2d Fu–Kane–Mele invariant as Wess–Zumino action of the sewing matrix

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Received: 23 June 2016 / Revised: 21 October 2016 / Accepted: 23 October 2016
Published online: 29 November 2016
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Abstract We show that the Fu–Kane–Mele invariant of the 2d time-reversal invariant crystalline insulators is equal to the properly normalized Wess–Zumino action of the so-called sewing-matrix field defined on the Brillouin torus. Applied to 3d, the result permits a direct proof of the known relation between the strong Fu–Kane–Mele invariant and the Chern–Simons action of the non-Abelian Berry connection on the bundle of valence states.

Keywords Topological insulators · Kane–Mele invariant · Bundle gerbes

Mathematics Subject Classification 53C08

1 Introduction

2d crystalline insulators invariant under odd time reversal are classified by the Kane–Mele $\mathbb{Z}_2$-valued bulk invariant introduced in [15] and rewritten in [8] in a form that will be used here. In [9], a similar $\mathbb{Z}_2$-valued invariant, called strong, was defined for 3d time-reversal invariant (TRI) crystalline insulators. The physical importance of such invariants relies on the fact that their nontrivial value guaranties the existence of robust massless edge modes in finite samples of TRI crystals. In [18], the origin of the $\mathbb{Z}_2$-valued invariants in 3d and 2d was traced back to the integer-valued 2nd Chern
number of the vector bundle of valence Bloch states of 4d crystalline insulators without TRI. This was achieved in a chain of dimensional reductions from 4 space dimensions to 3 and then from 3 to 2. In particular, a $\mathbb{Z}_2$-valued invariant of 3d TRI insulators was expressed as the Chern–Simons (CS) action functional of the non-Abelian Berry connection of the 3d valence bundle. The CS action, when normalized to change by even integers under gauge transformations, is forced by TRI to take integer values giving rise to a $\mathbb{Z}_2$-valued index. An indirect proof that such an index coincides with the strong invariant defined in [9] was given in [7], where it was shown that both represent the same $\mathbb{Z}_2$ subgroup of the Real K-theory group $KR^{-4}$ of the 3d Brillouin torus.

A continuation of the line of thought of [18] permits to define an invariant of 2d TRI crystals using another topological action functional: the Wess–Zumino action (WZ) of the so-called sewing-matrix field defined on the 2d Brillouin torus. When properly normalized, such an action is defined modulo even integers and it takes integer values when calculated on the sewing-matrix field. The main result of this note is a direct proof that the 2d $\mathbb{Z}_2$-valued invariant obtained this way coincides with the one defined in [8]. Section 2 is devoted to the precise statement of the result and Sect. 3 to its proof employing the technique of bundle gerbes, particularly suitable for the calculation of WZ actions. In Sect. 3, we show, basing on [18] that our 2d result also permits to directly prove the equality between the CS action of the 3d Berry connection and the strong $\mathbb{Z}_2$ invariant of [9].

2 Statements of the main result

Consider a smooth family of Hermitian $N \times N$ matrices $H(k)$ parameterized by $k \in \mathbb{R}^d$ and satisfying the relations

$$H(k + b) = H(k) \quad \text{for} \quad b \in 2\pi \mathbb{Z}^d,$$

$$\theta H(k) \theta^{-1} = H(-k) \quad \text{for an antiunitary} \quad \theta : \mathbb{C}^N \to \mathbb{C}^N \quad \text{with} \quad \theta^2 = -I.$$ 

Such families appear in the context of two-dimensional lattice tight-binding TRI systems where $H(k)$ describe the Bloch Hamiltonians. Condition (2.1) means that such Hamiltonians are effectively defined on the $d$-dimensional Brillouin torus $\mathbb{T}^d = \mathbb{R}^d / (2\pi \mathbb{Z}^d)$. The map $\theta$ realizes the odd time reversal and (2.2) expresses the time-reversal symmetry of the system. The existence of $\theta$ squaring to $-I$ requires that $N$ be even.

Suppose that there exists $\epsilon_F \in \mathbb{R}$ that is not in the spectrum of $H(k)$ for all $k$. Such a situation corresponds to systems that are insulators in the fermionic second-quantized ground state that fills all 1-particle eigenstates of $H(k)$ with eigenvalues $< \epsilon_F$, called the valence-band states. Let $P(k)$ be the spectral projectors of $H(k)$ corresponding to such eigenvalues. The projectors $P(k)$ depend smoothly on $k \mod 2\pi \mathbb{Z}$ and satisfy the relation

$$\theta P(k) \theta^{-1} = P(-k)$$

(2.3)
following from (2.2). The ranges of $P(k)$ form a vector subbundle $E$ of the trivial bundle $\mathbb{T}^d \times \mathbb{C}^N$ that will be called the valence subbundle. We shall denote by $n$ its rank, i.e., the dimension of the ranges of projectors $P(k)$. Necessarily, $n$ is even due to the time-reversal symmetry (2.3). In $d = 2, 3$, the vector bundle $E$ is trivializable. This follows from the vanishing of its first Chern number(s), another consequence of (2.3) [17]. The trivializability of $E$ means that there exists a smooth family $(e_i(k))_{i=1}^n$ of vectors in $\mathbb{C}^N$ such that $e_i(k) = e_i(k + b)$ for $b \in 2\pi \mathbb{Z}^d$, and for each $k$, $(e_i(k))$ form an orthonormal basis of the range of $P(k)$. In what follows, a prominent role will be played by the $n \times n$ unitary “sewing matrices” $w(k)$ with the entries

$$w_{ij}(k) = \langle e_i(-k)|\theta e_j(k)\rangle,$$  

(2.4)

depending smoothly on $k \mod 2\pi \mathbb{Z}$ and obeying the relation

$$w(-k) = -w(k)^T.$$  

(2.5)

That relation implies that the matrix $w(k)$ is antisymmetric at points of $\mathbb{T}^d$ where $k = -k \mod 2\pi \mathbb{Z}^d$, the so-called TRIM (time-reversal invariant (quasi-)momenta). There are $2^d$ such points in $\mathbb{T}^d$. It also follows from (2.5) that $\det w(k) = \det w(-k)$ for all $k$. The latter relation implies that $\det w(k)$ does not wind along the basic cycles of $\mathbb{T}^d$ so that one may define a smooth function $\ln \det w(k) = \ln \det w(-k)$ on $\mathbb{T}^d$ uniquely up to a global additive constant in $2\pi i \mathbb{Z}$. In particular, one may define smooth roots $\hat{\sqrt{\det w(k)}} = \exp\left[\frac{1}{p} \ln \det w(k)\right]$ over $\mathbb{T}^d$ up to a global factor equal to a $p$th root of unity.

In [15], Kane and Mele realized that in dimension $d = 2$, there is an obstruction $\text{KM} \in \mathbb{Z}_2$ whose non-zero value forbids that $(e_i(k))$ be composed of Kramers’ pairs satisfying the conditions

$$e_{2i}(-k) = \theta e_{2i-1}(k)$$  

(2.6)

for all $k$. Note that relations (2.6) demands that $w(k)$ be composed of $k$-independent $2 \times 2$ matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$  

(2.7)

placed diagonally. In [8], the Kane–Mele obstruction $\text{KM}$ to achieve such a form of $w(k)$ was expressed with the help of arbitrary family of sewing matrices via the multiplicative relation

$$(-1)^{\text{KM}} = \prod_{\text{TRIM} \in \mathbb{T}^2} \frac{\sqrt{\det w(k)}}{\text{pf} w(k)}$$  

(2.8)

where the product is over the four TRIM in $\mathbb{T}^2$, $\text{pf}$ denotes the pfaffian defined for antisymmetric matrices and $\sqrt{\det w(k)}$ is defined as above. Since $\sqrt{\det w(k)}/\text{pf} w(k)$
squares to 1, the right hand side of (2.8) is ±1. It is independent of the global sign ambiguity in the definition of $\sqrt{\det w(k)}$ and may be shown [7] to be independent of the choice of the trivialization $(e_i(k))$, determining uniquely $K_M \in \mathbb{Z}_2$. It was rigorously shown in [5,6] that $K_M$ is the only obstruction for trivializing the 2d TRI valence bundle with Kramers’ pairs.

The first part of the present note is devoted to the proof of the following result announced in [11].

**Theorem**

$$(-1)^{K_M} = \exp[i S_{WZ}(w)] \tag{2.9}$$

that establishes the equality between the Kane–Mele index $K_M \in \mathbb{Z}_2$ and the two-dimensional Wess–Zumino (WZ) action $S_{WZ}(w)$ divided by $\pi$ of the unitary-group-valued field $\mathbb{T}^2 \ni k \mapsto w(k) \in U(n)$. The WZ action in question is defined following Witten’s prescription [21]: one extends the field $w$ to a $U(n)$-valued map $W$ on an oriented three-dimensional manifold $B$ with boundary $\partial B = \mathbb{T}^2$, demanding that $W|_{\partial B} = w$, and one sets

$$S_{WZ}(w) = \int_B W^* H, \tag{2.10}$$

where $H$ is a closed bi-invariant 3-form on the unitary group $U(n)$,

$$H = \frac{1}{12\pi} \tr(g^{-1}dg)^3 \tag{2.11}$$

normalized so that its 3-periods are in $2\pi \mathbb{Z}$. An extension $W$ of $w$ always exists for a suitable $B$, see “Appendix”, and the right hand side of (2.10) is well defined modulo $2\pi$. This makes $\frac{1}{\pi} S_{WZ}(w)$ defined modulo 2 and the WZ Feynman amplitude $\exp[i S_{WZ}(w)]$ uniquely defined.

**Remark** It is proved in “Appendix” using the basic properties of the WZ action that the right hand side of (2.9) is equal to ±1, does not depend on the choice of the trivialization $(e_i(k))$ of the valence bundle $E$ and is invariant under smooth deformations of $w$ preserving the symmetry (2.5), so that the formula (2.9) renders more transparent the topological nature of the 2d Fu–Kane–Mele invariant.

It will be more convenient in the sequel to remove a $U(1)$ contribution from $w$ and to work with an $SU(n)$-valued field

$$\tilde{w}(k) = (\sqrt[\pi]{\det w(k)})^{-1}w(k) \tag{2.12}$$

on $\mathbb{T}^2$ using one of the smooth $n$th roots defined above. One may choose an extension $\tilde{W}$ of $\tilde{w}$ such that $\tilde{W} = D \tilde{W}$, where $D : B \mapsto U(1)$ extends $(\sqrt[\pi]{\det w(k)})^{-1}$ and $\tilde{W} : B \mapsto SU(n)$ extends $\tilde{w}$. By the formula (7.10) in “Appendix”,

$$W^* H = (D \tilde{W})^* H = D^* H + \tilde{W}^* H + 3d[(D^{-1}dD) \tr(\tilde{W}d\tilde{W}^{-1})] = \tilde{W}^* H \tag{2.13}$$

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because $D^* H = 0$ for dimensional reasons and $\text{tr}(\tilde{W} d \tilde{W}^{-1}) = 0$ because the 1-form $\tilde{W} d \tilde{W}^{-1}$ takes values in traceless matrices. It follows that

$$\exp[i S_{WZ}(w)] = \exp[i S_{WZ}(\tilde{w})].$$

Note that $\tilde{w}(k)$ still satisfies the relation (2.5), i.e.,

$$\tilde{w}(-k) = -\tilde{w}(k)^T$$

and that at the TRIM,

$$\text{pf} \tilde{w}(k) = \frac{\text{pf} w(k)}{\sqrt{\det w(k)}} = \frac{1}{\text{pf} \tilde{w}(k)}$$

so that

$$(-1)^{KM} = \prod_{\text{TRIM} \in T^2} \text{pf} \tilde{w}(k).$$

Hence, the Theorem above may be reduced to the following result that localizes the WZ amplitude of $\tilde{w}$ at the TRIM:

**Proposition 1**

$$\exp[i S_{WZ}(\tilde{w})] = \prod_{\text{TRIM} \in T^2} \text{pf} \tilde{w}(k).$$

### 3 Wess–Zumino amplitude as a gerbe holonomy

To prove Proposition 1, we shall reinterpret the WZ amplitude $\exp[i S_{WZ}(\tilde{w})]$ as the holonomy of a bundle gerbe $\mathcal{G}$ over the group $SU(n)$ [4,12,16]. That will provide a local expression for the WZ amplitude of $\tilde{w}$ with multiple cancelations, allowing at the end its localization at the TRIM. Loosely speaking, (bundle) gerbes are structures one degree higher than line bundles. Their holonomies are defined along closed surfaces rather than along loops and their curvatures are closed 3-forms rather than closed 2-forms. The gerbe $\mathcal{G}$ over $SU(n)$, called basic, is characterized, up to isomorphism, by its curvature equal to the 3-form $H$ of (2.11) (restricted to the special unitary group). We shall employ a construction of $\mathcal{G}$ from [12] that we briefly recall here, giving subsequently a local expression for the holonomy of $\mathcal{G}$.

Let $\lambda_i$, $i = 1, \ldots, n - 1$, be the standard choice for simple weights of the Lie algebra $su(n)$ given by the diagonal $n \times n$ matrices with the entries

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1. All bundle gerbes and line bundles considered below come equipped with a Hermitian structure and a Hermitian connection and their isomorphisms are assumed to respect those structures.
and let $\lambda_0 = 0$. Below, $\lambda_{ij}$ will stand for the difference $\lambda_j - \lambda_i$ for $i, j = 0, \ldots, n - 1$. One chooses a covering $(O_i), i = 0, \ldots, n - 1,$ of $\text{SU}(n)$ composed of open subsets

$$O_i = \left\{ \begin{array}{l} \gamma \in \text{SU}(n), \tau = \sum_{j=0}^{n-1} \tau_j \lambda_j \text{ with } 0 \leq \tau_j, \\ \sum_j \tau_j = 1, \tau_i > 0 \end{array} \right\}$$

(3.2)

equipped with smooth 2-forms

$$B_i(g) = \frac{1}{4\pi} \text{tr}((\gamma^{-1}d\gamma)e^{2\pi i\gamma^{-1}}(\gamma^{-1}d\gamma)e^{-2\pi i\gamma}) + i \text{tr}((\tau - \lambda_i)(\gamma^{-1}d\gamma)^2)$$

(3.3)

such that $dB_i = H|_{O_i}$. Let $O_i \cap O_j \equiv O_{ij}$ be a double intersection of sets of the covering. Then

$$B_{ij}(g) = B_j(g) - B_i(g) = -\text{tr}\lambda_{ij}(\gamma^{-1}d\gamma)^2$$

(3.4)

is a closed 2-form on $O_{ij}$. If

$$g = \gamma e^{2\pi i\gamma^{-1}} = \gamma\gamma_0^{-1}e^{2\pi i\gamma\gamma_0^{-1}} \in O_{ij}$$

(3.5)

then, necessarily, $\gamma_0 \in G_{ij}$, where

$$G_{ij} = \{ \gamma_0 \in \text{SU}(n) | \gamma_0\lambda_{ij}\gamma_0^{-1} = \lambda_{ij} \}. \quad (3.6)$$

All groups $G_{ij}$ are connected and they contain the Cartan subgroup $T \subset \text{SU}(n)$ consisting of the diagonal $\text{SU}(n)$ matrices. Let $\chi_{ij} : G_{ij} \to \mathbb{U}(1)$ be the character of $G_{ij}$ defined by the relations

$$\chi_{ij}(I) = 1, \quad d\ln \chi_{ij}(\gamma_0) = \text{tr}(\lambda_{ij}\gamma_0^{-1}d\gamma_0).$$

(3.7)

In particular, for a real traceless diagonal matrix $\phi$,

$$\chi_{ij}(e^{i\phi}) = e^{i\text{tr}(\lambda_{ij}\phi)}.$$

(3.8)

Over the double intersections $O_{ij}$, one considers the line bundles $L_{ij}$ composed of the equivalence classes $[\gamma, \zeta]_{ij}$ with $\zeta \in \mathbb{C}$ such that

$$(\gamma, \zeta) \sim_{ij} (\gamma\gamma_0^{-1}, \chi_{ij}(\gamma_0)\zeta)$$

(3.9)
for \( \gamma_0 \in G_{ij} \). The line bundle \( L_{ij} \) comes equipped with the Hermitian structure \( |[\gamma', \zeta]_{ij}| = |\zeta| \) and the Hermitian connection induced by the connection form

\[
A_{ij}(g) = \text{tr}(\lambda_{ij} \gamma'^{-1} d\gamma) \tag{3.10}
\]

whose curvature is given by the 2-form \( B_{ij} \). Finally, over the triple intersections \( O_{ijk} \), there exist line bundle isomorphisms \( L_{ij} \otimes L_{jk} \xrightarrow{t_{ijk}} L_{ik} \) defined by

\[
t_{ijk}([\gamma, \zeta]_{ij} \otimes [\gamma, \zeta']_{jk}) = [\gamma, \zeta \zeta']_{ik} \tag{3.11}
\]

that behave in an associative way over the quadruple intersections \( O_{ijkl} \) so that

\[
t_{ikl} \circ (t_{ijk} \otimes \text{id}_{L_{kl}}) = t_{ijl} \circ (\text{id}_{L_{ij}} \otimes t_{jkl}). \tag{3.12}
\]

The isomorphisms \( t_{ij} \) and \( t_{ji} \) permit to canonically identify \( L_{ii} \) with the trivial bundle \( O_i \times \mathbb{C} \) and \( L_{ji} \) with the line bundle \( L_{ij}^{-1} \) dual to \( L_{ij} \). The basic gerbe \( G \) over \( \text{SU}(n) \) is defined by the structure described above, i.e., \( G = ((O_i), (B_i), (L_{ij}), (t_{ijk})) \).

Bundle gerbes over a manifold \( M \) allow to define a \( U(1) \)-valued holonomy of smooth maps \( \phi : \Sigma \to M \) from a closed oriented surface \( \Sigma \) to \( M \). In particular, for the basic gerbe \( G \) over \( \text{SU}(n) \) the holonomy \( \text{Hol}_G(\phi) \) is identified with the WZ amplitude \( \exp[iS_{\text{WZ}}(\phi)] \), and the use of the gerbe structure allows to write a local expression for the latter as described in [12], see also the earlier works [1][10]. This is done in the following way. One chooses a triangulation \( \{(c), (b), (v)\} \) of \( \Sigma \), composed of triangles \( c \), edges \( b \) and vertices \( v \), that is sufficiently fine so that it is possible to fix indices \( i_c, i_b, i_v \) satisfying

\[
\phi(c) \subset O_{i_c}, \quad \phi(b) \subset O_{i_b}, \quad \phi(v) \in O_{i_v}. \tag{3.13}
\]

For each \( b \subset c \), let us denote by \( \text{hol}_{L_{ic};ib}(\phi|_b) \) the parallel transport in the line bundle \( L_{ic;ib} \) along the (open) curve \( \phi|_b \) for \( b \) oriented as the boundary edge of the triangle \( c \). Thus

\[
\text{hol}_{L_{ic;ib}}(\phi|_b) \in \bigotimes_{v \in b} (L_{ic;ib})_{\phi(v)}^{\pm 1}, \tag{3.14}
\]

where \( (L_{ij})_{g}^{\pm 1} \) denotes the fiber of \( L_{ij} \) over \( g \in O_{ij} \) or its dual, and on the right hand side the plus (the minus) sign is chosen if the vertex \( v \) is the end point (the starting point) of the oriented edge \( b \). The local expression for the gerbe holonomy takes the form

\[
\text{Hol}_G(\phi) = \exp \left[ i \sum_c \int_c \phi^* B_{ic} \right] \otimes \text{hol}_{L_{ic;ib}}(\phi|_b). \tag{3.15}
\]
As it stands, the right hand side is an element of the line

$$\otimes_{(b,c)} \left( \otimes_{v \in b} \left( L_{i(c,b)} \right) \right) = \otimes_{v} \left( \otimes_{(b,c)} \left( L_{i(c,b)} \right) \right)$$ (3.16)

that may be canonically identified with $\mathbb{C}$ using the isomorphisms $t_{ijk}$, see [12] or [11] for more details. Such an identification defines $\text{Hol}_G(\phi)$ as a (modulus 1) complex number that is independent of the choice of the triangulation of $\Sigma$ and the assignments $(i_c, i_b, i_v)$, as may be easily checked.

Above we assumed that $\partial \Sigma = \emptyset$. In the case when $\partial \Sigma \neq \emptyset$ and is composed of a closed loop, the right hand side of (3.15) cannot be canonically viewed as an element of $\mathbb{C}$ but only as an element of the line

$$\otimes_{v \in b \subset \partial \Sigma} \left( L_{i(c,v)} \right) \equiv (L^G)_{\phi|\partial \Sigma}$$ (3.17)

with the same sign convention that in (3.14). Such lines may be canonically identified for different triangulations of $\partial \Sigma$ (composed of edges $b$ and vertices $v$) defining the fibers $(L^G)_{\phi|\partial \Sigma}$ of the transgression line bundle $L^G$ over the loop group $\text{LSU}(n)$ canonically associated to the gerbe $G$ [12]. Hence,

$$\text{Hol}_G(\phi) \in (L^G)_{\phi|\partial \Sigma}$$ (3.18)

in this case. The lines $(L^G)_\phi$ related to loops $\varphi$ differing by an orientation-preserving reparameterization are canonically isomorphic and those related by an orientation-reversing reparameterization are canonically dual. More generally, if $\partial \Sigma = \sqcup S_a$ is composed of loops $S_a$ then

$$\text{Hol}_G(\phi) \in \otimes_a (L^G)_{\phi|S_a} \equiv (L^G)_{\phi|\Sigma},$$ (3.19)

where the rightmost term is the shorthand notation. This is, in fact, a part of a richer structure of a classical two-dimensional topological field theory.

Now suppose that $n = 2m$ is even and consider the involution $g \mapsto -g^T$ on $\text{SU}(n)$. We shall need some information on how the structure defining the basic gerbe $G$ over $\text{SU}(n)$ behaves under the $r$. First note that $r^*H = -H$ so that under $r$ the curvature 3-form of $G$ changes sign. If $g = e^{2\pi i \tau} \gamma^{-1} \in O_i$ then

$$r(g) = (\gamma^{-1})^T \omega^{-1} e^{2\pi i \tau} \omega \gamma^{-T} \quad \text{for} \quad \omega = \left( \begin{array}{cc} 0 & I_m \\ -I_m & 0 \end{array} \right),$$ (3.20)

where $I_m$ stands for the unit $m \times m$ matrix, and $e^{2\pi i \tau} = -e^{2\pi i \omega \tau_1}$ for $\tau' = \sum \tau_j \lambda^{j'}$, where $0 \leq j' \leq n - 1$, $j' = j + m \mod n$. The expression for $\tau'$ results from the relation

$$\omega \lambda_j \omega^{-1} = \lambda_{j'} - \lambda_m$$ (3.21)
that is straightforward to check. It follows that if \( g \in O_i \) then \( r(g) \in O_{ir} \), i.e., \( r \) maps \( O_i \) into \( O_{ir} \). Since the map \( j \mapsto j' \) is an involution on the set \{0, 1, \ldots, n - 1\}, \( r(O_i) = O_{ir} \). A simple calculation using the invariance of trace under the transposition shows that

\[
 r^*B_{ir} = \frac{1}{4\pi} \text{tr}\left((\omega\gamma^T d((\gamma^{-1})^T\omega^{-1}))e^{2\pi i\hat{\tau}}(\omega\gamma^T d((\gamma^{-1})^T\omega^{-1}))e^{-2\pi i\hat{\tau}}\right) + i \text{tr}\left((\tau - \lambda_{ir})(\omega\gamma^T d((\gamma^{-1})^T\omega^{-1}))^2\right)
\]

\[
= \frac{1}{4\pi} \text{tr}\left((\gamma^T d((\gamma^{-1}))e^{2\pi i\hat{\tau}}(\gamma^T d((\gamma^{-1})^T))e^{-2\pi i\hat{\tau}}\right) + i \text{tr}\left((\tau - \lambda_{ir})(\gamma^T d((\gamma^{-1})^T))^2\right)
\]

\[
= -\frac{1}{4\pi} \text{tr}\left((\gamma^{-1}d\gamma)e^{2\pi i\hat{\tau}}(\gamma^{-1}d\gamma)e^{-2\pi i\hat{\tau}}\right) - i \text{tr}\left(((\gamma^{-1})^2\gamma - \lambda_{ir})\right)
\]

\[
= -\frac{1}{4\pi} \text{tr}\left((\gamma^{-1}d\gamma)e^{2\pi i\hat{\tau}}(\gamma^{-1}d\gamma)e^{-2\pi i\hat{\tau}}\right) - i \text{tr}\left((\tau - \lambda_{ir})(\gamma^{-1}d\gamma)^2\right) = -B_i.
\]

**Lemma 1** There are line bundle isomorphisms \( \nu : L_{ji} \to r^*L_{ir}j' \) defined by

\[
\nu((\gamma, \zeta)_{ji}) = ((\gamma^{-1})^T\omega^{-1}, \zeta)_{ir}j'.
\]

that intertwine the groupoid multiplication.

**Proof of Lemma 1.** First, we have to check that the definition of \( \nu \) is independent of the choice of representatives of the equivalence classes. Indeed, for \((\gamma, \zeta) \sim_{ji} (\gamma_0^{-1}, \chi_{ji}(\gamma_0)\zeta)\),

\[
((\gamma\gamma_0^{-1})^{-1})^T\omega^{-1} = (\gamma^{-1})^T\gamma_0^T\omega^{-1} = (\gamma^{-1})^T\omega^{-1}\omega\gamma_0^T\omega^{-1}
\]

and

\[
(\omega\gamma_0^T\omega^{-1})^{-1}\lambda_{ir}j'\omega\gamma_0^T\omega^{-1} = \omega(\gamma_0^T)^{-1}\lambda_{ij}\gamma_0^T\omega^{-1}
\]

\[
= \omega(\gamma_0^T)^{-1}\lambda_{ij}^{-1}\omega^{-1}
\]

\[
= \lambda_{ir}j'
\]

so that \((\omega\gamma_0^T\omega^{-1})^{-1} \in G_{ir}j' \). Using (3.21) and the defining properties (3.7) of the characters \( \chi_{ij} \), one easily verifies that

\[
\chi_{ir}j'((\omega\gamma_0^T\omega^{-1})^{-1}) = \chi_{ji}(\gamma_0).
\]

Hence,

\[
((\gamma^{-1})^T\omega^{-1}, \zeta) \sim_{ir}j' ((\gamma^{-1})^T\omega^{-1}(\omega\gamma_0^T\omega^{-1}), \chi_{ir}j'((\omega\gamma_0^T\omega^{-1})^{-1}\zeta))
\]

\[
= ((\gamma\gamma_0^{-1})^{-1})^T\omega^{-1}, \chi_{ji}(\gamma_0)\zeta)
\]

(3.27)
implying that $\nu$ is well defined. The identity
\[
\text{tr}(\lambda_{ij'} \omega \gamma^T d(\gamma^{-1})^T \omega^{-1}) = \text{tr}(\lambda_{ij} \gamma^T d(\gamma^{-1}) \gamma) = \text{tr}(\lambda_{ji} \gamma^{-1} d\gamma)
\]
(3.28)
shows that $\nu$ intertwines the connections. Clearly, it also intertwines the Hermitian structures and the groupoid multiplication.

It follows from Lemma 1 that the isomorphisms $\nu$ intertwine also the parallel transport:
\[
\text{hol}_{L_{ij'}}(r \circ \phi|_b) = \text{hol}_{L_{ij'}}(\phi|_b) = \nu \left( \text{hol}_{L_{ji}}(\phi|_b) \right),
\]
(3.29)
where on the right hand side $\nu$ is understood to act on each tensor factor of an element of $\otimes_{v \in b} (L_{ji})_{\phi(v)}^{\pm 1}$.

**Remark** The line bundle isomorphisms $\nu$ of Lemma 1 may be completed to the so-called Jandl structure on the basic gerbe $G$ relative to the curvature-reversing involution $r$ of SU$(n)$ [20]. Such structures were originally designed to define WZ amplitudes over non-oriented surfaces, see also [13,14]. Below, we shall use only the $\nu$-part of the Jandl structure on $G$ to exhibit the cancelations in the local formula for the WZ amplitude of the sewing-matrix field $\tilde{w}$ defined on the oriented 2-torus $\mathbb{T}^2$.

### 4 Proof of Proposition 1

The proof will be done in two main steps. In the first step, equipped with the gerbe technology, we shall write a local expression for the holonomy of the basic gerbe $G$ over SU$(n)$ in the case when $\Sigma = \mathbb{T}^2$ and $\phi = \tilde{w}$ is the sewing-matrix field (2.12) without the $U(1)$ factor. The symmetry property (2.15) of $\tilde{w}$, that may be rewritten as the identity
\[
r \circ \tilde{w} = \tilde{w} \circ \vartheta
\]
(4.1)
for the involution $\vartheta$ of $\mathbb{T}^2$ induced by the map $k \mapsto -k$, will allow, in conjuction with Lemma 1, to exhibit cancelations in the local expression for Hol$_G(\tilde{w})$ reducing the latter to a product of contributions from TRIM. In the second step, we shall calculate such contributions reducing their product to the product of pfaffians of $\tilde{w}$ at TRIM.

We shall identify $\mathbb{T}^2$ with the square $[-\pi, \pi]^2$ with periodic identifications of the boundary points. Let $\mathbb{T}^2_\pm$ be the closure of the right half of $\mathbb{T}^2$, i.e., its part corresponding to points $k = (k_1, k_2)$ with $0 \leq k_1 \leq \pi$, and $\mathbb{T}^2_- = \vartheta(\mathbb{T}^2_+)$ be the closure of the complementary part of $\mathbb{T}^2$, see Fig. 1. Let us choose a (sufficiently fine) triangulation of $\mathbb{T}^2_\pm$ that is symmetric under $\vartheta$ and that restricts to triangulations of $\mathbb{T}^2_\pm$ and contains the TRIM as vertices. For such a triangulation, we shall choose a maximally symmetric assignment of indices satisfying
\[
i_{\vartheta(c)} = i^r_c, \quad i_{\vartheta(b)} = i^r_b, \quad i_{\vartheta(v)} = i^r_v
\]
(4.2)
\[\text{Springer}\]
Fig. 1 Triangulation of $T^2$ for the calculation of $\text{Hol}_G(\tilde{w})$

for all triangles $c$, all edges $b$, and for all vertices $v$ except for the TRIM for which such a choice would not be possible. Separating the contributions to $\text{Hol}_G(\tilde{w})$ on the right hand side of (3.15) into the ones coming from $T^2_+$ and $T^2_-$, we obtain

$$\text{Hol}_G(\tilde{w}) = \left( \exp \left[ \sum_{c \subset T^2_+} \int_c \tilde{w}^* B_{i_c} \right] \right. \left. \otimes \text{hol}_{L_{i,c}|_b}(\tilde{w}|_b) \right) \otimes \left( \exp \left[ \sum_{c \subset T^2_-} \int_c \tilde{w}^* B_{i_c} \right] \right. \left. \otimes \text{hol}_{L_{i,c}|_b}(\tilde{w}|_b) \right). \quad (4.3)$$

There are several cancelations between the two contributions on the right hand side. First, due to (4.1), (4.2) and (3.22),

$$\exp \left[ \sum_{c \subset T^2_+} \int_c \tilde{w}^* B_{i_c} \right] = \exp \left[ \sum_{c \subset T^2_+} \int_{\theta(c)} \tilde{w}^* B_{i_c} \right] = \exp \left[ \sum_{c \subset T^2_+} \int_c \theta^* \tilde{w}^* B_{i_c} \right] \quad (4.4)$$

so that the integrals over the triangles cancel out reducing formula (4.3) to

$$\text{Hol}_G(\tilde{w}) = \left( \otimes_{b \subset c \subset T^2_+} \text{hol}_{L_{i,c}|_b}(\tilde{w}|_b) \right) \otimes \left( \otimes_{b \subset c \subset T^2_-} \text{hol}_{L_{i,c}|_b}(\tilde{w}|_b) \right). \quad (4.5)$$
Now note that, as explained in Sect. 3,

\[
\bigotimes_{(b,c), b \subset c \subset T^2_+} \text{hol}_{L_{i^c.i^b}}(\tilde{w}|_b) \in (\mathcal{L}^G)_{\tilde{w}|_{T^2_+}},
\]

\[
\bigotimes_{(b,c), b \subset c \subset T^2_+} \text{hol}_{L_{i^c.i^b}}(\tilde{w}|_b) \in (\mathcal{L}^G)_{\tilde{w}|_{T^2_-}} = (\mathcal{L}^G)_{\tilde{w}|_{T^2_-}}^{-1},
\] (4.6)

see (3.19), permitting to view the right hand side of (4.5) as a number in a way consistent with the previous such interpretation based on the subsequent use of maps \(t_{ijk}\). From (4.1), (4.2) and (3.29), we further obtain:

\[
(L^G)_b^{-1} \tilde{w} \in \bigotimes_{(b,c), b \subset c \subset T^2_+} \text{hol}_{L_{i^c.i^b}}(\tilde{w}|_b) \bigotimes_{(b,c), b \subset c \subset T^2_+} \text{hol}_{L_{i^c.i^b}}(\tilde{w}|_b) \in (\mathcal{L}^G)_{\tilde{w}|_{T^2_-}},
\]

\[
= \nu \left( \bigotimes_{(b,c), b \subset c \subset T^2_+} \text{hol}_{L_{i^c.i^b}}(\tilde{w}|_b) \right).
\] (4.7)

Let us represent \(\partial T^2_+\) as \(\ell \cup \vartheta(\ell)\), where \(\ell\) is the union of two closed vertical intervals between the TRIM \((\pi, 0), (\pi, \pi)\) and \((0, \pi), (0, 0)\), respectively, see Fig. 1. For \(v \in \ell\), we shall write

\[
\tilde{w}(v) = \gamma_v e^{2\pi i \tau_v} \gamma^{-1}_v \in \bigcap_{b \subset \ell} O_{i^b.i^c}
\] (4.8)

so that \(\tilde{w}(\vartheta(v)) = r(\tilde{w}(v)) = (\gamma_v^{-1})^T \omega^{-1} e^{2\pi i \tau_v} \omega \gamma_v^T\), see (3.20). By the first of relations (4.6), the definition of the line \((\mathcal{L}^G)_{\tilde{w}|_{T^2_-}}\), see (3.17) and (3.19), and the \(\vartheta\)-symmetry of the triangulation of \(\partial T_+\) together with (4.2),

\[
\bigotimes_{(v,b), v \in b \subset \partial T^2_+} (L_{i^c.i^b})_{\pm 1}^{\tilde{w}(v)} \bigotimes_{(b,c), b \subset c \subset T^2_+} \text{hol}_{L_{i^c.i^b}}(\tilde{w}|_b) \bigotimes_{(v,b), v \in b \subset \partial \ell} (L_{i^c,i^b})_{v}^{\pm 1} \bigotimes_{(v,b), v \in \vartheta(\ell)} (L_{i^c,i^b})_{v}^{\pm 1}
\] (4.9)
for some $\zeta_v, \zeta'_v \in \mathbb{C}$, where $[\gamma, \zeta]_{ij}^{-1}$ denotes the element of $L_{ij}^{-1}$ dual to $[\gamma, \zeta]_{ij}$ in $L_{ij}$ that may be identified with $[\gamma, \zeta_{ij}]^{-1}$. Similarly, from (4.7),

$$
\otimes_{(v, b) \in b \subset \partial T^2_+} (L_{i,v}^T)^{-1}_{i'_v} \ni \otimes_{(b, c) \in c \subset T^2_-} \text{hol}_{L_{i,c}^T} (\bar{w}_{|b})
$$

$$
= v \left( \otimes_{(v, b) \in b \subset \ell, v \notin \partial \ell} \left( [\gamma_v, \zeta_v^{-1}]_{i'_v}^{\pm 1} \otimes \left[ (\gamma_v^{-1})^T \omega^{-1}, \bar{\zeta}_{v}^{-1}\right]_{i'_v}^{\pm 1} \right) \right) \otimes \left( \otimes_{(v, b) \in b \subset \ell, v \in \partial \ell} \left( [\gamma_v, \zeta_v^{-1}]_{i'_v}^{\pm 1} \otimes \left[ (\gamma_v^{-1})^T \omega^{-1}, \bar{\zeta}'_{v}^{-1}\right]_{i'_v}^{\pm 1} \right) \right) .
$$

(4.10)

Upon using (3.23) and the relation

$$
(((\gamma^{-1})^T \omega^{-1})^{-1})^T \omega^{-1} = (\omega \gamma^T)^T \omega^{-1} = \gamma \omega^T \omega^{-1} = \gamma (-I),
$$

(4.11)

this gives

$$
\otimes_{(b, c) \in c \subset T^2_-} \text{hol}_{L_{i,c}^T} (\bar{w}_{|b}) = \left( \otimes_{(v, b) \in b \subset \ell, v \notin \partial \ell} \left( [\gamma_v, \zeta_v^{-1}]_{i'_v}^{\pm 1} \otimes \left[ (\gamma_v^{-1})^T \omega^{-1}, \bar{\zeta}_{v}^{-1}\right]_{i'_v}^{\pm 1} \right) \right) \otimes \left( \otimes_{(v, b) \in b \subset \ell, v \in \partial \ell} \left( [\gamma_v, \zeta_v^{-1}]_{i'_v}^{\pm 1} \otimes \left[ (\gamma_v^{-1})^T \omega^{-1}, \bar{\zeta}'_{v}^{-1}\right]_{i'_v}^{\pm 1} \right) \right) .
$$

(4.12)

Since $\chi_{ij} (-I) = (-1)^{i-j}$, as may be easily seen from (3.8), this can be rewritten using (3.9) as

$$
\otimes_{(b, c) \in c \subset T^2_-} \text{hol}_{L_{i,c}^T} (\bar{w}_{|b}) = \left( \otimes_{(v, b) \in b \subset \ell, v \notin \partial \ell} \left( [\gamma_v, \zeta_v^{-1}]_{i'_v}^{\pm 1} \otimes \left[ (\gamma_v^{-1})^T \omega^{-1}, \bar{\zeta}_{v}^{-1}\right]_{i'_v}^{\pm 1} \right) \right) \otimes \left( \otimes_{(v, b) \in b \subset \ell, v \in \partial \ell} \left( [\gamma_v, \zeta_v^{-1}]_{i'_v}^{\pm 1} \otimes \left[ (\gamma_v^{-1})^T \omega^{-1}, \bar{\zeta}'_{v}^{-1}\right]_{i'_v}^{\pm 1} \right) \right) .
$$
\[
\left( \bigotimes_{v \in b \subseteq \ell \atop v \notin \partial \ell} \left( (\gamma_v^{-1})^T \omega^{-1}, \zeta_v^{-1} \right)^{i_{\ell i_b}^v}_{i_{\ell i_b}^v} \otimes [\gamma_v, \zeta_v'^{-1}]_{i_{\ell i_b}^v}^{\mp 1} \right) \right) \nonumber
\]

(4.13)

\[
\left( \bigotimes_{v \in b \subseteq \ell \atop v \notin \partial \ell} \left( (\gamma_v^{-1})^T \omega^{-1}, \zeta_v^{-1} \right)^{i_{\ell i_b}^v}_{i_{\ell i_b}^v} \otimes [\gamma_v, (\mp 1)^{i_{\ell i_b}^v} \zeta_v'^{-1}]_{i_{\ell i_b}^v}^{\mp 1} \right) \right).
\]

The factors \((-1)^{i_b}\) canceled because each \(b \subseteq \ell\) appeared two times, and similarly for the factors \((-1)^{i_v}\) for \(v \in \ell, v \notin \partial \ell\). The factors \((-1)^m\) also disappeared as they occurred four times. Substituting (4.9) and (4.13) to (4.5) and using the duality between pairs of factors in the tensor product, we observe that all the contributions from \(v \notin \partial \ell\) cancel out so that

\[
\text{Hol}_{G}(\tilde{w}) = \left( \bigotimes_{v \in b \subseteq \ell \atop v \notin \partial \ell} \left( [\gamma_v, \zeta_v]_{i_{i_b}^v}^{\pm 1} \otimes [(\gamma_v^{-1})^T \omega^{-1}, \zeta_v'^{-1}]_{i_{i_b}^v}^{\mp 1} \right) \right) \nonumber
\]

(4.14)

\[
\left( \bigotimes_{v \in b \subseteq \ell \atop v \notin \partial \ell} \left( [\gamma_v, (-1)^{i_{\ell i_b}^v} \zeta_v \zeta_v'^{-1}]_{i_{i_b}^v}^{\mp 1} \otimes [(\gamma_v^{-1})^T \omega^{-1}, \zeta_v \zeta_v'^{-1}]_{i_{i_b}^v}^{\mp 1} \right) \right),
\]

where the last line was obtained using the isomorphisms \(t_{i_{i_b}^v}^{i_{i_b}^v}\) and \(t_{i_{i_b}^v}^{i_{i_b}^v}\). But \(v \in \partial \ell\) are TRIM for which \(\tilde{w}(v) = r(\tilde{w}(v))\) implying that

\[
\gamma_v = (\gamma_v^{-1})^T \omega^{-1} e^{2\pi i r e^\gamma} \gamma_v^T \in O_{i_{i_b}^v}
\]

and consequently, that \(\tau = \tau^r\) and

\[
\gamma_{e0} = \gamma_v^{-1} (\gamma_v^{-1})^T \omega^{-1} \in \Gamma_{i_{i_b}^v}.
\]

Hence, by (3.9),

\[
\text{Hol}_{G}(\tilde{w}) = \left( \bigotimes_{v \in b \subseteq \ell \atop v \notin \partial \ell} \left( [\gamma_v, (-1)^{i_{\ell i_b}^v} \zeta_v \zeta_v'^{-1}]_{i_{i_b}^v}^{\mp 1} \otimes [\gamma_v \gamma_{e0}, \zeta_v \zeta_v'^{-1}]_{i_{i_b}^v}^{\mp 1} \right) \right) \nonumber
\]

(4.15)

(4.16)

\[
\left( \bigotimes_{v \in b \subseteq \ell \atop v \notin \partial \ell} \left( [\gamma_v, (-1)^{i_{\ell i_b}^v} \zeta_v \zeta_v'^{-1}]_{i_{i_b}^v}^{\mp 1} \otimes [\gamma_v \gamma_{e0}, \chi_{i_{i_b}^v}(\gamma_{e0}) \zeta_v \zeta_v'^{-1}]_{i_{i_b}^v}^{\mp 1} \right) \right) \nonumber
\]

(4.17)
Note that the right hand side is a product of contributions from TRIM. They are identified with the help of

**Lemma 2** For an antisymmetric $SU(n)$ matrix $\tilde{w} = \gamma e^{2\pi i r} \gamma^{-1} = r(\tilde{w}) = (\gamma^{-1})^T \omega^{-1} e^{2\pi i r} \omega \gamma^T \in O_{ir}$ and for $\gamma_0 = \gamma^{-1} (\gamma^{-1})^T \omega^{-1}$,

$$\text{pf } \tilde{w} = i^m n^2 (-1)^i \chi_{ir}(\gamma_0).$$

(4.18)

Assuming Lemma 2, the substitution of (4.18) for $\tilde{w} = \tilde{w}(v)$ and $v \in \partial \ell$ to the right hand side of (4.17) permits to complete the proof of Proposition 1 (recall that $\text{pf } \tilde{w}(v) = \pm 1$).

**Proof of Lemma 2.** Let us first suppose that $\tilde{w} \in O_{0m}$. Since $\tilde{w} = \gamma e^{2\pi i r} \gamma_0 \omega \gamma^T$ and $\gamma \in SU(n)$, it follows that $e^{2\pi i r} \gamma_0 \omega$ is an antisymmetric matrix and that

$$\text{pf } \tilde{w} = \det(\gamma) \text{ pf } (e^{2\pi i r} \gamma_0 \omega) = \text{ pf } (e^{2\pi i r} \gamma_0 \omega).$$

(4.19)

Under the assumption that $\tilde{w} \in O_{0m}$, $\tau = \sum_{j=0}^{n-1} \tau_j \lambda_j$ with $\tau_j = \tau_j^r$ and $\tau_0 = \tau_m > 0$. Due to (3.21),

$$\omega \tau \omega^{-1} = \sum_{j=0}^{n-1} \tau_j \lambda_j = \tau - \lambda_m.$$  

(4.20)

Recall that $\lambda_m = \text{diag}[\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}]$. If $2\pi \tau = \text{diag}[\phi_1, \ldots, \phi_n]$ then (4.20) means that

$$\text{diag}[\phi_{m+1}, \ldots, \phi_n, \phi_1, \ldots, \phi_m] = \text{diag}[\phi_1 - \pi, \ldots, \phi_m - \pi, \phi_{m+1} + \pi, \ldots, \phi_n + \pi],$$

(4.21)

i.e., $(\phi_{m+1}, \ldots, \phi_n) = (\phi_1 - \pi, \ldots, \phi_m - \pi)$. It follows that

$$\sum_{j=1}^{m} \phi_j = \frac{\pi m}{2}.$$  

(4.22)

Besides, as $\gamma_0 \in G_{0m}$ and $\lambda_{0m} = \lambda_m$, the matrix $\gamma_0$ has a block structure with $m \times m$ blocks:

$$\gamma_0 = \begin{pmatrix} \gamma_0' & 0 \\ 0 & \gamma_0'' \end{pmatrix}$$

(4.23)

and $\chi_{0m}(\gamma_0) = \det \gamma_0' = \det \gamma_0''^{-1}$ as is easy to see from (3.7). Since

$$\gamma^{-1} (\gamma^{-1})^T = \gamma_0 \omega = \begin{pmatrix} 0 & \gamma_0'' \\ -\gamma_0'' & 0 \end{pmatrix}$$

(4.24)
is a symmetric matrix, it follows that $\gamma'' = -(\gamma')^T$. Hence,

$$e^{2\pi i r} \gamma_0 = \left( \begin{array}{cc} \text{diag}[e^{i\varphi_1}, \ldots, e^{i\varphi_m}]_0 & \text{diag}[e^{i\varphi_1}, \ldots, e^{i\varphi_m}]_0' \\ \text{diag}[e^{i\varphi_{m+1}}, \ldots, e^{i\varphi_n}]_0 & 0 \end{array} \right)$$

$$\gamma_0'' = \left( \begin{array}{cc} \text{diag}[e^{i\varphi_1}, \ldots, e^{i\varphi_m}]_0 & 0 \\ 0 & \text{diag}[e^{i\varphi_1}, \ldots, e^{i\varphi_m}]_0' \\ \text{diag}[e^{i\varphi_{m+1}}, \ldots, e^{i\varphi_n}]_0 & 0 \end{array} \right)$$

(4.25)

and from the standard formula for the pfaffian of a block off-diagonal antisymmetric matrix,

$$\text{pf}(e^{2\pi i r} \gamma_0) = (-1)^{m(m-1)/2} \det(\text{diag}[e^{i\varphi_1}, \ldots, e^{i\varphi_m}]_0)$$

$$= (-1)^{m(m-1)/2} e^{i(\varphi_1+\cdots+\varphi_m)} \det(\gamma_0')$$

$$= (-1)^{m(m-1)/2} i^m \det(\gamma_0') = i^{m^2} e^{\pi 0 e^{2\pi i r}}$$

(4.26)

which together with (4.19), proves (4.18) for $i = 0$.

Suppose now that $\tilde{w} \in O_{i'r}$. In this case, $\tilde{w}' = z^{-i} \tilde{w} = -(\tilde{w}')^T \in O_{0m}$ for $z = e^{-2\pi i n}$ being the generator of the center of $SU(n)$. Let $\omega_1$ be the $n \times n$ matrix in $SU(n)$ such that $\omega_1 e_{i'} l = e^{\pi i/n} e_{i+1}$ in the action on the vectors of the canonical basis of $\mathbb{C}^n$ (with the identification $e_{n+1} = e_1$). Then

$$\omega_1 \lambda_j \omega_1^{-1} = \lambda_{j+i}, \quad \omega_1^j \lambda_j \omega_1^{-i} = \lambda_{j+i} - \lambda_i, \quad \omega = e^{-\pi i \lambda_m} \omega_1^m,$$

$$\omega_1^i e^{2\pi i \sum_{j=1}^{n-1} \tau_j \lambda_j - 2\pi i \lambda_i} = z^{-i} e^{2\pi i r},$$

(4.27)

(4.28)

where $j$ in $\lambda_j$ is taken modulo $n$. Hence, if $\tilde{w} = \gamma e^{2\pi i r} \gamma^{-1}$ then $\tilde{w}' = \gamma \omega_1^i e^{2\pi i \sum_{j=1}^{n-1} \tau_j \lambda_j - 2\pi i \lambda_i} \gamma^{-1}$. Let

$$\gamma_0' = (\gamma \omega_1^i)^{-1} (\gamma \omega_1^i)^{-1} \gamma \omega_1^i e^{\pi i \lambda_m}$$

$$= e^{\pi i i/m} \omega_1^{-i} \gamma^{-1} \gamma^{-1} \omega_1^{-i} e^{\pi i \lambda_m}$$

$$= e^{\pi i i/m} \omega_1^{-i} \gamma^{-1} \omega_1^{-i} e^{\pi i \lambda_m}$$

$$= e^{\pi i i/m} \omega_1^{-i} \gamma_0 e^{-\pi i \lambda_m} \omega_1^i e^{\pi i \lambda_m}$$

(4.29)

In other words, $\gamma_0' \in G_{0m}$ satisfies the relation

$$e^{\pi i i/m} \gamma_0' e^{-\pi i \lambda_m} = \omega_1^{-i} \gamma_0 e^{-\pi i \lambda_m} \omega_1^i.$$ 

(4.30)

As the adjoint action of $\omega_1^i$ sends $\lambda_0$ to $\lambda_{i'r}$, it maps $G_{0m}$ into $G_{ii'}$ and intertwines the characters $\chi_{0m}$ and $\chi_{ii'}$. From (4.30), it follows then that

$$\chi_{0m}(e^{\pi i i/m} \gamma_0' e^{-\pi i \lambda_m}) = (-1)^i \chi_{0m}(\gamma_0' e^{-\pi i \lambda_m}) = \chi_{ii'}(\gamma_0 e^{-\pi i \lambda_m}).$$

(4.31)
Relation (3.8) and the equalities $\text{tr}(\lambda_i \lambda_m) = \left\{ \frac{i/2}{n-i/2} \right\}$ for $\{i \leq m\}$ imply that

$$\chi_{0m}(e^{-\pi i \lambda_m}) = e^{-\pi m/2} = i^{-m}, \quad \chi_{ii^r}(e^{-\pi i \lambda_m}) = e^{\pi i \text{tr}(\lambda_i - \lambda_{i^r}) \lambda_m) = (-1)^{i-1}i^{-m}. \quad (4.32)$$

Hence, (4.31) reduces to the identity

$$\chi_{0m}(\gamma_i^0) = \chi_{ii^r}(\gamma_0). \quad (4.33)$$

Using (4.18) for $i = 0$, we infer then that

$$i^{-m^2} \text{pf}(\tilde{w}') = \chi_{0m}(\gamma_i^0) = \chi_{ii^r}(\gamma_0). \quad (4.34)$$

But $\text{pf} \tilde{w}' = \text{pf}(e^{2\pi i i/n} \tilde{w}) = (-1)^i \text{pf} \tilde{w}$ and (4.18) for general $i$ follows completing the proof of Lemma 2.

\[\square\]

5 3d Fu–Kane–Mele invariant and the Chern–Simons action of the Berry connection

We shall show here that the relation between the strong Fu–Kane–Mele invariant $\text{KM}^3 \in \mathbb{Z}_2$ for 3d TRI topological insulators [9] and the Chern–Simons (CS) action of the Berry connection of the valence bundle, first rigorously established in [7], is a corollary of the Theorem obtained in the previous sections. The invariant $\text{KM}^3$ was defined in [9] by the formula

$$(-1)^{\text{KM}^3} = \prod_{\text{TRIM} \in \mathbb{T}^3} \frac{\sqrt{\det w(k)}}{\text{pf} w(k)} \quad (5.1)$$

similar to (2.8) but with the product over the 8 TRIM in the 3d Brillouin torus. The non-Abelian Berry connection $A$ of the valence bundle $E$ over $\mathbb{T}^3$ is defined using an orthonormal frame $(e_i(k))$, $i = 1, \ldots, n$, globally trivializing $E$. It is a matrix-valued 1-form on $\mathbb{T}^3$ given by the formula

$$A_{ij}(k) = \{e_i(k)\, de_j(k)\} = -A_{ji}(k). \quad (5.2)$$

It determines the CS 3-form

$$\text{CS}(A) = \frac{1}{4\pi} \text{tr}(AdA + \frac{2}{3}A^3) \quad (5.3)$$

and the CS action

$$S_{\text{CS}}(A) = \int_{\mathbb{T}^3} \text{CS}(A). \quad (5.4)$$
Under the change of the trivializing frame,

\[ e_i(k) = \sum_{i'} U(k)_{ii'} e'_{i'}(k), \quad (5.5) \]

the connection form transforms by the gauge transformation \( A' = UAU^{-1} + UdU^{-1} \) and the CS 3-form by

\[ \text{CS}(A') = \text{CS}(A) + U^*H + \frac{1}{4\pi} d(\text{tr}(U^{-1}dU)A), \quad (5.6) \]

where the 3-form \( H \) on \( U(n) \) is given by (2.11). The transformation rule (5.6) results in the relation

\[ S_{\text{CS}}(A') = S_{\text{CS}}(A) + \int_{T^3} U^*H \quad (5.7) \]

which implies that the CS action \( S_{\text{CS}}(A) \) changes by multiples of \( 2\pi \) under gauge transformations (recall that the 3-periods of \( H \) are in \( 2\pi \mathbb{Z} \)) and that the CS Feynman amplitude \( \exp[iS_{\text{CS}}(A)] \) is gauge invariant. We shall show the following result:

**Proposition 2**

\[ (-1)^{KM'} = \exp\left[iS_{\text{CS}}(A)\right]. \quad (5.8) \]

*Proof.* The argument below closely follows [18], using at the end our Theorem from Sect. 2. Let us start by taking \( e'_i(k) = \theta e_i(-k) \) as the new trivialization of \( E \). One has

\[ e_i(k) = \sum_{i'} (\theta e_i'(-k)|e_i(k)) \theta e_i'(-k) = \sum_{i'} (e_i(k)|\theta e_i'(-k)) e_i'(k) \]

\[ = \sum_{i'} w_{ii'}(-k) e_{i'}(k) = -\sum_{i'} w_{i'i}(k) e_{i'}(k), \quad (5.9) \]

where we used the symmetry (2.5) of \( w \). It follows from the previous discussion that \( A' \) is a gauge transformation of \( A \),

\[ A' = \overline{w} A \overline{w}^{-1} + \overline{w} d \overline{w}^{-1}, \quad (5.10) \]

and from (5.6), that

\[ \text{CS}(A') = \text{CS}(A) + \overline{w}^*H + \frac{1}{4\pi} d \text{tr}(\overline{w}^{-1}d\overline{w})A \]

\[ = \text{CS}(A) + w^*H + \frac{1}{4\pi} d \text{tr}(A^T w^{-1}dw). \quad (5.11) \]
But by the antiunitarity of $\theta$,

$$
A'_{i'j'}(k) = \langle \theta e_{i'}(-k) | d\theta e_{j'}(-k) \rangle = \langle de_{j'}(-k) | e_{i'}(-k) \rangle = -\langle e_{j'}(-k) | de_{i'}(-k) \rangle = -A_{j'i'}(-k).
$$

(5.12)

or $A' = -\vartheta^* A^T$ for $\vartheta: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ induced by $k \mapsto -k$. This implies that $\text{CS}(A') = \vartheta^* \text{CS}(A)$ and, with the use of (5.11), that

$$
\vartheta^* \text{CS}(A) = \text{CS}(A) + w^* H + \frac{1}{4\pi} d \text{tr}(A^T w^{-1} dw).
$$

(5.13)

Integrating the latter identity over $\mathbb{T}^3$, remembering that $\vartheta$ reverses the orientation of $\mathbb{T}^3$, we obtain the relation [18]

$$
\text{CS}(A) = -\frac{1}{2} \int_{\mathbb{T}^3} w^* H.
$$

(5.14)

Let, similarly as in $2d$, $\mathbb{T}_3^{\pm}$ correspond to the parts of $\mathbb{T}^3$ with $\pm k_1 \in [0, \pi]$, see Fig. 2. Then

$$
\int_{\mathbb{T}^3} w^* H = \int_{\mathbb{T}_3^+} w^* H + \int_{\mathbb{T}_3^-} w^* H = \int_{\mathbb{T}_3^+} (w^* H - \vartheta^* w^* H) = \int_{\mathbb{T}_3^+} (w^* H - (w^T)^* H) = 2 \int_{\mathbb{T}_3^+} w^* H
$$

(5.15)

using the symmetry (2.5) and the relation $(w^T)^* H = -w^* H$. Hence,

$$
\text{CS}(A) = -\int_{\mathbb{T}_3^+} w^* H.
$$

(5.16)

Exponentiating the latter identity, one finally obtains

$$
\exp \left[ i \text{CS}(A) \right] = \exp \left[ -i \int_{\mathbb{T}_3^+} w^* H \right] = \frac{\exp \left[ i \text{WZ}(w|_{k_1=0}) \right]}{\exp \left[ i \text{WZ}(w|_{k_1=\pi}) \right]},
$$

(5.17)

where the second equality follows easily from the definition (2.10) of the WZ action. Denote by $\mathbb{T}_a^2$ the two-dimensional subtorus of $\mathbb{T}^3$ corresponding to $k_1 = a$ for $a = 0, \pi$, see Fig. 2. By our main Theorem of Sect. 2,

$$
\exp \left[ i \text{WZ}(w|_{k_1=a}) \right] = \prod_{\text{TRIM} \in \mathbb{T}_a^2} \frac{\sqrt{\det w(k)}}{\text{pf} w(k)}.
$$

(5.18)

Identity (5.1) follows now from (5.17) and (5.18), completing the proof of Proposition 2. 

\[ \square \]
6 Conclusions

We have proved that the Fu–Kane–Mele $\mathbb{Z}_2$-valued invariant of 2d time-reversal symmetric crystalline insulators can be expressed as the properly normalized Wess–Zumino action of the sewing-matrix field defined on the Brillouin torus $\mathbb{T}^2$. This was done by the localization of the WZ action in question at the four time-reversal invariant (quasi-)momenta, obtained using the bundle-gerbe technique. Our result shed new light on the 2d Fu–Kane–Mele invariant and its topological nature. Applied in the 3d setup, it also permitted a direct proof of the relation between the strong Fu–Kane–Mele invariant of TRI crystals and the Chern–Simons action of the non-Abelian Berry connection on the bundle of valence states over the Brillouin torus $\mathbb{T}^3$.

Previously, the present author with collaborators established different relations between the 2d and 3d Fu–Kane–Mele invariants and the Wess–Zumino action [2,3,11]. In particular, the 2d Fu–Kane–Mele invariant was described in terms of the properly defined square root of the Wess–Zumino amplitude of the unitary-group-valued field $U(k) = I - 2P(k)$ and the strong 3d Fu–Kane–Mele invariant in terms of a related $\pm 1$-valued index of the 3d version of the same field. Those constructions were extended to the case of periodically forced TRI crystalline systems allowing to define $\mathbb{Z}_2$-valued refinements of the $\mathbb{Z}$-valued dynamical indices that were introduced in [19] for Floquet systems without TRI. As discussed in [11], the geometric framework of those works involved $\mathbb{Z}_2$ equivariant structures on gerbes. A direct relation between the constructions presented there and the simpler ones described here is still missing. In particular, it is not clear whether it is possible to extend directly the static discussion of the present paper to the case of periodically forced crystals.

Appendix

We establish here (in the reversed order) the properties of the WZ amplitude $\exp[iS_{\text{WZ}}(w)]$ claimed in Remark in Sect. 2.

First, for a smooth family of fields $w_t(k)$ defined on $\mathbb{T}^2$ such that $w_t \circ \vartheta = -w^T_t$ for all $t$ (with $\vartheta$ induced by $k \mapsto -k$) the well known formula for the derivative of
the WZ action, easily following from the definition (2.10), gives:

\[ \frac{d}{dt} S_{WZ}(w(t)) = \frac{1}{4\pi} \int_{T^2} \text{tr} \left( (w_t^{-1} \partial_t w_t)(w_t^{-1} dw_t)^2 \right) \]

\[ = \frac{1}{4\pi} \int_{T^2} \vartheta^* \text{tr} \left( (w_t^{-1} \partial_t w_t)(w_t^{-1} dw_t)^2 \right) \]

\[ = \frac{1}{4\pi} \int_{T^2} \text{tr} \left( ((w_t^T)^{-1} \partial_t w_t^T)((w_t^T)^{-1} dw_t^T)^2 \right) \]

\[ = \frac{1}{4\pi} \int_{T^2} \text{tr} \left( ((w_t^T)^{-1} \partial_t w_t^T)((w_t^T)^{-1} dw_t^T)^2 \right)^T \]

\[ = -\frac{1}{4\pi} \int_{T^2} \text{tr} \left( (w_t^{-1} \partial_t w_t)(w_t^{-1} dw_t)^2 \right) = 0, \quad (7.1) \]

implying the invariance of the WZ amplitude \( \exp[i S_{WZ}(w)] \) under smooth deformations of \( w \) preserving the symmetry (2.5).

Next, let us consider a change of the trivialization of the valence bundle

\[ e_i(k) = \sum_{i'} U_{i'i}(k) e'_{i'}(k) \quad (7.2) \]

with unitary \( U(k) \). For the sewing matrices, this gives

\[ w_{ij}(k) = \langle e_i(-k) | \theta e_j(k) \rangle = \sum_{i',j'} U_{i'i}(-k) U_{j'j}(k) \langle e'_{i'}(-k) | \theta e'_{j'}(k) \rangle \]

\[ = \sum_{i',j'} U_{ii'}^{-1}(-k) U_{jj'}^{-1}(k) w'_{i'j'}(k) \quad (7.3) \]

so that \( w(k) = U^{-1}(-k)w'(k)(U^{-1}(k))^T \) or

\[ w'(k) = U(-k)w(k)U(k)^T. \quad (7.4) \]

The map \( T^2 \ni k \mapsto U(k) \in U(n) \) can be smoothly contracted to the one

\[ T^2 \ni k \mapsto U_{n_1,n_2}(k) = \text{diag}[e^{i(n_1 k_1 + n_2 k_2)}, 1, \ldots, 1], \quad (7.5) \]

where \( n_1, n_2 \in \mathbb{Z} \) are the winding numbers of \( \det U(k) \) along the basic cycles of \( T^2 \). By the previous argument, it is enough to check the relation \( \exp[i S_{WZ}(w')] = \exp[i S_{WZ}(w)] \) for \( U(k) = U_{n_1,n_2}(k) \). Besides, by an \( SL(2, \mathbb{Z}) \) change of variables \( k \), one may achieve that \( n_1 = 0 \). Let \( D \) be the unit disc in \( \mathbb{C} \). Then \( B = D \times S^1 \) is a 3-manifold with the boundary \( S^1 \times S^1 \cong T^2 \). Since \( \det w(k) \) has no windings, there exists a smooth contraction
\[ [0, 1] \times \mathbb{T}^2 \ni (r, k) \mapsto W(r, k) \in U(n) \quad (7.6) \]

such that \( W(1, k) = w(k) \) and \( W(r, k) = I \) for \( r \) close to zero. We shall identify \( W \) with a smooth map defined on \( B \) by setting
\[
W(re^{ik_1}, e^{ik_2}) = W(r, k). \quad (7.7)
\]

Consider two other smooth maps \( V_{1,2} : B \to U(n) \) given by
\[
V_1(re^{ik_1}, e^{ik_2}) = \text{diag}[e^{-in_2k_2}, 1, \ldots, 1], \quad V_2(re^{ik_1}, e^{ik_2}) = \text{diag}[e^{in_2k_2}, 1, \ldots, 1]. \quad (7.8)
\]

The product map \( W' = V_1WV_2 : B \to U(n) \) is a smooth extension of \( w'(k) = U_{0,n_2}(-k)w(k)U_{0,n_2}(k)^T \) to the interior of \( B \) so that, by Witten’s prescription,
\[
S_{WZ}(w') = \int_B (W')^*H = \int_B (V_1WV_2)^*H
= \int_B \left( V_1^*H + W^*H + V_2^*H + \frac{1}{4\pi} d\text{tr}
\left( (V_1^{-1}dV_1)WV_2d(WV_2)^{-1}
+ (W^{-1}dW)V_2dV_2^{-1}\right) \right), \quad (7.9)
\]

where we applied twice the formula
\[
(W_1W_2)^*H = W_1^*H + W_2^*H + \frac{1}{4\pi} d\text{tr}(W_1^{-1}dW_1)W_2dW_2^{-1} \quad (7.10)
\]

holding for two \( U(n) \)-valued maps \( W_{1,2} \) on the same domain. Since \( V_1^*H = 0 = V_2^*H \) for dimensional reasons, we infer that
\[
S_{WZ}(w') - S_{WZ}(w)
= \frac{1}{4\pi} \int_{\partial B} \text{tr}\left( (V_1^{-1}dV_1)W(V_2dV_2^{-1})W^{-1}
- (V_1^{-1}dV_1)(dW)W^{-1} + (W^{-1}dW)V_2dV_2^{-1}\right)
\]
\[
= \frac{1}{4\pi} \int_{\mathbb{T}^2} \text{tr}\left( \text{diag}[-in_2dk_2, 0, \ldots, 0] \ w \text{diag}[-in_2dk_2, 0, \ldots, 0] \ w^{-1}
- \text{diag}[-in_2dk_2, 0, \ldots, 0] (dw)w^{-1} + (w^{-1}dw) \text{diag}[-in_2dk_2, 0, \ldots, 0]\right). \quad (7.11)
\]

Changing the variables \( k \mapsto -k \) in the integral on the right hand side and using the symmetry (2.5) of \( w \) together with the invariance of trace under the transposition, one shows that the integral in question is equal to its negative, hence it vanishes.
Finally, note that $\tilde{W}(r, k) = -W(r, -k)^T$ is also a contraction of $w(k)$, and as $W$, it may be regarded as defined on $B$. Then
\[
S_{\text{WZ}}(w) = \int_B \tilde{W}^* H = \int_B (-W^T)^* H = -\int_B W^* H = -S_{\text{WZ}}(w)
\] (7.12)
modulo $2\pi$, implying that $\exp[i S_{\text{WZ}}(w)] = (\exp[i S_{\text{WZ}}(w)])^{-1} = \pm 1$.

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