The hysteresis limit in relaxation oscillation problems

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Abstract. A singularly perturbed differential equation with a small coefficient multiplying the derivative is shown to exhibit a limiting hysteresis behavior as the singular parameter tends to zero. The convergence takes place in the space of left-continuous regulated functions and is related to the generalized Helly selection principle for regulated functions established by Fraňková. Examples show that convergence cannot be expected in general if no regularity is assumed either for the forcing term or for the equilibrium set.

Introduction
We consider the problem
\[ \alpha \dot{x}_\alpha(t) = \Phi(x_\alpha(t), u(t)), \quad x_\alpha(0) = x_0, \] (0.1)
with given functions \( \Phi : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( u : [0, T] \rightarrow \mathbb{R} \), with initial condition \( x_0 \) and a (small) parameter \( \alpha > 0 \). We will study the dependence of \( x_\alpha \) on \( \alpha \), \( u \), and \( x_0 \). Our analysis will be based on properties of the equilibrium set
\[ E = \{(x, v) \in \mathbb{R}^2 ; \Phi(x, v) = 0\}, \] (0.2)
where a possible limit process as \( \alpha \downarrow 0 \) takes place. Typically, this limit can be expected to have a discontinuous hysteresis character as in Figure 1 below. Continuous functions of \( t \) therefore cannot constitute a suitable framework for its description, and an appropriate extension has to be taken into account.

We consider a fixed interval \([0, T]\) with some \( T > 0 \), and the space \( G(0, T) \) of regulated functions \([0, T] \rightarrow \mathbb{R}\). Recall that a function \( u : [0, T] \rightarrow \mathbb{R} \) is said to be regulated according to [1], if both one-sided limits \( u(t^+) \), \( u(t^-) \) exist at each point \( t \in [0, T] \) with the convention \( u(0^-) = u(0), \ u(T^+) = u(T) \).

We start with Section 1, where we briefly summarize some basic properties of regulated functions which we need in the sequel. Particular attention is paid to the concept of uniformly bounded oscillation which implies the existence of pointwise convergent subsequences in uniformly bounded sequences of regulated functions. The main result of Section 2 is Theorem 2.3, where we identify sufficient conditions on \( \Phi \) which ensure that the set of all solutions to (0.1) corresponding to uniformly bounded inputs \( u \) of uniformly bounded oscillation is itself uniformly bounded and has uniformly bounded oscillation independently of \( \alpha > 0 \). In Sections 3 and 4
we investigate the limiting behavior of the solution \( x_\alpha \) to (0.1) as \( \alpha \searrow 0 \) if the input \( u \) is kept fixed. We present examples showing that the limit exists only if additional regularity conditions are imposed either on \( u \) (Theorem 3.3) or on \( \Phi \) (Theorem 4.1). The mapping \( \Gamma \) which, with each \( u \) and initial condition \( x_0 \), associates the pointwise limit \( x(t) = \lim_{\alpha \to 0} x_\alpha(t) \) is shown to be a hysteresis operator (that is, causal and rate-independent) which acts discontinuously in the space \( G_L(0, T) \) of left-continuous regulated functions, but preserves the natural (pointwise) ordering in \( G_L(0, T) \). The last section is an Appendix, where we briefly outline some basic properties of the so-called play operator which is one of the basic elements of the theory of hysteresis and which we use here as a tool in the proof of Theorem 2.3.

1. Regulated functions

We introduce in \( G(0, T) \) a family of seminorms

\[
\|u\|_{[a,b]} = \sup \{|u(t)| : 0 \leq a \leq t \leq b \leq T\}.
\]

In particular, \( \|\cdot\|_{[0,T]} \) is a norm which transforms \( G(0, T) \) into a Banach space, see [1]. More about regulated functions can also be found in [2, 4].

We denote by \( G_L(0, T) \) the closed subspace of \( G(0, T) \) of left-continuous functions and by \( BV(0, T) \) the (dense) subset \( G(0, T) \) of functions of bounded variation. The closed subspace of \( G(0, T) \) with respect to the norm \( \|\cdot\|_{[0,T]} \) consisting of all continuous real functions on \( [0, T] \) will be denoted by \( C(0, T) \), and we set \( CBV(0, T) = C(0, T) \cap BV(0, T) \). Here and in the sequel, we denote by \( \mathbb{R}^+ \) the open interval \( ]0, +\infty[ \).

**Definition 1.1.** A set \( U \subset G(0, T) \) is said to have uniformly bounded oscillation if

(i) there exists a constant \( R > 0 \) such that

\[
|u(t) - u(s)| \leq R \quad \forall u \in U \quad \forall s, t \in [0, T],
\]

(ii) there exists a non-increasing function \( N : \mathbb{R}^+ \to \mathbb{R}^+ \) such that, for every \( r > 0 \) and every system \( \{a_k, b_k[ : k = 1, \ldots, m \} \) of pairwise disjoint intervals \( ]a_k, b_k[ \subset [0, T] \), the implication

\[
(u(b_k) - u(a_k)) \geq r \quad \forall k = 1, \ldots, m \quad \Rightarrow \quad m \leq N(r)
\]

holds for every \( u \in U \).

**Definition 1.2.** ([4]) A set \( U \subset G(0, T) \) is said to have uniformly bounded \( \varepsilon \)-variation if there exists a non-increasing function \( L : \mathbb{R}^+ \to \mathbb{R}^+ \) such that

\[
\forall \varepsilon > 0 \quad \forall u \in U \quad \exists \psi \in BV(0, T) : \quad \|u - \psi\|_{[0,T]} \leq \varepsilon, \quad \text{Var}_{[0,T]} \psi \leq L(\varepsilon).
\]

We mention without proof the following consequences of the concepts introduced above.

**Proposition 1.3.** A set \( U \subset G(0, T) \) has uniformly bounded \( \varepsilon \)-variation if and only if it has uniformly bounded \( \varepsilon \)-variation.

**Proposition 1.4.** Let \( \{u_n : n \in \mathbb{N}\} \) be a sequence in \( G(0, T) \).

(i) If \( \{u_n : n \in \mathbb{N}\} \) is bounded and has uniformly bounded \( \varepsilon \)-variation, then there exists \( u \in G(0, T) \) and a subsequence \( \{u_{n_k}\} \) such that \( u_{n_k}(t) \to u(t) \) for every \( t \in [0, T] \) as \( k \to \infty \).

(ii) If \( u_n \) converge uniformly to a function \( u \), then \( \{u_n : n \in \mathbb{N}\} \) has uniformly bounded oscillation.

We just note that Proposition 1.3 is a special case of [2, Theorem 2.2]. Part (i) of Proposition 1.4 was proved as a generalization of Helly’s Selection Principle in [4, Theorem 3.8], while part (ii) is an easy exercise. A more systematic discussion on this subject including detailed proofs and comments can also be found in [8].
2. Equations with small parameters

Let us consider Eq. (0.1) with a function $\Phi$ satisfying the following four conditions:

**Hypothesis 2.1.**

(i) $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $\Phi(\cdot, u) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous for all $u \in \mathbb{R}$;
(ii) For all $x, u, v \in \mathbb{R}$ we have $u > v \Rightarrow \Phi(x, u) > \Phi(x, v)$;
(iii) There exists $\bar{x} \in \mathbb{R}$ such that $\Phi(\bar{x}, 0) = 0$;
(iv) With $E$ given by (0.2) we denote

\[
E^x = \{ u \in \mathbb{R} ; (x, u) \in E \} \quad \text{for} \quad x \in \mathbb{R},
\]

\[
E_u = \{ x \in \mathbb{R} ; (x, u) \in E \} \quad \text{for} \quad u \in \mathbb{R},
\]

\[
z_\ast = \sup \{ x < \bar{x} ; E^x = \emptyset \},
\]

\[
z^\ast = \inf \{ x > \bar{x} ; E^x = \emptyset \},
\]

and assume that

(a) $E_u \cap ]z_\ast, z^\ast[$ is nowhere dense for any $u \in \mathbb{R}$,
(b) $E_u \cap ]x, z^\ast[ \neq \emptyset$ for every $u > 0$,
(c) $E_u \cap ]z_\ast, \bar{x}[ \neq \emptyset$ for every $u < 0$.

We start with the following easy result.

**Lemma 2.2.** Let Hypothesis 2.1 hold. Then we have $-\infty \leq z_\ast < \bar{x} < z^\ast \leq +\infty$ and there exists a continuous function $g : ]z_\ast, z^\ast[ \to \mathbb{R}$ such that

(i) $(x, u) \in E \iff u = g(x) \quad \forall x \in ]z_\ast, z^\ast[$,
(ii) $\limsup \Phi_{x \to z_\ast} g(x) = +\infty$, $\liminf \Phi_{x \to z_\ast} g(x) = -\infty$,
(iii) $g$ is non-constant in any interval $[a, b] \subset ]z_\ast, z^\ast[$.

**Proof.** Let $(x, u) \in E$ be arbitrary. Then, by Hypothesis 2.1(ii), we have for every $\varepsilon > 0$ that $\Phi(x, u + \varepsilon) > 0 > \Phi(x, u - \varepsilon)$, hence there exists $\delta > 0$ such that

\[
|y - x| < \delta \Rightarrow E^y \cap ]u - \varepsilon, u + \varepsilon[ \neq \emptyset. \tag{2.1}
\]

We thus have in particular $z_\ast < \bar{x} < z^\ast$. By virtue of Hypothesis 2.1(ii), $E^x$ contains for each $x \in ]z_\ast, z^\ast[$ exactly one point which we denote by $g(x)$. The function $g$ is continuous by (2.1), and its further properties follow immediately from the hypotheses. ☐

A typical shape of the function $g$ is shown on Fig. 1 below. We now state the main result of this section as an extension of [8, Theorem 2.6].

**Theorem 2.3.** Let $U \subset G_L(0, T)$ be a bounded set with uniformly bounded oscillation, let $\Phi$ satisfy Hypothesis 2.1, and let an interval $[x_\ast, x^\ast] \subset ]z_\ast, z^\ast]$ be given. Then for every $u \in U$, $x_0 \in [x_\ast, x^\ast]$, and $\alpha > 0$ there exists a unique solution $x_\alpha \in W^{1, \infty}(0, T)$ to (0.1), and the set $X$ of all such functions $x_\alpha$ is bounded in $G_L(0, T)$ and has uniformly bounded oscillation.

In the proof and in the sequel we make use of the following classical “barrier lemma”.

**Lemma 2.4.** Let $f : ]t_0, t_1[ \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, that is, $f(\cdot, w)$ integrable for all $w \in \mathbb{R}$, $f(t, \cdot)$ continuous for almost all $t \in ]t_0, t_1[$, and let the equation

\[
\dot{w}(t) = f(t, w(t)), \quad w(t_0) = w_0
\]

admit a unique solution $w : ]t_0, t_1[ \to \mathbb{R}$. Assume that $w_0 \leq 0$ and $f(t, 0) \leq 0$ a.e. in $]t_0, t_1[$. Then $w(t) \leq 0$ for all $t \in ]t_0, t_1[$.
Proof. Put \( f^*(t,w) = f(t,w) \) if \( w \leq 0 \), \( f^*(t,w) = f(t,0) \) if \( w > 0 \). Then \( f^* \) is a Carathéodory function and the equation

\[
\dot{w}_s(t) = f^*(t, w_s(t)), \quad w_s(t_0) = w_0
\]

admits a maximal solution \( w_s : [t_0, t^*] \to \mathbb{R} \) for some \( t^* \in [t_0, t_1] \). Testing (2.2) by \( w^+_s(t) = \max(0, w_s(t)) \) we obtain \( \frac{d}{dt}(w^+_s(t))^2 = w_s(t) w^+_s(t) \leq 0 \), hence \( w^+_s(t) \leq 0 \) for all \( t \in [t_0, t^*] \). From (2.2) it follows that \( w_s(t) = w(t) \) in \([t_0, t^*]\), hence \( t^* = t_1 \), and the proof is complete. \( \blacksquare \)

As straightforward consequences we have

Lemma 2.5. Let \( u \in G_L(0,T) \) and \( \alpha > 0 \) be given, let \([a, b] \subseteq [z_*, z^*] \) be an interval such that \( g(a) \leq g(b) \), and let \( x_\alpha \) be a solution of (0.1). Assume that there exists an interval \([t_0, t_1] \subseteq [0, T]\) such that

(i) \( x_\alpha(t_0) \in [a, b] \),
(ii) \( u(\tau) \in [g(a), g(b)] \) for every \( \tau \in [t_0, t_1] \).

Then \( x_\alpha(\tau) \in [a, b] \) for all \( \tau \in [t_0, t_1] \).

Proof. It suffices again to use Lemma 2.4 for \( w(t) = a - x_\alpha(t) \), \( f(t, w) = -\frac{1}{\alpha} \Phi(a - w, u(t)) \) and for \( w(t) = x_\alpha(t) - b \), \( f(t, w) = \frac{1}{\alpha} \Phi(b + w, u(t)) \). \( \blacksquare \)

Lemma 2.6. Let \( u, v \in G_L(0,T) \) and \( \alpha > 0 \) be given. Assume that there exists an interval \([t_0, t_1] \subseteq [0, T]\) such that \( u(t) \leq v(t) \) and

\[
\alpha \dot{x}_\alpha(t) = \Phi(x_\alpha(t), u(t)), \quad \alpha \dot{y}_\alpha(t) = \Phi(y_\alpha(t), v(t))
\]

for \( t \in [t_0, t_1] \), with initial conditions \( x_\alpha(t_0) \leq y_\alpha(t_0) \). Then \( x_\alpha(t) \leq y_\alpha(t) \) for all \( t \in [t_0, t_1] \).

Proof. It suffices again to use Lemma 2.4 with \( f(t, w) = \frac{1}{\alpha} (\Phi(x_\alpha(t), u(t)) - \Phi(x_\alpha(t) - w, v(t))) \) and \( w(t) = x_\alpha(t) - y_\alpha(t) \). \( \blacksquare \)

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. Let us fix real numbers \( z_* < A < B < z^* \) such that \( A \leq x_*, B \geq x^* \), \( g(A) \leq u(t) \leq g(B) \) for each \( u \in U \) and \( t \in [0, T] \) as in Fig. 3 below. For each \( \alpha > 0 \), \( x_0 \in [x_*, x^*] \), and \( u \in U \) we use the local Lipschitz continuity of \( g \) to construct a unique maximal solution \( x_\alpha : [0, t^*] \to \mathbb{R} \) to (0.1). From Lemma 2.5 we conclude that

\[
x_\alpha(t) \in [A, B] \quad \text{for every} \quad t \in [0, t^*],
\]

hence in particular \( t^* = T \) and \( x_\alpha \in W^{1,\infty}(0,T) \).

Now keeping \( \alpha > 0 \), \( x_0 \in [x_*, x^*] \), and \( u \in U \) fixed, we may write, for simplicity, \( x \) instead of \( x_\alpha \) to denote the solution of (0.1). To prove that the solutions have uniformly bounded \( \varepsilon \)-variation, we find uniform upper bounds as functions of \( r \) for the total variations of all outputs \( \eta = p_r[0,x] \) of the play operator (see Section Appendix A) for all values \( r > 0 \) independently of \( \alpha \), \( x_0 \), and \( u \).

Let us choose an arbitrary \( r > 0 \). Invoking Proposition A.4 we construct a partition

\[
0 = s_0 \leq t_0 < s_1 < t_1 < \ldots < s_m \leq t_m = T
\]

such that \( \eta \) is

(i) monotone in \([t_{k-1}, s_k]\) for \( k = 1, \ldots, m \),
(ii) constant in \([s_k, t_k]\) for \(k = 0, \ldots, m\),

(iii) non-monotone in \([t_{k-1}, t_k + \delta]\) for any \(\delta > 0\) and \(k = 1, \ldots, m - 1\).

We may assume that \(\eta\) is non-decreasing in \([s_0, s_1]\); the other case is analogous. Then \((-1)^{k-1}\eta\) is non-decreasing in \([s_{k-1}, s_k]\), hence in particular

\[
(-1)^{k-1}(\eta(t) - \eta(s_k)) \leq 0 \quad \text{for} \quad t \in [s_{k-1}, s_{k+1}], \quad k = 1, \ldots, m - 1. \tag{2.3}
\]

By (A.11) we have

\[
(-1)^{k-1}(x(s_k) - \eta(s_k)) = r \quad \text{for} \quad k = 1, \ldots, m - 1, \tag{2.4}
\]

and from (2.3) – (2.4) we obtain

\[
(-1)^{k-1}(x(s_k) - x(s_{k-1})) = (-1)^{k-1}(x(s_k) - \eta(s_k) + \eta(s_{k-1}) - x(s_{k-1})) \tag{2.5}
\]

\[
+ (-1)^{k-1}(\eta(s_k) - \eta(s_{k-1})) \geq 2r
\]

for \(k = 2, \ldots, m - 1\) analogous to (A.12). The elementary inequality \(|x(t) - \eta(t)| \leq r\) and (2.3) – (2.4) yield for \(t \in [s_{k-1}, s_{k+1}], \quad k = 1, \ldots, m - 1\) that

\[
(-1)^{k-1}(x(t) - x(s_k)) = (-1)^{k-1}(x(t) - \eta(t) + \eta(s_k) - x(s_k)) \tag{2.6}
\]

\[
+ (-1)^{k-1}(\eta(t) - \eta(s_k)) \leq 0.
\]

Integrating Eq. (0.1) we obtain from (2.6), for \(h > 0\) sufficiently small, that

\[
(-1)^{k-1}\frac{1}{h}\int_{s_k}^{s_k+h} \Phi(x(\tau), u(\tau)) \, d\tau = (-1)^{k-1}\frac{\alpha}{h}(x(s_k + h) - x(s_k)) \leq 0,
\]

\[
(-1)^{k-1}\frac{1}{h}\int_{s_k-h}^{s_k} \Phi(x(\tau), u(\tau)) \, d\tau = (-1)^{k-1}\frac{\alpha}{h}(x(s_k) - x(s_k - h)) \geq 0,
\]

and letting \(h \to 0^+\) we obtain for \(k = 1, \ldots, m - 1\) the inequalities

\[
(-1)^{k-1}\Phi(x(s_k), u(s_k)) \geq 0,
\]

\[
(-1)^{k-1}\Phi(x(s_k), u(s_k)) \leq 0,
\]

that is,

\[
(-1)^{k-1}(u(s_k) - g(x(s_k))) \leq 0, \tag{2.7}
\]

\[
(-1)^{k-1}(u(s_k) - g(x(s_k))) \geq 0. \tag{2.8}
\]

The next step consists in showing that the number \(m\) cannot be arbitrarily large. Set

\[
\mu(r) = \inf\{\max_{[a,b]} g - \min_{[a,b]} g : A \leq a < b \leq B, \quad b - a \geq 2r\}. \tag{2.9}
\]

By Lemma 2.2 (iii) we have \(\mu(r) > 0\), and we claim that:

For every \(j = 1, \ldots, [(m - 3)/4]\) there exists an interval \([a_j, b_j] \subset [s_{4j}, s_{4j+3}]\) such that \(|u(b_j) - u(a_j)| \geq \frac{1}{7}\mu(r)\). \tag{2.10}
Once (2.10) is proved, we conclude, using the hypothesis of uniformly bounded oscillation of $U$, that the number $m^*$ of intervals $[a_j, b_j]$, for which (2.10) holds, is at most $N\left(\frac{1}{5} \mu(r)\right)$. We may choose $[a_1, b_1 \subset [s_4, s_7],$ $[a_2, b_2 \subset [s_8, s_{11}], \ldots, [a_j, b_j \subset [s_{4j}, s_{4j+3}]$ for $j = 1, \ldots m^*$, as long as $4j + 3 \leq m - 1$. Putting

$$m^* = \max\{j \in \mathbb{N}; 4j + 3 \leq m - 1\},$$

we have

$$m \leq 4m^* + 7 \leq 4N\left(\frac{1}{5} \mu(r)\right) + 7.$$ 

Using Proposition A.4 and Corollary A.2, we obtain

$$\text{Var}_{[0,T]} \eta = \sum_{k=1}^{m} |\eta(s_k) - \eta(s_{k-1})| \leq \sum_{k=1}^{m} ||x(\cdot) - x(s_{k-1})||_{s_{k-1}, s_k} \leq m (B - A) \leq (B - A) \left(4N\left(\frac{1}{5} \mu(r)\right) + 7\right).$$

Since $||x - \eta||_{[0,T]} \leq r$, by definition of the play, and the above argument is valid for arbitrary $r > 0$, we conclude that $X$ has uniformly bounded $\varepsilon$-variation.

It remains to show that (2.10) holds. To this end, we put $x_0 = x(s_{4j})$, $x_1 = x(s_{4j+1})$, $x_2 = x(s_{4j+2})$. The assertion is easy provided

$$|g(x_1) - g(x_0)| \geq \frac{1}{4} \mu(r) \quad \text{or} \quad |g(x_2) - g(x_1)| \geq \frac{1}{4} \mu(r). \quad (2.11)$$

Indeed, then it follows from (2.7)–(2.8) that at least one of the following four cases necessarily occurs:

(i) $u(s_{4j+1}) - u(s_{4j}) \geq g(x_1) - g(x_0) \geq \mu(r)/4$,

(ii) $u(s_{4j+1}) - u(s_{4j+1}+) \geq g(x_0) - g(x_1) \geq \mu(r)/4$,

(iii) $u(s_{4j+1}) - u(s_{4j+2}) \geq g(x_1) - g(x_2) \geq \mu(r)/4$,

(iv) $u(s_{4j+2}) - u(s_{4j+1}+) \geq g(x_2) - g(x_1) \geq \mu(r)/4$.

We then obtain (2.10) by choosing $a_j, b_j$ sufficiently close to $s_{4j}$ or $s_{4j+1}$ or $s_{4j+2}$. Assume now that (2.11) fails to hold, that is,

$$|g(x_1) - g(x_0)| < \frac{1}{4} \mu(r), \quad |g(x_2) - g(x_1)| < \frac{1}{4} \mu(r).$$

As a consequence of (2.5) we have $\hat{x} := \max\{x_0, x_2\} \leq x_1 - 2r$, and (2.9) yields that

$$\max_{x_0, x_1} g \geq \max_{\hat{x}, x_1} g \geq \mu(r) + \min_{\hat{x}, x_1} g \geq \mu(r) + \min_{x_2, x_1} g. \quad (2.12)$$

Putting

$$y = \max\{\xi \in [x_0, x_1]; g(\xi) = \max_{[x_0, x_1]} g\},$$

$$z = \min\{\xi \in [x_2, x_1]; g(\xi) = \min_{[x_2, x_1]} g\}$$

we obtain from (2.12) that $g(y) - g(z) \geq \mu(r)$, hence either $g(y) - g(x_1) \geq \frac{1}{7} \mu(r)$ or $g(x_1) - g(z) \geq \frac{1}{7} \mu(r)$. We investigate the two cases separately.

A. $g(y) - g(x_1) \geq \frac{1}{7} \mu(r)$.
Then \( g(y) - g(x_0) \geq \frac{1}{4} \mu(r) > 0 \), hence \( y \in ]x_0, x_1[ \). Put \( \tau = \max \{ t \in ]s_{4j}, s_{4j+1}[ : x(t) = y \} \). For \( h > 0 \) sufficiently small we have \( x(\tau + h) \in ]y, x_1[ \), and

\[
0 \leq \alpha(x(\tau + h) - x(\tau)) = \int_{\tau}^{\tau+h} \Phi(x(t), u(t)) \, dt,
\]

hence \( \Phi(x(\tau), u(\tau+)) \geq 0 \), that is, \( u(\tau+) \geq g(x(\tau)) = g(y) \). Combining this inequality with (2.7) yields the desired result in the form

\[
u(\tau+) - u(s_{4j+1}+) \geq g(y) - g(x_1) \geq \frac{1}{2} \mu(r).
\]

B. \( g(x_1) - g(z) \geq \frac{1}{7} \mu(r) \).

Then \( g(z) - g(x_2) \leq -\frac{1}{4} \mu(r) < 0 \), hence \( z \in ]x_2, x_1[ \). Put \( \sigma = \max \{ t \in ]s_{4j+1}, s_{4j+2}[ : x(t) = z \} \). For \( h > 0 \) sufficiently small we have \( x(\sigma + h) \in ]x_2, z[ \), and

\[
0 \geq \alpha(x(\sigma + h) - x(\sigma)) = \int_{\sigma}^{\sigma+h} \Phi(x(t), u(t)) \, dt,
\]

hence \( \Phi(x(\sigma), u(\sigma+)) \geq 0 \), that is, \( u(\sigma+) \leq g(x(\sigma)) = g(z) \). Combining this inequality with (2.8) we obtain

\[
u(\sigma+) - u(s_{4j+1}) \leq g(z) - g(x_1) \leq -\frac{1}{2} \mu(r),
\]

and the proof of Theorem 2.3 is complete. \( \blacksquare \)

3. Singular hysteresis limit: The non-oscillatory case

We now consider Eq. (0.1) with some fixed \( u \in G_L(0, T) \) and \( x_0 \in ]z_*, z^*[ \), and let \( \alpha \) tends to 0. We will see in Example 3.5 below that the convergence of \( x_\alpha(t) \) as \( \alpha \to 0 \) for every \( t \in [0, T] \) can be problematic, so that further restrictions will have to be imposed either on \( u \) or on \( \Phi \). A natural and elegant description of the limit is possible if \( u \) has non-oscillatory behavior. It is based on the following statement.

Figure 1. The \( u \mapsto x \) trajectory for monotonically increasing and decreasing inputs.
Lemma 3.1. Let \( x_0 \in ]z_*, z^*[ \) and \( u_0 < u_1 \) be given such that \( \Phi(x_0, u_0) = 0 \), and put
\[
x_1 = \min \{ x \in [x_0, z^*]; \Phi(x, u_1) = 0 \} .
\]

Let \( u \in G_L(0, T) \) be a function such that
\[
u(t_0) = u_0, \quad u(t_1) = u_1, \quad u(t) \in [u_0, u_1] \quad \forall t \in [t_0, t_1],
\]
where \([t_0, t_1] \subset [0, T]\) is a given interval. Assume that there exists \( \alpha_0 > 0 \) such that \( \forall \alpha \in ]0, \alpha_0[ \) the solutions of Eq. (0.1) satisfy \( x_\alpha(t_0) \in [x_0, x_1] \). Then \( x_\alpha(t) \in [x_0, x_1] \) for all \( t \in [t_0, t_1] \) and \( \lim_{\alpha \to 0} x_\alpha(t_1) = x_1 \).

Proof. By Lemma 2.5 we have \( x_\alpha(t) \in [x_0, x_1] \) for all \( t \in [t_0, t_1] \) and \( \alpha \in ]0, \alpha_0[ \). Let \( \varepsilon \in ]0, u_1 - u_0[ \) be arbitrarily chosen. We find \( t_\varepsilon \in [t_0, t_1] \) such that \( u(t) \geq u_1 - \varepsilon \) for all \( t \in [t_\varepsilon, t_1] \), and set
\[
u_\varepsilon(t) = \begin{cases} u_0 & \text{for } t \in [t_0, t_\varepsilon), \\ u_1 - \varepsilon & \text{for } t \in [t_\varepsilon, t_1]. \end{cases}
\]

Thus we have \( x_\varepsilon = \min \{ x \in [x_0, z^*]; \Phi(x, u_1 - \varepsilon) = 0 \} \).

Let \( x^*_\alpha : [t_0, t_1] \to \mathbb{R} \) be the solution of the equation
\[
\alpha \dot{x}^*_\alpha(t) = \Phi(x^*_\alpha(t), u_\varepsilon(t)), \quad x^*_\alpha(0) = \min \{ x_\varepsilon, x_\alpha(t_0) \} . \tag{3.1}
\]

By virtue of Lemma 2.5 we have \( x^*_\alpha(t) \in [x_0, x_\varepsilon] \) for all \( t \in [t_0, t_1] \). We claim that
\[
\lim_{\alpha \to 0} x^*_\alpha(t_1) = x_\varepsilon . \tag{3.2}
\]

To prove (3.2), we fix any \( x^* \in [x_0, x_\varepsilon[ \), and put
\[
d = \min \{ \Phi(x, u_1 - \varepsilon); x \in [x_0, x^*]\} , \quad \alpha_1 = \min \left\{ \alpha_0, \frac{(t_1 - t_\varepsilon) d}{x_1 - x_0} \right\} . \tag{3.3}
\]

For an arbitrary \( \alpha \in ]0, \alpha_1[ \) and \( t \in ]t_\varepsilon, t_1[ \) we have by (3.1) that \( \dot{x}^*_\alpha(t) \geq 0 \), and we show that
\[
x^*_\alpha(t_1) \geq x^* . \tag{3.4}
\]

There is nothing to prove if \( x^*_\alpha(t) \geq x^* \) for all \( t \in [t_\varepsilon, t_1] \). If this is not the case, then there exists \( \tau \in ]t_\varepsilon, t_1[ \) such that \( x^*_\alpha(t) \geq x^* \) for \( t \in [t_\varepsilon, \tau] \). By (3.3) we have
\[
(t - t_\varepsilon) d \leq \int_{t_\varepsilon}^\tau \Phi(x^*_\alpha(t), u_1 - \varepsilon) dt = \alpha (x^*_\alpha(\tau) - x^*_\alpha(t_\varepsilon)) \leq \alpha (x^* - x_0) < (t_1 - t_\varepsilon) d .
\]

Hence \( \tau < t_1 \) and both (3.4) and (3.2) follow easily, as \( x^* < x_\varepsilon \) was arbitrary. By Lemma 2.6 we have \( x_\alpha(t) \geq x^*_\alpha(t) \) for all \( t \in [t_0, t_1] \), hence \( \liminf_{\alpha \to 0} x_\alpha(t_1) \geq x_\varepsilon \) for every \( \varepsilon > 0 \) sufficiently small. Letting \( \varepsilon \) tend to 0 we complete the proof.

In the same way we prove the following ‘dual’ statement the proof of which is left to the reader.
Lemma 3.2. Let \( x_1 \in ]z_s, z^*[ \) and \( u_2 < u_1 \) be given such that \( \Phi(x_1, u_1) = 0 \), and put
\[
x_2 = \max\{x \in ]z_s, x_1]\}; \Phi(x, u_2) = 0\}.
\]
Let \( u \in G_L(0, T) \) be a function such that
\[
u(t_1) = u_1, \quad u(t_2) = u_2, \quad u(t) \in [u_2, u_1] \quad \forall t \in [t_1, t_2],
\]
where \([t_1, t_2] \subset [0, T] \) is a given interval. Assume that there exists \( \alpha_0 > 0 \) such that for all \( \alpha \in ]0, \alpha_0] \) the solutions of Eq. (0.1) satisfy \( x_\alpha(t_1) \in [x_2, x_1] \). Then \( x_\alpha(t) \in [x_2, x_1] \) for all \( t \in [t_1, t_2] \) and \( \lim_{\alpha \to 0} x_\alpha(t_2) = x_2 \).

The above Lemmas enable us to extend the limit \( u \mapsto x \) relation depicted in Figure 1 to a mapping acting on every continuous function \( u \in C(0, T) \) such that
\[
u(t) \geq u(0) \quad \forall t \in [0, T].
\]
The same argument (we omit the details) works if
\[
u(t) \leq u(0) \quad \forall t \in [0, T].
\]
Assume that (3.5) holds. We set \( t_0 = 0 \) and for a fixed \( t \in [0, T] \) we define for \( i = 1, 2, \ldots \), recursively the sequence
\[
t_{2i−1} = \max\{\tau \in [t_{2i−2}, t]; u(\tau) = \max\{u(s); s \in [t_{2i−2}, \tau]\}\},
\]
\[
t_{2i} = \max\{\tau \in [t_{2i−1}, t]; u(\tau) = \min\{u(s); s \in [t_{2i−1}, \tau]\}\}
\]
until \( t_n = t \) for some \( n \in \mathbb{N} \). The sequence \( \{t_k\} \) can either be finite or infinite; in the latter case we have \( t_k \to t \), \( u(t_k) \to u(t) \) as \( k \to \infty \). Let \( x_0 \) be such that \( \Phi(x_0, u_0) = 0 \), and for \( j = 1, 2, \ldots \) put
\[
x_{2j−1} = \min\{x \in [x_{2j−2}, z^*]; \Phi(x, u_{2j−1}) = 0\},
\]
\[
x_{2j} = \max\{x \in ]z_s, x_{2j−1}]; \Phi(x, u_{2j}) = 0\}.
\]
We have \( x_{2j−1} > x_{2j+1} \) and \( x_{2j} < x_{2j+2} \) for all \( j \geq 1 \), and \( \Phi(x_k, u(t_k)) = 0 \) for all \( k = 0, 1, \ldots \).
If \( t = t_n \) for some \( n \in \mathbb{N} \), then we put \( x(t) = x_n \). If the sequence \( \{t_k\} \) is infinite, then \( \lim_{j \to \infty} x_{2j−1} = y^*, \lim_{j \to \infty} x_{2j} = y_*, y_* \leq y^* \), \( \Phi(x, u(t)) = 0 \) for all \( x \in [y_*, y^*] \). By hypothesis 2.1, the set \( E_{u(t)} \) does not contain any interval, hence we may put \( x(t) = y_* = y^* \).

A reader who is familiar with the Preisach hysteresis model has certainly recognized in (3.7) – (3.10) the construction algorithm for the Preisach memory curve, see e.g. [3, 7, 12]. Example 3.5 below however shows that the initial memory deletion property (see [3, Section 2.7]) is not fulfilled in general, so that the hysteresis model derived here does not belong to the Preisach-type class.

The following Theorem constitutes the main result of this section.

Theorem 3.3. Let Hypotheses 2.1 hold and let \( u \in C(0, T) \) be a function satisfying (3.5). Let \( x_\alpha \) be the solution to (0.1) with \( \Phi(x_0, u(0)) = 0 \), and let \( x(t) \) be defined as in the above. Then \( \lim_{\alpha \to 0} x_\alpha(t) = x(t) \) for every \( t \in [0, T] \).

Proof. Let \( \varepsilon > 0 \) and \( t \in [0, T] \) be given. We construct sequences \( 0 = t_0 < t_1 < \ldots, x_1 > x_3 > \ldots, x_0 \leq x_2 < x_4 < \ldots \) as in (3.7) – (3.10). By induction using alternatively
Lemmas 3.1 and 3.2 we find a sequence $\alpha_0 \geq \alpha_1 \geq \ldots$ such that for all available $k$ and all $\alpha \in [0, \alpha_k]$ we have

$$(-1)^k x_{\alpha}(t_k) \in \left[(-1)^k x_k, (-1)^k x_{k+1}\right], \quad |x_{\alpha}(t_k) - x_k| < \varepsilon.$$  

The assertion follows if $t = t_n$ for some $n$. Otherwise, we find $n$ such that $|x_n - x_{n+1}| < \varepsilon$ and use Lemma 2.5 to conclude that $(-1)^n x_{\alpha}(t) \in \left[(-1)^n x_n, (-1)^n x_{n+1}\right]$ for $\alpha < \alpha_n$, hence $|x_{\alpha}(t) - x(t)| \leq |x_n - x_{n+1}| < \varepsilon$ and the proof is complete.

**Remark 3.4.** The convergence $x_\alpha(t) \to x(t)$ does not hold in general if Condition (iv) (a) in Hypothesis 2.1 is violated. Assume that there exist $y_s < y^*$, $\varepsilon > 0$, and $\bar{u} \in \mathbb{R}$ such that $\Phi(x, \bar{u}) = 0$ for all $x \in [y_s - \varepsilon, y^* + \varepsilon]$. Let $x_0, u_0$ be such that $\Phi(x_0, u_0) = 0$, $u_0 < \bar{u}$, and let us consider the set

$$Y = \{u \in C(0, T); u(0) = u_0, u(t) = \bar{u}, \text{ and (3.5) holds}\}$$  

with a fixed $t \in [0, T]$. For a sequence $\alpha_j \searrow 0$ as $j \to \infty$ we denote by $x_j$ the solutions to the problem

$$\alpha_j \dot{x}_j = \Phi(x_j(t), u(t)), \quad x_j(0) = x_0.$$  

For $j \in \mathbb{N}$ we define the sets $A^+_{\alpha_j} = \{u \in Y; x_j(t) \leq y^*\}$, $A^-_{\alpha_j} = \{u \in Y; x_j(t) \geq y_s\}$, $B^+_{\alpha_j} = \bigcap_{j \geq 0} A^+_{\alpha_j}$. Then each $A^+_{\alpha_j}$, $A^-_{\alpha_j}$ is closed in $Y$, hence also $B^+_{\alpha_j}$, $B^-_{\alpha_j}$ are closed. The sets $B^+_{\alpha_j}$ and $B^-_{\alpha_j}$ are nowhere dense in $Y$. Indeed, for each $u \in Y$ and each $r \in (0, (\bar{u} - u_0)/2]$ we can find $r_0 \in (0, t]$ such that $|u(t) - u(\tau)| \leq r/2$ for $\tau \in [t, t']$, and set

$$u^{(r)}(\tau) = \begin{cases} \max\{u_0, u(\tau) \pm \frac{r}{t_0}\} & \text{for } \tau \in [0, t_0], \\ \frac{r}{t-t_0} u(t) + \frac{t-t_0}{r} (u(0) \pm r) & \text{for } t \in [t_0, t'], \\ u(\tau) & \text{for } \tau \in [t, T]. \end{cases}$$  

We have $|u(\tau) - u^{(r)}(\tau)| \leq 3r/2$, $u^{(r)}(t_0) \geq \bar{u} + r/2$, $u^{(r)}(t) \leq \bar{u} - r/2$ with $u^{(r)}$ monotone in $[t_0, T]$. By Theorem 3.3, the solutions $x^{(r)}_{i\pm}$ to the problem

$$\alpha_i x^{(r)}_{i\pm} = \Phi(x^{(r)}_{i\pm}(t), u^{(r)}(t)), \quad x^{(r)}_{i\pm}(0) = x_0$$  

satisfy $\lim_{i \to \infty} x_{i+}(t) \geq y^* + \varepsilon$, $\lim_{i \to \infty} x^{(r)}_{i-}(t) \leq y_s - \varepsilon$. In particular, each $u \in B^+_{\alpha_j}$ can be uniformly approximated by functions $u^{(r)}_+ \notin B^+_{\alpha_j}$ and each $u \in B^-_{\alpha_j}$ can be uniformly approximated by functions $u^{(r)}_- \notin B^-_{\alpha_j}$. By virtue of Baire’s Theorem (see [5, Chapter II, §3]), the set $Y_0 = Y \setminus \bigcup_{j=1}^\infty (B^+_{\alpha_j} \cup B^-_{\alpha_j})$ is non-empty. By construction, for $u \in Y_0$ we have

$$\forall j \in \mathbb{N} \left\{ \exists k \geq j : x_k(t) > y^*, \quad \exists k' \geq j : x_{k'}(t) < y_s \right\},$$  

hence the limit of $x_\alpha(t)$ as $\alpha \to 0$ does not exist.

**Example 3.5.** We now present an example showing the difficulties that may occur if the non-oscillatory property (3.5) or (3.6) is not fulfilled. Given a fixed parameter $c \in [0, 1]$, we consider
the Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$ defined to be affine in each interval $]-\infty, -c]$, $[-c^k, -c^{k+1}]$, $[c^{k+1}, c^k]$, $[c, \infty[$ for $k = 1, 2, \ldots$, and such that

$$
\begin{align*}
g(-c^{2i-1}) &= -c^{2i} \\
g(-c^{2i}) &= c^{2i-1} \\
g(c^{2i}) &= -c^{2i} \\
g(c^{2i+1}) &= c^{2i+1} \\
g(c) &= c, \\
g(\pm\infty) &= \pm\infty,
\end{align*}
$$

see Figure 2. We define a continuous function $u : [0, 1] \to \mathbb{R}$ associated with a sequence $1 = t_1 > t_2 > \ldots > 0$, $\lim_{k \to \infty} t_k = 0$ by the formula

$$
u(t_k) = (-1)^{k-1} c^k$$

monotonically interpolated in each interval $[t_{k+1}, t_k]$. For each $n \in \mathbb{N}$ we find $\tau_n \in ]t_{n+1}, t_n[$ such that $u(\tau_n) = 0$ and set

$$
u^{(n)}(t) = \begin{cases} 0 & \text{for } t \in [0, \tau_n[ , \\ u(t) & \text{for } t \in [\tau_n, 1]. \end{cases}$$

Let $x^{(n)}_\alpha$ be solutions of the problem

$$\alpha x^{(n)}_\alpha(t) = \nu^{(n)}(t) - g(x^{(n)}_\alpha(t)), \quad x^{(n)}_\alpha(0) = 0. \quad (3.11)$$
Following (3.9), (3.10) we can construct the limit functions \( x^{(n)}(t) = \lim_{\alpha \to 0} x^{(n)}_\alpha(t) \) explicitly for each \( n \in \mathbb{N} \) and easily obtain by induction that
\[
x^{(n)}(1) = \begin{cases} 
c & \text{if } n \text{ is odd}, \\
-c^2 & \text{if } n \text{ is even}.
\end{cases}
\]
This shows that the limit passage as \( \alpha \to 0 \) in the equation obtained from (3.11) for \( n \to \infty \)
\[
\alpha \dot{x}_\alpha(t) = u(t) - g(x_\alpha(t)), \quad x_\alpha(0) = 0
\]
is not straightforward and a singular behavior is to be expected. A rigorous analysis is however still awaited.

4. Piecewise monotone nonlinearity

We now remove all restrictions on the input \( u \in G_L(0, T) \). Having in mind Example 3.5, we reduce the class of admissible functions \( \Phi \) by assuming in addition to Hypothesis 2.1 that there exists an interval \([A, B] \subset [z_*, z^*] \) such that
\[
A \leq x_0 \leq B, \quad g(A) \leq u(t) \leq g(B) \quad \forall t \in [0, T],
\]
and such that \( g \) is a piecewise strictly monotone function in \([A, B] \) in the sense that there exist \( A = y_0 < y_1 < \ldots < y_{2n} < y_{2n+1} = B \) such that
\[
(-1)^{k-1} g \quad \text{is increasing in } [y_{k-1}, y_k] \quad \text{for } k = 1, \ldots, 2n + 1,
\]
as in Figure 3. We denote \( S_i = [y_{2i}, y_{2i+1}] \) for \( i = 0, \ldots, n \), \( J_i = [y_{2i-1}, y_{2i}] \) for \( i = 1, \ldots, n \).

The value of \( u(0) \) is not relevant for the analysis of Eq. (0.1), we may therefore set \( u(0) = g(x_0) \). On the other hand, we will assume that
\[
x_0 \in \bigcup_{i=1}^{n} J_i \implies u(0^+) \neq g(x_0).
\]
We will see in Remark 4.6 below that there is no convergence in general if the implication (4.3) does not hold, cf. also [11, Theorem 3.3].

Assume first that
\[
x_0 \in S_{i_0} \quad \text{for some } i_0 \in \{0, \ldots, n\},
\]
and put
\[
t_0 = \max\{t \in [0, T]; u(s) \in g(S_{i_0}) \forall s \in [0, t]\}.
\]
Then one of the following cases occurs:
(a) \( u(t_0^+) > g(y_{2i_0+1}), \exists h > 0 \forall \tau \in [t_0, t_0 + h]: u(\tau) \leq u(t_0^+) \),
(b) \( u(t_0^+) \geq g(y_{2i_0+1}), \exists \tau_\ell \setminus t_0 \forall \ell \in \mathbb{N}: u(\tau_\ell) > u(t_0^+) \),
(c) \( u(t_0^+) < g(y_{2i_0}), \exists h > 0 \forall \tau \in [t_0, t_0 + h]: u(\tau) \geq u(t_0^+) \),
(d) \( u(t_0^+) \leq g(y_{2i_0}), \exists \tau_\ell \setminus t_0 \forall \ell \in \mathbb{N}: u(\tau_\ell) < u(t_0^+) \).

If (4.4) does not hold, then
\[
x_0 \in J_{i_0} \quad \text{for some } i_0 \in \{1, \ldots, n\},
\]
and we set
\[
t_0 = 0
\]
with a similar case distinction as before, namely.
\[(a_0') \quad u(t_0 + ) > g(x_0), \exists h > 0 \forall \tau \in [t_0, t_0 + h] : u(\tau) \leq u(t_0 + ),
\]
\[(b_0') \quad u(t_0 + ) > g(x_0), \exists \tau \in [t_0, t_0 + h] : u(\tau) > u(t_0 + ),
\]
\[(c_0') \quad u(t_0 + ) < g(x_0), \exists \tau \in [t_0, t_0 + h] : u(\tau) \geq u(t_0 + ),
\]
\[(d_0') \quad u(t_0 + ) < g(x_0), \exists \tau \in [t_0, t_0 + h] : u(\tau) < u(t_0 + ).
\]

In each of the above cases we put respectively
\[(a_0) + (a_0') \quad i_1 = \min\{i \geq i_0 : u(t_0 + ) \leq g(y_{2i+1})\},
\]
\[(b_0) + (b_0') \quad i_1 = \min\{i \geq i_0 : u(t_0 + ) < g(y_{2i+1})\},
\]
\[(c_0) + (c_0') \quad i_1 = \max\{i < i_0 : u(t_0 + ) \geq g(y_{2i})\},
\]
\[(d_0) + (d_0') \quad i_1 = \max\{i < i_0 : u(t_0 + ) < g(y_{2i})\},
\]

and
\[t_1 = \max\{t \in [t_0, T] ; u(s) \in g(S_{i_1}) \forall s \in [t_0, t]\}.
\]

We can easily check in each case that we have \(t_1 > t_0\). Indeed, for \((a_0)\) this follows from the fact that \(i_1 > i_0\), \(u(\tau) \leq u(t_0 + ) \leq g(y_{2i+1})\) for \(\tau \in [t_0, t_0 + h]\) and \(u(t_0 + ) > g(y_{2i+1}) > g(y_{2i+1})\).

In case \((b_0')\) for instance we either have \(i_1 = i_0\) and \(g(y_{2i+1}) < g(x_0) < u(t_0 + ) < g(y_{2i+1})\) or \(i_1 > i_0\) with \(g(y_{2i+1}) > u(t_0 + ) \geq g(y_{2i+1}) > g(y_{2i+1})\). The other cases are analogous.

Assume now that we have constructed sequences \(0 \leq t_0 < t_1 < \ldots < t_k < T\) and \(i_0, i_1, \ldots, i_k\) for some \(k \geq 1\) such that
\[t_j = \max\{t \in [t_{j-1}, T] ; u(s) \in g(S_{i_j}) \forall s \in [t_{j-1}, t]\}\] (4.5)

for all \(j = 1, \ldots, k\). As above, we distinguish the cases
\[(a_k) \quad u(t_k + ) > g(y_{2i_{k+1}}), \exists h > 0 \forall \tau \in [t_k, t_k + h] : u(\tau) \leq u(t_k + ),
\]
\[(b_k) \quad u(t_k + ) \geq g(y_{2i_{k+1}}), \exists \tau \in [t_k, t_k + h] : u(\tau) > u(t_k + ),
\]
\[(c_k) \quad u(t_k + ) < g(y_{2i_{k+1}}), \exists h > 0 \forall \tau \in [t_k, t_k + h] : u(\tau) \geq u(t_k + ),
\]
\[(d_k) \quad u(t_k + ) \leq g(y_{2i_{k+1}}), \exists \tau \in [t_k, t_k + h] : u(\tau) < u(t_k + ),
\]

and put respectively
\[(a_k) \quad i_{k+1} = \min\{i > i_k : u(t_k + ) \leq g(y_{2i+1})\},
\]
\[(b_k) \quad i_{k+1} = \min\{i > i_k : u(t_k + ) < g(y_{2i+1})\},
\]
\[(c_k) \quad i_{k+1} = \max\{i < i_k : u(t_k + ) \geq g(y_{2i})\},
\]
\[(d_k) \quad i_{k+1} = \max\{i < i_k : u(t_k + ) \leq g(y_{2i})\},
\]

and
\[t_{k+1} = \max\{t \in [t_k, T] ; u(s) \in g(S_{i_{k+1}}) \forall s \in [t_k, t]\}.
\]

We put \(\Delta = \min\{|g(y_{p}) - g(y_{q})| ; p, q \in \{0, \ldots, 2n + 1\}, g(y_{p}) \neq g(y_{q})\} > 0\). We then claim that \(t_{k+1} > t_k\) and either \(t_{k+1} = T\) or
\[\forall \delta > 0 \exists \tau \in [t_k, t_k + \delta] , \exists \sigma \in [t_{k+1}, t_{k+1} + \delta] : |u(\tau) - u(\sigma)| \geq \Delta / 2.\] (4.6)

Once again, we use the above case distinctions to prove the claim. The fact that \(t_{k+1} > t_k\) is obvious, and (4.6) is based on the following argument.
\[(a_k) + (b_k) : \quad \text{We have } g(y_{2i_{k+1}+1}) > \max\{g(y_{2i_{k+1}+1})\} \text{ and either } g(y_{2i_{k+1}+1}) \geq u(t_k + ) \geq g(y_{2i_{k+1}+1}) > g(y_{2i_{k+1}+1}) \geq u(t_{k+1} + ).\]
Hence
\[ u(t_k+) - u(t_{k+1}+) > g(y_{2i+1} - y_{2i+1}) \geq \Delta, \]
or
\[ u(t_{k+1}+) \geq g(y_{2i+1}+1) > g(y_{2i+1}) \geq u(t_k), \]
and hence
\[ u(t_{k+1}+) - u(t_k) \geq g(y_{2i+1} + 1) - g(y_{2i+1}) \geq \Delta; \]

\((c_k) + (d_k):\) We have \(g(y_{2i+1}) < \min\{g(y_{2i+1}+2), g(y_{2i+1})\}\) and either
\[ g(y_{2i+1}) \leq u(t_k) \leq g(y_{2i+1}+2) < g(y_{2i+1}+1) \leq u(t_{k+1}). \]

Hence
\[ u(t_{k+1}+) - u(t_k) > g(y_{2i+1}+1) - g(y_{2i+1}+2) \geq \Delta, \]
or
\[ u(t_{k+1}+) \leq g(y_{2i+1}) < g(y_{2i+1}) \leq u(t_k), \]
and hence
\[ u(t_k) - u(t_{k+1}+) \geq g(y_{2i+1}) - g(y_{2i+1}) \geq \Delta, \]
and (4.6) follows. As \(u\) is regulated, we conclude from (4.6) that the sequence \(\{t_k\}\) is bounded, \(t_m = T\) for some \(m \in \mathbb{N}\), and (4.5) holds for all \(j = 1, \ldots, m\). We now define a function \(x \in G_L(0, T)\) by the formula

\[
x(t) = \begin{cases} 
  x_0 & \text{for } t = 0, \\
  \left(g \left| S_{t_0} \right. \right)^{-1} (u(t)) & \text{for } t \in [0, t_0], \\
  \left(g \left| S_{t_k} \right. \right)^{-1} (u(t)) & \text{for } t \in [t_{k-1}, t_k], \ k = 1, \ldots, m. 
\end{cases} \tag{4.7}
\]

The mapping \(\Gamma : ]z, +\) \times \(G_L(0, T) \rightarrow G_L(0, T) : (x_0, u) \mapsto x\) defined by (4.7), the diagram of which is depicted in Figure 3, is a hysteresis operator (that is, causal and rate-independent according to Visintin’s terminology in [14]) which we call a generalized relay generated by \(g\).

Our main result is as follows:

**Theorem 4.1.** Let (4.1) – (4.3) hold and let \(x\) be given by (4.7). Then the solutions \(x_\alpha\) to (0.1) fulfill \(\lim_{\alpha \rightarrow 0} x_\alpha(t) = x(t)\) for every \(t \in [0, T]\).

Theorem 4.1 together with (4.7) manifests the hysteresis character of the limit \(u \mapsto x\) relation, see Fig. 3. The intervals \(S_j \cap [g(A), g(B)]\) constitute the stability domain. Theorem 4.1 and Lemma 2.6 immediately imply that the mapping \(\Gamma\) is monotone in the following sense.

**Corollary 4.2.** Let \(x_0 \leq y_0\) and \(u_0, v_0 \in G_L(0, T)\) be such that \(u(t) \leq v(t)\) for all \(t \in [0, T]\), and put \(x = \Gamma(x_0, u), \ y = \Gamma(y_0, v)\). Then for every \(t \in [0, T]\) we have \(x(t) \leq y(t)\).

The proof of Theorem 4.1 is based on a series of Lemmas.

**Lemma 4.3.** Let Hypothesis 2.1 hold, and let \(z_* \leq a < b \leq c \leq z^*\) be such that \(g(a) \leq g(b) \leq g(c), \ g(x) \leq g(b)\) for all \(x \in [a, b]\). Assume that there exist \(\alpha_0 > 0\) and \(t_0 \in [0, T]\) such that

(i) \(x_\alpha(t_0) \in [a, b]\) for all \(\alpha \in [0, \alpha_0]\),
(ii) \( \exists \tau_{\ell} \setminus t_0 : u(\tau_{\ell}) > g(b) \) for all \( \ell \in \mathbb{N} \).

Then we have

\[
\forall t \in [t_0, T] \quad \exists \alpha_1 \in [0, \alpha_0] \quad \forall \alpha \in [0, \alpha_1] \quad \exists t_{\alpha} \in [t_0, t] : \quad x_{\alpha}(t_{\alpha}) \in [b, c].
\]

Proof. We have \( u(t_0^+) \geq g(b) \), hence there exists \( t^* \in [t_0, T] \) and such that \( u(\tau) \geq g(a) \) for all \( \tau \in [t_0, t^*] \). Let \( t \in [t_0, T] \) be given. We fix some \( \ell \in \mathbb{N} \) and \( \bar{v} \in \mathbb{R} \) such that \( \tau_{\ell} < \min\{t^*, t\} \) and

\[
g(b) < \bar{v} < \min\{u(\tau_{\ell}), g(c)\}.
\]

Then \( \Phi(y, \bar{v}) > 0 \) for all \( y \in [a, b] \). Let \( d > 0 \) be such that

\[
\Phi(y, \bar{v}) \geq 2d \quad \forall y \in [a, b] \quad u(\tau_{\ell}) - \bar{v} \geq d.
\]

Put

\[
c' = \min\{y \in [b, c] : \Phi(y, \bar{v}) = d\}.
\]

We have \( \Phi(b, \bar{v}) \geq 2d \), \( \Phi(c, \bar{v}) < 0 \), hence \( b < c' < c \) and

\[
\Phi(y, \bar{v}) \geq d \quad \forall y \in [a, c'].
\]

(4.8)

For \( v \in \mathbb{R} \) and \( y \in [z_s, z^*] \) set

\[
\tilde{\Phi}(y, v) = \begin{cases} \Phi(y, v) & \text{if } y \leq c', \\ \Phi(c', v) & \text{if } y > c'. \end{cases}
\]

We find \( s_{\ell} \in [t_0, t_{\ell}] \) such that \( u(\tau) \geq \bar{v} \) for every \( \tau \in [s_{\ell}, t_{\ell}] \) and consider the solution \( z_{\alpha} : [t_0, t^*] \rightarrow \mathbb{R} \) for \( \alpha \in [0, \alpha_0] \) to the problem

\[
\alpha \dot{z}_{\alpha}(\tau) = \tilde{\Phi}(z_{\alpha}(\tau), u(\tau)), \quad z_{\alpha}(t_0) = x_{\alpha}(t_0).
\]

From Lemma 2.4 for \( w(\tau) = a - z_{\alpha}(\tau) \) we obtain \( z_{\alpha}(\tau) \geq a \) for all \( \tau \in [t_0, t^*] \). On the other hand, by (4.8) we have for \( \tau \in [s_{\ell}, t_{\ell}] \) that

\[
\alpha \dot{z}_{\alpha}(\tau) \geq d, \quad z_{\alpha}(\tau) - z_{\alpha}(s_{\ell}) \geq \frac{d}{\alpha} (\tau - s_{\ell}).
\]
In particular, for $0 < \alpha \leq \alpha_1 := \min\{\alpha_0, (\tau_0 - s_0)\, d/(c' - a)\}$ we have $z_\alpha(\tau) \geq z_\alpha(s_\ell) + c' - a \geq c'_\alpha$, and we may find $t_\alpha \in [t_0, \tau_\ell]$ such that $\alpha(t) = \min\{\tau \in [t_0, \tau_\ell] : z_\alpha(\tau) = c'_\alpha\}$. Then for all $\tau \in [t_0, t_\alpha]$ we have $z_\alpha(\tau) \in [a, c']$, hence $\Phi(z_\alpha(\tau), u(\tau)) = \Phi(z_\alpha(\tau), u(\tau))$ and $z_\alpha(\tau) = x_\alpha(\tau)$. In particular, $z_\alpha(t_\alpha) = x_\alpha(t_\alpha) = c'$ and the proof is complete.\hfill\blacksquare

An analogous statement, Lemma 4.4, holds in the reverse case and we leave its proof to the reader.

**Lemma 4.4.** Let Hypothesis 2.1 hold, and let $z_\alpha < a < b < c < z_\alpha$ be such that $g(a) < g(b) < g(c)$, $g(x) \geq g(b)$ for all $x \in [b, c]$. Assume that there exist $\alpha_0 > 0$ and $t_0 \in [0, T]$ such that

(i) $x_\alpha(t_0) \in [b, c]$ for all $\alpha \in [0, \alpha_0]$,

(ii) $\exists \tau_\ell \setminus t_0 : \Phi(\alpha(t), u(\tau)) < g(b)$ for all $\ell \in \mathbb{N}$.

Then we have

$$\forall t \in [t_0, T] \quad \exists \alpha_1 \in [0, \alpha_0] \quad \forall \alpha \in [0, \alpha_1] \quad \exists t_\alpha \in [t_0, t] : \quad x_\alpha(t_\alpha) \in [a, b].$$

We now investigate the limit passage as $\alpha \to 0$.

**Lemma 4.5.** Let $[a, b] \subset ]z_\alpha, z_\alpha[$ and $[s, t] \subset [0, T]$ be intervals such that

(i) $g$ is increasing in $[a, b]$,

(ii) $x_\alpha(\tau) \in [a, b]$ for every $\tau \in [s, t]$ and for $\alpha$ sufficiently small.

Then we have

$$u(\tau) \in [g(a), g(b)], \quad \lim_{\alpha \to 0} x_\alpha(\tau) = \left(g\big|_{[a, b]}\right)^{-1}(u(\tau)) \quad \text{for every } \tau \in [s, t].$$

**Proof.** Assume, for instance, that $u(\tau) < g(a)$ for some $\tau \in [s, t]$. We fix $h \in ]0, \tau - s[$ and $\varepsilon > 0$ such that $\Phi(a, u(\sigma)) \leq -\varepsilon$ for $\sigma \in [\tau - h, \tau]$. Integrating Eq. (0.1) from $\tau - h$ to $\tau$ we obtain that $\alpha(x_\alpha(\tau) - x_\alpha(\tau - h)) \leq -h\varepsilon$ which is a contradiction for $\alpha$ sufficiently small. The same contradiction is obtained if $u(\tau) > g(b)$.

The system $\{x_\alpha : \alpha > 0\}$ is bounded and has uniformly bounded $\varepsilon$-variation, hence, according to Proposition 1.4, there exists a sequence $\alpha_n \to 0$ and a function $x^* \in G(0, T)$ such that

$$\lim_{n \to \infty} x_\alpha(\tau) = x^*(\tau) \quad \forall \tau \in [0, T]. \quad (4.9)$$

We therefore have $x^*(\tau) \in [a, b]$ for $\tau \in [s, t]$. Lemma 4.5 will be proved if we check that

$$g(x^*(\tau)) = u(\tau) \quad \text{for every } \tau \in [s, t]. \quad (4.10)$$

Indeed, then $x^*(\tau) = \left(g\big|_{[a, b]}\right)^{-1}(u(\tau))$ is the unique limit of $x_\alpha(\tau)$ as $\alpha \to 0$ independently of the subsequence $\alpha_n$.

For every test function $w \in W^{1,1}_\omega(0, T)$ we have

$$\alpha_n \int_0^T x_\alpha(\tau) \dot{w}(\tau) \, d\tau = -\int_0^T \Phi(x_\alpha(\tau), u(\tau)) w(\tau) \, d\tau,$$

and letting $\alpha_n \to 0$ we obtain

$$g(x^*(\tau)) = u(\tau) \quad \text{for a.e. } \tau \in [0, T].$$
To prove (4.10), we fix $\tau \in ]s,t[$, and an arbitrary $\varepsilon > 0$. Let $\delta > 0$ and $h \in ]0,\tau - s[\,$ be such that

\[
\delta = \min\{|g(y) - g(z)| : y, z \in [a, b], |y - z| \geq \varepsilon\},
\]

\[
\sigma \in [\tau - h, \tau] \Rightarrow |u(\sigma) - u(\tau)| < \frac{\delta}{2}.
\]

Let $q_k \not\in \tau$ be a sequence such that $0 < \tau - q_k < h$, and $g(x^*(q_k)) = u(q_k)$ for every $k \in \mathbb{N}$. For all $\sigma \in [q_k, \tau]$ and $k \in \mathbb{N}$ we then have

\[
u(\sigma) \in [g(a), g(b)] \cap [u(q_k) - \delta, u(q_k) + \delta] \subset [g(a), g(b)] \cap [g(x^*(q_k) - \varepsilon), g(x^*(q_k) + \varepsilon)].
\]

For each fixed $k \in \mathbb{N}$ we find $n(k)$ such that for $n \geq n(k)$ we have $|x_{\alpha_n}(q_k) - x^*(q_k)| < \varepsilon$. We now apply Lemma 2.5 for $\alpha = \alpha_n$ on intervals $[q_k, \tau]$ instead of $[s, t]$ and $[a, b] \cap [x^*(q_k) - \varepsilon, x^*(q_k) + \varepsilon]$ instead of $[a, b]$ and obtain

\[
x_{\alpha_n}(\sigma) \in [a, b] \cap [x^*(q_k) - \varepsilon, x^*(q_k) + \varepsilon] \quad \forall \sigma \in [q_k, \tau] \forall n \geq n(k).
\]

Then (4.9) yields that $|x^*(\tau) - x^*(q_k)| \leq \varepsilon$, and letting $k \to \infty$ we obtain that $|x^*(\tau) - x^*(\tau)| \leq \varepsilon$. Since $\varepsilon > 0$ has been chosen arbitrarily, we conclude that $x^*(\tau) = x^*(\tau)$ for every $\tau \in ]s, t[$. Consequently, $g(x^*(\tau))$ and $u(\tau)$ are left-continuous regulated functions which coincide almost everywhere. Hence (4.10) holds and Lemma 4.5 is proved.

We now conclude this section by proving Theorem 4.1.

**Proof of Theorem 4.1.** From Lemmas 2.5, 4.5 it follows that $x_{\alpha}(t) \in S_{t_0}$ and $x_{\alpha}(t) \to x(t)$ as $\alpha \to 0$ for all $t \in [0, t_0]$ whenever $t_0 > 0$. According to the above classification we consider separately the cases (A) $= (a_0) + (a_0') + (b_0) + (b_0')$ and (B) $= (c_0) + (c_0') + (d_0) + (d_0')$ and put

\[
(A) \quad \tilde{u} = \begin{cases} 
\max\{g(y_{2i+1}) : i = i_0, \ldots, i_1 - 1\} & \text{if } i_1 > i_0, \\
\min\{g(y_{2i+1})\} & \text{if } i_1 = i_0.
\end{cases}
\]

We have $g(y_{2i_1}) < g(y_{2i_1 - 1}) \leq \tilde{u} < g(y_{2i_1 + 1})$, hence we may use Lemma 4.3 with $a = x_0$ if $t_0 = 0$, $a = y_{2i_0}$ if $t_0 > 0$, $b = (g|_{S_{t_1}})^{-1}(\tilde{u})$, $c = y_{2i_1 + 1}$ and any $\alpha > 0$ to obtain that

\[
\forall t \in [t_0, t_1] \exists \alpha_1 \in ]0, \alpha_0[ \forall \alpha \in ]0, \alpha_1[ \exists \alpha_2 \in ]0, t_0, t_1[ : x_{\alpha}(t_0) \in S_{t_1}; \quad (4.11)
\]

(B) $\tilde{u} = \begin{cases} 
\min\{g(y_{2i+1}) : i = i_1 + 1, \ldots, i_0\} & \text{if } x_0 \geq y_{2i_0}, \\
\min\{g(y_{2i+1}) : i = i_1 + 1, \ldots, i_0-1\} & \text{if } x_0 < y_{2i_0}, \text{ or } i_1 \leq i_0 - 2, \\
g(x_0) & \text{if } x_0 < y_{2i_0}, \text{ or } i_1 = i_0 - 1.
\end{cases}
\]

Using in this case Lemma 4.4 with $c = x_0$ if $t_0 = 0$, $c = y_{2i_0+1}$ if $t_0 > 0$, $b = (g|_{S_{t_1}})^{-1}(\tilde{u}), a := y_{2i_1}$, we obtain again (4.11). We now continue in the same way and prove by induction for every $k = 1, \ldots, m$ that

\[
\forall t \in [t_{k-1}, t_k] \exists \alpha_{k} \in ]0, \alpha_{k-1}[ \forall \alpha \in ]0, \alpha_{k}[ \exists \alpha_{k} \in ]t_{k-1}, t_k[ : x_{\alpha}(t_0) \in S_{t_k}; \quad (4.12)
\]

For every $t \in [t_{k-1}, t_k]$ the convergence $x_{\alpha}(t) \to x(t)$ now follows from (4.12) and Lemmas 2.5, 4.5, and Theorem 4.1 is proved.
Remark 4.6. The assertion of Theorem 4.1 does not hold in general if \( x_0 \in J_i \) and \( u(0^+) = g(x_0) \). We argue as in Remark 3.4 and consider the set

\[ Y = \{ u \in G_L(0, T); u(0) = u(0^+) = g(x_0), u(t) \in [g(y_2i), g(y_{2i-1})] \, \forall t \in [0, T] \}, \]

and a sequence \( \alpha_j \searrow 0 \) as \( j \to \infty \). Let \( x_j \) for \( j \in \mathbb{N} \) be the solution to the problem

\[ \alpha_j \dot{x}_j = \Phi(x_j(t), u(t)), \quad x_j(0) = x_0. \]

For a fixed \( \bar{t} \in [0, T] \) and for \( j \in \mathbb{N} \) we define the sets \( A^j_+ = \{ u \in Y; x_j(\bar{t}) \leq y_{2i} - \alpha_j \} \), \( A^j_- = \{ u \in Y; x_j(\bar{t}) \geq y_{2i-1} + \alpha_j \} \), \( B^j = \bigcap_{i=1}^{\infty} A^j_i \). Then each \( A^j_+, A^j_- \) is closed in \( Y \), hence also \( B^j_+, B^j_- \) are closed. The sets \( B^j_+ \) (and analogously also \( B^j_- \)) are nowhere dense in \( Y \). Indeed, each \( u \in B^j_+ \) can be uniformly approximated by functions \( u^{(r)} \in Y \) of the form

\[ u^{(r)}(t) = \begin{cases} u(0) & \text{for } t \in [0, t_r] \\ \min\{u(t) + r, g(y_{2i-1})\} & \text{for } t \in [t_r, T] \end{cases} \]

for \( r \to 0^+ \), where \( t_r \in [0, \bar{t}] \) for \( r > 0 \) is chosen in such a way that \( |u(t) - u(0)| \leq r/2 \) for \( t \in [0, 2t_r] \). The solution \( x^{(r)}_i \) to the problem

\[ \alpha_i \dot{x}_i^{(r)} = \Phi(x_i^{(r)}(t), u^{(r)}(t)), \quad x_i^{(r)}(0) = x_0 \]

satisfies \( g(x_k^{(r)}(t_r)) = g(x_0) = u(0) < u^{(r)}(t_r+) \). We are thus in the situation of (4.11) with \( t_0 = t_r \) and \( t = \bar{t} \). Hence, for \( k \geq j \) sufficiently large, there exists \( t_k \in [t_r, \bar{t}] \) such that \( x_k^{(r)}(t_r) \in S_i \). From Lemma 2.5 it follows that \( x_k^{(r)}(\bar{t}) \in S_i \), and hence \( u^{(r)} \in Y \setminus B^j_+ \). From Baire’s Theorem we see that the set \( Y_0 = Y \setminus \bigcup_{j=1}^{\infty} (B^j_+ \cup B^j_-) \) is non-empty. By construction, for \( u \in Y_0 \) we have

\[ \forall j \in \mathbb{N} \left\{ \exists k \geq j : x_k(\bar{t}) > y_{2i} - \alpha_k, \quad \exists k' \geq j : x_k(\bar{t}) < y_{2i-1} + \alpha_{k'} \right\}. \]

hence \( \limsup_{j \to \infty} x_j(\bar{t}) \geq y_{2i} \), \( \liminf_{j \to \infty} x_j(\bar{t}) \leq y_{2i-1} \).

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Appendix A. The play operator

We give here a brief overview of results on the so-called play operator which constitutes one of the basic elements in the mathematical theory of hysteresis and is systematically used in our analysis. Consider the problem

Problem (P). For a given \( r > 0 \), \( x \in C(0, T), \) and \( z_0 \in [-r, r] \), find \( \eta \in CBV(0, T) \) such that

\[ x(t) - \eta(t) \in [-r, r] \quad \forall t \in [0, T], \quad \text{for } (A.1) \]
\[ x(0) - \eta(0) = z_0, \quad \text{for } (A.2) \]
\[ \int_0^T (x(\tau) - \eta(\tau) - y(\tau)) \, d\eta(\tau) \geq 0 \quad \forall y \in C(0, T), \quad \|y\|_{[0, T]} \leq r. \quad \text{for } (A.3) \]
The integral in (A.3) is for instance the Riemann-Stieltjes integral as a special case of the Young or Kurzweil integral. A vectorial counterpart of Problem (P) was investigated in [7] and the extension to regulated inputs was done in [10]. The following two results can be found in [6] or [3] as well as in Proposition II.1.1, Remark II.1.3, and Exercise I.3.2 of [7].

**Proposition A.1.** For every \( r > 0 \) and \((z_0, x) \in [-r, r] \times \mathbb{C}(0,T)\) there exists a unique \( \eta \in \text{CBV}(0,T) \) satisfying (A.1) – (A.3). Moreover, if \( \eta_1, \eta_2 \) are solutions of (A.1) – (A.3) corresponding to \((z_0^i, x_i) \in [-r, r] \times \mathbb{C}(0,T), i = 1, 2, \) respectively, then for every \( t \in [0,T] \) we have

\[
\| \eta_1 - \eta_2 \|_{[0,t]} \leq \max\{ |\eta_1(0) - \eta_2(0)|, \|x_1 - x_2\|_{[0,t]} \}. 
\]

**Corollary A.2.** Let \( \eta = p_r[z_0, x] \) for some \((z_0, x) \in [-r, r] \times \mathbb{C}(0,T)\). Then for every \( 0 \leq s < t \leq T \) we have

\[
|\eta(t) - \eta(s)| \leq \|x(\cdot) - x(s)\|_{[s,t]} .
\]

**Proposition A.3.** Let \((z_0, x) \in [-r, r] \times \mathbb{C}(0,T)\) be given, and let \( \eta \in \text{CBV}(0,T) \) satisfy (A.1) – (A.3). Then for every \( 0 \leq a < b \leq T \) we have

\[
\int_a^b (x(\tau) - \eta(\tau) - y(\tau)) \, d\eta(\tau) \geq 0 \quad \forall y \in \mathbb{C}(a,b), \|y\|_{[a,b]} \leq r.
\]

Proposition A.1 enables us to define the one-parametric family of solution operators

\[
p_r : [-r, r] \times \mathbb{C}(0,T) \rightarrow \text{CBV}(0,T) : (z_0, x) \mapsto \eta = p_r[z_0, x]
\]

of Problem (P) called the *play operators*. They were originally introduced in [6], and their various properties have been systematically studied e. g. in [3, 7, 13, 14]. The extension to arbitrary measurable inputs was done in [9] in a different setting. In Section 4 we made substantial use of the following characterization of the play which is typical for the scalar situation, see Fig. A1.

![Figure A1. A diagram of the play operator \( \eta = p_r[0, x] \).](image-url)
(i) monotone in \([t_{k-1}, s_k]\) for \(k = 1, \ldots, m\),
(ii) constant in \([s_k, t_k]\) for \(k = 0, \ldots, m\),
(iii) non-monotone in \([t_{k-1}, t_k + \delta]\) for any \(\delta > 0\) and \(k = 1, \ldots, m - 1\), and

\[
(x(s_k) - \eta(s_k))(x(t_k) - \eta(t_k)) = -r^2 \quad \text{for} \quad k = 1, \ldots, m - 1, \quad \text{(A.4)}
\]
\[
|x(s_k) - x(s_{k-1})| \geq 2r \quad \text{for} \quad k = 2, \ldots, m - 1, \quad \text{(A.5)}
\]
\[
\text{Var} \eta = \sum_{k=1}^{m} |\eta(s_k) - \eta(s_{k-1})|. \quad \text{(A.6)}
\]

Before giving to the proof of Proposition A.4, we start with the following auxiliary result.

**Lemma A.5.** Let \((z_0, x) \in [-r, r] \times C(0, T)\) be given, and let \(\eta = \varphi_r, [z_0, x]\). Then for every \(t \in [0, T]\) the following implications hold.

(i) If \(x(t) - \eta(t) > -r\), then there exists \(\delta > 0\) such that \(\eta\) is non-decreasing in the interval \([t - \delta, t + \delta] \cap [0, T]\);

(ii) If \(x(t) - \eta(t) < r\), then there exists \(\delta > 0\) such that \(\eta\) is non-increasing in the interval \([t - \delta, t + \delta] \cap [0, T]\);

**Proof.** The argument is the same in each of the cases (i), (ii). In (i) for instance, we find \(\delta > 0\) and \(\rho > 0\) such that \(x(\tau) - \eta(\tau) \geq -r + \rho\) for \(\tau \in [t - \delta, t + \delta] \cap [0, T]\). For every \([a, b] \subset [t - \delta, t + \delta] \cap [0, T]\) we define \(y(\tau) = x(\tau) - \eta(\tau) - \rho\) for \(\tau \in [a, b]\). Then \(-r \leq y(\tau) \leq x(\tau) - \eta(\tau) \leq r\) for every \(\tau \in [a, b]\), and Lemma A.3 yields

\[
\varphi(\eta(b) - \eta(a)) = \int_a^b (x(\tau) - \eta(\tau) - y(\tau)) \, d\eta(\tau) \geq 0.
\]

Hence \(\eta\) is non-decreasing in \([t - \delta, t + \delta] \cap [0, T]\). Case (ii) is similar. \(\blacksquare\)

**Proof of Proposition A.4.** If \(|x(\tau) - \eta(\tau)| < r\) for every \(\tau \in [0, T]\), then \(\eta\) is constant in \([0, T]\) and the assertion is trivial. If this is not the case, we put

\[
t_0 = \min\{t \in [0, T]; |x(t) - \eta(t)| = r\}.
\]

By Lemma A.5, we have \(\eta(\tau) = \eta(0)\) for all \(\tau \in [0, t_0]\). Assume for instance that \(x(t_0) - \eta(t_0) = r\); the other case is obtained by symmetry. By Lemma A.5, there exists \(\delta > 0\) such that \(\eta\) is non-decreasing in \([t_0, t_0 + \delta]\), and we may put

\[
t_1 = \sup\{t \in [t_0, T]; \eta\text{ is non-decreasing in } [t_0, t]\}.
\]

We stop the algorithm if \(t_1 = T\). Otherwise, we have by Lemma A.5 that \(x(t_1) - \eta(t_1) = -r\). We continue by induction and construct a sequence \(0 \leq t_0 < t_1 < t_2 < \ldots\) passing from \(t_k\) to \(t_{k+1}\) provided \(t_k < T\), with the properties

\[
x(t_k) - \eta(t_k) = (-1)^k r, \quad \text{(A.7)}
\]
\[
(-1)^{k-1} \eta \text{ is non-decreasing in } [t_{k-1}, t_k], \quad \text{(A.8)}
\]
\[
\eta \text{ is non-monotone in } [t_{k-1}, t_k + \delta] \text{ for any } \delta > 0. \quad \text{(A.9)}
\]

We fix any \(\ell \in \mathbb{N}\) such that \(t_\ell < T\), and for \(k = 1, 2, \ldots, \ell\) we define the points

\[
s_k = \min\{t \in [t_{k-1}, t_k]; \eta(t) = \eta(t_k)\}. \quad \text{(A.10)}
\]
Assuming for the moment that \((-1)^{k-1}(x(s_k) - \eta(s_k)) < r\), we may use Lemma A.5 to obtain that \((-1)^{k-1}\eta\) is non-increasing in \([s_k - \delta, s_k]\) for some \(\delta > 0\). Hence, by (A.8), \(\eta\) is constant in \([s_k - \delta, s_k]\) in contradiction with the definition of \(s_k\). Consequently,

\[ (-1)^{k-1}(x(s_k) - \eta(s_k)) = r, \quad t_{k-1} < s_k < t_k \quad \text{for} \quad k = 1, 2, \ldots, \ell. \quad (A.11) \]

For \(k = 2, \ldots, \ell\), we have by (A.8), (A.11) that

\[ (-1)^{k-1}(x(s_k) - x(s_{k-1})) = (-1)^{k-1}(x(s_k) - \eta(s_k) - x(s_{k-1}) + \eta(s_{k-1})) \geq 2r. \quad (A.12) \]

Since \(x\) is continuous, the number \(\ell\) cannot be arbitrarily large, and there exists necessarily \(m \in \mathbb{N}\) such that \(t_m = T, \ell \leq m - 1\). Properties (i) – (iii) (hence also (A.6)) then follow from (A.8) – (A.10). Furthermore, (A.4), (A.5) are direct consequences of (A.7), (A.11), and (A.12).

Proposition A.4 is proved. ■

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