We revisit the integral formulation (or Green’s function approach) of Einstein’s equations in the context of braneworlds. The integral formulation has been proposed independently by several authors in the past, based on the assumption that it is possible to give a reinterpretation of the local metric field in curved spacetimes as an integral expression involving sources and boundary conditions. This allows one to separate source-generated and source-free contributions to the metric field. As a consequence, an exact meaning to Mach’s Principle can be achieved in the sense that only source-generated (matter fields) contributions to the metric are allowed for; universes which do not obey this condition would be non-Machian. In this paper, we revisit this idea concentrating on a Randall-Sundrum-type model with a non-trivial cosmology on the brane. We argue that the role of the surface term (the source-free contribution) in the braneworld scenario may be quite subtler than in the 4D formulation. This may pose, for instance, an interesting issue to the cosmological constant problem.

Keywords: classical general relativity; gravity in more than four dimensions, Kaluza-Klein theory, unified field theories; alternative theories of gravity; cosmological constant; strings and branes

Mach’s Principle (MP) is often understood as the general idea that inertia arises from the interaction of matter with the rest of all matter in the universe. It is well known that this idea played a fundamental role in the developments of general relativity (GR). In 1917, Einstein added the so-called cosmological constant, \( \Lambda \), to his field equations in order to find a
static solution to his cosmological model, in which a direct connection of the mass density in the universe to its geometry could be achieved in accordance with Mach’s ideas. However, there are solutions to Einstein’s field equations that simply fail to be Machian. For instance, as early as in 1917, de Sitter \cite{2} proposed a static cosmological model including \( \Lambda \) but no matter fields as a solution to Einstein’s equations. Since then, the validity of MP remains an open question of GR \cite{3}. There are in fact dozens of possible formal interpretations for this principle in the context of different classical theories of gravitation (see the index on page 530 of Ref. \cite{3} for the various possible definitions of MP). At the same time, the so-called ‘cosmological constant problem’ is one of the most important unsolved issues in physics today (see reviews in, e.g., \cite{4}, \cite{5}). Therefore it is tempting to inquire whether MP and the cosmological constant could be deeply related in some manner \cite{6}.

On the other hand, it is important to realize that the question of the origin of inertia as well as the existence of a non-vanishing cosmological constant, as indicated by present-day cosmological observations \cite{7}, \cite{8}, \cite{9}, are not only a problem of classical theories but also a fundamental issue that any consistent quantum theory of gravity must address. Indeed, the construction of a consistent quantum theory of the gravitational field is the main goal of current theoretical physics. Presently, the two most relevant approaches are Loop Quantum Gravity \cite{10} and String/M-Theory (see reviews and introductory lectures in \cite{11}). The former focuses on the search for a relational notion of a quantum spacetime, whereas the latter is more appropriately regarded as an ambitious research program towards finding an unified, non-perturbative description of all fundamental interactions of nature \cite{12}. Although the literature on these new theories is vast, there has been very few discussions regarding the problem of the origin of inertia as advocated by MP in high energy physics (some work can be found in Ref. \cite{16}; interpretations of MP in the context of quantum gravity can be found in Ref. \cite{3}, Chap. 8).

In most quantum gravity theories, GR is supposed to break down at high energies (the Planck scale, at \( \sim 10^{19} \) GeV). In some string-inspired models (‘braneworld scenarios’), gravity emanates as a higher-dimensional theory and 4-dimensional Einstein’s GR is expected to be reproduced at low enough energies as an effective theory (e.g., \cite{13}). These braneworld models state in general that our observable universe (viz. all Standard Model fields) is confined to a ‘3-brane’, i.e., a (3+1)-dimensional subspace of a higher, five-dimensional (‘bulk’) space-time, where the gravitational field extends along a fifth-dimension. In the Randall-
Sundrum type 2 model (RS2) model, for instance, there is only one three-brane and a large extra-dimension. In this case, the general relativistic gravity is well reproduced in the low energy limit.

In the present letter, we briefly discuss such a model in the context of the integral formulation of GR, generalized here to five dimensions. The original formalism (see, e.g., Sciama, Waylen & Gilman, Al’tshuler and Lynden-Bell). The idea is that, despite the nonlinearity of Einstein’s field equations, a reinterpretation of the metric potentials in curved space-time as an integral expression involving sources and boundary conditions can be given. To that end, the information locked into the gravitational field of distant matter is propagated linearly through a self-consistent curved space-time (viz. the space-time derived from the sum of the contributions of all the matter in the universe). For instance, suppose that the 4D metric tensor $g_{\mu\nu}$, which ultimately determines the local inertial frames of reference, is completely derived from a joint influence of the matter-energy content in the universe, then the metric could be expressed as:

$$g_{\mu\nu}(x^\mu) = 2 \int_{\Omega} G^{\alpha' \beta'}_{\mu \nu}(x^\mu, x'^\mu) K_{\alpha' \beta'}(x'^\mu)[-g(x)]^{1/2} d^4 x' + \int_{\partial \Omega} \nabla_{\gamma'} G^{\alpha' \beta'}_{\mu \nu} g_{\beta' \alpha'}(x'^\mu)[-g(x)]^{1/2} dS_{\gamma'}.$$  

The source function is given by

$$K_{\alpha' \beta'}(x'^\mu) \equiv \kappa^2 \left[ T_{\alpha' \beta'}(x'^\mu) - 1/2 T_{\gamma' \gamma'}(x'^\mu) \delta_{\alpha' \beta'} \right] - \Lambda \delta_{\alpha' \beta'},$$

with $\kappa^2 \equiv 8\pi G$, and $\Lambda$ is the cosmological constant. The source function is therefore related to the energy-momentum tensor for the cosmic fluid and possible vacuum energy density contributions are allowed. The Green function, $G^{\alpha' \beta'}_{\mu \nu}(x^\mu, x'^\mu)$ is a second rank tensor at the two space-time points $x^\mu$ and $x'^\mu$, namely, a propagator operator of the gravitational field of a body located at $x'^\mu$ to coordinate $x^\mu$. The second integral of Eq. (1) is a surface term representing the contribution to the metric from the data specified on the intersection of the observer’s light cone $\Gamma^{-}_x$ and the surface $\delta \Omega$ bounding the proper volume $\Omega \equiv \sqrt{-g} d^4 x'$ of spacetime. The symbol $\nabla_{\gamma'}$ denotes the covariant derivative, and $dS_{\gamma'}$ is the coordinate surface element on $\delta \Omega$, which points outward from $\Omega$. Notice that in the classical GR, the Green function must sharply go to zero outside the past light cone of an observer at $x$, and therefore the first integration term must proceed over the inner region of space delimited...
by $\Gamma^\mu_{\nu \lambda}$ and $\delta \Omega$. Notice also that the metric in Eq. (1) is not itself a solution to Einstein’s equations, but an equivalent representation of them as integral equations.

The integral formalism has the advantage of formally separating source-generated and source-free contributions to the metric field. Gilman provides a classification scheme for interpreting cosmological models under MP in a strict sense, that is, the joint contribution of all mass-energy of matter in the universe fully specifies the metric tensor $g_{\mu\nu}$, up to diffeomorphisms. Notice that gravitational (geometrical) degrees of freedom do not enter in the above definition of MP. In these terms, a cosmological model is considered Machian from Gilman’s scheme if it does not contain source-free contributions [the surface term of Eq. (1) vanishes, the so-called ‘Gilman condition’]. In fact, the surface integral is to be considered as an integral representation of the solution to the homogeneous wave equation, namely, Einstein’s field equations with a null source term. In globally hyperbolic spacetimes, the integral formulation as a whole is a well-defined representation for both the homogeneous as well as the nonhomogeneous field equations. The boundary surface may be at an infinite proper past, or may well be formed by the the union of particle horizons for all points along the observer’s timelike worldline. When the volume of spacetime contains all the sources in the observer’s past light cone, then the surface term is to be interpreted as contributions to the local metric field that come from matter outside the volume of integration and/or from the data at the boundary surface, or, in the words of Gilman, as ‘contributions to the local field that cannot be attributed to any of the observable sources’ [22]. Notice that such boundary surface must be an initial surface where conditions must be specified. For instance, if we impose the following constraint equations on the initial surface, $\nabla_\gamma G^{\alpha \beta'}_{\mu \nu} = 0$, then these constraints are equivalent to the Machian requirement that the surface term, which includes source-free contributions to the local metric field, initially vanishes.

Gilman further assumes that, in a Machian cosmology, $\Lambda$ must be identically zero, since for the integral formulation to be valid (Eq. 1), $\Lambda$ must be treated as a source term, as is generically assumed in Eq. (2) above, but according to the strict definition of MP, it is not a Machian term (it unrelated to matter fields). Hence, the ‘Gilman condition’ for a strict Machian cosmology assumes that the surface term and the cosmological constant in Eq. (1) must be zero. In the present paper, we will relax the latter definition of MP above, in the sense that the energy-momentum tensor of the vacuum, $\langle T_{\mu\nu}^{\text{vac}} \rangle$, can be considered as a source term that contributes to the specification of the metric tensor as well. Explicitly, we
assume that (e.g., [4]): $\langle T_{\mu\nu}^{\text{vac}} \rangle = - \langle \rho^{\text{vac}} \rangle g_{\mu\nu}$, where $\langle \rho^{\text{vac}} \rangle$ is the energy density of the vacuum (the 4D metric tensor $g_{\mu\nu}$ has signature $-+++$ in our notation), in the four-dimensional universe. Such definition leaves the form of the classical Einstein’s equations,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = - \Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu},$$  \hspace{1cm} (3)

unaltered if the cosmological constant is redefined as: $\Lambda \rightarrow \Lambda + \kappa^2 \langle \rho^{\text{vac}} \rangle$. Such a setup is just one of the ‘many faces’ of the cosmological constant (see [2]).

The Einstein’s equations on the 3-brane world resulting from the Randall-Sundrum type 2 model [14], generalized to allow for matter fields on the brane were deduced by Shiromizu et al. [13], where higher-dimensional modifications to the standard Einstein’s equations were explicitly found. The induced field equations on the brane are:

$$\hat{G}_{\mu\nu} = - \Lambda_{\text{eff}} \hat{g}_{\mu\nu} + \kappa^2 \hat{T}_{\mu\nu},$$  \hspace{1cm} (4)

where

$$\hat{T}_{\mu\nu} \equiv T_{\mu\nu} + \frac{[\kappa(5)]^4}{\kappa^2} \pi_{\mu\nu} - \frac{1}{\kappa^2} E_{\mu\nu}$$  \hspace{1cm} (5)

is a redefined energy-momentum tensor, and the tensors $\pi_{\mu\nu}$ and $E_{\mu\nu}$ are correction terms that arise in the induced Einstein’s equations due to higher-dimensional effects. We see that these terms can all be absorbed into the redefined energy-momentum tensor, ultimately leaving the form of the induced Einstein’s equations on the brane unchanged from their standard (4D) expressions. The correspondences between the 4D and 5D quantities (scales) involved are the following: first, the induced cosmological constant on the brane is

$$\Lambda_{\text{eff}} = \frac{1}{2} [\kappa(5)]^2 \left( \Lambda^{(5)} + \frac{1}{6} [\kappa(5)]^2 \lambda^2 \right),$$  \hspace{1cm} (6)

and $\lambda = 6 \frac{\kappa^2}{[\kappa(5)]^4}$ is the so-called brane tension. Notice that $\Lambda_{\text{eff}}$ receives contributions from the bulk cosmological constant, $\Lambda^{(5)}$, and the brane tension, so that both could be fine-tuned in order to give a null cosmological constant on the brane [29]. For low enough energy densities (namely, lower than the brane tension), gravity is effectively 4D for the brane observer. Also, one important point is that Newton’s gravitational constant, $G$, depends on the brane tension. Second, the tensor $\pi_{\mu\nu}$ gives local corrections, quadratic in the energy-momentum tensor, due to matter fields on the brane (see [13]), and, finally, the influence of the free gravitational field in the bulk is expressed as the projection of the 5D Weyl tensor...
onto the brane, namely, $E_{\mu\nu} = (5)^{C^{A}_{BCD}} n^A n^C g_{\mu}^B g_{\nu}^D$, where $n^A$ is the unit vector normal to the brane.

Now we turn our attention to the integral formulation of GR in this braneworld scenario. It is immediately clear that, because of the fact that the form of the induced Einstein’s equations remain unchanged from their standard (4D) expressions, the same happens to the corresponding integral-induced expressions. In other words, a first trivial interpretation would be to consider the additional corrections to the classical field equations, represented by $\pi_{\mu\nu}$ and $E_{\mu\nu}$, simply as additional source terms to Eq. (2) in the case of the braneworld. The volume of integration would extend to the whole brane and include the higher-dimensional effects as additional source terms. Matter motions excite gravitational waves in the bulk, and, on the other hand, the excitations of the free gravitational field also perturb the dynamics of matter on the brane. These influences are all taken as source terms in the volume integral. However, what happens to the surface term in this case? Could it be interpreted in the usual manner, restrited to a surface at infinite past on the observer’s light cone on the brane?

In the usual (4D) interpretation, when the volume of spacetime contains all the sources in the observer’s past light cone, the surface term is to be interpreted as contributions to the local metric field that come from matter outside the volume of integration and/or from the data at the boundary surface. But we immediately see that, in the braneworld scenario, (i) the surface of integration must extend to the bulk; or (2) the higher-dimensional degrees of freedom must be included in the surface term computed on the brane. Either alterations represent non-negligible additional effects depending on the Machian conditions imposed to the braneworld observer.

In order to address these questions, a note of caution is necessary. As already mentioned, the metric in Eq. (1) does not represent a solution to Einstein’s field equations, but is only an equivalent representation of them as integral equations. Such equations are not supposed to be solved with any iteration scheme in RS2 cosmologies, because the induced Einstein’s equations derived by Shiromizu et al. are not closed. Notice that the tensor $E_{\mu\nu}$ is responsible for transmitting the effects of nonlocal gravitational degrees of freedom from the bulk to the brane. Solutions to the induced Einstein’s equations on the brane will generally depend on the evolution of the gravitational field in the bulk as well, in a somewhat complicated manner (see Shiromizu et al. [13]). Hence, in order to qualitatively elaborate the
role of MP in the braneworld scenario from the integral approach, we are led to assume a decomposition of the $E_{\mu\nu}$ tensor into a transverse-traceless part, $E^{TT}_{\mu\nu}$ (corresponding to the free gravitational degrees of freedom in 5D), and into a longitudinal part, $E^L_{\mu\nu}$ (corresponding to matter field contributions on the brane). Under such assumptions, the induced Einstein’s equations are completely closed with respect to the brane quantities if $E^{TT}_{\mu\nu}$ were zero \(^{[13]}\). This property leads us to conjecture the possibility of splitting the contribution of $E_{\mu\nu}$ into a part directly related to a source term on the brane ($E^L_{\mu\nu}$), and a source-free contribution ($E^{TT}_{\mu\nu}$) to the the surface term.

A simple example elaborates on further possibilities. We separate the contributions of the effective cosmological constant (Eq. 6) into a bulk, source-free contribution, to be included in the surface term, and a source-generated contribution from the tension of the brane, to be included in the volume integration term. With such a setup, from Gauss’ theorem, it is possible to impose, for instance, constraints such as:

$$\int_{\partial \text{brane}} \nabla \gamma \mathcal{G}^{\alpha\beta'} \hat{g}_{\alpha\beta'}(x_{\mu'}) \left[-\hat{g}(x)\right]^{1/2} dS' \equiv \int_{\text{brane}} \nabla^2 \mathcal{E}_{\mu\nu} \left[-\hat{g}(x)\right]^{1/2} d^4 x', \quad (7)$$

with

$$\mathcal{E}_{\mu\nu} \propto -\left(\kappa^{(5)}\right)^2 \Lambda^{(5)} \hat{g}_{\mu\nu} - E^{TT}_{\mu\nu}. \quad (8)$$

Notice that the bulk cosmological constant is not interpreted in this setup as a source term corresponding to the bulk vacuum energy density: we have isolated any contributions from the vacuum energy density to the brane tension. In other words, the bulk cosmological constant enter here only as a term that modifies the 5D gravity Lagrangian to $L_{\text{grav}} \propto R^{(5)} - 2\Lambda^{(5)}$. We have reinterpreted the surface term as to include source-free contributions solely from the bulk via the $\mathcal{E}_{\mu\nu}$ tensor defined above. In order that a Machian braneworld satisfies the ‘Gilman condition’, the surface term must go to zero as it tends to the infinite past, leading to the following constraint: $\nabla^2 \mathcal{E}_{\mu\nu} = 0$. In particular, if the bulk is purely AdS ($E^{TT}_{\mu\nu} = 0$) and Machian, the bulk cosmological constant must satisfy $\nabla^2 \Lambda^{(5)} \hat{g}_{\mu\nu} = 0$. In general, such a simple setup indicates that, for a Machian braneworld, the components of $\mathcal{E}_{\mu\nu}$ cannot increase or decrease in all directions from a given spacetime location at the initial hypersurface defined from the observer’s infinite past light-cone. Other considerations can be equally made on such grounds.

In summary, we believe that precise constraints for the behaviour of the bulk cosmological constant, as well as the strict condition $E^{TT}_{\mu\nu} = 0$, are needed for a Machian braneworld
universe. Such constraints could be intimately related to the initial conditions at the infinite past surface of the brane observer. We argue that if the induced Einstein’s equations on the brane are reinterpreted as integral equations, it is possible to arrive at conditions that are not evident in the usual differential field equations, in special, the surface term allows for a much richer interpretation of the interplay between MP, braneworlds, and the cosmological constant. Such developments are presently being explored.

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[1] Einstein, A. 1917, Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl. 142 [English translation in the *The Principle of Relativity* (Methuen, 1923, reprinted by Dover Publications), p. 177].
[2] de Sitter, 1917, *MNRAS*, 78, 3.
[3] Mach’s Principle: From Newton’s Bucket to Quantum Gravity 1995 ed J B Barbour and H Pfister (Boston: Birkhäuser)
[4] Weinberg, S. 1989, *Rev. Mod. Phys.*, 61, 1.
[5] Padmanabhan, T., 2003, *Phys. Rept.* 380, 235.
[6] Vishwakarma R G 2002 *Class. Quantum Grav.* 19 4747-4752, Preprint gr-qc/0205075
[7] Knop, R. A. et al. 2003, Preprint astro-ph/0309368
[8] Tonry, J. R. et al. 2003, *Ap. J.*, 594, 1.
[9] Spergel, D. N. et al. 2003, *Ap. J. Suppl.*, 148, 175.
[10] Rovelli C 1998 Loop Quantum Gravity *Liv. Rev. Rel.*
[11] Polchinski J 1994 Preprint hep-th/9411028
    Schwarz J H 1996 Preprint hep-th/9607201
    Polchinski J 1996 Preprint hep-th/9611050
    Kiritsis E 1997 Preprint hep-th/9709062
Schwarz J H 1998 Preprint hep-th/9807135
Schwarz J H 2000 Preprint hep-ex/0008017
Förste S 2001 Preprint hep-th/0110055
Szabo R J 2002 Preprint hep-th/0207142
Padilla A 2002 Preprint hep-th/0210217
Brax P and van de Bruck C 2003 Preprint hep-th/0303095
[12] Smolin L 2003 Preprint hep-th/0303185
[13] Arkani-Hamed N, Dimopoulos S and Dvali G 1998 Preprint hep-ph/9803315
Arkani-Hamed N, Dimopoulos S and Dvali G 1998 Preprint hep-ph/9807344
Antoniadis I, Arkani-Hamed N, Dimopoulos S, and Dvali G 1998 Preprint hep-ph/9804398
Shiromizu T, Maeda K and Sasaki M 1999 Preprint gr-qc/9910076
Sundrum R 1999 Phys. Rev. D 59, 085009
[14] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 4690 Preprint hep-th/9906064 (RS2)
[15] Randall L and Sundrum R 1999 Preprint hep-ph/9905221 (RS1)
[16] Horava P 1999 Phys. Rev. D 59 046004
Quiros I Preprint hep-th/0105167
[17] Sciama D W, Waylen P C and Gilman R C 1969 Phys. Rev. 187 1762
[18] Al’tshuler, B., 1967, Soviet Physics JETP, 24, 766.
[19] Lynden-Bell, D., 1967, MNRAS, 135, 453.
[20] Raine D J 1995 The Integral Formulation of Mach’s Principle Mach’s Principle: From Newton’s Bucket to Quantum Gravity ed J B Barbour and H Pfister (Boston: Birkhäuser) p 274
Nordtvedt K Machian Effects in Physical Law and the Field Paradigm of Modern Physics 1995 Mach’s Principle: From Newton’s Bucket to Quantum Gravity ed J B Barbour and H Pfister (Boston: Birkhäuser) p 422
Raine D J and Heller M 1981 The Science of Space-Time (Pachart Publishing House) chap 12
[21] Misner C W, Wheeler J A and Thorne K S 1973 Gravitation (W. H. Freeman & Co.) chap 21
Isenberg J 1995 Wheeler-Einstein-Mach Spacetimes Mach’s Principle: From Newton’s Bucket to Quantum Gravity ed J B Barbour and H Pfister (Boston: Birkhäuser) p 188
[22] Gilman, R.C., 1970, Phys. Rev. D, 2, 1400.
[23] Langlois, D., 2002, Preprint, gr-qc/0207047
[24] Youm D Preprint hep-th/0000414
[25] Will C M 1995 Testing Machian Effects in Laboratory and Space Experiments *Mach’s Principle: From Newton’s Bucket to Quantum Gravity* ed J B Barbour and H Pfister (Boston: Birkhäuser) p 365

[26] Hoyle C D et al 2001 *Phys. Rev. Lett.* **86** 1418

[27] Sciama D W 1953 *MNRAS* **113** 35

[28] The Randall-Sundrum type 1 model (RS1) [15], on the other hand, attempts to solve the hierarchy problem by a small extra dimension, with two three-branes, one with negative and the other with positive tension. This mechanism solves the hierarchy problem only if we live on the negative tension brane, but in this case gravity is not localized on this brane.

[29] Notice that in the original RS2 model, the cosmological constant on the brane is exactly zero.