Integer Laplacian eigenvalues of strictly chordal graphs

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Abstract
In this paper, we establish the relation between classic invariants of graphs and their integer Laplacian eigenvalues, focusing on a subclass of chordal graphs, the strictly chordal graphs, and pointing out how their computation can be efficiently implemented. Firstly we review results concerning general graphs showing that the number of universal vertices and the degree of false and true twins provide integer Laplacian eigenvalues and their multiplicities. Afterwards, we prove that many integer Laplacian eigenvalues of a strictly chordal graph are directly related to particular simplicial vertex sets and to the minimal vertex separators of the graph.

Keywords: integer Laplacian eigenvalue, strictly chordal graph, universal vertex, false and true twins, minimal vertex separator.

1 Introduction
Let $G = (V, E)$ (or $G = (V(G), E(G))$) be a connected graph, where $|E| = m$ is its size and $|V| = n$ is its order. The neighbors of a vertex $v \in V$ are denoted by $N(v) = \{w \in V; \{v, w\} \in E\}$ and its closed neighborhood by $N[v] = N(v) \cup \{v\}$. Two vertices $u, v \in V$ are false twins if $N(u) = N(v)$ and true twins if $N[u] = N[v]$. 


For any $S \subseteq V$, the subgraph of $G$ induced by $S$ is denoted $G[S]$. If $G[S]$ is a complete subgraph then $S$ is a clique in $G$. The complete graph of order $n$ is denoted by $K_n$. A vertex $v$ is said to be simplicial in $G$ when $N(v)$ is a clique in $G$; it is said to be universal when $N[v] = V$.

It is important to mention two kinds of cliques in a chordal graph $G$. A simplicial clique is a maximal clique containing at least one simplicial vertex. A simplicial clique $Q$ is called a boundary clique if there exists a maximal clique $Q'$ such that $Q \cap Q'$ is the set of non-simplicial vertices of $Q$.

For $i, 1 \leq i \leq n$, let $d_i$ be the degree of vertex $v_i$ of $G$. The Laplacian matrix of $G$ of order $n$ is defined as $L(G) = D(G) - A(G)$, where $D(G) = diag(d_1, \ldots, d_n)$ denotes the diagonal degree matrix and $A(G)$ the adjacency matrix of $G$. As $L(G)$ is symmetric, there are $n$ real eigenvalues and, as $L(G)$ is positive semidefinite, all these eigenvalues are non-negatives. We denote the eigenvalues of $L(G)$, called the Laplacian eigenvalues of $G$, by $\mu_1(G) \geq \cdots \geq \mu_n(G)$. All different Laplacian eigenvalues of $G$ together with their multiplicities form the Laplacian spectrum of $G$, denoted by $\text{Spec}_{L}(G)$. A graph is called Laplacian integral if its spectrum consists of integers; in the literature there are several articles on the subject [7], [8], [12], [13], [17], [18] and [22].

This paper resumes the subject already treated by the authors in [2] and [3]. We establish the relation between classic invariants of graphs and their integer Laplacian eigenvalues, pointing out how their computation can be efficiently implemented. We focus on a subclass of chordal graphs [14], the block duplicate graphs, introduced by Golumbic and Peled [10] and also defined by [15] as strictly chordal graphs, based on hypergraph properties [10] [16]; this class contains the classes of block graphs [14], block-indifference graphs [2], the generalized core-satellite graphs [6] and the $(k, t)$-split graphs [3]. In Section 2, we review results concerning general graphs, showing that the number of universal vertices and the degree of false and true twins can provide integer Laplacian eigenvalues and their multiplicities. In Section 3, we prove that many integer Laplacian eigenvalues of a strictly chordal graph are directly related to particular simplicial vertex sets and to the minimal vertex separators of the graph.

2 Universal vertices, twin vertices and integer Laplacian eigenvalues

This section is devoted to review known results from the literature concerning certain integer Laplacian eigenvalues based on classical invariants such as universal vertices and false and true twins. The goal is to show that such results as rewritten here allow us to determine these values in polynomial time.

The proof of the next theorem derives from Corollary 13.1.4 [11] which relates the universal vertices of $G$ with the connected components of its complement graph $\overline{G}$. It also derives from Theorem 4.1.8 [23] that states that if a graph $G$ is a connected graph of order $n$, then $n$ is a Laplacian eigenvalue of $G$ if and
only if \( G \) is a join of two graphs.

**Theorem 1** Let \( G \) be a connected non-complete graph of order \( n \). If \( G \) has \( k \) universal vertices, \( n \) is a Laplacian eigenvalue of \( G \) with multiplicity \( k \).

**Proof.** Since \( G \) can be expressed as the join of a graph induced by the set of universal vertices and the graph induced by the remaining vertices, the theorem mentioned above can be applied. \( \square \)

Let \( G = (V, E) \) be a connected graph and let \( F \subseteq V \) \((T \subseteq V)\) be a set of false twins (true twins) of order \( k \). We can observe that the graph induced by \( F \) is an independent set of size \( k \).

The next result can be found in [1] using the concept of clusters. We present here a different proof.

**Theorem 2** Let \( G \) be a connected graph with a set of false twins \( F \) such that each vertex has degree \( d \). Then, \( d \) is an integer Laplacian eigenvalue of \( L(G) \) with multiplicity at least \( |F| - 1 \).

**Proof.** Consider \( L(G) \) the Laplacian matrix of \( G \) with the set of false twins \( F \) labeled \( v_1, \ldots, v_k \). Label their neighbors consecutively by \( v_{k+1}, \ldots, v_{d+k} \) and the remaining vertices of the graph by \( v_{d+k+1}, \ldots, v_n \). If \( v_i, v_j \in F \) then \((v_i, v_j) \notin E\) and \( d_i = d_j = d \). So, the Laplacian matrix of \( G \) can be written as

\[
L(G) = \begin{bmatrix}
I_k & -J_k & 0 \\
-J_k & dI_k & 0 \\
0 & 0 & (n-k-d)I_k
\end{bmatrix},
\]

where undefined elements \((\ast)\) substitute the integer elements of the \((n-k) \times (n-k)\)-submatrix of \( L(G) \) after taking out the \( k \) first lines and columns of \( L(G) \). For \( 2 \leq i \leq k \), \( e_i \) are eigenvectors of \( L(G) \) associated to \( d \) where \( e_i \) is the \( i - th \) vector of the canonical basis of \( \mathbb{R}^n \). So, \( d \) is an integer Laplacian eigenvalue of \( L(G) \) with multiplicity at least \( k - 1 = |F| - 1 \). \( \square \)

The following theorem is a redraft of a result due to Grone and Merris [13].

**Theorem 3** Let \( G \) be a connected graph with a set of true twins \( T \) such that each vertex has degree \( d \). Then, \( d+1 \) is an integer Laplacian eigenvalue of \( L(G) \) with multiplicity at least \( |T| - 1 \).

**Proof.** The prove is similar to that given in Theorem 2 where it is enough to change the block \( dI_k \) in the Laplacian matrix \( L(G) \) by the block \((d+1)I_k\). \( \square \)

Figure 1 illustrates the computation of integer Laplacian eigenvalues of a graph by Theorems 2 and 3. \( F = \{g, h, i\} \) is a set of false twins; each vertex has degree 2 and, from Theorem 2, \( G \) has 2 as a Laplacian eigenvalue with multiplicity 2. \( T = \{k, l, j\} \) is a set of true twins; each vertex has degree 4. By Theorem 3, \( G \) has 5 as a Laplacian eigenvalue with multiplicity 2.

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2.1 Algorithmic aspects

If \( G = (V,E) \) has universal vertices, false or true twin sets, it is possible to efficiently recognize the existence of some of its integer Laplacian eigenvalues, by means of simple algorithms.

Given the set of vertices \( V \) and the set of edges \( E \), the following steps are needed:

Step 1: Perform a sequential search through the edges to build, for all \( v \in V \), \( N(v) \), stored in a list. The other lists \( N[v], v \in V \), are immediate since \( N[v] = N(v) \cup \{v\} \). This step takes \( O(m) \) time complexity.

Step 2: Apply a lexicographic ordering in both set of lists. All false twins and true twins appear together in the resulting lists. Using the radix sort \[24\], the result can be found in \( O(n^2) \) time complexity.

Observe that only Step 1 is necessary in order to determine the universal vertices, by adding just a test: if \( |N(v)| = n - 1 \) then vertex \( v \) is universal. The time complexity remains the same.

3 Results on strictly chordal graphs

The block duplicate graphs were introduced by Golumbic and Peled \[10\]: it is a graph obtained by adding zero or more true twins to the vertices of a block graph. The class, a subclass of chordal graphs, was also defined as strictly chordal graphs by \[15\] based on hypergraph properties and it was proved to be gem-free and dart-free \[10,16\] (see Figure 2).

In this section we present new results for this class. In Subsection 3.1, an important result due to Cardoso and Rojo \[6\] is rewritten in Theorem 4.2 taking into account only integer Laplacian eigenvalues. In Subsection 3.2, properties of strictly chordal graphs are stated. Based on them, in the last subsection,
we prove two new results. Theorem 6 shows how Theorem 3 can be applied directly to the minimal vertex separators of biconnected strictly chordal graphs and Theorem 7, our main result, gives the minimum number of integer Laplacian eigenvalues of a strictly chordal graph through particular simplicial vertices.

Figure 2: Gem and dart graphs

3.1 Clusters and some integer Laplacian eigenvalues

Let $G$ be a connected chordal graph of order $n$ with a set of $k$ false twins $F$. Let $S$ be the set of $\ell$ neighbors of each vertex of $F$. The pair $(F, S)$ is denoted a $(k, \ell)$-cluster or, simply, a cluster of $G$. It is immediate that $S$ is a minimal vertex separator of the graph. Cardoso and Rojo have defined clusters for any connected graph using the concept of co-neighbor vertices. Also, they built a new graph $H$ by insertion of edges between pairs of false twins vertices and proved that the intersection of the Laplacian spectrum of $H$ with the Laplacian spectrum of $G$ is not empty.

Let $G$ be a connected graph of order $n$ having a $(k, \ell)$-cluster $(F, S)$. Let $H$ be a graph of order $k$. Then $G(H)$ denotes the graph obtained from $G$ when the vertices of $H$ are identified with the vertices of $F$. The clusters $(F_1, S_1), (F_2, S_2)$ are disjoint in $G$ if $F_1 \cap F_2 = \emptyset$ and $S_1 \cap S_2 = \emptyset$. In the more general form, Cardoso and Rojo gave the following definition. Consider $G$ having $t$ pairwise disjoint clusters $(F_1, S_1), (F_2, S_2), \ldots, (F_t, S_t)$. For $1 \leq j \leq t$, let $H_j$ be a graph of order $|F_j|$. Then, $G(H_1, \ldots, H_t)$ denotes the graph obtained from $G$ where the vertices of each graph $H_j$ is identified with the vertices in $F_j$. It follows that $V(H_j) = F_j$, $V(G(H_1, \ldots, H_t)) = V(G)$ and $E(G(H_1, \ldots, H_t)) = E(G) \cup E(H_1) \cup \ldots \cup E(H_t)$. We denote $F = \bigcup_{j=1}^{t} F_j$ and $FS = \bigcup_{j=1}^{t} (F_j \cup S_j)$.

The next result shows that the Laplacian eigenvalues of $G(H_1, \ldots, H_t)$ remain the same, independently of the graphs $H_1, \ldots, H_t$, with the exception of $|F_1| + \ldots + |F_t| - t$ eigenvalues. Let $\tilde{L}(G - F)$ be the principal submatrix of $L(G)$ obtained after deleting the rows and columns of $L(G)$ with indices in $F$.

**Theorem 4** Let $G$ be a connected graph of order $n$ with $t \geq 1$ pairwise disjoint clusters $(F_1, S_1), \ldots, (F_t, S_t)$. For $j = 1, \ldots, t$, assume that $|F_j| = k_j$, $|S_j| = \ell_j$ and each graph $H_j$ is defined as above to obtain $G(H_1, \ldots, H_t)$ such that $L_j 1_{k_j} = \mu_{k_j} 1_{k_j}$ where $L_j$ is the Laplacian matrix of $H_j$. It follows that
\[
\det(\lambda I - L(G(H_1, \ldots, H_t))) = p_L(\lambda) \prod_{j=1, t} \prod_{i=1, k_j} (\lambda - (\ell_j + \mu_i(L_j))) \quad (1)
\]

where \(p_L(\lambda)\) is the characteristic polynomial of the matrix \(\tilde{L}(G - FS)\) whose degree is \(n - \sum_{j=1, t} k_j + t\). In particular, when the graphs \(H_1, \ldots, H_t\) are the empty graphs, \(G(H_1, \ldots, H_t) = G\) and

\[
\det(\lambda I - L(G)) = p_L(\lambda) \prod_{j=1, t} (\lambda - \ell_j)^{k_j - 1}. \quad (2)
\]

### 3.2 Chordal and strictly chordal graphs

A chordal graph is a graph in which every cycle of length four and greater has a cycle chord. Basic concepts about chordal graphs are assumed to be known and can be found in Blair and Peyton [4] and Golumbic [9]. Following, the most pertinent concepts are reviewed.

A subset \(S \subset V\) is a **separator** of \(G\) if at least two vertices in the same connected component of \(G\) are in two distinct connected components of \(G[V \setminus S]\). The set \(S\) is a **minimal separator** of \(G\) if \(S\) is a separator and no proper set of \(S\) separates the graph. The set of minimal separators of \(G\) is denoted by \(S\).

Let \(G = (V, E)\) be a chordal graph and \(u, v \in V\). A subset \(S \subset V\) is a **vertex separator** for non-adjacent vertices \(u\) and \(v\) (a **uv-separator**) if the removal of \(S\) from the graph separates \(u\) and \(v\) into distinct connected components. If no proper subset of \(S\) is a **uv-separator**, then \(S\) is a **minimal uv-separator**. When the pair of vertices remains unspecified, we refer to \(S\) as a **minimal vertex separator** (**mvs**). The set of minimal vertex separators of \(G\) is denoted by \(S\). A linear algorithm to determine the set of minimal vertex separators can be found in [20]. A minimal separator of \(G\) is always a minimal vertex separator but the converse is not true.

A **strictly chordal graph** is obtained by adding zero or more true twins to each vertex of a block graph \(G\). A new characterization based on minimal vertex separators was presented by Markenzon and Waga [21]. Based on the characterization theorem, a recognition algorithm becomes very simple.

**Theorem 5** Let \(G = (V, E)\) be a chordal graph and \(S\) be the set of minimal vertex separators of \(G\). The following statements are equivalent:

1. \(G\) is a strictly chordal graph.
2. For any distinct \(S, S' \in S\), \(S \cap S' = \emptyset\).
3. \(G\) is gem-free and dart-free.

Interesting properties of strictly chordal graphs can be stated.
Property 1 Let $G$ be a strictly chordal graph, $\mathcal{S}$ the set of minimal vertex separators of $G$ and $\mathcal{S}$ the set of minimal separators of $G$. Then $\mathcal{S} = \mathcal{S}$.

Proof. In [19], this property was proved for the non-inclusion chordal graphs. As strictly chordal graphs are a subclass of non-inclusion graphs, the result follows.

Property 2 Let $G$ be a strictly chordal graph. Then all boundary cliques contain exactly one minimal vertex separator.

Proof. By definition, a maximal clique $Q$ is called a boundary clique if there exists a maximal clique $Q'$ such that $Q \cap Q'$ is the set of non-simplicial vertices of $Q$. The set $Q \cap Q'$ is a clique; so it is a minimal vertex separator. As $G$ is a strictly chordal graph there is not proper containment of separators, then a boundary clique contains exactly one minimal vertex separator.

Property 3 Let $G$ be a strictly chordal graph and $S$ a minimal vertex separator of $G$. Then all vertices of $S$ are true twins.

Proof. By the definition, a strictly chordal graph is obtained by adding zero or more true twins to the vertices of a block graph. The separators of a block graph have cardinality one. Adding true twins to this vertex results in a separator of greater cardinality with the same set of neighbours.

3.3 New results

The next result shows how Theorem 5 can be applied directly to the minimal vertex separators of strictly chordal graphs.

Theorem 6 Let $G = (V, E)$ be a strictly chordal graph and let $\mathcal{S}$ be the set of minimal separators of $G$. Let $S \in \mathcal{S}$ and $v$ a vertex belonging to $S$. Then $d(v)+1$ is an integer Laplacian eigenvalue of $L(G)$ with multiplicity at least $|S| - 1$.

Proof. As $G$ is a chordal graph, set $S$, for all $S \in \mathcal{S}$, is a clique. By Theorem 5 the minimal vertex separators of $G$ do not intercept, and, as so, all vertices of $G$ have the same set of neighbours; thus they are true twins. By Theorem 5 the result follows.

Observe that, if $G$ is a biconnected strictly chordal graph, then there is at least one non zero integer eigenvalue of $L(G)$ for each minimal vertex separator of $G$.

Figure 3 illustrates how Theorem 6 can easily determine 6, 9 and $7(2)$ as integer Laplacian eigenvalues of the graph which spectrum is:

$SpecL(G) = [0; 1.18541; 2.61293; 3.72314; 5.64590; 6; 6.55734; 7^{(2)}; 9; 9.27527]$. 
Figure 3: Laplacian eigenvalues × separators in a strictly chordal graph

The next result, Theorem 7, gives the minimum number of integer Laplacian eigenvalues of a strictly chordal graph that derives from the quantity of some simplicial vertices of the graph.

Let $Q$ be the set of maximal cliques. For each minimal vertex separator $S$, $S \in \mathcal{S}$, let us denote by $B(S)$ the set of boundary cliques that contain $S$. Each maximal clique belonging to $B(S)$ can be partitioned into two subsets: $S$ and $P$, a set of simplicial vertices. If $|B(S)| > 1$ and each maximal clique has exactly one simplicial vertex, $B(S)$ is a cluster, as defined by Cardoso and Rojo [5].

**Theorem 7** Let $G$ be a strictly chordal graph and $S^*$ the set of minimal vertex separators of cardinality $t$, such that, for each $S_i \in S^*$, $|B(S_i)| > 1$. Let $\mathcal{P}_i$ be the set of simplicial vertices belonging to $B(S_i)$. Then, $G$ has at least $\sum_{i=1}^{t} (|\mathcal{P}_i| - 1)$ integer Laplacian eigenvalues.

**Proof.** Let $G$ be a strictly chordal graph and $S_i \in S^*$ a minimal vertex separator of $G$. Let $|S_i| = s_i$, $i = 1, \ldots, t$, $|B(S_i)| = b_i$ and $Q_{ij} \in B(S_i)$, $j = 1, \ldots, b_i$. $P_{ij}$ denotes the set of simplicial vertices of $Q_{ij}$, where $Q_{ij} = P_{ij} \cup S_i$. As $G$ is a chordal graph, the subgraph induced by $P_{ij}$ is the complete graph $K_{n_{ij}}$ of order $n_{ij}$. Moreover, the subgraph induced by $\mathcal{P}_i = \cup_{j=1,...,b_i} P_{ij}$ is $H_i = \cup_{j=1,...,b_i} K_{n_{ij}}$. So, $|\mathcal{P}_i| = \sum_{j=1,...,b_i} n_{ij}$.

Since $G$ is a strictly chordal graph, their minimal vertex separators are two by two disjoint sets. From Theorem 4 [1], for each $i, i = 1, \ldots, t$, all Laplacian eigenvalues of each $H_i$ are $s_i + n_{ij}, j = 1, \ldots, b_i$. Since $G$ has $t$ induced subgraphs $H_i$ with $|\mathcal{P}_i|$ simplicial vertices, the result follows. So, $G$ has at least $\sum_{i=1}^{t} (|\mathcal{P}_i| - 1)$ integer Laplacian eigenvalues.

As already mentioned, if $G$ is a strictly chordal graph, their minimal vertex separators are two by two disjoint sets. In order to simplify the notation in the remaining of the text, we work with only one minimal vertex separator $S \in S^*$.

**Corollary 7.1** Let $S \in S^*$, $B(S) = \{Q_1, \ldots, Q_k\}$ and $P_k$ the set of simplicial vertices of $Q_k$. Then the following values are some of the integer Laplacian eigenvalues of $G$:
1. \(|Q_k|\) with multiplicity \(|P_k| - 1\), \(\forall Q_k \in B(S)\);
2. \(|S|\) with multiplicity \(b - 1\).

**Corollary 7.2** Let \(S \in S^*\), \(B(S) = \{Q_1, \ldots, Q_b\}\) and \(P_k\) the set of simplicial vertices of \(Q_k\). Let \(\mathcal{P}\) be the set of simplicial vertices in \(B(S)\) and \(\mathcal{F}\), the set of false twins in \(\mathcal{P}\). The number of integer Laplacian eigenvalues uniquely provided by Theorem 4 is:

\[
\begin{cases} 
|\mathcal{P}| - 1 - \sum_{k=1}^{b}(|P_k| - 1) & \text{if } \mathcal{F} = \emptyset \\
|\mathcal{P}| - \sum_{k=1}^{b}(|P_k| - 1) - |\mathcal{F}| & \text{otherwise.}
\end{cases}
\]

The following example illustrates these results. Consider the graph \(G\) of Figure 4. There is a minimal vertex separator \(S = \{d\}\) of \(G\), such that \(B(S) = \{\{d, e, f\}, \{d, g, h\}, \{d, i, j\}, \{d, k, l, m\}\}\). So, \(\mathcal{P} = \{e, f, g, h, i, j, k, l, m\}\). Since \(|B(S)| > 1\), by Corollary 7.1 there are 8 integer Laplacian eigenvalues of \(G\): \(4^{(2)}, 3^{(3)}, 1^{(3)}\). Some of them, \(4^{(2)}, 3^{(3)},\) are already provided by Theorem 3. The remaining three integer eigenvalues are computed by Corollary 7.2 as can be seen in the spectrum below.

\[\text{Spec}_{L}(G) = [0; 0.23941; 1^{(3)}; 1.53342; 3^{(3)}; 3.21582; 4^{(2)}; 11.01135].\]

![Figure 4: Laplacian eigenvalues × boundary cliques](image)

A strictly chordal graph can have more simplicial vertices than the ones described in Theorem 7 which result from Theorem 3. Let \(Q \in Q\) and let \(P\) be the set of simplicial vertices belonging to \(Q\). If \(Q\) does not belong to any \(B(S)\) being \(S \in S^*\) then by Theorem 3 we can conclude that the value \(|Q|\) with multiplicity \(|P| - 1\) is an integer Laplacian eigenvalue of \(G\).
3.3.1 Algorithmic aspects

For the class of strictly chordal graphs, several integer Laplacian eigenvalues are directly deduced from its structural properties, as can be seen in Theorems 3, 6 and Corollaries 7.1 and 7.2. It is interesting to highlight the fact that the determination of these eigenvalues has linear time complexity. Given a strictly chordal graph $G = (V, E)$, the following steps are needed:

Step 1: determine the set of maximal cliques $Q$ and the set of minimal vertex separators $S$ of $G$. By Theorem 5, all minimal vertex separators are pairwise disjoint, so each vertex of $G$ can be labeled as a simplicial vertex or as belonging to exactly one minimal vertex separator $S$. This step can be accomplished in linear time complexity [20].

Step 2: test if $|S| > 1$, for each $S \in S$; in this case, by Theorem 6 $d(v) + 1$, for $v \in S$, with multiplicity $|S| - 1$ is a Laplacian eigenvalue. As each vertex belongs to at most one minimal vertex separator, $\sum_{S \in S} |S| < n$. This step has linear time complexity.

Step 3: determine $B(S)$ for each $S \in S$. By Property 2, all boundary cliques contain exactly one minimal vertex separator. So, a sequential search through the maximal cliques is enough to determine $B(S)$ for all the minimal vertex separators and to determine also the set of simplicial vertices $P$ that belongs to $B(S)$. This step must search through the vertices of all maximal cliques; the time complexity of this step is $O(n + m)$.

Step 4: For each $S \in S$, test if $|B(S)| > 1$. If yes, compute, for $B(S) = \{Q_1, \ldots, Q_b\}$, the set $P_i$ of simplicial vertices of $Q_i$. By Corollary 7.1 there are the following integer Laplacian eigenvalues in $G$:

1. $|Q_i|$ with multiplicity $|P_i| - 1, \forall Q_i \in B(S)$;
2. $|S|$ with multiplicity $b - 1$.

If no and $|B(S)| = 1$, then $B(S) = \{Q\}$ and $P$ is the set of simplicial vertices of $Q$. Thus $|Q|$ with multiplicity $|P| - 1$ is an integer Laplacian eigenvalue of $G$. This step must search all maximal cliques belonging to $B(S)$ for each $S \in S$. As $\cup_{S \in S} B(S) \subset Q$ the time complexity of this step is $O(n + m)$.

Step 5: determine the set of maximal cliques $Q$ that do not belong to any $B(S)$. For each one of these cliques, determine the set of simplicial vertices, $P$. By Theorem 3, $|Q|$, with multiplicity $|P| - 1$, is an integer Laplacian eigenvalue of $G$. As this step must also search all maximal cliques of the graph then its time complexity is $O(n + m)$.

4 Conclusions

In this paper, some important results already known in the literature were reviewed in terms of structural parameters of the graph. Theorems 11 and 2...
provide some integer Laplacian eigenvalues of a connected graph, enabling us to efficiently implement their computation. For the class of strictly chordal graphs, some new integer Laplacian eigenvalues are revealed, which can be seen in Theorem 7. Also interesting to highlight is the fact that the determination of a minimum number of integer Laplacian eigenvalues of a strictly chordal graph has linear time complexity, provided by resourceful graph algorithms.

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