EXAMPLES OF NON-TARGET-REPRESENTABLE SYMPLECTIC CAPACITIES

YANN GUGGISBERG AND FABIAN ZILTENER

Abstract. We give the first concrete examples of symplectic capacities that are not target-representable. This provides some answers to a question by Cieliebak, Hofer, Latschev, and Schlenk.

Contents

1. Main result and simpler but less interesting examples 1
2. Proofs of the results 9
Appendix A. General Cartan formula and Stokes’ Theorem for helicity 19
References 19

1. MAIN RESULT AND SIMPLER BUT LESS INTERESTING EXAMPLES

1.1. Introduction and main result. The results of this article are concerned with non-target-representable capacities on small weak differential form categories. These generalize symplectic capacities on small symplectic categories. To explain our results, we define the notion of a capacity in the following even more general setup.

Let $\mathcal{C}$ be a small category. We denote by $\mathcal{O}$ its set of objects. We denote $\mathbb{R}^+:=(0,\infty)$ and fix an $\mathbb{R}^+$-action$^1$ on $\mathcal{C}.$

Definition 1 (generalized capacity). A (generalized) capacity$^2$ on $\mathcal{C}$ is a map $c: \mathcal{O} \to [0,\infty]$ with the following properties:

(i) (monotonicity) If $A$ and $B$ are two objects in $\mathcal{O}$ such that there exists a $\mathcal{C}$-morphism from $A$ to $B$, then

$$c(A) \leq c(B).$$

---

1. Y. Guggisberg’s work on this publication is part of the project Symplectic capacities, recognition, discontinuity, and helicity (project number 613.009.140) of the research programme Mathematics Clusters, which is financed by the Dutch Research Council (NWO). We gratefully acknowledge this funding.

2. An action of a group $G$ on a small category $\mathcal{C}$ is a map $\rho: G \times \mathcal{C} \to \mathcal{C},$ such that for every $g, h \in G, \rho_g$ is a functor, $\rho_{gh} = \rho_g \rho_h,$ and $\rho_e$ is the identity functor, where $e$ denotes the neutral element of $G.$

---

In [CHLS07], a capacity is defined to be a generalized capacity that is non-trivial. Since we do not use this condition in this article, we will use the word capacity in the sense of generalized capacity.
(ii) (conformality) For every $A \in \mathcal{O}$ and every $a \in \mathbb{R}^+$ we have
\[ c(aA) = ac(A). \]

To define the notion of an embedding capacity, let $\hat{\mathcal{C}}$ be a category determined by a formula. By this we mean a category whose objects, morphisms, and pairs constituting the composition map are given by well-formed logical formulas.

**Remark 2.** This article is based on the Zermelo-Fraenkel axiomatic system with choice (ZFC). In most cases, the collection of objects and morphisms of a category are proper classes and not sets. Therefore, we need to be careful when talking about general categories. However, categories determined by a formula can be dealt with in ZFC.

We fix an $\mathbb{R}^+$-action on $\hat{\mathcal{C}}$ that is defined by a logical formula as in [GZ22, Appendix A, Definition 18]. Let $\mathcal{C}$ be a small $\mathbb{R}^+$-invariant subcategory of $\hat{\mathcal{C}}$. We denote by $\mathcal{O}$ its set of objects.

**Definition 3** (embedding capacity). For every object $A$ of $\hat{\mathcal{C}}$, we define the domain-embedding capacity for $A$ on $\mathcal{C}$ to be the map
\[ c_A := c_{\mathcal{O},\hat{\mathcal{C}}}^A : \mathcal{O} \to [0, \infty) \]
\[ c_A(B) := \sup \left\{ a \in (0, \infty) \mid \exists \text{\hat{\mathcal{C}}-morphism } aA \to B \right\}. \]

Similarly, for every object $B$ of $\hat{\mathcal{C}}$, we define the target-embedding capacity for $B$ on $\mathcal{C}$ to be the map
\[ c_B := c_{\mathcal{O},\hat{\mathcal{C}}}^B : \mathcal{O} \to [0, \infty) \]
\[ c_B(A) := \inf \left\{ b \in (0, \infty) \mid \exists \text{\hat{\mathcal{C}}-morphism } A \to bB \right\}. \] (1)

Here we use the conventions $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$.

The maps $c_A$ and $c_B$ are capacities on $\mathcal{C}$.

In [CHLS07], K. Cieliebak, H. Hofer, J. Latschev, and F. Schlenk defined a symplectic category to be a subcategory of the category of symplectic manifolds with symplectic embeddings as morphisms, such that for every object $(M, \omega)$ of the subcategory and every $a \in (0, \infty)$ the pair $(M, a\omega)$ is also an object of the subcategory. In [CHLS07, Problem 1], these authors asked the following question for symplectic categories.

---

3 Here, $aA$ denotes the action of $a$ on the object $A$.
4 By this, we mean pairs $((f, g), h)$, where $(f, g)$ is the input of the composition map and $h$ is the output, i.e. $f \circ g = h$.
5 As mentioned in Remark 2, arbitrary subcategories cannot be dealt with in ZFC. This can be resolved by considering isomorphism-closed categories as defined in [GZ22, Definition 16].
Question 4. Which capacities on a symplectic category can be represented as $c^{(X,\Omega)}$ for a connected symplectic manifold $(X,\Omega)$?

Remark. Let $C$ be the category whose objects are the connected open sets of $\mathbb{R}^{2n}$ and whose morphisms are the symplectic embeddings. Then, as explained in [CHLS07, Example 2], every capacity on $C$ can be represented by a possibly uncountable disjoint union $(X,\Omega)$ of manifolds. If this union is uncountable, then $(X,\Omega)$ itself is not a manifold, since its topology is not second countable.

For $n \geq 2$, the set of capacities on the category of symplectic manifolds of dimension $2n$ and symplectic embeddings that are representable by $c^{(X,\Omega)}$ for some symplectic manifold $(X,\Omega)$ has cardinality at most $\aleph_1$, see [JZ21, Corollary 58]. On the other hand, in [JZ21, Theorem 17], D. Joksimović and F. Ziltener showed that the cardinality of the set of all capacities on this category is $\aleph_2$. This means that almost no capacity is target-representable, even if we allow for disconnected targets. See [JZ21, theorem on page 8 and Corollary 20]. In particular, non-target-representable capacities exist.

The present article provides examples of capacities that are not target-representable in the following more general setting.

Let $m, k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

Definition 5 (universal form category). We define the universal form category

$$\Omega^{m,k}$$

as follows:

- Its objects are the pairs $(M,\omega)$, where $M$ is a (smooth) manifold of dimension $m$ and $\omega$ is a differential $k$-form on $M$.
- Its morphisms are (smooth) embeddings that intertwine the differential forms.

For two objects $(M,\omega)$ and $(M',\omega')$, we denote

$$(M,\omega) \leadsto (M',\omega')$$

iff there exists a morphism between them. We equip $\Omega^{m,k}$ with the following $\mathbb{R}^+$-action. For every $a \in \mathbb{R}$, we define $a \cdot$ to be the functor

\[ a \cdot (M,\omega) = (M, a \cdot \omega) \]

In that article, the authors used a different setup for capacities. Namely, they defined this notion on isomorphism-closed (as defined in Footnote 9) subcategories of the universal form category $\Omega^{m,k}$ (as in Definition 5). Such a category is not small, except if it is empty. The two theories of capacities are equivalent, if we only consider small weak form categories (as in Definition 6), such that for every pair of objects $A, B$, every $\Omega^{m,k}$-isomorphism from $A$ to $B$ is a morphism in the small weak form category.

All the manifolds in this article are smooth, finite-dimensional, and are allowed to have boundary.

We do not impose any condition involving the boundaries of the manifolds, here.
from $\Omega^{m,k}$ to itself that sends every object $M := (M, \omega)$ to

$$aM := (M, a\omega)$$

and every morphism $\varphi : (M, \omega) \hookrightarrow (M', \omega')$ to the same map viewed as a morphism $(M, a\omega) \hookrightarrow (M', a\omega')$.

**Remark.** The category $\Omega^{m,k}$ is determined by a logical formula. ♦

**Definition 6** (small weak $(m, k)$-form category). A *small weak $(m, k)$-form category* is a small $\mathbb{R}^+$-invariant subcategory of $\Omega^{m,k}$.

The present article provides examples of capacities that are not (closely) target-representable in the following sense.

**Definition 7** ((closely) target-representable capacity). Let $\mathcal{C}$ be a small weak $(m, k)$-form category. We denote its set of objects by $O$. Let $c$ be a capacity on $\mathcal{C}$. We call $c$ *target-representable* iff there exists an object $(X, \Omega)$ of $\hat{\mathcal{C}} := \Omega^{m,k}$, such that $c = c_{\hat{\mathcal{C}}, \Omega}^{(X, \Omega)}$ (defined as in (1)). We call $c$ *closely target-representable* iff there is such an $(X, \Omega)$ with $d\Omega = 0$.

To state our main result, we need the following notions. Let $k, n \in \mathbb{N}_0$, such that $n \geq 2$.

**Definition 8** (helicity). Let $N$ be a closed $(kn-1)$-dimensional manifold, $O$ an orientation on $N$, and $\sigma$ be an exact $k$-form on $N$. We define the helicity of $(N, O, \sigma)$ to be the integral

$$h(N, O, \sigma) := \int_{N, O} \alpha \wedge \sigma \wedge (n-1),$$

where $\alpha$ is an arbitrary primitive of $\sigma$ and $\int_{N, O}$ denotes the integral on $N$ with respect to the orientation $O$. By Lemma 21 in [GZ22] this number is well-defined.

**Remarks 9** (helicity). (See [JZ21, Remark 31].)

- *Helicity vanishes when $k$ is odd.*
- *Helicity is not well-defined for $n = 1.*$  

Let $M$ be a $kn$-dimensional (smooth) manifold with boundary, $O$ an orientation on $M$, and $\omega$ an exact $k$-form on $M$. We define

$$I_M := \{\text{connected components of } \partial M\}.$$ 

For every $i \in I_M$, we denote by $O_i$ the orientation induced on $i$ and by $\omega_i$ the pullback of $\omega$ to $i$.

Assume that $M$ is compact.

---

9Here, the word “weak” refers to the fact that the subcategory need not be isomorphism-closed. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is isomorphism-closed iff every isomorphism in $\mathcal{C}$ starting at some object of $\mathcal{C}'$ is also a morphism in $\mathcal{C}'$.

10This means compact and without boundary.
**Definition.** We define the *boundary helicity* of \((M,O,\omega)\) to be the map \(h_M := h_{M,O,\omega} : I_M \to \mathbb{R}\) given by
\[
h_{M,O,\omega}(i) := h(i,O_i,\omega_i).
\]

**Definition (maxipotency).** Assume that \(k,n \geq 1\). A \(k\)-form \(\omega\) on a \(kn\)-dimensional manifold \(M\) is called *maxipotent* iff \(\omega \wedge n\) is nowhere-vanishing. We call an object \((M,\omega)\) of \(\Omega^{kn,k}\) maxipotent iff \(\omega\) is maxipotent.

**Remark (maxipotent).** (i) If \(n \geq 2\) and \(k\) is odd, we have that \(\omega \wedge \omega = 0\), and hence there are no maxipotent \(k\)-forms.

(ii) A maxipotent \(k\)-form \(\omega\) on a \(kn\)-dimensional manifold induces an orientation on that manifold. The orientation is given by the \(kn\)-form \(\omega \wedge n\), which is nowhere-vanishing, and hence a volume form.

Let \(k,n \in \mathbb{N} := \{1,2,3,\ldots\}\), such that \(n \geq 2\). Let \(\mathcal{M}\) be an uncountable set of compact, 1-connected, exact, and maxipotent objects of \(\Omega^{kn,k}\) with induced volume 1. Assume that every element of \(\mathcal{M}\) has exactly three boundary components, one with helicity 4, and the other two with helicity \(\leq -1\). Moreover, assume that no two elements of \(\mathcal{M}\) are isomorphic.

**Remark.** By Remark 9, if \(k\) is odd, helicity vanishes. Since \(\mathcal{M}\) is a non-empty set of objects with non-zero boundary helicity, it follows that \(k\) is even.

For every set \(X\), we denote by \(\mathcal{P}(X)\) its power set. We define the set
\[
\mathcal{O}^{m,k}_0 := \{(M,\omega) \text{ object of } \Omega^{m,k} \mid \text{The set underlying } M \text{ is a subset of } \mathcal{P}(\mathbb{N}_0)\}.
\]

We denote by \(\widetilde{\mathcal{M}}\) the set of compact, 1-connected, exact, maxipotent elements of \(\mathcal{O}^{kn,k}_0\) of volume (strictly) greater than 1, that have exactly three boundary components, one with helicity at least 4, and two with negative helicity adding up to at least \(-3\).

**Theorem 10 (non-target-representable capacity).** Let \(C\) be a small weak \((kn,k)\)-form category. We denote by \(\mathcal{O}\) its set of objects. Assume that \(\mathcal{O}\) includes \(\mathcal{M} \cup \widetilde{\mathcal{M}}\). Then the capacity
\[
c_{\widetilde{\mathcal{M}}} := \sup_{\widetilde{M} \in \widetilde{\mathcal{M}}} c^{\mathcal{O}^{kn,k}}_{\widetilde{M}}
\]
is not closedly target-representable (as in Definition 7).

For a proof, see page 12.

---

11This means connected and simply connected.

12The reason for requiring the set underlying \(M\) to be included in \(\mathcal{P}(\mathbb{N}_0)\) is to ensure that \(\mathcal{O}^{m,k}_0\) is a set. Here we use the ZF axiom of restricted comprehension.
Remark. For every $\mathcal{M}$ as above, the full subcategory of $\Omega^{kn,k}$ consisting of all positive rescalings of elements of $\mathcal{M} \cup \tilde{\mathcal{M}}$ satisfies the hypotheses of Theorem 10.

Definition. A small weak symplectic category is a small weak $(2n,2)$-form category whose objects are symplectic manifolds.

Theorem 10 has the following immediate consequence.

Corollary. Let $\mathcal{C}$ be a small weak symplectic category whose set of objects includes $\mathcal{M} \cup \tilde{\mathcal{M}}$. Then $c|_{\tilde{\mathcal{M}}}$ is not equal to $c(X,\Omega)$ for any (even disconnected) symplectic manifold $(X,\Omega)$.

This provides some answer to Question 4. To our knowledge, this is the first result concerning this question, apart from [JZ21, theorem on page 8], which states that almost no normalized symplectic capacity on the category of all symplectic manifolds is domain or target-representable.

Example 11 (set $\mathcal{M}$ as in Theorem 10). Let $n \geq 2$. We equip $\mathbb{R}^{2n}$ with the standard symplectic form $\omega_{st}$. We give an example of a set $\mathcal{M}$ as above, whose elements are submanifolds of $\mathbb{R}^{2n}$. We choose a closed $2n$-dimensional “rounded rectangle” $R$ in $\mathbb{R}^{2n}$ whose volume with respect to $\omega_{st}$ is 4. For every $a \in \left[1, \frac{3}{2}\right]$, we choose two closed “rounded rectangles” $R_{0,a}$ and $R_{1,a}$ of volume $a$ and $3-a$, respectively, such that $R_{0,a} \cap R_{1,a} = \emptyset$ and $R_{0,a} \cup R_{1,a} \subseteq \text{Int}(R)$, where $\text{Int}(X)$ denotes the interior of a manifold $X$. We define

$$M_a := R \setminus (\text{Int}(R_{0,a}) \cup \text{Int}(R_{1,a})).$$

(See Figure 1 for an illustration.) We define

$$M := \left\{ (M_a,\omega_{st}|_{M_a}) \Big| a \in \left[1, \frac{3}{2}\right]\right\}.$$

Then $\mathcal{M}$ is an uncountable set of compact, 1-connected, exact objects of $\Omega^{2n,2}$ with volume 1. By Stokes’s theorem for helicity (see Lemma 23), for every $a \in \left[1, \frac{3}{2}\right]$, the boundary components of $M_a$ have helicity $4, -a$, and $-3+a$, respectively. If $a$ and $a'$ are such that $M_a$ and $M_{a'}$ are isomorphic, then $\partial M_a$ and $\partial M_{a'}$ are isomorphic as presymplectic manifolds and hence have the same helicities. It follows that $a = a'$. Hence $\mathcal{M}$ has the desired properties.

The idea of the proof of Theorem 10 is the following. Let $X$ be an object of $\Omega^{m,k}$. Without loss of generality, we may assume that $c^X$ is finite on $\mathcal{M}$. Since $\mathcal{M}$ is uncountable, $X$ has countably many connected components, and the elements of $\mathcal{M}$ are connected, it follows that two

---

13By this we mean a rectangle with rounded corners (of any dimension).
elements $M$ and $M'$ of $\mathcal{M}$ embed\footnote{For two objects $X_0, X_1$ of $\Omega^{m,k}$, we say that $X_0$ embeds into $X_1$ if there exists an $\Omega^{m,k}$-morphism from $X_0$ to $X_1$.} into the same connected component of $X$ after suitably rescaling them. Since no element of $\mathcal{M}$ is isomorphic to another one, one of these embeddings is non-surjective. Without loss of generality, assume that $M$ is the domain of this non-surjective embedding. By adding a bulge to the image of $M$, we construct an element $\tilde{M}_0$ of $\tilde{\mathcal{M}}$ that embeds into $X$ with the same rescaling factor as $M$. Figure 2 illustrates this construction. Stokes’ theorem for helicity (Lemma 23) implies that $c_{\tilde{\mathcal{M}}} (M)$ is small for every $M \in \mathcal{M}$. Comparing the values of $c_{\tilde{\mathcal{M}}}$ and $c^X$ on $\tilde{M}_0$ and the elements of $\mathcal{M}$, it follows that $c_{\tilde{\mathcal{M}}}$ and $c^X$ are not equal.

1.2. Simpler but less interesting examples of non-target-representable capacities. Our next two results provide examples of non-target-representable capacities that are simpler than the capacity in Theorem 10, but defined on categories including closed manifolds as objects.

Let $k, n \in \mathbb{N}$.

**Proposition 12** (non-target-representable capacity on closed manifolds). Let $M := (M, \omega)$ be a non-empty closed connected object of $\Omega^{kn,k}$, such that $\omega$ is maxipotent. Let $c$ be a capacity on some small weak $(kn,k)$-form category whose objects include all disjoint unions of two (positively) rescaled copies of $M$, such that $c$ is finite on each such union. Then $c$ is not target-representable.

For a proof, see page 16.

**Examples 13.** The following capacities satisfy the hypotheses of Proposition 12 and are therefore not target-representable on any category as in this Proposition:

![Figure 1. A manifold $M_a$ as in Example 11. The outside “rounded rectangle” has a volume of 4 while the two inner holes have a combined volume of 3.](image-url)
Figure 2. The construction of $\tilde{M}$ in the proof of Theorem 10. The dark blue manifold is $\varphi(M)$. The light blue bulge is added to construct $\tilde{M}$. The green manifold is $Y$.

- $c = c_Y$ for any non-empty object $Y$ of $\Omega_{kn,k}$.
- The volume capacity, if the given category consists of maxipotent objects.

The idea of the proof of Proposition 12 is the following. Assume that the capacity $c$ is target-represented by an object $X$. Since $c(aM \sqcup M)$ is finite for every $a \in (0, \infty)$, it follows that a rescaled copy of every such disjoint union imbeds into $X$. This implies that there is an uncountable set of rescaled copies of $M$ that embed into $X$. Since $M$ is closed, it follows that the image of each copy of $M$ under such an embedding is a connected component of $X$. Since every rescaled copy of $M$ has a different volume, it follows that no two rescaled copies get mapped to the same connected component. This implies that $X$ has uncountably many connected components. Therefore $X$ is not a (second countable) manifold. The statement follows.

Let $k \geq 1$. The next result gives an example of a non-target-representable capacity on a weak small $(2k, k)$-form category that consists of connected objects, which is not the case in Proposition 14.

**Proposition 14** (non-target-representable capacity on closed connected manifolds). Let $M_0$ and $M_1$ be non-empty closed connected $k$-dimensional manifolds, $\sigma_0$ and $\sigma_1$ be volume forms of volume 1 on $M_0$ and $M_1$, respectively, and $c$ be a capacity on some small weak $(2k, k)$-form category whose objects include the products $(M_0 \times M_1, a_0\sigma_0 \oplus a_1\sigma_1)$ for all $a_0, a_1 \in (0, \infty)$, such that $c$ is finite on each such product. Then $c$ is not target-representable.

For a proof, see page 17.

The idea of the proof of Proposition 14 is the following. It follows from the Degree Theorem that there exist uncountably many products...
of the form \((M_0 \times M_1, a_0\sigma_0 \oplus a_1\sigma_1)\) that are not isomorphic, even after rescaling. Since \(c\) is finite on these products, it follows as in the proof of Proposition 12 that any \(X\) that target-represents \(c\), has uncountably many connected components.

**Examples.** The capacities in Example 13 also fulfill the hypotheses of Proposition 14 and hence are not target-representable on any category as in this proposition.

The following remark provides an example of non-target-representable capacity that is even simpler, but defined on a non-full subcategory of \(\Omega^{2,2}\).

**Remark** (non-target-representable capacity on the restriction category of bounded open subsets of \(\mathbb{R}^2\)). We denote by \(\omega_{st}\) the standard symplectic form on \(\mathbb{R}^2\). We define a category \(C\) whose objects are the bounded open subsets of \(\mathbb{R}^2\) equipped with \(\omega_{st}\) and whose morphisms between two objects \(U\) and \(U'\) are given by the set
\[
\{ \varphi | U \mid \varphi \text{ is a symplectomorphism of } \mathbb{R}^2 \text{ and } \varphi(U) \subseteq U' \}.
\]
We define the **bounded hull** of an object \(U\) of \(C\) to be the union of \(U\) and the bounded components of \(\mathbb{R}^2 \setminus U\). For every object \(U\) of \(C\), we define \(c(U)\) to be the area (Lebesgue measure) of the bounded hull of \(U\). Since the bounded hull construction is monotone, \(c\) is a capacity on \(C\). For \(a > 0\), we denote by \(B(a)\) (\(\overline{B}(a)\)) the open (closed) ball of area \(a\) around 0 in \(\mathbb{R}^2\). The sets \(B(2) \setminus \overline{B}(1)\) and \(B(1) \setminus \{0\}\) are isomorphic in \(\Omega^{2,2}\). This implies that for every object \(X\) of \(\Omega^{2,2}\), we have that \(c^X(B(2) \setminus \overline{B}(1)) = c^X(B(1) \setminus \{0\})\). However, we have that \(c(B(2) \setminus \overline{B}(1)) = 2\) and \(c(B(1) \setminus \{0\}) = 1\). Hence \(c\) is not target-representable.

Here is a silly example of non-target-representable capacity.

**Remark** (non-target-representable capacity on a discrete category). Let \(A\) and \(B\) be two isomorphic objects of \(\Omega^{m,k}\). We consider the small weak \((m,k)\)-form category whose objects are rescaled copies of \(A\) and \(B\) and whose morphisms are the identity morphisms. Then any \(\mathbb{R}^+\)-equivariant function \(c\) on this small category is a capacity. However, if \(c(A) \neq c(B)\) then \(c\) is not target-representable, since \(c^X(A) = c^X(B)\) for every object \((X, \Omega)\) of \(\Omega^{m,k}\).

**Organization of the article.** In Section 2, we prove Theorem 10 and Propositions 12 and 14. Appendix A deals with the general Cartan formula and Stokes’ Theorem for helicity.

2. **Proofs of the results**

For the proof of Theorem 10 we need the following. For two objects \((M, \omega)\) and \((M', \omega')\) of \(\Omega^{m,k}\), we write
\[
(M, \omega) \cong (M', \omega')
\]
iff the two manifolds are isomorphic. We write

\[(M, \omega) \sim (M', \omega')\]

iff there exists a number \(a \in (0, \infty)\), such that \((M, a\omega) \cong (M', \omega')\). The main ingredient of the proof of Theorem 10 is the following proposition.

**Proposition 15** (criteria for non-representability). Let \(k, m \in \mathbb{N}_0\) and \(\mathcal{M}, \widetilde{\mathcal{M}}\) be sets consisting of objects of \(\Omega^{m,k}\) with the following properties:

(a) The elements of \(\mathcal{M}\) are connected.
(b) \(\mathcal{M}\) is uncountable.
(c) \(\forall M, M' \in \mathcal{M}: \text{if } M \sim M', \text{ then } M = M'\).
(d) \(\sup \left\{ c_{\widetilde{M}}(M) \right\} = \sup c_{\tilde{M}}(\widetilde{M}) \) < 1.
(e) For every connected object \(Y\) of \(\Omega^{m,k}\) whose second component is a closed differential form the following holds. If there exists an element \(M \in \mathcal{M}\) and a nonsurjective morphism \(\phi: M \rightarrow Y\), then there exists \(\tilde{M} \in \widetilde{\mathcal{M}}\) and a morphism \(\tilde{M} \rightarrow Y\).

Then the capacity

\[c_{\widetilde{\mathcal{M}}} := \sup_{\tilde{M} \in \widetilde{\mathcal{M}}} c_{\tilde{M}}\]

is not closely target-representable.

For a proof, see page 15.

**Remark 16** (criteria for non-representability). If condition (e) holds for every connected object \(Y\) (not only those with closed differential forms), then the same proof as that of Proposition 15 shows that \(c_{\widetilde{\mathcal{M}}}\) is not target-representable. However, we will not need this version of the proposition. \(\diamondsuit\)

**Remark.** If \(m = 0\), then \(k = 0\) by conditions (a) and (b). Condition (c) then implies that \(\mathcal{M}\) has at most three elements, which contradicts condition (b). It follows that the statement of Proposition 15 is void for \(m = 0\). \(\diamondsuit\)

**Remark 17.** Hypothesis (d) implies that for every manifold \(X\) that target-represents \(c_{\widetilde{\mathcal{M}}}\), we have

\[\sup_{M \in \mathcal{M}} c^X(M) = \sup \left\{ c_{\tilde{M}}(M) \mid M \in \mathcal{M}, \tilde{M} \in \widetilde{\mathcal{M}} \right\} < 1.\]

\(\diamondsuit\)

Let \(\varphi: M \rightarrow M'\) be an embedding between two topological manifolds. We denote

\[I := I_M := \{\text{connected components of } \partial M\};\]
\[I' := I_{M'};\]
\[P := M' \setminus \varphi(\text{Int}(M)).\]
We denote by 
\[ \psi : I \sqcup I' \to \mathcal{P}(P) \]
the map induced by \( \varphi \) on \( I \) and given by the inclusion on \( I' \). We define a partition
\[ \mathcal{P}^{\varphi} \]
on \( I \sqcup I' \) by declaring that two boundary components \( i, j \in I \sqcup I' \) lie in the same element of \( \mathcal{P}^{\varphi} \) if and only if there is a continuous path in \( P \) that starts in \( \psi(i) \) and ends in \( \psi(j) \).

The following three lemmas will also be used in the proof of Theorem 10.

**Lemma 18** (partition induced by an embedding). Assume that \( M \) and \( M' \) have the same dimension, are compact, connected and that \( M \neq \emptyset \). Suppose also that \( M' \) is 1-connected. Then we have for every \( J \in \mathcal{P}^{\varphi} \),
\[ |J \cap I| = 1. \]

**Proof of Lemma 18.** This follows from the proof of Lemma 47(i) in [JZ21]. \( \square \)

**Lemma 19** (helicity inequality). Let \( k, n \in \mathbb{N} \) with \( n \geq 2 \). Let \( M \) and \( M' \) be compact \( kn \)-dimensional smooth manifolds, and \( \omega \) and \( \omega' \) be maxipotent exact \( k \)-forms on \( M \) and \( M' \), respectively. Let \( a \) be a positive real number and \( \varphi : M \to M' \) a smooth orientation-preserving embedding that intertwines \( a\omega \) and \( \omega' \). We denote by \( O \) and \( O' \) the orientations of \( M \) and \( M' \) induced by \( \omega \) and \( \omega' \). Then, for every \( J \in \mathcal{P}^{\varphi} \) the following inequality holds:
\[ -a^n \sum_{i \in J \cap I} h_{M,O,\omega}(i) + \sum_{i' \in J \cap I'} h_{M',O',\omega'}(i') \geq 0. \]

**Proof.** This is Lemma 49 in [JZ21]. \( \square \)

**Lemma 20.** Let \( X \) and \( Y \) be topological manifolds, such that \( X \) is compact and non-empty and \( Y \) is connected. Let \( \varphi : X \to Y \) be a continuous injective map, such that \( \varphi(\partial X) \subseteq \partial Y \). Then \( \varphi \) is surjective.

**Proof.** Consider the equivalence relation \( \sim \) on \( X \sqcup X := (\{0,1\} \times X) \) given by \( (i,x) \sim (i,x), \forall x \in X, i = 0,1 \), and \( (0,x) \sim (1,x), (1,x) \sim (0,x), \forall x \in \partial X \). We define \( \hat{X} \) to be the quotient
\[ X \sqcup X / \sim. \]

We define \( \hat{Y} \) analogously. By [Lee13, Theorem 9.29], \( \hat{X} \) and \( \hat{Y} \) are canonically topological manifolds without boundary. We denote by \( \varphi \sqcup \varphi : X \sqcup X \to Y \sqcup Y \) the map induced by \( \varphi \) on the disjoint union. We define
\[ \hat{\varphi} : \hat{X} \to \hat{Y} \]
\[ [x] \mapsto [(\varphi \sqcup \varphi)(x)]. \]

\( ^{15} \)In this lemma, it is assumed that \( \partial M' \neq \emptyset \). However, this is not necessary.
By our assumption that $\varphi(\partial X) \subseteq \partial Y$, this map is well-defined. Since $\varphi$ is continuous, $\hat{\varphi}$ is continuous. By our assumption that $\varphi$ is continuous and injective and by Brouwer’s invariance of domain theorem, it follows that $\hat{\varphi}$ is injective. Since $\hat{X}$ and $\hat{Y}$ do not have boundary, it follows from the invariance of domain theorem that $\hat{\varphi} (\hat{X})$ is open in $\hat{Y}$. This implies that $\varphi(X)$ is open in $Y$. Since $X$ is compact, $\varphi(X)$ is compact. Since $Y$ is Hausdorff, $\varphi(X)$ is closed in $Y$. Since $Y$ is connected and $X$ non-empty, it follows that $\varphi(X) = Y$. This concludes the proof. □

Proof of Theorem 10. We show that $\mathcal{M}$ and $\tilde{\mathcal{M}}$ fulfill the conditions of Proposition 15. Conditions (a) and (b) hold by definition of $\mathcal{M}$. Condition (c) holds because every element of $\mathcal{M}$ has volume 1, but no two elements of $\mathcal{M}$ are isomorphic. Condition (d) follows from:

Claim 1. For every $\tilde{M} \in \tilde{\mathcal{M}}$, we have that

$$c_{\tilde{M}}(M) \leq a_0 := \sqrt[4]{\frac{3}{4}}.$$ 

Proof of Claim 1. Let $a \in (0, \infty)$ be such that there exists a morphism $\varphi : a\tilde{M} \to M$.[16] We denote by $I$ and $\tilde{I}$ the set of connected components of the boundary of $M$ and $\tilde{M}$, respectively, by $i_0 \in I$ the boundary component of $M$ with helicity 4, by $\tilde{i}_0 \in \tilde{I}$ the boundary component of $\tilde{M}$ with positive helicity, and by $J_0$ the partition element containing $\tilde{i}_0$.

Claim 2. We have that

(i) $i_0 \in J_0$,
(ii) $J_0 \setminus \{\tilde{i}_0, i_0\} \neq \emptyset$.

We denote by $h : I \to \mathbb{R}$ and $\tilde{h} : \tilde{I} \to \mathbb{R}$ the boundary helicity maps for $M$ and $\tilde{M}$ and by $I_-$ the set of boundary components of $M$ that have negative helicity. By Lemma 18 every partition element $J \in \mathcal{P}_\varphi$ (as defined in (2)) contains exactly one element of $\tilde{I}$. Hence, Lemma 19 with $J_0$ yields

$$-a^\circ \tilde{h}(\tilde{i}_0) + \sum_{i \in J_0 \cap I} h(i) \geq 0. \quad (3)$$

Proof of Claim 2. Since the first term on the left of (3) is negative, the sum on the left is positive, which implies that $i_0 \in J_0$. This proves (i). We denote by $\tilde{I}_-$ the set of boundary components of $\tilde{M}$ that have negative helicity. By our assumption on $\mathcal{M}$ and by definition of $\tilde{\mathcal{M}}$, we have that

$$\sum_{i \in \tilde{I}_-} \tilde{h}(\tilde{i}) = -3 \leq \sum_{i \in I_-} h(i), \quad (4)$$



[16]If no such $a$ exists, then $c_{\tilde{M}}(M) = 0$ and the claim holds.
Since the volume of $\tilde{M}$ is strictly bigger than that of $M$ and $a\tilde{M} \hookrightarrow M$ via $\varphi$, we have that $a < 1$. Combining this with equation (4), it follows that
\[
\sum_{i \in I_\varphi} h(i) < a^n \sum_{\tilde{i} \in I_\varphi} \tilde{h}(\tilde{i}).
\] (5)

By using Lemma 19 with every $J \in \mathcal{P}_J \setminus \{J_0\}$, summing up the inequalities, and using Claim 2(i), we get
\[
-a^n \sum_{\tilde{i} \in \tilde{I}} \tilde{h}(\tilde{i}) + \sum_{i \in I_\varphi \setminus \{J_0\}} h(i) \geq 0.
\]
Combining this with (5), it follows that
\[
\sum_{i \in I_\varphi} h(i) < \sum_{i \in I_\varphi \setminus \{J_0\}} h(i).
\]
It follows that $J_0 \cap I_\varphi \neq \emptyset$. Statement (ii) follows. This concludes the proof of Claim 2. □

By Lemma 18 applied with $\varphi : \tilde{M} \to M$ and $J_0$ and the fact that $J_0 \subseteq \tilde{I} \sqcup I$, we have that $J_0 \setminus \{\tilde{i}_0\} = J_0 \cap I$. It follows that
\[
a^n \leq \frac{h(i_0) + \sum_{i \in J_0 \setminus \{i_0, i_0\}} h(i)}{h(i_0)} \quad \text{(by (3) and Claim 2(i))}
\]
\[
\leq \frac{h(i_0) - 1}{h(\tilde{i}_0)} \quad \text{(by Claim 2(ii) and the definition of } \mathcal{M})
\]
\[
\leq \frac{3}{4} \quad \text{(since } \tilde{h}(\tilde{i}_0) \geq 4 = h(i_0))
\]
The statement of Claim 1 follows. □

We prove that condition (e) holds. Let $M \in \mathcal{M}$. Let $(Y, \Omega)$ be a connected object of $\Omega^{kn,k}$, such that $d\Omega = 0$ and there exists a non-surjective morphism $\varphi : M \to Y$. By Lemma 20, we have that there exists a point $x_0 \in \text{Int}(Y) \cap \varphi(\partial M)$. Since $\varphi$ is a smooth embedding, there exists a smooth parametrization $\psi : \mathbb{R}^{kn} \to U$ for $\text{Int}(Y)$ around $x_0$, such that $\Omega|_U$ is maxipotent, $\psi(0) = x_0$, $\psi(\mathbb{R}^{kn-1} \times \{0\}) = U \cap \varphi(\partial M)$ and $\psi(\mathbb{R}^{kn-1} \times (-\infty, 0]) = U \cap \varphi(M)$.

For $m \in \mathbb{N}$, a point $p \in \mathbb{R}^m$ and $r > 0$, we denote by $B^m_r(p)$ and $\overline{B}^m_r(p)$ the open and closed balls in $\mathbb{R}^m$ of radius $r$ around $p$. Let $\rho : B^{kn-1}_1(0) \to [0, \infty)$ be a smooth function with compact support, such that $\rho(0) > 0$.

We define
\[
K := \{(x_1, \ldots, x_{kn}) \in \mathbb{R}^{kn} \mid (x_1, \ldots, x_{kn-1}) \in \overline{B}^{kn-1}_1(0) \text{ and } 0 \leq x_{kn} \leq \rho(x_1, \ldots, x_{kn-1})\}.
\]
The set $K$ is closed and bounded in $\mathbb{R}^{kn}$ and hence compact. We define

$$\widetilde{M} := \psi(K) \cup \varphi(M).$$

This is a submanifold with boundary of $Y$. We equip $\widetilde{M}$ with the restriction of $\Omega$. By [JZ21, p.12, footnote 28], there exists a manifold $\widetilde{M}_0 \in \Omega_{kn,k}^0$ which is isomorphic to $\widetilde{M}$.

**Claim 3.** We have that $\widetilde{M}_0 \in \widetilde{M}$.

**Proof.** We check the corresponding conditions for $\widetilde{M}$. Since both $\psi(K)$ and $\varphi(M)$ are compact, it follows that $\widetilde{M}$ is compact.

**Claim 4.** $\widetilde{M}$ is 1-connected and $\Omega|_{\widetilde{M}}$ is exact.

**Proof of Claim 4.** We choose a smooth function $\eta : \mathbb{R}^{kn} \to [0,1]$ that is equal to 1 on $K$ and vanishes outside an open neighborhood of $K$. We define

$$h : [0,1] \times \mathbb{R}^{kn} \to \mathbb{R}^{kn},$$

$$(t, (x_1, \ldots, x_{kn})) \mapsto (x_1, \ldots, x_{n-1}, x_{kn} - \eta(x_1, \ldots, x_{kn})tx_{kn}).$$

The map $h$ is smooth and maps $K$ to $B_{kn-1}(0)$. We define

$$f : [0,1] \times \widetilde{M} \to \widetilde{M},$$

$$f_t := \begin{cases} 
\psi \circ h_t \circ \psi^{-1} & \text{on } U \\
\text{id} & \text{otherwise}
\end{cases}$$

The map $f_1$ is a smooth homotopy equivalence between $\widetilde{M}$ and $M$. It follows that $\widetilde{M}$ has the same homotopy type as $M$, and hence is 1-connected.

By Cartan’s general magic formula (Proposition 22) and our assumption that $d\Omega = 0$, there exists a smooth family $(\alpha_t)_{t \in [0,1]}$ of $(k-1)$-forms on $\widetilde{M}$, such that for every $t \in [0,1]$,

$$\frac{d}{dt} f_t^* (\Omega|_{\widetilde{M}}) = d\alpha_t.$$ 

It follows that

$$f_1^* (\Omega|_{\widetilde{M}}) - f_0^* (\Omega|_{\widetilde{M}}) = \int_0^1 d\alpha_t = d\int_0^1 \alpha_t.$$ 

(7)

By assumption, $\omega$ is exact on $M$. Therefore $\varphi_* \omega$ is exact on $\varphi(M)$. Since $\Omega|_{\varphi(M)} = \varphi_* \omega$ and $f_1(M) \subseteq \varphi(M)$, it follows that $f_1^* (\Omega|_{\widetilde{M}})$ is exact. Using equation (7), it follows that $f_0^* (\Omega|_{\widetilde{M}})$ is exact. Since $f_0 = \text{id}$, we have $f_0^* (\Omega|_{\widetilde{M}}) = \Omega|_{\widetilde{M}}$. This proves Claim 4. □

By construction, $\Omega|_{\widetilde{M}}$ is maxipotent.

We equip the boundary of $\widetilde{M}$ with the orientation induced by the form $\omega^{kn}$ on $M$. We show that $\widetilde{M}$ has the right volume and boundary.
helicity. By construction, the volume of $\tilde{M}$ is strictly greater than that of $M$, which is equal to 1. By Stokes’ Theorem for helicity (Lemma 23), the helicity of the boundary component of $\tilde{M}$ intersecting $\psi(K)$ has been increased by the difference between the volume of $\tilde{M}$ and the volume of $M$, while the other boundary components of $\tilde{M}$ have the same helicity as the corresponding boundary components of $M$. Hence, the positive boundary component has helicity at least 4 and the negative helicities sum up to at least -3.

This proves Claim 3. □

By construction, $\tilde{M} \subseteq Y$ and the inclusion is a morphism $(\tilde{M}, \Omega|_{\tilde{M}}) \hookrightarrow (Y, \Omega)$. By Claim 3, it follows that condition [e] holds. Thus, we may apply Proposition 15. It follows that $c_{\tilde{M}}$ is not closely target-representable. This concludes the proof of Theorem 10. □

Proof of Proposition 15. Let $X$ be an object of $\Omega^{m,k}$, such that

$$a_0 := \sup_{M \in \mathcal{M}} c^X(M) < \infty.$$  

We define

$$\varepsilon := 1 - \sup \left\{ c_{\tilde{M}}(M) \mid M \in \mathcal{M}, \tilde{M} \in \tilde{M} \right\}.$$  

By assumption (d) we have $\varepsilon > 0$. Let $M \in \mathcal{M}$. We have $c^X(M) \leq a_0$. Hence there exist $a_M \in (0, a_0 + \varepsilon)$ and a morphism $\varphi_M : M \hookrightarrow a_M X$.

Using hypothesis (a) the map $M \mapsto \varphi_M$ induces a map

$$\mathcal{M} \rightarrow \{\text{connected components of } X\}$$ $$M \mapsto \text{unique } Y \text{ containing } \varphi_M(M).$$

Since $X$ is second countable, it has countably many connected components. Using hypothesis (b) it follows that the map defined in (10) is not injective. Hence there exist $M, M' \in \mathcal{M}$ and a connected component $Y$ of $X$, such that $M \neq M'$ and $\varphi_M(M)$ and $\varphi_M'(M')$ are both included in $Y$.

Assume $\varphi_M'$ is surjective. Since $M \neq M'$, hypothesis (c) implies that $\varphi_M^{-1} \circ \varphi_M$ is not an isomorphism from $a_M M$ to $a_M' M'$. Since $\varphi_M'$ is an isomorphism, it follows that $\varphi_M$ is not an isomorphism, and hence is not surjective. It follows that $\varphi_M$ and $\varphi_M'$ are not both surjective.

Without loss of generality, assume that $\varphi_M$ is not surjective. This map is a morphism $M \hookrightarrow a_M Y$. Hence by hypothesis (e) there exists $\tilde{M}_0 \in \tilde{M}$, such that $\tilde{M}_0 \hookrightarrow a_M Y \subseteq a_M X$. It follows that

$$c^X(\tilde{M}_0) \leq a_M$$

We also have

$$c_{\tilde{M}}(\tilde{M}_0) \geq c_{\tilde{M}_0}(\tilde{M}_0) \geq 1.$$  

\begin{footnote}{17} By Remark 17, we do not need to consider the case $a_0 = \infty$. \end{footnote}
We compute
\[ c_{\widetilde{\mathcal{M}}} (\widetilde{M}_0) - c^X (\widetilde{M}_0) - \sup_{M \in \mathcal{M}} c_{\widetilde{\mathcal{M}}} (M) + \sup_{M \in \mathcal{M}} c^X (M) \geq 1 - a_M + \varepsilon - 1 + a_0 \] (by (12, 11, 9, 8))
\[ > 0. \] (since \( a_M \in (0, a_0 + \varepsilon) \))

It follows that \( c_{\widetilde{\mathcal{M}}} \neq c^X \). This concludes the proof of Proposition 15. \( \square \)

We now give the proofs of Propositions 12 and 14. The following observation is important for both proofs.

Remark 21 (image of a closed manifold under an embedding). Let \( M \) be a non-empty closed connected topological manifold and \( N \) a topological manifold of the same dimension. Let \( \varphi : M \hookrightarrow N \) be a topological embedding. The image of \( \varphi \) is a connected component of \( Y \). Indeed, by invariance of the domain, and using that \( M \) has no boundary, we have that \( \varphi (M) \) is open in \( N \). Since \( M \) is compact, \( \varphi (M) \) is compact. Since \( N \) is Hausdorff, it follows that \( \varphi (M) \) is closed in \( N \). Hence, since \( M \neq \emptyset \), we have that \( \varphi (M) \) is a connected component of \( N \). \( \diamondsuit \)

Proof of Proposition 12. Let \( X \) be an object of \( \Omega^{kn,k} \), where we omit the differential form from the notation. We show that \( c^X (aM \sqcup M) \) is infinite for some \( a \in (0, \infty) \). We denote by \( \pi_0 (X) \) the set of (path-)connected components of \( X \). We define the set
\[ S := \left\{ (a, b) \in (0, \infty)^2 \left| \frac{a}{b} M \hookrightarrow X, \frac{1}{b} M \hookrightarrow X \right. \right\}. \]

Claim 1. The set \( S \) is countable.

Proof of Claim. We define the set
\[ S' := \{ A \in (0, \infty) \mid \exists X_0 \in \pi_0 (X) : A M \cong X_0 \} \]
and the map
\[ f : S \rightarrow S' \times S', \]
\[ (a, b) \mapsto \left( \frac{a}{b}, \frac{1}{b} \right). \]

We show that the map \( f \) is well-defined. Let \((a, b) \in S \). We choose a morphism \( \varphi : \frac{a}{b} M \hookrightarrow X \). Since \( M \) is non-empty, closed, and connected, by Remark 21 there exists a connected component \( X_0 \) of \( X \), such that \( \varphi (M) = X_0 \). Hence \( \varphi \) is an isomorphism between \( \frac{a}{b} M \) and \( X_0 \). Similarly, there exists a connected component \( X_1 \) of \( X \), such that \( \frac{1}{b} M \cong X_1 \). This implies that the map \( f \) is well-defined.

A straight-forward argument shows that \( f \) is injective.

\(^{18}\) If \( S \) is empty, then Claim 1 holds.
Let \( A \in S' \). We choose a connected component \( X_0 \) of \( X \), such that \( AM \cong X_0 \). Since the form \( \omega \) on \( M \) is maxipotent, it follows that the form on \( X \) restricted to \( X_0 \) is maxipotent. Using \( AM \cong X_0 \) again, we have that \( A^n \text{Vol}(M) = \text{Vol}(X_0) \). It follows that \( S' \) is contained in the image of the map that sends each maxipotent connected component \( X_0 \) of \( X \) to \( \sqrt[n]{\frac{\text{Vol}(X_0)}{\text{Vol}(M)}} \). Since \( X \) is a manifold, \( \pi_0(X) \) is countable. It follows that \( S' \) is countable. Since \( f \) is injective, it follows that \( S \) is countable. This proves Claim 1.  

The set \[
\left\{ (a, b) \in (0, \infty)^2 \mid c^X(aM \sqcup M) < \infty \right\}
\]
is included in the image of \( S \) under the projection to the first component. Therefore, by Claim 1 it is countable. It follows that the complement of this set is uncountable, and hence non-empty. Hence \( c^X(aM \sqcup M) \) is infinite for some \( a \in (0, \infty) \). The statement of Proposition 12 follows.  

Proof of Proposition 14. Let \( X \) be an object of \( \Omega^{2k,k} \). We denote by \( \pi_0(X) \) the set of (path-) connected components of \( X \). We define the set \[
S := \left\{ (a, b) \in (0, \infty)^2 \mid \frac{a}{b} M_0 \times \frac{1}{b} M_1 \hookrightarrow X \right\}.
\]

Claim 1. The set \( S \) is countable.  

Proof of Claim 1. By Remark 21, since \( M_0 \) and \( M_1 \) are non-empty, closed and connected, we have that \[
(a, b) \in S \implies \exists X_0 \in \pi_0(X) : \frac{a}{b} M_0 \times \frac{1}{b} M_1 \cong X_0.
\] (13)

Let \( X_0 \) be a connected component of \( X \). We show that the set \[
S_{X_0} := \left\{ (a, b) \in (0, \infty)^2 \mid \frac{a}{b} M_0 \times \frac{1}{b} M_1 \cong X_0 \right\}
\]
is countable. We define the function \[
f := f_{X_0} : S_{X_0} \to \mathbb{R},
\]
\[
(a, b) \mapsto \frac{a}{b}.
\]

Assume that \( S_{X_0} \) is non-empty\(^{21} \). We choose \( (a, b) \in S_{X_0} \). We have that \( \frac{a}{b} M_0 \times \frac{1}{b} M_1 \cong X_0 \). Since \( M_0 \) and \( M_1 \) are maxipotent, it follows

\(^{19}\)If \( S' \) is empty, then \( S \) is also empty and Claim 1 holds.

\(^{20}\)Recall that a maxipotent form induces a volume form. We denote by \( \text{Vol}(M) \) the induced volume of \( M \).

\(^{21}\)Otherwise, \( S_{X_0} \) is countable, which is what we want to show.
that the restriction of the form on \( X \) to \( X_0 \) is maxipotent. We have that
\[
b = \frac{f(a, b)}{\text{Vol}(X_0)}.
\]
This implies that \( f \) is injective. Moreover, we have that
\[
\text{im}(f) \subseteq \left( \frac{a}{b} \mathbb{Z} + \frac{1}{b} \mathbb{Z} \right).
\]
This follows from the fact that \( \text{Vol}(M_i) = 1 \), for \( i = 0, 1 \), and from the following claim.

**Claim 2.** If \( a, b, a', b' \in (0, \infty) \) are such that \( \frac{a}{b} M_0 \times \frac{1}{b} M_1 \cong \frac{a'}{b'} M_0 \times \frac{1}{b'} M_1 \) then we have that
\[
\frac{a'}{b'} \in \frac{a}{b} \mathbb{Z} + \frac{1}{b} \mathbb{Z}.
\]

**Proof of Claim 2.** We denote
\[
\omega := \frac{a}{b} \sigma_0 + \frac{1}{b} \sigma_1 \quad \text{and} \quad \omega' := \frac{a'}{b'} \sigma_0 + \frac{1}{b'} \sigma_1,
\]
and by \( \text{pr}_i : M_0 \times M_1 \to M_i \) the canonical projection, \( i = 0, 1 \). We choose an isomorphism \( \varphi : \frac{a}{b} M_0 \times \frac{1}{b} M_1 \to \frac{a'}{b'} M_0 \times \frac{1}{b'} M_1 \) with respect to \( \omega' \) and \( \omega \). We fix a point \( x_1 \in M_1 \) and define
\[
\iota_0 : M_0 \to M_0 \times M_1 \quad \quad x \mapsto (x, x_1).
\]
We have that
\[
\frac{a'}{b'} = \text{Vol} \left( \frac{a'}{b} \sigma_0 \right) = \text{Vol}(\iota_0^* \omega') = \text{Vol}(\iota_0^* \varphi^* \omega)
\]
\[
= \int_{M_0} \left( \frac{a}{b} (\text{pr}_0 \circ \varphi \circ \iota_0)^* \sigma_0 + \frac{1}{b} (\text{pr}_1 \circ \varphi \circ \iota_0)^* \sigma_1 \right)
\]
\[
= \frac{a}{b} \deg(\text{pr}_0 \circ \varphi \circ \iota_0) \int_{M_0} \sigma_0 + \frac{1}{b} \deg(\text{pr}_1 \circ \varphi \circ \iota_0) \int_{M_1} \sigma_1 \quad \text{(Degree Thm)}
\]
\[
\in \frac{a}{b} \mathbb{Z} + \frac{1}{b} \mathbb{Z} \quad \quad \text{(since } \sigma_0 \text{ and } \sigma_1 \text{ have volume 1)}
\]
This proves Claim 2. \( \square \)

By [11] \( f \) has a countable image. Since \( f \) is injective, it follows that \( S_{X_0} \) is countable, as desired. Since \( \pi_0(X) \) is countable, it follows that \( \bigcup_{X_0 \in \pi_0(X)} S_{X_0} \) is countable. Using [13], it follows that \( S \) is countable. This proves Claim 4. \( \square \)

We have
\[
\{ a \in (0, \infty) \mid c^X(aM_0 \times M_1) < \infty \}
\]
\[
= \{ a \in (0, \infty) \mid \exists b \in (0, \infty) : (a, b) \in S \}.
\]
By Claim 1, this set is countable. Hence its complement is uncountable and thus non-empty. Hence $c_X$ is not finite on every product $(aM_0 \times M_1), a \in (0, \infty)$. The statement of Proposition 14 follows. 

**Appendix A. General Cartan Formula and Stokes’ Theorem for Helicity**

We state the general Cartan formula that we used in the proof of Theorem 10. Let $I$ be an interval, $M, N$ smooth manifolds, $f : I \times M \to N$ a smooth function, $k \in \mathbb{N}$, and $\omega \in \Omega^k(N)$. For every $t \in I$, we define $\alpha^t \in \Omega^{k-1}(M)$ and $\beta^t \in \Omega^k(M)$ by

$$
\alpha^t(x, v_1, \ldots, v_{k-1}) := \omega(\partial_t f_t(x), Df_t(x)v_1, \ldots, Df_t(x)v_{k-1}),
\beta^t(x, v_1, \ldots, v_k) := d\omega(\partial_t f_t(x), Df_t(x)v_1, \ldots, Df_t(x)v_k).
$$

**Proposition 22 (General Cartan formula).** We have

$$
\frac{d}{dt} f_t^* \omega = \beta^t + d\alpha^t, \quad \forall t \in I.
$$

This result follows from an elementary argument involving the Leibniz rule.

We state and prove Stokes’ theorem for helicity, which we used in Example 11 and the proof of Theorem 10. Let $k, n \in \mathbb{N}$, such that $n \geq 2$. Let $(M, O)$ be a compact, oriented manifold of dimension $kn$, and $\omega$ be an exact $k$-form on $M$. We denote by $\omega_{\partial M}$ the pullback of $\omega$ by the canonical inclusion of $\partial M$ into $M$ and by $O_{\partial M}$ the orientation of $\partial M$ induced by $\omega^{\wedge n}$.

**Lemma 23 (Stokes’ Theorem for Helicity).** The following equality holds:

$$
\int_{M,O} \omega^{\wedge n} = h(\partial M, O_{\partial M}, \omega_{\partial M}).
$$

**Proof of Lemma 23.** Let $\alpha$ be a primitive of $\omega$. Then, we have that $\omega^{\wedge n} = d(\alpha \wedge \omega^{\wedge (n-1)})$. Using Stokes’ Theorem, we get

$$
\int_{M,O} \omega^{\wedge n} = \int_{\partial M, O_{\partial M}} \alpha \wedge \omega^{\wedge (n-1)} = h(\partial M, O_{\partial M}, \omega_{\partial M}).
$$

This completes the proof. 

**References**

[CHLS07] Kai Cieliebak, Helmut Hofer, Janko Latschev, and Felix Schlenk, *Quantitative symplectic geometry*, Dynamics, Ergodic Theory, and Geometry (Boris Hasselblatt, ed.), Math. Sci. Res. Inst. Publ., vol. 54, Cambridge University Press, 2007.

[GZ22] Yann Guggisberg and Fabian Ziltener, *Recognition of objects through symplectic capacities*, Differential Geometry and its Applications 84 (2022), 101903.
[JZ21] Dušan Joksimović and Fabian Ziltener, \textit{Generating systems and representability for symplectic capacities}, 2021.

[Lee13] John M. Lee, \textit{Introduction to smooth manifolds}, 2 ed., Springer New York, NY, 2013.

\textbf{Affiliation of Y. Guggisberg:} Utrecht University, mathematics institute, Hans Freudenthalgebouw, Budapestlaan 6, 3584 CD Utrecht, The Netherlands
\textit{Email address:} y.b.guggisberg@uu.nl

\textbf{Affiliation of F. Ziltener:} Utrecht University, mathematics institute, Hans Freudenthalgebouw, Budapestlaan 6, 3584 CD Utrecht, The Netherlands
\textit{Email address:} f.ziltener@uu.nl