Abstract. We prove that on 2-connected closed oriented manifolds, the analytic and algebraic constructions of an IBL_{\infty} structure associated to a closed oriented manifold coincide. The corresponding structure is invariant under orientation preserving homotopy equivalences and induces on homology the involutive Lie bialgebra structure of Chas and Sullivan.

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1. Introduction

Let $M$ be a closed oriented manifold of dimension $n$ and $LM = C^\infty(S^1, M)$ its free loop space, equipped with the $S^1$-action by reparametrisation. In their seminal 1999 paper [6], Chas and Sullivan described algebraic structures on the non-equivariant and equivariant homology of $LM$ which are commonly known under the name string topology:

- The degree shifted homology $H_{*+n}(LM)$ carries the structure of a Batalin-Vilkovisky algebra. Moreover, the homology relative to the constant loops $H_*(LM, \text{const})$ carries a coproduct of degree $1-n$.
- The degree shifted equivariant homology relative to the constant loops $H_{S^1*+n-2}(LM, \text{const})$ carries the structure of an involutive Lie bialgebra (short IBL algebra), where the bracket has degree 0 and the cobracket has degree $2(2-n)$.

The products on $H_{*+n}(LM)$ and $H_{S^1*+n-2}(LM)$ are homotopy invariants [13, 22, 15], whereas the coproduct on $H_*(LM, \text{const})$ with $\mathbb{Z}$-coefficients is not [35].

Since the work of Stasheff in the 1960s [37], topologists have recognized the importance of studying the chain-level structures underlying operations on homology. So it is natural to ask for the chain-level structures underlying string topology operations. These structures are also important for applications in symplectic topology described by Fukaya and others, see, e.g., [19, 9]. In the non-equivariant case, the chain level structure underlying the Batalin-Vilkovisky structure on $H_{*+n}(LM)$ has been described and constructed by Irie [26]. Here and in the sequel we adhere to the Convention. The manifold $M$ is closed and oriented of dimension $n$, and all homology groups are with $\mathbb{R}$-coefficients.

In this paper we focus on the $S^1$-equivariant case. Here the chain level structure underlying an involutive Lie bialgebra structure on homology is that of an IBL$_\infty$ algebra introduced in [8]. We recall this notion in Section 3 below. See also [25] for further discussion of IBL$_\infty$ algebras from a properadic perspective.

If the manifold $M$ is simply connected, then it is known from Sullivan’s ground-breaking work in the 1970s (see, e.g., [38]) that the real homotopy type of $M$ is determined by the minimal model of its de Rham complex $(\Omega^*(M), d, \wedge)$. Work of Chen, Jones and others (see, e.g., [7, 27, 11]) expresses the homology of $LM$ in terms of the Hochschild and reduced cyclic cohomology of the de Rham complex,

\[(1) \quad HH^*(\Omega^*(M)) \cong H_*(LM), \quad \widetilde{CH}^*(\Omega^*(M)) \cong H_{S^1}^*(LM, \text{pt}).\]

In his thesis [2], Basu uses Sullivan’s minimal model to compute string topology operations.

In view of these results, it is natural to ask whether we can construct the IBL$_\infty$ algebra underlying equivariant string topology from the de Rham complex. It is argued in [8] that this should be possible if we enhance the de Rham complex by its intersection pairing $\langle a, b \rangle = \int_M a \wedge b$ to an oriented Poincaré DGA

\[(\Omega^*(M), d, \wedge, \langle \cdot, \cdot \rangle).\]
See Section 2 for the precise definition of an oriented Poincaré DGA; besides graded commutativity, the main requirement is that the pairing descends to a nondegenerate pairing on cohomology. A special case of this is a differential Poincaré duality algebra \((A, d, \wedge, (\cdot, \cdot))\) for which \(A\) is finite dimensional and the pairing is nondegenerate on \(A\). Without graded commutativity, this corresponds to a cyclic DGA in the terminology of [8]. In this algebraic situation we have the following result:

**Theorem 1.1** ([8]). Let \((A, d, \wedge, (\cdot, \cdot))\) be a cyclic DGA of degree \(n\) with cohomology \(H = H(A, d)\). Then \(B^{\text{cyc}} \mathbb{H}[2 - n] \) carries an IBL\(_\infty\) structure which is IBL\(_\infty\) homotopy equivalent to the twisted dIBL structure \(p^m\) on \(B^{\text{cyc}} A[2 - n]\). In particular, its homology equals the cyclic cohomology of \((A, d, \wedge)\).

See Section 3 for the notions appearing in this theorem such as the dual cyclic bar complex \(B^{\text{cyc}} A\). In view of the second isomorphism in (1), Theorem 1.1 suggests two approaches to chain level string topology.

**Algebraic approach:** Apply Theorem 1.1 to a suitable finite dimensional model of the de Rham complex \(\Omega^*(M)\).

**Analytic approach:** Extend Theorem 1.1 to the de Rham complex \(\Omega^*(M)\), replacing finite sums by integrals over configuration spaces.

The algebraic approach is based on the following theorem of Lambrechts and Stanley.

**Theorem 1.2** ([32]). For every simply connected Poincaré DGA \(A\) there exists a differential Poincaré duality algebra \(A'\) which is connected to \(A\) by a zigzag of quasi-isomorphisms of commutative DGAs.

Applying Theorem 1.2 to \(A = \Omega^*(M)\) for simply connected \(M\), and then Theorem 1.1 to the differential Poincaré duality algebra \(A'\), one obtains an IBL\(_\infty\) structure on \(B^{\text{cyc}} H^*_{\text{dr}}(M)[2 - n]\) whose reduced homology equals the reduced cyclic cohomology of \(\Omega^*(M)\) and therefore \(H^*_S^1(LM, \text{pt})\). The algebraic approach is now completed by the following theorem of Naef and Willwacher.

**Theorem 1.3** ([36]). Let \(M\) be a simply connected closed manifold. Then the involutive Lie bialgebra structure on the reduced cyclic homology of \(H = H^*_{\text{dr}}(M)\) induced by the IBL\(_\infty\) structure on \(B^{\text{cyc}} H^*_{\text{dr}}(M)[2 - n]\) obtained algebraically is isomorphic to the involutive Lie bialgebra structure on \(H^S_1(LM, \text{pt})\) due to Chas and Sullivan.

However, the algebraic approach raises the issue that the resulting IBL\(_\infty\) structure may depend on the chosen model and thus not be canonical. This issue has been addressed by the second author in [23]. To describe the results, let us abbreviate “differential Poincaré duality algebra” by “dPD algebra”. Two Poincaré DGAs are called weakly equivalent if they are connected by a zigzag of Poincaré DGA quasi-isomorphisms (see Section 2.3 for more details). Recall that a graded \(\mathbb{R}\)-vector space \(A = \bigoplus_{k \geq 0} A^k\) is called \(m\)-connected if \(A^0 = \mathbb{R}\) and \(A^1 = \cdots = A^m = 0\). Then we have the following improvement of Theorem 1.2.

**Theorem 1.4** ([23]). (a) Every Poincaré DGA \(A\) is weakly equivalent to a dPD algebra \(A_1\). If the homology of \(A\) is \(m\)-connected, then \(A_1\) can be chosen to be \(m\)-connected.

(b) Let \(A_1, A_2\) be 2-connected dPD algebras which are weakly equivalent as Poincaré DGAs. Then there exists a 1-connected dPD algebra \(A_3\) and quasi-isomorphisms of dPD algebras \(A_1 \leftarrow A_3 \to A_2\).

The analytic approach has been outlined in [8]. It is completed by the following result.
Theorem 1.5 (IBL). Let $M$ be a closed oriented manifold of dimension $n$ and $H = H_{dR}(M)$ its de Rham cohomology. Then the following hold.

(a) Integrals over configuration spaces give rise to an IBL$_\infty$ structure on $B^{\text{cyc}}H[2-n]$ whose homology equals the cyclic cohomology of the de Rham complex of $M$.

(b) This structure is independent of all choices up to IBL$_\infty$ homotopy equivalence.

Note that Theorem 1.5(b) resolves the issue of potential dependence on choices for the analytic approach.

Results of this paper. Our first new result is

Theorem 1.6. Let $A, A'$ be Poincaré DGAs with 2-connected homology and $A_1, A'_1$ associated 2-connected dPD algebras as in Theorem 1.4(a). Assume that $A$ and $A'$ are weakly equivalent. Then the IBL$_\infty$ structures on $B^{\text{cyc}}H[2-n]$ obtained by applying Theorem 1.5 to $A_1$ and $A'_1$ are IBL$_\infty$ homotopy equivalent.

In particular, we can apply this theorem to the de Rham complexes of closed oriented connected $n$-dimensional manifolds $M, M'$. We call a homotopy equivalence $f : M \to M'$ orientation preserving if the induced isomorphism $f_* : H_\ast(M) \to H_\ast(M')$ maps the fundamental class $[M]$ to $[M']$. This ensures that the pullback $f^* : \Omega^\ast(M') \to \Omega^\ast(M)$ is a quasi-isomorphism of Poincaré DGAs. Hence, Theorem 1.6 implies

Corollary 1.7. Let $M, M'$ be closed oriented connected $n$-dimensional manifolds with vanishing first and second de Rham cohomology. Let $A_1, A'_1$ be 2-connected dPD algebras associated to the de Rham complexes $\Omega^\ast(M), \Omega^\ast(M')$ as in Theorem 1.4(a). Assume that there exists an orientation preserving homotopy equivalence between $M$ and $M'$. Then the IBL$_\infty$ structures on $B^{\text{cyc}}H[2-n]$ obtained by applying Theorem 1.5 to $A_1$ and $A'_1$ are IBL$_\infty$ homotopy equivalent.

This accomplishes the algebraic construction of a canonical (up to IBL$_\infty$ homotopy equivalence) IBL$_\infty$ algebra associated to a closed oriented connected manifold with $H_{dR}^1(M) = H_{dR}^2(M) = 0$. Our second result says that this agrees with the analytic construction in Theorem 1.5.

Theorem 1.8. Let $M$ be a closed oriented connected manifold and $H = H_{dR}(M)$ its de Rham cohomology. If $H_{dR}^1(M) = H_{dR}^2(M) = 0$, then the IBL$_\infty$ structures on $B^{\text{cyc}}H[2-n]$ arising from Theorem 1.5 and Corollary 1.7 are IBL$_\infty$ homotopy equivalent.

This comparison result has the following immediate corollary.

Corollary 1.9. In the class of closed oriented connected manifolds with $H_{dR}^1 = H_{dR}^2 = 0$, the IBL$_\infty$ structure arising from the analytic construction in Theorem 1.5 is invariant (up to IBL$_\infty$ homotopy equivalence) under orientation preserving homotopy equivalences.

Combining this with Theorem 1.6, we obtain

Corollary 1.10. In the class of closed oriented simply connected manifolds with $H_{dR}^2 = 0$, the IBL$_\infty$ structure arising from Theorem 1.5 is invariant (up to IBL$_\infty$ homotopy equivalence) under orientation preserving homotopy equivalences. Moreover, the involutive Lie bialgebra structure on its reduced homology is isomorphic to the one on reduced equivariant loop space homology due to Chas and Sullivan.
Corollary 1.10 establishes, in the class of simply connected manifolds with $H^2_{dR} = 0$, the orientation preserving homotopy invariance of the IBL$_\infty$ structure underlying $S^1$-equivariant string topology. To our knowledge, this is the first result on homotopy invariance of string topology operations on the chain level. The homotopy invariance statement for the string cobracket appears to be new even on the level of homology.

Another corollary of Theorem 1.8 is the description of the IBL$_\infty$ structures in the case that $M$ is formal in the sense of rational homotopy theory (see §6.4 for the relevant definitions):

Corollary 1.11 (Formality implies IBL$_\infty$ formality). Let $M$ be a closed oriented connected manifold and $H = H_{dR}(M)$ its de Rham cohomology. Assume that $M$ is formal and $H^1_{dR}(M) = H^2_{dR}(M) = 0$. Then the analytic IBL$_\infty$ structure on $B^{cyc}H[2-n]$ in Theorem 1.5 is IBL$_\infty$ homotopy equivalent to the canonical dIBL structure $dIBL(H)$ which is only twisted by the triple intersection product.

Remark 1.12. Theorem 1.5 is in fact a consequence of a more refined statement: $B^{cyc}H[2-n]$ carries a Maurer–Cartan element for its untwisted dIBL structure which is independent of all choices up to IBL$_\infty$ gauge equivalence. Similarly, the IBL$_\infty$ structure in Theorem 1.1 is obtained by twisting with a Maurer–Cartan element. Combining the proof of Theorem 1.5 with Lemma 2.8 below, Theorem 1.6 can probably be upgraded to IBL$_\infty$ gauge equivalence of the corresponding Maurer–Cartan elements.

Remark 1.13. In the upcoming paper [12] it is proved analytically (using Chen’s iterated integrals) that, for a simply connected closed manifold, the involutive Lie bialgebra structure on the reduced homology of the IBL$_\infty$ structure from Theorem 1.5 is isomorphic to the one on reduced equivariant loop space homology due to Chas and Sullivan. Thus, the last assertion in Corollary 1.10 is true without the hypothesis $H^2_{dR} = 0$.

Remark 1.14. In the new version of [23] it is proved that in all the results above we can drop the hypothesis $H^2_{dR} = 0$ if the manifold has odd dimension. It would be interesting to know whether this also holds in the even dimensional case.

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2. Poincaré DGAs

2.1. Cochain complexes with pairing. Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of integers, $\mathbb{Q}$ the field of rational numbers, and $\mathbb{R}$ the field of real numbers.

We will work in the category of $\mathbb{Z}$-graded vector spaces $A = \bigoplus_{i \in \mathbb{Z}} A^i$ over $\mathbb{R}$ with morphisms

$\text{Hom}(A_1, A_2) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}^i(A_1, A_2),$

$\text{Hom}^i(A_1, A_2) := \{ f : A_1 \to A_2 \text{ linear homogenous, } \deg f = i \}.$

Here we say that $f : A_1 \to A_2$ is homogenous of degree $\deg f \in \mathbb{Z}$ if $f(A_1^i) \subset A_2^{i+\deg f}$ for all $i \in \mathbb{Z}$, and that $x \in A$ is homogenous of degree $\deg x \in \mathbb{Z}$ if $x \in A^{\deg x}$.

We say that $A$ is nonnegatively graded if $A = \bigoplus_{i \in \mathbb{N}_0} A^i$, and of finite type if $\dim A^i < \infty$ for all $i \in \mathbb{Z}$.
We point out the following facts which are easy to prove:

A pairing of degree \( n \in \mathbb{Z} \) on \( A \) is a bilinear form \( \langle \cdot, \cdot \rangle : A \times A \to \mathbb{R} \) which for all homogenous \( x, y \in A \) satisfies the degree condition:
\[
\langle x, y \rangle \neq 0 \implies \deg x + \deg y = n
\]
and graded symmetry
\[
\langle x, y \rangle = (-1)^{\deg x \deg y} \langle y, x \rangle.
\]
We write \( x \perp y \) if \( \langle x, y \rangle = 0 \) and say that \( x, y \) are orthogonal. The subspace of elements of \( A \) orthogonal to a given subspace \( B \subset A \) will be denoted by
\[
B^\perp := \{x \in A \mid x \perp B\}.
\]
We call a pairing \( \langle \cdot, \cdot \rangle : A \times A \to \mathbb{R} \) nondegenerate if the musical map
\[
b : A \to \text{Hom}(A, \mathbb{R})
\]
\[
x \mapsto x^\flat := \langle x, \cdot \rangle
\]
is injective, and perfect if it is an isomorphism. In that case we denote its inverse by
\[
\#: \text{Hom}(A, \mathbb{R}) \to A.
\]
Note that \( \flat \) and \( \# \) are linear homogenous maps of degrees \( \deg \flat = -n \) and \( \deg \# = n \).

We point out the following facts which are easy to prove:

(i) A pairing \( \langle \cdot, \cdot \rangle : A \times A \to \mathbb{R} \) is perfect if and only if it is nondegenerate and \( A \) is of finite type.

(ii) If \( A \) is nonegatively graded, then any nontrivial pairing \( \langle \cdot, \cdot \rangle : A \times A \to \mathbb{R} \) must have nonnegative degree \( n \in \mathbb{N}_0 \), and the existence of a nondegenerate pairing of degree \( n \) implies \( A = \bigoplus_{i=0}^\infty A^i \).

(iii) If \( A \) is nonegatively graded, then a pairing \( \langle \cdot, \cdot \rangle : A \times A \to \mathbb{R} \) is perfect if and only if it is nondegenerate and \( \dim A < \infty \).

**Convention.** From now on all pairings will be of degree \( n \in \mathbb{Z} \).

Consider now a cochain complex \((A, d)\), i.e., a graded vector space \( A \) together with a differential \( d \in \text{Hom}^1(A, A) \), \( d \circ d = 0 \). A pairing on \((A, d)\) is a pairing \( \langle \cdot, \cdot \rangle : A \times A \to \mathbb{R} \) satisfying for all homogenous \( x, y \in A \),
\[
\langle dx, y \rangle = (-1)^{1 + \deg x} \langle x, dy \rangle.
\]
Such a pairing descends naturally to a pairing \( \langle \cdot, \cdot \rangle_H : H(A) \times H(A) \to \mathbb{R} \) on cohomology \( (H(A) := H(A, d), d = 0) \). Following the terminology of \( \mathbb{R} \), we call a cochain complex with a perfect pairing \((A, d, \langle \cdot, \cdot \rangle)\) a cyclic cochain complex.

**Propagators and symmetric projections.** Let \((A, d, \langle \cdot, \cdot \rangle)\) be a cochain complex with pairing. A projection is a chain map \( \pi \in \text{Hom}^0(A, A) \) which satisfies \( \pi \circ \pi = \pi \). We say that \( P \in \text{Hom}^{-1}(A, A) \) is a homotopy operator if the map \( -d \circ P - P \circ d : A \to A \) is a projection. A homotopy operator \( P \) is called special if
\[
P \circ P = 0 \quad \text{and} \quad P \circ d \circ P = -P.
\]

Every homotopy operator \( P \) determines a projection
\[
\pi_P := 1 + d \circ P + P \circ d : A \to A
\]

\(^1\)Note that this is equivalent to \( \deg \langle \cdot, \cdot \rangle = -n \) in \( \text{Hom}(A^\oplus 2, \mathbb{R}) \).

\(^2\)In \( \mathbb{R} \) the authors use \( (-1)^{\deg x} \langle x, y \rangle \) instead of \( \langle x, y \rangle \).
which is a quasi-isomorphism, i.e., the induced map on cohomology \( H(\pi_P): H(A) \to H(A) \) is an isomorphism. Given a projection \( \pi: A \to A \) which is a quasi-isomorphism, we say that \( P \in \text{Hom}^{-1}(A, A) \) is a homotopy operator with respect to \( \pi \) if it is a homotopy operator and \( \pi_P = \pi \), so that
\[
\text{d} \circ P + P \circ \text{d} = \pi - I.
\]
Note: assuming (3) and (5), condition (4) on \( P \in \text{Hom}^{-1}(A, A) \) holds if and only if
\[
P \circ \pi = \pi \circ P = 0.
\]
We say that a homotopy operator \( P: A \to A \) is a propagator\(^3\) if it satisfies the symmetry property
\[
(Px, y) = (-1)^{\text{deg}x}(x, Py).
\]
The associated projection \( \pi_P: A \to A \) is then symmetric:
\[
\langle \pi_P x, y \rangle = \langle x, \pi_P y \rangle.
\]

**Lemma 2.1** ([2, Remark 2]). Any propagator \( P \) can be modified to a special propagator \( P_\lambda \) with respect to the same projection \( \pi \) by setting
\[
P_2 := (\pi - I) \circ P \circ (\pi - I), \quad P_3 := -P_2 \circ \text{d} \circ P_2.
\]
Note that a special propagator with respect to a given projection is not unique in general.

Given a subcomplex \( B \subset A \), we say that a projection \( \pi: A \to A \) is onto \( B \) if \( \text{im} \pi = B \) and identify \( \pi \) with the induced surjection \( \pi: A \to B \) in that case. One can show that a homotopy operator \( P: A \to A \) with respect to a projection \( \pi: A \to B \) exists if and only if \( \pi \) is a quasi-isomorphism (cf. the construction in the proof below). In the case with pairing, we have the following:

**Lemma 2.2.** Let \((A, d, \langle\cdot, \cdot\rangle)\) be a cyclic cochain complex and \( B \subset A \) a subcomplex. A propagator \( P \) with respect to a projection \( \pi: A \to B \) exists if and only if \( \pi \) is symmetric and a quasi-isomorphism.

**Proof.** Suppose that \( \pi: A \to B \) is a symmetric projection and a quasi-isomorphism. Then \( \ker \pi \subset A \) is an acyclic subcomplex, and thus there exists a subspace \( C \subset \ker \pi \) such that \( \ker \pi = C \oplus dC \). Consider the linear map \( P: A \to A \) defined by
\[
P(x) := \begin{cases} 
-x & \text{if } x = dy \text{ for some } y \in C, \\
0 & \text{for } x \in B \oplus C.
\end{cases}
\]
Then \( P \) is a homotopy operator with respect to \( \pi \). Due to the perfection of \( \langle\cdot, \cdot\rangle: A \times A \to \mathbb{R} \), the operator \( P: A \to A \) has a unique adjoint \( P^\dagger: A \to A \), i.e., a linear map such that \( \langle Px, y \rangle = (-1)^{\text{deg}x} \langle x, P^\dagger y \rangle \) for every \( x, y \in A \). Clearly, \( P^\dagger \) is also a homotopy operator with respect to \( \pi \), hence \( P_1 := \frac{1}{2}(P + P^\dagger) \) is a propagator with respect to \( \pi \).

We say that \( B \subset A \) is a quasi-isomorphic subcomplex if it is a subcomplex and the inclusion \( \iota: B \to A \) is a quasi-isomorphism. One can show that any quasi-isomorphic subcomplex \( B \subset A \) admits a projection \( \pi: A \to B \). Given a cochain complex with pairing \((A, d, \langle\cdot, \cdot\rangle)\), every subcomplex \( B \subset A \) such that \( A = B \oplus B^\perp \) admits a unique symmetric projection \( \pi_B: A \to B \), which is the projection with \( \ker \pi_B = B^\perp \). This occurs in particular when \( B \subset A \) is a cyclic subcomplex, i.e., the restriction \( \langle\cdot, \cdot\rangle|_B: B \times B \to \mathbb{R} \) is perfect, so that \((B, d, \langle\cdot, \cdot\rangle|_B)\) is a cyclic cochain complex. Lemmas 2.1 and 2.2 now imply the following:

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\(^3\)Thus a “propagator” is a partial inverse of the differential \( d \) and not of the Laplace operator.
Corollary 2.3. Let \((A, d, \langle \cdot, \cdot \rangle)\) be a cyclic cochain complex and \(B \subset A\) a quasi-isomorphic cyclic subcomplex. Then there exists a special propagator \(P: A \to A\) such that \(\im P = B\).

Harmonic subspaces and projections. Let \((A, d, \langle \cdot, \cdot \rangle)\) be a cochain complex with pairing. We say that \(H \subset A\) is a harmonic subspace if

\[
\ker d = H \oplus \im d.
\]

We call a projection \(\pi: H \subset A\) harmonic if it is symmetric and \(\im \pi \subset A\) is a harmonic subspace. Note that the last condition implies \(\im \pi = \pi \circ d = 0\). Given a harmonic subspace \(H \subset A\), the inclusion \(i: H \to A\) is a quasi-isomorphism, hence any harmonic projection \(\pi: A \to H\) is a quasi-isomorphism. If a harmonic projection onto \(H\) exists and \(\langle \cdot, \cdot \rangle|_H: H \times H \to \mathbb{R}\) is nondegenerate, then \(A = H \oplus H^\perp\) and every harmonic projection agrees with the unique symmetric projection \(\pi_H: A \to H\).

Lemma 2.4. Suppose that \((A, d, \langle \cdot, \cdot \rangle)\) is a cyclic cochain complex. Then every harmonic subspace \(H \subset A\) is a quasi-isomorphic cyclic subcomplex and the unique symmetric projection \(\pi_H: A \to H\) is harmonic.

Proof. Let \(\alpha: A \to A\) be a linear map of degree \(-1\) such that \(1 - \alpha \circ d\) is a projection onto \(\ker d\). Such \(\alpha\) can be constructed by choosing a complement \(C \subset A\) of \(\ker d\) and defining \(\alpha(dx) := x\) for \(x \in C\) and \(\alpha(x) := 0\) for \(x \in H \oplus C\). Given a homogenous element \(x \in A\), let \(x'\) denote the image of \((-1)^{\deg x}\langle x, \cdot \rangle \circ \alpha\) under the musical isomorphism \# : \(\text{Hom}(A, \mathbb{R}) \to A\). A straightforward computation shows that \(x \perp \ker d\) implies \(x = dx'\). Consequently, \(\langle \cdot, \cdot \rangle|_H: H \times H \to \mathbb{R}\) is nondegenerate, hence perfect since \(\langle \cdot, \cdot \rangle\) is perfect by assumption. The projection \(\pi_H\) is clearly harmonic because \(\im d = \ker H = \ker \pi_H\).

Lemma 2.3 implies that the cohomology \((H(A), d = 0, \langle \cdot, \cdot \rangle_H)\) of a cyclic cochain complex \((A, d, \langle \cdot, \cdot \rangle)\) equipped with the induced pairing is a cyclic cochain complex as well. Indeed, if \(H \subset A\) is a harmonic subspace, then \((H(A), d = 0, \langle \cdot, \cdot \rangle_H)\) can be canonically identified with \((H, d = 0, \langle \cdot, \cdot \rangle_H)\) via the quotient map

\[
\pi: H \subset \ker d \to H(A) = \ker d/\im d.
\]

Let \(P: A \to A\) be a propagator in a cochain complex with pairing \((A, d, \langle \cdot, \cdot \rangle)\). Then the associated projection \(\pi_P: A \to A\) satisfies

\[
\pi_P \circ d = d + d \circ P \circ d = d \circ \pi_P,
\]

and it follows that \(\pi_P\) is harmonic if and only if \(P\) satisfies condition \([14]\).

Hodge decompositions. Let \((A, d, \langle \cdot, \cdot \rangle)\) be a cochain complex with pairing. A Hodge decomposition of \((A, d, \langle \cdot, \cdot \rangle)\) is the data of a harmonic subspace \(H \subset A\) and a complement \(C\) of \(\ker d\) in \(A\) such that

\[
C \perp C \oplus H.
\]

Associated to each Hodge decomposition \(A = H \oplus \im d \oplus C\), there is a canonical harmonic projection \(\pi_{H,C}: A \to H\) defined by

\[
\pi_{H,C}(x) = \begin{cases} x \text{ for } x \in H, \\ 0 \text{ for } x \in \im d \oplus C, \end{cases}
\]

and a canonical special propagator \(P_{H,C}: A \to A\) with respect to \(\pi_{H,C}\) defined by

\[
P_{H,C}(x) = \begin{cases} -y \text{ if } x = dy \text{ for some } y \in C, \\ 0 \text{ for } x \in H \oplus C. \end{cases}
\]
Note that if in addition \( \langle \cdot, \cdot \rangle: A \times A \to \mathbb{R} \) is nondegenerate, then also the restriction
\[
\langle \cdot, \cdot \rangle|_{\mathcal{H}}: \mathcal{H} \times \mathcal{H} \to \mathbb{R}
\]
is nondegenerate and every harmonic projection \( \pi: A \to \mathcal{H} \) thus agrees with \( \pi_{\mathcal{H}, C} \). The proof of the following lemma is straightforward:

**Lemma 2.5.** Let \((A, d, \langle \cdot, \cdot \rangle)\) be a cochain complex with pairing. Then \( [9] \) defines a one-to-one correspondence between Hodge decompositions of \( A \) and special propagators \( P: A \to A \). Under this correspondence, the following holds:

\[
\mathcal{H} = \text{im} (1 + d \circ P + P \circ d), \quad C = \text{im} P, \quad \pi_{\mathcal{H}, C} = \pi_P.
\]

Following [18], we say that \((A, d, \langle \cdot, \cdot \rangle)\) is of Hodge type if it admits a Hodge decomposition. If \((A, d, \langle \cdot, \cdot \rangle)\) is of Hodge type and the induced pairing on cohomology is perfect, then [18, Remark 2.6] shows that for every harmonic subspace \( \mathcal{H} \subset A \) there is a subspace \( C \subset A \) such that \( A = \mathcal{H} \oplus \text{im} d \oplus C \) is a Hodge decomposition. This together with Lemma 2.8 implies the following:

**Corollary 2.6.** A cochain complex with pairing \((A, d, \langle \cdot, \cdot \rangle)\) is of Hodge type if and only if it admits a special propagator. Suppose that this is the case and the induced pairing on cohomology is perfect. Let \( \mathcal{H} \subset A \) be any harmonic subspace. Then there exists a unique harmonic projection \( \pi_{\mathcal{H}}: A \to \mathcal{H} \) and a (possibly nonunique) special propagator \( P: A \to A \) with respect to \( \pi_{\mathcal{H}} \).

**Lemma 2.7** ([8, Lemma 11.1]). Any cyclic cochain complex is of Hodge type.

Let us now discuss functoriality. We say that a chain map \( f: A \to A' \) respects Hodge decompositions \( A = \mathcal{H} \oplus \text{im} d \oplus C \) and \( A' = \mathcal{H}' \oplus \text{im} d' \oplus C' \) if

\[
f(\mathcal{H}) \subset \mathcal{H}' \quad \text{and} \quad f(C) \subset C'.
\]

This implies in particular

\[
f \circ \pi_{\mathcal{H}, C} = \pi_{\mathcal{H}', C'} \circ f.
\]

If \( P \) and \( P' \) are the associated special propagators, then \( [10] \) is equivalent to

\[
f \circ P = P' \circ f.
\]

**Relative Hodge decompositions.** Let \((A, d, \langle \cdot, \cdot \rangle)\) be a cochain complex with pairing, and let \( B \subset A \) be a subcomplex. We say that a Hodge decomposition \( A = \mathcal{H} \oplus \text{im} d \oplus C \) is relative to \( B \) if

\[
B = (B \cap \mathcal{H}) \oplus (B \cap \text{im} d) \oplus (B \cap C)
\]
is a Hodge decomposition of \((B, d, \langle \cdot, \cdot \rangle|_B)\). The inclusion \( \iota: B \hookrightarrow A \) is then a chain map which respects the Hodge decompositions. Let \( P^A: A \to A \) and \( P^B: B \to B \) be the special propagators corresponding to the Hodge decompositions of \( A \) and \( B \), respectively, and suppose that there is a symmetric projection \( \pi: A \to B \) which is a quasi-isomorphism. Then a simple computation shows that

\[
P := (1 - \pi) \circ P^A: A \to A
\]
is a propagator with respect to \( \pi \). If \( \pi \) respects the Hodge decompositions, then \( P = (1 - \pi) \circ P^A = P^B \circ (1 - \pi) \) is special.

**Lemma 2.8.** Let \((A, d, \langle \cdot, \cdot \rangle)\) be a cyclic cochain complex and \( B \subset A \) a quasi-isomorphic cyclic subcomplex. Then there exists a Hodge decomposition of \( A \) relative to \( B \) such that the unique symmetric projection \( \pi_B: A \to B \) respects the Hodge decompositions.
Proof. The subcomplex $B$ and its orthogonal complement $B^\perp := B^\perp$, both equipped with the restrictions of $\langle \cdot, \cdot \rangle$, are cyclic cochain complexes and hence admit Hodge decompositions $B = H \oplus dC \oplus C$ and $B' = H' \oplus dC' \oplus C'$ by Lemma 2.8. Setting $H'' = H \oplus H'$ and $C'' = C \oplus C'$, we obtain a Hodge decomposition $A = H \oplus dC'' \oplus C''$. The rest is a straightforward check. □

Hodge star. Let $(A, d, \langle \cdot, \cdot \rangle)$ be a nonnegatively graded cochain complex with pairing of degree $n \in \mathbb{N}_0$. A Hodge star on $A$ is a linear map $\star : A \to A$ which can be written as $\star = \sum_{k=0}^{n} k^k$ for linear maps $\star^k : A^k \to A^{n-k}$ satisfying $\star^k \circ \star^k = (-1)^{k(n-k)} 1 : A^k \to A^k$ such that the following bilinear form is a positive definite inner product:

$$\langle \cdot, \cdot \rangle := \langle \cdot, \star \cdot \rangle : A \times A \to \mathbb{R}.$$ 

We define the codifferential $d^* := \sum_{k=0}^{n} (-1)^{1+n(k-1)} \star^{n-k+1} \circ \star^k : A \to A$ which satisfies $\langle dx, y \rangle = \langle x, d^* y \rangle$. The nondegeneracy of $\langle \cdot, \cdot \rangle$ then implies that $\ker d$ is the orthogonal subspace to $\im d^*$ and $\ker d^*$ is the orthogonal subspace to $\im d$ with respect to $\langle \cdot, \cdot \rangle$. We define $H := \ker d \cap \ker d^*$ and suppose that the following condition is satisfied:

$$(\text{proj}) \quad \text{There exist projections } A \to \ker d \text{ and } \ker d \to H \text{ which are symmetric with respect to } \langle \cdot, \cdot \rangle.$$ 

Then we obtain the decomposition

$$A = H \oplus \im d \oplus \im d^*$$

which is orthogonal with respect to $\langle \cdot, \cdot \rangle$, and it follows from the previous relations that (11) is also a Hodge decomposition. Let $P_* : A \to A$ be the special propagator and $\pi_* : A \to H$ the harmonic projection canonically associated to (11). Since $\langle \cdot, \cdot \rangle : A \times A \to \mathbb{R}$ is nondegenerate, $\pi_*$ can be equivalently characterized as the unique symmetric projection $\pi_* = \pi_0 : A \to H$. The special propagator $P_*$ can also be equivalently characterized as the unique homotopy operator $P : A \to A$ with respect to $\pi_H$ such that $\im P \subseteq \im d^*$.

Two important examples which admit a Hodge star such that the condition (proj) above holds arise when $\dim A < \infty$, or when $A = \Omega$ is the de Rham complex of an oriented closed manifold (see Section 3).

The following lemma follows by inspection of the construction of the Hodge star in the proof of [3, Lemma 11.1]:

Lemma 2.9. Let $(A, d, \langle \cdot, \cdot \rangle)$ be a nonnegatively graded cyclic cochain complex and $B \subseteq A$ a cyclic subcomplex. Then there exists a Hodge star on $A$ which restricts to a Hodge star on $B$.

In the situation of this lemma, the corresponding Hodge decomposition of $A$ is relative to the corresponding Hodge decomposition of $B$, which gives an alternative proof of Lemma 2.8.

Nondegenerate quotient. Let $(A, d, \langle \cdot, \cdot \rangle)$ be a cochain complex with pairing. We consider the degenerate subspace $A_{\text{deg}} := \{ a \in A \mid a \perp A \}$ and define the nondegenerate quotient

$$Q(A) := A/A_{\text{deg}}.$$
Since $A_{\text{deg}} \subset A$ is a subcomplex, the differential $d$ descends to a differential $d_{Q}$ on $Q := \mathbb{Q}(A)$. Moreover, the pairing $\langle \cdot, \cdot \rangle_{A} : A \times A \to \mathbb{R}$ descends to a nondegenerate pairing $\langle \cdot, \cdot \rangle_{Q} : Q \times Q \to \mathbb{R}$, and the quotient map $\pi_{Q} : A \to Q$ is a chain map preserving the pairings.

Lemma 2.10 (23, Lemma 3.3, 24, Prop. 6.1.17]). Let $(A, d, \langle \cdot, \cdot \rangle_{A})$ be a cochain complex with pairing, and let $\langle \cdot, \cdot \rangle_{H} : H(A) \times H(A) \to \mathbb{R}$ be the induced pairing on cohomology. Consider the nondegenerate quotient $Q := \mathbb{Q}(A)$. Then:

(a) If $A$ is of Hodge type and $\langle \cdot, \cdot \rangle_{H}$ is nondegenerate, then $A_{\text{deg}} \subset A$ is an acyclic subcomplex and $\pi_{Q} : A \to Q$ is a quasi-isomorphism.

(b) If $A_{\text{deg}} \subset A$ is acyclic and $Q$ is of finite type, then $A$ is of Hodge type.

(c) Each Hodge decomposition $A = \mathcal{H} \oplus \text{im } d \oplus C$ induces a Hodge decomposition

$$Q = \pi_{Q}(\mathcal{H}) \oplus \text{im } d_{Q} \oplus \pi_{Q}(C),$$

and $\pi_{Q} : A \to Q$ respects these Hodge decompositions.

2.2. Differential graded algebras with pairing. A differential graded algebra (DGA) $(A, d, \wedge)$ is a cochain complex $(A, d)$ equipped with an associative product $\wedge : A \times A \to A$ of degree 0 satisfying the Leibniz identity

$$d(x \wedge y) = dx \wedge y + (-1)^{\deg x} x \wedge dy.$$

A commutative DGA (CDGA) satisfies in addition the graded commutativity

$$x \wedge y = (-1)^{\deg x \deg y} y \wedge x.$$

A nonzero element $1 \in A^{0}$ in a DGA $(A, d, \wedge)$ is called a unit if $1 \wedge x = x \wedge 1 = x$ for all $x \in A$. A DGA which admits a unit (which is then uniquely determined) is called a unital DGA. We say that a unital DGA $A$ is connected if it is nonnegatively graded and $A^{0} = \mathbb{R} \cdot 1$. Given $k \in \mathbb{N}$, we say that $A$ is $k$-connected if it is connected and $A^{i} = 0$ for all $i \in \{1, \ldots, k\}$. A 1-connected DGA is also called simply connected.

A pairing on a DGA $(A, d, \wedge)$ is a pairing $\langle \cdot, \cdot \rangle : A \times A \to \mathbb{R}$ on the cochain complex $(A, d)$ satisfying in addition

$$\langle x \wedge y, z \rangle = \langle x, y \wedge z \rangle. \tag{12}$$

Note: using (2), condition (12) is equivalent to the cyclicity condition

$$\langle x \wedge y, z \rangle = (-1)^{\deg z}(\deg x + \deg y)(z \wedge x, y), \tag{13}$$

which is considered in [8] and later in Subsection 5.22 in the context of $A_{\infty}$ algebras. Also note that (2) is implied by (12) if $A$ is commutative and by (13) if $A$ is unital.

Following [8], we call a DGA with a perfect pairing $(A, d, \wedge, \langle \cdot, \cdot \rangle)$ a cyclic DGA.

Orientations. Following [32], an orientation of degree $n \in \mathbb{N}_{0}$ on a nonnegatively graded cochain complex $(A, d)$ is a linear map $o : A \to \mathbb{R}$ with $\deg o = -n$ such that $o \circ d = 0$ and the induced map on cohomology $H(o) : H(A) \to \mathbb{R}$ is nontrivial. The triple $(A, d, o)$ is then called an oriented cochain complex.

If we view the cohomology as the trivial cochain complex $(H(A), d = 0)$, then for every orientation $\tilde{o}$ on $H(A)$ there is an orientation $o$ on $A$ such that $\tilde{o} = H(o)$, and such orientation is unique if and only if $dA^{0} = 0$.

An orientation $o$ on a DGA $(A, d, \wedge)$ is defined as an orientation on the underlying cochain complex $(A, d)$. It induces a pairing $\langle \cdot, \cdot \rangle : A \times A \to \mathbb{R}$ of degree $n$ via

$$\langle x, y \rangle := o(x \wedge y). \tag{14}$$

4 in [8] the authors use $(-1)^{\deg x}(x, y)$ instead of $\langle x, y \rangle$. 

Conversely, a pairing $\langle \cdot, \cdot \rangle : A \times A \rightarrow \mathbb{R}$ of degree $n$ induces a chain map $o : A \rightarrow \mathbb{R}$ with $\deg o = -n$ provided that $A$ is unital (see [24, Lemma 3.1]).

**Small subalgebra.** Consider a DGA of Hodge type $(A, d, \wedge, \langle \cdot, \cdot \rangle)$, and let $P : A \rightarrow A$ be a special propagator with associated harmonic subspace $\mathcal{H}_P \coloneqq \text{im} \pi_P$. Following [18], we call the smallest dg-subalgebra $S \subset A$ such that $\mathcal{H}_P \subset S$ and $P(S) \subset S$ the **small subalgebra** of $A$ associated to $P$ and denote it by $S_P$.

**Lemma 2.11** ([23, Lemma 3.4], [24, Prop. 6.1.15]). Let $(A, d, \wedge, \langle \cdot, \cdot \rangle)$ be a DGA of Hodge type, $P : A \rightarrow A$ a special propagator with associated harmonic subspace $\mathcal{H}_P \subset A$, and $S_P$ the associated small subalgebra. Then:

(a) The vector space $S_P$ is generated by iterated applications of $P$ and $\wedge$ to tuples of homogenous elements $h_1, \ldots, h_i \in \mathcal{H}_P$, $i \in \mathbb{N}$.

(b) If $A$ is unital and the cohomology $H(A)$ is simply connected and of finite type, then so is $S_P$.

(c) The restriction $P|_{S_P} : S_P \rightarrow S_P$ is a special propagator in the DGA with pairing $(S_P, d, \wedge, \langle \cdot, \cdot \rangle|_{S_P})$ and induces the Hodge decomposition $S_P = \mathcal{H}_P \oplus dS_P \oplus P(S_P)$ which is relative to the Hodge decomposition of $A$ associated to $P$. In particular, the inclusion $\iota : S_P \hookrightarrow A$ respects the Hodge decompositions and is a quasi-isomorphism.

**Remark 2.12.** The following description of $S_P$ in terms of Kontsevich–Soibelman-like evaluations of trees from [24] is of interest in the context of this paper. Let $T^\text{bin}_i$ denote the set of isotopy classes of planar embeddings of rooted binary trees with $i \in \mathbb{N}$ leaves (for $i = 1$ we include the trivial tree with only one edge). There is a natural orientation of edges of $T \in T^\text{bin}_i$ towards the root and a natural numbering of leaves of $T$ in the counterclockwise direction from the root. A labeling $L$ of $T$ is an assignment of either $P$ or $1_l$ to each edge (interior and exterior) and of $\wedge$ to each interior vertex. We denote the set of all labelings of $T$ by $L(T)$. For $L \in L(T)$, we interpret the labeled tree $(T, L)$ as a composition rule for the operations $1_l$, $P$, $\wedge$ and obtain a linear map (see Figure 1)

$$ev_{T,L} : A^{|L|} \rightarrow A.$$ (15)

Then we have

$$S_P = \sum_{i \in \mathbb{N}} \sum_{T \in T^\text{bin}_i} \sum_{L \in L(T)} ev_{T,L}(\mathcal{H}_P).$$

![Figure 1. Kontsevich–Soibelman-like evaluation of a labeled planar rooted binary tree.](image-url)
2.3. Poincaré DGAs and differential Poincaré duality models. In this subsection, we will work in the category of nonnegatively graded unital CDGAs with orientations of degree $n \in \mathbb{N}_0$. Following [32] we define:

**Definition 2.13.** A differential Poincaré duality algebra (dPD algebra) $(A, d, \wedge, o)$ of degree $n \in \mathbb{N}_0$ is a finite dimensional nonnegatively graded unital CDGA equipped with an orientation $o: A \to \mathbb{R}$ of degree $n$ such that the pairing $\langle \cdot, \cdot \rangle: A \times A \to \mathbb{R}$ corresponding to $o$ via [14] is nondegenerate.

Note: this is the same notion as that of a nonnegatively graded unital cyclic CDGA. Following [18] we define:

**Definition 2.14.** A Poincaré DGA (PDGA) $(A, d, \wedge, o_H)$ of degree $n \in \mathbb{N}_0$ is a nonnegatively graded unital CDGA whose cohomology $H(A)$ is equipped with an orientation $o_H: H(A) \to \mathbb{R}$ making it into a Poincaré duality algebra, i.e., a dPD algebra with trivial differential. An oriented PDGA $(A, d, \wedge, o)$ is a PDGA $(A, d, \wedge, o_H)$ equipped with an orientation $o: A \to \mathbb{R}$ such that $H(o) = o_H$.

Let us emphasize that a dPD algebra involves a perfect pairing on chain level, whereas a PDGA involves a perfect pairing only on cohomology.

A morphism of PDGAs $(A, d, \wedge, o_H)$ and $(A', d', \wedge', o'_H)$ (PDGA morphism) is a morphism of unital DGAs $f: (A, d, \wedge) \to (A', d', \wedge')$ such that the induced map on cohomology $H(f): H(A) \to H(A')$ satisfies

$$o'_H \circ H(f) = o_H.$$

Recall that $f$ is called a quasi-isomorphism if in addition $H(f)$ is an isomorphism. A morphism of oriented PDGAs $(A, d, \wedge, o)$ and $(A', d', \wedge', o')$ is a PDGA morphism $f: (A, d, \wedge, o_H) \to (A', d', \wedge', o'_H)$ such that

$$o' \circ f = o.$$

A morphism of dPD algebras is defined in the same way. Let us emphasize that a morphism of dPD algebras is injective on chain level whereas a morphism of PDGAs is injective only on cohomology.

Suppose that $dA^n = 0$, $d'A'^n = 0$ so that orientations on cohomology $o_H, o'_H$ are in one to one correspondence with orientations $o, o'$ on chain level. Then any PDGA morphism $f: (A, d, \wedge, o_H) \to (A', d', \wedge', o'_H)$ is also a morphism of oriented PDGAs $f: (A, d, \wedge, o) \to (A', d', \wedge', o')$. In particular, a PDGA morphism of two dPD algebras is automatically a morphism of dPD algebras.

A Hodge decomposition of an oriented PDGA $(A, d, \wedge, o)$ is defined as a Hodge decomposition of the corresponding cochain complex with pairing $(A, d, \langle \cdot, \cdot \rangle)$. We then say that $(A, d, \wedge, o)$ is of Hodge type if $(A, d, \langle \cdot, \cdot \rangle)$ is of Hodge type, and that a PDGA $(A, d, \wedge, o_H)$ is of Hodge type if $o_H$ is induced by a chain level orientation $o: A \to \mathbb{R}$ such that $(A, d, \wedge, o)$ is of Hodge type.

**Differential Poincaré duality models.** We say that two PDGAs $A$ and $A'$ are weakly equivalent if there exist PDGAs $A_1, \ldots, A_k$ for some $k \in \mathbb{N}$ and a zigzag of PDGA quasi-isomorphisms

$$A \xleftarrow{A_1} \xrightarrow{A_2} \cdots \xleftarrow{A_{k-1}} A_k \xrightarrow{A} A'.$$

A weak equivalence of oriented PDGAs resp. dPD algebras can be defined similarly by replacing the term “PDGA” with the terms “oriented PDGA” resp. “dPD algebra”.

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A differential Poincaré duality model of a PDGA $A$ is a dPD algebra $\mathcal{M}$ which is weakly equivalent to $A$ as a PDGA. We have the following existence and uniqueness theorem. Part (a) essentially corresponds to \cite[Theorem 1.1]{23}, and part (b) to \cite[Theorem 7.1]{32} under the additional hypotheses $n \geq 7$, $H^3(A) = H^3(A') = 0$ and $A^2 = A'^2 = 0$. In its present formulation, the theorem is proved in \cite{23}.

**Theorem 2.15** (Existence and uniqueness of dPD models \cite{23,32}).

(a) A PDGA $A$ whose cohomology $H(A)$ is simply connected admits a simply connected differential Poincaré duality model $\mathcal{M}$ in the form

$$\begin{array}{c}
A_1 \\
A \\
Q(A_1) =: \mathcal{M},
\end{array}$$

where the PDGA $A_1$ is simply connected, of Hodge type, and of finite type, and $Q$ denotes the nondegenerate quotient.

(b) Let $A, A'$ be simply connected dPD algebras which are weakly equivalent as PDGAs, and suppose that $H^2(A) = H^2(A') = 0$. Then there exists a simply connected dPD algebra $A_1$ and quasi-isomorphisms of dPD algebras

$$\begin{array}{c}
A_1 \\
A \\
A'.
\end{array}$$

In particular, $A$ and $A'$ are weakly equivalent as (simply connected) dPD algebras.

The PDGA $A_1$ in Theorem 2.15(a) is constructed as an extension of the Sullivan minimal model of $A$. In the case that $A$ is of Hodge type, we have the following more explicit construction of a differential Poincaré duality model which will be crucial in the sequel.

**Proposition 2.16** (\cite[Section 5]{23}). Let $(A, d, \wedge, o)$ be an oriented PDGA of Hodge type such that its cohomology $H(A)$ is simply connected. Let $P : A \to A$ be a special propagator and $S_P$ the associated small subalgebra. Then the nondegenerate quotient $Q_P := Q(S_P)$ is a differential Poincaré duality model of $A$ via the canonical zigzag of PDGA quasi-isomorphisms

$$\begin{array}{c}
S_P \\
A \\
Q_P.
\end{array}$$

Here both $S_P$ and $Q_P$ are simply connected, of finite type, and equipped with orientations and Hodge decompositions which are respected by $\iota, \pi_Q$.

**Proof.** The inclusion $\iota : S_P \hookrightarrow A$ and the quotient map $\pi_Q : S_P \twoheadrightarrow Q_P$ are PDGA morphisms by construction; they are quasi-isomorphisms (and hence PDGA quasi-isomorphisms) by Lemma 2.11(b) and Lemma 2.10(a), respectively. According to Lemma 2.11(b), a Hodge decomposition of $A$ induces a Hodge decomposition of $S_P$, which in turn induces a Hodge decomposition of $Q_P$ by Lemma 2.10(c), such that $\iota$ and $\pi_Q$ respect the Hodge decompositions. By Lemma 2.11(c), $S_P$ is simply connected and of finite type, and the same is clearly true for its nondegenerate quotient $Q_P$. The pairing on $Q_P$ is nondegenerate by construction, hence $Q_P$ is a dPD algebra. \qed
Remark 2.17. Examples in [23] show:
(a) The zigzag in Theorem 2.15(a) cannot generally be replaced by a single PDGA quasi-isomorphism: there exists a PDGA \( A \) with simply connected cohomology which cannot be connected to any of its Poincaré duality models \( M \) by a single PDGA quasi-isomorphism (neither \( M \to A \) nor \( A \to M \)).
(b) Theorem 2.15(b) fails without the hypothesis \( H^2(A_1) = H^2(A_2) = 0 \): there exist two simply connected dPD algebras \( A \) and \( A' \) such that there is no simply connected dPD algebra \( A_1 \) which would fit in (16). Nevertheless, we expect that \( A \) and \( A' \) are still weakly equivalent as dPD algebras (cf. the Conjecture at the end of [32]).
(c) The differential Poincaré duality model in Proposition 2.16 depends on the choice of \( P \): there exists an oriented PDGA \( A \) of Hodge type with simply connected cohomology and two special propagators \( P, P' : A \to A \) such that \( QP \) and \( QP' \) are not isomorphic as graded vector spaces.

3. IBL\(_\infty\) ALGEBRAS

3.1. Basic definitions and properties. In this subsection, we recall from [8] the basic notions of IBL\(_\infty\) algebras (over \( \mathbb{R} \)).

**IBL\(_\infty\) algebras.** Let \( C = \bigoplus_{c \in \mathbb{Z}} C^c \) be a \( \mathbb{Z} \)-graded \( \mathbb{R} \)-vector space. We denote the shifted grading in \( C[1] \) on a homogeneous element \( c \) by
\[
|c| := \deg c - 1,
\]
and use it to induce a grading \(|·|\) on all Cartesian and tensor products of \( C[1] \) and on their morphisms. For every \( k \in \mathbb{N} \), let
\[
E_k C := C[1]^{\otimes k} / \sim
\]
be the quotient of the \( k \)-fold tensor product (over \( \mathbb{R} \)) of \( C[1] \) under the action of the symmetric group \( S_k \) generated by the transposition \( c_1 \otimes c_2 \mapsto (-1)^{|c_1||c_2|} c_2 \otimes c_1 \).

Following [8], an IBL\(_\infty\) structure of degree \( d \in \mathbb{Z} \) on \( C \) is a collection of linear maps \( p_{k,\ell,g} : E_k C \to E_{\ell} C \), \( k \geq 1 \), \( \ell \geq 1 \), \( g \geq 0 \) of degrees
\[
|p_{k,\ell,g}| = -2d(k + g - 1) - 1
\]
which for every \( (k, \ell, g) \) satisfy the quadratic relation\(^3\)
\[
\sum_{h=1}^{\infty} \sum_{k_1+k_2-h=k} \sum_{\ell_1+\ell_2-h=\ell} p_{k_2,\ell_2,g_2} \circ_h p_{k_1,\ell_1,g_1} = 0,
\]
where \( \circ_h \) denotes the composition of \( h \) outputs of \( p_{k_1,\ell_1,g_1} \) with \( h \) inputs of \( p_{k_2,\ell_2,g_2} \) in all possible ways. We call the tuple
\[
(C, p := \{p_{k,\ell,g}\}_{k,\ell \geq 1, g \geq 0})
\]
an IBL\(_\infty\) algebra of degree \( d \).

We think of \( p_{k,\ell,g} \) as being encoded by a connected compact oriented surface of signature \( (k, \ell, g) \), meaning that it has \( k \) incoming boundary components, \( \ell \) outgoing boundary components, and genus \( g \), and interpret (20) as a sum over pairs of surfaces of signatures \( (k_1, \ell_1, g_1) \), \( (k_2, \ell_2, g_2) \) such that if we glue them at \( h \) common

\(^3\)Here, as well as in formula (21) below, the sums have only finitely many nonzero terms.
boundaries, then we obtain a connected surface of signature \((k,\ell,g)\). In fact, the left hand side of (20) is a special case of the connected composition
\[
(p_{k^-,\ell^-,g^-}; \ldots; p_{k^-,\ell^-,g^-}) \circ_{\text{conn}} (p_{k^+,\ell^+,g^+}; \ldots; p_{k^+,\ell^+,g^+})
\]
defined in [24, Definition D.4.2] with fixed total genus \(g\) and restricted to \(E_k C \to E_\ell D\). We will denote this modification of \(\circ_{\text{conn}}\) by \(\circ_{k,\ell,g}\).

An IBL\(_\infty\) algebra is called a dIBL algebra if the only possibly nontrivial operations are \(p_{1,1,0}, p_{2,1,0}, p_{1,2,0}\), and an IBL algebra if the only possibly nontrivial operations are \(p_{2,1,0}, p_{1,2,0}\). The data of an IBL algebra is the same as the data of a graded involutive Lie bialgebra up to signs, which can be explained by choosing a natural convention for degree shifts (see [24, Proposition 3.2.5]).

The relation \((1,1,0)\) of (20) implies that \(p_{1,1,0}: C \to C\) squares to zero and the relations \((2,1,0)\) resp. \((1,2,0)\) that the operations \(p_{2,1,0}\) resp. \(p_{1,2,0}\) descend to homology \(H := H(C,p_{1,1,0})\) where they constitute an IBL algebra. The Jacobi identity on \(H\) follows from \((3,1,0)\), the coJacobi identity from \((1,3,0)\), Drinfeld compatibility from \((2,2,0)\), and the involutivity from \((1,1,1)\).

**IBL\(_\infty\) morphisms.** Let \((C, p) = \{p_{k,\ell,g}\}\) and \((D, q) = \{q_{k,\ell,g}\}\) be two IBL\(_\infty\) algebras of the same degree \(d\). An IBL\(_\infty\) morphism \(f := \{f_{k,\ell,g}\}: C \to D\) is a collection of linear maps
\[
f_{k,\ell,g}: E_k C \to E_\ell D, \quad k,\ell \geq 1, g \geq 0
\]
of degrees
\[
|f_{k,\ell,g}| = -2d(k + g - 1)
\]
which for every \((k,\ell,g)\) satisfy
\[
\sum_{r \geq 1} \frac{1}{r+1} p_{k^-,\ell^-,g^-} \circ_{k,\ell,g} (p_{k^+,\ell^+,g^+}; \ldots; p_{k^+,\ell^+,g^+}) = \sum_{r \geq 1} \frac{1}{r+1} (p_{k^-,\ell^-,g^-}; \ldots; p_{k^-,\ell^-,g^-}) \circ_{k,\ell,g} (p_{k^+,\ell^+,g^+}; \ldots; p_{k^+,\ell^+,g^+}).
\]

Here and in all composition formulas below \(\sum\) means a summation over all free indices in \(N\) and genus in \(N_0\). The relation \((1,1,0)\) of (21) implies that \(f_{1,1,0}: (C^+, p^+) \to (C^-, p^-)\) is a chain map, and the relations \((2,1,0)\) resp. \((1,2,0)\) imply that the induced map on homology \(H(f_{1,1,0}): H(C^+) \to H(C^-)\) preserves the bracket resp. co-bracket.

An IBL\(_\infty\) quasi-isomorphism is an IBL\(_\infty\) morphism \(f = \{f_{k,\ell,g}\}: (C^+, p^+) \to (C^-, p^-)\) such that \(f_{1,1,0}: (C^+, p_{1,1,0}) \to (C^-, p_{1,1,0})\) is a quasi-isomorphism. In [8], the notion of a homotopy of IBL\(_\infty\) morphisms is defined and the following facts are proved using obstruction theory:

- **(Homotopy inverse)** Every IBL\(_\infty\) quasi-isomorphism is an IBL\(_\infty\) homotopy equivalence, i.e., it has an IBL\(_\infty\) homotopy inverse.
- **(Homotopy transfer)** Every IBL\(_\infty\) algebra \((C, p)\) induces an IBL\(_\infty\) structure \(q = \{q_{k,\ell,g}\}\) on its homology \(H := H(C,p_{1,1,0})\) together with an IBL\(_\infty\) homotopy equivalence \(f: (C, p) \to (H, q)\).

**Twisting with a Maurer–Cartan element.** The deformation theory of IBL\(_\infty\) algebras is formulated in terms of filtered IBL\(_\infty\) algebras.\(^6\) Similarly to IBL\(_\infty\) algebras they are governed by the relations (20), (21), with the difference that \(C\) is now a filtered graded vector space and the maps \(p_{k,\ell,g}: E_k C \to E_\ell C; f_{k,\ell,g}: E_k C \to E_\ell D\) are defined on suitable completions. Besides the degree \(d \in \mathbb{Z}\), the notion of a filtered IBL\(_\infty\) algebra also involves a filtration degree \(\gamma \in \mathbb{N}_0\). All the IBL\(_\infty\) algebras

\(^6\)All our filtered IBL\(_\infty\) algebras will be strict in the sense of [8].
considered in this paper are based on the dual cyclic bar complex, which has a natural filtration by word-length with $\gamma = 2$. We refer to [8, Section 8] and [24, Appendix D] for more details.

A *Maurer–Cartan element* in a filtered IBL$_\infty$ algebra $(C, p = \{p_{k,\ell,g}\})$ of degree $d$ and filtration degree $\gamma$ is a collection $m = \{m_{\ell,g}\}$ of elements

$$m_{\ell,g} \in \hat{E}_\ell C, \quad \ell \geq 1, g \geq 0$$

of degrees

$$|m_{\ell,g}| = -2d(g - 1)$$

and filtration degrees

$$\|m_{\ell,g}\| \geq \gamma(2 - 2g - \ell),$$

where a strict inequality is assumed for $(1, 0)$, $(2, 0)$, which satisfy the relations

$$\sum_{r+s = \ell} \frac{1}{r!} p_{k-\ell-g-}^r m_{\ell,g}^+ (m_{\ell+1,g}^+, \ldots, m_{\ell+r,g+r}^+) = 0$$

for all $(\ell, g) \in \mathbb{N} \times \mathbb{N}_0$. Here we view $m_{\ell,g}$ as a map $R \to \hat{E}_\ell C, 1 \mapsto m_{\ell,g}$ while applying the composition. Given a Maurer–Cartan element $m = \{m_{\ell,g}\}$ in a filtered IBL$_\infty$ algebra $(C, p = \{p_{k,\ell,g}\})$, the *twist of $p$ with $m$* is the collection $p^m = \{p^m_{k,\ell,g}\}$ of operations $p^m_{k,\ell,g}: \hat{E}_k C \to \hat{E}_\ell C$ for $k, \ell \geq 1, g \geq 0$ defined by

$$p^m_{k,\ell,g} := \sum_{r+s = \ell} \frac{1}{r!} p_{k-\ell-g-}^r m_{\ell,g}^+ (m_{\ell+1,g}^+, \ldots, m_{\ell+r,g+r}^+).$$

Given another filtered IBL$_\infty$ algebra $(D, q = \{q_{k,\ell,g}\})$ and a morphism $f: (C, p) \to (D, q)$, the *pushforward of $m$ along $f$* is the collection $f_* m = \{(f_* m)_{\ell,g}\}$ of elements

$$(f_* m)_{\ell,g} \in \hat{E}_\ell D$$

for $\ell \geq 1, g \geq 0$ defined by

$$(f_* m)_{\ell,g} = \sum_{r+s = \ell} \frac{1}{r!} (f^r_{k,\ell,g} m_{\ell,g}^+ (m_{\ell+1,g}^+, \ldots, m_{\ell+r,g+r}^+)).$$

Finally, the *twist of $f$ with $m$* is the collection $f^m = \{f^m_{k,\ell,g}\}$ of maps $f^m_{k,\ell,g}: \hat{E}_k C \to \hat{E}_\ell D$ for $k, \ell \geq 1, g \geq 0$ defined by

$$f^m_{k,\ell,g} = \sum_{r+s = \ell} \frac{1}{r!} (f^r_{k,\ell,g} m_{\ell,g}^+ (m_{\ell+1,g}^+, \ldots, m_{\ell+r,g+r}^+)).$$

**Proposition 3.1** ([8, Propositions 9.3, 9.6]). In the situation above, we have:

(i) The twist $p^m$ defines an filtered IBL$_\infty$ structure on $C$.

(ii) The pushforward $f_* m$ defines a Maurer–Cartan element in $(D, q)$.

(iii) The twist $f^m$ defines a filtered IBL$_\infty$ morphism $f^m: (C, p^m) \to (D, q^{f_* m})$.

(iv) If $f$ is a homotopy equivalence of filtered IBL$_\infty$ algebras, then so is $f^m$.

In the situation above, we will refer to $p^m$ as the *twisted IBL$_\infty$ structure*, to $f^m$ as the *twisted IBL$_\infty$ morphism*, and to $f_* m$ as the pushforward Maurer–Cartan element.

### 3.2. The dlBL structure associated to a cyclic cochain complex

In this and the following subsections, we recall the algebraic constructions from [8] which, taken all together, associate to a cyclic DGA $A$ of degree $n$ a filtered IBL$_\infty$ structure of degree $n - 3$ on the degree shifted dual cyclic bar complex of the cohomology $(B^{\text{conn}}H(A))[2 - n]$, whose homology is isomorphic to the cyclic cohomology of $A$ and which is defined naturally up to IBL$_\infty$ homotopy equivalence. The trickiest part of these constructions are the signs, which we will not spell out but refer to [8].
The dual cyclic bar complex and the dIBL functor. Let $A$ be a $\mathbb{Z}$-graded $\mathbb{R}$-vector space, and let $|\cdot|$ be the shifted grading on $A[1]$ as in [13]. For every $k \in \mathbb{N}$, we define
\[
B^\text{cyc}_k A := A[1]^{\otimes k} / \sim
\]
as the quotient of the $k$-fold tensor product of $A$ over $\mathbb{R}$ under the restriction of the action of $S_k$ from [10] to the cyclic permutations $\mathbb{Z}_k \subset S_k$. We will write the equivalence class of $a_1 \otimes \cdots \otimes a_k$ in $B^\text{cyc}_k A$ as $a_1 \cdots a_k$. Following [8], we define the cyclic bar complex
\[
B^\text{cyc} A := \bigoplus_{k \in \mathbb{N}} B^\text{cyc}_k A.
\]
We define the dual cyclic bar complex
\[
B^\text{cyc*,} A := (B^\text{cyc}_k A)^{\ast},
\]
\[
B^\text{cyc*,} A := (B^\text{cyc} A)^{\ast} = \prod_{k \in \mathbb{N}} B^\text{cyc*,} k A,
\]
where the upper * denotes the graded dual with respect to the degrees in $A[1]$. This definition differs from the one in [8] where $B^\text{cyc*,} A$ was defined as a direct sum. Our $B^\text{cyc*,} A$ is already complete with respect to the filtration of $\prod_{k \in \mathbb{N}} B^\text{cyc*,} k A$ by the tensor degree $k$, which avoids unnecessary completions later.

We consider the reversed shifted grading on $B^\text{cyc*,} A$, i.e., for homogenous $\varphi \in B^\text{cyc*,} A$ and $x \in B^\text{cyc*,} A$, we have
\[
\varphi(x) \neq 0 \implies |x| = |\varphi|.
\]

**Proposition 3.2** ([8, Proposition 10.4]). Let $(A, d, \langle \cdot, \cdot \rangle)$ be a cyclic cochain complex of degree $n \in \mathbb{Z}$. The degree shifted dual cyclic bar complex $C := (B^\text{cyc*,} A)[2-n]$ carries a canonical filtered dIBL structure $\{p_{1,1,0} = d^\ast, p_{2,1,0}, p_{1,2,0}\}$ of degree $n-3$.

**Sketch of proof.** Let us describe the operations. The coboundary operator $d$ extends as a graded derivation to $B^\text{cyc} A$, still denoted by $d$, and $p_{1,1,0} := d^\ast$ is just its dual. To define the other two operations $p_{2,1,0}, p_{1,2,0}$, we pick a homogeneous basis $(e_i)$ of $A$. Let $(e^i)$ be the dual basis of $A$ with respect to the pairing $\langle \cdot, \cdot \rangle : A \times A \to \mathbb{R}$, i.e., we require
\[
\langle e_i, e_j \rangle = \delta_i^j.
\]
We set
\[
g_{ij} := \langle e_i, e_j \rangle, \quad g^{ij} := \langle e^i, e^j \rangle.
\]
Let $\varphi \in B^\text{cyc\ast}_{k_1+1} A$, $\psi \in B^\text{cyc\ast}_{k_2+1} A$, $k_1, k_2 \geq 0$, $k_1 + k_2 \geq 1$. We define $p_{2,1,0}(\varphi, \psi) \in B^\text{cyc\ast}_{k_1+k_2} A$ on elements $x_i \in A$ by
\[
p_{2,1,0}(\varphi, \psi)(x_1, \ldots, x_{k_1+k_2})
= \sum_{a,b} \sum_{c=1}^{k_1+k_2} \pm g^{ab} \varphi(e_a, x_c, \ldots, x_{c+k_1-1}) \psi(e_b, x_{c+k_1}, \ldots, x_{c+k_2}).
\]
See [8] for the appropriate signs. Next, let $\varphi \in B^\text{cyc\ast}_k A$, $k \geq 4$. We define
\[
p_{1,2,0}(\varphi) \in \bigoplus_{k_1+k_2=k-2} B^\text{cyc\ast}_{k_1} A \otimes B^\text{cyc\ast}_{k_2} A
\]
\footnote{Note that our cyclic bar complex does not include a $k = 0$ term.}
on elements \( x_i, y_j \in A \) by
\[
p_{1,2,0}(\varphi)(x_1 \ldots x_k \otimes y_1, \ldots, y_k) = \sum_{a,b} \sum_{c=1}^{k_1} \sum_{c'=1}^{k_2} \pm g^{ab}_c \varphi(e_a, x_c, \ldots, x_{c-1}, e_b, y_{c'}, \ldots, y_{c'-1}),
\]
again with suitable signs. It is easy to see that operations \( p_{2,1,0}, p_{1,2,0} \) do not depend on the choice of the basis \((e_i)\). It is straightforward, though tedious, to verify that together with \( p_{1,1,0} \) they satisfy the relations of a dIBL algebra. \( \square \)

Following \[24\], we will denote the dIBL algebra from Proposition \[3.2\] by
\[ \text{dIBL}(A) := ((B^{\text{cyc}} A) [2 - n], p := (p_{1,1,0}, p_{2,1,0}, p_{1,2,0})) \]

This reflects the fact that it is part of a natural covariant functor from the category of cyclic cochain complexes to the category of dIBL algebras \footnote{A morphism of dIBL algebras is an IBL∞ morphism with \( f_{k,\ell,g} = 0 \) for \((k,\ell,g) \neq (1,1,0)\).} defined on morphisms as follows. A morphism of cyclic cochain complexes \( f: B \to A \) is automatically injective (because it preserves the nondegenerate pairings), hence it can be identified with an inclusion \( \iota: B \to A \). One can show that the pullback \( \pi_B: B^{\text{cyc}} B \to B^{\text{cyc}} A \) along the unique symmetric projection \( \pi_B: A \to B \) induces a dIBL morphism \( \text{dIBL}(f) := \pi_B^*: \text{dIBL}(B) \to \text{dIBL}(A) \).

### 3.3. Weak functoriality of the dIBL construction

We continue the discussion of functoriality of the dIBL construction from the previous subsection and ask now about its contravariant properties. Given an inclusion of cyclic cochain complexes \( \iota: B \to A \), the pullback \( \iota^* \) may not preserve the bracket and cobracket and hence may not define a morphism of dIBL algebras \( \text{dIBL}(A) \to \text{dIBL}(B) \). The following theorem extends \( \iota^* \) to an IBL∞ homotopy equivalence provided that \( \iota \) is a quasi-isomorphism.

**Theorem 3.3** ([8, Theorem 11.3]). Let \((A, d, \langle , \cdot, \rangle)\) be a cyclic cochain complex and \( B \subseteq A \) a quasi-isomorphic cyclic subcomplex. Let \( P: A \to B \) be a propogator with respect to the unique symmetric projection \( \pi_B: A \to B \). Then there is an IBL∞ homotopy equivalence
\[ f^P = \{ f^P_{k,\ell,g} \}: \text{dIBL}(A) \to \text{dIBL}(B) \]
which extends the pullback \( f^P_{1,1,0} = \iota^*: B^{\text{cyc}} A \to B^{\text{cyc}} B \) such that the value \( f^P_{k,\ell,g}(\varphi)(\alpha) \in \mathbb{R} \) of \( f^P_{k,\ell,g} \) on the tensor products
\[ \varphi := \varphi^1 \otimes \cdots \otimes \varphi^k, \quad \text{where} \ \varphi^i \in B^{\text{cyc}} A_i, \quad \text{and} \]
\[ \alpha := \alpha^1_1 \cdots \alpha^1_{s_1} \otimes \cdots \otimes \alpha^k_1 \cdots \alpha^k_{s_k}, \quad \text{where} \ s_i \in \mathbb{N} \ \text{and} \ \alpha^b_j \in B, \]
can be written as a sum over isomorphism classes of ribbon graphs \( \Gamma \) with \( k \) interior vertices, \( \ell \) boundary components, and genus \( g \), of contributions naturally associated to \( \Gamma \) by labeling the interior vertices with \( \varphi^1, \ldots, \varphi^k \), the interior edges with \( P \), and the exterior vertices on the \( i \)-th boundary component with \( \alpha^1_1, \ldots, \alpha^1_{s_1} \) (see Figure 2).

We refer to \( f^P \) from the theorem above as to the natural IBL∞ homotopy equivalence \( \text{dIBL}(A) \to \text{dIBL}(B) \) associated to \( P \).

The construction of \( f^P \) extends similar constructions for \( A_\infty \) or \( L_\infty \) structures based on planar rooted trees (see [30], [20, Subsection 5.4.2]) by considering general ribbon graphs. Before sketching the construction, we summarize our conventions on ribbon graphs.
A ribbon graph $\Gamma$ is a finite connected graph with a cyclic ordering of the adjacent edges at every vertex. We denote by $d(v)$ the degree of a vertex $v$, i.e., the number of edges adjacent to $v$. We suppose that $\Gamma$ is equipped with a decomposition $C_0(\Gamma) = C_0^{\text{int}}(\Gamma) \sqcup C_0^{\text{ext}}(\Gamma)$ of the set of vertices into interior and exterior vertices, where an exterior vertex has degree 1 and an interior vertex has degree at least 1. This induces a decomposition $C_1(\Gamma) = C_1^{\text{int}}(\Gamma) \sqcup C_1^{\text{ext}}(\Gamma)$ of the set of edges into interior and exterior edges, where an edge is called exterior if and only if it contains an exterior vertex. We denote by $\Sigma^0(\Gamma)$ the compact oriented surface with boundary obtained by thickening $\Gamma$ such that $\Gamma \cap \partial \Sigma^0 = C_0^{\text{ext}}(\Gamma)$. We require that each boundary component has precisely one marked exterior vertex.

The signature of $\Gamma$ is the triple $(k, \ell, g)$, where $k = \# C_0^{\text{int}}(\Gamma)$, $\ell$ is the number of boundary components of $\Sigma^0$, and $g$ is the genus of $\Sigma^0$. Given $k, \ell \geq 1$ and $g \geq 0$, we denote by $RG_{k,\ell,g}$ the set of isomorphism classes of ribbon graphs of signature $(k, \ell, g)$ with the additional data specified above.

Sketch of the construction of $f^P$ from Theorem 3.3. Given $\Gamma \in RG_{k,\ell,g}$, we will define the contribution $f_P^{\Gamma}(\varphi)(\alpha) \in \mathbb{R}$ of $\Gamma$ to $f_P$ such that we can write

$$f_P^{\Gamma}(\varphi)(\alpha) = \sum_{\Gamma \in RG_{k,\ell,g}} (-1)^{r_\Gamma} C_\Gamma f_P^{\Gamma}(\varphi)(\alpha),$$

where $C_\Gamma \in \mathbb{Q}^+$ is a combinatorial coefficient and $r_\Gamma \in \mathbb{Z}$ a sign exponent. The values of $C_\Gamma$ and $r_\Gamma$ can be deduced from [8] and we will not discuss them further. By an ordering $O$ on $\Gamma$ we mean an ordering of the interior vertices and an ordering of the boundary components of $\Sigma^0$. Having $O$ allows us to label the interior vertices by $\varphi^1, \ldots, \varphi^k$ and the exterior vertices on the $b$-th boundary component by $\alpha^1_b, \ldots, \alpha^s_b$, where $j = 1$ corresponds to the marked exterior vertex and $j \mapsto j+1$ goes in the positive direction of the boundary. Here we require implicitly that $s_b$ equals the number of exterior vertices at the $b$-th boundary component for every $b \in \{1, \ldots, \ell\}$, and define the contribution of the pair $(\Gamma, O)$ to be 0 otherwise.

Using the notation of [8], we define a flag of $\Gamma$ as a pair $(v, t)$ consisting of a vertex $v$ and an edge $t$ such that $v \in t$ (where an edge starting and ending at the same vertex gives rise to two flags). We have the decomposition

$$\text{Flag}(\Gamma) = \text{Flag}^{\text{int}}(\Gamma) \sqcup \text{Flag}^{\text{ext}}(\Gamma)$$

of the set of flags into interior and exterior flags, where a flag $(v, t)$ is called exterior if and only if $v$ is an exterior vertex. As in the proof of Proposition 3.2, we pick a
basis \((e_i)_{i \in I}\) of \(A\), denote by \((e^i)_{i \in I}\) the dual basis of \(A\) with respect to \(\langle \cdot, \cdot \rangle\), and set
\[
P_{ij} := \langle Pe_i, e_j \rangle, \quad P^{ij} := \langle Pe^i, e^j \rangle.
\]
Consider the set of maps \(\mathcal{J} := \text{Map}(\text{Flag}_{\text{int}}(\Gamma), I)\), where \(I\) is the index set of the basis. Given \(i \in \mathcal{J}\) and \((v, t) \in \text{Flag}(\Gamma)\), we define
\[
\epsilon_i(v, t) := \begin{cases} 
\epsilon_i(v, t) & \text{if } (v, t) \in \text{Flag}_{\text{int}}(\Gamma), \\
\alpha_i^b & \text{if } (v, t) \in \text{Flag}_{\text{ext}}(\Gamma) \text{ and } v \text{ is labeled by } \alpha_i^b.
\end{cases}
\]
Notice that \((v, t)\) in the subscript of \(\epsilon\) for \((v, t) \in \text{Flag}_{\text{ext}}(\Gamma)\) is not defined by itself. In order to write down a linear expression, we make the following additional choices: for every \(v \in C^n_{\text{int}}(\Gamma)\), we choose a bijection of the set of adjacent flags of \(v\) and the whole expression does not depend on the additional choices of the bijections provided that \(P\) and \(\varphi\) have the required symmetries. The contribution \(\sum_{i \in \mathcal{J}} \prod_{t \in C^n_{\text{int}}(\Gamma)} \varphi^{n}(\epsilon_i(v, 1), \ldots, \epsilon_i(v, d(v)))\)

\[
\sum_{i \in \mathcal{J}} \prod_{t \in C^n_{\text{int}}(\Gamma)} \varphi^{n}(\epsilon_i(v, 1), \ldots, \epsilon_i(v, d(v)))
\]
where the sign is obtained from a natural convention introduced in [8] such that the whole expression does not depend on the additional choices of the bijections provided that \(P\) and \(\varphi\) have the required symmetries. The contribution \(\sum_{i \in \mathcal{J}} \prod_{t \in C^n_{\text{int}}(\Gamma)} \varphi^{n}(\epsilon_i(v, 1), \ldots, \epsilon_i(v, d(v)))\)

3.4. The twisted dIBL structure associated to a cyclic DGA. Consider a nonnegatively graded cyclic DGA \((A, d, \wedge, \langle \cdot, \cdot \rangle)\) of degree \(n \in \mathbb{N}_0\) and the filtered dIBL algebra \(\text{dIBL}(A) = ((B^{2n+1} A)[2 - n], \mathfrak{p} = \{p_{1,1,0}, p_{2,1,0}, p_{1,2,0}\})\) of degree \(n - 3\) associated to the underlying cyclic cochain complex \((A, d, \langle \cdot, \cdot \rangle)\). Following [8], we will use the multiplication \(\mathfrak{m}_2(x, y) := (-1)^{\deg x} x \wedge y\) to obtain a canonical Maurer–Cartan element in \(\text{dIBL}(A)\). We define the triple intersection product \(\mathfrak{m}_2 \in B_{3}^{2n+1} A\) by
\[
\mathfrak{m}_2(x_0 x_1 x_2) := (-1)^{n + \deg x_1} (x_0 \wedge x_1, x_2).
\]

Proposition 3.4 ([8], Proposition 12.5]). Let \((A, d, \wedge, \langle \cdot, \cdot \rangle)\) be a nonnegatively graded cyclic DGA of degree \(n \in \mathbb{N}_0\). Then
\[
\mathfrak{m}_{\ell, g} := \begin{cases} 
\mathfrak{m}_2 & \text{if } (\ell, g) = (1, 0), \\
0 & \text{otherwise},
\end{cases}
\]
defines a Maurer–Cartan element \(\mathfrak{m} = \{\mathfrak{m}_{\ell, g}\}\) in \(\text{dIBL}(A)\). The twisted operations
\[
p^{\mathfrak{m}} = \{p^{\mathfrak{m}}_{1,1,0} = d^* + p_{2,1,0}(\mathfrak{m}_{1,0,\cdot}), p_{2,1,0}, p_{1,2,0}\}
\]
define a dIBL structure on \((B^{2n+1} A)[2 - n]\).

We call \(\mathfrak{m}\) in Proposition 3.4 the canonical Maurer Cartan element in \(\text{dIBL}(A)\) and denote it by \(\mathfrak{m}_{\text{can}}\).

Generally, for a Maurer–Cartan element \(\mathfrak{m}\) in \(\text{dIBL}(A)\), we denote the corresponding twisted IBL\(_\infty\) algebra by
\[
\text{dIBL}^{\mathfrak{m}}(A) := ((B^{2n+1} A)[2 - n], \mathfrak{p}^{\mathfrak{m}}).
\]
In the case of \( m = m_{A}^{\text{can}} \), we call it the canonical twisted dIBL algebra. It is straightforward to show that \( \mathfrak{p}_{2,1,0}(\langle m_{A}^{\text{can}} \rangle_{1,0}, \cdot) = b^{*} \) is the dual of the Hochschild differential \( b : B^{\text{cy}}A \rightarrow B^{\text{cy}}A \) for the algebra \( (A, \Lambda) \) which is given by

\[
b(x_1 \cdots x_k) = \sum_{i=1}^{k} (-1)^{|x_i| + \cdots + |x_i|} x_i \cdots x_{i-1}(x_i \Lambda x_{i+1})x_{i+2} \cdots x_k + (-1)^{|x_i|(|x_1| + \cdots + |x_{i-1}|) + 1} x_{i+1}(x_i \Lambda x_{i+1})x_{i+2} \cdots x_k.
\]

Therefore, the homology \( H(B^{\text{cy}}A, \mathfrak{p}_{m_{A}^{\text{can}}}) = d^{*} + b^{*} \) is a version of cyclic cohomology of the DGA \((A, d, \Lambda)\). We refer to [11] for a comparison of eight versions of cyclic cohomology (see also [24, Section 3.3]).

Note that the twist of a general filtered dIBL structure \( \mathfrak{p} = \{\mathfrak{p}_{1,1,0}, \mathfrak{p}_{2,1,0}, \mathfrak{p}_{1,2,0}\} \) with a general Maurer–Cartan element \( m = \{m_{\ell,g}\} \) has by [22] the form

\[
\mathfrak{p}^{m} = \begin{cases}
\mathfrak{p}_{1,1,0}^{m} = \mathfrak{p}_{1,1,0} + \mathfrak{p}_{2,1,0} \circ_{1} m_{1,0}, \\
\mathfrak{p}_{2,1,0}^{m} = \mathfrak{p}_{2,1,0}, \\
\mathfrak{p}_{1,2,0}^{m} = \mathfrak{p}_{1,2,0} + \mathfrak{p}_{2,1,0} \circ_{1} m_{2,0}, \\
\mathfrak{p}_{1,\ell,g}^{m} = \mathfrak{p}_{2,1,0} \circ_{1} m_{\ell,g} \text{ for } (\ell, g) \in \mathbb{N} \times \mathbb{N} \setminus \{(1, 0), (2, 0)\}
\end{cases}
\]

Consider now the cohomology \( H := H(A, d = 0, \langle \cdot, \cdot \rangle_H) \) as a cyclic cochain complex and write the associated canonical dIBL algebra as

\[
dIBL(H) = \left( (B^{\text{cy}}H)[2-n], q = \{q_{1,1,0} = 0, q_{2,1,0}, q_{1,2,0}\} \right).
\]

Let \( P : A \rightarrow A \) be a special propagator and \( \mathcal{H} \subset A \) the associated harmonic subspace. We identify the cyclic cochain complexes \( (H, d = 0, \langle \cdot, \cdot \rangle_H) \) and \( (H, d = 0, \langle \cdot, \cdot \rangle_{H}) \) via the quotient map \( [8] \). Proposition 3.2 then associates to \( P \) an IBL\( \infty \) homotopy equivalence \( \mathcal{I}^{\ell}: \text{dIBL}(A) \rightarrow \text{dIBL}(H) \), and Proposition 5.1 asserts that the pushforward

\[
\mathcal{I}^{\ell} \mathfrak{m}_{A}^{\text{can}} = \{(\mathcal{I}^{\ell} \mathfrak{m}_{A}^{\text{can}})_{\ell,g}\}
\]

is a Maurer–Cartan element in \( \text{dIBL}(H) \). We call \( \mathcal{I}^{\ell} \mathfrak{m}_{A}^{\text{can}} \) the pushforward Maurer–Cartan element in \( \text{dIBL}(H) \) associated to \( P \).

The IBL\( \infty \) algebras \( \text{dIBL}^{\ell} \mathfrak{m}_{A}^{\text{can}}(H) \) and \( \text{dIBL}^{\ell'} \mathfrak{m}_{A}^{\text{can}}(H) \) for different special propagators \( P \) and \( P' \) are IBL\( \infty \) homotopy equivalent because each of them is IBL\( \infty \) homotopy equivalent to \( \text{dIBL}^{m_{A}^{\text{can}}}(A) \) via the twisted morphisms

\[
(\mathcal{I}^{P})^{m_{A}^{\text{can}}}: \text{dIBL}^{m_{A}^{\text{can}}}(A) \rightarrow \text{dIBL}^{\ell} \mathfrak{m}_{A}^{\text{can}}(H).
\]

In fact, a stronger assertion holds: the Maurer–Cartan elements \( \mathcal{I}^{\ell} \mathfrak{m}_{A}^{\text{can}} \) and \( \mathcal{I}^{\ell'} \mathfrak{m}_{A}^{\text{can}} \) are gauge equivalent in the sense of [8, Section 9]. See [10] for a proof in the analytic case (to be introduced in Section 4), which can be adapted to work here as well.

The preceding discussion is summarized in the following theorem.

**Theorem 3.5** ([8, Theorem 12.10]). Let \( (A, d, \Lambda, \langle \cdot, \cdot \rangle) \) be a nonnegatively graded cyclic DGA of degree \( n \in \mathbb{N}_{0} \) and let \( H = H(A, d) \) be its cohomology. Then there is a natural IBL\( \infty \) structure on \( (B^{\text{cy}}H)[2-n] \) which is IBL\( \infty \) homotopy equivalent to the canonical twisted dIBL structure on \( (B^{\text{cy}}A)[2-n] \). In particular, the homology of \( B^{\text{cy}}H \) is isomorphic to the cyclic cohomology of \( (A, d, \Lambda) \).

---

9The twisted cobrackets alone form a version of a quantum co-L\( \infty \) algebra. We could thus study the structures in this paper up to homotopy of quantum co-L\( \infty \) algebras instead.

10A Maurer–Cartan element in \( \text{dIBL}(H) \) is equivalent to a quantum A\( \infty \) algebra on \( H \) (via a BV formalism due to the second author), and gauge equivalence corresponds to homotopy equivalence.
\[ \sum_{i_1, \ldots, i_6} \pm P^{i_1 i_2} P^{i_3 i_4} P^{i_5 i_6} m_1^+(\alpha_1^1 e_{i_6} e_{i_1}) m_2^+(\alpha_1^2 e_{i_3} e_{i_2}) m_2^+(\alpha_2^1 e_{i_4} e_{i_5}) \]

**Figure 3.** The contribution to \((f^* m_A^{can})_{L,g}(\alpha_1^1 \alpha_2^1 \alpha_1^2 \alpha_2^2)\) of a labeled ribbon graph immersed in the plane so that the cyclic order at interior vertices agrees with the counterclockwise orientation.

Inserting (26) and (27) in (23) leads to an explicit description of the pushforward Maurer–Cartan element \((f^* m_A^{can})_{L,g}\) in terms of trivalent ribbon graphs

\[ \text{RG}^3_{k,l,g} := \left\{ \Gamma \in \text{RG}_{k,l,g} \mid d(v) = 3 \text{ for all } v \in C^0_{\text{int}}(\Gamma) \right\}. \]

First of all, given \(\Gamma \in \text{RG}^3_{k,l,g}\), we denote by \(e := \# C^1_{\text{int}}(\Gamma)\) and \(s := \# C^0_{\text{ext}}(\Gamma)\) the numbers of interior edges and exterior vertices, respectively. Evaluating the Euler characteristic \(\chi(\Sigma_\Gamma)\) of the associated surface \(\Sigma_\Gamma\) in two ways gives then

\[ 2 + 2g - \ell = \chi(\Sigma_\Gamma) = k - e. \]

Counting flags of \(\Gamma\) in two ways, using trivalency, gives

\[ 2e + s = 3k. \]

Eliminating \(e\) from these equations yields

\[ k = 2(2g + \ell - 2) + s. \]

**Corollary 3.6.** The value of \((f^* m_A^{can})_{L,g}\) on the tensor product

\[ \alpha := \alpha_1^1 \cdots \alpha_s^1 \otimes \cdots \otimes \alpha_1^l \cdots \alpha_s^l, \quad \text{where } \alpha_j^b \in H, s_b \in \mathbb{N}, \]

can be written as

\[ (f^* m_A^{can})_{L,g}(\alpha) = \sum_{\Gamma \in \text{RG}^3_{k,l,g}} (-1)^{v_r} C_\Gamma(f^* m_A^{can})_\Gamma(\alpha), \]

where \(k\) is determined by (30) with \(s = \sum_{i=1}^l s_i\) and the numbers \((f^* m_A^{can})_\Gamma(\alpha)\) are defined similarly as in the construction from Theorem 3.3 using the following assignments (see Figure 3):

- to the \(j\)-th exterior vertex on the \(b\)-th boundary component we assign \(\alpha_j^b\);
- to each interior vertex we assign the triple intersection product \(m_2^+\);
- to each interior edge we assign the element

\[ P = \sum_{i,j} \pm P^{i j} e_i \otimes e_j \]

dual to the map \(x \otimes y \mapsto \langle Px, y \rangle\) (the Schwarz kernel of \(P\)).
4.1. Algebraic vanishing results. We are interested in conditions which imply the vanishing of the number \((f^P_m \mathcal{m}_A^\text{an})_R(\alpha)\) \(\in\mathbb{R}\) associated in Corollary 3.6 to a trivalent ribbon graph \(\Gamma \in RG_{k,\ell,g}^3\) and an element \(\alpha \in (B^\text{cyc}B)^{\otimes \ell}\).

We abbreviate the Euler characteristic of the graph \(\Gamma\) by
\[
\chi := \chi(\Gamma) = \chi(\Sigma\Gamma) \in \{1, 0, -1, -2, \ldots\}
\]
and define the number of loops
\[
\gamma := 1 - \chi \in \mathbb{N}_0.
\]

The following proposition is an algebraic analog of Proposition 5.4 in the analytic case which will be introduced in Section 5. Its proof is a straightforward adaption of the proof of [24, Propositions 4.4.1 and 4.4.2].

**Proposition 4.1.** Let \((A, d, \wedge, \langle \cdot, \cdot \rangle)\) be a nonnegatively graded unital cyclic DGA of degree \(n \in \mathbb{N}_0\). Let \(B \subset A\) be a quasi-isomorphic subcomplex, and let \(P: A \rightarrow A\) be a propagator with respect to a projection onto \(B\). Suppose that \(\Gamma \in RG_{k,\ell,g}^3\) is a trivalent ribbon graph which is not the Y-tree (see Figure 4) and \(\alpha \in (B^\text{cyc}B)^{\otimes \ell}\) a tensor product
\[
\alpha = \alpha_1^1 \cdots \alpha_{s_b}^1 \otimes \cdots \otimes \alpha_1^\ell \cdots \alpha_{s_b}^\ell, \quad \text{where } \alpha_i^b \in B, s_b \in \mathbb{N},
\]
such that
\[(f^P_m \mathcal{m}_A^\text{an})_R(\alpha) \neq 0.
\]
If \(P\) is special, then the following holds:

(i) If \(B^0 = \mathbb{R} \cdot 1\), then positivity of degrees holds:
\[
\deg(\alpha_i^b) > 0 \quad \text{for all } i \in \{1, \ldots, s_b\}, b \in \{1, \ldots, \ell\}.
\]

(ii) If \(B\) satisfies
\[B \wedge B \subset B,
\]
then \(\gamma \geq 1\), i.e., all trees vanish.

Positivity of degrees implies:

(iii) If \(\gamma = 1\) or \(n = 3\), then \(\deg(\alpha_i^b) = 1\) for all \(i \in \{1, \ldots, s_b\}, b \in \{1, \ldots, \ell\}.
\)

(iv) If \(n \geq 4\), then \(\gamma \leq 1\), i.e., all graphs with more than one loop vanish.

**Proof.** Recall that in order to write down \((f^P_m \mathcal{m}_A^\text{an})_R(\alpha)\), the construction in Theorem 3.3 associates basis elements \((e_i)\) to interior flags, elements \(\alpha_i^b\) to exterior flags,
and the triple intersection product $m_{2}^{+}$ to interior vertices. The crucial observation is that the degrees must add up to $\deg m_{2}^{+} = n$ around each interior vertex and $\deg \mathcal{F} = n - 1$ along each interior edge. This implies that the total degree

$$\deg \alpha = \sum_{b=1}^{\ell} \sum_{i=1}^{s_{b}} \deg(\alpha_{i}^{b})$$

satisfies

$$(31) \quad nk = (n - 1)e + \deg \alpha.$$ Expressing $(k, e)$ in terms of $(\chi, s)$ using (29) and (28) gives

$$\deg \alpha = (n - 3)\chi + s.$$

With this (i) immediately implies (iii) and (iv). Hence, only (i) and (ii) are left.

Following [24], we call an interior vertex an

- **A-vertex** if it has 1 adjacent interior edge;
- **B-vertex** if it has 2 adjacent interior edges;
- **C-vertex** if it has 3 adjacent interior edges (see Figure 5).

Since $\Gamma$ is not the $Y$-tree, each interior vertex is of type A, B or C.

(i) Since the expression for $(f_{\text{g}}^{3}m_{A}^{\text{can}})_{R}(\alpha)$ is homogenous in $\alpha_{i}^{b}$ and $B^{0} = R \cdot 1$, it suffices to show that none of the $\alpha_{i}^{b}$ equals 1.

For an A-vertex, let $e_{i_{1}}$ be the basis element associated to the adjacent interior flag, $e_{i_{2}}$, the basis element associated to the other flag on the corresponding interior edge, and $\alpha_{i}^{b}, \alpha_{j}^{b} \in B$ the elements associated to the adjacent exterior edges. Then $(f_{\text{g}}^{3}m_{A}^{\text{can}})_{R}(\alpha)$ involves the sum

$$\sum_{i_{1}} \langle \alpha_{i}^{b} \wedge \alpha_{j}^{b}, e_{i_{1}} \rangle P^{i_{1}i_{1}} = \langle \alpha_{i}^{b} \wedge \alpha_{j}^{b}, Pe^{i_{1}} \rangle = \pm (P(\alpha_{i}^{b} \wedge \alpha_{j}^{b}), e^{i_{1}}),$$

where we have used $\sum_{i_{1}} P^{i_{1}i_{1}} e_{i_{1}} = Pe^{i_{1}}$ and the symmetry of $P$. If $\alpha_{i}^{b}$ or $\alpha_{j}^{b}$ is equal to 1, then $\alpha_{i}^{b} \wedge \alpha_{j}^{b} \in B$, hence $P(\alpha_{i}^{b} \wedge \alpha_{j}^{b}) = 0$ because $P$ is special and thus $P(B) = 0$.

For a B-vertex, let $e_{i_{1}}, e_{i_{2}}$ be the basis elements associated to the adjacent interior flags, $e_{i_{3}}, e_{i_{4}}$, the basis elements associated to the other flags on the corresponding interior edges, and $\alpha_{i}^{b} \in B$ the element associated to the adjacent exterior edge. If $\alpha_{i}^{b} = 1$, then $(f_{\text{g}}^{3}m_{A}^{\text{can}})_{R}(\alpha)$ involves the sum

$$\sum_{i_{1}, i_{3}} \langle 1, e_{i_{1}} \wedge e_{i_{3}} \rangle P^{i_{1}i_{1}} P^{i_{3}i_{3}} = \langle 1, Pe^{i_{1}} \wedge Pe^{i_{3}} \rangle$$

$$= \pm (Pe^{i_{1}}, Pe^{i_{3}})$$

$$= \pm (P(\alpha_{i}^{b}), e^{i_{1}}),$$

which vanishes again because $P$ is special and thus $P \circ P = 0$.

(ii) If $B$ is a sub-DGA, then $\alpha_{i}^{b}, \alpha_{j}^{b} \in B$ implies $\alpha_{i}^{b} \wedge \alpha_{j}^{b} \in B$, so the proof of (i) shows that $\Gamma$ cannot have an A-vertex. This excludes each tree except the $Y$-tree, which is excluded by hypothesis. \hfill \Box

Notice that if $n \neq 3$, then $\alpha$ determines the signature $(k, \chi)$ of any $\Gamma \in RG_{k,e,g}^{3}$ whose contribution to $(f_{\text{g}}^{3}m_{A}^{\text{can}})_{R}(\alpha)$ might be nonzero as follows:

$$\begin{pmatrix} \deg \alpha \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} n - 1 \\ 2 \end{pmatrix} \begin{pmatrix} k \\ \chi \end{pmatrix} \begin{cases} \frac{k}{\chi} = \frac{1}{3 - n} & (\text{if } n \neq 3) \\ \frac{1}{n - 1} & (\text{if } n = 3) \end{cases} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} \deg \alpha \\ s \end{pmatrix}.$$

The conditions on $(k, \gamma)$ obtained by restricting to pairs $(\alpha, s)$ satisfying the bounds $s \leq \deg \alpha \leq ns$ are then equivalent to (iii) and (iv).
We remark that the formula for \( (\mathring{f}^P m^\text{can}_A)_{\Gamma}(\alpha) \in \mathbb{R} \) makes sense also for graphs with \( 0 \leq s < \ell \), i.e., when there is a boundary component with no exterior vertex, and the vanishing results still hold. However, such graphs do not naturally appear in our theory.

**Remark 4.2.** We summarize some facts from [24] (originally in the analytic case) about the low-degree cases in Proposition 4.1 immediately imply

\[(n = 0) \text{ We must have } P = 0, \text{ so all graphs vanish trivially.} \]

\[(n = 1) \text{ If } B^0 = \mathbb{R} \cdot 1 \text{ and } P \text{ is special, then every } \alpha^k \text{ has degree } 1 \text{ by (i). We then have } k = s \text{ by } (31) \text{ and } k = e \text{ by } (29). \text{ If } v \in B^1 \text{ is the element dual to 1, then } B = \text{span}_\mathbb{R}\{1, v\}, \text{ and hence there is no A-vertex by the proof of (ii). We deduce that } (\mathring{f}^P m^\text{can}_A)_{\Gamma}(\alpha) \text{ for } \Gamma \in RG^3_{k,\ell,g} \text{ does not necessarily vanish only if the underlying graph is the circular graph } C_s, \text{ i.e., the graph with } s \text{ interior vertices, } 1 \text{ loop, and no A-vertex, and } \alpha \text{ is a product of } s \text{ copies of } v \text{ (or its nonzero multiples). This value does not depend on the ribbon structure, can be computed explicitly, and in the de Rham case is related to Bernoulli numbers.} \]

\[(n = 2) \text{ Suppose that } B^0 = \mathbb{R} \cdot 1, B^1 = 0 \text{ and } P \text{ is special, so that } B = \text{span}_\mathbb{R}\{1, v\}, \text{ where } v \in B^2 \text{ is the element dual to 1. One can show that the only case when } (\mathring{f}^P m^\text{can}_A)_{\Gamma}(\alpha) \text{ for } \Gamma \in RG^3_{k,\ell,g} \text{ does not necessarily vanish is when the characteristic of the underlying graph satisfies } (k, \gamma) = (3s, s + 1) \text{ and } \alpha \text{ is the product of } s \text{ copies of } v \text{ (or its nonzero multiples). We were not able to prove vanishing in general except for small } s. \]

Proposition 4.1 and Remark 4.2 immediately imply

**Corollary 4.3.** Let \((A, d, \wedge, \langle \cdot, \cdot \rangle)\) be a nonnegatively graded unital cyclic DGA of degree \( n \in \mathbb{N}_0 \setminus \{2\} \) whose cohomology \( H := H(A, d) \) is simply connected. Let \( P: A \to A \) be a special propagator with associated harmonic subspace \( H \subset A, \) and let \( \mathring{f}^P: \text{dIBL}(A) \to \text{dIBL}(H) \) be the associated IBL\(_\infty\) homotopy. Then the only contributions to the pushforward Maurer–Cartan element \( \mathring{f}^P m^\text{can}_A \) come from trees, so that

\[(\mathring{f}^P m^\text{can}_A)_{\ell,g} = 0 \text{ for all } (\ell, g) \neq (1,0). \]

Consider \( H \) as a cyclic DGA \((H, d = 0, \wedge, \langle \cdot, \cdot \rangle_H)\), and let \( m^\text{can}_H \) be the canonical Maurer–Cartan element in \( \text{dIBL}(H) \). If in addition \( H \wedge H \subset H \), then the only contribution comes from the \( Y \)-tree, so that

\[\mathring{f}^P m^\text{can}_A = m^\text{can}_H.\]

**4.2. Weak functoriality of the twisted dIBL construction.** We have the following analog of Theorem 3.3 for the twisted dIBL algebra in the simply connected case:

**Proposition 4.4.** Let \((A, d, \wedge, \langle \cdot, \cdot \rangle)\) be a simply connected cyclic DGA of degree \( n \in \mathbb{N}_0, \) and let \( B \subset A \) be a quasi-isomorphic cyclic sub-DGA. Let \( P: A \to A \) be a special propagator with respect to the unique symmetric projection \( \pi_B: A \twoheadrightarrow B, \) and let \( \mathring{f}^P: \text{dIBL}(A) \to \text{dIBL}(B) \) be the associated IBL\(_\infty\) homotopy equivalence. Then the canonical Maurer–Cartan elements \( m^\text{can}_A \) in \( \text{dIBL}(A) \) and \( m^\text{can}_B \) in \( \text{dIBL}(B) \) satisfy

\[\mathring{f}^P m^\text{can}_A = m^\text{can}_B.\]

In particular, the twisted morphism \( (\mathring{f}^P) m^\text{can}_A \) is an IBL\(_\infty\) homotopy equivalence

\[(\mathring{f}^P) m^\text{can}_A: \text{dIBL}(m^\text{can}_A(A) \cong \text{dIBL}(m^\text{can}_B(B)).\]
Proof. Equation (32) for \( n \geq 3 \) follows from Proposition 4.1 using that \( P \) is special and \( B \) simply connected. Since \( A \) is simply connected, the case \( n = 1 \) is impossible, and for \( n = 2 \) we must have \( P = 0 \) for degree reasons. \( \square \)

Let \( f : B \to A \) be a morphism of cyclic DGAs. Consider the associated morphism of dIBL algebras \( \text{dIBL}(f) = \pi^* \text{dIBL}(B) \to \text{dIBL}(A) \) at the end of §3.2. It will in general not define a morphism \( \text{dIBL}^{m_{\infty}}(B) \to \text{dIBL}^{m_{\infty}}(A) \) as it need not preserve the Hochschild codifferential \( p^1_{1,0} = b^* \). On the other hand, there is a natural IBL∞ morphism in the opposite direction \( \text{dIBL}^{m_{\infty}}(A) \to \text{dIBL}^{m_{\infty}}(B) \) from Proposition 4.4 defined up to IBL∞ homotopy equivalence. It would be interesting to know whether the dIBL∞ construction induces a natural contravariant functor from the homotopy category of cyclic DGAs to the homotopy category of dIBL algebras.

4.3. Twisted dIBL algebras associated to Poincaré DGAs. We now explain how to extend the dIBL∞ construction to Poincaré DGAs using differential Poincaré duality models. Let \( A \) be a PDGA whose cohomology \( H := H(A) \) is 2-connected, i.e., \( H^0 = R \) and \( H^1 = H^2 = 0 \). By Theorem 2.15(a), \( A \) admits a simply connected dPD model \( M \). Proposition 3.4 associates to \( M \) the twisted dIBL algebra \( \text{dIBL}^{m_{\infty}}(M) \). Suppose that \( M' \) is another simply connected dPD model of \( A \). By Theorem 2.15(b), there exists a simply connected dPD algebra \( A_1 \) and quasi-isomorphisms of dPD algebras \( A_1 \cong M \) and \( M' \). Proposition 4.4 then extends the pullbacks to IBL∞ homotopy equivalences

\[
\begin{array}{ccc}
A_1 & \cong & \text{dIBL}^{m_{\infty}}(M) \\
\cong & & \cong \\
\text{dIBL}^{m_{\infty}}(A_1) & \cong & \text{dIBL}^{m_{\infty}}(M').
\end{array}
\]

Since IBL∞ homotopy equivalences are invertible and composable, this shows that \( \text{dIBL}^{m_{\infty}}(M) \) is independent of the simply connected dPD model \( M \) of \( A \) up to IBL∞ homotopy equivalence (provided that \( H \) is 2-connected). If \( A' \) is a PDGA weakly equivalent to \( A \), then a dPD model \( M' \) for \( A' \) is also one for \( A \), and hence the dIBL algebras associated to \( A \) and \( A' \) are IBL∞ homotopy equivalent by the previous discussion. We have thus proved the following result which corresponds to Theorem 1.6 in the Introduction:

**Theorem 4.5.** The map \( A \mapsto \text{dIBL}^{m_{\infty}}(M) \), where \( M \) is a simply connected differential Poincaré duality model of \( A \), assigns to each Poincaré DGA \( A \) of degree \( n \in \mathbb{N}_0 \) whose cohomology is 2-connected a dIBL algebra of degree \( n - 3 \) whose homology is the cyclic cohomology of the DGA \( A \), canonically up to IBL∞ homotopy equivalence. If two such Poincaré DGAs \( A \) and \( A' \) are weakly equivalent, then their associated dIBL algebras are IBL∞ homotopy equivalent. \( \square \)

**Application to the de Rham complex.** Let \( M \) be a connected closed oriented manifold of dimension \( n \). Consider the de Rham complex

\[
(\Omega := \Omega^*(M), d, \wedge)
\]
with the intersection pairing \( \langle \cdot , \cdot \rangle : \Omega \times \Omega \to \mathbb{R} \) of degree \( n \) defined by
\[
\langle \alpha, \beta \rangle := \int_M \alpha \wedge \beta.
\]
Stokes’ theorem implies that \( (\Omega, d, \wedge, \langle \cdot , \cdot \rangle) \) is a DGA with pairing in the sense of Section 2. The Poincaré duality theorem implies that the induced pairing \( \langle \cdot , \cdot \rangle_{H^{\text{dR}}} : \mathcal{H}^{\text{dR}} \times \mathcal{H}^{\text{dR}} \to \mathbb{R} \) on the de Rham cohomology \( \mathcal{H}^{\text{dR}} := \mathcal{H}^{\text{dR}}(M) = H(\Omega, d) \) is perfect, and so \( \Omega \) is an oriented PDGA of degree \( n \). It is of course unital with unit the constant 1.

A smooth homotopy equivalence \( f : M \to M' \) induces a DGA quasi-isomorphism \( f^* : \Omega^*(M') \to \Omega^*(M) \). So the map \( f^* \) is a DGA quasi-isomorphism if and only if the induced isomorphism \( H(f^*) : H_n(M; \mathbb{Z}) \to H_n(M'; \mathbb{Z}) \) intertwines the orientations, i.e maps the fundamental cycle \([M]\) to the fundamental cycle \([M']\). We will call such \( f \) an orientation preserving homotopy equivalence. Theorem 4.5 then implies the following result which corresponds to Corollary 1.7 in the Introduction:

**Corollary 4.6.** The map \( M \mapsto \mathrm{dIBL}^{\text{max}}_M(M) \), where \( M \) is a simply connected differential Poincaré duality model of the de Rham complex \( \Omega^*(M) \), assigns to each closed \( n \)-manifold \( M \) whose de Rham cohomology \( H^{\text{dR}}(M) \) is 2-connected a DGL algebra of degree \( n - 3 \) whose homology is the cyclic cohomology of \((\Omega, d, \wedge)\), canonically up to \( \text{IBL}_{\infty} \) homotopy equivalence. An orientation preserving homotopy equivalence between two such manifolds \( M \) and \( M' \) gives rise to an \( \text{IBL}_{\infty} \) homotopy equivalence between their associated DGL algebras.

**Remark 4.7.** On the class of manifolds \( M \) that admit an orientation reversing homotopy equivalence \( M \to M \), the specification “orientation preserving” can be dropped in Corollary 4.6. However, there are manifolds (such as the complex or quaternionic projective spaces \( \mathbb{C}P^m, \mathbb{H}P^m \) for even \( m \)) that do not admit an orientation reversing homotopy equivalence \( M \to M \). For further examples see [34].

## 5. The analytic construction

Throughout this section, \( M \) will be a connected closed oriented manifold of dimension \( n \) and \( (\Omega := \Omega^*(M), d, \wedge) \) its de Rham complex equipped with the intersection pairing \( \langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta \). We denote by \( H^{\text{dR}} := H^{\text{dR}}(M) \) the de Rham cohomology.

Given a Riemannian metric \( g \) on \( M \), let \( \ast_g : \Omega^* \to \Omega^{n-\bullet} \) be the induced Hodge star operator, \( d^*_g : \Omega^* \to \Omega^{n-1} \) the associated codifferential, and \( \mathcal{H}_g : \ker d \cap \ker d^*_g \) the corresponding harmonic subspace. The Hodge theorem asserts that
\[
\Omega = \mathcal{H}_g \oplus \text{im } d \oplus \text{im } d^*_g
\]
is a Hodge decomposition, and we conclude that \( \Omega \) is an oriented PDGA of Hodge type. Moreover, the pairing \( \langle \cdot , \cdot \rangle : \Omega \times \Omega \to \mathbb{R} \) is nondegenerate, so every harmonic subspace \( \mathcal{H} \subset \Omega \) admits a unique harmonic projection \( \pi_\mathcal{H} : \Omega \to \mathcal{H} \).

### 5.1. Analytic propagators

We call a propagator \( P : \Omega \to \Omega \) analytic if it can be written as
\[
(P\alpha)(x) = \int_{y \in M} P(x, y) \wedge \alpha(y)
\]
for a smooth \((n-1)\)-form \( P \) on the oriented real blow-up of the diagonal \( \Delta \subset M \times M \). This blow-up, which we denote by \( \text{Bl}_\Delta(M \times M) \), is obtained by replacing \( \Delta \) with the real oriented projectivization \( P^+N\Delta := N\Delta/\sim \) of the normal bundle \( N\Delta \to \Delta \), where \( v_1 \sim v_2 \) holds if and only if \( v_1 = \alpha v_2 \) for some \( \alpha \in (0, \infty) \), and promoting

\[\text{Here “analytic” does not stand for “real analytic” but for good analytic properties.}\]
polar coordinates around \( \Delta \) to boundary charts around \( P^+N\Delta \). This leads to a smooth compact manifold with boundary \( \partial BL_\Delta(M \times M) = P^+N\Delta \) whose interior is canonically identified with \( (M \times M) \setminus \Delta \) (see [11, 24] for details).

A version of the following lemma has been proved in dimension 3 in \([3, 5]\), and in arbitrary dimension (with essentially the same proof) in \([10, 24]\):

**Lemma 5.1.** Given a harmonic subspace \( \mathcal{H} \subset \Omega \), there exists an analytic propagator \( P : \Omega \to \Omega \) with respect to the harmonic projection \( \pi_H : \Omega \to \mathcal{H} \). The corresponding special propagator \( P_\delta \) from Lemma 2.1 is again analytic.

**Sketch of proof.** The integral kernel of \( \pi_H : \Omega \to \mathcal{H} \) is a closed smooth \( n \)-form on \( M \times M \) Poincaré dual to \( \Delta \). Therefore, it admits a primitive \( P \in \Omega^{n-1}(BL_\Delta(M \times M)) \) by Poincaré duality; we choose one and define \( P \) up to a sign by \([54]\). The homotopy equation \([55]\) for \( P \) then follows from Stokes’ theorem. The symmetry condition \([7]\) for \( P \) is equivalent to \( P(x, y) = \pm P(y, x) \), which can be always achieved by taking \( P_1(x, y) := \frac{1}{2}(P(x, y) \pm P(y, x)) \). The last assertion follows by translating the formulas in Lemma 2.1 to the integral kernels. \( \square \)

**Remark 5.2.** It has been frequently claimed that the special propagator \( P_\delta \) corresponding to the Hodge decomposition \([55]\) is analytic (see, e.g., \([5, 1]\)). However, we have been unable to find a proof of this assertion.

### 5.2. The analytic Maurer–Cartan element

The intersection pairing \( \langle \cdot, \cdot \rangle : \Omega \times \Omega \to \mathbb{R} \) is not perfect for \( n > 0 \), so that we cannot apply Propositions \([63, 64]\) to get “\( dIBL^{\text{can}}_\Omega(\Omega) \)”. Instead, one can construct a Maurer–Cartan element in

\[
dIBL(H_{dR}) = \{(B^{\text{cyc}}H_{dR})(2-n), q_{1,1,0} = 0, q_{2,1,0}, q_{1,2,0}\}
\]

by formally applying the homotopy transfer from Theorem 3.3 to the triple intersection product \( m_2^+ \in B^{\text{cyc}}_\Omega \) given by

\[
m_2^+(a_0a_1a_2) = (-1)^{\deg a_1+n} \int_M a_0 \wedge a_1 \wedge a_2.
\]

In analogy with Proposition 3.4 we can view \([35]\) as the “canonical Maurer–Cartan element in \( dIBL(\Omega) \)”, so that the twist of \( dIBL(H_{dR}) \) with its formal pushforward can be viewed as a model of “\( dIBL^{\text{can}}(\Omega) \)”, provided that the formal pushforward is a Maurer–Cartan element. This strategy has been proposed in \([8]\) and is carried out in \([10]\), leading to the following refinement of Theorem 1.5 in the Introduction:

**Theorem 5.3 (\([10]\)).** There is a Maurer–Cartan element \( m = \{m_{\ell,g}\} \) in \( dIBL(H_{dR}) \) which is defined naturally up to a gauge equivalence such that the homology of \( \langle B^{\text{cyc}}H_{dR}, q_{1,1,0}^m \rangle \) equals the cyclic cohomology of \( (\Omega, d, \wedge) \).

**Sketch of proof.** Let \( P : \Omega \to \Omega \) be a special analytic propagator with respect to the harmonic projection \( \pi_H : \Omega \to \mathcal{H} \), and let \( P \in \Omega^{n-1}(BL_\Delta(M \times M)) \) be its integral kernel. The value of \( m_{\ell,g} \) on the tensor product \( \alpha = \alpha_1^1 \cdots \alpha_1^{s_1} \otimes \cdots \otimes \alpha_\ell^1 \cdots \alpha_\ell^{s_\ell} \), \( \alpha_j^i \in H_{dR}, s_\ell \in \mathbb{N} \) is defined as a sum over trivalent ribbon graphs \( \Gamma \in RG_{k,\ell,g}^3 \) with \( k \) determined by \([20]\) similarly as in Corollary 3.6. The contribution of \( \Gamma \) is defined as the integral

\[
m_\Gamma(\alpha) := \int_{X_\Gamma} P_\Gamma \times \alpha,
\]

where \( X_\Gamma \) is a suitable configuration space of \( k \) points of \( M \) assigned to interior vertices, \( P_\Gamma \) is a wedge product of the \((n-1)\)-forms \( P \) assigned to interior edges, and \( \alpha \) is a product of harmonic representatives of \( \alpha_j^i \) assigned to exterior flags (see Figure 6 for a clarifying example).
The proof consists in overcoming the following four difficulties:

(1) Since $P$ is singular along the diagonal, it is a priori not clear that the integrals converge. This is resolved by taking for $X_\Gamma$ a Fulton-MacPherson type compactification [21] similar to the ones used in [1, 4].

(2) The compactification $X_\Gamma$ has additional codimension one boundary components, so-called “hidden faces”, which may obstruct the Maurer–Cartan equation for $m$. This is resolved by showing, via a symmetry argument similar to the one in [5] going back to [29] and [4], that the integrals over hidden faces vanish.

(3) One needs to sum (36) over $\Gamma \in RG_{k,\ell,g}^3$ with suitable signs and combinatorial coefficients in order to obtain a Maurer–Cartan element.

(4) One needs to produce a gauge equivalence between the Maurer–Cartan elements corresponding to different choices of a special analytic propagator $P$. This uses an alternative description of gauge equivalence in terms of the Weyl formalism from [17], and can also be reformulated in terms of an equivalence of BV actions corresponding to $m$ introduced in [24]. □

We denote the Maurer–Cartan element $m$ associated to a special analytic propagator $P$ in Theorem 5.3 by $m^\text{ana}_P$ and call it the analytic Maurer–Cartan element in $\text{dIBL}(H_{\text{dR}})$.

5.3. Analytic vanishing results. The following result is an analog of Proposition 4.1 in the analytic case and corresponds to [24] Propositions 4.4.1 and 4.4.2. Recall that $\gamma$ denotes the number of loops in $\Gamma$.

**Proposition 5.4.** Let $P: \Omega \to \Omega$ be a special analytic propagator, $\Gamma \in RG_{k,\ell,g}^3$ a trivalent ribbon graph which is not the Y-tree, and $\alpha \in (B^{\infty}(\Omega))^{\otimes \ell}$ a tensor product $\alpha = \alpha_1^1 \otimes \cdots \otimes \alpha_1^{s_1} \otimes \cdots \otimes \alpha_\ell^1 \cdots \alpha_\ell^{s_\ell}$ with $\alpha_i^b \in H_{\text{dR}}, s_i \in \mathbb{N}$ such that $$(m^\text{ana}_P)_{\Gamma}(\alpha) \neq 0.$$ Then the following holds:

(i) The positivity of degrees ($H_{\text{dR}}$ is connected by assumption):
$$\deg(\alpha_i^b) > 0 \quad \text{for all } i \in \{1, \ldots, s_i\}, b \in \{1, \ldots, \ell\}.$$
Then we have

\[ H \subset \Omega \]

The positivity of degrees implies:

(iii) If \( \gamma = 1 \) or \( n = 3 \), then \( \deg(\alpha_i^\ell) = 1 \) for all \( i \in \{1, \ldots, s_b\}, b \in \{1, \ldots, \ell\} \).

(iv) If \( n \geq 4 \), then \( \gamma \leq 1 \), i.e., all graphs with more than one loop vanish.

Recall that a manifold \( M \) is called geometrically formal if it admits a Riemannian metric \( g \) such that \( H_g \wedge H_g \subset H_g \). Remark \( \ref{rem:formal} \) on the cases \( n \in \{0, 1, 2\} \) applies here, too, and we conclude:

**Corollary 5.5.** Suppose that \( M \) is a connected closed oriented manifold such that \( H_{d\text{R}}^1 = 0 \) which is not diffeomorphic to \( S^2 \). Let \( P : \Omega \to \Omega \) be a special analytic propagator and \( m_P^{\text{ana}} \) the associated analytic Maurer–Cartan element in \( \text{dIBL}(H_{d\text{R}}) \).

Then we have

\[ (m_P^{\text{ana}})_{\ell, g} = 0 \quad \text{for all } \ell, g \neq (1, 0). \]

Consider \( H_{d\text{R}} \) as a cyclic DGA \( (H_{d\text{R}}, \d, \wedge, \langle \cdot, \cdot \rangle_{H_{d\text{R}}}) \), and let \( m_{H_{d\text{R}}}^{\text{can}} \) be the canonical Maurer–Cartan element in \( \text{dIBL}(H_{d\text{R}}) \). If \( M \) is in addition geometrically formal, then \( P \) can be chosen such that

\[ m_P^{\text{ana}} = m_{H_{d\text{R}}}^{\text{can}}. \]

6. Comparison of the algebraic and analytic constructions

As in the previous section, \( M \) will be a connected closed oriented manifold of dimension \( n \) and \( \Omega := \Omega^*(M), d, \wedge \) its de Rham complex equipped with the intersection pairing \( \langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta \). We denote by \( H_{d\text{R}} := H_{d\text{R}}(M) \) the de Rham cohomology of \( M \).

Suppose that the Poincaré DGA \( \Omega \) has a differential Poincaré duality model \( \mathcal{M} \). We choose a zig-zag of PDGA quasi-isomorphisms connecting \( \Omega \) to \( \mathcal{M} \) and identify the Poincaré duality algebras \( (H_{d\text{R}}, \d = 0, \wedge, \langle \cdot, \cdot \rangle_{H_{d\text{R}}}) \) and \( (H := H(M), \d = 0, \wedge, \langle \cdot, \cdot \rangle_H) \) via the induced isomorphism on cohomology. Consider the canonical Maurer–Cartan element \( m_{\mathcal{M}}^{\text{can}} \) in \( \text{dIBL}(\mathcal{M}) \), and let \( f^\mathcal{M}_\Omega : \text{dIBL}(\mathcal{M}) \to \text{dIBL}(H) \) be the \( A_\infty \) homotopy associated to a special propagator \( P^\mathcal{M} : \mathcal{M} \to \mathcal{M} \). The push-forward \( f^\mathcal{M}_\Omega m_{\mathcal{M}}^{\text{can}} \) then naturally induces a Maurer–Cartan element in \( \text{dIBL}(H_{d\text{R}}) \). Suppose now that \( H_{d\text{R}}^0 \) is \( 0 \), and let \( P : \Omega \to \Omega \) be a special propagator. Proposition \( \ref{prop:compare} \) provides a canonical differential Poincaré duality model of \( \Omega \),

\[ Q_P := Q(S_P(\Omega)), \]

where \( S_P \) is the small subalgebra associated to \( P \) and \( Q \) denotes the nondegenerate quotient. We also have the canonical zigzag of PDGA quasi-isomorphisms \( \ref{zigzag} \). In particular, the Poincaré duality algebras \( H_{d\text{R}} \) and \( H(Q_P) \) are canonically identified. Moreover, Lemmas \( \ref{lem:canonical} \) and \( \ref{lem:canonical2} \) equip \( S_P \) and \( Q_P \) canonically with special propagators induced from \( P \). Therefore, the construction in the previous paragraph for \( \mathcal{M} = Q_P \) provides a canonical Maurer–Cartan element in \( \text{dIBL}(H_{d\text{R}}) \). We denote it by

\[ m_P^{\text{alg}} \]

and call it the algebraic Maurer–Cartan element in \( \text{dIBL}(H_{d\text{R}}) \).

In the next subsections, we will compare \( m_P^{\text{ana}} \) to \( m_P^{\text{alg}} \) for a special analytic propagator \( P \). Our approach is to use the algebraic and analytic vanishing results to reduce the problem to a comparison of the corresponding cyclic \( A_\infty \) algebras.
6.1. $A_\infty$ algebras and homotopy transfer. In this subsection, we recall $A_\infty$ algebras and their homotopy transfer and prove its functoriality in a special case.

**$A_\infty$ algebras.** An $A_\infty$ algebra $(A, \{m_i\})$ consists of a $\mathbb{Z}$-graded vector space $A$ and a sequence of operations $m_i: A^{\otimes i} \to A$, $i \in \mathbb{N}$, of degrees $\deg m_i = 2 - i$ satisfying for each $r \in \mathbb{N}$ and $x_1, \ldots, x_r \in A$ the relation

$$
\sum_{i+j=r+1} \sum_{i,j \geq 1}^{r+1-j} (-1)^{|x_1|+\cdots+|x_{i-1}|} m_i(x_1, \ldots, m_j(x_c, \ldots, x_{c+j-1}), \ldots, x_r) = 0,
$$

where $|x| = \deg(x) - 1$ is the shifted degree. In particular, $d = m_1$ is a differential and $x \wedge y = (-1)^{\deg x} m_2(x, y)$ a product on $A$ which becomes associative on the cohomology $H(A, d)$. In fact, a DGA $(A, d, \wedge)$ corresponds naturally to an $A_\infty$ algebra $(A, \{m_1, m_2, m_i = 0 \text{ for } i \geq 3\})$ and vice versa via

$$
m_i = d, \quad m_2(x, y) = (-1)^{\deg x} x \wedge y.
$$

We refer to [20] for the notions of a morphism, quasi-isomorphism, and homotopy equivalence of $A_\infty$ algebras and their basic properties, such as the following result:

**Proposition 6.1 (Homotopy transfer for $A_\infty$ algebras, [30]).** Let $(A, \{m_i\})$ be an $A_\infty$ algebra and $B \subset A$ a quasi-isomorphic subcomplex with respect to $d = m_1$. Given a homotopy operator $P: A \to A$ with respect to a projection $\pi: A \to B$, there is a natural $A_\infty$ structure $\{m_i^B\}$ on $B$ extending $m_i^B = d|_B$ and an $A_\infty$ homotopy equivalence

$$
g = \{g_i: B^{\otimes i} \to A\}: (B, \{m_i^B\}) \hookrightarrow (A, \{m_i\})
$$

extending the inclusion $g_1 = 1: B \hookrightarrow A$.

**Sketch of proof.** Following [30], the maps $m_i^B: B^{\otimes i} \to B$ and $g_i: A^{\otimes i} \to B$ for $i \geq 2$ can be explicitly written as

$$
m_i^B := \sum_{T \in \mathcal{T}_i} m_T^B: B^{\otimes i} \to B \quad \text{and} \quad g_i := \sum_{T \in \mathcal{T}_i} g_T: B^{\otimes i} \to A,
$$

where $\mathcal{T}_i$ denotes the set of isotopy classes of planar embeddings of rooted trees with $i$ leaves similarly as in Remark 2.12. We orient the edges of $T \in \mathcal{T}_i$ towards the root and require in addition that each interior vertex has at least two incoming edges. We also order the leaves in the counterclockwise direction of the plane starting from the root. We then define $m_T^B: B^{\otimes i} \to B$ and $g_T: B^{\otimes i} \to A$ by labeling $T$ and interpreting it as a composition rule similarly as in [15]. Namely, we label each interior vertex with $j$ incoming edges by $m_j$, each interior edge by $P$, each leaf by $i$, and the root by $\pi$ in the case of $m_T^B$ and by $P$ in the case of $g_T$. It is now straightforward to check the desired properties. 

Note that if $m_i = 0$ for all $i \geq 3$, i.e., if $A$ is a DGA, then the construction of $m_i^B$ and $g_i$ in the proof above involves only binary trees $\mathcal{T}_i^{\text{bin}} \subset \mathcal{T}_i$.

Let $P: A \to A$ be a special propagator and $\mathcal{H} \subset A$ the associated harmonic subspace. Taking $\mathcal{H}$ for $B$ in Proposition 6.1 and identifying it with the cohomology $H := H(A)$ via the quotient map $[5]$ gives an $A_\infty$ structure $\{m_i^H\}$ on $H$ and an $A_\infty$ homotopy transfer $g: (H, \{m_i^H\}) \to (A, \{m_i\})$. 


Functoriality of $\Lambda_\infty$ homotopy transfer for DGAs. Consider a commutative diagram

$$
\begin{array}{c}
(A, d, \wedge) & \xrightarrow{f} & (\tilde{A}, \tilde{d}, \tilde{\wedge}) \\
\downarrow \iota & & \downarrow \tilde{\iota} \\
(B, d_B) & & (\tilde{B}, d_{\tilde{B}})
\end{array}
$$

where $A$ and $\tilde{A}$ are DGAs, $f$ is a DGA morphism, $B$ is a cochain complex, and the chain maps $\iota$ and $\tilde{\iota}$ are injective. Suppose that $P: A \to A$ and $\tilde{P}: \tilde{A} \to \tilde{A}$ are homotopy operators with respect to some projections $\pi: A \to B$ and $\tilde{\pi}: \tilde{A} \to \tilde{B}$, respectively, such that the following compatibility with $f$ is satisfied:

$$f \circ P = \tilde{P} \circ f.$$ 

Viewing $A$ and $\tilde{A}$ as $\Lambda_\infty$ algebras and $B$ as their subcomplex via the inclusions $\iota$ and $\tilde{\iota}$, respectively, Proposition 6.1 yields the $\Lambda_\infty$ homotopy transfers

$$g: (B, m_B) \xrightarrow{\sim} (A, m), \quad \tilde{g}: (B, \tilde{m}_B) \xrightarrow{\sim} (\tilde{A}, \tilde{m}).$$

**Lemma 6.2.** In the setup above, we have $m_B^i = \tilde{m}_B^i$ and $f \circ g_i = \tilde{g}_i$ for every $i \in \mathbb{N}$. In other words, the following diagram of $\Lambda_\infty$ morphisms commutes:

$$
\begin{array}{c}
(A, m) & \xrightarrow{f} & (\tilde{A}, \tilde{m}) \\
| & & | \\
(B, m_B^i) & \xrightarrow{\sim} & (\tilde{A}, \tilde{m}_B^i).
\end{array}
$$

**Proof.** For a planar rooted tree $T \in \mathcal{T}_i$, consider the contributions $m_B^T$ and $g_T$ resp. $\tilde{m}_B^T$ and $\tilde{g}_T$ from the proof of Proposition 6.1 which were obtained by labeling $T$ with the data of $m, P, \iota, \pi$ resp. $\tilde{m}, \tilde{P}, \tilde{\iota}, \tilde{\pi}$ and interpreting it as a composition rule. The claim then clearly follows from the commutation relations of this data with $f$. \qed

We apply the lemma above in the following two situations:

(i) $A$ is a DGA with pairing of Hodge type, $\tilde{P}: \tilde{A} \to \tilde{A}$ is a special propagator, $B := \mathcal{H}_{\tilde{P}} \subset \tilde{A}$ is the associated harmonic subspace, $A := S_{\tilde{P}}$ is the corresponding small subalgebra, $f: A \hookrightarrow \tilde{A}$ is the inclusion, and $P: A \to A$ is the induced special propagator from Lemma 2.11.

(ii) $A$ is a DGA with pairing of Hodge type such that the induced pairing on cohomology is nondegenerate, $P: A \to A$ is a special propagator, $B := \mathcal{H}_P \subset A$ is the associated harmonic subspace, $\tilde{A} := Q(A)$ is the nondegenerate quotient, $f: A \to \tilde{A}$ is the quotient map, and $\tilde{P}: \tilde{A} \to \tilde{A}$ is the corresponding special propagator from Lemma 2.10.

Combining (i) and (ii), we thus have:

**Proposition 6.3.** Let $A$ be a DGA with pairing of Hodge type such that the induced pairing on cohomology $H := H(A)$ is nondegenerate. Let $P: A \to A$ be a special propagator, $\mathcal{H} := \mathcal{H}_P \subset A$ the associated harmonic subspace, $\mathcal{S} := S_P$ the associated...
small subalgebra, and $Q := Q(S)$ the nondegenerate quotient of $S$. Consider the commutative diagram

$$
\begin{array}{ccc}
A & \xleftarrow{\iota} & S \\
\downarrow{\pi_Q} & & \downarrow{\pi_Q} \\
H & \xrightarrow{\pi_Q} & Q
\end{array}
$$

where $\pi_Q$ is the quotient map and the other maps are the obvious inclusions. Viewing the DGAs $A$, $S$, $Q$ as $\Lambda_\infty$ algebras and $H$ as their subcomplex via the vertical maps, Proposition 6.3 yields the $\Lambda_\infty$ homotopy transfers

$$
g_h : (H, m_H^\infty) \xrightarrow{\cong} (A, m), \quad g_S : (H, m_S^\infty) \xrightarrow{\cong} (S, m_S), \quad g_Q : (Q, m_Q^\infty) \xrightarrow{\cong} (Q, m_Q).
$$

The following diagram of $\Lambda_\infty$ morphisms then commutes:

$$
\begin{array}{cccc}
(A, m) & \xleftarrow{\iota} & (S, m_S) & \xrightarrow{\pi_Q} & (Q, m_Q) \\
\downarrow{g} & & \downarrow{g_S} & & \downarrow{g_Q} \\
(H, m) & \xrightarrow{\iota} & (H, m_S) & \xrightarrow{\pi_Q} & (H, m_Q)
\end{array}
$$

Corollary 6.4. Let $A$ be a DGA with pairing of Hodge type such that the induced pairing on cohomology $H := H(A)$ is nondegenerate. Let $P : A \to A$ be a special propagator and $m_H^\infty$, $m_S^\infty$ and $m_Q^\infty$ the three $\Lambda_\infty$ structures on $H$ obtained by the induced $\Lambda_\infty$ homotopy transfers from $A$, $S := S_P(A)$ and $Q := Q(S)$, respectively. Then all these $\Lambda_\infty$ structures on $H$ are equal:

$$
m_H^\infty = m_S^\infty = m_Q^\infty.
$$

6.2. Cyclic $\Lambda_\infty$ algebras and IBL$\infty$ Maurer Cartan elements.

Cyclic $\Lambda_\infty$ algebras. A pairing on an $\Lambda_\infty$ algebra $(A, \{m_i\})$ is a bilinear form $\langle \cdot, \cdot \rangle : A \times A \to \mathbb{R}$ such that if we define the cyclic pairing

$$
\langle x, y \rangle_{\text{cyc}} = (-1)^{\deg x \cdot (x, y)}
$$

and consider the shifted grading $|x| := \deg x - 1$, then the cyclicity condition

$$
\langle m_i(x_1, \cdots, x_i), x_0 \rangle_{\text{cyc}} = (-1)^{|x_0|(|x_1| + \cdots + |x_i|)} \langle m_i(x_0, x_1, \cdots, x_{i-1}, x_i) \rangle_{\text{cyc}}
$$

holds for all $i \in \mathbb{N}$. We call the triple $(A, \{m_i\}, \langle \cdot, \cdot \rangle)$ an $\Lambda_\infty$ algebra with pairing. Following [8], we call an $\Lambda_\infty$ algebra with a perfect pairing $(A, \{m_i\}, \langle \cdot, \cdot \rangle)$ a cyclic $\Lambda_\infty$ algebra.

The following proposition can be checked by a straightforward yet nontrivial computation using the expression [38] for the $\Lambda_\infty$ homotopy transfer in terms of trees.

Proposition 6.5 ([28, Theorem 5.15]). Let $(A, \{m_i\}, \langle \cdot, \cdot \rangle)$ be an $\Lambda_\infty$ algebra with pairing, and $H := H(A, d)$ its cohomology. Let $P : A \to A$ be a special propagator and $(H, \{m_H^\infty\})$ the associated homotopy transferred $\Lambda_\infty$ algebra. Then the induced map $\langle \cdot, \cdot \rangle_H : H \times H \to \mathbb{R}$ is a pairing on $(H, \{m_H^\infty\})$. In particular, if $(A, \{m_i\}, \langle \cdot, \cdot \rangle)$ is a cyclic $\Lambda_\infty$ algebra, then $(H, \{m_H^\infty\}, \langle \cdot, \cdot \rangle_H)$ is a cyclic $\Lambda_\infty$ algebra as well.

Cyclic $\Lambda_\infty$ algebras and IBL$\infty$ Maurer–Cartan elements. Following [8], we can recast cyclic $\Lambda_\infty$ algebras on a cyclic cochain complex $(A, d, \langle \cdot, \cdot \rangle)$ of degree $n$ in terms of Maurer–Cartan elements in dIBL$(A)$. For this, we associate to each linear map $f : A^{\otimes i} \to A$ its cyclization $f^+ : A^{\otimes (i+1)} \to \mathbb{R}$ given by

$$
f^+(x_0, x_1, \cdots, x_i) = (-1)^{n-2}(f(x_0, \cdots, x_{i-1}), x_i)_{\text{cyc}}.
$$

Since the pairing is perfect, we have a one-to-one correspondence $f \leftrightarrow f^+$.\footnote{In [8] the authors use $\langle \cdot, \cdot \rangle_{\text{cyc}}$ instead of $\langle \cdot, \cdot \rangle$.}
Consider the canonical filtered dIBL (42) p

The cyclic cochain complex

and a collection of such maps \( \{ \mathbf{m}_i \} \) together with \( \mathbf{m}_1 := \mathbf{d} \) satisfies the A

relations (37) if and only if \( \mathbf{m}_i^+ \in B^{cy^\ast} A \) with \( |\mathbf{m}_i^+| = 1 \), and a collection of such maps \( \{ \mathbf{m}_i \} \) satisfies the cyclicity condition (39) if and only if the element

satisfies the first part of the Maurer–Cartan equation for \( \{ \mathbf{m}_{1,0} := \mathbf{m}^+ \} \):

\[
\mathbf{p}_{1,1,0}(\mathbf{m}^+) + \frac{1}{2} \mathbf{p}_{2,1,0}(\mathbf{m}^+, \mathbf{m}^+) = 0.
\]

The second part of the Maurer–Cartan equation for \( \{ \mathbf{m}_{1,0} := \mathbf{m}^+ \} \),

\[
\mathbf{p}_{1,2,0}(\mathbf{m}^+) = 0,
\]

is not satisfied for a general cyclic A

algebra (see \[8, Remark 12.4\] for a discussion). However, it holds for every cyclic DGA because \( \mathbf{p}_{1,2,0}(B_3^{cy^\ast} A) = 0 \) by definition in Proposition 3.2 and because \( \mathbf{m}^+ = \mathbf{m}_2^+ \in B_3^{cy^\ast} A \) is the triple intersection product.

We now discuss another natural class of A

algebras for which (43) holds.

A nonzero element \( 1 \in A^0 \) in an A

algebra \( (A, \{ \mathbf{m}_i \}) \) is called a strict unit if

(i) \( \mathbf{m}_2(1, x) = (-1)^{\deg x} \mathbf{m}_2(x, 1) = x \), and

(ii) \( \mathbf{m}_i(x_1, \ldots, x_i) = 0 \) whenever \( x_j = 1 \) for some \( j \in \{ 1, \ldots, i \} \), for all \( i \geq 3 \).

An A

algebra which admits a strict unit is called a strictly unital A

algebra. We define connectedness and \( k \)-connectedness for a strictly unital A

algebra in the same way as for a unital DGA. We call a cyclic A

algebra strictly unital, connected, or \( k \)-connected if the underlying A

algebra has the respective property.

Let \( \{ \mathbf{m}_i \}, (\cdot, \cdot) \) be a connected strictly unital cyclic A

algebra. Consider the canonical filtered dIBL algebra dIBL(\( A \)) associated to the underlying cyclic cochain complex \( (A, d, (\cdot, \cdot)) \), and let \( \mathbf{m}^+ \in B^{cy^\ast} A \) be the element defined in (11). Then \( \mathbf{m} = \{ \mathbf{m}_{1,0} := \mathbf{m}^+ \} \) is a Maurer–Cartan element in dIBL(\( A \)).

Proof. For \( x = x_1^1 \cdots x_{s_1}^1 \otimes x_2^1 \cdots x_{s_2}^2 \in (B^{cy^\ast} A)^{\otimes 2}, s_1, s_2 \in \mathbb{N} \), we have

\[
\mathbf{p}_{1,2,0}(\mathbf{m}^+)(x_1^1 \cdots x_{s_1}^1 \otimes x_2^1 \cdots x_{s_2}^2)
\]

\[
= \sum_{c=1}^{s_1} \sum_{c'=1}^{s_2} \sum_{a} \pm (\epsilon_a, \mathbf{m}_{s_1+s_2+1}(x_c^2, \ldots, x_{c+1}^2, e^a, x_{c-1}^2 \cdots x_{c'-1}^2)).
\]

The pairing in the sum vanishes for degree reasons unless

\[
\deg(x) = -\deg(\mathbf{m}_{s_1+s_2+1}) = s_1 + s_2 - 1.
\]

But then some \( x_i^j \) must have degree 0, hence be a multiple of 1 by connectedness on \( A \), and the right hand side vanished by strict unitality since \( s_1 + s_2 + 1 \geq 3 \). Therefore, the second Maurer–Cartan equation (43) is satisfied\(^\ddagger\). The first equation (42) is satisfied by Proposition 6.6. \( \square \)

\(^\ddagger\)We used here only condition (ii) of strict unitarity as \( \mathbf{p}_{1,2,0}(B_3^{cy^\ast} A) = 0 \) holds trivially. For a unital version of our construction including a term \( B_0^{cy^\ast} A := \mathbb{R} \cdot 1 \), condition (i) would also be needed.
We assumed so far that we were given an $A_\infty$ algebra first. Suppose now that we are given a cyclic cochain complex $(A,d,\langle\cdot,\cdot\rangle)$ and a Maurer–Cartan element $m = \{m_i\}$ in $dIBL(A)$ instead. Proposition [6.6] then implies that the collection of linear maps $\{m_i; A^s \to A\} \geq 2$ corresponding via (40) to $m_{1,0} \in B^{cyc}A$ together with $m_1 = d$ and the pairing $\langle\cdot,\cdot\rangle$ constitutes a cyclic $A_\infty$ structure on $A$. Notice that the filtration degree condition from the definition of a Maurer–Cartan element implies $m_{1,0}(x) = m_{1,0}(x_1,x_2) = 0$, so $m_{1,0}$ does not contain any operation with less than three inputs.

**Kontsevich–Soibelman Maurer–Cartan element.**

**Proposition 6.8.** Let $(A,d,\bigwedge,\langle\cdot,\cdot\rangle)$ be a nonnegatively graded unital DGA with pairing, and let $B \subset A$ be a quasi-isomorphic cyclic subcomplex such that $B^0 = \mathbb{R} \cdot 1$. Let $P; A \to A$ be a special propagator with respect to a projection $\pi; A \to B$, and let $\mathfrak{m}^{B+} \in B^{cyc}B$ be the element associated in Proposition 6.6 to the homotopy transferred $A_\infty$ algebra $(B,\{\mathfrak{m}_i^{B}\})$ associated to $P$. Then

$$m := \{m_{1,0} := \mathfrak{m}^{B+}\}$$

is a Maurer–Cartan element in $dIBL(B)$.

**Proof.** We will show that 1 is a strict unit for the $A_\infty$ algebra $(B,\{\mathfrak{m}_i^{B}\})$. The proposition then follows from Lemma [6.7]. Let $i \geq 2$ and $b_1,\ldots, b_i \in B$, and consider the formula (35) for $\mathfrak{m}_i^B(b_1,\ldots, b_i)$ as a sum of contributions $\mathfrak{m}_i^B(b_1,\ldots, b_i)$ of planar binary trees $T \in T_i^{P,m}$ with interior edges labeled by $P$, interior vertices labeled by $\bigwedge$, and leaves labeled by $b_1,\ldots, b_i \in B$. For $i = 2$, we have $\mathfrak{m}_2^B(b_1, b_2) = (-1)^{deg b_1}b_1 \bigwedge b_2$, hence condition (i) of a strict unit in $(B,\{\mathfrak{m}_i^{B}\})$ for 1 follows from it being a unit in $(A,d,\bigwedge)$. For $i \geq 3$, suppose that some $b_j$ equals 1 and consider the interior vertex $v$ adjacent to the exterior edge labeled with $b_j = 1$. Denote by $e^m$ the other incoming edge and by $e^{out}$ the outgoing edge at $v$. Now there are 3 cases. If both $e^m$ and $e^{out}$ are interior edges, then they are labeled with $P$ and $\mathfrak{m}_i^B(b_1,\ldots, b_i) = 0$ because $P \circ P = 0$. If $e^m$ is exterior and $e^{out}$ is interior, then $\mathfrak{m}_i^B(b_1,\ldots, b_i) = 0$ because $P \circ \pi = 0$. If $e^m$ is interior and $e^{out}$ is exterior (hence the root edge), then $e^{out}$ is labeled with $\pi$ and $\mathfrak{m}_i^B(b_1,\ldots, b_i) = 0$ because $\pi \circ P = 0$. This proves condition (ii) of the strict unitality of $(B,\{\mathfrak{m}_i^{B}\})$. □

We call $m$ from Proposition 6.8 the Kontsevich–Soibelman Maurer–Cartan element in $dIBL(B)$ and denote it by

$$m_{KS}^B.$$ 

Now we will compare $m_{KS}^B$ to $f_*^P m_{can}^A$ in the algebraic case for a cyclic DGA $A$, and $m_{KS}^A$ to $m_{can}^A$ in the analytic case for the de Rham complex $A = \Omega$. The following lemmas are clear on the pictorial level and proved in detail in [39]:

**Lemma 6.9 (39).** Let $(A,d,\bigwedge,\langle\cdot,\cdot\rangle)$ be a nonnegatively graded unital cyclic DGA with connected cohomology $H := H(A)$. Consider the canonical filtered $dIBL(A)$ and $dIBL(H)$ and the canonical Maurer–Cartan element $m_{can}^{\infty}$ in $dIBL(A)$. Let $P; A \to A$ be a special propagator, and let $f^P; dIBL(A) \to dIBL(H)$ be the associated $IBL_\infty$ homotopy equivalence and $m_{KS}^B$ the associated Kontsevich–Soibelman Maurer–Cartan element in $dIBL(H)$. Then we have

$$(f_*^P m_{can}^A)_{1,0} = (m_{KS}^B)_{1,0}.$$ 

**Sketch of proof.** Let $\alpha = \alpha_1 \cdots \alpha_s \in B^{cyc}_s H$ with $s \geq 3$. Evaluated on $\alpha$, the left-hand side of (44) can be written as a sum over $\Gamma \in RG^3_{s-2,1,0}$ of contributions $(f_*^P m_{can}^A)_\Gamma(\alpha) \in \mathbb{R}$ by Corollary [3.6]. The right-hand side can be written as a sum

$$(m_{KS}^B)_{1,0}(\alpha) \in \mathbb{R}.$$
over $T \in \mathcal{T}^{\text{bin}}_{s-1}$ of contributions $m^H_+ (\alpha) \in \mathbb{R}$ as in the proof of Proposition 6.1. Identifying the marked exterior vertex of $\Gamma$ with the root of $T$ we see that

$$RG^3_{s-2,1,0} \simeq \mathcal{T}^{\text{bin}}_{s-1}.$$  

It remains to compare the contributions of $\Gamma \in RG^3_{s-2,1,0}$ and the corresponding $T \in \mathcal{T}^{\text{bin}}_{s-1}$. We orient the edges of $\Gamma$ towards the root and order the leaves in the positive direction of the boundary starting from the root. We choose an ordering of interior vertices and a basis $(e_i)$ of $A$ and consider the coordinate expression (26). Performing the sum in (26) iteratively starting at the leaves and using

$$Pe^i = \sum_j P^j e_j \quad \text{and} \quad x = \sum_i \langle x, e_i \rangle e_i = \sum_i \langle e_i, x \rangle e_i$$

gives easily

$$(f^* m^\text{ana}_{\mathcal{P}})_T (\alpha) = \pm \langle \alpha_1, m^H (\alpha_2, \ldots, \alpha_s) \rangle = \pm m^H_+ (\alpha).$$

A careful comparison of signs and combinatorial coefficients finishes the proof. □

**Lemma 6.10** (29). Let $M$ be a connected closed oriented manifold. Let $P : \Omega \rightarrow \Omega$ be a special analytic propagator, and let $m^\text{ana}_P$ and $m^K_S$ be the associated analytic and Kontsevich–Soibelman Maurer–Cartan elements in $d\text{IBL}(H_{\text{dR}})$, respectively. Then we have

$$(m^\text{ana})_{1,0} = (m^K_S)_{1,0}.$$  

**Sketch of proof.** The proof is the same as the proof of Lemma 6.9 on the structural level. Technically, instead of performing the sum in (26) iteratively one has to apply Fubini’s theorem for the compactified configuration spaces $X_T$ in (56). □

6.3. Comparison of the algebraic and analytic Maurer Cartan elements.

The main result of this section is the following theorem which immediately implies Theorem 1.8 in the Introduction:

**Theorem 6.11.** Let $M$ be a connected closed oriented manifold with $H^1_{\text{dR}} = 0$ which is not diffeomorphic to $S^2$. For a special analytic propagator $P : \Omega \rightarrow \Omega$, consider the following Maurer–Cartan elements in $d\text{IBL}(H_{\text{dR}})$ associated to $P$: the analytic Maurer–Cartan element $m^\text{ana}_P$, the algebraic Maurer–Cartan element $m^\text{alg}_P$, and the Kontsevich–Soibelman Maurer–Cartan element $m^K_S$. Then all three Maurer–Cartan elements are equal:

$$m^\text{ana}_P = m^\text{alg}_P = m^K_S.$$  

**Proof.** Suppose that $n = \dim (M) \geq 3$ (the cases $n \in \{1, 2\}$ are impossible due to the assumptions, and the case $n = 0$ is trivial). The algebraic vanishing result from Corollary 4.3 implies that $(m^\text{alg})_{\ell, g} = 0$ for all $(\ell, g) \neq (1, 0)$. The analytic vanishing result from Corollary 5.5 implies that $(m^\text{ana})_{\ell, g} = 0$ for all $(\ell, g) \neq (1, 0)$. Therefore, it remains to prove that

$$(m^\text{ana})_{1,0} = (m^K_S)_{1,0} = (m^\text{alg})_{1,0}.$$  

The first equality is Lemma 6.10. As for the second one, consider the nondegenerate quotient $Q_P := Q(S_P(\Omega))$ of the small subalgebra $S_P \subset \Omega$, and identify $H_{\text{dR}} \simeq H := H (Q_P)$ via the canonical zig-zag (17). Recall the definition $m^\text{alg}_{Q_P}$, where $f^* : d\text{IBL}(Q_P) \rightarrow d\text{IBL}(H_{\text{dR}})$ is the $d\text{IBL}_\infty$ homotopy equivalence associated to $P$ and $m^\text{alg}_{Q_P}$ is the canonical Maurer–Cartan element in $d\text{IBL}(Q_P)$. Lemma 6.9 asserts that $(f^* m^\text{alg}_{Q_P})_{1,0} = m^H_{\text{dR}}$, where $m^H_{\text{dR}} \in B^\text{cy} \simeq H_{\text{dR}}$ is associated via Proposition 6.4 to the $A_\infty$ homotopy transfer $Q_P \sim H_{\text{dR}}$ from Proposition 6.1. On the other hand, $(m^K_S)_{1,0} = m^H_{\text{dR}}$ is associated via Proposition 6.0.
to the $\Lambda_\infty$ homotopy transfer $\Omega \sim H_{dR}$ from Proposition 6.3. The comparison of $A_\infty$ homotopy transfers $\Omega \sim H_{dR}$ and $Q_P \sim H_{dR}$ in Corollary 6.3 gives $m^{H_{dR}+} = m^{Q_P+}$, and the theorem follows.

We summarize the situation for a connected closed oriented manifold $M$. We associate to $M$ up to $\text{IBL}_\infty$ homotopy equivalence the following $\text{IBL}_\infty$ algebras whose homology is the cyclic cohomology of $(\Omega, d, \wedge)$:

- the $\text{IBL}_\infty$ algebra $\text{dIBL}^{\text{ana}}(H_{dR})$ based on ribbon graphs of all genera;
- the $\text{dIBL}$ algebra $\text{dIBL}^{\text{KS}}(H_{dR})$ based on ribbon trees only.

If $H_{dR}^0 = 0$ and $M$ is not diffeomorphic to $S^2$, then Theorem 6.11 implies that these structures for a special analytic propagator $P$ are equal. A computation for $M = S^1$ in [24] shows on the one hand that

$$q_{1,2,0}^{\text{ana}} \neq q_{1,2,0}^{\text{KS}},$$

and on the other hand that the IBL structures induced on homology are equal. In fact, for any $M$ we have $H(B^{\text{cyc}}H_{dR}, q_{1,0}^{\text{KS}}) = H(B^{\text{cyc}}H_{dR}, q_{1,0}^{\text{ana}})$ by Lemma 6.10, so it is plausible that there should be an $\text{IBL}_\infty$ homotopy equivalence $\text{dIBL}^{\text{ana}}(H_{dR}) \simeq \text{dIBL}^{\text{KS}}(H_{dR})$ in the general case. One can also ask the stronger question whether $m^{\text{ana}}$ and $m^{\text{KS}}$ are gauge equivalent.

6.4. Formality. In rational homotopy theory (see, e.g., [38]), a manifold $M$ is called formal if its de Rham complex $\Omega$ is weakly equivalent as a CDGA to its de Rham cohomology $H_{dR}$, i.e., if there exists a zigzag of CDGA quasi-isomorphisms connecting $\Omega$ to $H_{dR}$.

Similarly, we say that $M$ is $\text{IBL}_\infty$ formal if the $\text{IBL}_\infty$ algebra $\text{dIBL}^{\text{ana}}(H_{dR})$ from Theorem 6.3 is weakly equivalent as an $\text{IBL}_\infty$ algebra to the $\text{dIBL}$ algebra $\text{dIBL}^{\text{ana}}(H_{dR})$ from Proposition 6.3.14 Recall that two $\text{IBL}_\infty$ algebras are weakly equivalent if and only if they are $\text{IBL}_\infty$ homotopy equivalent, i.e, if there exists an $\text{IBL}_\infty$ homotopy equivalence in either direction. The following result corresponds to Corollary 1.11 in the Introduction:

**Corollary 6.12** (Formality implies $\text{IBL}_\infty$ formality). Let $M$ be a closed connected oriented manifold such that $H^1_{dR} = H^2_{dR} = 0$. Then $M$ being formal implies $M$ being $\text{IBL}_\infty$ formal.

**Proof.** Let $P : \Omega \rightarrow \Omega$ be a special analytic propagator. We consider the following three Maurer–Cartan elements in $\text{dIBL}(H_{dR})$: the analytic Maurer–Cartan element $m^{\text{ana}}_P$ associated to $P$, the algebraic Maurer–Cartan element $m^{\text{alg}}_P$ associated to $P$, and the canonical Maurer–Cartan element $m^{\text{can}}_P$. Theorem 6.11 asserts that $m^{\text{ana}}_P = m^{\text{alg}}_P$. Let $Q_P := Q(S_P(\Omega))$ be the nondegenerate quotient of the small subalgebra $S_P \subset \Omega$ associated to $P$. By construction (cf. the proof of Theorem 6.11), there is an $\text{IBL}_\infty$ homotopy equivalence $\text{dIBL}^{\text{ana}}(H_{dR}) \simeq \text{dIBL}^{\text{can}}Q_P(\Omega)$. Recall from Proposition 2.10 that $Q_P$ is a differential Poincaré duality model of $\Omega$. On the other hand, if $M$ is formal, then $H_{dR}$ is also a differential Poincaré duality model of $\Omega$. Indeed, a simple argument from [24] Proposition 6.2.5 shows that a zig zag of CDGA quasi-isomorphisms $f : \Omega \sim H_{dR}$ induces a zig-zag of PDGA

\[\cdots\]
quasi-isomorphisms \( \hat{f}: \Omega \rightarrow H_{\text{dR}} \) (one orients the cohomologies of the intermediate CDGAs so that they become PDGAs and the maps between them PDGA quasi-isomorphisms, and composes \( \hat{f} \) with the inverse of the induced isomorphism on cohomology \( H(f)^{-1}: H_{\text{dR}} \rightarrow H_{\text{dR}} \) to obtain \( \hat{f} \)). Theorem 1.3 applied to \( Q_P \) and \( H_{\text{dR}} \) under the assumption that \( H^2_{\text{dR}} = 0 \) then implies the existence of an IBL\(_{\infty} \) homotopy equivalence \( d\text{IBL}^m\circ r(Q_P) \simeq d\text{IBL}^m_{\text{can}}(H_{\text{dR}}) \), and the corollary follows.

**Remark 6.13.** Consider a compact connected Lie group \( G \) with \( H^1_{\text{dR}}(G) = 0 \). Then \( G \) is formal and satisfies \( H^2_{\text{dR}}(G) = 0 \), hence it is IBL\(_{\infty} \) formal by Corollary 6.12. In fact, \( G \) is even geometrically formal, so Corollary 5.5 implies the stronger assertion that there exists a special analytic propagator \( P: \Omega \rightarrow \Omega \) such that \( m^\text{can} = m^\text{formal}_{\text{dR}} \) in \( \text{dBL}(H_{\text{dR}}) \). Finally, \( G \) is also simply connected, hence \( d\text{IBL}^m_{\text{can}}(H_{\text{dR}}) \) induces a chain model for the equivariant string topology of \( G \) by Theorem 1.3.

### 6.5. Relation to Massey products

Massey products are secondary cohomological operations that are trivial for formal manifolds, and thus give obstructions to formality. See [16] for some examples of non-formal manifolds. In this subsection we discuss the following question which would constitute a partial converse to Corollary 6.12.

**Question 6.14.** If a closed connected oriented manifold \( M \) is IBL\(_{\infty} \) formal, are then all its Massey products trivial?

**Massey products.** We begin by recalling some facts about Massey products, following the presentation in [31]. Let \((A,d)\) be a DGA and \( H_A \) its cohomology. Let us first describe the triple Massey product. Consider three cycles \( a_1, a_2, a_3 \in A \) of homogeneous degree such that \( a_1a_2 \) and \( a_2a_3 \) are exact. For each choice of primitives \( b, c \in A \) with

\[
db = a_1a_2, \quad dc = a_2a_3,
\]

the element

\[
ba_3 + (-1)^{\deg a_1+1}a_1c \in A^{\deg a_1 + \deg a_2 + \deg a_3 - 1}
\]

is closed. We define

\[
\langle a_1, a_2, a_3 \rangle \subset H_A^{\deg a_1 + \deg a_2 + \deg a_3 - 1}
\]

as the set of all cohomology classes \( [ba_3 + (-1)^{\deg a_1+1}a_1c] \) for primitives \( b, c \). One easily checks that this is an affine space over \([a_1]H_A + H_A[a_3]\) which depends only on the cohomology classes \( u_i = [a_i] \) of the \( a_i \). The set

\[
\langle u_1, u_2, u_3 \rangle := \langle a_1, a_2, a_3 \rangle \subset H_A^{\deg u_1 + \deg u_2 + \deg u_3 - 1}
\]

is called the **triple Massey product** of the cohomology classes \( u_1, u_2, u_3 \) with \( u_1u_2 = u_2u_3 = 0 \).

More generally, for each \( k \geq 3 \) and homogeneous cohomology classes \( u_1, \ldots, u_k \in H_A \) satisfying suitable conditions one obtains **Massey products**

\[
\langle u_1, \ldots, u_k \rangle \subset H_A^{\deg u_1 + \cdots + \deg u_3 + 2 - k}
\]

with the following properties:

(i) If \((A,d)\) has vanishing differential \( d = 0 \), then all Massey products are trivial in the sense that \( 0 \in \langle u_1, \ldots, u_k \rangle \).

---

\(^{16}\)The triple Massey product is often defined as an element of the quotient space \( H_A/(u_1H_A + H_Au_3) \), but the definition as a set is more suitable for the generalization to higher Massey products.
(ii) If \( f: A \to B \) is a DGA quasi-isomorphism, then
\[
f_* (u_1, \ldots, u_k) = (f_* u_1, \ldots, f_* u_k).
\]
Let us emphasize that, although the Massey products are subsets of \( H_A \), they depend on the DGA \( A \) and not just on its homology.

The following proposition allows us to extend property (ii) to \( A_\infty \) morphisms.

**Proposition 6.15.** Two DGAs \( A \) and \( B \) are \( A_\infty \) homotopy equivalent if and only if they can be connected by a zigzag of DGA quasi-isomorphisms.

**Proof.** We can describe DGAs as dg algebras over the nonsymmetric operad \( As \), and \( A_\infty \) algebras as infinity dg algebras over the operad \( As \), see [33, Chapter 10]. The result now follows immediately from [33, Theorem 11.4.9]. \( \square \)

**Proposition 6.15** together with the discussion above yields

**Lemma 6.16.** Let \( A \) and \( B \) be two DGAs which are \( A_\infty \) homotopy equivalent. If all Massey products are trivial on \( A \), then so they are on \( B \). \( \square \)

Let now \( M \) be a closed connected oriented manifold. By the discussion above, its de Rham complex \( \Omega^*(M) \) induces Massey products on its de Rham cohomology \( H_{dR}(M) \) which we call the *Massey products on \( M \). The homotopy transfer in Proposition 6.15 yields an \( A_\infty \) structure \( \{m^H_k \}_{k \geq 1} \) on \( H_{dR}(M) \) which is \( A_\infty \) homotopy equivalent to \( \Omega^*(M) \). On the other hand, we can consider \( (H_{dR}(M), d = 0, \wedge) \) as a DGA with trivial differential, and thus trivial Massey products by property (i) above. Therefore, Lemma 6.16 implies

**Corollary 6.17.** Let \( M \) be a closed connected oriented manifold. If the \( A_\infty \) algebra \( (H_{dR}(M), \{m^H_k \}_{k \geq 1}) \) is \( A_\infty \) homotopy equivalent to the DGA \( (H_{dR}(M), d = 0, \wedge) \), then all Massey products on \( M \) are trivial. \( \square \)

Let us now abbreviate \( H := H_{dR}(M) \) and choose a special analytic propagator \( P \). By Proposition 6.18, the \( A_\infty \) structure \( m^H = \{m^H_k \}_{k \geq 1} \) gives rise to an element \( m^H_+ \in B^{cyc} H \) and to the Kontsevich–Soibelman Maurer–Cartan element \( m^{KS} \) with \( m^H_0 = m^{KS} = m^H_+ \) and all other components zero. In view of Lemma 6.10 we have
\[
m^\text{ana}_{1,0} = m^{KS}_{1,0} = m^H_+
\]
for the analytic Maurer–Cartan element \( m^\text{ana} \) from Theorem 5.3. On the other hand, by Proposition 6.14 the DGA \( (H_{dR}(M), d = 0, \wedge) \) gives rise to the canonical Maurer–Cartan element \( m^\text{can} \) on \( B^{cyc} H \) with \( m^\text{can} = m^H_0 = m^H_+ \) and all other components zero. IBL formality of \( M \) means IBL\( A_\infty \) homotopy equivalence of the twisted IBL\( H \) structures on \( B^{cyc} H \) defined by \( m^\text{ana} \) and \( m^\text{can} \). In view of Corollary 6.17 Question 6.14 therefore comes down to understanding whether this condition implies \( A_\infty \) homotopy equivalence between the \( A_\infty \) structures encoded by the \( (1, 0) \) components of \( m^\text{ana} \) and \( m^\text{can} \).

### 6.6. Reduced version.

Let \( (A, d, \wedge, (\cdot, \cdot)) \) be a unital cyclic DGA of degree \( n \in \mathbb{Z} \) and \( dIBL(A) := (B^{cyc} A, d - n) \), \( p = \{p_{1,1,0}, p_{2,1,0}, p_{1,2,0}\} \) the canonical filtered dIBL algebra associated in Proposition 6.2 to the underlying cyclic cochain complex \( (A, d, (\cdot, \cdot)) \). We define the reduced dual cyclic bar complex of \( A \) by
\[
\overline{B^{cyc}} A := \{ \varphi \in B^{cyc} A \mid \varphi(1 \cdots) = 0 \} \subset B^{cyc} A.
\]
The operations \( p_{2,1,0}, p_{1,2,0} \) defined by (24), (25) clearly restrict to \( \overline{B^{cyc}} A \), and so does \( p_{1,1,0} = d^* \) because \( d(1) = 0 \). The restrictions of \( p_{1,1,0}, p_{2,1,0}, p_{1,2,0} \) then define
a filtered dIBL structure of degree \((n - 3)\) on \(\widetilde{B}^{\text{cyc}}A[2 - n]\). We will denote the corresponding filtered dIBL algebra by
\[
\text{dIBL}(A) := (\widetilde{B}^{\text{cyc}}A[2 - n], p = \{p_{1,1,0}, p_{2,1,0}, p_{1,2,0}\}).
\]
We call a Maurer–Cartan element \(m = \{m_{\ell,g}\}\) in dIBL\((A)\) strictly unital if:

(i) the cyclic \(A_\infty\) algebra \((A, \{m_1\}, \langle\cdot, \cdot\rangle)\) corresponding to \(m_{1,0}\) via Proposition 6.6 is strictly unital with unit 1 (see subsection 6.2 for the definition);

(ii) we have \(m_{\ell,g}(1 \cdots) = 0\) for all \((\ell, g) \neq (1, 0)\).

The following was observed in [24]:

Lemma 6.18. Let \((A, d, \wedge, \langle\cdot, \cdot\rangle)\) be a unital cyclic DGA of degree \(n \in \mathbb{Z}\), and let \(m = \{m_{\ell,g}\}\) be a strictly unital Maurer–Cartan element in dIBL\((A)\) = \((\widetilde{B}^{\text{cyc}}A[2 - n], p = \{p_{1,1,0}, p_{2,1,0}, p_{1,2,0}\})\). Then the twisted operations
\[
p_{1,\ell,g}^m, \hat{E}_1(B^{\text{cyc}}A)[2 - n] \rightarrow \hat{E}_1(B^{\text{cyc}}A)[2 - n]
\]
restrict to operations \(\hat{E}_1(B^{\text{cyc}}A)[2 - n] \rightarrow \hat{E}_1(B^{\text{cyc}}A)[2 - n]\) which together with the restriction of \(p_{2,1,0}\) define an IBL\(_\infty\) structure of degree \((n - 3)\) on \(\widetilde{B}^{\text{cyc}}A[2 - n]\).

Proof. The first two twisted operations can be written as
\[
p_{1,1,0}^m = p_{1,1,0} + p_{2,1,0} \circ m_{1,0},
p_{1,2,0}^m = p_{1,2,0} + p_{2,1,0} \circ m_{2,0},
\]
and the others as
\[
p_{1,\ell,g}^m = p_{2,1,0} \circ m_{\ell,g}.
\]
Given \(\varphi \in \hat{E}_1(B^{\text{cyc}}A)[2 - n]\) and \(\alpha = \alpha^1 \otimes \cdots \otimes \alpha^\ell, \alpha^i \in B^{\text{cyc}}A\), we have
\[
(45) \quad (p_{2,1,0} \circ m_{\ell,g})(\varphi)(\alpha) = \sum_{j=1}^\ell \sum_{i=1}^\ell \pm m_{\ell,g}^{(i)}(\alpha^1) \cdots \pm p_{2,1,0}(m_{\ell,g}^{(j)}(\varphi)(\alpha^j)) \cdots m_{\ell,g}^{(\ell)}(\alpha^\ell),
\]
where we used the following Sweedler’s notation for \(m_{\ell,g} \in \hat{E}_1(B^{\text{cyc}}A)[2 - n]\):
\[
m_{\ell,g} = \sum m_{\ell,g}^{(j)} \otimes \cdots \otimes m_{\ell,g}^{(\ell)}.
\]
Assumption (ii) on \(m\) gives
\[
m_{\ell,g}^{(j)} \in B^{\text{cyc}}A \quad \text{for all} \quad j \in \{1, \ldots, \ell\}, (\ell, g) \neq (1, 0),
\]
which implies that \(p_{1,\ell,g}^m\) restricts to \(\hat{E}_1(B^{\text{cyc}}A)[2 - n] \rightarrow \hat{E}_1(B^{\text{cyc}}A)[2 - n]\) for all \((\ell, g) \neq (1, 0)\). Given \(\varphi \in \hat{E}_1(B^{\text{cyc}}A)\) and \(x_2, \ldots, x_i \in A\), assumption (i) implies
\[
(p_{2,1,0} \circ m_{1,0})(\varphi)(1x_2 \cdots x_i) \quad = \pm p_{2,1,0}(m_{1,0,0})(1x_2 \cdots x_i) \quad = \pm (\varphi(m_2(1, x_2) x_3 \cdots x_i) + (-1)^{x_i} \varphi(x_2 \cdots m_2(x_i, 1))) = 0,
\]
hence \(p_{1,1,0}^m\) restricts to \(\hat{E}_1(B^{\text{cyc}}A)[2 - n] \rightarrow \hat{E}_1(B^{\text{cyc}}A)[2 - n]\) as well. \(\square\)

We denote the IBL\(_\infty\) algebra from the previous lemma by
\[
\text{dIBL}\tilde{\infty}(A) := (\widetilde{B}^{\text{cyc}}A[2 - n], p^m = \{p_{2,1,0}, p_{1,\ell,g}^m\} \text{ for } \ell \geq 1, g \geq 0\}).
\]
A useful observation from [24] is that if \(A\) is simply connected, then we have
\[
\hat{E}_\ell(B^{\text{cyc}}A[2 - n] \simeq E_\ell(B^{\text{cyc}}A[2 - n] \quad \text{for all } \ell \in \mathbb{N}_0.
\]
The following Maurer–Cartan elements are strictly unital:
(i) the canonical Maurer–Cartan element $m^\text{can}_A$ in the canonical filtered dIBL algebra $\text{dIBL}(A)$ associated to a unital cyclic DGA $A$;
(ii) the pushforward $i^!_B m^\text{can}_A$ in $\text{dIBL}(B)$ for a quasi-isomorphic cyclic cochain subcomplex $B \subset A$ of a nonegatively graded unital cyclic DGA $A$ satisfying $B^0 = \mathbb{R} \cdot 1$ along an IBL$\infty$ homotopy $i^!: \text{dIBL}(A) \to \text{dIBL}(B)$ associated to a special propagator $P: A \to A$ with respect to a projection $A \twoheadrightarrow B$ (by the positivity of degrees in Propositions 4.1(i)).
(iii) the Kontsevich–Soibelman Maurer–Cartan element $m^{\text{KS}}_B$ in $\text{dIBL}(B)$ for a quasi-isomorphic cyclic cochain subcomplex $B \subset A$ of a unital DGA $A$ with pairing $A$ satisfying $B^0 = \mathbb{R} \cdot 1$.

**Proposition 6.19.** Let $(A, d, \wedge, \langle \cdot, \cdot \rangle)$ be a nonegatively graded unital cyclic DGA, and let $B \subset A$ be a quasi-isomorphic cyclic cochain subcomplex such that $B^0 = \mathbb{R} \cdot 1$ and $B^1 = 0$. Let $P: A \to A$ be a special propagator with respect to a projection $A \twoheadrightarrow B$, and let $i^!: \text{dIBL}(A) \to \text{dIBL}(B)$ be the associated IBL$\infty$ homotopy equivalence. Consider the canonical Maurer–Cartan element $m^\text{can}_A$ in $\text{dIBL}(A)$ and its pushforward $i^!_B m^\text{can}_A$ in $\text{dIBL}(B)$. Let $m^{\text{KS}}_P$ be the Kontsevich–Soibelman Maurer–Cartan element in $\text{dIBL}(B)$ associated to $P$. Then we have:
$$\text{dIBL}^{i^!_B m^\text{can}_A}(B) = \text{dIBL}^{m^{\text{KS}}_P}(B).$$

**Proof.** For $n \geq 3$, the vanishing results in Proposition 4.1 imply that $i^!_B m^\text{can}_A = m^{\text{KS}}_P$. The case $n = 1$ is impossible and the case $n = 0$ is trivial ($P = 0$). Suppose therefore that $n = 2$ (the argument actually works for $n = 3$, too). Since $B$ is a cyclic cochain complex with $B^0 = \mathbb{R} \cdot 1$ and $B^1 = 0$, there is a unique $v \in B$ with $\deg v = 2$ and $\{1, v\} = 1$ such that $B = \text{span}\{1, v\}$. Consider the basis $e_1 = 1$, $e_2 = v$ of $B$. Then $g^{ij} = \langle e^i, e^j \rangle = 0$ unless $\{i, j\} = \{1, 2\}$ in $[24]$, hence the restriction $p_{2,1,0}: (B^{\text{sym}}B)^{\otimes 2} \to B^{\text{sym}}B$ vanishes (for the same reason $p_{1,2,0}$ vanishes). The restriction of the operation given by (16) for $(t, g) \neq (1, 0)$ to $B^{\text{sym}}B$ therefore vanishes for every strictly unital Maurer–Cartan element $m$ in $\text{dIBL}(B)$. The desired equality now follows from Lemma 6.9. □

The same proof works in the analytic case as well, and we have the following:

**Corollary 6.20.** Let $M$ be a closed oriented manifold such that its de Rham cohomology $H_{\text{dR}} := H_{\text{dR}}(M)$ is 1-connected, and let $P: \Omega := \Omega(M) \to \Omega$ be a special analytic propagator. Consider the following three Maurer–Cartan elements in $\text{dIBL}(H_{\text{dR}})$ associated to $P$: the analytic Maurer–Cartan element $m^\text{ana}_P$, the algebraic Maurer–Cartan element $m^{\text{KS}}_P$, and the Kontsevich–Soibelman Maurer–Cartan element $m^{\text{KS}}_P$. Then we have:
$$\text{dIBL}^{m^\text{ana}_P}(H_{\text{dR}}) = \text{dIBL}^{m^{\text{KS}}_P}(H_{\text{dR}}) = \text{dIBL}^{m^{\text{KS}}_P}(H_{\text{dR}}),$$
which is the dIBL algebra
$$\{B^{\text{sym}}H_{\text{dR}}[2 - n], q^{m^{\text{KS}}_P} = \{q_{1,1,0} = b^*, q_{2,1,0}, q_{1,2,0}\}\},$$
where $b: B^{\text{sym}}H_{\text{dR}} \to B^{\text{sym}}H_{\text{dR}}$ is the Hochschild differential associated to the homotopy transferred $A_{\infty}$ algebra $(H_{\text{dR}}, \{m\})$ associated to $P$.

In the situation of Corollary 6.20 we have moreover the following:

- If $M$ is in addition geometrically formal, then Corollary 5.5 implies that we can choose $P$ such that the dIBL algebra (46) equals the reduced canonical twisted dIBL algebra $\text{dIBL}^{m^\text{can}_A}(H_{\text{dR}})$, so that $b$ is the Hochschild differential for $(H_{\text{dR}}, \wedge)$. 

If \( M \) is formal and \( H_{dR} \) is 2-connected, then Corollary 6.12 implies that the \( dIBL \) algebra (46) is \( IBL_{\infty} \) homotopy equivalent to \( dIBL_{m} \) can \( H_{dR} \) (46).

Note that for \( n = \dim(M) \geq 3 \) we even have \( m_{m}^{\text{ana}} = m_{m}^{\text{alg}} = m_{m}^{\text{KS}} \) by Corollary 5.5.

The only reason why we cannot strengthen the equality (46) of reduced twisted \( IBL_{\infty} \) algebras to an equality of the nonreduced versions, or even to an equality of Maurer–Cartan elements, is the lack of vanishing results for \( M = S^{2} \).

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