Optimizing synchronizability of networks

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In this paper, we investigate the factors that affect the synchronization of coupled oscillators on networks. By using the edge-intercrossing method, we keep the degree distribution unchanged to see other statistical properties’ effects on network’s synchronizability. By optimizing the eigenratio $R$ of the coupling matrix with Memory Tabu Search (MTS), we observe that a network with lower degree of clustering, without modular structure and displaying disassortative connecting pattern may be easy to synchronize. Moreover, the optimal network contains fewer small-size loops. The optimization process on scale-free network strongly suggests that the heterogeneity plays the main role in determining the synchronizability.

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I. INTRODUCTION

Many real networked systems exhibit some common characteristics, including small-world effect [1] and scale-free property [2]. The network structure has significant impact on the dynamical processes taking place on it [3-6]. Recently, a particular issue is concerned the synchronization of individuals with coupling dynamics located at each node of a given network. It has been shown that the ability of a network to synchronize is generally improved in both small-world networks and scale-free networks as compared to regular graphs [7-9]. However, so far, a completely clear picture about the relationship between structure and synchronizability is lacking.

How does the network topology affects its synchronizability? Some previous works have shown that the average network distance $\langle d \rangle$ is one of the key factors that affect the network synchronizability. The smaller $\langle d \rangle$ will lead to better synchronizability [10-12], while other researchers found that the heterogeneity of degree can strongly influence a network’s synchronizability, that is, a more homogeneous network will show better synchronizability compared to the heterogeneous one, though the average distance of a homogeneous network is longer [13-15]. In addition, some recent works demonstrate that the disassortative networks synchronize easier than assortative ones [16], and the increasing clustering will hinder the global synchronizability [17-18].

The majority of previous works are based on the simulation results allowing tuning only one or a very few topological measures (see also some recently proposed analytical approaches [19-22]). If all the topological properties can simultaneously vary, what will happen? Should a network with better synchronizability have some particular properties? Besides the simulated and analytical approaches, a potential way to investigate the relation between structural and dynamical properties is to track the optimization process, which will lead to some networks with specific dynamical characters [23-25]. Concerning with network synchronizability, one pioneer work [26], the rewiring operation is used, thus for different initial configurations, the optimization process will lead to the same optimal result, named Entangled Network. In this paper, as a complementary work, we would like to see the optimal results of keeping the degree of each node unchanged. With this idea, we take the network synchronizability as the object and change the network structure with the constraint condition that the degree of each node could not be changed, which can be realized by edge-intercrossing operation [27, 28, 29]. As shown in Fig. 1, the procedure of edge-intercrossing operation is as fol-
The ratio of the maximum eigenvalue $\lambda$ to the smallest eigenvalue is zero for the rows of $L$, and there is no edge between $x_1$ and $x_3$ as well as $x_2$ and $x_3$. Interchange these two edges, that is, connect $x_1$ and $x_4$ as well as $x_2$ and $x_3$, and remove the edges $e_1$ and $e_2$ simultaneously.

This article is organized as follows: The concept of synchronizability and the optimization algorithm are briefly introduced in Sec. II and Sec. III, respectively. Then, in Sec. IV, we will give out the main simulation results for both homogenous and heterogenous networks. Finally, in Sec. V, we will summarize this work.

II. SYNCHRONIZABILITY

Network synchronizability can be quantified by the eigenratio of the Laplacian matrix $L$. Consider a network of $N$ identical systems with symmetric coupling between oscillators. The equations of motion for the system are:

$$\dot{x}_i = F(x_i) - \sigma \sum_{j=1}^{N} L_{ij} H(x_j),$$

where $x_i = F(x_i)$ governs the dynamics of individual oscillator, $H(x)$ is the output function and $\sigma$ is the coupling strength. The $N \times N$ Laplacian matrix $L$ is given by

$$L_{ij} = \begin{cases} k_i & \text{for } i = j \\ -1 & \text{for } j \in \Lambda_i \\ 0 & \text{otherwise} \end{cases}$$

where $\Lambda_i$ denotes the set of $i$'s neighbors. All the eigenvalues of Laplacian matrix $L$ are positive reals and the smallest eigenvalue $\lambda_1$ is zero for the rows of $L$ having zero sum. The eigenvalues are $0 = \lambda_1 \leq \lambda_2 \ldots \leq \lambda_N$. The ratio of the maximum eigenvalue $\lambda_N$ to the smallest nonzero eigenvalue $\lambda_2$ is widely used to measure the synchronizability of the network. If the eigenratio $R = \frac{\lambda_N}{\lambda_2} < \beta$, where $\beta$ is a constant depending on $F(x)$ and $H(x)$, then the network is synchronizable. The eigenratio $R$ depends only on the topology of interactions among oscillators. The impact of having a particular coupling topology on the network's synchronizability is represented by a single quantity $R = \frac{\lambda_N}{\lambda_2}$: The smaller the eigenratio $R$, the easier it is to synchronize the oscillators, and vice versa. Having reduced the problem of optimizing the network synchronizability to finding the smallest eigenratio $R$ of the matrix $L$, we shall seek the network with approximately best synchronizability by using edge-intercrossing operation.

III. ALGORITHM

To obtain the approximately minimal $R$, we combine the intercrossing processes with a heuristic algorithm, named Memory Tabu Search (MTS). The whole processes of the algorithm are (see also Refs. [32, 33, 34] for the details about MTS):

step 1: Generate an initial random graph $G_0$ with $N$ nodes, $E$ edges following a previously given degree distribution. Set $G^* = G_0$ and compute the eigenratio $R$ of the Laplacian matrix of $G^*$. $G^*$ and $R^*$ are used to record the current optimal configuration and the corresponding eigenratio, respectively.

step 2: Stop and output the current result $G^*$ and $R^*$ if a prescribed terminal condition is satisfied. Otherwise, do intercross two randomly selected edges of $G_k$ (denotes the time step, and is set as zero initially), and denote by $G'$ and $R'$ the resulting graph and its eigenratio.

step 3: Update the current optimal eigenratio $R^*$ and set the current optimal graph as $G^* = G'$ if the eigenratio $R'$ satisfies $R^* < R^*$.

step 4: Update the new iteration resolution $G_{k+1}$. If $R' < R_{G_k}$ or $G'$ does not satisfy the tabu condition, set $G_{k+1} = G'$; else set $G_{k+1} = G_k$.

step 5: Update the tabu list. Let $G_{k+1}$ enter into the tabu list. If $k \geq L$, where $L$ is the length of tabu list, delete $G_{k+1-L}$ from the tabu list.

step 6: Return to step 2 and set $k = k + 1$, where $k$ always denotes the current time step.

The following condition is used to determine if a move is tabu: $\frac{|R_{G_k} - R_{G'}|}{R_{G_k}} > \delta$, which is the percentage improvement or destruction that will be accepted if the new move is accepted. Thus, the new graph $G'$ at step 2 is assumed tabu if the total change in the objective function is higher than a percentage $\delta$. In this paper, $\delta$ is a random number following uniform distribution between 0.50 and 0.75 and the selection of $\delta$ will not affect the optimal result. The terminal condition is that the present step is getting to a

![FIG. 2: The eigenratio $R$ vs. iteration steps of a WS network with rewiring probability $p = 0.1$. The network size is $N = 400$ and the average degree is $(k) = 6$. Only the steps in which $R$ being reduced are recorded.](image-url)
modularity

In Fig. 3, these includes the average distance tendencies of some other topological measures are shown where $S$ is the clustering coefficient. (a) The average distance $\langle d \rangle$; (b) The average clustering coefficient $C$; (c) Modularity $M$; (d) The Pearson parameter of degree-degree coefficient $r$. As the same as shown in Fig. 2, the initial network is a WS network with $N = 400$, $\langle k \rangle = 6$ and $p = 0.1$.

the predefined maximal iteration steps.

IV. SIMULATION RESULTS

In this paper, four different kinds of extensively studied networks are used for simulation. These are scale-free networks proposed by Barabási and Albert (BA) [1], small-world networks proposed by Watts and Strogatz (WS) [2], random networks proposed by Erdös and Rényi (ER) [3], and the regular networks. The former one is heterogenous, while the latter three are homogenous. The simulation results of all the homogenous networks are almost the same, so, in this article, only the results of WS networks are shown.

Fig. 2 reports the eigenratio $R$ vs. the iteration steps of a WS network with rewiring probability $p = 0.1$. The tendencies of some other topological measures are shown in Fig. 3, these includes the average distance $\langle d \rangle$, the clustering coefficient $C$, the assortativity $r$ [37], and the modularity $M$ [37, 38, 39]. Based on Pimm’s work [37], $M$ is defined as

$$M = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1}^{N} S_{ij},$$

where $S_{ij}$ is the number of common neighbors of nodes $i$ and $j$ divided by their total number of neighbors.

One trivial result is that the networks with better synchronizability are of shorter average distance. Besides, we observe that the degree correlation coefficient decreases from positive reals to negative ones, that is, the disassortative networks are, in general, easier to synchronize. And, in the whole process, the average clustering coefficient reduces monotonically. The lower clustering and disassortative pattern of the optimal network are consistent with some previous results [16, 17, 18]. More interestingly, the modularity decreases too, which indicates that a network with strong modular structures may be more difficult to synchronize, which is also in accordance with some recent results [40].

It has been found that many biological and technological networks contain motifs, that is, some specific subgraphs appearing much more frequently than that observed in random graphs with the same degree sequence [11, 12]. Loop is one of the simplest but most significant subgraphs, for it accounts for the multiplicity of paths between any two nodes [43]. Here we will show the change of loop structure in the optimization process. Due to the limitation of computational ability, we only calculate the number of loops from size 3 to 5. From Fig. 4, one can observe that the number of loops drops drastically during the optimization process, which indicates that the dense loops may hinder the global synchronization. As shown in Fig. 5, in WS networks, the number of loops will increase linearly as the increasing of network size, however, the number of loops in the optimal networks is stable and much smaller than that in the initial networks.

All these results indeed state that the optimal networks belong to a class of networks in which there are few number of loops, which may be different from the majority of real biological and technological networks. This phenomenon may deserve deep explorations in local structures of both artificial and real networks.

As shown before, this algorithm is highly effective for homogenous networks. For example, the eigenratio of WS networks can be reduced to about 15% of the original value (see Fig. 2). Since the degree distributions of many
real networks are heterogenous, next, we will check if this algorithm is also effective for heterogenous networks. As shown in Fig. 6, this algorithm can also enhance the synchronizability of scale-free networks, however, the enhancement is very small compared with the case of homogenous networks. In the optimization process of BA network, the tendencies of average distance, clustering coefficient, modularity and assortativity are almost the same as those for WS network (not shown), but with larger fluctuation and smaller decrement. Different from the case of homogenous networks, as the increasing of network size, the number of loops in both the optimal network and original BA network increases (see Fig. 7). The number of loops in the optimal network is a little bit smaller than that in the original BA network.

V. CONCLUSION AND DISCUSSION

In summary, a heuristic algorithm, memory tabu search, is used to optimize the network synchronizability by changing the connection pattern between different pairs of nodes while keeping the degree of each node unchanged. For both the homogenous and heterogenous networks, the change of some topological measures is recorded during the optimization process, which strongly suggests that a network with shorter average distance, lower clustering, negative degree-degree correlation and weaker modular structure may be easier to synchronize. In fact, these results has been reported respectively by some previous works. However, most of those works are based on some specific network models and can only reveal the relation between synchronizability and one or two structural properties, while the present optimization algorithm can simultaneously lay out the whole picture. Here, we would like to emphasize that, besides the traditional simulated and analytical approaches, to track the optimization process is a powerful tool of analysis on the relation between structural and dynamic properties of networks.

In addition, we investigated the change of loop structure in the optimization process, and found that the number of loops will decrease as the increasing of synchronizability. That is to say, the networks with fewer loops will be easier to synchronize. More interestingly, in the original WS networks, the number of loops will increase linearly as the increasing of network size, while in the
optimal networks, this number keeps stable. This novel phenomenon deserves in-depth exploration. Since each node can only impact its neighbors (see Eq. (1)), if there is one path, of length l, between node i and j, then, along this path, it takes l steps transferring the synchronization signal from i to j (or from j to i). Therefore, if there are so many paths of different lengths between node i and j, the synchronization signal of i at a given time will arrive on j along different paths at different time, which may disturb each other. It may be the reason why dense loops will hinder the global synchronization. An extreme case is that for directed networks, the one with highest synchronizability (i.e. with eigenratio $R$ being equal to 1) is a tree structure without any loops [44]. Even if adding one loop of length 2 [45], the eigenratio will be doubled [16,17]. However, the conclusion in Ref. [44] may be not universal. Actually, in synchronization system consisted of non-identical oscillators, a counterexample against Ref. [44] is reported very recently [48]. We believe this work will be helpful for the in-depth understanding about the role of loops in network synchronization.

Many previous works focus on synchronization on heterogeneous networks, especially scale-free networks. A common cognition is that the heterogeneity will hinder the global synchronization. However, since most of those works are based on some specific models, it is not clear whether the homogeneity is a necessary condition for strong synchronizability. Note that, for a fixed degree sequence the present algorithm with edge-intercrossing operation is ergodic, thus the less efficiency of the present algorithm on scale-free networks indicate that the homogeneity should be a necessary condition. That is to say, within the framework of synchronization of identical oscillators with uniform coupling mode, the heterogeneous network is hard to synchronize in despite of its idiographic structure.

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