Competition/Enhancement of Two Probe Order Parameters in the Unbalanced Holographic Superconductor

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Abstract

We introduce and study a simple unbalanced holographic superconductor model with two scalar order parameters. The attention is focused on the possibility of coexisting orderings corresponding to the concomitant condensation of two scalar operators. Through a probe analysis we show that an attractive or repulsive direct interaction between the two bulk scalars leads respectively to competition and enhancement of the associated condensates. The system at hand is a toy model for studying generic multiple ordering in a strongly coupled context and some comments are given about its applicability to the ferromagnetic unconventional superconductors.

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1 Introduction

The $AdS$/CFT correspondence offers us the possibility of studying quantum field theory holographically, that is, in a dual perspective involving an appropriate gravity theory on a spacetime with higher dimensionality. According to the $AdS$/CFT conjecture, a classical gravitational theory on an asymptotically $AdS$ vacuum can be employed to compute correlation functions of the dual quantum field theory thought of as “living” on the conformal boundary of the asymptotic $AdS$ geometry. This is true in an appropriate dynamical regime involving small curvature and weak coupling on the gravity side and strong coupling in the dual quantum field theory.

Holography, as a framework to access the strongly coupled regime of quantum field theory, has received vast attention since it provides us with a new approach to study interesting toy models for real physical systems whose dynamics happens to be strongly coupled. These strongly coupled models are significant both in condensed matter and in QCD.

A particularly important field of application for holographic analysis is given by the physics associated with quantum phase transitions. Within this context, models for high-$T_c$ superconductors have been widely studied holographically. Their non-standard properties (i.e. not describable within the BCS framework) are thought of as being possibly
intimately related to the collective and strongly coupled dynamics of the system. The first model for a holographic superconductor has been introduced in [3,4], see [5,6] for a wider perspective. The ultimate and ambitious aim of the holographic approach to unconventional superconductors would be to shed light on the high-$T_c$ mechanism of superconduction or at least provide us with useful effective models. Indeed, already on a phenomenologically oriented level, holographic studies offer the possibility of handling quantitatively some macroscopic properties of strongly coupled superconducting systems, such as the thermodynamic and transport properties.

As studied in [7] it is possible to extend the minimal holographic superconductor [4,8] and introduce an additional gauge field in the gravitational theory. According to the holographic dictionary, such new field represents an effective way to account for a second chemical potential in the boundary theory. Indeed, the extended holographic model introduced in [7] is meant to study the strongly coupled physics of unbalanced mixtures where more than one (in that case two) chemical species are related to different chemical potentials.

Unbalanced mixtures are important both in the condensed matter context and in QCD. In the latter the two bulk Abelian gauge fields correspond to two boundary global $U(1)$ symmetries which can describe the baryon number and isospin symmetries. Albeit in this paper we are going to focus the attention (and the language) on the condensed matter context, it should be kept in the back of one’s mind that the holographic toy model under study allows for a wider range of applications.

According to the unbalanced superconductor interpretation of [7], the two chemical potentials are read as the sum and the difference of the chemical potentials for spin-up and spin-down electrons. In this sense, the first bulk gauge field is associated to a global “electric” symmetry while the second bulk gauge field accounts for an effective “spin” symmetry of the boundary theory. A non-vanishing difference among the chemical potentials for spin-up and spin-down electrons is what makes the system unbalanced. As pointed out in [11], in electronic systems at low energy the spin-orbit coupling is suppressed leading to an effective decoupling between spin and spacetime rotations. Therefore, even in a context where spatial rotations happen to be broken, the low-energy $SU(2)$ spin symmetry is a valid approximate symmetry of the IR system. The “magnetic” $U(1)$ in our holographic model can be thought of as, for instance, the $\sigma_3$ component of the $SU(2)$ spin symmetry.

Unbalanced systems are particularly interesting in view of studying mixed spin-electric (in one word “spintronic”) transport properties at strong coupling and the possible emer-

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1 A seminal paper employing a holographic approach to the study of unbalanced systems is [28].
2 In the present treatment the ferromagnetic-like order parameter is effectively described with a scalar, namely, without loss of generality, we understand the choice of a fixed spatial direction with respect to which spins are projected.
gence of inhomogeneous FFLO phases where the superconducting condensate acquires a non-trivial spatial modulation. FFLO phases at weak coupling were introduced and theoretically studied by Fulde, Ferrel, Larkin and Ovchinnikov \cite{Fulde}. The first suggestion about the possibility of FFLO-like phases in holography has been advanced in \cite{Shifman}; further holographic studies have been carried out in \cite{Susskind} and separately in \cite{Herzog, Herzog2}.

1.1 Motivations for a second order parameter

The purpose of the present paper is to generalize the minimal setup for an unbalanced holographic superconductor adding a second scalar field to the model introduced in \cite{Susskind}. A scalar operator which undergoes condensation describes the occurrence of a non-trivial order parameter. For instance in the minimal holographic setup the charged scalar breaks, upon condensing, the electric $U(1)$ symmetry and it is naturally interpreted as a superconducting order parameter\(^3\). In the presence of a second scalar we access the possibility of a more complicated phase diagram where two orderings can occur. The dynamics and mutual relation between the two condensates is one of the most interesting property to be studied; in other words, one would like to investigate whether the two orderings compete or enhance one another.

The point is so far very general. Before making it more specific it is important to underline the versatility of a toy model possessing two order parameters. By simple changes of the parameters (e.g. the charges and masses of the two scalars) one is able to describe different and interesting physical situations. Indeed, the toy model at hand could admit various physical interpretations and allows us to address the problem of ordering coexistence at strong coupling in a rather general fashion.

The focus of the present treatment is nevertheless on a specific case particularly relevant for condensed matter applications: We take the first scalar $\psi$ to be charged under the “electric” $U(1)$ and neutral to the “magnetic” $U(1)$ while for the second scalar $\lambda$ we consider the opposite charge assignment. Hence $\psi$ is meant to account for an electrically charged and magnetically neutral condensate, like for instance a condensate arising from an $s$-wave Cooper-like pairing. On the other hand, $\lambda$ could represent electrically neutral magnetization\(^4\).

A pivotal question in the experimental and theoretical research on high-$T_c$ superconductors has been the interplay between superconductivity and magnetic orderings (see for instance \cite{Bednyakov} and references therein). From the theoretical standpoint, standard ef-\(^3\)This superconductivity claim can be supported with direct analysis of the DC diverging conductivity below $T_c$, see for instance \cite{Sachdev} for further details.
\(^4\)Notice that we have just introduced a “magnetically charged” order parameter accounting for ferromagnetic-like ordering. One could also consider a neutral order parameter with the aim of describing antiferromagnetic-like orderings, see \cite{Shifman, Herzog}.
Effective techniques \`a la Landau-Ginzburg have usually been employed as, for instance, in the perturbative study of spin-density-waves in superconducting cuprates \cite{21}. Complementarily, we adopt here an effective and minimal approach which nevertheless describes an intrinsically strongly coupled medium which allows for coexistence of multiple order parameters.

The general standard picture arising at weak coupling tends to discourage the concomitant occurrence of superconducting and long-range ferromagnetic order \cite{24}. Moreover, in the weak coupling analysis, standard Fermi liquids present usually few instabilities and, more importantly here, the occurrence of an instability tends to disfavor further instability. This is intuitively due to the fact that when the Fermi liquid undergoes an instability, a mass gap is usually produced. Such mass gap tends to prevent, or at least discourage, further instabilities \cite{9}.

In a strongly correlated system however the picture might change radically. In particular, in the panorama of high-$T_c$ superconductors a wealth of coexisting orderings such as magnetic orderings or striped phases are possible. The recent discovery of materials where itinerant ferromagnetism and superconductivity could be even cooperative (see \cite{14} and references therein for details) is particularly interesting.

## 2 The minimal holographic unbalanced superconductor

We start introducing the holographic model for the minimal unbalanced superconductor in $2+1$ dimensions in a rather succinct way; we refer to \cite{7} for any further detail. The gravitational $3+1$ dimensional bulk action is:

$$S = \frac{1}{2\kappa_4^2} \int dx^4 \sqrt{-g} \left[ R + \frac{6}{L^2} - \frac{1}{4} F_{ab} F^{ab} - \frac{1}{4} Y_{ab} Y^{ab} - m_\psi^2 \psi^\dagger \psi - |\partial \psi - i q A \psi|^2 \right], \quad (2.1)$$

where we have an ordinary Abelian Higgs model with gauge field $A$ ($F = dA$) in the presence of a negative cosmological constant $-6/L^2$ plus a second gauge field $B$ whose field strength is $Y = dB$. We underline that the scalar $\psi$ is minimally coupled to $A$ and uncharged with respect to $B$.

We study the model considering the standard radial ansatz:

$$ds^2 = -g(r)e^{-\chi(r)} dt^2 + \frac{r^2}{L^2} (dx^2 + dy^2) + \frac{dr^2}{g(r)} \quad (2.2)$$

\footnote{A study of a Landau-Ginzburg approach in the case of two order parameters is analyzed in \cite{10}. In addition, the paper contains a list of interesting physical applications of the general problem of coexistence of multiple orderings.}
\[ \psi = \psi(r), \quad A_a dx^a = \phi(r) dt, \quad B_a dx^a = v(r) dt, \] 

and solve the equations of motion descending from (2.1). We skip all the analysis which has already been described in [7] and focus on some interesting results regarding the relation between the imbalance and the condensation of \( \psi \).

We remind the reader that the boundary values for the gauge fields \( A \) and \( B \) are respectively interpreted as the overall chemical potential \( \mu \) and the chemical potential imbalance \( \delta \mu \) of the two fermion families (namely “spin-up” and “spin-down” electrons). The thermodynamics of the finite temperature boundary field theory is mapped into the thermodynamics of the dual black hole and vice versa [29]. To simplify the following expressions we henceforth consider the \( \text{AdS}_3 + 1 \) curvature radius to be equal to one, namely \( L = 1 \); this can be done in full generality and amounts to a choice of length unit.

2.1 Phase diagram: the imbalance hinders the scalar condensation

The phase diagram of the unbalanced holographic superconductor in the case of \( m^2_\psi = -2 \) is plotted in Figure 1. As noted in [7] (where the diagram was obtained) it is possible to observe that, for this specific mass value, scalar condensation appears for any value of the imbalance. In other terms there is no Chandrasekhar-Clogston bound [19] or, equivalently, for any \( \delta \mu / \mu \) there is a finite critical temperature below which the system undergoes the superconducting transition. Note however that the bigger the imbalance, the lower the critical temperature and therefore a larger value for \( \delta \mu / \mu \) discourages condensation. This observation is in line with the interpretation of the condensate as formed by Cooper-like spin-singlet pairs and indeed it matches with the weak coupling expectation (see for instance [20]).

The instability leading to scalar hair formation is associated to the violation of the Breitenlohner-Freedman bound for a scalar field living on the near-horizon \( \text{AdS}_2 \) geometry [15]. The study of the near horizon geometry of the doubly charged Reissner-Nordström black hole solution at \( T = 0 \) leads to the following condition for instability\(^6\)

\[ \tilde{m}^2 \psi < -\frac{3}{2}, \] 

where \( \tilde{m}^2 \) is the effective IR mass of the scalar field given by

\[ \tilde{m}^2_\psi = \frac{m^2_\psi}{1 + \frac{\delta \mu^2}{\mu^2}}, \] 

\(^6\)Some details are given in Subsection 4.1 of [7].
Figure 1: On the left, phase diagram of the unbalanced superconductor; the shaded region corresponds to the superconducting condensed phase separated from the normal region by a line of second order phase transitions. Temperature is normalized with respect to the balanced case, i.e. $T_c(\delta \mu = 0)$. On the right we have a plot showing the difference between the square of the IR effective mass and the bare mass of the bulk scalar field; the plot corresponds to $q = 2$ and $\mu = 1$.

Since the model we consider has $m_\psi^2 = -2$, inequality (2.4) is always satisfied and, as already noted, the RN doubly charged black hole is always unstable towards hair formation at sufficiently low $T$. Nevertheless, let us underline that for larger values of the chemical imbalance the effective mass (2.5) is larger as well; then, increasing $\delta \mu / \mu$ the condition for instability, though always satisfied, is met with a smaller margin. This intuitively leads one to think that the $T = 0$ doubly charged RN is “less unstable” when the imbalance increases and accordingly the condensation requires a lower temperature.

3 The holographic unbalanced superconductor with two order parameters

We introduce a second scalar field $\lambda$ in the gravitational model (2.1). It is minimally coupled to the gauge field $B$ and neutral with respect to $A$; in addition, we consider a direct coupling $\gamma$ between the two scalars. In practice we add the following terms to the action (2.1),

$$S_\lambda = \frac{1}{2\kappa^2} \int dx^4 \sqrt{-g} \left[-m_\lambda^2 \lambda^\dagger \lambda - |\partial \lambda - iq_B B \lambda|^2 - \gamma \psi^\dagger \psi \lambda^\dagger \lambda \right] .$$

We take $m_\lambda^2 = -2$ so, in the specific case at hand, $\psi$ and $\lambda$ have the same mass value; this value is in agreement with many instances where one studies the $AdS_4/CFT_3$ cor-
respondence in a UV completed framework, namely considering string theory consistent truncations \([30, 31]\). In accordance with the ansatz \((2.3)\) we choose a radial ansatz for \(\lambda\) as well,

\[
\lambda = \lambda(r).
\]

Putting \(\gamma = 0\) we simply recover a “double-copy” of the standard superconductor \([4, 8]\). In other terms, we have two scalar/gauge vector sectors, namely \((\psi, A)\) and \((\lambda, B)\), both minimally coupled with gravity. Let us observe that gravity couples the two sectors but, as far as this paper is concerned, we will neglect the gravitational interaction (i.e. we work in the probe approximation, see Section \((3.1)\) and the direct coupling \(\gamma\) turns out to be crucial\(^7\).

According to the “magnetic” interpretation of \(U(1)_B\), the \(\lambda\) condensate describes a magnetic order parameter. Hence, the coexistence of the two condensates, \(\psi\) and \(\lambda\), corresponds to a ferromagnetic superconducting phase\(^8\). Without pretending to be particularly close to any specific realistic model, let us however observe that the “magnetic” degrees of freedom giving rise to a condensate resemble itinerant ferromagnetism. Said otherwise, the degrees of freedom that we describe in terms of the “magnetic” condensate are not localized or fixed to any spatial pattern or lattice; in this respect, recall that our ansatz does not present any specific spatial features, neither periodicity nor lattice-like structures\(^9\). A condensate is likely to describe a plasma of degrees of freedom rather than an array of localized spins; this can be thought of in analogy with the fact that \(\psi\) is interpreted as a condensate of (clearly itinerant) Cooper-like pairs at strong coupling.

The total action is obtained summing \((2.1)\) and \((3.1)\),

\[
S_{\text{tot}} = S + S_{\lambda}.
\]

The equations of motion descending from \(S_{\text{tot}}\) are

\[
\phi'' + \left(\frac{\chi'}{2} + \frac{2}{r}\right) \phi' - \frac{2q_A^2}{g} \psi^2 \phi = 0
\]

\[
\psi'' + \left(\frac{g'}{g} - \frac{\chi'}{2} + \frac{2}{r}\right) \psi' + \frac{q_A^2 \phi^2 e^\chi}{g^2} \psi - \frac{m^2}{g} + \frac{\gamma \lambda^2}{g} \psi = 0
\]

\(^7\)It would be interesting to consider instead, or in addition, also a direct coupling between the two gauge fields. We defer this to future work.

\(^8\)A treatment of ferromagnetic superconductors performed by means of Hubbard-like Hamiltonians is described in \([14]\); we refer to this paper and references therein also for a list of interesting and recently investigated ferromagnetic superconducting systems.

\(^9\)Any phenomenon of commensurate magnetism appears then to be outside of the present treatment. Indeed, with “commensurate” we mean that the spatial modulations of the order parameter correspond to the periodicity of an underlying lattice. However, there is an interesting interplay between possible modulated s-wave superconducting phases and incommensurate spin-density-wave ordering (see for instance \([23]\) ) which could be worth studying from a holographic viewpoint.
\[ \chi' + r\psi'^2 + r\lambda'^2 + \frac{rq_A^2\phi^2\psi^2 e^x}{g^2} + \frac{rq_B^2\lambda^2 e^x}{g^2} = 0 \] (3.6)

\[ \frac{1}{2} \psi'^2 + \frac{1}{2} \lambda'^2 + \frac{\phi'^2 e^x}{4g} + \frac{v'^2 e^x}{4g} + \frac{g'}{gr} + \frac{1}{r^2} - \frac{3}{g} \]
\[ + \frac{m^2_A}{2g} \psi^2 + \frac{\gamma}{2g} \lambda^2 \psi^2 + \frac{m^2_B}{2g} \lambda^2 + \frac{q^2_A \phi^2 e^x}{2g^2} + \frac{q^2_B \lambda^2 v^2 e^x}{2g^2} = 0 \] (3.7)

\[ \chi'' + \left( \frac{g'}{g} - \frac{\chi'}{2} + \frac{2}{r} \right) \chi' + \frac{q^2_B v^2 e^x}{g^2} \lambda - \frac{m^2_A + \gamma \psi^2}{g} \lambda = 0 \] (3.8)

\[ v'' + \left( \frac{\lambda'}{2} + \frac{2}{r} \right) v' - \frac{2q_B^2 e^x}{g} \lambda^2 v = 0 \] (3.9)

To solve them we consider the ansatz (2.3) and (3.2). We study black hole solutions and the horizon is defined by \( g(r_h) = 0 \). The equations of motion have the following scaling symmetry

\[ r \rightarrow ar, \quad (t, x, y) \rightarrow \frac{1}{a}(t, x, y), \quad g \rightarrow a^2g, \quad \phi \rightarrow a\phi, \quad v \rightarrow av. \] (3.10)

In an asymptotically \( AdS_4 \) background, the near boundary behavior of the gauge and scalar fields is

\[ \phi = \mu_A - \frac{\rho_A}{r} + \ldots, \quad v = \mu_B - \frac{\rho_B}{r} + \ldots, \quad \psi = \frac{C_1}{r} + \frac{C_2}{r^2} + \ldots, \quad \lambda = \frac{D_1}{r} + \frac{D_2}{r^2} + \ldots \] (3.11)

To underline the generality and symmetry of the present model, we have just adopted a less specific notation trading respectively \( \mu, \rho, \delta\mu \) and \( \delta\rho \) (used in the previous Sections) for \( \mu_A, \rho_A, \mu_B \) and \( \rho_B \).

Note that, under the scaling (3.10), the near-boundary quantities behave as follows:

\[ \mu_{A,B} \rightarrow a\mu_{A,B}, \quad \rho_{A,B} \rightarrow a^2\rho_{A,B}, \quad (C_1, D_1) \rightarrow a(C_1, D_1), \quad (C_2, D_2) \rightarrow a^2(C_1, D_1). \] (3.12)

We will be concerned in studying the spontaneous condensation of the scalars and we define the condensates as follows

\[ \mathcal{O}_x = \sqrt{2} \ C_x, \quad \mathcal{P}_x = \sqrt{2} \ D_x, \] (3.13)

where \( x = 1, 2 \) and the conventional factor \( \sqrt{2} \) is introduced to comply with the existing literature; the dimension of the two kinds of condensates is \([mass]^x\).
Figure 2: Examples of coexisting $\psi$ (black) and $\lambda$ (gray) condensates plotted by performing a sweep in temperature at constant charge densities. On the left/right plot the UV subleading/leading terms for the scalars have been respectively fixed to zero. The $\lambda$ (gray) plot is interrupted at a temperature value which is below its own condensation temperature because our sweeps are technically performed “moving” $\psi(r_H)$, i.e. the value of $\psi$ at the horizon; indeed, we have $\psi(r_H) = 0$ at and above the $\psi$ condensation temperature (see Section 4.2). The temperature is here normalized with respect to the critical temperature for $\psi$ condensation.

3.1 Probe analysis

We do not have analytic solutions for the system of equations of motion, then we tackle the problem numerically. We work in the probe approximation that consists in neglecting the effect of the gauge and scalar fields on the geometry. Such approximation becomes exact in the limit of large charges $q_A, q_B \to \infty$ with $q_A \psi, q_A A, q_B \lambda, q_B B$ kept fixed and finite. Indeed, in this limit, the terms involving the gauge and scalar fields inside the equations of motion for the metric become negligible $[8]$; the metric is therefore a fixed background describing an uncharged $AdS$-Schwarzschild black hole, namely

$$g(r) = r^2 \left( 1 - \frac{r_H^3}{r^3} \right), \quad \chi(r) = 0, \quad r_H = \frac{4}{3} \pi T,$$

where the temperature $T$ is linearly related to the black hole horizon radius $r_H$.

The equations of motion of the two gauge fields and the two scalars are ordinary second order differential equations, namely

$$\phi'' + \frac{2}{r} \phi' - \frac{2q_A^2}{g} \psi^2 \phi = 0, \quad v'' + \frac{2}{r} v' - \frac{2q_B^2}{g} \lambda^2 v = 0$$

$$\psi'' + \left( \frac{2}{r} + \frac{g'}{g} \right) \psi' + \frac{q_A^2}{g^2} \phi^2 \psi - \frac{m_\psi^2 + \gamma \lambda^2}{g} \psi = 0$$

(3.16)
\[ \lambda' + \left( \frac{2}{r} + \frac{g'}{g} \right) \lambda' + \frac{q_B^2}{g^2} v^2 \lambda - \frac{m_\lambda^2 + \gamma v^2}{g} \lambda = 0 \quad (3.17) \]

Albeit the space of solutions of the system is spanned by eight parameters, we need to impose some constraints dictated by consistency reasons. According to the definition of black hole horizon we have \( g(r_H) = 0 \) and then, in order to avoid unphysical divergent quantities we must impose \( A(r_H) = B(r_H) = 0 \) [6]. Smoothness of solutions at the horizon implies a further consistency constraint [6]: multiplying the scalar equations by \( g(r) \) and then considering them at \( r = r_H \) one obtains that the value of a scalar field and its derivative at \( r_H \) are not independent quantities.

All in all, we have a four-parameter family of consistent solutions; as the consistency requirements impose some conditions at the horizon, our numerical code is build to propagate solutions from the horizon towards the boundary. The four “free” inputs are therefore the following horizon quantities,

\[ \phi'(r_h) \ , \ v'(r_h) \ , \ \lambda(r_h) \ , \ \psi(r_h) \ . \quad (3.18) \]

Now we turn the attention to some physical features. In our analysis we always require that either the leading or the subleading term in the UV expansion of both bulk scalars is set to zero. We choose \( m^2 = -2 \) for both scalars, this implies that we fall inside the window where two quantizations are allowed [12]; on practical grounds this means that we have the possibility of interpreting either the leading fall-off as the source and the subleading as the VEV of the corresponding dual operator or vice versa. Even without invoking any more sophisticated argument, requiring zero scalar sources corresponds to ask for unsourced operators VEV’s, i.e. spontaneous condensations. Furthermore, we decide to work in the canonical ensemble and therefore the charge densities (i.e. the subleading terms in the UV behavior of the gauge fields, see Eq. (3.11)) are imposed from outside. The four parameters (3.18) are therefore fixed and the code implements such physical inputs by means of a shooting method\(^{10}\).

An important observation is in order: As already noted, because of the scaling (3.10), the physically meaningful quantities need to be scaling invariant. When solving the holographic system we want to study a specified thermodynamical state \( (\rho_A, \rho_B, T) \) but we are insensitive to an underlying scaling factor acting as in (3.10). Such feature can be read as emerging from the underlying conformal structure of the strongly coupled quantum field theory.

We define the following “physical” (i.e. scaling invariant) quantities

\[ \hat{T} = \frac{T}{\sqrt{\rho_A + \rho_B}} \ , \ \hat{O}_x = \frac{O_x}{(\sqrt{\rho_A + \rho_B})^x} \ , \ \hat{P}_x = \frac{P_x}{(\sqrt{\rho_A + \rho_B})^x} \ , \quad (3.19) \]

where \( x = 1, 2 \) and the condensates \( O_x \) and \( P_x \) have been defined in (3.13).

\(^{10}\)Some further technical detail is given in Section (4.2).
Figure 3: The plots correspond to the two possible $\psi$-condensates in the balanced $\rho_A = \rho_B$ case. The solid black line represent the non interacting setting ($\gamma = 0$); red dotted lines correspond to increasingly negative (repulsive) values of $\gamma$ while green dashed lines refer to increasingly positive (attractive) values of $\gamma$. The $\lambda$-condensate is not plotted. We have competition/enhancement corresponding respectively to attractive/repulsive interaction. In the balanced case the critical temperature is not affected.

4 Coexistence and mutual competition/enhancement of the two condensates

We want to study how the presence of the $\lambda$-condensate affects the condensation of $\psi$; in particular, we are interested in how $\gamma$, the direct coupling between $\psi$ and $\lambda$, influences the mutual competition/enhancement of the two order parameters. We plot the $\psi$-condensate as a function of the temperature $\tilde{T}$ in the canonical ensemble, namely keeping the charge densities $\rho_A$ and $\rho_B$ fixed, in various situations: with and without the $\lambda$-condensate and for different values of $\gamma$ (of either signs). In Figure 2 we show two examples of coexisting $\psi$ and $\lambda$ condensates.

In Figure 3 we show the effects of the $\gamma$ interaction on the plotted $\psi$-condensates (for both the two different quantization schemes) in a balanced case, namely $\rho_A = \rho_B$. The black solid line represents the $\gamma = 0$ case where the two condensates are completely independent; in this case the presence of a $\lambda$-condensate has no effect on the $\psi$-condensate. The green dashed condensates correspond to increasing attractive $^{11}$ (i.e. $\gamma > 0$) $\gamma$-couplings while the red dotted condensates correspond to increasingly repulsive interaction. In this balanced case the system is symmetric under the exchange of the “electric” and “magnetic” sectors and the $\lambda$-condensates (not plotted) are identical to their $\psi$ cousins. Attractive values of $\gamma$ disfavor the condensates attenuating their amplitudes while neg-

$^{11}$The attractive/repulsive character of the $\gamma$ interaction can be checked comparing to the case in which the interactions between $\psi$ and $\lambda$ are mediated by a would-be scalar (i.e. with even spin) particle.
Figure 4: Plots showing the two $\psi$-condensates in the unbalanced case. We observe again (as in the balanced cases plotted in Figure 3) competition/enhancement for attractive/repulsive interaction but here we have in addition that the critical temperature is accordingly affected. The plots are traced for $\rho_A \neq \rho_B$ with $\rho_A$ and $\rho_B$ kept fixed; the temperature is normalized with respect to $\tilde{T}_C$, i.e. the critical temperature for the $\gamma = 0$, non-interacting case.

Negative values of $\gamma$ have the opposite effect. Notice that in this balanced case there is no effect on the critical temperature at which condensation occurs.

Turning to an unbalanced case, $\rho_A \neq \rho_B$, we still have the shrinking/amplifying pattern already occurring in the balanced case but, in addition, also the critical temperature at which $\psi$ condenses is affected. In particular, attractive interactions lower the critical temperature while repulsive interactions raise it. This is in line with the idea of competition/enhancement already suggested in the balanced case. Indeed a higher value of the critical temperature corresponds to an easier condensation. We show our results for the unbalanced case in Figure 4.

In [9] it was conjectured that two holographic orders are competing or mutually enhancing when their interaction is respectively attractive or repulsive. The present model offers then a neat example where this conjecture is confirmed. In Eq. (3.16) and (3.17) the mixing terms containing $\gamma$ could be thought of as contributing to the effective mass of the scalars. As a consequence we have for instance that for $\psi$ the presence of a non-vanishing $\lambda$ affects its effective mass. We furthermore see that a negative $\gamma$ lowers the effective mass while a positive $\gamma$ has the opposite effect. In relation to the BF bound criterion for scalar instabilities we know that a lower mass value encourages the condensation (see Subsection 4.1).
4.1 Comments on stability

In [7], studying the unbalanced holographic superconductor with a single order parameter, the criterion for IR stability of a scalar field on the doubly charged $AdS$-Reissner-Nordström black hole was given\textsuperscript{12}. The supporting argument is based on the analysis of scalar fluctuations on the Reissner-Nordström background in the near-horizon region; there the scalar equation turns out to be that of a free scalar on $AdS_2$ geometry with appropriate effective mass. Eventually, the $AdS_2$ Breitenlohner-Freedman bound gives the criterion for IR stability of the scalar. An analogous approach appears to be not feasible here. In principle a similar near-horizon analysis could be performed, but here we encounter a problem: actually we would like to study one scalar on a hairy black hole background (namely on a solution where the other, say $\lambda$, scalar has already undergone condensation); even though we have an analytical expression for the fields of the hairy black hole in the IR region [15], on such background the equation for $\psi$ fluctuations does not reduce to a simple free scalar on an $AdS$ spacetime. In other words, here we cannot just apply a simple BF bound argument to study IR stability.

Another aspect related to the “stability” of our solutions in a broader sense concerns the numerical approach. In fact, the numerical computations performed to study the thermodynamics of the holographic system at hand are rather delicate and sensitive. Solutions with coexisting $\psi$ and $\lambda$ condensates are sometimes numerically unstable. An excessively low numerical accuracy can lead to unphysical “transitions” between a double condensate phase to a single condensate phase along the numerical sweep in temperature. A numerical instability must however not be confused with a thermodynamical one. Indeed, in the holographic framework, the numerical computations consist essentially in the numerical solution of a system of coupled differential equations, i.e. the classical equations of motion of the dual bulk gravity theory. Said otherwise, we are not performing any direct thermodynamical sampling of the partition function of the boundary theory.

The thermodynamical stability or metastability of the coexisting phase can be checked explicitly studying the value of the renormalized Euclidean on-shell action of the corresponding dual solutions. For the sake of simplicity we have checked successfully the case where the scalar condensates have dimension 2 (i.e. we chose the UV boundary conditions which set the fastest fall off to zero for both the scalars); in this circumstance the on-shell Euclidean action is finite [17].

Throughout this paper we have considered both signs for the quartic coupling $\gamma$. One could worry about the stability of the scalar potential and particularly about the fact that it could be unbounded from below. However, we work in a probe approximation where the fields are regarded as small fluctuations; consequently, our effective model can be thought of to be corrected by higher order terms in the potential which nevertheless

\textsuperscript{12}See [15] for the analogous study in the balanced case.
are negligible as long as the scalar fields are small fluctuation.

4.2 A technical comment on the code

In the first model of a probe holographic superconductor [4], the code employed kept the horizon radius (and consequently the temperature) fixed. The sweep in temperature was actually performed varying $\psi(r_H)$ and letting $\mu$ free; in fact in [4] the condensate is plotted with respect to the ratio $T/\mu$ and, taking advantage of a scaling analogous to (3.10), the plot is interpreted as a sweep at fixed $\mu$ and progressively lower temperature. We adopt a similar approach here. We fix the horizon radius $r_H$ to one and require to the shooting procedure to keep fixed the ratio between the two charge densities. In this way the plotted points corresponding to various values of the ratio $T/\sqrt{\rho_A + \rho_B}$ can be interpreted as a sweep performed varying the temperature and keeping the charge densities fixed.

5 Conclusion and future developments

We introduced a simple holographic model presenting the coexistence of two concomitant, non-trivial order parameters. The model represents a simple generalization of the standard holographic superconductor. Our focus was on the thermodynamical equilibrium properties of the system with particular attention on the mutual interaction of the two orderings. Contrarily to the general weak-coupling expectation, multiple orderings at strong coupling can coexist and even enhance one another. We observed the possibility of such an enhancement and, relying on a probe analysis, we found results in accordance with the conjecture advanced in [9] claiming that an attractive/repulsive interaction between the dual bulk fields leads to competition/enhancement of the corresponding order parameters. Our model possesses two charge densities associated to two Abelian symmetries $U(1)_A, U(1)_B$; working in the canonical ensemble we studied the coexistence of two orderings both in an unbalanced ($\rho_A \neq \rho_B$) and in a balanced ($\rho_A = \rho_B$) setting.

Ongoing analysis of the backreacted case indicates that in general, since it is an attractive interaction, gravity tends to render the two condensations competitive. However, still on a preliminary level, the situation could be more complicated: gravitational interactions are indeed not effectively accountable with a simple shift in the effective mass of the two scalars as it happened for the direct $\gamma$ interaction in the probe analysis.

One future point of interest consists in the study of transport properties of the unbalanced, backreacted holographic superconductor with two order parameters. According to [7], in the normal phase, the linear response of the system to external perturbations could be accounted for by a single frequency dependent “mobility” function. However, in the condensed or doubly condensed phases, the expectation is that possibly the introduction of a second mobility function could account for the transport properties of the
“normal” and “condensed” part of the holographic strongly coupled system. The system introduced in the present paper could offer non trivial checks of this expectation.

Another ongoing interesting perspective (both in relation to condensed matter and QCD) is to consider $\lambda$ to be charged under both gauge fields. In the QCD perspective this brings our model closer to model a quark-quark condensate and a chiral condensate. In the superconductor sense, instead, this situation could represent the coexistence of two superconducting condensates, one s-wave and the other effectively mimicking a p-wave order parameter; said otherwise, we could possibly have two coexisting Cooper-like pairings both in singlet and triplet spin state. Still on a rather speculative level, it could be interesting to further study whether the present model could furnish a simplified environment to study a bi-layered holographic superconductor where the two order parameters would represents both superconducting order parameters associated to two different layers.

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