Large density expansion of a hydrodynamic theory for self-propelled particles

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Abstract. Recently, an Enskog-type kinetic theory for Vicsek-type models for self-propelled particles has been proposed [T. Ihle, Phys. Rev. E 83, 030901 (2011)]. This theory is based on an exact equation for a Markov chain in phase space and is not limited to small density. Previously, the hydrodynamic equations were derived from this theory and its transport coefficients were given in terms of infinite series. Here, I show that the transport coefficients take a simple form in the large density limit. This allows me to analytically evaluate the well-known density instability of the polarly ordered phase near the flocking threshold at moderate and large densities. The growth rate of a longitudinal perturbation is calculated and several scaling regimes, including three different power laws, are identified. It is shown that at large densities, the restabilization of the ordered phase at smaller noise is analytically accessible within the range of validity of the hydrodynamic theory. Analytical predictions for the width of the unstable band, the maximum growth rate, and for the wave number below which the instability occurs are given. In particular, the system size below which spatial perturbations of the homogeneous ordered state are stable is predicted to scale with $\sqrt{M}$ where $M$ is the average number of collision partners. The typical time scale until the instability becomes visible is calculated and is proportional to $M$.

1 Introduction

The emergence of collective motion of living things, such as insects, birds, slime molds, bacteria, and fish is a fascinating far-from-equilibrium phenomenon which has attracted a great deal of cross-disciplinary attention [1]. The study of these systems falls under the broader umbrella of “active matter”. This term stands for collections of agents that are able to extract and dissipate energy from their surrounding to produce systematic motion [2,3]. Non-living examples of active matter are chemically powered nanorods [4], networks of actin fibers driven by molecular motors [5], and swarms of interacting robots [6,7]. In 1995, a minimal computational model that captures the essentials of collective motion without entering too much detail was introduced by Vicsek [8]. In this so-called Vicsek model (VM), agents are represented as point...
particles traveling at constant speed. The agents follow noisy interaction rules which aim to align the velocity vector of any given particle with its neighbors. Later, more sophisticated models that include additional interactions such as attraction and short-range repulsion were introduced [10,11,30,31]. On the theoretical side, coarse-grained descriptions of the dynamics in terms of phenomenological hydrodynamic equations have been proposed on the basis of symmetry and conservation law arguments. Well-known examples are the Toner-Tu theory for polar active matter [14,15] and the theory by Kruse et al. for active polar gels [16]. While being very successful, a disadvantage of these approaches is that no link between microscopic collision rules and the coefficients of the macroscopic equations is provided. In particular, since all coefficients must be related to only a few microscopic parameters, they cannot be varied independently, and the actual parameter space should be more restricted than the hydrodynamic equations might suggest. Furthermore, by just postulating equations there is the immanent danger that some terms or even entire equations might have been omitted, which could become relevant in new, previously untested, situations. To address the missing-link issue, several groups have derived hydrodynamic equations from the underlying microscopic rules and provided expressions for the transport coefficients in terms of microscopic parameters [17,18,28,37–39,47]. One of the first attempts to put the Toner-Tu theory on a microscopic basis was published by Bertin et al., who studied a Vicsek-type model with continuous time-dynamics and binary interactions by means of a Boltzmann approach [28,29]. While the authors recently clarified [25] that even at low density their underlying microscopic model is not identical to the Vicsek-model, many predictions agree qualitatively with the ones for the VM. This Boltzmann approach was later extended to systems of self-propelled particles with nematic and metric-free interactions, and hydrodynamic equations were derived [36,47]. Very recently, the very basis of these derivations – the Boltzmann equation and its validity in active matter – has been critically assessed [12,21,24]. In particular, it has been shown that, at least near the threshold to collective motion, the binary collision assumption is not valid in the VM at realistic particle speeds and even at very low densities [21]. Furthermore, it has been demonstrated that the mean-field assumption of molecular chaos is not justified near the threshold to collective motion in soft active colloids and that the Boltzmann theory must be amended by pre-collisional correlations [13]. A first attempt to rigorously include correlations by means of a ring-kinetic theory for Vicsek-type models was put forward by Chou et al. [34]. This theory is very complicated; work is still in progress to simplify it [46].

Nevertheless, arguments from ring-kinetic theory can be used to confirm the plausible presumption that mean-field theories become reliable in the VM at large particle speeds (or time steps) and/or at large particle densities [49]. Here, large density means that the average number \( M \) of collision partners of a given agent is much larger than one. The large density limit \( M \gg 1 \) is beyond the capability of Boltzmann approaches because Boltzmann equations are restricted to binary interactions. However, a recently proposed Enskog-like theory [18,20,21] has no limitation on density and has already been applied to a model with \( M = 2 \ldots 7 \) and metric-free interactions [26]. This Enskog-like theory is based on an exact equation for a Markov chain in phase space. In a previous paper [18], hydrodynamic equations were derived from this theory and its transport coefficients were given in terms of infinite series. In this paper, I analyze the transport coefficients in the large density limit, \( M \gg 1 \), and show that they take a simple form. This allows me to analytically evaluate the well-known density instability of the polarly ordered phase near the flocking threshold [28,32], and to obtain simple formulas and scaling laws for the dispersion relation. Note that a similar analysis for the opposite limit, \( M \ll 1 \), has already been performed by Bertin et al. [29].