COCHARACTERS FOR THE WEAK POLYNOMIAL IDENTITIES OF THE LIE ALGEBRA OF 3 $\times$ 3 SKEW-SYMMETRIC MATRICES

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Abstract. Let $so_3(K)$ be the Lie algebra of 3 $\times$ 3 skew-symmetric matrices over a field $K$ of characteristic 0. The ideal $I(M_3(K), so_3(K))$ of the weak polynomial identities of the pair $(M_3(K), so_3(K))$ consists of the elements $f(x_1, \ldots, x_n)$ of the free associative algebra $K\langle X \rangle$ with the property that $f(a_1, \ldots, a_n) = 0$ in the algebra $M_3(K)$ of all 3 $\times$ 3 matrices for all $a_1, \ldots, a_n \in so_3(K)$. The generators of $I(M_3(K), so_3(K))$ were found by Razmyslov in the 1980’s. In this paper the cocharacter sequence of $I(M_3(K), so_3(K))$ is computed. In other words, the $GL_p(K)$-module structure of the algebra generated by $p$ generic skew-symmetric matrices is determined. Moreover, the same is done for the closely related algebra of $SO_3(K)$-equivariant polynomial maps from the space of $p$-tuples of 3 $\times$ 3 skew-symmetric matrices into $M_3(K)$ (endowed with the conjugation action). In the special case $p = 3$ the latter algebra is a module over a 6-variable polynomial subring in the algebra of $SO_3(K)$-invariants of triples of 3 $\times$ 3 skew-symmetric matrices, and a free resolution of this module is found. The proofs involve methods and results of classical invariant theory, representation theory of the general linear group and explicit computations with matrices.

1. Introduction

This paper can be considered as a relative of the well-known paper of Procesi [P2] whose abstract says that “In a precise way the ring of $m$ generic 2 $\times$ 2 matrices and related rings are described.” In the present work we also describe the ring of $m$ generic 3 $\times$ 3 skew-symmetric matrices and a related ring in a precise way, but in somewhat different terms than [P2] (and we restrict to the case of a characteristic zero base field).

Take 3 $\times$ 3 generic skew-symmetric matrices

$$t_k = \begin{pmatrix} 0 & t_{12}^{(k)} & t_{13}^{(k)} \\ -t_{12}^{(k)} & 0 & t_{23}^{(k)} \\ -t_{13}^{(k)} & -t_{23}^{(k)} & 0 \end{pmatrix}, \quad k = 1, \ldots, p,$$

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where \( T_p = \{ t_{ij}^{(k)} \mid i, j = 1, 2, 3, k = 1, \ldots, p \} \) are commuting variables. Till the end of the paper we fix a field \( K \) of characteristic zero. Then \( t_1, \ldots, t_p \) are elements of the \( 3 \times 3 \) matrix algebra \( M_3(K[T_p]) \) over the polynomial ring \( K[T_p] \). As usual, we shall identify the elements of \( K[T_p] \) with polynomial maps \( so_3(K)^{\otimes p} \to K \) in the obvious way, where \( so_3(K) \) is the space of \( 3 \times 3 \) skew-symmetric matrices over \( K \), which is also the Lie algebra of the special orthogonal group \( SO_3(K) = \{ A \in K^{3 \times 3} \mid AA^T = I, \det(A) = 1 \} \). Accordingly, \( M_3(K[T_p]) \) is identified with the set of polynomial maps \( so_3(K)^{\otimes p} \to M_3(K) \). Denote by \( \mathcal{F}_p \) the associative \( K \)-subalgebra (with an identity element) of \( M_3(K[T_p]) \) generated by \( t_1, \ldots, t_p \) (so the identity matrix \( I \) is an element of \( \mathcal{F}_p \) by definition):

\[
\mathcal{F}_p = K\langle t_1, \ldots, t_p \rangle \subset M_3(K[T_p])
\]

The special orthogonal group \( SO_3(K) \) acts on \( so_3(K) \) by conjugation (the adjoint action of \( SO_3(K) \) on its Lie algebra), and \( SO_3(K) \) acts by simultaneous conjugation on \( so_3(K)^{\otimes p} \), the space of \( p \)-tuples of skew-symmetric \( 3 \times 3 \) matrices. Also \( SO_3(K) \) acts on \( M_3(K) \) by conjugation, and we write \( \mathcal{E}_p \) for the subset of \( M_3(K[T_p]) \) consisting of the \( SO_3(K) \)-equivariant polynomial maps \( so_3(K)^{\otimes p} \to M_3(K) \). Clearly \( \mathcal{E}_p \) is an associative \( K \)-subalgebra of \( M_3(K[T_p]) \), and \( \mathcal{E}_p \) contains \( \mathcal{F}_p \). It follows easily from known results (see Proposition 4.2) that

\[
\mathcal{E}_p = K\langle t_1, \ldots, t_p, \text{tr}(t_it_j)I, \text{tr}(t_kt_lt_m)I \mid i \leq j, k < l < m \rangle \subset M_3(K[T_p]),
\]

where \( I \) stands for the \( 3 \times 3 \) identity matrix throughout the paper.

In the present paper we aim at a combinatorial description of the algebras \( \mathcal{F}_p \) and \( \mathcal{E}_p \). The general linear group \( GL_p(K) \) acts on \( \mathcal{E}_p \) via graded \( K \)-algebra automorphisms. Note first that \( GL_p(K) \) acts (from the right) on \( so_3(K)^{\otimes p} \) as follows. For

\[
g = \begin{pmatrix} g_{11} & \cdots & g_{1p} \\ \vdots & \ddots & \vdots \\ g_{p1} & \cdots & g_{pp} \end{pmatrix}, \text{ and } a = (a_1, \ldots, a_p) \in so_3(K)^{\otimes p}
\]

we have

\[
(a_1, \ldots, a_p) \cdot g = \left( \sum_{i=1}^{p} g_{i1}a_i, \sum_{i=1}^{p} g_{i2}a_2, \ldots, \sum_{i=1}^{p} g_{ip}a_i \right).
\]

This induces a left action (via graded \( K \)-algebra automorphisms) of \( GL_p(K) \) on the algebra \( K[T_p] \) (respectively \( M_3(K[T_p]) \)) of polynomial maps \( so_3^{\otimes p} \to K \) (respectively \( so_3^{\otimes p} \to M_3(K[T_p]) \)) in the standard way (for a function \( f \), we have \( (g \cdot f)(a) = f(a \cdot g) \)). More explicitly, \( g \cdot f^{(k)} = \sum_{i=1}^{p} g_{ik} f^{(l)} \cdot i \), and for a matrix \( m = (m_{ij})_{i,j=1}^{3} \in M_3(K[T_p]) \), we have \( g \cdot m = (g \cdot m_{ij})_{i,j=1}^{3} \). The \( GL_3(K) \)-action on \( so_3(K)^{\otimes p} \) commutes with the \( SO_3(K) \)-action, hence \( \mathcal{E}_p \) is a \( GL_p(K) \)-submodule of \( M_3(K[T_p]) \). Obviously \( \mathcal{F}_p \) is a \( GL_p(K) \)-submodule in \( \mathcal{E}_p \).

We shall determine the \( GL_p(K) \)-module structure both for \( \mathcal{F}_p \) and \( \mathcal{E}_p \). Our Theorem 3.7 (see also Theorem 4.1 and Corollary 4.5 (together with Lemma 3.3) give the multiplicities of the irreducible \( GL_p(K) \)-modules as summands in \( \mathcal{F}_p \) and in \( \mathcal{E}_p \). In fact, in the course of the proofs highest weight vectors for each irreducible summand are explicitly provided. In the case of \( \mathcal{E}_p \) results from classical invariant theory allow to give upper bounds for these multiplicities. Then with explicit constructions we show that for almost all irreducibles these upper bounds are achieved.
2. Preliminaries

For a background on the mathematics used in this paper we recommend:

- On trace identities the paper by Procesi [P1] and the book by Razmyslov [Ra3, Chapter IV];
- On invariant theory the book by Weyl [W];
- On representation theory of the general linear group the book by Macdonald [Mc] and for the applications to algebras with polynomial identities the book by one of the authors [Dr2, Chapter 12].

2.1. Weak polynomial identities.

Definition 2.1. Let \( R \) be an associative algebra over a field \( K \) and let \( R^{(-)} \) be the Lie algebra with respect to the operation \([r_1, r_2] = r_1r_2 - r_2r_1, r_1, r_2 \in R\). Let \( L \) be a Lie subalgebra of \( R^{(-)} \) which generates \( R \) as an associative algebra, i.e., \( R \) is an associative enveloping algebra of \( L \). The polynomial \( f(x_1, \ldots, x_n) \) of the free associative algebra \( K\langle X \rangle = K\langle x_1, x_2, \ldots \rangle \) is called a weak polynomial identity for the pair \((R, L)\) if \( f(a_1, \ldots, a_n) = 0 \) in \( R \) for all \( a_1, \ldots, a_n \in L \). The ideal \( I(R, L) \) of the weak polynomial identities of \((R, L)\) is generated by the system \( B = \{f_j(x_1, \ldots, x_n) \mid j \in J\} \) (and \( B \) is called a basis of the weak polynomial identities of the pair \((R, L)\)) if \( I(R, L) \) is the minimal ideal of weak identities containing \( B \). Then \( I(R, L) \) is generated as an ideal by the polynomials \( f_j(u_1, \ldots, u_{n_j}), j \in J, \) where \( u_1, \ldots, u_{n_j} \) are Lie elements in \( K\langle X \rangle \).
Weak polynomial identities were introduced by Razmyslov [Ra1, Ra2] as a powerful tool in the solution of two important problems in the theory of PI-algebras. In [Ra1] Razmyslov found bases, over a field $K$ of characteristic 0, of the weak polynomial identities of the pair $(M_2(K), sl_2(K))$, the polynomial identities of the Lie algebra $sl_2(K)$ of traceless $2 \times 2$ matrices, and the polynomial identities of the associative algebra $M_2(K)$ of $2 \times 2$ matrices. Up till now, in the case of characteristic 0, the algebras $sl_2(K)$ and $M_2(K)$ are the only nontrivial simple Lie and associative algebras with known bases of their polynomial identities. (Another proof for the basis of the weak polynomial identities of $(M_3(K), sl_2(K))$ is given in [DrK].) In [Ra2] Razmyslov constructed, using weak polynomial identities of the pair $(M_q(K), sl_q(K))$, a central polynomial for the algebra $M_q(K)$ of $q \times q$ matrices, solving an old problem of Kaplansky [K1, K2]. The existence of central polynomials for $M_q(K)$ was established independently with other methods by Formanek [F]. (For more information on the polynomial identities and central polynomials for matrices see, e.g., [Dr2, DrF].)

Let $g \cong sl_2(\mathbb{C})$ be the three-dimensional complex simple Lie algebra and let $U(g)$ be its universal enveloping algebra. In [Ra3] Razmyslov showed that the ideal $I(U(g), g)$ satisfies the Specht property. It is finitely generated and the same holds for any ideal of weak polynomial identities which contains it. Later, in [Ra4] Theorem 38.1) page 251 in the Russian original and page 181 in the English translation he found an explicit basis of the weak polynomial identities of the pair $(M_3(\mathbb{C}), g)$, where $\hat{g} : g \to \text{End}_\mathbb{C}(V_g) \cong M_3(\mathbb{C})$ is a $q$-dimensional irreducible representation of $g$. The basis consists of three weak polynomial identities:

$$s_3(x_1, x_2, x_3)x_4 = x_4s_3(x_1, x_2, x_3),$$

where

$$s_3(x_1, x_2, x_3) = \sum_{\sigma \in S_3} \text{sign}(\sigma)x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$$

is the standard polynomial of degree 3,

$$\delta \sum_{\sigma \in S_3} \text{sign}(\sigma)[x_4, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = 2x_4s_3(x_1, x_2, x_3),$$

where the commutators are left normed, e.g., $[x_1, x_2, x_3] = [x_1, x_2], x_3]$, and $\delta = (q^2 - 1)/4$ is the value of the Casimir element in the representation $\hat{g}$, and one more identity in two variables

$$\text{ART}_q(x_1, x_2) = \text{ad}x_2 \prod_{i=1}^{q-1} \left( L_{x_2} - \left( i - \frac{1-q}{2} \right) \text{ad}x_2 \right)x_1 = 0.$$ 

Here $L_r : R \to R, r \in R$, is the operator of left multiplication of the algebra $R$, defined by $r' \mapsto rr'$, $r' \in R$, and $\text{ad}r(r') = [r, r'], r, r' \in R$. For $q = 2$ this gives that the weak polynomial identities of the pair $(M_2(\mathbb{C}), sl_2(\mathbb{C}))$ follow from the weak identity $[x_1^2, x_2] = 0$, which was established already in [Ra1]. The Lie algebra $sl_2(\mathbb{C})$ is isomorphic to the Lie algebra $so_3(\mathbb{C})$ of $3 \times 3$ skew-symmetric matrices and after easy computations the result from [Ra4] Theorem 38.1] gives:

Theorem 2.2. The weak polynomial identities of the pair $(M_3(\mathbb{C}), so_3(\mathbb{C}))$ follow from the weak identities

$$s_3(x_1, x_2, x_3)x_4 = x_4s_3(x_1, x_2, x_3),$$

where

$$s_3(x_1, x_2, x_3) = \sum_{\sigma \in S_3} \text{sign}(\sigma)x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$$

is the standard polynomial of degree 3,
\[
\sum_{\sigma \in S_3} \text{sign}(\sigma) [x_4, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = x_4 s_3(x_1, x_2, x_3),
\]

and

\[x_1[x_1, x_2]x_1 = 0.\]

(The result in [Ra4, Theorem 38.1] gives explicit bases of the weak polynomial identities also in the infinite dimensional cases.)

As in the case of ordinary polynomial identities the symmetric group \(S_n\) of degree \(n\) acts from the left on the vector space \(P_n \subset K \langle X \rangle\) of multilinear polynomials of degree \(n\) and for any ideal \(I(R, L)\) of weak polynomial identities \(P_n \cap I(R, L)\) is an \(S_n\)-submodule of \(P_n\). The sequence of \(S_n\)-characters \(\chi_n(R, L)\) of \(P_n/(P_n \cap I(R, L))\), \(n = 0, 1, 2, \ldots\), is called the cocharacter sequence of \(I(R, L)\). Then

\[
\chi_n(R, L) = \sum_{\lambda \vdash n} m_{\lambda} \chi_{\lambda},
\]

where \(\chi_{\lambda}\) is the irreducible character of \(S_n\) indexed by the partition \(\lambda\) of \(n\) and the nonnegative integer \(m_{\lambda}\) is the multiplicity of \(\chi_{\lambda}\) in \(\chi_n(R, L)\). By a result of Berele [B] and one of the authors [Dr1] the multiplicity \(m_{\lambda}\), \(\lambda = (\lambda_1, \ldots, \lambda_p) \vdash n\), is the same as the multiplicity of the irreducible polynomial \(GL_p(K)\)-module \(W_p(\lambda)\) in the \(GL_p(K)\)-module

\[
F_p(R, L) = K \langle X_p \rangle/(K \langle X_p \rangle \cap I(R, L)) \cong \sum_{\lambda} m_{\lambda} W_p(\lambda),
\]

where \(K \langle X_p \rangle = K \langle x_1, \ldots, x_p \rangle\), the general linear group \(GL_p(K)\) acts canonically on the vector space \(K X_p\) with basis \(X_p = \{x_1, \ldots, x_p\}\) and this action is extended diagonally on the whole algebra \(K \langle X_p \rangle\).

Our Theorem [Ca] gives explicitly the cocharacter sequence \(\chi_n(M_3(K), so_3(K))\), \(n = 0, 1, 2, \ldots\). The proof is based on a combination of classical invariant theory and representation theory of the general linear group. By standard arguments due to Regev [Re], since \(\dim(so_3(K)) = 3\), we work in the algebra \(F_3(M_3(K), so_3(K))\) considered as a \(GL_3(K)\)-module instead to work with \(P_n(M_3(K), so_3(K))\) and representations of \(S_n\). Using classical results from invariant theory we give upper bounds for the multiplicities \(m_{\lambda}\), \(\lambda = (\lambda_1, \lambda_2, \lambda_3)\) depending on the parity of the differences \(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3\). Then with explicit constructions we show that these upper bounds are achieved.

Since the adjoint representation of the three-dimensional Lie algebra \(sl_2(\mathbb{C})\) is isomorphic to the canonical three-dimensional representation of \(so_3(\mathbb{C})\), our main result holds also for the weak polynomial identities of \((M_3(K), ad(sl_2(K)))\). We mention that the cocharacter sequence of the pair \((M_2(K), sl_2(K))\) was found by Procesi [P2, see also [Dr2] Exercise 12.6.12].

2.2. Invariant theory of \(SO_3(K)\). Let \(V_d\) be the \(d\)-dimensional \(K\)-vector space with basis \(\{v_1, \ldots, v_d\}\) with the canonical action of the group \(GL(V_d)\) identified in the usual way with \(GL_d(K)\). The action of \(GL_d(K)\) on \(V_d\) induces an action on the algebra \(K[X_d] = K[x_1, \ldots, x_d]\) of polynomial functions on \(V_d\) (here \(x_1, \ldots, x_d\) is the dual basis in \(V_d^*\) to the basis chosen in \(V_d\)). If

\[
v = \alpha_1 v_1 + \cdots + \alpha_d v_d \in V_d, \quad f = f(X_d) \in K[X_d], \quad g \in GL_d(K),
\]

then

\[
f(v) = f(\alpha_1, \ldots, \alpha_d) \quad \text{and} \quad (g(f))(v) = f(g^{-1}(v)).
\]
For any subgroup \( G \) of \( \text{GL}_d(K) \) the algebra \( K[X_d]^G \) of \( G \)-invariants consists of all \( f(X_d) \in K[X_d] \) with the property \( g(f) = f \) for all \( g \in G \).

We equip the vector space \( V_d \) with a nondegenerate symmetric bilinear form. If

\[
v' = \alpha_1 v_1 + \cdots + \alpha_d v_d, \quad v'' = \beta_1 v_1 + \cdots + \beta_d v_d,
\]

then

\[
\langle v', v'' \rangle = \alpha_1 \beta_1 + \cdots + \alpha_d \beta_d.
\]

The special orthogonal group \( \text{SO}_d(K) \) acts canonically on the vector space \( V_d \) and consists of all matrices with determinant equal to 1 which preserve the symmetric bilinear form. The action of \( \text{SO}_d(K) \) can be extended to the direct sum \( V_d^{\oplus p} \) of \( p \) copies of \( V_d \). Write \( \{v_1, \ldots, v_d\} \) for the basis of the \( i \)th direct summand of \( V_d^{\oplus p} \) corresponding to the chosen basis \( \{v_1, \ldots, v_n\} \) of \( V_d \), and let \( y_{ik} \) be the polynomial (in fact linear) function which sends the vector \( v_{ik} \) to 1 and to 0 all other vectors of the fixed basis of \( V_d^{\oplus p} \). Set \( y_i = (y_{i1}, \ldots, y_{id}), \ i = 1, \ldots, p, \) and consider the scalar products

\[
\langle y_i, y_j \rangle = y_{i1} y_{j1} + \cdots + y_{id} y_{jd}, \quad 1 \leq i, j \leq p,
\]

the determinant

\[
\Delta_d(y_{j1}, \ldots, y_{jd}) = \det(y_{j1}, \ldots, y_{jd}) = \begin{vmatrix} y_{1j1} & y_{1j2} & \cdots & y_{1jd} \\ y_{2j1} & y_{2j2} & \cdots & y_{2jd} \\ \vdots & \vdots & \ddots & \vdots \\ y_{dj1} & y_{dj2} & \cdots & y_{djd} \end{vmatrix},
\]

\( 1 \leq j_1 < \cdots < j_d \leq p \), and the Gram determinant

\[
\Gamma_k(y_{i1}, \ldots, y_{ik}, y_{j1}, \ldots, y_{jk}) = \det(\langle y_{ik}, y_{j1} \rangle, \ldots, \langle y_{ik}, y_{jk} \rangle) = \begin{vmatrix} \langle y_{i1}, y_{j1} \rangle & \cdots & \langle y_{i1}, y_{jk} \rangle \\ \vdots & \ddots & \vdots \\ \langle y_{ik}, y_{j1} \rangle & \cdots & \langle y_{ik}, y_{jk} \rangle \end{vmatrix},
\]

\( 1 \leq i_1 < \cdots < i_k \leq p, \ 1 \leq j_1 < \cdots < j_k \leq p \).

The following classical theorems, see, e.g., [W, Theorems 2.9.A and 2.17.A], describe the generating set and the defining relations of the algebra

\[
K[Y_{pd}]^{\text{SO}_d(K)} = K[y_{ik} \mid i = 1, \ldots, p, k = 1, \ldots, d]^{\text{SO}_d(K)}
\]

of \( \text{SO}_d(K) \)-invariants of \( V_d^{\oplus p} \).

**Theorem 2.3** (First fundamental theorem for the invariants of \( \text{SO}_d(K) \)). (i) The algebra \( K[Y_{pd}]^{\text{SO}_d(K)} \) is generated by the scalar products \( \langle y_i, y_j \rangle, \ 1 \leq i, j \leq p \), and by the determinants \( \Delta_d(y_{j1}, \ldots, y_{jd}), \ 1 \leq j_1 < \cdots < j_d \leq p \).

(ii) The elements of \( K[Y_{pd}]^{\text{SO}_d(K)} \) are linear combinations of products

\[
\langle y_{i_1}, y_{j_1} \rangle \cdots \langle y_{i_r}, y_{j_r} \rangle \text{ and } \Delta_d(y_{k_1}, \ldots, y_{k_d}) \langle y_{i_1}, y_{j_1} \rangle \cdots \langle y_{i_r}, y_{j_r} \rangle,
\]

\( 1 \leq i_r, j_r \leq p, \ r = 1, \ldots, n, \ 1 \leq k_1 < \cdots < k_d \leq p \).

**Theorem 2.4** (Second fundamental theorem for the invariants of \( \text{SO}_d(K) \)). The defining relations of the algebra \( K[Y_{pd}]^{\text{SO}_d(K)} \) consist of

\[
\Gamma_d(y_{i_0}, y_{i_1}, \ldots, y_{i_d} \mid y_{j_0}, y_{j_1}, \ldots, y_{jd}) = 0,
\]

\( 1 \leq i_0 < i_1 < \cdots < i_d \leq p, \ 1 \leq j_0 < j_1 < \cdots < j_d \leq p \),

\[
\Delta_d(y_{i_1}, \ldots, y_{i_d}) \Delta_d(y_{j_1}, \ldots, y_{jd}) - \Gamma_d(y_{i_1}, \ldots, y_{i_d} \mid y_{j_1}, \ldots, y_{jd}) = 0,
\]

\( 1 \leq i_1 < \cdots < i_d \leq p, \ 1 \leq j_1 < \cdots < j_d \leq p \),
Theorem 2.6.\ Theorem 2.5.\ the so called polarized Pfaffians.

In [DoDr] we found a Grobner basis of the ideal of defining relations of the algebra \( K[Y_{pd}]^{SO_d(K)} \).

The general linear group \( GL_d(K) \) acts on the space \( M_d(K)^{\oplus p} \) of \( p \)-tuples of \( d \times d \) matrices by simultaneous conjugation:

\[ g \cdot (r_1, \ldots, r_p) = (gr_1g^{-1}, \ldots, gr_pg^{-1}), \quad g \in GL_d(K), \quad r_1, \ldots, r_p \in M_d(K). \]

The polynomial algebra corresponding to this action is in \( pd^2 \) variables:

\[ K[Z_p] = K[z_{ij}^{(k)} \mid 1 \leq i, j \leq d, \quad k = 1, \ldots, p]. \]

The action of \( GL_d(K) \) is defined in terms of generic \( d \times d \) matrices

\[ z_k = \begin{pmatrix} z_{11}^{(k)} & \cdots & z_{1d}^{(k)} \\ \vdots & \ddots & \vdots \\ z_{d1}^{(k)} & \cdots & z_{dd}^{(k)} \end{pmatrix}, \quad k = 1, \ldots, p. \]

If

\[ g^{-1} \left( z_{ij}^{(k)} \right) g = \left( w_{ij}^{(k)} \right), \quad g \in GL_d(K), \quad k = 1, \ldots, p, \]

then under the action of \( g \) the variable \( z_{ij}^{(k)} \) goes to \( w_{ij}^{(k)} \).

The algebra of invariants of the orthogonal group \( O_d(K) \subset GL_d(K) \) is described by Sibirskii [S] and Procesi [P1] Theorem 7.1:

**Theorem 2.5.** The algebra \( K[Z_p]^{O_d(K)} \) of invariants of the group \( O_d(K) \) acting by simultaneous conjugation on \( p \) copies of \( M_d(K) \) is generated by the traces

\[ \text{tr}(u_{k_1} \cdots u_{k_n}), \quad 1 \leq k_1, \ldots, k_n \leq p, \]

where \( u_{k_r} = z_{k_r} \) or \( u_{k_r} = z_{k_r}' \), the transpose of \( z_{k_r} \), \( r = 1, \ldots, n \).

The generators of the algebra \( K[Z_m]^{SO_d(K)} \) of invariants of \( SO_d(K) \) are given by Aslaksen, Tan, and Zhu [ATZ] Theorem 3:

**Theorem 2.6.** (i) For \( d \) odd the algebra \( K[Z_p]^{SO_d(K)} \) of \( SO_d(K) \)-invariants coincides with the algebra \( K[Z_p]^{O_d(K)} \) of \( O_d(K) \)-invariants.

(ii) For \( d \) even \( K[Z_p]^{SO_d(K)} \) is generated by the generators of \( K[Z_p]^{O_d(K)} \) and the so called polarized Pfaffians.

Now we define \( 3 \times 3 \) generic skew-symmetric matrices

\[ t_k = \begin{pmatrix} 0 & t_{12}^{(k)} & t_{13}^{(k)} \\ -t_{12}^{(k)} & 0 & t_{23}^{(k)} \\ -t_{13}^{(k)} & -t_{23}^{(k)} & 0 \end{pmatrix}, \quad k = 1, \ldots, p, \]

where \( T_p = \{ t_{ij}^{(k)} \mid i, j = 1, 2, 3; \quad k = 1, \ldots, p \} \) are commuting variables. The well-known generating system of the algebra of invariants \( K[T_p]^{SO_3(K)} \) of the special orthogonal group \( SO_3(K) \) acting by simultaneous conjugation on \( p \) copies of the Lie algebra \( so_3(K) \) of \( 3 \times 3 \) skew-symmetric matrices can be obtained as a consequence.
of Theorems 2.5 and 2.6. The same arguments hold for skew-symmetric matrices of any odd size.

**Corollary 2.7.** The algebra $K[T_p]^{SO_3(K)}$ is generated by the traces
\[ \text{tr}(t_{k_1} \cdots t_{k_n}), \quad 1 \leq k_1, \ldots, k_n \leq p. \]

**Proof.** Every generic $3 \times 3$ matrix $z_k$ is a sum of a skew-symmetric and a symmetric generic matrix,
\[ t_k = \frac{1}{2}(z_k - z_k^t) \quad \text{and} \quad r_k = \frac{1}{2}(z_k + z_k^t), \]
respectively. Since $z_k$ and $z_k^t$ can be expressed by $t_k$ and $r_k$, we can replace the generators $\text{tr}(u_{k_1} \cdots u_{k_n})$ of $K[Z_m]^{SO_3(K)}$ given in Theorems 2.5 and 2.6 by $\text{tr}(q_{k_1} \cdots q_{k_n})$, where $q_{k_1} = t_{k_1}$ or $q_{k_n} = r_{k_n}$. In our considerations $SO_3(K)$ acts on $p$ copies of the Lie algebra of the $3 \times 3$ skew-symmetric matrices. Hence $K[T_p]^{SO_3(K)}$ is generated by traces of products of skew-symmetric generic matrices only. \hfill \square

In the sequel we shall refer to the algebra $K[T_p]^{SO_3(K)}$ generated by traces of products of $3 \times 3$ generic skew-symmetric matrices as the generic trace algebra.

Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We define an $\mathbb{N}_0$-grading on the polynomial algebras $K[Y_{p,3}]$, $K[T_p]$ and on the algebra $F_p$, assuming that the variables $y_{k_1}, t_{ij}$ and the matrix $t_k$ are of degree $(0, \ldots, 0, 1, 0, \ldots, 0)$ (the $k$th coordinate is equal to 1 and all other coordinates are equal to 0). The generic trace algebra is an $\mathbb{N}_0$-graded subalgebra of $K[T_p]$.

**Proposition 2.8.** The algebras $K[Y_{p,3}]^{SO_3(K)}$ and $K[T_p]^{SO_3(K)}$ are isomorphic as $\mathbb{N}_0$-graded algebras.

**Proof.** The vector space $so_3(K)$ has a basis
\[ a_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \]

Denoting by $e_1, e_2, e_3$ the standard basis vectors in the space $K^3$ of column vectors, a straightforward calculation shows that the linear map
\[ so_3(K) \to K^3, \quad a_1 \mapsto e_3, \quad a_2 \mapsto -e_2, \quad a_3 \mapsto e_1 \]
is an isomorphism between the $SO_3(K)$-modules $so_3(K)$ and $K^3$, where $SO_3(K)$ acts via conjugation on $so_3(K)$ and via matrix multiplication on $K^3$. This isomorphism induces an isomorphism of the $SO_3(K)$-modules $so_3(K)^{\oplus p} \cong (K^3)^{\oplus p}$, their coordinate rings $K[T_p] \cong K[Y_{p,3}]$, and finally the $\mathbb{N}_0$-graded subalgebras $K[T_p]^{SO_3(K)} \cong K[Y_{p,3}]^{SO_3(K)}$ of $SO_3(K)$-invariants. For sake of completeness of the picture, we mention that the basis $\{a_1, a_2, a_3\}$ in $so_3(K)$ is orthonormal with respect to the nondegenerate, symmetric, $SO_3(K)$-invariant bilinear form defined by
\[ \langle a, b \rangle = -\frac{1}{2}\text{tr}(ab), \quad a, b \in so_3(K). \]

For a skew-symmetric $3 \times 3$ matrix $a$ and a symmetric $3 \times 3$ matrix $b$ we have $\text{tr}(ab) = 0$. It follows that $\text{tr}(t_{12}t_{23}) + \text{tr}(t_{23}t_{12}) = \text{tr}((t_{12}t_{23} + t_{23}t_{12})t_{3}) = 0$, so for any permutation $\pi \in S_3$ we have
\[ \text{tr}(t_{\pi(1)}t_{\pi(2)}t_{\pi(3)}) = \text{sign}(\pi)\text{tr}(t_{12}t_{23}). \]
Corollary 2.9. (i) The algebra \( K[T_3]^{SO_3(K)} \) is generated by the elements \( \text{tr}(t_i t_j) \), \( 1 \leq i \leq j \leq p \), and \( \text{tr}(t_k t_l t_m) \), \( 1 \leq k < l < m \leq p \).

(ii) The algebra \( K[T_3]^{SO_3(K)} \) is a rank two free module generated by \( \text{tr}(t_1 t_2 t_3) \) over its subalgebra generated by the algebraically independent elements \( \text{tr}(t_1^2) \), \( \text{tr}(t_2^2) \), \( \text{tr}(t_1 t_2) \), \( \text{tr}(t_1 t_3) \), \( \text{tr}(t_2 t_3) \).

Proof. (i) is an immediate consequence of Theorem 2.3. Corollary 2.7, Proposition 2.8. Taking into account also Theorem 2.4, we get (ii). □

2.3. Representation theory of \( \text{GL}_p(K) \). In what follows we assume that the general linear group \( \text{GL}_p(K) = \text{GL}(KX_p) \) acts canonically on the vector space \( KX_p \) with basis \( X_p \). If

\[
g = \begin{pmatrix} g_{11} & \cdots & g_{1p} \\ \vdots & \ddots & \vdots \\ g_{p1} & \cdots & g_{pp} \end{pmatrix},
\]

then \( g(x_j) = \sum_{i=1}^{p} g_{ji} x_i \), \( j = 1, \ldots, p \).

This action can be extended diagonally on the tensor algebra

\[
T(KX_p) = \sum_{n \geq 0} (KX_p)^{\otimes n} \cong K(X_p).
\]

In the sequel we shall identify \( T(KX_p) \) with the free associative algebra \( K(X_p) \) and \( (KX_p)^{\otimes n} \) with the homogeneous component \( K(XX_p)^{(n)} \) of degree \( n \) of \( K(X_p) \). The \( \text{GL}_p(K) \)-module \( K(X_p) \) is a direct sum of irreducible polynomial \( \text{GL}_p(K) \)-modules. The irreducible polynomial \( \text{GL}_p(K) \)-modules are indexed by partitions having not more than \( p \) parts and all they appear as summands in \( K(X_p) \). Let

\[
\lambda = (\lambda_1, \ldots, \lambda_p), \quad \lambda_1 \geq \cdots \geq \lambda_p \geq 0, \quad \lambda_1 + \cdots + \lambda_p = n,
\]

be a partition of \( n \) and let \( W_p(\lambda) \) be the corresponding \( \text{GL}_p(K) \)-module. The homogeneous component \( K(XX_p)^{(n)} \) of \( K(X_p) \) decomposes as

\[
K(XX_p)^{(n)} = \sum_{\lambda \vdash n} \text{deg}(\lambda) W_p(\lambda),
\]

where \( \text{deg}(\lambda) \) is the degree of the irreducible \( S_n \)-character \( \chi_{\lambda} \). The \( \text{GL}_p(K) \)-module \( W_p(\lambda) \subset K(XX_p)^{(n)} \) is generated by a polynomial \( w_p(\lambda) \) called the highest weight vector of \( W_p(\lambda) \) which is homogeneous of \( \mathbb{N}_0 \)-degree \( \lambda \) and can be described in the following way. The symmetric group \( S_n \) acts from the right on \( K(XX_p)^{(n)} \) by the rule

\[
(x_1, \ldots, x_n)^\tau = x_{i_{\tau(1)}}, \ldots, x_{i_{\tau(n)}}, \quad \tau \in S_n,
\]

and this action commutes with the action of \( \text{GL}_p(K) \) introduced before. Let \( [\lambda] \) be the Young diagram corresponding to the partition \( \lambda \) and let the lengths of the columns of \( [\lambda] \) be \( k_1, \ldots, k_{\lambda_1} \). Consider the product of standard polynomials

\[
w_\lambda(x_1, \ldots, x_{k_1}) = \prod_{j=1}^{\lambda_1} s_{k_j}(x_1, \ldots, x_{k_j}) = \prod_{j=1}^{\lambda_1} \left( \sum_{\sigma_j \in S_{k_j}} \text{sign}(\sigma_j) x_{\sigma_j(1)} \cdots x_{\sigma_j(k_j)} \right).
\]

Then every highest weight vector is of the form

\[
w = \sum_{\tau \in S_n} \alpha_\tau w_\lambda^\tau, \quad \alpha_\tau \in K.
\]
A $\lambda$-tableau is the Young diagram $[\lambda]$ whose boxes are filled with positive integers. We say that the tableau is of content $(n_1, \ldots, n_p)$ if $1, \ldots, p$ appear in it $n_1, \ldots, n_p$ times, respectively. The tableau is standard if its entries are the numbers $1, \ldots, n$, without repetition, arranged in such a way that they increase in rows (reading them from left to right) and in columns (reading from top to bottom). It is semistandard if its entries (allowing repetitions) do not decrease in rows and increase in columns.

Given a partition $\lambda$ of $n$, we set up a bijection between the set of $\lambda$-tableaux of content $(1, \ldots, 1)$ and $S_n$ as follows: we assign to the permutation $\varrho \in S_n$ the Young tableau $T_\lambda(\varrho)$ obtained by filling in the boxes of the first column of $[\lambda]$ with $\varrho^{-1}(1), \ldots, \varrho^{-1}(k_1)$, of the second column with $\varrho^{-1}(k_1+1), \ldots, \varrho^{-1}(k_1+k_2)$, etc. Then the highest weight vector $w_\lambda^\varrho$ has skew-symmetries in the positions listed in the first column of $T_\lambda(\varrho)$, skew-symmetries in the positions listed in the second column of $T_\lambda(\varrho)$, etc. For example, for $n = 5$, $\lambda = (2, 2, 1)$, and

$$\varrho^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 2 & 4 \end{pmatrix},$$

we have

$$T_\lambda(\varrho) = \begin{array}{c} 1 \hline 2 \\ 5 \hline 4 \end{array}, \quad w_\lambda^\varrho = \sum_{\sigma_1 \in S_3, \sigma_2 \in S_2} \text{sign}(\sigma_1) \text{sign}(\sigma_2) x_{\sigma_1(1)} x_{\sigma_2(1)} x_{\sigma_1(3)} x_{\sigma_2(2)} x_{\sigma_1(2)}.$$

It is known that the set of all $w_\lambda^\varrho$ corresponding to the standard $\lambda$-tableaux $T_\lambda(\varrho)$ is a basis of the vector space of the highest weight vectors of all $W_p(\lambda) \subset K \langle X_p \rangle^{(n)}$.

**Proposition 2.10.** (i) There is a one-to-one correspondence between an arbitrary $\mathbb{N}_0^p$-graded basis of a $\text{GL}_p(K)$-submodule $W_p(\lambda)$ of $K \langle X_p \rangle$ and the set of semistandard $\lambda$-tableaux filled in with $1, \ldots, p$, such that a basis vector of degree $(n_1, \ldots, n_p)$ corresponds to a semistandard tableau of content $(n_1, \ldots, n_p)$.

(ii) Suppose that there exists a mapping $\pi$ from an $\mathbb{N}_0^p$-graded basis of a $\text{GL}_p(K)$-submodule or factor module $W_p$ of $K \langle X_p \rangle$ into the set of semistandard $\lambda$-tableaux, such that a basis vector of degree $(n_1, \ldots, n_p)$ is mapped to a semistandard tableau of content $(n_1, \ldots, n_p)$, and for each partition $\lambda$, there exists a non-negative integer $m_\lambda$ such that every semistandard $\lambda$-tableau is the image of exactly $m_\lambda$ basis elements. Then $W_p$ decomposes as

$$W_p = \sum_\lambda m_\lambda W_p(\lambda).$$

**Proof.** The statement (i) follows immediately from the fact that the dimension of the homogeneous component $W_p^{(n_1, \ldots, n_p)}(\lambda)$ of degree $(n_1, \ldots, n_p)$ is equal to the coefficient of $\xi_1^{n_1} \cdots \xi_d^{n_d}$ of the Schur function $S_\lambda(\xi_1, \ldots, \xi_p)$. On the other hand this coefficient is equal to the number of semistandard $\lambda$-tableaux of content $(n_1, \ldots, n_d)$. For (ii) it is sufficient to apply the fact that the Schur functions play the role of characters of the representation of $\text{GL}_p(K)$ corresponding to the $\text{GL}_p(K)$-module $W_p(\lambda)$ and that the character of the direct sum of polynomial representations determines the decomposition of the corresponding $\text{GL}_p(K)$-module $W_p$. □

The decomposition of the $\text{GL}_p(K)$-module structure of the algebra of invariants of $\text{SO}_3(K)$ acting on $p$ copies of $V_3$ is given for example in [P2] Section 1.2] or in [LB] Chapter I, Theorem 4.3] in terms of semistandard tableaux.
Theorem 2.11. The algebra $K[Y_{p3}]^{SO_3(K)}$ has an $\mathbb{N}_0^n$-graded basis indexed (via a mapping $\pi$ as in Proposition 2.11 (ii)) by all semistandard $\lambda$-tableaux for all $\lambda = (2\mu_1, 2\mu_2, 2\mu_3)$ and $\lambda = (2\mu_1 + 1, 2\mu_2 + 1, 2\mu_3 + 1)$.

As an immediate consequence of Propositions 2.8 and 2.10 and Theorem 2.11 we obtain:

Corollary 2.12. As a $\text{GL}_p(K)$-module the algebra $K[T_p]^{SO_3(K)}$ of invariants of the action by simultaneous conjugation of $SO_3(K)$ on $p$ copies of $3 \times 3$ skew-symmetric matrices decomposes as

$$K[T_p]^{SO_3(K)} = \sum W_\rho(2\mu_1 + \delta, 2\mu_2 + \delta, 2\mu_3 + \delta),$$

where the summation runs on all $(\mu_1, \mu_2, \mu_3)$ and $\delta = 0$ or 1.

3. The cocharacter sequence and highest weight vectors

Our strategy will be the following. Since $\text{dim}(so_3(K)) = 3$, the cocharacter sequence of $I(M_3(K), so_3(K))$ is of the form

$$\chi_n(M_3(K), so_3(K)) = \sum_{\lambda \vdash n} m_{\lambda} \chi_{\lambda},$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a partition of $n$ in not more than three parts. This allows to replace the problem for the cocharacter sequence with the problem of the decomposition into a direct sum of irreducible components of the GL$_3(K)$-module $F_3(M_3(K), so_3(K))$. Then we use that $f(x_1, x_2, x_3) \in K[X_3]$ is a weak polynomial identity for the pair $(M_3(K), so_3(K))$ if and only if $\text{tr}(f(x_1, x_2, x_3)x_4x_5) = 0$ vanishes in $M_3(K)$ when evaluated on $so_3(K)$. We describe in the language of semistandard tableaux those elements of an $\mathbb{N}_0^n$-graded basis of the generic trace algebra $K[T_3]^{SO_3(K)}$ which are linear in $t_4$ and $t_5$. This gives an upper bound of the multiplicities $m_{\lambda}$ in $\chi_n(M_3(K), so_3(K))$. Then with explicit constructions we show that in all nontrivial cases these upper bounds are achieved.

Lemma 3.1. The polynomial $f(x_1, x_2, x_3) \in K[X_3]$ is a weak polynomial identity for the pair $(M_3(K), so_3(K))$ if and only if $\text{tr}(f(x_1, x_2, x_3)x_4x_5)$ is a weak trace identity, i.e., vanishes when evaluated on $so_3(K)$.

Proof. The matrix $a \in M_3(K)$ is equal to 0 if and only if $\text{tr}(ab) = 0$ for all matrices $b \in M_3(K)$. Since every matrix $b \in M_3(K)$ is a linear combination of products of two matrices $b_1, b_2 \in so_3(K)$, we obtain that $a = 0$ if and only if $\text{tr}(ab_1b_2) = 0$ for all $b_1, b_2 \in so_3(K)$. Applied to $f(x_1, x_2, x_3) \in K[X_3]$, this gives that it is a weak polynomial identity for the pair $(M_3(K), so_3(K))$ if and only if $\text{tr}(f(a_1, a_2, a_3)b_4b_5) = 0$ for all $a_1, a_2, a_3, b_4, b_5 \in so_3(K)$, i.e., when $\text{tr}(f(x_1, x_2, x_3)x_4x_5)$ is a weak trace identity.

Lemma 3.2. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (2\mu_1 + \delta, 2\mu_2 + \delta, 2\mu_3 + \delta)$, $\delta = 0$ or 1. Consider the set of semistandard $\lambda$-tableaux of content $(n_1, n_2, n_3, 1, 1)$. Deleting the boxes containing 4 and 5 from each tableau, we obtain a multiset of semistandard tableaux of content $(n_1, n_2, n_3)$. (i) The multiplicity of a semistandard $\nu$-tableau of content $(n_1, n_2, n_3)$ in this multiset is non-zero if and only if $\nu = (\nu_1, \nu_2, \nu_3), \nu_1 \geq \nu_2 \geq \nu_3 \geq 0$, and

$$\nu \in \{(\lambda_1 - 2, \lambda_2, \lambda_3), (\lambda_1, \lambda_2 - 2, \lambda_3), (\lambda_1, \lambda_2, \lambda_3 - 2), (\lambda_1 - 1, \lambda_2 - 1, \lambda_3), (\lambda_1 - 1, \lambda_2, \lambda_3 - 1), (\lambda_1, \lambda_2 - 1, \lambda_3 - 1)\}.$$
(ii) Moreover, the multiplicity is 2 if \( \nu = (\lambda_1 - 1, \lambda_2 - 1, \lambda_3) \) and \( \lambda_1 > \lambda_2 \), or \( \nu = (\lambda_1, \lambda_2 - 1, \lambda_3 - 1) \) and \( \lambda_2 > \lambda_3 \), or \( \nu = (\lambda_1 - 1, \lambda_2, \lambda_3 - 1) \).

(iii) All other positive multiplicities are equal to 1.

Proof. If \( \lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 \geq 2 \), then the semistandard \( \lambda \)-tableaux of content \((n_1, n_2, n_3, 1, 1)\) are of the following form:

If \( \lambda_1 = \lambda_2 > \lambda_3 \) or \( \lambda_2 = \lambda_3 > 0 \), then 4 and 5 may appear in the same column, and 4 is necessarily above 5. These observations clearly yield the statements (i), (ii), (iii).

Let \((K[T_5]|_{\text{SO}_3(K)})^{(n_0,n_0,n_0,1,1)}\) be the component of the generic trace algebra \(K[T_5]|_{\text{SO}_3(K)}\) which is linear in the generic skew-symmetric matrices \(t_4\) and \(t_5\). Embedding \(\text{GL}_3(K)\) into \(\text{GL}_5(K)\) by

\[
\begin{pmatrix}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
g_{11} & g_{12} & g_{13} & 0 & 0 \\
g_{21} & g_{22} & g_{23} & 0 & 0 \\
g_{31} & g_{32} & g_{33} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\in \text{GL}_5(K)
\]

we equip the vector space \((K[T_5]|_{\text{SO}_3(K)})^{(n_0,n_0,n_0,1,1)}\) with the structure of a \(\text{GL}_3(K)\)-module.

Lemma 3.3. The \(\text{GL}_3(K)\)-module \((K[T_5]|_{\text{SO}_3(K)})^{(n_0,n_0,n_0,1,1)}\) decomposes as

\[
(K[T_5]|_{\text{SO}_3(K)})^{(n_0,n_0,n_0,1,1)} = \sum_\nu m_\nu W_3(\nu), \quad \nu = (\nu_1, \nu_2, \nu_3),
\]

where:

(i) If \(\nu_1 \equiv \nu_2 \equiv \nu_3 \pmod{2}\), then

\[
m_\nu = \begin{cases} 
3, & \text{if } \nu_1 > \nu_2 > \nu_3; \\
2, & \text{if } \nu_1 = \nu_2 > \nu_3; \\
2, & \text{if } \nu_1 > \nu_2 = \nu_3; \\
1, & \text{if } \nu_1 = \nu_2 = \nu_3;
\end{cases}
\]

(ii) If \(\nu_1 \equiv \nu_2 \not\equiv \nu_3 \pmod{2}\), then

\[
m_\nu = \begin{cases} 
2, & \text{if } \nu_1 > \nu_2; \\
1, & \text{if } \nu_1 = \nu_2;
\end{cases}
\]

(iii) If \(\nu_1 \equiv \nu_3 \not\equiv \nu_2 \pmod{2}\), then \(m_\nu = 2\);
(iv) If $\nu_1 \neq \nu_2 \equiv \nu_3 \pmod{2}$, then
$$m_\nu = \begin{cases} 2, & \text{if } \nu_2 > \nu_3; \\ 1, & \text{if } \nu_2 = \nu_3. \end{cases}$$

**Proof.** Combining Proposition 2.8, Theorem 2.11 and Corollary 2.12 we obtain that as an $\mathbb{N}_0^3$-graded vector space the algebra $K[T_5]^{SO_3(K)}$ has a graded basis indexed by all semistandard $\lambda$-tableaux for all $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, such that $\lambda_1 \equiv \lambda_2 \equiv \lambda_3 \pmod{2}$. Hence the vector space $(K[T_5]^{SO_3(K)})^{(N_0,N_0,N_0,1,1)}$ has a basis indexed by the semistandard $\lambda$-tableaux of content $(n_1, n_2, n_3, 1, 1)$, $n_1, n_2, n_3 \in \mathbb{N}_0$. Deleting 4 and 5 from such a semistandard $\lambda$-tableau, we obtain the semistandard $\nu$-tableaux of content $(n_1, n_2, n_3)$ described in Lemma 3.2.

(i) If $\nu_1 \equiv \nu_2 \equiv \nu_3 \pmod{2}$ and $\nu_1 > \nu_2 > \nu_3$ then we can obtain the $\nu$-tableau only from the corresponding $\lambda$-tableaux for $\lambda = (\nu_1 + 2, \nu_2, \nu_3), (\nu_1, \nu_2 + 2, \nu_3), (\nu_1, \nu_2, \nu_3 + 2)$, i.e., $m_\nu = 3$. If $\nu_1 = \nu_2 > \nu_3$ we have to exclude the case $\lambda = (\nu_1, \nu_2 + 2, \nu_3)$ because $\nu_1 < \nu_2 + 2$. The other cases $\nu_1 < \nu_2 = \nu_3$ and $\nu_1 = \nu_2 = \nu_3$ are handled in a similar way.

(ii) If $\nu_1 \equiv \nu_2 \neq \nu_3 \pmod{2}$ and $\nu_1 > \nu_2$, then we can obtain the $\nu$-tableau from the two $\lambda$-tableaux for $\lambda = (\nu_1 + 1, \nu_2 + 1, \nu_3)$, i.e., $m_\nu = 2$. When $\nu_1 = \nu_2$ there is only one $\lambda$-tableau $\lambda = (\nu_1 + 1, \nu_2 + 1, \nu_3)$ when 4 and 5 are in the most right column of the $\lambda$-tableau.

The proofs of the other two cases (iii) and (iv) are similar. \hfill \Box

**Corollary 3.4.** The multiplicities of the irreducible components $W_3(\nu)$ in the decomposition of
$$F_3(M_3(K), so_3(K)) = \sum_\nu m_\nu(M_3(K), so_3(K))W_3(\nu)$$
are bounded from above by the integers $m_\nu$ in Lemma 3.3.

**Proof.** By Lemma 3.1 the $GL_3(K)$-module $F_3(M_3(K), so_3(K))$ is isomorphic to the $GL_3(K)$-module of all elements in the form $\text{tr}(f(t_1,t_2,t_3)t_4t_5)$ in the generic trace algebra $K[T_5]^{SO_3(K)}$. Clearly this module is a submodule of the $GL_3(K)$-module $(K[T_5]^{SO_3(K)})^{(N_0,N_0,N_0,0,1,1)}$ and hence the multiplicities of its irreducible components $m_\nu(M_3(K), so_3(K))$ are bounded from above by the multiplicities of the irreducible components of $(K[T_5]^{SO_3(K)})^{(N_0,N_0,N_0,0,1,1)}$ given in Lemma 3.3. \hfill \Box

For a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3) \vdash n$, a permutation $q \in S_n$, and for certain $q \in \{1, \ldots, n\}$ we define operations $\iota_{1q}, \iota_{2}, \iota_{3}$ on the highest weight vector $w(x_1, x_2, x_3) = w^q(x_1, x_2, x_3) \in K\langle X_3 \rangle$:

- If $\lambda_1 > \lambda_2$ and the integer $q$ is at the $r$th position in the first row of the tableau $T_{\lambda}(q)$, and $r > \lambda_2$, then $w(x_1, x_2, x_3)$ has the form
  $$w(x_1, x_2, x_3) = \sum \pm u' x_1 u'',$$
  where the summation runs on some monomials $u'$ and $u''$ of degree $q - 1$ and $n - q$, respectively, and we define
  $$\iota_{1q}(w(x_1, x_2, x_3)) = \sum \pm u'_1 x_1 u'';$$

- Let $\tau = (2, 2)$ and let
  $$w^{(2)}(x_1, x_2) = \sum_{\sigma_1, \sigma_2 \in S_2} \text{sign}(\sigma_1)\text{sign}(\sigma_2)x_{\sigma_1(1)}x_{\sigma_2(1)}x_{\sigma_1(2)}x_{\sigma_2(2)}$$
  with
Let the highest weight vector corresponding to the permutation \( \tau \)-tableau be the highest weight vector corresponding to the \( \tau \)-tableau \( \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 3 & 1 \\ 2 & 1 & 4 \end{array} \). Then we define

\[
\iota_2(w(x_1, x_2, x_3)) = w(x_1, x_2, x_3)u^{(2)}_{(2,2)}(x_1, x_2).
\]

\begin{itemize}
  \item We define
  \[
  \iota_3(w(x_1, x_2, x_3)) = w(x_1, x_2, x_3)s_3(x_1, x_2, x_3).
  \]
\end{itemize}

**Lemma 3.5.** Let \( \nu = (\nu_1, \nu_2, \nu_3) \vdash n \) and let \( w(x_1, x_2, x_3) = w^{\nu}(x_1, x_2, x_3) \) be the highest weight vector corresponding to the permutation \( \varrho \in S_n \). Then the polynomials \( \iota_1 q(w(x_1, x_2, x_3)), \iota_2(w(x_1, x_2, x_3)), \iota_3(w(x_1, x_2, x_3)) \) are highest weight vectors of the form \( w^\mu \), where \( \mu \) is the partition \( (\nu_1 + 2, \nu_2, \nu_3), (\nu_1 + 2, \nu_2 + 2, \nu_3), (\nu_1 + 1, \nu_2 + 1, \nu_3 + 1) \), respectively.

Let \( w^{(i)}(x_1, x_2, x_3) = w^{\sigma}(x_1, x_2, x_3), \rho_i \in S_n, i = 1, \ldots, m, \) and let \( \{a_1, a_2, a_3\} \) be the basis of \( s_{03}(K) \) defined above. If the matrices \( w^{(i)}(a_1, a_2, a_3) \) are linearly independent in \( M_3(K) \), then the matrices of each set

\[
\{\iota_q(u^{(i)}(a_1, a_2, a_3)) | i = 1, \ldots, m\}, \quad \{\iota_2(w^{(i)}(a_1, a_2, a_3)) | i = 1, \ldots, m\},
\]

\[
\{\iota_3(w^{(i)}(a_1, a_2, a_3)) | i = 1, \ldots, m\}
\]

are also linearly independent in \( M_3(K) \).

**Proof.** Applying \( \iota_q \) we insert \( x_1^2 \) between the \( q \)-th and \( (q + 1) \)-th positions of the monomials of \( w(x_1, x_2, x_3) \). So \( \iota_q(w^{\sigma}) = w^{\sigma} \), where \( \mu = (\nu_1 + 2, \nu_2, \nu_3) \) and the tableau \( T_{\mu}(\psi) \) is obtained from the tableau \( T_{\mu}(\rho) \) by adding 2 to each entry greater than \( q \), and writing \( q + 1, q + 2 \) in the two new boxes at the end of the first row of the Young diagram of \( \mu \).

Hence \( \iota_q(w(x_1, x_2, x_3)) \) is a highest weight vector corresponding to the partition \( (\nu_1 + 2, \nu_2, \nu_3) \). Similarly, \( \iota_2 \) multiplies \( w(x_1, x_2, x_3) \) by a polynomial with two skew-symmetries in \( x_1, x_2, i.e., \iota_2(w(x_1, x_2, x_3)) \) is also a highest weight vector corresponding to the partition \( (\nu_1 + 2, \nu_2 + 2, \nu_3) \). Finally, \( \iota_3 \) multiplies \( w(x_1, x_2, x_3) \) by the standard polynomial \( s_3(x_1, x_2, x_3) \) which is a skew-symmetric sum in three variables. Hence \( \iota_3(w(x_1, x_2, x_3)) \) is a highest weight vector corresponding to the partition \( (\nu_1 + 1, \nu_2 + 1, \nu_3 + 1) \). It is also clear that the resulting highest weight vectors are all of the form \( w^{\mu}_{\sigma} \) for some partition \( \mu \) and permutation \( \sigma \).

Direct computations show that

\[
a_1^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -a_1,
\]

\[
w^{(2)}_{(2,2)}(a_1, a_2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad s_3(a_1, a_2, a_3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\]

If the matrices \( \iota_q(w^{(i)}(a_1, a_2, a_3)), i = 1, \ldots, m, \) are linearly dependent, then the equality \( \iota_q(w^{(i)}(a_1, a_2, a_3)) = -w^{(i)}(a_1, a_2, a_3) \) implies the linear dependence for \( w^{(i)}(a_1, a_2, a_3) \) which is a contradiction. Similarly, since the matrices \( w^{(2)}_{(2,2)}(a_1, a_2) \) and \( s_3(a_1, a_2, a_3) \) are invertible, the linear dependence of \( \iota_2(w^{(i)}(a_1, a_2, a_3)) \) and of \( \iota_3(w^{(i)}(a_1, a_2, a_3)) , i = 1, \ldots, m, \) gives the linear dependence of \( w^{(i)}(a_1, a_2, a_3) \). \( \square \)
Lemma 3.6. For each of the following partitions $\nu$ the evaluations of the highest weight vectors $w^{(i)}(a_1, a_2, a_3), i = 1, \ldots, m_{\nu}$, are linearly independent if $m_{\nu} > 1$ and nonzero if $m_{\nu} = 1$:

(i) For $\nu = (4, 2)$ and the $\nu$-tableaux $\begin{array}{c} 1 \ 3 \\ 5 \\ 6 \\ 2 \\ 4 \end{array}$ and $\begin{array}{c} 1 \\ 3 \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \\ 6 \end{array}$, and $\begin{array}{c} 1 \\ 3 \\ 4 \\ 5 \end{array}$

$w^{(1)} = [x_1, x_2]^2 x_1^2, \quad w^{(2)} = [x_1, x_2](x_1^3 x_2 - x_2 x_1^3), \quad w^{(3)} = (x_1 x_2^2 - x_1 x_2 x_1 x_2 - x_2 x_1 x_2 x_1 + x_2^2 x_1^2)x_1$;

(ii) For $\nu = (2, 2)$

$w^{(1)} = [x_1, x_2]^2, \quad w^{(2)} = x_1^2 x_2^2 - x_1 x_2 x_1 x_2 - x_2 x_1 x_2 x_1 + x_2^2 x_1^2$;

(iii) For $\nu = (3, 1, 1)$ and the $\nu$-tableaux $\begin{array}{c} 1 \\ 4 \\ 5 \\ 2 \\ 3 \\ 4 \end{array}$ and $\begin{array}{c} 1 \\ 3 \\ 4 \end{array}$

$w^{(1)} = s_3(x_1, x_2, x_3)x_1^2, \quad w^{(2)} = \sum_{\sigma \in S_3} \text{sign}(\sigma)x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}^2$;

(iv) For $\nu = (0), w^{(1)} = 1$;

(v) For $\nu = (3, 1)$ and the $\nu$-tableaux $\begin{array}{c} 1 \\ 3 \\ 4 \\ 2 \\ 4 \end{array}$ and $\begin{array}{c} 3 \\ 1 \\ 2 \\ 4 \end{array}$

$w^{(1)} = [x_1, x_2]^2 x_1^2, \quad w^{(2)} = x_1^2 [x_1, x_2]$;

(vi) For $\nu = (1, 1)$ and the $\nu$-tableau $\begin{array}{c} 1 \\ 4 \end{array}$

$w^{(1)} = [x_1, x_2]$;

(vii) For $\nu = (2, 1)$ and the $\nu$-tableaux $\begin{array}{c} 1 \\ 3 \\ 2 \\ 4 \end{array}$ and $\begin{array}{c} 2 \\ 1 \end{array}$

$w^{(1)} = [x_1, x_2]x_1, \quad w^{(2)} = x_1[x_1, x_2]$;

(viii) For $\nu = (3, 2)$ and the $\nu$-tableaux $\begin{array}{c} 1 \\ 3 \\ 5 \end{array}$ and $\begin{array}{c} 2 \\ 4 \\ 1 \end{array}$

$w^{(1)} = [x_1, x_2]^2 x_1, \quad w^{(2)} = x_1[x_1, x_2]^2$;

(ix) For $\nu = (1)$ and the $\nu$-tableau $\begin{array}{c} 1 \end{array}$

$w^{(1)} = x_1$.

Proof. Direct computations show that:

(i) For $\nu = (4, 2)$

$w^{(1)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad w^{(2)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$w^{(3)}(a_1, a_2, a_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$;

(ii) For $\nu = (2, 2)$

$w^{(1)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad w^{(2)}(a_1, a_2, a_3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$;
(iii) For $\nu = (3, 1, 1)$

$$w^{(1)}(a_1, a_2, a_3) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad w^{(2)}(a_1, a_2, a_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix};$$

(iv) For $\nu = (0)$

$$w^{(1)}(a_1, a_2, a_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

(v) For $\nu = (3, 1)$

$$w^{(1)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad w^{(2)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

(vi) For $\nu = (1, 1)$

$$w^{(1)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix};$$

(vii) For $\nu = (2, 1)$

$$w^{(1)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad w^{(2)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

(viii) For $\nu = (3, 2)$

$$w^{(1)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad w^{(2)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

(ix) For $\nu = (1)$ and the $\nu$-tableau $\begin{array}{|c|c|} \hline 1 \\ \hline \end{array}$

$$w^{(1)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In all nine cases the matrices $w^{(i)}(a_1, a_2, a_3)$ are linearly independent if $m_\nu > 1$ and nonzero if $m_\nu = 1$. □

Now we state the main result of this section.

**Theorem 3.7.** Let $K$ be a field of characteristic 0 and let $I(M_3(K), so_3(K))$ be the ideal of the weak polynomial identities for the pair $(M_3(K), so_3(K))$. Then the cocharacter sequence of $I(M_3(K), so_3(K))$ is

$$\chi_n(M_3(K), so_3(K)) = \sum_{\nu \vdash n} m_\nu(M_3(K), so_3(K))\chi_\nu, \quad \nu = (\nu_1, \nu_2, \nu_3),$$

where the multiplicity $m_\nu(M_3(K), so_3(K))$ is equal to 1 if $\nu = (n)$, $n \geq 0$, and to the following if $\nu_2 > 0$:
Corollary 3.8. The cocharacter sequence of the ideal of weak polynomial identities of the pair $(M_3(K), b_3(K))$ is the same as the cocharacter sequence of $(M_3(K), so_3(K))$.
Note that this is multiplicity free, unlike the cocharacter sequence given in Theorem 3.7.

**Problem 3.10.** Let \( q : \mathfrak{sl}_2(K) \to \text{End}_K(V_q) \cong M_q(K) \) be a \( q \)-dimensional irreducible representation of \( \mathfrak{sl}_2(K) \), \( q > 2 \) (or \( q = \infty \)). Find the cocharacter sequence of the pair \((M_q(K), \mathfrak{sl}_2(K))\).

4. The embedding into the trace ring

Denote by \( \kappa : K\langle X_p \rangle \to F_p \) the \( K \)-algebra surjection with \( x_k \mapsto t_k \), \( k = 1, \ldots, p \). Clearly \( \ker(\kappa) = I(M_3(K), so_3(K)) \cap K\langle X_p \rangle \). We can therefore reformulate Theorem 3.7 as follows:

**Theorem 4.1.** For \( p \geq 3 \) we have the \( GL_p(K) \)-module isomorphism

\[
F_p \cong \bigoplus_{n=0}^{\infty} \bigoplus_{\nu=(\nu_1,\nu_2,\nu_3)} m_\nu(M_3(K), so_3(K))W_p(\nu),
\]

where the multiplicities \( m_\nu(M_3(K), so_3(K)) \) are given in Theorem 3.7. For \( p < 3 \) the summands labeled by partitions \( \nu \) with more than \( p \) non-zero parts have to be removed from the formula.

Consider the space \( so_3(K)^{\oplus p} \oplus M_3(K) \), on which \( SO_3(K) \) acts by simultaneous conjugation, and \( GL_p(K) \) acts on the right by

\[
(a_1, \ldots, a_p, b) \cdot g = (\sum_{i=1}^{3} g_{i1}a_i, \ldots, \sum_{i=1}^{3} g_{ip}a_i, b)
\]

for \( g = (g_{ij})_{i,j=1}^{3} \in GL_p(K) \). The coordinate ring of \( so_3(K)^{\oplus p} \oplus M_3(K) \) is \( K[T_p, Z] \), where \( Z = \{ z_{ij} \mid 1 \leq i, j \leq 3 \} \) is a set of commuting indeterminates over \( K[T_p] \). We have the \( K \)-linear embedding

\[
M_3(K[T_p]) \to K[T_p, Z]^{(N_0, \ldots, N_0, 1)}
\]

given by \( f \mapsto \text{tr}(fz) \), where \( z = (z_{ij})_{i,j=1}^{3} \) is a generic \( 3 \times 3 \) matrix as in Section 2.2 and \((N_0, \ldots, N_0, 1)\) in the exponent means that we take the component of \( K[T_p, Z] \) consisting of the polynomial functions that are linear on the summand \( M_3(K) \) of \( so_3(K)^{\oplus p} \oplus M_3(K) \).

**Proposition 4.2.** (i) The \( K \)-subalgebra \( \mathcal{E}_p \) of \( M_3(K[T_p]) \) is generated by the generic skew-symmetric matrices \( t_1, \ldots, t_p \), and the scalar matrices \( \text{tr}(t_it_j)I \) (\( 1 \leq i \leq j \leq p \)), \( \text{tr}(t_itMt_m)I \) (\( 1 \leq l < t < m \leq p \)).

(ii) The map \( f \mapsto \text{tr}(fz) \) gives a \( GL_p(K) \)-module isomorphism

\[
\iota : \mathcal{E}_p \cong (K[T_p, Z]^{SO_3(K)})^{(N_0, \ldots, N_0, 1)}.
\]

**Proof.** By standard properties of the trace, if \( f : so_3(K)^{\oplus p} \to M_3(K) \) is \( SO_3(K) \)-equivariant, then \( \text{tr}(fz) \) is \( SO_3(K) \)-invariant, so the restriction of the embedding maps \( \mathcal{E}_p \) into \((K[T_p, Z]^{SO_3(K)})^{(N_0, \ldots, N_0, 1)}\). On the other hand, \( \mathcal{E}_p \) contains the \( K \)-subalgebra generated by \( t_1, \ldots, t_p \), \( \text{tr}(t_it_j)I \), \( \text{tr}(t_itMt_m)I \), and the images of the elements of this subalgebra already exhaust \((K[T_p, Z]^{SO_3(K)})^{(N_0, \ldots, N_0, 1)}\) by Corollary 3.7. Thus \( \iota \) is also surjective, hence an isomorphism, implying in turn that the given elements indeed generate \( \mathcal{E}_p \). So both (ii) and (i) hold.

**Corollary 4.3.** For \( p \geq 3 \) we have \( \mathcal{E}_p = \langle \mathcal{E}_3 \rangle_{GL_p(K)} \).
Proof. Since \( \dim_K(\mathfrak{so}_3(K)) = 3 \), by Weyl’s Theorem on polarizations (cf. [W]) we have \( K[T_p, Z] = (K[T_3, Z])_{GL_p(K)} \), hence
\[
(K[T_p, Z]^{SO_3(K)})^{(N_0, \ldots, N_0, 1)} = ((K[T_3, Z]^{SO_3(K)})^{(N_0, N_0, N_0, 1)})_{GL_p(K)} .
\]
So the statement follows by the isomorphism \( \iota \) in Proposition 4.2.

Proposition 4.4. (i) We have \( \mathcal{E}_3 = \mathcal{F}_3 \otimes \bigoplus_{k=1}^{\infty} (\text{tr}(t_1^{2k})I)_{GL_3(K)} \).

(ii) The comorphism of
\[
\mu : \mathfrak{so}_3(K)^{\otimes 5} \to \mathfrak{so}_3(K)^{\otimes 3} \oplus M_3(K),
\]
\[
(a_1, a_2, a_3, a_4, a_5) \mapsto (a_1, a_2, a_3, a_4, a_5)
\]
gives a \( GL_3(K) \)-module isomorphism
\[
\mu^* : (K[T_3, Z])^{SO_3(K)}^{(N_0, N_0, N_0, 1)} \cong (K[T_3])^{SO_3(K)}^{(N_0, N_0, N_0, 1, 1)} .
\]

Proof. Since \( M_3(K) \) is spanned as a \( K \)-vector space by the image of the multiplication map \( \mathfrak{so}_3(K) \oplus \mathfrak{so}_3(K) \to M_3(K), (a, b) \mapsto a \cdot b \), the \( GL_3(K) \)-module homomorphism \( \mu^* \) is injective. Composing \( \mu^* \) and \( \iota \) we get
\[
\mathcal{E}_3/\mathcal{F}_3 \cong (\mu^* \circ \iota)(\mathcal{E}_3)/((\mu^* \circ \iota)(\mathcal{F}_3) \leq (K[T_3])^{SO_3(K)}^{(N_0, N_0, N_0, 1, 1)}/(\mu^* \circ \iota)(\mathcal{F}_3) .
\]
By Lemma 3.3 and Theorem 4.1 we have the \( GL_3(K) \)-module isomorphism
\[
(K[T_3])^{SO_3(K)}^{(N_0, N_0, N_0, 1, 1)}/(\mu^* \circ \iota)(\mathcal{F}_3) \cong \bigoplus_{k=1}^{\infty} W_p((2k)) .
\]
We conclude that \( \mathcal{E}_3/\mathcal{F}_3 \) is isomorphic to a \( GL_3(K) \)-submodule of \( \bigoplus_{k=1}^{\infty} W_p((2k)) \).

The multiplicity of \( W_3((2k)) \) in \( \mathcal{F}_3 \) is 1, and the corresponding highest weight vector is \( t_1^{2k} \in \mathcal{F}_3 \). A second highest weight vector of the same type in \( \mathcal{E}_3 \) is the scalar matrix \( \text{tr}(t_1^{2k})I \). Now these two highest weight vectors are linearly independent over \( K \), as one can easily see for example by making the substitution \( t_1 \mapsto \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \).

It follows that the multiplicity of \( W_3((2k)) \) in \( \mathcal{E}_3 \) is at least 2. On the other hand by the embedding \( \mu^* \) this multiplicity is bounded from above by the multiplicity of \( W_3((2k)) \) in \( (K[T_3])^{SO_3(K)}^{(N_0, N_0, N_0, 1, 1)} \), which by Lemma 3.3 equals 2. Hence both (i) and (ii) follow.

Corollary 4.5. (i) The \( GL_p(K) \)-module \( \mathcal{E}_p \) decomposes as
\[
\mathcal{E}_p \cong \bigoplus_{\nu = (\nu_1, \nu_2, \nu_3)} m_\nu W_p(\nu),
\]
where the value of \( m_\nu \) is the same as in Lemma 3.3 for \( p \geq 3 \); when \( p < 3 \), the summands labeled by partitions with more than \( p \) non-zero parts are removed.

(ii) For all \( p \) we have \( \mathcal{E}_p = \mathcal{F}_p \oplus \bigoplus_{k=1}^{\infty} (\text{tr}(t_1^{2k})I)_{GL_p(K)} \) (where \( I \) is the \( 3 \times 3 \) identity matrix).

Proof. This follows from Corollary 4.3 Proposition 4.4 and Lemma 3.3.

Remark 4.6. (i) The Cayley-Hamilton theorem gives that
\[
t_1^3 - \frac{1}{2}\text{tr}(t_1^2)t_1 = 0 \quad \text{and hence} \quad t_1^3 = \frac{1}{2}\text{tr}(t_1^2)t_1^2 .
\]
This easily implies that
\[
\text{tr}(t_1^{2k}) = \frac{1}{2k-1} \text{tr}(t_1^2) = 2(-1)^k((t_{12}^{(1)})^2 + (t_{13}^{(1)})^2 + (t_{23}^{(1)})^2)^k, \quad k \geq 1.
\]

(ii) The GL_p(K)-module structure of the algebra of GL_2(K)-equivariant polynomial maps \(sl_2(K)^{op}\rightarrow M_2(K)\), where GL_2(K) acts by (simultaneous) conjugation, is determined in [P2, Theorem 2.2]. This turns out to be multiplicity free, unlike our \(E_p\).

(iii) It follows from Corollary 4.3 (ii) that \(\text{tr}(t_1 t_2) t_3\) can be expressed as a \(K\)-linear combination of monomials in \(t_1, t_2, t_3\). Indeed, the explicit identity of this form is
\[
\text{tr}(t_1 t_2) \cdot t_3 = t_1 t_2 t_3 - t_2 t_3 t_1 + t_3 t_1 t_2 + t_2 t_1 t_3 + t_3 t_2 t_1 - t_1 t_3 t_2.
\]

**Remark 4.7.** The center \(C(E_p)\) (\(p \geq 3\)) of \(E_p\) (that is, the subalgebra of scalar matrices in \(E_p\)) is isomorphic to the generic trace algebra \(K[T_p]^{\text{SO}_3(K)}\), hence by Corollary 2.12 as a GL_p(K)-module it decomposes as
\[
(2) \quad C(E_p) \cong \bigoplus W_p(\lambda),
\]
where the summation runs on all \(\lambda = (\lambda_1, \lambda_2, \lambda_3)\) such that \(\lambda_1 - \lambda_2 \equiv \lambda_2 - \lambda_3 \equiv 0\) (mod 2). It follows from Proposition 4.3 (i) that the center \(C(F_p)\) of \(F_p\) is a direct sum of all \(W_p(\lambda)\) from (2) such that \(\lambda_2 > 0\). By analogy with the notion of a weak polynomial identity one may define weak central polynomials for the pair \((R, L)\) as elements of the free algebra \(K(X)\) which take central values in \(R\) when evaluated on \(L\). Hence we can restate the structure of \(C(F_p)\) as a GL_p(K)-module in the language of central cocharacter sequence
\[
\chi^n(\text{M}_3(K), \text{so}_3(K)) = \sum_{\lambda \vdash n} \chi_\lambda,
\]
where \(\lambda = (\lambda_1, \lambda_2, \lambda_3)\) is as in (2) and \(\lambda_2 > 0\). We shall write \(\lambda\) in the form
\[
\lambda = (2(\mu_1 + \mu_2) + \lambda_3, 2\mu_2 + \lambda_3, \lambda_3), \mu_2 + \lambda_3 > 0.
\]

Then direct computations show that the corresponding highest weight vectors are
\[
s_{3}^{3}(x_1, x_2, x_3)([x_1, x_2]^2 - (x_1^2x_2 - x_1x_2^2 - x_2^2x_1 + x_2^2x_1^2))\mu_2 - 1 w'(x_1, x_2),
\]
\[
w'(x_1, x_2) = [x_1, x_2]^{2\mu_1} + [x_1, x_2][x_1^{2\mu_1 + 1}x_2 - x_2^{2\mu_1 + 1}x_1]
\]
\[+(x_1^2x_2 - x_1x_2^2x_1x_2 - x_2x_1^2x_2 + x_2^2x_1^2)x_1^{2\mu_1 - 1}, \text{ if } \mu_1, \mu_2 > 0;
\]
\[
s_{3}^{3}(x_1, x_2, x_3)^{-1} w''(x_1, x_2),
\]
\[
w''(x_1, x_2) = s_3(x_1, x_2, x_3)x_1^{2\mu_1} + 2 \sum_{\sigma \in S_3} \text{sign}(\sigma)x_{\sigma(1)}x_{\sigma(2)}x_1^{2\mu_1}x_{\sigma(3)}, \text{ if } \mu_1 > 0, \mu_2 = 0;
\]
\[
s_{3}^{3}(x_1, x_2, x_3)([x_1, x_2]^2 - (x_1^2x_2 - x_1x_2^2 - x_2^2x_1 + x_2^2x_1^2))^{\mu_2}, \text{ if } \mu_1 = 0.
\]
5. The module of covariants

We saw above (cf. Corollary 4.3) that to a large extent, the analysis of $\mathcal{E}_p$ for arbitrary $p$ can be reduced to the special case $p = 3$. Our aim is to describe $\mathcal{E}_3$ as a module over the ring $K[T_3]^{|SO_3(K)|}$.

We set

$$C := \iota(\mathcal{E}_3),$$

where $\iota$ is defined in Proposition 4.2.

The space of $3 \times 3$ matrices has the decomposition

$$M_3(K) = K \oplus so_3(K) \oplus M_3(K)^{+}$$

as a direct sum of minimal $SO_3(K)$-invariant subspaces, where $I$ is the identity matrix and $M_3(K)^{+}$ is the space of trace zero symmetric $3 \times 3$ matrices. Accordingly we have the decomposition

$$C = C_1 \oplus C_2 \oplus C_3$$

(3)

into a direct sum of $K[so_3(K)]^{SO_3(K)}$-submodules (that are also $GL_3(K)$-submodules) of $C$. Namely

$$C_1 = K[T_3]|^{SO_3(K)} \cdot \text{tr}(z),$$

$$C_2 = (K[T_3, Z^-]|^{SO_3(K)})(N_0, N_0, N_0, 1),$$

and

$$C_3 \cong (K[T_3, Z_0^+]|^{SO_3(K)})(N_0, N_0, N_0, 1),$$

where

$$Z^- = \{ u_{ij} := \frac{1}{2}(z_{ij} - z_{ji}) \mid 1 \leq i < j \leq 3 \},$$

$$Z_0^+ = \{ s_{ij} := \frac{1}{2}(z_{ij} + z_{ji}), s_{kk} := z_{kk} - \frac{1}{3}(z_{11} + z_{22} + z_{33}) \mid 1 \leq i \leq j \leq 3, k = 1, 2, 3 \},$$

and $(N_0, N_0, N_0, 1)$ in the exponents above indicates that we take the component consisting of the polynomials that have total degree one in the variables belonging to $Z$.

Denote by $P$ the subalgebra of $K[T_3]|^{SO_3(K)}$ generated by $\text{tr}(t_1^2)$, $\text{tr}(t_2^2)$, $\text{tr}(t_3^2)$, $\text{tr}(t_1 t_2)$, $\text{tr}(t_1 t_3)$, $\text{tr}(t_2 t_3)$. Note that the six generators are algebraically independent over $K$. The algebra $K[T_3]|^{SO_3(K)}$ is a free $P$-module of rank two, generated by 1 and $\text{tr}(t_1 t_2 t_3)$ (see Corollary 2.9). It follows that the $N_0^3$-graded Hilbert series (or in other words, the formal $GL_3(K)$-character of $C_1$) is

$$H(C_1; \tau_1, \tau_2, \tau_3) = \frac{1 + \tau_1 \tau_2 \tau_3}{\prod_{1 \leq i \leq j \leq 3}(1 - \tau_i \tau_j)}.$$  

(5)

**Proposition 5.1.** The $N_0^3$-graded Hilbert series of $C_2$ and $C_3$ are the following:

$$H(C_2; \tau_1, \tau_2, \tau_3) = \frac{(S_{(1)} + S_{(1,1)})(\tau_1, \tau_2, \tau_3)}{\prod_{1 \leq i \leq j \leq 3}(1 - \tau_i \tau_j)}.$$  

(6)

$$H(C_3; \tau_1, \tau_2, \tau_3) = \frac{(S_{(2)} + S_{(2,1)} - S_{(2,2,1)} - S_{(2,2,2)})(\tau_1, \tau_2, \tau_3)}{\prod_{1 \leq i \leq j \leq 3}(1 - \tau_i \tau_j)}.$$  

(7)

where

$$S_{(1)}(\tau_1, \tau_2, \tau_3) = \tau_1 + \tau_2 + \tau_3,$$

$$S_{(2)}(\tau_1, \tau_2, \tau_3) = \sum_{1 \leq i \leq j \leq 3} \tau_i \tau_j,$$
\[ S_{(1,1)}(\tau_1, \tau_2, \tau_3) = \sum_{1 \leq i < j \leq 3} \tau_i \tau_j, \]
\[ S_{(2,1)}(\tau_1, \tau_2, \tau_3) = \sum_{i \neq j} \tau_i^2 \tau_j + 2 \tau_1 \tau_2 \tau_3, \]
\[ S_{(2,2,1)}(\tau_1, \tau_2, \tau_3) = \tau_1 \tau_2 \tau_3 S_{(1,1)}(\tau_1, \tau_2, \tau_3), \]
\[ S_{(2,2,2)}(\tau_1, \tau_2, \tau_3) = S_{(1,1,1)}(\tau_1, \tau_2, \tau_3)^2 = (\tau_1 \tau_2 \tau_3)^2 \]
are Schur polynomials (the formal characters of the \( GL_3(K) \)-modules \( W_3(1), W_3(2), W_3(1, 1), W_3(2, 1), W_3(2, 2, 1), W_3(2, 2, 2) \)).

**Proof.** It is well known that the Hilbert series in question are independent of the characteristic zero base field \( K \). Therefore we may assume \( K = \mathbb{C} \), the field of complex numbers. View the \( SO_3(\mathbb{C}) \)-module \( \mathbb{C}[T_3, Z_0^+] \) as an \( SL_2(\mathbb{C}) \)-module via the natural surjection \( SL_2(\mathbb{C}) \to SO_3(\mathbb{C}) \) with kernel consisting of the \( 2 \times 2 \) identity matrix and its negative. The maximal compact subgroup \( SU_2(\mathbb{C}) \) (the special unitary group) of \( SL_2(\mathbb{C}) \) has the same subspace of invariants in \( \mathbb{C}[T_3, Z_0^+] \) as \( SL_2(\mathbb{C}) \). We compute the Hilbert series of \( C_3 = (\mathbb{C}[T_3, Z_0^+]^{(N_0,N_0,N_0,1)})_{SU_2(\mathbb{C})} \) using standard methods. Namely, it can be expressed by the Molien-Weyl formula and the Weyl integration formula as an integral over a maximal torus of \( SU_2(\mathbb{C}) \) as follows. Consider the maximal torus \( T = \{ \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \mid |\rho| = 1 \} \) in \( SU_2(\mathbb{C}) \). The character of the multihomogenous components of \( \mathbb{C}[T_3, Z_0^+]^{(N_0,N_0,N_0,1)} \) as a \( T \)-module is given by the series
\[
\rho^4 + \rho^2 + 1 + \rho^{-2} + \rho^{-4} \prod_{j=1}^3 (1 - \rho^2 \tau_j)(1 - \tau_j)(1 - \rho^{-2} \tau_j).
\]
The roots of \( SU_2(\mathbb{C}) \) are \( \rho^2 \) and \( \rho^{-2} \), and the order of the Weyl group is 2. Therefore the Molien-Weyl formula combined with the Weyl integration formula yields
\[
H(C_3; \tau_1, \tau_2, \tau_3) = \frac{1}{2} \int_{|\rho|=1} \frac{(\rho^4 + \rho^2 + 1 + \rho^{-2} + \rho^{-4})(1 - \rho^2)(1 - \rho^{-2})}{\prod_{j=1}^3 (1 - \rho^2 \tau_j)(1 - \tau_j)(1 - \rho^{-2} \tau_j)} \, d\rho
\]
\[
= \frac{1}{2} \cdot \frac{1}{2\pi i} \int_{|\rho|=1} \frac{-\rho^{12} + \rho^{10} + \rho^{2} - 1}{\rho \prod_{j=1}^3 (1 - \rho^2 \tau_j)(1 - \tau_j)(\rho^2 - \tau_j)} \, d\rho.
\]
The above integral can be evaluated by residue calculus. Suppose that \( \tau_1, \tau_2, \tau_3 \) are non-zero complex numbers of absolute value less than 1. Then the integrand has poles inside the unit circle at \( \rho = \pm \sqrt{t_k}, \ k = 1, 2, 3, \) and at \( \rho = 0 \). The residue at \( \pm \sqrt{t_k} \) is
\[
-\frac{t_k^6 + t_k^3}{2}\tau_k(1 - \tau_k^3)(1 - \tau_k)\prod_{j \in \{1, 2, 3\} \setminus \{k\}}(1 - \tau_k \tau_j)(1 - \tau_j)(\tau_k - \tau_j),
\]
whereas the residue of the integrand at \( \rho = 0 \) is
\[
\frac{1}{\prod_{j=1}^3 (1 - \tau_j)\tau_j}.
\]
It follows that
\[
H(C_3; \tau_1, \tau_2, \tau_3) = \frac{1}{2} \left( \frac{1}{\prod_{j=1}^{3} (1 - \tau_j)} - \tau_k^3 + \tau_k^3 + \tau_k - 1 \right)
\]
\[
2 \sum_{k=1}^{3} 2 \tau_j (1 - \tau_k^3) (1 - \tau_k) \prod_{j \in \{1,2,3\} \setminus \{k\}} (1 - \tau_j \tau_k) (1 - \tau_j) (\tau_k - \tau_j).
\]

Bringing to common denominator the summands on the right hand side and after some cancellations we obtain (7).

Similarly,
\[
H(C_2; \tau_1, \tau_2, \tau_3) = \frac{1}{2} \int_{|\rho|=1} \frac{(\rho^2 + 1 + \rho^{-2})(1 - \rho^2)(1 - \rho^{-2})}{\prod_{j=1}^{3} (1 - \rho^2 \tau_j) (1 - \tau_j) (1 - \rho^2 - \tau_j) 2 \pi i \rho} \, d\rho
\]
\[
= \frac{1}{2} \cdot \frac{1}{2 \pi i} \int_{|\rho|=1} \frac{-\rho^9 + \rho^7 + \rho^3 - \rho}{\prod_{j=1}^{3} (1 - \rho^2 \tau_j) (1 - \tau_j) (\rho^2 - \tau_j)} \, d\rho
\]
\[
= \frac{1}{2} \sum_{k=1}^{3} \frac{-\tau_k^3 + \tau_k - 1}{\prod_{j \in \{1,2,3\} \setminus \{k\}} (1 - \tau_j \tau_k) (1 - \tau_j) (\tau_k - \tau_j),
\]
from which one gets (6) after bringing the summands to common denominator and cancelling certain factors. □

Remark 5.2. It would be possible to derive Corollary 4.3 (i) (giving the multiplicities of the irreducible GL_p(K)-module summands in E_p) using Proposition 5.1 and Corollary 4.3.

Proposition 5.3. Write s for the symmetric trace zero matrix whose entries above and in the diagonal are s_{ij}, 1 \leq i \leq j \leq 3, and write u for the skew-symmetric matrix whose entries above the diagonal are u_{ij}, 1 \leq i < j \leq 3 (where s_{ij}, u_{ij} were introduced in (4)). (i) tr(t_1 u) is a highest weight vector in the GL_3(K)-module C_2 generating a GL_3(K)-submodule isomorphic to W_3(1).

(ii) tr(t_1 t_2 u) is a highest weight vector in the GL_3(K)-module C_2 generating a GL_3(K)-submodule isomorphic to W_3(1,1).

(iii) tr(t_1 t_2 s) is a highest weight vector in the GL_3(K)-module C_3 generating a GL_3(K)-submodule isomorphic to W_3(2).

(iv) tr([t_1, t_2] s) is a highest weight vector in the GL_3(K)-module C_3 generating a GL_3(K)-submodule isomorphic to W_3(2,1).

Proof. The map K\langle X_3 \rangle \to K\langle T_3, Z_0^+ \rangle, f(x_1, x_2, x_3) \mapsto tr(f(t_1, t_2, t_3) s) is a GL_3(K)-module homomorphism. As explained in Section 2.3, x_1^3 is a highest weight vector in K\langle X_3 \rangle^{(2)} generating a GL_3(K)-submodule isomorphic to W_3(2), whereas [x_1^3, x_2] is a highest weight vector in K\langle X_3 \rangle^{(3)} generating a GL_3(K)-submodule isomorphic to W_3(2,1). Since the images of these highest weight vectors in C_3 are non-zero, they are also highest weight vectors as required. Similarly, the map K\langle X_3 \rangle \to K\langle T_3, Z^- \rangle, f(x_1, x_2, x_3) \mapsto tr(f(t_1, t_2, t_3) u) is a GL_3(K)-module homomorphism. Now x_1 \in K\langle X_3 \rangle^{(1)} is a highest weight vector generating a GL_3(K)-module isomorphic to W_3(1). Also \frac{1}{2} (x_1 x_2 - x_2 x_1) \in K\langle X_3 \rangle^{(2)} is a highest weight vector generating a GL_3(K)-module isomorphic to W_3(1,1), and its image in C_2 is tr(t_1 t_2 u), since t_1 t_2 + t_2 t_1 is a symmetric matrix, hence tr((t_1 t_2 + t_2 t_1) u) = 0. □
The $GL_3(K)$-submodule $(\text{tr}(t_1 u))_{GL_3(K)}$ generated by $\text{tr}(t_1 u)$ has the $K$-vector space basis $\{ \text{tr}(t_1 u), \text{tr}(t_2 u), \text{tr}(t_3 u) \}$, and the $GL_3(K)$-submodule $(\text{tr}(t_1 t_2 u))_{GL_3(K)}$ generated by $\text{tr}(t_1 t_2 u)$ has the $K$-vector space basis $\{ \text{tr}(t_1 t_2 u), \text{tr}(t_1 t_3 u), \text{tr}(t_2 t_3 u) \}$.

**Proposition 5.4.** $C_2$ is a rank 6 free $P$-module generated by $\text{tr}(t_1 u), \text{tr}(t_2 u), \text{tr}(t_3 u), \text{tr}(t_1 t_2 u), \text{tr}(t_1 t_3 u), \text{tr}(t_2 t_3 u)$.

**Proof.** The fact that the above 6 elements generate $C_2$ as a $K[T_3]^{SO_3(K)}$-module is an immediate consequence of Corollary 2.9. The following two relations hold as a consequence of Theorem 2.4 and the proof of Proposition 2.8. They (together with relations obtained by permuting $t_1, t_2, t_3$) show that the 6 elements in the statement in fact generate $C_2$ as a $P$-module:

\begin{align*}
(8) \quad & \text{tr}(t_1 t_2 t_3) \text{tr}(t_1 u) = \text{tr}(t_1^2) \text{tr}(t_2 t_3 u) - \text{tr}(t_1 t_2) \text{tr}(t_1 t_3 u) + \text{tr}(t_1 t_3) \text{tr}(t_2 t_2 u) \\
(9) \quad & \text{tr}(t_1 t_2 t_3) \text{tr}(t_1 t_2 u) = \frac{1}{8} \left( \text{tr}(t_1 t_3) \text{tr}(t_2^2) - \text{tr}(t_1 t_2) \text{tr}(t_2 t_3) \right) \text{tr}(t_1 u) \\
& \quad + \frac{1}{8} \left( \text{tr}(t_1^2) \text{tr}(t_2 t_3) - \text{tr}(t_1 t_2) \text{tr}(t_1 t_3) \right) \text{tr}(t_2 u) \\
& \quad + \frac{1}{8} \left( \text{tr}(t_1 t_2)^2 - \text{tr}(t_1^2) \text{tr}(t_2^2) \right) \text{tr}(t_3 u)
\end{align*}

Therefore denoting by $e_1, \ldots, e_6$ the standard generators of the free $P$-module $P^{\oplus 6}$, we have a $P$-module surjection

$$
\mu : P^{\oplus 6} \rightarrow C_2, \quad e_1 \mapsto \text{tr}(t_1 u), \quad e_2 \mapsto \text{tr}(t_2 u), \quad e_3 \mapsto \text{tr}(t_3 u), \\
(4) \quad e_4 \mapsto \text{tr}(t_1 t_2 u), \quad e_5 \mapsto \text{tr}(t_1 t_3 u), \quad e_6 \mapsto \text{tr}(t_2 t_3 u).
$$

This is a homomorphism of graded $P$-modules, where we endow $P^{\oplus 6}$ with the grading given by $\text{deg}(e_1) = \text{deg}(e_2) = \text{deg}(e_3) = 1$ and $\text{deg}(e_4) = \text{deg}(e_5) = \text{deg}(e_6) = 2$, and $C_2$ is endowed with the standard grading coming from the action of the subgroup of scalar matrices in $GL_3(K)$. The Hilbert series of $P^{\oplus 6}$ is $\frac{1 + 3z^2}{(1 - z)^3(1 - z^3)}$, and by Proposition 5.1 this agrees with the Hilbert series of $C_2$. It follows that $\mu$ is an isomorphism. 

The $GL_3(K)$-submodule $(\text{tr}(t_i^2 s))_{GL_3(K)}$ generated by $\text{tr}(t_i^2 s)$ has the basis

$$
e_{ij} := \text{tr}(t_i t_j s), \quad 1 \leq i \leq j \leq 3.
$$

The $GL_3(K)$-submodule $(\text{tr}(t_i^2 t_j s))_{GL_3(K)}$ generated by $\text{tr}(t_i^2 t_j s)$ has the basis

$$
e_{ij} := \text{tr}(t_i^2 t_j | s), \quad i \neq j \in \{1, 2, 3\},
$$

$$
e_{132} := \text{tr}(t_1 t_3 + t_3 t_1, t_2 s), \quad e_{123} := \text{tr}(t_1 t_2 + t_2 t_1, t_3 s).
$$

**Theorem 5.5.** (i) As a $P$-module, $C_3$ is generated by $$
\{ e_{ij}, e_{kl} \mid i \neq j \in \{1, 2, 3\}, \quad 1 \leq k \leq l \leq 3 \}.
$$

Moreover, it has the direct sum decomposition

$$
C_3 = C_3^{(0)} \oplus C_3^{(1)}, \quad \text{where} \quad C_3^{(0)} = P \cdot \langle e_{11} \rangle_{GL_3(K)}, \quad C_3^{(1)} = P \cdot \langle e_{112} \rangle_{GL_3(K)}.
$$

(ii) The $P$-module $C_3^{(0)}$ has the free resolution

$$
0 \rightarrow P \xrightarrow{\phi^{(0)}} P^{\oplus 6} \xrightarrow{\psi^{(0)}} C_3^{(0)} \rightarrow 0
$$
where denoting by $e_1, e_2, e_3, e_4, e_5, e_6$ the standard generators of $P^6$, $\varphi^{(0)}$ is the $P$-module homomorphism given by

$$
\varphi^{(0)} : e_1 \mapsto e_{11}, e_2 \mapsto e_{12}, e_3 \mapsto e_{13}, e_4 \mapsto e_{22}, e_5 \mapsto e_{23}, e_6 \mapsto e_{33},
$$

and $\psi^{(0)}$ maps the generator of the rank one $P$-module $P$ to

$$
\begin{pmatrix}
\frac{1}{2}(\text{tr}(t_2^3)\text{tr}(t_3^3) - \text{tr}(t_2t_3)^2) \\
\text{tr}(t_2t_3)\text{tr}(t_2t_3) - \text{tr}(t_1t_2)\text{tr}(t_2^2) \\
\text{tr}(t_1t_2)\text{tr}(t_1t_2) - \text{tr}(t_1t_3)\text{tr}(t_2^2) \\
\frac{1}{2}(\text{tr}(t_1t_3)\text{tr}(t_2^2) - \text{tr}(t_1t_3)^2) \\
\text{tr}(t_1t_2)\text{tr}(t_1t_3) - \text{tr}(t_1t_2)\text{tr}(t_2t_3) \\
\frac{1}{2}(\text{tr}(t_1t_2)\text{tr}(t_3^2) - \text{tr}(t_1t_2)^2)
\end{pmatrix} \in P^{\oplus 6}.
$$

(iii) The $P$-module $C^{(1)}_3$ has the free resolution

$$0 \rightarrow P^{\oplus 3} \xrightarrow{\psi^{(1)}} P^{\oplus 8} \xrightarrow{\varphi^{(1)}} C^{(1)}_3 \rightarrow 0$$

where denoting by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ the standard generators of $P^8$, $\varphi^{(1)}$ is the $P$-module homomorphism given by

$$
\varphi^{(1)} : e_1 \mapsto e_{11}, e_2 \mapsto e_{22}, e_3 \mapsto e_{33}, e_4 \mapsto e_{13}, e_5 \mapsto e_{23}, e_6 \mapsto e_{33},
$$

and $\psi^{(1)} : P^{\oplus 3} \rightarrow P^{\oplus 8}$ is given by the matrix below:

$$
\begin{pmatrix}
\text{tr}(t_2t_3) & 0 & -\text{tr}(t_3^2) \\
\text{tr}(t_1t_3) & -\text{tr}(t_3^2) & 0 \\
-\text{tr}(t_2^2) & 0 & \text{tr}(t_2t_3) \\
0 & -\text{tr}(t_2^2) & \text{tr}(t_1t_2) \\
-\text{tr}(t_3^2) & \text{tr}(t_1t_3) & 0 \\
0 & \text{tr}(t_1t_2) & -\text{tr}(t_1^2) \\
\text{tr}(t_1t_2) & -\text{tr}(t_2t_3) & 0
\end{pmatrix} \in P^{8 \times 3}
$$

Proof. (i) $C_3$ is spanned as a $K$-vector space by products

$$
\text{tr}(t_{i_1} \cdots t_{i_k}) \cdots \text{tr}(t_{j_1} \cdots t_{j_l}) \text{tr}(t_{a_1} \cdots t_{a_m})
$$

by Theorem 2.5 and Theorem 2.6. For $k \geq 4$, $\text{tr}(t_{i_1} \cdots t_{i_k})$ can be expressed as a polynomial in $\text{tr}(t_{i_1}t_{j_1})$ and $\text{tr}(t_{i_1}t_{j_2}t_{k_3})$ by Theorem 2.3 and Proposition 2.8. Moreover, $\text{tr}(t_{i_1}t_{j_1}t_{k_3})$ is non-zero only if $i, j, k$ are distinct.

Claim: for $k \geq 4$, $\text{tr}(t_{i_1} \cdots t_{i_k})$ can be expressed by products of traces of shorter products.

Indeed, one can easily verify the identity

$$
\text{tr}(t_1t_2t_3t_4s) = \frac{1}{2}(\text{tr}(t_1t_2)\text{tr}(t_3t_4s) + \text{tr}(t_3t_4)\text{tr}(t_1t_2s) - \text{tr}(t_1t_4)\text{tr}(t_2t_3s)),
$$

implying our claim for $k = 4$. Apply next the fundamental trace identity (see for example [DrF, p. 63, Theorem 5.2.4]) for the four $3 \times 3$ matrices $t_{1}t_{2}$, $t_{3}t_{4}$, $t_{5}$, $s$, ...
and take into account that 0 = tr(t_i) = tr(s) = tr(t_is) to get

\begin{equation}
0 = tr(t_1t_2t_3t_4t_5s) + tr(t_3t_4t_5t_1t_2s) + tr(t_5t_1t_2t_3t_4s) + tr(t_3t_4t_5t_1t_2s) + tr(t_5t_3t_4t_1t_2s) - tr(t_1t_2)tr(t_3t_4t_5s) - tr(t_1t_2)tr(t_5t_3t_4s) - tr(t_3t_4)tr(t_1t_2t_3s) - tr(t_3t_4)tr(t_5t_3t_4s) - tr(t_3t_4)tr(t_1t_2t_3s).
\end{equation}

For \( f, g \in C_3 \) write \( f \equiv g \) if \( f - g \in K[T_3]^{\text{SO}_3(K)}C_3 \), where \( K[T_3]^{\text{SO}_3(K)} \) stands for the sum of the positive degree homogeneous components of \( K[T_3]^{\text{SO}_3(K)} \). Since \([t_i, t_j] \) is a skew-symmetric matrix, the identity (13) implies that

\[ \text{tr}(t_1t_2t_3t_4t_5s) \equiv \text{tr}(t_{\pi(1)}t_{\pi(2)}t_{\pi(3)}t_{\pi(4)}t_{\pi(5)}s) \text{ for any permutation } \pi \in S_5. \]

Therefore (13) implies 6tr(\( t_1t_2t_3t_4t_5s \)) = 0. This settles our claim for \( k = 5 \). Finally, for \( k \geq 6 \), recall that \( \text{tr}(z_{12}z_{23}z_{45}z_{56}z_{27}) \) can be expressed by traces of shorter products where \( z_{1}, \ldots, z_{7} \) are arbitrary (not necessarily skew-symmetric or symmetric) \( 3 \times 3 \) matrices (see for example [DrF, p. 78, Theorem 6.1.6 and p. 79]), so our claim holds for \( k \geq 6 \) as well.

Thus we proved that \( C_3 \) is generated as a \( K[T_3]^{\text{SO}_3(K)} \)-module by

\[ V := \text{Span}_{K}\{ \text{tr}(t_it_js) \mid i, j, k \in \{1, 2, 3\}. \]

This is a \( \text{GL}_3(K) \)-submodule of \( C_3 \). Consider the surjective \( \text{GL}_3(K) \)-module homomorphism \( \rho : K(X_3)^{(2)} \oplus K(X_3)^{(3)} \to V \) given by \( \rho(f(x_1, x_2, x_3)) = \text{tr}(f(t_1, t_2, t_3) s) \). As a \( \text{GL}_3(K) \)-module, \( K(X_3)^{(2)} \) is generated by \( x_1^2 \) and \([x_1, x_2]\), whereas \( K(X_3)^{(3)} \) is generated by \( s_1, s_2, s_3 \) (see for example [DrF, p. 78, Theorem 6.1.6 and p. 79]). Now \( \rho([x_1, x_2]), \rho(x_1^2), \rho(s_1), \rho(s_2), \rho(s_3) \) are all zero. Hence we conclude

\[ V = \langle e_{11} \rangle_{\text{GL}_3(K)} \oplus \langle e_{112} \rangle_{\text{GL}_3(K)}. \]

Recall that \( K[T_3]^{\text{SO}_3(K)} \) is a rank two free \( P \)-module generated by \( 1 \) and \( \text{tr}(t_1t_2t_3) \). By Theorem 2.3 Theorem 2.4 and Proposition 2.3 Thus by \( C_3 = K[T_3]^{\text{SO}_3(K)} \cdot V \) we conclude that \( C_3 \) is generated as a \( P \)-module by \( V + \text{tr}(t_1t_2t_3)V \). Next we show that

\[ \text{tr}(t_1t_2t_3)V \subseteq PV. \]

Indeed, observe that \( \text{tr}(t_1t_2t_3) \) spans a 1-dimensional \( \text{GL}_3(K) \)-invariant subspace in \( K[T_3, Z_0^+] \). Therefore to prove (16), it suffices to show that the \( \text{GL}_3(K) \)-module generators \( e_{11} \) and \( e_{112} \) are multiplied by \( \text{tr}(t_1t_2t_3) \) into \( PV \). This follows from the following two equalities:

\begin{equation}
\text{tr}(t_1t_2t_3)e_{11} = \frac{1}{4}\text{tr}(t_1t_3)e_{112} - \frac{1}{4}\text{tr}(t_1t_2)e_{113} - \frac{1}{12}\text{tr}(t_1^2)e_{123} + \frac{1}{12}\text{tr}(t_1^2)e_{123}
\end{equation}

\begin{equation}
\text{tr}(t_1t_2t_3)e_{112} = \frac{1}{2}\left( \text{tr}(t_1t_3)e_{112}^2 - \text{tr}(t_1t_2)e_{123} \right) e_{11}
\end{equation}

\begin{equation}
+ \frac{1}{2}\left( \text{tr}(t_1^2)e_{123} - \text{tr}(t_1t_2)e_{113} \right) e_{12}
\end{equation}

\begin{equation}
+ \frac{1}{2}\left( \text{tr}(t_1t_2)^2 - \text{tr}(t_1^2)e_{123} \right) e_{13}
\end{equation}

So we proved

\[ C_3 = P(e_{11})_{\text{GL}_3(K)} + P(e_{112})_{\text{GL}_3(K)}. \]
The above sum is necessarily direct, as the polynomials in the first summand have odd total degree, whereas the polynomials in the second summand have even total degree. This finishes the proof of (i).

(ii) We proved above that \( \varphi^{(0)} \) is surjective onto \( C_3^{(0)} \). Using \texttt{CoCoA} we found the following relation:

\[
0 = \frac{1}{2} (\text{tr}(t_2^2)\text{tr}(t_3^2) - \text{tr}(t_2 t_3)^2)e_{11} + (\text{tr}(t_1 t_3)\text{tr}(t_2 t_3) - \text{tr}(t_1 t_2)\text{tr}(t_3^2))e_{12} + \frac{1}{2}(\text{tr}(t_1^2)\text{tr}(t_3^2) - \text{tr}(t_1 t_3)^2)e_{22} + \frac{1}{2}(\text{tr}(t_1^2)\text{tr}(t_2^2) - \text{tr}(t_1 t_2)^2)e_{33}.
\]

Hence we have established \( \psi^{(0)}(P) \subseteq \ker(\varphi^{(0)}) \). Taking into account the Hilbert series of \( C_3^{(0)} \) we may conclude the equality \( \psi^{(0)}(P) = \ker(\varphi^{(0)}) \). Indeed, by Proposition 5.1 we have that the univariate Hilbert series of \( C_3 \) with the standard \( \mathbb{N}_0 \)-grading (coming from the action of the subgroup of scalar matrices in \( \text{GL}_3(K) \)) is

\[
\frac{6\tau^2 - \tau^6}{(1 - \tau^2)^6}.
\]

The Hilbert series of the free module \( P^{\oplus 6} \) (endowed with the appropriate grading respected by \( \varphi^{(0)} \)) is \( \frac{6\tau^2}{(1 - \tau^2)^6} \). It follows that the Hilbert series of \( \ker(\varphi^{(0)}) \) is \( \frac{\tau^6}{(1 - \tau^2)^6} \), which obviously agrees with the Hilbert series of the rank one free \( P \)-submodule \( \psi^{(0)}(P) \) generated by a single element of degree 6.

(iii) In the proof of (i) we saw already that \( \varphi^{(1)} \) is surjective onto \( C_3^{(1)} \). Using \texttt{CoCoA} we found the relation

\[
0 = \text{tr}(t_2 t_3)e_{112} + \text{tr}(t_1 t_3)e_{221} - \text{tr}(t_1^2)e_{223} - \text{tr}(t_2^2)e_{113} + \text{tr}(t_1 t_2)e_{123}.
\]

This means that the first column of the \( 8 \times 3 \) matrix in the statement (iii) belongs to \( \ker(\varphi^{(1)}) \). Permuting cyclically the matrix variables \( t_1, t_2, t_3 \) in (19) we get two other relations, meaning that the second and third columns of the \( 8 \times 3 \) matrix in the statement (iii) belong to \( \ker(\varphi^{(1)}) \). So we have \( \psi^{(1)}(P^{\oplus 3}) \subseteq \ker(\varphi^{(1)}) \). As the upper \( 3 \times 3 \) minor of the \( 8 \times 3 \) matrix in the statement (iii) has non-zero determinant, we get that \( \psi^{(1)} \) is injective, and consequently the univariate Hilbert series of \( \psi^{(1)}(P^{\oplus 3}) \) agrees with \( \frac{3\tau^8}{(1 - \tau^2)^6} \), the Hilbert series of \( P^{\oplus 3} \) (graded appropriately). On the other hand, by Proposition 5.1 we know that the Hilbert series of \( C_3^{(1)} \) is \( \frac{8\tau^3 - 3\tau^5}{(1 - \tau^2)^6} \). The Hilbert series of \( P^{\oplus 8} \) (with the suitable grading) is \( \frac{8\tau^3}{(1 - \tau^2)^6} \), implying that the Hilbert series of \( \ker(\varphi^{(1)}) \) is \( \frac{3\tau^8}{(1 - \tau^2)^6} \), the same as the Hilbert series of \( \psi^{(1)}(P^{\oplus 3}) \). This proves the equality \( \text{im}(\psi^{(1)}) = \ker(\varphi^{(1)}) \).

\[\tag{19}0 = \text{tr}(t_2 t_3)e_{112} + \text{tr}(t_1 t_3)e_{221} - \text{tr}(t_1^2)e_{223} - \text{tr}(t_2^2)e_{113} + \text{tr}(t_1 t_2)e_{123}.\]

\[\textbf{Theorem 5.6.} \text{ (i) The } P-\text{module } \mathcal{E}_3 \text{ has the direct sum decomposition}
\]

\[\mathcal{E}_3 = \mathcal{E}_{3,1} \oplus \mathcal{E}_{3,2}^{(1)} \oplus \mathcal{E}_{3,2}^{(0)} \oplus \mathcal{E}_{3,3}^{(0)} \oplus \mathcal{E}_{3,3}^{(1)},\]
where 
\[
\begin{align*}
\mathcal{E}_{3,1} &= P \cdot I \oplus P \cdot \text{tr}(t_1 t_2 t_3) I = K[T_3]^{SO_3(K)} \cdot I \subset \mathcal{E}_3 \\
\mathcal{E}_{3,2}^{(1)} &= P \cdot \langle t_1 \rangle_{GL_3(K)} \\
\mathcal{E}_{3,2}^{(0)} &= P \cdot \langle [t_1, t_2] \rangle_{GL_3(K)} \\
\mathcal{E}_{3,3}^{(0)} &= P \cdot \langle [t_1^2 - \frac{1}{3} \text{tr}(t_1^2)] \rangle_{GL_3(K)} \\
\mathcal{E}_{3,3}^{(1)} &= P \cdot \langle [t_1^2, t_2] \rangle_{GL_3(K)}.
\end{align*}
\]

(ii) Both \( \mathcal{E}_{3,2}^{(1)} \) and \( \mathcal{E}_{3,2}^{(0)} \) are free \( P \)-modules of rank 3:
\[
\mathcal{E}_{3,2}^{(1)} = P \cdot t_1 \oplus P \cdot t_2 \oplus P \cdot t_3 \quad \text{and} \quad \mathcal{E}_{3,2}^{(0)} = P \cdot [t_1, t_2] \oplus P \cdot [t_1, t_3] \oplus P \cdot [t_2, t_3].
\]

(iii) The \( K \)-vector space \( \langle t_1^2 - \frac{1}{3} \text{tr}(t_1^2) I \rangle_{GL_3(K)} \) has the basis
\[
\{ f_{ij} = \frac{1}{2}(t_i t_j + t_j t_i) - \frac{1}{3} \text{tr}(t_i t_j) I \mid 1 \leq i \leq j \leq 3 \},
\]
and the \( P \)-module \( \mathcal{E}_{3,3}^{(0)} \) has the free resolution
\[
0 \longrightarrow P \xrightarrow{\mu^{(0)}} P^{\oplus 6} \xrightarrow{\eta^{(0)}} C_3^{(0)} \longrightarrow 0,
\]
where denoting by \( e_1, e_2, e_3, e_4, e_5, e_6 \) the standard generators of \( P^6 \), \( \eta^{(0)} \) is the \( P \)-module surjection given by
\[
\eta^{(0)} : e_1 \mapsto f_{11}, e_2 \mapsto f_{12}, e_3 \mapsto f_{13}, e_4 \mapsto f_{22}, e_5 \mapsto f_{23}, e_6 \mapsto f_{33},
\]
and \( \mu^{(0)} \) maps the generator of the rank one \( P \)-module \( P \) to the element of \( P^{\oplus 6} \) given in (12) in Theorem 5.5 (ii).

(iv) The \( K \)-vector space \( \langle [t_1^2, t_2] \rangle_{GL_3(K)} \) has the basis
\[
\{ f_{ij} = [t_i^2, t_j], \quad f_{132} = [t_1 t_3 + t_3 t_1, t_2], \quad f_{123} = [t_1 t_2 + t_2 t_1, t_3] \mid i \neq j \in \{1, 2, 3\} \},
\]
and the \( P \)-module \( C_3^{(1)} \) has the free resolution
\[
0 \longrightarrow P^{\oplus 3} \xrightarrow{\mu^{(1)}} P^{\oplus 8} \xrightarrow{\eta^{(1)}} C_3^{(1)} \longrightarrow 0,
\]
where denoting by \( e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \) the standard generators of \( P^8 \), \( \eta^{(1)} \) is the \( P \)-module surjection given by
\[
\eta^{(1)} : e_1 \mapsto f_{112}, e_2 \mapsto f_{221}, e_3 \mapsto f_{113}, e_4 \mapsto f_{331},
\]
\[
e_5 \mapsto f_{223}, e_6 \mapsto f_{332}, e_7 \mapsto f_{132}, e_8 \mapsto f_{123}
\]
and \( \mu^{(1)} : P^{\oplus 3} \rightarrow P^{\oplus 8} \) is given by the matrix in (13) in Theorem 5.5 (iii).

Proof. Consider the \( GL_3(K) \)-module isomorphism
\[
\iota : \mathcal{E}_3 \rightarrow (K[T_3, Z]^{SO_3(K)})^{(N_0, N_0, N_0, 1)}, \quad f \mapsto \text{tr}(f z)
\]
from Proposition 4.2. Write the generic matrix \( z \) as the sum
\[
z = \frac{1}{3} \text{tr}(z) I + s + u, \quad \text{with } s, u \text{ as in Proposition 5.3}.
\]
Moreover, we have

\[ 0 = \text{tr}(t_i) = \text{tr}([t_i, t_j]) = \text{tr}(t_i^2 - \frac{1}{3}\text{tr}(t_i^3)I) = \text{tr}([t_i^2, t_j]) \]
\[ 0 = \text{tr}(s) = \text{tr}(t_is) = \text{tr}([t_i, t_j]s) \]
\[ 0 = \text{tr}(u) = \text{tr}(t_i^2u) = \text{tr}([t_i^2, t_j]u) = \text{tr}((t_it_j + t jt_i)u). \]

These equalities show that

\[ \iota(1) = \text{tr}(z), \quad \iota(\text{tr}(t_1t_2t_3)) = \text{tr}(t_1t_2t_3)\text{tr}(z), \]
\[ \iota(t_i) = \text{tr}(t_iu) \quad (i = 1, 2, 3), \]
\[ \iota([t_i, t_j]) = \text{tr}([t_i, t_j]u) = 2\text{tr}(t_it_ju) \quad (1 \leq i < j \leq 3) \]
\[ \iota\left(\frac{1}{2}(t_it_j + t jt_i)\right) = \text{tr}(t_it_j) + \text{tr}(tjt_i)\text{tr}(z), \quad (1 \leq i < j \leq 3). \]

Moreover, we have

\[ \iota(f_{ij}) = e_{ij} \quad (1 \leq i \leq j \leq 3), \]
\[ \iota(f_{iij}) = e_{ij} \quad (i \neq j \in \{1, 2, 3\}), \]
\[ \iota(f_{132}) = e_{132}, \quad \iota(f_{123}) = e_{123} \]

(where \(e_{ij}, e_{ij}\) were defined in (10), (11)). It follows that \(\iota\) restricts to isomorphisms \(E_3 \xrightarrow{\cong} C_1, E_3^{(1)} + E_3^{(2)} \xrightarrow{\cong} C_2, E_3^{(0)} \xrightarrow{\cong} C_3^{(0)}, \) and \(E_3^{(1)} \xrightarrow{\cong} C_3^{(1)}\). Thus our statements immediately follow from (3), Corollary 2.9 Proposition 5.4 and Theorem 5.6.

We record a few relations in \(E_3\) that follow from (8), (9), (17), (18) by the proof of Theorem 5.6; these relations show the effect of multiplication by \(\text{tr}(t_1t_2t_3)\) on the \(P\)-module \(E_3\) written in the form as in Theorem 5.6.

**Proposition 5.7.** We have the following equalities: (i)

\[ \text{tr}(t_1t_2t_3) \cdot t_1 = \frac{1}{2} \left( \text{tr}(t_1^2) \cdot [t_2, t_3] - \text{tr}(t_1t_2) \cdot [t_1, t_3] + \text{tr}(t_1t_3) \cdot [t_1, t_2] \right) \]

(ii)

\[ \text{tr}(t_1t_2t_3) \cdot [t_1, t_2] = \frac{1}{4} \left( \text{tr}(t_1t_3)\text{tr}(t_2^2) - \text{tr}(t_1t_2)\text{tr}(t_2t_3) \right) \cdot t_1 \]
\[ + \frac{1}{4} \left( \text{tr}(t_2^2)\text{tr}(t_2t_3) - \text{tr}(t_1t_2)\text{tr}(t_1t_3) \right) \cdot t_2 \]
\[ + \frac{1}{4} \left( \text{tr}(t_1t_2)^2 - \text{tr}(t_1^2)\text{tr}(t_2^2) \right) \cdot t_3 \]

(iii)

\[ \text{tr}(t_1t_2t_3)f_{11} = \frac{1}{4}\text{tr}(t_1t_3)f_{112} - \frac{1}{4}\text{tr}(t_1t_2)f_{113} - \frac{1}{12}\text{tr}(t_1^2)f_{132} + \frac{1}{12}\text{tr}(t_1^2)f_{123} \]

(iv)

\[ \text{tr}(t_1t_2t_3)f_{112} = \frac{1}{8} \left( \text{tr}(t_1t_3)\text{tr}(t_2^2) - \text{tr}(t_1t_2)\text{tr}(t_2t_3) \right) f_{11} \]
\[ + \frac{1}{2} \left( \text{tr}(t_1t_2)^2 - \text{tr}(t_1^2)\text{tr}(t_2^2) \right) f_{13} \]
\[ + \frac{1}{8} \left( \text{tr}(t_1t_2)\text{tr}(t_2t_3) - \text{tr}(t_1t_2)\text{tr}(t_1t_3) \right) f_{112} \]
\[ + \frac{1}{8} \left( \text{tr}(t_1t_3)\text{tr}(t_2^2) - \text{tr}(t_1t_2)\text{tr}(t_2t_3) \right) f_{12} \]
For an arbitrary \( p \geq 3 \), denote by \( A_p \) the subalgebra of \( K[T_p]^{SO_p(K)} \) generated by \( \text{tr}(t_it_j) \), \( 1 \leq i \leq j \leq p \) (note that for \( p \geq 4 \), \( A_p \) is not a polynomial algebra). The algebra \( \mathcal{E}_p \) is naturally an \( A_p \)-module. Now Theorem 5.6 and Corollary 4.3 imply the following:

**Proposition 5.8.** For any \( p \geq 3 \), the \( A_p \)-module \( \mathcal{E}_p \) decomposes as
\[
\mathcal{E}_p = A_p \cdot I + A_p \cdot \text{tr}(t_1t_2t_3)I + A_p \cdot \langle (t_1)_{GL_p(K)} \rangle + A_p \cdot \langle (t_1, t_2) \rangle_{GL_p(K)} + A_p \cdot \langle (t_1^2 - \frac{1}{3}\text{tr}(t_1^2)I)_{GL_p(K)} \rangle + A_p \cdot \langle (t_1^2, t_2) \rangle_{GL_p(K)}.
\]
In particular, as an \( A_p \)-module, \( \mathcal{E}_p \) is generated by
\[
I, \text{tr}(t_1t_2t_3), t_i, t_it_j, t_it_jt_k \quad 1 \leq i, j, k \leq p.
\]

Proposition 5.8 implies that for \( m \geq 4 \), any product \( t_i t_j \cdots t_m \) is contained in \( A_p^+ \cdot \mathcal{E}_p \), where \( A_p^+ \) stands for the ideal in \( A_p \) generated by \( \text{tr}(t_it_j) \), \( 1 \leq i \leq j \leq p \).

A more direct explanation of this fact is given by the following identity:

**Proposition 5.9.** We have the equality
\[
t_1t_2t_3t_4 = \frac{1}{4} (\text{tr}(t_1t_4)\text{tr}(t_2t_3) - \text{tr}(t_1t_2)\text{tr}(t_3t_4)) I + \frac{1}{2} (\text{tr}(t_1t_2)t_3t_4 - \text{tr}(t_3t_4)t_1t_2 - \text{tr}(t_1t_4)t_3t_2).
\]

**Proof.** Proposition 5.8 implies that \( t_1t_2t_3t_4 \) must be a \( K \)-linear combination of \( \text{tr}(t_1t_2t_3t_4)I, \text{tr}(t_1t_2t_3)[t_1, t_2]_{GL_p(K)}, \text{tr}(t_1t_2t_3)[t_2, t_3]_{GL_p(K)}, \text{tr}(t_1t_2t_3)[t_3, t_4]_{GL_p(K)} \), \( \pi \in S_4 \).

The actual coefficients were found using [CoCoA]:
\[
t_1t_2t_3t_4 = \frac{1}{12} (\text{tr}(t_1t_2)\text{tr}(t_3t_4) + \text{tr}(t_1t_4)\text{tr}(t_2t_3)) I + \frac{1}{4} (\text{tr}(t_3t_4)[t_1, t_2] + \text{tr}(t_1t_4)[t_2, t_3] + \text{tr}(t_1t_2)[t_3, t_4]) + \frac{1}{2} (\text{tr}(t_1t_4)f_{12} - \text{tr}(t_1t_4)f_{23} + \text{tr}(t_1t_2)f_{34}).
\]

Plugging in the explicit expressions for \( f_{12}, f_{23}, f_{34} \) on the right hand side of the above formula, we obtain the desired statement. \( \square \)

**Remark 5.10.** Based on Theorem 5.6 and its proof, it is possible to give a normal form for the elements in \( \mathcal{E}_3 \). With an iterated use of (20) it is possible to rewrite the product of any two \( P \)-module generators of \( \mathcal{E}_3 \) in normal form. This way one obtains a normal form plus a rewriting algorithm for products of elements given in normal form. The result is complicated and technical, so we leave out the details.

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