ON THE DEPTH AND STANLEY DEPTH OF INTEGRAL CLOSURE OF POWERS OF MONOMIAL IDEALS

S. A. SEYED FAKHARI

Abstract. Let \( K \) be a field and \( S = K[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables over \( K \). Assume that \( G \) is a graph with edge ideal \( I(G) \). We prove that the modules \( S/I(G)^k \) and \( I(G)^k/I(G)^{k+1} \) satisfy Stanley’s inequality for every integer \( k \gg 0 \). If \( G \) is a non-bipartite graph, we show that the ideals \( I(G)^k \) satisfy Stanley’s inequality for all \( k \gg 0 \). For every connected bipartite graph \( G \) (with at least one edge), we prove that \( \text{sdepth}(I(G)^k) \geq 2 \), for any positive integer \( k \leq \text{girth}(G)/2 + 1 \). This result partially answers a question asked in [20]. For any proper monomial ideal \( I \) of \( S \), it is shown that the sequence \( \{ \text{depth}(S/I^m) \} \) is convergent and \( \lim_{m \to \infty} \text{depth}(S/I^m) = n - \ell(I) \), where \( \ell(I) \) denotes the analytic spread of \( I \). Furthermore, it is proved that for any monomial ideal \( I \), there exists an integer \( s \) such that

\[
\text{depth}(S/I^m) \leq \text{depth}(S/I),
\]

for every integer \( m \geq 1 \). We also determine a value \( s \) for which the above inequality holds. If \( I \) is an integrally closed ideal, we show that \( \text{depth}(S/I^m) \leq \text{depth}(S/I) \), for every integer \( m \geq 1 \). As a consequence, we obtain that for any integrally closed monomial ideal \( I \) and any integer \( m \geq 1 \), we have \( \text{Ass}(S/I) \subseteq \text{Ass}(S/I^m) \).

1. Introduction

Let \( K \) be a field and let \( S = K[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables over \( K \). Let \( M \) be a finitely generated \( \mathbb{Z}^n \)-graded \( S \)-module. Let \( u \in M \) be a homogeneous element and \( Z \subseteq \{ x_1, \ldots, x_n \} \). The \( K \)-subspace \( uK[Z] \) generated by all elements \( uv \), with \( v \) a monomial in \( K[Z] \), is called a Stanley space of dimension \( |Z| \), if it is a free \( K[Z] \)-module. Here, as usual, \( |Z| \) denotes the number of elements of \( Z \). A decomposition \( D \) of \( M \) as a finite direct sum of Stanley spaces is called a Stanley decomposition of \( M \). The minimum dimension of a Stanley space in \( D \) is called the Stanley depth of \( D \) and is denoted by \( \text{sdepth}(D) \). The quantity

\[
\text{sdepth}(M) := \max \{ \text{sdepth}(D) \mid D \text{ is a Stanley decomposition of } M \}
\]

is called the Stanley depth of \( M \). As a convention, we set \( \text{sdepth}(M) = \infty \), when \( M \) is the zero module. We say that a \( \mathbb{Z}^n \)-graded \( S \)-module \( M \) satisfies Stanley’s inequality if

\[
\text{depth}(M) \leq \text{sdepth}(M).
\]

2000 Mathematics Subject Classification. 13C15, 05E40, 13B22.

Key words and phrases. Depth, Edge ideal, Integral closure, Stanley depth, Stanley’s inequality.
In fact, Stanley [24] conjectured that every \( \mathbb{Z}^n \)-graded \( S \)-module satisfies Stanley’s inequality. For a reader friendly introduction to Stanley depth, we refer to [15] and for a nice survey on this topic, we refer to [7].

The Stanley’s conjecture has been recently disproved in [6]. The counterexample presented in [6] lives in the category of squarefree monomial ideals. Thus, one can still ask whether Stanley’s inequality holds for non-squarefree monomial ideals. Based on this observation, in [21, Question 1.1], we asked whether the high powers of any monomial ideal satisfy Stanley’s inequality. More explicit, we proposed the following question.

**Question 1.1.** ([21, Question 1.1]) Let \( I \) be a monomial ideal. Is it true that \( I^k \) and \( S/I^k \) satisfy Stanley’s inequality for every integer \( k \gg 0 \)?

This question was investigated for edge ideals in [2], [16] and [20] (see Section 2 for the definition of edge ideals). The most general results are obtained in [20]. In that paper, we proved that if \( G \) is a graph with \( n \) vertices and \( I(G) \) is its edge ideal, then \( S/I(G)^k \) satisfies Stanley’s inequality for every integer \( k \geq n - 1 \) [20, Corollary 2.5]. If moreover \( G \) is a non-bipartite graph, or at least one of the connected components of \( G \) is a tree with at least one edge, then \( I(G)^k \) satisfies Stanley’s inequality for every integer \( k \geq n - 1 \) [20, Corollary 3.6]. Also, in [19], we showed that Question 1.1 has the positive answer when \( I \) is the cover ideal of a bipartite graph.

In this paper, we ask whether the answer of Question 1.1 is positive if one replaces \( I^k \) by its integral closure. In other words, we pose the following question.

**Question 1.2.** Let \( I \) be a monomial ideal. Is it true that \( \overline{I^k} \) and \( S/\overline{I^k} \) satisfy Stanley’s inequality for every integer \( k \gg 0 \)?

Hoa and Trung [12, Lemma 1.5], prove that for every monomial ideal \( I \), we have \( \lim_{k \to \infty} \text{depth}(S/I^k) = n - \ell(I) \), where \( \ell(I) \) denotes the analytic spread of \( I \). Thus, Question 1.2 is equivalent to the following question.

**Question 1.3.** Let \( I \) be a monomial ideal. Is it true that \( \text{sdepth}(\overline{I^k}) \geq n - \ell(I) + 1 \) and \( \text{sdepth}(S/\overline{I^k}) \geq n - \ell(I) \), for every integer \( k \gg 0 \)?

In Section 3, we study this question for edge ideals of graphs. Note that for any graph \( G \), we have \( \ell(I(G)) = n - p \), where \( n \) is the number of vertices and \( p \) is the number of bipartite connected components of \( G \) (see e.g. [26, Page 50]). Before stating our results, we mention that Stanley depth of integral closure of powers of monomial ideals was studied in [17]. One of the main results of that paper asserts that if \( I_2 \subseteq I_1 \) are two monomial ideals, then there exists an integer \( s \geq 1 \), such that for every \( m \geq 1 \),

\[
\text{sdepth}(I_1^m/I_2^m) \leq \text{sdepth}(\overline{I_1}/\overline{I_2})
\]

(see Lemma 3.1). This inequality has a crucial role in this paper. As a consequence of this inequality, we will show in Theorem 3.2 that for any edge ideal \( I = I(G) \), the module \( S/\overline{I^k} \) satisfies Stanley’s inequality, for \( k \gg 0 \). We also, prove that if \( G \) is a
non-bipartite graph, then \( \overline{I(G)^k} \) satisfies Stanley’s inequality for every integer \( k \gg 0 \) (see Theorem 3.3).

Assume that \( G \) is a bipartite graph. By \cite{8} Theorem 1.4.6 and Corollary 10.3.17, we know that \( I(G) \) is a normal ideal. Thus, \( \overline{I(G)^k} \) satisfies Stanley’s inequality if and only if \( I(G)^k \) satisfies the Stanley’s inequality. We do not know whether for a bipartite graph \( G \), the ideal \( I(G)^k \) satisfies the Stanley’s inequality, for any integer \( k \gg 0 \). However, in \cite{20}, we noticed that it is sufficient to consider connected bipartite graphs. Indeed, we proved that \( I(G)^k \) satisfies the Stanley’s inequality, for every bipartite graph \( G \) and for any integer \( k \gg 0 \), provided that the answer of the following question is positive.

**Question 1.4.** \((\cite{20}, \text{Question 3.3})\) Let \( G \) be a connected bipartite graph (with at least one edge) and suppose \( k \geq 1 \) is an integer. Is it true that \( \text{sdepth}(I(G)^k) \geq 2 \)?

In \cite{20} Proposition 3.4, we showed that the answer of Question 1.4 is positive when \( G \) is a tree. In Theorem 3.4 we extend this result, by proving that for any connected bipartite graph \( G \) and every integer \( k \leq \text{girth}(G)/2 + 1 \), we have \( \text{sdepth}(I(G)^k) \geq 2 \). Assume that \( G \) is a (not necessarily connected) bipartite graph with at least one edge and let \( g \) be the maximum girth of the connected components of \( G \). As a consequence of Theorem 3.4, we conclude that for every integer \( k \leq g/2 + 1 \),

\[
\text{sdepth}(I(G)^k) \geq n - \ell(I(G)) = n - p,
\]

where \( n \) is the number of vertices and \( p \) is the number of connected components of \( G \) (see Corollary 3.5).

After studying the Stanley depth of \( \overline{I^k} \) and \( S/I^k \), we consider the modules of the form \( \overline{I^k}/I^{k+1} \). In order to determine whether \( \overline{I^k}/I^{k+1} \) satisfies Stanley’s inequality for \( k \gg 0 \), we need to know the asymptotic behavior of depth of these modules. We know from \cite{9} Theorem 1.2 that for every proper monomial ideal \( I \) of \( S \), the sequence \( \{\text{depth}(I^k/I^{k+1})\} \) is convergent and

\[
\lim_{k \to \infty} \text{depth}(S/I^k) = \lim_{k \to \infty} \text{depth}(I^k/I^{k+1}).
\]

In Theorem 4.1 we show that the same holds if one replaces the powers of \( I \) by their integral closure. In other words, the sequence \( \{\text{depth}(\overline{I^k}/I^{k+1})\} \) is convergent and moreover,

\[
\lim_{k \to \infty} \text{depth}(S/\overline{I^k}) = \lim_{k \to \infty} \text{depth}(\overline{I^k}/I^{k+1}).
\]

As mentioned above, by the result of Hoa and Trung \cite{12}, Lemma 1.5, we know that \( \lim_{k \to \infty} \text{depth}(S/\overline{I^k}) = n - \ell(I) \). Thus, in order to prove that \( \overline{I^k}/I^{k+1} \) satisfies Stanley’s inequality for every integer \( k \gg 0 \), we must show \( \text{sdepth}(\overline{I^k}/I^{k+1}) \geq n - \ell(I) \). We prove this for edge ideals in Theorem 3.7. We mention that the proof of Theorem 3.7 is also based on Lemma 3.1.

As we mentioned above, Lemma 3.1 has a crucial role in Section 3. As a particular case of this lemma, for every monomial ideal \( I \subseteq S \), there exists an integer \( s \geq 1 \) with
the property that
\[ \text{sdepth}(S/I^m) \leq \text{sdepth}(S/\mathcal{T}). \]

It is reasonable to ask whether this inequality is true, if one replaces sdepth by depth. In Theorem 4.5, we give a positive answer to this question and even more, we show that one can choose \( s \) to be \( \mu(I^\ell(I) - 1)! \), where for every monomial ideal \( J \), we denote by \( \mu(J) \) the number of minimal monomial generators of \( J \). The proof of Theorem 4.5 is based on a formula due to Takayama [25, Theorem 2.2] which is a generalization of the so-called Hochster’s formula and relates the local cohomology modules of a (non-squarefree) monomial ideal to reduced homologies of particular simplicial complexes.

Finally, assume that \( I \) is a squarefree monomial ideal. We know from the proof of [11, Theorem 2.6] that
\[ \text{depth}(S/I^m) \leq \text{depth}(S/I), \]
for every integer \( m \geq 1 \). In Theorem 4.6, we extend this inequality to the class integrally closed monomial ideals. As a consequence, we obtain that for any integrally closed monomial ideal and every integer \( m \geq 1 \), we have
\[ \text{Ass}(S/I) \subseteq \text{Ass}(S/I^m) \]
(see Corollary 4.7).

2. Preliminaries

In this section, we provide the definitions and basic facts which will be used in the next sections.

2.1. Notions from commutative algebra. Let \( \mathbb{K} \) be a field and \( S = \mathbb{K}[x_1, x_2, \ldots, x_n] \) be the polynomial ring in \( n \) variables over \( \mathbb{K} \). Assume that \( I \subset S \) is an arbitrary ideal. An element \( f \in S \) is integral over \( I \), if there exists an equation
\[ f^k + c_1 f^{k-1} + \ldots + c_k = 0 \]
with \( c_i \in I^i \).

The set of elements \( \mathcal{T} \) in \( S \) which are integral over \( I \) is the integral closure of \( I \). It is known that the integral closure of a monomial ideal \( I \subset S \) is a monomial ideal generated by all monomials \( u \in S \) for which there exists an integer \( k \) such that \( u^k \in I^k \) (see [8, Theorem 1.4.2]). The ideal \( I \) is integrally closed, if \( I = \mathcal{T} \), and \( I \) is normal if all powers of \( I \) are integrally closed. By [27, Theorem 3.3.18], a monomial ideal \( I \) is normal if and only if the Rees algebra \( \mathcal{R}(I) = S[IT] = \bigoplus_{n=0}^{\infty} I^n \) is a normal ring.

Let \( I \subset S \) be a monomial ideal. A classical result by Burch [5] states that
\[ \min_k \text{depth}(S/I^k) \leq n - \ell(I), \]
where \( \ell(I) \) is the analytic spread of \( I \), that is, the dimension of \( \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I) \), where \( \mathfrak{m} = (x_1, \ldots, x_n) \) is the maximal ideal of \( S \). By a theorem of Brodmann [3], depth \( (S/I^k) \) is constant for large \( k \). We call this constant value the limit depth of \( I \), and denote it by \( \lim_{k \to \infty} \text{depth}(S/I^k) \). Brodmann improved the Burch’s inequality by showing that
\[ \lim_{k \to \infty} \text{depth}(S/I^k) \leq n - \ell(I). \]
It is well-known [8 Proposition 10.3.2] that the equality occurs in the above inequality, if \( I \) is a normal ideal. As mention in introduction, recently, Hoa and Trung [12, Lemma 1.5] proved that for every monomial ideal \( I \),

\[
\lim_{k \to \infty} \text{depth}(S/I^k) = n - \ell(I).
\]

Let \( I \subseteq S \) be a monomial ideal. The set of minimal monomial generators of \( I \) is denoted by \( G(I) \) and we set \( \mu(I) := |G(I)| \). We also denote the set of associated primes of \( S/I \), by \( \text{Ass}(S/I) \). The associated graded ring of \( S \) with respect to \( I \) will be denoted by \( \text{gr}_I(S) \) and it is defined as \( \text{gr}_I(S) = \bigoplus_{k=0}^{\infty} I^k/I^{k+1} \).

2.2. Notions from combinatorics. Let \( G \) be a simple graph with vertex set \( V(G) = \{v_1, \ldots, v_n\} \) and edge set \( E(G) \). For every vertex \( v_i \in V(G) \), we denote by \( G \setminus v_i \), the graph with vertex set \( V(G \setminus v_i) = V(G) \setminus \{v_i\} \) edge set \( E(G \setminus v_i) = \{e \in E(G) \mid v_i \notin e\} \).

A tree is a connected graph which has no cycle. The girth of \( G \), denoted by \( \text{girth}(G) \) is the length of the shortest cycle in \( G \). We set \( \text{girth}(G) = \infty \), if \( G \) has no cycle. The graph \( G \) is bipartite if there exists a partition \( V(G) = U_1 \cup U_2 \) with \( U_1 \cap U_2 = \emptyset \) such that each edge of \( G \) is of the form \( \{v_i, v_j\} \) with \( v_i \in U_1 \) and \( v_j \in U_2 \). A subset \( A \) of \( V(G) \) is called an independent subset of \( G \) if there are no edges among the vertices of \( A \).

A simplicial complex \( \Delta \) on the set of vertices \( [n] := \{1, \ldots, n\} \) is a collection of subsets of \( [n] \) which is closed under taking subsets; that is, if \( F \in \Delta \) and \( F' \subseteq F \), then also \( F' \in \Delta \). By \( \tilde{H}_i(\Delta; \mathbb{K}) \), we mean the \( i \)th reduced homology of \( \Delta \) with coefficients \( \mathbb{K} \).

The independence simplicial complex of a graph \( G \) is defined by

\[
\Delta_G = \{A \subseteq V(G) \mid A \text{ is an independent set in } G\},
\]

and it is an important object in combinatorial commutative algebra.

2.3. Notions from combinatorial commutative algebra. One of the connections between the combinatorics and commutative algebra is via rings constructed from the combinatorial objects. Let \( \Delta \) be a simplicial complex on \( [n] \). For every subset \( F \subseteq [n] \), we set \( x_F = \prod_{i \in F} x_i \). The Stanley–Reisner ideal of \( \Delta \) over \( \mathbb{K} \) is the ideal \( I_\Delta \) of \( S \) which is generated by those squarefree monomials \( x_F \) with \( F \notin \Delta \). In other words, \( I_\Delta = (x_F \mid F \notin \Delta) \).

There is a natural correspondence between quadratic squarefree monomial ideals of \( S \) and finite simple graphs with \( n \) vertices. To every simple graph \( G \) with vertex set \( V(G) = \{v_1, \ldots, v_n\} \) and edge set \( E(G) \), one associates its edge ideal \( I(G) \) defined by

\[
I(G) = (x_i x_j : \{v_i, v_j\} \in E(G)) \subseteq S.
\]

On can easily check that \( I(G) = I_{\Delta_G} \). It is well-known that for any graph \( G \), we have \( \ell(I(G)) = n - p \), where \( n \) is the number of vertices and \( p \) is the number of bipartite connected components of \( G \) (see e.g. [26 Page 50]).
3. Stanley depth of integral closure of powers of edge ideals

In this section, we study the Stanley depth of integral closure of powers of edge ideals and their quotients. In [20], we proved that for every graph $G$ the modules $S/I(G)^k$ and $I(G)^k/I(G)^{k+1}$ satisfy Stanley’s inequality for every integer $k \gg 0$. In the same paper, we also proved that for any non-bipartite graph $G$, the ideal $I(G)^k$ satisfies Stanley’s inequality for every $k \gg 0$. In this section, we prove all these results are true, if one replaces the powers of $I(G)$ by their integral closure. The following lemma from [17] has a key role in this section.

**Lemma 3.1.** ([17, Theorem 2.8]) Let $I_2 \subseteq I_1$ be two monomial ideals in $S$. Then there exists an integer $s \geq 1$, such that for every $m \geq 1$,

$$\text{sdepth}(I_1^{sm}/I_2^{sm}) \leq \text{sdepth}(I_1/I_2).$$

The following two theorems are the first main results of this section and they follow from Lemma 3.1 and the results of [20].

**Theorem 3.2.** Let $G$ be a graph with edge ideal $I = I(G)$. Suppose that $p$ is the number of bipartite connected components of $G$. Then for every integer $k \geq 1$, we have $\text{sdepth}(S/T^k) \geq p$. In particular, $S/T^k$ satisfies Stanley’s inequality for every integer $k \gg 0$.

**Proof.** By Lemma 3.1, there exists an integer $s \geq 1$,

$$\text{sdepth}(S/I^k) \geq \text{sdepth}(S/I^{ks}).$$

On the other hand, it follows from [20, Theorem 2.3] that $\text{sdepth}(S/I^{ks}) \geq p$. This proves the first assertion. The last statement follows from [12, Lemma 1.5], together with the fact that $\ell(I) = n - p$. □

**Theorem 3.3.** Let $G$ be a non-bipartite graph with edge ideal $I = I(G)$. Suppose that $p$ is the number of bipartite connected components of $G$. Then for every integer $k \geq 1$, we have $\text{sdepth}(T^k) \geq p + 1$. In particular, $T^k$ satisfies Stanley’s inequality for every integer $k \gg 0$.

**Proof.** The proof is similar to the proof Theorem 3.2. The only difference is that one should use [20, Corollary 3.2] instead of [20, Theorem 2.3]. □

Assume that $G$ is a bipartite graph. By [8, Theorem 1.4.6 and Corollary 10.3.17], we know that $I(G)$ is a normal ideal. Thus, the study of the Stanley depth of $I(G)^k$ is nothing other than that of $I(G)^k$. We do not know whether for a bipartite graph $G$, the ideal $I(G)^k$ satisfies Stanley’s inequality, for any integer $k \gg 0$. However, we proved in [20, Corollary 3.6] that $I(G)^k$ satisfies Stanley’s conjecture for any integer $k \gg 0$, provided that $G$ has a connected component which is a tree (with at least one edge). In the same paper, we also proposed Question 1.4 and proved that $I(G)^k$ satisfies Stanley’s inequality, for every bipartite graph $G$ and for every integer $k \gg 0$, provided that the answer of Question 1.4 is positive. In [20, Proposition 3.4], we
gave a positive answer to this question in the case \( G \) is a tree. This result will be generalized in the following theorem.

**Theorem 3.4.** Let \( G \) be a connected bipartite graph (with at least one edge) and suppose that \( \text{girth}(G) = g \). Then for every positive integer \( k \leq g/2 + 1 \), we have \( \text{sdepth}(I(G)^k) \geq 2 \).

**Proof.** If \( g = \infty \), i.e., if \( G \) is a tree, the assertion follows from \([20]\) Proposition 3.4. Thus, assume that \( g \) is finite. As \( G \) is a bipartite graph, \( g \) is an even integer. Assume that \( g = 2r \) and let \( k \) be a positive integer with \( k \leq r + 1 \). We must prove that \( \text{sdepth}(I(G)^k) \geq 2 \). For \( k = 1 \), the desired inequality follows from \([18]\) Corollary 3.4. Thus, assume that \( k \geq 2 \). We use induction on the number of vertices of \( G \), say \( n \). Let \( C \) be a cycle of \( G \) of length \( g = 2r \). Without lose of generality, we assume that \( V(C) = \{v_1, \ldots, v_{2r}\} \) and

\[
E(C) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{2r-1}, v_{2r}\}, \{v_1, v_{2r}\}\}.
\]

Let \( S_1 = \mathbb{K}[x_2, \ldots, x_n] \) be the polynomial ring obtained from \( S \) by deleting the variable \( x_1 \) and consider the ideals \( I_1 = I(G)^k \cap S_1 \) and \( I'_1 = (I(G)^k : x_1) \).

Now \( I(G)^k = I_1 \oplus x_1 I'_1 \) (as vector spaces) and therefore by definition of the Stanley depth we have

\[
(1) \quad \text{sdepth}(I(G)^k) \geq \min\{\text{sdepth}_{S_1}(I_1), \text{sdepth}_{S}(I'_1)\}.
\]

Notice that \( I_1 = I(G \setminus v_1)^k \). Since

\[
k \leq \frac{\text{girth}(G)}{2} + 1 \leq \frac{\text{girth}(G \setminus v_1)}{2} + 1,
\]

the induction hypothesis implies that \( \text{sdepth}_{S_1}(I_1) \geq 2 \). Thus, using the inequality \((1)\), it is enough to prove that \( \text{sdepth}_{S}(I'_1) \geq 2 \).

For every integer \( i \) with \( 2 \leq i \leq 2k - 2 \), let \( S_i = \mathbb{K}[x_1, \ldots, x_i, x_{i+1}, \ldots, x_n] \) be the polynomial ring obtained from \( S \) by deleting the variable \( x_i \) and consider the ideals \( I'_i = (I'_{i-1} : x_i) \) and \( I_i = I'_{i-1} \cap S_i \).

**Claim.** For every integer \( i \) with \( 1 \leq i \leq 2k - 3 \) we have

\[
\text{sdepth}(I'_i) \geq \min\{2, \text{sdepth}(I'_{i+1})\}.
\]

**Proof of the Claim.** For every integer \( i \) with \( 1 \leq i \leq 2k - 3 \), we have \( I'_i = I_{i+1} \oplus x_{i+1} I'_{i+1} \) (as vector spaces) and therefore by definition of the Stanley depth we have

\[
(2) \quad \text{sdepth}(I'_i) \geq \min\{\text{sdepth}_{S_{i+1}}(I_{i+1}), \text{sdepth}_{S}(I'_{i+1})\},
\]

Notice that for every integer \( i \) with \( 1 \leq i \leq 2k - 3 \), we have \( I'_i = (I(G)^k : x_1 x_2 \ldots x_i) \).

Thus

\[
I_{i+1} = I'_i \cap S_{i+1} = ((I(G)^k \cap S_{i+1}) : S_{i+1} x_1 x_2 \ldots x_i).
\]

Hence, using \([14]\) Proposition 2 (see also \([22]\) Proposition 2.5), we conclude that

\[
(3) \quad \text{sdepth}_{S_{i+1}}(I_{i+1}) \geq \text{sdepth}_{S_{i+1}}(I(G)^k \cap S_{i+1}).
\]
Note that $I(G)^k \cap S_{i+1} = I(G \setminus v_i)^k$. Since $$k \leq \frac{\text{girth}(G)}{2} + 1 \leq \frac{\text{girth}(G \setminus v_i)}{2} + 1,$$
the induction hypothesis implies that $\text{sdepth}_{S_{i+1}}(I(G)^k \cap S_{i+1}) \geq 2$. Hence, the claim follows by inequalities (2), and (3).

It is clear that $I_{2k-2}' = (I(G)^k : x_1 x_2 \ldots x_{2k-2})$. Thus, by [11 Proposition 3.2], there exists a bipartite graph $G'$ with $V(G') = V(G)$ such that $I(G') = (I(G)^k : x_1 x_2 \ldots x_{2k-2})$. Therefore, [13 Corollary 3.4] implies that $$\text{sdepth}(I_{2k-2}') = \text{sdepth}((I(G)^k : x_1 x_2 \ldots x_{2k-2})) = \text{sdepth}(I(G')) \geq 2.$$ Therefore, using the claim repeatedly, we conclude that $\text{sdepth}(I_1') \geq 2$. This completes the proof of the theorem.

As a consequence of Theorem 3.4, we obtain the following corollary.

**Corollary 3.5.** Let $G$ be a bipartite graph with at least one edge. Suppose that $p$ is the number of connected components of $G$ and assume that $g$ is the maximum girth of connected components of $G$. Then for every positive integer $k \leq g/2 + 1$, we have $\text{sdepth}(I(G)^k) \geq p + 1$.

**Proof.** Let $H$ be a connected component of $G$ with $\text{girth}(H) = g$ and set $S' = \mathbb{K}[x_i \mid v_i \in V(H)]$. It follows from [20 Theorem 3.1] and Theorem 3.4 that for every positive integer $k \leq g/2 + 1$, $$\text{sdepth}(I(G)^k) \geq \min_{1 \leq i \leq k} \{\text{sdepth}_{S'}(I(H)^i)\} + p - 1 \geq p + 1.$$ 

Let $G$ be an arbitrary graph. Our next goal in this section is to study the Stanley depth of the modules in the form $I(G)^k / I(G)^{k+1}$. We will see in Corollary 3.8 that these modules satisfy Stanley’s inequality for every integer $k \gg 0$. The proof of this result is also based on Lemma 3.1. However, we first need the following lemma.

**Lemma 3.6.** Let $G$ be a graph with edge ideal $I = I(G)$. Suppose that $p$ is the number of bipartite connected components of $G$. Then for every pair of integers $s > t \geq 0$, we have $\text{sdepth}(I^t / I^s) \geq p$.

**Proof.** Note that $$I^t / I^s = \bigoplus_{k=t}^{s-1} I^k / I^{k+1}.$$ By the definition of Stanley depth we conclude that $$\text{sdepth}(I^t / I^s) \geq \min \{\text{sdepth}(I^k / I^{k+1}) \mid k = t, \ldots, s - 1\} \geq p,$$ where the last inequality follows from [20 Theorem 2.2].

In the next theorem we will show that the number of bipartite connected components of $G$ is a lower bound for the Stanley depth of $I(G)^k / I(G)^{k+1}$. 


Theorem 3.7. Let $G$ be a graph with edge ideal $I = I(G)$. Suppose that $p$ is the number of bipartite connected components of $G$. Then for every integer $k \geq 0$, we have $\text{sdepth}(I^k/I^{k+1}) \geq p$.

Proof. By Lemma 3.1, for every integer $k \geq 0$, there exists an integer $s \geq 1$ such that $\text{sdepth}(I^k/I^{k+1}) \geq \text{sdepth}(I^{sk}/I^{s(k+1)})$, (we set $m = 1$ in Lemma 3.1). Thus, Lemma 3.6 implies that $\text{sdepth}(I^k/I^{k+1}) \geq p$. □

In Corollary 4.2, we will prove that for any graph $G$ with $p$ bipartite connected components, $\lim_{k \to \infty} \text{depth}(I^k/I^{k+1}) = p$. Thus, as a consequence of Theorem 3.7, we obtain the following result.

Corollary 3.8. Let $G$ be a graph with edge ideal $I = I(G)$. Then $I^k/I^{k+1}$ satisfies Stanley’s inequality, for every integer $k \gg 0$.

4. Depth of integral closure of powers of monomial ideals

In this section, we study the depth of the integral closure of powers of monomial ideals. As we promised in Section 3, our first goal is to prove that for every graph $G$ with $p$ bipartite connected components, $\lim_{k \to \infty} \text{depth}(I^k/I^{k+1}) = p$.

In fact, we prove a more general result in Theorem 4.1. We show that for any monomial ideal $I \subset S$, the sequence $\{\text{depth}(I^k/I^{k+1})\}_{k=0}^{\infty}$ is convergent and $\lim_{k \to \infty} \text{depth}(I^k/I^{k+1}) = n - \ell(I)$.

Theorem 4.1. For any nonzero monomial ideal $I \subset S$, the sequence $\{\text{depth}(I^k/I^{k+1})\}_{k=0}^{\infty}$ is convergent and moreover, $\lim_{k \to \infty} \text{depth}(I^k/I^{k+1}) = n - \ell(I)$.

Proof. Note that $A = \text{gr}_I(S) = \bigoplus_{k=0}^{\infty} I^k/I^{k+1}$ is a finitely generated standard graded $S$-algebra. By [13, Proposition 5.3.4], there exists an integer $s \geq 1$ such that for all $k \geq s$ we have $I^k = I^{k-s}T_s$. This shows that $E = \bigoplus_{k=0}^{\infty} I^k/I^{k+1}$ is a finitely generated graded $A$-module. Hence, [2, Theorem 1.1] implies that the sequence $\{\text{depth}(I^k/I^{k+1})\}_{k=0}^{\infty}$ is convergent.

Let $k_0 \geq 1$ be an integer with the property that for every $k \geq k_0$ we have $
\text{depth}(I^k/I^{k+1}) = \lim_{k \to \infty} \text{depth}(I^k/I^{k+1}).$

As mentioned above, there exists an integer $s \geq 1$ such that for all $k \geq s$ we have $I^k = I^{k-s}T_s$. In particular, for every integer $k \geq 1$, we have $T_s^k \subseteq T_s^{k-s} = (I^{k-s})^s \subseteq (I^s)^{k-s} T_s = (T_s)^k$. 
Hence, $(\overline{T^s})^k = \overline{T^{ks}}$, for every integer $k \geq 1$. Let $k \geq k_0$ be an integer. For every integer $i$ with $ks \leq i \leq (k + 1)s - 2$, consider the following exact sequence.

$$0 \rightarrow \overline{T^{i+1}/T^{(k+1)s}} \rightarrow \overline{T^i/T^{(k+1)s}} \rightarrow \overline{T^i/T^{i+1}} \rightarrow 0$$

Applying the depth Lemma [4, Proposition 1.2.9] on the above exact sequence, we obtain that

$$\text{depth}(\overline{T^i/T^{i+1}}) \geq \min\{\text{depth}(\overline{T^{i+1}/T^{(k+1)s}}), \text{depth}(\overline{T^i/T^{i+1}})\}.$$  

Using this inequality repeatedly, we conclude that

$$\text{depth}(\overline{T^{ks}/T^{(k+1)s}}) \geq \min\{\text{depth}(\overline{T^{ks}/T^{(k+1)s}}) : i = ks, \ldots, (k + 1)s - 1\} = \lim_{k \rightarrow \infty} \text{depth}(\overline{T^k/T^{k+1}}),$$

where the last equality follows from the choice of $k$. Thus, we have

$$\lim_{k \rightarrow \infty} \text{depth}(\overline{T^{ks}/T^{k+1}}) = \lim_{k \rightarrow \infty} \text{depth}(\overline{T^{ks}/T^{(k+1)s}}) \geq \lim_{k \rightarrow \infty} \text{depth}(\overline{T^k/T^{k+1}}).$$

By [9, Theorem 1.2],

$$\lim_{k \rightarrow \infty} \text{depth}(\overline{T^{k}/T^{k+1}}) = n - \ell(\overline{T^s}),$$

and since $\ell(\overline{T^s}) = \ell(I^s) = \ell(I)$, we conclude that

$$\lim_{k \rightarrow \infty} \text{depth}(\overline{T^{k}/T^{k+1}}) \leq \lim_{k \rightarrow \infty} \text{depth}(\overline{T^{s^k}/T^{s^{k+1}}}) = n - \ell(I).$$

We now prove that $\lim_{k \rightarrow \infty} \text{depth}(\overline{T^{k}/T^{k+1}}) \geq n - \ell(I)$.

Consider the exact sequence

$$0 \rightarrow \overline{T^k/T^{k+1}} \rightarrow S/\overline{T^{k+1}} \rightarrow S/\overline{T^k} \rightarrow 0.$$

By [12, Lemma 1.5], there exists an integer $k_1 \geq 1$ such that for every $k \geq k_1$, we have $\text{depth}(S/\overline{T^k}) = n - \ell(I)$. Thus, applying the depth Lemma [4, Proposition 1.2.9] on the above exact sequence, we conclude that $\text{depth}(\overline{T^k/T^{k+1}}) \geq n - \ell(I)$, for every integer $k \geq k_1$. This completes the proof. \hfill \Box

Restricting to edge ideals, we obtain the following corollary.

**Corollary 4.2.** For any graph $G$, the sequence $\{\text{depth}(\overline{(I(G))^k/(I(G))^{k+1}})\}_{k=0}^{\infty}$ is convergent and

$$\lim_{k \rightarrow \infty} \text{depth}(\overline{(I(G))^k/(I(G))^{k+1}}) = p,$$

where $p$ is the number of bipartite connected components of $G$.

As we saw in Section 3, Lemma 3.1 has a key role in the proof of Theorems 3.2, 3.3 and 3.7. As a particular case of this lemma, for every monomial ideal $I$, there exists an integer $s \geq 1$ with the property that

$$\text{sdepth}(S/I^{sm}) \leq \text{sdepth}(S/I).$$
It is reasonable to ask whether this inequality is true, if one replaces sdepth by depth. In Theorem 4.3, we show this is the case. Our proof is base on a formula due to Takayama \cite{25}, which is presented as follows.

Let $I$ be a monomial ideal. As $S/I$ is a $\mathbb{Z}^n$-graded $S$-module, it follows that for every integer $i$, the local cohomology module $H^i_m(S/I)$ is $\mathbb{Z}^n$-graded too. For any vector $\alpha \in \mathbb{Z}^n$, we denote the $\alpha$-component of $H^i_m(S/I)$ by $H^i_m(S/I)_\alpha$. The co-support of the vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ is defined to be the set $\text{CS}(\alpha) = \{ i : \alpha_i < 0 \}$. For any subset $F \subseteq [n]$, let $S_F = S[x_i^{-1} : i \in F]$. Suppose that $\Delta(I)$ is the simplicial complex over $[n]$ with Stanley-Reisner ideal $I_{\Delta(I)} = \sqrt{I}$. For any vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we set

$$\Delta_\alpha(I) = \{ F \subseteq [n] \setminus \text{CS}(\alpha) : x^\alpha \notin IS_{F \cup \text{CS}(\alpha)} \},$$

where $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$. Takayama \cite{25} Theorem 2.2 proves that for every vector $\alpha \in \mathbb{Z}^n$ and for every integer $i$, we have

$$\dim F H^i_m(S/I)_\alpha = \dim F \tilde{H}^{t-|\text{CS}(\alpha)|-1}_m(\Delta_\alpha(I); \mathbb{K}).$$

Using this formula, we are able to prove the following proposition.

**Proposition 4.3.** Let $I$ be a monomial ideal in $S$. Assume that $s \geq 1$ is an integer with property the for every monomial $u \in T$, we have $u^s \in I^s$. Then for every $m \geq 1$,

$$\text{depth}(S/I^s \Delta(I)) \leq \text{depth}(S/I).$$

**Proof.** Set $t = \text{depth}(S/T)$. It follows that there exists a vector $\alpha \in \mathbb{Z}^n$ such that $H^t_m(S/T)_\alpha \neq 0$. Thus, equality (1), implies that

$$\tilde{H}^{t-|\text{CS}(\alpha)|-1}_m(\Delta_\alpha(T); \mathbb{K}) \neq 0.$$

Now it follows from the assumption that for every integer $m \geq 1$ and every monomial $u \in S$, we have $u \in T$ if and only if $u^s \in I^s$. We conclude that

$$\Delta_\alpha(T) = \{ F \subseteq [n] \setminus \text{CS}(\alpha) : x^\alpha \notin TS_{F \cup \text{CS}(\alpha)} \}$$

$$= \{ F \subseteq [n] \setminus \text{CS}(\alpha) : x^{s \alpha} \notin I^s S_{F \cup \text{CS}(\alpha)} \}$$

$$= \Delta_{s \alpha}(I^s),$$

where the last equality follows from the fact that $\text{CS}(s \alpha) = \text{CS}(\alpha)$. Therefore,

$$\tilde{H}^{t-|\text{CS}(s \alpha)|-1}_m(\Delta_{s \alpha}(I^s); \mathbb{K}) = \tilde{H}^{t-|\text{CS}(\alpha)|-1}_m(\Delta_\alpha(T); \mathbb{K}) \neq 0.$$

By equality (1), we deduce that $H^t_m(S/I^s)_{s \alpha} \neq 0$ and hence, $\text{depth}(S/I^s) \leq t$. \hfill \Box

Our next goal is determine an integer $s$ which satisfies the assumption of Proposition 4.3. The following lemma is the main step in this regard. The proof of this lemma is based on the determinantal trick which was suggested to us by Irena Swanson.

**Lemma 4.4.** Let $I$ be a monomial ideal with analytic spread $\ell = \ell(I)$. Assume that $u \in T$ is a monomial. Then there exists an integer $t \leq \mu(T^{t-1})$ with $u^t \in I^t$. 

Proof. Assume that \( G(\overline{T}^{-1}) = \{u_1, \ldots, u_m\} \) is the set of minimal monomial generators of \( \overline{T}^{-1} \). By [23, Theorem 5.1], there exists a monomial ideal \( J \subset I \), such that \( \overline{T} = J\overline{T}^{-1} \). In particular, \( \overline{T} \subseteq \overline{T}^{-1} \). Since \( u \in \overline{T} \), we conclude that \( u\overline{T}^{-1} \subseteq \overline{T} \subseteq \overline{T}^{-1} \).

As a consequence, for every integer \( i \) with \( 1 \leq i \leq m \), we have \( uu_i \in \overline{I}^{\ell-1} \). Thus, we may write \( uu_i = \sum_{j=1}^{m} c_{ij}u_j \), for some polynomial \( c_{ij} \in I \). Let \( A = (a_{ij})_{m \times m} \) be the matrix with \( a_{ij} = \delta_{ij}u - c_{ij} \), where \( \delta_{ij} \) denotes the Kronecker delta function. Set \( u = (u_1, \ldots, u_m)^T \). Then we have \( Au = 0 \). Multiplying by the adjoint of \( A \), we conclude that \( \det(A)u_i = 0 \), for every integer \( i \) with \( 1 \leq i \leq m \). This means that \( \det(A) = 0 \).

Obviously, \( \det(A) \) is a polynomial of the form

\[
\det(A) = u^m + f_1u^{m-1} + \cdots + f_{m-1}u + f_m,
\]

where for every integer \( i \) with \( 1 \leq i \leq m \), we have \( f_i \in \overline{I} \). In particular, we obtain that

\[
(5) \quad u^m + f_1u^{m-1} + \cdots + f_{m-1}u + f_m = 0.
\]

For every integer \( i \) with \( 1 \leq i \leq m \), let \( \alpha_i \in \mathbb{K} \) be the coefficient of \( u^i \) in the polynomial \( f_i \). Thus, equality (5) implies that

\[
u^m + \alpha_1u^m + \cdots + \alpha_{m-1}u^m + \alpha_mu^m = 0.\]

Hence, that there exists and integer \( t \) with \( 1 \leq t \leq m \) such that \( \alpha_t \neq 0 \). This means that \( u^t \) is one of the monomials appearing in the expansion of \( f_t \). Since \( f_t \in \overline{I} \) and \( \overline{I} \) is a monomial ideal, we conclude that \( u^t \in \overline{I} \). We also notice that \( t \leq m = \mu(\overline{T}^{-1}) \), which completes the proof. \( \square \)

As a consequence of Proposition 4.3 and Lemma 4.4, we obtain our next main result.

**Theorem 4.5.** Let \( I \subset S \) be a monomial ideal with analytic spread \( \ell = \ell(I) \). Set \( s = \mu(\overline{T}^{-1})! \). Then for every integer \( m \geq 1 \), we have

\[
\text{depth}(S/I^{sm}) \leq \text{depth}(S/\overline{T}).
\]

**Proof.** Let \( G(\overline{T}) = \{u_1, \ldots, u_r\} \) be the set of minimal monomial generators of \( \overline{T} \). By Lemma 4.4, for every \( 1 \leq i \leq r \), there exists integer \( k_i \leq \mu(\overline{T}^{-1}) \), such that \( u_i^{k_i} \in I^{k_i} \).

As \( k_i \) divides \( s \) for every \( i \), we conclude that \( u^s \in I^s \), for every monomial \( u \in \overline{T} \). The assertion now follows from Proposition 4.3. \( \square \)

Assume that \( I \) is a squarefree monomial ideal. We know from the proof of [11, Theorem 2.6] that

\[
\text{depth}(S/I^m) \leq \text{depth}(S/I),
\]

for every integer \( m \geq 1 \). In the following theorem, we show that the above inequality is true for any integrally closed monomial ideal.
Theorem 4.6. Let $I$ be an integrally closed monomial ideal in $S$. Then for every integer $m \geq 1$, we have
\[
\text{depth}(S/I^m) \leq \text{depth}(S/I).
\]

Proof. As $T = I$, for every $u \in T$ we have $u \in I$ and hence, Proposition 4.3 implies the assertion. \qed

Let $P = (x_{i_1}, \ldots, x_{i_r})$ be a monomial prime ideal in $S$, and $I \subseteq S$ any monomial ideal. Set $L = [n] \setminus \{x_{i_1}, \ldots, x_{i_r}\}$. We denote by $I(P)$ the monomial ideal in the polynomial ring $S(P) = \mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$, which is obtained from $I$ by applying the $\mathbb{K}$-algebra homomorphism $S \to S(P)$ defined by $x_i \mapsto 1$ for all $i \in L$ and $x_i \mapsto x_i$, otherwise. It is known that ([10, Lemma 1.3])
\[
\text{Ass}(S(P)/I(P)) = \{Q \in \text{Ass}(S/I) : x_i \notin Q \text{ for all } i \in L\}.
\]

Using this fact, we deduce the following result concerning the associated primes of powers of integrally closed monomial ideals.

Corollary 4.7. Let $I$ be an integrally closed monomial ideal in $S$. Then for every integer $m \geq 1$, we have
\[
\text{Ass}(S/I) \subseteq \text{Ass}(S/I^m).
\]

Proof. Let $P \in \text{Ass}(S/I)$ be a monomial prime ideal of $S$. Then by [10, Lemma 1.3], we have $P \in \text{Ass}(S(P)/I(P))$ and hence, $\text{depth}_{S(P)}(S(P)/I(P)) = 0$. It follows from [17, Lemma 3.1] that $I(P)$ is an integrally closed ideal and thus, Theorem 4.6 shows that $\text{depth}_{S(P)}(S(P)/I(P)^m) = 0$, for every integer $m \geq 1$. Therefore,
\[
P \in \text{Ass}(S(P)/I(P)^m) = \text{Ass}(S(P)/I^m(P)).
\]

Again [10, Lemma 1.3] implies that $P \in \text{Ass}(S/I^m)$. \qed

Acknowledgment

The author thanks Irena Swanson for suggesting determinantal trick in the proof of Lemma 4.4.

References

[1] A. Alilooee, A. Banerjee, Powers of edge ideals of regularity three bipartite graphs, J. Commut. Algebra, 9 (2017), 441–454.
[2] A. Alipour, S. A. Seyed Fakhari, S. Yassemi, Stanley depth of factor of polymatroidal ideals and edge ideal of forests, Arch. Math. (Basel), 105 (2015), 323–332.
[3] M. Brodmann, The asymptotic nature of the analytic spread, Math. Proc. Cambridge Philos. Soc. 86 (1979), no. 1, 35–39.
[4] W. Bruns, J. Herzog, Cohen–Macaulay Rings, Cambridge Studies in Advanced Mathematics, 39, Cambridge University Press, 1993.
[5] L. Burch, Codimension and analytic spread, Proc. Cambridge Philos. Soc. 72 (1972), 369–373.
[6] A. M. Duval, B. Goecckner, C. J. Klivans, J. L. Martin, A non-partitionable Cohen-Macaulay simplicial complex, Adv. Math. 299 (2016), 381–395.
[7] J. Herzog, A survey on Stanley depth. In ”Monomial Ideals, Computations and Applications”, A. Bigatti, P. Giménez, E. Sáenz-de-Cabezón (Eds.), Proceedings of MONICA 2011. Lecture Notes in Math. 2083, Springer (2013).
[8] J. Herzog, T. Hibi, Monomial Ideals, Springer-Verlag, 2011.
[9] J. Herzog, T. Hibi, The depth of powers of an ideal, J. Algebra 291 (2005), no. 2, 325–650.
[10] J. Herzog, A. Rauf, M. Vladoiu, The stable set of associated prime ideals of a polymatroidal ideal, J. Algebraic Combin., 37 (2013), 289–312.
[11] J. Herzog, Y. Takayama, N. Terai, On the radical of a monomial ideal, Arch. Math. (Basel) 85 (2005), no. 5, 397–408.
[12] L. T. Hoa, T. N. Trung, Stability of depth and Cohen-Macaulayness of integral closures of powers of monomial ideals, Acta Math. Vietnam. 43 (2018), 67–81.
[13] C. Huneke and I. Swanson, Integral Closure of Ideals Rings, and Modules, London Math. Soc., Lecture Note Series 336, Cambridge University Press, Cambridge, 2006.
[14] D. Popescu, Bounds of Stanley depth, An. St. Univ. Ovidius. Constanta, 19(2),(2011), 187–194.
[15] M. R. Pournaki, S. A. Seyed Fakhari, M. Tousi, S. Yassemi, What is . . . Stanley depth? Notices Amer. Math. Soc. 56 (2009), no. 9, 1106–1108.
[16] M. R. Pournaki, S. A. Seyed Fakhari, S. Yassemi, Stanley depth of powers of the edge ideal of a forest, Proc. Amer. Math. Soc. 141 (2013), no. 10, 3327–3336.
[17] S. A. Seyed Fakhari, Stanley depth of the integral closure of monomial ideals, Collect. Math. 64 (2013), 351–362.
[18] S. A. Seyed Fakhari, Stanley depth of weakly polymatroidal ideals and squarefree monomial ideals, Illinois J. Math., 57 (2013), no. 3, 871–881.
[19] S. A. Seyed Fakhari, Depth, Stanley depth and regularity of ideals associated to graphs, Arch. Math. (Basel) 107 (2016), 461–471.
[20] S. A. Seyed Fakhari, On the Stanley depth of powers of edge ideals, J. Algebra, 489 (2017), 463–474.
[21] S. A. Seyed Fakhari, Depth and Stanley depth of symbolic powers of cover ideals of graphs, J. Algebra, 492 (2017), 402–413.
[22] S. A. Seyed Fakhari, Stanley depth and symbolic powers of monomial ideals, Math. Scand. 120 (2017), 5–16.
[23] P. Singla, Minimal monomial reductions and the reduced fiber ring of an extremal ideal, Illinois J. Math., 51 (2007), no. 4, 1085–1102.
[24] R. P. Stanley, Linear Diophantine equations and local cohomology, Invent. Math. 68 (1982), no. 2, 175–193.
[25] Y. Takayama, Combinatorial characterizations of generalized Cohen-Macaulay monomial ideals, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 48 (2005), 327–344.
[26] W. Vasconcelos, Integral Closure, Rees Algebras, Multiplicities, Algorithms, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005.
[27] R. H. Villarreal, Monomial Algebras, Dekker, New York, N.Y., 2001.

S. A. SEYED FAKHARI, SCHOOL OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE, COLLEGE OF SCIENCE, UNIVERSITY OF TEHRAN, TEHRAN, IRAN.
E-mail address: aminfakhari@ut.ac.ir