Dynamics of the Chaplygin ball on a rotating plane

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Abstract This paper addresses the problem of the Chaplygin ball rolling on a horizontal plane which rotates with constant angular velocity. In this case, the equations of motion admit area integrals, an integral of squared angular momentum and the Jacobi integral, which is a generalization of the energy integral, and possess an invariant measure. After reduction the problem reduces to investigating a three-dimensional Poincaré map that preserves phase volume (with density defined by the invariant measure). We show that in the general case the system’s dynamics is chaotic.

Keywords: Chaplygin ball, nonholonomic mechanics, Poincaré map, invariant measure, reduction, first integrals, Jacobi integral, affine constraints.

Mathematics Subject Classification: 37J60
Introduction

1. Affine (inhomogeneous) constraints in nonholonomic mechanics are represented as

\[ f_\mu = (a_\mu(q), \dot{q}) + b_\mu(q) = 0, \]

where \( q \) and \( \dot{q} \) are the generalized coordinates and velocities of the system. The best understood case is that of constraints homogeneous in velocities (i.e., \( b_\mu(q) = 0 \)). These constraints arise, for example, when a convex rigid body rolls without slipping or when a rigid body with a fastened wheel pair (knife edge) moves on a fixed supporting surface (the Chaplygin sleigh). As was shown already by Hertz, equations of motion preserve the energy integral in the case of homogeneous constraints. For more detailed treatments of modern methods and problems of nonholonomic mechanics, see \([31, 15, 14, 3, 13]\).

As is well known, a system with affine constraints (\( b_\mu(q) \neq 0 \)) does generally not preserve the energy integral. Nevertheless, an example of constraints in which the system admits a generalization of the energy integral is provided by the motion of a rigid body on a supporting surface which rotates with constant angular velocity. In this case, after transition to a uniformly rotating coordinate system, terms linear in the velocity are added to the Lagrangian functions, and the constraints are reduced to homogeneous ones. For this reason there exists an additional integral, which was called the Jacobi integral in \([11]\) using an analogy with celestial mechanics. In \([25, 26]\), an analogous integral is called “moving energy”.

2. The best-known example of a system with affine constraints is the rolling motion of a homogeneous ball on a uniformly rotating plane. This problem was first considered by S. Earnshaw \([24]\) before nonholonomic mechanics grew into a separate discipline \([12]\). He showed that the trajectory of the center of the ball in absolute space is a circle the position of the center of which depends on initial conditions. Later this problem was considered in the textbooks of E. A. Milne \([34]\), Yu. I. Neimark and N. A. Fufaev \([41]\), and E. J. Routh \([35]\). The authors of \([28]\) discuss the following interesting demonstration based on this problem and presented in the Franklin Museum: a homogeneous ball rolling in a straight line gets onto a rotating table and then leaves it along the same straight line. Theoretical and experimental results on this system, called ANAIS billiard, are discussed in \([32, 29]\). For references to the literature on various generalizations of the Earnshaw problem, see \([11]\).

3. Another well-known integrable, but much more complex system of nonholonomic mechanics is the problem of a dynamically asymmetric balanced ball \([21]\) (Chaplygin ball) rolling on a fixed horizontal plane. In \([10]\), a detailed qualitative analysis of this system is presented and, in particular, conditions for boundedness and unboundedness of trajectories of the contact point are found.

This problem admits a number of integrable generalizations obtained by adding a gyrostatic term or a Brun field \([40]\). Another integrable nontrivial generalization is related to the Chaplygin ball rolling over a sphere \([38, 17]\). However, if the center of mass of the ball is displaced relative to the geometric center, then the system becomes
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nonintegrable [8] and exhibits the reversal phenomenon [7] and strange attractors typical of rattlebacks [16]. There is no detailed treatment of dynamics in this case, although it is of great interest, especially when it comes to verifying the nonholonomic model (that is, checking whether it agrees with experiments) and detecting new nontrivial dynamical phenomena. Particular interest in this problem has been stimulated recently by designs of spherical robots which move by displacing the center of mass [11, 19].

4. In this paper, we consider the motion of the Chaplygin ball on a horizontal plane which rotates with constant angular velocity. An explicit generalization of the energy integral in this case was presented in [27], although this integral can easily be obtained on the basis of the results of [11]. Moreover, this problem admits an area integral, an integral of squared angular momentum, and an invariant measure. After reduction the problem reduces to investigating a three-dimensional Poincaré map that preserves phase volume (with density defined by an invariant measure). In this paper we show that in the general case the dynamics of the system is chaotic. Similar three-dimensional maps were considered earlier in relation to chaotic advection in [20, 22].

Three-dimensional maps that are given by explicit expressions (and not generated by the phase flow) have recently been investigated in [30, 23] from the viewpoint of bifurcation and chaotic dynamics. Of interest is the generalization of methods which are developed in these studies for the problem we consider here. A number of phenomena that are similar to those described in [30, 23] and lead to the appearance of three-dimensional tori are presented in this paper. When there is axial symmetry, the problem can be reduced to investigating a two-dimensional area-preserving map where the dynamics can be analyzed in more detail.

5. We note that Tzénooff [36, 37] considered the problem of a dynamically symmetric body of revolution rolling on a rotating plane to illustrate a new form (proposed by him) of equations of nonholonomic mechanics. However, in these papers he failed to obtain any dynamical conclusions or results, and the equations of motion presented in them are intractable for further analysis. As will be shown below, he also made an incorrect statement concerning integrability of the case where the body has a spherical surface.
1. Equations of motion with an inhomogeneous nonholonomic constraint

Consider the problem of an inhomogeneous balanced ball rolling without slipping on a horizontal plane rotating with constant angular velocity $\Omega$. Let us choose two coordinate systems:

- $Oxyz$, a fixed coordinate system with a vertical axis $OZ$ coinciding with the axis of rotation of the plane;
- $Cx_1x_2x_3$, a moving coordinate system with origin $C$ at the center of the ball and with axes directed along the principal axes of inertia.

The position and orientation of the ball are given by coordinates $x$ and $y$ of the ball’s center $C$ on the plane $Oxy$ and by the matrix of rotation of the moving axes relative to the fixed axes $Q = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} \in SO(3)$.

In what follows, unless otherwise specified, all vectors are referred to the moving coordinate system $Ox_1x_2x_3$, and the following notation is used:

- $v = (v_1, v_2, v_3)$ — the velocity of the ball’s center $C$,
- $\omega = (\omega_1, \omega_2, \omega_3)$ — the angular velocity of the ball,
- $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$, $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ — the unit vectors of the fixed axes $Oxyz$, with $\gamma \parallel OZ$.

The constraint equations which express the no-slip constraint can be represented in the vector form

$$f = v - a \omega \times \gamma - \Omega \gamma \times R = 0,$$

where $R$ is the radius vector of the ball’s center $C$ in the moving axes and $a$ is the radius of the ball.

The kinematic equations governing the evolution of the position and orientation of the ball can be written as

$$\dot{\alpha} = \alpha \times \omega, \quad \dot{\beta} = \beta \times \omega, \quad \dot{\gamma} = \gamma \times \omega, \quad \dot{x} = (\alpha, v), \quad \dot{y} = (\beta, v).$$

As is well known, dynamical equations of a system are derived from the D’Alembert–Lagrange principle and have in this case the form [18]:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{v}} \right) + \omega \times \frac{\partial L}{\partial \omega} = \frac{\partial L}{\partial x} \alpha + \frac{\partial L}{\partial y} \beta + \sum_{\mu} N_{\mu} \frac{d f_{\mu}}{d v},$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\omega}} \right) + \omega \times \frac{\partial L}{\partial \omega} + v \times \frac{\partial L}{\partial v} =$$

$$= \frac{\partial L}{\partial \alpha} \times \alpha + \frac{\partial L}{\partial \beta} \times \beta + \frac{\partial L}{\partial \gamma} \times \gamma + \sum_{\mu} N_{\mu} \frac{d f_{\mu}}{d \omega},$$
where \( f_\mu \) are the components of the vector constraint equation (1) and \( N_\mu \) are the undetermined multipliers (constraint reactions).

Since the center of mass coincides with the geometric center, the Lagrangian function coincides with the kinetic energy of the ball:

\[
L = \frac{1}{2}mv^2 + \frac{1}{2}(\omega, I\omega),
\]

where \( m \) is the mass of the ball and \( I = \text{diag}(I_1, I_2, I_3) \) is its tensor of inertia relative to the center of mass. Substituting into (3), we obtain

\[
m\ddot{v} + m\omega \times v = N,
I\dot{\omega} + \omega \times I\omega = aN \times \gamma.
\]

From the constraint equation (1) we find the relations

\[
v = (a\omega - \Omega R) \times \gamma,
\dot{v} = (a\dot{\omega} - \Omega \dot{R}) \times \gamma + (a\omega - \Omega R) \times (\gamma \times \omega).
\]

Using these relations and eliminating the reaction \( N \) and the velocity \( v \) from (3) and (2), we obtain a complete system, which describes the dynamics, in the form

\[
I\ddot{\omega} + ma^2 \gamma \times (\dot{\omega} \times \gamma) = I\omega \times \omega - ma\Omega(\omega, \gamma)R \times \gamma - ma\Omega(\dot{R} \times \gamma) \times \gamma
\]

\[
\dot{\alpha} = \alpha \times \omega,
\dot{\beta} = \beta \times \omega,
\dot{\gamma} = \gamma \times \omega.
\]

\[
\dot{x} = -\Omega y + a(\omega, \beta),
\dot{y} = \Omega x - a(\omega, \alpha).
\]

These equations define the flow on the eight-dimensional phase space \( \mathcal{M}^8 = SO(3) \times \mathbb{R}^5 \).
2. Conservation laws

Let us restrict the Lagrangian function using (7) to the constraint (i.e., eliminate \( v \)):

\[
L^* = \frac{1}{2}(\omega, I\omega) + \frac{1}{2}ma^2(\omega \times \gamma, \omega \times \gamma) - ma\Omega(\omega, R) + \frac{1}{2}m\Omega^2(x^2 + y^2),
\]

where for simplicity we have used the identity \( \gamma \times (R \times \gamma) = R \). Let us define the angular momentum vector of the system

\[
M = \frac{\partial L^*}{\partial \omega} = I\omega + ma^2\gamma \times (\omega \times \gamma) - ma\Omega R.
\]

(11)

From equations (8) we find that its evolution is governed by the equation

\[
\dot{M} = M \times \omega.
\]

This equation implies that the vector \( M \) remains constant in the fixed coordinate system \( Oxyz \). As a consequence, we find that the system (8) admits three linear first integrals

\[
F_1 = (M, \alpha), \quad F_2 = (M, \beta), \quad F_3 = (M, \gamma).
\]

(12)

In addition, it was shown in [11] that in the system under consideration one can construct an integral similar to the Jacobi integral in mechanics

\[
E = \frac{1}{2}(\omega, I\omega + ma^2\gamma \times (\omega \times \gamma)) - \frac{m}{2}\Omega^2(x^2 + y^2).
\]

(13)

This integral was found explicitly in [27].

Another general invariant of the system (6) is the invariant measure \( \mu = \rho dxdyd^3\alpha d^3\beta d^3\gamma d^3\omega \), where density is given by the expression

\[
\rho = \frac{\det(I + ma^2E - ma^2\gamma \otimes \gamma)}{\sqrt{1 - ma^2(\gamma, (I + ma^2E)^{-1}\gamma)}}.
\]

Equations (8) also admit an obvious symmetry field corresponding to invariance under rotations of the plane of support:

\[
\hat{u}_s = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - \beta_1\frac{\partial}{\partial \alpha_1} + \alpha_1\frac{\partial}{\partial \beta_1} - \beta_2\frac{\partial}{\partial \alpha_2} + \alpha_2\frac{\partial}{\partial \beta_2} - \beta_3\frac{\partial}{\partial \alpha_3} + \alpha_3\frac{\partial}{\partial \beta_3}.
\]

(14)

3. Reduction

Eliminating the variable corresponding to the symmetry field (14), we obtain the reduced system describing the evolution of \( \omega, \gamma \) and \( R \):

\[
\tilde{\omega} = I\omega \times \omega - ma\Omega(\omega, \gamma)R \times \gamma - ma\Omega(\dot{R} \times \gamma) \times \gamma
\]

(15)

\[
\dot{\gamma} = \gamma \times \omega, \quad \dot{R} = R \times (\omega - \Omega\gamma) - a\gamma \times \omega,
\]

(16)

where \( \tilde{I} = I + ma^2(\gamma^2E - \gamma \otimes \gamma) \) is the tensor of inertia relative to the point of contact.

The system (15) admits two geometric integrals whose values are fixed:

\[
\gamma^2 = 1, \quad (R, \gamma) = 0.
\]
These relations define in $\mathbb{R}^9$ the seven-dimensional phase space of the reduced system $\mathcal{M}^7 \approx TS^2 \times \mathbb{R}^3$.

A special feature of reduction in this case is that the complete set of integrals turns out to be noninvariant under the action of the symmetry field (14):

$$\hat{u}_s(F_3) = 0, \quad \hat{u}_s(E) = 0, \quad \hat{u}_s(F_1) = -F_2, \quad \hat{u}_s(F_2) = F_1. \quad \text{(17)}$$

As a result, the system (15) admits only three additional first integrals (and not four, as one would expect), which are invariant under the action (17). For example, one can choose:

$$\tilde{F}_1 = F_2^1 + F_2^2 + F_2^3 = (M, M), \quad \tilde{F}_2 = F_3 = (M, \gamma)$$

$$E = \frac{1}{2}(M, \omega) + \frac{1}{2}ma\Omega(\omega, R) - \frac{m\Omega^2}{2}(R, R),$$

where $M$ is given by (11).

From the known solutions $\omega(t), \gamma(t), R(t)$ of the reduced system (15) at given values of the first integrals $\tilde{F}_1 = C^2, \tilde{F}_1 = M, \gamma$ and at $M \parallel \gamma$ the orientation of the ball and the motion of the contact point are defined by the relations

$$\dot{\alpha} = \frac{M \times \gamma}{\sqrt{C^2 - M^2}}, \quad \dot{\beta} = \frac{M - M\gamma}{\sqrt{C^2 - M^2}}, \quad x = (\alpha, R), \quad y = (\beta, R),$$

where $M$ is given by (11).

For the system (15) to be integrable, there must exist (for suitable system parameters) another pair of tensor invariants, for example, two integrals, or an integral and a symmetry field.

4. Poincaré map

For numerical analysis and illustration of the behavior of the trajectories of (15), it is convenient to use the method of Poincaré section [15, 6]. We describe here briefly its construction for the system considered.

We first restrict the system (15) to the level manifold of the first integrals

$$\mathcal{M}^5 = \{ (\omega, \gamma, R) \mid \gamma^2 = 1, \quad (R, \gamma) = 0, \quad \tilde{F}_1(z) = C^2, \quad \tilde{F}_2(z) = M, \gamma \}$$

and obtain a five-dimensional flow with the energy integral $\tilde{E} = E|_{\mathcal{M}^5}$. To parameterize $\mathcal{M}^5$, we use the variables $(K, r_1, r_2, l, g,)$, which we define as follows:

$$K = M_3, \quad r_1 = R_1\gamma_1 + R_2\gamma_2, \quad r_2 = R_1\gamma_2 - R_2\gamma_1, \quad \tan l = \frac{M_1}{M_2}, \quad \tan g = \frac{C(M_2\gamma_1 - M_1\gamma_2)}{M_7K - C^2\gamma_3},$$

where $l, g \in [0, 2\pi)$ are the angle variables.

Next, we fix the level set of the energy integral $\tilde{E} = h$ and thus obtain a one-parameter family of four-dimensional flows on the manifolds $\mathcal{M}^4_h$, and as a secant for this flow we choose a manifold given by the relation

$$g = g_0 = \text{const.}$$
Numerically integrating the system under consideration and finding intersections of the trajectories with the given section, we obtain a family of three-dimensional maps $P_{h,g_0}^3$, which we parameterize by the variables $(l, r_1, r_2)$, and define the variable $K$ from the energy integral.

Since the system (15) possesses an invariant measure, the three-dimensional Poincaré map preserves some volume form. Below we outline its main properties.

For the homogeneous ball ($I_1 = I_2 = I_3$) the equations of motion (15) possess additional symmetry fields corresponding to invariance of the system under rotation about each axis:

$$\dot{u}_1 = -\omega_2 \frac{\partial}{\partial \omega_3} + \omega_3 \frac{\partial}{\partial \omega_2} - \gamma_2 \frac{\partial}{\partial \gamma_3} + \gamma_3 \frac{\partial}{\partial \gamma_2} - R_2 \frac{\partial}{\partial R_3} + R_2 \frac{\partial}{\partial R_3},$$

$$\dot{u}_2 = -\omega_3 \frac{\partial}{\partial \omega_1} + \omega_1 \frac{\partial}{\partial \omega_3} - \gamma_3 \frac{\partial}{\partial \gamma_1} + \gamma_1 \frac{\partial}{\partial \gamma_3} - R_3 \frac{\partial}{\partial R_1} + R_3 \frac{\partial}{\partial R_1},$$

$$\dot{u}_3 = -\omega_2 \frac{\partial}{\partial \omega_1} + \omega_1 \frac{\partial}{\partial \omega_2} - \gamma_2 \frac{\partial}{\partial \gamma_1} + \gamma_1 \frac{\partial}{\partial \gamma_2} - R_2 \frac{\partial}{\partial R_1} + R_2 \frac{\partial}{\partial R_1}.$$

A Poincaré map for this case is shown in Fig. 1a, and the motion of the point is illustrated in Fig. 1b. As can be seen, the map is foliated by invariant curves, and the point of contact moves in a circle. For explicit integration of the system (15) in this case it is more convenient to pass to the fixed coordinate system (see [11] for details).

For a dynamically symmetric ball ($I_1 = I_2$) the equations of motion (15) possess only one symmetry field $\dot{u}_3$. In this case, the variable $l$ is cyclic (i.e., $\dot{u}_3(l) = 0$). Thereby the problem reduces to investigating a two-dimensional Poincaré map. As is evident from Fig. 2b, the map has in this case a chaotic trajectory, and hence there is no additional integral. This implies that the conclusion made by Tzénoff in [36] concerning integrability of this case by quadratures is incorrect.

A Poincaré map for a dynamically asymmetric ball is shown in Fig. 3. The presence of a chaotic trajectory in Fig. 3b shows the absence of two additional integrals. However, the map has invariant tori (see Fig. 3b). Numerical experiments show that in both cases the trajectory of the contact point remains bounded (see Fig. 4).

An interesting feature of the behavior of the contact point is that it is not only bounded, but also regular in a sense, at least visually. Indeed, although the reduced system has chaotic behavior, the latter is very difficult to detect from visual inspection of the trajectory of the contact point, which lies between two circles and, at first sight, has quasi-periodic behavior. As a result, the above-mentioned chaotic behavior is very difficult to detect experimentally. A similar phenomenon is observed in the behavior of the trajectory of the contact point of the Chaplygin top [8] and rattlebacks [16], the reduced phase space of which contains strange attractors. In this connection we refer the reader to the work [8], in which the authors actually doubt the widely spread belief that a long-range weather forecast is impossible due to the presence of a Lorenz attractor in the simplified hydrodynamical model. It turns out that, although some dynamical variables (such as velocities) may have “strongly chaotic” behavior, experimentally measurable variables can behave rather regularly. In particular, the trajectories in Fig. 4 are obtained.
from the reduced system (whose map is shown in Fig. 3a) by an additional quadrature, which is seen to have a “regularizing” character.

In contrast to the fixed plane, where the integrability of the system allows a complete analysis of the contact point [10], the reduced system (15) exhibits in this case chaotic trajectories, and so it does not seem possible to completely describe the motion of the contact point. The problem of finite motion of the contact point is also unresolved. It is possible that there exist unbounded trajectories due to the phenomenon of diffusion described for three-dimensional maps in the recent paper [33].

Рис. 1. A Poincaré map for a completely dynamically symmetric ball (a) and the trajectory of the contact point (b) for fixed parameters $m = 1, a = 5, I_1 = I_2 = I_3 = 3, H = 6, h = 5, G = 10, \Omega = 2$.

Рис. 2. A Poincaré map for a dynamically symmetric ball (a) and the trajectory of the contact point (b) for fixed parameters $m = 1, a = 5, I_1 = I_2 = 2 I_3 = 3, H = 6, h = 5, G = 10, \Omega = 2$.

5. The case $M \parallel \gamma$

1. Let the angular momentum $M$ be parallel to the normal vector $\gamma$:

$$M = \lambda \gamma, \quad \lambda = \tilde{F}_2 = \text{const.}$$ (18)

This case requires a separate analysis, since in this case the integrals $\tilde{F}_1$ and $\tilde{F}_2$ are dependent. If the plane is fixed ($\Omega = 0$), the system is integrable and, on a fixed level
Рис. 3. A Poincaré map for a dynamically asymmetric ball at different values a) $H = 2$ and b) $H = 6$, the other parameters have the values $m = 1, a = 5, I_1 = 2, I_2 = 3, I_3 = 4, h = 5, G = 10, \Omega = 2$.

Рис. 4. Trajectory of the contact point of a dynamically asymmetric ball for parameters corresponding to Fig. 3.

As a result, we obtain a closed system of equations governing the evolution of $K$ and $\gamma$ in the form

$$
\dot{K} = \Omega \gamma \times K + (K - d \Omega \gamma) \times \omega, \quad \dot{\gamma} = \gamma \times \omega.
$$

The first integrals of this system can be represented as

$$
\gamma^2 = 1, \quad (K, \gamma) = \lambda, \quad \tilde{E} = \frac{1}{2}(K, AK) - \frac{K^2}{2d} + \frac{d}{2(1 - d(\gamma, A \gamma))}(A K, \gamma)^2.
$$

In addition, this system possesses the invariant measure

$$
\mu = \rho dK d\gamma, \quad \rho = (1 - d(\gamma, A \gamma))^{-\frac{1}{2}}.
$$
We see that the first integrals and the invariant measure of the system \((22)\) are analogous to the invariants in the Chaplygin problem of a ball rolling on a fixed plane \([21, 39]\). This raises a natural question of the possibility of representing the equations of motion (22) in conformally Hamiltonian form (see [5, 2] for details). This problem will not be considered here.

2. For the system \((22)\) to be integrable, we need an additional integral. We show that in the general case it is absent. For this we investigate numerically the Poincaré map.

We define the Andoyer–Deprit variables \((L, G, l, g)\)
\[
K_1 = \sqrt{G^2 - L^2} \sin l, \quad K_2 = \sqrt{G^2 - L^2} \cos l, \quad K_3 = L,
\]
\[
\gamma_1 = \left(\frac{\lambda}{G}\sqrt{1 - \frac{L^2}{G^2}} + \frac{L}{G}\sqrt{1 - \frac{\lambda^2}{G^2}}\right) \sin l + \sqrt{1 - \frac{\lambda^2}{G^2}} \sin g \cos l,
\]
\[
\gamma_2 = \left(\frac{\lambda}{G}\sqrt{1 - \frac{L^2}{G^2}} + \frac{L}{G}\sqrt{1 - \frac{\lambda^2}{G^2}}\right) \cos l - \sqrt{1 - \frac{\lambda^2}{G^2}} \sin g \sin l,
\]
\[
\gamma_3 = \frac{\lambda L}{G} - \sqrt{1 - \frac{L^2}{G^2}} \sqrt{1 - \frac{\lambda^2}{G^2}} \cos g,
\]
where \(l, g \in [0, 2\pi)\) and \(L, G\) satisfy the inequality
\[
-1 \leq \frac{L}{G} \leq 1.
\]

On a fixed level set of the integral \(\tilde{E} = h\) the system \((22)\) describes a four-dimensional flow. We choose \(g = 0\) as a secant of this flow and obtain a two-dimensional Poincaré map, which we parameterize by the variables \((l, \frac{L}{G})\).

![Poincaré map](image)

**Рис. 5.** A Poincaré map of the system \((22)\) for fixed parameters \(a_1 = 1, a_2 = 2, a_3 = 3, d = 4, h = 100, \lambda = 2, \Omega = 0.5.\)

A typical view of the Poincaré map is presented in Fig. 5. In this case, the map exhibits chaotic trajectories and hence, in the general case, there is no additional integral.

3. If the case is dynamically symmetric \((a_1 = a_2)\), there exists an additional integral \(F_3\) (linear in \(K\)):
\[
F_3 = \rho \gamma_3(K_1 \gamma_1 + K_2 \gamma_2) - \rho \left(1 - \gamma_3^2 \frac{1}{a_1 d}\right) K_3 + \Omega \Psi(\gamma_3),
\]
where the function $\Psi(\gamma_3)$ is defined, depending on the moments of inertia, by the following relations:

If $a_3 > a_1$, then

$$
\Psi(\gamma_3) = \frac{\sqrt{d}}{2\sqrt{a_3 - a_1}} \left( 1 - \frac{1}{a_1 d} \right)^2 \arctan \left( \frac{1 - a_1 d - 2d(a_3 - a_1)\gamma_3^2}{2\gamma_3 \sqrt{d(a_3 - a_1)}} \right).
$$

If $a_3 < a_1$, then

$$
\Psi(\gamma_3) = \frac{\sqrt{d}}{4\sqrt{a_1 - a_3}} \left( 1 - \frac{1}{a_1 d} \right)^2 (\ln(\xi+1)-\ln(\xi-1)), \xi = \frac{1 - a_1 d - 2d(a_3 - a_1)\gamma_3^2}{4\gamma_3 \sqrt{d(a_1 - a_3)}}.
$$

Thus, in the case of a dynamically symmetric ball the system (22) is integrable by quadratures.

It can be shown that in the general case the fixed level set of the integral $	ilde{E}(K, \gamma) = h$ can, depending on the system parameters, define a noncompact surface in phase space. However, numerical experiments show that the trajectories of the system (22) and the motion of the point of contact of the ball are bounded. A rigorous analysis of the conditions of boundedness of the ball’s trajectories is a separate problem which can be solved by the methods of [4].
6. Discussion

We point out some questions that are related to this work and require additional research.

For the motion of the Chaplygin ball on a rotating plane one can present the same generalizations of the problem that were studied in detail for the Earnshaw problem of a homogeneous ball described in the Introduction. For example, it is possible to introduce additional forces and friction torques, inclination of the plane and the action of gravity and to consider the now nonintegrable scattering problem [32]. However, all these generalizations are rather complicated and can evidently be carried out only by numerical simulations. The study of these problems is of great importance for determining the scope of applicability of the nonholonomic model and the role of friction which can lead to new interesting phenomena in natural experiments.

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References

[1] Alves J and Dias J 2003 Design and Control of a Spherical Mobile Robot J. Syst. Control Eng. 217 457–467
[2] Balseiro P and García-Naranjo L C 2012 Gauge Transformations, Twisted Poisson Brackets and Hamiltonization of Nonholonomic Systems Arch. Ration. Mech. Anal. 205 267–310
[3] Bizyaev I A, Borisov A V and Mamaev I S 2017 The Chaplygin Sleigh with Parametric Excitation: Chaotic Dynamics and Nonholonomic Acceleration Regul. Chaotic Dyn. 22 955–975
[4] Bolsinov A V, Borisov A V and Mamaev I S 2010 Topology and Stability of Integrable Systems Russian Math. Surveys 65 259–318 see also: Uspekhi Mat. Nauk 65 71–132
[5] Bolsinov A V, Borisov A V and Mamaev I S 2015 Geometrisation of Chaplygin’s Reducing Multiplier Theorem Nonlinearity 28 2307–2318
[6] Borisov A V, Jahnine A Yu, Kuznetsov S P, Sataev I R and Sedova J V 2012 Dynamical Phenomena Occurring due to Phase Volume Compression in Nonholonomic Model of the Rattleback Regul. Chaotic Dyn. 17 512–532
[7] Borisov A V, Kazakov A O and Sataev I R 2014 The Reversal and Chaotic Attractor in the Nonholonomic Model of Chaplygin’s Top Regul. Chaotic Dyn. 19 718–733
[8] Borisov A V, Kazakov A O and Pivovarova E N 2016 Regular and Chaotic Dynamics in the Rubber Model of a Chaplygin Top. Regul. Chaotic Dyn. 21 885–901
[9] Borisov A V, Kilin A A and Mamaev I S 2012 Generalized Chaplygin’s Transformation and Explicit Integration of a System with a Spherical Support Regul. Chaotic Dyn. 17 170–190
[10] Borisov A V, Kilin A A and Mamaev I S 2013 The Problem of Drift and Recurrence for the Rolling Chaplygin Ball Regul. Chaotic Dyn. 18 832–859
Dynamics of the Chaplygin ball on a rotating plane

[11] Borisov A V, Mamaev I S and Bizyaev I A 2015 The Jacobi Integral in Nonholonomic Mechanics Regul. Chaotic Dyn. 20 383–400
[12] Borisov A V, Mamaev I S and Bizyaev I A 2016 Historical and Critical Review of the Development of Nonholonomic Mechanics: The Classical Period Regul. Chaotic Dyn. 21 455–476
[13] Borisov A V, Mamaev I S and Bizyaev I A 2017 Dynamical Systems with Non-Integrable Constraints: Vaconomic Mechanics, Sub-Riemannian Geometry, and Non-Holonomic Mechanics Russian Math. Surveys 72 783–840 see also: Uspekhi Mat. Nauk 72 3–62
[14] Borisov A V, Mamaev I S and Kilin A A 2002 Rolling of a Ball on a Surface: New Integrals and Hierarchy of Dynamics Regul. Chaotic Dyn. 7 201–219
[15] Borisov A V and Mamaev I S 2002 The Rolling Motion of a Rigid Body on a Plane and a Sphere: Hierarchy of Dynamics Regul. Chaotic Dyn. 7 177–200
[16] Borisov A V and Mamaev I S 2013 Topological Analysis of an Integrable System Related to the Rolling of a Ball on a Sphere Regul. Chaotic Dyn. 18 356–371
[17] Borisov A V and Mamaev I S 2015 Symmetries and Reduction in Nonholonomic Mechanics Regul. Chaotic Dyn. 20 553–604
[18] Camicia C, Conticelli F and Bicch A 2000 Nonholonomic Kinematics and Dynamics of the Sphericle in Proc. of the IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS, Takamatsu, Japan) 1 805–810
[19] Cartwright J H E, Feingold M and Piro O 1999 An Introduction to Chaotic Advection in Mixing: Chaos and Turbulence H Chaté, E Villermaux and J-MChomaz (Eds.) 1999 NATO ASI Series (Series B: Physics) 373 (Boston, Mass.: Springer) 307–342
[20] Chaplygin S A 2002 On a Ball’s Rolling on a Horizontal Plane Regul. Chaotic Dyn. 7 131–148 see also: Chaplygin S A 1903 On a Ball’s Rolling on a Horizontal Plane Math. Sb. 24 139–168
[21] Cheng C Q and Sun Y S 1989/90 Existence of Invariant Tori in Three-Dimensional Measure-Preserving Mappings Celest. Mech. Dynam. Astronom. 47 275–292
[22] Dullin H R and Meiss J D 2009 Quadratic Volume-Preserving Maps: Invariant Circles and Bifurcations, SIAM J. Appl. Dyn. Syst. 8 76–128
[23] Earnshaw S 1844 Dynamics, or an Elementary Treatise on Motion 3rd ed. (Cambridge: Deighton)
[24] Fassò F and Sansonetto N 2015 Conservation of Energy and Momenta in Nonholonomic Systems with Affine Constraints Regul. Chaotic Dyn. 20 449–462
[25] Fassò F and Sanz-Serna J M 2016 Conservation of Energy in Nonholonomic Systems with Affine Constraints and Integrability of Spheres on Rotating Surfaces J. Nonlinear Sci. 26 519–544
[26] Fassò F, García-Naranjo L C and Sansonetto N 2018 Moving Energies As First Integrals of Nonholonomic Systems with Affine Constraints Nonlinearity 31 755–782
[27] Gersten J, Soodak H and Tiersten M S 1992 Ball Moving on Stationary or Rotating Horizontal Surface Am. J. Phys. 60 43–47
[28] Ivanov A P 2016 The ANAIS Billiard Experiment Dokl. Phys. 61 285–287 2016, vol. 61, no. 6, pp. 285–287; see also: Dokl. Akad. Nauk 468 401–402
[29] Kozlov V V 2016 The Phenomenon of Reversal in the Euler–Poincaré–Suslov Nonholonomic Systems J. Dyn. Control Syst. 22 713–724
[30] Levy-Leblond J-M 1986 The ANAIS Billiard Table Eur. J. Phys. 7 252–258
[31] Meiss J D, Miguel N, Simó C and Vieiro A 2018 Accelerator Modes and Anomalous Diffusion in 3D Volume-Preserving Maps arXiv:1802.10484
[32] Milne E A 1948 Vectorial Mechanics (New York: Interscience)
[33] Routh E J 1955 The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies: Being Part II of a Treatise on the Whole Subject 6th ed. (New York: Dover)
Dynamics of the Chaplygin ball on a rotating plane

[36] Tzénoff I 1925 Quelques formes différentes des équations générales du mouvement des systèmes matériels Bull. Soc. Math. France 53 80–105
[37] Tzénoff I 1920 Sur les équations générales du mouvement des systèmes matériels non holonomes J. Math. Pures Appl. (8) 3 245–263
[38] Borisov A V, Fedorov Yu N 1995 On two modified integrable problems in dynamics Mosc. Univ. Mech. Bull. 50 16–18 see also: Vestnik Moskov. Univ. Ser. 1. Mat. Mekh. 6 102–105
[39] Borisov A V and Mamaev I S 2001 Chaplygin’s Ball Rolling Problem Is Hamiltonian Math. Notes 70 720–723 see also: Mat. Zametki 70 793–795
[40] Kozlov V V 1985 On the Theory of Integration of the Equations of Nonholonomic Mechanics Uspekhi Mekh. 8 85–107 (Russian)
[41] Neimark Ju I and Fufaev N A 1972 Dynamics of Nonholonomic Systems Trans. Math. Monogr. 33 (Providence, R.I.: AMS)