On inverse problem of dynamics

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February 13, 2004

Abstract

We study the question whether for a natural Hamiltonian system on a two-dimensional compact configuration manifold, a single trajectory of sufficiently high energy is almost surely enough to reconstruct a real analytic potential.

AMS subject classification 37J05, 70H12

Consider a compact configuration manifold $M^n$ (essentially we deal with the dimension $n = 2$) and a natural Hamiltonian system thereon, with Hamiltonian

$$H(p, q) = \langle p, p \rangle_q + U(q), \quad (p, q) \in T^*M^n.$$

Above, $\langle \cdot, \cdot \rangle_q$ is a Riemannian metric on $M^n$ and $U$ a potential. The direct problem of dynamics on $M^n$ is finding the trajectory $q(t) \subset M^n$, with initial conditions $q(0) = q_0$ and $\dot{q}(0) = v_0$, moving in the known force field $f(q) = -\nabla_q U(q)$ on $M^n$, where the gradient $\nabla_q$ has been associated with the metric $\langle \cdot, \cdot \rangle_q$.

Let us call the inverse problem of dynamics the problem of reconstruction of the potential by observing the system’s trajectories $q(t)$. The first problem of this type was explored in Newton’s Principia, in quest for a physical law determining the planetary motion compatible with observational data\(^1\). In the general case, knowledge of infinitely many trajectories is required to completely solve the problem. In this note we show that in the special case when $M^n$ is two-dimensional, compact and topologically non-trivial, a single trajectory with sufficiently large energy would almost surely suffice to reconstruct the potential.

In the sequel, we assume that $M^n$ as well as all the quantities involved are real-analytic. Also suppose, there is an a-priori estimate $|U(q)| < C_0, \forall q \in M^n$, and we consider only the trajectories $q(t)$ with total energy $E \geq C_0$, so $M^n$ coincides with the domain of possible motions.

**Theorem.** Let $n = 2$, suppose $M^2$ is not diffeomorphic to $S^2$ or $\mathbb{R}P^2$. Almost every trajectory $q(t), t \geq 0, \text{ with energy } E \geq C_0,$ suffices to reconstruct the potential $U$ as a real-analytic function on $M^2$.

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\(^1\)Note that the term “inverse problem of mechanics” has also been used to address the problem of deciding whether a given system of second order ODEs on $M^n$ has a Lagrangian, see e.g. [5].
Let us recall the definition of a key set, or set of uniqueness, see e.g. [3].

Definition. Let $D$ be a domain in $\mathbb{R}^n$ and $C^\omega(D)$ the class of real-analytic functions in $D$. A set $K \subset D$ is a key set if any $f \in C^\omega(D)$ vanishing identically on $K$, vanishes identically on $D$.

The theorem will follow from the following lemma.

Lemma. If a real-analytic dynamical system $A : \dot{x} = F(x)$ on a compact phase space $P$, is non-singular (i.e. for no $x \in P, F(x) = 0$) and has a positive measure of closed orbits, then all its orbits are closed.

Proof of the lemma

Since the manifold $P$ is compact and the vector field $F$ is non-singular, there exists a Riemannian metric $g$ on $P$, such that in this metric the vector field $F$ has unit length at every $x \in P$. Furthermore, by compactness of $P$, the curvature (associated with $g$) of integral trajectories of $A$ is bounded from above, and therefore there exists some $T_m > 0$, such that any periodic orbit of $A$ has period not smaller than $T_m$.

Let us partition the range $[T_m, \infty)$ of possible periods (henceforth periods stand for minimum periods) for closed orbits on intervals of some small length $\delta_1$ to be specified. Let $I_k = [T_m + k\delta_1, T_m + (k + 1)\delta_1) \equiv [T_k, T_{k+1})$, for $k = 0, 1, \ldots$. Let $\Gamma_k$ be the set of all closed orbits, whose periods lie in $I_k$. Then for some $k = k_s$ the set $\Gamma_s = \Gamma_{k_s}$, considered as a subset of $P$, has positive measure. (We refer to the sets $\Gamma$ either as point sets or sets of orbits, depending on the context. As there are only measure-theoretical considerations involved, this should not cause confusion.)

Let $D_{\delta_2}(x)$ be a codimension one disk in $P$, centered at some $x \in P$, with radius $\delta_2$ and perpendicular (in the sense of metric $g$) to the vector field $F(x)$ at the point $x$. Let $\delta_2$ be small enough, so that the vector field is transversal to $D_{\delta_2}(x)$ at every point of the disk. Clearly, $\delta_2$ can be taken as a universal constant independent of $x$.

Let $x_0$ be a Lebesgue point of the set $\Gamma_s$. Recall that at a Lebesgue point, the density of the set is one. Let $\gamma_0$ be the closed trajectory passing through $x_0$, so $\gamma_0 \in \Gamma_s$. Take a disk $D_{\delta_2}(x_0)$.

Then there is a well defined analytic Poincaré map $S$ from a disk $D_\epsilon(x_0)$ contained in $D_{\delta_2}(x_0)$, where $\epsilon < \frac{1}{C_1 T_m}\delta_2$, for some $C_1 = C_1(\delta_1)$, and $x_0$ is a fixed point of $S$. On the disk $D_\epsilon(x_0)$, points which are initial conditions for orbits from $\Gamma_s$ form a set of positive measure, as $x_0$ is a Lebesgue point. Besides, the quantity $\delta_1 < T_m$ can be chosen small enough to ensure that any $x \in D_\epsilon(x_0) \cap \Gamma_s$ is also a fixed point of the map $S$.

The union of all $x \in D_\epsilon(x_0) \cap \Gamma_s$ is a positive measure subset of $D_\epsilon(x_0)$, and hence is a key set (see [3] for the proof that every set of positive measure is a key set). Therefore, every point of $D_\epsilon(x_0)$ is an equilibrium of the map $S$, and hence an initial condition for a periodic orbit of $A$. Let $\Gamma_\epsilon$ be the union of all such orbits, with initial conditions in $D_\epsilon(x_0)$. Let $\Gamma \subseteq P$ be the maximum connected open set, which contains $\Gamma_\epsilon$ and is a union of periodic orbits.

To complete the proof, let us show that the set $\Gamma$ does not have a boundary, i.e. $\Gamma = P$. To show it, we use the following Gronwall type estimate.

Proposition. Let $\phi(s) \subset \Gamma$ be a curve of length $L$, where $s$ is a natural parameter with respect to the metric $g$. Then for all $s \in [0, L]$ and some absolute constant $C_2$,

$$T(\phi(s)) \leq T(\phi(0))e^{C_2 L},$$

where $T(\phi(s))$ is the period of the closed orbit passing through the point $\phi(s)$.
Indeed, the proposition follows immediately from the following infinitesimal estimate: for some \( C_2, \)
\[
\left| \frac{d}{ds} T(\phi(s)) \right| \leq C_2 T(\phi(s)).
\]

Returning to the proof of the lemma, suppose the boundary \( \partial \Gamma \) is non-empty. As \( \partial \Gamma \) is a compact set, the distance (in the sense of metric \( g \)) between \( \partial \Gamma \) and the above mentioned point \( x_0 \) attains its minimum at some point \( y \in \partial \Gamma \). Connect \( x_0 \) and \( y \) by a geodesic segment. Let the latter segment have length \( L \); clearly all its points, except \( y \) belong to \( \Gamma \). Let \( \gamma_y \) be the trajectory of \( A \) with initial condition \( y \).

For any point \( x_1 \neq y \) on the above geodesic segment, there is a uniform bound for the period of the corresponding closed orbit, by the proposition. Hence, for any such \( x_1 \), there exists a uniform \( \varepsilon \) (one can take \( \varepsilon = \varepsilon e^{-C_2 L} \), where \( \varepsilon \) has been defined earlier) such that an analytic Poincaré map can be defined in exactly the same way as \( S \) above, but now with the domain \( D_\varepsilon(x_1) \). Choosing \( x_1 \) such that the intersection \( \gamma_y \cap D_\varepsilon(x_1) \) is not empty and repeating the key set argument leads to contradiction: all orbits in some tubular neighborhood of \( \gamma_y \), including \( \gamma_y \) itself, are periodic.

Q.E.D.

**Proof of the theorem**

Consider a randomly chosen trajectory \( \gamma \) on some energy level \( H^{-1}(E), \ E \geq C_0 \), which is obviously a non-critical level.

According to the lemma, either (i) all the trajectories on \( H^{-1}(E) \) are periodic, or (ii) a randomly chosen initial condition \( (p_0, q_0) \in H^{-1}(E) \) results in a phase trajectory of infinite length almost surely.

The former case (i) may occur only if \( M^2 \) is a so-called \( P \)-manifold. Indeed, according to the Maupertuis principle, the phase trajectories of motions with total energy \( E \) project onto \( M^2 \) as geodesic lines of the corresponding (Riemannian) Jacobi metric. \( P \)-manifolds are Riemannian manifolds, all whose geodesics are closed, see [1]. Topological properties of \( P \)-manifolds are characterized in great detail in various dimensions, within the framework of the Bott-Samelson theorem. In our (simplest possible) case, it is easy to see that \( M^2 \) can only be diffeomorphic to either \( S^2 \) or \( \mathbb{R}P^2 \).

Indeed, the proof of the lemma implies that all the (closed) phase trajectories on the energy level \( E \) are homotopic to one another. Then their images on \( M^2 \) (under natural projection) are also homotopic. On the other hand, if \( M^2 \) is different from \( S^2 \) or \( \mathbb{R}P^2 \), the number of generators for its fundamental group equals at least two. As for any Riemannian metric there are closed geodesics in each free homotopy class, we would have a contradiction, unless \( M^2 \) is \( S^2 \) or \( \mathbb{R}P^2 \).

In the case (ii), let \( q(t) = (q_1(t), q_2(t)) \) be a randomly chosen trajectory: it almost surely has infinite length. Clearly, we can easily derive the gradient of \( U \) at every point of the trajectory from the Euler-Lagrange equations.

Thus to complete the proof of the theorem, let us show that any non-closed trajectory is a key set. Consider orbit segments \( \{q(t), t \in [k, k + 1]\}, \ k = 0, 1, \ldots \) in \( M^2 \). (Note that time can always be scaled to ensure that each segment is a simple curve in \( M^2 \), or shorter time intervals can be considered.) As \( M^2 \) is compact, there is a limit point \( q_* \) of the point sequence \( \{q(k + 1/2)\} \).

Consider a sufficiently small circle centered at \( q_* \). There are two options. Either the circle intersects the trajectory \( q(t) \) at infinitely many points, or at some point on the circle, the trajectory \( q(t) \) intersects itself (transversely as it is a geodesic) infinitely many times. In the later case, take the point of infinite self-intersection for \( q_* \), otherwise leave \( q_* \) as it is. In either case, there exists a point
\[q_*, \] with the property that any sufficiently small circle centered at \(q_*\) is intersected by the trajectory \(q(t)\) infinitely many times.

Therefore, the force \(f\) and the potential \(U\) can be uniquely reconstructed on any sufficiently small circle centered at \(q_*\) (an infinite point set on a circle is a key set for the circle), and therefore in some neighborhood of \(q_*\), and hence on the whole configuration manifold \(M^2\). \textit{Q.E.D.}

In conclusion, let us make several remarks.

1. The theorem can restated in terms of analytic geodesic flows on compact Riemannian 2-manifolds. Namely, if \(M^2\) is not diffeomorphic to a sphere or real projective plane, a randomly chosen geodesic suffices to reconstruct the metric, almost surely. Indeed, any Riemannian metric is locally conformal to the Euclidean one, i.e. can be locally associated with Hamiltonian
\[H(p,q) = e^{\rho(q_1,q_2)}(p_1^2 + p_2^2).\]
Our theorem enables one to reconstruct the real-analytic function \(\rho(q_1,q_2)\) locally near \(q_*\) from the Hamilton equations, with subsequent analytic continuation to get the metric globally on \(M^2\).

2. The theorem is essentially two-dimensional, as for \(n \geq 3\) the fact that a random trajectory has infinite length does not suffice to reconstruct the potential. Consider for instance the Euler top, where \(M^n = SO(3)\). In this case, the phase space is foliated by invariant two-tori, where the trajectories are in general conditionally periodic. Clearly, a projection of a single invariant two-torus onto the three-dimensional configuration space is not a key set.

3. The lemma does not apply to the case of invariant tori of dimension higher than one. Indeed, a particular case of KAM theorem tells one that a sufficiently small perturbation of a non-degenerate Liouville-integrable Hamiltonian system in \(T^*M^n\) yields a positive measure set of invariant \(n\)-tori, which do not fill the whole energy surface however.

4. In the special case \(M^2 = \mathbb{R}P^2\), the only possibility to have all the geodesic closed is the standard metric, by the theorem of L. Green (\[1\]). The case \(M^2 = S^2\) has been a subject of extensive research for over 100 years, arguably beginning with the doctoral thesis of O. Zoll (\[2\]). The reader is referred to the excellent book by L. Bess (\[1\]), which gives the issue a thorough treatment.

5. The condition \(E \geq C_0\) also seems unavoidable, as for small energies the domain of possible motions can be a disk, with the Jacobi metric degenerate on the boundary, in which case one may expect the scenario, similar to the case \(M^2 = S^2\).

6. Observe that in the exceptional case when \(M^2\) is a \(P\)-manifold, the geodesic flow thereon is completely integrable (see e.g. \[2\] for the proof of this fact). Hence our theorem implies that if \(n = 2\), it is sufficient for restoration of the potential almost surely from a single trajectory that the system possess no other analytic integrals of motion but energy. It seems likely that in the latter weaker formulation the theorem should extend to the case \(n > 2\), however we do not know how to prove it.
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