TWO WEIGHT $L^p$ INEQUALITIES FOR $\lambda$-FRACTIONAL VECTOR RIESZ TRANSFORMS AND DOUBLING MEASURES

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Abstract. If $R^\lambda$ denotes the $\lambda$-fractional vector Riesz transform on $\mathbb{R}^n$, $1 < p < \infty$, and $(\sigma, \omega)$ is a pair of doubling measures, then the two weight $L^p$ norm inequality,

$$\int_{\mathbb{R}^n} |R^\lambda (f\sigma)|^p \, d\omega \leq C_{\lambda, p} \int_{\mathbb{R}^n} |f|^p \, d\sigma, \quad f \in L^p(\sigma)$$

holds if and only if the following quadratic triple testing conditions of Hytönen and Vuorinen hold,

$$\int_{\mathbb{R}^n} \left( \sum_{j=1}^\infty (a_j 1_{\lambda I_j} R^\lambda 1_{I_j \sigma})^2 \right)^{\frac{p}{2}} \, d\omega \leq \left( \frac{T_{\lambda, p}^2}{T_{\lambda, p}^2} \right)^{\frac{p}{2}} \int_{\mathbb{R}^n} \left( \sum_{j=1}^\infty (a_j 1_{I_j})^2 \right)^{\frac{p}{2}} \, d\sigma,$$

$$\int_{\mathbb{R}^n} \left( \sum_{j=1}^\infty (a_j 1_{\lambda I_j} R^\lambda 1_{I_j \omega})^2 \right)^{\frac{p'}{2}} \, d\sigma \leq \left( \frac{T_{\lambda, p'}^2}{T_{\lambda, p'}^2} \right)^{\frac{p'}{2}} \int_{\mathbb{R}^n} \left( \sum_{j=1}^\infty (a_j 1_{I_j})^2 \right)^{\frac{p'}{2}} \, d\omega,$$

where the inequalities are taken over all sequences $\{I_j\}_{j=1}^\infty$ and $\{a_j\}_{j=1}^\infty$ of cubes and real numbers respectively. We also show that these quadratic triple testing conditions can be relaxed to local quadratic testing conditions, quadratic offset Muckenhoupt conditions, and a quadratic weak boundedness property.

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1. Introduction

The Nazarov-Treil-Volberg T1 conjecture on the boundedness of the Hilbert transform from one weighted space $L^2(\sigma)$ to another $L^2(\omega)$, was settled affirmatively in the two part paper [LaSaShUr3, Lac] when the measures have no common point masses, and this restriction was removed by Hytönen in [Hyt]. Since then there have been a number of generalizations of boundedness of Calderón-Zygmund operators from one weighted $L^2$ space to another, including

- to higher dimensional Euclidean spaces (see e.g. [SaShUr7], [LaWi] and [LaSaShUrWi]),
- to spaces of homogeneous type (see e.g. [DuLiSaVeWiYa]), and
- and to partial results for the case when both measures are doubling (see [AlSaUr]).

It had been known from work of Neugebauer [Neu] and Coifman and Fefferman [CoFe] some time ago that in the case of $A_\infty$ weights, the two weight norm inequality for a Calderón-Zygmund operator was implied by the classical two weight $A_p$ condition; see [AlSaUr2] for the elementary proof when $p = 2$, and [HyLa] for a sharp estimate on the characteristics. In addition there have been some generalizations to Sobolev spaces in place of $L^2$ spaces in the setting of a single weight (see e.g. [DiWiWi] and [KaLiPeWa]).

The purpose of this paper is to prove a two weight T1 theorem for $\lambda$-fractional vector Riesz transforms on weighted $L^p(\mathbb{R}^n)$ spaces with $1 < p < \infty$, in the special case when the measures are both doubling. In view of the $L^2$ result in [LaSaShUr3, Lac] one might suspect that the Hilbert transform $H$ is bounded from $L^p(\sigma)$ to $L^p(\omega)$ with general locally finite positive Borel measures $\sigma$ and $\omega$ if and only if the local testing conditions for $H$,

$$\int_I |H1_I\sigma|^p \, d\omega \lesssim |I|_\sigma \quad \text{and} \quad \int_I |H1_I\omega|^p \, d\sigma \lesssim |I|_\omega,$$

both hold, along with the tailed Muckenhoupt $A_p$ conditions,

$$\left(\int_I \frac{|I|}{|I| + \text{dist}(x,I)} d\omega \right)^\frac{1}{p} \left(\frac{|I|}{|I|} \right)^\frac{1}{q} \lesssim 1 \quad \text{and} \quad \left(\int_I \frac{|I|}{|I| + \text{dist}(x,I)} d\sigma \right)^\frac{1}{p} \left(\frac{|I|}{|I|} \right)^\frac{1}{q} \lesssim 1.$$

In fact this conjecture was already made in [LaSaUr1], see Conjecture 1.8, where the case of maximal singular integrals was treated when one of the measures was doubling, but with more complicated testing conditions. However, this conjecture fails for the Hilbert transform [AlLaSaUr], and even for pairs of doubling measures and Riesz transforms (including the Hilbert transform) [AlLaSaUr2].

Another stronger conjecture, but difficult nonetheless, has been put forward by Hytönen and Vuorinen [HyVu] pages 16-18, see also [Y1] and [Y2]. Namely, that $H$ is bounded from $L^p(\sigma)$ to $L^p(\omega)$ if and only if certain quadratic interval testing conditions for $H$ hold, along with corresponding quadratic Muckenhoupt conditions and a quadratic weak boundedness property. Here ‘quadratic’ refers to $\ell^2$-valued extensions of the familiar scalar conditions. More generally, these quadratic conditions can be formulated for fractional singular integrals $T^\lambda$ in higher dimensions in a straightforward way.

We emphasize that our doubling assumptions are in part offset by the fact that we characterize boundedness for all vector fractional Riesz transform operators, and in part due to the fact that we have obtained a two weight T1 theorem for $p \neq 2$ (for the first time). If one considers a matrix of Calderón-Zygmund operators and weight pairs such as,

| $T$ | $\sigma, \omega \in A_p$ | $\sigma, \omega \in A_\infty$ | $\sigma, \omega \in \text{doubling}$ | $\sigma, \omega \in \text{Borel}$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| Hilbert | * | * | * | * |
| Cauchy | * | * | * | * |
| Beurling | * | * | * | * |
| Riesz | * | * | * | * |
| General | * | * | * | * |

We have proved here for $1 < p < \infty$??
two features stand out,

(1) for general (locally finite positive) Borel measures, a two weight $T1$ characterization for $1 < p < \infty$ has been found in this matrix only for the Hilbert transform when $p = 2$,

(2) for fractional Riesz transform operators, a two weight $T1$ characterization for $1 < p < \infty$ has been found in this matrix only for pairs of doubling measures.

The starred entries in the matrix correspond to $T1$ characterizations that hold by virtue of the known results, and the question mark entries remain unknown for any $1 < p < \infty$ at this time. Of course there are other geometric restrictions on the measures that give rise to a $T1$ theorem, and these can be found in the references at the end of this paper. On the other hand, it appears quite challenging to find a natural class of measures $\mathcal{M}$, more general than doubling measures, for which a $T1$ theorem can be obtained for all $1 < p < \infty$, all fractional Riesz transforms, and all measure pairs in $\mathcal{M} \times \mathcal{M}$.

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1.1. Quadratic conditions of Hytönen and Vuorinen. For a $\lambda$-fractional singular integral operator $T^\lambda$ on $\mathbb{R}^n$, and locally finite positive Borel measures $\sigma$ and $\omega$, let $T^\lambda_{\omega} f = T^\lambda (f \, d\sigma)$ and $T^\lambda_{\omega} g = T^\lambda_{\omega} (g \, d\omega)$ (see below for definitions). The quadratic cube testing conditions of Hytönen and Vuorinen are

\begin{align}
(1.1) & \left\| \sum_{i=1}^{\infty} \left| a_i 1_{I_i} T^\lambda_{\omega} 1_{I_i} \right|^2 \right\| \leq \mathcal{X}^p_{T^\lambda, p}(\sigma,\omega) \left( \sum_{i=1}^{\infty} |a_i 1_{I_i}|^2 \right)^{\frac{1}{2}}_{L^p(\sigma)}, \\
& \left\| \sum_{i=1}^{\infty} \left| a_i 1_{I_i} T^\lambda_{\omega} 1_{I_i} \right|^2 \right\| \leq \mathcal{X}^p_{T^\lambda, p}(\sigma,\omega) \left( \sum_{i=1}^{\infty} |a_i 1_{I_i}|^2 \right)^{\frac{1}{2}}_{L^p'(\omega)},
\end{align}

taken over all sequences $\{I_i\}_{i=1}^{\infty}$ and $\{a_i\}_{i=1}^{\infty}$ of cubes and numbers respectively. The corresponding quadratic $\text{global}$ cube testing constants $\mathcal{X}^p_{T^\lambda, p}(\sigma,\omega)$ and $\mathcal{X}^p_{T^\lambda, p}(\omega,\sigma)$ are defined as in (1.1), but without the indicator $1_{I_i}$ outside the operator, namely with $1_{I_i} T^\lambda_{\omega} 1_{I_i}$ replaced by $T^\lambda_{\omega} 1_{I_i}$. The quadratic Muckenhoupt conditions of Hytönen and Vuorinen are

\begin{align}
(1.2) & \left\| \sum_{i=1}^{\infty} \left| \int_{\mathbb{R}^n \setminus I_i} \frac{f_i(y)}{|y - c_i|^{n-\lambda}} \, d\sigma(y) \right|^2 1_{I_i} \right\| \leq A^p_{\mathcal{X}^p_{T^\lambda, p}}(\sigma,\omega) \left( \sum_{i=1}^{\infty} |f_i|^2 \right)^{\frac{1}{2}}_{L^p(\sigma)}, \\
& \left\| \sum_{i=1}^{\infty} \left| \int_{\mathbb{R}^n \setminus I_i} \frac{f_i(y)}{|y - c_i|^{n-\lambda}} \, d\omega(y) \right|^2 1_{I_i} \right\| \leq A^p_{\mathcal{X}^p_{T^\lambda, p}}(\omega,\sigma) \left( \sum_{i=1}^{\infty} |f_i|^2 \right)^{\frac{1}{2}}_{L^p'(\omega)},
\end{align}

taken over all sequences $\{I_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$ of cubes and functions respectively. Note that $A^p_{\mathcal{X}^p_{T^\lambda, p}}(\sigma,\omega)$ is homogeneous of degree 1 in the measure pair $(\sigma,\omega)$, as opposed to the usual formulation with degree 2. Finally, the quadratic weak boundedness property of Hytönen and Vuorinen (not so named in [HyVu]) is

\begin{align}
(1.3) & \sum_{i=1}^{\infty} \left| \int_{\mathbb{R}^n} a_i T^\lambda_{\omega} 1_{I_i}(x) b_i 1_{J(I_i)}(x) \, d\omega(x) \right| \\
& \leq \mathcal{WBP}^p_{T^\lambda, p}(\sigma,\omega) \left( \sum_{i=1}^{\infty} |a_i 1_{I_i}|^2 \right)^{\frac{1}{2}}_{L^p(\sigma)} \left( \sum_{i=1}^{\infty} |b_i 1_{I_i}|^2 \right)^{\frac{1}{2}}_{L^p'(\omega)},
\end{align}

taken over all sequences $\{I_i\}_{i=1}^{\infty}$, $\{J(I_i)\}_{i=1}^{\infty}$, $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ of cubes and numbers respectively where $J(I_i)$ denotes any cube adjacent to $I_i$ with the same side length.
If the Calderón-Zygmund operator $T^\lambda$ is bounded from $L^p(\sigma)$ to $L^p(\omega)$, then the Hilbert space valued extension $(T^\lambda)^{\ell^2}$ is bounded from $L^p(\sigma; \ell^2)$ to $L^p(\omega; \ell^2)$, and it is now not hard to see that 
\[
T_{T^\lambda,p}^{\ell^2,\text{global}}(\sigma,\omega) + T_{T^\lambda,p'}^{\ell^2,\text{global}}(\omega,\sigma) + A_p^{\lambda,\ell^2}(\sigma,\omega) + A_p^{\lambda,\ell^2}(\omega,\sigma) \\
+ WBP_{T^\lambda,p}^{\ell^2}(\sigma,\omega) \lesssim \mathfrak{N}_{T^\lambda,p}(\sigma,\omega),
\]
where $\mathfrak{N}_{T^\lambda,p}(\sigma,\omega)$ denotes the operator norm of $H$ from $L^p(\sigma)$ to $L^p(\omega)$, more generally see (1.11) below.

For now the conjecture of Hytönen and Vuorinen for the Hilbert transform $H$ and more generally for $\lambda$-fractional vector Riesz transforms $R^\lambda$ on $\mathbb{R}^n$, in the case that the measures $\sigma$ and $\omega$ are both doubling. Moreover, we use certain ‘logically weaker’ quadratic conditions, which we now describe in the general setting of $\lambda$-fractional Calderón-Zygmund operators $T^\lambda$.

### 1.2. Weaker quadratic conditions for doubling measures.

First, we will use local scalar testing conditions,
\[
\| 1_I T^\lambda 1_I \|_{L^p(\omega)} \leq T_{T^\lambda,p}(\sigma,\omega) |I|^{\frac{1}{p}},
\]
\[
\| 1_I T^\lambda 1_I \|_{L^p(\sigma)} \leq T_{T^\lambda,p'}(\omega,\sigma) |I|^{\frac{1}{p'}},
\]
which do not involve any vector-valued extensions.

Second, we will typically use $\ell^2$ in a superscript instead of quad to indicate a ‘quadratic’ constant, and we will use quadratic offset Muckenhoupt conditions given by
\[
\left\| \left( \sum_{i=1}^{\infty} a_i \left| I_i^\sigma \right|^{\frac{1}{2}} \left| I_i^\omega \right|^{\frac{1}{2}} \right)^\frac{1}{2} \right\|_{L^p(\omega)} \leq A_p^{\lambda,\ell^2,\text{offset}}(\sigma,\omega) \left\| \left( \sum_{i=1}^{\infty} |a_i|^2 1_I^\omega \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)},
\]
where for each $i$, the minimums are taken over the finitely many dyadic cubes $I_i^\sigma$ such that $\ell(I_i^\sigma) = \ell(I_i^\omega)$ and $\text{dist}(I_i^\sigma, I_i^\omega) \leq C_0 \ell(I_i^\sigma)$ for some positive constant $C_0$. Of course, when the measures are doubling, we may take $I_i^\sigma = I_i^\omega$ so that (1.15) is equivalent to the following condition of Vuorinen [Vuo2] that was introduced in the context of dyadic shifts,
\[
\left\| \left( \sum_{i=1}^{\infty} a_i \left| I_i^\sigma \right|^{\frac{1}{2}} \left| I_i^\omega \right|^{\frac{1}{2}} \right)^\frac{1}{2} \right\|_{L^p(\omega)} \leq A_p^{\lambda,\ell^2,\text{offset}}(\sigma,\omega) \left\| \left( \sum_{i=1}^{\infty} |a_i|^2 1_I^\omega \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)}.
\]

We prove below that the offset constants $A_p^{\lambda,\ell^2,\text{offset}}(\sigma,\omega)$ in (1.15) are necessary for the norm inequality $\| T^\lambda f \|_{L^p(\omega)} \leq \mathfrak{N}_{T^\lambda}(\sigma,\omega) \| f \|_{L^p(\sigma)}$ when $\sigma$ and $\omega$ are doubling. Here we simply note that using the Fefferman-Stein vector-valued inequality for the maximal function $M_\sigma$ on a space of homogeneous type $(\mathbb{R}^n, |\cdot|, \sigma)$ [GrLYa], we see that $A_p^{\lambda,\ell^2,\text{offset}}(\sigma,\omega)$ is smaller than $A_p^{\lambda,\ell^2,\text{quad}}(\sigma,\omega)$ for doubling measures because
\[
\frac{|I_i^\sigma|}{|I_i|} \leq \int_{\mathbb{R}^n \setminus I_i} M_\sigma 1_{I_i^\sigma} (y) \frac{1}{|y-c_i|^{n-\lambda}} \sigma \left( \frac{y}{|y-c_i|} \right) dy, \quad \text{when } I_i^\sigma \cap I_i = \emptyset.
\]

Such use of the Fefferman-Stein vector-valued inequality occurs frequently in the sequel. Note again that $A_p^{\lambda,\ell^2,\text{offset}}(\sigma,\omega)$ is homogeneous of degree 1 in the measure pair $(\sigma, \omega)$.

\[\text{In applications one takes } C_0 \text{ sufficiently large depending on the Stein elliptic constant for the operator } T^\lambda. \text{ But if } \sigma \text{ is doubling the condition doesn’t depend on } C_0.\]
Third, we use a variant of the weak boundedness property \[1.3\] of Hytönen and Vuorinen given by

\[
(1.7) \quad \sum_{i=1}^{\infty} \sum_{I_i^* \in \text{Adj}(I_i)} \left| \int_{\mathbb{R}^n} a_i T_\omega^{\lambda} 1_{I_i} (x) b_i^* 1_{I_i^*} (x) \, d\omega (x) \right| 
\leq WBP_{T_\omega^{\lambda},p} (\sigma, \omega) \left\| \left( \sum_{i=1}^{\infty} |a_i 1_{I_i}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} \left\| \left( \sum_{i=1}^{\infty} \sum_{I_i^* \in \text{Adj}(I_i)} |b_i^* 1_{I_i^*}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)},
\]

where for \( I \in \mathcal{D} \), its \textit{adjacent} cubes are defined by

\[
(1.8) \quad \text{Adj}(I) \equiv \{ I^* \in \mathcal{D} : \overline{T} \cap T \neq \emptyset \text{ and } \ell(I^*) = \ell(I) \},
\]

and in particular include \( I \) itself.

Finally, we also define the stronger quadratic \textit{triple} testing constants by

\[
(1.9) \quad \left\| \left( \sum_{i=1}^{\infty} (a_i 1_{3I_i} T_\omega^{\lambda} 1_{I_i})^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \leq \mathcal{F}_{T_\omega^{\lambda},p} (\sigma, \omega) \left\| \left( \sum_{i=1}^{\infty} (a_i 1_{I_i})^2 \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)},
\]

\[
\left\| \left( \sum_{i=1}^{\infty} (a_i 1_{3I_i} T_\omega^{\lambda,*,p} 1_{I_i})^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \leq \mathcal{F}_{T_\omega^{\lambda,*,p},p'} (\omega, \sigma) \left\| \left( \sum_{i=1}^{\infty} (a_i 1_{I_i})^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(\omega)}.
\]

1.3. \textbf{Statement of the main theorem.} Our main theorem is restrict to \( \lambda \)-fractional vector Riesz transforms \( \mathbf{R}^\lambda \), but we will continue with general \( \lambda \)-fractional vector Calderón-Zygmund operators \( T^\lambda \) in describing the setup.

Denote by \( \Omega_{\text{dyad}} \) the collection of all dyadic grids in \( \mathbb{R}^n \), and let \( \mathcal{Q}^n \) denote the collection of all cubes in \( \mathbb{R}^n \) having sides parallel to the coordinate axes. A positive locally finite Borel measure \( \mu \) on \( \mathbb{R}^n \) is said to be doubling if there is a constant \( C_{\text{doub}} \), called the doubling constant, such that

\[
|2Q|_\mu \leq C_{\text{doub}} |Q|_\mu, \quad \text{for all cubes } Q \in \mathcal{Q}^n.
\]

For \( 0 \leq \lambda < n \) we define a smooth \( \lambda \)-fractional Calderón-Zygmund kernel \( K^\lambda (x, y) \) to be a function \( K^\lambda : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) satisfying the following fractional size and smoothness conditions

\[
(1.10) \quad |\nabla_x K^\lambda (x, y)| + |\nabla_y K^\lambda (x, y)| \leq C_{L, \lambda} |x - y|^{\lambda - j - n}, \quad 0 \leq j < \infty,
\]

and we denote by \( T^\lambda \) the associated \( \lambda \)-fractional singular integral on \( \mathbb{R}^n \). Following [Steb] (39) on page 210 as in [AlSaUr], we say that a \( \lambda \)-fractional Calderón-Zygmund kernel \( K^\lambda \) is \textit{elliptic in the sense of Stein} if there is a unit vector \( \mathbf{u}_0 \in \mathbb{R}^n \) and a constant \( c > 0 \) such that

\[
|K^\lambda (x, x + t \mathbf{u}_0)| \geq c |t|^{\lambda - n}, \quad \text{for all } t \in \mathbb{R}.
\]

1.3.1. \textbf{Defining the norm inequality.} As in [SaShUr9] see page 314, we introduce a family \( \left\{ \eta^\lambda_{\delta, R} \right\}_{0 < \delta < R < \infty} \) of smooth nonnegative functions on \([0, \infty)\) so that the truncated kernels \( K^\lambda_{\delta, R} (x, y) = \eta^\lambda_{\delta, R} (|x - y|) K^\lambda (x, y) \) are bounded with compact support for fixed \( x \) or \( y \), and uniformly satisfy \[1.10\]. Then the truncated operators

\[
T^\lambda_{\sigma, \delta, R} f (x) \equiv \int_{\mathbb{R}^n} K^\lambda_{\delta, R} (x, y) f (y) \, d\sigma (y), \quad x \in \mathbb{R}^n,
\]

are pointwise well-defined when \( f \) is bounded with compact support, and we will refer to the pair \( \left( K^\lambda, \left\{ \eta^\lambda_{\delta, R} \right\}_{0 < \delta < R < \infty} \right) \) as a \( \lambda \)-fractional singular integral operator, which we typically denote by \( T^\lambda \), suppressing the dependence on the truncations. For \( 1 < p < \infty \), we say that a \( \lambda \)-fractional singular integral operator \( T^\lambda = \left( K^\lambda, \left\{ \eta^\lambda_{\delta, R} \right\}_{0 < \delta < R < \infty} \right) \) satisfies the norm inequality

\[
(1.11) \quad \left\| T^\lambda f \right\|_{L^p(\omega)} \leq C_{T^\lambda} (\sigma, \omega) \left\| f \right\|_{L^p(\sigma)}, \quad f \in L^p(\sigma),
\]
where \( \mathcal{M}_{\lambda \sigma} (\sigma, \omega) \) denotes the best constant in (1.11), provided
\[
\| T_{\sigma, R}^\lambda f \|_{L^p(\omega)} \leq \mathcal{M}_{\lambda \sigma} (\sigma, \omega) \| f \|_{L^p(\sigma)}, \quad f \in L^p(\sigma), 0 < \delta < R < \infty.
\]
In the presence of the classical Muckenhoupt condition \( A_\sigma^{\lambda} \), it can be easily shown that the norm inequality is independent of the choice of truncations used - see e.g. [LaSaShUr] where rough operators are treated in the case \( p = 2 \), but the proofs can be modified. We can now state our main theorem. Note that the second two parts of the theorem apply to vector Riesz transforms only.

**Theorem 2.** Suppose that \( 1 < p < \infty \), that \( \sigma \) and \( \omega \) are locally finite positive Borel measures on \( \mathbb{R}^n \), and that \( T_\lambda \) is a smooth \( \lambda \)-fractional singular integral operator on \( \mathbb{R}^n \). Denote by \( \mathcal{M}_{\lambda \sigma} (\sigma, \omega) \) the smallest constant \( C \) in the two weight norm inequality
\[
(1.12) \quad \| T_\sigma^\lambda f \|_{L^p(\omega)} \leq C \| f \|_{L^p(\sigma)}.
\]
(1) Then
\[
\| T_\sigma^\lambda f \|_{L^p(\omega)} \leq \mathcal{M}_{\lambda \sigma} (\sigma, \omega) \| f \|_{L^p(\sigma)}.
\]
(2) Suppose in addition that \( \sigma \) and \( \omega \) are doubling measures on \( \mathbb{R}^n \), and that \( T_\lambda \) is replaced by the \( \lambda \)-fractional vector Riesz transform \( R_\lambda^\lambda \) on \( \mathbb{R}^n \). Then the two weight norm inequality (1.12) holds provided the quadratic weak boundedness property (1.7) holds, and the quadratic local testing conditions (1.9) hold, and moreover, in this case we have
\[
(1.13) \quad \| T_\sigma^\lambda f \|_{L^p(\omega)} \leq \mathcal{M}_{\lambda \sigma} (\sigma, \omega) \| f \|_{L^p(\sigma)}.
\]
(3) Suppose in addition that \( \sigma \) and \( \omega \) are doubling measures on \( \mathbb{R}^n \), and that \( T_\lambda \) is replaced by the \( \lambda \)-fractional vector Riesz transform \( R_\lambda^\lambda \) on \( \mathbb{R}^n \). Then the two weight norm inequality (1.12) holds if and only if the quadratic triple testing conditions (1.10) hold, and moreover,
\[
\| T_\sigma^\lambda f \|_{L^p(\omega)} \leq \mathcal{M}_{\lambda \sigma} (\sigma, \omega) \| f \|_{L^p(\sigma)}.
\]

The constants on the right hand side of (1.10) represent the most 'elementary' constants we were able to find that characterize the norm of the two weight inequality for Riesz transforms and doubling measures when \( p \neq 2 \).

**Remark 3.** In the case of equal measures \( \sigma = \omega \), the quadratic \( A_\sigma^{\lambda_1} \) and \( A_\sigma^{\lambda_2, \text{offset}} \) conditions trivially reduce to the scalar \( A_\sigma^{\lambda_1} \) and \( A_\sigma^{\lambda_2, \text{offset}} \) conditions respectively. We show in the appendix that \( A_\sigma^{\lambda_1} (\sigma, \omega) + A_\sigma^{\lambda_2, \text{offset}} (\omega, \sigma) \) is not controlled by \( A_\sigma^{\lambda_1} (\sigma, \omega) \) in general, but the case of doubling measures remains open.

We also note that our weak boundedness property (1.7) excludes the case \( I_1^* = I_1 \). Finally, we note that our proof shows that we can extend the theorem to include all smooth Stein elliptic Calderón-Zygmund operators if we assume the classical pivotal condition.

Part (3) is an easy corollary of parts (1) and (2). Indeed, it is trivial that
\[
\| T_\sigma^\lambda f \|_{L^p(\omega)} \leq \mathcal{M}_{\lambda \sigma} (\sigma, \omega) \| f \|_{L^p(\sigma)}, \quad f \in L^p(\sigma), 0 < \delta < R < \infty.
\]
And a simple exercise to see that for general measures,
\[
\| T_\sigma^\lambda f \|_{L^p(\omega)} \leq \mathcal{M}_{\lambda \sigma} (\sigma, \omega) \| f \|_{L^p(\sigma)}.
\]

**Notation 4.** In the interest of reducing notational clutter we will sometimes omit specifying the measure pair and simply write \( \mathcal{M}_{\lambda \sigma} (\sigma, \omega) \) and \( A_\sigma^{\lambda, \text{offset}} (\sigma, \omega) \) etc. especially when in line.
2. Organization of the Proof

We follow the overall outline of an argument for the case $p = 2$ given in [AlSaUr], but only for Haar wavelets which simplifies matters a bit, but also with a number of adaptations to the use of square functions. The proof of Theorem 2 is achieved by proving the bilinear form bound,

$$
\frac{\langle |R_\lambda a f|, g \rangle_{\lambda}}{\|f\|_{L^p(\mu)} \|g\|_{L^p(\mu)}} \lesssim \mathfrak{T}_{R_\lambda a,p} + \mathfrak{T}_{R_\lambda a,p'} + \mathfrak{WB}B_{\lambda, p} + A_{p}^{\lambda, \ell^2, \text{offset}} + A_{p}^{\lambda, \ell^2, \text{offset}} ,
$$

for good functions $f$ and $g$ in the sense of Nazarov, Treil and Volberg, see [NTV] for the treatment we use here\footnote{See also [SaShUr10] Subsection 3.1 for a treatment using finite collections of grids, in which case the conditional probability arguments are elementary.} Following the weighted Haar expansions as given by Nazarov, Treil and Volberg in [NTV], we write $f$ and $g$ in weighted Alpert wavelet expansions,

$$
\langle R_\lambda a f, g \rangle_{\lambda} = \left\langle \sum_{J \in \mathcal{D}} \Delta_J^\lambda f, \sum_{J \in \mathcal{D}} \Delta_J^\lambda g \right\rangle_{\lambda} = \sum_{J \in \mathcal{D} \text{ and } J \in \mathcal{D}} \langle \lambda a f, \lambda a g \rangle_{\lambda}.
$$

The sum is further decomposed, as depicted in the brief schematic diagram below, by first Cube Size Splitting, then using the Shifted Corona Decomposition, according to the Canonical Splitting. All of these ‘descriptive’ expressions will be defined as the proof proceeds.

Here is the brief schematic diagram as in [AlSaUr], summarizing the shifted corona decompositions as used in [AlSaUr] and [SaShUr] for Alpert and Haar wavelet expansions of $f$ and $g$, and where $T_\lambda$ is a smooth $\lambda$-fractional Calderón-Zygmund operator in $\mathbb{R}^n$. The parameter $\rho$ is defined below.

\[
\begin{array}{cccccc}
\langle T_\lambda f, g \rangle_{\lambda} & \downarrow & B_{\rho, \emptyset} (f, g) & + & B_{\rho, \emptyset} (f, g) & + & B_{\rho, \emptyset} (f, g) & + & B_{\rho, \emptyset} (f, g) & + & B_{\rho, \emptyset} (f, g) \\
\downarrow & \text{[duality]} & & & & & & & & & \\
\text{T_diagonal} (f, g) & + & T_{\text{far below}} (f, g) & + & T_{\text{far above}} (f, g) & + & T_{\text{disjoint}} (f, g) & & & & \\
\downarrow & & & & & & & & & & \\
B_{\rho, \emptyset}^1 (f, g) & + & T_{\text{far below}}^1 (f, g) & + & T_{\text{far above}}^1 (f, g) & + & T_{\text{disjoint}}^1 (f, g) & & & & \\
\downarrow & & & & & & & & & & \\
B_{\rho, \emptyset}^2 (f, g) & + & B_{\rho, \emptyset}^2 (f, g) & + & B_{\rho, \emptyset}^2 (f, g) & + & B_{\rho, \emptyset}^2 (f, g) & & & & \\
\downarrow & & & & & & & & & & \\
A_{\rho, \emptyset}^{\lambda, \ell^2, \text{offset}} & + & A_{\rho, \emptyset}^{\lambda, \ell^2, \text{offset}} & + & A_{\rho, \emptyset}^{\lambda, \ell^2, \text{offset}} & + & A_{\rho, \emptyset}^{\lambda, \ell^2, \text{offset}} & & & &
\end{array}
\]

The condition that is used to control the indicated form is given in square brackets directly underneath. Note that all forms are controlled solely by the quadratic offset Muckenhoupt condition, save for the adjacent form which uses only the weak boundedness property, and the paraproduct form which uses only the scalar testing condition.

There are however notable exceptions in our treatment here as compared to that in [AlSaUr]. For example, we use only the classical Calderón-Zygmund stopping time to bound the forms, and we control the stopping form by Muckenhoupt, Riesz testing and doubling conditions. We will bound the remaining forms using only the fact that for a doubling measure $\mu$, the Poisson averages reduce to ordinary averages in the presence of vector Riesz testing and the Muckenhoupt condition. Indeed, the Poisson kernel of order $\lambda$ is given by

$$
P^\lambda (Q, \mu) \equiv \int_{\mathbb{R}^n} \ell (Q) \left( \ell (Q) + |y - c_Q| \right)^{n+1-\lambda} d\mu (y),
$$

and a doubling measure $\mu$ has a ‘doubling exponent’ $\theta > 0$ and a positive constant $c$ that satisfy the condition,

$$
|2^{-j} Q|_{\mu} \geq c 2^{-j^\theta} |Q|_{\mu}, \quad \text{for all } j \in \mathbb{N}.
$$
Thus if $\mu$ has doubling exponent $\theta$ and $\kappa > \theta + \lambda - n$, we have

\begin{equation}
\begin{aligned}
P^\lambda (Q, \mu) &= \int_{\mathbb{R}^n} \frac{\ell (Q)}{\ell (Q) + |x - c_Q|} |x - c_Q|^{n+\lambda - \delta} d\mu (x) \\
&= \ell (Q)^{\lambda - n} \left\{ \int_Q + \sum_{j=1}^{\infty} \int_{2^j Q \setminus 2^{j-1} Q} \frac{1}{1 + |x - c_Q|} |x - c_Q|^{n+\lambda - \delta} d\mu (x) \right\} \\
&\approx |Q|^{\frac{\lambda - 1}{2}} \sum_{j=0}^{\infty} 2^{-j(n+\lambda - \delta)} |2^j Q|_\mu \approx |Q|^{\frac{\lambda - 1}{2}} \sum_{j=0}^{\infty} 2^{-j(n+\lambda - \delta)} \frac{1}{c_{2^{-j}Q}} |Q|_\mu \approx C_{n, \kappa, \lambda, \theta} |Q|^\frac{\lambda - 1}{2} |Q|_\mu .
\end{aligned}
\end{equation}

We now turn to defining the decompositions of the bilinear form $\langle T^\lambda_\sigma f, g \rangle_\omega$ used in the schematic diagram above. For this we first need some preliminaries. We introduce parameters $r, \varepsilon, \rho, \tau$ as in [AlSaUr] and [SaShUr7]. We will choose $\varepsilon > 0$ sufficiently small later in the argument, and then $r$ must be chosen sufficiently large depending on $\varepsilon$ in order to reduce matters to $(r, \varepsilon)$ good functions by the Nazarov, Treil and Volberg argument - see either [NTV4] or [SaShUr7] for details.

**Definition 5.** The parameters $\tau$ and $\rho$ are fixed to satisfy

$$\tau > r \text{ and } \rho > r + \tau,$$

where $\tau$ is the goodness parameter already fixed.

Let $\mu$ be a positive locally finite Borel measure on $\mathbb{R}^n$ that is doubling, let $\mathcal{D}$ be a dyadic grid on $\mathbb{R}^n$, and let $\{\Delta^\mu_Q\}_{Q \in \mathcal{D}}$ be the set of weighted Haar projections on $L^2 (\mu)$ and $E^\mu_T$ the associated set of projections (see [RaSaWi] for definitions). Recall also the following bound for the ‘average’ projections $E^\mu_I = (E^\mu_I f) 1_I$:

\begin{equation}
\|E^\mu_I f\|_{L^\infty (\mu)} \lesssim E^\mu_I |f| \leq \sqrt{\frac{1}{|I|}} \int_I |f|^2 d\mu, \text{ for all } f \in L^2_{\text{loc}} (\mu).
\end{equation}

In terms of the Haar coefficient vectors

$$\hat{f} (I) \equiv \{ \langle f, h^\mu_{I, a} \rangle \}_{a \in \Gamma_{I,n}}$$

for an orthonormal basis $\{h^\mu_{I, a}\}_{a \in \Gamma_{I,n}}$ of $L^2 (\mu)$ where $\Gamma_{I,n}$ is a convenient finite index set of size $d_Q$, we thus have

\begin{equation}
\|\Delta^\mu_I f\|_{L^2 (\mu)} \leq \|\Delta^\mu_I f\|_{L^\infty (\mu)} \sqrt{|I|}_\mu \lesssim \|\Delta^\mu_I f\|_{L^2 (\mu)} = |\hat{f} (I)|.
\end{equation}

**Notation 6.** We will write the equal quantities $|\hat{f} (I)|$ and $\|\Delta^\mu_I f\|_{L^2 (\mu)}$ interchangeably throughout the paper, depending on context.

### 2.1. The cube size and corona decompositions.

Now we can define the cube size decomposition in the second row of the diagram as given in [AlSaUr]. For a sufficiently large positive integer $\rho \in \mathbb{N}$, we let

\begin{equation}
\text{Adj}_\rho (I) \equiv \left\{ J \in \mathcal{D} : 2^{-\rho} \leq \frac{\ell (J)}{\ell (I)} \leq 2^\rho \text{ and } J \cap \overline{T} \neq \emptyset \right\}, \quad I \in \mathcal{D},
\end{equation}

be the finite collection of dyadic cubes of side length between $2^{-\rho} \ell (I)$ and $2^\rho \ell (I)$, and whose closures have nonempty intersection. We write $J \Subset_{\rho, \varepsilon} I$ to mean that $J \subset I$, $\ell (J) \leq 2^{-\rho} \ell (I)$ and $\text{dist} (J, \partial I) >
two weight $L^p$ inequalities

$2 \sqrt{\ell(J) \ell(I)} (I^J)^{1-\varepsilon}$. Then we write

$$\langle T_\sigma^\lambda f, g \rangle_\omega = \sum_{I, J \in D} \langle T_\sigma^\lambda \Delta^\sigma_I f, \Delta^\omega_J g \rangle_\omega$$

$$= \sum_{I, J \in D, J \in p, I} \langle T_\sigma^\lambda \Delta^\sigma_I f, \Delta^\omega_J g \rangle_\omega + \sum_{I, J \in D, J \in p, I} \langle T_\sigma^\lambda \Delta^\sigma_I f, \Delta^\omega_J g \rangle_\omega$$

$$+ \sum_{I, J \in D, | I \cap I \sigma | < 2^{-\varepsilon} \text{ or } | I \cap I \sigma | > 2^\varepsilon} \langle T_\sigma^\lambda \Delta^\sigma_I f, \Delta^\omega_J g \rangle_\omega$$

$$+ \sum_{I, J \in D, I \in Adj(p)} \langle T_\sigma^\lambda \Delta^\sigma_I f, \Delta^\omega_J g \rangle_\omega$$

$$\equiv B_{\omega, p} (f, g) + B_{\omega, p} (f, g) + B_{\omega} (f, g) + B_{\omega} (f, g) + B_{\omega} (f, g).$$

The disjoint and comparable forms $B_{\omega} (f, g)$ and $B_{\omega} (f, g)$ are controlled using only the quadratic offset Muckenhoupt condition, while the adjacent form $B_{\omega} (f, g)$ is controlled by the Alpert weak boundedness property. The above form $B_{\omega, p} (f, g)$ is handled exactly as is the below form $B_{\omega, p} (f, g)$ but interchanging the measures $\sigma$ and $\omega$, and the exponents $p$ and $p'$, as well as using the duals of the scalar testing and quadratic Muckenhoupt testing conditions. So it remains only to treat the below form $B_{\omega, p} (f, g)$, to which we now turn.

In order to describe the ensuing decompositions of $B_{\omega, p} (f, g)$, we first need to introduce the corona and shifted corona decompositions of $f$ and $g$ respectively. We construct the Calderón-Zygmund corona decomposition for a function $f$ in $L^p (\mu)$ (where $\mu = \sigma$ here, and where $\mu = \omega$ when treating $B_{\omega, p} (f, g)$) and that is supported in a dyadic cube $F_1^0$. Fix $\Gamma > 1$ and define $\mathcal{G}_0 = \{ F_1^0 \}$ to consist of the single cube $F_1^0$, and define the first generation $\mathcal{G}_1 = \{ F_k^1 \}$ of CZ stopping children of $F_1^0$ to be the maximal dyadic subcubes $I$ of $F_0$ satisfying

$$E_\sigma^\mu | f | \geq \Gamma E_{F_1^0}^\mu | f | .$$

Then define the second generation $\mathcal{G}_2 = \{ F_k^2 \}$ of CZ stopping children of $F_1^0$ to be the maximal dyadic subcubes $I$ of some $F_k^1 \in \mathcal{G}_1$ satisfying

$$E_\sigma^\mu | f | \geq \Gamma E_{F_k^1}^\mu | f | .$$

Continue by recursion to define $\mathcal{G}_n$ for all $n \geq 0$, and then set

$$\mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{G}_n = \{ F_k^n : n \geq 0, k \geq 1 \}$$

to be the set of all CZ stopping intervals in $F_1^0$ obtained in this way.

The $\mu$-Carleson condition for $\mathcal{F}$ follows from the first step,

$$\sum_{F' \in \mathcal{F} (F)} | F' |_{\mu} \leq \frac{1}{\Gamma} \sum_{F' \in \mathcal{F} (F)} \frac{1}{E_{F'}^\mu | f |} \int_{F'} | f | \mu \leq \frac{1}{\Gamma} | F |_{\mu} .$$

Moreover, if we define

$$\alpha_{\mathcal{F}} (F) = \sup_{F' \in \mathcal{F} : F' \subset F'} E_{F'}^\mu | f | ,$$

then in each corona

$$C_{\mathcal{F}} \equiv \{ I \in D : I \subset F \text{ and } I \not\subset F' \text{ for any } F' \in \mathcal{F} \text{ with } F' \not\subset F \} ,$$

we have, from the definition of the stopping times, the following average control

$$E_{I}^\mu | f | \leq \Gamma \alpha_{\mathcal{F}} (F) , \quad I \in C_{\mathcal{F}} \text{ and } F \in \mathcal{F} .$$

Finally, as in [NTV4], [LaSaShUr3] and [SaShUr7], we obtain the Carleson condition and quasiorthogonality inequality,

$$\sum_{F' \subset F} | F' |_{\mu} \leq C_0 | F |_{\mu} \text{ for all } F \in \mathcal{F} ; \quad \text{and } \quad \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}} (F)^2 | F |_{\mu} \leq C_0^2 \| f \|^2_{L^2 (\mu)} .$$
where \( \leq \) denotes the tree relation \( F' \subset F \) for \( F', F \in \mathcal{F} \). Moreover, there is the following useful consequence of (2.9) that says the sequence \( \{ \alpha_{F'}(F) 1_F \}_{F \in \mathcal{F}} \) has an additional *quasiorthogonal* property relative to \( f \) with a constant \( C'_0 \) depending only on \( C_0 \):

\[
(2.10) \quad \left\| \sum_{F \in \mathcal{F}} \alpha_{F'}(F) 1_F \right\|_{L^2(\mu)}^2 \leq C'_0 \| f \|_{L^2(\mu)}^2.
\]

Indeed, this is an easy consequence of a geometric decay in levels of the tree \( \mathcal{F} \), that follows in turn from the Carleson condition in the first inequality of (2.9).

This geometric decay asserts that there are positive constants \( C_1 \) and \( \varepsilon \), depending on \( C_0 \), such that if \( \mathcal{C}^{(n)}_F(F) \) denotes the set of \( n \)th generation children of \( F \) in \( \mathcal{F} \),

\[
(2.11) \quad \sum_{F' \in \mathcal{C}^{(n)}_F(F)} |F'|_\mu \leq (C_1 2^{-\varepsilon n})^2 |F|_\mu, \quad \text{for all } n \geq 0 \text{ and } F \in \mathcal{F}.
\]

To see this, let \( \beta_k(F) \equiv \sum_{F' \in \mathcal{C}^{(k)}_F(F)} |F'|_\mu \) and note that \( \beta_{k+1}(F) \leq \beta_k(F) \) implies that for any integer \( N \geq C \), we have

\[
(N + 1) \beta_N(F) \leq \sum_{k=0}^{N} \beta_k(F) \leq C |F|_\mu,
\]

and hence

\[
\beta_N(F) \leq \frac{C}{N+1} |F|_\mu < \frac{1}{2} |F|_\mu, \quad \text{for } F \in \mathcal{F} \text{ and } N = [2C].
\]

It follows that

\[
\beta_{\ell N}(F) \leq \frac{1}{2} \beta_{(\ell - 1) N}(F) \leq ... \leq \frac{1}{2^\ell} \beta_0(F) = \frac{1}{2^\ell} |F|_\mu, \quad \ell = 0, 1, 2, ...
\]

and so given \( n \in \mathbb{N} \), choose \( \ell \) such that \( \ell N \leq n < (\ell + 1) N \), and note that

\[
\sum_{F' \in \mathcal{C}^{(n)}_F(F)} |F'|_\mu = \beta_n(F) \leq \beta_{\ell N}(F) \approx C_1 2^{-\varepsilon n} |F|_\mu,
\]

which proves the geometric decay (2.11).

Now let \( \sigma \) and \( \omega \) be doubling measures and define the two corona projections

\[
P^\sigma_{C_F} = \sum_{I \in \mathcal{C}_F} \Delta^\sigma_I \quad \text{and} \quad P^{\omega}_{C_F-\text{shift}} = \sum_{J \in \mathcal{C}_F-\text{shift}} \Delta^\omega_J,
\]

where

\[
C_F-\text{shift} \equiv [C_F \setminus N'_{D}(F)] \cup \bigcup_{F' \in \mathcal{F}(F)} [N'_{D}(F') \setminus N'_{D}(F)];
\]

where \( N'_{D}(F) \equiv \{ J \in D : J \subset F \text{ and } \ell(J) > 2^{-\tau} \ell(F) \} \),

and note that \( f = \sum_{F \in \mathcal{F}} P^\sigma_{C_F} f \). Thus the corona \( C_F-\text{shift} \) has the top \( \tau \) levels from \( C_F \) removed, and includes the first \( \tau \) levels from each of its \( \mathcal{F} \)-children, except if they have already been removed.

### 2.2. The canonical splitting.

We can now continue with the definitions of decompositions in the schematic diagram above. To bound the below form \( B_{\in_{\rho, \sigma}}(f, g) \), we proceed with the *Canonical Splitting of*

\[
B_{\in_{\rho, \sigma}}(f, g) = \sum_{I, J \in \mathcal{D}} \langle T^\rho_{I} (\Delta^{\rho}_{I} f), (\Delta^{\rho}_{J} g) \rangle_{\omega}
\]
The final two forms \( T_{\text{far above}} (f, g) \) and \( T_{\text{disjoint}} (f, g) \) each vanish just as in \( \text{SaShUr7} \) and \( \text{AlSaUr} \), since there are no pairs \((I, J) \in \mathcal{C}_F \times \mathcal{C}_G\) with both (i) \( I \not\subset G \) and (ii) either \( F \not\subset G \) or \( G \cap F = \emptyset \). The far below form \( T_{\text{far below}} (f, g) \) is then further split into two forms \( T^1_{\text{far below}} (f, g) \) and \( T^2_{\text{far below}} (f, g) \) as in \( \text{SaShUr7} \) and \( \text{AlSaUr} \),

\[
T_{\text{far below}} (f, g) = \sum_{F \in \mathcal{F}} \sum_{G \subset F} \sum_{I \in \mathcal{C}_F \text{ and } J \in \mathcal{C}_{G^-}^-} \langle T^\lambda \triangle_I^f, \triangle_J^g \rangle \omega
\]

\[
= \sum_{F \in \mathcal{F}} \sum_{G \subset F} \sum_{I \in \mathcal{C}_F} \langle T^\lambda \triangle_I^f, \triangle_J^g \rangle \omega
\]

\[
- \sum_{F \in \mathcal{F}} \sum_{G \subset F} \sum_{I \in \mathcal{C}_F} \langle T^\lambda \triangle_I^f, \triangle_J^g \rangle \omega
\]

where for \( F \in \mathcal{F} \) we use the shorthand

\[
\langle T^\lambda \triangle_I^f, \triangle_J^g \rangle \omega = \sum_{I \in \mathcal{C}_F, J \in \mathcal{C}_G^-} \langle T^\lambda \triangle_I^f, \triangle_J^g \rangle \omega.
\]

The second far below form \( T^2_{\text{far below}} (f, g) \) satisfies

\[
|T^2_{\text{far below}} (f, g)| \lesssim \left( A^{\lambda, e^2, \text{offset}}_p (\sigma, \omega) + WBP_{\text{T}}^2 (\sigma, \omega) \right) \|f\|_{L^p(\sigma)} \|g\|_{L^p(\omega)},
\]

which follows in an easy way from (6.3) and (6.7) and their corollaries - see below. To control the first and main far below form \( T^1_{\text{far below}} (f, g) \), we will use some new quadratic arguments exploiting Carleson measure conditions to establish

\[
|T^1_{\text{far below}} (f, g)| \lesssim A^{\lambda, e^2, \text{offset}}_p (\sigma, \omega) \|f\|_{L^p(\sigma)} \|g\|_{L^p(\omega)}.
\]

To handle the diagonal term \( T_{\text{diagonal}} (f, g) \), we further decompose according to the stopping times \( F \),

\[
T_{\text{diagonal}} (f, g) = \sum_{F \in \mathcal{F}} B^F_{\rho, e} (f, g), \quad \text{where } B^F_{\rho, e} (f, g) = \langle T^\lambda \triangle^f_{\rho, e}, \triangle^g_{\rho, e} \rangle \omega,
\]

where we recall that in \( \text{AlSaUr} \) for \( p = 2 \), the following estimate was obtained,

\[
|B^F_{\rho, e} (f, g)| \lesssim \left( 2T^\lambda + \sqrt{A^2_2} \right) \left( \|E_{F, \rho} f\|_{L^\infty} \sqrt{|F| + \|P^\sigma_{C_F} f\|_{L^2(\sigma)}} + \|P^\sigma_{C_F} f\|_{L^2(\sigma)} \right) \|P^\omega_{C_F^-} \triangle g\|_{L^2(\omega)}.
\]

This was achieved by implementing the classical reach of Nazarov, Treil and Volberg using Haar wavelet projections \( \triangle^f_{\rho, e} \), where by ‘reach’ we mean the ingenious ‘thinking outside the box’ idea of the paraproduct / stopping / neighbour decomposition of Nazarov, Treil and Volberg \( \text{NTV4} \).
2.3. The Nazarov, Treil and Volberg reach. Here is the Nazarov, Treil and Volberg decomposition, or reach. We have that \( B^F_{\rho, \tau, \kappa} (f, g) \) equals
\[
\sum_{I \in \mathcal{C}_F \text{ and } J \in \mathcal{C}_F^{\rho, \tau, \kappa}} \langle T_\kappa^J (1_{I_J} \Delta_J^\rho f), \Delta_J^\rho g \rangle_{\omega} + \sum_{I \in \mathcal{C}_F} \sum_{J \in \mathcal{C}_F^{\rho, \tau, \kappa}} \langle T_\kappa^J (1_{\theta(I_J)} \Delta_J^\rho f), \Delta_J^\rho g \rangle_{\omega}
\]
\[
\equiv B^F_{\text{home}} (f, g) + B^F_{\text{neighbour}} (f, g),
\]
and we further decompose the home form using the constant \( \kappa \) for Theorem 7.

A quadratic Carleson measure inequality.

2.4. A quadratic Carleson measure inequality. We end this section with a quadratic Carleson measure inequality we will need for bounding the stopping form below.

**Theorem 7.** Suppose that the triple \((C_0, \mathcal{F}, \alpha_F)\) constitutes stopping data for a function \( f \in L^1_{\text{loc}} (\mu) \), and for \( \kappa \in \mathbb{Z}_+ \), set
\[
\alpha^\kappa_F (x) = \{ \alpha_F (F) 1_{F^\kappa} (x) \}_{F \in \mathcal{F}} \quad \text{where} \quad F^\kappa = \bigcup_{G \in \mathcal{C}_G^\kappa (F)} G.
\]
Then for $1 < p < \infty$,

\begin{equation}
\int_{\mathbb{R}^n} |\alpha_{\mathcal{F}}(x)|_p^p \, d\mu(x) = \int_{\mathbb{R}^n} \left( \sum_{F \in \mathcal{F}} |\alpha_{\mathcal{F}}(F)|^2 1_F(x) \right)^{\frac{p}{2}} \, d\mu(x) \leq C_\delta 2^{-\delta \kappa} \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F)^p |F|_\mu^p,
\end{equation}

where $\delta > 0$ is the constant in (2.11). The inequality can be reversed for $\kappa = 0$ and $2 \leq p < \infty$.

**Proof of Theorem 7.** We claim that for $1 < p < \infty$, i.e.

\begin{equation}
\int_{\mathbb{R}^n} \left( \sum_{F \in \mathcal{F}} |\alpha_{\mathcal{F}}(F)|^2 1_F(x) \right)^{\frac{p}{2}} \, d\mu(x) \leq C_\delta \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F)^p |F|_\mu^p.
\end{equation}

Indeed, for $1 < p \leq 2$ (and even for $0 < p \leq 2$), the inequality follows from the trivial inequality $\|\cdot\|_{\ell^q} \leq \|\cdot\|_{\ell^1}$ for $0 < q \leq 1$,

\begin{equation}
\int_{\mathbb{R}^n} \left( \sum_{F \in \mathcal{F}} |\alpha_{\mathcal{F}}(F)|^2 1_F(x) \right)^{\frac{p}{2}} \, d\mu(x) \leq \int_{\mathbb{R}^n} \sum_{F \in \mathcal{F}} |\alpha_{\mathcal{F}}(F)|^p 1_F(x) \, d\mu(x)
= \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F)^p |F|_\mu \leq C_\delta \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F)^p |F|_\mu^p,
\end{equation}

where $\delta > 0$ is the geometric decay in generations exponent in (2.11).

Now we turn to the case $p \geq 2$. When $p = 2m$ is an even positive integer, we will set

\[ \mathcal{F}_{2m} = \{(F_1, ..., F_{2m}) \in \mathcal{F} \times ... \times \mathcal{F} : F_i \subset F_j \text{ for } 1 \leq i \leq j \leq 2m, \text{ and } F_i = F_{i+1} \text{ for all odd } i\}, \]

and then by symmetry we can arrange the intervals below in nondecreasing order to obtain

\begin{equation}
\int_{\mathbb{R}^n} \left( \sum_{F \in \mathcal{F}} |\alpha_{\mathcal{F}}(F)|^2 1_F(x) \right)^{\frac{p}{2}} \, d\mu(x) = \int_{\mathbb{R}^n} \left( \sum_{F \in \mathcal{F}} |\alpha_{\mathcal{F}}(F)|^2 1_F(x) \right)^{\frac{p}{2}} \, d\mu(x)
= \int_{\mathbb{R}^n} \sum_{(F_1, ..., F_{2m}) \in \mathcal{F}_{2m}} \alpha_{\mathcal{F}}(F_1) ... \alpha_{\mathcal{F}}(F_{2m}) 1_{F_1 \cap ... \cap F_{2m}} \, d\mu(x)
= C_m \int_{\mathbb{R}^n} \sum_{(F_1, ..., F_{2m}) \in \mathcal{F}_{2m}} \alpha_{\mathcal{F}}(F_1) ... \alpha_{\mathcal{F}}(F_{2m}) 1_{F_1 \cap ... \cap F_{2m}} \, d\mu(x)
= C_m \int_{\mathcal{F}_{2m}} \sum_{(F_1, ..., F_{2m}) \in \mathcal{F}_{2m}} \alpha_{\mathcal{F}}(F_1) ... \alpha_{\mathcal{F}}(F_{2m}) |F_1|_\mu = C_m \operatorname{Int}(m),
\end{equation}

where from the geometric decay in (2.11), we obtain

\begin{equation}
\operatorname{Int}(m) \equiv \sum_{(F_1, ..., F_{2m}) \in \mathcal{F}_{2m}} \alpha_{\mathcal{F}}(F_1) ... \alpha_{\mathcal{F}}(F_{2m}) |F_1|_\mu \leq \operatorname{Int}(m),
\end{equation}

where $\operatorname{Int}(m) \equiv \sum_{(F_1, ..., F_{2m}) \in \mathcal{F}_{2m}} \alpha_{\mathcal{F}}(F_1) ... \alpha_{\mathcal{F}}(F_{2m}) |F_1|_\mu$.

We now set about showing that

\begin{equation}
\operatorname{Int}(m) \leq \sum_{F \in \mathcal{F}} |\alpha_{\mathcal{F}}(F)|^{2m} |F|_\mu^m.
\end{equation}

For this, we first prove (2.10) in order to outline the main idea. Using the geometric decay in (2.11) once more we obtain

\begin{equation}
\sum_{F' \in \mathcal{F}[F]} \alpha_{\mathcal{F}}(F') |F'|_\mu \leq \sum_{n=0}^{\infty} \sqrt{\sum_{F' \in \mathcal{F}[F]} \alpha_{\mathcal{F}}(F')^2 |F'|_\mu} C_n \sqrt{|F|_\mu} \leq C_n \sqrt{|F|_\mu} \sqrt{\sum_{F' \in \mathcal{F}[F]} \alpha_{\mathcal{F}}(F')^2 |F'|_\mu},
\end{equation}

where
and hence that
\[
\sum_{F \in \mathcal{F}} \alpha_F(F) \left\{ \sum_{F' \in \mathcal{F}} \alpha_F(F') |F'|_\mu \right\} \lesssim \sum_{F \in \mathcal{F}} \alpha_F(F) \sqrt{|F|_\mu} \left( \sum_{F' \in \mathcal{F}} \alpha_F(F')^2 |F'|_\mu \right) \lesssim \|f\|_{L^2(\mu)} \left( \sum_{F' \in \mathcal{F}} \alpha_F(F')^2 |F'|_\mu \right) \lesssim \|f\|_{L^2(\mu)}^2.
\]

This proves (2.11) since \( \|\sum_{F \in \mathcal{F}} \alpha_F(F) 1_F \|_{L^2(\mu)}^2 \) is dominated by twice the left hand side above.

We now adapt this last argument to apply to (2.21). For example, in the case \( m = 2 \), we have that
\[
\text{Int (2)} = \sum_{F_4 \in \mathcal{F}} \alpha_F(F_4) \sum_{F_3 \in \mathcal{F}} \alpha_F(F_3) \sum_{F_2 \in \mathcal{F}} \alpha_F(F_2) \sum_{F_1 \in \mathcal{F}} \alpha_F(F_1) |F_1|_\mu.
\]

which is at most (we continue to write \( m \) in place of 2 until the very end of the argument)
\[
C_\delta \sum_{n_3=0}^\infty \sum_{n_2=0}^\infty \sum_{n_1=0}^\infty \sum_{F_4 \in \mathcal{F}} \alpha_F(F_4) \sum_{F_3 \in \mathcal{F}^{(n_3)}(F_4)} \alpha_F(F_3)
\]
\[
\times \sum_{F_2 \in \mathcal{F}^{(n_2)}(F_3)} \alpha_F(F_2) \left( 2^{-\delta n_2} |F_2|_\mu \right)^{2m-1} \left( \sum_{F_1 \in \mathcal{F}^{(n_1)}(F_2)} \alpha_F(F_1)^{2m} |F_1|_\mu \right)^{\frac{1}{2m}}
\]
\[
= C_\delta \sum_{n_3=0}^\infty \sum_{n_2=0}^\infty \sum_{n_1=0}^\infty 2^{-\delta \frac{2m-1}{2m} n_1} \sum_{F_4 \in \mathcal{F}} \alpha_F(F_4) \sum_{F_3 \in \mathcal{F}^{(n_3)}(F_4)} \alpha_F(F_3)
\]
\[
\times \sum_{F_2 \in \mathcal{F}^{(n_2)}(F_3)} \alpha_F(F_2) |F_2|_\mu^{\frac{1}{2m}} \left( \sum_{F_1 \in \mathcal{F}^{(n_1)}(F_2)} \alpha_F(F_1)^{2m} |F_1|_\mu \right)^{\frac{1}{2m}} |F_2|_\mu^{-\frac{2m}{2m-2}},
\]

which is in turn dominated by
\[
C_\delta \sum_{n_3=0}^\infty \sum_{n_2=0}^\infty \sum_{n_1=0}^\infty 2^{-\delta \frac{2m-1}{2m} n_1} \sum_{F_4 \in \mathcal{F}} \alpha_F(F_4) \sum_{F_3 \in \mathcal{F}^{(n_3)}(F_4)} \alpha_F(F_3)
\]
\[
\times \left( \sum_{F_2 \in \mathcal{F}^{(n_2)}(F_3)} \alpha_F(F_2)^{2m} |F_2|_\mu \right)^{\frac{1}{2m}} \left( \sum_{F_2 \in \mathcal{F}^{(n_2)}(F_3)} \sum_{F_1 \in \mathcal{F}^{(n_1)}(F_2)} \alpha_F(F_1)^{2m} |F_1|_\mu \right)^{\frac{1}{2m}} |F_2|_\mu^{\frac{2m}{2m-2}} |F_3|_\mu^{\frac{2m-2}{2m-2}},
\]

where in the last line we have applied Hölder’s inequality with exponents \( (2m, 2m, \frac{2m}{2m-2}) \), and then used that \( \sum_{F_2 \in \mathcal{F}^{(n_2)}(F_3)} |F_2|_\mu \leq C_\delta 2^{-\delta n_2} |F_3|_\mu \).
Continuing in this way, we dominate the sum above by

\[
\leq \sum_{n_3=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} 2^{-\delta} \frac{2m-1}{2m} n_1 \sum_{F_4 \in \mathcal{F}} \alpha_{\mathcal{F}}(F_4) \sum_{F_3 \in \mathcal{C}_{\mathcal{F}}^{(n_3)}(F_4)} \alpha_{\mathcal{F}}(F_3) \\
\times \left( \sum_{F_2 \in \mathcal{C}_{\mathcal{F}}^{(n_2)}(F_3)} \alpha_{\mathcal{F}}(F_2)^{2m} |F_2|_{\mu} \right)^{-\frac{1}{2m}} \left( \sum_{F_2 \in \mathcal{C}_{\mathcal{F}}^{(n_2)}(F_3)} \sum_{F_1 \in \mathcal{C}_{\mathcal{F}}^{(n_1)}(F_2)} \alpha_{\mathcal{F}}(F_1)^{2m} |F_1|_{\mu} \right)^{\frac{1}{2m}} \left(2^{-\delta} |F_3|_{\mu}\right)^{1-\frac{m}{2m}} \]

\[
= \sum_{n_3=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} 2^{-\delta} \frac{1}{2m} n_1 |1-(1-\frac{2m}{2m})n_2| |1-(1-\frac{2m}{2m})n_3| \sum_{F_4 \in \mathcal{F}} \alpha_{\mathcal{F}}(F_4) \frac{1}{|F_4|_{\mu}} \left( \sum_{F_3 \in \mathcal{C}_{\mathcal{F}}^{(n_3)}(F_4)} \alpha_{\mathcal{F}}(F_3)^{2m} |F_3|_{\mu} \right)^{-\frac{1}{2m}} \\
\times \left( \sum_{F_2 \in \mathcal{C}_{\mathcal{F}}^{(n_2)}(F_3)} \sum_{F_1 \in \mathcal{C}_{\mathcal{F}}^{(n_1)}(F_2)} \alpha_{\mathcal{F}}(F_1)^{2m} |F_1|_{\mu} \right)^{\frac{1}{2m}} |F_3|_{\mu}^{-\frac{m}{2m}} \\
\times \left( \sum_{F_2 \in \mathcal{C}_{\mathcal{F}}^{(n_2)}(F_3)} \sum_{F_1 \in \mathcal{C}_{\mathcal{F}}^{(n_1)}(F_2)} \sum_{F_4 \in \mathcal{F}} \alpha_{\mathcal{F}}(F_4)^{2m} |F_4|_{\mu} \right)^{-\frac{1}{2m}} |F_4|_{\mu}^{2m-4} ,
\]

which is at most

\[
\leq \sum_{n_3=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} 2^{-\delta} \frac{1}{2m} n_1 |1-(1-\frac{2m}{2m})n_2| |1-(1-\frac{2m}{2m})n_3| \left( \sum_{F_4 \in \mathcal{F}} \alpha_{\mathcal{F}}(F_4)^{2m} |F_4|_{\mu} \right)^{-\frac{1}{2m}} \\
\times \left( \sum_{F_3 \in \mathcal{C}_{\mathcal{F}}^{(n_3)}(F_4)} \sum_{F_2 \in \mathcal{C}_{\mathcal{F}}^{(n_2)}(F_3)} \alpha_{\mathcal{F}}(F_2)^{2m} |F_2|_{\mu} \right)^{-\frac{1}{2m}} \left( \sum_{F_1 \in \mathcal{C}_{\mathcal{F}}^{(n_1)}(F_2)} \sum_{F_4 \in \mathcal{C}_{\mathcal{F}}^{(n_3)}(F_3)} \sum_{F_2 \in \mathcal{C}_{\mathcal{F}}^{(n_2)}(F_3)} \alpha_{\mathcal{F}}(F_2)^{2m} |F_2|_{\mu} \right)^{\frac{1}{2m}} \\
\times \left( \sum_{F_1 \in \mathcal{C}_{\mathcal{F}}^{(n_1)}(F_2)} \sum_{F_3 \in \mathcal{C}_{\mathcal{F}}^{(n_3)}(F_4)} \sum_{F_2 \in \mathcal{C}_{\mathcal{F}}^{(n_2)}(F_3)} \sum_{F_1 \in \mathcal{C}_{\mathcal{F}}^{(n_1)}(F_2)} \alpha_{\mathcal{F}}(F_1)^{2m} |F_1|_{\mu} \right)^{-\frac{1}{2m}} .
\]
Finally, since
\[
\sum_{F_1 \in \mathcal{F}} \sum_{F_3 \in \mathcal{F}} \sum_{F_2 \in \mathcal{C}_3} \alpha_{\mathcal{F}}(F_1) \leq \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F) 2^{m} |F|_{\mu},
\]
we obtain that \(|\text{Int}(2)|\) is dominated by
\[
\sum_{n_3=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} 2^{-\delta[(1-\frac{1}{d_0})n_1+(1-\frac{1}{d_0})n_2+(1-\frac{1}{d_0})n_3]} \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F) 2^{m} |F|_{\mu} = C_{\delta,p} \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F) 2^{m} |F|_{\mu}.
\]
This together with (2.21), proves
\[
\int_{\mathbb{R}^n} |\alpha_{\mathcal{F}}(x)|_{l^2}^4 \, d\mu(x) \lesssim \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F) |F|_{\mu}.
\]
Similarly, we can show for \(m \geq 3\) that
\[
\int_{\mathbb{R}^n} |\alpha_{\mathcal{F}}(x)|_{l^2}^2 \, d\mu(x) \lesssim \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F) 2^{m} |F|_{\mu}.
\]
Altogether then we have
\[
\int_{\mathbb{R}} |\alpha_{\mathcal{F}}(x)|_{l^p}^p \, d\mu(x) \lesssim \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F)^p |F|_{\mu}, \quad \text{for } p \in (0,2) \cup \{2m\}_{m \in \mathbb{N}},
\]
where \(\alpha_{\mathcal{F}}(x) = \{\alpha_{\mathcal{F}}(F) \mathbf{1}_F(x)\}_{F \in \mathcal{F}}\). Marcinkiewicz interpolation \cite{GaRu} Theorem 1.18 on page 480\) applied with the linear operator taking sequences of numbers \(\{\alpha_{\mathcal{F}}(F)\}_{F \in \mathcal{F}} \in \ell^p(\mathcal{F},|F|_{\mu})\) to sequences of functions \(\{\alpha_{\mathcal{F}}(F) \mathbf{1}_F(x)\}_{F \in \mathcal{F}} \in L^p(\ell^2;\omega)\), now gives this inequality for all \(1 < p < \infty\), and this completes the proof of (2.20), which is the inequality in (2.19).

For the reverse inequality when \(2 \leq p < \infty\), we have with \(\alpha_{\mathcal{F}}(x) = \alpha_{\mathcal{F}}^0(x)\) that
\[
\int_{\mathbb{R}^n} |\alpha_{\mathcal{F}}(x)|_{l^p}^p \, d\mu(x) = \int_{\mathbb{R}^n} \left( \sum_{F \in \mathcal{F}} |\alpha_{\mathcal{F}}(F) \mathbf{1}_F(x)|^2 \right)^{\frac{p}{2}} \, d\mu(x)
\]
\[\gtrsim \int_{\mathbb{R}^n} \sum_{F \in \mathcal{F}} |\alpha_{\mathcal{F}}(F) \mathbf{1}_F(x)|^p \, d\mu(x) = \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F)^p |F|_{\mu}.
\]
\[\square\]

3. Square functions and vector-valued inequalities

Recall that the Haar square function
\[
S_{\text{Haar}}f(x) = \left( \sum_{I \in \mathcal{D}} |\Delta_I^\mu f(x)|^2 \right)^{\frac{1}{2}}
\]
is bounded on \(L^p(\mu)\) for any \(1 < p < \infty\) and any locally finite positive Borel measure \(\mu\) - simply because \(S_{\text{Haar}}\) is the martingale difference square function of an \(L^p\) bounded martingale. We now extend this result to more complicated square functions.

Fix a \(\mathcal{D}\)-dyadic cube \(F_0\), let \(\mu\) be a locally finite positive Borel measure on \(F_0\), and suppose that \(\mathcal{F}\) is a subset of \(\mathcal{D}_{F_0} = \{I \in \mathcal{D} : I \subseteq F_0\}\). We say that \(F' \subseteq \mathcal{F}\) is an \(\mathcal{F}\)-child of \(F\) if \(F' \subseteq F\), and is maximal with respect to this inclusion. The collection \(\{\mathcal{C}_F\}_{F \in \mathcal{F}}\) of subsets \(\mathcal{C}_F \subseteq \mathcal{D}_{F_0}\) is defined by
\[
\mathcal{C}_F = \{I \in \mathcal{D} : I \subseteq F \text{ and } I \not\subseteq F' \text{ for any } \mathcal{F}\text{-child } F' \text{ of } F\}, \quad F \in \mathcal{F},
\]
and satisfy the properties

\[ C_F \] is connected for each \( F \in \mathcal{F} \),
\[ F \in C_F \text{ and } I \in C_F \implies I \subset F \text{ for each } F \in \mathcal{F} ,
\[ C_F \cap C_{F'} = \emptyset \text{ for all distinct } F, F' \in \mathcal{F} ,
\[ D_{F_0} = \bigcup_{F \in \mathcal{F}} C_F . \]

The subset \( C_F \) of \( D \) is referred to as the \( \mathcal{F} \)-corona with top \( F \). Define the Haar corona projections \( P_{\mu}^{\mu} = \sum_{I \in C_F} \Delta_I^{\mu} \) and group them together according to their depth in the tree \( \mathcal{F} \) into the projections

\[ P_{k}^{\mu} = \sum_{F \in \mathcal{E}^{(k)}_F (F_0)} P_{\mu}^{\mu} . \]

Note that the \( k \)th grandchildren \( F \in \mathcal{E}^{(k)}_F (F_0) \) are pairwise disjoint and hence so are the supports of the functions \( P_{\mu}^{\mu} f \) for \( F \in \mathcal{E}^{(k)}_F (F_0) \). Define the \( \mathcal{F} \)-square function \( S_{\mathcal{F}} f \) by

\[ S_{\mathcal{F}} f (x) = \left( \sum_{k=0}^{\infty} | P_{\mu}^{\mu} f (x) |^2 \right)^{\frac{1}{2}} = \left( \sum_{F \in \mathcal{F}} | P_{\mu}^{\mu} f (x) |^2 \right)^{\frac{1}{2}} = \left( \sum_{F \in \mathcal{F}} \sum_{I \in C_F} \Delta_I^{\mu} f (x) \right)^{\frac{1}{2}} . \]

Now note that the sequence \( \{ P_{\mu}^{\mu} f (x) \}_{F \in \mathcal{F}} \) of functions is the martingale difference sequence of the \( L^p \) bounded martingale \( \{ E_{\mathcal{F}}^\mu f (x) \}_{F \in \mathcal{F}} \) with respect to the increasing sequence \( \{ \mathcal{E}_k \}_{k=0}^{\infty} \) of \( \sigma \)-algebras, where \( \mathcal{E}_k \) is the \( \sigma \)-algebra generated by the ‘atoms’ \( F \in \mathcal{E}^{(k)}_F (F_0) \), i.e.

\[ \mathcal{E}_k \equiv \left\{ E \text{ Borel } \subset F_0 : E \cap F \in \{ \emptyset, F \} \text{ for all } F \in \mathcal{E}^{(k)}_F (F_0) \right\} , \]

and where

\[ E_{\mathcal{F}}^\mu f (x) = \begin{cases} E_{\mathcal{F}}^\mu f & \text{if } x \in F \text{ for some } F \in \mathcal{E}^{(k)}_F (F_0) \\ f (x) & \text{if } x \in F_0 \setminus \bigcup_{F \in \mathcal{E}^{(k)}_F (F_0)} F \end{cases} ; \]

where \( \bigcup_{F \in \mathcal{E}^{(k)}_F (F_0)} (F_0) = \bigcup_{F \in \mathcal{E}^{(k)}_F (F_0) \setminus F} F \).

Indeed, if \( E \in \mathcal{E}_{k-1} \), then

\[
\int_E E_{\mathcal{F}}^\mu f (x) \, d\mu (x) = \int_{E \setminus \bigcup_{F \in \mathcal{E}^{(k-1)}_F (F_0)} F} E_{\mathcal{F}}^\mu f (x) \, d\mu (x) + \sum_{F \in \mathcal{E}^{(k-1)}_F (F_0) : F \subset E} \int_F E_{\mathcal{F}}^\mu f (x) \, d\mu (x) = \int_{E \setminus \bigcup_{F \in \mathcal{E}^{(k-1)}_F (F_0)} F} E_{\mathcal{F}}^\mu f (x) \, d\mu (x) + \sum_{F \in \mathcal{E}^{(k-1)}_F (F_0) : F \subset E} \int_F f (x) \, d\mu (x) = \int_{E \setminus \bigcup_{F \in \mathcal{E}^{(k-1)}_F (F_0)} F} E_{\mathcal{F}}^\mu f (x) \, d\mu (x) + \sum_{F \in \mathcal{E}^{(k-1)}_F (F_0) : F \subset E} \int_F E_{\mathcal{F}}^\mu f (x) \, d\mu (x) \]

shows that \( \{ E_{\mathcal{F}}^\mu f (x) \}_{F \in \mathcal{F}} \) is a martingale. Finally, it is easy to check that the Haar support of the function \( P_{k}^{\mu} f = E_{\mathcal{F}}^\mu f - E_{\mathcal{F}}^\mu f_{k-1} \) is precisely \( \bigcup_{F \in \mathcal{E}^{(k)}_F (F_0)} C_F \), the union of the coronas associated to the \( k \)-grandchildren of \( F_0 \).

From Burkholder’s martingale transform theorem, for a nice treatment see Hyytinen [Hyt2], we obtain the inequality

\[
\left\| \sum_{k=0}^{\infty} v_k P_{k}^{\mu} f \right\|_{L^p (\mu)} \leq C_p \left( \sup_{0 \leq k < \infty} |v_k| \right) \| f \|_{L^p (\mu)} ,
\]
for all sequences $v_k$ of predictable functions. Now we take $v_k = \pm 1$ randomly on $\Delta^\mu_f \equiv 1_F P^\mu_k$ for $F \in \mathcal{E}_F^{(k)}(F_0)$, and then an application of Khintchine’s inequality, for which see [MuSc] Lemma 5.5 page 114] and [Wo] Proposition 4.5 page 28, allows us to conclude that the square function satisfies the following $L^p(\mu)$ bound,
\[
\|S_f f\|_{L^p(\mu)} \leq C_p \|f\|_{L^p(\mu)} \quad \text{for all } 1 < p < \infty.
\]

We now note that from this result, we can obtain the square function bounds we need for the nearby and paraproduct forms treated below, which include both of the square functions $S_F$ and
\[
S_{F^\tau - \text{shift}} f(x) \equiv \left( \sum_{\mathcal{F} \in \mathcal{F}} \left| \mathcal{P}^{\mu}_{\mathcal{F}, \tau - \text{shift}} f(x) \right|^2 \right)^{\frac{1}{2}}.
\]
Indeed, we first note that if we take $\mathcal{F} = \mathcal{D}F_0$, then we obtain the bound
\[
\|S_{\text{Haar}} f\|_{L^p(\mu)} \leq C_p \|f\|_{L^p(\mu)} \quad \text{for all } 1 < p < \infty;
\]

Then using,
\[
C_F \setminus C_{\mathcal{F}, \tau - \text{shift}} \subset N_F \quad \text{and} \quad C_{\mathcal{F}, \tau - \text{shift}} \setminus C_F \subset \bigcup_{F' \in \mathcal{E}_F(F)} N_{F'},
\]
we conclude that the symmetric difference of $C_F$ and $C_{\mathcal{F}, \tau - \text{shift}}$ is contained in $N_F \cup \bigcup_{F' \in \mathcal{E}_F(F)} N_{F'}$, where $N_F$ denotes the set of cubes $I$ near $F$ in the corona $C_F$, i.e. $\ell(I) \geq 2^{-\tau} \ell(F)$. But since the children $F' \in \mathcal{E}_F(F)$ are pairwise disjoint, and the cardinality of the nearby sets $N_F$ and $N_{F'}$ are each $2^{n\tau}$, we see that
\[
\|S_{F^\tau - \text{shift}} f\|_{L^p(\mu)} \leq \|S_F f\|_{L^p(\mu)} + C_{\tau, \kappa} \|S_{\text{Haar}} f\|_{L^p(\mu)},
\]
since each of the square functions $S_F$ and $S_{\text{Haar}}$ have already been shown to be bounded on $L^p(\mu)$. We have thus proved the following theorem.

**Theorem 8.** Suppose $\mu$ is a locally finite positive Borel measure on $\mathbb{R}^n$. Then for $1 < p < \infty$,
\[
\|S_{F^\tau - \text{shift}} f\|_{L^p(\mu)} \leq C_p \tau \|f\|_{L^p(\mu)}.
\]

Another square function that will arise in the nearby and related forms is
\[
S_{p, \delta} f(x) \equiv \left( \sum_{I \in \mathcal{D}} \left| P^{\mu}_{I, \delta} f(x) \right|^2 \right)^{\frac{1}{2}},
\]
where $P^{\mu}_{I, \delta} f(x) \equiv \sum_{J \in \mathcal{D}, 2^{-\rho \ell(I)} \leq \ell(J) \leq 2^{\rho \ell(I)}} 2^{-\delta \text{dist}(J, I)} \Delta^{\mu}_f f(x)$.

**Theorem 9.** Suppose $\mu$ is a locally finite positive Borel measure on $\mathbb{R}^n$, and let $0 < \rho, \delta < 1$. Then for $1 < p < \infty$,
\[
\|S_{p, \delta} f\|_{L^p(\mu)} \leq C_{p, \rho, \delta} \|f\|_{L^p(\mu)}.
\]

**Proof.** It is easy to see that $S_{p, \delta} f(x) \leq C_{p, \rho, \delta} S_{\text{Haar}} f(x)$, and the boundedness of $S_{p, \delta}$ now follows from the boundedness of the Haar square function $S_{\text{Haar}}$.

### 3.1. Alpert square functions.

This subsection will not be used in this paper, but we include it due to its likely use in extensions of the current paper, and its utility in other situations as well. We extend the Haar square function inequalities to Alpert square functions that use weighted Alpert wavelets in place of Haar wavelets, but only for doubling measures now. Recall from [RaSaWi] that if $E^\mu_H$ denotes orthogonal projection in $L^2(\mu)$ onto the finite dimensional space of restrictions to $I$ of polynomials of degree less than $\kappa$, then the weighted Alpert projection $\Delta^{\mu}_{I, \kappa}$ is given by
\[
\Delta^{\mu}_{I, \kappa} = \left( \sum_{I' \in \mathcal{D}(I)} E^\mu_{I', \kappa} \right) - E^\mu_{I, \kappa}.
These weighted Alpert projections \( \{ \Delta_{I,\kappa}^\mu \}_{I \in \mathcal{D}} \) are orthogonal and span \( L^2(\mu) \) for measures \( \mu \) that are infinite on all dyadic tops, and in particular for doubling measures, see \cite{RaSaWi} and \cite{AlSaUr} for terminology and proofs.

We begin by showing that the Alpert square function

\[
S_{\mathrm{Alpert};\kappa} f(x) = \left( \sum_{I \in \mathcal{D}} \left| \Delta_{I,\kappa}^\mu f(x) \right|^2 \right)^{1/2}
\]

is bounded on \( L^p(\mu) \) for any \( 1 < p < \infty \) and any doubling measure \( \mu \). We thank a referee of a previous version of this paper, for pointing out to us that it is **not** the case that \( S_{\mathrm{Alpert};\kappa} \) is a martingale difference square function of an \( L^p \) bounded martingale, and so we cannot apply Burkholder’s martingale transform theorem as we did for the Haar square function. On the other hand, in the case the measure \( \mu \) is doubling, the sequence of projections of Alpert wavelets satisfies all of the properties needed by Burkholder’s proof, as we now demonstrate. For the convenience of the reader, we repeat Burkholder’s beautiful argument, following the treatment in Hytönen \cite{Hyt2}.

Recall that \( \mathcal{D}_k \equiv \{ Q \in \mathcal{D} : \ell(Q) = 2^k \} \) is the tiling of \( \mathbb{R}^n \) with dyadic cubes of side length \( 2^k \). For each \( k \in \mathbb{Z} \) define the projections

\[
P_{k;\kappa}^\mu f(x) = \sum_{Q \in \mathcal{D}_k} 
\]

of \( f \) onto the linear space of functions whose restrictions to cubes in \( \mathcal{D}_k \) are polynomials of degree at most \( \kappa \). While there is no conditional expectation result in the current setting, we can show the key inequality needed by appealing to the properties of Alpert projections. Indeed, we show that the functions \( P_{k;\kappa}^\mu \) and \( P_{k+1;\kappa}^\mu \) have the same integral over all \( P \in \mathcal{D}_k \), and this holds because \( \Delta_{P;\kappa}^\mu f \) has vanishing mean on \( P \):

\[
\int_P P_{k+1;\kappa}^\mu f(x) \, d\mu(x) - \int_P P_{k;\kappa}^\mu f(x) \, d\mu(x) = \int_P \left( P_{k+1;\kappa}^\mu f(x) - P_{k;\kappa}^\mu f(x) \right) \, d\mu(x)
\]

\[
= \int_P \left( \sum_{Q \in \mathcal{D}_{k+1}} \mathbb{E}_{Q;\kappa}^\mu f - \sum_{Q \in \mathcal{D}_k} \mathbb{E}_{Q;\kappa}^\mu f \right) \, d\mu(x) = \int_P \sum_{Q \in \mathcal{D}_{k+1} : Q \subseteq P} \left( \mathbb{E}_{Q;\kappa}^\mu f - \mathbb{E}_{P;\kappa}^\mu f \right) \, d\mu(x)
\]

\[
= \int_P \left( \Delta_{P;\kappa}^\mu f(x) \right) \, d\mu(x) = 0.
\]

The \( L^p \) boundedness \( \| \mathbb{E}_{Q;\kappa}^\mu f \|_{L^p(\mu)} \leq C \) follows easily from the estimate \( \| \mathbb{E}_{Q;\kappa}^\mu f \|_{\infty} \lesssim E_Q^\mu |f| \) in \( \ref{inequalities19} \) for \( Q \in \mathcal{D}_k \).

We will in fact establish the analogue of Burkholder’s martingale transform theorem for a new class of what we call \( L^p(\mu) \)-quasimartingales, that share all of the formal properties of martingales except for the presence of sigma algebras and measurability.

**Definition 10.** We say that \( \{ f_k(x) \}_{k \in \mathbb{Z}} \subset L^p(\mu) \) is an \( L^p(\mu) \)-quasimartingale if there is a collection of \( L^2 \) projections \( \{ \mathbb{E}_I^\mu \}_{I \in \mathcal{F}} \) such that

\[
f_k(x) = \sum_{I \in \mathcal{F}_k} \mathbb{E}_I^\mu f(x), \quad k \in \mathbb{Z}, \text{ with convergence a.e. and in } L^p(\mu),
\]

\[
d_k(x) = f_k(x) - f_{k-1}(x) = \sum_{I \in \mathcal{D}_k} P_I^\mu f(x),
\]

\[
P_I^\mu P_J^\mu = \begin{cases} P_I^\mu & \text{if } I = J \\ 0 & \text{if } I \neq J \end{cases}, \quad I, J \in \mathcal{F},
\]

where \( P_I^\mu = \sum_{I' \in \mathcal{F}(I)} \mathbb{E}_{I'}^\mu f - 1_I \mathbb{E}_I^\mu f \).

For convenience, we restrict our attention from now on to the case \( \mathcal{F} = \mathcal{D} \), but the reader can easily extend the analysis below to the case of an arbitrary subset \( \mathcal{F} \subset \mathcal{D} \). Define

\[
T_{\beta I} f(x) = \sum_{I \in \mathcal{D}} \beta_I \Delta_{I,\kappa}^\mu f(x).
\]
Consider the $L^p(\mu)$-quasimartingale,
\[ \left\{ \mathbb{P}_{k,\kappa}^\mu f(x) \right\}_{k \in \mathbb{Z}} = \left\{ \sum_{Q \in \mathcal{D}_k} \mathbb{E}_{Q;\kappa}^\mu f(x) \right\}_{k \in \mathbb{Z}} = \left\{ \sum_{I \in \mathcal{D}: \ell(I) \geq 2^k} \Delta_{I;\kappa}^\mu f(x) \right\}_{k \in \mathbb{Z}}, \]
and its associated $L^p(\mu)$-quasimartingale difference sequence,
\[ \left\{ d_{k,\kappa}^\mu f(x) \right\}_{k \in \mathbb{Z}} = \left\{ \mathbb{P}_{k,\kappa}^\mu f(x) - \mathbb{P}_{k-1,\kappa}^\mu f(x) \right\}_{k \in \mathbb{Z}} = \left\{ \sum_{I \in \mathcal{D}: \ell(I) = 2^k} \Delta_{I;\kappa}^\mu f(x) \right\}_{k \in \mathbb{Z}}. \]

Note that
\[ \int \mathbb{P}_{k+1,\kappa}^\mu f(x) d\mu(x) - \int \mathbb{P}_{k,\kappa}^\mu f(x) d\mu(x) = \int \left( \Delta_{k;\kappa}^\mu f(x) \right) d\mu(x) = 0, \]
shows that,
\[ \int \mathbb{P}_{k+1,\kappa}^\mu T_\beta f(x) d\mu(x) - \int \mathbb{P}_{k,\kappa}^\mu T_\beta f(x) d\mu(x) = \int \left( \beta \Delta_{k;\kappa}^\mu f(x) \right) d\mu(x) = 0, \]
and so \( \left\{ \mathbb{P}_{k,\kappa}^\mu f(x) \right\}_{k \in \mathbb{Z}} \) is also an $L^p(\mu)$-quasimartingale.

**Definition 11.** We say that a sequence of functions \( \{v_k\}_{k \in \mathbb{Z}} \) is predictable if \( v_k \) is constant on every cube \( Q \in \mathcal{D}_k \).

**Notation 12.** In order to conform to the notation used in [Hyt2], we write
\[ v_k(x) = \sum_{Q \in \mathcal{D}_k} \beta_Q 1_Q(x), \]
\[ f_k(x) = \mathbb{E}_{k,\kappa}^\mu f(x) = \sum_{Q \in \mathcal{D}_k} \mathbb{E}_{Q;\kappa}^\mu f(x), \]
\[ d_k(x) = \mathbb{P}_{k,\kappa}^\mu f(x) - \mathbb{P}_{k-1,\kappa}^\mu f(x), \]
\[ (T_\beta f)_n(x) = \sum_{Q \in \mathcal{D}_k} \mathbb{E}_{Q;\kappa}^\mu T_\beta f(x) = T_\beta \sum_{Q \in \mathcal{D}_k} \mathbb{E}_{Q;\kappa}^\mu f(x). \]

**Theorem 13.** Let \( \{f_k\}_{k=0}^n \) be an $L^p(\mu)$-quasimartingale with \( \mu \) doubling, and let \( \{v_k\}_{k=0}^n \) be a bounded predictable sequence, and define numbers \( \beta_Q \) by \( v_k = \sum_{Q \in \mathcal{D}_k} \beta_Q 1_Q \). Then
\[ \|\mathbb{P}_{n,\kappa}^\mu T_\beta f\|_{L^p(\mu)} \leq C_p \|\mathbb{P}_{n,\kappa}^\mu f\|_{L^p(\mu)}, \]
and \( \|T_\beta f\|_{L^p(\mu)} \leq C_p \|f\|_{L^p(\mu)} \).

**Proof.** By interpolation and duality, it suffices to show that if the Theorem holds for some index \( p \in (0, \infty) \), then it also holds for the index \( 2p \). We start with
\[ (T_\beta f)_n^2 = \left( \sum_{k=0}^n v_k d_k \right)^2 = \sum_{k=0}^n v_k^2 d_k^2 + 2 \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} v_j d_j v_k d_k \]
\[ = \sum_{k=0}^n v_k^2 d_k^2 + 2 \sum_{k=0}^n (T_\beta f)_{k-1} v_k d_k, \]
which gives
\[ \| (T_\beta f)_n \|_{L^{2p}(\mu)}^2 = \| (T_\beta f)_n^2 \|_{L^p(\mu)} \leq \sum_{k=0}^n v_k^2 d_k^2 + 2 \sum_{k=0}^n (T_\beta f)_{k-1} v_k d_k \|_{L^{2p}(\mu)}. \]

Now we write
\[ (T_\beta f)_{k-1} v_k = \frac{(T_\beta f)_{k-1}}{(T_\beta f)_{k-1}^*) v_{(T_\beta f)_{k-1}^*)} \equiv u_k \cdot (T_\beta f)_{k-1}^*, \]
where
\[ (T_\beta f)_{k-1}^* = \max_{j \leq k-1} |(T_\beta f)_j|. \]
Note that \( u_k \) is predictable and bounded by 1, and that \((T_\beta f)^*_k\) is increasing in \( k \). Now \( \left\{ (T_\beta f)_{k-1} d_k \right\}_{k=1}^n \) is also a quasimartingale difference sequence, and so by our induction hypothesis,

\[
\left\| \sum_{k=0}^n (T_\beta f)_{k-1} v_k d_k \right\|_{L^p(\mu)} \leq \left\| \sum_{k=0}^n u_k \cdot (T_\beta f)^*_{k-1} d_k \right\|_{L^p(\mu)} \leq C_p \left\| \sum_{k=0}^n (T_\beta f)^*_{k-1} d_k \right\|_{L^p(\mu)} .
\]

We now consider the following pointwise estimate using summation by parts,

\[
\sum_{k=0}^n (T_\beta f)^*_{k-1} d_k = \sum_{k=0}^n (T_\beta f)^*_{k-1} (f_k - f_{k-1}) = \sum_{k=0}^n (T_\beta f)^*_{k-1} f_k - \sum_{k=1}^{n-1} (T_\beta f)^*_{k} f_k
\]

where the final term vanishes by our convention about the quasimartingale at \(-1\). Recalling that \((T_\beta f)^*_{k}\) is increasing in \( k \), we have

\[
\left\| \sum_{k=0}^n (T_\beta f)^*_{k-1} d_k \right\| \leq (T_\beta f)^*_{n-1} |f_n| + \sum_{k=0}^{n-1} \left[ (T_\beta f)^*_{k} - (T_\beta f)^*_{k-1} \right] |f_k|
\]

\[
\leq (T_\beta f)^*_{n-1} f_n + \sum_{k=0}^{n-1} \left[ (T_\beta f)^*_{k} - (T_\beta f)^*_{k-1} \right] f_n^*
\]

\[
\leq (T_\beta f)^*_{n-1} f_n + (T_\beta f)^*_{n-1} f_n^* \leq 2 (T_\beta f)^*_{n} f_n^* .
\]

We then have

\[
\left\| 2 (T_\beta f)^*_{n} f_n^* \right\|_{L^p(\mu)} \leq 2 \left\| (T_\beta f)^*_{n} \right\|_{L^{2p}(\mu)} \left\| f_n^* \right\|_{L^{2p}(\mu)} \leq 2 (A_p B_p)^2 \left\| (T_\beta f)^*_{n} \right\|_{L^{2p}(\mu)} \left\| f_n \right\|_{L^p(\mu)} ,
\]

by the dyadic maximal theorem, whose bound is \( A_p \), i.e.

\[
f_k^* (x) = \max_{k \leq j < k} |f_j (x)| = \max_{j \leq k-1} \left\{ \sum_{Q \in \mathcal{D}_j} \mathbb{E}^\mu_{Q:x} f (x) \right\} \leq \sup_{Q \in \mathcal{D}_x : x \in Q} \left\{ \mathbb{E}^\mu_{Q,x} f (x) \right\}
\]

\[
\leq B_p \sup_{Q \in \mathcal{D}_x} \frac{1}{Q} \int_Q |f (x)| \, d\mu (x) = B_p M^{dy}_f (x)
\]

by an inequality in [Saw6] since \( \mu \) is doubling, and where \( \| M^{dy}_\mu f \|_{L^p(\mu)} \leq A_p \| f \|_{L^p(\mu)} \) .

So far we have

\[
\left\| (T_\beta f)^*_{n} \right\|_{L^{2p}(\mu)} = \left\| (T_\beta f)^*_{n} \right\|_{L^{p}(\mu)} \leq \left\| \sum_{k=0}^n v_k^2 d_k^2 \right\|_{L^p(\mu)} + 2 \cdot 2 (A_p B_p)^2 C_p \left\| (T_\beta f)^*_{n} \right\|_{L^{2p}(\mu)} \left\| f_n \right\|_{L^p(\mu)} .
\]

To bound the remaining term \( \left\| \sum_{k=0}^n v_k^2 d_k^2 \right\|_{L^p(\mu)} \) on the right, we use \( |v_k| \leq 1 \), and then follow some of the earlier steps in reverse order to obtain,

\[
\sum_{k=0}^n v_k^2 d_k^2 \leq \sum_{k=0}^n d_k^2 = \left( \sum_{k=0}^n d_k \right)^2 - 2 \sum_{k=0}^{n-1} d_k d_{k+1} = f_n^2 - 2 \sum_{k=0}^n f_{k-1} d_k .
\]

Thus we have

\[
\left\| \sum_{k=0}^n v_k^2 d_k^2 \right\|_{L^p(\mu)} \leq \left\| f_n^2 \right\|_{L^p(\mu)} + 2 \left\| \sum_{k=0}^n f_{k-1} d_k \right\|_{L^p(\mu)} ,
\]

where the term \( \left\| \sum_{k=0}^n f_{k-1} d_k \right\|_{L^p(\mu)} \) is exactly the term \( \left\| \sum_{k=0}^n v_k d_k \right\|_{L^p(\mu)} \) handled above, but with \( v_k \equiv 1 \), so that

\[
\left\| \sum_{k=0}^n f_{k-1} d_k \right\|_{L^p(\mu)} \leq 2 (A_p B_p)^2 C_p \left\| f_n^2 \right\|_{L^{2p}(\mu)} .
\]
Altogether then we have
\[ \| (T_\beta f)_n \|_{L^{2p}(\mu)}^2 \leq \| f_n \|_{L^{2p}(\mu)}^2 + 4 (A_p B_p)^2 C_p \| f_n \|_{L^{2p}(\mu)}^2 + 4 (A_p B_p)^2 C_p \| (T_\beta f)_n \|_{L^{2p}(\mu)} \| f_n \|_{L^p(\mu)} \]
\[ = \left\{ 1 + 4 (A_p B_p)^2 C_p \right\} \| f_n \|_{L^{2p}(\mu)} + 4 (A_p B_p)^2 C_p \| (T_\beta f)_n \|_{L^{2p}(\mu)} \| f_n \|_{L^p(\mu)} , \]
and a simple divide and conquer argument, considering the dominant summand inside the braces, finishes the proof of the induction step. Indeed, if \[ 1 + 4 (A_p B_p)^2 C_p \] dominates, then
\[ \| (T_\beta f)_n \|_{L^{2p}(\mu)}^2 \leq 2 \left[ 1 + 4 (A_p B_p)^2 C_p \right] \| f_n \|_{L^{2p}(\mu)} , \]
while if \( 4 (A_p B_p)^2 C_p \| (T_\beta f)_n \|_{L^{2p}(\mu)} \) dominates, then
\[ \| (T_\beta f)_n \|_{L^{2p}(\mu)} \leq 4 (A_p B_p)^2 C_p \| (T_\beta f)_n \|_{L^{2p}(\mu)} \| f_n \|_{L^p(\mu)} . \]
Altogether then,
\[ \| (T_\beta f)_n \|_{L^{2p}(\mu)} \leq C_{2p} \| f_n \|_{L^{2p}(\mu)} , \]
where
\[ C_{2p} \equiv \sqrt{2} \max \left\{ \sqrt{1 + 4 (A_p B_p)^2 C_p}, 4 (A_p B_p)^2 C_p \right\} , \]
which proves the first line in the statement of the theorem, and the second line then follows by a limiting argument. \( \square \)

Now we can prove the Alpert square function equivalence for doubling measures in the standard way.

**Definition 14.** Given a quasimartingale \( f \sim \{ f_k \}_{k=0}^n \) with respect to a doubling measure \( \mu \), and with difference sequence \( \{ d_k \}_{k=0}^n \), define the associated square function by,
\[ S_\mu f (x) = \sqrt{\sum_{k=0}^n d_k (x)^2} . \]

**Corollary 15.** Let \( \{ f_k \}_{k=0}^n \) be an \( L^p(\mu) \)-quasimartingale with respect to a doubling measure \( \mu \), and let \( 1 < p < \infty \). Then for \( 1 < p < \infty \), we have the square function equivalence,
\[ \| S_\mu f \|_{L^p(\mu)} \approx \| f \|_{L^p(\mu)} , \quad f \in L^p(\mu) . \]

**Proof.** Combining Theorem 13 with Khintchine’s inequality, yields the boundedness
\[ \| S_\mu f \|_{L^p(\mu)} \lesssim \| f \|_{L^p(\mu)} , \]
of the square function \( S_\mu \).

For \( \beta_1 = \pm 1 \), we have
\[ \langle f, g \rangle_\mu = \int_{\mathbb{R}^n} \left( \sum_{I \in \mathcal{D}} \triangle^I f \right) \left( \sum_{J \in \mathcal{D}} \triangle^J g \right) d\mu = \int_{\mathbb{R}^n} \left( \sum_{I \in \mathcal{D}} (\triangle^I f) \cdot (\triangle^I g) \right) d\mu \]
\[ = \int_{\mathbb{R}^n} \left( \sum_{I \in \mathcal{D}} (\beta_1 \triangle^I f) \cdot (\beta_1 \triangle^I g) \right) d\mu = \int_{\mathbb{R}^n} \left( \sum_{I \in \mathcal{D}} \beta_1 \triangle^I f \right) \left( \sum_{J \in \mathcal{D}} \beta_1 \triangle^J g \right) d\mu = \langle T_\beta f, T_\beta g \rangle_\mu , \]
and so duality then gives
\[ \| f \|_{L^p(\mu)} = \sup_{\| g \|_{L^{p'}(\mu)} \leq 1} \| f, g \|_\mu = \sup_{\| g \|_{L^{p'}(\mu)} \leq 1} \| T_\beta f, T_\beta g \|_\mu \]
\[ \leq \| T_\beta f \|_{L^{p'}(\mu)} \sup_{\| g \|_{L^{p'}(\mu)} \leq 1} \| T_\beta g \|_{L^p(\mu)} \leq C_{p'} \| T_\beta f \|_{L^p(\mu)} , \]
independently of \( \beta_1 = \pm 1 \). Another application of Khintchine’s inequality gives the reverse inequality
\[ \| f \|_{L^p(\mu)} \lesssim \| S_\mu f \|_{L^p(\mu)} . \]
\( \square \)
Define the square functions
\[
S_{F^{t,-\text{shift};\kappa}} f(x) \equiv \left( \sum_{F \in F} \left| \mathcal{P}^{\mu}_{C_{F^{t,-\text{shift};\kappa}}} f(x) \right|^2 \right)^{\frac{1}{2}},
\]
and
\[
S_{p,\delta;\kappa} f(x) \equiv \left( \sum_{J \in \mathcal{D} : \tau \in J} \left| \mathcal{P}^\rho_{J,\kappa} f(x) \right|^2 \right)^{\frac{1}{2}},
\]
where \( \mathcal{P}^\rho_{J,\kappa} f(x) \equiv \sum_{J \in \mathcal{D} : \tau \in J} 2^{-\delta \text{dist}(J,1)} \Delta^\rho_{J,\kappa} f(x) \).

Altogether we have the following theorem.

**Theorem 16.** Suppose \( \mu \) is a doubling measure on \( \mathbb{R}^n \). Then for \( \kappa \in \mathbb{N} \), \( 1 < p < \infty \) and \( 0 < \rho, \delta < 1 \), we have
\[
\| S_{\text{Alpert};\kappa} f \|_{L^p(\mu)} + \| S_{F;\kappa} f \|_{L^p(\mu)} + \| S_{F^{t,-\text{shift};\kappa}} f \|_{L^p(\mu)} \leq C_{p, n, \kappa, \tau} \| f \|_{L^p(\mu)},
\]
\[
\| S_{p,\delta;\kappa} f \|_{L^p(\mu)} \leq C_{p, p, \delta, n, \kappa} \| f \|_{L^p(\mu)}.
\]

3.2. **Vector-valued inequalities.** We begin by reviewing the well-known \( L^2 \)-extension of a bounded linear operator. We include the simple proof here as it sheds light on the nature of the quadratic Muckenhoupt condition, in particular on its necessity for the norm inequality - namely that one must test the norm inequality over all functions \( f \) defined below.

Let \( M \in \mathbb{N} \) be a large positive integer that we will send to \( \infty \) later on. Suppose \( T \) is bounded from \( L^p(\sigma) \) to \( L^p(\omega) \), \( 0 < p < \infty \), and for \( f = \{f_j\}_{j=1}^M \), define
\[
T f \equiv \{T f_j\}_{j=1}^M.
\]
For any unit vector \( u = (u_j)_{j=1}^M \) in \( \mathbb{C}^M \) define
\[
f_u \equiv \langle f, u \rangle \quad \text{and} \quad T u f \equiv \langle T f, u \rangle = T(f, u) = T f_u
\]
where the final equalities follow since \( T \) is linear. We have
\[
\int_{\mathbb{R}^n} |T u f(x)|^p d\omega(x) = \int_{\mathbb{R}^n} |T f_u(x)|^p d\omega(x) \leq \|T\|^p_{L^p(\sigma) \rightarrow L^p(\omega)} \int_{\mathbb{R}^n} |f_u(x)|^p d\sigma(x),
\]
where
\[
T u f(x) = \langle T f(x), u \rangle = |T f(x)|_{\ell^2} \left\langle \frac{T f(x)}{|T f(x)|_{\ell^2}}, u \right\rangle = |T f(x)|_{\ell^2} \cos \theta,
\]
if \( \theta \) is the angle between \( \frac{T f(x)}{|T f(x)|_{\ell^2}} \) and \( u \) in \( \mathbb{C}^M \). Then using
\[
\int_{\mathbb{S}^{M-1}} |\langle u, v \rangle|^p d\mu = \gamma_p \quad \text{for} \quad \|v\| = 1,
\]
we have
\[
\int_{\mathbb{S}^{M-1}} \left\{ \int_{\mathbb{R}^n} |T u f(x)|^p d\omega(x) \right\} d\mu = \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{S}^{M-1}} |T u f(x)|^p d\mu \right\} d\omega(x)
\]
\[
= \int_{\mathbb{R}^n} |T f(x)|_{\ell^2} \left\{ \int_{\mathbb{S}^{M-1}} \cos \theta^p d\mu \right\} d\omega(x) = \gamma_p \int_{\mathbb{R}^n} |T f(x)|_{\ell^2}^p d\omega(x),
\]
and similarly,
\[
\int_{\mathbb{S}^{M-1}} \left\{ \int_{\mathbb{R}^n} |f_u(x)|^p d\sigma(x) \right\} d\mu = \gamma_p \int_{\mathbb{R}^n} |f(x)|_{\ell^2}^p d\sigma(x).
\]
Altogether then,
\[
\gamma_p \int_{\mathbb{R}^n} |T f(x)|_{\ell^2}^p d\omega(x) = \int_{\mathbb{S}^{M-1}} \left\{ \int_{\mathbb{R}^n} |T u f(x)|^p d\omega(x) \right\} d\mu = \int_{\mathbb{S}^{M-1}} \left\{ \int_{\mathbb{R}^n} |T f_u(x)|^p d\omega(x) \right\} d\mu
\]
\[
\leq \int_{\mathbb{S}^{M-1}} \left\{ \|T\|^p_{L^p(\sigma) \rightarrow L^p(\omega)} \int_{\mathbb{R}^n} |f_u(x)|^p d\sigma(x) \right\} d\mu = \gamma_p \|T\|^p_{L^p(\sigma) \rightarrow L^p(\omega)} \int_{\mathbb{R}^n} |f(x)|_{\ell^2}^p d\sigma(x),
\]
and upon dividing both sides by $\gamma_p$ we conclude that
\[
\int_{\mathbb{R}^n} |Tf(x)|^p_{\sigma} \, d\omega(x) \leq \|T\|_{L^p(\sigma) \to L^p(\omega)} \int_{\mathbb{R}^n} |f(x)|^p_{\sigma} \, d\sigma(x).
\]

Finally we can let $M \nrightarrow \infty$ to obtain the desired vector-valued extension,
\[
\left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} |Tf_j(x)|^2 \right)^{\frac{p}{2}} \, d\omega(x) \right)^{\frac{1}{p}} \leq \|T\|_{L^p(\sigma) \to L^p(\omega)} \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} |f_j(x)|^2 \right)^{\frac{p}{2}} \, d\sigma(x) \right)^{\frac{1}{p}}. \tag{3.2}
\]

4. Necessity of quadratic testing and $A_p$ conditions

We can use the vector-valued inequality (3.2) to obtain the necessity of the quadratic testing inequality, namely
\[
\left( \sum_{i=1}^{\infty} (a_i \mathbf{1}_{I_i} T_\sigma^\lambda \mathbf{1}_{I_i})^2 \right)^{\frac{1}{2}} \leq \mathcal{Q}_T^{\lambda^{2}}(\sigma, \omega) \left( \sum_{i=1}^{\infty} (a_i \mathbf{1}_{I_i})^2 \right)^{\frac{1}{2}}, \tag{4.1}
\]
for the boundedness of $T^\lambda$ from $L^p(\sigma)$ to $L^p(\omega)$, i.e. $\mathcal{Q}_T^{\lambda^{2}}(\sigma, \omega) \lesssim \|T^\lambda\|_{L^p(\sigma) \to L^p(\omega)}$. Indeed, we simply set $f_i \equiv a_i \mathbf{1}_{I_i}$ in (3.2) to obtain the global quadratic testing inequality,
\[
\left( \sum_{i=1}^{\infty} (a_i \mathbf{1}_{I_i} T_\sigma^\lambda \mathbf{1}_{I_i})^2 \right)^{\frac{1}{2}} \leq \mathcal{Q}_T^{\lambda^{2,\text{global}}}(\sigma, \omega) \left( \sum_{i=1}^{\infty} (a_i \mathbf{1}_{I_i})^2 \right)^{\frac{1}{2}}, \tag{4.2}
\]
and then simply note the pointwise inequality
\[
\sum_{i=1}^{\infty} (a_i \mathbf{1}_{I_i} T_\sigma^\lambda \mathbf{1}_{I_i})^2 (x) \leq \sum_{i=1}^{\infty} |a_i|^2 |T_\sigma^\lambda \mathbf{1}_{I_i}(x)|^2 \leq \sum_{i=1}^{\infty} |a_i|^2 |T_\sigma^\lambda \mathbf{1}_{I_i}(x)|^2,
\]
to obtain the local version (4.1).

Now we turn to the necessity of the quadratic offset $A_p^{\lambda, \delta, \text{offset}}$ condition, namely
\[
\left( \sum_{i=1}^{\infty} \left( a_i \mathbf{1}_{I_i} \frac{|I_i|_{\sigma}}{|I_i|^{1-\frac{\delta}{2}}} \right)^2 \right)^{\frac{1}{2}} \leq A_p^{\lambda, \delta, \text{offset}}(\sigma, \omega) \left( \sum_{i=1}^{\infty} |a_i \mathbf{1}_{I_i}|^2 \right)^{\frac{1}{2}}. \tag{4.3}
\]
Suppose that $T^\lambda$ is Stein elliptic, and fix appropriate sequences $\{I_i\}_{i=1}^{\infty}$ and $\{a_i\}_{i=1}^{\infty}$ of cubes and numbers respectively. Then there is a choice of constant $C$ and appropriate cubes $I_i^*$ such that

\[
|T_\sigma^\lambda \mathbf{1}_{I_i}(x)| \geq C \frac{|I_i|_{\sigma}}{|I_i|^{1-\frac{\delta}{2}}} \quad \text{for} \quad x \in I_i^*, \quad 1 \leq i \leq \infty.
\]

Now we simply apply (4.2) to obtain
\[
A_p^{\lambda, \delta, \text{offset}}(\sigma, \omega) \lesssim \mathcal{Q}_T^{\lambda^{2,\text{global}}}(\sigma, \omega) \lesssim \mathbf{N}_{T^\lambda \sigma}(\delta, \omega).
\]

It should be noticed that while the necessity of the quadratic Muckenhoupt condition $A_p^{\lambda, \delta}(\sigma, \omega)$ itself is easily shown for the Hilbert transform, the necessity for even nice operators in higher dimensions is much more difficult.

4.1. Quadratic Haar weak boundedness property. It is convenient in our proof to introduce the quadratic Haar weak boundedness property constant $\mathcal{HWBP}_{T^{\lambda, \rho}}^{\delta, \rho}(\sigma, \omega)$ as the least constant in the inequality,
\[
\left( \sum_{j \in \mathcal{D}} \sum_{I \in \mathcal{A}_j(I)} |\triangle_j T_\sigma^\lambda \triangle_j^\sigma f_j|^2 \right)^{\frac{1}{2}} \leq \mathcal{HWBP}_{T^{\lambda, \rho}}^{\delta, \rho}(\sigma, \omega) \|f\|_{L^p(\sigma)}.
\]
There is only one quadratic Haar inequality in the weak boundedness condition \(4.3\), since we show below in Proposition 17 that \(4.3\) is equivalent to the bilinear inequality

\[
(4.4) \quad \left| \sum_{I \in \mathcal{D}} \sum_{J \in \text{Adj}_\rho(I)} \left\langle T^\lambda_\sigma, \Delta_I^\sigma f, \Delta_I^\sigma g \right\rangle_\omega \right| \leq C \|f\|_{L^p(\sigma)} \|g\|_{L^p(\omega)}, \quad f \in L^p(\sigma), g \in L^p(\omega),
\]

which is then also equivalent to the inequality dual to that appearing in \(4.3\). In fact, this bilinear inequality is a ‘quadratic analogue’ of a scalar weak boundedness property, which points to the relative ‘weakness’ of \(4.3\). Of course, one can use the \(L^\infty\) estimates (2.5) on Haar wavelets, together with the vector-valued maximal theorem of Fefferman and Stein in a space of homogeneous type [GrLYa, Theorem 2.1], to show that for doubling measures, we actually have \(\mathcal{H}WB\mathcal{P}_{T^\lambda, p}^{\ell^2, p} < \mathcal{W}_{T^\lambda, p}^{\ell^2, p}\), but we will instead show a stronger result in Lemma 18 below.

The property \(4.3\) appears at first glance to be much stronger than the corresponding scalar testing and weak boundedness conditions, mainly because the standard proof of necessity of these conditions involves testing the scalar norm inequality over a dense set of functions \(\sum_{i=1}^\infty u_i a_i 1_k\), with \(\sum_{i=1}^\infty u_i^2 = 1\), see the subsection on vector-valued inequalities above. However, the minimal nature of the role of the quadratic Haar weak boundedness property \(4.3\) is demonstrated by considering the adjacent diagonal bilinear form \(B_{\text{Adj}, \rho}(f, g)\) associated with the form

\[
\left\langle T^\lambda_\sigma f, g \right\rangle_\omega = \left\langle T^\lambda_\sigma \left( \sum_{I \in \mathcal{D}} \omega \Delta_I^\sigma f \right), \sum_{J \in \mathcal{D}} \omega \Delta_J^\sigma g \right\rangle_\omega = \sum_{I, J \in \mathcal{D}} \left\langle T^\lambda_\sigma \Delta_I^\sigma f, \Delta_J^\sigma g \right\rangle_\omega,
\]

where \(f = \sum_{I \in \mathcal{D}} \omega \Delta_I^\sigma f\) and \(g = \sum_{J \in \mathcal{D}} \omega \Delta_J^\sigma g\) are the weighted Haar expansions of \(f\) and \(g\) respectively. Here the adjacent diagonal form \(B_{\text{Adj}, \rho}(f, g)\) is given by

\[
B_{\text{Adj}, \rho}(f, g) = \sum_{I \in \mathcal{D}} \sum_{J \in \text{Adj}_\rho(I)} \left\langle T^\lambda_\sigma \Delta_I^\sigma f, \Delta_J^\sigma g \right\rangle_\omega,
\]

where \(\text{Adj}_\rho(I)\) is defined in (2.5). We now demonstrate that the norm of \(B_{\text{Adj}, \rho}(f, g)\) as a bilinear form is comparable to the quadratic Alpert weak boundedness constant \(\mathcal{H}WB\mathcal{P}_{T^\lambda, p}^{\ell^2, p}\).

**Proposition 17.** Suppose \(1 < p < \infty\), \(0 \leq \rho < \infty\), and \(\sigma\) and \(\omega\) are positive locally finite Borel measures on \(\mathbb{R}^n\). If \(\mathcal{M}_{L^p(\sigma) \times L^p(\omega)}\) denotes the smallest constant \(C\) in the bilinear inequality

\[
|B_{\text{Adj}, \rho}(f, g)| \leq C \|f\|_{L^p(\sigma)} \|g\|_{L^p(\omega)},
\]

then

\[
\mathcal{M}_{L^p(\sigma) \times L^p(\omega)} \approx \mathcal{H}WB\mathcal{P}_{T^\lambda, p}^{\ell^2, p}(\sigma, \omega).
\]

**Proof.** We have

\[
\mathcal{M}_{L^p(\sigma) \times L^p(\omega)} = \sup_{\|f\|_{L^p(\sigma)} = \|g\|_{L^p(\omega)} = 1} |B_{\text{Adj}, \rho}(f, g)|
\]

\[
= \sup_{\|f\|_{L^p(\sigma)} = \|g\|_{L^p(\omega)} = 1} \left| \int_{\mathbb{R}^n} \left| \sum_{I \in \mathcal{D}} \sum_{J \in \text{Adj}_\rho(I)} \omega \Delta_I^\sigma T^\lambda_\sigma \Delta_J^\sigma f(x) g(x) \right| d\omega(x) \right|
\]

\[
= \sup_{\|f\|_{L^p(\sigma)} = 1} \left( \int_{\mathbb{R}^n} \left| \sum_{I \in \mathcal{D}} \sum_{J \in \text{Adj}_\rho(I)} \omega \Delta_I^\sigma T^\lambda_\sigma \Delta_J^\sigma f(x) \right|^p d\omega(x) \right)^{\frac{1}{p}}.
\]

Now we use the fact that Haar multipliers are bounded on \(L^p(\sigma)\) to obtain

\[
\left\| \sum_{I \in \mathcal{D}} \pm \Delta_I^\sigma f \right\|_{L^p(\sigma)} \approx \left\| \sum_{I \in \mathcal{D}} \Delta_I^\sigma f \right\|_{L^p(\sigma)} = \|f\|_{L^p(\sigma)}.
\]
Hence by the equivalence \( \|S_{\text{Har}} f\|_{L^p(\sigma)} \approx \|f\|_{L^p(\sigma)} \), we have

\[
\mathfrak{N}_{L^p(\sigma) \times L^p(\omega)} \approx \sup_{\|f\|_{L^p(\sigma)} = 1} \frac{1}{E_{\pm} \left( \int_{\mathbb{R}^n} \sum_{I' \in D} \sum_{J' \in \text{Adj}_p(I)} \sum_{m} T_{\sigma}^{\lambda} (\pm \triangle^f_j f) (x) \right)^p \sum_{\omega} \|d\omega(x)\|^p}
\]

\[
\approx \sup_{\|f\|_{L^p(\sigma)} = 1} \left( \int_{\mathbb{R}^n} \sum_{I' \in D} \sum_{J' \in \text{Adj}_p(I)} \sum_{m} \left| \triangle^f_j T_{\sigma}^{\lambda} (\pm \triangle^f_j f) (x) \right|^2 \sum_{\omega} \|d\omega(x)\|^p \right)^{\frac{1}{p}} = H_{\text{WBP}}^{p, \lambda}(\sigma, \omega).
\]

\[\square\]

Note that when \( \rho = 0 \), we have \( \text{Adj}_p(I) = \text{Adj}_0(I) = \text{Adj}(I) \) as defined in the introduction.

**Lemma 18.** Suppose that \( \sigma \) and \( \omega \) are doubling measures, \( 1 < p < \infty \), and that \( T^{\lambda} \) is a smooth \( \lambda \)-fractional Calderon-Zygmund operator. Then for \( 0 < \varepsilon < 1 \),

\[
H_{\text{WBP}}^{p, \lambda}(\sigma, \omega) \leq C_{\varepsilon} \left( \text{WBP}^{p, \lambda}(\sigma, \omega) + A^{\lambda, \ell^2, \text{offset}}_p (\sigma, \omega) \right) + \varepsilon \mathfrak{N}_{T^{\lambda, \rho}}(\sigma, \omega).
\]

**Proof.** Fix a dyadic cube \( I \). We can write

\[
\triangle^f_j f = \sum_{I' \in \mathcal{D}(I) J'' \in \mathcal{D}^{(m)}(I')} a_{I''} 1_{I''},
\]

\[
\triangle^g_j g = \sum_{J' \in \mathcal{D}(J) J'' \in \mathcal{D}^{(m)}(J')} b_{J''} 1_{J''},
\]

where the constants \( a_{I''} \) and \( b_{J''} \) are controlled by \( \|E_{\ell'} (\triangle^f_{j, x})\|_\infty \) and \( \|E_{\ell'} (\triangle^g_{j, x})\|_\infty \) respectively. Thus we have

\[
B_{\text{Adj}_p}(f, g) = \sum_{I' \in \mathcal{D}(I) J'' \in \mathcal{D}^{(m)}(I')} \langle T_{\sigma}^{\lambda} \triangle^f_{j, x} f, \triangle^g_{j, x} g \rangle_{\omega}
\]

\[
= \left\{ \sum_{I' \in \mathcal{D}(I) J'' \in \mathcal{D}^{(m)}(I')} \left\{ \sum_{J'' \in \mathcal{D}(I) J'' \in \mathcal{D}^{(m)}(J')} \left\{ \sum_{J'' \in \mathcal{D}(I) J'' \in \mathcal{D}^{(m)}(J')} \right\} a_{I''} b_{J''} \langle T_{\sigma}^{\lambda} 1_{I''}, 1_{J''} \rangle_{\omega} \right\} \right\} \equiv T_{\text{sep}} + T_{\text{touch}},
\]

where we have suppressed many of the conditions governing the dyadic cubes \( I'' \) and \( J'' \), including the fact that \( \ell(J'') = 2^{-m-1} \ell(J) = 2^{-m-1} \ell(I) = \ell(I'') \). Thus the cubes \( J'' \) and \( I'' \) arising in term \( T_{\text{sep}} \) are separated and it is then an easy matter to see that

\[
|T_{\text{sep}}| \leq C_m A^{\lambda, \ell^2, \text{offset}}_p (\sigma, \omega) \|f\|_{L^p(\sigma)} \|g\|_{L^{p'}(\omega)}.
\]

As for the term \( T_{\text{touch}} \), it is controlled by the weak boundedness constant,

\[
|T_{\text{touch}}| \leq C_{m, \rho} \text{WBP}^{p, \lambda}(\sigma, \omega) \|f\|_{L^p(\sigma)} \|g\|_{L^{p'}(\omega)},
\]

since since the cubes \( J'' \) and \( I'' \) are adjacent in this sum. \[\square\]

5. Forms requiring testing conditions

The three forms requiring conditions other than those of Muckenhoupt type, are the Haar adjacent diagonal form, which uses only the quadratic Haar weak boundedness constant \( H_{\text{WBP}}^{p, \lambda}(\sigma, \omega) \), and the two dual paraproduct forms, which each use only the appropriate scalar testing condition \( \mathbb{T}_{T^{\lambda, \rho}} \) or \( \mathbb{T}_{T^{\lambda, \cdot, \rho'}} \).
5.1. **Adjacent diagonal form.** Here we control the quadratic adjacent form by

\[
|B_{\text{Adj},\rho}(f,g)| = \left| \sum_{I \in \mathcal{D}, J \in \text{Adj}_I(I)} \langle T_\sigma^\lambda(\Delta_j^\tau f), (\Delta_j^\tau g) \rangle_\omega \right|
\]

\[
= \left| \int_{\mathbb{R}^n} \sum_{I \in \mathcal{D}, J \in \text{Adj}_I(I)} \Delta_j^\tau T_\sigma^\lambda(\Delta_j^\tau f)(x) \Delta_j^\tau g(x) \, d\omega(x) \right|
\]

\[
\leq \int_{\mathbb{R}^n} \left( \sum_{I \in \mathcal{D}, J \in \text{Adj}_I(I)} |\Delta_j^\tau T_\sigma^\lambda(\Delta_j^\tau f)(x)|^2 \right)^{\frac{1}{2}} \left( \sum_{I \in \mathcal{D}, J \in \text{Adj}_I(I)} |\Delta_j^\tau g(x)|^2 \right)^{\frac{1}{2}} \, d\omega(x)
\]

\[
\lesssim \left\| \left( \sum_{I \in \mathcal{D}, J \in \text{Adj}_I(I)} |\Delta_j^\tau T_\sigma^\lambda(\Delta_j^\tau f)(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \left\| S_{\text{Haar}} g \right\|_{L^{p'}(\omega)}.
\]

We have \( \|S_{\text{Haar}} g\|_{L^{p'}(\omega)} \approx \|g\|_{L^{p'}(\omega)} \) by a square function estimate, and using the quadratic Haar weak boundedness property, we obtain

\[
\left( \int_{\mathbb{R}^n} \left( \sum_{I \in \mathcal{D}, J \in \text{Adj}_I(I)} |\Delta_j^\tau T_\sigma^\lambda(\Delta_j^\tau f)(x)|^2 \right)^{\frac{1}{2}} \, d\omega(x) \right)^{\frac{1}{2}} \lesssim \mathcal{H}_{\text{WB}}^{\mathcal{P}^2,\rho}_{\mathcal{T},p}(\sigma,\omega) \|f\|_{L^p(\sigma)}
\]

and so altogether that

\[
|B_{\text{Adj},\rho}(f,g)| \lesssim \mathcal{H}_{\text{WB}}^{\mathcal{P}^2,\rho}_{\mathcal{T},p}(\sigma,\omega) \|f\|_{L^p(\sigma)} \|g\|_{L^{p'}(\omega)}.
\]

Recall that in Lemma\[18\] we have controlled the Haar weak boundedness property constant \( \mathcal{H}_{\text{WB}}^{\mathcal{P}^2,\rho}_{\mathcal{T},p}(\sigma,\omega) \) by the adjacent weak boundedness property constant \( \mathcal{WBP}^{\mathcal{P}^2,\rho}_{\mathcal{T},p}(\sigma,\omega) \), and the offset Muckenhoupt constant \( A_\mathcal{P}^{\mathcal{L},\mathcal{P},\text{offset}} \), plus a small multiple of the operator norm. This will be used at the end of the proof to eliminate the use of \( \mathcal{H}_{\text{WB}}^{\mathcal{P}^2,\rho}_{\mathcal{T},p} \).

5.2. **Paraproduct form.** Here we must bound the paraproduct form,

\[
\mathbf{B}_{\text{paraproduct}}(f,g) = \sum_{F \in \mathcal{F}} \mathbf{B}_{\text{paraproduct}}(f,g) = \sum_{F \in \mathcal{F}} \sum_{I \in C_F \text{ and } J \in C_F^{\text{shift}} \text{ and } \ell \in \mathbb{N} \text{ and } J \in \mathbb{N} \text{ and } I} \langle M_{\hat{J}} T_\sigma^\lambda 1_F, \Delta_j^\tau g \rangle_\omega
\]

\[
= \sum_{F \in \mathcal{F}} \sum_{J \in C_F^{\text{shift}}} \sum_{I \in C_F \text{ and } J \in \mathbb{N} \text{ and } I} \langle M_{\hat{J}} T_\sigma^\lambda 1_F, \Delta_j^\tau g \rangle_\omega
\]

\[
= \sum_{F \in \mathcal{F}} \sum_{J \in C_F^{\text{shift}}} \langle \left[ 1_{\hat{J}} \left[ \left( E_\sigma^\tau f \right) - \left( E_\sigma^\tau f \right) \right] \right] T_\sigma^\lambda 1_F, \Delta_j^\tau g \rangle_\omega,
\]

where \( \hat{J} \) is the smallest \( I \in C_F \) for which \( J \in \mathbb{N} \), in Theorem\[2\] Note that because of the projection \( \Delta_j^\tau g \) the telescoping sum in the second line above is restricted to \( J \). Define \( \bar{g} = \sum_{J \in \mathcal{D}} \frac{1}{\mathcal{J}_F[J]} \frac{1}{\mathcal{J}_F[J]} \Delta_j^\tau g \), and
noting that $|E^j f| + |E^j f| \lesssim E^j |f|$ by (5.1), we obtain

$$|B_{\text{paraproduct}}(f, g)| = \left| \sum_{F \in \mathcal{F}} B^F_{\text{paraproduct}}(f, g) \right| = \left| \sum_{F \in \mathcal{F}} \sum_{j \in C^\ell_F - \text{shift}} \left\langle \left( 1_{j_F} \left[ (E^j f) - (E^j f) \right] \right) T^F_{\lambda_j} 1_F, \Delta^\omega_j g \right\rangle \omega \right|$$

$$= \left| \sum_{F \in \mathcal{F}} \sum_{j \in C^\ell_F - \text{shift}} \left\langle T^F_{\lambda_j} 1_F, \left( 1_{j_F} \left[ (E^j f) - (E^j f) \right] \right) \Delta^\omega_j g \right\rangle \omega \right|$$

$$= \left| \sum_{F \in \mathcal{F}} \sum_{j \in C^\ell_F - \text{shift}} \left\langle T^F_{\lambda_j} 1_F, \left( 1_{j_F} \left[ (E^j f) - (E^j f) \right] \right) \Delta^\omega_j g \right\rangle \omega \right|$$

where $P_j$ is the constant $1_{j_F} \left[ (E^j f) - (E^j f) \right]$. The final term above is dominated by

$$\left( \int_{\mathbb{R}^n} \left( \sum_{F \in \mathcal{F}} E^j |f|^2 \right) \left( \int_{\mathbb{R}^n} \left( \sum_{F \in \mathcal{F}} \frac{P_j}{E^j f} \Delta^\omega_j g \right) (x) \right)^2 \omega \right)^{1/2}$$

The first factor above is controlled by the local quadratic testing characteristic,

$$\left( \int_{\mathbb{R}^n} \left( \sum_{F \in \mathcal{F}} (E^j |f|)^2 \right) \left( \int_{\mathbb{R}^n} \sum_{F \in \mathcal{F}} \frac{P_j}{E^j f} \Delta^\omega_j g \right) (x) \right)^2 \omega \left( \int_{\mathbb{R}^n} \left( \sum_{F \in \mathcal{F}} (E^j |f|)^2 \right) \left( \int_{\mathbb{R}^n} \sum_{F \in \mathcal{F}} \frac{P_j}{E^j f} \Delta^\omega_j g \right) (x) \right)^{1/2} \omega$$

and the second factor above is controlled by the square function estimates and the inequality $|\frac{P_j}{E^j f}| \lesssim 1$.

Indeed with $\tilde{g} = \sum_{J \in C^\ell_F - \text{shift}} \frac{P_j}{E^j f} \Delta^\omega_j g (x)$ we have

$$(5.1) \quad \int_{\mathbb{R}^n} \left( \sum_{F \in \mathcal{F}} \sum_{J \in C^\ell_F - \text{shift}} \frac{P_j}{E^j f} \Delta^\omega_j (g) \right) (x)^2 \omega \lesssim \| \tilde{g} \|_{L^{p'}(\omega)} \| g \|_{L^{p'}(\omega)}, \quad 1 < p' < \infty.$$
Lemma 19. Suppose that $J \subset I \subset K$ and that $\text{dist} (J, \partial I) > 2\sqrt{n} \ell (J)^{\gamma} \ell (I)^{1-\gamma}$. Then

\begin{equation}
\text{P}^\lambda (J, \sigma 1_{K \setminus I}) \leq \left( \frac{\ell (J)}{\ell (I)} \right)^{1-\gamma(n+1-\lambda)} \text{P}^\lambda (I, \sigma 1_{K \setminus I}).
\end{equation}

Lemma 20 (Pivotal Lemma). Let $J$ be a cube in $D$, and let $\Psi_J$ be an $L^2 (\omega)$ function supported in $J$ with vanishing $\omega$-mean. Let $\nu$ be a positive measure supported in $\mathbb{R}^n \setminus \gamma J$ with $\gamma > 1$, and let $\text{T}^\lambda$ be a standard $\lambda$-fractional singular integral operator with $0 \leq \lambda < n$. Then we have the 'pivotal' bound

\begin{equation}
| \text{T}^\lambda (\varphi \nu), \Psi_J |_{L^2 (\omega)} \lesssim C, \text{P}^\lambda (J, \nu) \| \Psi_J \|_{L^1 (\omega)} \leq C, \text{P}^\lambda (J, \nu) \sqrt{|J|^\omega} \| \Psi_J \|_{L^2 (\omega)},
\end{equation}

for any function $\varphi$ with $|\varphi| \leq 1$.

This form of the lemma is proved in many places in the literature, but usually with only the far right estimate. However, all of the proofs can be stopped one line short to give the first estimate, see e.g. \[NTV4\] where it originates.

6.1. Disjoint form. We decompose the disjoint form into two pieces,

$$B \cap (f, g) = \sum_{I, J \in D : \text{dist}(I, J) \leq \delta} \langle T_\sigma^\lambda (\Delta_I^\gamma f), (\Delta_J^\gamma g) \rangle \omega$$

$$= \left\{ \sum_{I, J \in D : \text{dist}(I, J) \leq \delta} \sum_{\ell (I) < 2^{-\gamma} \ell (J)} + \sum_{I, J \in D : \text{dist}(I, J) \leq \delta} \sum_{\ell (I) > 2^{-\gamma} \ell (J)} \langle T_\sigma^\lambda (\Delta_I^\gamma f), (\Delta_J^\gamma g) \rangle \omega \right\}
$$

$$B \cap (f, g) = B_{\text{down}} (f, g) + B_{\text{up}} (f, g).$$

Since the up form is dual to the down form, we consider only $B_{\text{down}} (f, g)$, and we will prove the following estimate:

\begin{equation}
|B_{\text{down}} (f, g)| \lesssim A_\psi^{\lambda, \epsilon, \text{offset}} (\sigma, \omega) \| f \|_{L^p (\omega)} \| g \|_{L^{p'} (\omega)}.
\end{equation}

Porism: It is important to note that from the proof given, we may replace the sum $\sum_{I, J \in D : \text{dist}(I, J) \leq \delta}$ in the left hand side of (6.3) with a sum over any subset of the pairs $I, J$ arising in $B_{\text{down}} (f, g)$. A similar remark of course applies to $B_{\text{up}} (f, g)$.

Proof of (6.3). Denote by $\text{dist} (x, y)$ the $\infty$ distance in $\mathbb{R}^n$: $\text{dist} (x, y) = \max_{1 \leq j \leq n} |x_j - y_j|$. We now estimate separately the long-range and mid-range cases in $B_{\text{down}} (f, g)$ where $\text{dist} (J, I) \geq \ell (I)$ holds or not, and we decompose $B_{\text{down}} (f, g)$ accordingly:

$$B_{\text{down}} (f, g) = A_{\text{long}} (f, g) + A_{\text{mid}} (f, g).$$

The long-range case: We begin with the case where $\text{dist} (J, I)$ is at least $\ell (I)$, i.e. $J \cap 3I = \emptyset$. With $A (f, g) = A_{\text{long}} (f, g)$ we have

$$A (f, g) = \sum_{I, J \in D : \text{dist}(J, I) \geq \ell (I)} \langle T_\sigma^\lambda (\Delta_I^\gamma f), (\Delta_J^\gamma g) \rangle \omega = \sum_{s=1}^{\infty} \sum_{m=1}^{\infty} A_{s, m} (f, g),$$

where

$$A_{s, m} (f, g) = \sum_{I, J \in D : \text{dist}(J, I) \geq \ell (I)^m} \langle T_\sigma^\lambda (\Delta_I^\gamma f), (\Delta_J^\gamma g) \rangle \omega = \sum_{I \in D} \sum_{J \in F_{s, m} (I)} \langle T_\sigma^\lambda (\Delta_J^\gamma f), (\Delta_J^\gamma g) \rangle \omega = \sum_{I \in D} \langle T_\sigma^\lambda (Q_J^s, m f), (\Delta_J^\gamma g) \rangle \omega,$$
with
\[ F_{s,m}(J) \equiv \{ I \in D : \text{dist}(J,I) \approx 2^m \ell(J) \}, \ell(I) = 2^s \ell(J) \} \text{ and } Q_{s,m}^J \equiv \sum_{I \in F_{s,m}(J)} \Delta_J^\sigma.

Then from the Pivotal Lemma \[2\] we have
\[
\left| \left< T_\lambda^\sigma Q_{s,m}^J f, \Delta_J^\sigma g \right> \right|_\omega \approx P^\lambda(J, |Q_{s,m}^J f| \sigma) \int_J |\Delta_J^\sigma g| \, d\omega,
\]
where
\[
P^\lambda(J, |Q_{s,m}^J f| \sigma) = \int_{\mathbb{R}^n} \frac{\ell(J)}{\ell(J) + \text{dist}(y,J)^{n+1-\lambda}} |Q_{s,m}^J f(y)| \, d\sigma(y) \lesssim 2^{-(s+m)} \int_{\mathbb{R}^n} \frac{1}{\ell(J) + \text{dist}(y,J)^{n+1-\lambda}} |Q_{s,m}^J f(y)| \, d\sigma(y),
\]
by the definition of \( Q_{s,m}^J \) since
\[
\ell(J) = 2^{-s} \ell(I) \approx 2^{-s-m} \text{dist}(y,J).
\]
Thus we have
\[
\left| A_{s,m}(f,g) \right| \lesssim 2^{-(s+m)} \int_{\mathbb{R}^n} \sum_{J \in D} \left( \int_{J \in D} \frac{1}{|c_J - y|^{n-\lambda}} |Q_{s,m}^J f(y)| \, d\sigma(y) \right)^2 \int_J |\Delta_J^\sigma g(x)| \, d\omega(x)
\lesssim 2^{-(s+m)} \int_{\mathbb{R}^n} \left( \sum_{J \in D} \left( \int_{J \in D} \frac{1}{|c_J - y|^{n-\lambda}} |Q_{s,m}^J f(y)| \, d\sigma(y) \right)^2 \right)^\frac{1}{2} \left( \sum_{J \in D} |\Delta_J^\sigma g(x)|^2 \right)^\frac{1}{2} \, d\omega(x)
\leq 2^{-(s+m)} \left( \int_{\mathbb{R}^n} \left( \sum_{J \in D} \left( \int_{J \in D} \frac{1}{|c_J - y|^{n-\lambda}} |Q_{s,m}^J f(y)| \, d\sigma(y) \right)^2 \right)^\frac{1}{2} \right)^\frac{\|S_{\text{Haar}}^J g\|_{L^p(\omega)}}{\|f\|_{L^p(\sigma)}}.
\]
Now \( S_{\text{Haar}} \) is bounded on \( L^p(\omega) \), and so by the geometric decay in \( s \) and \( m \), it remains to show that for each \( s, m \in \mathbb{N} \),
\[
\left( \int_{\mathbb{R}^n} \left( \sum_{J \in D} \left( \int_{J \in D} \frac{1}{|c_J - y|^{n-\lambda}} |Q_{s,m}^J f(y)| \, d\sigma(y) \right)^2 \int_J |\Delta_J^\sigma g(x)| \, d\omega(x) \right)^\frac{1}{2} \right)^\frac{1}{2} \lesssim A_p \|f\|_{L^p(\sigma)}.
\]
For this we use \[3\] to write
\[
\int_{\mathbb{R}^n} \frac{1}{|c_J - y|^{n-\lambda}} |Q_{s,m}^J f(y)| \, d\sigma(y) \lesssim \frac{1}{(2^{(s+m)} \ell(J))^{n-\lambda}} \int_{\mathbb{R}^n} |Q_{s,m}^J f(y)| \, d\sigma(y),
\]
and then obtain with \( K_{s,m}(J) \approx \bigcup_{j=1}^{2^n} K_{s,m}(J) \) roughly equal to the support of \( Q_{s,m}^J \), that
\[
\int_{\mathbb{R}^n} \left( \sum_{J \in D} \left( \int_{J \in D} \frac{1}{|c_J - y|^{n-\lambda}} |Q_{s,m}^J f(y)| \, d\sigma(y) \right)^2 \int_J |\Delta_J^\sigma g(x)| \, d\omega(x) \right)^\frac{1}{2} \lesssim \int_{\mathbb{R}^n} \left( \sum_{J \in D} \left( \frac{1}{(2^{(s+m)} \ell(J))^{n-\lambda}} \int_{K_{s,m}(J)} |Q_{s,m}^J f(y)| \, d\sigma(y) \right)^2 \right)^\frac{1}{2} \, d\omega(x)
\lesssim \int_{\mathbb{R}^n} \left( \sum_{K \in D} \sum_{J \in D : J \subset K} \sum_{J : K \subset J} \sum_{J \in D} \left( \frac{1}{\ell(K)^{n-\lambda}} \int_{K} |Q_K^J f(y)| \, d\sigma(y) \right)^2 \right)^\frac{1}{2} \, d\omega(x),
where $Q_K^s \equiv \sum_{K_{\cdot, m}(J)=K} Q_{k}^{s, m}$ and $\bar{K} = \bigcup_{j=1}^{2^n} \bar{K}^j (J)$ is a union of dyadic cubes $\bar{K}^j$ surrounding $K$ with $0 < \text{dist} (\bar{K}^j, K) \lesssim \ell (K) \approx \ell (\bar{K}^j)$. Now we use first the quadratic offset condition $A_{p, \ell}^{\lambda, \epsilon, \text{offset}} (\sigma, \omega)$, and then the Fefferman-Stein vector-valued inequality for the maximal function, to obtain the following vector-valued inequality for each fixed $s, m \in \mathbb{N}$,

$$
\int_{\mathbb{R}^n} \left( \sum_{K \in D} 1_{\bar{K}} (x) \left( \frac{|K|_x}{\ell (K) \cdot |K|_x} \int_K |Q_K^{s, m} f (y)| \, d\sigma (y) \right)^2 \right)^{\frac{p}{2}} \, d\omega (x) \\
\lesssim A_{p, \ell}^{\lambda, \epsilon, \text{offset}} (\sigma, \omega)^p \int_{\mathbb{R}^n} \left( \sum_{K \in D} 1_{\bar{K}} (x) \left( \frac{1}{|K|_x} \int_K |Q_K^{s, m} f (y)| \, d\sigma (y) \right)^2 \right)^{\frac{p}{2}} \, d\sigma (x) \\
\lesssim A_{p, \ell}^{\lambda, \epsilon, \text{offset}} (\sigma, \omega)^p \int_{\mathbb{R}^n} \left( \sum_{K \in D} |Q_K^{s, m} f (x)|^2 \right)^{\frac{p}{2}} \, d\sigma (x) \lesssim A_{p, \ell}^{\lambda, \epsilon, \text{offset}} (\sigma, \omega)^p \| f \|_{L^p (\sigma)} \, .
$$

As mentioned above, this completes the proof of the long range case by the geometric decay in $s$ and $m$.

The mid range case: Let

$$
P = \{ (I, J) \in D \times D : J \text{ is good, } \ell (J) \leq 2^{-\xi} \ell (I), J \subset 3I \setminus I \} \, .
$$

Now we pigeonhole the lengths of $I$ and $J$ and the distance between them by defining

$$
P^i_d = \{ (I, J) \in D \times D : J \text{ is good, } \ell (I) = 2^i \ell (J), J \subset 3I \setminus I, 2^{d-1} \ell (J) \leq \text{dist} (I, J) \leq 2^d \ell (J) \} \, .
$$

Note that the closest a good cube $J$ can come to $I$ is determined by the goodness inequality, which gives this bound:

$$
2^d \ell (J) \geq \text{dist} (I, J) \geq \frac{1}{2} \epsilon (I) \ell (I)^{1-\epsilon} \ell (J)^{\epsilon} = \frac{1}{2} \sigma^{(1-\epsilon)} \ell (J) \; ;
$$

which implies $d \geq t (1-\epsilon) - 1$.

We write

$$
\sum_{(I, J) \in P} \langle T_\sigma^\lambda (\Delta^\omega_j f), \Delta^\omega_j g \rangle_\omega = \sum_{i=0}^{\infty} \sum_{d=N-d-1}^N \sum_{(I, J) \in P^i_d} \langle T_\sigma^\lambda (\Delta^\omega_j f), \Delta^\omega_j g \rangle_\omega \, ,
$$

and for fixed $t$ and $d$, we estimate

$$
\left| \sum_{(I, J) \in P^i_d} \langle T_\sigma^\lambda (\Delta^\omega_j f), \Delta^\omega_j g \rangle_\omega \right| = \left| \int_{\mathbb{R}^n} \sum_{(I, J) \in P^i_d} T_\sigma^\lambda (\Delta^\omega_j f) (x) \, T_\sigma^\lambda (\Delta^\omega_j g) (x) \, d\omega (x) \right| \\
= \left| \int_{\mathbb{R}^n} \sum_{J \in D} \Delta^\omega_j T_\sigma^\lambda \left( \sum_{(I, J) \in P^i_d} \Delta^\omega_j f \right) (x) \, \Delta^\omega_j g (x) \, d\omega (x) \right| \\
\leq \left| \int_{\mathbb{R}^n} \sum_{J \in D} \Delta^\omega_j T_\sigma^\lambda \left( \sum_{(I, J) \in P^i_d} \Delta^\omega_j f \right) (x) \, d\omega (x) \right| \\
\lesssim \left\{ \int_{\mathbb{R}^n} \left( \sum_{J \in D} \left| \Delta^\omega_j T_\sigma^\lambda \left( \sum_{(I, J) \in P^i_d} \Delta^\omega_j f \right) (x) \right|^2 \right)^{\frac{p}{2}} \, d\omega (x) \right\}^{\frac{1}{p}} \times \left\{ \int_{\mathbb{R}^n} \left( \sum_{J \in D} \left| \Delta^\omega_j g (x) \right|^2 \right)^{\frac{p}{2}} \, d\omega (x) \right\}^{\frac{1}{p}} \, .
$$
Now we use the fact that for a fixed \( J \), there are only boundedly many \( I \in \mathcal{D} \) with \( (I, J) \in \mathcal{P}_d^\prime \), which without loss of generality we can suppose is a single cube \( I [J] \), together with (6.6) to obtain the estimate

\[
|\Delta_J^\sigma T^\sigma_I (\Delta_I^\sigma f) (x)| \lesssim P^\lambda (J, |\Delta_I^\sigma f| \sigma) 1_J (x) = \int \frac{\ell (J)}{\ell (J) + |y - c_J|} |\Delta_I^\sigma f (y)| d\sigma (y) 1_J (x)
\]

\[
\lesssim \frac{\ell (J)}{(24\ell (J))^{n-1-\lambda}} \sum_{I' \in \mathcal{D}_I (I [J])} E_{I'} T_{I'} (\Delta_{I'}^\sigma f) |I'| |I| 1_J (x)
\]

\[
\lesssim \frac{\ell (J)}{(24\ell (J))^{n-1-\lambda}} \sum_{I' \in \mathcal{D}_I (I [J])} E_{I'} T_{I'} (\Delta_{I'}^\sigma f) |I'| |I| 1_J (x),
\]

since

\[
\frac{\ell (J)}{(24\ell (J))^{n-1-\lambda}} = \frac{2^{-t} (t-d)n+1-\lambda}{\ell (I [J])^{n-1-\lambda}} \leq \frac{2^{-t} (t-1)(n+1-\lambda)}{\ell (I [J])^{n-1-\lambda}} = \frac{2^{n+1-\lambda} 2^{-t} (n+1-\lambda)}{\ell (I [J])^{n-1-\lambda}}.
\]

Thus we have

\[
\left\{ \int_{\mathbb{R}^n} \left( \sum_{J' \in \mathcal{D}} \left| \Delta_{J'}^\sigma T^\sigma_{I'} \left( \sum_{I, J \in \mathcal{P}_d^\prime} \Delta_I^\sigma f \right) \right|^2 \right)^{\frac{p}{2}} d\omega (x) \right\}^{\frac{1}{p}}
\]

\[
\lesssim 2^{-t[1-(n+1-\lambda)]} \left( \int_{\mathbb{R}^n} \left( \sum_{J' \in \mathcal{D}_I (I [J])} \sum_{I' \in \mathcal{D}_I (I [J])} E_{I'} T_{I'} (\Delta_{I'}^\sigma f) \left| I' \right| |I^\sigma| 1_I (x) \right)^2 d\omega (x) \right)^{\frac{1}{2}}
\]

\[
\lesssim 2^{-t[1-(n+1-\lambda)]} \left( \int_{\mathbb{R}^n} \left( \sum_{J' \in \mathcal{D}_I (I [J])} \sum_{I' \in \mathcal{D}_I (I [J])} E_{I'} T_{I'} (\Delta_{I'}^\sigma f) \left| I' \right| |I^\sigma| 1_I (x) \right)^2 d\omega (x) \right)^{\frac{1}{2}}
\]

\[
\lesssim 2^{-t[1-(n+1-\lambda)]} A_p^{1, \ell^2, \text{offset} (\sigma, \omega)} \left( \int_{\mathbb{R}^n} \left( \sum_{J' \in \mathcal{D}_I (I [J])} \sum_{I' \in \mathcal{D}_I (I [J])} \left( E_{I'} (\Delta_{I'}^\sigma f)^2 \right) 1_I (x) \right)^{\frac{p}{2}} d\sigma (x) \right)^{\frac{1}{p}}
\]

and provided \( 0 < \varepsilon < \frac{1}{n+1-\lambda} \), we can sum in \( t \) to complete the proof of (6.3). \( \square \)

6.2. **Comparable form.** We decompose

\[
B^{\prime} (f, g) = B^{\prime \text{below}} (f, g) + B^{\prime \text{above}} (f, g);
\]

where \( B^{\prime \text{below}} (f, g) = \sum_{J \in \mathcal{D} : 2^{-\rho} \leq |J| \leq 1} \langle T^\sigma_J (\Delta_J^\sigma f), (\Delta_J^\sigma g) \rangle \omega \)

\[
= \sum_{J \in \mathcal{D} : 2^{-\rho} \leq |J| \leq 1} \langle T^\sigma_J (\Delta_J^\sigma f), (\Delta_J^\sigma g) \rangle \omega
\]

\[
+ \sum_{J \in \mathcal{D} : 2^{-\rho} \leq |J| \leq 1} \langle T^\sigma_J (\Delta_J^\sigma f), (\Delta_J^\sigma g) \rangle \omega
\]

\[
= B^{\text{below near} (f, g)} + B^{\text{below far} (f, g)}.
\]
The second form $B_{below \, far}^\lambda (f, g)$ is handled in the same way as the disjoint far form $B_{far}^\lambda (f, g)$ in the previous subsection, and for the first form $B_{below \, near}^\lambda (f, g)$, we write

$$
|B_{below \, near}^\lambda (f, g)| = \left| \int_{\mathbb{R}^n} \sum_{I, J \in D: 2^{-\rho} \leq \frac{|I|}{|J|} \leq 1} \sum_{J \cap T = \emptyset \text{ and } |J| \leq 1} T^\lambda_{\alpha} (\Delta^\alpha_I f) (x) \, (\Delta^\omega_J g) (x) \, d\omega (x) \right|
$$

$$
\lesssim \int_{\mathbb{R}^n} \sum_{I, J \in D: 2^{-\rho} \leq \frac{|I|}{|J|} \leq 1} \left( \frac{1}{|I|} \int_I |\Delta^\alpha_I f| \, d\sigma \right) \left| \Delta^\omega_J g (x) \right| \, d\omega (x)
$$

$$
\lesssim \int_{\mathbb{R}^n} \sum_{I, J \in D: 2^{-\rho} \leq \frac{|I|}{|J|} \leq 1} \left( \frac{1}{|I|} \int_I \left| \int_I |\Delta^\alpha_I f| \, d\sigma \right| \frac{|I|_\sigma}{|I|^{1-\frac{\sigma}{n}}} \right) \left| \Delta^\omega_J g (x) \right| \, d\omega (x),
$$

and so by the Cauchy-Schwarz inequality, we have

$$
|B_{below \, near}^\lambda (f, g)| \lesssim \int_{\mathbb{R}^n} \left( \sum_{I, J \in D: 2^{-\rho} \leq \frac{|I|}{|J|} \leq 1} \left( \frac{1}{|I|} \int_I |\Delta^\alpha_I f| \, d\sigma \right) \frac{|I|_\sigma}{|I|^{1-\frac{\sigma}{n}}} \right)^2 \left| \Delta^\omega_J g (x) \right| \, d\omega (x)
$$

and

$$
\left\| \left( \sum_{I, J \in D: 2^{-\rho} \leq \frac{|I|}{|J|} \leq 1} \left( \frac{1}{|I|} \int_I |\Delta^\alpha_I f| \, d\sigma \right) \frac{|I|_\sigma}{|I|^{1-\frac{\sigma}{n}}} \right)^2 \left| \Delta^\omega_J g (x) \right| \right\|_{L^p(\omega)}
$$

$$
\lesssim A_p^{\lambda^2, \text{offset}} (\sigma, \omega) \left\| \left( \sum_{I, J \in D: 2^{-\rho} \leq \frac{|I|}{|J|} \leq 1} \left( \frac{1}{|I|} \int_I |\Delta^\alpha_I f| \, d\sigma \right) \frac{2}{|I|_\sigma} \left| \Delta^\omega_J g (x) \right| \right)^2 \right\|_{L^p(\sigma)}
$$
Finally, the form \(B_{\text{stop}}\) (6.7) quadratic testing stopping form argument due to Nazarov, Treil and Volberg \[NTV4\] to bound the stopping form by local estimates, we have controlled the norms of the below forms and by the Fefferman-Stein maximal inequality in the space of homogeneous type \([\text{GrLiYa}]\).

\[
\left\| \sum_{I,J \in \mathcal{D}: 2^{-n} \leq \frac{\|I\|}{\|J\|} \leq 1 \atop \mathcal{T} \cap \mathcal{T} = \emptyset, J \subset 3I} \left( \frac{1}{|3I|} \int_{3I} |\nabla^\sigma f| \, d\sigma \right)^2 1_{3I}(x) \right\|_{L^p(\sigma)} \lesssim \left\| \sum_{I,J \in \mathcal{D}: 2^{-n} \leq \frac{\|I\|}{\|J\|} \leq 1 \atop \mathcal{T} \cap \mathcal{T} = \emptyset, J \subset 3I} [\mathcal{M}_\sigma |\nabla^\sigma f| (x)]^2 \right\|_{L^p(\sigma)} \]

\[
\lesssim \left\| \sum_{I,J \in \mathcal{D}: 2^{-n} \leq \frac{\|I\|}{\|J\|} \leq 1 \atop \mathcal{T} \cap \mathcal{T} = \emptyset, J \subset 3I} |\nabla^\sigma f (x)|^2 \right\|_{L^p(\sigma)} \lesssim \|S_{\text{Haar}}f\|_{L^p(\sigma)}.
\]

Altogether, since both \(\|S_{\text{Haar}}f\|_{L^p(\sigma)} \approx \|f\|_{L^p(\sigma)} \) and \(\|S_{\text{Haar}}g\|_{L^p(\omega)} \approx \|g\|_{L^p(\omega)}\) by square function estimates, we have controlled the norms of the below forms \(B_{\text{below}}\) near \((f, g)\) and \(B_{\text{below}}\) far \((f, g)\) by the quadratic offset Muckenhoupt constant \(A_p^{\lambda,\ell^2,\text{offset}} (\sigma, \omega)\), hence

\[
|B_{\text{below}}(f, g)| \lesssim A_p^{\lambda,\ell^2,\text{offset}} (\sigma, \omega) \|f\|_{L^p(\sigma)} \|g\|_{L^p(\omega)}.
\]

Finally, the form \(B_{\text{above}}(f, g)\) is handled in dual fashion to \(B_{\text{below}}(f, g)\).

**Porism:** It is important to note that from the proof given, we may replace the sum

\[
\sum_{I,J \in \mathcal{D}: 2^{-n} \leq \frac{\|I\|}{\|J\|} \leq 1 \atop \mathcal{T} \cap \mathcal{T} = \emptyset \text{ and } J \subset 3I}
\]

in the left hand side of (6.7) with a sum over any subset of the pairs \(I, J\) arising in \(B_{\text{below}}(f, g)\). A similar remark of course applies to \(B_{\text{above}}(f, g)\).

### 6.3. Stopping form

We assume that \(\sigma\) and \(\omega\) are doubling measures. We will use a variant of the Haar stopping form argument due to Nazarov, Treil and Volberg \[NTV4\] to bound the stopping form by local quadratic testing \(T_{B_{\text{stop}}, p}^{\lambda, \ell^2, \text{loc}} (\sigma, \omega)\) and offset Muckenhoupt \(A_p^{\lambda,\ell^2,\text{offset}} (\sigma, \omega)\) constants defined in \ref{1.1} and \ref{1.0} respectively. We start the proof by pigeonholing the ratio of side lengths of \(I\) and \(J\) in the local stopping forms:

\[
B_{\text{stop}}^F (f, g) \equiv \sum_{I \in C_F} \sum_{I' \in \mathcal{D}(I)} \sum_{J \in C_{F-\text{shift}}^{\ell} \atop J \subset I'} \langle 1_{I'}, \nabla^\sigma f, T_\sigma^\lambda 1_{F \setminus I'}, \nabla^\omega g \rangle_\omega
\]

\[
= \sum_{I \in C_F} \sum_{I' \in \mathcal{D}(I)} \sum_{J \in C_{F-\text{shift}}^{\ell} \atop J \subset I'} \langle \nabla^\sigma f, T_\sigma^\lambda 1_{F \setminus I'}, \nabla^\omega g \rangle_\omega
\]

\[
= \sum_{s=0}^{\infty} \sum_{I \in C_F} \sum_{I' \in \mathcal{D}(I) \atop J \in C_{F-\text{shift}}^{\ell} \text{and } \ell(J) = 2^{-s} \ell(I)} \sum_{J \subset I'} \langle \nabla^\omega \left[ (\nabla^\sigma f) T_\sigma^\lambda 1_{F \setminus I'} \right], \nabla^\omega g \rangle_\omega.
\]

Now we write \(J \prec I'\) when \(\pi_D I' \in C_{F-\text{shift}}^{\ell}\) and

\[
J \in C_{F-\text{shift}}^{\ell}, \ell(J) = 2^{-s} \ell(I), J \subset I' \text{ and } J \Subset_{p, \ell} I,
\]
so that we have
\[
\mathcal{B}_{\text{stop}}(f, g) = \sum_{F \in \mathcal{F}} \mathcal{B}^F_{\text{stop}}(f, g)
\]
\[
= \sum_{s=0}^{\infty} \sum_{F \in \mathcal{F}} \left( \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{D}(I) \cap \mathcal{C}_{F}^{-\text{shift}}} \sum_{J \subseteq I'} \sum_{j \in \mathbb{S}_I} \langle \triangle^s_j \left[ (\triangle^s_j f) \ T^s_\sigma 1_{F' \setminus I} \right], \triangle^s_j g \rangle \omega \right)
\]
\[
= \sum_{s=0}^{\infty} \sum_{J \in \mathcal{D}} \left( \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{D}(I) \cap \mathcal{C}_{F}^{-\text{shift}}} \sum_{J \subseteq I'} \triangle^s_j \left[ (\triangle^s_j f) \ T^s_\sigma 1_{F' \setminus I} \right] \right) \omega \int_{\mathbb{R}^n} \sum_{J \in \mathcal{D}} \sum_{s=0}^{\infty} \mathcal{B}_{\text{stop}, s}(f, g) \, d\omega(x)
\]
But now we observe that if \( J \subset I' \) then \( \triangle^s_j f \) is a constant on \( J \) and so \( \mathcal{B}^s_{\text{stop}} \) and \( \mathcal{B}^s_{\text{stop}} \), together with the observation that \( \triangle^s_j \left[ (\triangle^s_j f) \ T^s_\sigma 1_{F' \setminus I} \right] \) has a vanishing moment, yield the following inequality,
\[
\left| \triangle^s_j \left[ (\triangle^s_j f) \ T^s_\sigma 1_{F' \setminus I} \right] \right| \leq \left\| E^s_{\sigma} \triangle^s_j f \right\|_\infty \mathcal{P}^{\lambda} \left( J, 1_{F' \setminus I'} \right) 1_J(x) \leq E^s_{\sigma} \left| \triangle^s_j f \right| \mathcal{P}^{\lambda} \left( J, 1_{F' \setminus I'} \right) 1_J(x).
\]
Now we can obtain geometric decay in \( s \). Indeed, applying Cauchy-Schwarz we obtain for each \( s \),
\[
\mathcal{B}_{\text{stop}, s}(f, g) = \int_{\mathbb{R}^n} \sum_{J \in \mathcal{D}} \left( \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{D}(I) \cap \mathcal{C}_{F}^{-\text{shift}} \cap \mathcal{C}_{F}^{-\text{shift}}} \sum_{J \subseteq I'} \triangle^s_j \left[ (\triangle^s_j f) \ T^s_\sigma 1_{F' \setminus I} \right] \right) \, d\omega(x)
\]
\[
\leq \left\| \left( \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{D}(I) \cap \mathcal{C}_{F}^{-\text{shift}}} \sum_{J \subseteq I'} E^s_{\sigma} \left| \triangle^s_j f \right| \mathcal{P}^{\lambda} \left( J, 1_{F' \setminus I} \right) 1_J(x) \right\|_{L^p(\omega)} \left( \sum_{J \in \mathcal{D}} \left| \triangle^s_j g(x) \right|^2 \right)^{\frac{1}{2}}
\]
\[
= \left\| S(x) \right\|_{L^p(\omega)} \left( \sum_{J \in \mathcal{D}} \left| \triangle^s_j g(x) \right|^2 \right)^{\frac{1}{2}}
\]
where \( S(x)^2 \equiv \sum_{J \in \mathcal{D}} \left( \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{D}(I) \cap \mathcal{C}_{F}^{-\text{shift}}} \sum_{J \subseteq I'} E^s_{\sigma} \left| \triangle^s_j f \right| \mathcal{P}^{\lambda} \left( J, 1_{F' \setminus I} \right) 1_J(x) \right)^2.
\]
For fixed \( x \in J \), the pigeonholing above yields \( I = \pi^J_D J \) and \( F = \pi^J_D \pi^J_D J \), and thus we obtain
\[
S(x)^2 = \sum_{J \in \mathcal{D}} \left( \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{D}(I) \cap \mathcal{C}_{F}^{-\text{shift}}} \sum_{J \subseteq I'} E^s_{\sigma} \left| \triangle^s_j f \right| \mathcal{P}^{\lambda} \left( J, 1_{F' \setminus I} \right) 1_J(x) \right)^2
\]
\[
\lesssim \sum_{J \in \mathcal{D}} \left( E^s_{\sigma} \left| \triangle^s_j f \right| \mathcal{P}^{\lambda} \left( J, 1_{F' \setminus I} \right) 1_J(x) \right)^2,
\]
and now using the Poisson inequality with
\[
\eta \equiv 1 - \varepsilon (n + 1 - \lambda) > 0,
\]
we obtain
\[
S(x)^2 \lesssim 2^{-2s} \sum_{J \in \mathcal{D}} \left( E^s_{\sigma} \left| \triangle^s_j f \right| \mathcal{P}^{\lambda} \left( J, 1_{F' \setminus I} \right) 1_J(x) \right)^2 \lesssim 2^{-2s} \sum_{J \in \mathcal{D}} \left| \mathcal{F}(\pi^J_D J) \right|^2 \frac{1}{\mathcal{P}^{\lambda} \left( J, 1_{F' \setminus I} \right) 1_J(x)} \mathcal{P}^{\lambda} \left( J, 1_{F' \setminus I} \right) 1_J(x).
\]
Since \( E_f^\gamma |\Delta_f^\gamma| \lesssim E_f^\gamma |\Delta_f^\gamma f| \) by (2.4) and (2.5), we have
\[
(6.9) \quad S(x)^2 \lesssim 2^{-2\eta s} \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{C}_P(I)} \left| \hat{f}(I) \right|^2 \frac{1}{|I'|_\sigma} P^\lambda \left( I', 1_{F \setminus I'} \right)^2 1_{I'}(x),
\]

Then from inequality (6.12) below we get,
\[
(6.10) \quad \|S(x)\|_{L^p(\omega)} \lesssim 2^{-\eta s} \left( 2^{\ell_0,\text{loc}} (\sigma, \omega) + A_p^{\lambda,\ell_0,\text{offset}} (\sigma, \omega) \right) \|f\|_{L^p(\sigma)}.
\]

Finally then, by Hölder’s inequality we obtain
\[
|B\text{\textit{stop}}(f, g)| \lesssim \|S(x)\|_{L^p(\omega)} \left\| \sum_{\ell \in \mathcal{D}} |\Delta_f^\sigma g(x)|^2 \right\|_{L^\sigma(\omega)}^{\frac{1}{2}} \lesssim 2^{-\eta s} \left( 2^{\ell_0,\text{loc}} (\sigma, \omega) + A_p^{\lambda,\ell_0,\text{offset}} (\sigma, \omega) \right) \|f\|_{L^p(\sigma)} \|g\|_{L^{p'}(\omega)},
\]

and provided \( \varepsilon < \frac{1}{n+1-\lambda} \), i.e. \( \eta > 0 \), summing in \( s \) gives
\[
|B\text{\textit{stop}}(f, g)| \leq \sum_{s = 0}^{\infty} |B\text{\textit{stop}}(f, g)| \lesssim C_{n, \lambda} \left( 2^{\ell_0,\text{loc}} (\sigma, \omega) + A_p^{\lambda,\ell_0,\text{offset}} (\sigma, \omega) \right) \|f\|_{L^p(\sigma)} \|g\|_{L^{p'}(\omega)}.
\]

It remains to justify (6.10). Since \( E_f^\gamma |\Delta_f^\gamma| \leq \|1_J|\Delta_f^\gamma|\|_\infty \leq |1_I| |\Delta_f^\gamma f|\|_\infty \leq E_f^\gamma |\Delta_f^\gamma f| \) by (2.4) and (2.5), we have
\[
S(x)^2 \lesssim 2^{-2\eta s} \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{C}_P(I)} (E_f^\gamma |\Delta_f^\gamma f|^2)^2 P^\lambda \left( I', 1_{F \setminus I'} \right)^2 1_{I'}(x).
\]

We claim that
\[
(6.11) \quad \left\| \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{C}_P(I)} (E_f^\gamma |\Delta_f^\gamma f|^2)^2 P^\lambda \left( I', 1_{F \setminus I'} \right)^2 1_{I'}(x) \right\|_{L^p(\omega)} \lesssim \left( 2^{\ell_0,\text{loc}} (\sigma, \omega) + A_p^{\lambda,\ell_0,\text{offset}} (\sigma, \omega) \right) \|f\|_{L^p(\sigma)}.
\]

With this established we will then obtain,
\[
(6.12) \quad \|S(x)\|_{L^p(\omega)} \lesssim 2^{-\eta s} \left\| \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{C}_P(I)} (E_f^\gamma |\Delta_f^\gamma f|^2)^2 P^\lambda \left( I', 1_{F \setminus I'} \right)^2 1_{I'}(x) \right\|_{L^p(\omega)} \lesssim 2^{-\eta s} \left( 2^{\ell_0,\text{loc}} (\sigma, \omega) + A_p^{\lambda,\ell_0,\text{offset}} (\sigma, \omega) \right) \|f\|_{L^p(\sigma)},
\]

which gives (6.11).

In order to prove (6.11), we will need a stronger notion of energy reversal, which we now describe. But first we recall the definition of strong energy reversal from SaShUr9. We say that a vector \( T^\lambda = \{T^\lambda_{\ell}\}_{\ell = 1}^2 \) of \( \lambda \)-fractional transforms has strong reversal of \( \omega \)-energy on a cube \( J \) if there is a positive constant \( C_0 \) such that for all \( 2 \leq \gamma \leq 2^{(1-\varepsilon)} \) and for all positive measures \( \mu \) supported outside \( \gamma J \), we have the inequality
\[
(6.13) \quad E_f^\gamma \left( (x - E_f^\gamma x)^2 \right) \left( \frac{P^\lambda (J, \mu)}{|J|^\frac{\gamma}{\omega}} \right)^2 = E_J (\omega \mu) P^\lambda (J, \mu)^2 \leq C_0 E_f^\gamma \|T^\lambda \mu - E_f^\gamma T^\lambda \mu\|^2.
\]

We now introduce a stronger notion of energy reversal which we call extreme energy reversal. We say that a vector \( T^\lambda = \{T^\lambda_{\ell}\}_{\ell = 1}^2 \) of \( \lambda \)-fractional transforms has extreme reversal of \( \omega \)-energy on a cube \( J \) if there is a Haar function \( h_f^\gamma(x) \) and a positive constant \( C_0 \), such that for all \( 2 \leq \gamma \leq 2^{(1-\varepsilon)} \) and for all positive measures \( \mu \) supported outside \( \gamma J \), we have the inequality,
\[
(6.14) \quad E_f^\gamma \left( (x - E_f^\gamma x)^2 \right) \left( \frac{P^\lambda (J, \mu)}{|J|^\frac{\gamma}{\omega}} \right)^2 |J|_\omega = E_J (\omega \mu) P^\lambda (J, \mu)^2 |J|_\omega \leq C \int_J \int_{R^\gamma \setminus J} \left[ K^\lambda (x, y) - K^\lambda (c_f, y) \right] h_f^\gamma (x) d\mu (y) d\omega (x),
\]
and $K^\lambda$ is the kernel of $T^\lambda$. Note that (6.14) is weaker than (6.13) in that the there is no absolute value inside the integral, and the difference of kernels $K^\lambda(x, y) - K^\lambda(c, y)$ is multiplied by a single Haar function $h^\sigma_j(x)$.

Clearly extreme reversal of energy implies strong reversal of energy. We prove below that extreme reversal of energy holds for the vector $\lambda$-fractional Riesz transform $R^\lambda_n$ in $\mathbb{R}^n$. But first we will use extreme reversal of energy to prove (6.11).

**Lemma 21.** Suppose $\sigma$ and $\omega$ are doubling measures on $\mathbb{R}^n$. Then (6.11) holds for $1 < p < \infty$, $0 < \lambda < n$ and $f$ and $F$ as above.

**Proof.** For each $I'$ we first write $F \setminus I' = (\gamma I' \setminus I') \cup (F \setminus \gamma I')$ and $P^\lambda (I', 1_{F \setminus I'}) = P^\lambda (I', 1_{\gamma I' \setminus I'}) + P^\lambda (I', 1_{F \setminus \gamma I'})$. For convenience, we sometimes write $a_I = E_{I'} |\Delta f|$, and only use $E_{I'} |\Delta f|$ when it matters. Because $\sigma$ is doubling we have $P^\lambda (I', 1_{\gamma I' \setminus I'}) \approx \frac{|I'|}{|\gamma I' \setminus I'|}$, and

$$
\left\| \left( \sum_{F \in \mathcal{F}} \sum_{I' \in \mathcal{F}_I} \sum_{I \in \mathcal{D}(I)} a_{I'}^2 P^\lambda (I', 1_{F \setminus \gamma I'})^2 1_{I'}(x) \right)^\frac{1}{2} \right\|_{L^p(\omega)} \lesssim \left\| \left( \sum_{F \in \mathcal{F}} \sum_{I' \in \mathcal{F}_I} \sum_{I \in \mathcal{D}(I)} a_{I'}^2 \left( \frac{|I'|}{|\gamma I' \setminus I'|} \right)^2 1_{I'}(x) \right)^\frac{1}{2} \right\|_{L^p(\omega)} \lesssim A_{\lambda, I'}\text{ offset } (\sigma, \omega) \|f\|_{L^p(\sigma)}.
$$

To handle the remaining term involving $P^\lambda (I', 1_{F \setminus I'})$ we will use extreme reversal of energy inequality for the vector $\lambda$-fractional Riesz transform $R^\lambda_n$ in $\mathbb{R}^n$. Since $\omega$ is doubling, we have $E (I', \omega) \approx 1$, and so by the Fefferman-Stein vector valued maximal inequality,

$$
\left\| \left( \sum_{F \in \mathcal{F}} \sum_{I' \in \mathcal{F}_I} \sum_{I \in \mathcal{D}(I)} a_{I'}^2 P^\lambda (I', 1_{F \setminus I'})^2 1_{I'} \right)^\frac{1}{2} \right\|_{L^p(\omega)} \approx \left\| \left( \sum_{F \in \mathcal{F}} \sum_{I' \in \mathcal{F}_I} \sum_{I \in \mathcal{D}(I)} a_{I'}^2 P^\lambda (I', 1_{F \setminus I'})^2 E (I', \omega)^2 1_{I'} \right)^\frac{1}{2} \right\|_{L^p(\omega)} \lesssim A_{\lambda, I'} \|f\|_{L^p(\sigma)}.
$$

which is at most

$$
(6.15) \quad \left\| \left( \sum_{F \in \mathcal{F}} \sum_{I' \in \mathcal{F}_I} \sum_{I \in \mathcal{D}(I)} a_{I'}^2 \left( \frac{R^\lambda_n 1_{F \setminus I'}}{\sqrt{|I'|}} \frac{h^\sigma_j(x)}{\sqrt{|I'|}} \right)^2 1_{I'} \right)^\frac{1}{2} \right\|_{L^p(\omega)} + \left\| \left( \sum_{F \in \mathcal{F}} \sum_{I' \in \mathcal{F}_I} \sum_{I \in \mathcal{D}(I)} a_{I'}^2 \left( \frac{R^\lambda_n 1_{F \setminus I'}}{\sqrt{|I'|}} \frac{h^\sigma_j(x)}{\sqrt{|I'|}} \right)^2 1_{I'} \right)^\frac{1}{2} \right\|_{L^p(\omega)},
$$

which is at most

$$
A + B.
$$
Then using the Fefferman-Stein vector valued maximal inequality in \([GrLiYa]\), first applied to the dyadic operator \(M^\text{dy}_\sigma\), followed by the quadratic testing condition, and finally another application of the Fefferman-Stein vector valued maximal inequality applied to the classical operator \(M_\sigma\), we obtain

\[
A = \left\| \left( \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{E}_D(I)} a_{I'}^2 \left\langle R^\lambda_\sigma 1_{I'}, \frac{h^\omega_{I'}}{\|h^\omega_{I'}\|_{L^\infty(\omega)}} \right\rangle \right)^2 1_{I'} \right\|_{L^p(\omega)}^{\frac{1}{2}} \]

\[
\lesssim \left\| \left( \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{E}_D(I)} a_{I'}^2 \left| M^\text{dy}_\sigma 1_F R^\lambda_\sigma 1_{I'} \right|^2 1_{I'} \right) \right\|_{L^p(\omega)} \lesssim \left\| \left( \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{E}_D(I)} a_{I'}^2 \left| 1_F R^\lambda_\sigma 1_{I'} \right|^2 1_{I'} \right) \right\|_{L^p(\omega)} \]

In order to estimate term \(B\) in (6.15), we use \(|E^\omega_{I'} \Delta^\omega_{I'} f| \lesssim \alpha_F(F)\) for \(I' \in \mathcal{C}_D(I)\) and \(I \in \mathcal{C}_F\), which holds since \(\sigma\) is doubling, and the inequality \(|P_C^\omega 1_F R^\lambda_\sigma 1_F| \lesssim M^\text{dy}_\sigma R^\lambda_\sigma 1_F\), and the Fefferman-Stein vector valued maximal inequality in \([GrLiYa]\), to obtain

\[
B = \left\| \left( \sum_{F \in \mathcal{F}} \alpha_F(F)^2 \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{E}_D(I)} \left\langle R^\lambda_\sigma 1_F, \frac{h^\omega_{I'}}{\|h^\omega_{I'}\|_{L^\infty(\omega)}} \right\rangle \right)^2 1_{I'} \right\|_{L^p(\omega)}^{\frac{1}{2}} \]

\[
\lesssim \left\| \left( \sum_{F \in \mathcal{F}} \alpha_F(F)^2 \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{E}_D(I)} \left\langle R^\lambda_\sigma 1_F, \frac{h^\omega_{I'}}{\|h^\omega_{I'}\|_{L^\infty(\omega)}} \right\rangle \right)^2 1_{I'} \right\|_{L^p(\omega)} \]

\[
= \left\| \left( \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{E}_D(I)} \frac{1}{\|h^\omega_{I'}\|_{L^\infty(\omega)}} \alpha_F(F) \Delta^\omega_{I'} R^\lambda_\sigma 1_F \right)^2 \right\|_{L^p(\omega)}^{\frac{1}{2}} \],

which is approximately

\[
\approx \left\| \left( \sum_{F \in \mathcal{F}} \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{E}_D(I)} \left| \alpha_F(F) \Delta^\omega_{I'} R^\lambda_\sigma 1_F \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \approx \left\| \sum_{F \in \mathcal{F}} \alpha_F(F) \sum_{I \in \mathcal{C}_F} \sum_{I' \in \mathcal{E}_D(I)} \Delta^\omega_{I'} R^\lambda_\sigma 1_F \right\|_{L^p(\omega)}
\]

\[
\lesssim \left\| \sum_{F \in \mathcal{F}} \alpha_F(F)^2 \left| M^\text{dy}_\sigma 1_F (R^\lambda_\sigma 1_F) \right|^2 \right\|_{L^p(\omega)} \lesssim \left\| \sum_{F \in \mathcal{F}} \alpha_F(F)^2 \left| 1_F R^\lambda_\sigma (1_F) \right|^2 \right\|_{L^p(\omega)} \]

\[
\lesssim \mathcal{T}^{p^2;\text{loc}}_{\mathcal{T}^\lambda_\sigma, p} (\sigma, \omega) \left\| \sum_{F \in \mathcal{F}} \alpha_F(F)^2 1_F \right\|_{L^p(\omega)} \lesssim \mathcal{T}^{p^2;\text{loc}}_{\mathcal{T}^\lambda_\sigma, p} (\sigma, \omega) \|f\|_{L^p(\omega)},
\]

where the final inequality follows from Theorem [7].
In order to show that extreme reversal of energy holds for the vector Riesz transform, we will model our argument on some of the material from [SashUr0], beginning with a calculation of the Laplacian of powers of $|x|$. An earlier, and somewhat similar and simpler, argument can be found in [LaWi], but we do not see how to immediately adapt that argument to the setting of $L^p$.

6.3.1. Fractional Riesz transforms. Now we compute for $\beta$ real that

$$\Delta |x|^\beta = \nabla \cdot \nabla |x|^{2\beta} = \nabla \cdot \left\{ \frac{\beta}{2} |x|^{\frac{2\beta}{\beta-2}} \right\} = \frac{\beta}{2} \left\{ |x|^{\frac{2\beta}{\beta-2}} + x \cdot \nabla |x|^{\frac{2\beta}{\beta-2}} \right\} = \frac{\beta}{2} \left\{ n |x|^{\frac{\beta}{\beta-2}} + x \cdot \frac{\beta-2}{2} |x|^{\frac{2}{\beta-2}} \right\}$$

The case of interest for us is when $\beta = \alpha - n + 1$, since then

$$\Delta |x|^\beta = \nabla \cdot \nabla |x|^{\alpha-n+1} = \nabla \cdot \nabla |x|^{\alpha-n+1} = c_{\alpha,n} \nabla \cdot K^{\alpha,n}(x),$$

where $K^{\alpha,n}$ is the vector convolution kernel of the $\alpha$-fractional Riesz transform $R^{\alpha,n}$. We conclude that $\Delta |x|^\beta$ is of one sign for all $x$, provided $\beta \neq 0$ and $n + \beta - 2 \neq 0$, i.e. $\alpha \notin \{1, n-1\}$. The case $\alpha = 1$ is not included since $|x|^{\alpha-n+1} = |x|^{\alpha-n}$ is the fundamental solution of the Laplacian for $n > 2$ and constant for $n = 2$. The case $\alpha = n - 1$ is not included since $|x|^{\alpha-n+1} = 1$ is constant.

Thus $z \in J$, we have from (6.16) with $I^{\alpha+1,n}(z) \equiv \int_{R^n} |z - y|^{\alpha+1-n} d\mu(y)$ denoting the convolution of $|x|^{\alpha+1-n}$ with $\mu$, that

$$|\nabla R^{\alpha,n}(z)|| \geq \text{trace} \nabla R^{\alpha,n}(z) = |\Delta I^{\alpha+1,n}(z)| \approx \int_{R^n} |y - z|^{\alpha-n-1} d\mu(y) \approx \frac{P^{\alpha}(J,\mu)}{\ell(J)},$$

where we assume that the positive measure $\mu$ is supported outside the expanded cube $J$. Recall that the trace of a matrix is invariant under conjugation by rotations, and hence is the sum of the eigenvalues of a symmetric matrix. We now claim that for every $z \in J$, the full matrix gradient $\nabla R^{\alpha,n}(z)$ has at least $1$ eigenvalue of size at least $cP^{\alpha}(J,\mu)\ell(J)$. Indeed, if all eigenvalues of the matrix $\nabla R^{\alpha,n}(z)$ have size at most $cP^{\alpha}(J,\mu)\ell(J)$, then $|\nabla R^{\alpha,n}(z)| \leq cP^{\alpha}(J,\mu)\ell(J)$, which contradicts (6.17) if $c$ is chosen small enough. This proves our claim, and moreover, it satisfies the quantitative quadratic estimate

$$|\xi \cdot \nabla R^{\alpha,n}(z)| \leq cP^{\alpha}(J,\mu)\ell(J) |\xi|^2, \quad \xi \in S_z, \quad z \in J.$$
6.3.2. **Extreme reversal of energy.** We now show that (6.14) holds for the vector Riesz transform $R^{n,n}$.

**Lemma 22.** Let $0 \leq \alpha < n$ and suppose $\omega$ doubling. Then the $\alpha$-fractional Riesz transform $R^{\alpha,n} = \{R^{\alpha,n}_I\}_{I=1}^n$ has extreme reversal of $\omega$-energy (6.14) on all cubes $J$ provided $\gamma$ is chosen large enough depending only on $n$ and $\alpha$, i.e.,

$$
\mathbb{E}^\omega J \left[ (x - \mathbb{E}^\omega_J x)^2 \right] \left( \frac{P^\alpha (J, \mu)}{|J|^\frac{1}{2}} \right)^2 |J|_\omega \leq C \int_J \int_{\mathbb{R}^n \setminus J} \left| K^\alpha (x, y) - K^\alpha (c_J, y) \right| \frac{h^\gamma_J (x)}{|J|_\omega} d\mu (y) d\omega (x).
$$

**Proof.** It suffices to show that (6.20) holds with

$$
h^\omega_J (x) = \sum_{K \in (J)} a_K 1_K (x), \text{ where } \begin{cases} a_K > 0 & \text{if } K \text{ lies to the right of center} \\ a_K < 0 & \text{if } K \text{ lies to the left of center} \end{cases},
$$

and without loss of generality $v_{c_J} = e_1$. To see this we compute,

$$
\int_J \left[ K^\alpha_n (x, y) - K^\alpha_n (c_J, y) \right] h^\gamma_J (x) \frac{d\omega (x)}{|J|_\omega} = \int_J \left[ \frac{x_1 - y_1}{|x - y|^{n-\alpha+1}} - \frac{(c_J)_1 - y_1}{|c_J - y|^{n-\alpha+1}} \right] \frac{h^\gamma_J (x)}{|J|_\omega} d\omega (x)
$$

$$
= \int_J \left( x_1 - y_1 \right) \left\{ \frac{1}{|x - y|^{n-\alpha+1}} - \frac{1}{|c_J - y|^{n-\alpha+1}} \right\} \frac{h^\gamma_J (x)}{|J|_\omega} d\omega (x) + \int_J \left( c_J - y \right) \left\{ \frac{1}{|x - y|^{n-\alpha+1}} - \frac{1}{|c_J - y|^{n-\alpha+1}} \right\} \frac{h^\gamma_J (x)}{|J|_\omega} d\omega (x) \equiv A + B.
$$

Now in term $B$ we have $(x_1 - (c_J)_1) \frac{h^\gamma_J (x)}{|J|_\omega}$ is of one sign and so

$$
|B| = \int_J \left( x_1 - (c_J)_1 \right) \frac{h^\gamma_J (x)}{|J|_\omega} d\omega (x) = \int_J \left| x_1 - (c_J)_1 \right| \frac{h^\gamma_J (x)}{|J|_\omega} d\omega (x) \geq c \frac{\ell (J)}{|c_J - y|^{n-\alpha+1} \sqrt{|J|_\omega}},
$$

because $\omega$ is doubling. On the other hand,

$$
|A| \leq \int_J \left| x_1 - y_1 \right| \frac{1}{|x - y|^{n-\alpha+1}} - \frac{1}{|c_J - y|^{n-\alpha+1}} \left| h^\gamma_J (x) \frac{d\omega (x)}{|J|_\omega} \right|
$$

$$
\leq \frac{\ell (J)^2}{|c_J - y|^{n-\alpha+2} \sqrt{|J|_\omega}} = \frac{\ell (J)}{|c_J - y|} \frac{\ell (J)}{|c_J - y|^{n-\alpha+1} \sqrt{|J|_\omega}} \leq C \frac{1}{\gamma} \frac{\ell (J)}{|c_J - y|^{n-\alpha+1} \sqrt{|J|_\omega}}
$$

and so for $\gamma > 1$ chosen sufficiently large, we obtain

$$
\int_J \left[ K^\alpha_n (x, y) - K^\alpha_n (c_J, y) \right] \frac{h^\gamma_J (x)}{|J|_\omega} d\omega (x) \geq |B| - |A| \geq \left( c - C \frac{1}{\gamma} \right) \frac{\ell (J)}{|c_J - y|^{n-\alpha+1} \sqrt{|J|_\omega}}
$$

$$
\geq \frac{c \ell (J)}{2 |c_J - y|^{n-\alpha+1} \sqrt{|J|_\omega}}.
$$

Since $\int_J [K^\alpha_n (x, y) - K^\alpha_n (c_J, y)] \frac{h^\gamma_J (x)}{|J|_\omega} d\omega (x)$ is also of one sign, it follows that

$$
\left| \int_J \int_{\mathbb{R}^n \setminus J} v_{c_J} \cdot [K^\alpha (x, y) - K^\alpha (c_J, y)] \frac{h^\gamma_J (x)}{|J|_\omega} d\mu (y) d\omega (x) \right|
$$

$$
= \int_{\mathbb{R}^n \setminus J} \left| \int_J [K^\alpha_n (x, y) - K^\alpha_n (c_J, y)] \frac{h^\gamma_J (x)}{|J|_\omega} d\omega (x) \right| d\mu (y)
$$

$$
\geq \int_{\mathbb{R}^n \setminus J} \frac{c \ell (J)}{2 |c_J - y|^{n-\alpha+1} \sqrt{|J|_\omega}} d\mu (y) = \frac{c}{2} \sqrt{|J|_\omega} \mathcal{P}^\alpha (J, 1_{\mathbb{R}^n \setminus J}),
$$

which proves the extreme reversal of energy. \( \square \)
6.4. Far below form. Recall that we decomposed the far below form $T_{\text{far below}}(f, g)$ as $T^1_{\text{far below}}(f, g) + T^2_{\text{far below}}(f, g)$, where we claimed that the second form $T^2_{\text{far below}}(f, g)$ was controlled by the disjoint, comparable and adjacent forms and $B_{\cap}(f, g)$, $B_{\lor}(f, g)$ and $B_{adj, \rho}(f, g)$, upon noting the porisms following (6.3) and (6.7). Indeed, if $\Delta J g$ is not identically zero, then $J$ must be good, and in that case the condition "$J \subset I$ but $J \not\subset E_p, \Delta I$" implies that the pair of cubes $I, J$ is included in either the sum defining the disjoint down form $B^\text{down}_p(f, g)$ or in the sum defining the comparable below form $B^\text{below}_p(f, g)$ or in the sum defining the adjacent below form $B^\text{adj, \rho}_p(f, g)$. The first far below form $T^1_{\text{far below}}(f, g)$ is handled by the following Intertwining Proposition.

**Proposition 23 (The Intertwining Proposition).** Suppose $\sigma, \omega$ are positive locally finite Borel measures on $\mathbb{R}^n$, that $\sigma$ is doubling, and that $\mathcal{F}$ satisfies a $\sigma$-Carleson condition. Then for a smooth $\lambda$-fractional singular integral $T^\lambda$, and for good functions $f \in L^2(\sigma) \cap L_p(\sigma)$ and $g \in L^2(\omega) \cap L_p^p(\omega)$, and with $k \geq 1$ sufficiently large, we have the following bound for $T_{\text{far below}}(f, g) = \sum_{F \in \mathcal{F}} \sum_{I \supset F} T^\alpha_{\sigma} \Delta J f, f, g\rangle_{\omega}$:

$$
|T^1_{\text{far below}}(f, g)| \lesssim A^{\lambda, k} \cdot \|f\|_{L_p^p(\sigma)} \|g\|_{L_p^p(\omega)}.
$$

**Proof.** For any dyadic cube $I$, let $\theta(I)$ denote any of the dyadic siblings of $I$, namely the children of the dyadic parent $\pi I$ other than $I$ itself. We write

$$
f_F = \sum_{I : I \supset F} \Delta J f = \sum_{m=1}^{\infty} \sum_{I : \pi^m F \subset I \subset \pi^{m+1} F} \Delta J f
$$

$$
= \sum_{m=1}^{\infty} \sum_{I : \pi^m F \subset I \subset \pi^{m+1} F} \theta(I) \left( E^\sigma_I f - E_{\pi^m F}^\sigma f \right)
$$

$$
= \sum_{m=1}^{\infty} \sum_{\pi^m F \subset I \subset \pi^{m+1} F} \theta(I) \left( E^\sigma_I f - \sum_{m=1}^{\infty} \sum_{\pi^m F \subset I \subset \pi^{m+1} F} \pi^m F \pi^m F \left( E_{\pi^m F}^\sigma f \right) \right)
$$

$$
= \beta_F - \gamma_F,
$$

and then

$$
\sum_{F \in \mathcal{F}} \langle T^\lambda_{\sigma} f, g\rangle_{\omega} = \sum_{F \in \mathcal{F}} \langle T^\lambda_{\sigma} \beta_F, g\rangle_{\omega} - \sum_{F \in \mathcal{F}} \langle T^\lambda_{\sigma} \gamma_F, g\rangle_{\omega}.
$$

Now we use the Poisson inequality (6.10), namely

$$
P^\lambda(J, \sigma \mathbf{1}_{K^c \setminus I}) \lesssim \left( \frac{\ell(J)}{\ell(I)} \right)^{1-\varepsilon(n+1-\lambda)} P^\lambda(I, \sigma \mathbf{1}_{K^c \setminus I}),
$$

to obtain that

$$
\left| \sum_{F \in \mathcal{F}} \langle T^\lambda_{\sigma} \gamma_F, g\rangle_{\omega} \right| = \left| \sum_{F \in \mathcal{F}} \int_{\mathbb{R}^n} T^\lambda_{\sigma} \left( \sum_{m=1}^{\infty} \sum_{\pi^m F \subset I} \pi^m F \left( E_{\pi^m F}^\sigma f \right) \right) (x) \sum_{J \in C_{\text{shif}}} \Delta^\sigma J g (x) \right| dx
$$

$$
= \left| \int_{\mathbb{R}^n} \sum_{F \in \mathcal{F}} \sum_{\pi^m F \subset I} \Delta^\sigma J T^\lambda_{\sigma} \left( \sum_{m=1}^{\infty} \sum_{\pi^m F \subset I} \pi^m F \left( E_{\pi^m F}^\sigma f \right) \right) (x) \Delta^\sigma J g (x) \right| dx
$$

$$
\leq \left( \sum_{F \in \mathcal{F}} \left| \sum_{\pi^m F \subset I} \Delta^\sigma J T^\lambda_{\sigma} \left( \sum_{m=1}^{\infty} \sum_{\pi^m F \subset I} \pi^m F \left( E_{\pi^m F}^\sigma f \right) \right) (x) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{F \in \mathcal{F}} \right) \left| \Delta^\sigma J g (x) \right|^2 \frac{1}{2} dx
$$

$$
\leq \left( \sum_{F \in \mathcal{F}} \left| \sum_{\pi^m F \subset I} \Delta^\sigma J T^\lambda_{\sigma} \left( \sum_{m=1}^{\infty} \sum_{\pi^m F \subset I} \pi^m F \left( E_{\pi^m F}^\sigma f \right) \right) (x) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{F \in \mathcal{F}} \right) \left| \Delta^\sigma J g (x) \right|^2 \frac{1}{2} \right|_{L_p^p(\omega)},
where the second factor is equivalent to \( \|g\|_{L_p'(\omega)} \), and then using the Pivotal Lemma \( 2 \), the first factor \( S \) is dominated by

\[
S \lesssim \left\| \sum_{J \in D} \sum_{F \in \mathcal{F}} \sum_{m=1}^{\infty} \mathbb{P}^\lambda \left( J, 1_{\pi F \setminus \pi f F} \left| \mathbb{E}^\sigma_{\pi F \setminus \pi f F} f \right|_{\sigma} \right)^2 1_J \right\|_{L_p'(\omega)}
\]

\[
= \left\| \sum_{J \in D} \sum_{m=1}^{\infty} \sum_{F \in \mathcal{F}^{\tau,\text{shift}}} \left\| \mathbb{E}^\sigma_{\pi F \setminus \pi f F} f \right\|_\infty \left( \frac{\ell(J)}{\ell(\pi F)} \right)^{1-\varepsilon(n+1)} \mathbb{P}^\lambda \left( \pi F, 1_{\pi F \setminus \pi f F} \right)^2 1_J \right\|_{L_p'(\omega)}
\]

\[
\lesssim \sum_{m=1}^{\infty} \left( \sum_{J \in D} \sum_{F \in \mathcal{F}^{\tau,\text{shift}}} \left\| \mathbb{E}^\sigma_{\pi F \setminus \pi f F} f \right\|_\infty \left( \frac{\ell(J)}{\ell(\pi F)} \right)^{1-\varepsilon(n+1)} \frac{\left| \pi F \right|_{\sigma}}{\left| \pi F \right|_{1-\frac{\varepsilon}{n}}} 1_J(x) \right\|^\frac{1}{2} \right\|_{L_p'(\omega)}
\]

where in the last line we have used \( 3 \). Now we note that for each \( J \in D \) the number of cubes \( F \in \mathcal{F} \) such that \( J \in \mathcal{C}^{\tau,\text{shift}}_F \) is at most \( \tau \). So without loss of generality, we may simply suppose that there is just one such cube denoted \( F[J] \). Thus for each \( m \in \mathbb{N} \), the above norm is at most

\[
(2^{-m})^{1-\varepsilon(n+1)} \left( \sum_{J \in D} \left\| \mathbb{E}^\sigma_{\pi F \setminus \pi f F} f \right\|_\infty \left( \frac{\ell(J)}{\ell(\pi F)} \right)^{1-\varepsilon(n+1)} \frac{\left| \pi F \right|_{\sigma}}{\left| \pi F \right|_{1-\frac{\varepsilon}{n}}} 1_J(x) \right\|^\frac{1}{2} \right\|_{L_p'(\omega)}
\]

and the sum inside the parentheses equals

\[
\sum_{F \in \mathcal{F}^{\tau,\text{shift}}} \sum_{x \in F} \left( \frac{\ell(J)}{\ell(\pi F)} \right)^{1-\varepsilon(n+1)} \left\| \mathbb{E}^\sigma_{\pi F \setminus \pi f F} f \right\|_\infty \frac{\left| \pi F \right|_{\sigma}}{\left| \pi F \right|_{1-\frac{\varepsilon}{n}}} 1_J(x)
\]

\[
\lesssim \sum_{F \in \mathcal{F}^{\tau,\text{shift}}} \sum_{x \in F} \left( \frac{\ell(J)}{\ell(\pi F)} \right)^{1-\varepsilon(n+1)} \left\| \mathbb{E}^\sigma_{\pi F \setminus \pi f F} f \right\|_\infty \frac{\left| \pi F \right|_{\sigma}}{\left| \pi F \right|_{1-\frac{\varepsilon}{n}}} 1_J(x)
\]

\[
\lesssim \sum_{F \in \mathcal{F}} \left\| \mathbb{E}^\sigma_{\pi F \setminus \pi f F} f \right\|_\infty \frac{\left| \pi F \right|_{\sigma}}{\left| \pi F \right|_{1-\frac{\varepsilon}{n}}} 1_J(x).
\]

Altogether then, using the quadratic offset \( A^\lambda_{p,2,\text{offset}} \) condition and doubling, we have

\[
S \lesssim \sum_{m=1}^{\infty} (2^{-m})^{1-\varepsilon(n+1)} \left( \sum_{F \in \mathcal{F}} \left\| \mathbb{E}^\sigma_{\pi F \setminus \pi f F} f \right\|_\infty \frac{\left| \pi F \right|_{\sigma}}{\left| \pi F \right|_{1-\frac{\varepsilon}{n}}} 1_F(x) \right\|^\frac{1}{2} \right\|_{L_p'(\omega)}
\]

\[
\lesssim A^\lambda_{p,2,\text{offset}} (\sigma,\omega) \sum_{m=1}^{\infty} (2^{-m})^{1-\varepsilon(n+1)} \left( \sum_{F \in \mathcal{F}} \left| \Delta^\sigma_{\pi F \setminus \pi f F} f(x) \right| 1_F(x) \right)^\frac{1}{2} \right\|_{L_p(\sigma)}
\]
and we can continue with

\[
\begin{align*}
&= A_p^{\lambda} \ell^2, \text{offset} (\sigma, \omega) \sum_{m=1}^{\infty} (2^{-m})^{1-(n+1-\lambda)} \left\| \left( \sum_{G \in F} \sum_{F \in F : \pi_{\sigma} f = G} |\Delta_{\sigma}^G f(x)|^2 1_F (x) \right)^{1/2} \right\|_{L^p(\sigma)} \\
&\leq A_p^{\lambda} \ell^2, \text{offset} (\sigma, \omega) \sum_{m=1}^{\infty} (2^{-m})^{1-(n+1-\lambda)} \left\| \left( \sum_{G \in F} |\Delta_{\sigma}^G f(x)|^2 1_G (x) \right)^{1/2} \right\|_{L^p(\sigma)} \\
&\leq A_p^{\lambda} \ell^2, \text{offset} (\sigma, \omega) \sum_{m=1}^{\infty} (2^{-m})^{1-(n+1-\lambda)} \|f\|_{L^p(\sigma)} = C_{\epsilon, \lambda} A_p^{\lambda} \ell^2, \text{offset} (\sigma, \omega) \|f\|_{L^p(\sigma)}.
\end{align*}
\]

Thus provided \(1 - \epsilon > \epsilon (n - \lambda)\), we have proved the estimate

\[
\left\| \sum_{F \in F} \langle T_{\sigma}^\lambda, \beta_F, g_F \rangle \right\|_{L^p(\omega)} \lesssim A_p^{\lambda} \ell^2, \text{offset} (\sigma, \omega) \|f\|_{L^p(\sigma)} \|g\|_{L^p'(\omega)}.
\]

It remains to bound \(\sum_{F \in F} \langle T_{\sigma}^\lambda, \beta_F, g_F \rangle \omega\) where

\[
\beta_F = \sum_{m=1}^{\infty} \sum_{I \in D} \sum_{\pi_{\omega} F \subseteq I \subseteq \pi_{\sigma} F} 1_{\theta(I)} \left( E_{\Omega I}^\sigma \right) \text{ and } g_F (x) = \sum_{J \in C_{\omega, \tau} \text{-shift}} \Delta_{J}^\omega g(x).
\]

The difference between the previous estimate and this one is that the averages \(1_{\pi_{\sigma} F \cap \pi_{\omega} F} \left[ \sum_{I \in D} \sum_{\pi_{\omega} F \subseteq I \subseteq \pi_{\sigma} F} 1_{\theta(I)} \left( E_{\Omega I}^\sigma \right) \right] \text{ and inside the Poisson kernel have been replaced with the sum of averages } \sum_{I \in D} \sum_{\pi_{\omega} F \subseteq I \subseteq \pi_{\sigma} F} 1_{\theta(I)} \left( E_{\Omega I}^\sigma \right),\) but where the sum is taken over pairwise disjoint sets \(\{\theta(I)\}_{\pi_{\omega} F \subseteq I \subseteq \pi_{\sigma} F}.\) Just as in the previous estimate we start with

\[
\left| \sum_{F \in F} \langle T_{\sigma}^\lambda, \beta_F, g_F \rangle \right| = \left| \sum_{F \in F} \int_{\mathbb{R}^n} T_{\sigma}^\lambda \left( \sum_{m=1}^{\infty} \sum_{I \in D} \sum_{\pi_{\omega} F \subseteq I \subseteq \pi_{\sigma} F} 1_{\theta(I)} \left( E_{\Omega I}^\sigma \right) \right) (x) \sum_{J \in C_{\omega, \tau} \text{-shift}} \Delta_{J}^\omega g(x) \right| dx.
\]

\[
\begin{align*}
&\leq \left| \sum_{F \in F} \sum_{J \in D} \Delta_{J}^\omega T_{\sigma}^\lambda \left( \sum_{m=1}^{\infty} \sum_{I \in D} \sum_{\pi_{\omega} F \subseteq I \subseteq \pi_{\sigma} F} 1_{\theta(I)} \left( E_{\Omega I}^\sigma \right) \right) (x) \sum_{J \in C_{\omega, \tau} \text{-shift}} | \Delta_{J}^\omega g(x) | \right| dx \\
&\leq \left\| \left( \sum_{F \in F} \sum_{J \in D} \Delta_{J}^\omega T_{\sigma}^\lambda \left( \sum_{m=1}^{\infty} \sum_{I \in D} \sum_{\pi_{\omega} F \subseteq I \subseteq \pi_{\sigma} F} 1_{\theta(I)} \left( E_{\Omega I}^\sigma \right) \right) (x) \left| \sum_{J \in C_{\omega, \tau} \text{-shift}} | \Delta_{J}^\omega g(x) | \right| \right\|_{L^p(\omega)} \\
&\leq \left\| \left( \sum_{J \in D} \left( \sum_{F \in F} \sum_{m=1}^{\infty} \sum_{I \in D} \sum_{\pi_{\omega} F \subseteq I \subseteq \pi_{\sigma} F} \left| \sum_{J \in C_{\omega, \tau} \text{-shift}} | \Delta_{J}^\omega g(x) | \right| \right) (x) \right\|_{L^p(\omega)} \\
&\leq \left\| \sum_{J \in D} \left( \sum_{m=1}^{\infty} \sum_{I \in D} \sum_{\pi_{\omega} F \subseteq I \subseteq \pi_{\sigma} F} \left| 1_{\theta(I)} \left( E_{\Omega I}^\sigma \right) \right| \right) (x) \right\|_{L^p(\omega)}.
\end{align*}
\]

The second factor is equivalent to \(\|g\|_{L^p'(\omega)},\) and the first factor \(S\) is dominated by

\[
S \leq \left\| \sum_{J \in D} \left( \sum_{m=1}^{\infty} \sum_{I \in D} \sum_{\pi_{\omega} F \subseteq I \subseteq \pi_{\sigma} F} \left| 1_{\theta(I)} \left( E_{\Omega I}^\sigma \right) \right| \right) (x) \right\|_{L^p(\omega)}.
\]
Then we use

$$
\sum_{I: \pi m F I \subset \pi m+1 F} \|E_J^f\|_{\infty} P^\lambda (J, 1_{\theta(1)}\sigma) \leq \left( \sup_{I: \pi m F I \subset \pi m+1 F} \|E_J^f\|_{\infty} \right)^2 P^\lambda (J, 1_{\pi m+1 F \setminus \pi m F \sigma}) = \left( \sup_{I: \pi m F I \subset \pi m+1 F} \|E_J^f\|_{\infty} \right)^2 P^\lambda (J, 1_{\pi m+1 F \setminus \pi m F \sigma}) \),
$$

and obtain that

$$
S \lesssim \sum_{m=1}^{\infty} \left( \sum_{J \in \mathcal{D}} \sum_{F \in \mathcal{F}: J \in \psi_F, \tau \text{-shift}} \left( \sup_{I: \pi m F I \subset \pi m+1 F} \|E_J^f\|_{\infty} \right)^2 P^\lambda (J, 1_{\pi m+1 F \setminus \pi m F \sigma}) \right) 1_J \right) \right)^{1/2} \|_{L^p(\omega)}.
$$

Now we define $G_m[F] \in \pi m F, \pi m+1 F$ so that $\sup_{I: \pi m F I \subset \pi m+1 F} \|E_J^f\|_{\infty} = \|E_{G_m[F]}^f\|_{\infty}$, and dominate $S$ by

$$
\lesssim \sum_{m=1}^{\infty} \left( \sum_{J \in \mathcal{D}} \sum_{F \in \mathcal{F}: J \in \psi_F, \tau \text{-shift}} \left( \sup_{I: \pi m F I \subset \pi m+1 F} \|E_J^f\|_{\infty} \right)^2 \left( \frac{\ell(J)}{\ell(F)} \right)^{\eta} P^\lambda (G_m[F], 1_{\pi m+1 F \setminus \pi m F \sigma}) \right) 1_J \right) \right)^{1/2} \|_{L^p(\omega)},
$$

where $\eta = 1 - \varepsilon (n + 1 - \lambda)$ is the constant appearing in \[\text{[101]}\].

Just as above we note that for each $J \in \mathcal{D}$ the number of cubes $F \in \mathcal{F}$ such that $J \in \psi_F, \tau \text{-shift}$ is at most $\tau$. So without loss of generality, we may simply suppose that there is just one such cube denoted $F[J]$. Thus for each $m \in \mathbb{N}$, the above norm is at most

$$
\left\| \sum_{J \in \mathcal{D}} \left( \frac{\ell(J)}{\ell(F[J])} \right)^{\eta} \left( \frac{G_m[F]_{[J]}_{\sigma}}{|G_m[F]_{[J]}|^{1-\frac{\lambda}{2}}} \right)^2 1_J \right) \right\|_{L^p(\omega)},
$$

and the sum inside the parentheses equals

$$
\sum_{J \in \mathcal{D}} \left\| E_{G_m[F]}^f \right\|_{\infty} \left( \frac{|G_m[F]_{[J]}|_{\sigma}}{|G_m[F]_{[J]}|^{1-\frac{\lambda}{2}}} \right)^2 \left( \frac{\ell(J)}{\ell(F[J])} \right)^{2\eta} 1_J(x) \lesssim \left\| E_{G_m[F]}^f \right\|_{\infty} \left( \frac{|G_m[F]_{[J]}|_{\sigma}}{|G_m[F]_{[J]}|^{1-\frac{\lambda}{2}}} \right)^2 1_{G_m[F]}(x).
$$
Altogether then, using the quadratic offset $A_p^{λ,ℓ,offset}$ condition and doubling, we have

$$S \lesssim A_p^{λ,ℓ,offset} (σ, ω) \frac{∞}{m=1} 2^{-mη} \left\| \left( \sum_{F ∈ F} \left| \left| E_{G_m[F]} f \right| \right| _∞ \frac{|G_m[F]|_σ}{|G_m[F]|^{1-θ \over 2}} 1_{G_m[F]} (x) \right\| _{L^p(σ)} \right.$$  

$$\lesssim A_p^{λ,ℓ,offset} (σ, ω) \frac{∞}{m=1} 2^{-mη} \left\| \left( \sum_{G ∈ G} \sum_{F ∈ F: G_m[F]=G} \left| A^σ_G f (x) \right|^2 1_G (x) \right\| _{L^p(σ)} \right.$$  

$$\lesssim A_p^{λ,ℓ,offset} (σ, ω) \frac{∞}{m=1} 2^{-mη} \left\| \left| 1_F \right| \right\| _{L^p(σ)} \left\| f_{L^p(σ)} \right\| \left\| g_{L^p(σ)} \right\|,$$

provided $η = 1 - ε (n + 1 - λ) > 0$ holds. Thus we have proved the estimate

$$\left| \sum_{F ∈ F} \langle T^λ σ, β_F, g_F \rangle_ω \right| \lesssim A_p^{λ,ℓ,offset} (σ, ω) \left\| f_{L^p(σ)} \right\| \left\| g_{L^p(σ)} \right\|,$$

which together with the corresponding estimate for $\sum_{F ∈ F} \langle T^λ σ, γ_F, g_F \rangle_ω$ proved above, completes the proof of the Intertwining Proposition.

Thus we have controlled both the first and second far below forms $T^1_{far below} (f, g)$ and $T^2_{far below} (f, g)$ by the quadratic offset Muckenhoupt constant $A_p^{λ,ℓ,offset}$.

### 6.5. Neighbour form.

We begin with $M_I = 1_{I^c} \Delta^σ_I f$ to obtain

$$B_{neighbour} (f, g) = \sum_{F ∈ F} B^{F}_{neighbour} (f, g)$$

$$= \sum_{F ∈ F} \sum_{I ∈ C_F} \sum_{J ∈ C_F^{shift} \cap (I, I)} \int \mathbb{R}^n T^λ σ (1_{θ(I)}} \Delta^σ_I f \right\rangle (x) \Delta^ω_J g \right\rangle (x) dω \right\rangle (x)$$

$$= \int \mathbb{R}^n \sum_{J ∈ D} \left\{ \sum_{F ∈ F} \sum_{I ∈ C_F} \sum_{J ∈ C_F^{shift} \cap (I, I)} \right\} T^λ σ (1_{θ(I)}} \Delta^σ_I f \right\rangle (x) \Delta^ω_J g \right\rangle (x) dω \right\rangle (x)$$

where for $J ∈ D$ we write $I > J$ if $I$ satisfies

there is $F ∈ F$ such that $I ∈ C_F, J ∈ C^{shift}_F$ and $J ⊇ p, x I.$
Applying the Cauchy-Schwarz and Hölder inequalities gives
\[
|B_{\text{neighbour}}(f, g)| \leq \int_{D} \left( \sum_{j \in I, J} \left| \sum_{n \in D} \Delta^\omega NJ\sigma (1_{\theta(J)} \Delta^\gamma f) (x) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{j \in I, J} \left| \sum_{n \in D} \Delta^\gamma g (x) \right|^2 \right)^{\frac{1}{2}} d\omega (x)
\]
\[
\leq \left\| \left( \sum_{j \in I, J} \sum_{n \in D} \left| \Delta^\omega NJ\sigma (1_{\theta(J)} \Delta^\gamma f) (x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \left\| \left( \sum_{j \in I, J} \sum_{n \in D} \left| \Delta^\gamma g (x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)},
\]
where the final factor is dominated by \( \|g\|_{L^p(\omega)} \). Using the Pivotal Lemma (20), and the estimate \( \|M\|_{L^\infty(\sigma)} \approx \frac{1}{\sqrt{|I|_{\sigma}}} \), we have
\[
|\Delta^\gamma T^\lambda (M_1) (x)| \lesssim P^\lambda \left( J, \|M\|_{L^\infty(\sigma)} \right) 1_J (x)
\]
\[
\lesssim \frac{1}{\sqrt{|I|_{\sigma}}} \|f(I)|P^\lambda \left( J, 1, \|\sigma\| \right) 1_J (x).
\]
Now we pigeonhole the side lengths of \( I \) and \( J \) by \( \ell (J) = 2^{-s} \ell (I) \) and use goodness, followed by (23), to obtain
\[
\left\| \left( \sum_{j \in I, J} \sum_{n \in D} \left| \Delta^\omega NJ\sigma (1_{\theta(J)} \Delta^\gamma f) (x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \lesssim \left\| \left( \sum_{j \in I, J} \sum_{n \in D} \left( \frac{1}{\ell (J)^{p-1}} \right) \left| f(I)|P^\lambda \left( J, 1, \|\sigma\| \right) 1_J (x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \lesssim 2^{-\eta s} A^\lambda p, \omega (\sigma, \|f\|_{L^p(\sigma)} \|g\|_{L^p(\omega)}).
\]
Summing in \( s \geq 0 \) proves the required bound for the neighbour norm,

\[
\langle T^\lambda f, g \rangle \lesssim A^\lambda p, \omega (\sigma, \|f\|_{L^p(\sigma)} \|g\|_{L^p(\omega)}).
\]

6.6. Conclusion of the proof. An examination of the schematic diagram at the beginning of the section on organization of the proof, together with all the estimates proved so far, completes the proof that
\[
\langle T^\lambda f, g \rangle \lesssim \left[ \Gamma^\lambda_{\theta, p} + \mathcal{H}B\mathcal{F}_{\omega, \sigma} \right] \|f\|_{L^p(\sigma)} \|g\|_{L^p(\omega)} + \alpha \mathcal{N}_{\omega, \sigma} + \beta \mathcal{N}_{\sigma, \omega}.
\]
where the constant \( \Gamma_{\theta, p}^\lambda \) is the sum of the scalar testing and quadratic Muckenhoupt offset conditions
\[
\Gamma_{\theta, p}^\lambda \equiv \mathcal{T}_{\theta, p} (\sigma, \omega) + \mathcal{T}_{\theta, \omega} (\omega, \sigma) + A^\lambda p, \omega (\sigma, \omega) + \alpha A^\lambda p, \omega (\sigma, \sigma).
\]
Now we invoke Lemma (18) to obtain that for all \( 0 < \varepsilon < 1 \), there is a constant \( C_{\varepsilon} \) such that
\[
\langle T^\lambda f, g \rangle \lesssim \left\{ C_{\varepsilon} \left[ \Gamma_{\theta, p}^\lambda + \mathcal{H}B\mathcal{F}_{\omega, \sigma} \right] + \varepsilon \mathcal{N}_{\omega, \sigma} + \varepsilon \mathcal{N}_{\sigma, \omega} \right\} \|f\|_{L^p(\sigma)} \|g\|_{L^p(\omega)}.
\]
from which we conclude that
\[
\mathcal{N}_{T, p} (\sigma, \omega) \lesssim \left\{ C \left[ \mathcal{N}^2_{T, p} + \mathcal{WBP}_{T, p}^{2} (\sigma, \omega) \right] + \varepsilon \mathcal{N}_{T, p} (\sigma, \omega) \right\} \| f \|_{L^p(\sigma)} \| g \|_{L^p'(\omega)}.
\]

At this point, a standard argument using the definition of the two weight norm inequality [1.11], for which see e.g. [AlSaUr, Section 6], shows that for any smooth truncation of \( T^\lambda \), we can absorb the term \( \varepsilon \mathcal{N}_{T, p} (\sigma, \omega) \| f \|_{L^p(\sigma)} \| g \|_{L^p'(\omega)} \) into the left hand side and obtain [1.13].

\[
\mathcal{N}_{T, p} (\sigma, \omega) \lesssim \left[ \mathcal{N}^2_{T, p} + \mathcal{WBP}_{T, p}^{2} (\sigma, \omega) \right] \| f \|_{L^p(\sigma)} \| g \|_{L^p'(\omega)}.
\]

This completes the proof of Theorem 2.

7. Appendix

7.1. A counterexample. Regarding the quadratic Muckenhoupt condition in the case \( p = 2 \), we clearly we have

\[
A_2^{\lambda,\ell^2} (\sigma, \omega) + A_2^{\lambda,\ell^2} (\omega, \sigma) \leq A_2^\lambda (\sigma, \omega),
\]

for any pair of locally finite positive Borel measures. However, this fails when \( 1 < p < \infty \), \( \lambda = 0 \) and \( p \neq 2 \) as we now show.

Let \( 1 < p < \infty \), \( 0 < \alpha \leq 1 \) and define

\[
f (x) = \frac{1}{x (\ln \frac{1}{x})^{1+\alpha}} 1_{(0, \frac{1}{2})} (x),
\]

and note that

\[
M f (x) 1_{(0, \frac{1}{2})} (x) \approx \frac{1}{x (\ln \frac{1}{x})^{\alpha}} 1_{(0, \frac{1}{2})} (x).
\]

Then define

\[
v (x) = f (x)^{1-p} \, dx = \left[ x \left( \ln \frac{1}{x} \right)^{1+\alpha} \right]^{p-1} 1_{(0, \frac{1}{2})} (x) \, dx,
\]

\[
w (x) = M f (x)^{1-p} \approx \left[ x \left( \ln \frac{1}{x} \right)^{\alpha} \right]^{p-1} 1_{(0, \frac{1}{2})} (x) \, dx,
\]

so that

\[
\int_0^{\frac{1}{2}} |f (x)|^p \, v (x) \, dx = \int_0^{\frac{1}{2}} f (x) \, dx = \int_0^{\frac{1}{2}} \frac{1}{x (\ln \frac{1}{x})^{1+\alpha}} < \infty,
\]

\[
\int_0^{\frac{1}{2}} |M f (x)|^p \, w (x) \, dx = \int_0^{\frac{1}{2}} M f (x) \, dx \approx \int_0^{\frac{1}{2}} \frac{1}{x (\ln \frac{1}{x})^{\alpha}} = \infty.
\]

On the other hand, using \((p-1)(1-p') = -1\) we have for \( 0 < r < \frac{1}{2} \),

\[
\left( \frac{1}{r} \int_0^r w (x) \, dx \right) \left( \frac{1}{r} \int_0^r v (x)^{1-p'} \, dx \right)^{p-1}
= \left( \frac{1}{r} \int_0^r \left[ x \left( \ln \frac{1}{x} \right)^{\alpha} \right]^{p-1} \, dx \right) \left( \frac{1}{r} \int_0^r \frac{1}{x (\ln \frac{1}{x})^{1+\alpha}} \, dx \right)^{p-1}
\approx \left( \frac{1}{r^\alpha} \left( \ln \frac{1}{r} \right)^{(\alpha-1)} \right) \left( \frac{1}{r} \left( \ln \frac{1}{r} \right)^{-\alpha} \right)^{p-1} = 1,
\]

and it follows easily that

\[
\sup_{I \subset (0, \frac{1}{2})} \left( \frac{1}{|I|} \int_I w (x) \, dx \right) \left( \frac{1}{|I|} \int_I v (x)^{1-p'} \, dx \right)^{p-1} < \infty.
\]
Thus if we set
\[
d\omega_{p,\alpha}(x) \equiv \left[ x \left( \frac{1}{x} \right)^{\alpha} \right] ^{p-1} \mathbf{1}_{(0, \frac{1}{2})}(x) \, dx,
\]
\[
d\sigma_{p,\alpha}(x) \equiv \frac{1}{x \left( \frac{1}{x} \right)^{1+\alpha}} \mathbf{1}_{(0, \frac{1}{2})}(x) \, dx,
\]
then we have both finiteness of the Muckenhoupt constant \( A_{p}^{\text{local}}(\sigma, \omega) \) localized to \((0, \frac{1}{2})\), and failure of the norm inequality
\[
\int_{\mathbb{R}} |M(f \sigma)(x)|^{p} \, d\omega(x) \lesssim \int_{\mathbb{R}} |f(x)|^{p} \, d\sigma(x).
\]
Now we investigate the local quadratic Muckenhoupt constant
\[
A_{p}^{q,\text{local}}(\sigma, \omega) + A_{p'}^{q,\text{local}}(\omega, \sigma)
\]
when \( \lambda = 0 \), i.e. where
\[
\left\| \left( \sum_{k=1}^{\infty} |a_{k}|^{2} \frac{|I_{k}|}{|I_{k}|} \right)^{\frac{1}{2}} \right\|_{L^{p}(\omega)} \leq A_{p}^{q,\text{local}}(\sigma, \omega) \left\| \left( \sum_{k=1}^{\infty} |a_{k}|^{2} \mathbf{1}_{I_{k}} \right)^{\frac{1}{2}} \right\|_{L^{p}(\sigma)},
\]
\[
\left\| \left( \sum_{k=1}^{\infty} |a_{k}|^{2} \frac{|I_{k}|}{|I_{k}|} \right)^{\frac{1}{2}} \right\|_{L^{p'}(\sigma)} \leq A_{p'}^{q,\text{local}}(\omega, \sigma) \left\| \left( \sum_{k=1}^{\infty} |a_{k}|^{2} \mathbf{1}_{I_{k}} \right)^{\frac{1}{2}} \right\|_{L^{p'}(\omega)},
\]
for all sequences \( \{I_{i}\}_{i=1}^{\infty} \) of intervals in \( I_{i} \) and all sequences \( \{a_{i}\}_{i=1}^{\infty} \) of numbers. We have
\[
|[0, r]|_{\sigma} = \int_{0}^{r} \frac{1}{x \left( \frac{1}{x} \right)^{1+\alpha}} \, dx \approx \frac{1}{\left( \frac{1}{x} \right)^{\alpha}},
\]
\[
|[0, r]|_{\omega} = \int_{0}^{r} \left[ x \left( \frac{1}{x} \right)^{\alpha} \right] ^{p-1} \, dx \approx r^{p} \left( \frac{1}{x} \right)^{\alpha(p-1)}.
\]
Thus if we take \( I_{k} = (0, 2^{-k}) \), the inequality becomes
\[
\left\| \left( \sum_{k=1}^{\infty} |a_{k} 2^{-k} \frac{1}{k^{\alpha}} \right)^{2} \mathbf{1}_{(0, 2^{-k})} \right\|_{L^{p}(\omega)} \leq A_{p}^{q,\text{local}}(\sigma, \omega) \left\| \left( \sum_{k=1}^{\infty} |a_{k}|^{2} \mathbf{1}_{(0, 2^{-k})} \right)^{\frac{1}{2}} \right\|_{L^{p}(\sigma)}.
\]
Now the \( p^{th} \) power of the right hand side is
\[
\int_{0}^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} |a_{k}|^{2} \mathbf{1}_{(0, 2^{-k})}(x) \right)^{\frac{1}{2}} \frac{1}{x \left( \ln \frac{1}{x} \right)^{1+\alpha}} \, dx = \sum_{k=1}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \left( \sum_{\ell=1}^{k} |a_{\ell}|^{2} \right)^{\frac{1}{2}} \frac{1}{x \left( \ln \frac{1}{x} \right)^{1+\alpha}} \, dx
\]
\[
\approx \sum_{k=1}^{\infty} \left( \sum_{\ell=1}^{k} |a_{\ell}|^{2} \right)^{\frac{1}{2}} \left( \frac{1}{k^{\alpha}} - \frac{1}{(k+1)^{\alpha}} \right) \approx \sum_{k=1}^{\infty} \left( \sum_{\ell=1}^{k} |a_{\ell}|^{2} \right)^{\frac{1}{2}} \frac{1}{k^{1+\alpha}},
\]
and the \( p^{th} \) power of the left hand side is
\[
\int_{0}^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} |a_{k} 2^{-k} \frac{1}{k^{\alpha}} \right)^{2} \mathbf{1}_{(0, 2^{-k})}(x) \left[ x \left( \ln \frac{1}{x} \right)^{\alpha} \right] ^{p-1} \, dx
\]
\[
= \sum_{k=1}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \left( \sum_{\ell=1}^{k} |a_{\ell} 2^{-\ell} \frac{1}{\ell^{\alpha}} |^{2} \right) \frac{1}{x \left( \ln \frac{1}{x} \right)^{\alpha}} \, dx \approx \sum_{k=1}^{\infty} \left( \sum_{\ell=1}^{k} |a_{\ell} 2^{-\ell} \frac{1}{\ell^{\alpha}} |^{2} \right) \frac{1}{x \left( \ln \frac{1}{x} \right)^{\alpha}} \, dx \approx 2^{-kp} k^{\alpha(p-1)}.
\]
Thus the right hand side will be finite if
\[
a_{\ell} = \ell^{\eta}, \quad \text{where} \ 2\eta + 1 = (\alpha - \varepsilon) \frac{n}{p} > 0,
\]
since then
\[ \sum_{\ell=1}^{k} |a_\ell|^2 \leq \sum_{\ell=1}^{k} \ell^{2\eta} \approx k^{2\eta+1} = k^{(\alpha-\varepsilon)\frac{p}{2}} \] and hence
\[ \left( \sum_{\ell=1}^{k} |a_\ell|^2 \right)^{\frac{p}{2}} = \frac{k^{1+\alpha}}{k^{1+\varepsilon}}, \]
and so
\[ \sum_{k=1}^{\infty} \left( \sum_{\ell=1}^{k} |a_\ell|^2 \right)^{\frac{p}{2}} \frac{1}{k^{1+\alpha}} = \sum_{k=1}^{\infty} \frac{1}{k^{1+\varepsilon}} < \infty. \]
On the other hand, with this choice of \( a_\ell \), the \( p^{th} \) power of the left hand side is
\[ \sum_{k=1}^{\infty} \left( \sum_{\ell=1}^{k} |a_\ell|^{2\eta} \right)^{\frac{p}{2}} 2^{-kp_k^{\alpha(p-1)}} = \sum_{k=1}^{\infty} \left( \sum_{\ell=1}^{k} \ell^{2\eta-\varepsilon} \right)^{\frac{p}{2}} 2^{-kp_k^{\alpha(p-1)}} \]
\[ \approx \sum_{k=1}^{\infty} \left( \ell^{2} k^{\alpha-\varepsilon} \right)^{\frac{p}{2}} 2^{-kp_k^{\alpha(p-1)}} = \sum_{k=1}^{\infty} 2kp_k^{(\alpha-\varepsilon)p} 2^{-kp_k^{\alpha(p-1)}} = \sum_{k=1}^{\infty} 2^{kp_k^{\alpha(p-1)}} k^{\alpha p-\alpha} \]
which will be infinite if \( \eta p - \alpha > -1 \), and since \( 2\eta + 1 = (\alpha - \varepsilon)\frac{p}{2} \), this will be the case provided
\[ 1 < \eta p - \alpha = \left[ \frac{(\alpha - \varepsilon)\frac{p}{2} - 1}{2} \right] \]
\[ p - \alpha = (\alpha - \varepsilon) - \frac{p}{2} - \alpha = -\varepsilon - \frac{p}{2}, \]
i.e. \( 0 < \varepsilon < \frac{2-p}{2} \).
Thus we have a counterexample to the implication \( A_p^{\text{local}}(\sigma, \omega) \implies A^p_{\ell}^{\text{local}}(\sigma, \omega) + A_{\ell'}^{p'}_{\text{local}}(\omega, \sigma) \) when \( 1 < p < 2 \), provided we choose \( (\sigma, \omega) = (\sigma_{p,\alpha}, \omega_{p,\alpha}) \) with \( 0 < \alpha \leq 1 \).

**Proposition 24.** Let \( p \in (1, \infty) \setminus \{2\} \). There is a weight pair \( (\sigma, \omega) \) such that
\[ A_p^{\text{local}}(\sigma, \omega) < \infty, \]
\[ A^p_{\ell}^{\text{local}}(\sigma, \omega) + A_{\ell'}^{p'}_{\text{local}}(\omega, \sigma) = \infty. \]

**Proof.** Let \( (\sigma_{p,\alpha}, \omega_{p,\alpha}) \) be the weight pair constructed above. If \( 1 < p < 2 \), we can take \( (\sigma, \omega) = (\sigma_{p,1}, \omega_{p,1}) \). If \( 2 < p < \infty \), then \( 1 < p' < 2 \) and we can take \( (\sigma, \omega) = (\omega_{p',1}, \sigma_{p',1}) \). \( \square \)

**Remark 25.** If we take \( 0 < \alpha \leq 1 \), then the two weight norm inequality for the maximal function fails with weights \( \sigma_{p',\alpha} \) and \( \omega_{p',\alpha} \).

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