On the Maximum Satisfiability of Random Formulas

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Abstract

Maximum satisfiability is a canonical NP-hard optimization problem that appears empirically hard for random instances. In particular, its apparent hardness on random $k$-CNF formulas of certain densities was recently suggested by Feige as a starting point for studying inapproximability. At the same time, it is rapidly becoming a canonical problem for statistical physics. In both of these realms, evaluating new ideas relies crucially on knowing the maximum number of clauses one can typically satisfy in a random $k$-CNF formula. In this paper we give asymptotically tight estimates for this quantity. Specifically, let us say that a $k$-CNF is $p$-satisfiable if there exists a truth assignment satisfying $1 - 2^{-k} + p2^{-k}$ of all clauses (observe that every $k$-CNF is 0-satisfiable). Also, let $F_k(n, m)$ denote a random $k$-CNF on $n$ variables formed by selecting uniformly and independently $m$ out of all $2^k \binom{n}{k}$ possible $k$-clauses.

Let $\tau(p) = 2^k \ln 2/(p + (1 - p) \ln(1 - p))$. It is easy to prove that for every $k \geq 2$ and every $p \in (0, 1]$, if $r \geq \tau(p)$ then the probability that $F_k(n, rn)$ is $p$-satisfiable tends to 0 as $n \to \infty$. We prove that there exists a sequence $\delta_k \to 0$ such that if $r \leq (1 - \delta_k)\tau(p)$ then the probability that $F_k(n, rn)$ is $p$-satisfiable tends to 1 as $n \to \infty$. The sequence $\delta_k$ tends to 0 exponentially fast in $k$. Indeed, even for moderate values of $k$, e.g. $k = 10$, our result gives very tight bounds for the number of satisfiable clauses in a random $k$-CNF. In particular, for $k > 2$ it improves upon all previously known such bounds.

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1 Introduction

Given a Boolean CNF formula $F$, the Satisfiability problem is to determine whether there exists a truth assignment that satisfies $F$. When $F$ has exactly $k$ literals in each clause, Satisfiability is known as $k$-SAT and is NP-complete \cite{Coo71} for all $k \geq 3$. A natural generalization of satisfiability is determining whether there exists a truth assignment that satisfies a given number of clauses in $F$. For $k$-CNF this problem is known as Max $k$-SAT and is NP-complete for all $k \geq 2$ (see \cite{Coo71}).

Optimization problems with random inputs are pervasive in operations research (e.g., the travelling salesman problem and variants), in statistical physics (determining ground states of spin glasses) and in computer science. An interesting source of Max $k$-SAT instances comes from considering $k$-CNF chosen uniformly at random (see below). Historically, the motivation for studying such formulas has been the desire to understand the hardness of “typical” instances. Random $k$-CNF are by now the most studied generative model for random formulas and have been a very popular benchmark for testing and tuning satisfiability algorithms. In fact, some of the better practical ideas in use today come from insights gained by studying the performance of algorithms on random $k$-CNF \cite{SLM92, SK93, GSCK00}.

A natural starting point for considering Max $k$-SAT is the observation that for every $k$-CNF formula there exists a truth assignment satisfying at least $(1 - 2^{-k})$ of all clauses. Indeed, if such a formula has $m$ clauses, the average over all $2^m$ truth assignments of the number of satisfied clauses is precisely $(1 - 2^{-k})m$. With this in mind, we will say that a $k$-CNF formula is $p$-satisfiable, where $p \in [0, 1]$, if there exists a truth assignment satisfying $1 - 2^{-k} + p2^{-k}$ of all clauses.

To consider random $k$-CNF formulas, let $C_k$ denote the set of all $(2n)^k$ possible disjunctions of $k$ literals on some canonical set of $n$ Boolean variables. To form a random $k$-CNF formula $F_k(n, m)$ with $m$ clauses we select uniformly, independently and with replacement $m$ clauses from $C_k$ and take their conjunction\footnote{Our discussion and results hold in all common models for random $k$-CNF, e.g. when clause replacement is not allowed and/or when each $k$-clause is formed by selecting $k$ distinct, non-complementary literals with/without ordering. The model defined here is best suited for our calculations. We further comment on its relationship to other models in the end of Section 4.}

We will say that a sequence of random events $\mathcal{E}_n$ occurs with high probability (w.h.p.) if $\lim_{n \to \infty} \Pr[\mathcal{E}_n] = 1$ and with uniformly positive probability if $\lim \inf_{n \to \infty} \Pr[\mathcal{E}_n] > 0$. We emphasize that throughout the paper $k$ is arbitrarily large but fixed, while $n \to \infty$. For every $k \geq 2$ and $p \in (0, 1]$, let

$$r_k(p) \equiv \sup \{ r : F_k(n, rn) \text{ is } p\text{-satisfiable w.h.p.} \} \leq \inf \{ r : F_k(n, rn) \text{ is not } p\text{-satisfiable w.h.p.} \} \equiv r_k^*(p).$$

One of the most intriguing aspects of random formulas is the Satisfiability Threshold Conjecture which asserts that $r_k(1) = r_k^*(1)$ for every $k \geq 3$. Much work has been done to bound $r_k(1)$ and $r_k^*(1)$. Currently, the best rigorous bounds for general $k \geq 3$, from \cite{AP03, DB97} respectively, are: $2^k \ln 2 - O(k) < r_k \leq r_k^* < 2^k \ln 2 - O(1)$. For $p < 1$, the bounds for $r_k(p), r_k^*(p)$ were much further apart.

The state of the art for general $k$ was presented in an important recent paper by Coppersmith, Gamarnik, Hajiaghayi, and Sorkin \cite{CGHS03}, where it was proved (see \cite{CGHS03} for a more precise formulation) that there exists an absolute constant $c > 0$ such that for all $k$ and all $p \in (0, p_0(k)]$,

$$\frac{c}{k} \frac{2^{k+1} \ln 2}{p^2} \leq r_k(p) \leq r_k^*(p) \leq \frac{2^{k+1} \ln 2}{p^2(1 + o(1))}. \quad (1)$$

The upper bound in (1) was proved via the first moment method, while the lower bound is algorithmic. For small $k$ the two are reasonably close, but the ratio between them tends to infinity as $k$ grows; this naturally raises the question which bound is closer to the truth. Our main result resolves this question by pinpointing the values of $r_k(p)$ and $r_k^*(p)$ with relative error that tends to zero exponentially fast in $k$. For every $p \in (0, 1)$ denote

$$T_k(p) = \frac{2^k \ln 2}{p + (1 - p) \ln(1/p)}, \quad (2)$$

and let $T_k(1) = 2^k \ln 2$ so that $T_k(\cdot)$ is continuous on $(0, 1)$.
Theorem 1. There exists a sequence \( \delta_k = O(k2^{-k/2}) \), such that for all \( k \geq 2 \) and \( p \in (0,1] \),

\[
(1 - \delta_k) T_k(p) < r_k(p) \leq r^*_k(p) \leq T_k(p).
\] (3)

The upper bound in (3) follows from well-known tail estimates. Taylor expansion gives that as \( p \to 0 \),

\[
T_k(p) = \frac{2^k \ln 2}{p^2/2 + O(p^3)},
\]

so as \( p \to 0 \), we can sharpen (3) to

\[
(1 - \delta_k) \frac{2^{k+1} \ln 2}{p^2 + O(p^3)} \leq r_k(p) \leq r^*_k(p) \leq \frac{2^{k+1} \ln 2}{p^2 + O(p^3)}.
\] (4)

Our proof of Theorem 4 actually yields an explicit lower bound for \( r_k(p) \) for each \( k \geq 2 \). For \( k = 2 \), i.e. Max 2-SAT, the algorithm presented in [CGHS03] dominates our lower bound uniformly, i.e. for every density it satisfies a greater fraction of all clauses. Already for \( k \geq 3 \), though, our methods yield a better bound, as indicated by the following plots.

![Figure 1. Upper and lower bounds for the density \( r \) as a function of \( q = 1 - p \).](image-url)

Our approach in proving Theorem 4 is non-algorithmic, based instead on a delicate application of the second moment method to a random generating function in two variables. It is notoriously difficult to obtain precise asymptotics from such random multivariable generating functions; the fact that this is possible for random Max \( k \)-SAT is technically due to the surprising cancellation of four terms of equal magnitude in our analysis, leaving only lower order terms. This cancellation hints at the existence of some unexpected hidden structure in random Max \( k \)-SAT; characterizing this structure combinatorially (rather than just analytically) appears to us worthy of further study.
1.1 Background

For a random formula $F_k(n, m)$, denote by $s_k(n, m)$ the random variable equal to the maximum (over all truth assignments $\sigma$) of the number of clauses satisfied by $\sigma$. Perhaps the first rigorous study of random Max $k$-SAT appeared in the work of Frieze, Broder and Upfal [BFU93] where it was shown that $s_k(n, m)$ is sharply concentrated around its mean. Specifically, 

**Theorem 2 ([BFU93]).** $\Pr \left[ |s_k(n, m) - \mathbb{E}[s_k(n, m)]| > t \right] < 2\exp(-2t^2/m)$.

The following corollary allows us to infer high probability results from positive probability results.

**Corollary 1.** If $F_k(n, rn)$ is $p_0$-satisfiable with uniformly positive probability, then for every constant $p < p_0$, $F_k(n, rn)$ is $p$-satisfiable w.h.p.

*Proof.* Let $S \equiv (1 - 2^{-k} + p_02^{-k})rn$. Since $F_k(n, rn)$ is $p_0$-satisfiable with uniformly positive probability, $\mathbb{E}[s_k(n, rn)] > S - n^{2/3}$. For, otherwise, Theorem 2 would imply that the probability of $p_0$-satisfiability is exponentially small. By the same token, $\Pr[s_k(n, rn) < S - 2n^{2/3}] = o(1)$, implying the claim. \qed

Thus it will suffice to find, for every $p \in (0, 1]$, a value $r = r(p)$ such that $F_k(n, rn)$ is $p$-satisfiable with uniformly positive probability and rely on Corollary 1 to get a high probability result.

Regarding the mean in Theorem 2, in view of the a priori bound $s_k(n, m) \geq (1 - 2^{-k})m$, it is natural to consider $\Phi_k(n, m) = \mathbb{E}[s_k(n, m)] - (1 - 2^{-k})m$, measuring how much the optimum truth assignment does better than the a priori bound in expectation (over random $k$-CNF). In [CGHS03] it was shown that for all $k$, for sufficiently large $r$, as $n \to \infty$ one has in $F_k(n, rn)$

$$\frac{2}{k+1} \sqrt{\frac{k}{\pi 2^k}} \times \sqrt{r} - O(1) \leq \frac{\Phi_k(n, rn)}{n} \leq \sqrt{\frac{2(2^k - 1) \ln 2}{2^{2k} - 1}} \times \sqrt{r}.$$  \hspace{1cm} (5)

This is equivalent to the assertion that for $p$ sufficiently small,

$$\frac{k^{2k+2}}{\pi (k + 1)^2} \times p^{-2} - O(p^{-1}) \leq r_k(p) \leq r_k^*(p) \leq 2(2^k - 1) \ln 2 \times p^{-2},$$  \hspace{1cm} (6)

which is a more precise formulation of (5).

Since for $k = 2$, the threshold for satisfiability is known, namely $r_2(1) = r_2^*(1) = 1$, in [CGHS03] very fine results were derived for $s_k(n, rn)$ when $r \approx 1$. In particular, when $r = 1 + \varepsilon$ one has $\mathbb{E}[s_2(n, m)] = (1 + \varepsilon - O(\varepsilon^3))n$, while for large $r > 1$ the bound in (4) can be improved to

$$\frac{\sqrt{8} - 1}{3\sqrt{\pi}} \times \sqrt{r} - O(1) \leq \frac{\Phi_2(n, rn)}{n} \leq \sqrt{\frac{3 \ln 2}{8}} \times \sqrt{r}.$$  \hspace{1cm} (5)

Another intriguing aspect of random $k$-CNF formulas is their proof complexity. In a seminal paper, Chvátal and Szemerédi [CSS88] proved that for all $k \geq 3$ and $r > 2^k \ln 2$ there exists $\varepsilon = \varepsilon(r)$ such that w.h.p. every resolution refutation of $F_k(n, rn)$ contains at least $(1 + \varepsilon)n^k$ clauses. Since then there have been a number of extensions of this result [BP96, BKPS02] and it is widely believed that random $k$-CNF are hard for many stronger proof systems than resolution. Indeed, recently, Feige [Fei02] showed that a hypothesis asserting that proving unsatisfiability of random $k$-CNF with $r \gg 2^k \ln 2$ is hard, implies a number of strong inapproximability results. A closely related hypothesis is that approximating Max $k$-SAT for such formulas is also hard for all $k \geq 2$. Recent work by Fernandez De la Vega and Karpinski [FdlVK02] proves that one can approximate Max 3-SAT on $F_3(n, rn)$ within $9/8$ which is better than the trivial $8/7$ bound.
2 Outline

2.1 Understanding correlation sources in MAX k-SAT

The following easy consequence of the Cauchy-Schwarz inequality underlies the second moment method.

Lemma 1. For any non-negative random variable $X$,

$$\Pr[X > 0] \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$  \hspace{1cm} (7)

Thus, for any fixed $p \in (0,1]$ one can let $X$ denote the number of $p$-satisfying assignments and apply (1) to bound $\Pr[X > 0]$ from below. Unfortunately, it turns out that for every $r > 0$, there exists a constant $\beta = \beta(k,r) > 1$ such that $\mathbb{E}[X^2] > \beta^n \mathbb{E}[X]^2$. As a result, this straightforward approach only gives a trivial lower bound on the probability of $p$-satisfiability.

In [AP03], it was shown that in the case $p = 1$ a major factor in the excessive correlations behind the above failure is the following form of populism: leaning toward the majority vote truth assignment. To see this, first observe that truth assignments that satisfy more literal occurrences than average, have higher probability of being satisfying. At the same time, in order to satisfy many literal occurrences such assignments tend to agree with each other (and the majority truth assignment) on more than half the variables. As a result, the successes of such assignments tend to be highly correlated, thus dominating $\mathbb{E}[X^2]$. In order to avoid this pitfall, we would like, as in [AP03], to apply the second moment method to truth assignments that satisfy, approximately, half of all literal occurrences; we call such truth assignments “balanced”. In the context of $p$-satisfiability, however, there are new obstacles to overcome before obtaining a lower bound for $r_k(p)$ that asymptotically matches the upper bound. To capture the behavior of balanced truth assignments we begin by defining two “fitness” gauges.

Given any $k$-CNF formula $F$ on $n$ variables and any truth assignment $\sigma \in \{0,1\}^n$ let

1. $H = H(\sigma,F)$ be the number of satisfied literal occurrences in $F$ under $\sigma$, minus the number of unsatisfied literal occurrences in $F$ under $\sigma$.

2. $U = U(\sigma,F)$ be the number of unsatisfied clauses in $F$ under $\sigma$.

We would like to focus on truth assignments that are balanced and $p$-satisfying, up to fluctuations one would expect from a central limit theorem, i.e., truth assignments $\sigma$ such that

$$|H(\sigma,F)| \leq A\sqrt{m},$$  \hspace{1cm} (8)

$$|U(\sigma,F)(1-p)2^{-k}m| \leq A\sqrt{m}. $$  \hspace{1cm} (9)

To do this let us write $u_0 \equiv (1-p)2^{-k}$ and fix some $\gamma, \eta < 1$. Now, for a random $k$-CNF formula $F$, consider the weighted sum $F$

$$X = X(\gamma, \eta) = \sum_{\sigma} \gamma^{H(\sigma,F)} \eta^{U(\sigma,F)-u_0m} .$$

Since $\gamma, \eta < 1$ we see that in $X$ the truth assignments $\sigma$ for which $H(\sigma,F) > 0$ or $U(\sigma,F) > u_0m$ are suppressed exponentially, whereas the rest are rewarded exponentially. Decreasing $\gamma, \eta \in [0,1)$ makes this phenomenon more and more acute, with the limiting case $\gamma, \eta = 0$ corresponding to a 0-1 weighting scheme (we adopt the convention $0^0 \equiv 1$). Indeed, applying the second moment method to $X$ with $\eta = 0$ corresponds to the approach of [AP03] for the random $k$-SAT threshold, where only satisfying assignments receive non-zero weight $\gamma^{H(\sigma,F)}$. A key step in our analysis, presented in Subsection 2.3, is the tuning of the parameters $\gamma, \eta$ to focus on truth assignments $\sigma$ for which (8) and (9) hold. Before doing that, we establish the upper bound in Theorem 1.
2.2 The upper bound in Theorem 1

This upper bound can be readily established by using the entropic-form Chernoff bound for the Binomial (see Lemma A.10 in [AS91] or Lemma 3.8 in [DM95]), but it is more informative to give a self-contained argument. Recall the definition of $T_k(\cdot)$ from [2].

**Lemma 2.** For all $k \geq 2$ and $p \in (0, 1]$, if $q = 1 - p$ then

$$r_k^*(p) \leq \frac{2^k \ln 2}{q \ln q - (2^k - q) \ln \left(\frac{2^k - 1}{2^k - q}\right)} \leq T_k(p). \tag{10}$$

**Proof.** The right hand inequality of (10) follows from the inequality $\ln t \leq t - 1$ applied to $t = \frac{2^k - 1}{2^k - q}$, so we just need to verify the left hand inequality. To do that, write $u_0 = 2^k q$. Let $\eta \in (0, 1)$, and observe that if $F$ is $p$-satisfiable, then $U(\sigma, F) \leq u_0 m$ for some $\sigma$, whence

$$X(1, \eta) = \sum_{\sigma} \eta^{U(\sigma, F) - u_0 m} \geq 1.$$

From [20] in the next section we have that

$$P[X(1, \eta) \geq 1] \leq E[X(1, \eta)] = 2^n \eta^{-q \eta^{-k} - k} \left(1 - (1 - \eta)^{-2^{-k}}\right)^{r^n}. \tag{11}$$

Thus, the probability of $p$-satisfiability decays exponentially in $n$ if the the $n$-th root of the RHS of (11) is strictly smaller than 1. Taking $\eta = q(2^{-k})/(2^{-k} - q)$ yields the lemma. \qed

2.3 Tuning parameters and truncation

When $\eta > 0$, attempting to apply the second moment method to $X$ we encounter two major problems.

The first problem is that while $X > 0$ implies satisfiability when $\eta = 0$, when $\eta > 0$ having $X > 0$ does not imply $p_0$-satisfiability: in principle, $X$ could be positive due to the contribution of assignments falsifying many more clauses than $u_0 m$. This necessitates restricting the sum defining $X$ to truth assignments falsifying at most $u_0 m + O(\sqrt{m})$ clauses, i.e. truncating $X$.

The second, more severe, problem is that with or without this truncation, $E[X]^2/E[X^2]$ becomes exponentially small when $r$ is only, roughly, half the (asymptotically optimal) lower bound of Theorem 1. Rather counterintuitively, we will be able to delay this explosion until $r$ is within $1 - o(1)$ of the upper bound by also removing from the sum those “hercic” truth assignments falsifying fewer than $u_0 m$ clauses. This affords us much tighter control of pairs of assignments that agree on nearly all variables, which turn out to be the dominant contributors to $E[X^2]$ as we approach the upper bound. The idea behind this sacrifice is motivated by Cramer’s classical “change of measure” technique in large deviation theory. The corresponding “adaptive weighting” scheme requires an extremely sharp asymptotic analysis, involving a number of rather miraculous cancellations. Due to space limitations this analysis appears entirely in the Appendix.

Specifically, for some fixed $A > 0$ let

$$\mathcal{S}^* = \{\sigma \in \{0, 1\}^n : H(\sigma, F) \geq 0 \text{ and } U(\sigma, F) \in [u_0 m, u_0 m + A\sqrt{m}]\}.$$ 

Moreover, given $u_0$, let $\gamma_0, \eta_0$ be defined by

$$1 - \eta_0 = (1 - \gamma_0^2) (1 + \gamma_0^2)^{k-1} \tag{12}$$

$$u_0 = \frac{\eta_0}{(1 + \gamma_0^2)^k - (1 - \eta_0)}.$$ 

These two equations are designed so that the main contribution in the sum defining $X$ comes from truth assignments for which [8] and [9] holds. The connection is made in equations [24] and [26] in Section 3.
We define
\[ X_* = \sum_{s \in S_*} H(s,F) U(s,F)^{u_0 m} . \]
Note that, by definition, when \( X_* > 0 \) at least one truth assignment must falsify at most \( u_0 m + A \sqrt{m} \) clauses. Thus, if for a given \( p_0 \) we can prove that there exists a constant \( D > 0 \) such that \( E[X_*^2] < D \times E[X_*]^2 \) then, by Corollary \( \ref{corollary} \) it follows that \( F_k(n, rn) \) is w.h.p. \( p \)-satisfiable for all \( p < p_0 \).

Bounding the second moment of \( X_* \), will be accomplished in the following lemmata. For \( \alpha \in [0,1] \), let
\[
f(\alpha, \gamma, \eta) = \eta^{-2u_0} \left[ \alpha \left( \frac{\gamma^2 + \gamma^{-2}}{2} + 1 - \alpha \right)^k - 2(1 - \eta) \left( \frac{\alpha \gamma^{-2} + (1 - \alpha)}{2}\right)^k + (1 - \eta^2) \left( \frac{\alpha \gamma^{-2}}{2} \right)^k \right] ,
\]
and
\[
g_r(\alpha, \gamma, \eta) = \frac{f(\alpha, \gamma, \eta)^r}{\alpha^\alpha(1 - \alpha)^{1 - \alpha}} .
\]
In all of the following lemmata \( k \geq 2 \) is a fixed integer and \( r > 0 \).

- Lemma \( \ref{lemma3} \) with \( \gamma = \gamma_0 \) and \( \eta = \eta_0 \) gives us \( E[X(\gamma_0, \eta_0)]^2 \) which is \( E[X_*]^2 \) but for the truncation.
- Lemma \( \ref{lemma4} \) asserts that for every value of \( u_0 \), \( E[X_*] \) is a constant fraction of \( E[X(\gamma_0, \eta_0)] \). Thus, combined with Lemma \( \ref{lemma3} \) it gives us \( E[X_*]^2 \) up to a constant factor (which is all we need).
- Lemma \( \ref{lemma5} \) expresses \( E[X_*^2] \) as a sum with \( n + 1 \) terms, the \( z \)-th term capturing the contribution of the \( 2^n \binom{n}{z} \) pairs of truth assignments with overlap \( z \). The contribution of each such pair is then bounded by \( f(z/n, \gamma, \eta)^r \) where \( \gamma, \eta \) are allowed to depend on \( z \), subject only to \( \gamma \geq \gamma_0 \) and \( \eta \geq \eta_0 \) respectively. In other words, Lemma \( \ref{lemma5} \) allows us to adapt \( \gamma \) and \( \eta \) to \( \alpha \), which is crucial when \( p < 1 \).
- Lemma \( \ref{lemma6} \) is based on the fact that for any “smooth” choice of sequences \( \gamma(z), \eta(z) \), the sum in Lemma \( \ref{lemma5} \) will be dominated by the contribution of the \( \Theta(n^{1/2}) \) terms around the maximum term. Specifically, if \( \chi, \omega \) express our adaptive scheme for \( \gamma, \eta \), then we can use the Laplace method to get that the maximum of \( g_r(\alpha, \chi(\alpha), \omega(\alpha)) \) over \( \alpha \in (0,1) \), characterizes the sum in Lemma \( \ref{lemma5} \) up to a constant factor.

**Lemma 3.** For every \( u_0, \gamma, \eta \in [0,1) \),
\[
E[X]^2 = \left( 2 g_r(1/2, \gamma, \eta) \right)^n .
\]

**Lemma 4.** For every \( u_0 \), there exists \( \theta = \theta(k, A) > 0 \) such that as \( n \to \infty \),
\[
\frac{E[X_*]}{E[X(\gamma_0, \eta_0)]} \to \theta .
\]

**Lemma 5.** Let \( \gamma(z), \eta(z) \) be arbitrary sequences such that \( \gamma(z) \geq \gamma_0 \) and \( \eta(z) \geq \eta(0) \) for every \( 0 \leq z \leq n \). Then, for every \( u_0 \),
\[
E[X_*^2] \leq 2^n \sum_{z=0}^{n} \binom{n}{z} f(z/n, \gamma(z), \eta(z))^r .
\]

**Lemma 6.** Let \( \chi : [0,1] \to [\gamma_0,1) \) and \( \omega : [0,1] \to [\eta_0,1) \) be arbitrary piecewise-smooth functions and let \( g_r(\alpha) = g_r(\alpha, \chi(\alpha), \omega(\alpha)) \). If there exists \( \alpha_{\text{max}} \in (0,1) \) such that \( g_r(\alpha_{\text{max}}) \equiv g_{\text{max}} > g_r(\alpha) \) for all \( \alpha \neq \alpha_{\text{max}} \), and \( g''_r(\alpha_{\text{max}}) < 0 \), then there exists a constant \( D = D_{\chi, \omega}(k, r, u_0) > 0 \) such that for all sufficiently large \( n \),
\[
E[X_*^2] < D \times \left( 2 g_{\text{max}} \right)^n .
\]
Combining Lemmata 9, 10 we see that if for a given $u_0$ and $r$ there exist $\chi, \omega$ such that for all $\alpha \neq 1/2$

$$g_r \left( 1/2, \gamma_0, \eta_0 \right) > g_r \left( \alpha, \chi(\alpha), \omega(\alpha) \right)$$

then $\mathbb{E}[X_r^2] < D\theta^{-2} \times \mathbb{E}[X_r]^2$, yielding the desired conclusion $\mathbb{E}[X_r^2] = O(\mathbb{E}[X_r]^2)$.

Indeed, to prove Theorem 1 we will show that for every $p \in (0,1)$ and for the stated $r = r(p)$, there exist functions $\chi, \omega$ for which (10) holds. To simplify the asymptotic analysis, we use the crudest possible such functions, paying the price of this simplicity in the value of $k_0$ in Proposition 7 below. We note that by choosing a more refined (and more cumbersome) adaptation of $\gamma, \eta$ to $\alpha$ this value can be improved greatly. Moreover, we emphasize that for any fixed value of $k$, one can get a sharper lower bound (such as those reported in the Introduction) by partitioning $[0,1]$ to a large number of intervals and numerically finding a good value of $\gamma, \eta$ for each one. We discuss this point further in Section 3. Finally, we note that general large deviations considerations imply that for every $k$ and $p$, the condition (10) is sharp for our method. That is, no better lower bound can be derived by considering balanced assignments and, in fact, by any argument that classifies assignments according to their number of satisfied literal occurrences in the formula.

Definition 1. Let $q = 1 - p_0 = u_0 2^k$ and let

$$t_k = \frac{2^k \ln 2}{1 - q + q \ln q} \left( 1 - 20k 2^{-k \varphi(q)} \right)$$

where $\varphi(q) = \frac{(1 - \sqrt{q})^2}{1 - q + q \ln q}$. (16)

Theorem 1 will follow from the following Proposition.

Proposition 7. Let

$$G_r(\alpha) = \begin{cases} 
    g_r(\alpha, \gamma_0, \eta_0) & \text{if } \alpha \in [\frac{3 \ln k}{k}, 1 - \frac{3 \ln k}{k}] \\
    g_r(\alpha, \sqrt{\gamma_0}, \sqrt{\eta_0}) & \text{otherwise.}
\end{cases}$$

(17)

For all $k \geq k_0$, if $r \leq t_k$ then $G_r'(1/2) < 0$ and $G_r(1/2) > G_r(\alpha)$ for all $\alpha \neq 1/2$.

The proof of Proposition 1 itself, will be decomposed into three lemmata of increasing difficulty. The first lemma holds for any $\gamma, \eta$ and reduces the proof to the case $\alpha \geq 1/2$. The second lemma reflects the behavior of $f$ (and thus $g_r$) around $\alpha = 1/2$, motivating the judicious choice $\eta = \eta_0$ and $\gamma = \gamma_0$ for $G_r$. The third lemma deals with $\alpha$ near 1. That case needs a lot more work in order to handle the unique local maximum of $g_r$ in that region. The condition $r \leq t_k$ and the change to $\gamma = \sqrt{\gamma_0}, \eta = \sqrt{\eta_0}$ aims precisely at keeping the value of $g_r$ at this other local maximum smaller than $g_r(1/2, \gamma_0, \eta_0)$.

Lemma 8. For every $0 < x \leq \frac{1}{2}$, $G_r(1/2 + x) > G_r(1/2 - x)$.

Lemma 9. For all $k \geq k_0$, if $r \leq \frac{2^k \ln 2}{1 - q + q \ln q}$ then $G_r'(1/2) < 0$ and $G_r$ is strictly decreasing on $[\frac{1}{2}, 1 - \frac{3 \ln k}{k}]$.

Lemma 10. For all $k \geq k_0$, if $r \leq t_k$ then for every $\alpha \in [1 - \frac{3 \ln k}{k}, 1]$, $G_r(1/2) > G_r(\alpha)$.

In the following sections we prove Lemmata 3-6 while Lemmata 8-10 are proven in the appendix. Before delving into the probabilistic calculations involved in proving Lemmata 3-6 a couple of remarks are in order.

Relationship to other $k$-CNF models: Recall that the $m$ clauses of $F_k(n,m)$ are chosen independently with replacement among the $(2m)^k$ possibilities. Thus, the $m$ clauses $\{c_i\}_{i=1}^m$ are i.i.d. random variables, each $c_i$ being the conjunction of $k$ i.i.d. random variables $\{\ell_{ij}\}_{j=1}^k$, each $\ell_{ij}$ being a uniformly random literal. This viewpoint of the formula as a sequence of $km$ i.i.d. random literals will be very handy for our calculations.

Clearly, in this model some clauses might be improper, i.e. they might contain repeated and/or contradictory literals. At the same time, though, observe that the probability that any given clause is improper is
smaller than $k^2/n$ and, moreover, the proper clauses are uniformly selected among all such clauses. Therefore w.h.p. the number of improper clauses is $o(n)$ implying that if for a given $r$, $F_k(n, rn)$ is $p$-satisfiable w.h.p. then for $m = rn - o(n)$, the same is true in the model where we only select among proper clauses. The issue of selecting clauses without replacement is completely analogous as w.h.p. there are $o(n)$ clauses that contain the same $k$ variables as some other clause.

**Notation:** In the ensuing probabilistic calculations it will be convenient to write $\sigma \not\models F$ to denote that the truth assignment $\sigma$ violates the formula $F$ where $F$ can be a literal, a clause, or an entire CNF.

### 3 The first moment and proof of Lemma 3

By linearity of expectation and since the $m = rn$ clauses $c_1, c_2, \ldots, c_m$ are chosen independently we have

$$
\eta^{u_0} \mathbb{E}[X] = \sum_{\sigma} \gamma^{H(\sigma, F)} \eta^{U(\sigma, F)} = \sum_{\sigma} \prod_{c_i} \gamma^{H(\sigma, c_i)} \eta^{U(\sigma, c_i)} = \sum_{\sigma} \prod_{c_i} \mathbb{E} \left[ \gamma^{H(\sigma, c_i)} \eta^{U(\sigma, c_i)} \right].
$$

(18)

Observe now that since the clauses are identically distributed, by symmetry, it suffices to consider the expectation in (18) for a single random clause $c = \ell_1 \lor \cdots \lor \ell_k$ and a fixed truth assignment $\sigma$. Moreover, observe that if we write $\gamma^{H(\eta, U)}$ as $\gamma^{H} + \gamma^{H(\eta, U) - 1}$ we see that the second expression is non-zero only when $U > 0$, i.e. when $c$ is violated by $\sigma$. So, since the literals $\ell_1, \ldots, \ell_k$ are i.i.d. we get

$$
\mathbb{E} \left[ \gamma^{H(\sigma, c)} \eta^{U(\sigma, c)} \right] = \mathbb{E} \left[ \gamma^{H(\sigma, c)} - \gamma^{H(\sigma, c)} \left( 1 - \eta^{U(\sigma, c)} \right) \right]
= \mathbb{E} \left[ \gamma^{H(\sigma, c)} \right] - \mathbb{E} \left[ \gamma^{H} \eta^{U(\sigma, c)} \right] - \mathbb{E} \left[ \gamma^{H} \eta^{U(\sigma, c)} \right] 1_{\sigma \not\models c}
= \mathbb{E} \left[ \prod_{\ell_i} \gamma^{H(\sigma, \ell_i)} \right] - 2^{-k} \gamma^{-k}(1 - \eta)
= \prod_{\ell_i} \mathbb{E} \left[ \gamma^{H(\sigma, \ell_i)} \right] - 2^{-k} \gamma^{-k}(1 - \eta)
= \left( \frac{\gamma + \gamma^{-1}}{2} \right)^k - (2\gamma)^{-k}(1 - \eta)
= Z(\gamma, \eta).
$$

(19)

Thus,

$$
\mathbb{E}[X] = \eta^{-u_0} r^n Z(\gamma, \eta)^r n.
$$

(20)

Observe now that

$$
(\eta^{-u_0} Z(\gamma, \eta))^2 = f(1/2, \gamma, \eta).
$$

Therefore,

$$
\mathbb{E}[X]^2 = (\eta^{-u_0} r^n Z(\gamma, \eta)^r n)^2 = \left[ (\eta^{-u_0} 2 Z(\gamma, \eta)^r)^n \right]^2 = [4 f(1/2, \gamma, \eta)^r]^n = [2 g_r(1/2, \gamma, \eta)]^n.
$$
4 Proof of Lemma 4

By linearity of expectation, it suffices to prove that there exists some $\theta = \theta(k, A) > 0$ such that for the values of $\gamma_0, \eta_0$ satisfying \[12\] and every truth assignment $\sigma$, we have
\[
\frac{\mathbb{E} \left[ \gamma_0^H(\sigma, F) U(\sigma, F) \mathbf{1}_{\sigma \in S^*(F)} \right]}{\mathbb{E} \left[ \gamma_0^H(\sigma, F) U(\sigma, F) \right]} \to \theta .
\]

Recalling that formulas in our model are sequences of i.i.d. random literals $\ell_1, \ldots, \ell_{km}$, let $P(\cdot)$ denote the probability assigned by our distribution to any such sequence, i.e. $(2n)^{-km}$. Now, fix any truth assignment $\sigma$ and consider an auxiliary distribution $P_\sigma$ on $k$-CNF formulas where the $m$ clauses $c_1, \ldots, c_m$ are again i.i.d. among all $(2n)^k$ clauses, but where now for any fixed clause $\omega$
\[
P_\sigma(c_i = \omega) = \frac{\gamma_0^H(\sigma, \omega) U(\sigma, \omega)}{Z(\gamma_0, \eta_0)} P(\omega),
\]
where
\[
Z(\gamma_0, \eta_0) = \mathbb{E} \left[ \gamma_0^H(\sigma, c) U(\sigma, c) \right],
\]
was defined in \[20\]. (Since each fixed clause $\omega$ receives probability proportional to $\gamma_0^H(\sigma, \omega) U(\sigma, \omega)$, indeed $Z(\gamma_0, \eta_0)$ provides the correct normalization to a probability distribution.) So, whereas under $P(\cdot)$ every $k$-CNF formula $F$ with $m$ clauses had the same probability $P(F) = (2n)^{-km}$, under $P_\sigma$ its probability is
\[
P_\sigma(F) = \frac{\gamma_0^H(\sigma, F) U(\sigma, F) P(F)}{Z(\gamma_0, \eta_0)^m}.
\]

Let $E_\sigma$ be the expectation operator corresponding to $P_\sigma$. A calculation similar to that leading to \[21\], adding the equal contributions from the $k$ literals, gives that for a single random clause $c$
\[
Z(\gamma_0, \eta_0) E_\sigma[H(\sigma, c)] = k(\gamma_0 - \gamma_0^{-1}) \left( \frac{\gamma_0 + \gamma_0^{-1}}{2} \right)^{k-1} + k(2\gamma_0)^{-k}(1 - \eta_0).
\]
Moreover,
\[
Z(\gamma_0, \eta_0) E_\sigma[U(\sigma, c)] = (2\gamma_0)^k \eta_0.
\]
Thus \[12\] ensures that $E_\sigma[H(\sigma, c)] = 0$ and also that $E_\sigma[U(\sigma, c) - u_0] = 0$.

Next, we apply the multivariate central limit theorem (see, e.g. \[Pol02\], page 182) to the i.i.d. mean-zero random vectors $(H(\sigma, c_i), U(\sigma, c_i) - u_0)$ for $i = 1, \ldots, m$. Observe that, since $k \geq 2$, the common law of these random vectors is not supported on a line. We deduce that as $n \to \infty$
\[
P_\sigma[\sigma \in S^*(F)] = P_\sigma \left[ H(\sigma, F) \geq 0 \text{ and } U(\sigma, F) \in [m u_0, m u_0 + A \sqrt{m}] \right] \to \theta(k, A) > 0 .
\]
Here, the right hand side is the probability that a certain nondegenerate bivariate normal law assigns to a certain open set. Its exact value is unimportant for our purpose. By \[24\], this is equivalent to \[24\].

5 Proof of Lemma 5

Linearity of expectation implies
\[
\eta_0^2 u_0 m \mathbb{E}[X^2] = \mathbb{E} \left[ \left( \sum_{\sigma} \gamma_0^H(\sigma, F) U(\sigma, F) \mathbf{1}_{\sigma \in S^*(F)} \right)^2 \right] = \sum_{\sigma, \tau} \mathbb{E} \left[ \gamma_0^H(\sigma, F) + H(\tau, F) U(\sigma, F) + U(\tau, F) \mathbf{1}_{\sigma, \tau \in S^*(F)} \right].
\]

10
Observe now that since $\sigma \in S^*$ implies $H(\sigma, F) \geq 0$ and $U(\sigma, F) \geq u_0 m$, we get that for every pair $\sigma, \tau$ and any $\gamma \geq \gamma_0$ and $\eta \geq \eta_0$,
\[
\mathbb{E} \left[ \gamma^0 (H(\sigma, F) + H(\tau, F)) \eta \left( U(\sigma, F) + U(\tau, F) \right) 1_{\sigma, \tau \in S^*(F)} \right] \leq \mathbb{E} \left[ \gamma^0 (H(\sigma, F) + H(\tau, F)) \eta \left( U(\sigma, F) + U(\tau, F) \right) 1_{\sigma, \tau \in S^*(F)} \right] \leq \mathbb{E} \left[ \gamma^0 (H(\sigma, F) + H(\tau, F)) \eta U(\sigma, F) + U(\tau, F) \right].
\]
(28)

In other words, when using the right hand side of (28) to bound each term of the sum in (29), we are allowed to adapt the value of $\gamma$ and $\eta$ to the pair $\sigma, \tau$, the only restrictions being $\gamma \geq \gamma_0$ and $\eta \geq \eta_0$. This is a crucial point and we will exploit it heavily when bounding the contribution of pairs with large overlap.

To estimate the right hand side of (29) for any pair $\sigma, \tau$ we first observe that since the $m$ clauses $c_1, c_2, \ldots, c_m$ are i.i.d., letting $c$ be a single random clause we have
\[
\mathbb{E} \left[ \gamma (H(\sigma, c) + H(\tau, c)) \eta U(\sigma, c) + U(\tau, c) \right] = \prod_{c_i} \mathbb{E} \left[ \gamma (H(\sigma, c) + H(\tau, c)) \eta U(\sigma, c) + U(\tau, c) \right] = \left( \mathbb{E} \left[ \gamma (H(\sigma, c) + H(\tau, c)) \eta U(\sigma, c) + U(\tau, c) \right] \right)^m.
\]
(29)

Next, we observe that for every pair $\sigma, \tau$, by symmetry, the expectation in (29) depends only on the number of variables to which $\sigma, \tau$ assign the same value. So, let $\sigma, \tau$ be any pair of truth assignments that agree on exactly $z = \alpha n$ variables, i.e. have overlap $z$. By first rewriting (again) $\gamma^0 (H(\sigma, c) + H(\tau, c)) \eta U(\sigma, c) + U(\tau, c)$ and then observing that $U(\sigma, c)$ is distributed identically with $U(\sigma, c)$ we get
\[
\mathbb{E} \left[ \gamma (H(\sigma, c) + H(\tau, c)) \eta U(\sigma, c) + U(\tau, c) \right] = \mathbb{E} \left[ \gamma (H(\sigma, c) + H(\tau, c)) \left( 1 - \eta U(\sigma, c) \right) \left( 1 - \eta U(\tau, c) \right) \right]
\]
\[
= \mathbb{E} \gamma (H(\sigma, c) + H(\tau, c)) - 2 \mathbb{E} \gamma (H(\sigma, c) + H(\tau, c)) \left( 1 - \eta U(\sigma, c) \right) + \mathbb{E} \gamma (H(\sigma, c) + H(\tau, c)) \left( 1 - \eta U(\tau, c) \right)
\]
\[
= \mathbb{E} \gamma (H(\sigma, c) + H(\tau, c)) - 2(1 - \eta) \mathbb{E} \gamma (H(\sigma, c) + H(\tau, c)) 1_{\sigma \neq \tau} + 2^{-k} \alpha^k \gamma^2 (1 - \eta)^2.
\]
(30)

Now, to estimate (30) we note that since the literals $\ell_1, \ell_2, \ldots, \ell_k$ comprising $c$ are i.i.d. we have
\[
\mathbb{E} \gamma (H(\sigma, c) + H(\tau, c)) = \mathbb{E} \prod_i \gamma (H(\sigma, \ell_i) + H(\tau, \ell_i)) = \prod_i \mathbb{E} \gamma (H(\sigma, \ell_i) + H(\tau, \ell_i)) = \left( \alpha^2 + \gamma^2 \right) / 2 + 1 - \alpha
\]
and, similarly,
\[
\mathbb{E} \gamma (H(\sigma, c) + H(\tau, c)) 1_{\sigma \neq \tau} = \mathbb{E} \prod_i \gamma (H(\sigma, \ell_i) + H(\tau, \ell_i)) 1_{\sigma \neq \ell_i} = \prod_i \mathbb{E} \gamma (H(\sigma, \ell_i) + H(\tau, \ell_i)) 1_{\sigma \neq \ell_i} = \left( \alpha^2 + (1 - \alpha) \right) / 2
\]

Substituting these last two equations in (30) we get
\[
\eta^2 u_0 \mathbb{E} \gamma (H(\sigma, c) + H(\sigma, c)) \eta (H(\tau, c) + H(\tau, c))
\]
\[
= \eta^2 u_0 \left( \alpha^2 + \gamma^2 \right) / 2 - 2(1 - \eta) \left( \alpha^2 + (1 - \alpha) \right) / 2 + (1 - \eta)^2 \left( \alpha^2 / 2 \right)
\]
\[
= f(\alpha, \gamma, \eta).
\]
(31)
So, in conclusion, since the number of ordered pairs with overlap \( z \) is \( 2^n \binom{n}{z} \) we get that

\[
E[X^2] \leq 2^n \sum_{z=0}^{n} \binom{n}{z} f(z/n, \gamma(z), \eta(z))^n,
\]

for any set of choices for \( \gamma(z), \eta(z) \) such that \( \gamma(z) \geq \gamma_0 \) and \( \eta(z) \geq \eta_0 \) for all \( 0 \leq z \leq n \).

### 5.1 Proof of Lemma 6

If \( \chi : [0, 1] \to [\gamma_0, 1] \) and \( \omega : [0, 1] \to [\gamma_0, 1] \) are piecewise smooth, then from the definition of \( f \) we see that \( f(\alpha, \chi(\alpha), \omega(\alpha)) \) is also piecewise smooth. Thus, we can decompose the sum in Equation 32 into a fixed number of sums such that \( f(\alpha, \chi(\alpha), \omega(\alpha)) \) is smooth in the range of each sum. To bound each such sum, then, we use the following lemma whose proof is implied by the proof of Lemma 2 in [AM02] (that lemma is stated with the requirement that \( f \) is analytic, a condition not needed for the proof; in fact, it suffices for \( f \) to only be twice differentiable.) The idea is that each of these sums is dominated by the contribution of \( \Theta(n^{1/2}) \) terms around the maximum term. Since the number of sums is finite the lemma follows.

**Lemma 11.** Let \( \phi \) be any real, positive, twice-differentiable function on \([0, 1] \) and let

\[
S_n = \sum_{z=0}^{n} \binom{n}{z} \phi(z/n)^n.
\]

Letting \( 0^0 \equiv 1 \), define \( g \) on \([0, 1] \) as

\[
g(\alpha) = \frac{\phi(\alpha)}{\alpha^\alpha (1 - \alpha)^{1-\alpha}}.
\]

If there exists \( \alpha_{\text{max}} \in (0, 1) \) such that \( g(\alpha_{\text{max}}) \equiv g_{\text{max}} > g(\alpha) \) for all \( \alpha \neq \alpha_{\text{max}} \), and \( g''(\alpha_{\text{max}}) < 0 \), then there exist constants \( B, C > 0 \) such that for all sufficiently large \( n \)

\[
B \times g_{\text{max}}^n \leq S_n \leq C \times g_{\text{max}}^n.
\]

### 6 Bounds for finite \( k \)

As mentioned in Section 2 for small values of \( k \) the simple adaptation scheme of Proposition 7 does not yield the best possible lower bound for \( p \)-satisfiability afforded by our method. For that, one has to use a significantly more refined adaptation of \( \gamma, \eta \) with respect to \( \alpha \). Our lower bounds reported in Figure 1 are, indeed, the result of performing such optimization of \( \gamma, \eta \) numerically (for both the upper bound plots and the plots of the lower bound from [CGHS03] we used the explicit formulas).

Specifically, to create the plots of the lower bounds we computed a lower bound for 100 equally spaced values of \( p \) on the horizontal axis (and then had Maple’s Red94 plotting function “connect the dots”). For each of these values of \( p \), to prove the corresponding lower bound for \( r \) we had to establish that there exist a choice of functions \( \chi, \omega \) as in Lemma 6 such that for all \( \alpha \in (1/2, 1] \) we have \( g_r(1/2, \gamma_0, \eta_0) > g_r(\alpha, \chi(\alpha), \omega(\alpha)) \). To that end, we partitioned \((1/2, 1]\) to 10,000 points and for each such point we searched for values of \( \gamma \geq \gamma_0 \) and \( \eta \geq \eta_0 \) such that this condition holds with a bit of room. (For \( k > 4 \) we solved 12, defining \( \gamma_0 \) and \( \eta_0 \) numerically to 10 digits of accuracy. For the optimization we exploited convexity to speed up the search.) Having determined such values, we (implicitly) extended the functions \( \chi, \omega \) to all \((1/2, 1]\) by assigning to every not-chosen point the value at the nearest chosen point. Finally, we computed a (crude) upper bound on the derivative of \( g_r \) with respect to \( \alpha \) in \((1/2, 1]\). This bound on the derivative, along with our room factor, then implied that for every point that we did not check, the value of \( g_r \) was sufficiently close to its value at the corresponding chosen point to also be dominated by \( g_r(1/2, \gamma_0, \eta_0) \).
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References

[AM02] Dimitris Achlioptas and Cristopher Moore, The asymptotic order of the random $k$-SAT threshold, 43th Annual Symposium on Foundations of Computer Science (Vancouver, BC, 2002), IEEE Comput. Soc. Press, Los Alamitos, CA, 2002, pp. 779–788.

[AP03] Dimitris Achlioptas and Yuval Peres, The random $k$-SAT threshold is $2^k \ln 2 - O(k)$, 35th Annual ACM Symposium on Theory of Computing (San Diego, CA), 2003, to appear.

[AS91] Noga Alon and Joel H. Spencer, The Probabilistic Method, Wiley 1991.

[BFU93] Andrei Z. Broder, Alan M. Frieze, and Eli Upfal, On the satisfiability and maximum satisfiability of random 3-CNF formulas, Proc. 4th Annual ACM-SIAM Symposium on Discrete Algorithms, 1993, pp. 322–330.

[BKPS02] Paul Beame, Richard Karp, Toniann Pitassi, and Michael Saks, The efficiency of resolution and davis-putnam procedures, SIAM J. Comput. 31 (2002), no. 4, 1048–1075.

[BP96] Paul W. Beame and Toniann Pitassi, Simplified and improved resolution lower bounds, Proceedings 37th Annual Symposium on Foundations of Computer Science (Burlington, VT), IEEE, October 1996, pp. 274–282.

[CGHS03] Don Coppersmith, David Gamarnik, Mohammad T. Hajiaghayi, and Gregory B. Sorkin, Random MAX 2-SAT and MAX CUT, 14th Annual ACM-SIAM Symposium on Discrete Algorithms (Baltimore, MD, 2003), ACM, New York, 2003.

[Coo71] Stephen A. Cook, The complexity of theorem-proving procedures, 3rd Annual ACM Symposium on Theory of Computing (Shaker Heights, OH, 1971), ACM, New York, 1971, pp. 151–158.

[CS88] Vašek Chvátal and Endre Szemerédi, Many hard examples for resolution, J. Assoc. Comput. Mach. 35 (1988), no. 4, 759–768.

[DM95] Paul Deheuvels and David M. Mason On the Fractal Nature of Empirical Increments, The Annals of Probability 23 (1995), 355–387.

[DB97] Olivier Dubois and Yacine Boufkhad, A general upper bound for the satisfiability threshold of random $r$-SAT formulae, J. Algorithms 24 (1997), no. 2, 395–420.

[FdlVK02] W. Fernandez de la Vega and Marek Karpinski, 9/8-approximation algorithm for random max-3sat, Technical Report TR02-070, Electronic Colloquium on Computational Complexity (2002).

[Fei02] Uriel Feige, Relations between average case complexity and approximation complexity, 34th Annual ACM Symposium on Theory of Computing (Montreal, QC), 2002, pp. 534 – 543.

[GJ79] Michael R. Garey and David S. Johnson, Computers and intractability, Freeman, San Francisco, CA, 1979.

[GSCK00] C. P. Gomes, B. Selman, N. Crato, and H. Kautz, Heavy-tailed phenomena in satisfiability and constraint satisfaction problems, J. Automat. Reason. 24 (2000), no. 1-2, 67–100. MR 2000k:68070

[Pol02] David Pollard, A user’s guide to measure theoretic probability, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge, 2002. MR 2002k:60003
[Red94] Darren Redfern, *The Maple Handbook: Maple V Release 3*, third ed., Springer Verlag, New York, 1994.

[SK93] Bart Selman and Henry Kautz, *Domain-independent extensions to GSAT: Solving large structured satisfiability problems*, Proc. 13th International Joint Conference on Artificial Intelligence, 1993, pp. 290–295.

[SLM92] Bart Selman, Hector Levesque, and D. Mitchell, *A new method for solving hard satisfiability problems*, Proc. 10th National Conference on Artificial Intelligence, 1992, pp. 440–446.
A Building up an arsenal

In this section we collect some basic inequalities and identities that we will use in the proofs of Lemmas 3 and 10. For readability, in this Appendix we have replaced \( q \) of Definition 1 with the letter \( y \).

Fact 1. For all sufficiently large \( k \),

\[
\frac{2(1 - y)}{2^k - k - 1} - \frac{4k(1 - y)^2}{2^{2k}} \leq \varepsilon_0 \leq \frac{2(1 - y)}{2^k - k - 1},
\]

and

\[
\eta_0 \leq \min \left\{ \frac{(k + 1)(1 - y)}{2^k - k - 1} + \frac{4k(1 - y)^2}{2^k} \right\}.
\]
A.1 Proof of Fact 1

By the first equation in (12) we have that \( \eta_0 = 1 - \varepsilon_0(2 - \varepsilon_0)^{k-1} \). Plugging this into the second equation in (12) we find that

\[
\frac{y}{2^k} = \frac{1 - \varepsilon_0(2 - \varepsilon_0)^{k-1}}{(2 - \varepsilon_0)^k - \varepsilon_0(2 - \varepsilon_0)^{k-1}} = \frac{1 - \varepsilon_0(2 - \varepsilon_0)^{k-1}}{2(1 - \varepsilon_0)(2 - \varepsilon_0)^{k-1}} = \frac{1}{2} \left[ 1 - \sum_{j=1}^{k-1} \frac{1}{(2 - \varepsilon_0)^j} \right].
\]

(39)

Hence, if we denote

\[
\psi(t) = \sum_{j=1}^{k-1} \frac{1}{(2 - t)^j} = \frac{(2 - t)^{k-1} - 1}{(1 - t)(2 - t)^{k-1}},
\]

then we require that \( \psi(\varepsilon_0) = 1 - \frac{y}{2^k - 1} \). We collect below some useful properties of \( \psi \).

Lemma 12. \( \psi \) is increasing on \([0, 1]\). Furthermore, for \( k \) large enough and every \( t \leq \frac{1}{2k} \),

\[
1 - \frac{1}{2k-1} + t - \frac{(k+1)t}{2k} + \frac{t^2}{2} \leq \psi(t) \leq 1 - \frac{1}{2k-1} + t - \frac{(k+1)t}{2k} + 2t^2 .
\]

(41)

Proof. The fact that \( \psi \) is increasing follows immediately from the first formula in (10). To prove the inequalities in (41), observe that

\[
\psi(t) = \frac{(1 - \frac{t}{2})^{k-1} - \frac{1}{2^{k-1}}}{(1 - t)(1 - \frac{t}{2})^{k-1}} = \frac{1}{1 - t} - \frac{1}{2^{k-1}(1 - (1 - \frac{t}{2})^{k-1})}.
\]

To estimate \( \psi \) from below, we use the inequalities \( 1 + a + a^2 \leq 1/(1 - a) \leq 1 + a + 2a^2 \) and \((1 - a)^{k-1} \geq 1 - (k-1)a\), valid for all \( 0 \leq a \leq 1/2 \), to show that whenever \( a \leq 1/(2k) \)

\[
\psi(t) \geq 1 + t + t^2 - \frac{1}{2^{k-1}(1 - t)} \left( 1 - \left( \frac{k-1}{2} \right)^t \right) \\
\geq 1 + t^2 - \frac{1}{2^{k-1}} \left( 1 + t + 2t^2 \right) \left( 1 + \frac{(k-1)t}{2} + 2 \left( \frac{k-1}{2} \right)^{2t^2} \right) \\\n\geq 1 - \frac{1}{2^{k-1}} + t - \frac{(k+1)t}{2k} + \frac{t^2}{2},
\]

for all \( k \) sufficiently large.

The reverse inequality is just as simple

\[
\psi(t) \leq 1 + t + 2t^2 - \frac{1}{2^{k-1}(1 + t)} \left( 1 + \left( \frac{t}{2} \right)^{k-1} \right) \\
\leq 1 + t + 2t^2 - \frac{1}{2^{k-1}} \left( 1 + \left( \frac{k-1}{2} \right)^t \right) \\\n\leq 1 - \frac{1}{2^{k-1}} + t - \frac{(k+1)t}{2k} + 2t^2 .
\]

We are now in position to conclude the proof of Fact 1. Since \( \psi \) is increasing and \( \psi(\varepsilon_0) = 1 - \frac{y}{2^k - 1} \), the inequalities in (37) will be proved once we show that

\[
\psi \left( \frac{2(1 - y)}{2^k - k - 1} - \frac{16(1 - y)^2}{2^{2k}} \right) \leq 1 - \frac{y}{2^{k-1}} \leq \psi \left( \frac{2(1 - y)}{2^k - k - 1} \right).
\]

(42)
To prove the right-hand inequality in (42), set \( t = \frac{2(1-y)}{2^k - k - 1} \) and observe that for \( k \) large enough, \( t \leq 1/(2k) \). Hence, by Lemma 12
\[
\psi(t) \geq 1 - \frac{1}{2^{k-1}} + t - \frac{(k+1)t}{2^k} = 1 - \frac{1}{2^{k-1}} + \frac{2(1-y)}{2^k - k - 1} \cdot \frac{(k+1)}{2^k} = 1 - \frac{y}{2^{k-1}}. 
\]

The left-hand inequality in (42) is equally simple. In this case we apply Lemma 12 with \( t = \frac{2(1-y)}{2^k - k - 1} \) and get that
\[
\psi(t) \leq 1 - \frac{1}{2^{k-1}} + t - \frac{(k+1)t}{2^k} + 2t^2 
\leq 1 - \frac{1}{2^{k-1}} + \frac{2(1-y)}{2^k - k - 1} - \frac{(k+1)}{2^k} - \frac{2(1-y)}{2^k - k - 1} - \frac{16(1-y)^2}{2^{2k}} + 2 \cdot \frac{4(1-y)^2}{(2^k - k - 1)^2} 
\leq 1 - \frac{y}{2^{k-1}},
\]
as long as \( k \) is sufficiently large.

To prove the estimate (38) observe that the function \( s \mapsto s(2-s)^{k-1} \) is increasing on \([0, 2/k]\). Since we have shown that for sufficiently large \( k \), \( \varepsilon_0 \leq \frac{2(1-y)}{2^k - k - 1} \leq \frac{2}{k} \), the lower bound in (37) yields
\[
\eta_0 = 1 - \varepsilon_0(2 - \varepsilon_0)^{k-1} 
\leq 1 - \frac{1}{2^{k-1}} \frac{2(1-y)}{2^k - k - 1} - \frac{16(1-y)^2}{2^{2k}} \left(1 - \frac{1}{2^k - k - 1} - \frac{8(1-y)^2}{2^{2k}}\right)^{k-1} 
\leq 1 - \left(\frac{2(1-y)}{2^k - k - 1} - \frac{8(1-y)^2}{2^{2k}}\right) \left(1 - \frac{(k-1)(1-y)}{2^k - k - 1} - \frac{8(k-1)(1-y)}{2^{2k}}\right) 
\leq \frac{y}{2^k} \frac{(k+1)(1-y)}{2^k - k - 1} + \frac{4k(1-y)^2}{2^k},
\]
provided \( k \) is large enough. The inequality \( \eta_0 \leq y \) is simpler. By (39),
\[
\frac{y}{2^k} = \frac{1 - \varepsilon_0(2 - \varepsilon_0)^{k-1}}{2(1-\varepsilon_0)(2-\varepsilon_0)^{k-1}} = \frac{\eta_0}{2^k(1-\varepsilon_0)(1-\varepsilon_0/2)^{k-1}} \geq \frac{\eta_0}{2^k}.
\]

## B Proof of Lemma 8

Since the function \( \alpha \mapsto \alpha^\alpha(1-\alpha)^{1-\alpha} \) is symmetric around 1/2, it suffices to prove that for every \( x \in (0, 1/2) \),
\[
f \left( \frac{1}{2} + x, \gamma, \eta \right) > f \left( \frac{1}{2} - x, \gamma, \eta \right).
\]

To this end, fix \( x \in [-1/2, 1/2] \) and \( \gamma, \eta > 0 \). Denote \( \varepsilon = 1 - \gamma^2 \). Plugging this notation and \( \alpha = 1/2 + x \) into (13), we find that the following identity holds
\[
\eta^{\varepsilon/2^{k-1}} 2^{2k} \gamma 2^k f \left( \frac{1}{2} + x, \gamma, \eta \right) = [2\varepsilon x^2 + (2 - \varepsilon)^2]k - 2(1-\eta)[(2-\varepsilon) + 2\varepsilon]k + (1-\eta)^2(1+2x)^k 
= \sum_{j=0}^{k} k \binom{k}{j} 2^j x^j [\varepsilon^j (2-\varepsilon)^{2(j-1)} - 2(1-\eta)\varepsilon^j (2-\varepsilon)^{k-j} + (1-\eta)^2] 
= \sum_{j=0}^{k} k \binom{k}{j} 2^j x^j [\varepsilon^j (2-\varepsilon)^{k-j} - (1-\eta)^2].
\]

This shows that we can write \( f(1/2 + x, \gamma, \eta) = \sum_{j=0}^{k} a_j x^j \) for some \( a_j \geq 0 \), such that at most one of the \( a_j \)'s is zero. Since for every \( x > 0 \) and odd \( j \), \( x^j - (-x)^j > 0 \), (13) follows.
C Proof of Lemma \[\text{9}\]

In this section we will use the normalization \[\text{33}\]. To prove the first assertion of Lemma \[\text{9}\] our goal is to show that \(g_0(\alpha) < 0\) for \(\frac{1}{2} < \alpha \leq 1 - \frac{3\ln k}{k}\). Observe that

\[
g_0(\alpha) = \frac{f_0(\alpha)^{-1}\{rf'_0(\alpha) + f_0(\alpha)[\ln(1 - \alpha) - \ln \alpha]\}}{\alpha^{\alpha(1 - \alpha)^{1 - \alpha}}}.
\] (45)

Differentiating \[\text{34}\] at \(x = 0\) we find that

\[
f'_0\left(\frac{1}{2}\right) = 2k\varepsilon_0^2(2 - \varepsilon_0)^{2k-2} - 4\varepsilon_0^2(2 - \varepsilon_0)^{2k-2} + 2\varepsilon_0^2(2 - \varepsilon_0)^{2k-2} = 0.
\]

Since, by \[\text{44}\], \(f_0(\alpha) > 0\) it is enough to show that the following function is decreasing on \([\frac{1}{2}, 1 - \frac{3\ln k}{k}]\)

\[
\psi(\alpha) = rf_0''(\alpha) + f_0(\alpha)[\ln(1 - \alpha) - \ln \alpha].
\]

Now,

\[
\psi'(\alpha) = rf_0''(\alpha) + f_0'(\alpha)[\ln(1 - \alpha) - \ln \alpha] - f_0(\alpha) \left(\frac{1}{\alpha} + \frac{1}{1 - \alpha}\right).
\]

Since for \(1/2 < \alpha \leq 1\), \(\ln(1 - \alpha) < \ln \alpha\) and, by \[\text{44}\], \(f'_0 > 0\) on \((1/2, 1]\), it is thus enough to prove that

\[
rf_0''(\alpha) \leq f_0(\alpha) \left(\frac{1}{\alpha} + \frac{1}{1 - \alpha}\right).
\]

Now, \(\frac{1}{\alpha} + \frac{1}{1 - \alpha} \geq 4\) and, from \[\text{35}\], we get that for \(\alpha \geq 1/2\),

\[
f_0(\alpha) \geq f_0\left(\frac{1}{2}\right) = 4(1 - \varepsilon_0)^2(2 - \varepsilon_0)^{2k-2} \geq (2 - \varepsilon_0)^{2k-2},
\]

where we also used that, by \[\text{34}\], \(\varepsilon_0 \leq 1/2\) for \(k\) large enough. Thus, it suffices to prove

\[
rf_0''\left(\frac{1}{2} + x\right) \leq 4(2 - \varepsilon_0)^{2k-2}.
\] (46)

Now, using that \(x \leq \frac{1}{2} - \frac{3\ln k}{k}\), we differentiate \[\text{34}\] twice to get

\[
rf_0''\left(\frac{1}{2} + x\right) = 4k(k - 1) \left\{\varepsilon_0^4(2x\varepsilon_0^2 + (2 - \varepsilon_0)^2)^{k-2} - 2\varepsilon_0^3(2 - \varepsilon_0)^{k-1}(2 - \varepsilon_0 + 2x\varepsilon_0)^{k-2} + \varepsilon_0^2(2 - \varepsilon_0)^{2k-2}(1 + 2x)^{k-2}\right\}
\]

\[
\leq 4k^2 \varepsilon_0^4(2 - \varepsilon_0)^{2k-4} \left(1 + \frac{2x\varepsilon_0}{(2 - \varepsilon_0)^2}\right)^{k-2} + \varepsilon_0^2(2 - \varepsilon_0)^{2k-2}(1 + 2x)^{k-2}
\]

\[
\leq 4k^2 \varepsilon_0^4(2 - \varepsilon_0)^{2k-4}(1 + 2x)^{k-2} + \varepsilon_0^2(2 - \varepsilon_0)^{2k-2}(1 + 2x)^{k-2}
\]

\[
\leq 8k^2\varepsilon_0^2(2 - \varepsilon_0)^{2k-2}(1 + 2x)^k
\]

\[
\leq 8k^2\varepsilon_0^2(2 - \varepsilon_0)^{2k-2}\left(2 - \frac{6k}{k}\right)^k
\]

\[
\leq 8k^2\varepsilon_0^2(2 - \varepsilon_0)^{2k-2}\frac{2 - 6\ln k}{k^3}
\]

\[
\leq \frac{2^{k+3}}{k} \left(\frac{4(1 - y)}{2^k}\right)^2 (2 - \varepsilon_0)^{2k-2},
\]

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where in the last line we used the fact that for \( k \) large enough, \( 37 \) implies \( \varepsilon_0 \leq 4(1 - y)/2^k \).
Combining this estimate with \( 46 \), we see that we must show that for sufficiently large \( k \)
\[
\frac{128 \ln 2}{1 - y + y \ln y} \cdot \frac{(1 - y)^2}{k} \leq 4 ,
\]
and this is indeed the case by \( 40 \).

It remains to show that \( g''(1/2) < 0 \). Denoting \( \zeta(\alpha) = \alpha^{-\alpha}(1 - \alpha)^{1-\alpha} \), we see from \( 44 \) that \( g''(\alpha) = f_0'(\alpha)^{\alpha-1}\psi(\alpha)\zeta(\alpha) \). Since \( f_0(1/2) = 0 \), \( \psi'(1/2) = 0 \), and we have just verified that \( \psi''(1/2) < 0 \), the required result follows.

\[\text{D Proof of Lemma 10}\]

Our goal is to show that for any \( 0 < r \leq t_k \), and \( 1 - \frac{4\ln k}{k} < \alpha \leq 1 \),
\[
\left[ \frac{f(\alpha, \sqrt{\gamma_0}, \sqrt{\eta_0})}{f(1/2, \gamma_0, \eta_0)} \right]^r \leq 2 \alpha^\alpha(1 - \alpha)^{1-\alpha} .
\] (47)

The following lemma gives an upper bound for the left-hand side of (47).

**Lemma 13. For all sufficiently large \( k \),**
\[
\frac{f(\alpha, \sqrt{\gamma_0}, \sqrt{\eta_0})}{f(1/2, \gamma_0, \eta_0)} \leq y^{y/2k} \left[ 1 + 2(1 - y) - 2(1 - \sqrt{y}) + \frac{(1 - \sqrt{y})^2 \alpha^k}{2^k} + \frac{60(1 - y)^2}{2^{2k}} \right] .
\] (48)

**Proof.** Denote \( \varepsilon_1 = 1 - (\sqrt{\gamma_0})^2 = 1 - \sqrt{1 - \varepsilon_0} \). For \( 1 - \frac{4\ln k}{k} < \alpha \leq 1 \) write \( x = \alpha - 1/2 \). Analogously to \( 14 \) we have the following identity
\[
\eta_0^{y/2k} 2^{2k} \gamma_0^k f \left( \frac{1}{2} + x, \sqrt{\gamma_0}, \sqrt{\eta_0} \right)
= [2x\varepsilon_1^2 + (2 - \varepsilon_1)^2] - 2(1 - \sqrt{\eta_0})[(2 - \varepsilon_1) + 2x\varepsilon_1]^k + (1 - \sqrt{\eta_0})^2(1 + 2x)^k .
\] (49)

Our first goal is replace \( \eta_0 \) in the right-hand side of \( 49 \) by its upper bound from \( 38 \). To this end consider the function
\[
\rho(b) = [2x\varepsilon_1^2 + (2 - \varepsilon_1)^2]^k - 2b[(2 - \varepsilon_1) + 2x\varepsilon_1]^k + b^2(1 + 2x)^k ,
\] (50)
and observe the right-hand side of \( 49 \) equals \( \rho \left( 1 - \sqrt{\eta_0} \right) \). So, that is it enough to show that \( \rho \) is decreasing on \([0, 1]\). Since \( \rho \) is convex and quadratic, this would follow once we show that \( \rho'(1) \leq 0 \). This is equivalent to \( 1 + 2x \leq 2 - \varepsilon_1 + 2x\varepsilon_1 \), which is true since \( x \leq 1/2 \). Hence
\[
\eta_0^{y/2k} 2^{2k} \gamma_0^k f \left( \alpha, \sqrt{\gamma_0}, \sqrt{\eta_0} \right) = \rho(1 - \sqrt{\eta_0})
\leq \rho(1 - \sqrt{z}) = [2x\varepsilon_1^2 + (2 - \varepsilon_1)^2]^k - 2(1 - \sqrt{z})[(2 - \varepsilon_1) + 2x\varepsilon_1]^k + (1 - \sqrt{z})^2(1 + 2x)^k ,
\] (51)

Where \( z \) is the upper bound for \( \eta_0 \) from \( 38 \), i.e.,
\[
z = \min \left\{ y, y - \frac{(k + 1)(1 - y)}{2^k - k - 1} + \frac{4k(1 - y)^2}{2^k} \right\} .
\] (52)
Hence using $\eta_0 \leq z$ and the identity (50) we bound the ratio in (47) as follows

$$f(\alpha, \sqrt{\gamma_0}, \sqrt{\gamma_0})$$

$$= \frac{\eta_0^{y/2^k} \cdot \gamma_0^{y/2^k} \cdot \gamma_0^{2k \cdot k \cdot f(\alpha, \sqrt{\gamma_0}, \sqrt{\gamma_0})}}{\eta_0^{y/2^k} \cdot \gamma_0^{2k \cdot k \cdot f(1/2, \gamma_0, \eta_0)}}$$

$$= \frac{\eta_0^{y/2^k} (1 - \varepsilon_0)^{y/2}}{4(1 - \varepsilon_0)^2 (2 - \varepsilon_0)^2 k - 2 \cdot f(1/2, \gamma_0, \eta_0)}$$

$$= \frac{\eta_0^{y/2^k} (1 - \varepsilon_0)^{y/2}}{4(1 - \varepsilon_0)^2 (2 - \varepsilon_0)^2 k - 2} \cdot \{ (x \varepsilon_1^2 + (2 - \varepsilon_1)^2)(1 - \varepsilon_1) + 2x \varepsilon_1 + (1 - \sqrt{z})^2 (1 + 2x)^k \}$$

$$= \varepsilon_0^{y/2^k} \left[ 1 + \frac{\varepsilon_0}{2(1 - \varepsilon_0)} \right]^2 \frac{\sqrt{1 - \varepsilon_0}}{\sqrt{1 - \varepsilon_0/2^k}}$$

$$\leq \varepsilon_0^{y/2^k} \left[ 1 + \frac{(k + 1)(1 - y)}{2(2^k - k - 1)} \right] + \frac{(k + 1)(1 - y)^2}{2(2^k - k - 1)}$$

$$\leq \varepsilon_0^{y/2^k} \left[ 1 + \frac{(k + 1)(1 - y)}{2(2^k - k - 1)} \right] + \frac{8k(1 - y)^2}{2^k}.$$ (53)

We will bound the various terms in (53) separately. First of all, using (52) and the inequality $e^z \leq 1 + a + a^2$, which is valid for $0 \leq a \leq 1$, we get that

$$\varepsilon_0^{y/2^k} \leq \varepsilon_0^{y/2^k} \left[ \left[ 1 + \frac{\varepsilon_0}{2(1 - \varepsilon_0)} \right]^2 \leq 1 + \varepsilon_0 + \varepsilon_0^2. (55)$$

Next, using the inequality $1/(1 - a) \leq 1 + 2a$, valid for $0 \leq a \leq 1/2$ we get

$$\left[ 1 + \frac{\varepsilon_0}{2(1 - \varepsilon_0)} \right]^2 \leq 1 + \frac{\varepsilon_0}{2} (1 + 2\varepsilon_0) \leq 1 + \varepsilon_0 + 5\varepsilon_0^2. (55)$$

Next, using the inequality $\sqrt{1 - x} \leq 1 - x/2$, the inequality $1/(1 - a) \leq 1 + a + a^2$, valid for $0 \leq a \leq 1/2$, and the inequality $(1 + a)^k \leq 1 + ka + k^2 a^2/2$, which is valid for all $a \leq 1/(4k^2)$, we get that since for $k$ large enough $\varepsilon_0 \leq 1/(4k^2)$,

$$\frac{\sqrt{1 - \varepsilon_0}}{(1 - \varepsilon_0/2^k)^2} \leq \frac{1}{(1 - \varepsilon_0)^2} \leq \left[ 1 + \frac{\varepsilon_0}{2(1 - \varepsilon_0)} \right]^2 \leq 1 + \varepsilon_0 + \varepsilon_0^2. (55)$$

Hence, for $k$ large enough

$$\left[ 1 + \frac{\varepsilon_0}{2(1 - \varepsilon_0)} \right]^2 \left[ \sqrt{1 - \varepsilon_0} \right] \leq 1 + \left( 1 + \frac{k}{2} \right) \varepsilon_0 + k^2 \varepsilon_0^2 + \varepsilon_0^2. (57)$$

Next, using the inequality $x/2 \leq 1 - \sqrt{1 - x} \leq x/2 + x^2$, which is valid for $0 \leq x \leq 1/2$, we get that

$$\frac{\varepsilon_0}{2} \leq \varepsilon_1 = 1 - \sqrt{1 - \varepsilon_0} \leq \frac{\varepsilon_0}{2} + \varepsilon_0^2 \leq \varepsilon_0. (58)$$

Observe that since $0 \leq x \leq 1/2$ and $\varepsilon_1 < 1/2$, the function $\varepsilon_1 \mapsto \frac{x\varepsilon_1^2}{4} + (1 - \varepsilon_1)^2$ is decreasing in $\varepsilon_1$. Hence, the lower bound in (58), together with another application of the inequality $(1 + a)^k \leq 1 + ka + k^2 a^2/2$,
valid for all \( a \leq 1/(4k^2) \), implies that for sufficiently large \( k \)

\[
\left[ \frac{xk^2}{4} + \left( 1 - \frac{\varepsilon}{2} \right)^2 \right]^k \leq \left[ \frac{\varepsilon_0^2}{32} + \left( 1 - \frac{\varepsilon_0}{4} \right)^2 \right]^k \leq \left[ 1 - \frac{\varepsilon_0}{2} + \frac{\varepsilon_0^2}{10} \right]^k \leq 1 - \frac{k\varepsilon_0}{2} + \frac{k^2\varepsilon_0^2}{8} + k\varepsilon_0^2. \tag{59}
\]

The second term in the brackets of \( 59 \) appears with a minus sign, so we bound it from below, using the fact that \( z \leq y \) and \( \varepsilon_1 \leq \varepsilon_0 \).

\[
\frac{2(1 - \sqrt{z})[1 - \varepsilon_1(1 - \alpha)]}{2k^2} \geq \frac{2(1 - \sqrt{z})}{2k} - \frac{2(1 - \sqrt{z})k\varepsilon_1(1 - \alpha)}{2k^2} \geq \frac{2(1 - \sqrt{z})}{2k} - \frac{2(1 - \sqrt{y})k\varepsilon_0}{2k^2} \geq \frac{2(1 - \sqrt{z})}{2k} - \frac{2(1 - y)k}{2k^2} - \frac{2(1 - y)}{2k} \geq \frac{2(1 - \sqrt{z})}{2k} - \frac{8k(1 - y)^2}{2k^2}, \tag{60}
\]

where we have used the upper bound in \( 58 \).

Combining \( 58, 59, 57, 55 \) and \( 60 \) we get

\[
f(\alpha, \sqrt{\varepsilon_0}, \sqrt{\eta_0}) \leq y^{\nu/2k} \left[ 1 - \frac{(k + 1)(1 - y)}{2k(2^k - k - 1)} + \frac{8k(1 - y)^2}{2^{2k}} \right] \left[ 1 + \left( 1 + \frac{k\varepsilon_0^2}{2} \right) \varepsilon_0 + \frac{k^2\varepsilon_0^2}{2} + k\varepsilon_0^2 \right].
\]

\[
\left\{ 1 - \frac{k\varepsilon_0}{2} + \frac{k^2\varepsilon_0^2}{8} + k\varepsilon_0^2 \right\} - \frac{2(1 - \sqrt{z})}{2k} + \frac{8k(1 - y)^2}{2^{2k}} + \frac{(1 - \sqrt{z})^2\alpha^k}{2k^2} \left\{ 1 + \frac{(k + 1)(1 - y)^2}{2k(2^k - k - 1)} - \frac{2(1 - \sqrt{y})^2\alpha^k}{2k} + \frac{k(1 - y^2)\varepsilon_0}{2k^2} + \frac{8k^2(1 - y)^2\varepsilon_0}{2^{2k}} + k^3\varepsilon_0^3 \right\}
\]

\[
\leq y^{\nu/2k} \left[ 1 - \frac{(k + 1)(1 - y)}{2k(2^k - k - 1)} + \frac{8k(1 - y)^2}{2^{2k}} \right] \left\{ 1 + \frac{(k + 1)(1 - y)^2}{2k(2^k - k - 1)} + \frac{2(1 - y)}{2^k} - \frac{2(1 - \sqrt{z})^2\alpha^k}{2k^2} + \frac{k(1 - y^2)\varepsilon_0}{2k^2} + \frac{8k^2(1 - y)^2\varepsilon_0}{2^{2k}} + k^3\varepsilon_0^3 \right\}
\]

\[
\leq y^{\nu/2k} \left\{ 1 + \frac{(k + 1)(1 - y)^2}{2k(2^k - k - 1)} - \frac{(k + 1)(1 - y)}{2k(2^k - k - 1)} + \frac{(1 - \sqrt{y})^2\alpha^k}{2k} - \frac{2(1 - \sqrt{z})^2\alpha^k}{2k} + \frac{50k(1 - y)^2}{2k^2} \right\}, \tag{61}
\]

where we have used the upper bound in \( 57 \), \( 55 \) and the fact that \( k \) is large enough.

Now, we claim that for every \( \alpha \in [0, 1] \),

\[
(1 - \sqrt{z})^2\alpha^k - 2(1 - \sqrt{z}) \leq (1 - \sqrt{y})^2\alpha^k - 2(1 - \sqrt{y}) + z - y. \tag{62}
\]

Indeed, since by \( 52 \), \( z \leq y \), the left-hand side minus the right-hand side of \( 52 \) is an increasing function, which vanishes at 1. Moreover, by \( 52 \),

\[
z - y \leq -\frac{(k + 1)(1 - y)}{2k - k - 1} + \frac{4k(1 - y)^2}{2k},
\]

so that \( 52 \) becomes

\[
(1 - \sqrt{z})^2\alpha^k - 2(1 - \sqrt{z}) \leq (1 - \sqrt{y})^2\alpha^k - 2(1 - \sqrt{y}) - \frac{(k + 1)(1 - y)}{2k - k - 1} + \frac{4k(1 - y)^2}{2k}.
\]

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Plugging this into (43) we get

\[
\frac{f(\alpha, \sqrt{\gamma_0}, \sqrt{\eta_0})}{f(1/2, \gamma_0, \eta_0)} \leq y^{\gamma/\alpha} \left\{ 1 + \frac{2(1-y)}{2^k - k - 1} - \frac{2(k+1)(1-y)}{2^k (2^k - k - 1)} + \frac{(1-\sqrt{y})^2 \alpha^k}{2^k} - \frac{2(1-\sqrt{y})}{2^k} + \frac{60k(1-y)^2}{2^{2k}} \right\}
\]

\[
= y^{\gamma/2k} \left\{ 1 + \frac{2(1-y) + (1-\sqrt{y})^2 \alpha^k - 2(1-\sqrt{y}) + 60k(1-y)^2}{2^k} \right\}.
\]

(63)

This concludes the proof of Lemma 13.

Denote \( h(\alpha) = -\alpha \ln \alpha - (1-\alpha) \ln(1-\alpha) \). Taking logarithms of (44), and using (48) and the inequality \( \ln(1+x) \leq x \), we see that our goal is reduced to showing that for all \( r \leq t_k \),

\[
\frac{r}{2^k} \left[ y \ln y + 2(1-y) - 2(1-\sqrt{y}) + (1-\sqrt{y})^2 \alpha^k + \frac{60k(1-y)^2}{2^k} \right] \leq \ln 2 - h(\alpha).
\]

(64)

For simplicity denote:

\[
\begin{align*}
A &= (1-\sqrt{y})^2 \\
B &= y \ln y + 2(1-y) - 2(1-\sqrt{y}) + \frac{60k(1-y)^2}{2^k}
\end{align*}
\]

(65)

With this notation (44) becomes

\[
\frac{r}{2^k} \leq \frac{\ln 2 - h(\alpha)}{A \alpha^k + B} \equiv M(\alpha),
\]

and this should hold for all \( \alpha \geq 1 - \frac{3\ln k}{k} \). We are therefore interested in the minimal value of \( M \) on the interval \([1 - \frac{3\ln k}{k}, 1]\). The derivative of \( M \) is

\[
M'(\alpha) = \frac{(A \alpha^k + B) \cdot [\ln \alpha - \ln(1-\alpha)] - k A \alpha^{k-1} [\ln 2 - h(\alpha)]}{(A \alpha^k + B)^2}.
\]

(66)

In particular, \( M'(1) = \infty \), so that the minimum of \( M \) cannot occur at \( \alpha = 1 \). We rule out the possibility of the minimum being at \( 1 - \frac{3\ln k}{k} \) in the following claim.

**Claim 1.** If \( k \) is large enough then for every \( 1 - \frac{3\ln k}{k} \leq \alpha \leq 2^{-1/k}, M(\alpha) > M(1) \).

**Proof.** Observe that for every \( \beta \in [0, 1] \),

\[
h(\beta) = \beta \ln(1/\beta) - (1-\beta) \ln(1-\beta)
\]

\[
\leq \beta \left( \frac{1}{\beta} - 1 \right) - (1-\beta) \ln(1-\beta) = 1 - \beta - (1-\beta) \ln(1-\beta).
\]

(67)

Hence, since \( \alpha \geq 1 - \frac{3\ln k}{k} \),

\[
h(\alpha) \leq h \left( 1 - \frac{3\ln k}{k} \right) \leq \frac{3\ln k}{k} + \frac{3\ln k}{k} \ln \left( \frac{k}{3\ln k} \right) \leq \frac{4(\ln k)^2}{k}.
\]

(68)

Using the fact that \( \alpha \leq 2^{-1/k} \), it follows that

\[
M(\alpha) \geq \frac{\ln 2 - \frac{5(\ln k)^2}{k}}{A \left( 2^{-1/k} \right)^k + B} \geq \frac{\ln 2 \left( 1 - \frac{10(\ln k)^2}{k} \right)}{A \left( 2^{-1/k} \right)^k + B}.
\]

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On the other hand, \( M(1) = \ln \frac{2}{(A + B)} \), so that it is enough to show that

\[
1 - \frac{10(\ln k)^2}{k} \geq \frac{\frac{4}{A+B}}{A+B} = 1 - \frac{\frac{4}{A}}{A+B},
\]

which is equivalent to

\[
\frac{A}{A+B} \geq 20(\ln k)^2.
\] (69)

Observe that since \( 1 - \sqrt{y} \geq (1-y)/2 \), \( A \geq (1-y)^2/4 \). On the other hand, using (66) we get that for sufficiently large \( k \),

\[
A + B = 1 - y + y \ln y + \frac{60k(1-y)^2}{2k} \leq (1-y)^2 + \frac{60k(1-y)^2}{2k} \leq 2(1-y)^2
\]

It follows that the left-hand side in (69) is at least 1/8, so that (69) provided \( k \) large enough.

By Claim 1 it remains to bound \( M(\alpha) \) from below when \( \alpha > 2^{-1/k} \) and \( M'(\alpha) = 0 \). In this case, by (66),

\[
-\ln(1 - \alpha) = -\ln \alpha + \frac{kAe^{x-1}}{Ae^k + B} [\ln \frac{2 - h(\alpha)}{k}]
\]

(70)

From the lower bound \( \alpha > 2^{-1/k} \) and (67) it follows that \( h(\alpha) \leq \frac{4\ln k}{k} \). Hence (70), together with our assumption that \( k \) is large, implies

\[
-\ln(1 - \alpha) > \frac{kA}{A+B} \left[ \ln 2 - \frac{4\ln k}{k} \right] \geq \frac{k}{5} \cdot \frac{A}{A+B}
\]

(71)

As we have seen in the proof of Claim 1 \( A \geq (1-y)^2/4 \) and \( A + B \leq 2(1-y)^2 \). Plugging these inequalities into (71), we get that \( -\ln(1 - \alpha) > k/40 \), i.e. \( \alpha \geq 1 - e^{-k/40} \). Plugging this into (66) once more, we get that

\[
-\ln(1 - \alpha) \geq \frac{kA[1 - 2k/(e^{k/40})]}{A+B} [\ln 2 - 2/(e^{k/40})] \geq \frac{kA \ln 2}{A+B} \left( 1 - \frac{6}{e^{k/40}} \right)
\]

Finally, we have shown that

\[
\alpha \geq 1 - \exp \left[ -\frac{kA \ln 2}{A+B} \left( 1 - \frac{6}{e^{k/40}} \right) \right].
\] (72)

We are now ready to bound \( M(\alpha) \) from below. We start by recalling that

\[
A + B = 1 - y + y \ln y + \frac{60k(1-y)^2}{2k} \equiv 1 - y + y \ln y + P.
\]

Using the inequality \( 1/(1 + x) \geq 1 - x \), we get

\[
\frac{1}{A+B} \geq \frac{1}{1 - y + y \ln y} \left[ 1 - \frac{P}{1 - y + y \ln y} \right] \geq \frac{1}{1 - y + y \ln y} \left[ 1 - \frac{120k}{2k} \right],
\]

(73)

where the last inequality used (39). Of course, we also know that \( A + B \geq 1 - y + y \ln y \).

Now, using (67), (72), (73), and the fact that \( (1 - \sqrt{y})^2/(1 - y + y \ln y) \leq 1 \), we get

\[
h(\alpha) \leq 2kA \ln 2 \left( 1 - \frac{6}{e^{k/40}} \right) \exp \left[ -\frac{kA \ln 2}{A+B} \left( 1 - \frac{6}{e^{k/40}} \right) \right]
\]

\[
\leq 2k \ln 2 \exp \left[ -\frac{k(1 - \sqrt{y})^2 \ln 2}{1 - y + y \ln y} \left( 1 - \frac{120k}{2k} \right) \left( 1 - \frac{6}{e^{k/40}} \right) \right]
\]

\[
\leq 2k \ln 2 \exp \left[ -\frac{k(1 - \sqrt{y})^2 \ln 2}{1 - y + y \ln y} \left( 1 - \frac{120k}{2k} - \frac{6}{e^{k/40}} \right) \right]
\]

\[
\leq 10k \ln 2 \cdot 2^{-k \varphi(y)},
\]
where as in Proposition\(7), \(\varphi(y) = \frac{(1-\sqrt{y})^2}{1-y+y\ln y}\).

So, using (73), we get

\[
M(\alpha) = \frac{\ln 2 - h(\alpha)}{A\alpha^k + B} \geq \frac{\ln 2}{A + B} (1 - 10k2^{-k\varphi(y)}) \geq \frac{\ln 2}{1 - y + y\ln y} \left[ 1 - \frac{120k}{2^k} \right] (1 - 10k2^{-k\varphi(y)}) \geq \frac{\ln 2}{1 - y + y\ln y} \left[ 1 - 20k2^{-k\varphi(y)} \right],
\]

where we have used the fact that \(\varphi(y) \geq 1/2\) and that \(k\) is sufficiently large.

This concludes the proof of Lemma 10.