Numerical and theoretical explorations in helical and fan-beam tomography

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Abstract. Katsevich’s inversion formula for helical tomography is explored in the limit of vanishing pitch, yielding a general reconstruction formula for fan-beam tomography. The relationship of this formula to other formulas in the literature is explored and a rigorous proof of a rebinning formula relating parallel-beam and fan-beam tomography is given. For the case of curved detector coordinates several numerical implementations of this formula and a related fan-beam formula are proposed, numerically implemented, and compared with the standard fan-beam algorithm. This gives insight into some numerical questions also encountered in the three-dimensional case, including a theoretical explanation of the usefulness of a shift in the convolution kernel for removal of ringing artifacts. A new discretization scheme for the derivatives is suggested and shown to be promising in both two and three dimensions. Numerical experiments with simulated as well as real data are presented.

1. Introduction

Three-dimensional tomography with sources on a curve is an increasingly popular field of applications and research. In this paper we explore some theoretical and numerical questions originating from the case where the source curve is a helix, the reconstruction is based on Katsevich’s exact inversion formula [1], and curved detector coordinates are being used. In the limit of vanishing helical pitch we obtain a related and formally entirely analogous reconstruction formula for 2D fan-beam tomography (Theorem 1) and clarify its relationship to other such formulas in the literature, in particular a general formula found by Noo, Defrise, Clackdoyle and Kudo [2] and an early formula found by Herman and Naparstek [3]. Similar to [2] our proof is based on the ‘rebinning formula’ given in Theorem 2 which is a very useful connection between fan-beam and parallel-beam geometry, and for which we contribute a rigorous proof. For our numerical considerations we first implement the early formula first derived by Herman and Naparstek [3, Eq. (10)], which is our equation (10) below. While this formula has not received much attention it does share some numerical features with Katsevich’s 3D formula and it can be used to study in detail some numerical questions that have arisen in 3D tomography. Several different numerical implementations of this formula will be compared with the standard fan-beam filtered backprojection algorithm. A new discretization scheme will be suggested that increases image quality in both fan-beam and helical CT and reduces an anisotropy in the resolution observed with some of the other schemes. A theoretical explanation will be provided for the effectiveness of a shift in the convolution kernel that has been employed in a similar fashion the 3D case. This explanation clarifies that while the shift is very useful in reducing ringing artifacts, it does not increase the intrinsic resolution of the algorithm, assuming that...
resolution is measured by the highest frequency that can be resolved. Numerical experiments will also be conducted for the 2D analogue of Katsevich’s formula derived in Theorem 1. In some cases the algorithms based on this formula can outperform the standard filtered backprojection algorithm. The experiments will involve simulated data from different mathematical phantoms as well as real data, thus revealing different performance aspects of the algorithms.

Let the function \( f \) denote the x-ray absorption coefficient of the object to be imaged. The mathematical model for x-ray tomography with sources on a curve is the divergent beam x-ray transform of \( f \), \( \mathcal{D}f(y, \theta) \). It gives the integral of \( f \) over the ray with vertex \( y \) and direction \( \theta \), i.e.,

\[
\mathcal{D}f(y, \theta) = \int_{0}^{\infty} f(y + t\theta) \, dt.
\]

Physically \( y \) is the position of the x-ray source. The goal of tomography is to reconstruct \( f(x) \) from measurements of \( \mathcal{D}f(y, \theta) \) with \( y \) lying on a certain curve and \( \theta \) in an appropriate subset of the unit sphere. Here we consider tomography with an x-ray source traveling on a helical curve \( y(s) \) given by

\[
y(s) = \left( R \cos(s), R \sin(s), \frac{P}{2\pi} s \right),
\]

where \( R \) and \( P \) denote the radius and the pitch of the helix, respectively. The helical axis coincides with the \( x_3 \)-axis. The object to be imaged is assumed to be contained in a cylinder of radius \( r < R \) and axis equal to the \( x_3 \)-axis as well.

Katsevich [1] found the following remarkable inversion formula for smooth functions \( f \).

\[
f(x) = \frac{1}{2\pi^2} \int_{I_{\mathcal{D}}(x)} \frac{1}{|x - y(s)|} \int_{0}^{2\pi} \frac{\partial}{\partial q} \mathcal{D}f(y(q), \cos \gamma \beta + \sin \gamma \beta^\perp)|_{q=s} \frac{1}{\sin \gamma} \, d\gamma \, ds.
\]

The meaning of the various quantities and notations in this formula is as follows; cf. Figure 1. For each point \( x \) inside the helix there are uniquely determined \( s_b \) and \( s_t \) such that \( s_b < s_t < s_b + 2\pi \) and \( x \) lies on the so-called PI-line segment connecting \( y(s_b) \) and \( y(s_t) \), as illustrated in Figure 1.
Figure 2. Illustration of the detector surface \( d(s, \alpha, w) \) and the detector coordinates \( \alpha \) and \( w \). The common vertex of the rays shown is the source point \( y(s) \).

The parametric interval \( I_{PF}(x) \) is given by \( I_{PF}(x) = \{ s : s_b \leq s \leq s_I \} \). The unit vector \( \beta \) is given by \( \beta = \beta(s, x) = (x - y(s))/|x - y(s)| \). Furthermore, as shown in [1], for each \( s \in I_{PF}(x) \) there exists a uniquely determined \( s_2 \in I_{PF}(x) \) such that \( x \) lies in the plane containing \( y(s) \), \( y(s_1) \), and \( y(s_2) \) with \( s_1 = (s + s_2)/2 \). This plane is called a \( \kappa \)-plane in [4]. The unit vector \( \beta^\perp \) is chosen to be orthogonal to \( \beta \) and such that the \( \kappa \)-plane is spanned by \( \beta \) and \( \beta^\perp \). Hence the inner integral in (2) only involves integrals over rays that lie in the \( \kappa \)-plane. When comparing (2) with [1, Equation (2.13)] please note that \( \beta^\perp = -e(s, x) \) with \( e(s, x) \) defined in [1, Theorem 1].

For further details, analysis, and generalizations see, e.g., [1, 4–8] and the references provided there.

It is convenient [4, §2.2] to introduce a coordinate system that rotates with the source and is given by the three unit vectors

\[
e_u(s) = (-\sin s, \cos s, 0), \quad e_v(s) = -(\cos s, \sin s, 0), \quad e_w = (0, 0, 1).
\]

The source curve (1) can then be written as

\[
y(s) = -Re_v + \frac{Ps}{2\pi}e_w.
\]

We assume a measurement geometry with a curved detector array (indicated as a shaded area in Figure 1) where x-rays emitted from a source at \( y(s) \) and passing through the object are recorded by detectors located on the rotating curved surface

\[
d(s, \alpha, w) = y(s) + D \sin \alpha e_u + D \cos \alpha e_v + w e_w, \quad |\alpha| \leq \alpha_{\text{max}}, \quad |w| \leq w_{\text{max}}, \quad D \geq R.
\]

Geometrically, \( D \) is the distance between \( y(s) \) and the surface \( d(s, \cdot, \cdot) \), \( \alpha \) is the angle between \( e_u \) and the projection onto the \( x_3 = 0 \) plane of the ray connecting the source \( y(s) \) with a detector located at \( d(s, \alpha, w) \), and \( w \) is the difference between the \( x_3 \)-coordinates of the detector position \( d(s, \alpha, w) \) and the source position \( y(s) \); cf. Figure 2.
The measured data are then given by
\[ g(s, \alpha, w) = \mathcal{D}f(y(s), \theta(s, \alpha, w)) \]
with \( y(s) \) as in (3) and
\[ \theta(s, \alpha, w) = \frac{1}{\sqrt{D^2 + w^2}} (D\sin \alpha \, e_u + D\cos \alpha \, e_w + w \, e_w). \]

It can be shown (cf. [4, §4], [9, Theorem 4]) that with these curved detector coordinates Katsevich’s inversion formula (2) reads as follows:
\[
f(x) = \frac{1}{2\pi^2} \int_{l_{P}} \frac{\cos \alpha^*}{R + \langle x, e_v \rangle} \int_{0}^{2\pi} \frac{D}{\sqrt{D^2 + w'^2}} \left( \frac{\partial g}{\partial s} + \frac{\partial g}{\partial \alpha} \right) \frac{1}{\sin(\alpha^* - \alpha)} \, d\alpha \, ds.
\]

The notation \( \langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i} u_i v_i \) denotes the usual scalar product of two vectors \( \mathbf{u} \) and \( \mathbf{v} \), and \( w' = w'(s, \alpha, \mathbf{x}) \) is such that \( \langle \alpha, w' \rangle \) parametrizes the intersection of the \( \kappa \)-plane with the detector surface \( d(s, \cdot, \cdot) \) given in (4). The derivatives \( \partial g/\partial s, \partial g/\partial \alpha \) are evaluated at \( (s, \alpha, w') \). The ray from \( y(s) \) through \( x \) intersects the detector surface at the detector coordinates \( (\alpha^*, w'(s, \alpha^*, x)) \), with
\[
\alpha^* = \alpha^*(s, x) = \arctan \left( \frac{\langle x, e_u \rangle}{R + \langle x, e_v \rangle} \right), \quad w'(s, \alpha^*, x) = \frac{D \cos \alpha^*}{R + \langle x, e_v \rangle} \left( x_3 - \frac{P \, s}{2\pi} \right);
\]

(6) cf. [4, p. 3796]. These formulas can also be expressed in terms of the unit vector \( \mathbf{\beta} = (x - y(s))/(|x - y(s)|) \). Observing that
\[
\langle x - y(s), e_u \rangle = \langle x, e_u \rangle, \quad \langle x - y(s), e_v \rangle = R + \langle x, e_v \rangle, \quad \langle x - y(s), e_w \rangle = x_3 - \frac{Ps}{2\pi},
\]
one obtains with \( \beta_u = \langle \mathbf{\beta}, e_u \rangle, \beta_v = \langle \mathbf{\beta}, e_v \rangle, \) and \( \beta_w = \langle \mathbf{\beta}, e_w \rangle, \)
\[
\alpha^* = \arctan \left( \frac{\beta_u}{\beta_v} \right), \quad \cos \alpha^* = \frac{\beta_v}{\sqrt{\beta_u^2 + \beta_v^2}} = \frac{\beta_u}{\sqrt{1 - \beta_w^2}} \quad w'(s, \alpha^*, x) = \frac{D \beta_w}{\sqrt{1 - \beta_w^2}}
\]
(7)

2. The limit of vanishing pitch

We now consider the limit of \( P \to 0 \) in (5) for \( x \) in the plane \( x_3 = 0 \). For \( x \) in the \( x_3 = 0 \) plane it follows from (1) and the definition of \( I_{PF}(x) \) that \( I_{PF}(x) \subset [-2\pi, 2\pi] \). Hence, for \( s \in I_{PF}(x) \) the \( x_3 \)-components of \( y(s), y(s_1), \) and \( y(s_2) \) all lie in the interval \([-P, P]\). Therefore, as \( P \to 0 \), the plane containing \( y(s), y(s_1), y(s_2) \) will converge to the \( x_3 = 0 \) plane and thus \( w'(s, \alpha, x) \to 0 \). It is not readily apparent what will be \( \lim_{P \to 0} I_{PF}(x) \). Kyle Champley [10] derived the following equations for \( s_0 \) and \( s_1 \) (see also [9] or alternatively [8, Eq. (11)]) which will provide an immediate answer to this question.

Let \( e_u = e_u(s_b), e_v = e_v(s_b) \) and \( \alpha = \arctan \left( \frac{\langle x, e_u \rangle}{R + \langle x, e_v \rangle} \right) \). Then
\[
\langle x, e_w \rangle - \frac{P}{2\pi} \left( s_b + (\pi - 2\alpha) \left( 1 + \frac{\langle x, e_u \rangle^2 + \langle x, e_v \rangle^2 - R^2}{2R(R + \langle x, e_v \rangle)} \right) \right) = 0,
\]
\[ s_t = s_b + \pi - 2\alpha. \]

Note that in the plane \( x_3 = \langle x, e_w \rangle = 0 \) the first equation simplifies to
\[
\left( s_b + (\pi - 2\alpha) \left( 1 + \frac{\langle x, e_u \rangle^2 + \langle x, e_v \rangle^2 - R^2}{2R(R + \langle x, e_v \rangle)} \right) \right) = 0
\]
which is independent of the pitch \( P \). It follows that for \( x \) in the plane \( x_3 = 0 \) the interval \( I_{Pf}(x) = [s_b, s_l] \) is independent of the pitch \( P \).

Hence by taking the limit \( P \to 0 \) we obtain from (5) for sufficiently smooth \( f \)

\[
f(x) = \frac{1}{2\pi^2} \int_{I_{Pf}(x)} \frac{\cos \alpha^*}{R + (x, e_v)} \int_0^{2\pi} \left( \frac{\partial g}{\partial s} + \frac{\partial g}{\partial \alpha} \right) \frac{d\alpha}{\sin(\alpha^* - \alpha)}, \quad x = (x_1, x_2, 0) \tag{8}
\]

where now \( g \) and its derivatives are evaluated at \((s, \alpha, 0)\). We also note that in the plane \( x_3 = 0 \) one has the relation \( \frac{\cos \alpha^*}{R + (x, e_v)} = \frac{1}{|x-y(s)|} \).

The limit of vanishing pitch has been investigated from a slightly different perspective in [5, Theorem 3]. There it was shown that in the limit \( P \to 0 \) the 3D inversion formula can be transformed into the classical 2D parallel-beam inversion formula, i.e., equation (9) below, but it was not explicitly pointed out that the limit also leads to explicit inversion formulas for fan-beam tomography, such as formula (8) or the more general formula (12) in Theorem 1 below. Nevertheless, the proof of Theorem 1 given in the appendix uses ideas that are closely related to those in [5, Theorem 2] as well as to [2].

3. Relationship with inversion formulas for 2D fan-beam tomography

Consider the two-dimensional (2D) x-ray transform in the plane \( z = 0 \),

\[
\mathcal{P}f(\varphi, p) = \int f(p \omega^\perp + t\omega) \, dt, \quad p \in \mathbb{R},
\]

\[
\omega = (\cos \varphi, \sin \varphi, 0), \quad \omega^\perp = (-\sin \varphi, \cos \varphi, 0).
\]

A change of variables gives the fan-beam transform

\[
g(s, \alpha) = \begin{cases} 
\mathcal{P}f(s - \alpha, R \sin \alpha) & \text{for } |\alpha| \leq \frac{\pi}{2}, \\
0 & \text{for } |\alpha| > \frac{\pi}{2},
\end{cases} \quad \alpha \in [-\pi, \pi),
\]

assuming that \( f(x) \) vanishes for \(|x| \geq R\) and that \( g(s, \alpha) \) is \( 2\pi \)-periodic with respect to both \( s \) and \( \alpha \). One has

\[
g(s, \alpha) = \mathcal{D}f(y(s), \theta) = \int_0^\infty f(y(s) + t\theta) \, dt
\]

with \( y(s) = -Re_v, \quad \theta = \cos \alpha \, e_v + \sin \alpha \, e_n. \)

The same change of variables in the classical 2D inversion formula for \( \mathcal{P}f \) gives [3]

\[
f(x) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\partial \mathcal{P}f(\varphi, p)}{\partial p} \frac{1}{(x, \omega^\perp) - p} \, dp \, d\varphi
\]

\[
= \frac{1}{4\pi^2} \int_0^{2\pi} \frac{\cos \alpha^*}{R + (x, e_v)} \int_0^{2\pi} \left( \frac{\partial g(s, \alpha)}{\partial s} + \frac{\partial g(s, \alpha)}{\partial \alpha} \right) \frac{1}{\sin(\alpha^* - \alpha)} \, d\alpha \, ds. \tag{10}
\]

To see the equivalence of (10) with [3, Eq. (10)] one uses the relation \( \frac{\cos \alpha^*}{R + (x, e_v)} = \frac{1}{|x-y(s)|} \) which holds in the plane \( x_3 = 0 \).

Equations (8) and (10) have strong formal similarities. Their difference lies in the interval of integration with respect to \( s \). In (10) the integration is over the entire interval \( [0, 2\pi) \) while in (8) it is only over the parametric interval \( I_{Pf}(x) \). For a given point \( x \) and in the limit \( P \to 0 \) the PI-line segment connecting \( y(s_b) \) and \( y(s_l) \) divides the circle \(|y| = R\) into two arcs. One of these arcs corresponds to source positions \( y(s) \) with \( s \in I_{Pf}(x) \). Formula (8) uses only data
from sources located on this arc, while formula (10) can be interpreted as the average of the contributions from both arcs, as will become clear from Theorem 1 below.

It is now natural to ask the question if equation (8) only holds for the particular PL-line segments obtained from the helix in the limit of vanishing pitch, or if \( I_{\text{PL}}(x) \) can be replaced by any interval \([s_b, s_t]\) such that \( x \) lies on the line segment connecting \( y(s_b) \) and \( y(s_t) \). The following Theorem states that this is indeed the case.

**Theorem 1** Let \( f \in C^\infty(\mathbb{R}^2) \) with \( f(x) = 0 \) for \( |x| > r > 0 \). Let \( y(s) = R \omega(s) \) with \( R > r \) and \( \omega(s) = (\cos(s), \sin(s)) \) for \( s \in \mathbb{R} \). For \( x \in \mathbb{R}^2, |x| < R \) let \( I_{\text{PL}}(x) \) be an interval \( s_b \leq s \leq s_t \) with \( s_b, s_t \) such that the line segment connecting \( y(s_b) \) and \( y(s_t) \) contains \( x \). Then

\[
f(x) = \frac{1}{2\pi^2} \int_{I_{\text{PL}}(x)} \frac{1}{|x - y(s)|} \int_0^{2\pi} \left( \frac{\partial g(s, \alpha)}{\partial s} + \frac{\partial g(s, \alpha)}{\partial \alpha} \right) \frac{d\alpha}{\sin(\alpha^* - \alpha)} \, \, ds \, \, d\alpha,
\]

where \( \beta = \beta(s, x) = (x - y(s)) / |x - y(s)|, \alpha^* \) such that \( \beta = (s - \alpha^* - \pi), \) and \( \beta^\perp = (-\beta_2, \beta_1) = \omega(s - \alpha^* - \pi/2) \).

The proof of Theorem 1 is given in the appendix. Inspection of the proof of the inversion formula (12) reveals that it does not depend on the curve \( y(s) \) being a circle. All that is required is that \( y(s) \) does not pass through \( x \) and as \( s \) varies from \( s_b \) to \( s_t \) the polar angle \( \varphi(s) \) of \( \beta(s, x) \) changes smoothly and strictly monotone. The hypothesis that the line segment connecting \( y(s_b) \) and \( y(s_t) \) contains \( x \) implies that \( \varphi \) traverses a region of \( \pi \) radians as \( s \) runs through \( I_{\text{PL}}(x) \). Thus (12) appears to be a very general formula that is valid for a large class of curves \( y(s) \) and is formally completely analogous to (2). The natural question if equation (2) itself can be generalized to hold for curves other than the helix has very recently been answered to the affirmative [7].

From the observation that (9) involves integrals over all lines in the plane while (11) and (12) only involve lines that meet the arc from \( y(s_b) \) to \( y(s_t) \) it is clear that Theorem 1 cannot be proven by merely making a change in the parametrization of the lines. The key is the following Theorem which gives a relation between the fan-beam and parallel-beam transforms. We first introduce some notation. Let \( \mathcal{H} \) denote the Hilbert transform,

\[
\mathcal{H}g(t) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g(p)}{t - p} \, dp,
\]

and \( \mathcal{R} \) the 2D Radon transform,

\[
\mathcal{R}_\theta f(p) = \int_{\mathbb{R}} f(p \theta + t\theta^\perp) \, dt, \quad p \in \mathbb{R},
\]

\[
\theta = (\cos \varphi, \sin \varphi), \quad \theta^\perp = (-\sin \varphi, \cos \varphi),
\]

where \( f \) is a function of two variables. The 2D Radon transform differs from the 2D x-ray transform only by the way in which the lines are parametrized. One has \( P f(\varphi, p) = \mathcal{R}_\theta f(p) \). Furthermore, let \( S_1 \) denote the unit sphere in \( \mathbb{R}^2 \).

**Theorem 2** Suppose that \( f \) has compact support in \( \mathbb{R}^2 \) and is uniformly Hölder continuous of order \( \alpha \) for some \( \alpha > 0 \) ( \( |f(x) - f(z)| \leq C||x - z||^\alpha \) for all \( x, z \in \mathbb{R}^2 \)). If \( y \in \mathbb{R}^2 \) is outside the support of \( f \), then

\[
\lim_{\epsilon \to 0^+} \int_{S_1 \cap |\omega, \theta| > \epsilon} \frac{\mathcal{R}_\theta f(y, \omega)}{\langle \omega, \theta \rangle} \, d\omega = -\pi(\mathcal{H}(\mathcal{R}_\theta f))(y, \theta).
\]
The usefulness of the relation (15) and its analogues in higher dimensions for inversion formulas has been recognized in [11, p. 188]; see also [12]. In the 2D case a formal proof has been given in [2]. The Theorem as stated above and its proof given in the appendix are rigorous and include a hypothesis about the smoothness of $f$.

4. Numerical implementation

We wish to explore numerical questions related to the 2D inversion formula (10) with the further goal of thereby gaining insight into the behavior of algorithms based on the more complex 3D formula (5). In particular we will investigate questions related to discretizing the derivatives $\partial g/\partial s$ and $\partial g/\partial \alpha$. While it is also possible to remove at least the derivatives with respect to $s$ in (5) by an integration by parts [1, Eq. (2.14)], the resulting formula may lead to a slower algorithm; cf. [4, p. 3794]. Therefore the implementation in [4] is based on (5) and it is useful to explore these questions. A different approach for implementing the derivatives has very recently been proposed in [13].

The formula (10), although it is an exact inversion formula and was known early on, has not played a prominent role in the subsequent development of fan-beam tomography. The numerical implementation in the original paper [3] is based on a formula where both the derivatives with respect to $s$ and $\alpha$ have been removed via integration by parts [3, Eq. (22)]. A similar but more compact formula can be found by first regularizing the parallel-beam inversion formula (9) and only then making the change of variables to fan-beam coordinates; see, e.g., [14, §3.4.1], [15, §5.1.3], [16, §V.1.2]. This approach has been most widely used. The derivation in [14] leads to the approximate reconstruction formula

$$f(x) \simeq f_b(x) = R \int_0^{2\pi} \int_0^{2\pi} k_b(\alpha^s - \alpha) \left(\frac{\alpha^s - \alpha}{\sin(\alpha^s - \alpha)}\right)^2 g(s, \alpha) \cos \alpha \, d\alpha \, ds$$

with

$$k_b(t) = \frac{1}{8\pi^2} \int_{-b}^{b} \phi(\sigma) |e^{i\sigma t}| \, d\sigma.$$  

Choosing the window function $\phi(\sigma) = 1$ gives the so-called ramp filter, while $\phi(\sigma) = \text{sinc}(\sigma \pi / (2b))$ yields the Shepp-Logan filter; cf. [16, p. 111], where $\text{sinc}(t) = \sin(t) / t$. Formula (16) is the basis for what we consider the standard filtered backprojection algorithm for fan-beam tomography.

In the following we will describe several different numerical implementations of (10) and compare them with each other as well as with the standard algorithm based on (16). The implementations of (10) will differ in the discretizations of the derivatives $\partial g/\partial s$ and $\partial g/\partial \alpha$.

We assume that measurements $g(s_j, \alpha_l)$ with $s_j = j \Delta s$ and $l \Delta \alpha$ are available. Since $f(x)$ vanishes for $|x| > r$ we have $g(s, \alpha) = 0$ for $|\alpha| \geq \alpha_0 = \arcsin(r/R)$.

For the discretization of the singular term $1/\sin(\alpha^s - \alpha)$ which occurs in the inner integral in (10) we follow an idea that has already been used in [3] and later in [14]. It consists in first writing

$$\frac{1}{\sin(\alpha^s - \alpha)} = \frac{1}{\alpha^s - \alpha} \frac{\alpha^s - \alpha}{\sin(\alpha^s - \alpha)}.$$  

Now the first factor is (up to a constant factor) the convolution kernel of the Hilbert transform while the second is a smooth function for $|\alpha| \leq \alpha_0$ since $\alpha_0 < \pi/2$.

The Hilbert transform kernel $1/\alpha$ is regularized using a bandlimited approximation. The Fourier transform (in the sense of distributions) of the Hilbert transform is given by

$$\left(\frac{1}{\alpha}\right)^\wedge(\sigma) = -i \frac{\pi}{2} \text{sgn}(\sigma).$$
where \( \text{sgn}(\sigma) \) equals 1 for \( \sigma > 0 \) and \(-1\) for \( \sigma < 0 \). Accordingly, we obtain the bandlimited approximation
\[
\frac{1}{\alpha} \approx (2\pi)^{-1/2} \int_{-b}^{b} \left(-i \sqrt{\frac{\pi}{2} \text{sgn}(\sigma)} \right) e^{i\sigma\alpha} d\sigma = \frac{1 - \cos(b \alpha)}{\alpha} \tag{17}
\]
with a cut-off frequency \( b \) to be chosen later. Thus we obtain the approximation
\[
\int_{0}^{2\pi} \left( \frac{\partial g}{\partial s} + \frac{\partial g}{\partial \alpha} \right) \frac{1}{\sin(\alpha^* - \alpha)} d\alpha \approx \int_{0}^{2\pi} \left( \frac{\partial g}{\partial s} + \frac{\partial g}{\partial \alpha} \right) k(\alpha^* - \alpha) d\alpha = G(s, \alpha^*)
\]
with
\[
k(\alpha) = \frac{1 - \cos(b \alpha)}{\alpha} \frac{\alpha}{\sin(\alpha^*)} = \frac{1 - \cos(b \alpha)}{\sin(\alpha^*)}. \tag{18}
\]
In practice the convolution \( G(s, \alpha^*) \) is not evaluated for every occurring value of \( \alpha^* \). Instead, one first computes \( G \) on an equispaced grid, \( G(s, n\Delta \alpha), -\alpha_0 \leq n\Delta \alpha \leq \alpha_0 \), and then approximates \( G(s, \alpha^*) \) by (usually linear) interpolation. Discretizing the integral with the trapezoidal rule yields
\[
G(s, n\Delta \alpha) = \int_{0}^{2\pi} \left( \frac{\partial g}{\partial s} + \frac{\partial g}{\partial \alpha} \right) k(n\Delta \alpha - \alpha) d\alpha \\
\simeq \Delta \alpha \sum_{l} \left( \frac{\partial g}{\partial s}(s, l\Delta \alpha) + \frac{\partial g}{\partial \alpha}(s, l\Delta \alpha) \right) k((n - l)\Delta \alpha) \tag{19}
\]
In this case the convolution kernel \( k \) is discretized by its samples \( k_{m} = k(m\Delta \alpha) \).

The derivatives can be discretized in various ways. A simple and straightforward possibility are central differences, such as
\[
\frac{\partial g}{\partial s}(s_{j}, \alpha_{l}) \simeq \frac{g(s_{j+1}, \alpha_{l}) - g(s_{j-1}, \alpha_{l})}{2\Delta s}
\]
\[
\frac{\partial g}{\partial \alpha}(s_{j}, \alpha_{l}) \simeq \frac{g(s_{j}, \alpha_{l+1}) - g(s_{j}, \alpha_{l-1})}{2\Delta \alpha} \tag{20}
\]
This approximates the derivatives at \((s_{j}, \alpha_{l})\) and the discrete convolution can be computed according to (19). Reconstructions using this first method of implementation have been labeled as M1 in the figures.

For the second and third methods of implementation we use a smaller discretization step size and approximate the derivatives at the points \((s_{j+1/2}, \alpha_{l+1/2}) = (s_{j} + \Delta s/2, \alpha_{l} + \Delta \alpha/2)\) via
\[
\frac{\partial g}{\partial s}(s_{j+1/2}, \alpha_{l+1/2}) \simeq \frac{1}{2\Delta s} [g(s_{j+1}, \alpha_{l}) - g(s_{j}, \alpha_{l}) + g(s_{j+1}, \alpha_{l+1}) - g(s_{j}, \alpha_{l+1})]
\]
\[
\frac{\partial g}{\partial \alpha}(s_{j+1/2}, \alpha_{l+1/2}) \simeq \frac{1}{2\Delta \alpha} [g(s_{j}, \alpha_{l+1}) - g(s_{j}, \alpha_{l}) + g(s_{j+1}, \alpha_{l+1}) - g(s_{j+1}, \alpha_{l})] \tag{21}
\]
The analogous scheme for the 3D formula (5) has been suggested in [4, Equ. (46)] and is also used in [6, Equ. (6.2)]. In [13] it is referred to as a ‘blended chain-rule scheme’.

In this case there are two natural choices for implementing the convolution and backprojection. First one could use the same discrete convolution kernel as in (19) and compute \( G \) on the shifted grid \((s_{j+1/2}, \alpha_{n+1/2})\) according to
\[
G(s_{j+1/2}, \alpha_{n+1/2})
\]
\[
= \int_{0}^{2\pi} \left( \frac{\partial g}{\partial s}(s_{j+1/2}, \alpha + \Delta \alpha/2) + \frac{\partial g}{\partial \alpha}(s_{j+1/2}, \alpha + \Delta \alpha/2) \right) k((\alpha_{n+1/2} - \alpha - \Delta \alpha/2) d\alpha \\
\simeq \Delta \alpha \sum_{l} \left( \frac{\partial g}{\partial s}(s_{j+1/2}, \alpha_{l+1/2}) + \frac{\partial g}{\partial \alpha}(s_{j+1/2}, \alpha_{l+1/2}) \right) k((\alpha_{n+1/2} - \alpha_{l+1/2}) \tag{22}
\]

8
Figure 3. Direct reconstructions with the Shepp-Logan phantom. Parameters used for the reconstructions were $R = 3$, $r = 1$, $\Delta s = 2\pi / P = 2\pi / 256$, $\Delta \alpha = \pi / Q = \pi / 942$. Relative root mean-square errors are indicated under each image.

noting that $k(\alpha_{n+1/2} - \alpha_{l+1/2}) = k((n - l)\Delta \alpha) = k_{n-l}$ gives the same discrete kernel as in (19). The shift in the grid where $G$ is computed can then be taken into account during the evaluation of the outer integral in (10), the so-called backprojection step. Reconstructions using this second method have been labeled M2 in the figures.

The third method consists in incorporating the shift in $\alpha$ into the discrete convolution kernel, according to the equation

$$G(s_{j+1/2}, \alpha_n) = \int_0^{2\pi} \left( \frac{\partial g(s_{j+1/2}, \alpha + \Delta \alpha/2)}{\partial \alpha} k(\alpha_n - \alpha - \Delta \alpha/2) + \frac{\partial g(s_{j+1/2}, \alpha + \Delta \alpha/2)}{\partial s} \right) k(\alpha_n - \alpha - \Delta \alpha/2) d\alpha$$

$$\simeq \Delta \alpha \sum_l \left( \frac{\partial g(s_{j+1/2}, \alpha_{l+1/2})}{\partial s} + \frac{\partial g(s_{j+1/2}, \alpha_{l+1/2})}{\partial \alpha} \right) k(\alpha_n - \alpha_{l+1/2})$$

which leads to the discrete convolution kernel $k(\alpha_n - \alpha_{l+1/2}) = k((n - l - 1/2)\Delta \alpha) = \tilde{k}_{n-l}$.

Reconstructions with this method have been labeled M3. In helical CT this incorporation of the shift into the convolution kernel has been the preferred method [4, Equ. (53)], [6, p. 865].

Our first comparison of the three methods for implementation of equation (10) and the standard filtered backprojection algorithm uses the Shepp-Logan phantom. The reconstructions and relative root mean-square errors ($l_2$-errors) are shown in Figure 3.

We call these direct reconstructions because the data have been used directly in the reconstruction algorithm without a prior interpolation to a denser grid. Such an interpolation has been found to be advantageous for high-accuracy reconstructions [17] but does not make a significant difference in this case where the error is still quite high. We see that the standard reconstruction has the lowest relative $l_2$-error; that the M1 reconstruction, while having the largest relative $l_2$-error, is the smoothest of the three implementations of (10); and that the M2
reconstruction shows considerable ‘ringing’ artifacts while the M3 image does not. We will now give an explanation for this difference which is based on the continuity of the Fourier transform of the linearly interpolated filtered data.

5. An explanation for the ringing artifact.

It can be shown (see, e.g., [18, p. 84]) that the effect of the linear interpolation can be taken into account by assuming that the convolution kernel is piecewise linear between the values \( k(n\Delta\alpha) = k_n \). Let \( K(t) \) and \( \hat{K}(t) \) be the piecewise linear functions that satisfy 

\[
K(n\Delta\alpha) = k(n\Delta\alpha) = k_n \quad \text{and} \quad \hat{K}(n\Delta\alpha) = \hat{k}_n = k(n\Delta\alpha - \Delta\alpha/2) = \tau_{\Delta\alpha/2}k(n\Delta\alpha),
\]

respectively, where \( \tau_{\Delta\alpha/2}k(t) = k(t - \Delta\alpha/2) \). Let \( Q(t) \) denote the discrete convolution 

\[
Q(t) = \Delta\alpha \sum_l K(t - l\Delta\alpha) C_{jl}
\]

where \( C_{jl} \) denotes the discrete approximation of \( \left( \frac{\partial^s}{\partial s^{s+1/2}}(s_{j+1/2}, \alpha l + 1/2) + \frac{\partial^s}{\partial s^{s+1/2}}(s_{j+1/2}, \alpha l + 1/2) \right) \) according to (21). The Fourier transform of \( Q \) is given by 

\[
\hat{Q}(\sigma) = \Delta\alpha \sum_l \hat{K}(\sigma) e^{-i\sigma l\Delta\alpha} C_{jl}.
\]

A jump discontinuity in \( \hat{Q} \) may cause the ringing artifact so we investigate the continuity of \( \hat{Q} \), which in turn depends on the continuity of the Fourier transform of \( K \). It is given by 

\[
\hat{K}(\sigma) = \text{sinc}^2(\Delta\alpha \sigma/2) \sum_{m \in \mathbb{Z}} \hat{k} \left( \sigma - \frac{2\pi m}{\Delta\alpha} \right) = \text{sinc}^2(\pi\sigma/(2b)) \sum_{m \in \mathbb{Z}} \hat{k} (\sigma - 2mb)
\]

(cf. [16, p. 61]) with \( k \) as in (18), \( \text{sinc}(t) = \frac{\sin t}{t} \) and the cut-off frequency \( b = \pi/\Delta\alpha \). Ignoring the slowly varying factor \( \alpha/\sin\alpha \) in \( k \) we have that \( \hat{k}(\sigma) \simeq -i \sqrt{\frac{2}{\pi}} \text{sgn}(\sigma) \) for \( |\sigma| \leq b \) and \( \hat{k}(\sigma) = 0 \) otherwise.

The sum \( \sum_{m \in \mathbb{Z}} \hat{k}(\sigma - 2mb) \) is a periodic function of \( \sigma \) with period \( 2b \). Since we assume \( \hat{k}(\sigma) = 0 \) for \( |\sigma| > b = \pi/\Delta\alpha \), only one term in the sum can be non-zero. For \( 0 < \sigma < b \) this is \( \hat{k}(\sigma) \) and for \( b < \sigma < 2b \) the non-vanishing term is \( \hat{k}(\sigma - 2b) \).

There is clearly a discontinuity of \( \hat{K} \) at \( \sigma = 0 \) and potential discontinuities at \( \sigma = \pm b \). While the discontinuity at \( \sigma = 0 \) is intrinsic to the inversion, a discontinuity at \( \sigma = \pm b \) would cause a ringing artifact. We therefore proceed to investigate the continuity of \( \hat{K}(\sigma) \) at \( \sigma = b \).

From the above considerations it follows that the left-sided limit \( \lim_{\sigma \to -b^-} \hat{K}(\sigma) \) equals 

\[
\lim_{\sigma \to -b^-} \text{sinc}^2(\pi\sigma/(2b)) \hat{k}(\sigma) = \lim_{\sigma \to -b^-} -i \sqrt{\frac{2}{\pi}} \text{sgn}(\sigma) \text{sinc}^2(\pi\sigma/(2b)) = -i(2/\pi)^{3/2}
\]

while the right-sided limit gives 

\[
\lim_{\sigma \to b^+} \hat{K}(\sigma) = \lim_{\sigma \to b^+} \text{sinc}^2(\pi\sigma/(2b)) \hat{k}(\sigma - 2b) = +i(2/\pi)^{3/2}.
\]

Since these limits are non-zero and have opposite sign, \( \hat{K} \) is not continuous at the cut-off frequency, which explains the ringing artifact. The situation is different when the convolution
is computed with the shifted kernel, where \( K \) is replaced by \( \tilde{K} \). We have

\[
\tilde{K}(\sigma) = \text{sinc}^2(\Delta \alpha \sigma/2) \sum_{m \in \mathbb{Z}} (\tau_{m/2})^\wedge \left( \sigma - \frac{2\pi m}{\Delta \alpha} \right)
\]

\[
= \text{sinc}^2(\pi \sigma/(2b)) \sum_{m \in \mathbb{Z}} \hat{k}(\sigma - 2mb) e^{-i(\sigma - 2mb)\pi/(2b)}
\]

\[
= \text{sinc}^2(\pi \sigma/(2b)) \sum_{m \in \mathbb{Z}} (-1)^m \hat{k}(\sigma - 2mb) e^{-i\sigma\pi/(2b)}
\]

Hence

\[
\lim_{\sigma \to -b^-} \tilde{K}(\sigma) = \lim_{\sigma \to -b^-} \text{sinc}^2(\pi \sigma/(2b)) \hat{k}(\sigma) e^{-i\sigma\pi/(2b)} = -(2/\pi)^{3/2}
\]

and

\[
\lim_{\sigma \to -b^+} \tilde{K}(\sigma) = \lim_{\sigma \to -b^+} \text{sinc}^2(\pi \sigma/(2b)) (-1) \hat{k}(\sigma - 2b) e^{-i\sigma\pi/(2b)} = -(2/\pi)^{3/2}.
\]

Hence \( \tilde{K} \) is continuous at the cut-off frequency which avoids the ringing artifact for the corresponding implementation M3. Investigation of the continuity of \( \hat{K} \) and \( \tilde{K} \) at \( \sigma = -b \) gives the same results as for \( \sigma = b \). The inclusion of the factor \( \alpha/\sin \alpha \) in \( k \) will not materially influence this reasoning. While it will smooth the jump of \( \tilde{K} \) at the cut-off frequency, there will still be a rapid change of \( \tilde{K} \) that results in the observed ringing artifact.

We may conclude that the combined effects of the interpolation, the choice of \( b = \pi/\Delta \alpha \), and the shift in the convolution kernel result in the continuity of the Fourier transform of \( \tilde{K} \) and thereby avoid the ringing artifact.

We now conclude this section with numerical experiments using a strictly bandlimited phantom which will reveal some additional aspects of the algorithms.

Most functions \( f \) encountered in tomography are non-negative, which means that their Fourier transform attains its maximum at the origin and decays quite rapidly away from the origin. To test the resolution of reconstruction algorithms it is useful to also use functions whose Fourier transform does not decay rapidly away from the origin. We consider a function \( f \) given by

\[
f(x) = \frac{J_1(b_0|x - x_0|)}{b_0|x - x_0|}, \quad x \in \mathbb{R}^2, \quad b_0 = 100, \quad x_0 = (0.4, 0.7)
\]

(23)

where \( J_1 \) denotes the Bessel function of the first kind of order one. The Fourier transform of \( f \) satisfies

\[
|\hat{f}(\xi)| = \begin{cases} 1/b_0^2 & \text{if } |\xi| \leq b_0 \\ 0 & \text{otherwise} \end{cases}, \quad \xi \in \mathbb{R}^2.
\]

Thus the function \( f \) is strictly bandlimited but its Fourier transform does not decay for frequencies less than the cut-off frequency \( b_0 \). For our first experiments we use \( b_0 = 100 \) and set the cut-off frequency \( b \) for the convolution kernel in equation (18) to the same value, i.e., \( b = b_0 = 100 \). For a radius \( R = 3 \) Shannon sampling theory [17] yields that \( P = 175 \) sources and \( Q = 300 \) rays distributed over \( |\alpha| \leq \pi/2 \) should provide sufficient information for an accurate reconstruction.

While in case of the Shepp-Logan phantom an optically pleasing image is considered more important than lower root mean-square error, our main goal with regard to the present phantom is to achieve accurate reconstructions in the sense of low error. While in the case of the Shepp-Logan phantom the smoothing caused by the linear interpolation in the backprojection step is desirable, for this phantom this smoothing effect would significantly increase the error. Therefore
the convolution is computed on a grid of step size $\Delta \alpha/8$ instead of $\Delta \alpha$. Nevertheless, in the numerical experiments shown in Figure 4 the errors for all four methods are quite large.

Again the standard algorithm shows the smallest error while method M1 has the largest.

A striking observation is that methods M2 and M3 yield identical results for this phantom (up to round-off error). This can be explained by the fact that the phantom is strictly bandlimited and the cut-off frequency for the Hilbert transform kernel was chosen to be equal to the bandwidth of the phantom. The discontinuity of the Fourier transform of the convolution kernel at this cut-off frequency is now immaterial since the Fourier transform of the phantom itself is discontinuous there. This also shows that shifting the convolution kernel, while helpful in avoiding unwanted ringing artifacts in some phantoms, does not increase the intrinsic resolution of the algorithm if resolution is measured by the highest frequency that can be accurately reconstructed.

It is known [17, 19] that the standard algorithm does not achieve optimal resolution. One way around this that was suggested in [17] is to use Shannon sampling theory to interpolate the data to a denser grid prior to reconstruction. This has been done in the experiments shown in Figure 5. We see that interpolating the data prior to reconstruction reduces the error for the standard algorithm by a factor of about 10, while the reductions for the other algorithms are more modest, the factors being slightly below 2 for method M1, and slightly above 2 for methods M2 and M3, the latter two methods again yielding images that are identical up to round-off error.

6. Improving the resolution

The experiments above indicate that methods M1-M3 are not fully competitive with the standard algorithm. Methods M2 and M3 also show a distinct anisotropy in the achieved resolution. As can be seen most clearly in Figure 4 the circular peak of the original function appears to be elliptical in the reconstructions with methods M2 and M3. In [20] such an effect has been described as the resolution in radial direction being different (in this case better) than in tangential direction; cf. [20, Fig. 8]. Neither the standard nor the M1 reconstructions exhibit this undesirable anisotropy. A heuristic explanation is as follows. The derivative with respect
to $\alpha$ contributes the dominant part of the reconstruction. Indeed, it can be shown that the contribution of $\partial g/\partial s$ in (10) actually vanishes; cf. [20, Equ. (17)]. When inspecting the scheme (21) one observes that $\frac{\partial g}{\partial \alpha}$ is approximated at $s = s_{j+1/2}$ by averaging the derivatives computed at $s = s_j$ and $s = s_{j+1}$. This then results in the entire reconstruction being an average of two slightly rotated images, thus causing a blur in the tangential direction. The following new scheme, labeled M4, is a straightforward attempt to overcome this deficiency. We compute the derivatives at the point $(s_j, \alpha_{l+1/2})$ via

$$
\frac{\partial g}{\partial \alpha}(s_j, \alpha_{l+1/2}) \simeq \frac{1}{\Delta \alpha} [g(s_j, \alpha_{l+1}) - g(s_j, \alpha_l)]
$$

$$
\frac{\partial g}{\partial s}(s_j, \alpha_{l+1/2}) \simeq \frac{1}{4\Delta s} [(g(s_{j+1}, \alpha_l) - g(s_{j-1}, \alpha_l)) + (g(s_{j+1}, \alpha_{l+1}) - g(s_{j-1}, \alpha_{l+1}))]
$$

This scheme retains the shorter stepsize and thereby the accuracy for $\frac{\partial g}{\partial \alpha}$ while avoiding the averaging process. As a tradeoff a larger stepsize is used for $\frac{\partial g}{\partial s}$ in order to compute this derivative at the same point $(s_j, \alpha_{l+1/2})$. We retain the shift of the convolution kernel as in method M3. Our numerical experiments, some of which are reported below, indicate that method M4 appears preferable to M3 both in fan-beam and in helical tomography. A different approach to improve the discretization of the derivatives has been developed in [13].

Another question in this context is if the relatively low-order difference quotient used for $\partial g/\partial \alpha$ will significantly reduce the resolution compared to the standard algorithm. We explore this by comparing the Fourier transforms of the exact shifted derivative with the Fourier transform of the difference quotient. Let $\tilde{g}(t) = g(t + \Delta t/2)$ and $Dg(t) = (g(t + \Delta t) - g(t))/\Delta t$. The Fourier transforms satisfy

$$
\hat{D}g(\sigma) = \left(\frac{dg}{dt}\right)^\wedge (\sigma) \text{sinc}(\sigma \Delta t/2), \quad \text{sinc}(x) = \frac{\sin(x)}{x}.
$$

Applying this relation to the $\alpha$-derivative in (24) shows that the difference quotient gives a filtered version of the derivative where the window function multiplying the Fourier transform is

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image}
\caption{Reconstructions of the bandlimited phantom (23) from interpolated data.}
\end{figure}
Figure 6. Helical CT implementing equation (5) with methods M3 and M4. Upper row: Strictly bandlimited phantom \( f(x) = J_3/2(b_0|x - x_0|)/((b_0|x - x_0|)^{3/2}) \). Lower row: Smooth, compactly supported phantom \( f(x) = (1 - |x - x_0|^2/100)^{3/2} \). In both cases method M4 appears preferable.

given by \( \text{sinc}(\sigma \Delta \alpha /2) \). If the cut-off frequency is chosen as \( b = \pi / \Delta \alpha \) this becomes \( \text{sinc}(\sigma \pi / (2b)) \) which is precisely the same window as for the popular Shepp-Logan filter for the standard algorithm that was mentioned in conjunction with equation (16) above. One may therefore expect that method M4 will give results comparable to the standard algorithm with the Shepp-Logan filter. The Shepp-Logan filter is usually a good choice but in cases like the bandlimited phantom above the straight ramp filter will give more accurate results. If such a filter is desired the convolutions for \( \partial g / \partial \alpha \) and \( \partial g / \partial s \) can be treated separately as has been done in [20].

We will present experiments for fan-beam CT in the next section, but will conclude this section with a first numerical experiment comparing discretizations M3 and M4 for helical tomography based on equation (5). We reconstruct in the xy-plane and use a 3D version of the strictly bandlimited phantom as well as a smooth, compactly supported phantom given by the function \( f(x) = (1 - |x - x_0|^2/100)^{3/2} \). As Figure 6 shows, the experiment indicates that method M4 gives a more circular peak and fewer artifacts for both of these phantoms.

7. Numerical comparisons for fan-beam algorithms.

In this section we present numerical experiments comparing four fan-beam algorithms: the discretization of (10) with method M4, labeled M4 in the figures; the standard algorithm using the Shepp-Logan filter; and two implementations of formula (11) that also use method M4 but differ in the choice of \( I_{P1}(x) \). Unlike the helical case where \( I_{P1}(x) \) is uniquely determined there are infinitely many ways to choose \( I_{P1}(x) \) in (11). Here we focus on two choices. The first is obtained by choosing the longer of the two arcs whose endpoints are the intersections of the circle with the line that passes through the point \( x \) and is orthogonal to \( x \). This choice represents the largest possible interval \( I_{P1}(x) \) and is labeled ‘orthogonal-long’ in the figures. The second choice, labeled ‘Helix-xy’ are the intervals for points in the xy-plane obtained from the helix in the limit of vanishing pitch, as discussed in §2 above.

Figure 7 shows the results for the bandlimited phantom (23). It is at first glance remarkable
that the reconstruction labeled 'PI-line orthogonal-long' is the most accurate although it uses fewer data than the standard algorithm and method M4 for equation (10). A heuristic explanation could be that the full scan using equation (10) is an average over the reconstructions with equation (11) from the long arc and the short arc. Since the long arc used in the image in the upper left has more sources, it is not unreasonable to expect it to give a better image than the short arc, or even the average between the two.

For this phantom the standard algorithm shows a larger error with the Shepp-Logan kernel than it had with the straight ramp kernel in Figure 5. This is because the Shepp-Logan kernel attenuates the higher frequencies to some degree which increases the error significantly for this particular phantom. The error for the three other methods that all use discretization M4 are significantly smaller than the errors for methods M1-M3 reported in Figure 5. Also the anisotropy in the resolution does no longer occur.

The reconstructions of the Shepp-Logan phantom shown in Figure 8 show very similar errors and images for all four methods.

Our last two figures show reconstructions from real data. The data came from an old generation Siemens hospital scanner. Reconstructions of the human pelvis shown in Figure 9 appear roughly comparable for all four methods, although the images in the bottom row seem preferable upon closer inspection.

Figure 10 shows images from real data for a physical resolution phantom. Here the reconstructions from the standard algorithm and from method M4 for equation (10) are good and of comparable quality. The methods based on equation (11) show distinct artifacts near the white rectangular block. A jump discontinuity along a straight edge causes a stronger singularity in the x-ray data that appears to be more challenging for these methods.

Note that in all experiments the algorithm based on equation (10) with discretization method M4 gave similar results to the standard algorithm with the Shepp-Logan filter as had been expected.
8. Conclusions
We considered Katsevich \textquoteleft s inversion formula for helical tomography and investigated the limit of vanishing helical pitch. It was shown that in this limit one obtains a reconstruction formula for fan-beam tomography (equations (11) and (12)) which is formally completely analogous to the 3D formula. This limiting formula was shown to hold for arbitrary PI-lines in the fan-beam case, thus establishing a certain universality of Katsevich \textquoteleft s formula. Our proof turned out to be related to the method employed in [2], and the 2D version of Katsevich \textquoteleft s formula can indeed also be obtained as a special case of the general formula derived there. An alternative method of
proof can be found in [21]. In this work we contribute a rigorous proof of the rebinning formula (Theorem 2) which provides a highly useful connection between fan-beam and parallel-beam tomography.

For curved detector coordinates the relationship of the 2D version of Katsevich’s formula to some other fan-beam inversion formulas in the literature was explored. This included the rediscovery of equation (10), an early formula by Herman and Naparstek [3]. This formula shares some numerical features with Katsevich’s 3D formula and it was numerically implemented in order to study some numerical questions that have also arisen in 3D tomography. A theoretical explanation was provided for the effectiveness of a shift in the convolution kernel that is similar to one employed in the 3D case. This explanation clarifies that while the shift is very useful in exploiting the smoothing effect of the linear interpolation to reduce ringing artifacts, it does not increase the maximal resolution of the algorithm, if resolution is measured by the highest frequency that can be resolved.

A new discretization scheme (24) for the derivatives occurring in formulas (5), (10), and (11) was suggested and shown in numerical experiments to lead to improvements in image quality and a significant reduction if not removal of the anisotropy of the resolution both in fan-beam and helical CT.

The numerical tests comparing the standard algorithm with implementations of both formula (10) and formula (11) indicated that for smooth phantoms the algorithm based on (11) and PI-line choice ‘orthogonal-long’ may perform best. For objects with moderate singularities such as the Shepp-Logan phantom four algorithms yielded almost equivalent results. For objects with somewhat stronger singularities such as the the bones in the human pelvis and especially the resolution phantom our implementations of the formula (11) showed artifacts.

Finally, our implementation of formula (10) using the new discretization scheme (24) yielded comparable results to the standard algorithm with the Shepp-Logan kernel throughout all tests, thus indicating that this is a competitive method.
Appendix. Proofs of Theorems 1 and 2

Proof of Theorem 2. For simplicity of notation we first prove the result when \( \theta = (0, 1) \), \( \omega = (\cos \alpha, \sin \alpha) \), \( f \) has support in the closed unit ball \( B(0, 1) \) and \( y = (a, 0) \), \( a < -1 \), in which case the right hand side of equation (15) becomes \( \pi(\mathcal{H}_{\theta} f)(0) \). With these notations, switching from polar coordinates centered at \((a,0)\) to rectangular coordinates we have

\[
\int_{S_1 \cap \{(\omega', \theta)\}} \frac{Df(y, \omega)}{\langle \omega, \theta \rangle} \, d\omega = \int_{|\sin \alpha|\geq \epsilon} \int_{0}^{\infty} f(a + t \cos \alpha, t \sin \alpha) \frac{1}{\sin \alpha} \, dt \, d\alpha = \int \int_{\Gamma_{a,\epsilon}} \frac{f(x, y)}{y} \, dx \, dy,
\]

where \( \Gamma_{a,\epsilon} \) is the complement of the closed cone with vertex at \((a,0)\) and vertex angle \( \epsilon \).

The ray emanating from \((a,0)\), \( a < -1 \), and making angle \( \epsilon \) with the horizontal intersects the unit ball at points with vertical coordinates

\[
y_{e-} = (-a \cos \epsilon - \sqrt{1 - a^2 \sin^2 \epsilon}) \sin \epsilon, \quad y_{e+} = (-a \cos \epsilon + \sqrt{1 - a^2 \sin^2 \epsilon}) \sin \epsilon.
\]

The last integral may be rewritten as

\[
\int \int_{y \geq y_{e+}} \frac{f(x, y)}{y} \, dx \, dy + \int \int_{y \leq -y_{e+}} \frac{f(x, y)}{y} \, dx \, dy + \int_{\Gamma_{a,\epsilon} \cap [y_{e-}, y_{e+}]} \frac{f(x, y)}{y} \, dx \, dy + \int_{\Gamma_{a,\epsilon} \cap [-y_{e+}, -y_{e-}]} \frac{f(x, y)}{y} \, dx \, dy.
\]

Integrating first with respect to \( x \) the first two integrals combine to give

\[
\int_{|y|\geq y_{e+}} \frac{\mathcal{R}_\theta f(y)}{y} \, dy,
\]

which converges to \( -\pi(\mathcal{H}_{\theta} f)(0) \) as \( \epsilon \to 0 \). The singular integral converges because \( \mathcal{R}_\theta f \) satisfies a H"older condition of order \( \alpha > 0 \) since \( f \) does. It remains to show that the remaining integrals vanish as \( \epsilon \to 0 \).

Replacing \( y \) by \(-y\) in the second of the remaining integrals we obtain

\[
\left| \int_{\Gamma_{a,\epsilon} \cap [y_{e-}, y_{e+}]} \frac{f(x, y) - f(x, -y)}{y} \, dx \, dy \right| \leq \int_{\Gamma_{a,\epsilon} \cap [y_{e-}, y_{e+}]} \frac{|f(x, y) - f(x, 0)|}{y} \, dx \, dy + \int_{\Gamma_{a,\epsilon} \cap [y_{e-}, y_{e+}]} \frac{|f(x, 0) - f(x, -y)|}{y} \, dx \, dy
\]

Using that \( f \) satisfies a H"older condition of order \( \alpha > 0 \) we have

\[
\int_{\Gamma_{a,\epsilon} \cap [y_{e-}, y_{e+}]} \frac{|f(x, y) - f(x, 0)|}{y} \, dx \, dy \leq \int_{y_{e-}}^{y_{e+}} \int_{-1}^{1} \frac{C}{y^{1-\alpha}} \, dx \, dy = 2C \frac{y_{e+}^{\alpha} - y_{e-}^{\alpha}}{\alpha} \to 0
\]

since \( \alpha > 0 \). The remaining integral is handled in the same way, completing the proof under the original choice of coordinates.

In general, if \( \theta = (\cos \beta, \sin \beta) \) by rotating coordinates centered at \( y \) we may obtain \( \theta = (0, 1) \) at the expense of a rotation of the unit ball so that its center is no longer at the origin of the rectangular coordinate system. This loss of symmetry leads to more complicated expressions for \( y_{e+}, y_{e-} \), but that is the only change.
Proof of Theorem 1. We first show that the change of variables \( \gamma = \alpha^* - \alpha \) will transform (11) into (12).

To begin we note that \( g(s, \alpha) \) is the integral of \( f \) over the ray with vertex at \( y(s) \), subverting an angle \( \alpha \) with \( \omega(s) \), and direction such that it passes through the disk \(|x| < R\). Hence

\[
g(s, \alpha) = Df(y(s), \omega(s - \alpha - \pi)).
\]

It follows that \( Df(y(q), \omega(\alpha')) = g(q, q - \alpha' - \pi) \), and therefore

\[
\frac{\partial}{\partial q} Df(y(q), \omega(\alpha')) \bigg|_{q=s} = \frac{\partial}{\partial q} g(q, q - \alpha' - \pi) \bigg|_{q=s} = \frac{\partial g}{\partial s}(s, s - \alpha' - \pi) + \frac{\partial g}{\partial \alpha}(s, s - \alpha' - \pi)
\]

(25)

Now letting \( \gamma = \alpha^* - \alpha \) and choosing \( \alpha' = s - \alpha - \pi = s + \gamma - \alpha^* - \pi \) gives

\[
\frac{\partial q}{\partial s}(s, \alpha) + \frac{\partial g}{\partial \alpha}(s, \alpha) = \frac{\partial}{\partial q} Df(y(q), \omega(s + \gamma - \alpha^* - \pi)) \bigg|_{q=s} = \frac{\partial}{\partial q} Df(y(q), \cos \gamma \beta + \sin \gamma \beta^\perp) \bigg|_{q=s}
\]

(26)

For the last equation we used that

\[
\omega(s - \alpha - \pi) = \omega(s + \gamma - \alpha^* - \pi) = \cos \gamma \beta + \sin \gamma \beta^\perp.
\]

This shows that the change of variables \( \gamma = \alpha^* - \alpha \) will transform (11) into (12). The next step will be to show that (12) is equivalent to (9). From the observation that (9) involves integrals over all lines in the plane while (12) only involves lines that meet the arc from \( y(s_i) \) to \( y(s_t) \) it is clear that this cannot be shown by merely making a change in the parametrization of the lines. The key is Theorem 2. Let \( \theta = (\cos \varphi, \sin \varphi) \in S_1 \) and \( \theta^\perp = (-\sin \varphi, \cos \varphi) \). Then Theorem 2 yields

\[
\int_0^{2\pi} \frac{Df(y(q), \omega(\alpha'))}{\langle \omega(\alpha'), \theta^\perp \rangle} d\alpha' = -\int_\mathbb{R} \frac{Pf(\varphi, p)}{\langle y(q), \theta^\perp \rangle - p} dp = -\int_\mathbb{R} Pf(\varphi, \langle y(q), \theta^\perp \rangle - u) \frac{1}{u} du
\]

(27)

where the relation \( R_{\theta^\perp}f(p) = Pf(\varphi, p) \) has been used and the singular integrals are understood as principal values. Differentiating with respect to \( q \) gives

\[
\int_0^{2\pi} \frac{\partial}{\partial q} Df(y(q), \omega(\alpha')) \frac{1}{\langle \omega(\alpha'), \theta^\perp \rangle} d\alpha' = -\langle \frac{\partial y(q)}{\partial q}, \theta^\perp \rangle \int_\mathbb{R} \frac{\partial Pf(\varphi, p)}{\partial p} \frac{1}{\langle y(q), \theta^\perp \rangle - p} dp
\]

Now we choose \( \theta = \beta = \beta(s, x) \) and \( \gamma \) such that \( \omega(\alpha') = \cos \gamma \beta + \sin \gamma \beta^\perp \). Then \( \langle \omega(\alpha'), \theta^\perp \rangle = \langle \cos \gamma \beta + \sin \gamma \beta^\perp, \beta^\perp \rangle = \sin \gamma \). We also observe that \( \langle y(s), \beta^\perp \rangle = \langle x, \beta^\perp \rangle \) and obtain

\[
\int_0^{2\pi} \frac{\partial}{\partial q} Df(y(q), \cos \gamma \beta + \sin \gamma \beta^\perp) \bigg|_{q=s} \frac{1}{\sin \gamma} d\gamma = -\langle \frac{\partial y(s)}{\partial s}, \beta^\perp \rangle \int_\mathbb{R} \frac{\partial Pf(\varphi(s), p)}{\partial p} \frac{1}{\langle x, \beta^\perp \rangle - p} dp
\]

(28)
where $\varphi(s) = s - \alpha^* - \pi$ is the polar angle of $\beta$. Hence we have that the right-hand side of (12) is equal to

$$I = -\frac{1}{2\pi^2} \int_{\phi(x)}^{\varphi(s)+\pi} \frac{\partial f(\varphi, p)}{\partial p} \frac{1}{\langle \mathbf{x}, \beta^\perp \rangle - p} dp d\varphi.$$

Since $\beta(s_t, x) = -\beta(s_b, x)$ the angle $\varphi(s)$ traverses a range of $\pi$ radians as $s$ changes from $s_b$ to $s_t$. Changing the variable of integration from $s$ to $\varphi$ we use

$$\frac{\partial \beta}{\partial s} = \frac{d}{ds}(\cos(\varphi(s), \sin(\varphi(s))) = (\sin(\varphi(s), \cos(\varphi(s))) \frac{d\varphi}{ds} = \beta^\perp \frac{d\varphi}{ds}$$

together with the relation $y(s) = x - |x - y(s)| \beta$ to obtain that

$$\frac{\langle \mathbf{y}'(s), \beta^\perp \rangle}{|x - y(s)|} = \frac{d\varphi}{ds}.$$

Hence

$$I = \frac{1}{2\pi^2} \int_{\phi(x)}^{\varphi(s_b)+\pi} \frac{\partial f(\varphi, p)}{\partial p} \frac{1}{\langle \mathbf{x}, \beta^\perp \rangle - p} dp d\varphi$$

$$= \frac{1}{2\pi^2} \int_{0}^{\pi} \frac{\partial f(\varphi, p)}{\partial p} \frac{1}{\langle \mathbf{x}, \beta^\perp \rangle - p} dp d\varphi, \quad \beta^\perp = (\sin(\varphi), \cos(\varphi)),$$

(29)

where the change of the limits of integration in the last step is justified due to the symmetry relation $\mathcal{P} f(\varphi, p) = \mathcal{P} f(\varphi + \pi, -p)$. This same relation also yields that (29) is equivalent to (9), i.e., $I = f(x)$. \[\blacksquare\]

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