A CATEGORIZATION OF THE CARTAN-EILENBERG FORMULA

JUN MAILLARD

Abstract. We prove a categorification of the stable elements formula of Cartan and Eilenberg. Our formula expresses the derived category and the stable module category of a group as a bilimit of the corresponding categories for the p-subgroups.

A classical theorem by Cartan and Eilenberg [CE56, Theorem 10.1] presents the mod p cohomology of a finite group G as a subalgebra of the cohomology of any p-Sylow subgroup S of G. The formalism of fusion systems provides a compact formula to express this result in terms of the fusion category $F_S(G)$ (definition 2.1):

$$H^*(G; \mathbb{F}_p) \simeq \lim_{P \in F_S(G)^{op}} H^*(P; \mathbb{F}_p)$$

This formula actually holds [Par17] for any cohomological Mackey functor $M$:

$$M(G) \simeq \lim_{P \in F_S(G)^{op}} M(P)$$

We prove a categorified Cartan-Eilenberg formula (theorem 3.9) for any p-monadic Mackey 2-functor $\mathbb{M}$ (definitions 3.2 and 3.7):

$$\mathbb{M}(G) \cong \lim_{P \in F_S(G)^{op}} \mathbb{M}(P)$$

For instance, the left-hand side can stand for the categories of group representations $\mathbb{M}(G) = \text{mod}(kG)$, the stable categories of group representations $\mathbb{M}(G) = \text{stmod}(kG)$ or the derived categories of the group algebras $\mathbb{M}(G) = D(kG)$. (There are other examples, see remark 3.8.) We can let $G$ range over all finite groupoids and these mappings then describe the values on objects of 2-functors $\mathbb{M} : (\text{gpd}^{f \text{op}}) \to \text{Cat}$ from the 2-category of finite groupoids, with faithful functors as 1-morphisms, to the 2-category of small categories.

This categorification requires us to replace the fusion category $F_S(G)$ by an extended transporter category $\hat{T}_S(G)$ (definition 2.4) and the classical limit by a pseudo bilimit (definition 2.5) taken in the 2-category of categories.

Through a 2-finality argument, using the criterion of [Mai21b], the bilimit in theorem 3.9 can be reinterpreted as a descent-shaped bilimit. Theorem 3.9 thus states that the functor $\mathbb{M}$ is a 2-sheaf, which allows us to recover the main theorem of [Bal15]. More details on this point can be found in [Mai21a].

The article is organized as follows. In section 1 we recall classical definitions concerning bicategories and relevant notations. In section 2 we give a quick summary of the classical categorical formalism of fusion in groups and introduce our generalization. In section 3 we state and prove the main theorem. In section 4 we explain how to retrieve the classical Cartan-Eilenberg formula, and similar results, from our categorified formula.

2010 Mathematics Subject Classification. 18A05, 18A30, 18C15, 18C20, 18D05, 20C05, 20L05.

Key words and phrases. Categorification, fusion of finite groups, bilimits.

Author supported by Project ANR ChroK (ANR-16-CE40-0003) and Labex CEMPI (ANR-11-LABX-0007-01).
Acknowledgements. This work is a part of a PhD thesis under the supervision of Ivo Dell’Ambrogio.

1. Bicategorical notions

1.1. Usual definitions and conventions. We will follow the naming conventions of [Y21] for bicategorical notions. In particular, the terms 2-category and 2-functor denote the strict ones; bicategory and pseudofunctor refer to the strong but (possibly) non-strict variants. We will use the term (2, 1)-category for a 2-category with only invertible 2-morphisms, and the term (2, 1)-functor for a 2-functor between (2, 1)-categories.

We recall some usual constructions and properties of 2-categories we will use, and introduce some notations.

1.1. Notation. We use the symbol \(\simeq\) to denote an isomorphism between two objects (in a 1-category) and the symbol \(\cong\) to denote an equivalence between two objects (in a 2-category).

1.2. Definition. Let \(C\) be a 2-category. The opposite 2-category \(C^{op}\) of \(C\) is the 2-category with:

- **Objects:** the objects of \(C\)
- **Hom-categories:** \(C^{op}(A,B) = C(B,A)\)

1.3. Notation. We use the notation \([A,B]\) for the 2-category of pseudofunctors, pseudonatural transformations and modifications, between two 2-categories \(A\) and \(B\).

1.4. Notation. Let \(I, C\) be 2-categories and \(T\) be an object of \(C\). The constant 2-functor \(I \rightarrow C\) with value \(T\) is denoted by \(\Delta_T\).

1.5. Definition. Let \(I, C\) be 2-categories and \(D: I \rightarrow C\) be a 2-functor. A (pseudo) bilimit of \(D\) is an object \(L\) of \(C\) and a family of equivalences \(\Psi_T: C(T,L) \cong [I,C](\Delta T, D)\) pseudonatural in \(T\). When it exists, the bilimit of \(D\) is unique up to equivalence and the object \(L\) is noted \(\text{bilim}_{I} D\).

We call the canonical pseudonatural transformation \(\Psi_L(\text{Id}_L)\) the standard cone of the bilimit \(L\).

1.6. Definition. Let \(C\) be a 2-category. Let \(f: a \rightarrow c\) and \(g: b \rightarrow c\) be two 1-morphisms of \(C\) with the same target. A 2-pullback of \(f\) and \(g\) is a bilimit of the diagram:

\[
\begin{array}{ccc}
    a & \rightarrow & c & \leftarrow & b \\
    f & \downarrow & \ & \downarrow & g \\
    (f|g) & \rightarrow & \ & \rightarrow & (f|g)
\end{array}
\]

1.7. Notation. We write \((f|g)\) for the 2-pullback of \(f\) and \(g\). We will use the following naming scheme for the structural 1- and 2-morphisms:

1.8. Definition. Let \(C\) be a (2, 1)-category. Fix a (2, 1)-functor \(F: I \rightarrow C\) and an object \(c\) of \(C\). The slice \(F/c\) is the (2, 1)-category with:
• Objects: the pairs \((i, f)\) consisting of an object \(i\) of \(I\) and a morphism \(f: i \to c\).

• Morphisms \((i, f) \to (i', f')\): the pairs \((u, \mu)\) consisting of a morphism \(u: i \to i'\) of \(I\) and a 2-isomorphism \(\mu: f \Rightarrow f'\) of \(C\):

\[
\begin{array}{ccc}
F_i & \xrightarrow{\mu} & F_i' \\
\downarrow f & & \downarrow f' \\
C & \xleftarrow{\mu} & C
\end{array}
\]

• 2-Morphisms \((u, \mu) \Rightarrow (v, \nu)\): the 2-morphisms \(\alpha: u \Rightarrow v\) of \(I\) satisfying:

\[
F_i F_i' \circ \mu \sim F_{i'} F_i \circ \nu = F_i F_{i'} \circ \alpha
\]

Compositions are induced by the compositions of \(I\) and \(C\).

A slice 2-category \(\mathcal{F}/c\) is endowed with a canonical forgetful 2-functor:

\[
\begin{cases}
\mathcal{F}/c & \rightarrow \ I \\
(i, f) & \mapsto i \\
(u, \mu) & \mapsto u \\
\alpha & \mapsto \alpha
\end{cases}
\]

1.2. **Adjunctions and monads.** We recall the definitions of adjunctions and monads, and some of their basic properties in \(\text{Cat}\).

1.9. **Definition.** Let \(C, D\) be two categories. An adjunction between \(C\) and \(D\) is a quadruple \((\ell, r, \eta, \epsilon)\) of a functor \(\ell: C \to D\), a functor \(r: D \to C\), a natural transformation \(\eta: \text{Id}_C \Rightarrow r\ell\) and a natural transformation \(\epsilon: \ell r \Rightarrow \text{Id}_D\), such that:

\begin{align*}
\text{(1.10)} & \quad \text{Id}_\ell = \epsilon \ell \circ \ell \eta \\
\text{(1.11)} & \quad \text{Id}_r = r \epsilon \circ \eta r
\end{align*}

We write an adjunction \(\ell \dashv r\), with the natural transformations \(\eta\) and \(\epsilon\) omitted. The natural transformation \(\eta\) is called the *unit* of the adjunction and the natural transformation \(\epsilon\) is called the *counit*.

1.12. **Definition.** Let \(C\) be a category. A monad on \(C\) is a monoid object in the monoidal category \(\text{Cat}(C, C)\) of endofunctors of \(C\), that is, a triple \((T, \eta, \mu)\) consisting of a functor \(T: C \to C\), a natural transformation \(\eta: \text{Id}_C \Rightarrow T\) and a natural transformation \(\mu: T \circ T \Rightarrow T\) such that the following diagrams commute:

1.13. **Proposition.** Let \(C, D\) be two categories and \((\ell: C \to D, r: D \to C, \eta, \epsilon)\) be an adjunction \(\ell \dashv r\) between \(C\) and \(D\). Then the triple \((r\ell, \eta, r\epsilon)\) defines a monad on \(C\).

1.14. **Definition.** Let \(C\) be a category and \(T\) be a monad on \(C\). The *Eilenberg-Moore category* \(C^T\) of \(T\) is the category with:
• **Objects:** the pairs \((c, f)\) with \(c\) an object of \(C\) and \(f : Tc \rightarrow c\) a morphism of \(C\).

• **Morphisms** \((c, f) \rightarrow (c', f')\): the morphisms \(g : c \rightarrow c'\) such that the following square commutes:

\[
\begin{array}{c}
\begin{array}{c}
\text{TC} \\
\downarrow f \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{TC}' \\
\downarrow f' \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
c \\
\downarrow g \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
c' \\
\end{array}
\end{array}
\end{array}
\]

• **Composition** is induced by the composition of \(C\).

There is a canonical functor

\[
U : \begin{cases} 
C^T & \rightarrow C \\
(c, f) & \mapsto c \\
g & \mapsto g 
\end{cases}
\]

and a natural transformation \(\bar{\mu} : TU \Rightarrow U\) with components

\[
\bar{\mu}_{(c, f)} = f
\]

1.15. **Definition.** Let \(C\) be a category and \((T, \eta, \mu)\) be a monad on \(C\). A **left module** on \(T\), or left \(T\)-module, is a category \(D\) endowed with a functor \(V : D \rightarrow C\) and a natural transformation \(\nu : TV \Rightarrow V\) such that the following diagrams commute:

\[
\begin{array}{c}
\begin{array}{c}
\text{TTV} \\
\downarrow \tau_T \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{TV} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{V} \\
\downarrow \nu \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{TV} \\
\downarrow \nu \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{V} \\
\end{array}
\end{array}
\end{array}
\]

1.16. **Definition.** Let \(C\) be a category and \((T, \eta, \mu)\) be a monad on \(C\). The **2-category of left \(T\)-modules** is the category with:

• **Objects:** the left \(T\)-modules,

• **1-morphisms** \((D, V, \nu) \rightarrow (E, W, \omega)\): the pairs \((X, \chi)\) consisting of a functor \(X : D \rightarrow E\) and a natural isomorphism \(\chi : V \Rightarrow WX\) such that:

\[
\chi \circ \nu = \omega X \circ T \chi
\]

• **2-morphisms** \((X, \chi) \Rightarrow (X', \chi')\): the natural transformations \(\psi : X \Rightarrow X'\) such that:

1.17. **Proposition.** Let \(C\) be a category and \(T\) be a monad on \(C\). The Eilenberg-Moore category \(C^T\) endowed with \(U\) and \(\bar{\mu}\) (definition 1.14) is a left module on \(T\) (definition 1.13).

Moreover it is terminal among the left modules on \(T\) in the following sense: if \((D, V, \nu)\) is another left module on \(T\), there is a unique functor \(K : D \rightarrow C^T\) such that \(V = UK\) and \(\nu = \bar{\mu}K\).
1.18. Remark. The left $T$-module $(C^T, U, \bar{\mu})$ of proposition 1.17 is pseudo biterminal in the 2-category of left $T$-modules (definition 1.16): for any other left $T$-module $(D, V, \nu)$, there is a 1-morphism $(K, \kappa): (D, V, \nu) \to (C^T, U, \bar{\mu})$, unique up to a unique 2-isomorphism.

1.3. String diagrams. The proof of our main result is an extensive manipulation of string diagrams. We recall the general ideas of string diagrams, and introduce our notations. A fully detailed description of string diagrams can be found in [JY21, §3.7].

String diagrams are used to depict natural transformations. They are the dual diagrams of the “usual” pasting diagrams.

- An object $A$ is represented by a labeled surface

\[
\begin{array}{c}
A
\end{array}
\]

- A 1-morphism $f: A \to B$ is represented by a labeled vertical edge

\[
\begin{array}{c|c}
A & B \\
\hline
f
\end{array}
\]

with the source on the left and the target on the right.

- A 2-morphism $\phi: f \Rightarrow g: A \to B$ is represented by a labeled vertex

\[
\begin{array}{c|c}
A & B \\
\hline
f & g \\
\phi
\end{array}
\]

with the source above and the target below. The identity 1-morphisms may be omitted.

- We will occasionally represent identity 2-morphisms by a white dot, that is, the string diagram

\[
\begin{array}{c|c}
A & B \\
\hline
f & g \\
\phi
\end{array}
\]

is the identity between the 1-morphisms $f$ and $g$.

- When dealing with an adjunction $\ell \dashv r$, units and counits will be represented by black dots. For instance, the string diagram

\[
\begin{array}{c}
\ell \\
\hline
\phi
\end{array}
\]

is the unit of the adjunction $\ell \dashv r$. 

1.19. Remark. The notation for the unit and the counit of an adjunction $\ell \dashv r$ gives a simple interpretation of the unit-counit laws (eq. (1.10) and eq. (1.11)):

\[
\ell \begin{array}{c}
\ell \\
\ell
\end{array} = \begin{array}{c}
\ell \\
\ell
\end{array} \quad r \begin{array}{c}
r \\
r
\end{array} = \begin{array}{c}
r \\
r
\end{array}
\]

1.4. 1-categories as 2-categories. A 1-category can be seen as a (2, 1)-category, with only identities as 2-morphisms, and a 1-functor between 1-categories can itself be viewed as a 2-functor. Hence all the constructions on 2-categories can be applied to 1-categories and 1-functors. Reciprocally there are several ways to extract 1-categories from bicategories.

1.20. Definition. Let $C$ be a bicategory with only invertible 2-morphisms. The truncated category $\tau_1 C$ is the category where:

- **Objects** are the objects $c$ of $C$.
- **Morphisms** $c \to c'$ are the classes of 1-morphisms $c \to c'$ of $C$ up to 2-isomorphism.
- **Composition** is induced by the composition of $C$.

There is a canonical projection pseudofunctor $\pi : C \to \tau_1 C$.

1.21. Proposition. Let $C$ be a bicategory with only invertible 2-morphisms and $D$ be a 1-category seen as a 2-category. Let $\tilde{F} : C \to D$ be a pseudofunctor. Then there is a unique 1-functor $\tau_1 \tilde{F} : \tau_1 C \to D$ factoring $\tilde{F}$ through $\pi$:

\[
\begin{array}{c}
C \\
\pi \\
\tau_1 C
\end{array} \quad \quad \tau_1 \tilde{F} \quad \begin{bmatrix} \quad & \tilde{F} \quad \\
\tau_1 C & \to & D
\end{bmatrix}
\]

1.22. Definition. Let $C$ be a (2, 1)-category. The underlying category $C^{(1)}$ is the category where:

- **Objects** are the objects $c$ of $C$.
- **Morphisms** $c \to c'$ are the 1-morphisms of $C$.
- **Composition** is induced by the composition of $C$.

There is a canonical inclusion 2-functor $\iota : C^{(1)} \to C$.

1.23. Proposition. Let $C$ be a (2, 1)-category and $D$ be a 1-category. Let $F : C \to D$ be a 2-functor. Then the bilimit of $F$, if it exists, can be expressed as either of the following two ordinary limits in $D$:

\[
\text{bilim}_C F \simeq \lim_{\tau_1 C} \tau_1 F \simeq \lim_{C^{(1)}} F \circ \iota
\]

2. A 2-categorical framework for fusion theory

2.1. The classical Cartan-Eilenberg formula.

2.1. Definition. Let $G$ be a finite group and $S$ be a $p$-Sylow subgroup of $G$. The fusion system $F_S(G)$ of $G$ is the category with:

- **Objects**: the subgroups $P$ of $S$.
- **Morphisms** $P \to Q$: the group morphisms $P \to Q$ induced by the conjugation action of $G$:

$F_S(G)(P, Q) = \{c_g : P \to gPg^{-1} \mid g \in G, gPg^{-1} \subset Q\}$.

- **Composition** is the composition of group morphisms.
The fusion system $\mathcal{F}_S(G)$ of a group $G$ is canonically endowed with a forgetful functor to the category $\text{gp}$ of finite groups:

$$U : \begin{cases} \mathcal{F}_S(G) & \to \text{gp} \\ P & \mapsto P \\ f & \mapsto f \end{cases}$$

The Cartan-Eilenberg formula \cite[Theorem 10.1]{CE56} expresses the cohomology (with trivial coefficients) of a group $G$ as a limit over the fusion system $\mathcal{F}_S(G)$ of $G$ in the category $\mathcal{F}_p\text{Alg}^{gr}$ of graded $\mathbb{F}_p$-algebras:

2.2. Proposition. For any finite group $G$ and $p$-Sylow $S$ of $G$:

$$H^*(G; \mathbb{F}_p) \simeq \lim_{\mathcal{F}_S(G)^{op}} H^*(-; \mathbb{F}_p) \circ U$$

2.3. Definition. Let $G$ be a finite group and $S$ be a $p$-Sylow of $G$. The transporter category $\mathcal{T}_S(G)$ is the category with:

- **Objects**: the subgroups $P$ of $S$.
- **Morphisms** $P \to Q$: the elements $g$ of $G$ such that $g Pg^{-1} \subset Q$:

$$\mathcal{T}_S(G)(P,Q) = \{g \in G \mid g Pg^{-1} \subset Q\}.$$  

- **Composition** is given by the multiplication of $G$.

The transporter category $\mathcal{T}_S(G)$ of a group $G$ is also canonically endowed with a forgetful functor to the category $\text{gp}$ of finite groups:

$$U : \begin{cases} \mathcal{T}_S(G) & \to \text{gp} \\ P & \mapsto P \\ g & \mapsto c_g : p \mapsto gPg^{-1} \end{cases}$$

This forgetful functor $U : \mathcal{T}_S(G) \to \text{gp}$ factors through $\mathcal{F}_S(G)$. The comparison functor $\pi : \mathcal{T}_S(G) \to \mathcal{F}_S(G)$ is the identity on objects and full.

$$\mathcal{T}_S(G) \xrightarrow{\pi} \mathcal{F}_S(G) \xrightarrow{\text{gp}} \mathcal{F}_S(G)$$

Note that the functor $\pi$ is a final functor, in the sense of \cite[§IX.3]{Mac71}, since it is the identity on objects and full. Hence, there is a canonical isomorphism:

$$\lim_{\mathcal{F}_S(G)^{op}} H^*(-; \mathbb{F}_p) \circ U \simeq \lim_{\mathcal{T}_S(G)^{op}} H^*(-; \mathbb{F}_p) \circ U$$

In particular, the Cartan-Eilenberg formula can equivalently be stated as:

$$H^*(G; \mathbb{F}_p) \simeq \lim_{\mathcal{T}_S(G)^{op}} H^*(-; \mathbb{F}_p) \circ U$$

2.2. The extended transporter $(2,1)$-category of a group. We present a 2-categorification $\hat{\mathcal{T}}_S(G)$ of the transporter category. This 2-category will have a role similar to the one the fusion category $\mathcal{F}_S(G)$ and the transporter category $\mathcal{T}_S(G)$ have in the classical Cartan-Eilenberg formula.

Mostly for convenience, rather than restricting our attention to the category of finite groups $\text{gp}$, we should consider the $(2,1)$-category of finite groupoids $\text{gpd}$. It inherits all its structure of 2-category from the usual structure of 2-category on $\text{Cat}$, the category of small categories, functors and natural transformations. The finite groups are viewed in $\text{gpd}$ as the groupoids with exactly one object. We should also note that the 2-morphisms of $\text{gpd}$ are relevant from the point of view of the fusion
in groups. Indeed, given two parallel morphisms \( \phi, \psi : H \to G \) between groups, a 2-morphism \( \phi \Rightarrow \psi \) is precisely an element \( g \) of \( G \) such that:
\[
\forall h \in H, \phi(h) = g \psi(h) g^{-1}
\]
The 2-full subcategory of \( \text{gpd} \) with faithful functors as 1-morphisms is denoted by \( \text{gpd}^f \).

2.4. Definition. Let \( G \) be a group and \( S \) be a \( p \)-Sylow of \( G \). Denote by \( i : S \to G \) the inclusion, seen as a 1-morphism of \( \text{gpd}^f \). The extended transporter category \( \hat{T}_S(G) \) is the \((2,1)\)-category with:

- **Objects**: the pairs \( (P, j_P : P \to S) \) of a finite groupoid \( P \) and a faithful functor \( j_P : P \to S \).
- **1-Morphisms** \( (P, j_P) \to (Q, j_Q) \): the pairs \((a, \alpha)\) of a faithful functor \( a : P \to Q \) and 2-morphisms \( \alpha : i_Qa \Rightarrow ij_P \) in \( \text{gpd}^f \).
- **2-Morphisms** \( (a, \alpha) \Rightarrow (b, \beta) \): the 2-morphisms \( \phi : \alpha \Rightarrow \beta \) in \( \text{gpd}^f \), such that:

\[
\begin{array}{ccc}
P & \xrightarrow{\phi} & Q \\
i_j P & \downarrow^{a} & \downarrow^{b} \\
 & G & \xrightarrow{i} & G \\
g & \xrightarrow{ij_P} & \alpha & \xrightarrow{ij_Q} & \beta \\
 & P & \xrightarrow{\phi} & Q
\end{array}
\]

- **Compositions** are induced by the obvious pasting of diagrams in \( \text{gpd}^f \).

The extended transporter category is endowed with a forgetful 2-functor:
\[
\mathbb{U} : \hat{T}_S(G) \to \text{gpd}^f
\]

\[
\begin{array}{ccc}
\hat{T}_S(G) & \xrightarrow{\mathbb{U}} & \text{gpd}^f \\
(P, i_P) & \xrightarrow{\mathbb{U}} & P \\
(a, \alpha) & \xrightarrow{\mathbb{U}} & a \\
\phi & \xrightarrow{\mathbb{U}} & \phi
\end{array}
\]

2.5. Remark. Since \( i : S \to G \) is a faithful functor between 1-object groupoids, the factorization of a functor \( i_P : P \to G \) through \( i \), if it exists, is unique. Hence the objects of the transporter 2-category \( \hat{T}_S(G) \) could equivalently be described as pairs \( (P, i_P : P \to G) \) such that the faithful functor \( i_P \) factors through \( i \):

\[
i_P = ij_P \text{ for some } j_P : P \to S
\]

2.6. Remark. The underlying 1-category \( \hat{T}_S(G)^{(1)} \) is quite similar to the classical \( T_S(G) \). Actually, the main difference is the addition of finite coproducts. In section \( \ref{section:co} \) we choose not to distinguish them. Since we are working with product-preserving functors (over the opposite categories), this is not really a problem.

2.7. Remark. Biequivalently, the extended transporter category \( \hat{T}_S(G) \) can be defined as follows, using generic constructions of bicategories. Consider the slice \((2,1)\)-category \( \text{gpd}^f/G \). Then we can define the extended transporter category \( \hat{T}_S(G) \) as the 2-category of subobjects of \((S,i)\) in \( \text{gpd}^f/G \), that is, the full, 2-full subcategory of \( \text{gpd}^f/G \) over objects \((P, i_P : P \to G)\) such that \( i_P \) factors (up to some isomorphism) through \( i : S \to G \):

\[
P \xrightarrow{i_P} S \xleftarrow{i} G
\]

In this construction, the forgetful 2-functor
\[
\mathbb{U} : \hat{T}_S(G) \to \text{gpd}^f
\]
appears as the restriction of the forgetful 2-functor $\text{gpd}^f/G \to \text{gpd}^f$.

2.8. Remark. Note that there is a canonical way to recover the group $G$ from the 2-functor $U$ as a pseudo bicolimit:

$$\mathcal{T}_S(G) \cong \text{bicolim} U \mathcal{F}(\{e\})$$

Indeed there is a canonical way to endow $G$ with a cocone structure over $U$, hence to define a comparison morphism from the bilimit to $G$. Then, by noticing that the endomorphisms of the trivial group $\langle \{e\}, \{e\} \to S \rangle$ in $\mathcal{T}_S(G)$ is precisely the group $G$, one can construct another morphism from $G$ to the bilimit, which is a pseudoinverse of the canonical comparison morphism. A detailed proof can be found in [Mai21a, Proposition 4.2.9].

3. Group representations as bilimits

3.1. Group representations as Mackey 2-functors. In this section, $k$ is a commutative $\mathbb{Z}(\rho)$-algebra (that is, every prime integer different from $p$ is invertible in $k$), for instance a field of characteristic $p$.

3.1. Notation. Let $G$ be a finite group.

- The category $\text{mod}(kG)$ is the category of $k$-linear representations of finite dimension of the group $G$.
- The category $\text{stmod}(kG)$ is the stable category of $k$-linear representations of finite dimension of the group $G$, for $k$ a field of characteristic $p$.
- The category $D(kG)$ is the (bounded) derived category of the finite dimensional $kG$-modules.

The three mappings $G \mapsto \text{mod}(kG)$, $G \mapsto \text{stmod}(kG)$ and $G \mapsto D(kG)$ extend to 2-functors $\mathcal{M}: (\text{gpd}^f)^{\text{op}} \to \text{Add}$ from the opposite of the 2-category of finite groupoids (with faithful 1-morphisms), to the 2-category of small additive categories. The image $\mathcal{M}(f)$ of a group morphism $f: H \to G$, noted $f^*$, is the restriction along $f$; similarly, given a 2-morphism $\alpha: f \Rightarrow g$ in $\text{gpd}^f$, $\alpha^*: f^* \Rightarrow g^*$ denotes the image $\mathcal{M}(\alpha)$. We can see that these three 2-functors are cohomological Mackey 2-functors with values in $k$-linear idempotent-complete categories, as we now explain.

3.2. Definition. A Mackey 2-functor is a 2-functor $\mathcal{M}: (\text{gpd}^f)^{\text{op}} \to \text{Add}$ satisfying the following four axioms:

- (Mack 1) Additivity: If $i_G: G \to G \uplus H$ and $i_H: H \to G \uplus H$ are the canonical inclusions into a coproduct of groupoids, then the induced functor

$$\mathcal{M}(i_G^*), \mathcal{M}(i_H^*): \mathcal{M}(G \uplus H) \to \mathcal{M}(G) \times \mathcal{M}(H)$$

is an equivalence.

- (Mack 2) If $f: H \to G$ is a 1-morphism in $\text{gpd}^f$, its image $f^*$ has both a left adjoint $f_!$ and a right adjoint $f_*$:

$$f_! \dashv f^* \dashv f_*$$

- (Mack 3) Beck-Chevalley property: For any 2-pullback square in $\text{gpd}^f$ (see notation [L7])

$$\begin{array}{ccc}
K & \xrightarrow{f} & G \\
\downarrow & \searrow & \downarrow \\
(f|g) & \xrightarrow{(f|g)} & L
\end{array}$$

is a pullback square in $\mathcal{M}$.\[\]
the mates \( \lambda_! : (f|g)_! (f|g)^* \Rightarrow g^* f_! \) of \( \lambda^* \) and \( (\lambda^{-1})_* : g^* f_* \Rightarrow (f|g)_* (f|g)^* \) of \( (\lambda^{-1})^* \) are invertible, where:

\[
\begin{align*}
(3.3) \quad \lambda_! &= \begin{array}{c}
\xymatrix{
\lambda^* & \lambda^* \ar[l] \ar[r] & \lambda^*}
\end{array} \\
\lambda_* &= \begin{array}{c}
\xymatrix{
(\lambda^{-1})^* & (\lambda^{-1})^* \ar[r] & (\lambda^{-1})^*}
\end{array}
\end{align*}
\]

(Mack 4) **Ambidexterity:** For any 1-morphism \( f : H \to G \) in \( \text{gpd}^\text{f} \), the left and right adjoints of \( f^* \) are isomorphic:

\[ f_! \simeq f_* \]

3.4. **Notation.** A 2-functor \( \mathbb{M} : (\text{gpd}^\text{f})^\text{op} \to \text{Cat} \) which only satisfies the axioms [Mack 2] and [Mack 3] is said to have the Beck-Chevalley property.

A 2-functor \( \mathbb{M} : (\text{gpd}^\text{f})^\text{op} \to \text{Cat} \) which only satisfies one half of the axioms [Mack 2] and [Mack 3] is said to have the left Beck-Chevalley property or right Beck-Chevalley property, accordingly.

3.5. **Definition.** A cohomological Mackey 2-functor is a Mackey 2-functor \( \mathbb{M} \) such that for any inclusion of groups \( i : H \to G \) the composite natural transformation

\[ \text{Id}_{\mathbb{M}(G)} \xrightarrow{\eta} i_* i^* \xRightarrow{\epsilon} \text{Id}_{\mathbb{M}(G)} \]

is equal to \( [G : H] \sigma \), for some 2-isomorphism \( \sigma \). In particular, if \( \mathbb{M} \) takes values in \( k \)-linear categories, the composite natural transformation \( \epsilon \eta \) above is invertible whenever \( [G : H] \) is coprime to \( p \), since \( k \) is a \( \mathbb{Z}(p)^{\text{op}} \)-algebra.

The following comparison theorem between \( \mathbb{M}(G) \) and the associated Eilenberg-Moore category appears in [BD21]; we reproduce here its short proof.

3.6. **Theorem.** Let \( \mathbb{M} \) be a cohomological Mackey 2-functor (definition 3.2 and definition 3.5) with values in \( k \)-linear idempotent-complete categories.

Then, for any finite group \( G \) and subgroup \( i : H \to G \) with \( [G : H] \) coprime to \( p \), the adjunction \( i_! i^* \) is monadic. That is, the canonical comparison functor

\[ \mathbb{M}(G) \to \mathbb{M}(H)^T \]

is an equivalence, where \( T = i^* i_! \) is the monad induced by \( i \).

**Proof.** Recall that \( \epsilon : i_* i^* \Rightarrow \text{Id}_{\mathbb{M}(G)} \) is the counit of the adjunction \( i_! \dashv i^* \) and \( \eta : \text{Id}_{\mathbb{M}(G)} \Rightarrow i_* i^* \) is the unit of the adjunction \( i^* \dashv i_* \).

Since \( \mathbb{M} \) is cohomological and \( [G : H] \) is coprime to \( p \), the natural transformation \( \epsilon \eta \) is invertible. Hence \( \epsilon \) admits a section, and the adjunction \( i_! \dashv i^* \) is monadic, by [Bal15, Lemma 2.10].

This naturally leads to the following definition:

3.7. **Definition.** A \( p \)-monadic Mackey 2-functor is a Mackey 2-functor \( \mathbb{M} \) such that, for any inclusion of groups \( i : H \to G \) with index coprime to \( p \), the canonical comparison functor

\[ \mathbb{M}(G) \to \mathbb{M}(H)^T \]

is an equivalence, where \( T = i^* i_! \) is the monad induced by \( i \).
3.8. Remark. By theorem 3.6, any cohomological Mackey 2-functor with values in \(k\)-linear idempotent-complete categories is p-monadic. This includes the 2-functors \(\text{mod}(k)\), \(\text{stmod}(k)\) and \(D(k)\); other examples can be found in [BD20, Chapter 4].

For instance, the mapping \(G \mapsto \text{coMack}_k(G)\) associating to a group \(G\) the category of cohomological \(G\)-local Mackey 1-functors extends to a cohomological Mackey 2-functor. This is a corollary of [BD20, Proposition 7.3.2] applied to the cohomological Mackey 2-functor \(S(G) = \text{perm}_k(G)\) of permutations \(k\)-modules and the Yoshida theorem [Web00], characterizing the category of cohomological Mackey 1-functors as

\[
\text{coMack}_k(G) \cong \text{Fun}_+(\text{perm}_k(G), \text{Ab}).
\]

3.9. Theorem. Let \(\mathcal{M}: (\text{gpd})^{\text{op}} \to \text{Add}\) be a p-monadic Mackey 2-functor (definition 3.7). Then, for any finite group \(G\) and any p-Sylow \(S\) of \(G\), there is an equivalence

\[
\mathcal{M}(G) \cong \text{bilim}_{T_S(G)^{\text{op}}} \mathcal{M} \circ U
\]

where \(T_S(G)\) is the extended transporter 2-category of \(G\) (definition 2.4).

Using the p-monadicity, theorem 3.9 is a corollary of the following result:

3.10. Theorem. Let \(\mathcal{M}: (\text{gpd})^{\text{op}} \to \text{Cat}\) be a 2-functor with the left Beck-Chevalley property (notation 3.4).

For any finite group \(G\) with a p-Sylow subgroup \(i: S \to G\), there is an equivalence

\[
\mathcal{M}(S)^T \cong \text{bilim}_{T_S(G)^{\text{op}}} \mathcal{M} \circ U
\]

where \(T = i^*i^!\) is the monad induced by \(i\).

3.11. Remark. The equivalence of theorem 3.10 is reminiscent of the Bénabou-Roubaud theorem [BR70]. This is actually an instance of a generalization to 2-categories; a more precise statement is in [Mai21a, Theorem 3.2.1]. Nonetheless, the essential arguments of the proof are already present in this article.

We will now construct the equivalence of theorem 3.10. Let \(L\) be the bilimit:

\[
L = \text{bilim}_{T_S(G)^{\text{op}}} \mathcal{M} \circ U
\]

Its standard cone (definition 1.5) is denoted by \(L_*\). In the rest of this section, we will:

- construct a functor \(A: L \to \mathcal{M}(S)^T\),
- construct a functor \(B: \mathcal{M}(S)^T \to L\), and
- show that the functors \(A\) and \(B\) are mutual pseudoinverses.

This will prove theorem 3.10.

3.3. Construction of a comparison functor \(L \to \mathcal{M}(S)^T\). Recall that \(\mathcal{M}(S)^T\) has a canonical structure of left module on \(T\) consisting of the forgetful functor \(U: \mathcal{M}(S)^T \to \mathcal{M}(S)\) and a natural transformation \(\mu: TU \Rightarrow U\). Moreover, it is terminal among the left modules on \(T\) (proposition 1.17).

One can define a structure of left \(T\)-module for the bilimit \(L\) as follows. There is an obvious functor \(L \to \mathcal{M}(S)\), namely the component \(L_S\) of the standard cone.
The definition of the action $\nu$ use the invertibility of the mate $\lambda_!$ (see [Mack 3]):

\[
\mathcal{L}_S \overset{\nu}{\longrightarrow} \mathcal{L}_S \quad \text{with} \quad \lambda^{-1} = (i_i^! i_i^! *) \lambda^! \mathcal{L}_S
\]

where $\lambda = (i_i^! i_i^! *)$ is defined by the following 2-pullback square in $\text{gpd}^f$

\[
\begin{array}{ccc}
G & \overset{i}{\longrightarrow} & S \\
\downarrow & \searrow & \downarrow \\
S & \overset{\Delta}{\longrightarrow} & S \\
\downarrow & \nearrow & \downarrow \\
\ominus i \ominus & \overset{(i_i^! i_i^! *)}{\longrightarrow} & \ominus i \ominus
\end{array}
\]

and $\mathcal{L}_\lambda$ is the composite 2-morphism

\[
(i_i^! i_i^! *) \mathcal{L}_S \xrightarrow{\mathcal{L}_\lambda(i_i^! i_i^! *)} \mathcal{L}_\lambda \mathcal{L}_S \xrightarrow{\mathcal{L}_\lambda^{-1} (i_i^! i_i^! *)} \mathcal{L}_S
\]

3.13. **Proposition.** The action $\nu$ is unital.

**Proof.** First, note that there is a diagonal morphism $\Delta: S \rightarrow (i_i^! i_i^! *)$ satisfying $(\ominus i \ominus) \Delta = (i_i^! i_i^! *) \Delta = \text{Id}_S$ and $\lambda \Delta = \text{Id}_i$. We thus have the following relation:

\[
\begin{array}{ccc}
\overset{i_i^! i_i^! ^*}{\longrightarrow} & \lambda^* & \overset{\Delta^*}{\longrightarrow} \\
\overset{i_i^! i_i^! ^*}{\longrightarrow} & \lambda^* & \overset{\Delta^*}{\longrightarrow}
\end{array}
\]

\[
\begin{array}{ccc}
\overset{i_i^! i_i^! ^*}{\longrightarrow} & \lambda^* & \overset{\Delta^*}{\longrightarrow} \\
\overset{i_i^! i_i^! ^*}{\longrightarrow} & \lambda^* & \overset{\Delta^*}{\longrightarrow}
\end{array}
\]
Hence, we have:

\[
\lambda \nu = \lambda_i^{-1} \Delta^* = \lambda_i \Delta^* = \lambda_i
\]

3.15. **Proposition.** The action \( \nu \) is associative.

**Proof.** We introduce another 2-pullback square

\[
\begin{align*}
(i|i) & \overset{\kappa}{\longrightarrow} (i|i) \\
\downarrow v & \quad \downarrow w \\
S & \overset{(i|\lambda)}{\longrightarrow} (i|\lambda)
\end{align*}
\]

The universal property of \((i|i)\) allows us to define a morphism \( \nabla: X \to (i|i) \) such that

\[
(3.16)
\]

\[
\nu (i|\lambda) i = \nabla (i|\lambda) i = \lambda
\]

\[
\lambda (i|\lambda) i = \lambda
\]
Applying the 2-functor \( \mathcal{M} \) on eq. (3.16) yields

\[
\begin{align*}
\lambda^* & \quad \kappa^* \\
\lambda^* & \quad \kappa^* \\
i^* \quad (i|\bar{i})^* & \quad v^* \\
& = \\
i^* \quad (i|\bar{i})^* & \quad v^*
\end{align*}
\]

Similarly, the compatibility between the cone \( \mathcal{L} \) and the 2-morphisms of \( \hat{T}_2(G) \) provides the equation

\[
\begin{align*}
\mathcal{L}_S \quad (i|\bar{i})^* & \quad v^* \\
\mathcal{L}_\lambda & \quad \kappa^* \\
\mathcal{L}_S \quad (i|\bar{i})^* & \quad w^* \\
& = \\
\mathcal{L}_S \quad (i|\bar{i})^* & \quad v^* \\
\mathcal{L}_\lambda & \quad \kappa^* \\
\mathcal{L}_S \quad (i|\bar{i})^* & \quad w^*
\end{align*}
\]

We can now prove the associativity of \( \nu \). We want to prove:

\[
\begin{align*}
\mathcal{L}_S \quad i! \quad i^* \quad i! \quad i^* & \quad \nu \\
\nu & = \\
\mathcal{L}_S \quad i! \quad i^* \quad i! \quad i^* & \quad \nu \\
\mathcal{L}_\lambda & \quad \kappa^* \\
\mathcal{L}_S \quad (i|\bar{i})^* & \quad w^*
\end{align*}
\]

Consider the mate \( \kappa! \) of \( \kappa^* \):

\[
\kappa! = \quad \kappa^* \\
(i|\bar{i})^* \quad (i|\bar{i})^*
\]
Since $\lambda_1$ and $\kappa_1$ are invertible, eq. (3.19) is equivalent to the equation:

\[
\begin{align*}
\mathcal{L}_S (i|i)^* v^* w_1 (i|\tilde{i}) &= \mathcal{L}_S (i|i)^* v^* w_1 (i|\tilde{i}) \\
\nu &= \nu \\
\mathcal{L}_S &= \mathcal{L}_S
\end{align*}
\]  

(3.20)

Expanding the definitions of $\nu$ and $\kappa_1$ on the left-hand side of eq. (3.20) leads to the following computation:
To simplify the right-hand side of eq. (3.20), first note that we have the following relation:

\[
\mathcal{L}_S (i|j)^* \nu^* w_1 (i|j)^* = \mathcal{L}_S (i|j)^* \nu^* w_1 (i|j)^* \]

\[(3.12)\]

Then, by expanding the definitions of \(\lambda_i\) and \(\kappa_i\) in the right-hand side of eq. (3.20), we get:

\[
\mathcal{L}_S (i|j)^* v^* w_1 (i|j)^* = \mathcal{L}_S (i|j)^* v^* w_1 (i|j)^* \]

\[(3.21)\]
Thus eq. (3.20) holds, and so does eq. (3.19).

By the universal property of $\mathcal{M}(S)^T$ (proposition 1.17) applied to the left $T$-module $(L, \mathcal{L}_S, \nu)$, we have:

3.22. Proposition. There is a unique functor $A: L \to \mathcal{M}(S)^T$ fitting in the diagram

\[
\begin{array}{c}
L \\
\xrightarrow{A} \mathcal{M}(S)^T \\
\mathcal{M}(S)
\end{array}
\]

and satisfying $\bar{\mu}A = \nu$.

3.4. The comparison functor $\mathcal{M}(S)^T \to L$. In the reverse direction, giving a functor $B: \mathcal{M}(S)^T \to L$ is equivalent (definition 1.5) to giving a pseudonatural transformation $\phi: \Delta\mathcal{M}(S)^T \Rightarrow \mathcal{M} \circ U$ between 2-functors $\tilde{T}_S(G) \to \text{Cat}$. Define the components of $\phi$ as follows:

- for any object $(P, j_P)$ of $\tilde{T}_S(G)$,

\[
\phi_{(P, j_P)}: \mathcal{M}(S)^T \xrightarrow{U} \mathcal{M}(S) \xrightarrow{j_P} \mathcal{M}(P)
\]
• for any 1-morphism \((a, \alpha): (P, j_P) \to (Q, j_Q)\), that is a 1-morphism \(a: P \to Q\) and a 2-morphism \(\alpha: i_{j_Q}a \Rightarrow i_{j_P}\) in \(\text{gpd}^f\),

\[ \phi_{(a, \alpha)} = \begin{array}{c}
\text{for any 1-morphism (a, \alpha): (P, j_P) \to (Q, j_Q), that is a 1-morphism a: P \to Q and a 2-morphism \alpha: i_{j_Q}a \Rightarrow i_{j_P} in gpd^f,}
\end{array} \]

\[ (3.23) \]

3.24. **Proposition.** The natural transformation \(\phi_{(a, \alpha)}\) is invertible.

**Proof.** Consider

\[ \psi_{(a, \alpha)} = \begin{array}{c}
\end{array} \]

We can check, using the associativity and the unitality of the action \(\bar{\mu}\), and the definition of the multiplication \(\mu\) of \(T\), that \(\psi_{(a, \alpha)} \circ \phi_{(a, \alpha)} = \text{Id}\) and \(\phi_{(a, \alpha)} \circ \psi_{(a, \alpha)} = \text{Id}\). \(\square\)

3.25. **Proposition.** The family of functors and natural transformations \(\phi\) is a pseudonatural transformation \(\phi: \Delta \mathcal{M}(S)^T \Rightarrow \mathcal{M} \circ \U\).

**Proof.** This is done by the following straightforward computations.

\[ \phi_{(\text{Id}, \text{Id})} = \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} = \text{Id}_{j_P^* U} \]

For any composable morphisms \((a, \alpha): (P, j_P) \to (Q, j_Q)\) and \((b, \beta): (Q, j_Q) \to (R, j_R)\), using the associativity of the action \(\bar{\mu}\) and the definition of the multiplication \(\mu\) of
the monad $T$:

$$
\phi_{(a,\alpha)} \circ a^* \phi_{(b,\beta)} = \phi_{(b,\beta)} \circ \phi_{(a,\alpha)}
$$

For any 2-morphism $\zeta: (a, \alpha) \Rightarrow (b, \beta)$ between parallel morphisms $(P, j_P) \to (Q, j_Q)$ in $\hat{T}_S(G)$, we compute:

These equalities show the pseudonaturality of $\phi$. □

Hence, by the universal property of the bilimit:

3.26. **Proposition.** There exists a unique (up-to 2-isomorphism) functor

$$B: \mathcal{M}(S)^T \to L$$

and a modification $m: \phi \cong L \cdot B$.

3.5. **The functors** $A$ **and** $B$ **are mutually pseudoinverse.**

3.27. **Proposition.** The composite $AB: \mathcal{M}(S)^T \to \mathcal{M}(S)^T$ is isomorphic to the identity functor.
Proof. The universal property (remark 1.18) of $\mathcal{M}(\mathcal{S})^T$ guarantees that $(\text{Id}_{\mathcal{M}(\mathcal{S})^T}, \text{Id}_U)$ is the unique 1-endomorphism of the left $T$-module $(\mathcal{M}(\mathcal{S})^T, U, \bar{\mu})$, up to a 2-isomorphism.

Hence, the existence of a 2-isomorphism $\xi : U \xrightarrow{\sim} UAB$ such that $(AB, \xi) : (\mathcal{M}(\mathcal{S})^T, U, \bar{\mu}) \rightarrow (\mathcal{M}(\mathcal{S})^T, U, \bar{\mu})$ is a 1-morphism of left $T$-modules (definition 1.16) implies that $\text{Id}_{\mathcal{M}(\mathcal{S})^T}$ and $AB$ are isomorphic. Let us find such a $\xi$.

First, as $\phi_S = j^*SU = U$ we can take the 2-isomorphism $m_S : UAB = LSB \cong \phi_S = U$ given by proposition 3.26 as our 2-isomorphism $\xi$. Moreover, since $m : LB \rightarrow \phi$ is a modification,

\[ m_S \circ \bar{\mu}AB \overset{3.26}{=} m_S \circ \nu B \overset{1.12}{=} \]

This equality shows that $(AB, m_S) : (\mathcal{M}(\mathcal{S})^T, U, \bar{\mu}) \rightarrow (\mathcal{M}(\mathcal{S})^T, U, \bar{\mu})$ is a 1-morphism of left $T$-modules. Since such a 1-morphism is unique up to 2-isomorphism, $AB$ is isomorphic to $\text{Id}_{\mathcal{M}(\mathcal{S})^T}$. $\square$

3.28. Proposition. The composite $BA : L \rightarrow L$ is isomorphic to the identity functor.

Proof. Similarly we will show that the induced cone $\tilde{\mathcal{C}} := \mathcal{L} : (BA)$ is isomorphic to the standard cone $\mathcal{L}$, that is, that there is an invertible modification $\eta$ between
On any object \((P,j_P)\) define \(n_P\) as
\[
\begin{align*}
n_P : \tilde{L}_P &= \mathcal{L}_P B A \xrightarrow{m_P A} \phi_P A = j_P^* U A = j_P^* \mathcal{L}_S = \mathcal{L}_P
\end{align*}
\]

The family of morphisms \((n_P)\) is an invertible modification between the cones \(\tilde{L}\) and \(\phi A\). Thus, it suffices to check that \(\phi A\) and \(\mathcal{L}\) are indeed the same cones. Their components at any object \((P,j_P)\) coincide; we still have to check that the components at any morphism \((a,\alpha)\) are equal.

At the morphism \((\text{Id},\lambda) : ((i|i),(i|i)) \to ((i|i),(i|i))\) of \(\hat{T}_S(G)\), we have:
\[
\phi((\text{Id},\lambda)) A = A U (\hat{i}i)^* \xrightarrow{(3.22)} \lambda^* \mu
\]
\[
\xrightarrow{(i|i)^*} \lambda^* \nu = \lambda^* \mu
\]
\[
\xrightarrow{\mathcal{L}_\lambda} \mathcal{L}_S (\hat{i}i)^* = \mathcal{L}_\lambda
\]

For any morphism \((a,\alpha) : (P,j_P) \to (Q,j_Q)\) there is a (unique) morphism of groupoids \(\nabla_\alpha : P \to (i|i)\) such that
\[
(\hat{i}i) \nabla_\alpha = j_Q a \quad (i|i) \nabla_\alpha = j_P a \quad \lambda \nabla_\alpha = \alpha
\]

Hence,
\[
\phi((a,\alpha)) A = A (\hat{j}_Q a)^* \xrightarrow{(3.12)} \alpha^* \mu
\]
\[
\xrightarrow{\mathcal{L}_\alpha} \lambda^* \nabla_\alpha^* = \lambda^* \nabla_\alpha
\]
Thus, \( BA \) is isomorphic to the identity functor. \( \square \)

Putting together proposition \[ \text{3.27} \] and proposition \[ \text{3.28} \] we get:

**3.29. Theorem.** The categories \( L \) and \( M(S)^T \) are equivalent.

### 4. Applications

In this section, we describe two different ways of extracting the classical Cartan-Eilenberg formula from theorem \[ \text{3.9} \].

#### 4.1. Extracting Hom-sets

In this subsection, we will express the Hom-sets of a bilimit in \( \text{Cat} \) for a strict 2-functor as a certain limit, and show how it can be used to extract more concrete information from the formula of theorem \[ \text{3.9} \].

We should first give an explicit description of pseudo bilimits in \( \text{Cat} \).

**4.1. Definition.** Let \( I \) be a (small) 2-category and \( D : I \to \text{Cat} \) be a 2-functor. Let \( L_D \) be the category with:

- **Objects:** The pairs of families \( d := ((d_i), (d_f)_f) \) such that:
  - for each object \( i \in I \), \( d_i \) is an object of \( D_i \)
  - for each 1-morphism \( f : i \to j \) in \( I \), \( d_f \) is an isomorphism \( D_f(d_i) \sim d_j \) in \( D_j \).
  - for any 2-morphism \( \phi : f \Rightarrow g \) in \( I \), with \( f, g : i \to j \),
    \[ d_g \circ (D\phi)_{di} = d_f \]
  - for any composable 1-morphisms \( f : i \to j \) and \( g : j \to k \) in \( I \),
    \[ d_{gf} = d_g \circ (Dg)(df) \]

- **Morphisms** \( ((d_i), (d_f)_f) \to ((d'_i), (d'_f)_f) : ((\delta_i) : d_i \to d'_i)_{i \in \text{Ob}(I)} \): the families of morphisms such that, for any morphism \( f : i \to j \) in \( I \), the following square commutes:

\[
\begin{array}{ccc}
\nabla f(d_i) & \xrightarrow{df} & d_j \\
\n\nabla f(\delta_i) & \downarrow & \delta_j \\
\n\nabla f(d'_i) & \xrightarrow{d'_f} & d'_{j}
\end{array}
\]

(4.2)

- **Composition** is induced componentwise by the compositions of the categories \( (D_i)_i \).

The category \( L_D \) is endowed with a canonical cone \( \phi \) over \( D \) with components:

\[
\phi : \begin{cases} \
L_D & \Rightarrow D_i \\
((d_i), (d_f)_f) & \mapsto d_i \text{ for any object } i \in I \\
(f_i) & \mapsto f_i \\
(\phi f)((d_i), (d_f)_f) & = df \text{ for any morphism } f : i \to j \in I
\end{cases}
\]
4.3. **Proposition.** The category $L_D$, with its canonical cone, is a bilimit of $\mathbb{D}$:

$$L_D \cong \operatorname{bilim}_1 \mathbb{D}$$

**Proof.** The category $L_D$ is an explicit description of the category $[\Delta 1, \mathbb{D}]$ of the (pseudo)cones over $\mathbb{D}$ with vertex 1, the category with exactly one object and its identity morphism.

In turn, $[\Delta 1, \mathbb{D}]$ is a model of the bilimit of $\mathbb{D}$ by the following classical equivalence, pseudonatural in $X \in \text{Cat}$:

$$\text{Cat}(X, [\Delta 1, \mathbb{D}]) \cong \text{Cat}(1, [\Delta X, \mathbb{D}]) \cong [\Delta X, \mathbb{D}]$$

$\square$

We now fix a 2-diagram $\mathbb{D}: I \to \text{Cat}$ and two objects $d, d'$ of $L_D$. By definition:

$$L_D(d, d') = \{ (\delta_i : d_i \to d'_i)_{i, 1} \mid \text{the squares } \square \text{ commute} \}$$

We can define an associated 1-diagram on the underlying 1-category $I^{(1)}$ of $I$, obtained by forgetting the 2-morphisms, as follows:

4.5. **Definition.** The diagram $\mathbb{D}_{d, d'} : I^{(1)} \to \text{Set}$ is:

$$\mathbb{D}_{d, d'} : \begin{cases} i \mapsto \mathbb{D}(d_i, d'_i) \\ f : i \to j \mapsto \mathbb{D}(d_f, d'_j) \circ \mathbb{D}f \end{cases}$$

4.6. **Proposition.** There is an isomorphism:

$$L_D(d, d') \cong \lim_{\mathbb{D}_{d, d'}}$$

**Proof.** The limit $\lim_{\mathbb{D}_{d, d'}} \mathbb{D}_{d, d'}$ has an explicit description, as a limit in $\text{Set}$:

$$\lim_{\mathbb{D}_{d, d'}} = \{ (\delta_i \in \mathbb{D}_{d, d'}(i))_{1} \mid \forall f : i \to j, \mathbb{D}_{d, d'}(f)(\delta_i) = \delta_j \}$$

Unfolding the definitions, we precisely get back the set of (1.4). $\square$

4.7. **Remark.** The expression of the diagram $\mathbb{D}_{d, d'}$ simplifies when the objects $d$ and $d'$ are such that, for all $f : i \to j$, $d_f = d'_f = \text{Id}$:

$$\mathbb{D}_{d, d'} : \begin{cases} i \mapsto \mathbb{D}(d_i, d'_i) \\ f : i \to j \mapsto \mathbb{D}f \end{cases}$$

4.8. **Example.** We can use these results to recover the Cartan-Eilenberg formula for the Tate cohomology $\hat{H}^*(G; k)$ of a finite group $G$ over a field $k$ of characteristic $p$. Indeed, for any finite group $G$, the Tate cohomology groups can be recovered as Hom-sets of $\text{stmod}(kG)$:

$$\text{stmod}(kG)(\Omega^n k, k) = \hat{H}^n(G; k)$$

Fix a finite group $G$ and a $p$-Sylow $S$ of $G$. We consider the 2-diagram:

$$\mathbb{D} = \text{stmod}(k(-)) \circ U : T_5^G(\text{op}) \to \text{Cat}$$

Note that, for any $n$, $\Omega^n k \in \text{stmod}(kG)$ can be seen as an object of $L_D$ satisfying the conditions of remark 4.7. Hence we have:

$$\hat{H}^n(G; k) = \text{stmod}(kG)(\Omega^n k, k) \simeq \lim_{P \in T_5^G(\text{op})} \text{stmod}(kP)(\Omega^n k, k) = \lim_{P \in T_5^G(\text{op})} \hat{H}^n(P; k)$$

The Cartan-Eilenberg formula for the usual group cohomology $H^*(G; k)$ can be similarly recovered from the 2-functor $D(k(-))$. 

4.2. Categorical invariants. Another way to exploit theorem 3.9 is to try to factor $\mathcal{M}$ as a composite of two 2-functors

$$\mathcal{M}: (\text{gpd})^{\text{op}} \overset{\tilde{\mathcal{M}}}{\rightarrow} C \overset{W}{\rightarrow} \text{Cat}$$

where $\tilde{\mathcal{M}}$ is product preserving, $W: C \rightarrow \text{Cat}$ reflects bilimits, and to find a bilimit-preserving 2-functor $H: C \rightarrow \text{D}$ to a 1-category D. In this case, for any finite group $G$ with $p$-Sylow $S$:

$$H \circ \tilde{\mathcal{M}}(G) \simeq H(\text{bilim}_{\mathfrak{T}_S(G)^{\text{op}}} \circ U)$$

(4.9)

$$\simeq \text{bilim}_{\mathfrak{T}_S(G)^{\text{op}}} H \circ \tilde{\mathcal{M}} \circ U$$

$$\simeq \lim_{\mathfrak{T}_S(G)^{\text{op}}} H \circ \tilde{\mathcal{M}} \circ U \quad \text{(by proposition 1.23)}$$

4.10. Example. We sketch an example of application, taking $\mathcal{M} = \text{stmod}(k^-)$. We can factor it through the 2-category $\text{Add}^\otimes$ of graded additive categories with a distinguished object, choosing the trivial module $k$ in each $\text{stmod}(kG)$ and graduating by the Heller shift $\Omega$. The forgetful functor $\text{Add}^\otimes \rightarrow \text{Cat}$ reflects bilimits. Then consider the well-defined bilimit-preserving 2-functor to the 1-category of rings:

$$H: \begin{cases} \text{Add}^\otimes(C, \bullet_\mathcal{C}) &\rightarrow \text{Ring} \\ (F: C \rightarrow D, u : F(\bullet_\mathcal{C}) \overset{\sim}{\rightarrow} \bullet_D) &\rightarrow D(u^{-1}, u) \circ F \\ \alpha &\rightarrow \text{Id} \end{cases}$$

Note that, for any finite group $G$:

$$H \circ \tilde{\mathcal{M}}(G) = \text{stmod}(kG)^*\langle k, \underline{k} \rangle = \hat{H}^*(G; k)$$

Hence by eq. (4.9), we obtain the same Cartan-Eilenberg formula as in example 4.8:

$$\hat{H}^*(G; k) = H \circ \tilde{\mathcal{M}}(G)$$

$$\simeq \lim_{\mathfrak{T}_S(G)^{\text{op}}} H \circ \tilde{\mathcal{M}} \circ U$$

$$= \lim_{P \in \mathfrak{T}_S(G)} \hat{H}^*(P; k)$$

It would be interesting to find other factorizations and bilimit-preserving 2-functors $H$ with value in a 1-category for the various $p$-monadic Mackey 2-functors: this would give possibly new “Cartan-Eilenberg formulas”.

References

[Bal15] Paul Balmer. “Stacks of Group Representations”. In: Journal of the European Mathematical Society 17.1 (2015), pp. 189–228. issn: 1435-9855. DOI: [10.4171/JEMS/501]

[BD20] Paul Balmer and Ivo Dell’Ambrogio. Mackey 2-Functors and Mackey 2-Motives. Zuerich, Switzerland: European Mathematical Society Publishing House, July 31, 2020. isbn: 978-3-03719-209-2. DOI: [10.4171/209]

[BD21] Paul Balmer and Ivo Dell’Ambrogio. Cohomological Mackey 2-Functors. 2021. arXiv: 2103.03974 [math.CT]. Preprint.

[BR70] Jean Bénabou and Jacques Roubaud. “Monades et Descente”. In: C. R. Acad. Sci. Paris Sér. A-B 270 (1970), A96–A98. issn: 0151-0509.

[CE56] Henri Cartan and Samuel Eilenberg. Homological Algebra. Princeton University Press, Princeton, N. J., 1956, pp. xv+390.
[JY21] Niles Johnson and Donald Yau. 2 Dimensional Categories. New York: Oxford University Press, 2021. isbn: 978-0-19-887137-8.

[Mac71] Mac Lane Saunders. Categories for the Working Mathematician. Graduate Texts in Mathematics. New York: Springer-Verlag, 1971. isbn: 978-0-387-90036-0.

[Mai21a] Jun Maillard. “Finite Group Representations through 2-Sheaves”. PhD thesis. Université de Lille, 2021.

[Mai21b] Jun Maillard. On 2-Final 2-Functors. 2021. arXiv:2101.08727 [math.CT] Preprint, 16 pages.

[Par17] Sejong Park. “Mislin’s Theorem for Fusion Systems through Mackey Functors”. In: Communications in Algebra 45.4 (Apr. 3, 2017), pp. 1409–1415. ISSN: 0092-7872, 1532-4125. DOI:10.1080/00927872.2016.1175607

[Web00] Peter Webb. “A Guide to Mackey Functors”. In: Handbook of Algebra, Vol. 2. Vol. 2. Handb. Algebr. Elsevier/North-Holland, Amsterdam, 2000, pp. 805–836.

Univ. Lille, CNRS, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France
Email address: jun.maillard@gmail.com
URL: http://math.univ-lille1.fr/~jmaillard