On Monotonous Separately Continuous Functions

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Let $T = (T, \leq)$ and $T_1 = (T_1, \leq_1)$ be linearly ordered sets and $\mathcal{X}$ be a topological space. The main result of the paper is the following:

If function $f(t, x) : T \times \mathcal{X} \mapsto T_1$ is continuous in each variable ("$t$" and "$x$") separately and function $f_x(t) = f(t, x)$ is monotonous on $T$ for every $x \in \mathcal{X}$, then $f$ is continuous mapping from $T \times \mathcal{X}$ to $T_1$, where $T$ and $T_1$ are considered as topological spaces under the order topology and $T \times \mathcal{X}$ is considered as topological space under the Tychonoff topology on the Cartesian product of topological spaces $T$ and $\mathcal{X}$.

1 Introduction

In 1910 W.H. Young had proved the following theorem.

**Theorem A** (see [11]). Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be separately continuous. If $f(\cdot, y)$ is also monotonous for every $y$, then $f$ is continuous.

In 1969 this theorem was generalized for the case of separately continuous function $f : \mathbb{R}^d \mapsto \mathbb{R}$ ($d \geq 2$):

**Theorem B** (see [7]). Let $f : \mathbb{R}^{d+1} \mapsto \mathbb{R}$ ($d \in \mathbb{N}$) be continuous in each variable separately. Suppose $f(t_1, \ldots, t_d, \tau)$ is monotonous in each $t_i$ separately ($1 \leq i \leq d$). Then $f$ is continuous on $\mathbb{R}^{d+1}$.

Note that theorems A and B were also mentioned in the overview [2]. In the papers [8,9] authors investigated functions of kind $f : T \times \mathcal{X} \mapsto \mathbb{R}$, where $(T, \leq)$ is linearly ordered set equipped by the order topology, $(\mathcal{X}, \mathcal{S}_\mathcal{X})$ is any topological space and the function $f$ is monotonous relatively the first variable as well continuous (or quasi-continuous) relatively the second variable. In particular in [9] it was proven that each separately quasi-continuous and monotonous relatively the first variable function $f : \mathbb{R} \times \mathcal{X} \mapsto \mathbb{R}$ is quasi-continuous relatively the set of variables. The last result may be considered as the abstract analog of Young’s theorem (Theorem A) for separately quasi-continuous functions.

However, we do not know any direct generalization of Theorem A (for separately continuous and monotonous relatively the first variable function) in abstract topological spaces at the present time. In the present paper we prove the generalization of theorems A and B for the case of (separately continuous and monotonous relatively the first variable) function $f : T \times \mathcal{X} \mapsto T_1$, where $(T, \leq)$, $(T_1, \leq_1)$ are linearly ordered sets equipped by the order topology and $\mathcal{X}$ is any topological space.

2 Preliminaries

Let $T = (T, \leq)$ be any linearly (ie totally) ordered set (in the sense of [1]). Then we can define the strict linear order relation on $T$ such, that for any $t, \tau \in T$ the correlation $t < \tau$.
holds if and only if \( t \leq \tau \) and \( t \neq \tau \). Together with the linearly ordered set \( T \) we introduce the linearly ordered set

\[
T_{[\pm\infty]} = (T \cup \{ -\infty, +\infty \}, \leq),
\]

where the order relation is extended on the set \( T \cup \{ -\infty, +\infty \} \) by means of the following clear conventions:

(a) \(-\infty < +\infty\);
(b) \( \forall t \in T \) \(-\infty < t < +\infty\).

Recall \([1]\) that every such linearly ordered set \( T = (T, \leq) \) can be equipped by the natural “internal” order topology \( \mathcal{F}_{\pi} [T] \), generated by the base consisting of the open sets of kind:

\[
(\tau_1, \tau_2) = \{ t \in T \mid \tau_1 < t < \tau_2 \}, \quad \text{where} \quad \tau_1, \tau_2 \in T \cup \{ -\infty, +\infty \}, \quad \tau_1 < \tau_2. \tag{1}
\]

Let \((\mathcal{X}, \mathcal{G}_x)\), \((\mathcal{Y}, \mathcal{G}_y)\) and \((\mathcal{Z}, \mathcal{G}_z)\) be topological spaces, where \( \mathcal{G}_x \subseteq 2^\mathcal{X} \) is the topology on the topological space \( \mathcal{X} \) \((\mathcal{X} \in \{ \mathcal{X}, \mathcal{Y}, \mathcal{Z} \})\). By \( C(\mathcal{X}, \mathcal{Y}) \) we denote the collection of all continuous mappings from \( \mathcal{X} \) to \( \mathcal{Y} \). For a mapping \( f : \mathcal{X} \times \mathcal{Y} \mapsto \mathcal{Z} \) and a point \((x, y) \in \mathcal{X} \times \mathcal{Y} \) we write

\[
f^x(y) := f_y(x) := f(x, y).
\]

Recall \([5]\) that the mapping \( f : \mathcal{X} \times \mathcal{Y} \mapsto \mathcal{Z} \) is referred to as **separately continuous** if and only if \( f^x \in C(\mathcal{Y}, \mathcal{Z}) \) and \( f_y \in C(\mathcal{X}, \mathcal{Z}) \) for every point \((x, y) \in \mathcal{X} \times \mathcal{Y} \) (see also \([8,10]\)). The set of all separately continuous mappings \( f : \mathcal{X} \times \mathcal{Y} \mapsto \mathcal{Z} \) is denoted by \( C(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) \) \([5,8,10]\).

Let \( T = (T, \leq) \) and \( T_1 = (T_1, \leq_1) \) be linearly ordered sets. We say that a function \( f : T \mapsto T_1 \) is **non-decreasing (non-increasing)** on \( T \) if and only if for every \( t, \tau \in T \) the inequality \( t \leq \tau \) leads to the inequality \( f(t) \leq f(\tau) \) (\( f(\tau) \leq f(t) \)) correspondingly. Function \( f : T \mapsto T_1 \), which is non-decreasing or non-increasing on \( T \) is called by **monotonous**.

### 3 Main Results

Let \((\mathcal{X}_1, \mathcal{G}_{x_1}), \ldots, (\mathcal{X}_d, \mathcal{G}_{x_d}) \ (d \in \mathbb{N})\) be topological spaces. Further we consider \( \mathcal{X}_1 \times \cdots \times \mathcal{X}_d \) as a topological space under the Tychonoff topology \( \mathcal{G}_{x_1 \times \cdots \times x_d} \) on the Cartesian product of topological spaces \( \mathcal{X}_1, \ldots, \mathcal{X}_d \). Recall \([6]\) Chapter 3 that topology \( \mathcal{G}_{x_1 \times \cdots \times x_d} \) is generated by the base of kind:

\[
\{ U_1 \times \cdots \times U_d \mid U_j \in \mathcal{G}_{x_j} \ (\forall j \in \{1, \ldots, d\}) \}.
\]

**Theorem 1.** Let \( T = (T, \leq) \) and \( T_1 = (T_1, \leq_1) \) be linearly ordered sets and \((\mathcal{X}, \mathcal{G}_x)\) be a topological space.

If \( f \in C(\mathcal{T} \times \mathcal{X}, \mathcal{T}_1) \) and function \( f_x(t) = f(t, x) \) is monotonous on \( T \) for every \( x \in \mathcal{X} \), then \( f \) is continuous mapping from the topological space \( (\mathcal{T} \times \mathcal{X}, \mathcal{G}_{\mathcal{T} \times \mathcal{X}}) \) to the topological space \( (T_1, \mathcal{G}_{\mathcal{T}_1}) \).

**Proof.** Consider any ordered pair \((t_0, x_0) \in T \times \mathcal{X}\). Take any open set \( V \subseteq T_1 \) such that \( f(t_0, x_0) \in V \). Since the sets of kind \([1]\) form the base of topology \( \mathcal{F}_{\pi} [T]\), there exist \( \tau_1, \tau_2 \in T_1 \cup \{ -\infty, +\infty \} \) such that \( \tau_1 < f(t_0, x_0) < \tau_2 \) and \( (\tau_1, \tau_2) \subseteq V \), where \( \tau_1 \) is the strict linear order, generated by (non-strict) order \( \leq_1 \) (on \( T_1 \cup \{ -\infty, +\infty \} \)). The function \( f \) is separately continuous. So, since the sets of kind \([1]\) form the base of topology \( \mathcal{F}_{\pi} [T]\), there exist \( t_1, t_2 \in T \cup \{ -\infty, +\infty \} \) such that

\[
t_1 < t_2 \quad \text{and} \quad f(t_1, x_0, t_2) \subseteq (\tau_1, \tau_2). \tag{2}
\]

Further we need the some additional denotations.
In the case, where \((t_1, t_0) \neq \emptyset\) we choose any element \(\alpha_1 \in T\) such that \(t_1 < \alpha_1 < t_0\) and denote \(\tilde{\alpha}_1 := \alpha_1\). In the opposite case we denote \(\alpha_1 := t_0, \tilde{\alpha}_1 := t_1\).

In the case \((t_0, t_2) \neq \emptyset\) we choose any element \(\alpha_2 \in T\) such that \(t_0 < \alpha_2 < t_2\) and denote \(\tilde{\alpha}_2 := \alpha_2\). In the opposite case we denote \(\alpha_2 := t_0, \tilde{\alpha}_2 := t_2\).

It is not hard to verify, that in the all cases the following conditions are performed:

\[
\begin{align*}
\alpha_1, \alpha_2 & \in T, \quad \tilde{\alpha}_1, \tilde{\alpha}_2 \in T \cup \{-\infty, +\infty\}; \\
\alpha_1 & \leq \alpha_2; \\
\tilde{\alpha}_1 & < \tilde{\alpha}_2; \\
[\alpha_1, \alpha_2] & \subseteq (t_1, t_2), \quad \text{where} \ [\alpha_1, \alpha_2] = \{t \in T \mid \alpha_1 \leq t \leq \alpha_2\}; \\
t_0 & \in (\tilde{\alpha}_1, \tilde{\alpha}_2) \subseteq [\alpha_1, \alpha_2].
\end{align*}
\]

According to (4), \(\alpha_1, \alpha_2 \in (t_1, t_2)\). Hence, according to (3), interval \((\tau_1, \tau_2)\) is the open neighborhood of the both points \(f(\alpha_1, x_0)\) and \(f(\alpha_2, x_0)\). Since the function \(f\) is separately continuous on \(T \times \mathcal{X}\), then there exist an open neighborhood \(U \in \mathcal{S}_{\mathcal{X}}\) of the point \(x_0\) (in the space \(\mathcal{X}\)) such, that:

\[
\begin{align*}
f(\alpha_1, U) & \subseteq (\tau_1, \tau_2); \\
f(\alpha_2, U) & \subseteq (\tau_1, \tau_2).
\end{align*}
\]

The set \((\tilde{\alpha}_1, \tilde{\alpha}_2) \times U\) is the open neighborhood of the point \((t_0, x_0)\) in the topology \(\mathcal{S}_{T \times \mathcal{X}}\) of the space \(T \times \mathcal{X}\). Now our aim is to prove that

\[
\forall (t, x) \in (\tilde{\alpha}_1, \tilde{\alpha}_2) \times U \ (f(t, x) \in (\tau_1, \tau_2) \subseteq V).
\]

So, chose any point \((t, x) \in (\tilde{\alpha}_1, \tilde{\alpha}_2) \times U\). According to the condition (3), we have \((t, x) \in [\alpha_1, \alpha_2] \times U\), that is \(\alpha_1 \leq t \leq \alpha_2\) and \(x \in U\). In accordance with (3), (4), we have \(f(\alpha_1, x) \in (\tau_1, \tau_2)\) and \(f(\alpha_2, x) \in (\tau_1, \tau_2)\). Hence, since the function \(f(x) = f(t, x)\) is monotonous on \(T\) and \(\alpha_1 \leq t \leq \alpha_2\), we deduce \(f(t, x) \in (\tau_1, \tau_2) \subseteq V\). Thus, the correlation (3) is proven. Hence, the function \(f\) is continuous in (every) point \((t_0, x_0) \in T \times \mathcal{X}\).

Theorem A is a consequence of Theorem 1 in the case \(T = \mathcal{X} = \mathbb{R}\), where \(\mathbb{R}\) is considered together with the usual linear order on the field of real numbers and usual topology.

**Corollary 1.** Let \(T_0 = (T_0, \leq_0), T_1 = (T_1, \leq_1), \ldots, T_d = (T_d, \leq_d) \ (d \in \mathbb{N})\) be linearly ordered sets, and \((\mathcal{X}, \mathcal{S}_{\mathcal{X}})\) be a topological space.

If the function \(f : T_1 \times \cdots \times T_d \times \mathcal{X} \mapsto T_0\) is continuous in each variable separately and \(f(t_1, \ldots, t_d, \tau)\) is monotonous in each \(t_i\) separately \((1 \leq i \leq d)\) then \(f\) is a continuous mapping from the topological space \((T_1 \times \cdots \times T_d \times \mathcal{X}, \mathcal{S}_{T_1 \times \cdots \times T_d \times \mathcal{X}})\) to the topological space \((T_0, \mathcal{S}_{\mathcal{X}i[T_0]}))\).

**Proof.** We will prove this corollary by induction. For \(d = 1\) the corollary is true by Theorem 1. Assume, that the corollary is true for the number \(d - 1\), where \(d \in \mathbb{N}, d \geq 2\). Suppose, that function \(f : T_1 \times \cdots \times T_d \times \mathcal{X} \mapsto T_0\) is satisfying the conditions of the corollary. Then we may consider this function as a mapping from \(T_1 \times \mathcal{X}(d)\) to \(T_0\), where \(\mathcal{X}(d) = T_2 \times \cdots \times T_d \times \mathcal{X}\). According to inductive hypothesis, function \(f(t_1, \cdot)\) is continuous on \(\mathcal{X}(d)\) for every fixed \(t_1 \in T_1\). So \(f\) is a separately continuous mapping from \(T_1 \times \mathcal{X}(d)\) to \(T_0\). Moreover, \(f\) is monotonous relatively the first variable (by conditions of the corollary). Hence, by Theorem 1 \(f\) is continuous on \(T_1 \times \mathcal{X}(d)\).

Theorem B is a consequence of Corollary 1 in the case \(T_0 = T_1 = \cdots = T_d = \mathcal{X} = \mathbb{R}\), where \(\mathbb{R}\) is considered together with the usual linear order on the field of real numbers and usual topology. In the case \(T_0 = \mathbb{R}, T_j = (a_j, b_j), \mathcal{X} = (a_{d+1}, b_{d+1})\) where \(a_j, b_j \in \mathbb{R}\) and \(a_j < b_j\) \((j \in \{1, \ldots, d+1\})\) and intervals \((a_j, b_j)\) are considered together with the usual linear order and topology, induced from the field of real numbers, we obtain the following corollary.
Corollary 2. If the function $f : (a_1, b_1) \times \cdots \times (a_d, b_d) \times (a_{d+1}, b_{d+1}) \rightarrow \mathbb{R}$ ($d \in \mathbb{N}$) is continuous in each variable separately and $f(t_1, \ldots, t_d, \tau)$ is monotonous in each $t_i$ separately ($1 \leq i \leq d$) then $f$ is a continuous mapping from $(a_1, b_1) \times \cdots \times (a_{d+1}, b_{d+1})$ to $\mathbb{R}$.

Remark 1. In fact in the paper [7] the more general result was formulated, in comparison with Theorem B. Namely the author of [7] had considered the real valued function $f(t_1, \ldots, t_d, \tau)$ defined on an open set $G \subseteq \mathbb{R}^{d+1}$, $d \in \mathbb{N}$ such, that $f$ is continuous in each variable separately and monotonous in each $t_i$ separately ($1 \leq i \leq d$). But this result of [7] can be delivered from Corollary 2, because for each point $t = (t_1, \ldots, t_d, \tau) \in G$ in the open set $G$ there exists the set of intervals $(a_1, b_1), \ldots, (a_{d+1}, b_{d+1})$ such, that $t \in (a_1, b_1) \times \cdots \times (a_{d+1}, b_{d+1}) \subseteq G$.

Notes on applications in abstract kinematics. Main results of the paper are expected to be applied in the theory of universal kinematics for establishing some additional properties of coordinate transform operators (between reference frames), separately continuous relatively space and time variables. For more details about the theory of universal kinematics see [3,4] and other papers, reference to which you can find in [3,4].

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