Oscillatory criteria for the second order linear ordinary differential equations in the marginal sub extremal and extremal cases

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Abstract. The Riccati equation method is used to establish three new oscillatory criteria for the second order linear ordinary differential equations in the marginal, sub extremal and extremal cases. We show that the first of these criteria implies the J. Deng’s oscillatory criterion. An extremal oscillatory condition for the Mathieu’s equation is obtained. The obtained results are compared with some known oscillatory criteria.

Key words: Riccati equation, Mathieu equation, normal and extremal solutions, interval and integral oscillatory criteria, marginal, sub extremal and extremal cases.

1. Introduction. Let \( q(t) \) be a real valued locally integrable function on \([t_0; +\infty)\). Consider the equation

\[
\phi''(t) + q(t)\phi(t) = 0, \quad t \geq t_0.
\]  

Definition 1. The equation (1) is said to be oscillatory, if its every solution has arbitrary large zeroes.

Study of the oscillatory behavior of Eq. (1) is an important problem of the qualitative theory of differential equations and many works are devoted to it (see. [1] and cited works therein, [2 - 9]). Study of the oscillatory property of Eq. (1) by properties of its coefficients was developed (and now is being developed) in general in two directions. The goal of the first direction is to study the oscillatory property of Eq. (1) on the finite interval (interval oscillatory criteria: Sturm, J. S. W. Wong [2], J. G. Sun, C. H. Ou and J. S. W. Wong [3], Q. Kong [4]). Then oscillation of Eq. (1) follows from its oscillation on the countable set of finite intervals. The second one studies correlation between oscillatory property of Eq. (1) and properties of the function \( q(t) \) on the whole half axes (integral oscillatory criteria: Fite, Wintner [5], Ph. Hartman [3, Theorem 52], Leighton [1], I. V. Kamenev [6], J. Yan [7], J. Deng [8], A. Elbert [9]). Recently new integral oscillatory criteria were obtained,
describing a wide classes of oscillatory second order linear ordinary differential equations in terms of their coefficients (see [5], [8], [9]). Among them note the following important result due to J. Deng (see [8]) and Q. Kong (see [4]).

**Theorem 1** [8, Theorem 1]. If for large \( t \in R \)

\[
\int_{t}^{+\infty} q(\tau)d\tau \geq \frac{\alpha_0}{t} \quad \text{where} \quad \alpha_0 > \frac{1}{4},
\]

then Eq. (1) is oscillatory. □

**Theorem 2** [4, Theorem 2.3]. Eq. (1) is oscillatory provided that for each \( r \geq t_0 \) and for some \( \lambda > 1 \), either

(i) The following two inequalities hold:

\[
\limsup_{t \to +\infty} \frac{1}{t^{\lambda-1}} \int_{r}^{t} (t-s)^{\lambda} q(s)ds > \frac{\lambda^2}{4(\lambda-1)}
\]

and

\[
\limsup_{t \to +\infty} \frac{1}{t^{\lambda-1}} \int_{r}^{t} (s-r)^{\lambda} q(s)ds > \frac{\lambda^2}{4(\lambda-1)}, \quad \text{or}
\]

(ii) The following inequality holds

\[
\limsup_{t \to +\infty} \frac{1}{t^{\lambda-1}} \int_{r}^{t} (t-s)^{\lambda}[q(s) + q(2t-s)]ds > \frac{\lambda^2}{2(\lambda-1)}.
\]

□

In this work we use the Riccati equation method for establishing three new oscillatory criteria for Eq. (1) in three different directions having relations to the cases when: a) \( \lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{\tau} q(s)ds = \lambda \neq \pm \infty \) (marginal case, see below Theorem 3 and Corollary 1); b) \( \liminf_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{\tau} q(s)ds = -\infty \) (sub extremal case, see below Theorem 4 and Corollary 2); c) \( \limsup_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{\tau} q(s)ds = -\infty \) ( extremal case, see below Theorem 5 and Corollary 3). Note that the oscillation of Eq. (1) in the remaining (main) cases when:
d) \( \lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{\tau} q(s) ds = +\infty \) (regular case); e) \( -\infty < \liminf_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{\tau} q(s) ds < \limsup_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{\tau} q(s) ds \) (irregular case) was discovered by A. Wintner (see [2, Theorem 51]) and Ph. Hartman (see [2, Theorem 52]) respectively. We show that Corollary 1 (see below) implies Theorem 1. We obtain an oscillatory condition for the Mathieu's equation. This condition contains extremal cases for the Mathieu's equation. We compare the obtained results with some known oscillatory criteria.

2. Main results. Denote by \( \Omega \) the set of all positive and absolutely continuous on \([t_0; +\infty)\) functions \( f(t) \) such that the functions \( f'(t)^2 \) are locally integrable on \([t_0; +\infty)\).

**Theorem 3.** Let for some \( f \in \Omega, \lambda \in R, \alpha \geq 1 \) the following conditions be satisfied:

1) \( +\infty \int_{t_0}^{t} \exp \left\{ \int_{t_0}^{\tau} \left[ 2f(s)q(s) - \frac{f'(s)^2}{f(s)} \right] ds \right\} dt = +\infty; \)

2) \( \liminf_{t \to +\infty} \left\{ \frac{1}{t} \int_{t_0}^{t} \left[ 4f(\tau)q(\tau) - \frac{f'(\tau)^2}{f(\tau)} \right] d\tau - 4 \int_{t_0}^{t} q(\tau)d\tau \right\} < +\infty; \)

3) \( +\infty \int_{t_0}^{t} \exp \left\{ -4\lambda t + 4 \int_{t_0}^{\tau} q(s)ds \right\} dt < +\infty; \)

4) \( \limsup_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} (t-\tau)^{\alpha} q(\tau)d\tau \geq \lambda. \)

Then Eq. (1) is oscillatory. \( \square \)

**Remark 1.** For \( f(t) \equiv 1 \) the condition 2) of Theorem 3 is always satisfied.

**Remark 2.** The condition 4) of Theorem 3 can be replaced by one of the following conditions

\[ \int_{t_0}^{+\infty} \left[ \lambda - \int_{t_0}^{\tau} q(s)ds \right]^2 d\tau < +\infty, \quad \int_{A_\lambda} d\tau = +\infty, \]

where \( A_\lambda \equiv \{ t \geq t_0 : \int_{t_0}^{t} q(\tau)d\tau \geq \lambda \} \). Indeed, when one of these conditions takes place we have:

\[ \int_{t_0}^{+\infty} \left[ y_* (\tau) + \lambda - \int_{t_0}^{\tau} q(s)ds \right]^2 d\tau = +\infty. \]

From here and from (14) it follows (13) (see below the proof of Theorem 3).

**Corollary 1.** Let for some \( \lambda \in R \) the following conditions be satisfied:

A) \( \lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{\tau} q(s) ds = \lambda; \)
B) \( \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{\tau} q(s)ds \leq \lambda - \frac{\ln Q_0(t)}{4t} + \frac{Q_1(t)}{t} \),

for enough large \( t \), where \( Q_k(t) \) (\( k = 0, 1 \)) are some real continuous functions on \([t_0; +\infty)\), satisfying the conditions: \( Q_0(t) > 0, \ t \geq t_0; \int_{t_0}^{+\infty} \frac{dr}{Q_0(r)} < +\infty, \ \sup_{t \geq t_0} Q_1(t) < +\infty. \)

Then Eq. (1) is oscillatory. □

Let us show that Theorem 1 is a consequence of Corollary 1. Suppose the condition of Theorem 1 holds. Then from the convergence of the integral \( \int_{t_0}^{+\infty} q(\tau)d\tau \) it follows existence of the finite limit

\[
\lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{\tau} q(s)ds \overset{\text{def}}{=} \lambda. \tag{2}
\]

(Since by L'Hospital’s rule \( \lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{\tau} q(s)ds = \lim_{t \to +\infty} \int_{t_0}^{+\infty} q(s)ds = 0 \). Therefore condition \( A \) of Corollary 1 takes place. According to the condition of Theorem 1 chose \( \eta_0 > 0 \) so large that \( \int_{t_0}^{+\infty} q(\tau)d\tau \geq \frac{\alpha_0}{t}, \ t \geq \eta_0. \) Then taking into account (2) we will have:

\[
\frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{\tau} q(s)ds = \frac{1}{t} \int_{t_0}^{\eta_0} \int_{t_0}^{\tau} q(s)ds + \frac{1}{t} \int_{t_0}^{+\infty} \int_{t_0}^{\tau} q(s)ds - \frac{1}{t} \int_{\eta_0}^{+\infty} \int_{t_0}^{\tau} q(s)ds \leq \lambda - \frac{\alpha_0 \ln t}{t} + \frac{\alpha_0}{t}, \ t \geq \eta_0, \ \text{where} \ c_1 \equiv \int_{t_0}^{+\infty} \int_{t_0}^{\tau} q(s)ds = \lambda \eta_0 + \alpha_0 \ln \eta_0. \]

Therefore condition \( B \) of Corollary 1 takes place. Thus the Theorem 1 is a consequence of Corollary 1. The next example shows that Corollary 1 does not follow (even with additional hypothesis of convergence of integral \( \int_{t_0}^{+\infty} q(\tau)d\tau \)) from Theorem 1.

**Example 1.** Consider the equation

\[
\phi''(t) + \left[ \frac{\alpha_0}{t^2} + \frac{\alpha \cos(\beta t)}{t^\gamma} \right] \phi(t) = 0, \quad t \geq t_0 > 0. \tag{3}
\]

where \( \alpha_0, \ \alpha, \ \beta, \ \gamma \) are some real constants, \( \alpha_0 > \frac{1}{4}, \ \gamma > 0, \ \alpha \beta \neq 0. \) Without loss of generality we will assume that

\[
\sin(\beta t_0) = \frac{\gamma \cos(\beta t_0)}{t_0} - \gamma(\gamma + 1) t_0^\gamma \int_{t_0}^{+\infty} \frac{\cos(\beta \tau)}{\tau^{\gamma+2}} d\tau \overset{\text{def}}{=} \xi(t_0) \ (\text{since} \ \xi(t_0) \to 0 \ \text{for} \ t_0 \to \infty). \]
Then
\[\int_{t_0}^{t} \left[ \frac{\alpha_0}{t^2} + \frac{\alpha \cos(\beta \tau)}{\tau^\gamma} \right] d\tau = \frac{\alpha_0}{t_0} - \frac{\alpha_0}{t} + \frac{\alpha \sin(\beta t)}{\beta t^\gamma} - \frac{\alpha \sin(\beta t_0)}{\beta t_0^\gamma} + \frac{\alpha \gamma}{\beta} \int_{t_0}^{t} \frac{\sin(\beta \tau)}{\tau^\gamma + 1} d\tau =\]
\[= \frac{\alpha_0}{t_0} - \frac{\alpha_0}{t} + \frac{\alpha \sin(\beta t)}{\beta t^\gamma} - \frac{\alpha \sin(\beta t_0)}{\beta t_0^\gamma} + \frac{\alpha \gamma}{\beta} \int_{t_0}^{t} \frac{\sin(\beta \tau)}{\tau^\gamma + 1} d\tau - \frac{\alpha \gamma}{\beta} \int_{t_0}^{t} \frac{\sin(\beta \tau)}{\tau^\gamma + 1} d\tau =\]
\[= \frac{\alpha_0}{t_0} - \frac{\alpha_0}{t} + \frac{\alpha \sin(\beta t_0)}{\beta t_0^\gamma} + \frac{\alpha \gamma \cos(\beta t_0)}{\beta^2} \int_{t_0}^{t} \frac{\sin(\beta \tau)}{\tau^\gamma + 1} d\tau - \frac{\alpha \gamma (\gamma + 1)}{\beta^2} \int_{t_0}^{t} \frac{\cos(\beta \tau)}{\tau^\gamma + 2} d\tau =\]
\[= \frac{\alpha_0}{t_0} - \frac{\alpha_0}{t} + \frac{\alpha \sin(\beta t_0)}{\beta t_0^\gamma} + \frac{\alpha \gamma}{\beta} \int_{t_0}^{t} \frac{\sin(\beta \tau)}{\tau^\gamma + 1} d\tau - \frac{\alpha \gamma}{\beta} \int_{t_0}^{t} \frac{\sin(\beta \tau)}{\tau^\gamma + 1} d\tau, \quad t \geq t_0.\]

Therefore
\[\int_{t}^{t_0} d\tau \int_{t_0}^{t} \left[ \frac{\alpha_0}{s^2} + \frac{\alpha \cos(\beta s)}{s^\gamma} \right] ds = \frac{\alpha_0(t-t_0)}{t_0} - \alpha_0 \ln t + \alpha_0 \ln t_0 + \]
\[+ \frac{\alpha}{\beta} \int_{t_0}^{t} \frac{\sin(\beta \tau)}{\tau^\gamma} d\tau - \frac{\alpha \gamma}{\beta} \int_{t_0}^{t} \frac{\sin(\beta \tau)}{\tau^\gamma + 1} d\tau = \lambda.\]

Since \[\frac{\sin(\beta t)}{t^\gamma} - \gamma \int_{t_0}^{t} \frac{\sin(\beta \tau)}{\tau^\gamma + 1} d\tau \to 0 \text{ for } t \to \infty \text{ by L'Hospital's rule}\]
\[\lim_{t \to \infty} \left[ \frac{1}{t} \int_{t_0}^{t} \frac{\sin(\beta \tau)}{\tau^\gamma} d\tau - \frac{\gamma}{t} \int_{t_0}^{t} \frac{\sin(\beta \tau)}{\tau^\gamma + 1} d\tau \right] = 0\]
\[\text{From here and from (4) it follows } \lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \frac{\alpha_0}{s^2} + \frac{\alpha \cos(\beta s)}{s^\gamma} ds = \frac{\alpha_0}{t_0} \text{ def } \lambda. \text{ Therefore condition A) of Corollary 1 is satisfied. From (4) we have}\]
\[\frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{t} \left[ \frac{\alpha_0}{s^2} + \frac{\alpha \cos(\beta s)}{s^\gamma} \right] ds = \lambda - \alpha_0 \ln t + \alpha_0 \ln t_0 + \]
\[\frac{\alpha}{\beta} \int_{t_0}^{t} \frac{\sin(\beta \tau)}{\tau^\gamma} d\tau - \frac{\alpha \gamma}{\beta} \int_{t_0}^{t} \frac{\sin(\beta \tau)}{\tau^\gamma + 1} d\tau = \lambda.\]
From here it follows that for Eq. (3) condition $B$ of Corollary 1 is satisfied too. So Eq. (3) is oscillatory. It is easy to show that

$$
\int_{t}^{+\infty} \left[ \frac{\alpha}{t^2} + \frac{\alpha \cos(\beta \tau)}{\tau^\gamma} \right] d\tau = \frac{\alpha_0}{t} - \frac{\alpha \sin(\beta t)}{\beta t^\gamma} + O(t), \quad t \geq t_0,
$$

where $O(t)$ is a bounded function on $[t_0; +\infty)$. Hence it is clear that for $0 < \gamma < 1$ as well as for $\gamma = 1$, $|\frac{\alpha}{\beta}| > \alpha_0$ Theorem 1 is not applicable to Eq. (3).

**Remark 3.** The conditions of Corollary 1 exclude the condition of Ph. Hartman’s oscillatory criterion [2, Theorem 52] and the condition of I. V. Kamenev’s oscillatory criterion.

**Example 2.** Consider the equation

$$
\phi''(t) + \left[ \frac{1}{4t^2} + \frac{1}{4t^2 \ln t} + \ldots + \frac{1 + \varepsilon}{4t^2 \ln r \ldots \ln t} + \frac{\alpha \sin(\beta t)}{t \ln t} \right] \phi(t) = 0, \quad \ln r t > 0. \quad (5)
$$

Here $\ln_1 t = \ln t, \ldots, \ln_r t = \ln \ln_{r-1} t, \ r = 2, 3, \ldots, \varepsilon, \alpha, \beta$ are some real constants, $\varepsilon > 0$, $\beta \neq 0$. Using the identity $\frac{1}{t} \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} q(s) ds = \int_{t_0}^{t} q(s) ds - \frac{1}{t} \int_{t_0}^{t} sq(s) ds, \ t \geq t_0$, one can readily check that for Eq. (5) all conditions of Corollary 1 are fulfilled. Therefore Eq. (5) is oscillatory. It is not difficult to verify that Theorem 1 and Theorem 2 are not applicable to Eq. (5).

**Theorem 4.** Let for some $\varepsilon > 0$, $\alpha \geq 1$ and $\lambda \in R$ the following conditions be satisfied:

5) $\int_{t_0}^{+\infty} \exp \left\{ -4\lambda t + 4 \phi(t) \int_{t_0}^{t} q(s) ds \right\} dt = +\infty$;

6) $\liminf_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} (t - \tau)^\alpha q(\tau) d\tau \leq \lambda - \varepsilon$.

Then Eq. (1) is oscillatory. □

Denote: $B_\lambda \equiv \{ t \geq t_0 : \int_{t_0}^{t} q(s) ds \geq \lambda t \}$.

**Corollary 2.** Let for some $\lambda \in R$ and $\varepsilon > 0$ the following conditions be satisfied:

C) $\int_{B_\lambda} d\tau = +\infty$;
D) \( \lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} q(s) ds \leq \lambda - \varepsilon. \)

Then Eq. (1) is oscillatory. □

Remark 4. From the conditions of Corollary 2 is seen that

\[
\lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} q(s) ds < \lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} q(s) ds
\]

However Corollary 2 is not a consequence of Ph. Hartman’s oscillatory criterion. Indeed in particular case when

\[
\int_{t_0}^{t} d\tau \int_{t_0}^{\tau} q(s) ds = \begin{cases} t \sin^3 t, & \sin t \geq 0; \\ t^2 \sin^3 t, & \sin t < 0, \end{cases}
\]

\( t \geq t_0, \) for \( \lambda = 0, \) \( \varepsilon = 1 \) all conditions of Corollary 2 are satisfied, whereas for this case the conditions of Ph. Hartman’s criterion are not fulfilled (since \( \lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} q(s) ds = -\infty. \))

Theorem 5. Let for some \( f \in \Omega \) the condition 1) and the condition

7) \( \lim_{t \to +\infty} \left\{ \frac{1}{f(t_0)} \int_{t_0}^{t} [2f(\tau)q(\tau) - \frac{f'(\tau)}{2f(\tau)}] d\tau - 2 \int_{t_0}^{t} q(\tau) d\tau \right\} = -\infty \)

be satisfied. Then Eq. (1) is oscillatory. □

Consider the Matheu’s equation (see [10], p. 111)

\[ \phi''(t) + (\delta + \varepsilon \cos(2t))\phi(t) = 0, \quad t \geq t_0. \quad (6) \]

Here \( \delta \) and \( \varepsilon \) are some real constants, \( \varepsilon \neq 0. \) Set:

\[ F_\varepsilon(\mu) \equiv -\frac{\pi |\varepsilon| \mu}{4(\pi + \mu)} + \frac{\mu^2}{\pi + \mu} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2(2t)}{1 + \mu \cos(2t)} dt, \]

\( \mu \geq 0, \) \( m(\varepsilon) \equiv \inf_{\mu \geq 0} F_\varepsilon(\mu). \) Obviously \( m(\varepsilon) < 0. \)

Corollary 3. If \( \delta > m(\varepsilon) \) then Eq. (6) is oscillatory. □

Remark 5. For \( \delta \in (m(\varepsilon); 0) \) we have the extremal case of Eq. (1): \( \int_{t_0}^{+\infty} q(t) dt = -\infty. \)

In this case Theorem 2 is not applicable to Eq. (6).

Remark 6. Using standard methods for integration of trigonometric functions it is
easy to calculate the value of the integral, presenting in the expression for $F_\varepsilon(\mu)$:

$$
\frac{\mu^2}{\pi + \mu} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2(2t)}{1 + \mu \cos(2t)} \, dt =
\begin{cases}
\frac{\pi - 2\mu}{2(\pi + \mu)} - \frac{2\sqrt{1-\mu^2}}{\pi + \mu} \arctan \frac{\sqrt{1-\mu^2}}{1+\mu}, & 0 \leq \mu < 1; \\
\frac{\pi - 2}{2(\pi + 1)}, & \mu = 1; \\
\frac{\pi - 2\mu}{2(\pi + \mu)} + \frac{\sqrt{\mu^2 - 1}}{\pi + \mu} \ln \frac{\sqrt{\mu + 1} + \sqrt{\mu - 1}}{\sqrt{\mu + 1} - \sqrt{\mu - 1}}, & \mu > 1.
\end{cases}
$$

**Example 3.** It is not difficult to verify that $m(4) < F_4(1) = -\frac{\pi + 2}{2(\pi + 1)}$. Therefore by Corollary 3 the equation

$$
\phi''(t) + \left[-\frac{\pi + 2}{2(\pi + 1)} + 4 \cos(2t)\right] \phi(t) = 0, \quad t \geq t_0. \tag{7}
$$

is oscillatory.

**Remark 7.** It is not difficult to verify that the Sturm’s comparison criterion (comparison of Eq. (7) with an equation $\phi''(t) + q_0 \phi(t) = 0$, where $q_0$ is any constant $> 0$) is not applicable to Eq. (7).

One can readily verify that the criteria of Ph. Hartman [2, Theorem 52] and I. V. Kamenev [6] are not applicable to Eq. (3). For $\gamma \geq 1$ the J. Yan’s criterion [7, p. 277, THEOREM] is not applicable to Eq. (3) too. None of Theorems 1 - 9 of work [5] is applicable to Eq. (3), and for $\gamma \geq \frac{1}{2}$ Theorem 11 of work [5] does not applicable to Eq. (3) as well. Theorem 1 as well as the criteria of Ph. Hartman [2, Theorem 52], I. V. Kamenev [6], and J. Yan [7, p. THEOREM] are not applicable to Eq. (7).

3. **Proof of the main results.** To prove the main results we need in some auxiliary propositions.

3.1. **Auxiliary propositions.** Let $a(t)$ and $b(t)$ be real valued locally integrable functions on $[t_0; +\infty)$. Consider the Riccati equation

$$
y'(t) + y^2(t) + a(t)y(t) + b(t) = 0, \quad t \geq t_0. \tag{8}
$$

**Definition 2.** A solution of Eq. (8) is said to be $t_1$-regular, if it exists on $[t_1; +\infty)$ ($t_1 \geq t_0$).

**Definition 3.** A $t_1$-regular solution $y(t)$ of Eq. (8) is said to be $t_1$-normal, if there exists $\delta > 0$ such that every solution $y_1(t)$ of Eq. (8) with $y_1(t_1) \in (y(t_1) - \delta; y(t_1) + \delta)$ is $t_1$-regular, otherwise it is said to be $t_1$-extremal.

Denote by $reg(t_1)$ the set of all $y(t_0) \in R$, for which the solution $y(t)$ of Eq. (8) with $y(t_1) = y(t_0)$ is $t_1$-regular.
Lemma 1. If Eq. (8) has a $t_1$-regular solution then it has the unique $t_1$-extremal solution $y_*(t)$, moreover $\text{reg}(t_1) = [y_*(t_1); +\infty)$. This lemma is proved in [11] for the case of continuous $a(t)$ and $b(t)$. For the case of locally integrable $a(t)$ and $b(t)$ the proof by analogy.

Lemma 2. Suppose Eq. (8) has a $t_1$-regular solution and $b(t) \geq 0$, $t \geq t_0$. Then if:

I) $\int_{t_0}^{+\infty} \exp \left\{ -2 \int_{t_0}^{t} a(\tau) d\tau \right\} dt < +\infty$, then $y_*(t) < 0$, $t \geq t_2$ for some $t_2 \geq t_1$;

II) $\int_{t_0}^{+\infty} \exp \left\{ -2 \int_{t_0}^{t} a(\tau) d\tau \right\} dt = +\infty$, then $y_*(t) \geq 0$, $t \geq t_1$, where $y_*(t)$ is the unique $t_1$-extremal solution of Eq. (8).

This lemma is proved in [12] for the case of continuous $a(t)$ and $b(t)$. For the case of locally integrable $a(t)$ and $b(t)$ the proof by analogy.

Let $y(t)$ be a $t_1$-regular solution of Eq. (8). Consider the integral

$$\nu_y(t) \equiv \int_{t}^{+\infty} \exp \left\{ - \int_{t}^{\tau} [2y(s) + a(s)] ds \right\} dt, \quad t \geq t_1.$$

Theorem 6 [11, Theorem 2.A]. The integral $\nu_y(t)$ converges for all $t \geq t_1$ if and only if $y(t)$ is $t_1$-normal. □

This theorem is proved in [11] for the case of continuous $a(t)$ and $b(t)$. For the case of locally integrable $a(t)$ and $b(t)$ the proof by analogy.

3.2. Proof of the main results. Proof of Theorem 3. Suppose Eq. (1) is not oscillatory. Then the equation

$$x'(t) + x(t)^2 + q(t) = 0, \quad t \geq t_0,$$

has a $t_1$-regular solution for some $t_1 \geq t_0$ (see [13], p. 332). In this equation make the substitution

$$x(t) = y(t) + \lambda - \int_{t_0}^{t} q(\tau) d\tau, \quad t \geq t_0. \quad (10)$$

We obtain:

$$y'(t) + y(t)^2 + 2 \left( \lambda - \int_{t_0}^{t} q(\tau) d\tau \right) y(t) + \left( \lambda - \int_{t_0}^{t} q(\tau) d\tau \right)^2 = 0, \quad t \geq t_0. \quad (11)$$
Since Eq. (9) has a $t_1$-regular solution, from (10) it follows that the last equation has a $t_1$-regular solution too. Then since $\left(\lambda - \int_{t_0}^{t} q(\tau)d\tau\right)^2 \geq 0$, $t \geq t_0$, by virtue of Lemma 2.1) from condition 3) it follows that

$$y_*(t) < 0, \quad t \geq t_2,$$

for some $t_2 \geq t_1$, where $y_*(t)$ is the unique $t_1$-extremal solution of Eq. (11). Let us show that

$$y_*(t) \to -\infty \quad \text{for } t \to +\infty$$

By virtue of (11) we have:

$$y_*(t) = y_*(t_1) - \int_{t_1}^{t} \left[ y_*(\tau) + \lambda - \int_{t_0}^{\tau} q(s)ds \right]^2 d\tau, \quad t \geq t_1. \quad (14)$$

Suppose that the relation (13) is not true. Then from (12) and (14) it follows that $y_*(t)$ is a decreasing function on $[t_1; +\infty)$ with a negative finite limit:

$$y_*(+\infty) \equiv \lim_{t \to +\infty} y_*(t) < 0 \quad (y_*(t) \downarrow y_*(+\infty) > -\infty) \quad (15)$$

From here and (14) it follows that $\int_{+\infty}^{t_1} \left[ y_*(\tau) + \lambda - \int_{t_0}^{\tau} q(s)ds \right]^2 d\tau < +\infty$. Then

$$0 \leq \lim_{t \to +\infty} \frac{1}{t^\alpha} \int_{t_1}^{t} \left( t - \tau \right)^{\alpha-1} \left[ y_*(\tau) + \lambda - \int_{t_0}^{\tau} q(s)ds \right]^2 d\tau \leq$$

$$\leq \lim_{t \to +\infty} \frac{1}{t^\alpha} \int_{t_1}^{+\infty} \left( t - \tau \right)^{\alpha-1} \left[ y_*(\tau) + \lambda - \int_{t_0}^{\tau} q(s)ds \right]^2 d\tau = 0. \quad (16)$$

Set: $\rho(t) \equiv y_*(t) - y_*(+\infty)$, $t \geq t_1$; $I \equiv \limsup_{t \to +\infty} \frac{1}{t^\alpha} \int_{t_1}^{t} \left( t - \tau \right)^{\alpha-1} \left[ y_*(\tau) + \lambda - \int_{t_0}^{\tau} q(s)ds \right]^2 d\tau$. It is evident that

$$\rho(t) \to 0 \quad \text{for } t \to +\infty. \quad (17)$$

We have:

$$I = \limsup_{t \to +\infty} \frac{1}{t^\alpha} \int_{t_1}^{t} \left( t - \tau \right)^{\alpha-1} \left[ y_*(+\infty) + \lambda - \int_{t_0}^{\tau} q(s)ds + \rho(\tau) \right]^2 d\tau =$$
\[
\limsup_{t \to +\infty} \left[ \frac{y^2_*(+\infty)}{t^\alpha} \int_{t_1}^t (t - \tau)^{\alpha-1} d\tau + \frac{2y_*(+\infty)}{t^\alpha} \int_{t_1}^t (t - \tau)^{\alpha-1} \rho(\tau) d\tau + \right.
\]
\[
\left. + \frac{2y_*(+\infty)}{t^\alpha} \int_{t_1}^t (t - \tau)^{\alpha-1} \left( \lambda - \int_{t_0}^\tau q(s) ds \right) d\tau + \frac{1}{t^\alpha} \int_{t_0}^t \left( \lambda - \int_{t_0}^\tau q(s) ds + \rho(\tau) \right)^2 d\tau \right] \geq
\]
\[
\limsup_{t \to +\infty} \left[ \frac{y^2_*(+\infty)}{t^\alpha} \int_{t_1}^t (t - \tau)^{\alpha-1} d\tau + \frac{2y_*(+\infty)}{t^\alpha} \int_{t_1}^t (t - \tau)^{\alpha-1} \rho(\tau) d\tau + \right.
\]
\[
\left. + \frac{2y_*(+\infty)}{t^\alpha} \int_{t_1}^t (t - \tau)^{\alpha-1} \left( \lambda - \int_{t_0}^\tau q(s) ds \right) d\tau \right]. \quad (18)
\]

From the condition 4) and from (15) it follows that
\[
\lim_{n \to +\infty} \frac{2y_*(+\infty)}{\theta_n^\alpha} \int_{t_1}^{\theta_n} (\theta_n - \tau)^{\alpha-1} \left( \lambda - \int_{t_0}^\tau q(s) ds \right) d\tau \geq 0, \quad (19)
\]

for some infinitely large sequence \( \{\theta_n\}_{n=1}^{+\infty} \). Obviously by virtue of (17)\linebreak \( \lim_{t \to +\infty} \frac{1}{t^\alpha} \int_{t_1}^t (t - \tau)^{\alpha-1} \rho(\tau) d\tau = 0 \). From here, from (18) and (19) it follows that\linebreak \( I \geq \frac{y^2_*(+\infty)}{\lambda} > 0 \), which contradicts (16). The obtained contradiction proves (13). It follows from the condition 2) that there exists a infinitely large sequence \( \{\xi_n\}_{n=1}^{+\infty} \) such that \( S \equiv \sup_{n \geq 1} \left\{ \frac{1}{f(\xi_n)} \int_{t_0}^{\xi_n} \left[ 4f(\tau) q(\tau) - \frac{f'(\tau)^2}{f(\tau)} \right] d\tau - 4 \int_{t_0}^{\xi_n} q(\tau) d\tau \right\} < +\infty \). In view of this and relation (9) we chose \( t_3 = \xi_{n_0} \) so large that\linebreak \( y_*(t_3) + \lambda + S/4 < 0 \). \quad (20)

Show that the solution \( x_0(t) \) of Eq. (9) with
\[
x_0(t_3) = \frac{1}{f(t_3)} \int_{t_0}^{t_3} \left[ \frac{f'(\tau)^2}{4f(\tau)} - f(\tau) q(\tau) \right] d\tau \quad (21)
\]
is $t_3$-normal. By 5) from (20) it follows $x_*(t_3) = y_*(t_3) - \int_{t_0}^{t_3} q(\tau)d\tau \leq x_0(t_3)$. By virtue of Lemma 1 from here it follows that $x_0(t)$ is $t_3$-normal. Since $x_0(t)$ is a $t_3$-regular solution of Eq. (9), we have

$$f(t)x_0'(t) + f(t)x_0^2(t) + f(t)q(t) = 0, \quad t \geq t_3.$$ 

Let us integrate this equality from $t_3$ to $t$. We get

$$f(t)x_0(t) = \int_{t_3}^{t} \left[f(\tau)x_0^2(\tau) - f'(\tau)x_0(\tau)\right]d\tau = f(t_3)x_0(t_3) - \int_{t_3}^{t} f(\tau)q(\tau)d\tau, \quad t \geq t_3.$$ 

After completing the square under the integral on the left-hand side of this equality and dividing both sides of it on $f(t)$ we obtain

$$x_0(t) = \frac{1}{f(t)} \int_{t_3}^{t} f(\tau) \left[x_0(\tau) - \frac{f'(\tau)}{2f(\tau)}\right]^2 d\tau = \frac{c}{f(t)} - \frac{1}{f(t)} \int_{t_0}^{t} \left[\frac{f'(\tau)^2}{f(\tau)} - f(\tau)q(\tau)\right]d\tau, \quad (22)$$

where $c \equiv f(t_3)x_0(t_3) - \int_{t_0}^{t_3} \left[\frac{f'(\tau)^2}{f(\tau)} - f(\tau)q(\tau)\right]d\tau$. By virtue of (21) we have $c = 0$. Therefore from (22) we get:

$$-2x(t) \geq \frac{1}{f(t)} \int_{t_0}^{t} \left[2f(\tau)q(\tau) - \frac{f'(\tau)^2}{f(\tau)}\right]d\tau, \quad t \geq t_3.$$ Then

$$\int_{t_3}^{+\infty} \exp\left\{-2\int_{t_3}^{t} x_0(\tau)d\tau\right\}dt \geq M \int_{t_3}^{+\infty} \exp\left\{\int_{t_3}^{\tau} \frac{d\tau}{f(\tau)} \int_{t_0}^{t} \left[2f(s)q(s) - \frac{f'(s)^2}{2f(s)}\right]ds\right\}dt, \quad (23)$$

where $M \equiv \exp\left\{\int_{t_0}^{t_3} \frac{d\tau}{f(\tau)} \int_{t_0}^{\tau} \left[2f(s)q(s) - \frac{f'(s)^2}{2f(s)}\right]ds\right\} = const > 0$. Since $x_0(t)$ is $t_3$-normal, by virtue of Theorem 2 the left-hand side of the inequality (23) is finite. Whereas from the condition 1) it follows, that its right-hand side equals to $+\infty$. The obtained contradiction proves the theorem.

**Proof of Corollary 1.** At first we prove the corollary in the particular case when $\lambda > 0$. Take: $f(t) \equiv 1$. Then from A) it follow 1) and 4), and by virtue of Remark 1 the condition 2) is fulfilled. From B) it follows:

$$\int_{t_0}^{+\infty} \exp\left\{-4\lambda + 4\int_{t_0}^{t} q(s)ds\right\}dt \leq c_0 + \int_{t_0}^{+\infty} \exp\left\{-t\left|4\lambda - \int_{t_0}^{t} q(s)ds\right|\right\}dt \leq$$

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\[
\leq c_0 + \int_{\eta_0}^{+\infty} \exp\left\{-\ln Q_0(t) + Q_1(t)\right\} dt \leq c_0 + M \int_{\eta_0}^{+\infty} \frac{d\tau}{Q_0(\tau)} < +\infty,
\]

where \( c_0 \equiv \int_{t_0}^{\eta_0} \exp\left\{-t \left| 4\lambda - \frac{4}{t} \int_{t_0}^{t} q(s) ds \right| \right\} dt \), \( M \equiv \exp\{\sup_{t \geq t_0} Q_1(t)\} < +\infty \), and \( \eta_0 \) is a enough large number such that for all \( t \geq \eta_0 \) the inequality of condition B) holds. So for \( \lambda > 0 \) all conditions of Theorem 3 are fulfilled, and for this particular case the corollary is proved. Suppose \( \lambda \leq 0 \).

Consider the equation
\[
\phi''(t) + \tilde{q}(t)\phi(t) = 0, \quad t \geq t_0.
\]

where \( \tilde{q}(t) = q(t) + \Delta q(t), \quad \Delta q(t) \) is a continuous function on \([t_0; +\infty)\) such that \( \Delta q(t) \geq 0 \), \( t \in [t_0; t_0 + 1] \), \( \Delta q(t) = 0, \quad t \geq t_0 + 1 \), \( \int_{t_0}^{t_0 + 1} \Delta q(\tau)d\tau > |\lambda| \). Then

\[
\lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} \tilde{q}(s) ds = \lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} q(s) ds + \frac{1}{t} \int_{t_0}^{t_0 + 1} d\tau \int_{t_0}^{\tau} \Delta q(s) ds + \frac{1}{t} \int_{t_0}^{t_0 + 1} d\tau \int_{t_0}^{\tau} \Delta q(s) ds = \lambda + \int_{t_0}^{t_0 + 1} \Delta q(\tau) d\tau > 0.
\]

Thus

\[
\tilde{A}) \quad \lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} \tilde{q}(s) ds = \lambda > 0
\]

From B) it follows:

\[
\lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} \tilde{q}(s) ds = \frac{1}{t} \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} q(s) ds + \frac{1}{t} \int_{t_0}^{t_0 + 1} d\tau \int_{t_0}^{\tau} \Delta q(s) ds \leq \lambda - \frac{\ln Q_0(t)}{4t} + \frac{Q_1(t)}{t} + \frac{1}{t} \int_{t_0}^{t_0 + 1} d\tau \int_{t_0}^{\tau} \Delta q(s) ds
\]

for all enough large \( t \). From here it follows:

\[
\tilde{B}) \quad \frac{1}{t} \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} \tilde{q}(s) ds \leq \lambda - \frac{\ln Q_0(t)}{4t} + \tilde{Q}_1(t)
\]
where \( \tilde{Q}_1(t) \equiv Q_1(t) + \int_{t_0}^{t_0 + 1} \int_{t_0}^{\tau} \Delta q(s) ds - (t_0 + 1) \int_{t_0}^{t_0 + 1} \Delta q(s) ds \). Obviously \( \sup_{t \geq t_0} \tilde{Q}_1(t) < +\infty \).

Then by already proven from \( \tilde{A} \) and \( \tilde{B} \) it follows that Eq. (20) is oscillatory. Since \( \tilde{Q}(t) \) differs from \( q(t) \) only at most on \([t_0; t_0 + 1] \), from the oscillation of Eq. (20) it follows the oscillation of Eq. (1). The corollary is proved.

**Proof of Theorem 4.** Let \( y_*(t) \) be the same \( t_1 \)-extremal solution of Eq. (11), as in the proof of Theorem 3. By virtue of Lemma 2. II) from nonnegativity of the function

\[
(\lambda - \int_{t_0}^{t} q(\tau)d\tau)^2, \quad t \geq t_0,
\]

and from 5) it follows that \( y_*(t) \) is a nonnegative decreasing function with finite limit:

\[
y_*(+\infty) \equiv \lim_{t \to +\infty} y_*(t) \geq 0 \quad (y_*(t) \downarrow y_*(+\infty) \geq 0). \tag{25}
\]

From here and from (14) it follows, that

\[
\lim_{t \to +\infty} \frac{1}{t^{\alpha}} \int_{t_1}^{t} (t - \tau)^{\alpha-1} \left[ y_*(\tau) + \lambda - \int_{t_0}^{\tau} q(s) ds \right]^2 d\tau = 0. \tag{26}
\]

Set (as above): \( \rho(t) \equiv y_*(t) - y_*(+\infty), \quad t \geq t_1 \). Then we have

\[
\lim_{t \to +\infty} \frac{1}{t^{\alpha}} \int_{t_1}^{t} (t - \tau)^{\alpha-1} \left[ y_*(\tau) + \lambda - \int_{t_0}^{\tau} q(s) ds \right]^2 d\tau =
\]

\[
= \lim_{t \to +\infty} \frac{(y_*(+\infty) + \varepsilon)^2}{t^{\alpha}} \int_{t_1}^{t} (t - \tau)^{\alpha-1} d\tau + 2 \frac{(y_*(+\infty) + \varepsilon)}{t^{\alpha}} \int_{t_1}^{t} (t - \tau)^{\alpha-1} (\lambda -
\]

\[
- \varepsilon - \int_{t_0}^{\tau} q(s) ds) d\tau + \frac{1}{t^{\alpha}} \int_{t_1}^{t} (t - \tau)^{\alpha-1} (\lambda - \varepsilon - \int_{t_0}^{\tau} q(s) ds + \rho(\tau))^2 d\tau \geq
\]

\[
\geq \lim_{t \to +\infty} \frac{(y_*(+\infty) + \varepsilon)^2}{t^{\alpha}} \int_{t_1}^{t} (t - \tau)^{\alpha-1} d\tau + 2 \frac{(y_*(+\infty) + \varepsilon)}{t^{\alpha}} \int_{t_1}^{t} (t - \tau)^{\alpha-1} \rho(\tau) d\tau +
\]

\[
+ 2 \frac{(y_*(+\infty) + \varepsilon)}{t^{\alpha}} \int_{t_1}^{t} (t - \tau)^{\alpha-1} (\lambda - \varepsilon - \int_{t_0}^{\tau} q(s) ds) d\tau \tag{27}
\]

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Since \( \rho(t) \to 0 \) for \( t \to +\infty \), we have

\[
\lim_{t \to +\infty} \frac{1}{t^\alpha} \int_{t_1}^{t} (t - \tau)^{\alpha-1} \rho(\tau) d\tau = 0. \tag{28}
\]

From the condition 6) it follows, that there exists an infinitely large sequence \( \{t_n\}_{n=2}^{+\infty} \) such that

\[
\lim_{n \to +\infty} \frac{1}{t_n^\alpha} \int_{t_1}^{t_n} (t_n - \tau)^{\alpha-1} d\tau \int_{t_0}^{\tau} q(s) ds \leq \frac{\lambda - \varepsilon}{\alpha}.
\]

From here, from (27) and (28) it follows that

\[
\limsup_{t \to +\infty} \frac{1}{t^\alpha} \int_{t_1}^{t} (t - \tau)^{\alpha-1} \left[ y^*(\tau) + \lambda - \int_{t_0}^{\tau} q(s) ds \right]^2 d\tau \geq \frac{|y^*(+\infty) + \varepsilon|^2}{\alpha} > 0,
\]

which contradicts (26). The obtained contradiction proves the theorem.

**Proof of Corollary 2.** From the condition C) it follows

\[
\int_{t_0}^{+\infty} \exp \left\{ -4\lambda t + 4 \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} q(s) ds \right\} dt \geq \int_{B_\lambda} \exp \left\{ -4\lambda t + 4 \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} q(s) ds \right\} dt = +\infty.
\]

Therefore the condition 5) of Theorem 4 is satisfied. It follows from D) that for \( \alpha = 1 \) the condition 6) of Theorem 4 is satisfied too. The corollary is proved.

**Proof of Theorem 5.** Suppose Eq. (1) is not oscillatory. Then Eq. (9) has a \( t_1 \)-regular solution for some \( t_1 \geq t_0 \). Then by (10) the equation

\[
y'(t) + y^2(t) + 2 \left( \lambda - \int_{t_0}^{t} q(\tau) d\tau \right) y(t) + \left( \lambda - \int_{t_0}^{t} q(\tau) d\tau \right)^2 = 0, \quad t \geq t_0. \tag{29}
\]

has also a \( t_1 \) - regular solution. Let then \( y_*(t) \) be the \( t_1 \) - extremal solution of Eq. (29). We have

\[
y_*(t) = y_*(t_1) - \int_{t_1}^{t} \left[ y_*(\tau) + \lambda - \int_{t_0}^{\tau} q(s) ds \right]^2 d\tau, \quad t \geq t_1.
\]
Due to the equality (7) chose $t_3 \geq t_1$ such that

$$\frac{1}{f(t_3)} \int_{t_0}^{t_3} \left[ f(\tau)q(\tau) - \frac{f'(\tau)^2}{4f(\tau)} \right] d\tau - \int_{t_0}^{t_3} q(\tau) d\tau < -\lambda - y_*(t_1). \quad (30)$$

Let $x_0(t)$ be the solution of Eq. (28) with

$$x_0(t_3) = \frac{1}{f(t_3)} \int_{t_0}^{t_3} \left[ \frac{f'(\tau)^2}{4f(\tau)} - f(\tau)q(\tau) \right] d\tau.$$

Show that $x_0(t)$ is $t_3$-normal. By (30) we have:

$$x_*(t_3) = y_*(t_3) = \lambda - \int_{t_0}^{t_3} q(\tau) d\tau \leq y_*(t_1) = \lambda - \int_{t_0}^{t_3} q(\tau) d\tau < -\lambda - y_*(t_1).$$

Therefore $x_0(t)$ is $t_3$-normal. Further the continuation of the proof is similar to the proof of Theorem 3. The theorem is proved.

**Proof of Corollary 3.** It is not difficult to verify that $F_\varepsilon(\mu)$ reaches its minimum on $(0; +\infty)$. Let then $m(\varepsilon) = F_\varepsilon(\mu_0)$ for some $\mu_0 > 0$ and let $g(t)$ be a periodic function of period $\pi$, defined by formula

$$g(t) = \begin{cases} 1 + \mu_0 \cos 2t, & t \in [-\frac{\pi}{4}; \frac{\pi}{4}]; \\ 1, & t \in (\frac{\pi}{4}; \frac{3\pi}{4}]. \end{cases}$$

From the condition of Corollary 3 it follows that

$$\int_{-\pi}^{\pi} \left[ 2g(t)(\delta + \varepsilon \cos 2t) - \frac{g'(t)^2}{g(t)} \right] dt > 0, \quad (31)$$

Consider the sequence of functions: $h_1(t), \ldots, h_k(t), \ldots$, defined by formulae

$$h_k(t) \equiv \begin{cases} k(t + \frac{1}{k}), & k \in [-\frac{1}{k}; 0]; \\ -k(t - \frac{1}{k}), & k \in (0; \frac{1}{k}); \\ 0, & k \not\in [-\frac{1}{k}; \frac{1}{k}], \end{cases}$$
\[ k = 1, 2, \ldots \text{. It is easy to show that} \]
\[
\int_{-t}^{t} \frac{h_k'(t)^2}{1 + h_k(t)} dt > k, \quad k = 1, 2, \ldots . \tag{32}
\]

Let \( k_0 < k_1 < \ldots \) be a increasing sequence of natural numbers (yet arbitrary). Define the function \( f(t) \) on \([-\frac{\pi}{4}; +\infty)\) as follows

\[ f(t) = g(t) + h_{k_j+1}\left(t + \frac{\pi}{4} - \pi k_j + \frac{2}{k_{j+1}}\right), \quad t \in \left[-\frac{\pi}{4} + \pi k_{j-1}; -\frac{\pi}{4} + \pi k_j\right], \quad j = 1, 2, \ldots . \]

Using (31) we chose \( k_1 \) so large that

\[
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{d\tau}{f(\tau)} \int_{-\frac{\pi}{4}}^{\tau} \left[2f(s)(\delta + \varepsilon \cos 2s) - \frac{f'(s)^2}{2f(s)}\right] ds > 0, \quad t \in \left[-\frac{\pi}{4} + \pi (k_1 - 1); -\frac{\pi}{4} + \pi k_1 - \frac{2}{k_1}\right],
\]

and using (32) chose \( k_2 > k_1 \) so large that

\[
\frac{1}{f(-\frac{\pi}{4} + \pi k_1)} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4} + \pi k_1} \left[2f(\tau)(\delta + \varepsilon \cos 2\tau) - \frac{f'(\tau)^2}{2f(\tau)}\right] d\tau - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4} + \pi k_1} (\delta + \varepsilon \cos 2\tau) d\tau < -1.
\]

Using (31) we chose \( k_3 > k_2 \) so large that

\[
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{d\tau}{f(\tau)} \int_{-\frac{\pi}{4}}^{\tau} \left[2f(s)(\delta + \varepsilon \cos 2s) - \frac{f'(s)^2}{2f(s)}\right] ds > 0, \quad t \in \left[-\frac{\pi}{4} + \pi (k_3 - 1); -\frac{\pi}{4} + \pi k_3 - \frac{2}{k_3}\right],
\]

and using (32) chose \( k_4 > k_3 \) so large that

\[
\frac{1}{f(-\frac{\pi}{4} + \pi k_3)} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4} + \pi k_3} \left[2f(\tau)(\delta + \varepsilon \cos 2\tau) - \frac{f'(\tau)^2}{2f(\tau)}\right] d\tau - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4} + \pi k_3} (\delta + \varepsilon \cos 2\tau) d\tau < -2,
\]

so on ... We see that for such a chose of \( k_1 < k_2 < \ldots \) for the constructed function \( f(t) \) all the conditions of Theorem 5 for Eq. (6) are fulfilled. The corollary is proved
References

1. C. A. Swanson, Comparison and oscillation theory of linear differential equations. Academic press, New York and London, 1968.
2. Q. Kong, M. Pasic, Second Order Differential Equations: Some Significant Results Due to James S. W. Wong. Differential Equations and Applications, Vol. 6. Number 1(2014) 99 - 163.
3. J. G. Sun, C. H. Ou and J. S. W. Wong, Interval Oscillation Theorems for a Second Order Linear Differential Equations. Computers and Mathematics with Applications. 48 (2004) 1693 - 1699.
4. Q. Kong, Interval Criteria for Oscillation of Second-Order Linear Ordinary Differential Equations, Journal of Mathematical analysis and applications, 229, 258 - 270 (1999).
5. M. K. Kwong, Integral criteria for second- order linear oscillation. Electronic journal of qualitative theory of differential equations, 2006, No 10, pp. 1 - 28. http// WWW. math.u- szeged.hu/ ejqtde/
6. I. V. Kamenev, On oscillation criterion for the second order linear differential equations, Matematicheskie zametki, vol 23, N° 2, 1978, pp. 249 - 251.
7. J. Yan, Oscillation theorems for second order linear differential equations with damping. Proceedings of the American Mathematical Society, vol. 98, N° 2, 1986, pp. 276 - 282.
8. J. Deng, Oscillation criteria for second order linear differential equations. Journal of Mathematical Analysis and Applications 271(2002) 283 - 287.
9. A. Elbert, Oscillation/Nonoscillation Criteria for Linear Second Order Differential Equations. Journal ournal of Mathematical Analyis and Applications, 226, 207 - 219 (1998).
10. L. Cezari, Asymptotic Behavior and Stability Problems in Ordinary Differential equations, Berlin 1959. Translated under the title Asimptoticheskoie povedenie i ustoichivost' reshenii obiknovennich differentialnih uravnenii, Moskow, 1964.
11. G. A. Grigorian, Properties of Solutions of Riccati Equation. Journal of Contemporary Mathematical analysis. 2007, vol. 42, No. 4, pp.184 - 197.
12. G. A. Grigorian, On the Stability of Systems of Two First-Order Linear Ordinary Differential Equations. Differential Equations (Original Russian Differential'nie Uravnenia) 2015, Vol. 51, No 3, pp. 1 -10.
13. Ph. Hartman, Ordinary differential equations, Second edition, The Jhon Hopkins University, Baltimore, Merilend, 1982.