Exponential–Weierstrass type, exponential–Jacobi type and solitary type solutions to some conformable fractional equations

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Abstract

A new algebraic method to find two special types of exact traveling wave solutions and the solitary type solutions to some conformable fractional partial differential equations is proposed. The two special types of solutions given by the product of exponential and Weierstrass elliptic functions, and the product of exponential and Jacobi elliptic functions are new and they cannot be obtained by using the existing algebraic methods.

Key Words: fractional derivative, conformable fractional derivative, fractional partial differential equation, traveling wave solution.

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1 Introduction

In recent years, the problem of finding exact solutions to nonlinear fractional partial differential equations (FPDEs) has become an active research area in nonlinear science due to the wide applicability of nonlinear FPDEs to describe many important phenomena and dynamic processes in fluid mechanics, biological and chemical processes, signal processing, control systems and so on. Based on the product rule and the chain rule for fractional derivatives, several direct methods have been applied to find the exact solutions to some nonlinear FPDEs [1, 2, 6, 7, 8]. However, the recent studies showed that the fractional derivatives, such as the Riemann–Liouville derivative and the Caputo derivative etc. do not satisfy the product rule and the chain rule [13, 14]. Therefore, the previous works of using these fractional derivatives to solve those nonlinear FPDEs were not correct. But a so-called α–order conformable fractional derivative (CFD) of a function $f : [0, \infty) \to R$ given by

$$D_t^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad t > 0, \alpha \in (0, 1],$$

(1.1)

which was defined by Khalil et al. [10] just satisfies the required properties such as [10, 16]

$$D_t^\alpha (af + bg) = aD_t^\alpha f + bD_t^\alpha g, \forall x \in R, \text{(Linearity)}$$

(1.2)

$$D_t^\alpha t^\beta = \beta t^{\beta - \alpha},$$

(1.3)

$$D_t^\alpha (fg) = gD_t^\alpha f + fD_t^\alpha g, \text{(Product rule)},$$

(1.4)

$$D_t^\alpha f(g(t)) = f'(g(t))D_t^\alpha g, \text{(Chain rule)},$$

(1.5)

$$D_t^\alpha f = t^{1-\alpha} f'(t), f \text{ is differentiable.}$$

(1.6)

Therefore, it is an acceptable choice to use CFD to solve the conformable fractional partial differential equations (CFPDEs). As a matter of fact, the CFD (1.1) with its properties (1.4), (1.5)
and (1.6) were successfully used to solve some CFPDEs by means of the Riccati equation expansion [3,15], the sine–cosine method [4], the exp(ϕ ε)–expansion [9], the modified Kudryashov method [9,12] and the Jacobi elliptic function expansion method [18], etc. However, we find that all these existing direct methods cannot be used to find two types of exact traveling wave solutions of CFPDEs when their original equations have the non–integrable background. The first type solution, namely the exponential–Weierstrass type solution, is expressed by the product of exponential function and Weierstrass elliptic function. The second type solution, namely the exponential–Jacobi type solution, is expressed by the product of exponential function and Jacobi elliptic function. Motivated by this problem, in this paper, we shall propose a direct algebraic method for constructing these two special types of solutions for CFPDEs. It is seen that the method introduced in this paper is effective to find the exponential–Weierstrass type, the exponential–Jacobi type and the solitary type solutions of the CFPDEs.

This paper is organized as follows. In Sec. 2 by introducing two new second order auxiliary nonlinear ordinary differential equations and their solutions we shall propose a new algebraic method for solving CFPDEs. In Sec. 3 we shall use our suggested method to find the exact traveling wave solutions of some CFPDEs. Finally, Sec. 4 offers a discussion.

2 Description of the method

Now let us simplify describe our direct algebraic method for finding traveling wave solutions for CFPDEs. Suppose that a CFPDE is given by

\[ P(u, D_u^\alpha u, D_x^\beta u, D_t^\gamma u, D_{xt}^\delta u, \ldots) = 0, \]  

(2.7)

where \( P \) is a polynomial of its unknowns and \( 0 < \alpha, \beta \leq 1 \). The key steps of our method are outlined as follows.

**Step 1:** Making the wave transformation

\[ u(x,t) = u(\xi), \quad \xi = x^\beta t^\alpha + \omega t^{\alpha \beta}, \]  

(2.8)

and using the properties of CFD we can convert Eq. (2.7) into the following ODE

\[ Q(u, \frac{du}{d\xi}, \frac{d^2 u}{d\xi^2}, \ldots) = 0. \]  

(2.9)

**Step 2:** (a) To seek the exponential–Weierstrass type and solitary type exact traveling wave solutions, we assume that Eq. (2.9) has the solution of the form

\[ u(\xi) = a_0 + a_1 F(\xi), \]  

(2.10)

where \( a_0, a_1 \) are constants, \( F(\xi) \) satisfies the following second order auxiliary ODE [19]

\[ F''(\xi) = b F^2(\xi) - 6a^2 F(\xi) + 5a F'(\xi). \]  

(2.11)

This equation admits the following solutions

\[ F(\xi) = \begin{cases} 
\frac{6}{b}e^{2a\xi} \varphi \left( \frac{1}{a} e^{a\xi} + c_1, 0, g_3 \right), \\
\frac{3a^2}{2b} \left[ 1 + \tanh \left( \frac{a}{2} \xi \right) \right]^2, \\
\frac{3a^2}{2b} \left[ 1 + \coth \left( \frac{a}{2} \xi \right) \right]^2,
\end{cases} \]  

(2.12)
(b) To find the exponential–Jacobi type and the solitary type exact traveling wave solutions, we can take the solution of Eq. (2.9) of the form

\[ u(\xi) = a_1 F(\xi), \]  

(2.13)

where \( a_1 \) is an undermined constant, \( F(\xi) \) satisfies the following second order auxiliary ODE

\[ F''(\xi) = cF^3(\xi) - 2a^2 F(\xi) - 3aF'(\xi). \]  

(2.14)

This equation has the following solutions

\[ F(\xi) = \begin{cases} 
\varepsilon ae^{-a\xi} \left( e^{-a\xi} + c_2, \frac{\sqrt{2}}{2} \right), & c = 2, \\
\varepsilon ae^{-a\xi} nc \left( \sqrt{2}e^{-a\xi} + c_2, \frac{\sqrt{2}}{2} \right), & c = 2, \\
\frac{\varepsilon a}{2} \left[ 1 - \tanh \left( \frac{a}{2} \xi \right) \right], & c = 2, \\
\frac{\varepsilon a}{2} \left[ 1 - \coth \left( \frac{a}{2} \xi \right) \right], & c = 2, \\
\varepsilon ae^{-a\xi} \left( \sqrt{2}e^{-a\xi} + c_3, \frac{\sqrt{2}}{2} \right), & c = -2, \\
\sqrt{2} \varepsilon ae^{-a\xi} \left( \sqrt{2}e^{-a\xi} + c_3, \frac{\sqrt{2}}{2} \right), & c = -2, 
\end{cases} \]  

(2.15)

where \( \wp \) expresses the Weierstras elliptic function, \( ds, nc, cn, sd \) are the Jacobi eliptic functions, \( \varepsilon = \pm 1 \), and \( a, b, c_1, c_2, c_3, g_3 \) are constants.

Step 3: Substituting (2.10) with (2.11), and (2.13) with (2.14) separately into Eq. (2.9) and setting the coefficients of like powers of \( F^i(F')^j \) to zero, we get a set of algebraic equations for unknowns \( a_0, a_1, a, b, \omega \), and for unknowns \( a_1, a, \omega \), respectively. The system of algebraic equations is solved by using a computer algebraic system, then the values of these unknowns can be obtained.

Step 4: The exact traveling wave solutions of the Eq. (2.7) can be obtained by putting the values of unknowns obtained in Step 3 with (2.12) into (2.10), and (2.15) into (2.13), respectively.

3 Applications of the method

Now we consider some illustrative examples to show the effectiveness of our proposed method.

Example 1 The space–time fractional KdV–Burgers equation

\[ D_t^\alpha u + \lambda u D_x^\beta u + \mu D_x^{2\beta} u + \nu D_x^{3\beta} u = 0, \]  

(3.16)

in which \( \lambda, \mu, \nu \) are constants.

Taking (2.8) into (3.16) we obtain the following ODE

\[ \omega u' + \lambda uu' + \mu u'' + \nu uu''' = 0. \]  

(3.17)

Substituting (2.10) with (2.11) into (3.17) and setting the coefficients of \( F, F^2, F', FF' \) to be zero, we obtain a set of algebraic equations

\[
\begin{cases}
2bva_1 + \lambda a_1^2 = 0, \\
-30a^3 v a_1 - 6a^2 \mu a_1 = 0, \\
5aba_1 + b u a_1 = 0, \\
19a^2 v a_1 + 5a \mu a_1 + \lambda a_0 a_1 + \omega a_1 = 0.
\end{cases}
\]
It solves that

\[ a = -\frac{\mu}{5\nu}, a_0 = \frac{6\mu^2 - 25\omega\nu}{25\lambda\nu}, a_1 = -\frac{2b\nu}{\lambda}. \]  

Inserting (2.12) with (3.18) into (2.10) we get the exponential–Weierstrass type and solitary type exact traveling wave solutions of Eq. (3.16) as following

\[
\begin{align*}
    u_1(x,t) &= -\frac{12}{\lambda} e^{\frac{2\beta}{\lambda}} \left( e^{\frac{\beta}{\lambda} + \omega \alpha} \right) \left( \frac{5\mu}{\mu} e^{-\frac{\beta}{\lambda} \left( \frac{\beta}{\lambda} + \omega \alpha \right)} + c_1, 0, g_3 \right) + \frac{6\mu^2 - 25\omega\nu}{25\lambda\nu}, \\
    u_2(x,t) &= -\frac{3\mu^2}{25\lambda\nu} \left( 1 - \tanh \frac{\mu}{10\nu} \left( \frac{\beta}{\beta} + \omega \alpha \right) \right)^2 + \frac{6\mu^2 - 25\omega\nu}{25\lambda\nu}, \\
    u_3(x,t) &= -\frac{3\mu^2}{25\lambda\nu} \left( 1 - \coth \frac{\mu}{10\nu} \left( \frac{\beta}{\beta} + \omega \alpha \right) \right)^2 + \frac{6\mu^2 - 25\omega\nu}{25\lambda\nu},
\end{align*}
\]

where \( c_1, g_3 \) are arbitrary constants.

**Example 2** The time fractional Fisher equation \[17\]

\[ D_t^\alpha u = u_{xx} + 6u (1 - u). \]  

Substituting (2.8) with \( \beta = 1 \) into (3.19), we can convert the Eq. (3.19) into the following ODE

\[ \omega u' = u'' + 6u (1 - u), \]  

Substituting (2.10) and (2.11) into (3.20) and setting the coefficients of \( F^j \) for \( j = 0, 1, 2 \) to zero, we have

\[
\begin{align*}
    6a_0^2 - 6a_0 &= 0, \\
    6a_1^2 - a_1b &= 0, \\
    -5a_0a_1 + a_1\omega &= 0, \\
    6a_2^2a_1 + 12a_0a_1 - 6a_1 &= 0.
\end{align*}
\]

This algebraic equations is solved that

\[
\begin{align*}
    a_0 &= 0, a_1 = \frac{b}{6}, a = 1, \omega = 5, \quad (3.21) \\
    a_0 &= 0, a_1 = \frac{b}{6}, a = -1, \omega = -5. \quad (3.22)
\end{align*}
\]

Now the exponential–Weierstrass type and the solitary type exact traveling wave solutions for Eq. (3.19) can be obtained by taking (2.12) with (3.21) and (3.22) into (2.10), respectively, they are

\[
\begin{align*}
    u_1(x,t) &= e^{2x + \frac{10\alpha}{\alpha}} \left( e^{x + \frac{5\alpha}{\alpha}} + c_1, 0, g_3 \right), \\
    u_2(x,t) &= \frac{1}{4} \left[ 1 + \tanh \frac{1}{2} \left( x + \frac{5\alpha}{\alpha} \right) \right]^2, \\
    u_3(x,t) &= \frac{1}{4} \left[ 1 + \coth \frac{1}{2} \left( x + \frac{5\alpha}{\alpha} \right) \right]^2, \\
    u_4(x,t) &= e^{-2x + \frac{10\alpha}{\alpha}} \left( e^{-x + \frac{5\alpha}{\alpha}} + c_1, 0, g_3 \right), \\
    u_5(x,t) &= \frac{1}{4} \left[ 1 - \tanh \frac{1}{2} \left( x - \frac{5\alpha}{\alpha} \right) \right]^2, \\
    u_6(x,t) &= \frac{1}{4} \left[ 1 - \coth \frac{1}{2} \left( x - \frac{5\alpha}{\alpha} \right) \right]^2,
\end{align*}
\]

where \( c_1, g_3 \) are arbitrary constants.

**Example 3** The time fractional RLW–Burgers equation \[12\]

\[ D_t^\alpha u + pu_x + quu_x + ru_{xx} + su_{xxt} = 0. \]  

(3.23)
Inserting (2.8) with \( \beta = 1 \) into (3.23) leads the following ODE
\[
(\omega + p) u' + q u'' + ru'' + \omega s u''' = 0. \tag{3.24}
\]
Taking (2.10) with (2.11) into (3.24) and setting the coefficients of \( F, F', FF' \) to be zero, then we obtain
\[
\begin{align*}
-30a^3\omega sa_1 - 6a^2ra_1 &= 0, \\
2b\omega sa_1 + qa_1^2 &= 0, \\
5ab\omega sa_1 + bra_1 &= 0, \\
19a^2\omega sa_1 + 5ara_1 + qa_0a_1 + (\omega + p)a_1 &= 0.
\end{align*}
\]
It solves that
\[
a = -\frac{r}{5\omega}, a_0 = -\frac{25s\omega^2 + 25ps\omega - 6r^2}{25q\omega}, a_1 = -\frac{2bs\omega}{q}. \tag{3.25}
\]
The exponential–Weierstrass type and the solitary type exact solutions of the Eq. (3.23) are obtained by substituting (2.12) with (3.25) into (2.10), they are the following ODE
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{25s\omega^2 + 25ps\omega - 6r^2}{25q\omega} = 1 \text{ into (3.23)} \text{ leads the following ODE} \\
\frac{25s\omega^2 + 25ps\omega - 6r^2}{25q\omega} = 0 \text{ into (3.26)} \text{ we get the following ODE}
\end{array} \right.
\end{align*}
\]
Substituting (2.13) with (2.14) into (3.27) and setting the coefficients of \( F, F', FF' \) to zero gives the following ODE
\[
\begin{align*}
\left\{ \begin{array}{l}
2a^2a_1 - a_1 &= 0, \\
a_1^3 - ca_1 &= 0, \\
3aa_1 + \omega a_1 &= 0.
\end{array} \right.
\end{align*}
\]
Solving this algebraic equation we obtain that
\[
a = \pm \frac{\sqrt{2}}{2}, a_1 = \pm \sqrt{c}, \omega = \pm \frac{3\sqrt{2}}{2}. \tag{3.28}
\]
The exponential–Jacobi type and the solitary type exact solutions of Eq. (3.26) are obtained by taking (3.28) and (2.15) with \( c = 2 \) into (2.13), they are now given by
\[
\begin{align*}
u_1(x, t) &= e^{\frac{\sqrt{2}}{2}(x + \frac{3\sqrt{2}}{2a}t)} ds \left( e^{\frac{\sqrt{2}}{2}(x + \frac{3\sqrt{2}}{2a}t)} + c_2, \frac{\sqrt{2}}{2} \right), \\
u_2(x, t) &= e^{\frac{\sqrt{2}}{2}(x + \frac{3\sqrt{2}}{2a}t)} nc \left( \sqrt{2}e^{\frac{\sqrt{2}}{2}(x + \frac{3\sqrt{2}}{2a}t)} + c_2, \frac{\sqrt{2}}{2} \right), \\
u_3(x, t) &= \frac{c}{\sqrt{2}} \left[ 1 + \tanh \frac{\sqrt{2}}{4} \left( x + \frac{3\sqrt{2}}{2a}t \right) \right], \\
u_4(x, t) &= \frac{c}{\sqrt{2}} \left[ 1 + \coth \frac{\sqrt{2}}{4} \left( x + \frac{3\sqrt{2}}{2a}t \right) \right],
\end{align*}
\]
where \( c_2 \) is a free parameter.
Example 5 The space–time fractional mKdV–Burgers equation [15]

\[ D^\alpha_t u + \lambda u^2 D^\alpha_x u + r D^2_x u + s D^3_x u = 0. \]  
(3.29)

Inserting (2.8) with \( \beta = \alpha \) into (3.29) leads the following ODE

\[ \omega u' + \lambda u^2 u' + ru'' + su''' = 0. \]  
(3.30)

Taking (2.13) with (2.14) into (3.30) and setting the coefficients of \( F, F^2, F', F^2F' \) to zero, we obtain a set of algebraic equations

\[
\begin{align*}
6a^2sa_1 - 2a^2ra_1 &= 0, \\
\lambda a_1^2 + 3csa_1 &= 0, \\
-3acs a_1 + cr a_1 &= 0, \\
7a^2sa_1 - 3ara_1 + \omega a_1 &= 0.
\end{align*}
\]

Its solution is found to be

\[ a = \frac{r}{3s}, a_1 = \pm \sqrt{-\frac{3cs}{\lambda}}, \omega = \frac{2r^2}{9s}. \]  
(3.31)

The exponential–Jacobi type and the solitary type exact traveling wave solutions of Eq. (3.29) obtained by taking (3.31) and (2.15) with \( c = 2 \) into (2.13) are

\[
u_1(x, t) = \frac{r^2}{3s} \sqrt{-\frac{6a}{\lambda}} e^{-\frac{r^2}{3s}} \left( \frac{\alpha^2}{\alpha^2} + \frac{2s}{9s} \right) \sin \left( \frac{\sqrt{2e - \frac{r^2}{3s}}}{\frac{\alpha^2}{\alpha^2} + \frac{2s}{9s}} + c_2, \frac{\sqrt{2}}{2} \right), s\lambda < 0,
\]

\[
u_2(x, t) = \frac{r^2}{3s} \sqrt{-\frac{6a}{\lambda}} e^{-\frac{r^2}{3s}} \left( \frac{\alpha^2}{\alpha^2} + \frac{2s}{9s} \right) \csc \left( \frac{\sqrt{2e - \frac{r^2}{3s}}}{\frac{\alpha^2}{\alpha^2} + \frac{2s}{9s}} + c_2, \frac{\sqrt{2}}{2} \right), s\lambda < 0,
\]

\[
u_3(x, t) = -\frac{r^2}{6s} \sqrt{-\frac{6a}{\alpha^2}} \left( 1 - \tanh \frac{r^2}{6s} \left( \frac{\alpha^2}{\alpha^2} + \frac{2s}{9s} \right) \right), s\lambda < 0,
\]

\[
u_4(x, t) = -\frac{r^2}{6s} \sqrt{-\frac{6a}{\alpha^2}} \left( 1 - \coth \frac{r^2}{6s} \left( \frac{\alpha^2}{\alpha^2} + \frac{2s}{9s} \right) \right), s\lambda < 0,
\]

where \( c_2 \) is an arbitrary constant.

By inserting (3.31) and (2.15) with \( c = -2 \) into (2.13) we get the following exponential–Jacobi type and solitary type traveling wave solutions of Eq. (3.26)

\[
u_5(x, t) = \frac{r^2}{3s} \sqrt{-\frac{6s}{\lambda}} \left( \frac{\alpha^2}{\alpha^2} + \frac{2s}{9s} \right) \csc \left( \sqrt{2e - \frac{r^2}{3s}} \left( \frac{\alpha^2}{\alpha^2} + \frac{2s}{9s} \right) + c_2, \frac{\sqrt{2}}{2} \right), s\lambda > 0,
\]

\[
u_6(x, t) = \frac{r^2}{3s} \sqrt{-\frac{6s}{\lambda}} \left( \frac{\alpha^2}{\alpha^2} + \frac{2s}{9s} \right) \sec \left( \sqrt{2e - \frac{r^2}{3s}} \left( \frac{\alpha^2}{\alpha^2} + \frac{2s}{9s} \right) + c_2, \frac{\sqrt{2}}{2} \right), s\lambda < 0,
\]

in which \( c_2 \) is a free parameter.

Example 6 The space–time fractional telegraph equation [15]

\[ D^\alpha_t u - D^\beta_x u + D^\gamma_x u + \mu u + \nu u^3 = 0. \]  
(3.32)

Taking (2.8) with \( \beta = \alpha \) into (3.32) we obtain the following ODE

\[(\omega^2 - 1) u'' + \omega u' + \mu u + \nu u^3 = 0. \]  
(3.33)
Substituting (2.13) with (2.14) into (3.33) and setting the coefficients of $F,F^3,F'$ to zero we obtain

\[
\begin{align*}
-3a\omega^2 a_1 + 3aa_1 + \omega a_1 &= 0, \\
-2a^2\omega^2 a_1 + 2a^2 a_1 + \mu a_1 &= 0, \\
\eta a_1^2 + c\omega^2 a_1 - ca_1 &= 0.
\end{align*}
\]

Solutions of this algebraic equations are found to be

\[
a = \frac{9\mu - 2}{2} \sqrt{\frac{\mu}{9\mu - 2}}, a_1 = \pm \sqrt{\frac{2c}{\nu(9\mu - 2)}}, \omega = 3 \sqrt{\frac{\mu}{9\mu - 2}},
\]

(3.34)

\[
a = -\frac{9\mu - 2}{2} \sqrt{\frac{\mu}{9\mu - 2}}, a_1 = \pm \sqrt{\frac{2c}{\nu(9\mu - 2)}}, \omega = -3 \sqrt{\frac{\mu}{9\mu - 2}},
\]

(3.35)

When $c = 2$ and $c = -2$,by substituting (3.34) and (2.15) into (2.13),respectively,we get the exponential–Jacobi type and the solitary type traveling wave solutions of Eq.(3.32) as following

\[
\begin{align*}
\eta^+ &= \frac{9\mu - 2}{2} \sqrt{\frac{\mu}{9\mu - 2}} \left( \frac{x}{\alpha} + 3 \sqrt{\frac{\mu}{9\mu - 2}} \right).
\end{align*}
\]

When $c = 2$ and $c = -2$,by substituting (3.35) and (2.15) into (2.13),respectively,we get the exponential–Jacobi type and solitary type traveling wave solutions of Eq.(3.32) as following

\[
\begin{align*}
\eta^- &= \frac{9\mu - 2}{2} \sqrt{\frac{\mu}{9\mu - 2}} \left( \frac{x}{\alpha} - 3 \sqrt{\frac{\mu}{9\mu - 2}} \right).
\end{align*}
\]

4 Discussion

The examples in Sec.3 showed that our method is very effective to find the exponential–Weierstrass type, the exponential–Jacobi type and solitary type exact traveling wave solutions for CFPDEs. In
our method, the auxiliary equations (2.11) and (2.14) are new and they have not been previously used in any direct methods. Therefore, all our exponential–Weierstrass type and exponential–Jacobi type traveling wave solutions are new and cannot be found by using the existing direct algebraic methods. It is also pointed out that the original integer order nonlinear partial differential equations of the CFPDEs considered in our examples are non–integrable. Therefore, our method is restricted to solve those CFPDEs whose original equations are non–integrable.

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