ENERGY-MOMENTUM CONSERVATION LAWS
IN AFFINE-METRIC GRAVITATION THEORY.

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Abstract
The Lagrangian formulation of field theory does not provide any universal energy-momentum conservation law in order to analyze that in gravitation theory. In Lagrangian field theory, we get different identities involving different stress energy-momentum tensors which however are not conserved, otherwise in the covariant multimomentum Hamiltonian formalism. In the framework of this formalism, we have the fundamental identity whose restriction to a constraint space can be treated the energy-momentum transformation law. This identity remains true also for gravity. Thus, the tools are at hand to investigate the energy-momentum conservation laws in gravitation theory. The key point consists in the feature of a metric gravitational field whose canonical momenta on the constraint space are equal to zero.

1 Introduction
In Hamiltonian mechanics, there is the conventional energy transformation law
\[ \frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} \] (1)
on solutions of the Hamilton equations, otherwise in field theory.

The standard Hamiltonian formalism has been applied to field theory. In the straightforward manner, it takes the form of the instantaneous Hamiltonian formalism when canonical variables are field functions at a given instant of time. The corresponding phase space is infinite-dimensional, so that the Hamilton equations in the bracket form are not the familiar differential equations, adequate to the Euler-Lagrange field equations.

In Lagrangian field theory, we have no conventional energy-momentum transformation law. One gets different identities which involve different stress energy-momentum tensors, in particular, different canonical energy-momentum tensors. Moreover, one can not say a priori what is really conserved.

We follow the generally accepted geometric description of classical fields by sections of fibred manifolds \( Y \rightarrow X \). Their dynamics is phrased in terms of jet spaces \([2, 7, 10, 13, 15]\). Given a fibred manifold \( Y \rightarrow X \), the \( k \)-order jet space \( J^kY \) of \( Y \) comprises the equivalence
classes \( j^k \), \( x \in X \), of sections \( s \) of \( Y \) identified by the first \((k + 1)\) terms of their Taylor series at a point \( x \). It is a finite-dimensional smooth manifold. Recall that a \( k \)-order differential operator on sections of a fibred manifold \( Y \), by definition, is a morphism of \( J^kY \) to a vector bundle over \( X \). As a consequence, the dynamics of field systems is played out on finite-dimensional configuration and phase spaces.

In field theory, we can restrict ourselves to the first order Lagrangian formalism when the configuration space is \( J^1Y \). Given fibred coordinates \((x^\mu, y^i)\) of \( Y \), the jet space \( J^1Y \) is endowed with the adapted coordinates \((x^\mu, y^i, y^i_\mu)\):

\[
y^i_\lambda = \left( \frac{\partial y^i}{\partial y^j} y^j_\mu + \frac{\partial y^i}{\partial x^\mu} \right) \frac{\partial x^\mu}{\partial x^\lambda},
\]

A first order Lagrangian density on the configuration space \( J^1Y \) is represented by a horizontal exterior density

\[
L = \mathcal{L}(x^\mu, y^i, y^i_\mu) \omega, \quad \omega = dx^1 \wedge ... \wedge dx^n, \quad n = \dim X.
\]

The corresponding first order Euler-Lagrange equations for sections \( \pi \) of the fibred jet manifold \( J^1Y \to X \) read

\[
\partial_\lambda \pi^i = \pi^i_\lambda, \\
\partial_\lambda \mathcal{L} - \left( \partial_\lambda + \pi^j_\lambda \partial_j + \partial_\lambda \pi^j_\mu \partial_j^\mu \right) \partial_i^\lambda \mathcal{L} = 0.
\]  

(2)

We consider the Lie derivatives of Lagrangian densities in order to obtain differential conservation laws. Let

\[
u = u^\mu \partial_\mu + u^i \partial_i
\]

be a vector field on a fibred manifold \( Y \) and \( \pi \) its jet lift \((\ref{Lift})\) onto the fibred jet manifold \( J^1Y \to X \). Given a Lagrangian density \( L \) on \( J^1Y \), let us computer the Lie derivative \( \mathcal{L}_\pi \).

\[
\mathcal{L}_\pi L = \frac{d}{dx^\lambda} [\pi^\lambda_\iota \mathcal{L}(\pi) (u^i - u^\mu \pi^i_\mu)] \omega, \quad \pi^\mu_\iota = \partial^\mu_\iota \mathcal{L}.
\]  

(3)

In particular, if \( u \) is a vertical vector field such that \( \mathcal{L}_\pi L = 0 \), the equality \((\ref{Lift})\) takes the form of the current conservation law

\[
\frac{d}{dx^\lambda} [u^i \pi^\lambda_\iota (\pi)] = 0.
\]  

(4)

In gauge theory, this conservation law is exemplified by the Noether identities. Let

\[
\tau = \tau^\lambda \partial_\lambda
\]
be a vector field on $X$ and

$$u = \tau_\Gamma = \tau^\mu (\partial_\mu + \Gamma^i_\mu \partial_i)$$

its horizontal lift onto the fibred manifold $Y$ by a connection $\Gamma$ on $Y$. In this case, the

equality (3) takes the form

$$\mathbf{s}^\star L_\tau L = -\frac{d}{dx^\lambda}[\tau^\mu T^\lambda_\mu (\mathbf{s})] \omega$$

where

$$T^\lambda_\mu (\mathbf{s}) = \pi^\lambda_i (s^\mu - \Gamma^i_\mu) - \delta^\lambda_\mu \mathcal{L}$$

is the canonical energy-momentum tensor of a field $\mathbf{s}$ with respect to the connection $\Gamma$ on $Y$. The tensor (3) is the particular case of the stress energy-momentum tensors [1, 3, 6].

In particular, when the fibration $Y \to X$ is trivial, one can choose the trivial connection $\Gamma^i_\mu = 0$. In this case, the tensor (3) is precisely the standard canonical energy-momentum tensor, and if

$$L_\tau \mathcal{L} = 0$$

for all vector fields $\tau$ on $X$ (e.g., $X$ is the Minkowski space), the conservation law (3) comes to the well-known conservation law

$$\frac{d}{d x^\lambda} T^\lambda_\mu (\mathbf{s}) = 0$$

of the canonical energy-momentum tensor.

In general, the Lie derivative $L_\tau L$ fails to be equal to zero as a rule, and the equality (3) is not the conservation law of a canonical energy-momentum tensor. For instance, in

in gauge theory of gauge potentials and scalar matter fields in the presence of a background world metric $g$, we get the covariant conservation law

$$\nabla_\lambda t^\lambda_\mu = 0$$

of the metric energy-momentum tensor.

In Einstein’s General Relativity, the covariant conservation law (7) issues directly from gravitational equations. But it is concerned only with zero-spin matter in the presence

of the gravitational field generated by this matter itself. The total energy-momentum conservation law for matter and gravity is introduced by hand. It reads

$$\frac{d}{d x^\mu} [(-g)^N (t^\lambda_\mu + T^\lambda_\mu)] = 0$$

where the energy-momentum pseudotensor $T^\lambda_\mu$ of a metric gravitational field is defined to satisfy the relation

$$(-g)^N (t^\lambda_\mu + T^\lambda_\mu) = \frac{1}{2\kappa} \partial_\sigma \partial_\alpha [(-g)^N (g^{\lambda\mu} g^{\sigma\alpha} - g^{\mu\alpha} g^{\lambda\sigma})$$

3
on solutions of the Einstein equations. The conservation law (8) is rather satisfactory only in cases of asymptotic-flat gravitational fields and a background gravitational field. The energy-momentum conservation law in the affine-metric gravitation theory and the gauge gravitation theory was not discussed widely [5].

Thus, the Lagrangian formulation of field theory does not provide us with any universal procedure in order to analyze the energy-momentum conservation law in gravitation theory, otherwise the covariant multimomentum Hamiltonian formalism. In the framework of this formalism, we get the fundamental identity (32) whose restriction to the Lagrangian constraint space can be treated the energy-momentum transformation law in field theory [10, 14].

Lagrangian densities of field models are almost always degenerate and the corresponding Euler-Lagrange equations are underdetermined. To describe constraint field systems, the multimomentum Hamiltonian formalism can be utilized [9, 11, 12]. In the framework of this formalism, the finite-dimensional phase space of fields is the Legendre bundle

\[ \Pi = \bigwedge^n T^* X \otimes T X \otimes V^* Y \] (9)

over \( Y \) into which the Legendre morphism \( \hat{L} \) associated with a Lagrangian density \( L \) on \( J^1 Y \) takes its values. This phase space is provided with the fibred coordinates \((x^\lambda, y^i, p^\lambda_i)\) such that

\[ (x^\mu, y^i, p^\mu_i) \circ \hat{L} = (x^\mu, y^i, \pi^\mu_i). \]

The Legendre bundle (9) carries the multisymplectic form

\[ \Omega = dp^\lambda_i \wedge dy^i \wedge \omega \otimes \partial_\lambda. \] (10)

In case of \( X = \mathbb{R} \), this form recovers the standard symplectic form in analytical mechanics.

Building on the multisymplectic form \( \Omega \), one can develop the so-called multimomentum Hamiltonian formalism of field theory where canonical momenta correspond to derivatives of fields with respect to all world coordinates, not only the temporal one. On the mathematical level, this is the multisymplectic generalization of the standard Hamiltonian formalism in analytical mechanics to fibred manifolds over an \( n \)-dimensional base \( X \), not only \( \mathbb{R} \). We say that a connection \( \gamma \) on the fibred Legendre manifold \( \Pi \to X \) is a Hamiltonian connection if the form \( \gamma \rfloor \Omega \) is closed. Then, a Hamiltonian form \( H \) on \( \Pi \) is defined to be an exterior form such that

\[ dH = \gamma \rfloor \Omega \] (11)

for some Hamiltonian connection \( \gamma \). Every Hamiltonian form admits splitting

\[ H = p^\lambda_i dy^i \wedge \omega_\lambda - p^\lambda_i \Gamma^i_\lambda \omega - \tilde{\mathcal{H}}_\Gamma \omega = p^\lambda_i dy^i \wedge \omega_\lambda - \mathcal{H}, \quad \omega_\lambda = \partial_\lambda \rfloor \omega, \] (12)

where \( \Gamma \) is a connection on \( Y \to X \). Given the Hamiltonian form \( H \) (12), the equality (11) comes to the Hamilton equations

\[ \partial_\lambda y^i(x) = \partial_i^\lambda \mathcal{H}, \quad \partial_\lambda p^\lambda_i(x) = -\partial_i \mathcal{H} \] (13)
for sections of the fibred Legendre manifold $\Pi \to X$.

The Hamilton equations (13) are the multimomentum generalization of the standard Hamilton equations in mechanics. The energy-momentum transformation law (32) which we suggest is accordingly the multimomentum generalization of the conventional energy transformation law (1). Its application to the Hamiltonian gauge theory in the presence of a background world metric recovers the familiar metric energy-momentum transformation law (7) [10, 14].

The identity (32) remains true also in the gravitation theory. The tools are now at hand to examine the energy-momentum transformation for gravity. In this work, we restrict our consideration to the affine-metric gravitation theory. The key point consists in the feature of a metric gravitational field whose canonical momenta on the constraint space are equal to zero [13].

2 Technical preliminary

A fibred manifold

$$\pi : Y \to X$$

is provided with fibred coordinates $(x^\lambda, y^i)$ where $x^\lambda$ are coordinates of the base $X$. A locally trivial fibred manifold is termed the bundle. We denote by $VY$ and $V^*Y$ the vertical tangent bundle and the vertical cotangent bundle of $Y$ respectively. For the sake of simplicity, the pullbacks $Y \times_T X$ and $Y \times_T T^*X$ are denoted by $TX$ and $T^*X$ respectively.

On fibred manifolds, we consider the following types of differential forms:

(i) exterior horizontal forms $Y \to \wedge T^*X$,

(ii) tangent-valued horizontal forms $Y \to \wedge T^*X \otimes TY$ and, in particular, soldering forms $Y \to T^*X \otimes VY$,

(iii) pullback-valued forms

$$Y \to \wedge T^*Y \otimes TX, \quad Y \to \wedge T^*Y \otimes T^*X.$$

Horizontal $n$-forms are called horizontal densities.

Given a fibred manifold $Y \to X$, the first order jet manifold $J^1Y$ of $Y$ is both the fibred manifold $J^1Y \to X$ and the affine bundle $J^1Y \to Y$ modelled on the vector bundle $T^*X \otimes_Y VY$.

We identify $J^1Y$ to its image under the canonical bundle monomorphism

$$\lambda : J^1Y \to T^*X \otimes_T TY,$$

$$\lambda = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i). \quad (14)$$

5
Given a fibred morphism of $\Phi : Y \to Y'$ over a diffeomorphism of $X$, its jet prolongation $J^1\Phi : J^1Y \to J^1Y'$ reads

$$y'_\mu \circ J^1\Phi = (\partial_\lambda \Phi^i + \partial_j \Phi^i y^i_j) \frac{\partial x^\lambda}{\partial x'^\mu}.$$ 

Every vector field

$$u = u^\lambda \partial_\lambda + u^i \partial_i$$
on a fibred manifold $Y \to X$ gives rise to the projectable vector field

$$\bar{u} = u^\lambda \partial_\lambda + u^i \partial_i + (\partial_\lambda u^i + y^i_\lambda \partial_j u^i - y^i_\mu \partial_\lambda u^\mu) \partial^i_\lambda,$$

(15)
on the fibred jet manifold $J^1Y \to X$ where $J^1TY$ is the jet manifold of the fibred manifold $TY \to X$.

The canonical morphism (14) gives rise to the bundle monomorphism

$$\hat{\lambda} : J^1Y \times TX \ni \partial_\lambda \mapsto \hat{\partial}_\lambda = \partial_\lambda \lambda \in J^1Y \times TY,$$ 

$$\hat{\partial}_\lambda = \partial_\lambda + y^i_\lambda \partial_i.$$

This morphism determines the canonical horizontal splitting of the pullback

$$J^1Y \times TY = \hat{\lambda}(TY) \oplus VY,$$

(16)
on the fibred jet manifold $J^1Y \to X$ with respect to the connection $\Gamma$ on $Y$. In other words, over $J^1Y$, we have the canonical horizontal splitting of the tangent bundle $TY$.

Building on the canonical splitting (16), one gets the corresponding horizontal splittings of a projectable vector field

$$u = u^\lambda \partial_\lambda + u^i \partial_i = u_H + u_V = u^\lambda (\partial_\lambda + y^i_\lambda \partial_i) + (u^i - u^\lambda y^i_\lambda) \partial_i$$

(17)
on a fibred manifold $Y \to X$.

Given a fibred manifold $Y \to X$, there is the 1:1 correspondence between the connections on $Y \to X$ and global sections

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i)$$
on the affine jet bundle $J^1Y \to Y$. Substitution of such a global section $\Gamma$ into the canonical horizontal splitting (16) recovers the familiar horizontal splitting of the tangent bundle $TY$ with respect to the connection $\Gamma$ on $Y$. These global sections form the affine space modelled on the linear space of soldering forms on $Y$.

Every connection $\Gamma$ on $Y \to X$ yields the first order differential operator

$$D_\Gamma : J^1Y \to T^*X \otimes VY,$$ 

$$D_\Gamma = (y^i_\lambda - \Gamma^i_\lambda) dx^\lambda \otimes \partial_i,$$
on $Y$ which is called the covariant differential relative to the connection $\Gamma$.

The repeated jet manifold $J^1J^1Y$, by definition, is the first order jet manifold of $J^1Y \to X$. It is provided with the adapted coordinates $(x^\lambda, y^i, y^i_\lambda, y^i_{\mu\lambda}, y^i_{\mu\lambda})$. Its subbundle $\tilde{J}^2Y$ with $y^i_{(\lambda)} = y^i_\lambda$ is called the sesquiholonomic jet manifold. The second order jet manifold $J^2Y$ of $Y$ is the subbundle of $\tilde{J}^2Y$ with $y^i_{\lambda\mu} = y^i_{\mu\lambda}$.

3 Lagrangian formalism

Let $Y \to X$ be a fibred manifold and $L = \mathcal{L}\omega$ a Lagrangian density on $J^1Y$. With $L$, the jet manifold $J^1Y$ carries the unique associated Poincaré-Cartan form

$$\Xi_L = \pi^i_\lambda dy^i \wedge \omega_\lambda - \pi^i_\lambda y^i_\lambda \omega + \mathcal{L}\omega$$

and the Lagrangian multisymplectic form

$$\Omega_L = (\partial_j \pi^i_\lambda dy^i + \partial_j^\mu \pi^i_\lambda dy^i_{\mu}) \wedge dy^i \wedge \omega \otimes \partial_\lambda.$$ 

Using the pullback of these forms onto the repeated jet manifold $J^1J^1Y$, one can construct the exterior form

$$\Lambda_L = d\Xi_L - \lambda|\Omega_L = [y^i_{(\lambda)} - y^i_\lambda) d\pi^i_\lambda + (\partial_i - \tilde{\partial}_\lambda \partial^\lambda_1)\mathcal{L}dy^i] \wedge \omega,$$

$$\lambda = dx^\lambda \otimes \tilde{\partial}_\lambda, \quad \tilde{\partial}_\lambda = \partial_\lambda + y^i_\lambda \partial_i + y^i_{\mu\lambda} \partial^\mu_i,$$

on $J^1J^1Y$. Its restriction to the second order jet manifold $J^2Y$ of $Y$ reproduces the familiar variational Euler-Lagrange operator

$$\mathcal{E}_L = [\partial_i - (\partial_\lambda + y^i_\lambda \partial_i + y^i_{\mu\lambda} \partial^\mu_i) \partial^\lambda_1]\mathcal{L}dy^i \wedge \omega.$$ 

The restriction of the form (19) to the sesquiholonomic jet manifold $\tilde{J}^2Y$ defines the sesquiholonomic extension $\mathcal{E}'_L$ of the Euler-Lagrange operator (20). It is given by the expression (21), but with nonsymmetric coordinates $y^i_{\mu\lambda}$.

Let $\bar{s}$ be a section of the fibred jet manifold $J^1Y \to X$ such that its first order jet prolongation $J^1\bar{s}$ takes its values into $\text{Ker} \mathcal{E}'_L$. Then, $\bar{s}$ satisfies the first order differential Euler-Lagrange equations (3). They are equivalent to the second order Euler-Lagrange equations

$$\partial_i \mathcal{L} - (\partial_\lambda + \partial_\lambda s^j \partial_j + \partial_\lambda \partial_{\mu} s^j \partial^\mu_j) \partial^\lambda_1 \mathcal{L} = 0.$$ 

for sections $s$ of $Y$ where $\bar{s} = J^1s$.

We have the following conservation laws on solutions of the first order Euler-Lagrange equations.

Let

$$u = u^\mu \partial_\mu + u^i \partial_i$$
be a vector field on a fibred manifold $Y$ and $\pi$ its jet lift \((15)\) onto the fibred jet manifold $J^1Y \to X$. Given a Lagrangian density $L$ on $J^1Y$, let us compute the Lie derivative $L_{\pi}L$. We have

\[
L_{\pi}L = \left[ \hat{\partial}_\lambda (\pi_i^\lambda (u^i - u^\mu y_i^\mu) + u^\lambda L) + (u^i - u^\mu y_i^\mu)(\partial_i - \hat{\partial}_\lambda \partial_i^\lambda) L \right] \omega, \tag{22}
\]

\[
\hat{\partial}_\lambda = \partial_\lambda + y_i^\lambda \partial_i + y_{\mu \lambda}^i \partial_i^\mu.
\]

On solutions $\gamma$ of the first order Euler-Lagrange equations, the equality \((22)\) comes to the conservation law \((3)\).

In particular, let

\[
u = \tau = \tau^\mu (\partial_\mu + \Gamma_i^\mu \partial_i)
\]

be the horizontal lift of a vector field

\[
\tau = \tau^\lambda \partial_\lambda
\]

on $X$ onto the fibred manifold $Y$ by a connection $\Gamma$ on $Y$. In this case, the equality \((3)\) takes the form \((23)\) where $T_{\Gamma}^\lambda \mu (\gamma)$ \((\gamma)\) are coefficients of the $T^*X$-valued form

\[
T_{\Gamma}(\gamma) = -(\Gamma \circ \Xi_L) \circ \gamma = [\pi_i^\lambda (\gamma_i^\mu - \Gamma_i^\mu) - \delta_i^\lambda \gamma_e^i] dx^\mu \otimes \omega^\lambda \tag{23}
\]

on $X$. One can think on this form as being the canonical energy-momentum tensor of a field $\gamma$ with respect to the connection $\Gamma$ on $Y$.

\section{Multimomentum Hamiltonian formalism}

Let $\Pi$ be the Legendre bundle \((9)\) over a fibred manifold $Y \to X$. It is provided with the fibred coordinates $(x^\lambda, y^i, p_i^\lambda)$:

\[
p_i^{\lambda} = \frac{\partial y^i}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x^\lambda} p_j^\mu, \quad J^{-1} = \det(\frac{\partial x^{\mu}}{\partial x^\lambda}).
\]

By $J^1\Pi$ is meant the first order jet manifold of $\Pi \to X$. It is coordinatized by

\[
(x^\lambda, y^i, p_i^\lambda, \gamma_{(i)}, p_{(i)}^\lambda).
\]

We call by a momentum morphism any bundle morphism $\Phi : \Pi \to J^1Y$ over $Y$. For instance, let $\Gamma$ be a connection on $Y$. Then, the composition $\hat{\Gamma} = \Gamma \circ \pi_{HY}$ is a momentum morphism. Conversely, every momentum morphism $\Phi$ determines the associated connection $\Gamma_\Phi = \Phi \circ \partial_{HY}$ on $Y \to X$ where $\partial_{HY}$ is the global zero section of $\Pi \to Y$. Every connection $\Gamma$ on $Y$ gives rise to the connection

\[
\hat{\Gamma} = dx^\lambda \otimes [\partial_\lambda + \Gamma_i^\lambda (y) \partial_i + (-\partial_j \Gamma_j^\lambda (y)p_i^\mu - K^\mu_{\nu \lambda} (x)p_j^\nu + K^\nu_{\alpha \lambda} (x)p_j^\mu) \partial_i^\lambda]
\]

on $\Pi \to X$ where $K$ is a linear symmetric connection on $T^*X$. 8
The Legendre manifold $\Pi$ carries the multimomentum Liouville form

$$\theta = - p^i_\lambda dy^i \wedge \omega \otimes \partial_\lambda$$

and the multisymplectic form $\Omega$ [10].

The Hamiltonian formalism in fibred manifolds is formulated intrinsically in terms of Hamiltonian connections which play the role similar to that of Hamiltonian vector fields in the symplectic geometry.

We say that a connection $\gamma$ on the fibre Legendre manifold $\Pi \to X$ is a Hamiltonian connection if the exterior form $\gamma \rfloor \Omega$ is closed. An exterior $n$-form $H$ on the Legendre manifold $\Pi$ is called a Hamiltonian form if there exists a Hamiltonian connection satisfying the equation (11).

Let $H$ be a Hamiltonian form. For any exterior horizontal density $\tilde{H} = \tilde{\hat{H}} \omega$ on $\Pi \to X$, the form $H - \tilde{H}$ is a Hamiltonian form. Conversely, if $H$ and $H'$ are Hamiltonian forms, their difference $H - H'$ is an exterior horizontal density on $\Pi \to X$. Thus, Hamiltonian forms constitute an affine space modelled on a linear space of the exterior horizontal densities on $\Pi \to X$.

Let $\Gamma$ be a connection on $Y \to X$ and $\tilde{\Gamma}$ its lift (24) onto $\Pi \to X$. We have the equality

$$\tilde{\Gamma} \rfloor \Omega = d(\hat{\Gamma} \rfloor \theta).$$

A glance at this equality shows that $\tilde{\Gamma}$ is a Hamiltonian connection and

$$H_\Gamma = \tilde{\Gamma} \rfloor \theta = p^i_\lambda dy^i \wedge \omega_\lambda - p^i_\lambda \Gamma^i_\lambda \omega$$

is a Hamiltonian form. It follows that every Hamiltonian form on $\Pi$ can be given by the expression (12), where $\Gamma$ is some connection on $Y \to X$. Moreover, a Hamiltonian form has the canonical splitting (12) as follows. Given a Hamiltonian form $H$, the vertical tangent morphism $V H$ yields the momentum morphism

$$\tilde{H} : \Pi \to J^1 Y, \quad y^i_\lambda \circ \tilde{H} = \partial^i_\lambda \mathcal{H},$$

and the associated connection $\Gamma_H = \tilde{H} \circ \hat{0}$ on $Y$. As a consequence, we have the canonical splitting

$$H = H_{\Gamma_H} - \tilde{H}.$$

The Hamilton operator $\mathcal{E}_H$ for a Hamiltonian form $H$ is defined to be the first order differential operator

$$\mathcal{E}_H = dH - \tilde{\Omega} = [[y^i_\lambda - \partial^i_\lambda \mathcal{H})dp^\lambda_i - (p^i_\lambda \lambda + \partial_\lambda \mathcal{H})dy^i] \wedge \omega,$$

(25)

where $\tilde{\Omega}$ is the pullback of the multisymplectic form $\Omega$ onto $J^1 \Pi$.

For any connection $\gamma$ on $\Pi \to X$, we have

$$\mathcal{E}_H \circ \gamma = dH - \gamma \rfloor \Omega.$$
It follows that $\gamma$ is a Hamiltonian jet field for a Hamiltonian form $H$ if and only if it takes its values into $\text{Ker} \mathcal{E}_H$, that is, satisfies the algebraic Hamilton equations

$$\begin{align*}
\gamma^\lambda_i & = \partial^\lambda_i H, \\
\gamma^\lambda_{i\lambda} & = -\partial_i H.
\end{align*}$$  (26)

Let a Hamiltonian connection $\gamma$ has an integral section $r$ of $\Pi \to X$, that is, $\gamma \circ r = J^1 r$. Then, the Hamilton equations (26) are brought into the first order differential Hamilton equations (13).

Now we consider relations between Lagrangian and Hamiltonian formalisms on fibred manifolds in case of semiregular Lagrangian densities $L$ when the preimage $\hat{L}^{-1}(q)$ of each point of $q \in Q$ is the connected submanifold of $J^1 Y$.

Given a Lagrangian density $L$, the vertical tangent morphism $VL$ of $L$ yields the Legendre morphism

$$\hat{L} : J^1 Y \to \Pi, \quad p^\lambda_i \circ \hat{L} \circ \pi^\lambda_i.$$  

We say that a Hamiltonian form $H$ is associated with a Lagrangian density $L$ if $H$ satisfies the relations

$$\begin{align*}
\hat{L} \circ \hat{H} |_{Q} & = \text{Id}, \quad Q = \hat{L} (J^1 Y), \\
H & = H_{\hat{H}} + L \circ \hat{H},
\end{align*}$$  (27a)

or in the coordinate form

$$\begin{align*}
\partial^\mu_i L(x^\lambda, y^j, \partial^\lambda_j) & = p^\mu_i, \quad p^\mu_i \in Q, \\
L(x^\lambda, y^j, \partial^\lambda_j) & = p^\mu_i \partial^\mu_i H - H.
\end{align*}$$  (27b)

Note that different Hamiltonian forms can be associated with the same Lagrangian density.

Let a section $r$ of $\Pi \to X$ be a solution of the Hamilton equations (13) for a Hamiltonian form $H$ associated with a semiregular Lagrangian density $L$. If $r$ lives on the constraint space $Q$, the section $\bar{s} = \hat{H} \circ r$ of $J^1 Y \to X$ satisfies the first order Euler-Lagrange equations (2). Conversely, given a semiregular Lagrangian density $L$, let $\bar{s}$ be a solution of the first order Euler-Lagrange equations (2). Let $H$ be a Hamiltonian form associated with $L$ so that

$$\hat{H} \circ \hat{L} \circ \bar{s} = \bar{s}.$$  (28)

Then, the section $r = \hat{L} \circ \bar{s}$ of $\Pi \to X$ is a solution of the Hamilton equations (13) for $H$. For sections $\bar{s}$ and $r$, we have the relations

$$\bar{s} = J^1 s, \quad s = \pi_{HY} \circ r$$

where $s$ is a solution of the second order Euler-Lagrange equations (21).

We shall say that a family of Hamiltonian forms $H$ associated with a semiregular Lagrangian density $L$ is complete if, for each solution $\bar{s}$ of the first order Euler-Lagrange
equations (2), there exists a solution $r$ of the Hamilton equations (13) for some Hamiltonian form $H$ from this family so that

$$r = \tilde{L} \circ \tilde{s}, \quad \tilde{s} = \tilde{H} \circ r, \quad \tilde{s} = J^1(\pi_{HY} \circ r).$$

(29)

Such a complete family exists iff, for each solution $\tilde{s}$ of the Euler-Lagrange equations for $L$, there exists a Hamiltonian form $H$ from this family so that the condition (28) holds.

The most of field models possesses affine and quadratic Lagrangian densities. Complete families of Hamiltonian forms associated with such Lagrangian densities always exist [10, 11].

5 Energy-momentum conservation laws

In the framework of the multimomentum Hamiltonian formalism, we get the fundamental identity whose restriction to the Lagrangian constraint space recovers the familiar energy-momentum conservation law [10, 14].

Let $H$ be a Hamiltonian form on the Legendre bundle $\Pi$ over a fibred manifold $Y \to X$. Let $r$ be a section of of the fibred Legendre manifold $\Pi \to X$ and $(y^i(x), p_i^\alpha(x))$ its local components. Given a connection $\Gamma$ on $Y \to X$, we consider the following $T^*X$-valued $(n-1)$-form on $X$:

$$T_\Gamma(r) = - (\Gamma \cdot H) \circ r,$$

$$T_\Gamma(r) = [p_i^\lambda (y_\mu^i - \Gamma^i_\mu) - \delta_\mu^\lambda (p_\alpha^i (y_\alpha^i - \Gamma^i_\alpha) - \tilde{H}_\Gamma)] dx^\mu \otimes \omega_\lambda,$$

(30)

where $\tilde{H}_\Gamma$ is the Hamiltonian density in the splitting (12) of $H$ with respect to the connection $\Gamma$.

Let

$$\tau = \tau^\lambda \partial_\lambda$$

be a vector field on $X$. Given a connection $\Gamma$ on $Y \to X$, it gives rise to the vector field

$$\tilde{\tau}_\Gamma = \tau^\lambda \partial_\lambda + \tau^\lambda \Gamma^i_\lambda \partial_i + (-\tau^\mu p_i^\lambda \partial_i \Gamma^i_\mu - p_i^\lambda \partial_\mu \tau^\mu + p_i^\mu \partial_\mu \tau^\lambda) \partial_\lambda$$

on the Legendre bundle $\Pi$. Let us calculate the Lie derivative $L_{\tilde{\tau}_\Gamma} \tilde{H}_\Gamma$ on a section $r$. We have

$$(L_{\tilde{\tau}_\Gamma} \tilde{H}_\Gamma) \circ r = p_i^\lambda R^i_{\lambda \mu} \partial_\mu + d[\tau^\mu T^\lambda_\Gamma_{\mu} (r) \omega_\lambda] - (\tilde{\tau}_\Gamma \cdot \tilde{\tau}_\Gamma) \circ r$$

(31)

where

$$R = \frac{1}{2} R_{\lambda \mu}^i d\lambda \wedge dx^\mu \otimes \partial_i =$$

$$\frac{1}{2} (\partial_\lambda \Gamma^i_\mu - \partial_\mu \Gamma^i_\lambda + \Gamma^j_\lambda \partial_j \Gamma^i_\mu - \Gamma^j_\mu \partial_j \Gamma^i_\lambda) dx^\lambda \wedge dx^\mu \otimes \partial_i.$$
is the curvature of the connection $\Gamma$, $\mathcal{E}_H$ is the Hamilton operator (25) and

$$\tilde{\tau}_{\Gamma V} = \tau^\lambda (\Gamma^i_\lambda - y^i_\lambda) \partial_i + (-\tau^\mu p^\lambda_j \partial_i \Gamma^j_\mu - p^\lambda_i \partial_\mu \tau^\mu + p^\mu_i \partial_\mu \tau^\lambda - \tau^\mu p^\lambda_\mu) \partial^i_\lambda$$

is the vertical part of the canonical horizontal splitting (17) of the vector field $\tilde{\tau}_\Gamma$ on $\Pi$ over $J^1\Pi$. If $r$ is a solution of the Hamilton equations, the equality (31) comes to the identity

$$(\partial_\mu + \Gamma^i_\mu \partial_i - \partial_i \Gamma^j_\mu p^\lambda_j \partial^i_\lambda) \tilde{H}_\Gamma = \frac{d}{dx^\lambda} T^\lambda_\mu (r) + p^i_\lambda R^i_\mu.$$  (32)

On solutions of the Hamilton equations, the form (30) reads

$$T^\Gamma_\mu (r) = [p^\lambda_i \partial^i_\mu \tilde{H}_\Gamma - \delta^\lambda_\mu (p^\alpha_i \partial^i_\alpha \tilde{H}_\Gamma - \tilde{H}_\Gamma)] dx^\mu \otimes \omega_\lambda.$$  (33)

One can verify that the identity (32) does not depend upon choice of the connection $\Gamma$.

For instance, if $X = \mathbb{R}$ and $\Gamma$ is the trivial connection, then $T^\Gamma_\mu (r) = \mathcal{H} dt$ where $\mathcal{H}$ is a Hamiltonian function and the identity (32) consists with the familiar energy transformation law (1).

To clarify the physical meaning of (32) when $n > 1$, we turn to the Lagrangian formalism. Let a multimomentum Hamiltonian form $H$ be associated with a semiregular Lagrangian density $L$. Let $r$ be a solution of the Hamilton equations for $H$ which lives on the Lagrangian constraint space $Q$ and $\mathbf{\pi}$ the associated solution of the first order Euler-Lagrange equations for $L$ so that $r$ and $\mathbf{\pi}$ satisfy the conditions (29). Then, we have

$$T^\Gamma_\mu (r) = T^\Gamma_\mu (\mathbf{\pi})$$

where $T^\Gamma_\mu (\mathbf{\pi})$ is the Lagrangian canonical energy-momentum tensor (23). It follows that the form (33) may be treated as a Hamiltonian canonical energy-momentum tensor with respect to a background connection $\Gamma$ on the fibred manifold $Y \to X$ (or a Hamiltonian stress energy-momentum tensor). At the same time, the identity (32) in gauge theory turns out to be precisely the covariant energy-momentum conservation law for the metric energy-momentum tensor, not the canonical one (14).

In the Lagrangian formalism, the metric energy-momentum tensor is defined to be

$$\sqrt{-g} t_{\alpha\beta} = 2 \frac{\partial L}{\partial g^{\alpha\beta}}.$$  (34)

In case of a background world metric $g$, this object is well-behaved. In the framework of the multimomentum Hamiltonian formalism, one can introduce the similar tensor

$$\sqrt{-g} t^\alpha_\beta = 2 \frac{\partial \mathcal{H}}{\partial g_{\alpha\beta}}.$$  (34)
Recall the useful relation
\[
\frac{\partial}{\partial g^{\alpha \beta}} = -g^{\alpha \mu} g^{\beta \nu} \frac{\partial}{g_{\mu \nu}}.
\]

If a multimomentum Hamiltonian form \( H \) is associated with a semiregular Lagrangian density \( L \), we have the equalities
\[
t_H^{\alpha \beta}(q) = -g^{\alpha \mu} g^{\beta \nu} t_{\mu \nu}(x^\lambda, y^i, \partial_i \mathcal{H}(q)),
\]
\[
t_H^{\alpha \beta}(x^\lambda, y^i, \pi^\lambda_i(z)) = -g^{\alpha \mu} g^{\beta \nu} t_{\mu \nu}(z)
\]
where \( q \in Q \), \( z \in J^1 Y \) and
\[
\widehat{H} \circ \widehat{L}(z) = z.
\]
In view of these equalities, we can think of the tensor (34) restricted to the Lagrangian constraint space \( Q \) as being the Hamiltonian metric energy-momentum tensor. On \( Q \), the tensor (34) does not depend upon choice of a Hamiltonian form \( H \) associated with \( L \). Therefore, we shall denote it by the common symbol \( t \). Set
\[
t^\lambda_\alpha = g_{\alpha \nu} t^{\lambda \nu}.
\]

In the presence of a background world metric \( g \), the identity (32) takes the form
\[
t^\lambda_\alpha \{^{\alpha}_{\lambda \mu}\} \sqrt{-g} + (\Gamma^i_\mu \partial_i - \partial_i \Gamma^j_\mu \partial^j_\lambda) \mathcal{H}_\lambda = \frac{d}{dx^\lambda} T^\lambda_\mu + p^i_\lambda R^i_\lambda
\]
where
\[
\frac{d}{dx^\lambda} = \partial_\lambda + \partial_\lambda y^j \partial_i + \partial_\lambda p^j_\mu \partial^j_\mu
\]
and by \( \{^{\alpha}_{\lambda \mu}\} \) are meant the Cristoffel symbols of the world metric \( g \).

When applied to the Hamiltonian gauge theory in the presence of a background world metric, the identity (33) recovers the familiar metric energy-momentum transformation law (10) \[14\].

6 Energy-momentum conservation laws in affine-metric gravitation theory

The contemporary concept of gravitation interaction is based on the gauge gravitation theory with two types of gravitational fields. These are tetrad gravitational fields and Lorentz gauge potentials. At present, all Lagrangian densities of classical and quantum gravitation theories are expressed in these variables. They are of the first order with respect to these fields. Only General Relativity without fermion matter sources utilizes traditionally the Hilbert-Einstein Lagrangian density \( L_{HE} \) which is of the second order with respect to a pseudo-Riemannian metric. One can reduce its order by means of the Palatini variables when the Levi-Civita connection is regarded on the same footing as a pseudo-Riemannian metric.
Here we consider the affine-metric gravitation theory when there is no fermion matter and gravitational variables are both a pseudo-Riemannian metric $g$ on a world manifold $X^4$ and linear connections $K$ on the tangent bundle of $X^4$. We call them a world metric and a world connection respectively. Given a world metric, every world connection meets the well-known decomposition in the Cristoffel symbols, contorsion and the nonmetricity term. We here are not concerned with the matter interacting with a general linear connection, for it is non-Lagrangian and hypothetical as a rule.

In the rest of the article, $X$ is an oriented 4-dimensional world manifold which obeys the well-known topological conditions in order that a gravitational field exists on $X^4$.

Let $LX \rightarrow X^4$ be the principal bundle of linear frames in the tangent spaces to $X^4$. The structure group of $LX$ is the group

$$GL_4 = GL^+(4, \mathbb{R})$$

of general linear transformations of the standard fibre $\mathbb{R}^4$ of the tangent bundle $TX$. The world connections are associated with the principal connections on the principal bundle $LX \rightarrow X^4$. Hence, there is the 1:1 correspondence between the world connections and the global sections of the principal connection bundle

$$C = J^1LX/GL_4.$$  \hspace{1cm} (36)

Therefore, we can apply the standard procedure of the Hamiltonian gauge theory in order to describe the configuration and phase spaces of world connections \cite{11, 12, 13}.

There is the 1:1 correspondence between the world metrics $g$ on $X^4$ and the global sections of the bundle $\Sigma_g$ of pseudo-Riemannian bilinear forms in tangent spaces to $X^4$. This bundle is associated with the $GL_4$-principal bundle $LX$. The 2-fold covering of the bundle $\Sigma_g$ is the quotient bundle

$$\Sigma = LX/SO(3, 1)$$  \hspace{1cm} (37)

where by $SO(3, 1)$ is meant the connected Lorentz group.

Thus, the total configuration space of the affine-metric gravitational variables is represented by the product of the corresponding jet manifolds:

$$J^1C \times J^4\Sigma.$$  \hspace{1cm} (38)

Given a holonomic bundle atlas of $LX$ associated with induced coordinates of $TX$ and $T^*X$, this configuration space is provided with the adapted coordinates

$$(x^\mu, g^{\alpha\beta}, k^{\alpha}_{\beta\mu}, g^{\alpha\beta}_{\lambda}, k^{\alpha}_{\beta\mu\lambda}).$$

Also the total phase space $\Pi$ of the affine-metric gravity is the product of the Legendre bundles over the above-mentioned bundles $C$ and $\Sigma$. It is coordinatized by the corresponding canonical coordinates

$$(x^\mu, g^{\alpha\beta}, k^{\alpha}_{\beta\mu}, p^{\alpha\beta}_{\lambda}, p^{\alpha}_{\beta\mu\lambda}).$$
On the configuration space (38), the Hilbert-Einstein Lagrangian density of General Relativity reads

$$L_{HE} = -\frac{1}{2\kappa}g^{\beta\lambda}\mathcal{F}^\alpha_{\beta\alpha\lambda}\sqrt{-g}\omega,$$

(39)

$$\mathcal{F}^\alpha_{\beta\nu\lambda} = k^\alpha_{\beta\lambda\nu} - k^\alpha_{\beta\nu\lambda} + k^\alpha_{\varepsilon\nu}k^\varepsilon_{\beta\lambda} - k^\alpha_{\varepsilon\lambda}k^\varepsilon_{\beta\nu}.$$ 

The corresponding Legendre morphism is given by the expressions

$$p^\alpha_{\beta\lambda} \circ \hat{L}_{HE} = 0,$$

$$p^\alpha_{\beta\nu\lambda} \circ \hat{L}_{HE} = \pi^\alpha_{\beta\nu\lambda} = \frac{1}{2\kappa}(\delta^\nu_{\alpha}g^\beta\lambda - \delta^\lambda_{\alpha}g^\beta\nu)\sqrt{-g}.$$ 

(40)

These relations define the constraint space of General Relativity in multimomentum canonical variables.

Building on the set of connections on the bundle $C \times \Sigma$, one can construct the complete family of multimomentum Hamiltonian forms associated with the Lagrangian density (39). To minimize it, we consider the following subset of these connections.

Let $K$ be a world connection and

$$S_{K}^\alpha_{\beta\nu\lambda} = \frac{1}{2}[k^\alpha_{\varepsilon\nu}k^\varepsilon_{\beta\lambda} - k^\alpha_{\varepsilon\lambda}k^\varepsilon_{\beta\nu} + \partial_\lambda K^\alpha_{\beta\nu} + \partial_\nu K^\alpha_{\beta\lambda}$$

$$-2K^\varepsilon_{(\nu\lambda)}(K^\alpha_{\beta\varepsilon} - k^\alpha_{\beta\varepsilon}) + K^\varepsilon_{\beta\lambda}k^\alpha_{\varepsilon\nu} + K^\varepsilon_{\beta\nu}k^\alpha_{\varepsilon\lambda}$$

$$-K^\alpha_{\varepsilon\lambda}k^\varepsilon_{\beta\nu} - K^\alpha_{\varepsilon\nu}k^\varepsilon_{\beta\lambda}]$$

the corresponding connection on the bundle $C$ (33). Let $K'$ be another symmetric world connection which induces an associated connection on the bundle $\Sigma$. On the bundle $C \times \Sigma$, we consider the following connection

$$\Gamma^\alpha_{\beta\lambda} = -K^\alpha_{\varepsilon\lambda}g^{\varepsilon\beta} - K^\varepsilon_{\beta\lambda}g^{\alpha\varepsilon},$$

$$\Gamma^\alpha_{\beta\nu\lambda} = S_{K}^\alpha_{\beta\nu\lambda} - \frac{1}{2}R^\alpha_{\beta\nu\lambda}$$

(41)

where

$$R^\alpha_{\beta\nu\lambda} = K^\alpha_{\beta\nu\lambda} - K^\alpha_{\beta\lambda\nu} + K^\alpha_{\varepsilon\lambda}K^\varepsilon_{\beta\nu} - K^\alpha_{\varepsilon\nu}K^\varepsilon_{\beta\lambda}$$

is the curvature of the connection $K$. The corresponding multimomentum Hamiltonian form is given by the expression

$$H_{HE} = (p^\beta_{\alpha\lambda}dg^{\alpha\beta} + p^\alpha_{\beta\nu\lambda}dk^\alpha_{\beta\nu}) \wedge \omega_\lambda - \mathcal{H}_{HE}\omega,$$

$$\mathcal{H}_{HE} = -p^\beta_{\alpha\lambda}(K^\alpha_{\varepsilon\lambda}g^{\varepsilon\beta} + K^\varepsilon_{\beta\lambda}g^{\alpha\varepsilon}) + p^\beta_{\alpha\nu\lambda}S^\alpha_{\beta\nu\lambda}$$

$$-\frac{1}{2}R^\alpha_{\beta\nu\lambda}(p^\beta_{\alpha\nu\lambda} - \pi^\alpha_{\beta\nu\lambda}).$$

(42)

It is associated with the Lagrangian density $L_{HE}$. We shall justify that the multimomentum Hamiltonian forms (42) parameterized by all the world connections $K$ and $K'$ constitute the complete family.
Given the multimomentum Hamiltonian form $H_{HE}$ (42) plus that $H_M$ of matter, the corresponding covariant Hamilton equations for General Relativity read

$$\partial_\lambda g^{\alpha\beta} + K^\epsilon_{\alpha\lambda} g^{\epsilon\beta} + K^\epsilon_{\epsilon\lambda} g^{\alpha\epsilon} = 0, \quad (43a)$$

$$\partial_\lambda k^\alpha_{\beta\nu} = S^\alpha_{\beta\nu\lambda} - \frac{1}{2} R^\alpha_{\beta\nu\lambda}, \quad (43b)$$

$$\partial_\lambda p_{\alpha\beta} = p_{\epsilon\beta} K^\epsilon_{\alpha\sigma} + p_{\epsilon\alpha} K^\epsilon_{\beta\sigma} - \frac{1}{2} \kappa \left( R^\alpha_{\beta\nu\lambda} - \frac{1}{2} g^\alpha_{\beta\nu\lambda} R \right) \sqrt{-g} - \partial H_M \frac{\partial g^{\alpha\beta}}{\partial g^{\alpha\beta}}, \quad (43c)$$

$$\partial_\lambda p_{\beta\nu\lambda} = -p_{\alpha}^{\epsilon[\nu\gamma]} k^\epsilon_{\beta\gamma} + p_{\epsilon}^{\beta[\nu\gamma]} k^\epsilon_{\alpha\gamma} - p_{\alpha}^{\beta\gamma\nu} K^\nu_{(\gamma)} - p_{\alpha}^{\epsilon(\nu\gamma)} K^\nu_{\gamma} + p_{\epsilon}^{\beta(\nu\gamma)} K^\epsilon_{\alpha\gamma}. \quad (43d)$$

The Hamilton equations (43a) and (43b) are independent of canonical momenta and so, reduce to the gauge-type condition. The gauge-type condition (43b) breaks into two parts

$$F^{\alpha\beta\lambda\nu} = R^{\alpha\beta\lambda\nu}, \quad (44)$$

$$\partial_\nu (K^\alpha_{\beta\lambda} - k^\alpha_{\beta\lambda}) + \partial_\lambda (K^\alpha_{\beta\nu} - k^\alpha_{\beta\nu}) - 2 K^\epsilon_{(\nu\lambda)} (K^\alpha_{\beta\epsilon} - k^\alpha_{\beta\epsilon}) + K^\epsilon_{\beta\lambda} k^\epsilon_{\alpha\nu} + K^\epsilon_{\beta\nu} k^\epsilon_{\alpha\lambda} - K^\epsilon_{\epsilon\lambda} k^\epsilon_{\beta\nu} - K^\epsilon_{\epsilon\nu} k^\epsilon_{\beta\lambda} = 0. \quad (45)$$

It is readily observed that, for a given world metric $g$ and a world connection $k$, there always exist the world connections $K'$ and $K$ such that the gauge-type conditions (43a), (44) and (45) hold (e.g. $K'$ is the Levi-Civita connection of $g$ and $K = k$). It follows that the multimomentum Hamiltonian forms (42) constiute really the complete family.

Being restricted to the constraint space (40), the Hamilton equations (43c) and (43d) comes to

$$\frac{1}{\kappa} (R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R) \sqrt{-g} = -\frac{\partial H_M}{\partial g^{\alpha\beta}}, \quad (46)$$

$$D_\alpha (\sqrt{-g} g^{\mu\beta}) - \delta_\alpha^\mu D_\lambda (\sqrt{-g} g^{\lambda\beta}) + \sqrt{-g} g^{\nu\beta} (k^\nu_{\alpha\lambda} - k^\lambda_{\nu\alpha}) + g^{\lambda\beta} (k^\nu_{\lambda\alpha} - k^\nu_{\rho\alpha}) + \delta_\alpha^\nu g^{\lambda\beta} (k^\mu_{\mu\lambda} - k^\mu_{\mu\lambda}) = 0 \quad (47)$$

where

$$D_\lambda g^{\alpha\beta} = \partial_\lambda g^{\alpha\beta} + k^\alpha_{\mu\lambda} g^{\beta\mu} + k^\beta_{\mu\lambda} g^{\alpha\mu}. \quad (48)$$

Substituting Eq.(44) into Eq.(46), we obtain the Einstein equations

$$\frac{1}{\kappa} F_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} F = -t_{\alpha\beta} \quad (47)$$

where $t_{\alpha\beta}$ is the metric energy-momentum tensor of matter. It is easy to see that Eqs.(47) and (48) are the familiar equations of gravitation theory phrased in terms of the generalized Palatini variables. In particular, the former is the equation for the torsion and the
nonmetricity term of the connection $k^{\alpha \beta \nu}$. In the absence of matter sources of a general linear connection, it admits the well-known solution

$$k^{\alpha \beta \nu} = \{^\alpha _{\beta \nu}\} - \frac{1}{2} \delta^\alpha _\nu V_\beta,$$

$$D_\alpha g^{\beta \gamma} = V_\alpha g^{\beta \gamma}$$

where $V_\alpha$ is an arbitrary covector field corresponding to the well-known projective freedom.

Turn now to the identity (32). It takes the form

$$\left( \partial_\mu + \Gamma^{\alpha \beta}_\mu \partial_\alpha \beta - \partial_\mu \Gamma^i_\mu p^j_\lambda \partial_\lambda \right)(\tilde{\mathcal{H}}_{HE} + \tilde{\mathcal{H}}_M) =$$

$$\frac{d}{dx^\lambda} (T^{\lambda \mu}_\mu + T^M_\lambda^\mu) + p_\alpha^\beta \nu \lambda^{\alpha \beta \nu \lambda} + p_\lambda^\lambda R^\lambda_i \lambda \mu,$$  \hspace{1cm} (49)

where $T$ and $T_M$ are the canonical energy-momentum tensors of affine-metric gravity and matter respectively.

The energy-momentum tensor (33) of affine-metric gravity on solutions of the Hamilton equations reads

$$T^{\lambda \mu}_\mu = \delta^{\lambda}_\mu \tilde{\mathcal{H}}_{HE} = \frac{1}{2} \delta^{\lambda}_\mu \pi^{\alpha \beta \nu}_\alpha R^\alpha_{\beta \nu \lambda} = \frac{1}{2} \delta^{\lambda}_\mu \pi P \sqrt{-g}.$$ \hspace{1cm} (50)

It follows that

$$\left( \partial_\mu + \Gamma^{\alpha \beta}_\mu \partial_\alpha \beta \right) \tilde{\mathcal{H}}_{HE} = \frac{d}{dx^\lambda} T^{\lambda \mu}_\mu.$$ \hspace{1cm} (51)

One can verify also that, on solutions of the Hamilton equations, the curvature of the connection (41) vanishes. Then, the identity (49) takes the form (33). In gauge theory, it is reduced to the familiar conservation law (7). This is however a very particular case because the Hilbert-Einstein Hamiltonian density $\tilde{\mathcal{H}}_{HE}$ is independent of gravitational momenta.

Since the canonical momenta $p_{\alpha \beta}^\lambda$ of the world metric are equal to zero and the Hamilton equation (43c) comes to

$$\partial_\alpha \beta (\tilde{\mathcal{H}}_{HE} + \tilde{\mathcal{H}}_M) = 0,$$

the equality (49) can be rewritten

$$\pi^{\beta \nu \lambda \mu}_\alpha \partial_\mu R^\alpha_{\beta \nu \lambda} + (\Gamma^i_\mu \partial_\lambda - \partial_\mu \Gamma^i_\mu p^j_\lambda \partial_\lambda) \tilde{\mathcal{H}}_M = \frac{d}{dx^\lambda} (T^{\lambda \mu}_\mu + T^M_\lambda^\mu) + p_\lambda^\lambda R^\lambda_i \lambda \mu.$$ \hspace{1cm} (51)

In gauge theory, it takes the form

$$\pi^{\beta \nu \lambda \mu}_\alpha \partial_\mu R^\alpha_{\beta \nu \lambda} - p_\lambda^\lambda R^\lambda_i \lambda \mu = \frac{d}{dx^\lambda} (T^{\lambda \mu}_\mu + T^M_\lambda^\mu).$$ \hspace{1cm} (52)

It is the form of the energy-momentum transformation law which we observe also in the case of the quadratic Lagrangian densities of affine-metric gravity.
As a test case, let us consider the sum
\[ L = \left( -\frac{1}{2\kappa} g_{\beta\gamma} \mathcal{F}^{\alpha}_{\beta\gamma} \right) + \frac{1}{4\varepsilon} g_{\alpha\gamma} g^{\beta\sigma} g^{\gamma\lambda} \mathcal{F}^{\alpha}_{\beta\nu\lambda} \mathcal{F}^{\gamma}_{\sigma\mu\varepsilon} \sqrt{-g} \omega \]  
(53)
of the Hilbert-Einstein Lagrangian density and the Yang-Mills one. The corresponding Legendre morphism reads
\[ p_{\alpha\beta} \circ \hat{L} = 0, \]  
(54a)
\[ p_{\alpha}^{\beta(\nu\lambda)} \circ \hat{L} = 0, \]  
(54b)
\[ p_{\alpha}^{\beta[\nu\lambda]} \circ \hat{L} = \pi_{\alpha}^{\beta[\nu\lambda]} + \frac{1}{\varepsilon} g_{\alpha\gamma} g^{\beta\sigma} g^{\gamma\lambda} g^{\mu\nu} g^{\lambda\varepsilon} \mathcal{F}^{\alpha}_{\beta\nu\lambda} \mathcal{F}^{\gamma}_{\sigma\mu\varepsilon} \sqrt{-g}. \]  
(54c)

To construct the complete family of multimomentum Hamiltonian forms associated with the Lagrangian density (53), we consider the following connection
\[ \Gamma_{\alpha\beta}^{\lambda} = -K_{\alpha}^{\kappa \lambda} \varepsilon_{\kappa\beta} - K_{\beta}^{\kappa \lambda} \varepsilon_{\alpha\kappa}, \]
\[ \Gamma_{\alpha\beta}^{\nu\lambda} = S_{\alpha\beta}^{\nu\lambda} \]  
(55)
on the bundle \( C \times \Sigma \) where we utilize the notations of the expression (41). Then the multimomentum Hamiltonian forms
\[ H = \left( p_{\alpha\beta}^{\lambda} \lambda g^{\alpha\beta} + p_{\alpha}^{\beta\nu\lambda} dk_{\beta\nu} \right) \wedge \omega_{\lambda} - \mathcal{H}, \]
\[ \mathcal{H} = -p_{\alpha\beta}^{\lambda} (K_{\alpha}^{\kappa \lambda} \varepsilon_{\kappa\beta} + K_{\beta}^{\kappa \lambda} \varepsilon_{\alpha\kappa} \varepsilon_{\kappa\sigma} + S_{\alpha\beta}^{\nu\lambda} g^{\gamma\lambda} g^{\mu\nu} g^{\lambda\varepsilon} \pi_{\alpha}^{\beta[\nu\lambda]} - \pi_{\alpha}^{\beta[\nu\lambda]} \pi_{\gamma}^{\beta[\mu\varepsilon]} \right) \]  
(56)
are associated with the Lagrangian density (53) and constitute the complete family.

The corresponding Hamilton equations read
\[ \partial_{\alpha} g^{\alpha\beta} + K_{\alpha}^{\kappa \beta} \varepsilon_{\kappa\beta} + K_{\beta}^{\kappa \alpha} \varepsilon_{\alpha\kappa} \varepsilon_{\kappa\sigma} = 0, \]  
(57a)
\[ \partial_{\alpha} k_{\beta\nu} = S_{\alpha\beta\nu} + \varepsilon_{\alpha\gamma} g_{\beta\nu} g_{\gamma\lambda} (p_{\gamma}^{\beta[\mu\varepsilon]} - \pi_{\gamma}^{\beta[\mu\varepsilon]}), \]  
(57b)
\[ \partial_{\alpha} p_{\beta\nu}^{\lambda} = -\partial_{\alpha} \mathcal{H} - \partial_{\mathcal{H}} \partial_{\mathcal{H}} g_{\alpha\beta} - \partial_{\mathcal{H}} g_{\alpha\beta}, \]  
(57c)
\[ \partial_{\alpha} p_{\beta\nu}^{\lambda} = -p_{\alpha}^{\varepsilon[\nu]} k_{\beta\varepsilon} - p_{\varepsilon[\nu]} k_{\alpha\varepsilon} - p_{\varepsilon[\nu]} k_{\alpha\varepsilon} - p_{\varepsilon[\nu]} k_{\alpha\varepsilon}, \]  
(57d)
The equation (57b) breaks into the equation (54a) and the equation (45). It is readily observed that, for a given world metric \( g \) and a world connection \( k \), there always exist the world connections \( K' \) and \( K \) such that the gauge-type conditions (57a) and (45) hold (e.g. \( K' \) is the Levi-Civita connection of \( g \) and \( K = k \)). It follows that the multimomentum Hamiltonian forms (56) constitute really the complete family.

Substituting Eq. (57b) into Eq. (57c) on the constraint space (54a), we get the Einstein equations. On the constraint space (54a) - (54c), the equation (57d) is the Yang-Mills generalization
\[ D_{\alpha} p_{\beta\nu}^{\lambda} = \partial_{\alpha} p_{\beta\nu}^{\lambda} + p_{\alpha}^{\varepsilon[\nu]} k_{\beta\varepsilon} - p_{\varepsilon[\nu]} k_{\alpha\varepsilon} = 0 \]
of the equation (47).

Turn now to the energy-momentum conservation law. For the sake of simplicity, let us consider the splitting of the multimomentum Hamiltonian form (52) with regard to the connection (41) where the Hamiltonian density is

$$\tilde{H}_\Gamma = \tilde{H} + \frac{1}{2} p_\alpha^{\beta \nu \lambda} R^\alpha_{\beta \nu \lambda}.$$ 

We have the relation

$$\frac{\partial \tilde{H}_\Gamma}{\partial \beta^\nu \lambda} = R^\alpha_{\beta \nu \lambda}$$

on solutions of the Hamilton equations. The canonical energy-momentum tensor associated with the Hamiltonian density \(\tilde{H}_\Gamma\) is written

$$T^\Gamma_{\lambda \mu} = \frac{1}{2} p_\alpha^{\beta \nu \lambda} R^\alpha_{\beta \nu \mu} + \frac{\varepsilon}{2} g^{\alpha \gamma} g_{\beta \delta \mu \nu} p_\alpha^{\beta [\gamma \lambda]} (p_\gamma^{\sigma \delta \varepsilon} - \pi_\sigma^{\gamma \lambda \varepsilon})$$

$$- \delta^\mu_\lambda \left( \tilde{H} + \frac{\varepsilon}{2} g^{\alpha \gamma} g_{\beta \delta \mu \nu} \pi_\alpha^{\beta [\gamma \lambda]} (p_\gamma^{\sigma \delta \varepsilon} - \pi_\sigma^{\gamma \lambda \varepsilon}) \right) = $$

$$1 \varepsilon R^\alpha_{\beta \nu \lambda} R^\alpha_{\beta \nu \mu} + \pi^\alpha_{\beta \nu \lambda} R^\alpha_{\beta \nu \mu} - \delta^\lambda_\mu \left( \frac{1}{4 \varepsilon} R^\alpha_{\gamma \nu \lambda} R^\alpha_{\gamma \nu \mu} + \frac{1}{2 \kappa} R \right).$$

The identity (32) takes the form

$$(\partial^\mu + \Gamma^\alpha_\mu \partial_\alpha + \Gamma^i_\mu \partial_i - \partial_i \Gamma^j_\mu p^j_\alpha \partial^i_\lambda - p^\beta_\gamma \partial^\alpha - \partial^\alpha_\gamma (\tilde{H}_\Gamma + \tilde{H}_M) = $$

$$\frac{d}{dx^\lambda} (p^\lambda_\beta \gamma \lambda^\mu) + p^\beta_\alpha \gamma \lambda^\mu + p^\lambda_\alpha \beta^i \mu$$

On solutions of the Hamilton equations, we have

$$\Gamma^i_\mu \partial_i - \partial_i \Gamma^j_\mu p^j_\gamma \partial^i_\lambda \tilde{H}_M - p^\beta_\alpha \gamma \lambda^\mu \frac{\partial}{\partial k^\sigma_\gamma \delta} S^\sigma_\beta \nu \mu \frac{\partial}{\partial \sigma^\gamma \delta \lambda} \tilde{H}_\Gamma \frac{d}{dx^\lambda} (T^\lambda_\mu + T^\lambda_\beta \mu) + p^\lambda_\beta R^\lambda_\beta \mu$$

where the term

$$p^\alpha_\beta \gamma \lambda \frac{\partial}{\partial k^\sigma_\gamma \delta} S^\sigma_\beta \nu \mu \frac{\partial}{\partial \sigma^\gamma \delta \lambda} \tilde{H}_\Gamma = k^\gamma_\beta \lambda (\pi^\alpha_\beta \gamma \lambda^\mu - \pi^\alpha_\gamma \mu \lambda^\beta - k^\gamma_\beta \lambda^\mu) p^\lambda_\beta R^\lambda_\gamma \lambda$$

does not vanish. In case of gauge theory and gravity without non-metricity, we obtain

$$- \frac{1}{\varepsilon} K^\gamma_\beta \lambda R^\gamma_\beta R^\lambda_\alpha \nu \gamma \lambda + \frac{1}{2 \kappa} K^\gamma_\beta \lambda R^\gamma_\beta \lambda = \frac{d}{dx^\lambda} (T^\lambda_\mu + T^\lambda_\beta \mu).$$

(58)

Let us choose the local geodetic coordinate system at a point \(x \in X\). Relative to this coordinate system, the equality (58) at \(x\) comes to the conservation law

$$\frac{d}{dx^\lambda} (T^\lambda_\mu + t^\lambda_\mu) = 0.$$
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