mappings having unique minimizers are dense in $H^1(\partial B^3)$. Theorem 3.2 and the trick used in its proof, will play an important role in our construction.

**Theorem 3.2.** Suppose that $p$ is a point in $\partial B^3$, $\varepsilon > 0$, and that $\varphi : \partial B^3 \to S^2$ is a boundary mapping with $\partial E(\varphi) < \infty$. Then there is another mapping $\varphi^* : \partial B^3 \to S^2$ which coincides with $\varphi$ except possibly on that part of $\partial B^3$ within the ball $B(p, \varepsilon)$, which differs from $\varphi$ in $H^1(\partial B^3)$ norm by no more than $\varepsilon$, and for which there is exactly one minimizer $u^* : B^3 \to S^2$ having boundary mapping $\varphi^*$.

The key observation in the proof of the above theorem is the following lemma which follows form the fact that harmonic maps into $S^2$ are real analytic away from their singular points.

**Lemma 3.3.** Suppose $\Omega$ is a proper subdomain of a larger domain $\Omega^*$ and $u$ is any minimizer (not necessary unique) in $\Omega^*$. Then the restriction $u|_{\Omega}$ of $u$ to $\Omega$ is the unique minimizer for its boundary mapping.

### 3.2. Construction of $\tilde{\varphi}$.

We start with the observation that if $\text{deg} \varphi = 0$ then there exist two antipodal points $p$, $-p \in S^2$ such that $\varphi(p) = \varphi(-p)$. For the existence of such $\pm p \in S^2$, see for instance Granas and Dugundji [6, Theorem 6.1]. For the convenience of the reader, we give here the gist of a quick argument: assume on the contrary that $\varphi(p) \neq \varphi(-p)$ for all $p \in S^2$; one then easily constructs a homotopy from $\varphi$ to another map $\varphi_0$ which preserves the antipodes, i.e. $\varphi_0(p) = -\varphi_0(-p)$ for each $p \in S^2$. This is done as follows: for a given $p \in S^2$, if we already have $\varphi(p) = -\varphi(-p)$ for some $p \in S^2$, then the homotopy changes nothing; if $\varphi(p) \neq -\varphi(-p)$, then the two distinct points $\varphi(\pm p) \in S^2$ determine a unique arc $\gamma$ of the great circle such that the length of $\gamma$ is smaller than $\pi$, and we let $\varphi(\pm p)$ travel at equal, constant speeds towards two antipodal points $\pm q$ on that great circle (note that $\gamma$ is located symmetrically on one of the half-circles joining $\pm q$). However, it is well known that a map which preserves the antipodes must be of odd degree, a contradiction.

In the remaining part of this section, we simply say that $\pm p \in S^2$ are the antipodal points of $\varphi$. First, we perturb $\varphi$ slightly by making it constant close to $\pm p$. 


Lemma 3.5. For each choice of $\delta > 0$ such that $H^2 (\varphi(B(p, 2\delta)) \cup \varphi(B(-p, 2\delta))) < 4\pi$, we let $\varphi_1 : S^2 \to S^2$ denote any intermediate smooth mapping such that

1. $\varphi_1(x) \equiv \varphi(p)$ for $x \in S^2 \cap (B(p, \delta) \cup B(-p, \delta))$;
2. $\varphi_1(x) = \varphi(x)$ on $S^2 \setminus (B(p, 2\delta) \cup B(-p, 2\delta))$;
3. On each of the two annuli $B(\pm p, 2\delta) \setminus B(\pm p, \delta)$ the map $\varphi_1$ is given by a composition of $\varphi$ with a smooth diffeomorphism from the annulus to a punctured disc.

The parameter $\delta$ will be important in our further estimates. Therefore, we explain the choice of $\delta$ in the following lemma.

Lemma 3.5. For each $\varepsilon > 0$ there is a $\delta > 0$ such that the map $\varphi_1$ specified in Definition 3.4 above has $\deg(\varphi_1) = 0$ and $\| \varphi - \varphi_1 \|_{H^{1/2}} < \frac{\varepsilon}{3}$.

Proof. By Sard’s theorem (and the assumption that $\varphi(B(p, 2\delta)) \cup \varphi(B(-p, 2\delta))$ is not of full measure) we may choose a regular value $y$ of $\varphi_1$ such that $y \not\in \varphi(B(\pm p, 2\delta))$; by definition, the preimages of $y$ under $\varphi_1$ are the same as its preimages under $\varphi$, so that $\deg(\varphi_1) = \deg(\varphi) = 0$. We also have

\begin{equation}
\| \varphi - \varphi_1 \|_{H^{1/2}(\partial B^3)} \leq C \| u - u_1 \|_{H^1(B^3)} \leq M \| \varphi - \varphi_1 \|_{H^{1/2}(\partial B^3)}^{1/2},
\end{equation}

where $u$ and $u_1$ are minimizers for boundary mappings $\varphi$, $\varphi_1$ respectively. Here, the first equality is just the standard property of traces, and the second estimate is a known fact that the Dirichlet integral energy for a ball is dominated by the square root of boundary Dirichlet integral energy (see e.g. [2] Theorem 1.1]), with a constant $M$ which is independent of $\delta$, $\varphi$, $\varphi_1$. Since $\varphi \in C^\infty(S^2, S^2)$, we have $\max_{x \in S^2} |\nabla T \varphi(x)| < \infty$ and

\begin{equation}
\frac{1}{2} \| \varphi - \varphi_1 \|^2_{H^{1/2}(\partial B^3)} \leq \int_{S^2 \cap B(p, 2\delta)} |T \varphi(x)|^2 d\sigma + \int_{S^2 \setminus (B(p, 2\delta) \cup B(p, 2\delta))} |T \varphi_1(x)|^2 d\sigma
+ \int_{S^2 \cap B(-p, 2\delta)} |T \varphi_1(x)|^2 d\sigma + \int_{S^2 \setminus (B(-p, 2\delta) \cup B(-p, 2\delta))} |T \varphi_1(x)|^2 d\sigma
\leq 2(4\pi \delta^2 + 3\pi \delta^2) \max_{x \in S^2} |T \varphi(x)|^2 + 16\pi \delta^2
\leq 16\pi \delta^2 \max_{x \in S^2} (|T \varphi(x)|^2 + 1).
\end{equation}

Thus combining (3.1) and (3.2) and choosing $\delta$ such that

$$\delta < \frac{\varepsilon^2}{3^2 \cdot M^2 \cdot \sqrt{32} \pi \max_{x \in S^2} (|T \varphi(x)|^2 + 1)}$$

we obtain $\| \varphi - \varphi_1 \|_{H^{1/2}} < \frac{\varepsilon}{3}$.

\[\square\]
We now introduce a perturbation of $\varphi_1$ and define a new intermediate map $\varphi_2 : S^2 \to S^2$. Let $\alpha = 4 \arcsin \frac{\delta}{2}$ denote the length of the arc $\gamma \cap B(p, \delta)$, where $\gamma$ is any great circle through $p$. Without loss of generality suppose from now on that $p = (0, 0, 1) \in S^2$. Roughly speaking, we are going to insert $2N$ appropriately small bubbles into $\varphi_1$, at points $\pm \xi_i$ close to $\pm p$, preserving the degree but forcing the minimizers to be singular at many points.

Recall also that $S(a, 2/j) \equiv B(a, 2/j) \cap S^2$ denotes a spherical cap centered at $a$.

**Definition 3.6.** Let $\xi_i = (0, \sin \left( \frac{i \alpha}{N+1} \right), \cos \left( \frac{i \alpha}{N+1} \right)) \in B(p, \delta)$ for $i = 1, \ldots, N$. For sufficiently large $j$’s, with $2/j \ll \delta/2N$, we define $\varphi_2 : S^2 \to S^2$ as follows:

1. $\varphi_2(x) = [\varphi_1]_{\xi_i,2/j}(x)$ for $x \in S(\xi_i, 2/j)$;
2. $\varphi_2(x) = [\varphi_1]_{\xi_i,2/j}(-x)$ for $x \in S(-\xi_i, 2/j)$;
3. $\varphi_2 \equiv \varphi_1$ on $S^2 \setminus \left( \bigcup_{i=1}^{N} S(\xi_i, 2/j) \cup S(-\xi_i, 2/j) \right)$,

where $[\psi]_{a,b}$ is the modification of $\psi$ in the spherical cap $S(a,b)$, see Definition 2.1.

Note that $\varphi_2$ on each cap $S(\xi_i, 2/j)$ is either an orientation-preserving (degree 1) or an orientation-reversing (degree $-1$) map onto $S^2$, while on $S(-\xi_i, 2/j)$ it is of opposite orientation (respectively degree $-1$ or degree 1) map onto $S^2$. Since $\deg(\varphi_1) = 0$ we also have $\deg(\varphi_2) = 0$.

**Lemma 3.7.** Fix $\delta_1 > 0$ sufficiently small. One may modify $\varphi_2 : S^2 \to S^2$ in a spherical cap of radius $\delta_1$, located away from all $S(\xi_i, 2/j)$, obtaining a new map $\varphi_3 : S^2 \to S^2$ such that $\|\varphi_2 - \varphi_3\|_{H^1(\partial B^3)} < 10\delta_1$ and $\deg(\varphi_3) = 0$, for which there is exactly one minimizer $\tilde{u} : B^3 \to S^2$ with $\tilde{u} |_{\partial B^3} = \varphi_3$.

We essentially repeat Almgren and Lieb’s proof of Theorem 3.2; the only important difference is that we have to show that our $\varphi_3$ is of degree 0. For the sake of completeness we state the argument in full.

**Proof.** We extend the ball $B^3$ slightly to obtain a new smooth domain $\Omega \supset B^3$, so that $\Omega$ coincides with $B^3$ except in a ball $B(p^*, \delta_1)$, where $\delta_1 < \frac{1}{4} \text{dist}(p, p^*)$ and, to fix the ideas, we choose

$$p^* = \left(0, -\sin \left( \frac{\alpha}{2N} \right), \cos \left( \frac{-\alpha}{2N} \right) \right)$$

away from all the $\xi_i$ and from the caps where the bubbles are inserted into $\varphi_1$. Roughly speaking, the new $\Omega$ is the union of $B^3$ and of a tiny and very flat bump of width $2\delta_1$ and height $\delta_1^5$, which is added close to $p^*$. It is convenient to imagine $\partial \Omega$ as the graph of a smooth nonnegative function $\theta : S^2 \to [0, \infty)$ such that $\theta$ vanishes on $S^2 \setminus B(p^*, \delta_1)$ and close to $p^*$, after we flatten the sphere locally,

$$\theta(\cdot) = \delta_1^5 \eta \left( \frac{\cdot}{\delta_1} \right) : \mathbb{R}^2 \to [0, \infty),$$

where $\eta$ is a smooth nonnegative cutoff function supported in the unit disc with $\eta(0) > 0$. Formally, we let $T : S^2 \setminus \{-p^*\} \to \mathbb{R}^2$ be a stereographic projection such that $T(p^*) = 0$.
and set

$$\Omega = \mathbb{B}^3 \cup \left\{ y : T(\Pi(y)) \in B(0, \delta_1) \subseteq \mathbb{R}^2 \text{ and dist}(y, S^2) < \delta_5 \eta \left( \frac{T(\Pi(y))}{\delta_1} \right) \right\},$$

where $\Pi$ stands for the nearest point projection from $\partial \Omega$ to $\partial \mathbb{B}^3$. Multiplying $\eta$ by a positive constant, we may obviously assume that each partial derivative up to order 3 of $\delta_5 \eta (\cdot) \delta_1$ does not exceed $\delta_2 \delta_1$ in absolute value.

Next we define a new mapping on the boundary of $\Omega$, $\varphi^* : \partial \Omega \rightarrow S^2$, by setting $\varphi^*(x) = \varphi(x)$ for $x \in B(p^*, 4\delta_1)$. In particular each partial derivative of $\varphi^*$ is equal to 0 on that set and therefore does not exceed $\delta_2 \delta_1$ in absolute value.

Let $u^* : \Omega \rightarrow S^2$ be any minimizer for $\varphi^*$. Then, $u^*|_{\mathbb{B}^3} : \mathbb{B}^3 \rightarrow S^2$ is the unique minimizer for its boundary mapping $\varphi_3 := u^*|_{\partial \mathbb{B}^3}$, by Lemma 3.3. Note that by Theorem 3.1 $u^*$ is of class $C^2$ up to the boundary of $B(p^*, 2\delta_1) \cap \Omega$. This regularity assertion can easily be improved. To this end, we fix any smooth bounded domain $V \subset B(p^*, 2\delta_1) \cap \Omega$ with, say, $V \supset \Omega \cap B(p^*, \frac{3}{2}\delta_1)$, and with a $C^\infty$ boundary $\partial V \supset B(p^*, \frac{3}{2}\delta_1) \cap \partial \Omega$. An easy inductive argument using linear Schauder theory, see [5, Thm. 6.19], applied to $u^*|_{V}$ and the elliptic system $-\Delta u = |\nabla u|^2 u \equiv f$ on $V$, shows that in fact $u^*$ is of class $C^\infty(V)$. Therefore $\varphi_3$ is of class $C^\infty(S^2, S^2)$.

Next we show that $\deg(\varphi_3) = 0$. By the Uniform Boundary Regularity Theorem 3.1, the energy minimizer $u^*$ is two times continuously differentiable at least on $B(p^*, \delta_1)$ and each of its partial derivatives does not exceed $\delta_1$, so that $|u^*(x) - u^*(y)| \leq \sqrt{3} \delta_1 |x - y|$ by the mean value theorem. Thus, if $x \in \partial \Omega \cap B(p^*, \delta_1)$ and $y \in S^2 \cap B(p^*, \delta_1)$, then

$$|\varphi(p) - \varphi_3(y)| = |u^*(x) - u^*(y)| \leq \sqrt{3} \delta_1 |x - y| < 4\delta_1^2.$$

**Figure 3.** $\Omega$ is the union of a ball and a small flat bump.
To compute the degree of \( \varphi_3 \), choose any regular value of \( \varphi_3 \) away from \( S^2 \cap B(\varphi(p), 4\delta_1^2) \).
Its preimages under \( \varphi_3 \) will be the same as those under \( \varphi_2 \). Thus, the degree of \( \varphi_3 \) must be the same as that of \( \varphi_2 \), i.e. equal to zero.

Finally, since by Theorem 3.1 each partial derivative of \( u^* \) does not exceed \( \delta_1 \) on the set \( B(p^*, \delta_1) \) and on the set \( \{ \varphi_2 \neq \varphi_3 \} \) the mapping \( \varphi_2 \) is constant, we have the estimate
\[
\| \varphi_2 - \varphi_3 \|_{H^1(\partial B^3)}^2 = \int_{\{ \varphi_2 \neq \varphi_3 \}} (|\nabla_T \varphi_2 - \nabla_T \varphi_3|^2 + |\varphi_2 - \varphi_3|^2) \, d\sigma
\leq 2 \int_{\{ \varphi_2 \neq \varphi_3 \}} |\nabla_T \varphi_3|^2 \, d\sigma + 2^2 \mathcal{H}^2(\{ \varphi_2 \neq \varphi_3 \})
< 2 \int_{B(p^*, \delta_1)} |\nabla u^*|^2 \, dx + 4\pi \delta_1^2 < 30\pi \delta_1^2 \quad \text{for } \delta_1 < 1.
\]
Therefore, for sufficiently small \( \delta_1 \) we conclude that \( \| \varphi_2 - \varphi_3 \|_{H^1(\partial B^3)} < 10\delta_1. \) \( \square \)

**Proof of Theorem 1.1.** We first prove that the mapping \( \varphi_3 \) given in Lemma 3.7 has the properties (i)–(iii) and then to finish the proof we will choose an approximation \( \tilde{\varphi} \) of \( \varphi_3 \) so that \( \tilde{\varphi} \in C^\infty(S^2, S^2). \)

(i) and (iii): By Lemma 3.7, a minimizer \( u_3 \) for the boundary condition \( \varphi_3 \) is unique and of degree 0. The proof that \( u_3 \) has at least \( 2N \) singularities is essentially the same as in Theorem 2.2, therefore we skip it.

(ii): Fix \( \varepsilon > 0 \). We now attune \( \delta_1 \) and \( j \) to obtain \( \| \varphi - \varphi_3 \|_{H^{1/2}} < \varepsilon. \) We first choose \( \delta > 0 \) as in the proof of Lemma 3.5 and then \( j \) so that \( \| \varphi_1 - [\varphi_1]_{\xi, 2/j} \|_{H^{1/2}} < \frac{\varepsilon}{6N} \) (this is possible in view of the strong convergence in Theorem 2.2). Finally, fixing \( \delta_1 < \frac{\varepsilon}{30} \) we obtain
\[
\| \varphi - \varphi_3 \|_{H^{1/2}} < \| \varphi - \varphi_1 \|_{H^{1/2}} + \| \varphi_1 - \varphi_2 \|_{H^{1/2}} + \| \varphi_2 - \varphi_3 \|_{H^{1/2}}
< \frac{1}{3} \varepsilon + 2N \cdot \| \varphi_1 - [\varphi_1]_{\xi, 2/j} \|_{H^{1/2}} + 10\delta_1
< \frac{1}{3} \varepsilon + \frac{1}{3} \varepsilon + \frac{1}{3} \varepsilon = \varepsilon.
\]
To see that \( \mathcal{H}^2(\{ x \in S^2 : \varphi(x) \neq \varphi_3(x) \}) < \varepsilon \) we recall that \( \varphi \) differs from \( \varphi_3 \) only on the two spherical caps \( S^2 \cap B(\pm p, 2\delta) \) whose \( \mathcal{H}^2 \) measure is \( 8\pi \delta^2 \). Shrinking \( \delta \) if necessary, we conclude the whole proof. \( \square \)

4. A REMARK ON THE NONUNIQUENESS IN THE CLASS OF MINIMIZING MAPPINGS

In the following we would like to remark how does the boundary mapping constructed in Theorem 1.1 lead to a nonuniqueness example, similar (in the construction) to that of [8] Section 5.

**Remark 4.1.** There exist a mapping \( \varphi_\tau \in C^\infty(S^2, S^2), \deg(\varphi_\tau) = 0, \) which serves as a boundary data for at least two energy minimizing maps from \( B^3 \) to \( S^2 \) having different number of singularities.
Indeed, let $\psi \in C^\infty(S^2, S^2)$ be any mapping having exactly $M \in \mathbb{N}$ singular points such that $\deg(\psi) = 0$ and for which there exists unique energy minimizer $w \in W^{1,2}(\mathbb{B}^3, S^2)$. We construct $\tilde{\psi} \in C^\infty(S^2, S^2)$ as in Theorem 1.1 for which $\deg(\tilde{\psi}) = 0$ and there exists precisely one energy minimizing mapping $\tilde{w} \in W^{1,2}(\mathbb{B}^3, S^2)$ with at least $M + 2$ singularities.

Since the mappings $\psi$ and $\tilde{\psi}$ are homotopic, there exist a smooth family of smooth mappings $\{\varphi_t\}_{t \in [0,1]}$ such that $\varphi_0 = \psi$ and $\varphi_1 = \tilde{\psi}$.

From the Stability Theorem obtained in [7] we deduce that for $t$ sufficiently close to 0 each energy minimizer with boundary data $\varphi_t$ has exactly $M$ singular points. Let $\tau = \sup\{t : \text{each energy minimizer with boundary data } \varphi_t \text{ has at most } M \text{ singular points}\}$, then $0 < \tau < 1$.

We may choose a sequence $s_i \nearrow \tau$ and a sequence of energy minimizing maps $u_i \in W^{1,2}(\mathbb{B}^3, S^2)$ having at most $M$ singular points such that $u_i|_{S^2} = \varphi_{s_i}$. Similarly we choose $t_i \searrow \tau$ with a sequence of minimizing mappings $v_i \in W^{1,2}(\mathbb{B}^3, S^2)$ having at least $M + 2$ singularities, $v_i|_{S^2} = \varphi_{t_i}$. (Since we consider boundary maps of degree zero, and it is known that the degree of a minimizing harmonic map on a small sphere around a singular point is $\pm 1$, the number of singular points must jump at least by 2.) Passing to subsequences, without changing notation we obtain $u_i \to u$ and $v_i \to v$, the convergence is strong in $W^{1,2}$ and $u|_{S^2} = \varphi_\tau = v|_{S^2}$.

The mapping $u$ has at most $M$ singularities; possibly $M = 0$. (It is plausible that one may exclude the possibility $M = 0$ by choosing the homotopy appropriately.) Indeed, assume $u$ has at least $M + 2$ singular points. Then, by [2, Theorem 1.8 (2)], in an arbitrarily small ball around each singularity of $u$ there would be a singularity of $u_i$ for $i$ sufficiently large, a contradiction.

On the other hand, $v$ has at least $M + 2$ point singularities. Recall that each $v_i$ has at least $M + 2$ singularities and again by [2, Theorem 1.8] we know that singular points converge to singular points. To see that $v$ has at least $M + 2$ singularities we must exclude the possibility that some singularities of the $v_i$’s come together and cancel. By [2, Theorem 2.1] there exists a universal constant $C$ such that if $D$ denotes the distance from a singularity to the boundary of the ball then there is no other singularity within distance $CD$ from the singularity. Thus, the singularities of $v_i$ cannot merge in the interior of $\mathbb{B}^3$. Moreover, by Theorem 3.1 there is a neighborhood of the boundary which contains no singularities of $v$ and of the $v_i$’s sufficiently close to $v$ (as the $\varphi_{t_i}$’s and $\varphi_\tau$ are close to each other in $C^\infty$). This precludes the case of singularities merging in the limit at the boundary.

References

[1] F. Almgren, W. Browder, and E. H. Lieb. Co-area, liquid crystals, and minimal surfaces. In Partial differential equations (Tianjin, 1986), volume 1306 of Lecture Notes in Math., pages 1–22. Springer, Berlin, 1988.

[2] Frederick J. Almgren, Jr. and Elliott H. Lieb. Singularities of energy minimizing maps from the ball to the sphere: examples, counterexamples, and bounds. Ann. of Math. (2), 128(3):483–530, 1988.
[3] F. Bethuel, H. Brezis, and J.-M. Coron. Relaxed energies for harmonic maps. In *Variational methods (Paris, 1988)*, volume 4 of *Progr. Nonlinear Differential Equations Appl.*, pages 37–52. Birkhäuser Boston, Boston, MA, 1990.

[4] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[5] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[6] Andrzej Granas and James Dugundji. *Fixed point theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[7] Robert Hardt and Fang-Hau Lin. Stability of singularities of minimizing harmonic maps. *J. Differential Geom.*, 29(1):113–123, 1989.

[8] Robert Hardt and Fang-Hua Lin. A remark on $H^1$ mappings. *Manuscripta Math.*, 56(1):1–10, 1986.

[9] Mohammad Reza Pakzad. Existence of infinitely many weakly harmonic maps from a domain in $\mathbb{R}^n$ into $S^2$ for non-constant boundary data. *Calc. Var. Partial Differential Equations*, 13(1):97–121, 2001.

[10] Tristan Rivière. Everywhere discontinuous harmonic maps into spheres. *Acta Math.*, 175(2):197–226, 1995.

[11] Richard Schoen and Karen Uhlenbeck. A regularity theory for harmonic maps. *J. Differential Geom.*, 17(2):307–335, 1982.

[12] Richard Schoen and Karen Uhlenbeck. Boundary regularity and the Dirichlet problem for harmonic maps. *J. Differential Geom.*, 18(2):253–268, 1983.

**Katarzyna Mazowiecka**
Instytut Matematyki
Uniwersytet Warszawski
ul. Banacha 2
PL-02-097 Warsaw
POLAND
E-mail: K.Mazowiecka@mimuw.edu.pl

**Paweł Strzelecki**
Instytut Matematyki
Uniwersytet Warszawski
ul. Banacha 2
PL-02-097 Warsaw
POLAND
E-mail: pawelst@mimuw.edu.pl