The flight of the bumblebee: vacuum solutions of a gravity-induced spontaneous Lorentz symmetry breaking

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We study the vacuum solutions of a gravity model where Lorentz symmetry is spontaneously broken once a vector field acquires a vacuum expectation value. Results are presented for the purely radial Lorentz symmetry breaking (LSB), radial/temporal LSB and axial/temporal LSB. The purely radial LSB result corresponds to new black hole solutions. When possible, Parametrized Post-Newtonian (PPN) parameters are computed and observational boundaries used to constrain the Lorentz symmetry breaking scale.

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I. INTRODUCTION

Lorentz invariance is possibly one of the most fundamental symmetries of Nature. It is theoretically sound, it has been extensively tested, and so far no clear cut experimental evidence has emerged to bridge its validity (see e.g. Refs. [1, 2]). It is therefore no surprise that most theories of gravity encompass this symmetry, and little attention has been paid in understanding the implications of the breaking of Lorentz invariance in this context.

There is, however, a window of opportunity for fiddling with a less stringent approach: as for gauge symmetries in field theory, many relevant after effects can arise if one assumes a spontaneous breaking of Lorentz invariance [3, 4]. In the context of gravity, the breaking of Lorentz invariance can be implemented if a vector field ruled by a potential exhibiting a minimum rolls to its vacuum expectation value (vev), in the fashion of the Higgs mechanism [5]. This “bumblebee” vector thus acquires an explicit (four-dimensional) orientation, and preferred-frame effects may arise.

The possibility of violation of the this fundamental symmetry has been widely discussed in the recent literature (see Refs. [1, 2]). Indeed, string/Mtheory allow for a spontaneous breaking of Lorentz symmetry, due to the existence of non-trivial solutions in string field theory [3, 4], loop quantum gravity [6, 7], quantum gravity inspired spacetime foam scenarios [8] and noncommutative field theories [9, 10]. Also, LSB could result from spacetime variation of fundamental coupling constants [11]. Experimental tests of this symmetry breaking could be achieved, for instance, in ultra-high energy cosmic rays [12].

Efforts to quantify an hypothetical breaking of Lorentz invariance have been mainly concerned with the phenomenology of such spontaneous Lorentz symmetry breaking (LSB) in particle physics. Only recently its implications for gravity have been more thoroughly explored [5, 13]. In that work, the framework for the LSB gravity model is set up, developing the action and using the vielbein formalism.

In this study, we focus on the consequences of such scenario. For this, one assumes three relevant cases: the bumblebee field acquiring a purely radial vev, a mixed radial and temporal vev and a mixed axial and temporal vev. For prompt comparison with experimental tests, we shall write our results in terms of the PPN parameters, when possible.

The action of the bumblebee model is written as

\[
S = \int d^4x \sqrt{-g} \left( R + \frac{\kappa}{2} B^\mu B^\nu R_{\mu \nu} \right) - \frac{1}{4} B^{\mu \nu} B_{\mu \nu} - V(B^\mu B_\mu + b^2) \right] ,
\]

where \( \kappa = 8 \pi G \), \( \xi \) is a real coupling constant and \( b^2 \) is a real positive constant. The potential \( V \) driving Lorentz and/or CPT violation can be chosen to have a minimum at \( B^\mu B_\mu + b^2 = 0 \).

II. PURELY RADIAL LSB

In this section we develop a method to obtain the exact solution for the purely radial LSB. For this, we assume a static, spherically symmetric spacetime, with a metric \( g_{\mu \nu} = diag(-c^2, e^{2\phi}, r^2, r^2 \sin^2 \theta) \), where \( \phi \) and \( \rho \) are functions of \( r \). Also, we admit that the vector field \( B_\mu \) has relaxed onto its expectation value \( b_\mu \). It is imposed that \( D_\mu b_\nu = 0 \). This enables us to calculate \( b_\mu \), using the affine connection derived from the metric \( g_{\mu \nu} \) for this, and since the only non trivial covariant derivative is with respect to the radial coordinate, it is assumed that \( b_\mu = (0, b(r), 0, 0) \). Hence, from

\[
D_\mu b_\nu = \partial_\mu b_\nu - \Gamma^\alpha_{\nu \mu} b_\alpha = 0 ,
\]
follows that

\[ b(r) = \xi^{-1/2} b_\rho e^\rho \]  

where the \( \sqrt{\xi} \) is introduced for later convenience. It can be immediately understood that, as expected, \( b^2 = b\rho b_\mu = b_0^2 \xi^{-1} \) is constant. The action can be thus written as

\[ S_s = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa} + (g^{rr})^2 b^2(r) R_{rr} \right] , \]

where the subscript \( s \) stands for the spatial quantities. With the assumed metric, the determinant is given by \( \sqrt{-g} = r^2 e^{\rho + \phi} \); the scalar curvature and the relevant Ricci tensor non-vanishing component are given by

\[ R = \frac{2 \left[ (2r^\rho - 1)e^{-2\rho} \right]}{r^2} , \quad R_{rr} = \frac{2\rho'}{r} , \]

where the prime stands for derivative with respect to \( r \) and we have integrated over the angular dependence. Also, \( b^r \) is the (contravariant) radial component of \( b_\rho \). We now introduce the field redefinition \( \Psi = (1 - e^{-2\rho})^{-1/2} \), so that

\[ \Psi' = \frac{2\rho' e^{-2\rho}}{r^2} - \frac{2\Psi}{r} \]

and thus

\[ \frac{2\rho' e^{-2\rho}}{r} = 2\Psi + r\Psi' \]  

\[ R = 2(3\Psi + r\Psi') \]  

\[ (b^r)^2 R_{rr} = b_0^2 (2\Psi + r\Psi') . \]

We can now work with the action (see [16] and references therein):

\[ S_s = \frac{2}{\kappa} \int dr e^{\rho + \phi} r^2 \left[ (3 + b_0^2)\Psi + (1 + b_0^2) r\Psi' \right] . \]

Variation with respect to \( \phi \) yields the equation of motion

\[ (3 + b_0^2)\Psi + (1 + b_0^2) r\Psi' = 0 \]

which admits the solution \( \Psi(r) = \Psi_0 r^{-3+L} \), where we define

\[ 3 - L = \frac{3 + b_0^2}{1 + \frac{b_0^2}{2}} \approx 3 - \frac{b_0^2}{2} \]

and hence \( L \approx b_0^2/2 \). From the definition of \( \Psi \), we obtain

\[ g_{rr} = e^{2\rho} = (1 - \Psi_0 r^{-1+L})^{-1} \]

Comparing with the usual Schwarzschild metric, one can write

\[ g_{rr} = \left( 1 - \frac{2G_L m}{r} \right)^{-1} , \]

\[ g_{00} = -1 + \frac{2G_L m}{r} , \]

where \( G_L \) has dimensions \( [G_L] = L^2 - L \) (in natural units, where \( c = \hbar = 1 \)). One can define \( G_L = G/r_0^L \), where \( r_0 \) is an arbitrary distance. The \( L \to 0 \) limit yields \( G_L \to G \) and the usual geometrical mass, \( M \equiv Gm \), with dimensions of length. From now on we express all results in terms of \( M \). The event horizon condition is given by

\[ g_{00} = -1 + \frac{2M}{r} = 0 , \]

thus \( r_s = (2Mr_0^{-L})^{1/(1-L)} \). One can compute the scalar invariant \( I = R_{\mu
u\rho\lambda} R^{\mu
u\rho\lambda} \) (the norm-square of the Riemann tensor), obtaining

\[ I = 48 \left[ 1 - \frac{5}{3}L + \frac{17}{12} L^2 - \frac{1}{2} L^3 + \frac{1}{12} L^4 \right] M^2 \left( \frac{r}{r_0} \right)^{2L} r^{-6} \]

\[ \simeq \left( 1 - \frac{5}{3}L \right) \left( \frac{r}{r_0} \right)^{2L} I_0 , \]

where \( I_0 = 48M^2 r^{-6} \) is the usual scalar invariant in the limit \( L \to 0 \). Hence, since \( I(r = r_s) \) is finite, one sees that the singularity at \( r = r_s \) is removable. By the same token, the singularity at \( r = 0 \) is intrinsic, as given by the divergence of the scalar invariant. Thus, an axial LSB gravity model admits new black hole solutions whose singularity is well protected within a horizon of radius \( r_s \). One expects a Hawking temperature given by

\[ T = \frac{\hbar}{2\pi k_B r_s} = (2Mr_0^{-L})^{-L/(1-L)}T_0 \simeq (2Mr_0^{-L})^{-L}T_0 , \]

where \( T_0 = \hbar/8\pi k_B M \) is the usual Hawking temperature, which is recovered in the limit \( L \to 0 \).

Of course, this exact solution does not allow for a PPN expansion, as the obtained metric cannot be expanded in powers of \( U = M/r \). One can, however, compare with results for deviations from Newtonian gravity [17], usually stated in terms of a Yukawa potential of the form
\[ V_Y(r) = \frac{G_Y m}{r} \left( 1 + \alpha e^{-r/\lambda} \right). \]  
\( \text{(17)} \)

Equating this potential to the one arising from \( g_{00} \) in Eq. (13), one gets
\[ G_L r^L = G_Y \left( 1 + \alpha e^{-r/\lambda} \right). \]  
\( \text{(18)} \)

Expanding to first order around \( r = r_0 \) yields
\[ G_L r^L \left( 1 + L \frac{r}{r_0} \right) = G_Y \left( 1 + \alpha - \alpha \frac{r}{\lambda} \right), \]  
\( \text{(19)} \)

which allows us to identify \( \lambda = r_0 \) and \( \alpha = -L \) (with \( G_Y (1 - L) = G_L r^L = G \)). This states that the effects of a radial LSB, probed at a distance \( r_0 \) from the source, can be interpreted as due to a Yukawa potential of coupling strength \( \alpha = -L \) and range \( \lambda = r_0 \). The negative sign of the coupling shows that the radial LSB yields a “repulsive” component (for \( r > r_0 \)), as can be seen from Eq. (13).

Notice that this identification of the LSB effect with a Yukawa potential constraints the length \( \lambda \) to be equal to the distance \( r_0 \) at which one tests deviations from the inverse square law. This is not the case with a “true” Yukawa perturbation, where each test of gravity at a distance \( r_0 \) yields an allowed range for both \( \alpha \) and \( \lambda \) (although, of course, the test is sensible to deviations only at a scale \( \lambda \) close to \( r_0 \)). Hence, to obtain a bound on \( L \) one must look at the allowed value of \( \alpha \) for the fixed \( \lambda = r_0 \) at which an experiment is carried out. The most stringent bound is derived from planetary tests to Kepler’s law, with Venus yielding \( \lambda = r_0 = 0.723 \) \( AU \) and \( L = |\alpha| \leq 2 \times 10^{-9} \) \[17\].

### III. RADIAL/TEMPORAL LSB

We consider now the mixed radial and temporal Lorentz symmetry breaking. As before, we assume that the bumblebee field \( B_0 \) has relaxed to its vacuum expectation value. Assuming temporal variations to be of order of the age of the Universe \( H^{-1}_0 \), where \( H_0 \) is the Hubble constant, one can, as before, consider a Birkhoff static, radially symmetric metric \( g_{\mu\nu} = \text{diag}(e^{-2\phi}, e^{2\phi}, r^2, r^2 \sin^2(\theta)) \). Imposing as (physical) gauge choice the vanishing of the covariant derivative of the field \( B_{\mu} \), one gets \( b_0(r) = \xi^{1/2} A_0 e^{\phi} \) and, similarly, \( b_0(r) = \xi^{1/2} A_0 e^{\phi} \), with \( A_0 \) and \( A_r \) dimensionless constants. One immediately sees that, as expected, \( b^2 = b^{\mu} b_{\mu} = (A_0^2 - A_r^2) \xi^{-1} \) is constant.

Since one now has both a radial and a time component for the vector field \( v^{\mu} \), the symmetry \( \phi = -\rho \) does not hold. Therefore, one cannot use the previous spatial action formalism depicted in Eq. (9). Instead, one resorts to the full Einstein equations,
\[ G_{\mu\nu} = \xi \left[ \frac{1}{2} (b^\alpha)^2 R_{\alpha\alpha} g_{\mu\nu} - b_\mu b^\nu R_{\mu\nu} - b_\mu b^\rho R_{\mu\nu} \right]. \]  
\( \text{(20)} \)

The additional equation of motion for the vector field vanishes, since it has relaxed to its \( v^{\nu} \) and therefore both the field strength and the potential term are null. Introducing the metric Ansatz and the expressions for \( b_\mu \) one gets, after a little algebra,
\[ G_{00} = \frac{1}{2} \left[ 3 A_0^2 f_0 - A_r^2 e^{2(\phi - \rho)} R_{\rho\rho} \right], \]
\[ G_{rr} = \frac{1}{2} \left[ A_0^2 e^{2(\phi - \rho)} R_{00} - 3 A_r^2 R_{rr} \right], \]  
\( \text{(21)} \)

We now write \( G_{00} = g_0(r) e^{2(\phi - \rho)} \), \( G_{rr} = g_r(r), R_{00} = f_0(r) e^{2(\phi - \rho)} \) and \( R_{rr} = f_r(r), \) where
\[ f_0(r) = \frac{(2 - r') \phi'}{r} + \phi'^2 + \phi'' \],
\[ f_r(r) = \frac{(2 + r') \phi'}{r} - \phi'^2 - \phi'' \],
\[ g_0(r) = \frac{-1 + e^{2\phi}}{e^{r'}} + \frac{2 \phi'}{r^2} \],
\[ g_r(r) = \frac{1 - e^{2\phi}}{e^{r'}} + \frac{2 \phi'}{r^2} \]  
\( \text{(22)} \)

Inserting the above into the Einstein equations, it follows that
\[ g_0(r) = \frac{1}{2} \left[ 3 A_0^2 f_0(r) - A_r^2 f_r(r) \right], \]
\[ g_r(r) = \frac{1}{2} \left[ A_0^2 f_0(r) - 3 A_r^2 f_r(r) \right]. \]  
\( \text{(23)} \)

Hence, one must solve this set of coupled second order differential equations, with boundary conditions given by \( \phi(\infty) = \rho(\infty) = \phi'(\infty) = \rho'(\infty) = 0 \).

Before continuing, we point out that LSB is clearly exhibited: noticing that \( g_0 + g_r = f_0 + f_r \), one has \((1 - 2 A_0^2) f_0 = -(1 + 2 A_r^2) f_r \), which is an explicit manifestation of the breaking of Lorentz symmetry. In the unperturbed case \( A_0 = A_r = 0, f_0 = -f_r \) and one recovers the Schwarzschild solution \( \phi = -\rho \) from the equation \( g_0 + g_r = 0 \). This symmetry does not hold in the perturbed case, which yields \( f_0 \approx -(1 + 2 A_0^2 + 2 A_r^2) f_r \).

To solve the set of coupled differential equations Eqs. (23) one considers an expansion of the metric in terms of \( \phi = \phi_0 + \delta \phi \) and \( \rho = -\phi_0 - \delta \rho \), where \( \phi_0 \) is given by the usual Schwarzschild metric:
\[ \phi_0 = \frac{1}{2} \ln \left( 1 - \frac{2 M}{r} \right), \]  
\( \text{(24)} \)

and \( \delta \rho, \delta \phi \) are assumed to be small perturbations. Hence, one gets to first order
\[ f_0(r) = \frac{2}{r} \delta \phi' + \frac{M}{1 - \frac{2M}{r}} (3 \delta \phi' + \delta \rho') + \delta \phi'' , \tag{25} \]

and

\[ f_r(r) = -\frac{2}{r} \delta \rho' - \frac{M}{1 - \frac{2M}{r}} (3 \delta \phi' + \delta \rho') - \delta \phi'' . \tag{26} \]

As expected, the above quantities are homogeneous in \( \delta \rho, \delta \phi \) and their derivatives.

For the calculus of \( g_0(r) \) and \( g_r(r) \), one first computes the contribution of the exponential term:

\[ \frac{1 - e^{2\rho}}{r^2} = \frac{1 - e^{-2(\phi_0 + \delta \phi)}}{r^2} \]
\[ \simeq \frac{1 - e^{-2\phi_0}(1 - 2\delta \rho)}{r^2} = \frac{\frac{2M}{r}}{1 - \frac{2M}{r}} + \frac{2}{r^2} \delta \rho . \]

Thus, one finds

\[ g_0(r) = -\frac{2}{r} \left( \frac{1}{1 - \frac{2M}{r}} \delta \rho + \delta \phi' \right) , \tag{28} \]

and

\[ g_r(r) = \frac{2}{r} \left( \frac{1}{1 - \frac{2M}{r}} \delta \rho + \delta \phi' \right) , \tag{29} \]

which are also homogeneous.

To solve Eqs. (23), one first obtains a relation between \( \delta \rho \) and \( \delta \phi \). For this, one sums both coupled differential equations, which yields

\[ \frac{2}{r} (\delta \phi' - \delta \rho') = 4 \left[ \frac{2}{r} (A \delta \phi' + B \delta \rho') \right. \]
\[ + (A + B) \left. \frac{M}{1 - \frac{2M}{r}} (3 \delta \phi' + \delta \rho') + (A + B) \delta \phi'' \right] , \tag{30} \]

where the simplifying notation \( A = A_0^2, B = A_2^2 \) has been used (\( A \) and \( B \) are dimensionless). Since the LSB is presumably a small effect, one expects \( A \ll 1 \) and \( B \ll 1 \). Hence, one has

\[ \delta \rho' \left[ 1 + 4B + \frac{A + B}{2} \frac{M}{1 - \frac{2M}{r}} \right] \]
\[ = \delta \phi' \left[ 1 - 4A - \frac{3(A + B)}{2} \frac{M}{1 - \frac{2M}{r}} \right] - \frac{A + B}{2} r \delta \phi'' . \tag{31} \]

Dropping terms smaller than \( O(A), O(B), \) this expression simplifies to

\[ \delta \rho' = \frac{1 - 4A}{1 + 4B} \delta \phi' - \frac{A + B}{2(1 + 4B)} r \delta \phi'' , \tag{32} \]

which, after integration, yields

\[ \delta \rho = \frac{2 + B - 7A}{2(1 + 4B)} \delta \phi - \frac{A + B}{2(1 + 4B)} r \delta \phi' . \tag{33} \]

One can now introduce this expression in one of the Eqs. (23) and obtain, after a somewhat tedious computation, the following ordinary differential equation

\[ -C_1 r^2 \delta \phi'' + C_2 r \delta \phi' + C_3 \delta \phi = 0 , \tag{34} \]

with

\[ C_1 = A + 3B + AB + 9B^2 \simeq A + 3B \, , \]
\[ C_2 = 2 + B - 3A + 16AB \simeq 2 \, , \]
\[ C_3 = 2 + B - 7A \simeq 2 . \tag{35} \]

This equation has the solution

\[ \delta \phi = K_+ r^{\frac{C_1 + C_2 + \sqrt{(C_1 + C_2)^2 + 4C_1 C_3}}{2C_1}} \]
\[ + K_- r^{\frac{C_1 + C_2 - \sqrt{(C_1 + C_2)^2 + 4C_1 C_3}}{2C_1}} , \tag{36} \]

where \( K_+ \) and \( K_- \) are integration constants. Since all \( C_i \) are positive, one obtains that \( \sqrt{(C_1 + C_2)^2 + 4C_1 C_3} > C_1 + C_2 \), and hence the second power-law term diverges when \( r \to \infty \). Since one demands that \( \delta \phi \to 0 \), this implies that \( K_+ = 0 \). The remaining first power-law term automatically satisfies the conditions \( \delta \phi \to 0 \) and \( \delta \phi' \to 0 \) for all \( K_- \equiv K \). Hence, the perturbation is simply given by \( \delta \phi = K r^{-\alpha}, \) where

\[ \alpha = -C_1 - C_2 + \frac{\sqrt{(C_1 + C_2)^2 + 4C_1 C_3}}{2C_1} > 0 \, , \tag{37} \]

and \( K \) has dimensions of \( [K] = L^\alpha \). One can linearize this exponent with respect to \( C_1 \ll 1 \), that is

\[ \alpha \simeq \frac{C_3}{C_1 + C_2} = \frac{2 - 7A + B}{2 + A + 3B} \, , \tag{38} \]

so that \( \alpha \simeq 1 \). One can now compute \( \delta \rho \), obtaining

\[ \delta \rho = \frac{2 + B - 7A}{2(1 + 4B)} \delta \phi - \frac{A + B}{2(1 + 4B)} r \delta \phi' \]
\[ \simeq \left[ 1 + \alpha \frac{(A + B)}{2} \right] K r^{-\alpha} . \tag{39} \]

Hence, the non-trivial components of the metric read
\[ g_{tt} = -e^{2(\phi_0 + \delta \rho)} = -e^{2K r^{-\alpha}} \left( 1 - \frac{2M}{r} \right), \quad (40) \]

\[ g_{rr} = e^{-2(\phi_0 + \delta \rho)} \frac{e^{-(2\alpha(A+B))K r^{-\alpha}}}{1 - \frac{2M}{r}} = e^{-2K r^{-\alpha}} \frac{1 - \frac{2M}{r}}{1 - \frac{2M}{r}}, \]

where one defines \( K_r = [1 + \alpha(A + B)/2]K \approx [1 + (A + B)/2]K \). To compute the PPN parameters, one first performs a Lorentz transformation to an isotropic coordinate system, that is, to one on which all spatial metric components are equal. Since the angular coordinates are not affected by the LSB dynamics, this amounts only to a change of the radial parameter, \( r \to \xi = \xi(r) \). Thus, instead of explicitly deriving the associated metric \( \bar{g}_{\mu \nu} \) through its transformation properties, one resorts to the invariance of the interval \( ds^2 \), obtaining

\[ g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega^2 \]

\[ = g_{tt}(\xi)d\xi^2 + \lambda(\xi)(d\xi^2 + \xi^2 d\Omega^2) \quad (41) \]

Equating the angular part, one obtains \( \lambda(\xi) = r^2/\xi^2 \). Since the temporal part is equal on both sides of the equation (merely expressed in terms of \( r \) or \( \xi \)), one also obtains \( g_{rr}dr^2 = \lambda(\xi)d\xi^2 \) and

\[ \frac{g_{rr}^{1/2}}{\xi} dr = d\xi, \quad (42) \]

implying that

\[ \int \frac{e^{-K r^{-\alpha}}}{r\sqrt{1 - \frac{2M}{r}}} dr = \log \xi + \text{const}. \quad (43) \]

Since the perturbation is assumed to be small, one can consider that both \( K, K_r \ll r^\alpha \). Hence, one expects \( \xi \) also to be a small perturbation to the unperturbed isotropic coordinate \( \xi_0 \); indeed, this is obtained setting \( K_r = 0 \), from which follows

\[ \frac{dr}{\sqrt{r^2 - 2Mr}} = \frac{d\xi_0}{\xi_0} \quad (44) \]

and

\[ r = \xi_0 \left( 1 + \frac{M}{2\xi_0} \right)^2, \]

\[ \xi_0 = \frac{1}{2} \left[ r \left( 1 + \sqrt{1 - \frac{2M}{r}} \right) - M \right]. \quad (45) \]

Hence, to solve the perturbed \( K_r \neq 0 \) case, one performs the coordinate transformation \( r \to \xi_0 \), obtaining

\[ \int \frac{e^{-K r^{-\alpha}}}{r\sqrt{1 - \frac{2M}{r}}} d\xi = \int \frac{e^{-K r^{-\alpha}(\xi_0)}}{\xi_0} d\xi_0. \quad (46) \]

Expanding the integrand around \( \alpha = 1 \) (through \( r^\alpha = r + (\alpha - 1)r \log(r) \)) and \( K_r = 0 \), it can be easily seen that the contribution of the \( \alpha \) term amounts to corrections of second-order \( O((\alpha - 1)K_r) \), which we shall disregard. This is equivalent to set \( \alpha = 1 \) in the above expressions. Hence, one obtains

\[ \log \xi = \int \frac{e^{-K r^{-\alpha}(\xi_0)}}{\xi_0} d\xi_0 \simeq \int 1 - \frac{K_r}{r(\xi_0)} d\xi_0 \]

\[ = \log(\xi_0) + \frac{K_r}{\xi_0} \oint \frac{\xi_0}{\xi_0 + \frac{K_r}{\xi_0}}. \quad (47) \]

Solving for \( \xi \), yields

\[ \xi = \xi_0 e^{\frac{K_r}{\xi_0 + \frac{K_r}{\xi_0}}}. \quad (48) \]

Obviously, setting \( K_r = 0 \) gives \( \xi = \xi_0 \), that is, one recovers the Schzschild isotropic coordinates. One can now read the (isotropic) spatial component of the metric \( r^{-2}g_{\theta\theta} = r^{-2}\sin^{-2}\theta \bar{g}_{\phi\phi} = g_{rr} \):

\[ \bar{g}_{\xi \xi} = \lambda(\xi) e^{\frac{2K_r}{\xi_0 + \frac{K_r}{\xi_0}}} \left( 1 + \frac{M}{2\xi_0} \right)^4. \quad (49) \]

Recall that the Lorentz transformation maps \( r \to \xi \), while \( \xi_0 \) is just a convenient integration variable. Hence, one must now write \( \xi_0 \) in terms of \( \xi \). For this, one should invert the relation

\[ 1 \simeq \xi - K_r \left( 1 - \frac{M}{2\xi} \right). \quad (51) \]

Inserting this expression into the above equation and expanding to first order in \( U = M/\xi \), one gets

\[ \bar{g}_{\xi \xi} e^{\frac{2K_r}{\xi_0 + \frac{K_r}{\xi_0}}} \left( 1 + \frac{M}{2\xi_0} \right)^4 \simeq 1 + 2 \left( 1 - \frac{K_r}{M} \right) U. \quad (52) \]

Through a similar procedure, the \( g_{tt} \) component can be found:
\[ \tilde{g}_{tt} = -e^{2Kr^-\alpha(\xi)} \left( 1 - \frac{2M}{r(\xi)} \right) \]
\[ = -e^{2K\xi^{-1}(1+\frac{2M}{r})} \left( 1 - \frac{M}{2\xi} \right)^2 \left( 1 + \frac{M}{2\xi} \right)^2 \]
\[ \approx -1 + 2 \left( 1 - \frac{K}{M} \right) U - 2 \left( 1 - \frac{3K + K_r}{M} \right) U^2 , \]

where, as before, we have taken \( \alpha = 1 \) (so that \( K, K_r \) have dimension of length). Both \( \tilde{g}_{tt} \) and \( \tilde{g}_{\xi\xi} \) reduce to the usual isotropic metric components in the limit \( K, K_r \to 0 \).

To read the PPN parameters one must now transform to a quasi-cartesian referential, where the metric reads

\[ \eta_{tt} = -1 + 2U - 2\beta U^2 , \]
\[ \eta_{ij} = (1 + 2\gamma U) \delta_{ij} . \]  \( \text{(54)} \)

This is achieved by a (spatial) coordinate change \( \xi \to \xi' \) so that \( \xi = (1 - K/M)\xi' \). This transforms the metric to

\[ \eta_{tt} = g_{tt} = -1 + 2\beta U^2 - \frac{1 - 2K}{1 - \frac{M}{2\xi}} \left( \frac{M}{\xi} \right)^2 \]
\[ \approx -1 + 2U - \left( 1 - \frac{K + K_r}{M} \right) U^2 , \]
\[ \eta_{\xi\xi'} = \left( \frac{\partial \xi}{\partial \xi'} \right)^2 g_{\xi\xi} \]
\[ = \left( 1 - \frac{M}{K} \right)^2 \left[ 1 + 2 \left( 1 - \frac{2K_r}{K} \right) \frac{M}{1 - K/M} \right] \]
\[ \approx 1 + \left( 1 - \frac{K + 2K_r}{M} \right) U . \]  \( \text{(55)} \)

Hence, one obtains

\[ \beta \approx 1 - \frac{K + K_r}{M} , \]
\[ \gamma \approx 1 - \frac{K + 2K_r}{M} . \]  \( \text{(56)} \)

Inverting, one finds that

\[ \frac{K}{M} \approx 1 - 2\beta + \gamma , \]
\[ \frac{K_r}{M} \approx \beta - \gamma . \]  \( \text{(57)} \)

A drawback of these results is the dependence of these PPN parameters on \( K \) and \( K_r \), which are integration constant (more precisely, \( K_r \) is defined as proportional to \( K \), which is free valued). Therefore, these do not depend on the physical parameters associated with the breaking of Lorentz invariance. This reflects the linearization of the Einstein’s equation that was used in order to obtain the radially symmetric Birkhoff metric. In any case, one can conclude that the effect of temporal/radial LSB manifests itself linearly on the PPN parameters \( \beta \) and \( \gamma \).

The current bounds, arising from the Nordvedt effect, \( |\beta - 1| \leq 6 \times 10^{-4} \) [18] and the Cassini-Huygens experiment, \( \gamma = 1 + (2.1 \pm 2.3) \times 10^{-5} \) [19], can be used to constraint the parameter space \((K, K_r)\). Taking the Sun’s geometrical mass \( M = 1.5 \ km \), one obtains the constraints

\[ |K + K_r| < 0.9 \ m , \]
\[ K + 2K_r = (-3.1 \pm 3.4) \times 10^{-2} \ m . \]  \( \text{(58)} \)

Also, by definition

\[ K_r = \left[ 1 + \alpha \frac{(A + B)}{2} \right] K , \]  \( \text{(59)} \)

with \( \alpha \approx 1, A, B \ll 1 \). Hence, one expects deviations of \( K_r \) from \( K \) to be small. Thus, considering for instance the constraint \( |1 - K_r/K| \lesssim 0.1 \), the resulting range of allowed values for these parameters is depicted in Figures 1 and 2. Notice that considering \( K \sim K_r \) immediately yields \( K \approx (-1 \pm 1.1) \times 10^{-2} \), indicating that the perturbation has a very short range. Indeed, since this lies well inside the Sun, one should work with the interior Schwarzschild solution, and this value merely indicates that the effect outside the Sun is negligible.

Returning to Eqs. (41) one notices that, in the limit \( M \to 0 \), one gets \( \tilde{g}_{tt} = -e^{2K\xi}/\xi \) and \( \tilde{g}_{rr} = -e^{2K_r}/\xi \). This allows establishing an analogy with Rosen’s bimetric theory, by interpreting these changes as due to a background metric \( \eta_{\mu\nu} \) [14]. Notice that, in the presence of a central mass, the vector field no longer rolls to a radial \( \vec{v} \), since this spatial symmetry is inherited from the vanishing covariant derivative, that is, from the presence of
waves, on the other hand, is shifted by an amount $c\alpha$ of light remains equal to the expected effect on the metric.

The radius of the Sun. This also reflects the smallness of $\times 4 \times 10^4$ considering the spin precession constraint arising from which exhibits a radial dependency. Assuming $\alpha \simeq 1$ and one cannot directly consider the act, while the radial/temporal results are not. Therefore, one cannot directly consider the $A \to 0$ limit for comparison of the results obtained.

IV. AXIAL/TEMPORAL LSB

One now treat the anisotropic LSB case. For definitiveness, one assumes that the bumblebee field is stabilized at its vacuum expectation value and exhibits both a temporal and a spatial coordinate, assumed to lie on the x-axis, that is, $b_i = \kappa^{-1}(a, b, 0, 0)$. One calculates the metric perturbations $h_{\mu\nu}$ to the flat Minkowsky metric. To first order in $h_{\mu\nu}$, one has

$$R_{00} = -\frac{1}{2} \nabla^2 h_{00} \quad (63)$$

$$R_{0i} = -\frac{1}{2} \left( \nabla^2 h_{0i} - h_{k0,ik} \right) \quad ,$$

$$R_{ij} = -\frac{1}{2} \left( \nabla^2 h_{ij} - h_{00,ij} + h_{kk,ij} - h_{ki,kj} - h_{kj,ki} \right) \quad ,$$

where time derivatives were neglected, since one assumes that $v \ll c$.

The axial LSB breaks the radial symmetry, and one must consider the Einstein equations: $G_{\mu\nu} = T_{\mu\nu} + T_{B_{\mu\nu}}$, where $T_{B_{\mu\nu}}$ is the stress-energy tensor for the bumblebee field,

$$T_{B_{\mu\nu}} = \left[ \frac{1}{2} b^\alpha b^\beta R_{\alpha\beta\mu\nu} - b^\alpha b^\alpha R_{\alpha\mu\nu} - b^\beta b^\beta R_{\nu\mu\nu} \right] \quad (64)$$

Its trace clearly vanishes and hence, from the trace of the Einstein equations one obtains

$$R_{\mu\nu} = \kappa \left[ T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} T + T_{B_{\mu\nu}} \right] \quad (65)$$

First one calculates $h_{00}$ to first order of the potential $U$; for that one writes

$$R_{00} = -\frac{1}{2} \left( a^2 R_{00} + b^2 R_{11} + 2abR_{10} \right) - 2a \left( aR_{00} + bR_{10} \right) \quad .$$

Since

$$R_{00} = -\frac{1}{2} \nabla^2 h_{00} \quad , \quad R_{0i} = 0 \quad , \quad R_{ij} = \frac{1}{2} h_{00,ij} \quad ,$$

one has

$$R_{00} = -\frac{1}{2} \left( a^2 R_{00} + b^2 R_{11} \right) - 2a^2 R_{00} \quad ,$$

and hence

$$\left( 1 + \frac{5a^2}{2} \right) \nabla^2 h_{00} - \frac{b^2}{2} h_{00,11} = 0 \quad .$$

One can rewrite this as

$$\left( 2 + \frac{5a^2 - b^2}{2 + 5a^2} \right) h_{00,11} + h_{00,22} + h_{00,33} = 0 \quad .$$

This equation admits the solution

FIG. 2: Detail of Fig. 1, showing only the allowed region.
where $c_0^2 = (2 + 5a^2)/(2 + 5a^2 - b^2)$.

One now computes the components $h_{ii}$ ($i \neq 1$) using this result:

$$R_{ii} = \frac{1}{2} \left( a^2 R_{00} + b^2 R_{11} + 2ab R_{01} \right),$$

hence

$$R_{00} = -\frac{1}{2} \nabla^2 h_{00},$$
$$R_{01} = 0,$$
$$R_{ii} = -\frac{1}{2} \left( \nabla^2 h_{ii} - h_{00,ii} + h_{ii,ii} \right),$$
$$R_{11} = \frac{1}{2} h_{00,11},$$

and therefore (for $j \neq i, j \neq 1$)

$$\frac{-1}{2} \left( \nabla^2 h_{ii} - h_{00,ii} + h_{ii,ii} \right)$$
$$= \frac{a^2}{2} \left( -\frac{1}{2} \nabla^2 h_{00} \right) + \frac{b^2}{2} \left( \frac{1}{2} h_{00,11} - \frac{1}{2} h_{ii,11} \right).$$

One rewrites this equation as

$$(2 - b^2) h_{ii,11} + 4 h_{ii,ii} + 2 h_{ii,jj}$$

$$= (a^2 - b^2) h_{00,11} + (2 + a^2) h_{00,ii} + a^2 h_{00,jj}.$$ 

As an Ansatz for the solution, one takes $h_{ii}(x, y, z) = h_{00}(\alpha_1 x, \alpha_2 y, \alpha_3 z)$. Notice that

$$h_{ii,jj} = \alpha_i^2 h_{00,jj}(\alpha_1 x, \alpha_2 y, \alpha_3 z) \simeq \alpha_i^2 h_{00,jj}(x, y, z)$$

and that

$$h_{00,11} = -2Mc_0^2 \frac{-2c_0^2 x^2 + y^2 + z^2}{(c_0^2 x^2 + y^2 + z^2)^5},$$
$$h_{00,jj} = -2M \frac{-2x_j^2 + c_0^2 x^2 + x_j^2}{(c_0^2 x^2 + y^2 + z^2)^5}.$$ 

Substituting in Eq. (76) one obtains, after some calculation (see Appendix I), the coefficients $\alpha_i$:

$$\alpha_i^2 = \frac{-2(2 + 5a^2 - b^2)\alpha_i^4 + 2(1 + 2a^2)b^2}{(2 + 5a^2)(-2 + b^2)},$$
$$\alpha_i^2 = \frac{1 + \alpha_i^2}{2},$$

for free $\alpha_j$. In order to match with the unperturbed case $r^2 = x,x'$ one chooses $\alpha_j = 1$ to get

$$\alpha_i^2 = 1 - \frac{(2 - a^2)b^2}{(2 + 5a^2)(2 - b^2)}, \quad \alpha_i = \alpha_j = 1.$$ 

Hence,

$$h_{ii}(x, y, z) = h_{00}(\alpha_1 x, \alpha_2 y, \alpha_3 z) \equiv \frac{2M}{\sqrt{c_0^2 x^2 + y^2 + z^2}}.$$ 

We now compute the $h_{11}$ component:

$$R_{11} = \frac{1}{2} \left( a^2 R_{00} + b^2 R_{11} \right) - 2b^2 R_{11},$$

leading to

$$\left( 1 + \frac{3b^2}{2} \right) R_{11} = \frac{a^2}{2} R_{00},$$

and

$$R_{00} = -\frac{1}{2} \nabla^2 h_{00},$$
$$R_{11} = -\frac{1}{2} \left( \nabla^2 h_{11} - h_{00,11} + h_{kk,11} - 2h_{kk,11} \right).$$ 

As before, one can combine these equations into

$$h_{11,22} + h_{11,33} = \frac{2 + a^2 + 3b^2}{2 + 3b^2} h_{00,11}$$
$$+ \frac{a^2}{2 + 3b^2} (h_{00,22} + h_{00,33}) - 2h_{22,11}.$$ 

Introducing the expression for $h_{00}$ and $h_{22}$ in the r.h.s. term, one gets

$$h_{11,22} + h_{11,33} = 2M \left[ \frac{a^2(c_0^2 - 1)}{2 + 3b^2} + c_0^2 \right] \frac{2c_0^2 x^2 - y^2 + z^2}{(c_0^2 x^2 + y^2 + z^2)^{3/2}} - 2c_0^2 \frac{2c_0^2 x^2 - y^2 - z^2}{(c_0^2 x^2 + y^2 + z^2)^{3/2}}.$$

\[ (78) \]
It is clear that the solution is a linear combination of $h_{00}$ and $h_{22}$. Indeed, one has

$$h_{11}(x, y, z) = -\left(\frac{a^2(c_0^2 - 1)}{2 + 3b^2} + c_0^2\right) h_{00}(x, y, z) + 2c_0^2 h_{22}(x, y, z) .$$  \hspace{1cm} (87)

One now searches for the off-diagonal component $h_{10}$. The component

$$R_{10} = -(a^2 + b^2) R_{10} - ab(R_{00} + R_{11}) ,$$  \hspace{1cm} (88)

leads to

$$(1 + a^2 + b^2) R_{10} = -ab(R_{00} + R_{11}) ,$$  \hspace{1cm} (89)

and

$$R_{00} = -\frac{1}{2} \nabla^2 h_{00} ,$$  \hspace{1cm} (90)

$$R_{10} = -\frac{1}{2} \left(\nabla^2 h_{10} - h_{10,11}\right) ,$$

$$R_{11} = -\frac{1}{2} \left(\nabla^2 h_{11} - h_{00,11} + h_{kk,11} - 2h_{11,11}\right) .$$

Repeating the above procedure, one obtains

$$h_{01,22} + h_{01,33} = -\frac{ab}{1 + a^2 + b^2}$$

$$\times \left[ h_{00,22} + h_{00,33} + h_{11,22} + h_{11,33} - 2h_{22,11} \right] .$$

Writing $h_{01} = -ab/(1 + a^2 + b^2)(h_{00} + h_{11} + \delta h_{01})$, one finds

$$\delta h_{01,22} + \delta h_{01,33} = 2h_{22,11}$$

$$= \frac{4Mc_0^2(2c_0^2x^2 - y^2 - z^2)}{(c_0^2x^2 + y^2 + z^2)^{5/2}} ,$$  \hspace{1cm} (92)

and hence

$$\delta h_{01}(x, y, z) = -2c_0^2 h_{22}(x, y, z) .$$  \hspace{1cm} (93)

Therefore,

$$h_{01} = \frac{ab}{1 + a^2 + b^2} \left[ \frac{a^2(c_0^2 - 1)}{2 + 3b^2} + c_0^2 \right] h_{00} .$$  \hspace{1cm} (94)

Finally, one computes $h_{00}$ to second order. The results are developed in the Appendix II. It is found that one gets only a correction to the first order term $h^{(1)}_{00}$:

$$h_{00} = \frac{2c_0^2[6 + 9b^2 + (15 + 22b^2)a^2] + a^2b^2}{c_0^2(6 + 15a^2 + b^2)(2 + 3b^2)} h^{(1)}_{00}$$

$$\simeq \left(1 - \frac{b^2}{6}\right) h^{(1)}_{00} .$$  \hspace{1cm} (95)

Since the LSB clearly turns the metric anisotropic, the usual PPN parametrization cannot be straightforwardly used to ascertain its effects. This is so as the PPN formalism relies on a quasi-cartesian frame of reference which, by definition, has all diagonal metric components $g_{ii}$ equal. One might argue that there is a transformation to such a isotropic frame, but the obtained PPN parameters would undoubtedly be unphysical. However, one can extract some PPN-like parameters from the results. First, one notes that

$$\frac{1}{\sqrt{1 + 2k x^2 + y^2 + z^2}} \simeq \frac{1}{r} (1 - k \cos^2 \theta) .$$  \hspace{1cm} (96)

For $h_{00}$, one gets

$$h_{00} = \left(1 - \frac{b^2}{6}\right) 2M \frac{1 - (c_0^2 - 1) \cos^2 \theta}{r} ,$$  \hspace{1cm} (97)

where no $r^{-2}$ correction appears. Thus, one cannot obtain the PPN parameter $\beta$. However, since $h_{11} \neq h_{22} = h_{33}$, the same reasoning allows us to compute two parameters analogous to the $\gamma$ PPN parameter. Recalling that

$$h_{11}(x, y, z) = -\left[ \frac{a^2(c_0^2 - 1)}{2 + 3b^2} + c_0^2 \right] h_{00}(x, y, z)$$

$$+ 2c_0^2 h_{22}(x, y, z) ,$$  \hspace{1cm} (98)

one gets (after neglecting the normalization with respect to $h_{00}$),

$$\gamma_1 = 1 + \cos^2 \theta$$

$$\times \left[ \left[ \frac{a^2(c_0^2 - 1)}{2 + 3b^2} + c_0^2 \right] (1 - c_0^2) + 2c_0^2(1 - c_0^2) \right]$$

$$\simeq 1 + \frac{b^2}{2} \cos^2 \theta ,$$

$$\gamma_2 = 1 + (1 - c_0^2) \cos^2 \theta \simeq 1 - \left(\frac{ab}{2}\right)^2 \cos^2 \theta .$$  \hspace{1cm} (99)

As expected, the effect of the $x$-axis LSB is mostly felt on the $h_{11}$ component. Direct comparison with the PPN parameter $\gamma$ is troublesome, given that the present case is obviously anisotropic. Hence, no clear connection can be derived to link $\gamma$ with $\gamma_1$ and $\gamma_2$.

However, the measured value of $\gamma$ should be of the same order of magnitude as the average of $\gamma_1$ and $\gamma_2$, integrated over one orbit:
\[ \gamma - 1 \simeq \frac{1}{2} (\gamma_1 + \gamma_2) - 1 \simeq \frac{b^2}{4} (\cos^2 \theta) \simeq \frac{b^2}{8} (1 - e^2) \ , \]  

(100)

where \( e \) is the orbit eccentricity. For low values of \( e \) such as those found in the Solar System, one gets \( \gamma \simeq b^2/8 \). Taking \( e \simeq 0 \), the constraint \( \gamma = 1 + (2.1 \pm 2.3) \times 10^{-5} \) now yields \(|b| \leq 1.9 \times 10^{-2}\).

As stated above, the standard PPN analysis fails in the present scenario, which is clearly anisotropic. A discussion involving anisotropy of inertia and its effect in the width of resonance lines can be found in Ref. [20] and references therein (see also Ref. [21]). Presented as a test between Mach’s principle and the equivalence principle, it relies on the hypothetical effect the proximity to the large mass of the galactic core could have on the proton’s mass. By comparison, we note that a radial LSB with the galactic core as source would be perceived as an axial LSB in a region such as the Solar System. In Ref. [22] the bound \( \Delta m/m \leq 3 \times 10^{-22} \) can be found, \( m \) being the proton mass. Comparing with Eq. (97) gives

\[ \frac{\Delta m}{m} = \left( 1 - \frac{b^2}{6} \right) (c_0^2 - 1) \simeq \frac{b^2}{2} \leq 3 \times 10^{-22} \ , \]

(101)

which yields \(|b| \leq 2.4 \times 10^{-11} \), a much more stringent bound than the obtained above.

V. CONCLUSIONS

In this study we have obtained the solutions of a gravity model coupled to a vector field where Lorentz symmetry is spontaneously broken. We have analyzed three cases: purely radial, temporal/radial and temporal/axial LSB.

In the first case, we have found a new black hole solution; we showed that, as in the standard scenario, this solution has a removable singularity at a horizon of radius \( r_s = (2M r_0^{-L})^{1/1-L} \), which is slightly perturbed with respect to the usual Scharzschild radius \( r_s = 2M \). This horizon has an associated Hawking temperature of \( T = (2Mr_0^{-L})^{-LT_0} \), and protects a real singularity at \( r = 0 \). Deviations from Kepler’s law yield \( L \leq 2 \times 10^{-9} \).

The temporal/radial case yields a slightly perturbed metric which leads to PPN parameters \( \beta \approx 1 - (K + K_r)/M \) and \( \gamma \approx 1 - (K + 2K_r)/M \), directly proportional to the strength of the induced effect (given by \( K \) and \( K_r \sim K \)). Unfortunately, since \( K \) and \( K_r \) are integration constants, one cannot constrain the physical parameters from the observed limits on the PPN parameters. Also, analogously to Rosen’s bimetric theory, one can obtain the PPN parameter \( \gamma \approx (A + B)\xi \), where \( \xi \) is the distance to the central body and \( A \) and \( B \) rule the temporal and radial components of the vector field \( v_{\mu v} \).

In the temporal/axial case one gets, as expected, a breakdown of isotropy, and hence a standard PPN analysis is not feasible. However, we have defined the direction-dependent “PPN” parameters \( \gamma_1 \simeq b^2 \cos^2 \theta/2 \) and \( \gamma_2 \simeq a^2 b^2 \cos^2 \theta/4 \) (considering the LSB occurs in the direction of \( x_1 \)). Naturally, \( \gamma_1 \ll \gamma_2 \). A crude estimation of the measurable PPN parameter \( \gamma \) yields \( \gamma \approx b^2 (1 - e^2)/4 \), where \( e \) is the orbit’s eccentricity. This final case requires further study, as its effects cannot be properly accounted for an isotropic formalism such as the parametrized post-Newtonian one. However, comparison with experiments concerning the anisotropy of inertia yields \(|b| \leq 2.4 \times 10^{-11} \).

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VI. APPENDIX I

Substituting Eqs. (76) in Eq. (76), one gets the following relation:

\[ ((2 - b^2)\alpha_1^2 + (b^2 - a^2)) h_{00,11} + (4\alpha_2^2 - (2 + a^2)) h_{00,ii} + (2\alpha_2^2 - a^2) h_{00, jj} = 0 \ , \]

(102)

and hence

\[ ((2 - b^2)\alpha_1^2 + (b^2 - a^2)) c_0^2 (-2\alpha_1^2 x^2 + y^2 + z^2) + (4\alpha_2^2 - (2 + a^2)) (-2x_i^2 + c_0^2 x^2 + x_k^2) + (2\alpha_2^2 - a^2) (-2x_j^2 + c_0^2 x^2 + x_M^2) = 0 \ . \]

(103)

For this equality to hold, the coefficients of each coordinate must vanish, leading to

\[ 2(2 + 5a^2)[-2 + 5a^2)(1 + 2a_1^2 - 2a_2^2 - a_2^2) + 2(1 + 2a_1^2 - 4 + 5a_2^2) - 2a_1^2 - a_2^2]b^2 = 0 \ , \]

(104)
These equations admit the solution

\[ \alpha_1^2 = \frac{-2(2 + 5a^2 - b^2)\alpha_1^2 + 2(1 + 2a^2)b^2}{(2 + 5a^2)(-2 + b^2)} \quad \text{for a free } \alpha_j. \]

\[ \alpha_i^2 = \frac{1 + \alpha_i^2}{2}, \] (105)

for a free \( \alpha_j \).

\**VII. APPENDIX II**

To obtain \( h_{00} \) to second order, one writes \( h_{00} = h_{00}^{(1)} + h_{00}^{(2)} \) and \( R_{00} = R_{00}^{(1)} + R_{00}^{(2)} \), \( R_{ij} = R_{ij}^{(1)} + R_{ij}^{(2)} \). Notice that

\[ R_{00}^{(1)} = R_{00}^{(1)}, \] since it contains no \( h_{00} \) term. The equation for \( R_{00}^{(1)} \) is

\[ R_{00}^{(1)} = \frac{1}{2} \left( a^2 R_{00}^{(1)} + b^2 R_{11}^{(1)} + 2ab R_{10} \right) - 2a^2 R_{00}^{(1)} - 2ab R_{10} \] (106)

One obtains

\[ R_{00}^{(1)} + R_{00}^{(2)} = \frac{1}{2} \left( a^2 R_{00}^{(1)} + b^2 R_{11}^{(1)} + 2ab R_{10} \right) - 2a^2 R_{00}^{(1)} - 2ab R_{10} + \frac{1}{2} \left( a^2 R_{00}^{(2)} + b^2 R_{11}^{(2)} \right) h_{00}^{(1)} + \frac{1}{2} \left( a^2 R_{00}^{(2)} + b^2 R_{11}^{(2)} \right) \left( -1 + h_{00}^{(1)} \right) - 2a^2 R_{00}^{(2)} \]

\[ = \left( (1 + 2a^2) R_{00}^{(1)} + 2ab R_{10} \right) h_{00}^{(1)} + a^2 \left( -\frac{5}{2} + h_{00}^{(1)} \right) R_{00}^{(2)} + \frac{b^2}{2} \left( -1 + h_{00}^{(1)} \right) R_{11}^{(1)}. \] (107)

Hence

\[ \left[ 1 + a^2 \left( \frac{5}{2} - h_{00}^{(1)} \right) \right] R_{00}^{(2)} = \left( (1 + 2a^2) R_{00}^{(1)} + 2ab R_{10} \right) h_{00}^{(1)} + \frac{b^2}{2} R_{11}^{(2)} \left( -1 + h_{00}^{(1)} \right), \] (108)

and

\[ \left[ 1 + a^2 \left( \frac{5}{2} - h_{00}^{(1)} \right) \right] \left( -\frac{1}{2} \nabla^2 h_{00}^{(2)} \right) = \left( 1 + 2a^2 \right) \left[ -\frac{1}{2} \nabla^2 h_{00}^{(1)} + 2ab \left( -\frac{1}{2} \left( \nabla^2 h_{10} - h_{10, 11} \right) \right) \right] h_{00}^{(1)} \]

\[ + \frac{b^2}{2} \left[ -\frac{1}{2} \left( \nabla^2 h_{11} - h_{00, 11}^{(2)} + 2h_{22, 11} - h_{11, 11} \right) \right] \left( -1 + h_{00}^{(1)} \right). \] (109)

Dropping the \((-1 + h_{00}^{(1)})\) term yields

\[ \left( 1 + \frac{5a^2}{2} \right) \nabla^2 h_{00}^{(2)} = -\frac{b^2}{2} \left[ -\frac{1}{2} \left( \nabla^2 h_{11} - h_{00, 11}^{(2)} + 2h_{22, 11} - h_{11, 11} \right) \right]. \] (110)

It follows that

\[ \left( 1 + \frac{5a^2 + b^2}{2} \right) h_{00, 11}^{(2)} + \left( 1 + 5a^2 \right) \left( h_{00, 22}^{(2)} + h_{00, 33}^{(2)} \right) = -\frac{b^2}{2} \left( h_{11, 22} + h_{11, 33} + 2h_{22, 11} \right). \] (111)

Taking the Ansatz \( h_{00}^{(2)} = Ah_{00}^{(1)} \), one finds
\[ \frac{2MAC_0^2(6 + 15a^2 + b^2)}{2 + 5a^2} \frac{2c_0^2x^2 - y^2 - z^2}{(c_0^2x^2 + y^2 + z^2)^{5/2}} = -2M \frac{b^2[a^2(c_0^2 - 1) + c_0^2(2 + 3b^2)]}{(2 + 5a^2)(2 + 3b^2)} \frac{2c_0^2x^2 - y^2 - z^2}{(c_0^2x^2 + y^2 + z^2)^{5/2}}, \] (112)

and

\[ A = -\frac{b^2[a^2(c_0^2 - 1) + c_0^2(2 + 3b^2)]}{c_0^2(6 + 15a^2 + b^2)(2 + 3b^2)}. \] (113)

Thus, one obtains a small correction to \( h_{00}^{(1)} \), but no change in behavior. One can write

\[ h_{00} = h_{00}^{(1)} + h_{00}^{(2)} = \frac{2c_0^2[6 + 9b^2 + (15 + 22b^2)a^2] + a^2b^2}{c_0^2(6 + 15a^2 + b^2)(2 + 3b^2)} h_{00}^{(1)} \simeq \left(1 - \frac{b^2}{6}\right) h_{00}^{(1)}. \] (114)

[1] “CPT and Lorentz Symmetry II”, V. A. Kostelecký (World Scientific, Singapore, 2002).
[2] O. Bertolami, Nucl. Phys. Proc. Suppl. 88, 49 (2000); O. Bertolami in “Decoherence and Entropy in Complex Systems” (Springer-Verlag, Berlin, 2004).
[3] V. A. Kostelecký and S. Samuel, Phys. Rev. D 39, 683 (1989); Phys. Rev. Lett. 66, 1811 (1991).
[4] V. A. Kostelecký and R. Potting, Phys. Rev. D 51, 3923 (1995).
[5] V. A. Kostelecký, Phys. Rev. D 69, 105009 (2004).
[6] R. Gambini, J. Pullin, Phys. Rev. D 59, 124021 (1999).
[7] J. Alfaro, H.A. Morales-Tecotl, L.F. Urrutia, Phys. Rev. Lett. 84, 2183 (2000).
[8] L.J. Garay, Phys. Rev. Lett. 80, 2508 (1998).
[9] S.M. Carroll, J.A. Harvey, V.A. Kostelecký, C.D. Lane, T. Okamoto, Phys. Rev. Lett. 87, 141601 (2001).
[10] O. Bertolami, L. Guisado, Phys. Rev. D 67, 025001 (2003).
[11] V.A. Kostelecký, R. Lehnert, M. J. Perry, Phys. Rev. D 68, 123511 (2003) ; O. Bertolami, R. Lehnert, R. Potting, A. Ribeiro, Phys. Rev. D 69, 083513 (2004).
[12] H. Sato, T. Tati, Progr. Theor. Phys. 47, 1788 (1972); S. Coleman, S.L. Glashow, Phys. Lett. B 405, 249 (1997); Phys. Rev. D 59, 116008 (1999); L. Gonzalez-Mestres, hep-ph/9905430; O. Bertolami, C.S. Carvalho, Phys. Rev. D 61, 103002 (2000); G. Amelino-Camelia, T. Piran, Phys. Lett. B 497, 265 (2001); O. Bertolami, Gen. Relativity and Gravitation 34 707 (2002); R. Lehnert, hep-ph/0312093.
[13] R. Bluhm and V. A. Kostelecký, hep-th/0412320.
[14] N. Rosen, J. Gen. Rel. and Grav. 4, 435 (1973).
[15] C.M. Will, “Theory and Experiment in Gravitational Physics”, C.M. Will (Cambridge U. P., 1993).
[16] M. C. Bento and O. Bertolami, Phys. Lett. B 228, 348 (1999).
[17] “The search for non-Newtonian gravity”, E. Fischbach, C.L. Talmadge (Springer, New York 1999).
[18] C. M. Will, Living Rev. Rel. 4, 4 (2001).
[19] B. Bertotti, L. Iess and P. Tortora, Nature 425, 374 (2003).
[20] S. Weinberg, “Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity” (John Wiley and Sons, New Jersey, 1972).
[21] V. A. Kostelecký and C. D. Lane, J. Math. Phys. 40 6245 (1999).
[22] S. K. Lamoreaux, J. P. Jacobs, B. R. Heckel, F. J. Raab, and E. N. Fortson, Phys. Rev. Lett. 58, 746 (1987).