Existence of Bogomol’nyi solitons via Floer theory

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March 29, 2022

Abstract

In this paper we use Floer homology to prove existence of solutions of a generalized Abelian Higgs equation introduced by Schroers.

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1 Introduction

A Floer homology is the Morse homology of an action functional \( A \) defined on the infinite dimensional space \( C^\infty(N, M) \) of smooth maps from a manifold \( N \) to a target manifold \( M \). The Floer chain complex is generated by critical points of \( A \) which are smooth maps from \( N \) to \( M \) solving a PDE (respectively an ODE if \( N \) is 1-dimensional). The Floer boundary operator is defined by counting gradient flow lines of \( A \). These are smooth maps from \( N \times \mathbb{R} \) to \( M \) again solving a PDE. In the special case where \( N = S^1 \) is the circle and \( A \) is the classical Hamiltonian action functional the critical points of \( A \) are the periodic orbits of the Hamiltonian system. This was used by Floer in his proof of the Arnold conjecture \([4, 5, 6, 7]\).

Floer flow lines satisfy a Bogomol’nyi type selfduality equation and hence the solutions of various problems in physics can be interpreted as Floer flow lines. In this paper we prove existence of solutions of a generalized Abelian Higgs Equation discovered by Schroers \([12]\). This equation is a special case of the symplectic vortex equations introduced by Cieliebak, Gaio, Mundet, and Salamon \([2, 3]\).

The method of our proof is the following. We consider an action functional \( A \) which is invariant under the action of a finite group \( \Gamma \) which acts freely on the critical set of \( A \). This gives rise to two Floer homologies where the chain complex of the first one is generated by critical points of \( A \) and the chain complex of the second one is generated by \( \Gamma \)-orbits of critical points of \( A \). We show that the first Floer homology is trivial and the second one is non-trivial which implies the existence of Floer flow lines.

This paper is organized as follows. In Section 2 we show how Floer flow lines can be interpreted as selfduality equations and how they are related to problems in physics. In Section 3 we introduce our Floer action functional and state our main theorem. In Section 4 we first explain how one can find Floer flow lines if one is able to compute two Floer homologies. We then introduce our two Floer homologies and compute them. We finally consider the classical vortex equations as an example.

Acknowledgements: I would like to thank Dietmar Salamon for useful discussions. This paper was written during my stay at FIM of ETH Zürich. I wish to thank FIM for its kind hospitality.

2 Floer flow lines as selfduality equations

Assume that \((M, g)\) is a Riemannian manifold and \( f \in C^\infty(M) \) is a Morse function on \( M \). For two critical points \( c_-, c_+ \in \text{crit}(f) \) let

\[
\Omega(c_-, c_+) = \{ x \in C^\infty(\mathbb{R}, M) : \lim_{s \to \pm \infty} x(s) = c_\pm \}
\]
be the space of paths from \(c_-\) to \(c_+\). The energy functional \(E: \Omega(c_-, c_+) \to \mathbb{R} \cup \{\infty\}\) is defined by

\[
E(x) = \frac{1}{2} \int_{\mathbb{R}} \left( \|\partial_s x\|^2 + \|\nabla_g f(x)\|^2 g \right) ds.
\]

The energy functional satisfies the following Bogomol’nyi equation [1],

\[
E(x) = \frac{1}{2} \int_{\mathbb{R}} \|\partial_s x + \nabla_g f(x)\|^2 ds + f(c_-) - f(c_+) \quad (1)
\]

It follows from (1) that downwards and upwards gradient flow lines, i.e. solutions of the following first order ODE

\[
\partial_s x = \pm \nabla_g f(x) \quad (2)
\]

are the absolute minimizers of the energy functional. Note that the Euler-Lagrange equations for \(E\) are given by the following second order ODE

\[
\nabla^2_s x = H_f(x) \nabla_g f(x) \quad (3)
\]

where \(H_f\) denotes the Hessian of \(f\). Of course solutions of (2) are also solutions of (3). This can be directly checked by differentiating (2). Equations describing the absolute minima of a functional which are first order while the Euler-Lagrange equations of the functional are second order are called (anti)-selfduality equations.

Selfduality equations are of great importance in physics. We show how some of the selfduality equations physicists are interested in can be interpreted as gradient flow lines of an action functional on an infinite dimensional space. The basic example are the classical vortex equations on the cylinder, see [11] as a basic reference. To describe them recall that the standard \(S^1\)-action on the complex plane \(\mathbb{C}\) given by

\[
z \mapsto e^{i\theta} z, \quad e^{i\theta} \in S^1
\]

is Hamiltonian with respect to the standard symplectic structure \(\omega = dx \wedge dy\) on \(\mathbb{C}\). A moment map for this action is given by

\[
\mu(z) = -\frac{1}{2} |z|^2 + \frac{i}{2} \in i\mathbb{R} = \text{Lie}(S^1).
\]

The vortex equations on the cylinder for a triple \((v, \eta, \zeta) \in C^\infty(\mathbb{R} \times S^1, \mathbb{C} \times i\mathbb{R} \times i\mathbb{R})\) are

\[
\begin{align*}
\partial_s v + \zeta v + i(\partial_v v + \eta v) &= 0, \\
\partial_s \eta - \partial_v \zeta + \mu(v) &= 0.
\end{align*}
\]

(4)

If one thinks of \(A = \zeta ds + \eta dt\) as a connection on the trivial bundle over the cylinder then the second equation in (4) reads

\[
* F_A + \mu(v) = 0
\]
where $*$ denotes the Hodge operator on the cylinder $\mathbb{R} \times S^1$ and $F_A$ is the curvature of the connection $A$. The gauge group
\[ \mathcal{G} = C^\infty(\mathbb{R} \times S^1, S^1) \]
acts on the solutions of (4) via
\[ g_*(v, \eta, \zeta) = (gv, \eta - g^{-1}\partial_t g, \zeta - g^{-1}\partial_s g). \]
In particular, every solution of (4) is gauge equivalent to a solution of (4) with $\zeta \equiv 0$, i.e. a pair $(v, \eta) \in C^\infty(\mathbb{R} \times S^1, \mathbb{C} \times i \mathbb{R})$ satisfying
\[
\begin{align*}
\partial_s v + i(\partial_t v + \eta v) &= 0, \\
\partial_s \eta + \mu(v) &= 0.
\end{align*}
\]
(5)
The equations (5) are called vortex equations in temporal gauge. The gauge group
\[ \mathcal{H} = C^\infty(S^1, S^1) \]
acts on solutions of (5) by
\[ h_*(v, \eta) = (hv, \eta - h^{-1}\partial_t h) \]
and the map $(v, \eta) \mapsto (v, \eta, 0)$ from solutions of (5) to solutions of (4) induces a bijection
\[
\{ \text{solutions of (5)} \}/\mathcal{H} \cong \{ \text{solutions of (4)} \}/\mathcal{G}.
\]
The equation (5) can be interpreted as the gradient flow equation of an action functional $A$ defined on an infinite dimensional space $\mathcal{L}$. Set
\[ \mathcal{L} = C^\infty(S^1, \mathbb{C} \times i \mathbb{R}) \]
and define $A: \mathcal{L} \to \mathbb{R}$ as
\[ A(v, \eta) = \int_0^1 \lambda(v)\partial_t v + \int_0^1 \langle \mu(v), \eta \rangle dt \]
where $\lambda = ydx$ is the Liouville one-form satisfying $d\lambda = -\omega$. Then the gradient flow lines of $A$ with respect to the $L^2$-metric $g_{L^2}$ on $\mathcal{L}$, i.e.
\[
\partial_s (v, \eta) + \nabla_{g_{L^2}} A(v, \eta) = 0
\]
are precisely the equations (5).

Another example are the self-dual Chern-Simons vortices discovered by Hong-Kim-Pac and Jackiw-Weinberg, see [9, 10]. They read
\[
\begin{align*}
\partial_s v + \zeta v + i(\partial_t v + \eta v) &= 0, \\
\partial_s \eta - \partial_t \zeta + |v|^2 \mu(v) &= 0.
\end{align*}
\]
(6)
In temporal gauge they are given by
\[
\begin{align*}
\partial_s v + i(\partial_t v + \eta v) &= 0, \\
\partial_s \eta + |v|^2 \mu(v) &= 0.
\end{align*}
\]
(7)
The equations (7) are again gradient flow lines of $A$, but with respect to a different metric, namely the following warped product metric on $\mathcal{L} = C^\infty(S^1, \mathbb{C}) \times C^\infty(S^1, \times i\mathbb{R})$ given for $(v, \eta) \in \mathcal{L}$ and $(\hat{v}_1, \hat{\eta}_1), (\hat{v}_2, \hat{\eta}_2) \in T_{(v, \eta)} \mathcal{L} \cong \mathcal{L}$ by the formula

$$g(v, \eta)((\hat{v}_1, \hat{\eta}_1), (\hat{v}_2, \hat{\eta}_2)) = \int_0^1 \langle \hat{v}_1, \hat{v}_2 \rangle dt + \int_0^1 \frac{1}{\|v\|^2} \langle \hat{\eta}_1, \hat{\eta}_2 \rangle dt.$$  

### 3 Statement of the main result

Assume that for $k \leq n$ the torus $T^k = \{ e^{iv} : v \in \mathbb{R}^k \}$ acts on the complex vector space $\mathbb{C}^n$ via the action

$$\rho(e^{iv}) z = e^{iAv} z, \quad z \in \mathbb{C}^n, \ v \in \mathbb{R}^k$$

for some $(n \times k)$-matrix $A$ with integer entries. We endow the Lie algebra of the torus

$$\text{Lie}(T^k) = t^k = i\mathbb{R}^k$$

with its standard inner product. The action of the torus on $\mathbb{C}^n$ is Hamiltonian with respect to the standard symplectic structure $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. Let $\tau$ be an element of the Lie algebra $t^k$. Denoting by $A^T$ the transposed matrix of $A$ a moment map $\mu : \mathbb{C}^n \to t^k$ is given by

$$\mu(z) = -iA^T w - \tau, \quad w = \frac{1}{2} \begin{pmatrix} |z_1|^2 \\ \vdots \\ |z_n|^2 \end{pmatrix},$$

i.e.

$$d\langle \mu, \xi \rangle = i_{X_{\xi}} \omega, \quad \xi \in t^k$$

for the vector field $X_{\xi}$ on $\mathbb{C}^n$ given by the infinitesimal action

$$X_{\xi}(z) = \dot{\rho}(\xi)(z), \quad z \in \mathbb{C}^n.$$  

We will assume throughout this paper the following hypothesis,

(H) The moment map $\mu$ is proper, $\mu^{-1}(0)$ is not empty, and $T^k$ acts freely on it.

It follows from (H) that the Marsden-Weinstein quotient

$$\mathbb{C}^n / T^k = \mu^{-1}(0) / T^k$$

is a compact symplectic manifold of dimension

$$\dim(\mathbb{C}^n / T^k) = 2(n - k),$$

where the symplectic structure is induced from the standard symplectic structure on $\mathbb{C}^n$. 

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Let $H_t \in C^\infty(\mathbb{C}^n)$ for $t \in [0, 1]$ be a smooth family of $T^k$-invariant Hamiltonians. We denote by
\[ dH_t = -\omega(X_{H_t}, \cdot) \]
the Hamiltonian vector field of $H_t$. Furthermore, let $J_t$ for $t \in [0, 1]$ be a smooth family of $T^k$-invariant $\omega$-compatible almost complex structures, i.e.
\[ g_t(\cdot, \cdot) = \omega(\cdot, J_t \cdot) \]
is a Riemannian metric on $\mathbb{C}^n$. In this paper we are proving existence of solutions $(v, \eta) \in C^\infty(\mathbb{R} \times [0, 1], \mathbb{C}^n \times T^k)$ of the following PDE
\[
\begin{align*}
\partial_s v + J_t(v)(\partial_t v + X_\eta(v) - X_{H_t}(v)) &= 0, \\
\partial_s \eta + \mu(v) &= \tau,
\end{align*}
\]
\[ v(0), v(1) \in \mathbb{R}. \tag{9} \]
Schroers \[12\] discovered these equations for the special case where $J_t$ is the standard complex structure given by multiplication with $i$ and the Hamiltonian $H$ vanishes identically, see also \[13\]. Note that for the standard circle action on 1-dimensional complex space they reduce to the classical vortex equations. They are examples of the symplectic vortex equations in temporal gauge on the cylinder. The symplectic vortex equations can be defined more generally for Hamiltonian group actions on a symplectic manifold. They were introduced by Cieliebak, Gaio, Mundet, and Salamon in \[2, 3\].

Abbreviate
\[ T^k_{\mathbb{R}^n} = \{ g \in T^k : \rho(g)\mathbb{R}^n = \mathbb{R}^n \} \]
the isotropy subgroup of the Lagrangian submanifold $\mathbb{R}^n$. Since by our standing assumption (H) there is a point $x \in \mathbb{R}^n$ which is fixed only by the identity in $T^k$ it follows that
\[ T^k_{\mathbb{R}^n} \cong \mathbb{Z}_2^k. \]
The gauge group
\[ \mathcal{H} = \{ h \in C^\infty([0, 1], T^k) : h(0), h(1) \in T^k_{\mathbb{R}^n} \} \]
acts on the solutions of \eqref{9} by
\[ h_*(v, \eta) = (\rho(h)v, \eta - h^{-1}\partial_t h). \]
We define the energy of a solution of \eqref{9} by
\[ E(v, \eta) = \int_{\mathbb{R} \times S^1} (g_t(\partial_s v, \partial_s v) + \langle \partial_s \eta, \partial_s \eta \rangle) ds dt. \]
We finally introduce the space of admissible families of Hamiltonians $\mathcal{H}$ and the space of admissible families of almost complex structures $\mathcal{J}$. The space $\mathcal{H}$ consists of smooth families of $T^k$ invariant Hamiltonians $H_t$ for which there exists a compact set $K = K(H) \subset \mathbb{C}^n$ such that the support of $H_t$ for every
$t \in [0, 1]$ is contained in $K$. The space $J$ consists of smooth families of $T^k$-invariant $\omega$-compatible almost complex structures $J_t$ such that there exists a compact set $K = K(J) \subset \mathbb{C}^n$ such that $J_t$ equals outside of $K$ the standard complex structure, i.e. the multiplication by $i$.

Our main result is the following theorem.

**Theorem A:** For generic pairs $(H, J) \in \mathcal{H} \times J$ there exists at least two not gauge equivalent solutions of (9) whose energy is positive and finite.

4 Proof of Theorem A

4.1 A general principle for finding Floer flow lines

Let $L$ be a space on which a finite group $\Gamma$ acts and let $\mathcal{A}: L \to \mathbb{R}$ be a $\Gamma$-invariant action functional all whose critical points are nondegenerate. Assume that the action of $\Gamma$ on $\text{crit}(\mathcal{A})$ is free. We then can define two Floer chain complexes $CF(L, \mathcal{A})$ and $CF(L/\Gamma, \mathcal{A})$ where the first one is generated by critical points of $\mathcal{A}$ and the second one is generated by $\Gamma$-orbits of critical points of $\mathcal{A}$. Obviously,

$$\text{rk}(CF(L, \mathcal{A})) = |\Gamma| \cdot \text{rk}(CF(L/\Gamma, \mathcal{A})).$$

Suppose that we are able to define Floer homologies $HF(L, \mathcal{A})$ and $HF(L/\Gamma, \mathcal{A})$ out of the chain complexes above. We are now in position to formulate our principle.

**Principle:** Assume that $\text{rk}(HF(L, \mathcal{A})) \neq |\Gamma| \cdot \text{rk}(HF(L/\Gamma, \mathcal{A}))$. Then there exists a Floer flow line of $\mathcal{A}$ of positive, finite energy.

To illustrate our principle we consider an example were $L$ is a finite dimensional manifold. Let $L$ be the two-sphere $S^2$ and let $\Gamma = \mathbb{Z}_2$ act on $S^2$ as the antipodal involution. Then the quotient $S^2/\Gamma$ is $\mathbb{RP}^2$. We consider the Morse homology with $\mathbb{Z}_2$-coefficients. In this case the assumption of our principle is satisfied, i.e.

$$\text{rk}(HM(S^2; \mathbb{Z}_2)) \neq |\Gamma| \cdot \text{rk}(HM(\mathbb{RP}^2; \mathbb{Z}_2)).$$

To understand what is going on we consider the standard Morse function on $\mathbb{RP}^2$ which has one maximum, one critical point of index one, and one minimum, two Morse flow lines from the maximum to the critical point of index one as well as two Morse flow lines from the critical point of index one to the minimum. Since we are considering Morse homology with $\mathbb{Z}_2$-coefficients the boundary operator on the Morse chain complex on $\mathbb{RP}^2$ is zero. However, on $S^2$ the two flow lines split off and the boundary operator of the Morse complex on $S^2$ is not trivial anymore.
4.2 Two Floer homologies

We first show how solution of \( \mathcal{P} \) arise as flow lines of an action functional. We introduce the following path space

\[ \mathcal{P} = \{ (v, \eta) \in C^\infty([0, 1], C^n \times t^k) : v(0), v(1) \in \mathbb{R}^n \}. \]

For a smooth family of \( T^k \)-invariant Hamiltonians \( H_t \) we define the action functional \( A_H : \mathcal{P} \to \mathbb{R} \) by

\[ A_H(v, \eta) = \int_0^1 \lambda(v) \partial_t v - \int_0^1 H_t(v(t)) \, dt + \int_0^1 \langle \mu(v), \eta \rangle \, dt \]

where \( \lambda = \sum_{i=1}^n y_i \, dx_i \) is the Liouville 1-form satisfying \( d\lambda = -\omega \). Recall the gauge group

\[ \mathcal{H} = \{ h \in C^\infty([0, 1], T^k) : h(0), h(1) \in T^k_{\mathbb{R}^n} \}. \]

It acts on \( \mathcal{P} \) by

\[ h_* (v, \eta) = (\rho(h)v, \eta - h^{-1} \partial_t h). \]

Note that the differential of \( A \) is invariant under the action of \( \mathcal{H} \) on \( \mathcal{P} \). For a smooth family of \( T^k \)-invariant \( \omega \)-compatible almost complex structures \( J_t \) define the metric \( g_J \) on \( \mathcal{P} \) for \( (v, \eta) \in \mathcal{P} \) and \( (\dot{v}_1, \dot{\eta}_1), (\dot{v}_2, \dot{\eta}_2) \in T_{(v, \eta)} \mathcal{P} \equiv \mathcal{P} \) by

\[ g_J(v, \eta)((\dot{v}_1, \dot{\eta}_1), (\dot{v}_2, \dot{\eta}_2)) = \int_0^1 \omega(\dot{v}_1, J_t(v)\dot{v}_2) \, dt + \int_0^1 \langle \dot{\eta}_1, \dot{\eta}_2 \rangle \, dt. \]
Then the gradient flow lines of $A_H$ with respect to $g_J$ are precisely the solutions of $(9)$.

Recall that $\mathcal{H}$ denotes the space of families of admissible Hamiltonians, i.e. torus invariant Hamiltonians with compact support, and $\mathcal{J}$ denotes the space of families of admissible almost complex structures, i.e families of torus invariant $\omega$-compatible almost complex structures which agree outside of a compact set with the standard complex structure on $\mathbb{C}^n$. For generic $(H, J) \in \mathcal{H} \times \mathcal{J}$ it was shown in [8] that the Floer homologies $HF(\mathcal{P}, A_H, g_J; \mathbb{Z}_2)$ and $HF(\mathcal{P}/T^k_{\mathbb{R}^n}, A_H, g_J; \mathbb{Z}_2)$ are well defined. Theorem A follows from the following theorem.

**Theorem 4.1** $HF(\mathcal{P}, A_H, g_J; \mathbb{Z}_2) = 0$ and $HF(\mathcal{P}/T^k_{\mathbb{R}^n}, A_H, g_J; \mathbb{Z}_2) \neq 0$.

**Proof:** That $HF(\mathcal{P}/T^k_{\mathbb{R}^n}, A_H, g_J; \mathbb{Z}_2)$ is not zero follows from the results in [8]. There it was shown that the homology $HF(\mathcal{P}/T^k_{\mathbb{R}^n}, A_H, g_J; \mathbb{Z}_2)$ is given by the Morse homology with coefficients in $\mathbb{Z}_2$ of the induced Lagrangian in the Marsden-Weinstein quotient

$$\bar{L} = T^k(\mu^{-1}(0) \cap \mathbb{R}^n)/T^k \cong (\mu^{-1}(0) \cap \mathbb{R}^n)/T^k,$$

tensored with a Novikov ring which is itself an infinite dimensional vector space over the field $\mathbb{Z}_2$. Since $\mu^{-1}(0)$ is not empty it follows that $\bar{L}$ is not empty and hence $HF(\mathcal{P}/T^k_{\mathbb{R}^n}, A_H, g_J; \mathbb{Z}_2)$ is not zero.

To show that $HF(\mathcal{P}, A_J, g_J; \mathbb{Z}_2)$ is zero we make use of the fact that there exists a Hamiltonian isotopy $\phi_H$ of $\mathbb{C}^n$ generated by a smooth family of Hamiltonians $H_t$ with support in a compact set $K \subset \mathbb{C}^n$ such that

$$\phi_H(\mathbb{R}^n) \cap \mu^{-1}(0) = \emptyset.$$

To define the corresponding Floer homotopy care has to be taken since the Hamiltonians $H_t$ are not $T^k$-invariant anymore. We therefore define the Floer homotopy not on $\mathcal{P}$ but on a Coulomb section $\mathcal{P}_c \subset \mathcal{P}$.

Here is how this works. There is a natural splitting

$$\mathcal{H} \cong \mathcal{H}_0 \times \mathbb{Z}^k \times T^k_{\mathbb{R}^n}$$

where the contractible infinite dimensional Lie group $\mathcal{H}_0$ is given by elements $h = \exp(\xi) \in \mathcal{H}$ satisfying $\xi(0) = \xi(1) = 0 \in \mathbb{t}^k$. The gauge group $\mathcal{H}_0$ acts freely on $\mathcal{P}$. Moreover, for each path $\eta \in C^\infty([0, 1], \mathbb{t}^k)$ there is a unique element $h_\eta \in \mathbb{t}^k$ which puts $\eta$ into Coulomb gauge

$$d^*((h_\eta)_*\eta) = -\partial_t((h_\eta)_*\eta) = 0,$$

i.e. $(h_\eta)_*\eta$ is constant. Hence we may think of

$$\mathcal{P}_c = \{v \in C^\infty([0, 1], \mathbb{C}^n) : v(0), v(1) \in \mathbb{R}^n\} \times \mathbb{t}^k$$

as a global section in the principal $\mathcal{H}_0$-bundle $\mathcal{P}$, i.e. the following diagram is commutative

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where \( \iota \) is the canonical inclusion and \( \phi_c \) is the isomorphism given by Coulomb gauge.

The \( L^2 \)-metric \( g_{L^2} \) on \( \mathcal{P} \) is \( H_0 \)-invariant, hence it induces a quotient metric \([g_{L^2}] \) on \( \mathcal{P}/H_0 \) and we denote by

\[
g_c = \phi_c^*[g_{L^2}]
\]

its pullback on \( \mathcal{P}_c \). \( H_0 \)-equivalence classes of gradient flow lines of \( A_H \) with respect to \( g_{L^2} \) are in natural one-to-one correspondence with gradient flow lines of the restriction of \( A_H \) to \( \mathcal{P}_c \) with respect to the metric \( g_c \). To compute the metric we introduce the following notation. For \( z \in \mathbb{C}^n \) denote by \( \mathcal{L}_z : t^k \rightarrow T_z \mathbb{C}^n \sim = \mathbb{C}^n \) the linear map defined by

\[
\mathcal{L}_z \eta = X_\eta(z)
\]

and by \( \mathcal{L}_z^* \) its dual with respect to the inner products on \( t^k \) and on \( \mathbb{C}^n \). Then for \( (\dot{v}_1, \dot{\eta}_1), (\dot{v}_2, \dot{\eta}_2) \in T_{(v, \eta)} \mathcal{P}_c \) where \( (v, \eta) \in \mathcal{P}_c \) the metric \( g_c \) is given by

\[
g_c(v, \eta)((\dot{v}_1, \dot{\eta}_1), (\dot{v}_2, \dot{\eta}_2)) = \int_0^1 \langle \dot{v}_1 - L_v \xi_{v, \dot{v}_1}, \dot{v}_2 \rangle dt + \langle \dot{\eta}_1, \dot{\eta}_2 \rangle
\]

where \( \xi_{v, \dot{v}} \) is determined by

\[
L_{\dot{v}}^* \dot{v} - L_v \xi_{v, \dot{v}} + \partial_\xi \xi_{v, \dot{v}} = 0, \quad \xi_{v, \dot{v}}(0) = \xi_{v, \dot{v}}(1) = 0.
\]

By abuse of notation we will denote by \( A_H \) also the restriction of \( A_H \) to the Coulomb section \( \mathcal{P}_c \). In the following we will drop the assumption that \( H_t \) is \( T^k \)-invariant. To compute the gradient of \( A_H \) with respect to \( g_c \) it is useful to abbreviate for \( v \in C^\infty([0, 1], \mathbb{C}^n) \)

\[
\bar{\mu}(v) = \int_0^1 \mu(v(t)) dt,
\]

\[
\kappa_H(v)(t) = \int_0^t L^*_v H_v(v(\tau)) d\tau,
\]

\[
\bar{\kappa}_H(v) = \int_0^1 \kappa_H(v)(t) dt.
\]
Note that $\kappa_H$ and $\bar{\kappa}_H$ vanish identically if $H$ is $T^k$-invariant. Using this notation we get for the gradient
\[
\nabla_{g_c} \mathcal{A}_H(v, \eta) = \left( L_v \xi_v + i(\partial_t v + L_v \eta - X_H(v)) \right) \bar{\mu}(v)
\]
where $\xi_v$ is given by
\[
\xi_v(t) = \int_0^t \left( \mu(v(\tau)) + \kappa_H(v)(\tau) \right) d\tau - t(\bar{\mu}(v) + \bar{\kappa}_H(v)).
\]
Hence gradient flow lines satisfy the following equation
\[
\begin{align*}
\partial_s v + \dot{\rho}(\xi_v) + i(\partial_t v + \dot{\rho}(\eta)) &= 0, \\
\partial_s \eta - \partial_t \xi_v + \mu(v) + \kappa_H(v) - \bar{\kappa}_H(v) &= 0.
\end{align*}
\] (10)

Up to the factor $\kappa_H(v) - \bar{\kappa}_H(v)$ which measures how far is $H$ from being $T^k$-invariant these equations are examples of symplectic vortex equations. However, since $H$ has compact support there exists a constant $c = c(H) < \infty$ not depending on $v$ such that
\[
||\kappa_H(v) - \bar{\kappa}_H(v)|| \leq c(H).
\]
So the compactness result in [2] can be adapted to our situation to prove that the Floer homotopy is well defined. In our situation the proof simplifies considerably, though. This is firstly due to the fact that our gauge group is abelian so that we do not need the full strength of Uhlenbeck’s compactness theorem. Secondly thanks to the now established abstract perturbation techniques we do not have to bother about transversality any more and can work with the standard complex structure on $\mathbb{C}^n$. Since there are no derivatives of the standard complex structure the bubbling analysis can be avoided by directly using the elliptic estimate for the $\bar{\partial}$-operator. This proves the theorem. □

4.3 An example: The vortex equations

We consider the special case of the standard $S^1$-action on $\mathbb{C}$. We set the Hamiltonian equal to zero and consider the standard complex structure on $\mathbb{C}$. Then the gradient flow lines are solutions of the classical vortex equations on the strip with Lagrangian boundary condition. The connected components of the critical set of $\mathcal{A}$ are parametrised by $\mathbb{Z} \times \mathbb{Z}_2$. It was shown by Jaffe and Taubes [11] that between critical components whose difference in the $\mathbb{Z}$-factor is one there is exactly one solution up to gauge equivalence.
The Floer chain complex on the path space modulo isotropy group.

Hence if one considers the Floer complex modulo $\mathbb{Z}_2$, the isotropy group of the Lagrangian $\mathbb{R}$, then the Floer boundary operator vanishes and the homology is nontrivial. On the other if one does not mod out by $\mathbb{Z}_2$, then the same phenomenon as in the case of the standard Morse function on $\mathbb{R}^2$ happens. The flow lines split off and the Floer homology is zero.

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