Research Article

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An admissible Hybrid contraction with an Ulam type stability

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Abstract: In this manuscript, we introduce a new hybrid contraction that unify several nonlinear and linear contractions in the set-up of a complete metric space. We present an example to indicate the genuine of the proved result. In addition, we consider Ulam type stability and well-posedness for this new hybrid contraction.

Keywords: admissible mappings, hybrid contractions, fixed point, metric space

MSC: 47H10, 54H25, 46J10

1 Introduction and preliminaries

In the last three-four decades, there is a blown out in the number of publications in metric fixed point theory. This fact forces researchers to find a way to combine, unify and merge the existing results in a proper way. In this paper, we aim to give an interesting example for this trend. We introduce a new hybrid contraction which not only combine and unify the several existing linear and nonlinear contractions but also extend these results.

Let \( \Psi \) be the set of functions \( \psi : [0, \infty) \to [0, \infty) \) such that

\[
\begin{align*}
\Psi_1 & : \psi \text{ is non-decreasing}; \\
\Psi_2 & : \text{there are } i_0 \in \mathbb{N} \text{ and } \delta \in (0, 1) \text{ and a convergent series } \sum_{i=1}^{\infty} v_i \text{ such that } v_i \geq 0 \text{ and } \\
\Psi_i & (t) \leq \delta \psi_i (t) + v_i,
\end{align*}
\]

for \( i \geq i_0 \) and \( t \geq 0 \).

Each \( \psi \in \Psi \) is called a \((c)\)-comparison function (see [1, 2]).

Lemma 1.1. [1] If \( \psi \in \Psi \), then

(i) \( \psi^n (t) \) converges to 0 as \( n \to \infty \) for \( t \geq 0 \);
(ii) \( \psi (t) < t \), for any \( t \in \mathbb{R}^+ \);
(iii) \( \psi \) is continuous at 0;
(iv) the series \( \sum_{k=1}^{\infty} \psi^k (t) \) is convergent for \( t \geq 0 \).

Let \( \alpha : X \times X \to [0, \infty) \) be a function. We say that a mapping \( f : X \to X \) is \( \alpha \)-orbital admissible ([3]) if

\[
\alpha \big( x, f(x) \big) \geq 1 \Rightarrow \alpha \big( f(x), f^2(x) \big) \geq 1, \ \forall x \in X.
\]
An \( \alpha \)-orbital admissible mapping \( f \) is called triangular \( \alpha \)-orbital admissible ([3]) if

\[
a(\alpha, y) \geq 1 \quad \text{and} \quad a(y, f y) \geq 1 \quad \Rightarrow \quad a(\alpha, y) \geq 1,
\]

for every \( \alpha, y \in X \).

**Lemma 1.2.** Suppose that for a triangular \( \alpha \)-orbital admissible mapping \( f : X \to X \) there exists \( x_0 \in X \) such that \( a(x_0, f x_0) \geq 1 \). Then

\[
a(x_n, x_m) \geq 1, \quad \text{for all} \quad n, m \in \mathbb{N},
\]

where the sequence \( \{x_n\} \) is defined by \( x_{n+1} = fx_n, n \in \mathbb{N} \).

**Definition 1.3.** Let \( \alpha : X \times X \to [0, \infty) \) be a mapping. The set \( X \) is called regular with respect to \( \alpha \) if for a sequence \( \{x_n\} \) in \( X \) such that \( a(x_n, x_{n+1}) \geq 1 \), for all \( n \) and \( x_0 \to x \in X \) as \( n \to \infty \) we have \( a(x_0, x) \geq 1 \) for all \( n \).

### 2 Main results

We start with a definition of a new notion, namely "admissible hybrid contraction":

**Definition 2.1.** Let \( (X, d) \) be a metric space. A self-mapping \( f \) is called an admissible hybrid contraction, if there exist \( \psi \in \Psi \) and \( \alpha : X \times X \to [0, \infty) \) such that

\[
a(\alpha, y)d(f x, y) \leq \psi \left( \Re_\alpha^d(\alpha, y) \right),
\]

where \( q \geq 0 \) and \( \lambda_i \geq 0, i = 1, 2, 3, 4, 5 \) such that \( \sum_{i=1}^{5} \lambda_i = 1 \) and

\[
\Re_\alpha^d(\alpha, y) = \begin{cases} 
\left[ \lambda_1 d^4(\alpha, y)(\alpha, y) + \lambda_2 d^4(\alpha, f \alpha) + \lambda_3 d^4(\alpha, f y) + \lambda_4 \left( \frac{d(f y, f \alpha)(1 + d(y, f \alpha))}{1 + d(\alpha, y)} \right)^q \right]^{\frac{1}{q}}, & \text{for} \quad q > 0, \alpha, y \in X \\
\left[ d(\alpha, y)^{\lambda_4} \cdot (d(\alpha, f \alpha))^{\lambda_2} \cdot (d(y, f y))^{\lambda_3} \cdot \frac{d(f y, f \alpha)(1 + d(y, f \alpha))}{1 + d(\alpha, y)} \right]^{\lambda_4} \cdot \frac{d(f y, f \alpha)(1 + d(y, f \alpha))}{2}, & \text{for} \quad q = 0, \alpha, y \in X \setminus \text{Fix}_f(X) 
\end{cases}
\]

(Here \( \text{Fix}_f(X) = \{ x \in X : f x = x \} \).)

The concept of "admissible hybrid contraction" is inspired from the notion of "interpolative contractions", see e.g. [4–9]. The main results of this manuscript is the following theorem:

**Theorem 2.2.** Let \( (X, d) \) be a complete metric space and let \( f \) be an admissible hybrid contraction, Suppose also that:

(i) \( f \) is triangular \( \alpha \)-orbital admissible;
(ii) there exists \( x_0 \in X \) such that \( a(x_0, f x_0) \geq 1 \);
(iii) either, \( f \) is continuous, or
(iv) \( f^2 \) is continuous and \( a(f \alpha, \alpha) \geq 1 \) for any \( \alpha \in \text{Fix}_f(X) \).

Then \( f \) has a fixed point.

**Proof.** Starting from an arbitrary point \( x_0 \) in \( X \) we recursively set-up the sequence \( \{x_n\} \), as \( x_{n+1} = f x_n \) for all \( n \in \mathbb{N} \). Supposing that there exists some \( m \in \mathbb{N} \) such that \( f^n x_0 = x_m \), we find that \( x_m \) is a fixed point of \( f \) and the proof is finished. So, we can presume from now on that \( x_n \neq x_{n-1} \) for any \( n \in \mathbb{N} \). Under the assumption (i), \( f \) is admissible hybrid contraction, if we substituting in (5) \( x \) by \( x_0 \) and \( y \) by \( x_0 \) we get

\[
a(x_{n+1}, x_n) \geq 1 \quad \Rightarrow \quad a(x_{n+1}, x_n) \geq 1.
\]
Taking into account that $f$ is triangular $\alpha$–orbital admissible, together with (4) holds and the above inequality becomes
\[ d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n) d(f x_{n-1}, f x_n) < \psi(\mathcal{R}^2(x_{n-1}, x_n)). \] (8)

**Case 1.** For the case $g > 0$ we have
\[
\mathcal{R}^2(x_{n-1}, x_n) = [\lambda_1 d^2(x_{n-1}, x_n) + \lambda_2 d^2(x_{n-1}, f x_{n-1}) + \lambda_3 d^2(x_n, f x_n) + \lambda_4 (\frac{d(\psi, f x_n)(1 + d(x_{n-1}, f x_n))}{1 + d(x_{n-1}, x_n)})^q]
+ \lambda_5 \left( \frac{d(\psi, f x_n)(1 + d(x_{n-1}, f x_n))}{1 + d(x_{n-1}, x_n)} \right) \right]^{1/q}
= [\lambda_1 d^2(x_{n-1}, x_n) + \lambda_2 d^2(x_{n-1}, x_n) + \lambda_3 d^2(x_n, x_{n+1}) + \lambda_4 (d(x_n, x_{n+1}))^q]
+ \lambda_5 \left( \frac{d(\psi, f x_n)(1 + d(x_{n-1}, f x_n))}{1 + d(x_{n-1}, x_n)} \right) \right]^{1/q}
= [(\lambda_1 + \lambda_2) d^2(x_{n-1}, x_n) + (\lambda_3 + \lambda_4) d^2(x_n, x_{n+1})]^{1/q},
\]
and from (8) we get
\[
d(x_n, x_{n+1}) < \alpha(x_{n-1}, x_n) d(f x_{n-1}, f x_n)
< \psi(\mathcal{R}^2(x_{n-1}, x_n))
= \psi([(\lambda_1 + \lambda_2) d^2(x_{n-1}, x_n) + (\lambda_3 + \lambda_4) d^2(x_n, x_{n+1})]^{1/q}).
\] (9)

If we suppose that $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$, since $\psi$ is a nondecreasing function,
\[
d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n) d(f x_{n-1}, f x_n)
\leq \psi([(\lambda_1 + \lambda_2) d^2(x_{n-1}, x_n) + (\lambda_3 + \lambda_4) d^2(x_n, x_{n+1})]^{1/q})
\leq \psi([(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) d^2(x_n, x_{n+1})]^{1/q})
\leq \psi(x_{n-1}, x_n)) \leq (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) d^2(x_n, x_{n+1})
\leq d(x_n, x_{n+1}),
\]
which is a contradiction. Therefore, for every $n \in \mathbb{N}$ we have
\[ d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \]
and the inequality (8) yields
\[
d(x_n, x_{n+1}) \leq \psi([(\lambda_1 + \lambda_2) d^2(x_{n-1}, x_n) + (\lambda_3 + \lambda_4) d^2(x_n, x_{n+1})]^{1/q})
\leq \psi([(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) d^2(x_n, x_{n+1})]^{1/q})
\leq \psi(x_{n-1}, x_n)) \leq \psi^2(d(x_{n-2}, x_{n-1}))
\ ...
\leq \psi^n(d(x_0, x_1)).
\] (11)

Let now, $m, p \in \mathbb{N}$ such that $p > m$. By the triangle inequality and since $d(x_m, x_{m+1}) < \psi^m(d(x_0, x_1))$ for any $m \in \mathbb{N}$, we have
\[
d(x_m, x_p) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{p-1}, x_p)
= \sum_{j=m}^{p-1} d(x_j, x_{j+1}) \leq \sum_{j=0}^{p-1} \psi(d(x_0, x_1)).
\]

Since $\psi$ is a $c$–comparison function the series $\sum_{j=0}^{\infty} \psi(d(x_0, x_1))$ is convergent, so that, denoting by $S_n = \sum_{j=0}^{n} \psi(d(x_0, x_1))$ the above inequality becomes:
\[
d(x_m, x_p) \leq S_{p-1} - S_{m-1}.
\]
and as \( m, p \to \infty \) we get
\[
d(x_0, x_p) \to 0,
\]
which tells us that \( \{x_n\} \) is a Cauchy sequence on a complete metric space, so that, there exists \( z \) such that
\[
\lim_{n \to \infty} d(x_n, z) = 0. \tag{13}
\]
We will prove that this point \( z \) is a fixed point of \( f \). If \( f \) is continuous, (due to assumption (iii))
\[
\lim_{n \to \infty} d(x_{n+1}, f z) = \lim_{n \to \infty} d(x_0, f x_n) = 0,
\]
so, we get that \( f z = z \), that is, \( z \) is a fixed point of \( f \).

In the alternative hypothesis, that \( f^2 = \lim_{n \to \infty} f^2 x_n = z \) and we want to show that \( f z = z \). Supposing that, on the contrary, \( f z \neq z \), we have from (5)
\[
d(f z, z) = d(f^2 z, f z) \leq d(f z, z) d(f z, z) \\
\leq \psi(R_f^0(f z, z)) < R_f^0(f z, z)
\]
\[
= [\lambda_1 d^4(f z, z) + \lambda_2 d^4(f z, f z) + \lambda_3 d^4(z, f z) + \lambda_4 \left( \frac{d(z, f z)(1 + d(f z, f z))}{1 + d(f z, z)} \right)^{1 + \lambda_5} + \lambda_5 \left( \frac{d(z, f z)(1 + d(f z, f z))}{1 + d(f z, z)} \right)^{1 + \lambda_4}]
\]
\[
= \left( [\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4] d(f z, z) \right)^{1 + \lambda_5}
\]
\[
\leq \lambda_5 d(f z, z).
\]
This is a contradiction, so that \( f z = z \).

**Case 2.** For the case \( q = 0 \) taking \( x = x_{n-1} \) and \( y = x_n \) we have
\[
R_f^q(x_{n-1}, y_n) = [d(x_{n-1}, x_n)]^{\lambda_5} \cdot [d(x_{n-1}, f x_{n-1})]^{\lambda_5} \cdot [d(x_n, f x_n)]^{\lambda_4} \cdot \left[ \frac{d(x_n, f x_n)(1 + d(x_{n-1}, f x_{n-1}))}{1 + d(x_n, f x_n)} \right]^{\lambda_5}
\]
\[
\leq [d(x_{n-1}, x_n)]^{\lambda_5} \cdot [d(x_{n-1}, f x_{n-1})]^{\lambda_5} \cdot [d(x_n, f x_n)]^{\lambda_4} \cdot \left[ \frac{d(x_n, f x_n)(1 + d(x_{n-1}, f x_{n-1}))}{1 + d(x_n, f x_n)} \right]^{\lambda_5}
\]
\[
\leq [d(x_{n-1}, x_n)]^{\lambda_5} \cdot [d(x_{n-1}, f x_{n-1})]^{\lambda_5} \cdot [d(x_n, f x_n)]^{\lambda_4} \cdot \left[ \frac{d(x_n, f x_n)(1 + d(x_{n-1}, f x_{n-1}))}{1 + d(x_n, f x_n)} \right]^{\lambda_5}
\]
\[
\leq [d(x_{n-1}, x_n)]^{\lambda_5} \cdot [d(x_n, f x_n)]^{\lambda_4} + [d(x_{n-1}, f x_{n-1})]^{\lambda_4} \cdot \left[ \frac{d(x_n, f x_n)(1 + d(x_{n-1}, f x_{n-1}))}{1 + d(x_n, f x_n)} \right]^{\lambda_5}
\]
and from (5)
\[
d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n) d(f x_{n-1}, f x_n) \leq \psi(R_f^q(x_{n-1}, y_n)). \tag{14}
\]
As in the first case, we have that \( d(x_{n-1}, x_n) > d(x_n, x_{n+1}) \) since in the contrary case we have a contradiction. Indeed, if we suppose \( \text{ad absurdum} \) that \( d(x_{n-1}, x_n) \leq d(x_n, x_{n+1}) \), we have
\[
d(x_n, x_{n+1}) < \psi(R_f^q(x_{n-1}, y_n)) < [d(x_n, x_{n+1})]^{\lambda_5} \cdot [d(x_{n-1}, f x_{n-1})]^{\lambda_5} \cdot [d(x_{n-1}, f x_{n-1})]^{\lambda_4} \cdot \left[ \frac{d(x_n, f x_n)(1 + d(x_{n-1}, f x_{n-1}))}{1 + d(x_n, f x_n)} \right]^{\lambda_5}
\]
which is a contradiction. Then from (14) we obtain
\[
d(x_n, x_{n+1}) \leq \psi(R_f^q(x_{n-1}, y_n)) < \psi(d(x_{n-1}, x_n))
\]
and inductively we get
\[
d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)). \tag{15}
\]
By using the same arguments as the case \( q > 0 \) we shall easily obtain that \( \{x_n\} \) is a Cauchy sequence in a complete metric space and so, there exists \( z \) such that \( \lim_{n \to \infty} x_n = z \).
We claim that $z$ is a fixed point of $f$.
Under the assumption that $f$ is continuous we have
\[ \lim_{n \to \infty} d(x_{n+1}, f z) = \lim_{n \to \infty} d(f x_n, f z) = 0, \]
and together with the uniqueness of limit, $f z = z$. Also, if $f^2$ is continuous, as in case (1) we have that $f z = z$ and then
\[
d(z, f z) = d(f^2 z, f z) \leq a(f z, z)d(f^2 z, f z) \leq \psi(qf^2(f z, f z)) \leq \psi(d(z, f z))^{1 + \lambda_1 + \lambda_3 + \lambda_4 + k_1} < d(z, f z).
\]
This contradiction shows us that $z = f z$.

**Example.** Let $X = [0, 2]$, $d : X \times X \to [0, \infty)$ be the usual metric, $d(x, y) = |x - y|$ for all $x, y \in X$ and the mapping $f : X \to X$ be defined by $f(x) = \begin{cases} 2/3, & \text{if } x \in [0, 1] \\ x/2, & \text{if } x \in (1, 2] \end{cases}$. Consider also a function $a(x, y) = \begin{cases} 2, & \text{if } x, y \in [0, 1] \\ 1, & \text{if } x = 0, y = 2 \end{cases}$ and the comparison function $\psi : [0, \infty) \to [0, \infty)$, $\psi(t) = t/5$. We can easily observe that the assumptions (i) and (ii) are satisfied and since $f^2(x) = 2/3$ is continuous, the assumption (iv) is also verified. For any $x, y \in [0, 1]$ we have $d(f x, f y) = 0$ so, the inequality (5) holds. For $x = 0$ and $y = 2$, we have
\[
a(0, 2)d(f(0), f(2)) = a(0, 2)d(2/3, 1) = \frac{1}{3} \sqrt{\frac{505}{81}} = \frac{1}{3} \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9} + \frac{3}{9}} = \frac{1}{3} \left[\frac{1}{2} d^2(0, 2) + \frac{1}{6} d^2(0, 0) + \frac{1}{6} d^2(2, 2) + \frac{1}{6} (\frac{d(2, 0)(1 + d(0, 0))}{1 + d(0, 2)})^2\right]^{1/2}.
\]
In all other cases, $a(x, y) = 0$ and (5) is obviously satisfied. Thus, letting $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = \frac{1}{4}$, $\lambda_4 = 0$ and $q = 2$ we obtain that $f$ is an admissible hybrid contraction which satisfies the assumptions (i), (ii), (iv) of Theorem 2.2 and then $x = 0$ is the fixed point of $f$.

**Theorem 2.3.** Let $(X, d)$ be a complete metric space and let $f$ be an admissible hybrid contraction. Suppose also that:

1. $f$ is triangular $a$–orbital admissible;
2. there exists $x_0 \in X$ such that $a(x_0, f x_0) \geq 1$;
3. $(X, d)$ is regular with respect to $a$.

Then $f$ possesses a fixed point.

**Proof.** Following the lines in the proof of Theorem 2.2, we already know that for any $q \geq 0$, the sequence $\{x_n\}$ is Cauchy, and due to the completeness of the metric space $(X, d)$, there exists a point $z$ such that $\lim_{n \to \infty} d(x_n, z) = 0$. Since the space $X$ is regular with respect to $a$, inequality (5) together with the triangular inequality gives us
\[
d(z, f z) = d(z, x_{n+1}) + d(x_{n+1}, f z) \leq a(x_n, z)d(f x_n, f z) \leq \psi(qf^2(x_n, z)) \leq R_f^2(x_n, z).
\]
Again, we have to consider two separate cases. For the case $p > 0$,
\[
R_f^2(x_n, z) = \left[\lambda_1 d^2(x_n, z) + \lambda_2 d^2(x_n, f x_n) + \lambda_3 d^2(z, f z) + \lambda_4 \left(\frac{d(z, f z)(1 + d(f x_n, f z))}{1 + d(x_n, z)}\right)^q\right]^{1/2} + \lambda_5 \left(\frac{d(z, f z)(1 + d(f x_n, f z))}{1 + d(x_n, z)}\right)^q.
\]
Since $\lim_{n \to \infty} R_f^2(x_n, z) = (\lambda_3 + \lambda_4)d(z, f z)$, letting $n \to \infty$ in (16) we obtain $d(z, f z) < d(z, f z)$ which implies that $f z = z$.
Similarly, for the case $q = 0$, we get $\lim_{n \to \infty} R_f^2(x_n, z) = 0$ and then $d(z, f z) = 0$. 

\[\square\]
Corollary 2.4. Let \((X, d)\) be a complete metric space and the functions \(\psi \in \Psi\) and \(\alpha : X \times X \to [0, \infty)\). Let \(f\) be a self map on \(X\) such that:

(i) \(f\) is triangular \(\alpha\)-orbital admissible;

(ii) there exists \(x_0 \in X\) such that \(\alpha(x_0, f(x_0)) \geq 1\);

(iii) either, \(f\) is continuous, or

(iv) \(f^2\) is continuous and \(\alpha(f^2(x, x), x) \geq 1\) for any \(x \in \text{Fix}_f(X)\).

If one of the below conditions \((c_1)-(c_3)\) is satisfied, then \(f\) has a fixed point \(z \in X\), that is, \(fz = z\).

\((c_1)\) \(\alpha(x, y)d(x, y) \leq \psi(\lambda f(x, y)), \) where \(a_1, a_2, a_3, a_4\) are non-negative real such that \(a_1 + a_2 + a_3 + a_4 = 1\) and

\[
\lambda_q^f(x, y) = \begin{cases} 
[ a_1 d^f(x, y) + a_2 d^f(x, f x) + a_3 d^f(y, f y) + a_4 \left( \frac{d(f(x), f(y))(1 + d(x, y))}{1 + d(x, y)} \right) ]^{1/2}, & \text{for } q > 0, x, y \in X \\
[d(x, y)]^{a_1} \cdot [d(x, f x)]^{a_2} \cdot [d(y, f y)]^{a_3} \cdot \left( \frac{d(f(x), f(y))(1 + d(x, y))}{1 + d(x, y)} \right)^{\frac{a_4}{2}}, & \text{for } q = 0, x, y \in X \setminus \text{Fix}_f(X) 
\end{cases}
\]  

\((c_2)\) \(\alpha(x, y)d(x, y) \leq \psi(\beta f(x, y)), \) where \(b_1, b_2, b_3\) are non-negative real such that \(b_1 + b_2 + b_3 = 1\) and

\[
\beta_q^f(x, y) = \begin{cases} 
[b_1 d^f(x, y) + b_2 d^f(x, f x) + b_3 d^f(y, f y)]^{1/2}, & \text{for } q > 0, x, y \in X \\
[d(x, y)]^{b_1} \cdot [d(x, f x)]^{b_2} \cdot [d(y, f y)]^{b_3}, & \text{for } q = 0, x, y \in X \setminus \text{Fix}_f(X) 
\end{cases}
\]  

\((c_3)\) \(\alpha(x, y)d(x, y) \leq \psi(\gamma f(x, y)), \) where \(c_1, c_2\) are non-negative real numbers such that \(c_1 + c_2 = 1\) and

\[
\gamma_q^f(x, y) = \begin{cases} 
[c_1 d^f(x, f x) + c_2 d^f(y, f y)]^{1/2}, & \text{for } q > 0, x, y \in X \\
[d(x, f x)]^{c_1} \cdot [d(y, f y)]^{c_2}, & \text{for } q = 0, x, y \in X \setminus \text{Fix}_f(X) 
\end{cases}
\]  

We can get a series of corollaries, considering in Corollary 2.4 by assigning \(\psi \in \Psi\) properly, for example, by taking \(\psi(t) = kt\) for any \(t \geq 0\) with \(k \in [0, 1)\), and/or \(\alpha(x, y) = 1\) or both. Since it is apparent we skip the details.

**Theorem 2.5.** If in Theorems 2.2 and 2.3, in the case \(q > 0\), we assume supplementary that

\[
\alpha(x, y) \geq 1
\]

for any \(x, y \in \text{Fix}_f(X)\) then the fixed point of \(f\) is unique.

**Proof.** Let \(v \in X\) be another fixed point of \(f\), different from \(z\). By replacing in (5), and taking into account the additional hypotheses, we have

\[
d(z, v) \leq \alpha(z, v)(f z, f v) \leq \psi(\lambda f(z, v)) < \psi(\lambda f(z, v)) = \frac{\lambda_1 d^f(x, v)\lambda_2 d^f(z, x) + \lambda_3 d^f(v, f v) + \lambda_4 \left( \frac{d(v, f v)(1 + d(z, v))}{1 + d(z, v)} \right)}{1 + d(z, v)}^{1/2}
\]

which is a contradiction. Thus, \(z = v\), so that \(f\) possesses exactly one fixed point. \(\square\)

**Example.** Let \(X = \{a, b, c, d\}\) and \(d : X \times X \to [0, \infty)\) such that \(d(x, y) = d(y, x), d(x, x) = 0\) for any \(x, y \in X\) and

\[
d(x, y) = \begin{cases} 
1, & \text{if } (x, y) \in \{(a, a), (b, b), (c, c), (d, d)\} \\
2, & \text{if } (x, y) \in \{(a, c), (b, d)\} \\
3, & \text{if } (x, y) \in \{(a, d)\}
\end{cases}
\]
On metric space \((X, d)\) let us define the self-mapping \(f\) by \(f(a) = f(\beta) = a\), \(f(\zeta) = d\), \(f(d) = \beta\). Consider also a function \(a : X \times X \rightarrow [0, \infty)\), where \(a(x, a) = a(a, x) = 3\) for any \(x \in X\), \(a(\beta, d) = 1\), \(a(x, y) = 0\) otherwise and the comparison function \(\psi : [0, \infty) \rightarrow [0, \infty), \psi(t) = \sqrt[4]{t}\). Since neither \(f\), nor \(f^2\) are continuous, Theorem 2.2 cannot be applied. On the other hand, is easy to see that \(f\) is triangular \(a\)-orbital admissible and also the assumptions (2), (3) from Theorem 2.3 are satisfied. Considering \(q = 0\), \(\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1/4\) and \(\lambda_5 = 0\) and taking into account the definition of function \(a\), we remark that the only interesting case is for \(x = \beta\) and \(y = d\). We have in this case:

\[
\begin{align*}
\alpha(\beta, d)d(f \beta, f d) &= d(a, \beta) = 1 < \sqrt{2} = \sqrt{\frac{3}{4} \cdot 2^{1/4} \cdot 1 \cdot 2^{1/4} \cdot (\frac{4}{9})^{1/4}} \\
&= \frac{4}{9} [d(\beta, d)]^{1/4} \cdot [d(\beta, f \beta)]^{1/4} \cdot [d(d, f d)]^{1/4} \cdot \left[ \frac{d(\beta, f \beta)(1 + d(\beta, f \beta))}{1 + d(\beta, d)} \right]^{1/4} \\
&= \psi \left( [d(\beta, d)]^{1/4} \cdot [d(\beta, f \beta)]^{1/4} \cdot [d(d, f d)]^{1/4} \cdot \left[ \frac{d(\beta, f \beta)(1 + d(\beta, f \beta))}{1 + d(\beta, d)} \right]^{1/4} \right).
\end{align*}
\]

Consequently, the map \(f\) has a fixed point, that is \(x = a\).

### 3 Ulam type stability

Considered as a type of data dependence, the notion of Ulam stability was started by Ulam [10, 11] and developed by Hyers [12], Rassias [13], etc. In this section we investigate the general Ulam type stability in sense of a fixed point problem.

Suppose that \(f : X \rightarrow X\) is a self-mapping on a metric space \((X, d)\). The fixed point problem

\[
x = f x,
\]

has the general Ulam type stability if and only if there exists an increasing function \(\rho : [0, \infty) \leftrightarrow [0, \infty)\), continuous at 0 with \(\rho(0) = 0\) such that for every \(\epsilon > 0\) and for each \(y' \in X\) which satisfies the inequality

\[
d(y', f y') \leq \epsilon
\]

there exists a solution \(z \in X\) of (20) such that

\[
d(z, y') \leq \rho(\epsilon).
\]

In case that for \(C > 0\), we consider \(\rho(t) = Ct\) for all \(t \geq 0\) then the fixed point equation (20) is said to be Ulam type stable.

On a metric space \((X, d)\), the fixed point problem (20), where \(f : X \rightarrow X\), is said to be well-posed if the following assumptions are satisfy:

1. \(f\) has a unique fixed point \(z\) in \(X\);
2. \(d(x_n, z) = 0\) for each sequence \(\{x_n\} \in X\) such that \(\lim_{n \to \infty} d(x_n, f x_n) = 0\).

**Theorem 3.1.** Let \((X, d)\) be a complete metric space. If we add the condition \(\lambda_1 + \lambda_2 < \frac{1}{c(q)}\), where \(c(q) = \max \{1, 2^{q-1}\}\), to the assumptions of Theorem 2.5, then the following affirmations hold:

(i) the fixed point equation (20) is Ulam-Hyers stable if \(a(u, v) \geq 1\) for any \(u, v\) satisfying the inequality (21);

(ii) the fixed point equation (20) is well-posed if \(a(x_n, z) \geq 1\) for any sequence \(\{x_n\} \in X\) such that \(\lim_{n \to \infty} d(x_n, f x_n) = 0\) and \(\text{Fix}_f(x) = z\).

**Proof.** (i) Since from Theorem 2.5 we know that there is an unique \(z \in X\) such that \(f z = z\), let \(y^* \in X\) such that

\[
d(y^*, f y^*) \leq \epsilon, \quad \text{for} \ \epsilon > 0.
\]
Obvious, \( z \) verifies (21) so we have that \( a(y^*, z) \geq 1 \) and then by using the triangular inequality we get

\[
d(z, y^*) \leq d(fz, f(y^*)) + d(f(y^*), y^*) \\
\leq \psi(y^*, z) + d(f(y^*), y^*) < \mathcal{D}^{\psi}(y^*, z) + d(f(y^*), y^*) \\
\leq [\Lambda_1 d^\psi(z, y^*) + \Lambda_2 d^\psi(y^*, f(y^*)) + \Lambda_3 d^\psi(z, f(z)) + \Lambda_4 \left( \frac{d(z, f(z))(1+d(f(z), f(y^*))}{1+d(y^*, z)} \right)]^{\frac{1}{4}} + d(f(y^*), y^*) \\
+ \Lambda_5 \frac{d(z, f(z))(1+d(f(z), f(y^*))}{1+d(y^*, z)} \right)^{\frac{1}{4}} + d(f(y^*), y^*) \\
\leq \left[ \Lambda_1 d^\psi(z, y^*) + \Lambda_2 \epsilon + \Lambda_5 d^\psi(z, f(y^*)) \right]^{\frac{1}{4}} + \epsilon \\
\leq \left[ \Lambda_1 d^\psi(z, y^*) + \Lambda_2 \epsilon + \Lambda_5 (d(z, y^*) + d(y^*, f(y^*))\right]^{\frac{1}{4}} + \epsilon \\
\leq \left[ \Lambda_1 d^\psi(z, y^*) + \Lambda_2 \epsilon + \Lambda_5 (d(z, y^*) + \epsilon)^{\frac{1}{4}} + \epsilon \\
\leq \left[ \Lambda_1 d^\psi(z, y^*) + \Lambda_2 \epsilon + \Lambda_5 (d(z, y^*) + \epsilon)^{\frac{1}{4}} + \epsilon \right]^{\frac{1}{4}} + \epsilon.
\]

Therefore,

\[
d^\psi(z, y^*) \leq c(q) \left[ \Lambda_1 d^\psi(z, y^*) + \Lambda_2 \epsilon + \Lambda_5 (d(z, y^*) + \epsilon)^{\frac{1}{4}} + \epsilon \right]^{\frac{1}{4}} + \epsilon
\]

where \( c(q) = \max \{1, 2^{q-1}\} \). By simple calculation, from the above inequality we have

\[
d^\psi(z, y^*) \leq \frac{1 + \Lambda_2 + c(q)\Lambda_5 \epsilon}{1 - c(q)\Lambda_1 - c^2(q)\Lambda_5} \epsilon
\]

which is equivalent with

\[
d(z, y^*) \leq C \epsilon,
\]

where \( C = \left( \frac{1 + \Lambda_1 + c(q)\Lambda_5 \epsilon}{1 - c(q)\Lambda_1 - c^2(q)\Lambda_5} \right)^{\frac{1}{2}} \), for any \( q > 0 \) and \( \Lambda_1, \Lambda_5 \in [0, 1) \) such that \( \Lambda_1 + \Lambda_5 < \frac{1}{c(q)\Lambda_1} \).

(ii) Taking into account the supplementary condition and since \( \text{Fix}_f(x) = z \) we have

\[
d(x_n, z) \leq d(x_n, f(x_n)) + d(f(x_n), f(z)) \\
\leq d(x_n, f(x_n)) + a(x_n, z) d(f(x_n), f(z)) \\
\leq d(x_n, f(x_n)) + \psi(y^*, z) \\
< d(x_n, f(x_n)) + \mathcal{D}^{\psi}(x_n, f(x_n)) \\
\leq [\Lambda_1 d^\psi(x_n, z) + \Lambda_2 d^\psi(x_n, f(x_n)) + \Lambda_3 d^\psi(z, f(z)) \\
+ \Lambda_4 \left( \frac{d(x_n, f(x_n))(1+d(x_n, f(z)))}{1+d(x_n, f(z))} \right)]^{\frac{1}{4}} + d(x_n, f(x_n)) \\
\leq [\Lambda_1 d^\psi(x_n, z) + \Lambda_2 d^\psi(x_n, f(x_n)) + \Lambda_3 d^\psi(z, f(x_n)) \right]^{\frac{1}{4}} + d(x_n, f(x_n)) \\
\leq \left[ \Lambda_1 d^\psi(x_n, z) + \Lambda_2 d^\psi(x_n, f(x_n)) + \Lambda_5 (d(z, x_n) + d(f(x_n), f(x_n))) \right]^{\frac{1}{4}} + d(x_n, f(x_n)) \\
\leq \left[ \Lambda_1 d^\psi(x_n, z) + \Lambda_2 d^\psi(x_n, f(x_n)) + \Lambda_5 c(q) d^\psi(z, x_n) + d^\psi(x_n, f(x_n)) \right]^{\frac{1}{4}} + d(x_n, f(x_n)) \\
\leq \left[ \Lambda_1 d^\psi(x_n, z) + \Lambda_2 d^\psi(x_n, f(x_n)) + \Lambda_5 c(q) d^\psi(z, x_n) + d^\psi(x_n, f(x_n)) \right]^{\frac{1}{4}} + d(x_n, f(x_n)),
\]

or,

\[
d(x_n, z)^q \leq \frac{1 + \Lambda_2 + c(q)\Lambda_5 \epsilon}{1 - c(q)\Lambda_1 - c^2(q)\Lambda_5} \epsilon \cdot d(x_n, f(x_n)).
\]

Letting \( n \to \infty \) in the above inequality and keeping in mind that \( \lim_{n \to \infty} d(x_n, f(x_n)) = 0 \), we obtain

\[
\lim_{n \to \infty} d(x_n, z) = 0
\]

that is, the fixed point equation (20) is well-posed.
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