Approaches to constructing two-dimensional interpolation formulas in the presence of boundary layers

A I Zadorin

1 Sobolev Institute of Mathematics, pr. Koptyuga,4, 630090, Novosibirsk, Russia

E-mail: zadorin@ofim.oscsbras.ru

Abstract. An overview of approaches to the construction of two-dimensional interpolation formulas for a function of two variables with large gradients in the boundary layer regions is given. The problem is that the use of polynomial interpolation formulas on a uniform grid can lead to errors of the order of $O(1)$. It is shown that the use of polynomial interpolation formulas on the Shishkin and Bakhvalov grids leads to the fact that the error becomes uniform in a small parameter. More accurate results are obtained by using the Bakhvalov grid. It is shown that on a uniform grid it is possible to successfully apply interpolation formulas that are exact on the singular components responsible for large function gradients in boundary layers. Numerical results are given.

1. Introduction

On the basis of singularly perturbed problems, various convective-diffusion processes with predominant convection are modeled. The solution of the singularly perturbed problem has large gradients in the boundary layer region, which leads to the loss of convergence of classical difference schemes. For the numerical solution of such problems, two main approaches have been developed: refining the grid in the boundary layer region [1], [2] and constructing difference schemes on a uniform grid based on fitting to the singular component responsible for large gradients of the solution in region of the boundary layer [3]. The application of the considered approaches makes it possible to ensure the convergence of the difference scheme, uniform in a small parameter. The error of the difference schemes constructed in this way is uniform in large solution gradients in the boundary layer region.

The problem of constructing interpolation formulas for functions with large gradients in the boundary layer is also relevant. The problem is that the error of the polynomial interpolation formula can be of the order of $O(1)$ in the presence of a boundary layer [4].

To construct workable interpolation formulas, one can apply the same approaches as when constructing difference schemes. In [4] it is proposed to use the boundary layer component and construct interpolation formulas on a uniform grid that are exact on this component. In [4] such an interpolation formula is constructed with an arbitrarily specified number of interpolation nodes. The estimates of the error of such formula, uniform in a small parameter, were obtained in [5].

Let us dwell on the use of meshes that are condensed in the boundary layer. In [6], the uniform in a small parameter estimate of the error of Lagrange polynomial of an arbitrary degree on the
Shishkin mesh [2] is obtained. In [7], an error estimate is obtained for the formula of piecewise linear interpolation of order \(O(1/N^2)\) uniformly in a small parameter on the Bakhvalov mesh [1], [8].

In this paper, we will analyze approaches to the construction of interpolation formulas in the presence of boundary layers in the two-dimensional case.

In [9] investigated a two-dimensional interpolation formula constructed on the basis of fitting the Lagrange polynomial on the Shishkin mesh was obtained in the two-dimensional case.

By \(C\) and \(C_j\) we mean positive constants independent of \(\varepsilon\) and the number of grid steps \(N\). We will use one constant \(C\) to estimate various values, if this does not cause confusion.

### 2. Interpolation formulas on refined grids

Let us investigate the application of two-dimensional polynomial interpolation formulas on the grids of Bakhvalov [1] and Shishkin [2].

We assume that the decomposition is valid for the function \(u(x, y)\):

\[
  u(x, y) = p(x, y) + E_1(x, y) + E_2(x, y) + E_{1,2}(x, y),
\]

where \((x, y) \in \bar{\Omega}, \ \bar{\Omega} = [0, 1]^2\). We assume that \(p(x, y)\) is a regular component with bounded derivatives up to a certain order, \(E_1, E_2, E_{1,2}\) are boundary layer components with large gradients in the boundary layers. All components in representation (1) are not specified explicitly, but for some constant \(C\) the following estimates hold

\[
  \left| \frac{\partial^j p(x, y)}{\partial x^i} \right| \leq C, \quad \left| \frac{\partial^j p(x, y)}{\partial y^j} \right| \leq C, \quad 0 \leq i, j \leq J,
\]

where \(\alpha > 0, \beta > 0\) are separated from zero, \(\varepsilon \in (0, 1]\). The constant \(J\) will be set. The derivatives of the functions \(E_1, E_2, E_{1,2}\) in the boundary layer regions can grow indefinitely with decreasing parameter \(\varepsilon\).

According to [2, 11], the representation (1) is valid for the solution of following elliptic problem:

\[
  \varepsilon u_{xx} + \varepsilon u_{yy} + a_1(x)u_x + a_2(y)u_y - c(x, y)u = f(x, y), \quad (x, y) \in \Omega; \quad (6)
\]

\[
  u(x, y) = g(x, y), \quad (x, y) \partial \Omega, \quad (7)
\]

where \(a_1(x) \geq \alpha > 0, \ a_2(y) \geq \beta > 0, \ \varepsilon \in (0, 1]\), the functions \(a_1(x), a_2(x), c(x, y), g(x, y)\) are sufficiently smooth.

Difference schemes on Shishkin and Bakhvalov grids are widely used to solve the problem (6) - (7). Let’s set these grids. In general, the grid looks as

\[
  \Omega^h = \{(x_i, y_j), \ i = 0, 1, \ldots, N_1, \ j = 0, 1, \ldots, N_2, \ h_i = x_i - x_{i-1}, \ \tau_j = y_j - y_{j-1},
\]

\[
  x_0 = 0, x_{N_1} = 1, \ y_0 = 0, y_{N_2} = 1. \}
\]

**Bakhvalov mesh.** We define the Bakhvalov mesh in the region \(\Omega\) according to [1], [8].
Let $k_1$ be the number of nodes of interpolation by $x$ and $k_2$ be the number of nodes of interpolation by $y$. Let’s set the nodes and steps of the grid. Let

$$
\sigma_1 = \min \left\{ \frac{1}{2}, \frac{k_1 \varepsilon}{\alpha} \ln \varepsilon \right\}. \quad (8)
$$

We set $\sigma_1 = 1/2$ if $\varepsilon > \varepsilon^{-1}$.

For $\sigma_1 = 1/2$ we define the grid $\Omega^h$ uniform in $x$ with steps $h_i = 1/N_1$.

For $\sigma_1 < 1/2$, we set

$$
x_i = -\frac{k_1 \varepsilon}{\alpha} \ln \left[ 1 - 2(1 - \varepsilon)i/N_1 \right], \quad i = 0, 1, \ldots, \frac{N_1}{2}, \quad (9)
$$

$$
x_i = \sigma_1 + (2i/N_1 - 1)(1 - \sigma_1), \quad N_1/2 \leq i \leq N_1. \quad (10)
$$

For the variable $y$, we set the grid nodes in the same way. Let be

$$
\sigma_2 = \min \left\{ \frac{1}{2}, \frac{k_2 \varepsilon}{\beta} \ln \varepsilon \right\}. \quad (11)
$$

For $\varepsilon > \varepsilon^{-1}$, we set $\sigma_2 = 1/2$.

For $\sigma_2 = 1/2$, we define the grid $\Omega^h$ uniform in $y$ with steps $\tau_j = 1/N_2$. For $\sigma_2 < 1/2$, we set

$$
y_j = -\frac{k_2 \varepsilon}{\beta} \ln \left[ 1 - 2(1 - \varepsilon)j/N_2 \right], \quad j = 0, 1, \ldots, \frac{N_2}{2}, \quad (12)
$$

$$
y_j = \sigma_2 + (2j/N_2 - 1)(1 - \sigma_2), \quad N_2/2 \leq j \leq N_2. \quad (13)
$$

**Shishkin mesh.** According to [2], let us

$$
h_i = \frac{2\sigma_1}{N_1}, \quad 1 \leq i \leq \frac{N_1}{2}; \quad h_i = \frac{2(1 - \sigma_1)}{N_1}, \quad \frac{N_1}{2} < i \leq N_1,
$$

$$
\tau_j = \frac{2\sigma_2}{N_2}, \quad 1 \leq j \leq \frac{N_2}{2}; \quad \tau_j = \frac{2(1 - \sigma_2)}{N_2}, \quad \frac{N_2}{2} < j \leq N_2, \quad (14)
$$

$$
\sigma_1 = \min \left\{ \frac{1}{2}, \frac{k_1 \varepsilon}{\alpha} \ln N_1 \right\}, \quad \sigma_2 = \min \left\{ \frac{1}{2}, \frac{k_2 \varepsilon}{\beta} \ln N_2 \right\}. \quad (15)
$$

**Two-dimensional polynomial interpolation.** We will interpolate the function $u(x, y)$ in rectangular cells $K_{i,j} = [x_i, x_{i+k_1-1}] \times [y_j, y_{j+k_2-1}]$, forming a coverage of the original domain $\bar{\Omega}$. We assume that $N_1$ is a multiple of $2(k_1 - 1)$ and $N_2$ is a multiple of $2(k_2 - 1)$. Then each cell $K_{i,j}$ is entirely in the area of the boundary layer or outside it. To interpolate $u(x, y)$ in the cell $K_{i,j}$ we use the Lagrange polynomial with $k_1$ interpolation nodes by $x$ and $k_2$ interpolation nodes by $y$.

To interpolate by $x$, we use the Lagrange polynomial [12]:

$$
L_{k_1}(u, x, y) = \sum_{m=i}^{i+k_1-1} u(x_m, y) \prod_{k=i, k \neq m}^{i+k_1-1} \frac{x - x_k}{x_m - x_k}. \quad (16)
$$

Similarly, for a given $x$, we interpolate by $y$

$$
L_{k_2}(u, x, y) = \sum_{m=j}^{j+k_2-1} u(x, y_m) \prod_{k=j, k \neq m}^{j+k_2-1} \frac{y - y_k}{y_m - y_k}. \quad (17)
$$
Based on (16), (17), we write down a two-dimensional interpolation formula:

\[ L_{k_1,k_2}(u, x, y) = L_{k_2}(L_{k_1}(u, x, y), x, y). \] (18)

To estimate the error of the two-dimensional interpolation formula (18), we use the relation:

\[ \left| L_{k_1,k_2}(u, x, y) - u(x, y) \right| \leq \left| L_{k_2}(L_{k_1}(u, x, y) - u(x, y), x, y) \right| + \left| L_{k_2}(u, x, y) - u(x, y) \right|. \] (19)

On the basis of (19), obtaining an estimate of the error of a two-dimensional interpolation formula is reduced to estimating the error of one-dimensional interpolation formulas and the stability of one-dimensional interpolants to a disturbance of the interpolated function.

Let us dwell on the Bakhvalov mesh. In [7], an error estimate is obtained for the formula of piecewise linear interpolation on the Bakhvalov mesh

\[ \left| L_2(u, x, y) - u(x, y) \right| \leq \frac{C}{N^2}. \] (20)

Using the stability of the linear interpolation to the disturbance of the interpolated function, from (19), (20) we get

\[ \left| L_{2,2}(u, x, y) - u(x, y) \right| \leq \frac{C}{N^2}. \] (21)

The cases of other values of \( k_1, k_2 \) can be considered similarly.

Let us estimate the interpolation error on the Shishkin mesh.

**Lemma 1** Let \( \Omega^h \) be Shishkin mesh. Let us in (2)-(5) \( J \geq k_1, k_2 \). Then for some constant \( C \) and any cell \( K_{i,j} \) the following estimate is fullfiled

\[ \left| L_{k_1,k_2}(u, x, y) - u(x, y) \right| \leq C\left[ \left( \frac{\ln N_1}{N_1} \right)^{k_1} + \left( \frac{\ln N_2}{N_2} \right)^{k_2} \right], (x, y) \in K_{i,j}. \] (22)

**Proof.** We use the well-known estimate

\[ \left| L_{k_1,k_2}(u, x, y) - u(x, y) \right| \leq C_1 \left[ M_{k_1,x}h_1^{k_1} + M_{k_2,y}h_2^{k_2} \right], (x, y) \in K_{i,j}, \] (23)

where

\[ M_{k_1,x} = \max_{(x,y) \in K_{i,j}} \left| \frac{\partial^{k_1}u(x,y)}{\partial x^{k_1}} \right|, \quad M_{k_2,y} = \max_{(x,y) \in K_{i,j}} \left| \frac{\partial^{k_2}u(x,y)}{\partial y^{k_2}} \right|. \]

Further, in (23) we apply the estimates (2) - (5) for the derivatives and use the values of the steps of the Shishkin mesh (14). As a result, we get the estimate (22).

### 3. Interpolation fitting formula on uniform grid

Let us a function \( u(x, y) \) be smooth enough with the following representation:

\[ u(x, y) = p(x, y) + d_1(y)\Phi(x) + d_2(x)\Theta(y) + d_3\Phi(x)\Theta(y), \] (24)

where \((x, y) \in \bar{\Omega}, \ \bar{\Omega} = [0, 1]^2\). We suppose that functions \( p, d_1, d_2 \) have bounded derivatives up to some order and are not known, \( d_3 \) is not given. Boundary layer functions \( \Phi(x), \Theta(y) \) are known, but have large gradients. The representation (24) takes a place for the solution of a singular perturbed elliptic problem (6) - (7) with regular boundary layers [13].

We assume that the grid \( \Omega^h \) is uniform with steps \( h_1 \) and \( h_2 \) in \( x \) and \( y \).

In an arbitrary rectangular cell \( K_{i,j} \) with \( k_1 \) nodes along \( x \) and with \( k_2 \) nodes along \( y \), we construct an interpolation formula for functions of the form (24), exact on the boundary.
layer components $\Phi(x), \Theta(y)$. First, for a given $y$, we define an interpolation in $x$, exact on the component $\Phi(x)$:

$$L_x(u, x, y) = L_{k_1-1}(u, x, y) + \frac{[x_i, \ldots, x_i+k_1-1]u}{[x_i, \ldots, x_i+k_1-1]} [\Phi(x) - L_{k_1-1}(x, y)].$$  (25)

In (25) $L_{k_1-1}(u, x, y)$ corresponds to the interpolation by $x$ of the function $u(x, y)$ by the Lagrange polynomial with interpolation nodes $x_i, x_{i+1}, \ldots, x_{i+k_1-2}$ for a given value of $y$, $[x_i, \ldots, x_i+k_1-1]u$ is divided difference by $x$ of function $u(x, y)$ [12].

By analogy with (25), we define an interpolation by $y$, exact on $\Theta(y)$:

$$L_y(u, x, y) = L_{k_2-1}(u, x, y) + \frac{[y_j, \ldots, y_j+k_2-1]u}{[y_j, \ldots, y_j+k_2-1]} [\Theta(y) - L_{k_2-1}(y, x)].$$  (26)

Using (25), (26), after interpolation by $x$, we interpolate by $y$

$$L_{\Phi, \Theta, k_1, k_2}(u, x, y) = L_{k_2-1}(L_x(u, x, y), x, y) + \frac{[y_j, \ldots, y_j+k_2-1]L_x(u, x, y)}{[y_j, \ldots, y_j+k_2-1]} [\Theta(y) - L_{k_2-1}(y, x)].$$  (27)

So, we have constructed a two-dimensional interpolation formula (27).

The formula (27) is correct if

$$\Phi^{(k_1-1)}(x) \neq 0, x \in (x_i, x_{i+k_1-1}), \quad \Theta^{(k_2-1)}(y) \neq 0, y \in (y_j, y_{j+k_2-1}).$$  (28)

It is easy to prove that the interpolation formula (27) is exact on functions:

$$x^i, x^j \Theta(y), \quad i = 0, 1, \ldots, k_1 - 2, \quad y^i, y^j \Phi(x), \quad j = 0, 1, \ldots, k_2 - 2, \quad \Phi(x) \Theta(y).$$

**Lemma 2** Let

$$\Phi^{(k_1-1)}(x) \neq 0, \quad \Phi^{(k_1)}(x) \neq 0, \quad k_1 \geq 2, \quad x \in (x_i, x_{i+k_1-1}),$$  (29)

$$\Theta^{(k_2-1)}(y) \neq 0, \quad \Theta^{(k_2)}(y) \neq 0, \quad k_2 \geq 2, \quad y \in (y_j, y_{j+k_2-1}).$$  (30)

Then for an arbitrary cell $K_{i,j}$, for $k_1, k_2 \geq 2$ for some constant $C$

$$|u(x, y) - L_{\Phi, \Theta, k_1, k_2}(u, x, y)| \leq C [h_1^{k_1-1} + h_2^{k_2-1}], \quad (x, y) \in K_{i,j}.\quad (31)$$

The proof is based on an estimate of the error of one-dimensional interpolation formulas (25), (26). The error of such formulas is estimated in [5]. The conditions (29), (30) are fulfilled, for example, in the cases of exponential and power boundary layers.

4. Numerical results

Consider a function of the form (1)

$$u(x, y) = (1 - e^{-x/\varepsilon})(1 - e^{-2y/\varepsilon})(1 - x)(1 - y) + \cos \frac{\pi x}{2} e^{-y}, \quad \varepsilon > 0, x, y \in [0, 1].$$

We consider the case $k_1 = k_2 = k$, $N_1 = N_2 = N$.

We will calculate the error at the nodes $(\tilde{x}_i, \tilde{y}_j)$ of the 10 times thickened grid $\Omega^h$.

Table 1 shows the error of the formula (18)

$$\Delta_{\varepsilon,N} = \max_{i,j} |L_{2,2}(u, \tilde{x}_i, \tilde{y}_j) - u(\tilde{x}_i, \tilde{y}_j)|.$$
Table 1. The error of the interpolation formula on the uniform grid, \( k_1 = k_2 = 2 \)

| \( \varepsilon \) | 16     | 32     | 64     | 128    | 256    | 512    |
|-------------------|--------|--------|--------|--------|--------|--------|
| 1                 | 1.34e-3| 3.37e-4| 8.47e-5| 2.12e-5| 5.31e-6| 1.33e-6|
| \( 10^{-1} \)     | 8.28e-2| 2.76e-2| 8.01e-3| 2.16e-3| 5.62e-4| 1.43e-4|
| \( 10^{-2} \)     | 6.77e-1| 5.09e-1| 2.98e-1| 1.40e-1| 4.98e-2| 1.50e-2|
| \( 10^{-3} \)     | 7.19e-1| 7.35e-1| 7.42e-1| 7.26e-1| 5.95e-1| 3.66e-1|
| \( 10^{-4} \)     | 7.19e-1| 7.35e-1| 7.42e-1| 7.46e-1| 7.48e-1| 7.49e-1|
| \( 10^{-5} \)     | 7.19e-1| 7.35e-1| 7.42e-1| 7.46e-1| 7.48e-1| 7.49e-1|

Table 2. The error of the interpolation formula on Shishkin mesh, \( k_1 = k_2 = 2 \)

| \( \varepsilon \) | 16     | 32     | 64     | 128    | 256    | 512    |
|-------------------|--------|--------|--------|--------|--------|--------|
| 1                 | 1.34e-3| 3.37e-4| 8.47e-5| 2.12e-5| 5.31e-6| 1.33e-6|
| \( 10^{-1} \)     | 3.58e-2| 1.48e-2| 5.72e-3| 2.04e-3| 5.62e-4| 1.43e-4|
| \( 10^{-2} \)     | 4.30e-2| 1.87e-2| 7.34e-3| 2.63e-3| 8.88e-4| 2.86e-4|
| \( 10^{-3} \)     | 4.52e-2| 1.95e-2| 7.57e-3| 2.70e-3| 9.05e-4| 2.91e-4|
| \( 10^{-4} \)     | 4.55e-2| 1.96e-2| 7.62e-3| 2.72e-3| 9.13e-4| 2.94e-4|
| \( 10^{-5} \)     | 4.55e-2| 1.97e-2| 7.63e-3| 2.72e-3| 9.14e-4| 2.94e-4|

Table 3. The error of the interpolation formula on Bakhvalov mesh, \( k_1 = k_2 = 2 \)

| \( \varepsilon \) | 16     | 32     | 64     | 128    | 256    | 512    |
|-------------------|--------|--------|--------|--------|--------|--------|
| 1                 | 1.34e-3| 3.37e-4| 8.47e-5| 2.12e-5| 5.31e-6| 1.33e-6|
| \( 10^{-1} \)     | 1.30e-2| 3.63e-3| 9.85e-4| 2.59e-4| 6.67e-5| 1.69e-5|
| \( 10^{-2} \)     | 1.49e-2| 3.98e-3| 1.11e-3| 3.21e-4| 9.32e-5| 2.62e-5|
| \( 10^{-3} \)     | 1.54e-2| 3.87e-3| 9.77e-4| 2.50e-4| 6.59e-5| 1.81e-5|
| \( 10^{-4} \)     | 1.55e-2| 3.90e-3| 9.75e-4| 2.44e-4| 6.12e-5| 1.54e-5|
| \( 10^{-5} \)     | 1.56e-2| 3.90e-3| 9.76e-4| 2.44e-4| 6.10e-5| 1.53e-5|
Table 4. The error of the interpolation formula fitted to boundary layer components, $k = 2$

| $\varepsilon$ | 8    | 16   | 32   | 64   | 128  |
|---------------|------|------|------|------|------|
| 1             | 8.65e-3 | 2.20e-3 | 5.52e-4 | 1.38e-4 | 3.46e-5 |
| 2^{-2}        | 3.58e-2 | 9.24e-3 | 2.33e-3 | 5.85e-4 | 1.46e-4 |
| 2^{-3}        | 7.07e-2 | 1.93e-2 | 4.99e-3 | 1.26e-3 | 3.17e-4 |
| 2^{-4}        | 1.22e-1 | 3.73e-2 | 1.02e-2 | 2.63e-3 | 6.68e-4 |
| 2^{-5}        | 1.66e-1 | 6.27e-2 | 1.92e-2 | 5.22e-3 | 1.36e-3 |
| 2^{-6}        | 1.89e-1 | 8.60e-2 | 3.17e-2 | 9.68e-3 | 2.64e-3 |
| 2^{-7}        | 1.92e-1 | 9.81e-2 | 4.37e-2 | 1.59e-2 | 4.86e-3 |
| 2^{-8}        | 1.92e-1 | 1.00e-1 | 5.00e-2 | 2.20e-2 | 7.97e-3 |

Table 5. The error of the interpolation formula on Bakhvalov mesh, $k_1 = k_2 = 3$

| $\varepsilon$ | 16   | 32   | 64   | 128  | 256  | 512  |
|---------------|------|------|------|------|------|------|
| 1             | 8.86e-5 | 1.14e-5 | 1.45e-6 | 1.82e-7 | 2.29e-8 | 2.86e-9 |
| 10^{-1}       | 2.96  | 2.98  | 2.99  | 2.99  | 3.00  | 3.00  |
| 10^{-2}       | 9.43e-3 | 1.54e-3 | 2.19e-4 | 2.93e-5 | 3.78e-6 | 4.81e-7 |
| 10^{-3}       | 2.62  | 2.81  | 2.90  | 2.95  | 2.98  | 2.98  |
| 10^{-4}       | 1.81e-2 | 1.74e-3 | 1.69e-4 | 1.71e-5 | 1.86e-6 | 2.22e-7 |
| 10^{-5}       | 3.38  | 3.36  | 3.31  | 3.20  | 3.07  | 3.07  |
| 10^{-6}       | 3.93e-2 | 4.10e-3 | 4.21e-4 | 4.24e-5 | 4.19e-6 | 4.10e-7 |
| 10^{-7}       | 3.26  | 3.28  | 3.31  | 3.34  | 3.35  | 3.35  |
| 10^{-8}       | 6.22e-2 | 6.83e-3 | 7.50e-4 | 8.10e-5 | 8.60e-6 | 8.93e-7 |
| 10^{-9}       | 3.19  | 3.19  | 3.21  | 3.24  | 3.27  | 3.27  |
| 10^{-10}      | 8.56e-2 | 9.65e-3 | 1.10e-3 | 1.24e-4 | 1.39e-5 | 1.53e-6 |
| 10^{-11}      | 3.15  | 3.14  | 3.15  | 3.16  | 3.18  | 3.18  |

Table 6. The error of the interpolation formula fitted to boundary layer components, $k = 3$

| $\varepsilon$ | 8    | 16   | 32   | 64   | 128  |
|---------------|------|------|------|------|------|
| 1             | 4.57e-4 | 5.90e-5 | 7.41e-6 | 9.26e-7 | 1.16e-7 |
| 2^{-2}        | 1.96e-3 | 3.39e-4 | 4.99e-5 | 6.78e-6 | 8.84e-7 |
| 2^{-3}        | 4.67e-3 | 1.20e-3 | 2.16e-4 | 3.24e-5 | 4.44e-6 |
| 2^{-4}        | 7.27e-3 | 2.66e-3 | 7.07e-4 | 1.23e-4 | 1.95e-5 |
| 2^{-5}        | 3.40e-3 | 4.14e-3 | 1.47e-3 | 3.94e-4 | 7.21e-5 |
| 2^{-6}        | 6.83e-3 | 1.96e-3 | 2.20e-3 | 7.77e-4 | 2.10e-4 |
| 2^{-7}        | 8.08e-3 | 2.11e-3 | 5.35e-4 | 1.34e-4 | 5.33e-5 |
| 2^{-8}        | 8.08e-3 | 2.11e-3 | 5.35e-4 | 1.34e-4 | 3.97e-5 |

in the case $k = 2$ and of the uniform grid. From table 1 implies the unacceptability of using the uniform grid for $\varepsilon \leq h = 1/N$.  

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Table 2 shows the error $\Delta_{\varepsilon,N}$ and the calculated order of accuracy

$$M_{\varepsilon,N} = \log_2 \frac{\Delta_{\varepsilon,N}}{\Delta_{\varepsilon,2N}}$$

of the formula (18) in the case $k = 2$ and the Shishkin mesh. The calculation results agree with the estimate (22).

Table 3 shows the error and the order of accuracy of the formula (18) in the case of the Bakhvalov mesh for $k = 2$. The calculation results give an error of the order of $O(1/N^2)$, which is consistent with the estimate (21).

Table 4 shows the error of the formula of fitting to the boundary layer component (27) in the case $k_1 = k_2 = 2$. The order of accuracy decreases with decreasing $\varepsilon$ from the second to the first, which is consistent with the estimate (31).

Table 5 shows the interpolation error and the order of accuracy of the formula (18) in the case of the Bakhvalov mesh for $k_1 = k_2 = 3$. Numerical results give the errors of the order of $O(1/N^3)$.

Table 6 shows the error of the formula of fitting to the boundary layer component (27) in the case of $k_1 = k_2 = 3$. The order of accuracy decreases from the third to the second with decreasing $\varepsilon$, which is consistent with the estimate (31).

5. Conclusion
A survey of the author’s approaches to the construction of two-dimensional interpolation formulas for the function corresponding to the solution of an elliptic singularly perturbed problem is presented. It is shown that the use of polynomial interpolation formulas on a uniform grid is unacceptable, since as the parameter $\varepsilon$ decreases, the error becomes of the order of $O(1)$.

The use of Shishkin and Bakhvalov grids, which are condensed in the boundary layers, leads to the fact that the error of the polynomial interpolation formulas becomes uniform in a small parameter, that is, uniform in large gradients of the function in the boundary layers. More accurate results are obtained by using the Bakhvalov grid.

It is shown that in the case of a uniform grid, one can successfully apply interpolation formulas that are exact on the singular components responsible for large gradients of the function in the boundary layers. The error of such formulas is uniform in a small parameter and has order $O(1/N^{k-1})$, where $k$ is the number of interpolation nodes by $x$ and $y$. Application of the polynomial interpolation formula on the Bakhvalov grid gives an error of the order of $O(1/N^k)$.

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