Some moduli spaces of 1-dimensional sheaves on an elliptic ruled surface

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Abstract
We shall study moduli spaces of stable 1-dimensional sheaves on an elliptic ruled surface. In particular we shall compute the generating series of Hodge numbers of some moduli spaces of stable 1-dimensional sheaves.

Keywords Elliptic ruled surfaces · Stable sheaves · Fourier-Mukai · Hodge numbers

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1 Introduction
Let $X$ be a smooth projective surface over $\mathbb{C}$. Moduli spaces of stable sheaves of rank $r$ on $X$ are studied by many people. In particular, by analysing the chamber structure on the ample cone, topological invariants of the moduli spaces are extensively studied if $X$ is a surface of Kodaira dimension $-\infty$ and $r > 0$ (cf. [2, 3, 8–11, 14–16]). For the moduli spaces of stable 1-dimensional sheaves on surfaces, topological properties are also studied for $\mathbb{P}^2$, a quadratic surface and an elliptic surface. In this note, we shall compute the Hodge numbers of some moduli spaces on an elliptic ruled surface, that is, a $\mathbb{P}^1$-bundle over an elliptic curve $C$.

Let $\sigma : X \to C$ be the structure morphism of the $\mathbb{P}^1$-fibration. Let $g$ be a fiber of $\sigma$ and $C_0$ a minimal section. Then $\text{NS}(X) = \mathbb{Z}C_0 + \mathbb{Z}g$ with $(C_0 \cdot g) = 1$, $(g^2) = 0$ and $(C_0^2) = -e$. The canonical divisor $K_X = -2C_0 - eg$ satisfies $(K_X^2) = 0$ and $K_X$ is nef if and only if $e = 0, -1$. In each cases, the nef cone is generated by $g$ and $-K_X$ [6, Prop. 2.20, 2.21]. Let $M_H(0, \xi, \chi)$ be the moduli space of stable 1-dimensional sheaves $E$ such that $c_1(E) = \xi$ and $\chi(E) = \chi$.

Theorem 1.1 Let $\sigma : X \to C$ be an elliptic ruled surface with $e = -1$. Let $(\xi, \chi)$ be a pair of $\xi \in \text{NS}(X)$ with $(\xi \cdot K_X) < 0$ and $\chi \in \mathbb{Z}$ with $\chi \neq 0$.

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(1) Assume that \( H \) is a general polarization. Then \( M_H(0, \xi, \chi) \) is a smooth projective manifold of dimension \( (\xi^2) + 1 \).

(2) The Hodge numbers of \( M_H(0, \xi, \chi) \) is independent of a general choice of \( H \). If \( (\xi \cdot K_X) = -1 \), then the generating function is given by

\[
\sum_{-\xi \cdot K_X = 1} \left( \sum_{p, q} (-1)^{p+q} h^{p, q}(M_H(0, \xi, \chi)) x^p y^q \right) q^{\frac{|\xi|^2}{4}} = (x - 1)^2 (y - 1)^2 q^2 \prod_{n > 0} \frac{(1 - x^{-1} (x^2 y^2 q)^n)^2 (1 - y^{-1} (x^2 y^2 q)^n)^2 (1 - x (x^2 y^2 q)^n)^2 (1 - y (x^2 y^2 q)^n)^2}{(1 - (xy)^{-1} (x^2 y^2 q)^2) (1 - (xy) (x^2 y^2 q)^2)}.
\]

\[(1.1)\]

The smoothness of \( M_H(0, \xi, \chi) \) is an easy consequence of the deformation theory of a coherent sheaf. The computation of the Hodge numbers is our main result. We note that there is an elliptic fibration \( \pi : X \to \mathbb{P}^1 \) with three multiple fibers of multiplicity 2. Then the assumption \( (\xi \cdot K_X) = -1 \) means that the support \( D \) of \( E \in M_H(0, \xi, \chi) \) is a double cover of \( \mathbb{P}^1 \). We shall also treat the case where \( e = 0 \) (Theorem 3.11).

For the proof, we shall use Fourier-Mukai transform associated to the elliptic fibration \( [1] \). Since \( D \) is a double cover of \( \mathbb{P}^1 \), the computation is reduced to the computation of Hodge numbers of the moduli spaces of stable sheaves of rank two, which is computed in \( [2] \) or \( [15] \). We also use indefinite theta function in \( [5] \) to get the product expression. Since the Betti numbers of moduli spaces for higher rank cases are computed in \( [11] \), it is possible to get the Betti numbers of \( M_H(0, \xi, a) \) for a general \( \xi \) in principle.

We would like to remark that the same method works for a 9 points blow-ups \( X \) of \( \mathbb{P}^2 \). Thus if \( -K_X \) is nef, then by using Fourier-Mukai transforms on a rational elliptic surface and the deformation invariance of the Hodge numbers, we can compute the Hodge numbers from the computations for positive rank cases. In particular we can derive an explicit form of Euler characteristics of \( M_H(0, \xi, \chi) \) from the computations in \( [16] \), where \( (\xi \cdot K_X) = -2 \).

2 Preliminaries

Notation. Let \( X \) be a smooth projective surface. For two divisors \( D_1, D_2 \) on \( X \), \( D_1 \equiv D_2 \) means \( D_1 \) is algebraically equivalent to \( D_2 \). \( (D_1 \cdot D_2) \) denotes the intersection number of \( D_1, D_2 \). We set \( (D_1^2) := (D_1 \cdot D_1) \).

For a smooth projective variety \( X, D(X) := D(\text{Coh}(X)) \) denotes the bounded derived category of the category \( \text{Coh}(X) \) of coherent sheaves on \( X \). For \( E \in D(X), E^\vee := \text{RHom}_{\mathcal{O}_X}(E, \mathcal{O}_X) \) denotes the derived dual of \( E \). For the Grothendieck group \( K(X) \) of \( X \), we set \( K(X)_{\text{top}} := K(X) / \ker \text{ch} \), where \( \text{ch} : K(X) \to H^*(X, \mathbb{Q}) \) is the Chern character map.

For an algebraic set \( Y, e(Y) := \sum_{p, q} \sum_k (-1)^k h^{p, q}(H^k(Y, \mathbb{Q}))(x^p y^q) \) denotes the virtual Hodge polynomial of \( Y \). If \( Y \) is a smooth projective manifold, then \( e(Y) \) is the Hodge polynomial of \( Y \).

Assume that \( \sigma : X \to C \) is an elliptic ruled surface. Thus \( C \) is an elliptic curve and \( \sigma \) is a \( \mathbb{P}^1 \)-bundle morphism. \( C_0 \) denotes a minimal section of \( \sigma \) and \( g \) a fiber of \( \sigma \). We set \( e := -(C_0^2) \). Then we have \( (g^2) = 0, (g \cdot C_0) = 1 \) and \( (C_0^2) = -e \).
2.1 Basic facts

Let $X$ be a smooth projective surface. Let $H$ be an ample divisor on $X$ and $\alpha$ a $\mathbb{Q}$-divisor on $X$. For a coherent sheaf $E$ on $X$, an $\alpha$-twisted Euler characteristic $\chi_{\alpha}(E)$ of $E$ is defined by $\chi(E(-\alpha)) = \int_X ch(E)e^{-\alpha}td_X$. Matsuki and Wentworth [12] defined the $\alpha$-twisted stability of a torsion free sheaf $E$ by using twisted Hilbert polynomial $\chi_{\alpha}(E(nH))$. It is generalized to 1-dimensional sheaf in [18]. For $(r, \xi, \chi) \in \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z}$, $M^\alpha_H(r, \xi, \chi)$ denotes the moduli space of $\alpha$-twisted stable sheaves $E$ on $X$ with $(rk E, c_1(E), \chi(E)) = (r, \xi, \chi)$ and $\overline{M}^\alpha_H(r, \xi, \chi)$ the projective compactification by adding $S$-equivalence classes of $\alpha$-twisted semi-stable sheaves (see [12] for $r > 0$ and [18, Thm. 4.7] for $r = 0$).

Then the Hilbert polynomials of the Hilbert schemes $\text{Hilb}^n_X$ of $n$ points on $X$ are give by

$$\sum_n e(\text{Hilb}^n_X)q^n = \prod_{a > 0} Z_{x,y}(X, (xy)^{-1}(xyq)^a)$$

(see [4]).

**Proposition 2.1** Let $\pi : X \to C$ be an elliptic ruled surface such that $-K_X$ is nef. Assume that $\gcd(r, \xi, \chi) = 1$ and $H$ is general with respect to $(r, \xi, \chi)$.

1. $M_H(r, \xi, \chi)$ is a smooth projective manifold with

$$\dim M_H(r, \xi, \chi) = -2r \chi - r(\xi \cdot K_X) + (\xi^2) + 1$$

unless $r = (\xi \cdot K_X) = 0$.

2. Assume that $r \neq 0$ or $(\xi \cdot K_X) \neq 0$. Then the deformation class of $M_H(r, \xi, \chi)$ is independent of $X$. In particular $e(M_H(r, \xi, \chi))$ is independent of the choice of $X$.

**Proof** (1) It is sufficiently to prove $\text{Ext}^2(E, E) = 0$ for $E \in M_H(r, \xi, \chi)$. If $r > 0$, then $(K_X \cdot H) < 0$ implies the claim. If $r = 0$, the the claim will be proved in Proposition 2.7.

(2) Let $T$ be a smooth curve and $\phi : \mathcal{E}$ a family of semi-stable vector bundles $\mathcal{E}_t$ ($t \in T$) of rank 2 and degree 0 or $-1$ on $C$. Then we have a family of $\mathbb{P}^1$-bundles $\mathcal{X} := \mathbb{P}(\mathcal{E}) \to C \times T \to T$. Let $\mathcal{L} := \mathcal{O}(\mathcal{E})(1)$ be the tautological line bundle of the projective bundle. Then $\text{NS}(\mathcal{X}_t) = \mathbb{Z}(c_1(\mathcal{L})_t - zg, (c_1(\mathcal{L})_t - g) = 1$ and $(c_1(\mathcal{L})^2_t) = -e$. For a family of ample divisors $\mathcal{H} = xc_1(\mathcal{L}) + yg$, the nefness of $K_{\mathcal{X}_t}$ and the proof of (1) imply that we have a
smooth family of moduli spaces \( M(\mathcal{X}, \mathcal{H})/T(r, \xi, \chi) \rightarrow T \). In particular \( e(M_{\mathcal{H}}(r, \xi, \chi)) \) is independent of \( t \in T \).

**Remark 2.2** Since \(-K_X\) is numerically equivalent to an effective divisor for any elliptic ruled surface, if \( r > 0 \), then the claim (1) holds without assuming the nefness of \(-K_X\).

**Definition 2.3** Let \( X \) be an elliptic ruled surface such that \(-K_X\) is nef. Then \( M_{-K_X + k\mathcal{H}}(r, \xi, \chi) \) is independent of \( k \gg 0 \). We set \( M(r, \xi, \chi) := M_{-K_X + k\mathcal{H}}(r, \xi, \chi) (k \gg 0) \).

### 2.2 Moduli of stable 1-dimensional sheaves.

Let \( X \) be a smooth projective surface.

**Lemma 2.4** Assume that \(-K_X\) is nef. For a \( \mathbb{Q} \)-divisor \( \alpha \), let \( E \) and \( F \) be \( \alpha \)-twisted semi-stable sheaves of dimension 1 such that
\[
\frac{\chi_\alpha(E)}{(c_1(E) \cdot H)} \geq \frac{\chi_\alpha(F)}{(c_1(F) \cdot H)}.
\]
Then \( \text{Hom}(E, F(K_X)) = 0 \) if one of the following conditions hold:

1. \( \frac{\chi_\alpha(E)}{(c_1(E) \cdot H)} > \frac{\chi_\alpha(F)}{(c_1(F) \cdot H)} \).
2. \(-K_X\) is ample.
3. \( E \) is \( \alpha \)-twisted stable and \((c_1(E) \cdot K_X) \neq 0\).
4. \( F \) is \( \alpha \)-twisted stable and \((c_1(F) \cdot K_X) \neq 0\).

**Proof** Assume that there is a non-trivial homomorphism \( \varphi : E \rightarrow F(K_X) \). We set \( G := \varphi(E) \). Then
\[
\frac{\chi_\alpha(E)}{(c_1(E) \cdot H)} \leq \frac{\chi_\alpha(G)}{(c_1(G) \cdot H)}.
\]
Since \( G(-K_X) \) is a subsheaf of \( F \), we also have
\[
\frac{\chi_\alpha(G)}{(c_1(G) \cdot H)} = \frac{\chi_\alpha(G(-K_X))}{(c_1(G) \cdot H)} \leq \frac{\chi_\alpha(F)}{(c_1(F) \cdot H)}.
\]
Since \((c_1(G) \cdot K_X) \leq 0\), we get \((c_1(G) \cdot K_X) = 0\) and
\[
\frac{\chi_\alpha(E)}{(c_1(E) \cdot H)} = \frac{\chi_\alpha(G)}{(c_1(G) \cdot H)} = \frac{\chi_\alpha(F)}{(c_1(F) \cdot H)}.
\]
In particular cases (1) and (2) do not occur. If \( E \) is \( \alpha \)-twisted stable, then \( E \cong G \), which implies \((c_1(E) \cdot K_X) = 0\). If \( F \) is \( \alpha \)-twisted stable, then \( G \cong F \), which implies \((c_1(F) \cdot K_X) = 0\). Thus cases (3) and (4) do not occur. Therefore \( \text{Hom}(E, F(K_X)) = 0 \).

**Proposition 2.5** (cf. [18, Prop. 2.7]) Assume that \(-K_X\) is nef. Then \( e(M_{\mathcal{H}}^0(0, \xi, \chi)) \) is independent of the choice of a general \((H, \eta)\).

**Proof** Thanks to Lemma 2.4, we can show that the claim of [18, Prop. 2.6] holds. Hence the claim holds (see [18, Prop. 2.7]).
Remark 2.6 If $\chi \neq 0$, then there is a general $(H, \eta)$ with $\eta = 0$ (see [17, Lem. 1.2]).

Proposition 2.7 Assume that $-K_X$ is nef and $(\xi \cdot K_X) \neq 0$. Then $\text{Ext}^2(E, E) = 0$ for $E \in M_H^0(0, \xi, \chi)$.

Proof For an $\alpha$-twisted stable sheaf $E \in M_H^0(0, \xi, \chi)$, Lemma 2.4 implies $\text{Ext}^2(E, E) = \text{Hom}(E, (K(X))^\vee) = 0$.

Remark 2.8 Assume that $-K_X$ is nef and $(\xi \cdot K_X) = 0$. Then $E \in M_H^0(0, \xi, \chi)$ satisfies $\chi(E(K_X)) = \chi$. In this case, $\text{Ext}^2(E, E) = 0$ if and only if $E \not\cong E(K_X)$.

For a purely 1-dimensional sheaf $E$, Div$(E)$ denotes the scheme-theoretic support of $E$. We have a morphism

$$M_H^0(0, \xi, \chi) \to \text{Hilb}_X^\xi E \mapsto \text{Div}(E).$$

Assume that $h^2(O_X) = 0$ and $\xi - K_X$ is ample. Then Kodaira vanishing theorem implies $h^1(O_X(D)) = 0$ if $c_1(O_X(D)) = \xi$. Hence we see that Hilb$_X^\xi$ is smooth, by $h^1(O_D(D)) = 0$ ($D \in \text{Hilb}_X^\xi$).

Remark 2.9 Let $E$ be a stable 1-dimensional sheaf with $(K_X \cdot \text{Div}(E)) < 0$. If $(K_X \cdot C) \leq 0$ for all irreducible components $C$ of Div$(E)$, then the proof of Proposition 2.7 implies $\text{Ext}^2(E, E) = 0$. On the other hand we have an example of $E$ such that $\text{Ext}^2(E, E) \neq 0$. Thus the nefness is a reasonable condition to ensure the smoothness of the moduli space.

Example 2.10 Assume that $X$ is a ruled surface with a minimal section $C_0$. Assume that $e := -(C_0^2) > 0$. Let $E$ be a stable 1-dimensional sheaf such that Div$(E) = C_0 + C$ and $C_0$ intersects $C$ properly. Then $L := E|_{C_0}/(\text{torsion})$ is a line bundle on $C_0$. Assume that $h^0(O_{C_0}(K_X - C)) = h^0(K_{C_0}(-C_0 - C)) \neq 0$. We set $F := \ker(E \to E|_{C_0}/(\text{torsion}))$. $F$ is a line bundle on $C_0$. Then we have an injective homomorphism $L(-C) \to F$. Hence we get a non-zero homomorphism $E \to L \to F(C) \to F(K_X) \to E(K_X)$.

3 Moduli spaces of stable 1-dimensional sheaves on $X$.

3.1 An elliptic ruled surface with $e = -1$.

Let $\sigma : X \to C$ be an elliptic ruled surface with $e = -1$. Thus we have $(C_0^2) = -e = 1$. Then $-K_X \equiv 2C_0 - g$ and $| -2K_X|$ defines an elliptic fibration $\pi : X \to \mathbb{P}^1$. $\pi$ has three multiple fibers $\Pi_1, \Pi_2, \Pi_3$ of multiplicity 2 and $O_X(K_X) \cong \pi^*(O_{\mathbb{P}^1}(-2))(\Pi_1 + \Pi_2 + \Pi_3)$. We set $f_0 := \Pi_1$. Then $f_0 \equiv -K_X \equiv 2C_0 - g$. We also have $(f_0 \cdot C_0) = 1$, $(f_0 \cdot g) = 2$ and

$$\text{NS}(X) = \mathbb{Z}C_0 + \mathbb{Z}g = \mathbb{Z}C_0 + \mathbb{Z}f_0.$$

As we mentioned in [15, sect. 0], we know the Hodge numbers of $M_H(2, c_1, \chi)$ with $(c_1 \cdot g) = 1$. Thus we get the generating function

$$\sum_{\chi} e(M_H(2, c_1, \chi)) q^{-\frac{1}{2}(c_1 \cdot K_X) + \frac{(c_1^2)}{2}}$$

for $c_1 = C_0 - g$ and $c_1 = C_0$. 

Proposition 3.1

\[
\sum_m e(M(2, C_0 - g, m))q^{-m - \frac{3}{4}} = \frac{(x - 1)^2(y - 1)^2}{xy - 1} \left( \sum_{a \geq 0, 2b - a \geq 0} (x^2y^2q)^{\frac{(4b + 1 - 2a)(2a + 1)}{4}} (xy)^{\frac{(4b + 1 - 2a)}{2}} \right) - \sum_{a < 0, 2b - a < 0} (x^2y^2q)^{\frac{(4b + 1 - 2a)(2a + 1)}{4}} (xy)^{\frac{(4b + 1 - 2a)}{2}} \times \prod_{a \geq 1} Z_{x,y}(X, x^{-1}y^{-1}(x^2y^2q)^a)^2
\]

and

\[
\sum_m e(M(2, C_0, m))q^{-m + \frac{1}{4}} = \frac{(x - 1)^2(y - 1)^2}{xy - 1} \left( \sum_{a \geq 0, 2b - a \geq 0} (x^2y^2q)^{\frac{(4b + 1 - 2a)(2a + 1)}{4}} (xy)^{\frac{(4b + 1 - 2a)}{2}} \right) - \sum_{a < 0, 2b - a < 0} (x^2y^2q)^{\frac{(4b + 1 - 2a)(2a + 1)}{4}} (xy)^{\frac{(4b + 1 - 2a)}{2}} \times \prod_{a \geq 1} Z_{x,y}(X, x^{-1}y^{-1}(x^2y^2q)^a)^2
\]

**Proof** Let us briefly explain our computation. For a stable sheaf \( E \in M_H(2, c_1, \chi) \), we set \( c_2 := c_2(E) \). Then \( -\chi - \frac{1}{2}(c_1 \cdot K_X) + \frac{(c_1^2)}{4} = c_2 - \frac{(c_1^2)}{4} \) by (2.1). Assume that \( 2 \nmid (c_1 \cdot g) \) and \( 2 \nmid (c_1 \cdot f_0) \). Then

\[
((2D - c_1) \cdot f_0) \equiv ((2D - c_1) \cdot g) \equiv 1 \mod 2 \quad (3.3)
\]

for all \( D \in \text{NS}(X) \). By the proof of [15, Prop. A.4, Thm. A.5], [7, 14] and (3.3), we get

\[
e(M_H(2, c_1, \chi)) = \sum_{l, f} \left( \sum_{d_D = 0} (xy)^{d_D} + \sum_{d_D > 0} (xy)^{d_D + ((2D - c_1) \cdot K_X)} \right) e(\text{Pic}^0(X))^2 (1 - xy) e(\text{Hilb}^l_\chi) e(\text{Hilb}^l_{\chi})
\]

where

\[
l + l' = c_2 + (D \cdot (D - c_1)) = c_2 - \frac{(c_1^2)}{4} + \frac{(2D - c_1)^2}{4}.
\]
For $D' = -D + c_1, 2D' - c_1 = -(2D - c_1)$ and $d_{D'} = d_D + ((2D - c_1) \cdot K_X)$. Hence

$$\sum_{((2D-c_1) \cdot g) > 0} (xy)^{d_D + ((2D-c_1) \cdot K_X)} = \sum_{((2D-c_1) \cdot g) < 0} (xy)^{d_{D'}}.$$  

(3.6)

We write $D = -aC_0 + bf \ (a, b \in \mathbb{Z})$. By using the formula (2.3) of Göttche and Soergel [4], we get (3.1) and (3.2).

\[\Box\]

**Remark 3.2** The relation $l + l' = c_2 - (D, D + F)$ in the statement of [15, Thm. A.5] should be $l + l' = c_2 + (D, D + F)$, where $c_1 = -F$.

**Remark 3.3** Göttche independently computed Hodge numbers by using virtual Hodge polynomials and chamber structures of polarizations [2, Thm. 4.4].

We define two injective maps

$$v_i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, \ (i = 1, 2)$$

by

$$v_1(a, b) := (a, 2b - a), \ v_2(a, b) := (a, 2b - a - 1).$$

(3.7)

We note that $(a, n) \in \text{im} \ v_1$ if and only if $n \equiv a \mod 2$ and $(a, n) \in \text{im} \ v_2$ if and only if $n \not\equiv a \mod 2$. Hence we get

$$\text{im} \ v_1 \cap \text{im} \ v_2 = \emptyset, \ \text{im} \ v_1 \cup \text{im} \ v_2 = \mathbb{Z} \times \mathbb{Z}.$$  

(3.8)

If $(a, n) \in \text{im} \ v_1$, then $2n + 1 = 4b - 2a + 1$ and $n \geq 0$ if and only if $2b - a \geq 0$. If $(a, n) \in \text{im} \ v_2$, then $2n + 1 = 4b - 2a - 1$ and $n \geq 0$ if and only if $2b - a > 0$. By summing (3.1) and (3.2), we get

$$\sum_m e(M(2, C_0, m))q^{-m - \frac{a}{2}} + \sum_m e(M(2, C_0 - g, m))q^{-m - \frac{a}{2}}$$

$$= \frac{(x - 1)^2(y - 1)^2}{xy - 1} \left( \sum_{\substack{a \geq 0 \\ n \geq 0}} (x^2y^2q)^{\frac{(2n+1)(2a+1)}{4}}(xy)^{\frac{(2n+1)}{2}} - \sum_{\substack{a < 0 \\ n < 0}} (x^2y^2q)^{\frac{(2n+1)(2a+1)}{4}}(xy)^{\frac{(2a+1)}{2}} \right)$$

$$\times \prod_{a \geq 1} Z_{x, y}(X, x^{-1}y^{-1}(x^2y^2q)^a)^2.$$  

(3.9)

### 3.2 An autoequivalence of $X$.

In this subsection, we shall relate the Hodge numbers of the moduli spaces of stable 1-dimensional sheaves to (3.9) by the relative Fourier-Mukai transform of Bridgeland [1]. We note that $2f_0$ is algebraically equivalent to a smooth fiber $f$ of the elliptic fibration $\pi$. We set $Y := M_H(0, 2f_0, 1)$. Then $Y$ is a fine moduli space, which is a smooth projective surface. $Y$ has an elliptic fibration $Y \to C$ which is a compactification of the relative Picard scheme $\text{Pic}^1_{X'/C'} \to C'$, where $X' = X \setminus \bigcup_i \Pi_i$ and $C' = \pi(X')$. Let $P$ be a universal family on $X \times Y$. We set $Q := P^\vee[1]$. Then $P$ and $Q$ are coherent sheaves on $X \times Y$ and they are flat.
over $X$ and $Y$ ([1, Lem. 5.1]). We note that $Q_{|X \times \{y\}} \in M_H(0, 2f_0, -1)$ for all $y \in Y$. Let us consider a Fourier-Mukai transform $\Phi^P_{X \to Y} : D(X) \to D(Y)$ defined by

$$\Phi^P_{X \to Y}(E) := \mathbb{R} p_Y \ast (\mathcal{O}_Y \otimes p_X^*(E)), \quad E \in D(X)$$

where $p_X$ and $p_Y$ are projections from $X \times Y$ to $X$ and $Y$ respectively.

By [13, Thm. 1.1], we have an identification $K \cong X$ as elliptic surfaces over $C$. We denote divisors on $Y$ corresponding to $C_0, f_0, f, g \subset X$ via the identification $X \cong Y$ by the same symbols $C_0, f_0, f, g$.

We note that $\text{Hom}(P_{|X \times \{y\}}, \mathcal{O}_X) = 0$ and $\text{Ext}^1(P_{|X \times \{y\}}, \mathcal{O}_X) = H^0(X, Q_{|X \times \{y\}}) = 0$ for all $y \in Y$. Hence $\Phi^P_{X \to Y}(\mathcal{O}_X)[2]$ is a line bundle on $Y$. Replacing the family $P$, we may assume that $\Phi^P_{X \to Y}(\mathcal{O}_X)[2] = \mathcal{O}_Y$. We note that $K(X)_{\text{top}}$ is generated by $\mathcal{O}_X, \mathcal{O}_X(C_0), \mathcal{O}_X(f_0), C_x$. For these generators, we get the following.

**Lemma 3.4** (1) We have the following relation in $K(Y)_{\text{top}}$.

$$\begin{align*}
\Phi^P_{X \to Y}(\mathcal{O}_X) &= \mathcal{O}_Y \\
\Phi^P_{X \to Y}(\mathcal{O}_X(C_0)) &= -\mathcal{O}_Y(-C_0 + f_0) \\
\Phi^P_{X \to Y}(\mathcal{O}_X(f_0)) &= \mathcal{O}_Y(f_0) \\
\Phi^P_{X \to Y}(C_x) &= -\mathcal{O}_f + C_x.
\end{align*}$$

(3.10)

(2) If $\text{ch}(E) = (r, sC_0 + tf_0, a)$, then

$$\text{ch} \Phi^P_{X \to Y}(E) = (r - 2s, sC_0 + (t - 2a)f_0, a).$$

In particular

$$(c_1(E) \cdot K_X) = (c_1(\Phi^P_{X \to Y}(E)) \cdot K_Y).$$

**Proof** We only prove (1). Since $\chi(\mathcal{O}_X, C_x) = \chi(\mathcal{O}_Y[-2], P_{|X \times \{y\}}) = -\chi(Q_{|X \times \{y\}}, -1$. Thus $Q_{|X \times \{y\}}$ is a line bundle of degree $-1$ on $f$ if $x \in f$.

Since $\text{Ext}^2(P_{|X \times \{y\}}, \mathcal{O}_X(C_0)) = \text{Hom}(\mathcal{O}_X(C_0), P_{|X \times \{y\}}) = 0$ for all $y \in Y$, $\Phi^P_{X \to Y}(\mathcal{O}_X(C_0))[1]$ is a line bundle on $Y$. Thus we can write $\Phi^P_{X \to Y}(\mathcal{O}_X(C_0))[1] = \mathcal{O}_Y(D)$ for a divisor $D$. By

$$1 = \chi(\mathcal{O}(\mathcal{O}_X(C_0), C_x) - \chi(\mathcal{O}_Y(D), Q_{|X \times \{y\}}) = -(D \cdot f) - 1,$$

we get $(D \cdot f_0) = -1$. Since $1 = \chi(\mathcal{O}(\mathcal{O}_X(C_0))) = \chi(\mathcal{O}_Y(D))$, we get $D \equiv -C_0 + f_0$. By $\Phi^P_{X \to Y}(\mathcal{O}_X(-K_X)) = \mathcal{O}_Y(-K_Y)$, we get $\Phi^P_{X \to Y}(\mathcal{O}_X(f_0)) = \mathcal{O}_Y(f_0)$.

**Proposition 3.5** For a sufficiently large $k$, $\Phi^P_{X \to Y}^{[2]}$ induces an isomorphism

$$M(2p, pC_0 + (l + 2k)p_0, \chi + kp) \cong M^\eta(0, 4\Delta C_0 + \frac{p - 4\Delta}{2}g, \chi + kp),$$

(3.12)

where $\eta$ is a suitable $\mathbb{Q}$-divisor and $\dim M(2p, pC_0 + (l + 2k)p_0, \chi + kp) = 4p\Delta + 1$.

**Proof** Since we use [19, Prop. 3.4.5], let us explain some notations and definitions. First of all, $G$-twisted stability is the stability defined by using $G$-twisted Hilbert polynomial $\chi(G^\vee \otimes E(nH))$ instead of the Hilbert polynomial $\chi(E(nH))$, where $G \in K(X)$ satisfies $\text{rk} G > 0$. This stability is equivalent to the $\alpha$-twisted stability if $\alpha = c_1(G)/\text{rk} G$. If $H$ is a general polarization, then the $G$-twisted stability for a torsion free sheaf is the same as the
usual Gieseker stability. In particular $G_1$-twisted stability in [19, Prop. 3.4.5] is the same as the usual stability.

We next explain the functor $\Psi$ and the sheaves $G_1, G_2$. By [19, 3.2],

$$\Psi(E)[1] = \mathbf{R}\text{Hom}_{p_Y}(p_X^*(E), \mathbf{P})[1] = (\Phi_{X \to Y}^\vee(E(K_Y))[2])^\vee[1]$$

$$= (\Phi_{X \to Y}^\vee(E)[1])^\vee(-K_Y).$$

(3.13)

$G_1$ is a locally free sheaf on $X$ such that $(\text{rk } G_1, c_1(G_1)) = (2(H \cdot f), H)$ (which shows $\chi(G_1, \mathbf{P}|_{X \times \{y\}}) = 0$) and

$$G_2 = \Psi(\mathcal{O}_X)[1] = (\Phi_{X \to Y}(\mathcal{O}_C)[1])^\vee(-K_Y),$$

where $C \in |H|$.

We set $\alpha_2 := c_1(G_2)/\text{rk } G_2$. Then [19, Prop. 3.4.5] and Lemma 3.4 imply that we have an isomorphism

$$M(2p, pC_0 + (l + 2kp)f_0, \chi + kp) \to M^{\alpha_2}(0, 4\Delta C_0 + \frac{p-4\Delta}{2}g, -(\chi + (k-1)p))$$

$$E \quad \leftrightarrow \quad (\Phi_{X \to Y}^\vee(E)[1])^\vee(-K_Y)$$

(3.14)

provided $\chi(G_2, (\Phi_{X \to Y}^\vee(E)[1])^\vee(-K_Y)) < 0$. In particular we can apply this result for a sufficiently large $k$. For a purely 1-dimensional sheaf $F'$ on $Y$, $F := F^\vee(-K_Y)[1]$ is a purely 1-dimensional sheaf, $F^\vee(-K_Y)[1] = F'$ and

$$\chi(G_2, F') = \chi(F'^\vee, G_2^\vee) = -\chi(G_2^\vee(-2K_Y), F).$$

(3.15)

Hence we have an isomorphism

$$M^\eta(0, 4\Delta C_0 + \frac{p-4\Delta}{2}g, \chi + (k-1)p) \to M^{\alpha_2}(0, 4\Delta C_0 + \frac{p-4\Delta}{2}g, -(\chi + (k-1)p))$$

$$F \quad \leftrightarrow \quad F'^\vee[1](-K_Y),$$

(3.16)

where $\eta := -\frac{c_1(G_2)}{\text{rk } G_2} - 2K_Y$. By the isomorphisms (3.14) and (3.16), we get our claim. \(\Box\)

**Remark 3.6** We set $\xi := 4\Delta C_0 + \frac{p-4\Delta}{2}g$. If $\Delta > 0$, then $\xi$ is ample by [6, Prop. 2.21]. Hence $\text{Hilb}_X^\xi$ is smooth of dimension $\frac{4\Delta+1}{2}$.

Assume that $p = 1$ and $4\Delta \geq 3$. We take $D \in \text{Hilb}_X^\xi$. For all smooth fiber $f$ of $\pi$, $h^1(\mathcal{O}_X(D - f)) = 0$. Hence $H^0(X, \mathcal{O}_X(D)) \to H^0(f, \mathcal{O}_f(D))$ is surjective. Therefore the base point of $|D|$ is a subset of multiple fibers. By the theorem of Bertini (see [6, Rem. 10.9.2]), a general member of $D \in \text{Hilb}_X^\xi$ is smooth on $X \setminus \bigcup_{i=1}^3 \Pi_i$. Since $(D \cdot \Pi_i) = 1$, $D$ is smooth in a neighborhood of $\bigcup_{i=1}^3 \Pi_i$. Therefore a general member of $D \in \text{Hilb}_X^\xi$ is a smooth curve on $X$. In particular a general fiber of $M_0^\eta(0, \xi, \chi) \to \text{Hilb}_X^\xi$ is the Picard variety $\text{Pic}^\chi(D)$ of a smooth curve $D$ parameterizing line bundles $L$ on $D$ with $\chi(L) = \chi$.

We note that $4\Delta C_0 + \frac{p-4\Delta}{2}g = pC_0 + \frac{4\Delta-p}{2}f_0$. Since $(C_0 \cdot f_0) = 1$, by using Proposition 3.5 and Proposition 2.5, we get

$$e(M(2, C_0 + lg, n)) = e(M(0, C_0 + \frac{4\Delta-1}{2}f_0, n)) = e(M_H(0, C_0 + \frac{4\Delta-1}{2}f_0, \chi)),$$

where $\chi$ is an arbitrary non-zero integer and $H$ is a general polarization (see also Proposition 3.12). Hence we get the following result from (3.9).
Proof

For a convenience sake, we write a proof. We set

\[ (\xi, f_0) = (21, 5) \]

Lemma 3.8 (see [5, 3.1]). Then we see that

\[
123
\]

\[ Z_{x, y}(X, x^{-1}y^{-1}(x^2 y^2 q)^a)^2. \]

(3.17)

In order to get a product expression of (3.17), we quote the following formula.

\[
\sum_{n, m \geq 0} q^{(n+\frac{1}{2})(m+\frac{1}{2})} x^{n+\frac{1}{2}} - \sum_{n, m < 0} q^{(n+\frac{1}{2})(m+\frac{1}{2})} x^{n+\frac{1}{2}}
= \frac{\eta(q)^4}{\eta(q^{\frac{1}{2}})^2} q^{\frac{1}{2}t} \prod_{n > 0} \frac{(1 - q^n t)(1 - q^n t^{-1})}{(1 - q^{n-1} t)(1 - q^{n-1} t^{-1})}.
\]

Proof For a convenience sake, we write a proof. We set

\[
G(\tau, x, y) := \sum_{n \geq 0, m > 0} q^{nm} e^{2\pi \sqrt{-1}(nx - my)} - \sum_{n > 0, m \geq 0} q^{nm} e^{2\pi \sqrt{-1}(nx + my)}
= \frac{\eta(\tau)^3 \theta_{11}(\tau, x + y)}{\theta_{11}(\tau, x) \theta_{11}(\tau, y)}
\]

(3.19)

(see [5, 3.1]). Then we see that

\[
q^{-\frac{1}{4}} e^{2\pi \sqrt{-1}(\tau + \frac{1}{2}, y - \frac{1}{2})} G(\tau, x + \frac{1}{2}, \tau + \frac{1}{2})
= q^{-\frac{1}{2}} e^{2\pi \sqrt{-1}(\tau + \frac{1}{2}, y + \frac{1}{2})} \left( \sum_{n \geq 0, m > 0} q^{nm} e^{2\pi \sqrt{-1}(nx - my + \frac{1}{2})} \right)
- \sum_{n > 0, m \geq 0} q^{nm} e^{2\pi \sqrt{-1}(n(x - \frac{1}{2}) + m(y + \frac{1}{2}))}
\]

(3.20)

\[
= \sum_{n \geq 0, m > 0} q^{(n+\frac{1}{2})(m+\frac{1}{2})} e^{2\pi \sqrt{-1}(x(n+\frac{1}{2}) - y(m - \frac{1}{2}))}
- \sum_{n > 0, m \geq 0} q^{(n+\frac{1}{2})(m+\frac{1}{2})} e^{2\pi \sqrt{-1}(x(n+\frac{1}{2}) + y(m + \frac{1}{2}))}
\]

\[
= - \left( \sum_{n \geq 0, m \geq 0} q^{(n+\frac{1}{2})(m+\frac{1}{2})} e^{2\pi \sqrt{-1}(x(n+\frac{1}{2}) + y(m + \frac{1}{2}))} \right) + \left( \sum_{n \geq 0, m > 0} q^{(n+\frac{1}{2})(m+\frac{1}{2})} e^{2\pi \sqrt{-1}(x(n+\frac{1}{2}) - y(m - \frac{1}{2}))} \right).
\]
Since
\[ q^{\frac{1}{2}} e^{2\pi \sqrt{-1} \tau} \theta_{11}(\tau, x + \frac{1}{2}) = -\theta_{01}(\tau, x), \]
\[ q^{\frac{1}{2}} e^{2\pi \sqrt{-1} \tau} \theta_{11}(\tau, y - \frac{1}{2}) = \theta_{01}(\tau, y), \]
we get
\[ \sum_{n \geq 0, m \geq 0} q^{(n+\frac{1}{2})(m+\frac{1}{2})} e^{2\pi \sqrt{-1}(x(n + \frac{1}{2}) + y(m + \frac{1}{2}))} \]
\[ \quad - \sum_{n < 0, m < 0} q^{(n+\frac{1}{2})(m+\frac{1}{2})} e^{2\pi \sqrt{-1}(x(n + \frac{1}{2}) + y(m + \frac{1}{2}))} = \frac{\eta(\tau)^3 \theta_{11}(\tau, x + y)}{\theta_{01}(\tau, x) \theta_{01}(\tau, y)}. \] (3.22)

Substituting \( y = 0 \) and setting \( t = e^{2\pi \sqrt{-1} \tau} \), we have
\[ \sum_{n \geq 0, m \geq 0} q^{(n+\frac{1}{2})(m+\frac{1}{2})} t^{(n+\frac{1}{2})} - \sum_{n < 0, m < 0} q^{(n+\frac{1}{2})(m+\frac{1}{2})} t^{(n+\frac{1}{2})} = \frac{\eta(\tau)^3 \theta_{11}(\tau, x)}{\theta_{01}(\tau, x) \theta_{01}(\tau, 0)}. \] (3.23)

By the triple multiple formula of theta functions (cf. [3, cf. (2.5)]), we get our claim. \( \square \)

**Proof of Theorem 1.1.** By using Lemma 3.8, we get Theorem 1.1 from Lemma 3.7:
\[ \sum_{(\xi, \varphi) = \mathbb{I}} e(M_H(0, \xi, \chi)) q^{a_1} \]
\[ = \frac{(x - 1)^2(y - 1)^2}{xy - 1} \frac{(x - 1)^2(y - 1)^2}{xy - 1} (x^2 y^2 q^2)^2 \prod_{n > 0} \frac{(1 - (x^2 y^2 q^2)^n)}{(1 - (xy q^2)^n)^2} \]
\[ \times \prod_{n > 0} \frac{(1 - (x^2 y^2 q^2)^n (xy)) (1 - (x^2 y^2 q^2)^n (xy)^{-1})}{(1 - (xy q^2)^n)^2} \prod_{n > 0} \frac{(1 - (x^2 y^2 q^2)^n (xy)) (1 - (x^2 y^2 q^2)^n (xy)^{-1})}{(1 - (xy q^2)^n)^2} \]
\[ = (x - 1)^2(y - 1)^2 q^2 \prod_{n > 0} \frac{(1 - (x^2 y^2 q^2)^n)}{(1 - (xy q^2)^n)^2} \prod_{n > 0} \frac{(1 - (x^2 y^2 q^2)^n (xy)) (1 - (x^2 y^2 q^2)^n (xy)^{-1})}{(1 - (xy q^2)^n)^2} \]
\[ \times \prod_{n > 0} \frac{(1 - (x^2 y^2 q^2)^n (xy)) (1 - (x^2 y^2 q^2)^n (xy)^{-1})}{(1 - (xy q^2)^n)^2} \]
\[ = (x - 1)^2(y - 1)^2 q^2 \prod_{n > 0} \frac{1 - (x^{-1} (x^2 y^2 q^2)^n)^2 (1 - y^{-1} (x^2 y^2 q^2)^n)^2 (1 - (x^2 y^2 q^2)^n)^2 (1 - (x^2 y^2 q^2)^n)^2}{(1 - (xy)^{-1} (x^2 y^2 q^2)^2) (1 - (xy) (x^2 y^2 q^2)^2)} \]. (3.24)

**Remark 3.9** By Theorem 1.1, we see that \( h^{0,0}(M_H(0, \xi, \chi)) = 1 \) for all \( \xi \). In particular they are irreducible. Let \( S \) be the open subscheme of \( \text{Hilb}^2_X \) consisting of smooth curves \( D \) and \( D \subset S \times X \) the universal family. By Remark 3.6, \( M_H(0, \xi, \chi) \) contains the relative Picard scheme \( \text{Pic}^2_{D/S} \) as an open dense subscheme. Hence the birational equivalence class of \( M_H(0, \xi, \chi) \) is independent of the choice of \( (H, \alpha) \).

### 3.3 An elliptic ruled surface with \( e = 0 \)

We shall treat the case where \( e = 0 \). We first assume that \( X = C \times \mathbb{P}^1 \). Then the projection \( \pi : X \to \mathbb{P}^1 \) is an elliptic fibration and \( K_X = -2C_0 \). We may assume that \( g \) is a 0-section of \( \pi \). We set \( z = C_0 \cap g \).

We set \( Y := M_H(0, C_0, 0) \). Then there is a universal family \( \mathcal{P} \) on \( X \times Y \). We may assume that \( \mathcal{P}|_{g \times Y} \cong \mathcal{O}_Y \). Moreover we can identify \( Y \) with \( X \). Then we define \( C_0, g, z \subset Y \) via the
identification $X \cong Y$. Let $\Phi^P_{Y \to X} : D(Y) \to D(X)$ be the Fourier-Mukai transform whose kernel is $P$. Then we get

$$
\begin{align*}
\Phi^P_{Y \to X}(O_Y) &= O_g[-1], \\
\Phi^P_{Y \to X}(O_g) &= O_X, \\
\Phi^P_{Y \to X}(O_{C_0}) &= O_C[-1], \\
\Phi^P_{Y \to X}(O_C) &= O_{C_0}.
\end{align*}
$$

(3.25)

If $\text{ch}(E) = (x, rg + yC_0, a)$, then $\text{ch}(\Phi^P_{Y \to X}(E)) = (r, -xg + aC_0, -y)$. Thus we get the following by using [19, Prop. 3.4.5].

**Proposition 3.10** $\Phi^P_{X \to Y}$ induces an isomorphism $M(p, aC_0, -l) \to M(0, pg + lC_0, a + p)$.

By using Proposition 2.1, we get the following result.

**Theorem 3.11** Let $X$ be an elliptic ruled surface with $e = 0$.

$$
\sum_n e(M_H(0, g + nC_0, 1))q^n = (x - 1)(y - 1) \sum_n e(\text{Hilb}_X^n)q^n
$$

$$
= (x - 1)(y - 1) \prod_{n>0} \frac{(1 - x^{-1}(x^2y^2q)^2)(1 - y^{-1}(x^2y^2q)^2)(1 - x(x^2y^2q)^2)(1 - y(x^2y^2q)^2)}{(1 - x^{-1}y^{-1}(x^2y^2q)^2)(1 - (x^2y^2q)^2)(1 - xy(x^2y^2q)^2)}.
$$

(3.26)

$$
\sum_n e(M_H(0, 2g + nC_0, 3))q^n
= \frac{(x - 1)^2(y - 1)^2}{xy - 1} \left( \sum_{a \geq 0, b > 0} (x^2y^2q)^{b(2a+1)}(xy)^{2b} - \sum_{a < 0, b < 0} (x^2y^2q)^{b(2a+1)}(xy)^{2b} \right)
\times \prod_{a \geq 1} Z_{x,y}(X, x^{-1}y^{-1}(x^2y^2q)^a).
$$

(3.27)

**Proposition 3.12** Assume that $\gcd(r, n) = 1$. Then $e(M_H(0, rg + nC_0, \chi))$ is independent of the choice of $\chi$, where $H$ is a general polarization.

**Proof** Since $\gcd(r, n) = 1$, there is a divisor $\eta$ such that $(\eta \cdot (rg + nC_0)) = 1$. Then $M_H(0, rg + nC_0, 1) \cong M_H^{\eta^{-1}}(0, rg + nC_0, \chi)$. Since $e(M_H^{\eta^{-1}}(0, rg + nC_0, \chi)) = e(M_H(0, rg + nC_0, \chi))$ by Proposition 2.5, we get our claim. \hfill \Box

**Corollary 3.13** Assume that $\gcd(r, n) = 1$. Then $e(M(r, dC_0, n))$ is independent of the choice of $d$.

**Remark 3.14** By using Proposition 2.1 and [18, Thm. 0.2], we have

$$
e(M(r, pg + dC_0, n)) = e(\text{Hilb}_X^{-rn+rp+pd})e(C)
$$

if $\gcd(r, p) = 1$. 

\[ Springer \]
Appendix

Assume that $X$ is an elliptic ruled surface with $e = -1$. By using Fourier-Mukai transforms, we can derive the Hodge numbers of some moduli spaces of stable sheaves of rank $r > 0$ from Theorem 1.1 and (2.3).

**Theorem 4.1** Assume that $\gcd(r, d_1) = 1$.

1. If $r$ is even, then
   \[ e(M(r, d_1 C_0 + d_2 f_0, \chi)) = e(M(0, \xi, \chi)), \]
   where $(\xi \cdot K_X) = -1$ and $(\xi^2) = -2r \chi + rd_1 + d_1^2 + 2d_1d_2$.

2. If $r$ is odd, then
   \[ e(M(r, d_1 C_0 + d_2 f_0, \chi)) = e(\text{Hilb}^n_X \times \text{Pic}^0(X)) = e(\text{Hilb}^n_X) e(C), \]
   where $2n = -2r \chi + rd_1 + d_1^2 + 2d_1d_2$.

**Proof** (1) We set $Y := M_H(0, r f_0, d_1)$. Then $Y$ is a fine moduli space and $Y \cong X$. Let $\Phi_{X \rightarrow Y} : D(X) \rightarrow D(Y)$ be the Fourier-Mukai transform defined by a universal family $P$. Then a similar claim to Proposition 3.5 holds. By Proposition 2.5, we get the claim (1).

(2) is a consequence of the birational correspondence in [20].

**Remark 4.2** By Remark 3.9 and the proof of Theorem 4.1, we also see that $M(r, d_1 C_0 + d_2 f_0, \chi)$ is birationally equivalent to $M(0, \xi, \chi)$ if $r$ is even. Moreover by the computations in (3.1) and (3.2), we see that they are projective bundles over $\text{Pic}^0(X) \times \text{Pic}^0(X)$. If $r$ is odd, then we also remark that Bridgeland [1] proved that $M(r, d_1 C_0 + d_2 f_0, \chi)$ is birationally equivalent to $\text{Hilb}^n_X \times \text{Pic}^0(X)$. Thus the birational equivalence class of $M(r, d_1 C_0 + d_2 f_0, \chi)$ is determined by $2r \chi + rd_1 + d_1^2 + 2d_1d_2$ if $\gcd(r, d_1) = 1$.

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