POINTWISE MULTIPLICATION
OF BESOV AND TRIEBEL–LIZORKIN SPACES

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ABSTRACT. It is shown that para-multiplication applies to a certain product $\pi(u,v)$ defined for appropriate $u$ and $v \in S'(\mathbb{R}^n)$. Boundeness of $\pi(\cdot,\cdot)$ is investigated for the anisotropic Besov and Triebel–Lizorkin spaces — i.e., for $B^{s_0,s_1}_{p,q}$ and $F^{s_0,s_1}_{p,q}$ with $s \in \mathbb{R}$ and $p$ and $q \in [0,\infty]$ (though $p < \infty$ in the $F$-case) — with a treatment of the generic as well as various borderline cases.

When $\max\{s_0,s_1\} > 0$ the spaces $B^{s_0,s_1}_{p,q} \oplus B^{s_2,s_3}_{p,q}$ and $F^{s_0,s_1}_{p,q} \oplus F^{s_2,s_3}_{p,q}$ to which $\pi(\cdot,\cdot)$ applies are determined. For generic $F^{s_0,s_1}_{p,q} \oplus F^{s_2,s_3}_{p,q}$ the receiving $F^{s_0,s_1}_{p,q}$ spaces are characterised.

It is proved that $\pi(f,g) = f \cdot g$ holds for functions $f$ and $g$ when $f,g \in L^1_{\text{loc}}$, roughly speaking. In addition, $\pi(f,u) = fu$ when $f \in C^M$ and $u \in S'$.

Moreover, for an arbitrary open set $\Omega \subset \mathbb{R}^n$, a product $\pi_M(\cdot,\cdot)$ is defined by lifting to $\mathbb{R}^n$. Boundeness of $\pi$ on $\mathbb{R}^n$ is shown to carry over to $\pi_\Omega$ in general.

1. Introduction

For the pointwise multiplication of functions, given as

$$\mu(f,g)(x) = f \cdot g(x) = f(x)g(x),$$

the differentiability and integrability properties of $\mu(f,g)$ are examined and expressed in terms of such properties of the two factors $f$ and $g$.

To exemplify this, note that on one hand $\mu(f,g)$ can have integrability properties determined by $f$ and $g$, since for $0 < p,q \leq \infty$ Hölder’s inequality shows the implication

$$f \in L_p(\mathbb{R}^n), \quad g \in L_q(\mathbb{R}^n), \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \quad \implies \quad \mu(f,g) \in L_r(\mathbb{R}^n).$$

On the other hand $\mu(f,g)$ can have differentiability properties determined by $f$ and $g$, since it follows from Leibniz’ rule that

$$f \in C^s(\mathbb{R}^n), \quad g \in C^t(\mathbb{R}^n), \quad r = \min(s,t) \quad \implies \quad \mu(f,g) \in C^r(\mathbb{R}^n),$$

where $C^s(\mathbb{R}^n)$ with $s \in \mathbb{R}_+ \setminus \mathbb{N}$ is the following Hölder space, with global properties suited for Fourier analysis,

$$C^s(\mathbb{R}^n) = \{ u \in C^{[s]}(\mathbb{R}^n) \mid \sum_{|\alpha| = [s]} \sup_{x \neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^{s-|\alpha|}} < \infty \};$$

see Section 2 for the notation.

More generally one can describe both properties simultaneously, even with fractional derivatives in the $L_p$-sense. To do so we follow M. Yamazaki and W. Sickel, cf. [Yam86a] and [Sic87], in their use of Fourier analysis and para-multiplication in the framework of the scale of anisotropic Besov spaces $B^{s_0,s_1}_{p,q}(\mathbb{R}^n)$, for $s \in \mathbb{R}$ and $0 < p,q \leq \infty$, and the scale of anisotropic Triebel–Lizorkin spaces $F^{s_0,s_1}_{p,q}(\mathbb{R}^n)$; the latter are only considered for $0 < p < \infty$. See Section 2.1 for details on these spaces.
However, for technical reasons it is convenient to replace \( \mu(\cdot, \cdot) \) by a product \( \pi(u, v) \) defined for \( u \) and \( v \in \mathcal{S}'(\mathbb{R}^n) \) as

\[
\pi(u, v) = \lim_{k \to \infty} \mathcal{F}^{-1}(\psi_k \mathcal{F}u) \cdot \mathcal{F}^{-1}(\psi_k \mathcal{F}v),
\]

(1.5)

when the limit exists in \( \mathcal{D}'(\mathbb{R}^n) \) for each \( \psi \in C_0^\infty(\mathbb{R}^n) \) that equals 1 on a neighbourhood of 0; the limit is required to be independent of the functions \( \psi \). Hereby \( \psi_k(\xi) = \psi(2^{-kM} \xi) \) denotes a quasi-homogeneous dilation, cf. Section 2.

An advantage of \( \pi(\cdot, \cdot) \) is that it allows the use of para-multiplication, cf. Section 3 but it is a drawback that one has to examine

(I) whether \( \pi(f, g) = \mu(f, g) = f \cdot g \), whenever \( f \) and \( g \) are functions, and

whether \( \pi(\cdot, \cdot) \) in general has properties similar to those of \( \mu(\cdot, \cdot) \).

In this paper the analysis of \( \pi(\cdot, \cdot) \) is centred around the following

**Main questions:**

For each \( j = 0, 1, 2 \), let \( A_j \) denote either a Besov space \( B^{s_j}_{p_j,q_j}(\mathbb{R}^n) \) or a Triebel–Lizorkin space \( F^{s_j}_{p_j,q_j}(\mathbb{R}^n) \).

(II) Which conditions on \( (s_0,p_0,q_0) \) and \( (s_1,p_1,q_1) \) are necessary and sufficient for \( \pi(\cdot, \cdot) \) to be a bounded bilinear operator

\[
\pi(\cdot, \cdot) : A_0 \oplus A_1 \to A
\]

(1.6)

for some Besov or Triebel–Lizorkin space \( A \)?

(III) And in the affirmative case, which conditions on \( (s_2,p_2,q_2) \) are necessary and sufficient for obtaining (1.6) with \( A = A_2 \)?

For convenience, any case where, e.g., \( A_0 \) and \( A_1 \) are Besov spaces and \( A_2 \) is a Triebel–Lizorkin space is referred to as a BBF case. It is also practical to let “•” denote a space which can be either a Besov or a Triebel–Lizorkin space. In this terminology question (III) above has a version for each of the • • • cases, whereas (II) has BB•, BF• and FF• versions.

A solution to the problem for \( \mu(\cdot, \cdot) \) in (1.1) ff. above is gathered by establishing answers to (I), (II) and (III). In these directions it is obtained in this article that:

1. Para-multiplication allows an almost exhaustive discussion of (II) and (III) in the BBB and FFF cases.

   In more details, in Section 4 below the set of necessary conditions is enlarged, and afterwards, in Section 6 para-multiplication is used to show that the new set of conditions is sufficient too, except in some borderline cases.

   In fact, for \( \max(s_0, s_1) > 0 \) a complete answer is given to question (II) in the BB• and FF• cases. For the isotropic FFF cases the receiving \( A_2 \) spaces in (III) is completely characterised for generic \( A_0 \) and \( A_1 \). In the general generic BBB and FFF cases a few gaps remain open concerning (III).

2. The identity \( \pi(f_0, f_1) = f_0 \cdot f_1 \) holds when \( f_j \in L_{p_j,\text{loc}} \cap \mathcal{S}' \) for \( p_j \in [0, \infty] \) such that \( \frac{1}{p_0} + \frac{1}{p_1} \leq 1 \) (i.e., when (1.2) gives rise to a product in \( L_{1,\text{loc}} \cap \mathcal{D}' \)).

   Moreover, \( \pi(f, u) = fu \) for \( f \in \mathcal{O}_M \) and \( u \in \mathcal{S}' \).

3. For an arbitrary open set \( \Omega \subset \mathbb{R}^n \), a product, \( \pi_\Omega(\cdot, \cdot) \), on \( \Omega \) can be defined by lifting to \( \mathbb{R}^n \), that is to say, by letting

\[
\pi_\Omega(u, v) = \lim_{k \to \infty} r_\Omega(f_k \mathcal{F}u') \cdot \mathcal{F}^{-1}(\psi_k \mathcal{F}v'),
\]

(1.7)
when the limit exists in $D'(\Omega)$ for $u'$ and $v' \in S'(\mathbb{R}^n)$ such that $r_{\Omega} u' = u$ and $r_{\Omega} v' = v$ (cf. Definition 7.1 below).

As a consequence boundedness of $\pi(\cdot, \cdot)$ as in (1.6) generally implies boundedness of $\pi_{\Omega}(\cdot, \cdot)$ in the corresponding spaces over $\Omega$.

(4) When $\omega \subset \Omega$ is an open set, then $\pi_{\Omega}(u, v) = 0$ in $\omega$, if either $r_{\omega} u = 0$ or $r_{\omega} v = 0$ (which trivially holds for $\mu$).

Concerning earlier contributions to this subject the paper [Yam86a] deals with the situation where $p_0 = p_1 = p_2$, whereas [Sic87] treats the cases with $p_0 \neq p_1$ and Sobolev spaces $W^s_p$ (of functions with values in Banach spaces); and in some cases only the closures of $S$ in these spaces are covered. In the latter paper the full scale of $F^s_p$ spaces is considered. However, these references do not provide any new necessary conditions for the general problem.

In a recent work of Sickel and H. Triebel [ST] the case $p_0 \neq p_1$ is also studied and a rather complete set of necessary conditions is given. However, there the scope is restricted to the isotropic situation where $0 \leq s_0 = s_1 = s_2$ and $s_j - \frac{n}{p_j} \in ] - n, 0 [$ holds for $j = 0$ and 1.

It should be mentioned, that the questions (II) and (III) have been considered much earlier even for $p_0 \neq p_1$. In fact, for Sobolev spaces $W^s_p$ there is a treatment (for $m$ factors) by R. Palais [Pal68], and Besov (as well as Sobolev) spaces were considered by J. L. Zolesio in [Zol77]. B. Hanouzet [Han85] treated Besov spaces with $p$ and $q \in [1, \infty]$. As a general reference the recent monograph by M. Oberguggenberger [Obe92] is mentioned. The more classical multiplier subject is treated by V. A. Maz'ya and T. O. Shaposhnikova [MS85] and in R. S. Strichartz’ paper [Str67], e.g.

The definition of $\pi(\cdot, \cdot)$, cf. [Joh93], has been introduced independently in [Sic91] and [Joh93] but without the $\psi$-independence. In a related context, this requirement has been shown to be necessary by J. F. Colombeau and Oberguggenberger [CO90], cf. also Remark 3.5 below.

The results in (2)–(4) — that address (I) above — are important for the applications of para-multiplication, in particular for problems on domains $\Omega \subset \mathbb{R}^n$. In [Sic87] the result in (2) was observed in the rather restricted case with global spaces for which $\frac{1}{p} + \frac{1}{q} = 1$. Seemingly (3) and (4) are unprecedented (at least when $\max(p, q) = \infty$ or when $\Omega$ is non-smooth). (4) is used to show (2) and (3).

In comparison with [Ama91] and [Sic91] the present article treats anisotropic spaces (though only for $m = 2$) and it gives sufficient conditions which in

- the BBB cases generalise those in [Ama91], since we allow $p$ and $q$ to be arbitrary in $[0, \infty]$ and treat the full spaces (instead of closures of $S(\mathbb{R}^n)$), and in addition our statements on the sum-exponents $q$ are sharper,
- the BBB and FFF cases cover various borderline cases in (II) above, whereas [Sic91] does not deal with these at all. (Sickel, however, also treats intersections like $F^s_{p,q} \cap L_\infty$.)

The sufficient conditions here are supplemented by a set of necessary conditions which in the generic BBB and FFF cases leaves only a few open questions, cf. (1).
When restricted to the case $0 < s_0 = s_1 = s_2$, our sufficient conditions coincide with those contained in [ST], whereas the necessary conditions there are slightly sharper, in fact also sufficient. Moreover, they include a study of the case $0 = s_0 = s_1 = s_2$, cf. Remark 6.13 below.

Altogether new sufficient conditions for the BBB and FFF cases are given here. In addition we include a rather sharp set of necessary conditions, valid for general anisotropic problems. Moreover, the results (2), (3) and (4) above are proved.

An overview of the necessary and sufficient conditions for multiplication, cf. (II) and (III), is given in the beginning of Section 6 below. Comments on the applications can be found in Section 8.

Thanks are due to M. Yamazaki and to W. Sickel for conversations that have led to improvements of the results.

2. Notation and preliminaries

For a normed or quasi-normed space $X$ we denote by $\| x | X \|$ the norm of the vector $x$. (Recall that $X$ is quasi-normed when the triangle inequality is weakened to $\| x + y | X \| \leq c(\| x | X \| + \| y | X \|)$ for some $c \geq 1$ independent of $x$ and $y$. The prefix “quasi-” is omitted when confusion is unlikely to occur.) For $X_1 \times X_2$ the quasi-norm $\| x_1 | X_1 \| + \| x_2 | X_2 \|$ is used, and considered in this way we write $X_1 \oplus X_2$.

As simple examples there is $L_p(\mathbb{R}^n)$ and $\ell_p := \ell_p(\mathbb{N}_0)$ for $p \in [0, \infty]$, where $c = 2^{p-1}$ is possible for $p < 1$. However, it is a stronger fact that

$$\| f + g | L_p \| \leq (\| f | L_p \|^p + \| g | L_p \|^p)^{\frac{1}{p}}, \quad \text{for } 0 < p \leq 1,$$  

which has an exact analogue for the $\ell_p$ spaces.

For a bilinear operator $B(\cdot, \cdot): X_1 \oplus X_2 \to Y$, continuity is equivalent to the existence of a constant $c$ such that $\| B(x_1, x_2) | Y \| \leq c \| x_1 | X_1 \| \| x_2 | X_2 \|$ and to boundedness. A map $T: X \to \cap Y_j$ is continuous, if $T: X \to Y_j$ is continuous for each $j$.

The space of compactly supported smooth functions is denoted by $C_c^\infty(\Omega)$ or $\mathcal{D}(\Omega)$, when $\Omega \subset \mathbb{R}^n$ is open; then $\mathcal{D}'(\Omega)$ is the dual space of distributions on $\Omega$. $\langle u, \varphi \rangle$ denotes the duality between $u \in \mathcal{D}'(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$.

The Schwartz space is denoted by $S(\mathbb{R}^n)$, and the tempered distributions by $S'(\mathbb{R}^n)$. The seminorms on $S(\mathbb{R}^n)$ are taken to be $\| \varphi | S, \alpha, \beta \| = \sup\{ | x^\alpha D^\beta \varphi | \mid x \in \mathbb{R}^n \}$ for $\alpha, \beta \in \mathbb{N}_0^n$, or equivalently $\| \varphi | S, N \| = \max\{ \| \varphi | S, \alpha, \beta \| \mid | \alpha |, | \beta | \leq N \}$ for $N \in \mathbb{N}_0$.

The Fourier transform is denoted by $\mathcal{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) \, dx$, and the notation $\mathcal{F}^{-1}v(x) = \hat{v}(x)$ is used for its inverse. As customary in Fourier analysis

$$C(\mathbb{R}^n) = \{ f \in L_\infty(\mathbb{R}^n) \mid f \text{ is uniformly continuous } \}$$  

and $\| f | C(\mathbb{R}^n) \| = \sup | f |$. Moreover, $C^k(\mathbb{R}^n) = \{ f \mid D^\alpha f \in C(\mathbb{R}^n), | \alpha | \leq k \}$ with the seminorms $\| f | C^k \| = \sup\{ | D^\alpha f | \mid x \in \mathbb{R}^n, | \alpha | \leq k \}$.

When $\Omega \subset \mathbb{R}^n$ is open, the restriction $r\Omega: \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\Omega)$ is the transpose of the extension by $0$ outside of $\Omega$, denoted $e\Omega: C_0^\infty(\Omega) \to C_0^\infty(\mathbb{R}^n)$.

For $t \in \mathbb{R}$, $t_\pm = \max(0, \pm t)$ and $\lceil t \rceil$ denotes the largest integer $\leq t$, whereas $\lfloor t \rfloor$ is the smallest integer $\geq t$. Moreover, $\min^+(s, t) := \min(s, t, s + t)$. 

For each given assertion we shall follow D. E. Knuth’s suggestion in [Knu92] and let \([\text{assertion}]\) denote 1 respectively 0 when the assertion is true respectively false.

2.1. The spaces. To make the considerations in this paper more applicable the anisotropic versions of the Besov and Triebel–Lizorkin spaces are treated (at a marginal extra cost). The reader can easily specialise to the isotropic case, if desired, since then \(M = (1, \ldots, 1)\), \(|M| = n\) and \([x] = |x|\) below.

The definitions are recalled from [Yam86a]: First each coordinate \(x_j\) in \(\mathbb{R}^n\) is given a weight \(m_j \geq 1\), such that \(\min m_j = 1\), and \(M = (m_1, \ldots, m_n)\) with \(|M| = m_1 + \cdots + m_n\). The action of \(t \in \mathbb{R}_+ = [0, \infty[\) on \(x \in \mathbb{R}^n\) is defined by \(t^M x = (t^{m_1} x_1, \ldots, t^{m_n} x_n)\), and \(t^s M x = (t^s)^M x\) for \(s \in \mathbb{R}\), so that \(t^{-M} x = (t^{-1})^M x\).

The anisotropic numerical value \([x]\) associated with \(M\) is introduced for \(x = 0\) as \([0] = 0\) and otherwise as the unique positive \(t\) such that \(t^{-M} x \in S^{n-1}\), i.e., such that

\[
\left( \frac{x_1}{t^{m_1}} \right)^2 + \cdots + \left( \frac{x_n}{t^{m_n}} \right)^2 = 1. \tag{2.3}
\]

See for example [Yam86a] for properties of and remarks on \([x]\). As examples one could let \(M = (1, \ldots, 1, 2)\) in a treatment of, say, the Navier–Stokes equations or of the parabolic operator \(\partial_t - (\partial_t^2 + \cdots + \partial_n^2)\); then \([x] = (\frac{1}{2}(|x'|^2 + (|x'|^4 + 4x_n^2)^{\frac{1}{2}}))^{\frac{1}{2}}\) for \(|x'|^2 = x_1^2 + \cdots + x_{n-1}^2\).

Secondly a partition of unity, \(1 = \sum_{j=0}^{\infty} \Phi_j\), is constructed: From a fixed \(\Psi \in C^\infty(\mathbb{R})\), such that \(\Psi(t) = 1\) for \(0 \leq t \leq \frac{11}{10}\) and \(\Psi(t) = 0\) for \(\frac{13}{10} \leq t\), the functions

\[
\Psi_j(\xi) = [j \in \mathbb{N}_0] \Psi(2^{-j} |\xi|) \tag{2.4}
\]

are introduced and used to define

\[
\Phi_j(\xi) = \Psi_j(\xi) - \Psi_{j-1}(\xi), \quad \text{for} \quad j \in \mathbb{Z}. \tag{2.5}
\]

Thirdly there is then a decomposition, with (weak) convergence in \(S'\),

\[
u = \sum_{j=0}^{\infty} F^{-1} \Phi_j F u, \quad \text{for every} \quad u \in S'. \tag{2.6}
\]

Here it is understood that \(F^{-1} \psi F u = F^{-1}(\psi \hat{u})\) for \(\psi \in S\) and \(u \in S'\). Moreover, \(u_k := F^{-1} \Phi_k F u\) and \(u^k := F^{-1} \Psi_k F u\), and also for general \(\psi \in S\) we write \(u^k = F^{-1} \psi_k F u\) when \(\psi_k = \psi(2^{-k} \cdot)\).

Now the anisotropic Besov space, \(B_{p,q}^{M,s}(\mathbb{R}^n)\), with weight \(M\), smoothness index \(s \in \mathbb{R}\), integral-exponent \(p \in [0, \infty]\) and sum-exponent \(q \in [0, \infty]\), is defined as

\[
B_{p,q}^{M,s}(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) \mid \| 2^j \| F^{-1} \Phi_j F u(\cdot) \|_{L_p} \|_{\ell_q} \| < \infty \}, \tag{2.7}
\]

and the anisotropic Triebel–Lizorkin space, \(F_{p,q}^{M,s}(\mathbb{R}^n)\), with weight \(M\), smoothness index \(s \in \mathbb{R}\), integral-exponent \(p \in [0, \infty]\) and sum-exponent \(q \in [0, \infty]\) as

\[
F_{p,q}^{M,s}(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) \mid \| 2^j F^{-1} \Phi_j F u \|_{L_p} \|_{\ell_q} \| < \infty \}. \tag{2.8}
\]

For the history of (the isotropic versions of) the spaces we refer to [Tri83] [Tri92].

The spaces \(B_{p,q}^{M,s}\) and \(F_{p,q}^{M,s}\) are quasi-Banach spaces with the quasi-norms given by the finite expressions in (2.7) and (2.8). Concerning an analogue of (2.1) one has

\[
\| f + g \| B_{p,q}^{M,s} \| \leq (\| f \| B_{p,q}^{M,s} \|^ \lambda + \| g \| B_{p,q}^{M,s} \|^ \lambda)^{\frac{1}{\lambda}}, \quad \text{for} \quad \lambda = \min(1, p, q), \tag{2.9}
\]

with a similar result for the Triebel–Lizorkin spaces.
2.2. Embeddings. In the rest of this subsection the explicit mention of the restriction \( p < \infty \) concerning the \( F_{p,q}^s \) spaces is omitted. E.g., (2.10) below should be read with \( p \in [0, \infty) \) in the Besov part, and with \( p \in [0, \infty[ \) in the Triebel–Lizorkin part.

The spaces \( B_{p,q}^{M,s}(\mathbb{R}^n) \) and \( F_{p,q}^{M,s}(\mathbb{R}^n) \) are complete, for \( p \) and \( q \geq 1 \) they are Banach spaces, and in any case \( S(\mathbb{R}^n) \hookrightarrow B_{p,q}^{M,s}(\mathbb{R}^n), F_{p,q}^{M,s}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n) \) are continuous.

By a modification of the proof in [Tri83] of the cases with \( M = (1, \ldots, 1) \) and \( p < \infty \), the image of \( S(\mathbb{R}^n) \) is dense in \( B_{p,q}^{M,s}(\mathbb{R}^n) \) and in \( F_{p,q}^{M,s}(\mathbb{R}^n) \) for \( p \) and \( q < \infty \), and similarly \( C^\infty(\mathbb{R}^n) \) is dense in \( B_{p,q}^{M,s}(\mathbb{R}^n) \) for \( q < \infty \).

The definitions imply that \( B_{p,p}^{M,s}(\mathbb{R}^n) = F_{p,p}^{M,s}(\mathbb{R}^n) \), and they imply the existence of simple embeddings for \( s \in \mathbb{R}, p \in [0, \infty] \) and \( o \) and \( q \in [0, \infty] \):

\[
\begin{align*}
B_{p,q}^{M,s}(\mathbb{R}^n) &\hookrightarrow B_{p,o}^{M,s}(\mathbb{R}^n), & F_{p,q}^{M,s}(\mathbb{R}^n) &\hookrightarrow F_{p,o}^{M,s}(\mathbb{R}^n), & \text{when } q \leq o, \\
B_{p,q}^{M,s}(\mathbb{R}^n) &\hookrightarrow B_{p,q}^{M,s-\varepsilon}(\mathbb{R}^n), & F_{p,q}^{M,s}(\mathbb{R}^n) &\hookrightarrow F_{p,q}^{M,s-\varepsilon}(\mathbb{R}^n), & \text{when } \varepsilon > 0, \\
F_{p,q}^{M,s}(\mathbb{R}^n) &\hookrightarrow F_{p,q}^{M,s}(\mathbb{R}^n) & B_{p,q}^{M,s}(\mathbb{R}^n) &\hookrightarrow B_{p,q}^{M,s}(\mathbb{R}^n). & \text{(2.12)}
\end{align*}
\]

There are Sobolev embeddings if \( s - \frac{|M|}{p} \geq t - \frac{|M|}{r} \) and \( r > p \), cf. [Yam86a],

\[
\begin{align*}
B_{p,q}^{M,s}(\mathbb{R}^n) &\hookrightarrow B_{r,o}^{M,l}(\mathbb{R}^n), & \text{provided } q \leq o \text{ when } s - \frac{|M|}{p} = t - \frac{|M|}{r}, \\
F_{p,q}^{M,s}(\mathbb{R}^n) &\hookrightarrow F_{r,o}^{M,l}(\mathbb{R}^n), & \text{for any } o \text{ and } q \in [0, \infty].
\end{align*}
\]

Furthermore, Sobolev embeddings also exist between the two scales, in fact under the assumptions \( \infty \geq p_1 > p > p_0 > 0 \) and \( s_0 = \frac{|M|}{p_0} = s - \frac{|M|}{p} = s_1 = \frac{|M|}{p_1} \),

\[
B_{p_0,q_0}^{M,s_0}(\mathbb{R}^n) \hookrightarrow B_{p,q}^{M,s}(\mathbb{R}^n) \hookrightarrow B_{p_1,q_1}^{M,s_1}(\mathbb{R}^n),
\]

for \( q_0 < p < q_1 \) and for \( q_0 = p \leq q \) and \( q \leq p = q_1 \), if \( M \neq (1, \ldots, 1) \); and for \( q_0 \leq p \) and \( p \leq q_1 \), if \( M = (1, \ldots, 1) \).

This is obtained from (2.13), (2.14) and (2.12) except for the sharpened results for \( M = (1, \ldots, 1) \), which are interpolation results due to J. Franke, [Fra86], and B. Jawerth, [Jaw77], respectively.

Concerning relations to other spaces, one has that \( B_{\infty,\infty}^{M,s}(\mathbb{R}^n) = C^{M,s}(\mathbb{R}^n) \) when \( s > 0 \) and \( \frac{1}{m_k} \notin \mathbb{N} \) for \( k = 1, \ldots, n \) (the anisotropic Hölder spaces), and that \( L_p(\mathbb{R}^n) = F_{p,2}^{M,0}(\mathbb{R}^n) \) for \( 1 < p < \infty \). Moreover, the \( F_{p,2}^{M,s} \) equal the anisotropic Bessel-potential spaces \( H_{p,s}^{M,s} \), for \( s \in \mathbb{R} \) and \( 1 < p < \infty \), hence—in the isotropic case, which is indicated by omission of \( M \)—that the \( F_{p,2}^{M,s}(\mathbb{R}^n) \) equal the classical Sobolev spaces \( W_p^m(\mathbb{R}^n) \) for \( m \in \mathbb{N} \), and \( H_2(\mathbb{R}^n) = F_{2,2}^{M,0}(\mathbb{R}^n) = B_2^2(\mathbb{R}^n) \) for \( s \in \mathbb{R} \). See [Yam86a], [Yam86b] Rem. 4.4 and [Tri83] [Tri92] for these and other identifications.

Furthermore, one finds by use of (2.6), (2.7) and (2.13), when \( 0 < p, q \leq \infty \), that

\[
B_{p,q}^{M,s}(\mathbb{R}^n) \hookrightarrow B_{\infty,\infty}^{M,0}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \hookrightarrow B_{\infty,\infty}^{M,0}(\mathbb{R}^n),
\]

if \( s > \frac{|M|}{p} \), or if \( s = \frac{|M|}{p} \) and \( q \leq 1 \).
Then (2.15) gives for the Triebel–Lizorkin spaces that for $0 < q \leq \infty$
\[ F_{p,q}^{M,s}(\mathbb{R}^n) \hookrightarrow B_{\infty,1}^{M,0}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n), \]
if $s > \frac{|M|}{p}$, or $s = \frac{|M|}{p}$ and either $p < 1$ or $p = 1 \geq q$;
(2.17)
when $M = (1, \ldots, 1)$ also for $s = \frac{|M|}{p}$ with $p \leq 1$.
Moreover, when $|M|\left(\frac{1}{p} - 1\right)_+ \leq s < \frac{|M|}{p}$ one has, with $\frac{|M|}{p} = \frac{|M|}{p} - s$, that
\[ F_{p,q}^{M,s}(\mathbb{R}^n) \hookrightarrow \bigcap \{ L_r(\mathbb{R}^n) \mid p \leq r \leq t \}, \]
provided $q \leq 1 + \lceil 1/p \rceil$ if $s = 0$.
(2.18)
Indeed, when $s > |M|\left(\frac{1}{p} - 1\right)_+$ and $r = t$ one can use (2.14) and the fact that $F_{1,2} = L_1$. Hence (2.6) is a series of functions in $L_t$, that converges in $L_t$ since $F_{t,1} \hookrightarrow L_t$. Then, from $F_{p,q}^{M,s} \hookrightarrow F_{p,1}^{M,0}$, it follows that there is also convergence in $L_p$, the limits being the same a.e. Thus (2.13) follows for $r = p$ and therefore also for the intermediate values. This extends to $s = |M|\left(\frac{1}{p} - 1\right)$ when $s \geq 0$, since $F_{p,q}^{M,s} \hookrightarrow B_{1,1}^{M,0} \hookrightarrow L_1$ then.
Likewise (2.15) implies for $|M|\left(\frac{1}{p} - 1\right)_+ \leq s < \frac{|M|}{p}$ and $0 < p, q \leq \infty$ that, with $t$ as above,
\[ B_{p,q}^{M,s}(\mathbb{R}^n) \hookrightarrow \bigcap \{ L_r(\mathbb{R}^n) \mid p \leq r < t \} \]
(2.19)
Here $r = t$ can be included in general when $q < t$, and if either $t \leq 2$ or $M = (1, \ldots, 1)$ also when $q = t$. For $s = 0$, one has $B_{p,q}^{M,0} \hookrightarrow L_p$ for $q \leq \min(2, p)$.
(Cf. [Tri92] p. 97) for the pitfalls in the case $p < 1$.)

The ‘intermediate value property’ for the $L_p$ spaces (used above) gives that
\[ F_{p,q}^{M,s}(\mathbb{R}^n) \cap F_{p,q}^{M,s}(\mathbb{R}^n) \subset \bigcap \{ F_{r,q}^{M,s}(\mathbb{R}^n) \mid p_0 \leq r \leq p_1 \}, \]
holds for $p_0 \leq p_1$, and similarly for the Besov spaces.

2.3. Convergence theorems. As a basic tool Yamazaki’s theorems are recalled.
They will be applied to (the series defining) the para-multiplication operators $\pi_1$, $\pi_2$ and $\pi_3$ in Theorem 5.1 below.

**Theorem 2.1.** Let $s \in \mathbb{R}$, let $p$ and $q \in [0, \infty]$ and suppose $u_j \in S'(\mathbb{R}^n)$ satisfies
\[ \text{supp } \hat{u}_j \subset \{ \xi \mid j \geq 0 \} A^{-1}2^j \leq |\xi| \leq A2^j \}, \quad \text{for } j \in \mathbb{N}_0, \]
for some $A > 0$. Then the following holds, if $p < \infty$ in (2):

1. If $\| \{2^j \| u_j \|_{L_p}\}_{j=0}^{\infty} \|_{L_q} = B < \infty$, then the series $\sum_{j=0}^{\infty} u_j$ converges in $S'(\mathbb{R}^n)$ to a limit $u \in B_{p,q}^{M,s}(\mathbb{R}^n)$ and the estimate $\| u \|_{B_{p,q}^{M,s}} \leq CB$ holds for some constant $C = C(n, M, A, s, p, q)$.

2. If $\| \| \{2^j u_j\}_{j=0}^{\infty} \|_{L_q} \|_{L_p} = B < \infty$, then the series $\sum_{j=0}^{\infty} u_j$ converges in $S'(\mathbb{R}^n)$ to a limit $u \in F_{p,q}^{M,s}(\mathbb{R}^n)$ and the estimate $\| u \|_{F_{p,q}^{M,s}} \leq CB$ holds for some constant $C = C(n, M, A, s, p, q)$.

The second of these theorems states that the spectral conditions on the series $\sum_{j=0}^{\infty} u_j$ can be relaxed if the smoothness index $s$ is sufficiently large.
Theorem 2.2. Let \( s \in \mathbb{R} \), let \( p \) and \( q \in [0, \infty] \) and suppose \( u_j \in S'(\mathbb{R}^n) \) satisfies
\[
\text{supp } \hat{u}_j \subset \{ \xi \mid |\xi| \leq A 2^j \}, \quad \text{for } j \in \mathbb{N}_0, \tag{2.22}
\]
for some \( A > 0 \). Then the following holds, if \( p < \infty \) in (2):

1. If \( s > |M| \left( \frac{1}{p} - 1 \right)_+ \) and if \( \| 2^s j \| |U|_p^q \| \| \ell_q \| = B < \infty \), then the series \( \sum_{j=0}^{\infty} u_j \) converges in \( S'(\mathbb{R}^n) \) to a limit \( u \in B_{p,q}^m(\mathbb{R}^n) \) and the estimate \( \| u |B_{p,q}^m| \| \leq CB \) holds for some constant \( C = C(n, M, A, s, p, q) \).

2. If \( s > \frac{1}{\min(p,q)} - 1 \), and if \( \| 2^s j u_j \| |\ell|_q \| \| L_p \| = B < \infty \), then the series \( \sum_{j=0}^{\infty} u_j \) converges in \( S'(\mathbb{R}^n) \) to a limit \( u \in F_{p,q}^m(\mathbb{R}^n) \) and the estimate \( \| u |F_{p,q}^m| \| \leq CB \) holds for some constant \( C = C(n, M, A, s, p, q) \).

For the proofs of Theorems 2.1 and 2.2 the reader is referred to Yamada. In part Theorem 2.2 is based on Yamada [Lemma 3.8], which we for later reference shall state for \( s < 0 \) in a slightly generalised version (that is proved similarly):

Lemma 2.3. For each \( s < 0 \) and \( q \) and \( r \in [0, \infty] \) there exists \( c < \infty \) such that
\[
\| \{ 2^s j (\sum_{k=0}^{\infty} a_k) \} \| |\ell_q \| \leq c \| \{ 2^s j a_j \} \| |\ell_q \|
\]
holds for any sequence \( \{ a_j \} \) of complex numbers (with modification for \( r = \infty \)).

We shall also pay attention to the cases with \( s = |M| \left( \frac{1}{p} - 1 \right)_+ \).

Example 2.4. In case (1) of Theorem 2.2 above it is not possible to relax the condition \( s > |M| \left( \frac{1}{p} - 1 \right)_+ \) much:

On one hand, when \( s = |M| \left( \frac{1}{p} - 1 \right) \) and \( q > 1 \) the sequence \( \{ k^{-1} \psi_k \} \) has finite norm as required there, but the series \( \sum_{k=1}^{\infty} k^{-1} \psi_k \) is not convergent in \( S'(\mathbb{R}^n) \) since \( \langle \psi_1 + \cdots + k^{-1} \psi_k, \phi \rangle \) equals \( \varphi(0)(1 + \cdots + k^{-1}) \) when \( \text{supp } \hat{\varphi} \subset \{ \xi \mid |\xi| = 1 \} \).

On the other hand, for \( s = 0 \) and \( q > 1 \) one may consider \( \{ k^{-1} \psi_k \} \). Again \( \{ k^{-1} |\psi_k| |L_p| \} \) is in \( \ell_q \), and with \( \varphi \) as above \( \langle (1 + \cdots + k^{-1}) \psi_0, \varphi \rangle \) diverges when \( \varphi(0) \neq 0 \). Obviously the second example applies also to case (2) of Theorem 2.2.

The cases with \( s = 0 \) may be partly covered, even without spectral conditions:

Proposition 2.5. Let \( q \leq 1 \leq p \leq \infty \) and let \( u_j \in S'(\mathbb{R}^n) \) for \( j \in \mathbb{N}_0 \).

1. If \( \{ \sum_{j=0}^{\infty} |u_j| |L_p| \}^{1/p}_q \leq B < \infty \), then \( \sum_{j=0}^{\infty} u_j \) converges in \( L_p(\mathbb{R}^n) \) to a limit \( u \) with \( \| u |L_p| \| \leq B \).

2. If, for \( p < \infty \), \( \{ \sum_{j=0}^{\infty} |u_j| |\ell|_q \| \| L_p \| \}^{1/p}_q \leq B < \infty \), then \( \sum_{j=0}^{\infty} u_j \) converges in \( L_p(\mathbb{R}^n) \) to a limit \( u \) with \( \| u |L_p| \| \leq B \).

Proof. In case (1) the embedding \( \ell_q \rightarrow \ell_1 \) implies that \( \sum_{j=0}^{\infty} |u_j| |L_p| \| \leq B \). Then the completeness of \( L_p \) gives the convergence of \( u_j \). Concerning (2) one has that \( \sum_{j \in J} u_j \leq \| \sum_{j \in J} |u_j| \| |L_p| \| \) when \( J \subset \mathbb{N}_0 \). For \( J \) of the form \( \{ m, \ldots, m + k \} \) this gives the convergence by majorisation with \( \{ \sum_{j=0}^{\infty} |u_j| \}^{1/p}_q \). Then \( J = \{ 0, \ldots, m \} \) gives \( \| u |L_p| \| \leq B \).

It is seen from Example 2.4 above that this result can not be extended to higher values of \( q \). For the borderline cases with \( s = |M| \left( \frac{1}{p} - 1 \right) \) in Theorem 2.2 we have
Example 3.2. One has $R\Omega = \ldots$ in order that Definition 7.1 below gives back Definition 3.1 when it is applied to $\Pi$. Proof. Let $\pi$ be as in Definition 3.1. When the limit exists in $D'(\mathbb{R}^n)$, the delta measure at the origin in $F\pi$ differs significantly from $\mu$ as one may verify similarly to [Obe92, Ex. 2.3]. Thus it may be said that $\pi'(\cdot, \cdot)$ differs significantly from $\mu(\cdot, \cdot)$, cf. (1) in the introduction.

It would be interesting and useful to know if the limits in Proposition 2.6 belong to $B_{p,q}^{M,s}$ and $F_{p,q}^{M,s}$, respectively.

3. PRODUCTS OF TEMPERED DISTRIBUTIONS

The results for $\mu(f, g)$ will be obtained from the more general product $\pi(u, v)$ defined — for each $M$ — as follows:

Definition 3.1. Let $\psi \in C^\infty_0(\mathbb{R}^n)$ satisfy $\psi(x) = 1$ for $x$ in a neighbourhood of $x = 0$, and let $\psi_k(x) = \psi(2^{-kM}x)$. Denote then, for $u$ and $v \in S'(\mathbb{R}^n)$,

$$\pi \psi(u, v) = \lim_{k \to \infty} F^{-1}(\psi_k F u) \cdot F^{-1}(\psi_k F v),$$

(3.1)

when the limit exists in $\mathcal{D}'(\mathbb{R}^n)$.

The product $\pi(u, v)$ is defined as $\pi(u, v) = \pi \psi(u, v)$, when $\pi \psi(u, v)$ exists for all such $\psi$ and is independent of $\psi$.

Since $F^{-1}\psi F u$ is a smooth function by the Paley–Wiener theorem, the multiplication on the right hand side of (3.1) makes sense. The limit is taken in $\mathcal{D}'(\mathbb{R}^n)$ in order that Definition 2.1 below gives back Definition 3.1 when it is applied to $\Omega = \mathbb{R}^n$.

Example 3.2. One has $\pi(\chi, \delta_0) = \frac{1}{M} \delta_0$ (for any $M$), by a direct computation, when $\chi(x) = [x_n > 0]$ denotes the characteristic function of the half-space $\mathbb{R}^n_+$ and $\delta_0$ is the delta measure at the origin in $\mathbb{R}^n$.

On the real line, $\pi(x^{-\frac{1}{2}}, x_\pm^{-\frac{1}{2}}) = \pi \delta_0$ for the locally integrable functions $x_\pm^{-\frac{1}{2}} := [x \leq 0]|x|^{-\frac{1}{2}}$, as one may verify similarly to [Obe92, Ex. 2.3]. Thus it may be said that $\pi'(\cdot, \cdot)$ differs significantly from $\mu(\cdot, \cdot)$, cf. (1) in the introduction.

For the analysis of $\pi(u, v)$ it is convenient to introduce the para-multiplication operators $\pi_1(\cdot, \cdot)$, $\pi_2(\cdot, \cdot)$ and $\pi_3(\cdot, \cdot)$. Let $\psi$ be as in Definition 3.1.
With \( u_j = \mathcal{F}^{-1} \varphi_j \mathcal{F} u \) — where \( \varphi_j = \psi_j - \psi_{j-1} \) (and \( \psi_j \equiv 0 \) for \( j < 0 \)) similarly to (2.5) above — the operators \( \pi_1, \pi_2 \) and \( \pi_3 \) are defined as follows, cf. [Yam86a],

\[
\pi_1(u, v) = \sum_{j=0}^{\infty} (u_0 + \cdots + u_{j-2}) \cdot v_j, \tag{3.2}
\]

\[
\pi_2(u, v) = \sum_{j=0}^{\infty} (u_{j-1} \cdot v_j + u_j \cdot v_j + u_j \cdot v_{j-1}), \tag{3.3}
\]

\[
\pi_3(u, v) = \sum_{j=0}^{\infty} u_j \cdot (v_0 + \cdots + v_{j-2}) = \pi_1(v, u). \tag{3.4}
\]

This proves (3.5).

**Lemma 3.3.** The limit \( \pi_\psi(u, v) \) exists and

\[
\pi_\psi(u, v) = \pi_1(u, v) + \pi_2(u, v) + \pi_3(u, v), \tag{3.5}
\]

whenever the series defining \( \pi_j(u, v) \) converges in \( \mathcal{D}'(\mathbb{R}^n) \) for \( j = 1, 2 \) and \( 3 \) (for a \( \psi \) as in Definition 3.1).

**Proof.** It is found by the construction of the \( \varphi_j \) that \( \varphi_0 + \cdots + \varphi_k = \psi_k \), so

\[
\pi_1(u, v) + \pi_2(u, v) + \pi_3(u, v) = \lim_{k \to \infty} \sum_{j=0}^{k} ((u_0 + \cdots + u_j) \cdot v_j + u_j \cdot (v_0 + \cdots + v_{j-1})) \tag{3.6}
\]

\[
= \lim_{k \to \infty} \mathcal{F}^{-1}(\psi_k \mathcal{F} u) \cdot \mathcal{F}^{-1}(\psi_k \mathcal{F} v).
\]

This proves (3.5). \( \square \)

Hence, when each \( A_0, A_1 \) and \( A_2 \) is chosen independently as a Besov space or as a Triebel–Lizorkin space and the \( \pi_j(\cdot, \cdot) \) are continuous \( A_0 \oplus A_1 \to A_2 \), it follows that \( \pi_\psi(\cdot, \cdot) : A_0 \oplus A_1 \to A_2 \) is a continuous bilinear operator.

Then when \( A_0 \hookrightarrow L_p \) and \( A_1 \hookrightarrow L_q \) with \( 0 \leq \frac{1}{p} + \frac{1}{q} \leq 1 \), e.g., Proposition 3.8 below yields \( \pi_\psi = \mu \) (and hence \( \psi \)-independence), and thus \( \mu : A_0 \oplus A_1 \hookrightarrow A_2 \) is continuous.

In general it is necessary to verify that the results obtained for \( \pi_\psi \) by use of (3.5) do not depend on \( \psi \). In Section 6.3 below it is seen that when the \( A_j \) are Besov or Triebel–Lizorkin spaces, \( \pi_\psi(\cdot, \cdot) : A_0 \oplus A_1 \to A_2 \) is an extension by continuity (or a restriction of) \( \mu \) or of the product on \( \mathcal{O}_M \times \mathcal{S}' \). For this reason we shall not emphasise the \( \psi \)-dependence of \( \pi_1, \pi_2 \) and \( \pi_3 \) in the following.

The applications of Theorems 2.1 and 2.2 to the \( \pi_j \) are contained in Section 5 below. However, here it is observed that the spectral conditions in Theorems 2.1 and 2.2 are satisfied by the terms in the sums in (3.2), (3.3) and (3.4) (recall 2.6 ff.).
Pointwise Multiplication

If $|\xi| \leq r \Rightarrow \psi(\xi) = 1$ and $|\xi| > R \Rightarrow \psi(\xi) = 0$, then the identity $t^M|\xi| = t|\xi|$ gives $supp \psi_k \subset \{ \xi | [\xi] \leq R^{2^k} \}$ while $supp \phi_k \subset \{ \xi | r^{2^{k-1}} \leq [\xi] \leq R^{2^k} \}$. So for $u$ and $v \in S'$ and $j \in \mathbb{N}_0$,

\[
\begin{align*}
supp F(u^{j-2}v_j + u_jv^{j-2}) \subset \{ \xi | |\xi|^j - \frac{j}{2} |^{2^k} \leq [\xi] \leq \frac{5R}{2}2^k \},
\end{align*}
\]

since $supp F(u_j \cdot v_k) \subset supp \phi_j \hat{u} + supp \phi_k \hat{v}$ and $|[\xi]-[\eta]| \leq [\xi + \eta] \leq [\xi + [\eta]$. Note that $R \neq 2r$ yields $|\frac{R}{2} - \frac{r}{2}| > 0$.

**Remark 3.4.** It will be convenient later on to assume that $\psi = \Psi_0$, where $\Psi_0$ is defined in (2.4). For this purpose it is observed that Theorem 2.1 gives the inequality

\[
\| \{2^{nj} \| F^{-1}\varphi_j F u \| L_p \| \}_{j=0}^\infty |\ell_q| \leq c \| u \| B_{p,q}^{M,s}\]

for a constant $c$ independent of $u$ and a similar inequality for the $F_{p,q}^{M,s}$ spaces.

**Remark 3.5.** The product $\pi(u, v)$ is by definition obtained by a simultaneous regularisation of both factors. On one hand it may be seen as in [Oba92, Ex. 2.3] that regularisation of one factor only gives a limit depending on $\psi$ when one considers the product of $x^{-\frac{1}{r}}$ and $x^{-\frac{1}{s}}$.

On the other hand, this dependence may occasionally disappear by regularising both factors by means of the same $\psi$ (as is the case with the product of $x^{-\frac{1}{r}}$ and $x^{-\frac{1}{s}}$), but even then the limit $\pi_{uv}(u, v)$ may still depend on $\psi$ in some cases. For this reason $\psi$-independence is required in Definition 3.1.

For a thorough discussion of such questions for the Antosik–Mikusiński–Sikorski product one may consult [Oba92, Ch. 2]. For this product there is in [CO90] given an example of $\psi$-dependence, namely when multiplying $H \otimes \delta_0$ and $\delta_0 \otimes H$ in dimension $n = 2$, but the example does not carry over with its conclusions to $\pi(\cdot, \cdot)$.

### 3.1. Relations to other products

Recall that the usual product $\mathcal{E}(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$ restricts to a bilinear operator

\[
\mathcal{O}_M(\mathbb{R}^n) \times S'(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n),
\]

where $\mathcal{O}_M$ denotes the spaces of slowly increasing functions. With $\langle x \rangle = (1+|x|^2)^\frac{1}{2}$

\[
\mathcal{O}_M(\mathbb{R}^n) = \{ f \in \mathcal{E}(\mathbb{R}^n) \mid \forall \alpha \in \mathbb{N}_0^n \exists a, c > 0 : |D^a f(x) | \leq c \langle x \rangle^a \}.
\]

($\mathcal{O}_M(\mathbb{R}^n)$ consists of the pointwise multipliers of $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$.)

**Proposition 3.6.** Let $\psi \in S(\mathbb{R}^n)$ with $\psi(0) = 1$ and a weight $M$ be given. Then

\[
F^{-1}(\psi_k F f) \cdot F^{-1}(\psi_k F u) \underset{k \rightarrow \infty}{\longrightarrow} f u \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n)
\]

for every $f \in \mathcal{O}_M(\mathbb{R}^n)$ and every $u \in S'(\mathbb{R}^n)$.

In particular $\pi(f, u)$ is defined and equal to $f u$. 

Proof. From $\langle x - y \rangle^s \leq c_s(x)^s(y)^s$, valid for $s > 0$, it follows that $f^k = \tilde{\varphi}_k * f$ is an element of $\mathcal{O}_M(\mathbb{R}^n)$, so $f^k u^k = (\tilde{\varphi}_k * f) \cdot \mathcal{F}^{-1}(\tilde{\varphi}_k \hat{u})$ makes sense in $\mathcal{S}'(\mathbb{R}^n)$.

For each $\varphi \in C_0^\infty(\mathbb{R}^n)$, it suffices for the convergence

$$\langle f^k u^k, \varphi \rangle - \langle f u, \varphi \rangle = (u^k, f^k \varphi - f \varphi) + \langle u^k - u, f \varphi \rangle \to 0, \quad k \to \infty,$$

(3.13)

that $f^k \varphi \rightarrow f \varphi$ in $\mathcal{S}(\mathbb{R}^n)$ for $k \to \infty$, for the family $\{u^k\}_{k \in \mathbb{N}}$ of operators $\mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is equicontinuous. Thus we have arrived at a “linear” problem.

However, that $f^k \varphi \rightarrow f \varphi$ in $\mathcal{S}(\mathbb{R}^n)$ follows if $\sup \{|f^k(x) - f(x)| \mid x \in K\} \rightarrow 0$ for $k \to \infty$ for every $f \in \mathcal{O}_M$, when $K := \text{supp} \varphi$. Concerning the latter convergence we may assume $f$ to be real so that

$$\tilde{\varphi}_k * f(x) - f(x) \leq \int |\tilde{\varphi}(z)| \text{grad} f(x - \theta(z)2^{-k M}z) \cdot 2^{-k M}z) | dz,$$

(3.14)

where $|2^{-k M}z| \leq 2^{-k}|z|$. Since $\theta(z) \in ]0,1[$, the estimate $|\text{grad} f(x - \theta(z)2^{-k M}z) | \leq c_N (x)^N \langle \theta(z)2^{-k M}z \rangle^N \leq c_N (x)^N \langle z \rangle^N$ holds for a big $N \in \mathbb{N}$. Hence

$$\sup_{x \in K} \langle f^k(x) - f(x) \rangle \leq \sup_{x \in K} \langle x \rangle^N \int \langle z \rangle^N |\tilde{\varphi}(z)| dz,$$

(3.15)

and it is seen that the left hand side tends to zero for $k \to \infty$. \hfill \square

The result above generalises the observation made in [Sic87] that $\pi(f, u) = f u$ when $f \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$.

Since $u \text{ and } v$ in Definition 3.4 are assumed only to lie in $\mathcal{S}'(\mathbb{R}^n)$, one may now ask for stricter conditions on $u$ which allows $f$ to be more general than an element of the space $\mathcal{O}_M(\mathbb{R}^n)$.

Carried to the extreme, when $u \in L_{p,loc} \cap \mathcal{S}'$ it is possible to take $f \in L_{q,loc} \cap \mathcal{S}'$ provided only that $1 \leq p \leq \frac{1}{q} \leq 1$. Before we show this in Proposition 3.8 below, the next result on a local property (of $\pi$) is included as a preparation.

Proposition 3.7. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\psi(0) = 1$ and let $\Omega$ be a weight.

If $u$ and $v \in \mathcal{S}'(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ is an open set such that either $r_\Omega u = 0$ or $r_\Omega v = 0$, then

$$r_\Omega \mathcal{F}^{-1}(\tilde{\varphi}_k \mathcal{F}u) \cdot \mathcal{F}^{-1}(\tilde{\varphi}_k \mathcal{F}v) \to 0 \quad \text{in } \mathcal{D}'(\Omega) \quad \text{for } k \to \infty.$$

(3.16)

In particular, $r_\Omega \pi(u, v) = 0$ when $\pi(u, v)$ is defined.

Proof. It can be assumed that $r_\Omega u = 0$, for if not the roles of $u$ and $v$ can be interchanged. It suffices for each $\varphi \in C_0^\infty(\Omega)$ to show the convergence

$$\langle r_\Omega u^k v^k, r_\Omega \varphi \rangle = \langle v^k, u^k \varphi \rangle \to 0, \quad k \to \infty.$$

(3.17)

By equicontinuity and the relation $\langle v^k, u^k \varphi \rangle = \langle v^k - v, u^k \varphi \rangle + \langle v, u^k \varphi \rangle$, (3.17) follows if $u^k \varphi \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ for $k \to \infty$.

For completeness’ sake we supply a proof of the fact that $(\tilde{\varphi}_k * u) \cdot \varphi \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ for $k \to \infty$, when $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ are arbitrary such that $r_\Omega u = 0$, and $\psi_k(x) = 2^{\kappa M} |\psi(2^{k M} x)|$. It suffices to show, for $K = \text{supp} \varphi \subset \Omega$, that

$$\sup \{|\tilde{\varphi}_k * u| \mid x \in K\} \rightarrow 0 \quad \text{for } k \to \infty,$$

(3.18)

for then $\| (\tilde{\varphi}_k * u) \cdot \varphi \|_{\mathcal{S}, \alpha, \beta} \rightarrow 0$ for every $\alpha$ and $\beta$. 

With closed sets $F_1$ and $F_2$ satisfying $K \subseteq F_1 \subseteq F_1^c \subset F_2 \subset F_2^c \subset \Omega$, we take $\eta \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $\eta = 1$ on $\mathbb{R}^n \setminus F_2^c$ and $\eta = 0$ on $F_1$. Then

$$|\psi_k \ast u| = |\langle u, \eta \cdot \psi_k(x - \cdot) \rangle| \leq c \| \eta \psi_k(x - \cdot) | \mathcal{S}, \alpha, \beta \| (3.19)$$

For each $x \in K$ it follows here that $\eta(y)\psi_k(x - y) = 0$ when $|x - y| < a := \text{dist}(K, \mathbb{R}^n \setminus F_2^c)$. Because $2^k |.| \leq 2^{2k} |.|$ one has $|z| < 2^ka \Rightarrow |x - y| < a$, when $z = 2^{2k}M(x - y)$. Moreover $2^kb|x - y|b \leq |z|^b$ holds for $b > 0$, so for each $\gamma \leq \beta$

$$|y^\alpha D_y^{\beta - \gamma} \eta D_y^\gamma \psi_k(x - \cdot)| \leq |D^{\beta - \gamma} \eta| \cdot 2^{\alpha}(|x - y|^{\alpha} + |x|^{\alpha}) \times 2^{k(\alpha + \beta - \gamma)} |D^\gamma \psi(z)\| \|z\| \geq 2^k a$$

$$\leq 2^{\alpha} a^{- |\alpha| - \beta + 1} |\eta| C^{\beta} \| \| \psi | \mathcal{S}, \alpha, + \beta - \gamma + |\beta| + 1\| \times (2^{-\alpha} + \sup_{x \in K} |x|^{\alpha}) \sup_{z \in \mathbb{R}^n} |z|^{-1} \|z\| \geq 2^k a.$$ 

(3.20)

Because the last factor $\to 0$ for $k \to \infty$ the convergence in (3.18) is inferred from Leibniz’ formula, (3.20) and (3.19).

In Section 7 the full generality of the result above will be used to define the product $\pi$ on an open set $\Omega$. Here Proposition 3.7 is used to prove the next result, where the simple case with $f \in L_p$, $g \in L_q$ and $\frac{1}{p} + \frac{1}{q} = 1$ is included in [St87].

**Proposition 3.8.** Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\psi(0) = 1$ and a weight $\gamma$ be given.

For $f \in L_{p,\text{loc}}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ and $g \in L_{q,\text{loc}}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ such that $\frac{1}{p} := \frac{1}{p} + \frac{1}{q} \leq 1$ there is convergence $f_k g_k \to fg$ in $\mathcal{D}'(\mathbb{R}^n)$. In particular $\pi(f, g)$ is defined and

$$\pi(f, g) = f(x) \cdot g(x) \in L_{r,\text{loc}}$$

(3.21) for such $f$ and $g$.

**Proof.** First we assume that $f \in L_p$, $g \in L_q$ and that $p$ and $q < \infty$. It follows (by inspection of the usual convolution proofs, where $\gamma = (1, \ldots, 1)$ and $\mathcal{F}^{-1} \psi \geq 0$) that $f_k \to f$ in $L_p$, when $f \in L_p$ for $1 \leq p < \infty$. Similarly $g_k \to g$ in $L_q$. Then

$$f_k g_k - fg = (f_k - f)(g_k - g) + (f_k - f)g + f(g_k - g)$$

(3.22)

and (1.2) imply that $\pi(f, g)$ is defined as an element of $L_r \to \mathcal{D}'(\mathbb{R}^n)$.

In general, let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be given and take $\eta_0 \in C_0^\infty(\mathbb{R}^n)$ such that $\eta_0 = 1$ on $\Omega \supset \text{supp} \varphi$, where $\Omega$ is open and bounded. Then it is found with $\eta_1 = 1 - \eta_0$ that

$$r_{\Omega} f_k g_k = r_{\Omega} (\eta_0 f)(k \eta_0 g) + r_{\Omega} (\eta_1 f)(\eta_0 g) + f_k g_k$$

(3.23)

where the second term on the right hand side goes to zero in $\mathcal{D}'(\Omega)$ by Proposition 3.7, while the first term converges to $r_{\Omega} (\eta_0 f \cdot \eta_0 g)$ in virtue of the special case treated above. Hence $(f_k g_k, \varphi) \to (f, g, \varphi)$.

**Remark 3.9.** In Definition 3.1 above the mollifiers $\psi$ were required to have compact support and to equal 1 on a neighbourhood of the origin. This is because the validity of the spectral conditions in formulae (3.7)–(3.8) (cf. the numbers $r$ and $R$ there) is crucial for the application of Yamazaki’s theorems to the operators $\pi_j (\cdot, \cdot)$ in Sections 5 and 6 below, cf. also Lemma 3.3.
An introduction of further restrictions on the \( \psi \)’s, say, positivity or dependence on \(|\xi|\) alone, leads to a product defined on a larger subset of \( S'(\mathbb{R}^n) \times S'(\mathbb{R}^n) \), of course. However, (3.7) and (3.8) remain valid, so the method of para-multiplication applies to such more general products also.

In contrast to this, by taking \( \psi \) merely in \( S(\mathbb{R}^n) \) or just satisfying \( \psi(0) = 1 \) one obtains a product that is a restriction of \( \pi(\cdot, \cdot) \). To this restriction Propositions 3.6 and 3.8 apply, but since (3.7) and (3.8) do not hold, it is not clear whether para-multiplication may be used to analyse such products.

The observations above serve as a justification of the definition of \( \pi(\cdot, \cdot) \), which may also be seen as an \( S' \)-version of the Antosik–Mikusiński–Sikorski model product. The relation to this product, or to its restrictions such as the duality product, the wave front product etc., is not clear. The reader is referred to Oberguggenberger’s book [Obe92] and A. Kaminski [Kam82] and the references there.

4. Necessary conditions for multiplication

In this section conditions necessary for boundedness of \( \mu \) and \( \pi \) as bilinear operators \( A_0 \oplus A_1 \rightarrow A_2 \) are proved. But first we include a lemma concerning the existence of auxiliary functions with convenient norms.

To prepare for this, observe that there exist \( \rho \), \( \theta \) and \( \omega \) in \( S(\mathbb{R}^n) \setminus \{0\} \) for which:

\[
\begin{align*}
supp F\theta &\subset \{ \xi \mid |\xi| \leq \frac{1}{10} \}, \quad \text{and} \quad \theta(0) = 1, \\
supp F\rho &\subset \{ \xi \mid \frac{1}{4} \leq |\xi| \leq 1 \}, \quad \text{and} \quad \rho \text{ is real valued,} \\
supp F\omega &\subset \{ \xi \mid \frac{3}{4} \leq |\xi| \leq 1 \} \cap B, \quad \text{and} \quad \omega(0) = 1,
\end{align*}
\]

for \( B = \{ \xi \mid |\xi - \zeta| \leq \frac{1}{10} \} \), with \( \zeta = (\zeta_j)_{j=1,\ldots,n} \) for \( \zeta_j = [j = j_0] \), where \( j_0 \) is chosen so that \( m_{j_0} = 1 \) (cf. the assumptions on \( M \)).

The functions may, for example, be constructed as \( \hat{\theta}(\xi) = \Psi(26[\xi])(\int \Psi(26[\cdot]))^{-1} \),

\( \hat{\rho}(\xi) = \Psi(\frac{11}{12}[\xi]) - \Psi(\frac{11}{24}[\xi]) \) and \( \hat{\omega}(\xi) = \chi(\xi)\hat{\rho}(\xi) \Psi(\frac{11}{8}[\xi - \zeta]) (\int \chi \hat{\rho} \Psi(\frac{11}{8}[\cdot - \zeta]))^{-1} \)

for some \( \chi \in C_0^\infty(\mathbb{R}^n) \). Indeed, \( \rho \) is real-valued if and only if \( \hat{\rho}(-\xi) = \hat{\rho}(\xi) \). If \( 0 < \Psi(t) < 1 \) whenever \( t \) is in \( \left[ \frac{11}{24}, \frac{11}{12} \right] \), as we may assume, this condition is satisfied, and for \( \xi = \frac{11}{12}\zeta \) the factor \( \hat{\rho}(\xi) \) is \( > 0 \) and \( \Psi(\frac{11}{8}[\xi - \zeta]) = 1 \), so \( \int \chi \hat{\rho} \Psi(\frac{11}{8}[\cdot - \zeta]) \neq 0 \) for some \( \chi \in C_0^\infty(\mathbb{R}^n) \).

For such \( \rho \), \( \omega \) and \( \theta \) we denote

\[
\begin{align*}
\rho_k(x) &= \rho(2^{km}x) \quad (k \in \mathbb{Z}), \\
\omega_k(x) &= \omega(2^{km}x) \quad (k \in \mathbb{N}), \\
\theta_k(x) &= \theta(x) \exp( i \text{sgn } k 2^{km} x_{j_0} ) \quad (k \in \mathbb{Z}),
\end{align*}
\]

whereby \( \theta_0 \) should be read as \( \theta \).
Lemma 4.1. 1° For any admissible \( s, p \) and \( q \), the functions \( \rho_k, \omega_k \) and \( \theta_k \) satisfy

\[
\| \rho_k |F_{p,q}^{M,s}| \| = \| \rho_k |B_{p,q}^{M,s}| \| = \| \rho |L_p|2^{k(s-(\frac{M}{p}))}, \quad (k \in \mathbb{N})
\]
\[
\| \rho_k |F_{p,q}^{M,s}| \| = \| \rho_k |B_{p,q}^{M,s}| \| = \| \rho |L_p|2^{-k(\frac{M}{p})}, \quad (-k \in \mathbb{N})
\]
\[
\| \rho_k^2 |F_{p,q}^{M,s}| \| = \| \rho_k^2 |B_{p,q}^{M,s}| \| = \| \rho^2 |L_p|2^{-k(\frac{M}{p})}, \quad (-k \in \mathbb{N})
\]
\[
\| \omega_k |F_{p,q}^{M,s}| \| = \| \omega_k |B_{p,q}^{M,s}| \| = \| \omega |L_p|2^{k(s-(\frac{M}{p}))}, \quad (4.3)
\]
\[
\| \theta_k |F_{p,q}^{M,s}| \| = \| \theta_k |B_{p,q}^{M,s}| \| = \| \theta |L_p|2^{k(s)}
\]
\[
\| \omega_k^2 |F_{p,q}^{M,s}| \| = \| \omega_k^2 |B_{p,q}^{M,s}| \| = \| \omega^2 |L_p|2^{(s-(\frac{M}{p}))}+s,
\]
\[
\| \theta_k^2 |F_{p,q}^{M,s}| \| = \| \theta_k^2 |B_{p,q}^{M,s}| \| = \| \theta^2 |L_p|2^{2k(s)},
\]
\[
\| \theta_k \omega_k |F_{p,q}^{M,s}| \| = \| \theta \omega_k |B_{p,q}^{M,s}| \| = \| \theta \omega |L_p|2^{(s-(\frac{M}{p}))}.
\]

and for any \( p \in [0, \infty) \) one has moreover

\[
\lim_{k \to \infty} 2^{k|M|}F^{-1} \Phi_0 F(\rho_k^2)(x) = \| \rho_k^2 |L_1|F^{-1} \Phi_0(x) \quad \text{in} \quad L_p(\mathbb{R}^n). \quad (4.4)
\]

2° The functions \( \theta^{(l)}_{N,\pm}, \rho^{(l)}_{N,\pm}, \omega^{(l)}_N \) and \( \Omega^{(l)}_N \), given by \( \theta^{(l)}_{N,\pm} = \sum_{k=1}^{N} 2^{-kt}\theta_{\pm k} \) and

\[
\rho^{(l)}_{N,\pm} = \sum_{k=l}^{l+N} 2^{-kt}\rho_{k}, \quad \omega^{(l)}_N = \sum_{k=N+1}^{2N} 2^{-kt}\omega_k, \quad \Omega^{(l)}_N = \sum_{k=2}^{N+1} 2^{-2k}t\omega_{2k}, \quad (4.5)
\]

have for the indicated values of \( t \) norms with the characterisations (for \( N \geq 1 \) respectively \( N \geq 4 \) in the last line)

\[
\| \rho^{(s-(\frac{M}{p}))}_{N,\pm} |B_{p,q}^{M,s}| \| = \| \rho |L_p|N^\frac{2}{p},
\]
\[
\| \omega^{(s-(\frac{M}{p}))}_N |B_{p,q}^{M,s}| \| = \| \Omega^{(s-(\frac{M}{p}))}_N |B_{p,q}^{M,s}| \| = \| \omega |L_p|N^\frac{1}{q},
\]
\[
\| \theta^{(s)}_{N,\pm} |F_{p,q}^{M,s}| \| = \| \theta^{(s)}_{N,\pm} |B_{p,q}^{M,s}| \| = \| \theta |L_p|N^\frac{2}{p},
\]
\[
\| \theta^{(s)}_{N,\pm} |F_{p,q}^{M,s}| \| = \| \theta^{(s)}_{N,\pm} |B_{p,q}^{M,s}| \| = \| \theta^2 |L_p|N^\frac{1}{q},
\]
\[
\| \omega^{(2N)}_N |F_{p,q}^{M,s}| \| = \| \omega^{(2N)}_N |B_{p,q}^{M,s}| \| = \| \omega \sum_{k=N}^\infty \omega(2^{-kM} |L_p| 2^{2k(s-(\frac{M}{p}))}).
\]

Furthermore, for \( 2 \leq k \leq N + 1 \) respectively for each \( p \in [0, \infty) \)

\[
F^{-1} \Phi_{2^{k+1}}(\Omega^{(t_0)}_N, \Omega^{(t_1)}_N) = 2^{-2(t_0+t_1)} \omega_{2^{k}} \quad (4.7)
\]
\[
\lim_{N \to \infty} \omega N^{-1} \sum_{k=N}^{2N-1} \omega(2^{-kM}) = \omega \quad \text{in} \quad L_p, \quad (4.8)
\]

and under the conditions \( t_0 + t_1 = -|M| \) respectively \( s_0 + s_1 = 0 \), with \( N \in \mathbb{N} \),

\[
\lim_{l \to \infty} F^{-1} \Phi_0 F(\theta^{(s_0)}_{N,\pm} \cdot \rho^{(s_1)}_{N,\pm}) = N \| \rho |L_2|^2 F^{-1} \Phi_0 \quad \text{in} \quad L_p,
\]
\[
F^{-1} \Phi_0 F(\theta^{(s_0)}_{N,\pm} \cdot \rho^{(s_1)}_{N,\pm}) = N \theta^2. \quad (4.9)
\]
Proof. 1° The main point is to show the relations

\[
\text{supp } F\rho_k \subset \{ \xi \mid |\hat{\omega}_k| \leq 2^k \} \quad \subset \{ \xi \mid \Phi_k(\xi) = 1 \},
\]

\[
\text{supp } F\rho_k^2 \subset \{ \xi \mid |\hat{\omega}_k| \leq 2^{k+1} \} \quad \subset \{ \xi \mid \Phi_k(\xi) = 1 \},
\]

\[
\text{supp } F\omega_k \subset \{ \xi \mid |\hat{\omega}_k| \leq 2^k \} \quad \subset \{ \xi \mid \Phi_k(\xi) = 1 \},
\]

\[
\text{supp } F\theta_k \subset \{ \xi \mid |\hat{\omega}_k| \leq 2^{k+1} \} \quad \subset \{ \xi \mid \Phi_k(\xi) = 1 \},
\]

\[
\text{supp } F\omega_k^2 \subset \{ \xi \mid |\hat{\omega}_k| \leq 2^{k+1} \} \quad \subset \{ \xi \mid \Phi_k(\xi) = 1 \},
\]

\[
\text{supp } F(\theta_k) \subset \{ \xi \mid |\hat{\omega}_k| \leq 2^k \} \quad \subset \{ \xi \mid \Phi_k(\xi) = 1 \}, \quad \text{for } k \neq 0,
\]

\[
\text{supp } F(\theta_k) \subset \{ \xi \mid |\hat{\omega}_k| \leq 2^{k+1} \} \quad \subset \{ \xi \mid \Phi_k(\xi) = 1 \}.
\]

(4.10)

Indeed, (4.10) gives \( F^{-1}(\hat{\omega}_k) = [k = l] \omega_k \) etc., so that \( \| \omega_k \|_{F_{p,q}^{M,s}} = \| 2^k \omega_k \|_{L^p} \) etc. Taking the dilations or the exponential factors into account (4.3) follows.

The support conditions on \( \rho_k \) and \( \omega_k \) follow from (4.1) since \([M\xi] = t[\xi] \) and \( \hat{\rho}_k(\xi) = 2^{-|M|\xi} \hat{\rho}(2^{-kM}\xi) \); and by (2.5a) \( \{ \xi \mid \Phi_k(\xi) = 1 \} \supset \{ \xi \mid |\hat{\omega}_k| \leq 2^k \} \) for \( k \geq 1 \). For \( \omega_k^2 \) and \( \theta_k \) one can use that \( \text{supp } F(\hat{\psi}) \subset \text{supp } \hat{\rho} \) and that \( |[\xi] - [\eta]| \leq |\xi + \eta| \leq |\xi| + |\eta| \). Indeed, for \( \xi, \xi' \in \text{supp } \hat{\omega}_k \) one has \( 2^{-kM}(\xi(\eta)) = \xi + \eta' \) with \( \eta' \in B \), hence \( |\xi' + \xi''| \leq |\xi'| + |\xi''| \leq 2^k+1 \) and \( |\xi' + \xi''| \geq 2^k(2\xi' - 2\xi'') \geq 2^k(2 - 2\xi') = \frac{1}{2}2^k \). (The definition of \( \zeta \) gives \( 2\xi' = 2 \) here.) \( \theta_k \) and \( \theta_k \) are treated along the latter lines, since their spectra are obtained from the case \( k = 0 \) by translation in both directions along the \( \xi_{3\theta} \)-axis.

To obtain (4.4) note that there is strong convergence \( 2^{k|M} \rho_k^2(\int \rho^2)^{-1} \cdot 1 \) on \( C(\mathbb{R}^n) \) (\( \int \rho^2 > 0 \) follows since \( \rho \) is real). This gives (4.4) for \( p = \infty \). For \( p < \infty \) we use the pointwise convergence thus shown together with \( \| \Phi_0 \|_{L^p} \leq 2^{p+1} |\hat{\Phi_0}|_{L^p} \) and the following majorisation,

\[
|2^{k|M} \Phi_0 F(\rho_k^2)(x)| \leq \langle x \rangle^{-\frac{n+1}{p}} \sup_x |F^{-1}(1 - \Delta \xi)^N \Phi_0 2^{k|M} F(\rho_k^2)(x)|
\]

\[
\leq \langle x \rangle^{-\frac{n+1}{p}} \sum_{|\alpha|,|\beta| \leq 2N} c_{\alpha,\beta} \| x^\beta \rho^2 \|_{L^1} \int |D^\alpha \Phi_0| d\xi, \quad (4.11)
\]

where \( \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}} \) and \( N \geq \frac{n+1}{2p} \).

2° (4.6) is obtained like (4.3) from (4.10) and the norm definitions. In particular the \( k \)th summand in \( \theta(s)_{N,\pm} \) has spectrum in \( \{ \xi \mid \Phi_k(\xi) = 1 \} \), implying, e.g.,

\[
\| \theta(s)_{N,\pm} |B_{p,q}^{M,s} \| = \left( \sum_{k=1}^{N} 2^{ks} \| 2^{-ks} \theta^2 e^{2^k x_n} \|_{L^p} \| \right)^{\frac{1}{s}} = \| \theta^2 \|_{L^p} N^{\frac{1}{s}}. \quad (4.12)
\]

Concerning \( \omega_{2N} \omega_k \) of each term \( \omega_{2N} \omega_k \) with \( N + 1 \leq k \leq 2N \) can be treated like \( \omega_k^2 \) in (4.10) and thus \( F(\omega_{2N} \omega_k) \subset \{ \xi \mid 2^{3N}(\frac{1}{2N^2} - 2^{k-3N}) \leq |\xi| \leq 2^{3N}(1 + 2 - 3N) \} \). This set is contained in \( \{ \xi \mid \Phi_{2N}(\xi) = 1 \} \) for \( N \geq 4 \), so (4.10) follows.

The product \( \Omega_1^{(\xi)} \Omega_1^{(\tau)} \) consists in part of terms \( 2^{-2k(t+\tau)} \omega^2(2^k)^M x \) with spectrum in \( \{ \xi \mid \Phi_{2k+1}(\xi) = 1 \} \), cf. (4.10), and in part of terms stemming from \( \omega_j \omega_k \) with \( j < k \). Since \( 2^k \geq 2^k+1 \geq 2^j + 4 \) (because \( j \geq 2 \)) it is found that
Indeed, for any element of supp we have for while those with since the terms with by use of (4.4) that in the topology of \( L_p \),

\[
F^{-1}\Phi_0 F(\rho_{N,l}^{(t_0)} \rho_{N,l}^{(t_1)}) = \sum_{j=l+1}^{l+N} 2^{jM} |F^{-1}\Phi_0 F\rho_j^2| \to N \| \rho \|_2^2 F^{-1}\Phi_0. \quad (4.13)
\]

Indeed, for \( j \neq k \) the spectra of \( \rho_j \rho_k \) and supp \( \Phi_0 \) are disjoint: E.g., for \( j \geq k + 1 \) any element of supp \( F(\rho_j \rho_k) \) is of the form \( \xi_j + \xi_k \), with \( \xi_m \in \text{supp} F_{\rho_m} \), for which \( |\xi_j + \xi_k| \geq 2^{j} - 2^{k} \geq 1/2^{k} \geq 13/10 \) for \( k \geq 2 \). For \( s_0 + s_1 = 0 \) it is seen that

\[
F^{-1}\Phi_0 F(\theta_{N,+}^{(s_0)} \theta_{N,-}^{(s_1)}) = F^{-1}\Phi_0 F\left( \sum_{k,l=1}^{N} 2^{-k s_0 - l s_1} e^{i((2^k 2^l x)_{j_0})} \theta^2 \right) = N \theta^2, \quad (4.14)
\]

since the terms with \( l \neq k \) in the sum have their spectrum disjoint from supp \( \Phi_0 \), while those with \( l = k \) have their spectrum in \( \{ \xi \mid \Phi_0(\xi) = 1 \} \).

The present versions of the functions \( \rho_k, \theta_k \) and \( \theta_{N,\pm}^{(t)} \), that via [Sic87] and [Fra86] go back at least to [Tri83, 2.3.9], are introduced in order to obtain greater clarity via characterisations of the norms rather than estimates. The other functions are introduced to show some of the new parts of Theorem 4.2 below, and so is the technique of considering the limits in \( (4.4) \), \( (4.8) \) and \( (4.9) \).

In the next result \( (1), (1') \), \( (2) \) and \( (2') \) set limits for the admissible spaces \( A_0 \oplus A_1 \), while \( (3) \)–\( (7) \) etc. restrict the best obtainable \( A_2 \), cf. \( (II) \) and \( (III) \).

**Theorem 4.2.** If there exists a constant \( c < \infty \) such that the inequality

\[
\| f \cdot g \|_{A_2} \leq c \| f \|_{A_0} \| g \|_{A_1} \quad \text{holds for all } f \text{ and } g \in \mathcal{S}(\mathbb{R}^n),
\]

where \( A_j = B_{p_j,q_j}^{s_j}(\mathbb{R}^n) \) or \( A_j = L_{p_j,q_j}^{s_j}(\mathbb{R}^n) \) for \( j = 0, 1 \) and \( 2 \), then it follows that

\[
\begin{align*}
(1) \quad & s_0 + s_1 \geq |M| \left( \frac{1}{p_0} + \frac{1}{p_1} - 1 \right), \\
(2) \quad & s_0 + s_1 \geq 0, \\
(3) \quad & s_2 \leq \min(s_0, s_1), \\
(4) \quad & \frac{s_2}{p_2} \leq \frac{1}{p_0} + \frac{1}{p_1}, \\
(5) \quad & s_2 - \frac{|M|}{p_2} \leq \min(s_0 - \frac{|M|}{p_0}, s_1 - \frac{|M|}{p_1}), \\
(6) \quad & s_2 - \frac{|M|}{p_2} \leq s_0 + s_1 - |M| \left( \frac{1}{p_0} + \frac{1}{p_1} \right), \\
(7) \quad & s_2 - \frac{|M|}{p_2} = s_1 - \frac{|M|}{p_1} \quad \text{and} \quad s_0 = \frac{|M|}{p_0}
\end{align*}
\]

implies

\[
\begin{align*}
\{ q_0 & \leq 1 \text{ in } B^{**} \text{ cases}, \\
\{ p_0 & \leq 1 \text{ in } F^{**} \text{ cases}.
\end{align*}
\]
Furthermore it also follows, for $j = 0$ respectively $j = 1$, that

\begin{align*}
(1') \quad s_0 + s_1 &= \frac{|M|}{p_0} + \frac{|M|}{p_1} - |M| \quad \text{implies } \left\{ \frac{1}{q_0} + \frac{1}{q_1} \geq 1 \text{ in } \text{BB}^* \text{ cases}, \right. \\
(2') \quad s_0 + s_1 &= 0 \quad \text{implies } \frac{1}{q_0} + \frac{1}{q_1} \geq 1, \\
(3') \quad s_2 &= s_j \quad \text{implies } q_2 \geq q_j, \\
(5') \quad s_2 - \frac{|M|}{p_2} &= s_j - \frac{|M|}{p_j} \quad \text{implies } q_2 \geq q_j \text{ in } \text{BB} \text{ resp. } \text{BB}^* \text{ cases}, \\
(6') \quad s_2 - \frac{|M|}{p_2} &= s_0 - \frac{|M|}{p_0} + s_1 - \frac{|M|}{p_1} \quad \text{implies } q_2 \geq (\frac{1}{q_0} + \frac{1}{q_1})^{-1} \text{ in } \text{BBB} \text{ cases.}
\end{align*}

By Proposition 3.6, the same conclusions can be drawn for $\pi$ when it satisfies (4.15).

Proof. Observe first, that in any case one has $\|F^{-1}\Phi_0 F u \|_{L^{p_2}} \leq \|u \|_{A_2}$ by the definition of $\| \cdot \|_{B^{M,s}_p}$ and $\| \cdot \|_{F^{M,s}_p}$.

By application of (4.13) to (4.15) it follows that

\begin{equation}
\|F^{-1}\Phi_0 F r_k^2 \|_{L^{p_2}} \leq \| r_k^2 \|_{A_2} \leq c \| r_k \|_{A_0} \| r_k \|_{A_1} \leq c \| r \|_{L^{p_0}} \| r \|_{L^{p_1}} \left\| 2k(s_0 + s_1 - |M|)(\frac{1}{q_0} + \frac{1}{q_1}) \right\|, 
\end{equation}

and taken together, since $\rho \neq 0$, (4.13) and (4.18) show that

\begin{equation}
0 < \|F^{-1}\Phi_0 \|_{L^{p_2}} \leq c \| \rho \|_{L^{p_0}} \leq c \| r \|_{L^{p_0}} \leq c \| \rho \|_{L^{p_1}} \leq c \| \rho \|_{L^{p_1}} \left\| 2k(s_0 + s_1 - |M|)(\frac{1}{q_0} + \frac{1}{q_1} - 1) \right\|. 
\end{equation}

Here $s_0 + s_1 - |M|(\frac{1}{q_0} + \frac{1}{q_1} - 1) < 0$ would be absurd, so (1) in (4.16) follows.

For $s_0 + s_1 = |M|(\frac{1}{q_0} + \frac{1}{q_1} - 1)$ we conclude from (4.15) that, with $t_j = s_j - \frac{|M|}{p_j}$,

\begin{equation}
\| F^{-1} \Phi_0 \|_{L^{p_2}} \leq \| r_k \|_{A_0} \| r_k \|_{A_1} \leq c \| \rho \|_{L^{p_0}} \leq c \| \rho \|_{L^{p_1}} \| N^{\frac{1}{q_0} + \frac{1}{q_1}} 
\end{equation}

when $A_0$ and $A_1$ are Besov spaces. By (4.19) there exists for each $N$ an $l$ such that

\begin{equation}
\| F^{-1} \Phi_0 \|_{L^{p_2}} \leq \| F^{-1} \Phi_0 \|_{L^{p_2}} N \leq \| F^{-1} \Phi_0 \|_{L^{p_2}} L \leq c \| \rho \|_{L^{p_1}} \leq c \| \rho \|_{L^{p_1}} \leq c \| \rho \|_{L^{p_1}} \leq c \| \rho \|_{L^{p_1}} \left\| 2k(s_0 + s_1 - |M|)(\frac{1}{q_0} + \frac{1}{q_1} - 1) \right\|, 
\end{equation}

and therefore (4.20) and (4.21) give a contradiction for a big $N$ unless $\frac{1}{q_0} + \frac{1}{q_1} \geq 1$.

When $A_0$ is a Besov space and $A_1 = F^{p_0, q_1}_{1, t} \dot{F}^{p_1, q_1}_{1, t}$ the embedding $B^{M,t}_{r_1, t} \hookrightarrow \dot{F}^{M,t}_{r_1, t}$ holds for every $r < p$ when $t - \frac{|M|}{r} = s_1 - \frac{|M|}{p_1}$. Then (4.15) holds with $A_1$ replaced by $B^{M,t}_{r_1, t}$. Since $s_0 + t = \frac{|M|}{p_0} + \frac{|M|}{r} - |M|$ the statement on the $BB^*$ cases gives that

\begin{equation}
\frac{1}{q_0} + \frac{1}{r} \geq 1. \text{ Then } \frac{1}{q_0} + \frac{1}{r} = \inf\{ \frac{1}{q_0} + \frac{1}{r} \ | r < p_1 \} \geq 1. \text{ This proves (1').}
\end{equation}

The proof of (2), (2'), (3) and (3') is due to Franke, who treated some of the eight cases with $M = (1, \ldots, 1)$ in [Fra86]. (2) is found from (4.15) with $\rho = \theta_k$, $g = \theta_k$ and the fact that $\theta_k\theta_k = \theta^2$, and (3) together with (4.9) gives (2').

To show (3) one can take $f$ and $g$ equal to $\theta_k$ and $\theta_k$ respectively $\theta$ and $\theta_k'$, and (3') is obtained with $f$ and $g$ equal to $\theta_k$, and $\theta_k'$ respectively $\theta$ and $\theta_k'$.

(4) is due to Sickel, [Sic87]. The proof consists of an insertion of $f = g = \rho_k$ for $-k \in \mathbb{N}$ into (4.15) and an application of (4.3).
Concerning (5) for \( j = 0 \) one has \( \omega \theta(2^{-M}) \to \omega \) in \( L_p \) for \( l \to \infty \) (by a majorisation), so for \( k \) large enough

\[
0 < \frac{1}{\varpi} \left\| \omega \right\|_{L_{p_2}} 2^{k(s_2 - \frac{|M|}{p_2})} \leq c \left\| \omega \right\|_{L_{p_0}} \left\| \theta \right\|_{L_{p_1}} 2^{k(s_0 - \frac{|M|}{p_0})} \tag{4.22}
\]

by \((4.3)\); for \( j = 1 \) the roles of \( \omega_k \) and \( \theta \) can be interchanged.

\((5')\) is obtained analogously from \( \omega^{(s_0 - \frac{|M|}{p_0})} \) and \( \theta \) respectively and \( \omega^{(s_1 - \frac{|M|}{p_1})} \).

Condition (6) can be shown by insertion of \( f = g = \omega_k \) into \((4.13)\) followed by use of \((4.9)\). For \((6')\) formula \((4.7)\) leads to the inequalities, where \( t_j = s_j - \frac{|M|}{p_j} \),

\[
2^{s_2} \left\| \omega^2 \right\|_{L_{p_2}} \|N\| \leq \left| \Omega^{(t_0)} \right| \left| \Omega^{(t_1)} \right| \left| B^{M,s_2}_{p_2,q_2} \right| \leq c \left\| \omega \right\|_{L_{p_0}} \left\| \omega \right\|_{L_{p_1}} \|N\| 2^{s_0 + \frac{|M|}{p_0}} \tag{4.23}
\]

when only terms with \( M \neq _{s_0} = \frac{|M|}{p_0} \) are kept in the \( B^{M,s_2}_{p_2,q_2} \) norm.

Concerning (7) in the \( B^{\infty} \) cases it is found from \((4.3)\), \((4.8)\) with a large \( N \) and the assumption \( s_0 = \frac{|M|}{p_0} \) that

\[
\frac{1}{2} 2^{3N(s_2 - \frac{|M|}{p_2})} N \left\| \omega \right\|_{L_{p_2}} \leq c \left\| \omega \right\|_{L_{p_0}} \left\| \omega \right\|_{L_{p_1}} \|N\| 2^{3N(s_0 - \frac{|M|}{p_0})}. \tag{4.24}
\]

The second assumption, \( s_2 - \frac{|M|}{p_2} = s_1 - \frac{|M|}{p_1} \), then leads to the conclusion \( q_0 \leq 1 \). The \( F^{\infty} \) cases can be reduced to the \( B^{\infty} \) cases. Indeed, if \( p_0 > 1 \) is assumed, there is a Sobolev embedding \( B^{M,t}_{r,r} \hookrightarrow B^{M,s_0}_{p_0,q_0} \) with \( r \) and \( 1 < o < p_0 \) according to \((4.13)\). But then \((4.13)\) holds with \( A_0 = F^{M,s_0}_{p_0,q_0} \) replaced by \( B^{M,t}_{r,r} \), hence \( o \leq 1 \) is necessary and the assumption \( p_0 > 1 \) is absurd. \( \square \)

When (5) and (6) in Theorem 4.2 are taken together, they may be written

\[
s_2 - \frac{|M|}{p_2} \leq \min(s_0 - \frac{|M|}{p_0}, s_1 - \frac{|M|}{p_1}, s_0 - \frac{|M|}{p_0} + s_1 - \frac{|M|}{p_1}) := \min^+(s_0 - \frac{|M|}{p_0}, s_1 - \frac{|M|}{p_1}) =: \min^+(s_j - \frac{|M|}{p_j}) \tag{4.25}
\]

For later reference it is observed that for \( s_2 = s_1 \) formula \((4.25)\) is equivalent to

\[
\frac{|M|}{p_2} \geq \max(s_0 - \frac{|M|}{p_0} + s_1 - \frac{|M|}{p_1}, s_0 - \frac{|M|}{p_0} + \frac{|M|}{p_1} - s_0) \tag{4.26}
\]

Remark 4.3. When the embeddings \( A_j \hookrightarrow L_{t_j} \) hold for \( j = 0 \) and \( 1 \) with \( -\frac{|M|}{t_j} = s_j - \frac{|M|}{p_j} \), cf. \((2.15)\) and \((2.19)\), \((1)\) in \((4.16)\) amounts to \( \frac{1}{t_2} := \frac{1}{t_0} + \frac{1}{t_1} \leq 1 \). Then Proposition 3.8 shows that \( \pi \) equals \( \mu \) on \( A_0 \oplus A_1 \) and that \( \pi(A_0 \oplus A_1) \subset L_{t_2} \) where \( t_2 \geq 1 \). We may therefore interprete \((1)\) in \((4.16)\) as a condition assuring that \( \pi(A_0 \oplus A_1) \) is a distribution space.

Remark 4.4. Applied to the situation where \( A_0 = A_1 = A_2 \) the condition (6) in \((4.16)\) amounts to \( s \geq \frac{|M|}{p} \), and for the borderline case \( s = \frac{|M|}{p} \) condition (7) gives \( q \leq 1 \) and \( p \leq 1 \) in the \( BBB \) respectively \( FFF \) cases.

For \( M = (1, \ldots, 1) \) these conditions are known to be necessary (and sufficient too for \( s > 0 \)) for \( B^{M,s}_{p,q} \) respectively \( F^{M,s}_{p,q} \) to be algebras. The proof of the necessity given here, for general \( M \), seems simpler than those in [Tri78] and [Fra86].

Remark 4.5. The conditions \((1), (1'), (5), (5'), (6)\) and \((6')\) above have seemingly not been published before, but from a personal conversation the author knows that W. Sickel has obtained some of these independently.
Condition (7) generalises [Sic87] Rem. III.13, where \( s_0 = s_1 = \frac{n}{p_0} = \frac{n}{p_1} \) is assumed. (However, the intersection of one factor with \( L_\infty \) is included there.)

**Remark 4.6.** As an exercise one may use Lemma 4.1 to analyse the optimality of the linear embeddings in Section 2.2. See also [ST] for sharp results in the isotropic case.

In particular, if \( B_{p,q}^{s,q} \hookrightarrow L_\infty \), the \( \rho_k \)-part of (4.3) yields \( s \geq \frac{|M|}{p} \). For \( s = \frac{|M|}{p} \) the inequality \( \| x \cdot |L_\infty| \leq c \| \cdot |B_{p,q}^{M,s} \| \) gives with \( \omega_N^{(0)} \) inserted that

\[
N = N \omega(0) \leq c \| |L_p| \| N^\frac{1}{q},
\]

so \( q \leq 1 \) follows. This shows the optimality of (2.16).

Similarly it is found from the properties of \( \rho_k \) that \( F_{p,q}^{M,s} \hookrightarrow L_\infty \) imply that \( s \geq \frac{|M|}{p} \). And as in the proof of (7) in Theorem 4.2 above it is found for \( s = \frac{|M|}{p} \) that \( p \leq 1 \) is necessary. Hence (2.17) is optimal for \( M = (1, \ldots, 1) \). Otherwise it is open whether \( F_{p,q}^{M,|M|} \) with \( 1 < q \leq \infty \) is embedded into \( L_\infty \) or not.

These counterexamples concerning \( L_\infty \) are not only valid for general \( M \), but they also seem simpler than the arguments for the isotropic cases in [Fra86, Tri78].

## 5. Estimates of para-multiplication operators

First the basic consequences of Yamazaki’s theorems are collected. The approach is essentially known since it is adopted from [Yam86a] and [Sic87]. However, these references are inadequate for our purposes, so we state and prove Theorem 5.1.

It should be noted that Sichel for the isotropic versions of (5.1)–(5.2) below has introduced a shorter formulation by means of the local Hardy spaces \( h_p = F_{p,2}^0 \) \((0 < p < \infty)\), cf. [Sic91, ST], but the proof becomes less elementary, then.

**Theorem 5.1.** Let \( s, s_0 \) and \( s_1 \in \mathbb{R} \) be given together with \( p_0, p_1, q, q_0 \) and \( q_1 \) in \([0, \infty]\), and let \( s_2 = s_0 + s_1, \frac{1}{p_2} = \frac{1}{p_0} + \frac{1}{p_1} \), and \( \frac{1}{q_2} = \frac{1}{q_0} + \frac{1}{q_1} \). Then the operators

\[
\pi_1: L_{p_0}(\mathbb{R}^n) \oplus B_{p_1,q}^{s,q}(\mathbb{R}^n) \to B_{p_2,q}^{s,q}(\mathbb{R}^n), \quad \text{for} \quad 1 \leq p_0 \leq \infty,
\]

\[
\pi_1: L_{p_0}(\mathbb{R}^n) \oplus F_{p_1,q}^{s,q}(\mathbb{R}^n) \to F_{p_2,q}^{s,q}(\mathbb{R}^n), \quad \text{for} \quad 1 < p_0 \leq \infty, \quad p_1 < \infty,
\]

\[
\pi_1: F_{p_0,1}^{s_0,q}(\mathbb{R}^n) \oplus B_{p_1,q}^{s,q}(\mathbb{R}^n) \to B_{p_2,q}^{s,q}(\mathbb{R}^n), \quad \text{for} \quad 0 < p_0 < \infty, \quad p_1 < \infty,
\]

\[
\pi_1: F_{p_0,1}^{s_0,1}(\mathbb{R}^n) \oplus F_{p_1,q}^{s,q}(\mathbb{R}^n) \to F_{p_2,q}^{s,q}(\mathbb{R}^n), \quad \text{for} \quad 0 < p_0 < \infty, \quad p_1 < \infty
\]

are bounded, and \( \pi_3 \) has similar properties when the summands are interchanged (since \( \pi_3(u,v) = \pi_1(v,u) \)).

For \( s_0 + s_1 > |M| \max(0, \frac{1}{p_0} + \frac{1}{p_1} - 1) \), i.e., \( s_2 > |M|(\frac{1}{p_2} - 1)_+ \), the operator

\[
\pi_2: B_{p_0,q_0}^{s_0}(\mathbb{R}^n) \oplus B_{p_1,q_1}^{s_1}(\mathbb{R}^n) \to B_{p_2,q_2}^{s_2}(\mathbb{R}^n),
\]

is bounded, and if \( s_2 > |M|(\frac{1}{\min(p_2,q_2)} - 1)_+ \) and both \( p_0 \) and \( p_1 < \infty \), so is

\[
\pi_2: F_{p_0,q_0}^{s_0}(\mathbb{R}^n) \oplus F_{p_1,q_1}^{s_1}(\mathbb{R}^n) \to F_{p_2,q_2}^{s_2}(\mathbb{R}^n).
\]

Furthermore, if \( s_0 < 0 \) the operator \( \pi_1 \) is continuous

\[
\pi_1: B_{p_0,q_0}^{s_0}(\mathbb{R}^n) \oplus B_{p_1,q_1}^{s_1}(\mathbb{R}^n) \to B_{p_2,q_2}^{s_2}(\mathbb{R}^n),
\]

\[
\pi_1: F_{p_0,q_0}^{s_0}(\mathbb{R}^n) \oplus B_{p_1,q_1}^{s_1}(\mathbb{R}^n) \to B_{p_2,q_2}^{s_2}(\mathbb{R}^n),
\]

for \( p_0 \) and \( p_1 < \infty \),

and for \( s_1 < 0 \) the operator \( \pi_3 \) has similar properties.
Proof. It may be assumed that $\psi_j = \Psi_j$ and $\varphi_j = \Phi_j$ for if not the estimates below are valid with $\|2^{sj} v_j |L_{p_j}\| \|\ell_q\|$ instead of $\|u|B^{|s|}_{p,q}\|$ etc., and then Remark 3.4 applies. We begin with (5.7).

For $u \in B^{|s|}_{p_0,q_0}$ and $v \in B^{|s|}_{p_1,q_1}$ one has, when $r_0 = \min(1,p_0)$ and $\alpha_j = \|u_j |L_{p_0}\|$, 
\[
\| \{ 2^{s_j} v_j |L_{p_2}\} \|_j = 0 \|\ell_q\| \leq \| \{ 2^{s_j} v_j |L_{p_2}\} \|_j = 0 \|\ell_q\| \leq \|2^{s_j} \{ u_0 + \cdots + a_j \} \| \ell_{q_0}\| \|v|B^{|s|}_{p_1,q_1}\|_j = 0, \tag{5.9}
\]
by use of Hölder’s inequality, (2.1) and Lemma 2.3. Since (3.7) and (3.9) show that the conditions in Theorem 2.1 are satisfied, (5.7) follows. Similarly one can prove (5.8) and the analogous properties of $\pi_3$ when $s_1 < 0$.

For the treatment of $\pi_2$ one can use the estimate 
\[
\| \{ 2^{s_j} u_j |L_{p_2}\} \|_j = 0 \|\ell_q\| \leq \|2^{s_j} \{ 2^{s_j} u_j \} \| \ell_{q_0}\| \|2^{s_j} v_j \| \ell_{q_1}\| |L_{p_2}| \leq 2^{s_0} \| \{ 2^{s_j} u_j \} \| \|v|B^{|s|}_{p_0,q_0}\| \|v|B^{|s|}_{p_1,q_1}\| \leq c \|u|B^{|s|}_{p_0,q_0}\| \|v|B^{|s|}_{p_1,q_1}\|, \tag{5.10}
\]
along with similar estimates of $u_j v_j$ and $u_j v_{j-1}$ to conclude that 
\[
\| \{ 2^{s_j} (u_j - v_j + u_j v_j + u_j v_{j-1}) \} \| \ell_{q_2}\| |L_{p_2}| \leq c \|u|B^{|s|}_{p_0,q_0}\| \|v|B^{|s|}_{p_1,q_1}\|, \tag{5.11}
\]
where $c$ is proportional to $2^{s_0} + 1 + 2^{s_1}$. In view of (3.8) and Theorem 2.2, this proves (5.6). (5.7) is proved similarly.

The formulae (5.1) and (5.3) are deduced from the estimate 
\[
\| \{ 2^{s_j} |L_{p_2}| \} \|_j = 0 \|\ell_q\| \leq \sup_k \|u^k |L_{p_0}\| \|2^{s_j} v_j \| \ell_{q}\| |L_{p_2}| \tag{5.12}
\]
while (5.2) and (5.4) are based on a version of (5.12) with $\|\|u^k \| |L_{p_0}\|$. Indeed, in (5.12) we can introduce the estimates, where $a = \|\Psi |L_1\|$, 
\[
\sup_k \|u^k |L_{p_0}\| \leq \sup_k \|\Psi_k |L_1\| \|u |L_{p_0}\| \leq a \|u |L_{p_0}\|, \tag{5.13}
\]
\[
\sup_k \|u^k |L_{p_0}\| \leq \sup_k \|u_0 + \cdots + u_k |L_{p_0}\| = \|u|F^{|s|}_{p_0,1}\|, \tag{5.14}
\]
which by application of Theorem 2.1 shows (5.1) and (5.3), respectively.

In the $F$ case one can estimate, for $0 < p_0 < \infty$ respectively $1 < p_0 \leq \infty$, 
\[
\|\|u^k |L_{p_0}\| \leq \|\|u_0 + \cdots + u_k |L_{p_0}\| \leq \|u |F^{|s|}_{p_0,1}\|, \tag{5.15}
\]
\[
\|\|u^k |L_{p_0}\| \leq c \|u |L_{p_0}\|, \tag{5.16}
\]
where $c = a$ for $p_0 = \infty$, while for $1 < p_0 < \infty$ Corollary 2.9 in Yam86a applies.

Next we include in Corollary 5.2 various properties that are directly applicable to the sufficient conditions with $p_0 \neq p_1 \neq p_2$ in Section 6 below.

In the sequel, $s_0 \geq s_1$ is assumed for simplicity (as we may by commutativity of $\pi$). For a sum-exponent $t$, the requirement $q_0 [s_0 = s_1] \leq t \leq \infty$ reduces to $0 < t \leq \infty$ for $s_0 \neq s_1$, since only $t > 0$ is allowed. A similar remark applies to integral-exponents in the $F$ case. Recall the $s_2$, $p_2$ and $q_2$ notation of Theorem 5.1.
Corollary 5.2. Let \( s_j \in \mathbb{R} \), \( p_j \in [0, \infty) \) and \( q_j \in [0, \infty] \) be given for \( j = 0 \) and \( 1 \) such that \( s_0 + s_1 > \frac{|M|}{|m|} \), \( \frac{1}{p_1} + \frac{1}{p_0} - 1 > 1 \) and \( s_0 \geq s_1 \).

Then, if \( p_0 \) and \( p_1 < \infty \), the bilinear operators

\[
F_{p_0, q_0}^{M, s_0} \oplus F_{p_1, q_1}^{M, s_1} \overset{\pi_1}{\to} \bigcap \left\{ F_{r, q_1}^{M, s_1} \mid \left( \frac{|M|}{p_1} + (s_0 - \frac{|M|}{p_0}) \right)_- < \frac{|M|}{r} \right\}, \tag{5.17}
\]

\[
F_{p_0, q_0}^{M, s_0} \oplus F_{p_1, q_1}^{M, s_1} \overset{\pi_2}{\to} \bigcap \left\{ F_{r, t}^{M, s_1} \mid 0 < t \leq \infty, \left( \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0 \right)_+ \leq \frac{|M|}{r} \leq \frac{|M|}{p} \right\}, \tag{5.18}
\]

\[
F_{p_0, q_0}^{M, s_0} \oplus F_{p_1, q_1}^{M, s_1} \overset{\pi_3}{\to} \bigcap \left\{ F_{r, t}^{M, s_1} \mid q_0 \left[ s_0 = s_1 \right] \leq t \leq \infty, \left( \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0 \right)_+ + \left( s_1 - \frac{|M|}{p_1} \right)_+ \right\}, \tag{5.19}
\]

are bounded. In addition the value \( \frac{|M|}{r} = \frac{|M|}{p_1} + (s_0 - \frac{|M|}{p_0})_- \) may be included in \( 5.17 \) under the condition that \( F_{p_0, q_0}^{M, s_0} \mapsto L_\infty \) holds if \( s_0 = \frac{|M|}{p_0} \). Similarly \( \frac{|M|}{r} = \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0 + \left( s_1 - \frac{|M|}{p_1} \right)_+ \) may be included in \( 5.19 \) provided \( F_{p_1, q_1}^{M, s_1} \mapsto L_\infty \) holds if \( s_1 = \frac{|M|}{p_1} \).

Furthermore, there is boundedness of the operators

\[
B_{p_0, q_0}^{M, s_0} \oplus B_{p_1, q_1}^{M, s_1} \overset{\pi_1}{\to} \bigcap \left\{ B_{r, q_1}^{M, s_1} \mid \left( \frac{|M|}{p_1} + (s_0 - \frac{|M|}{p_0}) \right)_- < \frac{|M|}{r} \right\}, \tag{5.20}
\]

\[
B_{p_0, q_0}^{M, s_0} \oplus B_{p_1, q_1}^{M, s_1} \overset{\pi_2}{\to} \bigcap \left\{ B_{r, t}^{M, s_1} \mid 0 < t \leq \infty, \left( \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0 \right)_+ < \frac{|M|}{r} \leq \frac{|M|}{p} \right\}, \tag{5.21}
\]

\[
B_{p_0, q_0}^{M, s_0} \oplus B_{p_1, q_1}^{M, s_1} \overset{\pi_3}{\to} \bigcap \left\{ B_{r, t}^{M, s_1} \mid q_0 \left[ s_0 = s_1 \right] \leq t \leq \infty, \left( \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0 \right)_+ + \left( s_1 - \frac{|M|}{p_1} \right)_+ \leq \frac{|M|}{r} \leq \frac{|M|}{p} \right\}. \tag{5.22}
\]

When \( \frac{|M|}{t_1} - s_j \) one can include \( B_{r, q_1}^{M, s_1} \) with \( \frac{|M|}{r} = \frac{|M|}{p_1} + (s_0 - \frac{|M|}{p_0})_- \) in the intersection in \( 5.20 \) if

\[
s_0 > \frac{|M|}{p_0}, \tag{5.23}
\]

\[
s_0 = \frac{|M|}{p_0} \quad \text{and} \quad q_0 \leq 1, \tag{5.24}
\]

\[
s_0 < \frac{|M|}{p_0} \quad \text{and} \quad q_0 < t_0, \tag{5.25}
\]

\[
s_0 < \frac{|M|}{p_0} \quad \text{and} \quad q_0 \leq t_0 \quad \text{and either} \quad M = (1, \ldots, 1) \quad \text{or} \quad t_0 \leq 2. \tag{5.26}
\]

In \( 5.21 \) the space \( B_{r, q_0}^{M, s_1} \) may be included when \( \frac{|M|}{r} = \left( \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0 \right)_+ \) may be included in \( 5.22 \) if

\[
s_1 > \frac{|M|}{p_1}, \tag{5.27}
\]

\[
s_1 = \frac{|M|}{p_1} \quad \text{and} \quad q_1 \leq 1, \tag{5.28}
\]

\[
0 < s_1 < \frac{|M|}{p_1} \quad \text{and} \quad q_1 < t_1, \tag{5.29}
\]

\[
0 < s_1 < \frac{|M|}{p_1} \quad \text{and} \quad q_1 \leq t_1 \quad \text{and either} \quad M = (1, \ldots, 1) \quad \text{or} \quad t_1 \leq 2, \tag{5.30}
\]

and \( B_{r, q_2}^{M, s_1} \) with \( \frac{|M|}{r} = \left( \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0 \right)_+ \) may be included if

\[
s_1 \leq 0. \tag{5.31}
\]
Proof. Suppose \( s_2 > |M|(\frac{1}{\min(p_2, q_2)} - 1)_+ \). From (5.6) and \( F^{M, s_2}_{p_2, q_2} \hookrightarrow F^{M, s_1}_{p_1, t} \), it is inferred that \( \pi_2(F^{M, s_0}_{p_0, q_0} \oplus F^{M, s_1}_{p_1, q_1}) \subset F^{M, s_1}_{p_1, t} \) holds for \( \frac{|M|}{r} = \frac{|M|}{p_1} + \frac{|M|}{p_0} + s_0 \) and any \( t \). Using Sobolev embeddings \( F^{M, s_2}_{p_2, q_2} \hookrightarrow F^{M, s_1}_{p_1, t} \) it is not only required that \( \frac{|M|}{r} \geq \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0 \), also \( \frac{|M|}{r} > 0 \) must hold. Then (2.20) gives the intermediate values. — Observe that when \( s_2 \neq |M|(\frac{1}{p_2} - 1) \) one can consider \( \pi_2 \) on the larger space \( F^{M, s_0}_{p_0, \infty} \oplus F^{M, s_1}_{p_1, \infty} \), the embedding procedure above gives the same result, eventually.

To prove (5.17), one can combine \( F^{M, s_0}_{p_0, q_0} \hookrightarrow L_\infty \cap F^{M, 0}_{p_0, 1} \) (when this holds) with (5.2) and (5.4), and it is seen that even \( \frac{|M|}{r} = \frac{|M|}{p_1} \) is possible. For \( s_0 \leq \frac{|M|}{p_0} \), use of (5.3) together with (2.14) gives the lower bound \( \frac{1}{|M|} = \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0 \), except when \( s_0 = \frac{|M|}{p_0} \) where only ‘\(<’ is obtained.

When \( F^{M, s_1}_{p_1, q_1} \hookrightarrow L_\infty \) application of the \( \pi_3 \) version of (5.2) and (5.4) gives \( \pi_3(F^{M, s_0}_{p_0, q_0} \oplus F^{M, s_1}_{p_1, q_1}) \subset F^{M, s_0}_{r_0, q_0} \) for \( \frac{|M|}{r} \leq \frac{|M|}{p_1} \leq \frac{|M|}{p_0} \). From (5.19) holds even with ‘\(<’ replaced by ‘\(\leq\)’. For \( 0 < s_1 < \frac{|M|}{p_1} \) there is an inclusion \( \pi_3(F^{M, s_0}_{p_0, q_0} \oplus F^{M, s_1}_{p_1, q_1}) \subset F^{M, s_0}_{r_0, q_0} \) for \( \frac{|M|}{r} = \frac{|M|}{p_0} + \frac{|M|}{p_1} - s_1 \). Hence (5.19) holds with ‘\(\leq\’) for \( 0 < s_1 < \frac{|M|}{p_1} \) and with ‘\(<’ for \( s_1 = \frac{|M|}{p_1} \). When \( s_1 < 0 \) the last statement in Theorem 5.1 gives that

\[
F^{M, s_0}_{p_0, q_0} \oplus F^{M, s_1}_{p_1, q_1} \xrightarrow{\pi_3} F^{M, s_0 + s_1}_{p_0, q_0} \cap F^{M, s_0}_{t, q_0}
\]

\[
\hookrightarrow \cap \{ F^{M, s_1}_{r, q_0} \mid \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0 \leq \frac{|M|}{t} \leq \frac{|M|}{p_0}, 0 < t \leq \infty \}
\]

(5.32)

The inclusion \( F^{M, 0}_{p_1, q_1} \hookrightarrow \cap \{ F^{M, t, q_0}_{r, q_0} \mid \frac{|M|}{p_1} - \frac{q_0}{2} \leq \frac{|M|}{t} \leq \frac{|M|}{p_0} \} \) combined with the techniques for \( s_1 < 0 \) can be used to show that \( \pi_3(F^{M, s_0}_{p_0, q_0} \oplus F^{M, s_1}_{p_1, q_1}) \) is contained in the space on the right hand side of (5.32) for \( s_1 = 0 \).

Formulæ (5.20)–(5.22) are proved in the same manner: Since (5.5) holds under fewer conditions than (5.6), (5.18) carry over to the Besov case with the modification that \( t \geq q_2 \) is necessary when \( \frac{|M|}{r} = \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0 \geq 0 \), cf. (2.13).

(5.20) and (5.23)–(5.26) are proved in the same way as (5.17) using (5.1), (5.3) and (2.15). The properties (5.22) and (5.27)–(5.31) follow from Theorem 5.1 and (2.19) analogously to (5.19).□

6. SUFFICIENT CONDITIONS FOR MULTIPLICATION

Before we establish the sufficient conditions, an overview of the results in Sections 1 and 3 pertinent to the questions (II) and (III) is given. Recall that \( A_j \), for each \( j = 0, 1, \) and \( 2 \), denotes either \( B^{M, s_j}_{p_j, q_j} \) or \( F^{M, s_j}_{p_j, q_j} \). (In this section all spaces are over \( \mathbb{R}^n \), so for simplicity \( \mathbb{R}^n \) is omitted here.)

Concerning question (II) in Section 1 it is necessary for \( \pi: A_0 \oplus A_1 \to A_2 \) to be bounded that \( (s_j, p_j, q_j)_{j=0,1} \) satisfy (1) and (2) in (4.16), i.e., it is necessary that

\[
s_0 + s_1 \geq |M|(\frac{1}{p_0} + \frac{1}{p_1} - 1)_+.
\]

(6.1)

Here one can denote by \( \mathcal{D}(\pi, BB) \) the domain of parameters \( (s_j, p_j, q_j)_{j=0,1} \) such that there is continuity of \( \pi: B^{M, s_0}_{p_0, q_0} \oplus B^{M, s_1}_{p_1, q_1} \to A_2 \) for some Besov or Triebel–Lizorkin space \( A_2 \); in a similar way one can define domains \( \mathcal{D}(\pi, BF) \) etc. (The
notation should remind one that these domains consists of numbers and not of vectors.)

In this section we shall show that in any of the \( \bullet \bullet \bullet \) cases the sharp inequality in (6.1) is sufficient for the continuity of \( \pi : A_0 \oplus A_1 \rightarrow A_2 \), and we suggest to write

\[
(s_j, p_j, q_j)_{j=0,1} \in \mathbb{D}(\pi) \quad \text{if} \quad s_0 + s_1 > |M| (\frac{1}{p_0} + \frac{1}{p_1} - 1)_+,
\]

\[
(s_j, p_j, q_j)_{j=0,1} \in \mathbb{D}(\pi) \quad \text{if} \quad s_0 + s_1 \geq |M| (\frac{1}{p_0} + \frac{1}{p_1} - 1)_+.
\]

(with a possible omission of “\( j = 0, 1 \)”). Here \( \mathbb{D}(\pi) \subset \mathbb{D}(\pi, \bullet \bullet) \subset \mathbb{D}(\pi) \) (cf. the abovementioned sufficiency and (6.1)). In these terms (1') and (2') in (4.17) state that \( \mathbb{D}(\pi, BB) \not= \mathbb{D}(\pi) \) respectively \( \mathbb{D}(\pi, \bullet \bullet) \not= \mathbb{D}(\pi) \).

We say that \( (s_j, p_j, q_j)_{j=0,1} \in \mathbb{D}(\pi) \) is a pair of generic parameters.

Moreover, when \( \max(s_0, s_1) > 0 \) a complete characterisation of \( \mathbb{D}(\pi, BB) \) and \( \mathbb{D}(\pi, FF) \) is found, cf. Corollary 6.12 below.

When \( \pi \) is defined on \( A_0 \oplus A_1 \) it is a question for which \( A_2 \) there is continuity of \( \pi : A_0 \oplus A_1 \rightarrow A_2 \), cf. (III). To have a convenient notation for this we define the set

\[
\mathbb{P}(A_0, A_1) = \{ (t, r, o) \mid \pi : A_0 \oplus A_1 \rightarrow A_2 \text{ is bounded when } A_2 \text{ has the parameter } (t, r, o) \}.
\]

\( \mathbb{P}(A_0, A_1) \) refers to two specified spaces on which \( \pi(\cdot, \cdot) \) makes sense, cf. (II). To distinguish between the various \( \bullet \bullet \bullet \) and \( \bullet \bullet B \) cases one could write \( \mathbb{P}(A_0, A_1; B) \) and \( \mathbb{P}(A_0, A_1; F) \), respectively, but usually it is unnecessary.

For each \( A_0 \oplus A_1 \) with \( (s_j, p_j, q_j) \in \mathbb{D}(\pi, \bullet \bullet) \) one may wish to determine the set \( \mathbb{P}(A_0, A_1) \), and this will be done below for \( (s_j, p_j, q_j) \in \mathbb{D}(\pi) \) in the isotropic \( FFF \) cases, whereas in the \( BBB \) cases a certain ‘vertex’ question remains open.

However, in the general case much information on \( \mathbb{P}(A_0, A_1) \) is contained in Theorem 4.2 already. Indeed, if \( (s_2, p_2, q_2) \in \mathbb{P}(A_0, A_1) \), then (3)–(7) in (4.16) hold — regardless of which of the \( \bullet \bullet \bullet \) cases that are under consideration. The set that contains \( (s_2, |M|_{p_2}) \) for all \( (s_2, p_2, q_2) \) satisfying (3)–(6) is pictured in Figure 1.

The dashed line in the figure, that corresponds to \( s = \frac{|M|}{p_0} + \min_{j=0,1} (s_j - \frac{|M|}{p_j}) \), cf. (4.20) for the notation, should remind one that it is not always possible to obtain \( (s_2, |M|_{p_2}) \) here for a parameter \( (s_2, p_2, q_2) \in \mathbb{P}(A_0, A_1) \). In fact (7) in (4.16) is a necessary condition on \( A_0 \oplus A_1 \) for this. The “\( \circ \)" at the left vertex has been used to indicate that this point is subject to two set of necessary conditions (the open question for the generic \( BBB \) cases also concerns this point).

For all the \( \bullet \bullet \bullet \) cases it is found when \( (s_j, p_j, q_j) \in \mathbb{D}(\pi) \) that \( \mathbb{P}(A_0, A_1) \) contains all \( (s_2, p_2, q_2) \) with \( (s_2, |M|_{p_2}) \) in the interior of the set in Figure 1 and \( q_2 \in [0, \infty] \).

The possibility of having \( (s_2, p_2, q_2) \in \mathbb{P}(A_0, A_1) \) with \( (s_2, |M|_{p_2}) \) on the boundary is restricted somewhat by the necessary conditions in (7), (3'), (5') and (6').

In the following it is verified that (7) and (3') are both necessary and sufficient for this in the generic isotropic \( FFF \) cases.

For the \( BBB \) cases the four conditions (7), (3'), (5') and (6') are also necessary and sufficient in this respect, except at “\( \circ \)" in Figure 1. At this particular point it is in general further required that \( q_j < \frac{|M|}{p_j} (\frac{|M|}{p_j} - s_j)^{-1} \) for \( j = 0 \) and or 1 when \( s_j < \frac{|M|}{p_j} \), cf. Theorem 6.6. The necessity of these requirements constitute the abovementioned open ‘vertex’ question for the generic \( BBB \) cases.
6.1. The $FFF$ cases. In the rest of this section we assume that $s_0 \geq s_1$ as we may because $\pi$ is commutative on $S' \times S'$. Despite Definition 3.1 above of $\pi(\cdot, \cdot)$ we shall not pay attention to the independence of $\psi$ here, for this will be obtained afterwards in Section 6.4 below.

When $s_0 \geq s_1$ the notation

$$\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}, \quad q = \max(q_0, s_0 = s_1), q_1)$$

will be useful in this and the following subsection.

The next result concerns $F_{p_0, q_0}^{M, s_0} \otimes F_{p_1, q_1}^{M, s_1}$ with $(s_j, p_j, q_j)_{j=0,1} \in \mathbb{D}(\pi)$:

**Theorem 6.1.** Let $M$ and the numbers $s_0, s_1 \in \mathbb{R}$, $p_0, p_1 \in [0, \infty[$ and $q_0, q_1 \in [0, \infty]$ be given and suppose that $s_0 + s_1 > |M| \max(0, \frac{1}{p_0} + \frac{1}{p_1} - 1)$.

For $s_0 \geq s_1$ and with $p, p_1^*$ and $q$ given by (6.5), the product $\pi(\cdot, \cdot)$ is continuous

$$F_{p_0, q_0}^{M, s_0} \otimes F_{p_1, q_1}^{M, s_1} \xrightarrow{\pi(\cdot, \cdot)} \bigcap \{ F_{r, q}^{M, s_1} \mid \frac{|M|}{p_1^*} < \frac{|M|}{r} \leq \frac{|M|}{p} \}. \quad (6.6)$$
Furthermore one can include $|M| = |M|_{p_1}$ when $F_{p_0,q_0}^{M,s_0} \to L_{\infty}$ holds in addition to one of the following conditions:

\begin{align*}
\text{(1a)} & \quad s_1 - \frac{|M|}{p_1} > s_0 - \frac{|M|}{p_0}, \\
\text{(1b)} & \quad s_1 - \frac{|M|}{p_1} \leq s_0 - \frac{|M|}{p_0}, \quad \text{and } F_{p_1,q_1}^{M,s_1} \to L_{\infty} \quad \text{holds},
\end{align*}

and under each of the conditions

\begin{align*}
\text{(2a)} & \quad s_1 > \frac{|M|}{p_1}, \\
\text{(2b)} & \quad s_1 < \frac{|M|}{p_1}, \quad \text{and } F_{p_0,q_0}^{M,s_0} \to L_{\infty} \quad \text{holds if } s_0 = \frac{|M|}{p_0}, \\
\text{(2c)} & \quad s_1 = \frac{|M|}{p_1} \quad \text{and } F_{p_1,q_1}^{M,s_1} \to L_{\infty}, \quad \text{and } F_{p_0,q_0}^{M,s_0} \to L_{\infty} \quad \text{holds if } s_0 = \frac{|M|}{p_0},
\end{align*}

the value $\frac{|M|}{p_1} = \frac{|M|}{p_2}$ can be included when $0 < s_0 \leq \frac{|M|}{p_0}$.

\begin{proof}
It is clear from (5.17)–(5.19) that $A_2$ can be obtained with sum-exponent $q_2 = \min(q_0,q_1,q_1)$. Hence we focus on the $r$-values. When $0 < s_0 \leq \frac{|M|}{p_0}$ the lower bounds for $\frac{|M|}{r}$ in (5.17) and (5.18) are equal to $\frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0$, and they satisfy the inequality

\[0 < \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0 \leq \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0 + (s_1 - \frac{|M|}{p_1}),\]

so the continuity of $\pi_1, \pi_2$ and $\pi_3$ follows under the assumptions in (6.6) and (6.8).

For the case $F_{p_0,q_0} \to L_{\infty}$, note that

\[\left(\frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0\right) \leq \frac{|M|}{p_1} \leq \left(\frac{|M|}{p_1} + (s_1 - \frac{|M|}{p_1} - (s_0 - \frac{|M|}{p_0}))\right),\]

and that the quantities are the lower bounds for $\frac{|M|}{r}$ in (5.18), (5.17) and (6.6) respectively. When $s_1 - \frac{|M|}{p_1} - (s_0 - \frac{|M|}{p_0}) > 0$ the right hand side of (6.10) is equal to the lower bound of $\frac{|M|}{r}$ in (5.19), so this case of (6.6) and (1a) is proved. When $s_1 - \frac{|M|}{p_1} - (s_0 - \frac{|M|}{p_0}) \leq 0$, $\frac{|M|}{p_1}$ is the largest of the lower bounds of $\frac{|M|}{r}$. For $s_0 > \frac{|M|}{p_0}$ it is seen that the lower bound of $\frac{|M|}{r}$ in (5.19) is $\leq \frac{|M|}{p_1}$ with equality only if $s_1 = \frac{|M|}{p_1} = s_0 - \frac{|M|}{p_0}$. But then, since $s_0 > \frac{|M|}{p_0}$, the $L_{\infty}$-condition in (5.19) ff. is satisfied. For $s_0 = \frac{|M|}{p_0}$ the lower bound in (5.19) equals $\frac{|M|}{p_1}$, but it can be included since $F_{p_1,q_1}^{M,s_1} \to L_{\infty}$ if $s_1 = \frac{|M|}{p_1}$.

\end{proof}

\begin{remark}
Concerning the spaces on the right hand side of (6.6) and concerning (6.7) and (6.8) it should be observed explicitly that

- $s_1$ is largest possible index that can occur there by (3) in Theorem 1.2 and the assumption $s_0 \geq s_1$;
- the integral-exponent $r$ must satisfy $\frac{|M|}{r} \leq \frac{|M|}{p}$ according to (4) in (4.16), and for $s_2 = s_1$ it follows from (5) and (6) there that $\frac{|M|}{r} \geq \frac{|M|}{p}$, cf. (4.26);
- for $s_2 = s_1$ the sum-exponent $q_2 = q$ is best possible according to (3'),
- for the cases of (1b), (2b) and (2c) with $s_j = \frac{|M|}{p_j}$, condition (7) in (4.16) gives that $p_j \leq 1$ if $\frac{|M|}{r} = \frac{|M|}{p_1}$ is to be obtained. But for $p_j \leq 1$ one has $F_{p_j,q_1}^{M,s_1} \to L_{\infty}$ except for $M \neq (1, \ldots, 1)$, cf. Remark 4.6. Hence the $L_{\infty}$-conditions in (6.7) and (6.8) are optimal for the isotropic $FFF$ cases.
\end{remark}
Also every $A_2$ space with $s_2 < s_1$ can be obtained from Theorem 6.1. When
the result in the theorem is combined with Sobolev embeddings, it is seen that $A_2$
can be obtained with $(s_2, \frac{|M|}{p^s})$ at (or arbitrarily close to) each point on the line
characterised by $s - \frac{|M|}{p} = \min_{j=0,1}^+ (s_j - \frac{|M|}{p_j})$, and simple embeddings gives that
the set $A_2$ ranges through is unbounded below. Figure 2 illustrates this.

According to (4)–(7) in (4.16) this procedure gives essentially all the possible
cases with $s_2 < s_1$; only when $M \neq (1, \ldots, 1)$ and the relevant $(s_j, p_j, q_j)$ equals
$(|M|, 1, q)$ for $1 < q \leq \infty$ is it open whether some of the $L_\infty$-conditions stemming
from (6.7) and (6.8) above can be relaxed for the $A_2$ spaces with $s_2 < s_1$. For
$M = (1, \ldots, 1)$ they cannot.

These additional results are summed up as follows:

**Theorem 6.3.** With assumptions as in Theorem 6.1 the product $\pi(\cdot, \cdot)$ is bounded
for each $s_2 < s_1$ and $o \in [0, \infty]$

$$F_{p_0,q_0}^{M,s_0} \oplus F_{p_1,q_1}^{M,s_1} \xrightarrow{\pi(\cdot, \cdot)} \bigcap \left\{ F_{r,o}^{M,s_2} \mid \frac{|M|}{p^r} - (s_2) > 0 \right\} \subseteq L_\infty$$

In addition, when $\frac{|M|}{p^1} - s_1 + s_2 > 0$, this value of $\frac{|M|}{p^1}$ may be included when
$F_{p_0,q_0}^{M,s_0} \xrightarrow{\pi(\cdot, \cdot)} L_\infty$ and $0 < s_0 \leq \frac{|M|}{p^0}$ under each of the conditions in (6.7) and (6.8)
respectively.

Altogether it can be concluded for the isotropic $FFF$ cases, where $M = (1, \ldots, 1)$,
that the set $\mathcal{P}(F_{p_0,q_0}^{M,s_0}, F_{p_1,q_1}^{M,s_1})$ is completely described by Theorems 6.1 and 6.3 when
$(s_j, p_j, q_j)_{j=0,1} \in \mathbb{D}(\pi)$. 

![Figure 2. Embeddings giving the structure of $\mathbb{P}(\cdot, \cdot)$](image-url)
For the borderline cases with $s_0 = -s_1 = s > 0$ condition (1) in Theorem 4.2 reduces to $1 \geq \frac{1}{p_0} + \frac{1}{p_1}$. Concerning the notation in (6.5) we observe that $p_1^* = p_1$ for $F_{p_0 q_0}^{M, s_0} \to L_\infty$ whilst $\frac{|M|}{p_1} = \frac{|M|}{p_1^*} + \frac{|M|}{p_0} - s$ for $0 < s_0 \leq \frac{|M|}{p_0}$. Moreover, for $s = \frac{|M|}{p_0} \geq |M|$ there is not any space $F_{p_1 q_1}^{M, -s}$ such that (1) in Theorem 4.2 holds (since $0 < p_1 < \infty$); hence the case with $F_{p_0 q_0}^{M, s} \to L_\infty$ is reduced to $s > \frac{|M|}{p_0}$.

**Theorem 6.4.** Let $(s, p_0, q_0)$ and $(-s, p_1, q_1)$ satisfy the three inequalities $s > 0$, $1 \geq \frac{1}{p_0} + \frac{1}{q_0}$ and $\frac{1}{p_1} + \frac{1}{q_1} \geq 1$. Then $\pi(\cdot, \cdot)$ is bounded

$$F_{p_0 q_0}^{M, s} \oplus F_{p_1 q_1}^{M, -s} \xrightarrow{\pi(\cdot, \cdot)} \bigcap \{ F_{r, q_1}^{M, -s} \mid \frac{|M|}{p_1} < \frac{|M|}{r} \leq \frac{|M|}{p} \}.$$  \hspace{1cm} (6.12)

When $s > \frac{|M|}{p_0}$, or when $s < \frac{|M|}{p_0}$ and $\frac{1}{p_0} + \frac{1}{p_1} < 1$, the space $F_{p_1 q_1}^{M, -s}$ may be included, whereas when $s < \frac{|M|}{p_0}$ and $\frac{1}{p_0} + \frac{1}{p_1} = 1$ the space $B_{p_1, \infty}^{M, -s}$ can be used instead.

**Proof.** We have by Proposition 2.12 continuous mappings

$$F_{p_0 q_0}^{M, s} \oplus F_{p_1 q_1}^{M, -s} \xrightarrow{\pi(\cdot, \cdot)} L_p \hookrightarrow \begin{cases} F_{r, q_1}^{M, -s} & \text{for } p > 1, \\ B_{r, \infty}^{M, -s} & \text{for } p = 1, \end{cases} \hspace{1cm} (6.13)$$

when $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$, $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}$ and $\frac{1}{r_0} = \frac{|M|}{p_1} = \frac{|M|}{p_0} + \frac{|M|}{p_0} - s$ (since the estimate there is shown as in (6.11) ff.). Moreover $L_p \hookrightarrow \bigcap \{ F_{r, q_1}^{M, -s} \mid \frac{|M|}{r} < \frac{|M|}{p} \leq \frac{|M|}{p} \}$ for $p \geq 1$. Since $s_1 < 0$ the same result holds for $\pi_3(\cdot, \cdot)$, and when this is combined with (6.1) for $\pi_1(\cdot, \cdot)$ the theorem follows.

The optimality of the receiving spaces above may be verified in the same way as for Theorem 6.1 in Remark 6.2 if, except for the statement on $B_{p_1, \infty}^{M, -s}$. Also in this case does one get $A_2$ spaces with $s_2 < s_1$ by means of embeddings.

Concerning earlier treatments of $FFF$ cases with $s_0 + s_1 = 0$ we mention [GS91], where $F_{2,2}^{2,2} \oplus F_{2,2}^{M, -s}$ with $M = (1, \ldots, 1, m_n)$ is given a treatment directly based on the Fourier transformation.

For the borderline cases with $s_0 + s_1 = \frac{|M|}{p_0} + \frac{|M|}{p_1} - |M|$ one can assume that $s_0 + s_1 > 0$ since the case with $s_0 + s_1 = 0$ is covered above (for $s_0 > 0$ at least). Then $\frac{1}{p} := \frac{1}{p_0} + \frac{1}{p_1} > 1$. Here $p_1^* = p_1$ for $F_{p_0 q_0}^{M, s_0} \to L_\infty$ while $\frac{|M|}{p_1} = \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0$ for $0 < s_0 \leq \frac{|M|}{p_0}$.

**Theorem 6.5.** Let $s_0 + s_1 = \frac{|M|}{p_0} + \frac{|M|}{p_1} - |M| > 0$ and suppose that $s_1 < 0$. Then $\pi(\cdot, \cdot)$ is bounded

$$F_{p_0 q_0}^{M, s_0} \oplus F_{p_1 q_1}^{M, s_1} \xrightarrow{\pi(\cdot, \cdot)} \bigcap \{ F_{r, q_1}^{M, s_1} \mid \frac{|M|}{r_0} < \frac{|M|}{r} \leq \frac{|M|}{p_0} \}. \hspace{1cm} (6.14)$$

When $F_{p_0 q_0}^{M, s_0} \to L_\infty$ the space $F_{p_1 q_1}^{M, s_1}$ may be included for $s_0 > \frac{|M|}{p_0}$, and $B_{p_1, \infty}^{M, s_1}$ can receive for $s_0 = \frac{|M|}{p_0}$. Similarly $B_{p_1, \infty}^{M, s_1}$ may be included for $0 < s_0 < \frac{|M|}{p_0}$.

**Proof.** For $\pi_2$ one has continuous mappings (with $\frac{|M|}{r_0} = \frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0$)

$$F_{p_0 q_0}^{M, s_0} \oplus F_{p_1 q_1}^{M, s_1} \xrightarrow{\pi_2} L_1 \hookrightarrow B_{r_0, \infty}^{M, s_1} \cap \bigcap \{ F_{r, q_1}^{M, s_1} \mid r_0 \geq r \geq 1 \} \hspace{1cm} (6.15)$$
we focus on the modifications that lead to (6.18) and (6.19). In general $q \leq 1$ may be achieved from Sobolev embeddings of the $F_{p_j,q_j}^{M,s_j}$. Since $s_1 < 0$ the proof may be conducted along the lines of that of Theorem 6.3.

On one hand it is not clear whether the $B_{p_1,\infty}^{M,s_1}$ are optimal, on the other hand — by a Sobolev embedding of $F_{p_j,q_j}^{M,s_j}$ — the cases with $s_0 \geq s_1 \geq 0$ and

$$s_0 + s_1 = \frac{|M|}{p_0} + \frac{|M|}{p_1} - |M| > 0$$

are covered too, but only with receiving $A_2$ spaces for which $s_2 < \min(s_0, s_1)$.

However, this suffices to see that $(s_j, p_j, q_j)_{j=0,1} \in \mathbb{D}(\pi, FF)$ when (6.16) holds.

6.2. The $BBB$ cases. We shall now modify the arguments in the subsection above and obtain analogous results for the Besov spaces.

**Theorem 6.6.** Let $M$ and the numbers $s_0$ and $s_1 \in \mathbb{R}$ and $p_0, p_1, q_0$ and $q_1 \in [0, \infty]$ be given such that $s_0 + s_1 > |M| \max(0, \frac{1}{p_0} + \frac{1}{p_1} - 1)$.

For $s_0 \geq s_1$ and with $p, p_1^*$ and $q$ given by (6.50), the product $\pi(\cdot, \cdot)$ is continuous

$$B_{p_0,q_0}^{M,s_0} \oplus B_{p_1,q_1}^{M,s_1}(\pi(\cdot)) \cap \{ B_{p,q}^{M,s_1} \mid \frac{|M|}{p_1} < \frac{|M|}{p} \leq \frac{|M|}{p_0} \}.$$ (6.17)

Furthermore one can include $B_{p_1,\infty}^{M,s_1}$ on the right hand side of (6.17) when $B_{p_0,q_0}^{M,s_0} \rightarrow L_\infty$ holds in addition to one of the following conditions:

1. $s_1 - \frac{|M|}{p_1} \geq s_0 - \frac{|M|}{p_0}, \quad o \geq \max(q_0, q_1), \quad$ and $B_{p_1,q_1}^{M,s_1} \rightarrow L_\infty$ if $s_1 = \frac{|M|}{p_1}$,
2. $0 \leq s_1 - \frac{|M|}{p_1} < s_0 - \frac{|M|}{p_0}, \quad o \geq q$,
3. $s_1 - \frac{|M|}{p_1} < s_0 - \frac{|M|}{p_0}, \quad 0 < s_1 < \frac{|M|}{p_1}, \quad o \geq q$,
   and $q_1 < t_1$ if $s_1 = s_0 = \frac{|M|}{p_0}$,
4. $s_1 - \frac{|M|}{p_1} < s_0 - \frac{|M|}{p_0}, \quad s_1 \leq 0, \quad s_1 < \frac{|M|}{p_1}$ and $o \geq q_1$,

(6.18)

where $t_j = |M|(\frac{|M|}{p_j} - s_j)^{-1}$ for $j = 0$ and 1, and under each of the conditions

1. $s_1 > \frac{|M|}{p_1}$ and $o \geq \max(q_0, q_1)$,
2. $s_1 = \frac{|M|}{p_1}, \quad B_{p_1,q_1}^{M,s_1} \rightarrow L_\infty, \quad s_0 < \frac{|M|}{p_0}, \quad o \geq \max(q_0, q_1)$ and $q_0 < t_0$,
3. $0 < s_1 < \frac{|M|}{p_1}, \quad s_0 < \frac{|M|}{p_0}, \quad s_0 > s_1, \quad o \geq q_1$ and $q_0 < t_0$,
4. $0 < s_1 < \frac{|M|}{p_1}, \quad s_0 = s_1, \quad o \geq q_j$ and $q_j < t_j$ for $j = 0, 1$,
5. $s_1 < \frac{|M|}{p_1}, \quad s_0 < \frac{|M|}{p_0}, \quad s_1 \leq 0, \quad o \geq q_1$ and $q_0 < t_0$,

(6.19)

the space $B_{p_1,\infty}^{M,s_1}$ can be included when $0 < s_0 \leq \frac{|M|}{p_0}$.

In (6.18) and (6.19) it suffices with $q_j = t_j$ if either $t_j \leq 2$ or $M = (1, \ldots, 1)$.

**Proof.** Since (6.17) may be obtained analogously to the corresponding $FFF$ cases, we focus on the modifications that lead to (6.18) and (6.19).

In (1a) one can use that $B_{p_j,q_j}^{M,s_j} \rightarrow L_\infty$ for $j = 0$ and 1, whereby the restriction $o \geq q_1$ occurs in the treatment of $\pi_1(\cdot, \cdot)$ and $o \geq q_0$ comes from $\pi_2(\cdot, \cdot)$, cf. (5.20), (5.23) (5.24), respectively (5.27) (5.28). To treat (1b) for $s_1 = \frac{|M|}{p_1} > 0$ is simpler...
since \((\frac{|M|}{p_1} + \frac{|M|}{p_0} - s_0)_+ < \frac{|M|}{p_1}\) (so that it suffices to obtain \(\frac{|M|}{p_1}\) equal to the lower bound in (6.20) only). For \(s_1 = \frac{|M|}{p_1} = 0\) one can apply (5.31) for the estimate of \(\pi_3\).

For \(s_0 > \frac{|M|}{p_0}\) condition (1c) is easy since (5.23), (5.21) and (5.22) provide the necessary results. However, for \(s_0 = \frac{|M|}{p_0}\) the three lower bounds of \(\frac{|M|}{p_1}\) in (5.20)–(5.22) all coincide with \(\frac{|M|}{p_1}\) but at least for \(s_1 = s_0\) one can apply (5.29).

The case with \(s_1 < s_0 = \frac{|M|}{p_0}\) requires a special treatment of \(\pi_3\) (to allow \(q_1 \geq t_1\)). Note that \(A_1 \hookrightarrow B^{M, -\varepsilon}_{p_1, q_1}\) provided \(s_1 - \frac{|M|}{p_1} \geq -\varepsilon - \frac{|M|}{p_0} + \frac{s_0}{p_0} + \frac{s_1}{p_1} + \frac{s_0}{p_0}\), so that under the restriction \(\varepsilon \in [0, \frac{|M|}{p_1} - s_1]\) this embedding exists for \(\frac{|M|}{p_1} = \frac{|M|}{p_1} - s_1 - \varepsilon\). We shall now take \(\varepsilon < s_0 - s_1\) and show that, with \(\frac{|M|}{r_0} = \frac{|M|}{p_0} + \frac{|M|}{p_1}\) and \(\frac{1}{q_2} = \frac{1}{q_0} + \frac{1}{q_1}\),

\[
A_0 \oplus A_1 \hookrightarrow A_0 \oplus B^{M, -\varepsilon}_{r_0, q_1} \hookrightarrow B^{M, s_0 - \varepsilon}_{r_0, q_2} \hookrightarrow B^{M, s_1}_{r_0, q_2}. \tag{6.20}
\]

According to Theorem 6.1 it is sufficient to verify the last embedding, i.e.,

\[
s_0 - \varepsilon - \frac{|M|}{r_0} \geq s_1 - \frac{|M|}{p_1}, \quad \frac{|M|}{r_0} \geq \frac{|M|}{p_1}. \tag{6.21}
\]

Note that \(\frac{|M|}{r_0} = \frac{|M|}{p_0} + \frac{|M|}{r_1} - s_1 - \varepsilon\), so the former inequality reduces to \(s_0 \geq \frac{|M|}{p_0}\) and the latter to \(\frac{|M|}{p_1} - s_1 \geq \varepsilon\).

Concerning (1d) it is seen that (5.22) suffices for \(\pi_3\) when \(s_0 > \frac{|M|}{p_0}\), whereas (5.31) applies for \(s_0 = \frac{|M|}{p_0}\).

(2a) is based on (5.27) and (2b) combines (5.24) and (5.28). Concerning (2c) the cases \(q_1 \geq t_1\) are included as in (1c) above, except that the embedding \(B^{M, s_0 - \varepsilon}_{r_0, q_2} \hookrightarrow B^{M, s_1}_{r_0, q_2}\) is wanted. However, this is obtained since \(\frac{|M|}{r_1} = \frac{|M|}{r_0} + s_1 + \varepsilon - s_0 < \frac{|M|}{r_0}\) (using \(\varepsilon < s_0 - s_1\)) and since \(s_0 - \varepsilon - \frac{|M|}{r_0} = s_0 - \varepsilon - \frac{|M|}{p_0} - \frac{|M|}{p_1} - s_1 - \varepsilon = s_1 - \frac{|M|}{p_1}\).

From (5.25) and (5.29) one derives (2d), and (5.31) may be used for (2e).

The final statement in the theorem is based on (5.26) and (5.30). \(\Box\)

Remark 6.7. As for the \(FFF\) cases it is seen that the spaces on the right hand sides of (6.17) can neither have smoothness indices larger than \(s_1 = \min(s_0, s_1)\) nor have integral-exponents outside \(\frac{|M|}{p_1}, \frac{|M|}{p_0}\) when the index is \(s_1\). And, still for index \(s_1\), the sum-exponent can not be lower than \(q\).

The possibility of obtaining \(\frac{|M|}{r_1}\) equal to \(\frac{|M|}{p_1}\) is more delicate. Observe that the conditions on \(s_j - \frac{|M|}{p_j}\) and \(s_j\) in (1a)–(1d) and (2a)–(2e) exhaust the possibilities. (In (1d) the subcase \(s_1 = 0 = \frac{|M|}{p_1}\) of (1b) is omitted, and in (2b)–(2e) the possibility \(s_0 = \frac{|M|}{p_0}\) is excluded because of overlap with the case \(B^{M, s_0}_{p_0, q_0} \rightarrow L_\infty\).) Concerning the optimality of (6.18) and (6.19) we have that

- \(o \geq \max(q_0, q_1)\) in (1a), (2a) and (2b) is necessary by (3') and (5') in Theorem 4.2 and in (2d) by (3') applied for \(j = 0\) and \(j = 1\). Similarly \(o \geq q\) is necessary in (1b) and (1c) and \(o \geq q_1\) in (1d), (2c) and (2e);
- the \(L_\infty\)-conditions in (1a) and (2b) are unremovable by (7) in (4.10) and Remark 4.6.
- the conditions \(q_j < t_j\) occurring in (1c) and (2b)–(2e) are not in the present paper shown to be necessary.
For case (2d) there are counterexamples in [ST] that show for $M = (1, \ldots, 1)$ that $q_0 \leq t_0$ and $q_1 \leq t_1$ must hold, but for the other cases it seems to be an open problem whether these conditions are necessary.

**Remark 6.8.** The conditions $q_j < t_j$ are connected to the question of having $A_2$ represented by the “$\circ$” in Figure 14 and the answer depends on $A_0 \oplus A_1$ instead of $A_2$, contrary to the rest of the line segment with $s = \min(s_0, s_1)$. It should be observed that such conditions apply only for one value of $j$, except for (2d) where $s_0 = s_1$.

For this reason the cases with $s_0 > s_1 \geq 0$ can not be reduced to those with $s_0 = s_1$ by means of a Sobolev embedding $B^{M,s_0}_{p_0,q_0} \hookrightarrow B^{M,s_1}_{p_1,q_1}$.

Because of the condition on the sum-exponents in the Sobolev embeddings for the Besov spaces, the question of having $A_2$ with $s_2 < s_1 = \min(s_0, s_1)$ and $|M| = s_2 - \min^+(s_j - |M|/p_j)$ is more complicated for the BBB cases than for the $FFF$ cases.

However it suffices to use embeddings of $B^{M,s_0}_{p_0,q_0}$ and $B^{M,s_1}_{p_1,q_1}$.

**Theorem 6.9.** With assumptions as in Theorem 6.6, the product $\pi(\cdot, \cdot)$ is bounded for each $s_2 < s_1$ and $o \in [0, \infty]$\[ B^{M,s_0}_{p_0,q_0} \oplus B^{M,s_1}_{p_1,q_1} \overset{\pi(\cdot, \cdot)}{\longrightarrow} \bigcap \{ B^{M,s_2}_{r,o} \mid \|M/p_1 - (s_1 - s_2)\| < |M|/r \leq |M|/p \} \quad (6.22) \]

Moreover, when $|M|/p_1 - s_1 + s_2 \geq 0$ the space $B^{M,s_2}_{r,o}$ with $|M|/r = |M|/p_1 - s_1 + s_2$ may be included if $B^{M,s_0}_{p_0,q_0} \rightarrow L_\infty$ holds in addition to one of the conditions

\begin{align*}
(1a) & \quad s_1 - |M|/p_1 > s_0 - |M|/p_0, \quad o \geq q_0, \\
(1b) & \quad s_1 - |M|/p_1 = s_0 - |M|/p_0, \quad o \geq \max(q_0, q_1), \quad \text{and } B^{M,s_1}_{p_1,q_1} \rightarrow L_\infty \quad \text{if } s_1 = |M|/p_1, \\
(1c) & \quad s_1 - |M|/p_1 < s_0 - |M|/p_0, \quad o \geq q_1, \quad (6.23)
\end{align*}

and when $|M|/p_1 - s_1 + s_2 \geq 0$ each of the conditions

\begin{align*}
(2a) & \quad s_1 > |M|/p_1 \quad \text{and } o \geq q_0, \\
(2b) & \quad s_1 = |M|/p_1, \quad B^{M,s_1}_{p_1,q_1} \rightarrow L_\infty, \quad s_0 < |M|/p_0, \quad o \geq q_0, \\
(2c) & \quad s_1 < |M|/p_1, \quad s_0 < |M|/p_0, \quad o \geq (\frac{q_0}{q_0 + q_1})^{-1} \quad (6.24)
\end{align*}

allow the space $B^{M,s_2}_{r,o}$ with $|M|/r = |M|/p_1 - s_1 + s_2$ to be included when $0 < s_0 \leq |M|/p_0$.

**Proof.** Formula (6.22) is obtained from Theorem 6.6 by means of embeddings, cf. Figure 2. Hence it remains to show the sufficiency of the conditions (1a)–(2c).

In case (1a) the receiving space $A_2 = B^{M,s_2}_{r,o}$ may be obtained for $|M|/r = |M|/p_1 - s_1 + s_2$ and $o = \max(q_0, q_1)$ by means of a Sobolev embedding of the space covered by Theorem 6.6. To obtain $o$ as low as $q_0$ it suffices to improve the estimate of $\pi_1$, for the $\pi_2$ estimate gives $o = (\frac{q_0}{q_0 + q_1})^{-1}$ or better.

It is enough to treat $s_2 = s_1 - \varepsilon$ for arbitrarily small $\varepsilon > 0$. By (5.20)\[ A_0 \oplus A_1 \hookrightarrow B^{M,s_0}_{p_0,q_0} \oplus B^{M,s_1}_{p_1,q_1 - \varepsilon} \overset{\pi(\cdot, \cdot)}{\longrightarrow} \bigcap \{ B^{M,s_2}_{r,q_0} \mid |M|/p_1 < |M|/r \leq |M|/p_0 + |M|/p_1 \} \quad (6.25) \]
Taking $\varepsilon$ so small that $s_1 - \varepsilon - \frac{|M|}{p_1} > s_0 - \frac{|M|}{p_0}$ the value $\frac{|M|}{p_1} = \frac{|M|}{p_0} - \varepsilon = \frac{|M|}{p_0} - s_0 + s_1 - \varepsilon$ is included on the right hand side. This proves (1a) and (1b) is evident from Theorem 6.6.

For the treatment of (1c) it is necessary to modify the estimate of $\pi_3(\cdot\cdot\cdot)$. Consider $s_2 = s_1 - \varepsilon$ for small $\varepsilon \in [0, \frac{|M|}{p_1} - s_1]$ and take $r \in [p_1, \infty]$ such that $s_1 - \frac{|M|}{p_1} = -\varepsilon - \frac{|M|}{r}$. Then $B_{p_1,q_1}^{M,s_1} \hookrightarrow B_{r,q_1}^{s_1-\varepsilon}$. From the last statement in Theorem 5.4 it is seen that $\pi_3$ maps $B_{p_0,q_0}^{M,s_0} \oplus B_{r,q_1}^{M,s_1} \hookrightarrow B_{r,q_0}^{s_0-\varepsilon}$ when $\frac{|M|}{r_0} = \frac{|M|}{p_0} + \frac{|M|}{r}$ and $o = \left(\frac{1}{p_1} + \frac{1}{q_1}\right)^{-1}$. But

$$ s_0 - \varepsilon - \frac{|M|}{r_0} = s_0 - \frac{|M|}{p_0} + s_1 - \frac{|M|}{p_1} = s_1 - \frac{|M|}{p_1} = s_2 - \frac{|M|}{p_1} - s_1 + s_2, \quad (6.26) $$

so $\pi_3$ maps $B_{p_0,q_0}^{M,s_0} \oplus B_{p_1,q_1}^{M,s_1}$ into $B_{r,q_0}^{M,s_2}$ for $\frac{|M|}{p_1} = s_1 + s_2$.

For (2a) the treatment of $\pi_1$ in case (1a) is easily modified by taking $\varepsilon$ such that $s_1 - \varepsilon > \frac{|M|}{p_1}$. In (2b) one can handle $\pi_1$ by means of (6.7): For $s_2 = s_1 - \varepsilon$ with a small $\varepsilon \in [0, \frac{|M|}{p_0} - s_0]$ the number $r \in [p_0, \infty]$ such that $s_0 - \frac{|M|}{p_0} = -\varepsilon - \frac{|M|}{r}$ is considered. Then $\pi_1(B_{p_0,q_0}^{M,s_0} \oplus B_{p_1,q_1}^{M,s_1}) \subset B_{r,q_0}^{M,s_0-\varepsilon}$ for $\frac{|M|}{r_0} = \frac{|M|}{p_0} + \frac{|M|}{r}$ and $o = \left(\frac{1}{p_0} + \frac{1}{q_0}\right)^{-1}$. Here $\frac{|M|}{r_0} = \frac{|M|}{p_1} - \varepsilon$ is equal to the value of $\frac{|M|}{r_1}$ pertinent to (6.24).

In (2c) the case with $s_1 < 0$ is easy concerning $\pi_3$ since we already have a receiving space (for this operator) with sum-exponent $(\frac{1}{p_0} + \frac{1}{q_0})^{-1}$, cf. (5.31). Observe that $\pi_1$ can be treated as in case (2b), and that for $s_1 \geq 0$ one may treat $\pi_3$ along these lines too. Indeed, if $\varepsilon \in [0, \frac{|M|}{p_0} - s_1]$ and $s_1 - \frac{|M|}{p_1} = -\varepsilon - \frac{|M|}{r}$ it is found that $\pi_1(B_{p_0,q_0}^{M,s_0} \oplus B_{p_1,q_1}^{M,s_1}) \subset B_{r,q_0}^{M,s_0-\varepsilon}$ for $\frac{|M|}{r_0} = \frac{|M|}{p_0} + \frac{|M|}{r}$. Here $\frac{|M|}{r_0} = \frac{|M|}{p_1} - \varepsilon + s_0 - s_1$, and hence $B_{r,q_0}^{M,s_0-\varepsilon} \hookrightarrow B_{r,q_0}^{M,s_1-\varepsilon}$ for $\frac{|M|}{r_1} = \frac{|M|}{p_1} - \varepsilon$.

The optimality of (6.22)–(6.24) is seen from (4)–(7), (5’), and (6’) in Theorem 4.2, cf. also (4.26).

Thus Theorem 6.9 gives a complete description of the $(s_2,p_2,q_2) \in F(B_{p_0,q_0}^{M,s_0}, B_{p_1,q_1}^{M,s_1})$ with $s_2 < \min(s_0,s_1)$ when $(s_j,p_j,q_j)_{j=0,1} \in D(\pi)$.

The borderline cases with $s_0 + s_1 = \frac{|M|}{p_0} + \frac{|M|}{p_1} - |M|$, are now given a treatment analogous to the one in Section 6.1.

**Theorem 6.10.** Let $(s_{p_0,q_0})$ and $(-s_{p_1,q_1})$ satisfy the three inequalities $s > 0$, $1 \geq \frac{1}{p_0} + \frac{1}{p_1}$ and $\frac{1}{p_0} + \frac{1}{p_1} \geq 1$, and recall (6.3). Then $\pi(\cdot\cdot\cdot)$ is bounded

$$ B_{p_0,q_0}^{M,s} \oplus B_{p_1,q_1}^{M,-s} \xrightarrow{\pi(\cdot\cdot\cdot)} \bigcap \{ B_{r,q_1}^{M,s} \mid \frac{|M|}{p_1} < \frac{|M|}{r} \leq \frac{|M|}{p_1} \}. \quad (6.27) $$

In (6.27) the space $B_{p_1,q_1}^{M,-s}$ may be included if $B_{p_0,q_0}^{M,s} \hookrightarrow L_\infty$ holds together with one of the conditions

(1a) $s > \frac{|M|}{p_0}$ and $o \geq q_1$, 
(1b) $s = \frac{|M|}{p_0}$, $p > 1$ and $o \geq \max(q_1,p + \varepsilon)$ for $\varepsilon > 0$, 
(1c) $s = \frac{|M|}{p_0}$, $p = 1$ and $o = \infty$. 


and if one of the following conditions (where \( t_0 = |M| \left( |M| - s \right)^{-1} \))

\[
\begin{align*}
(2a) \quad & s < \frac{|M|}{p_0}, \quad q_0 < t_0, \quad p > 1 \quad \text{and} \quad o \geq \max(q_1, p + \varepsilon), \\
(2b) \quad & s < \frac{|M|}{p_0}, \quad q_0 < t_0, \quad p = 1 \quad \text{and} \quad o = \infty,
\end{align*}
\]

(6.29)

holds, the space \( B_{\pi_{q_0}}^{M,-s} \) can receive when \( 0 < s < \frac{|M|}{p_0} \).

It suffices with \( \varepsilon = 0 \) above if either \( p \geq 2 \) or \( M = (1, \ldots, 1) \), and it suffices with \( q_0 \leq t_0 \) if either \( t_0 \leq 2 \) or \( M = (1, \ldots, 1) \).

Proof. The property analogous to the one in (6.13) is

\[
B_{\pi_{q_0}}^{M,s} \oplus B_{\pi_{q_1}}^{M,-s} \pi_2(\cdot, \cdot) \xrightarrow{L_p} \bigcap \{ B_{\pi_{q_1}}^{M,s} \mid \frac{|M|}{p_0} < \frac{|M|}{r} \leq |M| \}. \tag{6.30}
\]

for \( \varepsilon > 0 \); even \( \varepsilon = 0 \) is possible if either \( p \geq 2 \) or \( M = (1, \ldots, 1) \). \( \square \)

It is not clear whether the sum-exponents \( o \) in (1a)–(1c) and (2a)–(2b) are optimal. For this reason, we shall not treat \( A_2 \) spaces with \( s_2 < \min(s_0, s_1) \) here; results may be obtained by use of the same methods as for Theorem 6.9 when needed.

Theorem 6.11. Let \( s_0 + s_1 = \frac{|M|}{p_0} + \frac{|M|}{p_1} - |M| > 0 \) and suppose that \( s_1 < 0 \) and \( \frac{1}{q_0} + \frac{1}{q_1} \geq 1 \). Then \( \pi(\cdot, \cdot) \) is bounded

\[
B_{\pi_{q_0}}^{M,s_0} \oplus B_{\pi_{q_1}}^{M,s_1} \pi_2(\cdot, \cdot) \xrightarrow{L_1} \bigcap \{ B_{\pi_{q_1}}^{M,s_1} \mid r_0 > r \geq 1 \}. \tag{6.31}
\]

Moreover, when \( B_{\pi_{q_0}}^{M,s_0} \rightarrow L_\infty \) the space \( B_{\pi_{q_1}}^{M,s_1} \) can receive in (6.31) for \( s_0 > \frac{|M|}{p_0} \), while \( B_{\pi_{q_1}}^{M,s_1} \) can do so for \( s_0 = \frac{|M|}{p_0} \).

Similarly \( B_{\pi_{q_1}}^{M,s_1} \) may be included when \( 0 < s_0 < \frac{|M|}{p_0} \) holds in addition to \( q_0 < \frac{|M|}{p_0} = |M| - s_0^{-1} \) (or just \( q_0 \leq t_0 \) if \( t_0 \leq 2 \) or if \( M = (1, \ldots, 1) \)).

Proof. For the treatment of \( \pi_2 \) one can use the continuity of

\[
B_{\pi_{q_0}}^{M,s_0} \oplus B_{\pi_{q_1}}^{M,s_1} \pi_2 \xrightarrow{L_1} B_{\pi_{q_1}}^{M,s_1} \cap \bigcap \{ B_{\pi_{q_1}}^{M,s_1} \mid r_0 > r \geq 1 \}. \tag{6.32}
\]

that one has in view of the condition \( \frac{1}{q_0} + \frac{1}{q_1} \geq 1 \), cf. Proposition 2.6. \( \square \)

It is seen by use of embeddings that all the borderline cases with

\[
s_0 + s_1 = \frac{|M|}{p_0} + \frac{|M|}{p_1} - |M| \quad \text{and} \quad \frac{1}{q_0} + \frac{1}{q_1} \geq 1 \tag{6.33}
\]

allow application of \( \pi(\cdot, \cdot) \), that is, (6.33) implies that \((s_j, p_j, q_j)_{j=0,1} \in \mathbb{D}(\pi, BB)\).

However, for \( 0 < s_1 < s_0 \) the resulting \( A_2 \) spaces with \( s_2 < 0 \leq \min(s_0, s_1) \) are not optimal, cf. [Ama91 Thm. 4.1].

Altogether we have now in Sections 6.1 and 6.2 given a fairly complete description of \( \mathbb{D}(\pi, BB) \) and \( \mathbb{D}(\pi, FF) \). In fact, using (6.9) in Theorem 6.1 for the generic \( FF^* \) cases, (6.12) for the cases with \( s_0 + s_1 = 0 \) and (6.10) ff. — and analogously for the \( BB^* \) cases — we have found

**Corollary 6.12.** When \( \max(s_0, s_1) > 0 \) in the \( BB^* \) or in the \( FF^* \) cases, the simultaneous fulfillment of the conditions (1), (1′), (2) and (2′) in Theorem 4.2 is necessary and sufficient for the boundedness of \( \pi(\cdot, \cdot) \) in (1.6).
Remark 6.13. The cases with \(s_0 = 0 = s_1\) are treated in the preprint [ST], where it is shown that \(A_j \hookrightarrow L_{p_j}\) for \(j = 0\) and \(1\) together with \(L_0 \hookrightarrow A\) must hold when (1.6) and \(s_0 = 0 = s_1\) do so. By Hölder’s inequality these embeddings conversely imply (1.6) in this case, but the author is unable to follow the present proof in [ST] for the cases with \(p_j = 1\). Hence a complete description of \(D(\pi, BB)\) and \(D(\pi, FF)\) is left for the future.

6.3. The mixed cases. Concerning the remaining \(BB, BF, BF^\bullet, FB^\bullet\) and \(FF\) cases we shall verify the sufficiency claimed above (6.2): Theorem 6.1 shows that \((s_j, p_j, q_j) \in D(\pi, FF)\) if \((s_j, p_j, q_j) \in D(\pi)\). But then \((s_j, p_j, q_j) \in D(\pi, \bullet, \bullet)\), so that there is in any case a simple embedding \(A_0 \oplus A_1 \hookrightarrow F^0_{p_0, \infty} - \varepsilon + F^0_{p_1, \infty}\), where \((s_j - \varepsilon, p_j, 1)_{j=0,1} \in D(\pi)\) when \(\varepsilon > 0\). Thus \(D(\pi) \subset D(\pi, \bullet, \bullet)\).

In addition it is clear from this that (by taking \(\varepsilon > 0\) sufficiently small) \(A_2\) can be obtained with any parameter \((s_2, p_2, q_2)\) for which \(s_2 < s_1\), \(\frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p}\) and \(s_2 - |M| = 0\). Thus \(D(\pi) \subset D(\pi, \bullet, \bullet)\), cf. the interior of the region sketched in Figure 1.

6.4. \(\psi\)-independence. Strictly speaking, boundedness has in this section only been obtained for \(\pi_{\psi}((\cdot, \cdot))\) with \(\psi\) as an arbitrary function entering in Definition 3.1.

For \((s_j, p_j, q_j)_{j=0,1} \in D(\pi, \bullet, \bullet)\) and \((s_2, p_2, q_2) \in \mathcal{P}(A_0, A_1)\) we shall now verify that the action of \(\pi_{\psi}((\cdot, \cdot))\) on \(A_0 \oplus A_1 \rightarrow A_2\) does not depend on \(\psi\) (except for a \(BF^\bullet\) borderline case with \(p_1 = 1, q_0 = q_1 = \infty\) that is left open for \(M \neq (1, \ldots, 1)\)).

When applying \(\pi_{\psi}((\cdot, \cdot))\) to spaces with \((s_j, p_j, q_j)_{j=0,1} \in D(\pi)\) we may assume that \(q_0\) and \(q_1 < \infty\) so that either \(S(\mathbb{R}^n)\) or \(C^\infty(\mathbb{R}^n)\) is dense in \(A_0\) and \(A_1\). When \(A_j = B^{s_j, p_j}_{\infty, \infty}(\mathbb{R}^n)\) for \(j = 0, 1\), it is found from Proposition 3.9 that \(\pi_{\psi}((\cdot, \cdot))\) on \(A_0 \oplus A_1\) is an extension by continuity of the \(\psi\)-independent restriction of \(\mu(\cdot, \cdot)\) to \(C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n)\). The argument carries over to the situation where one or both of the \(p_j < \infty\).

For the borderline cases with \(s_0 + s_1 = 0\) the inequality \(\frac{1}{q_0} + \frac{1}{q_1} \geq 1\) assures that \(q_j < \infty\) for \(j = 0\) or \(1\). So on a dense subset of \(A_0 \oplus A_1\) the bilinear operator \(\pi_{\psi}((\cdot, \cdot))\) coincides with a restriction of the product on \(\mathcal{O}_M \times \mathcal{S}'\) by Proposition 3.6.

When \(s_0 + s_1 = \frac{|M|}{p_0} + \frac{|M|}{p_1} - |M|\) one can for the \(BB^\bullet\) cases use the argument above, since the inequality \(\frac{1}{q_0} + \frac{1}{q_1} \geq 1\) holds according to Theorem 4.2. When \(A_j = F^{s_j, p_j}_{p_j, q_j}\) for \(j = 0, 1\), one can reduce to this situation: Let \(\varepsilon = s_0 + s_1 = \frac{|M|}{p_0} + \frac{|M|}{p_1} - |M|\) and \(\varepsilon_0 + \varepsilon_1 < \varepsilon\). Because

\[t_0 + t_1 = \frac{|M|}{p_0} + \frac{|M|}{p_1} - |M| = \varepsilon - \varepsilon_0 - \varepsilon_1 > 0\]

it is found for the sum-exponents of \(B^{s_j, t_j}_{r_j, r_j}\) that \(\frac{1}{p_0} + \frac{1}{p_1} > 1\). In connection with Theorem 6.1 it is seen that \(\pi((\cdot, \cdot))\) is defined on \(B^{s_0, t_0}_{r_0, r_0} \oplus B^{s_1, t_1}_{r_0, r_1} \subset A_0 \oplus A_1\), and the \(\psi\)-independence on \(A_0 \oplus A_1\) follows.

In the \(BF^\bullet\) cases the inequality \(\frac{1}{q_0} + \frac{1}{q_1} \geq 1\) holds, so the possibility \(q_0 = \infty\) is excluded for \(p_1 > 1\); hence either \(S\) or \(C^\infty\) is dense in \(A_0\), so Proposition 3.6 applies. For arbitrary \(p_1 < \infty\) we may define \(\varepsilon\) and \(\varepsilon_1\) as above (and take \(\varepsilon_0 = 0\)). When \(\varepsilon > 0\) and \(p_1 < 1\) a small \(\varepsilon_1\) yields \(r_1 < 1\) so that \(\frac{1}{p_0} + \frac{1}{p_1} > 1\) and \(\psi\)-independence follows. The case \(p_1 = 1\) poses a problem only when \(q_0 = \infty = q_1\), but for \(M = (1, \ldots, 1)\) one may use that \(F^{s_1, p_1}_{\infty, \infty} \rightarrow B^{s_1, t_1}_{r_1, r_1}\) for \(s_1 - |M| = t - |M|\) when \(r > 1\) and note that \(\pi((\cdot, \cdot))\) is defined on \(B^{s_0, t_0}_{p_0, q_0} \oplus B^{s_1, t_1}_{p_1, r_1}\).
Altogether this shows the $\psi$-independence (with the mentioned exception), and thus the formulation in Section 6 of results for $\pi(\cdot,\cdot)$ has been justified.

7. **Multiplication on open sets**

For an arbitrary open set $\Omega \subset \mathbb{R}^n$ the product $\pi_\Omega(\cdot,\cdot)$ is defined by lifting to $\mathbb{R}^n$.

**Definition 7.1.** The product $\pi_\Omega(u,v)$ is defined for $u$ and $v \in \mathcal{D}'(\Omega)$ when there exists $u'$ and $v' \in \mathcal{S}'(\mathbb{R}^n)$ such that $r_\Omega u' = u$ and $r_\Omega v' = v$ and such that for every $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(x) = 1$ for $x$ in a neighbourhood of $x = 0$ the sequence

$$r_\Omega(\mathcal{F}^{-1}(\psi u') \cdot \mathcal{F}^{-1}(\psi v'))$$

converges in $\mathcal{D}'(\Omega)$ with a $\psi$-independent limit.

In the affirmative case $\pi_\Omega(u,v) = \lim_{k \to \infty} r_\Omega(\mathcal{F}^{-1}(\psi_k u') \cdot \mathcal{F}^{-1}(\psi_k v'))$.

Naturally this definition needs to be justified. Let $u_j$ and $v_j \in \mathcal{S}'(\mathbb{R}^n)$ satisfy $r_\Omega u_j = u$ and $r_\Omega v_j = v$ for $j = 1$ and 2 and let $r_\Omega u^k_j v_k^j := r_\Omega(u^k_1 v^k_1)$ converge in $\mathcal{D}'(\Omega)$. Then one can use the identity

$$r_\Omega u^k_1 v^k_1 - r_\Omega u^k_2 v^k_2 = r_\Omega(u^k_1 - u^k_2)v^k_1 + r_\Omega u^k_2(v^k_1 - v^k_2)$$

(7.2)

to infer from Proposition 3.7 that also $r_\Omega u^k_2 v^k_2$ converges, and that $\lim r_\Omega u^k_2 v^k_2$ is equal to $\lim r_\Omega u^k_1 v^k_1$. Thus the existence of a limit is independent of how $u'$ and $v'$ are chosen, and there is $\psi$-independence for every pair $(u', v')$ if there is for one. Observe also that in Definition 7.1 $\pi(u', v')$ need not be defined, and that we get back the definition of $\pi$ itself when $\Omega = \mathbb{R}^n$.

In the sequel, $B_{p,q}^{M,s}(\Omega) = r_\Omega B_{p,q}^{M,s}(\mathbb{R}^n)$, e.g., is equipped with the infimum quasinorm.

**Theorem 7.2.** Let $\pi(\cdot,\cdot) : A_0 \oplus A_1 \to A_2$ be bounded for spaces $A_j$ that for $j = 0$, 1 and 2 satisfy either $A_j = B_{p,q}^{M,s}(\mathbb{R}^n)$ or $A_j = F_{p,q}^{M,s}(\mathbb{R}^n)$.

When $\Omega \subset \mathbb{R}^n$ is open, one has boundedness of

$$\pi_\Omega(\cdot,\cdot) : A_0(\Omega) \oplus A_1(\Omega) \to A_2(\Omega).$$

(7.3)

Moreover, if $f \in r_\Omega(L_{p,\text{loc}}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n))$ and $g \in r_\Omega(L_{q,\text{loc}}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n))$ then

$$\pi_\Omega(f,g) = f(x) \cdot g(x) \in L_{r,\text{loc}}(\Omega),$$

(7.4)

when $0 \leq \frac{1}{p} = \frac{1}{r} + \frac{1}{q} \leq 1$.

**Proof.** When $(u,v) \in A_0(\Omega) \oplus A_1(\Omega)$ any lift $(u',v') \in A_0 \oplus A_1$ admits application of $\pi$, so a fortiori $\pi_\Omega(u,v)$ exists and equals $r_\Omega \pi(u',v')$. From

$$\|\pi_\Omega(u,v)\|_{A_2(\Omega)} \leq \|\pi(u',v')\|_{A_2} \leq \|\pi\| \|u\|_{A_0} \|v\|_{A_1}$$

(7.5)

it follows by taking the infimum over $u'$ and $v'$ that $\pi_\Omega$ is bounded. Concerning $f$ and $g$ one can simply apply Proposition 3.8 to $e_\Omega f$ and $e_\Omega g$. □

In [Tri83] and [Tri92] there is not given any precise definition of the restriction to $\Omega$, but, because $\pi$ can not be identified with $\mu$ in general, and because $q = \infty$ excludes denseness of smooth functions, it is a point to show that $\pi_\Omega(u,v)$ does not depend on the actual choice of the lift. The approach used in Definition 7.1 takes care of this in a general way. (By doing it for each space $A_0(\Omega) \oplus A_1(\Omega)$ a consistency problem arises, since $u$ and $v$ may belong to other spaces.)
In the particular case when $\Omega$ is of finite measure the inclusion $L_p(\Omega) \hookrightarrow L_p(\Omega)$, valid for $\infty \geq r \geq p > 0$, carries over to the scales $B_{r,q}^M(\Omega)$ and $F_{p,q}^M(\Omega)$ when $\Omega$ is suitably ‘nice’, cf. Lemma 7.3 below. Consequently — when the results of Section 6 are carried over to $\pi_\Omega(\cdot, \cdot)$ by means of Theorem 7.2 above — the restriction $\frac{|M|}{p_0} \leq \frac{|M|}{p_1}$ may be removed when $\Omega$ is bounded, e.g.

This observation raises the question whether the other conditions in Theorem 4.2 can be shown to hold for $\pi_\Omega$ or not. The methods in Section 4 are only applicable for $\Omega = \mathbb{R}^n$, and it remains open how to proceed in general when $\Omega \neq \mathbb{R}^n$. Even so it would be rather surprising if other modifications were necessary.

In Lemma 7.3 if. below we address in particular the embeddings $B_{r,q}^M(\Omega) \hookrightarrow B_{r,q}^M(\Omega)$ and $F_{p,q}^M(\Omega) \hookrightarrow F_{p,q}^M(\Omega)$ etc. that are shown to be valid provided $\infty \geq r \geq p > 0$ when $\Omega$ is a suitable set with meas$(\Omega) < \infty$.

For isotropic spaces over a bounded $\Omega$ the results are identical to [Tri83] 3.3.1, except that $q = \infty$ is not included there in the $F$ case for $t = s$. Seemingly the technique here gives a simpler proof of all cases with $r \geq p$.

**Lemma 7.3.** Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and let $(s, p, q)$ and $(t, r, o)$ belong to $\mathbb{R} \times [0, \infty] \times [0, \infty]$.

Then there is an embedding $B_{r,o}^M(\Omega) \hookrightarrow B_{p,q}^M(\Omega)$ when

$$ t \geq s \text{ and } t - \frac{|M|}{p} \geq s - \frac{|M|}{p} \text{ are satisfied,} $$

$$ \text{together with } o \leq q \text{ if } t = s, \text{ or if } t - \frac{|M|}{p} = s - \frac{|M|}{p},$$

(7.6)

and when $r$ and $p < \infty$ and moreover

$$ t \geq s \text{ and } t - \frac{|M|}{p} \geq s - \frac{|M|}{p} \text{ are satisfied,} $$

$$ \text{together with } o \leq q \text{ if } t = s, $$

(7.7)

an embedding $F_{r,o}^M(\Omega) \hookrightarrow F_{p,q}^M(\Omega)$ exists.

**Proof.** We show for $s \in \mathbb{R}$ and $\infty > r \geq p > 0$ the existence of an embedding

$$ F_{r,q}^M(\Omega) \hookrightarrow F_{p,q}^M(\Omega), \text{ for any } q \in [0, \infty],$$

(7.8)

by use of the result that on $\mathbb{R}^n$ one has, cf. Section 6

$$ F_{r,q}^{M,s_0} + F_{r,q}^{M,s} \xrightarrow{\pi'(\cdot)} \bigcap \{ F_{r,q}^{M,s} \mid \frac{1}{r} \leq \frac{1}{s} \leq \frac{1}{s} + \frac{1}{2} \},$$

(7.9)

for $s_0 > \frac{|M|}{2}$, when $s_0 > s$ and $s_0 + s > \max(0, \frac{|M|}{2} - \frac{|M|}{2})$.

Let $N$ be a natural number such that $p \geq (\frac{1}{s} + \frac{N}{2})^{-1}$ and let $\Omega_j$ be open bounded sets such that $\Omega = \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_N \subset \Omega_{N+1}$. For each $j = 0, \ldots, N$ we pick $\psi_j \in C^\infty(\mathbb{R}^n)$ such that $\psi_j \equiv 1$ on $\Omega_j$, and supp$\psi_j \subset \Omega_{j+1}$.

Now let $v \in F_{r,q}^{M,s}(\Omega)$. Since $r_\Omega \psi_0 w = v$ when $r_\Omega w = v$ and since $\psi_k \cdots \psi_0 w = \psi_0 w$, a repeated application of (7.9) gives, with $\frac{1}{r_j} = \frac{1}{s} + \frac{1}{2}$,

$$ \| v \| F_{r,k+1,q}^{M,s}(\Omega) \leq \inf \{ \| \psi_k \cdots \psi_0 w \| F_{r,k+1,q}^{M,s} \mid w \in F_{r,q}^{M,s}, r_\Omega w = v \}$$

$$ \leq c \| \psi_k F_{r,\frac{1}{2}}^{M,s} \cdots \psi_0 \| F_{r,\frac{1}{2}}^{M,s} \| v \| F_{r,q}^{M,s}(\Omega),$$

(7.10)

for some $c < \infty$, if $s_0, \ldots, s_k$ are big enough. For some $k \leq N$ it is even possible to take the $F_{r,q}^{M,s}$-norm on the left hand side of (7.10), cf. (7.9) and the definition of $N$. 


A similar procedure works for \( t = s \) in the Besov-case. The full statements now follow by use the embeddings in Section 2.2. \( \square \)

Obviously one could equally well work with sets \( \Omega \) of finite measure for which there exists open sets \( \Omega_j \) and \( \psi_j \in \mathcal{S}(\mathbb{R}^n) \) such that
\[
\Omega = \Omega_0 \subset \cdots \subset \Omega_j \subset \Omega_{j+1} \subset \cdots, \\
\text{meas}(\Omega_j) < \infty \quad \text{for all} \quad j \in \mathbb{N}_0, \\
\psi_j \equiv 1 \text{ on } \Omega_j, \quad \text{supp } \psi_j \subset \Omega_{j+1}.
\]

(7.11)

Indeed, \( \psi_k \in W^{[s_k]}_2(\mathbb{R}^n) \hookrightarrow F^{s_k}_{2,2}(\mathbb{R}^n) \) since \( \text{meas}(\Omega_{k+1}) < \infty \). Thus we have

**Corollary 7.4.** When \( \Omega \subset \mathbb{R}^n \) is open with \( \text{meas}(\Omega) < \infty \) and when (7.11) holds for some \( \psi_j \) and \( \Omega_j \), then Lemma 7.3 holds for \( \Omega \).

8. Applications

Already in Lemma 7.3 ff. above there is a theoretical application of the results for \( p_0 \neq p_1 \), cf. (7.9).

For boundary problems for non-linear partial differential equations Theorem 7.2 is a tool, which allows one to treat, say, products in \( B^{s}_{p,q} \) and \( F^{s}_{p,q} \) spaces. For the particular cases with \( p_0 = p_1 = p_2 \) one can hereby apply the \( \mathbb{R}^n \)-results in [Yam86a, Thm. 6.1] to such problems. In the author’s thesis [Joh93] this approach has been used for the stationary Navier–Stokes equations. In particular the non-linear terms are estimated in the \( B^{s}_{p,q} \) and \( F^{s}_{p,q} \) spaces in this way.

Moreover, a rather satisfactory set of regularity properties for these equations have been deduced there, and in the particular case of boundary conditions of class 2 (similar to those in [GS91]), the product results for \( p_0 = p_1 \neq p_2 \) enter in a decisive way. See [Joh93, Thm. 5.5.3] for further details.

The general anisotropic results with \( M \neq (1, \ldots, 1) \) are applicable to the time-dependent Navier–Stokes equations, cf. [GS91] and [Gru95], and to other non-linear parabolic problems, that are considered on a cylinder like \( \Omega \times [0, T] \) with the time variable running in the interval \( [0, T] \). For such problems it is most convenient that the open sets in Definition 7.1 and Theorem 7.2 are allowed to have non-smooth boundaries.

The \( BF^* \) and \( FB^* \) cases, that are only given a minimal treatment on \( \mathbb{R}^n \) here, should be relevant for differentials of non-linear operators, and hence for certain stability questions. As an example one could take the differential \( d\pi \) of the product \( \pi(\cdot, \cdot) \) at a point \( u \in B^{s_{k}}_{p,q} \); this is the linear operator \( v \mapsto d\pi(u)v = \pi(u, v) + \pi(v, u) \), that may act between \( F^{s}_{p,q} \) spaces.

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