CONSTRUCTING HOPF BRACES

A. L. AGORE

ABSTRACT. We investigate Hopf braces, a concept recently introduced by Angiono, Galindo and Vendramin ([3]) in connection to the quantum Yang-Baxter equation. More precisely, we propose two methods for constructing Hopf braces. The first one uses matched pairs of Hopf algebras while the second one relies on category theoretic tools.

INTRODUCTION

The Yang-Baxter equation first appeared in the field of statistical mechanics in independent papers by Yang ([16]) and respectively Baxter ([4]). Given a vector space $V$, a linear map $R : V \otimes V \to V \otimes V$ is called a solution of the quantum Yang-Baxter equation if
\begin{equation}
R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}
\end{equation}
in $\text{End}(V^{\otimes 3})$, where $R^{ij}$ denotes $R$ acting on the $i$-th and respectively $j$-th components. It is well-known that (1) is equivalent to the braid equation in the sense that $R$ is a solution of the quantum Yang-Baxter equation if and only if $R\tau$ is a solution of the braid equation, where $\tau$ denotes the flip map (see for instance [5, Proposition 114]).

Although the problem of finding solutions to the Yang-Baxter equation is widely open, the interest it generated has led to the development of many fields and even to the appearance of new ones such as the theory of quantum groups. Due to its connections with various seemingly unrelated areas of mathematics and physics (e.g. knot theory, non-commutative geometry, conformal field theory, quantum groups, integrable systems, etc.), the Yang-Baxter equation was intensively studied from many different points of view and using a broad range of techniques.

Following an idea of Drinfel’d ([9]), set-theoretical solutions of the Yang-Baxter equation are being studied quite intensively. An important class of solutions are the non-degenerate involutive set-theoretical solutions which led to the introduction of braces by Rump ([13]). The equivalent definition of a brace proposed in [8] opened the way for generalizing this concept to the non-commutative setting: skew-braces were introduced in [7] as a tool for studying the non-involutive set-theoretical solutions. This newly
introduced concepts allow for methods from both group theory and ring theory to be used in studying set-theoretical solutions of the Yang-Baxter equation. Hopf braces were recently considered in [3] as the quantum version of skew-braces and, as expected, they provide solutions to the quantum Yang-Baxter equation. The purpose of this paper is to further investigate this new structures; more specifically we focus on constructing new examples of (cocommutative) Hopf braces. This is achieved on the one hand by using matched pairs of Hopf algebras and on the other hand by pursuing a category theoretic path. Although the idea of using matched pairs of groups or Hopf algebras for constructing solutions of the Yang-Baxter equation has been used before (see for instance [6], [3, Proposition 3.2]) our approach is different from the existing ones.

The outline of the paper is as follows. In Section 1 we collect some auxiliary material needed in the sequel. Section 2 presents two methods of constructing new (cocommutative) Hopf braces from a given (cocommutative) Hopf brace which is a part of a matched pair of (cocommutative) Hopf algebras (Theorem 2.1, Theorem 2.5). One notable consequence is Corollary 2.6 which proves that any matched pair between two cocommutative Hopf algebras $A$ and $H$ gives rise to a cocommutative Hopf brace on the tensor product coalgebra $A \otimes H$. Furthermore, any matched pair of groups or Lie algebras induces a cocommutative Hopf brace on the tensor product coalgebra of the corresponding group algebras, respectively the corresponding universal enveloping algebras (Example 2.7). Several explicit examples are also presented. Finally, Section 3 proves the completeness of the categories of Hopf braces and respectively cocommutative Hopf braces (Theorem 3.1).

1. Preliminaries

Throughout this paper, $k$ will be a field. Unless specified otherwise, all vector spaces, tensor products, homomorphisms, algebras, coalgebras, bialgebras, Lie algebras, Hopf algebras live over the field $k$. For a coalgebra $C$, we will use Sweedler’s Σ-notation $\Delta(c) = c_{(1)} \otimes c_{(2)}$ with suppressed summation sign. We use the classical notations for opposite and coopposite structures: $A^{\text{op}}$ is the opposite of the algebra $A$ while $C^{\text{coop}}$ stands for the coopposite of the coalgebra $C$. We refer to [15] for further details concerning Hopf algebras and to [5, Chapter 5] for a thorough background on the Yang-Baxter equation.

Without going into great detail, we collect here some useful notions and results which will be used in the sequel. Recall that a Hopf brace ([3]) over a coalgebra $(H, \Delta, \varepsilon)$ consists of two Hopf algebra structures on $H$, denoted by $(H, \cdot, 1, \Delta, \varepsilon, S)$ and respectively $(H, \circ, 1_\circ, \Delta, \varepsilon, T)$, compatible in the sense that for all $x, y, z \in H$ we have:

$$x \circ (yz) = (x_{(1)} \circ y)S(x_{(2)})(x_{(3)} \circ z)$$

where the multiplication of the Hopf algebra $(H, \cdot, 1, \Delta, \varepsilon, S)$ is denoted by juxtaposition. In order to keep the notation simple, we will denote the two Hopf algebra structures of a Hopf brace simply by $H$ and respectively $H_\circ$. By setting $x = y = 1_\circ$ and respectively $x = z = 1_\circ$ in (2), it can be easily seen that in any Hopf brace we have $1 = 1_\circ$ (see [3]). A Hopf brace will be called cocommutative if its underlying coalgebra is cocommutative.
CONSTRUCTING HOPF BRACES

Given two Hopf braces \((H, \cdot, \circ)\) and \((H', \cdot', \circ')\), a \(k\)-linear map \(f\) between the two underlying vector spaces is called a \textit{morphism of Hopf braces} if both \(f : H \to H'\) and \(f : H \circ \to H' \circ\) are morphisms of Hopf algebras. Hopf braces (resp. cocommutative Hopf braces) together with morphisms of Hopf braces form a category which we denote by \(k\)-HopfBr (resp. by \(k\)-(co)HopfBr). In what follows \(k\)-CoAlg stands for the category of coalgebras.

The main feature of cocommutative Hopf braces is that they provide solutions to the braid equation (\(3,\) Theorem 2.3, Corollary 2.4). More precisely, if \((H, \cdot, \circ, 1, \Delta, \varepsilon, S, T)\) is a cocommutative Hopf brace then the operator \(c : H \otimes H \to H \otimes H\) defined below provides a solution to the braid equation:

\[
c(x \otimes y) = S(x_{(1)})(x_{(2)} \circ y_{(1)}) \otimes T(S(x_{(3)})(x_{(4)} \circ y_{(2)})^\sim) \circ x_{(5)} \circ y_{(3)}
\]

\textbf{Examples 1.1.} 1) Any Hopf algebra \((H, \cdot, 1, \Delta, \varepsilon, S)\) gives rise to a Hopf brace by considering \(x \circ y = xy\) for all \(x, y \in H\). If \(H\) is cocommutative, the corresponding solution of the braid equation is given by:

\[
c(x \otimes y) = y_{(1)} \otimes S(y_{(2)})xy_{(3)}
\]

2) If \((H, \cdot, 1, \Delta, \varepsilon, S)\) is a Hopf algebra such that \(S^2 = \text{Id}_H\) then we can define a new multiplication on \(H\) by \(x \circ y = yx\), for all \(x, y \in H\), which together with the underlying coalgebra structure of \(H\) form a new Hopf algebra. Moreover, it is straightforward to check that (2) is trivially fulfilled and thus \((H, \cdot, \circ, 1, \Delta, \varepsilon, S, S)\) is a Hopf brace. If \(H\) is cocommutative (which automatically implies \(S^2 = \text{Id}_H\)) then we obtain the following solution of the braid equation:

\[
c(x \otimes y) = S(x_{(1)})yx_{(2)} \otimes x_{(3)}
\]

3) An important class of examples can be obtained from group theory and respectively from Lie algebra theory. More precisely, in light of our previous example, any group algebra and respectively any universal enveloping algebra of a Lie algebra is an example of a cocommutative Hopf brace.

Since the first method we use for constructing Hopf braces relies on matched pairs of Hopf algebras and the corresponding bicrossed product, it will be worth our while to recall these notions.

\textbf{Definition 1.2.} A \textit{matched pair} of Hopf algebras is a quadruple \((A, H, \triangleleft, \triangleright)\), where \(A\) and \(H\) are Hopf algebras, \(\triangleleft : H \otimes A \to H, \triangleright : H \otimes A \to A\) are coalgebra maps such that \((A, \triangleright)\) is a left \(H\)-module coalgebra, \((H, \triangleleft)\) is a right \(A\)-module coalgebra and the following compatibilities hold for any \(a, b \in A, x, y \in H:\)

\[
(3) \quad x \triangleright 1_A = \varepsilon_H(x)1_A, \quad 1_H \triangleleft a = \varepsilon_A(a)1_H \\
(4) \quad x \triangleright (ab) = (x_{(1)} \triangleright a_{(1)})((x_{(2)} \triangleleft a_{(2)}) \triangleright b) \\
(5) \quad (xy) \triangleleft a = (x \triangleleft (y_{(1)} \triangleright a_{(1)}))(y_{(2)} \triangleleft a_{(2)}) \\
(6) \quad x_{(1)} \triangleleft a_{(1)} \otimes x_{(2)} \triangleright a_{(2)} = x_{(2)} \triangleleft a_{(2)} \otimes x_{(1)} \triangleright a_{(1)}
\]
If $(A, H, \triangleleft, \triangleright)$ is a matched pair of Hopf algebras then the $k$-module $A \otimes H$ with the tensor coalgebra structure and the multiplication defined as follows for all $a, c \in A$, $x, y \in H$:

$$(a \otimes x)(b \otimes y) := a(x(1) \triangleright b(1)) \otimes (x(2) \triangleleft b(2))y$$

is a Hopf algebra called the bicrossed product of $A$ and $H$ and will be denoted by $A \triangleright \triangleleft H$.

The antipode on $A \triangleright \triangleleft H$ is given by:

$$(8) \quad S_{A \triangleright \triangleleft H}(a \triangleleft \triangleright x) = S_H(x(2)) \triangleright S_A(a(2)) \triangleright S_H(x(1)) \triangleleft S_A(a(1))$$

for all $a \in A$ and $x \in H$.

Examples of bicrossed products include the semi-direct (smash) product as defined by Molnar ([12]) in the cocommutative case. More precisely, assume that $(A, \triangleleft)$ is a left $H$-module bialgebra such that for all $a \in A$ and $x \in H$ we have:

$$(9) \quad x(1) \otimes x(2) \triangleright a = x(2) \otimes x(1) \triangleright a$$

Then $(A, H, \triangleleft, \triangleright)$ is a matched pair of Hopf algebras where $\triangleleft : H \otimes A \to B$ is the trivial action (i.e. $x \triangleleft a = x \epsilon(a)$ for all $a \in A$, $x \in H$) and the multiplication on the corresponding bicrossed product takes the form:

$$(10) \quad (a \# x)(b \# y) := a (x(1) \triangleright b) \# x(2)y$$

2. Constructing Hopf braces from matched pairs of Hopf algebras

In this section we use matched pairs of (cocommutative) Hopf algebras in order to construct (cocommutative) Hopf braces. More precisely, we obtain new Hopf braces by starting with a given Hopf brace which is part of a matched pair of Hopf algebras.

**Theorem 2.1.** Let $(A, \cdot, \circ)$ be a Hopf brace and $H$ a commutative, cocommutative Hopf algebra. If $(A, \triangleright)$ is a left $H$-module algebra and $(A_0, H, \triangleright, \triangleleft)$ is a matched pair of Hopf algebras such that for all $a, a' \in A$, $x \in H$ we have:

$$(11) \quad x \triangleright aa' = (x(1) \triangleright a)S(x(2))(x(3) \triangleright a')$$

then $A \otimes H$ together with the tensor product coalgebra and the following algebra structures:

$$(12) \quad (a \otimes x)(b \otimes y) = ab \otimes xy$$

$$(13) \quad (a \otimes x) \circ (b \otimes y) = a \circ (x(1) \triangleright b(1)) \otimes (x(2) \triangleleft b(2))y$$

is a Hopf brace.

**Proof.** The multiplication on $A \otimes H$ defined by (12) together with the tensor product of coalgebras is obviously a Hopf algebra. Furthermore, since $(A_0, H, \triangleright, \triangleleft)$ is a matched pair of Hopf algebras then $A \otimes H$ with the multiplication defined by (13) and the tensor product of coalgebras is also a Hopf algebra. Thus, we are left to prove that (2) holds
true. Indeed, for all \(a, b, c \in A\) and \(x, y, z \in H\) we have:

\[
(a \otimes x) \circ ((b \otimes y)(c \otimes z)) = a \circ (x_{(1)} \triangleright b_{(1)}c_{(1)}) \otimes (x_{(2)} \triangleright b_{(2)}c_{(2)})yz
\]

\[
= a \circ ((x_{(1)} \triangleright b_{(1)})(x_{(2)} \triangleright c_{(1)})) \otimes (x_{(3)} \triangleright b_{(2)}c_{(2)})yz
\]

(2)  

\[
(a_{(1)} \circ (x_{(1)} \triangleright b_{(1)}))S(a_{(2)})(a_{(3)} \circ (x_{(2)} \triangleright c_{(1)})) \otimes (x_{(3)} \triangleright b_{(2)})S(x_{(4)}(x_{(5)} \triangleright c_{(2)})yz
\]

(16)  

\[
= (a_{(1)} \circ (x_{(1)} \triangleright b_{(1)}))S(a_{(2)})(a_{(3)} \circ (x_{(4)} \triangleright c_{(1)})) \otimes (x_{(2)} \triangleright b_{(2)})S(x_{(3)}(x_{(5)} \triangleright c_{(2)})yz
\]

(12)  

\[
= (a_{(1)} \circ (x_{(1)} \triangleright b_{(1)}) \otimes (x_{(3)}(1) \triangleright (x_{(3)(2)} \triangleright c_{(2)}))z
\]

(13)  

\[
= (a_{(1)} \otimes x_{(1)}) \circ (b \otimes y))S(a_{(2)} \otimes x_{(2)})((a_{(3)} \otimes x_{(3)}) \circ (c \otimes z))
\]

where in the second equality we used the fact that \((A, \triangleright)\) is a left \(H\)-module algebra while the fifth and respectively the sixth equality follow from the cocommutativity and commutativity of \(H\).

**Corollary 2.2.** Let \((A, \cdot, \circ)\) be a Hopf brace and \(H\) a commutative, cocommutative Hopf algebra. If \((A, \triangleright)\) is a left \(H\)-module algebra and \((A_\circ, \triangleright)\) is a left \(H\)-module bialgebra then \(A \otimes H\) together with the tensor product coalgebra and the following algebra structures:

\[
(a \otimes x)(b \otimes y) = ab \otimes xy
\]

\[
(a \otimes x) \circ (b \otimes y) = a \circ (x_{(1)} \triangleright b_{(1)}) \otimes x_{(2)}y
\]

is a Hopf brace.

**Proof.** We apply Theorem 2.1 for the trivial right \(A_\circ\)-module coalgebra on \(H\), i.e. \(x \triangleright a = x \varepsilon(a)\) for all \(a \in A, x \in H\). Since \((A_\circ, \triangleright)\) is a left \(H\)-module bialgebra and (10) is fulfilled due to the cocommutativity of \(H\), the tensor product of coalgebras \(A \otimes H\) with the multiplication defined by (15) is a Hopf algebra. Now the conclusion follows from Theorem 2.1 by noticing that in this case the compatibility condition (16) is trivially fulfilled.

**Corollary 2.3.** Let \((A, \cdot, \circ)\) be a cocommutative Hopf brace and \(H\) a commutative, cocommutative Hopf algebra. If \((H, \triangleright)\) is a right \(A_\circ\)-module bialgebra such that for all \(a, a' \in A, x \in H\) we have:

\[
x \triangleright aa' = (x_{(1)} \triangleright a)S(x_{(2)})(x_{(3)} \triangleright a')
\]

then \(A \otimes H\) together with the tensor product coalgebra and the following algebra structures:

\[
(a \otimes x)(b \otimes y) = ab \otimes xy
\]

\[
(a \otimes x) \circ (b \otimes y) = a \circ b_{(1)} \otimes (x \triangleright b_{(2)})y
\]

is a Hopf brace.
Proof. We use again Theorem 2.1 for the trivial left \( H \)-module coalgebra on \( A \), i.e. \( x \triangleright a = a \in \mathcal{C}(x) \) for all \( a \in A \), \( x \in H \). As \( (H, \cdot) \) is a right \( A \)-module bialgebra then the tensor product of coalgebras \( A \otimes H \) with the multiplication defined by (15) is a Hopf algebra. Indeed, it is straightforward to see that the compatibility conditions (3)-(6) hold; for instance (6) is trivially fulfilled due to the cocommutativity of \( A \). The conclusion now follows from Theorem 2.1.

\[ \square \]

Example 2.4. Consider the cyclic groups \( C_3 \) and \( C_6 \) and let \( a \), respectively \( b \), denote the generators of the aforementioned groups. Then \( (k[C_3], k[C_6], \triangleright, \triangleleft) \) is a matched pair of cocommutative Hopf algebras, where:

\[
1 \triangleright a^j = a^j, \quad b^i \triangleright 1 = 1, \quad b^i \triangleright a = \begin{cases} a, & \text{if } i \text{ is even} \\ a^2, & \text{if } i \text{ is odd} \end{cases}, \quad b^i \triangleright a^2 = \begin{cases} a^2, & \text{if } i \text{ is even} \\ a, & \text{if } i \text{ is odd} \end{cases},
\]

\[
b^i \triangleleft 1 = b^i, \quad 1 \triangleleft a^j = 1, \quad b^i \triangleleft a = \begin{cases} b^i, & \text{if } i \text{ is even} \\ b^{i+2}, & \text{if } i \text{ is odd} \end{cases}, \quad b^i \triangleleft a^2 = \begin{cases} b^i, & \text{if } i \text{ is even} \\ b^{i+4}, & \text{if } i \text{ is odd} \end{cases}
\]

for all \( i \in \{1, 2, \ldots, 5\} \), \( j \in \{1, 2\} \). Moreover it can be easily seen by a straightforward computation that \( \triangleleft : k[C_6] \otimes k[C_3] \to k[C_6] \) defined above satisfies compatibility (16) which in this case comes down to proving that for all \( g \in C_6 \) and \( h, h' \in C_3 \) we have \( g \triangleleft hh' = (g \triangleleft h)g^{-1}(g \triangleleft h') \). Indeed, for instance we have:

\[
b^{2k+1} \triangleleft a^2 = b^{2k+6} = b^{2k+3}b^{2k-1}b^{2k+3} = (b^{2k+1} \triangleleft a)b^{2k-1}(b^{2k+1} \triangleleft a)
\]

\[
b^{2k+1} \triangleleft a^3 = b^{2k+7} = b^{2k+1}b^{2k-1}b^{2k+5} = (b^{2k+1} \triangleleft a)b^{2k-1}(b^{2k+1} \triangleleft a^2)
\]

all \( k \in \mathbb{N} \). Therefore, we have a cocommutative Hopf brace on \( k[C_3] \otimes k[C_6] \) as in Theorem 2.1.

Theorem 2.5. Let \( A \) be a Hopf algebra and \( (H, \cdot, \circ) \) a cocommutative Hopf brace. If \( (A, H, \triangleright, \triangleleft) \) is matched pair of Hopf algebras and \( (A, \triangleright) \) is a left \( H_\circ \)-module bialgebra such that for all \( a \in A \), \( x, x' \in H \) we have:

\[
(19) \quad x \triangleright (x' \triangleright a) = [(x_{(1)} \circ x')S(x_{(2)}) \triangleright (x_{(3)} \triangleright a)]
\]

\[
(20) \quad x \circ (x' \triangleleft a) = [(x_{(1)} \circ x')S(x_{(2)}) \triangleleft (x_{(3)} \triangleright a)]x_{(4)}
\]

then \( A \otimes H \) together with the tensor product coalgebra and the following algebra structures:

\[
(21) \quad (a \otimes x)(b \otimes y) = a(x_{(1)} \triangleright b_{(1)}) \otimes (x_{(2)} \triangleleft b_{(2)})y
\]

\[
(22) \quad (a \otimes x) \circ (b \otimes y) = a(x_{(1)} \triangleright b) \otimes x_{(2)} \circ y
\]

is a Hopf brace.

Proof. Since \( (A, H, \triangleright, \triangleleft) \) is a matched pair of Hopf algebras then \( A \otimes H \) with the multiplication defined by (21) and the tensor product of coalgebras is a Hopf algebra with antipode given by (8). Furthermore, as \( (A, \triangleright) \) is a left \( H_\circ \)-module bialgebra then the tensor product of coalgebras \( A \otimes H \) with the multiplication defined by (22) is again a Hopf algebra. Indeed, notice that (9) is trivially fulfilled by the cocommutativity assumption on \( H \). Therefore, we are left to prove that (2) holds true. Indeed, for all \( a, b \),
c \in A \text{ and } x, y, z \in B \text{ we have:}

\[(a_1 \otimes x_1) \circ (b \otimes y)] S(a_2 \otimes x_2) [(a_3 \otimes x_3) \circ (c \otimes z)]

= a_1(x_1) \triangleright b \left[ (x_2 \circ y_1) S(x_7) \triangleright S(a_4) \right] \left[ (x_3 \circ y_2) S(x_6) \triangleleft S(a_3) \right] \circ a_5(x_8) \triangleright c_1)

\otimes \left[ \left( (x_4 \circ y_3) S(x_5) \triangleleft S(a_2) \right) \circ a_6 \left( x_9 \triangleright c_2 \right) \right] \left( x_{10} \circ z \right)

= a_1(x_1) \triangleright b \left[ (x_2 \circ y_1) S(x_5) \triangleright S(a_3) a_4(x_6) \triangleright c_1 \right]

\otimes \left[ \left( (x_3 \circ y_2) S(x_4) \triangleleft S(a_2) \right) \circ a_5 \left( x_7 \triangleright c_2 \right) \right] \left( x_8 \circ z \right)

= a_1(x_1) \triangleright b \left[ (x_2 \circ y_1) S(x_5) \triangleright (x_6) \triangleright c_1 \right]

\otimes \left[ \left( (x_3 \circ y_2) S(x_4) \triangleleft S(a_2) \right) \circ a_3 \left( x_7 \triangleright c_2 \right) \right] \left( x_8 \circ z \right)

= a_1(x_1) \triangleright b \left[ (x_2 \circ y_1) S(x_5) \triangleright (x_6) \triangleright c_1 \right]

\otimes \left[ \left( (x_3 \circ y_2) S(x_4) \triangleleft S(a_2) \right) \circ S(a_3) \left( x_7 \triangleright c_2 \right) \right] \left( x_8 \circ z \right)

= a_1(x_1) \triangleright b \left[ (x_2 \circ y_1) S(x_5) \triangleright (x_6) \triangleright c_1 \right]

\otimes \left[ \left( (x_3 \circ y_2) S(x_4) \triangleleft S(a_2) \right) \circ (x_7) \triangleright c_2 \right] \left( x_8 \circ z \right)

= a_1(x_1) \triangleright b \left[ (x_2 \circ y_1) S(x_5) \triangleright (x_6) \triangleright c_1 \right]

\otimes \left[ \left( (x_3 \circ y_2) S(x_4) \triangleleft S(a_2) \right) \circ (x_7) \triangleright c_2 \right] \left( x_8 \circ z \right)

= a_1(x_1) \triangleright b \left[ (x_2 \circ y_1) S(x_5) \triangleright (x_6) \triangleright c_1 \right]

\otimes \left[ \left( (x_3 \circ y_2) S(x_4) \triangleleft S(a_2) \right) \circ (x_7) \triangleright c_2 \right] \left( x_8 \circ z \right)

= a_1(x_1) \triangleright b \left[ (x_2 \circ y_1) S(x_5) \triangleright (x_6) \triangleright c_1 \right]

\otimes \left[ \left( (x_3 \circ y_2) S(x_4) \triangleleft S(a_2) \right) \circ (x_7) \triangleright c_2 \right] \left( x_8 \circ z \right)

= a_1(x_1) \triangleright b \left[ (x_2 \circ y_1) S(x_5) \triangleright (x_6) \triangleright c_1 \right]

\otimes \left[ \left( (x_3 \circ y_2) S(x_4) \triangleleft S(a_2) \right) \circ (x_7) \triangleright c_2 \right] \left( x_8 \circ z \right)

\overset{(4)}{=} a_1(x_1) \triangleright b \left[ (x_2 \circ y_1) S(x_5) \triangleright S(a_3) a_4(x_6) \triangleright c_1 \right]

\otimes \left[ \left( (x_3 \circ y_2) S(x_4) \triangleleft S(a_2) \right) \circ a_5 \left( x_7 \triangleright c_2 \right) \right] \left( x_8 \circ z \right)

\overset{(5)}{=} a_1(x_1) \triangleright b \left[ (x_2 \circ y_1) S(x_5) \triangleright (x_6) \triangleright c_1 \right]

\otimes \left[ \left( (x_3 \circ y_2) S(x_4) \triangleleft S(a_2) \right) \circ a_3 \left( x_7 \triangleright c_2 \right) \right] \left( x_8 \circ z \right)

\overset{(20)}{=} a_1(x_1) \triangleright b \left[ (x_2 \circ y_1) S(x_3) \triangleright (x_4) \triangleright c_1 \right]

\otimes \left[ (x_5 \circ y_2) S(x_6) \triangleleft (x_7) \triangleright c_2 \right] \left( x_9 \circ z \right)

\overset{(2)}{=} a_1(x_1) \triangleright b \left[ (x_2 \circ y_1) S(x_3) \triangleright (x_4) \triangleright c_1 \right]

\otimes x_5 \circ \left[ (y_2 \triangleleft c_2) z \right]

\overset{(19)}{=} a_1(x_1) \triangleright b \left[ (x_2 \triangleright (y_1 \triangleright c_1)) \otimes x_3 \triangleleft (y_2 \triangleleft c_2) z \right]

\overset{(18)}{=} a_1(x_1) \triangleright b(y_1 \triangleright c_1) \otimes x_2 \triangleright (y_2 \triangleleft c_2) z

\overset{(a \otimes x) (b \otimes y)}{=} (a \otimes x) \circ (b \otimes y)

\overset{(a \otimes x) \circ (b \otimes y)}{=} (ab \otimes xy)

\text{where the sixth equality follows by the cocommutativity of } H \text{ while in the eleventh equality we used the fact that } (A, \triangleright) \text{ is a left } H_0 \text{-module algebra.}

\textbf{Corollary 2.6.} If } (A, H, \triangleright, \triangleleft) \text{ is a matched pair of Hopf algebras with } H \text{ cocommutative then the tensor product coalgebra } A \otimes H \text{ with the following algebra structures:}

\[(a \otimes x)(b \otimes y) = (a(x_1) \triangleright b_1) \otimes (x_2 \triangleleft b_2) y \]

\[(a \otimes x) \circ (b \otimes y) = (ab \otimes xy)\]
is a Hopf brace.

Proof. Indeed, this is an immediate consequence of Theorem 2.5 by seeing $H$ as a cocommutative Hopf brace with $x \circ x' = x' x$ (see Example 1.1, 2)) and considering the left $H\sigma$-module bialgebra on $A$ to be the trivial one, i.e. $x \triangleright a = \varepsilon(x) a$ for all $a \in A$, $x \in H$. It is now straightforward to see that in this case the compatibility conditions (19) and (20) are trivially fulfilled and the desired conclusion follows. □

Example 2.7. Corollary 2.6 allows for a plethora of examples of cocommutative Hopf braces obtained from matched pairs of groups ([10, Definition IX.1.1]) and respectively matched pairs of Lie algebras ([11, Definition 8.3.1]). More precisely, any matched pair of groups $(G, H, \rho, \lambda)$ extends uniquely to a matched pair of cocommutative Hopf algebras between the corresponding group Hopf algebras $(k[G], k[H], \rho, \lambda)$ ([10, Example 1, pg. 207]. Similarly, any matched pair of Lie algebras $(g, h, \lhd, \rhd)$ can be uniquely extended to a matched pair of cocommutative Hopf algebras between the corresponding universal enveloping algebras $(U(g), U(h), \lhd, \rhd)$.

Since both the group Hopf algebras and the enveloping universal algebras are cocommutative Hopf algebras, we can conclude that any matched pair of groups gives rise to a cocommutative Hopf brace on $k[G] \otimes k[H]$ and any matched pair of Lie algebras gives rise to a cocommutative Hopf brace on $U(g) \otimes U(h)$ as in Corollary 2.6.

Example 2.8. Consider $k[T]$ to be the polynomial Hopf algebra and let $H$ be the Hopf algebra generated by $X$, $Y$ and $Z$ subject to the following relations:

$$XY - YX = Z, \quad XZ = ZX, \quad YZ = ZY,$$

and the coalgebra structure given by:

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \Delta(Y) = Y \otimes 1 + 1 \otimes Y, \quad \Delta(Z) = Z \otimes 1 + 1 \otimes Z,$$

$$\varepsilon(X) = \varepsilon(Y) = \varepsilon(Z) = 0.$$

It can be easily seen that $k[T]$ is the universal enveloping algebra of the 1-dimensional abelian Lie algebra while $H$ is the universal enveloping algebra of the 3-dimensional Heisenberg Lie algebra. Then, for any $\alpha, \beta \in k$ we have a matched pair of cocommutative Hopf algebras $(k[T], H, \triangleright, \lhd)$ defined as follows for all $n \geq 1$:

$$X \triangleright T^n = Z \triangleright T^n = 0, \quad Y \triangleright T^n = \alpha \sum_{i=0}^{n-1} T^{n-i}(T - \alpha\beta)^i,$$

$$X^n \lhd T = n\beta X^{n-1}Z, \quad Y^n \lhd T = (Z - \alpha\beta Y) \sum_{i=0}^{n-1} Y^{n-1-i}(Y + \alpha)^i, \quad Y^n \lhd T = 0$$

Using Corollary 2.6 we obtain a cocommutative Hopf brace on $k[T] \otimes H$.

Corollary 2.9. Let $A$, $H$ be two Hopf algebras with $H$ cocommutative. If $(A, \triangleright)$ is a left $H^{op}$-module bialgebra and $(A, \triangleright)$ is a left $H$-module bialgebra such that for all $a \in A$, $x, x' \in H$ we have:

$$(23) \quad x \triangleright (x' \triangleright a) = x' \triangleright (x \triangleright a)$$

by seeing $H$ as a co-commutative Hopf brace with $x \circ x' = x' x$ (see Example 1.1, 2)) and considering the left $H\sigma$-module bialgebra on $A$ to be the trivial one, i.e. $x \triangleright a = \varepsilon(x) a$ for all $a \in A$, $x \in H$. It is now straightforward to see that in this case the compatibility conditions (19) and (20) are trivially fulfilled and the desired conclusion follows. □

Example 2.7. Corollary 2.6 allows for a plethora of examples of cocommutative Hopf braces obtained from matched pairs of groups ([10, Definition IX.1.1]) and respectively matched pairs of Lie algebras ([11, Definition 8.3.1]). More precisely, any matched pair of groups $(G, H, \rho, \lambda)$ extends uniquely to a matched pair of cocommutative Hopf algebras between the corresponding group Hopf algebras $(k[G], k[H], \rho, \lambda)$ ([10, Example 1, pg. 207]. Similarly, any matched pair of Lie algebras $(g, h, \lhd, \rhd)$ can be uniquely extended to a matched pair of cocommutative Hopf algebras between the corresponding universal enveloping algebras $(U(g), U(h), \lhd, \rhd)$.

Since both the group Hopf algebras and the enveloping universal algebras are cocommutative Hopf algebras, we can conclude that any matched pair of groups gives rise to a cocommutative Hopf brace on $k[G] \otimes k[H]$ and any matched pair of Lie algebras gives rise to a cocommutative Hopf brace on $U(g) \otimes U(h)$ as in Corollary 2.6.

Example 2.8. Consider $k[T]$ to be the polynomial Hopf algebra and let $H$ be the Hopf algebra generated by $X$, $Y$ and $Z$ subject to the following relations:

$$XY - YX = Z, \quad XZ = ZX, \quad YZ = ZY,$$

and the coalgebra structure given by:

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \Delta(Y) = Y \otimes 1 + 1 \otimes Y, \quad \Delta(Z) = Z \otimes 1 + 1 \otimes Z,$$

$$\varepsilon(X) = \varepsilon(Y) = \varepsilon(Z) = 0.$$

It can be easily seen that $k[T]$ is the universal enveloping algebra of the 1-dimensional abelian Lie algebra while $H$ is the universal enveloping algebra of the 3-dimensional Heisenberg Lie algebra. Then, for any $\alpha, \beta \in k$ we have a matched pair of cocommutative Hopf algebras $(k[T], H, \triangleright, \lhd)$ defined as follows for all $n \geq 1$:

$$X \triangleright T^n = Z \triangleright T^n = 0, \quad Y \triangleright T^n = \alpha \sum_{i=0}^{n-1} T^{n-i}(T - \alpha\beta)^i,$$

$$X^n \lhd T = n\beta X^{n-1}Z, \quad Y^n \lhd T = (Z - \alpha\beta Y) \sum_{i=0}^{n-1} Y^{n-1-i}(Y + \alpha)^i, \quad Y^n \lhd T = 0$$

Using Corollary 2.6 we obtain a cocommutative Hopf brace on $k[T] \otimes H$.
then the tensor product coalgebra $A \otimes H$ with the following algebra structures:

\[
\begin{align*}
(a \otimes x)(b \otimes y) &= (a(x(1) \triangleright b) \otimes x(2)y) \\
(a \otimes x) \circ (b \otimes y) &= (a(x(1) \blacktriangleright b) \otimes yx(2))
\end{align*}
\]

is a Hopf brace.

**Proof.** This can be easily derived from Theorem 2.5 by seeing $H$ as a cocommutative Hopf brace with $x \circ x' = x'x$ (see Example 1.1, 2) and considering the right $A$-module coalgebra structure on $H$ to be trivial, i.e. $x \triangleright a = x\varepsilon(a)$ for all $a \in A$, $x \in H$. In this case the compatibility condition (20) is trivially fulfilled while (19) amounts to (23) as desired. \qed

**Example 2.10.** Consider $k[C]$ to be the group Hopf algebra of the infinite cyclic group in multiplicative notation and let $d$ be a generator of $C$. If $k[X]$ denotes the polynomial Hopf algebra then given any $\alpha, \beta \in k^*$ it is easy to see that $\triangleright, \blacktriangleright: k[C] \otimes k[X] \rightarrow k[X]$ defined below are left $kC$-module bialgebras:

\[
d^i \triangleright X^t = \alpha^{it}X^t, \quad d^i \blacktriangleright X^t = \beta^{it}X^t, \quad i \in \mathbb{Z}, \ t \in \mathbb{N}
\]

and, moreover, (23) is fulfilled. Thus, we obtain a cocommutative Hopf brace on $k[X] \otimes k[C]$ as in Corollary 2.9.

**Example 2.11.** Let $k$ be a field of characteristic $\neq 2$. $C_n$ denotes the cyclic group of order $n$ generated by $c$ and $H_4$ is the Sweedler’s 4-dimensional Hopf algebra having $\{1, g, x, gx\}$ as a basis subject to the relations:

\[
g^2 = 1, \quad x^2 = 0, \quad xg = -gx
\]

with the coalgebra structure and antipode given by:

\[
\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad S(g) = g, \quad S(x) = -gx.
\]

If $\omega, \lambda$ are $n$-th roots on unity in $k$, then $(H_4, \triangleright)$ and respectively $(H_4, \blacktriangleright)$ are left $k[C_n]$-module bialgebras ([2, Proposition 5.3]) where $\triangleright, \blacktriangleright: k[C_n] \otimes H_4 \rightarrow H_4$ are defined as follows:

\[
c^i \triangleright g = g, \quad c^i \triangleright x = \omega^i x, \quad c^i \blacktriangleright g = g, \quad c^i \blacktriangleright x = \lambda^i x,
\]

for all $i \in \mathbb{N}$. Moreover, it can be easily seen that (23) is fulfilled for the two left $k[C_n]$-module bialgebras defined above. Therefore, we have a Hopf brace on $H_4 \otimes k[C_n]$ as in Corollary 2.9.

**Corollary 2.12.** Let $A, H$ be two Hopf algebras with $H$ cocommutative and consider $(A, \blacktriangleright)$ a left $H^{op}$-module bialgebra and $(H, \triangleright)$ a right $A$-module bialgebra such that for all $a \in A$, $x, x' \in B$ we have:

\[
(x' \triangleright a)x = (x' \triangleright (x(1) \blacktriangleright a))x(2)
\]

Then the tensor product coalgebra $A \otimes H$ with the following algebra structures:

\[
\begin{align*}
(a \otimes x)(b \otimes y) &= (ab_{(1)} \otimes (x \triangleright b_{(2)}y)) \\
(a \otimes x) \circ (b \otimes y) &= (a(x(1) \blacktriangleright b) \otimes yx(2))
\end{align*}
\]

is a Hopf brace.
Proof. We apply again Theorem 2.5; \( H \) will be seen as a comultiplicative Hopf brace with \( x \circ x' = x'x \) (see Example 1.1, 2) and the left \( A \)-module coalgebra structure on \( H \) will be the trivial one, i.e. \( a \triangleright x = x \varepsilon (a) \) for all \( a \in A, x \in H \). Then (19) is trivially fulfilled while (20) comes down to (24).

We end this section with some generic examples of (cocommutative) Hopf braces and the corresponding solutions of the braid equation.

Examples 2.13. 1) Let \( A, H \) be two Hopf algebras with \( H \) cocommutative. If \((A, \triangleright)\) is a left \( H \)-module bialgebra then by Corollary 2.9 the tensor product coalgebra \( A \otimes H \) with the following algebra structures:

\[
(a \otimes x)(b \otimes y) = (a(x_{(1)} \triangleright b) \otimes x_{(2)}y), \quad (a \otimes x) \circ (b \otimes y) = (ab \otimes yx)
\]

is a Hopf brace. If \( A \) is cocommutative as well we obtain a cocommutative Hopf brace on \( A \otimes H \) and a solution of the braid equation given by:

\[
c\left( (a \otimes x) \otimes (b \otimes y) \right) =
\left( S(x_{(2)} \triangleright b_{(1)}) \otimes S(x_{(1)}y_{(1)}x_{(3)}) \otimes T(S(x_{(3)} \triangleright b_{(2)})) a_{(5)}b_{(3)} \otimes T(S(x_{(4)})) \right)
\]

2) Let \( A, H \) be two Hopf algebras with \( H \) cocommutative. If \((A, \triangleright)\) is a left \( H^{\text{op}} \)-module bialgebra then by Corollary 2.12 the tensor product coalgebra \( A \otimes H \) with the following algebra structures:

\[
(a \otimes x)(b \otimes y) = (ab \otimes xy), \quad (a \otimes x) \circ (b \otimes y) = (a(x_{(1)} \triangleright b) \otimes yx_{(2)})
\]

is a Hopf brace. If \( A \) is cocommutative as well we obtain a cocommutative Hopf brace on \( A \otimes H \) and a solution of the braid equation given by:

\[
c\left( (a \otimes x) \otimes (b \otimes y) \right) =
\left( x_{(2)} \triangleright b_{(1)} \otimes S(x_{(1)}y_{(1)}x_{(3)}) \otimes \right)
\left( T(x_{(5)}T(y_{(2)})T(S(x_{(4)}))) \triangleright T(x_{(7)} \triangleright b_{(2)}) a_{(8)} \triangleright b_{(3)} \otimes T(S(x_{(5)})) \right)
\]

3) Let \( A, H \) be two Hopf algebras with \( H \) cocommutative. If \((H, \triangleleft)\) is a right \( A \)-module bialgebra then by Corollary 2.12 the tensor product coalgebra \( A \otimes H \) with the following algebra structures:

\[
(a \otimes x)(b \otimes y) = (ab_{(1)} \otimes (x \triangleleft b_{(2)})y), \quad (a \otimes x) \circ (b \otimes y) = (ab \otimes yx)
\]

is a Hopf brace. If \( A \) is cocommutative as well we obtain a cocommutative Hopf brace on \( A \otimes H \) and a solution of the braid equation given by:

\[
c\left( (a \otimes x) \otimes (b \otimes y) \right) =
\left( b_{(1)} \otimes (S(x_{(1)} \triangleleft b_{(2)})y_{x_{(3)}}) \otimes \right)
\left( T(b_{(3)}ab_{(5)} \otimes T(S(x_{(3)} \triangleleft b_{(4)})) \right)
\]

Example 2.14. Let \( \mathfrak{g} \) be a finite dimensional Lie algebra with \( k \)-basis \( \{x_1, x_2, \cdots, x_n\} \). Then the adjoint representation of \( \mathfrak{g} \) uniquely induces a left \( U(\mathfrak{g}) \)-module bialgebra structure on \( U(\mathfrak{g}) \) given as follows for all \( i, j \in 1, 2, \cdots, n \) and \( k \in \mathbb{N}^* \):

\[
x_i \triangleright x_j^k = \sum_{t=0}^{k-1} x_j^t(x_ix_j - x_jx_i)x_j^{k-1-t}.
\]
Hence, we obtain a cocommutative Hopf brace on $U(g) \otimes U(g)$ as in Example 2.13, 1).

### 3. On the category of Hopf braces

In this section we consider some category theoretic properties of Hopf braces. More precisely, we will prove that the category of Hopf braces (resp. cocommutative Hopf braces) is complete. This allows for the construction of new Hopf braces. Throughout this section it is convenient to denote the two algebra structures of a Hopf brace $H$ by $m$, respectively $\overline{m}$. Using this notation, (2) can be written equivalently as follows:

\[(25) \; \overline{m}(I \otimes m) = m(m \otimes 1)(\overline{m} \otimes S \otimes \overline{m})(1 \otimes \tau \otimes I)(I \otimes I \otimes \tau \otimes I)(\Delta \otimes I \otimes I)(\Delta \otimes I \otimes I)\]

where $\tau : H \otimes H \to H \otimes H$ denotes the flip map, i.e. $\tau(x \otimes y) = y \otimes x$ for all $x, y \in H$.

**Theorem 3.1.** The category $k$-HopfBr is complete, i.e. it has all small limits.

**Proof.** As the construction of limits in the category of Hopf braces is rather cumbersome we will restrict ourselves to products and equalizers; this ensures the existence of all small limits in the aforementioned category. We start by constructing products. To this end, let $(H_i, m_i, \eta_i, \pi_i, u_i, \Delta_i, \varepsilon_i, S_i, T_i)_{i \in I}$ be a family of Hopf braces, i.e. $(H_i, m_i, \eta_i, \Delta_i, \varepsilon_i, S_i)_{i \in I}$ and respectively $(H_i, \overline{m}_i, u_i, \Delta_i, \varepsilon_i, T_i)_{i \in I}$ are Hopf algebras such that the compatibility condition (25) is fulfilled. Consider $((H, \Delta, \varepsilon), (\pi_i)_{i \in I})$ to be the product (in $k$-CoAlg) of the underlying family of coalgebras (see [1, Theorem 1.1] for the explicit construction), where $\pi_i : H \to H_i$ are coalgebra maps for all $i \in I$. Since each $m_i, \eta_i, \overline{m}_i, u_i$ is a coalgebra map, the universality of the coproduct in $k$-CoAlg yields unique coalgebra maps $\eta : k \to H$ and $m, \overline{m} : H \otimes H \to H$ such that the following diagrams are commutative for all $i \in I$:

\[
\begin{array}{ccc}
\eta & \downarrow \pi_i & H_i \\
H & \pi_i & H_i \\
\end{array}
\]

\[
\begin{array}{ccc}
u & \downarrow \pi_i & H_i \\
H & \pi_i & H_i \\
\end{array}
\]

\[
\begin{array}{ccc}
m & \downarrow m_i(\pi_i \otimes \pi_i) & H_i \\
H & \pi_i & H_i \\
\end{array}
\]

\[
\begin{array}{ccc}
\overline{m} & \downarrow \overline{m}_i(\pi_i \otimes \pi_i) & H_i \\
H & \pi_i & H_i \\
\end{array}
\]

(26) \quad i.e. \quad $\pi_i \eta = \eta_i$, $\pi_i u = u_i$

(27) \quad $\pi_i m = m_i(\pi_i \otimes \pi_i)$

(28) \quad $\pi_i \overline{m} = \overline{m}_i(\pi_i \otimes \pi_i)$

Since proving that $(H, m, \eta, \Delta, \varepsilon)$ and respectively $(H, \overline{m}, u, \Delta, \varepsilon)$ are actually bialgebras goes essentially in the same vain as the proof of [1, Theorem 1.5] we will not include the details here; we refer the reader to [1].

Next we construct antipodes for the two bialgebras $(H, m, \eta, \Delta, \varepsilon)$ and $(H, \overline{m}, u, \Delta, \varepsilon)$. Using again the universality of the product in the category $k$-CoAlg we obtain two unique coalgebra morphisms $S, T : H \to H^{op, cop}$ such that the following diagrams are
Indeed, for all $i \in I$:

\[
\begin{array}{ccc}
S & \xrightarrow{\pi_i} & H_i \xleftarrow{\pi_i} S_i \\
H & \xrightarrow{\pi_i} & H_i
\end{array}
\]

(29) i.e. $\pi_i S = S_i \pi_i$ and $\pi_i T = T_i \pi_i$

In order to prove that $S, T : H^{op,cop} \to H$ are also algebra maps it will suffice to show that for all $i \in I$ we have $\pi_i m(S \otimes S) \tau = \pi_i S m$, and respectively $\pi_i \overline{m}(T \otimes T) \tau = \pi_i T \overline{m}$. Indeed, having in mind that $S_i : H_i^{op,cop} \to H_i$ is an algebra map for all $i \in I$ we obtain:

\[
\begin{align*}
\pi_i S m &= S_i \overline{m} \pi_i (\pi_i \otimes \pi_i) = \overline{m}(S_i \otimes S_i) \tau (\pi_i \otimes \pi_i) \\
&= m_i(S_i \otimes S_i)(\pi_i \otimes \pi_i) \tau = m_i(S_i \overline{m} \otimes S_i \overline{m}) \tau \\
&= m_i(\pi_i S \otimes \pi_i S) \tau = m_i(\pi_i \otimes \pi_i)(S \otimes S) \tau \\
&= \pi_i m(S \otimes S) \tau
\end{align*}
\]

as desired. A similar computation shows that $T$ is also an algebra map.

Now exactly as in the proof of [1] one can prove that $(H, m, \eta, \Delta, \varepsilon, S, \overline{m}, u, \Delta, \varepsilon, T)$ are actually Hopf algebras. We are left to prove that (25) holds. As before, it will suffice to show that for all $i \in I$ the following compatibility holds true:

\[
\pi_i \overline{m}(I \otimes m) = \pi_i m(m \otimes I)(\overline{m} \otimes S \otimes \overline{m})(I \otimes \tau \otimes I \otimes I)(I \otimes I \otimes \tau \otimes I)(\Delta \otimes I \otimes I \otimes I)(\Delta \otimes I \otimes I)
\]

Indeed, for all $i \in I$, we have:

\[
\begin{align*}
\pi_i m(m \otimes I)(\overline{m} \otimes S \otimes \overline{m})(I \otimes \tau \otimes I \otimes I)(I \otimes I \otimes \tau \otimes I)(\Delta \otimes I \otimes I \otimes I)(\Delta \otimes I \otimes I) \\
&= m_i(\pi_i \otimes \pi_i)(m \otimes I)(\overline{m} \otimes S \otimes \overline{m})(I \otimes \tau \otimes I \otimes I)(I \otimes I \otimes \tau \otimes I)(\Delta \otimes I \otimes I \otimes I)(\Delta \otimes I \otimes I) \\
&= m_i(\pi_i \otimes \pi_i)(\overline{m} \otimes S \otimes \overline{m})(I \otimes \tau \otimes I \otimes I)(I \otimes I \otimes \tau \otimes I)(\Delta \otimes I \otimes I \otimes I)(\Delta \otimes I \otimes I) \\
&= m_i(m_i \otimes I)(\pi_i \otimes \pi_i \otimes \pi_i)(\overline{m} \otimes S \otimes \overline{m})(I \otimes \tau \otimes I \otimes I)(I \otimes I \otimes \tau \otimes I)(\Delta \otimes I \otimes I \otimes I)(\Delta \otimes I \otimes I)
\end{align*}
\]
Now let $f, g : A \to B$ be two morphisms of Hopf braces. Consider $S = \{ a \in A \mid f(a) = g(a) \}$ and let $C$ be the sum of all subcoalgebras of $A$ contained in $S$. It can be easily seen (see the proof of [1, Theorem 1.7]) that $C$ is actually a Hopf subalgebra with respect to both Hopf algebra structures of $A$ and is obviously a Hopf brace. Thus, $(C, i)$ is the equalizer of the pair of morphisms $(f, g)$ in $k$-HopfBr, where $i : C \to A$ is the canonical inclusion.

\[ \text{Corollary 3.2. The category } k\text{-}(co)\text{HopfBr is complete, i.e. it has all limits.} \]

\[ \text{Proof. The constructions are similar to those in Theorem 3.1. Indeed, we only point out that the product in the category of cocommutative coalgebras is simply the tensor product of the given family of coalgebras. Therefore, the product of a given family of cocommutative Hopf braces is the tensor product of the underlying coalgebras with the Hopf brace structure defined exactly as in the proof of Theorem 3.1.} \]

\[ \text{References} \]

[1] A.L. Agore - Limits of coalgebras, bialgebras and Hopf algebras, Proc. Amer. Math. Soc. 139 (2011), 855–863.
[2] A.L. Agore, C.G. Bontea, G. Militaru - Classifying bicrossed products of Hopf algebras. *Algebr. Represent. Theory* **17** (2014), 227–264.

[3] I. Angiono, C. Galindo, L. Vendramin - Hopf braces and Yang-Baxter operators, *Proc. Amer. Math. Soc.* **145** (2017), 1981–1995.

[4] R. J. Baxter - Partition function of the eight-vertex lattice model, *Ann. Physics* **70** (1972), 193–228.

[5] S. Caenepeel, G. Militaru, S. Zhu - Frobenius and Separable Functors for Generalized Module Categories and Nonlinear Equations, *Lect. Notes Math.* **1787** Springer Verlag, Berlin, 2002.

[6] T. Gateva-Ivanova, S. Majid - Matched pairs approach to set theoretic solutions of the Yang-Baxter equation, *J. Algebra* **319** (2008), 1462–1529.

[7] L. Guarnieri, L. Vendramin - Skew braces and the Yang-Baxter equation, *Math. Comp.* **86** (2017), 2519–2534.

[8] F. Cedó, E. Jespers, J. Okniński - Braces and the Yang-Baxter Equation, *Commun. Math. Phys.* **327** (2014), 101–116.

[9] V. G. Drinfel’d - On some unsolved problems in quantum group theory, Quantum groups (Leningrad, 1990), *Lecture Notes in Math.* **1510**, Springer, Berlin, 1992, 1–8.

[10] C. Kassel - Quantum groups, Graduate Texts in Mathematics 155. Springer-Verlag, New York, 1995.

[11] S. Majid - Foundations of quantum groups theory, Cambridge University Press, 1995.

[12] R.K. Molnar - Semi-direct products of Hopf algebras, *J. Algebra* **47** (1977), 29 – 51.

[13] W. Rump - Braces, radical rings, and the quantum Yang-Baxter equation, *J. Algebra* **307** (2007), 153–170.

[14] A. Smoktunowicz, L. Vendramin - On skew braces, arXiv:1705.06958v2.

[15] M.E. Sweedler - Hopf Algebras, Benjamin New York, 1969.

[16] C. N. Yang - Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, *Phys. Rev. Lett.* **19** (1967), 1312–1315.

Faculty of Engineering, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium and "Simion Stoilow" Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

E-mail address: ana.agore@vub.be and ana.agore@gmail.com