Hausdorff dimension,  
Mean quadratic variation of  
infinite self-similar measures*  

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Abstract: Under weaker condition than that of Riedi & Mandelbrot, the Hausdorff  
(and Hausdorff-Besicovitch) dimension of infinite self-similar set $K \subset \mathbb{R}^d$ which is the  
invariant compact set of infinite contractive similarities $\{S_j(x) = \rho_j R_j x + b_j\}_{j \in \mathbb{N}}$ ($0 < \rho_j < 1$, $b_j \in \mathbb{R}^d$, $R_j$ orthogonal) satisfying open set condition is obtained. It is proved  
(under some additional hypotheses) that the $\beta$-mean quadratic variation of infinite self-  
similar measure is of asymptotic property (as $t \to 0$).

Key Words: Hausdorff (and Hausdorff-Besicovitch) dimension, infinite self-similar  
set/measure, mean quadratic variation.

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1 Introduction

In this paper, we denote $\mathbb{R}^d$ the $d$-dimensional Euclidean space, $\mathbb{N}$ the set of natural numbers and $\mathbb{Z}$ the set of integer numbers.

For given finite contractive similarities $\{S_j(x) = \rho_j R_j x + b_j\}_{j=1}^m$ of $\mathbb{R}^d$, where $0 < \rho_j < 1$, $b_j \in \mathbb{R}^d$, $R_j$ orthogonal, J.E.Hutchinson [1] proved that there exists unique compact set $K_1$ satisfying

$$K_1 = \bigcup_{j=1}^m S_j(K_1).$$

$K_1$ is called self-similar set. If there exists an open set $O_1$ satisfying $S_j(O_1) \subset O_1$ and $S_i(O_1) \cap S_j(O_1) = \emptyset$ ($i \neq j$), we call that $\{S_j\}_{j=1}^m$ satisfy open set condition. We call that they satisfy strong open set condition if the sets $S_j(O)$ are disjoint. Then

**Theorem A** (Hutchinson) If $\{S_j\}_{j=1}^m$ satisfy open set condition, then the Hausdorff dimension $s'$ of $K_1$ is the unique solution of the equation $\sum_{j=1}^m \rho_j s' = 1$.

In [1], he also proved that for given probability vector $P = (P_1, P_2, \cdots, P_m)$ satisfying $\sum_{j=1}^m P_j = 1$, there exists unique probability measure $\mu_1$ on $\mathbb{R}^d$ satisfying

$$\mu_1(\cdot) = \sum_{j=1}^m P_j \mu_1(S_j(\cdot))$$

and the support set of $\mu_1$ is $K_1$. $\mu_1$ is called self-similar measure and $\{P_j\}_{j=1}^m$ is called weights of $\mu_1$.

Ka-Sing Lau and Jian-rong Wang [2], and R.S.Strichartz [3-7] have done much study on Fourier analysis of self-similar measure. R.S.Strichartz in [3] (or [7]) discussed many fractal numbers and set of integer numbers.

In [1], he also proved that for given probability vector $P = (P_1, P_2, \cdots, P_m)$ satisfying $\sum_{j=1}^m P_j = 1$, there exists unique probability measure $\mu_1$ on $\mathbb{R}^d$ satisfying

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It is proved in [5] that if $\{S_j\}_{j=1}^m$ satisfies the strong open set condition, then for the self-similar measure $\mu$ defined by natural weights (i.e. $P_j = \rho_j^{\beta'}$, $\beta' = s'$)

$$\frac{1}{r^{d-\beta'}} \int_{|x| \leq r} |(\mu f)(x)|^2 dx = q(r) \int |f|^2 d\mu + E(r) \quad \forall f \in L^2(d\mu),$$

(*)

where $E(r) \rightarrow 0$ as $r \rightarrow +\infty$, and $q(r)$ is a multiplicative periodic function or a positive constant.

Let $\mu$ be a $\sigma$-finite measure on $\mathbb{R}^d$, for $0 \leq \alpha \leq d$, let

$$V_\alpha(t; \mu) = \frac{1}{t^{d+\alpha}} \int_{\mathbb{R}^d} |\mu(B_t(x))|^2 dx,$$
where $B_t(x)$ is the ball of radius $t$, centered at $x$. We will call $\limsup_{t\to 0} V_\alpha(t; \mu)$ the 
upper $\alpha$-mean quadratic variation (m.q.v.) of $\mu$, and simply call it $\alpha$-m.q.v. if the limit 
exists.

If $\mu$ is a self-similar measure on $\mathbb{R}^d$, Ka-sing Lau and Jian-rong Wang [2] proved the 
following two Theorems

**Theorem B** ([2]) Under some additional conditions, we have

$$
\lim_{t \to 0} \left| V_{\beta'}(t; \mu) - p(t) \right| = 0.
$$

where $p(t)$ is a multiplicative periodic function or a positive constant and $\beta'$ is defined as 
above.

**Theorem C** ([2]) If the self-similar measure $\mu$ defined by natural weights (i.e. $P_j = \rho_j^{\beta'}$, $\beta' = s'$), under some additional hypotheses

$$
\lim_{t \to 0} \frac{1}{t^{d+\beta'}} \int_{\mathbb{R}^d} |\mu_f(B_t(x))|^2 dx - p(t) \int |f|^2 d\mu = 0 \quad \text{for} \quad \forall f \in L^2(d\mu),
$$

where $p(t)$ is the function in Theorem B.

R.H.Riedi and B.B.Mandelbrot [8] introduced infinite self-similar sets and infinite self-similar 
measures on $\mathbb{R}^d$ (definitions see later of this paper), discussed multifractal formalism for infinite self-similar measures and the Hausdorff dimension of infinite self-similar sets (under some additional conditions). In this paper, under weaker condition than that of Riedi & Mandelbrot, we extend Theorem A to the infinite self-similar case. If $\mu$ is infinite self-similar measure and the equation $\sum_{j=1}^{\infty} P_j \rho_j^{-\beta} = 1$ has finite solution $\beta$, then under some additional hypotheses, R.S.Strichartz [5] obtained the asymptotic property of function $H(r)$ and conclusion (*). In this paper, we also extend Theorem B,C to the infinite self-similar case.

2 Hausdorff (and Hausdorff-Besicovitch) dimension of infinite self-similar set.

For given infinite contractive similarities $\{S_j(x) = \rho_j R_j x + b_j\}_{j \in \mathbb{N}}$ of $\mathbb{R}^d$, where $0 < \rho_j < 1$, $b_j \in \mathbb{R}^d$, $R_j$ orthogonal, from [8], there exists unique compact set $K$ satisfying

$$
K = \bigcup_{j=1}^{\infty} S_j(K).
$$

$K$ is called infinite self-similar set. $K$ can be constructed as following. Let $E_0 \subset \mathbb{R}^d$ be 
a compact set, denote $E_{j_1\cdots j_k} = S_{j_1} \circ \cdots \circ S_{j_k}(E_0)$, then

$$
K = \bigcap_{k=0}^{\infty} \bigcup_{j_1,\cdots,j_k \in \mathbb{N}} E_{j_1\cdots j_k}.
$$

For given probability sequence $(P_1, P_2, \cdots)$ with $\sum_{j=1}^{\infty} P_j = 1$, from [8], there exists unique 
probability measure $\mu$ on $\mathbb{R}^d$ satisfying

$$
\mu(\cdot) = \sum_{j=1}^{\infty} P_j \mu(S_j(\cdot)).
$$
We call $\mu$ infinite self-similar measure and $\{P_j\}_{j=1}^\infty$ weights of $\mu$. Its support set is $K$.

**Definition 1** We call $\{S_j(x)\}_{j \in \mathbb{N}}$ satisfying open set condition if there exists a bounded open set $O \subset \mathbb{R}^d$ such that $S_j(O) \subset O$ and $S_i(O) \cap S_j(O) = \emptyset$ ($i \neq j$).

For any subset $A \subset \mathbb{R}^d$ and $0 \leq s < \infty$, let $\mathcal{M}_s^\delta(A) = \inf \sum_{i=1}^\infty |A_i|^s$, where $A = \cup_{i=1}^\infty A_i$ is a countable decomposition of $A$ into subsets of diameter $|A_i| < \delta$ ($>0$). We set $|A_i|^0 = 0$ if $A_i$ is empty and $|A_i|^0 = 1$ otherwise. The $s$-dimensional measure of $A$ is defined to be

$$\mathcal{M}_s^\delta(A) = \sup_{\delta > 0} \mathcal{M}_s^\delta(A).$$

The Hausdorff-Besicovitch dimension of $A$ is

$$\dim_M(A) = \sup\{0 \leq s < \infty : \mathcal{M}_s^\infty(A) > 0\}.$$

**Remark:** It is easy to see that in the definition of $\mathcal{M}_s^\delta(A)$, we can replace $|A_i|$ by $|\overline{A_i}|$. From the definition of fractal dimension, $\dim_H(A)$, we can see that

$$\dim_H(A) \leq \dim_M(A). \quad (1)$$

**Theorem 1** If the equation $\sum_{j=1}^\infty \rho_j^s = 1$ has finite solution $s$, and $\{S_j\}_{j=1}^\infty$ satisfy open set condition, $K$ is the infinite self-similar set, then the Hausdorff-Besicovitch dimension $\dim_M(K)$ and Hausdorff dimension $\dim_H(K)$ of $K$ is $s$.

**Remark.** Our condition is weaker than Riedi & Mandelbrot’s [8] condition: there exist numbers $r$, $R$ such that $-\infty < \log r \leq (1/j) \log \rho_j \leq \log R < 0 \forall j$.

**Proof of Theorem 1** To get the upper bound. Let $K = \cup_{i=1}^\infty A_i$ be any decomposition of $K$ into subsets of diameter $< \delta$, then a new decomposition is provided by $K = \cup_{i=1}^\infty \cup_{j=1}^\infty A_{ij}$, where $A_{ij} = \varphi_j(A_i)$. Because

$$\sum_{i=1}^\infty \sum_{j=1}^\infty |A_{ij}|^s \leq \sum_{i=1}^\infty \sum_{j=1}^\infty |\rho_j|^s |A_i|^s \leq \sum_{j=1}^\infty \rho_j^s \sum_{i=1}^\infty |A_i|^s,$$

it follows that whenever $\sum_{j=1}^\infty \rho_j^s < 1$ we must have $\mathcal{M}_s^\delta(K) = 0$, then $\mathcal{M}_s^\infty(K) = 0$. As $\dim_M(K) = \inf\{s : \mathcal{M}_s^\infty(K) = 0\}$, hence $\dim_M(K) \leq s$ where $\sum_{j=1}^\infty \rho_j^s = 1$. From (1), we have $\dim_H(K) \leq s$.

To get the lower bound. We let $K^{(m)}$ be the self-similar set generated by $\{S_j\}_{j=1}^m$, then from Theorem 8 of ref.[11], we have

$$\dim_M(K^{(m)}) \geq \min\{d, s^{(m)}\}, \quad (2)$$

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where $s^{(m)}$ is the positive solution of $\sum_{j=1}^{m} \rho_j^{s^{(m)}} = 1$. Using Theorem 4.13 of ref.[10], similar to the proof of Theorem 8 of ref.[11], we can obtain
\[
\dim_H(K^{(m)}) \geq \min\{d, s^{(m)}\}. \tag{3}
\]
Then from Lemma 8 of ref.[8], we have $\lim_{m \to \infty} s^{(m)} = s$, where $\sum_{j=1}^{\infty} \rho_j^s = 1$. Since for any $m$, $K^{(m)} \subset K$, we have $\dim_M(K) \geq \dim_M(K^{(m)})$ and $\dim_H(K) \geq \dim_H(K^{(m)})$. From open set condition, we have $s < d$, then from (2) and (3), we have
\[
\dim_M(K) \geq s^{(m)} \tag{4}
\]
and
\[
\dim_H(K) \geq s^{(m)}. \tag{5}
\]
Take limit from (4) and (5), we have $\dim_M(K) \geq s$ and $\dim_H(K) \geq s$. 

The method used in proof of Theorem 1 can be used to estimate the Hausdorff (and Hausdorff-Besicovitch) dimension of the limit set of infinite non-similar contractive maps.

**Corollary 1** Let $\{\varphi_j\}_{j=1}^{\infty}$ be infinite contractive maps with
\[|\varphi_j(x) - \varphi_j(y)| \leq c_j|x - y|, \quad x, y \in \mathbb{R}^d, \quad j = 1, 2, \ldots,\]
and satisfying open set condition, and denote $E$ their contractive-invariant set. If the equation $\sum_{j=1}^{\infty} c_j^u = 1$ has finite solution $u$, then $\dim_H(E) \leq \dim_M(E) \leq u$.

**Corollary 2** Let $\{\varphi_j\}_{j=1}^{\infty}$ be infinite contractive maps with
\[|\varphi_j(x) - \varphi_j(y)| \geq b_j|x - y|, \quad x, y \in \mathbb{R}^d, \quad j = 1, 2, \ldots,\]
and satisfying open set condition, and denote $E$ their contractive-invariant set. If the equation $\sum_{j=1}^{\infty} b_j^l = 1$ has finite solution $l$, then $\dim_M(E) \geq \dim_H(E) \geq \min\{d, l\}$.

Proof. Since $\{\varphi_j\}$ are non-similar maps, we can not obtain $l^{(m)} \leq d$ from open set condition, where $l^{(m)}$ satisfies $\sum_{j=1}^{m} b_j^{l^{(m)}} = 1$. then similar to proof of Theorem 1, this conclusion holds. 

## 3 Mean quadratic variations of infinite self-similar measures.

We define
\[H(r) = \frac{1}{r^{d-\beta}} \int_{|x| \leq r} |F(x)|^2dx,\]
where $F(x)$ is the Fourier transform of $\mu$.

If $\mu$ is a Borel measure on $\mathbb{R}^d$, for every $\mu$-measurable function $f$, we use $\mu_f$ to denote the measure $\mu_f(E) = \int_E f d\mu$ for any Borel subset $E$. 

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Definition 2. If in addition to the definition of open set condition, the sets $S_j(O)$ are mutually disjoint and $O$ intersects $K$, we call $\{S_j\}_{j \in \mathbb{N}}$ satisfy strong open set condition.

We assume $\{S_j\}_{j=1}^\infty$ satisfy strong open set condition. Let $d_{jk}$ denote the distance between $S_j(O)$ and $S_k(O)$ which is positive for $j \neq k$ by strong open set condition. We assume

$$\sum_{j \neq k} P_jP_kd_{jk}^{-\beta} < \infty. \tag{6}$$

Denote $q(\lambda) = \sum_{\rho_j \leq \lambda} P_j^2 \rho_j^{-\beta}$, we assume

$$q(\varepsilon \lambda) \leq \delta q(\lambda) \tag{7}$$

for some $0 < \varepsilon < 1$ and $0 < \delta < 1$.

Under the conditions (6) and (7), R.S. Strichartz [5] (P357-P358) obtained the asymptotic property (as $r \to +\infty$) of the function $H(r)$ and conclusion (*) for infinite self-similar measures.

We use $J = (j_1, j_2, \ldots, j_k)$ to denote the multi-index, $|J| = k$ its length, and $\Lambda$ the set of all such multi-indices, where $j_i \in \mathbb{N}, \ i = 1, \ldots, k$ and $k \in \mathbb{N}$. We set

$$P_J = P_{j_1}P_{j_2}\cdots P_{j_k}, \quad \rho_J = \rho_{j_1}\cdots \rho_{j_k}, \quad E_J = E_{j_1j_2\cdots j_k}$$

For any $0 < t < 1$, we denote

$$\Lambda(t) = \{J \in \Lambda : \rho_J = \sup \rho_{J'}, \ \rho_{J'} < t\},$$

and for fixed parameter $\varepsilon$ (given in condition (7)), we denote

$$\Lambda_1(t) = \{J \in \Lambda(t) : \rho_J \geq \varepsilon t\}.$$

Then we have

Theorem 2 Let $\mu$ be infinite self-similar measure, we assume that the condition (7) holds, then $V_\beta(t; \mu)$ is bounded below by a positive constant on $0 < t \leq 1$.

Proof. Since $\sum_{j=1}^\infty P_j^2 \rho_j^{-\beta} = 1$, then $\sum_{J \in \Lambda(t)} P_J^2 \rho_J^{-\beta} = 1$. When $J \in \Lambda_1(t)$, we have $\varepsilon t \leq \rho_J < t$. Hence

$$t^{-\beta} < \rho_J^{-\beta} \leq (\varepsilon t)^{-\beta}.$$

From the condition (7) and similar to ref.[5](P358), we can prove

$$\sum_{J \in \Lambda_1(t)} P_J^2 \rho_J^{-\beta} \geq (\delta^{-1} - 1) \sum_{J \in \Lambda(t)} P_J^2 \rho_J^{-\beta}.$$

Hence

$$\begin{align*}
(\delta^{-1} - 1) &= (\delta^{-1} - 1) \sum_{J \in \Lambda(t)} P_J^2 \rho_J^{-\beta} \leq \sum_{J \in \Lambda_1(t)} P_J^2 \rho_J^{-\beta} \\
&\leq \sum_{J \in \Lambda_1(t)} P_J^2 (\varepsilon t)^{-\beta} \leq \sum_{J \in \Lambda(t)} P_J^2 (\varepsilon t)^{-\beta},
\end{align*}$$
hence
\[ \frac{1}{t^\beta} \sum_{J \in \Lambda(t)} P_j^2 \geq (\delta^{-1} - 1)\varepsilon^\beta. \]

Without loss of generality we assume \(|E_0| = 1\). We denote \(\omega_d\) the Lebesgue measure. Note that \(\mu\) is supported by \(\cup \{E_J : J \in \Lambda(t)\}\) and \(\mu(E_J) = P_J\). Hence
\[
V_\beta(t; \mu) = \frac{1}{t^{d+\beta}} \int \int \chi_{B_t(x)}(\xi) \chi_{B_t(x)}(\eta) d\mu(\xi) d\mu(\eta) dx
\]
\[= \frac{1}{t^{d+\beta}} \int \int \omega_d(B_t(\xi) \cap B_t(\eta)) d\mu(\xi) d\mu(\eta)
\]
\[\geq \frac{1}{t^{d+\beta}} \sum_{J \in \Lambda(t)} \int_{\xi, \eta \in E_J} \omega_d(B_t(\xi) \cap B_t(\eta)) d\mu(\xi) d\mu(\eta). \]

Since \(|E_J| = \rho_J \leq t\), hence \(B_t(\xi) \cap B_t(\eta)\) contains a ball of radius \(t/2\) whenever \(\xi, \eta \in E_J\). It follows that
\[
V_\beta(t; \mu) \geq \frac{c}{t^\beta} \sum_{J \in \Lambda(t)} \int_{\xi, \eta \in E_J} d\mu(\xi) d\mu(\eta)
\]
\[\geq \frac{c}{t^\beta} \sum_{J \in \Lambda(t)} P_j^2 \geq c(\delta^{-1} - 1)\varepsilon^\beta, \]

where \(c(\delta^{-1} - 1)\varepsilon^\beta\) is a positive constant. #

From the asymptotic property of \(H(r)\) of infinite self-similar measure ([5]), Theorem 4.10 and Corollary 4.12 of [2] and our Theorem 2, we have

**Theorem 3** Let \(\mu\) be infinite self-similar measure. Assume conditions (6) and (7) hold, then
\[
\lim_{t \rightarrow 0} (V_\beta(t; \mu) - P(t)) = 0
\]
for some \(P > 0\) such that the following holds.

(i) If \(\{ - \ln \rho_j : j \in \mathbb{N}\}\) is non-arithmetic, then \(P(t) = c'\) for some constant \(c'\).

(ii) Otherwise, let \((\ln \rho)\mathbb{Z}, \rho > 1\) be the lattice generated by \(\{ - \ln \rho_j : j \in \mathbb{N}\}\), then \(P(\rho t) = P(t)\).

From the conclusion (*) of infinite self-similar measure ([5]), Theorem 4.10 and Corollary 4.12 of [2], if the equation \(\sum_{j=1}^{\infty} \rho_j^s = 1\) has finite solution \(s\), then

**Theorem 4** Let \(\mu\) be infinite self-similar measure with natural weights \(P_j = \rho_j^\beta\), where \(\beta = s\) is the finite solution of equation \(\sum_{j=1}^{\infty} \rho_j^s = 1\), we assume conditions (6) and (7) holds, then for any \(f \in L^2(d\mu)\) we have
\[
\lim_{t \rightarrow 0} \frac{1}{t^{d+\beta}} \int |\mu_f(B_t(x))|^2 dx - P(t) \int |f|^2 d\mu = 0,
\]
where \(P\) defined in Theorem 3.
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