The Peakon Limit of the \( N \)-Soliton Solution of the Camassa-Holm Equation

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We show that the analytic \( N \)-soliton solution of the Camassa-Holm (CH) shallow-water model equation converges to the nonanalytic \( N \)-peakon solution of the dispersionless CH equation when the dispersion parameter tends to zero. To demonstrate this, we develop a novel limiting procedure and apply it to the parametric representation for the \( N \)-soliton solution of the CH equation. In the process, we use Jacobi’s formula for determinants as well as various identities among the Hankel determinants to facilitate the asymptotic analysis. We also provide a new representation of the \( N \)-peakon solution in terms of the Hankel determinants.

KEYWORDS: Camassa-Holm equation, soliton, peakon, parametric representation

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1. Introduction

The Camassa-Holm (CH) equation is a model equation describing the unidirectional propagation of nonlinear shallow-water waves over a flat bottom.\(^1\) It may be written in a dimensionless form as

\[
\frac{u_t}{2} + 2\kappa^2 u_x - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx},
\]

where \(u = u(x,t)\) is the fluid velocity, \(\kappa\) is a positive parameter related to the phase velocity of the linear dispersive wave and the subscripts \(t\) and \(x\) appended to \(u\) denote partial differentiation. Like the Korteweg-de Vries equation in shallow-water theory, the CH equation is a completely integrable nonlinear partial differential equation with a rich mathematical structure. Because of this fact, a large number of works have been devoted to the study of the equation from both physical and mathematical points of view. We do not provide a comprehensive review on various consequences established for the equation. Instead, we refer to an excellent survey given in ref. 5). Our main concern here is solutions to the CH equation for both \(\kappa \neq 0\) and \(\kappa = 0\). Although equation (1.1) can be transformed to the corresponding equation with \(\kappa = 0\) by a Galilean transformation \(x' = x + \kappa^2 t, t' = t, u' = u + \kappa^2\), solutions under the same vanishing boundary condition at infinity have quite different characteristics. In fact, in the case of \(\kappa \neq 0\), it typically exhibits analytic multisoliton solutions (the so-called \(N\)-soliton solution with \(N\) being an arbitrary positive integer) like the usual soliton equations.\(^6\)\(^\text{--}\)\(^15\) We also remark that it has singular cusp soliton solutions as well as solutions consisting of an arbitrary number of solitons and cusp solitons.\(^16\)\(^\text{--}\)\(^19\) When \(\kappa = 0\), on the other hand, eq. (1.1) becomes a dispersionless version of the original CH equation. It admits a new kind of solitary waves which have a discontinuous slope at their crest.\(^1\)\(^,\)\(^2\) For this unique feature, they are now termed peakons. Hence, the peakon has a nonanalytic nature unlike the smooth soliton. One of its remarkable properties is that the motion of peakons is described by a finite dimensional completely integrable dynamical system. This fact was used in refs. 20 and 21 to construct the general \(N\)-peakon solution which represents the interaction of \(N\) peakons. The careful inspection of the interaction of peakons reveals that it occurs elastically in pairs and the total effect of the collision is the phase shift whose explicit
formula has been given in a closed form.\textsuperscript{2,9,21} See also a recent paper concerning the detailed investigation of the dynamics of two peakons.\textsuperscript{22} An important issue about the $N$-peakon solution is how one can reduce it from the analytic $N$-soliton solution by taking the singular limit $\kappa \to 0$. Although this problem has received considerable attention, it has been resolved only partially. Indeed, in the case of $N = 1$, the convergence of the single soliton to the single peakon has been demonstrated using the explicit form of the 1-soliton solution.\textsuperscript{2,5} Also, an analysis without recourse to the explicit form of the soliton solution has been carried out in a general context by employing an abstract theory of dynamical systems.\textsuperscript{23,24} However, the treatment of the general $N$-soliton case has remained open. Quite recently, the case $N = 2$ was completed by two different ways. One will be given in ref. 25 while the other is included here as a special version of the general $N$-soliton case.

The purpose of this paper is to demonstrate the convergence of the $N$-soliton solution to the $N$-peakon solution for the general $N$ by using the explicit $N$-soliton solution of the CH equation. In §2, we present a parametric representation for the $N$-soliton solution of the CH equation which offers a relevant form to the subsequent asymptotic analysis. In §3, the convergence of the 1-soliton solution to the 1-peakon solution is exemplified to explain the new idea used in the limiting procedure. In §4, we perform the corresponding limit for the $N$-soliton solution and show that it reproduces the $N$-peakon solution given by Beals et al.\textsuperscript{21} We also obtain a new representation of the $N$-peakon solution in terms of the Hankel determinants. Section 5 is devoted to the concluding remarks. In Appendix A, we give a proof of the formula which enables us to rewrite the tau-functions for the $N$-soliton solution of the CH equation in terms of the Hankel determinants. In Appendix B, we establish various identities among the Hankel determinants which are used effectively to simplify the limiting waveform of the $N$-soliton solution.

2. The $N$-Soliton Solution of the CH Equation

The $N$-soliton solution of the CH equation (1.1) may be represented in a parametric form\textsuperscript{11,15}

\begin{align}
  u(y, t) &= \left( \ln \frac{f_2}{f_1} \right)_t, \quad (2.1a) \\
  x(y, t) &= \frac{y}{\kappa} + \ln \frac{f_2}{f_1} + d, \quad (2.1b)
\end{align}
with

\[
f_1 = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^{N} \mu_i (\xi_i + \phi_i) + \sum_{1 \leq i < j \leq N} \mu_i \mu_j \gamma_{ij} \right], \tag{2.2a}
\]

\[
f_2 = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^{N} \mu_i (\xi_i - \phi_i) + \sum_{1 \leq i < j \leq N} \mu_i \mu_j \gamma_{ij} \right], \tag{2.2b}
\]

where

\[
\xi_i = k_i (y - \kappa c_i t - y_{i0}), \quad c_i = \frac{2\kappa^2}{1 - \kappa^2 k_i^2}, \quad (i = 1, 2, ..., N), \tag{2.3a}
\]

\[
e^{-\phi_i} = \frac{1 - \kappa k_i}{1 + \kappa k_i}, \quad (0 < \kappa k_i < 1), \quad (i = 1, 2, ..., N), \tag{2.3b}
\]

\[
e^{\gamma_{ij}} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad (i, j = 1, 2, ..., N; i \neq j). \tag{2.3c}
\]

Here, \(k_i\) and \(y_{i0}\) are soliton parameters characterizing the amplitude and phase of the \(i\)th soliton, respectively, \(c_i\) is the velocity of the \(i\)th soliton in the \((x, t)\) coordinate system and \(d\) is an integration constant. We assume that \(c_i > 0\) and \(c_i \neq c_j\) for \(i \neq j\) \((i, j = 1, 2, ..., N)\). In (2.2), the notation \(\sum_{\mu=0,1}\) implies the summation over all possible combination of \(\mu_1 = 0, 1, \mu_2 = 0, 1, ..., \mu_N = 0, 1\). The following coordinate transformation \((x, t) \rightarrow (y, t')\) has been introduced to parametrize the \(N\)-soliton solution

\[
dy = rdx - urdt, \quad dt' = dt, \tag{2.4a}
\]

where the variable \(r\) is defined by the relation

\[
r^2 = u - u_{xx} + \kappa^2. \tag{2.4b}
\]

Note in (2.1) that the time variable \(t'\) is identified with the original time variable \(t\) by virtue of (2.4a).

The 1-soliton solution is given by

\[
u(y, t) = \frac{2\kappa^2 c_1 k_1^2}{1 + \kappa^2 k_1^2 + (1 - \kappa^2 k_1^2) \cosh \xi_1}, \tag{2.5a}
\]

\[
x(y, t) = \frac{y}{\kappa} + \ln \left[ \frac{(1 - \kappa k_1)e^{\xi_1} + 1 + \kappa k_1}{((1 + \kappa k_1)e^{\xi_1} + 1 - \kappa k_1)} \right] + d. \tag{2.5b}
\]
It represents a solitary wave travelling to the right with the amplitude \( c_1 - 2\kappa^2 \) and velocity \( c_1 \). The property of the 1-soliton solutions has been explored in detail as well as its peakon limit. See, for example ref. 5. For completeness, we write the explicit 1-peakon solution of equation (1.1) with \( \kappa = 0 \). It reads

\[
    u(x, t) = c_1 e^{-|x - c_1 t - x_{10}|},
\]

where \( x_{10} \) represents the initial position of the peakon. In view of the invariance property of the dispersionless version of the CH equation under the transformation \( x' = x, t' = -t, u' = -u \), it also admits a peakon solution (2.6) with a negative amplitude. It represents a peakon with depression propagating to the left.

3. The Peakon Limit of the 1-Soliton Solution

We first consider the limiting procedure for the 1-soliton solution. This will be helpful to understand the basic idea in developing the procedure for the general \( N \)-soliton solution. We start with the tau-functions for the 1-soliton solution which are the most important constituents in our analysis. These are given by (2.2) with \( N = 1 \). Before proceeding to the limit, we find it convenient to shift the phase constant \( y_{10} \) as \( y_{10} \rightarrow y_{10} + \phi_1/k_1 \) or in terms of the phase variable \( \xi_1 \rightarrow \xi_1 - \phi_1 \) so that

\[
    f_1 = 1 + e^{\xi_1}, \quad \text{(3.1a)}
\]

\[
    f_2 = 1 + \nu_1^2 e^{\xi_1}, \quad \text{(3.1b)}
\]

where we have put \( \nu_1 = e^{-\phi_1} \). The limit \( \kappa \rightarrow 0 \) is taken in such a way that the amplitudes of the soliton and peakon given respectively by \( c_1 - 2\kappa^2 \) and \( c_1 \) coincide and remain finite.\(^5\) This can be attained by taking the limit \( \kappa \rightarrow 0 \) with the soliton velocity \( c_1 \) in the \((x, t)\) coordinate system being fixed. We see from (2.3a) that the appropriate limit of the wavenumber \( k_1 \) is carried out by taking \( \kappa k_1 \rightarrow 1 \). The limiting procedure described here may be called the peakon limit. It now follows from (2.3) that various wave parameters have the following leading-order asymptotics in the peakon limit

\[
    \kappa k_1 \sim 1 - \frac{\kappa^2}{c_1}, \quad \nu_1 \sim \frac{\kappa^2}{2c_1} = \frac{\lambda_1}{4}\kappa^2, \quad \text{(3.2a)}
\]
\[ e^{\xi_1} = e^{\kappa_1\left(\frac{y}{c_1} - c_1 t - x_{10}\right)} \sim \frac{f_1}{f_2} e^{x_c - c_1 t - x_{10}}, \quad (3.2b) \]

where \( \lambda_1 = 2/c_1 \) and \( x_{10} = y_{10}/\kappa \). In passing to the last line of (3.2b), we have used (2.1b) to eliminate the \( y \) variable and the constant \( d \) has been absorbed in the phase constant \( x_{10} \). Substituting (3.2b) into (3.1), the leading terms of \( f_1 \) and \( f_2 \) are found to be as

\[ f_1 \sim 1 + \frac{f_1}{f_2} z_1, \quad (3.3a) \]

\[ f_2 \sim 1 + \epsilon^2 \lambda_1^2 \frac{f_1}{f_2} z_1, \quad (3.3b) \]

where we have put

\[ z_1 = e^{x_c - c_1 t - x_{10}}, \quad \epsilon = \frac{\kappa^2}{4}, \quad (3.3c) \]

for simplicity. If we introduce the new quantity \( f \) by \( f = f_2/f_1 \), we can deduce from (3.3) that

\[ f \sim \frac{f + \epsilon^2 \lambda_1^2 z_1}{f + z_1}. \quad (3.4) \]

The expression (3.4) yields the following quadratic equation for \( f \):

\[ f^2 + (z_1 - 1)f - \epsilon^2 \lambda_1^2 z_1 + O(\epsilon^3) = 0. \quad (3.5) \]

If we solve this equation, we can express \( f \) as a function of \( x \) and \( t \). At this instant, it is crucial to observe that \( f > 0 \) since both \( f_1 \) and \( f_2 \) are positive quantities. A positive solution is then substituted into (2.1a) to obtain \( u \). In performing the differentiation, however, we must replace the \( t \) derivative \( \partial/\partial t \) by \( \partial/\partial t + u \partial/\partial x \) in accordance with the coordinate transformation (2.4a). With this notice in mind, we can rewrite (2.1a) as

\[ u = \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) \ln f. \quad (3.6) \]

Solving (3.6) with respect to \( u \) to express it in terms of a single variable \( f \) and its derivatives, we have

\[ u = \frac{f_t}{f - f_x}. \quad (3.7) \]

The derivatives \( f_t \) and \( f_x \) in (3.7) are obtained simply if one differentiates (3.5) by \( t \) and \( x \), respectively. To be more specific, as \( \epsilon \to 0 \)

\[ f_t \sim \frac{c_1 z_1 f - \epsilon^2 c_1 \lambda_1^2 z_1}{2f + z_1 - 1}, \quad (3.8a) \]
\[ f_x \sim -z_1 f + \epsilon^2 \lambda_1^2 z_1 \left( \frac{2f + z_1 - 1}{f + z_1} \right), \]  

where we have used the relations \( z_{1,t} = -c_1 z_1 \) and \( z_{1,x} = z_1 \) which are derived from (3.3c).

After a manipulation using (3.5) and (3.8), we arrive at the expression of \( u \) in terms of \( f \):

\[ u \sim -c_1 f + c_1 \left( \frac{f}{f + z_1} \right). \]  

(3.9)

The final step of the limiting process is to solve (3.5) under the condition \( f > 0 \) and then take the limit \( \kappa \to 0 \) after substituting a positive solution into (3.9). Although the analytical expression is obtained for \( f \) by quadrature, we need only the series solution. This fact is crucial in developing the peakon limit of the general \( N \)-soliton solution where the equation corresponding to (3.5) becomes an algebraic equation of degree \( N + 1 \) whose analytical solution is in general not available.

Now, we expand \( f \) in powers of \( \epsilon \) as

\[ f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + ..., \]  

(3.10)

and insert this expression into (3.5). Comparing the coefficients of \( \epsilon^n (n = 0, 1, ...) \), we obtain a system of algebraic equations for \( f^{(n)} \), the first two of which read

\[ f^{(0)}^2 + (z_1 - 1)f^{(0)} = 0, \]  

(3.11a)

\[ 2f^{(0)}f^{(1)} + (z_1 - 1)f^{(1)} = 0, \]  

(3.11b)

\[ (z_1 - 1)f^{(2)} - \lambda_1^2 z_1 = 0. \]  

(3.11c)

The above equations can be solved immediately to obtain positive solutions. Indeed, if \( z_1 \leq 1 \ (x - c_1 t - x_{10} \leq 0) \), then

\[ f \sim f^{(0)} = 1 - z_1. \]  

(3.12a)

Substitution of this result into (3.9) yields

\[ u \sim c_1 z_1 = c_1 e^{x - c_1 t - x_{10}}. \]  

(3.12b)

If, on the other hand, \( z_1 > 1 \) (or \( x - c_1 t - x_{10} > 0 \), then

\[ f^{(0)} = f^{(1)} = 0, \]  

\[ f \sim f^{(2)} \epsilon^2 = \lambda_1^2 z_1 / (z_1 - 1) \epsilon^2, \]  

(3.13a)
\[ u \sim c_1 z_1^{-1} = c_1 e^{-(x-c_1 t-x_{10})}. \]  \hfill (3.13b)

It follows from (3.12) and (3.13) that in the limit \( \kappa \to 0 \) (or equivalently \( \epsilon \to 0 \) by (3.3c)) \( u \) has a limiting waveform
\[ u = c_1 e^{-|x-c_1 t-x_{10}|}. \]  \hfill (3.14)

This expression is just the 1-peakon solution (2.6) of the CH equation with \( \kappa = 0 \).

4. The Peakon Limit of the \( N \)-Soliton Solution

4.1 Formulas for determinants

The peakon limit of the general \( N \)-soliton solution can be taken along the lines of the 1-soliton case. However, the calculation involved is quite formidable. To perform the calculation in an effective manner, we first define the following determinants which are closely related to the \( N \)-peakon solution:

\[ \Delta_n(i_1, i_2, \ldots, i_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{i_1} & \lambda_{i_2} & \cdots & \lambda_{i_n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{i_1}^{n-1} & \lambda_{i_2}^{n-1} & \cdots & \lambda_{i_n}^{n-1} \end{vmatrix}^2 \]

\[ = \prod_{1 \leq l < m \leq n} (\lambda_{i_l} - \lambda_{i_m})^2, \quad (n \geq 2), \]  \hfill (4.1)

\[ D_{(m)}^n = \begin{vmatrix} A_m & A_{m+1} & \cdots & A_{m+n-1} \\ A_{m+1} & A_{m+2} & \cdots & A_{m+n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m+n-1} & A_{m+n} & \cdots & A_{2(n-1)+m} \end{vmatrix}. \]  \hfill (4.2a)

Here

\[ A_m = \sum_{i=1}^{N} \lambda_i^m E_i, \]  \hfill (4.2b)

\[ E_i = e^{\frac{2}{\kappa} x_{i0},} \quad x_{i0} = \frac{y_{i0}}{\kappa}, \quad \lambda_i = \frac{2}{c_i}, \quad (i = 1, 2, \ldots, N), \]  \hfill (4.2c)

where in (4.2) \( m \) is an arbitrary integer and \( n \) is a nonnegative integer less than or equal to \( N \). For \( n \) greater than \( N \), \( D_{(m)}^n = 0 \). We use the convention \( \Delta_1(i_1) = 1, D_0^{(m)} = 1, D_1^{(m)} = A_m \). The quantity \( \Delta_n \) is the square of the Vandermonde determinant whereas \( D_{(m)}^n \) is the determinant of a symmetric matrix. It is a Hankel determinant. We use some properties of Hankelians in the following analysis.
Let $D_n^{(m)}(i_1, i_2, ..., i_p; j_1, j_2, ..., j_q)$ ($i_1 < i_2 < ... < i_p$, $j_1 < j_2 < ... < j_q$, $1 \leq p, q < N$) be a determinant which is obtained from $D_n^{(m)}$ by deleting rows $i_i, i_2, ..., i_p$ and columns $j_1, j_2, ..., j_q$, respectively. Then, the following Jacobi formula holds which will play a central role in the present analysis:

$$D_{n+2}^{(m)}(1, n+2; 1, n+2) = D_{n+2}^{(m)}(1; 1) D_{n+2}^{(m)}(n+2; n+2) - D_{n+2}^{(m)}(1; n+2) D_{n+2}^{(m)}(n+2; 1).$$

(4.3)

By virtue of the definition (4.2), we see that

$$D_{n+2}^{(m)}(1; 1) = D_{n+1}^{(m+2)},$$

(4.4a)

$$D_{n+2}^{(m)}(1; 1) = D_{n+1}^{(m+2)},$$

(4.4b)

$$D_{n+2}^{(m)}(n+2; n+2) = D_{n+1}^{(m)},$$

(4.4c)

$$D_{n+2}^{(m)}(1; n+2) = D_{n+2}^{(m)}(n+2; 1) = D_{n+1}^{(m+1)}.$$  

(4.4d)

Hence, (4.3) can be rewritten in the form

$$D_{n+2}^{(m)} D_{n+2}^{(m+2)} = D_{n+1}^{(m+2)} D_{n+1}^{(m)} - \left(D_{n+1}^{(m+1)}\right)^2.$$  

(4.5)

The determinant $D_n^{(m)}$ has an alternative expression in the form of a finite sum

$$D_n^{(m)} = \sum_{1 \leq i_1 < i_2 < ... < i_n \leq N} \Delta_n(i_1, i_2, ..., i_n)(\lambda_{i_1} \lambda_{i_2} ... \lambda_{i_n})^m E_{i_1} E_{i_2} ... E_{i_n}, \quad (n = 1, 2, ..., N).$$

(4.6)

This formula is very useful in rewriting the tau-functions. Note that $D_n^{(m)}$ ($n = 1, 2, ..., N$) are positive definite since $\Delta_n > 0$ and $E_i > 0$ ($i = 1, 2, ..., N$). We give a simple proof of (4.6) in Appendix A.

4.2 The peakon limit of the N-soliton solution

We first shift the phase variables as $\xi_i \rightarrow \xi_i - \phi_i (i = 1, 2, ..., N)$ in (2.2) and take the peakon limit $\kappa \kappa_i \rightarrow 1$ with fixed $c_i (i = 1, 2, ..., N)$. Using (3.2) and the asymptotic $e^{\gamma_{ij}} \sim (\lambda_i - \lambda_j)^2 \kappa^4 / 16$, the leading-order asymptotics of the tau-functions $f_1$ and $f_2$ can be written in the form

$$f_1 \sim 1 + \sum_{n=1}^{N} e^{n(n-1)} \left(\frac{f_1}{f_2}\right)^n \sum_{1 \leq i_1 < i_2 < ... < i_n \leq N} \Delta_n(i_1, i_2, ..., i_n) z_{i_1} z_{i_2} ... z_{i_n},$$

(4.7a)
\[ f_2 \sim 1 + \sum_{n=1}^{N} \epsilon^{n+1} \left( \frac{f_1}{f_2} \right)^n \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N} (\lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_n})^2 \Delta_n(i_1, i_2, \ldots, i_n) z_{i_1} z_{i_2} \ldots z_{i_n}, \]  

(4.7b)

where

\[ z_i = e^{x - c_i t - x_0} = e^x E_i^{-1}, \quad (i = 1, 2, \ldots, N). \]

(4.7c)

Furthermore, to compare the limiting form of \( u \) resulting from the peakon limit with the \( N \)-peakon solution given by ref. 21), we shift the phase constant appropriately, so that

\[ z_i \rightarrow \frac{\prod_{i=1}^{N} \lambda_i^2}{2 \prod_{i \neq j}^{N} (\lambda_i - \lambda_j)^2 \lambda_i^2} z_i, \quad (i = 1, 2, \ldots, N). \]

(4.8)

We substitute (4.8) into (4.7) and use (4.6) to modify them into the form

\[ f_1 \sim 1 + \sum_{n=1}^{N} \epsilon^{n+1} \left( \frac{f_1}{f_2} \right)^n d_n e^{nx} D^{(2)}_{N-n}, \]

(4.9a)

\[ f_2 \sim 1 + 2 \sum_{n=1}^{N} \epsilon^{n+1} \left( \frac{f_1}{f_2} \right)^n d_{n+1} e^{nx} D^{(0)}_{N-n}, \]

(4.9b)

where the positive coefficients \( d_n \) are defined by

\[ d_n = d_n(t) = \frac{\prod_{i=1}^{N} \lambda_i^{2(n-1)}}{2^n \Delta_N \prod_{i=1}^{N} E_i}, \quad (n = 1, 2, \ldots, N). \]

(4.9c)

Thus, equation corresponding to (3.5) becomes an algebraic equation of degree \( N + 1 \)

\[ \sum_{n=0}^{N+1} \epsilon^{n(n-1)} h_n f^{N-n+1} + O(\epsilon^{N(N+1)+1}) = 0. \]

(4.10a)

Here, as in the 1-soliton case \( f = f_2/f_1 \) and the coefficients \( h_n \) are defined by the relations

\[ h_0 = 1, \]

(4.10b)

\[ h_n = d_n e^{(n-1)x} \left( e^{x} D^{(2)}_{N-n} - 2D^{(0)}_{N-n+1} \right), \quad (n = 1, 2, \ldots, N), \]

(4.10c)

\[ h_{N+1} = -\frac{\prod_{i=1}^{N} \lambda_i^{2N} e^{Nx}}{2^N \Delta_N \prod_{i=1}^{N} E_i}. \]

(4.10d)
To evaluate \( u \) using (3.7), we need the derivatives \( f_t \) and \( f_x \). They are derived simply from (4.10) by differentiation. Explicitly

\[
 f_t \sim -\frac{\sum_{n=1}^{N+1} \epsilon^{n(n-1)} h_{n,t} f^{N-n+1}}{\sum_{n=0}^{N} \epsilon^{n(n-1)}(N - n + 1) f^{N-n}}, \quad (4.11a)
\]

\[
 f_x \sim -\frac{\sum_{n=1}^{N+1} \epsilon^{n(n-1)} h_{n,x} f^{N-n+1}}{\sum_{n=0}^{N} \epsilon^{n(n-1)}(N - n + 1) f^{N-n}}, \quad (4.11b)
\]

It follows from (4.2c) and (4.10) that

\[
 h_{n,x} = (n - 1)h_n + d_n e^{nx} D_n^{(2)} \quad (n = 1, 2, \ldots, N), \quad (4.12a)
\]

\[
 h_{N+1,x} = Nh_{N+1}, \quad (4.12b)
\]

\[
 h_{n,t} = -\left(\sum_{i=1}^{N} c_i\right)h_n + d_n e^{(n-1)x} \left( e^{x} D_n^{(2)} - 2 D_n^{(0)} \right), \quad (n = 1, 2, \ldots, N), \quad (4.13a)
\]

\[
 h_{N+1,t} = -\left(\sum_{i=1}^{N} c_i\right) h_{N+1}. \quad (4.13b)
\]

The expression of \( u \) in terms of \( f \) now follows from (3.7) and (4.10)-(4.13). After an elementary calculation, we find that

\[
 u \sim -\left(\sum_{i=1}^{N} c_i\right) f^N + \sum_{n=1}^{N} \epsilon^{n(n-1)} d_n e^{(n-1)x} \left( e^{x} D_n^{(2)} - 2 D_n^{(0)} \right) f^{N-n} \div f^N + \sum_{n=1}^{N} \epsilon^{n(n-1)} d_n e^{nx} D_n^{(2)} f^{N-n} \quad (4.14)
\]

The unknown \( f \) is obtained by solving the algebraic equation (4.10). The procedure for constructing solution can now be carried out straightforwardly. Here, we summarize the result.

We seek the series solution of the form

\[
 f = f^{(0)} + \epsilon^2 f^{(2)} + \epsilon^4 f^{(4)} + \ldots, \quad (4.15)
\]

under the conditions \( f^{(n)} > 0 \ (n = 0, 2, 4, \ldots) \). Note that the odd powers of \( \epsilon \) have been dropped in the above expansion. An inspection shows that this can be justified since eq. (4.10) includes only the even powers of \( \epsilon \). See also the 1-peakon case in §3. We find
that the leading-order asymptotic of the positive solution of equation (4.10) is expressed simply as
\[ f \sim f^{(2n-2)} \epsilon^{2(n-1)} = -\frac{h_n}{h_{n-1}} \epsilon^{2(n-1)}, \] (4.16a)
if the following inequalities hold for \( h_n \ (n = 1, 2, \ldots, N + 1) \)
\[ h_1 > 0, \ h_2 > 0, \ldots, \ h_{n-1} > 0, \ h_n \leq 0, \ h_{n+1} < 0, \ldots, \ h_{N+1} < 0. \] (4.16b)
To obtain (4.16a), we assume that \( f \) has a leading-order asymptotic of the form \( f \sim \epsilon^{2m} f^{(2m)} \) and substitute this into (4.10). Then, (4.10) is expanded in powers of \( \epsilon \) as
\[ [h_m \{ f^{(2m)} \}^{N-m+1} + h_{m+1} \{ f^{(2m)} \}^{N-m}] \epsilon^{m(2N-m+1)} + O(\epsilon^{m(2N-m+1)+2}) = 0. \] (4.17)
By taking the coefficient of \( \epsilon^{m(2N-m+1)} \) zero, we find that \( f^{(2m)} = -h_{m+1}/h_m \ (h_m \neq 0) \).
According to (4.16b) and the requirement \( f^{(2m)} > 0 \), the integer \( m \) is determined uniquely as \( m = n - 1 \), which immediately leads to (4.16a). Note from (4.10c) that the coordinate \( x = x_n \) giving rise to the equality \( h_n = 0 \) can be specified as
\[ x_n = \ln \left[ \frac{2D^{(0)}_{N-n+1}}{D^{(2)}_{N-n}} \right], \quad (n = 1, 2, \ldots, N), \] (4.18)
and \( h_n \geq 0 \) for \( x \geq x_n \ (n = 1, 2, \ldots, N) \). An important consequence deduced from (4.18) is the notable inequalities
\[ x_{j-1} < x_j, \quad (j = 1, 2, \ldots, N + 1), \] (4.19)
where we have used the convention \( x_0 = -\infty \) and \( x_{N+1} = +\infty \). In fact, we substitute (4.18) into (4.19) to rewrite them in the following alternative forms
\[ D^{(0)}_{N-j+1}D^{(2)}_{N-j+1} - D^{(0)}_{N-j+2}D^{(2)}_{N-j} > 0, \quad (j = 1, 2, \ldots, N + 1). \] (4.20)
The left-hand side of (4.20) equals to \( \left( D^{(1)}_{N-j+1} \right)^2 \) by Jacobi’s identity (4.5) with \( m = 0 \) and \( n = N - j \) and consequently it always has a positive value, which proves (4.20). We see from (4.10c), (4.18) and (4.19) that under the inequalities (4.16b), \( x \) must lie in the interval \( x_{n-1} < x \leq x_n \ (n = 1, 2, \ldots, N + 1) \).
Let $u_n$ be the waveform of $u(x, t)$ for $x$ in the interval $x_{n-1} < x \leq x_n (n = 1, 2, ..., N)$ at any instant $t$. We substitute (4.16) into (4.14) and see that both the denominator and numerator of (4.14) have a leading-order asymptotic of order $\epsilon^{(n-1)(2N-n)}$. Consequently, the limit exists when $\epsilon$ tends to zero. It turns out that $u_n$ has a limiting waveform given by

$$u_n = \frac{G_n}{F_n},$$  \hspace{1cm} (4.21a)

where

$$F_n = -d_{n-1}D_{N-n+1}^{(2)}h_n + d_ne^x D_{N-n}^{(2)}h_{n-1},$$  \hspace{1cm} (4.21b)

$$G_n = d_{n-1}e^{-x} \left( e^x D_{N-n+1,t}^{(2)} - 2D_{N-n+2,t}^{(0)} \right) h_n - d_n \left( e^x D_{N-n,t}^{(2)} - 2D_{N-n+1,t}^{(0)} \right) h_{n-1}. \hspace{1cm} (4.21c)$$

Inserting $h_n$ from (4.10c), $F_n$ becomes

$$F_n = 2d_{n-1}d_ne^{(n-1)x} \left( D_{N-n+1}^{(2)}D_{N-n+1}^{(0)} - D_{N-n}^{(2)}D_{N-n+2}^{(0)} \right). \hspace{1cm} (4.22)$$

We use the formula (4.5) with $m = 0$ and $n$ replaced by $N - n$ in (4.22). Then, $F_n$ simplifies to

$$F_n = 2d_{n-1}d_ne^{(n-1)x} \left[ D_{N-n+1}^{(1)} \right]^2. \hspace{1cm} (4.23)$$

By a similar calculation, $G_n$ is transformed to

$$G_n = 2d_{n-1}d_ne^{(n-1)x} \left[ \frac{e^x}{2} \left\{ D_{N-n+1,t}^{(2)}D_{N-n}^{(2)} - D_{N-n,t}^{(2)}D_{N-n+1}^{(2)} \right\} 
+ D_{N-n,t}^{(2)}D_{N-n+2}^{(0)} + D_{N-n+1,t}^{(2)}D_{N-n+1}^{(2)} - D_{N-n+1,t}^{(2)}D_{N-n+1}^{(0)} - D_{N-n+2,t}^{(2)}D_{N-n}^{(0)} 
+ 2e^{-x} \left\{ D_{N-n+2,t}^{(0)}D_{N-n+1}^{(0)} - D_{N-n+1,t}^{(0)}D_{N-n+2}^{(0)} \right\} \right]. \hspace{1cm} (4.24)$$

The following identities among determinants are verified with the aid of the Jacobi identity (see Appendix B):

$$D_{N-n+1,t}^{(2)}D_{N-n}^{(2)} - D_{N-n,t}^{(2)}D_{N-n+1}^{(2)} = 2D_{N-n+1}^{(1)}D_{N-n}^{(3)}, \hspace{1cm} (4.25a)$$

$$D_{N-n+2,t}^{(0)}D_{N-n+1}^{(0)} - D_{N-n+1,t}^{(0)}D_{N-n+2}^{(0)} = 2D_{N-n+2}^{(-1)}D_{N-n+1}^{(1)}, \hspace{1cm} (4.25b)$$

$$D_{N-n,t}^{(2)}D_{N-n+2}^{(0)} - D_{N-n+1,t}^{(2)}D_{N-n+1}^{(0)} = -2D_{N-n+2}^{(-1)}(1; 2)D_{N-n+1}^{(1)}. \hspace{1cm} (4.25c)$$
Substituting (4.25) into (4.24), we can reduce (4.24) considerably, giving rise to

\[ G_n = 2d_{n-1}d_n e^{(n-1)x} D_{N+n-1}^{(1)} \left[ e^x D_{N+n}^{(3)} + 4e^{-x} D_{N+n+2}^{(-1)} \right]. \] (4.26)

The expression of \( u_n \) now follows from (4.21a), (4.23) and (4.26). It is expressed compactly in terms of the Hankel determinants as

\[ u_n = \frac{e^x D_{N+n}^{(3)} + 4e^{-x} D_{N+n+2}^{(-1)}}{D_{N+n+1}^{(1)}}, \quad (n = 1, 2, ..., N). \] (4.27)

Note that when \( n = 1 \), (4.27) reduces to \( u_1 = e^x D_{N-1}^{(3)}/D_N^{(1)} \) due to the relation \( D_{N+1}^{(-1)} = 0 \) which represents the waveform of \( u \) in the range \( x \leq x_1 \). Using (4.6), this expression can be written in the form

\[ u_1 = \sum_{i=1}^{N} b_i z_i, \] (4.28a)

with

\[ b_i = \prod_{j=1}^{N} \lambda_j^2 \prod_{j \neq i}^{N} (\lambda_i - \lambda_j)^{-2} \lambda_i^{-3}, \quad (i = 1, 2, ..., N), \] (4.28b)

where \( \lambda_i \) and \( z_i \) are defined respectively by (4.2c) and (4.7c).

For \( x \) in the range \( x_N \leq x \), we find that \( u = u_{N+1} \) has a particularly simple form

\[ u_{N+1} = 2e^{-x} D_{1,t}^{(0)} = 2 \sum_{i=1}^{N} c_i z_i^{-1}, \] (4.29)

where we have used \( D_{1,t}^{(0)} = A_0 \), (4.2b) and (4.7c).

4.3 Comparison with the N-peakon solution

Here, we compare the expression of the peakon limit arising from the \( N \)-soliton solution with the \( N \)-peakon solution given by Beals et al.\cite{21} If we identify the quantities \( \Delta_m^{n} \) \( (m > 0) \) and \( \tilde{\Delta}_n^0 \) introduced in ref. 21 with \( D_{n}^{(m)} \) \( (m > 0) \) and \( D_{n}^{(0)} \), respectively, then we can write the latter solution in the form

\[ u(x, t) = \sum_{i=1}^{N} m_i(t) e^{-|x-x_i(t)|}, \] (4.30a)
\[ m_i = \frac{2D_{N-i+1}^{(0)}D_{N-i}^{(2)}}{D_{N-i+1}^{(1)}D_{N-i}^{(1)}}, \quad (i = 1, 2, \ldots, N), \quad (4.30b) \]

\[ x_i = \ln \left[ \frac{2D_{N-i+1}^{(0)}D_{N-i}^{(2)}}{D_{N-i}^{(2)}} \right], \quad (i = 1, 2, \ldots, N). \quad (4.30c) \]

When \( x \) lies in the interval \( x_{n-1} < x \leq x_n \), then \( u = u_n \) takes the form

\[
\begin{align*}
  u_n & = \sum_{i=1}^{n-1} m_i e^{-(x-x_i)} + \sum_{i=n}^{N} m_i e^{-(x_i-x)} \\
  & = e^{-x} \sum_{i=N-n+1}^{N-1} \frac{4(D_{i+1}^{(0)})^2}{D_{i+1}^{(1)}D_{i}^{(1)}} + e^{x} \sum_{i=0}^{N-n} \frac{(D_i^{(2)})^2}{D_{i+1}^{(1)}D_{i}^{(1)}}. \\
  & = e^{-x} \sum_{i=N-n+1}^{N-1} \frac{4(D_{i+1}^{(0)})^2}{D_{i+1}^{(1)}D_{i}^{(1)}} + e^{x} \sum_{i=0}^{N-n} \frac{(D_i^{(2)})^2}{D_{i+1}^{(1)}D_{i}^{(1)}}. \\
  \end{align*}
\]

(4.31)

The following formulas are particularly useful to simplify (4.31):

\[
\begin{align*}
  \sum_{i=0}^{N-n} \frac{(D_i^{(2)})^2}{D_{i+1}^{(1)}D_{i}^{(1)}} & = \frac{D_{N-n}^{(3)}}{D_{N-n+1}^{(1)}}, \quad (n = 0, 1, \ldots, N), \quad (4.32a) \\
  \sum_{i=N-n+1}^{N-1} \frac{(D_{i+1}^{(0)})^2}{D_{i+1}^{(1)}D_{i}^{(1)}} & = \frac{D_{N-n+2}^{(-1)}}{D_{N-n+1}^{(1)}}, \quad (n = 2, 3, \ldots, N + 1). \quad (4.32b) \\
\end{align*}
\]

To prove (4.32a), we write Jacobi’s formula (4.5) with \( m = 1 \) and \( n = i - 1 \). It reads

\[
\left( D_i^{(2)} \right)^2 = D_i^{(3)} D_i^{(1)} - D_{i+1}^{(1)} D_{i-1}^{(3)}. \quad (4.33)
\]

Dividing (4.33) by \( D_{i+1}^{(1)}D_{i}^{(1)} \) and summing up the resultant expression from \( i = 1 \) to \( i = N - n \), we obtain (4.32a) by taking account of the relations \( D_0^{(m)} = 1 \) \((m = 1, 2, 3)\). By a similar calculation using (4.5) with \( m = -1 \) and \( n = i \), formula (4.32b) follows immediately upon noting the relation \( D_{N+1}^{(1)} = 0 \). If we introduce (4.32) into (4.31), we find that the resultant expression coincides with the peakon limit of the \( N \)-soliton solution (4.27).

Last, for \( x_N \leq x \), (4.30) becomes

\[
\begin{align*}
  u & = u_{n+1} = 4e^{-x} \sum_{i=0}^{N-1} \frac{(D_{i+1}^{(0)})^2}{D_{i+1}^{(1)}D_{i}^{(1)}}. \\
  \end{align*}
\]

(4.34)
Using (4.32b) with \( n = N + 1 \), we can recast (4.34) to

\[
u_{N+1} = 4e^{-x}D_1^{(-1)} = 4e^{-x} \sum_{i=1}^{N} \lambda_i^{-1} E_i,
\]

which is in agreement with (4.29) by (4.2c) and (4.7c). In conclusion, we have completed the proof that the \( N \)-soliton solution of the CH equation reduces to the \( N \)-peakon solution of the dispersionless CH equation in the peakon limit.

5. Concluding Remarks

We have developed a novel limiting procedure for recovering the nonanalytic \( N \)-peakon solution from the analytic \( N \)-soliton solution of the CH equation. In the process, the asymptotic analysis has been performed which expands all the wave parameters in powers of the small dispersion parameter. We have employed the key identity, i.e. Jacobi’s formula (4.5) for determinants by which the significant part of the calculations has been carried out in quite a transparent manner. In view of this formula, we were also able to obtain the new formulas (4.27) and (4.28) which represent the waveform of the \( N \)-peakon solution in a specified interval of the space variable. Our approach is quite different from that used by Beals et al.\(^{20,21}\) which relies on the classical moment problem originally studied by Stieltjes. In a future work, we have a plan to apply our method to the \( N \)-soliton solution\(^{27,28}\) of the Degasperis-Procesi (DP) equation to show that it recovers the \( N \)-peakon solution already given by Lundmark and Szmigielski.\(^{29,30}\) While both the CH and DP equations have a different mathematical structure, the \( N \)-soliton solution of the DP equation can be written in a parametric form just like (2.1) and (2.2), the only difference being the structure of the tau-functions \( f_1 \) and \( f_2 \). This observation will make it possible to perform the peakon limit of the \( N \)-soliton solution in a manner similar to that developed here for the CH equation.

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Appendix A: Proof of (4.6)

We introduce the \( n \times N \) matrix \( P \) and the \( N \times n \) matrix \( Q \)

\[
P = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_N \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^\ell & \lambda_2^\ell & \cdots & \lambda_N^\ell
\end{pmatrix}, \tag{A.1}
\]

\[
Q = \begin{pmatrix}
\lambda_1^m E_1 & \lambda_1^{m+1} E_1 & \cdots & \lambda_1^{m+n-1} E_1 \\
\lambda_2^m E_2 & \lambda_2^{m+1} E_2 & \cdots & \lambda_2^{m+n-1} E_2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_N^m E_N & \lambda_N^{m+1} E_N & \cdots & \lambda_N^{m+n-1} E_N
\end{pmatrix}. \tag{A.2}
\]

We calculate \( \det(PQ) \) by two ways. First, multiplying \( P \) and \( Q \) and using (4.2), we obtain

\[
\det(PQ) = D_n^{(m)}. \tag{A.3}
\]

On the other hand, the determinant is expanded by means of the Binet-Cauchy formula as\(^{31}\)

\[
\det(PQ) = \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N} \begin{vmatrix}
1 & 1 & \cdots & 1 \\
\lambda_{i_1} & \lambda_{i_2} & \cdots & \lambda_{i_n} \\
\lambda_{i_1}^{\ell} & \lambda_{i_2}^{\ell} & \cdots & \lambda_{i_n}^{\ell} \\
\lambda_{i_1}^m E_{i_1} & \lambda_{i_1}^{m+1} E_{i_1} & \cdots & \lambda_{i_1}^{m+n-1} E_{i_1} \\
\lambda_{i_2}^m E_{i_2} & \lambda_{i_2}^{m+1} E_{i_2} & \cdots & \lambda_{i_2}^{m+n-1} E_{i_2} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{i_n}^m E_{i_n} & \lambda_{i_n}^{m+1} E_{i_n} & \cdots & \lambda_{i_n}^{m+n-1} E_{i_n}
\end{vmatrix}.
\]

\[
= \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N} \Delta_n(i_1, i_2, \ldots, i_n)(\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n})^m E_{i_1} E_{i_2} \cdots E_{i_n}. \tag{A.4}
\]

In passing to the last line of (A.4), we have extracted the factor \( \lambda_{i_l}^m E_{i_l} \) from the \( l \)th row \( (l = 1, 2, \ldots, n) \) and used (4.1). Formula (4.6) follows immediately from (A.3) and (A.4).

Appendix B: Formulas for Determinants

Here, we prove (4.25). We define the quantity \( D \) by the relation

\[
D = D_n^{(2)} + D_{n+1}^{(2)} - D_{n+1}^{(2)} D_n^{(2)}. \tag{B.1}
\]

We differentiate \( D_n^{(2)} \) by \( t \) with use of the relation \( dA_m/dt = 2A_{m-1} \) which is a consequence of (4.2b). The differentiation of the \( j \)th column of \( D_n^{(2)} \) is then proportional to the \( (j-1) \)th
Thus, it vanishes identically except for \( j = 1 \). Hence

\[
D_{n,t}^{(2)} = 2 \begin{vmatrix}
A_1 & A_3 & \cdots & A_{n+1} \\
A_2 & A_4 & \cdots & A_{n+2} \\
\vdots & \vdots & \ddots & \vdots \\
A_n & A_{n+2} & \cdots & A_{2n}
\end{vmatrix}.
\]

Using (B·2) and the relations (4.4), (B·1) becomes

\[
D = 2D_{n+2}^{(1)}(n+2; 2)D_{n+2}^{(1)}(n+1, n+2; 1, n+2) - 2D_{n+2}^{(1)}(n+2; 1)D_{n+2}^{(1)}(n+1, n+2; 2, n+2).
\]

It follows by using Jacobi's formula (4.3) and a fact \( D_{n+2}^{(1)} > 0 \) (see (4.6)) that

\[
D_{n+2}^{(1)}(n+1, n+2; 1, n+2) = \frac{1}{D_{n+2}^{(1)}} \left[ D_{n+2}^{(1)}(n+1; 1)D_{n+2}^{(1)}(n+2; n+2) - D_{n+2}^{(1)}(n+1; n+2)D_{n+2}^{(1)}(n+2; 1) \right],
\]

\[
D_{n+2}^{(1)}(n+1, n+2; 2, n+2) = \frac{1}{D_{n+2}^{(1)}} \left[ D_{n+2}^{(1)}(n+1; 2)D_{n+2}^{(1)}(n+2; n+2) - D_{n+2}^{(1)}(n+1; n+2)D_{n+2}^{(1)}(n+2; 2) \right].
\]

Substituting (B·4) and (B·5) into (B·3), we obtain

\[
D = 2D_{n+2}^{(1)}(n+2; n+2) \left[ D_{n+2}^{(1)}(n+1; 1)D_{n+2}^{(1)}(n+1; 2) - D_{n+2}^{(1)}(n+1; 2)D_{n+2}^{(1)}(n+1; 1) \right].
\]

We apply Jacobi's formula to the right-hand side of (B·6) and use the relations (4.4). Then, (B·6) is recast to a simple form

\[
D = 2D_{n+1}^{(1)}D_{n}^{(3)}.
\]

Formula (4.24a) is a consequence of (B·1) and (B·7) with \( n \) replaced by \( N - n \). Formula (4.24b) follows from (4.24a). To show this, we may simply replace \( n \) by \( n - 1 \) and \( E_i \) by \( \lambda_i^{-2}E_i(i = 1, 2, \ldots, N) \), respectively in (4.24a).
By repeating the similar calculation, we can show that

\[ D_{n,t}^{(2)} D_{n+2}^{(0)} - D_{n+1,t}^{(2)} D_{n+1}^{(0)} = -2D_{n+3}^{(-1)}(1, n+3; 2, n+3)D_{n+3}^{(-1)}(n+2, n+3; 1, 2) \]

\[ = -2D_{n+2}^{(-1)}(1; 2)D_{n+1}^{(1)}. \]  \hspace{1cm} (B.8)

If we replace \( n \) by \( N - n \) in (B.8), we obtain (4.25c). Formula (4.25d) is derived by the same way.
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