Asphericity and small cancellation theory for rotation family of groups.

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Abstract

Using small cancellation for rotating families of groups, we construct new examples of aspherical polyhedra.

Contents

1 Hyperbolic spaces ........................................ 3
   1.1 Ultra-limit of hyperbolic spaces ..................... 3
   1.2 Quasi-convexity ..................................... 5
   1.3 Asphericity ........................................ 6

2 Cone over a metric space .................................. 7
   2.1 Definition ........................................ 7
   2.2 Cone and hyperbolicity ............................. 9
   2.3 Group acting on a cone ............................ 10

3 Cone-off over a metric space ............................. 10
   3.1 Definition ........................................ 10
   3.2 Uniform approximation of the distance on the cone-off ........................ 12
   3.3 Hyperbolicity of the cone-off over an R-tree .................. 13
   3.4 Hyperbolicity of the cone-off over a hyperbolic space .................. 14
   3.5 Length structure on the cone-off ................... 17

4 Small cancellation theory ................................. 18
   4.1 Orbifold ........................................ 18
   4.2 Statement of the very small cancellation theorem .......... 19
   4.3 Proof of the very small cancellation theorem .......... 20

5 Examples of aspherical complexes ....................... 21

Introduction

The goal of this article is to produce new examples of aspherical polyhedra. The example we have in mind is the following: let $P$ be an aspherical simplicial complex and $Q$ a subcomplex of $P$. We define $\tilde{P}$ by attaching on $P$ a cone of base $Q$. We are looking for some conditions under which $\tilde{P}$ remains aspherical. This kind of situation was already studied by J. H. C. Whitehead in the 1940's. In [Whi1] and [Whi2] he studied the second homotopy group of a space $\tilde{P}$ obtained by attaching 2-cells on a cell complex $P$. He proved that $\pi_2(\tilde{P})$ exactly describes the identities between the relators that define the projection $\pi_1(P) \twoheadrightarrow \pi_1(\tilde{P})$.

Our main example of application is the following. Let $H_n(C)$ be the complex, $n$-dimensional, hyperbolic space. We consider $SO(n, 1)$ as the stabilizer of the real hyperbolic space $H_n(R)$ in $H_n(C)$. Let $G \subset SU(n, 1)$ be a real lattice, i.e. a lattice of $SU(n, 1)$ such that $H = G \cap SO(n, 1)$ is still a lattice of $SO(n, 1)$. We want to study the group $G/ \ll H \gg$.

**Theorem.** There exists a finite index subgroup $G'$ of $G$ with the following property. Let $H'$ be the group $G' \cap SO(n, 1)$. Let $\tilde{P}$ be the space obtained by attaching on $H_n(C)/G'$ a cone of base $H_n(R)/H'$. The complex $\tilde{P}$ is a classifying space for the group $G' = G'/ \ll H' \gg$. 
This situation is similar to a result of N. Bergeron in [Ber03] who proved that the map from the homology associated with $H_1(R)/G \cap SO(n,1)$ in the one of $H_1(C)/G$ is one to one. We also establish some analogue statements to our theorem for the pairs $(SO(n,1), SO(k,1))$, $(SU(n,1), SU(k,1))$ and $(Sp(n,1), Sp(k,1))$.

Our strategy consists to endow the space $\bar{P}$ with a local hyperbolic geometry and to apply a version of the Cartan-Hadamard theorem. This will prove that the universal cover $\bar{X}$ of $\bar{P}$ is globally hyperbolic. We deduce then the asphericity from a kind of Rips’ theorem (cf. Th. [1.3.4]); if $\bar{X}$ is a hyperbolic simplicial complex, it is sufficient to prove that $\bar{X}$ is locally contractible. This last local assumption follows from the property of the developing map.

The hyperbolic structure on $\bar{P}$ is constructed as follows. Given a subcomplex $Q$ of $P$, we endow the cone of base $Q$ with a metric modelled on the hyperbolic disc. An argument of Berestovskii tells us that if $Q$ is CAT(1) then the cone is CAT(-1) (see [BH99]). In particular it is hyperbolic. The problem consists then to find some conditions so that the complex $\bar{P}$ remains locally hyperbolic.

With this aim in mind, we explore an idea of M. Gromov (see [Gro03]) to extend the small cancellation theory to a so called “rotation family” of groups. If $X$ is a metric space and $G$ a group acting on $X$ by isometries, a rotation family is a pairwise distinct collection $(Y_i, H_i)_{i \in I}$ such that

- $H_i$ is a subgroup of $G$ stabilizing $Y_i \subset X$.
- There is an action of $G$ on $I$ compatible with the one on $X$ (i.e. for all $g \in G$, for all $i \in I$, $Y_{gi} = gY_i$ and $H_{gi} = gH_i g^{-1}$).

In order to study such a family, we introduce two quantities that respectively play the role of the smallest piece and the largest relator in the usual small cancellation theory. The constant $\Delta$ measures the overlap between two $Y_i$’s whereas $\rho$ is the minimal translation length of an element that belongs to a $H_i$.

Given a rotation family, we define the cone-over off $X$, denoted by $\bar{X}$, by attaching over $X$ cones of base $Y_i$. We consider the space $\bar{P} = \bar{X}/\bar{G}$ where $\bar{G} = G/\ll H_i \gg$. The small cancellation provides us a framework where it is possible to endow $\bar{P}$ with a local hyperbolic geometry. Moreover this theory recovers the usual small cancellation (see [LS77] or [Ol01]) and the small cancellation with graphs as well (see [Gro03] or [Ol06]).

We describe the space $\bar{P}$ as an orbifold using two kind of charts: the cones and the cone-off. The second part of the article is dedicated to the study of the geometry of the cones. Adapting an argument of Berestovskii, we prove that, the cone over a hyperbolic space remains hyperbolic (cf. Th. [2.3.2]). The third part deals with the cone-off $\bar{X}$. In particular we prove the following fact. Under small cancellation assumptions, the cone-off over a hyperbolic space is still hyperbolic (cf. Th. [3.4.2]). We choose for this proof an asymptotical point of view that involves ultra-limits as in [Dru02]. The main technical lemma consists to switch the cone-off construction and the ultra-limit: given a sequence of metric spaces $X_n$, we prove (see Cor. [3.4.9]) that there exists a local isometry between the cone-off over the ultra-limit of $X_n$ and the ultra-limit of $\bar{X}_n$. The fourth part mixes all the previous ingredients in order to obtain the following theorems.

**Theorem (cf Th. [4.2.2]).** There exist two positive numbers $\delta_0$ and $\Delta_0$ satisfying the following property.

Let $X$ be a geodesic, simply connected, $\delta$-hyperbolic space and $G$ a group acting properly, by isometries on $X$. Let $(Y_i, H_i)_{i \in I}$ be a rotation family, such that each $Y_i$ is strongly-quasi-convex.

Let $N$ be the normal subgroup of $G$ generated by the $H_i$’s and $\bar{G}$ the quotient group $G/N$. Assume also that

$$\frac{\delta}{\rho} \leq \delta_0 \quad \text{and} \quad \frac{\Delta}{\rho} \leq \Delta_0$$

Then, $\bar{X}$ is simply and compactly connected and hyperbolic and $\bar{G}$ acts properly by isometries on $\bar{X}$.

Moreover if $G$ (respectively $H_i$) acts co-compactly on $X$ (respectively $Y_i$) and $I/G$ is finite, then $\bar{X}/\bar{G}$ is compact. In particular $\bar{G}$ is hyperbolic.

**Theorem (cf Th. [4.3.7]).** Under the same hypotheses, if $X$ is a $n$-dimensionnal simplicial complex such that every closed ball of $X$ or $Y_i$ is homotopic to zero, then $\bar{X}$ is contractible.
Remark: In some cases, the space \( \bar{P} = X/G \) may be endowed with a sharper geometry that the hyperbolic one. For instance, M. Gromov constructed in a similar way CAT(-1) polyhedra in order to produce infinite torsion groups (see [GdlH90 Chap 12]). In [Grot01b] M. Gromov introduced the notion of CAT(-1,\( \varepsilon \)) spaces, a \( \varepsilon \)-perturbation of CAT(-1)-spaces. This provides another framework to study small cancellation constructions that is not asymptotic.

In the last part, we explain how to construct examples of rotation family that satisfy the small cancellation assumptions. To that end, we use the geometry of lattices and a result of N. Bergeron about the profinite topology of finitely generated linear groups [Ber00]. It leads to these new examples of aspherical polyhedra.

**Question:** Let \( G \) be as previously a real lattice of \( SU(n,1) \) and \( H \) the subgroup \( G \cap SO(n,1) \). Does \( G = G/ \ll H \gg \) have the Kazhdan property (T) ?

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## 1 Hyperbolic spaces

Let \( X \) be a metric space. If \( x \) and \( x' \) are two points of \( X \), we denote by \(|x' - x|_X \) (or simply \(|x' - x|\)) the distance between them. Although it may not be unique, we denote by \([x, x']\) a geodesic joining \( x \) and \( x' \). Given a base point \( x \), the Gromov product of two points \( y \) and \( z \) is defined by

\[
\langle y, z \rangle_x = \frac{1}{2} \left( |y - x| + |z - x| - |z - y| \right)
\]

Let \( \delta \) be a non negative number. The space \( X \) is \( \delta \)-hyperbolic if for all \( x, y, z, t \in X \), we have

\[
\langle x, z \rangle_t \geq \min \{ \langle x, y \rangle_t, \langle y, z \rangle_t \} - \delta.
\]

R-trees are very special examples of hyperbolic spaces.

**Proposition-definition 1.0.1** ([GdlH90 Chap 2 Prop 6] or [CDS00 Chap. 3 Th. 4.1]), An R-tree is a geodesic space such that every two points are connected by a unique topological arc. A metric space is an R-tree if and only if it is geodesic and \( \delta \)-hyperbolic.

**Definition 1.0.2** (Quasi-isometry). Let \( \eta \) be a non negative number. A \((1, \eta)\)-quasi-isometry is a map \( f : X \to Y \) between two metric spaces such that for all \( x, x' \in X \), we have

\[
|x' - x| - \eta \leq |f(x') - f(x)| \leq |x' - x| + \eta
\]

The next result is a very easy case of the stability of quasi-geodesics. An asymptotic proof of this fact for a general \((\lambda, k)\)-quasi-isometry can be founded in [CDS08].

**Proposition 1.0.3.** Let \( \delta \) be a non negative number. For all \( \delta' > \delta \), there exists \( \eta > 0 \) satisfying the following property. Let \( X \) be a metric space and \( Y \) a \( \delta \)-hyperbolic space. If there exists a \((1, \eta)\)-quasi-isometry from \( X \) to \( Y \), then \( X \) is \( \delta' \)-hyperbolic.

**Proof.** Let \( f : X \to Y \) be a \((1, \eta)\)-quasi-isometry. For all \( x, y, z \in X \) we have

\[
\langle f(x), f(y) \rangle_{f(x)} - \frac{3}{2} \eta \leq \langle x, y \rangle_x \leq \langle f(x), f(y) \rangle_{f(x)} + \frac{3}{2} \eta
\]

It follows that for all \( x, y, z, t \in X \),

\[
\langle x, z \rangle_t \geq \langle f(x), f(z) \rangle_{f(t)} - \frac{3}{2} \eta
\]

\[
\geq \min \{ \langle f(x), f(y) \rangle_{f(t)}, \langle f(y), f(z) \rangle_{f(t)} \} - \frac{3}{2} \eta
\]

\[
\geq \min \{ \langle x, y \rangle_t, \langle y, z \rangle_t \} - (\delta + 3\eta)
\]

Hence \( X \) is \((\delta + 3\eta)\)-hyperbolic.

\[\square\]

### 1.1 Ultra-limit of hyperbolic spaces

Let us recall the definition of the ultra-limit of a sequence of pointed metric spaces and its link with hyperbolicity. For more details about this point of view see [Dr101], [Dr102] or [DS03].
A non-principal ultra-filter is a finite additive map $\omega : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ which vanishes on every finite subset of $\mathbb{N}$. A property $P_n$ is $\omega$-almost sure if $\omega\{ n \in \mathbb{N} / P_n \text{ is true} \} = 1$. A real sequence $(u_n)$ is $\omega$-essentially bounded if there exists $M \in \mathbb{R}$ such that $|u_n| \leq M$, $\omega$-almost surely. If $\omega$ is a real number, we say that the $\omega$-limit of $(u_n)$ is $\omega$ and write $\lim_\omega u_n = \omega$, if for all $\varepsilon > 0$, $|u_n - \omega| < \varepsilon$, $\omega$-almost surely. In particular, any $\omega$-essentially bounded sequence admits an $\omega$-limit (cf. [Bon71]).

Let $(X_n, x_n^0)$ be a sequence of pointed metric spaces. We introduce

$$\Pi_\omega X_n = \{(x_n)/\forall n \in \mathbb{N}, x_n \in X_n \text{ and } (|x_n - x_n^0|) \text{ is } \omega\text{-essentially bounded}\}.$$  

We endow this space with a pseudo-metric defined as follows: $|(y_n) - (x_n)| = \lim_\omega |y_n - x_n|$.

**Definition 1.1.1** (Ultra-limit of metric spaces). Let $(X_n, x_n^0)$ be a sequence of pointed metric spaces and $\omega$ a non-principal ultra-filter. The $\omega$-limit of $(X_n, x_n^0)$, denoted by $\lim_\omega (X_n, x_n^0)$ (or simply $\lim_\omega X_n$) is the quotient of $\Pi_\omega X_n$ by the equivalence relation which identifies the points at distance zero.

The pseudo-distance on $\Pi_\omega X_n$ induces a distance on $\lim_\omega X_n$.

**Notation:**

(i) If $(x_n)$ is an element of $\Pi_\omega X_n$, its image in $\lim_\omega X_n$ is denoted by $\lim_\omega x_n$.

(ii) For all $n \in \mathbb{N}$, let $Y_n$ be a subset of $X_n$. The set $\lim_\omega Y_n$ is defined by

$$\lim_\omega Y_n = \left\{ \lim y_n / (|y_n - x_n^0|) \text{ is } \omega\text{-essentially bounded and } y_n \in Y_n \text{ } \omega\text{-almost surely} \right\}.$$  

**Proposition 1.1.2.** Let $\omega$ be a non-principal ultra-filter. Let $(\delta_n)$ be a sequence of non-negative numbers which admits a $\omega$-limit $\delta$. Let $(X_n, x_n^0)$ be a sequence of pointed metric spaces. If for all $n \in \mathbb{N}$, $X_n$ is $\delta_n$-hyperbolic, then the limit space $\lim_\omega X_n$ is $\delta$-hyperbolic.

**Proof.** Let $x \equiv \lim_\omega x_n$, $y \equiv \lim_\omega y_n$, $z \equiv \lim_\omega z_n$ and $t \equiv \lim_\omega t_n$ be four points of $\lim_\omega X_n$. Since $X_n$ is $\delta_n$-hyperbolic we have for all $n \in \mathbb{N}$, $\langle x_n, y_n, z_n \rangle_{t_n} \leq \min\{\langle x_n, y_n \rangle_{t_n}, \langle y_n, z_n \rangle_{t_n}\} - \delta_n$. Taking the $\omega$-limit, we obtain $\langle x, y, z \rangle_{t} \leq \min\{\langle x, y \rangle_{t}, \langle y, z \rangle_{t}\} - \delta$. Thus $\lim_\omega X_n$ is $\delta$-hyperbolic. $\square$

**Corollary 1.1.3.** Let $\omega$ be a non-principal ultra-filter and $(\delta_n)$ a sequence of non-negative numbers such that $\lim_\omega \delta_n = 0$. Let $(X_n, x_n^0)$ be a sequence of pointed geodesic spaces. If for all $n \in \mathbb{N}$, $X_n$ is $\delta_n$-hyperbolic, then the limit space $\lim_\omega X_n$ is an $R$-tree.

**Proof.** The $\omega$-limit of a sequence of geodesic spaces is still geodesic (cf. [Cap96]). It follows that $\lim_\omega X_n$ is a geodesic, 0-hyperbolic metric space. Hence $\lim_\omega X_n$ is an $R$-tree. $\square$

**Proposition 1.1.4.** Let $\omega$ be a non-principal ultra-filter and $\delta$ a non negative number. Let $(X_n, x_n^0)$ be a sequence of pointed metric spaces whose diameter is bounded. If $\lim_\omega X_n$ is $\delta$-hyperbolic, then for all $\delta' > \delta$, $X_n$ is $\delta'$-hyperbolic $\omega$-almost surely. In particular there exists $n \in \mathbb{N}$ such that $X_n$ is $\delta'$-hyperbolic.

**Proof.** Assume that the proposition is false. Then $\lim_\omega X_n$ is $\delta$-hyperbolic. Nevertheless there exists $\delta' > \delta$ such that $X_n$ is $\omega$-almost sure not $\delta'$-hyperbolic. Thus we can find four sequences $(x_n), (y_n), (z_n)$ and $(t_n)$ satisfying the following properties:

(i) for all $n \in \mathbb{N}$, $x_n, y_n, z_n, t_n \in X_n$

(ii) $\langle x_n, y_n \rangle_{t_n} \leq \min\{\langle x_n, y_n \rangle_{t_n}, \langle y_n, z_n \rangle_{t_n}\} - \delta', \omega$-almost surely

Since $(\text{diam}(X_n))$ is bounded, these four sequences define four points of $\lim_\omega X_n$, respectively $x, y, z$ and $t$. After taking the $\omega$-limit in the previous inequality we obtain

$$\langle x, z \rangle_{t} \leq \min\{\langle x, y \rangle_{t}, \langle y, z \rangle_{t}\} - \delta' < \min\{\langle x, y \rangle_{t}, \langle y, z \rangle_{t}\} - \delta$$

Hence $\lim_\omega X_n$ is not $\delta$-hyperbolic. Contradiction. $\square$
1.2 Quasi-convexity

If \( X \) is a geodesic space, there is an other way to characterize the hyperbolicity using geodesic triangles. Let \( \delta \) be a non negative number. A geodesic triangle is \( \delta \)-thin if each one of its sides is contained in the \( \delta \)-neighbourhood of the union of the two others.

**Proposition 1.2.1** (cf. [CDP90 Chap. 1 Prop. 3.6] or [GdlH90 Chap. 3 §2]). Let \( \delta \) be a non negative number. Consider a geodesic space \( X \).

(i) If \( X \) is \( \delta \)-hyperbolic, then every geodesic triangle of \( X \) is \( 4\delta \)-thin.

(ii) If every geodesic triangle of \( X \) is \( \delta \)-thin, then \( X \) is \( 8\delta \)-hyperbolic.

**Corollary 1.2.2.** Let \( x, x', y \) and \( y' \) be four points of a geodesic \( \delta \)-hyperbolic space \( X \). If \( u \) is a point of \([x, x']\) such that \(|u - x| > |y - x| + 8\delta \) and \(|u - x'| > |y' - x'| + 8\delta \), then \( u \) lies in the \( 8\delta \)-neighbourhood of \([y, y']\).

*Proof.* Since the triangles \([x, y, y']\) and \([x, x', y']\) are \( 4\delta \)-thin, we can find a point \( v \) in \([x, y] \cup [y, y'] \cup [y', x']\) such that \(|v - u| \leq 8\delta \). Assume that \( v \in [x, y] \) (the other case is symmetric).

The triangle inequality gives

\[ |u - x| \leq |u - v| + |v - x| \leq |y - x| + 8\delta \]

Contradiction. Finally \( u \) lies in the \( 8\delta \)-neighbourhood of \([y, y']\). \( \Box \)

**Definition 1.2.3** (Quasi-convexity). Let \( \alpha \) be a non negative number. A subset \( Y \) of a geodesic metric space \( X \) is \( \alpha \)-quasi-convex if every geodesic between two points of \( Y \) is contained in the \( \alpha \)-neighbourhood of \( Y \).

**Notation:** We denote by \( Y^{+\alpha} \) the \( \alpha \)-neighbourhood of \( Y \).

**Proposition 1.2.4** (compare [DG85 Lemma 2.2.2]). Let \( \delta, \alpha \geq 0 \). Let \( X \) be a geodesic \( \delta \)-hyperbolic space. If \( Y \) and \( Z \) are two \( \alpha \)-quasi-convex subsets of \( X \), then for all \( A \geq 0 \) we have

\[ \text{diam} \left( Y^{+A} \cap Z^{+A} \right) \leq \text{diam} \left( Y^{+\alpha+10\delta} \cap Z^{+\alpha+10\delta} \right) + 2A + 20\delta \]

*Proof.* Let \( x \) and \( x' \) be two points of \( Y^{+A} \cap Z^{+A} \) and assume that \(|x' - x| \geq 2A + 20\delta \). We introduce

(i) two points \( t \) and \( t' \) of \([x, x']\) such that \(|t - x| = |t' - x'| = A + 10\delta \)

(ii) two points \( y \) and \( y' \) of \( Y \) such that \(|y - x|, |y' - x'| \leq A + \delta \)

Applying the corollary 1.2.2 \( t \) belongs to the \( 8\delta \)-neighbourhood of \([y, y']\). Since \( Y \) is \( \alpha \)-quasi-convex, \([y, y']\) lies in the \( \alpha \)-neighbourhood of \( Y \). Hence \( t \in Y^{+\alpha+10\delta} \). We prove by the same way that \( t \in Z^{+\alpha+10\delta} \). The same fact holds for \( t' \). Thus

\[ |x' - x| - 2A - 20\delta = |t' - t| \leq \text{diam} \left( Y^{+\alpha+10\delta} \cap Z^{+\alpha+10\delta} \right) \]

The above inequality is true for all \( x, x' \in Y^{+A} \cap Z^{+A} \), hence

\[ \text{diam} \left( Y^{+A} \cap Z^{+A} \right) \leq \text{diam} \left( Y^{+\alpha+10\delta} \cap Z^{+\alpha+10\delta} \right) + 2A + 20\delta \]

\( \Box \)

**Corollary 1.2.5.** Let \( \omega \) be a non-principal ultra-filter, and \( (\delta_n) \) a real sequence such that \( \lim_n \delta_n = 0 \). For all \( n \in \mathbb{N} \), let \((X_n, x_n^0)\) be a pointed, geodesic \( \delta_n \)-hyperbolic space and \( Y_n, Z_n \) two \( 10\delta_n \)-quasi-convex subsets of \( X_n \). Let \( X = \lim_\omega (X_n, x_n^0) \), and \( Y = \lim_\omega Y_n, Z = \lim_\omega Z_n \). We have

\[ \text{diam}(Y \cap Z) \leq \omega \lim \text{diam} \left( Y_n^{+20\delta_n} \cap Z_n^{+20\delta_n} \right) \]

*Proof.* Let \( x \) and \( x' \) be two points of \( Y \cap Z \). Since \( x, x' \in Y \), we can find two sequences \((y_n)\) and \((y'_n)\) such that \( x = \lim_n y_n, x' = \lim_n y'_n \) and \( y_n, y'_n \in Y_n \) \( \omega \)-almost surely. Moreover \( x \) and \( x' \) belong to \( Z \), thus if \( A > 0 \) is given \( y_n \) and \( y'_n \) belong to \( Z^{+A} \) \( \omega \)-almost surely. Using the proposition 1.2.4 we have

\[ |y'_n - y_n| \leq \text{diam} \left( Y_n^{+A} \cap Z_n^{+A} \right) \leq \text{diam} \left( Y_n^{+20\delta_n} \cap Z_n^{+20\delta_n} \right) + 2A + 20\delta_n \]

By taking the \( \omega \)-limit, we obtain \(|x' - x| \leq \lim_n \text{diam} \left( Y_n^{+20\delta_n} \cap Z_n^{+20\delta_n} \right) + 2A \). This inequality is true for all \( A > 0 \) and \( x, x' \in Y \cap Z \), thus \( \text{diam}(Y \cap Z) \leq \lim_n \text{diam} \left( Y_n^{+20\delta_n} \cap Z_n^{+20\delta_n} \right) \). \( \Box \)
We need in the third part a little stronger condition than the quasi-convexity.

**Definition 1.2.6.** Let $X$ be a $\delta$-hyperbolic space. A subset of $X$ is strongly quasi-convex if for all $x, x' \in Y$ there exist $p, p' \in Y$ such that $|x - p|, |x' - p'| \leq 10\delta$ and the path $[x, p] \cup [p, p'] \cup [p', x']$ lies in $Y$.

**Remark:** Since any geodesic triangle of $X$ is $4\delta$-thin, any strongly quasi-convex space is $10\delta$-quasi-convex. Given a $10\delta$-quasi-convex subset $Y$ of $X$, there is a way to find a subset of $X$, a little larger than $Y$ that is strongly quasi-convex. To that end, we introduce the cylinder of a subset.

**Definition 1.2.7.** Let $Y$ be a subset of a geodesic $\delta$-hyperbolic space $X$. The cylinder of $Y$, denoted by $\text{cyl}(Y)$, is the set of all points which are in the $10\delta$-neighbourhood of a geodesic of $X$ joining two points of $Y$.

**Lemma 1.2.8.** Let $Y$ be a $10\delta$-quasi-convex subset of a geodesic, $\delta$-hyperbolic space $X$. The set $\text{cyl}(Y) \subseteq Y^{+10\delta}$ is strongly quasi-convex.

**Proof.** By definition of quasi-convexity, any geodesic joining two points of $Y$ lies in $Y^{+10\delta}$. It follows that $\text{cyl}(Y) \subseteq Y^{+10\delta}$. Let $x$ and $x'$ be two points of $\text{cyl}(Y)$. By definition there exist two points of $Y$, $y_1$ and $y_2$ (respectively $y'_1$ and $y'_2$) such that $x$ (respectively $x'$) belongs to the $10\delta$-neighbourhood of $\{y_1, y_2\}$ (respectively $\{y'_1, y'_2\}$). We introduce $p$ and $p'$ the respective projections of $x$ and $x'$ on $[y_1, y_2]$ and $[y'_1, y'_2]$.

- By construction the geodesic segments $[x, p]$ and $[p', y']$ are contained in $\text{cyl}(Y)$ and smaller than $10\delta$.

- Since the triangles $[y_2, p, p']$ and $[y'_2, y'_2, p']$ are $4\delta$-thin, $[p, p']$ stays on the $8\delta$-neighbourhood of $[p, y_2] \cup [y_2, y'_2] \cup [y'_2, p']$. However these segments are parts of geodesics between two points of $Y$. Thus $[p, p'] \subseteq \text{cyl}(Y)$.

Finally $[x, p] \cup [p, p'] \cup [p', x']$ lies in $\text{cyl}(Y)$.

\[\square\]

### 1.3 Asphericity

**Notation:** If $X$ is a simplicial complex, we denote by $X^{(k)}$ its $k$-skeleton.

In this part we prove a version of the famous Rips’ theorem: a hyperbolic simplicial complex which is locally aspherical is globally aspherical. Let $X$ be a metric space and $d$ a positive number. The Rips’ polyhedron of $X$ denoted by $P_d(X)$ is a simplicial complex defined as follows. The simplices of $P_d(X)$ are the finite subsets of $X$ which diameter are less than $d$. It is known (\cite{Gro87} Section 2.2) that if $X$ is geodesic $\delta$-hyperbolic, then for all $d \geq 4\delta$ the polyhedron $P_d(X)$ is contractible. More precisely, we have the following proposition.

**Proposition 1.3.1** (cf. \cite{CDP00} Chap. 5 Prop 1.1). Let $X$ be a geodesic, $\delta$-hyperbolic space. Let $d \geq 4\delta$ and $n \in \mathbb{N}$. The polyhedron $P_d^{(n+1)}(X)$ is $n$-connected.

Before studying the case of an arbitrary simplicial complex, we prove the following proposition.

**Proposition 1.3.2.** Let $X$ be a $n$-dimensional simplicial complex. Let $d \geq 1$. Assume that for all $r \leq 2n$ and for all $x \in X$ the closed ball $B(x, r + d)$ is homotopic to zero in $B(x, r + d)$, then there exist two maps: $f : X \rightarrow P_d^{(n+1)}(X^{(0)})$ and $g : P_d^{(n+1)}(X^{(0)}) \rightarrow X$ such that $g \circ f$ is homotopic to id$_X$.

**Proof.** In this proof we denote by $P$ the $(n+1)$-skeleton of the Rips’ polyhedron $P_d(X^{(0)})$. We define $f : X^{(0)} \rightarrow P$ by $f(x) = \{x\}$. Let $k \leq n$. If $\sigma$ is a $k$-simplex of $X$, its diameter is less than $1$. Thus the set of its vertices defines a $k$-simplex of $P$. Hence, $f$ induces a simplicial map from $X$ in $P$. We now define by induction a map $g : P \rightarrow X$.

First we define a map $g^{(0)} : P^{(0)} \rightarrow X$ by $g^{(0)}(\{x\}) = x$. 

6
Assume now, that there is a continuous map \(g^{(k)} : P^{(k)} \to X\) with the following property: for all \(k\)-simplex \(\sigma\) of \(P^{(k)}\), there is a vertex \(x\) of \(\sigma\) such that \(g^{(k)}(\sigma)\) is contained in \(B(g^{(k)}(x), 2d)\). Let \(\sigma\) be a \((k + 1)\)-simplex of \(P^{(k+1)}\) whose faces are \(\sigma_0, \ldots, \sigma_{k+1}\). Choose a vertex \(x\) of \(\sigma\). The application \(g^{(k)}\) maps the boundary \(\partial \sigma = \bigcup_{i=0}^{k+1} \sigma_i\) onto a \(k\)-sphere of \(X\) contained in \(B(g^{(k)}(x), 2k + 1)\). However this sphere is contractible in \(B(g^{(k)}(x), 2k + 1)\).

We define \(g^{(k+1)}(\sigma)\) by choosing a homotopy which contracts \(g^{(k)}(\partial \sigma)\) to a point. This defines a continuous map \(g^{(k+1)} : P^{(k+1)} \to P\) which coincides with \(g^{(k)}\) on \(P^{(k)}\) and satisfies the following property: for all \(k\)-simplex \(\sigma\) of \(P^{(k+1)}\), there is a vertex \(x\) of \(\sigma\) such that \(g^{(k)}(\sigma)\) is contained in \(B(g^{(k)}(x), 2d)\). Finally we define \(g = g^{(n+1)}\).

**Lemma 1.3.3.** For all \(k \leq n\) there is a continuous map \(H^{(k)} : X^{(k)} \times [0, 1] \to X\) satisfying the following properties:

- \(H^{(k)}|_{X^{(k)} \times \{0\}} = \text{id}_{X^{(k)}}\) and \(H^{(k)}|_{X^{(k)} \times \{1\}} = g \circ f|_{X^{(k)}}\)
- for all \(k\)-simplex \(\sigma\) of \(X^{(k)}\), there is a vertex \(x\) of \(\sigma\) such that \(H^{(k)}(\sigma \times [0, 1])\) is contained in \(B(x, (2l + 1)d)\).

**Proof.** We prove this result by induction on \(k\). The restriction of \(g \circ f\) to \(X^{(0)}\) is the identity, thus the proposition is obvious for the 0-skeleton.

Assume now that the lemma is true for \(k \leq n - 1\). Consider a \((k + 1)\)-simplex \(\sigma\) of \(X^{(k+1)}\). We chose a vertex \(x\) of \(\sigma\). By definition of \(g\) the set \(g \circ f(\sigma)\) is contained in \(B(x, (2k + 2)d)\). Moreover, the induction assumption gives that, \(H^{(k)}(\partial \sigma \times [0, 1]) \subset B(x, (2k + 2)d)\). Thus the subset \(\sigma \cup g \circ f(\sigma)\) is a \((k + 1)\)-sphere of \(X\) contained in \(B(x, (2k + 2)d)\). This sphere is therefore contractible in \(B(x, (2k + 3)d)\). By choosing a homotopy which contracts it to a point, we define a map \(H^{(k+1)} : \sigma \times [0, 1] \to X\) such that:

(i) \(H^{(k+1)}|_{\sigma \times \{0\}} = \text{id}_\sigma\) and \(H^{(k+1)}|_{\sigma \times \{1\}} = g \circ f|_\sigma\)
(ii) \(H^{(k+1)}|_{\partial \sigma \times [0, 1]} = H^{(k)}|_{\partial \sigma \times [0, 1]}\)
(iii) \(H^{(k+1)}(\sigma \times [0, 1]) \subset B(x, (2k + 3)d)\).

This defines a map \(H^{(k+1)} : X^{(k+1)} \times [0, 1]\) which satisfies the properties of the lemma.

**End of the proof of the proposition 1.3.2** The map \(H^{(n)} : X \times [0, 1] \to X\) is a homotopy between \(g \circ f\) and \(g_{id_X}\).

**Theorem 1.3.4.** Let \(X\) be a \(\delta\)-hyperbolic, \(n\)-dimensional, simplicial complex. Assume that for all \(r \leq 4\delta\) and for all \(x \in X\), the ball \(B(x, r)\) is homotopic to zero in \(B(x, r + 4\delta)\), then all homotopy groups of \(X\) are trivial. Hence \(X\) is contractible.

**Proof.** We fix \(d = 4\delta\). Using the proposition 1.3.3 the Rips’ polyhedron \(X = P_{\delta}^{(n+1)}(X^{(0)})\) is \(n\)-connected. Moreover, the fact that the small balls are aspherical gives two maps \(f : X \to P\) and \(g : P \to X\) such that \(g \circ f\) is homotopic to \(id_X\). It follows that \(X\) is also \(n\)-connected.

Since \(X\) is \(n\)-dimensional, all the higher homotopy groups of \(X\) are trivial.

## 2 Cone over a metric space

In this section we prove an asymptotic version of the Berestovskii’s theorem concerning the hyperbolicity of a cone with a locally hyperbolic base. From now on, we fix a positive number \(r_0\) which value will be precise in section 4

### 2.1 Definition

**Definition 2.1.1.** Let \(Y\) be a metric space. The cone over \(Y\), denoted by \(C(Y)\) is the quotient of \(Y \times [0, r_0]\) by the equivalence relation defined as follows. Two points \((y_1, r_1)\) and \((y_2, r_2)\) are equivalent if \(r_1 = r_2 = 0\) or \((y_1, r_1) = (y_2, r_2)\).

**Notation :** The equivalence class of \((y, 0)\), called the vertex of the cone, is denoted \(v_Y\) (or simply \(v\)). The equivalence class of any other point \((y, r)\) is still denoted by \((y, r)\).
**Hyperbolic metric on a cone**

We define a metric on $C(Y)$ as M. Bridson and A. Haefliger do in [BH99] Chap I.5. If $y$ and $y'$ are two points of $Y$, we introduce the angle $\theta (y, y')$ defined by $\theta (y, y') = \min \left\{ \pi, \frac{|y' - y|}{\sinh r_0} \right\}$.

**Proposition 2.1.2** ([BH99] Chap. I.5 Prop. 5.9). The following formula defines a distance on the cone $C(Y, r_0)$.

$$|(y', r') - (y, r)| = \arccosh \left( \cosh r \cosh r' - \sinh r \sinh r' \cos \theta (y, y') \right)$$  \hspace{1cm} (1)

**Remarks :**

- The distance on the cone has the following interpretation. Given two points $(y, r)$ and $(y', r')$ of $C(Y)$, the distance between them is the distance between two points of the hyperbolic disc respectively distant from the centre of $r$ and $r'$ such that the central angle between them is $\theta (y, y')$.

- It is important to notice that $| (y', r') - (y, r) |$ is a continuous function of $y, y', r$ and $r'$.

- The cone $C(Y)$ is the ball of centre $v$ and of radius $r_0$ of the space $C_{-1} (\frac{y}{\max \{ r_0 \}})$ defined in [BH99] Chap. I.5.

**Proposition 2.1.3** ([BH99] Chap. I.5 Prop. 5.10). Let $(y, r)$ and $(y', r')$ be two points of $C(Y, r_0)$.

(i) If $r, r' > 0$ and $\theta (y, y') < \pi$, then there is a bijection between the set of geodesics joining $y$ and $y'$ in $Y$ and the set of geodesics joining $(y, r)$ and $(y', r')$ in $C(Y)$.

(ii) In all other cases, there is a unique geodesic joining $(y, r)$ and $(y', r')$.

**Examples :**

(i) If $Y$ is a circle, endowed with its length metric, which perimeter is $2 \pi \sinh r_0$, then the cone $C(Y)$ is the hyperbolic disc of radius $r_0$.

(ii) If $Y$ is isometric to a line, then $C(Y) \setminus \{ v \}$ is the universal cover of the punctured hyperbolic disc of radius $r_0$.

**Relation between the cone and its base**

In order to compare the cone $C(Y)$ and its base $Y$, we introduce two maps:

- $i : Y \to C(Y)$
- $p : C(Y) \setminus \{ v \} \to Y$
- $y \to (y, r_0)$
- $(y, r) \to y$

**Proposition 2.1.4.** Let $(y, r)$ and $(y', r')$ be two points of $C(Y)$.

$$(r + r') \frac{\theta (y, y')}{\pi} \leq | (y', r') - (y, r) | \leq | r' - r | + \sqrt{\sinh r \sinh r'} \theta (y, y')$$

In particular, if $x$ is a point of $C(Y)$ whose distance to $v$ is at most $\frac{\pi}{2}$, then for every point $x'$ in the ball $B (x, \frac{\pi}{2})$ we have $|p(x') - p(x)|_Y \leq \frac{2 \sinh r_0}{\pi} |x' - x|_{C(Y)}$.

**Proof.** The first inequality follows from the facts that

- the map $t \to \arccosh (1 + a(1 - \cos t))$ is concave
- for all $t \geq 0$, $\arccosh (a + t) \leq \arccosh (a) + \sqrt{2t}$

Consider now a point $x = (y, r)$ of $C(Y)$ whose distance to $v$ is at most $\frac{\pi}{2}$. If $x' = (y', r')$ belongs to the ball $B (x, \frac{\pi}{2})$, then $|x' - x| \leq \frac{\pi}{2} < r < r + r'$. It follows that $\theta (y, y') < \pi$.

Using the previous inequality, we obtain $\frac{t}{\pi \sinh r_0} |y' - y| \leq \frac{\pi}{\sinh r_0} |y' - y| \leq |x' - x|$. \hfill $\Box$

**Proposition 2.1.5.** Let $\mu : \mathbb{R}^+ \to \mathbb{R}^+$, be the map defined by $\forall y, y' \in Y$, $|\mu (y') - \mu (y)| = \mu (|y' - y|)$. Then $\mu$ is non decreasing, continuous, concave map satisfying the following properties.

(i) $\forall t, t' \geq 0$, $\mu (t + t') \leq \mu (t) + \mu (t')$ (subadditivity)

(ii) $\forall t \geq 0$, $\frac{2r_0}{\pi \sinh r_0} \min \{ \pi \sinh r_0, t \} \leq \mu (t) \leq t$
Proof. By construction, the map \( \mu \) is defined by

\[
\mu(t) = \text{arccosh} \left( \cosh^2 r_0 - \sinh^2 r_0 \cos \left( \pi \frac{t}{\sinh r_0} \right) \right)
\]

The properties of \( \mu \) follow from its concavity (cf Fig. 1).

\[ \Box \]

2.2 Cone and hyperbolicity

Lemma 2.2.1. Let \( \omega \) be a non-principal ultra-filter. Let \((Y_n, y_n^0)\) be a sequence of pointed metric spaces. For all \( n \in \mathbb{N} \), we denote by \( \tilde{Y}_n \) the space \( Y_n \) endowed with the following metric,

\[
|y - y'|_{\tilde{Y}_n} = \min \{ \pi \sinh r_0, |y - y'|_{Y_n} \}.
\]

The spaces \( C\left( \lim_{\omega} (Y_n, y_n^0) \right) \) and \( \lim_{\omega} (C(Y_n), \tau_n(y_n^0)) \) are isometries.

Proof. Let denote by \( \tilde{Y} \) the space \( \lim_{\omega} \left( \tilde{Y}_n, y_n^0 \right) \). We introduce a map \( f : C(\tilde{Y}) \to \lim_{\omega} C(Y_n) \) defined by \( f(\lim_{\omega} y_n, r) = \lim_{\omega} (y_n, r) \). Since the formula \( (1) \) giving the distance in a cone is continuous, the map \( f \) preserves the distances. Consider now a point \( x = \lim_{\omega} (y_n, r_n) \) of \( \lim_{\omega} C(Y_n) \). By construction, the sequences \( (r_n) \) and \( \left( |y_n - y_n^0|_{Y_n} \right) \) are bounded. Thus, we may introduce the real number \( r = \lim_{\omega} r_n \) and the point \( y = \lim_{\omega} y_n \) of \( \tilde{Y} \). Furthermore,

\[
|f(y, r) - x| = \lim_{\omega} |(y_n, r) - (y_n, r_n)| = \lim_{\omega} |r - r_n| = 0
\]

It follows that \( f(y, r) = x \). Hence \( f \) is onto.

\[ \Box \]

Lemma 2.2.2. Let \( Y \) be a metric space. If every ball of radius \( \pi \sinh r_0 \) of \( Y \) is an \( \mathbb{R} \)-tree, then the cone \( C(Y) \) is \( \text{CAT}(1) \). In particular this cone is \( \text{in}3\)-hyperbolic.

Proof. Let \( T \) be a geodesic triangle of \( Y \) whose diameter is less than \( 2\pi \sinh r_0 \). It is contained in a ball of radius \( \pi \sinh r_0 \). It follows that \( T \) is \( 0 \)-thin. Consequently the rescaled space \( \frac{Y}{\sinh r_0} \) is \( \text{CAT}(1) \) (cf [BH99]). Using a Berestovskiĭ’s theorem (cf. [BH99 Chap. II.3 Th. 3.14]), the cone \( C(Y) \) is \( \text{CAT}(1) \). In particular it is \( 3 \)-hyperbolic.

\[ \Box \]

Proposition 2.2.3. Let \( \varepsilon > 0 \). There exists \( \delta > 0 \) satisfying the following property. Let \( X \) be a geodesic space such that each ball of radius \( \pi \sinh r_0 \) of \( Y \) is \( \delta \)-hyperbolic. Let \( Y \) be a \( 10\delta \)-quasi-convex subset of \( X \). The cone \( C(Y) \) is \( (\text{in}3 + \varepsilon) \)-hyperbolic.

Proof. Assume that the proposition is false. For all \( n \in \mathbb{N} \), we can find

- a geodesic space \( X_n \), whose balls of radius \( \pi \sinh r_0 \) are \( \delta_n \)-hyperbolic, with \( \delta_n = o(1) \)
- a \( 10\delta_n \)-quasi-convex subset \( Y_n \) of \( X_n \)

such that the cone \( C(Y_n) \) is not \( (\text{in}3 + \varepsilon) \)-hyperbolic. We denote by \( \tilde{X}_n \), the space \( X_n \) endowed with the following metric, \( |x' - x|_{\tilde{X}_n} = \min \{ \pi \sinh r_0, |x' - x|_{X_n} \} \). The set \( Y_n \) viewed as a subspace of \( X_n \) is denoted by \( \tilde{Y}_n \). We introduce a non-principal ultra-filter \( \omega \), the limit space \( \tilde{X} = \lim_{\omega} \tilde{X}_n \) and the subspace \( \tilde{Y} = \lim_{\omega} \tilde{Y}_n \). Each ball of radius \( \pi \sinh r_0 \) of \( \tilde{Y} \) is \( \delta \)-hyperbolic. Hence each ball of radius \( \pi \sinh r_0 \) of \( \tilde{Y} \) is \( 0 \)-hyperbolic. Moreover \( Y_n \) is \( 10\delta_n \)-quasi-convex. It follows that, for all \( y \in \tilde{Y} \), the set \( \tilde{Y} \cap B(y, \pi \sinh r_0) \) is an \( \mathbb{R} \)-tree. Using the previous lemma 2.2.2, \( C(\tilde{Y}) \) is \( \text{in}3 \)-hyperbolic. However, the lemma 2.2.1 tells us that \( C(\tilde{Y}) \) and \( \lim_{\omega} C(Y_n) \) are isometries. Thus there exists \( n \in \mathbb{N} \) such that \( C(Y_n) \) is \( (\text{in}3 + \varepsilon) \)-hyperbolic. Contradiction.

\[ \Box \]
2.3 Group acting on a cone

**Definition 2.3.1.** Let $X$ be a metric space and a $G$ group acting on $X$ by isometries. For all $g \in G$ the translation length of $g$, denoted by $[g]_X$ (or simply $[g]$) is

$$[g]_X = \inf_{x \in X} [gx - x]$$

The injectivity radius of $G$ on $X$ is $r_{inj}(G, X) = \inf_{g \in G \setminus \{1\}} [g]_X$.

Let $Y$ be a metric space. We fix a group $H$ acting by isometries on $Y$. We assume that this action is proper, that is for all $y \in Y$ and $r > 0$ the set $\{h \in H/h.B(y, r) \cap B(y, r) \neq \emptyset\}$ is finite. We denote by $\bar{Y}$ the quotient $Y/H$ and by $\bar{y}$ the image in $\bar{Y}$ of a point $y \in Y$. Since $H$ acts properly on $Y$, the quotient $\bar{Y}$ may be endowed with a metric defined by $[\bar{g} - \bar{y}]_{\bar{Y}} = \inf_{h \in H} [hy - \bar{y}]_{\bar{Y}}$.

We extend the action of $H$ to the cone $C(Y)$ by homogeneity: If $x = (y, r)$ is a point of $C(Y)$ and $h$ an element of $H$, then $hx$ is defined by $hx = (hy, r)$. The group $H$ acts by isometries on $C(Y)$. Note that, if $Y$ is not compact, this action is no more proper. However the relation $[\bar{x} - \bar{x}] = \inf_{h \in H} [hx - x]_{C(Y)}$ still defines a metric on the quotient $C(Y)/H$. Moreover $C(Y/H)$ and $C(Y)/H$ are isometric.

**Theorem 2.3.2** (First hyperbolicity theorem). Let $\varepsilon > 0$. There exists $\delta > 0$ satisfying the following property. Let $X$ be a $\delta$-hyperbolic, geodesic space and $Y$ a $10\delta$-quasi-convex subset of $X$. Assume that $H$ is a group acting by isometries on $X$ such that $H$ stabilizes $Y$ and $r_{inj}(H, Y) \geq 2\pi \sinh r_0$. Then the space $C(Y)/H$ is $(\ln 3 + \varepsilon)$-hyperbolic.

**Proof.** We introduce the constant $\delta > 0$ given by the proposition 2.2.3. Let $X$ be a $\delta$-hyperbolic, geodesic space and $Y$ a $10\delta$-quasi-convex subset of $X$. Assume that $H$ is a group acting by isometries on $X$ such that $H$ stabilizes $Y$ and $r_{inj}(H, Y) \geq 2\pi \sinh r_0$. The spaces $X/H$ and $Y/H$ satisfy the assumptions of the proposition 2.2.3. It follows that $C(Y/H)$, which is isometric to $C(Y)/H$, is $(\ln 3 + \varepsilon)$-hyperbolic. \hfill \Box

3 Cone-off over a metric space

The goal of this part is to study the large scale geometry of the cone-off. We give a detailed proof that under some small cancellation assumptions, the cone-off of a hyperbolic space is locally hyperbolic. We recall that $r_0$ is a fix positive number which value will be precise in section 4.

3.1 Definition

Let $X$ be a metric space and $Y = (Y_i)_{i \in I}$ a collection of subsets of $X$. For each $i \in I$, we introduce the following objects.

- $C_i$ is the cone $C(Y_i)$ and $v_i$ its vertex.
- $\iota_i : Y_i \to C_i$ and $\nu_i : C_i \setminus \{v_i\} \to Y_i$ are the maps between the cone and its base defined in the previous part.

**Definition 3.1.1** (Cone-off over a metric space). The cone-off over $X$ relatively to $Y$ is the space obtained by gluing each cone $C_i$ on $X$ along $Y_i$ according to $\iota_i$. We denote it by $\hat{X}(Y)$ (or simply $\hat{X}$).

**Metric on the cone-off**

In this paragraph, we define a metric on the cone-off such that its restriction to a cone is the metric previously defined. To this end, one defines the metric as the lower bound of the length of chains joining two points. By this way we obtain a pseudo-metric. The goal here is to prove that it is also positive.
Let \( x \) and \( y \) be two points of \( \hat{X} \), we introduce
\[
\|y - x\| = \inf \left\{ \|y' - x'\|_{X, j} / x, y' \in \left( \bigcup_{i \in I} C_i \right) \cup X \text{ such that } x = x', y = y' \text{ in } \hat{X} \right\}
\]

**Definition 3.1.2.** Let \( x \) and \( x' \) be two points of \( \hat{X} \).

- A chain between \( x \) and \( x' \) is a sequence \( C = (z_1, \ldots, z_n) \) of points of \( \hat{X} \), such that \( z_1 = x \) and \( z_n = x' \). Its length is \( l(C) = \sum_{j=1}^{n-1} \|z_{j+1} - z_j\| \).
- Furthermore we introduce \( |x' - x|_X = \inf \{l(C)/C \text{ a chain between } x \text{ and } x'\} \).

Recall that \( \mu \) is the function defined in the proposition [2.1.3] by
\[
\mu(t) = \arccosh \left( \cosh^2 r_0 - \sinh^2 r_0 \cos \left( \min \left\{ \pi, \frac{t}{\sinh r_0} \right\} \right) \right)
\]

It has the following interpretation. If \( y \) and \( y' \) are two points of \( Y_i \), then the distance between \((y, r_0)\) and \((y', r_0)\) in \( C_i \) is \( \mu(|y' - y|_X) \). Obviously, if \( \cdot \) is a pseudo-metric. The following lemma is a consequence of the definition of \( \| \cdot \| \).

**Lemma 3.1.3.** Let \( x \) and \( x' \) be two points of \( \hat{X} \).

(i) If there is \( i \in I \) such that \( x, x' \in C_i \), then \( \|x' - x\| = \|x' - x\|_{C_i} \). In particular, if \( x, x' \in Y_i \), then \( \|x' - x\| = \mu(|x' - x|_X) \).

(ii) If \( x, x' \in X \), but there is no \( i \in I \) such that \( x, x' \in Y_i \), then \( \|x' - x\| = \|x' - x\|_X \).

(iii) In all other cases, we have \( \|x' - x\| = +\infty \).

In particular, for all \( x, x' \in X \), \( \|x' - x\| \geq \mu(|x' - x|_X) \).

**Lemma 3.1.4.** Let \( x \) and \( x' \) be two points of \( \hat{X} \). For all \( \varepsilon > 0 \) there is a chain \( C = (z_1, \ldots, z_m) \) between \( x \) and \( x' \) such that \( l(C) \leq \|x - x\|_X + \varepsilon \) and for all \( j \in \{2, \ldots, m-1\} \), \( z_j \in X \).

**Proof.** Let \( \varepsilon > 0 \). By definition there exists a chain \( C = (z_1, \ldots, z_m) \) a chain between \( x \) and \( x' \) such that \( l(C) \leq \|x' - x\|_X + \varepsilon \). Assume that there is \( j \in \{2, \ldots, m-1\} \) such that \( z_j \) does not belong to \( X \). It follows that \( z_j \) is strictly contained in a cone, that is there exists \( i \in I \) such that \( z_j \in C_i \setminus \iota_i(Y_i) \). In particular there is only one point of \( \{\bigcup_{i \in I} C_i\} \cup X \) whose image in \( X \) is \( z_j \). Using the triangle inequality, we have \( \|z_{j+1} - z_{j-1}\| \leq \|z_{j+1} - z_j\| + \|z_j - z_{j-1}\| \). Thus, the sequence \( C' \) obtained by removing the point \( z_j \) from \( C \) is a chain between \( x \) and \( x' \) shorter than \( C \). Hence after removing all points of \( C' \) which are not in \( X \) we obtain a chain satisfying the properties of the lemma.

**Lemma 3.1.5.** For all \( i \in I \), \( \cdot \) \( C_i \), and \( \cdot \) \( X \) locally coincide: for all \( x = (y, r) \in C_i \setminus \iota_i(Y_i) \) if \( x' \) is a point of \( \hat{X} \) such that \( \|x' - x\|_{C_i} \leq \frac{1}{2}|r_0 - r| \), then \( x' \in C_i \) and \( \|x' - x\|_X = \|x' - x\|_{C_i} \).

**Proof.** Let \( i \in I \) and \( x = (y, r) \) be a point of \( C_i \setminus \iota_i(Y_i) \). Since \( x \notin \iota_i(Y_i) \), \( r_0 - r > 0 \). Let \( x' \) be a point of \( \hat{X} \) such that \( \|x' - x\|_X \leq \frac{1}{2}|r_0 - r| \). Let \( \eta \in \left[0, \frac{1}{2}|r_0 - r| \right] \). Using the previous lemma, there is a chain \( C = (z_1, \ldots, z_m) \) between \( x \) and \( x' \) such that \( l(C) \leq \|x' - x\|_X + \eta \) and for all \( j \in \{2, \ldots, m-1\} \), \( z_j \in X \). Assume that \( m \geq 3 \). Since \( x \in C_i \setminus \iota_i(Y_i) \), we have
\[
|z_j - z_{j-1}| \leq l(C) \leq \|x' - x\|_X + \eta \leq \frac{1}{2}|r_0 - r|
\]
Contradiction. Thus \( m = 2 \) and \( \|x' - x\| = l(C) \leq \frac{1}{2}|r_0 - r| \). Consequently \( x' \) belongs to \( C_i \) and \( \|x' - x\| = \|x' - x\|_{C_i} \). Finally for all \( \eta \in \left[0, \frac{1}{2}|r_0 - r| \right] \), we have
\[
\|x' - x\|_{C_i} \leq \|x' - x\|_{C_i} \leq \|x' - x\|_X + \eta
\]
It follows that \( \|x' - x\|_X = \|x' - x\|_{C_i} \).

**Lemma 3.1.6.** For all \( x, x' \in X \), we have \( \|x' - x\|_X \geq \mu(|x' - x|_X) \).

**Proof.** Let \( \varepsilon > 0 \). Using the lemma 3.1.4 there exists a chain \( C = (z_1, \ldots, z_m) \) between \( x \) and \( x' \) such that \( l(C) \leq \|x' - x\|_X + \varepsilon \) and for all \( j \in \{1, \ldots, m\} \), \( z_j \in X \). The subadditivity of \( \mu \) gives
\[
\mu(|x' - x|_X) \leq \sum_{j=1}^{m-1} \mu(|z_{j+1} - z_j|_X) \leq \sum_{j=1}^{m-1} \|z_{j+1} - z_j\| = l(C)
\]
Thus for all \( \varepsilon > 0 \) we have \( \mu(|x' - x|_X) \leq |x' - x|_X + \varepsilon \). It follows that \( |x' - x|_X \geq \mu(|x' - x|_X) \).
Proposition 3.1.7. \( |. \cdot |_X \) defines a metric on \( X \).

Proof. The only point to prove is the positivity of \( |. \cdot |_X \). Consider two points \( x \) and \( x' \) of \( X \) such that \( |x' - x|_X = 0 \). There are two cases.

(i) If there is \( i \in I \) such that \( x \in C_i \setminus \iota_i(Y_i) \) or \( x' \in C_i \setminus \iota_i(Y_i) \), then, using the lemma 3.1.5, \( x \) and \( x' \) both belong to \( C_i \). Moreover \( |x' - x|_{C_i} = |x' - x|_X = 0 \). Thus \( x = x' \).

(ii) If \( x \) and \( x' \) are both elements of \( X \), then the lemma 3.1.6 gives \( \mu(|x' - x|_X) \leq |x' - x|_X = 0 \). Hence \( |x' - x|_X = 0 \). It follows that \( x = x' \).

\( \square \)

Projection of the cone-off on its base

We introduce a map \( p \) from \( \hat{X} \setminus \{v_i, i \in I\} \) onto \( X \) whose restriction to a cone \( C_i \setminus \{v_i\} \) is \( p_i \) and whose restriction to \( X \) is the identity.

Proposition 3.1.8. Consider a point \( x \) of \( \hat{X} \) such that the distance between \( x \) and any vertex of \( \hat{X} \) is at most \( \frac{\pi}{2} \). For all \( x' \in B(x, \frac{\pi}{2}) \), we have \( |p(x') - p(x)|_X \leq \frac{2\pi \sinh r_0}{r_0} |x' - x|_X \).

Proof. Let \( \varepsilon \in ]0, \frac{\pi}{2}[ \). Consider a point \( x' \) of \( B(x, \frac{\pi}{2}) \). Using the lemma 3.1.4, there exists a chain \( C = (z_1, \ldots, z_m) \) such that for all \( j \in \{2, \ldots, m - 1\} \), \( z_j \in X \) and \( l(C) \leq |x' - x|_X + \varepsilon \). We chose \( j \in \{2, \ldots, m - 1\} \). The lemma 2.1.3 gives

\[ r_0 \geq l(C) \geq \|z_{j+1} - z_j\| \geq \mu(|z_{j+1} - z_j|_X) \geq \frac{2r_0}{\pi \sinh r_0} \min \{ \pi \sinh r_0, |z_{j+1} - z_j|_X \} \]

Thus, \( |p(z_{j+1}) - p(z_j)|_X = |z_{j+1} - z_j|_X \leq \frac{2\pi \sinh r_0}{r_0} \|z_{j+1} - z_j\| \).

If \( x = z_1 \) is a point of \( X \), the same proof gives \( |p(z_2) - p(z_1)|_X \leq \frac{2\pi \sinh r_0}{r_0} \|z_2 - z_1\| \).

On the other hand, if \( x \) belongs to a cone \( C_i \), then the lemma 2.1.3 gives the same inequality. By the same way we obtain \( |p(z_m) - p(z_{m-1})|_X \leq \frac{2\pi \sinh r_0}{r_0} \|z_m - z_{m-1}\| \). Finally, we have

\[ |p(x') - p(x)|_X \leq \sum_{j=1}^{m-1} |p(z_{j+1}) - p(z_j)|_X \leq \frac{2\pi \sinh r_0}{r_0} \sum_{j=1}^{m-1} \|z_{j+1} - z_j\| \]

\[ = \frac{2\pi \sinh r_0}{r_0} l(C) \leq \frac{2\pi \sinh r_0}{r_0} \left( |x' - x|_X + \varepsilon \right) \]

Hence \( |p(x') - p(x)|_X \leq \frac{2\pi \sinh r_0}{r_0} |x' - x|_X \).

\( \square \)

3.2 Uniform approximation of the distance on the cone-off

In order to study the ultra limit of a sequence of cone-off, we need to approximate the distance between two points of the cone-off \( \hat{X} \) by a chain such that the number of points involved in the chain only depends on the error and not on the base space \( X \). This point was already noted by M. Gromov in [Gro91D]. More precisely, in this section we prove the following result:

Proposition 3.2.1. Let \( A \geq 1 \). There exists a constant \( M \), depending only on \( A \) and not on \( r_0 \), with the following property. Let \( \varepsilon \in ]0, 1[ \), \( X \) be a metric space and \( Y = (Y_i)_{i \in I} \) a collection of subsets of \( X \). If \( x \) and \( x' \) are two points of the cone-off \( \hat{X}(Y) \) such that \( |x' - x|_X \leq A \), then there exists a chain \( C \) between \( x \) and \( x' \) with less than \( \frac{M}{\varepsilon} \) points and such that \( l(C) \leq |x' - x|_X + \varepsilon \).

Proof. Let \( \varepsilon \in ]0, 1[ \). Let \( x \) and \( x' \) be two points of \( \hat{X} \) such that \( |x' - x|_X \leq A \). Using the lemma 3.1.4, there is a chain \( C = (z_1, \ldots, z_m) \) between \( x \) and \( x' \) such that \( l(C) \leq |x' - x|_X + \varepsilon \) and for all \( j \in \{2, \ldots, n - 1\} \), \( z_j \in X \). We fix \( \eta \in ]0, 1[ \), and construct a subchain of \( C \) between \( x \) and \( x' \), denoted by \( C_\eta = (z_{j_1}, \ldots, z_{j_m}) \).

(i) \( j_1 = 1 \) and \( j_2 = 2 \)

(ii) Let \( k \geq 2 \). We construct \( j_{k+1} \) from \( j_k \).

- If \( j_k \leq n - 1 \) and \( |z_{j_k+1} - z_{j_k}|_X > \eta \), then \( j_{k+1} = j_k + 1 \).
- If \( j_k \leq n - 1 \) and \( |z_{j_k+1} - z_{j_k}|_X \leq \eta \), then \( j_{k+1} \) is the largest \( j \in \{j_k, \ldots, n - 1\} \) such that \( |z_j - z_{j_k}|_X \leq \eta \).
- If \( j_k = n - 1 \), then \( j_{k+1} = n \) and the process stops.

This construction removes from \( C \) the small parts of the chain which may be contained in a cone. We prove now that it hardly changes the length of the chain.

**Lemma 3.2.2** (Comparison between the two chains). The chains \( C_\eta \) and \( C \) satisfy the following inequality: \( l(C_\eta) \leq l(C) + m\eta^3 \)

**Proof.** We consider an integer \( k \in \{1, \ldots, m - 2\} \). There are two cases:

**First case:** Assume that \( |z_{j_k+1} - z_{j_k}| \leq \eta \). The function \( \mu \) given by the proposition 2.1.5 has the following property: \( \forall t \in [0, \pi \sinh r_0], \mu(t) \geq t - t^3 \). Thus using the subadditivity of \( \mu \) we obtain

\[
\sum_{j=j_k}^{j_k+1-1} \| z_{j+1} - z_j \| \geq \sum_{j=j_k}^{j_k+1-1} \mu(|z_{j+1} - z_j|_X) \geq \mu(|z_{j_k+1} - z_{j_k}|_X) \geq |z_{j_k+1} - z_{j_k} - |z_{j_k+1} - z_{j_k}|_X^3
\]

Thus we have \( \sum_{j=j_k}^{j_k+1-1} \| z_{j+1} - z_j \| \geq \| z_{j_k+1} - z_{j_k} \| - \eta^3 \)

**Second case:** Assume that \( |z_{j_k+1} - z_{j_k}|_X > \eta \). By construction, we have \( j_{k+1} = j_k + 1 \). Hence the last inequality remains true.

After summing over \( k \) these inequalities, we finally obtain \( l(C) \geq l(C_\eta) - m\eta^3 \).

**Lemma 3.2.3** (Estimation of \( m \)). If \( \eta \leq \frac{1}{2} \), then \( m \leq 100\frac{A^2}{\eta} \).

**Proof.** Let \( k \in \{2, \ldots, m-3\} \). The two inequalities \( |z_{j_k+1} - z_{j_k}|_X \leq \frac{1}{2} \eta \) and \( |z_{j_k+2} - z_{j_k+1}|_X \leq \frac{1}{2} \eta \) cannot be both true. Indeed, if it was the case, \( j_{k+1} \) will not be the largest \( j \in \{j_k, \ldots, n-1\} \) such that \( |z_j - z_{j_k}|_X \leq \eta \). Assume that \( |z_{j_{k+1}+1} - z_{j_k}|_X > \frac{1}{2} \eta \) (The other case is symmetric.) Using the same estimation of \( \mu \) that the one in previous lemma, we obtain

\[
\| z_{j_{k+1}} - z_{j_k} \| \geq \mu(|z_{j_{k+1}} - z_{j_k}|_X) \geq \mu \left( \frac{1}{2} \eta \right) \geq \frac{1}{2} \eta - \frac{1}{8} \eta^3
\]

Thus \( \| z_{j_{k+1}} - z_{j_k} \| + \| z_{j_k+2} - z_{j_{k+1}} \| \geq \frac{1}{2} \eta - \frac{3}{8} \eta^3 \). After summing over \( k \), the previous lemma gives

\[
\left( \frac{m-4}{2} \right) \left( \frac{1}{2} \eta - \frac{1}{8} \eta^3 \right) \leq l(C_\eta) \leq l(C) + m\eta^3 \leq |x' - x|_X + \frac{1}{2} \varepsilon + m\eta^3
\]

Finally we have the following inequality

\[
m \left( 4 - 17\eta^2 \right) \leq 50 \frac{A}{\eta}
\]

Hence, if \( \eta \leq \frac{1}{2} \), then \( 4 - 17\eta^2 \geq \frac{1}{2} \). It follows that \( m \) must be bounded by \( 100\frac{A^2}{\eta} \).

**End of the proof of the proposition 3.2.1** Combining the two previous lemmas, we obtain

\[
l(C_\eta) \leq l(C) + m\eta^3 \leq |x' - x|_X + \frac{1}{2} \varepsilon + 100A\eta^2.
\]

If we chose \( \eta = \frac{1}{m} \sqrt{\frac{A}{\varepsilon}} \), then we have \( l(C_\eta) \leq |x' - x|_X + \varepsilon \). Moreover the number \( m \) of points of \( C_\eta \) is less than \( 2000A\sqrt{\frac{2A}{\varepsilon}} \).

**3.3 Hyperbolicity of the cone-off over an R-tree**

**Proposition 3.3.1.** Let \( X \) be an R-tree and \( Y = (Y_i)_{i \in I} \) a collection of subtrees of \( X \) such that two distinct elements of \( Y \) share no more than one point. The cone-off \( \hat{X}(Y) \) is \( \ln 3 \)-hyperbolic.

**Remark:** In fact \( \hat{X}(Y) \) is a CAT(-1) space. But we should not use this point.
This result is a consequence of the more particular case where $Y$ is finite.

**Lemma 3.3.2.** Consider a $\mathbb{R}$-tree, $X$ and a finite collection $Y = (Y_i)_{i \in I}$ of subtrees of $X$ such that two distinct elements of $Y$ share no more than one point. The cone-off $\hat{X}(Y)$ is CAT(-1).

In particular it is in 3-hyperbolic.

**Proof.** Each $Y_i$ is a $\mathbb{R}$-tree, thus all the cones $C(Y_i)$ are CAT(-1) (cf. lemma 2.2.2). Consequently the cone-off $\hat{X}$ is obtained by gluing a finite number of CAT(-1)-spaces that share no more than one points. This spaces are the cones and the remaining parts of $X$ on which no cone is glued. It follows that $X$ is CAT(-1) (cf. [BH99 Chap II.11 Th. 11.1]).

**Proof of the proposition 3.3.3.** Let $x, y, z$ and $t$ be four points of $\hat{X}$. For all $n \in \mathbb{N}$, we can find a subtree $X_n$ of $X$ and a finite collection $Y^n$ of subtrees of $X_n$ such that
- two distinct elements of $Y^n$ share no more than one point
- $x, y, z, t$ belongs to $\hat{X}_n(Y^n)$
- for all $u, v \in \{x, y, z, t\}$, we have $\lim_{n \to +\infty} |v - u|_{X_n} = |v - u|_X$

Since $X_n$ is in 3-hyperbolic (lemma 3.3.2), $x, y, z, t$ satisfy in these spaces the hyperbolicity condition. After taking the limit we obtain in $\hat{X}$, $\langle x, z \rangle_{\hat{X}} \geq \min \{\langle x, y \rangle_{\hat{X}}, \langle y, z \rangle_{\hat{X}}\} - \ln 3$.

**3.4 Hyperbolicity of the cone-off over a hyperbolic space**

In this section, we generalize the previous proposition to the case where the base $X$ is a hyperbolic space. Let $\delta$ be a positive number. We consider a geodesic $\delta$-hyperbolic space $X$ and a collection $Y = (Y_i)_{i \in I}$ of closed 10\$\delta$-quasi-convex subsets of $X$. In order to estimate the hyperbolicity of $\hat{X}(Y)$, we introduce a constant which controls how much two elements of $Y$ overlap.

**Definition 3.4.1.** The largest piece of $Y$, denoted by $\Delta(Y)$ is the quantity

$$\Delta(Y) = \sup_{i \neq j} \text{diam} \left( Y_i^{+2\delta} \cap Y_j^{-2\delta} \right)$$

Assume that $X$ is an $\mathbb{R}$-tree, then $\Delta(Y)$ is zero if and only if two distinct elements of $Y$ share no more than one point.

**Theorem 3.4.2** (Second hyperbolicity theorem). Let $\varepsilon > 0$. There exist $\delta, \Delta > 0$ satisfying the following properties. Consider a geodesic, $\delta$-hyperbolic space $X$ and a collection $Y = (Y_i)_{i \in I}$ of closed 10\$\delta$-quasi-convex subsets of $X$, such that $\Delta(Y) \leq \Delta$. If $x_0$ is a point of the cone-off $\hat{X}(Y)$ whose distance to any vertex is greater than $2\delta$, then the ball $B(x_0, 2\delta)$ is $(\ln 3 + \varepsilon)$-hyperbolic.

**Remark :** This theorem is an extension of the proposition 3.3.3 for $\delta$-perturbed spaces. Thus it is possible to prove that $\hat{X}(Y)$ that satisfies the CAT(-1) conditions with a small error, that depends only on $\delta$, $\Delta$ and $\rho$. M. Gromov introduced in [Gro11b] the notion of CAT(-1, $\varepsilon$)-spaces that formalizes this idea. It was also developed in [BG08]. Since we are only interested in the hyperbolicity of $\hat{X}(Y)$, we will not use it.

In [Gro11b Hyperbolic coning lemma], M. Gromov give a quantitative proof of this result. Here, we use an asymptotic point of view, that provides an qualitative version of the theorem. The strategy is the following. Assuming that this theorem is false gives a family $X_n$ of $\delta_n$-hyperbolic counter-examples with $\delta_n$ tending to zero. Taking the limit gives the cone off over a $\mathbb{R}$-tree which we already know that it is in 3-hyperbolic. This is a contradiction according to the corollary 1.1.4. The point is to construct a local isometry between the cone-off over the ultra-limit of $(X_n)$ and the ultra-limit of the cones-off over $(X_n)$.

**Proof.** Assume that the theorem is false. Then for all $n \in \mathbb{N}$, we can find

(i) a geodesic, $\delta_n$-hyperbolic space $X_n$, with $\delta_n = o(1)$
(ii) a collection $Y_n = (Y_n,i)_{i \in I_n}$ of 10$\delta_n$-quasi-convex subsets of $X_n$, with $\Delta(Y_n) = o(1)$
(iii) a point $x_n^0 \in X_n(Y_n)$ whose distance to any vertex is greater than $2\delta_n$ and such that the ball $B(x_n^0, 2\delta_n)$ is not $(\ln 3 + \varepsilon)$-hyperbolic.

We fix a non-principal ultra-filter $\omega$ in order to study the limit space $\lim_\omega (\hat{X}_n)$.

First we introduce several objects.
- \( x^0 = \lim_\omega x_n^0 \)
- \( X = \lim_\omega (X_n, p_n (x_n^0)) \)
- \( I = \Pi_{n \in N} I_n / \sim \) where \( \sim \) is the equivalence relation defined by \( i \sim j \) if \( i_n = j_n \), \( \omega \)-almost surely
- If \( i \) is a sequence of \( \Pi_{n \in N} I_n \), we define \( Y_i = \lim_\omega Y_{n,i} \)

**Lemma 3.4.3.** Let \( i = (i_n) \) and \( j = (j_n) \) be two sequences of \( \Pi_{n \in N} I_n \). If \( i_n = j_n \), \( \omega \)-almost surely, then \( Y_i = Y_j \). Otherwise, \( \text{diam} (Y_i \cap Y_j) = 0 \).

**Proof.** If \( i_n = j_n \), \( \omega \)-almost surely, the equality \( Y_i = Y_j \) follows from the definition of the \( \omega \)-limit of a sequence of subsets. In the other case, we have \( i_n \neq j_n \), \( \omega \)-almost surely. Hence \( \text{diam} \left( Y_{i,n} + 20 \delta_n \cap Y_{j,n} + 20 \delta_n \right) \leq \Delta_n \), \( \omega \)-almost surely. Thus, the corollary 3.2.3 gives that \( \text{diam} (Y_i \cap Y_j) \leq \lim_\omega \text{diam} \left( Y_{i,n} + 20 \delta_n \cap Y_{j,n} + 20 \delta_n \right) = 0 \).

Thanks to the previous lemma, we may consider the collection \( Y = (Y_i)_{i \in I} \).

**Lemma 3.4.4.** The cone-off \( \tilde{X}(Y) \) is \( \text{CAT}(4) \), hence in 3-hyperbolic.

**Proof.** Since for all \( n \in N \), \( X_n \) is geodesic and \( \delta_n \)-hyperbolic with \( \delta_n = o(1) \), \( X \) is an R-tree. Furthermore, any \( Y_{n,i,n} \) is a \( 10 \delta_n \)-quasi-convex subset of \( X_n \). Thanks to the previous lemma, \( Y \) is a collection of subtrees such that two distinct elements of \( Y \) share no more than one point. Applying the proposition 3.3.1, \( \tilde{X}(Y) \) is in 3-hyperbolic.

The next step consists to produce a local isometry between \( \tilde{X}(Y) \), the cone-off over \( \lim_\omega X_n \) and \( \lim_\omega \tilde{X}_n \). For this reason we introduce the following maps,

\[
\psi : X \to \lim_\omega \tilde{X}_n \quad \psi : C(Y_i) \to \lim_\omega \tilde{X}_n \\
\lim_\omega x_n \to \lim_\omega \tilde{x}_n \quad (\lim_\omega y_n, r) \to \lim_\omega \tilde{y}_n((y_n, r))
\]

These maps induce an application \( \tilde{\psi} \) from \( \tilde{X} \) to \( \lim_\omega \tilde{X}_n \), such that its restriction to \( X \) (respectively \( C(Y_i) \)) is \( \psi \) (respectively \( \psi_i \)). In a first time we prove that this map is 1-lipschitz, then we prove that it induces a local isometry.

**Lemma 3.4.5.** Let \( x \) and \( x' \) two points of \( \tilde{X} \). We have \( \|x' - x\| \geq \|\tilde{\psi}(x') - \tilde{\psi}(x)\| \).

**Proof.** We distinguish three cases.

(i) Assume that there is \( i \in I \) such that \( x, x' \in C(Y_i) \). Then we can write \( x = (y, r) \) and \( (x', r) \) where \( y = \lim_\omega y_n \) and \( y' = \lim_\omega y'_n \) are two points of \( Y_i \). In this situation we have

\[
\|x' - x\| = \|x' - x\|_{C(Y_i)} = \arccosh (\cosh r \cosh r' - \sinh r \sinh r' \cos \theta (y, y'))
\]

By continuity, it gives,

\[
\|x' - x\| = \lim_\omega \arccosh (\cosh r \cosh r' - \sinh r \sinh r' \cos \theta (y_n, y'_n))
\]

\[
= \lim_\omega (\|y_n - y'_n\|_{C(Y_i, i_n)})
\]

\[
\geq \lim_\omega (\|y_n - y'_n\|_{X_n})
\]

\[
= \|\tilde{\psi}(x') - \tilde{\psi}(x)\|
\]

(ii) Assume \( x = \lim_\omega x_n \) and \( x' = \lim_\omega x'_n \) belong to \( X \), but there is no \( i \in I \) such that \( x, x' \in Y_i \). In this case \( \|x' - x\| = \|x' - x\|_X = \lim_\omega \|x'_n - x_n\|_{X_n} \). However, for all \( n \in N \), we have \( |x'_n - x_n|_{X_n} \geq |x'_n - x_n|_{X_n} \). Thus \( \lim_\omega \|x' - x\| \geq \lim_\omega \|x'_n - x_n\|_{X_n} = \|\tilde{\psi}(x') - \tilde{\psi}(x)\| \).

(iii) In all other cases, \( \|x' - x\| = +\infty \). There is nothing to prove.

**Corollary 3.4.6.** The map \( \tilde{\psi} : X \to \lim_\omega \tilde{X}_n \) is 1-Lipschitz, where \( X \) is the cone-off over \( \lim_\omega X_n \).
Proof. Let $x$ and $x'$ be two points of $\hat{X}$. Consider a chain $C = (z_1, \ldots, z_m)$ of points of $\hat{X}$ between $x$ and $x'$. Using the previous lemma, we have

$$\left| \hat{\psi}(x') - \hat{\psi}(x) \right| \leq \sum_{j=1}^{m-1} \left| \hat{\psi}(z_j) - \hat{\psi}(z_{j+1}) \right| \leq \sum_{j=1}^{m-1} \left\| \hat{\psi}(z_j) - \hat{\psi}(z_{j+1}) \right\| = l(C).$$

After taking the infimum over all chains between $x$ and $x'$, we obtain $\left| \hat{\psi}(x') - \hat{\psi}(x) \right| \leq \left| x' - x \right|_X$.

We now construct a partial inverse function of $\hat{\psi}$.

Lemma 3.4.7. There is a map $\hat{\psi} : B \left( x_0, \frac{2\pi}{\omega} \right) \subset \lim_{\omega} \hat{X}_n \to X$ such that $\hat{\psi}$ induces a bijection onto the ball $B \left( \hat{\varphi}(x_0), \frac{2\pi}{\omega} \right)$, whose inverse is $\hat{\psi}$.

Proof. Let $x = \lim_{\omega} x_n$ be a point of $B \left( x_0, \frac{2\pi}{\omega} \right)$. By construction, the distance between $x_n$ and any vertex of $\hat{X}_n$ is greater than $\frac{2\pi}{\omega}$. Thus applying the lemma 3.1.8 we have

$$\left| p_n(x_n) - p_n(x_0) \right|_X \leq \frac{2\pi \sinh r_0}{r_0} \left| x_n - x_0 \right|_X.$$

It follows that $\left( p_n(x_n) - p_n(x_0) \right|_X$ is $\omega$-essentially bounded. Hence $\lim_{\omega} p_n(x_n)$ defines a point in $X$. We now distinguish two cases

(i) If there is $i \in I$ such that $x_n \in C(Y_{n,i}, x)$ $\omega$-almost surely, then we can write $x_n = (p_n(x_n), r_n)$ $\omega$-almost surely. Since $(r_n)$ is bounded, we may introduce $r = \lim_{\omega} r_n$. We define $\hat{\varphi}(x)$ as the point $(\lim_{\omega} p_n(x_n), r)$ of $C(Y_i)$. 

(ii) If $x_n$ belongs to $X_n$, $\omega$-almost surely, then we define $\hat{\varphi}(x)$ as the point $\lim_{\omega} p_n(x_n)$ of $X$.

The properties of $\hat{\varphi}$ are satisfied.

□

Lemma 3.4.8. Let $x = \lim_{\omega} x_n$ and $x' = \lim_{\omega} x'_n$ be two points of $B \left( x_0, \frac{2\pi}{\omega} \right)$ such that $\omega$-essentially bounded. Then $\lim_{\omega} \left( x'_n - x_n \right) \geq \left( \hat{\varphi}(x') - \hat{\varphi}(x) \right)$.

Proof. We distinguish two cases

(i) If there is $i \in I$ such that $x_n$ and $x'_n$ belongs to $C(Y_{n,i}, x)$ $\omega$-almost surely, then $\left| x'_n - x_n \right| = \left| x'_n - x_n \right|_{C(Y_{n,i}, x)}$ $\omega$-almost surely. By continuity we have

$$\lim_{\omega} \left| x'_n - x_n \right|_{C(Y_{n,i}, x)} = \left| \hat{\varphi}(x') - \hat{\varphi}(x) \right|_{C(Y_{i}, x)} \geq \left| \hat{\varphi}(x') - \hat{\varphi}(x) \right|.$$

Thus $\lim_{\omega} \left| x'_n - x_n \right| \geq \left| \hat{\varphi}(x') - \hat{\varphi}(x) \right|.$

(ii) If $x_n, x'_n \in X_n$, $\omega$-almost surely, but there is no $i \in I$ such that $x_n$ and $x'_n$ belongs to $C(Y_{n,i}, x)$ $\omega$-almost surely, then $\left| x'_n - x_n \right| = \left| x'_n - x_n \right|_{X_n}$ $\omega$-almost surely. In this case we have

$$\lim_{\omega} \left| x'_n - x_n \right|_{X_n} = \left| \hat{\varphi}(x') - \hat{\varphi}(x) \right|_{X_n} \geq \left| \hat{\varphi}(x') - \hat{\varphi}(x) \right|.$$

Thus $\lim_{\omega} \left| x'_n - x_n \right| \geq \left| \hat{\varphi}(x') - \hat{\varphi}(x) \right|.$

□

The following corollary is the one that uses the uniform approximation of the distance on the cone-off. Indeed, if $x = \lim_{\omega} x_n$ and $x' = \lim_{\omega} x'_n$ are two points $B \left( x_0, \frac{2\pi}{\omega} \right)$, we can find for each $n$ a chain $C_n$ of $\hat{X}_n$ that approximates $\left| x'_n - x_n \right|$ with a given error. However it is difficult to give a sense to the $\omega$-limit of $C_n$ if the number of points of $C_n$ is not uniformly bounded. That is why we previously proved the proposition 3.2.1.

Corollary 3.4.9. The restriction of $\hat{\varphi}$ to the ball $B \left( x_0, \frac{2\pi}{\omega} \right)$ is $1$-Lipschitz

Proof. Consider two points $x = \lim_{\omega} x_n$ and $x' = \lim_{\omega} x'_n$ of $B \left( x_0, \frac{2\pi}{\omega} \right)$. Let $\varepsilon \in \left] 0, \frac{2\pi}{\omega} \right[$. Applying the proposition 3.2.1 there is a number $m$ depending only of $r_0$ such that for all $n \in \mathbb{N}$, there is a chain $C_n = (z_n, \ldots, z_m)$ between $x_n$ and $x'_n$ with $l(C_n) \leq \left| x'_n - x_n \right|_{X_n} + \varepsilon$.  

16
Notice that for all $1 \leq j \leq m$, $|z_j - x_0^0|_{X_n} \leq |x_n - x_0^0|_{X_n} + l(C_n) \leq \frac{C}{2}$. Thus we can introduce the points $z^j = \lim_{\omega} z^j_\omega$. The previous lemma gives

$$|\varphi(x') - \varphi(x)|_X \leq \sum_{j=1}^{m-1} \left| \varphi(z^{j+1}) - \varphi(z^j) \right| \leq \lim_{\omega} l(C_n) \leq \lim_{\omega} |x_n' - x_n|_{X_n} + \varepsilon$$

Hence for all $\varepsilon \in [0, \frac{C}{2}]$, we have $|\varphi(x') - \varphi(x)|_X \leq |x' - x| + \varepsilon$. Finally $\varphi$ is 1-Lipschitz.

\[ \square \]

**Corollary 3.4.10.** The map $\varphi$ induces an isometry from the ball $B(x^0, \frac{C}{2})$ onto its image.

**Proof.** We already know that $\varphi$ is a 1-Lipschitz bijection from $B(x^0, \frac{C}{2})$ onto its image. However its inverse function $\psi$ is also 1-Lipschitz. Finally $\varphi$ preserves the distances.

\[ \square \]

**End of the proof of the theorem** We just have proved that $B(x^0, \frac{C}{2})$ is isometric to a subset of $\tilde{X}(Y, r_0)$. Since $X(Y, r_0)$ is in 3-hyperbolic, so is $B(x^0, \frac{C}{2}) = \lim_{\omega} B(x_0^0, \frac{C}{2})$. Consequently there exists $n \in \mathbb{N}$ such that $B(x_0^n, \frac{C}{2})$ is $(\ln 3 + \varepsilon)$-hyperbolic. Contradiction.

\[ \square \]

### 3.5 Length structure on the cone-off

In order to apply the Cartan-Hadamard theorem, we need a length structure on the cone-off. But the metric $|\cdot|_X$ is not necessary a length metric. We study here the difference between $|\cdot|_X$ and the length metric $d_{X_n}$ induced by $|\cdot|_X$. We will see that $d_{X_n}$ hardly change the geometry of $X$. Thus if $(X, |\cdot|_X)$ is hyperbolic, then so is $(X, d_{X_n})$.

From now on, $X$ is a geodesic, $\delta$-hyperbolic space, and $Y = (Y_i)_{i \in I}$ a collection of strongly quasi-convex subsets of $X$. We recall that a strongly quasi-convex set $Y_i$ satisfies the following property: for all $x, x' \in Y_i$, there exist $y, y' \in Y_i$ such that the path $[x, y] \cup [y, y'] \cup [y', x'] \subset Y_i$ and $|y - y'|, |y' - x'| \leq 10 \delta$. In particular this condition is satisfied if $Y_i$ is a cylinder (proposition 1.2.8) $X(Y_i)$ is the cone-off constructed as in section 3.

**Lemma 3.5.1.** Let $i \in I$. For all $x, x' \in C(Y_i)$, we have $d(x, x') \leq |x' - x|_{C(Y_i)} + 40 \delta$.

**Proof.** We denote by $x = (y, r)$ and $x' = (y', r')$ two points of the cone $C(Y_i)$. We have assumed that $Y_i$ was strongly quasi-convex, thus we can find two points $z$ and $z'$ in $Y_i$ such that the geodesics $[y, z], [z, x']$ and $[z', y']$ are contained in $Y_i$ and $|z - y|_X, |z' - y'|_X \leq 10 \delta$.

Thanks to the proposition 2.1.3 we can find a geodesic $c_0$ (respectively $c', c''$) between $(z, r)$ and $(z', r')$ (respectively between $(y, r)$ and $(z, r)$, between $(y', r')$ and $(z', r')$). Since $|z - y|_X, |z' - y'|_X \leq 10 \delta$, we have $|z - y|_{C(Y_i)} = m_{C(Y_i)}((z', r') - (y', r'))_{C(Y_i)} \leq 10 \delta$. It follows that $|z - y|_{C(Y_i)} \leq |x' - x|_{C(Y_i)} + 20 \delta$.

Thus, by composing the geodesics $c, c_0$ and $c'$, we obtain a path from $x$ to $x'$ whose length is no more than $|x' - x|_{C(Y_i)} + 40 \delta$.

\[ \square \]

**Corollary 3.5.2.** For all $x, x' \in \tilde{X}$, we have $d(x, x') \leq |x' - x| + 40 \delta$.

**Proof.** Let $x$ and $x'$ be two points of $X$. We distinguish three cases:

- If there exists $i \in I$ such that $x, x' \subset C(Y_i)$, then $|x' - x| = |x' - x|_{C(Y_i)}$. Thus the inequality is given by the previous lemma.

- If $x, x' \subset X$, but there is no $i \in I$ such that $x, x' \subset C(Y_i)$. In this case, $|x' - x| = |x' - x|_X$. There is a geodesic of $X$ between $x$ and $x'$. It gives a path in $\tilde{X}$, whose length is no more than $|x' - x|_X$. It follows that $d(x, y) \leq |x' - x|_X$.

- In all the other cases, $|x' - x| = +\infty$. There is nothing to prove.

\[ \square \]

**Proposition 3.5.3.** Let $A \geq 1$. There exists a constant $c(A)$, depending only on $A$ and not on $r_0$, with the following property. Let $\delta \in [0, 1]$. If $X$ is a geodesic, $\delta$-hyperbolic space, and $Y = (Y_i)_{i \in I}$ a collection of strongly quasi-convex subsets of $X$. Then for all $x, x' \subset X(Y)$, such that $|x' - x|_X \leq A$, we have:

$$|x' - x|_X \leq d(x, x') \leq |x' - x|_X + c(A)\sqrt{\delta}$$
Proof. Let \( x \) and \( x' \) be two points of \( X \) such that \( |x' - x|_X \leq A \). Using the proposition 3.2.1 there exists a constant \( M(A) \), an integer \( m \leq \frac{M(A)}{\sqrt{c}} \), and a chain \( C = (z_1, \ldots, z_m) \) between \( x \) and \( x' \) such that \( l(C) \leq |x' - x|_X + \delta \). Thanks to the corollary 3.5.2 we have
\[
  d(x, x') \leq \sum_{j=1}^{m-1} d(z_j, z_{j+1}) \leq l(C) + 40m\delta \leq |x' - x|_X + \delta + 40M(A)\sqrt{\delta}.
\]
The other inequality \( |x' - x|_X \leq d(x, x') \) is just a consequence of the definition of the length structure.

Finally the identity map from \((X, |.|)\) onto \((\hat{X}, \hat{d})\) is a local \((1, c(A)\sqrt{\delta})\)-quasi-isometry. Hence, the proposition 1.0.3 tells us that if \((\hat{X}, |.|)\) is \(\delta\)-hyperbolic, then \((X, \hat{d})\) is \(\delta'\)-hyperbolic with \(\delta' = \delta + 3c(A)\sqrt{\delta}\).

4 Small cancellation theory

4.1 Orbifold

We introduce in the section vocabulary about orbifolds. For more details about these objects see [BH99] Part III G

Definition and length structure

**Definition 4.1.1 (Rigidity).** The action of a group \( G \) on a topological space \( X \) is rigid if it satisfies the following property: for all \( g \in G \) if there is an open \( U \subset X \) such that \( g|_U = \text{id}_U \) then \( g = 1 \).

**Definition 4.1.2 (Orbifold).** Let \( Q \) be a topological space. We say that \( Q \) is an orbifold if there exists a collection \( (U_i, \varphi_i)_{i \in I} \), where \( U_i \) is a topological space and \( \varphi_i \), a continuous map from \( U_i \) into \( Q \) satisfying the following properties.

(i) \( Q = \bigcup_{i \in I} \varphi_i(U_i) \).

(ii) For all \( y \in \varphi_i(U_i) \), for all \( x \in \varphi_i^{-1}(\{y\}) \), there exists a finite, rigid group of homeomorphisms of \( U_i \), \( G_x \), fixing \( x \), such that for all \( g \in G_x \), \( \varphi_i \circ g = \varphi_i \) and such that the restriction of \( \varphi_i \) to a neighbourhood of \( x \) \( V_x \), induces a homeomorphism from \( V_x/G_x \) onto its image.

(iii) For all \( x_i \in U_i \) and \( x_j \in U_j \), such that \( \varphi_i(x_i) = \varphi_j(x_j) \), there exists a homeomorphism \( \theta_{j,i} \) from a neighbourhood of \( x_i \) onto a neighbourhood of \( x_j \) such that \( \varphi_i = \varphi_j \circ \theta_{j,i} \).

(iv) For all \( i \in I \), \( \varphi_i \) lifts paths and homotopies, i.e. if \( c : [0,1] \to Q \) (respectively \( H : [0,1] \times [0,1] \to Q \)) is a continuous path (respectively a homotopy), there exists \( 0 = t_0 < \cdots < t_p = 1 \) a subdivision of \([0,1]\) (respectively \( 0 = t_0 < \cdots < t_p = 1 \) and \( 0 = u_0 < \cdots < u_q = 1 \) subdivisions of \([0,1]\)) such that \( c_{\varphi_i^{-1}((t_i, t_{i+1}))} \) (respectively \( H_{\varphi_i^{-1}([u_i, u_{i+1}])} \)) lifts in one of the \( U_i \).

**Vocabulary:** \( (U_i, \varphi_i) \) is called a chart of \( Q \). The set of charts is an atlas. The map \( \theta_{j,i} \) is a transition map, and the group \( G_x \) is an isotropy group.

**Definition 4.1.3 (Length structure).** The orbifold defined as above is endowed with a length structure if

(i) The spaces \( U_i \) are endowed with a length structure.

(ii) For all \( x \in U_i \), the isotropy group \( G_x \) is an isometry group.

(iii) The transition maps \( \theta_{j,i} \) are isometries.

In this case, we can measure the length of a path, by measuring the length of its lift.

**Definition 4.1.4 (\( \sigma \)-useful length structure).** Let \( \sigma \) be a positive number. The length structure defined as above is said to be \( \sigma \)-useful if for all \( y \in Q \), there exists a chart \( (U_i, \varphi_i) \) and a point \( x \in \varphi_i^{-1}(\{y\}) \) such that

(i) the restriction \( \varphi_i : B(x, \sigma) \to B(y, \sigma) \) is onto.

(ii) this restriction lifts the paths starting in \( x \), which lengths are less than \( \frac{\sigma}{2} \).
(iii) this restriction lifts the homotopies $H : [0, 1] \times [0, 1] \to Q$ satisfying $H(0, 0) = x$, and for all $t_0 \in [0, 1]$ (respectively $u_0 \in [0, 1]$) the length of the path $u \to H(t_0, u)$ (respectively $t \to H(t, u_0)$) is shorter than $\frac{\delta}{2}$.

$(U, \varphi, x)$ is called a $\sigma$-useful chart.

**Definition 4.1.5 ($\sigma$-locally $\delta$-hyperbolic length structure).** Let $\sigma > 0$ and $\delta > 0$. The $\sigma$-useful length structure defined as above is said to be $\sigma$-locally $\delta$-hyperbolic if for all $y \in Q$, there exists a $\sigma$-useful chart $(U, \varphi, x)$ such that the ball $B(x, \sigma)$ is $\delta$-hyperbolic.

**Topology of orbifolds**

If $Q$ is an orbifold, we can define the $G$-paths and the homotopy of two $G$-paths. (cf. [BH99] or [DG08]). This leads to the definition of the fundamental group the orbifold $Q$ denoted by $\pi_1^\text{orb}(Q)$. We may also define the notion of covering and universal covering of $Q$ in the sense of orbifolds. (cf. [BH99])

**Example:** Let $X$ be a geodesic space and $G$ a group whose action on $X$ is rigid and proper. We denote by $Q$ the quotient $X/G$, and by $q : X \to Q$ the canonical projection. $Q$ may be endowed with an orbifold structure with one chart $(X, q)$. Indeed, for all $x \in X$, the isotropy group $G_x$ is necessary finite. Moreover, $q$ induces a local isometry from $X/G_x$ onto its image. If $X$ is simply connected, the map $q : X \to Q$ is also the universal cover of $Q$ and $G = \pi_1^\text{orb}(Q)$. Such an orbifold is said to be developable.

**Cartan Hadamard Theorem**

**Theorem 4.1.6 ([DG08 The. 4.3.1]).** Let $\delta > 0$ and $\sigma > 10^5 \delta$. Consider an orbifold $Q$ with a $\sigma$-locally $\delta$-hyperbolic length structure.

- $Q$ is developable and its universal cover $X$ is $200\delta$-hyperbolic.
- Let $(U, \varphi, x)$ be a $\sigma$-useful chart. If $z$ is a pre image in $X$ of the point $y = \varphi(x)$, then the developing map $(U, x) \to (X, z)$ induces an isometry from $B(x, \frac{\rho}{\sigma})$ onto its image.

**4.2 Statement of the very small cancellation theorem**

**Notation:** If $G$ is a group acting on a space $X$, and $Y$ a subset of $X$, we denote by $\text{Stab}(Y)$ the subgroup of $G$ that preserves $Y$, i.e.

$$\text{Stab}(Y) = \{g \in G : gY = Y\}$$

**Definition 4.2.1 (Rotation family).** Let $(H_i)_{i \in I}$ be a family of subgroups of $G$ and $(Y_i)_{i \in I}$ a pairwise distinct collection of subspaces of $X$. We say that $(Y_i, H_i)_{i \in I}$ is a rotation family if

(i) for all $i \in I$, $H_i$ is a finite index normal subgroup of $\text{Stab}(Y_i)$

(ii) there is an action of $G$ on $I$ which is compatible with the one on $X$, that is for all $g \in G$ and $i \in I$, we have $Y_{gi} = gY_i$ and $H_{gi} = gH_i g^{-1}$.

**Theorem 4.2.2 (Very small cancellation theorem).** There exist two positive numbers $\delta_0$ and $\Delta_0$ satisfying the following property.

Let $X$ be a geodesic, simply connected, $\delta$-hyperbolic space and $G$ a group acting properly, by isometries on $X$. Let $(Y_i, H_i)_{i \in I}$ be a rotation family, such that each $Y_i$ is strongly-quasi-convex. Let $\rho = \min_{i \in I} \min_{x \in Y_i} (H_i, X)$, $N$ the normal subgroup of $G$ generated by the $H_i$’s and $\bar{G}$ the quotient group $G/N$. Assume also that

$$\frac{\delta}{\rho} \leq \delta_0 \quad \text{and} \quad \frac{\Delta(Y)}{\rho} \leq \Delta_0$$

Then, there exists a simply connected, hyperbolic, metric space $\bar{X}$ such that $\bar{G}$ acts properly by isometries on $\bar{X}$.

Moreover if $G$ (respectively $H_i$) acts co-compactly on $X$ (respectively $Y_i$) and $I/G$ is finite, then $X/G$ is compact. In particular $\bar{G}$ is hyperbolic.

**Remark:** In this theorem, $\Delta(Y)$ and $\rho$ respectively play the role of the length of the largest piece and the length of the smallest relation in the usual small cancellation theory.

It is important to notice that the constants $\delta_0$ and $\Delta_0$ are independent from the space $X$. This is useful in order to construct by iteration a sequence of hyperbolic groups, as it is done in [Gro03], [DG08] or [AD].
4.3 Proof of the very small cancellation theorem

Construction of an orbifold

First we have to fix several constants in order to apply the Cartan-Hadamard theorem \(4.1.6\) and the two hyperbolicity theorems \(2.3.2\) and \(3.4.2\).

We consider a positive number \(\varepsilon\) and chose a radius \(r_0\) such that \(r_0 > 10^6 \ln 3 + \varepsilon\). With such constants we can apply the Cartan-Hadamard theorem to a \(\frac{\ln 3}{\varepsilon}\)-locally \((\ln 3 + \varepsilon)\)-hyperbolic orbifold. Using the proposition \(1.0.3\), there exists a positive number \(\eta\) with the following property. Consider two metric spaces \(X, Y\) and a \((1, \eta)\)-quasi-isometry \(f : X \to Y\). If \(Y\) is \((\ln 3 + \frac{\varepsilon}{2})\)-hyperbolic then \(X\) is \((\ln 3 + \varepsilon)\)-hyperbolic. From now on we will work with the rescaled metric space \(X_\rho = \frac{2\pi \sinh r_0}{\rho} X\). Thus for all \(i, r_{\text{rel}}(H_i, X_\rho) \geq 2\pi \sinh r_0\).

We can find \(\delta_0, \Delta_0 > 0\) only depending of \(r_0\) and \(\varepsilon\) such that if \(\frac{\delta}{\rho} \leq \delta_0\) and \(\frac{\Delta(Y)}{\rho} \leq \Delta_0\), then

1. Assume that \(x_0\) is a point of \(\tilde{X}_\rho(Y)\), whose distance to a vertex is at least \(\frac{\delta}{\rho}\), then the ball \(B(x_0, \frac{\delta}{\rho})\) of \((\tilde{X}_\rho, |\cdot|_{\tilde{X}_\rho})\) is \((\ln 3 + \frac{\varepsilon}{2})\)-hyperbolic. (Theorem \(2.3.2\))
2. For all \(i \in I\) the cone \(C(Y_i)/H_i\) is \((\ln 3 + \frac{\varepsilon}{2})\)-hyperbolic. (Theorem \(3.4.2\))
3. The identity map from \((\tilde{X}_\rho, d)\) onto \((\tilde{X}_\rho, |\cdot|)\) restricted to any ball of radius \(r_0\) is a \((1, \eta)\)-quasi-isometry. (Proposition \(3.5.3\)).

Thus if \(x_0\) is a point of \(\tilde{X}_\rho(Y)\), whose distance to a vertex is at least \(\frac{\delta}{\rho}\), the ball \(B(x_0, \frac{\delta}{\rho})\) of \((\tilde{X}_\rho, d)\) is \((\ln 3 + \varepsilon)\)-hyperbolic and for all \(i \in I\), the cones \(C(Y_i)/H_i\) with the length metric are \((\ln 3 + \varepsilon)\)-hyperbolic.

**Lemma 4.3.1.** The action of \(G\) on \(X_{\rho}\) extends to an action by isometries of \(G\) on \(\tilde{X}_\rho\).

**Proof.** We define this action such that its restriction to \(X_{\rho}\) is the action of \(G\) on \(X\). Let \(i \in I\). Consider a point \(x = (y, r)\) of \(C(Y_i)\) and an element \(g\) of \(G\). We define \(gx, r = \psi(g, y, r)\) of \(C(Y_i)\). We check that for all \(x, x' \in X_{\rho}\) and for all \(g \in G\) we have \(\|gx' - gx\| = \|x' - x\|\).

It follows that the action of \(G\) preserves the distances \(|\cdot|_{\tilde{X}_\rho}\) and \(d\). \(\square\)

From now on, we denoted by \(Q\) the quotient space \(\tilde{X}_\rho(Y)/G\) endowed with the quotient topology. The canonical projection \(\tilde{X}_\rho \to Q\) is denoted by \(q\). Then we introduce two kind of charts - The first one is \((U, q)\) where \(U\) is the cone-off \(X_{\rho}\) from which we have removed the vertices.
- Let \(i \in I\), we define \(U_i = (C(Y_i) \setminus i(Y_i)) / H_i\). The composition \(C(Y_i) \to \tilde{X}_{\rho} \to Q\) induces an application \(q_i : U_i \to Q\). The second type of charts is \((U_i, q_i)\).

**Lemma 4.3.2.** The charts defined previously endow \(Q\) with a structure of orbifold.

**Proof.** The action of \(G\) on \(U\) is proper. Moreover the stabilizer of the vertex \(v_i\) of the cone \(U_i\) is exactly the finite group \(\text{Stab}(Y_i) / H_i\). We check that the atlas \((U, q), (U_i, q_i)\) defines a structure of orbifold on \(Q\). \(\square\)

**Properties of the orbifold \(Q\)**

**Lemma 4.3.3.** The structure of orbifold on \(Q\) defined as above is \(\frac{\ln 3}{\varepsilon}\)-locally \((\ln 3 + \varepsilon)\)-hyperbolic.

**Proof.** It is a consequence of the two hyperbolicity theorem: the constants \(\delta_0\) and \(\Delta_0\) have been chosen in such a way, that the structure of orbifold is \(\frac{\ln 3}{\varepsilon}\)-locally \((\ln 3 + \varepsilon)\)-hyperbolic. \(\square\)

**Corollary 4.3.4.** The orbifold \(Q\) is developable and its universal cover \(\tilde{X}\) is \(\delta\)-hyperbolic, with \(\delta = 200(\ln 3 + \varepsilon)\).

**Proof.** This is an application of the Cartan-Hadamard theorem \(4.1.6\) to the orbifold \(Q\) with \(\frac{\ln 3}{\varepsilon}\)-locally \((\ln 3 + \varepsilon)\)-hyperbolic length structure. \(\square\)

**Proposition 4.3.5.** The group \(\hat{G}\) acts properly by isometries on \(\tilde{X}\).
Hence there exists a finite index normal subgroup $N$ of $G$. The charts $U$ and $U_i$ are simply connected. Hence $\pi^{orb}_1(U/G) = G$ and $\pi^{orb}_1(U_i \cap (\text{Stab}(Y) / H_i)) = \text{Stab}(Y_i) / H_i$. Applying to $Q$ the Van Kampen's theorem for orbifolds, we obtain $\pi^{orb}_1(Q) = \text{Stab}(Y_i) / H_i \ast_{\text{Stab}(Y)} G = \bar{G}$. Thus $\bar{G}$ acts properly on $\bar{X}$.

**Proposition 4.3.6.** If $G$ (respectively $H_i$) acts co-compactly on $X$ (respectively $Y_i$) and $I/G$ is finite, then $X/G$ is compact. In particular $G$ is hyperbolic.

**Proof.** Since there are, up to a translation by an element of $G$, a finite number of $Y_i$, $Q$ is obtained by gluing a finite number of compact cones, on the compact space $X/G$. Hence $Q = X/G$ is compact. Finally the action of $G$ on $\bar{X}$ is proper, co-compact.

**Theorem 4.3.7.** Assume that $X$ is a $n$-dimensional simplicial complex such that $n < \frac{\ln 3 + \varepsilon}{10\delta}$.

If every ball of $B(x,r)$ of $X$ and $Y_i$ is homotopic to zero in $B \left(x, r + \frac{4\delta r}{\sinh r} \right)$ then $X$ is contractible.

**Proof.** We have seen that $\bar{X}$ is $\bar{\delta}$-hyperbolic with $\bar{\delta} = 200(\ln 3 + \varepsilon)$. Using the corollary 1.3.4 it is sufficient to prove that for all $r \leq 4\delta$ and for all $x \in \bar{X}$, the closed ball $B(\bar{x}, r)$ is homotopic to zero in $\bar{B}(\bar{x}, r + 4\delta)$. Let $B(\bar{x}, r)$ be such a ball. Thanks to the Cartan-Hadamard theorem, the developing map $(V, x) \to (\bar{X}, \bar{x})$ induces a homeomorphism of $\bar{B}(\bar{x}, \frac{4\delta r}{\sinh r})$ onto its image. Thus the ball $B(\bar{x}, r)$ fits in one of the charts. It follows from the additional conditions on $X$ that this ball is homotopic to zero in $B \left(\bar{x}, r + 4\delta \right)$.

5 Examples of aspherical complexes

In this section we explain how to construct examples of rotation families satisfying the very small cancellation assumptions. Let $X$ be a proper geodesic $\delta$-hyperbolic space and $Y$ a closed convex subset of $X$. Let $G$ be a group acting properly co-compactly by isometries on $X$, such that $\text{Stab}(Y)$ acts co-compactly on $Y$.

We are interested in the following rotation family $(gY, gHg^{-1})_{g \in G \setminus \text{Stab}(Y)}$ where $H$ is a subgroup of $\text{Stab}(Y)$.

In concrete situations, $\{ \text{diam} (gY^{+20\delta} \cap Y^{+20\delta}), g \in G \setminus \text{Stab}(Y) \}$ may not be bounded. Nevertheless, in many situations, this assumption can be achieved by replacing $G$ by a finite index subgroup of $G$. This uses, as explained in the next lemma, the profinite topology of groups.

**Lemma 5.1.** If the subgroup $\text{Stab}(Y)$ is closed in $G$ for the profinite topology, then for all $\Delta > 0$ there exists a finite index subgroup $G'$ of $G$ containing $\text{Stab}(Y)$ such that for all $g \in G'$, $g \in \text{Stab}(Y)$ if and only if $\text{diam} (gY^{+20\delta} \cap Y^{+20\delta}) > \Delta$.

**Proof.** Let $K$ be a compact fundamental domain for the action of $\text{Stab}(Y)$ on $X$. Since the action of $G$ is proper, the following set is finite.

$$E = \{ g \in G \mid \text{diam} (gK^{+\Delta + 20\delta} \cap K^{+\Delta + 20\delta}) > \Delta \}$$

We assumed that $\text{Stab}(Y)$ was closed in $G$ for the profinite topology. In other words

$$\text{Stab}(Y) = \bigcap_{[G:N] < \infty} \text{Stab}(Y) \cdot N$$

Hence there exists a finite index normal subgroup $N$ of $G$ such that $E \cap \text{Stab}(Y).N \subset \text{Stab}(Y)$. We denote by $G'$ the set $\text{Stab}(Y) \cdot N$. It is a finite index subgroup of $G$ containing $\text{Stab}(Y)$. Consider now $g \in G'$, such that $\text{diam} (gY^{+20\delta} \cap Y^{+20\delta}) > \Delta$. Since $K$ is a fundamental domain of $Y$, there exists four points of $K^{+\Delta + 20\delta}$, $x, x', y$ and $y'$ and two elements of $\text{Stab}(Y)$, $h$ and $h'$ with the following properties : $hx = gh'x'$, $hy = gh'y'$ and $|y - x| = |y' - x'| > \Delta$. Thus, $h^{-1}gh'$ belongs to $E \cap G'$. But $h$ and $h'$ both belong to $\text{Stab}(Y)$. Hence $g \in \text{Stab}(Y)$.

In this context, the following result of N. Bergeron will be useful.

**Proposition 5.2** [Ber00, Lemme principal]. Let $\Lambda$ be an algebraic subgroup of $\text{GL}_n(\mathbb{R})$ and $G$ a finitely generated subgroup of $\text{GL}_n(\mathbb{R})$. Then $\Lambda \cap G$ is closed in $G$ for the profinite topology.
The second lemma explain how to find a subgroup \( H \) of \( \text{Stab}(Y) \) with an injectivity radius as large as desired.

**Lemma 5.3.** If the group \( G \) is residually finite, then for all \( \rho > 0 \) there exists a finite index normal subgroup \( H \) of \( \text{Stab}(Y) \), such that \( r_{\text{inf}}(H,X) \geq \rho \).

**Proof.** Let \( K \) be a compact fundamental domain for the action of \( G \) on \( X \). Since the action of \( G \) is proper, the following set is finite.

\[
E = \{ g \in G/gK^{+\rho} \cap K^{+\rho} \neq \emptyset \}
\]

Moreover, \( G \) is residually finite. Hence there exists a finite index normal subgroup \( N \) of \( G \) such that \( E \cap N = \{1\} \). Consider \( g \in N \setminus \{1\} \) and \( x \in X \). By definition there exists \( h \in G \) such that \( hx \in K \). But \( hgh^{-1} \) belongs to \( N \setminus \{1\} \), thus \( hgh^{-1}K^{+\rho} \cap K^{+\rho} = \emptyset \). It follows that \( |gx - x| = |(hgh^{-1})hx - hx| \geq \rho \). Hence \([g] \geq \rho \). Finally, we take for \( H \) the group \( N \cap \text{Stab}(Y) \).

**Theorem 5.4.** Let \( H_n \) denotes the real (respectively complex, quaternionic) hyperbolic space, and \( \delta \) its hyperbolicity constant. We consider \( \Lambda_k = SO(k,1) \) (respectively \( SU(k,1) \), \( Sp(k,1) \)) as the stabilizer of \( H_k \) in \( H_n \). Let \( G \) be a uniform lattice of \( \Lambda_n = SO(n,1) \) (respectively \( SU(n,1) \), \( Sp(n,1) \)). We assume that \( G \cap \Lambda_k \) is a uniform lattice of \( \Lambda_k \).

(i) There exists a finite index subgroup \( G' \) of \( G \) such that the following set is bounded

\[
\{ \text{diam}(gH_k^{+20\delta} \cap H_k^{+20\delta}) , g \in G' \setminus \Lambda_k \}
\]

(ii) Let \( \bar{Q} \) be the space obtained by gluing a cone of base \( H_k/G' \cap \Lambda_k \) over \( H_k/G' \). There is a finite index subgroup \( H \) of \( G' \setminus \Lambda_k \) and a contractible hyperbolic space \( \bar{X} \) such that \( G' = G'/\approx H \approx \) acts properly co-compactly on \( \bar{X} \), and \( \bar{Q} = \bar{X}/G' \).

**Proof.** Applying the proposition 5.2, \( G \cap \Lambda_k \) is closed in \( G \) for the profinite topology. The first point follows from the lemma 6.1.

We denote by \( \Delta \) the upper bound of \( \{ \text{diam}(gH_k^{+20\delta} \cap H_k^{+20\delta}) , g \in G \setminus \Lambda_k \} \). It is known that a finitely generated subgroup of \( \Lambda_n \) is residually finite. Using the lemma 5.5, there exists a finite index normal subgroup \( H \) of \( G \cap \Lambda_k \), which injectivity radius is \( \rho \), and such that \( \delta \leq \delta_0 \rho \) and \( \Delta \leq \Delta_0 \rho \), where \( \delta_0 \) and \( \Delta_0 \) are the constant given by the very small cancellation theorem 4.2.2. It follows that the rotation family \( \{ gH_k, ghg^{-1} \} \) \( g \in G/G' \setminus \Lambda_k \) satisfies the hypotheses of the very small cancellation theorem. Thus there exists a hyperbolic space \( \bar{X} \) such that \( G = G'/\approx H \approx \) acts properly by isometries on \( \bar{X} \) and \( \bar{Q} = \bar{X}/G' \). Moreover since \( G \) (respectively \( H \)) is a uniform lattice of \( \Lambda_k \) (respectively \( \Lambda_k \)) the action of \( G \) (respectively \( H \)) on \( H_n \) (respectively \( H_k \)) is co-compact. It follows that the action of \( G \) on \( \bar{X} \) is co-compact. Finally every ball in \( H_n \) or \( H_k \) is homotopic to a point. Thus by applying the theorem 4.3.7 \( \bar{X} \) is contractible. \( \square \)

The next result is proved by the same way, it only uses an other kind of convex subset of \( H_n \).

**Theorem 5.5.** Let \( H_n(C) \) denotes the complex hyperbolic space, and \( \delta \) its hyperbolicity constant. We consider \( SO(n,1) \) as the stabilizer of the \( n \)-dimensional hyperbolic space \( H_n(C) \) in \( H_n(C) \). Let \( G \) be a uniform lattice of \( SU(n,1) \) such that \( G \cap SO(n,1) \) is also a uniform lattice of \( SO(n,1) \).

(i) There exists a finite index subgroup \( G' \) of \( G \) such that the following set is bounded

\[
\{ \text{diam}(gH_n(R)^{+20\delta} \cap H_n(R)^{+20\delta}) , g \in G' \setminus SO(n,1) \}
\]

(ii) We denote by \( \bar{Q} \) the space obtained by gluing a cone of base \( H_n(R)/G' \cap SO(n,1) \) over \( H_n/G' \). There is a finite index subgroup \( H \) of \( G' \cap SO(n,1) \) and a contractible hyperbolic space \( \bar{X} \) such that \( G' = G'/\approx H \approx \) acts properly co-compactly on \( \bar{X} \), and \( \bar{Q} = \bar{X}/G' \).

**Remark :** In this last situation we may wonder if the group \( G' = G'/\approx H \approx \), that we have produced, has or not the Kazhdan property (T).
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