A MODEL FOR NONEXPONENTIAL RELAXATION AND AGING IN DISSIPATIVE SYSTEMS

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(10/03/04date)

Abstract

The nonexponential relaxation and aging inherent to complex dynamics manifested in a wide variety of dissipative systems is analyzed through a model of diffusion in phase space in the presence of a nonconservative force. The action of this force establishes a heat flow which maintains the system away from equilibrium. The inability of the system to find its equilibrium state becomes apparent through the presence of an effective temperature field. This is the temperature of the stationary nonequilibrium state reached by the system satisfying a generalized version of the fluctuation-dissipation theorem. The presence of a nonequilibrium temperature leads to a hierarchy of relaxation times responsible for the aging phenomena and to a relation similar to the Vogel-Fulcher-Tammann law.

PACS numbers: 64.70.Pf, 05.10.Gg, 05.40.-a
I. INTRODUCTION

Under certain conditions dissipative systems give rise to an often complex dynamics. This is the case of glassy systems which constitute an ubiquitous example of dissipative systems following a complex dynamics. Glass is a nonequilibrium system whose dynamics is nonlinear and has nonexponential relaxation functions which leads to the aging phenomena. Similar to glass is the process of protein folding which involves crossing a large number of energy barriers. Anomalous transport processes and strange kinetics based on fractional generalizations of the diffusion and Fokker-Planck equations constitute other typical examples. Hence, complex dynamics is the object of active research due to its multiplicity of manifestations. Here, we will focus on the nonexponential character of the relaxation functions and on aging in complex dynamics. The mechanisms underlying slow relaxation and aging still lack a clear and definitive elucidation. In glasses and proteins for example, several experiments and computer simulations have been done which support the explanation of these relaxation phenomena in the framework of the energy landscape paradigm as the result of activated diffusion through a rough energy landscape of valleys and peaks.

To contribute to the understanding of what these mechanisms are, we propose here a simple model to show a possible origin of nonexponential relaxation and aging based on the idea of the energy landscape and nontrivial energy barriers. This model which consists of the diffusion in phase space, provides a direct link between the phase space dynamics and the slow relaxation of the functions of the configuration of the system in the corresponding energy landscape. The slowing down of the dynamics clearly appears as a consequence of the freezing of some degrees of freedom which takes the system out of equilibrium. This fact is indicated by the presence of an effective temperature field incorporating the information of the suppressed degrees of freedom and depending on the equilibrium temperature at the moment the quench was applied.

The paper is distributed as follows. In the second section we develop the thermodynamic framework in which our model is embedded. In section three, we write the Fokker-Planck equation governing the dynamics and study the relaxation. Section four is devoted to deriving the generalized diffusion equation and obtaining the relaxation function and the hierarchy of relaxation times in some particular cases. Section five deals with the computation of the generalized fluctuation-dissipation theorem GFDT. Finally in section seven, we discuss our
main conclusions.

II. THERMODYNAMIC FRAMEWORK

We model the nonexponential relaxation phenomenon in a dissipative system as the Brownian motion of a particle of unit mass in a nonperiodic potential $V(x)$, able to constitute a schematic representation of a rough energy landscape. Moreover, we will apply the methods of nonequilibrium thermodynamics at a mesoscopic level which consists of computing the entropy production from the Gibbs entropy postulate and postulating the linear phenomenological equations relating the fluxes and the thermodynamic forces occurring therein. This constitutes a mesoscopic generalization of the macroscopic formalism of the nonequilibrium thermodynamics, the mesoscopic nonequilibrium thermodynamics (MNET).

Thus, our thermodynamic study arises from the assumption that we are dealing with a physical ensemble of Brownian particles, a thermodynamic system to which thermodynamic relations can be applied. We consider that the ensemble is initially at equilibrium in a heat bath at temperature $T_o$, and thus, is distributed according to the canonical distribution

$$\psi_{eq} \sim \exp\left(-\frac{H(\Gamma)}{k_B T_o}\right),$$

where $H(\Gamma) = 1/2u^2 + V(x)$ is the Hamiltonian, and $\Gamma = (x, u)$ represents a point of the one-particle phase space, with $x$ being the position and $u$ the velocity. At a certain moment ($t = 0$), we apply a nonconservative force $f(t)$ quenching the system. It is assumed that the nonequilibrium entropy of the system is given through the Gibbs entropy postulate

$$S(t) = -k \int \psi \ln \frac{\psi}{\psi_{eq}} d\Gamma + S_{eq},$$

where $S_{eq}$ is the equilibrium entropy. Variations in the probability density $\psi(\Gamma, t)$ imply changes in the entropy which can be obtained from Eq. (2). These are given by

$$\delta S = -\int \left(k \ln \frac{\psi}{\psi_{eq}} + \frac{\mu_{eq}}{T_o}\right) \delta \psi d\Gamma,$$

where we have taken into account that $\delta S_{eq} = -\int \mu_{eq}/T_o \delta \psi d\Gamma$, with $\mu_{eq} = -V(x) + \mu_o$ being the equilibrium chemical potential per particle of the Brownian gas and $\mu_o$ a constant. On the other hand, we assume local equilibrium in the phase space $\Gamma$ consisting in postulating
the Gibbs equation at a local level in the phase space$^{10,11}$, which in the case of a homogeneous bath, reduces to

$$T_o \delta S = - \int \mu(\Gamma, t) \delta \psi(\Gamma, t) d\Gamma ,$$

(4)

where the thermodynamically conjugated of the density $\psi(\Gamma, t)$ nonequilibrium chemical potential $\mu(\Gamma, t)$ appears. The expression of the nonequilibrium chemical potential is obtained by comparing Eqs. (3) and (4), thus

$$\mu(\Gamma, t) = kT_o \frac{\psi}{\psi_{eq}} + \mu_{eq}.$$  

(5)

A gradient of the chemical potential induces a diffusion process in phase space in order to restore the equilibrium state. Throughout this process, the distribution of probability changes according to the following Generalized Liouville equation

$$\frac{\partial}{\partial t} \psi(\Gamma, t) + V_T(\Gamma, t) \cdot \nabla \psi(\Gamma, t) = - \frac{\partial}{\partial u} \psi(\Gamma, t) f(t) - \frac{\partial}{\partial u} J(\Gamma, t) ,$$

(6)

where $V_T(\Gamma, t) = (u, -\nabla V(x))$ is the velocity corresponding to the Hamiltonian flow, $J(\Gamma, t)$ constitutes a diffusion current and $\nabla \Gamma = (\frac{\partial}{\partial u}, \nabla)$, where $\nabla$ stands for the spatial derivative. In order to completely characterize the diffusion process, one has to find the expression of the diffusion current by following the rules of nonequilibrium thermodynamics$^{14}$. In this framework, through the computation of the entropy production one can establish phenomenological relations between currents and conjugated thermodynamic forces. Thus, from Eq. (4) we can determine the rate of change of the entropy

$$T_o \frac{dS}{dt} = - \int \mu(\Gamma, t) \frac{\partial}{\partial t} \psi(\Gamma, t) d\Gamma ,$$

(7)

which after substituting Eq. (6) and by using Eq. (5) becomes

$$T_o \frac{dS}{dt} = \langle (f(t) - \nabla V(x)) u \rangle - \int J(\Gamma, t) \frac{\partial}{\partial u} \mu(\Gamma, t) d\Gamma ,$$

(8)

obtained after partial integration. Here,

$$\langle (f(t) - \nabla V(x)) u \rangle = \int (f(t) - \nabla V(x)) u \psi(\Gamma, t) d\Gamma$$

(9)

is the power supplied by the net force which is dissipated in the system as heat. Thus, the first term on the right hand side of Eq. (8) can be interpreted as the rate of heat exchange with the surroundings

$$\frac{dq}{dt} = - \langle (f(t) - \nabla V(x)) u \rangle ,$$

(10)
with $dq$ being the amount of heat released in a time $dt$. However, the second term on the right hand side of Eq. (8) constitutes the entropy production $\sigma \geq 0$, according to the second law, containing the current and its conjugated thermodynamic force $(\partial/\partial u)\mu(\Gamma,t)$, which by postulate of the nonequilibrium thermodynamics are related through the phenomenological law

$$J(\Gamma,t) = -\frac{L}{T_o} \frac{\partial}{\partial u} \mu(\Gamma,t), \quad (11)$$

where $L$ is a phenomenological coefficient. Therefore, Eq. (8) may be rewritten in a more significant form

$$\frac{dS}{dt} = \frac{1}{T_o} \frac{dq}{dt} + \sigma, \quad (12)$$

expressing the entropy balance between the exchange of heat with the surroundings and the entropy generated in the irreversible processes established in the system when it is removed from its equilibrium state. By using the expression of the nonequilibrium chemical potential in (11), one obtains

$$J(\Gamma,t) = -\gamma \left( k_B T_o \frac{\partial}{\partial u} + u \right) \psi, \quad (13)$$

where we have performed the identification $L/\psi T_o = \gamma$, with $\gamma$ being the friction coefficient of the Brownian particle so that

$$\sigma = \frac{\gamma}{T_o} \int \frac{(k_B T_o (\partial/\partial u) \psi(\Gamma,t) + \psi(\Gamma,t) u)^2}{\psi(\Gamma,t)} d\Gamma. \quad (14)$$

One can conclude that is possible the existence of a stationary nonequilibrium state ($dS/dt = 0$) in the system, i.e. not given by an equilibrium maxwellian, for which

$$\frac{1}{T_o} \frac{1}{dt} = -\sigma_{st}. \quad (15)$$

### III. FOKKER-PLANCK EQUATION AND RELAXATION

The Fokker-Planck equation governs the dissipative dynamics when the description of the state of the system is given in terms of the probability density $\psi(\Gamma,t)$. This equation is obtained after substituting the expression of the current into Eq. (6) leading to

$$\frac{\partial}{\partial t} \psi(\Gamma,t) = -\frac{\partial}{\partial x} u \psi(\Gamma,t) + \frac{\partial}{\partial u} \psi(\Gamma,t) (\nabla V(x) - f(t))$$

$$+ \gamma \frac{\partial}{\partial u} \left( \beta^{-1} \frac{\partial}{\partial u} + u \right) \psi(\Gamma,t). \quad (16)$$
We will assume that \( u \) is the fast variable, thus the dynamical processes in the system are associated to configurational changes related to \( x \), the slow variable. Thus, we write

\[
\psi(\Gamma, t) = \phi_x(u, t)\rho(x, t) ,
\]

where \( \phi_x(u, t) \) is the conditional probability. The probability density is \( \rho(x, t) = \int \psi(\Gamma, t)du \) evolves according to

\[
\frac{\partial}{\partial t} \rho(x, t) = -\nabla \int u\psi(\Gamma, t)du ,
\]

obtained by partial integration of Eq. (16), which defines the current \( J(x, t) = \int u\psi(\Gamma, t)du \).

This current satisfies the equation

\[
\frac{\partial}{\partial t} J(x, t) + \gamma J(x, t) = -\{\rho(x, t) (\nabla V(x) - f(t)) + k_B \nabla \rho(x, t) T(x, t)\} .
\]

which can be obtained after multiplying Eq. (16) by the velocity \( u \) and performing the partial integration, by using the decoupling (17). Here, \( k_B T(x, t) = \int u^2 \phi_x(u, t)du \) is the second moment of the conditional distribution \( \phi_x(u, t) \), playing the role of an effective temperature which contains information on the frozen degrees of freedom. This temperature field introduces thermal barriers in the system. The presence of these thermal barriers is a nonequilibrium effect that disappears when the system is at equilibrium. Over long time, when the transient regime has died out, Eq. (19) becomes

\[
J(x, t) = -\tau_0 \{\rho(x, t) (\nabla V(x) - f(t)) + k_B \nabla \rho(x, t) T(x, t)\} ,
\]

where \( \tau_0 = \gamma^{-1} \).

On the other hand, in Eqs. (19) and (20) the current \( J(x, t) \) appears coupled to the temperature field \( T(x, t) \) which evolves according to

\[
\frac{\partial}{\partial t} \rho(x, t) T(x, t) = -\frac{2}{k_B} \nabla \rho(x, t) h(x, t) \\
-2 (\nabla V(x) - f(t)) \frac{J(x, t)}{k_B} - \frac{2}{k_B \tau_0} \rho(x, t) (T(x, t) - T_o) ,
\]

obtained by multiplying Eq. (13) by \( u^2 \) and integrating in \( u \). Here, we have defined the heat flow

\[
h(x, t) = \frac{1}{2} \int u^3 \phi_x(u, t)du .
\]
For times $t \gg \tau_0$, Eq. (21) reduces to

$$\rho(x,t)k_B (T(x,t) - T_o) = -\tau_0 \nabla \rho(x,t)h(x,t)$$  \hspace{1cm} (23)

$$-\tau_0 (\nabla V(x) - f(t)) J(x,t) .$$

In the particular case of equilibrium, $\phi_x(u,t)$ would be an equilibrium Maxwellian, thus $J(x,t) = h(x,t) = 0$ which would lead to $T(x,t) = T_o$. If, on the other hand, we integrate Eq. (23) we obtain

$$k_B (T(t) - T_o) = \tau_0 \frac{dq}{dt} ,$$  \hspace{1cm} (24)

with $T(t) = \langle T(x,t) \rangle$. This last result together with Eq. (10) leads to the interesting conclusion that even if $f(t) = 0$ the partial elimination of the fast variable would introduce a nonequilibrium temperature\textsuperscript{16}. Only for the particular case of a soft energy landscape with small gradients $\nabla V(x)$, the difference $T(t) - T_o$ would be negligible in comparison to $T_o$. We can make this statement more precise by defining a critical length scale $L = \Delta V/\nabla V(x)$, where $\Delta V$ is the characteristic variation of the potential, such that if $(\gamma L)^2 / k_B T_o \ll 1$, then $T(x,t) = T_o$.

In fact the Brownian motion as described by the Klein-Kramers equation (16) is a problem with multiple time scales as was pointed out in refs.\textsuperscript{17,18} and implicitly included in the generalization of the Smoluchowski equation found previously in\textsuperscript{20,19}. Here, we have embedded all of these scales in the nonequilibrium temperature $T(x,t)$ or $T(t)$, constituting a kind of mean field approach which leads to the nonexponential relaxation and aging we will show in the following sections.

To sum up, in this section we have obtained the evolution equations for the first moments of the conditional probability density which are of interest for the thermodynamic description of the Brownian gas. More detail and some consequences will be shown in the following sections.

**IV. GENERALIZED DIFFUSION EQUATION**

Here, we will establish the generalized diffusion equation and examine its consequences in respect to the relaxation phenomena.
After defining the effective potential \( \Phi(x, t) = V(x) + k_B T(x, t) \), Eq. (20) can be expressed

\[
J(x, t) = -D(x, t) \nabla \rho(x, t) - \tau_0 \rho(x, t) (\nabla \Phi(x, t) - f(t)) \\
\equiv Q(x, t) + \tau_0 \rho(x, t) f(t),
\]

where \( D(x, t) = \tau_0 k_B T(x, t) \) is the generalized diffusion coefficient and \( Q(x, t) \) corresponds to the current of the purely relaxational system \( \text{i.e. in absence of nonconservative forces} \). Thus, by substituting Eq. (25) into Eq. (18) and assuming \( \tau_0 = 1 \) \( \text{i.e. rescaling the time} \), this last equation becomes the generalized diffusion equation

\[
\frac{\partial}{\partial t} \rho(x, t) + \nabla \rho(x, t) f(t) = \nabla Q(x, t).
\]

In addition, as a consequence of the elimination of degrees of freedom, the dynamics becomes non-Markovian and depends on the equilibrium temperature \( T_0 \) at the quench time. In the frame of reference defined through the transformation \( x^* = x - \int f(v) dv \), the current (25) reduces to \( Q(x^*, t) \). Thus, in this frame one has a purely relaxational process for which

\[
\frac{\partial}{\partial t} \rho(x^*, t) = \nabla Q(x^*, t).
\]

This equation admits a quasi-equilibrium solution \( \rho_{qe}(x, t) \sim \exp \left\{ -\int x^2 \frac{1}{k_B T(x, t)} \nabla' \Phi(x', t) dx' \right\} \) for which \( Q(x, t) = 0 \), where to simplify the notation we have omitted the superscript * on the \( x \).

**A. Population dynamics and relaxation function**

To examine the problem of the population dynamics with its origin in the generalized diffusion equation (27) and to find the corresponding relaxation function we will assume that at a point \( x_2 \) there is a sink where \( \rho(x_2, t) = 0 \) and a source at \( x_1 \). Hence, a quasi-stationary current, \( Q(t) \) can be established in the system, thus, integration of \( Q(x, t) \) gives us the quasi-stationary probability density in terms of \( Q(t) \)

\[
\rho(x, t) = Q(t) \rho_{qe}(x, t) \int_{x}^{x_2} \frac{dy}{D(y, t) \rho_{l,qe}(y, t)}.
\]

On the other hand, a second integration leads to

\[
n(t) = Q(t) \int_{x_1}^{x_2} dx \rho_{qe}(x, t) \int_{x}^{x_2} \frac{dy}{D(y, t) \rho_{l,qe}(y, t)},
\]
with \( n(t) = \int_{x_1}^{x_b} \rho(x, t) dx \) being the population at the left of the sink. Thus, after integration of Eq. (18) we obtain

\[
\frac{d}{dt} n(t) = - \{Q(x_b, t) - Q(x_1, t)\} = -Q(t) , \tag{30}
\]

which leads to the following rate equation

\[
\frac{d}{dt} n(t) = -K(t; x_b)n(t) , \tag{31}
\]

where

\[
K(t; x_b)^{-1} = \int_{x_1}^{x_b} dx \rho_{q}(x, t) \int_{x}^{x_2} dy \frac{D(y, t; t_0)\rho_{q}(y, t)}{\rho_l qe(y, t)} \tag{32}
\]

is the rate constant, and \( x_b \) is the position of the barrier between \( x_1 \) and \( x_2 \). The relaxation equation (31) admits the solution

\[
n(t) = n(t_0) \exp \left\{ - \int_{t_0}^{t} K(t'; x_b)dt' \right\} \equiv n(t_0) \exp \left\{ -g(t) \right\} \tag{33}
\]

or

\[
\hat{n}(t) = \hat{n}(t_0) \exp \left\{ -g(t/t_0) \right\} , \tag{34}
\]

where \( \hat{n}(t) = n(t/t_0) \). Both Eqs. (33) and (34) constitute an interesting result since we obtain a hierarchy of relaxation times starting from a Markovian equation, the Fokker-Planck equation (16). Here, the function \( g(t/t_0) \) might be interpreted as an algebraic function \( (t/t_0)^\beta \) obtaining a stretched exponential behavior\(^{23} \) or as a logarithmic function \(-\alpha \log At \) which leads to a power law behavior\(^{24} \) that characterizes anomalous diffusion\(^{25} \). In addition, another reading of our result is possible because assuming the form of \( g(t/t_0) \) is equivalent to assuming the form of the distribution of residence times \( \psi(t) \equiv -\frac{d}{dt}n(t) \)\(^{26} \). Thus, an in-depth analysis of the implications of the supposition of a nonexponential distribution of residence times usually performed in Continuous Time Random Walk models, reveals that this hypothesis might also be rooted in the nonequilibrium character of the dynamics inherent to the energy landscape picture as we have shown.

B. Hierarchy of relaxation times

Inherent to the nonexponential relaxation Eqs. (33) and (34) there is a hierarchy of relaxation times. To show this clearly here, we will derive the hierarchy of relaxation times
for the particular case of a bistable potential and the state of quasi-equilibrium. Thus, let us assume here that $V(x)$ is a bistable potential with one well much deeper than the other. In this case, we can obtain the dynamic of the population of the shalower well through an equation analogous to Eq. (31), where now according to the Kramers formula

$$K = f_0 \exp \left\{ -\frac{\Delta}{k_BT(t)} \right\}, \quad (35)$$

where we have defined the frequency $f_0 = \frac{1}{2} \sqrt{V''(x_1)}|V''(x_b)|$, and the barrier height $\Delta = V(x_b) - V(x_1)$. Thus,

$$n(t) = n(t_0) \exp \left\{ - \int_{t_0}^{t} \frac{dt'}{\tau(t')} \right\}, \quad (36)$$

with

$$\tau(t) = \frac{1}{f_0} \exp \left\{ \frac{\Delta}{k_BT(t)} \right\} \quad (37)$$

From Eq. (24), we can write

$$T(t) - T(t_0) = f(t)^2 - f(t_0)^2, \quad (38)$$

where now $T(t) = \langle T(x,t) \rangle_{qe}$ and $\langle \nabla V(x)u \rangle_{qe} = 0$. Therefore,

$$\tau(t) = \frac{1}{f_0} \exp \left\{ \frac{\Delta/k_B}{f(t)^2 - f(t_0)^2 + T(t_0)} \right\}, \quad (39)$$

or equivalently

$$\tau(t) = \frac{1}{f_0} \exp \left\{ \frac{\Delta/k_B}{T^*(t) - T^*(t_0)} \right\} \exp \left\{ \frac{1}{1 + T^*(t)/T^*(t_0)} \right\}, \quad (40)$$

where $T^*(t) = T(t) - T_o$, i.e. a scale of temperature whose zero point coincides with the temperature of the heat bath. Here, assuming that $\frac{T^*(t) - T^*(t_0)}{T(t_0)} \ll 1$ we can neglect the second exponential on the right hand side of Eq. (40), leading to the result

$$\tau(t) = \frac{1}{f_0} \exp \left\{ \frac{\Delta/k_B}{T^*(t) - T^*(t_0)} \right\} \quad (41)$$

similar to the celebrated Vogel-Tammann-Fulcher equation. In the next order of approximation we would have

$$\tau(t) = \frac{1}{f_0} \left( \frac{T^*(t) - T^*(t_0)}{T(t_0)} \right) \exp \left\{ \frac{\Delta/k_B}{T^*(t) - T^*(t_0)} \right\} \quad (42)$$

providing an expression for the relaxation time which is finite when $T^*(t) = T^*(t_0)$. 
V. FLUCTUATION-DISSIPATION THEOREM

Here, we want to derive the fluctuation-dissipation theorem for the nonequilibrium system, which is locally satisfied and provides us with a way to compute the nonequilibrium temperature. In order to highlight the main details and avoid lengthy calculations, we will homogenize Eq. (27) introducing a preaveraging condition with respect to the quasi-equilibrium distribution

\[
\frac{\partial}{\partial t} \rho = \frac{\partial}{\partial x} \left\{ D(t) \frac{\partial}{\partial x} \rho(x,t) + \rho(x,t) \frac{\partial}{\partial x} \Phi(x,t) \right\},
\]

where \( D(t) = k_B T(t) \) is the preaveraged diffusion coefficient. Thus, now the quasi-equilibrium density becomes a local Boltzmann probability density \( \rho_{qe}(x,t) \sim \exp \left\{ -\Phi(x,t)/k_B T(t) \right\} \).

Consider a time-dependent external field \( F(t) = F \Theta(t-t_w) \) applied to the system Eq. (43) at the waiting time \( t_w \) when this system is in the quasi-equilibrium state described by \( \rho_{qe} \). Here, \( \Theta(t-t_w) \) is the step function. This force modifies the potential, and we assume that the perturbed potential is \( \tilde{\Phi}(x,t) = \Phi(x,t) - F(t)B(x) \). If the field is weak, the deviation with respect to stationarity of the average value of any function \( A(x) \) of the configuration of the system is a linear functional of the field

\[
\langle A(t) \rangle_F - \langle A(t) \rangle_{qe} = \int_{t_w}^t R(t,t')F(t')dt',
\]

which introduces the response function \( R(t,t') \). Now \( \langle A(t) \rangle_F \) can be calculated once the perturbed distribution function \( \tilde{\rho}(x,t) \) is known:

\[
\langle A(t) \rangle_F = \int A(x) \tilde{\rho}(x,t)dx.
\]

Since the system is in the nonequilibrium stationary state at the time \( t_w \), \( \tilde{\rho}(x,t) \) is related to \( \rho_{qe}(x,t_w) \) through the perturbed Green function \( \tilde{G}(x,t \mid x',t_w) \)

\[
\tilde{\rho}(x,t) = \int \tilde{G}(x,t \mid x',t_w)\rho_{qe}(x',t_w)dx',
\]

and

\[
\tilde{\rho}_{qe}(x,t) = \frac{\exp \left( FB(x)/k_B T(t) \right)}{\langle \exp \left( FB(x)/k_B T(t) \rangle \rangle_{qe} \rho_{qe}(x,t) \right.}
\]

To compute \( \langle A(t) \rangle_F \) by using Eq. (45) with the help of Eq. (46) we have to know the relation between \( \tilde{G}(t,x \mid x',t_w) \) and the corresponding unperturbed Green function \( G(t,x \mid x',t_w) \).
This relation can be established from the identity

\[ \tilde{\rho}_{qe}(x, t) = \int \tilde{G}(t, x | x', t_w) \tilde{\rho}_{qe}(x', t_w) dx', \]  

(48)

by applying the relation (47) at both sides of Eq. (48), leading to

\[ G(t, x | x', t_w) = \exp \left( \frac{FB(x)}{k_B T(t_w)} \right) \tilde{G}(t, x | x', t_w) \langle \exp \left( \frac{FB(x)}{k_B T(t_w)} \right) \rangle_{qe}. \]  

(49)

By using Eqs. (45), (46) and (49), after expanding the exponentials with respect to F, one then obtains

\[ \langle A(t) \rangle_F - \langle A(t) \rangle_{qe} = F \left\{ \frac{1}{k_B T(t)} C_{A,B}(t, t) - \frac{1}{k_B T(t_w)} C_{A,B}(t, t_w) \right\}, \]  

(50)

where \( C_{A,B}(t, t') = \langle A(t) B(t') \rangle - \langle A(t) \rangle \langle B(t') \rangle \). We assume that the state of the system varies slowly enough so that it is safe to replace \( T(t) \) for \( T(t_w) \). Thus, one achieves

\[ \langle A(t) \rangle_F - \langle A(t) \rangle_{qe} = \frac{1}{k_B T(t_w)} \int_{-\infty}^{t} \frac{\partial}{\partial t'} C_{A,B}(t, t') F(t') dt', \]  

(51)

which defines the nonequilibrium response function

\[ R(t, t_w) = \frac{1}{k_B T(t_w)} \frac{\partial}{\partial t_w} C_{A,B}(t, t_w), \]  

(52)

for \( t > t_w \), constituting a local version of the fluctuation-dissipation theorem, the GFDT.

This section shows that the kind of effects related to the nonequilibrium nature of the solutions arised in the Fokker-Planck equation can be tested by means of linear response theory.

**VI. CONCLUSIONS**

Here, we have learned that the dynamics of a brownian gas under the action of a nonconservative force slowly varying over time is representative of the complex dynamics observed in dissipative systems. After quenching, the system is taken out of its initial equilibrium and then a heat flow is established. We model the effects of the quench of the system through the elimination of the fast degrees of freedom from the dynamics described by the Fokker-Planck equation. The departure from equilibrium and consequently the exchange of heat is maintained by means of the nonconservative force. We have also considered the existence of a nonperiodic rough potential in order to incorporate activated processes into our analysis.
According to the energy landscape paradigm inherent to several complex systems. A series of facts constituting the main results of our analysis follows.

First, the frozen degrees of freedom are manifested in the constrained dynamics through an effective temperature field different from the equilibrium temperature. The average of this temperature field plays the role of the temperature that characterizes the non-Markovian diffusion equation obtained from the Fokker-Planck equation, which describes the dynamic of the slow degrees of freedom. We also derive the relaxation equation for the effective temperature field from the Fokker-Planck equation which enables us to give a sound base to the nonequilibrium temperature as a result of the balance between the added heat and the heat generated during the relaxation process in the system. In addition, we show that the nonequilibrium temperature computed using the quasi-equilibrium distribution, i.e. the stationary solution of the non-Markovian diffusion equation, characterizes the generalized fluctuation-dissipation theorem we have derived, thus coinciding with the effective temperature referred to by several authors in the literature.

On the other hand, we find a hierarchy of relaxation times as a consequence of the non-Markovian character of the constrained dynamics of the system. In this respect, our model also describes stretched exponential relaxation as well as anomalous diffusion and other Continuous Time Random Walk problems. Likewise, as a consequence of the nonexponential relaxation, we obtain an equation similar to the celebrated Vogel-Tammann-Fulcher law which is, in our opinion, an important result.

Other approaches based on the energy landscape, considering the relaxation as a series of activated processes, assume that the activation energies are independent from the temperature but with a certain random spatial distribution. Averaging over this distribution of energy barrier gives one the corresponding relaxation. In the literature another types of models undergoing nonexponential relaxation can be found, as for example, disorder models.

We conclude that our model incorporating the multiplicity of time scales embedded in the Fokker-Planck equation, although simple, contains the main features in the dynamical slowing down observed in a wide variety of complex systems. On the other hand, with our approach we reach more insight about the physical meaning of the corrections to the Smoluchowski equation found in previous works as those referred to in this contribution, stating this problem in the modern language of the complex dynamics. In addition, our
model constitutes a way to study Brownian motion in disordered space leading to results similar to those reached through the approach based on fractal Fokker-Planck equations. Therefore, we think this work will contribute to the understanding of some aspects of complex dynamics in several dissipative systems.

**Acknowledgement 1** This work was supported by DGICYT of the Spanish Government under Grant No. PB2002-01267.

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