Erratum: Sufficient condition for a quantum state to be genuinely quantum non-Gaussian (2018 New J. Phys. 20 023046)

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The following errors was introduced during the production process. In section 1.3. Outline the final two sentences should be interchanged and read ‘In appendix C we derive the asymptotic behavior of the maximum mean value $F_{SG} = F_{SG}(c)$ for the coherent superposition of two squeezed Gaussian states. We conclude in appendix D by describing the model of the dissipation of a quantum state via interaction with a reservoir at zero temperature.’ instead of ‘We conclude in appendix D by describing the model of the dissipation of a quantum state via interaction with a reservoir at zero temperature. In appendix C we derive the asymptotic behavior of the maximum mean value $F_{SG} = F_{SG}(c)$ for the coherent superposition of two squeezed Gaussian states.’

Also, reference [14] has an error in the year of publication. It should read ‘Straka I, Lachman L, Hloušek J, Miková M, Mičuda M, Ježek M and Filip R 2018 npj Quantum Inf. 4 4’ not ‘Straka I, Lachman L, Hloušek J, Miková M, Mičuda M, Ježek M and Filip R 2016 npj Quantum Inf. 4 4’.
Sufficient condition for a quantum state to be genuinely quantum non-Gaussian

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Abstract
We show that the expectation value of the operator $\hat{\mathcal{O}} = \exp(-c\hat{x}^2) + \exp(-c\hat{p}^2)$ defined by the position and momentum operators $\hat{x}$ and $\hat{p}$ with a positive parameter $c$ can serve as a tool to identify quantum non-Gaussian states, that is states that cannot be represented as a mixture of Gaussian states. Our condition can be readily tested employing a highly efficient homodyne detection which unlike quantum-state tomography requires the measurements of only two orthogonal quadratures. We demonstrate that our method is even able to detect quantum non-Gaussian states with positive-definite Wigner functions. This situation cannot be addressed in terms of the negativity of the phase-space distribution. Moreover, we demonstrate that our condition can characterize quantum non-Gaussianity for the class of superposition states consisting of a vacuum and integer multiples of four photons under more than 50 % signal attenuation.

1. Introduction

Gaussian states play a major role in the field of quantum information with continuous variables [1]. They are not only readily accessible in experiment, for example in the form of squeezed states, but also allow a compact description in theory [2]. On the other hand, non-Gaussian states and non-Gaussian operations are known to provide an advantage in quantum information processing such as the distillation of quantum entanglement [3] and enhanced performance in quantum teleportation [4]. They are even a critical requirement for some quantum tasks such as universal quantum computation [5], nonlocality tests [6] and entanglement distillation [7].

In this regard, the problem of distinguishing between Gaussian and non-Gaussian states has recently attracted considerable attention, in particular, detecting a state of genuine quantum non-Gaussian character that cannot be represented by a mixture of Gaussian states. Quantum non-Gaussianity is in a sense a stronger notion than nonclassicality as the latter refers to states that cannot be represented by a mixture of coherent states, which are strictly a subset of the whole Gaussian states. Nonclassicality and quantum non-Gaussianity provide crucial resources for quantum information processing. For instance, it is known that every single-mode nonclassical state, when mixed with a vacuum state at a beam splitter, becomes a quantum entangled state [8]. This fact also implies that a single-mode quantum non-Gaussian state can be turned to a useful non-Gaussian entangled state by passive transformations.

A well-known criterion [9] for a pure state to be a Gaussian one is the positivity of the corresponding Wigner function [10] at any point of phase space. However, this elementary but fundamental criterion of Gaussianity is not useful in practice as the states we usually deal with are contaminated with noise and thus are mixed ones.
Instead, one may look into the statistics of photon-number distributions to obtain a practically useful criterion on quantum non-Gaussianity. In particular, Filip and Mišta, established a Gaussian bound $P_G(n)$ on the photon-number probability for each $n = 1, 2, \ldots$. Provided a given state shows a probability $P(n)$ larger than the Gaussian bound, it demonstrates that the state cannot be expressed as a mixture of Gaussian states.

This approach was further developed to identify the non-Gaussianity of a noisy single-photon state including multi-mode contributions to the signal as well, which requires detection of two output-photons without resolving photon numbers after a beam-splitting operation. This technique was experimentally realized to demonstrate quantum non-Gaussianity of an imperfect single-photon state possessing even a positive Wigner function [12] and to study the resilience of quantum non-Gaussianity under dissipation [13]. Moreover, [14] introduced an experimental criterion to identify quantum non-Gaussianity of multi-photon states of light.

Later, Genoni and co-workers [15] suggested another criterion based on the value of the Wigner function at the origin of phase space addressing the number-parity of the state, which was further extended to the generalized quasi-distributions including the Husimi–Q function [16].

More recently, Park et al established a Bell-type test in phase space to address nonclassicality and quantum non-Gaussianity for single-mode fields [17], while another efficient formalism was also developed to adopt the marginal distributions of the Wigner function [18].

In contrast to the previous methods, which mostly rely on the particle nature, i.e. photon-number statistics, we here present a formalism to adopt continuous observables such as the quadrature amplitudes of a light field or position and momentum of a massive particle, to identify quantum non-Gaussianity. Our method provides not only a fundamentally different approach but also a practically efficient tool to test quantum non-Gaussianity.

In particular, the homodyne detection to measure quadrature amplitudes in quantum optics, while requiring some conditions such as mode-matching with the local oscillator field and the narrow-band filtering of the fields to be measured, is highly efficient achieving efficiency around or above 90%. It can thus provide a practical advantage over photon-counting measurement with lower efficiency [15, 16], e.g. the detection of negativity in phase space requiring the measurement of photon-number parity can be challenging [15].

As an illustration, we show that our condition is useful to detect quantum non-Gaussianity for the class of states

$$|\psi_{\alpha n}\rangle \equiv \sum_{n=0}^{\infty} C_n |4n\rangle,$$

representing a superposition of a vacuum and an integer multiple of four photons. This set of states includes a generalized Schrödinger-cat state consisting of four coherent states

$$|\psi_{\alpha n}\rangle = \mathcal{N}_{\alpha}(|\alpha e^{i\pi}\rangle + |\alpha e^{i\pi}\rangle + |\alpha e^{i\pi}\rangle + |\alpha e^{i\pi}\rangle).$$

A recent proposal suggested to use the generalized cat-state as a component of logical qubits dynamically protected under photon loss for universal quantum computation, together with its experimental scheme [19]. Furthermore, such a cat-state mixed with a vacuum using a beam splitter produces a two-mode state that yields a useful resource for quantum metrology, e.g. enhanced sensitivity in phase estimation [20].

1.1. In a nutshell

We first briefly describe our condition on quantum non-Gaussianity. In particular, we consider the mean value

$$f_{\hat{\rho}}(c) \equiv \langle \hat{O}(c) \rangle \equiv \text{Tr}[\hat{O}(c) \hat{\rho}]$$

of the operator

$$\hat{O}(c) \equiv \exp(-c\hat{x}^2) + \exp(-c\hat{p}^2)$$

as a function of the non-negative parameter $c$. Here $\hat{x}$ and $\hat{p}$ denote the dimensionless operators of coordinate and momentum, respectively.

Our condition relies on the function

$$E_{\hat{\rho}}(c) \equiv \max_{\hat{\rho}} f_{\hat{\rho}}(c)$$

of the maximum value of $\langle \hat{O}(c) \rangle$ defined by equation (3) with respect to all quantum states from a set $\rho$ represented by the density operator $\hat{\rho}$.

We first maximize the mean value, equation (3), over all Gaussian states, that is we establish the value of Gaussian bound $F_G = F_G(c)$ with respect to the parameter $c$, depicted in figure 1 by the black solid line. Then we consider all normalized quantum states in infinite-dimensional Hilbert space and obtain the corresponding
maximum mean value $F_{H} = F_{H}(c)$ presented in figure 1 by a red dotted line. Having found both functions, we can formulate a sufficient condition to detect quantum non-Gaussianity as follows.

**Theorem.** Let $\hat{\rho}$ be the density operator of a quantum state. If there exists a constant $c > 0$, for which

$$F_{G}(c) < \text{Tr} [\hat{\mathcal{O}}(c) \hat{\rho}] \leq F_{H}(c),$$

then $\hat{\rho}$ describes a quantum non-Gaussian state that cannot be represented by a mixture of Gaussian states.

Here we emphasize three points: (i) the word ‘sufficient’ indicates that every state corresponding to a point caught between the two lines $F_{G} = F_{G}(c)$ and $F_{H} = F_{H}(c)$ is indeed a quantum non-Gaussian state. (ii) Nevertheless, points below the curve $F_{G} = F_{G}(c)$ can also represent non-Gaussian states as discussed in detail in section 5. (iii) Although we obtain the two lines $F_{G} = F_{G}(c)$ and $F_{H} = F_{H}(c)$ for pure states our theorem also holds true for mixed states.

### 1.2. Measurement scheme

Next we briefly discuss questions related to an experimental test of our condition. We start by outlining a method to measure the expectation value $\langle \hat{\mathcal{O}} \rangle$. Then we address the choice of $c$ which is intimately connected to the fact that our condition is sufficient.

Since $\hat{\mathcal{O}}$ is the sum of two operators, $\exp(-c\hat{x}^2)$ and $\exp(-c\hat{p}^2)$, $\langle \hat{\mathcal{O}} \rangle$ is the sum of the two corresponding expectation values. Thus, we do not have to measure both operators simultaneously, but it suffices to obtain $\langle \exp(-c\hat{x}^2) \rangle$ and $\langle \exp(-c\hat{p}^2) \rangle$ separately and finally add them.

In *quantum optics*, homodyne detection [21] is a well-established and highly efficient technique to measure the quadrature amplitudes. They readily facilitate the evaluation of the expectation values of the two orthogonal quadratures $\hat{x}$ and $\hat{p}$. Suppose we obtain a data set of values $\{x_1, x_2, \cdots, x_N\}$ and $\{p_1, p_2, \cdots, p_N\}$ from each measurement. Then, we can establish the expectation values as

$$\langle \exp(-c\hat{x}^2) \rangle \sim \frac{1}{N} \sum_{j=0}^{N} e^{-c\xi_j^2},$$

and

$$\langle \exp(-c\hat{p}^2) \rangle \sim \frac{1}{N} \sum_{j=0}^{N} e^{-c\eta_j^2}.$$  

In *matter-wave optics*, the operator $\exp(-c\hat{x}^2)$ can be realized by applying a purely imaginary optical potential [22–25]. The measurement of $\exp(-c\hat{p}^2)$ follows from that of $\exp(-c\hat{x}^2)$ by a $\pi/2$-rotation of the quantum state.

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7 Unfortunately, we can derive an exact analytical expression only for the class of Gaussian states. In the case of an arbitrary pure state of Hilbert space we have to resort to a numerical optimization.
\begin{equation}
\exp(-\hat{c}^\dagger \hat{c}) = \exp(i\hat{n}\pi/2)\exp(-\hat{c}^2)\exp(-i\hat{n}\pi/2).
\end{equation}

Here \(\hat{n}\) denotes the number operator, \(\hat{n} \equiv (\hat{c}^\dagger \hat{c} + \hat{c}^2 - 1)/2\).

Figure 1 also indicates the importance of the choice of \(c\). Indeed, a value \(\langle \hat{O} \rangle = 1.1\) confirms a non-Gaussian state in the range of \(2 \leq c \leq 9\). On the other hand, it is inconclusive if \(0 < c \leq 2\). In this domain the underlying state can still be non-Gaussian but it is not recognized by \(\langle \hat{O} \rangle\). Likewise, a value of \(\langle \hat{O} \rangle < 1\) does not necessarily imply a Gaussian state. Only when the value \(\langle \hat{O} \rangle\) constructed from experimental data for a fixed \(c\) is in the shaded red domain of figure 1, our condition given by equation (6) can identify a quantum non-Gaussian state. In this sense it is a sufficient condition.

### 1.3. Outline

Our article is organized as follows. In section 2 we introduce a geometrical interpretation of the mean value, equation (3), of the operator \(\hat{O}\) defined by equation (4) as an overlap integral in phase space. We then find in section 3 exact analytical expressions for the optimal Gaussian state and the corresponding function \(F_{G} = F_{G}(c)\). In section 4 we cast the optimization problem into an eigenvalue problem and numerically determine the function \(F_{II} = F_{II}(c)\). In contrast to the previous sections this approach relies on position representation rather than phase space functions. We also analytically consider the superposition of two squeezed Gaussian states and show that its mean value \(F_{GD} = F_{GD}(c)\) is very close to that of the exact optimal state. In section 5, we apply our condition to the class of states \(|\psi_{bn}\rangle = \sum_{n=0}^{\infty} c_{n} |4n\rangle\) under dissipation and demonstrate that our condition is powerful in detecting those states even if the attenuation is larger than 50%. Finally, in section 6 we conclude and provide an outlook.

In order to keep our article self-contained but focused on the central ideas, we present lengthy calculations in appendices. We find in appendix A the mean value given by equation (3) for a general Gaussian state and obtain an analytical formula for the function \(F_{G} = F_{G}(c)\). Appendix B presents approximate but analytical results for the function \(F_{II} = F_{II}(c)\) and the optimal quantum state. We conclude in appendix D by describing the model of the dissipation of a quantum state via interaction with a reservoir at zero temperature. In appendix C we derive the asymptotic behavior of the maximum mean value \(F_{GD} = F_{GD}(c)\) for the coherent superposition of two squeezed Gaussian states.

### 2. Optimization of state means maximization of phase space overlap

In the preceding section we have defined the optimization problem for the operator \(\hat{O}\). We now use the Wigner function to cast the relevant expectation value into an overlap integral which provides us with a guide to the optimization.

For this purpose we express the mean value, equation (3), as the overlap integral [10]

\begin{equation}
\langle \hat{O}(c) \rangle = \int dx \int dp \; \mathcal{O}_{W}(x, p) W_{\rho}(x, p)
\end{equation}

between the Weyl–Wigner symbol

\begin{equation}
\mathcal{O}_{W}(x, p) \equiv \exp(-\alpha x^{2}) + \exp(-\alpha p^{2}),
\end{equation}

corresponding to the operator \(\hat{O}(c)\), equation (4), and the Wigner function \(W_{\rho}(x, p)\) of the density operator \(\hat{\rho}\). This representation has the advantage to provide us with a geometrical point of view on the maximization procedure. Indeed, maximizing the mean value \(\langle \hat{O}(c) \rangle\) is equivalent to maximizing the overlap between \(\mathcal{O}_{W}(x, p)\) and \(W_{\rho}(x, p)\), which occurs when the Wigner function of a given state matches \(\mathcal{O}_{W}(x, p)\) in an optimal way.

To underline this idea, we show in figure 2 contour-plots of the phase space representation \(\mathcal{O}_{W} = \mathcal{O}_{W}(x, p)\), equation (11), and the Wigner function \(W_{\rho} = W_{\rho}(x, p)\) of an arbitrary Gaussian state. Hence, in this case the optimization problem reduces to finding the appropriate parameters of \(W_{\rho}\) to maximize the overlap.

### 3. Gaussian states

In this section we pursue this phase space approach to find the lower border \(F_{G} = F_{G}(c)\) of the domain of non-Gaussianity shown in figure 1. From figure 2 we can immediately conclude that the Wigner function must (i) be centered at the origin of the phase space, and (ii) be aligned along the \(x\)- or \(p\)-axis as shown in more detail in appendix A.

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8 This approach is similar to [26] where an optimally focusing wave packet moving in the absence of a potential was constructed.
Motivated by these prerequisites, we start with the wave function
\[ \psi(x) = \left( \frac{s}{\pi} \right)^{1/4} \exp \left( -\frac{x^2}{2} \right) \] (12)
of a squeezed state with its Wigner function
\[ W_s(x, p) = \frac{1}{\pi} \exp \left( -sx^2 - \frac{1}{s}p^2 \right). \] (13)
We are thus left with only a single parameter to vary, that is the squeezing parameter \( s = s(c) \geq 0 \), which we determine by maximizing the overlap integral, equation (10), given by
\[ f(c, s) = \frac{1}{\pi} \int dx \int dp \left[ \exp(-cx^2) + \exp(-cp^2) \right] \exp \left( -sx^2 - \frac{1}{s}p^2 \right). \] (14)
After a direct integration, we obtain
\[ f(c, s) = \frac{1}{\sqrt{1 + cs}} + \frac{1}{\sqrt{1 + cs}}. \] (15)

The symmetry of the Weyl–Wigner symbol \( \mathcal{O}_W(x, p) \), equation (11), with respect to the exchange of the \( x \)- and \( p \)-coordinates reflects itself in equation (15) in the symmetry relation
\[ f(c, s^{-1}) = f(c, s). \] (16)
Therefore, the two optimal values \( s(c) \) and \( s^{-1}(c) \) give rise to the same value of \( F_G(c) \).
In appendix A we show that the optimal values of \( s(c) \) read
\[ s_{\pm}(c) = \begin{cases} 
\frac{1}{\pm \sqrt{c^2 - 4}} & \text{for } 0 \leq c \leq 2 \\
\frac{1}{\pm \sqrt{c^2 - 4}} & \text{for } c > 2 
\end{cases} \] (17)
and note the identity
\[ s_{-}(c) = \frac{1}{s_{+}(c)}. \] (18)
for any non-negative \( c \), in complete agreement with the symmetry relation, equation (16).
In the domain \( c \leq 2 \), the two solutions \( s_{+} \) and \( s_{-} \) coincide, that is \( s_{+} = 1 \), and describe the rotationally symmetric vacuum state. Indeed, for small values of \( c \), the Weyl–Wigner symbol \( \mathcal{O}_W = \mathcal{O}_W(x, p) \) of the operator \( \hat{O} \), presented in figure 3(a), is very broad and there is no preferred orientation of the Wigner function, resulting in the fact that the vacuum state, presented in figure 3(b), is maximizing the overlap integral, equation (10).
At \( c = 2 \) a bifurcation occurs and for \( c > 2 \) two solutions emerge, corresponding to squeezed states in \( x \)-direction \( (s_{+}) \) and \( p \)-direction \( (s_{-}) \). With increasing value of \( c \), the Weyl–Wigner symbol \( \mathcal{O}_W = \mathcal{O}_W(x, p) \) of

Figure 2. Formulation of the optimization problem in Wigner phase space spanned by the coordinate \( x \) and the momentum \( p \). In the contour-plots of the Weyl–Wigner symbol \( \mathcal{O}_W = \mathcal{O}_W(x, p) \) given by equation (11), left, and the Wigner function \( W_G = W_G(x, p) \) of an arbitrary Gaussian state (right), darker colors represent higher values. We maximize the overlap integral, equation (10), by varying the parameters of \( W_G \), such as its displacement, purity, orientation, and squeezing. Obviously the optimal overlap arises when \( W_G \) is located at the origin and its axes are aligned with the axes of phase space. Hence, the optimization problem reduces to finding the optimal value of the width of the Gaussian.
the operator $\hat{\mathcal{O}}$, depicted in the left column of figure 3, becomes more localized along the $x$- and $p$-axes of the phase space. In order to maximize the overlap integral, equation (10), the Wigner function $W^{(G)}_{\hat{\mathcal{O}}} = W^{(G)}_{\hat{\mathcal{O}}}(x, p)$, equation (13), of the optimal Gaussian state (middle column), and the Wigner function $W^{(H)}_{\hat{\mathcal{O}}} = W^{(H)}_{\hat{\mathcal{O}}}(x, p)$ of the optimal non-Gaussian state (right column), for the two values $c = 0.5$ (first row) and $c = 4$ (second row). For small values of $c$ of the optimal state is mainly determined by the dominant maximum of $\hat{\mathcal{O}}$ at the origin. For larger values of $c$ such as $c = 4$ the Gaussian Wigner function can only accommodate one of the ridges of $\hat{\mathcal{O}}$ along the axes of phase space. In contrast, the optimal state represented by $W^{(H)}_{\hat{\mathcal{O}}}$ fits both.

When we substitute the optimal squeezing parameter $s_\pm (c)$ given by equation (17) into the expression equation (15) for the overlap, the mean value $\langle \hat{\mathcal{O}}(c) \rangle$ maximized over all Gaussian states reads

$$F_G(c) = \begin{cases} 2(1 + c)^{-1/2} & \text{for } 0 \leq c \leq 2 \\ \left[ 1 + \frac{c}{\sqrt{c^2 - 4}} \right]^{-1/2} + \left[ 1 + \frac{c}{\sqrt{c^2 - 4}} \right]^{-1/2} & \text{for } c > 2. \end{cases} \quad (19)$$

This result defines the lower border in the condition equation (6) of quantum non-Gaussianity and is presented in figure 1 by the solid line. For any Gaussian state, the mean value $\langle \hat{\mathcal{O}}(c) \rangle$ has to be on, or below this line.

From equation (19) we note the asymptotic behavior

$$F_G(c) \approx \begin{cases} 2 - c & \text{for } c \ll 1 \\ 1 + \frac{1}{2c} & \text{for } c \gg 1 \end{cases} \quad (20)$$

in the two limits $c \ll 1$ and $c \gg 1$. Hence, the curve $F_G = F_G(c)$ starts from $F_G = 2$ and first decays linearly with $c$. However, for large values of $c$ it approaches unity from above decaying with $1/(2c^2)$.

We conclude by emphasizing that while we have obtained the Gaussian bound, equation (19), for single pure Gaussian states, it is actually the maximum for all incoherent mixtures

$$\hat{\rho}_{\mathcal{G}} = \sum_j w_j \hat{\rho}_j^{(G)} \quad (21)$$

of Gaussian states $\hat{\rho}_j^{(G)}$, with probabilities $0 \leq w_j \leq 1$.

In this case the corresponding mean value

$$\langle \hat{\mathcal{O}} \rangle_{\mathcal{G}} = \sum_j w_j \langle \hat{\mathcal{O}} \rangle_j^{(G)} \quad (22)$$

assumes its maximum value, provided $\langle \hat{\mathcal{O}} \rangle_j^{(G)}$ assume their maximum values.
As we have shown above and in appendix A, there exist only the two pure optimal states, \( |\psi_+\rangle \langle \psi_+| \) and \( |\psi_-\rangle \langle \psi_-| \), which can fulfill this requirement. Hence, the maximum value of \( \langle \hat{O} \rangle \), i.e. \( F_G(c) \), is obtained also for an incoherent mixture

\[
\hat{\rho}_{IG} = w_+ |\psi_+\rangle \langle \psi_+| + w_- |\psi_-\rangle \langle \psi_-|
\]

with \( w_+ + w_- = 1 \).

4. Non-Gaussian states

In section 3 we have found the lower border \( F_G = F_G(c) \) of the condition, equation (6), corresponding to the optimal Gaussian state. In this section we calculate the upper border \( F_H = F_H(c) \) corresponding to the optimal quantum state.

Central to our approach is the eigenvalue equation

\[
\hat{O}|\psi_n\rangle = \lambda_n |\psi_n\rangle
\]

of the operator \( \hat{O} \), given by equation (4). Since \( \hat{O} \) is hermitian the eigenstates \( |\psi_n\rangle \) corresponding to the eigenvalues \( \lambda_n \) are complete and orthonormal, that is

\[
\sum_n |\psi_n\rangle \langle \psi_n| = \mathbf{1}
\]

with

\[
\langle \psi_m|\psi_n\rangle = \delta_{mn}.
\]

We start by showing that the problem of finding \( F_H \) is equivalent to finding the largest eigenvalue of \( \hat{O} \). Then, we discuss (i) an exact but numerical solution of the variational problem representing \( \hat{O} \) in the basis of energy eigenstates of a harmonic oscillator, and (ii) an approximate but analytical approach considering a superposition of two squeezed states. Moreover, an analysis of the asymptotic behavior of the numerical results based on the eigenvalue equation, equation (24), in position representation is presented in appendix B. We demonstrate that our approximate result is in excellent agreement with the exact one.

4.1. Largest eigenvalue determines optimal value

We employ the eigenstates \( |\psi_n\rangle \) of \( \hat{O} \) together with equations (24)–(26) to express the expectation value \( \langle \hat{O} \rangle \) in the form

\[
f_\beta = \text{Tr} [\hat{O}\hat{\rho}] = \sum_{m,n} \langle \psi_m|\hat{O}|\psi_n\rangle \langle \psi_n|\hat{\rho}|\psi_m\rangle = \sum_n \lambda_n p_n,
\]

with the probabilities \( p_n \equiv \langle \psi_n|\hat{\rho}|\psi_n\rangle \).

To obtain the maximum of \( f_\beta \), equation (27), we are thus left with varying only the probabilities \( p_n \) subject to the conditions \( 0 \leq p_n \leq 1 \) and \( \sum p_n = 1 \). Since \( \hat{O} \) is positive and bounded, the eigenvalues \( \lambda_n \) are bounded as well, that is \( 0 \leq \lambda_n \leq 2 \). As a result, equation (27) assumes its maximum if \( p_n = \delta_{n,n_0} \), where \( n_0 \) denotes the index of the largest eigenvalue \( \lambda_{n_0} \) of the operator \( \hat{O} \).

Therefore, we can write

\[
F_H(c) = \max_{p_n} \sum_n \lambda_n p_n = \lambda_{n_0} \cdot 1,
\]

and the problem of finding the maximum mean value in equation (3) is reduced to finding the largest eigenvalue \( \lambda_{n_0} \) of the operator \( \hat{O} \). Indeed, the eigenvalues of an hermitian operator describe all possible measurement outcomes. The largest outcome is thus equal to the largest eigenvalue of the operator, which is assumed with certainty if the considered state is the eigenstate corresponding to this eigenvalue.

In the derivation of equation (28) we have assumed that \( \lambda_{n_0} \) is non-degenerate. When \( \lambda_{n_0} \) is degenerate the maximum mean value is not only achieved for a single eigenstate of the operator \( \hat{O} \), but for any arbitrary mixture of all eigenstates corresponding to the largest eigenvalue \( \lambda_{n_0} \). However, for finding \( \lambda_{n_0} \) which is equivalent to the upper border \( F_H = F_H(c) \) of the condition, equation (6), it is sufficient to only obtain one of the eigenstates.

4.2. Largest eigenvalue: numerical approach via Fock states

In order to find numerically the largest eigenvalue \( \lambda_{n_0} \) of the operator \( \hat{O} \) given by equation (4) we choose the representation of \( \hat{O} \) in terms of the eigenfunctions
\[ u_n(x) \equiv \frac{1}{\sqrt{2^{n} n! \sqrt{\pi}}} H_n(x) e^{-\frac{x^2}{2}} \]  

(29)

denotes the harmonic oscillator, with \( H_n(x) \) being the \( n \)th Hermite polynomial.

This choice allows for an analytical calculation of the matrix elements

\[ \mathcal{O}_{mn} = \int_{-\infty}^{\infty} dx \ u_m^*(x) e^{-x^2} u_n(x) + \int_{-\infty}^{\infty} dp \ \hat{u}_m^*(p) e^{-\frac{p^2}{2}} \hat{u}_n(p) \]  

(30)

of \( \hat{O} \), equation (4), involving the functions \( u_m \) and their Fourier transform

\[ \hat{u}_m(p) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \ e^{-ipx} u_n(x). \]  

(31)

Due to the relation

\[ \hat{u}_n(p) = (-i)^n u_n(x = p) \]  

(32)

this choice of basis states makes the two terms in equation (30) identical up to a prefactor \( i^{n-m} \). Indeed, when we substitute the expression, equation (29), for \( u_m \) into the matrix element \( \mathcal{O}_{mn} \), we find [27]

\[ \mathcal{O}_{mn}(c) = (1 + i^{n-m}) \left( \frac{2^{m+n}}{\sqrt{\pi n! m!}} \right) \Gamma \left( \frac{n+m+1}{2} \right) \frac{1}{\sqrt{1+c}} \] 

\[ \times \left( -c \frac{1}{1+c} \right)^{(m+n)/2} \left( \frac{1}{2} F_1 \left( -m, -n, \frac{1 - m - n}{2}, \frac{1 + c}{2} \right) \right) \]  

(33)

for \( m + n = 2, 4, 6, \ldots \) Here \( \Gamma \) and \( \left( \frac{1}{2} F_1 \right) \) denote the Gamma function and the hypergeometric function [28], respectively. In the case \( m + n = 1, 3, 5, \ldots \) the matrix element vanishes. Additionally, the elements satisfying \( n - m = 2, 6, 10, \ldots \) vanish due to the first term in equation (33).

Taking into account the first 30 000 functions \( u_m \), we have numerically obtained the maximum mean value \( F_{G1} \) as the largest eigenvalue of the matrix \( \mathcal{O}_{mn} \), equation (33). The function \( F_{G1} = F_{G1}(c) \) indicated by dots in figure 1 determines the upper border, which cannot be exceeded by any normalized quantum state, be it a mixed or pure one.

4.3 Near-optimal non-Gaussian states: superposition of two squeezed states

Inspired by the analysis of section 3 we now find analytically a near-optimal non-Gaussian state to achieve \( F_{G1}(c) \).

In the case of Gaussian states we have shown in section 3 that there exist two solutions \( s(c) \) and \( s^{-1}(c) \) for the squeezing parameter \( s \) leading to the same maximal mean value \( F_{G1}(c) \), given by equation (19). With this idea in mind we may optimize the overlap integral, equation (10), by considering the normalized non-Gaussian wave function

\[ \psi_{sG}(x) \equiv N(s) [\psi_s (x) + \psi_{1/s} (x)] \]  

(34)

consisting of a coherent superposition of two Gaussians \( \psi_s \) and \( \psi_{1/s} \) given by equation (12), where

\[ N^2 \equiv \frac{1}{2} + \frac{1}{\sqrt{2 \pi}} \]  

(35)

is the normalization constant.

For \( \psi_{sG} \) defined by equation (34), the Wigner function reads

\[ W_{sG}(x, p) = N^2(s) \left\{ W^{(1)}_{G}(x, p) + W^{(1/s)}_{G}(x, p) \right\} \] 

\[ + \frac{2}{\sqrt{2 \pi}} \left\{ \frac{2s}{s^2 + 1} \cos \left( \frac{2s^2 - 1}{s^2 + 1} \right) \exp \left[ -\frac{2s}{s^2 + 1} (x^2 + p^2) \right] \right\}, \]  

(36)

where \( W^{(s)}_{G}(x, p) \) is the Wigner function of the single squeezed Gaussian state given by equation (13).

When we substitute the expression for \( W_{sG} \), equation (36), together with the Weyl–Wigner symbol given by equation (11) into the overlap integral, equation (10), and perform the integration, we obtain

\[ f_{sG}(c, s) = \frac{1}{1 + \sqrt{2s/(s^2 + 1)}} \left( \frac{1}{\sqrt{1 + cs}} + \frac{1}{\sqrt{1 + c/s}} + \frac{2}{\sqrt{c + (s^2 + 1)/(2s)}} \right) \]  

(37)
The optimal value $s_0(c)$ of the parameter $s$ follows from the equation

$$\frac{\partial}{\partial s} f_{sG}(c, s) \bigg|_{s=s_0(c)} = 0. \tag{38}$$

The solution $s_0(c)$ to equation (38) is found only numerically and, when substituted into the expression (37) for $f_{sG}$, yields the function $F_{sG} = F_{sG}(c)$ displayed in figure 4 by the blue solid line.

It is worth mentioning that the curve $F_{sG} = F_{sG}(c)$ turns out to be very close to the curve $F_{H} = F_{H}(c)$ represented by red dots. Indeed, the relative deviation $\delta F$ of $F_{sG}(c)$ from $F_{H}(c)$ is less than 0.5% as depicted by the dashed line and the right-hand axis of figure 4.

This result reflects the fact that the non-Gaussian state $\psi_{sG}$ given by equation (34), is very close to the optimal one. Indeed, the corresponding Wigner function $W_{sG}$ in equation (36), presented in figure 5 together with the Wigner function $W_{H}$ of the numerically obtained optimal quantum state for $c = 0.5$ (the upper row) and $c = 4$ (the lower row) are almost indistinguishable.

In order to gain deeper insight into the curve $F_{sG} = F_{sG}(c)$ we obtain in appendix C the asymptotic expression

$$s_0(c) \approx \begin{cases} 
1 + \sqrt{c} & \text{for } c \ll 1 \\
\frac{1}{2} c^2 \left(1 - \frac{2}{\sqrt{c}} \right) & \text{for } c \gg 1 
\end{cases} \tag{39}$$

for $s_0$ with the corresponding maximal mean value

$$F_{sG}(c) \equiv f_{sG}[c, s_0(c)] \approx \begin{cases} 
2 - c & \text{for } c \ll 1 \\
1 + \frac{1}{c} & \text{for } c \gg 1.
\end{cases} \tag{40}$$

We note that for $c \ll 1$ the optimal value $F_{sG}$ displays the same behavior as the optimal Gaussian in equation (20). However, for $c \gg 1$ it decays as $1/c$ which is slower than the decay $1/2c^2$ given by equation (20) corresponding to the single Gaussian.

5. Applicability of our condition to practical states

In the previous sections we have found the two borders, $F_{sG}$ and $F_{H}$, to formulate our condition, equation (6). Here we highlight the features of our condition and illustrate its power as well as its limitations by applying our condition to several non-Gaussian states. In particular, we consider (i) the superposition of Gaussian states $\psi_{sG}$ introduced in equation (34) and (ii) the superposition $|\psi_{a}\rangle$ of four coherent states, defined by equation (3) as our examples under a dissipative channel.

5.1. Superposition of Gaussian states

We emphasize that our condition is a sufficient one. Indeed, the state $\psi_{sG}$, equation (34), given by a superposition of two squeezed states with squeezing parameters $s = 3$ and $s = 1/3$, is non-Gaussian. The corresponding mean value $\langle \hat{O} \rangle$ is represented by a blue line in figure 6. For values of $c \gtrsim 3.5$ the blue curve drops
below the Gaussian one and our condition fails to observe the non-Gaussianity of this superposition. Therefore, we are able to detect quantum non-Gaussianity for this state only by using the parameter $c$ in the range $0 < c \leq 3.5$.

Figure 5. Comparison between the Wigner functions $W_H$ and $W_{sG}$ of the optimal quantum state (left column) and the superposition of two squeezed Gaussians (right column) for the two values $c = 0.5$ (first row) and $c = 4$ (second row). For both choices of $c$ the two Wigner functions are almost indiscernible. Nevertheless, there are small deviations as expressed by figures $5a$ and $5b$ reflecting the fact that $\psi_{sG}$ is not an eigenfunction of $\hat{O}$. Moreover, we note that already for $c = 0.5$ the star symmetry is lurking in the background although the Wigner function is still dominated by the rotational symmetry of the peak at the origin of phase space. Only for larger values of $c$ such as $c = 4$ is the star symmetry the dominant feature determining the expectation value $\langle \hat{O} \rangle$.

Figure 6. Detection of quantum non-Gaussianity for the superposition of two squeezed states, $\psi_{sG}$, equation (34) with $s = 3$, under a dissipative channel with transmittance $\eta$. Here we depict the two borders of our condition, $F_G$ (black line) and $F_H$ (red dots), together with the mean value $f_{sG}$, equation (37), before dissipation ($\eta = 1$, blue line) and after dissipation ($\eta = 0.5$, orange line). The dissipation leads to a decrease in the mean value and henceforth also decreases our ability to detect quantum non-Gaussianity, however even for $\eta = 0.5$ we note a positive gap between the orange and the black curves around $c \approx 2$. This gap means we are still able to detect the non-Gaussianity of the state $\psi_{sG}$, even though half of the intensity has dissipated. This figure also displays the sufficiency of our condition: indeed, the blue curve which corresponds to the non-dissipated version of this superposition, can still be located in the Gaussian domain. For example, when $c > 3.5$ the mean value $f_{sG}$ falls below the line $F_G$. Nevertheless, the state is non-Gaussian.
One way to experimentally detect the non-Gaussianity of a given state is to obtain its Wigner function via tomography [29]. Specifically, negative values of the corresponding Wigner function immediately indicate that the state is quantum non-Gaussian. In general, however, any measurement is subject to losses\(^9\) which reduce the ability to detect negative values of the Wigner function. Indeed, if the attenuation of a state is higher than 50%, i.e. a dissipation parameter \(\eta < 0.5\), it is experimentally impossible to detect any negativity via tomography or number-parity measurement. In contrast, our condition does not require \(\eta > 0.5\) to detect the non-Gaussianity of a state. This is illustrated in figure 6, where the blue and orange lines display the mean values corresponding to the superposition \(\psi_{3\alpha}\) with \(s = 3\), without \(\eta = 1\) and with \(\eta = 0.5\) dissipation, respectively. Considering losses of \(1 - \eta = 0.5\), we are still able to detect the non-Gaussian nature of the state, as the orange line exceeds the black curve \(F_G = F_\mathcal{C}(c)\) in the region around \(c = 2\). The minimal value of \(\eta\) (corresponding to maximal losses \(1 - \eta\)) for which this happens is found to be around \(\eta_{\min} \approx 0.401\). This demonstrates that our condition works in a regime where most conventional methods fail. It is worth noting however that \(\eta_{\min}\) strongly depends on the state under consideration and it might also lie above 0.5.

5.2. Superposition of coherent states

Next we test our condition to detect a superposition

\[
|\psi_{\alpha}\rangle = \mathcal{N}_\alpha (|\alpha\rangle + |\alpha e^{i\pi}\rangle + |\alpha e^{i\pi/2}\rangle)
\]

(41)
of four coherent states with \(\mathcal{N}_\alpha\) being a normalization constant.

In figure 7(a) we show the contour plot of the corresponding Wigner function \(W(x, p)\) for the case of \(\alpha = 1\).

This generalized Schrödinger-cat state was recently suggested as a logical qubit dynamically protected under photon loss thus suitable for fault-tolerant quantum computation [19]. In addition, it was found to be a useful resource state for enhanced quantum phase estimation [20].

For each state \(|\psi_{\alpha}\rangle\) undergoing dissipation, we maximize the ratio of \(F_\alpha(c) \equiv \langle \psi_{\alpha} | \hat{\mathcal{O}}(c) | \psi_{\alpha}\rangle\) to the Gaussian bound \(F_\mathcal{C}(c)\) over the parameter \(0 < c \leq 10\) and display the maximum value of \(F_\alpha / F_\mathcal{C}\) as a function of \(\alpha\) in figure 7(b). For the ideal case \((\eta = 1)\), the quantum non-Gaussianity is successfully detected for \(0.18 \leq \alpha \leq 1.4\). On the other hand, with a non-zero dissipation \((0 < \eta < 1)\), this detection range shrinks with decreasing \(\eta\). Nevertheless our condition is capable of detecting those states even below \(\eta = 0.5\). In particular, quantum non-Gaussianity is successfully detected around \(\eta < 0.3\) for certain values of \(\alpha\) of the superposition states \(|\psi_{\alpha}\rangle\), as depicted in figure 7(c).

5.3. Discussion

Note that the two discussed examples belong to the class of states \(|\psi_{\ell\hbar}\rangle\), equation (1), consisting of a superposition of the vacuum and an integer multiple of four photons. We have numerically tested other states as well and found that our condition is able to detect quantum non-Gaussianity broadly for this class.

 Naturally, the efficiency of our condition is crucially determined by the state of interest. Given another class of states one might think of variations of the operator \(\hat{\mathcal{O}}\). e.g. changing the functional form from Gaussian to non-Gaussian including polynomials, to build up the condition which is then particularly suited for detecting the non-Gaussianity of those states, but less useful for other ones. We may not anticipate that there is a single best operator for all states, however a more extensive analysis goes beyond the scope of the present article.

6. Conclusions and outlook

In this article we have proposed a sufficient condition to distinguish quantum non-Gaussian states from a mixture of Gaussian ones. Our condition relies on the expectation value of an operator \(\hat{\mathcal{O}}\) consisting of the sum of two exponentials containing the squares of the appropriately scaled coordinate and momentum operators which are both multiplied by a parameter \(c\).

In classical physics it is possible to maximize the expectation value of \(\hat{x}^2\) and \(\hat{p}^2\) simultaneously by a phase space distribution function representing the underlying state which consists of the product of two delta functions centered around the origin. This choice leads to the value \(\langle \hat{\mathcal{O}} \rangle_{\delta} = 2\). However, in quantum mechanics a state has to take up a non-zero area in phase space. This fundamental restriction leads to a suppression of \(\langle \hat{\mathcal{O}} \rangle\) below the classical value \(\langle \hat{\mathcal{O}} \rangle_{\delta} = 2\). Indeed, for small values of \(\epsilon\) the operator \(\hat{\mathcal{O}}\) reduces to the difference of the classical result and the product of \(\epsilon\) with the familiar Hamiltonian of a harmonic oscillator. Due to this difference, the optimal state maximizing \(\langle \hat{\mathcal{O}} \rangle\) is the lowest state of the oscillator, that is the vacuum state.

However, for larger values of \(\epsilon\) the Weyl–Wigner symbol is concentrated mainly along the x- and p-axis in phase space. As a result, the optimal state when expressed by the corresponding Wigner function also has to

\(^9\)The losses can be modeled by coupling the system to a thermal bath. We refer to appendix D for a more detailed analysis.
reflect this symmetry. Indeed, we have found a star symmetry of these optimal states combined with a narrow peak at the origin. The latter contribution is the consequence of the sum of the operators $\exp(-c\hat{x}^2)$ and $\exp(-c\hat{p}^2)$ leading to a maximum in the Weyl–Wigner symbol of $\hat{\varphi}$ at the origin. In this sense it is a remnant of the optimal state, i.e. the vacuum state, for small values of $c$.

It is also interesting to note that there is a transition of the Weyl–Wigner symbol of $\hat{\varphi}$ from a rotational symmetry, at least in the neighborhood of the origin, for small values of $c$, to a star structure along the axes of phase space for larger values of $c$. As a result, the decay of the expectation value of $\hat{\varphi}$ changes from one linear in $c$ to one linear in $1/c$. Moreover, for Gaussian states a bifurcation occurs since a single Gaussian can accommodate either the $x$- or the $p$-axis of phase space, but not both. It is for this reason that a superposition of two appropriately squeezed Gaussian states can provide an excellent approximation to the optimal state. In particular, it also leads to an interference peak at the origin capturing the overlap of the two Gaussians in the Weyl–Wigner symbol of $\hat{\varphi}$ at the origin. Despite the close similarity between $\psi_{\alpha G}$ and the exact wave function $\psi_{11}$ solving the optimization problem, there are differences and $\psi_{\alpha G}$ is not a rigorous eigenfunction of $\hat{\varphi}$.
We have applied our condition to the class of states $|\psi_{\text{in}}\rangle$ and demonstrated its power by detecting those states even under strong dissipation of the signal below $\eta = 0.5$. Our condition can be efficiently tested, e.g. using highly efficient homodyne detection in quantum optics, which readily achieves a detection efficiency above 90%. Our current approach can be further extended to a different functional form of the test operator $\hat{O}$ to be adapted to other class of non-Gaussian states, which will be investigated in future.

We conclude by noting that the eigenvalue problem of $\hat{O}$ is fundamentally different from the familiar Schrödinger equation since for large values of $c$ the operator $\exp \left( -c\hat{p}^2 \right)$ involves infinitely many derivatives larger than two. Indeed, in this case the eigenvalue equation takes the form of an integral equation and we have presented here approximate but analytical expressions for the largest eigenvalue and the corresponding eigenfunction. However, it is interesting to study in this formulation the behavior of all eigenvalues as a function of $c$. Indeed, for large values of $c$ they display a rather curious behavior: the largest eigenvalue separates from all other ones which asymptotically cluster around unity. Unfortunately, this analysis goes beyond the scope of the present article and has to be postponed to a future publication \[30\].

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**Appendix A. Phase space overlap for a Gaussian state**

In this appendix we derive the upper bound for the expectation value $f_G$ of the operator $\hat{O}$ in the presence of the optimal Gaussian state. For this purpose we employ the Wigner approach and pursue in three steps: (i) we first obtain an analytical expression for $f_G$ which depends on the displacement, the purity, the orientation and the amount of squeezing of the Gaussian state. (ii) Then we find the Gaussian state which makes this expression largest by locating, concentrating and orientating the Gaussian appropriately, and finally (iii) we derive the optimal squeezing.

**A.1. Analytical expression for expectation value**

In this appendix we calculate the mean value

$$
  f_G \equiv \int \! \! dx \int \! \! dp \; \mathcal{O}_W(x, p)\; W_G(x, p)
$$

(A.1)

of the operator $\mathcal{O}_c$, equation (4), with the Weyl–Wigner symbol

$$
  \mathcal{O}_W(x, p) \equiv \exp(-c x^2) + \exp(-c p^2)
$$

(A.2)

for a general Gaussian state with the Wigner function \[31, 32\]

$$
  W_G(X) \equiv \frac{1}{\pi \sqrt{\det \sigma}} \exp\left\{ -(X - R)^T \sigma^{-1} (X - R) \right\},
$$

(A.3)

where

$$
  X \equiv \begin{pmatrix} x \\ p \end{pmatrix}
$$

(A.4)

is the vector of the phase space coordinates, that is of the position $x$ and the momentum $p$.

The Wigner function $W_G$, equation (A.3), is completely characterized by the vector

$$
  R \equiv \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}
$$

(A.5)

of the first moments, and the covariance matrix

$$
  \sigma \equiv \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}.
$$

(A.6)
When we substitute equations (A.2) and (A.3) into equation (A.1), we arrive at the expression

\[ f_G = \int dx \int dp \frac{1}{\pi^{3/2} \sqrt{\det \sigma}} \exp \left[ -(X - R)^T \sigma^{-1} (X - R) \right] \left( e^{-\alpha^2} + e^{-\beta^2} \right), \]  

(A.7)

which with the new integration variable \( X' \equiv X - R \) reads

\[ f_G = \frac{1}{\pi^{3/2} \sqrt{\det \sigma}} \int dx' dp' \left[ \exp \left( -X'^T A_x X' + X'^T b_x - c x_0^2 \right) 
+ \exp \left( -X'^T A_p X' + X'^T b_p - c p_0^2 \right) \right], \]  

(A.8)

where

\[ A_x \equiv \sigma^{-1} + \begin{pmatrix} \frac{c}{0} \\ 0 \end{pmatrix}, \quad b_x \equiv -2c \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \]  

(A.9)

and

\[ A_p \equiv \sigma^{-1} + \begin{pmatrix} 0 \\ \frac{c}{0} \end{pmatrix}, \quad b_p \equiv -2c \begin{pmatrix} 0 \\ p_0 \end{pmatrix}. \]  

(A.10)

With the help of the relation

\[ \int d^n x \exp \left( -x^T A x + x^T b \right) = \sqrt{\frac{\pi^n}{\det A}} \exp \left( \frac{1}{4} b^T A^{-1} b \right), \]  

(A.11)

for a \( n \)-dimensional Gaussian integral with a matrix \( A \) and a vector \( b \), we can perform the integration in equation (A.8) and obtain the expression

\[ f_G = \frac{1}{\sqrt{\det \sigma}} \left[ \frac{1}{\sqrt{\det A_x}} \exp \left( c^2 x_0^2 (A_x^{-1})_{11} - c x_0^2 \right) 
+ \frac{1}{\sqrt{\det A_p}} \exp \left( c^2 p_0^2 (A_p^{-1})_{22} - c p_0^2 \right) \right], \]  

(A.12)

or

\[ f_G(c, x_0, p_0, \sigma_{11}, \sigma_{12}, \det \sigma) = \frac{1}{\sqrt{1 + \sigma_{11}}} \exp \left( -c x_0^2 \right) 
+ \frac{1}{\sqrt{1 + \sigma_{12} / \sigma_{11}}} \exp \left( -c p_0^2 \right), \]  

(A.13)

Here we have used the definitions given by equations (A.9) and (A.10), and represented the element

\[ \sigma_{22} = \frac{\det \sigma + \sigma_{12}^2}{\sigma_{11}} \]  

(A.14)

of the covariance matrix \( \sigma \) in terms of its determinant, \( \det \sigma \), and the matrix elements \( \sigma_{11} \) and \( \sigma_{12} \) determining the Gaussian Wigner function, equation (A.3).

A.2. Variation of Gaussian

In the next steps we obtain the maximum value of the function \( f_G \), given by equation (A.13), by varying the five parameters \( x_0, p_0, \sigma_{11}, \sigma_{12}, \) and \( \det \sigma \).

Displacements. The dependence of \( f_G \) on the displacement parameters \( x_0 \) and \( p_0 \) in equation (A.13) is only in the exponential functions, which obviously have their maxima at

\[ x_0 = p_0 = 0. \]  

(A.15)

Hence, the Wigner function \( W_G \), equation (A.3), maximizing the mean value \( f_G \), equation (A.1), is centered at the origin of the phase space. In other words, the Wigner function \( W_G \) has its maximum value at the point \( (x_0, p_0) \) and this point should match the point \( (0, 0) \) of maximum value of the Weyl–Wigner symbol \( \mathcal{O}_W(x, p) \), equation (A.2).
This insight significantly simplifies equation (A.13), leading us to the expression
\[
\mathcal{f}_G(c, \sigma_{11}, \sigma_{12}, \det \sigma) = \frac{1}{\sqrt{1 + c \sigma_{11}}} + \frac{1}{\sqrt{1 + c (\det \sigma + \sigma_{12}^2) / \sigma_{11}}}
\] (A.16)
which still depends on four parameters.

**Purity.** For a Gaussian state \(\hat{\rho}_G\), the value of \(\det \sigma\) determines the purity
\[
\text{Tr}(\hat{\rho}_G^2) \equiv \int \! dx \int \! dp \, W_G^2(x, p) = \frac{1}{\sqrt{\det \sigma}}
\] (A.17)
of the state. Since for a physical state we have the condition, \(\text{Tr}(\hat{\rho}_G^2) \leq 1\), we find the condition
\[
1 \leq \det \sigma
\] (A.18)
and the minimal value of \(\det \sigma\) corresponds to a pure state with \(\text{Tr}(\hat{\rho}_G^2) = 1\).

The parameter \(\sigma_{11}\) is always positive since it corresponds to the variance in \(x\)-direction. Hence, the function \(\mathcal{f}_G\) given by equation (A.16) decreases in a monotonic way for increasing \(\det \sigma\). Finally by replacing \(\det \sigma\) by its minimal value, that is \(\det \sigma = 1\), we arrive at
\[
\mathcal{f}_G(c, \sigma_{11}, \sigma_{12}) = \frac{1}{\sqrt{1 + c \sigma_{11}}} + \frac{1}{\sqrt{1 + c (1 + \sigma_{12}^2) / \sigma_{11}}}.
\] (A.19)

As a result, the optimal Gaussian state is a pure one and occupies a minimal area\(^{10}\) in the phase space.

**Parameter of phase space correlations.** Again, since \(\sigma_{11} > 0\) the function \(\mathcal{f}_G\) given by equation (A.19) is maximal for the smallest value of \(\sigma_{12}^2\). The optimal value
\[
\sigma_{12} = 0
\] (A.20)
corresponds to the Wigner function of the state oriented along the \(x\)- or \(p\)-axis in the phase space. Since the function \(O_W(x, p)\), equation (A.2), is localized mainly along the \(x\)- and \(p\)-axis, as depicted in figure 2, the Wigner function should be focused there as well.

Finally equation (A.19) takes the simple form
\[
\mathcal{f}_G(c, \sigma_{11}) = \frac{1}{\sqrt{1 + cs}} + \frac{1}{\sqrt{1 + c/s}},
\] (A.21)
where for convenience we have introduced the notation \(\sigma_{11} = s\).

**A.3. Variation of the squeezing parameter**

At an extremum the derivative of the function \(\mathcal{f}_G\), equation (A.21), with respect to \(s\) vanishes, that is
\[
\frac{\partial \mathcal{f}_G(c, s)}{\partial s} = -\frac{c}{2} (1 + cs)^{-\frac{3}{2}} + \left(1 + \frac{c}{2}ight)^{-\frac{3}{2}} \left(\frac{c}{2s^2}\right) = 0,
\] (A.22)
giving rise to the algebraic equation
\[
z^4 \left(1 + \frac{c}{2z^2}\right) - (1 + cz^3) = 0
\] (A.23)
for \(z = s^{1/3}\).

The roots of equation (A.23) are given by the expressions
\[
z_{1,2} = \pm 1 \quad \text{and} \quad z_{3,4} = \frac{1}{2} \left(c \pm \sqrt{c^2 - 4}\right),
\] (A.24)
which correspond to the values
\[
 s_{(1)} = 1 \quad \text{and} \quad s_{(3,4)} = \left(c \pm \sqrt{c^2 - 4}\right)^{\frac{3}{2}}.
\] (A.25)

The solution corresponding to the root \(z_2 = -1\) is not taken into account as we consider only positive values of \(s\).

---

\(^{10}\) This result can also be easily obtained from the overlap integral, equation (10). Indeed, according to equations (A.17) and (A.18), the Wigner function of a mixed state is broader in all directions than the Wigner function of a pure state. Therefore, in order to obtain the maximum overlap, the Wigner function of the state should not only be centered at the origin, but it should also be concentrated there.
Moreover, due to the symmetry relations
\[ z_3 = \frac{1}{z_4} \] (A.26)
and
\[ f_G(c, s) = f_G(c, s^{-1}), \] (A.27)
between the roots \( z_3 \) and \( z_4 \) as well as the expectation values \( f_G \) at the values \( s \) and \( 1/s \), it suffices to investigate only the two roots \( s^{(1)} \) and \( s^{(3)} \) given by equation (A.25).

For this purpose, we consider the second derivative
\[ \frac{\partial^2 f_G(c, s)}{\partial s^2} = \frac{c}{4} \left[ \frac{3c}{(1 + cs)^2} - \frac{4s + c}{(s^2 (c + s)^2)} \right] \] (A.28)
of \( f_G \) with respect to \( s \) and study its sign at the extremal points \( s^{(1)} \) and \( s^{(3)} \).

For \( s = s^{(1)} \), the second derivative
\[ \frac{\partial^2 f_G(c, s)}{\partial s^2} \bigg|_{s = s^{(1)}} = \frac{c(c - 2)}{2(1 + c)^2} \] (A.29)
is negative only for \( c < 2 \).

For \( s = s^{(3)} \), the second derivative
\[ \frac{\partial^2 f_G(c, s)}{\partial s^2} \bigg|_{s = s^{(3)}} = -\frac{c(c^2 - 4)}{4(s^{(3)})^2(1 + 2cs^{(3)})^2} \] (A.30)
is negative for \( c > 2 \).

As a result, we find that for the choice
\[ s_{cl}(c) \equiv \begin{cases} 
1 & \text{for } 0 \leq c \leq 2 \\
\frac{1}{s} \left( c \pm \sqrt{c^2 - 4} \right)^3 & \text{for } c > 2
\end{cases} \] (A.31)
of the parameter \( s \) the function \( f_G(c, s) \) assumes its maximum value.

**Appendix B. Largest eigenvalue and corresponding eigenfunction**

In section 4.3 we have presented an educated guess for the optimal non-Gaussian state which turned out to be very close to the one obtained numerically. However, within that analysis we have varied the state depending on the resulting mean value, which does not necessarily require the resulting state to be an eigenstate of \( \hat{\mathcal{O}} \). For this reason we now start from the eigenvalue equation, equation (24), and derive analytic approximations for the maximum eigenvalue and the corresponding eigenfunction of \( \hat{\mathcal{O}} \).

For small values of \( c \) the operator \( \hat{\mathcal{O}} \) given by equation (4) reduces to the form
\[ \hat{\mathcal{O}} \approx 2 - c(\hat{x}^2 + \hat{p}^2), \] (B.1)
and the optimal state maximizing \( \langle \hat{\mathcal{O}} \rangle \) is thus the lowest eigenstate of the operator \( \hat{x}^2 + \hat{p}^2 \), that is the ground state of a harmonic oscillator with eigenvalue \( k = 1 \). As a result, in the regime \( c \ll 1 \) the maximum eigenvalue \( \lambda_{n_0} \) takes the form
\[ F_{H}(c) = \lambda_{n_0} \approx 2 - c. \] (B.2)

We recall that this behavior also emerges for the optimal Gaussian state, equation (20), and the optimal superposition of two Gaussians, equation (40).

In [30] we investigate in great detail the eigenvalue equation, equation (24), of \( \hat{\mathcal{O}} \). In particular, we derive the integral equation
\[ e^{-\alpha x} \psi_\lambda(x) + \frac{1}{\sqrt{4\pi \alpha}} \int_{-\infty}^{\infty} dx' \exp\left[-\frac{(x - x')^2}{4\alpha}\right] \psi_\lambda(x') = \lambda \psi_\lambda(x) \] (B.3)
for the position representation \( \psi_\lambda(x) \equiv \langle x|\psi_\lambda \rangle \) of the eigenstate \( |\psi_\lambda \rangle \).
From this eigenvalue equation we obtain in \( \text{(30)} \) for large values of \( c \) the asymptotic behavior of the maximum eigenvalue \( F_{\tilde{H}} \) together with the corresponding eigenfunction \( \psi_{\tilde{H}} \).

\[
F_{\tilde{H}}(c) \approx 1 + \frac{\pi}{2} \frac{1}{c}, \tag{B.4}
\]

of the maximum eigenvalue \( F_{\tilde{H}}(c) \) together with the corresponding eigenfunction

\[
\psi_{\tilde{H}}(x) \approx \frac{N_{\tilde{H}}}{F_{\tilde{H}}(c)} \exp \left[ -\frac{(x - \mu)^2}{2} \right] \int_{-\infty}^{\infty} dq \frac{\exp(\imath q)}{\exp(\imath q^2) - 1/F_{\tilde{H}}(c)}, \tag{B.5}
\]

where \( N_{\tilde{H}} \) is a normalization constant.

When we compare \( F_{\tilde{H}} \) given by equation (B.4) to the corresponding expression, equation (40), for the non-Gaussian state \( \psi_{\tilde{G}} \) we note that for \( c \to \infty \) both decay with the inverse of \( c \). However, they differ by their decay constants. Whereas for \( F_{\tilde{G}} \) the slope is determined by unity, for \( F_{\tilde{H}} \) it is given by \( \pi/2 \). This difference stands out most clearly in figure B1 where we display the different scalings of the functions \( F_{\tilde{G}} = F_{\tilde{G}}(c) \) and \( F_{\tilde{H}} = F_{\tilde{H}}(c) \) together with the numerical result.

Moreover, this difference corresponding to the relative error \( \delta F \) shown in figure 4 indicates that \( \psi_{\tilde{G}} \) is not an eigenfunction of \( \hat{\cal{O}} \). This feature is apparent from figure B2 where we show the corresponding wave functions in position representation. We note slight differences between \( \psi_{\tilde{G}} \) and the optimal eigenfunction \( \psi_{\tilde{H}} \) of \( \hat{\cal{O}} \).
Appendix C. Asymptotics of the superposition of two squeezed Gaussians

In this appendix we obtain the asymptotic behavior of the maximum expectation value $f_{SG}$ of $\langle \hat{O} \rangle$ for the superposition of two appropriately squeezed Gaussians. For this purpose we have to find the value $s_0 = s_0(c)$ maximizing the mean value

$$f_{SG}(c, s) = \frac{1}{1 + 2s/(s^2 + 1)} \left( \frac{1}{\sqrt{1 + cs}} + \frac{1}{\sqrt{1 + c/s}} + \frac{2}{\sqrt{c + (s^2 + 1)/(2s)}} \right)$$

as a function of $c$ and $s$. We pursue this task for the limits $c \to \infty$ and $c \to 0$.

C.1. Large values of $c$

First we consider the regime of very large values of $c$. In order to find a solution for $s_0(c)$ as $c \to \infty$, we assume $s > c^{5/3}$. Under this assumption we can approximate equation (C.1) by

$$f_{SG}(c, s) \approx \frac{1}{1 + \sqrt{2}/s} \left( \frac{1}{\sqrt{\alpha s}} + 1 - \frac{c}{2s} + \frac{2}{\sqrt{c}} \right),$$

or

$$f_{SG}(c, s) \approx 1 + \frac{\alpha}{\sqrt{s}} - \frac{\beta}{s}$$

with

$$\alpha \equiv \sqrt{2} + \frac{1}{\sqrt{c}} > 0$$

and

$$\beta \equiv \frac{c}{2} + 4 + \frac{7}{\sqrt{c}} > 0.$$  

Maximizing $f_{SG}(c, s)$ in equation (C.3) requires a vanishing first derivative with respect to $s$, leading us to the condition

$$-\frac{1}{2} \frac{\alpha}{s^{3/2}} + \frac{\beta}{s^2} = 0.$$  

Since both $\alpha$ and $\beta$ are positive, this expression can only vanish if both terms cancel each other. To lowest order $\alpha \approx \sqrt{2}$ and $\beta \approx c/2$, and we finally end up with the equation

$$s_0(c) \approx \frac{c^2}{2}.$$  

This result which is consistent with our initial assumption $s > c^{5/3}$ describes to lowest order the behavior of $s_0$ as a function of $c$ for $c \to \infty$.

C.2. Small values of $c$

In the regime of $c$ going to zero, we expect the root $s_0(c)$ of the first derivative of $f_{SG}$ with respect to $s$ to approach unity due to the symmetry of $f_{SG}$ with respect to the exchange of $s$ and $1/s$, leading us to the asymptotic relations $cs \to 0$ and $c/s \to 0$.

In this limit, we expand equation (C.1) up to first order in $c$ and get

$$f_{SG}(c, s) \approx 2 - c \frac{\gamma + \gamma^{-3/2}}{1 + \gamma^{-1/2}}$$

with

$$\gamma \equiv \frac{1}{2} \left( s + \frac{1}{s} \right).$$

Since $\gamma \geq 1$, the function $f_{SG}(c, s)$ achieves its maximum if the second term in equation (C.9) is minimal. We find the optimal value to be $\gamma = 1$, corresponding to

This result which is consistent with our initial assumption $s > c^{5/3}$ describes to lowest order the behavior of $s_0$ as a function of $c$ for $c \to \infty$. 


for \( c \to 0 \).

### C.3. Summary

Together with equation (C.8), we arrive at the expression

\[
s_0(c) \simeq \begin{cases} 
1 + \sqrt{c} & \text{for } c \ll 1 \\
\frac{1}{2} c^2 \left( 1 - \frac{\sqrt{c}}{\sqrt{1-c}} \right) & \text{for } c \gg 1
\end{cases}
\]

for the optimal squeezing parameter.

When we insert equation (C.12) into equation (C.1) we finally obtain the asymptotic behavior

\[
\bar{B}_G(c) \equiv f_G[c, s_0(c)] \simeq \begin{cases} 
2 - c & \text{for } c \ll 1 \\
1 + \frac{c}{2} & \text{for } c \gg 1
\end{cases}
\]

of the mean value.

### Appendix D. Dissipation of a superposition of Fock states

In this appendix we consider the effect of the amplitude damping (attenuation) of a given state \( \hat{\rho} \) due to a coupling to a reservoir of linear oscillators at zero temperature.

For this decoherence process with the dissipation parameter \( \eta < 1 \), the density operator

\[
\hat{\rho}(\eta) = \sum_{n,m=0}^{\infty} \rho_{nm}(\eta) |n\rangle \langle m|
\]

with

\[
\rho_{nm}(\eta) = \eta^{n+m/2} \frac{(1-\eta)^{l}}{l!} \sqrt{\frac{(n+f)!(m+f)!}{n!m!}} \rho_{n+l,m+l}(1),
\]

is determined \(^{33}\) by the matrix elements \( \rho_{nm}(1) = \rho_{nm} = 1 \) corresponding to no dissipation, that is \( \eta = 1 \).

Here, \( n! \) is the factorial of \( n \).

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