ON ASYMPTOTIC DIMENSION AND A PROPERTY OF NAGATA

J. HIGES AND A. MITRA

Abstract. In this note we prove that every metric space \((X, d)\) of asymptotic dimension at most \(n\) is coarsely equivalent to a metric space \((Y, D)\) that satisfies the following property of Nagata:

For every \(x, y_1, \cdots, y_{n+2} \in Y\) there exist \(i, j \in \{1, \cdots, n+2\}\) with \(i \neq j\), such that \(D(y_i, y_j) \leq D(x, y_i)\).
This solves problem 1400 of [1].

Contents

1. Introduction 1
2. Main theorem 2
References 3

1. INTRODUCTION

Nagata introduced two properties to characterize metric spaces with finite topological dimension (see [3], [6] and [7]). Such properties are generalizations to higher dimensions of the notion of ultrametric space. The definition of the properties are the following ones:

Definition 1.1. A metric space \((X, d)\) is said to satisfy the property \((N_1)_n\) if for every \(r > 0\) and every \(x, y_1, \cdots, y_{n+2} \in X\) such that \(d(y_i, B(x, r)) < 2 \cdot r\) then there exists \(i, j \in \{1, \cdots, n+2\}\) with \(i \neq j\) such that \(d(y_i, y_j) < 2 \cdot r\).

Definition 1.2. A metric space \((X, d)\) is said to satisfy property \((N_2)_n\) if for every \(x, y_1, \cdots, y_{n+2} \in X\) there exists \(i, j \in \{1, \cdots, n+2\}\) with \(i \neq j\), such that \(d(y_i, y_j) \leq d(x, y_i)\).

In [3] Dranishnikov and Zarichnyi showed that every proper metric space \((X, d)\) of asymptotic dimension at most \(n\) is coarsely equivalent to a proper metric space that satisfies \((N_1)_n\). So a natural question is if the same statement is true for the second property. Such problem appeared in [1] as problem 1400. In this paper we solve it for general metric spaces using a technique of [2].
2. Main theorem

The notion of asymptotic dimension was introduced by Gromov in [4] and it has been the focus of intense research in recent years. To give a definition we need to recall that a family of subsets $\mathcal{U}$ of a metric space $(X, d)$ is said to be $r$-disjoint if for every two different $U \in \mathcal{U}$ and $V \in \mathcal{U}$ then $d(U, V) > r$.

**Definition 2.1.** A metric space $(X, d)$ is said to be of asymptotic dimension at most $n$ ($asdim(X, d) \leq n$) if for every $r > 0$ there exists a uniformly bounded covering $\mathcal{U}$ of subsets of $X$ such that $\mathcal{U}$ splits in a union of the form $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}^i$ where each $\mathcal{U}^i$ is an $r$-disjoint family of subsets.

To simplify we will say that a covering $\mathcal{U}$ of a metric space $(X, d)$ has **Lebesgue number** at least $r$ ($L(\mathcal{U}) \geq r$) if for every $x \in X$ the ball $B(x, r)$ is contained in one set of the covering. Next lemma characterizes asymptotic dimension in a nice way.

**Lemma 2.2.** A metric space $(X, d)$ is of asymptotic dimension at most $n$ if and only if for every $r > 0$ there exists a uniformly bounded covering $\mathcal{U}$ where $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}^i$ such that each $\mathcal{U}^i$ is an $r$-disjoint family and the Lebesgue number of $\mathcal{U}$ is at least $r$.

**Proof.** Only one implication is not trivial. Suppose $asdim(X, d) \leq n$ and let $r$ be a positive number. As $asdim(X, d) \leq n$ there exists a uniformly bounded covering $\mathcal{V}' = \bigcup_{i=1}^{n+1} \mathcal{V}^i$ such that each $\mathcal{V}^i$ is a $3 \cdot r$-disjoint family. Let $\mathcal{U}^i$ be the family of subsets given by $\mathcal{U}^i = \{N(V, r) \mid V \in \mathcal{V}^i\}$ where $N(V, r) = \{x \mid d(x, V) < r\}$. Now if $U, U' \in \mathcal{U}^i$ are different elements of $\mathcal{U}^i$ then clearly $d(U, U') > r$. Therefore the family $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}^i$ is a covering of mesh at most $mesh(\mathcal{V}) + 2 \cdot r$ so it satisfies the conditions of the lemma.

Next proposition is a version of proposition 3.6. of [3]. We give a proof to make the paper self-contained.

**Proposition 2.3.** Let $(X, d)$ be a metric space such that $asdim(X, d) \leq n$. Then there exist a sequence of uniformly bounded coverings $\{\mathcal{U}_k\}_{k=1}^{\infty}$ and an increasing sequences of numbers $\{d_k\}_{k=1}^{\infty}$ such that:

1. For every $k$, $\mathcal{U}_k = \bigcup_{i=1}^{n+1} \mathcal{U}^i_k$ and each $\mathcal{U}^i_k$ is a $d_k$-disjoint family.
2. $L(\mathcal{U}_k) \geq d_k$.
3. $d_{k+1} > 2 \cdot m_k$ where $m_k = mesh(\mathcal{U}_k)$.
4. For every $i, k, l$ with $k < l$ and every $U \in \mathcal{U}^i_k$, $V \in \mathcal{U}^i_l$, if $U \cap V \neq \emptyset$, then $U \subset V$.

**Proof.** Let us construct the sequence by induction on $k$. For $k = 1$ the result is just an application of lemma 2.2 for $d_1 = 1$. Suppose we have a finite sequence of uniformly bounded coverings $\{\mathcal{U}_k\}_{k=1}^{i}$ and a finite sequence of numbers $\{d_k\}_{k=1}^{i}$ that satisfy properties (1)-(4). Let $d_{i+1} > 0$ be a positive number such that $d_{i+1} > 2 \cdot m_i$ and define $D_{i+1} = d_i + 2 \cdot m_i$. Hence by Lemma 2.2 there exists an uniformly bounded covering $\mathcal{V}_{i+1} = \bigcup_{i=1}^{n+1} \mathcal{V}^i_{i+1}$ such that each $\mathcal{V}^i_{i+1}$ is a $D_{i+1}$-disjoint and $L(\mathcal{V}_{i+1}) \geq D_{i+1}$. Now for every $i \in \{1, ..., n+1\}$ we define the family of subsets $\mathcal{U}^i_{i+1} = \{U \mid V \in \mathcal{V}^i_{i+1}\}$ where $U$ is defined as the union of $V$ with all the subsets $W \in \mathcal{U}^i_k$ with $k \in \{1, ..., i\}$ such that $W \cap V \neq \emptyset$. We claim that the covering given by $\mathcal{U}_{i+1} = \bigcup_{i=1}^{n+1} \mathcal{U}^i_{i+1}$ satisfies the required conditions. Clearly the mesh of $\mathcal{U}_{i+1}$ is bounded by $2 \cdot m_i + mesh(\mathcal{V}_{i+1})$. Also we have $L(\mathcal{U}_{i+1}) \geq L(\mathcal{V}_{i+1}) \geq L(\mathcal{V}^i_{i+1}) \geq d_{i+1}$.
\(D_{t+1} \geq d_{t+1}\). Let \(i \in \{1, \ldots, n+1\}\) be a fix number. For every \(U_V, U_W \in \mathcal{U}_{t+1}\) we have \(d(U_V, U_W) > d(V, W) - 2 \cdot m_l = d_{t+1}\) so each \(\mathcal{U}_{t+1}\) is \(d_{t+1}\)-disjoint. The unique property that we need to check is (4). Let \(k, l\) be two numbers such that \(k < l \leq t+1\). If \(l < t+1\) the fourth condition follows from the induction hypothesis. Let us suppose \(l = t+1\). Let \(W \in \mathcal{U}_k\) and \(U_V \in \mathcal{U}_{t+1}\) such that \(W \cap U_V \neq \emptyset\). This implies there exist an \(s \leq t\) and a \(V' \in \mathcal{U}_s\) such that \(V' \cap V \neq \emptyset\) and \(W \cap V' \neq \emptyset\). By the induction hypothesis the last codition implies \(W \subset V'\) or \(V' \subset W\). In both cases we can conclude \(W \subset U_V\).

\[ \square \]

A map \(f : (X, d_X) \to (Y, d_Y)\) between metric spaces is said to be a coarse map if for every \(\delta > 0\) there is an \(\epsilon > 0\) such that for every two points \(a, b \in X\) that satisfy \(d_X(a, b) \leq \delta\) then \(d_Y(f(a), f(b)) \leq \epsilon\). If there exist also a coarse map \(g : (Y, d_Y) \to (X, d_X)\) and a constant \(C > 0\) such that for every \(x \in X\) and every \(y \in Y\), \(d_X(x, g(f(x))) \leq C\) and \(d_Y(y, f(g(y))) \leq C\) then \(f\) is said to be a coarse equivalence and the metric spaces \(X\) and \(Y\) are said to be coarse equivalent.

**Theorem 2.4.** Every metric space \((X, d)\) with \(\text{asdim} X \leq n\) is coarsely equivalent to a metric space that satisfies property \((N2)_n\).

**Proof.** Let \(\{\mathcal{U}_k\}_{k=1}^{\infty}\) be a sequence of coverings as in proposition 2.3. For every \(x, y\) in \(X\) we define:

\[ D(x, y) = \min\{k | \text{there exists } U \in \mathcal{U}_k \text{ such that } x, y \in U\} \]

Notice that by the second and third properties of \(\mathcal{U}_k\) we can easily deduce that for every \(x, y, z \in X\):

\[ D(x, y) \leq \max\{D(x, z), D(y, z)\} + 1 \]

This implies \((X, D)\) is a metric space. Moreover if \(D(x, y) \leq \delta\) and \(k\) is the minimum natural number such that \(\delta \leq k\) then \(d(x, y) \leq m_k\). In a similar way let us assume \(d(x, y) \leq \delta\). Let \(d_r\) be the minimum number of the sequence \(\{d_k\}_{k=1}^{\infty}\) such that \(\delta \leq d_r\). We have \(L(\mathcal{U}_r) > d_r\), this implies there exists an \(U \in \mathcal{U}_r\) such that \(x \in U\) and \(y \in U\) which means \(D(x, y) \leq r\). We have shown \((X, D)\) is coarsely equivalent to \((X, d)\).

Now let \(y_1, \ldots, y_{n+2}, x \in X\) and define for every \(i \in \{1, \ldots, n+2\}\) the number \(r(i) = D(y_i, x)\). By definition of \(D\) we get that for every \(i\) there exists a \(V_i \in \mathcal{U}_{r(i)}\) such that \(y_i \in V_i\) and \(x \in V_i\). For every \(i\) there exists \(k(i)\) such that \(V_i \in \mathcal{U}_{k(i)}\). By the pigeon principle there are two \(i, j\) with \(i \neq j\) such that \(k(i) = k(j)\). As \(x \in V_i \cap V_j\) by the fourth property of proposition 2.3 we conclude \(V_i \subseteq V_j\) or \(V_j \subseteq V_i\). The first case means \(D(y_i, y_j) \leq D(x, y_j)\) and the second one means \(D(y_j, y_i) \leq D(x, y_i)\). Therefore \((X, D)\) satisfies \((N2)_n\). \(\square\)

**References**

[1] T. Banakh, B. Bokalo, I. Guran, T. Radul, M. Zarichnyi, *Problems from the Lviv topological seminar*, in Open problems in Topology II. (E. Pearl ed.) Elsevier, 2007. P. 655–667.

[2] N. Brodskiy, J. Dydak, J. Higes, A. Mitra, *Assouad-Nagata dimension via Lipschitz extensions*, to appear in Israel Journal of Mathematics.

[3] A. Dranishnikov, M. Zarichnyi, *Universal spaces for asymptotic dimension*, Topology Appl. 140 (2004), no.2-3, 203–225.

[4] M. Gromov, *Asymptotic invariants for infinite groups*, in Geometric Group Theory, vol. 2, 1–295, G.Niblo and M.Roller, eds., Cambridge University Press, 1993.

[5] J. Nagata, *Note on dimension theory for metric spaces*, Fund Math. 45 (1958), 143-181.
[6] J. Nagata, *On a special metric characterizing a metric space of dim ≤ n*, Proc. Japan Acad. **39** (1963), 278-282.

[7] J. Nagata, *On a special metric and dimension*, Fund Math. **55** (1964), 181-194.

Departamento de Geometría y Topología, Facultad de CC. Matemáticas. Universidad Complutense de Madrid. Madrid, 28040 Spain

E-mail address: josemhiges@yahoo.es

University of South Florida, St. Petersburg, FL 33701, USA

E-mail address: atish@stpt.usf.edu