Equidistribution of high-rank polynomials with variables restricted to subsets of $\mathbb{F}_p$

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Abstract

Let $p$ be a prime and let $S$ be a non-empty subset of $\mathbb{F}_p$. Generalizing a result of Green and Tao on the equidistribution of high-rank polynomials over finite fields, we show that if $P : \mathbb{F}_p^n \to \mathbb{F}_p$ is a polynomial and its restriction to $S^n$ does not take each value with approximately the same frequency, then there exists a polynomial $P_0 : \mathbb{F}_p^n \to \mathbb{F}_p$ that vanishes on $S^n$, such that the polynomial $P - P_0$ has bounded rank. Our argument uses two black boxes: that a tensor with high partition rank has high analytic rank and that a tensor with high essential partition rank has high disjoint partition rank.

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1 Introduction

Our starting point in this paper is the following theorem of Green and Tao [6, Theorem 1.7], which broadly speaking states that if a multivariable polynomial over \( \mathbb{F}_p \) does not take each value with approximately the same frequency, then it can be expressed in terms of a bounded number of polynomials of lower degree. We write \( \omega_p \) for \( \exp(2\pi i/p) \).

Theorem 1.1. Let \( p \geq 2 \) be a prime and let \( d \) be an integer such that \( 0 < d < p \). Then there exists a function \( B_{p,d} : (0, 1] \to [0, \infty) \) such that for every \( \epsilon > 0 \), if \( P : \mathbb{F}_p^n \to \mathbb{F}_p \) is a polynomial with degree \( d \) such that

\[
|\mathbb{E}_{x \in \mathbb{F}_p^n} \omega_p^{P(x)}| \geq \epsilon
\]

then there exists \( k \leq B_{p,d}(\epsilon) \), polynomials \( P_1, \ldots, P_k : \mathbb{F}_p^n \to \mathbb{F}_p \) each with degree at most \( d - 1 \) and a function \( F : \mathbb{F}_k^n \to \mathbb{F}_p \) such that \( P = F(P_1, \ldots, P_k) \).

The assumption \( d < p \) was later removed [10] by Kaufman and Lovett. The quantity \( \mathbb{E}_{x \in \mathbb{F}_p^n} \omega_p^{P(x)} \) from Theorem 1.1 is often referred to as the bias of the polynomial \( P \), and the smallest possible nonnegative integer \( k \) in its conclusion is often called the rank of \( P \).

Note that for any \( t \in \mathbb{F}_p \) and any function \( f : \mathbb{F}_p^n \to \mathbb{F}_p \),

\[
\mathbb{P}[f(x) = t] = \mathbb{E}_{x \in \mathbb{F}_p^n} \mathbb{1}_{f(x) = t} = \mathbb{E}_{x \in \mathbb{F}_p^n} \mathbb{E}_{s \in \mathbb{F}_p} \omega_p^{s(f(x) - t)} = p^{-1} + \mathbb{E}_{s \neq 0} \mathbb{E}_{x \in \mathbb{F}_p^n} \omega_p^s(f(x) - t).
\]

It follows that if some value is taken by \( f \) with a probability that differs from \( p^{-1} \) by at least \( \delta \), then there exists \( s \neq 0 \) such that \( |\mathbb{E}_{x \in \mathbb{F}_p^n} \omega_p^{sf(x)}| \geq \delta \). In particular, if the values of a polynomial \( P \) are not approximately uniformly distributed, then some non-zero multiple of \( P \) has large bias.

As we shall discuss in more detail later in this introduction, it has subsequently been shown that the function \( F \) that appears in the statement of Theorem 1.1 can be taken to be of a particular form: under the assumptions of the theorem, there are polynomials \( Q_1, \ldots, Q_k \) and \( R_1, \ldots, R_k \) of degree at most \( d - 1 \) such that \( P = Q_1R_1 + \cdots + Q_kR_k \).

Our aim in this paper is to generalize Theorem 1.1 to a statement concerning a restricted alphabet. Let \( S \) be a proper subset of \( \mathbb{F}_p \). We shall say that \( P : \mathbb{F}_p^n \to \mathbb{F}_p \) is biased on \( S^n \) if it does not take each value with approximately the same frequency. We wish to formulate and prove a statement to the effect that if \( P \) is biased on \( S^n \), then it has low rank on \( S^n \).
Proving results for restricted alphabets is already a mini-theme in additive combinatorics. For example, Bourgain, Dilworth, Ford, Konyagin and Kutzarova have proved a lower bound [1 Theorem 5] for the size of the sumset $A + B$ in $\mathbb{Z}^n$ if $A$ and $B$ are both subsets of $\{0, 1, \ldots, M - 1\}^n$ (which has obvious consequences for subsets of $S^n \subset \mathbb{F}_p^n$ if $S$ is Freiman isomorphic to a subset of $\mathbb{Z}$). There has also been work [3, 8] on minimizing the additive energy of a subset of $\{0, 1\}^n$ of given density.

For us the motivation for considering restricted alphabets came from density Hales-Jewett type problems. We shall not explain it in full here, but suppose, for example, that one wishes to find conditions on three sets $A, B, C \subset \{0, 1, 2\}^n$ that will ensure that there is a combinatorial line $(x, y, z)$ with $x \in A, y \in B$ and $z \in C$. (This is a set of points such that for each coordinate $i$, the triple $(x_i, y_i, z_i)$ is equal to $(0, 0, 0), (1, 1, 1), (2, 2, 2), \text{ or } (0, 1, 2)$.) If we write $|x|$ for the sum of the coordinates of $x$ (in $\mathbb{Z}$), then we see that $|x|, |y|$ and $|z|$ form an arithmetic progression, so we can define sets such as $A = B = \{x : |x| \equiv 0 \mod(5)\}$ and $C = \{z : |z| \equiv 1 \mod 5\}$. In this way, it becomes natural to regard $\{0, 1, 2\}$ as a subset of $\mathbb{F}_5$ and consider the restriction of the linear form $|x|$ to $\{0, 1, 2\}^n$. And for longer combinatorial lines it becomes natural in a similar way to look at restrictions of polynomials of higher degree.

In particular, one can use polynomials over $\mathbb{F}_p$ to define “Bohr-like” sets on $\{0, 1, \ldots, k - 1\}^n$. First, one regards $\{0, 1, \ldots, k - 1\}$ as a subset of $\mathbb{F}_p$. Then given polynomial functions $P_1, \ldots, P_r : \mathbb{F}_p^n \to \mathbb{F}_p^t$ for some $t$, and subsets $E_1, \ldots, E_r$ of $\mathbb{F}_p^t$, one can define a set $B((P_1, E_1), \ldots, (P_r, E_r))$ to be the set of all $x \in \{0, 1, \ldots, k - 1\}^n$ such that $P_i(x) \in E_i$ for each $i$. Let us (just for this paragraph) call this an $\mathbb{F}_p$-polynomial Bohr set in $[k]^n$ with parameters $(t, r)$. In a separate paper [4], we shall use the results of this paper to prove that every $\mathbb{F}_p$-polynomial Bohr set in $[k]^n$ with parameters $(t, r)$ can be approximated to within a density $\delta$ by an $\mathbb{F}_p$-polynomial Bohr set in $[k]^n$ with parameters $(t', r')$, where $t'$ and $r'$ are bounded above by functions of $\delta, p, k$ and $t$. (The main point here is that the upper bounds for $t'$ and $r'$ do not depend on $r$, so if a dense set is defined by a large number of polynomial conditions, then it can be approximated by a set defined by a bounded number of polynomial conditions.)

To understand the formulation of our main result, it will help to have examples such as the following in mind. Let $P(x) = \prod_{i=1}^n x_i(x_i - 1)$. Then $P$, considered as a polynomial on $\mathbb{F}_p^n$, has very high rank and is consequently very close to being equidistributed. However, its restriction to $\{0, 1\}^n$ is identically zero, so is as unevenly distributed as possible. We would like to be able to say that this restriction has low rank in some sense.

We do this in a fairly obvious way: we say that $P$ has low rank on $S^n$ if there is a low-rank polynomial $Q$ on $\mathbb{F}_p^n$ such that $P - Q$ vanishes on $S^n$. For instance, in the example above, we can simply take $Q$ to be the zero polynomial.

We now make this precise. We start by defining appropriate versions of bias and rank in the context of restricted alphabets.

**Definition 1.2.** Let $\mathbb{F}$ be a finite field, let $S$ be a subset $\mathbb{F}$, and let $f : \mathbb{F}^n \to \mathbb{F}$ be a function. For $\chi : \mathbb{F} \to \mathbb{C}$ a non-trivial character we define the bias of the function $f$ with
respect to the character $\chi$ and the set $S$ by
\[ \text{bias}_{\chi,S}(f) = \mathbb{E}_{x \in S^n} \chi(f(x)). \]

More generally, if $D$ is a probability distribution on $\mathbb{F}_p$, we define the bias of $F$ with respect to $\chi$ and $D$ by
\[ \text{bias}_{\chi,D}(f) = \mathbb{E}_{x \sim D^n} \chi(f(x)). \]

When $S = \mathbb{F}$ (or equivalently $D$ is the uniform distribution on $\mathbb{F}$), we write $\text{bias}_{\chi,S}(f)$ for $\text{bias}_{\chi,D}(f)$.

When $\mathbb{F} = \mathbb{F}_p$ the non-trivial characters are the functions $\chi_t : x \mapsto \omega_p^{xt}$ for each $t \in \mathbb{F}_p^*$. In this case we write $\text{bias}_{\chi_t,S}(f)$ for $\text{bias}_{\chi,S}(f)$.

**Definition 1.3.** Let $\mathbb{F}$ be a field, let $d$ be a non-negative integer, and let $P : \mathbb{F}^n \to \mathbb{F}$ be a polynomial of degree at most $d$. The degree-$d$ rank of $P$, denoted by $\text{rk}_d(P)$, is defined as follows.

1. If $d = 0$, then $\text{rk}_d P = 0$.

2. If $d = 1$, then $\text{rk}_d P$ is the number of $i$ such that $a_i \neq 0$ in the unique representation of $P$ in the form $P(x) = c + \sum_{i=1}^n a_i x_i$, where $c, a_1, \ldots, a_n \in \mathbb{F}_p$.

3. If $d \geq 2$, then $\text{rk}_d P$ is the smallest nonnegative integer $k$ such that there exist polynomials $Q_1, R_1, \ldots, Q_k, R_k : \mathbb{F}^n \to \mathbb{F}$, each with degree strictly smaller than $d$, and with $\deg(Q_i) + \deg(R_i) \leq d$ for each $i$, such that
\[ P = Q_1 R_1 + \cdots + Q_k R_k. \] (2)

If $S$ is a non-empty subset of $\mathbb{F}$, then the degree-$d$ rank of $P$ with respect to $S$, denoted by $\text{rk}_{d,S}(P)$, is $\min_{P_0} \text{rk}_d(P - P_0)$, where the minimum is taken over all polynomials $P_0 : \mathbb{F}^n \to \mathbb{F}$ such that $P_0(S^n) = \{0\}$. Equivalently, it is the minimum degree-$d$ rank of any polynomial $Q$ that agrees with $P$ on $S^n$.

If $P$ has degree exactly $d$, then we define the rank of $P$ to be $\text{rk}_d(P)$, and denote it by $\text{rk}_d P$.

Finally, if $S$ is a non-empty subset of $\mathbb{F}$, then we define $\text{rk}_S P$ to be the minimum of $\text{rk}_{d,P'}(P')$ over all polynomials $P'$ that agree with $P$ on $S^n$.

When $S = \mathbb{F}_p$, this definition is very similar to that of Green and Tao. One difference is that in the case $d \geq 2$, we ask for a decomposition $P = Q_1 R_1 + \cdots + Q_k R_k$ rather than one of the form $P = F(P_1, \ldots, P_k)$. This arises naturally from the notion of partition rank, as we shall explain later in this introduction.

A second difference is our definition when $d = 1$, which may at first seem a little strange, since there is a readily available notion of rank coming from linear algebra, according to which the degree-1 rank of $P$ would be 1 if $P$ is non-constant and 0 if $P$ is constant. If $P$ is defined on all of $\mathbb{F}_p^n$, then this is a satisfactory definition, since then the values of $P$
will be exactly equidistributed if $P$ is non-constant, and very unevenly distributed if $P$ is constant. However, these simple facts clearly do not continue to hold when $P$ is defined on $S^n$ for some proper subset $S \subset \mathbb{F}_p$. For example, if $P(x) = x_1$, then the values of $P$ are all contained in $S$. However, a fairly simple Fourier-analysis argument can be used to show that if $P$ has large degree-1 rank in the sense defined above, then its restriction to $S^n$ will be approximately equidistributed (provided $|S| \geq 2$).

Note also that when $d = 2$ the definition is similar, but not identical, to the usual definition of the rank of a quadratic form as the rank of the associated symmetric bilinear form. For example, the rank of the form $x \mapsto x_1x_2$ is obviously 1 when defined as above, but the associated symmetric bilinear form has rank 2 (over a field of odd characteristic).

The definition of the rank of a polynomial is slightly unnatural in that it is not subadditive, since if $P$ and $Q$ are low-rank polynomials of degree $d$, it may be that $P + Q$ is a high-rank polynomial of degree less than $d$. However, for certain statements it is nevertheless a convenient definition to have. One reason for this is that if $S$ is a proper subset of $\mathbb{F}_p$, then a polynomial $P$ of degree $d$ on $\mathbb{F}_p^n$ can agree on $S^n$ with a polynomial $Q$ of lower degree. In such a situation, the rank of $Q$ affects how well $P$ is equidistributed on $S^n$. The definition we have given allows us to express this in a concise way.

We now state our main theorem. Like Green and Tao, we assume that the field size is greater than the degree, but it is likely that the assumption is not necessary.

**Theorem 1.4.** Let $d$ be a positive integer and let $p > d$ be a prime. Let $S$ be a non-empty finite subset of $\mathbb{F}_p$, and let $P$ be a polynomial of degree $d$. Then there exists a function $H_{p,d,S} : (0, 1] \to [0, +\infty)$ such that if $P : \mathbb{F}_p^n \to \mathbb{F}$ is a polynomial with degree $d$ such that there exists a non-trivial character $\chi : \mathbb{F}_p \to \mathbb{C}$ for which $|\text{bias}_{\chi,S}P| \geq \epsilon$, then $\text{rk}_{d,S}P \leq \text{rk}_S P \leq H_{p,d,S}(\epsilon)$.

### 1.1 Background results on tensors

In our proof, we shall appeal to two known results, which we shall use as black boxes. The first is an analogue of Theorem 1.1 for tensors, and the second is a result of the second author that allows us, under suitable conditions, to restrict a high-rank tensor to a product of disjoint sets in such a way that it remains of high rank. This “decoupling” will play an important role in the reduction of the polynomial statement to the tensor statement.

By an order-$d$ tensor over a field $\mathbb{F}$ we mean simply a function $T : X_1 \times \cdots \times X_d \to \mathbb{F}$ for some finite sets $X_1, \ldots, X_d$. If $T$ is an order-$d$ tensor, we can associate with it a $d$-linear form $m : \mathbb{F}^{X_1} \times \cdots \times \mathbb{F}^{X_d} \to \mathbb{F}$, defined by the formula

$$m(x^1, \ldots, x^d) = \sum_{(i_1, \ldots, i_d) \in X_1 \times \cdots \times X_d} T(i_1, \ldots, i_d)x_{i_1}^1 \cdots x_{i_d}^d.$$

The analogues for tensors of the bias and of rank of polynomials that will be relevant to us are respectively the analytic rank introduced by Gowers and Wolf in [2] and the partition rank introduced by Naslund in [14].
Definition 1.5. Let \( d \geq 2 \) be a positive integer, let \( \mathbb{F} \) be a finite field and let \( T : X_1 \times \cdots \times X_d \to \mathbb{F} \) be an order-\( d \) tensor, and let \( m \) be the \( d \)-linear form associated with \( T \). The bias of \( m \) is defined to be

\[
\text{bias } m = \mathbb{E}_{x_1 \in \mathbb{F} \times x_1, \ldots, x_d \in \mathbb{F} \times x_d} \chi(m(x_1, \ldots, x_d))
\]

for any arbitrary non-trivial character \( \chi : \mathbb{F} \to \mathbb{C} \) of \( \mathbb{F} \). The analytic rank of \( T \), denoted by \( \text{ar } T \), is defined to be \( -\log_{|\mathbb{F}|} \text{bias } m \).

The right-hand side of (3) is independent of the non-trivial character \( \chi \). Indeed, the only property we require of \( \chi \) is that if \( \lambda : \mathbb{F} \to \mathbb{F} \) is a linear map, then \( \mathbb{E}_x \chi(\lambda(x)) = 1 \) if \( \lambda \) is the zero map and 0 otherwise. From this property, it follows easily that \( \text{bias}(T) \) is always a positive real number: indeed, it is equal to the probability that if we randomly restrict \( x_1, \ldots, x_{d-1} \), then the resulting linear map from \( \mathbb{F}^{x_d} \) to \( \mathbb{F} \) is identically zero. (Of course, we could randomly restrict any \( d-1 \) of the coordinates and get the same result.) However, the definition in terms of characters is more convenient for the purposes of generalization.

In the next definition we write \([d]\) for the set \( \{1,2,\ldots,d\} \), as is customary. If \( x = (x_1, \ldots, x_d) \in X_1 \times \cdots \times X_d \) and \( I \subset [d] \), then we write \( x(I) \) for the restriction of \( x \) to \( I \) – that is, for the element \( y \) of \( \prod_{i \in I} X_i \) such that \( y_i = x_i \) for each \( i \in I \).

Definition 1.6. Let \( d \geq 2 \) be a positive integer, let \( \mathbb{F} \) be a field and let \( T : X_1 \times \cdots \times X_d \to \mathbb{F} \) be an order-\( d \) tensor. Then \( T \) has partition rank at most 1 if there exists a non-trivial partition of \([d]\) into sets \( I, J \), and functions \( a : \prod_{i \in I} X_i \to \mathbb{F} \) and \( b : \prod_{i \in J} X_i \to \mathbb{F} \), such that \( T(x) = a(x(I))b(x(J)) \) for every \( x \in X_1 \times \cdots \times X_d \). The partition rank of \( T \), denoted \( \text{pr } T \), is the smallest non-negative integer \( k \) such that \( T \) is a sum of \( k \) tensors of partition rank at most 1. If \( m \) is the multilinear form associated with \( T \), then the partition rank \( \text{pr } m \) of \( m \) is defined to be \( \text{pr } T \).

Since the paper \[6\] of Green and Tao there has been significant interest in comparing these two notions of rank. Our first black box is that when \( \mathbb{F} \) and \( d \) are fixed, an order-\( d \) tensor over \( \mathbb{F}_p \) with large partition rank necessarily has a large analytic rank.

Theorem 1.7. Let \( d \geq 2 \) be a positive integer and let \( \mathbb{F} \) be a finite field. Then there exists a function \( A_{d,\mathbb{F}} : [0, \infty) \to [0, \infty) \) such that \( \text{pr } T \leq A_{d,\mathbb{F}}(\text{ar } T) \) for every order-\( d \) tensor \( T \) over \( \mathbb{F} \).

In the regime where \( \mathbb{F} \) has fixed size, which is the case that we will consider through this paper, the best bounds for the function \( A_{d,\mathbb{F}} \) are due to Janzer \[7\] and Milici\'evic \[13\]. For each \( r \geq 1 \), Janzer obtains \[7\] Theorem 1.10] the bound \( A_{d,\mathbb{F}}(r) = (c \log |\mathbb{F}|)^{c'(d)(d)} \) for \( c \) an absolute constant and \( c'(d) = 4^d \) and Milici\'evic obtains \[13\] Theorem 3] the bound \( A_{d,\mathbb{F}}(r) = 2^{O(r^2)}(r^{2O(d^2)} + 1) \).

We shall deduce from Theorem 1.7 a similar statement for restricted alphabets. Let \( d \) be a positive integer and let \( p \) be a prime, let \( X_1, \ldots, X_d \) be finite sets, let \( m : (\mathbb{F}^{X_1} \times \cdots \times}
\(F^X \rightarrow \mathbb{F}\) be a \(d\)-linear form, and let \(\chi : \mathbb{F} \to \mathbb{C}\) be a non-trivial character. Then given non-empty subsets \(S_1, \ldots, S_d\) of \(\mathbb{F}_p\), we define

\[
\text{bias}_{\chi,(S_1,\ldots,S_d)} m := \mathbb{E}_{(x_1,\ldots,x_d) \in S_1 \times \cdots \times S_d} \chi(m(x_1,\ldots,x_d)).
\]

Similarly, if \(D_1,\ldots,D_d\) are probability distributions on \(\mathbb{F}_p\), then we define

\[
\text{bias}_{\chi,(D_1,\ldots,D_d)} m := \mathbb{E}_{(x_1,\ldots,x_d) \sim D_1 \times \cdots \times D_d} \chi(m(x_1,\ldots,x_d)).
\]

Here \(D_i^{X_i}\) stands for the distribution on \(\mathbb{F}_i^{X_i}\) where each coordinate is chosen independently according to the distribution \(D_i\).

Note that it is no longer the case that the bias must be a positive real number or that it is independent of the choice of \(\chi\). However, we do not need these properties in the formulation of the next result.

**Proposition 1.8.** Let \(p\) be a prime, let \(d \geq 2\) and let \(S_1,\ldots,S_d\) be subsets of \(\mathbb{F}_p\) each with size at least 2. Then there exists a function \(A_{d,p,(S_1,\ldots,S_d)} : [0,\infty) \to [0,\infty)\) such that whenever \(T\) is an order-\(d\) tensor and the \(d\)-linear form \(m\) associated with \(T\) satisfies

\[
|\text{bias}_{\chi,(S_1,\ldots,S_d)} m| \geq \epsilon
\]

for some \(t \in \mathbb{F}_p^*, \text{ then } \text{pr} T \leq A_{d,p,(S_1,\ldots,S_d)}(-\log_p(\epsilon)).\)

In the special case \(p = 2\) the statement of Proposition 1.8 is the same as that of Theorem 1.7. We shall therefore assume that \(p \geq 3\) in the remainder of the paper. We also remark here that two of our later lemmas, Lemma 3.2 and a generalization of it, Lemma 4.1 do not hold when \(p = 2\).

The reader will notice that we have not formulated a notion of partition rank for restricted alphabets, and that the conclusion of Proposition 1.8 concerns the usual partition rank of \(T\). The reason is that a multilinear map defined on \(\mathbb{F}_p^{X_1} \times \cdots \times \mathbb{F}_p^{X_d}\) is determined by its values on \(S_1^{X_1} \times \cdots \times S_d^{X_d}\). This is easy to prove inductively: a linear map defined on \(\mathbb{F}_i^{X_i}\) is determined by its values on \(S_1^{X_1}\) (since \(S_1^{X_1}\) spans \(\mathbb{F}_i^{X_i}\)), so for each \(x_1,\ldots,x_{d-1} \in S_1^{X_1} \times \cdots \times S_{d-1}^{X_{d-1}}\), if two multilinear maps agree on \(\{x_1\} \times \cdots \times \{x_{d-1}\} \times S_d^{X_d}\), then they agree on \(\{x_1\} \times \cdots \times \{x_{d-1}\} \times \mathbb{F}_d^{X_d}\). Repeating for each coordinate, we get that they agree everywhere. (Here we made use of the assumption that each \(S_i\) has size at least 2.)

As Janzer and Milićević explain, we can deduce Theorem 1.1 from Theorem 1.7 using a connection with uniformity norms. It is known that the bias of a function \(f\) is bounded above in modulus by its uniformity norm \(\|f\|_{U_k}\) for any \(k\). But if \(f\) is of the form \(\chi \circ P\) for a polynomial \(P\) of degree \(d\), then \(\|f\|_{U_d}\) can be shown to equal bias\((m)\), where \(m\) is a multilinear form naturally associated with \(P\) that satisfies an identity of the form \(P(x) = (d!)^{-1}m(x,x,\ldots,x) + W(x)\) for a polynomial \(W\) of degree strictly less than \(d\). Since \(m\) has large bias, it has small partition rank, and this translates into an upper bound for the degree-\(d\) rank of \(P\).

However, we cannot use this method to deduce Theorem 1.4 from Proposition 1.8 for multilinear forms, because it relies on several identities that do not apply when we restrict
the alphabet. For example, $E_{x,y \in \mathbb{F}_p} f(x)f(x + y) = |E_{x \in \mathbb{F}_p} f(x)|^2$, but $E_{x,y \in \mathbb{S}} f(x)f(x + y)$ is not related in any simple way to $|E_{x \in \mathbb{F}_p} f(x)|^2$.

Instead, we shall combine a slight generalization of Proposition 1.8 with the second black box, the purpose of which is to decouple the variables of a polynomial, so that polynomials are reduced to multilinear forms in our arguments.

To see roughly how this will work, consider the quadratic case. If $P(x) = \sum_{i,j=1}^n a_{ij} x_i x_j$ (for simplicity we look at the homogeneous case here, but we shall prove the result in general), then we can write it in the form $\sum_i a_{ii} x_i^2 + \sum_{i \neq j} a_{ij} x_i x_j$. Consider now the bilinear form $\beta(x,y) = \sum_{i \neq j} a_{ij} x_i y_j$. If $\beta + \delta$ has high rank for every diagonal bilinear form $\delta$ (that is, one given by a formula $\delta(x) = \sum_i c_i x_i^2$), then our second black box (which is a simple result in the quadratic case, but much less so for higher degree) allows us to conclude that there is a partition of $[n]$ into sets $X$ and $Y$ such that the restriction $\beta'$ of $\beta$ to $\mathbb{F}_p^X \times \mathbb{F}_p^Y$ still has high rank. If we now regard $P$ as a 2-variable function defined on $\mathbb{F}_p^X \times \mathbb{F}_p^Y$, a standard argument bounds the $t$-bias of $P$ by the box norm of the function $\omega_p^{tP}$, which is equal to the expectation $E_{x,y \in \mathbb{F}_p^X} \beta'(x,y)$. Since $\beta'$ has high rank, this expectation is small, so $P$ has small $t$-bias for every $t \in \mathbb{F}_p$. Taking the contrapositive, if we assume that $P$ has large bias for some $t$, then we deduce that $\beta'$ also has large bias, and therefore that it has small partition rank, which in this case is just the rank, which implies that $\beta + \delta$ has small rank for some diagonal bilinear form $\delta$, by how we chose $\beta'$. This allows us to write $P(x, y)$ as the sum of a low-rank quadratic form and a diagonal form. It is then not too hard to prove that for $P$ to have large bias, the diagonal form must have small support.

We have not mentioned the set $S$ in the above sketch, but it turns out that, unlike with the proof based on uniformity norms, all the elements of the argument carry over for restricted alphabets, as we shall see in the next section.

To complete this section, we give a precise statement of the theorem that forms the second black box. Let $d \geq 2$ and $n_1, \ldots, n_d$ be positive integers and for each $i$ let $X_i$ be a subsets of $[n_i]$. Given an order-$d$ tensor $T : [n_1] \times \cdots \times [n_d] \to \mathbb{F}$ we write $T(X_1 \times \cdots \times X_d)$ for the restriction of $T$ to $X_1 \times \cdots \times X_d$. Similarly, given a multilinear form $m : \mathbb{F}^{n_1} \times \cdots \times \mathbb{F}^{n_d} \to \mathbb{F}$ a $d$-linear form we write $m(\mathbb{F}^{X_1} \times \cdots \times \mathbb{F}^{X_d})$ for the restriction of $m$ to $\mathbb{F}^{X_1} \times \cdots \times \mathbb{F}^{X_d}$.

**Definition 1.9.** Let $d \geq 2$ be an integer, let $\mathbb{F}$ be a finite field, and let $T : [n]^d \to \mathbb{F}$ be an order-$d$ tensor. Let $E \subset [n]^d$ be the set of all $d$-tuples with at least one pair of equal coordinates. The essential partition rank of $T$ is the quantity

$$\text{epr } T = \min_V \text{pr } (T + V),$$

where the minimum is taken over all order-$d$ tensors $V : [n]^d \to \mathbb{F}$ that are supported in $E$.

The disjoint partition rank is the quantity

$$\text{dpr } T = \max_{X_1, \ldots, X_d} \text{pr } (T(X_1 \times \cdots \times X_d)),$$

where the maximum is taken over all sequences of disjoint subsets $X_1, \ldots, X_d$ of $[n]$. 

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Since restricting a tensor cannot increase its partition rank, and since adding a tensor supported in \(E\) has no effect on the disjoint partition rank (because \(X_1 \times \cdots \times X_d\) and \(E\) are disjoint if the \(X_i\) are disjoint), we see immediately that the disjoint partition rank is bounded above by the essential partition rank. The result we shall use, due to the second author [9], provides a bound in the other direction.

**Theorem 1.10.** For every positive integer \(d \geq 2\) there exists a function \(\Lambda_d : \mathbb{N} \to \mathbb{N}\) such that if \(T\) is an order-\(d\) tensor satisfying \(\text{epr} \ T \geq \Lambda_d(l)\) then \(\text{dpr} \ T \geq l\).

The rest of the paper is organized as follows. In Section 2 we shall prove Theorem 1.4 in the cases \(d = 1\) and \(d = 2\). In Section 3 we shall then prove Theorem 1.4 in the case of alphabets \(S\) of size 2, where the proof is significantly simpler than in the general case. After this, in Section 4 we shall state and prove a more complicated version of Proposition 1.8 which, together with Theorem 1.10, will then allow us to deduce Theorem 1.4. This we shall do in Section 5.

### 2 The linear, bilinear and quadratic cases

#### 2.1 The linear case

If \(l : \mathbb{F}_p^n \to \mathbb{F}_p\) is a linear form, then bias \(l\) is equal to 1 if \(l \equiv 0\), and equal to 0 otherwise. As mentioned in the introduction, in the case of restricted subsets of \(\mathbb{F}_p\), the modulus of the bias of \(l\) instead decreases, as it turns out exponentially, with the order-1 rank of \(l\), which we defined earlier to be the size of the support of \(l\) when it is considered as a vector in \(\mathbb{F}_p^n\), in the usual way.

We begin with a technical lemma.

**Lemma 2.1.** Let \(p\) be a prime, let \(0 < c \leq 1/2\) be a real number, and let \(D\) be a probability distribution on \(\mathbb{F}_p\) such that \(D(x) \leq 1 - c\) for every \(x \in \mathbb{F}_p\). Then \(|\mathbb{E}_{x \sim D} x^y| \leq 1 - c/n^2/p^2\).

**Proof.** Let \(\alpha = \|D\|_\infty\) (that is, \(\max_x D(x)\)). We claim first that \(D\) is a convex combination of distributions \(D'\) with \(\|D'\| \leq \max\{\alpha, 1/2\}\) and of support size at most 2.

If \(\alpha \geq 1/2\), then let \(x\) be such that \(D(x) = \alpha\). Then for each \(y\) let \(D_y(x) = \alpha\), let \(D_y(z) = 0\) for all other \(z\). Then \(D = (1 - \alpha)^{-1} \sum_y D(y)D_y\).

If \(\alpha < 1/2\), then observe first that every distribution that is uniform on a subset \(A\) of \(\mathbb{F}_p\) of size at least 2 is a convex combination of distributions of the required form. (These are now distributions that are uniform on a two-point subset – just take the average over all two-point subsets of \(A\).) Now let \(x\) be such that \(D(x) = \alpha\) and let \(A\) be the support of \(D\). Then subtract a multiple of \(1_A\) from \(D\) to obtain a non-negative function \(f_1\) such that either \(f_1(x) = \sum_{y \neq x} f_1(y)\) or the support of \(f_1\) is strictly contained in \(A\). In the first case, we can finish by using the result for \(\alpha = 1/2\) and in the second case we can finish by using induction on the size of \(A\).

Now note that if \(D'\) has support size 2 and maximum \(\alpha\), then \(|\mathbb{E}_{x \sim D} x^y| \leq |\alpha + (1 - \alpha)\omega_p|\), and also that \(|\alpha + (1 - \alpha)\omega_p|\) decreases on \([0, 1/2]\) and increases on \([1/2, 1]\). Therefore, by the triangle inequality, \(|\mathbb{E}_{x \sim D} x^y| \leq |\beta + (1 - \beta)\omega_p|\), where \(\beta = \max\{\alpha, 1/2\}\).
Finally, we note that
\[
|\alpha + (1 - \alpha)\omega_p|^2 = \alpha^2 + 2\alpha(1 - \alpha)\cos(2\pi/p) + (1 - \alpha)^2
= 1 - 2\alpha(1 - \alpha)(1 - \cos(2\pi/p)).
\]

Since \(1 - \cos \theta \geq \theta^2/2 - \theta^4/24\), which is at least \(\theta^2/4\) when \(\theta \leq 2\pi/3\), we obtain an upper bound of \(1 - 2\beta(1 - \beta)\pi^2/p^2\). If \(\beta > 1/2\), this is at most \(1 - (1 - \alpha)\pi^2/p^2\), which is at most \(1 - c\pi^2/p^2\). If \(\beta = 1/2\), then the upper bound is \(1 - \pi^2/2p^2\), which again is at most \(1 - c\pi^2/p^2\) since \(c \leq 1/2\).

Recall that we are assuming that \(p \geq 3\), which is why we could assume that \(\theta \leq 2\pi/3\). However, with small changes the above lemma is clearly true for \(p = 2\) as well – in this case \(|\alpha + (1 - \alpha)\omega_p| = |2\alpha - 1|\), and we end up with a bound of \(1 - 2c\).

Given the lemma, the proof is very simple.

**Proposition 2.2.** Let \(p\) be a prime and let \(c \in (0, 1/2]\) be a positive real number. If \(D\) is a probability distribution on \(\mathbb{F}_p\) such that \(D(x) \leq 1 - c\) for each \(x \in \mathbb{F}_p\), and \(l : \mathbb{F}_p^* \to \mathbb{F}_p\) is a linear form, then
\[
|\text{bias}_{l,D} l| \leq (1 - c\pi^2/p^2)^{rk(l)}
\]
for every \(t \in \mathbb{F}_p^*\).

**Proof.** Writing \(l : x \to \sum_{z=1}^n a_z x_z\), we write \(\text{bias}_{l,D}\) as a product \(\prod_{z=1}^n \mathbb{E}_{x_z \sim D} \omega_p l(a_z l(x_z))\). For each \(z \in [n]\) the inner expectation is equal to 1 if \(a_z = 0\), and its modulus is otherwise always at most \(1 - c\pi^2/p^2\), by Lemma 2.1.

The result follows. \(\square\)

Again, the result also holds for \(p = 2\), but with a bound of \((1 - 2c)^{rk(l)}\).

### 2.2 The bilinear case

We next consider the case of a bilinear form \(b : \mathbb{F}_p^{n_1} \times \mathbb{F}_p^{n_2} \to \mathbb{F}_p\).

**Proposition 2.3.** Let \(p\) be a prime, let \(c_1, c_2 \in (0, 1/2]\), and let \(D_1, D_2\) be distributions on \(\mathbb{F}_p\) such that \(D_i(x) \leq 1 - c_i\) for all \(x \in \mathbb{F}_p\) and for \(i = 1, 2\). Let \(n_1, n_2\) be positive integers and let \(b : \mathbb{F}_p^{n_1} \times \mathbb{F}_p^{n_2} \to \mathbb{F}_p\) be a bilinear form of rank \(k\). Then \(|\text{bias}_{l,(D_1,D_2)} b| \leq 3e^{-c_1c_2\pi^2k/2p^2\log(2ep/c_1)}\) for every \(t \in \mathbb{F}_p\).

**Proof.** Since \(b\) has rank \(k\) we can find subsets \(X \subseteq [n_1]\) and \(Y \subseteq [n_2]\) of size \(k\) such that the restriction of \(b\) to \(\mathbb{F}_p^X \times \mathbb{F}_p^Y\) has rank \(k\). For each \(x \in \mathbb{F}_p^{n_1}\) let us write it as \((x_1, x_2)\), where \(x_1 \in \mathbb{F}_p^X\) and \(x_2 \in \mathbb{F}_p^{n_1\setminus X}\). We also let \(L_x : \mathbb{F}_p^Y \to \mathbb{F}_p\) be the linear form \(y \mapsto b(x, y)\) and let \(a_x \in \mathbb{F}_p^Y\) be the vector such that \(L_x(y) = \sum_{j \in [n_2]} (a_x)_j y_j\) for every \(y \in \mathbb{F}_p^Y\).

Let us now fix \(x_2 \in \mathbb{F}_p^{[n_1\setminus X]}\). Then the map \(x_1 \mapsto a_{(x_1, x_2)}\) is an affine map from \(\mathbb{F}_p^X\) to \(\mathbb{F}_p^Y\), with linear part \(a_{(x_1,0)}\) of full rank. Therefore, it is a bijection. It follows that for every
Lemma 2.3. Let $p$ be a prime, let $c_1, c_2 \in (0, 1/2]$, and let $D_1, D_2$ be distributions on $\mathbb{F}_p$ such that $D_i(x) \leq 1 - c_i$ for all $x \in \mathbb{F}_p$ and for $i = 1, 2$. Let $\epsilon > 0$, let $n_1, n_2$ be positive integers, and let $b : \mathbb{F}_{p^{n_1}} \times \mathbb{F}_{p^{n_2}} \to \mathbb{F}_p$ be a bilinear form with $|\text{bias}_{t, (D_1, D_2)} b| \geq \epsilon$ for some non-zero $t \in \mathbb{F}_p$. Then $b$ has rank at most $K_{p, c_1, c_2}(\epsilon)$, where

$$K_{p, c_1, c_2}(\epsilon) = 2c_1^{-1}c_2^{-1}\pi^{-2}p^2 \log(2ep/c_1) \log(3/\epsilon).$$

Proof. This is just a back-of-envelope calculation, but for the convenience of the reader we include it. By Proposition 2.3, we have the inequality

$$3e^{-c_1c_2\pi^2}rk b/2p^2 \log(2ep/c_1) \geq \epsilon.$$

It follows that

$$c_1c_2\pi^2 rk b/2p^2 \log(2ep/c_1) \leq \log(3/\epsilon),$$

and from that we obtain the bound stated. □
2.3 The quadratic case

We now begin the proof of Theorem 1.4 in the case of a polynomial of degree 2. First, we recall the definition of box norms and a standard fact about them, which for convenience we give in full, since it may be hard to find a proof of the precise formulation we give.

**Definition 2.5.** Let $W_1$ and $W_2$ be finite sets and let $f: W_1 \times W_2 \to \mathbb{C}$. Let $U_1$ and $U_2$ be independent random variables with $U_i$ taking values in $W_i$ for $i = 1, 2$. Let $U_1'$ and $U_2'$ be copies of $U_1$ and $U_2$, respectively, with $U_1, U_1', U_2$ and $U_2'$ all independent. The box norm of $f$ with respect to $U_1$ and $U_2$ is the quantity $\|f\|_\square$ defined by the formula

$$\|f\|_\square = \mathbb{E}f(U_1, U_2)f(U_1', U_2)f(U_1', U_2')f(U_1, U_2').$$

The standard fact we shall use is the following.

**Lemma 2.6.** Let $f, U_1, U_1', U_2, U_2'$ be as above. Then $\|f\|_\square \geq |\mathbb{E}f|$. 

**Proof.** By Cauchy-Schwarz,

$$|\mathbb{E}_{U_1, U_2}f(U_1, U_2)|^2 \leq \mathbb{E}_{U_1}|\mathbb{E}_{U_2}f(U_1, U_2)|^2 = \mathbb{E}_{U_1, U_2, U_2'}f(U_1, U_2)f(U_1, U_2').$$

Squaring both sides and applying Cauchy-Schwarz again, this time taking $U_2$ and $U_2'$ outside the modulus sign and keeping $U_1$ inside, we conclude that $|\mathbb{E}f|^4 \leq \|f\|_\square^4$. 

We will also use Theorem 1.10 for $d = 2$, for which a linear bound is available.

**Proposition 2.7** ([9], Proposition 5.1). Let $F$ be a field, let $A$ be an $n \times n$ matrix over $F$, and let $k$ be a positive integer. If for each $n \times n$ diagonal matrix $D$ we have $\text{rk}(A + D) \geq k$ then there exist disjoint $X, Y \subset [n]$ such that $\text{rk} A(X \times Y) \geq k/3$.

Our third preparatory result provides us with a useful sufficient condition for a function to have small bias.

We now introduce an analogue for quadratic forms of the notion of essential rank from Definition 1.9 for bilinear forms.

**Definition 2.8.** Let $q: \mathbb{F}_{p^n} \to \mathbb{F}_p$ be a quadratic form. The essential rank of the quadratic form is the quantity $\text{erk} q = \min \text{rk}(q + q_\Delta)$, where $\text{rk}$ is the notion of rank introduced in Definition 1.3, and where the minimum is taken over all diagonal quadratic forms $q_\Delta: \mathbb{F}_{p^n} \to \mathbb{F}_p$, that is, forms of the type $x \mapsto \sum_{i=1}^n a_i x_i^2$ for some coefficients $a_1, \ldots, a_n \in \mathbb{F}_p$.

**Lemma 2.9.** Let $q: \mathbb{F}_{p^n} \to \mathbb{F}_p$ be a quadratic form and let $b: \mathbb{F}_{p^n} \times \mathbb{F}_{p^n} \to \mathbb{F}_p$ be a bilinear form such that $q(x) = b(x, x)$ for every $x \in \mathbb{F}_{p^n}$. Then $\text{erk} q \leq \text{erk} b$. 

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Proof. If \( \text{erk} b \leq k \), then we can find a diagonal form \( b_\Delta \) such that \( \text{rk}(b + b_\Delta) \leq k \). Let \( q_\Delta(x) = b_\Delta(x, x) \) for each \( x \). Then \( q_\Delta \) is a diagonal quadratic form, and \( q(x) + q_\Delta(x) = b(x, x) + b_\Delta(x, x) \) for every \( x \). Since \( b + b_\Delta \) has rank at most \( k \), we can write \( b(x, y) + b_\Delta(x, y) \) in the form \( \sum_{i=1}^{k} f_i(x)g_i(y) \). But then \( q(x) + q_\Delta(x) = \sum_{i=1}^{k} f_i(x)g_i(x) \), so by the definition of rank we are using, \( q + q_\Delta \) has rank at most \( k \).

The next result is a simple technical lemma, which we state separately from the proof of the main result of the section because it will be used twice more in the paper.

**Lemma 2.10.** Let \( p \) be a prime, let \( D \) be a distribution on \( \mathbb{F}_p \), let \( f_0, f_1, \ldots, f_k : \mathbb{F}_p^n \to \mathbb{F}_p \) and let \( F : \mathbb{F}_p^k \to \mathbb{F}_p \). If

\[
|\text{bias}_{t,D}(f_0 + a_1f_1 + \cdots + a_kf_k)| \leq p^{-k}\epsilon
\]

for each \( t \in \mathbb{F}_p^* \) and for all \( (a_1, \ldots, a_k) \in \mathbb{F}_p^k \), then

\[
|\text{bias}_{t,D}(f_0 + F \circ (f_1, \ldots, f_k))| \leq \epsilon
\]

for each \( t \in \mathbb{F}_p^* \).

**Proof.** By definition,

\[
\text{bias}_{t,D}(f_0 + F \circ (f_1, \ldots, f_k)) = \mathbb{E}_{x \sim D^n} \omega_p^{tf_0(x)} \omega_p^{tF(f_1(x), \ldots, f_k(x))}.
\]

Let \( \phi(u_1, \ldots, u_k) = \omega_p^{tf(u_1, \ldots, u_k)} \) for each \( u_1, \ldots, u_k \). Then by the Fourier inversion formula,

\[
\phi(u_1, \ldots, u_k) = \sum_{a_1, \ldots, a_k} \hat{\phi}(a_1, \ldots, a_k) \omega_p^{a_1u_1 + \cdots + a_ku_k}.
\]

Noting that \( |\hat{\phi}(a_1, \ldots, a_k)| \leq \|\phi\|_\infty \leq 1 \) for every \( a_1, \ldots, a_k \), we may conclude that

\[
|\mathbb{E}_{x \sim D^n} \omega_p^{tf_0(x)} \omega_p^{tF(f_1(x), \ldots, f_k(x))}| \leq \sum_{a_1, \ldots, a_k} \mathbb{E}_{x \sim D^n} |\omega_p^{t(a_1f_1 + \cdots + a_kf_k)}|
\]

\[
= \sum_{a_1, \ldots, a_k} |\text{bias}_{t,D}(f_0 + t^{-1}(a_1f_1 + \cdots + a_kf_k))|.
\]

By our assumption, each summand in the last expression is at most \( p^{-k}\epsilon \), so the result follows. \( \square \)

We are now ready to start the proof of Theorem 1.4 for polynomials of degree 2.

**Proposition 2.11.** Let \( p \geq 3 \) be a prime, let \( S \) be a subset of \( \mathbb{F}_p \) with \( |S| \geq 2 \), let \( P : \mathbb{F}_p^n \to \mathbb{F}_p \) be a polynomial of degree 2. If \( |S| \geq 3 \), then

\[
|\text{bias}_{t,S} P| \leq 3e^{-rk_2P/2^2p^4(log p)^2}
\]

for every \( t \neq 0 \), while if \( |S| = 2 \), then

\[
|\text{bias}_{t,S} P| \leq 3e^{-(rk_2P-2)/64p^2 \log p}.
\]
Proof. Let \( t \in \mathbb{F}_p^n \) be fixed throughout the proof. We write \( P = q + l + c \), where \( q, l, c \) are the quadratic, linear and constant parts of \( P \), respectively. Let \( b : \mathbb{F}_p^n \times \mathbb{F}_p^n \rightarrow \mathbb{F}_p \) be the unique symmetric bilinear form such that \( q(x) = b(x, x) \) for every \( x \in \mathbb{F}_p^n \). Note that we have the polarization-type identity

\[
q(x + y) - q(x + y') - q(x' + y) + q(x' + y') = 2b(x - x', y - y'),
\]

from which it follows that the same identity holds with \( P \) replacing \( q \).

Let \( r = \text{erk} \, q \). Then by Lemma 2.29 we have \( \text{erk} \, b \geq r \). By Proposition 2.7 there exists a bipartition \( \{ X, Y \} \) of \([n]\) such that the restriction \( b(\mathbb{F}^X \times \mathbb{F}^Y) \) has rank at least \( r/3 \).

For \( x \) chosen uniformly at random from \( S^n \) let \( U_1, U'_1 \) be copies of \( x(X) \) and let \( U_2, U'_2 \) be copies of \( x(Y) \), with all four random variables being independent. Let \( f(U_1, U_2) = \omega_p^{tP(U_1, U_2)} \), where we are writing \( (U_1, U_2) \) for the vector \( x \in \mathbb{F}_p^n \) such that \( x(X) = U_1 \) and \( x(Y) = U_2 \). Then we have that \( \text{bias}_{t, S} P = \mathbb{E} f \). By Lemma 2.30 we also know that \( |\mathbb{E} f| \leq \|f\|_\square \). But

\[
|\mathbb{E} f|^4 = \mathbb{E} \omega_p^{4(P(U_1, U_2)-P(U_1, U'_2)-P(U'_1, U_2)+P(U'_1, U'_2))} = \mathbb{E} \omega_p^{2b(U_1-U'_1, U_2-U'_2)},
\]

where the last equality comes from the polarization identity mentioned earlier.

Let \( \mu_S \) be the uniform distribution on \( S \) and let \( D = \mu_S * \mu_{-S} \). That is, \( D \) is the distribution on \( \mathbb{F}_p \) where \( D(u) = \mathbb{P}[v - w = u] \) when \( v, w \) are chosen uniformly from \( S \).

Note that since \( |S| \geq 2 \), the probability \( \mathbb{P}[v - w = u] \) is at most 1/2.

The last expression above can be rewritten as

\[
\mathbb{E}_{x \sim DX, y \sim DY} \omega_p^{2b(x,y)} = \text{bias}_{3t(D,D)} b.
\]

Since \( \max_u D(u) \leq 1/2 \) we can apply Proposition 2.33 with \( c_1 = c_2 = 1/2 \) and \( k = r/3 \) to deduce that

\[
|\text{bias}_{3t(D,D)} b| \leq 3e^{-\pi r/24p^2 \log(4ep)}.
\]

From this it follows that

\[
|\text{bias}_{t, S} P| \leq 3e^{-\pi r/96p^2 \log(4ep)}.
\]  (4)

This is not yet what we want, because \( r = \text{erk} \, q \), which is not necessarily the same as \( \text{rk}_2 \, P \). To complete the proof, we observe first that we can write \( P \) in the form

\[
P(x) = d(x) + \sum_{j=1}^r \phi_j(x) \psi_j(x) + \theta(x),
\]

where \( d \) is a diagonal quadratic form, the \( \phi_j \) and \( \psi_j \) are linear forms, and \( \theta \) is an affine form (all the forms being from \( \mathbb{F}_p^n \) to \( \mathbb{F}_p \)).

By Lemma 2.10 \( |\text{bias}_{t, S} P| \) is at most \( p^{2r+1} \) times the maximum over all \( a_1, \ldots, a_{2r+1} \) of

\[
|\text{bias}_{t, S}(d + a_1 \phi_1 + a_2 \psi_1 + \cdots + a_{2r-1} \phi_r + a_2 \psi_r + a_{2r+1} \theta)|,
\]

which is equal to

\[
|\mathbb{E}_{x \in \mathbb{F}_p^n} \omega_p t d(x) + a_1 \phi_1(x) + a_2 \psi_1(x) + \cdots + a_{2r-1} \phi_r(x) + a_2 \psi_r(x) + a_{2r+1} \theta(x)|.
\]
In this section we prove our main theorem in the case where \(|S| = 2\). We now consider the cases \(|S| \geq 3\) and \(|S| = 2\) separately.

Write \(\text{supp}(d)\) for the support of \(d\), meaning that if \(d(x) = \sum_i \lambda_i x_i^2\), then \(\text{supp}(d) = \{i : \lambda_i \neq 0\}\). Then for each \(i \in d\) the polynomial \(P_i\) has degree exactly 2 and therefore takes each value at most twice. In particular, if \(|S| \geq 3\), then we can apply Lemma \([2.1]\) with \(c = 1/3\), to deduce that \(|E_{x_i \in S} P_{\mathbb{F}_p}(x_i)| \leq 1 - \pi^2/3p^2\). Therefore,

\[
|E_{x \in S^n} P_{\mathbb{F}_p}| \leq p^{2r+1}(1 - \pi^2/3p^2)^{|\text{supp}(d)|} \leq e^{(2r+1)\log p - \pi^2/3p^2}.
\]

It follows that

\[
|\text{bias}_{t,S} P| \leq p^{2r+1}(1 - \pi^2/3p^2)^{|\text{supp}(d)|} \leq e^{(2r+1)|\log p - \pi^2/3p^2|}.
\]

Choosing \(s\) to be \(r_k^{2} P/2\) and \(r\) to be \(\pi^2 r_k^{2} P/32p^2\), we obtain an upper bound of \(3e^{-\pi^2 r/96p^2\log(4ep)}, e^{(2r+1)|\log p - \pi^2/3p^2|}\) after a back-of-envelope calculation.

Now let us assume that \(|S| = 2\). Let \(Q\) be a polynomial that agrees with \(P\) on \(S^n\) and is such that \(rk Q = rk S P\). Let \(Q = q + l + c\), where \(q, l\) and \(c\) are the quadratic, linear and constant parts of \(Q\). If \(Q\) has degree 2 (as opposed to degree 1, which is also a possibility even if \(P\) has degree 2), then let \(d\) be a diagonal quadratic form such that \(rk(q + d) = \text{erk} q\).

Then

\[
rk S P \leq \text{rk} S(P + d) + 1 = \text{rk} S(Q + d) + 1 \leq \text{rk} S(q + d) + 2 \leq \text{erk} (q + d) + 2 = \text{erk} q + 2.
\]

Therefore, \(\text{erk} q \geq \text{rk} S P - 2\), so from \([4]\) it follows that \(|\text{bias}_{t,S} P| \leq 3e^{-\pi^2 (\text{rk} S P - 2)/96p^2\log(4ep)}\), which a simple calculation shows is at most \(3e^{-(\text{rk} S P - 2)/96p^2\log p}\).

If \(Q\) has degree 1, then \(rk Q = rk S P\), so by Proposition \([2.2]\) with \(D\) the uniform distribution on \(S\) (and hence with \(c = 1/2\)) we have \(|\text{bias}_{t,S} P| \leq (1 - \pi^2/2p^2)\text{rk} S P\), which is smaller than the bound just proved when \(Q\) has degree 2. This completes the proof. \(\square\)

## 3 The proof in the case of an alphabet with size 2

In this section we prove our main theorem in the case where \(|S| = 2\). Our argument can be summarized as follows. We begin by proving in particular that if \(S\) is a non-empty subset of \(\mathbb{F}_p\) with \(|S| \geq 2\) and \(A\) is a subset of \(S^n\) dense inside \(S^n\), then \((p - 1)A\) is dense in \(\mathbb{F}_p^n\). Using a small strengthening of this fact, and using the fact that the bias of a \(d\)-linear form

\[
\]
Later in the proof we shall find ourselves in a situation where we have a\(3.1\) Lemmas on the density of sumsets

sue.\(G\) uniform probability distribution on\(G\) of \(A\) with respect to \(D\) by \(G\) distributions on \(D\) by \((D + D')(x) = \sum_{y,z \in G : y + z = x} D'(y)D''(z)\) for all \(x \in G\) – that is, for the convolution of \(D\) and \(D'\), or equivalently for the distribution of the sum of two independent random variables, one with distribution \(D\) and the other with distribution \(D'\). We write \(D - D'\) for the distribution \(D + (-D')\). Given a probability distribution \(D\) on \(G\) and a positive integer \(C\), we shall write \(CD\) for the probability distribution \(D + \cdots + D\) (\(C\) times) on \(G\). If \(A\) is a subset of \(G\) and \(D\) is a probability distribution on \(G\) then we define the density of \(A\) with respect to \(D\) by \(\sum_{x \in A} D(x)\). In particular, the density of \(A\) with respect to the uniform probability distribution on \(G\) is the density \(|A|/|G|\) of \(A\) inside \(G\) in the usual sense.

3.1 Lemmas on the density of sumsets

Later in the proof we shall find ourselves in a situation where we have a \(d\)-linear form \(m : (\mathbb{F}_p^n)^d \to \mathbb{F}_p\) and a dense set \(A\) of \(u \in \mathbb{F}_p^n\) such that the \((d - 1)\)-linear form \(m_u : (\mathbb{F}_p^n)^{d-1} \to \mathbb{F}_p\) defined by \(m_u(x_2, \ldots, x_d) = m(u, x_2, \ldots, x_d)\) has low partition rank. However, this density will be with respect to a distribution \(D^n\), whereas we would prefer the uniform distribution. The results of this section show that we can achieve this at the cost of passing to a suitable subset of \(A\), which, by the subadditivity of partition rank, will not be a problem for us. (This trick of passing to a subset and exploiting subadditivity is often used to obtain a more structured set than \(A\), though we shall not need the extra structure here.)

Let \(p\) be a prime. We begin by proving a result which, when iterated \(p - 1\) times, shows that if \(A\) is a dense subset of \(\{0, 1\}^n\) then \((p - 1)A\) is a dense subset of \(\mathbb{F}_p^n\).

**Proposition 3.1.** Let \(r \geq 1\), \(n \geq 1\) be positive integers, let \(A \subset \{0, 1\}^n\) with density \(\alpha\) inside \(\{0, 1\}^n\) and let \(B \subset \{0, \ldots, r\}^n\) with density \(\beta\) inside \(\{0, \ldots, r\}^n\). Then \(A + B\) has density at least \(\alpha \beta\) inside \(\{0, \ldots, r + 1\}^n\).

**Proof.** For \(f : \{0, 1\}^n \to [0, \infty)\), \(g : \{0, \ldots, r\}^n \to [0, \infty)\), the max-convolution \(f \circ g : \{0, \ldots, r + 1\}^n \to [0, \infty)\) is defined by

\[
f \circ g(z) = \max_{x + y = z} f(x)g(y).
\]
It suffices to show that
\[ \mathbb{E}_z (f \circ g)(z) \geq \left( \mathbb{E}_x f(x) \right) \left( \mathbb{E}_y g(y) \right) \]
where the expectations are taken over \( \{0, \ldots, r+1\}^n \times \{0, 1\}^n \), and \( \{0, \ldots, r\}^n \), respectively, since if \( f = 1_A \) and \( g = 1_B \), then \( f \circ g = 1_{A+B} \). The more general result is, however, convenient as an inductive hypothesis.

If \( f \) and \( g \) are constant functions equal to \( \alpha \) and to \( \beta \) respectively, then \( f \circ g \) is the constant function equal to \( \alpha \beta \), so in this case \((5)\) holds. For each \( i \in [n] \) let \( T_i \) be the operator that averages over the \( i \)th coordinate. That is, if \( f : \{0,1\}^n \to [0,\infty) \) and \( g : \{0,1,\ldots,r\}^n \to [0,\infty) \), then
\[ T_i f(x) = \mathbb{E}_{u \in \{0,1\}} f(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n) \]
and
\[ T_i g(y) = \mathbb{E}_{u \in \{0,\ldots,r\}} g(y_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n). \]

For each \( i \in [n] \) the right-hand side of \((5)\) remains unchanged after replacing \( f \) and \( g \) by \( T_i f \) and \( T_i g \), so to prove the inequality it suffices to show that we always have the inequality
\[ \mathbb{E}_z (T_i f \circ T_i g)(z) \leq \mathbb{E}_z (f \circ g)(z). \]

Applying inequality \((5)\) successively for \( i = 1, \ldots, n \) then proves inequality \((6)\).

To prove inequality \((6)\), we begin by showing that it follows from the one-dimensional case. Without loss of generality we can assume \( i = n \). For each \( z \in \{0, \ldots, r+1\}^n \), let \( z' = (z_1, \ldots, z_{n-1}) \). Then we can write
\[ (f \circ g)(z', z_n) = \max_{x' + y' = z'} \max_{x_n + y_n = z_n} f(x', x_n) g(y', y_n) \]
and
\[ (T_n f \circ T_n g)(z', z_n) = \max_{x' + y' = z'} \mathbb{E}_{x_n} f(x', x_n) \mathbb{E}_{y_n} g(y', y_n) \]
for all \( z \in \{0, \ldots, r+1\}^n \). For a fixed \( z' \in \{0, \ldots, r+1\}^{n-1} \) let \( (x', y') \in \{0,1\}^{n-1} \times \{0, \ldots, r\}^{n-1} \) such that \( x' + y' = z' \) and \( \mathbb{E}_{x_n} f(x', x_n) \mathbb{E}_{y_n} g(y', y_n) \) is maximized, and therefore equal to \( \mathbb{E}_{z_n} (T_n f \circ T_n g)(z', z_n) \) for every \( z_n \). If inequality \((6)\) holds when \( n = 1 \), then
\[ \mathbb{E}_{x_n} f(x', x_n) \mathbb{E}_{y_n} g(y', y_n) \leq \mathbb{E}_{z_n} \max_{x_n + y_n = z_n} f(x', x_n) g(y', y_n). \]

For each \( z_n \), the maximum on the right-hand side is at most \( (f \circ g)(z', z_n) \), so we deduce that
\[ \mathbb{E}_{z_n} (T_n f \circ T_n g)(z', z_n) \leq \mathbb{E}_{z_n} (f \circ g)(z', z_n). \]
Averaging over all \( z' \in \{0, \ldots, r+1\}^{n-1} \) we obtain inequality \((6)\).

In the \( n = 1 \) case, writing \( a_j = f(j) \) for each \( j \in \{0,1\} \) and \( b_j = g(j) \) for each \( j \in \{0, \ldots, r\} \), inequality \((6)\) can be rewritten as
\[ \frac{r + 2}{2(r+1)} (a_0 + a_1) (b_0 + \cdots + b_r) \leq a_0 b_0 + (a_1 b_0 \vee a_0 b_1) + \cdots + (a_1 b_{r-1} \vee a_0 b_r) + a_1 b_r. \]

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Without loss of generality we can assume $a_1 \leq a_0$ and that $a_0 = 1$. We write $a_1 = \mu a_0$ for some $\mu \leq 1$. We now begin a second reduction where we show that it suffices to prove the inequality (7) in the case where $(b_0, \ldots, b_r)$ is a geometric progression with ratio $\mu$. Indeed, assume that there exists $j \in \{0, r−1\}$ such that for all $j' \in \{0, j−1\}$, $b_{j+1} = \mu b_{j'}$ but $b_{j+1} \neq \mu b_{j'}$. Inequality (7) then simplifies to

$$\frac{r+2}{2(r+1)}(1 + \mu)(b_0 + \cdots + b_r) \leq b_0 + \cdots + b_j + (\mu b_j \lor b_{j+1}) + \cdots + (\mu b_{r-1} \lor b_r) + \mu b_r. \quad (8)$$

If $b_{j+1} > \mu b_j$ then there exists $t > 0$ such that after decreasing $b_{j+1}$ by $t$ and increasing each of the quantities $b_0, \ldots, b_j$ in such a way that $b_0 + \cdots + b_j$ is increased by $t$ and $(b_0, \ldots, b_j)$ remains a geometric progression with ratio $\mu$ after the modification, we have $b_{j+1} = \mu b_j$. This modification leaves the left-hand side of (8) unchanged. On the right-hand side the sum $b_0 + \cdots + b_j$ increases by $t$, the term $\mu b_j \lor b_{j+1}$ decreases by $t$, the term $\mu b_{j+1} \lor b_{j+2}$ (or instead, if $j = r−1$, the term $\mu b_r$) cannot increase, and all other terms are left unchanged, so the right-hand side cannot increase.

If $b_{j+1} < \mu b_j$ then there exists $t > 0$ such that after increasing $b_{j+1}$ by $t$ and decreasing each of the quantities $b_0, \ldots, b_j$ in such a way that $b_0 + \cdots + b_j$ is decreased by $t$ and $(b_0, \ldots, b_j)$ remains a geometric progression with ratio $\mu$ after the modification, we have $b_{j+1} = \mu b_j$. The left-hand side of (8) is again unchanged. On the right-hand side the sum $b_0 + \cdots + b_j$ decreases by $t$, the term $\mu b_j \lor b_{j+1}$ remains unchanged, the term $\mu b_{j+1} \lor b_{j+2}$ (or instead, if $j = r−1$, the term $\mu b_r$) increases by at most $\mu t \leq t$, and all other terms are left unchanged, so the right-hand side cannot increase.

Iterating this process at most $r$ times we obtain $b_0, \ldots, b_r$ in geometric progression with ratio $\mu$. In this case, assuming without loss of generality that $b_0 = 1$, the inequality (8) becomes

$$\frac{r+2}{2(r+1)}(1 + \mu)(1 + \mu + \cdots + \mu^r) \leq 1 + \mu + \cdots + \mu^{r+1}$$

which simplifies to

$$2(\mu + \cdots + \mu^r) \leq r(1 + \mu^{r+1}).$$

This inequality holds, since by the weighted AM-GM inequality, for each $j \in [r]$ we have $\mu^j \leq \frac{(r+1-j)\mu^{j+1}}{r+1}$ and moreover $\sum_{j=1}^{r} \frac{j}{r+1} = r/2$. \hfill \square

We shall use the next two lemmas to obtain a connection between the density of a subset $B$ of $S^n$ inside $S^n$ for some subset $S$ of $\mathbb{F}_p$ and the density of $B$ with respect to $D^n$ for some distribution $D$ on $\mathbb{F}_p$.

**Lemma 3.2.** Let $p \geq 3$ be a prime, let $U$ be the distribution on $\mathbb{F}_p$ that assigns probability $1/2$ to $0$ and $1$, and let $D$ be a probability distribution on $\mathbb{F}_p$ such that $|D(x) - p^{-1}| \leq p^{-2}$ for each $x \in \mathbb{F}_p$. Then there exists a probability distribution $E$ on $\mathbb{F}_p$ such that $D = U + E$.

**Proof.** Given $x \in \mathbb{F}_p$, write $[x]$ for the residue of $x$ in $\{0, 1, \ldots, p−1\}$. Then the convolution of $U$ with the function $x \mapsto (-1)^{[x]}$ takes the value $1$ at $0$ and $0$ everywhere else. By translating this example, we can show that every function that takes the value $1$ in one
Lemma 3.3. Let $p \geq 3$ be a prime and let $c > 0$. Let $D$ be a probability distribution on $\mathbb{F}_p$ and suppose that $D(x) \leq 1 - c$ for every $x \in \mathbb{F}_p$. Then for every $M \geq 2p^2 \log p/c\pi^2$, the distribution $MD$ satisfies $|MD(x) - p^{-1}| \leq p^{-2}$ for each $x \in \mathbb{F}_p$.

Proof. By the convolution identity and the Fourier inversion formula,

$$MD(x) = \mathbb{E}_r(\hat{D}(r))^m \omega^{-rx},$$

where for each $r \in \mathbb{F}_p^n$, we define $\hat{D}(r)$ to be $\sum_y D(y) \omega^{ry}$.

If $r = 0$, then $\hat{D}(r) = 1$, while otherwise, by Lemma 2.1 it has absolute value at most $1 - c\pi^2/p^2$. It follows that $|MD(x) - p^{-1}| \leq (1 - c\pi^2/p^2)^M$, which is at most $e^{-c\pi^2M/p^2}$. The result follows. □

Proposition 3.4. Let $p \geq 3$ be a prime, let $c > 0$, let $M = 2p^2 \log p/c\pi^2$, and let $D$ be a probability distribution on $\mathbb{F}_p$ such that $|D(x)| \leq 1 - c$ for each $x \in \mathbb{F}_p$. Then if $A$ is a subset of $\mathbb{F}_p^n$ with density $\epsilon$ inside $\mathbb{F}_p^n$ with respect to $D^n$, the subset $(p-1)MA$ has density at least $\epsilon(p-1)^M$ inside $\mathbb{F}_p^n$ with respect to the uniform distribution on $\mathbb{F}_p^n$.

Proof. Lemma 3.2 implies that $|(MD)(\{x\}) - p^{-1}| \leq p^{-2}$ for every $x \in \mathbb{F}_p$. Applying Lemma 3.2 to $MD$ we obtain a probability distribution $E$ on $\mathbb{F}_p$ such that $MD = U + E$. Because $A$ has density at least $\epsilon$ with respect to $D^n$, the set $MA$ has density at least $\epsilon^M$ in $\mathbb{F}_p^n$ with respect to the distribution $(MD)^n$, since if $x_1, \ldots, x_M$ are chosen independently according to the distribution $D$, the probability that $x_1 + \cdots + x_M \in MA$ is at least the probability that each $x_i$ belongs to $A$.

Suppose now that we choose $y$ and $z$ independently at random from $\mathbb{F}_p^n$, according to the distributions $U^n$ and $E^n$, respectively. Then $y + z$ is distributed according to $(U + E)^n = (MD)^n$, so the probability that $y + z \in MA$ is at least $\epsilon^M$. It follows that there exists $z \in \mathbb{F}_p^n$ such that the density with respect to $U^n$ of the set $\{y \in \mathbb{F}_p^n : y + z \in MA\} = MA - z$ is at least $\epsilon^M$. In other words, letting $Y = (MA - z) \cap \{0, 1\}^n$, we have that $Y$ has density at least $\epsilon^M$ inside $\{0, 1\}^n$. Applying Proposition 3.2 $p - 2$ times we obtain that $(p - 1)Y$
has density at least $\epsilon^{(p-1)M}$ inside $\mathbb{F}_p^n$ with respect to the uniform distribution on $\mathbb{F}_p^n$. Since $(p-1)(Y+z)$ is contained in $(p-1)MA$, it follows that $(p-1)MA$ also has density at least $\epsilon^{(p-1)M}$.

\[ \square \]

### 3.2 Equidistribution of multilinear forms

We are now ready to prove a result about the equidistribution of multilinear forms which in particular implies Proposition 1.8 which bounds the partition rank of a tensor in terms of its bias with respect to a restricted alphabet. For this result we shall not need the hypothesis that $|S| = 2$, so it applies for general alphabets. Throughout the remainder of the paper we will restrict attention to $d$-linear forms from $(\mathbb{F}^n)^d$ to $\mathbb{F}$, because only these will be relevant to the proof of Theorem 1.4. However, all the results and proofs that we provide for multilinear forms $(\mathbb{F}^n)^d \to \mathbb{F}$ can be generalized easily to multilinear forms from $\mathbb{F}^{m_1} \times \ldots \times \mathbb{F}^{m_d}$ to $\mathbb{F}$.

As in the previous subsection, if we are given a $d$-linear form $m : (\mathbb{F}^n)^d \to \mathbb{F}$ and an element $u \in \mathbb{F}_p^n$, we write $m_u$ for the $(d-1)$-linear form from $(\mathbb{F}^n)^{d-1}$ to $\mathbb{F}$ defined by

$$ m_u(x_2, \ldots, x_d) = m(u, x_2, \ldots, x_d). $$

Our proof will appeal to Theorem 1.7, the result we quoted earlier that bounds partition rank in terms of analytic rank when the alphabet is unrestricted. We continue to write $A_{d,F}$ for the best function such that $pr T \leq A_{d,F}(ar T)$ for every degree-$d$ tensor $T$ over the field $\mathbb{F}$. For a fixed prime $p$ and a fixed $c > 0$, we define a family of functions $B_{d,p,c} : (0, 1) \to (0, 1)$ for all $d \geq 2$ by $B_{2,p,c} = K_{p,c}$ and for all $d \geq 3$,

$$ B_{d,p,c}(\epsilon) = A_{d,F_p}(p-1)M(p,c)B_{d-1,p,c}(\epsilon/2) + (p-1)M(p,c)\log_p((\epsilon/2)^{-1}), $$

where $M(p,c) = 2p^2 \log p/c\pi^2$ is the constant $M$ from Proposition 3.4. Note that $B_{2,p,c}(\epsilon)$ is the bound arising from Corollary 2.4 in the case $c_1 = c_2 = c$: that is, if two distributions $D_1, D_2$ on $\mathbb{F}_p$ both take maximum values at most $1-c$ and $b : \mathbb{F}_p^n \times \mathbb{F}_p^n \to \mathbb{F}_p$ is a bilinear form with bias at least $\epsilon$, then $rk b \leq B_{2,p,c}(\epsilon)$.

**Proposition 3.5.** Let $p \geq 3$ be a prime, let $d \geq 2$ be a positive integer, let $0 < c \leq 1/2$ and let $D_1, \ldots, D_d$ be distributions on $\mathbb{F}_p$ such that for each $1 \leq i \leq d$ and each $x \in \mathbb{F}_p$ we have $D_i(x) \leq 1-c$. Let $\epsilon > 0$. Let $m : (\mathbb{F}^n)^d \to \mathbb{F}_p$ be a $d$-linear form such that there exists $t \neq 0$ for which $|\text{bias} t(D_1, \ldots, D_d) m| \geq \epsilon$. Then the partition rank of $m$ is at most $B_{d,p,c}(\epsilon)$.

**Proof.** We proceed by induction on $d$. The result holds for $d = 2$ by our choice of $B_{2,p,c}$ (as discussed just above). Now let $d \geq 3$ and assume that the result holds for $d-1$. For each $u \in \mathbb{F}_p^n$ such that $pr m_u \geq B_{d-1,p,c}(\epsilon/2)$, the inductive hypothesis guarantees that

$$ |\text{bias} t(D_2, \ldots, D_d)(m_u)| \leq \epsilon/2 $$

for every $t \neq 0$. It follows that the set of such $u \in \mathbb{F}_p^n$ has density at most $1-\epsilon/2$ with respect to $D^n_1$, since otherwise we would have

$$ |\text{bias} t(D_1, \ldots, D_d) m| < \epsilon/2 + \epsilon/2 = \epsilon, $$

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contradicting our assumption. By the subadditivity of the partition rank and Proposition 3.4 the set of $u \in \mathbb{F}_p^n$ such that $pr m_u \leq (p-1)M(p,c)B_{d-1,p,c}(\epsilon/2)$ has density at least $(\epsilon/2)^{(p-1)M(p,c)}$ with respect to the uniform distribution on $\mathbb{F}_p^n$.

A theorem of Lovett [12, Theorem 1.7(i)], proved independently by Kazhdan and Ziegler [11, Lemma 2.2], states that the analytic rank of a tensor is bounded above by its partition rank. Therefore,

$$am_u \leq (p-1)M(p,c)B_{d-1,p,c}(\epsilon/2)$$

for all such $u \in \mathbb{F}_p^n$. Since

$$bias m = \mathbb{E}_{u \in \mathbb{F}_p^n} bias(m_u) = \mathbb{E}_{u \in \mathbb{F}_p^n} p^{-am_u},$$

we obtain the inequality

$$bias m \geq (\epsilon/2)^{(p-1)M(p,c)}p^{-(p-1)M(p,c)B_{d-1,p,c}(\epsilon/2)},$$

and therefore

$$am \leq (p-1)M(p,c)B_{d-1,p,c}(\epsilon/2) + (p-1)M(p,c)\log_p((\epsilon/2)^{-1}).$$

It follows that $pr m \leq B_{d,p,c}(\epsilon)$, as desired.

Proposition 1.8 follows as a special case of Proposition 3.5 by taking $A_{d,p,(S_1,\ldots,S_d)} : [0,\infty) \to [0,\infty)$ to be defined by

$$A_{d,p,(S_1,\ldots,S_d)}(r) = B_{d,p,1/2}(p^{-r})$$

for all $r \geq 0$.

### 3.3 Equidistribution of polynomials

We now turn to polynomials. Our strategy is broadly the same as it was for the quadratic case proved in the last section: we obtain a $d$-linear form $m$ from $P$ and disjoint sets $X_1,\ldots,X_d$ such that when we restrict $m$ to $\mathbb{F}_p^{X_1} \times \cdots \times \mathbb{F}_p^{X_d}$, the rank of the restriction tends to infinity with the rank of $m$ itself. We then apply the result for multilinear forms to $m$ and deduce from it the corresponding result for $P$. However, at one point we shall make critical use of the assumption that $|S| = 2$.

Let $d \geq 2$ be a positive integer and let $\epsilon$ be an element of $\{-1,1\}^d$. We shall write $N(\epsilon)$ for the number of indices $1 \leq i \leq d$ such that $\epsilon_i = -1$. For each $k \geq 1$, we say that a monomial $\prod_{a=1}^n x_a^{s_a}$ with $a \in \mathbb{F}_p^*$ involves at least (resp. at most) $k$ pairwise distinct variables if the set $\{u \in [n] : s_u \geq 1\}$ has size at least (resp. at most) $k$. The next proposition is a generalization of Lemma 2.6 to $d$ variables, except that we state it only for functions of the form $\omega_{f(U_1,\ldots,U_d)}$, as these are the functions that concern us. Since the result is standard, we give only a sketch of the proof.
Proposition 3.6. Let $p$ be a prime, let $d \geq 2$ be a positive integer, let $W_1, \ldots, W_d$ be finite sets, let $U_1, \ldots, U_d$ be jointly independent random variables taking values in $W_1, \ldots, W_d$ respectively and let $f : W_1 \times \cdots \times W_d \to \mathbb{F}_p$ be a function. Then

$$|E_{x \sim D} \omega_p f(U_1, \ldots, U_d)|^{2d} \leq E_{x \sim D} \sum_{\epsilon \in \{-1, 1\}^d} (-1)^{N(\epsilon)} f(U_{1, \epsilon_1}, \ldots, U_{d, \epsilon_d})$$

where $U_{i,-1}$ and $U_{i,1}$ have the same distribution as $U_i$ for each $i \in [d]$, and the $2d$ variables $U_{i,-1}, U_{i,1}$ with $i \in [d]$ are jointly independent.

Proof sketch. The proof is basically the same as that of Lemma 2.6 except that the Cauchy-Schwarz inequality is now applied to each of the $d$ variables instead of just to two variables. \hfill \square

We now establish the connection that we shall use to deduce the approximate equidistribution of a polynomial from that of a suitable associated multilinear form. Given a polynomial $P : \mathbb{F}_p^n \to \mathbb{F}_p$ and a partition $\{X_1, \ldots, X_d\}$ of $[n]$, we write $P_{\{X_1, \ldots, X_d\}}$ for the polynomial obtained from $P$ by keeping only the monomials $a \prod_{u=1}^n x_u^{s_u}$ with $a \neq 0$ such that for each $1 \leq i \leq d$ there exists $u \in X_i$ with $s_u \geq 1$.

Proposition 3.7. Let $p \geq 3$ be a prime, let $d \geq 2$ be a positive integer, let $D$ be a distribution on $\mathbb{F}_p$, let $P : \mathbb{F}_p^n \to \mathbb{F}_p$ be a polynomial, and let $\{X_1, \ldots, X_d\}$ be a partition of $[n]$. If $P$ has degree at most $d$, then

$$|E_{x \sim D} \omega_p P(x)|^{2d} \leq E_{x \sim D} \omega_p P_{\{X_1, \ldots, X_d\}}(x)^d.$$ 

Proof. We apply Proposition 3.6. Consider first the special case where $P$ is a monomial $P(x) = \prod_{u=1}^n x_u^{s_u}$. For each $1 \leq i \leq d$ let $U_i$ be the random variable $\prod_{u \in X_i} x_u^{s_u}$, where $x \sim D^n$, so that $P(x) = U_1 \ldots U_d$.

We now look at the behaviour of the quantity

$$\sum_{\epsilon \in \{-1, 1\}^d} (-1)^{N(\epsilon)} f(U_{1, \epsilon_1}, \ldots, U_{d, \epsilon_d}).$$

If there exists $i$ such that $s_u = 0$ for every $u \in X_i$, then $f(U_{1, \epsilon_1}, \ldots, U_{d, \epsilon_d})$ is independent of $\epsilon_i$, from which it follows that the whole sum is zero. The only other possibility, since $P$ has degree at most $d$, is if for each $i$ there is exactly one $u_i$ such that $s_{u_i} = 1$, and all other $s_u$ are zero. In that case for each $i$ let $U_{i,1} = x_{u_i}$ and $U_{i,-1} = x_{u_i}'$, where $x'$ is an independent copy of $x$. Then the quantity is equal to $\prod_{i=1}^d (U_{i,1} - U_{i,-1}) = \prod_{i=1}^d (x_{u_i} - x_{u_i}')$.

By linearity it follows more generally that

$$\sum_{\epsilon \in \{-1, 1\}^d} (-1)^{N(\epsilon)} f(U_{1, \epsilon_1}, \ldots, U_{d, \epsilon_d}) = P_{\{X_1, \ldots, X_d\}}(x - x').$$

The result therefore follows from Proposition 3.6. \hfill \square
Let $S$ be a subset of $\mathbb{F}_p$ of size 2, which will remain fixed until the end of the section. Before we start the main proof we need a few more results about polynomials and their connections to multilinear forms.

For the next three lemmas let $P : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ be a polynomial of degree $d$, and let $P = Q + R$ be the unique decomposition such that $Q$ is a linear combination of monomials of degree at most 1 in each variable separately and of total degree $d$, and $R$ is a linear combination of monomials such that either at least one variable has degree greater than 1 or the monomial has total degree less than $d$.

We shall make use of the following decomposition: for $p$ a prime, for $d \geq 2$ a positive integer, and for $P : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ a polynomial of degree $d$ over $\mathbb{F}_p$, we can write $P = Q + R$, where $Q$ is a linear combination of monomials $x_{i_1} \cdots x_{i_d}$ with $i_1, \ldots, i_d$ distinct elements of $[n]$, and $R$ is a linear combination of monomials $x_{i_1} \cdots x_{i_{d'}}$ such that either $d' < d$, or else $d' = d$ and $i_1, \ldots, i_d$ are not distinct.

The next lemma is where we shall use the assumption that $|S| = 2$: it is false in general for $|S| \geq 3$.

**Lemma 3.8.** The polynomial $R$ coincides on $S^n$ with a polynomial of degree at most $d - 1$.

**Proof.** Each monomial in $R$ can be written as $x_{i_1}^{s_1} \cdots x_{i_{d'}}^{s_{d'}}$ for some nonnegative integer $0 \leq d' \leq d - 1$, some $i_1, \ldots, i_{d'} \in [n]$, and some positive integers $s_1, \ldots, s_{d'} \geq 1$. If all powers $s_1, \ldots, s_{d'}$ are equal to 1, then this monomial has degree at most $d - 1$. If on the other hand one of the powers $s_1, \ldots, s_{d'}$ is at least 2, then without loss of generality this power is $s_1$. Since $S$ has size 2, there exist $a, b \in \mathbb{F}_p$ such that $x_{i_1}^{s_1} = ax + b$ for each of the two $x \in S$, we obtain that the monomial $x_{i_1}^{s_1} \cdots x_{i_{d'}}^{s_{d'}}$ coincides on $S^n$ with the polynomial

$$(ax_{i_1} + b)x_{i_1}^{s_2} \cdots x_{i_{d'}}^{s_{d' - 1}},$$

which has degree at most $d - 1$. \hfill \Box

The next lemma is a polarization identity for $d$-linear forms.

**Lemma 3.9.** There exists a unique symmetric $d$-linear form $m : (\mathbb{F}_p^n)^d \rightarrow \mathbb{F}_p$ such that

$$Q(y) = m(y, \ldots, y)$$

for every $y \in \mathbb{F}_p^n$.

**Proof.** Let $Q$ be given by the formula

$$Q(x) = \sum_{\{i_1 < \cdots < i_d\}} a_{\{i_1, \ldots, i_d\}} x_{i_1} \cdots x_{i_d}.$$

Then we can define $m$ by the formula

$$m(y_1, \ldots, y_d) = \sum_{j_1, \ldots, j_d} b_{j_1, \ldots, j_d} (y_1)_{j_1} \cdots (y_d)_{j_d},$$

where $a_{\{i_1, \ldots, i_d\}} = b_{j_1, \ldots, j_d}$.
where \( b_{j_1,\ldots,j_d} = (d!)^{-1} a_{i_1,\ldots,i_d} \) if \((j_1,\ldots,j_d)\) is a permutation of \((i_1,\ldots,i_d)\) and is zero if \(j_1,\ldots,j_d\) are not all distinct. Then it is not hard to check that \( Q(y) = m(y, y, \ldots, y) \) for every \( y \in \mathbb{F}_p^n \).

The uniqueness of \( m \) follows from the fact that a non-zero polynomial of degree less than \( p \) does not take the value zero everywhere, combined with the symmetry of \( m \) and the fact that

\[
a_{i_1,\ldots,i_d} = \sum_{\sigma \in S_d} b_{i_{\sigma(1)},\ldots,i_{\sigma(d)}}
\]

for every \( i_1,\ldots,i_d \).

Let \( V_d \) be the class of polynomials that are linear combinations of monomials \( x_{i_1} \ldots x_{i_d} \) for which \( i_1,\ldots,i_d \) are not all distinct. We define the essential rank of a homogeneous polynomial \( Q \) of degree \( d \) to be \( \min_{V \in V_d} \text{rk}(Q + V) \), and we denote it by \( \text{erk} Q \). Note that this agrees with our previous definition when \( d = 2 \).

**Lemma 3.10.** Let \( m \) be the symmetric multilinear form \( m \) defined in Lemma 3.9. Then the essential partition rank of \( m \) is at least the essential rank of \( Q \).

**Proof.** Suppose that \( \text{epr} \ m \leq k \). Then there exists a \( d \)-linear form \( m' : (\mathbb{F}_p^n)^d \to \mathbb{F}_p \) such that \( m'_{(i_1,\ldots,i_d)} = 0 \) whenever \( i_1,\ldots,i_d \in [n] \) are distinct and such that \( \text{pr}(m + m') \leq k \). We evaluate

\[
(m + m')(y, \ldots, y) = m(y, \ldots, y) + m'(y, \ldots, y)
\]

for all \( y \in \mathbb{F}_p^n \); the first term of the right-hand side is \( Q(y) \), and by the condition on the \( m'_{(i_1,\ldots,i_d)} \) with \( i_1,\ldots,i_d \in [n] \) pairwise distinct, the second term \( m'(y, \ldots, y) \) is a linear combination of monomials of the type \( y_{i_1}^a \ldots y_{i_d}^a \) with \( i_1,\ldots,i_d \) not pairwise distinct, so we can write it as \( V(y) \) for some polynomial \( V \) spanned by these monomials, which shows that \( \text{erk} Q \leq \text{rk}(Q + V) \).

Because \( \text{pr}(m + m') \leq k \) there exist for each \( i \in [k] \) a bipartition \( \{ J_i, J'_i \} \) of \([d]\) (with \( J_i, J'_i \) both non-empty) and multilinear forms \( M_{i,1} : (\mathbb{F}_p^n)^{J_i} \to \mathbb{F}_p, M_{i,2} : (\mathbb{F}_p^n)^{J'_i} \to \mathbb{F}_p \) such that

\[
(m + m') : (y^1, \ldots, y^d) \mapsto \sum_{i=1}^k M_{i,1}(y(J_i)) M_{i,2}(y(J'_i)).
\]

For each \( i \in [k], \) let the polynomials \( Q_i, R_i \) be defined by \( Q_i(y) = M_{i,1}(y, \ldots, y) \) and \( R_i(y) = M_{i,2}(y, \ldots, y) \). Then

\[
(Q + V)(y) = \sum_{i=1}^k Q_i(y) R_i(y)
\]

for every \( y \in \mathbb{F}_p^n \). Because the sets \( J_i, J'_i, i \in [k] \) are all strict subsets of \([d]\) we have \( \deg Q_i, \deg R_i < \deg Q \) for each \( i \in [k] \). Therefore, \( \text{erk} Q \leq k \). \( \square \)
In the previous section, where we proved Theorem 1.4 in the case where the subset $S$ of $\mathbb{F}_p$ had size 2, part of the proof of Proposition 3.11 relied heavily on Claim 3.8, which ensured that if $P$ had high rank, then its part $Q$ made up of monomials of the form $x_{i_1} \cdots x_{i_d}$

We are now ready to prove the main result of this section. In the proof we shall make use of Theorem 1.10. Recall that for each $d$, this theorem yields a function $\Lambda_d : \mathbb{N} \to \mathbb{N}$ such that if the essential partition rank of a tensor is at least $\Lambda_d(l)$, then the disjoint partition rank is at least $l$.

**Proposition 3.11.** Let $p \geq 3$ be a prime, let $d \geq 2$ be a positive integer, let $S$ be a subset of $\mathbb{F}_p$ of size 2, and let $\epsilon > 0$. Let $P : \mathbb{F}_p^\ast \to \mathbb{F}_p$ be a polynomial of degree $d$ and suppose that there exists $t \in \mathbb{F}_p^\ast$ with $|\text{bias}_{t,S} P| \geq \epsilon$. Then $\text{rk}_S P \leq \Lambda_d(B_{d,p,1/2}(\epsilon^{2d})) + 1$.

**Proof.** If $P$ agrees with a lower-degree polynomial $P_1$ on $S^n$, then we can replace $P$ by $P_1$ and apply induction on $d$. Otherwise, we note first that $\text{rk}_S P \leq (\text{erk} Q) + 1$, since if $R_0$ is a linear combination of monomials of the type $y_{i_1} \cdots y_{i_d}$ with $i_1, \ldots, i_d$ not all distinct then by applying Lemma 3.8 to $R + R_0$ we have $\text{rk}_S(R + R_0) \leq 1$, which implies that

$$\text{rk}_S P \leq \text{rk}(Q - R_0) + \text{rk}_S(R + R_0) \leq \text{rk}(Q - R_0) + 1.$$  

(The assumption that $P$ does not agree with a lower-degree polynomial on $S^n$ gives us the subadditivity here.) Choosing $R_0$ such that $\text{erk} Q = \text{rk}(Q - R_0)$ gives the bound claimed.

If $\text{erk} Q \geq \Lambda_d(B_{d,p,1/2}(\epsilon^{2d}))$ then by Lemmas 3.8 and 3.10 there exists a symmetric $d$-linear form $m : (\mathbb{F}_p^\ast)^d \to \mathbb{F}_p$ such that $Q(y) = m(y, \ldots, y)$ for every $y \in \mathbb{F}_p^\ast$, and such that $\text{erp} m \geq \Lambda_d(B_{d,p,1/2}(\epsilon^{2d}))$. By Theorem 1.10 we can find pairwise disjoint subsets $X_1, \ldots, X_d \subset [n]$ such that $\text{pr} m(\mathbb{F}_p^{X_1} \times \cdots \times \mathbb{F}_p^{X_d}) \geq B_{d,p,1/2}(\epsilon^{2d})$. We now apply Proposition 3.7. Letting $D$ be the distribution on $\mathbb{F}_p^\ast$ defined by $D(x) = |S|^{-2} \sum_{y,z \in S} S(y)S(z)1_{y-z=x}$, we have

$$|\mathbb{E}_{x \in S} \omega_\mathbb{F}_p l^{P(x)}(x)|^{2d} \leq \mathbb{E}_{x \sim D} \omega_\mathbb{F}_p l^{P(x)}(x)$$

(12)

for each $t \in \mathbb{F}_p^\ast$. We can write

$$P_{\{X_1, \ldots, X_d\}}(y) = \sum_{\sigma \in S_d} m(\mathbb{F}_p^{X_{\sigma(1)}} \times \cdots \times \mathbb{F}_p^{X_{\sigma(d)}})(y(X_{\sigma(1)}), \ldots, y(X_{\sigma(d)}))$$

$$= d!m(\mathbb{F}_p^{X_1} \times \cdots \times \mathbb{F}_p^{X_d})(y(X_1), \ldots, y(X_d))$$

where the second equality follows from the symmetry of $m$. Since $D$ has maximum value at most $1/2$, and $\text{pr} m(\mathbb{F}_p^{X_1} \times \cdots \times \mathbb{F}_p^{X_d}) \geq B_{d,p,1/2}(\epsilon^{2d})$, by Proposition 3.5 and (12) we get $|\text{bias}_{t,S} P| < \epsilon$ for all $t \in \mathbb{F}_p^\ast$, which is incompatible with our assumption. So $\text{erk} Q \leq \Lambda_d(B_{d,p,1/2}(\epsilon^{2d}))$. The result follows. \[\square\]

4 Equidistribution of combinations of multilinear forms with several choices of powers

In the previous section, where we proved Theorem 1.4 in the case where the subset $S$ of $\mathbb{F}_p$ had size 2, part of the proof of Proposition 3.11 relied heavily on Claim 3.8 which ensured that if $P$ had high rank, then its part $Q$ made up of monomials of the form $x_{i_1} \cdots x_{i_d}$
necessarily had high essential rank. For $|S| \geq 3$ this fact becomes false, so we have to consider the case where $P$ has high rank but $Q$ has bounded essential rank.

Let us very briefly sketch the argument that will follow. Given a polynomial $P : \mathbb{F}_p^n \to \mathbb{F}_p$ of degree $d$ and an alphabet $S$ of size at least $d + 1$ (if it has size at most $d$ then we can replace $P$ by a polynomial of degree at most $d - 1$ that takes the same values on $S^n$), we write $P$ as a linear combination of monomials, and then split it up according to the forms of those monomials – that is, the sequence of indices used, in non-increasing order. Given a non-increasing sequence $s = (s_1, \ldots, s_k)$ of positive integers with $s_1 + \cdots + s_k \leq d$, we write $P_s$ for the polynomial obtained when we retain just the linear combination of monomials of the form $x_1^{s_1} \cdots x_k^{s_k}$, where $i_1, \ldots, i_k$ are distinct numbers between 1 and $n$.

We also write $|s|$ for the length of the sequence $s$, that is, for the number $k$.

With each polynomial $P_s$ with $|s| = k$ we can associate a polynomial $Q_s$ in $k$ variables $x(1), \ldots, x(k) \in \mathbb{F}_p^n$ of the form $Q_s(x(1), \ldots, x(k)) = b_s(x(1)^{s_1}, \ldots, x(k)^{s_k})$, such that $b_s$ is a $k$-linear form, $P_s(x) = Q_s(x, x, \ldots, x)$ for every $x$, and $b_s$ is symmetric in $x(i)$ and $x(j)$ whenever $s_i = s_j$. A key lemma, which we shall prove in this section, will be that if any one of the $|s|$-linear forms $b_s$ has high rank (the notion of essential rank does not arise here because we regard the variables $x(1), \ldots, x(k)$ as belonging to distinct copies of $\mathbb{F}_p^n$), then $P$ has small bias on $S^n$. Therefore, if $P$ has large bias on $S^n$, we may conclude that the multilinear forms $b_s$ all have low rank. In the next section, with the help of Theorem 1.10, we shall deduce from this that $P$ agrees with a low-rank polynomial on $S^n$.

We now begin adapting some of the results that led to the proof of Proposition 3.3 which stated that multilinear forms that are significantly biased with respect to product distributions that are not too close to being atomic have low partition rank. Those results concerned a distribution $D$ on $\mathbb{F}_p$, that is not concentrated at a single point. We now need to generalize them to results concerning a distribution on $\mathbb{F}_p^k$ that is not concentrated on a proper affine subspace of $\mathbb{F}_p^k$.

The first result we shall adapt is Lemma 3.2.

**Lemma 4.1.** Let $p \geq 3$ be an odd integer, let $k \geq 1$ be a positive integer, let $U$ be the uniform distribution on the subset $\{0, 1\} \subset \mathbb{F}_p$, and let $D$ be a distribution on $\mathbb{F}_p^k$ such that for each $x \in \mathbb{F}_p^k$, $|D(x) - p^{-k}| \leq p^{-2k}$. Then there exists a probability distribution $E$ on $\mathbb{F}_p^k$ such that $D = U^k + E$.

**Proof.** We saw in the proof of Lemma 3.2 that there is a $\pm 1$-valued function $\phi$ on $\mathbb{F}_p$ such that $U * \phi$ takes the value 1 at 0 and 0 everywhere else. It follows that $U^k * \phi^k$ takes the value 1 at 0 (where now 0 is an element of $\mathbb{F}_p^k$) and 0 everywhere else, where by $\phi^k$ we mean the function $\phi^k(x) = \phi(x_1) \cdots \phi(x_k)$.

Just as in the one-dimensional case, it follows that for every function $f : \mathbb{F}_p^k \to \mathbb{R}$ there is a function $g : \mathbb{F}_p^k \to \mathbb{R}$ with $\|g\|_\infty \leq \sum_x |f(x)|$ such that $f = g * U$. We apply this to the function $f(x) = D(x) - p^{-k}$, noting that $\sum_x |f(x)| \leq p^{-k}$ by our hypothesis. Then $g * U = D - p^{-k}$, from which it follows that $(g + p^{-k}) * U = D$. Since $\|g\|_\infty \leq p^{-k}$, the function $g + p^{-k}$ is a probability distribution. \qed
Proposition 3.1 will be used as is, and we start by adapting Lemma 3.2, Lemma 3.3 and Proposition 3.4. For \( k \geq 1 \) a positive integer and \( p \geq 3 \) a prime, let \( U_{0,1},k,p \) be the distribution on \( \mathbb{F}_p^k \) defined by \( U_{0,1},k,p(x) = 2^{-k} \) if \( x_1, \ldots, x_k \in \{0,1\} \) and \( U_{0,1},k,p(x) = 0 \) otherwise.

Now we shall modify Lemma 3.3. For \( p \) a prime, for \( k \) a positive integer, and for \( c \geq 0 \), let \( M(p,c,k) = 2k \log p / \log(C(p,c)^{-1}) = kM(p,c) \).

Lemma 4.2. Let \( p \) be a prime, let \( k \geq 1 \) be a positive integer, and let \( c \geq 0 \). If \( D \) is a distribution on \( \mathbb{F}_p^k \) such that \( D(W) \leq 1 - c \) for every strict affine subspace \( W \) of \( \mathbb{F}_p^k \) then for all \( M \geq 2kp^2 \log p/c\pi^2 \) the distribution \( MD \) satisfies \( |MD(x) - p^{-k}| \leq p^{-2k} \) for each \( x \in \mathbb{F}_p^k \).

**Proof.** The proof is essentially the same as that of Lemma 3.3. The one thing we need to observe is that the condition on affine subspaces implies an upper bound on the size of each non-trivial Fourier coefficient of \( D \). Indeed,

\[
\hat{D}(r) = \sum_y D(y)\omega^r y = \sum_t \left( \sum_{r,y=t} D(y) \right) \omega^t \leq 1 - c \pi^2/p^2,
\]

where the inequality follows from Lemma 2.1 and the fact that \( \sum_{r,y=t} D(y) \leq 1 - c \) for each \( r, t \), by hypothesis.

This time, the probability that \( r = 0 \) is \( p^{-k} \), so we deduce that \( |MD(x) - p^{-k}| \leq (1 - c \pi^2/p^2)^M \leq e^{-c\pi^2M/p^2} \). The result follows from our assumed lower bound on \( M \).

We now generalize Proposition 3.3. Again, the generalization is straightforward, but we write it out in full, just to be clear about the details of the small changes needed.

Proposition 4.3. Let \( p \geq 3 \) be a prime, let \( k \geq 1 \) be a positive integer, let \( c \geq 0 \), let \( M = 2kp^2 \log p/c\pi^2 \), and let \( D \) be a probability distribution on \( \mathbb{F}_p^k \) such that \( D(W) \leq 1 - c \) for every strict affine subspace \( W \) of \( \mathbb{F}_p^k \). Then if \( A \) is a subset of \( (\mathbb{F}_p^k)^n \) with density \( \epsilon \) inside \( (\mathbb{F}_p^k)^n \) with respect to the distribution \( D^n \), then \( (p-1)MA \) has density at least \( \epsilon^{(p-1)M} \) inside \( (\mathbb{F}_p^k)^n \) with respect to the uniform distribution on \( (\mathbb{F}_p^k)^n \).

**Proof.** Lemma 1.2 implies that \( |(MD)(\{x\}) - p^{-k}| \leq p^{-2k} \) for every \( x \in \mathbb{F}_p^k \). Applying Lemma 1.1 to \( MD \) we obtain a probability distribution \( E \) on \( \mathbb{F}_p \) such that \( MD = U^k + E \). Because \( A \) has density at least \( \epsilon \) with respect to \( D^n \), the set \( MA \) has density at least \( \epsilon^M \) in \( (\mathbb{F}_p^k)^n \) with respect to the distribution \( (MD)^n \), since, as before, if \( x_1, \ldots, x_M \) are chosen independently according to the distribution \( D \), the probability that \( x_1 + \cdots + x_M \in MA \) is at least the probability that each \( x_i \) belongs to \( A \).

Suppose now that we choose \( y \) and \( z \) independently at random from \( (\mathbb{F}_p^k)^n \), according to the distributions \( (U^k)^n \) and \( E^n \), respectively. Then \( y + z \) is distributed according to \( (U^k + E)^n = (MD)^n \), so the probability that \( y + z \in MA \) is at least \( \epsilon^M \). It follows that there exists \( z \in (\mathbb{F}_p^k)^n \) such that the density with respect to \( (U^k)^n \) of the set \( \{ y \in (\mathbb{F}_p^k)^n : y + z \in MA \} = MA - z \) is at least \( \epsilon^M \). In other words, letting \( Y = (MA - z) \cap \{0,1\}^n \), we have that \( Y \) has density at least \( \epsilon^M \) inside \( \{0,1\}^n \). Identifying this with \( \{0,1\}^{kn} \) and
\((\mathbb{F}_p^k)^n\) with \(\mathbb{F}_p^kn\), we can apply Proposition \(3.1\) \(p - 2\) times to obtain the conclusion that \((p - 1)Y\) has density at least \(\epsilon(p - 1)M\) inside \((\mathbb{F}_p^k)^n\) with respect to the uniform distribution on \((\mathbb{F}_p^k)^n\). Since \((p - 1)(Y + z)\) is contained in \((p - 1)MA\), it follows that \((p - 1)MA\) also has density at least \(\epsilon(p - 1)M\).

In what follows, we shall often consider a non-empty finite set \(\Sigma\) and linearly independent functions \(\pi_1, \ldots, \pi_k : \Sigma \to \mathbb{F}_p\) that do not contain any non-zero constant function in their linear span. An important special case of this, which we shall need in Section 5, is when \(\Sigma\) is a subset \(\pi\) to us to prove our results in the more general set-up below.

Then for each \((a_1, \ldots, a_k) \in \mathbb{F}_p^k\) with \(a_1 \neq 0\), and each \(t \in \mathbb{F}_p^*\), it follows that \((p - 1)MA\) also has density at least \(\epsilon(p - 1)M\).

Proof. Let \((a_1, \ldots, a_k) \in \mathbb{F}_p^k\) be such that \(a_1 \neq 0\) and let \(t \in \mathbb{F}_p^*\). For each \(1 \leq i \leq k\) let the coefficients of \(l_i\) be \(l_i, \ldots, l_{i_n}\) so \(l_i(x) = \sum_{j=1}^{n} l_{ij}x_j\). Then

\[
(a_1l_1 \circ \pi_1^n + \cdots + a_kl_k \circ \pi_k^n)(x) = \sum_{j=1}^{n} (a_1l_1 \pi_1(x_j) + \cdots + a_kl_k \pi_k(x_j)).
\]

It follows that \(\text{bias}_t(a_1l_1 \circ \pi_1^n + \cdots + a_kl_k \circ \pi_k^n)\) factors as

\[
\prod_{j=1}^{n} E_{x_j \in \Sigma} \omega_p^{t(a_1l_1 \pi_1(x_j) + \cdots + a_kl_k \pi_k(x_j))}.
\]

Let \(j\) be such that \(l_{ij}\) is non-zero, and therefore such that \(a_1l_{ij}\) is non-zero. Then by our assumption about the functions \(\pi_i\), the function \(a_1l_{ij} \pi_1 + \cdots + a_kl_{kj} \pi_k\) is non-constant on \(\Sigma\), which implies that it does not take any value with probability more than \(1 - C_0^{-1}\). By Lemma \(2.1\) it follows that

\[
|E_{x_j \in \Sigma} \omega_p^{t(a_1l_{ij} \pi_1(x_j) + \cdots + a_kl_{kj} \pi_k(x_j))}| \leq 1 - \pi^2/C_0 p^2.
\]

Taking the product over all \(j\) such that \(l_{ij} \neq 0\), we conclude the desired inequality. \(\square\)
Remark. It follows from Proposition 4.4 that if \( 1 \leq d \leq p - 1 \) is a positive integer, \( S \) is a subset of \( \mathbb{F}_p \) with size at least \( d + 1 \), and \( P \) is a polynomial with degree \( d \) of the type \( P = \sum_i P_i(x_i) \) for some polynomials \( P_i : x \mapsto \sum_{j=0}^d a_{ij} x_i^j \) of degree at most \( d \) then

\[
|\text{bias}_{t,S} P| \leq (1 - \pi^2/p^3)|\{i : a_{ij} \neq 0\}| \leq (1 - \pi^2/p^3)^{rk P - 1}
\]

for every \( t \in \mathbb{F}_p^* \).

We next generalize Proposition 3.5. Recall that if \( \pi : \Sigma \to \mathbb{F}_p \) and \( x \in \Sigma^n \), then we write \( \pi^n(x) \) for \( (\pi(x_1), \ldots, \pi(x_n)) \).

**Proposition 4.5.** Let \( p \geq 3 \) be a prime, let \( C_0, d, k \) and \( l \) be positive integers with \( d \geq 2 \), let \( \Sigma \) be a non-empty set of size at most \( C_0 \), and let \( \pi_1, \ldots, \pi_k : \Sigma \to \mathbb{F}_p \) be linearly independent maps that do not contain a non-zero constant map in their linear span. Let \( m_1, \ldots, m_k : ((\mathbb{F}_p^n)^d) \to \mathbb{F}_p \) be \( d \)-linear forms, let \( (a_1, \ldots, a_k) \in \mathbb{F}_p^k \) be such that \( a_i \neq 0 \) and let \( \epsilon > 0 \) be a positive real number. If for a proportion at least \( \epsilon \) of the \( x \in \Sigma^n \) the \((d - 1)\)-linear map

\[
(y_2, \ldots, y_d) \mapsto a_1 m_1(\pi_1^n(x), y_2, \ldots, y_d) + \cdots + a_k m_k(\pi_k^n(x), y_2, \ldots, y_d)
\]

has partition rank at most \( l \), then

\[
\text{rk}_1 m_1 \leq 2(p - 1) \log_{p/2}(p) M l + (p - 1) M \log_{p/2} \epsilon^{-1} \text{ if } d = 2 \text{ and }
\]

\[
\text{pr } m_1 \leq A_d p^d(p - 1) M l + (p - 1) M \log_2 \epsilon^{-1} \text{ if } d \geq 3
\]

where \( M = 2kC_0 p^2 \log p/\pi^2 \).

**Proof.** Without loss of generality we can assume that \( a_i \neq 0 \) for each \( i \in [k] \): if this is not the case, then we proceed with a smaller \( k \).

As earlier in the paper, given \( y^1 \in \mathbb{F}_p^n \) we write \((m_1)_y^1 : (\mathbb{F}_p^n)^{d-1} \to \mathbb{F}_p^n \) for the \((d - 1)\)-linear form defined by the formula

\[
(m_1)_y^1(y_2, \ldots, y_d) = m_1(y^1, y_2, \ldots, y_d).
\]

We shall carry out the proof in the case \( d \geq 3 \). In the case \( d = 2 \) the proof is the same except that we use the bound \( E_p p^{-\text{rk}_1(m_1)} \leq (2/p)\text{rk}^m \) rather than \( E_p p^{-\text{ar} m} = p^{-\text{ar}^m} \).

For each \( y = (y^1, \ldots, y^k) \in (\mathbb{F}_p^n)^k \), we shall also write \( a.m_y : (\mathbb{F}_p^n)^{d-1} \to \mathbb{F}_p^n \) for the \((d - 1)\)-linear form defined by the formula

\[
a.m_y(y_2, \ldots, y_d) = a_1 m_1(y^1, y_2, \ldots, y_d) + \cdots + a_k m_k(y^k, y_2, \ldots, y_d).
\]

Note that the \((d - 1)\)-linear map specified in the statement of the proposition is \( a.m_{\pi^n(x)} \), where \( \pi^n(x) \) is shorthand for \( (\pi_1^n(x), \ldots, \pi_k^n(x)) \).

Let \( X = \{ x \in \Sigma^n : \text{pr}(a.m_{\pi^n(x)}) \leq l \} \) and suppose that \( X \) has density at least \( \epsilon \) inside \( \Sigma^n \) with respect to the uniform probability measure on \( \Sigma^n \). Write \( \pi : \Sigma \to \mathbb{F}_p^k \) for the map \( x \mapsto (\pi_1(x), \ldots, \pi_k(x)) \), and let \( D \) be the measure on \( \mathbb{F}_p^k \) defined by \( D(B) = \mathbb{P}[\pi(x) \in B] \),

\[
29
\]
where \( x \) is chosen uniformly from \( \Sigma \). Then the density of \( \pi^n(X) \) with respect to the measure \( D^n \) at least \( \epsilon \), since it is equal to \( \mathbb{P}[\pi^n(x) \in \pi^n(X)] \), which is at least \( \mathbb{P}[x \in X] \), which is at least \( \epsilon \) by hypothesis. Let \( A = \pi^n(X) \).

The statement that the \( \pi_i \) are linearly independent and do not span a non-zero constant function can be expressed as follows: if \( \lambda_1, \ldots, \lambda_k \) and \( w \in \mathbb{F}_p \) are such that \( \sum_i \lambda_i \pi_i(u) = w \) for every \( u \in \Sigma \), then \( \lambda_1 = \cdots = \lambda_k = 0 \). This tells us that there is no proper affine subspace that contains all the functions \( \psi_u : [k] \to \mathbb{F}_p \) defined by \( \psi_u(i) = \pi_i(u) \).

In particular for any such subspace \( W \), \( D(W) \leq 1 - 1/|\Sigma| \leq 1 - 1/C_0 \), so applying Proposition 4.3, the set \( B = (p - 1)MA \) has density at least \( \epsilon^{(p-1)M} \) inside \( \mathbb{F}_p^n \). By averaging, there exists \( (y^1, \ldots, y^k) \in (\mathbb{F}_p^n)^{k-1} \) such that the set

\[
Y^1 = \{y^1 \in \mathbb{F}_p^n : (y^1, y^2, \ldots, y^k) \in B\}
\]

has density at least \( \epsilon^{(p-1)M} \) inside \( \mathbb{F}_p^n \). For each \( y^1 \in Y^1 \) we have by subadditivity of the partition rank that \( \text{pr}(a.m(y^1, \ldots, y^k)) \leq (p - 1)MI \). Let \( y^1_0 \) be a fixed element of \( Y^1 \). For each \( y^1 \in Y^1 \) the map \( (m_1)_{y^1 - y^1_0} \) can be rewritten as the difference \( a_1^{-1}(a.m(y^1, y^2, \ldots, y^k) - a.m(y^1_0, y^2, \ldots, y^k)) \), so \( \text{pr}(m_1)_{y^1 - y^1_0} \leq 2(p - 1)MI \) by subadditivity. By construction the set \( Y^1 - \{y^1_0\} \) has density at least \( \epsilon^{(p-1)M} \) and for each \( y^1 \in Y^1 - \{y^1_0\} \), \( \text{pr} m^1 \leq 2(p - 1)MI \).

For each \( y^1 \in Y^1 - \{y^1_0\} \), using the definition \( b(m_1)_{y^1} = p^{-\text{ar}(m_1)_{y^1}} \) of the analytic rank and Theorem 1.7 from [12], which states that analytic rank is bounded above by partition rank, we have \( b(m_1)_{y^1} \geq p^{-\text{pr}(m_1)_{y^1}} \), so since \( \text{pr}(m_1)_{y^1} \leq 2(p - 1)MI \) we obtain the lower bound \( b(m_1)_{y^1} \geq p^{-2(p-1)MI} \).

We now use the fact that \( b(m_1) = \mathbb{E}_{y^1 \in \mathbb{F}_p^n} b(m_1)_{y^1} \). Since \( Y^1 \) has density at least \( \epsilon^{(p-1)M} \) inside \( \mathbb{F}_p^n \) and \( b(m_1) > 0 \) for for each \( y^1 \in \mathbb{F}_p^n \) we obtain that \( b(m_1) \geq \epsilon^{(p-1)M/p - 2(p-1)MI} \). Therefore,

\[
\text{pr} m_1 \leq A_{d,\mathbb{F}_p}(\text{ar} m_1) \leq A_{d,\mathbb{F}_p}(2(p - 1)MI + (p - 1)M \log_p \epsilon^{-1})
\]

as desired. \( \square \)

We are now ready to prove a result that will have as a consequence that if the multilinear form associated with one “piece” of a polynomial has high rank, then the whole polynomial has small bias. For fixed \( p, k, C_0 \) we define a sequence of functions \( B_{d,p,k,C_0} \) by

\[
B_{1,p,k,C_0} = C_0 p^2 \log \epsilon^{-1}/\pi^2
\]

and for all \( d \geq 2, \)

\[
B_{d,p,k,C_0}(\epsilon) = A_{d,\mathbb{F}_p}(2(p - 1)(2kC_0p^2 \log p/\pi^2)(2B_{d-1,p,k,C_0}(\epsilon/2) + \log_p(\epsilon/2)^{-1})).
\]

**Proposition 4.6.** Let \( p \geq 3 \) be a prime, let \( C_0, d \) and \( k \) be positive integers, let \( \Sigma \) be a non-empty set of size at most \( C_0 \), let \( \pi_1, \ldots, \pi_k : \Sigma \to \mathbb{F}_p \) be linearly independent maps that do not span a non-zero constant function, and let \( \epsilon > 0 \). For each \( (i_1, \ldots, i_d) \in [k]^d \)
let \( m_{(i_1,\ldots,i_d)} : (\mathbb{F}_p^n)^d \to \mathbb{F}_p \) be a \( d \)-linear form. If there exists \((i'_1,\ldots,i'_d) \in [k]^d\) such that \( \text{pr } m_{(i'_1,\ldots,i'_d)} \geq B_{d,p,k,C_0}(\epsilon) \), then every linear combination \( a.m^n : (\Sigma^n)^d \to \mathbb{F}_p \) defined by

\[
a.m^n : (x_1,\ldots,x_d) \mapsto \sum_{(i_1,\ldots,i_d) \in [k]^d} a_{(i_1,\ldots,i_d)} m_{(i_1,\ldots,i_d)}(\pi^n(x_1),\ldots,\pi^n(x_d))
\]

with \( a_{(i'_1,\ldots,i'_d)} \neq 0 \) satisfies that

\[|\text{bias}_t a.m^n| \leq \epsilon\]

for all \( t \in \mathbb{F}_p^n\).

**Proof.** We proceed by induction on \( d \). The \( d = 1 \) case holds by Proposition 4.1. We now assume \( d \geq 2 \). Let \( a \in \mathbb{F}_p^{[k]^d} \) with \( a_{(i'_1,\ldots,i'_d)} \neq 0 \) be fixed throughout. Since \( \text{pr } m_{(i'_1,\ldots,i'_d)} \geq B_{d,p,k,C_0}(\epsilon) \) and \( a_{(i'_1,\ldots,i'_d)} \neq 0 \), by Proposition 4.5 there exists a subset \( X \subset \Sigma^n \) with density at most \( \epsilon/2 \) in \( \Sigma^n \) and such that for all \( x^1 \in \Sigma^n \setminus X \), the \((d-1)\)-linear form

\[
(y_2,\ldots,y_d) \mapsto \sum_{i_1=1}^k a_{(i_1,i'_2,\ldots,i'_d)} m_{(i_1,i'_2,\ldots,i'_d)}(\pi^n(x^1),y_2,\ldots,y_d)
\]

has partition rank (or rather support size in the case \( d = 2 \)) at least \( B_{d-1,p,k,C_0}(\epsilon/2) \). Let \( t \in \mathbb{F}_p^n \) be fixed. For each \( x^1 \in \Sigma^n \setminus X \), applying Proposition 4.5 for \( d-1 \) to the \((d-1)\)-linear forms

\[
(y_2,\ldots,y_d) \mapsto \sum_{i_1=1}^k a_{(i_1,i_2,\ldots,i_d)} m_{(i_1,i_2,\ldots,i_d)}(\pi^n(x^1),y_2,\ldots,y_d)
\]

with \((i_2,\ldots,i_d) \in [k]^{d-1}\) we get

\[|\mathbb{E}_{(x_2,\ldots,x_d) \in (\Sigma^n)^{d-1}} \omega_p^{a.m^n(x^1,x_2,\ldots,x_d)}| \leq \epsilon/2.
\]

Because \( X \) has density at most \( \epsilon/2 \) in \( \Sigma^n \) we conclude that \(|\text{bias}_t a.m^n| \leq \epsilon/2 + \epsilon/2 = \epsilon\).

**5 The general polynomial case**

Let \( d \geq 1 \) be a positive integer and let \( P \) be a polynomial of degree exactly \( d \). For a given number \( d' \) of pairwise distinct variables and for a given total degree \( t \in \{d',\ldots,d\} \), let \( S(d',t) \) be the set of \( d' \)-tuples of positive integers \((s_1,\ldots,s_{d'})\) with \( s_1 \geq s_2 \geq \cdots \geq s_{d'} \geq 1 \) and \( s_1 + \cdots + s_{d'} = t \). We can decompose

\[
P = \sum_{d'=0}^d \sum_{t=d'}^{d} \sum_{s \in S(d',t)} P_s
\]

where \( P_s \) is the part of \( P \) that consists of monomials of the type \( x_1^{s_1} \cdots x_{i_{d'}}^{s_{d'}} \) with \( x_{i_1},\ldots,x_{i_{d'}} \) distinct. We make the following definition.
Definition 5.1. The essential rank of a part $P_s$, denoted by $\text{erk} \ P_s$, is $\min_Q \text{rk}(P_s - V_s)$, where the minimum is taken over all parts $P_s$ that are linear combinations of monomials $x_{i_1}^{s_1} \ldots x_{i_{d'}}^{s_{d'}}$, for which $x_{i_1}, \ldots, x_{i_{d'}}$ are not all distinct.

If $s = (s_1, \ldots, s_{d'})$, then each $P_s$ can be written in the form

$$\sum_{i_1, \ldots, i_{d'}} a_{i_1, \ldots, i_{d'}} x_{i_1}^{s_1} \ldots x_{i_{d'}}^{s_{d'}}$$

with $a_{i_1, \ldots, i_{d'}} = 0$ unless $i_1, \ldots, i_{d'}$ are distinct. Let us partition the set $[d']$ into sets $I_1, \ldots, I_r$ according to the value of $s_i$. Then for any permutation $\sigma$ of $[d']$ that leaves the sets $I_j$ invariant we have that $x_{i_1}^{s_1} \ldots x_{i_{d'}}^{s_{d'}} = x_{i_{\sigma(1)}}^{s_{\sigma(1)}} \ldots x_{i_{\sigma(d')}}^{s_{\sigma(d')}}$ so if we replace each coefficient $a_{i_1, \ldots, i_{d'}}$ by the average of the coefficients $a_{s_{\sigma(1)}, \ldots, s_{\sigma(d')}}$ over all such permutations, we obtain the same polynomial $P_s$, and now the coefficients have the symmetry property that $a_{i_1, \ldots, i_{d'}} = a_{s_{\sigma(1)}, \ldots, s_{\sigma(d')}}$ whenever $\sigma$ is such a permutation. We therefore have a representation of $P_s$ in the form

$$P_s(x) = m_s(x^{s_1}, \ldots, x^{s_{d'}}),$$

where $m_s$ is a $d$-linear form that is symmetric under all permutations of the variables that leave the sets $I_j$ invariant, and if $x = (x_1, \ldots, x_n)$, then we write $x^{s_i}$ for the vector $(x_1^{s_i}, \ldots, x_n^{s_i})$. (It is not hard to show that this multilinear form is unique, using the fact that a non-zero polynomial of degree less than $p$ over $\mathbb{F}_p$ must take non-zero values, but we shall not need this.)

Lemma 5.2. Let $s = (s_1, \ldots, s_{d'})$ and suppose that $P_s \neq 0$. Then $\text{erk} \ P_s \leq \text{epr} \ m_s$.

Proof. Assume that $\text{epr} \ m_s \leq k$ for some nonnegative integer $k$. Then there exists a $D$-linear form $m' : (\mathbb{F}_p^n)^D \to \mathbb{F}_p$ such that the coefficient $m'_{(i_1, \ldots, i_D)} = 0$ whenever $i_1, \ldots, i_D \in [n]$ are distinct, and such that $\text{pr}(m_s - m') \leq k$. Then

$$(m_s - m')(y^{s_1}, \ldots, y^{s_{d'}}) = m_s(y^{s_1}, \ldots, y^{s_{d'}}) - m'(y^{s_1}, \ldots, y^{s_{d'}})$$

for all $y \in \mathbb{F}_p^n$. The first term of the right-hand side is equal to $P_s(y)$, by the choice of $m_s$. The second term $m'(y^{s_1}, \ldots, y^{s_{d'}})$ is a linear combination of monomials of the type $y_{i_1}^{s_1} \ldots y_{i_{d'}}^{s_{d'}}$ with $i_1, \ldots, i_{d'}$ not distinct, so we can write it as $V_s(y)$ for some polynomial $V_s$ spanned by these monomials. It follows that $\text{erk} \ P_s \leq \text{rk}(P_s - V_s)$.

Because $\text{pr}(m_s - m') \leq k$, for each $i \in [k]$ there exist a bipartition $\{J'_i, J''_i\}$ of $[D]$ with $J'_i, J''_i$ both non-empty and multilinear forms $M_{i,1} : (\mathbb{F}_p^n)^{J'_i} \to \mathbb{F}_p$, $M_{i,2} : (\mathbb{F}_p^n)^{J''_i} \to \mathbb{F}_p$ such that

$$(m_s - m') : (z_1, \ldots, z_D) \mapsto \sum_{i=1}^{k} M_{i,1}(z(J'_i))M_{i,2}(z(J''_i)).$$

(Note that here $z_1, \ldots, z_D$ are $D$ elements of $\mathbb{F}_p^n$ and $z = (z_1, \ldots, z_D)$ is an element of $(\mathbb{F}_p^n)^D$.) For each $i \in [k]$ let the polynomials $Q_i, R_i$ be defined by

$$Q_i(y) = M_{i,1}(y^{s}(J'_i))$$

and
and
\[ R_i(y) = M_{i,s}(y^s(J_i^d)), \]
where we write \( y^s \) as shorthand for \( (y^{s_1}, \ldots, y^{s_D}) \). Then for every \( y \in \mathbb{F}_p^n \) we have
\[ (P_s - V_s)(y) = \sum_{i=1}^k Q_i(y) R_i(y). \]

Because \( s_1, \ldots, s_D \geq 1 \), for any strict subset \( J \) of \([D]\) we have \( \sum_{j \in J} s_j < \sum_{j=1}^D s_j \), so \( \deg Q_i, \deg R_i < \deg P \) for each \( i \in [k] \). Therefore, \( \text{rk} P \leq k \).

Before starting the proof we note the following simple reduction for polynomials defined on restricted alphabets, which we shall use repeatedly.

**Lemma 5.3.** Let \( p \) be a prime, let \( 1 \leq d \leq p - 1 \) be a positive integer and let \( S \) be a non-empty finite subset of \( \mathbb{F}_p \). If \( P \) is a polynomial of degree \( d \) then \( P \) coincides on \( S^n \) with a linear combination of monomials of the type \( \prod_i x_i^{s_i} \) with \( s_i \leq |S| - 1 \) for all \( i \in [n] \).

**Proof.** Whenever a monomial \( \prod_{i=1}^n x_i^{s_i} \) contains a power \( x_i^{s_i} \) with \( s_i \geq |S| \) we can rewrite \( x_i^{s_i} \) as a linear combination of the \( x_i^{s_i'} \) with \( s_i' < s_i \), and hence rewrite the monomial. Each time a replacement is performed the difference between the previous monomial and the new monomial only takes the value 0 on \( S^n \). After all replacements we obtain a polynomial \( P - P_0 \) which is spanned by the monomials \( \prod_i x_i^{s_i} \) with \( s_i \leq (|S|) - 1 \) for all \( i \in [n] \) and which coincides with \( P \) on \( S^n \).

We note further that when we initially replace each monomial individually, every monomial from the new polynomial involves at most as many distinct variables as the monomial in the original polynomial did, and has degree at most that of the original monomial.

We are now ready to prove our main theorem.

**Proof of Theorem 1.4.** Let \( \epsilon > 0 \) and let \( P \) be a polynomial of degree \( d \) such that \( \text{bias}_{t,S} P \geq \epsilon \) for some \( t \in \mathbb{F}_p^n \).

We first apply Lemma 5.3 to \( P \), and, still writing \( P \) for the resulting polynomial, we decompose \( P \) into its pieces as in (13), fix \( D \) to be the highest value of \( d' \in [d] \) such that there exist \( t' \in \{d', \ldots, d\}, s \in S(d', t') \) with \( P_s \neq 0 \), and set \( T \) to be the largest such value of \( t' \in \{d', \ldots, d\} \). That is, \( D \) is the largest number of variables involved in a monomial of \( P \), and \( T \) is the largest degree of a monomial that involves that number of variables (which is necessarily at least \( D \) but may be less than \( d \)).

We prove the result by a double induction: the outer induction takes place on the degree \( d \), and for a fixed \( d \), we will use an inner induction with respect to the lexicographic order on the pairs \( (D,T) \) with \( 1 \leq D \leq T \leq d \). We will construct functions \( (H_{p,d,S})_{(D,T)} : (0,1] \rightarrow [0,\infty) \) such that if the relevant pair for \( P \) is at most \( (D,T) \) for this order and \( \text{bias}_{t,S} P \geq \epsilon \) for some \( t \in \mathbb{F}_p^n \) and some \( \epsilon > 0 \) then \( \text{rk}_S P \leq (H_{p,d,S})_{(D,T)}(\epsilon) \).
The base case of the induction is the case where \( D = 1 \): We can write \( P = \sum_{i=1}^{n} P_i(x_i) \) for some polynomials \( P_i \) with degree at most \( d \), and the result follows from the remark just before Proposition 4.5 with \((H_{p,d,S})_{\leq 1}(T) = p^3 \log \epsilon^{-1}/\pi^2 + 1\) for all \( T \in [d] \).

Now let \( D \geq 2 \). We distinguish two cases. Let

\[
\kappa_{p,D,d,S}(\epsilon) = \Lambda_D(B_{D,p,d,|S|^2}(\epsilon^{2^D})).
\]

Here \( \Lambda_D \) is the function coming from Theorem 1.10 if an order-\( D \) tensor has essential partition rank at least \( \Lambda_D(l) \) then it has disjoint partition rank at least \( l \). As for \( B_{D,p,d,|S|^2} \), it comes from Proposition 4.6, the main result of the previous section. If there exists a polynomial \( P_0 : \mathbb{F}_p^n \to \mathbb{F}_p \) such that \( P_0(S^n) = \{0\} \) and \( \deg(P - P_0) < \deg P \) then we can conclude by the outer inductive hypothesis on \( \deg P \), so we may assume without loss of generality that

\[
\deg(P - P_0) \geq \deg P
\]

for every polynomial \( P_0 : \mathbb{F}_p^n \to \mathbb{F}_p \) such that \( P_0(S^n) = \{0\} \).

**Case 1.** For each \( s \in S(D,T) \), we have \( \text{erk} P_s \leq \kappa_{p,D,d,S}(\epsilon) \), and we can hence write

\[
P_s = V_s + \sum_{i=1}^{\kappa_{p,D,d,S}(\epsilon)} Q_{s,i} R_{s,i}
\]

where \( V_s \) is as in the definition of essential rank (Definition 5.1), and for each \( i \in [\kappa_{p,D,d,S}(\epsilon)] \), \( \deg Q_{s,i}, \deg R_{s,i} \leq T - 1 \). Moreover we can require that for each \( i \in [\kappa_{p,D,d,S}(\epsilon)] \), all monomials of the polynomials \( Q_{s,i} \) and \( R_{s,i} \) involve at most \( D \) pairwise distinct variables: if one of these polynomials, say \( Q_{s,i} \), contains a monomial with at least \( D + 1 \) variables, then all the contributions of this monomial to \( \sum_{i=1}^{\kappa_{p,D,d,S}(\epsilon)} Q_{s,i} R_{s,i} \) necessarily have to be cancelled by contributions from other \( Q_{s,i'} R_{s,i'} \) with \( i' \neq i \), as multiplication by any monomial other than 0 cannot decrease the number of pairwise distinct variables in a monomial. Let

\[
P_{\text{new}} = P - \sum_{s \in S(D,T)} (P_s - V_s) = \sum_{s \in S(D,T)} V_s + \sum_{0 \leq d' \leq D, d' \leq d} \sum_{s \in S(d',T')} P_s.
\]

Since \( P_{\text{new}} + \sum_{s \in S(D,T)} \sum_{i=1}^{\kappa_{p,D,d,S}(\epsilon)} Q_{s,i} R_{s,i} \), and since \( |\text{bias}_{s,S} P| \geq \epsilon \), Lemma 2.10 implies that there exist \( a_{s,i}, b_{s,i} \in \mathbb{F}_p \) for each \( s \in S(D,T) \) and each \( 1 \leq i \leq \kappa_{p,D,d,S}(\epsilon) \), such that the bias with respect to \( t \) of the polynomial

\[
P'_{\text{new}} = P_{\text{new}} + \sum_{s \in S(D,T)} \sum_{i=1}^{\kappa_{p,D,d,S}(\epsilon)} (a_{s,i} Q_{s,i} + b_{s,i} R_{s,i})
\]

is at least \( p^{-2\kappa_{p,D,d,S}(\epsilon)} \epsilon \). The polynomial \( P'_{\text{new}} \) has the two following properties.

1. Each of its monomials has degree at most \( T \) and also involves at most \( D \) distinct variables.
2. Each of its monomials has degree at most $T - 1$, or involves at most $D - 1$ distinct variables.

The second property follows from the second expression for $P_{\text{new}}$ given above, together with the fact that the monomials in $V_s$ involve fewer than $D$ distinct variables, and the fact that the monomials $Q_{s,i}$ and $R_{s,i}$ have degree less than $T$.

These two properties ensure that the pair $(D', T')$ associated with the polynomial $P'_{\text{new}}$ is less than $(D, T)$ in lexicographical order, which will allow us to apply the inductive hypothesis. First, however, we apply Lemma 5.3 to $P'_{\text{new}}$, obtaining a polynomial $P''_{\text{new}}$ such that every monomial $\prod_{i=1}^{n} x_i^{s_i}$ of $P''_{\text{new}}$ satisfies $s_i \leq |S| - 1$ for all $i \in [n]$, and such that the two properties above are still satisfied. Now using the inductive hypotheses we deduce that

\[
\text{rk}_S P''_{\text{new}} \leq H_{p,d,S}(p^{-(2\kappa_{p,D,d,S}(\epsilon)+1)}\epsilon) \quad \text{if } \deg P''_{\text{new}} < d
\]

\[
\text{rk}_S P''_{\text{new}} \leq (H_{p,d,S})_{(D,T-1)}(p^{-(2\kappa_{p,D,d,S}(\epsilon)+1)}\epsilon) \quad \text{if } \deg P''_{\text{new}} = d \text{ and } T > D
\]

\[
\text{rk}_S P''_{\text{new}} \leq (H_{p,d,S})_{(D-1,d)}(p^{-(2\kappa_{p,D,d,S}(\epsilon)+1)}\epsilon) \quad \text{if } \deg P''_{\text{new}} = d \text{ and } T = D.
\]

Using our assumption (15), the fact that $P$ coincides with $P''_{\text{new}} + (P - P_{\text{new}}) + (P_{\text{new}} - P'_{\text{new}})$ on $S^n$ (since $P'_{\text{new}}$ and $P''_{\text{new}}$ agree on $S^n$), the decomposition

\[
P - P_{\text{new}} = \sum_{s \in S(D,T)} \sum_{i=1}^{\kappa_{p,D,d,S}(\epsilon)} Q_{s,i}R_{s,i}
\]

and the fact that $P'_{\text{new}} - P_{\text{new}}$ is a linear combination of polynomials of degree strictly smaller than $\deg P$ we have

\[
\text{rk}_S P \leq (\text{rk}_S P''_{\text{new}}) + |S(D,T)|\kappa_{p,D,d,S}(\epsilon) + 1.
\]

It follows that $\text{rk}_S P \leq (H_{p,d,S})_{(D,S)}(\epsilon)$ with

\[
(H_{p,d,S})_{(D,S)} = \max \left\{ H_{p,d-1,S}(p^{-(2\kappa_{p,D,d,S}(\epsilon)+1)}\epsilon), (H_{p,d,S})_{(D,T-1)}(p^{-(2\kappa_{p,D,d,S}(\epsilon)+1)}\epsilon), (H_{p,d,S})_{(D-1,d)}(p^{-(2\kappa_{p,D,d,S}(\epsilon)+1)}\epsilon) \right\} + |S(D,T)|\kappa_{p,D,d,S}(\epsilon) + 1. \quad (16)
\]

This concludes Case 1.

We define the desired function $H_{p,d,S}$ to be $(H_{p,d,S})_{(d,d)}$. As we shall show, Case 2 will lead to a contradiction and this function is therefore suitable for Theorem 1.4.

Case 2. There exists $s_0 \in S(D, T)$ such that

\[
erk P_{s_0} \geq \Lambda_D(B_{D,p,d,|S|^2}(\epsilon^{2D})).
\]
We start with the decomposition

\[ P = \sum_{d' = 0}^{D} \sum_{t' = d'}^{d} \sum_{s \in S(d', t')} P_s. \]  

(17)

For each \( t' \in \{ D, \ldots, d \} \) and \( s \in S(D, t') \) let \( m_s \) be a \( D \)-linear form of the form (14). That is, \( P_s(x) = m_s(x^{s_1}, \ldots, x^{s_D}) \), and \( m \) is symmetric in \( i \) and \( j \) whenever \( s_i = s_j \).

By our assumption on \( s_0 \) and by Claim 5.2, we have

\[ \text{epr } m_{s_0} \geq \Lambda_D(B_{D,p,d,|S|^{2}}(\epsilon^{2D})). \]

By Theorem 1.10 applied to \( m_{s_0} \), there exist disjoint subsets \( X_1, \ldots, X_D \subset [n] \) such that

\[ \text{pr } m_{s_0}(\mathbb{F}^{X_1} \times \cdots \times \mathbb{F}^{X_D}) \geq B_{D,p,d,|S|^{2}}(\epsilon^{2D}). \]

We now apply the argument from Proposition 3.7. Although we no longer obtain the inequality (10) (as we only know that all monomials of \( P \) involve at most \( D \) pairwise distinct variables rather than that all monomials of \( P \) have degree at most \( D \)), using Proposition 3.6 and following the first half of the proof of Proposition 3.7 shows that

\[ \sum_{\nu \in \{-1, 1\}^D} N(\nu) P_{\{X_1, \ldots, X_D\}}(y_{\nu_1}(x_1), \ldots, y_{\nu_D}(x_D)) \leq \sum_{s \in E} \sum_{t' = D}^{d} \sum_{s \in S(D, t')} (P_s)_{\{X_1, \ldots, X_D\}} \]

(18)

\[ \sum_{t' = D}^{d} \sum_{s \in S(D, t')} (P_s)_{\{X_1, \ldots, X_D\}} \]

(19)

as the contribution of the terms from (17) obtained from \( d' < D \) is zero. For a fixed \( s \), we define an equivalence relation on \( S_D \), the set of permutations of \([D]\), by taking two permutations \( \sigma_1, \sigma_2 \) to be equivalent if and only if \( \sigma_2\sigma_1^{-1} \) leaves the intervals \( I_j \) (the intervals on which \( s \) is constant) invariant. In other words, \( \sigma_1 \) and \( \sigma_2 \) are equivalent if the sequences \((s_{\sigma_1(1)}, \ldots, s_{\sigma_1(D)})\) and \((s_{\sigma_2(1)}, \ldots, s_{\sigma_2(D)})\) are equal. Let \( E \) be the set of equivalence classes for this relation, and for each equivalence class \( E \in E\) let us pick a representative \( \sigma_E \in E\). For each \( t' \in \{ D, \ldots, d \} \), \( s \in S(D, t') \) and \( y \in \mathbb{F}^n_p \), we have that \( (P_s)_{\{X_1, \ldots, X_D\}}(y) \) is
equal to
\[
\sum_{\sigma \in \mathcal{S}_D} m_s(\mathbb{F}^{X_{\pi(1)}} \times \cdots \times \mathbb{F}^{X_{\pi(D)}})(y(X_{\sigma(1)})^{s_1}, \ldots, y(X_{\sigma(D)})^{s_D})
\]
\[
= \sum_{\sigma \in \mathcal{S}_D} m_s(\mathbb{F}^{X_1} \times \cdots \times \mathbb{F}^{X_D})(y(X_1)^{s_{\sigma(1)}}, \ldots, y(X_D)^{s_{\sigma(D)}})
\]
\[
= \sum_{\sigma \in \mathcal{S}_D} \sum_{E \in \mathcal{E}_s} m_s(\mathbb{F}^{X_1} \times \cdots \times \mathbb{F}^{X_D})(y(X_1)^{s_{\sigma(1)}}, \ldots, y(X_D)^{s_{\sigma(D)}})
\]
\[
= |I_1(s)|! \ldots |I_r(s)|! \sum_{E \in \mathcal{E}_s} m_s(\mathbb{F}^{X_1} \times \cdots \times \mathbb{F}^{X_D})(y(X_1)^{s_{E(1)}}, \ldots, y(X_D)^{s_{E(D)}})
\]

using the symmetry of \( m_s \). For all \( y \in \mathbb{F}_p^n \), using \([19]\) we obtain that \( P_{\{X_1, \ldots, X_D\}}(y) \) is equal to
\[
\sum_{t'=D}^{d} \sum_{s \in \mathcal{S}(D,t')} |I_1(s)|! \ldots |I_r(s)|! \sum_{E \in \mathcal{E}_s} m_s(\mathbb{F}^{X_1} \times \cdots \times \mathbb{F}^{X_D})(y(X_1)^{s_{E(1)}}, \ldots, y(X_D)^{s_{E(D)}}) \quad (20)
\]
and the exponent on the right-hand side of \([18]\) can therefore be rewritten
\[
\sum_{t'=D}^{d} \sum_{s \in \mathcal{S}(D,t')} |I_1(s)|! \ldots |I_r(s)|! \sum_{E \in \mathcal{E}_s} (m_s(\mathbb{F}^{X_1} \times \cdots \times \mathbb{F}^{X_D})
\]
\[
(y_1(X_1)^{s_{E(1)}} - y_{-1}(X_1)^{s_{E(1)}}, \ldots, y_1(X_D)^{s_{E(D)}} - y_{-1}(X_D)^{s_{E(D)}})) \quad (21)
\]

We finally apply the main result of the previous section, Proposition \([4.6]\). We apply it to the set \( \Sigma = S^2 \) and to the functions \( \pi_i : \Sigma \to \mathbb{F}_p \) defined by \( \pi_i(x', x'') = (x')^i - (x'')^i \) for \( 1 \leq i \leq d - 1 \). These are linearly independent and do not span a non-zero constant function, as can be seen by fixing \( x'' \), and moreover we have \( \text{pr} m_{s_0} \geq B_{D,p,d,|s|^2} \epsilon^{2D} \) and \( |I_1(s_0)|! \ldots |I_r(s_0)|! \neq 0 \). Therefore, the assumptions of Proposition \([4.6]\) is satisfied. Applying the proposition and \([18]\) then shows that \( |\text{bias}_t P| < \epsilon \) for all \( t \in \mathbb{F}_p^*, \) which is incompatible with our assumption at the start of the proof. This finishes the proof. \( \Box \)

### 6 Surjectivity of multilinear forms on subsets of finite prime fields

Proposition \([1.8]\) states that having a high partition rank is a sufficient condition for a multilinear form \( m \) over \( \mathbb{F}_p \) (of fixed order and for a fixed prime \( p \)) to be equidistributed on a product \( S_1^n \times \cdots \times S_d^n \) with \( S_1, \ldots, S_d \) subsets of \( \mathbb{F}_p \) each containing at least two elements.
In this section we show that to ensure that the restriction of \(m\) to \(S_1^n \times \ldots \times S_d^n\) is surjective, it suffices to fulfill the qualitatively weaker condition that the multilinear form \(m\) has high tensor rank.

**Definition 6.1.** Let \(d \geq 2\) be a positive integer, let \(\mathbb{F}\) be a field, and let \(T: [n_1] \times \cdots \times [n_d] \to \mathbb{F}\). The tensor rank of the tensor \(T\), denoted by \(\text{tr}(T)\), is the smallest nonnegative integer \(k\) such that there exist functions \(a_{i,\alpha}: [n_\alpha] \to \mathbb{F}\) for all \(\alpha \in [d]\) and all \(i \in [k]\) such that we can write

\[
T(x_1, \ldots, x_d) = \sum_{i=1}^{k} a_{i,1}(x_1) \cdots a_{i,d}(x_d)
\]

for every \((x_1, \ldots, x_d) \in [n_1] \times \cdots \times [n_d]\).

We will use the following result which follows from repeatedly applying Proposition 11.4 from [9] (by performing the iterations in a way similar to those of the proof of Corollary 11.8 there). For \(T\) an order-\(d\) tensor, for \(I\) a subset of \([d]\), for \(y \in I^c\) and \(z \in I\) let \(T((y,z))\) be the value \(T(x)\), where \(x_a = y_a\) for all \(\alpha \in I^c\) and \(x_a = z_a\) for all \(\alpha \in I\).

**Proposition 6.2.** Let \(d \geq 2\) be a positive integer, let \(\mathbb{F}\) be a field, let \(T: [n_1] \times \cdots \times [n_d] \to \mathbb{F}\), and let \(l \geq 1\) be a positive integer. If \(\text{pr}T \leq l\) and for every subset \(I\) of \([d]\) with \(2 \leq |I| \leq d-1\) and all \(y \in \prod_{\alpha \in I^c} [n_\alpha]\), the order \(|I|\) slice \(T_y: x(I) \to T((x(I), y))\) has order \((d-|I|)\) partition rank at most \(l\), then \(\text{tr}T \leq (4l^2)^{2^d}\).

We define a sequence \(\Theta(p, d)\) for all \(d \geq 2\) by

\[
\Theta(p, 2) = K_{p,p^{-1},p^{-1}}((2p)^{-1})\quad \text{and for all } d \geq 3, \quad \Theta(p, d) = (4(A_{d,\mathbb{F}_p}(p\Theta(p, d-1))))^{3/2^d},
\]

where \(K_{p,p^{-1},p^{-1}}\) has been defined in Corollary 2.3.

**Proposition 6.3.** Let \(p\) be a prime and let \(d \geq 2\) be a positive integer. There exists \(\Theta(d, p)\) such that whenever \(T: [n_1] \times \cdots \times [n_d] \to \mathbb{F}_p\) is an order-\(d\) tensor such that \(\text{tr}T \geq \Theta(d, p)\), then whenever \(S_1, \ldots, S_d\) are subsets of \(\mathbb{F}_p\) each containing at least two elements, the \(d\)-linear form \(m: (\mathbb{F}_p^n)^d \to \mathbb{F}_p\) associated with \(T\) is surjective.

**Proof.** We prove the result by induction on \(d\). The result holds for \(d = 2\) by Proposition 2.3. We now take \(d \geq 3\), and assume \(\text{tr}T \geq \Theta(p, d)\).

**Case 1:** There exists an order-\((d-1)\) slice of \(T\), which without loss of generality we can assume to be the slice \(T_a: (x_2, \ldots, x_d) \to T(a, x_2, \ldots, x_d)\), with order-\((d-1)\) tensor rank at least \(A_{d,\mathbb{F}_p}(p\Theta(p, d-1)) \geq p\Theta(p, d-1)\).

Because \(S_1\) contains at least two elements, there exist \(b_1, \ldots, b_p, c_1, \ldots, c_p \in S_1\) such that \(b_1 + \cdots + b_p = 1\) and \(c_1 + \cdots + c_p = 0\). Writing the identity

\[
1_a = (b_11_a + c_1(1-1_a)) + \cdots + (b_p1_a + c_p(1-1_a))
\]

between elements of \(\mathbb{F}_p^{n_1}\) and using subadditivity of the tensor rank there exists \(i \in [p]\) such that \(\text{tr}(b_i1_a + c_i(1-1_a)) \geq \Theta(p, d-1)\). Letting \(u\) the element of \(\mathbb{F}_p^{n_1}\) defined by
generally assign $d$ in Theorem 1.4 to obtain that $d$.

Our results still leave open a number of questions. We have used the assumption $d < p$
In Theorem 1.4 to obtain that $d! \neq 0$, which allowed us to assign a unique underlying $d$-linear form to a homogeneous polynomial of degree $d$, and later allowed us to more generally assign $D$-linear forms to the polynomials $P_i$. However it seems likely to us that

\begin{equation}
\begin{aligned}
& u := b_{1_a} + c_i(1 - 1_a), \text{ by the inductive hypothesis we have } m(\{u\} \times S^n_1 \times \cdots \times S^n_d) = \mathbb{F}_p. \\
& \text{By construction } u \in S^n_1, \text{ so in particular we have } m(S^n_1 \times S^n_2 \times \cdots \times S^n_d) = \mathbb{F}_p.
\end{aligned}
\end{equation}

Case 2: We are not in Case 1. Then for every subset $I$ of $[d]$ with $2 \leq |I| \leq d - 1$ and all $y \in \prod_{\alpha \in I} |n_{\alpha}|$,

\[ p r T_y \leq \text{tr} T_y \leq \text{tr} T_{(y_0)} \leq A_{d,p} (p \Theta(p,d - 1)) \]

where $T_{(y_0)}$ is an order $(d - 1)$ slice of $T$ with domain containing the domain of $T_y$.

By our assumption $\text{tr} T \geq \Theta(p,d)$ and Proposition 6.2 we necessarily have $p \text{tr} T \geq A_{d,p} (p \Theta(p,d - 1))$. Therefore, $ar T \geq p \Theta(p,d - 1)$. Defining for each $u \in \mathbb{F}^n_p$ the order $d - 1$ tensor $u.T : [n_2] \times \cdots \times [n_d] \rightarrow \mathbb{F}_p$ by

\[ (u.T)(x_2, \ldots, x_d) = \sum_{x_1=1}^{n_1} u(x_1)T(x_1, x_2, \ldots, x_d) \]

for every $(x_2, \ldots, x_d) \in [n_2] \times \cdots \times [n_d]$ and using that $p^{-ar} T = E_{u \in \mathbb{F}^n_p} p^{-ar} u.T$ we can find $u \in \mathbb{F}^n_p$ such that

\[ \text{tr} u.T \geq pr u.T \geq ar u.T \geq p \Theta(p,d - 1). \]

The $(d - 1)$-linear form associated with the tensor $u.T$ is $(y_2, \ldots, y_d) \mapsto m(u, y_2, \ldots, y_d)$. Because $S$ has size at least 2, every element of $\mathbb{F}_p$ can be written as a sum of at most $p$ elements of $S$, so for each $x_1 \in [n]$ there exist $u_1(x_1), \ldots, u_p(x_1) \in S$ such that we can write $u(x_1) = u_1(x_1) + \cdots + u_p(x_1)$. By subadditivity $\text{tr} u.T \leq \sum_{i=1}^p \text{tr} u_i.T$, so there exists $i \in [p]$ such that $\text{tr} u_i.T \geq \Theta(p,d - 1)$. Since $u_i \in S$ and by the inductive hypothesis we have $m(\{u_i\} \times S^n_2 \times \cdots \times S^n_d) = \mathbb{F}_p$, we conclude that $m(S^n_1 \times S^n_2 \times \cdots \times S^n_d) = \mathbb{F}_p$. \hfill \[ \square \]

In summary we have shown that for a fixed prime $p$, a fixed positive integer $d \geq 2$, and fixed non-empty subsets $S_1, \ldots, S_d$ of $\mathbb{F}_p$, the behaviour of the range and distribution of a $d$-linear form $m : (\mathbb{F}_p)^d \rightarrow \mathbb{F}_p$, is as follows.

1. If $\text{pr} T$ is high (and hence $\text{tr} T$ is high) then $m$ is approximately uniformly distributed on $S^n_1 \times \cdots \times S^n_d$.
2. If $\text{pr} T$ is low but $\text{tr} T$ is high, then $m$ is not necessarily approximately uniformly distributed on $S^n_1 \times \cdots \times S^n_d$ but $m(S^n_1 \times \cdots \times S^n_d) = \mathbb{F}_p$.
3. If $\text{tr} T$ is low (and hence $\text{pr} T$ is low) then the image $m(S^n_1 \times \cdots \times S^n_d)$ is not necessarily the whole of $\mathbb{F}_p$.

7 Open problems

Our results still leave open a number of questions. We have used the assumption $d < p$

Conjecture 7.1. Theorem 1.4 still holds for \( d \geq p \).

We can next ask try to improve our bounds.

Conjecture 7.2. Let \( p \) be a prime, let \( d \geq 2 \) be a positive integer, and let \( S \) be a non-empty subset of \( \mathbb{F}_p \). Then there exists a constant \( C_{p,d,S}^{br} > 0 \) such that for every \( d \)-linear form \( m : (\mathbb{F}_p^n)^d \to \mathbb{F}_p \), if \( \max_{t \in F_p} |\text{bias}_{t,S} m| \geq \epsilon \) then \( \text{pr} m \leq C_{p,d,S}^{br} \log \epsilon^{-1} \).

Conjecture 7.3 specialises in the case \( S = \mathbb{F}_p \) to the well-known conjecture that partition and analytic rank are equal up to a constant, and which was recently established in the large fields case by Cohen and Moshkovitz [2]. As we explained (following Janzer and Miličević) in the introduction, in the case \( S = \mathbb{F}_p \) a bound \( \text{pr} \leq A_{d,p}(ar) \) translates into a bound of the type \( O_{p,d}(1) A_{d,p}(ar) \) in Theorem 1.4. Although the proof of this implication no longer holds for an arbitrary non-empty subset \( S \) of \( \mathbb{F}_p \), it seems at least plausible to us that if Conjecture 7.2 is true then we can take linear bounds in \( \log \epsilon^{-1} \) in Theorem 1.4.

Conjecture 7.3. Let \( p \) be a prime, let \( d \geq 2 \) be a positive integer, and let \( S \) be a non-empty subset of \( \mathbb{F}_p \). Then there exists a constant \( C_{p,d,S}^{rk} > 0 \) such that for every polynomial \( P : \mathbb{F}_p^n \to \mathbb{F}_p \) with \( \deg P = d \), if \( \max_{t \in F_p} |\text{bias}_{t,S} P| \geq \epsilon \) then \( \text{rk} P \leq C_{p,d,S}^{rk} \log \epsilon^{-1} \).

Even if Conjecture 7.2 is true, it would still not guarantee linear bounds in \( \log \epsilon^{-1} \) in Theorem 1.4. It is conjectured in [9] that the disjoint partition rank and essential partition rank are also equal up to a constant. If this were proved then the bounds of Theorem 1.4 would significantly improve, but owing to the inductive structure of our proof of Theorem 1.4 this would still not yield linear bounds in \( \log \epsilon^{-1} \). We therefore expect that arguments significantly different from those that we have used in our inductive proof would be required to prove Conjecture 7.3 if it is true.

In another direction we can ask for qualitative strengthenings of Theorem 1.4; even in the case \( S = \mathbb{F}_p \), having high rank is merely a sufficient, and not a necessary condition for a polynomial to be approximately uniformly distributed. For instance, if \( I, J \) are two disjoint subsets of \([n]\), and \( P_1, P_2 \) are two polynomials both with degree at least 2 such that \( P_1 \) is a polynomial in the variables \( x_i, i \in I \) and \( P_2 \) is a polynomial in the variables \( x_j, j \in J \), then it suffices that the rank of either of the individual polynomials \( P_1, P_2 \) is large for their sum \( P_1 + P_2 \) to be approximately uniformly distributed, but if \( \deg P_1 < \deg P_2 \) and \( \text{rk} P_2 = 1 \), then \( \text{rk} P_1 + P_2 = 2 \).

Question 7.4. Let \( p \) be a prime, let \( d \geq 2 \) be a positive integer, let \( S \) be a non-empty subset of \( \mathbb{F}_p \), and let \( \epsilon > 0 \). Let \( P : \mathbb{F}_p^n \to \mathbb{F}_p \) be a polynomial such that \( \max_{t \in F_p} |\text{bias}_{t,S} P| \geq \epsilon \). Can we describe the structure of \( P \) over and above the fact that there exists a polynomial \( P_0 : \mathbb{F}_p^n \to \mathbb{F}_p \) such that \( P_0(S^n) = \{0\} \) and \( P - P_0 \) has bounded rank?
In fact, to our knowledge not much is known about Question 7.4 even in the case $S = \mathbb{F}_p$. Theorem 1.4 is an analogue of Proposition 1.8 for polynomials, and we can also ask whether we can obtain an analogue of Proposition 6.3 for polynomials.

**Conjecture 7.5.** Let $p$ be a prime, let $d \geq 2$ be a positive integer, and let $S$ be a non-empty subset of $\mathbb{F}_p$. Then there exists a positive integer $R(p,d)$ such that if $P : \mathbb{F}_p^n \to \mathbb{F}_p$ is a polynomial and the restriction of $P$ to $S^n$ is not surjective, then there exists a polynomial $P_0 : \mathbb{F}_p^n \to \mathbb{F}_p$ such that $P_0(S^n) = \{0\}$ and we can write

$$P - P_0 = \sum_{i=1}^{R(p,d)} l_{i,1} \ldots l_{i,d}$$

for some linear forms $l_{i,j} : \mathbb{F}_p^n \to \mathbb{F}_p$.

Proposition 3.7 no longer seems to be of any help for the task of proving Conjecture 7.5, but it is conceivable that some similar decoupling strategy could work to reduce Conjecture 7.5 to Proposition 6.3.

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