Chern classes for representations of reductive groups

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Introduction

Let $G$ be a complex connected reductive group, and let $R(G)$ be its representation ring. As an abelian group $R(G)$ is spanned by the finite-dimensional representations $[V]$ of $G$, with the relations $[V \oplus W] = [V] + [W]$; the ring structure is defined by the tensor product. Moreover the exterior product of representations gives rise to a sequence of operations $\lambda^p : R(G) \to R(G)$ which make $R(G)$ into a $\lambda$-ring – see (1.1) below for the definition.

In the course of his work on the Riemann-Roch theorem, Grothendieck had the remarkable insight that this purely algebraic structure is enough to define Chern classes, without any reference to a cohomology or Chow ring. He associated to any $\lambda$-ring $R$ a filtration of $R$, the $\gamma$-filtration; the Chern classes take value in the associated graded ring $\text{gr} R$, and the Chern character is a ring homomorphism $\text{ch} : R \to \prod_p \text{gr}_p^\mathbb{Q} R$ where $\text{gr}_Q R = \text{gr} R \otimes \mathbb{Q}$. When applied to the Grothendieck ring $K(X)$ of vector bundles on a smooth algebraic variety $X$ these definitions give back the classical ones, at least modulo torsion: the graded ring $\text{gr}_Q K(X)$ coincides with the Chow ring $\text{CH}(X)_\mathbb{Q}$, and the Chern classes with the usual ones.

The aim of this note is to compute the Chern classes for the representation ring $R(G)$ – a simple exercise which I have been unable to find in the literature. Let $g$ be the Lie algebra of $G$; we will assume that $G$, and therefore $g$, are defined over $\mathbb{Q}$ (alternatively, we could without any loss take our Chern classes in $\text{gr}_C^\mathbb{Q} R(G)$). We denote by $\text{Pol}(g)^\text{inv}$ the ring of polynomial functions on $g$ which are invariant under the action of the adjoint group.

Theorem. – a) The graded ring $\text{gr}_Q^\mathbb{Q} R(G)$ is canonically isomorphic to the ring $\text{Pol}(g)^\text{inv}$ of invariant polynomial functions on $g$.

b) Let $\rho$ be a representation of $G$, and $L\rho$ the corresponding representation of $g$. The total Chern class $c(\rho)$ is equal to the invariant function $\det(1 + L\rho)$, and the Chern character $\text{ch}(\rho)$ to $\text{Tr}(\exp L\rho)$.

We have of course an explicit description of the ring $R(G)$. Let $T$ be a maximal torus of $G$; the Weyl group $W$ acts on the ring $R(T)$, and the restriction map $R(G) \to R(T)$ identifies $R(G)$ with the invariant sub-ring $R(T)^W$. As an abelian
group \( R(T) \) is spanned by one-dimensional elements (we will say that it is a *split \( \lambda \)-ring), which makes easy to compute its \( \gamma \)-filtration and Chern classes. The theorem follows easily once we know that the \( \gamma \)-filtration of \( R(T) \) induces that of \( R(G) \). This turns out to be a general fact for invariant sub-rings of split \( \lambda \)-rings; we will deduce it from the behaviour of the \( \gamma \)-filtration under the Adams operations (Proposition 1.6).

In section 3 we discuss an application. Let \( P \) be a principal \( G \)-bundle over a base \( B \) (which may be a variety, or an arbitrary topos); the Theorem provides a simple definition of the characteristic classes of \( P \) in the graded ring \( \text{gr}_Q K(B) \), and a simple way of computing the Chern classes of the associated vector bundles.

1. Generalities on \( \lambda \)-rings

In this section we recall the definition and basic properties of \( \lambda \)-rings. Standard references are [SGA6] or [F-L]; we follow the terminology of [SGA6], Exposé V.

(1.1) A \( \lambda \)-*ring* \( R \) is a commutative ring with two more pieces of structure:

- An augmentation, that is a ring homomorphism \( \varepsilon : R \to \mathbb{Z} \);
- A \( \lambda \)-structure, that is a sequence of maps \( \lambda^i : R \to R \) such that, for any \( x, y \) in \( R \) and \( n \in \mathbb{N} \),

\[
\lambda^0(x) = 1 \quad \lambda^1(x) = x \quad \lambda^n(x + y) = \sum_{p+q=n} \lambda^p(x) \lambda^q(y).\]

If we put \( \lambda_t(x) = \sum_p \lambda^p(x) t^p \in R[[t]] \), the last condition is equivalent to

\[
\lambda_t(x + y) = \lambda_t(x) \lambda_t(y).\]

Moreover we want formulas giving \( \lambda^p(xy) \) and \( \lambda^p(\lambda^q(x)) \) as polynomials in \( \lambda^1(x), \ldots, \lambda^p(x); \lambda^1(y), \ldots, \lambda^p(y) \) and \( \lambda^1(x), \ldots, \lambda^{pq}(x) \) respectively. A convenient way of expressing these is to introduce a sequence \( (\psi^k)_{k \geq 1} \) of additive endomorphisms of \( R \), the *Adams operations*, defined by

\[
\sum_{k \geq 1} \psi^k(x) (-t)^{k-1} = \lambda_t(x)^{-1} \frac{d}{dt} \lambda_t(x).\]

Then the condition on the \( \lambda \)-structure means that the \( \psi^k \) are ring endomorphisms, and satisfy \( \psi^k \circ \psi^\ell = \psi^{k\ell} \) for \( k, \ell \geq 1 \).

(1.2) Grothendieck associates to this situation a second \( \lambda \)-structure, defined by \( \gamma_t(x) = \lambda_{\frac{1}{t}}(x) \), and a decreasing filtration of \( R \), the \( \gamma \)-*filtration* \( (\Gamma^p(R))_{p \geq 0} \); we put \( \Gamma^0 = R \), \( \Gamma^1 = \ker \varepsilon \), and \( \Gamma^p \) is spanned by the elements \( \gamma^i_1(x_1) \ldots \gamma^i_k(x_k) \)
with $x_1, \ldots, x_k \in \Gamma^1$ and $i_1 + \cdots + i_k = p$. Let $\text{gr} R = \bigoplus_{p \geq 0} \Gamma^p / \Gamma^{p+1}$ be the associated graded ring. The Chern classes $(c_p(x))_{p \geq 0}$ of an element $x \in R$ are defined by

$$c_p(x) = \gamma^p(x - \varepsilon(x)) \text{ in } \Gamma^p / \Gamma^{p+1}.$$  

(1.3) A crucial point in what follows will be the behaviour of the $\gamma$-filtration with respect to the Adams operations. Let us say that an element $x$ of $R$ has $\lambda$-dimension $n$ if it satisfies $\varepsilon(x) = n$ and $\lambda^i(x) = 0$ for $i > n$; we say that $R$ is $\mathbb{Q}$-finite $\lambda$-dimensional if the $\mathbb{Q}$-vector space $R \otimes \mathbb{Q}$ is spanned by elements of finite $\lambda$-dimension. For such a $\lambda$-ring we have, for each $k \geq 1$ and $x \in \Gamma^p(R)$:

$$\psi^k(x) \equiv k^p x \mod. \mathbb{Q}\Gamma^{p+1}(R)$$  

([F-L], III, Proposition 3.1).

(1.5) Let us say that the $\lambda$-ring $R$ is split if it is generated as an abelian group by elements of $\lambda$-dimension 1. In this case we have $\Gamma^p(R) = \mathfrak{r}^p$, where $\mathfrak{r}$ is the augmentation ideal of $R$: indeed we have $\mathfrak{r}^p \subseteq \Gamma^p$ because $\gamma^1$ is the identity, and $\gamma_t(x) \in \sum_{p \geq 0} \mathfrak{r}^p t^p$ for all $x \in \mathfrak{r}$ because this holds for $x + y$ if it holds for $x$ and $y$, and $\gamma_t(\xi - 1) = 1 + (\xi - 1)t$ if $\xi$ has $\lambda$-dimension 1.

**Proposition 1.6**.— Let $R$ be a split $\lambda$-ring, and $G$ a finite group of automorphisms of the $\lambda$-ring $R$. Then the $G$-invariant elements form a sub-$\lambda$-ring $S$ of $R$. The filtrations $(\Gamma^p(S))_{p \geq 0}$ and $(\Gamma^p(R) \cap S)_{p \geq 0}$ coincide in $S \otimes \mathbb{Q}$.

**Proof**: Since the action of $G$ commutes with the $\lambda^i$ the $\lambda$-structure of $R$ induces a $\lambda$-structure on $S$. Moreover the $\mathbb{Q}$-vector space $S \otimes \mathbb{Q}$ is spanned by the elements $\sum_{g \in G} g \xi$, where $\xi$ is an element of $R$ of $\lambda$-dimension 1; these elements have $\lambda$-dimension $|G|$, thus $S$ is $\mathbb{Q}$-finite $\lambda$-dimensional.

To alleviate the notation we write $R, S$ instead of $R \otimes \mathbb{Q}$ and $S \otimes \mathbb{Q}$, and $\Gamma^p(R), \Gamma^p(S)$ instead of $\mathbb{Q}\Gamma^p(R), \mathbb{Q}\Gamma^p(S)$. We have $\Gamma^p(S) \subset \Gamma^p(R) \cap S$ for each $p$. Let us first prove that the two filtrations define the same topology. Let $\mathfrak{r} = \Gamma^1(R)$ be the augmentation ideal of $R$, and $\mathfrak{s} = S \cap \mathfrak{r}$. Since $R$ is a finite $S$-module, the ring $R / \mathfrak{s}R$ is artinian; thus there exists an integer $\nu$ such that $\mathfrak{r}^\nu \subset \mathfrak{s}R$, and therefore $\mathfrak{r}^n \subset \mathfrak{s}^n R$ for all $n$.

As $S$-modules $S$ is a direct summand of $R$ (consider the projector $r \mapsto \frac{1}{|G|} \sum_{g \in G} gr$). Thus for every ideal $\mathfrak{a}$ of $S$, the induced homomorphism $S / \mathfrak{a} \to R / \mathfrak{a}R$ is injective, which means that $\mathfrak{a}R \cap S = \mathfrak{a}$. Using (1.5) we obtain

$$\Gamma^{p\nu}(R) \cap S = \mathfrak{r}^{p\nu} \cap S \subset \mathfrak{s}^p R \cap S = \mathfrak{s}^p \subset \Gamma^p(S).$$
Let us prove now the inclusion $\Gamma^p(R) \cap S \subset \Gamma^p(S)$ by induction on $p$, the case $p = 0$ being obvious. We have just seen that there exists an integer $N$ such that $\Gamma^N(R) \cap S \subset \Gamma^p(S)$; let $n$ be the smallest integer with that property. Assume $n > p$. Let $x \in \Gamma^{n-1}(R) \cap S$, and let $k \geq 2$ be an integer; by (1.4) we have

$$\psi^k(x) \equiv k^{n-1}x \pmod{\Gamma^n(R) \cap S}.$$ 

On the other hand we have $x \in \Gamma^{p-1}(S)$ by the induction hypothesis; since $S$ is $\mathbb{Q}$-finite $\lambda$-dimensional (1.4) gives

$$\psi^k(x) \equiv k^{p-1}x \pmod{\Gamma^p(S)}.$$ 

Since $\Gamma^n(R) \cap S \subset \Gamma^p(S)$ we get $x \in \Gamma^p(S)$ for all $x \in \Gamma^{n-1}(R) \cap S$, contradicting the choice of $n$. Therefore we have $n \leq p$, hence $\Gamma^p(R) \cap S = \Gamma^p(S)$.

2. The $\lambda$-ring $R(G)$

(2.1) Let $T$ be a connected multiplicative group, and $X$ its character group; this is a free finitely generated abelian group. The representation ring $R(T)$ is isomorphic to the group algebra $\mathbb{Z}[X]$; we denote by $([\alpha])_{\alpha \in X}$ its canonical basis. The augmentation and the $\lambda$-structure are characterized by $\varepsilon([\alpha]) = 1$ and $\lambda_t([\alpha]) = 1 + t\alpha$.

Let $\mathfrak{r}$ be the augmentation ideal in $R(T)$. We have a canonical isomorphism $X \otimes \mathbb{Q} \to \mathfrak{h}^*$ which maps a character $\alpha$ to the class of $[\alpha] - 1$ (observe that $[\alpha\beta] - 1 \equiv ([\alpha] - 1) + ([\beta] - 1) \pmod{\mathfrak{r}^2}$). Since the rings $R(T)$ and $R(T)/\mathfrak{r} \cong \mathbb{Z}$ are regular, the canonical map $S(\mathfrak{r}/\mathfrak{r}^2) \to \bigoplus_{p \geq 0} \mathfrak{r}^p/\mathfrak{r}^{p+1}$ is bijective; using (1.5) we get a canonical isomorphism

$$\varphi : S(X) \cong \varphi \to \text{gr} R(T).$$

Under this isomorphism the element $\gamma^1([\alpha] - 1)$ of $\mathfrak{r}/\mathfrak{r}^2$ corresponds to $\alpha \in X \subset S(X)$. Thus the total Chern class $c(\rho)$ of an element $\rho = [\alpha_1] + \cdots + [\alpha_d]$ is given by $c(\rho) = \prod_i (1 + \alpha_i)$.

Let $\mathfrak{h}$ be the Lie algebra of $T$, viewed as a vector space over $\mathbb{Q}$. We have a canonical isomorphism $X \otimes \mathbb{Q} \to \mathfrak{h}^*$, which associates to a character $\alpha : T \to \mathbb{G}_m$ its derivative $L\alpha : \mathfrak{h} \to \mathbb{Q}$. Thus we can identify $S(X) \otimes \mathbb{Q}$ to the algebra $\text{Pol}(\mathfrak{h})$ of polynomial maps $\mathfrak{h} \to \mathbb{Q}$; the above formula becomes $c(\rho) = \prod_i (1 + L\alpha_i)$ in $\text{Pol}(\mathfrak{h})$.

For $\alpha \in X$, let $C_\alpha$ denote the one-dimensional representation with character $\alpha$. The element $\rho = [\alpha_1] + \cdots + [\alpha_d]$ of $R(T)$ is the class of the representation $V = C_{\alpha_1} \oplus \cdots \oplus C_{\alpha_d}$. In the corresponding representation $L\rho$ of $\mathfrak{h}$, an element $H$
of \( \mathfrak{h} \) acts through the diagonal matrix \( \text{diag}(L\alpha_1(H), \ldots, L\alpha_d(H)) \). Thus \( c(\rho) \) is equal to the function \( \det(1 + L\rho) \) in \( \widehat{\text{Pol}}(\mathfrak{h}) \).

The Chern character gives an homomorphism \( \text{ch} \) of \( R(T) \) into the ring \( \widehat{\text{Pol}}(\mathfrak{h}) \) of formal series on \( \mathfrak{h} \). We have \( \text{ch}([\alpha]) = e^{c_1([\alpha])} = e^{L\alpha} \), hence

\[
\text{ch}(\rho) = e^{L\alpha_1} + \ldots + e^{L\alpha_d} = \text{Tr}(\exp L\rho) \quad \text{in} \quad \widehat{\text{Pol}}(\mathfrak{h}) .
\]

**Remark 2.2.**— The exponential morphism of formal groups \( \exp : \widehat{\mathfrak{h}} \to \widehat{T} \) induces an injective homomorphism \( \exp^* : \mathcal{O}_{\widehat{T},1} = \mathbb{Q}[[X]] \to \mathcal{O}_{\widehat{\mathfrak{h}},0} = \widehat{\text{Pol}}(\mathfrak{h}) \), which maps \([\alpha]\) to \( e^{L\alpha} \). The Chern character is the composition of this map with the injection \( R(T) = \mathbb{Z}[X] \hookrightarrow \mathbb{Q}[X] \).

(2.3) Let \( G \) be a complex connected reductive group, \( \mathfrak{g} \) its Lie algebra, \( T \) a maximal torus of \( G \) and \( \mathfrak{h} \) its Lie algebra; we can assume that \( G, T, \mathfrak{g} \) and \( \mathfrak{h} \) are defined over \( \mathbb{Q} \). The Weyl group \( W \) acts on \( T \), hence on the ring \( R(T) \); restriction to \( T \) induces a homomorphism of \( \lambda \)-rings \( R(G) \to R(T) \), whose image is the invariant sub-ring \( R(T)^W \) ([SGA6], Exposé 0 App., Th. 1.1). By Proposition 1.6, the graded ring \( \text{gr}_\mathbb{Q} R(G) \) is isomorphic to \( (\text{gr}_\mathbb{Q} R(T))^W \), that is to the ring \( \text{Pol}(\mathfrak{h})^W \) of invariant polynomials on \( \mathfrak{h} \) (2.1).

Let \( \rho \) be a representation of \( G \). From the commutative diagrams

\[
\begin{array}{ccc}
R(T) & \xrightarrow{c} & \widehat{\text{Pol}}(\mathfrak{h}) \\
\uparrow & & \uparrow \\
R(G) & \xrightarrow{c} & \widehat{\text{Pol}}(\mathfrak{h})^W \\
\end{array}
\quad
\begin{array}{ccc}
R(T) & \xrightarrow{\text{ch}} & \widehat{\text{Pol}}(\mathfrak{h}) \\
\uparrow & & \uparrow \\
R(G) & \xrightarrow{\text{ch}} & \widehat{\text{Pol}}(\mathfrak{h})^W \\
\end{array}
\]

we obtain

\[
c(\rho) = \det(1 + L\rho) \quad \text{in} \quad \text{Pol}(\mathfrak{h})^W ; \quad \text{ch}(\rho) = \text{Tr}(\exp L\rho) \quad \text{in} \quad \widehat{\text{Pol}}(\mathfrak{h})^W .
\]

The restriction map \( \text{Pol}(\mathfrak{g}) \to \text{Pol}(\mathfrak{h}) \) induces an isomorphism \( \text{Pol}(\mathfrak{g})^{inv} \to \text{Pol}(\mathfrak{h})^W \) ([B], §8, Th. 1). Since the functions \( \det(1 + L\rho) \) and \( \text{Tr}(\exp L\rho) \) are invariant, the above equalities hold as well in \( \text{Pol}(\mathfrak{g})^{inv} \) and \( \widehat{\text{Pol}}(\mathfrak{g})^{inv} \) respectively. This proves the theorem stated in the introduction.

### 3. Application: Chern classes of associated bundles

(3.1) Let \( B \) be an algebraic variety, and \( P \) a principal \( G \)-bundle over \( B \) (for what follows \( B \) could be as well a scheme, or even a topos). To any representation
\( \rho : G \to \text{GL}(V) \) is associated a vector bundle \( P^\rho = P \times^G V \) on \( B \); we define in this way a homomorphism of \( \lambda \)-rings \( b^*_P : \text{R}(G) \to \text{K}(B) \)\(^1\). In view of the isomorphism described above, it induces a homomorphism of graded rings

\[
c_P : \text{Pol}(\mathfrak{g})^{\text{inv}} \to \text{gr}_{\mathbb{Q}}\text{K}(B),
\]
called the characteristic homomorphism.

Let \( \ell = \dim T \). Recall that there are homogeneous functions \( I_1, \ldots, I_\ell \) in \( \text{Pol}(\mathfrak{g})^{\text{inv}} \) such that \( \text{Pol}(\mathfrak{g})^{\text{inv}} = \mathbb{Q}[I_1, \ldots, I_\ell] \) ([B], §8, Théorème 1). The elements \( c_P^{(i)} := c_P(I_i) \) may be called the characteristic classes of \( P \) (but note that they depend on the choice of the generating sequence \( (I_1, \ldots, I_\ell) \)). If \( B \) is a smooth variety, the graded ring \( \text{gr}_{\mathbb{Q}}\text{K}(B) \) is canonically isomorphic to the rational Chow ring \( \text{CH}_{\mathbb{Q}}(B) \) ([SGA6], Exposé XIV, n°4); our characteristic classes correspond under this isomorphism to those defined in [V] and [E-G].

**Proposition 3.2.**— Let \( \rho \) be a representation of \( G \); write \( \det(1 + L\rho) = F(I_1, \ldots, I_\ell) \), where \( F \) is a polynomial in \( \ell \) indeterminates. Let \( P \) be a principal \( G \)-bundle on \( B \), with characteristic classes \( c_P^{(1)}, \ldots, c_P^{(\ell)} \) in \( \text{gr}_{\mathbb{Q}}\text{K}(B) \). Then the total Chern class in \( \text{gr}_{\mathbb{Q}}\text{K}(B) \) of the associated bundle \( P^\rho \) is

\[
c(P^\rho) = F(c_P^{(1)}, \ldots, c_P^{(\ell)}).
\]

Similarly, if \( \text{Tr}(\exp L\rho) = G(I_1, \ldots, I_\ell) \), with \( G \in \mathbb{Q}[[T_1, \ldots, T_\ell]] \), we have \( \text{ch}(P^\rho) = G(c_P^{(1)}, \ldots, c_P^{(\ell)}) \).

**Proof:** This follows from the Theorem, the commutative diagram

\[
\begin{array}{ccc}
\text{R}(G) & \xrightarrow{b^*_P} & \text{K}(B) \\
\downarrow c & & \downarrow c \\
\text{Pol}(\mathfrak{g})^{\text{inv}} & \xrightarrow{c_P} & \text{gr}_{\mathbb{Q}}\text{K}(B)
\end{array}
\]

and the corresponding diagram for the Chern character. ■

**Examples 3.3.**— The Proposition provides a way of computing the characteristic classes in terms of Chern classes, at least for classical groups. We use the standard generating system \( (I_1, \ldots, I_\ell) \) given for instance in [B], §13. We denote by \( E_P \) the vector bundle on \( B \) associated to \( P \) through the standard representation of \( \text{GL}(n), \text{SO}(n) \) or \( \text{Sp}(n) \) in \( \mathbb{C}^n \).

\(^1\) In fancy terms, the principal bundle \( P \) corresponds to a morphism \( b_P \) of \( B \) into the classifying topos (or stack) \( BG \), and \( b^*_P \) is just the pull-back map.
a) For $G = \text{GL}(\ell)$, we have $I_p(A) = \text{Tr}(\Lambda^p A)$ for $A \in \mathfrak{gl}(\ell)$ and $1 \leq p \leq \ell$; this gives $c_p^{(p)} = c_p(E_P)$ for $1 \leq p \leq \ell$.

b) For $G = \text{Sp}(2\ell)$ or $\text{SO}(2\ell + 1)$, we have $I_p(A) = \text{Tr}(\Lambda^{2p} A)$; this gives $c_p^{(p)} = c_{2p}(E_P)$ for $1 \leq p \leq \ell$ (the vector bundle $E_P$ is isomorphic to its dual, thus its odd Chern classes with rational coefficients vanish).

c) For $G = \text{SO}(2\ell)$, we realize $\mathfrak{g}$ as the space of skew-symmetric matrices in $\mathfrak{gl}(2\ell)$; we have $I_p(A) = \text{Tr}(\Lambda^{2p} A)$ for $p < \ell$, and $I_{2\ell}(A) = \text{Pf}(A)$. We obtain $c_p^{(p)} = c_{2p}(E_P)$ for $1 \leq p \leq \ell - 1$, and $c_{2\ell}^{(\ell)}$ is a class in $\text{gr}^Q K(B)$ with square $c_{2\ell}(E_P)$.

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