Optimal continuous variable quantum teleportation protocol for realistic settings

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We show the optimal setup that allows Alice to teleport coherent states $|\alpha\rangle$ to Bob giving the greatest fidelity (efficiency) when one takes into account two realistic assumptions. The first one says that in any actual implementation of the continuous variable teleportation protocol (CVTP) Alice and Bob necessarily share non-maximally entangled states (two-mode finitely squeezed states). The second one assumes that Alice’s pool of possible coherent states to be teleported to Bob does not cover the whole complex plane ($|\alpha| < \infty$). The optimal strategy is achieved by tuning three parameters in the original CVTP, namely, the beam splitter transmittance and the displacements in position and momentum implemented on the teleported state. These slight changes in the protocol are currently easy to be implemented and, as we show, give considerable gain in performance for a variety of possible pool of input states with Alice.

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I. INTRODUCTION

The extension of the quantum teleportation protocol from discrete (finite dimensional Hilbert spaces) to continuous variable (CV) (infinite dimension) systems was a landmark to CV quantum communication. The main goal of teleportation is to make sure that at the end of the whole protocol a quantum state originally describing Alice’s system turns out to describe a quantum system with Bob at a different location. Moreover, no direct transmission of the system from Alice to Bob is done and the knowledge of Alice’s system’s state is not needed at all to accomplish such a task. These two properties clearly illustrate why teleportation is so powerful a tool. Indeed, for quantum teleportation take place Alice and Bob necessarily share non-maximally entangled states (Bell states) that now describes Bob’s system.

In principle a perfect teleportation only occurs when Alice and Bob share a maximally entangled state. By perfect teleportation we mean that at the end of the protocol and with probability one Bob’s system will be exactly described by the state that originally described Alice’s system. For discrete systems, and in particular for qubits, such maximally entangled states (Bell states) that Alice and Bob must share can be experimentally generated in the laboratory. For CV-systems, however, the perfect implementation of the teleportation protocol requires a maximally entangled state (the Einstein–Podolsky–Rosen (EPR) state) that cannot be generated in the laboratory. In modern quantum optics terminology, one needs an entangled two-mode squeezed state with infinite squeezing ($r \to \infty$). For finite squeezing ($r < \infty$), the teleported quantum state at Bob’s is never identical to the original one at Alice’s.

Another assumption is related to the pool of input states available to Alice, i.e., to the states that Alice might choose to teleport to Bob. For example, consider the simplest discrete system, a qubit. In this case it is assumed that Alice’s input is given by $|\psi\rangle = a|0\rangle + b|1\rangle$, with $a$ and $b$ random complex numbers satisfying the normalization condition $|a|^2 + |b|^2 = 1$. For CV systems, and in particular for coherent states $|\alpha\rangle$, with $\alpha$ complex, it is often assumed that Alice’s pool of states cover the entire complex plane. From a theoretical point of view, either for a qubit or a coherent state, these assumptions are the proper ones in order to determine the strictest conditions guaranteeing a “truly” quantum teleportation, i.e., the conditions where no purely classical protocol can achieve the same efficiency as those predicted by the quantum ones. From a practical point of view, however, these assumptions are only valid for qubits, being unrealistic for CV systems. Indeed, the energy of a coherent state is proportional to $|\alpha|^2$ and in order to cover the entire complex plane we would need states with infinite energy. Also, the greater $|\alpha|$ the less quantum a coherent state becomes and other techniques than quantum teleportation such as a direct transmission of the state may be better suited in this case.

With these two realistic assumptions in mind, namely, Alice and Bob share a non-maximally entangled state and Alice’s pool of states are more likely to be near $|\alpha| = 0$, a natural question then arises. Is it possible to further improve the efficiency of the standard CV teleportation protocol (CVTP) by taking into account in any modification of the original setup these two facts? In other words, tackling these two limitations at once, can we transform the handicap of a quantum channel given by a finite squeezed entangled state ($r < \infty$) to an advantage?

For a pool of input coherent states with Alice described by a Gaussian distribution centered at the vacuum state and when Alice is always teleporting a fixed single state, the answer to the previous question is af-
firmative. The modification introduced in $[14, 15]$ was a single new parameter that Bob is free to choose, namely, the gain $g$ that he might apply equally to the quadratures of his mode at the end of the protocol (see Fig. 1). In the original setup $g = 1$, while in the modified versions it was tuned as a function of the input states and of the squeezing of the channel to increase the efficiency of CVTP. An identical strategy was employed to improve the efficiency of CV entanglement swapping $[15, 16]$, where the optimal $g$ was tuned for a specific input state.

What would happen if we go beyond a Gaussian probability distribution centered at the vacuum and use instead uniform distributions or distributions centered in the coherent state $|\beta\rangle$, $\beta \neq 0$? More important, what are the optimal conditions for CVTP if we introduce more than one free parameter in the modified version of CVTP $[16]$, where either Alice or Bob can change the protocol? Our goal here is to investigate these last two questions in detail and without assuming we know the state to be teleported $[16, 17]$. The only knowledge we have is the probability of Alice picking a particular coherent state $|\alpha\rangle$ according to a predefined probability distribution.

We show that it is possible to achieve further significant increases in performance with extra free parameters that introduce, however, minimal changes to the original scheme, modifications of which can already be implemented in the laboratory. Also, we investigate several probability distributions (see Fig. 2) describing the input states and we show the optimal modifications to each one of them. Moreover, the optimal parameters change appreciably if we work with either uniform or Gaussian distribution or if those distributions are centered or not on the vacuum state. And as expected, the changes on the original CVTP not only depend on the specific probability distribution associated to Alice’s input states but also on the entanglement of the channel.

II. FORMALISM

A. Qualitative analysis

Before diving into the mathematical details of our calculations, it is worth presenting the bigger picture, i.e., the choices we made from the start in order to modify the original setup and the strategy employed to determine the optimal teleportation protocol.

In the original proposal (see Fig. 1), a two-mode squeezed state with squeezing $r$, our entanglement resource, is shared between Alice and Bob. Mode 2 goes to Alice and mode 3 to Bob. The state with Alice to be teleported is represented by mode 1, which can be any coherent state $|\alpha\rangle$. To proceed with the teleportation, Alice combines modes 1 and 2 in a 50:50 beam splitter (BS) and afterwards measures the position and momentum (quadratures of the electromagnetic field) of modes $u$ and $v$, respectively, whose results $x_u$ and $p_v$ are then classically communicated to Bob. With this information he displaces in position $(x_3 \rightarrow x_3 + g\sqrt{2}\tilde{x}_u)$ and momentum $(p_3 \rightarrow p_3 + g\sqrt{2}\tilde{p}_v)$ his mode to get the right teleported state. The displacements and gain $(g = 1)$ are chosen and fixed as the ones yielding perfect teleportation when $r \rightarrow \infty$ (maximal entanglement).

![Fig. 1](image)

FIG. 1: (Color online) In the original proposal $[3]$ we have $\theta = \pi/4$ (50 : 50 BS), $g_u = g_v = g\sqrt{2}$, with $g = 1$, and position and momentum displacements given by $x_3 \rightarrow x_3 + g_u\tilde{x}_u$ and $p_3 \rightarrow p_3 + g_v\tilde{p}_v$. These choices yield an average fidelity $F_{av}$, independent of whatever pool of states (labeled by $\lambda$) is available to Alice. Here, for fixed $\lambda$ and squeezing $r$, the optimization to get the optimal $F_{av}$ is implemented over three free parameters, $\theta, g_u$, and $g_v$, leading to $F_{av}$ that depends both on $r$ and $\lambda$. See text for details.

A priori there is no guarantee that the previous choices for beam splitter transmittance ($\cos^2 \theta = 1/2$), displacements and gain are the optimal ones for all combinations of finite squeezing $r$ and probability distribution for the pool of states available to Alice. Therefore, in order to search for the optimal protocol for a given squeezing $r$ and probability distribution we allow the beam splitter to have an arbitrary transmittance (BS($\theta$)), with $0 < \theta < \pi/2$ (see Fig. 1). Furthermore, the quadratures’ displacements and gain $g$ from the transformation of classical photocurrent to complex field amplitude that Bob must implement after Alice informs him of her measurement results ($\tilde{x}_u$ and $\tilde{p}_v$) $[3, 10]$ are also independently chosen in order to optimize the protocol. Formally, Bob’s displacements are given by $x_3 \rightarrow x_3 + g_u\tilde{x}_u$ and $p_3 \rightarrow p_3 + g_v\tilde{p}_v$, with $g_u$ and $g_v$ chosen in order to optimize the efficiency of CVTP.

We employ two figures of merit to decide the optimality of the protocol, the computation of which requires the specification of the presumed probability of available states with Alice, assumed fixed throughout the many runs of the protocol. One is the average fidelity $F_{av}$ $[13]$ (see Sec. 2B). The fidelity measures how close the output state with Bob at the end of the protocol is to the input state employed by Alice. Here the average is taken over the fidelities of each input state and its respective output, with the weight of each state given by its prob-
ability to be picked out of the states available to Alice. Our strategy consists, therefore, in choosing \( \theta, g_v, \) and \( g_u \) in order to maximize \( F_{av} \).

Since we are working with an average, there might be states not only with higher but with lower fidelities than that given by the original CVTP setup \([3]\). The other figure of merit fixes this possible problem and we call it “no state left behind”, or NSLB condition for short. This condition is such that the optimal triple of parameters \( \theta, g_v, \) and \( g_u \) is the one in which all states of a given distribution have fidelities greater than that predicted by the original CVTP. Of course, for distributions covering the whole complex plane, we weaken the condition and allow a predefined number of states to be left behind.

B. Quantitative analysis

1. The protocol

In what follows we present the details of the mathematical analysis of the modified CVTP, where the three parameters described above are incorporated into the protocol. We use interchangeably the words kets, states, and modes to refer to the same object, namely, the quantized electromagnetic modes \([10]\). Also, when we refer to position \( \hat{x}_k = (\hat{a}_k + \hat{a}_k^\dagger)/2 \) and momentum \( \hat{p}_k = (\hat{a}_k - \hat{a}_k^\dagger)/2i \), with \( \hat{a}_k \) and \( \hat{a}_k^\dagger \) annihilation and creation operators, we mean the quadratures of mode \( k \), with commutation relation \([\hat{x}_k, \hat{p}_k] = i/2\).

An arbitrary input state with Alice can be written in the position basis as

\[
|\varphi\rangle = \int dx_1 \varphi(x_1) |x_1\rangle,
\]

where \( \varphi(x_1) = \langle x_1|\varphi \rangle \) and the integration runs over the entire real line. The entangled state shared between Alice and Bob can also be written in the position basis as

\[
|\Phi\rangle = \int dx_2 dx_3 \Phi(x_2, x_3) |x_2, x_3\rangle,
\]

with \( \Phi(x_2, x_3) = \langle x_2, x_3|\Phi \rangle \) and \( |x_2, x_3\rangle = |x_2\rangle \otimes |x_3\rangle \).

Unless stated otherwise, we keep the ordering of the kets fixed, i.e., the notation \(|x, y, z\rangle\) means that the first two modes/kets are with Alice and the third one with Bob. Using Eqs. (1) and (2) we can write the initial state describing all modes before the beginning of the teleportation protocol as \(|\Psi\rangle = |\varphi\rangle \otimes |\Phi\rangle\), or more explicitly,

\[
|\Psi\rangle = \int dx_1 dx_2 dx_3 \varphi(x_1) \Phi(x_2, x_3) |x_1, x_2, x_3\rangle.
\]

The first step in the protocol consists in sending mode 1 (input state) and mode 2 (Alice’s share of the two-mode squeezed state) to a BS with transmittance \( \cos^2 \theta \) (see Fig. 1). Calling \( \hat{B}_{12}(\theta) \) the operator representing the action of the BS we have in the position basis \([10]\)

\[
\hat{B}_{12}(\theta)|x_1, x_2\rangle = |x_1 \sin \theta + x_2 \cos \theta, x_1 \cos \theta - x_2 \sin \theta\rangle.
\]

Inserting Eq. (1) into (3) and making the following variable changes, \( x_v = x_1 \sin \theta + x_2 \cos \theta \) and \( x_v = x_1 \cos \theta - x_2 \sin \theta \), we have

\[
|\Psi'\rangle = \int dx_v dx_u dx_3 \varphi(x_v, \sin \theta + x_u \cos \theta) \\
\times \Phi(x_v \cos \theta - x_u \sin \theta, x_3) |x_v, x_u, x_3\rangle,
\]

for the total state after modes 1 and 2 go through \( BS(\theta) \).

The next step of the protocol consists in measuring the momentum and position of modes \( v \) and \( u \), respectively.

For the quantized electromagnetic mode, this is achieved by homodyne detectors yielding classical photocurrents that assign real numbers for the quadratures \( \hat{p}_v \) and \( \hat{x}_u \).

Since Alice will project mode \( v \) onto the momentum basis, it is convenient to rewrite Eq. (5) using the Fourier transformation relating the position and momentum eigenstates,

\[
|x_v\rangle = \frac{1}{\sqrt{\pi}} \int dp_v e^{-2ix_v p_v} |p_v\rangle.
\]

Thus, inserting Eq. (6) into (5) we have

\[
|\Psi'\rangle = \frac{1}{\sqrt{\pi}} \int dp_v dx_v dx_u dx_3 \varphi(x_v, \sin \theta + x_u \cos \theta) \\
\times \Phi(x_v \cos \theta - x_u \sin \theta, x_3) \sqrt{p_v} \sqrt{p_v + x_u} |p_v, x_u, x_3\rangle.
\]

In the second step of the protocol, Alice measures the momentum of mode \( v \) and the position of mode \( u \) (See Fig. 1). Assuming her measurement results are \( \tilde{p}_v \) and \( \tilde{x}_u \), the total state at the end of the measurement is simply obtained applying the measurement postulate of quantum mechanics, \( |\Psi''\rangle = P_{\tilde{p}_v, \tilde{x}_u} |\Psi'\rangle / \sqrt{p_v \tilde{p}_v + x_u \tilde{x}_u} \), where \( P_{\tilde{p}_v, \tilde{x}_u} = |\langle \tilde{p}_v, \tilde{x}_u| \tilde{p}_v, \tilde{x}_u \rangle \rangle \) is the projector describing the measurements, with \( |\rangle \rangle \) being the identity operator acting on mode 3, and \( p_v \tilde{p}_v + x_u \tilde{x}_u \) the total trace. Specifying to the position basis and noting that \( \langle \tilde{p}_v|\tilde{p}_v\rangle = \delta(\tilde{p}_v - \tilde{p}_v) \) and \( \langle x_u|\tilde{x}_u\rangle = \delta(x_u - \tilde{x}_u) \) we have

\[
|\Psi''\rangle = |\tilde{p}_v, \tilde{x}_u\rangle \otimes |\chi'\rangle,
\]

where Bob’s state is
\[ |\chi'\rangle = \frac{1}{\sqrt{\pi p(\hat{p}_v, \hat{x}_u)}} \int dx_v dx_3 \varphi(x_v \sin \theta + \hat{x}_u \cos \theta) \Phi(x_v \cos \theta - \hat{x}_u \sin \theta, x_3) e^{-2ix_v \hat{p}_v} |x_3\rangle. \]  

Here

\[ \varphi(\hat{p}_v, \hat{x}_u) = \int dx_3 |\psi'(\hat{p}_v, \hat{x}_u, x_3)|^2 \]  

and

\[ \Psi'(\hat{p}_v, \hat{x}_u, x_3) = \langle \hat{p}_v, \hat{x}_u, x_3 | \psi' \rangle = \frac{1}{\sqrt{\pi}} \int dx_v \varphi(x_v \sin \theta + \hat{x}_u \cos \theta) \Phi(x_v \cos \theta - \hat{x}_u \sin \theta, x_3) e^{-2ix_v \hat{p}_v}, \]  

where Eq. (11) was obtained using (7).

The third step of the protocol consists in Alice sending to Bob via a classical channel (photocurrents) her measurement results. With this information Bob is able to implement the fourth and last step of the protocol, namely, he displaces his mode quadratures according to the following rule, \( x_3 \rightarrow x_3 + g_u \hat{x}_u \) and \( p_3 \rightarrow p_3 + g_v \hat{p}_v \). Mathematically this corresponds to the application of the displacement operator \( \hat{D}(\alpha) = e^{i\alpha \hat{x} - \alpha^* \hat{p}} = e^{-2iRe[\alpha] \hat{p} + 2iIm[\alpha] \hat{x}}, \) with \( \alpha = g_u \hat{x}_u + ig_v \hat{p}_v \). Since \( \hat{x} \) and \( \hat{p} \) commute with their commutator we can apply Glauber’s formula to obtain \( \hat{D}(\alpha) = e^{iRe[\alpha] \hat{p} + 2iIm[\alpha] \hat{x}}. \) This leads to

\[ \hat{D}(g_u \hat{x}_u + ig_v \hat{p}_v) |x_3\rangle = e^{ig_u g_v \hat{x}_u \hat{p}_v} e^{2i(g_v \hat{p}_v x_3 + g_u \hat{x}_u)} |x_3\rangle. \]  

The final state with Bob, \( |\chi\rangle = \hat{D}(g_u \hat{x}_u + ig_v \hat{p}_v) |\chi'\rangle \), can be put as follows if we use Eq. (12) and make the variable change \( x_3 \rightarrow x_3 - g_u \hat{x}_u, \)

\[ |\chi\rangle = \frac{e^{-ig_u g_v \hat{x}_u \hat{p}_v}}{\sqrt{\pi p(\hat{p}_v, \hat{x}_u)}} \int dx_v dx_3 \varphi(x_v \sin \theta + \hat{x}_u \cos \theta) \Phi(x_v \cos \theta - \hat{x}_u \sin \theta, x_3 - g_u \hat{x}_u) e^{-2i(x_v - g_u x_3) \hat{p}_v} |x_3\rangle \]

\[ = \int dx_3 \left( \frac{e^{-ig_u g_v \hat{x}_u \hat{p}_v}}{\sqrt{\pi p(\hat{p}_v, \hat{x}_u)}} \right) \int dx_v \varphi(x_v \sin \theta + \hat{x}_u \cos \theta) \Phi(x_v \cos \theta - \hat{x}_u \sin \theta, x_3 - g_u \hat{x}_u) e^{-2i(x_v - g_u x_3) \hat{p}_v} |x_3\rangle \]

\[ = \int dx_3 \chi(x_3) |x_3\rangle. \]  

Note that \( e^{-ig_u g_v \hat{x}_u \hat{p}_v} \) in Bob’s final state is an irrelevant global phase and could be suppressed.

It is worth mentioning at this point that Eq. (13), together with Eqs. (10) and (11), are quite general. They allow us to get the teleported state with Bob for any input state and any entangled state (channel) shared between Alice and Bob. For instance, if we use a maximally entangled state (EPR state) we have \( \Phi(x_2, x_3) \propto \delta(x_2 - x_3) \) in Eq. (2). Using this channel in Eq. (13) leads to \( \chi(x_3) = \varphi(x_3) \) if we set \( g_u = g_v = \sqrt{2} \) and \( \theta = \pi/4 \), i.e., we have a perfect teleportation.

2. Fidelity

If the density matrix describing the output state at Bob after one single run of the protocol is \( \hat{\rho}_B = |\chi\rangle \langle \chi| \) the fidelity is defined as

\[ F(|\varphi\rangle, \hat{\rho}_B, \hat{x}_u) = \langle \varphi | \hat{\rho}_B | \varphi \rangle, \]  

where we highlight that the fidelity depends on the input state \( |\varphi\rangle \) and on the measurement outcomes obtained by Alice. For the moment we leave implicit the dependence on the other parameters, i.e., what channel/entanglement we have and \( \theta, g_v, \) and \( g_u \). The fidelity is a good measure of the similarity between the input and output states since it achieves its maximal value (one) only if we have a flawless teleportation \( (\hat{\rho}_B = \hat{\rho}_{out}) \) and its minimal one (zero) if the output is orthogonal to the input.

At each run of the protocol, Alice will measure \( \hat{p}_v \) and \( \hat{x}_u \) with probability \( p(\hat{p}_v, \hat{x}_u) \). Hence, Bob’s final ensemble of states, averaged over all possible measurement results for a fixed input state \( |\varphi\rangle \), is

\[ F(|\varphi\rangle) = \int d\hat{p}_v d\hat{x}_u p(\hat{p}_v, \hat{x}_u) F(|\varphi\rangle, \hat{\rho}_v, \hat{x}_u). \]  

Furthermore, to properly search the optimal configura-
tion for a probability distribution of input states $P(|\varphi\rangle)$ with Alice, another averaging is needed, this time over the pool of states available to her [11],

$$F_{av}(\theta, g_v, g_u) = \int d|\varphi\rangle P(|\varphi\rangle) F(|\varphi\rangle).$$

(16)

The first strategy to optimize the teleportation protocol, once we know $P(|\varphi\rangle)$, is the search for the triple of points $(\theta, g_v, g_u)$ maximizing $F_{av}(\theta, g_v, g_u)$. Therefore, for fixed entanglement, we either solve analytically (if possible) or numerically the following three equations for $\theta, g_v$, and $g_u$,

$$\frac{\partial F_{av}}{\partial \theta} = 0, \quad \frac{\partial F_{av}}{\partial g_v} = 0, \quad \frac{\partial F_{av}}{\partial g_u} = 0.$$ (17)

The obtained solutions are then inserted into (16) and slightly varied about their actual values in order to be sure we have a global maximum, ruling out possible local maxima, minima, and saddle points.

The second optimization procedure is the NSLB condition. Here we work with $F(|\varphi\rangle)$, Eq. (15), and search for a single triple of points $\theta, g_v$, and $g_u$ such that all states $|\varphi\rangle$, within a bounded distribution $P(|\varphi\rangle)$, have $F(|\varphi\rangle) > F_{orig}$, where $F_{orig}$ is the fidelity predicted by the original CVTP [10]. For unbounded $P(|\varphi\rangle)$, Gaussian distributions for example, we require that $F(|\varphi\rangle) > F_{orig}$ for states lying within a circular region such that $\int_{region} P(|\varphi\rangle) d|\varphi\rangle = 95\%$, i.e., a region containing 95% of the randomly picked input states $|\varphi\rangle$ at the end of several runs of the protocol.

### III. Results

In the rest of this paper we particularize to pool of input states at Alice’s giving by coherent states, $|\varphi\rangle = |\alpha\rangle$, with $\alpha$ complex, and to the entanglement shared between Alice and Bob given by two-mode squeezed vacuum states, $|\Phi\rangle = |\psi\rangle = \sqrt{1 - \tan^2 r} \sum_{n=0}^{\infty} \tan^n r |n\rangle_A \otimes |n\rangle_B$, where $|n\rangle_A(B)$ are Fock number states at Alice’s (Bob’s). Also, $r$ is the squeezing parameter and for $r = 0$ we have $|0\rangle$, the vacuum state, and for $r \to \infty$ the unphysical maximally entangled EPR state. Both coherent and two-mode squeezed states of the quantized electromagnetic field are easily generated in the laboratory today and they were the ingredients employed in the experimental implementation of the original CVTP [4].

These two states are represented in the position basis as [10]

$$\phi(x_1) = \langle x_1 |\alpha\rangle = \left(\frac{2}{\pi}\right)^{1/4} e^{-x_1^2 + 2ax_1} |\alpha|^2 / 2 - a^2 / 2$$ (18)

and

$$\Phi(x_2, x_3) = \langle x_2, x_3, \psi\rangle = \sqrt{\frac{2}{\pi}} \exp \left[ -e^{-2r}(x_2 + x_3)^2 / 2 - e^{2r}(x_2 - x_3)^2 / 2 \right].$$ (19)

Equations (18) and (19) allow us to explicitly compute Eq. (15).

$$F(|\alpha\rangle) = f_2(\theta, g_v) - \frac{1}{2} \exp \left[ -\frac{f_1(\theta, g_v)}{f_2(\theta, g_v)} Im[\alpha]^2 \right] \times f_2(\theta - \frac{\pi}{2}, g_u) - \frac{1}{2} \exp \left[ -\frac{f_1(\theta - \frac{\pi}{2}, g_u)}{f_2(\theta - \frac{\pi}{2}, g_u)} Re[\alpha]^2 \right],$$ (20)

where

$$f_1(\theta, g_v) = (1 - g_v \sin \theta)^2,$$ (21)

$$f_2(\theta, g_v) = [(2 + g_v^2) \cos^2 r + g_v^2 \cos(2\theta) \sinh^2 r - 2g_v \cos \theta \sinh(2r)] / 2$$ (22)

Looking at Eq. (20) we see that if we want to have $F(|\alpha\rangle)$ independent of $|\alpha\rangle$ we have to choose $g_v$ and $g_u$ such that $f_1(\theta, g_v) = f_1(\theta + \pi/2, g_u) = 0$. This is accomplished if $g_v = \csc \theta$ and $g_u = \sec \theta$. Inserting these values for $g_v$ and $g_u$ into Eq. (20) and maximizing it we get $\theta = \pi/4$ and $g_v = g_u = \sqrt{2}$, as the optimal parameter configuration, and $F(|\alpha\rangle) = 1/(1 + e^{-2\pi})$ for the optimal fidelity. These are the configuration and fidelity of the original CVTP. However, when we maximize the fidelity taking into account a specific $|\alpha\rangle$, or a pool of states $|\alpha\rangle$ with Alice, the optimal setting necessarily changes. See Appendix A for the calculation of the optimal $F(|\alpha\rangle)$ for all $|\alpha\rangle$ with $|\alpha| \leq 5$.

In what follows we work with several different probability distributions $P(|\alpha\rangle)$ for Alice’s pool of states, whose normalization condition reads

$$\int P(|\alpha\rangle) d|\alpha\rangle = \int_{-\infty}^{\infty} P(\alpha) d\alpha |Re[\alpha]| dIm[\alpha] = \int_{0}^{2\pi} \int_{0}^{\infty} P(\alpha) |\alpha| d|\alpha| d\omega = 1,$$ (23)

where $\alpha = Re[\alpha] + i Im[\alpha] = |\alpha| e^{i\omega}$. A pictorial representation of the several $P(\alpha)$ we deal with are given in Figs. 2 and 17.

#### A. Purely real or imaginary states

The first two distributions we work with confine the states $|\alpha\rangle$ to be given by either real or imaginary $\alpha$. We assume the states to be uniformly distributed along the real or imaginary axis from $-R$ to $R$, where $R > 0$.

The real $\alpha$ distribution is given by

$$P_r(\alpha) = \delta(Im[\alpha]) \Theta(R^2 - Re[\alpha]^2) / 2R,$$ (24)

with $\delta(x)$ being the Dirac delta function and $\Theta(x)$ the Heaviside theta function ($\Theta(x) = 0$ if $x < 0$ and $\Theta(x) = 1$ for $x \geq 0$). The imaginary $\alpha$ distribution reads

$$P_i(\alpha) = \delta(Re[\alpha]) \Theta(R^2 - Im[\alpha]^2) / 2R.$$ (25)
Inserting Eq. (24) into (16) we obtain for the average fidelity of states on the real line,

$$F_{av}^r(\theta, g_v, g_u) = \sqrt{\frac{\pi}{2}} \frac{\operatorname{Erf}[R \sqrt{f_1(\theta + \pi/2, g_u)f_2(\theta - \pi/2, g_u)}]}{f_1(\theta + \pi/2, g_u)f_2(\theta - \pi/2, g_u)}^{1/2},$$

(26)

where $\operatorname{Erf}[x] = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ is the error function.

Since the dependence of (26) on $g_v$ is simply given by $f_2(\theta, g_v)^{-1/2}$, it is straightforward to solve for $\partial F_{av}^r/\partial g_v = 0$, leading to the following optimal $g_v$,

$$g_v^{opt} = \frac{\sinh(2r) \cos \theta^{opt}}{\cosh^2 r + \cos(2\theta^{opt}) \sinh^2 r}.$$  

(27)

Now, substituting Eq. (27) into (26) we obtain the optimal average fidelity by solving the remaining two equations in (17). However, due to the presence of the error function in (26), we cannot get a closed solution for $g_u^{opt}$ and $\theta^{opt}$. Thus, we must rely on numerical solutions once the squeezing $r$ and the range $R$ of the distribution are specified.

Repeating the same calculations above for a uniform distribution of states on the imaginary line, ranging from $-R$ to $R$, we have

$$F_{av}^i(\theta, g_v, g_u) = \sqrt{\frac{\pi}{2}} \frac{\operatorname{Erf}[R \sqrt{f_1(\theta - \pi/2, g_u)f_2(\theta + \pi/2, g_u)}]}{f_1(\theta - \pi/2, g_u)f_2(\theta + \pi/2, g_u)}^{1/2},$$

(28)

The roles of $g_v$ and $g_u$ are now interchanged and the extremum condition $\partial F_{av}^i/\partial g_u = 0$ is easily solved and gives the following optimal $g_u$,

$$g_u^{opt} = \frac{\sinh(2r) \sin \theta^{opt}}{\cosh^2 r - \cos(2\theta^{opt}) \sinh^2 r}. \quad (29)$$

The remaining two extremum conditions, similarly to what happened to the real uniform distribution, must be numerically computed due to the presence of the error function.

In Fig. 3 we show the optimal $F_{av}^r(\theta, g_v, g_u)$ and $F_{av}^i(\theta, g_v, g_u)$ for several distributions with $|\alpha| \leq R$ as a function of the squeezing $r$ of the channel. The first thing we note is that the optimal average fidelities are the same for the real and imaginary distributions. Second, the lower the range $R$ of the distribution the greater the efficiency. This is expected since as we decrease $R$ the states available to Alice become more and more similar. Therefore, the optimal parameters giving the best average fidelity approach the optimal parameters giving the highest fidelity for each one of the states within the distribution. Third, as we increase $R$ the optimal fidelity decreases. However, it rapidly tends to its asymptotic limit (maroon/circle line in Fig. 3), which is still by far superior than the fidelity given by the original CVTP (dashed line in Fig. 3). Figure 4 gives the optimal $\theta$, $g_v$, and $g_u$ leading to the optimal average fidelities shown in Fig. 3.

We have also compared how much we gain in efficiency optimizing $F_{av}(\theta, g_v, g_u)$ considering $\theta$, $g_v$, and $g_u$ as free parameters against the optimization of $F_{av}(\pi/4, g, g)$, where only the gain $g$ is free [14, 15, 19]. As can be seen in Fig. 5 we obtain a considerable gain in efficiency when we allow the three parameters to be freely adjusted for a given distribution when compared to the single parameter scenario. Also, we have checked that the greater
FIG. 4: (Color online) Parameters giving the optimal average fidelities shown in Fig. 3 for states lying on the real line. Note that the many curves for $g^{\text{opt}}$ are very close to each other. For $g^{\text{opt}}$ the range $R$ increases from bottom to top while for $\theta$ it increases from top to bottom. The dashed black curves give the values used in the original CVTP ($g_u = g_v = \sqrt{2} \approx 1.41$ and $\theta = \pi/4 \approx 0.79$). For the imaginary distribution, $g_u \leftrightarrow g_v$ and $\theta \rightarrow \pi/2 - \theta$ in the graphics above.

$R$ the less efficient is the single parameter optimization. For not too big $R$ it approaches the original CVTP fidelity $F_{\text{orig}}$ and we must resort to the three parameter optimization to get effective efficiency gains.

![Graph showing optimal $g_u$ and $g_v$](image)

FIG. 5: (Color online) The curves above were calculated considering either a real or imaginary uniform distribution with $|\alpha| \leq R = 5.0$. It is clear from the figure that in order to get an expressive gain in efficiency for quantum channels with low degree of entanglement it is crucial to optimize the average fidelity over the three free parameters.

In Fig. 6 we show the dependence of the optimal average fidelity as a function of the range $R$ of the distribution describing the pool of input states with Alice. It is clear from the data that for all ranges $R$ we have a better performance working with the optimized CVTP and that the lower the degree of entanglement of the channel the greater the gains in efficiency.

Finally, Fig. 7 shows $F(|\alpha\rangle)$, Eq. (20), where in order to compute $F(|\alpha\rangle)$ for each state $|\alpha\rangle$ we have employed the parameters $\theta^{\text{opt}}, g_v^{\text{opt}}$, and $g_u^{\text{opt}}$ giving the optimal average fidelity for a real/imaginary distribution ranging from $-R$ to $R = 2.0$ and squeezing $r = 0.3$. As can be seen in Fig. 4 all states within the range of the distribution have $F(|\alpha\rangle) > F_{\text{orig}}$. Thus, $\theta^{\text{opt}}, g_v^{\text{opt}}$, and $g_u^{\text{opt}}$ are also a set of parameters satisfying the NSLB condition. Note that a considerable amount of states outside the range of the distribution still have $F(|\alpha\rangle) > F_{\text{orig}}$. Moreover, we have carried out the previous analysis for many other combinations of $R$ and $r$ and always obtained the same features highlighted in Fig. 7.

B. States lying on a circumference or a disk

For a set of states $|\alpha\rangle$ available to Alice having the same amplitude $|\alpha| = R$ and phases $\omega$ given by a uniform distribution, where $0 \leq \omega < 2\pi$, we have in the complex plane representation of $|\alpha\rangle$ a circumference of radius $R$ centered in the vacuum (See Fig. 2). We can thus write its probability distribution as

$$P_c(\alpha) = \delta(|\alpha| - R)/(2\pi R).$$

(30)

Inserting Eq. (30) into (16) we have the average fidelity for states uniformly lying on a circumference of radius $R$ as follows,

$$F_{av}^c(\theta, g_v, g_u) = \frac{e^{-h_+ (\theta, g_v, g_u) R^2} I_0[h_+ (\theta, g_v, g_u) R^2]}{\sqrt{f_2(\theta, g_v) f_2(\theta - \pi/2, g_u)}}.$$  

(31)
of entanglement of the channel. In Fig. 9 we show to a single variable optimization problem. A direct com-

inserting the previous parameters into Eq. (31), we can recast the determination of the optimal average fidelity

\[ F_{\text{opt}}(g) = \frac{2 \exp \left[ - \frac{(\sqrt{2} g - 2)^2 R^2 \text{sech} r}{2(g^2 + 2) \cosh^2 r - \sqrt{2} g \sinh(2r)} \right]}{(g^2 + 2) \cosh^2 r - \sqrt{2} g \sinh(2r)}, \]  

where the optimal \( g \) is given by solving the cubic equation

\[ \sqrt{2}(e^r \sinh(2r) \cosh r + 2R^2) - ge^r(3 \cosh(2r) - 1) \cosh r - g^2 \sqrt{2}(R^2 - 3e^r \sinh r \cosh^2 r) - g^3 e^r \cosh^3 r = 0. \]

In Fig. 8 we plot the optimal \( F_{\text{opt}}(\theta, g_v, g_u) \), or equivalently \( F_{\text{opt}}(g) \), for several uniform distributions with states lying on circumferences of radius \( R \) as a function of the entanglement of the channel. In Fig. 9 we show \( F_{\text{opt}}(g) \) as a function of the radius \( R \) for fixed values of entanglement.

Looking at Fig. 8 we see that the smaller the radius \( R \) of the distribution the greater the efficiency. Also, for \( R = 0 \) we have \( F_{\text{opt}}(g) = 1 \) for any value of squeezing since in this case we just have one state to teleport, the vacuum state. In this scenario Bob has complete knowledge of what state will be sent and the teleportation is trivial. Now, as we increase \( R \) the optimal fidelity decreases, and contrary to the pure real and imaginary distributions, it does not tend to an asymptotic limit (maroon/circle line in Fig. 8) yielding a better performance than that given by the original CVTP. Actually, \( F_{\text{opt}}(g) \)
approaches the fidelity of the original CVTP (dashed line in Fig. 8).

We have also tested for the behavior of \( F(\vert \alpha \rangle) \), with \( \vert \alpha \rangle \) lying on a given circumference of radius \( R \), using the optimal parameters \( \theta^{\text{opt}} = \pi/4, g_v^{\text{opt}} = g_u^{\text{opt}} \) that give \( F^{\text{opt}}(g) \) for this distribution. We have noticed that \( F(\vert \alpha \rangle) = F^{\text{opt}}(g) \) for all \( \vert \alpha \rangle \) on this distribution. In other words, the optimal average fidelity is exactly the fidelity of any state belonging to the circumference distribution, which implies that \( \theta^{\text{opt}}, g_v^{\text{opt}}, g_u^{\text{opt}} \) also belong to a set of parameters satisfying the NSLB condition.

Assuming now that both the amplitude and phase are given by independent uniform distributions, i.e., the input states are contained in a disk of radius \( R \) (\( |\alpha| \leq R \) and \( 0 \leq \omega < 2\pi \)), we have

\[
P_d(\alpha) = \Theta(R - |\alpha|)/(\pi R^2).
\]

Inserting Eq. (34) into (10) we get

\[
F_{av}^d(\theta, g_v, g_u) = \frac{2}{R^2} \int_0^R R' F_{av}^d(R')dR', \tag{35}
\]

with \( F_{av}^d(R') \) being Eq. (31) thought as a function of the radius \( R' \). The last integration in Eq. (35) cannot be analytically computed and we must work numerically to obtain the optimal average fidelity.

We have made a systematic numerical study for several values of squeezing \( r \) and discs with radius \( R \) obtaining, in all cases, the optimal parameters such that \( \theta^{\text{opt}} = \pi/4 \) and \( g_v^{\text{opt}} = g_u^{\text{opt}} = g \). Hence, going back to Eq. (35) with these parameters the last integration can be computed and we get for the optimal average fidelity

\[
F_{av}^{\text{opt}}(g) = 2 \left( 1 - \exp \left[ -\frac{(\sqrt{2}-g)^2}{(2+g^2) \cosh^2(r) - \sqrt{2}g \sinh(2r)} \right] \right)/R^2.
\]

The optimal \( g \) is obtained solving \( \partial F_{av}^{\text{opt}}(g)/\partial g = 0 \), a transcendental equation too cumbersome to be shown here. However, since we have a one parameter problem, its numerical solution is trivially obtained by standard methods and we can easily get \( F_{av}^{\text{opt}}(g) \) once \( r \) and \( R \) are specified.

Figures 10 and 11 show, respectively, the optimal average fidelities as functions of the entanglement/squeezing \( r \) of the channel and as functions of the radius \( R \) of the disc whereon the input states are uniformly distributed. They possess the same qualitative features already explained for the distribution of states on a circumference. Quantitatively, however, we have a better performance for a given disc of radius \( R \) when compared to a circumference of the same radius. This is understood noting that a disc is the union of all circumferences with radius lower than or equal to \( R \). And since we have shown that the smaller \( R \) the greater the fidelity for a circumference distribution, it is clear that the optimal average fidelity of a disc should outperform the optimal one for a circumference with the same \( R \).

In Fig. 12 we show \( F(\vert \alpha \rangle) \), Eq. (20), where in the calculation of \( F(\vert \alpha \rangle) \) for every state \( \vert \alpha \rangle \) we have employed the parameters \( \theta^{\text{opt}}, g_v^{\text{opt}}, g_u^{\text{opt}} \) giving the optimal average fidelities for the circumference (blue/square) and the disc (red/circle) distributions with \( R = 2.0 \). The squeezing was set to \( r = 0.5 \) for the upper panel and \( r = 1.0 \) for the lower one. As can be seen in Fig. 12 all states within the disc and on the border has \( F(\vert \alpha \rangle) > F_{\text{orig}} \) either employing the optimal parameters for the circumference or the disc. Hence, these two sets of parameters satisfy the NSLB condition.
Moreover, similarly to the real and imaginary line distributions, several states outside the range of the distributions still have $F(|\alpha\rangle) > F_{\text{orig}}$. This property is more evident as we increase the squeezing $r$ and if we use the parameters giving the optimal average fidelity for the states on the border (blue/square curves). This same feature occurs for other combinations of $R$ and $r$. However, as we increase $R$ and/or $r$, all curves tend to the one for the original CVTP (black/dashed curve).

![Graph showing optimal average fidelity for various distributions](image)

**FIG. 12:** (Color online) The blue/square curves are $F(|\alpha\rangle) = F(|\text{Re}|^\alpha\rangle)$, Eq. 20, as a function of the radius $R$ where $\theta, g_v$, and $g_u$ were chosen such that they optimize the average fidelity for a circumference distribution with $R = 2.0$, highlighted by the dotted/green line. The red/circle curves give $F(|\alpha\rangle)$ with $\theta, g_v$, and $g_u$ such that they optimize the average fidelity for a disc distribution with $R = 2.0$. The black/dashed curve is the original CVTP fidelity. Note that $F(|\text{Re}|^\alpha\rangle) = F(|R\rangle)$ and hence the curves above are valid for all coherent states having the same amplitude.

If we want to work with states lying on a circumference or disk centered at the state $|\beta\rangle$, we simply replace $\alpha$ with $\alpha - \beta$ in the right hand sides of Eqs. (20) and (24). The expressions for the optimal average fidelities cannot be put in a simple analytical form for arbitrary $\beta$ and we must work numerically in order to find the optimal configuration of parameters. The key features for these distributions are shown in Figs. 13–16.

For the displaced circumference and disk distributions we observed that whenever $|\text{Re}|\beta\rangle = |\text{Im}|\beta\rangle$ the optimal parameters are such that $g_v = g_u$ and $\theta = \pi/4$, while $g_v \neq g_u$ and $\theta \neq \pi/4$ when $|\text{Re}|\beta\rangle \neq |\text{Im}|\beta\rangle$ (lower panels of Figs. 13 and 14). This result shows that for the great majority of the distributions here investigated the single parameter optimization strategy is not enough to achieve the highest efficiency possible.

We have also noted that as we increase the squeezing $r$ and the radius $R$ of the distribution we approach the original CVTP fidelity for distributions centered at any $\beta$. However, and quite surprising, if we displace the distribution along either the real or imaginary axis, the optimal average fidelity has an asymptotic limit as we increase $|\beta|$ considerably better than the one predicted for the original CVTP (left panels of Figs. 13 and 15).

As we move the center of the distribution away from the real/imaginary axis, the asymptotic optimal average fidelity starts to approach the fidelity given by the original CVTP. For $|\beta| = 30^\circ$ we already have the asymptotic optimal average fidelity indistinguishable from the original CVTP fidelity (right panels of Figs. 13 and 15).

![Graph showing optimal average fidelity for various distributions](image)

**FIG. 13:** (Color online) The plots show the optimal average fidelity as a function of the squeezing $r$ for several circular distributions of radius $R = 0.5$ displaced by $\beta = |\beta|e^{i\phi(\beta)}$. $|\beta|$ increases from top to bottom (solid curves) and $\arg(\beta)$ are shown in the graphics and illustrated by the insets. The dashed curve gives the fidelity of the original CVTP. For low squeezing, note that as we increase $|\beta|$ for $\arg(\beta) = 0$ or $\pi/2$ (left panels) the optimal average fidelity tends to values far superior than that predicted by the original CVTP. This interesting fact does not happen if the center of the distribution moves away from the real or imaginary axis (right panels), where the curves for the original CVTP and $|\beta| = 10$ fidelities cannot be distinguished.

### C. States given by Gaussian distributions

We now consider that the pool of input states with Alice is described by a Gaussian distribution with variance $1/(2\lambda)$ and mean $\beta$,

$$P_\beta(\alpha) = (\lambda/\pi) \exp(-\lambda|\alpha - \beta|^2).$$

(37)

Here, when $\beta = 0$ the distribution is centered at the vacuum state and for $\beta \neq 0$ it is centered at the coherent state $|\beta\rangle$ (see Fig. 17). Also, as we increase $\lambda$ (decrease the variance) the distribution approaches a single point,
$g$ and $u$ are found here. The dashed curve gives the bottom (solid curves) and $\arg(\beta)$ optimal average fidelities. Left lower panel: The optimal $g_o$ and $g_u$, the values of which are in quadrature (90° dephasing). Note that they are only equal for $\arg(\beta) = \pm 45^\circ$ and $\arg(\beta) = \pm 135^\circ$, i.e., when $|\text{Re}(\beta)| = |\text{Im}(\beta)|$. Right lower panel: The optimal $\theta$, which equals $\theta = 45^\circ$ at the same points where $g_u = g_o$. All dashed curves represent the corresponding values for the original CVTP.

$\beta$, and for $\lambda \to 0$ we have a uniform distribution covering the entire complex plane.

Inserting Eq. 37 into 16 we can readily compute the average fidelity for an input of coherent states distributed...
according to a Gaussian centered at $\beta$, 

$$F_{av}^g(\theta, g_v, g_u) = \frac{\sqrt{\lambda} \exp \left[ \frac{-\lambda f_1(\theta + \pi/2, g_u) \text{Re}(\beta^2)}{f_1(\theta + \pi/2, g_u) + f_2(\theta - \pi/2, g_u)} \right]}{\sqrt{f_1(\theta, g_v) + f_2(\theta, g_v)}} \times \frac{\sqrt{\lambda} \exp \left[ \frac{-\lambda f_1(\theta, g_v) \text{Im}(\beta^2)}{f_1(\theta, g_v) + f_2(\theta, g_v)} \right]}{\sqrt{f_1(\theta, g_v) + f_2(\theta, g_v)}}. \quad (38)$$

For Gaussians with $\beta = 0$ it is not difficult to show that the optimal average fidelity is such that $\theta^{opt} = \pi/4$ and $g_v = g_u = g$, where $g$ is 14

$$g = \frac{2\sqrt{2} + \lambda \sqrt{2} \sinh(2\tau)}{2 + \lambda + \lambda \cosh(2\tau)}. \quad (39)$$

In Fig. 18 we show how the optimal average fidelity depends on the entanglement of the channel for several Gaussians with different variances. As expected, as we decrease $\lambda$, covering the entire complex plane, we recover the results of the original CVTP.

FIG. 18: (Color online) Solid curves give the optimal average fidelities as functions of the entanglement of the channel (squeezing $r$) for a pool of input states given by Gaussians centered at the origin with variance 1/(2$\lambda$), with $\lambda$ increasing from bottom to top. Dashed curve: average fidelity given by the original CVTP, which is indistinguishable from the corresponding one for the Gaussian with $\lambda = 0.01$. Upper inset: Density plots of the several Gaussian distributions. Their variance ($\lambda$) decreases (increases) from left to right. Lower inset: The optimal gain $g_v = g_u = g$ giving the optimal fidelities shown in the main graph. Here, $\lambda$ increases from top to bottom and the dashed curve is $g$ according to the original CVTP, indistinguishable from the optimal one for the Gaussian with $\lambda = 0.01$. The optimal $\theta$ is always $\pi/4$.

If we work with Gaussian distributions displaced by $\beta \neq 0$, we cannot get a simple closed solution to the optimization problem and we must rely on numerical methods. In Figs. 19 and 20 we show numerical computations giving the optimal average fidelity and the optimal settings for Gaussians displaced in a variety of ways from the origin of the complex plane. The qualitative behaviours are very similar to the ones already discussed for the circumference and disk distributions. A more detailed analysis can be found in the captions of Figs. 19 and 20.

FIG. 19: (Color online) Optimal average fidelity as a function of the squeezing $r$ of the channel for Gaussian distributions with variance 1/(2$\lambda$), $\lambda = 2.0$, and mean $\beta = |\beta|e^{i\arg(\beta)}$. $|\beta|$ increases from top to bottom (solid curves) and $\arg(\beta)$ are shown in the graphics and illustrated in the insets. The dashed curve gives the fidelity of the original CVTP. For low squeezing, note that as we increase $|\beta|$ for $\arg(\beta) = 0$ or $\pi/2$ (left panels) the optimal average fidelity tends to values far superior than that predicted by the original CVTP. This interesting fact does not happen if the center of the distribution moves away from the real or imaginary axis (right panels), where the curves for the original CVTP and $|\beta| = 10$ fidelities cannot be distinguished.

We have also checked if the optimal parameters coming from the optimization of the average fidelity for Gaussian distributions satisfy the NSLB condition for unbounded distributions, as explained before in Sec. IIIB. For Gaussians with not to big variances these parameters are indeed a set of parameters satisfying the NSLB condition. But as we increase the variance they no longer meet that requirement. However, we can always get a set of parameters satisfying the NSLB condition by using the parameters giving the optimal average fidelity of a circumference distribution whose radius $R$ is such that $\int_0^{2\pi} \int_0^R P(|\alpha|) |\alpha| d|\alpha| d\omega = 95\%$, with $P(|\alpha|)$ being the Gaussian distribution.

IV. DISCUSSIONS AND CONCLUSION

We have extensively studied how one can modify the original continuous variable teleportation protocol in order to increase its efficiency in teleporting coherent states by taking into account two facts inherently present in any actual implementation. The first one is the fact that Alice and Bob always deal with non-maximally entangled resources (finitely squeezed two-mode states) and...
FIG. 20: (Color online) We show a Gaussian distribution with variance $1/(2\lambda)$, $\lambda = 2.0$, and mean $\beta = |\beta|e^{i\text{arg}(\beta)}$ with $|\beta| = 1.5$. The squeezing of the channel is $r = 0.2$. Top panel: Optimal average fidelity as a function of arg($\theta$) (solid curve). The inset shows the Gaussians with greatest (centered on the real and imaginary axis) and lowest (arg($\beta$) = ±45° and arg($\beta$) = ±135°) optimal average fidelities. Left lower panel: The optimal $g_v$ and $g_u$, the values of which are in quadrature (90° dephasing). Note that they are only equal for arg($\beta$) = ±45° and arg($\beta$) = ±135°, i.e., when $|\text{Re}(\beta)| = |\text{Im}(\beta)|$. Right lower panel: The optimal $\theta$, which equals $\theta = 45^\circ$ at the same points where $g_u = g_v$. All dashed curves represent the corresponding values for the original CVTP.

the second one is related to the fact that Alice’s pool of possible coherent states to be teleported cannot cover the entire complex plane.

After studying several different probability distributions for the pool of input states with Alice, we showed that considerable gains in efficiency are achieved for all distributions if we introduce two slight modifications in the original setup. The first modification was the use of a beam splitter whose transmittance could be changed at our will. This beam splitter was employed to mix the state to be teleported with Alice’s share of the entangled resource instead of the usual 50 : 50 beam splitter. The other modification was the possibility to freely choose the displacements in the quadratures, i.e., in position and momentum, of the output (teleported) state with Bob. By allowing these three actions to be independently adjusted once the entanglement of the channel and the pool of input states are known, we were able to achieve considerable gains in efficiency when compared to that predicted by the original protocol.

We have also compared the three parameters optimization strategy against the standard one parameter strategy [14, 15, 19], where the position and momentum gains are not independently chosen. For certain types of distributions for the pool of input states with Alice, those centered at the vacuum state and with circular symmetry, we have shown that the three parameter strategy reduces to the one parameter case. However, when the circular symmetry is broken, the three parameter strategy is crucial in order to obtain a more efficient teleportation protocol. Indeed, we have shown that for circular symmetry broken distributions, the one parameter case does not give significant gains in efficiency when compared to the original teleportation protocol, while at the same conditions the three parameter strategy gives considerable gains.

In addition to important gains in efficiency with the three parameter strategy, we were also able to identify an interesting feature for distributions off-centered from the origin but with the circular symmetry point lying on either the real or imaginary axis. We have shown that these distributions achieve the highest gain in performance when compared to the equivalent distributions with symmetry points centered away from the real and imaginary axis. Also, for distributions with circular symmetry points lying on the real and imaginary axis, we have shown that as we increase the distance of the circular symmetry point from the origin, the optimal efficiency tends to a limiting value that is greater than the efficiency of the original protocol. This effect is more expressive for channels with a low degree of entanglement and is absent for distributions with the symmetry point not belonging to the real and imaginary axis. We believe these interesting properties might be useful in the implementation of continuous variable quantum key distributions based on coherent states [20, 21], where instead of transmitting the coherent state between the parties involved in the key distribution scheme, with the transmission process adding noise and degrading the signal, one teleports it using the optimal strategy here presented.

Furthermore, the calculations we made assumed a two-mode vacuum squeezed state as our entanglement resource and a pool of input states given by coherent states. These choices were dictated by the fact that the usual resources employed in actual implementations of continuous variable teleportation are described by these states [3, 7]. However, the formalism presented in Sec. III is quite general and can be easily adapted to any input state and any type of quantum channel. These changes are mathematically implemented by simply substituting the input and entangled states’ expansion coefficients in the position basis, Eqs. (1), (2), (18), and (19), with the corresponding ones for the new states.

Finally, we would like to point out to a particular extension of the research here presented that might prove fruitful. It is an extensive analysis of the three parameter optimization strategy for several pool of input states assuming that Alice and Bob share quantum channels given by mixed states. Since decoherence, noise, and attenuation drive pure entangled states to mixed ones after a sufficient exposure time, it would be interesting to investigate whether the same techniques here presented can be helpful in improving the efficiency of a continuous variable teleportation protocol that employs non pure channels.
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Appendix A: Optimal settings for a specific $|\alpha\rangle$

Looking at Eq. (20) we see that it depends on the entanglement/squeezing of the channel ($r$) shared between Alice and Bob, Alice’s choice for the BS transmittance ($\cos^2 \theta = g_v$, $g_u$, and the input state $|\alpha\rangle$). In the original CVTP [8] the choices for $\theta, g_v$, and $g_u$ are such that the fidelity becomes independent of the input state (see the discussion in Sec. 11). This particular arrangement of the experiment is best suited when the pool of states with Alice covers all coherent states $|\alpha\rangle$.

Restricting our attention to a specific $|\alpha\rangle$, the following two minor changes to the original setup increases the efficiency of the protocol. The first one keeps the BS transmittance as given by the original CVTP and assumes that $g_v = g_u = g$, with $g$ chosen such that one gets the best fidelity possible [14, 15]. The second one assumes that the BS transmittance, $g_v$, and $g_u$ are freely chosen in order to get the greatest fidelity. Here we analyze both options for a given $|\alpha\rangle$ and show that the latter strategy gives the best results.

Working with the first strategy, i.e. BS transmittance $\cos^2 \theta = 1/2$ and $g_v = g_u = g$, we computed the optimal $F(|\alpha\rangle)$ fixing for definiteness the squeezing of the channel as $r = 0.5$. The results are shown in Fig. 21 where the left panel gives the optimal fidelity and the right panel the optimal $g$. We clearly see that the optimal fidelity and optimal $g$ have radial symmetry, which means that every coherent state with the same amplitude $|\alpha\rangle$ achieves the same optimal fidelity with the same $g$. Also, the optimal fidelity has its maximum value, $F(|\alpha\rangle) = 1$, at the vacuum state. When we move away from the vacuum state, i.e. increase $|\alpha\rangle$, the optimal fidelity and the optimal $g$ tend to the respective values of the original CVTP.

It is interesting to note that since we have radial symmetry for $F(|\alpha\rangle)$, the optimal $g$ is actually the optimal one for the average fidelity for states lying on a circular distribution (see Sec. 11). Therefore, this optimal setting of parameters also satisfies the NSLB condition (see Secs. 11 and 11). This point is illustrated in Fig. 22 where one can see that any state inside a circle of radius $|\alpha\rangle = 3.0$ has fidelity greater than that predicted by the original CVTP if one uses the optimal $g$ for the states on the border to compute all fidelities within the circle.

We now move to the second strategy and let $\theta, g_v$, and $g_u$ be independently chosen for each input state $|\alpha\rangle$ in order to get its optimal fidelity. These calculations are shown in Figs. 23 and 24 where the optimal $F(|\alpha\rangle)$ and the optimal parameters as a function of $|\alpha\rangle$ are respectively shown.

The first thing we note is that the radial symmetry is lost when the three parameters are independently tuned.
FIG. 23: (Color online) The plots show the optimal fidelity $F(|\alpha\rangle)$ as function of the real and imaginary parts of the coherent state $|\alpha\rangle$ for channels with squeezing (a) $r = 0$, (b) $r = 0.5$, and (c) $r = 1$ (left to right). The parameters $g_v$, $g_u$, and $\theta$ are chosen such that they optimize $F(|\alpha\rangle)$ for each state $|\alpha\rangle$. We show the optimal $F(|\alpha\rangle)$ in 3D plots (up) with its respective density plots (bottom). The planes just beneath the 3D plots are the fidelity predicted by the original CVTP.

FIG. 24: (Color online) From left to right we have the density plots of the optimal parameters $\theta$, $g_v$, and $g_u$ giving the optimal fidelities shown in Fig. 23. The z-axis represents the squeezing $r$, which increases from bottom to top ($r = 0, 0.5, \ and \ 1.0$).

in the calculation of the optimized fidelity. Also, we can note that for states along the real and imaginary axis we get a very significant increase in the fidelity of the teleported state. As we move away from these axis the optimal fidelity decreases and approaches the value given by the original CVTP. The optimal values of $\theta$, $g_v$, and $g_u$ also tend to those of CVTP.

For low squeezing $r$, the optimal set of parameters satisfying the NSLB condition are such that $g_v \neq g_u$ and $\theta \neq \pi/4$, with the exception of those states within the lines forming $\pm 45^\circ$ with the abscissa, where the optimal $g_v$ and $g_u$ are exactly equal and $\theta = \pi/4$. In Fig. 22 we compare the optimal fidelities for states on the real and imaginary lines given by the two strategies in order to illustrate this point. Finally, as we approach the $\pm 45^\circ$ lines or as we increase $r$, $g_v \rightarrow g_u$ and $\theta \rightarrow \pi/4$.

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