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INEQUALITIES FOR THE NORM AND NUMERICAL RADIUS OF COMPOSITE OPERATORS IN HILBERT SPACES

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Abstract. Some new inequalities for the norm and the numerical radius of composite operators generated by a pair of operators are given.

1. Introduction

Let \((H, \langle \cdot, \cdot \rangle)\) be a complex Hilbert space. The numerical range of an operator \(T\) is the subset of the complex numbers \(\mathbb{C}\) given by [4, p. 1]:

\[
W(T) = \{\langle Tx, x \rangle, \ x \in H, \ |x| = 1 \}.
\]

It is well known that (see [4]):

(i) The numerical range of an operator is convex;

(ii) The spectrum of an operator is contained in the closure of its numerical range;

(iii) \(T\) is self-adjoint if and only if \(W(T)\) is real.

The numerical radius \(w(T)\) of an operator \(T\) on \(H\) is defined by [4, p. 8]

\[
w(T) := \sup \{ |\lambda|, \lambda \in W(T) \} = \sup \{ |\langle Tx, x \rangle|, |x| = 1 \}.
\]

It is well known that \(w(\cdot)\) is a norm on the Banach algebra \(B(H)\) of all bounded linear operators acting on \(H\) and the following inequality holds true:

\[
w(T) \leq ||T|| \leq 2w(T).
\]

We recall some classical results involving the numerical radius of two linear operators \(A, B\).

The following general result for the product of two operators holds [4, p. 37]:

**Theorem 1.** If \(A, B\) are two bounded linear operators on the Hilbert space \((H, \langle \cdot, \cdot \rangle)\), then

\[
w(AB) \leq 4w(A)w(B).
\]

In the case that \(AB = BA\), then

\[
w(AB) \leq 2w(A)w(B).
\]

The following results are also well known [4, p. 38].
Theorem 2. If $A$ is a unitary operator that commutes with another operator $B$, then
\[(1.6)\quad w(AB) \leq w(B) .\]
If $A$ is an isometry and $AB = BA$, then (1.6) also holds true.

We say that $A$ and $B$ double commute if $AB = BA$ and $AB^* = B^*A$.

The following result holds [4, p. 38].

Theorem 3 (Double commute). If the operators $A$ and $B$ double commute, then
\[(1.7)\quad w(AB) \leq w(B) \|A\| .\]

As a consequence of the above, we have [4, p. 39]:

Corollary 1. Let $A$ be a normal operator commuting with $B$. Then
\[(1.8)\quad w(AB) \leq w(A) w(B) .\]

For other results and historical comments on the above see [4, p. 39–41]. For more results on the numerical radius, see [5].

The main aim of this paper is to establish some new inequalities for composite operators generated by a pair of operators $(A, B)$ under various assumptions. Namely, in one side, several inequalities involving the norm$\left\| A^*A + B^*B \right\|_2$ and the numerical radius $w(B^*A)$ are established. On the other side, upper bounds for the nonnegative quantities $\|A\| \|B\| - w(B^*A)$ and $\|A\|^2 \|B\|^2 - w^2(B^*A)$ under special conditions for the operators involved are also given.

2. The Results

The following result may be stated:

Theorem 4. Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$. If $r > 0$ and
\[(2.1)\quad \|A - B\| \leq r ,\]
then
\[(2.2)\quad \left\| \frac{A^*A + B^*B}{2} \right\| \leq w(B^*A) + \frac{1}{2} r^2 .\]

Proof. For any $x \in H$, $\|x\| = 1$, we have from (2.1) that
\[(2.3)\quad \|Ax\|^2 + \|Bx\|^2 \leq 2 \text{Re} \langle Ax, Bx \rangle + r^2 .\]

However
\[\|Ax\|^2 + \|Bx\|^2 = \langle (A^*A) x, x \rangle + \langle (B^*B) x, x \rangle = \langle (A^*A + B^*B) x, x \rangle \]
and by (2.3) we obtain
\[(2.4)\quad \langle (A^*A + B^*B) x, x \rangle \leq 2 |\langle (B^*A) x, x \rangle| + r^2 \]
for any $x \in H$, $\|x\| = 1$. 

Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.4) we get
\begin{equation}
(2.5)
\quad w(A^*A + B^*B) \leq 2w(B^*A) + r^2
\end{equation}
and since the operator $A^*A + B^*B$ is self-adjoint, hence
\[ \quad w(A^*A + B^*B) = \|A^*A + B^*B\| \]
and by (2.5) we deduce the desired inequality (2.2). 

**Remark 1.** We observe that, from the proof of the above theorem, we have the inequalities
\begin{equation}
(2.6)
0 \leq \left\| \frac{A^*A + B^*B}{2} \right\| - w(B^*A) \leq \frac{1}{2} \|A - B\|^2,
\end{equation}
provided that $A, B$ are bounded linear operators in $H$.

The second inequality in (2.6) is obvious while the first inequality follows by the fact that
\[ \langle (A^*A + B^*B)x, x \rangle = \|Ax\|^2 + \|Bx\|^2 \geq 2 \|Ax\| \|Bx\| \geq 2 |\langle (B^*A)x, x \rangle| \]
for any $x \in H$.

The inequality (2.2) is obviously a reach source of particular inequalities of interest.

Indeed, if we assume, for $\lambda \in \mathbb{C}$ and a bounded linear operator $T$, that we have
\begin{equation}
(2.7)
\quad \|T - \lambda T^*\| \leq r,
\end{equation}
for a given positive number $r$, then by (2.6) we deduce the inequality
\begin{equation}
(2.8)
\quad 0 \leq \left\| \frac{T^*T + |\lambda|^2 TT^*}{2} \right\| - |\lambda| w(T^2) \leq \frac{1}{2} r^2.
\end{equation}

Now, if we assume that for $\lambda \in \mathbb{C}$ and a bounded linear operator $V$ we have that
\begin{equation}
(2.9)
\quad \|V - \lambda I\| \leq r,
\end{equation}
where $I$ is the identity operator on $H$, then by (2.2) we deduce the inequality
\[ \quad 0 \leq \left\| \frac{V^*V + |\lambda|^2 I}{2} \right\| - |\lambda| w(V) \leq \frac{1}{2} r^2. \]

As a dual approach, the following result may be noted as well:

**Theorem 5.** Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space $H$. Then
\begin{equation}
(2.10)
\quad \left\| \frac{A + B}{2} \right\|^2 \leq \frac{1}{2} \left\| \frac{A^*A + B^*B}{2} \right\| + w(B^*A).
\end{equation}

**Proof.** We obviously have
\[ \quad \|Ax + Bx\|^2 = \|Ax\|^2 + 2 \text{Re} \langle Ax, Bx \rangle + \|Bx\|^2 \leq \langle (A^*A + B^*B)x, x \rangle + 2 |\langle (B^*A)x, x \rangle| \]
for any $x \in H$. 

Taking the supremum over \( x \in H, \|x\| = 1 \), we get
\[
\|A + B\|^2 \leq w(A^*A + B^*B) + 2w(B^*A) = \|A^*A + B^*B\| + 2w(B^*A),
\]
from where we get the desired inequality (2.10).

**Remark 2.** The inequality (2.10) can generate some interesting particular results such as the following inequality
\[
\left\| \frac{T + T^*}{2} \right\|_2^2 \leq \frac{1}{2} \left[ \left\| T^*T + TT^* \right\|_2 + w(T^2) \right],
\]
holding for each bounded linear operator \( T : H \to H \).

The following result may be stated as well.

**Theorem 6.** Let \( A, B : H \to H \) be two bounded linear operators on the Hilbert space \( H \) and \( p \geq 2 \). Then
\[
\left\| \frac{A^*A + B^*B}{2} \right\|^\frac{2}{p} \leq \frac{1}{4} \left[ \|A - B\|^p + \|A + B\|^p \right].
\]

**Proof.** We use the following inequality for vectors in inner product spaces obtained by Dragomir and Sándor in [2]:
\[
2 (\|a\|^p + \|b\|^p) \leq \|a + b\|^p + \|a - b\|^p
\]
for any \( a, b \in H \) and \( p \geq 2 \).

Utilising (2.13) we may write
\[
2 (\|Ax\|^p + \|Bx\|^p) \leq \|Ax + Bx\|^p + \|Ax - Bx\|^p
\]
for any \( x \in H \).

Now, observe that
\[
\|Ax\|^p + \|Bx\|^p = \left( \|Ax\|^2 \right)^\frac{p}{2} + \left( \|Bx\|^2 \right)^\frac{p}{2}
\]
and by the elementary inequality:
\[
\frac{\alpha^q + \beta^q}{2} \geq \left( \frac{\alpha + \beta}{2} \right)^q, \quad \alpha, \beta \geq 0 \quad \text{and} \quad q \geq 1
\]
we have
\[
\left( \|Ax\|^2 \right)^\frac{p}{2} + \left( \|Bx\|^2 \right)^\frac{p}{2} \geq 2^{1 - \frac{p}{2}} \left( \|Ax\|^2 + \|Bx\|^2 \right)^\frac{p}{2}
\]
\[
= 2^{1 - \frac{p}{2}} \left[ (A^*A + B^*B) x, x \right]^\frac{p}{2}.
\]

Combining (2.14) with (2.15) we get
\[
\frac{1}{4} \left[ \|Ax - Bx\|^p + \|Ax + Bx\|^p \right] \geq \left[ \left( \frac{A^*A + B^*B}{2} \right) x, x \right]^\frac{p}{2}
\]
for any \( x \in H, \|x\| = 1 \). Taking the supremum over \( x \in H, \|x\| = 1 \), and taking into account that
\[
w \left( \frac{A^*A + B^*B}{2} \right) = \left\| \frac{A^*A + B^*B}{2} \right\|,
\]
we deduce the desired result (2.12).
Remark 3. If \( p = 2 \), then we have the inequality:

\[
(2.17) \quad \left\| \frac{A^* A + B^* B}{2} \right\| \leq \left\| \frac{A - B}{2} \right\|^2 + \left\| \frac{A + B}{2} \right\|^2,
\]

for any \( A, B \) bounded linear operators. This result can also be obtained directly on utilising the parallelogram identity.

We also should observe that for \( A = T \) and \( B = T^* \), \( T \) a normal operator, the inequality (2.12) becomes

\[
(2.23) \quad \|T\|^p \leq \frac{1}{4} [\|T - T^*\|^p + \|T + T^*\|^p],
\]

where \( p \geq 2 \).

The following result may be stated as well.

Theorem 7. Let \( A, B : H \to H \) be two bounded linear operators on the Hilbert space \( H \) and \( r \geq 1 \). If \( A^* A \geq B^* B \) in the operator order or, equivalently, \( \|Ax\| \geq \|Bx\| \) for any \( x \in H \), then:

\[
(2.18) \quad \left\| \frac{A^* A + B^* B}{2} \right\|^r \leq \|A\|^{r-1} \|B\|^{r-1} w(B^* A) + \frac{1}{2} r^2 \|A\|^{2r-2} \|A - B\|^2.
\]

Proof. We use the following inequality for vectors in inner product spaces due to Goldstein, Ryff and Clarke [3]:

\[
(2.19) \quad \|a\|^{2r} + \|b\|^{2r} \leq 2 \|a\|^{r-1} \|b\|^{r-1} \text{Re} \langle a, b \rangle + r^2 \|a\|^{2r-2} \|a - b\|^2,
\]

where \( r \geq 1, a, b \in H \) and \( \|a\| \geq \|b\| \).

Utilising (2.19) we can state that:

\[
(2.20) \quad \|Ax\|^{2r} + \|Bx\|^{2r} \leq 2 \|Ax\|^{r-1} \|Bx\|^{r-1} |\langle Ax, Bx \rangle| + \frac{1}{2} r^2 \|Ax\|^{2r-2} \|Ax - Bx\|^2,
\]

for any \( x \in H \).

As in the proof of Theorem 6, we also have

\[
(2.21) \quad 2^{1-r} \left[ \left( (A^* A + B^* B) x, x \right) \right]^r \leq \|Ax\|^{2r} + \|Bx\|^{2r},
\]

for any \( x \in H \).

Therefore, by (2.20) and (2.21) we deduce

\[
(2.22) \quad \left[ \left( \frac{A^* A + B^* B}{2} \right) x, x \right]^r \leq \|Ax\|^{r-1} \|Bx\|^{r-1} |\langle Ax, Bx \rangle| + \frac{1}{2} r^2 \|A\|^{2r-2} \|Ax - Bx\|^2
\]

for any \( x \in H \).

Taking the supremum in (2.22) we obtain the desired result (2.18). \( \blacksquare \)

Remark 4. Following [4, p. 156], we recall that the bounded linear operator \( V \) is hyponormal, if

\[
\|V^* x\| \leq \|V x\| \quad \text{for all} \ x \in H.
\]

Now, if we choose in (2.18) \( A = V \) and \( B = V^* \), then, on taking into account that for hyponormal operators \( w(V^2) = \|V\|^2 \), we get the inequality

\[
(2.23) \quad \left\| \frac{V^* V + V V^*}{2} \right\|^r \leq \|V\|^{2r-2} \left[ \|V\|^2 + \frac{1}{2} r^2 \|V - V^*\|^2 \right].
\]
holding for any hyponormal operator \( V \) and any \( r \geq 1 \).

3. Further Inequalities for an Invertible Operator

In this section we assume that \( B : H \to H \) is an invertible bounded linear operator and let \( B^{-1} : H \to H \) be its inverse. Then, obviously,

\[
\| Bx \| \geq \frac{1}{\| B^{-1} \|} \| x \| \quad \text{for any} \quad x \in H,
\]  

where \( \| B^{-1} \| \) denotes the norm of the inverse \( B^{-1} \).

The following result holds true:

**Theorem 8.** Let \( A, B : H \to H \) be two bounded linear operators on \( H \) and \( B \) is invertible such that, for a given \( r > 0 \),

\[
\| A - B \| \leq r.
\]

Then:

\[
\| A \| \leq \| B^{-1} \| \left[ w(B^*A) + \frac{1}{2}r^2 \right].
\]

**Proof.** The condition (3.2) is obviously equivalent to:

\[
\| Ax \|^2 + \| Bx \|^2 \leq 2 \text{Re} \langle (B^*A)x, x \rangle + r^2
\]

for any \( x \in H, \| x \| = 1 \).

Since, by (3.1),

\[
\| Bx \|^2 \geq \frac{1}{\| B^{-1} \|^2} \| x \|^2, \quad x \in H
\]

and \( \text{Re} \langle (B^*A)x, x \rangle \leq |\langle (B^*A)x, x \rangle| \), hence by (3.4) we get

\[
\| Ax \|^2 + \frac{\| x \|^2}{\| B^{-1} \|^2} \leq 2 |\langle (B^*A)x, x \rangle| + r^2
\]

for any \( x \in H, \| x \| = 1 \).

Taking the supremum over \( x \in H, \| x \| = 1 \) in (3.5), we have

\[
\| A \|^2 + \frac{1}{\| B^{-1} \|^2} \leq 2w(B^*A) + r^2.
\]

By the elementary inequality

\[
\frac{2\| A \|}{\| B^{-1} \|^2} \leq \| A \|^2 + \frac{1}{\| B^{-1} \|^2}
\]

and by (3.6) we then deduce the desired result (3.3). \( \square \)

**Remark 5.** If we choose above \( B = \lambda I, \lambda \neq 0 \), then we get the inequality

\[
(0 \leq) \| A \| - w(A) \leq \frac{1}{2|\lambda|}r^2,
\]

provided \( \| A - \lambda I \| \leq r \). This result has been obtained in the earlier paper [1].

Also, if we assume that \( B = \lambda A^* \), \( A \) is invertible, then we obtain

\[
\| A \| \leq \| A^{-1} \| \left[ w(A^2) + \frac{1}{2|\lambda|}r^2 \right],
\]

provided \( \| A - \lambda A^* \| \leq r, \lambda \neq 0 \).
Theorem 9. Let \( A, B : H \to H \) be two bounded linear operators on \( H \). If \( B \) is invertible and for \( r > 0 \),

\[
\|A - B\| \leq r,
\]

then

\[
\begin{align*}
(0 \leq) & \quad \|A\| \|B\| - w(B^*A) \leq \frac{1}{2} r^2 + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{\|B^{-1}\|^2}.
\end{align*}
\]

Proof. The condition (3.10) is obviously equivalent to

\[
\|Ax\|^2 + \|Bx\|^2 \leq 2 \text{Re} \langle Ax, Bx \rangle + r^2
\]

for any \( x \in H \), which is clearly equivalent to

\[
\|Ax\|^2 + \|B\|^2 \leq 2 \text{Re} \langle B^*Ax, x \rangle + r^2 + \|B\|^2 - \|Bx\|^2.
\]

Since

\[
\text{Re} \langle B^*Ax, x \rangle \leq |\langle B^*Ax, x \rangle|, \quad \|Bx\|^2 \geq \frac{1}{\|B^{-1}\|^2} \|x\|^2
\]

and

\[
\|Ax\|^2 + \|B\|^2 \geq 2 \|B\| \|Ax\|
\]

for any \( x \in H \), hence by (3.12) we get

\[
\begin{align*}
2 \|B\| \|Ax\| & \leq 2 |\langle B^*Ax, x \rangle| + r^2 + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{\|B^{-1}\|^2}
\end{align*}
\]

for any \( x \in H, \|x\| = 1 \).

Taking the supremum over \( x \in H, \|x\| = 1 \) we deduce the desired result (3.11).

Remark 6. If we choose in Theorem 9, \( B = \lambda A^* \), \( \lambda \neq 0 \), \( A \) is invertible, then we get the inequality:

\[
(0 \leq) \|A\|^2 - w(A^2) \leq \frac{1}{2 |\lambda|} r^2 + |\lambda| \cdot \frac{\|A\|^2 \|A^{-1}\|^2 - 1}{\|A^{-1}\|^2}
\]

provided \( \|A - \lambda A^*\| \leq r \).

The following result may be stated as well.

Theorem 10. Let \( A, B : H \to H \) be two bounded linear operators on \( H \). If \( B \) is invertible and for \( r > 0 \) we have

\[
\|A - B\| \leq r < \|B\|
\]

then

\[
\|A\| \leq \frac{1}{\sqrt{\|B\|^2 - r^2}} \left( w(B^*A) + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{2 \|B^{-1}\|^2} \right).
\]

Proof. The first part of condition (3.15) is obviously equivalent to

\[
\|Ax\|^2 + \|Bx\|^2 \leq 2 \text{Re} \langle Ax, Bx \rangle + r^2
\]

for any \( x \in H \), which is clearly equivalent to

\[
\|Ax\|^2 + \|B\|^2 - r^2 \leq 2 \text{Re} \langle B^*Ax, x \rangle + \|B\|^2 - \|Bx\|^2.
\]
Since
\[ \text{Re} \langle B^*Ax, x \rangle \leq |\langle B^*Ax, x \rangle|, \]
\[ \|Bx\|^2 \geq \frac{1}{\|B^{-1}\|^2} \|x\|^2, \]
and, by the second part of (3.15),
\[ \|Ax\|^2 + \|B\|^2 - r^2 \geq 2\sqrt{\|B\|^2 - r^2} \|Ax\|, \]
for any \( x \in H \), hence by (3.17) we get
\[ (3.18) \quad 2 \|Ax\| \sqrt{\|B\|^2 - r^2} \leq 2|\langle B^*Ax, x \rangle| + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{\|B^{-1}\|^2}, \]
for any \( x \in H, \|x\| = 1 \).

Taking the supremum over \( x \in H, \|x\| = 1 \) in (3.18), we deduce the desired inequality (3.16).

**Remark 7.** The above Theorem 10 has some particular cases of interest. For instance, if we choose \( B = \lambda I \), with \(|\lambda| > r\), then (3.15) is obviously fulfilled and by (3.16) we get
\[ (3.19) \quad \|A\| \leq \frac{w(A)}{\sqrt{1 - \left(\frac{r}{|\lambda|}\right)^2}}, \]
provided \( \|A - \lambda I\| \leq r \). This result has been obtained in the earlier paper [1].

On the other hand, if in the above we choose \( B = \lambda A^* \) with \( \|A\| \geq \frac{r}{|\lambda|} \) (\( \lambda \neq 0 \)), then by (3.16) we get
\[ (3.20) \quad \|A\| \leq \frac{1}{\sqrt{\|A\|^2 - \left(\frac{r}{|\lambda|}\right)^2}} \left[ w(A^2) + |\lambda| \cdot \frac{\|A\|^2 \|A^{-1}\|^2 - 1}{2 \|A^{-1}\|^2} \right], \]
provided \( \|A - \lambda A^*\| \leq r \).

The following result may be stated as well.

**Theorem 11.** Let \( A, B \) and \( r \) be as in Theorem 8. Moreover, if
\[ (3.21) \quad \|B^{-1}\| < \frac{1}{r}, \]
then
\[ (3.22) \quad \|A\| \leq \frac{\|B^{-1}\|}{\sqrt{1 - r^2 \|B^{-1}\|^2}} w(B^*A), \]

**Proof.** Observe that, by (3.6) we have
\[ (3.23) \quad \|A\|^2 + \frac{1 - r^2 \|B^{-1}\|^2}{\|B^{-1}\|^2} \leq 2w(B^*A). \]

Utilising the elementary inequality
\[ (3.24) \quad 2 \frac{\|A\|}{\|B^{-1}\|} \sqrt{1 - r^2 \|B^{-1}\|^2} \leq \|A\|^2 + \frac{1 - r^2 \|B^{-1}\|^2}{\|B^{-1}\|^2}, \]
which can be stated since (3.21) is assumed to be true, hence by (3.23) and (3.24) we deduce the desired result (3.22).

**Remark 8.** If we assume that \( B = \lambda A^* \) with \( \lambda \neq 0 \) and \( A \) an invertible operator, then, by applying Theorem 11, we get the inequality:

\[
\|A\| \leq \frac{\|A^{-1}\| w(A^2)}{\sqrt{|\lambda|^2 - r^2 \|A^{-1}\|^2}}.
\]

provided \( \|A - \lambda A^*\| \leq r \) and \( \|A^{-1}\| \leq \frac{|\lambda|}{r} \).

The following result may be stated as well.

**Theorem 12.** Let \( A, B : H \to H \) be two bounded linear operators. If \( r > 0 \) and \( B \) is invertible with the property that \( \|A - B\| \leq r \) and

\[
\frac{1}{\sqrt{r^2 + 1}} \leq \|B^{-1}\| < \frac{1}{r},
\]

then

\[
\|A\|^2 \leq w^2(B^*A) + 2w(B^*A) \cdot \frac{\|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\|}.
\]

**Proof.** Let \( x \in H, \|x\| = 1 \). Then by (3.5) we have

\[
\|Ax\|^2 + \frac{1}{\|B^{-1}\|^2} \leq 2 |\langle B^*Ax, x \rangle| + r^2,
\]

and since

\[
\frac{1}{\|B^{-1}\|^2} - r^2 > 0,
\]

we can conclude that \( |\langle B^*Ax, x \rangle| > 0 \) for any \( x \in H, \|x\| = 1 \).

Dividing in (3.28) with \( |\langle B^*Ax, x \rangle| > 0 \), we obtain

\[
\frac{\|Ax\|^2}{|\langle B^*Ax, x \rangle|} \leq 2 + \frac{r^2}{|\langle B^*Ax, x \rangle|} - \frac{1}{\|B^{-1}\|^2 |\langle B^*Ax, x \rangle|}.
\]

Subtracting \( |\langle B^*Ax, x \rangle| \) from both sides of (3.29), we get

\[
\frac{\|Ax\|^2}{|\langle B^*Ax, x \rangle|} - |\langle B^*Ax, x \rangle| \leq 2 + \frac{r^2}{|\langle B^*Ax, x \rangle|} - \frac{1}{\|B^{-1}\|^2 |\langle B^*Ax, x \rangle|}.
\]

\[
\leq 2 - \frac{|\langle B^*Ax, x \rangle| - \frac{1 - r^2 \|B^{-1}\|^2}{|\langle B^*Ax, x \rangle|}}{\|B^{-1}\|^2} + \left( \sqrt{|\langle B^*Ax, x \rangle|} - \frac{\sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\|} \right)^2
\]

\[
\leq 2 \left( \frac{\|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\|} \right)^2,
\]

which gives:

\[
\|Ax\|^2 \leq |\langle B^*Ax, x \rangle|^2 + 2 \frac{|\langle B^*Ax, x \rangle| - \frac{1 - r^2 \|B^{-1}\|^2}{|\langle B^*Ax, x \rangle|}}{\|B^{-1}\|}.$$
We also remark that, by (3.26) the quantity
\[ \|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2} \geq 0, \]
hence, on taking the supremum in (3.31) over \( x \in H, \|x\| = 1 \), we deduce the desired inequality. \( \blacksquare \)

**Remark 9.** It is interesting to remark that if we assume \( \lambda \in \mathbb{C} \) with \( 0 < r \leq |\lambda| \leq \sqrt{r^2 + 1} \) and \( \|A - \lambda I\| \leq r \), then by (3.2) we can state the following inequality:

(3.32) \[ \|A\|^2 \leq \lambda^2 w(A^2) + 2|\lambda| \left( 1 - \sqrt{|\lambda|^2 - r^2} \right) w(A). \]

Also, if \( \|A - A^*\| \leq r \), \( A \) is invertible and \( \frac{1}{\sqrt{r^2 + 1}} \leq \|A^{-1}\| \leq \frac{1}{r} \), then by (3.27) we also have

(3.33) \[ \|A\|^2 \leq w^2(A^2) + 2w(A) \cdot \frac{\|A^{-1}\| - \sqrt{1 - r^2 \|A^{-1}\|^2}}{\|A^{-1}\|}. \]

One can also prove the following result.

**Theorem 13.** Let \( A, B : H \to H \) be two bounded linear operators. If \( r > 0 \) and \( B \) is invertible with the property that \( \|A - B\| \leq r \) and \( \|B^{-1}\| \leq \frac{1}{r} \), then

(3.34) \[ (0 \leq) \frac{\|A\|^2 \|B\|^2 - w^2(B^*A)}{\|B\|^2} \leq 2w(B^*A) \cdot \frac{\|B\| \|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B\| \|B^{-1}\|}. \]

**Proof.** We subtract the quantity \( \frac{|\langle B^*Ax, x \rangle|}{\|B\|^2} \) from both sides of (3.29) to obtain

(3.35) \[ 0 \leq \frac{|\langle B^*Ax, x \rangle|}{\|B\|^2} \leq 2 - \frac{|\langle B^*Ax, x \rangle|}{\|B\|^2} - \frac{1 - r^2 \|B^{-1}\|^2}{\|B\| \|B^{-1}\|} \]
\[ = 2 - 2 \cdot \frac{\sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B\| \|B^{-1}\|} \left( \frac{\sqrt{|\langle B^*Ax, x \rangle|}}{\|B\|} - \sqrt{1 - r^2 \|B^{-1}\|^2} \right)^2 \]
\[ \leq 2 \cdot \frac{\|B\| \|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B\| \|B^{-1}\|}, \]

which is equivalent with

(3.36) \[ (0 \leq) \|A\|^2 \|B\|^2 - |\langle B^*Ax, x \rangle|^2 \]
\[ \leq 2 \cdot \frac{\|B\| \|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B\| \|B^{-1}\|} \]

for any \( x \in H, \|x\| = 1 \).
The inequality (3.36) also shows that \( \|B\| \|B^{-1}\| \geq \sqrt{1 - r^2 \|B^{-1}\|^2} \) and then, by (3.36), we get

\[
\|Ax\|^2 \|B\|^2 \leq |\langle B^*Ax, x \rangle|^2 \]

\[
+ 2 \frac{\|B\|}{\|B^{-1}\|} |\langle B^*Ax, x \rangle| \left( \|B\| \|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2} \right)
\]

for any \( x \in X, \|x\| = 1 \).

Taking the supremum in (3.37) we deduce the desired inequality (3.34).

**Remark 10.** The above Theorem 13 has some particular instances of interest as follows. If, for instance, we choose \( B = \lambda I \) with \( |\lambda| \geq r > 0 \) and \( \|A - \lambda I\| \leq r \), then by (3.34) we obtain the inequality

\[
0 \leq \|A\|^2 - w^2 (A)
\]

\[
\leq 2 |\lambda| w (A) \left( 1 - \sqrt{1 - \frac{r^2}{|\lambda|^2}} \right).
\]

Also, if \( A \) is invertible, \( \|A - \lambda A^*\| \leq r \) and \( \|A^{-1}\| \leq \frac{|\lambda|}{r} \), then by (3.34) we can state:

\[
0 \leq \|A\|^4 - w^2 (A^2)
\]

\[
\leq 2 |\lambda| w (A^2) \cdot \frac{\|A\|}{\|A^{-1}\|} \left( \|A\| \|A^{-1}\| - \sqrt{1 - \frac{r^2}{|\lambda|^2} \|A^{-1}\|^2} \right).
\]

**References**

[1] S.S. DRAGOMIR, Reverse inequalities for the numerical radius of linear operators in Hilbert spaces, *RG Mia Res. Rep. Coll.*, 8(2005), Supplement, Article 9 [ONLINE: http://rgmia.vu.edu.au/v8(E).html].

[2] S.S. DRAGOMIR and J. SÁNDOR, Some inequalities in prehilbertian spaces, *Studia Univ. “Babeş-Bolyai”- Mathematica*, 32(1) (1987), 71-78.

[3] A. GOLDSTEIN, J.V. RYFF and L.E. CLARKE, Problem 5473, *Amer. Math. Monthly*, 75(3) (1968), 309.

[4] K.E. GUSTAFSON and D.K.M. RAO, *Numerical Range*, Springer-Verlag, New York, Inc., 1997.

[5] P.R. HALMOS, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, Chelsea Pub. Comp, New York, N.Y., 1972.

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