Spectra of alternating Hilbert operators

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Dedicated to Professor Toshikazu Sunada for his sixtieth birthday.

Abstract

Spectra of real alternating operators seem to be quite interesting from the viewpoint of explaining the Riemann Hypothesis for various zeta functions. Unfortunately we have not sufficient experiments concerning this theme. Necessary works would be to supply new examples of spectra related to zeros and poles of zeta functions. A century ago Hilbert (1907) considered a kind of operators representing quadratic forms of infinitely many variables. Demonstrating the calculation of spectra for alternating Hilbert operators we hope to present a novel scheme in this paper. Authors expect this study encourages experts for further studies.

1 Introduction

In 1907 Hilbert studied the alternating infinite matrix

\[ A = A_\infty = \left( \frac{1}{m-n} \right)_{m,n \geq 1} \]

as an interesting example relating to quadratic forms of infinitely many variables; see Weyl [W] (1908). Hilbert studied the symmetric infinite matrix

\[ \left( \frac{1}{m+n-1} \right)_{m,n \geq 1} \]

also. A few years later Schur [Sc] (1911) gave a good upper estimate \( \pi \) for their spectral radius.

We hope to report on our discovery of a periodic nature of the spectra of the finite segment

\[ A_N := \left( \frac{1}{m-n} \right)_{m,n=1,...,N} \]

(zero on the diagonal), as in the following conjecture. It seems that there is no literature concerning this theme in contrast to the symmetric case where we know many studies.

We make the following basic conjecture.

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Conjecture 1 Spect($A_N$) is asymptotically periodic in the following sense:
For $N$ even,

$$\text{Spect}(A_N) = \{ \pm i\lambda_1^{(N)}, \ldots, \pm i\lambda_{N/2}^{(N)} \}$$

with $0 < \lambda_1^{(N)} < \cdots < \lambda_{N/2}^{(N)}$ satisfying

$$\lambda_k^{(N)} \sim \frac{2\pi}{N} \left( k - \frac{1}{2} \right)$$
as $N \to \infty$.

For $N$ odd,

$$\text{Spect}(A_N) = \{0\} \cup \{ \pm i\lambda_1^{(N)}, \ldots, \pm i\lambda_{(N-1)/2}^{(N)} \}$$

with $0 < \lambda_1^{(N)} < \cdots < \lambda_{(N-1)/2}^{(N)}$ satisfying

$$\lambda_k^{(N)} \sim \frac{2\pi}{N} k$$
as $N \to \infty$.

The next conjecture is a quantum analogue ($q$-analogue at $q = \zeta_N$) of Conjecture 1.

Conjecture 2 Let

$$A_N^{\text{quant}} = \left( \frac{\sin \frac{\pi m}{N}}{\sin \frac{\pi}{N}(m-n)} \right)_{m,n=1,\ldots,N}.$$

Then

$$\text{Spect}(A_N^{\text{quant}}) = \begin{cases} 
\{ \pm i2(\sin \frac{\pi}{N})(k - \frac{1}{2}) \mid k = 1, \ldots, \frac{N}{2} \} & \text{for } N: \text{ even} \\
\{0\} \cup \{ \pm i2(\sin \frac{\pi}{N})k \mid k = 1, \ldots, \frac{N-1}{2} \} & \text{for } N: \text{ odd}.
\end{cases}$$

Note that Conjecture 2 might imply Conjecture 1. For an integer $n \in \mathbb{Z}$, a $q$-integer $[n] = [n]_q$ is defined to be $[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$. This is a polynomial in $q^{1/2}$ and the limit $\lim_{q \to 1}[n]_q = n$. In this sense, $[n]_q$ is understood to be a deformation (quantization) of an integer $n$. In our case, $q$ is taken to be the $N$-th primitive root of unity, $q = \exp(2\pi i/N)$. Then $[n] = \frac{\sin \frac{\pi n}{N}}{\sin \frac{\pi}{N}}$. A native deformation of $A_N = \left( \frac{1}{m-n} \right)$ will be $\left( \frac{\sin \frac{\pi n}{N}}{\sin \frac{\pi}{N}} \right)$, which is equal to $A_N^{\text{quant}}$.

We also consider an oscillatory version. For $0 \leq \theta \leq \frac{\pi}{2}$, let

$$A_N(\theta) := \left( \frac{\cos(m-n)\theta}{m-n} \right)_{m,n=1,\ldots,N}$$

(zero on diagonal), and

$$B_N(\theta) := \left( \frac{\sin(m-n)\theta}{m-n} \right)_{m,n=1,\ldots,N}$$

($\theta$ on diagonal).
Conjecture 3

(1) The set \( \text{Spect}(A_N(\theta)) \) splits into the following two subsets:

- \( \lfloor (1 - \frac{\theta}{\pi}) N \rfloor \) spectra, which are “major” periodic in the interval \([−i(\pi − \theta), i(\pi − \theta)]\).
- \( \lfloor \frac{\theta}{\pi} N \rfloor \) spectra, which are “minor” periodic in the interval \([−i\theta, i\theta]\).

(2) The set \( \text{Spect}(B_N(\theta)) \) splits into the following two subsets:

- \( \lfloor \frac{\theta}{\pi} N \rfloor \) spectra, which are “almost” \( \pi \).
- The remaining \( \lfloor (1 - \frac{\theta}{\pi}) N \rfloor \) spectra, which are “almost” zero.

First we notice elementary confirmations of Conjectures 1 – 3 concerning \( \text{trace}(A_N^2) \).

Theorem 1

\[
\lim_{N \to \infty} \frac{\text{trace}(A_N^2)}{N} = -\frac{\pi^2}{3}.
\]

Theorem 2

\[
\text{trace}(A_N^{\text{quant}})^2 = -\left(\sin \frac{\pi}{N}\right)^2 \frac{(N-1)N(N+1)}{3}.
\]

Theorem 3

(1) \[
\lim_{N \to \infty} \frac{\text{trace}(A_N(\theta)^2)}{N} = -\left(\frac{\pi^2}{3} + \theta^2 - \pi \theta\right).
\]

(2) \[
\lim_{N \to \infty} \frac{\text{trace}(B_N(\theta)^2)}{N} = \pi \theta.
\]

We remark that these Theorems are compatible with Conjectures.

Conjecture 1 ⇒ Theorem 1

\[
\text{trace}(A_N)^2 \sim -\left(\frac{\pi}{N}\right)^2 \sum_{k=1}^{N} (N-2k+1)^2 = -\left(\frac{\pi}{N}\right)^2 \frac{(N-1)N(N+1)}{3}.
\]

Conjecture 2 ⇒ Theorem 2

\[
\text{trace}(A_N^{\text{quant}})^2 = -\left(\sin \frac{\pi}{N}\right)^2 \sum_{k=1}^{N} (N-2k+1)^2 = -\left(\sin \frac{\pi}{N}\right)^2 \frac{(N-1)N(N+1)}{3}.
\]

Conjecture 3(1) ⇒ Theorem 3(1):

\[
\text{trace}(A_N(\theta)^2) \sim -2 \sum_{k=1}^{\lfloor (1-\theta/\pi)N/2 \rfloor} \left(\frac{(\pi-\theta)k}{(1-\theta/\pi)N/2}\right)^2 - 2 \sum_{l=1}^{\lfloor \theta/\pi N/2 \rfloor} \left(\frac{\theta}{((\theta/\pi)N/2)}\right)^2
\]

\[
\sim -\frac{(\pi-\theta)^2}{3\pi} N - \frac{\theta^3}{3\pi} N
\]

\[
= -\left(\frac{\pi^2}{3} + \theta^2 - \pi \theta\right) N.
\]

Conjecture 3(2) ⇒ Theorem 3(2):

\[
\text{trace}(B_N(\theta)^2) \sim \pi^2 \times \lfloor \frac{\theta}{\pi} N \rfloor \sim \pi \theta N.
\]

We report also the proof of Conjecture 2:
**Theorem 4**  Conjecture 2 is valid.

Closing this Introduction, we briefly explain a possible connection to zeta functions. For a prime \( p \), let

\[
A(p) = \left( \frac{1}{m \log p - n \log p} \right)_{m,n \geq 1} = \frac{1}{\log p} \left( \frac{1}{m - n} \right)_{m,n \geq 1}.
\]

Then Conjecture 1 implies

\[
\lim_{N \to \infty, N \text{ odd}} \text{Spect}(N A(p)^N) = \frac{2\pi i}{\log p} \mathbf{Z}.
\]

This coincides with the set of poles of the zeta function

\[
\zeta(s, F_p) = (1 - p^{-s})^{-1}
\]

of the finite field \( F_p \). This fact may indicate that zeros and poles of a zeta function

\[
Z(s) = \prod_P (1 - N(P)^{-s})^{-1}
\]

are explained by the spectra of the alternating matrix

\[
A = \left( \left( \frac{1}{m \log N(P) - n \log N(Q)} \right)_{m,n \geq 1} \right)_{P,Q}.
\]

# 2 Proof of Theorems 1 to 3

## 2.1 Proof of Theorem 1

\[
\text{trace}(A^2_N) = -2 \sum_{1 \leq m < n \leq N} \frac{1}{(n-m)^2} = -2 \sum_{k=1}^{N-1} \frac{N - k}{k^2} = -2N \sum_{k=1}^{N-1} \frac{1}{k^2} + 2 \sum_{k=1}^{N-1} \frac{1}{k} \sim -N \frac{\pi^2}{3}.
\]
2.2 Proof of Theorem 2

\[
\text{trace}(A_{N}^{\text{quant}}^2) = - \sum_{m,n=1}^{N} \frac{\sin^2 \frac{\pi}{N} (m-n)}{\sin^2 \frac{\pi}{N}}
\]
\[
= -2 \sin^2 \frac{\pi}{N} \sum_{k=1}^{N-1} \frac{N-k}{N} \frac{k}{\sin^2 \frac{\pi}{N} k}
\]
\[
= - \sin^2 \frac{\pi}{N} \sum_{k=1}^{N-1} \left( \frac{N-k}{\sin^2 \frac{\pi}{N} k} + \frac{k}{\sin^2 \frac{\pi}{N} k} \right)
\]
\[
= -N \sin^2 \frac{\pi}{N} \sum_{k=1}^{N-1} \frac{1}{\sin^2 \frac{\pi}{N} k}
\]
\[
= -N \sin^2 \frac{\pi}{N} \frac{N^2-1}{3}
\]
\[
= - \sin^2 \frac{\pi}{N} \frac{(N-1)N(N+1)}{3}.
\]

2.3 Proof of Theorem 3(1)

\[
\text{trace}(A_N(\theta)^2) = -2 \sum_{1 \leq m < n \leq N} \frac{\cos^2(n-m)\theta}{(n-m)^2}
\]
\[
= -2 \sum_{k=1}^{N-1} \frac{\cos^2(k\theta)}{k^2} (N-k)
\]
\[
= -2 \left\{ N \sum_{k=1}^{N-1} \frac{\cos^2(k\theta)}{k^2} - \sum_{k=1}^{N-1} \frac{\cos^2(k\theta)}{k} \right\}.
\]

Here from

\[
\sum_{k=1}^{N-1} \frac{\cos^2(k\theta)}{k} = O(\log N),
\]

\[
\lim_{N \to \infty} \frac{\text{trace}(A_N(\theta)^2)}{N} = -2 \sum_{k=1}^{\infty} \frac{\cos^2(k\theta)}{k^2}
\]
\[
= - \sum_{k=1}^{\infty} \frac{1 + \cos(2k\theta)}{k^2}
\]
\[
= - \left\{ \frac{\pi^2}{6} + \left( \frac{\pi^2}{6} - \pi \theta + \theta^2 \right) \right\}
\]
\[
= - \left( \frac{\pi^2}{3} + \theta^2 - \pi \theta \right).
\]
2.4 Proof of Theorem 3(2)

\[
\text{trace}(B_N(\theta)^2) - \text{trace}(A_N(\theta)^2) = 2 \sum_{1 \leq m < n \leq N} \frac{1}{(m-n)^2} + \theta^2 N
\]

\[
= 2 \sum_{k=1}^{N-1} \frac{N-k}{k^2} + \theta^2 N.
\]

This shows

\[
\lim_{N \to \infty} \frac{\text{trace}(B_N(\theta)^2) - \text{trace}(A_N(\theta)^2)}{N} = 2 \sum_{k=1}^{\infty} \frac{1}{k^2} + \theta^2 = \frac{\pi^2}{3} + \theta^2.
\]

3 Proof of Theorem 4

We set \(\zeta_{2N} = \exp(\pi i/N)\) and \(\zeta_{N} = \zeta_{2N}^2 = \exp(2\pi i/N)\). For \(k = 1, 2, \ldots, N\), we have

\[
\sum_{1 \leq n \leq N, n \neq m} \frac{\sin \frac{\pi}{N} n(2k-1)}{\sin \frac{\pi}{N} (m-n)} \zeta_{2N}^{n(2k-1)} = \sum_{1 \leq n \leq N, n \neq m} \frac{(\zeta_{2N} - \zeta_{2N}^{-1})\zeta_{2N}^m}{\zeta_{2N}^n - \zeta_{2N}^{n-m}} \zeta_{2N}^{kn}
\]

\[
= (\zeta_{2N} - \zeta_{2N}^{-1}) \sum_{1 \leq n \leq N, n \neq m} \frac{\zeta_{2N}^m}{\zeta_{2N}^n - \zeta_{2N}^{n-m}} \zeta_{2N}^{kn}
\]

\[
= (\zeta_{2N} - \zeta_{2N}^{-1}) \sum_{1 \leq n \leq N-1} \frac{\zeta_{2N}^m}{1 - \zeta_{2N}^n} \zeta_{2N}^{kn}
\]

\[
= (\zeta_{2N} - \zeta_{2N}^{-1}) \sum_{1 \leq n \leq N-1} \frac{\zeta_{2N}^m}{1 - \zeta_{2N}^n} (k - \frac{N+1}{2}).
\]

Note that the last equality follows from

\[
\sum_{n=1}^{N-1} \frac{1}{1 - \zeta_{2N}^n} = \frac{N-1}{2},
\]

and

\[
\sum_{n=1}^{N-1} \frac{1 - \zeta_{2N}^{kn}}{1 - \zeta_{N}^n} = N - k \quad \text{for } k = 1, 2, \ldots, N - 1.
\]

We define an invertible matrix \(P_N \in GL(N, \mathbb{C})\) and a diagonal matrix \(D_N \in M(N, \mathbb{C})\) by

\[
P_N = \left( \zeta_{2N}^{m(2n-1)} \right)_{m,n=1,\ldots,N}, \quad D_N = \left( 2i(\sin \frac{\pi}{N})(n - \frac{N+1}{2}) \delta_{mn} \right)_{m,n=1,\ldots,N}
\]

then we have an equality \(A_N^{\text{quant}} P_N = P_N D_N\). This proves

\[
\text{Spect}(A_N^{\text{quant}}) = \left\{ 2i(\sin \frac{\pi}{N})(k - \frac{N+1}{2}) \mid k = 1, \ldots, N \right\}.
\]
4 Discussion

4.1 Szegö’s Theorem and Conjecture 3

Szegö [Sz] (1920) proved the uniform distribution property of the eigenvalues \( \lambda_k^{(N)} \) \( (k = 1, \ldots, N) \) of the hermitian Toeplitz operator

\[
T_N = \begin{pmatrix}
  c_0 & c_{-1} & \cdots & c_{-(N-1)} \\
  c_1 & c_0 & & \\
  \vdots & \ddots & \ddots & \vdots \\
  c_{N-1} & \cdots & c_1 & c_0
\end{pmatrix}
\]

in the following form:

\[
\lim_{N \to \infty} \frac{F(\lambda_1^{(N)}) + \cdots + F(\lambda_N^{(N)})}{N} = \frac{1}{2\pi} \int_0^{2\pi} F(f(x)) \, dx,
\]

where \( F \) is a suitable test function, and \( f(x) = \sum_n c_n e^{inx} \). We explain below that Conjecture 3 is compatible with Szegö’s theorem.

4.1.1 For \( A_N(\theta) \)

We check the relation with Conjecture 3(1) and Szegö’s theorem for a hermitian Toeplitz matrix \( iA_N(\theta) = (c_{m-n})_{m,n=1,\ldots,N} \). In this case the entries \( c_n = i \cos n\theta/n \), and then

\[
f(x) = \begin{cases} 
  x & \text{for } 0 < x < \theta \\
  x - \pi & \text{for } \theta < x < 2\pi - \theta \\
  x - 2\pi & \text{for } 2\pi - \theta < x < 2\pi.
\end{cases}
\]

This shows

\[
\lim_{N \to \infty} \frac{F(\lambda_1^{(N)}) + \cdots + F(\lambda_N^{(N)})}{N} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(m(x)) \, dx,
\]

where

\[
m(x) = \begin{cases} 
  2 & \text{for } |x| < \theta \\
  1 & \text{for } \theta < |x| < \pi - \theta \\
  0 & \text{for } \pi - \theta < |x| < \pi.
\end{cases}
\]

This is compatible with Conjecture 3(1).

4.1.2 For \( B_N(\theta) \)

In this case the hermitian Toeplitz matrix is \( B_N(\theta) = (c_{m-n})_{m,n=1,\ldots,N} \) with \( c_n = \sin n\theta/n \). Using the formula

\[
\sum_{n=1}^\infty \frac{\sin nx}{n} = \frac{\pi - x}{2} \quad \text{for } 0 < x < 2\pi,
\]

(1)
we have
\[ f(x) = \begin{cases} 
\pi & \text{for } 0 < x < \theta \\
0 & \text{for } \theta < x < 2\pi - \theta \\
\pi & \text{for } 2\pi - \theta < x < 2\pi.
\end{cases} \]
This shows
\[
\lim_{N \to \infty} \frac{F(\lambda_1(N)) + \cdots + F(\lambda_N)}{N} = \frac{\theta}{\pi} F(\pi) + \frac{\pi - \theta}{\pi} F(0).
\]
This is compatible with Conjecture 3(2).

4.2 Expected relation to zeta functions

We explain a possible way to reach
\[ A = \left( \frac{1}{m \log N(P) - n \log N(Q)} \right)_{m,n \geq 1} \]
from a zeta function
\[ Z(s) = \prod_p (1 - N(p)^{-s})^{-1}. \]
As a typical example of a zeta function we take up the Riemann zeta function \( \zeta(s) \). Suppose that we have a determinant expression
\[ \hat{\zeta}(s) \equiv \frac{\text{Det}(A - (s - \frac{1}{2}))}{s(s-1)} \]
for the completed Riemann zeta function
\[ \hat{\zeta}(s) = \zeta(s)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) \]
with a real alternating matrix \( A \). Then we will have a proof of Riemann Hypothesis as suggested by Hilbert and Polya around 1915.

Now, on the other hand we have
\[ \hat{\zeta}(s) = \frac{e^{as+b}}{s(s-1)} \prod_{\rho} \left(1 - s - \frac{1}{s-\frac{1}{\rho}}\right) e^{s/\rho}, \]
where \( \rho \) runs over essential zeros. The logarithmic derivation gives
\[ \frac{\zeta'(s)}{\zeta(s)} - \frac{1}{2} \log \pi + \frac{1}{2} \Gamma'(s) = a - s - \frac{1}{s-1} + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right). \]
Hence we have
\[ \left( \frac{\zeta'(s)}{\zeta(s)} \right)' + \frac{1}{4} \left( \Gamma'(s) \right)' = \frac{1}{s^2} + \frac{1}{(s-1)^2} - \sum_{\rho} \frac{1}{(s-\rho)^2}. \]
In particular,
\[ \sum_{\rho} \frac{1}{(\rho - \frac{1}{2})^2} = - \left( \frac{\zeta'(s)}{\zeta(s)} \right)' - \frac{1}{4} \left( \Gamma'(s) \right)' + \frac{1}{4} + 8. \]
Hence, elementary calculation shows that
\[
\left( \Gamma \frac{1}{2} \right)^{'} \left( \frac{1}{4} \right) = \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{4})^2} = 16 \sum_{n=0}^{\infty} \frac{1}{(4n + 1)^2} = 16 \cdot \frac{1}{2} \left\{ (1 - 2^{-s})\zeta(2) + L(2, \chi_{-4}) \right\} = \pi^2 + 8G,
\]
where
\[
G = L(2, \chi_{-4}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^2}
\]
is the Catalan constant. Thus
\[
\sum_{\rho} \frac{1}{(\rho - \frac{1}{2})^2} = - \left( \frac{\zeta'}{\zeta} \right)^{'} \left( \frac{1}{2} \right) - \frac{\pi^2}{4} - 2G + 8.
\]
Here it might be suggestive to write this as
\[
\sum_{\rho} \frac{1}{(\rho - \frac{1}{2})^{2k}} = - \frac{1}{(2k - 1)!} \sum_{p,m \geq 1} (\log p)^2 mp^{-ms} - \frac{\pi^2}{4} - 2G + 8
\]
since we see
\[
\left( \frac{\zeta'}{\zeta} \right)^{'}(s) = \sum_{p,m} (\log p)^2 mp^{-ms}
\]
at least for \( \Re(s) > 1 \). Of course we have similarly the formula
\[
\sum_{\rho} \frac{1}{(\rho - \frac{1}{2})^{2k}} = - \frac{1}{(2k - 1)!} \sum_{p,m} (\log p)^2 m^{2k-1}p^{-ms} + \alpha_k
\]
with
\[
\alpha_k = - \frac{1}{2}(2^{2k} - 1)\zeta(2k) - 2^{2k-1}L(2k, \chi_{-4}) + 2^{2k+1}
\]
for \( k = 1, 2, 3, \ldots \). On the other hand, this should be equal to the trace
\[
\text{Trace}(A^{-2k})
\]
because of our determinant expression. Thus we should have
\[
\text{Trace}(A^{-2k}) = \frac{1}{(2k - 1)!} \sum_{p,m} (\log p)^2 m^{2k-1}p^{-ms} + \alpha_k
\]
for \( k = 1, 2, 3, \ldots \). [And moreover we see easily that this trace formula is essentially equivalent to the determinant expression supposed first.]

Thus, we reach to the basis problem: determine the index set (basic set) \( X \)
representing
\[
A = (a(i, j))_{i,j \in X}.
\]
Since, formally we may write
\[ A^{-2k} = (a_{-2k}(i, j))_{i,j \in X} \]
and
\[ \text{Trace}(A^{-2k}) = \sum_{i \in X} a_{-2k}(i, i), \]
we would have
\[ \sum_{i \in X} a_{-2k}(i, i) = \frac{1}{(2k-1)!} \sum_{p^m} (\log p)^{2k} m^{2k-1} p^{-\frac{m}{2}} + \alpha_k. \]

Hence, we reach to an obvious suggestion
\[ X = \{ p^m \mid p : \text{prime}, m \geq 1 \}, \]
and a simple
\[ A = \left( \frac{1}{\log(p^m) - \log(q^n)} \right)_{p^m,q^n \in X} \]
in considering the case of \( \zeta(s, \mathbb{F}_p) \). Similarly we might expect that
\[ A = \left( \frac{1}{\log(N(P)^m) - \log(N(Q)^n)} \right)_{p^m,Q^n} \]
would explain central essential zeros and poles of
\[ Z(s) = \prod_p (1 - N(P)^{-s})^{-1} \]
in the sense
\[ Z(s) \cong \text{Det} \left( A - \left( s - \frac{\text{dim}}{2} \right) \right)^{(-1)^{\text{dim}+1}}, \]
where \( \text{dim} \) is the Kronecker dimension, for example:
\[ \text{dim} = \begin{cases} 1 & \text{for } \zeta(s) \\ 0 & \text{for } \zeta(s, \mathbb{F}_p). \end{cases} \]

4.3 Symmetric case

In the case of the symmetric Hilbert matrix
\[ S_N = \left( \frac{1}{m+n-1} \right)_{m,n=1,...,N} \]
the spectra \( \text{Spect}(S_N) \) is contained in \((0, \pi)\) and is not periodic. But curiously enough a quantization
\[ S_N^{\text{quant}} = \left( \frac{\sin \frac{\pi}{N}}{\sin \frac{\pi}{N}(m+n-1)} \right)_{m,n=1,...,N} \]
(zero on the anti-diagonal) has the periodic spectra. In fact we have
**Theorem 5**

\[ \text{Spect}(S_N^{\text{quant}}) = \{(N-1)\sin \frac{\pi}{N}, (N-3)\sin \frac{\pi}{N}, \ldots, (3-N)\sin \frac{\pi}{N}, (1-N)\sin \frac{\pi}{N}\}. \]

Proof. Let \( C_N^{\text{quant}} = \left( \frac{1}{\sin \frac{\pi}{N}(m+n-1)} \right)_{m,n=1,\ldots,N} \) (zero on the diagonal). We will prove

\[ \text{Spect}(C_N^{\text{quant}}) = \{ n \in \mathbb{Z} \mid n - N \text{ is odd, and } |n| < N \} = \{ N-1, N-3, \ldots, 3-N, 1-N \}, \]

which is the set of weights of the irreducible \( N \)-dimensional representation of \( SL_2 \).

We define a matrix

\[ Q_N = \left( \cos \left( \frac{\pi}{2N} (2m-1)(2n-1) - \frac{\pi}{4} \right) \right)_{m,n=1,\ldots,N} \]

and a diagonal matrix

\[ D_N' = ((N+1-2m)\delta_{mn})_{m,n=1,\ldots,N}. \]

We claim that \( Q_N \) is invertible and we have an equality \( C_N^{\text{quant}} Q_N = Q_N D_N' \).

We use

\[ \sum_{n=1}^{N-1} \frac{\cos(\frac{\pi}{N} n(2k-1))}{\sin \frac{\pi}{N} n} = 0, \]
\[ \sum_{n=1}^{N-1} \frac{\sin(\frac{\pi}{N} n(2k-1))}{\sin \frac{\pi}{N} n} = N + 1 - 2k \]

for \( k = 1, 2, \ldots, N \).

Lastly, we notice that \( \det Q_N = \pm (N/2)^{N/2} \), where the sign – for \( N+1 \in 4\mathbb{Z} \) and the sign + for otherwise.

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