Macroscopic objects in quantum mechanics: A combinatorial approach

Itamar Pitowsky

Department of Philosophy, The Hebrew University,
Mount Scopus, Jerusalem 91905, Israel.

(Accepted Physical Review A)

Abstract

Why we do not see large macroscopic objects in entangled states? There are two ways to approach this question. The first is dynamic: the coupling of a large object to its environment causes any entanglement to decrease considerably. The second approach, which is discussed in this paper, puts the stress on the difficulty to observe a large scale entanglement. As the number of particles $n$ grows we need an ever more precise knowledge of the state, and an ever more carefully designed experiment, in order to recognize entanglement. To develop this point we consider the family of observables, called witnesses, which are designed to detect entanglement. A witness $W$ distinguishes all the separable (unentangled) states from some entangled states. If we normalize the witness $W$ to satisfy $|\text{tr}(W\rho)| \leq 1$ for all separable states $\rho$, then the efficiency of $W$ depends on the size of its maximal eigenvalue in absolute value; that is, its operator norm $\|W\|$. It is known that there are witnesses on the space of $n$ qubits for which $\|W\|$ is exponential in $n$. However, we conjecture that for a large majority of $n$-qubit witnesses $\|W\| \leq O(\sqrt{n \log n})$. Thus, in a non ideal measurement, which includes errors, the largest eigenvalue of a typical witness lies below the threshold of detection. We prove this conjecture for the family of extremal witnesses introduced by Werner and Wolf (Phys. Rev. A 64, 032112 (2001)).
I. INTRODUCTION

Why do we not see large macroscopic objects in entangled states? There are two ways to approach this question. The first is dynamic: the coupling of a large object to its surroundings and its constant random bombardment from the environment cause any entanglement that ever existed to disentangle. The second approach - which does not in any way conflict with the first - puts the stress on the word *we*. Even if the particles composing the object were all entangled and insulated from the environment, we shall still find it hard to observe the superposition. The reason is that as the number of particles $n$ grows we need an ever more precise knowledge of the state, and an ever more carefully designed experiment in order to recognize the entangled character of the state of the object.

In this paper I examine the second approach by considering entanglements of multi-particle systems. For simplicity, the discussion concentrates mostly on two-level systems, that is, $n$-qubit systems where $n$ is large. An observable $W$ that distinguishes all the unentangled states from some entangled states is called a *witness*. For our purpose it is convenient to use the following definition: A witness $W$ satisfies $\max |\text{tr}(W\rho)| = 1$, where the maximum is taken over all separable states $\rho$; while $\|W\| > 1$, where $\|W\|$ is the operator norm, the maximum among the absolute values of the eigenvalues of $W$. It is easy to see that this definition is equivalent to the usual one [1].

For each $n \geq 2$ there is a witness $W$, called the Mermin-Klyshko operator [2], whose norm, $\|W\| = \sqrt{2^n - 1}$, increases exponentially with $n$. The eigenvector at which this norm obtains is called the generalized GHZ state; it represents a maximally entangled system of $n$ qubits. However, we shall see that this large norm is the exception, not the rule. My aim is to show that the norm of a *majority* of the witnesses grows with $n$ very slowly, $\|W\| \leq O(\sqrt{n \log n})$, slower than the growth of the measurement error. We shall prove this with respect to a particular family of witnesses, and formulate it as a conjecture in the general case. This means that unless the system has been very carefully prepared in a specific state, there is a very little chance that we shall detect multiparticle entanglement, even if it is there.
II. WITNESSES AND QUANTUM CORRELATIONS

A. The two particles case

Let $A_0, A_1, B_0, B_1$ be four Hermitian operators in a finite dimensional Hilbert space $\mathbb{H}$ such that $A_i^2 = B_j^2 = 1$. On $\mathbb{H} \otimes \mathbb{H}$ define the following operator

$$W = \frac{1}{2} A_0 \otimes B_0 + \frac{1}{2} A_0 \otimes B_1 + \frac{1}{2} A_1 \otimes B_0 - \frac{1}{2} A_1 \otimes B_1$$

(1)

It is easy to see that $|\text{tr}(W \rho)| \leq 1$ for any separable state $\rho$ on $\mathbb{H} \otimes \mathbb{H}$. This is, in fact, the Clauser Horne Shimony and Holt (CHSH) inequality \[4\]. Indeed, if $X_0, X_1, Y_0, Y_1$ are any four random variables taking the values $\pm 1$ then

$$-1 \leq \frac{1}{2} X_0 Y_0 + \frac{1}{2} X_0 Y_1 + \frac{1}{2} X_1 Y_0 - \frac{1}{2} X_1 Y_1 \leq 1$$

(2)

as can easily be verified by considering the 16 possible cases. Hence $c_{ij} = \mathcal{E}(X_i Y_j)$, the correlations between $X_i$ and $Y_j$, also satisfy the inequalities

$$-1 \leq \frac{1}{2} c_{00} + \frac{1}{2} c_{01} + \frac{1}{2} c_{10} - \frac{1}{2} c_{11} \leq 1$$

(3)

If $\rho$ is a separable state on $\mathbb{H} \otimes \mathbb{H}$ we can represent $\text{tr}(\rho A_i \otimes B_j)$ as correlations $c_{ij}$ between such $X_i$'s and $Y_j$'s. In other words, the correlations can be recovered in a local hidden variables model.

From Eq (2) three other conditions of the form Eq (3) can be obtained by permuting the two $X$ indices or the two $Y$ indices. The resulting constraints on the correlations form a necessary and sufficient condition for the existence of a local hidden variables model \[5, 6\]. One way to see this is to consider the four-dimensional convex hull $C_2$ of the sixteen real vectors in $\mathbb{R}^4$:

$$(X_0 Y_0, X_0 Y_1, X_1 Y_0, X_1 Y_1) \quad X_i = \pm 1, \ Y_j = \pm 1$$

(4)

and prove that an arbitrary four-dimensional real vector $(c_{00}, c_{01}, c_{10}, c_{11})$ is an element of $C_2$ if, and only if, the $c_{ij}$ satisfy $-1 \leq c_{ij} \leq 1$ and all the conditions of type Eq (3). The inequalities, then, are the facets of the polytope $C_2$.

By contrast with the classical case let $\rho$ be a pure entangled state on $\mathbb{H} \otimes \mathbb{H}$. Then for a suitable choice of $A_i$'s, and $B_j$’s in Eq (1) we have $|\text{tr}(W \rho)| > 1$, so the operators $W$ of this
type are sufficient as witnesses for all pure entangled bipartite states. To obtain a geometric representation of the quantum correlations let \( \rho \) be any state, denote \( q_{ij} = \tr(\rho A_i \otimes B_j) \), and consider the set \( Q_2 \) of all four dimensional vectors \((q_{00}, q_{01}, q_{10}, q_{11})\) which obtain as we vary the Hilbert space and the choice of \( \rho, A_i \), and \( B_j \). The set \( Q_2 \) is convex, \( Q_2 \supset C_2 \) but it is not a polytope. The shape of this set has been the focus of a great deal of interest [8, 9, 10, 11, 12]. To get a handle on its boundary we can check how the value of \( \|W\| \) changes with the choice of \( A_i \)'s, and \( B_j \)'s. Cirel’son [8, 9] showed that \( \|W\| \leq \sqrt{2} \). This is a tight inequality, and equality obtains already for qbits. In this case, \( H = C_2 \) and \( A_i = \sigma(a_i), B_j = \sigma(b_j) \) are spin operators, with \( a_0, a_1, b_0, b_1 \) four directions in physical space. There is a choice of directions such that \( \|W\| = \sqrt{2} \), and the eigenvectors \( |\phi\rangle \in C^2 \otimes C^2 \), corresponding to this value are the maximally entangled states.

**B. The n particles case - Werner Wolf operators**

Some of these results can be extended to \( n \) particle systems, provided that the operators are restricted to two binary measurements per particle. To account for classical correlations consider \( 2n \) random variables \( X_1^1, X_1^2, X_2^1, \ldots, X_0^n, X_1^n \), each taking the two possible values \( \pm 1 \). We shall parametrize the coordinates of a vector in the \( 2^n \)-dimensional real space \( \mathbb{R}^{2^n} \) by sequences \( s = (s_1, \ldots, s_n) \in \{0, 1\}^n \). Now, consider the set of \( 2^{2n} \) real vectors in \( \mathbb{R}^{2^n} \)

\[
(a(0, \ldots, 0), \ldots, a(s_1, \ldots, s_n), \ldots, a(1, \ldots, 1)), \quad a(s_1, \ldots, s_n) = X_1^{s_1}X_2^{s_2}\ldots X_n^{s_n}
\]  

(5)

Their convex hull in \( \mathbb{R}^{2^n} \), denoted by \( C_n \), is the range of values of all possible classical correlations for \( n \) particles and two measurements per site. Werner and Wolf [3], and independently Zukowski and Brukner [13] showed that \( C_n \) is a hyper-octahedron and derived the inequalities of its facets. These are \( 2^{2^n} \) inequalities of the form

\[
-1 \leq \sum_{s_1, \ldots, s_n = 0,1} \beta_f(s_1, \ldots, s_n)X_1^{s_1}X_2^{s_2}\ldots X_n^{s_n} \leq 1
\]  

(6)

where each inequality is determined by an arbitrary function \( f : \{0, 1\}^n \to \{-1, 1\} \) with

\[
\beta_f(s_1, \ldots, s_n) = \frac{1}{2^n} \sum_{\varepsilon_1, \ldots, \varepsilon_n = 0,1} (-1)^{\varepsilon_1 s_1 + \ldots + \varepsilon_n s_n} f(\varepsilon_1, \ldots, \varepsilon_n)
\]  

(7)
In other words, to each choice of function $f$ there corresponds a choice of coefficients $\beta_f$. Since $\beta_f$ is the inverse Fourier transform of $f$ on the group $\mathbb{Z}_2^n$ we have by Plancherel’s theorem:

$$\sum_s |\beta_f(s)|^2 = \frac{1}{2^n} \sum_\varepsilon |f(\varepsilon)|^2 = 1$$ (8)

Using the analogy with the bipartite case let $A^1_0, A^1_1, ..., A^n_0, A^n_1$ be $2n$ arbitrary Hermitian operators in a Hilbert space $\mathbb{H}$, satisfying $(A^i_j)^2 = 1$. The quantum operators corresponding to the classical facets in Eq (6) are the Werner Wolf operators on $\mathbb{H}^\otimes n$ given by:

$$W_f = \sum_{s_1, ..., s_n \in \{0,1\}} \beta_f(s_1, ..., s_n) A^1_{s_1} \otimes ... \otimes A^n_{s_n}$$ (9)

It is easy to see from Eq (6) that $|tr(\rho W_f)| \leq 1$ for every separable state $\rho$ and all the $f$’s. However, the inequalities may be violated by entangled states. Let $Q_n$ be the set of all vectors in $\mathbb{R}^{2^n}$ whose coordinates have the form $q(s_1, ..., s_n) = tr(\rho A^1_{s_1} \otimes ... \otimes A^n_{s_n})$ for some choice of state $\rho$ and operators $A^i_j$ as above. The set $Q_n$ is the range of possible values of quantum correlations, and it is not difficult to see that $Q_n$ is convex and $Q_n \supset C_n$. To obtain information about the boundary of $Q_n$ we can examine how $\|W_f\|$ varies as we change the $A^i_j$’s. In this case too, it was shown [3] that for each fixed $f$, the maximal value of $\|W_f\|$ is already obtained when we choose $\mathbb{H} = \mathbb{C}^2$, and the $A^i_j$’s to be spin operators. Therefore, without loss of generality, consider

$$W_f = \sum_{s_1, ..., s_n \in \{0,1\}} \beta_f(s_1, ..., s_n) \sigma(a^1_{s_1}) \otimes ... \otimes \sigma(a^n_{s_n})$$ (10)

Where $a^1_0, a^1_1, ..., a^n_0, a^n_1$ are $2n$ arbitrary directions. We can calculate explicitly the eigenvalues of $W_f$ [3, 15]. Let $z_j$ be the direction orthogonal to the vectors $a^j_0, a^j_1, j = 1, ..., n$. Denote by $|-1\rangle_j$ and $|1\rangle_j$ the states “spin-down” and “spin-up” in the $z_j$-direction; so the vectors $|\omega_1, \omega_2, ..., \omega_n\rangle, \omega = (\omega_1, \omega_2, ..., \omega_n) \in \{-1, 1\}^n$ form a basis for the $n$-qbits space. Let $x_j$ be orthogonal to $z_j$ and let $\theta^j_s$ be the angle between $a^j_s$ and $x_j$, $s = 0, 1$. For each $f$ there are $2^n$ eigenvectors of $W_f$ which have the generalized GHZ form

$$|\Psi_f(\omega)\rangle = \frac{1}{\sqrt{2}} (e^{i\theta(\omega)} |\omega_1, \omega_2, ..., \omega_n\rangle + |\omega_1, -\omega_2, ..., -\omega_n\rangle)$$ (11)
and the corresponding eigenvalue

$$\lambda_f(\omega) = e^{i\Theta(\omega)} \sum_{s_1, \ldots, s_n \in \{0,1\}} \beta_f(s_1, \ldots, s_n) \exp i \left( \omega_1 \theta_{s_1}^1 + \ldots + \omega_n \theta_{s_n}^n \right)$$

(12)

where \(\Theta(\omega)\) in Eqs (11,12) is chosen so that \(\lambda_f(\omega)\) is a real number. Hence

$$\|W_f\| = \max_{\omega} |\lambda_f(\omega)|$$

(13)

As in the CHSH case we can check how large \(\|W_f\|\) can become as \(a_0^j, a_1^j\) range over all possible directions. Using Eqs (12,13) we see that

$$\max_{a_0^j, a_1^j} \|W_f\| = \max_{\theta_1^1, \ldots, \theta_n^1} \sum_{s_1, \ldots, s_n \in \{0,1\}} \beta_f(s_1, \ldots, s_n) \exp i \left( \theta_{s_1}^1 + \ldots + \theta_{s_n}^n \right)$$

(14)

with the maximum on the left is taken over all possible choices of directions \(a_1^1, a_1^1, \ldots, a_0^n, a_1^n\).

The Mermin-Klyshko operators [2], mentioned previously, correspond to a particular choice of \(f_0 : \{0,1\}^n \rightarrow \{-1,1\}\) and \(a_0^j, a_1^j\), with the result that \(\|W_{f_0}\| = \sqrt{2^{n-1}}\). This is the maximal value of Eq (14) possible. The maximum value is attained by a small minority of the operators \(W_f\), only those which are obtained from \(W_{f_0}\) by one of the \(n!2^{2n+1}\) symmetry operations of the polytope \(C_n\) (as compared with the total of \(2^{2n}\) of facets in Eq (6)).

In any case, for most \(f\)'s there is a choice of angles such that \(W_f\) is a witness. This means that in addition to the fact that \(|\text{tr}(\rho W_f)| \leq 1\) for every separable state \(\rho\), we also have \(\|W_f\| > 1\). The \(2^n\) exceptional cases are those in which the inequality in Eq (6) degenerates into the trivial condition \(1 \leq X_{s_1}^1 X_{s_2}^2 \cdots X_{s_n}^n \leq 1\). All the other \(W_f\)'s are witnesses. The reason is that all inequalities of type Eq (6) are obtained from the basic inequalities for \(C_2\) (including the trivial ones) by iteration [3]. If the iteration contains even one instance of the type Eq (2) the corresponding operator can be chosen to violate the CHSH inequality.

### III. RANDOM WITNESSES

#### A. Typical behavior of \(\max_{a_0^j, a_1^j} \|W_f\|\)

Although the norm of \(\|W_f\|\) can reach as high as \(\sqrt{2^{n-1}}\) this is not the rule but the exception. Our aim is to estimate the typical behavior of \(\max_{a_0^j, a_1^j} \|W_f\|\) as we let \(f\) range over all its values (and the maximum taken over all directions \(a_0^j, a_1^j\)). To do that, consider
the set of all $2^n$ functions $f : \{0,1\}^n \rightarrow \{-1,1\}$ as a probability space with a uniform probability distribution $\mathcal{P}$, which assigns probability $2^{-2^n}$ to each one of the $f$’s. Then, for each set of fixed directions $a_i^j$ we can look at $\|W_f\|$ as a random variable defined on the space of $f$’s. Likewise, also $\max_{a_i^0, a_i^1} \|W_f\|$ in Eq (14) is a random variable on the space of $f$’s, for which we have

**Theorem 1** There is a universal constant $C$ such that

$$\mathcal{P} \{ f ; \max_{a_i^0, a_i^1} \|W_f\| > C\sqrt{n \log n} \} \rightarrow 0 \text{ as } n \rightarrow \infty$$  \hspace{1cm} (15)

The proof of this result is based on the theorem of Salem, Zygmund, and Kahane \[16\] and is given in the appendix. This means that for the vast majority of the $f$’s the violation of the classical inequalities Eq (6) is small. Accordingly, the boundary of $Q_n$ is highly uneven about the facets of $C_n$; it does not extend far above most of the facets of $C_n$, but occasionally it has an extended exponential hump.

The expected growth of $|\lambda_f|$ is even slower when the directions $a_i^j$ (or the angles $\theta_i^j$) are fixed. As a direct consequence of Tchebychev’s inequality \[17\] we get:

**Proposition 2** For $\lambda_f$ in Eq (12) we have for all $M > 1 : \mathcal{P} \{ f ; |\lambda_f| > M \} \leq \frac{1}{M^2}$.

(See the appendix for details). This means that most of the eigenvalues of the $W_f$’s are bounded within a small sphere. The application of a randomly chosen $W_f$ to any of its eigenstates $|\Psi_f(\omega)\rangle$ in Eq (12) is unlikely to reveal a significant violation of Eq (6).

**B. The random witness conjecture**

For appropriate choices of angles the Werner-Wolf operators Eq (10) are, with very few exceptions, entanglement witnesses on the space of $n$ qbits. They are very special witnesses for two reasons: firstly, they are local operators. This means that if we possess many copies of a system made of $n$-qbits, all in the same state $|\Phi\rangle$, we can measure the expectation $\langle \Phi | W_f | \Phi \rangle$ by performing separate measurements on each qbit of the system. Secondly, even as local observables the Werner Wolf operators are special, because of the restriction to two measurements per particle. Indeed, one would have liked to extend the results beyond this restriction, and obtain all the inequalities for any number of measurement per site, but this problem is $NP$-hard even for $n = 2$, see \[18\].
However, the $W_j$’s are the most likely to be violated among the local operators with two measurements per site, because they are derived from the facets of $C_n$. Moreover, we already noted that all the norm estimates are also valid for the wider family given in Eq (9), with the $A_j$’s acting on any finite dimensional space, and satisfying $(A_j)^2 = I$. Hence, the estimate of theorem 1 includes many more witnesses than those given in Eq (10). To an $n$ qubits system we can add auxiliary particles and use quantum and classical communication protocols. As long as our overall measurement is in the closed convex hull of operators of the form

$$W = \sum_{s_1, \ldots, s_k \in \{0,1\}} \beta f(s_1, \ldots, s_k)A_{s_1}^1 \otimes \cdots \otimes A_{s_k}^k$$

with the $A_j$’s satisfying $(A_j)^2 = I$, and $k \leq O(n)$, the estimate of theorem 1 holds.

Hence, there is a reason to suspect that the typical behavior of the Werner Wolf operators is also typical of general random witnesses. A random witness is an observable drawn from the set of all witnesses $W$ with uniform probability. It is easy to give an abstract description of $W$: Consider the space of Hermitian operators on $(\mathbb{C}^2)^\otimes n$, it has dimension $d_n = 2^{n-1}(2^n + 1)$. For an Hermitian operator $A$ define the norm $\|[A]\| = \sup \|A \alpha_1 \cdots \alpha_n\|$ where the supremum ranges over all choices of unit vectors $\alpha_i \in \mathbb{C}^2$. Denote the unit sphere in this norm by $K_1 = \{A; \|[A]\| = 1\}$. It is a hypersurface of dimension $d_n - 1$, which is equipped with the uniform (Lebesgue) measure, and its total hyper-area is finite. Now, denote by $K_2$ the normal unit sphere $K_2 = \{A; \|A\| = 1\}$. Here, as usual, $\|A\| = \sup \|A \Phi\|$ is the operator norm, where the supremum is taken over all unit vectors $\Phi$. The set of witnesses is $W = K_1 \setminus K_2$. Note that if $A \in W$ then necessarily $\|A\| > 1$. It is not difficult to see that $W$ is relatively open in $K_1$, and therefore has a non zero measure in $K_1$. We consider the set of all witnesses $W$ on $(\mathbb{C}^2)^\otimes n$ and the normalized Lebesgue measure $P$ on it.

**Conjecture 3** There is a universal constant $C$ such that

$$P \{W \in W; \|W\| > C \sqrt{n \log n} \} \to 0 \text{ as } n \to \infty.$$  

(17)

Let $|\Phi\rangle$ be a unit vector in $(\mathbb{C}^2)^\otimes n$. Consider the measure of entanglement defined by:

$$E(|\Phi\rangle) = \sup \{\|W |\Phi\rangle\|; W \in W\}.$$  

(18)
Hence, $\mathcal{E}(|\Phi\rangle)$ is the least upper bound on the expectation values that any witness can have in the state $|\Phi\rangle$. Because we have normalized the witnesses by $[|W\rangle] = 1$ the measure $\mathcal{E}(|\Phi\rangle)$ is finite for all $|\Phi\rangle$. The ”dual” of Eq (17) is

**Conjecture 4** There is a universal constant $C$ such that

$$\mathcal{M}\{|\Phi\rangle;\; \mathcal{E}(|\Phi\rangle) > C\sqrt{n\log n}\} \to 0 \; \text{as } n \to \infty \quad (19)$$

Where $\mathcal{M}$ is the normalized Lebesgue measure on the unit sphere of $(\mathbb{C}^2)^\otimes n$, and the supremum is taken over all unit vectors $|\Phi\rangle \in (\mathbb{C}^2)^\otimes n$.

**Conjecture 3** is based on the assumption that the witnesses of the type $W_f$ form a sufficiently dense mesh in the $|[|W\rangle]|$-norm on $W$. If this is the case the conjectures may be proved as a concentration effect of the kind expressed by Levy’s lemma. This lemma has been recently used in the theory of quantum information [19, 20].

**C. Discussion**

Assume that conjecture 3 is valid and consider the following highly ideal situation: A macroscopic object (a single copy of it) is prepared in a state unknown to us and is carefully kept insulated from environmental decoherence. Since the state is unknown we choose randomly a witness $W$ to examine it. Now, suppose that by pure luck the system happens to be in an eigenstate of $W$; in fact, the eigenstate corresponding to its maximum eigenvalue (in absolute value). This means, in particular, that the state of the system is entangled; but can we detect this fact using $W$? A measurement of a witness on a single copy is a very complicated affair (just think about any one of the $W_f$’s in Eq (10)). Such a measurement invariably involves manipulations of the individual particles. If we make the reasonable assumption that each such manipulation introduces a small independent error, we obtain a total measurement error that grows exponentially with $n$. By the random witness conjecture, Eq (17), this means that we are unlikely to see a clear non-classical effect. The typical witness is a poor witness.

A natural question to ask is why should we consider the uniform measure over witnesses as the correct probability measure. In other words, why is a witness chosen at random according to the uniform measure typical? The answer is implicit in the situation just
described; we assume that we lack any knowledge of the state, and therefore have no reason
to give a preference to one witness over another. The point is that our choice is not going
to work even if we were lucky. At any rate, cases in which very little is known about the
quantum state of a macroscopic object are not rare.

Moreover, there is a reason to believe that a result about the uniform distribution is
also relevant to the case where partial information about the state is available. In this
case we should modify the uniform distribution and condition it on the additional available
constraints. However, there is an exponential gap between the typical witness norm in
Eq (17) and the norm required for a successful measurement. This means that the additional
information should be pretty accurate to be of any help. Moreover, it is possible that we
will not be able to witness the entanglement of many precisely known states, because, quite
likely, even the best witnesses for such states have small norms.

Conjecture 4, if true, is even more relevant to our ability to observe macroscopic entangle-
ment. There are two types of macroscopic or mesoscopic states whose entanglement might
be witnessed, and the conjecture concerns the second case:

1. There may be relatively rare cases in which the entanglement witness happens to be
a thermodynamic observable, that is, an observable whose measurement does not require
manipulation of individual particles but only the observation of some global property of the
system. There are some indications that this may be the case for some spin chains and
lattices [21].

2. Cases of very strong entanglement, like the GHZ state in Eq (11), which do require
many manipulations of individual particles to witness the entanglement of a single copy of
it; however, the value of $\mathcal{E}(|\Phi\rangle)$ is large enough to give significant results that rise above the
measurement errors.

This means that the answer to the question ”why don’t I see cats in superposition” is
twofold: decoherence surely; but even if we could turn it off, there is the combinatorial
possibility that ”seeing” something like this is nearly impossible. All this, luckily, does not
prevent the existence of exotic macroscopic superpositions that can be recorded.

There is some analogy between the present approach to multiparticle systems and the
point made by Khinchin on the foundations of classical statistical mechanics [22, 23]. While
thermodynamic equilibrium has its origins in the dynamics of the molecules, much of the
observable qualities of multiparticle systems can be explained on the basis of the law of large
numbers. The tradition which began with Boltzmann identifies equilibrium with ergodicity. The condition of ergodicity ensures that every integrable function has identical phase-space and long-time averages. However, Khinchine points out that this is an overkill, because most of the integrable functions do not correspond to macroscopic (that is, thermodynamic) observables. If we concentrate on thermodynamic observables, which involve averages over an enormous number of particles, weaker dynamical assumptions will do the job.

I believe that a similar answer can be given to our original question, namely, why we do not normally see large macroscopic objects in entangled states. Since decoherence cannot be “turned off” the multiparticle systems that we encounter are never maximally entangled. But even if the amount of entanglement that remains in them is still significant, we cannot detect it, because the witnesses are simply too weak and their states not sufficiently entangled.

IV. APPENDIX

In the proof of theorem 1 we shall rely on a theorem in Fourier analysis due to Salem, Zygmund and Kahane [16]. Our aim is to consider random trigonometric polynomials. So let \((\Omega, \Sigma, P)\) be a probability space, where \(\Omega\) is a set, \(\Sigma\) a \(\sigma\)-algebra of subsets of \(\Omega\), and \(P : \Sigma \rightarrow [0,1]\) a probability measure. For a random variable \(\xi\) on \(\Omega\) denote by \(E(\xi) = \int_{\Omega} \xi(\omega) dP(\omega)\) the expectation of \(\xi\). A real random variable \(\xi\) is called subnormal if \(E(\exp(\lambda \xi)) \leq \exp(\lambda^2/2)\) for all \(-\infty < \lambda < \infty\).

A trigonometric polynomial in \(r\) variables is a function on the torus \(\mathbb{T}^r\) given by

\[
g(t_1, t_2, ..., t_r) = \sum b(k_1, k_2, ..., k_r) e^{i(k_1 t_1 + k_2 t_2 + ... + k_r t_r)} \tag{20}
\]

where the sum is taken over all negative and nonnegative integers \(k_1, k_2, ..., k_r\) which satisfy \(|k_1| + |k_2| + ... + |k_r| \leq N\). The integer \(N\) is called the degree of the polynomial. Denote \(\|g\|_\infty = \max_{t_1, ..., t_r} |g(t_1, t_2, ..., t_r)|\).

**Theorem 5 (Salem, Zygmund, Kahane)** Let the \(\xi_j(\omega), j = 1, 2, ..., J\) be a finite sequence of real, independent, subnormal random variables on \(\Omega\). Let \(g_j(t), j = 1, 2, ..., J\) be a sequence of trigonometric polynomials in \(r\) variables whose degree is less or equal \(N\), and such that \(\sum_j |g_j(t)|^2 \leq 1\) for all \(t\). Then

\[
P \left\{ \omega; \left\| \sum_{j=1}^{J} \xi_j(\omega)g_j(t) \right\|_\infty > C \sqrt{r \log N} \right\} \leq \frac{1}{N^2 e^{r}} \tag{21}
\]
for some universal constant $C$.

Note that the formulation here is slightly different from that in [16], but the proof is identical. Our probability space $\Omega$ is the set of all functions $f : \{0,1\}^n \to \{-1,1\}$ with the uniform distribution which assigns each such function $f$ a weight $2^{-2n}$. On this space consider the $2^n$ random variables $\xi_\varepsilon(f)$ defined for each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n$ by

$$\xi_\varepsilon(f) = f(\varepsilon) \quad (22)$$

Now, note that $f \in \Omega$ iff $-f \in \Omega$ hence for each fixed $\varepsilon$ we get $E(\xi_\varepsilon) = 2^{-2n} \sum_f f(\varepsilon) = -2^{-2n} \sum_f f(\varepsilon) = -\mathcal{E}(\xi_\varepsilon)$, and therefore $\mathcal{E}(\xi_\varepsilon) = 0$, similarly, for $\varepsilon, \varepsilon'$ we have $\mathcal{E}(\xi_\varepsilon \xi_{\varepsilon'}) = \delta(\varepsilon, \varepsilon')$ and so on; the $2^n$ random variables $\xi_\varepsilon(f)$ are independent. Now, by a similar argument

$$\mathcal{E}(\exp(\lambda \xi_\varepsilon)) = 2^{-2n} \sum_f \exp(\lambda f(\varepsilon)) = 2^{-2n} \sum_f \frac{1}{2}[\exp(\lambda f(\varepsilon)) + \exp(-\lambda f(\varepsilon))] = \frac{1}{2}(e^\lambda + e^{-\lambda}) \leq e^{\frac{\lambda^2}{2}}$$

To define the trigonometric polynomials note that by Eqs (7,12)

$$\sum_{s_1, \ldots, s_n \in \{0,1\}} \beta_f(s_1, \ldots, s_n) \exp i (t^1_{s_1} + \ldots + t^n_{s_n}) = \frac{1}{2^n} \sum_{\varepsilon} f(\varepsilon) \sum_s (-1)^{s_1 \varepsilon_1 + \ldots + s_n \varepsilon_n} \exp i (t^1_{s_1} + t^2_{s_2} + \ldots + t^n_{s_n}) = \sum_{\varepsilon} f(\varepsilon) \frac{1}{2^n} \prod_{j=1}^n (\exp it^j_0 + (-1)^{s_j} \exp it^j_1) = \sum_{\varepsilon} \xi_\varepsilon(f) g_\varepsilon(t) \quad (24)$$

with

$$g_\varepsilon(t) = 2^{-n} \prod_j (\exp i t^j_0 + (-1)^{s_j} \exp i t^j_1) \quad (25)$$

The polynomials $g_\varepsilon(t)$ do not depend on $f$, have $2n$ variables $t^j_0$, $t^j_1$, $j = 1,2,\ldots,n$, and their degree is $n$. We shall prove that $\sum_\varepsilon |g_\varepsilon(t)|^2 = 1$ for all $t$. Indeed, $|g_\varepsilon(t)|^2 = 2^{-2n} \left| \prod_j (1 + (-1)^{s_j} \exp (i \phi_j)) \right|^2$, with $\phi_j = t^j_1 - t^j_0$. But, $|1 + \exp i \phi_j|^2 = 4 \cos^2 \left( \frac{\phi_j}{2} \right)$ and $|1 - \exp i \phi_j|^2 = 4 \sin^2 \left( \frac{\phi_j}{2} \right)$ and therefore $\sum_\varepsilon |g_\varepsilon(t)|^2 = \prod_j \left( \cos^2 \left( \frac{\phi_j}{2} \right) + \sin^2 \left( \frac{\phi_j}{2} \right) \right) = 1.$
From Eqs (14,24) we get

$$\max_{a_0^j,a_1^j} \|W_f\| = \left\| \sum_\epsilon \xi_\epsilon(f)g_\epsilon(t) \right\|_\infty$$ \hspace{1cm} (26)

Hence, we can apply the Salem Zygmund Kahane inequality Eq (19) to the present case, with $N = n$ and $r = 2n$, to obtain theorem 1.

To prove proposition 2 consider $|\lambda_f|$ as a random variable on the space of $f$’s. By Eq (12) we get

$$|\lambda_f| = \left| \sum_{s_1,...,s_n \in \{0,1\}} \beta_f(s_1,...,s_n) \exp i \left( t_{s_1}^1 + ... + t_{s_n}^n \right) \right| \hspace{1cm} (27)$$

with $t_{s_j}^j = \omega_j \theta_{s_j}^j$. By Eq (??) we get $|\lambda_f| = |\sum_\epsilon \xi_\epsilon(f)g_\epsilon(t)|$. But $\mathcal{E}(\sum_\epsilon \xi_\epsilon g_\epsilon) = 0$, and $\mathcal{E}(|\sum_\epsilon \xi_\epsilon g_\epsilon|^2) = \sum_\epsilon |g_\epsilon|^2 = 1$. Therefore, by Tchebishev’s inequality [17] we have

$$\mathcal{P} \{ f; \ |\lambda_f| > M \} \leq \frac{1}{M^2} \text{ for all } M > 1.$$

**Acknowledgement 6** The research for this paper was done while I was visiting the Perimeter Institute, I am grateful for the hospitality. This research is supported by the Israel Science Foundation, grant number 879/02.

[1] B. M. Terhal, *J. Theor. Comp. Sci.* **287**, 313 (2002).
[2] N. D. Mermin, *Phys. Rev. Lett.* **65**, 1838 (1990).
[3] R. F. Werner and M. M. Wolf, *Phys. Rev. A* **64**, 032112 (2001).
[4] J.F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, *Phys.Rev. Lett.* **23**, 880 (1969).
[5] A. Fine, *Phys. Rev. Lett.* **48**, 291 (1982).
[6] I. Pitowsky, *Quantum Probability-Quantum Logic* (Springer, Berlin, 1989).
[7] N. Gisin, and A. Peres, *Phys. Lett. A* **162**, 15 (1992).
[8] B. S. Cirel’son, *Lett. Math. Phys.* **4**, 93 (1980).
[9] B. S. Tsirel’son, *J. Sov. Math.* **36**, 557 (1987).
[10] S. Phillip and K, Svozil, *Phys. Rev. A* **63**, 032101 (2004).
[11] A. Cabello, *Phys. Rev. Lett.* **92** 060403 (2004).
[12] F. A. Bovino and G. Castagnoli, *Phys.Rev.Lett.* **92**, 060404 (2004).
[13] M. Zukowski and C. Brukner *Phys. Rev. Lett.* **88** 210401 (2002).
[14] W. Rudin, *Fourier Analysis on Groups* (Interscience Publishers, New York, 1960).

[15] V. Scarani and N. Gisin, e-print quant-ph/0103068.

[16] J. P. Kahane, *Some Random Series of Functions* (Cambridge University Press, Cambridge, second edition, 1985)

[17] Y. S. Chow and H. Teicher, *Probability Theory, Independence, Interchangeability, Martingales* (Springer, Berlin 1978).

[18] I. Pitowsky, *Math. Programming A* 50, 395 (1991).

[19] S. O. Bartlett, P. Hayden and R.W. Speckens quant-ph/0506260 (2005).

[20] S. Popescu, A. J. Short and A. Winter quant-ph/0511225 (2005).

[21] C. Brukner and V. Vedral, quant-ph/0406040 (2004).

[22] A. I. Khinchin, *Mathematical Foundations of Statistical Mechanics* (Dover, New-York 1949).

[23] I. Pitowsky, *Studies Hist. Phil. of Mod. Phys.*, 32, 595 (2001).