Relativistic anisotropic pair plasmas

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ABSTRACT
The properties of waves able to propagate in a relativistic pair plasma are at the basis of the interpretation of several astrophysical observations. For instance, they are invoked in relation to radio emission processes in pulsar magnetospheres and to radiation mechanisms for relativistic radio jets. In such physical environments, pair plasma particles probably have relativistic, or even ultrarelativistic, temperatures. Besides, the presence of an extremely strong magnetic field in the emission region constrains the particles to one-dimensional motion: all the charged particles strictly move along magnetic field lines.

We take anisotropic effects and relativistic effects into account by choosing one-dimensional relativistic Jüttner–Synge distribution functions to characterize the distribution of electrons and/or positrons in a relativistic, anisotropic pair plasma. The dielectric tensor, from which the dispersion relation associated with plane wave perturbations of such a pair plasma is derived, involves specific coefficients that depend on the distribution function of particles. A precise determination of these coefficients, using the relativistic one-dimensional Jüttner–Synge distribution function, allows us to obtain the appropriate dispersion relation. The properties of waves able to propagate in anisotropic relativistic pair plasmas are deduced from this dispersion relation. The conditions in which a beam and a plasma, both ultrarelativistic, may interact and trigger off a two-stream instability are obtained from this same dispersion relation. Two astrophysical applications are discussed.

Key words: instabilities – plasmas – radiation methods: non-thermal – waves – pulsars: general – galaxies: jets.

1 INTRODUCTION
Relativistic pair plasmas are present in several astrophysical sites: in the environments of black holes and neutron stars, pulsar magnetospheres, extragalactic relativistic jets on the pc-scale and sources for gamma-ray bursts. Thus, knowledge of the properties of such plasmas is essential to the description of mechanisms such as radio emission processes in pulsar magnetospheres, radio radiation from relativistic jets and relativistic expansion of the strongly magnetized pair plasma involved in gamma-ray burst models. In all these physical situations, the magnetic field is very strong and plays a central role: all the electrons and positrons forming the pair plasma move along magnetic field lines. Consequently, astrophysical relativistic pair plasmas may be considered as one-dimensional in the direction of the magnetic field. In addition, there are two ways in which pair plasmas must be considered as relativistic: first, if their motion relative to an ideal observer is such that their velocities are relativistic, so that the associated Lorentz factors are important; secondly, if the temperature of the particles is comparable to their rest-mass energy, or even greater.

Characteristic bulk velocities (and associated Lorentz factors) of pair plasma particles, namely electrons and positrons, are deduced from specific pair creation processes involved in standard models (e.g. for either pulsar magnetospheres or for jet formation in the environment of black holes). However, the temperature of pair plasma particles is completely unknown in all the physical situations mentioned: the choice of a relativistic or non-relativistic character for the distribution function of the involved particles is presumed to be correct as long as it does not contradict the coherence of the models. Thus, the real shape of the distribution function of electrons and positrons in the pair plasma, together with their dependence on the respective temperatures, is intimately related to the process of formation of the pairs. In the hypothesis where the cascade process of pair production results from the interaction between gamma-ray photons and the...
extremely strong magnetic field, it appears quite improbable that all the photons will be emitted at the same place by particles with the same energy. One expects a relative spread in the distribution function for the photons, and subsequently a similar character for the distribution functions of resulting pairs of electrons and positrons. For instance, in view of the analysis of the reparation of particles in pulsar magnetospheres, proposed by Daugherty & Harding (1982, 1983), the width in energy of their distribution function should be of the order of the mean energy of particles, i.e. \( \langle \Delta \gamma \rangle \approx \langle \gamma \rangle \), so that the temperature of particles in the secondary pair plasma may be considered as relativistic, or even ultrarelativistic.

Characteristics of the distribution functions for pair plasma particles must be specified in relation to the expected relativistic effects. Here, we consider relativistic one-dimensional Jüttner–Synge distribution functions to characterize pair plasma particles. We derive a new dispersion relation that is adequate to study properties of the different types of waves able to propagate in such relativistic one-dimensional pair plasmas. We apply our results to investigate the particular system formed by a beam and a plasma and to obtain the conditions in which their interaction is efficient. Different alternatives were recently considered in the literature that concerned such an astrophysical context. For instance, Gedalin, Melrose & Gruman (1998), Lyutikov (1998), Melrose et al. (1999) and Melrose & Gedalin (1999) consider ultrarelativistic pair plasmas and use identical temperatures for electrons and positrons, while Asseo (1993) and Weatherall (1994) use different temperatures for each species, i.e. ultrarelativistic fast particles and weakly relativistic slow particles.

The paper is organized as follows. In Section 2, we recall the background material needed for our analysis. We choose a distribution function in Section 2.1, then derive the dielectric tensor coefficients in Section 2.2. The main component of this tensor depends on a function \( F_a(k||, \omega) \), which we present in Section 2.3. We conclude this section by deriving the dispersion relation in Section 2.4. The two next sections are the core of the paper. First, in Section 3, we study very carefully the function \( F_a(k||, \omega) \). We perform some analytical expansion of this function in two limits (Section 3.1 and Section 3.2), and give the exact expression for the coefficients of this expansion in Section 3.3. Some calculations are skipped and are detailed in Appendix A. We end this section by giving some simple solutions to the dispersion relation in Section 3.4. Secondly, in Section 4.1, we use a simpler analytical formalism to study \( F_a(k||, \omega) \). Some calculations are given in Appendices B and C. We use these results in Section 4.2 to study the relativistic two-stream instability, and we derive some conclusions about the feasibility of such a mechanism in explaining pulsar radio emission as well as the physical properties in the relativistic jets present around active galactic nuclei (AGN). The results are summarized in Section 5.

2 SPECIFIC DISTRIBUTION FUNCTIONS, AND DERIVATIONS OF THE DIELECTRIC TENSOR AND THE DISPERSION RELATION

2.1 Distribution functions

2.1.1 One-dimensional distribution function of relativistic particles

Two different specificities must be included in the choice of the distribution function that characterizes relativistic particles forming the pair plasma immersed in a strong magnetic field.

At first, the important radiation reaction of these particles moving in a strong magnetic field will modify the character of the starting distribution function of relativistic particles, and will reduce the distribution function to a one-dimensional function in the direction of the local magnetic field. Such an effect results from the synchrotron radiation of relativistic particles, which ultimately leads to a loss of their transverse momentum (transverse relative to the direction of the local magnetic field). The motion and radiation reaction of relativistic particles immersed in a strong magnetic field, together with the relaxation of their distribution function to a one-dimensional distribution function, were treated by Suvorov & Chugunov (1973). Initially, both photons at the origin of the formation of pairs and relativistic particles in the pair plasma have isotropic distribution functions. Nevertheless, particles that form pairs rapidly lose their transverse momentum through synchrotron radiation: their initially isotropic distribution function is transformed into a one-dimensional anisotropic distribution function. According to Suvorov & Chugunov (1973), the ultimate one-dimensional anisotropic distribution function for the species \( \alpha \) takes the form

\[
\hat{f}_a(p) = \hat{f}_a(p_{||}) \delta(p_{\perp}) = \hat{f}_a(p_{||}) \delta(p_{\perp})/2\pi p_{\perp},
\]

and the dependence on the parallel momentum is obtained from an integration over transverse directions where \( \delta(p_{\perp}) \) represents the Dirac function relative to the variable \( p_{\perp} \). The || and \( \perp \) directions are defined relative to the direction of the local magnetic field.

Secondly, although it is quite probable that relativistic particles in the pulsar magnetosphere do not have enough time to be thermalized, the choice of a relativistic distribution function equivalent to a thermalized distribution function of particles allows the study of different cases. One may assume that pair plasma particles are thermalized in the magnetic field direction and choose a one-dimensional relativistic distribution function. Thus, even for a non-thermal one-dimensional distribution function, it is useful to introduce a parameter that behaves like the temperature. This parameter, defined as the temperature parameter \( \mu \), is related to the mean kinetic energy of particles; then for particles of species \( \alpha \), either electrons or positrons,

\[
m_a c^2 \gamma - 1 \alpha = m_a c^2 \mu_a^{-1},
\]

where \( c \) is the velocity of light, \( \gamma \) is the Lorentz factor of particles of species \( \alpha \), electrons or positrons. Any quantity \( X \) within brackets, \( \langle X \rangle_\alpha \),
The relativistic one-dimensional Jüttner–Synge distribution function, the relativistic equivalent of the Maxwellian distribution function, is defined in the one-dimensional momentum space as

\[ f_{\alpha}^{1}(p_{\parallel}) = C_{1}(\mu_{\alpha}) \exp[-\mu_{\alpha} \gamma(p_{\parallel})], \]

with

\[ C_{1}(\mu_{\alpha}) = \frac{n_{\alpha}}{2mcK_{1}(\mu_{\alpha})} \quad \text{and} \quad \mu_{\alpha} = \frac{mc^{2}}{k_{B}T_{\alpha}}. \]

\[ \mu_{\alpha} \quad \text{is the temperature parameter; } \mu_{\alpha} \ll 1 \quad \text{characterizes an ultrarelativistic plasma and } \mu_{\alpha} \gg 1 \quad \text{characterizes a non-relativistic plasma.} \]

The transition between the non-relativistic case, with \( \mu_{\alpha} \ll 1 \), and the ultrarelativistic case, with \( \mu_{\alpha} \gg 1 \), is clear in the case of a one-dimensional Jüttner–Synge distribution function \( f_{\alpha}^{1}(p_{\parallel}) = C_{1}(\mu_{\alpha}) \exp[-\mu_{\alpha} \gamma(p_{\parallel})] \). If we change the variable \( p_{\parallel} \) into the variable \( \gamma_{\parallel} \), it is then straightforward to see that the maximum of \( f_{\alpha}(\gamma_{\parallel}) = f_{\alpha}^{1}(p_{\parallel})|p_{\parallel}|d|p_{\parallel}|/d\gamma_{\parallel} \) occurs at

\[ \gamma_{\parallel} = 0 \quad \text{if} \quad \mu_{\alpha} \gg 3 \]

and

\[ \gamma_{\parallel} = \pm \sqrt{1 - \mu_{\alpha}^{2}/9} \quad \text{if} \quad \mu_{\alpha} \leq 3. \]

The transition between the non-relativistic and relativistic regimes therefore occurs at \( \mu_{\alpha} = 3 \). More details about the distribution functions and their properties are given in Appendix B.

## 2.2 The dielectric tensor

### 2.2.1 Derivation

The physical situation is described locally. We consider that the pair plasma is pervaded by an intense straight and uniform magnetic field \( B \) (neglecting the curvature of the magnetic field in either the pulsar magnetosphere or in the description of curved jets). A cartesian geometry is adopted; we assume that \( B \) is aligned along the \( z \)-axis, whereas the wave vector \( k \), is in the \( x-z \) plane.

The dielectric tensor is simply derived from the relativistic Vlasov and Poisson equations, written for a distribution function of particles of species \( \alpha, f_{\alpha}(r, p, t) \). The pair plasma is assumed to be neutral initially and the initial electric field is assumed to equal zero. The dielectric tensor is obtained after linearization, Fourier transformation and integration of the relativistic Vlasov and Poisson equations in the momentum space.

### 2.2.2 Coefficients of the dielectric tensor

In the specified cartesian geometry, the dielectric tensor can be written in a compact form (see for instance Krall & Trivelpiece 1973). \( q_{\alpha}, m_{\alpha} \) and \( n_{\alpha} \) are, respectively, the charge, mass and number density of the species \( \alpha \). Besides, the different components of the tensor involve several quantities: the plasma frequency \( \omega_{\text{pl}} = \sqrt{q_{\alpha}^{2}n_{\alpha}/m_{\alpha}e} \); the refractive index of the plasma \( n = c|k|/\omega \) (where \( \omega \) and \( k \) are the

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1. We will, of course, also consider the case where the observer is not in the rest frame of the particles; in which case, the distribution function will be centred around \( p_{\alpha} \) instead of the origin.

2. Hereafter, we will set \( c = 1 \).
frequency and wavenumber with which we are concerned; the cyclotron frequency \( \omega_{\text{c}} = |q_{e}B/m_{e} | \) (where \( B \) is the strength of the magnetic field); the angle \( \theta \) between the wave vector \( \mathbf{k} \) and the magnetic field \( \mathbf{B} \). In addition we define the following quantities: \( \beta_{\parallel} = v_{\parallel}/c \) is the ratio of the parallel velocity of a given particle to the velocity of light; \( \gamma \) is the associated Lorentz factor;
\[
S = \sum_{\alpha} \left( \frac{\omega_{\alpha}}{\omega_{\text{c}}} \right)^{2}; \quad r = \frac{\omega_{\alpha}}{\omega_{\text{c}}}; \quad a = 1 - n_{\beta_{\parallel}} \cos \theta; \quad b = n_{\beta_{\parallel}} \sin \theta; \quad D = (1 - n_{\beta_{\parallel}} \cos \theta)^2 - \left( \frac{\omega_{\alpha}}{\omega_{\text{c}}} \right)^2 = a^2 - r^2. \tag{8}
\]

With these quantities, the coefficients of the dielectric tensor are
\[
-\varepsilon_{xy} = \varepsilon_{yx} = -iS \left( \frac{1}{\gamma D} \right)_{a}, \quad -\varepsilon_{yz} = \varepsilon_{zy} = -S \left( \frac{1}{\gamma D} \right)_{a}, \quad -\varepsilon_{xx} = \varepsilon_{yy} = -iS \left( \frac{1}{\gamma D} \right)_{a}, \quad \varepsilon_{zz} = 1 - S \left( \frac{1}{\gamma D} \right)_{a}
\]
and
\[
\varepsilon_{xc} = 1 + \sum_{a} F_{a}(k_{\parallel}, \omega) + S \left( \frac{ab^{2}}{D} \right)_{a}. \tag{9}
\]

where
\[
F_{a}(k_{\parallel}, \omega) = \left( \frac{\omega_{\alpha}}{k_{\parallel}} \right)^{2} n_{\alpha} \frac{1}{v_{\parallel} - v_{\parallel}} \frac{\partial f_{a}(p_{\parallel})}{\partial p_{\parallel}} dp_{\parallel}.
\]

2.2.3 Simplified expressions for the coefficients

In the case of a very strong magnetic field, most coefficients of the dielectric tensor can be simplified and, as the plasma is one-dimensional because of the extreme strength of the magnetic field, the only coefficient of interest is \( \varepsilon_{zz} \). One easily shows that the validity of the simplified expressions for the coefficients of the dielectric tensor is limited to very small angles of propagation \( \theta \leq \theta_{\text{crit}} \), where \( \theta \) is the angle between the wavevector and the local direction of the magnetic field and where \( \theta_{\text{crit}} = 0.03 \) rad is a value obtained using standard pulsar parameters. Let us note that, the value of \( \theta_{\text{crit}} \) being very small, the processes of emission described by the means of these simplified coefficients approximately occur in the direction of magnetic field lines. Assuming that relativistic particles in the pair plasma have Lorentz factors in the range \( 10^{2}–10^{3} \), emission at a maximum angle \( \theta_{\text{crit}} \) roughly corresponds to emission at an angle equal to a few times \( 1/\gamma \), a result somewhat reminiscent of the emission of individual relativistic particles moving along magnetic field lines. Thus, as long as the mechanism is studied not too far from the centre of the star in the context of pulsars, or not too far from the centre of the central engine in the context of relativistic radio jets, most terms in the symmetrical tensor \( \varepsilon \) can be simplified:
\[
\varepsilon_{yy} = \varepsilon_{yx} = 0, \quad \varepsilon_{zz} = \varepsilon_{zy} = 0, \quad \varepsilon_{xx} = \varepsilon_{xy} = 0, \quad \varepsilon_{xy} = \varepsilon_{xx} = 1 \quad \text{and} \quad \varepsilon_{xc} = 1 + \sum_{a} F_{a}(k_{\parallel}, \omega).
\]

In these conditions, the dispersion relation only involves the component \( \varepsilon_{zz} \) of this tensor. Such a result completely depends on the fact that the magnetic field, although not apparent in the equations, is responsible for the necessity to have a one-dimensional treatment because particles are constrained to move along magnetic field lines once their transverse momentum has been radiated. In the hypothesis where the geometry assumes straight magnetic field lines and an infinite extent of the plasma or beam, this is clear from the expression defining the function \( F_{a}(k_{\parallel}, \omega) \), as the usual term \( (\omega - k \cdot v) \) is reduced to \( (\omega - k_{\parallel}v_{\parallel}) \), in the denominator of the integrand. Different consequences for wave propagation and the two-stream instability derived from such a dispersion relation were recently investigated by Magneville (1990b), Asseo & Melikidze (1998), Gedalin et al. (1998), Melrose & Gedalin (1999) and Melrose et al. (1999).

2.3 The function \( F_{a}(k_{\parallel}, \omega) \)

As a first step in the derivation of the dispersion relation, we pay attention to the function \( F_{a}(k_{\parallel}, \omega) \):
\[
F_{a}(k_{\parallel}, \omega) = \left( \frac{\omega_{\alpha}}{k_{\parallel}} \right)^{2} n_{\alpha} \frac{1}{v_{\parallel} - v_{\parallel}} \frac{\partial f_{a}(p_{\parallel})}{\partial p_{\parallel}} dp_{\parallel}.
\]

For the case of a one-dimensional Jüttner–Syngre distribution function, the derivative is simply related to the distribution function itself,
\[
\frac{\partial f_{a}(p_{\parallel})}{\partial p_{\parallel}} = -\mu_{a} \gamma_{a} (v_{\parallel} - v_{a}) f_{a}(p_{\parallel}). \tag{13}
\]

where \( v_{a} \) is the velocity of the centre of mass of the species \( \alpha \), or else the velocity of the reference frame in which the species \( \alpha \) is at rest, and \( \gamma_{a} \) is its associated Lorentz factor. It is not possible to obtain an analytical expression for the function \( F_{a}(k_{\parallel}, \omega) \) from equation (12) because of the singularity in the factor \( (v_{\parallel} - v_{\parallel})^{-1} \) at the point where \( v_{\parallel} = v_{\parallel} \). However, the ratio of velocities can be expressed in terms of
the variable
\[ w = \frac{v_\parallel - v_\alpha}{v_\alpha - v_\parallel}, \]
\[ (14) \]
and a limited expansion in terms of this variable \( w \) can be used to obtain an expression for the function \( F_\alpha(k_\parallel, \omega) \), and to further analyse the dispersion relation (as in Magneville’s method given in Magneville 1990a,b). As \( (v_\parallel - v_\alpha)/(v_\alpha - v_\parallel) = w/(1 - w) \), the function \( F_\alpha(k_\parallel, \omega) \) can be rewritten as
\[ F_\alpha(k_\parallel, \omega) = -\frac{1}{n_\alpha} \mu_{\alpha} \gamma_{\alpha} \left( \frac{\omega_{\alpha}}{k_\parallel} \right)^2 \int_{0}^{2\pi} \frac{w}{1 - w} f_\alpha(p_\parallel) \, dp_\parallel. \]
\[ (15) \]
It is then possible to obtain asymptotic expansions of the integrand in equation (15) in two different limits according to the relative ordering of the phase velocity relatively to the width of the equilibrium distribution function. This will be presented in Section 3.

2.4 The dispersion relation
Classically, the dispersion relation is obtained in terms of the components of the dielectric tensor associated with wavenumber components and frequency using the relation
\[ \left( k_\parallel^2 \delta_{ij} - k_i k_j \right) \frac{1}{\omega^2} - \epsilon_i = 0. \]
\[ (16) \]
Within the above mentioned simplifying hypothesis it can be written, in terms of the function \( F_\alpha(k_\parallel, \omega) \) (defined above), of the frequency and of the wavenumber components relative to the local direction of the magnetic field, as
\[ (\omega^2 - k_\parallel^2)(\omega^2 - k_i^2) + \sum_\alpha F_\alpha(k_\parallel, \omega)(\omega^2 - k_\parallel^2) = 0. \]
\[ (17) \]
This dispersion relation describes both transverse waves and quasi-electrostatic longitudinal waves: the factor \( (\omega^2 - k^2) \) describes transverse waves, the second factor within brackets describes quasi-electrostatic longitudinal waves. For strict parallel propagation along magnetic field lines, thus taking \( k_\perp = 0 \) (that is \( \theta = 0 \)), this second factor is simply written as
\[ 1 + \sum_\alpha F_\alpha(k_\parallel, \omega) = 0, \]
\[ (18) \]
which is exactly the dispersion relation for purely longitudinal electrostatic waves, as is clear from comparison with previous works (e.g. Asseo, Pelletier & Sol 1990a; Magneville 1990a,b). For oblique propagation, the dispersion relation is somewhat more complex:
\[ (\omega^2 - k_\parallel^2) + \sum_\alpha F_\alpha(k_\parallel, \omega)(\omega^2 - k_\parallel^2) = 0. \]
\[ (19) \]
However, the influence of the oblique propagation can be separated in the dispersion relation. In the case where the transverse component of the wavenumber is non-zero, the dispersion relation can be written as
\[ 1 + \sum_\alpha F_\alpha(k_\parallel, \omega) = \frac{k_\parallel^2}{\omega^2 - k_\parallel^2}. \]
\[ (20) \]
The importance of the presence of the right-hand side of this term results from the possibility that there may be a resonant contribution at small angles of propagation of the waves, and at frequencies close to \( k_\parallel \) when both the numerator and the denominator of the right-hand side of the above equation are small, but when their ratio is finite. This particular point has already been explored and emphasized in Asseo et al. (1990a) for the hydrodynamical instability of quasi-longitudinal Alfvén waves arising in the inner pulsar magnetosphere, and in Pelletier, Sol & Asseo (1988) for an analogous two-stream instability that develops between the relativistic radio jet observed on the pc-scale and the wind from the accretion disc associated with an AGN. Starting from Vlasov and Poisson equations written for specified relativistic distribution functions for the interaction of relativistic pair plasma clouds moving in a pulsar magnetosphere, a recent analysis by Asseo & Melikidze (1998) for the case of strict parallel propagation of waves along magnetic field lines concludes that such interactions are important. Such an analysis is to be extended to the case of oblique propagation in future work.

3 General analysis of the function \( F_\alpha(k_\parallel, \omega) \)
In this section we study the general behaviour of the function \( F_\alpha(k_\parallel, \omega) \) in two different limits, depending on the ordering of the thermal velocity \( v_\alpha \) associated with the distribution function relative to the phase velocity \( v_\parallel \) of the wave.

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3.1 ‘Broad’ distribution function: \( v_\phi \ll v_\text{th} \)

First, if \( v_\phi = v_a \), and as long as the distribution function slightly varies in the vicinity of the singularity (thus for a large width of the distribution function in the reference frame of particles, which is when the phase velocity is very small compared with the thermal velocity of the particles, \( v_\phi \ll v_\text{th} \)), the fraction in equation (15) can be approximated by \((-1)\). Then

\[
F_a(k||, \omega) = \left( \frac{\omega_p \omega_a}{k||} \right)^2 \gamma_a \mu_a \chi_a.
\]  

(21)

where

\[
\chi_a = \frac{1}{n_a} \int f_a dp = \frac{1}{n_a} \int f_a(p||) dp|| = 1.
\]  

(22)

This equality to 1 holds in the case of a relativistic one-dimensional Jüttner–Synge distribution function. The dependence on the temperature parameter at the lowest order in this expression can be intuitively understood considering the character of the integrated quantities and their asymptotic properties, as explained in Appendix C.

3.2 ‘Narrow’ distribution function: \( v_\phi \gg v_\text{th} \)

When the singularity is far from the domain in which the distribution function takes non-negligible values, which is when the phase velocity is large compared with the thermal velocity of the particles, \( v_\phi \gg v_\text{th} \), a limited expansion of the fraction in terms of \( w \) is possible as consequently \( w \ll 1 \). One therefore obtains

\[
F_a(k||, \omega) = -\mu_a \gamma_a \left( \frac{\omega_p \omega_a}{k||} \right)^2 \sum_{n=1}^\infty \left( \frac{w}{a} \right)^n \gamma_a.
\]  

(23)

An expression for the function \( F_a(k||, \omega) \) is obtained after long calculations of the successive averaged quantities (see Appendix A). For a relativistic fluid moving at the velocity \( v_a \), relative to the observer’s frame, equation (15) can be approximated by

\[
F_a(k||, \omega) = -\left( \frac{\omega_p \omega_a}{k||} \right)^2 \frac{1}{(v_\phi - v_a)^2} \left[ \xi_1 + \frac{1}{(v_\phi - v_a)} \eta_1 + \frac{1}{(v_\phi - v_a)^2} \xi_1 + \cdots \right],
\]  

(24)

where the coefficients \( \xi_1, \eta_1 \) and \( \xi_1 \) depend on the temperature parameter (they will be calculated in the next section). In this case, the dispersion relation (16) is written as

\[
1 - \sum_a \left( \frac{\omega_p \omega_a}{k||} \right)^2 \left( \frac{1}{v_\phi - v_a} \right)^2 \left[ \xi_1 + \frac{1}{v_\phi - v_a} \eta_1 + \frac{1}{(v_\phi - v_a)^2} \xi_1 + \cdots \right] = \frac{k^2}{\omega^2 - k^2}.
\]  

(25)

Thus the complete dispersion relation (25) does depend on: the temperature parameter, through the coefficients \( \xi_1, \eta_1 \) and \( \xi_1 \); the phase velocity \( v_\phi = \omega/k|| \); and the averaged motion of the fluid of velocity, \( v_a \), and the associated Lorentz factor \( \gamma_a \).

The coefficients differ according to the relativistic or non-relativistic character of the plasma. We now turn to an analytical determination of these coefficients for either non-relativistic or ultrarelativistic particles.

3.3 Evaluation of the coefficients \( \xi_1, \eta_1 \) and \( \xi_1 \)

3.3.1 Expansion in terms of the temperature parameter

The above coefficients \( \xi_1, \eta_1 \) and \( \xi_1 \) are calculated using a long and rather complicated derivation described in Appendix A for a one-dimensional relativistic Jüttner–Synge equilibrium distribution function. The expansions are available up to high orders in the temperature parameter, either \( \mu_a \) in the ultrarelativistic case or \( 1/\mu_a \) in the weakly relativistic case, and in terms of the velocity \( v_a \) and in the associated Lorentz factor \( \gamma_a \). In what follows the index UR denotes the ultrarelativistic case and the index NR denotes the weakly relativistic case.

In the ultrarelativistic case \( \mu_a \ll 1 \), and we obtain

\[
\xi_1^\text{UR} = \mu_a \frac{\pi}{2} \mu_a^2 \gamma_a + \mu_a^3 \left[ 1 + 2v_a^2 + O(v_a^4) \right] - \gamma_a \left[ \gamma + \ln(\mu_a/2) \right] + O(\mu_a^4),
\]  

(26)

\[
\eta_1^\text{UR} = v_a \left[ 2\mu_a - \frac{3\pi}{2} \mu_a^2 \gamma_a + \mu_a^3 \left[ 2 - \frac{4}{3} v_a^2 + O(v_a^4) + 4\gamma_a \left[ \gamma + \ln(\mu_a/2) \right] \right] \right] + O(\mu_a^4),
\]  

(27)

and

\[
\xi_1^\text{UR} = \mu_a (1 + 3v_a) - \pi \mu_a^2 \left[ \frac{3}{4} + \frac{27}{8} v_a^2 + O(v_a^4) \right] + \mu_a^3 \left[ 1 + v_a^2 + O(v_a^4) + (10 - 12\gamma_a^2) \left[ \gamma + \ln(\mu_a/2) \right] \right] + O(\mu_a^4),
\]  

(28)
(where we have used the Euler constant \( \gamma = 0.5772 \)) while in the weakly relativistic case \( \mu_a \gg 1 \),
\[
\xi_1^{NR} = \frac{1}{\gamma_a^6} \left[ 1 + \frac{3}{\mu_a} \left( -\frac{1}{2} + \gamma_a^2 \right) + \frac{3}{\mu_a^2} \left( 1 - 5 \gamma_a^2 \right) + \frac{3}{\mu_a^3} \left( \frac{7}{2} - 65 \gamma_a^2 \right) - \frac{1}{8} \right] + \mathcal{O} \left( \frac{1}{\mu_a^4} \right),
\]
(29)
\[
\eta_1^{NR} = 6 \nu_a \gamma_a^4 \left[ \frac{1}{\mu_a} + \frac{2}{\mu_a^2} \left( -3 + 5 \gamma_a^2 \right) + \frac{3}{\mu_a^3} \left( 12 - 45 \gamma_a^2 \right) \right] + \mathcal{O} \left( \frac{1}{\mu_a^3} \right)
\]
(30)
and
\[
\hat{\xi}_1^{NR} = 3 \gamma_a^{-6} \left[ \frac{1}{\mu_a} + \frac{6}{\mu_a^2} \left( -1 + 5 \gamma_a^2 \right) + \frac{3}{\mu_a^3} \left( 12 - 135 \gamma_a^2 \right) \right] + \mathcal{O} \left( \frac{1}{\mu_a^4} \right)
\]
(31)
Note that the covariance of the above formulae is not obvious, see Magneville (1990a) for interesting comments about this.

3.3.2 Validity of the expansion

The convergence of the obtained series is to be analysed. First, considering the non-relativistic case, and leaving apart the factors \( \gamma_a \), the coefficients depend on the temperature parameter as follows:
\[
\xi_1^{NR} = 1 + \cdots,
\]
(32)
\[
\eta_1^{NR} = \frac{\nu_a}{\mu_a},
\]
(33)
and
\[
\hat{\xi}_1^{NR} = 1/\mu_a.
\]
(34)
Such results can be justified, starting with the expansion of the function \( F_a(k, \omega) \),
\[
F_a(k, \omega) = -\left( \frac{\omega_{th} \gamma}{k^2} \right)^2 \frac{1}{(\nu_a - \nu_a) \gamma_a^3} \left[ \xi_1^{NR} + \frac{1}{(\nu_a - \nu_a) \gamma_a^3} \eta_1^{NR} + \frac{1}{(\nu_a - \nu_a) \gamma_a^3} \hat{\xi}_1^{NR} + \frac{X_3}{(\nu_a - \nu_a)^3} + \frac{X_4}{(\nu_a - \nu_a)^4} + \cdots \right].
\]
(35)
The terms \( X_1 = \eta_1^{NR} \), \( X_3 \) and \( X_5 \) are all proportional to \( \nu_a \) and thus they are zero in the rest frame of particles where \( \nu_a = 0 \). This is related to properties of parity, where terms of the type \( (\nu_a^{2n+1}) \) tend to zero if the distribution function is even. On the reverse, it is easy to check that the terms \( X_0 = \xi_1^{NR} \), \( X_2 = \hat{\xi}_1^{NR} \), \( X_4 \) and \( X_6 \) are never zero and take the form \( X_2n = \mu_a^{-n} \). Thus, the expansion is obtained in successive powers of \( \mu_a^{-n} \).

In particular, using the reference frame where particles are at rest, so that \( \nu_a = 0 \), the sum takes the form \( 1/(\nu_a \gamma_a^{1/2})^{2n} \) in the non-relativistic case. Indeed, in the non-relativistic case we consider here, the thermal velocity is defined as \( \nu_{th} = \mu_a^{-1/2} \) and the expansion is obtained in successive powers of \( \nu_{th}/\nu_a \). As expected.

In the relativistic case, all the terms \( \xi_1^{UR}, \eta_1^{UR} \) and \( \hat{\xi}_1^{UR} \) are proportional to \( \mu_a/\nu_a \nu_a = \nu_a \gamma_a \). In the case where, for example, \( \nu_a = 0 \) and \( \nu_a \gamma_a > 1 \), convergence of the series is straightforward. In some other cases, i.e. when \( \nu_a > 1 \), \( \nu_a < 1 \) and \( \nu_a - \nu_a < 1 \), one must recall that the result is obtained by performing an expansion in powers of \( w = (\nu_{th} - \nu_a)/(\nu_a - \nu_a) \). In order for this derivation to be valid, \( \nu_a \gamma_a \) must be smaller than unity. Supposing that in the rest frame of the particles, the distribution function takes a non-zero value only in the interval \( [-\nu_{th}, +\nu_{th}] \), then the phase velocity must satisfy the following constraint:
\[
[\nu_a - \nu_a] > \frac{\nu_{th}(1 + \nu_a^2)}{1 - \nu_{th} \nu_a}. \tag{36}
\]
If both \( \nu_a \) and \( \nu_{th} \) are non-relativistic, this is almost equivalent to
\[
\nu_a < \nu_{th} \quad \text{or} \quad \nu_a > \nu_{th} + \nu_{th}. \tag{37}
\]
On the contrary, if both \( \nu_a \) and \( \nu_{th} \) are relativistic, equation (36) becomes
\[
\left[ \nu_a - \nu_a \right] > \frac{2}{1 + \nu_a^2 \nu_{th}^2}, \tag{38}
\]
so that the result is not necessarily valid if \( |\nu_a| < 3 \). For this reason, we shall use other analytical estimates of the different integrals involved in the calculations of the dispersion relation in order to derive some results when both \( \nu_a = 1 \) and \( \mu_a \ll 1 \). This will be detailed in Appendix C, and applied to the two-stream instability in Section 4.

3.3.3 Comparison with three-dimensional analysis

Extending our calculations in the case of a three-dimensional Jüttner–Synge distribution function, we recover the results of Magneville (1990a) exactly. Consequently, it is of interest to point out the differences between the one-dimensional case and the three-dimensional case. A comparison between the values obtained for these coefficients in the rest frame of the particles (i.e. where \( \nu_a = 0 \) and \( \gamma_a = 1 \) is
performed for the coefficients listed in Tables 1 and 2. In the ultrarelativistic case with \( \mu_a \ll 1 \), we obtain the expressions listed in Table 1; whereas in the weakly relativistic case with \( \mu_a \gg 1 \) one obtains the expressions listed in Table 2.

These tables show a similar dependence of the coefficients on the temperature parameter (as well as a slightly different dependence on the dynamical velocity \( v_a \) and on its associated Lorentz factor \( \gamma_a \)). More precisely, the leading order of each term always has the same power in \( \mu_a \), and in the non-relativistic limit, the numerical coefficients of the leading orders are identical. However, in the ultrarelativistic limit, the numerical factor of the leading order for \( \xi_{UR} \) and \( \xi_{NR} \) are slightly different. This implies that the behaviour of a one-dimensional pair plasma is qualitatively the same as for a three-dimensional plasma, but that the exact dispersion relation will differ by numerical coefficients.

Note also that many of these results can be intuitively understood considering the areas involved in the diverse integrals, as explained in Appendix C.

### 3.4 Applications: some solutions of the dispersion relation

Here, we give some simple solutions of the dispersion equation that describes quasi-longitudinal waves. As our analysis shows that the differences between a one-dimensional and a three-dimensional plasma do not lead to strong modifications of the dispersion relation (at least in the limits we have considered here), one can follow ‘standard’ plasma physics in order to find solutions of the dispersion relation.

#### 3.4.1 Broad distribution functions

In the case considered in Section 3.1, the dispersion relation for quasi-longitudinal waves is straightforwardly obtained using equation (21) for the function \( F_{\parallel}(k_\parallel, \omega) \) as

\[
1 + \sum_\alpha \left( \frac{\omega_{p\alpha}}{k_\parallel} \right)^2 \gamma_\alpha \mu_\alpha = \frac{k_\parallel^2}{\omega^2 - k_\parallel^2},
\]

in the general case of oblique propagation of the wave relative to the direction of the local magnetic field.

Characteristics for quasi-longitudinal waves able to propagate in a specified pair plasma are obtained by solving the above dispersion relation for the pair plasma alone. This dispersion relation is very simple and allows us to obtain an exact solution. In the case where one assumes strict parallel propagation of the waves, the dispersion relation does not depend on the frequency. No plane waves are able to propagate. The wave is an evanescent wave, the solution for the wavevector being purely imaginary and the frequency arbitrary. For a pair plasma seen in the observer’s frame, with identical densities, Lorentz factors and temperatures for its electrons and positrons, namely \( n_- = n_+ = n_p, \gamma_- = \gamma_+ = \gamma_p \) and \( \mu_- = \mu_+ = \mu_p \), respectively,

\[
k_\parallel = \pm i \sum_\alpha \gamma_\alpha \omega_{p\alpha} \sqrt{\mu_\alpha} = \pm i 2 \gamma_p \omega_p \sqrt{\mu_p}.
\]

In the case of an oblique propagation of the waves, the second member of the dispersion relation does depend on the frequency, and the solution in terms of the frequency is

\[
\omega = \pm k_\parallel \sqrt{1 + \frac{k_\parallel^2}{k_\parallel^2} \left( 1 + \sum_\alpha \omega_{p\alpha}^2 \gamma_\alpha \mu_\alpha / k_\parallel^2 \right)^2} = \pm k_\parallel \sqrt{1 + \frac{k_\parallel^2}{k_\parallel^2} \left( 1 + 2 \omega_{p\alpha}^2 \gamma_\alpha \mu_\alpha / k_\parallel^2 \right)^2}
\]

Assuming \( k_\perp \ll k_\parallel \), there is a real solution for the frequency very close to the frequency of the transverse mode. The wave is able to propagate as long as the condition \( \gamma_\parallel \ll \gamma_\perp \) is realized. Thus, in this case, propagation of quasi-longitudinal waves is allowed.

3 Let us note that this solution is valid only if \( k_\perp \neq 0 \).
3.4.2 Narrow distribution functions: plasma described in its own rest frame where \( v_\alpha = 0 \)

In the rest frame of the particles, the coefficients at lowest order for the ultrarelativistic plasma are written as \( \xi_1^\text{UR} = \mu_\alpha, \eta_1^\text{UR} = 0 \) and \( \xi_1^\text{UR} = \mu_\alpha \); whereas for the non-relativistic plasma \( \xi_1^\text{NR} = 1 - 3/(2\mu_\alpha), \eta_1^\text{NR} = 0 \) and \( \xi_1^\text{NR} = 3/\mu_\alpha \). The dispersion relation is written as

\[
1 - 2\left( \frac{\alpha_p}{\omega} \right)^2 \left( \xi_1^p + 1 \right) - \frac{k_\perp^2}{\omega^2 - k_\parallel^2} = 0. \tag{42}
\]

For the case of parallel propagation, assuming that the parallel component of the wavenumber \( k_\parallel \) is real and the perpendicular component of the wavenumber \( k_\perp \) is zero, we obtain, after an iteration, purely real solutions for the frequency, namely

\[
\omega = \pm \omega_p \sqrt{2\xi_1^p \left[ 1 + \frac{1}{2} \frac{\xi_1^p k_\parallel^2}{\omega_p^2} \right]} \tag{43}
\]

so that a wave with this frequency is able to propagate (as long as the iteration process is valid). In contrast, the dispersion relation has a more complicated expression for oblique propagation dependent on \( \omega^6 \) instead of \( \omega^4 \). Upon simplification this becomes a cubic equation in terms of \( \omega^2 \), which admits one real solution and two complex ones. Thus, a wave with frequency equal to the real root of the dispersion relation, and associated with real wavenumbers \( k_\parallel \) and \( k_\perp \), is able to propagate.

3.4.3 Narrow distribution functions: plasma moving with non-zero velocity \( v_\alpha \) relative to the observer’s reference frame

In this case the expressions for the coefficients are slightly different. At lowest order \( \xi_1^\text{UR} = \mu_\alpha, \eta_1^\text{UR} = 2\mu_\alpha v_\alpha \) and \( \xi_1^\text{UR} = \mu_\alpha (1 + 3v_\alpha) \), whereas \( \xi_1^\text{NR} = \gamma_\alpha^2 \), \( \eta_1^\text{NR} = 6\mu_\alpha \gamma_\alpha^4 / \mu_\alpha \) and \( \xi_1^\text{NR} = 3\gamma_\alpha^6 / \mu_\alpha \). The corresponding dispersion relation differs from the one obtained above in the plasma rest frame because of the non-zero coefficient \( \eta_1 \) and because of the shifted phase velocity. In the dispersion relation the quantity \( Y = v_\alpha - v_\alpha \) replaces \( v_\alpha \). The solution can be obtained in terms of \( Y \) and further in terms of the frequency. There is a possibility of wave propagation.

3.4.4 Narrow distribution functions: plasma composed of two species with identical thermal velocities

Starting from the dispersion relation for the plasma and beam, both with relativistic motions and temperatures, in the case of parallel propagation:

\[
1 - \frac{\omega_p^2 \xi_p}{\gamma_p^3 (\omega - kv_p)^2} - \frac{\omega_p^2 \xi_b}{\gamma_b^3 (\omega - kv_b)^2} = 0. \tag{44}
\]

We assume (for simplicity) that both the beam and the plasma species have identical plasma frequencies, temperatures and Lorentz factors such that \( \omega_p^2 \xi_p / \gamma_p^3 = \omega_p^2 \xi_b / \gamma_b^3 = \omega_b^2 \xi_b \). Then the four solutions are

\[
\omega = k_\parallel \frac{v_b + v_p}{2} \pm \left\{ \frac{1}{4} k_\parallel^2 (v_b - v_p)^2 + \alpha_b^2 \xi_b \pm \sqrt{\alpha_b^2 \xi_b [k_\parallel^2 (v_b - v_p)^2 + \alpha_b^2 \xi_b]} \right\}^{1/2}, \tag{45}
\]

where \( v_b \) and \( v_p \) are, respectively, the plasma and beam velocities relative to the observer’s reference frame. Two of these solutions are complex conjugate in the case where \( 8\omega_b^2 \xi_b > k_\parallel^2 (v_b - v_p)^2 \), one of these solutions is unstable and corresponds to a two-stream instability with resonant frequency

\[
\omega = k_\parallel \frac{v_b + v_p}{2}, \tag{46}
\]

and with a maximal growth rate obtained when \( 3\omega_b^2 \xi_b = k_\parallel^2 (v_b - v_p)^2 \) and equal to

\[
\omega = \frac{1}{2\sqrt{3}} k_\parallel |v_b - v_p| = \frac{1}{2} \omega_p \sqrt{\xi_b}. \tag{47}
\]

This is a well known particular solution for the two-stream instability that applies to counterstreaming beams (see e.g. Krall & Treivelpiece 1973; Schmidt 1979).

4 TWO-STREAM INSTABILITY FOR TWO RELATIVISTIC SPECIES

4.1 A simplified derivation

The results of the last section are difficult to exploit mainly for two reasons. First, they deal with complicated formulae and secondly, they are obtained by using a specific (thermal) distribution function. Whether they remain valid for other distribution functions remains an open question. For these two reasons, we focus on some more generic features of the function \( F_\alpha \) in various limits, and show that for any reasonably well-defined distribution function, the results of the last section still hold. Also, we can improve the validity of the expansions that we have performed. The idea is to suppose that the distribution function of the particles is only parametrized by the width of the...
distribution function, given by the averaged (possibly thermal) energy of the particles. Interestingly enough, this simple ansatz is sufficient to perform many first-order estimates of various quantities appearing in the dispersion relation.

The details of this new formalism are presented in Appendices B and C. As an application of these results, we consider relativistic plasma and beam particles and assume that they have a relativistic temperature, with temperature parameters $\mu_p$, $\mu_b \ll 1$. We treat the problem in the plasma rest frame, where plasma particles of species $p$ have a zero average velocity ($v_p = 0$). Relative to this frame, beam particles have an average velocity $v_b$. Plasma frequencies for the plasma and the beam in their respective rest frames are, respectively, $\omega_{pp}$ and $\omega_{pb}$. The dispersion relation is simply obtained by adding the corresponding functions $F_a(k, \omega)$ for the plasma and for the beam, i.e. (considering longitudinal waves):

$$1 + F_{\text{plasma}}(k, \omega) + F_{\text{beam}}(k, \omega) = 0.$$  \hfill (48)

Obviously, very different physical situations could exist so that the final point is to decide whether plasma and beam particles are either weakly relativistic or ultrarelativistic, respectively. However, here we focus on understanding whether a two-stream instability can arise in the case where the two species are relativistic.

As explained in Appendix B, the two distribution functions that respectively describe a non-relativistic plasma and a non-relativistic beam are written:

$$f_p = \delta(v_b)$$ \hfill (49)

and

$$f_b = \gamma_b^{-2} \delta(v_b - v_b).$$ \hfill (50)

Thus, whatever the value of the beam velocity $v_b$, there is always a clear separation between the plasma and the beam distribution functions. In the relativistic case, the distribution function that describes plasma particles simplifies to

$$f_p = \mathcal{O}(\mu_p)[Y(v + v_p^b)Y(v - v)]$$ \hfill (51)

whereas the distribution function that describes the beam depends on the relative value of $v_b$ and of the thermal velocities of the beam $v_b^\text{th}$, or else on the relative values of the corresponding Lorentz factors, namely $\gamma_b$ and $\gamma_b^\text{th}$. Indeed,

$$f_b = \begin{cases} \mathcal{O}(\mu_p) [Y(v + v_b^\text{th})Y(v - v)] & \text{if } \gamma_b \ll 2 \gamma_b^\text{th}^2, \\ \mathcal{O}(\mu_p^{-1}) \gamma_b^{-2} \delta(v - v_b) & \text{if } \gamma_b \gg 2 \gamma_b^\text{th}^2. \end{cases}$$ \hfill (52)

If $\gamma_b \ll 2 \gamma_b^\text{th}^2$, then both the distribution functions for beam and plasma particles have comparable widths and occupy the same region in phase space. It is therefore not possible to have a two-stream instability because there are not two separate ‘streams’. On the contrary, if $\gamma_b \gg 2 \gamma_b^\text{th}^2$, the plasma and the beam always have distinct velocities and widths. In what follows, we suppose that

$$\gamma_b \gg \frac{2}{\mu_b}$$ \hfill (53)

In these conditions, and as derived in Appendix C, the dispersion relation is written

$$1 - \mathcal{O}(\mu_p) \frac{\omega_p^2}{\alpha^2(1 - v_p^\text{th}^2/v_p^2)} - \mathcal{O}(1/\mu_p) \gamma_b^{-3} \left(\frac{\omega_{pb}}{\omega - kv_b}\right)^2 = 0.$$ \hfill (54)

The term for the plasma (denoted by subscript $p$) arises from equation (C14), whereas the term for the beam (denoted by subscript $b$) arises by using the distribution function given by equation (B11). In this expression, a supplementary factor $\gamma_b^{-1}$ has been added to take into account the fact that the density is not the same in its own rest frame as in the frame moving with a velocity $v_b$ with respect to its rest frame. With the notation

$$\tilde{\omega}_p^2 = \mathcal{O}(\mu_p) \frac{v_p^2}{1 - v_p^\text{th}^2/v_p^2} \quad \text{and} \quad \tilde{\omega}_b^2 = \mathcal{O}(1/\mu_b) \gamma_b^{-3} \omega_{pb}^2,$$ \hfill (55)

one obtains the same dispersion relation as in the non-relativistic case. As is well known, there are resonant unstable solutions in the limit where the relativistic beam density is less than the relativistic plasma density, $(\tilde{\omega}_b/\tilde{\omega}_p)^2 \ll 1$. Then

$$\omega = \tilde{\omega}_p + \delta \omega, \quad kv_b = \tilde{\omega}_p \quad \text{and} \quad \frac{\delta \omega}{\omega}_p = \left[\frac{1}{2} \left(\frac{\omega_p}{\tilde{\omega}_p}\right)^2\right]^{\frac{1}{3}} \exp(2im\pi/3), \quad n = 0, 1, 2.$$ \hfill (56)

For the unperturbed quantities, this yields

$$\frac{\alpha^2}{k^2} = v_p^2 = v_b^2 = v_p^\text{th}^2 + \frac{\omega_{pb}^2 \mathcal{O}(\mu_p)}{k^2}.$$ \hfill (57)
Moreover, the species being relativistic, $v_{th}^2 = 1 - O(\mu_p^2)$. As a consequence,
\[
\frac{\omega_p^2 O(\mu_p)}{k^2} = \frac{\omega^2}{v_{th}^2} - v_{th}^2 = O(\mu_p^2) \quad \text{and} \quad \omega = k = \omega_p O(\mu_p^{-1/2}).
\] (58)

Equivalently, this implies
\[
1 - \frac{v_{th}^2}{v_p^2} = O(\mu_p^2).
\] (59)

By injecting these results into the formula (56), one obtains
\[
\frac{\delta \omega}{\omega_p} = \left[ \frac{1}{2} \frac{\alpha p^2}{\gamma_p^2} O(\mu_p) \right]^{\frac{1}{3}} \exp(2i\pi/3).
\] (60)

From equation (53), it is clear that this correction is small. If we look at the unstable (complex) solution we see that the frequency shift is negative, $R(\delta \omega/\omega_p) < 0$, and is almost equal to the growth rate of the unstable wave, $I \omega(\delta \omega/\omega_p) = \sqrt{3}R(\delta \omega/\omega_p)$. The phase velocity corresponding to the unstable solution is therefore
\[
v_p + \delta v_p = \frac{\delta \omega}{k} = v_h \left( 1 + \frac{\delta \omega}{\omega_p} \right) = v_b \left[ 1 - \frac{1}{243} \frac{\alpha b^2}{\alpha p^2} O(\mu_p^{1/3}) \right] O(\gamma_b).
\] (61)

Numerical estimates for the frequency shift and for the growth rate, together with the determination of the ordering of the different velocities involved, $v_{th}^2$, $v_b$ and $v_w$, can be obtained for particular sets of parameters.

### 4.2 Applications

The physics of the interaction between a relativistic beam of electrons (or positrons) and a relativistic pair plasma, is at the basis of the interpretations of radio observations from pulsar emission regions and from the environment of AGN.

#### 4.2.1 Pulsar magnetospheres

The possibility that a two-stream instability could be responsible for pulsar radio radiation has been investigated for the three typical interactions that are able to develop in a pulsar magnetosphere. These are the interaction between the primary beam and the secondary pulsar pair plasma (Ruderman & Sutherland 1975), the interaction between electrons and positrons of the pair plasma itself, that acquire relativistic temperatures of the particles. Thus, as mentioned in the introduction, pulsar pair plasmas are relativistic in two ways: first, because of the highly relativistic motion of pair plasma particles and secondly, because of the relativistic temperatures of the electrons and positrons that form the pair plasma. In addition, they are anisotropic due to the extreme strength of the pulsar magnetic field. Consequently, we can use relativistic one-dimensional Jüttner–Synge distribution functions to characterize flows of relativistic particles in a pulsar magnetosphere. We obtain analytical expressions of the dispersion relation for quasi-longitudinal waves as a succession of terms the coefficients of which ($\xi$, $\eta$, and $\xi$ introduced above) depend on the temperature, velocities and Lorentz factors of the flows. In order to use simplified forms of the relativistic one-dimensional Jüttner–Synge distribution functions for the plasma and beam, the constraint of equation (53) must be satisfied (as derived in Appendix C and explained above). Such a constraint should be easily verified for the interaction arising between a beam and a pair plasma with very different energies and densities, but with not-too-high relativistic temperatures.

Let us first consider the interaction between the primary beam and the secondary pulsar pair plasma: the expected primary beam (hereafter denoted pb) and secondary pair plasma (hereafter denoted sp) have very different Lorentz factors and densities. Usual values for the Lorentz factors in the observer’s frame (hereafter denoted obs), $\gamma_{pb}^{obs} = 10^6$ and $\gamma_{sp}^{obs} = 10^4$, correspond to $\gamma_{pb}^0 = 500$ in the plasma rest frame (hereafter denoted pf). In this case the above constraint of equation (53) is easily fulfilled, assuming that the temperature of the secondary pair plasma is not too high. As an example, one may suppose that $\mu_p = 10^{-1}$. For such values of the parameters, the resonant frequency obtained in the plasma rest frame differs from the plasma frequency as it depends on the temperature parameter of the plasma, $\omega_p^0 = \omega_{sp} \mu_p^{-1/2}$. On the other hand, if one assumes that the temperature parameters of the primary beam and of the secondary pair plasma

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are alike, the relative growth rate $\delta_{\text{sp}}^{\text{obs}} / \omega_p$ (as derived from equation 60) only depends on the multiplication factor $\kappa$, characteristic of the ratio of densities of the secondary pair plasma and of the primary beam, which belongs to the domain $\kappa \in [10^3, 10^5]$, and on the Lorentz factor of the beam, $\gamma_{p,\text{sp}}^{\text{obs}}$. In this case, one obtains the correct ordering for the velocities, as equation (53) is satisfied. Subsequently, it is possible to obtain a solution of the above equations in the plasma rest frame that represent unstable waves in the radio domain at some distance from the surface of the star.

However, going to the observer’s frame through Lorentz transformation modifies this very solution: using the same set of pulsars parameters the frequency is significantly enhanced, it is no longer in the radio domain, and the growth rate is reduced, so that the wave is not sufficiently amplified along its whole trajectory in the presumed emission region. Indeed, as both the beam and the plasma have high Lorentz factors with respect to the observer’s frame,

$$\omega_{k,\text{obs}}^\text{sp} = 2 \gamma_p \omega_{k,\text{sp}}^\text{obs}, \quad k_{k,\text{obs}}^\text{sp} = 2 \gamma_p k_{k,\text{sp}}^\text{obs}, \quad \delta\omega_{k,\text{obs}}^\text{sp} = \gamma_p \delta\omega_{k,\text{sp}}^\text{obs} \quad \text{and} \quad k_{q,\text{obs}}^\text{sp} = \gamma_p v_p \delta\omega_{k,\text{sp}}^\text{obs}. \quad (62)$$

From this, and as a consequence of the invariance of the phase factor of the perturbed wave through a Lorentz transformation, one obtains the result that the amplification factor for the propagation of a plane wave in the observer’s frame over the extent $L$ of the instability region, $\exp(\delta\omega_{k,\text{sp}}^\text{obs} / L \gamma_p v_p)$, is too small for the wave to reach a sufficient order of magnitude. Consequently, at this step, temperature and anisotropy effects do not allow us to obtain resonant waves in the radio domain with a sufficient growth rate from the two-stream instability that arises between the primary beam and the secondary plasma: resonant unstable ‘Langmuir’ waves are probably not responsible for pulsar radio events, in analogy with the conclusions of Cheng & Ruderman (1977) and with the analyses of Melrose & Gedalin (1999), Melrose et al. (1999).

Let us now consider the interaction between drifting electrons and positrons that form the ultrarelativistic pulsar pair plasma itself present in a pulsar magnetosphere. As the pair plasma is supposed to have an ultrarelativistic character, the electrons and the positrons, considered as forming a ‘beam’ and a ‘plasma’ in interaction, should also be ultrarelativistic with temperature parameters in the same domain of values, probably $[10^{-1}, 10^{-4}]$. Moreover, in the plasma rest frame the ‘beam’ formed with the fastest particles of the pair plasma has a Lorentz factor of the order $[2.5, 3]$ (Cheng & Ruderman 1977). In these conditions, the necessary constraint to treat simplified one-dimensional Jüttner–Synge distribution functions, equation (53), cannot be fulfilled so that the instability cannot be described under the prescribed hypothesis. However, recent numerical analysis using relativistic one-dimensional Gaussian distribution functions to characterize the distribution functions of drifting electrons and positrons that form the pair plasma, and also by considering reasonable pulsar parameters together with a temperature parameter $\mu_p = 1$ that keeps the equivalence with relativistic one-dimensional distribution functions in the ultrarelativistic case, shows the possibility for an efficient two-stream instability to be raised up (Asseo & Melikidze 1998).

Finally, and similarly to the above interaction, the interaction between electrons and positrons that form successive moving pair plasma clouds, which overlap and interact on a very short time-scale associated with the non-stationary character of particle flows in successively emitted clouds, as a result of the ‘sparking’ phenomenon arising above the surface of the star (Usov 1987), cannot be treated analytically here. Using a specific model, and assuming reasonable pulsar quantities and a temperature parameter $\mu = 1$, it has been possible to obtain from numerical analysis that on the short time-scale, the overlapping and interaction of particles from successive pair plasma clouds, which have a large spectrum of energies, also leads to the beginning of an efficient ‘two-stream’-like instability (Asseo & Melikidze 1998).

### 4.2.2 Application to the physics in extragalactic relativistic jets

In view of very long baseline interferometry (VLBI) observations, the material source of relativistic jets from AGN may be a relativistic pair plasma, the apparent superluminal jet motions being associated with intrinsically relativistic motions of a tenuous pair plasma immersed in the strong magnetic field probably present in the immediate vicinity of AGN. In addition, relativistic pair plasmas, or relativistic plasmas dominated by electron–positron pairs, are presumed to be at the origin of observed radio emissions from jets. Whether the involved radiation process is a result of incoherent synchrotron emission of particles in the existing magnetic field, or is due to an intrinsic plasma radiation mechanism, must be tested. Indeed, both processes can similarly reproduce the essential features of radio emissions observed from relativistic jets (Benford 1992).

In the two-flow model proposed by Sol, Pelletier & Asseo (1989), the superluminal relativistic jet observed on the VLBI pc-scale is supposed to be a beam of relativistic particles, presumably electrons and positrons issued from the innermost part of the accretion disc with a high bulk Lorentz factor: it is taken as one of the flows. The jet observed on the kpc-scale, which represents the main contribution to the energy budget of the extended radio source, is supposed to be a classical or weakly relativistic wind of electrons and protons coming out from the whole accretion disc: it forms the other flow. Both the relativistic beam of electrons and positrons and the slow wind of particles interact as they stream along the magnetic field present in the environment of AGN, and eventually further along jets. This ‘two-stream interaction’, which gives rise to unstable quasi-longitudinal Langmuir waves at the linear stage, arises at a frequency associated with the plasma frequency of the wind and allows observed radio frequencies to be explained in the vicinity of AGN, as long as the electron gyrofrequency is greater than the ambient plasma frequency, $\omega_B \gg \omega_p$. Ultimately, it evolves non-linearly towards a stable Langmuir turbulence (Sol et al. 1989).

Such a result only accounts for relativistic hydrodynamic motions and for weakly relativistic temperatures for the beam and plasma wind. However, in analogy with the cascade process of pair creation at the surface of neutron stars, the cascade process of pair creation around black holes results in the creation of pairs of electrons and positrons that reflect the repartition of photon energies, as they initially
share the energy of initiating photons. As one expects a large spread in the energies of the photons, the involved relativistic beam particles should have a relativistic temperature. Consequently, further evolution of the ultrarelativistic pair plasma moving out from the central engine should be reconsidered taking into account relativistic thermal effects (Pelletier et al. 1988; Pelletier & Marcowitch 1998). This is performed here in the context of the two-flow model, using the dispersion relation derived above, which is valid on the pc scale close to the central object, as the magnetic field is extremely strong there and most often parallel to the jet motion, and specifying the particular quantities such as the densities \( n_p, n_b \), temperatures \( T_p, T_b \), velocities \( v_p, v_b \), Lorentz factors \( \gamma_p, \gamma_b \) and plasma frequencies \( \omega_{pp}, \omega_{pb} \), respectively, associated with the relativistic beam and plasma wind.

In the conditions of the two-flow model, the perpendicular electric wave field component (also named the ‘electromagnetic’ component), the transverse magnetic wave field component and the electromagnetic energy available in the emission region are quantities easy to evaluate in terms of the parallel electric wave field component (usually named the ‘electrostatic’ component). However, let us note that indications on the relative importance of the circular polarization to the linear polarization cannot be obtained from such calculations: as the magnetic field is very strong, all contributions in which cyclotron effects intervene are neglected, thus right- and left-handed polarized waves do not exist as such, and only effects related to the direction of the magnetic field can be evaluated.

We define the directions for wavevectors and velocities of the flows, namely \( \| \), \( \perp \) and transverse relative to the direction of the background magnetic field. For perturbations with frequency \( \omega \) and with a zero transverse wavevector \( \mathbf{k} \), which is to say with components \( (k_\parallel,k_\perp,0) \), the electric components of quasi-longitudinal perturbations verify the following coupled equations in an arbitrary frame of reference (see Asseo et al. 1990a):

\[
(\omega^2 - k_\parallel^2)E_\parallel + k_\parallel k_\perp E_\perp = 0 \quad \text{and} \quad (\omega^2 - k_\perp^2)E_\perp + k_\parallel k_\perp E_\parallel = 0.
\]

For the interaction between a weakly relativistic wind plasma with \( \mu_p \gg 1 \) and a relativistic beam with \( \mu_b \ll 1 \), the dispersion relation that includes thermal effects is written in the observer’s frame as

\[
e_\parallel = \frac{k_\parallel^2}{\omega^2 - k_\parallel^2},
\]

with

\[
e_\parallel = 1 - \frac{\omega_{pp}^2}{\gamma_p^2(\omega - k_v p)^2} - \frac{\omega_{pb}^2 \mathcal{O}(1/\mu_b)^2}{\gamma_b^2(\omega - k_v b)^2}.
\]

As we assume that the plasma wind is weakly relativistic, the emission caused by the two-stream interaction arises independently of relativistic temperature effects, as in Sol et al. (1989), at the resonant frequency \( \omega^{\text{res}} = k_\parallel^{\text{res}}c_b = k_\parallel^{\text{res}} = 2\omega_{pp}\sqrt{\gamma_p} \), where the parallel resonant wavenumber is

\[
k_\parallel^{\text{res}} = \frac{\omega_{pp}}{\gamma_p^{3/2}} \frac{1}{\Delta v}.
\]

with \( \Delta v = v_b - v_p \). Assuming relativistic motions for the plasma wind and for the relativistic beam, with \( v_b \gg v_p \), \( \Delta v = 1/2\gamma_b^2 \).

On the reverse, the characteristics of the instability, namely the frequency shift, the growth rate and the instability windows, obtained from the above equations in analogy of the derivation given in Section 4, are modified by the inclusion of the relativistic temperature of the beam. Indeed, \( \omega = k_\parallel v_b + \delta \omega_k \),

\[
\delta \omega_k = - \frac{1}{2} \frac{\omega_{pp}^{1/3} \omega_{pb}^{2/3}}{\gamma_p^{2/3} \gamma_b^{1/3}} \left[ \frac{1}{\mathcal{O}(\mu_b)} \right]^{1/3}
\]

and

\[
\delta \omega_\parallel = \pm i \sqrt{3} \delta \omega_k.
\]

The instability occurs in a restricted domain of perpendicular wavenumbers, \( k_\perp \leq k_\perp^{\text{crit}} \), with

\[
k_\perp^{\text{crit}} = \frac{\omega_{pp} \gamma_b^{3/2}}{\gamma_p^{3/2} \sqrt{\Delta v}} \alpha^{1/3} 2^{1/3} \sqrt{3}.
\]

Here,

\[
\alpha = \frac{\omega_{pp} \mathcal{O}(1/\mu_b) \gamma_p^2}{\omega_{pb} \gamma_b^2}.
\]

Consequently, excepting the resonant frequency and resonant parallel wavenumber, the characteristics of the instability are enhanced as a result of the relativistic temperature of the interacting beam.

An evaluation of the ‘electromagnetic’ wave field component in terms of the ‘electrostatic’ one yields in the observer’s frame,
according to equations (63 and 64),

\[ E_\perp = - \frac{\omega^2 \epsilon_0 k_\perp^2}{k_\parallel k_\perp} E_\parallel, \]  

(72)

or equivalently,

\[ E_\perp = - \frac{k_\parallel k_\perp}{\omega^2 - k_\parallel^2} E_\parallel. \]  

(73)

The transverse component of the magnetic wave field, obtained from Maxwell’s equations, expressed in terms of \( E_\parallel \) is written

\[ B_\perp = \left( k_\parallel / \omega \right) \left( E_\perp - \frac{k_\perp}{k_\parallel} E_\parallel \right) = - \frac{k_\parallel \omega}{\omega^2 - k_\parallel^2} E_\parallel. \]  

(74)

Therefore, the inclusion of relativistic temperature effects in the expression of \( \epsilon_\parallel \) involves modifications in both \( E_\perp \) and \( B_\perp \), because \( \epsilon_\parallel = \frac{\omega^2}{\omega^2 - k_\parallel^2}. \)

Close to resonance, where \( \omega^2 = k_\parallel^2 \epsilon_\parallel^2 = k_\parallel^2 \), the mixed electrostatic-electromagnetic mode is characterized by a highly intense non-zero perpendicular electric wave field component, \( E_\perp \), together with a highly intense magnetic wave field component, \( B_\perp \), as these components have the factor \( (\omega^2 - k_\parallel^2) \) in their denominator, which is close to zero at resonance.

For instance, using the characteristics of the instability defined above,

\[ \left| \frac{E_\perp}{E_\parallel} \right| \leq \frac{k_\parallel}{k_\parallel - k_\parallel^2} \frac{k_\parallel^2 \gamma_\parallel^2}{\gamma_\parallel - 2 \gamma_\parallel \delta \omega / k_\parallel}, \]  

(75)

Depending on the relative values of the quantities \( \gamma_\parallel^2 \) and \( 2 \gamma_\parallel \delta \omega / k_\parallel \), we obtain

\[ \left| \frac{E_\perp}{E_\parallel} \right| \leq \frac{k_\parallel^2 \gamma_\parallel^2}{k_\parallel - k_\parallel^2}, \]  

or else

\[ \left| \frac{E_\perp}{E_\parallel} \right| \leq \frac{k_\parallel^2 \gamma_\parallel^2}{2 \gamma_\parallel \delta \omega / k_\parallel}. \]  

(76)

Assuming conventional values for the Lorentz factors of the relativistic beam and pair plasma as given by models, \( \gamma_\parallel = 200 \) and \( \gamma_\parallel = 10 \), the Lorentz factor of the beam in the plasma rest frame \( \gamma_\parallel^0 = 10 \), is compatible with the constraints deduced for the parameters of extragalactic radio jets (see Pelletier & Sol 1992). In this case, the second term in the denominator is easily preponderant over the first one [as \( 1 / (\gamma_\parallel^2) = 2.5 \times 10^{-5} \), whereas \( (2 \gamma_\parallel \delta \omega / k_\parallel \gamma_\parallel) = 2 \times 10^{-1} \)]. Finally,

\[ \left| \frac{E_\perp}{E_\parallel} \right| \leq \frac{k_\parallel^2 \gamma_\parallel^2}{2 \gamma_\parallel \delta \omega / k_\parallel}, \]  

(77)

Therefore, the critical value of the ratio of the ‘electromagnetic’ to the ‘electrostatic’ electric wave field components indicates a relatively high value for \( E_\perp \) as compared with \( E_\parallel \). It is essentially dependent upon the difference in velocity of the two flows, but independent of their temperatures as \( k_\parallel^2 \) and \( \delta \omega \) depend in the same way on the temperature parameter. Of course, for a given \( k_\parallel \), this ratio effectively depends on the temperature of the relativistic beam. Similar conclusions are obtained by comparing \( B_\perp \) and \( E_\parallel \).

The fact that close to resonance, the perpendicular electric wave field component, together with the transverse magnetic wave field component, can reach very high values in the observer’s frame implies important energy flows in the direction of the magnetic field. This can be tested as the electromagnetic energy available in the emission region can be related to the flux of the Poynting vector through a surface characteristic of the emission zone.

Let us first calculate the Poynting vector. With usual conventions its Fourier transform is defined as

\[ \Pi(k, \omega) = \frac{1}{8 \pi} \mathcal{R}(E \times B)(k, \omega). \]  

(78)

Thus, in terms of the perpendicular and transverse components of the perturbation, the Poynting vector is obtained as

\[ \Pi_\parallel(k, \omega) = \frac{1}{8 \pi} \mathcal{R}(E_\parallel B_\parallel) = \frac{1}{8 \pi} E_\parallel^2 \frac{k_\parallel \omega k_\parallel^2}{(\omega^2 - k_\parallel^2)}, \]  

(79)

\[ \Pi_\perp(k, \omega) = \frac{1}{8 \pi} \mathcal{R}(-E_\parallel B_\perp) = \frac{1}{8 \pi} E_\parallel^2 \frac{k_\parallel \omega}{(\omega^2 - k_\parallel^2)} \]  

(80)

and

\[ \Pi_t(k, \omega) = 0, \]  

(81)

where again the relativistic temperature effects are included through the term \( \epsilon_\parallel = \frac{k_\parallel \omega}{(\omega^2 - k_\parallel^2)} \). Obviously, the Poynting vector is very important close to resonance as the denominator of the involved components contains the factor \( (\omega^2 - k_\parallel^2) \), which is zero close to the frequency of the transverse wave. It is significantly enhanced by the inclusion of the relativistic temperature parameter of the beam. The
electromagnetic energy flow depends on the Poynting vector integrated over the frequency spectrum and wavenumber domain. As the most important contribution comes from the resonant domain, and with the choice $k_l = 0$, we can write

$$W_l(k, \omega) = \frac{(2\pi)^3}{2\pi} W_l(k) \delta(\omega - \mathcal{R}(\omega_{res}) \delta(k_{||} - k_{||,res}) \delta(k_l),$$

where $\delta(x)$ is the Dirac function. Thus, as long as $k_l$ belongs to the instability window of the perpendicular wavenumber, we expect to have $W_l(k_{||}) \neq \delta(k_{||})$: we define $W_l(k_{||}) = 0$ in the domain where $k_{||} \gg k_{||,res}$, whereas $W_l(k_{||}) = W_l/2k_{||,res}^2$ in the domain where $0 \leq k_{||} \leq k_{||,res}^2$, where $W_l$ stands for $(\mathcal{E}_l)^2/(8\pi)$.

Then, 

$$\langle \Pi_l \rangle = \left[ \frac{d\omega}{2\pi} \right] \left[ \frac{dk_{||}}{2\pi} \right] \int_{-k_{res}}^{k_{res}} \int_{-k_{res}}^{k_{res}} \frac{dk}{2\pi} \Pi_l(k, \omega)$$

(83)

and 

$$\langle \Pi_l \rangle = \frac{(E_l)^2 (k_{||,res}^2)^2}{8\pi} \frac{k_{res} \omega_{res}}{3 \left( (\omega_{res})^2 - (k_{res}^2)^2 \right)^2}.$$ 

(84)

On the other hand, as $\Pi_{||}(k, \omega)$ is an odd function in terms of the variable $k_{||}$, $\langle \Pi_{||} \rangle = 0$. Thus, the flux of the Poynting vector through a surface $S_{||}$ characteristic of the emission zone,

$$P_{||} = \langle \Pi_{||} \rangle S_{||},$$

(85)

represents the electromagnetic energy that flows in the direction parallel to the strong magnetic field direction. As it involves $k_{||,res}$, the available electromagnetic energy depends on the relativistic temperature of the flowing beam.

Let us note that, using relativistic fluid and Maxwell equations, the parallel ‘electrostatic’ wave field component can also be expressed in terms of the density fluctuations. However, at the linear stage the level of density fluctuations in the emission zone is quite arbitrary and it is not possible to determine the true value of the energy flow. A real determination requires a non-linear analysis to be performed.

5 SUMMARY

Our linear analysis here restricts us to relativistic one-dimensional astrophysical pair plasmas, which have both relativistic motion and relativistic temperatures. Indeed, as implied by standard pulsar models for instance, the pair creation process that arises in the emission region of a pulsar does result in a significant spread of the energies of the created pairs, so that pair plasma particles probably have relativistic temperatures. In addition, the presence of an extremely strong magnetic field in a pulsar magnetosphere constrains the particles to one-dimensional motion, giving an anisotropic character to the pair plasma moving there. Similar features of the created pair plasma characterize pair plasma flows associated with relativistic extragalactic radio jets close to AGN, the magnetic field being very strong in this region and the pair creation process alike.

We derive the dielectric tensor and the dispersion relation for such an anisotropic relativistic pair plasma, using distribution functions that are cold in the transverse direction and relativistically hot in the parallel direction (relative to the direction of the magnetic field). Namely, we include these anisotropic and relativistic features, choosing one-dimensional relativistic Jüttner–Synge distribution functions to characterize the distribution functions of the electrons and positrons that form the pair plasma. The dispersion relation, associated with plane wave perturbations of such a relativistic anisotropic pair plasma, involves coefficients that specifically depend on the distribution function of its particles. To get simple analytical forms of dispersion relations, and to derive conclusions about the stability of waves, we use approximate forms of the relativistic one-dimensional Jüttner–Synge distribution functions. A precise determination of these coefficients specifies their dependence on the relativistic temperature and on the relativistic velocity of the flows, and allows one to obtain the appropriate dispersion relation.

A few solutions for the propagation characteristics of quasi-longitudinal waves together with the conditions for the unstable interaction of a beam and a plasma, both ultrarelativistic, are deduced from this dispersion relation. Relativistic and anisotropic effects modify the frequency of the waves able to propagate and lead to different instability growth rates.

The analytical writing of the dispersion relation for broad and narrow relativistic one-dimensional Jüttner–Synge distribution functions, adequate to describe a relativistic anisotropic pair plasma, or else the interaction of a relativistic (or weakly relativistic) beam and plasma system, represents our main result. Applications of such calculations to astrophysical situations, like the emission region in pulsar magnetospheres or in the proximity of AGN, are new with respect to previous analyses as they include relativistic temperature effects.

In the emission region of a pulsar magnetosphere, there are three different interactions that lead to the development of a two-stream instability: the interaction between the primary beam and the secondary plasma, the interaction between drifting electrons and positrons that form the pair plasma and the interaction between particles in different plasma clouds. Our analytical derivation is adequate to discuss the fate of the first interaction: using parameters to characterize the primary beam and the secondary pair plasma, we conclude that due to relativistic temperature effects it is not possible to obtain waves with frequencies in the radio domain and with sufficient instability growth rates. Consequently, we recover the conclusions of Cheng & Ruderman (1977) and Melrose & Gedalin (1999): that such a two-stream instability between the primary beam and the secondary pair plasma is insufficient in accounting for the high level of observed radio
radiation. Let us mention that our approximate forms of the relativistic one-dimensional Jüttner–Synge distribution functions are not appropriate to treat the second and third cases, because in these cases it is impossible to fulfil the necessary constraint that links the Lorentz factor of the beam (relative to the plasma rest frame) and the temperature parameters of the plasma and beam. In other words, the schematized distribution functions we use are not general enough to represent the true, slightly shifted, distribution functions of drifting electrons and positrons, or the interaction between particles in moving clouds. However, we know from a recent numerical treatment (Asseo & Melikidze 1998) that such interactions lead to consequent instabilities. Thus, relativistic effects do not lead us to modify the relative importance of the three interactions that involve two-stream instabilities in a pulsar emission region.

Similar derivations are used to propose implications of the two-flow model introduced by Sol et al. (1989) for the interpretation of radio features in extragalactic radio jets, in the case where relativistic temperature effects are taken into account.

In the case of an electron-positron plasma immersed in an extremely strong magnetic field, perturbations are always 100 per cent linearly polarized: using the description provided by the two-flow model in the case of extragalactic jets, it is only the ratio of the perpendicular and parallel components of the electric wave field, or else the ratio of the transverse magnetic wave field and parallel electric wave field components, relative to the direction of the local magnetic field, which can be evaluated. From this we are able to express the available electromagnetic energy in terms of the flux of the Poynting vector through a surface characteristic of the emission zone. For resonant frequencies, characteristic in the two-flow model, the available electromagnetic energy is significantly enhanced as a result of relativistic temperature effects. However, it is not possible to reach definite conclusions from a comparison between observed quantities and results numerically deduced from our linear analysis, as there is one degree of freedom left for the magnitude of the perturbation. With relativistic thermal effects included, definite conclusions could only be reached from a comparison between dispersive and non-linear effects. Such a non-linear analysis is beyond the scope of our present work.

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APPENDIX A: DETERMINATION OF THE COEFFICIENTS $\xi$, $\eta$ AND $\zeta$

To solve the dispersion relation (25), an evaluation of the function $F_{\psi}(k_{\parallel}, \omega)$, given in equation (12), is required. As a consequence, it is necessary to integrate the integrand over the whole domain of the variations of the parallel momentum $p_{\parallel}$ (where parallel here means along
the magnetic field line at the point where the process is considered). The integrand is formed of two factors, namely the derivative \(\frac{\partial f_a(p_\parallel)}{\partial p_\parallel}\) and the ratio \(1/(v_\parallel - v_\alpha)\). For a relativistic one-dimensional Jüttner–Synge distribution function, the derivative of the distribution function is proportional to the distribution function itself. In the reference frame where \(v_\alpha\) is the velocity of the centre of mass of the particles and \(\gamma_\alpha = (1 - v_\alpha^2)^{-1/2}\) (its associated Lorentz factor), the derivative is expressed as

\[
\frac{\partial f_a(p_\parallel)}{\partial p_\parallel} = -\mu_\alpha \gamma_\alpha (v_\parallel - v_\alpha) f_a(p_\parallel),
\]

(A1)

and the ratio that involves the velocities can be expressed in terms of the variable \(w = (v_\parallel - v_\alpha)/(v_\phi - v_\alpha)\). The function \(F_a(k_\parallel, \omega)\) in equation (12) is then written as

\[
F_a(k_\parallel, \omega) = -\sum_\alpha \mu_\alpha \gamma_\alpha \left(\frac{\omega_\parallel}{k_\parallel}\right)^2 \frac{1}{n_\alpha} \int \frac{v_\parallel - v_\alpha}{v_\phi - v_\parallel} f_a(p_\parallel) \, dp_\parallel,
\]

(A2)

\[
= -\sum_\alpha \mu_\alpha \gamma_\alpha \left(\frac{\omega_\parallel}{k_\parallel}\right)^2 \frac{1}{n_\alpha} \int \frac{w}{1 - w} f_a(p_\parallel) \, dw.
\]

(A3)

As long as the phase velocity is of the order of the velocity of the particles, i.e. as long as \(v_\phi = v_\alpha\), and for a distribution function that does not vary too much in the vicinity of the singularity (which is a broad distribution function with \(v_\phi \ll v_\parallel\) in the reference frame of particles), the expression \(v_\parallel - v_\alpha/v_\parallel - v_\phi\) can be approximated by unity. Then

\[
F_a(k_\parallel, \omega) = \left(\frac{\omega_\parallel}{k_\parallel}\right)^2 \mu_\alpha \gamma_\alpha.
\]

(A4)

On the reverse, if the distribution function is narrow, i.e. if the phase velocity is large as compared with the thermal velocity in the reference frame of the centre of momentum of the particles, namely \(v_\phi \gg v_\parallel\), the value of the integral can be obtained using a limited expansion in \(w\) as \(w \ll 1\). We obtain

\[
\int \frac{w}{1 - w} f_a(p_\parallel) \, dw = \sum_{n=0}^{\infty} (w^n)_n,
\]

(A5)

where the averaged value of an arbitrary quantity \(X\) is defined as usual using

\[
\langle X \rangle_\alpha = \frac{1}{n_\alpha} \int X f_a(p_\parallel) \, dp_\parallel,
\]

(A6)

\[
= \frac{1}{n_\alpha} \int \gamma_\alpha f_a(p_\parallel) \, dv_\parallel.
\]

(A7)

The integrals and the series are absolutely convergent on the domain \([1 - \tilde{I}, 1]\), and it is possible to exchange the integration and the summation. The integrals involving the distribution function are calculated in an easier way by going to the reference frame of the centre of mass of the particles: \(v_\parallel\) is transformed into \((v_\parallel + v_\alpha)/(1 + v_\parallel v_\alpha)\) and \(dp\) is transformed into \(\gamma_\alpha/(1 + v_\parallel v_\alpha) \, dp\). The mean values are

\[
(w^n)_n = \frac{\gamma_\alpha}{(v_\phi - v_\alpha)^n} \sum_{k=0}^{\infty} \binom{n}{k} \left(\frac{v_\alpha}{1 + v_\parallel v_\alpha}\right)^k \left(\frac{v_\parallel + v_\alpha}{1 + v_\parallel v_\alpha}\right)^{n-k}.
\]

(A8)

where the index 0 denotes a mean value calculated in the reference frame of the centre of mass of the particles. The ratio in equation (A8) can be calculated as

\[
\frac{(v_\parallel + v_\alpha)^n}{(1 + v_\parallel v_\alpha)^{n-k}} = c_0(v_\alpha) v_\parallel + b_0(v_\alpha) + \sum_{k=1}^{n-1} a_{n,k}(v_\alpha) \frac{(k - 1)!(-v_\parallel)^{k-1}}{(1 + v_\parallel v_\alpha)^k}.
\]

(A9)

The above coefficients are

\[
c_0(v_\alpha) = \frac{1}{v_\alpha^{n-1}},
\]

(A10)

\[
b_0(v_\alpha) = \frac{1 - n \gamma_\alpha v_\alpha}{v_\alpha^n},
\]

(A11)

and

\[
a_{n,k-1}(v_\alpha) = \frac{(-1)^{n-k} \gamma_\alpha^{2n}}{(n-2)!} \frac{v_\alpha}{v_\alpha^2},
\]

(A12)

and the others are deduced in a recursive manner. Also, as the distribution function in the reference frame of the centre of mass of the

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particles does not depend on $v_a$, the velocity of the centre of mass of the species $\alpha$, we have:

$$\left\langle (k - 1)(-v)_{k-1} \right\rangle = \frac{1}{(1 + v_\parallel v_a)} \left( \frac{1}{v_\parallel v_a} \right)_0.$$ (A13)

The last mean value is calculable by performing another limited expansion in $v_\alpha v_\parallel$ (the uniform convergence of the series in $v_\parallel$ over the domain $[-1, 1]$ allows this operation to be performed). The final result is

$$\left\langle \frac{1}{1 + v_\parallel v_a} \right\rangle = C_\alpha \sum_{d=0}^\infty A_d 2^d v_\alpha^{2d} I_{d-1} 2 \cdot 2 \cdot 2 A(m_\alpha).$$ (A14)

with

$$A_{1,\alpha} = 2$$ (A15)

and

$$A_{3,\alpha} = 2\pi \int_0^\pi \cos^\alpha \theta \sin \theta d\theta.$$ (A16)

Returning to the expression (A8) for $\langle w \rangle^\alpha$, the coefficients $\zeta$, $\eta$ and $\xi$ are obtained after expansions of the functions $f(\mu_\alpha)$ in either the parameter $\mu_\alpha$ for a relativistic beam or plasma, or $\mu_\alpha^{-1}$ for a non-relativistic beam or plasma (see e.g. Abramowitz & Stegun 1965).

**APPENDIX B: APPROXIMATE FORMS OF NON-RELATIVISTIC AND ULTRARELATIVISTIC ONE-DIMENSIONAL DISTRIBUTION FUNCTIONS**

Simple approximations to one-dimensional distribution functions can be obtained for either the non-relativistic or the ultrarelativistic regimes.

In the non-relativistic limit, the distribution function can be taken as a Dirac function because it is very narrow [the temperature being defined as $k_B T = mc^2/\mu = m(v_\parallel^2/2) = mv_\parallel^2/2$, so that the width of the distribution function is of the order $O(\mu^{-1/2})$]. On the contrary, in the ultrarelativistic limit most particles have a velocity close to the velocity of light, approximately equal to $\pm (1 - e)$, so that their distribution function can be approximated by two Dirac functions. Finally, the mildly relativistic case is an intermediate state between these two regimes for which the distribution function is flat. Thus,

$$\tilde{f}(v_\parallel) = n \delta(v_\parallel) \quad \text{if} \quad \mu \gg 3,$$ (B1)

$$\tilde{f}(v_\parallel) = n \frac{1}{2} [\delta(v_\parallel - v_\parallel^+) + \delta(v_\parallel - v_\parallel^-)] \quad \text{if} \quad \mu \ll 3,$$ (B2)

where $\delta$ is the Dirac function and $n$ is the particle density. Let us emphasize that this result does not depend on the exact shape of the distribution function [although the exact value of $v_\parallel^\pm$ does, but it is always of the form $v_\parallel^\pm = \pm (1 - O(\mu^2))]$. These two limits are shown in Fig. B1.

In the non-relativistic regime ($\mu \gg 3$), the one-dimensional Jüttner–Syngle distribution function asymptotically behaves like a Gaussian distribution function in the velocity space. In the intermediate case where $\mu \sim 3$, the one-dimensional Jüttner–Syngle distribution function behaves like a uniform distribution function. In the ultrarelativistic regime ($\mu \ll 3$), the one-dimensional Jüttner–Syngle distribution function can simply be represented by the sum of two Dirac functions, namely $\delta(v_\parallel^+ + 1) + \delta(v_\parallel - 1)$, multiplied by a constant.

Such a distribution function describes two beams, moving in opposite directions at a velocity comparable to the velocity of light. Intuitively, this can be understood observing that in the ultrarelativistic case, all the relativistic particles have a velocity close to the velocity of light, so that any distribution function can be approximated by a sum of two Dirac functions $\delta(|v| \pm 1)$, multiplied by a constant.

In terms of the velocity variable, the shape of the function $f(p_\parallel v_\parallel)$ [hereafter simply denoted by $f(v_\parallel)$] is deduced from $f(p_\parallel)$ by shrinking the latter so that its non-relativistic part ($p_\parallel = v_\parallel \ll 1$) is not affected, but its relativistic part $p_\parallel \gg 1$ is shrunk into the domain $[-1, 1]$. This corresponds to the fact that one divides the function $f(v_\parallel)$ by $\gamma(v_\parallel)$ when changing from the momentum to the velocity variable. In the non-relativistic limit, the Lorentz factor $\gamma(v_\parallel)$ being very close to unity in the region where the distribution function takes non-negligible values, one obtains the same expression as for $\tilde{f}(v_\parallel)$. In the ultrarelativistic limit, the function $f(v_\parallel)$ is very different from the function $f(v_\parallel)$. Indeed, the distribution function is expected to have a width of $\sqrt{2} p_\parallel$ in the momentum space, where $p_\parallel$ is of the same order of magnitude than $p_\parallel = O(1/\mu_\parallel)$. As the integral of $f(p_\parallel)$ is 1, the height of the function $f(p_\parallel)$ is of order $O(\mu_\parallel)$. In the velocity $v_\parallel$ space, the function $f(v_\parallel)$ retains the same height, but its width is now shrunk to $2v_\parallel$, where $v_\parallel = \sqrt{1 - v_\parallel^2} = 1 - O(\mu_\parallel^2)$. Thus,

$$f(v_\parallel) = n \delta(v_\parallel) \quad \text{if} \quad \mu \gg 3,$$ (B3)

$$f(v_\parallel) = nO(\mu)Y(v_\parallel + v_\parallel^+)Y(v_\parallel - v_\parallel^-) \quad \text{if} \quad \mu \ll 3,$$ (B4)

where $Y$ is the Heavyside function. These two limits have been represented on Fig. B2.

Such a relation is valid because it is straightforward to derive the function to be integrated, the absolute convergence of the function to be integrated and of its derivative relative to $v_\parallel$. 

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Finally, a distribution function seen by an observer moving away with a velocity $(2v_0)$ with respect to the rest frame of particles is changed according to the Lorentz transformation law for momenta, i.e.

$$f(p_k) \rightarrow f\left(p_k \gamma (v_0 - v_0)\right).$$

(B5)

In addition, since the velocities $\pm v_{th}$ transform as

$$\pm v_{th} \rightarrow \frac{v_0 \pm v_{th}}{1 \pm v_{th}v_0},$$

(B6)

the width of the distribution function transforms as

$$2v_{th} \rightarrow 2v_{th} \frac{1 - v_0^2}{1 - (v_{th}v_0)^2}.$$  

(B7)

There are several possible cases. For a non-relativistic distribution function and a non-relativistic velocity $v_0$, the width of the distribution function (B3) remains unchanged:

$$f(v_{\parallel}) \rightarrow n \delta(v_{\parallel} - v_0), \quad v_0, v_{th} \ll 1,$$

(B8)

whereas the width of the distribution function is shrunk by a factor $\gamma_0^{-2}$ if the velocity $v_0$ is relativistic,

$$f(v_{\parallel}) \rightarrow n \gamma_0^{-2} \delta(v_{\parallel} - v_0), \quad v_0, v_{th} \ll 1, \quad \gamma_0 \gg 1.$$  

(B9)

For a relativistic distribution function and a non-relativistic velocity $v_0$, the distribution function (B4) will also remain unchanged.

---

**Figure B1.** Jüttner–Synge distribution function $\tilde{f}(v_k)$ for $\mu = 300$ (solid line, non-relativistic case) and $\mu = 0.1$ (dashed line, ultrarelativistic case). For a better visibility, the two functions have not been represented on the same vertical scale.

**Figure B2.** The function $f(v_k)$ for $\mu = 300$ (solid line, non-relativistic case) and $\mu = 0.1$ (dashed line, ultrarelativistic case). For the non-relativistic case, the function is almost the same as $f(v_k)$ of Fig. B1, whereas they differ strongly in the ultrarelativistic case. For a better visibility, the two functions have not been represented on the same vertical scale.
Table B1. Analytical results of Appendix B.

| temperature parameter | observer’s velocity versus thermal velocity | \( \frac{f(\nu)}{\nu} \) |
|-----------------------|---------------------------------------------|-----------------|
| \( \mu \gg 1 \)      | \( [v_{\parallel}] \ll v_{\parallel} \)   | \( \delta(v_{\parallel} - v_{\parallel}) \) |
| \( \mu \ll 1 \)      | \( \gamma_0 \ll \gamma_{\parallel} \)   | \( O(\mu)\gamma(v_{\parallel} + v_{\parallel})(v_{\parallel} - v_{\parallel}) \) |
| \( \mu \gg 1 \)      | \( [v_{\parallel}] \gg v_{\parallel} \)   | \( \gamma_{\parallel}^2 \delta(v_{\parallel} - v_{\parallel}) \) |
| \( \mu \ll 1 \)      | \( \gamma_0 \gg \gamma_{\parallel} \)   | \( \gamma_{\parallel}^2 O(\mu)^{-1} \delta(v_{\parallel} - v_{\parallel}) \) |

namely

\[
f(v_{\parallel}) \rightarrow nO(\mu)\gamma(v_{\parallel} + v_{\parallel})(v_{\parallel} - v_{\parallel}), \quad \gamma_0 \ll 2\gamma_{\parallel}^2, \quad \gamma_{\parallel} \gg 1.
\]

On the contrary, if the velocity \( v_0 \) is also relativistic, the distribution function can be shrunk to a very small width, of the order of \( v_{\parallel} \gamma_{\parallel}^2 / \gamma_0^2 \) if \( \gamma_0 \gg 2\gamma_{\parallel}^2 \), so that the distribution function will look like a Dirac function:

\[
f(v_{\parallel}) \rightarrow n[\gamma_0^2 O(\mu)]^{-1} \delta(v_{\parallel} - v_0), \quad \gamma_0 \gg 2\gamma_{\parallel}^2, \quad \gamma_{\parallel} \gg 1.
\]

The above condition, \( \gamma_0 \gg 2\gamma_{\parallel}^2 \), can be understood as follows: in order for the distribution function to look like a Dirac function, one must perform a Lorentz boost in order to put particles with negative velocity around the region where particles with positive velocity are located. This can be realized by performing two successive Lorentz boosts of Lorentz factor \( \gamma_{\parallel} \), which is almost equivalent to performing a Lorentz boost of Lorentz factor \( 2\gamma_{\parallel}^2 \) because \( v_{\parallel} \) is relativistic. In this case, because the height of the distribution function is of order \( O(\mu) \) (see equation B4), the distribution function behaves like \( [\gamma_0^2 O(\mu)]^{-1} \delta(v_{\parallel} - v_0) \).

Indeed, a comparison between the curves representing the one-dimensional Jüttner–Synge distribution functions of relativistic particles in the one-dimensional velocity space, for different values of the temperature parameter, clearly shows that according to the value of this parameter, the distribution functions have quite a different shape in the velocity space. The main results of this section are summarized in Table B1.

APPENDIX C: APPROXIMATE FORMS OF THE FUNCTION \( F_a(k_{\parallel}, \omega) \)

From geometric considerations and general properties of the involved relativistic one-dimensional Jüttner–Synge distribution functions in the different cases, we can reach a better and more direct understanding of our results. According to the above discussion, the shape of a one-dimensional distribution function as seen by a moving observer can be approximated by either simple Dirac or Heaviside functions. Such simplifications are used in the non-relativistic or in the ultrarelativistic cases to obtain estimates of the function \( F_a(k_{\parallel}, \omega) \), involved in the calculation of the appropriate dispersion relations.

In the rest frame of the particles, the function \( F_a(k_{\parallel}, \omega) \) can be written as

\[
F_a(k_{\parallel}, \omega) = \frac{1}{n_a} \frac{\alpha_{\omega}}{k_{\parallel}^2} \int_{-\infty}^{\infty} \frac{\partial f_a(p_{\parallel})}{\partial p_{\parallel}} \frac{1}{v_{\parallel} - v_{\parallel}} dp_{\parallel},
\]

or, equivalently,

\[
F_a(k_{\parallel}, \omega) = \frac{1}{n_a} \frac{\alpha_{\omega}}{k_{\parallel}^2} \int_{-1}^{1} \frac{\partial f_a[p_{\parallel}(v_{\parallel})]}{\partial v_{\parallel}} \frac{1}{v_{\parallel} - v_{\parallel}} dv_{\parallel},
\]

where in the last integral, we must use the function \( f_a[p_{\parallel}(v_{\parallel})] \) and not the function \( \tilde{f}_a(v_{\parallel}) \).

C1 Phase velocity smaller than the thermal velocity

In this case the phase speed is ‘inside’ the distribution, which is to say that the phase velocity is smaller than the thermal velocity. However, as we assume that \( v_{\parallel} \ll v_{\parallel} \), calculations can be carried out in the case of a vanishing phase velocity.

C1.1 Non-relativistic case

The non-relativistic distribution function has a width of order \( v_{\parallel} = O(\mu^{-1/2}) \). After normalization of the distribution function its integral is equal to 1, therefore, its height is of the order of \( O(\mu^{1/2}) \). Besides, the derivative of the distribution function \( -f[p_{\parallel}(v_{\parallel})] \) involved in the integrand exhibits a single peak centred around \( v_{\parallel} \). This peak has an approximate width of order \( 2v_{\parallel} \). By definition, the area contained under the peak is equal to

\[
- \frac{df[p_{\parallel}(v_{\parallel})]}{dv_{\parallel}} dv_{\parallel} = f(0) = O(\mu^{1/2}).
\]
Therefore, the height of the peak of the derivative of the distribution function is of the order of

$$\frac{df[p_\parallel(v_\parallel)]}{dv_\parallel} \bigg|_{v_\parallel = v_\parallel = \mathcal{O}(\mu^{1/2})} = \mathcal{O}(\mu).$$  \hfill (C4)

Let us now consider the function

$$g = -\frac{1}{v_\parallel} \frac{df[p_\parallel(v_\parallel)]}{dv_\parallel}. \hfill (C5)$$

The region around the centre of the peak of $v_\parallel g$ is boosted by a factor $1/v_\parallel = \mathcal{O}(\mu^{1/2})$, so that the peak height of $g$ is now of order $\mathcal{O}(\mu^{3/2})$. Around 0 the function $g$ can be approximated by $f''[p_\parallel(v_\parallel)](0)$ [the second derivative of $f[p_\parallel(v_\parallel)]$]. Now, a function of width $v_{\text{th}}$ can (crudely) be approximated by

$$f(v_\parallel) = f(0) \left( 1 - \frac{v_\parallel^2}{v_{\text{th}}^2} \right), \hfill (C6)$$

so that the second derivative of $f$ behaves as

$$f''[p_\parallel(v_\parallel)](0) = -f(0)v_{\text{th}}^2 = \mathcal{O}(\mu^{1/2}). \hfill (C7)$$

In conclusion, the function $g$ exhibits one peak of width $\mathcal{O}(\mu^{-1/2})$ and of height $\mathcal{O}(\mu^{3/2})$. Its integral is therefore

$$\int g(v_\parallel) dv_\parallel = \mathcal{O}(\mu). \hfill (C8)$$

Injecting this result in equation (C2), we obtain

$$F_\alpha(k_\parallel, \omega) = \left( \frac{\omega_{\text{th}}}{k_\parallel} \right)^2 \mathcal{O}(\mu), \hfill (C9)$$

in agreement with equation (21). 5

C1.2 Relativistic case

As we are interested in the case of a null phase velocity, we do not need the exact shape of the distribution function. The function $f(v_\parallel)$ in the velocity $v_\parallel$-space is like a Heaviside function:

$$f(v_\parallel) = n\mathcal{O}(\mu)[Y(v_{\text{th}} + v_\parallel)Y(v_{\text{th}} - v_\parallel)]. \hfill (C10)$$

In analogy of the above derivation, we then derive the distribution function, divide it by $v_\parallel$ and finally integrate over $v_\parallel$ in the domain $[-1, 1]$. Near the Dirac function, $v_\parallel$ is of the order of 1, consequently, the contribution of relativistic species is of the order of $\mu$. The contribution of the non-relativistic part of the distribution function is much smaller: using the same method as in the last section, one finds that $g = \mathcal{O}(\mu^3)$, and therefore the contribution of the non-relativistic part of the distribution is of order $\mathcal{O}(\mu^5)$.

Thus, finally, the contribution of the non-relativistic part of the distribution function $f(v_\parallel)$ is negligible in comparison with the contribution of the relativistic part, and we obtain

$$F_\alpha(k_\parallel, \omega) = \left( \frac{\omega_{\text{th}}}{k_\parallel} \right)^2 \mathcal{O}(\mu), \hfill (C11)$$

which is the same expression as in the non-relativistic case given by equation (C9), (although for different reasons as $\mu$ does not have the same value), and also agrees with equation 21.

C2 Phase velocity greater than the thermal velocity

C2.1 Non-relativistic case

As long as $(v_\parallel - v_{\text{th}})^{-1}$ can be considered as constant over the width of the distribution function, the function $F_\alpha(k_\parallel, \omega)$ can be written as

$$F_\alpha(k_\parallel, \omega) = \left( \frac{\omega_{\text{th}}}{k_\parallel} \right)^2 \int \frac{1}{v_\parallel - v_{\text{th}}} \delta'(v_\parallel) dv_\parallel. \hfill (C12)$$

Therefore, by integrating by parts, one obtains

$$F_\alpha(k_\parallel, \omega) = -\left( \frac{\omega_{\text{th}}}{\omega} \right)^2, \hfill (C13)$$

which agrees with the limit obtained for $\xi_{\text{NR}}^1$, when one takes $\mu \to \infty$.

5 Note that for a Gaussian distribution function such results are easily recovered.

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## C2.2 Relativistic case

Using the same method as in the above reverse relativistic case with \( v_w < v_{th} \), we obtain:

\[
F_{a}(k_{\parallel}, \omega) = -\left(\frac{\omega_{pa}}{k_{\parallel}}\right)^2 \frac{c(\mu)}{v_{ph} - v_{th}}^2 O(\mu),
\]  

(C14)

an expression that diverges when \( v_w \rightarrow v_{th} \) as expected. However, such an expression remains valid both when \( v_w > v_{th} \) and when \( v_w < v_{th} \).

By setting \( v_w \rightarrow 0 \), one recovers the result given by equation (C11). By setting \( v_w \gg v_{th} \) and \( \omega \rightarrow 0 \), one obtains

\[
F_{a}(k_{\parallel}, \omega) = -\left(\frac{\omega_{pa}}{\omega}\right)^2 \frac{c(\mu)}{v_{ph} - v_{th}}^2 O(\mu),
\]  

(C15)

which must be compared with equation (24) in the relativistic case. Indeed, by looking at the relativistic case, one sees that in the case \( v_w = 0 \) and \( k_{\parallel}^{\text{LR}} = k_{\parallel}^{\text{UR}} = \mu \), whereas \( \eta_{1}^{\text{LR}} \rightarrow 0 \) so that equation (24) reduces to

\[
F_{a}(k_{\parallel}, \omega) = -\left(\frac{\omega_{pa}}{k_{\parallel}}\right)^2 \frac{c(\mu)}{v_{ph} - v_{th}}^2 \sum_n \frac{1}{v_{ph}^{2n}} - \left(\frac{\omega_{pa}}{k_{\parallel}}\right)^2 \frac{c(\mu)}{v_{ph}^2 - 1},
\]  

(C16)

in agreement with equation (C14).

## C3 Summary

We summarize in Table C1 the results of this appendix, which we will use in Section 4 where we study the two-stream instability.

| phase speed versus thermal velocity | temperature parameter | \( F_{a}(k_{\parallel}, \omega) \) |
|------------------------------------|------------------------|-----------------------------|
| \( |v_{w}| \ll v_{th} \)             | \( \mu \gg 1 \)        | \( c(\mu)\left(\frac{v_{w}}{v_{th}}\right)^2 \) |
| \( |v_{w}| \ll v_{th} \)             | \( \mu \ll 1 \)        | \( c(\mu)\left(\frac{v_{w}}{v_{th}}\right)^2 \) |
| \( |v_{w}| \gg v_{th} \)             | \( \mu \gg 1 \)        | \( -\left(\frac{v_{w}}{v_{th}}\right)^2 \) |
| \( |v_{w}| \gg v_{th} \)             | \( \mu \ll 1 \)        | \( -\cfrac{c(\mu)}{v_{w}^2}\left(\frac{v_{w}}{v_{th}}\right)^2 \) |

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