From single to many Impurities in one-dimensional systems

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We examine the effects of disorder in one-dimensional systems. We link the case of a few impurities, typical of a short quantum wire, to that of a finite density of scatterers more appropriate for a long wire or a macroscopic system. Finally we investigate the effects of long-range interactions on the transport in 1D systems. We predict in that case a conductivity behaving as $\sigma(T) \sim T^2$.

I. INTRODUCTION

Since the discovery of Anderson localization, impurity effects in electronic systems have always been a fascinating subject. When interactions among the electrons are present, the effect of disorder is enhanced and leads to various singular behaviors. This is particularly true in one-dimensional systems, in which interactions themselves have a dramatic effect on the physical properties of the system and lead to a non-Fermi liquid behavior.

Prompted by the existing experimental situation of one-dimensional conductors [1], namely the organic conductors, and by analogy with the situation in higher dimensions, studies of such systems mainly looked in two directions. First the interactions were considered as short-range, a reasonable assumption for a material with many chains where the long range Coulomb interactions could easily be screened. In that case the interactions lead to the so-called Luttinger liquid behavior [2–4]. Second, a macroscopic system with many weak impurities was considered. Even in this situation the disorder has drastic effects and it was rapidly recognized that the effects of disorder are strongly affected by the interactions [5–12].

More recently progress in nanotechnologies have made it possible to directly realize a one-dimensional channel of electrons [13–20]. Edge states of the fractional quantum Hall effect [21–23] have also been shown to be realizations of Luttinger liquids. Such systems present a quite different situation than the organic conductors. First they can be realized of arbitrary length, and besides the existing disorder artificial impurities can be made by creating constrictions. For such systems it is natural to examine the effects of a few, weak or strong impurities on the conductance [24–29].

The question that arises is therefore how to link these two extreme situations, respectively of one or two impurities in a system and of a uniform distribution of impurities. We address such a question in section [1] and link the methods used for these two cases.

For realistic wires it is also important to worry about the nature of the interactions. In a quantum wire, a single channel of electrons exists, and unless an external gate or metallic plate provides screening, one can expect a drastic effect of the long-range Coulomb interactions [30,24]. It is necessary to reexamine the effects of disorder in the presence of such long-range forces. In the pure system these forces lead to a dramatic modification of the physical properties and transform the Luttinger liquid into the one-dimensional equivalent of a Wigner crystal.
We examine how impurities pin this Wigner crystal in section III, as well as the observable consequences.

II. IMPURITIES IN A LUTTINGER LIQUID

Let us consider a one-dimensional interacting system. We restrict ourselves in this section to the case of short-range interactions. The effects of long-range Coulomb interactions will be examined in section III. In that case the system can be described by the so-called Luttinger Liquid (LL) [2–4], quite different from the normal Fermi liquid occurring in higher dimensions. To deal with such an interacting system, it is very convenient to reexpress the excitations of the electron gas in terms of collective excitations (charge and spin fluctuations). This technique, known as bosonization is by now familiar and we will just recall its salient features. More details about the technique can be found in [2–4]: for each type of fermions with spin \( s = \uparrow, \downarrow \), one introduces a field \( \phi_s \) related to the density \( \rho_s(x) = -\nabla \phi_s(x)/\pi \). It is convenient to introduce the sum \( \phi \) and the difference \( \phi_{\sigma} \), describing charge and spin density fluctuations. In terms of such fields the low energy properties of the system can be expressed for any microscopic Hamiltonian with short-range interactions as \( H = H_\rho + H_\sigma \) with

\[
H_\nu = \frac{1}{2\pi} \int dx \left[ (u_\nu K_\nu)(\pi \Pi_\nu)^2 + \left( \frac{u_\nu}{K_\nu} \right)(\partial_\nu \phi_\nu)^2 \right]
\]

(1)

where \( \Pi_\nu \) is the momentum conjugate to \( \phi_\nu \). All interaction effects are absorbed in the parameters \( u_\nu \) (the velocity of charge or spin excitations) and \( K_\nu \). \( K_\rho = 1 \) in the absence of interactions, \( K_\rho > 1 \) for attractive interactions and \( K_\rho < 1 \) for repulsive ones. In fact the spin Hamiltonian is slightly more complicated and contains, in addition to the quadratic part, a sine-Gordon term

\[
\frac{2g}{(2\pi \alpha)^2} \int dx \cos(\sqrt{8} \phi_\sigma)
\]

(2)

which describes the backscattering of electrons with opposite spins. For spin isotropic interactions \( K_\sigma - 1 \sim g/(\pi u_\sigma) \) and [2] renormalizes the spin part of (1) to impose asymptotically \( K_\sigma^* = 1 \) and \( g^* = 0 \) (for repulsive interactions). For attractive interactions a gap opens in the spin excitations and the spin degrees of freedom can be discarded.

The main physical manifestation of the Luttinger liquid is the nonuniversal decay of the correlation functions. For example the density-density correlation behaves as

\[
\langle \rho(x)\rho(0) \rangle = \frac{K_\rho}{(\pi x)^2} + A_1 \cos(2k_F x) x^{-1 - K_\rho} \ln^{-3/2}(x) + A_2 \cos(4k_F x) x^{-4K_\rho} + ...
\]

(3)

For the case of spin-anisotropy the exponent of the \( 2k_F \) part is given by \(- (K_\rho + K_\sigma)\), and in (3) the log corrections arise from the renormalization of \( K_\sigma \) to one [3]. For not too repulsive interactions the \( 2k_F \) fluctuation is the strongest. The \( 4k_F \) term comes from the interplay of the density and the backscattering term [2] and has extremely important effects for Coulomb interactions, as will be seen in section III. In what follows we will note by \(- \mu \) the exponent of the density-density correlation function (for a Luttinger liquid with moderately repulsive interactions \( \mu = K_\rho + K_\sigma \)).

This nonuniversal decay of the density correlations greatly affects the coupling to disorder. Disorder can be introduced in the system by putting impurities at positions \( R_i \). If for simplicity we take an identical potential \( V \) for each impurity, the disorder term becomes
\[ H_{\text{dis}} = \sum_i \int dr V(r - R_i) \rho(r) \] (4)

For macroscopic systems, the impurities are randomly distributed, and transport is best described by the conductivity, as a function of either temperature or frequency. In the case of a short system, or for a few impurities, it is more convenient to compute the conductance.

A. Many impurities

Let us first examine the macroscopic system with many random impurities. For a macroscopic system, and for not too strong disorder, one usually approximates the disorder by a random potential \( H_{\text{dis}} = \int dr V(r) \rho(r) \) where \( V(x) \) is Gaussian correlated \( V(x)V(x') = D_\xi \delta(x-x') \).

Although the main motivation in doing so is probably theoretical simplicity, such a substitution is quite good for weak disorder. Indeed it assumes that the concentration of impurities becomes infinite \( n_i \to \infty \), but that the potential of each impurity becomes weak \( V \to 0 \), so that the product \( n_i V^2 = D \), remains a constant. In the process one loses a parameter: the average distance between impurities (or the strength of the potential on one impurity). Such an approximation is in fact equivalent to looking only at the lowest order in the self energy of the electrons [32].

Various methods have been proposed to treat such a disorder. We will explain here the results obtained by a renormalization group method since it allows both to treat finite disorder and to make connection with the single impurity case (for more references on the other methods see e.g. [12]). To use the RG one expands in powers of the coupling to disorder \( D_\xi \). The perturbation is divergent, and one can derive [11,12] renormalization equations upon rescaling of the short distance cutoff \( \alpha \to \alpha e^{l} \)

\[
\frac{dK_\rho}{dl} = -\frac{2u_\rho}{u_\sigma} D \quad (5)
\]
\[
\frac{dD}{dl} = D(3 - \mu - g) \quad (6)
\]

where \( \mu \) is the exponent of the density-density correlation function and \( g \) is defined in (3). \( D \) is a dimensionless parameter related to the disorder \( D = \frac{2D_\xi}{\pi u_\sigma} \left( \frac{u_\sigma}{u_\rho} \right) K_\rho \). There are similar renormalization equations for the spin part but since we focus on transport we will ignore them for the moment. The \((3 - \mu)\) term in equation (3) for the disorder is easy to understand and comes from the dimension of the second order term in disorder

\[
\int dx d\tau \int dx' d\tau' D_\xi \delta(x-x') \langle \rho(x,\tau) \rho(x',\tau') \rangle \quad (7)
\]

Note that the backscattering term \( g \) changes this factor into \((3 - \mu - g)\). Very often this term \( g \) is incorrectly neglected. Two reasons for this: (i) it is difficult to obtain since it results from contractions in higher order in the RG, although it is in fact of the same order as the terms giving \( \mu \). No naive derivation of the RG equations will give it. (ii) For repulsive interactions \( g \) scales to zero and \( K_\rho \to 1 \), and it is argued that to get the correct asymptotic physics one can merely insert the renormalized values of the parameters in (3). This is perfectly correct to obtain the phase diagram, but not if one wants to describe the physics at intermediate length scales (such as conductivity at finite length or finite temperatures). This term traduces the fact that a local repulsion among opposite spins, although unable to block density fluctuations at large length scales (namely \( K_\rho^* \to 1 \)) will kill short distance density fluctuations, making the system much harder to pin on disorder. At intermediate length scales, repulsive interactions
tend to make the disorder less relevant and decrease localization (let us again emphasize that this effect is only at intermediate length scales). This has dramatic consequences on the effects of interactions on persistent currents \[33\]. We will not explore these effects further (more details and consequences can be found in [12,33]) and denote by $\tilde{\mu} = \mu + g$, the correct factor in (6).

From equation (6) is is easy to see that the disorder is relevant when $\mu < 3$. For repulsive interactions between spins ($g > 0$) this corresponds to $K_\rho < 2$, and for attractive interactions between spins to $K_\rho < 3$. (Let us recall that $K_\rho = 2$ or 3 correspond to extremely attractive interactions among charges.) \[9–12\]. The equation (6) traduces the renormalization of the Born amplitude of disorder by the interactions and has been derived under various forms in the past \[3–7\]. In a diagrammatic language it corresponds \[3,34\] to diagram (a) of Figure 1.

FIG. 1. Diagrams describing the renormalization of the disorder by the interactions (a) and the renormalization of the interactions by the disorder (b). Solid and dotted lines are fermions with $\pm k_F$, the wiggly line is the interaction and the cross is the impurity scattering.

In fact, in the RG treatment the equation (6) is not sufficient to correctly describe the physics of the problem as we will see below, and should be complemented by equation (5). This first equation (5) traduces the effect of disorder on the exponents of the correlation functions. It has also a diagrammatic representation given in Figure 1 (b) \[34\].

Using the RG one can compute the finite temperature (or finite frequency) conductivity of the system \[11,12\]. The idea is simply to renormalize until the cutoff is of the order of the thermal length $l_T \sim u/T$ corresponding to $e^{\hat{\nu}} \sim l_T/\alpha$. At this length scale the disorder can be treated in the Born approximation. As the conductivity is a physical quantity it is not changed under renormalization and we have:

$$
\sigma(n(0), D(0), 0) = \sigma(n(l), D(l), l) = \sigma_0 \frac{n(l)D(0)}{n(0)D(l)} = \sigma_0 e^{\hat{\nu}D(0)/D(l)}
$$

where $\sigma(n(l), D(l), l) = \sigma(l)$ and $n(l)$ are respectively the conductivity and the electronic density at the scale $l$. $\sigma_0 = e^2 v_F^2 / 2\pi \hbar D_\xi$ is the conductivity in the Born approximation, expressed with the initial parameters. To compute the conductivity the full integration of the two equations (5-6) is required \[11,12\]. A simple expression can be obtained for infinitesimal disorder $D \to 0$ since one can neglect the renormalization of the exponents (5). In that case one can trivially integrate (6) to obtain, using (8)

$$
\sigma(T) \sim \frac{1}{n_i V^2} T^{2-\tilde{\mu}}
$$

This result is quite ancient \[3–7\] and corresponds simply to the renormalization of the effective disorder by the interactions. One immediately sees that (8) alone would lead to a paradox since (8) gives a localized-delocalized boundary at $\mu = 3$ whereas (6) gives perfect conductivity above $\mu = 2$ (i.e. the noninteracting point). This apparent contradiction is solved when (5) is correctly taken into account. For $\mu < 3$ (including the noninteracting point) any small but
finite disorder grows, renormalizing the exponents and ultimately leading to a decrease of the conductivity, even if one started initially from $\mu > 2$. A crude way of taking into account both equations (5-6) would be to say that one can still use (9) but with scale dependent exponents (see [11,12]).

$$\sigma(T) \sim T^{2-\mu(T)}$$

(10)

This renormalization of exponents and the faster decay of conductivity is in fact the signature of Anderson localization. The whole RG scheme breaks down when $D \sim 1$, at a length scale corresponding to the localization length of the system [11,12]. A reasonable guess of the temperature dependence below this length scale is an exponentially activated conductivity.

**B. Single impurity**

Let us now examine the case of a single impurity. The coupling to disorder is simply

$$H = V \rho(x = 0)$$

(11)

and since scattering occurs only in a finite (here one point) portion of the sample, the conductance is the more appropriate way to describe transport. For weak $V$ one can use the same renormalization method expanding in the interaction $V$. One obtains the RG equations [25,26]

$$\frac{dK_\rho}{dl} = 0$$

(12)

$$\frac{dV^2}{dl} = V^2(2 - \tilde{\mu})$$

(13)

The first difference in (12) compared to the finite density of impurities is the absence of renormalization of the exponents. The second equation (13) is seemingly different from the one for Gaussian disorder (6): The factor 2, instead of 3, now comes from the fact that the impurity only acts at $x = 0$ leaving only a double integral over time. In fact this difference is only apparent and (13) and (6) are in fact the same equation giving the renormalization of the Born amplitude of disorder by the interactions. The difference in dimension can be accounted for by the fact that $D$ is not just the impurity potential but $D_\xi = n_i V^2$ where $n_i$ is the impurity concentration, and contains also the inverse of a length. One can define $\tilde{D} = e^l D$. Upon renormalization $\tilde{D}$ follows the RG equation (13), but of course would lead to a modified equation (6), preserving the physics for the case of a uniform disorder.

For a single impurity the only effect of interactions is therefore to change the Born amplitude of scattering (for weak disorder). The conductance is therefore given by the effective scattering at the scale $l = \ln(E_F/T)$. Integrating (13) one gets [25,26]

$$G_0 - G(T) = -\delta G(T) \propto \delta R(T) = V^2 T^{\tilde{\mu}-2}$$

(14)

where $\delta R$ is the scattering produced by one impurity and $G_0$ the conductance of a pure wire. Of course here since only renormalization of the disorder is present the transition between zero/infinite conductivity (equivalently zero/finite conductance) occurs at $\mu = 2$, (i.e. in the vicinity of the noninteracting point).

When the disorder is relevant $\mu < 2$, the weak coupling RG scheme ceases to be valid when the Born amplitude is of order one. Contrarily to Gaussian disorder where this indicates the localization, here one has a different strong coupling fixed point. The pinning on the impurity becomes strong and one has to consider weak tunneling through the impurity site. This fixed point has been analyzed in [25,26] and still gives a power law for the conductance, but with a different exponent than in the weak coupling case.
C. Impurities vs Impurity

A summary of the various equations and physical behaviors for a single and many impurities is given in Table 2.

| disorder | D  | dK_p/dl | dD/dl | transition          | σ or δG              | strong coupling          |
|----------|----|---------|-------|---------------------|-----------------------|--------------------------|
| Gaussian | $n_i V^2$ | $-D(l)$ | $3 - \mu(l)$ | strong attraction | $\sigma \sim T^{2-\mu(l)}$ | Anderson loc. ($\sigma \sim e^{-T/T^?}$) |
| single   | $V^2$ | 0       | $2 - \mu(l)$ | non-interacting    | $\delta G \sim T^{\mu-2}$ | strong barrier ($G \sim T^{1/\kappa_p-1}$) |

As is obvious from Table 2, the two systems offer striking similarities as well as some important physical differences. As can be expected on physical grounds since $D$ has the meaning of $n_i V^2$ at weak coupling, the behavior of a single impurity corresponds exactly to taking the limit $D \to 0$ in the equations (5-6) (and therefore formally to the $n_i \to 0$ limit i.e. a single impurity). This limit is non-trivial since in principle the Gaussian approximation corresponds to infinitely dense (very weak) impurities $n_i \to \infty$. However if one takes $D \to 0$ first, since the equation (6) does not depend on the strength of $D$, the only effect is to neglect the renormalization of the exponents by the disorder. To match the conductivity and conductance one notices that in the limit $n_i \to 0$ the impurities are well separated enough to be considered as independent scatterers and their scatterings add up. For $N_i$ impurities in a wire of length $L$ then the conductivity would be, if $G$ is the conductance of the wire and using $G = \sigma(\omega = 0)L$

$$\sigma(T) = LG \propto \frac{L}{N_i}(\delta R)^{-1} = \frac{1}{n_i}(-\delta G)^{-1}$$

showing again that (14) and the approximation (9) are identical. As we already pointed out, the collective effects of impurities (i.e. the fact that they cannot be considered as independent scatterers) manifest themselves in the renormalization of the exponents. This leads to a faster decay of the conductivity/conductance when the temperature decreases compared to that of independent impurities.

For Gaussian disorder collective effects manifest themselves at any length scale since again formally $n_i \to \infty$. On the other hand the scattering on a single impurity can never become strong. For realistic (Poissonian) disorder corresponding to many impurities with both an average distance $a$ between the impurities and a strength of the potential of one impurity $V$, one can therefore expect a competition between two length scales: the temperature $T_2$ for which the thermal length $u/T$ is equal to the distance between impurities and the crossover temperature $T_{1,cr}$ for which one impurity goes to strong coupling.

As long as the thermal length $u/T$ remains smaller than the distance between impurities $a$, one cannot have any collective effect $[35,36]$. There is no renormalization of the exponents, even for many impurities and the system is described by the equations (12-13). The conductivity of the system is given by (3) i.e. with fixed exponents. Two cases are then possible:

(i) Either the collective effects between impurities will become important before each impurity can reach strong coupling ($T_2 > T_{1,cr}$). This is the case if the impurities are weak and/or dense enough. In that case below $T_2$ one can use the Gaussian representation of the disorder (3-4), and the exponents start to be renormalized. The conductivity becomes (11) i.e. the same type of temperature dependence, with varying exponents. Finally at the scale $T_{loc}$ the disorder...
becomes of order one and the system becomes localized. Below $T_{\text{loc}}$ we expect an activated conductivity, but up to our knowledge no theoretical method allows to investigate this regime unambiguously. Experimental measurements of the conductivity in this regime would therefore prove to be extremely useful. Note that already at the scale $T_{\text{loc}}$ the conductivity is a strongly decreasing function of the temperature since $\mu \sim 0$ and therefore $\sigma(T) \sim T^2$.

(ii) Impurities are dilute enough or strong enough to reach individually strong coupling before the collective effects can take place. One then crosses over to the tunnelling behavior. Collective effects will occur at a lower temperature but since each impurity corresponds to a strong potential, they will have to be examined by quite different methods, leading to a very different localization transition. This fascinating problem is quite complicated and has only been solved up to now for two barriers, but a solution for many impurities is still lacking.

These two situations are reminiscent of the “weak pinning” and “strong pinning” cases of the classical charge density waves, and could in principle be observed in quantum wires. The various situations are summed up on figure 3.

![FIG. 3. Temperature dependence of the conductivity. The dashed line is the prolongation of the $T^2-\mu$ law. (a) “Weak pinning”: $T_{1,\text{cr}} < T_2$. Collective effects occur before each impurity can reach strong coupling. Below $T_2$, renormalization of the exponents gives a faster decay of the conductivity. (b) “Strong pinning”: the opposite situation, $T_{1,\text{cr}} > T_2$.](image)

III. LONG-RANGE COULOMB INTERACTIONS

Let us now focus on the effects of long-range interactions on a one-dimensional system. For quantum wires, it is a priori important to retain the long-range nature of the Coulomb interaction. Whether this is screened or not in a given experimental system is an heavily debated question. At least for some experimental situations there is evidence that Coulomb interactions play an important role. We will show here how transport properties of the system allow to answer this important question and show characteristic features of the Coulomb interaction.

It is by now well known that in the presence of long-range forces the physical properties of the system are quite different from those of a LL. As we saw from (3) density-density fluctuations have power law decay, and for not too strong repulsion (i.e. $\tilde{K}_p > 1/3$), dominant density fluctuations are the $2k_F$ ones. When long-range interactions are present, the electrons form a Wigner “crystal” (WC): indeed the more slowly decaying correlation functions are now the $4k_F$ ones corresponding to the distance between electrons. The decay is slower than any
power law, of the form \[ \rho_{4k_F}(x) \rho_{4k_F}(0) \sim e^{-\ln^{1/2}(x)} \] (16)

The \(2k_F\) charge and spin correlations still decay as power laws. Due to the change in nature of the dominant correlations, one can expect quite different transport properties than in a LL \([3, 12]\). Pinning on a single impurity \([23, 38]\) or two impurities \([33, 40]\) already shows these differences and lead to extremely interesting new behaviors for the conductance. Here we confine ourselves to the study of pinning on many weak impurities, such a study being relevant for long wires.

The WC and the pinning on disorder can be described using again a bosonization representation similar to \([1]\). We will not dwell here on the technical details, that can be found in \([2]\), and concentrate on the results. In the following we will drop the subscript \(\rho\) in \(K_\rho\) and \(\Phi_\rho\). The WC can be described as a modulation of the charge density \(\rho(x) \sim \rho_0 \cos(Qx + 2\sqrt{2}\Phi)\) where \(\rho_0\) is the uniform amplitude of the charge density, \(Q = 4k_F\) its wave vector. \(\Phi\) describes here the location and the motion of the Wigner crystal. Coupling to disorder is described again by a random potential of the form \([3]\). Since in the WC the \(4k_F\) density fluctuation is the dominant one (i.e. the one with the slowest decay), transport will be dominated by \(4k_F\) scattering on impurities, contrarily to the case of a LL where \(2k_F\) scattering is usually the dominant one. This has important consequences for electrons with spins: even in the presence of long-range charge interaction or very strongly repulsive short-range interactions \(K_\rho \to 0\) spin isotropy still imposes \(K^s_\rho = 1\), and the \(2k_F\) density-density correlation still decays at best as \(1/r\). Considering the \(2k_F\) scattering on impurities as was done in \([1, 4]\) amounts to underestimating seriously the scattering on disorder and gives incorrect exponents for the temperature dependence of the conductivity. Here we retain the dominant \(4k_F\) scattering only \([3]\).

In the presence of impurities the Wigner crystal is pinned: the phase \(\Phi(x)\) adjusts to the impurity potential on a scale given by \(L_0\) the pinning length (which corresponds to the localization length of the electron system). This process of pinning is analogous to what happens in charge density waves (CDW) \([13, 14]\), but with important differences: (i) since we are dealing with electrons, quantum effects are a priori important contrary to what happened for charge density waves; (ii) the long-range Coulomb interactions have to be taken into account. To study such effects one uses techniques similar to \([13, 14]\) suitably modified to take into account (i) and (ii) \([3]\). The pinning length is given by

\[
L_0 = \left(\frac{8e^2/\kappa}{\alpha\pi^2V_0\rho_0\gamma n_i}\right)^{\frac{4}{3}} \ln^{\frac{2}{3}} \left(\frac{1}{d}\left(\frac{8e^2/\kappa}{\alpha\pi^2V_0\rho_0\gamma n_i}\right)^{\frac{4}{3}}\right)
\] (17)

where \(\gamma = e^{-4(\Phi^2)} \approx e^{-4\Phi^2}\ln^{1/2}V_0\) and \(\bar{K} = \sqrt{\pi\alpha K_\rho}\). \(V_0\) is the strength of the impurity potential, \(n_i\) their concentration, \(d\) the width of the wire, \(\kappa = 4\pi\epsilon\) the dielectric constant and \(K\) the LL parameter, taking into account the short-range part of the Coulomb interaction, a number typically of order \(0.5 - 1\). In the above expression we have neglected log(log) corrections, and, estimating numerically the relative contributions of the elastic (short-range part of the interaction \(q \sim 2k_F\)) and Coulomb terms (the long-range part \(q \sim 0\)) using typical values \(u = 3 \times 10^7 cm.s^{-1}\) and \(K = 0.5\), we have kept only the dominant Coulomb term. For comparison the localization length is \(L_0 \approx \left(\frac{v_F}{\alpha\pi V_0\rho_0 n_i}\right)^{2/3}\) for a charge density wave and \(L_0 \approx \left(\frac{v_F}{\alpha\pi V_0\rho_0 n_i}\right)^{2/3} / (3-K_\rho-K_\sigma)\) for a LL to the same degree of approximation (for the LL, due to the strong quantum fluctuations, this expression is only valid for very small disorder and far from the transition. For a more complete formula see \([1, 2]\)).
Coulomb interactions have two effects. First they give the logarithmic factor enhancing the rigidity of the system. Secondly they kill the anomalous exponents coming from the quantum fluctuations (the $K_\rho + K_\sigma$ in the LL) and drive the system to a classical limit. This can be traced back to the fact that the correlation functions decay much more slowly than in a LL ($e^{-\ln^{3/2}(r)}$ instead of a power law), therefore the system is much more ordered and the fluctuations around the ground state are much less important. Although they do not lead any more to anomalous exponents, quantum effects are still important: they strongly reduce the effective disorder seen by the WC since $V \rightarrow V\gamma$. This effect can be quantitatively very important (since $L_0$ is very large for dilute impurities), and contributes to making the system more likely to be in the weak pinning regime. Let us emphasize again that this limit cannot be reproduced naively by taking the LL and just letting the charge interactions becoming large $K_\rho \rightarrow 0$, due to the $K_\sigma$ term. It is crucial to consider the $4k_F$ scattering for which spin fluctuations are absent.

Another important length scale comes from the competition between the short-range and long-range parts of the Coulomb interaction. In addition to the long-range part responsible for the formation of the WC at large distances, the short distance repulsion also gives rise to LL effects. Below a certain length scale $L_{cr}$, the short-range part is dominant, and the system can be described by $2k_F$ scattering on impurities, power law type correlation functions and standard LL transport. Above this length scale the effect of the long-range part of the Coulomb interactions is dominant and one recovers the WC behavior. For a wire with a single mode and no external screening, we estimated for reasonable parameters $L_{cr} \sim d$ where $d$ is the width of the wire and for practical purposes the whole behavior should be WC. For wires where the screening of the Coulomb interactions is more efficient, one could observe a crossover between LL and WC behavior.

The most interesting quantity to measure is of course the conductivity (or conductance). The above two length scales can of course be converted into either frequency or temperature by using the dispersion relation $\omega_L = \epsilon(q = 1/L)$, and define a pinning frequency $\omega_{pin}$ (or temperature $T_{pin}$) and a crossover frequency $\omega_{cr}$ (or temperature $T_{cr}$). The frequency dependence of the real part of the conductivity is shown on figure 4.

![FIG. 4. Frequency dependence of the conductivity. For simplicity the regime above $\omega_{cr}$ is not shown. $y$ stands for the rescaled frequency $\omega/\omega_{pin}$. $\zeta$ is a constant depending on the parameters of the wire. The dash-dotted curve shows an $y^2$ law.](image)
At small frequencies $\sigma(\omega) \sim \omega^2$, a similar result than for the LL [15–17] (up to log corrections). The low frequency conductivity is to be contrasted with the previous result of Shklovskii and Efros [48] who find $\sigma(\omega) \sim |\omega|$. They derived this result in a very different physical limit when the localization length is much smaller than the interparticle distance. In that case, the phase $\phi$ consists of a series of kinks of width $l$ the localization length and located at random positions (with an average spacing $k_F^{-1} \gg l$). The low-energy excitations correspond to soliton-like excitations. In the physical limit we are considering $k_F^{-1} \ll L_0$, the phase $\phi$ has no kink-like structure but rather smooth distortions between random values at a scale of order $L_0$. To get the dynamics, the approximation we are using only retains the small “phonon” like displacements of the phase $\phi$ relative to the equilibrium position and no “soliton” like excitations. In the absence of Coulomb interactions the phonon-like excitations alone, when treated exactly in the classical limit $K \to 0$ are known [46] to give the correct frequency dependence of the conductivity $\omega^2 \ln^2(1/\omega)$. When Coulomb interactions are included and in the limit where the localization length is much larger than the interparticle distance, it is not clear whether soliton-like excitations similar to those considered by Efros and Shklovskii have to be taken into account, but in the classical limit $K \to 0$, phonon modes have a much lower energy than soliton excitations, and should dominate the physical behavior of the system. We would therefore argue that the conductivity is given correctly by our result (up to possible log corrections) and to behave in $\omega^2$. If our assumption is correct the crossover towards the Efros and Shklovskii result when the disorder becomes stronger would be very interesting to study.

At higher frequencies a crossover occurs above the pinning frequency $\omega_{\text{pin}}$ (typical pinning frequencies are $\omega_{\text{pin}} \sim 10^{12} - 10^{14}$Hz, but precise values depend of course on the disorder). Above the pinning frequency, the WC is not strongly pinned any more but still feels the scattering on disorder. At the opposite of the LL case, one now has a universal power-law $\sigma(\omega) \sim 1/\omega^4$. Density fluctuations give only subdominant corrections [36]. Above the crossover frequency one recovers the nonuniversal power-law of the LL $\sigma(\omega) \sim (1/\omega)^{4-K_\rho[\omega]-K_\sigma[\omega]-g[\omega]}$ (with scale dependent exponents since one has many impurities).

The similar three regimes can be found on the temperature dependence of the conductivity/conductance shown on figure 5.
For the strongly pinned regime \((T < T_{\text{pin}})\) one again can naively expect an exponentially activated conductivity \([36]\). The main characteristic of Coulomb interactions is therefore to give, for temperatures above the pinning length \(T_{\text{pin}}\), a conductivity going down with decreasing \(T\) roughly as \(T^2\) (up to the subdominant corrections). Such a behavior is characteristic of pinning of the \(4k_F\) density fluctuation i.e. the WC. Indeed if only \(2k_F\) scattering was considered, due to the spin fluctuations, even extremely repulsive charge interactions leading to \(K_\rho \to 0\), would lead to \(\sigma(T) \sim T\), i.e. less scattering on the impurities. Analyzing the temperature dependence of the conductance/conductivity in one-dimensional wires should therefore provide useful information on the nature and importance of the interactions, as well as check the above theories. It is noteworthy that in some experiments in one dimensional wires, a dependence \(G \sim T^2\) was indeed observed \([13]\). To probe the correlations in the WC or the LL noise experiments would prove useful. In a similar way than for a CDW, such an experiment would provide information on the periodic nature of the WC, and on its correlation functions.

**IV. CONCLUSIONS**

We have examined the effects of impurities in a one-dimensional system. For short-range interactions we have shown that the old results obtained for long systems with a finite density of impurities and the more recent results on the conductance of a LL with a single barrier are in fact identical for weak disorder if one formally lets the effective scattering go to zero for the macroscopic system. Taking into account both the distance between impurities and the collective effects leads to an interesting temperature dependence of the conductivity, that could in principle be tested in quantum wires. We also looked at the effects of Coulomb interactions, that lead to the formation of a WC. Contrarily to the case of the LL, the conductivity now behaves as \(\sigma(T) \sim T^2\), an experimentally testable prediction.

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Impureté et impuretés dans les systèmes unidimensionnels

Nous examinons les effets du désordre dans les systèmes unidimensionnels. Nous établissons le lien entre une situation avec un petit nombre d’impuretés, propre à décrire un fil quantique court, et celle d’un système contenant une densité finie de diffuseurs, i.e. un fil long ou un système macroscopique. Enfin nous étudions les effets des interactions Coulombiennes sur les propriétés de transport des fils 1D. Nous trouvons une conductivité se comportant en fonction de la température comme $\sigma(T) \sim T^2$. 