The Principle of Relativity and the Special Relativity Triple

Han-Ying Guo\textsuperscript{1,*} Hong-Tu Wu\textsuperscript{2,†} and Bin Zhou\textsuperscript{3‡}

\textsuperscript{1} Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China, \textsuperscript{2} Department of Mathematics, Capital Normal University, Beijing 100048, China, and \textsuperscript{3} Department of Physics, Beijing Normal University, Beijing 100875, China.

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Based on the principle of relativity and the postulate on universal invariant constants \((c,l)\) as well as Einstein’s isotropy conditions, three kinds of special relativity form a triple with a common Lorentz group as isotropy group under full Umov-Weyl-Fock-Lorentz transformations among inertial motions.

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I. INTRODUCTION

Recent observations in precise cosmology show that our universe is accelerated expanding and quite possibly asymptotic to a Robertson-Walker-like de Sitter \((dS)\) spacetime with a tiny cosmo-
logical constant \( \Lambda \), rather than the flat Minkowski (\( M_{\text{ink}} \)) spacetime [1, 2]. With lots of puzzles, these greatly challenge Einstein’s theory of relativity as a foundation of the cosmic scale physics characterized by \( \Lambda \). In order to face the challenges, it would be better to re-examine the principles of Einstein’s theory from the very beginning.

As is well known, like the Galilei principle of relativity in Newtonian mechanics the Poincaré principle of relativity plays the fundamental role in Einstein’s special relativity. It requires: “For a set of inertial reference frames, in which free particles including light signals move with uniform velocities along straight lines, the laws of nature hold good in their simplest form invariant under the transformations among the frames” (see, e.g. [3, 4]). There is a simple and important problem for the principle: What are the most general transformations among the inertial motions in the inertial frames?

Long ago, Umov [5], Weyl [6] and Fock [7] investigated this problem. It turns out that those transformations are linear fractional ones with common denominators (\( \text{LFTs} \)), called the Umov-Weyl-Fock-transformations denoted by \( T \). Since all physical coordinates should have right dimensions in general, in order to make the time coordinate has the same dimension with three spacial ones, there should be two universal invariant constants: the speed of signal \( c \) and the length \( l \), which are universal for all inertial frames and invariant under their transformations. Thus, there should be a postulate on universal invariant constants \((c, l)\) for the transformations among the inertial frames, although in Galilei transformations they do not appear explicitly and neither does \( l \) in Poincaré transformations. Clearly, these transformations form a Lie group with twenty four parameters as the transformation group among inertial motions in inertial frames denoted by \( IM(4) \), whose algebra \( \mathfrak{im}(4) \) may be called the inertial motion algebra. In previous investigations it is immediately turned to the transformations of Poincaré group \( ISO(1, 3) \) on the \( M_{\text{ink}} \)-spacetime. Such a group \( IM(4) \) that contains the Lorentz group \( SO(1, 3) \) as isotropy group is denoted by \( IM_L(4) \) and its transformations called the Umov-Weyl-Fock-Lorentz transformations \( T_L \). However, there are ten generators for the Poincaré transformations only, which form merely a subset of \( T_L \). What is the role played by the rest transformations of \( IM_L(4) \)? This problem had been ignored for long time.

Later, the \( dS/\text{anti-de Sitter} (AdS) \) special relativity has been proposed [8–21], based on the principle of relativity [8, 9] and the postulate on two universal constants \((c \text{ and } l)\)[12]. The \( dS/AdS \) special relativity is invariant under the \( \text{LFTs} \) of \( dS/AdS \)-group \( SO(1, 4)/SO(2, 3) \) among the inertial motions in the Beltrami model [22] of \( dS/AdS \)-spacetime [12, 21] with curvature radius \( l \), respectively. Since there is also a Lorentz group as isotropy group in both, the \( \text{LFTs} \) of \( dS/AdS \)-group are also subsets of the \( T_L \). However, there are thirty parameters for three kinds of special relativity in total, this seems another puzzle. How to describe them at the same time in view of the \( IM_L(4) \)?

In this Letter, we show that three kinds of special relativity form such a special relativity triple in \( IM_L(4) \) that there are a common isotropy Lorentz group and some of the Umov-Weyl-Fock-Lorentz transformations among them under \( IM_L(4) \).

These can be shown by means of the algebraic analysis and specially the linear combinations in the Lie algebraic space of \( \mathfrak{im}_L(4) \). In fact, if the Lorentz algebra \( \mathfrak{so}(1, 3) \) as the isotropy algebra, the Poincaré/\( dS/AdS \)-algebra \( \mathfrak{iso}(1, 3)/\mathfrak{so}(1, 4)/\mathfrak{so}(2, 3) \) as subalgebras simultaneously should share the common isotropy algebra \( \mathfrak{so}(1, 3) \). Further, the translation generators of Poincaré algebra
iso(1, 3) can be given by the linear combination (plus) of the ‘translations’ generators of $dS/AdS$-algebra $\mathfrak{so}(1, 4)/\mathfrak{so}(2, 3)$ in the space of $\mathfrak{im}_L(4)$. And rest generators including four of Cartan subalgebra $\mathfrak{h}$ transform either the ‘translations’ in $dS/AdS$-algebra $\mathfrak{so}(1, 4)/\mathfrak{so}(2, 3)$ from each other or the translations in Poincaré algebras $\mathfrak{iso}(1, 3)$ among themselves. In this sense, three subalgebras $\mathfrak{iso}(1, 3), \mathfrak{so}(1, 4),$ and $\mathfrak{so}(2, 3)$ as well as their groups form a triple under $\mathfrak{IM}_L(4)$. Since these symmetries correspond to three kinds of special relativity, respectively, three kinds of special relativity do form the special relativity triple under $\mathcal{T}_L$. In addition, the linear combination (minus) of $dS/AdS$’s ‘translation’ operators may lead to another set of four generators that with generators of $\mathfrak{so}(1, 3)$ forms the second Poincaré algebra and whereas the linear combinations of these generators of two Poincaré algebras may also lead to the $dS/AdS$ algebra, too. With the help of the linear combinatorial in the space of $\mathfrak{im}_L(4)$, which is different from İnönü-Wigner’s contraction procedure [26], we would like to show the special relativity triple step by step more physically in this Letter.

This Letter is arranged as follows. In section II, we recall the Umov-Weyl-Fock transformations $\mathcal{T}$ for the principle of relativity with two universal invariant constants $(c, l)$ and how to get the symmetries of three kinds of special relativity. In section III, we show the linear combinatorial relations between the ‘translations’ of $dS/AdS$ algebra and those of two Poincaré algebras. And we also show that under the Umov-Weyl-Fock-Lorentz transformations $\mathcal{T}_L$ three kinds of special relativity do form the special relativity triple. Finally, we end with some remarks.

II. THE PRINCIPLE OF RELATIVITY AND THREE KINDS OF SPECIAL RELATIVITY

A. The Principle of Relativity and Most General Transformations for Inertial Motions

For the principle of relativity, in the inertial coordinate frames $\mathcal{F} := \{S(x)\}$, a free particle takes inertial motion described by

$$x^i = x_0^i + v^i(t - t_0), \quad v^i = \frac{dx^i}{dt} = \text{consts.} \quad i = 1, 2, 3. \quad (1)$$

What are the most general transformations $\mathcal{T} := \{T\}$

$$\mathcal{T} \ni T : \quad x'^\mu = f^\mu(x, T), \quad x^0 = ct, \quad \mu = 0, \cdots, 3, \quad (2)$$

that keep Eq. (1) invariant? Here, $c$ is invariant under $\mathcal{T}$.

It can be proved [5–7] that the most general form of the transformations (2), which transform a uniform-velocity, straight-line motion (1) in $S(x)$ to a motion of the same nature in $S'(x')$, are that the four functions $f^\mu$ are ratios of linear functions, all with the same denominators, i.e. the LFT's

$$T : \quad l^{-1}x'^\mu = \frac{A^\mu_{\nu}l^{-1}x^\nu + b^\mu}{c\lambda l^{-1}x^\lambda + d} \quad (3)$$

with

$$\det T = \begin{vmatrix} A & b^i \\ c & d \end{vmatrix} = 1.$$
where \( l \) is another universal length parameter invariant under \( T \in \mathcal{T} \) and \( A = (A_{\mu}^{\nu}) \) is a \( 4 \times 4 \) matrix, 
\( b, c \) \( 1 \times 4 \) matrixes, \( d \in \mathbb{R} \), up-index \(^t\) denotes the transpose. Thus, together with the principle of relativity, there is a postulate on universal invariant constants: *For all the inertial frames, there are two universal constants \((c, l)\) with dimension of speed and length invariant under \( T \), respectively.* A simpler proof for Umov-Weyl-Fock transformations is given in appendix of [23].

It is clear that an Umov-Weyl-Fock transformation \( T \in \mathcal{T} \) (3) is represented by a non-singular \( 5 \times 5 \) matrix with twenty four independent entries, all these matrices \( \forall T \in \mathcal{T} \) form the inertial motion transformation group \( IM(4) \) with generators defined by

\[
T_a := \left. \frac{\partial x^{\mu}(x, \tau^a)}{\partial \tau^a} \right|_{\tau^a = \epsilon} \frac{\partial}{\partial x^{\mu}}, \quad \tau^a \in T, \quad a = 1, \cdots, 24, \tag{4}
\]

forming the inertial motion algebra \( \text{im}(4) \) as Lie algebra of \( IM(4) \). Actually, the inertial motion (1) may be viewed as a straight line and the most general transformations among straight lines may form a group, which may be referred to the real projective transformations of the real projective group \( RP(4) \) (see, e.g. [10, 11]). But, care should be taken for the orientation [12].

However, Umov, Weyl and Fock did not go further rather immediately went back to Poincaré transformations by requiring, say, the plane wave front equation of light to be invariant [7].

**B. From Einstein’s Isotropy Conditions to Einstein’s Special Relativity**

As is well known, the principle of relativity connotatively assume that there should be isotropy and homogeneity for all points in a given frame \( S(x) \). We may first consider the isotropy by studying the inertial motions passing through the origin \( O(\phi^\mu = 0) \) of \( S(x) \). This leads to the Lorentz group as isotropy group at the origin. Then we consider the homogeneous property among all points by studying how to transit the origin \( O(\phi^\mu = 0) \) to the other point \( A(\forall \phi^\mu \neq 0) \) in \( S(x) \) such that the point \( A \) becomes the origin \( O'(\phi^\mu = 0) \) in the transformed frame \( S'(x') \) by some transformations generated by other four generators of \( T_L \).

Let us consider the isotropy first as Einstein did in 1905.

“At the time \( t = \tau = 0 \), when the origin of the co-ordinates is common to the two frames, let a spherical wave be emitted therefrom, and be propagated with the velocity \( c \) in system \( K \). If \((x, y, z)\) be a point just attained by this wave, then

\[
x^2 + y^2 + z^2 = c^2 t^2.
\]

Transforming this equation with the aid of our equations of transformation we obtain after a simple calculation

\[
\xi^2 + \eta^2 + \zeta^2 = c^2 \tau^2
\]

The wave under consideration is therefore no less a spherical wave with velocity of propagation \( c \) when viewed in the moving system. This shows that our two fundamental principles are compatible.
Actually, Einstein considered the motions of light signals passing through the common origin of the inertial frames, and introduced the light-cone at the origin may be called Einstein’s light-cone

\[ D_0 : \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu|_O := \eta_{\mu\nu} x^\mu x^\nu|_O = 0, \quad x^0 = ct, \quad \mu, \nu = 0, \ldots, 3, \]

\[ J := (\eta_{\mu\nu})_{\mu,\nu} = \text{diag}(1,-1,-1,-1), \]

where, \( \Delta x^\mu := (0, x^\mu) \). For an infinitesimal time duration and spacial intervals \( dx^\mu = \lim_{\Delta x^\nu \rightarrow 0} \Delta x^\mu \), it follows the infinitesimal form of Einstein’s light-cone

\[ D_{0, dx} : \quad ds^2|_O := \eta_{\mu\nu} dx^\mu dx^\nu|_O = 0. \]  

For other free massive particles or “tachyons” passing through the origin with uniform velocity

\[ v^i := \frac{dx^i}{dt} = \text{consts}, \]

they should satisfy \( x^2 + y^2 + z^2 = v^2 t^2 \lesssim c^2 t^2 \), respectively. Thus, the time-like/null/space-like condition at the origin or its infinitesimal form reads, respectively

\[ \eta_{\mu\nu} x^\mu x^\nu|_O \gtrless 0, \quad ds^2|_O = \eta_{\mu\nu} dx^\mu dx^\nu|_O \gtrless 0, \]

which may be called Einstein’s isotropy conditions and Einstein’s metric, respectively.

Actually, from the invariance of Einstein’s isotropy conditions and Einstein’s metric (8) under the transformations (3), it follows a set of transformations \( L^\nu_{\mu} \subset A^\nu_{\mu} \) of Lorentz group as isotropy group

\[ \mathcal{L} : \quad \eta_{\mu\nu} = \eta_{\mu'\nu'} L^\mu_{\mu'} L^\nu_{\nu'}, \quad L^\nu_{\mu} \in SO(1,3) \equiv \mathcal{L} \subset \mathcal{T}. \]

Then, \( \mathcal{T} \) should be the Umov-Weyl-Fock-Lorentz transformations \( \mathcal{T}_L \). Up to now, there are no restrictions on the rest eighteen parameters of \( \mathcal{T}_L \).

Einstein also required: “The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion.”[4] This means that the 4-d translation group \( R(1,3) \) holds for the transitivity. From this requirement, it follows Poincaré transformations of \( ISO(1,3) \) as the semidirect product of translations and Lorentz transformations \( R(1,3) \ltimes \mathcal{L} \)

\[ \mathcal{P} : \quad x^\mu \rightarrow x'^\mu = L^\mu_{\nu}(x^\nu - a^\nu). \]  

Both isotropy and homogeneity make the Mink-spacetime as a 4-d homogeneous space \( M = \mathcal{P}/\mathcal{L} \). And Einstein’s isotropy conditions (8) hold simultaneously at all points on it

\[ \eta_{\mu\nu} x^\mu x^\nu \gtrless 0, \quad ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \gtrless 0. \]

\[ ^1 \] The equations of the Lorentz transformation may be more simply deduced directly from the condition that in virtue of those equations the relation \( x^2 + y^2 + z^2 = c^2 t^2 \) shall have as its consequence the second relation \( \xi^2 + \eta^2 + \zeta^2 = c^2 \tau^2 \).
In view of the $T_L$, the Poincaré group is a subgroup of $IM_L(4)$ with generators

$$P_\mu = \partial_\mu, \quad x_\mu := \eta_{\mu\nu}x^\nu,$$

(12)

$$L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \in \mathfrak{so}(1, 3)$$

(13)

forming an $\mathfrak{so}(1, 3)$ algebra as a subalgebra of $\mathfrak{im}_L(4)$

$$[P_\mu, P_\nu] = 0, \quad [L_{\mu\nu}, P_\kappa] = \eta_{\kappa\nu}P_\mu - \eta_{\kappa\mu}P_\nu, \quad [L_{\mu\nu}, L_{\kappa\lambda}] = \eta_{\kappa\nu}L_{\mu\lambda} - \eta_{\kappa\lambda}L_{\mu\nu} + \eta_{\mu\lambda}L_{\nu\kappa} - \eta_{\mu\kappa}L_{\nu\lambda}.$$  

(14)

C. From Einstein’s Isotropy Conditions to Other Two Kinds of Special Relativity

In order to get other subsets of inertial motions including Lorentz transformations for Einstein’s isotropy conditions (8), it should be relaxed Einstein’s assumptions on Poincaré translations. This is also deeply motivated by recent observations in precise cosmology [1, 2].

In order to do so, we may first rewrite Eq. (5) to get

$$D_\pm: \quad \sigma_\pm(x) := \sigma_\pm(x, x) = 1 \mp l^2 \eta_{\mu\nu}x^\mu x^\nu > 0,$$

(15)

which denote two regions on Mink-spacetime with boundary as ‘pseudosphere’, respectively (see, e.g. [24])

$$\partial(D_\pm): \quad \sigma_\pm(x) = 1 \mp l^2 \eta_{\mu\nu}x^\mu x^\nu = 0.$$  

(16)

However, if the flatness of spacetime is relaxed, the conditions (15) would mean that the sets of all points in the frames $\{S(x)\}$ with Einstein’s isotropy conditions (8) at the origin, but without the Poincaré translations $R(1, 3)$. Therefore, we need to find the other four generators for transitivity. And together with the Lorentz isotropy group they should also form a group with ten generators, respectively. It is clear both the $dS/AdS$-group should be the candidates.

Actually, conditions (15) without flatness are just the same as the domain conditions of inertial frames in the Beltrami model of $dS/AdS$-spacetime with radius $l$, respectively [12, 21]. If we regard the inertial coordinates $x^\mu$ as the Beltrami coordinates in a chart $U_4$ [12, 21], say, of the Beltrami atlas$^2$

$$x^\mu = \frac{l^2}{\xi^4}, \quad \xi^4 > 0,$$

(17)

the domain conditions (15) become

$$H_\pm: \quad \eta_{\mu\nu}\xi^\mu \xi^\nu \pm (\xi^4)^2 \leq 0.$$  

(18)

These are just the $dS/AdS$-hyperboloid $H_\pm$, respectively.

$^2$ In order to preserve the orientations, the antipodal identifications should not be taken as was just mentioned [12, 21].
From the Beltrami model of $dS/AdS$-spacetime [12, 21], it follows that there are the LFTs of $dS/AdS$-group $SO(1, 4)/SO(2, 3) \subset T_L$,

\[
S_\pm: \ x^\mu \rightarrow x'^\mu = \pm \sigma_\pm^{1/2}(a)\sigma_\pm^{-1}(a, x)(x^\nu - a^\nu)D_\pm^{\nu},
\]

\[
D_\pm^{\nu} = L_\nu^\mu \pm l^2 \eta_{\mu\lambda} a^\lambda a^\kappa (\sigma_\pm(a) + \sigma_\pm^{1/2}(a))^{-1}L_\kappa^\mu, \tag{19}
\]

\[L := (L_\nu^\mu) \in SO(1, 3),\]

which transform a point $A(\forall a^\mu \neq 0)$ with $\sigma_\pm(a^\mu) > 0$ in the system $S(x)$ to the origin of $S'(x')$ and keep the domain conditions (15) and the Beltrami metrics invariant, respectively

\[
d_{S_\pm}^2 = [\eta_{\mu\nu}\sigma_\pm^{-1}(x) \pm l^2 \eta_{\mu\lambda} \eta_{\nu\kappa} x^\lambda x^\kappa \sigma_\pm^{-2}(x)]dx^\mu dx^\nu. \tag{20}
\]

It is straightforward to see that at the origin of frames, both $d_{S_\pm}^2|_0$ become Einstein’s metric in Eq. (8). Therefore, the LFTs $S_\pm$ of $dS/AdS$-group do form a subset of Umov-Weyl-Fock-Lorentz transformations, $S_\pm \subset T_L$, respectively. In fact, the domain conditions (15), the Beltrami-metrics (20) and the light-cones [12, 21]

\[
\mathcal{F}_\pm := \{\sigma_\pm(a, x) - [\sigma_\pm(a)\sigma_\pm(x)]^{1/2}\} \geq 0. \tag{21}
\]

can be deduced by LFTs (19) from Einstein’s isotropy conditions (8), respectively. Further, due to the transitivity of LFTs (19), the Beltrami-$dS/AdS$ spacetime $\mathcal{B}_\pm$ with the domain condition is also homogeneous, i.e. $\mathcal{B}_\pm \cong S_\pm/L$, respectively. It is definitely true globally chart by chart for the entire Beltrami-$dS/AdS$ spacetime (see, e.g.[12, 21]).

From the LFTs $S_\pm$ (19), it is straightforward to get their generators, which are also the Killing vectors with respect to the Beltrami metrics (20) up to some coefficients, as follows

\[
P_\mu^\pm = (\delta_\mu^\nu \mp l^2 x_\mu x^\nu)\partial_\nu, \quad x_\mu := \eta_{\mu\nu} x^\nu, \tag{22}
\]

\[
L_{\mu\nu}^\pm = x_\mu P_{\nu\mu}^\pm - x_\nu P_{\mu\nu}^\pm = x_\mu \partial_\nu - x_\nu \partial_\mu = L_{\mu\nu}^\pm \in \mathfrak{so}(1, 3), \tag{23}
\]

which form the $\mathfrak{so}(1, 4)/\mathfrak{so}(2, 3) \subset \mathfrak{im}(4)$ algebra

\[
[P_{\mu}^\pm, P_{\nu}^\pm] = \pm l^2 L_{\mu\nu}, \quad [L_{\mu\nu}, P_{\kappa}^\pm] = \eta_{\kappa\nu} P_{\mu}^\pm - \eta_{\kappa\mu} P_{\nu}^\pm,
\]

\[
[L_{\mu\nu}, L_{\kappa\lambda}] = \eta_{\kappa\nu} L_{\mu\lambda} - \eta_{\kappa\mu} L_{\nu\lambda} + \eta_{\mu\lambda} L_{\nu\kappa} - \eta_{\mu\kappa} L_{\nu\lambda}. \tag{24}
\]

Here the generators $L_{\mu\nu}^\pm = L_{\mu\nu}^\pm$ of Lorentz algebra $\mathfrak{so}(1, 3)$ are the same as that of the Poincaré algebra (14).

Thus, this confirms that on the $dS/AdS$ spacetime of positive/negative constant curvature of radius $l$, there are indeed other two kinds of the Beltrami inertial frames with inertial motions for the $dS/AdS$ special relativity, respectively[8–21].

**III. THREE KINDS OF SPECIAL RELATIVITY AS A TRIPLE**

It is important to notice from our consideration in last section that for the Lorentz algebra $\mathfrak{so}(1, 3)$ of $dS/AdS$ algebra $\mathfrak{so}(1, 4)/\mathfrak{so}(2, 3)$ (24), the generators (23) are the same as that of the Poincaré algebra $\mathfrak{iso}(1, 3)$ (14). This would mean that there should be certain relations among $\mathfrak{iso}(1, 3)/\mathfrak{so}(1, 4)/\mathfrak{so}(2, 3) \subset \mathfrak{im}_L(4)$ for three kinds of special relativity, if their symmetries are subsets of $\mathfrak{im}_L(4)$ at the same time. Let us consider this issue further.
A. The Special Relativity Triple with Isotopy Lorentz Group

Firstly, it is also important to notice that in the space of $\mathfrak{im}_L(4)$, not only the isotropy algebra $\mathfrak{so}(1, 3)$ is common, but the translation’s generators $P^\mu$ of $\mathfrak{iso}(1, 3)$ in (12) can be directly given by the generators $P^\pm_\mu$ of $\mathfrak{so}(1, 4)/\mathfrak{so}(2, 3)$ in (22):

$$P^\mu := \frac{1}{2} (P^+_\mu + P^-_\mu) = \partial_\mu.$$  \hspace{1cm} (25)

Then, the Poincaré algebra $\mathfrak{iso}(1, 3)$ (14) can be given by $dS$-algebra $\mathfrak{so}(1, 4)$ and $AdS$-algebra $\mathfrak{so}(2, 3)$ in (24) with same $l$. Thus, for three kinds of inertial transformations of Poincaré/$dS/AdS$-invariant special relativity, the total number of generators is only fourteen. In $T_L$, there are still other ten generators. In order to get the entire algebra $\mathfrak{im}_L(4)$, we may take the infinitesimal transformations and their generators (4) or we may also equivalently start with $P^\pm_\mu$ in $dS/AdS$-algebra as the subalgebras of $\mathfrak{im}_L(4)$ to get it:

$$[P^\mu, P^\nu] = \begin{cases} -l^{-2}R^\mu_{\nu\rho} & \mu \neq \nu, \\ 2l^{-2}\eta^\nu_{(\nu)}M^\rho_{(\rho)} & \mu = \nu, \end{cases}$$ \hspace{1cm} (26)

where no summation for repeated indexes in brackets and

$$R^\mu_{\nu\rho} = x^\mu \partial_\rho + x^\rho \partial_\mu, \quad \mu \neq \nu,$$

$$M^\mu_\mu = - \left( x^{(\mu)} \partial_{(\mu)} + \sum_\kappa x^{\kappa} \partial_{\kappa} \right).$$  \hspace{1cm} (27)

It is straightforward to check that twenty four generators ($P^\pm_\mu, L^\mu_{\nu}, R^\mu_{\nu\rho}, M^\mu_\mu$) form the inertial motion algebra $\mathfrak{im}_L(4)$. In addition to the $dS/AdS$-algebraic relations (24), the non-vanishing relations include

$$[P^\pm_\mu, M^\nu] = -P^\mp_\mu - \delta_{\mu(\nu)}P^\pm_{(\nu)}, \quad [P^\pm_\mu, R^\rho_{\nu\sigma}] = \eta^\rho_{\nu\sigma}P^\pm_\rho + \eta^\mu_{\nu\sigma}P^\pm_\mu,$$

$$[L^\mu_{\nu\rho}, M^\rho] = \delta_{\mu(\rho)}R^\nu_{\nu\rho} - \delta_{\nu(\rho)}R^\mu_{\nu\rho}, \quad [L^\mu_{\nu\rho}, R^\sigma_{\rho\sigma}] = \begin{cases} 2\eta^\mu_{\nu\rho}(M^\rho_{(\mu)} - M^\rho_{(\nu)}) & \text{if } \mu = \rho, \nu = \sigma, (\mu = \sigma, \nu = \rho), \\
\eta^\nu_{\rho\sigma}R^\mu_{\rho\sigma} + \eta^\mu_{\nu\sigma}R^\rho_{\mu\sigma} - \eta^\mu_{\nu\rho}R^\rho_{\nu\sigma} & \text{etc.} \end{cases}$$  \hspace{1cm} (28)

It is clear that in the full algebra $\mathfrak{im}_L(4)$ there are the $dS$, $AdS$ and Poincaré algebras as subalgebras generated by ($P^\pm_\mu, L^\mu_{\nu}$) and their combination ($P^\mu_\mu, L^\mu_{\nu}$) for three kinds of special relativity, respectively, and the rest generators ($R^\mu_{\nu\rho}, M^\mu_\mu$) do combine three kinds of special relativity as a whole. In fact, four generators $M^\mu_\mu$ form a Cartan subalgebra $\mathfrak{h}$ that exchange the ‘translations’ $P^\mu_\mu$ in $dS/AdS$-algebra from each other and also transform $P^\mu_\mu$ of $\mathfrak{iso}(1, 3)$ among themselves

$$[P^\mu_\mu, M^\nu] = -P^\mu_\mu - \delta_{\mu(\nu)}P^\pm_{(\nu)}.$$ \hspace{1cm} (29)

And the rest six generators $R^\mu_{\nu\rho}$ play similar roles: transforming $P^\pm_\mu$ of $dS/AdS$-algebra from each other and $P^\mu_\mu$ of $\mathfrak{iso}(1, 3)$ among themselves

$$[P^\mu_\mu, R^\rho_{\nu\sigma}] = \eta^\mu_{\nu\rho}P^\sigma_\rho + \eta^\mu_{\nu\sigma}P^\rho_\mu.$$  \hspace{1cm} (30)
Thus, in addition to having a common Lorentz group, three kinds of special relativity act as a whole within $I\!M_L(4)$. It is called the special relativity triple.

From Eq. (25), it is natural to get the other linear combination

$$P'_\mu := \frac{1}{2} (p^+ - p^-) = -l^{-2} x_{\mu} \sum x_{\kappa} \partial_{\kappa}.$$  

Actually, the generators $P'_\mu, L_{\mu\nu}$ form the other $\text{iso}(1,3)$ algebra isomorphic to the Poincaré algebra (14). This algebra may be called the second Poincaré algebra. It can be proved that this algebra preserves the light-cone at the origin.

It is clear that in the Lie algebraic space of $\text{Im}_L(4)$, the set of generators of the $dS/AdS$ algebra and two sets of generators of the Poincaré algebras can be transferred from each other by linear combinations between the generators as bases.

\subsection{B. On Other Symmetries in the Special Relativity Triple}

There are very rich substructures in the special relativity triple. Let us consider some of them at Lie algebraic level.

It is easy to check that sixteen generators $(L_{\mu\nu}, R_{\mu\nu}, M_{\mu})$ form a subalgebra $\mathfrak{gl}(4)$. Although there is Lorentz algebra for isotropy, there are no generators for the transformation of the origin to other points. Actually, if in such kind subalgebras of $\text{Im}_L(4)$ that there are no generators for transitivity, which may be related to parameters $b^\mu$ in (3), this kind of subalgebras may not be related to the spacetime physics. Among this kind of subalgebras, the second Poincaré algebra just mentioned is an interesting one, which keeps the light cone at the origin. Therefore, for the spacetime physics, it is concerned with not only the subsets of the $T_L$, but also those subsets with transitivity for all points in the frames. It is clear that the special relativity triple should contain all these subsets.

If we further consider within the special relativity triple the algebras of ten generators with a common space rotation algebra $\mathfrak{so}(3) \subset \mathfrak{so}(1,3)$ as the isotropy algebra rather than the Lorentz algebra $\mathfrak{so}(1,3)$, it may follow all algebras of possible kinematics other than Poincaré/$dS/AdS$-algebra, such as Newton-Hooke algebras $\mathfrak{n}^\pm$, Galilei algebra, para-Poincaré algebra, para-Galilei algebra as well as Carroll and static algebras [25]. In addition, their space-times may follow, too. This may be seen from algebraic analysis including the root system of $\text{Im}(4)$. It is worthwhile to mention that all these kinematics should be based on the principle of relativity and appear as subalgebras of $\text{Im}(4)$ rather than by contractions under different limits. This is different from the ‘deformation/contraction’ approach [25, 26].

In addition, there are also very rich discrete symmetries for the inertial motion transformation group $I\!M_L(4)$. Related to the 4-d spacetime physics, the CPT and so on should be considered.

\section{IV. CONCLUDING REMARKS}

We have shown that based on the principle of relativity and the postulate for universal invariant constants, three kinds of special relativity form the special relativity triple with common isotropy
Lorentz group under the Umov-Weyl-Fock-Lorentz transformations $T_L$.

Since the inertial motion algebra $\mathfrak{im}(4)$ is very closely related to the algebra $\mathfrak{rp}(4)$ of the 4-d real projective group $\mathbb{RP}(4)$ on 4-d real projective space $\mathbb{RP}^4$, the projective geometry approach may be applied (see, e.g., [10, 11]) not only locally but also globally. However, care should be taken, since $\mathbb{RP}^4$ is not orientable. In our previous approach to the $dS/AdS$ special relativity, in order to preserve the orientations the antipodal identifications should not be taken (see, e.g.[12, 21]). This is also the case for the special relativity triple. Actually, the $\text{Mink}/dS/AdS$-triple may also be implied in view of algebraic geometry. In fact, two quadratic forms $\mathcal{H}_\pm$ the $dS/AdS$-hyperboloid (18) share a common quadratic form $\eta_{\mu\nu}\xi^\mu\xi^\nu$, which can be given by the sum of quadratic forms of $\mathcal{H}_\pm$, invariant under Poincaré transformations. In this Letter, we have mainly concerned with the local properties. The global aspects are of course important issues.

All these approaches may be applied to three kinds of classical geometries. For instance, if the isotropy algebra $\mathfrak{so}(1,3)$ is replaced by an $\mathfrak{so}(4)$ algebra, the algebra $\mathfrak{iso}(4)/\mathfrak{so}(5)/\mathfrak{so}(1,4)$ for 4-d Euclid/Riemann/Lobachevsky geometry may follows, respectively. The rest generators of $\mathfrak{rp}(4)$ also transform among themselves so that there is a Euclid-Riemann-Lobachevsky-triple of three kinds of classical geometries. In this case, there are also rich sub-geometry structures.

It should be emphasized that the algebra of twenty four generators can also be regarded as an entire ‘conformal’ algebra for three kinds of special relativity or three kinds of classical geometry of one dimension lower. This can be seen partially by the boundary of a 4-d $AdS$-spacetime as the conformal extensions of 3-d $\text{Mink}/dS/AdS$-spacetime, respectively [17].

All these issues can be extended to any dimensions.

We have only considered the kinematic aspects of the special relativity triple, the dynamic aspects are of course very important, for which there are lots of issues to be studied.

Although all these kinematics are based on the principle of relativity with inertial frames involved, the algebraic combinatory approach may also be employed for other kinds of coordinate frames. For instance, in the special relativity triple, the proper time coordinate may be taken. Thus, it follows the triple for the $\text{Mink}$-space, the Robertson-Walker ($RW$)-like $dS$-space with an accelerated expanding 3-sphere and the $RW$-like $AdS$-space with an oscillating 3-pseudosphere[20, 21]. Our universe should prefer the $RW$-like $dS$-cosmos as its fate.

For the special relativity triple, there is no gravity at all. In order to describe gravity, the principle of localization of special relativity [20, 21, 23] should be applied to the special relativity triple. In our universe, the special relativity triple should be reduced to the $dS$ special relativity and its $RW$-like $dS$-cosmos. Once gravity could be introduced, this might be closely related to the reduction of the structure group $IM(4)$ to its $dS$ sub-group as structure group. How to realize the localization, symmetry breaking or reduction are also important issues.

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[1] A. G. Riess et al., Astron. J. 116 (1998), 1009, astro-ph/9805201; S. Perlmutter et al., Astrophys. J. 517 (1999), 565, astro-ph/9812133; A. G. Riess et al., Astrophys. J. 536 (2000), 62, astro-ph/0001384; A. G. Riess et al., Astrophys. J. 560 (2001), 49, astro-ph/0104455.

[2] C. L. Bennett et al., Astrophys. J. (Suppl.) 148 1 (2003); D.N. Spergel et al., Astrophys. J. (Suppl.) 170 377 (2007); J. Dunkley et al., arXiv:0803.0586v1 [astro-ph].

[3] H. Poincaré, Revue générale des sciences pures et appliquées 11 (1900) 1163-1175; Rendiconti del Circolo matematico di Palermo 21 (1906) 129-176.

[4] A. Einstein, Ann. d. Phys., 17 (1905) 891.

[5] N.A. Umow, Physikalische Zeitschrift, 11 (1910) 905.

[6] H. Weyl, Mathemathische Analyse des Raumproblems, Berlin, Splinger, 1923.

[7] V. Fock, The Theory of Space-Time and Gravitation, Pergamon Press, 1964. And references therein.

[8] Q.H. Look (Q.K. Lu), Why the Minkowski metric must be used? (1970) An unpublished note.

[9] K.H. Look, C.L. Tsou (Z.L. Zou) and H.Y. Kuo (H.-Y. Guo), Acta Phys. Sinica 23 (1974) 225; Nature (Shanghai, Suppl.), Mod. Phys. 1 (1980) 97; H.Y. Kuo, Kexue Tongbao (Chinese Sci. Bull.) 22 (1977) 487 (all in Chinese); H.Y. Kuo, Proc. 2nd Marcel Grossmann Meeting on GR, ed. by R. Ruffini, (North-Holland Pub. 1982) 801; H.-Y. Guo, Nucl. Phys. B (Proc. Suppl.) 6 (1989) 381.

[10] L.G. Hua (1974), “Uniform velocity straight-line motions and projective geometry”. An unpublished manuscript.

[11] G. Arcidiacono, Gen. Rel. Grav. 7 (1976) 885-889. And references therein.

[12] H.-Y. Guo, C.-G. Huang, Z. Xu and B. Zhou, Mod. Phys. Lett. A19 (2004) 1701; Phys. Lett. A331 (2004) 1.

[13] H.-Y. Guo, C.-G. Huang and B. Zhou, Europhys. Lett. 72 [6] (2005) 2494.

[14] H.-Y. Guo, C.-G. Huang, Z. Xu and B. Zhou, Chinese Phys. Lett. 22 (2005) 2477.

[15] C.-G. Huang, H.-Y. Guo, Y. Tian, Z. Xu and B. Zhou, Intl. J. Mod. Phys. A22 (2007) 2535; Y. Tian, H.-Y. Guo, C.-G. Huang, Z. Xu and B. Zhou, Phys. Rev. D71 (2005) 044030.

[16] Q. K. Lu, Commun. Theor. Phys. 44 (2005) 389.

[17] H.-Y. Guo, B. Zhou, Y. Tian and Z. Xu, Phys. Rev. D75 (2007) 026006.

[18] M.L. Yan, N.C. Xiao, W. Huang, S. Li, Comm. Theor. Phys. 48 (2007) 27.

[19] H.-Y. Guo, C.-G. Huang, Y. Tian, Z. Xu and B. Zhou, hep-th/0405137.

[20] H.-Y. Guo, Phys. Lett. B653 (2007) 88.

[21] H.-Y. Guo, Scien. in China A51 (2008) 588.

[22] E. Beltrami, Opere Mat., 1 (1868) 374-405.

[23] H.-Y. Guo, C.-G. Huang, Y. Tian, H-T Wu, Z. Xu and B. Zhou, Class. Quan. Grav. 24 (2007) 4009-4035; hep-th/0703078v2.

[24] H. Minkowski, (1909) Physikalische Zeitschrift 10 104-111; See also J.L. Synge, Relativity: The Special
[25] H. Bacry and J.M. Lévy-Leblond, *J. Math. Phys.* 9 (1968) 1605; H. Bacry and J. Nuyts, *J. Math. Phys.* 27 (1986) 2455; J.R. Derome and J.G. Dubois, *IL Nuovo Cimento* 9 (1972) 351.

[26] E. İnönü and E.P. Wigner, *Proc. Nat. Acad. Sci.* 39 (1953) 510-524.