DIRAC-TYPE CONDITIONS FOR SPANNING BOUNDED-DEGREE HYPERTREES

MATÍAS PAVEZ-SIGNÉ, NICOLÁS SANHUEZA-MATAMALA, AND MAYA STEIN

ABSTRACT. We prove that for fixed $k$, every $k$-uniform hypergraph on $n$ vertices and of minimum codegree at least $n/2 + o(n)$ contains every spanning tight $k$-tree of bounded vertex degree as a subgraph. This generalises a well-known result of Komlós, Sárközy and Szemerédi for graphs. Our result is asymptotically sharp. We also prove an extension of our result to hypergraphs that satisfy some weak quasirandomness conditions.

1. INTRODUCTION

Forcing spanning substructures with minimum degree conditions is a central topic in extremal graph theory. For instance, a classic result of Dirac [Dir52] from 1952 asserts that any graph on $n \geq 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle. In the same spirit, Bollobás [Bol78] conjectured in the 1970s that graphs on $n$ vertices with minimum degree at least $n/2 + o(n)$ contain every $n$-vertex tree of bounded maximum degree as a subgraph. Komlós, Sárközy and Szemerédi [KSS95] proved this conjecture in 1995.

Theorem 1.1 (Komlós, Sárközy and Szemerédi [KSS95]). For all $\gamma > 0$ and $\Delta \in \mathbb{N}$, there is $n_0$ such that every graph $G$ on $n \geq n_0$ vertices with $\delta(G) \geq (1/2 + \gamma)n$ contains every $n$-vertex tree $T$ with $\Delta(T) \leq \Delta$.

In recent years, many efforts have been made to extend Dirac’s theorem to $k$-uniform hypergraphs, also called $k$-graphs. The minimum codegree of a $k$-graph $H$, denoted $\delta_{k-1}(H)$, is the largest number $m$ such that every set of $k-1$ vertices from $H$ is contained in at least $m$ edges of $H$. A notable result by Rödl, Ruciński and Szemerédi [RRS08] states that $k$-graphs on $n$ vertices and minimum codegree at least $n/2 + o(n)$ contain a tight Hamilton cycle, where a tight Hamilton cycle consists of a cyclic ordering of the vertices of the $k$-graph such that every $k$ consecutive vertices in this ordering form an edge. More Dirac-type results for Hamilton cycles can be found in the survey [SS19] by Simonovits and Szemerédi and the references therein.

In the present paper, we extend the Komlós–Sárközy–Szemerédi theorem to $k$-graphs (Theorem 1.2). To the best of our knowledge, our result is the first Dirac-type result for tightly connected spanning structures other than tight paths, tight cycles, or triangulations of 2-spheres. The structures considered in our result are referred as to hypertrees or tight $k$-trees, which are defined next.

A tight $k$-tree is a $k$-graph defined iteratively as follows: a single $k$-uniform edge is a tight $k$-tree; any $k$-graph obtained from a tight $k$-tree $T$ by adding a new vertex $v$ and a new edge $e$, such that $v \in e$ and $|e \cap e'| = k-1$ for some edge $e' \in E(T)$, is also a tight $k$-tree. Observe that tight 2-trees are the usual trees in graphs, since trees in graphs can be defined by successively adding leaves. Also, the well-known $k$-uniform tight paths are tight $k$-trees. Since no other kinds of trees will be considered here, we usually just write $k$-tree to refer to a tight $k$-tree.

Extremal problems for $k$-trees have a long history. In 1984, Kalai conjectured [FF87, Conjecture 3.6] that every $k$-graph on $n$ vertices with more than $\binom{n-1}{k-1}$ edges contains every $k$-tree with $t$ edges. For all $k$, this conjecture is tight for infinitely many $t$ and $n$. In general, Kalai’s conjecture is open, but there are partial and asymptotic results for special families of

MPS was supported by ANID Doctoral scholarship ANID-PFCHA/Doctorado Nacional/2017-21171132 while he was affiliated to the Universidad de Chile. NSM acknowledges support by the Czech Science Foundation, grant number GA19-087408 with institutional support RVO: 67985807. MS was supported by ANID Regular Grant 1221905, by MathAmSud 20MATH-01, by FAPESP-ANID Investigación Conjunta grant 2019/13364-7, and by ANID/BASAL ACE210010 y FB210005.
k-trees [FF87; FJ15; Für+19; Für+20b; Für+20a], among these are the (linear sized) tight paths [All+17], for k-partite host k-graphs [Ste19], and for the case k = 2 (see [Ste20] for references), where Kalai’s conjecture reduces to the Erdős–Sós conjecture [Erd64a]. Regarding spanning k-trees, it is also worth to mention the work of Georgakopoulos, Haslegrave, Montgomery, and Narayanan [Geo+22], who proved that large n-vertex 3-graphs with minimum codegree at least n/3 + o(n) have a spanning triangulation of a 2-sphere, which in particular contains some spanning 3-tree. In contrast, they also show that there are 3-graphs with minimum codegree at least \([n/3] − 1\) where the largest tight 3-tree has size at most 2\([n/3]\).

Before stating our main result, we need a definition. For a k-graph \(H\), the maximum 1-degree of \(H\), denoted \(Δ_1(H)\), is the maximum number \(m\) such that some vertex of \(H\) is contained in \(m\) edges.

**Theorem 1.2.** For all \(k, Δ ≥ 2\) and \(γ > 0\), there is \(n_0\) such that every k-graph \(H\) on \(n ≥ n_0\) vertices with \(δ_{k-1}(H) ≥ (1/2 + γ)n\) contains every k-tree \(T\) on \(n\) vertices with \(Δ_1(T) ≤ Δ\).

The condition imposed on \(δ_{k-1}(H)\) in Theorem 1.2 is best possible up to the term \(γn\) and an additive term depending on \(k\), as shown by the next proposition (we postpone its proof until Section 3).

**Proposition 1.3.** For every \(k ≥ 2\) and for every k-tree \(T\) on \(n ≥ k\) vertices, there is a k-graph \(H\) on \(n\) vertices not containing \(T\), with \(δ_{k-1}(H) ≥ \lfloor n/2 \rfloor − f(T)\), where \(f(T) ≤ 2k + k − 1\). Moreover, there are k-trees \(T\) with \(f(T) = k − 1\).

Theorem 1.2 generalises to host graphs that have certain quasirandom properties. For a k-graph \(H\) and a set \(F\) of distinct \((k−1)\)-subsets of \(V(H)\), we define the joint degree of \(F\) as

\[
\deg_H(F) = |\{v ∈ V(H) : f ∪ \{v\} ∈ H \text{ for each } f ∈ F\}|.
\]

We say that \(H\) is \((g, h, ε, τ)-typical\) if \(|\deg_H(F) − g|F|n| ≤ εn\) for every set \(F\) of \((k−1)\)-sets such that \(|F| ≤ h\). We show that, for suitable choices of \(g\) and \(ε\), every large \((g, 2, ε)-typical\) k-graph contains every spanning k-tree with bounded degree.

**Theorem 1.4.** For all \(k, Δ ≥ 2\) and \(g > 0\), there are \(n_0\) and \(ε_0 > 0\) such that the following holds for all \(0 < ε ≤ ε_0\). If \(H\) is a \((g, 2, ε)-typical\) k-graph on \(n ≥ n_0\) vertices, then \(H\) contains every k-tree \(T\) on \(n\) vertices with \(Δ_1(T) ≤ Δ\).

Ehard and Joos [EJ22] showed very general results for finding (almost perfect packings of) spanning bounded-degree hypergraphs in host hypergraphs satisfying certain strong quasirandom conditions. However, their results are incomparable with ours, as our quasirandomness conditions are much weaker.

We deduce Theorem 1.4 from a slightly more general statement (Corollary 11.2). Weaker notions of quasirandomness and minimum 1-degree (defined in analogy to the maximum 1-degree) of order \(Θ(n^{k−1})\) are not sufficient to guarantee the existence of any spanning k-tree in dense k-graphs on \(n\) vertices. This follows from examples of Araújo, Piga and Schacht [APS22] for \(k ≥ 3\). See Section 11 for more details.

The paper is organised as follows. In Section 2, we introduce some notation and terminology that we will use throughout the paper. In Section 3, we prove Proposition 1.3 and in Section 4 we give an overview of the proof of Theorem 1.2 (which can be read independently of Sections 2 and 3). In Section 5, we prove several results about tight k-trees which are higher-uniformity analogues of well-known results for trees. In Section 6, we introduce some tools that will be used in the proof of our main result (Theorem 1.2). In Section 7, we prove that k-graphs with large codegree are ‘well-connected’, meaning that pairs of disjoint \((k−1)\)-sets can be connected by many short walks of fixed length. In Section 8, we prove an embedding result for k-trees into dense k-partite k-graphs, and in Section 9 we introduce a suitable absorption method for k-trees with bounded degree. In Section 10, we use the results form Sections 5-9 to prove Theorem 1.2, and in Section 11 we extend Theorem 1.2 to hypergraphs satisfying certain quasirandomness conditions, in particular, proving Theorem 1.4. Section 12 contains some concluding remarks and open questions.
2. Notation

We introduce here some basic notation used throughout the paper. More specific notions will be introduced where first needed. Throughout this section, let $H$ be a $k$-graph, with $k \geq 2$.

**Hypertrees.** As stated in the introduction, a $k$-tree is a $k$-graph which can be defined iteratively as follows:

(i) a single $k$-uniform edge is a $k$-tree;

(ii) any $k$-graph obtained from a $k$-tree $T$ by adding a new vertex $v$ and a new edge $e$, such that $v \in e$ and $|e \cap e'| = k - 1$ for some edge $e' \in E(T)$, is also a $k$-tree.

By definition, every $k$-tree with $n$ vertices has $n - k + 1 \geq 1$ edges, and hence $n \geq k$. Also by definition, every $k$-tree $T$ on $n$ vertices has orderings $e_1, \ldots, e_{n-k+1}$ and $v_1, \ldots, v_n$ of its edges and vertices, respectively, such that $e_1 = \{v_1, \ldots, v_k\}$ and, for all $i \in \{k + 1, \ldots, n\}$,

\[(T1) \; \{v_i\} = e_{i-k+1} \setminus \bigcup_{1 \leq j < i-k+1} e_j, \text{ and}
\]

\[(T2) \; \text{there exists } j \in \{i - 1\} \text{ such that } e_{i-k+1} \setminus \{v_i\} \subseteq e_j,
\]

hold. Any ordering of $E(T)$ or $V(T)$ satisfying properties $(T1)$ and $(T2)$ will be called a valid ordering. Sometimes, while referring to a valid ordering of $E(T)$, we will also understand an ordering of $V(T)$ as implicitly given by the ordering of $E(T)$, except for the ordering of the first $k$ vertices, which can be arbitrary. Similarly, we may refer to a valid ordering of $V(T)$, and then the ordering of $E(T)$ is implicit.

If $j \in \{i - 1\}$ is the smallest index for which $(T2)$ holds for $e_{i-k+1}$ and $e_j$, then we say that $e_j$ is the parent of $e_{i-k+1}$ and that $e_{i-k+1}$ a child of $e_j$. For $i \geq k$, the anchor of $v_i$ is $\alpha(v_i) := e_{i-k+1} \setminus \{v_i\}$.

A $k$-subtree of $T$ is a $k$-tree $T'$ such that $T' \subseteq T$. For instance, $e_1, \ldots, e_r$ induces a $k$-subtree of $T$, for any $1 \leq r \leq n - k + 1$. Also, the tree $T - v_n$ obtained by removing $v_n$ and $e_{n-k+1}$ from $T$ is a $k$-subtree.

**$\ell$-partition** We say that $H$ is $\ell$-partite if there is a partition $\{V_1, \ldots, V_\ell\}$ of $V(H)$ such that $|e \cap V_i| \leq 1$ for every $e \in E(H)$ and $i \in [\ell]$. It is easy to show (by induction on the number of vertices) that every $k$-tree is $\ell$-partite and, moreover, the $\ell$-partition of its vertices is unique.

**Homomorphisms and embeddings.** If $H_1, H_2$ are hypergraphs, a hypergraph homomorphism of $H_1$ in $H_2$ is a function $\varphi : V(H_1) \to V(H_2)$ that preserves edges. If furthermore, $H_1 \subseteq H'_1$ and $\varphi' : V(H'_1) \to V(H'_2)$ is a hypergraph homomorphism such that $\varphi'$ agrees with $\varphi$ restricted to $V(H_1)$, then we call $\varphi'$ an extension of $\varphi$ to $H'_1$. If $\varphi$ is injective, we say that $\varphi$ is an embedding of $H_1$ into $H_2$. An extension of $\varphi$ is then an extension which is also an embedding.

**Shadows and ordered shadows.** The shadow of $H$, denoted $\partial H$, is the $(k-1)$-graph on vertex set $V(H)$ and whose edges are all the $(k-1)$-sets which are contained in some edge of $H$. The ordered shadow of $H$, denoted $\partial^o H$, is defined as the set of all tuples $v = (v_1, \ldots, v_{k-1})$ with $\{v_1, \ldots, v_{k-1}\} \in \partial H$, and we will use bold letters to denote its elements. If $\varphi : \partial^o H \to \partial^o H'$ is a function, $a = (a_1, \ldots, a_{k-1}) \in \partial^o H$ and $b = (b_1, \ldots, b_{k-1}) \in \partial^o H'$, then $\varphi(a) = b$ means that $\varphi(a_i) = b_i$ for all $i \in [k-1]$. Furthermore, elements from $\partial^o H$ will be used as their underlying set for set-theoretical operations. For instance, if $a = (a_1, \ldots, a_{k-1}), b = (b_1, \ldots, b_{k-1}) \in \partial^o H$, then $a \cup b = \{a_1, \ldots, a_{k-1}\} \cup \{b_1, \ldots, b_{k-1}\}$.

**Neighbourhoods and degrees.** For $S \subseteq V(H)$, the neighbourhood of $S$ in $H$ is defined as $N_H(S) = \{F \subseteq V(H) \setminus S : S \cup F \in H\}$. If $x_1, \ldots, x_\ell \in V(H)$, we will write $N_H(x_1, \ldots, x_\ell)$ instead of $N_H((x_1, \ldots, x_\ell))$. The degree of $f \in \partial H$, denoted $\deg_H(f)$, is the number of edges of $H$ containing $f$ and equals $|N_H(f)|$.

**Walks.** An ordered sequence $(x_1, \ldots, x_n)$ of vertices from $H$ is a walk if every $k$ consecutive vertices form an edge of $H$. We will often just write $x_1 \cdots x_n$ instead of $(x_1, \ldots, x_n)$ when using walks. The length of a walk is the number of its edges, e.g. a walk on $n$ vertices has length $n - k + 1$. The order of the vertices is important: $x_1 x_2 \cdots x_n$ will generally be a different walk than $x_n \cdots x_2 x_1$ (even though they use the same vertices and edges). Note that a walk in which every vertex appears exactly once is a tight path in $H$. At some points we will use that a walk corresponds naturally to a subgraph of $H$ (instead of a sequence of vertices).
Let $W = x_1 \cdots x_n$ be a walk in $H$. The start of $W$ is denoted by $\text{sta}(W) := (x_1, \ldots, x_{k-1})$, and the end of $W$ is denoted $\text{ter}(W) := (x_{n-k+1}, \ldots, x_n)$. Both $\text{sta}(W)$ and $\text{ter}(W)$ belong to $\partial^* H$. If $\text{sta}(W) = a$ and $\text{ter}(W) = b$, we also say that $W$ goes from $a$ to $b$. The interior of $W$, denoted $\text{int}(W)$, is the set $V(W) \setminus (\text{sta}(W) \cup \text{ter}(W))$. Thus $|\text{int}(W)| \leq n - 2k - 2$, strict inequality holds if $W$ is not a path. If $|\text{int}(W)| = q$, we also say that $W$ has $q$ internal vertices. If $\text{int}(W) \cap S = \emptyset$, we call $W$ internally disjoint from $S$.

**Numbers and hierarchies.** Given real numbers $x, y, z$, we write $x = y \pm z$ to denote that $x \in [y - z, y + z]$. Also, we write $a \ll b$ to mean that for $b > 0$, there exists $a_0 > 0$ such that for all $a \leq a_0$ the subsequent statements hold. Hierarchies with more constants are defined analogously, and should always be read from right to left. Implicitly, we assume that all constants appearing in a hierarchy are positive, and moreover if $1/m$ appears in a hierarchy then $m$ is an integer.

### 3. Extremal example

In this short section, we prove Proposition 1.3. The construction which witnesses the lower bound is similar in flavour to many other constructions in extremal hypergraph theory, as it is an example of a standard ‘parity obstruction’. We will use the following family of $k$-graphs.

**Definition 3.1.** For disjoint sets $A, B$, and $0 \leq i \leq k$, let $H_i := \{e \subseteq A \cup B : |e| = k, |e \cap A| = i\}$, and $I := \{i \in \{0, \ldots, k\} : i \not\equiv [k/2] \mod 2\}$. Define $H(A, B) := \bigcup_{i \in I} H_i$.

Assuming that $|A \cup B| \geq k$, note that $\delta_{k-1}(H(A, B)) \geq \min\{|A|, |B|\} - k + 1$. The following lemma asserts that there are not too many ways to embed a $k$-tree into $H(A, B)$. Recall that each $k$-tree admits a unique $k$-partition of its vertices.

**Proposition 3.2.** Let $k, n \in \mathbb{N}$, let $H(A, B)$ be as in Definition 3.1, with $|A \cup B| = n \geq k$. Let $T$ be a $k$-tree with $k$-partition $V_1 \cup \cdots \cup V_k$, and an embedding $\varphi : V(T) \to V(H(A, B))$. Then, for each $i \in [k]$ either $\varphi(V_i) \subseteq A$ or $\varphi(V_i) \subseteq B$.

**Proof.** We proceed by induction on $|E(T)|$; the base case $|E(T)| \leq 1$ is trivial. Let $v$ and $e$ be the last vertex and edge of some valid ordering of $T$, let $e'$ be the parent edge of $e$ and let $v'$ be the vertex in $e' \setminus e$. Note that there exists $j \in [k]$ such that $v, v' \in V_j$. Applying the induction hypothesis to $T - v$ and to $\varphi$ restricted to $V(T')$, we see that we only need to show that $\varphi(v) \in A$ if and only if $\varphi(v') \in A$. This is true, for otherwise, $|A \cap e| - |A \cap e'| = 1$, which contradicts the definition of $H(A, B)$. \hfill $\square$

Now we are ready for the proof of Proposition 1.3.

**Proof of Proposition 1.3.** Given $T$, together with the unique $k$-partition $\{V_1, \ldots, V_k\}$ of $V(T)$, choose $a(T)$ as the largest integer such that $a(T) \leq n/2$ and $a(T) \not= |\bigcup_{j \in J} V_j|$ for all $J \subseteq [k]$. Since $a(T)$ needs to avoid at most $2^k$ different values, it holds that $a(T) \geq \lfloor n/2 \rfloor - 2^k$. Set $f(T) = \lfloor n/2 \rfloor - a(T) + k - 1$.

Let $A, B$ be disjoint sets such that $|A| = a(T)$ and $|A \cup B| = n$, and consider the $k$-graph $H(A, B)$ as in Definition 3.1. Then $\delta_{k-1}(H(A, B)) \geq a(T) - k + 1 = \lfloor n/2 \rfloor - f(T)$ (by the observation after Definition 3.1), and $T$ does not embed into $H(A, B)$ because of Proposition 3.2 and by the choice of $a(T)$. \hfill $\square$

### 4. Overview of the proof of Theorem 1.2

Let $H$ be an $n$-vertex $k$-graph with $\delta_{k-1}(H) \geq (1/2 + \gamma)n$, and let $T$ be an $n$-vertex $k$-tree with $\Delta_1(T) \leq \Delta$. We start by partitioning $T$ into three edge-disjoint subgraphs $T_1, T_2, T_3$ such that

- (i) $T_1$ and $T_2$ are subtrees of $T$,
- (ii) $|V(T_1)| \approx \alpha n$ and $|V(T_3)| \approx \nu n$, for some $0 < \nu \ll \alpha \ll \gamma$,
- (iii) $V(T_1) \cap V(T_2) \in \partial T$, and
- (iv) $T_2$ is obtained from $T - T_1$ by removing ‘leaves’ one by one.
We call $T_2$ the bulk of $T$, which is the subgraph containing most vertices from $T$.

Building the absorbing structures: We use $T_1$ to build gadgets in $H$ which will allow us to extend a partial embedding of $T$ by adding vertices one by one. As we will see in Section 5, the link graph of a vertex in $T$ is a $(k-1)$-tree with $O(\Delta_1(T))$ vertices (Proposition 5.1). Since $|V(T_1)| \approx \alpha n$, there is a $(k-1)$-tree $X$ such that linearly many vertices from $T_1$ have $X$ as its link graph (here is crucial that $\Delta_1(T) = O(1)$). Our gadgets in $H$, called $X$-tuples, consists of a copy $\tilde{X}$ of $X$ and a special vertex $u^*$ such that $\tilde{X}$ is contained in the link graph of $u^*$. The idea behind this gadget is that any vertex whose link graph contains $\tilde{X}$ can be swapped with $u^*$in a potential embedding of $T_1$ (see Section 9 for details).

Using the large codegree in $H$, we can embed $T_1$ while covering a set of $\delta n$ disjoint $X$-tuples, with $\nu \ll \delta \ll \alpha$, which will be possible since $T_1$ contains linearly many vertices with $X$ as its link graph. Each $X$-tuple will be capable to 'absorb' one arbitrary vertex at the time, and so, in total, this family will be able to absorb one by one any sequence of $\delta n$ vertices.

Decomposing the bulk of $T$: In this step, we decompose $T_2$ into a constant number of smaller subtrees in a similar way as it has been done for trees in graphs [AKS95]. This is accomplished in Section 5, where, in particular, we discuss rooting a $k$-tree at a $(k-1)$-set of its vertices and also develop the notion of layerings of hypertrees, which resemble BFS-layerings of rooted graphs. Using these notions, we show (Lemma 5.15) that for any $\beta > 0$ one can decompose $T_2 = D_1 \cup \cdots \cup D_p$, with $p = O(\beta^{-1})$, so that each $D_i$ is a $k$-tree of size $O(\beta n)$. Moreover, these parts are edge-disjoint and $V(D_i) \cap V(D_j)$ is either empty or is an element of $\partial T_2$.

Embedding of $T_2$: The parts $D_1, ..., D_p$ can be ordered and each of them can be rooted so that the first $\ell$ parts, for any $\ell \leq p$, form a subtree of $T_2$ containing the ‘root’ of part $\ell + 1$. We will then embed the parts successively, embedding in each step one part (except its root, which is already embedded). Each $D_i$ will be embedded into a suitable part of the host hypergraph using the regularity method. Fortunately, the weak regularity lemma for hypergraphs (Theorem 6.1) is sufficient for our purposes here, which simplifies the technical details of the proof and also gives better bounds for $n_0$. We only use this lemma in order to find an almost perfect matching $\cal M$ in the corresponding reduced graph, which is a vertex disjoint collection of dense ‘regular $k$-tuples’ covering most of $H$.

Suppose we are about to embed the part $D_i$ which has its root already embedded. We first find an edge of $\cal M$ with sufficient free space, which spans a dense $k$-partite $k$-graph $F_i$ in $H$ where we will embed most of $D_i$. That is, we embed all but the first few layers of $D_i$ into $F_i$, because these layers will be needed to make the connection between the already embedded root of $D_i$ and $F_i$. It will be crucial here the bound on $\Delta_1(T)$. This will ensure that the number of vertices in the first few layers of $D_i$ is small and so most of $D_i$ is embedded into $F_i$.

In order to connect the root of $D_i$ with $F_i$, we will use a part of $H$ that we have separated earlier, before applying regularity, and that we will only use for the connections. This is the reservoir, a very small set $R \subseteq V(H)$ having (amongst others) the property that every $(k-1)$-set has many neighbours in $R$. The reservoir is found using a standard probabilistic argument (Lemma 6.7). A connecting lemma (Lemma 7.1) allows us to find many short walks between arbitrary pairs of ordered $(k-1)$-sets, whose internal vertices are all inside the reservoir, and an enhanced version of this lemma (Lemma 8.1) allows us to embed not only walks or paths, but instead bounded-size $k$-trees of bounded degree into the reservoir, joining given pairs of $(k-1)$-sets. This is what we need to finish the embedding described in the previous paragraph.

Absorption: Recall that $T_2$ was obtained from $T - T_1$ by removing leaves one by one, which implies that $T_3$ is spanned by the last $\nu n$ vertices in a valid ordering of $T - T_1$.

In order to embed $T_3$, we will use the collection of $X$-tuples we covered at the beginning of the proof, which is capable to absorb any sequence of $\delta n$ vertices. Since $T_3$ is formed by
a sequence of $\nu n \ll \delta n$ vertices, we can incorporate the vertices of $T_3$ one by one, using one $X$-tuple at the time, and thus finishing the embedding of $T$.

5. Hypertrees

In this section, we establish some structural results about hypertrees. Most importantly, we show any large hypertree can be decomposed into smaller hypertrees (see Lemma 5.15).

5.1. Link graph of a k-tree. For a $k$-graph $H$ and $v \in V(H)$, the restricted link graph of $v$ with respect to $H$, denoted $H(v)$, is the $(k-1)$-graph whose vertex set is $\bigcup\{e \setminus \{v\} : v \in e\}$ and its edge set is $\{e \setminus \{v\} : v \in e\}$.

**Proposition 5.1.** Let $k \geq 2$, let $T$ be a $k$-tree, and let $v \in V(T)$. Then $T(v)$ is a $(k-1)$-tree on at most $\Delta_1(T) + k - 1$ vertices.

**Proof.** Let $e_1, \ldots, e_m$ be a valid ordering of the edges of $T$, and let $I = \{i \in [m] : v \in e_i\}$. Then $E(T(v)) = \{e_1 \setminus \{v\} : i \in I\}$, and $I$ induces a valid ordering of $E(T(v))$, with $\alpha(v)$ being the first edge. So $T(v)$ is a $(k-1)$-tree. Since $v$ belongs to at most $\Delta_1(T)$ edges in $T$, we know that $T(v)$ has at most $\Delta_1(T)$ edges, and thus at most $\Delta_1(T) + k - 1$ vertices. \qed

5.2. Layerings. It is often convenient to root a 2-tree $T$ at some vertex $r \in V(T)$, which gives rise to a rooted tree $(T, r)$. Then one can define the $i$-th layer $L_i$ of $(T, r)$ as the set of all vertices at distance exactly $i$ from $r$ in $T$. The layers partition $V(T)$, and any vertex in layer $i + 1$ is joined to some vertex in layer $i$.

We now introduce a generalisation of these notions to higher uniformities.

**Definition 5.2.** A rooted $k$-tree is a pair $(T, x)$ where $T$ is a $k$-tree and $x \in \partial^0 T$.

**Definition 5.3** (Layering). Let $(T, x)$ be a rooted $k$-tree with $x = (x_1, \ldots, x_{k-1})$. A layering for $(T, x)$ is a tuple $\mathcal{L} = (L_1, \ldots, L_m)$, for some $m \in \mathbb{N}$, such that $\{L_1, \ldots, L_m\}$ is a partition of $V(T)$, and

- (L1) $x \cap L_i = \{x_i\}$ for all $i \in [k - 1]$, and $L_1 = \{x_1\}$,
- (L2) for each $v \in L_{i+1}$ with $1 \leq i < m$ there are $w \in L_i$, $e \in E(T)$ such that $\{v, w\} \subseteq e$, and
- (L3) for each $e \in E(T)$, there is $j \in [m]$ such that $|e \cap L_i| = 1$ for each $j \leq i < j + k$.

We call the tuple $(T, x, \mathcal{L})$ a layered $k$-tree.

Note that a layering $(L_1, \ldots, L_m)$ of $(T, x)$ is the preimage of the tight path on $m$ vertices under a homomorphism that maps all of $L_i$ to the $i$-th vertex of the tight path.

![Figure 1](image-url) **Figure 1.** On the left, a 3-tree $T$ with valid ordering $v_1, \ldots, v_{12}$. On the right, a table shows the layering $\mathcal{L} = (L_1, \ldots, L_6)$ of $T$ when it is rooted at $x = (v_1, v_2)$.

**Lemma 5.4.** Every rooted $k$-tree $(T, x)$ has a unique layering.
Proof. The proof is by induction on \(|E(T)|\). Let us note that if \(T\) is a single edge, then the first \(k - 1\) levels of the layering of \((T, x)\) correspond to \(x\), and the last level corresponds to the unique vertex in \(T - x\). So, we assume that \(|E(T)| \geq 2\) and, in some valid ordering of \(V(T)\), let \(v\) and \(e\) be the last vertex and last edge, respectively. Let \(e'\) be the parent of \(e\), and let \(w\) be the only vertex in \(e' \setminus e\).

We shall argue first about the existence of the layering for \((T, x)\), we will argue about uniqueness later. Let \(x = (x_1, ..., x_{k-1})\). We consider three cases. Suppose first that \(v \notin x\). Let \(x' = x\).

By induction, \((T - v, x')\) has a unique layering \(L' = (L'_1, ..., L'_{m})\) which we can extend to a layering of \((T, x)\) by either adding \(v\) to the layer \(L'_i\) containing \(w\), or (if all other vertices of \(e'\) lie in later layers than \(w\)) by adding \(v\) to \(L'_{i+k}\). Next, suppose \(v = x_j\) for some \(j \in [k-1] \setminus \{1\}\). Then we set \(x' = (x'_1, ..., x'_{k-1}) \in \partial^e T\), where \(x'_\ell = x_\ell\) for \(\ell \neq j\) and \(x'_j = w\). By induction, \((T - v, x')\) has a unique layering \((L'_1, ..., L'_m)\). We extend this layering to a layering of \((T, x)\) by adding \(v\) to the layer that hosts \(w\). It is easy to check that \((L1)\)–\((L3)\) hold for our layering of \(T\). Finally, suppose \(v = x_1\). In this case, set \(x' = (x_2, x_3, ..., x_{k-1}, w)\). Again, by induction \((T - v, x')\) has a unique layering \((L'_1, ..., L'_m)\). We set \(L_1 = \{v\}\) and \(L_i = L'_{i-1}\) for all \(2 \leq i \leq m + 1\). Again, \((L1)\)–\((L3)\) hold for \((L_1, ..., L_{m+1})\).

It is also straightforward to check that, in all cases, the obtained layering \(L\) must be unique, as the layering obtained from \(L\) by removing \(v\) will yield a layering of \((T - v, x')\), which is must be unique by induction.

Note that Definition 5.3 (L2) gives that \(|L_{i+1}| \leq \Delta_1(T)|L_i|\) for all \(i \in [m - 1]\). So, by \((L1)\) we have the following easy observation.

Proposition 5.5. Let \((T, x, L)\) be a layered \(k\)-tree with \(L = (L_1, ..., L_m)\). Then, \(|L_i| \leq (\Delta_1(T))^{i-1}\) for all \(i \in [m]\).

Recall that \(k\)-trees are \(k\)-partite, and that each \(k\)-tree \(T\) admits a unique \(k\)-partition \(\{V_1, ..., V_k\}\) of \(V(T)\). Given the layering \(L = (L_1, ..., L_m)\) of \((T, x)\), it is clear from Definition 5.3 (L3) that (after relabelling the partition classes) each \(V_i\) contains all layers \(L_{i+k}\). We use this to deduce that the sizes of the partition classes of a \(k\)-tree cannot differ too much, as detailed in the following lemma.

Proposition 5.6. Let \(\Delta, k \geq 2\) and let \(T\) be a \(k\)-tree with \(k\)-partition \(\{V_1, ..., V_k\}\) and with \(\Delta_1(T) \leq \Delta\). Then \(|V_i| \leq \Delta^k (1 + |V_j|)\) for each \(i, j \in [k]\).

Proof. Let \(L = (L_1, ..., L_m)\) be a layering of \(T\). By \((L1)\) and \((L2)\) we have that \(|L_1| = 1\) and \(|L_{i+1}| \leq \Delta_{i}(T)|L_i| \leq \Delta|L_i|\) for all \(i \in [m - 1]\). We can assume that \(V_1, ..., V_k\) are so that for each \(i \in [k]\), the set \(V_i\) contains precisely the layers \(\{L_{i+jk} : j \geq 0\}\).

So, for \(1 \leq i \leq k\), we have \(|V_i| = \sum_{j \geq 0} |L_{i+jk}| \leq \sum_{j \geq 0} |L_{i-1+jk}| = \Delta|V_{i-1}|\). Secondly, note that \(|V_i| = \sum_{j \geq 0} |L_{i+jk}| = |L_1| + \sum_{j \geq 1} |L_{i+jk}| \leq 1 + \sum_{j \geq 1} \Delta|L_{jk}| = 1 + \Delta|V_k|\). The desired bound follows by applying these inequalities repeatedly.

5.3. Pseudopaths in \(k\)-trees. A basic fact about \(2\)-trees is that every two vertices are joined by a unique path. We now introduce pseudopaths, which play a similar role in hypertrees.

Definition 5.7 (Pseudopath). A \(k\)-tree \(P\) is a pseudopath (of uniformity \(k\)) if there exists a valid ordering \(e_1, ..., e_t\) of \(E(P)\) such that for every \(i < t\), the only child of edge \(e_i\) is \(e_{i+1}\), in which case we say that \(e_1, ..., e_t\) is a path-ordering for \(P\).

Observe that a pseudopath \(P\) can have many different valid orderings of its edges which make it a \(k\)-tree, but not necessarily all valid orderings will be path-orderings. As an example, consider the \(k\)-tree \(F_{k,t}\) (see Figure 2) with vertex set \(\{v_0, v_1, ..., v_t\}\) and edges \(e_i = \{v_0, v_i, ..., v_{i+k-2}\}\), for \(1 \leq i \leq t - k + 2\). We easily see that \(e_1, ..., e_{i-k+2}\) is a path-ordering for \(F_{k,t}\). However, rooting \(F_{k,t}\) at any \((k - 1)\)-set \(f \subseteq e_i\), for \(2 \leq i \leq t - k - 3\), gives a valid ordering which is not a path-ordering.

Definition 5.8. Given a \(k\)-graph \(H\) and distinct \(f, f' \in \partial H\), an \((f, f')\)-pseudopath in \(H\) is a pseudopath \(P \subseteq H\) with a path-ordering \(e_1, ..., e_t\) such that \(f \subseteq e_i\) and \(f' \subseteq e_j\) if and only if \((i, j) = (1, t)\).
Then, setting \( r \) (Let \( x, y, z \) be the last vertex and last edge in a valid ordering of \( V(T) \). If \( v \notin f \cup f' \), then by induction, the tree \( T - v \) contains a unique \((f, f')\)-pseudopath \( P \). This path remains unique in \( T \). Indeed, note that in any \((f, f')\)-pseudopath \( P' \) in \( T \) with at least two edges the only vertices with degree 1 in \( P \) must be included in the first or last edge, and included in \( f \cup f' \). Since \( v \) has degree 1 in \( T \), if there were a \((f, f')\)-pseudopath \( P' \) in \( T \) with at least two edges including \( v \), this would imply that \( v \) is in the first or last edge of \( P' \), and also \( v \in f \cup f' \), a contradiction. It also cannot happen that \( P' \) is an \((f, f')\)-pseudopath consisting of a single edge and including \( v \), since then the edge is equal to \( f \cup f' \) and again it would imply that \( v \in f \cup f' \).

So assume \( v \in f \cup f' \). If \( v \in f \cap f' \), then \( f, f' \subseteq e \), and therefore, \( e \) is a \((f, f')\)-pseudopath, and it is unique. We can thus suppose that \( v \in f \setminus f' \), which implies \( f \subseteq e \). By induction, \( T - v \) contains a unique \((e \setminus \{v\}, f')\)-pseudopath \( P' \), which can be extended to an \((f, f')\)-pseudopath \( P \) by adding \( e \). Since \( v \) has degree 1 in \( T \), any \((f, f')\)-pseudopath in \( T \) contains \( e \). So, as \( P' \) was unique, \( P \) is unique too. \( \square \)

A set \( f \) of \( k - 1 \) vertices is said to lie on a pseudopath \( P \), if either \( P \) is an \((f, f')\)-pseudopath for some \( f' \), or \( f \) is contained in exactly two of the edges of \( P \).

**Definition 5.10** (Distance). Given a \( k \)-tree \( T \) and distinct \( f, f' \in \partial T \), the distance \( d_T(f, f') \) between \( f \) and \( f' \) is the number of edges in the unique \((f, f')\)-pseudopath connecting \( f \) with \( f' \).

If \( f = f' \), we let \( d_T(f, f') = 0 \).

Note that \( d_T(f, f') \geq 1 \) for all distinct \( f, f' \in \partial T \), with equality if and only if \( f \cup f' \in E(T) \).

Given tuples \( x, y \in \partial^o T \) and \( f \in \partial T \), we write \( d_T(x, y) \) for the distance between the underlying \((k - 1)\)-sets of \( x \) and \( y \), and let \( d_T(x, f) \) denote the distance between \( f \) and the underlying set of \( x \).

**Lemma 5.11.** Let \( P \) be an \((f, f')\)-pseudopath of uniformity \( k \) with a path-ordering \( e_1, \ldots, e_t \). Let \( x \in \partial^o P \) be such that \( x \subseteq e_1 \), and let \( \mathcal{L} = (L_1, \ldots, L_m) \) be the unique layering of \((P, x)\).

Then, setting \( r(j) = \min \{i : L_i \cap e_j \neq \emptyset \} \) for \( j = 1, \ldots, t \), we have

1. \( r(j + 1) - r(j) \in \{0, 1\} \) for all \( j \in [t - 1] \), and
2. \( |L_i| \leq k \Delta_1(P) \) for all \( i \in [m] \).

**Proof.** We begin by describing explicitly how can one construct \( \mathcal{L} \) by adding edges iteratively, as follows. First, start with all \( L_1, \ldots, L_m \) empty. Let \( x = (x_1, \ldots, x_{k-1}) \) and \( x_k \) be the unique vertex in \( e_1 \setminus x \). Begin by adding \( x_i \) to \( L_i \) for all \( 1 \leq i \leq k \). Now, given \( 2 \leq j \leq t \), assume that \( e_j \) already has been included, it has one vertex in each of the layers \( L_{i+1}, \ldots, L_{i+k} \), and we need to allocate \( e_{j+1} \). Let \( x, y \) be the unique vertices in \( e_j \setminus e_{j+1} \) and \( e_{j+1} \setminus e_j \), respectively. If \( x \notin L_{i+1} \), we add \( y \) to the same layer which contains \( x \); otherwise we add \( y \) to \( L_{i+k+1} \). It is straightforward to show by induction that this construction satisfies (L1)–(L3), and since there is a unique layering by Lemma 5.4, this construction precisely describes \( \mathcal{L} \).
Now, we show that (i) holds. Let $1 \leq j < t$. Note that since $|e_j \cap e_{j+1}| = k - 1$, together with (L3) it must hold that $|r(j+1) - r(j)| \leq 1$. Thus we only need to show that $r(j+1) \geq r(j)$, but this follows immediately from the iterative construction for $L$ which we described before.

For (ii), set $\Delta := \Delta_1(T)$ and observe that since $P$ is a pseudopath, for every $x \in V(P)$ there are $j \leq |E(P)|$ and $d < \Delta$ such that $x \in e_i$ if and only if $j \leq i \leq j + d$. In particular, because of (i) and (L3), we have
\[
e_i \cap L_{r(j)} = \emptyset \text{ for all } 1 \leq j \leq t \text{ and all } j + \Delta \leq i \leq t.
\] (5.1)

Now assume for contradiction that there is an index $i \in [m]$ with $|L_i| > k\Delta$. Note that each vertex in $L_i$ belongs to an edge that by (L3) meets the $k$ levels $L_{\ell}, L_{\ell+1}, \ldots, L_{\ell+k-1}$ for some $\ell \in \{i-k+1, \ldots, i\}$. So, there is an index $\ell \in \{i-k+1, \ldots, i\}$ such that more than $\Delta$ edges meet all of the levels $L_{\ell}, L_{\ell+1}, \ldots, L_{\ell+k-1}$. Let $j \in [t]$ be minimum with the property that $r(j) = \ell$. Then by (5.1), only edges $e_j, e_{j+1}, \ldots, e_{j+\Delta-1}$ may meet $L_{\ell}$. As these are only $\Delta$ edges, we arrive at a contradiction, as desired.

5.4. Cutting $k$-trees. We will now show how to partition a $k$-tree into smaller $k$-subtrees of controlled size. Given a layered $k$-tree $(T, x, \mathcal{L})$, with $\mathcal{L} = (L_1, \ldots, L_m)$, and given $s = (s_1, \ldots, s_{k-1}) \in \partial^* T$, we say $s$ is $\mathcal{L}$-layered if $s \cap L_i = \{s_i\}$ for each $i = j, \ldots, j + k - 2$ for some $j \in [m]$, that is, $s$ meets $k - 1$ consecutive layers of $\mathcal{L}$. In that case we say that $j$ is the rank of $s$.

**Definition 5.12** (Induced $k$-subtree). Let $(T, x, \mathcal{L})$ be a layered $k$-tree, with $\mathcal{L} = (L_1, \ldots, L_m)$, and let $s \in \partial^* T$ be $\mathcal{L}$-layered. The tree $T_s$ induced by $s$ is the $k$-subtree of $T$ spanned by $\bigcup_{i \geq 0} E_i$ where $E_0 := \{s \cup \{v\} : \alpha(v) = s\}$ and $E_{i+1}$ contains all children of edges in $E_i$. Observe that $T_s$ might be edgeless. Write $T-T_s$ for the tree obtained from $T$ by deleting all edges in $E(T_s)$, and all vertices in $V(T_s) \setminus s$.

Clearly, if $T$ is rooted at $x$, then $T_x = T$. Observe that if $(T_s, s)$ is an induced $k$-subtree of $(T, x)$, and $s' \in \partial^* T_s$ is $\mathcal{L}$-layered, then the induced $k$-subtree $((T_s)_{s'}, s')$ of $(T_s, s)$ is also an induced $k$-subtree of $(T, x)$, and we have $((T_s)_{s'}, s') = (T_{s'}, s')$. Note that for each $f \in \partial T_s$, the underlying set of $s$ lies on the unique pseudopath from $f$ to the root in $T$. Moreover, $T_s$ inherits a valid ordering and a layering from $(T, x, \mathcal{L})$, with layers $L_j \cap V(T_s)$, which we call the inherited layering of $(T_s, s)$ and denote by $\mathcal{L}^s$.

The following observation will be useful in a moment.

**Proposition 5.13.** Let $(T, x, \mathcal{L})$ be a layered $k$-tree with $\mathcal{L} = (L_1, \ldots, L_m)$, $\Delta_1(T) \leq \Delta$ and $k \geq 2$. Let $F \subseteq E(T)$ be the set of all edges meeting $L_1$ and and let $S \subseteq \partial^* T$ consist of all the $\mathcal{L}$-layered tuples whose unordered vertices are in $\{e \setminus L_1 : e \in F\}$. Then,

(i) each $s \in S$ is $\mathcal{L}$-layered and has rank 2,

(ii) $|F| = |S| \leq \Delta$, and

(iii) $E(T) = F \cup \bigcup_{s \in S} E(T_s)$.

**Proof.** As $\Delta_1(T) \leq \Delta$ and $|L_1| = 1$, we have $|F| \leq \Delta$. The other properties are easy to see.

The next definition captures the previously mentioned partition of a $k$-tree. Intuitively, it ensures that the small $k$-trees in the partition are of controlled size (no small $k$-tree is too large, and there are not many $k$-trees in the partition). Also, the roots of each $k$-tree are “far apart” from each other, as measured by their rank.

**Definition 5.14** (($\beta, d$)-decomposition). Let $\Delta, k \geq 2$, and let $(T, x, \mathcal{L})$ be a layered $k$-tree. For $\beta \in (0, 1)$ and $d \geq 1$, a ($\beta, d$)-decomposition of $(T, x, \mathcal{L})$ is a tuple $(D_i, s_i)_{1 \leq i \leq m}$ of rooted $k$-subtrees of $T$ such that

(i) $m \leq 2\Delta^d/\beta$,

(ii) $E(T) = \bigcup_{1 \leq i \leq m} E(D_i)$,

(iii) $|E(D_i)| \leq \beta |E(T)|$ for each $1 \leq i \leq m$,

(iv) $s_i = x$ and each $s_i$ is $\mathcal{L}$-layered,

(v) $(V(D_l) \setminus s_l) \cap V(D_i) = \emptyset$ for all $1 \leq i < l \leq m$, and
Lemma 5.15. Let $\Delta, k \geq 2$, $d \geq 1$, $\beta \in (0,1)$, and let $(T,x,\mathcal{L})$ be a layered $k$-tree with $t \geq 2\Delta^d \beta^{-1}$ edges satisfying $\Delta_1(T) \leq \Delta$. Then $T$ has a $(\beta,d)$-decomposition.

Proof. We will find the trees $(D_i,s_i)_{1 \leq i \leq m}$ inductively. At the end of each step $j \geq 0$, we will found trees $D_1, D_2, \ldots, D_j$ fulfilling properties (iii)--(vi) from Definition 5.14, with $m$ replaced by $j$. Moreover, there will be a set $S_j \subseteq \partial^o T$ such that

(a) $E(T) = \bigcup_{1 \leq i \leq j} E(D_i) \cup \bigcup_{s \in S_j} E(T_s)$,

(b) for each $s \in S_j \setminus \{x\}$, there is a unique $i \leq j$ such that $s \in \partial^o D_i$, the rank of $s$ in $D_i$ is at least $d$, and $(V(T_s) \setminus s) \cap \bigcup_{1 \leq i \leq j} V(D_i) = \emptyset$,

(c) $|E(D_i)| \geq \beta t/(2\Delta^d)$ for each $1 \leq i \leq j$, and

(d) $|E(T_s)| \geq \beta t/(2\Delta^d)$ for each $s \in S_j$.

Note that (c) guarantees that we stop in some step $m \leq 2\Delta^d/\beta$ with $S_m = \emptyset$. This, together with (a), ensures (i) and (ii) hold.

We start the procedure setting $S_0 = \{x\}$, with all properties trivially fulfilled. Now assume we are in step $j \geq 1$. Choose any $s \in S_{j-1}$. By (d), we have $|E(T_s)| \geq \beta t/(2\Delta^d)$. If $|E(T_s)| \leq \beta t$, then set $D_j := T_s$, $s_j := s$ and $S_j := S_{j-1} \setminus \{s\}$ and end step $j$. Otherwise, apply Proposition 5.13 to $(T_s, s)$, obtaining a set $F_1$ of edges, and a set $S_j'$ of $\Delta$-layered elements of $\partial^o T_s \subseteq \partial^o T$ of rank 2 in $T_s$ (that is, $F_1$ and $S_j'$ are the sets $F$ and $S$ from the statement of Proposition 5.13). Apply Proposition 5.13 to all trees $T'$ with $s' \in S_1$, thus generating a set $F_2$ of edges and a set $S_2$, such that each $s' \in S_2$ is $\Delta$-layered and has rank 3 in $T_s$. Continue in this manner until reaching a set $S_{d-1}$ of $\Delta$-layered elements of rank $d$, and set $F := \bigcup_{1 \leq i \leq d-1} F_i$. Note that $|S_{d-1}| \leq |F| \leq \Delta^d$ and the edges in $F$ span a $k$-tree $T_F$ rooted at $s$. Next, for each $s' \in S_d'$, in order, consider the tree $T_{s'}$. If $|E(T_{s'})| < \beta t/(2\Delta^d)$, then add $T_{s'}$ to $T_F$ and delete $s'$ from $S_d'$, and continue to examine the next $s' \in S_d'$. At the end of this process, we obtain a tree $B_1 \supseteq T_F$ and a set $Z_1 \subseteq S_d'$. Note that

$$|E(B_1)| \leq |F| + |S_{d-1}|(\beta t/(2\Delta^d)) \leq |F|(1 + \beta t/(2\Delta^d)) \leq \beta t < |E(T_s)|,$$

which implies that $Z_1 \neq \emptyset$. Moreover, we have that $|E(T_s)| \geq \beta t/(2\Delta^d)$ for each $z \in Z_1$.

Let us note here that if $|B_1| \geq \beta t/2$, then we could set $D_j := B_1$, $s_j := s$, and $S_j = (S_{j-1} \cup Z_1) \setminus \{s\}$, thus completing the inductive step. So, let us suppose that $|B_1| < \beta t/2$. In what follows next we will, gradually, add edges to $B_1$ to make it have size between $\beta t/2$ and $\beta t$. To do this, successively, for $i \geq 1$, choose any $z \in Z_i$ and apply Proposition 5.13 to $T_z$. Add the resulting edges given by Proposition 5.13 to $B_1$, obtaining the set $B_i \supseteq B_i$, and let $S$ be the subset of $\partial T$ from the lemma. For each $s' \in S$, check whether $|E(T_{s'})| < \beta t/(2\Delta^d)$, and if this is the case, then add $T_{s'}$ to $B_i$ and delete $s'$ from $S$. After processing all $s' \in S$, this results in a set $S'$, and a tree $B_{i+1}$. Set $Z_{i+1} := (Z_i \cup S') \setminus \{z\}$. Then $|E(B_{i+1})| \leq |E(B_i)| + |S'| (\beta t/(2\Delta^d)) \leq |E(B_i)| + |S' / \Delta^d_{i-1}| + |E(T_s)| \geq \beta t/(2\Delta^d)$ for each $z \in Z_i$.

We continue until we reach the first index $h$ with $|E(B_h)| \geq \beta t/2$ (this must happen at some point, since in each step, at least one edge from $E(T_s)$ is added to $E(B_i)$, and $|E(T_s)| > \beta t$). Then $|E(B_h)| \leq \beta t$. Set $D_j := B_h$, $s_j := s$, and set $S_j := \{S_{j-1} \cup Z_h\} \setminus \{s\}$. By construction, (a)--(d) and (iii)--(vi) from Definition 5.14 hold for $S_j$ and $D_1, \ldots, D_j$. \qed

6. Tools

In this section, we collect some tools that will be needed for the proof of Theorem 1.2.

6.1. The weak hypergraph regularity lemma. Let $H$ be a $k$-graph and let $V_1, \ldots, V_k$ be pairwise disjoint subsets of $V(H)$. Let $H[V_1, \ldots, V_k]$ be the $k$-partite subhypergraph of $H$ induced by all edges that intersect all sets $V_i$. The density of $H[V_1, \ldots, V_k]$ is defined as

$$d(V_1, \ldots, V_k) := \frac{e_H(V_1, \ldots, V_k)}{|V_1| \cdots |V_k|},$$
where \( \epsilon_H(V_1, ..., V_k) \) denotes the number of edges in \( H[V_1, ..., V_k] \). For \( \epsilon, d > 0 \), we say a \( k \)-tuple \((V_1, ..., V_k)\) of pairwise disjoint non-empty subsets of \( V(H) \) is \((\epsilon, d)\)-regular if
\[
|d(W_1, ..., W_k) - d| \leq \epsilon
\]
for all \( k \)-tuples of subsets \( W_i \subseteq V_i \) satisfying \(|W_1| \cdots |W_k| \geq \epsilon |V_1| \cdots |V_k|\). A \( k \)-tuple \((V_1, ..., V_k)\) will be called \( \epsilon \)-regular if it is \((\epsilon, d)\)-regular for some \( d > 0 \).

The weak regularity lemma for hypergraphs ensures that the vertex set of every \( k \)-graph can be partitioned into a bounded number of clusters, such that almost all \( k \)-tuples of these clusters are \( \epsilon \)-regular. We will use the lemma in the following form (see [Koh+10, Theorem 9]).

**Theorem 6.1 (Weak Hypergraph Regularity Lemma).** Let \( k \geq 2 \) and let \( 1/n, 1/T_0 \ll 1/t_0, 1/k, \epsilon \). For every \( k \)-graph \( H \) on \( n \) vertices there exists a partition \( \{V_0, V_1, ..., V_t\} \) of \( V(H) \) such that

1. \( t_0 \leq t \leq T_0 \),
2. \( |V_0| \leq \epsilon n \) and \( |V_1| = \cdots = |V_t| \), and
3. for all but at most \( \epsilon(t) \) sets \( \{i_1, ..., i_k\} \subseteq \{t\} \), the \( k \)-tuple \((V_{i_1}, ..., V_{i_k})\) is \( \epsilon \)-regular.

Any partition \( \mathcal{P} = \{V_0, V_1, ..., V_t\} \) of \( V(H) \) satisfying (i)–(iii) will be called an \( \epsilon \)-regular partition of \( H \). Given \( d > 0 \), we define the \( d \)-reduced \( k \)-graph \( R_d(H) \) of \( H \) with respect to \( \mathcal{P} \) as follows. Its vertex set is \( [t] = \{1, ..., t\} \), and its edges are the \( k \)-sets \( \{i_1, ..., i_k\} \) such that \( d_H(V_{i_1}, ..., V_{i_k}) \geq d \) and \((V_{i_1}, ..., V_{i_k})\) is \( \epsilon \)-regular. We will also refer to \( R_d(H) \) as “the” \( d \)-reduced \( k \)-graph of \( H \). (Even if \( R_d(H) \) depends on the choice of \( \mathcal{P} \), we omit explicit reference to \( \mathcal{P} \) in the notation for simplicity.)

We will need to find almost-perfect matchings in the reduced \( k \)-graph. For \( k = 2 \), it is easy to find one using graph regularity, and for \( k \geq 3 \) its existence may be deduced from Claims 4.4 and 4.5 in [RRS08].

**Lemma 6.2.** Let \( k \geq 2 \), \( 0 < 1/n \ll 1/t \ll \epsilon \ll 1/k, \gamma, \eta \), and let \( d \ll \gamma \). Let \( H \) be a \( k \)-graph on \( n \) vertices with \( d_{k-1}(H) \geq (1/2 + \gamma)n \). Let \( \mathcal{P} = \{V_0, V_1, ..., V_t\} \) be an \( \epsilon \)-regular partition of \( V(H) \). Then the \( d \)-reduced \( k \)-graph \( R_d(H) \) has a matching covering at least \((1 - \eta)t \) vertices.

6.2. Degenerate hypergraphs and extensible edges. Given \( k \geq 2 \) and \( s \in \mathbb{N} \), let \( K^{(k)}(s) \) denote the complete \( k \)-partite \( k \)-graph with each class of size \( s \). To be precise, \( V(K^{(k)}(s)) \) is partitioned in \( k \) clusters \( V_1, ..., V_k \) of size \( s \) each, and its edges are precisely the \( k \)-sets which intersect each \( V_i \) exactly once. The following result, due to Erdős [Erd64b], is a hypergraph version of the classical Kővári–Sós–Túran theorem [KTS54].

**Lemma 6.3.** Let \( 1/n \ll 1/k, 1/s, \epsilon \). Let \( H \) be a \( k \)-graph with \( n \) vertices and at least \( \epsilon n^k \) edges. Then \( H \) contains a copy of \( K^{(k)}(s) \) as a subgraph.

Note that for any \( k \geq 2 \), the complete \( k \)-partite \( k \)-graph \( K^{(k)}(2) \) has \( 2k \) vertices and \( 2^k \) edges. Given a \( k \)-graph \( H \) and an edge \( e \in H \), let \( d_H^K(e) \) be the number of copies of \( K^{(k)}(2) \) in \( H \) in which \( e \) participates. Note that \( d_H^K(e) \leq \binom{n-k}{k} \) always.

**Definition 6.4 (\( \theta \)-extensible edge).** Given an \( n \)-vertex \( k \)-graph \( H \) and \( \theta > 0 \), we say an edge \( e \in H \) is \( \theta \)-extensible if \( d_H^K(e) \geq \theta \binom{n-k}{k} \).

Extensible edges will be useful in our embedding of \( k \)-trees. We show that in an appropriately dense \( k \)-graph most edges are extensible.

**Lemma 6.5.** Let \( 1/n, \theta \ll \epsilon, 1/k \). In any \( k \)-graph on \( n \) vertices, all but at most \( \epsilon \binom{n}{k} \) edges are \( \theta \)-extensible.

**Proof.** Lemma 6.3 implies that the Turán density of \( K^{(k)}(2) \) is zero. Hence, by standard supersaturation arguments [Kee11, Lemma 2.1], there exist \( n_0 \) and \( \alpha > 0 \) such that every \( k \)-graph on \( n \geq n_0 \) vertices and at least \( \epsilon \binom{n}{k} \) edges has at least \( \alpha (\binom{n}{k}^e) \) copies of \( K^{(k)}(2) \). To prove the lemma we shall use \( n \geq n_0 \) and \( \theta \ll (k!)^2 2^{2k} \alpha/(2k!) \).

Indeed, let \( H \) be any \( k \)-graph on \( n \) vertices and let \( H' \subseteq H \) be the \( k \)-graph formed by the non-\( \theta \)-extensible edges of \( H \). To reach a contradiction, suppose that \( H' \) has at least \( \epsilon \binom{n}{k} \) edges. By
the choice of \(n_0\) and \(\alpha\), we know that \(H'\) contains at least \(\alpha \binom{n}{2k}\) copies of \(K^{(k)}(2)\). Note that \(H'\) has at most \(\binom{n}{k}\) edges and recall that each copy of \(K^{(k)}(2)\) has \(2^k\) edges. Therefore, a double-counting argument shows that some edge \(e\) in \(H'\) participates in at least \(2^k \alpha \binom{n}{2k}/(\binom{n}{k}) \geq \theta(n^{-k})\) copies of \(K^{(k)}(2)\). So, \(d^H_H(e) \geq \theta(n^{-k})\), or in other words, \(e\) is a \(\theta\)-extensible edge of \(H'\), and therefore of \(H\). This contradicts the definition of \(H'\).

6.3. Reservoirs. Let \(H\) be a \(k\)-graph \(H\), and let \(F \subseteq \partial H\). Recall that \(\deg_H(F)\) denotes the joint degree of \(F\), as defined in (1.1). For \(U \subseteq V(H)\), we let

\[
\deg_H(F, U) = |\{v \in U : f \cup \{v\} \in H \text{ for each } f \in F\}|	ag{6.1}
\]

Similarly, recalling that previously we defined \(d^H_H(e)\) as the number of copies of \(K^{(k)}(2)\) in the \(k\)-graph \(H\) that contain the edge \(e \in E(H)\), we define \(d^H_H(e, U)\) as the number of copies of \(K^{(k)}(2)\) in \(H[U \cup e]\) that contain \(e\).

**Definition 6.6** (Reservoir). Let \(H\) be a \(k\)-graph on \(n\) vertices, and let \(\gamma, \mu > 0\). We say that a set \(U \subseteq V(H)\) is a \((\gamma, \mu, h)\)-reservoir for \(H\) if

\begin{itemize}
  \item[(i)] \(|U| = (\gamma \pm \mu)n\),
  \item[(ii)] for every \(F \subseteq \partial H\) with \(|F| \leq h\) we have \(\deg_H(F, U) \geq (\deg_H(F)/n - \mu)|U|\), and
  \item[(iii)] for every \(e \in H\), we have \(d^H_H(e, U) \geq (d^H_H(e)/(\binom{n}{k} - \mu))(|U|^{k-1})\).
\end{itemize}

**Lemma 6.7** (Reservoir Lemma). Let \(1/n < \mu \leq \gamma, 1/h \leq 1\). Then every \(k\)-graph \(H\) on \(n\) vertices has a \((\gamma, \mu, h)\)-reservoir.

The proof of Lemma 6.7 is probabilistic, and we will use the following standard concentration inequalities for random variables.

**Theorem 6.8** (Chernoff’s inequality [JLR00, Theorem 2.1]). Let \(0 < \alpha < 3E[X]/2\) and \(X \sim \text{Bin}(n, p)\) be a binomial random variable. Then \(Pr(|X - E[X]| > \alpha) < 2 \exp(-\alpha^2/(3E[X]))\).

**Theorem 6.9** (McDiarmid’s inequality [McD89]). Suppose \(X_1, ..., X_m\) are independent Bernoulli random variables and \(b_1, ..., b_m \in [0, B]\). Suppose \(X\) is a real-valued random variable determined by \(X_1, ..., X_m\) such that changing the outcome of \(X_i\) changes \(X\) by at most \(b_i\) for all \(1 \leq i \leq m\). Then, for all \(\lambda > 0\), we have

\[
Pr(|X - E[X]| > \lambda) \leq 2 \exp\left(-\frac{2\lambda^2}{B \sum_{i=1}^m b_i}\right).
\]

**Proof of Lemma 6.7.** Choose a set \(U \subseteq V(H)\) randomly by independently including each vertex of \(V(H)\) with probability \(p = \gamma\). With non-zero probability \(U\) will satisfy all of the properties (i)–(iii) simultaneously, which shows the desired set \(U\) exists.

Indeed, \(E[|U|] = pn = \gamma n\). Thus, using Chernoff’s inequality (Theorem 6.8) with \(\alpha = (n^{1/3}\gamma)^{-1}\) we get that \(|U| = \gamma n \pm n^{2/3}\) fails to hold with probability at most \(2 \exp(-n^{1/3}/(3\gamma))\).

Since \(n\) is sufficiently large, \(|U| = \gamma n \pm n^{2/3}\) holds with probability at least \(1 - 1/n\) and we will assume those bounds on \(|U|\) from now on. Note also this implies (i) holds for \(U\).

Now we verify (ii) holds. Let \(F \subseteq \partial H\) of size at most \(h\) and note that \(E[\deg_H(F, U)] = p \deg_H(F)\). If \(\deg_H(F, U) < \mu n\) then there is nothing to show, so we assume otherwise. In particular, \(E[\deg_H(F, U)] \geq \mu n\). If \(\deg_H(F, U) < (\deg_H(F)/n - \mu)|U|\), then \(\deg_H(F, U) \leq (\deg_H(F)/n - \mu)(\gamma n + n^{2/3}) \leq E[\deg_H(F, U)] - \mu \gamma n/2\) since \(n\) is large. Apply Chernoff’s inequality with \(\alpha = \mu \gamma n/2 \leq 3E[\deg_H(F, U)]/2\) to get

\[
Pr[\deg_H(F, U) < (\deg_H(F)/n - \mu)|U|] \leq Pr[|\deg_H(F, U) - E[\deg_H(F, U)]| > \mu \gamma n/2] \leq 2 \exp\left(-\frac{(\mu \gamma)^2 n}{12}\right).
\]

Since \(|\partial H| \leq n^{k-1}\) and \(|F| \leq h\), there are at most \(n^{h(k-1)}\) possible choices for \(F\). Then a union bound shows that (ii) fails to hold with probability at most \(2n^{h(k-1)} \exp(-\beta \gamma^2 n/12) < 1/n\), where the last inequality holds since \(n\) is large.
To see (iii), fix an edge $e \in H$. If $d^K_H(e) < \mu(n/k)$ then there is nothing to show, so assume otherwise. Let $X = d^K_H(e, U)$ and note that $E[X] = p^k d^K_H(e)$. Since $|U| = pn \pm n^{2/3}$ and $1/n \ll 1/k$, we have $(|U| - k)/k = (1 + o(1)) \gamma(n/k)$. The presence of a vertex in $U$ can affect $X$ by at most $n^{k-1}$. Thus, we can apply McDiarmid’s inequality (Theorem 6.9) with $m = n$ and $B = b_1 = n^{k-1}$ for all $1 \leq i \leq n$ to see that

$$\Pr \left[ \deg^K_H(e, U) < \left( \frac{n - k}{k} \right) - \mu \left( \frac{|U| - k}{k} \right) \right] \leq \Pr \left[ X < E[X] - \mu \gamma(k/n) \right] / 2 \leq 2 \exp \left( -\mu^2 \gamma^2 n^{2k} / (2k^{2k} n^{2k-1}) \right),$$

where in the last inequality we also used $(\binom{n}{k}) \geq (n/k)^k$. The last term is less than $1/n^{k+1}$ since $n$ is sufficiently large. Since there are at most $n^k$ edges in $H$, we see (iii) fails with probability at most $1/n$, as required.

7. Connections

For this section, the following notion will be essential. We say $k$-graph $H$ is $\ell$-large if every two $f, f' \in \partial H$ have at least $\ell$ common neighbours. For instance, $k$-graphs $H$ on $n$ vertices with minimum codegree at least $(1/2 + \gamma)n$ are $2\gamma n$-large, and $(\rho^2, \varepsilon)$-typical graphs $H$ on $n$ vertices are $(\rho^2 - \varepsilon)n$-large. Note that a $k$-graph can be $\ell$-large and have isolated vertices (i.e. vertices which do not lie in any edge), since the property only says something about tuples in $\partial H$. For $U \subseteq V(H)$, we say that $H$ is $(\ell, U)$-large if every two $f, f' \in \partial H$ have at least $\ell$ common neighbours in $U$ (Later, $U$ will be a reservoir).

The main result of the current section is Lemma 7.1, which essentially says that in any $\Omega(n)$-large $k$-graph $H$ we can connect any two elements of $\partial H$ by a walk (actually, many such walks) of length exactly $\ell$, where $\ell$ is a number only depending on $k$. We observe that Lemma 7.1 can be seen as a strengthening of the ‘Connecting lemma’ of Rödl, Ruciński and Szemerédi [RRS08, Lemma 2.4]. For our approach, however, it is crucial that we can control the precise length of the walk (instead of only having an upper bound). We will also control the number of internal vertices of the walk, and the order of the elements $f, f'$ we are connecting.

Recall that $\partial^\ell H$ denotes the ordered shadow of $H$, and that the interior $\text{int}(W)$ of a walk $W$ corresponds to the set of vertices $V(W) \setminus \text{sta}(W) \cup \text{ter}(W)$.

Lemma 7.1 (Connecting Lemma). For integers $k, \ell$ with $k \geq 2$ and $\ell \geq (2k + 1)k/2 + 2k$, there exists $q \leq \ell$ with the following property. Let $1/n \ll \gamma \ll 1/k, 1/\ell$. Let $H$ be a $k$-graph on $n$ vertices which is $(2\gamma n, U)$-large for some $U \subseteq V(H)$, and let $x, x' \in \partial^\ell H$. Then there are at least $(\gamma n)^q$ many walks going from $x$ to $x'$, each of length $\ell$ and with $q$ internal vertices all from $U \setminus (x \cup x')$.

To prove Lemma 7.1, it will be useful to find a walk which swaps the order in which the vertices of a given edge appear. This will be achieved in the following two short lemmas. The first lemma effectively swaps the position of two vertices in an ordered edge $(b_j$ and $b_{k-j+1}$ in the statement below).

Lemma 7.2. Let $k \geq 2$ and let $(b_1, \ldots, b_k)$ be an ordered edge in a $k$-graph $H$. Let $1 \leq j \leq [k/2]$, and let $u \in N_H(b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_k) \cap N_H(b_1, b_{k-j}, b_{k-j+2}, \ldots, b_k)$. Then $H\left[ \bigcup_{1 \leq i \leq k} \{b_i\} \cup \{u\} \right]$ contains a walk of length $2k + 1$ from $(b_1, \ldots, b_k)$ to $(b_1, \ldots, b_{j-1}, b_{k-j+1}, b_{j+1}, b_{j+2}, \ldots, b_{k-j}, b_j, b_{k-j+2}, \ldots, b_k)$.

Proof. It suffices to consider the walk (each $k$ consecutive vertices form an edge)

$$b_1 \cdot b_k b_1 \cdot b_{j-1} \cdot b_{j+1} \cdots b_{k-j} b_{k-j+2} \cdots b_k \cdots b_{j-1} b_{k-j+1} b_{j+1} \cdots b_{k-j} b_j b_{k-j+2} \cdots b_k,$$

which uses $3k$ vertices and thus has length $2k + 1$.

Lemma 7.3. Let $k \geq 2$ and let $(a_1, \ldots, a_k)$ be an edge in a $k$-graph $H$. For each $j \leq [k/2]$, let $u_j \in N_H(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k) \cap N_H(a_1, \ldots, a_{k-j}, a_{k-j+2}, \ldots, a_k)$. Then $H\left[ \bigcup_{1 \leq i \leq k} \{a_i\} \cup \bigcup_{1 \leq j \leq [k/2]} \{u_j\} \right]$ contains a walk of length $[k/2](2k + 1)$ from $(a_1, \ldots, a_k)$ to $(a_k, \ldots, a_1)$.
Proof. We use Lemma 7.2 successively, for all \( j \leq \lfloor k/2 \rfloor \), thus swapping the vertices \( a_j \) and \( a_{k-j+1} \) in the walk until we reach \((a_k, ..., a_1)\). This gives a walk of length \( \lfloor k/2 \rfloor (2k+1) \).  

Now we can prove Lemma 7.1.

Proof of Lemma 7.1. Let \( x = (x_1, ..., x_{k-1}) \) and \( x' = (x'_1, ..., x'_{k-1}) \), and set \( \ell_0 := (2k+1)\lfloor k/2 \rfloor + 2k \). Let \( x''_R = (x''_{k-1}, ..., x'_1) \) be the reverse of the tuple \( x' \). Greedily construct a tight path \( P_1 \) of length \( \ell - \ell_0 \), starting at \( x''_R \), ending at some \( y = (y_1, ..., y_{k-1}) \), and using only vertices in \( U \setminus x \). This can be done: as every \( (k-1) \)-set has at least \( 2\gamma n \) neighbours in \( U \), at each step we need to avoid at most \( |x \cup x'| - (\ell - \ell_0) \leq 2k + \ell \) vertices, and so \( \ell_0 \ll 1/k, 1/\ell \) implies there are at least \( \gamma n \) choices at each step. Set \( Z := V(P_1) \setminus x \) and \( q := k + \lfloor k/2 \rfloor + \ell - \ell_0 = k + \lfloor k/2 \rfloor + |Z| \).

Building on \( P_1 \), we will first construct a single walk as promised in the lemma, afterwards we will estimate in how many ways this can be done. Let \( a_1 \in (N_H(x) \cap N_H(x'') \cap U) \setminus (Z \cup x \cup x') \).

Having defined \( a_1, ..., a_j \) for some \( j \in [k-1] \), we choose an arbitrary unused vertex

\[
a_{j+1} \in N_H(x_{j+1}, ..., x_{k-1}, a_1, ..., a_j) \cap N_H(y_{j+1}, ..., y_{k-1}, a_1, ..., a_j) \cap U.
\]

Clearly \( P_2 = x_1 \cdots x_{k-1} \cdots a_1 \cdots a_k \cdot a_k \cdots a_{k-1} \cdots y_{k-1} \cdots y_1 \) are tight paths. Applying Lemma 7.3, we find a walk \( P_3 \) which goes from \((a_1, ..., a_k)\) to \((a_k, a_{k-1}, ..., a_1)\) only occupying unused vertices \( u_1, ..., u_{|k/2|} \) from \( U \) (the vertices \( u_j \) exist because \( H \) is \((2\gamma n, U)\)-large). Concatenating the walks \( P_2, P_3, P_4 \) and \( P_1 \) (the latter traversed in reverse order) gives a walk \( W \) from \( x \) to \( x' \). Set \( Q = Z \cup \{a_i : 1 \leq i \leq k\} \cup \{u_j : 1 \leq j \leq \lfloor k/2 \rfloor\} \). Then \( Q = \text{int}(W) \) and \( |Q| = q \), and the length of \( W \) is \( |E(P)| = k + 2k\lfloor k/2 \rfloor + \lfloor k/2 \rfloor + k + \ell - \ell_0 = \ell \), so \( W \) is a walk which satisfies the required properties.

![Figure 3](image-url)  

Figure 3. This figure shows how to construct the walk \( x_1x_2a_1a_2a_4a_3a_2a_1y_2y_1 \) connecting \((x_1, x_2)\) with \((y_2, y_1)\) in the 3-uniform case.

Note that by construction, each vertex in the interior of \( P \) is chosen as an arbitrary unused vertex in the neighbourhood of one or two \((k-1)\)-sets. Since \( H \) is \((2\gamma n, U)\)-large, the common neighbourhoods in \( U \) have size at least \( 2\gamma n \), and thus (here we use \( 1/n \ll 1/k, 1/\ell \)) in every step we have at least \( 2\gamma n - q \geq \gamma n \) possible choices in \( U \). Thus by the previous discussion there are at least \( (\gamma n)^9 \) many different walks with the required properties.

8. Embedding large hypertrees

In this section, we prove an embedding result (Lemma 8.1) that will allow us to embed a small rooted bounded degree tree into a dense \( k \)-partite graph, while at the same time controlling the location of the root and the bulk of the tree quite accurately.

Given positive integers \( \ell_1 \leq \ell_2 \leq m \), and given a layering \( L = (L_1, ..., L_m) \) of a rooted \( k \)-tree \((T, x)\), we say that \( V_{[\ell_1, \ell_2]}(L) := \bigcup_{i = \ell_1}^{\ell_2} L_i \) is the \([\ell_1, \ell_2]\)-interval of \( T \). If the layering is clear from the context, we just write \( V_{[\ell_1, \ell_2]} \). If, moreover, \( |L_i| \leq M \) for each \( \ell_1 \leq i \leq \ell_2 \), we say \( V_{[\ell_1, \ell_2]} \) that is \( M \)-bounded. We write \( x \in \partial_T V_{[\ell_1, \ell_2]} \) if there is a \( j \in [\ell_1, \ell_2 - k + 1] \) such that \( x = (x_1, ..., x_{k-1}) \in \partial_T \) with \( x_i \in L_j \) for all \( i \in [k-1] \).

We can now state the Embedding Lemma.
Lemma 8.1 (Embedding Lemma). Let $\Delta, k \geq 2$, let $\ell \geq \lceil k/2 \rceil (2k + 1) + 2k$ and let $1/n, \mu \ll \beta, \theta \ll 1/k, 1/\Delta, c, \gamma, d$. Let $H$ be a $\gamma n$-large $k$-graph on $n$ vertices with a $(\gamma, \mu, 2)$-reservoir $R$. Let $W_1, \ldots, W_k \subseteq V(H) \setminus R$ be all pairwise disjoint, and such that $d(W_1, \ldots, W_k) \geq d$ and $|W_i| \geq cn$ for each $i \in [k]$. Let $(T, x, L)$ be a layered $k$-tree on at most $\beta n$ vertices with $L = (L_1, \ldots, L_m)$ and $\Delta_1(T) \leq \Delta$. Then, for any $\theta$-extensible edge $e \in H$, $f \subseteq e$ of size $k - 1$ and any ordering $f$ of $f$, there exists an embedding $\phi : V(T) \to f \cup R \cup W_1 \cup \cdots \cup W_k$ such that

1. $\phi(x) = f$,
2. $\phi^{-1}(R \cup f) = \bigcup_{i=1}^\ell L_i$,
3. $\phi(V_{[\ell+1, m]}(f)) \subseteq W_1 \cup \cdots \cup W_k$, with $|\phi^{-1}(W_1)| \geq \cdots \geq |\phi^{-1}(W_k)|$, and
4. $\phi(e')$ is $\theta$-extensible, for each $e' \in E(T[V_{[\ell+1, m]}])$.

Roughly speaking, Lemma 8.1 states that we can embed any sufficiently small tree of bounded degree into any large $k$-graph using vertices only from a given extensible edge, a given reservoir, and a given dense $k$-partite subgraph. Property (E1) says that we can map the root of the tree into any $(k - 1)$-subset of an extensible edge of the host graph, in any order. Property (E2) says that we only embed a fixed number of layers of $T$ in the reservoir (and thus only use a constant number of its vertices). Property (E3) ensures the remaining levels of the tree are embedded into the $k$-partite subgraph $(W_1, \ldots, W_k)$ and, moreover, we can decide which of these receives most (second most, etc) of the vertices from $T$. Finally, Property (E4) states that all the edges from $T[V_{[\ell+1, m]}]$ are mapped to $\theta$-extensible edges of $H$.

In Section 8.1, we will gather some tools for the proof of Lemma 8.1, which is postponed to Section 8.2.

8.1. Tools for embedding. We will use a hypergraph version of the fact that every graph of average degree at least $d$ contains a subgraph of minimum degree at least $d/2$. This is achieved by the next lemma.

Proposition 8.2 (Cleaning the graph). Let $k, m \geq 2$, let $d \in (0, 1)$ and let $H$ be a $k$-partite $k$-graph, with partition classes each of size at most $m$. If $H$ has at least $dmk$ edges, then $H$ has a non-empty subgraph $H'$ such that $\text{deg}_{H'}(f) \geq dm/k$ for every $f \in \partial H'$.

Proof. Starting with $H_1 := H$, proceed as follows for $i \geq 1$. If there is an $f \in \partial H_i$ with $\text{deg}_{H_i}(f) < dm/k$, then obtain $H_{i+1}$ from $H_i$ by removing all edges containing $f$. If there is no such $f$, we stop and set $H' := H_i$. It only remains to show that $H' \neq \emptyset$. For this, observe that the total number of deleted edges is less than $|\partial H|dm/k \leq dmk$. \hfill $\square$

In the proof of Lemma 8.1, we will use Proposition 8.2 to clean $H[W_1, \ldots, W_k]$ to find a subgraph $H' \subseteq H''$ such that every $f \in \partial H'$ has large codegree. The next lemma states that in such a $k$-graph, one can extend any (correctly located) partial embedding of a $k$-tree to a larger $k$-tree. The proof proceeds by mapping the remaining vertices successively, using the codegree condition.

Proposition 8.3 (Extending a partial tree embedding). Let $\Delta, k, m, n \in \mathbb{N}$ with $k \geq 2$, and let $\delta, \beta > 0$ with $\beta < \delta/2$. Let $H$ be a $k$-graph and let $W_1, \ldots, W_k \subseteq V(H)$ be pairwise disjoint, and of size at most $n$ each. Let $H' := H[W_1, \ldots, W_k]$ such that for each $f \in \partial H'$ we have $\text{deg}_{H'}(f) \geq \delta n$. Let $(T, x, L)$ be a layered $k$-tree with $|V(T)| \leq \beta n$, $\Delta_1(T) \leq \Delta$ and $L = (L_1, \ldots, L_m)$. Let $1 \leq \ell \leq m - k + 1$, and suppose there is an embedding $\varphi_1$ of $T_1 := T[V_{[\ell+1, k-1]}]$ in $H$ such that

(i) $\varphi_1(L_{\ell+i}) \subseteq W_i$ for every $i \in [k - 1]$, and
(ii) if $f \in \partial T[V_{[\ell+1, k-1]}]$ then $\varphi_1(f)$ has codegree at least $\delta m$ in $H'$.

Then there is an embedding $\varphi$ of $T$ which extends $\varphi_1$, such that for each $i \in [m - \ell]$, $\varphi(L_{\ell+i}) \subseteq W_j$ if and only if $j \equiv i \mod k$.

Proof. In each $W_i$, at most $|V(T)| \leq \beta n \leq \delta n/2$ vertices are used by the partial embedding $\varphi_1$, and at most $\delta n/2$ new vertices need to be embedded in each $W_i$. Because of our condition on the codegrees of $H'$, we can extend $\varphi_1$ greedily, embedding the vertices of $V(T) \setminus V(T_1)$ one by one, following any valid ordering, and choosing an unused vertex in each step. \hfill $\square$
The next lemma (Lemma 8.4) is designed to find an embedding of a short sequence of consecutive layers of a layered $k$-tree, while fixing the location of the initial and final segments. Its output will be the input for Proposition 8.3, which is then used to prove Lemma 8.1.

**Lemma 8.4** (Embedding the trunk of a tree). Let $1/n \ll 1/k, 1/q, 1/\ell, 1/M, \delta, \alpha$ and $2 \leq k \leq (\ell - 1)/2$. Let $\mathcal{L} = (L_1, \ldots, L_m)$ be a layering of a rooted $k$-tree $(T, x)$, and assume $V_{[t,t+\ell]}$ is $M$-bounded. If $T_I \subseteq T|_{[t,t+\ell]}$, $H$ is a $k$-graph on $n$ vertices, $U \subseteq V(H)$ and $F_1, F_2 \subseteq \partial^o H$ are such that

(I) $|F_1|, |F_2| \geq \delta n^{k-1}$, and

(II) for every $v_1 \in F_1$ and $v_2 \in F_2$ there are at least $\alpha n^q$ many walks going from $v_1$ to $v_2$, each of length $\ell - k + 1$, each with $q$ internal vertices all from $U \setminus (v_1 \cup v_2)$,

then there exists an embedding $\varphi : V(T_I) \to V(H)$ such that

(i) $\varphi(v) \in F_1$ for each $v \in \partial^o T_I|_{[t,t+k-2]}$,

(ii) $\varphi(v) \in F_2$ for each $v \in \partial^o T_I|_{[t+k-1,t+k+\ell]}$, and

(iii) $\varphi(c) \in U$ for each $c \in V_{[t+k-1,t+k+\ell]}$.

The proof of Lemma 8.4 can be summarised as follows. In a first step, we define an auxiliary hypergraph $\mathcal{H}'$ whose edges correspond to the interior vertices of a walk from some $v_1 \in F_1$ to some $v_2 \in F_2$; our assumptions will ensure that $\mathcal{H}'$ is sufficiently dense. Secondly, we discard some edges of $\mathcal{H}'$ to ensure that the remaining hypergraph $\mathcal{H}$ encodes only walks which use $q$ distinct vertices and also use repeated vertices in precisely the same positions of the walk. This is done by defining a $q$-vertex-colouring and only keeping the (edges which correspond to) walks which are $q$-coloured according to this colouring, in such a way that the colouring codifies the order of each walk. In a final step, we use supersaturation in $\mathcal{H}$ in order to find a copy of a large complete multipartite subgraph $K$ of $\mathcal{H}$. Now, as the walks were encoded in the colouring, we can use $K$ to embed $T_I$ into the walks corresponding to the edges of $K$.

**Proof of Lemma 8.4.** Define $k_1 = q + 2k - 2$ and $k_2 = \ell - k + 1$.

**Step 1: Defining an auxiliary hypergraph.** We begin by defining an auxiliary $k_1$-graph $\mathcal{H}'$. The vertices of $\mathcal{H}'$ are the vertices of $H$. The vertex set of a walk $W$ of length $k_2$ in $H$ is declared an edge of $\mathcal{H}'$ if there exist $v_1, v_2 \in \partial^o H$ such that the following conditions hold:

- $\text{sta}(W) = v_1 \in F_1$ and $\text{ter}(W) = v_2 \in F_2$,
- $\text{int}(W) \subseteq U$ and $|\text{int}(W)| = q$, and
- $\text{sta}(W), \text{ter}(W)$ and $\text{int}(W)$ (viewed as sets) are pairwise disjoint.

We claim that

$$\mathcal{H}'$$

has at least $\frac{\delta^2 \alpha}{2!} n^{k_1}$ edges. \hspace{1cm} (8.1)

Indeed, by (I) there are at least $|F_1| \geq \delta n^{k-1}$ possible choices for $v_1$, which will correspond to the start of a walk $W$ defining an edge of $\mathcal{H}'$. Each such $v_1$ intersects at most $(k-1)n^{k-2}$ elements of $\partial H$. So, as $|F_2| \geq \delta n^{k-1}$ and $n$ is large, there are at least $|F_2| - (k-1)n^{k-2} \geq \delta n^{k-1}/2$ ways to select an end $v_2 \in F_2$ disjoint from $v_1$. Having chosen $v_1$ and $v_2$, by (II) there are at least $\alpha n^q$ many walks $W$ going from $v_1$ to $v_2$ that could define an edge of $\mathcal{H}'$. Having fixed $v_1$ and $v_2$, since all the given walks $W$ have length $k_2$, the set $\text{int}(W)$ could coincide for at most $k_2! < \ell!$ many of the given walks, and thus at most $\ell!$ different walks from $v_1$ to $v_2$ yield the same edge of $\mathcal{H}'$. Thus the number of edges in $\mathcal{H}'$ is at least $\delta n^{k-1} \times (\delta n^{k-1}/2) \times \alpha n^q \times (\ell!)^{-1}$, as claimed.

**Step 2: Cleaning the auxiliary hypergraph.** We now define a colouring $c : V(H) \to \{1, \ldots, k_1\}$ by choosing a colour for each vertex independently and uniformly at random. Let $\mathcal{H}' \subseteq \mathcal{H}'$ be spanned by all edges $X \in \mathcal{H}'$ whose corresponding walk $W$ has the following properties:

(a) if $\text{sta}(W) = (x_1, \ldots, x_{k-1})$, then $c(x_i) = i$ for all $i \leq k - 1$,

(b) if $\text{ter}(W) = (y_1, \ldots, y_{k-1})$, then $c(y_i) = k_1 - k + i$ for all $i \leq k - 1$, and

(c) no two vertices of $\text{int}(W)$ have the same colour.
For a fixed $X \in \mathcal{H}'$, the probability of belonging to $\mathcal{H}'$ is at least $k_1^{-k_1}$. So the expected size of $\mathcal{H}'$ is at least $|E(\mathcal{H}')|/k_1^{k_1} \geq \delta^2\alpha n^{k_1}/(2!k_1^{k_1})$ (where we used (8.1)). We can thus fix a coloring $c$ and $\mathcal{H}' \subseteq \mathcal{H}'$ having properties (a)-(c) and also fulfilling

$$|\mathcal{H}'| \geq \delta^2\alpha n^{k_1}/(2!k_1^{k_1}).$$

(8.2)

Now we further restrict $\mathcal{H}'$ to make sure that, for all edges $X \in E(\mathcal{H}')$ corresponding to a walk $W$, the interiors of the walks are all consistently coloured. All of the walks $W$ have length $k_2$, and thus (seeing $W$ as a sequence of vertices), the vertices outside of $\text{sta}(W)$ and $\text{ter}(W)$ correspond to $k_2 - k + 1$ vertices with possible repetitions, whose underlying set $\text{int}(W)$ is coloured with different colours from $\{k, \ldots, k_1 - k\}$. So tracking the colours received by the vertices of the walk defines a sequence of colours, with possible repetitions, chosen among $\{k, \ldots, k_1 - k\}$. As all walks corresponding to edges of $\mathcal{H}'$ are $k$-colours. As there are at most $(k_1 - 2k + 1)^{k_2 - k + 1} \leq k_1^{k_2}$ such sequences, and because of (8.2), the pigeon-hole principle gives a subset $E(\mathcal{H}) \subseteq E(\mathcal{H}')$ of size at least

$$|E(\mathcal{H}')|/k_1^{k_2} \geq \delta^2\alpha n^{k_1}/(2!k_1^{k_1}k_2),$$

(8.3)

such that each walk $W$ corresponding to an edge $X$ of $\mathcal{H}$ is coloured in exactly the same way under $c$.

Step 3: Using supersaturation. Let $\beta = \delta^2\alpha/(2!k_1^{k_1+k_2})$. By our assumptions, $1/n \ll \beta$ and by (8.3), we know that $\mathcal{H}$ is a $k_1$-graph with at least $\beta n^{k_1}$ edges. So, we can apply Lemma 6.3, with $k_1$ and $M\ell$ playing the roles of $k$ and $s$, to find that $\mathcal{H}$ contains a copy $\mathcal{K}$ of $K^{(k_1)}(M\ell)$, the complete $k_1$-partite $k_1$-graph with classes of size $M\ell$. Now, take any edge in $\mathcal{K}$, and recall it gives rise to a walk $W = v_1v_2 \cdots v_{k_2}$ in $H$. For all $i$, let $V_i$ denote the partition class of $\mathcal{K}$ that contains $v_i$. Note that by construction, $V_i = V_j$ is only possible for $i, j \in \{k, \ldots, \ell - k + 1\}$. As all walks corresponding to edges of $\mathcal{K}$ are coloured in the same way, each of them passes through the sets $V_i$ in the same order.

Consider any injective function $h : V(T_1) \rightarrow V(\mathcal{K})$ which maps all of $L_{\ell + i - 1}$ to $V_i$, for each $i \in \{1, \ldots, \ell\}$. Such a function exists, since by assumption, each $L_{\ell + i - 1}$ has size at most $M$, and since $W$ repeats each vertex at most $k_2 \leq \ell$ times, whereas each $V_i$ has $M\ell$ vertices. As $\mathcal{K}$ is $k_1$-partite, we have found the desired embedding of $T_1$ in $H$.

8.2. Proof of Lemma 8.1. The proof of Lemma 8.1 proceeds by separating the input $T$ in $T_1$ induced by the first $\ell + k - 1$ layers, which we call the “trunk of $T$”, and the remaining $T_2 = T \setminus T_1$, which we call the “crown of $T$”.

The proof is separated into three steps. In the first step, we will prepare the host graph $H$ for the embedding. This will be done by removing non-extensible edges, or edges with undesirable codegree properties from $H[W_1, \ldots, W_k]$ to get to a ‘cleaned’ subgraph $H' \subseteq H$, and finding a suitable ordering of the clusters $W_1, \ldots, W_k$ so that the final embedding satisfies the required properties. In a second step, we will apply Lemma 8.4 (Embedding the trunk of a tree) to embed the trunk $T_1$. In the third and final step we extend the embedding of $T_1$ to an embedding of the whole tree, which is done using Proposition 8.3 (Extending a partial tree embedding). Now come the details.

Proof of Lemma 8.1. To begin, we introduce a new constant $\varepsilon > 0$ such that $\theta \ll \varepsilon \ll d, c, 1/k$, and set $\delta = dc^k/(2k)$.

Step 1: Preparing $H$ and $T$ for the embedding. Obtain $H''$ from $H[W_1, \ldots, W_k]$ by removing all non-$\theta$-extensible edges. Since $1/n, \theta \ll \varepsilon$, Lemma 6.5 implies that there are at most $\varepsilon n^k$ non-$\theta$-extensible edges. Hence, by our choice of $\varepsilon \ll d, c, 1/k$, we have

$$e(H''[W_1, \ldots, W_k]) \geq d|W_1| \cdots |W_k| - \varepsilon n^k \geq dc^k n^k - \varepsilon n^k \geq \frac{dc^k}{2} n^k.
$$

Use Proposition 8.2 to find a subgraph $H' \subseteq H''$ such that for every $f \in \partial H'$, $\deg_{H'}(f) \geq \frac{dc^k}{2} n = \delta n$, by definition of $\delta$.  

"
Now turn to $T$. For each $i \in [k]$, let $\mathcal{T}_i = \bigcup_{j \geq 0} L_{e+j+k+i}$. Note that $\{\mathcal{T}_i\}_{i \in [k]}$ partitions $V_{[e+1,m]}$. Let $\sigma : [k] \to [k]$ be a permutation with $|\mathcal{T}_{\sigma^{-1}(1)}| \geq \cdots \geq |\mathcal{T}_{\sigma^{-1}(k)}|$. Our plan is to embed $\mathcal{T}_i$ into $W_{\sigma(i)}$ for all $i \in [k]$, as this will ensure (E3).

**Step 2: Embedding the trunk of the tree.** Let $T_1 := T[V_{[1,e+k-1]}]$. We will embed $T_1$, using vertices in $R$, starting from $f \leq e$ and ending in $W_1 \cup \cdots \cup W_k$. Formally, our goal is to find an embedding $\varphi_1$ of $T_1$ such that

- $(T1)$ $\varphi_1(x) = f$,
- $(T2)$ $\varphi_1(V_{[1,\ell]} \setminus x) \subseteq R$,
- $(T3)$ for every $i \in \{1, \ldots, k-1\}$, $L_{e+i}$ is embedded in $W_{\sigma(i)}$, and
- $(T4)$ if $f \in \partial T_1[V_{[e+1,e+k-1]}]$ then $\deg_H(\varphi_1(f)) \geq \delta n$.

We initially set $\varphi_1(x) = f$, thus ensuring (T1). In order to embed the remaining vertices in $T_1 = T \setminus x$, we will use Lemma 8.4, which we will apply in a suitably defined subgraph. For this, say that a $k$-tuple $e' = (x'_1, \ldots, x'_k) \in R^k$ is $e$-good if $e \cup e'$ induces a copy of $K^{(k)}(2)$ with partition classes $\{x_1, x'_1\}, \ldots, \{x_k, x'_k\}$. Define sets $F_1, F_2 \subseteq \partial^0 H$ as

- $F_1 = \{(x_2', \ldots, x_k') \in \partial^0 H : (x'_1, x_2', \ldots, x'_k)$ is an $e$-good $k$-tuple for some $x'_1 \in R\}$, and
- $F_2 = \{(y_1, \ldots, y_k-1) \in W_{\sigma(1)} \times \cdots \times W_{\sigma(k-1)} : \deg_H(\{(y_1, \ldots, y_k-1)\}) \geq \delta n\}$.

Next, we show that between every $h, h' \in \partial^0 H$ there are many short walks of fixed length which pass through $R$. To this end, note that $H$ is $k$-large and $R$ is an $(\gamma, \mu, 2)$-reservoir, and therefore $H$ is $(\gamma^2 - 2\mu)n$-large. Moreover, by the choice of $\mu \ll \gamma$, $H$ is actually $(\gamma^2 n/2)$-large. By $1/n \ll \gamma \ll 1/k$, we can apply Lemma 7.1 with input $k, \gamma^2/2, \ell$ and $\bar{\ell}$ in place of $k, \gamma, \ell$, to see the following.

**Claim 8.5.** For all $h, h' \in \partial^0 H$, there are $(\gamma^2 n/4)^q$ many walks $W$ of length $\ell$ in $H[h \cup R \cup h']$ from $h$ to $h'$, each with $q$ internal vertices all in $R \setminus (h \cup h')$.

We wish to apply Lemma 8.4 with $F_1, F_2$, and $R$ in place of $U$. Let us check that its hypotheses are satisfied. First, we show that $|F_1|$ is large. Since $e$ is $\theta$-extendible and $R$ is an $(\gamma, \mu, 2)$-reservoir, there are at least $(\theta - \mu)(\ell - k)^k \geq (\theta \gamma^2/(2k!))n^k$ $e$-good tuples in $R$ (we have used $\mu \ll \theta$). Keeping the last $(k-1)$-vertices of any such tuple, we deduce that $|F_1| \geq (\theta \gamma^2/(2k!))n^{k-1}$. Secondly, we show that $|F_2|$ is large. For this, recall that each $(k-1)$-tuple in $\partial^0 H$ has codegree at least $\delta n$. So, we see that $F_2 \geq \delta n^{k-1}$. Finally, by Claim 8.5, for each choice $v_1 \in F_1$ and $v_2 \in F_2$ there are at least $(\gamma^2 n/4)^q$ walks $W$ of length $\ell$ going from $v_1$ to $v_2$ in $H$, each satisfying $|\text{int}(W)| = q$, internally disjoint from $v_1 \cup v_2$ and int($W$) $\subseteq R$, as required.

Note that $T_1 \subseteq T[V_{[2,e+k-1]}]$. Since $L = \{V_1, \ldots, V_m\}$ is a layering of $(T, x)$, and $\Delta(T) \leq \Delta$, Proposition 5.5 implies that $|L_i| \leq \Delta^k$ for all $1 \leq i \leq \ell - k + 1$, which implies that $V_{[2,e+k-1]}$ is $\Delta^k$-bounded. Since $1/n \ll 1/k, 1/\Delta, \theta, \gamma, d, c$, we can apply Lemma 8.4 (Embedding the trunk of a tree), with $T'_1, H, R, \{\langle \gamma \theta \gamma^2/(2k!)\rangle, \delta^{k-1}\}, \Delta^{k-1}, 2, \ell + k - 3$ playing the roles of $T_1, H, U, \delta, M, t$ and $\ell$, respectively. By doing so, this gives an embedding $\varphi_0 : V(T_1) \to V(H)$. By construction, the union of $\varphi_0$ and $\varphi_0'$ gives an embedding $\varphi_1$ of $T_1$ satisfying (T1)–(T4).

**Step 3: Embedding the crown of the tree.** We need to extend the embedding $\varphi_1$ of $T_1$ to an embedding of all of $T$ in $H'[W_1, \ldots, W_k]$. By (T4), we know that every $f \in \{\partial T[V_{[x+1,e+k-1]}]\}$ has codegree at least $\delta n$ in $H'[W_{\sigma(1)}, \ldots, W_{\sigma(k-1)}]$. Since $\beta \ll d, c, 1/k$ and the definition of $\delta$, we can assume $\beta \leq \delta/2$. Thus, we can use Proposition 8.3 (Extending a partial tree embedding), with $H', W_{\sigma(1)}, \ldots, W_{\sigma(k)}$ playing the role of $H, W_1, \ldots, W_k$, to find an embedding $\varphi$ of $T$ which extends $\varphi_1$ and such that, for each $i \in [m - \ell], L_{e+i}$ is embedded in $W_j$, where $i \equiv (j - 1)k$ mod $k$.

Now we verify that $\varphi$ satisfies (E1)–(E4). Since $\varphi$ extends $\varphi_1$ and since $\tau \subseteq V(T_1)$, (E1) follows from (T1). Since all of $V(T_2) = V(T) \setminus V(T_1)$ was embedded in $W_1 \cup \cdots \cup W_k$, which is disjoint from $R$, we have $\varphi^{-1}(R) = \varphi^{-1}(R)$. From (T2)–(T3), we deduce that (E2) holds. The properties of $\varphi$ imply that, for each $i \in [k]$, $\varphi(\mathcal{T}_i) \subseteq W_{\sigma(i)}$, and therefore $\varphi^{-1}(W_{\sigma(i)}) = \mathcal{T}_i$. By the choice of $\sigma$, (E3) holds. Finally, since $H'[W_1, \ldots, W_k]$ only contains $\theta$-extendible edges, (E4) holds.

□
9. Absorption

In this section, we state and prove lemmas which will allow us to complete the embedding of an almost spanning tree. This technique is similar to the one used in [Bőt+19; Bőt+20]. Let us first define useful structures both for the $k$-tree we want to embed, and for the host graph which is used to embed the $k$-tree.

**Definition 9.1** (Absorbing $X$-tuple). Let $k \geq 3$ and let $X$ be a $(k-1)$-tree on $h \geq k-1$ vertices, with a fixed valid ordering $x_1, \ldots, x_h$. For a $k$-tree $T$, we say that an $(h+1)$-tuple $(v_1, \ldots, v_k)$ of vertices of $T$ is an $X$-tuple if

(i) $V(T(v^*)) = \{v_1, \ldots, v_k\}$, and

(ii) the map $x_i \mapsto v_i$ is a hypergraph isomorphism between $X$ and $T(v^*)$.

Let $H$ be a $k$-graph on $n$ vertices and let $(v_1, \ldots, v_k)$ be a $k$-tuple of distinct vertices of $H$. An absorbing $X$-tuple for $(v_1, \ldots, v_k)$ is an $(h+1)$-tuple $(u_1, \ldots, u_k, u^*)$ of vertices of $H$ such that

(A) $\{v_1, \ldots, v_{k-1}, u^*\} \subseteq H$, and

(B) there exists a copy $X$ of $X$ on $\{u_1, \ldots, u_k\}$ such that $X \subseteq H(v_k) \cap H(u^*)$.

Furthermore, we write $\Lambda_X(v_1, \ldots, v_k)$ for the set of absorbing $X$-tuples for $(v_1, \ldots, v_k)$, and we let $\Lambda_X$ denote the set of all absorbing $X$-tuples in $H$, that is, $\Lambda_X$ is the union of $\Lambda_X(v_1, \ldots, v_k)$ over all $k$-tuples $(v_1, \ldots, v_k)$ of distinct vertices of $V(H)$. Suppose there exists an embedding $\varphi : V(T) \to V(H)$ and let $(u_1, \ldots, u_k, u^*)$ be an $(h+1)$-tuple of vertices of $H$. We say $(u_1, \ldots, u_k, u^*)$ is $X$-covered by $\varphi$ if there exists an $X$-tuple $(v_1, \ldots, v_k, v^*)$ of vertices of $T$ such that $\varphi(v^*) = u^*$ and $\varphi(v_i) = u_i$ for all $i \in [h]$.

The idea behind the definition of this gadget (the $X$-tuple) is that it will allow us to extend the embedding of a tree by iteratively adding leaves. We illustrate an $X$-tuple and how the extension step works in Figure 4.

![Figure 4](image-url)  

**Figure 4.** An illustration of an $X$-tuple and how this is used to extend an embedding of a $k$-tree. In this case, $k = 3$, $H$ is a 3-graph, $(v_1, v_2, v_3)$ is a tuple of distinct vertices of $H$, and $X$ is a star with 3 leaves (which is a 2-tree). The tuple $(u_1, u_2, u_3, u_4, u^*)$ is an $X$-tuple for $(v_1, v_2, v_3)$, so $(u_1, u_2, u_3, u_4)$ induces a copy of $X$ in $H(v^* \cap H(v_2))$, and $(v_1, v_2, v^*)$ is an edge of $H$, as shown in the left picture. Suppose, in addition, that $(u_1, u_2, u_3, u_4, u^*)$ is $X$-covered by a embedding $\varphi$ of some 3-uniform tree $T$, and $(v_1, v_2) \in \partial \varphi(T)$ and $v_3 \notin V(\varphi(T))$. To find an embedding of the tree $T + v$ obtained by attaching a new vertex to $(v_1, v_2)$, we can modify $\varphi$ by switching $u^*$ with $v_3$ and then adding the edge $(v_1, v_2, u^* \to v_3)$, as shown in the right picture. This gives an embedding of $T + v$ where $v$ is copied to $u^*$.

The following lemma is the heart of our absorbing method.

**Lemma 9.2** (Absorbing Lemma). Let $n \geq h \geq k \geq 3$ and let $0 < \delta < \alpha$. Let $T$ be a $k$-tree on $n$ vertices with a valid ordering of $V(T)$ given by $v_1, \ldots, v_n$, and let $T_0 = T[v_1, \ldots, v_n]$ be a $k$-subtree of $T$ on $n' \geq (1 - \delta)n$ vertices. Let $H$ be a $k$-graph on $n$ vertices, and suppose there
exists an embedding $\varphi_0 : V(T_0) \to V(H)$, a $(k - 1)$-tree $X$ on $h$ vertices and a family $A \subseteq \Lambda_X$ of $(h + 1)$-tuples of vertices of $H$ with the following properties:

(i) the tuples in $A$ are pairwise vertex-disjoint,

(ii) every tuple in $A$ is $X$-covered by $\varphi_0$, and

(iii) $|A_X(v_1, \ldots, v_k) \cap A| \geq \alpha n$ for every $k$-tuple of distinct vertices of $H$ such that $\{v_1, \ldots, v_{k - 1}\} \in \partial H$.

Then there exists an embedding of $T$ in $H$.

Proof. Let $m = n - n'$ and let $\{x_1, \ldots, x_m\}$ be an arbitrary enumeration of $V(H) \setminus V(\varphi_0(T_0))$. For every $i \in [m]$, we set $T_i := T[\{v_{i - 1} + 1, \ldots, v_{i + n'}\}]$. Iteratively, for each $0 \leq i \leq m$, we will find an embedding $\varphi_i : V(T_i) \to V(H)$ and subset $A_i \subseteq A$ with the following properties:

(a) $\varphi_i(V(T_i)) = \varphi_0(T_0) \cup \{x_1, \ldots, x_i\}$,

(b) $|A_i| \leq i$, and

(c) for every $(u_1, \ldots, u_h, u^*) \in A \setminus A_i$, $\varphi_i^{-1}(u^*) = \varphi_0^{-1}(u^*)$ and $\varphi_i^{-1}(u_j) = \varphi_0^{-1}(u_j)$ for every $j \in [h]$.

It is very easy to see that for $i = 0$ the properties hold for $\varphi_0$ and $A_0 := \emptyset$. Suppose that for some $0 \leq i \leq m - 1$ we have defined $\varphi_i$ and $A_i$ satisfying (a)–(c). We shall construct $\varphi_{i+1}$ and $A_{i+1}$ satisfying $(a_{i+1})$–$(c_{i+1})$.

Since $v_1, \ldots, v_n$ is a valid ordering for $T$, there exists a unique $(k-1)$-set $\{v_{i-1} + 1, \ldots, v_{i + n'}\} \subseteq T_i$ such that $\{v_{i-1} + 1, \ldots, v_{i + n'}\} \subseteq T_i$. Let $v_{i-1} + 1, \ldots, v_{i + n'} \in V(H)$ be an arbitrary labelling of $\varphi_i(\{v_{i-1} + 1, \ldots, v_{i + n'}\})$ and define $w_i := x_i + 1$. Note that $v_{i-1} + 1, \ldots, v_{i + n'} \in \partial H$, so by assumption $|A_X(v_{i-1}, \ldots, v_{i + n'}) \cap A| \geq \alpha n$, $i \leq m \leq \delta n$ and $\delta < \alpha$. Thus, by (b), we have

$$|A_X(w_1, \ldots, w_k) \cap A| \geq \alpha n - |A_i| = \alpha n - i \geq (\alpha - \delta) m > 0.$$ 

Now we can select an arbitrary absorbing $X$-tuple $(u_1, \ldots, u_h, u^*) \in A \setminus A_i$, and define $A_{i+1} := A_i \cup \{(u_1, \ldots, u_h, u^*)\}$. Note this definition of $A_{i+1}$ satisfies (b$_{i+1}$). Since $(u_1, \ldots, u_h, u^*)$ is an $X$-tuple for $(w_1, \ldots, w_k)$ in $A \setminus A_i$, then $(u_1, \ldots, u_h, u^*)$ is $X$-covered by $\varphi_0$ and, because of (c), it is also $X$-covered by $\varphi_i$. For every $x \in V(T_{i+1})$, define

$$\varphi_{i+1}(x) := \begin{cases} u_k & \text{if } x = \varphi_i^{-1}(u^*), \\ u^* & \text{if } x = v_{i + n'}, \\ \varphi_i(x) & \text{otherwise}. \end{cases}$$

Note that the function $\varphi_{i+1}$ is injective and it satisfies $(a_{i+1})$ and $(c_{i+1})$. To finish, we check that $\varphi_{i+1}$ is an embedding of $V(T_{i+1})$. Indeed, if $e \in T_{i+1}$ does not contain $v_{i + n'}, \varphi_i^{-1}(u^*)$, then $e \in T_i$ and $\varphi_i^{-1}(e) = \varphi_i(e) \in H$ since $\varphi_i$ is an embedding of $T_i$. If $e \in T_{i+1}$ contains $v_{i + n'}$, then $e = \{v_{i-1} + 1, \ldots, v_{i + n'}\}$ and because of (A) (in the definition of absorbing $X$-tuples) we know that $\varphi_{i+1}(e) = \{w_1, \ldots, w_{i + n'}, u^*\} \subseteq H$. If $e \in T_{i+1}$ contains $\varphi_i^{-1}(u^*)$, then there exists a $(k - 1)$-edge $e'$ in $T(\varphi_i^{-1}(u^*))$ such that $e = e' \cup \{\varphi_i^{-1}(u^*)\}$ and $u^* \cup \varphi_i(e') \subseteq H$. Note that $\varphi_{i+1}(e) = w_k \cup \varphi_i(e')$. Since $(u_1, \ldots, u_h, u^*)$ is an $X$-tuple for $(w_1, \ldots, w_k)$, then by (B) we know that $w_k \cup \varphi_i(e') \subseteq H$ and therefore $\varphi_{i+1}(e) \subseteq H$, as desired. Thus $\varphi_{i+1}$ is an embedding of $T_{i+1}$.

Following this process for $m$ steps, we find an embedding $\varphi_m$ of $T_m = T$, as desired.

In the remainder of this section we will prove a series of lemmas that allow us to build a partial embedding of a tree $T$ in which properties (i)–(iii) of Lemma 9.2 are fulfilled.

9.1. Finding separated tuples in $k$-trees. Let $T_{k,[h]}$ be the family of all non-labelled $k$-trees on at most $h$ vertices, up to isomorphism. For our purposes, we need to bound $|T_{k,[h]}|$ in terms of $h$ and $k$. Let $\mathcal{T}_{k,h}$ be the family of all $k$-trees on exactly $h$ vertices. We will bound $|\mathcal{T}_{k,h}|$ by the number of labelled $k$-trees on $h$ vertices. For all such labelled trees $T$, recall that all but the first $k$ vertices have an anchor. Since for each vertex in $T$ we have at most $\binom{h}{k-1}$ options for its anchor, we thus have $|T_{k,h}| \leq \binom{h}{k-1} h^{h-k} \leq h^{h(k-1)}$, which in turn implies

$$|T_{k,[h]}| \leq h^{hk}.$$ 

(9.1)
Recall that the distance between \((k-1)\)-tuples of the shadow of a \(k\)-tree was given in Definition 5.10. Given a \(k\)-tree \(T\), a \((k-1)\)-tree \(X\), and \(\ell \geq 0\), we say that a set \(B\) of \(X\)-tuples of \(T\) is \(\ell\)-separated if there are \(\ell\) pairwise at a distance at least \(\ell\), that is, for each distinct \(B_i, B_j \in B\), and each \(f_i \in \partial T[B_i], f_j \in \partial T[B_j]\), we have \(d_T(f_i, f_j) \geq \ell\).

We now show that every bounded-degree tree contains a large \(\ell\)-separated set of \(X\)-tuples, for some \((k-1)\)-tree \(X\).

**Proposition 9.3.** Suppose \(0 < \mu \ll 1/\Delta, 1/k, 1/\ell\) and \(k \geq 2\). Let \(T\) be a \(k\)-tree on \(n\) vertices such that \(\Delta_1(T) \leq \Delta\). Then there exists a \((k-1)\)-tree \(X \in T_{k-1, \Delta+k-1}\) and an \(\ell\)-separated set \(B\) of \(X\)-tuples of \(T\) with \(|B| \geq \mu n\).

**Proof.** By Proposition 5.1 and the bound \(\Delta_1(T) \leq \Delta\), for every vertex \(v \in V(T), T(v)\) is a \((k-1)\)-tree which is in \(T_{k-1, \Delta+k-1}\). By the pigeon-hole principle, there exists a \((k-1)\)-tree \(X \in T_{k-1, \Delta+k-1}\) and a subset \(W' \subseteq V(T)\) of size at least \(n/(|T_{k-1, \Delta+k-1}|)\) such that \(T(w') \cong X\) for every \(w \in W'\). Note that each \(w \in W'\) yields an \(X\)-tuple \(B_w\) in \(T\) which uses precisely the vertices in \(T(w)\). Let \(W \subseteq W'\) be maximal so that for all distinct \(w_1, w_2 \in W\), the \((k-1)\)-tuples in \(V(T(w_1))\) and \(V(T(w_2))\) are at distance at least \(\ell\). Set \(B := \{B_w : w \in W\}\). It remains to show that \(|B| \geq \mu n\).

The assumption \(\Delta_1(T) \leq \Delta\) implies that for every \(w' \in W'\), there are at most \((\Delta + k)^\ell + 1\) other vertices \(w'' \in W'\) such that \(T(w'_i)\) and \(T(w''_j)\) have distance less than \(\ell\). We deduce that

\[
|B| = |W| \geq \frac{|W'|}{(\Delta + k)^\ell + 1} \geq \frac{n}{|T_{k-1, \Delta+k-1}|(\Delta + k)^\ell + 1} \geq \mu n,
\]

where the last inequality follows from (9.1) and the assumption that \(\mu \ll 1/\Delta, 1/k, 1/\ell\). \(\square\)

**9.2. Finding absorbing tuples in the host graph.** Many copies of an \(X\)-tuple in a tree \(T\) will indicate parts of \(T\) that are ‘flexible enough’ to be interchanged. This can be used to extend a partial embedding of \(T\) into an embedding of all of \(T\). In the following proposition we will show the existence of many absorbers for each \(k\)-tuple. We remark that here the condition of \((\gamma n)\)-large is not enough, as we cannot construct absorbers if there are isolated vertices; but forbidding isolated vertices in addition to being \((\gamma n)\)-large will be enough.

**Proposition 9.4.** Let \(1/n \ll \beta \ll \gamma, 1/h, 1/k\) with \(h \geq k \geq 2\). Let \(H\) be a \(k\)-graph on \(n\) vertices which is \(\gamma n\)-large and has no isolated vertices. Let \(X\) be a \((k-1)\)-tree on \(h\) vertices and let \((v_1, ..., v_k)\) a \(k\)-tuple of distinct vertices of \(H\) such that \(\{v_1, ..., v_{k-1}\} \in \partial H\). Then \(|A_X(v_1, ..., v_k)| \geq \beta n^{k+1} \).

**Proof.** We construct an absorbing \(X\)-tuple for \((v_1, ..., v_k)\) by choosing vertices one by one. First, select an arbitrary vertex \(u_{h+1} \in N_H(\{v_1, ..., v_{k-1}\}) \setminus \{v_k\}\), and note that, since \(\{v_1, ..., v_{k-1}\} \subseteq \partial H\), there are at least \(\beta n^{k+1}\) possible choices for \(u_{h+1}\).

Define the \((k-1)\)-graph \(H' := H(v_k) \cap H(u_{h+1})\). First, we show that \(H'\) is non-empty. Since each vertex is contained at least in one edge, there must exist \(k\)-edges \(e_1, e_2\) in \(H\) which contain \(v_k\) and \(u_{h+1}\) respectively. Among all pairs \((e_1, e_2)\) of edges in \(H\) such that \(e_1, e_2\) contain \(v_k\) and \(u_{h+1}\) respectively, choose a pair such that \(|e_1 \cap e_2|\) is maximum. If \(|e_1 \cap e_2| < k-1\), then select \(Y_1 \subseteq e_1\) of size \(k-1\) containing \((e_1 \cap e_2) \cup \{v_k\}\), and \(Y_2 \subseteq e_2\) of size \(k-1\) containing \((e_1 \cap e_2) \cup \{u_{h+1}\}\). Then \(Y_1, Y_2 \in \partial H\), and since \(H\) is \(\gamma n\)-large, there exists \(u' \in N(Y_1) \cap N(Y_2)\). Then \(u' \cap Y_1\) and \(u' \cap Y_2\) are two edges in \(H\) containing \(v_k\) and \(u_{h+1}\) respectively and with larger intersection than \(e_1, e_2\), a contradiction. Thus \(|e_1 \cap e_2| = k-1\), which implies that \(e_1, e_2 \in H'\), and therefore \(H'\) is non-empty, as desired.

Now, observe that every \((k-2)\)-set in \(\partial H'\) has at least \(\gamma n\) neighbours (which follows since \(H\) is \(\gamma n\)-large). First, select an arbitrary \((k-1)\)-set \(\{x_1, ..., x_{k-1}\} \subseteq H'\). Using this, we can select \(u_1, ..., u_h\) iteratively in increasing order, as follows. First, fix a valid ordering \(\{t_1, ..., t_h\}\) of the vertices of \(X\). For each \(1 \leq i \leq k-1\), having chosen \(u_1, ..., u_{i-1}\) already, we select \(u_i\) as a neighbour of the \((k-2)\)-set \(\{x_{i+1}, ..., x_{k-1}, u_1, u_2, ..., u_{i-1}\} \subseteq \partial H'\), with the additional assumption that \(u_i \notin \{u_1, ..., u_{i-1}\}\). Then, successively for \(i = k-1, ..., h+1\), select \(u_{i+1}\) in the following way: if \(\{t_{j_1}, ..., t_{j_{k-2}}\}\) is the anchor of \(t_{i+1}\), then choose \(u_{i+1} \in N_{H'}(\{u_1, ..., u_{k-2}\}) \setminus \{u_1, ..., u_i\}\) arbitrarily. By construction, \(\{u_1, ..., u_{\ell+1}\}\) is an absorbing \(X\)-tuple for \((v_1, ..., v_k)\).
Finally, as in each step there are at least \( \gamma n/2 \) possibilities to choose the next vertex \( u_i \), there are at least \( (\gamma n/2)^{h+1} \geq \beta n^{h+1} \) absorbing \( X \)-tuples for \( (v_1, ..., v_k) \), as desired.

Now we would like to find a linear-sized family \( A \subseteq \Delta_X \) of vertex-disjoint \( X \)-tuples such that every \( k \)-tuple has many absorbing \( X \)-tuples in \( A \) (as to satisfy properties (i) and (iii) of Lemma 9.2). The following lemma can be proved by selecting independently each \((\ell + 1)\)-tuple in \( \Delta_X \) with probability \( p = \Theta(n^{-\ell}) \) and showing it satisfies the required properties with positive probability, which can be done using Chernoff’s inequality (Theorem 6.8) and Markov’s inequality. As this strategy is standard by now and has appeared in many other absorption-based proofs (e.g. [RRS08, Claim 3.2]), we leave the details of the proof to the reader.

**Lemma 9.5.** Let \( 1/n < \alpha < \beta < 1/k, 1/h \) with \( h \geq k \geq 2 \). Let \( H \) be a \( k \)-graph on \( n \) vertices and let \( X \) be a \((k - 1)\)-tree on \( h \) vertices. Suppose \( |\Delta_X(v_1, ..., v_k)| \geq \beta n^{h+1} \) for every \( k \)-tuple \((v_1, ..., v_k)\) of vertices of \( H \) such that \( \{v_1, ..., v_{k_1}\} \in \partial H \). Then there is a set \( A \subseteq \Delta_X \) of at least \( (h+1) \)-tuples of vertices of \( H \) such that \( |\Delta_X(v_1, ..., v_k) \cap A| \geq \beta n/8 \) for every \( k \)-tuple \((v_1, ..., v_k)\) of distinct vertices of \( H \) such that \( \{v_1, ..., v_{k-1}\} \in \partial H \).

**9.3. Covering \( X \)-tuples with a partial tree embedding.** Our final step in order to use the Absorbing Lemma is to show that we can cover a large family of absorbing tuples.

**Lemma 9.6** (Embedding pseudographs). Let \( 1/n \leq 1/\Delta, 1/k, 1/\ell, 1/\gamma \) satisfying \( \Delta, k \geq 2 \), and also \( \ell \geq (2k + 1)[k/2] + 2k \). Let \( H \) be a \( \gamma n \)-large \( k \)-graph on \( n \) vertices. Let \( P \) be a \( k \)-uniform \((f, f')\)-pseudograph \( P \), and let \( f, f' \) be any ordering of \( f \) and \( f' \) respectively. Moreover, suppose \( e(P) \geq \Delta k(\ell + 3k) \) and \( \Delta_1(P) \leq \Delta \). Then, given any pair of disjoint \((k - 1)\)-tuples \( x, y \in \partial^p H \), there exists an embedding \( \varphi : V(P) \to V(H) \) such that \( \varphi(f) = x \) and \( \varphi(f') = y \).

**Proof.** Suppose \( P \) has \( t \) edges \( e_1, ..., e_t \) such that \( f \leq e_1 \) and \( f' \leq e_t \). Now let \( L = (L_1, ..., L_m) \) be the (unique) layering for \((P, f)\). By Lemma 9.1(ii), \( |L_i| \leq k \Delta \) for all \( i \in [m] \). Thus the number \( m \) of layers of \( L \) satisfies \( m \geq |V(P)|/(k \Delta) \geq |E(P)|/(k \Delta) \geq \ell + 3k \).

We start our construction of the embedding by setting \( \varphi(f) = x \) and \( \varphi(f') = y \). As a next step, we will embed greedily the first \( k \) layers of \( L \) into \( R \). Recall the definition of \( r(j) \) from Lemma 9.1(i), and let \( j_1 \) be the maximum \( j \) such that \( r(j) \leq k \). Lemma 9.1(i) implies that \( r(j_1) = k \), \( r(j) \leq k \) for all \( j \leq j_1 \), and \( r(j) > k \) for all \( j > j_1 \). Let \( P_1 \) be the subgraph of \( P \) spanned by the edges \( e_1, ..., e_{j_1} \). Then \( P_1 \) is the ‘minimum’ subpath of \( P \) which contains all the \( k \)-tuples touching the first \( k \) layers.

Let \( s_1 = e_{j_1} \setminus (L_1 \cup \cdots \cup L_k) \), ordered according to the increasing layering order. Since \( r(j_1) = k \), \( s_1 \) has size \( k - 1 \). Now we embed \( P_1 \setminus s_1 \) making sure there are ‘many’ possible extensions available for \( s_1 \). First, in a greedy fashion, we count the number of embeddings of \( e_1 \) which extend \( \varphi(f) \) simply by completing \( f \) to \( e_1 \) to an unassigned vertex outside \( x \cup y \). Since \( H \) is \( \gamma n \)-large, this can be done in at least \( \gamma n - |x \cup y| \geq \gamma n/2 \) ways. Next, we iteratively count the extensions of the embedding of \( e_1 \) to an embedding of \( P_1 \), which can be done again in a greedy fashion, again having \( \gamma n/2 \) choices for an unused vertex each time. Letting \( n_1 = |V(P_1)| - (k - 1) \), we deduce that there are at least \( (\gamma n/2)^{n_1} \) embeddings of \( P_1 \) which extend \( \varphi(f) \). Now, an averaging argument entails that there is an embedding of \( P_1 \setminus s_1 \) which extends \( \varphi(f) \) and can be extended to at least \( (\gamma n/2)^{n_1}/n_1^{n_1 - |s_1|} = (\gamma n/2)^{n_1 - k + 1} \) embeddings of \( P_1 \). Let \( \varphi(P_1 \setminus s_1) \) be such an extension of \( \varphi(f) \) and let \( F_1 \subseteq V(H)^{k-1} \) be the set of all ordered \((k - 1)\)-tuples which are valid extensions of \( \varphi(P_1 \setminus s_1) \) to an embedding of \( P_1 \). Then \( |F_1| \geq (\gamma n/2)^{n_1 - k + 1} \).

Observe that if we consider the edges of \( P \) in reverse ordering (i.e. we consider \( e_t \) to be the first, \( e_1 \) to be the last edge), the resulting \( k \)-tree is an \((f', f)\)-pseudograph. So, defining \( j_2 \) as the minimum \( j \leq t \) such that \( r(j) \geq m - 2k + 2 \), and defining \( s_2 \) and \( P_2 \) accordingly, we can proceed as in the previous paragraph, to obtain an embedding \( \varphi(P_2 \setminus s_2) \) which extends \( \varphi(f') \) and can be extended to at least \( (\gamma n/2)^{n_2} \) embeddings of \( P_2 \). Letting \( F_2 \subseteq V(H)^{k-1} \) denote the set of all ordered \((k - 1)\)-tuples giving valid extensions of \( \varphi(P_2 \setminus s_2) \) to an embedding of \( P_2 \), we have \( |F_2| \geq (\gamma n/2)^{n_2 - k + 1} \).

We complete the embedding by using Lemma 8.4. Let \( \delta > 0 \) be such that \( 1/n \leq \delta < \gamma \). Since \( m - 3k \geq \ell \geq (2k + 1)[k/2] + 2k \) and \( H \) is \( (\gamma n, V(H)) \)-large, Lemma 7.1 outputs \( q \leq m - 3k \).
such that for each pair of disjoint tuples \( v_1 \in F_1 \) and \( v_2 \in F_2 \) there are at least \((\gamma n/2)^2\) walks of length \( m-3k \) connecting \( v_1 \) and \( v_2 \), each having \( q \) internal vertices, and internally disjoint from \( v_1 \cup v_2 \). By the choice of \( \delta \) we have \(|F_1|, |F_2| \geq \delta n^{k-1}\), and note that the remaining vertices to be embedded correspond to \( P \setminus (P_1 \cup P_2) \), whose set of vertices is completely in \( V_{[k+1, m-k]} \), which is \( k\Delta \)-bounded. So, we can use Lemma 8.4 with \( P, P \setminus (P_1 \cup P_2), k+1, m-2k-1, k\Delta \) playing the roles of \( T, T_j, t, \ell, M \) to obtain an embedding \( \varphi' \) of the remaining vertices from \( L_{k+1} \cup \cdots \cup L_{m-k} \), which extends the embedding \( \varphi \) to an embedding of all of \( P \).

\[\text{Lemma 9.7 (Covering Lemma).} \]

Let \( 1/n < \alpha < \mu < \nu = \gamma, 1/h \) with \( h, \Delta, k \geq 2 \) and \( \ell \geq (2k+1) / |k/2| + 2k \). Let \( X \) be a \((k-1)\)-tree on \( n \) vertices. Let \( H \) be a \( \gamma n \)-large \( k \)-graph on \( n \) vertices, and let \( A \subseteq \Delta X \) be a set of at most \( m/n \) pairwise disjoint \((h+1)\)-tuples of vertices of \( H \). Let \( T \) be a \( k \)-tree on \( \nu n \) vertices, with \( \Delta_1(T) \leq \Delta \), and let \( B \) be a \( 2\Delta k(\ell + 3k) \)-separated set of size at least \( \mu n \) of \( X \)-tuples of vertices of \( T \). Then, for any \( y \in \partial H, x \in \partial T \), there is an embedding \( \varphi : V(T) \to V(H) \) such that \( \varphi(x) = y \) and every tuple in \( A \) is \( X \)-covered by \( \varphi \).

\[\text{Proof.}\]

Write \( A = \{A_1, \ldots, A_t\} \), where \( 1 \leq t \leq an \). We will abuse notation by treating \( X \)-tuples \( B \in B \) as subgraphs of \( T \), consisting of the corresponding edges forming the \( X \)-tuple. We claim that there are \( B_1, \ldots, B_t \in B \) such that, defining \( d_T(x, B_i) \) as the minimum of \( d_T(x, b) \) over all \((k-1)\)-sets \( b \in \partial T[B_i] \), we have

\[\Delta k(\ell + 3k) \leq d_T(r, B_j) \leq d_T(r, B_i) \text{ for all } 1 \leq j < i \leq t. \quad (9.2)\]

To see this, order the elements of \( B \) as \( B'_1, B'_2, \ldots \) so that \( d_T(r, B'_i) \) increases. As \( B \) is \( 2\Delta k(\ell + 3k) \)-separated, and using the triangle inequality, we see that

\[2\Delta k(\ell + 3k) \leq d_T(B'_1, B'_2) \leq d_T(B'_1, x) + d_T(B'_2, x) \leq 2d_T(B'_2, x)\]

and thus \( d_T(B'_2, x) \geq \Delta k(\ell + 3k) \). Since \(|B| \geq \mu n \gg an \geq t\), we can delete \( B'_t \) from \( B \) if necessary, and delete more sets \( B'_i \) until size exactly \( t \) is reached. After relabelling, we arrive at sets \( B_1, \ldots, B_t \) satisfying (9.2).

Set \( T_0 := \emptyset \). Given \( i \in [t] \) and \( T_{i-1} \subseteq T \), define \( T_i \) as follows. Let \( t_i \in \partial T_{i-1} \) and \( b_i \in \partial B_i \) such that \( d_T(t_i, b_i) \) is minimised (if \( i = 1 \), select \( t_1 = r \) instead). Let \( P_i \) be the unique \((t_i, b_i)\)-pseudopath in \( T \), and let \( T_i = T_{i-1} \cup P_i \cup B_i \). Then \( T_0, T_1, \ldots, T_t \) satisfy the following properties.

\[\text{Claim 9.8.} \]

For all \( 1 \leq i \leq t, T_i \) is a subtree of \( T \), and

- \((Q1)\) \( T_i \subseteq T_{i+1} \) if \( i < t \),
- \((Q2)\) \( B_1 \cup \cdots \cup B_t \subseteq T_t \), and
- \((Q3)\) \( T_i \cap (B_{i+1} \cup \cdots \cup B_t) = \emptyset \).

Indeed, the only property which is not immediate from construction is \((Q3)\). Suppose the property failed, and let \( i \) be a minimum integer for which it fails. Then there exists a \( j > i \) such that \( B_j \cap T_i \neq \emptyset \). By minimality of \( i \), \( B_j \cap T_{i-1} = \emptyset \), and since \( B \) are pairwise disjoint subgraphs, \( B_j \cap B_i = \emptyset \). Thus \( B_j \cap (P_i \cup B_i) = \emptyset \). Since \( P_i \) was the unique minimum-length pseudopath between \( T_{i-1} \) and \( B_i \) in \( T \), this implies \( d_T(x, B_j) < d_T(x, B_i) \), contradicting (9.2).

For \( i \geq 0 \), we will now construct embeddings \( \varphi_i : V(T_i) \to V(H) \setminus (A_{i+1} \cup \cdots \cup A_t) \) such that for \( i \geq 1 \), \( \varphi_i \) extends \( \varphi_{i-1} \), and such that \( A_i = \varphi_i(B_i) \) is \( X \)-covered by \( \varphi_i \). We start by setting \( \varphi_0(x) = y \). Now assume that \( i \geq 1 \), and suppose we have embedded \( T_{i-1} \) with the embedding \( \varphi_{i-1} \). By \((Q3)\), the image of \( B_i \) has not been defined in \( \varphi_{i-1} \). We begin by setting \( \varphi_i'(B_i) = A_i \), in a way that \( A_i \) is \( X \)-covered by \( \varphi_i' \). Recall that, by definition, \( T_i \) is the union of \( T_{i-1}, B_i \), and a \((t_i, b_i)\)-pseudopath \( P_i \), for some \( b_i \in \partial B_i \) and some \( t_i \in \partial T_i \) (or \( t_i = x \) if \( i = 1 \)). Note that \( H_i := H \setminus (A_{i+1} \cup \cdots \cup A_t) \) has at least \( n' = n - (t-1)(h+1) \geq (\gamma - \alpha h) n \geq (1 - \gamma / 3)n \) vertices, as by assumption, \( ah \ll \gamma \). A similar calculation entails that \( H_i \) is \((\gamma n'/2)\)-large. We claim that

\[d_T(t_i, b_i) \geq \Delta k(\ell + 3k). \quad (9.3)\]

Then, we can use Lemma 9.6 to find an embedding \( \varphi_i : V(P_i) \to V(H_i) \) which extends both \( \varphi_{i-1} \) and \( \varphi_i' \), and this completes step \( i \).

So let us show (9.3). Assume otherwise, and let \( 1 \leq j < i - 1 \) be the minimum index such that \( t_i \in \partial T_j \). By minimality, we have \( t_i \in \partial (P_j \cup B_j) \). If \( t_i \in \partial B_j \), then \( P_i \) is a pseudopath from
$B_i$ to $B_j$, and since $B$ is $2\Delta k(\ell + 3k)$-separated we then have $d_T(b_i, t_i) \geq 2\Delta k(\ell + 3k)$, and we are done. Assume instead that $t_i \in \partial P_j \setminus \partial B_j$. Let $b_j \in \partial T_j$ be such that $d_T(b_j, x) = d_T(B_j, x)$. Note that $t_i$ must lie on the unique pseudopath in $T$ from $b_j$ to $x$, and thus

$$d_T(B_j, t_i) + d_T(t_i, x) = d_T(B_j, x) \leq d_T(B_i, x) \leq d_T(b_i, t_i) + d_T(t_i, x),$$

where the second inequality comes from (9.2). As furthermore

$$2\Delta k(\ell + 3k) \leq d_T(B_j, B_i) \leq d_T(b_i, t_i) + d_T(B_j, t_i) < 2\Delta k(\ell + 3k) + d_T(B_j, t_i),$$

we obtain $d_T(b_i, t_i) \geq d_T(B_j, t_i) > 2\Delta k(\ell + 3k)$, contrary to our assumption. Thus (9.3) holds.

Having defined all $\varphi_i$, we note that each absorbing tuple in $A$ is $X$-covered by $\varphi_i$. We extend $\varphi_i$ to an embedding $\varphi$ of all of $V(T)$. We can get from $T_i$ to $T$ by iteratively adding leaves, thus the embedding can be found in a greedy fashion since $\delta_{k-1}(H) \geq \gamma n$, and at each step there are at most $|V(T)| \leq \nu n \leq \gamma n/2$ used vertices.

10. Proof of the Main Theorem

We now assemble all results from the previous sections in order to prove Theorem 1.2. The proof is divided into three main steps.

We start the first step by finding a small subtree $T' \subseteq T$ of linear size which we use to build the absorbing structures. First, using Proposition 9.3 we find a $(k-1)$-tree $X$ such that $T'$ contains linearly many well-separated $X$-tuples, and then, with the help of Proposition 9.4 and Lemma 9.5, we find a family $A$ of disjoint absorbing $X$-tuples in the host graph $H$ such that every $k$-tuple in $H$ has linearly many absorbing $X$-tuples in $A$. We then use the Covering Lemma (Lemma 9.7) to embed $T'$ in $H$, covering every tuple in $A$.

In the second step, we will find an almost spanning subtree $T'' \subseteq T-T'$ and embed it following the regularity method. The Weak Hypergraph Regularity Lemma (Theorem 6.1) gives a regular partition of the vertices of $H$ and using Lemma 6.2, we find an almost spanning matching $\mathcal{M}$ in the corresponding reduced graph. We find a $(\beta, d)$-decomposition (Definition 5.14) of $T''$ and use the Embedding Lemma (Lemma 8.1) to map the small parts of this decomposition into edges of $\mathcal{M}$. In the third and last step, we finish the embedding by using the Absorbing Lemma (Lemma 9.2).

We begin with a lemma which essentially covers all of the second step outlined above. First, we need a definition. Let $H$ be a $k$-graph on $m$ vertices and let $\varepsilon, d > 0$. We say that $H$ is $(\varepsilon, d)$-uniformly dense if for all pairwise disjoint sets $W_1, \ldots, W_k \subseteq V(H)$ with $|W_i| = h \geq \varepsilon m$, $i \in [k]$, we have

$$e(W_1, \ldots, W_k) \geq dh^k. \quad (10.1)$$

If $H$ is $k$-partite, with parts $V_1, \ldots, V_k$, we say that $H$ is $k$-partite $(\varepsilon, d)$-uniformly dense if for all sets $W_i \subseteq V_1, \ldots, W_k \subseteq V_k$ with $|W_i| = h \geq \varepsilon m$, $i \in [k]$, the bound (10.1) holds.

Definition 10.1 (Uniformly dense perfect matching). For $\varepsilon, d > 0$ and $t \in \mathbb{N}$, we say $\mathcal{M} = \{(V_1^t, \ldots, V_k^t)\}_{i \in [t]}$ is an $(\varepsilon, d)$-uniformly dense perfect matching of a $k$-graph $H$ if

(M1) $\{V_i^t\}_{a \in [k], i \in [t]}$ partitions $V(H)$,

(M2) $|V_a^t| = |V_b^t|$ for all $i, j \in [t], a, b \in [k]$, and

(M3) $H[V_1^t, \ldots, V_k^t]$ is $k$-partite $(\varepsilon, d)$-uniformly dense for each $i \in [t]$.

The following proposition will be used in the second step of the proof of Theorem 1.2, providing us with an embedding of an almost spanning tree in any graph having an $(\varepsilon, d)$-uniformly dense perfect matching.

Proposition 10.2 (Embedding almost spanning trees). Let $\Delta, k \geq 2$, let $\theta \ll d, 1/T_0$, and let $1/n \ll \mu \ll \theta \ll \varepsilon, \delta \ll \alpha, \gamma, 1/k, 1/\Delta$. Let $H$ be a $\gamma n$-large $k$-graph on $n$ vertices with a $(\delta, \mu, 2)$-reservoir $R$. Let $\mathcal{M}$ be an $(\varepsilon, d)$-uniformly dense perfect matching of $H-R$ with $|\mathcal{M}| \leq T_0$. Let $(T, x)$ be a rooted $k$-tree with $|V(T)| \leq (1-\alpha)n$ and $\Delta_1(T) \leq \Delta$. Then $H$
contains a copy of $T$. Moreover, for any $\theta$-extensible edge $e$, and for any $f \in \partial H$ with $f \subseteq e$, any ordering $f \in \partial^p H$ of $f$ can be chosen as the image of $x$.

Proof. Let $\ell = \lfloor k/2 \rfloor (2k + 1) + 2k + 1$. Introduce new constants $c, \beta$ satisfying $\mu \ll \beta \ll \varepsilon \ll c \ll 1/k$ and $\beta \ll 1/T_0, d$.

Let $\mathcal{L}$ be the layering for $(T, x)$, which exists by Lemma 5.4. We invoke Lemma 5.15, with parameters $\beta, \Delta$ and $\ell$, to obtain a $(\beta, \ell)$-decomposition of $(T, x, \mathcal{L})$ into $p \leq 2\Delta^\ell/\beta$ parts. That is, we find a collection of rooted trees $\{(D_i, s_i)\}_{1 \leq i \leq p}$ such that

\begin{enumerate}
\item[(D1)] $E(T) = \bigcup_{i \in [p]} E(D_i),$
\item[(D2)] $|E(D_i)| \leq \beta n$ for each $i \in [p],$
\item[(D3)] $s_i = x$ and $s_i$ is $\mathcal{L}$-layered,
\item[(D4)] $(V(D_j) \setminus s_j) \cap V(D_i) = \emptyset$ for all $1 \leq i < j \leq p,$
\item[(D5)] for each $2 \leq j \leq p$ there is a unique $i < j$ such that $s_j \in \partial D_i$ and the rank of $s_j$ in $D_i$ is at least $\ell$ (in the inherited layering of $D_i$ from $\mathcal{L}$).
\end{enumerate}

Let $e \in H$ be an arbitrary $\theta$-extensible edge in $H$, let $f \subseteq e$ and let $f$ be an ordering of $f$. Say $\mathcal{M} = \{(V_1^h, ..., V_k^h)\}_{h \in [t]}$ are the clusters of the given uniformly dense perfect matching and set $m := |V_1^h|.$

We begin our embedding by setting $\varphi_0(x) := f$. Now we will construct successively, for all $i \in [p]$, an embedding $\varphi_i : V(D_1 \cup \cdots \cup D_i) \rightarrow V(H)$ such that $\varphi_i$ extends $\varphi_{i-1}$, and

(P1) if defined, $\varphi_i(s_j)$ is contained in a $\theta$-extensible edge for all $j > i$,

(P2) $|\varphi_i^{-1}(R)| \leq i\Delta^{\ell+1},$

(P3) for every $j \in [t]$, $\max_{a,b \in [k]} ||\varphi_j^{-1}(V_j^a)| - |\varphi_j^{-1}(V_j^b)|| \leq cm$.

Having done this, then $\varphi_\ell$ will be the desired embedding of $T$.

In step $i$, assume we have constructed $\varphi_{i-1}$ satisfying (P1)–(P3). Our aim is to embed $(D_i, s_i)$. We claim that $s_i$ is already embedded into some $(k - 1)$-tuple $\varphi_{i-1}(s_i)$ contained in some $\theta$-extensible edge. Indeed, if $i = 1$, we have $s_1 = x$ by (D3) and we are done by the choice of $\varphi_0$. Otherwise, if $i \geq 2$, then (D5) implies $\varphi_{i-1}(s_i)$ is defined and we are done by (P1).

We claim that there is a $j \in [t]$ such that for each $a \in [k]$,

\begin{equation}
|V_j^a| \geq |\varphi_{i-1}^{-1}(V_j^a)| + cm.
\end{equation}

Indeed, otherwise, for each $j \in [t]$ there is an $a \in [k]$ such that (10.2) does not hold. Then, by (P3), also all $V_j^a$ are almost full, and we calculate

\begin{align*}
|\varphi_{i-1}^{-1}(V(H))| & \geq \sum_{j \in [t], a \in [k]} |\varphi_{i-1}^{-1}(V_j^a)| - \sum_{j \in [t], a \in [k]} (|V_j^a| - 2cn) = n - |R| - 2cmtk \geq (1 - \alpha)n,
\end{align*}

a contradiction as $|V(T)| \leq (1 - \alpha)n$. So, (10.2) holds for $j$, say, index $j$.

Let $\sigma$ be a bijection satisfying

\begin{equation}
|W_1| \geq \cdots \geq |W_k|,
\end{equation}

where $W_a := V_j^a \setminus \varphi_{i-1}(V(T))$. Because of (10.2), $|W_i| \geq cm$ for each $i \in [k]$, so we can select $W_i^a \subseteq W_i$ of size exactly $cm$ for each $i$. Therefore, using that $H[V_1^j, ..., V_k^j]$ is $k$-partite $(\varepsilon, d)$-uniformly dense on $km$ vertices and $\varepsilon \ll c, d, 1/k$, we have

\begin{equation}
e(H[W_1, ..., W_k]) \geq e(H[W'_1, ..., W'_k]) \geq d(cm)^k.
\end{equation}

Also, note that since $m = |V(H) - R|/(kt_0)$ and $R$ is a $(\delta, \mu, 2)$-reservoir, we have $m \geq n/(2K_0)$ and thus $|W_a| \geq cn/(2K_0)$ for each $1 \leq a \leq k$.

Set $R_{i-1} := R \setminus \varphi_{i-1}(D_1 \cup \cdots \cup D_{i-1}).$ Recall that $i \leq p \leq 2\Delta^\ell\beta^{-1}$. Using (P2), we get $|R_{i-1}| \leq |R| - p\Delta^{\ell+1} \geq |R| - 2\Delta^{2\ell+1}\beta^{-1}$. Since $R$ is a $(\delta, \mu, 2)$-reservoir for $H$ and $1/n \ll 1/\Delta, 1/k, \beta, \mu, \delta$, we deduce that $R_{i-1}$ is a $(\delta, 2\mu, 2)$-reservoir for $H$. Let $L_i = (L_1, ..., L_k)$ be the inherited layering $\mathcal{L}^h$ for $(D_i, s_i)$ from $\mathcal{L}$, which is well-defined since $s_i$ is $\mathcal{L}$-layered by (D3).

Use Lemma 8.1 (Embedding Lemma) with

| object/parameter | playing the role of |
|------------------|------------------|
| $\ell - 1$      | $R_{i-1}$       |
| $\ell$          | $R$             |
| $\gamma$        | $\mu$           |
| $\ell - 1/2$    | $c/(2K_0)$      |
| $\ell$          | $d/2$           |
| $\mu$           | $c$             |
| $\Delta$        | $d$             |
| $\mu$           | $\ell$          |
| $\gamma$        | $\Delta$        |
| $\mu$           | $\ell$          |
| $\Delta$        | $\ell$          |
| $\Delta$        | $\ell$          |
| $\Delta$        | $\ell$          |
to find an embedding $\varphi'_i$ of $D_i$ such that

\begin{align*}
(\text{E1}) & \quad \varphi'_i(s_i) = \varphi'_{i-1}(s_i), \\
(\text{E2}) & \quad (\varphi'_i)^{-1}(R_{i-1} \cup \varphi'_{i-1}(s_i)) = \bigcup_{r=1}^{\ell} L_r, \\
(\text{E3}) & \quad \varphi'((V_{(\ell)})) \subseteq W_1 \cup \cdots \cup W_\ell, \text{ with } |(\varphi'_i)^{-1}(W_i)| \geq \cdots \geq |(\varphi'_i)^{-1}(W_k)|, \\
(\text{E4}) & \quad \varphi'_i(e') \text{ is } \theta\text{-extensible, for each } e' \in E(D_i|\varphi'_{i-1}(s_i))).
\end{align*}

By (D4)-(D5) and (E1), we get that $\varphi_i = \varphi_{i-1} \cup \varphi'_i$ is an extension of $\varphi_{i-1}$ which embeds $T[V(D_i \cup \cdots \cup D_{\ell})]$. It is only left to check that $\varphi_i$ satisfies (P1)-(P3).

We check (P1) holds for $\varphi_i$. Since (P1) holds for $i-1$, we only need to consider those $j > i$ with $s_j \in \partial D_i$. By (D5), each $s_j \in \partial D_i$ with $j > i$ has rank at least $\ell$ in $L_i$, namely it must be contained in $V_{(\ell)}$. Thus, by (E4), we know that all such $\varphi_i(s_j)$ are contained in a $\theta$-extensible edge, as required.

To see (P2) holds, we note that since (P2) holds for $i-1$, $|\varphi'_i^{-1}(R \setminus R_{i-1})| = |\varphi^{-1}_i(R \setminus R_{i-1})| \leq (i-1)\Delta^{\ell+1}$, so it only remains to show that $|\varphi^{-1}_i(R_{i-1})| \leq \Delta^{\ell+1}$. Note that by Proposition 5.5 we have that $|L_r| \leq \Delta^{\ell+1}$ holds for all $1 \leq r \leq \ell - 1$, and together with (E2) we get $|\varphi^{-1}_i(R_{i-1})| \leq \sum_{r=1}^{\ell-1} |L_r| \leq \sum_{r=1}^{\ell-1} \Delta^{\ell+1} \leq \Delta^{\ell+1}$, as required.

Finally, for (P3), observe that (E3) implies that

$$|V(D_i)| \geq |\varphi^{-1}_i(W_i) \setminus \varphi^{-1}_{i-1}(W_i)| \geq \cdots \geq |\varphi^{-1}_i(W_k) \setminus \varphi^{-1}_{i-1}(W_k)|.$$  

Using (10.3), we get, for any $a, b \in [k]$ with $a < b$,

$$||\varphi^{-1}_i(V_j)| \setminus |\varphi^{-1}_{i-1}(V_j)|| \leq \max \left\{ |(\varphi^{-1}_i(V_j)) \setminus |\varphi^{-1}_{i-1}(V_j)|, |V(D_i)| \right\}.$$  

Since (P3) holds for $i-1$, the first term in the maximum is bounded from above by $cn$, and $|V(D_i)| \leq 3n \leq cn$, follows from (D2) and $\beta \ll c$.

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** We begin by noting that $\delta_{k-1}(H) \geq (1/2 + \gamma)n$ implies that $H$ is $(2\gamma)n$-large, and also that every vertex is contained in an edge.

**Step 1: Finding the absorbing structures.** For the remainder of the proof, we choose $\ell = (2k + 1)|k/2| + 2k$ and also

$$1/n_0 \ll \nu_2 \ll \nu_2 \ll \delta, \nu_1 \ll \alpha \ll \theta_0 \ll \gamma, 1/k, 1/\Delta, 1/\ell.$$  

Let $T$ be any given tree on $n \geq n_0$ vertices such that $\Delta_1(T) \leq \Delta$. We choose an arbitrary tuple $x \in \partial T$ as the root of $T$ and, using Lemma 5.4, we find a layering $\mathcal{L} = (L_1, \ldots, L_m)$ for $(T, x)$. Let $x' \in \partial^* T$ be an $\mathcal{L}$-layered tuple such that $|E(T_{x'})| \geq an/(2\Delta)$ with the highest possible rank. Then, using Proposition 5.13 for $(T_{x'}, x', \mathcal{L}')$, and by the maximality of $x'$ we deduce that

$$\frac{an}{2\Delta} \leq |E(T_{x'})| \leq an. \quad (10.5)$$  

Since $\delta \ll 1/k, 1/\Delta, 1/\ell$, we can use Proposition 9.3 with $2\Delta k(\ell + 3k)$ and $\delta$ in place of $\ell$ and $\mu$, respectively. This yields a $(k-1)$-tree $X$ with $h \leq \Delta + k - 1$ vertices, and a $2\Delta k(\ell + 3k)$-separated set $B$ of $X$-tuples of $T_{x'}$ such that $|B| \geq \delta n$. Recall that $H$ is $(2\gamma)n$-large, there are no isolated vertices, and additionally $1/n \ll \nu_1 \ll \gamma, 1/k, 1/\Delta$ and $h \leq \Delta + k - 1$. Thus, Proposition 9.4 (with $\nu_1$ playing the role of $\beta$) tells us that $|\Lambda_X(v_1, \ldots, v_k)| \geq \nu_1 h^k$ for every $k$-tuple of distinct vertices $(v_1, \ldots, v_k) \in V(H)^k$. The hierarchy $\nu_2 \ll \nu_1, 1/k, 1/h$ allows us to use Lemma 9.5, with parameters $\nu_1, \nu_2$ in place of $\beta, \alpha$, respectively, to deduce the existence of a family $\mathcal{A} \subseteq \Lambda_X$ of size at most $\nu_2 n$, such that

$$\text{for every } k\text{-tuple } (v_1, \ldots, v_k) \text{ we have } |\Lambda_X(v_1, \ldots, v_k) \cap \mathcal{A}| \geq \frac{\nu_1 \nu_2}{8} n \geq \nu_3 n, \quad (10.6)$$  

where the last inequality follows from $\nu_3 \ll \nu_2, \nu_1$.

Bounding very crudely, we deduce that $H$ has at least $\gamma(\nu^n)$ edges. Since $\theta_0 \ll \gamma, 1/k$, an application of Lemma 6.5 implies the existence of a $\theta_0$-extensible edge $e_0$ which is vertex-disjoint of all the subgraphs in $\mathcal{A}$. Let $f_0 \subseteq e_0$ be an arbitrary $(k-1)$-set and set $f_0$ be an ordering of $f_0$. Finally, since $1/n \ll \nu_2 \ll \delta \ll \alpha \ll \gamma, 1/h$, we can use Lemma 9.7, with $\nu_2, \delta, \alpha, 2\gamma, x', f_0$
playing the roles of $\alpha, \mu, \nu, \gamma, x, y$ respectively, to find an embedding $\varphi': V(T_{x'}) \to V(H) \setminus (e_0 \setminus f_0)$ which $X$-covers each tuple in $\mathcal{A}$, such that $\varphi'(x') = f_0 \subseteq e_0$, and $e_0 \setminus f_0$ is disjoint from $\varphi'(V(T_{x'}))$.

**Step 2: Finding an almost spanning tree.** We introduce new constants by letting

$$1/n \leq \mu \leq \eta \leq 1/T_0 \leq 1/t_0 \ll \varepsilon' \ll \eta \ll \varepsilon' \ll \varepsilon \ll \alpha \ll \nu_3.$$ 

Let $H' = H - \varphi'(V(T_{x'})) + e_0$ and $n' = |V(H')|$. Then, by (10.5) we have $|V(T_{x'})| \leq \alpha n + k - 1 \leq 2\alpha n$, and thus $n' \geq (1 - 2\alpha)n$. Note that $e_0 \subseteq V(H')$.

The choice of $\alpha \ll \gamma$ ensures that $H'$ has minimum codegree at least $(1/2 + 2\gamma/3)n'$. Using $1/n \leq \mu \ll \gamma$, Lemma 6.7 provides us with a $(\gamma', \mu, 2)$-reservoir $R$ for $H'$. Now set $H'' = H' - R$ and $n'' = |V(H'')|$. Since $\gamma' \ll \alpha$, we deduce that $n'' \geq (1 - 3\alpha)n$.

Now we prepare the setup to apply regularity tools. Since $1/n, 1/T_0 \ll 1/t_0, 1/k, \varepsilon'$, an application of the Weak Hypergraph Regularity Lemma (Theorem 6.1) to $H''$, with parameters $\varepsilon'$ and $t_0$ as input, yields an $\varepsilon'$-regular partition $\mathcal{P} = \{V_0, V_1, \ldots, V_{t_0}\}$ of $V(H'')$, for some $t_0 \leq t \leq T_0$. Using Lemma 6.2, $1/t_0 \ll \varepsilon' \ll 1/k, \gamma, \eta$ and $2d \ll \gamma$, we know that the $(\varepsilon', 2d)$-reduced graph $R_{2d}(H'')$ contains a matching $\mathcal{M}$ covering at least $(1 - \eta)t$ clusters. Since each edge $(V_{i_1}, \ldots, V_{i_k}) \in \mathcal{P}$ is $(\varepsilon', d')$-regular for some $d' \geq 2d$, by our choice of $\varepsilon' \ll \varepsilon$, we deduce that $H'''[V_{i_1}, \ldots, V_{i_k}]$ is $k$-partite $(\varepsilon, 2d - \varepsilon)$-uniformly dense, and thus $(\varepsilon, d)$-uniformly dense. Let $V_{\mathcal{M}} \subseteq V(H'')$ consist of the union of clusters covered by $\mathcal{M}$ in the reduced graph. Thus, $H'''[V_{\mathcal{M}}]$ has an $(\varepsilon, d)$-uniformly dense perfect matching with $p \leq t_0/k$ edges.

We wish to apply Proposition 10.2. For this we need a $k$-graph having a reservoir and a uniformly dense perfect matching, which we plan to be $R$ and $V_{\mathcal{M}}$, respectively. We also need the $k$-graph to contain our desired root $e_0$, which is why we set $R' = (R \cup e_0) \setminus V_{\mathcal{M}}$. Let $H^*$ be the induced subgraph of $H'$ on $R' \cup V_{\mathcal{M}}$ and set $n^* = |V(H^*)|$.

We now check that the requirements of Proposition 10.2 are satisfied with this choice of $H^*, R', V_{\mathcal{M}}$. Clearly $\{R', V_{\mathcal{M}}\}$ partitions $V(H^*)$ and $e_0 \subseteq V(H^*)$. As $\mathcal{P}$ is an $\varepsilon'$-regular partition of $H'' = H' - R$, and $\mathcal{M}$ leaves at most $\eta t$ clusters uncovered, each having size at most $n''/t$, we see that

$$n' - n^* \leq |V_0| + \sum_{V_i \notin V(\mathcal{M})} |V_i| \leq \varepsilon'n'' + \eta n''/t \leq \varepsilon n',$$

where we used $\varepsilon', \nu \ll \varepsilon$ and $n'' \leq n'$ in the last inequality. This implies $n^* \geq (1 - \varepsilon)n'$. Since $\varepsilon \ll \gamma$ and since $\delta_k - 1(H'') \geq (1/2 + 2\gamma/3)n'$, we deduce that $H^*$ has minimum codegree at least $(1/2 + 2\gamma/3)n'$. In particular, $H^*$ is $\gamma n^*$-large. Since $R$ is a $(\gamma, \mu, 2)$-reservoir for $H'$, we use $n^* \geq (1 - \varepsilon)n'$ and $1/n \ll \mu, 1/k$ to deduce that $R'$ has size $\gamma'n' \pm (\mu n' + k)$ and then $|R'| = (\gamma^* \pm 2\mu)n^*$, for some $\gamma^* \leq 2\gamma$. Thus $R'$ is a $(\gamma^*, 2\mu, 2)$-reservoir for $H^*$. Finally, recall that $e_0$ is $\theta_0$-extensible in $H$. As $1/n \ll \varepsilon \ll \alpha \ll \theta_0$, we have $n - n^* \leq 4\alpha n$, implying that $e_0$ is $\theta_0$-extensible in $H^*$.

We set $T' = T - (T_{x'} - x')$ and root $T'$ at $x'$. Let $v_1, \ldots, v_{|V(T)|}$ be a valid ordering of $(T', x')$ with $x' = (v_1, \ldots, v_{k-1})$. Let $m^* = [(1 - \nu_2)n^*]$ and $T^* = T'[v_1, \ldots, v_{m^*}]$, such that $|V(T^*)| = m^*$.

An application of Proposition 10.2 with

| object/parameter | $n^*$ | $2\mu$ | $\gamma' \delta$ | $v_{3/2}$ | $d$ | $H^*$ | $R'$ | $x'$ | $e_0$ | $f_0$ |
|-------------------|-------|-------|----------------|----------|-----|--------|------|------|------|------|
| playing the role of | $n$ | $\mu$ | $\alpha$ | $d$ | $H$ | $R$ | $x$ | $e$ | | |

yields an embedding $\varphi^*: V(T^*) \to V(H^*)$ with $\varphi^*(x') = f_0 = \varphi'(x')$.

**Step 3: Finishing the embedding.** The embedding $\varphi' \cup \varphi^*$ of $T_{x'} \cup T^*$ fulfills all the conditions of Lemma 9.2 (as $|V(T_{x'} \cup T^*)| \geq m^* + |V(T_{x'})| - k - 1 \geq (1 - \nu_3)n$ and by (10.6)). So, we can use the Absorbing lemma (Lemma 9.2) to embed the remaining vertices $v_{m^*+1}, \ldots, v_{|V(T)|}$.

## 11. Spanning trees in quasirandom hypergraphs

In this section, we prove Theorem 1.4. Actually, we prove a slightly stronger statement since we only need the weaker notion of being uniformly dense (given right before Definition 10.1).
Theorem 11.1. Let $0 < 1/n \ll \delta \ll \eta, \gamma, 1/\Delta, 1/k$ and let $H_1$ and $H_2$ be $k$-graphs on the same vertex set of size $n$. Assume that $H_1$ is $(\eta, \delta)$-uniformly dense, and that $H_2$ is $\gamma n$-large and has no isolated vertices. Then, $H := H_1 \cup H_2$ contains a copy of every $k$-tree $T$ on $n$ vertices with $\Delta_1(T) \leq \Delta$.

Proof (sketch). Since the proof is essentially the same as the proof of Theorem 1.2, we only outline the major differences. In particular, step 1 (construction of absorbers) is virtually the same, noting that the only property used there is that the host graph was that $H$ is $2\gamma n$-large and has no isolated vertices, while here $H_2$ is $\gamma n$-large and has no isolated vertices.

The difference in step 2 is that instead of applying regularity to obtain the uniformly dense perfect matching, we use that $H_1$ is uniformly dense. Indeed, suppose we have already found the equivalent of $H'$ with a reservoir $R$, and we wish to find a uniformly dense matching covering most vertices in $H'' = H' - R$. Move at most $k-1$ vertices from $H''$ to $R$ we can assume $|V(H'')|$ is divisible by $k$. Fix any partition $V_1, \ldots, V_k$ of $V(H'')$ into equal-sized parts. Since $H_1$ is $(\eta, \delta)$-uniformly dense, $\mathcal{M} = \{(V_1, \ldots, V_k)\}$ is an $(\eta, \delta)$-uniformly dense perfect matching with only one edge. Having found $\mathcal{M}$, then we can proceed with the remainder of step 2 and step 3 exactly as before.

Let us now show how Theorem 11.1 implies the same result for host hypergraphs satisfying certain quasirandom properties, namely weak quasirandomness (Corollary 11.2) and typicality (Theorem 1.4).

The study of quasirandomness for graphs evolved in the late 1980’s, a milestone being the seminal result of Chung, Graham and Wilson [CGW89] relating uniform edge distribution to other ‘random-like’ properties. For the history of quasirandomness for hypergraphs we refer to the exposition in [Aig+18]. The weakest form of quasirandomness studied in the literature is as follows. Call a $k$-graph $H$ weakly $(\eta, \delta)$-quasirandom if for every $U \subseteq V(H)$, the number $e(U)$ of edges entirely contained in $U$ satisfies

$$\left| e(U) - \eta \left(\frac{|U|}{k}\right) \right| \leq \delta nk.$$  \hspace{1cm} (11.1)

An inclusion-exclusion argument shows that every weakly $(\eta, \delta)$-quasirandom $k$-graph is $(\eta, 2^k \delta)$-uniformly dense. Thus the following is an immediate corollary of Theorem 11.1.

Corollary 11.2. Let $0 < 1/n \ll \delta \ll \eta, \gamma, 1/\Delta, 1/k$ and let $H_1$ and $H_2$ be $k$-graphs on the same vertex set of size $n$. Assume that $H_1$ is weakly $(\eta, \delta)$-quasirandom, and that $H_2$ is $\gamma n$-large and has no isolated vertices. Then, $H := H_1 \cup H_2$ contains a copy of every $k$-tree $T$ on $n$ vertices with $\Delta_1(T) \leq \Delta$.

As mentioned in the introduction, being $\gamma n$-large in Corollary 11.2 cannot be replaced with a lower bound of type $\Omega(n^{k-1})$ on the minimum degree of $H$, as evidenced by an example by Araújo, Piga and Schacht [APS22, Section 8.2].

Let us now turn to the notion of typicality, which was defined in the introduction. Although weakly quasirandomness does not imply typicality, one can show that the converse is true: any $(\eta, 2, \delta)$-typical $k$-graph is weakly $(\eta, \delta')$-quasirandom, for some constant $\delta'$. This is the only ingredient we need in order to prove Theorem 1.4.

Proof of Theorem 1.4. Let $1/n_0 \ll \varepsilon' \ll \varepsilon$. As discussed above, $H$ is weakly $(\varrho, \varepsilon')$-quasirandom. Also, note that every $(\varrho, 2, \varepsilon)$-typical graph on $n$ vertices is $(\varrho^2 - \varepsilon)n$-large, and cannot have isolated vertices. Thus the theorem follows from Corollary 11.2, by taking $H = H_1 = H_2$.  \hspace{1cm} $\Box$
12. Further questions

12.1. Degree variations. For any $1 \leq j < k - 1$, one can define the minimum $j$-degree $\delta_j(H)$ of a graph $H$ in analogy to the minimum codegree $\delta_{k-1}(H)$ as defined in the introduction. Also, define $st_j(k)$ as the smallest $\delta > 0$ such that for every $\Delta, k \geq 2$ and $\mu > 0$, every large enough $k$-graph $H$ with $\delta_j(H) \geq (\delta + \mu)\binom{n-j}{k-j}$ contains every $k$-tree $T$ of the same order and with $\Delta_1(T) \leq \Delta$. Then, by Theorem 1.2, we have $st_{k-1}(k) \leq 1/2$ for all $k \geq 2$, and by Proposition 1.3 this is tight.

It would be interesting to understand $st_j(k)$ for $j < k - 1$. If instead of a spanning tree we are looking for a spanning cycle, then some results are known for the analogous problem. In $k$-graphs, recall that a tight Hamilton cycle in a $k$-graph $H$ on $n$ vertices is a sequence of distinct vertices $v_1, \ldots, v_n$ such that, for each $i \in \{1, \ldots, n\}$, the edge $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\}$ is present in $H$. Then $hc_j(k)$ is defined accordingly. It is known that $hc_{k-2}(k) = 5/9$ for all $k \geq 3$ [Pol+21; LS22]. For general $j, k$ satisfying $1 \leq j \leq k - 3$, the current best lower [HZ16] and upper [LS22] bounds are

$$1 - \frac{1}{\sqrt{k-j}} \leq hc_j(k) \leq 1 - \frac{1}{2(k-j)}.$$

Furthermore, the result of Rödl, Ruciński and Szemerédi [RRS08] mentioned in the introduction states that $hc_{k-1}(k) = 1/2$ for all $k \geq 2$. Thus $hc_{k-1}(k) = st_{k-1}(k) = 1/2$ for all $k \geq 2$, and we believe the same should be true in general.

Conjecture 12.1. For all $k > j \geq 1$, we have $hc_j(k) = st_j(k)$.

12.2. Trees of larger maximum degree. Another possible generalisation of Theorem 1.2 would be to relax the condition on $\Delta_1(T) = O(1)$, allowing it to grow with $n$. In the graph case, Komlós, Sárközy, and Szemerédi [KSS01] strengthened their earlier result, Theorem 1.1, considerably by showing that, with the same minimum degree conditions, one can find all trees of maximum degree at most $cn/\log n$, for a fixed $c > 0$ depending on the approximation $\gamma$ only. They also gave an example showing this bound is tight up to a multiplicative factor.

A natural adaptation of their example shows that, for $k \geq 3$, the bound on $\Delta_1(T)$ in Theorem 1.2 cannot be larger than $O(n/\log n)$. Indeed, let $T$ be the $k$-tree consisting of a vertex $v$ whose link graph is a $(k-1)$-tight path $P$ of length $c \log n$, for some sufficiently small constant $c > 0$. For each set $S$ of $k-1$ consecutive vertices in $P$, we add $n/(c \log n)$ new vertices adjacent to $S$. Then $T$ has maximum degree $\Theta(n/\log n)$, and a straightforward calculation shows that, with high probability, the binomial random $k$-graph of edge density $p = 0.9$ does not contain an ordered set $U$ of size $c \log n$ such that each vertex outside $U$ is adjacent to some $k-1$ consecutive vertices in $U$.

Acknowledgment

The authors are very grateful to Esteban Quiroz Camarasa for many valuable discussions in the beginning of this project, and to Yanitsa Pehova for useful comments.

References

[Aig+18] Elad Aigner-Horev, David Conlon, Hiêp Hán, Yury Person and Mathias Schacht. ‘Quasirandomness in hypergraphs’. Electron. J. Combin. 25.3 (2018), Paper No. 3.34, 22. DOI: 10.3736/7537.

[AKS95] Miklós Ajtai, János Komlós and Endre Szemerédi. ‘On a conjecture of Loebl’. Graph theory, combinatorics, and algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992). Wiley-Intersci. Publ. Wiley, New York, 1995, 1135–1146.

[All+17] Peter Allen, Julia Böttcher, Oliver Cooley and Richard Mycroft. ‘Tight cycles and regular slices in dense hypergraphs’. J. Combin. Theory Ser. A 149 (2017), 30–100. DOI: 10.1016/j.jcta.2017.01.003.

[APS22] Pedro Araújo, Simón Piga and Mathias Schacht. ‘Localized codegree conditions for tight Hamilton cycles in 3-uniform hypergraphs’. SIAM J. Discrete Math. 36.1 (2022), 147–169. DOI: 10.1137/21M1408531.

[Bol78] Béla Bollobás. ‘Extremal graph theory’. Vol. 11. London Mathematical Society Monographs. Academic Press, Inc., London-New York, 1978, xx+488.
[Ste20] Maya Stein. ‘Tree containment and degree conditions’. *Discrete mathematics and applications*. Vol. 165. Springer Optim. Appl. Springer, Cham, 2020, 459–486. doi: 10.1007/978-3-030-55857-4_19.

[Tow17] Henry Towsner. ‘σ-algebras for quasirandom hypergraphs’. *Random Structures Algorithms* 50.1 (2017), 114–139. doi: 10.1002/rsa.20641.

(M. Pavez-Signé) Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK
Email address: matias.pavez-signe@warwick.ac.uk

(N. Sanhueza-Matamala) Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción.
Email address: nicolas@sanhueza.net

(M. Stein) Departamento de Ingeniería Matemática y Centro de Modelamiento Matemático (CNRS IRL 2807), Universidad de Chile, Beauchef 851, Santiago, Chile.
Email address: mstein@dim.uchile.cl