INTERPOLATION BY ENTIRE FUNCTIONS WITH GROWTH CONDITIONS

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INTRODUCTION

Let \( p : \mathbb{C} \rightarrow [0, +\infty] \) be a weight (see Definition 1.1) and \( A_p(\mathbb{C}) \) the vector space of all entire functions satisfying \( \sup_{z \in \mathbb{C}} |f(z)| \leq \exp(-Bp(z)) < \infty \) for some constant \( B > 0 \). For instance, if \( p(z) = |z| \), then \( A_p(\mathbb{C}) \) is the space of all entire functions of exponential type.

Following [3], the interpolation problem we are considering is: let \( V = \{(z_j, m_j)\}_j \) be a multiplicity variety, that is, \( \{z_j\}_j \) is a sequence of complex numbers diverging to \( \infty \), \( |z_j| \leq |z_{j+1}| \) and \( \{m_j\}_j \) is a sequence of strictly positive integers. Let \( \{w_{j,l}\}_{j,0 \leq l < m_j} \) be a doubly indexed sequence of complex numbers.

Under what conditions does there exist an entire function \( f \in A_p(\mathbb{C}) \) such that

\[
\frac{f^{(l)}}{l!}(z_j) = w_{j,l}, \quad \forall j, \quad \forall 0 \leq l < m_j.
\]

In other words, if we denote by \( \rho \) the restriction operator defined on \( A_p(\mathbb{C}) \) by

\[
\rho(f) = \left\{ \frac{f^{(l)}}{l!}(z_j) \right\}_{j,0 \leq l < m_j},
\]

what is the image of \( A_p(\mathbb{C}) \) by \( \rho \)?

We say that \( V \) is an "interpolating variety" when \( \rho(A_p(\mathbb{C})) \) is the space of all doubly indexed sequence \( W = \{w_{j,l}\} \) satisfying the growth condition

\[
|w_{j,l}| \leq A \exp(Bp(z_j)) \quad \forall j, \quad \forall 0 \leq l < m_j,
\]

for certain constants \( A, B > 0 \).

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Let us mention the important following result:

**Theorem 0.1.** [2 Corollary 4.8]

A variety $V$ is an interpolating variety for $A_p(\mathbb{C})$ if and only if the following conditions hold:

(i) $\forall R > 0, \ N(0, R) \leq Ap(R) + B$

(ii) $\forall j \in \mathbb{N}, \ N(z_j, |z_j|) \leq Ap(z_j) + B,$

for some constants $A, B > 0.$

Here, $N(z, r)$ denotes the integrated counting function of $V$ in the disc of center $z$ and radius $r$ (see Definition 1.3 below).

In [3], Berenstein and Taylor describe the space $\rho(A_p(\mathbb{C}))$ in the case where there exists a function $g \in A_p(\mathbb{C})$ such that $V = g^{-1}(0)$. They used groupings of the points of $V$ with respect to the connex components of the set $\{ |g(z)| \leq \varepsilon \exp(-Bp(z)) \}$, for some $\varepsilon, B > 0$ and the divided differences with respect to this grouping.

The main aim of this paper is to determine more explicitly the space $\rho(A_p(\mathbb{C}))$ in the more general case where condition (i) is satisfied. It is clear that it is the case when $V$ is not a uniqueness set for $A_p(\mathbb{C})$, that is, when there exists $f \in A_p(\mathbb{C})$ not identically equal to zero such that $V \subset f^{(-1)}(0)$.

We refer to [6] and [10] for similar results in the case where $p(z) = |z|^\alpha$.

As in [3] and [10], the divided differences will be important tools. Our condition will involve the divided differences with respect to the intersections of $V$ with discs centered at the origin. To be more precise, the main theorem, stated in the case where all the multiplicities are equal to one, for the sake of simplicity, is the following:

**Theorem 0.2.** Assume that $V$ verifies condition (i). Then $W = \{ w_j \}_j \in \rho(A_p(\mathbb{C}))$ if and only if for all $R > 0$,

$$\left| \sum_{|z_k| < R} w_k \prod_{|z_m| < R, m \neq k} R/(z_k - z_m) \right| \leq A \exp Bp(R),$$

where $A, B > 0$ are positive constants only depending on $V$ and $W$. 
We will denote by $\tilde{A}_p(V)$ the space of sequences $W = \{w_j\}$ satisfying the above condition. We will show that in general $\rho(A_p(\mathbb{C})) \subset \tilde{A}_p(V)$, thus, we can consider $\rho : A_p(\mathbb{C}) \to \tilde{A}_p(V)$. In this context, the theorem states that condition (i) implies the surjectivity of $\rho$.

On the other hand, we will prove that condition (i) is actually equivalent to saying that $V$ is not a uniqueness set or, in other words, it is equivalent to the non-injectivity of $\rho$.

As a corollary of the main theorem, we will find the sufficiency in the geometric characterization of interpolating varieties given in Theorem 0.1.

The difficult part of the proof of the main theorem is the sufficiency. As in [4, 7, 11], we will follow a Bombieri-Hörmander approach based on $L^2$-estimates on the solution to the $\bar{\partial}$-equation. The scheme will be the following: the condition on $W$ gives a smooth interpolating function $F$ with a good growth, using a partition of the unity and Newton polynomials (see Lemma 2.5). Then we are led to solve the $\bar{\partial}$ equation: $\bar{\partial}u = -\bar{\partial}F$ with $L^2$-estimates, using Hörmander theorem [8]. To do so, we need to construct a subharmonic function $U$ with a convenient growth and with prescribed singularities on the points $z_j$ (see Lemma 2.6). Following Bombieri [5], the fact that $e^{-U}$ is not summable near the points $\{z_j\}$ forces $u$ to vanish on the points $z_j$ and we are done by defining the interpolating entire function by $u + F$.

A final remark about the notations:

$A$, $B$ and $C$ will denote positive constants and their actual value may change from one occurrence to the next.

$A(t) \lesssim B(t)$ means that there exists a constant $C > 0$, not depending on $t$ such that $A(t) \leq CB(t)$. $A \simeq B$ means that $A \lesssim B \lesssim A$.

The notation $D(z, r)$ will be used for the euclidean disk of center $z$ and radius $r$. We will denote $\partial f = \frac{\partial f}{\partial z}$, $\bar{\partial} f = \frac{\partial f}{\partial \bar{z}}$. Then $\Delta f = 4\partial \bar{\partial} f$ denotes the laplacian of $f$.

1. Preliminaries and definitions.

Definition 1.1. A subharmonic function $p : \mathbb{C} \to \mathbb{R}_+$, is called a weight if, for some positive constants $C$,

(a) $\ln(1 + |z|^2) \leq Cp(z)$.
(b) $p(z) = p(|z|)$.

(c) there exists a constant $C > 0$ such that $p(2z) \leq Cp(z)$.

Property (c) is referred to as the "doubling property of the weight $p". It implies that $p(z) = O(|z|^\alpha)$ for some $\alpha > 0$.

Let $A(\mathbb{C})$ be the set of all entire functions, we consider the space

$$\mathcal{A}_p(\mathbb{C}) = \left\{ f \in A(\mathbb{C}), \, \forall z \in \mathbb{C}, \, |f(z)| \leq Ae^{Bp(z)} \text{ for some } A > 0, B > 0 \right\}.$$  

Remark 1.2.  
(i) Condition (a) implies that $\mathcal{A}_p(\mathbb{C})$ contains all polynomials.

(ii) Condition (c) implies that $\mathcal{A}_p(\mathbb{C})$ is stable under differentiation.

Examples:

- $p(z) = \ln(1 + |z|^2)$. Then $\mathcal{A}_p(\mathbb{C})$ is the space of all the polynomials.
- $p(z) = |z|$. Then $\mathcal{A}_p(\mathbb{C})$ is the space of entire functions of exponential type.
- $p(z) = |z|^{\alpha}, \, \alpha > 0$. Then $\mathcal{A}_p(\mathbb{C})$ is the space of all entire functions of order $\leq \alpha$ and finite type.

Let $V = \{(z_j, m_j)\}_{j \in \mathbb{N}}$ be a multiplicity variety.

For a function $f \in A(\mathbb{C})$, we will write $V = f^{-1}(0)$ when $f$ vanishes exactly on the points $z_j$ with multiplicity $m_j$ and $V \subset f^{-1}(0)$ when $f$ vanishes on the points $z_j$ (but possibly elsewhere) with multiplicity at least equal to $m_j$.

We will say that $V$ is a uniqueness set for $\mathcal{A}_p(\mathbb{C})$ if there is no function $f \in \mathcal{A}_p(\mathbb{C})$, except the zero function, such that $V \subset f^{-1}(0)$.

We need to recall the definitions of the counting functions and the integrated counting functions:

Definition 1.3. Let $V = \{(z_j, m_j)\}_{j}$ be a multiplicity variety. For $z \in \mathbb{C}$ and $r > 0$,

$$n(z, r) = \sum_{|z-z_j| \leq r} m_j,$$

$$N(z, r) = \int_0^r \frac{n(z,t) - n(z,0)}{t} dt + n(z,0) \ln r = \sum_{0<|z-z_j| \leq r} m_j \ln \frac{r}{|z-z_j|} + n(z,0) \ln r,$$
An application of Jensen’s formula in the disc $D(0, R)$ shows that, if $V$ is not a uniqueness set for $A_p(\mathbb{C})$, then the following condition holds:

\[
\exists A, B > 0, \forall R > 0, \quad N(0, R) \leq A p(R) + B
\]

We will lately show that the converse property holds.

By analogy with the spaces $A(\mathbb{C})$ and $A_p(\mathbb{C})$, we define the following spaces

\[
A(V) = \{ W = \{ w_{j,l} \} j, 0 \leq l < m_j \subset \mathbb{C} \}
\]

and

\[
A_p(V) = \left\{ W = \{ w_{j,l} \} j, 0 \leq l < m_j \subset \mathbb{C}, \quad \forall j, \sum_{l=0}^{m_j-1} |w_{j,l}| \leq A e^{B p(z)} \text{ for some } A > 0, B > 0 \right\}.
\]

The space $A_p(\mathbb{C})$ can be seen as the union of the Banach spaces

\[
A_{p,B}(\mathbb{C}) = \{ f \in A(\mathbb{C}), \quad \| f \|_B := \sup_{z \in \mathbb{C}} |f(z)| e^{-B p(z)} < \infty \}
\]

and has a structure of an (LF)-space with the topology of the inductive limit. The analog is true about $A_p(V)$.

**Remark 1.4.** (see [1, Proposition 2.2.2])

Let $f$ be a function in $A_p(\mathbb{C})$. Then, for some constants $A > 0$ and $B > 0$,

\[
\forall z \in \mathbb{C}, \quad \sum_{k=0}^{\infty} \left| \frac{f^{(k)}(z)}{k!} \right| \leq A e^{B p(z)}.
\]

As a consequence of this remark, we see that the restriction map:

\[
\rho : A(\mathbb{C}) \rightarrow A(V)
\]

\[
f \rightarrow \{ \frac{f^{(l)}(z_j)}{l!} \} j, 0 \leq l \leq m_j - 1
\]

maps $A_p(\mathbb{C})$ into $A_p(V)$, but in general, the space $A_p(V)$ is larger than $\rho(A_p(\mathbb{C}))$. It is clear that $\rho$ is injective if and only if $V$ is a uniqueness set for $A_p(\mathbb{C})$. 
When \( \rho(A_p(C)) = A_p(V) \), we say that \( V \) is an interpolating variety for \( A_p(C) \). As mentioned in the introduction, Berenstein and Li gave a geometric characterization of these varieties:

**Theorem 1.5.** [2, Corollary 4.8]

\( V \) is an interpolating variety for \( A_p(C) \) if and only if conditions (1) and

\[
\exists A > 0, \exists B > 0 \forall j \in \mathbb{N}, \ N(z_j, |z_j|) \leq A p(z_j) + B
\]

hold.

In this paper, we are concerned by determining the subspace \( \rho(A_p(C)) \) of \( A(V) \) in the case where condition (1) is verified.

To any \( W = \{ w_{j,l} \}_{j,0 \leq l \leq m_j - 1} \in A(V) \), we associate the sequence of divided differences \( \Phi(W) = \{ \phi_{j,l} \}_{j,0 \leq l \leq m_j - 1} \) defined by induction as follows:

We will denote by

\[
\Pi_q(z) = \prod_{k=1}^{q} (z - z_k)^{m_k}, \text{ for all } q \geq 1.
\]

\[
\phi_{1,l} = w_{1,l}, \text{ for all } 0 \leq l \leq m_1 - 1,
\]

\[
\phi_{q,0} = \frac{w_{q,0} - P_{q-1}(z_q)}{\Pi_{q-1}(z_q)},
\]

\[
\phi_{q,l} = \frac{w_{q,l} - \frac{P_{q-1}^{(l)}(z_q)}{l!} - \sum_{j=0}^{l-1} \frac{1}{(l-j)!} \Pi_{q-1}^{(l-j)}(z_q) \phi_{q,j}}{\Pi_{q-1}(z_q)} \text{ for } 1 \leq l \leq m_q - 1
\]

where

\[
P_{q-1}(z) = \sum_{j=1}^{q-1} \left( \sum_{l=0}^{m_j - 1} \phi_{j,l}(z - z_j)^l \prod_{t=1}^{j-1} (z - z_t)^{m_t} \right).
\]

**Remark 1.6.** Actually, \( P_q \) is the polynomial interpolating the values \( w_{j,l} \) at the points \( z_j \) with multiplicity \( m_j \), for \( 1 \leq j \leq q \). It is the unique polynomial of degree \( m_1 + \cdots + m_q - 1 \) such that

\[
\frac{P_{q}^{(l)}(z_j)}{l!} = w_{j,l}
\]

for all \( 1 \leq j \leq q \) and \( 0 \leq l \leq m_j - 1 \).
Examples.

• Let \( W_0 = \{\delta_{1,j}\delta_{l,m_1-1}\}_{j,0 \leq l < m_j} \). Using the fact that \( P_j(z) \) must coincide with \( (z - z_1)^{m_1-1} \prod_{k=2}^{j-1} (z - z_j)^{m_j} \) and identifying the coefficient in front of \( z^{m_1+\cdots+m_{j-1}+l-1} \), we find:

\[
\phi_{1,1} = \phi_{1,2} = \cdots = \phi_{1,m_1-2} = 0, \quad \phi_{1,m_1-1} = 1,
\]

and, for \( j \geq 2, 0 \leq l \leq m_j - 1 \),

\[
\phi_{l,j} = (z_1 - z_j)^{-l-1} \prod_{k=2}^{j-1} (z_1 - z_k)^{-m_k}.
\]

• In the special case where \( m_j = 1 \) for all \( j \) and \( W = \{w_j\}_j \), we have for all \( j \geq 1 \),

\[
\phi_j = \sum_{k=1}^{j} w_k \prod_{1 \leq l \leq j, l \neq k} (z_k - z_l)^{-1}.
\]

To compute the coefficients, we may use the fact that \( P_j(z) \) must coincide with the Lagrange polynomial \( \sum_{n=1}^{j} w_n \prod_{1 \leq k \leq j, k \neq n} \frac{(z - z_k)}{(z_n - z_k)} \) and identify the coefficient in front of \( z^{j-1} \).

Let us denote by \( \tilde{A}_p(V) \) the subspace of \( A(V) \) consisting of the elements \( W \in A(V) \) such that the following condition holds:

\[
\text{for all } n \geq 0, |z_j| \leq 2^n \text{ and } 0 \leq l \leq m_j - 1, \quad |\phi_{j,l}|^{2^{n(l+m_1+\cdots+m_{j-1})}} \leq A \exp(Bp(2^n)),
\]

where \( A \) and \( B \) are positive constants only depending on \( V \) and \( W \).

We have chosen to use a covering of the complex plane by discs \( D(0, 2^n) \), but we can replace \( 2^n \) by any \( R^n \) with \( R > 1 \).

Lemma 1.7. Assume \( z_1 = 0 \). Then, condition \( (\ref{eq:condition}) \) holds if and only if

\[
W_0 = \{\delta_{1,j}\delta_{l,m_1-1}\}_{j,0 \leq l < m_j} \in \tilde{A}_p(V).
\]
Proof. Suppose that (1) is verified. Let \( n \in \mathbb{N} \), \( 0 < |z_j| \leq 2^n \) and \( 0 \leq l \leq m_j - 1 \). We have by definition,

\[
N(0, 2^n) = \sum_{0 < |z_k| \leq 2^n} m_k \ln \frac{2^n}{|z_k|} + m_1 \ln(2^n) \geq \ln \left( 2^{n(m_1 + \cdots + m_j)} \prod_{k=2}^j |z_k|^{-m_k} \right),
\]

\[
|\phi_{j,l}| = |z_j|^{m_j-l-1} \prod_{k=2}^j |z_k|^{-m_k} \leq 2^{n(m_j-l-1)} \prod_{k=2}^j |z_k|^{-m_k} \leq \exp(N(0, 2^n))2^{-n(m_1 + \cdots + m_{j-1} + l + 1)}.
\]

We readily obtain the estimate (3), using that \( N(0, 2^n) \leq Ap(2^n) + B \).

Conversely, let \( n \) be an integer. Using the estimate (3) when \( j \geq 2 \) is the number of distinct points \( \{z_k\} \) in \( D(0, 2^n) \) and \( l = m_j - 1 \), we have

\[
N(0, 2^n) = \ln \left( 2^{n(m_1 + \cdots + m_j)} \prod_{k=2}^j |z_k|^{-m_k} \right) = \ln(2^{n(m_1 + \cdots + m_j)}|\phi_{j,m_j-1}|) \leq Ap(2^n) + B.
\]

Then, we deduce the estimate for \( N(0, R) \) using the above one with \( 2^{n-1} \leq R < 2^n \) and the doubling property of \( p \).

We define the following norm:

\[
\|W\|_B = \sup_n \|W^{(n)}\|_n \exp(-Bp(2^n))
\]

where

\[
\|W^{(n)}\|_n = \sup_{|z_j| \leq 2^n} \sup_{0 \leq l \leq m_j - 1} |\phi_{j,l}| 2^{-n(l+m_1+\cdots+m_{j-1})}.
\]

The space \( \hat{A}_p(V) \) can also be seen as an (LF)-space as an inductive limit of the Banach spaces

\[
\hat{A}_{p,B}(V) = \{W \in A(V), \|W\|_B < \infty\}.
\]

We are now ready to state the main results.

**Proposition 1.8.** The restriction operator \( \rho \) maps continously \( A_p(C) \) into \( \hat{A}_p(V) \).

**Proposition 1.9.** Under the assumption of condition (1), \( \hat{A}_p(V) \) is a subspace of \( A_p(V) \).

**Proposition 1.10.** If conditions (1) and (2) are verified, then \( \hat{A}_p(V) = A_p(V) \).
Theorem 1.11. If condition (7) holds, then

\[ \tilde{A}_p(V) = \rho(A_p(\mathbb{C})). \]

In other words, condition (7) implies that the map \( \rho : A_p(\mathbb{C}) \to \tilde{A}_p(V) \) is surjective.

The combination of Proposition 1.10 and Theorem 1.11 shows easily the sufficiency in Theorem 1.5.

Using the results given so far, we can deduce next theorem:

Theorem 1.12. The following assertions are equivalent:

(i) \( V \) is not a uniqueness set for \( A_p(\mathbb{C}) \).

(ii) The map \( \rho \) is not injective.

(iii) \( V \) verifies condition (7).

(iv) The sequence \( W_0 = \{ \delta_{1,j} \delta_{l,m_j - 1} \}_{j,0 \leq l < m_j} \) belongs to \( \rho(A_p(\mathbb{C})) \).

In particular, it shows that condition (7) is equivalent to the existence of a function \( f \in A_p(\mathbb{C}) \) such that \( V \subset f^{-1}(0) \). Combined with Theorem 1.11 it shows that, if \( \rho \) is not injective, then it is surjective and that, if the image contains \( W_0 \), then it contains the whole \( \tilde{A}_p(V) \).

Proof of Theorem 1.12. As we mentioned before, it is clear that (i) is equivalent to (ii) and that (i) implies (iii).

(iv) implies (i) : We have a function \( f \in A_p(\mathbb{C}) \) not identically equal to 0 such that \( f^{(l)}(z_j) = 0 \) for all \( j \neq 1 \) and for all \( 0 \leq l < m_j \). The function \( g \) defined by \( g(z) = (z - z_1)^{m_1} f(z) \) belongs to \( A_p(\mathbb{C}) \), thanks to property (i) of the weight \( p \), and vanishes on every \( z_j \) with multiplicity at least \( m_j \).

(iii) implies (iv) :

Up to a translation, we may suppose that \( z_1 = 0 \). By Lemma 1.7 we know that \( W_0 \in \tilde{A}_p(\mathbb{C}) \). By Theorem 1.11 \( W_0 \in \rho(A_p(\mathbb{C})) \).
2. Proof of the main results.

Proof of Theorem 1.8 We will first recall some definitions about the divided differences and Newton polynomials. We refer the reader to [1, Chapter 6.2] or [9, Chapter 6] for more details.

Let \( f \in A(\mathbb{C}) \) and \( x_1, \ldots, x_q \) be distinct points of \( \mathbb{C} \). The \( q \)th divided difference of the function \( f \) with respect to the points \( x_1, \ldots, x_q \) is defined by

\[
\Delta_{q-1} f(x_1, \ldots, x_q) = \sum_{j=1}^{q} f(z_j) \prod_{1 \leq k \leq q, k \neq j} (x_j - x_k)^{-1}
\]

and the Newton polynomial of \( f \) of degree \( q - 1 \) is

\[
P(z) = \sum_{j=1}^{q} \Delta_{j-1} f(x_1, \ldots, x_j) \prod_{k=0}^{j-1} (z - x_k).
\]

It is the unique polynomial of degree \( q - 1 \) such that \( P_q(z) = f(x_j) \) for all \( 1 \leq j \leq q \).

When \( x_j, 1 \leq j \leq q \) are each one repeated \( l_j \) times, the divided differences are defined by

\[
\Delta^{l_1+\cdots+l_q-1} f(x_1, \ldots, x_q) = \frac{1}{l_1! \cdots l_q!} \frac{\partial^{l_1+\cdots+l_q}}{\partial x_1^{l_1} \cdots \partial x_q^{l_q}} \Delta_{q-1} f(x_1, \ldots, x_q).
\]

The corresponding Newton polynomial is the unique polynomial of degree \( l_1 + \cdots + l_q - 1 \) such that, for all \( 0 \leq j \leq q \) and \( 0 \leq l \leq l_j - 1 \),

\[
P^{(l)}(x_j) = f^{(l)}(x_j).
\]

We have the following estimate

**Lemma 2.1.** [1, Lemma 6.2.9.]

Let \( f \in A(\mathbb{C}), \Omega \) an open set of \( \mathbb{C} \), \( \delta > 0 \) and \( x_1, \cdots, x_k \in \Omega_0 = \{ z \in \Omega : d(z, \Omega^c) > \delta \} \). Then

\[
|\Delta_{k-1} f(x_1, \ldots, x_k)| \leq \frac{2^{k-1}}{\delta^{k-1}} \sup_{z \in \Omega} |f(z)|.
\]

Let \( B > 0 \) be fixed and \( f \in A_{p,B}(\mathbb{C}) \).

Let \( n \) be a fixed integer. Let \( |z_j| \leq 2^n \) and \( 0 \leq l \leq m_j - 1 \). We consider the divided differences of \( f \) with respect to the points \( z_1, \cdots, z_j \), each \( z_k, 1 \leq k \leq j - 1 \) repeated \( m_k \) times and \( z_j \) repeated \( l \) times.
Denote by $M_{j,l} = m_1 + \cdots + m_{j-1} + l$, the divided differences are

$$\phi_{j,l} = \Delta^{M_{j,l}} f(z_1, \ldots, z_1, \ldots, z_{j-1}, \ldots, z_{j-1}, z_j, \ldots, z_j).$$

Using Lemma 2.1 with $\Omega = D(0, 2^{n+2}), \delta = 2^{n+1}, k = M_{j,l} + 1$, we have

$$|\phi_{j,l}| \leq 2^{-nM_{j,l}} \|f\|_B \exp(Bp(2^{n+2})) \leq 2^{-nM_{j,l}} \|f\|_B \exp(B'p(2^n)).$$

Thus,

$$\|\rho(f)\|_{B'} \leq \|f\|_B$$

and this concludes the proof of Proposition 1.8.

Before proceeding with the proofs of the main results, we need the following lemmas:

**Lemma 2.2.** Condition (1) implies that there exist constants $A, B > 0$ such that, for all $R > 0$,

$$n(0, R) \leq Ap(R) + B.$$

**Proof.** Using property (c) of the weight, we have

$$n(0, R) \leq 2 \int_R^{2R} \frac{n(0, t)}{t} dt \leq 2N(0, 2R) \leq Ap(2R) + B \leq Ap(R) + B.$$

**Lemma 2.3.** Let $W$ be an element of $A(V)$ and $q$ be in $\mathbb{N}^*$. We suppose that for all $1 \leq j \leq q$, for all $n \in \mathbb{N}$ such that $|z_q| \leq 2^n$ and for all $0 \leq l \leq m_j - 1$, we have

$$|\phi_{j,l}|2^{n(l+m_1+\ldots+m_{j-1})} \leq A \exp(Bp(2^n)),$$

where $A$ and $B$ are positive constants only depending on $V$ and $W$. 
Then, there exist constants $A, B > 0$ only depending on $V$ and $W$, such that, for all $n \in \mathbb{N}$ and $|z| \leq 2^n$,

$$
\sum_{l=0}^{+\infty} \frac{|P_q^{(l)}(z)|}{l!} \leq A \exp(Bp(2^n)) \sum_{j=1}^{q} 2^{2(m_1+\cdots+m_j)},
$$

and

$$
\sum_{l=0}^{+\infty} \frac{|\Pi_q^{(l)}(z)|}{l!} \leq 2^{(n+2)(m_1+\cdots+m_q)}.
$$

**Proof.** If $|z| \leq 2^{n+1}$, then for $j = 1, \ldots, q$, $|z - z_j| \leq 2^{n+2}$,

$$
|P_q(z)| \leq \sum_{j=1}^{q} 2^{(n+2)(m_1+\cdots+m_{j-1})} \sum_{l=0}^{m_j-1} |\phi_{j,l}| 2^{(n+2)l} 
\leq A \exp(Bp(2^n)) \sum_{j=1}^{q} 2^{2(m_1+\cdots+m_j)}
$$

and

$$
|\Pi_q(z)| = \prod_{j=1}^{q} |z - z_j|^{m_j} \leq 2^{(n+2)(m_1+\cdots+m_q)}.
$$

Now for $|z| \leq 2^n$, if $|z-w| \leq 2$, then $|w| \leq 2^{n+1}$. By the preceding inequalities and Cauchy inequalities, for all $l \geq 0$,

$$
\frac{|P_q^{(l)}(z)|}{l!} \leq \frac{1}{2^l} \max_{|z-w| \leq 2} |P_q(w)| \leq \frac{1}{2^l} A \exp(Bp(2^n)) \sum_{j=1}^{q} 2^{2(m_1+\cdots+m_j)}.
$$

We readily obtain the desired estimate for $P_q$. Using Cauchy estimates once again for the function $\Pi_q$ we obtain the second inequality.

**Proof of Proposition 1.9** We assume that condition (1) holds. Let $W = \{w_{j,l}\}_{j,0 \leq l \leq m_j-1} \in \tilde{A}_p(V)$. Let $q \geq 1$ and $n$ be the integer such that $2^{n-1} \leq |q| < 2^n$. We know that $\frac{P_q^{(l)}(z_n)}{l!} = w_{q,l}$ for every $0 \leq l \leq m_q-1$. By the preceding lemma,

$$
\sum_{l=0}^{m_q-1} |w_{q,l}| \leq \sum_{l=0}^{+\infty} \frac{|P_q^{(l)}(z_n)|}{l!} \leq A \exp(Bp(2^n)) \sum_{j=1}^{q} 2^{2(m_1+\cdots+m_j)}.
$$

By Lemma 2.2 $m_1 + \cdots + m_j \leq n(0, |z_j|) \leq Ap(z_j) + B$. Using that $q \leq n(0, |z_q|) \leq Ap(z_q) + B$, we obtain

$$
\sum_{l=0}^{m_q-1} |w_{q,l}| \leq A \exp(Bp(2^n)) \leq A \exp(Bp(z_q)),
$$

that is $W \in A_p(V)$. 

\[ \square \]
Proof of Theorem 1.10. We assume that conditions (1) and (2) are fulfilled. We already have \( \hat{A}_p(V) \subset A_p(V) \) by Proposition 1.9.

Before proving the reverse inclusion, we need some useful consequences of (1) and (2):

**Lemma 2.4.** There exist constants \( A, B > 0 \) such that, for all \( j \in \mathbb{N}^* \) and for all \( n \in \mathbb{N} \) such that \( |z| \leq 2^n \), we have

(i) \( 2^{n m_j} \leq A |z|^{m_j} \exp(B p(2^n)) \), \( 2^{n m_1+\cdots+m_j} \leq A |z|^{m_1+\cdots+m_j} \exp(B p(2^n)) \).

(ii) \( |z|^m_j \leq A \exp(B p(z_j)) \).

(iii) \( \prod_{k=1}^{j-1} |z_j - z_k|^{-m_k} \leq A \exp(B p(2^n)) 2^{-n(m_1+\cdots+m_{j-1})} \).

**Proof.** (i) For \( 0 < |z_j| \leq 2^n \), we have

\[
N(0, 2^n) \geq \sum_{0 < |z_k| \leq 2^n} m_k \ln \frac{2^n}{|z_k|} \geq m_j \ln \frac{2^n}{|z_j|}.
\]

We readily obtain the result by condition (1).

The second inequality is obtained in the same way, noting that

\[
N(0, 2^n) \geq \sum_{k=1}^{j} m_k \ln \frac{2^n}{|z_k|} \geq \left( \ln \frac{2^n}{|z_j|} \right) \sum_{k=1}^{j} m_k.
\]

(ii) It is a simple consequence of condition (2):

\[
m_j \ln |z_j| \leq N(z_j, |z_j|) \leq Ap(z_j) + B.
\]

(iii) It is also a consequence of condition (2):

\[
\sum_{k=1}^{j-1} m_k \ln \frac{|z_j|}{|z_j - z_k|} = N(z_j, |z_j|) \leq Ap(z_j) + B.
\]

We deduce that

\[
\prod_{k=1}^{j-1} |z_j - z_k|^{-m_k} \leq A \exp(B p(z_j)) |z_j|^{-(m_1+\cdots+m_{j-1})} \leq A 2^{-n(m_1+\cdots+m_{j-1})} \exp(B p(2^n))
\]

using (i). ■
Let $W = \{w_{j,l}\}_{j,0 \leq l \leq m_j - 1}$ be in $A_p(V)$. In order to show that $W$ verifies (3), we are going to use Lemma 2.3 and show by induction on $q \geq 1$ the following property:

For all $n \in \mathbb{N}$ such that $|z_q| \leq 2^n$ and for all $0 \leq l \leq m_q - 1$,

$$|\phi_{q,l}|2^{n(l+m_1+\cdots+m_{q-1})} \leq A \exp(Bp(2^n)),$$

where $A$ and $B$ are positive constants only depending on $V$ and $W$.

$q = 1$: for $|z_1| \leq 2^n$ and $0 \leq l \leq m_1 - 1$, we have

$$|\phi_{1,l}| = |w_{1,l}| \leq A \exp(Bp(z_1)) \leq A \exp(Bp(z_1))2^{-nl}2^{n1} \leq A \exp(Bp(2^n))2^{-nl}$$

using Lemma 2.4 (i) and (ii).

Suppose the property true for $1 \leq j \leq q - 1$. Let $n \in \mathbb{N}$ be such that $|z_q| \leq 2^n$.

Again, we proceed by induction on $l$, $0 \leq l \leq m_q - 1$.

$l = 0$: by Lemmas 2.3 and 2.2 we have

$$|P_{q-1}(z_q)| \leq A \exp(Bp(2^n)) \sum_{j=1}^{q-1} 2^{2(m_1+\cdots+m_j)} \leq (q-1)2^{2(m_1+\cdots+m_{q-1})} \leq A \exp(Bp(2^n)).$$

By Lemma 2.4(iii),

$$|\Pi_{q-1}(z_q)|^{-1} = \prod_{k=1}^{q-1} |z_q - z_k|^{-m_k} \leq A \exp(Bp(2^n))2^{-n(m_1+\cdots+m_{q-1})}$$

We deduce that

$$|\phi_{q,0}| \leq A \exp(Bp(2^n))2^{-n(m_1+\cdots+m_{q-1})}.$$

Suppose the estimate true for $0 \leq j \leq l - 1$, using both inequalities of Lemma 2.3 and Lemma 2.2 we have

$$\sum_{j=0}^{l-1} |\frac{\Pi_{q-1}^{(l-j)}(z_q)}{(l-j)!}| \phi_{q,j} | \leq A \exp(Bp(2^n))$$

and

$$|\frac{P_{q-1}^{(l)}(z_q)}{l!}| \leq A \exp(Bp(2^n)).$$

As for $l = 0$, we use Lemma 2.4(iii) to complete the proof.
Proof of Theorem 1.11. We already showed the necessity in Theorem 1.8. Let us prove the sufficiency:

We assume condition (i). Let \( W = \{ w_{j,l} \}_{j,0 \leq l \leq m_j - 1} \) be an element of \( \hat{A}_p(V) \).

Let \( \mathcal{X} \) be a smooth cut-off function such that \( \mathcal{X}(x) = 1 \) if \( |x| \leq 1 \) and \( \mathcal{X}(x) = 0 \) if \( |x| \geq 4 \).

Set \( \mathcal{X}_n(z) = \mathcal{X}(\frac{|z|^2}{n^2}) \), for \( n \in \mathbb{N} \), \( \rho_0 = \mathcal{X}_0 \) and \( \rho_{n+1} = \mathcal{X}_n - \mathcal{X}_{n+1} \). It is clear that the family \( \{ \rho_n \}_n \) form a partition of the unity, that the support of \( \rho_n \) is contained in the disk \( |z| \leq 2^{n+1} \) and that the support of \( \rho_n \) is contained in the annulus \( \{ 2^{n-1} \leq |z| \leq 2^{n+1} \} \) for \( n \geq 1 \).

We will denote by \( q_n \) the number of distinct points \( z_j \) in \( D(0, 2^n) \), that is: \( q_n = \sum_{|z_j| \leq 2^n} 1 \).

Lemma 2.5. There exists a \( C^\infty \) function \( F \) on \( \mathbb{C} \) such that, for certain constants \( A, B > 0 \),

(i) \( \frac{F^{(l)}(z_j)}{l!} = w_{j,l} \) for all \( j \in \mathbb{N}, 0 \leq l \leq m_j - 1 \).

(ii) for all \( z \in \mathbb{C}, |F(z)| \leq Ae^{B|z|} \),

(iii) \( \partial F = 0 \) on \( D(0, 1) \) and for any \( n \geq 2 \) and \( 2^{n-2} \leq |z| < 2^{n-1} \),

\[ |\partial F(z)| \leq A 2^{-n(m_1 + \ldots + m_q)} \prod_{k=1}^{q_n} |z - z_k|^{m_k} e^{B|z|} \]

Proof. We set

\[ F(z) = \sum_{n \geq 2} \rho_{n-2}(z) P_{q_n}(z). \]

where

\[ P_{q_n}(z) = \sum_{j=1}^{q_n} \left( \sum_{l=0}^{m_j-1} \phi_{j,l}(z - z_j)^l \right) \prod_{k=1}^{j-1} (z - z_k)^{m_k}. \]

It is the Newton polynomial we mentioned in Remark 1.6.

(i) For all \( j \geq 1 \) and \( 0 \leq l \leq m_j - 1 \), if \( z_j \) is in the support of \( \rho_{n-2} \), then \( P_{q_n}^{(l)}(z_j) = l! w_{j,l} \). Thus

\[ F^{(l)}(z_j) = \sum_{n \geq 2} \left( \sum_{k=0}^{l} C^k_l \rho_{n-2}^{(l-k)}(z_j) k! w_{j,k} \right) \]

\[ = \sum_{k=0}^{l} C^k_l k! w_{j,k} \left( \sum_{n} \rho_n^{(l-k)}(z_k) \right) = l! w_{j,l}. \]

(ii) For \( z \geq 1 \), let \( n \geq 2 \) be the integer such that \( 2^{n-2} \leq |z| < 2^{n-1} \). Then, we have:

\[ F(z) = \rho_{n-2}(z) P_{q_n}(z) + \rho_{n-1}(z) P_{q_{n+1}}(z). \]
For all $0 \leq j \leq q_n$, we have $|z_j| \leq 2^n$ and $|z - z_j| \leq 2^{n+1}$. Using Lemmas 2.3 condition (1) and property (c) of the weight, we have

$$|P_{q_n}(z)| \lesssim \exp(Bp(2^n)) \leq A \exp(Bp(2^n)) \leq A \exp(Bp(z)).$$

The same estimation holds for $P_{q_n+1}$ thus,

$$|F(z)| \lesssim \exp(Bp(z)).$$

(iii) Now, we want to estimate $\bar{\partial}F$.

It is clear that $F(z) = P_{q_2}(z)$ on $D(0, 1)$.

Let $|z| \geq 1$ and $n$ the integer such that $2^{n-2} \leq |z| < 2^{n-1}$. We have

$$\bar{\partial}F(z) = \bar{\partial}\rho_{n-2}(z)P_{q_n}(z) + \bar{\partial}\rho_{n-1}(z)P_{q_n+1}(z).$$

Since $z$ is outside the supports of $\bar{\partial}\mathcal{X}_{n-3}$ and of $\bar{\partial}\mathcal{X}_{n-1}$, we have

$$\bar{\partial}F(z) = -\bar{\partial}\mathcal{X}_{n-2}(z)(P_{q_n+1}(z) - P_{q_n}(z)) = \prod_{k=1}^{q_n} (z - z_k)^{m_k} G_n(z)$$

where

$$G_n(z) = -\bar{\partial}\mathcal{X}_{n-2}(z) \sum_{j=q_n+1}^{q_{n+1}} \prod_{k=q_n+1}^{j-1} (z - z_k)^{m_k} \left( \sum_{l=0}^{m_j-1} \phi_{j,l}(z - z_j)^l \right).$$

For $k \leq q_{n+1}$, $|z - z_k| \leq 2^{n+2}$, thus, using the estimate given by (3) then Lemma 2.2 we show that

$$|G_n(z)| A \exp(Bp(2^n)) 2^{-n(m_1 + \cdots + m_{q_n})} \sum_{j=q_n+1}^{q_{n+1}} 2^{m_{q_n+1} + \cdots + m_j} \lesssim \exp(Bp(2^n)) 2^{-n(m_1 + \cdots + m_{q_n})}.$$

We readily obtain the desired estimate.

Now, when looking for a holomorphic interpolating function of the form $f = F + u$, we are led to the $\bar{\partial}$-problem

$$\bar{\partial}u = -\bar{\partial}F,$$

which we solve using Hörmander’s theorem [8, Theorem 4.2.1].
The interpolation problem is then reduced to the following lemma.

**Lemma 2.6.** There exists a subharmonic function $U$ such that, for certain constants $A, B > 0$,

(i) $U(z) \simeq m_j \log |z - z_j|^2$ near $z_j$,

(ii) $U(z) \leq Ap(z) + B$ for all $z \in \mathbb{C}$.

(iii) $|\partial F(z)|^2 e^{-U(z)} \leq Ae^{Bp(z)}$ for all $z \in \mathbb{C}$.

Admitting this lemma for a moment, we proceed with the proof of the theorem.

From Hörmander theorem \[8, Theorem 4.4.2\], we can find a $C^\infty$ function $u$ such that $\partial u = -\partial F$ and, denoting by $d\lambda$ the Lebesgue measure,

$$
\int_{\mathbb{C}} |u(w)|^2 e^{-U(w) - Ap(w)} (1 + |w|^2)^2 d\lambda(w) \leq \int_{\mathbb{C}} |\partial F|^2 e^{-U(w) - Ap(w)} d\lambda(w).
$$

By the property (a) of the weight $p$, there exists $C > 0$ such that

$$
\int_{\mathbb{C}} e^{-Cp(w)} d\lambda(w) < \infty.
$$

Thus, using (ii) of the lemma, and the estimate on $|\partial F(z)|^2$, we see that the last integral is convergent if $A$ is large enough. By condition (iii), near $z_j$, $e^{-U(w)}(w - z_j)^l$ is not summable for $0 \leq l \leq m_j - 1$, so we have necessarily $u^{(l)}(z_j) = 0$ for all $j$ and $0 \leq l \leq m_j - 1$ and consequently, $\frac{f^{(l)}(z_j)}{l!} = w_j^l$.

Now, we have to verify that $f$ has the desired growth.

By the mean value inequality,

$$
|f(z)| \lesssim \int_{D(z,1)} |f(w)| d\lambda(w) \lesssim \int_{D(z,1)} |F(w)| d\lambda(w) + \int_{D(z,1)} |u(w)| d\lambda(w).
$$

Let us estimate the two integrals that we denote by $I_1$ and $I_2$.

For $w \in D(z, 1)$,

$$
|F(w)| \lesssim e^{Bp(w)} \lesssim e^{Cp(z)}.
$$

Then,

$$
I_1 \lesssim e^{Cp(z)}
$$

To estimate $I_2$, we use Cauchy-Schwarz inequality,
where
\[ J_1 = \int_{D(z,1)} |u(w)|^2 e^{-U(w)-Bp(w)} \, d\lambda(w), \quad J_2 = \int_{D(z,1)} e^{U(w)+Bp(w)} \, d\lambda(w). \]

We have
\[ J_1 \lesssim \int_{\mathbb{C}} |u(w)|^2 e^{-U(w)-Bp(w)} \, d\lambda(w) \lesssim \int_{\mathbb{C}} \frac{|u(w)|^2 e^{-U(w)}}{1+|w|^2} \, d\lambda(w) < +\infty, \]
by property (a) of \( p \), if \( B > 0 \) is chosen big enough.

To estimate \( J_2 \), we use the condition (i) of the lemma and the property (b) of the weight \( p \). For \( w \in D(z,1) \),
\[ e^{U(w)+Bp(w)} \leq e^{Cp(w)} \lesssim e^{Ap(z)}. \]

We easily deduce that \( J_2 \lesssim e^{Ap(z)} \) and, finally, that \( f \in A_p(\mathbb{C}) \).

\[ \text{Proof of Lemma 2.6.} \]
For the sake of simplicity and up to a homotethy, we may assume that \( |z_k| > 2 \) for all \( z_k \neq 0 \). Besides, in the definition of the following functions \( V_n \), we will assume \( z_1 \neq 0 \), otherwise, we may add the term \( m_1 \ln|z| \) to each \( V_n \). We set
\[ V_n(z) = \sum_{0<|z_j| \leq 2^n} m_j \log \frac{|z-z_j|^2}{|z_j|^2}, \]
then
\[ V(z) = \sum_{n \geq 2} \rho_{n-2}(z)V_n(z). \]

First, we will show that \( V \) verifies (i), (ii) and (iii). Then, we will estimate \( \Delta V \) from below and add a correcting term \( W \). The subharmonic function \( U \) will be of the form \( V + W \).

(i) Let \( |z_k| \) be such that \( 2^{m-1} < |z_k| < 2^{m+1} \). For \( 2^{m-1} < |z| < 2^{m+1} \),
\[ V(z) = \rho_{m-1}(z)V_{m+1}(z) + \rho_{m}(z)V_{m+2}(z) + \rho_{m+1}(z)V_{m+3}(z). \]
As the \( \rho_n \)’s form a partition of the unity, it is clear that \( \Delta V(z) - m_k \ln|z-z_k|^2 \) is continuous in a neighborhood of \( z_k \).

Note that \( V \) is smooth on \( \{|z| \leq 2\} \) since we have assumed that all \( |z_j| > 2 \).
(ii) Let \( n \geq 2 \) and \( 2^{n-2} \leq |z| < 2^{n-1} \). then

\[
V(z) = \rho_{n-2}(z)V_n(z) + \rho_{n-1}(z)V_{n+1}(z).
\]

For all \( |z_j| < 2^n \), we have \( |z - z_j| < 2^{n+1} \). Thus,

\[
V_n(z) \leq \sum_{|z_j| \leq 2^n} m_j \log \frac{2^{n+1}}{|z_j|} \leq N(0, 2^{n+1}).
\]

Finally, we obtain that

\[
V(z) \leq N(0, 2^{n+1}) + N(0, 2^{n+2}) \lesssim p(2^n) \lesssim p(z)
\]

by condition (1) and property (c) of the weight.

(iii) We have

\[
-\frac{V(z)}{2} = \sum_{|z_j| \leq 2^n} m_j \log \frac{|z_j|}{|z - z_j|} + \rho_{n+1}(z) \sum_{2^n < |z_j| \leq 2^{n+1}} m_j \log \frac{|z_j|}{|z - z_j|}.
\]

Note that for all \( 2^n < |z_j| \leq 2^{n+1} \), we have \( |z - z_j| > 2^n - 2^{n-1} = 2^{n-1} \). We obtain

\[
-\frac{V(z)}{2} \leq \sum_{|z_j| \leq 2^n} m_j \log \frac{2^n}{|z - z_j|} + \ln 4 \sum_{2^n < |z_j| \leq 2^{n+1}} m_j
\]

\[
\leq \ln \left( 2^{n(m_1 + \cdots + m_n)} \prod_{j=1}^{n} |z - z_j|^{-m_j} \right) + \ln(A \exp(Bp(2^n)))
\]

for certain constants \( A, B > 0 \) using Lemma 2.2. Finally, combining this inequality with (iii) of Lemma 2.3 we obtain

\[
|\bar{\partial}F(z)| \exp(\frac{-V(z)}{2}) \lesssim \exp(Bp(2^n)) \lesssim \exp(Bp(z)).
\]

Now, in order to get a lower bound of the laplacian, we compute \( \Delta V(z) \):

\[
\Delta V = \sum_{n \geq 2} \rho_{n-2} \Delta V_n + 2 \text{Re} \left( \sum_{n} \bar{\partial} \rho_{n-2} \partial V_n \right) + \sum_{n \geq 2} \partial \bar{\partial} \rho_{n-2} V_n.
\]

The first sum is positive since every \( V_k \) is subharmonic.

Let us estimate the second and the third sums, that we will denote respectively by \( B(z) \) and \( C(z) \). For \( n \geq 2 \) and \( 2^{n-2} \leq |z| < 2^{n-1} \), since \( z \) is outside the supports of \( \bar{\partial} \chi_{n-3} \) and of \( \partial \chi_{n-1} \), we have
\[ B(z) = 2 \text{Re} \left[ \partial \bar{\partial} x_{n-2}(z) \partial (V_n(z) - V_{n+1}(z)) \right], \]

\[ C(z) = \partial \bar{\partial} x_{n-2}(z) (V_n(z) - V_{n+1}(z)). \]

\[ V_n(z) - V_{n+1}(z) = \sum_{2^n < |z_j| \leq 2^{n+1}} m_j \log \frac{|z - z_j|^2}{|z_j|^2}, \]

\[ \partial (V_n(z) - V_{n+1}(z)) = \sum_{2^n < |z_j| \leq 2^{n+1}} m_j \frac{1}{z - z_j}, \]

and

\[ |\partial \bar{\partial} x_{n-2}(z)| \lesssim \frac{1}{2^n}, \quad |\partial \bar{\partial} x_{n-2}(z)| \lesssim \frac{1}{2^{2n}}. \]

For \( z \) in the support of \( \partial \bar{\partial} x_{n-2} \), we have \( |z| \leq 2^{-n-1} \), and for \( 2^n \leq |z_j| < 2^{n+1}, 2^{n-1} \leq |z - z_j| \leq 2^{n+2} \).

Thus, we obtain that

\[ |\partial \bar{\partial} x_{n-2}(z) (V_{n+1}(z) - V_n(z))| \lesssim \frac{n(0, 2^{n+1}) - n(0, 2^n)}{2^{2n}}, \]

and

\[ |\partial \bar{\partial} x_{n-2}(z) \partial (V_{n+1}(z) - V_n(z))| \lesssim \frac{n(0, 2^{n+1}) - n(0, 2^n)}{2^{2n}}. \]

Finally,

\[ \Delta V(z) \gtrsim -\frac{n(0, 2^{n+1}) - n(0, 2^n)}{2^{2n}} \gtrsim \frac{n(0, 2^3|z|) - n(0, 2|z|)}{|z|^2}. \]

To construct the correcting term, \( W \), we begin by putting

\[ f(t) = \int_0^t n(0, s)ds, \quad g(t) = \int_0^t \frac{f(s)}{s^2} ds \text{ and } W(z) = g(2^3|z|). \]

The following inequalities are easy to see:

\[ f(t) \leq tn(0, t), \quad g(t) \leq \int_0^t \frac{n(0, s)}{s} ds = N(0, s). \]

Thus, by condition (1) and property (c),

\[ W(z) \leq N(0, 2^3|z|) \lesssim p(2^3z) \lesssim p(z) \]

Finally, to estimate the laplacian of \( W \), we will denote \( t = 2^3|z| \).

\[ \Delta W(z) = \frac{1}{t} g'(t) + g''(t) = \frac{1}{t^2} (f'(t) - \frac{f(t)}{t}). \]
\[ f(t) = \int_0^t n(0, s)ds = \int_0^{t/4} n(0, s)ds + \int_{t/4}^t n(0, s)ds \leq \frac{t}{4}n(0, \frac{t}{4}) + t(1 - \frac{1}{4})n(0, t). \]

Thus,
\[ f'(t) - \frac{f(t)}{t} = n(0, t) - \frac{f(t)}{t} \geq \frac{1}{4}(n(0, t) - n(0, \frac{t}{4})) \]

and
\[ \Delta W(z) \gtrsim \frac{n(0, 2^3|z|) - n(0, 2|z|)}{|z|^2}. \]

Now, the desired function will be of the form
\[ U(z) = V(z) + \alpha W(z), \]

where \( \alpha \) is a positive constant chosen big enough.

\[ \blacksquare \]

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