Higher Derived Brackets

Ezra Getzler

Abstract. We show that there is a sequence of operations on the positively graded part of a differential graded algebra $L_\bullet$ making it into an $L_\infty$-algebra. The formulas for the higher brackets involve Bernoulli numbers. The construction generalizes the derived bracket for Poisson manifolds, and the Lie 2-algebra associated to a Courant algebroid constructed by Roytenberg and Weinstein.

The Poisson bracket is a Lie bracket on the vector space $C^\infty(M)$ of functions on a manifold $M$ associated to a Poisson tensor $P$ on $M$. Koszul [2] showed how the construction of this bracket could be interpreted in terms of differential graded Lie algebras. The Poisson tensor induces a differential $\delta_P$ on the graded Lie algebra $L(M)$ of multivector fields on $M$ (the Schouten algebra):

$$\delta_P X = [P, X] : L_i(M) \to L_{i-1}(M).$$

Here, $L_i(M)$ is the space of smooth sections of the vector bundle $\Lambda^{1-i}TM$, $-n < i \leq 1$. The Poisson bracket $\{f, g\}_P$ on $L_1(M) \cong C^\infty(M)$ is given by the formula

$$\{f, g\}_P = [\delta_P f, g].$$

In fact, for any differential graded Lie algebra $L$ such that $L_i = 0$ if $i > 1$, this formula induces a Lie bracket on the vector space $L_1$.

In this note, we extend this construction, removing the condition that $L$ vanish above degree 1. We show that there is a sequence of operations on $L = \tau_{>0} L$ making it into an $L_\infty$-algebra. Note that we adopt the convention that each of the operations in an $L_\infty$-algebra lowers degree by 1. Often, in applications of $L_\infty$-algebras, the $n$th bracket is taken to have degree $n - 2$; the two conventions differ by a suspension.

Definition 1. An $L_\infty$-algebra is a graded vector space $L_\bullet$ with operations

$$\{a_0, \ldots, a_k\} : L^\otimes k+1 \to L, \quad k \geq 0,$$

satisfying the following conditions.

1. The operation $\{a_0, \ldots, a_k\}$ is graded symmetric: for all $1 \leq i \leq k$,

$$\{a_0, \ldots, a_{i-1}, a_i, \ldots, a_k\} = (-1)^{|a_{i-1}||a_i|}\{a_0, \ldots, a_i, a_{i-1}, \ldots, a_k\}.$$

2. The operation $\{a_0, \ldots, a_k\}$ has degree $-1$:

$$|\{a_0, \ldots, a_k\}| = |a_0| + \cdots + |a_k| - 1.$$
(3) For each $n \geq 0$, the $n$th Jacobi rule holds:

\[
\sum_{k=0}^{n} \sum_{\begin{subarray}{c} I=\{i_0<\ldots<i_k\} \\ J=\{j_1<\ldots<j_{n-k}\} \\ I\cup J=\{0,\ldots,n\}\end{subarray}} (-1)^{\varepsilon} \{\{a_{i_0}, \ldots, a_{i_k}\}, a_{j_1}, \ldots, a_{j_{n-k}}\} = 0.
\]

Here, $(-1)^{\varepsilon}$ is the sign associated by the Koszul sign convention to the action of $\pi$ on the elements $(a_0, \ldots, a_n)$ of $L$.

Let $L$ be an $L_\infty$-algebra. By the 0th Jacobi rule

\[
\{\{a\}\} = 0,
\]

the operation $x \mapsto \{x\}$ is seen to give the graded vector space $L$ the structure of a chain complex.

**Definition 2.** A Lie $n$-algebra is an $L_\infty$-algebra $L$ concentrated in degrees $[1, \ldots, n]$:

\[
0 \longrightarrow L_n \xrightarrow{\delta} \ldots \xrightarrow{\delta} L_1 \longrightarrow 0.
\]

In particular, a Lie 1-algebra is just a vector space $L = L_1$ with a Lie bracket $\{a, b\}$.

**Theorem 3.** Let $L$ be a differential graded Lie algebra, with differential $\delta$ and bracket $[a, b]$. Let $D$ be the operator on $L$ which equals $\delta$ on $L_1$, and vanishes in other degrees. Let $L$ be the positively graded chain complex $L_i = \left\{ \begin{array}{ll} L_i, & i > 0, \\ 0, & i \leq 0. \end{array} \right.$

Then $L$ is an $L_\infty$-algebra, with brackets

\[
\{a\} = \left\{ \begin{array}{ll} \delta a, & |a| > 1, \\ 0, & |a| = 1, \end{array} \right.
\]

and, for $n > 0$,

\[
\{a_0, \ldots, a_n\} = b_n \sum_{\pi \in S_{n+1}} (-1)^{\varepsilon}[\ldots [[Da_{\pi_0}, a_{\pi_1}], a_{\pi_2}], \ldots, a_{\pi_n}],
\]

Here, $(-1)^{\varepsilon}$ is the sign associated to the action of the permutation $\pi$ on the tensor product

\[
a_0 \otimes \ldots \otimes a_n \in L^{\otimes n+1}
\]

by the Koszul sign convention, and

\[
b_n = \frac{(-1)^n B_n}{n!}.
\]

If $L_i = 0$ for $i > n$, then $L$ is a Lie $n$-algebra.
We have the following explicit formulas for the first two brackets of this $L_\infty$-structure:
\[
\{a_0, a_1\} = \frac{1}{2} \left( [Da_0, a_1] - (-1)^{|a_0|} [a_0, Da_1] \right),
\]
\[
\{a_0, a_1, a_2\} = \frac{1}{12} \left( [[Da_0, a_1], a_2] - (-1)^{|a_1|} [[a_0, Da_1], a_2]
+ (-1)^{|a_0|(|a_1|+|a_2|)} [[Da_1, a_2], a_0] - (-1)^{|a_0|(|a_1|+|a_2|)+|a_1|} [[a_1, Da_2], a_0]
+ (-1)^{(|a_0|+|a_1|)|a_2|} [[Da_2, a_0], a_1] - (-1)^{(|a_0|+|a_1|)|a_2|+|a_2|} [[a_2, Da_0], a_1] \right).
\]

In the case where $L^i$ for $i > 2$, this theorem is due to Roytenberg and Weinstein \[3\]. They formulate their results in the setting of Courant algebroids, but their approach goes through with no change at all for any differential graded algebra vanishing above degree 2. Our proof in essence generalizes the direct calculations of Roytenberg and Weinstein.

Bernoulli numbers first arose in the study of differential graded Lie algebras in the work of Ran \[5\], whose results have been considerably clarified by Fiorenza and Manetti \[1\]. After the appearance of an earlier version of this note, D. Calaque observed to the author that Theorem 3 is a corollary of the main result of \[1\]. Actually, our methods would also yield a direct proof of the theorem of Fiorenza and Manetti.

Fiorenza and Manetti prove that if $\phi : K \to L$ is a morphism of differential graded Lie algebras, then there is a natural $L_\infty$-structure on the mapping cone $C_\phi[-1] = K[-1] \oplus L$. (Our conventions for $L_\infty$-algebras differ from theirs by a shift in degree of 1; in their paper, they use $C_\phi = K \oplus L[1]$.)

Their construction departs from the differential graded Lie algebra
\[
C_\phi = \{(x, f(t) + g(t) \, dt) \in K \oplus L[t, dt] \mid f(0) = 0, f(1) = x\}.
\]
Here, $L[t, dt]$ is the module over the free differential graded algebra $C[t, dt]$ generated by $t$ and its differential $dt$: that is, it is the space of $L$-valued differential forms on the affine line.

There is an inclusion of $C_\phi$ into $C_\phi$, which sends $(x, a) \in K \oplus L[1]$ to $(x, a \, dt)$. There is a chain homotopy $h$ on the complex $C_\phi$ which yields a contraction to the subcomplex $C_\phi$:
\[
h(x, f(t) + g(t) \, dt) = (0, \int_0^t g(s) \, ds - t \int_0^1 g(s) \, ds + x).
\]
This contraction induces an $L_\infty$-structure on $C_\phi[-1]$ in a standard way, via homological perturbation theory. The brackets for this structure are sums over binary trees, and a calculation involving Bernoulli polynomials yields the following explicit formulas: for $x, y \in K[-1]$ and $a_i \in L$,
\[
\{a\} = \delta a,
\]
\[
\{x\} = \phi(x) - \delta x,
\]
\[
\{x, y\} = (-1)^{|x|} [x, y],
\]
\[
\{x, a_1, \ldots, a_n\} = b_n \sum_{\pi \in S_n} (-1)^{\varepsilon} \left[\big[\ldots \left[ x, a_{\pi_1} \right], a_{\pi_2} \right], \ldots, a_{\pi_n} \right],
\]
while all other brackets vanish.

Let $\phi$ be the inclusion of $K = \tau_{\leq 0} L$ into $L$. There is a natural quasi-isomorphism from $\mathbb{L} = \tau_{>0} L$ to the mapping cone $C_\phi[-1]$, which sends $a$ to $(Da, a)$. The $L_\infty$-structure thereby induced on $\mathbb{L}$ is identical to the one in our theorem.
We now present our direct proof of Theorem 3, which relies on the following lemma. The proof is a straightforward application of the graded Jacobi relation for $L$.

**Lemma 4.** For $j, k \geq 0$, and $j + k < n$, consider the expression

$$Z_{n,j,k} = \sum_{\pi \in S_{n+1}} (-1)^{\varepsilon + |a_1| + \cdots + |a_j|}$$

$$\ldots \left[ \ldots [Da_{\pi_0}, a_{\pi_2}], \ldots, a_{\pi_2}], \ldots [Da_{\pi_j+1}, a_{\pi_j+2}], \ldots, a_{\pi_{j+1+k}}] \ldots \right], a_{\pi_n}],$$

Then $Z_{n,j,k} = Z_{n,k,j}$ and if $j + k + 1 < n$, $Z_{n,j,k} = Z_{n,j+1,k} + Z_{n,j,k+1}$.

**Corollary 5.** The expression

$$F = \sum_{i,j} a_{ij} Z_{n,i,j}$$

vanishes if $f(s,t) + f(t,s)$ lies in the ideal generated by $s + t = 1$, where $f$ is the polynomial in two variables

$$f(s,t) = \sum_{i,j} a_{ij} s^i t^j \in \mathbb{C}[s,t].$$

**Proof (of Theorem 3).** The cases $n = 0$ and $n = 1$ of the Jacobi rule Eq. (1) are easily checked directly, so from now on, we assume that $n > 1$.

The contribution of the terms with $k = 0$ and $k = n$ to the $n$th Jacobi rule is

$$\{ \{a_0, \ldots, a_n\} \} + \sum_{i=0}^{n} (-1)^{|a_0| + \cdots + |a_{i-1}|} \{ \{a_0, \ldots, a_i\}, \ldots, a_n\}$$

$$= b_n \sum_{\pi \in S_{n+1}} (-1)^{\varepsilon} \left\{ \delta[\ldots [Da_{\pi_0}, a_{\pi_2}], \ldots, a_{\pi_2}], a_{\pi_n}] \right. - \sum_{i=1}^{n} (-1)^{|a_{\pi_1}| + \cdots + |a_{\pi_{i-1}}|} \left[ \ldots [Da_{\pi_0}, a_{\pi_1}], \ldots, (\delta - D)a_{\pi_i}], \ldots, a_{\pi_n}] \right\}$$

$$= b_n \sum_{i=0}^{n-1} Z_{n,i,0}.$$  

Here, we have used that $Da_{\pi_0}$ vanishes unless $|a_{\pi_0}| = 1$: this is the source of the minus sign on the third line.

Next, we calculate the contribution of the terms with $k = 1$:

$$\sum_{\substack{\{i_0 < i_1\} \\
J = \{j_1 < \ldots < j_{n-1}\} \\
I \cup J = \{0, \ldots, n\}} \} (-1)^{\varepsilon} \{ \{a_{i_0}, a_{i_1}\}, a_{j_1}, \ldots, a_{j_{n-1}}\}$$

$$= b_1 b_{n-1} \left( \sum_{\pi \in S_{n+1}} (-1)^{\varepsilon} \left[ \ldots [Da_{\pi_0}, a_{\pi_2}], \ldots, a_{\pi_n}] - \sum_{i=0}^{n-2} Z_{n,i,1} \right] \right)$$

$$= b_1 b_{n-1} \left( Z_{n,0,0} - \sum_{i=0}^{n-2} Z_{n,i,1} \right).$$
When $n = 2$, we see that the Jacobi identity becomes
\[ b_2 Z_{2,0,0} + b_2 Z_{2,1,0} + b_1^2 Z_{2,0,0} - b_1^2 Z_{2,0,1} = 0. \]
Here, $b_2 = \frac{1}{12}$, and $b_1 = \frac{1}{2}$, while $Z_{2,0,0} = 2Z_{2,1,0}$ by Lemma 4, and so the whole expression does indeed sum to 0.

At last, we calculate the contribution of the terms with $1 < k < n$ to Eq. (??):
\[
\sum_{\substack{I = \{i_0, \ldots, i_k\} \subset \{0, \ldots, n\} \setminus \{i_{k+1}, \ldots, n\} \\
J = \{j_1, \ldots, j_{n-k}\} \subset \{0, \ldots, n\} \setminus \{i_0, \ldots, i_k\} \setminus \{j_{k+1}, \ldots, n\} \\
I \cup J = \{0, \ldots, n\}} (-1)^{\varepsilon} \{\{a_{i_0}, \ldots, a_{i_k}\}, a_{j_1}, \ldots, a_{j_{n-k}}\} = -b_k b_{n-k} \sum_{j=0}^{n-k} Z_{n,j,k-1} \\
= b_k b_{n-k} (Z_{n,n-k,k-1} - Z_{n,0,k-1}).
\]

When $n$ is odd, only $k = 1$ and $k = n - 1$ contribute to the Jacobi identity, which becomes
\[ b_1 b_{n-1} (Z_{n,0,0} - \sum_{i=0}^{n-3} Z_{n,i,1} - Z_{n,0,n-2}) = 0. \]
This identity holds by Corollary 5.

\[ f(s, t) = b_1 b_{n-1} (1 - \sum_{i=0}^{n-3} s^i t - t^{n-2}) \\
= 1 - \left( \frac{1 - s^{n-2}}{1 - s} \right) t - t^{n-2}. \]
This polynomial is congruent to $s^{n-2} - t^{n-2}$ modulo $s + t - 1$, and hence satisfies the necessary condition of Corollary 5.

When $n > 2$ is even, the identity becomes
\[ b_n \sum_{i=0}^{n-1} Z_{n,i,0} + \sum_{k=2}^{n-2} b_k b_{n-k} (Z_{n,0,k-1} - Z_{n,n-k,k-1}) = 0. \]
Replacing $Z_{n,i,j}$ by $s^i t^j$, we obtain the power series
\[ b_n \sum_{i=0}^{n-1} s^i + \sum_{k=2}^{n-2} b_k b_{n-k} (t^{k-1} - s^{n-k} t^{k-1}). \]
Modulo $s + t - 1$, this is congruent to
\[ \sum_{k=0}^{n} b_k b_{n-k} (1 - s^{n-k}) t^{k-1}. \]
Multiplying by $x^n$ and summing over $n$, we obtain
\[ A(s, t, x) = (f(x) - f(sx)) f(tx)/t, \]
where
\[ f(x) = 1 + \sum_{n=2}^{\infty} \frac{B_n x^n}{n!} = \frac{x/2}{\tanh(x/2)} = \frac{x}{e^x - 1} + \frac{x}{2}. \]
By Corollary 5, we see that the identity will follow if $A(s, t, x) + A(t, s, x)$, lies in the ideal generated by $s + t - 1$ in $(st)^{-1} \mathbb{Q}[s, t][[x]]$ up to terms of degree 2 in $x$. But

$$A(s, t, x) + A(t, s, x) = \frac{x^2}{4} \left\{ 1 + \frac{4e^x(e^{(s-1)x} - 1)}{(e^x - 1)(e^{sx} - 1)(e^{tx} - 1)} - (s + t - 1)\left(\frac{e^{sx} + 1}{e^{sx} - 1}\frac{e^{tx} + 1}{e^{tx} - 1}\right)\right\},$$

proving the theorem.

□

REFERENCES

[1] Fiorenza, D., and Manetti, M.: $L_\infty$ structures on mapping cones, Algebra Number Theory 1, 301-330 (2007). arXiv:math/0601312
[2] Koszul,J.-L.: Crochet de Schouten-Nijenhuis et cohomologie, In: “The mathematical heritage of Élie Cartan (Lyon, 1984),” Astérisque, Numero Hors Serie, 257-271 (1985).
[3] Ran, Z.: Lie atoms and their deformations, Geom. Funct. Anal. 18, 184-221, (2008). arXiv:math/0412204
[4] Roytenberg, D., and Weinstein, A.: Courant algebroids and strongly homotopy Lie algebras, Lett. Math. Phys. 46, 81-93 (1998). arXiv:math/9802118