Small-time solvability of a flow of forward-backward stochastic differential equations

Yushi Hamaguchi

Abstract

Motivated from time-inconsistent stochastic control problems, we introduce a new type of coupled forward-backward stochastic systems, namely, flows of forward-backward stochastic differential equations. They are systems consisting of a single forward SDE and a continuum of BSDEs, which are defined on different time-intervals and connected via an equilibrium condition. We formulate a notion of equilibrium solutions in a general framework and prove small-time well-posedness of the equations. We also consider discretized flows and show that their equilibrium solutions approximate the original one, together with an estimate of the convergence rate.

Keywords: Flow of forward-backward stochastic differential equations; equilibrium solution; time-inconsistency; stochastic control; backward stochastic Volterra integral equation.

1 Introduction

In this paper, we introduce a new type of coupled forward-backward stochastic systems, namely, flows of forward-backward stochastic differential equations. They are coupled systems consisting of a single forward stochastic differential equation (SDE) and a continuum of backward stochastic differential equations (BSDEs), which are defined on different time-intervals and connected via an equilibrium condition. The solution, which we call the equilibrium solution, of a flow of forward-backward SDEs consists of a family of processes \((X, \mathcal{Y}, \{Y^t, Z^t\}_{t \in [0,T]})\) where \(X\) and \(\mathcal{Y}\) are adapted processes defined on an interval \([0, T]\), \((Y^t, Z^t)\) is a pair of adapted processes defined on \([t, T]\) for each \(t \in [0, T]\), and they satisfy

\[
\begin{align*}
  dX_s &= B(s, X_s, \mathcal{Y}_s) \, ds + \Sigma(s, X_s, \mathcal{Y}_s) \, dW_s, \quad s \in [0, T], \\
  X_0 &= x_0, \\
  dY^t_s &= -F(t, X_t, s, X_s, \mathcal{Y}_s, Y^t_s, Z^t_s) \, ds + Z^t_s dW_s, \quad s \in [t, T], \quad t \in [0, T], \\
  Y^t_T &= G(t, X_t, \mathbb{E}_t[X_T], X_T), \\
  \mathcal{Y}_s &= Y^t_s, \quad \text{Leb-a.e.} \ s \in [0, T].
\end{align*}
\]

(1.1)
Here, $B$, $\Sigma$, $F$ and $G$ are given random functions, $x_0$ is a given initial condition for $X$, and $\mathbb{E}_t$ denotes the conditional expectation at time $t$. The equation consisting of the first and second lines of (1.1) is an SDE which determines the time-evolution of the process $X$ and, for each $t \in [0, T]$, the equation consisting of the third and fourth lines is a BSDE defined on $[t, T]$ which determines the pair of processes $(Y^t, Z^t)$. Finally, the fifth line of (1.1) represents the equilibrium condition which makes all the above equations be coupled. The arguments of $F$ and $G$ in the BSDEs depend on the current time $t$, the current state $X_t$ and the conditional expectation $\mathbb{E}_t[X_T]$. Thus, the system (1.1) is regarded as a generalization of classical forward-backward SDEs to a time-inconsistent setting.

This type of systems of equations appear in time-inconsistent stochastic control problems and characterize their subgame perfect Nash equilibria. Time-inconsistent control problems are recently studied by Ekeland and Lazrak (2010) [6], Yong (2012) [20], Björk, Murgoci and Zhou (2014) [2], Hu, Jin and Zhou (2012, 2017) [8, 9], Djehiche and Huang (2016) [5], Björk, Khapko and Murgoci (2017) [1], Wei, Yong and Yu (2017) [18], Ni, Zhang and Krstic (2018) [13], among others. Time-inconsistency occurs for example when a non-exponential discount rate is considered or when the cost functional is a nonlinear function of (conditional) expectation of a state process such as dynamic mean-variance control problems. Unlike classical control problems the so-called Bellman principle does not hold in these cases. In other words, a strategy which is optimal at a given starting point is no longer optimal when viewed from a later date and different state. Thus, we have to reconsider the concept of “optimality”. An alternative concept of optimality in time-inconsistent problems is the subgame perfect Nash equilibrium, which is a game theoretic concept. We overview its definition and connection to flows of forward-backward SDEs by informal arguments below. For a detailed discussion of subgame perfect Nash equilibria for time-inconsistent stochastic control problems, see [1, 5, 8, 9].

Let $u$ be a control process taking values in a Borel subset $U$ of a Euclidean space, and $x^u$ be the corresponding controlled state process defined as the unique solution of the SDE

\[
\begin{aligned}
\begin{cases}
\frac{dx^u_s}{ds} = b(s, x^u_s, u_s) \, ds + \sigma(s, x^u_s) \, dW_s, \quad s \in [0, T], \\
x^u_0 = x_0.
\end{cases}
\end{aligned}
\]

Define the player’s cost functional that is viewed at time $t \in [0, T]$ by

\[
J_t(u; x^u_t) := \mathbb{E}_t \left[ \int_t^T f(t, x^u_t, s, x^u_s, u_s) \, ds + g(t, x^u_t, x^u_T) + h(t, x^u_t, \mathbb{E}_t[x^u_T]) \right].
\]

(1.2)

Here, we assume that all given functions $b$, $\sigma$, $f$, $g$, $h$ are deterministic, one-dimensional and sufficiently smooth for simplicity. In the following sections, we consider multi-dimensional and random coefficients.

The player’s objective is to search for an “optimal” strategy through the time-interval $[0, T]$. This problem is time-inconsistent since (i) the cost functional depends on the current time and state $(t, x^u_t)$, and (ii) the second term of the right hand side of (1.2) is a (nonlinear) function of the conditional expectation of the terminal state. We call $\hat{u}$ a subgame perfect Nash equilibrium strategy and $\hat{x} := x^{\hat{u}}$ the corresponding equilibrium state process if it
satisfies
\[
\liminf_{\epsilon \to 0} \frac{J_t(u^{t,\epsilon,v}; \hat{x}_t) - J_t(\hat{u}; \hat{x}_t)}{\epsilon} \geq 0
\]
for any \( t \in [0, T) \) and control \( v \), where \( u^{t,\epsilon,v} \) is the “spike variation” of \( \hat{u} \) at time \( t \) with respect to \( v \), namely, \( u^{t,\epsilon,v}_s := v_s \) if \( s \in [t, t + \epsilon) \) and \( u^{t,\epsilon,v}_s := \hat{u}_s \) otherwise. Then, by a version of the stochastic maximum principle (see \([3, 5]\)), a subgame perfect Nash equilibrium strategy \( \hat{u} \) must satisfy the relation
\[
H(t, \hat{x}_t, t, \hat{x}_t, \hat{u}_t, p^1_t, q^1_t) \leq H(t, \hat{x}_t, t, \hat{x}_t, v, p^1_t, q^1_t)
\]
for any \( t \in [0, T) \) and \( v \in U \). Here, the function \( H \) is the Hamiltonian defined by
\[
H(t, \xi, s, x, u, p, q) := b(s, x, u)p + \sigma(s, x)q + f(t, \xi, s, x, u)
\]
for \( 0 \leq t \leq s \leq T \) and \( \xi, x, p, q \in \mathbb{R} \) and, for each \( t \in [0, T] \), \((p^t_s, q^t_s)_{s \in [t, T]}\) is the solution of the corresponding (first-order) adjoint equation, namely, the BSDE
\[
\begin{align*}
\frac{d\hat{x}_s}{dt} & = -\partial_x H(t, \hat{x}_t, s, \hat{x}_s, \hat{u}_s, p^1_s, q^1_s) + \frac{\partial_x \hat{u}_s}{\partial s} dW_s, \quad s \in [t, T], \\
p^1_T & = \partial_x g(t, \hat{x}_T) + \partial_x h(t, \hat{x}_T, \mathbb{E}_t[\hat{x}_T]), \\
\partial_x H(t, \hat{x}_t, s, \hat{x}_s, \hat{u}_s, p^1_s, q^1_s) & = b(s, \hat{x}_s, \hat{u}_s, p^1_s, q^1_s) - \partial_x \hat{u}_s \partial_u g(s, \hat{x}_s, \hat{u}_s, p^1_s, q^1_s),
\end{align*}
\]
where \( \partial_x H \) and \( \partial_x g \) are the partial derivatives of \( H \) and \( g \) with respect to the \( x \)-variables (the fourth variable of \( H \) and the third variable of \( g \), respectively) and \( \partial_x h \) is the partial derivative of \( h \) with respect to the \( x \)-variable (the third variable of \( h \)).

If the function \( U \ni u \mapsto H(t, x, t, x, u, p, q) \in \mathbb{R} \) has a unique minimizer \( \hat{u}(t, x, p) \) for each \( t \in [0, T) \) and \( x, p, q \in \mathbb{R} \), which is independent of \( q \) since the volatility \( \sigma \) is uncontrolled in this case, and \( \hat{u}(t, x, p) \) satisfies an appropriate regularity condition, then \( \hat{u}_t = \hat{u}(t, \hat{x}_t, p^1_t) \), \( t \in [0, T] \), and the subgame perfect Nash equilibrium is characterized (at least formally) by the following:
\[
\begin{align*}
\frac{d\hat{x}_s}{dt} & = b(s, \hat{x}_s, \hat{u}(s, \hat{x}_s, p^1_s)) + \sigma(s, \hat{x}_s), \quad s \in [0, T], \\
\hat{x}_0 & = x_0, \\
\frac{d\hat{u}_s}{dt} & = -\partial_x H(t, \hat{x}_t, s, \hat{x}_s, \hat{u}(s, \hat{x}_s, p^1_s)) + \frac{\partial_x \hat{u}(s, \hat{x}_s, p^1_s)}{\partial x} dW_s, \quad s \in [t, T], \\
p^1_T & = \partial_x g(t, \hat{x}_T) + \partial_x h(t, \hat{x}_T, \mathbb{E}_t[\hat{x}_T]), \\
p^1_t & = \partial_x H(t, \hat{x}_t, s, \hat{x}_s, \hat{u}(s, \hat{x}_s, p^1_s), p^1_s, q^1_s) + \frac{\partial_x \hat{u}(s, \hat{x}_s, p^1_s)}{\partial x} dW_s + q^1_s dW_s, \quad s \in [t, T], \\
p^1_t & = \partial_x g(t, \hat{x}_T) + \partial_x h(t, \hat{x}_T, \mathbb{E}_t[\hat{x}_T]), \\
p^1_T & = \partial_x g(t, \hat{x}_T) + \partial_x h(t, \hat{x}_T, \mathbb{E}_t[\hat{x}_T]),
\end{align*}
\]
This system is a special case of our equations (1.1). In this paper, we investigate its small-time solvability in a more general setting.

Although characterizations of subgame perfect Nash equilibria by flows of forward-backward SDEs are suggested in some papers, there are only a few studies about solvability of the equations. Hu, Jin and Zhou \([8, 9]\) studied linear-quadratic time-inconsistent stochastic control problems. They derived a flow of affine forward-backward SDEs with random coefficients characterizing the subgame perfect Nash equilibria and solved it by using Riccati-like equations only when the state is one-dimensional and all the coefficients are deterministic. Djehiche and Huang \([3]\) studied time-inconsistent mean-field stochastic control problems and characterized the subgame perfect Nash equilibria by a flow of forward-backward SDEs, while
their models are assumed to be deterministic and solvability of the equations were not discussed. In this paper, in contrast to the above-mentioned papers, we investigate a flow of forward-backward SDEs with general and random coefficients and solve it by using a priori estimates of SDEs and BSDEs when the time-interval is sufficiently small. Our idea to prove small-time solvability of (1.1) is to consider discrete flows of forward-backward SDEs (2.1). We can prove existence and uniqueness of the (discrete) equilibrium solution of a discrete flow directly by regarding the flow as a system consisting of finitely many forward-backward SDEs constructed by a backward induction and then applying the fixed point argument. Then, we show uniform estimates and convergence of discrete-equilibrium solutions. Lastly, we prove the limit of discrete-equilibrium solutions is the equilibrium solution of the original flow of forward-backward SDEs. Moreover, we provide an estimate of its convergence rate in Theorem 2.4. We hope that our approximation results provide a new insight to calculate a flow of forward-backward SDEs.

The system (1.1) is also regarded as a generalization of backward stochastic Volterra integral equations (BSVIEs) that were introduced by Lin (2002) [11] and studied by Yong (2006) [19], Shi and Wang (2012) [13], Li, Wu and Wang (2014) [10], Shi, Wang and Yong (2015) [15], Wang and Zhang (2017) [17], Wang and Yong (2019) [16], among others. Indeed, if the processes \( X \) and \( Y^t \), for each \( t \in [0, T] \), are given and the equilibrium condition is satisfied, the third and fourth lines of (1.1) become a (Type-I) BSVIE for the processes \( \{Y, \{Z^t\}_{t \in [0, T]}\} \) of the following form:

\[
Y_t = G(t, X_t, \mathbb{E}_t[X_T], X_T) + \int_t^T F(t, X_t, s, X_s, Y_s, Y^s_t, Z^s_t) \, ds - \int_t^T Z^s_t \, dW_s, \quad t \in [0, T].
\]

Thus, a flow of forward-backward SDEs (1.1) can be regarded as a fully coupled system consisting of an SDE, a BSVIE and a continuum of BSDEs. We remark on this matter in Section 4.4.

Our paper is organized as follows: In Section 2 we state the notations and our main results (Theorem 2.4). Section 3 is devoted to the proof of the main theorem. In Section 4 we investigate further properties of equilibrium solutions, and give an alternative proof of small-time solvability of flows of forward-backward SDEs by applying the fixed point argument directly to (1.1). Lastly, we provide some remarks and future problems.

### 2 A flow of forward-backward SDEs

#### 2.1 Notations

In this subsection, we summarize the notations we use throughout the paper.

\( W = (W_t)_{t \geq 0} \) is a \( d \)-dimensional Brownian motion defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \) is the augmentation of the filtration generated by \( W \). We sometimes omit the dependency of \( \omega \). For a set \( A \), we denote by \( 1_A \) the indicator function of \( A \). \( \text{Leb} \) denotes the Lebesgue measure on an interval.

For $0 \leq t \leq T < \infty$ and $D \in \mathbb{N}$, we define $\mathbb{E}_t[\cdot] := \mathbb{E}[^{\cdot}|\mathcal{F}_t]$, $\Delta_{[0,T]} := \{(t, s) \mid 0 \leq t \leq s \leq T\}$,

$$S_{[t,T]}^{2,D} := \left\{ \chi = (\chi_s)_{s \in [t,T]} \mid \chi \text{ is } \mathbb{R}^D\text{-valued, } (\mathcal{F}_s)_{s \in [t,T]}\text{-progressively measurable, continuous and satisfies } \mathbb{E}\left[\sup_{t \leq s \leq T} |\chi_s|^2\right] < \infty. \right\},$$

and

$$\mathbb{H}_{[t,T]}^{2,D} := \left\{ \chi = (\chi_s)_{s \in [t,T]} \mid \chi \text{ is } \mathbb{R}^D\text{-valued, } (\mathcal{F}_s)_{s \in [t,T]}\text{-progressively measurable and satisfies } \mathbb{E}\left[\int_t^T |\chi_s|^2 \, ds\right] < \infty. \right\}.$$

Note that $S_{[t,T]}^{2,D}$ and $\mathbb{H}_{[t,T]}^{2,D}$ are Banach spaces with respect to appropriate norms.

For each partition $\Pi = \{t_n \mid n = 0, 1, \ldots, N\}$ with $0 = t_0 < t_1 < \cdots < t_N = T$ of a compact interval $[0, T]$, we denote by $\|\Pi\|$ the mesh of $\Pi$, namely,

$$\|\Pi\| := \max_{n=1,\ldots,N} (t_n - t_{n-1}).$$

### 2.2 Statements of the main theorem

For $T > 0$, we consider the system (1.1). We call this system a **flow of forward-backward stochastic differential equations** and we use the notation $\text{FFBSDE}(T)$, where $T > 0$ represents the terminal time of the system and the term “FFBSDE” stands for a “Flow of Forward-Backward Stochastic Differential Equations”. The system (1.1) consists of a single forward SDE (the first and second lines) and a continuum of BSDEs (the third and fourth lines), which are coupled via the equilibrium condition (the fifth line).

We impose the following assumptions on the coefficients.

**Assumption. (A1)** $x_0 \in \mathbb{R}^d$. The mappings

$$\Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \ni (\omega, s, x, \eta) \mapsto B(\omega, s, x, \eta) \in \mathbb{R}^d,$$

$$\Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \ni (\omega, s, x, \eta) \mapsto \Sigma(\omega, s, x, \eta) \in \mathbb{R}^{d \times d}$$

are $(\mathcal{F}_s)_{s \in [0,T]}$-progressively measurable. Moreover, for each $t \in [0, T]$, the mapping

$$\Omega \times \mathbb{R}^d \times [t, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \ni (\omega, \xi, s, x, \eta, y, z) \mapsto F(\omega, t, \xi, s, x, \eta, y, z) \in \mathbb{R}^m$$

is $(\mathcal{F}_s)_{s \in [t,T]}$-progressively measurable and the mapping

$$\Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \ni (\omega, \xi, \bar{x}, x) \mapsto G(\omega, t, \xi, \bar{x}, x) \in \mathbb{R}^m$$

is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable.

**Assumption. (A2)**

$$R := \mathbb{E}\left[\int_0^T \left(|B(s, 0, 0)|^2 + |\Sigma(s, 0, 0)|^2\right) \, ds\right] + \sup_{t \in [0,T]} \mathbb{E}\left[\int_t^T |F(t, 0, s, 0, 0, 0)|^2 \, ds + |G(t, 0, 0, 0)|^2\right] < \infty.$$
(A3) There exists a constant $L > 0$ such that:

(i) For $\mathbb{P} \otimes \mathrm{Leb}$-a.e. $(\omega, s) \in \Omega \times [0, T]$, it holds that

$$|B(\omega, s, x, \eta) - B(\omega, s, x', \eta')| + |\Sigma(\omega, s, x, \eta) - \Sigma(\omega, s, x', \eta')| \leq L(|x - x'| + |\eta - \eta'|)$$

for any $x, x' \in \mathbb{R}^d$ and $\eta, \eta' \in \mathbb{R}^m$.

(ii) For any $t \in [0, T]$, for $\mathbb{P} \otimes \mathrm{Leb}$-a.e. $(\omega, s) \in \Omega \times [t, T]$, it holds that

$$|F(\omega, t, \xi, s, x, \eta, y, z) - F(\omega, t, \xi', s, x', \eta', y', z')|$$

$$\leq L(|\xi - \xi'| + |x - x'| + |\eta - \eta'| + |y - y'| + |z - z'|)$$

for any $\xi, \xi', x, x' \in \mathbb{R}^d$, $\eta, \eta', y, y' \in \mathbb{R}^m$ and $z, z' \in \mathbb{R}^{m \times d}$.

(iii) For any $t \in [0, T]$, for $\mathbb{P}$-a.e. $\omega \in \Omega$, it holds that

$$|G(\omega, t, \xi, \bar{x}, x) - G(\omega, t, \xi, \bar{x}', x')| \leq L(|\xi - \xi'| + |\bar{x} - \bar{x}'| + |x - x'|)$$

for any $\xi, \xi', \bar{x}, \bar{x}', x, x' \in \mathbb{R}^d$.

(A4) There exists an increasing function $\rho: [0, \infty) \to [0, \infty)$ with $\lim_{t \downarrow 0} \rho(t) = \rho(0) = 0$ such that, for $\mathbb{P}$-a.e. $\omega \in \Omega$, it holds that

$$|F(\omega, t, \xi, s, x, \eta, y, z) - F(\omega, t', \xi, s, x, \eta, y, z)| + |G(\omega, t, \xi, \bar{x}, x) - G(\omega, t', \xi, \bar{x}, x)|$$

$$\leq \rho(|t - t'|)(1 + |\xi| + |x| + |\bar{x}| + |\eta| + |y| + |z|)$$

for any $s \in [0, T]$ for any $t, t' \in [0, s]$, $\xi, x, \bar{x} \in \mathbb{R}^d$, $\eta, y \in \mathbb{R}^m$ and $z \in \mathbb{R}^{m \times d}$.

**Definition 2.1.** For each $T > 0$, we call a family of processes $\{X, Y, \{Y^t, Z^t\}_{t \in [0, T]}\}$ an equilibrium solution of FFBSDE($T$) if $X \in \mathcal{S}^{2, m}_{[0, T]}$, $Y \in \mathcal{H}^{2, m}_{[0, T]}$, $(Y^t, Z^t) \in \mathcal{H}^{2, m}_{[0, T]} \times \mathcal{H}^{2, m \times d}_{[0, T]}$ for any $t \in [0, T]$, the process $(Y^s)_{s \in [0, T]}$ is progressively measurable and they satisfy equations in (1.1) $\mathbb{P}$-a.s., for any $t \in [0, T]$. We say that the solution is unique if, for any other equilibrium solution $\{ \tilde{X}, \tilde{Y}, \{\tilde{Y}^t, \tilde{Z}^t\}_{t \in [0, T]} \}$, it holds that

$$\mathbb{E} \left[ \sup_{0 \leq s \leq T} \left| X_s - \tilde{X}_s \right|^2 + \int_0^T \left| Y_s - \tilde{Y}_s \right|^2 \, ds \right] = 0,$$

$$\mathbb{E} \left[ \sup_{t \leq s \leq T} \left| Y^t_s - \tilde{Y}^t_s \right|^2 + \int_t^T \left| Z^t_s - \tilde{Z}^t_s \right|^2 \, ds \right] = 0, \quad \forall t \in [0, T].$$

**Remark 2.2.** Since the system (1.1) has a continuum of backward equations, we need to be careful for “$\mathbb{P}$-a.s.” validity of equations and measurability of the process $(Y^s)_{s \in [0, T]}$. Our definition of equilibrium solutions imposes that, for each $t \in [0, T]$, $(Y^t_s, Z^t_s)_{s \in [t, T]}$ solves the BSDE on $[t, T]$ $\mathbb{P}$-a.s., its null set being allowed to depend on $t \in [0, T]$, and $(Y^s)_{s \in [0, T]}$ is progressively measurable. In fact, under Assumptions (A1)–(A4), for any $X \in \mathcal{S}^{2, d}_{[0, T]}$ and $Y \in \mathcal{H}^{2, m}_{[0, T]}$, the process $(Y^t)_{t \in [0, T]}$ where each random variable $Y^t$ is defined as the time-$t$ value of the solution of the BSDE

$$dY^t_s = -F(t, X_t, s, Y_s, Y^t_s, Z^t_s) \, ds + Z^t_s \, dW_s, \quad s \in [t, T],$$

$$Y^t_T = G(t, X_t, \mathbb{E}_t[X_T], X_T),$$

has a progressively measurable version; see Lemma 3.6.
We also consider a discrete flow of forward-backward stochastic differential equations. For $T > 0$ and a partition $\Pi = \{t_n \mid n = 0, 1, \ldots, N\}$ with $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0, T]$, the discrete flow of forward-backward SDEs is defined by the following:

\[
\begin{aligned}
&\left\{
\begin{array}{l}
\d X_s^\Pi = B\left(s, X_s^\Pi, Y_s^\Pi\right) \, ds + \Sigma\left(s, X_s^\Pi, Y_s^\Pi\right) \, dW_s, \ s \in [0, T], \\
\d X_0^\Pi = x_0, \\
\end{array}
\right.
\end{aligned}
\]

This system is denoted by FFBSDE$_\Pi(T)$.

**Definition 2.3.** For each $T > 0$ and a partition $\Pi$ of $[0, T]$, we call a sequence of processes $(X^\Pi, Y^\Pi, \{Y_n^\Pi, Z_n^\Pi\}_{n=1,\ldots,N})$ a discrete-equilibrium solution of FFBSDE$_\Pi(T)$ if $X^\Pi \in \mathbb{S}^d_{[0,T]}$, $Y^\Pi \in \mathbb{H}^{2,m}_{[0,T]}$, $(Y_n^\Pi, Z_n^\Pi) \in \mathbb{S}^2_{(t_{n-1},T)} \times \mathbb{H}^{2,m \times d}_{(t_{n-1},T)}$ for any $n = 1, \ldots, N$ and equation (2.1) holds $\mathbb{P}$-a.s. We say that the solution is unique if, for any other discrete-equilibrium solution $(\tilde{X}^\Pi, \tilde{Y}^\Pi, \{\tilde{Y}_n^\Pi, \tilde{Z}_n^\Pi\}_{n=1,\ldots,N})$, it holds that

\[
\begin{aligned}
&\mathbb{E} \left[ \sup_{0 \leq s \leq T} \left| X_s^\Pi - \tilde{X}_s^\Pi \right|^2 + \int_0^T \left| Y_s^\Pi - \tilde{Y}_s^\Pi \right|^2 \, ds \right] = 0, \\
&\mathbb{E} \left[ \sup_{t_{n-1} \leq s \leq T} \left| Y_n^\Pi - \tilde{Y}_n^\Pi \right|^2 + \int_{t_{n-1}}^T \left| Z_s^\Pi - \tilde{Z}_s^\Pi \right|^2 \, ds \right] = 0, \forall n = 1, \ldots, N.
\end{aligned}
\]

The following theorem is our main result in this paper.

**Theorem 2.4.** The following assertions hold:

(I) Under Assumptions (A1)–(A3), there exists a constant $T_0 > 0$ which depends only on $L$ such that, for any $0 < T \leq T_0$ and partition $\Pi = \{t_n \mid n = 0, 1, \ldots, N\}$ of $[0, T]$, there exists a unique discrete-equilibrium solution $(X^\Pi, Y^\Pi, \{Y_n^\Pi, Z_n^\Pi\}_{n=1,\ldots,N})$ of FFBSDE$_\Pi(T)$.

(II) Under Assumptions (A1)–(A4), there exists a constant $T_0 > 0$ which depends only on $L$ such that, for any $0 < T \leq T_0$:

(a) There exists a unique equilibrium solution $(X, Y, \{Y^\prime, Z^\prime\}_{t \in [0,T)})$ of FFBSDE(T).

(b) There exists a constant $C > 0$ which depends only on $L$ such that

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} \left| X_s^\Pi - X_s \right|^2 + \int_0^T \left| Y_s^\Pi - Y_s \right|^2 \, ds \right] \leq C \left( R + \left| x_0 \right|^2 \right) (T \rho (\left\| \Pi \right\|)^2 + \left\| \Pi \right\|)
\]

for any partition $\Pi$ of $[0, T]$. 


3 Proof of Theorem 2.4

We use the following standard lemmas about SDEs (Lemma 3.1) and BSDEs (Lemma 3.2) several times.

**Lemma 3.1.** Under Assumptions (A1)–(A3), for any $0 < T \leq T_0$ and $\mathcal{Y} \in \mathbb{H}^{2,m}_{[0,T]}$, there exists a unique solution $X \in \mathbb{S}^{2,d}_{[0,T]}$ of the SDE

$$
\begin{aligned}
&dX_s = B(s, X_s, \mathcal{Y}_s) \, ds + \Sigma(s, X_s, \mathcal{Y}_s) \, dW_s, \quad s \in [0, T], \\
&X_0 = x_0.
\end{aligned}
$$

Moreover, for any $0 < T \leq T_0$, $\mathcal{Y}^1, \mathcal{Y}^2 \in \mathbb{H}^{2,m}_{[0,T]}$, $x^1_0, x^2_0 \in \mathbb{R}^d$ and coefficients $(B^1, \Sigma^1)$, $(B^2, \Sigma^2)$ satisfying Assumptions (A1)–(A3) with constants $(R_1, L_1)$ and $(R_2, L_2)$ respectively, there exists a constant $C > 0$ which depends only on $T_0$ and $L_1$ such that the solutions $X^1, X^2 \in \mathbb{S}^{2,d}_{[0,T]}$ of the SDEs (3.1) with $(x^1_0, B^1, \Sigma^1)$ and $(x^2_0, B^2, \Sigma^2)$, respectively, satisfy

$$
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |X^1_s - X^2_s|^2 \right] \leq C \left( |x^1_0 - x^2_0|^2 + \mathbb{E} \left[ \int_0^T |\mathcal{Y}^1_s - \mathcal{Y}^2_s|^2 \, ds + \int_0^T \left( (B^1 - B^2, \Sigma^1 - \Sigma^2)(s, X^1_s, \mathcal{Y}^1_s) \right)^2 \, ds \right] \right).
$$

**Proof.** For each $\mathcal{Y} \in \mathbb{H}^{2,m}_{[0,T]}$, the coefficients $b(\omega, s, x) := B(\omega, s, x, \mathcal{Y}_s(\omega))$ and $\sigma(\omega, s, x) := \Sigma(\omega, s, x, \mathcal{Y}_s(\omega))$ satisfy:

(i) $b(\cdot, \cdot, 0), \sigma(\cdot, \cdot, 0) \in \mathbb{H}^{2,d}_{[0,T]}$;

(ii) $|b(\omega, s, x) - b(\omega, s, x')| + |\sigma(\omega, s, x) - \sigma(\omega, s, x')| \leq L|x - x'|$, for any $x, x' \in \mathbb{R}^d$, for $\mathbb{P} \otimes \text{Leb-a.e.} \ (\omega, s) \in \Omega \times [0, T]$.

Hence the SDE (3.1) has a unique solution in $\mathbb{S}^{2,d}_{[0,T]}$; see for example Chapter 3 of the textbook [21]. Now we shall prove the estimate (3.2). By using the Burkholder–Davis–Gundy inequality and $L_1$-Lipschitz continuity of $(B^1, \Sigma^1)$, we easily see that, for each $t \in [0, T]$,

$$
\begin{align*}
&\mathbb{E} \left[ \sup_{0 \leq r \leq t} |X^1_r - X^2_r|^2 \right] \\
&\quad \leq C \left( |x^1_0 - x^2_0|^2 + \mathbb{E} \left[ \int_0^t \left( (B^1, \Sigma^1)(s, X^1_s, \mathcal{Y}^1_s) - (B^1, \Sigma^1)(s, X^2_s, \mathcal{Y}^2_s) \right)^2 \, ds \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \int_0^t \left( (B^1 - B^2, \Sigma^1 - \Sigma^2)(s, X^1_s, \mathcal{Y}^1_s) \right)^2 \, ds \right] \right) \\
&\quad \leq C \left( |x^1_0 - x^2_0|^2 + \mathbb{E} \left[ \int_0^T |\mathcal{Y}^1_s - \mathcal{Y}^2_s|^2 \, ds + \int_0^T \left( (B^1 - B^2, \Sigma^1 - \Sigma^2)(s, X^1_s, \mathcal{Y}^1_s) \right)^2 \, ds \right] \right) \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \int_0^t \mathbb{E} \left[ \sup_{0 \leq r \leq s} |X^1_r - X^2_r|^2 \right] \, ds \right),
\end{align*}
$$

8
where $C > 0$ is a constant which depends only on $T_0$ and $L_1$ and is allowed to vary from line to line. By Gronwall’s inequality, we obtain (3.2).

We refer to [7] for the following lemma, which is a well-known fact of BSDE theory.

**Lemma 3.2.** Suppose that we are given $0 \leq t \leq T \leq T_0$, $\eta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ and a measurable function $\psi: \Omega \times [t, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^m$. Assume that $\psi(\cdot, \cdot, 0, 0) \in \mathbb{H}^{2,m}_{[t, T]}$ and $\psi$ is uniformly Lipschitz, i.e., there is a constant $K > 0$ such that, for $\mathbb{P} \otimes \text{Leb-a.e.} \ (\omega, s) \in \Omega \times [t, T]$, it holds that

$$|\psi(\omega, s, y^1, z^1) - \psi(\omega, s, y^2, z^2)| \leq K \left(|y^1 - y^2| + |z^1 - z^2|\right)$$

for any $y^1, y^2 \in \mathbb{R}^m$ and $z^1, z^2 \in \mathbb{R}^{m \times d}$. Then, there exists a unique solution $(Y, Z) \in \mathcal{S}^{2,m}_{[t, T]} \times \mathbb{H}^{2,m \times d}_{[t, T]}$ of the BSDE

$$\begin{cases}
    dY_s = -\psi(s, Y_s, Z_s) \, ds + Z_s \, dW_s, \ s \in [t, T], \\
    Y_T = \eta.
\end{cases} \quad (3.3)$$

Moreover, for any $0 \leq t \leq T \leq T_0$, $\eta^1, \eta^2 \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ and functions $\psi^1$ and $\psi^2$ satisfying the above assumptions with Lipschitz constants $K_1, K_2 > 0$, respectively, there exists a constant $C > 0$ which depends only on $T_0$ and $K_1$ such that the solutions $(Y^1, Z^1), (Y^2, Z^2) \in \mathcal{S}^{2,m}_{[t, T]} \times \mathbb{H}^{2,m \times d}_{[t, T]}$ of the BSDEs (3.3) with $(\eta^1, \psi^1)$ and $(\eta^2, \psi^2)$, respectively, satisfy

$$\mathbb{E} \left[ \sup_{t \leq s \leq T} |Y^1_s - Y^2_s|^2 + \int_t^T |Z^1_s - Z^2_s|^2 \, ds \right] \leq C \mathbb{E} \left[ |\eta^1 - \eta^2|^2 + \int_t^T (\psi^1 - \psi^2) (s, Y^2_s, Z^2_s)^2 \, ds \right].$$

Now we prove the first statement of Theorem 2.4, namely, small-time well-posedness of discrete flows of forward-backward SDEs. The method of proof is to regard FFBSDE$_\Pi(T)$ as a system consisting of finitely many forward-backward SDEs constructed by a backward induction and apply the fixed point argument.

**Proof of Theorem 2.4 (I).** Fix $T > 0$ and $\Pi = \{t_n \mid n = 0, 1, \ldots, N\}$ with $0 = t_0 < t_1 < \cdots < t_N = T$. If $N = 1$, FFBSDE$_\Pi(T)$ is a standard (mean-field) forward-backward SDE and hence the assertion holds; see the textbook [4]. So we assume that $N \geq 2$. For a given $X \in \mathcal{S}^{2,d}_{[0, T]}$, define $\Phi^{T, \Pi}_N(X) \in \mathcal{S}^{2,m}_{[t_{n-1}, T]} \times \mathbb{H}^{2,m \times d}_{[t_{n-1}, T]}$, $n = 1, \ldots, N$, $\Psi^{T, \Pi}_N(X) \in \mathbb{H}^{2,m}_{[0, T]}$ and $\Xi^{T, \Pi}_N(X) \in \mathcal{S}^{2,d}_{[0, T]}$ by the following procedure:

(i) Define $\Phi^{T, \Pi}_N(X)$ by $\Phi^{T, \Pi}_N(X) := (Y^N, Z^N)$, where $(Y^N, Z^N) \in \mathcal{S}^{2,m}_{[t_{n-1}, T]} \times \mathbb{H}^{2,m \times d}_{[t_{n-1}, T]}$ is a solution of the BSDE

$$\begin{cases}
    dY^N_s = -F \left(t_{n-1}, X_{t_{n-1}}, s, X_s, Y^N_s, Y^N_s, Z^N_s\right) \, ds + Z^N_s \, dW_s, \ s \in [t_{n-1}, T], \\
    Y^N_T = G \left(t_{n-1}, X_{t_{n-1}}, [X_{t_{n-1}}], [X_{t_{n-1}}], X_T\right).
\end{cases} \quad (3.4)$$

Note that, under our assumptions, the BSDE (3.4) has a unique solution in $\mathcal{S}^{2,m}_{[t_{n-1}, T]} \times \mathbb{H}^{2,m \times d}_{[t_{n-1}, T]}$; see Lemma 3.2.
By the above procedure, we can construct the mappings \( \Phi^{T,\Pi}_T = 1 \times X \) only if \( SYL > 0 \) \( \) (iii) Let \( \Phi^{T,\Pi}_n(X) \) be the sequence of pairs of processes obtained by the backward induction procedure (i) and (ii). Then, define \( \Psi^{T,\Pi}_n(X) \) by \( \Phi^{T,\Pi}_n(X) := (Y^n, Z^n) \), where \( (Y^n, Z^n) \in S^{2, m}_{[t_{n-1}, T]} \times H^{2, m \times d}_{[t_{n-1}, T]} \) is a solution of the BSDE

\[
\begin{cases}
dY^n_s = -F\left(t_{n-1}, X_{t_{n-1}}, s, X_s, Y^n_s \mathbb{1}_{(t_{n-1}, t_n)}(s) + \sum_{j=n+1}^N Y^j_s \mathbb{1}_{(t_{j-1}, t_j)}(s), Y^n_s, Z^n_s \right) ds \\
+ Z^n_s dW_s, \ s \in [t_{n-1}, T], \\
Y^n_T = G\left(t_{n-1}, X_{t_{n-1}}, \mathbb{E}_{t_{n-1}}[X_T], X_T \right).
\end{cases}
\]

Again, we can show that the BSDE (3.5) has a unique solution in \( S^{2, m}_{[t_{n-1}, T]} \times H^{2, m \times d}_{[t_{n-1}, T]} \).

(iii) Let \( \{ \Phi^{T,\Pi}_n(X) = (Y^n, Z^n) \}_{n=1}^N \) be the sequence of pairs of processes obtained by the backward induction procedure (i) and (ii). Then, define \( \Psi^{T,\Pi}_n(X) \) by \( \Phi^{T,\Pi}_n(X) := \sum_{n=1}^N Y^n_s \mathbb{1}_{(t_{n-1}, t_n)}(s), s \in [0, T] \).

(iv) Finally, define \( \Xi^{T,\Pi}(X) \) by \( \Xi^{T,\Pi}(X) := \tilde{X} \), where \( \tilde{X} \in S_{0, T}^{2, d} \) is a solution of the (forward) SDE

\[
\begin{cases}
d\tilde{X}_s = B\left(s, \tilde{X}_s, \Psi^{T,\Pi}(X)_s \right) ds + \Sigma\left(s, \tilde{X}_s, \Psi^{T,\Pi}(X)_s \right) dW_s, \ s \in [0, T], \\
\tilde{X}_0 = x_0.
\end{cases}
\]

Here, by Lemma 3.1, the SDE (3.6) has a unique solution in \( S_{0, T}^{2, d} \).

By the above procedure, we can construct the mappings \( \Phi^{T,\Pi}_n : S_{0, T}^{2, d} \to S_{[t_{n-1}, T]}^{2, m} \times H^{2, m \times d}_{[t_{n-1}, T]} \), \( n = 1, \ldots, N \), \( \Psi^{T,\Pi} : S_{0, T}^{2, d} \to H^{2, m}_{[0, T]} \) and \( \Xi^{T,\Pi} : S_{0, T}^{2, d} \to S_{0, T}^{2, d} \). Note that a sequence of processes \( (X, \mathcal{Y}, \{Y^n, Z^n\}_{n=1}^N) \) is a discrete-equilibrium solution of FFBSDE\( \Pi(T) \) if and only if \( X \in S_{0, T}^{2, d} \) is a fixed point of the mapping \( \Xi^{T,\Pi}, (Y^n, Z^n) = \Phi^{T,\Pi}_n(X), n = 1, \ldots, N \), and \( \mathcal{Y} = \Psi^{T,\Pi}(X) \). Hence, it suffices to prove that there exists a constant \( T_0 > 0 \) which depends only on \( L \) such that, for any \( 0 < T \leq T_0 \) and any partition \( \Pi \) of \( [0, T] \), \( \Xi^{T,\Pi} \) is a contraction mapping on \( S_{0, T}^{2, d} \).

In order to prove that, let \( 0 < T \leq 1 \) and \( \Pi = \{t_n \mid n = 0, 1, \ldots, N\} \) with \( 0 = t_0 < t_1 < \cdots < t_N = T \) be fixed. For given \( X^1, X^2 \in S_{0, T}^{2, d} \), let \( (Y^{i, n}, Z^{i, n}) = \Phi^{T,\Pi}_n(X^i), n = 1, \ldots, N \), \( \mathcal{Y}^i = \Psi^{T,\Pi}(X^i) \) and \( \tilde{X}^i = \Xi^{T,\Pi}(X^i) \), for \( i = 1, 2 \), respectively. In the inequalities below, \( C > 0 \) denotes a constant which depends only on \( L \) and is allowed to vary from line to line.
By Lemma 3.2 we have, for each $n = 1, \ldots, N$,

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t_{n-1} \leq s \leq t_n} |Y_s^{1,n} - Y_s^{2,n}|^2 \right] & \leq C \mathbb{E} \left[ |G(t_{n-1}, X_{t_{n-1}}^1, \mathbb{E}_{t_{n-1}}[X_T^1], X_T^1) - G(t_{n-1}, X_{t_{n-1}}^2, \mathbb{E}_{t_{n-1}}[X_T^2], X_T^2)|^2 \right] \\
& \quad + \int_{t_{n-1}}^T \left[ F \left( t_{n-1}, X_{t_{n-1}}^1, s, X_s^1, Y_{t_{n-1}}^1 \mathbb{1}_{(t_{n-1},t_n)}(s) + \sum_{j=n+1}^N Y_{s}^{1,j} \mathbb{1}_{[t_{j-1},t_j]}(s), Y_s^{1,n}, Z_s^{1,n} \right) \\
& \quad \quad \quad \quad \quad - F \left( t_{n-1}, X_{t_{n-1}}^2, s, X_s^2, Y_{t_{n-1}}^1 \mathbb{1}_{(t_{n-1},t_n)}(s) + \sum_{j=n+1}^N Y_{s}^{2,j} \mathbb{1}_{[t_{j-1},t_j]}(s), Y_s^{1,n}, Z_s^{1,n} \right) \right] ds \\
& \leq C \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s^1 - X_s^2|^2 + \int_0^T |Y_s^1 - Y_s^2|^2 ds \right].
\end{align*}
\]

Hence, we have

\[
\begin{align*}
\mathbb{E} \left[ \int_0^T |Y_s^1 - Y_s^2|^2 ds \right] & = \sum_{n=1}^N \mathbb{E} \left[ \int_{t_{n-1}}^{t_n} |Y_{s}^{1,n} - Y_{s}^{2,n}|^2 ds \right] \\
& \leq \sum_{n=1}^N (t_n - t_{n-1}) \mathbb{E} \left[ \sup_{t_{n-1} \leq s \leq t_n} |Y_{s}^{1,n} - Y_{s}^{2,n}|^2 \right] \\
& \leq \sum_{n=1}^N (t_n - t_{n-1}) C \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_{s}^1 - X_{s}^2|^2 + \int_0^T |Y_s^1 - Y_s^2|^2 ds \right] \\
& = CTE \left[ \sup_{0 \leq s \leq T} |X_{s}^1 - X_{s}^2|^2 + \int_0^T |Y_s^1 - Y_s^2|^2 ds \right].
\end{align*}
\]

Therefore, if $T_0 = T_0(L) > 0$ is small enough and $0 < T \leq T_0$, we obtain

\[
\mathbb{E} \left[ \int_0^T |Y_s^1 - Y_s^2|^2 ds \right] \leq CTE \left[ \sup_{0 \leq s \leq T} |X_{s}^1 - X_{s}^2|^2 \right].
\]

On the other hand, by Lemma 3.1 with $(x_{0}^1, B^1, \Sigma^1) = (x_{0}^2, B^2, \Sigma^2) = (x_0, B, \Sigma)$, we have

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |\tilde{X}_{s}^1 - \tilde{X}_{s}^2|^2 \right] \leq C \mathbb{E} \left[ \int_0^T |Y_s^1 - Y_s^2|^2 ds \right].
\]

Hence, by letting $T_0 = T_0(L) > 0$ be smaller if needed, we have, when $0 < T \leq T_0$,

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |\tilde{X}_{s}^1 - \tilde{X}_{s}^2|^2 \right] \leq \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_{s}^1 - X_{s}^2|^2 \right],
\]

and hence $\Xi^{T, \Pi}$ is a contraction mapping on $S^{2,d}_{[0,T]}$. \qed
Before going to the proof of the second statement of Theorem 2.4, we shall show some important lemmas that provide uniform estimates for partitions \( \Pi \) of \([0, T] \) (if \( T \) is sufficiently small).

**Lemma 3.3.** Suppose that Assumptions (A1)–(A3) hold. Then, there exist constants \( C > 0 \) and \( T_0 > 0 \) that depend only on \( L \) such that, for any \( 0 < T \leq T_0 \) and any partition \( \Pi = \{ t_n \mid n = 0, 1, \ldots, N \} \) of \([0, T] \), the discrete-equilibrium solution \((X^{\Pi}, Y^{\Pi}, \{Y^{\Pi,n}, Z^{\Pi,n}\}_{n=1, \ldots, N})\) of FFBSDE\(_{\Pi}(T)\) satisfies

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s^{\Pi}|^2 \right] + \sup_{0 \leq s \leq T} \mathbb{E} \left[ |Y_s^{\Pi}|^2 \right] + \max_{n \in \{1, \ldots, N\}} \mathbb{E} \left[ \sup_{t_{n-1} \leq s \leq T} |Y_s^{\Pi,n}|^2 + \int_{t_{n-1}}^T |Z_s^{\Pi,n}|^2 \, ds \right] \\
\leq C \left( R + |x_0|^2 \right),
\]

\[
\mathbb{E} \left[ |X_u - X_v|^2 \right] \leq C \left( R + |x_0|^2 \right) |u - v|, \quad \forall u, v \in [0, T].
\]

**Proof.** In this proof, we again denote by \( C > 0 \) a constant which depends only on \( L \) and is allowed to vary from line to line. Let \( 0 < T \leq T_0 \) with \( T_0 = T_0(L) > 0 \) satisfying the assertion of Theorem 2.4 (I). By Lemma 3.2, we have, for each \( n = 1, \ldots, N \),

\[
\mathbb{E} \left[ \sup_{t_{n-1} \leq s \leq T} |Y_s^{\Pi,n}|^2 + \int_{t_{n-1}}^T |Z_s^{\Pi,n}|^2 \, ds \right] \\
\leq C \mathbb{E} \left[ \left| G \left( t_{n-1}, X_{t_{n-1}}, \mathbb{E}_{t_{n-1}}[X_T] \right) \right|^2 + \int_{t_{n-1}}^T \left| F \left( t_{n-1}, X_{t_{n-1}}, s, X_s \right) \mathbb{I}_{[\Pi]}(s), 0, 0 \right|^2 \, ds \right] \\
\leq C \left( R + \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s^{\Pi}|^2 + \int_0^T |Y_s^{\Pi}|^2 \, ds \right] \right), \quad (3.7)
\]

and hence

\[
\mathbb{E} \left[ \int_0^T |Y_s^{\Pi}|^2 \, ds \right] = \sum_{n=1}^N \mathbb{E} \left[ \int_{t_{n-1}}^{t_n} |Y_s^{\Pi,n}|^2 \, ds \right] \\
\leq \sum_{n=1}^N (t_n - t_{n-1}) C \left( R + \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s^{\Pi}|^2 + \int_0^T |Y_s^{\Pi}|^2 \, ds \right] \right) \\
= C T \left( R + \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s^{\Pi}|^2 + \int_0^T |Y_s^{\Pi}|^2 \, ds \right] \right). \quad (3.8)
\]

On the other hand, by Lemma 3.1 with \((x_0^2, B^2, \Sigma^2) = (0, 0, 0)\), we have

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s^{\Pi}|^2 \right] \leq C \left( R + |x_0|^2 + \mathbb{E} \left[ \int_0^T |Y_s^{\Pi}|^2 \, ds \right] \right). \quad (3.9)
\]

Combining the estimates (3.8) and (3.9) yields that

\[
\mathbb{E} \left[ \int_0^T |Y_s^{\Pi}|^2 \, ds \right] \leq C T \left( R + |x_0|^2 + \mathbb{E} \left[ \int_0^T |Y_s^{\Pi}|^2 \, ds \right] \right). \quad (3.9)
\]
Hence, if \( T_0 = T_0(L) > 0 \) is sufficiently small, we have
\[
\mathbb{E} \left[ \int_0^T |\mathcal{Y}_s^\Pi|^2 \, ds \right] \leq CT \left( R + |x_0|^2 \right),
\] (3.10)
and hence
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s^\Pi|^2 \right] \leq C \left( R + |x_0|^2 \right). \tag{3.11}
\]
By (3.7), (3.10) and (3.11), we obtain
\[
\max_{n \in \{1, \ldots, N\}} \mathbb{E} \left[ \sup_{t_{n-1} \leq s \leq T} |Y_{s}^{\Pi,n}|^2 + \int_{t_{n-1}}^T |Z_{s}^{\Pi,n}|^2 \, ds \right] \leq C \left( R + |x_0|^2 \right).
\]
In particular, by the equilibrium condition \( \mathcal{Y}_s^\Pi = Y_s^{\Pi,n} \) for \( s \in [t_{n-1}, t_n) \), \( n = 1, \ldots, N \), we have
\[
\sup_{s \in [0, T]} \mathbb{E} \left[ |\mathcal{Y}_s^\Pi|^2 \right] \leq C \left( R + |x_0|^2 \right).
\]
Then, by the linear-growth properties of the coefficients \( B \) and \( \Sigma \) (which follow by Assumptions (A2) and (A3)), we see that, for any \( 0 \leq u \leq v \leq T \),
\[
\mathbb{E} \left[ |X_v^\Pi - X_u^\Pi|^2 \right] \\
\leq 2 \mathbb{E} \left[ \left( \int_u^v B \left( s, X_s^\Pi, \mathcal{Y}_s^\Pi \right) \, ds \right)^2 + \left( \int_u^v \Sigma \left( s, X_s^\Pi, \mathcal{Y}_s^\Pi \right) \, dW_s \right)^2 \right] \\
\leq 2 \mathbb{E} \left[ (v - u) \int_u^v |B \left( s, X_s^\Pi, \mathcal{Y}_s^\Pi \right)|^2 \, ds + \int_u^v |\Sigma \left( s, X_s^\Pi, \mathcal{Y}_s^\Pi \right)|^2 \, ds \right] \\
\leq C \left( R + \sup_{0 \leq s \leq T} \mathbb{E} \left[ |X_s^\Pi|^2 + |\mathcal{Y}_s^\Pi|^2 \right] \right) (v - u) \\
\leq C \left( R + |x_0|^2 \right) (v - u).
\]
This completes the proof. \( \square \)

The following lemma is crucial to the proof of the second statement of Theorem 2.4. Here we impose Assumption (A4) in addition to Assumptions (A1)–(A3).

**Lemma 3.4.** Suppose that Assumptions (A1)–(A4) hold. Then, there exist constants \( C > 0 \) and \( T_0 > 0 \) that depend only on \( L \) such that, for any \( 0 < T \leq T_0 \) and two partitions \( \Pi \) and \( \bar{\Pi} \) of \([0, T] \) with \( \Pi \subset \bar{\Pi} \), the corresponding discrete-equilibrium solutions \( \{X^\Pi, \mathcal{Y}^\Pi, \{Y^{\Pi,n}, Z^{\Pi,n}\}_n\} \) and \( \{X^{\bar{\Pi}}, \mathcal{Y}^{\bar{\Pi}}, \{Y^{\bar{\Pi},n}, Z^{\bar{\Pi},n}\}_n\} \) of FFBSDE\(_\Pi\)(\(T\)) and FFBSDE\(_{\bar{\Pi}}\)(\(T\)), respectively, satisfy
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} \left| X_s^{\bar{\Pi}} - X_s^\Pi \right|^2 + \int_0^T \left| \mathcal{Y}_s^{\bar{\Pi}} - \mathcal{Y}_s^\Pi \right|^2 \, ds \right] \leq C \left( R + |x_0|^2 \right) \left( T \rho(\|\Pi\|)^2 + \|\Pi\| \right).
\]
Proof. As before, we denote by $C > 0$ a constant which depends only on $L$ and is allowed to vary from line to line. Let $0 < T \leq T_0$ with $T_0 = T_0(L) > 0$ satisfying the assertions of Theorem 2.4 (I) and Lemma 3.3. Let $\Pi = \{t_n \mid n = 0, 1, \ldots, N\}$ with $0 = t_0 < t_1 < \cdots < t_N = T$ and $\bar{\Pi} = \{\theta_{n,j} \mid j = 0, 1, \ldots, J_n, \ n = 1, \ldots, N\}$ with $t_{n-1} = \theta_{n,0} < \theta_{n,1} < \cdots < \theta_{n,J_n} = t_n$ for each $n = 1, \ldots, N$. Denote by $(X, \mathcal{Y}, \{Y^n, Z^n\}_{n=1,\ldots,N})$ and $\left(\tilde{X}, \tilde{\mathcal{Y}}, \{\tilde{Y}^{n,j}, \tilde{Z}^{n,j}\}_{j=1,\ldots,J_n, n=1,\ldots,N}\right)$ the corresponding discrete-equilibrium solutions of $\text{FFBSDE}_\Pi(T)$ and $\text{FFBSDE}_{\bar{\Pi}}(T)$, respectively. Namely, they satisfy the following equations $\mathbb{P}$-a.s.:

\[
\begin{align*}
\left\{
\begin{array}{l}
dX_s = B(s, X_s, \mathcal{Y}_s) \, ds + \Sigma(s, X_s, \mathcal{Y}_s) \, dW_s, \ s \in [0, T], \\
X_0 = x_0,
\end{array}
\right.
\end{align*}
\]

\[
\begin{align*}
\left\{
\begin{array}{l}
dY^n_s = -F(t_{n-1}, X_{t_{n-1}}, s, X_s, \mathcal{Y}_s, Y^n_s, Z^n_s) \, ds + Z^n_s \, dW_s, \ s \in [t_{n-1}, T], \\
Y^n_T = G(t_{n-1}, X_{t_{n-1}}, \mathbb{E}_{t_{n-1}}[X_T], X_T), \\
Y_s = \sum_{n=1}^N Y^n_s \mathbb{1}_{[t_{n-1}, t_n)}(s), \ s \in [0, T],
\end{array}
\right.
\end{align*}
\]

and

\[
\begin{align*}
\left\{
\begin{array}{l}
d\tilde{X}_s = B(s, \tilde{X}_s, \tilde{\mathcal{Y}}_s) \, ds + \Sigma(s, \tilde{X}_s, \tilde{\mathcal{Y}}_s) \, dW_s, \ s \in [0, T], \\
\tilde{X}_0 = x_0,
\end{array}
\right.
\end{align*}
\]

\[
\begin{align*}
\left\{
\begin{array}{l}
d\tilde{Y}^{n,j}_s = -F(\theta_{n,j-1}^{\ast}, \tilde{X}_{\theta_{n,j-1}^{\ast}}, s, \tilde{X}_s, \tilde{\mathcal{Y}}_s, \tilde{Y}^{n,j}_s, \tilde{Z}^{n,j}_s) \, ds + \tilde{Z}^{n,j}_s \, dW_s, \ s \in [\theta_{n,j-1}^{\ast}, T], \\
\tilde{Y}^{n,j}_T = G(\theta_{n,j-1}^{\ast}, \tilde{X}_{\theta_{n,j-1}^{\ast}}, \mathbb{E}_{\theta_{n,j-1}^{\ast}}[\tilde{X}_T], \tilde{X}_T), \\
\tilde{Y}_s = \sum_{n=1}^N \sum_{j=1}^{J_n} \tilde{Y}^{n,j}_s \mathbb{1}_{[\theta_{n,j-1}^{\ast}, \theta_{n,j}^{\ast})}(s), \ s \in [0, T].
\end{array}
\right.
\end{align*}
\]

We shall show that, when $T_0 = T_0(L) > 0$ is sufficiently small and $0 < T \leq T_0$, it holds that

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |\tilde{X}_s - X_s|^2 + \int_0^T |\tilde{Y}_s - Y_s|^2 \, ds \right] \leq C \left( R + |x_0|^2 \right) \left( T \rho(\|\Pi\|)^2 + \|\Pi\| \right). \tag{3.12}
\]

Note that

\[
\mathbb{E} \left[ \int_0^T |\tilde{Y}_s - Y_s|^2 \, ds \right] = \sum_{n=1}^N \sum_{j=1}^{J_n} \mathbb{E} \left[ \int_{\theta_{n,j-1}^{\ast}}^{\theta_{n,j}^{\ast}} |\tilde{Y}^{n,j}_s - Y^n_s|^2 \, ds \right]. \tag{3.13}
\]

For the time being, we fix $n \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, J_n\}$. Then, by Lemma 3.2 we have

\[
\mathbb{E} \left[ \sup_{\theta_{n,j-1}^{\ast} \leq s \leq \theta_{n,j}^{\ast}} |\tilde{Y}^{n,j}_s - Y^n_s|^2 \right] \leq C \mathbb{E} \left[ \left| G(\theta_{n,j-1}^{\ast}, \tilde{X}_{\theta_{n,j-1}^{\ast}}, \mathbb{E}_{\theta_{n,j-1}^{\ast}}[\tilde{X}_T], \tilde{X}_T) - G(t_{n-1}, X_{t_{n-1}}, \mathbb{E}_{t_{n-1}}[X_T], X_T) \right|^2 \\
+ \int_{\theta_{n,j-1}^{\ast}}^T \left| F(\theta_{n,j-1}^{\ast}, \tilde{X}_{\theta_{n,j-1}^{\ast}}, s, \tilde{X}_s, \tilde{\mathcal{Y}}_s, \tilde{Y}^{n,j}_s, \tilde{Z}^{n,j}_s) - F(t_{n-1}, X_{t_{n-1}}, s, X_s, \mathcal{Y}_s, Y^n_s, Z^n_s) \right|^2 \, ds \right].
\]
\[
\begin{align*}
&\leq C \left\{ \mathbb{E}\left[ \left( G(\theta_{n,j-1}, \tilde{X}_{\theta_{n,j-1}}, \mathbb{E}_{\theta_{n,j-1}}[\tilde{X}_T], \tilde{X}_T) - G(t_{n-1}, \tilde{X}_{\theta_{n,j-1}}, \mathbb{E}_{\theta_{n,j-1}}[\tilde{X}_T], \tilde{X}_T) \right)^2 \right. \\
&\quad + \int_{\theta_{n,j-1}}^{T} \left| \mathbb{E}_{\theta_{n,j-1}}[F(\theta_{n,j-1}, \tilde{X}_{\theta_{n,j-1}}, s, \tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s^{n,j}) - F(t_{n-1}, \tilde{X}_{\theta_{n,j-1}}, s, \tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s^{n,j})] \right|^2 ds \\
&\left. + \int_{\theta_{n,j-1}}^{T} \mathbb{E}_{\theta_{n,j-1}}[|F(\theta_{n,j-1}, \tilde{X}_{\theta_{n,j-1}}, s, \tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s^{n,j}) - F(t_{n-1}, \tilde{X}_{\theta_{n,j-1}}, s, \tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s^{n,j})|^2 ds \right) \\
&= C (I_1 + I_2 + I_3). \\
\end{align*}
\]

By Assumption (A4) and the first estimate of Lemma 3.3, the first expectation \(I_1\) (the difference with respect to the \(t\)-variable) can be estimated as follows:

\[
I_1 \leq C \rho \left( \theta_{n,j-1} - t_{n-1} \right)^2 \left( 1 + \mathbb{E}_{s \in [0,T]} |\tilde{X}_s|^2 + \int_0^T |\tilde{Y}_s|^2 ds + \int_{\theta_{n,j-1}}^{T} (|\tilde{Y}_s^{n,j}|^2 + |\tilde{Z}_s^{n,j}|^2) ds \right)
\leq C \left( R + |x_0|^2 \right) \rho \left( \theta_{n,j-1} - t_{n-1} \right)^2
\leq C \left( R + |x_0|^2 \right) \rho (||\Pi||)^2.
\]

By Assumption (A3) and the second estimate of Lemma 3.3, the second expectation \(I_2\) (the difference with respect to the \((\xi, \tilde{x})\)-variables) can be estimated as follows:

\[
I_2 \leq C \mathbb{E}\left[ |\tilde{X}_{\theta_{n,j-1}} - \tilde{X}_{t_{n-1}}|^2 + \left| \mathbb{E}_{\theta_{n,j-1}}[\tilde{X}_T] - \mathbb{E}_{t_{n-1}}[\tilde{X}_T] \right|^2 \right]
\leq C \left( R + |x_0|^2 \right) \left( \theta_{n,j-1} - t_{n-1} \right) + \sum_{l=1}^{d} \mathbb{E}\left[ \langle \tilde{M}^{(l)} \rangle_{\theta_{n,j-1}} - \langle \tilde{M}^{(l)} \rangle_{t_{n-1}} \right]
\leq C \left( R + |x_0|^2 \right) ||\Pi|| + \sum_{l=1}^{d} \mathbb{E}\left[ \langle \tilde{M}^{(l)} \rangle_{t_{n-1}} - \langle \tilde{M}^{(l)} \rangle_{t_{n-1}} \right],
\]

where \(\tilde{M}^{(l)}\) is the square-integrable martingale defined by

\[
\tilde{M}_s^{(l)} := \mathbb{E}_s \left[ \tilde{X}_T^{(l)} \right], \ s \in [0, T],
\]

where \(\tilde{X}_T^{(l)}\) denotes the \(l\)-th component of \(\tilde{X}_T\) for each \(l = 1, \ldots, d\). For the third expectation
where, in the second inequality, we used the estimate

\[ E \left[ \sup_{0 \leq s \leq T} |\tilde{X}_s - X_s|^2 \right] \leq C E \left[ \int_0^T |\tilde{Y}_s - Y_s|^2 ds \right], \quad (3.17) \]

which follows by Lemma 3.1 with \((x_1, B^1, \Sigma^1) = (x_0, B^2, \Sigma^2) = (x_0, B, \Sigma)\). Therefore, by (3.14), (3.15), (3.16) and (3.17), we have

\[
E \left[ \sup_{\theta_{n,j-1} \leq s \leq \theta_{n,j}} |\tilde{Y}_s^{n,j} - Y_n|^2 \right] 
\leq C \left( (R + |x_0|^2) (\rho (||\Pi||)^2 + ||\Pi||) + \sum_{l=1}^d E \left[ \langle \tilde{M}^{(l)} \rangle_{t_n} - \langle \tilde{M}^{(l)} \rangle_{t_{n-1}} \right] + E \left[ \int_0^T |\tilde{Y}_s - Y_s|^2 ds \right] \right) \] 

(3.19)

for each \(n \in \{1, \ldots, N\}\) and \(j \in \{1, \ldots, J_n\}\). Note that the right hand side of (3.19) is independent of \(j\). By (3.13) and (3.19), we obtain

\[
E \left[ \int_0^T |\tilde{Y}_s - Y_s|^2 ds \right] 
\leq \sum_{n=1}^N \sum_{j=1}^{J_n} (\theta_{n,j} - \theta_{n,j-1}) E \left[ \sup_{\theta_{n,j-1} \leq s \leq \theta_{n,j}} |\tilde{Y}_s^{n,j} - Y_n|^2 \right] 
\leq C \sum_{n=1}^N (t_n - t_{n-1}) \left( (R + |x_0|^2) (\rho (||\Pi||)^2 + ||\Pi||) \right.

\left. + \sum_{l=1}^d E \left[ \langle \tilde{M}^{(l)} \rangle_{t_n} - \langle \tilde{M}^{(l)} \rangle_{t_{n-1}} \right] + E \left[ \int_0^T |\tilde{Y}_s - Y_s|^2 ds \right] \right) \] 

\[
\leq C \left( T (R + |x_0|^2) (\rho (||\Pi||)^2 + ||\Pi||) + ||\Pi|| \sum_{l=1}^d E \left[ \langle \tilde{M}^{(l)} \rangle_T \right] + T E \left[ \int_0^T |\tilde{Y}_s - Y_s|^2 ds \right] \right) \]

\[
\leq C \left( (R + |x_0|^2) (T \rho (||\Pi||)^2 + ||\Pi||) + T E \left[ \int_0^T |\tilde{Y}_s - Y_s|^2 ds \right] \right),
\]

where, in the last inequality, we used the inequality

\[
\sum_{l=1}^d E \left[ \langle \tilde{M}^{(l)} \rangle_T \right] = E \left[ |\tilde{X}_T|^2 \right] \leq C (R + |x_0|^2),
\]
which follows by Lemma 3.3. Therefore, if \( T_0 = T_0(L) > 0 \) is sufficiently small and \( 0 < T \leq T_0 \), we have
\[
\mathbb{E} \left[ \int_0^T |\tilde{Y}_s - Y_s|^2 ds \right] \leq C \left( R + |x_0|^2 \right) \left( T \rho(\|\Pi\|)^2 + \|\Pi\| \right).
\]
By this estimate and (3.18), we obtain (3.12).

**Remark 3.5.** By the arguments in the above proof, for any \( 0 < T \leq T_0 \) and partitions \( \Pi \subset \tilde{\Pi} \) of \([0, T]\), the difference
\[
I := \mathbb{E} \left[ \sup_{0 \leq s \leq T} \left| X^{\Pi}_s - X^{\tilde{\Pi}}_s \right|^2 + \int_0^T \left| Y^{\Pi}_s - Y^{\tilde{\Pi}}_s \right|^2 ds \right]
\]
can be estimated as follows:

(i) If \( G \) is independent of the \( \bar{x} \)-variable, then the second term of the last line in (3.16) vanishes and we obtain
\[
I \leq CT \left( R + |x_0|^2 \right) \left( \rho(\|\Pi\|)^2 + \|\Pi\| \right).
\]

(ii) If \( F \) is independent of the \( \xi \)-variable and \( G \) is independent of the \( (\xi, \bar{x}) \)-variables, then \( I_2 \) in the above proof vanishes and we obtain
\[
I \leq CT \left( R + |x_0|^2 \right) \rho(\|\Pi\|)^2.
\]

(iii) If \( F \) and \( G \) are independent of the \( t \)-variable, then \( I_1 \) in the above proof vanishes and we obtain
\[
I \leq C \left( R + |x_0|^2 \right) \|\Pi\|.
\]

(iv) If \( F \) is independent of the \( t \)-variable and \( G \) is independent of the \( (t, \bar{x}) \)-variables, we obtain
\[
I \leq CT \left( R + |x_0|^2 \right) \|\Pi\|.
\]

(v) If \( F \) is independent of the \( (t, \xi) \)-variables and \( G \) is independent of the \( (t, \xi, \bar{x}) \)-variables, then both \( I_1 \) and \( I_2 \) in the above proof vanish and we obtain
\[
I = 0.
\]

The next lemma justifies the statements in Remark 2.2.

**Lemma 3.6.** Suppose that Assumptions (A1)–(A4) hold. Fix arbitrary \( T > 0 \), \( X \in S^{2,d}_{[0, T]} \) and \( Y \in \mathbb{H}^{2,m}_{[0, T]} \). For each \( t \in [0, T] \), let \( (Y^t, Z^t) \in S^{2,m}_{[t, T]} \times \mathbb{H}^{2,m \times d}_{[t, T]} \) be the unique (up to a null set) solution of the BSDE
\[
\begin{cases}
\quad dY^t_s = -F(t, X^t_s, s, X_s, Y^t_s, Y^t_s, Z^t_s) \, ds + Z^t_s \, dW_s, \quad s \in [t, T], \\
\quad Y^t_T = G(t, X^t, \mathbb{E}_t[X_T], X_T).
\end{cases}
\]

Then, the process \((Y^t_s)_{s \in [0, T]}\) has a progressively measurable version.
Proof. In this proof, $C > 0$ represents a constant which is independent of $(t, s) \in \Delta_{[0,T]}$ and allowed to vary from line to line. By Lemma 3.2, we have
\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} |Y_s^t|^2 + \int_t^T |Z_s|^2 \, ds \right] \leq C \mathbb{E} \left[ |G(t, X_t, \mathbb{E}_t[X_T], X_T)|^2 + \int_t^T |F(t, X_t, s, X_s, \mathcal{Y}_s, 0, 0)|^2 \, ds \right]
\leq C \left( R + \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s|^2 + \int_0^T |\mathcal{Y}_s|^2 \, ds \right] \right)
\leq C
\]
for any $t \in [0, T]$. Hence, as in the proof of Lemma 3.4, we obtain
\[
\mathbb{E} \left[ (Y_s^t - Y_s^0)^2 \right] \leq C \mathbb{E} \left[ |G(s, X_s, \mathbb{E}_s[X_T], X_T) - G(t, X_t, \mathbb{E}_t[X_T], X_T)|^2 \right]
+ \int_s^T |F(s, X_s, r, X_r, \mathcal{Y}_r, Y_r^s, Z_r^s) - F(t, X_t, r, X_r, \mathcal{Y}_r, Y_r^s, Z_r^s)|^2 \, dr
\leq C \left( \rho(s-t)^2 \left( 1 + \mathbb{E} \left[ \sup_{0 \leq r \leq T} |X_r|^2 + \int_0^T |\mathcal{Y}_r|^2 \, dr + \int_s^T (|Y_r^s|^2 + |Z_r^s|^2) \, dr \right] \right) + \mathbb{E} \left[ |X_s - X_t|^2 \right] + \mathbb{E} \left[ |\mathbb{E}_s[X_T] - \mathbb{E}_t[X_T]|^2 \right] \right)
\leq C \left( \rho(s-t)^2 + \mathbb{E} \left[ |X_s - X_t|^2 \right] + \mathbb{E} \left[ |\mathbb{E}_s[X_T] - \mathbb{E}_t[X_T]|^2 \right] \right)
\]
(3.20)
for any $(t, s) \in \Delta_{[0,T]}$. Note that the last line of (3.20) tends to zero as $s$ tends to $t$ from above uniformly in $t$. Hence, for each $k \in \mathbb{N}$, there exists a partition $\Pi^k = \{ t_n^k \mid n = 0, 1, \ldots, N_k \}$ of $[0, T]$ such that
\[
\mathbb{P} \left\{ \left| Y_s^t - Y_s^{t_n^k -1} \right| \geq \frac{1}{2^k} \right\} \leq \frac{1}{2^k} \text{ if } s \in [t_{n-1}^k, t_n^k), n = 1, \ldots, N_k.
\]
Define
\[
\eta^k_s(\omega) := \begin{cases} Y_s^{t_n^k -1}(\omega) & \text{if } s \in [t_{n-1}^k, t_n^k), n = 1, \ldots, N_k, \\ Y_T^{t_n^k}(\omega) & \text{if } s = T, \end{cases}
\]
for each $(s, \omega) \in [0, T] \times \Omega$ and $k \in \mathbb{N}$. Then, the processes $\eta^k = (\eta^k_s)_{s \in [0,T]}$ are progressively measurable and so is the set $A := \{(s, \omega) \in [0, T] \times \Omega \mid \exists \lim_{k \to \infty} \eta^k_s(\omega)\}$. Hence, the process $\eta = (\eta_s)_{s \in [0,T]}$ defined by
\[
\eta_s(\omega) := \begin{cases} \lim_{k \to \infty} \eta^k_s(\omega) & \text{if } (s, \omega) \in A, \\ 0 & \text{if } (s, \omega) \notin A, \end{cases}
\]
is also progressively measurable. Fix an arbitrary $s \in [0, T]$. Then, the Borel–Cantelli lemma yields that, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\left| Y_s^s(\omega) - \eta_s(\omega) \right| \leq 2^{-k}$ holds for any sufficiently large $k \in \mathbb{N}$, and hence $(\eta_s)_{s \in [0,T]}$ is a version of $(Y_s^s)_{s \in [0,T]}$. \qed
We are ready to prove the second statement of Theorem \[2.4\] We shall state it in more detail in the following proposition.

**Proposition 3.7.** Suppose that Assumptions (A1)–(A4) hold. Then, there exists a constant $T_0 > 0$ which depends only on $L$ such that, for any $0 < T \leq T_0$, the following assertions hold:

(i) For any increasing sequence of partitions $\Pi^k = \{t^k_n \mid n = 0, 1, \ldots, N_k\}$, $k \in \mathbb{N}$, of $[0, T]$ such that $\|\Pi^k\| \to 0$ as $k$ tends to infinity, if we denote by $(X^k, Y^k, \{Y^k_n, Z^k_n\}_{n=1,\ldots,N_k})$ the unique discrete-equilibrium solution of FFBSDE$(k)$ for each $k \in \mathbb{N}$, then $\{(X^k, Y^k)\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $S_{[0,T]}^{2,d} \times \mathbb{H}_{[0,T]}^{2,m}$.

(ii) Denote the limit by $(X, Y) \in S_{[0,T]}^{2,d} \times \mathbb{H}_{[0,T]}^{2,m}$ and define $\{(Y^t, Z^t)_{t \in [0,T]}\}$ as a family of processes such that, for each $t \in [0, T]$, $(Y^t, Z^t) \in S_{[t,T]}^{2,m} \times \mathbb{H}_{[t,T]}^{2,m}$ is the unique solution of the BSDE

$$
\begin{align*}
dY^t_s &= -F(t, X_t, s, X_s, Y_s, Y^t_s, Z^t_s) \, ds + Z^t_s \, dW_s, \ s \in [t, T], \\
Y^t_T &= G(t, X_t, \mathbb{E}[X_T], X_T),
\end{align*}
$$

and that $(Y^t_s)_{s \in [0,T]}$ is progressively measurable; see Lemma \[3.6\]. Then, $(X, Y, \{Y^t, Z^t\}_{t \in [0,T]})$ is the unique equilibrium solution of FFBSDE$(T)$. In particular, it is independent of the choice of the sequence $\{\Pi^k\}_{k \in \mathbb{N}}$. Moreover, it holds that

$$
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |X^k_s - X_s|^2 + \int_0^T |Y^k_s - Y_s|^2 \, ds \right] \leq C \left( R^2 + |x_0|^2 \right) \left( T \rho^2 \|\Pi^k\|^2 + \|\Pi^k\| \right)
$$

for all $k \in \mathbb{N}$, where $C > 0$ is a constant which depends only on $L$.

**Proof.** Fix $0 < T \leq T_0$ with $T_0 = T_0(L) > 0$ satisfying the assertions in Theorem \[2.4\] (I) and Lemma \[3.3\]. Then, (i) is an immediate consequence of Lemma \[3.4\]. Define $(X, Y, \{Y^t, Z^t\}_{t \in [0,T]})$ as in (ii), which depends on the sequence $\{\Pi^k\}_{k \in \mathbb{N}}$ at this step of the argument. By Lemma \[3.3\], we easily see that $(3.21)$ holds. Thus, it remains to prove that $(X, Y, \{Y^t, Z^t\}_{t \in [0,T]})$ is the unique equilibrium solution of FFBSDE$(T)$. Below, we denote by $C > 0$ a constant which depends only on $L$ and is allowed to vary from line to line.

First, we show that $X \in S_{[0,T]}^{2,d}$ solves the SDE

$$
\begin{align*}
\left\{ \begin{array}{l}
\quad dX_s = B(s, X_s, Y_s) \, ds + \Sigma(s, X_s, Y_s) \, dW_s, \ s \in [0, T], \\
\quad X_0 = x_0.
\end{array} \right.
\end{align*}
$$

Indeed, for each $k \in \mathbb{N}$, $(X^k, Y^k)$ satisfies

$$
X^k_s = x_0 + \int_0^s B(r, X^k_r, Y^k_r) \, dr + \int_0^s \Sigma(r, X^k_r, Y^k_r) \, dW_r, \ \forall s \in [0, T],
$$

(3.23)
\( \mathbb{P} \)-a.s. Since \( \lim_{k \to \infty} (X^k, Y^k) = (X, Y) \) in \( \mathbb{S}^{2,d}_{[0,T]} \times \mathbb{H}^{2,m}_{[0,T]} \), by the Burkholder–Davis–Gundy inequality and Lipschitz continuity of \( B \) and \( \Sigma \), we see that the right hand side of (3.23) tends to

\[
x_0 + \int_0^s B (r, X_r, Y_r) \, dr + \int_0^s \Sigma (r, X_r, Y_r) \, dW_r, \quad s \in [0, T],
\]

in \( \mathbb{S}^{2,d}_{[0,T]} \) as \( k \) tends to infinity. Hence \( X \) is the solution of the SDE (3.22).

Second, we show that

\[
Y_t = Y^t \text{ for Leb-a.e. } t \in [0, T], \quad \mathbb{P}\text{-a.s.}
\]

In order to prove (3.24), fix \( k \in \mathbb{N} \) and \( n \in \{1, \ldots, N_k\} \) and take an arbitrary \( t \in [t_{n-1}^k, t_n^k] \). Then, using the same estimate as in the proof of Lemma 3.4 yields that

\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} |Y_{t,n}^k - Y_{s}^t|^2 \right] 
\leq C \mathbb{E} \left[ \left| G \left( t_{n-1}^k, X_{t_{n-1}^k}^k, \mathbb{E}_{t_{n-1}^k} [X_T^k], X_T^k \right) - G \left( t, X_t, \mathbb{E}_t [X_T], X_T \right) \right|^2 
+ \int_t^T \left| F \left( t_{n-1}^k, X_{t_{n-1}^k}^k, s, X_s^k, Y_s^k, Y_{t,n}^k, Z_{t,n}^k \right) - F \left( t, X_t, s, X_s, Y_s, Y_{t,n}^k, Z_{t,n}^k \right) \right|^2 \, ds \right]
\leq C \left( \rho (t - t_{n-1}^k)^2 \left( 1 + \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s^k|^2 + \int_0^T |Y_s|^2 \, ds + \int_{t_{n-1}^k}^T \left( |Y_{s,n}^k|^2 + |Z_{s,n}^k|^2 \right) \, ds \right) \right)
+ \mathbb{E} \left[ |X_t^k - X_{t_{n-1}^k}|^2 \right] + \mathbb{E} \left[ \left| \mathbb{E}_t [X_T^k] - \mathbb{E}_{t_{n-1}^k} [X_T^k] \right|^2 \right] + \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_{s}^k - X_s|^2 + \int_0^T |Y_s^k - Y_s|^2 \, ds \right]\right).
\]

Since \( Y_{t,n}^k = Y^k_t \) \( \mathbb{P}\text{-a.s.} \), we have, in particular,

\[
\mathbb{E} \left[ |Y_t^k - Y_t^t|^2 \right] \leq C \left( (R + |x_0|^2) \left( \rho \left( \left| \Pi^k \right| \right)^2 + \left| \Pi^k \right| \right) + \mathbb{E} \left[ \left| \mathbb{E}_{t_n^k} [X_T^k] - \mathbb{E}_{t_{n-1}^k} [X_T^k] \right|^2 \right] \right.
+ \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_{s}^k - X_s|^2 + \int_0^T |Y_s^k - Y_s|^2 \, ds \right] \right) .
\]

Note that the right hand side above is independent of \( t \) as long as it lies in \( [t_{n-1}^k, t_n^k] \). Noting that \( (Y_t^k)_{t \in [0,T]} \) is a (progressively) measurable process, integrating both sides from 0 to \( T \) yields that

\[
\mathbb{E} \left[ \int_0^T |Y_t^k - Y_t^t|^2 \, dt \right] \leq C \left( T (R + |x_0|^2) \left( \rho \left( \left| \Pi^k \right| \right)^2 + \left| \Pi^k \right| \right) + \left| \Pi^k \right| \mathbb{E} \left[ |X_T^k|^2 \right] \right.
+ T \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_{s}^k - X_s|^2 + \int_0^T |Y_s^k - Y_s|^2 \, ds \right] \right) .
\]

20
Note that the sequence \( \{ \mathbb{E}[|X^k_t|^2] \}_{k \in \mathbb{N}} \) is bounded. Hence, we see that \( \lim_{k \to \infty} Y^k = (Y^k_t)_{t \in [0,T]} \) in \( \mathbb{H}^2_{[0,T]} \). Since \( \lim_{k \to \infty} Y^k = Y \) in \( \mathbb{H}^2_{[0,T]} \), (3.24) holds.

As a consequence, we see that \( (X, Y, \{ Y^t, Z^t \}_{t \in [0,T]} ) \) is an equilibrium solution of FFBSDE\( (T) \).

In order to prove uniqueness, take another equilibrium solution \( (\tilde{X}, \tilde{Y}, \{ \tilde{Y}^t, \tilde{Z}^t \}_{t \in [0,T]} ) \) of FFBSDE\( (T) \). Then, by Lemma 3.1 with \((x^1_0, B^1, \Sigma^1) = (x^2_0, B^2, \Sigma^2) = (x_0, B, \Sigma) \), we have

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s - \tilde{X}_s|^2 \right] \leq C \mathbb{E} \left[ \int_0^T |Y_s - \tilde{Y}_s|^2 ds \right].
\]

By using this estimate and Lemma 3.2, we see that

\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} |Y^t_s - \tilde{Y}^t_s|^2 \right] \leq C \mathbb{E} \left[ \int_0^T |Y_s - \tilde{Y}_s|^2 ds \right]
\]

for any \( t \in [0, T] \). Hence, we obtain

\[
\mathbb{E} \left[ \int_0^T |Y_s - \tilde{Y}_s|^2 ds \right] = \mathbb{E} \left[ \int_0^T |Y^t_s - \tilde{Y}^t_s|^2 ds \right] \leq C \mathbb{E} \left[ \int_0^T |Y_s - \tilde{Y}_s|^2 ds \right].
\]

Therefore, if \( T_0 = T_0(L) > 0 \) is sufficiently small and \( 0 < T \leq T_0 \), we have

\[
\mathbb{E} \left[ \int_0^T |Y_s - \tilde{Y}_s|^2 ds \right] = 0,
\]

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s - \tilde{X}_s|^2 \right] = 0.
\]

By uniqueness of the solutions of the BSDEs, we have

\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} |Y^t_s - \tilde{Y}^t_s|^2 + \int_t^T |Z^t_s - \tilde{Z}^t_s|^2 ds \right] = 0, \, \forall \, t \in [0, T].
\]

Hence the equilibrium solution of FFBSDE\( (T) \) is unique. \( \square \)

This completes the proof of Theorem 2.4.

**Remark 3.8.** By Remark 3.5, for any \( 0 < T \leq T_0 \) and partition \( \Pi \) of \([0, T] \), the difference

\[
I := \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X^\Pi_s - X_s|^2 + \int_0^T |Y^\Pi_s - Y_s|^2 ds \right],
\]

with the notations in Definitions 2.1 and 2.3, can be estimated as follows:

(i) If \( G \) is independent of the \( \bar{x} \)-variable, we obtain

\[
I \leq CT \left( R + |x_0|^2 \right) \left( \rho(||\Pi||)^2 + ||\Pi|| \right).
\]
(ii) If $F$ is independent of the $\xi$-variable and $G$ is independent of the $(\xi, \bar{x})$-variables, we obtain

$$I \leq C T (R + |x_0|^2) \rho(\|\Pi\|)^2.$$  

(iii) If $F$ and $G$ are independent of the $t$-variable, we obtain

$$I \leq C (R + |x_0|^2) \|\Pi\|.$$  

(iv) If $F$ is independent of the $t$-variable and $G$ is independent of the $(t, \bar{x})$-variables, we obtain

$$I \leq C T (R + |x_0|^2) \|\Pi\|.$$  

(v) If $F$ is independent of the $(t, \xi)$-variables and $G$ is independent of the $(t, \xi, \bar{x})$-variables, we obtain

$$I = 0.$$  

In particular, in this case, we have $(X, Y) = (X', Y')$, where $(X', Y', Z') \in S_{[0, T]}^{2, d} \times S_{[0, T]}^{2, m} \times \mathbb{H}_{[0, T]}^{2, m \times d}$ is the unique solution of the (classical) forward-backward SDE

$$\begin{cases} dX'_s = B(s, X'_s, Y'_s) \, ds + \Sigma(s, X'_s, Y'_s) \, dW_s, & s \in [0, T], \\ X'_0 = x_0, \\ dY'_s = -F(s, X'_s, Y'_s, Y'_s, Z'_s) \, ds + Z'_s \, dW_s, & s \in [0, T], \\ Y'_T = G(X'_T). \end{cases}$$

4 Some remarks on flows of forward-backward SDEs

4.1 From FFBSDE$(T)$ to FFBSDE$_\Pi(T)$

In Section 3, we showed that, at least when $T_0 = T_0(L) > 0$ is sufficiently small and $0 < T \leq T_0$, the sequence of discrete-equilibrium solutions of FFBSDE$_\Pi(T)$ for increasing partitions converges to the equilibrium solution of FFBSDE$(T)$ in an appropriate sense. In this subsection, we consider the opposite direction. At first, we introduce a notion of a discrete-\(\epsilon\)-equilibrium solution of FFBSDE$_\Pi(T)$.

**Definition 4.1.** For $T > 0$, $\Pi = \{t_n \mid n = 0, \ldots, N\}$ with $0 = t_0 < t_1 < \cdots < t_N = T$ and $\epsilon > 0$, we call a sequence of processes $(X, Y, \{Y^n, Z^n\}_{n=1}^{N})$ a discrete-\(\epsilon\)-equilibrium solution of FFBSDE$_\Pi(T)$ if $X \in S_{[0, T]}^{2, d}$, $Y \in \mathbb{H}_{[0, T]}^{2, m}$, $(Y^n, Z^n) \in S_{[t_{n-1}, T]}^{2, m} \times \mathbb{H}_{[t_{n-1}, T]}^{2, m \times d}$, $n = 1, \ldots, N$, and it holds that

$$\begin{cases} dX_s = B(s, X_s, Y_s) \, ds + \Sigma(s, X_s, Y_s) \, dW_s, & s \in [0, T], \\ X_0 = x_0, \\ dY^n_s = -F(t_{n-1}, X_{t_{n-1}}, s, X_s, Y_s, Y^n_s, Z^n_s) \, ds + Z^n_s \, dW_s, & s \in [t_{n-1}, T], \\ Y^n_T = G(t_{n-1}, X_{t_{n-1}}, \mathbb{E}_{t_{n-1}}[X_T], X_T), & n = 1, \ldots, N, \end{cases}$$
Proposition 4.2. Suppose that Assumptions (A1)–(A4) hold. Let 0 < T \leq T_0 with T_0 = T_0(L) > 0 satisfying the assertions in Theorem 2.4 and (X, Y, \{Y^t, Z^t\}_{t \in [0, T]} ) be the unique equilibrium solution of FFBSDE(T). Fix an arbitrary partition \Pi of \[0, T\] and denote it by \Pi = \{t_n \mid n = 0, 1, \ldots, N\}, 0 = t_0 < t_1 < \cdots < t_N = T. Then, (X, Y, \{Y^{t_n}, Z^{t_n}\}_{n=1, \ldots, N}) is a discrete-\epsilon-equilibrium solution of FFBSDE_\Pi(T), where \epsilon := C (R + \|x_0\|^2) (T \rho (\|\Pi\|)^2 + \|\Pi\|) for some constant C > 0 which depends only on L.

**Proof.** As before, we denote by C > 0 a constant which depends only on L and is allowed to vary from line to line. It suffices to prove that

\[
\sum_{n=1}^{N} \mathbb{E} \left[ \int_{t_{n-1}}^{t_n} |Y_s - Y_s^{t_n-1}|^2 \, ds \right] \leq C (R + \|x_0\|^2) (T \rho (\|\Pi\|)^2 + \|\Pi\|).
\]

Let \((X^{\Pi}, Y^{\Pi}, \{Y^{\Pi,n}, Z^{\Pi,n}\}_{n=1, \ldots, N})\) be the unique discrete-equilibrium solution of FFBSDE_\Pi(T). Then, we have

\[
\sum_{n=1}^{N} \mathbb{E} \left[ \int_{t_{n-1}}^{t_n} |Y_s - Y_s^{t_n-1}|^2 \, ds \right] \leq 2 \left( \mathbb{E} \left[ \int_0^T |Y_s - Y_s^{\Pi}|^2 \, ds \right] + \sum_{n=1}^{N} \mathbb{E} \left[ \int_{t_{n-1}}^{t_n} |Y_s^{\Pi,n} - Y_s^{t_n-1}|^2 \, ds \right] \right).
\]

By Theorem 2.4 (II-b), we have

\[
\mathbb{E} \left[ \int_0^T |Y_s - Y_s^{\Pi}|^2 \, ds \right] \leq C (R + \|x_0\|^2) (T \rho (\|\Pi\|)^2 + \|\Pi\|).
\]

On the other hand, by Lemma 3.2 and Theorem 2.4 (II-b), we have

\[
\mathbb{E} \left[ \sup_{t_{n-1} \leq s \leq t_n} |Y_s^{\Pi,n} - Y_s^{t_n-1}|^2 \right]
\leq C \mathbb{E} \left[ |G(t_{n-1}, X^{\Pi}_{t_{n-1}}, \mathbb{E}_{t_{n-1}}[X^{\Pi}_{T}], X^{\Pi}_{T}) - G(t_{n-1}, X_{t_{n-1}}, \mathbb{E}_{t_{n-1}}[X_T], X_T)|^2 \right.
\left. + \int_{t_{n-1}}^{T} |F(t_{n-1}, X^{\Pi}_{t_{n-1}}, s, X_s^{\Pi}, Y_s^{\Pi}, Y_s^{\Pi,n}, Z_s^{\Pi,n}) - F(t_{n-1}, X_{t_{n-1}}, s, X_s, Y_s, Y_s^{\Pi,n}, Z_s^{\Pi,n})|^2 \, ds \right]
\leq C \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s^{\Pi} - X_s|^2 + \int_0^T |Y_s^{\Pi} - Y_s|^2 \, ds \right]
\leq C (R + \|x_0\|^2) (T \rho (\|\Pi\|)^2 + \|\Pi\|)
\]

for each \(n = 1, \ldots, N\). In particular, we obtain

\[
\sum_{n=1}^{N} \mathbb{E} \left[ \int_{t_{n-1}}^{t_n} |Y_s^{\Pi,n} - Y_s^{t_n-1}|^2 \, ds \right] \leq CT (R + \|x_0\|^2) (T \rho (\|\Pi\|)^2 + \|\Pi\|),
\]

and finish the proof.\[\square\]
4.2 Stability of equilibrium solutions

In this subsection, we investigate stability of equilibrium solutions of flows of forward-backward SDEs. In the following proposition, we assume that small-time solvability and Assumptions (A1)–(A3) hold for two flows of forward-backward SDEs.

**Proposition 4.3.** Let \((x_0^i, B^i, \Sigma^i, F^i, G^i)\) and \((x_0^2, B^2, \Sigma^2, F^2, G^2)\) be coefficients satisfying Assumptions (A1)–(A3) with constants \((R_1, L_1)\) and \((R_2, L_2)\), respectively. Assume that there exists a constant \(\bar{T} > 0\) such that, for any \(0 < T \leq \bar{T}\) and \(i = 1, 2\), there exists a unique equilibrium solution \((X^i, Y^i, \{Z^i_s\}_t^T)\) of FFBSDE\((T)\) with the coefficients \((x_0^i, B^i, \Sigma^i, F^i, G^i)\). Then, there exist constants \(0 < T_0 \leq \bar{T}\) and \(C > 0\) that depend only on \(L_1\) such that, for each \(0 < T \leq T_0\), it holds that

\[\mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s^1 - X_s^2|^2 \right] \leq C \left( |x_0^1 - x_0^2|^2 + \mathbb{E} \left[ \int_0^T |(\delta B, \delta \Sigma)(s, X_s^1, Y_s^1)|^2 ds \right] + T \sup_{0 \leq t \leq T} \mathbb{E} \left[ |\delta F(t, X_t^1, s, X_s^1, Y_s^1, Y_s^{2,t}, Z_s^{2,t})|^2 ds \right] \right),\]

\[\text{(4.1)}\]

\[\mathbb{E} \left[ \int_0^T |Y_s^1 - Y_s^2|^2 ds \right] \leq CT \left( |x_0^1 - x_0^2|^2 + \mathbb{E} \left[ \int_0^T |(\delta B, \delta \Sigma)(s, X_s^2, Y_s^2)|^2 ds \right] + \sup_{0 \leq t \leq T} \mathbb{E} \left[ |\delta G(t, X_t^2, \mathbb{E}_t[X_t^2], X_t^2)|^2 + \int_t^T |\delta F(t, X_t^2, s, X_s^2, Y_s^2, Y_s^{2,t}, Z_s^{2,t})|^2 ds \right] \right),\]

\[\text{(4.2)}\]

\[\sup_{0 \leq t \leq T} \mathbb{E} \left[ \sup_{t \leq s \leq T} |Y_{s,t}^1 - Y_{s,t}^2|^2 + \int_t^T |Z_{s,t}^1 - Z_{s,t}^2|^2 ds \right] \leq C \left( |x_0^1 - x_0^2|^2 + \mathbb{E} \left[ \int_0^T |(\delta B, \delta \Sigma)(s, X_s^2, Y_s^2)|^2 ds \right] + \sup_{0 \leq t \leq T} \mathbb{E} \left[ |\delta G(t, X_t^2, \mathbb{E}_t[X_t^2], X_t^2)|^2 + \int_t^T |\delta F(t, X_t^2, s, X_s^2, Y_s^2, Y_s^{2,t}, Z_s^{2,t})|^2 ds \right] \right),\]

\[\text{(4.3)}\]

where \(\delta \Phi := \Phi^1 - \Phi^2\) for \(\Phi = B, \Sigma, F, G\).

**Proof.** In this proof, we denote by \(C\) a positive constant which depends only on \(L_1\) and is allowed to vary from line to line. Let \(0 < T \leq \bar{T} \wedge 1\). By Lemma 3.2 we have, for each
Let \( t \in [0, T] \),

\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} |Y^1_{s,t} - Y^2_{s,t}|^2 + \int_t^T |Z^1_{s,t} - Z^2_{s,t}|^2 \, ds \right] 
\leq C \mathbb{E} \left[ \left| G^1 (t, X^1_t, \mathbb{E}_t [X^1_T], X^1_T) - G^2 (t, X^2_t, \mathbb{E}_t [X^2_T], X^2_T) \right|^2 
+ \int_t^T |F^1 (t, X^1_t, s, X^1_s, Y^1_s, Y^2_s, Z^1_s, Z^2_s) - F^2 (t, X^2_t, s, X^2_s, Y^2_s, Y^2_s, Z^2_s)|^2 \, ds \right]
\]

\[
\leq C \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X^1_s - X^2_s|^2 + \int_0^T |Y^1_s - Y^2_s|^2 \, ds 
+ \left| \delta G (t, X^2_t, \mathbb{E}_t [X^2_T], X^2_T) \right|^2 + \int_t^T \left| \delta F (t, X^2_t, s, X^2_s, Y^2_s, Y^2_s, Z^2_s) \right|^2 \, ds \right] . \tag{4.4}
\]

On the other hand, by Lemma 3.1, we have

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |X^1_s - X^2_s|^2 \right] 
\leq C \left( |x^1_0 - x^2_0|^2 + \mathbb{E} \left[ \int_0^T \left| \delta B (s) \right|^2 \, ds + \int_0^T \left| \delta \Sigma (s) \right|^2 \, ds \right] \right) . \tag{4.5}
\]

By (4.4), (4.5) and the equilibrium condition \( Y^i_s = Y^i_{s,s} \) Leb-a.e. \( s \in [0, T] \), \( \mathbb{P} \)-a.s., for \( i = 1, 2 \), we obtain

\[
\mathbb{E} \left[ \int_0^T |Y^1_s - Y^2_s|^2 \, ds \right] 
\leq T \sup_{0 \leq s \leq T} \mathbb{E} \left[ |Y^1_{s,s} - Y^2_{s,s}|^2 \right] 
\leq CT \left( |x^1_0 - x^2_0|^2 + \mathbb{E} \left[ \int_0^T \left| \delta B (s) \right|^2 \, ds + \int_0^T \left| \delta \Sigma (s) \right|^2 \, ds \right] \right) 
+ \sup_{0 \leq t \leq T} \mathbb{E} \left[ \left| \delta G (t, X^2_t, \mathbb{E}_t [X^2_T], X^2_T) \right|^2 + \int_t^T \left| \delta F (t, X^2_t, s, X^2_s, Y^2_s, Y^2_s, Z^2_s) \right|^2 \, ds \right] .
\]

Hence, if \( T_0 = T_0(L_1) > 0 \) is sufficiently small and \( 0 < T \leq T_0 \), we obtain the second estimate (4.2) and hence the first estimate (4.1). By these two estimates and (4.4), we also obtain the third estimate (4.3).

\[\square\]

### 4.3 A Direct proof of small-time solvability of FFBSDE\((T)\)

In this subsection, we give an alternative proof of small-time solvability of FFBSDE\((T)\). The method of the proof is to apply the fixed point argument for the mapping \( \mathcal{Y} \mapsto (Y^*_{s})_{s \in [0, T]} \).

**Direct proof of Theorem 2.4 (II-a).** Let \( 0 < T \leq 1 \). In this proof, we denote by \( C \) a positive constant which depends only on \( L \) and is allowed to vary from line to line. For each \( \mathcal{Y} \in \mathbb{H}^{2,m}_{[0, T]} \), define \( \Phi (\mathcal{Y}) \in \mathbb{H}^{2,m}_{[0, T]} \) by the following procedure:
Remark 4.4. In the above proof, we used Assumption (A4) only for justifying existence of a progressively measurable version of the process \((Y^s_s)_{s \in [0,T]}\) in (iii).
4.4 Concluding remarks and future problems

We conclude this paper by discussing three future problems.

The first problem is solvability of flows of forward-backward SDEs on arbitrary time-intervals \([0, T]\). This is a difficult problem since even in the case of classical forward-backward SDEs Lipschitz continuity of the coefficients is insufficient for well-posedness of the equation defined on an arbitrary time-interval; see the textbooks \([4, 12]\). In classical forward-backward SDE theory, the so-called decoupling field which is a function connecting the backward and forward components of the equation is a good tool to treat the case where the time-interval is arbitrary. Since there are two time variables \((t, s)\) and two state variables \((X_t, X_s)\) in the case of flows of forward-backward SDEs, we have to generalize the concept of decoupling fields in order to take these variables into account.

The second problem is a generalization to more intricately coupled systems. In order to treat the time-inconsistent stochastic control problems where the volatility of the state process is also controlled, we should consider more general forms of forward-backward systems, namely, the following form of flows:

\[
\begin{align*}
    dX_s &= B(s, X_s, Y_s, Z_s) \, ds + \Sigma (s, X_s, Y_s, Z_s) \, dW_s, \quad s \in [0, T], \\
    X_0 &= x_0, \\
    dY^t_s &= -F(t, X_t, s, X_s, Y_s, Z_s, Y^t_s, Z^t_s) \, ds + Z^t_s \, dW_s, \quad s \in [t, T], \\
    Y^t_T &= G(t, X_t, E_t[X_T], X_T), \\
    Y_s &= Y^s_t, \quad Z_s = Z^s_t, \ \text{Leb-a.e.} \ s \in [0, T].
\end{align*}
\]

Unfortunately, our arguments in this paper are insufficient to treat this generalized system. In this case, we have to estimate the term \(Z\) in more detail, which is yet to be investigated.

The third problem is about relationships between flows of forward-backward SDEs and classical forward-backward SDEs. We are interested in whether the equilibrium solution of a flow of forward-backward SDEs can be written as the solution of a classical forward-backward SDE or not. In our notation, the problem is to seek for a (backward) equation which determines the time-evolution of the process \(Y\). This problem can be restated in terms of control theory as follows: when is the subgame perfect Nash equilibrium of a time-inconsistent stochastic control problem rewritten as the optimal strategy of another time-consistent stochastic control problem? This is an interesting problem from viewpoints of both stochastic analysis and control theory.

We shall make two comments on the third problem. First, Remark 3.8 (v) says that, if \(F\) is independent of the \((t, \xi)\)-variables and \(G\) is independent of the \((t, \xi, \bar{x})\)-variables, then the equilibrium solution of FFBSDE(\(T\)) for \(0 < T \leq T_0\) is characterized by the solution of the corresponding classical forward-backward SDE. Second, if FFBSDE(\(T\)) has a unique equilibrium solution \((X, Y, \{Y^t, Z^t\}_{t \in [0, T]})\) and there exists a (random) function \(\phi\) such that \(Y^t_s = \phi(t, X_t, s, X_s)\) (or, more generally \(Y^t_s = \phi(t, X_t, s, X_s, Y_s)\)) for any \((t, s) \in \Delta_{[0, T]}\), then the equilibrium solution is characterized (at least formally) by the processes
\((X, \mathcal{Y}, \{Z_s^t\}_{(t,s) \in \Delta_{[0,T]}})\) satisfying

\[
\begin{aligned}
  &dX_s = B(s, X_s, \mathcal{Y}_s) \, ds + \Sigma(s, X_s, \mathcal{Y}_s) \, dW_s, \ s \in [0, T], \\
  &X_0 = x_0, \\
  &\mathcal{Y}_t = G(t, X_t, \mathbb{E}_t[X_T], X_T) + \int_t^T \tilde{F}(t, X_t, s, X_s, \mathcal{Y}_s, Z_s^t) \, ds - \int_t^T Z_s^t \, dW_s, \ t \in [0, T],
\end{aligned}
\]

with an appropriate function \(\tilde{F}\). The third line of the above system is a (Type-I) BSVIE with respect to \((\mathcal{Y}, \{Z_s^t\}_{(t,s) \in \Delta_{[0,T]}})\); see [10]. The function \(\phi\) plays the role of a “decoupling field” of the system (1.1) in some sense. What matters is to characterize \(\phi\) and this is also a future problem.

### Acknowledgments

I would like to thank Professor Masanori Hino, who is my supervisor, and Professor Ichiro Shigekawa, for helpful discussions. I am also grateful to Professor Jun Sekine for insightful discussions and the suggestion of the third problem in Section 4.4.

This work was supported by JSPS KAKENHI Grant Number JP18J20973.

### References

[1] T. Björk, M. Khapko, and A. Murgoci. On time-inconsistent stochastic control in continuous time. *Finance Stoch.*, 21(2):331–360, 2017.

[2] T. Björk, A. Murgoci, and X. Y. Zhou. Mean-variance portfolio optimization with state-dependent risk aversion. *Math. Finance*, 24(1):1–24, 2014.

[3] R. Buckdahn, B. Djehiche, and J. Li. A general stochastic maximum principle for SDEs of mean-field type. *Appl. Math. Optim.*, 64(2):197–216, 2011.

[4] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. II*, volume 84 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2018.

[5] B. Djehiche and M. Huang. A characterization of sub-game perfect equilibria for SDEs of mean-field type. *Dyn. Games Appl.*, 6(1):55–81, 2016.

[6] I. Ekeland and A. Lazrak. The golden rule when preferences are time inconsistent. *Math. Financ. Econ.*, 4(1):29–55, 2010.

[7] N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Math. Finance*, 7(1):1–71, 1997.

[8] Y. Hu, H. Jin, and X. Y. Zhou. Time-inconsistent stochastic linear-quadratic control. *SIAM J. Control Optim.*, 50(3):1548–1572, 2012.
[9] Y. Hu, H. Jin, and X. Y. Zhou. Time-inconsistent stochastic linear-quadratic control: characterization and uniqueness of equilibrium. *SIAM J. Control Optim.*, 55(2):1261–1279, 2017.

[10] W. Li, R. Wu, and K. Wang. Existence and uniqueness of M-solutions for backward stochastic Volterra integral equations. *Electron. J. Differential Equations*, pages No. 178, 16, 2014.

[11] J. Lin. Adapted solution of a backward stochastic nonlinear Volterra integral equation. *Stochastic Anal. Appl.*, 20(1):165–183, 2002.

[12] J. Ma and J. Yong. *Forward-backward stochastic differential equations and their applications*, volume 1702 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999.

[13] Y. H. Ni, J. F. Zhang, and M. Krstic. Time-inconsistent mean-field stochastic LQ problem: open-loop time-consistent control. *IEEE Trans. Automat. Control*, 63(9):2771–2786, 2018.

[14] Y. Shi and T. Wang. Solvability of general backward stochastic Volterra integral equations. *J. Korean Math. Soc.*, 49(6):1301–1321, 2012.

[15] Y. Shi, T. Wang, and J. Yong. Optimal control problems of forward-backward stochastic Volterra integral equations. *Math. Control Relat. Fields*, 5(3):613–649, 2015.

[16] T. Wang and J. Yong. Backward stochastic Volterra integral equations—representation of adapted solutions. *Stochastic Process. Appl.*, 2019.

[17] T. Wang and H. Zhang. Optimal control problems of forward-backward stochastic Volterra integral equations with closed control regions. *SIAM J. Control Optim.*, 55(4):2574–2602, 2017.

[18] Q. Wei, J. Yong, and Z. Yu. Time-inconsistent recursive stochastic optimal control problems. *SIAM J. Control Optim.*, 55(6):4156–4201, 2017.

[19] J. Yong. Backward stochastic Volterra integral equations and some related problems. *Stochastic Process. Appl.*, 116(5):779–795, 2006.

[20] J. Yong. Time-inconsistent optimal control problems and the equilibrium HJB equation. *Math. Control Relat. Fields*, 2(3):271–329, 2012.

[21] J. Zhang. *Backward stochastic differential equations*, volume 86 of *Probability Theory and Stochastic Modelling*. Springer, New York, 2017. From linear to fully nonlinear theory.