Explaining $\epsilon$ in differential privacy through the lens of information theory

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Abstract. The study of leakage measures for privacy has been a subject of intensive research and is an important aspect of understanding how privacy leaks occur in computer programs. Differential privacy has been a focal point in the privacy community for some years and yet its leakage characteristics are not completely understood. In this paper we bring together two areas of research – information theory and the $g$-leakage framework of quantitative information flow (QIF) – to give an operational interpretation for the epsilon parameter of differential privacy. We find that epsilon emerges as a capacity measure in both frameworks; via (log)-lift, a popular measure in information theory; and via max-case $g$-leakage, which describes the leakage of any system to Bayesian adversaries modelled using “worst-case” assumptions under the QIF framework. Our characterisation resolves an important question of interpretability of epsilon and consolidates a number of disparate results covering the literature of both information theory and quantitative information flow.

Keywords: Differential privacy · log-lift · information leakage · $g$-leakage · quantitative information flow

1 Introduction

Over the past two decades, characterising and limiting privacy leakage in data sharing systems has been the subject of intensive research. Information-theoretic privacy [1,2], differential privacy [3] and quantitative information flow (QIF) [4] are three main frameworks that have been developed and studied for this purpose.

Information-theoretic privacy is concerned with measuring privacy leakage using well-known information-theoretic quantities, such as Shannon mutual information [5]:

$$I(X;Y) := \sum_{x \in X, y \in Y} p(x, y) \log \frac{p(x, y)}{p(x)p(y)},$$

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where $X$ denotes a secret and $Y$ denotes the observable, with $p(x, y), p(x)$ and $p(y)$ denoting the joint and marginal distributions of $X$ and $Y$, respectively. The channel $C$ is characterised by the conditional probability distribution $p(y|x)$ and is assumed fixed by the data sharing system and publicly known. For any prior $p(x)$, a joint $p(x, y)$ and a marginal distribution $p(y)$ are induced by $C$.

Papers such as [4,6] discuss some of the limitations of Shannon mutual information in properly quantifying or differentiating various adversarial threats to privacy. Sibson and Arimoto mutual information of order $\alpha$ have recently been proposed to measure a spectrum of generalised threats to privacy [7]. It also turns out that the “nucleus” of Shannon mutual information, namely the information density variable: $i(x, y) := \log \frac{p(x,y)}{p(x)p(y)}$ can represent a much stronger notion of privacy leakage than its average. We call $\ell(x, y) := \frac{p(x,y)}{p(x)p(y)}$ the lift variable, the exponential of information density. Information density and lift have been studied, albeit under various names, in works such as [1,8,9,10,11].

Differential privacy (DP) is a well-known privacy framework [3] describing a worst-case attack scenario, in which an attacker with knowledge of all individuals but one in a dataset learns little information about the unknown individual upon an information release. Specifically, for a given $\epsilon$, any $E \subset Y$ and any two neighbouring datasets $x, x'$ differing in one individual and any observation $Y \in Y$, DP requires

$$p(Y \in E|x) \leq e^{\epsilon} p(Y \in E|x').$$

This definition is sometimes referred to as the central model for differential privacy, referring to its dependence on a trusted centralised data curator. Alternatively, in the local model for differential privacy [12,13], each individual first applies noise to their data before sending it to an untrusted data curator, who releases some aggregate information about the (noisy) data. Local DP is more stringent than DP as it requires the inequation above be satisfied for all $x, x' \in X$.

In parallel to the above works, an important development has been the introduction of operationally relevant leakage measures to provide information about the relative security of different probabilistic data sharing systems. Foundational work in this area by Smith [4] has led to the study of adversarial gain functions and $g$-leakage under the umbrella of QIF [14,15]. The $g$-leakage model is built on Smith’s insight that leakage measures relevant to security should correspond to an attacking threat; and that, surprisingly, traditional measures such as Shannon entropy do not provide good security assessments. Notably, the so-called Bayes capacity [15, Ch 7]

$$\mathcal{M}^{\infty}_{\alpha}(D, C) := \sum_{y \in Y} \max_{x \in X} p(y|x)$$

quantifies the maximum multiplicative leakage in the average sense for adversarial models using gain functions.

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3 Where $C$ refers to a channel whose components are conditional probabilities $p(y|x)$. 
The theory of QIF has been extended to include \textit{max-case} leakage measures. Max-case measures aim to quantify the worst-case leakage to an adversary after making an observation, regardless of the probability of that observation itself occurring \cite{15,16}. The max-case perspective is in part inspired by differential privacy, and indeed differential privacy can be modelled using max-case leakage notions from QIF \cite{16}.

1.1 Existing linkages between frameworks and open questions

Perhaps not surprisingly and underpinned by the foundational theory of information, many linkages have already been established between information-theoretic, differential privacy and QIF frameworks \cite{16,17}. Here, we only review two results that are most pertinent to this work.

First, the logarithm of the Bayes capacity is known as the Sibson mutual information of order $\alpha = \infty$ in information theory \cite{18} and was recently shown in \cite{6} to measure the worst-case leakage of adversaries wanting to guess arbitrary randomised functions of secret $X$. It turns out that this identity is no coincidence and a recent work \cite{19} proves that there is no advantage in generalising the class of adversaries from those who use deterministic gain functions to those who guess randomised functions.

Second, it is not difficult to prove that an upper bound on the information density $i(x, y)$ bounds the Sibson mutual information of order infinity \cite{6}, aka the logarithm of Bayes capacity in QIF language. Moreover, works such as \cite{1,8,9} link upper and lower bounds on lift to (local) differential privacy measures \cite{3,12,13} and vice versa.

Despite establishment of the aforementioned relations, several questions remain open at the intersection of these frameworks. For example, it is still not clear whether lift has an operational significance as a measure of privacy leakage or is capable of robustly quantifying maximum privacy leakage in strong adversarial threat models via the QIF framework. In addition, the current theory of max-case vulnerability in QIF leaves open the question of how best to model the max-case adversarial threats or how to characterise max-case capacity over all priors and a wide class of adversarial threats. These questions are addressed in this paper.

1.2 Contributions and summary of results

From a high level, the first contribution of this work is establishing a link between lift in information-theoretic privacy and a notion of max-case $g$-leakage, which we introduce in this paper via the QIF framework. Specifically, in Defn \ref{def:max-case leakage} we introduce a max-case multiplicative $g$-leakage, denoted by $L_{\max}^g(\pi, C)$, using the standard notion of $g$-vulnerability in QIF framework. We then show via Thm \ref{thm:lift_upper_bound} that lift upper bounds $L_{\max}^g(\pi, C)$ with respect to any gain function $g$ for any given prior $\pi$. In Thm \ref{thm:lift_realisable} we show that lift is indeed realisable as a max-case $g$-leakage for an adversary who chooses a suitable prior-dependent gain function.
In addition, we establish an important result, linking all three information-theoretic privacy, differential privacy and QIF frameworks in an operationally robust manner. Specifically in the information-theoretic privacy framework, we define in Defn 4 the supremum of lift over all priors for a channel $C$ as *lift capacity* and denote it by $\mathcal{M}_{\text{Lift}}(C)$. We then show in Thm 3 and Cor 2 that lift capacity is equivalent to $\epsilon$ in the local DP framework. Combining this result with Thm 1 and Thm 2 we conclude that lift capacity, aka $\epsilon^\epsilon$, is also equal to the max-case $g$-leakage capacity in the QIF framework. This gives lift a robust operational interpretation in terms of strong max-case adversarial threats and explains $\epsilon$ in DP as a capacity measure from the lens of information theory.

Table 1 visualises existing results in the literature and our contributions. As seen from the Table, this paper bridges between several results in the information-theoretic privacy, differential privacy and QIF literature and establishes new relations between lift and other measures of privacy. Our new results, together with our consolidated summary of existing results, depicts a fuller picture on deep connections between the information-theoretic, differential privacy and QIF frameworks.

| $\mathcal{L}_g^X(\pi, C)$ | $\leq$ | $\mathcal{L}_g^{\text{max}}(\pi, C)$ | $\leq$ | Lift($\pi, C$) |
|--------------------------|-------|-------------------------------|-------|---------------|
| $\mathcal{M}\mathcal{L}_g^X(\mathbb{D}, C)$ | $\leq$ | $\mathcal{M}_{\text{Lift}}(C)$ | $\leq$ | $\mathcal{M}_{\text{Lift}}(C)$ |

Table 1. Relationship between leakages (top row) and capacities (bottom row). The coloured text indicates new contributions of this paper.

## 2 Information-Theoretic Foundations for Privacy

### 2.1 The channel model for quantitative information flow

We adopt notation from QIF [15], a mathematical framework for studying and quantifying information leaks with respect to adversarial attack scenarios.

A probabilistic channel $C$ maps inputs (secrets) $x \in \mathcal{X}$ to observations $y \in \mathcal{Y}$ according to a distribution $\mathbb{D}\mathcal{Y}$. In the discrete case, such channels are $\mathcal{X} \times \mathcal{Y}$ matrices $C$ whose row-$x$, column-$y$ element $C_{x,y}$ is the probability that input $x$ produces observation $y$. The $x$-th row $C_{x,-}$ is thus a discrete distribution in $\mathbb{D}\mathcal{Y}$. We write $\mathcal{X} \rightarrow \mathbb{D}\mathcal{Y}$ for the type of the channel $C$.

We can use Bayes rule to model an adversary who uses their observations from a channel to (optimally) update their knowledge about the secrets $\mathcal{X}$. Given a prior distribution $\pi : \mathbb{D}\mathcal{X}$ (representing an adversary’s prior knowledge) and
channel \( C \), we can compute a joint distribution \( J : \mathcal{X} \times \mathcal{Y} \) where \( J_{x,y} = \pi_x C_{x,y} \). Marginalising down columns yields the \( y \)-marginals \( p(y) = \sum_x \pi_x C_{x,y} \) each having a posterior over \( \mathcal{X} \) corresponding to the posterior probabilities \( P_{X \mid y}(x) \), computed as \( J_{x,y}/p(y) \) (when \( p(y) \) is non-zero). We denote by \( \delta^y \) the posterior distribution \( P_{X \mid y}(X \mid y) \) corresponding to the observation \( y \). The set of posterior distributions and the corresponding marginals can be used to compute the adversary’s posterior knowledge after making an observation from the channel.

### 2.2 Local differential privacy and lift

Local differential privacy (LDP), as applied by an individual, can be defined as a property of a channel \( C : \mathcal{X} \rightarrow \mathcal{Y} \) taking data \( \mathcal{X} \) to noisy outputs \( \mathcal{Y} \).

**Definition 1.** We say that channel \( C \) satisfies \( \epsilon \)-LDP if

\[
C_{x,y} \leq e^\epsilon C_{x',y}, \quad \forall x, x' \in \mathcal{X}, y \in \mathcal{Y}.
\]

A central quantity in this paper, which we call lift, was defined in [6] as:

**Definition 2.** Given a channel \( C \) and prior \( \pi : \mathcal{X} \rightarrow \mathbb{D} \), lift is defined as

\[
\text{Lift}(\pi, C) := \max_{x \in \mathcal{X}, y \in \mathcal{Y}} \frac{C_{x,y}}{p(y)}, \quad (1)
\]

Two alternative expressions for the lift are

\[
\text{Lift}(\pi, C) := \max_{x \in \mathcal{X}, y \in \mathcal{Y}} \frac{\delta^y_{x,y}}{\pi_x}, \quad (2)
\]

\[
\text{Lift}(\pi, C) := \max_{x \in \mathcal{X}, y \in \mathcal{Y}} \frac{J_{x,y}}{\pi_x p(y)}, \quad (3)
\]

Notably, the intuition behind lift as expressed in Eqn. (2) is that it measures the adversary’s change in knowledge, through (multiplicative) comparison of her prior and posterior beliefs for each secret and observation. The observation providing the biggest “knowledge gap” or lift thereby produces the most leakage.

Note that the argument of maximisation \( \frac{J_{x,y}}{\pi_x p(y)} \) in Eqn. (3) is indeed

\[
\ell(x, y) = \frac{p(x, y)}{p(x)p(y)},
\]

in plain probability notation as described in §1 and its logarithm is known as the information density. Here, we provide a brief account of some works which have used such quantities in defining privacy measures. In [9], a mechanism is said to provide \( \epsilon \)-local information privacy (LIP) if

\[
e^{-\epsilon} \leq \frac{\delta^y}{\pi_x} \leq e^\epsilon, \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}.
\]

(4)
A similar definition was earlier studied in [1] for sensitive features of datasets. In [10,11], $\epsilon$-lift was said to be satisfied if the logarithm of the above inequalities held true for a sensitive variable $S$ and useful variable $X$ according to the Markov chain $S \rightarrow X \rightarrow Y$.

An important distinction between our definition of lift and the above works is that they imposed an additional lower bound, $e^{-\epsilon}$, on the ratio of the posterior to prior beliefs. Whereas in this work, we are only concerned with the largest lift ratio (and without the logarithm), which we simply call lift. Our definition captures the notion of maximum realisable leakage [6], which is proved in [6, Theorem 13] to be equal to the lift as stated in our Definition 2. This also coincides with the notion of almostsure pointwise maximal leakage in [19, Definition 4].

In [9,1,10,8], it is proven that Eqn (4) implies $2\epsilon$-LDP. This bound can be improved through optimisation as shown in [9]. However, we highlight that Eqn (4) and the aforementioned relations to LDP depend on the prior $\pi$. In §3.2, we will establish robust results strongly and directly linking a prior-independent notion of lift capacity to $\epsilon$-LDP. The reverse direction, linking $\epsilon$-LDP to Eqn (4) is already strong; it has been shown that in works such as [18,9] that $\epsilon$-LDP implies Eqn (4).

### 2.3 Operational scenarios and the $g$-leakage framework

An important development over the past decade in security has been the use of operationally relevant leakage measures to provide information about the relative security of different probabilistic systems. Foundational work in this area by Geoffrey Smith [4] has led to the study of $g$-leakage under the umbrella of QIF [14,15].

In QIF, adversaries are modelled as Bayesian: they are equipped with a prior $\pi : \mathbb{D}\mathcal{X}$ over secrets $\mathcal{X}$ and a gain function $g : \mathcal{W} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ (over actions $\mathcal{W}$ and secrets $\mathcal{X}$), which models the gain to the adversary upon taking action $w \in \mathcal{W}$ when the true value of the secret is $x \in \mathcal{X}$. Before observing an output from the leaky system, the adversary’s prior expected gain, which we call the expected prior vulnerability of the secret, can be expressed as

$$V_g(\pi) := \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \pi_x g(w, x) . \quad (5)$$

The adversary can use their knowledge of the system (modelled as a channel $C : \mathcal{X} \rightarrow \mathbb{D}\mathcal{Y}$) to maximise their expected gain after making an observation. This is called the expected posterior vulnerability and is expressed as

$$V_g[\pi \triangleleft C] := \sum_{y \in \mathcal{Y}} \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \pi_x C_{x,y} g(w, x) \quad (6)$$

$$= \sum_{y \in \mathcal{Y}} p(y) \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \delta_{x,y} g(w, x) \quad (7)$$

$$= \sum_{y \in \mathcal{Y}} p(y) V_g(\delta^y) , \quad (8)$$
where in the last equality, we have used the definition of vulnerability in Eqn. (6) for the posterior distribution, denoted $\delta^\nu$. Finally, the difference between the prior and posterior vulnerabilities gives a measure of the leakage of the system to this adversary; the greater the difference, the better is the adversary able to use the transmitted information to infer the value of the secret. This can be computed multiplicatively as

$$L_g^\pi (\pi, C) := \frac{V_g[\pi \triangleright C]}{V_g(\pi)}. \quad (9)$$

The $g$-leakage framework models a wide variety of attack scenarios, including guessing the secret in $n$ tries, guessing a property of the secret or guessing a value close to the secret. Moreover, attached to each leakage measure is an operational scenario given by the gain function and prior which describes a specific adversarial threat.

We also recall the notion of capacity, which measures the maximum leakage quantified over all priors and/or gain functions. Of particular note is the Bayes capacity, defined as

$$\mathcal{ML}_g^\pi (D, C) := \sup_{\pi} \sup_g L_g^\pi (\pi, C) \quad (10)$$

$$= \sup_{\pi} \sum_y \max_x \pi \chi C_{x,y} \quad (11)$$

$$= \sum_y \sup_x C_{x,y}, \quad (12)$$

where the first equality is proved in [14] and the last equality is due to the fact that the capacity is realised under the uniform prior [14] “Miracle Theorem”.

The Bayes capacity has an important operational significance: it is a tight upper bound on the multiplicative leakage of any channel in the average sense, quantified over all priors and gain functions [15]. In other words, there is no adversarial scenario, modelled using the expected gain of an adversary with any prior knowledge, for which the channel leakage exceeds the amount given by the Bayes capacity.

### 2.4 Connecting $g$-leakage and lift

We conclude this section by showing the connection between Bayes capacity and the information-theoretic measure lift: it turns out that lift is an upper bound on the Bayes capacity. The following lemma has been expressed in [6], Corollary of Theorem 13; our contribution here is an alternative proof of this result in a QIF formulation.

**Lemma 1.** Given a channel $C : \mathcal{X} \rightarrow \mathcal{Y}$, for all priors $\pi : \mathcal{D} \mathcal{X}$ it holds that

$$\mathcal{ML}_g^\pi (D, C) \leq \text{Lift}(\pi, C).$$

\footnote{In fact, it has been shown that any convex vulnerability function is expressible as a $g$-vulnerability [20].}
Proof. We reason as follows:

\[
\text{Lift}(\pi, C) = \sup_{x,y} C_{x,y} p(y) \quad \text{"Eqn (1)"}
\]

\[
= \sup_{x,y} C_{x,y} \sum_{y'} p(y') \quad \text{"Since } \sum_{y'} p(y') = 1 \text{"}
\]

\[
= \sum_{y'} p(y') \sup_{x,y} C_{x,y} \quad \text{"sup independent of } y' \"}
\]

\[
\geq \sum_{y'} p(y') \sup_{x} C_{x,y} \quad \text{"Taking sup over each column instead of channel"}
\]

\[
= \sum_{y} \sup_{x} C_{x,y} \quad \text{"Rearranging"}
\]

\[
= ML_{\pi}^*(D, C) \quad \text{"Eqn (12)"}
\]

Further, we show that the bound is strict, in that lift can be strictly greater than Bayes capacity (see example in Appendix A). Recalling that Bayes capacity is an upper bound on average-case $g$-leakage measures, this result then shows that lift cannot be represented as an average-case $g$-leakage measure.

Next, we study the relationship between lift and max-case measures of leakage.

3 On Max-case $g$-leakage Measures, Lift and Local DP

The QIF measures we have introduced thus far have been for average-case attacks; that is, we model the expected gain of an attacker. Differential privacy and lift, however, are max-case notions; they describe the worst-case gain for an adversary after making an observation, regardless of its probability of occurring. Max-case measures provide an alternative perspective to average-case measures: the average-case can be seen as the perspective of a data curator whose interest is in protecting attacks against the entire dataset, whereas the max-case provides the perspective of an individual in the dataset whose concern is their particular data point.

The theory of QIF has been extended to include max-case measures which quantify the gain to an adversary interested in only the worst-case leakage [13,16]. To model this, the max-case posterior vulnerability is defined as

\[
V^{\text{max}}[\pi\|C] := \max_y V(\delta^y). 
\]

This says that the max-case posterior vulnerability $V^{\text{max}}$ is the maximum vulnerability $V$ computed over the posteriors, where $V$ is a vulnerability function defined on distributions. The theory of max-case vulnerability leaves open the question of how best to define $V$, aside from a technical result which says that $V$ should be quasi-convex in order to satisfy basic axioms such as the data processing inequality and monotonicity [21].

We will choose $V$ in Eqn (13) to be the standard $g$-vulnerability function $V_g$ defined in Eqn (5), noting that $V_g$ is convex [15], and therefore also quasi-convex. This yields the following definition:
Definition 3 (Max-case \( g \)-leakage). Given a channel \( C \), prior \( \pi \) and gain function \( g \), the max-case \( g \)-leakage of \( C \) is defined as

\[
L^{\text{max}}_{g}(\pi, C) := \frac{V^{\text{max}}_{g}[\pi \circ C]}{V_{g}(\pi)} = \max_{y} \frac{V_{g}(\delta^{y})}{V_{g}(\pi)},
\]

where \( V_{g} \) is the prior vulnerability function given in Eqn (7).

We will leave further discussion on this choice of \( V \) to §4.

3.1 Lift and max-case \( g \)-leakage

Using Defn 3 above, we now show that lift can be expressed as a max-case \( g \)-leakage.

Theorem 1 (Lift as max-case \( g \)-leakage). For discrete domain \( X \), define the set of actions \( W = X \) and the gain function \( g_{\pi}: W \times X \rightarrow \mathbb{R}_{\geq 0} \) as:

\[
g_{\pi}(w, x) = \begin{cases} \frac{1}{\pi_{x}}, & \text{if } w = x \\ 0, & \text{otherwise} \end{cases}
\]

where \( \pi \) is the (full support) prior of the adversary using the gain function \( g_{\pi} \). Then the max-case \( g \)-leakage of any channel \( C \) given a prior \( \pi \) is equal to its lift. That is, \( L^{\text{max}}_{g_{\pi}}(\pi, C) = \text{Lift}(\pi, C) \).

Proof. For the gain function given above, the prior vulnerability \( V_{g_{\pi}}(\pi) \) is:

\[
V_{g_{\pi}}(\pi) = \max_{w} \sum_{x} \pi_{x} g_{\pi}(w, x) = 1 \quad \text{for all } w
\]

and the max-case posterior vulnerability is:

\[
V^{\text{max}}_{g_{\pi}}[\pi \circ C] = \max_{y} \frac{V_{g_{\pi}}(\delta^{y})}{V_{g_{\pi}}(\pi)} \\
= \max_{w} \max_{x} \sum_{y} \pi_{x} C_{x,y} g_{\pi}(w, x) \frac{1}{p(y)} \\
= \max_{w} \max_{x} C_{x,w} \frac{\pi_{x}}{p(y)} \\
= \text{Lift}(\pi, C)
\]

Thus the max-case leakage is \( L^{\text{max}}_{g_{\pi}}(\pi, C) = \frac{V^{\text{max}}_{g_{\pi}}[\pi \circ C]}{V_{g_{\pi}}(\pi)} = \text{Lift}(\pi, C) \). \qed

The significance of Thm 1 is that it gives an operational meaning to lift in terms of the adversarial model defined by the \( g \)-leakage framework. To give some intuition, for some distribution \( \sigma \), \( V_{g_{\pi}}(\sigma) \) is given by \( \max_{x} \frac{\sigma_{x}}{\pi_{x}} \), i.e. It measures the maximum “surprise” to the adversary, achieved on the secret \( x \) for which the probability \( \sigma_{x} \) most differs (multiplicatively) from the adversary’s prior \( \pi_{x} \).\(^5\)

\(^5\) We remark that the gain function \( g_{\pi} \) is also known as the distribution-reciprocal gain function [15 Ch 3].
We find, however, that lift has a much stronger property: it is in fact an upper bound on any max-case leakage measure (that is, using any gain function) with the minor restriction that the gain function must be non-negative.

**Theorem 2 (Lift bounds max-case g-leakage).** Define the max-case g-leakage $L^\text{max}_g$ as per Defn 3. Then for any non-negative gain function $g$, any prior $\pi$ and any channel $C$, it holds that $L^\text{max}_g(\pi, C) \leq \text{Lift}(\pi, C)$.

**Proof.** We reason as follows:

\[
\max_y V_g(\delta^y) = \max_y (\sup_w \sum_x \frac{\pi_x C_{x,y} g(w, x)}{p(y)}) \geq \max_y (\sup_w \sum_x (\max_x C_{x,y} \pi_x g(w, x)) = \max_y (\max_x C_{x,y} \sup_w (\sum_x \pi_x g(w, x))) = \text{Lift}(\pi, C)V_g(\pi) \tag{Eqn (5), Lift}
\]

The result follows using Defn 3.

Thm 2 tells us that lift is a surprisingly robust measure; for any prior, the lift is a measure of the maximum g-leakage of a channel wrt any (non-negative) gain function. Moreover, this upper bound holds regardless of whether we consider average-case leakage or max-case leakage: the max-case we proved above (Thm 2); the average-case follows from the fact that lift upper bounds Bayes capacity (Lem 1) and Bayes capacity is, by definition, an upper bound on any average-case g-leakage, for any prior and any (non-negative) gain function.

### 3.2 Robustness results: lift capacity and epsilon

Typically, leakage measures (as defined e.g., in QIF) are not robust – that is, they depend on the specifics of the adversary (their prior and gain function), and therefore may not provide a reliable measure of the safety of a channel against different (Bayesian) adversaries having different prior knowledge and intent. A more robust way to characterise a channel’s leakage properties is to measure its maximum leakage – that is, quantified over all priors, or over all gain functions, or both. Such quantities are termed capacities (see §2.3), and have

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6. Note that this non-negativity restriction is also required in the case of Bayes capacity as an upper bound on average-case leakage.

7. Note the distinction between maximum leakage and max-case leakage: the former quantifies over all priors and gain functions; the latter uses a max-case posterior vulnerability measure.
been previously studied in the context of average-case leakage \cite{22}: the Bayes capacity (recall Eqn (10)) is the most well-known and provides robust upper bounds on the average-case leakage of any channel.

There has, to date, been no study of max-case capacities. However, the result of Thm 2 implies an immediate result in this direction—it shows that lift can be seen as an example of a max-case capacity, since it provides an upper bound on max-case leakages quantified over all gain functions. It follows also that, by quantifying over all priors as well, the resulting lift capacity is an upper bound on all max-case $g$-leakages for all priors.

**Definition 4.** We define the lift capacity for a channel $C$ as

$$M_{\text{Lift}}(C) := \sup_{\pi: \mathcal{X} \to \mathcal{Y}} \text{Lift}(\pi, C) = \sup_{\pi: \mathcal{X}} \max_{x, y \in \mathcal{Y}} \frac{C_{x,y}}{p(y)}.$$  

Immediate from this definition and Thm 2 we get the following:

**Corollary 1.** For any channel $C$ and prior $\pi$ it holds that

$$M_{\text{Lift}}(C) \geq L_{\text{max}}^{g}(\pi, C).$$

We now show that this so-called lift capacity also has an important relationship with the epsilon of local differential privacy.

**Theorem 3.** Let $C: \mathcal{X} \to \mathcal{Y}$ be a channel and $\epsilon \geq 0$. Then

$$M_{\text{Lift}}(C) \leq \epsilon$$

iff

$$\frac{C_{x,y}}{C_{x',y}} \leq \epsilon$$

for all $x, x' \in \mathcal{X}, y \in \mathcal{Y}$. Moreover, the bounds are tight, in that $M_{\text{Lift}}(C) = \epsilon$ precisely when

$$\max_{x,x',y} \frac{C_{x,y}}{C_{x',y}} = \epsilon$$

and the lift capacity is attained on a point prior (in the limit).

**Proof.** For the forward direction, note that we can rewrite $p(y)$ as $\sum_{x, \pi_x C_{x,y}}$. Then we have:

$$\begin{align*}
\sup_{\pi} \text{Lift}(\pi, C) &= \sup_{\pi} \max_{x, y} \frac{C_{x,y}}{p(y)} \\
&= \sup_{\pi} \max_{x, y} \frac{C_{x,y}}{\sum_{x'} \pi_{x'} C_{x',y}} \quad \text{“Substituting for $p(y)$ as noted above”} \\
&= \max_{x, y} \sup_{\pi} \frac{C_{x,y}}{\sum_{x'} \pi_{x'} C_{x',y}} \\
&\geq \max_{x, y} \frac{C_{x,y}}{\min_{x'} C_{x',y}} \quad \text{“Rearranging”} \\
&\geq \frac{C_{x,x}}{C_{x',y}} \quad \text{for all $x, x', y$} \\
\end{align*}$$

And so $M_{\text{Lift}}(C) \leq \epsilon$ implies $\frac{C_{x,y}}{C_{x',y}} \leq \epsilon$ for all $x, x', y$. The reverse direction has previously been shown to hold \cite{1, 8, 9}.

Observe from Definition 4 that $M_{\text{Lift}}(C)$ is a supremum over all full support priors. Let $x^* = \arg \min_{x} C_{x',y}$ as above; now define a sequence of priors

$$\pi_{x}^n = 1 - \frac{1}{n} \quad \text{and} \quad \pi_{x}^n = \frac{1}{n(|\mathcal{X}| - 1)} \quad \text{for $x \neq x^*$}.$$
We see that \( \pi^n \) is full support, and also
\[
\lim_{n \to \infty} \sum_x \pi_x C_{x,y} = \min_x C_{x,y}.
\]
Thus we deduce that in fact \( \mathcal{M}_{\text{Lift}}(C) = \sup_{x,x',y} \frac{C_{x,y}}{C_{x',y}} \).
\[\square\]

**Corollary 2.** The lift capacity coincides with the \( \epsilon \) value of an LDP channel. That is, \( \mathcal{M}_{\text{Lift}}(C) = e^\epsilon \).

Thm 3 and Cor 2 establish the equivalence between the lift capacity and the \( \epsilon \) parameter of local differential privacy, and by Thm 1 and Thm 2 this gives that \( \epsilon \) is a tight upper bound on the max-case \( g \)-leakage of any channel. That is, \( e^\epsilon \) is the max-case \( g \)-leakage capacity, quantified over all priors and gain functions. This result connecting \( \epsilon \) with the \( g \)-leakage framework provides the first robust, operational interpretation for local differential privacy in terms of Bayesian adversarial threats. We note that \( \epsilon \) has previously been interpreted as a capacity under QIF [16]. However, in that work the vulnerability function chosen was not expressible as a \( g \)-vulnerability, and therefore did not carry with it the operational interpretation attached to the \( g \)-leakage framework. This was Smith’s original motivation for a \( g \)-leakage based model for secure information leakage measurements.

We also remark that the relationship between lift and local differential privacy has been established previously (e.g. [6,23]), but these results have not been tied back to the \( g \)-leakage framework, and did not establish the operational significance of \( \epsilon \) in terms of max-case \( g \)-leakage measures.

### 3.3 Max-case Dalenius leakage

Up until this point we have been satisfied with computing leakage measures with respect to deterministic functions of the domain \( \mathcal{X} \) (described by a prior \( \pi \)). As pointed out in [6], one may also be concerned about potentially randomised functions of \( \mathcal{X} \). The QIF theory of Dalenius leakage [15, Ch 10] accounts for such functions by modelling the leakage that a channel \( \mathcal{C} : \mathcal{X} \to \mathcal{Y} \) induces on a secret \( \mathcal{Z} \) through an unknown correlation \( J : \mathcal{D} (Z \times X) \). In [15] it is shown that the multiplicative Bayes capacity of the combined system \( D \cdot \mathcal{C} \) is in fact bounded from above by the multiplicative Bayes capacity of \( \mathcal{C} \); and thus considering arbitrary functions of the secret \( \mathcal{X} \) is not necessary for quantifying the maximum multiplicative leakage of a channel.

Here we confirm that this property also holds for the max-case leakage capacity, and that therefore the capacity results proven in this paper also hold for randomised functions of \( \mathcal{X} \).

We recall the Dalenius leakage context: given a channel \( \mathcal{C} \) taking secrets \( \mathcal{X} \) to observations \( \mathcal{Y} \), we ask: what is the maximum leakage caused by \( \mathcal{C} \) on secrets \( \mathcal{Z} \) through some unknown correlation \( J : \mathcal{D}(Z \times X) \)? Given such a correlation \( J \), we can factorise it into a prior \( \rho : \mathcal{D} \mathcal{Z} \) and a channel \( \mathcal{D} : \mathcal{Z} \to \mathcal{D} \mathcal{X} \) so that \( J_{z,x} = \rho_z \times D_{z,x} \). Then the max-case \( g \)-leakage of \( \mathcal{Z} \) caused by \( J \) and \( \mathcal{C} \) can be written as \( \mathcal{L}^\max_g (\rho, D \cdot \mathcal{C}) \). We now have the following:

**Theorem 4.** Let \( \mathcal{C} : \mathcal{X} \to \mathcal{D} \mathcal{Y} \) and \( \mathcal{D} : \mathcal{Z} \to \mathcal{D} \mathcal{X} \) be channels. Then
\[
\mathcal{M}_{\text{Lift}}(DC) \leq \min \{ \mathcal{M}_{\text{Lift}}(D), \mathcal{M}_{\text{Lift}}(C) \}.
\]
Proof. Referring to Thm 4, write \( g_\pi \) for the gain function that realises lift, i.e., \( \text{Lift}(\pi, C) = \mathcal{L}^\pi_{g_\pi}(\pi, C) \). The data processing inequality for max-case leakage and Cor 1 give that
\[
\mathcal{L}^\max_{g_\pi}(\rho, DC) \leq \mathcal{L}^\max_{g_\pi}(\rho, D) \leq \mathcal{MLift}(D)
\]
for any \( \rho \). Thus,
\[
\mathcal{MLift}(DC) = \sup_{\rho} \text{Lift}(\rho, DC) = \sup_{\rho} \mathcal{L}^\max_{g_\pi}(\rho, DC) \leq \mathcal{MLift}(D).
\]
We next have that:
\[
\mathcal{MLift}(DC) = \sup_{\rho \geq 0} \sup_{Z, X, Y} \text{Lift}(\rho, DC) = \sup_{\rho \geq 0} \sup_{Z, X, Y} \mathcal{L}^\max_{g_\pi}(\rho, DC).
\]

And thus it follows that \( \mathcal{MLift}(DC) \leq \min\{\mathcal{MLift}(D), \mathcal{MLift}(C)\} \). \( \square \)

As a corollary we have the desired result: that the max-case \( g \)-leakage of secrets \( Z \) via a channel \( C \) and arbitrary correlation \( J \) is bounded from above by the lift capacity of \( C \).

**Corollary 3.** For any channel \( C: \mathcal{X} \rightarrow \mathcal{Y} \), non-negative gain function \( g \) and correlation \( J \) given by \( J_{z,x} = \rho_z D_{z,x} \) we have that
\[
\mathcal{L}^\max_g(\rho, DC) \leq \mathcal{MLift}(C).
\]

**Proof.** By Cor 1, \( \mathcal{L}^\max_g(\rho, DC) \leq \mathcal{MLift}(DC) \). The result follows from Thm 4.

### 4 Additional Results on Max-case Leakage and Technical Discussion

In this section we provide further technical details on the max-case leakage definition of Defn 3.

Firstly, the following result shows that the max-case leakage of a channel is at least as large as the average-case leakage. This result completes Table I.
Lemma 2. Given a channel $C$ prior $\pi$ and gain function $g$, it holds that

$$\mathcal{L}_g^\pi(\pi, C) \leq \mathcal{L}_g^{\text{max}}(\pi, C).$$

Proof. We reason as follows:

\[
\begin{align*}
V_g[\pi \trianglerighteq C] &= \sum_y p(y) V_g(\delta^y) \\
&\leq \sum_y p(y) \max_j V_g(\delta^j) \\
&= \max_j V_g(\delta^j) \sum_y p(y) \\
&= V_g^{\text{max}}[\pi \trianglerighteq C]
\end{align*}
\]

Thus the corresponding leakages are ordered (since prior vulnerability is common).

Next, we remark that we chose (in Defn 3) to model max-case leakage using the prior vulnerability function $V_g$ representing the adversary’s expected gain using knowledge only of their prior. It may be preferable (for a max-case adversarial model) to instead choose a prior vulnerability function modelling the adversary’s max-case gain; i.e., we could define:

$$V_g^{\text{max}}(\pi) := \max_{w,x} \pi_x g(w, x). \quad (16)$$

That is, the adversary’s prior gain is computed using the secret $x$ which maximises their gain. However, we show now that this choice is equivalent to choosing the usual expected gain-based vulnerability function.

Lemma 3. Let $C$ be a channel, $\pi : \mathbb{X} \to \mathcal{D}$ be a prior and $g$ be a gain function. Then there exists a gain function $g^\star$ such that $V_g^{\text{max}}(\pi) = V_{g^\star}(\pi)$ and $\max_y V_{g^{\text{max}}}(\delta^y) = \max_y V_{g^{\text{max}}(\pi)} = \max_y V_{g^{\text{max}}(\delta^y)} = \max_y V_{g^{\text{max}}(\pi)}$.

Proof (Sketch). Observe that $V_g^{\text{max}}$ (Eqn (16)) is convex in $\pi$ and so it can be expressed as a convex $g$-vulnerability \[11\], Ch 11, Thm 11.5; i.e., there exists a gain function $g^\star$ such that $V_{g^\star}(\pi) = V_g^{\text{max}}(\pi)$. This also means that $\max_y V_g^{\text{max}}(\delta^y) = \max_y V_{g^\star}(\delta^y)$ and thus

$$\frac{\max_y V_g^{\text{max}}(\delta^y)}{V_g^{\text{max}}(\pi)} = \frac{\max_y V_{g^\star}(\delta^y)}{V_{g^\star}(\pi)}.$$ 

In Appendix \[12\] we show the construction of such a $g^\star$.

Finally, we recall the result of \[21\] which shows that, for reasonable axioms to hold under a max-case definition of leakage, then the prior vulnerability function should be quasi-convex in the prior. However, as we have seen, both the average-case and max-case prior vulnerability functions are convex. An open question in the community has been: are there any strictly quasi-convex functions which produce leakage measures of interest? Previous work \[16\] showed that the $\epsilon$
of differential privacy can be expressed (as a capacity) using a quasi-convex vulnerability, suggesting that this was indeed one such example. In this paper we have resolved this question, showing that in fact $\epsilon$ in local DP can be expressed using a convex vulnerability. This justifies our restriction of max-case leakage to $g$-leakage measures, and leaves open the question of the usefulness of max-case leakage defined over the full scope of quasi-convex vulnerabilities.

5 Conclusion

In this paper we have investigated the relationships between traditional information theory and the $g$-leakage framework of quantitative information flow. The connections are summarised Table 1. Overall we observe that the two notions converge wrt. robustness, namely via the capacity $\mathcal{MLift}(C)$, which we find is equivalent to $e^\epsilon$ of differential privacy.

Significant also is that differential privacy’s $\epsilon$ parameter can now be understood through the lens of QIF. In particular we see that it is also a measure of robustness in that it behaves as a capacity – that is, independent of any particular prior. Moreover, it represents a tight upper bound on all max-case $g$-leakages. This is the first time that $\epsilon$ has been explained as a robust measure of information leakage in the QIF framework.

From the perspective of information theory, Lift is also explained as a leakage measure, but interestingly we discovered that it has “capacity-like-properties” (§3.2). Table 1 clarifies how these measures relate to the leakages and capacities well-known in QIF.

References

1. F. du Pin Calmon and N. Fawaz, “Privacy against statistical inference,” in 50th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pp. 1401–1408, IEEE, 2012.
2. A. Makhdoumi, S. Salamatian, N. Fawaz, and M. Médard, “From the information bottleneck to the privacy funnel,” in 2014 IEEE Information Theory Workshop (ITW 2014), pp. 501–505, Nov 2014.
3. C. Dwork, F. McSherry, K. Nissim, and A. Smith, “Calibrating noise to sensitivity in private data analysis,” in Thr of Crypto. Conf., pp. 265–284, Springer, 2006.
4. G. Smith, “On the Foundations of Quantitative Information Flow,” in FOSSACS, vol. 5504 of LNCS, pp. 288–302, Springer, 2009.
5. M. Thomas and A. T. Joy, Elements of information theory. Wiley-Interscience, 2006.
6. I. Issa, A. B. Wagner, and S. Kamath, “An operational approach to information leakage,” IEEE Transactions on Information Theory, vol. 66, no. 3, pp. 1625–1657, 2020.
7. J. Liao, O. Kosut, L. Sankar, and F. d. P. Calmon, “Tunable measures for information leakage and applications to privacy-utility tradeoffs,” vol. 65, no. 12, pp. 8043–8066, 2019.
8. S. Salamatian, F. P. Calmon, N. Fawaz, A. Makhdoumi, and M. Médard, “Privacy-utility tradeoff and privacy funnel,” Unpublished preprint, https://www.mit.edu/salmansa/files/privacy_TIFS.pdf, 2020.
9. B. Jiang, M. Seif, R. Tandon, and M. Li, “Context-aware local information privacy,” IEEE Transactions on Information Forensics and Security, vol. 16, pp. 3694–3708, 2021.
10. H. Hsu, S. Assoodeh, and F. P. Calmon, “Information-theoretic privacy watchdogs,” in 2019 IEEE International Symposium on Information Theory (ISIT), pp. 552–556, IEEE, 2019.
11. P. Sadeghi, N. Ding, and T. Rakotoariveloh, “On properties and optimization of information-theoretic privacy watchdog,” in IEEE Inf. Theory Workshop (ITW), pp. 1–5, Apr. 2020.
12. S. P. Kasiviswanathan, H. K. Lee, K. Nissim, S. Raskhodnikova, and A. Smith, “What can we learn privately?,” SIAM Journal on Computing, vol. 40, no. 3, pp. 793–826, 2011.
13. J. C. Duchi, M. I. Jordan, and M. J. Wainwright, “Local privacy and statistical minimax rates,” in Proc. IEEE 54th Annu. Symp. Found. Comput. Sci., pp. 429–438, 2013.
14. M. S. Alvim, K. Chatzikokolakis, C. Palamidessi, and G. Smith, “Measuring information leakage using generalized gain functions,” in IEEE CSF, pp. 265–279, June 2012.
15. M. S. Alvim, K. Chatzikokolakis, A. McIver, C. Morgan, C. Palamidessi, and G. Smith, The Science of Quantitative Information Flow. Springer, 2020.
16. K. Chatzikokolakis, N. Fernandes, and C. Palamidessi, “Comparing systems: Max-case refinement orders and application to differential privacy,” in Proc. CSF, IEEE Press, 2019.
17. P. Cuff and L. Yu, “Differential privacy as a mutual information constraint,” in CSS, pp. 43–54, 2016.
18. R. Sibson, “Information radius,” Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, vol. 14, no. 2, pp. 149–160, 1969.
19. S. Saeidian, G. Cervia, T. J. Oechtering, and M. Skoglund, “Pointwise maximal leakage,” arXiv preprint arXiv:2205.04935, 2022.
20. A. McIver, C. Morgan, G. Smith, B. Espinoza, and L. Meinicke, “Abstract channels and their robust information-leakage ordering,” in Principles of Security and Trust - Third International Conference, POST 2014, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2014, Grenoble, France, April 5-13, 2014, Proceedings (M. Abadi and S. Kremer, eds.), vol. 8414 of Lecture Notes in Computer Science, pp. 83–102, Springer, 2014.
21. M. S. Alvim, K. Chatzikokolakis, A. McIver, C. Morgan, C. Palamidessi, and G. Smith, “Axioms for information leakage,” in 2016 IEEE 29th Computer Security Foundations Symposium (CSF), pp. 77–92, IEEE, 2016.
22. M. S. Alvim, K. Chatzikokolakis, A. McIver, C. Morgan, C. Palamidessi, and G. Smith, “Additive and Multiplicative Notions of Leakage, and Their Capacities,” in IEEE CSF, pp. 308–322, IEEE, 2014.
23. M. E. Andrés, N. E. Bordenabe, K. Chatzikokolakis, and C. Palamidessi, “Geo-indistinguishability: Differential privacy for location-based systems,” in Proceedings of the 2013 ACM SIGSAC conference on Computer & communications security, pp. 901–914, 2013.
Appendices

A Results supporting Section 2

A.1 Lift is a strict upper bound on Bayes capacity

The following example shows that lift can be strictly greater than the Bayes capacity. Consider the following channel:

|   | $y_1$ | $y_2$ | $y_3$ |
|---|-------|-------|-------|
| $x_1$ | $\frac{2}{3}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |
| $x_2$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $x_3$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |

For this channel, using the uniform prior $\nu$, we calculate that

$$\text{Lift}(\nu, C) = \max_{x,y} \frac{C_{x,y}}{p(y)} = \frac{2}{3}/\frac{7}{18} = \frac{12}{7}$$

whereas

$$\mathcal{ML}_1^\nu(D, C) = \sum_y \sup_x C_{x,y} = \frac{5}{3} < \frac{12}{7}$$

B Max-case $g$-leakage using average-case $g$-vulnerability

In this section, we complete the proof of Lem 3 showing a gain function $g^*$ which produces the same average-case leakage as a max-case leakage defined using a $g$. That is, we will prove that for any gain function $g$ it is possible to construct a gain function $g^*$ such that $V_{g^*}^{\text{max}}(\pi) = V_{g^*}(\pi)$ where $V_{g^*}(\pi)$ is defined in the usual way as:

$$V_{g^*}(\pi) := \max_w \sum_x \pi_x g^*(w, x) \tag{17}$$

and $V_{g^*}^{\text{max}}(\pi)$ is defined as

$$V_{g}^{\text{max}}(\pi) := \max_{w,x} \pi_x g(w, x) \tag{18}$$

Note that our construction assumes that the domain $\mathcal{X}$ is discrete, although the proof does not rely on this assumption.
For \( g: \mathcal{W} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0} \) we can define the set of actions \( \mathcal{W}^* \) such that for each \( w \in \mathcal{W} \) we have a set \( \{ w_x \in \mathcal{W}^* : x \in \mathcal{X} \} \), and a mapping \( g^*: \mathcal{W}^* \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0} \) satisfying

\[
g^*(w_x, x) = \begin{cases} g(w, x), & \text{if } x = x_i \\ 0, & \text{otherwise} \end{cases}
\]

This means that \( \max_x g^*(w_x, x) = \max_x g(w, x) \). And therefore:

\[
V_g^{\max}(\pi) = \max_{w \in \mathcal{W}}, x \in \mathcal{X} \pi x g(w, x) \quad \text{“From Eqn (18)”}
\]
\[
= \max_{w \in \mathcal{W}^*}, x \in \mathcal{X} \pi_x g^*(w, x) \quad \text{“From Eqn (19)”}
\]
\[
= \max_{w \in \mathcal{W}^*} \sum_x \pi_x g^*(w, x) \quad \text{“Since } g^*(w, x) = 0 \text{ everywhere else”}
\]
\[
= V_{g^*}(\pi) \quad \text{“Eqn (17)”}
\]

And this gives that \( L_g^{\max}(\pi, \mathcal{C}) = \frac{\max_{\delta^y} V_g^{\max}(\delta^y)}{V_g^{\max}(\pi)} = \frac{\max_{\delta^y} V_{g^*}(\delta^y)}{V_{g^*}(\pi)} \).