Enlarged Controllability of Riemann–Liouville Fractional Differential Equations

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ABSTRACT
We investigate exact enlarged controllability for time fractional diffusion systems of Riemann–Liouville type. The Hilbert uniqueness method is used to prove exact enlarged controllability for both cases of zone and pointwise actuators. A penalization method is given and the minimum energy control is characterized.

Keywords: Regional controllability; Time fractional diffusion processes; Fractional calculus; Enlarged controllability; Minimum energy; Optimal control.

Mathematics Subject Classification 2010: 26A33; 49J20; 93B05; 93C20.

1 Introduction
The purpose of fractional calculus is to generalize standard derivatives into non-integer order operators. As well acknowledged in the literature, many dynamical systems are best characterized by dynamic models of fractional order, based on the notion of non-integer order differentiation or integration. The study of fractional order systems is, however, more delicate. Indeed, fractional systems are, on one hand, memory systems, notably for taking into account the initial conditions, and on the other hand they present much more complex dynamics [1,2,3].

To backtrack the root of fractional calculus, one needs to go back to the XVII century (more accurately to 1695), when L’Hôpital was questioning Leibniz about the possible meaning of half order differentiation. This question has attracted the...
interest of many well-known mathematicians, including Euler, Liouville, Laplace, Riemann, Grünwald, Letnikov, and many others. Since the XIX century, the theory of fractional calculus developed rapidly, mostly as a foundation for a number of mathematical disciplines, including fractional geometry, fractional differential equations (FDE), and fractional dynamics. Since 1974, when the first international conference in the field took place, fractional calculus has been intensively developed with respect to practical applications. Applications of fractional calculus are nowadays very wide and most disciplines of modern engineering and science rely on tools and techniques of fractional calculus. For example, fruitful applications can be found in rheology, viscoelasticity, acoustics, optics, chemical and statistical physics, robotics, control theory, electrical and mechanical engineering, and bioengineering. In a fractional derivative sub-diffusion equation model is proposed to describe prime number distribution. Based on experimental results, develops a fractional damping wave equation for the acoustic propagation in a porous media. For a survey of time fractional diffusion models under different initial and boundary conditions we refer the reader to [19].

The problem of steering a system to a target state has a vast literature, which has essentially begun with the introduction of the notion of actuators and sensors in the 1980s. However, in many real world applications, one is concerned with those cases where the target states of the studied problem are defined in a given subregion of the whole space domain. This leads to the idea of regional controllability.

Controllability problems for integer order systems have been widely studied and many techniques have been developed for solving such problems. Problems with hard constraints on the state or control have attracted several authors in the last three decades, mostly for their importance in various applications in optimal control. Indeed, it is well known that by proving the existence of a Lagrange multiplier associated with the constraints in the state, we can derive optimality conditions. For instance, Barbu and Precupanu and Laseicka, derived the existence of a Lagrange multiplier for some optimal control problems with integral state constraints. Bergounioux used a penalization method to prove existence of a multiplier and to derive optimality conditions for elliptic equations with state constraints. Here, we solve the problem of enlarged controllability, also so-called controllability with constraints on the state, using the Hilbert Uniqueness Method (HUM) of Lions.

We consider a time fractional diffusion system where the traditional first-order time derivative is replaced by the Riemann–Liouville time fractional derivative, which can be used to well characterize sub-diffusion processes. The problem is defined in Section 2. Some basic knowledge of fractional calculus and some preliminary results for our problem are given in Section 3. Then, in Section 4 we characterize the enlarged controllability of the system. Our main results on enlarged controllability are proved in Section 5 in two different cases: for zone and pointwise actuators. In Section 6 we prove that the control that steers the system to the final state is of minimum energy. We end with Section 7 of conclusions and some interesting open questions that deserve further investigations.

### 2 The Time Fractional Diffusion System

Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \), \( n = 1, 2, 3 \), with a regular boundary \( \partial \Omega \). For \( T > 0 \), denote \( Q = \Omega \times [0, T] \) and \( \Sigma = \partial \Omega \times [0, T] \). We consider the following time fractional order diffusion system:

\[
\begin{aligned}
&\partial_t^\alpha y(x,t) = Ay(x,t) + Bu(t) \quad \text{in } Q, \\
y(x,0) = 0 \quad \text{on } \Sigma, \\
\lim_{t\to 0^+} \partial_t^{1-\alpha} y(x,t) = y_0(x) \quad \text{in } \Omega,
\end{aligned}
\]

where \( \partial_t^\alpha \) and \( \partial_t^{1-\alpha} \) denote, respectively, the Riemann–Liouville fractional order derivative and integral with respect to time \( t \). For details on these operators, see e.g. [30, 31]. Here we just recall their definition:

\[
\partial_t^{1-\alpha} y(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(x,s) ds
\]

and

\[
\partial_t^\alpha y(x,t) = \frac{\partial}{\partial t} \partial_t^{1-\alpha} y(x,t),
\]

where \( 0 < \alpha < 1 \). The second order operator \( A \) in (1) is linear with dense domain such that the coefficients do not depend on \( t \). It generates a \( C_0 \)-semi-group \( (S(t))_{t\geq 0} \) on the Hilbert space \( Y := L^2(\Omega) \). We refer the reader to Engel and Nagel and Renardy and Rogers for more properties on operator \( A \). The initial datum \( y_0 \) is in \( Y \). The operator \( B : \mathbb{R}^m \to Y \) is the control operator, which depends on the number \( m \) of actuators and \( u \in L^2(0,T;\mathbb{R}^m) \).
3 Preliminaries

In this section, we recall some notions and facts needed in the sequel. We begin with the concept of mild solution, that has been used in fractional calculus in several different contexts [34, 35, 36].

Definition 1 (See, e.g., [37, 38, 39]). For any given function \( f \in L^2(0,T;Y) \) and \( \alpha \in (0,1) \), we say that function \( g \in L^2(0,T;Y) \) is a mild solution of the system

\[
\begin{aligned}
0D_\alpha^t g(t) &= \mathcal{A}g(t) + f(t), \quad t \in [0,T], \\
\lim_{t \to 0^+} t^{-\alpha}g(t) &= g_0 \in Y,
\end{aligned}
\]

if it satisfies

\[
g(t) = K_\alpha(t)g_0 + \int_0^t (t-s)^{\alpha-1}K_\alpha(t-s)f(s)ds,
\]

where

\[
K_\alpha(t) = \alpha \int_0^\infty \theta \phi_\alpha(\theta)S(t\theta)d\theta
\]

with

\[
\phi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\alpha/\alpha} \phi_\alpha(\theta^{-1/\alpha})
\]

and \( \phi_\alpha \) is the probability density function defined by

\[
\phi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta > 0.
\]

Remark 1. The probability density function satisfies

\[
\int_0^\infty \phi_\alpha(\theta)d\theta = 1.
\]

Moreover,

\[
\int_0^\infty \theta^v \phi_\alpha(\theta)d\theta = \frac{\Gamma(1+v)}{\Gamma(1+\alpha v)}, \quad v \geq 0.
\]

For results on existence and uniqueness of mild solutions for a class of fractional neutral evolution equations with nonlocal conditions, we refer to Zhou and Jiao [40]. Here we note that a mild solution of system (1) can be written as

\[
y(x,T;u) = T^{\alpha-1}K_\alpha(T)g_0 + \int_0^T (T-s)^{\alpha-1}K_\alpha(T-s)Bu(s)ds.
\]

(2)

Remark 2. Throughout the paper, \( y(x,t) \) is the variable of system (1). It depends on time \( t \) and space \( x \). We use \( y(x,t;u) \) as the solution of system (1) when it is excited with a control \( u \). We also denote it formally by \( y(u) \).

Let us define the operator \( H : L^2(0,T;\mathbb{R}^m) \rightarrow Y \) by

\[
Hu = \int_0^T (T-s)^{\alpha-1}K_\alpha(T-s)Bu(s)ds
\]

(3)
for all $u \in L^2(0,T;\mathbb{R}^m)$ and assume that $(S^\ast(t))_{t \geq 0}$ is a strongly continuous semi-group generated by the adjoint operator of $A$ on the state space $Y$. Let $\langle \cdot, \cdot \rangle$ be the duality pairing of space $Y$. It is easy to see that $\langle Hu, v \rangle = \langle u, H^\ast v \rangle$. Indeed, for all $v \in Y$ one has

$$\langle Hu, v \rangle = \left\langle \int_0^T (T-s)^{\alpha-1} K_\alpha(T-s) Bu(s) ds, v \right\rangle = \int_0^T \langle (T-s)^{\alpha-1} K_\alpha(T-s) Bu(s), v \rangle ds$$

$$= \int_0^T \langle u(s), B^\ast(T-s)^{\alpha-1} K_\ast^\alpha(T-s)v \rangle ds = \langle u, H^\ast v \rangle. \quad (4)$$

For any $v \in Y$, it follows from (4) that

$$H^\ast v = B^\ast(T-s)^{\alpha-1} K_\ast^\alpha(T-s)v,$$

where $B^\ast$ is the adjoint operator of $B$, and

$$K_\ast^\alpha = \alpha \int_0^\infty \theta \phi_\alpha(\theta) S^\ast(t^\alpha \theta) d\theta.$$

Let $\omega \subset \Omega$ be a given region of positive Lebesgue measure. We define the restriction operator $\chi_\omega$ and its adjoint $\chi_\omega^\ast$ by

$$\chi_\omega : L^2(\Omega) \rightarrow L^2(\omega)$$

$$y \mapsto \chi_\omega y = y|_\omega$$

and

$$(\chi_\omega^\ast y)(x) = \begin{cases} y(x) & x \in \omega \\ 0 & x \in \Omega \setminus \omega. \end{cases}$$

In order to prove our results, the following lemma is needed.

**Lemma 1** (See [41]). Let the reflection operator $Q$ on the interval $[0,T]$ be defined by

$$Qh(t) := h(T-t),$$

for some function $h$ which is differentiable and integrable in the Riemann–Liouville sense. Then the following relations hold:

$$Q_0D_0^\alpha h(t) = iD_0^\alpha Qh(t), \quad Q_0I_0^\alpha h(t) = iI_0^\alpha Qh(t)$$

and

$$0I_0^\alpha Qh(t) = Q_0I_0^\alpha h(t), \quad 0D_0^\alpha Qh(t) = Q_0D_0^\alpha h(t).$$

### 4 Enlarged Controllability and Characterization

We define exact enlarged controllability (EEC) as follows.

**Definition 2.** Given a final time $T > 0$ and a suitable functional space $G$, we say that system (1) is exact enlarged controllable if, for every $y_0$ in a suitable functional space, there exists a control $u$ such that

$$\chi_{\omega} y(\cdot, T; u) \in G. \quad (5)$$
Remark 3. The concept of exact enlarged controllability depends, obviously, on G. If G = {0}, then one gets from Definition 2 the classical notion of controllability.

Remark 4. If there is exact enlarged controllability depending on G, then there exists an infinity number of controls $u$ that verify (5). However, exact enlarged controllability relatively to G does not implies exact controllability.

Example 1. Let $\omega$ be an open subset of $\Omega$ and let us search $u$ such that $\partial D^\alpha y(x,T;u) = 0$ in $\omega$. In this case, the space $G$ can be given as $G = G_0 \times \{\text{the whole domain}\}$, where $G_0$ is the space of null functions in $\omega$.

In our case, we take $G$ as a sub-vectorial closed space of $Y := L^2(\Omega)$. The following result holds.

Theorem 2. Our system is exact enlarged controllable (i.e., system (1) is exactly G-controllable in $\omega$ in the sense of Definition 3) if and only if

$$G - \{\chi_\omega T^{\alpha-1}K_\alpha(T)y_0\} \cap \text{Im} \chi_\omega H \neq \emptyset. \quad (6)$$

Proof. Suppose (6) holds. Then, there exists

$$z \in G - \{\chi_\omega T^{\alpha-1}K_\alpha(T)y_0\}$$

such that $z \in \text{Im} \chi_\omega H$. So, there exists $u \in L^2(0,T;\mathbb{R}^m)$ such that $z = \chi_\omega Hu$. Hence,

$$z = \chi_\omega Hu \in G - \{\chi_\omega T^{\alpha-1}K_\alpha(T)y_0\}.$$

Therefore, $y(u) \in G$ and we have EEC in $\omega$. Conversely, assume that one has EEC of (1) in $\omega$, which means that $\chi_\omega y(u) \in G$. Using (2) and (5), we have

$$\chi_\omega y(u) = \chi_\omega T^{\alpha-1}K_\alpha(T)y_0 + \chi_\omega Hu.$$

Let us denote $w = \chi_\omega y(u) - \chi_\omega T^{\alpha-1}K_\alpha(T)y_0 = \chi_\omega Hu$. Then, one has $w \in \text{Im} \chi_\omega H$ and

$$w \in G - \{\chi_\omega T^{\alpha-1}K_\alpha(T)y_0\}.$$

We just proved (6).

5 Fractional HUM Approach

We now extend the Hilbert uniqueness method (HUM), introduced by Lions in [29], to the fractional setting (1) and try to compute the (optimal) control that steers system (1) into $G$. First of all, we prove under what conditions we can find enlarged controllability. Then we compute the (optimal) control that steers our system into $G$. The reader interested in the HUM approach, in the context of fractional calculus, is referred to [42,43,44,45].

5.1 The Case of a Zone Actuator

Let us consider system (1) excited with a zone actuator $(D,f)$, where $D \subseteq \Omega$ is the support of the actuator and $f$ its spatial distribution. For details about actuators, we refer to [20,40]. System (1) can be written as follows:

$$\begin{cases}
0 D^\alpha_\nu z(x,t) = A z(x,t) + \chi_\omega f(x)u(t) & \text{in } Q \\
 z(\xi,t) = 0 & \text{on } \Sigma \\
 \lim_{t \to 0^+} 0 D^\alpha_\nu z(x,t) = z_0(x) & \text{in } \Omega.
\end{cases} \quad (7)$$

The main question we answer is the following one. Does there exist a (minimum norm) control $u \in L^2(0,T;\mathbb{R}^m)$ such that $\chi_\omega y(x,T;u) \in G$?
Let us introduce \( G^\circ \) as the polar space of \( G \). Then, \( \varphi_0 \in G^\circ \), that is, \( \langle \varphi_0, \phi \rangle = 0 \) for all \( \phi \in G \), where \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( Y \). For \( \varphi_0 \in G^\circ \), we consider the following backward system:

\[
\begin{aligned}
Q_t D^a_t \Phi(x, t) &= \mathcal{A}^* \Phi(x, t) & \text{in } \Omega \\
\Phi(x, 0) &= 0 & \text{on } \Sigma \\
\lim_{t \to 0^+} Q_t I^{1-a} \Phi(x, t) &= \chi^*_a \Phi_0(x) & \text{in } \Omega,
\end{aligned}
\]

where \( Q \) is the reflection operator on the interval \([0, T]\) defined in Lemma 1. Hence, system (8) can be rewritten as

\[
\begin{aligned}
0D^a_t Q \Phi(x, t) &= \mathcal{A}^* Q \Phi(x, t) & \text{in } \Omega \\
\Phi(x, 0) &= 0 & \text{on } \Sigma \\
\lim_{t \to 0^+} 0I^{1-a} Q \left[(T-t)^{1-a} \Phi(x, t)\right] &= \chi^*_a \Phi_0(x) & \text{in } \Omega.
\end{aligned}
\]

System (9) has a unique mild solution given by

\[
\Phi(x, t) = (T-t)^{\alpha-1} K^\alpha(T-t) \chi^*_a \Phi_0(x).
\]

Moreover, we define the following semi-norm on \( G \):

\[
\varphi_0 \in G^\circ \rightarrow \| \varphi_0 \|_{G^\circ}^2 = \int_0^T \langle f, \Phi(\cdot, t) \rangle_{L^2(\Omega)} dt.
\]

**Lemma 3.** Equation (10) defines a norm.

**Proof.** To prove that (10) is a norm, we show that \( \| \varphi_0 \|_{G^\circ}^2 = 0 \Leftrightarrow \varphi = 0 \). Writing \( \| \varphi_0 \|_{G^\circ}^2 = 0 \), is equivalent to \( \langle f, \Phi(\cdot, t) \rangle = 0 \). It follows from the uniqueness theorem in [47] that \( \varphi = 0 \). □

We also consider the problem:

\[
\begin{aligned}
0D^a_t \Psi(x, t) &= \mathcal{A} \Psi(x, t) + \chi_\alpha f(x) u(t) & \text{in } \Omega \\
\Psi(x, 0) &= 0 & \text{on } \Sigma \\
\lim_{t \to 0^+} 0I^{1-\alpha} \Psi(x, t) &= z_0(x) & \text{in } \Omega.
\end{aligned}
\]

We obtain \( \Psi \) such that \( \Psi : [0, T] \rightarrow Y \) is continuous. If we find \( \Phi_0 \) such that \( \Phi_0 \in G^\circ \) and

\[
\chi^*_a \Psi(T) \in G,
\]

then

\[
u = \langle f, \Phi(\cdot, t) \rangle_{L^2(\Omega)}.
\]

The control (12) ensures EEC and we have \( \Psi = y(u) \), where \( y(u) \) is a formal notation of the solution \( y(x, T; u) \). In order to easily explain (11) with an equation, we define

\[
M \Phi_0 = \chi^*_a \Psi(T),
\]

where \( M \) is an affine operator from \( G^\circ \) to the orthogonal \( G^\perp \) of \( G \). Thus, all return to solve the equation \( M \Phi_0 = 0 \). Let us decompose \( M \) into a linear part and a constant one:

\[
M \Phi_0 = \chi^*_a (\Psi_0(T) + \Psi_1(T)).
\]
Proof. From Lemma 3 we have that (10) is a norm. Now, we show that (15) admits a unique solution in \( G^\circ \). For any \( \varphi_0 \in G^\circ \), by (14) it follows that

\[
\langle \varphi_0, \Lambda \varphi_0 \rangle = \langle \varphi_0, \chi_{\omega} \psi_0(T) \rangle \\
= \langle \varphi_0, \chi_{\omega} \int_0^T (T - s)^{\alpha - 1} K_\alpha(T - s) \chi_{\omega} f(\cdot)(f, \varphi(\cdot, s))_{L^2(D)} ds \rangle \\
= \int_0^T \langle (T - s)^{\alpha - 1} K_\alpha(T - s) \chi_{\omega} \varphi_0, \chi_{\omega} f(\cdot)(f, \varphi(\cdot, s))_{L^2(D)} \rangle ds \\
= \int_0^T \tilde{\chi}_{\omega} f(\cdot) \varphi(\cdot, s), (f, \varphi(\cdot, s))_{L^2(D)} \rangle ds \\
= \int_0^T \langle f, \varphi(\cdot, s) \rangle_{L^2(D)}, (f, \varphi(\cdot, s))_{L^2(D)} \rangle ds \\
= \int_0^T \| (f, \varphi(\cdot, s))_{L^2(D)} \|^2 ds \\
= \| \varphi_0 \|^2_{G^\circ}.
\]

Existence of a unique solution follows from [23, Theorem 1.1].
5.2 The Case of a Pointwise Actuator

Consider now system (1) with a pointwise internal actuator, which can be written in the form

\[
\begin{align*}
0D^\alpha_t y(x,t) &= Ay(x,t) + \delta(x - b)u(t) \quad \text{in} \ Q \\
y(\xi, t) &= 0 \quad \text{on} \ \Sigma \\
\lim_{t \to 0^+} D_t^{1-\alpha} y(x, t) &= y_0(x) \quad \text{in} \ \Omega,
\end{align*}
\]

(16)

where \( b \) is the position of the actuator. For any \( \phi_0 \in G^\circ \), we consider system (8) and we define the following semi-norm:

\[
\phi_0 \in G^\circ \rightarrow \| \phi_0 \|_{G^\circ}^2 = \int_0^T \langle \phi(b, t) \rangle^2_{L^2(D)} dt.
\]

Using similar arguments as in Section 5.1, we can easily prove that this semi-norm is indeed a norm. Let us consider \( u(t) = \phi(b, t) \) and the following two systems:

\[
\begin{align*}
0D^\alpha_t \psi_0(x,t) &= A\psi_0(x,t) + \delta(x - b)\phi(b, t) \quad \text{in} \ Q \\
\psi_0(\xi, t) &= 0 \quad \text{on} \ \Sigma \\
\lim_{t \to 0^+} D_t^{1-\alpha} \psi_0(x, t) &= 0 \quad \text{in} \ \Omega.
\end{align*}
\]

and

\[
\begin{align*}
0D^\alpha_t \psi_1(x,t) &= A\psi_1(x,t) \quad \text{in} \ Q \\
\psi_1(\xi, t) &= 0 \quad \text{on} \ \Sigma \\
\lim_{t \to 0^+} D_t^{1-\alpha} \psi_1(x, t) &= y_0(x) \quad \text{in} \ \Omega.
\end{align*}
\]

Then, the exact enlarged controllability problem is equivalent to solve equation

\[
\Lambda \phi_0 = -\chi_\omega \psi_1(T).
\]

Similarly to the proof of Theorem 4, for the case of a zone actuator, we prove exact enlarged controllability for a pointwise actuator.

**Theorem 5.** System (16) is exact enlarged controllable relatively to \( G \). Moreover, the control

\[
u^*(t) = \phi(b, t)
\]

steers the system into \( G \).

6 Minimum Energy Control

The study of fractional optimal control problems is a subject under strong development: see \([48,49,50,51]\) and references therein. In this section, inspired by the results of \([52,53]\), we show that the steering controls found in Section 5 are minimizers of a suitable optimal control problem. For that, let us consider the following minimization problem:

\[
\begin{align*}
\inf_u J(u) &= \frac{1}{2} \int_0^T \| u \|^2_{U} dt \\
u \in U_{ad},
\end{align*}
\]

(17)

where

\[
U_{ad} = \{ u \in U = L^2(0, T; \mathbb{R}^m) \mid \chi_\omega y(u) \in G \}.
\]

The proof of our Theorem 6 is based on a penalization method \([54,55]\).
Theorem 6. Assume that $U_{ad}$ is nonempty with an exact enlarged controllable system relatively to $G$ given by (7) or (16). Then the optimal control problem (17) has a unique solution given by $u^*(t) = \langle \phi(t), \psi(t) \rangle$, in case of a zone actuator, and by $u^*(t) = \psi(b,t)$, in case of a pointwise actuator. Such control ensures the transfer of system (1) into $G$ with a minimum energy cost in the sense of $J$.

Proof. Let $\varepsilon > 0$. Suppose that there is EEC relatively to $G$ and consider the following problem:

$$\begin{cases}
0 D_t^\alpha z(x,t) - Az(x,t) - \chi_{\partial O} f(x) u(t) \in L^2(Q) \\
z(\xi,t) = 0 \quad \text{on } \Sigma \\
\lim_{t \to 0^+} D_t^{1-\alpha} z(x,t) = 0 \quad \text{in } \Omega \\
z(T,u) \in G.
\end{cases}$$

(18)

The set $S$ of pairs $(z,u)$ verifying (18) is nonempty. Consider the penalized problem of (17) given by

$$\begin{cases}
\inf J_\varepsilon(u,z) \\
(u,z) \in S,
\end{cases}$$

(19)

where

$$J_\varepsilon = \frac{1}{2} \int_0^T u^2(t) dt + \frac{1}{2\varepsilon} \int_Q (0 D_t^\alpha z(x,t) - Az(x,t) - \chi_{\partial O} f(x) u(t))^2 dQ.$$

Let $\{u_\varepsilon, z_\varepsilon\}$ be the solution of (19) and let us define

$$p_\varepsilon = -\frac{1}{\varepsilon} (0 D_t^\alpha z_\varepsilon(x,t) - Az_\varepsilon(x,t) - \chi_{\partial O} f(x) u_\varepsilon(t)).$$

Because we did assume that $U_{ad}$ is nonempty, we have

$$0 \leq J_\varepsilon(u_\varepsilon, z_\varepsilon) = \inf_j J_\varepsilon(u,z) < \inf J_\varepsilon(u) < \infty \quad \text{for } u \in U_{ad},$$

where $J_\varepsilon(u) = \frac{1}{2} \int_0^T u^2(t) dt$. Therefore,

$$\begin{cases}
\|u_\varepsilon\| \leq C, \\
\|0 D_t^\alpha z(x,t) - Az(x,t) - \chi_{\partial O} f(x) u(t)\| \leq C\sqrt{\varepsilon},
\end{cases}$$

(20)

where $C$ represents various positive constants independent of $\varepsilon$. It follows from (20) that

$$\|0 D_t^\alpha z(x,t) - Az(x,t)\| \leq C(1 + \sqrt{\varepsilon}).$$

Hence, when $\varepsilon \to 0$, we have that $u_\varepsilon$ is bounded and we can extract a sequence such that

$$u_\varepsilon \rightharpoonup \tilde{u} \quad \text{weakly in } U$$

$$z_\varepsilon \rightharpoonup z \quad \text{weakly in } L^2(Q).$$

Using the semi-continuity of $J$, one has

$$J(u^*) \leq \liminf J_\varepsilon(u_\varepsilon) \leq \liminf J_\varepsilon(u_\varepsilon, z_\varepsilon).$$
Then,
\[ f(u^*) = \inf f(u), \quad u \in U_{ad}, \]
and \( u^* = \bar{u} \). With respect to problem \( [19] \), we have
\[ \int_0^T u_e(t)u(t)dt + \int_Q \langle p_e, 0D^\alpha_t \eta(x,t) - \mathcal{A}\eta(x,t) \rangle dQ = - \int_Q \langle p_e, f(x) \rangle u(t)dQ. \]
For \( u \in U_{ad} \) and \( \eta \) such that
\[ \begin{cases} 0D^\alpha_t \eta(x,t) - \mathcal{A}\eta(x,t) = \chi_D f(x)u(t) & \text{in } Q \\ \eta(\xi,t) = 0 & \text{on } \Sigma \\ \lim_{t \to 0^+} 0I_t^\alpha \eta(x,t) = 0 & \text{in } \Omega \\ \eta(T) \in G, \end{cases} \]
we deduce that \( p_e \) verifies
\[ \begin{cases} 0D^\alpha_t p_e(x,t) - \mathcal{A}p_e(x,t) = \chi_D f(x)\langle p_e, f \rangle_{L^2(\Omega)} & \text{in } Q \\ p_e(\xi,t) = 0 & \text{on } \Sigma \\ \lim_{t \to 0^+} 0I_t^\alpha p_e(x,t) = 0 & \text{in } \Omega \end{cases} \]
with \( \langle \eta(T), p_e(T) \rangle = 0 \) for all \( \eta \) such that \( \eta(T) \in G \). Then, \( p_e(T) \in G^c \). If we suppose that
\[ \int_0^T \langle p_e, f \rangle^2 dt \geq c \| p_e(T) \|^2_{H^\alpha(\Omega)}, \]
then we can switch to the limit when \( \varepsilon \to 0 \). Moreover, if we have exact enlarged controllability relatively to \( G \), then
\[ \begin{cases} 0D^\alpha_t z(x,t) - \mathcal{A}z(x,t) = \chi_D f(x)u(t) & \text{in } Q \\ z(\xi,t) = 0 & \text{on } \Sigma \\ \lim_{t \to 0^+} 0I_t^\alpha z(x,t) = z_0(x) & \text{in } \Omega \\ 0D^\alpha_T p(x,t) - \mathcal{A}p(x,t) = \chi_D f(x)\langle p, f \rangle_{L^2(\Omega)} & \text{in } Q \\ p(\xi,t) = 0 & \text{on } \Sigma \\ p(T) \in G^c. \end{cases} \]
Thus, we take \( p(T) \in G^c \) and we introduce the solution \( \phi \) of \( [9] \). Then, \( \psi = z \) if \( \psi(T) \in G \), which proves that \( [13] \) has a unique solution for \( \phi_0 \in G^c \).

7 Conclusions
This paper deals with the notion of regional exact enlarged controllability for Riemann–Liouville time fractional diffusion systems. Our results extend the ones in \( [42, 56, 57] \). They can be extended to complex fractional-order distributed parameter dynamic systems. Other difficult questions are still open and deserving further investigations, e.g., the problem of boundary enlarged controllability for fractional systems; and the problem of gradient enlarged controllability/observability for fractional order distributed parameter systems. These and other questions, as to give numerical results and a real application to support our theoretical analysis, are being considered and will be addressed elsewhere.
Acknowledgements
This research is part of first author’s Ph.D., which is carried out at Moulay Ismail University, Meknes. It was essentially finished during a one-month visit of Karite to the Department of Mathematics of University of Aveiro, Portugal, June 2017. The hospitality of the host institution and the financial support of Moulay Ismail University, Morocco, and CIDMA, Portugal, are here gratefully acknowledged. Boutoulout was supported by Hassan II Academy of Science and Technology; Torres by Portuguese funds through CIDMA and FCT, within project UID/MAT/04106/2013. The authors are very grateful to two anonymous referees, for their suggestions and invaluable comments.

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