Global existence of weak solutions to the drift-flux system for general pressure laws

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Abstract The initial value problem of the multi-dimensional drift-flux model for two-phase flow is investigated in this paper, and the global existence of weak solutions with finite energy is established for general pressure-density functions without the monotonicity assumption.

Keywords two-phase flow, drift-flux model, global weak solution, non-monotone pressure law, quantitative regularity estimate

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1 Introduction

The drift-flux model for two-phase flow can be used widely in many applied scientific areas, such as chemical engineering, petroleum industry, biotechnology, microtechnology, and sprays [3,16,18,27,31,32,46,59]. In this paper, we consider the initial value problem (IVP) for the drift-flux system in the periodic domain $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ ($d \geq 2$):

\begin{alignat}{1}
&\partial_t \rho + \text{div}(\rho u) = 0, \\
&\partial_t n + \text{div}(nu) = 0, \\
&\partial_t ((\rho + n)u) + \text{div}((\rho + n)u \otimes u) + \nabla P(\rho, n) = \mu \Delta u + (\mu + \lambda) \nabla \text{div}u, \quad x \in \mathbb{T}^d, \quad t > 0
\end{alignat}

with the initial data

\begin{equation}
(\rho, n, (\rho + n)u)(x, 0) = (\rho_0, n_0, m_0)(x), \quad x \in \mathbb{T}^d,
\end{equation}

where $\rho = \rho(x, t) \in \mathbb{R}_+$ and $n = n(x, t) \in \mathbb{R}_+$ denote the densities of two fluids, respectively, $u = u(x, t) \in \mathbb{R}^d$ stands for the mixed velocity, the shear viscosity coefficient $\mu$ and the bulk viscosity coefficient $\lambda$ satisfy

\begin{equation}
\mu > 0, \quad 2\mu + \lambda > 0,
\end{equation}

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and the pressure \( P(\rho, n) \) is assumed to be the general laws

\[
\begin{aligned}
P(\rho, n) & \in C^1(\mathbb{R}_+ \times \mathbb{R}_+), \\
\frac{1}{C_0}(\rho^\gamma + n^\alpha) - C_1 & \leq P(\rho, n) \leq C_0(\rho^\gamma + n^\alpha) + C_1, \\
|\partial_\rho P(\rho, n)| & \leq C_2(\rho^{\gamma-1} + 1), \quad |\partial_n P(\rho, n)| \leq C_2(n^{\alpha-1} + 1), \\
P(\rho, n) & \leq C_3(\rho^\beta + n^\beta) \quad \text{for} \quad 0 \leq \rho, n \leq 1
\end{aligned}
\]  

(1.4)

with the constants \( C_i \) \((i = 0, 1, 2, 3)\), \( \tilde{\gamma}, \tilde{\alpha} \) and \( \beta \) satisfying

\[
C_0 \geq 1, \quad C_1 \geq 0, \quad C_2 > 0, \quad C_3 > 0, \quad \tilde{\gamma}, \tilde{\alpha} \geq 1, \quad 0 < \beta < 1. \tag{1.5}
\]

There is much significant progress made recently on the mathematical analysis of global solutions to the drift-flux system (1.1) (see [11, 18–20, 22, 28, 29, 41], [44, 47, 49, 52–56] and [57, 58]). Among them, for the pressure

\[ P(\rho, n) = C(-b(\rho, n) + \sqrt{b^2(\rho, n) + c(\rho, n)}) \]

the global existence and asymptotical behavior of solutions to (1.1) have been established in [19, 22] for one-dimensional general initial data and in [28, 29, 53, 54, 57] for multi-dimensional small initial data around the constant equilibrium state. For the pressure

\[ P(\rho, n) = C_{\rho}\gamma \left( \frac{n}{\rho} - \rho \right)^\gamma, \]

the well-posedness and dynamics of global weak solutions to the one-dimensional free boundary value problem were studied in [18, 20, 55, 56]. As for the pressure

\[ P(\rho, n) = \rho^\gamma + n^\alpha, \]

the existence of global weak solutions with finite energy to the three-dimensional drift-flux equations (1.1) was investigated by Bresch et al. [11] for \( \gamma, \alpha > 1 \) in a semi-stationary Stokes regime, by Vasseur et al. [44] for \( \gamma > \frac{9}{5} \) and \( \alpha \) satisfying that either \( \alpha \) is close to \( \gamma \) enough or \( \alpha \geq 1 \) if

\[ \varepsilon \rho_0 \leq n_0 \leq \varepsilon \rho_0 \]

(1.6)

with two constants \( 0 < \varepsilon \leq \varepsilon \leq \varepsilon < \infty \), and then by Wen [49] for two independent adiabatic constants \( \gamma, \alpha \geq \frac{9}{5} \). For more general pressure laws \( P(\rho, n) \) which can be non-monotone on a compact set of the two variables \( \rho \) and \( n \), the global weak solutions to (1.1) have been obtained by Novotný and Pokorný [41] in 3D under the assumption (1.6) with two constants \( 0 \leq \varepsilon \leq \varepsilon \leq \varepsilon < \infty \) and by Wen and Zhu [51] in 1D without the assumption (1.6). In addition, there are many interesting studies on the global existence of weak solutions to some models related to (1.1), such as the compressible Oldroyd-B model [1], the two-dimensional non-resistive compressible magnetohydrodynamic (MHD) model [35], the compressible Navier-Stokes equations with entropy transport [38] and the compressible Navier-Stokes-Vlasov-Fokker-Planck system [40]. The reader can refer to the review papers [50, 52].

The drift-flux system (1.1) can be viewed as a simplified model of the compressible non-conservative two-fluid equations (see [6, 7, 32, 50]):

\[
\begin{aligned}
\alpha^+ + \alpha^- &= 1, \\
D(\alpha^+ \rho^+ u^+) + \text{div}(\alpha^+ \rho^+ u^+) &= 0, \\
D(\alpha^+ \rho^+ u^+) + \text{div}(\alpha^+ \rho^+ u^+ \otimes u^+) + \alpha^+ \nabla P^+(\rho^+) &= \text{div}(\alpha^+ \tau^+) + \sigma^+ \alpha^+ \rho^+ \nabla \Delta(\alpha^+ \rho^+),
\end{aligned}
\]  

(1.7)

where \( \alpha^\pm, \rho^\pm, u^\pm, P^\pm(\rho^\pm) \) and \( \tau^\pm \) denote the volume fractions, densities, velocities, pressures and stress tensors of two fluids, respectively. The global existence of weak solutions to (1.7) with degenerate viscosity...
coefficients was proved in [6, 9] for \( P^+(\rho^+) = P^- (\rho^-) \), and the global well-posedness and optimal time-decay rates of strong solutions near the constant equilibrium state to (1.7) were studied in [21, 48] for \( P^+(\rho^+) \neq P^- (\rho^-) \).

The drift-flux system (1.1) for either \( \rho = 0 \) or \( \rho = n \) can reduce to the barotropic compressible Navier-Stokes equations

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho v) &= 0, \\
\partial_t (\rho v) + \text{div}(\rho v \otimes v) + \nabla \pi(\rho) &= \mu \Delta v + (\mu + \lambda) \nabla \text{div} v,
\end{align*}
\]

(1.8)

where \( \rho, v \) and \( \pi(\rho) \) denote the density, velocity and pressure, respectively. There are many important investigations about global weak solutions to the compressible Navier-Stokes equations (1.8) with constant viscosity coefficients [10, 23, 26, 30, 33, 37, 43]. For example, concerning the \( \gamma \)-law \( \pi(\rho) = \rho^\gamma \), the existence of global weak solutions with finite energy to (1.8) has been established by Lions [37] for \( \gamma \geq \frac{3d}{d+2} \) \((d = 2, 3)\) and \( \gamma > \frac{d}{2} \) \((d \geq 4)\), then by Feireisl et al. [26] for \( \gamma > \frac{d}{4} \) \((d \geq 2)\), by Jiang and Zhang [33] for \( \gamma > 1 \) subject to spherically symmetric initial data, and by Plotnikov and Weigant [43] for \( \gamma = 1 \) in 2D. Indeed, the concentration phenomenon of the convective term \( \rho u \otimes u \) may occur for \( \gamma \in [1, \frac{d}{2}] \) (see [30]). Feireisl [23] proved the global existence of weak solutions to (1.8) for the pressure laws allowed to be non-monotone on a compact set. Bresch and Jabin [10] developed new compactness tools to obtain global weak solutions to (1.8) for more general stress tensors, including thermodynamically unstable pressure laws and anisotropic viscosity coefficients. Moreover, some important progress has been made about global weak solutions to compressible Navier-Stokes equations with degenerate viscosity coefficients (see [4, 5, 8, 12, 34, 39, 45] and the references therein).

However, there are not any results about the global existence problem of the multi-dimensional drift-flux model (1.1) without the monotonicity assumption of the pressure as we know so far. The purpose of this paper is to extend the compactness techniques in [10] to the two-phase case and establish the global existence of weak solutions to the IVP (1.1)–(1.2) for the general pressure laws (1.4).

First, we give the definition of global weak solutions to the IVP (1.1)–(1.2) as follows.

**Definition 1.1.** \((\rho, n, (\rho+n)u)\) is said to be a global weak solution to the IVP (1.1)–(1.2) if for any time \( T > 0 \), the following properties hold:

1. It holds that

\[
\begin{align*}
0 \leq \rho &\in C([0, T]; L^\infty_{\text{weak}}(\mathbb{T}^d)), \\
0 \leq n &\in C([0, T]; L^0_{\text{weak}}(\mathbb{T}^d)), \\
u \in L^2(0, T; H^1(\mathbb{T}^d)), & \quad \sqrt{\rho + n u} \in L^\infty(0, T, L^2(\mathbb{T}^d)), \\
(\rho + n)u &\in C([0, T]; L^0_{\text{weak}}(\mathbb{T}^d)), \\
(\rho, n, (\rho+n)u)(x, 0) &= (\rho_0, n_0, 0)(x), \quad \text{a.e. } x \in \mathbb{T}^d.
\end{align*}
\]

2. The continuity equations (1.1.1)–(1.1.2) are satisfied in the sense of renormalized solutions:

\[
\begin{align*}
\partial_t \rho(\rho) + \text{div}(b(\rho)u) + [b'(\rho)\rho - b(\rho)] \text{div} u &= 0 \quad \text{in } \mathcal{D}(\mathbb{T}^d \times (0, T)), \\
\partial_t (\rho u) + \text{div}(b(u)u) + [b'(u)n - b(n)] \text{div} u &= 0 \quad \text{in } \mathcal{D}(\mathbb{T}^d \times (0, T)),
\end{align*}
\]

where \( b \in C^1(\mathbb{R}) \) satisfies \( b'(z) = 0 \) for all \( z \in \mathbb{R} \) large enough.

3. The momentum equation (1.1.3) is satisfied in \( \mathcal{D}'(\mathbb{T}^d \times (0, T)) \).

4. The energy inequality holds:

\[
\int_{\mathbb{T}^d} \frac{1}{2} (\rho + n) |u|^2 + G(\rho, n) \, dx + \int_0^t \int_{\mathbb{T}^d} [\mu |\nabla u|^2 + (\mu + \lambda)(\text{div} u)^2] \, dx \, d\tau 
\leq \int_{\mathbb{T}^d} \left[ \frac{1}{2} \frac{|m_0|^2}{\rho_0 + n_0} + G(\rho_0, n_0) \right] \, dx, \quad \text{a.e. } t \in (0, T),
\]

(1.9)

where the Helmholtz free energy function \( G(\rho, n) \) is defined by

\[
G(\rho, n) := \begin{cases} 
\rho \int_1^\rho P\left( s, \frac{n}{\rho} \right) s^{-2} \, ds, & \text{if } 0 \leq n \leq \tau \rho, \quad 0 < \tau < \infty, \\
\rho \int_0^\rho P\left( s, \frac{n}{\rho} \right) s^{-2} \, ds, & \text{if } \rho \geq 0, \quad n \geq 0, \quad C_1 = 0.
\end{cases}
\]

(1.10)
Then under the assumption (1.6), we have the global existence of weak solutions to the IVP (1.1)–(1.2) below.

**Theorem 1.2.** Suppose that the initial data $(\rho_0, n_0, m_0)$ satisfies

$$
\begin{align*}
0 \leq \varepsilon \rho_0(x) & \leq n_0(x) \leq \tilde{\varepsilon} \rho_0(x), \quad \text{a.e. } x \in \mathbb{T}^d, \\
\frac{m_0(x)}{\sqrt{\rho_0 + n_0(x)}} & = 0, \quad \text{if } (\rho_0 + n_0)(x) = 0, \quad \text{a.e. } x \in \mathbb{T}^d, \\
(\rho_0, n_0, \frac{m_0}{\sqrt{\rho_0 + n_0}}) & \in L^7(\mathbb{T}^d) \times L^6(\mathbb{T}^d) \times L^2(\mathbb{T}^d)
\end{align*}
$$

(1.11)

with two constants $0 < \varepsilon \leq \bar{\varepsilon} < \infty$. Then, if it holds for the pressure $P(\rho, n)$ given by (1.4) that

$$
\begin{align*}
\gamma & \geq \frac{3d}{d+2} (d = 2, 3), \quad \gamma \geq \frac{d}{2} (d \geq 4), \\
1 \leq \tilde{\gamma}, \tilde{\alpha} & \leq \max \{\gamma, \alpha\} + \theta, \quad \theta := \frac{2}{d} \max \{\gamma, \alpha\} - 1 > 0,
\end{align*}
$$

(1.12)

the IVP (1.1)–(1.2) admits a global weak solution $(\rho, n, (\rho + n)u)$ in the sense of Definition 1.1.

**Remark 1.3.** The assumption (1.4) implies $P(0, 0) = 0$ and the continuity of $G(\rho, n)$ near $(0, 0)$.

**Remark 1.4.** There are many interesting pressure laws satisfying (1.4). For example, we have the following:

1) We can take

$$
P(\rho, n) = \rho^\gamma + n^\alpha + \sum_{i=1}^N c_i \rho^{\gamma_i} n^{\alpha_i},
$$

where the constants $N, \gamma_i$ and $\alpha_i$ ($i = 1, \ldots, N$) satisfy $N \geq 1, 0 \leq \gamma_i < \gamma, 0 \leq \alpha_i < \alpha$ and $\gamma_i + \alpha_i < \max \{\gamma, \alpha\}$, and the constants $c_i$ ($i = 1, \ldots, N$) are allowed to be negative such that $P(\rho, n)$ can be non-monotone. In particular, one can choose the monotone pressure

$$
P(\rho, n) = \rho^\gamma + n^\alpha.
$$

2) Let $\Pi_1(\rho)$ and $\Pi_2(n)$ be two non-monotone pressure laws for compressible Navier-Stokes equations given in [10], for example, the virial expansion with high order terms, which can be thermodynamically unstable. Then the pressure

$$
P(\rho, n) = \Pi_1(\rho) + \Pi_2(n)
$$

satisfies (1.4). In particular, the pressure function $P(\rho, n) = \Pi_1(\rho) + n^2$ corresponds to the two-dimensional non-resistive compressible MHD equations [35].

3) An example of pressure laws is

$$
P(\rho, n) = \rho^\gamma (1 + 2 \cos \rho) + n^\alpha (1 + 2 \cos n),
$$

which is oscillatory even for large $\rho$ and $n$.

**Remark 1.5.** If (1.6) holds, then one can show $0 \leq \varepsilon n(x, t) \leq \rho(x, t) \leq \tilde{\varepsilon} \rho(x, t)$ for a.e. $(x, t) \in \mathbb{T}^d \times (0, T)$ due to the comparison principle for (1.1)$_1$–(1.1)$_2$ so that $G(\rho, n)$ satisfies

$$
\frac{1}{C_{\varepsilon_1}^\gamma} (\rho + n)^{\max \{\gamma, \alpha\}} - C_{\varepsilon_1}^\gamma \leq G(\rho, n) \leq C_{\varepsilon_1}^\gamma (\rho + n)^{\max \{\gamma, \alpha\}} + C_{\varepsilon_1}^\gamma,
$$

(1.13)

where $C_{\varepsilon_1}^\gamma \geq 1$ is a constant. Indeed, we can prove under the assumptions (1.11)–(1.12) that both $\rho$ and $n$ are bounded in $L^{\max \{\gamma, \alpha\} + \theta}(0, T; L^{\max \{\gamma, \alpha\} + \theta}(\mathbb{T}^d))$ for $\max \{\gamma, \alpha\} + \theta \geq 2$ so as to apply the arguments of renormalized solutions.

Without the condition (1.6), we also have the global existence of weak solutions to the IVP (1.1)–(1.2) as follows.
Theorem 1.6. Suppose that the initial data \((\rho_0, n_0, m_0)\) satisfies
\[
\begin{align*}
\rho_0(x) &\geq 0, \quad n_0(x) \geq 0, \quad \text{a.e. } x \in \mathbb{T}^d, \\
\frac{m_0(x)}{\sqrt{\rho_0 + n_0}(x)} &= 0, \quad \text{if } (\rho_0 + n_0)(x) = 0, \quad \text{a.e. } x \in \mathbb{T}^d, \\
(\rho_0, n_0, \frac{m_0}{\sqrt{\rho_0 + n_0}}) &\in L^\gamma(\mathbb{T}^d) \times L^n(\mathbb{T}^d) \times L^2(\mathbb{T}^d).
\end{align*}
\] (1.14)

Then, if it holds for the pressure \(P(\rho, n)\) given by (1.4) that
\[
\begin{align*}
C_1 &= 0, \\
\gamma, \alpha &\geq \frac{3d}{d + 2} \quad (d = 2, 3), \quad \gamma, \alpha > \frac{d}{2} \quad (d \geq 4), \\
1 &\leq \tilde{\gamma} \leq \gamma + \theta_1, \quad \theta_1 := \frac{2}{d} - \frac{\gamma}{\min \{\gamma, \alpha\}} > 0, \\
1 &\leq \tilde{\alpha} \leq \alpha + \theta_2, \quad \theta_2 := \frac{2}{d} - \frac{\alpha}{\min \{\gamma, \alpha\}} > 0,
\end{align*}
\] (1.15)

the IVP (1.1)–(1.2) admits a global weak solution \((\rho, n, (\rho + n)u)\) in the sense of Definition 1.1.

Remark 1.7. Theorem 1.6 is the first result on the global existence of weak solutions to the multi-dimensional drift-flux model (1.1) for non-monotone pressure laws without any requirement on the relation between \(\rho_0\) and \(n_0\) as in (1.6). As pointed in [22,49], the system (1.1) without the condition (1.6) is more realistic in some physical situations and has more “two-phase” properties from mathematical points of view.

Remark 1.8. The adiabatic constants \(\gamma\) and \(\alpha\) can take \(\frac{3d}{d + 2}\) for \(d = 2, 3\), which extends the previous work by Bresch and Jabin [10] about the global existence of weak solutions to the compressible Navier-Stokes equations (1.8) for thermodynamically unstable pressure laws.

Remark 1.9. If \((1.4)\) holds with \(C_1 = 0\), then we have
\[
\frac{1}{C_0} \left( \frac{\rho^\gamma}{\gamma - 1} + \frac{n^\alpha}{\alpha - 1} \right) \leq G(\rho, n) \leq C_0 \left( \frac{\rho^\gamma}{\gamma - 1} + \frac{n^\alpha}{\alpha - 1} \right). \tag{1.16}
\]

This implies under the assumptions (1.14)–(1.15) that \(\rho\) and \(n\) belong to \(L^{\gamma + \theta_1}(0, T; L^{\gamma + \theta_1}(\mathbb{T}^d))\) for \(\gamma + \theta_1 \geq 2\) and \(L^{\alpha + \theta_2}(0, T; L^{\alpha + \theta_2}(\mathbb{T}^d))\) for \(\alpha + \theta_2 \geq 2\), respectively.

Remark 1.10. By (1.10), the condition \(C_1 = 0\) in Theorem 1.6 can be removed provided that \(n_0 \leq \sigma \rho_0\) for some constant \(\sigma > 0\).

We explain the main strategies to prove Theorems 1.2 and 1.6. By the Faedo-Galerkin approximation and vanishing artificial viscosity, one can construct the approximate sequence \((\rho_\delta, n_\delta, (\rho_\delta + n_\delta)u_\delta)\) of the IVP (1.1)–(1.2) for \(\delta \in (0, 1)\) with the artificial pressure.

Then we pass to the limit as \(\delta \to 0\). As emphasized in many related papers [41,44,52], the key point is to show the strong convergence of the two densities \(\rho_\delta\) and \(n_\delta\). Due to the possible non-monotonicity of \(P(\rho, n)\) in the two variables \(\rho\) and \(n\), it is difficult to apply the techniques by Bresch and Jabin [10] to estimate \(\rho_\delta\) and \(n_\delta\), separately. To overcome this difficulty, we first make use of the ideas inspired by [44] to deduce
\[
(\rho_\delta, n_\delta) \to (\rho, n) \quad \text{in } L^1(0, T; L^1(\mathbb{T}^d)) \times L^1(0, T; L^1(\mathbb{T}^d))
\]
\[
\Leftrightarrow \rho_\delta + n_\delta \to \rho + n \quad \text{in } L^1(0, T; L^1(\mathbb{T}^d)) \quad \text{as } \delta \to 0.
\]

By virtue of the compactness criterion introduced in [2,10], one needs to derive the quantitative regularity estimate of the sum \(\delta_\delta := \rho_\delta + n_\delta\):
\[
\frac{1}{||K_h||_{L^1}} \int_0^T \int_{\mathbb{T}^d} K_h(x - y) |\partial_t^\delta \phi - \partial_t^\delta \psi| dx dy dt \to 0 \quad \text{uniformly in } \delta \text{ as } h \to 0
\]
with $\mathcal{K}_h$ the periodic symmetric kernel given by (4.15). By the structures of the continuity equations, we turn to estimate the term $\text{div}u_3^\delta - \text{div}u_3^\varepsilon$, which can be decomposed into the pressure part and the effective viscous flux part.

The key pressure part can be analyzed as follows:

$$
\begin{aligned}
P_\delta(\rho_j^\delta, n_j^\delta) - P_\delta(\rho_j^\varepsilon, n_j^\varepsilon) & = P_\delta(\rho_j^\delta, n_j^\delta) - P_\delta(A_{\rho^\varepsilon, n^\varepsilon} \cdot \nabla \rho_j^\varepsilon, B_{\rho^\varepsilon, n^\varepsilon} \cdot \nabla n_j^\varepsilon) \\
& + P_\delta(A_{\rho^\delta, n^\delta} \cdot \nabla \rho_j^\delta, B_{\rho^\delta, n^\delta} \cdot \nabla n_j^\delta) - P_\delta(\rho_j^\varepsilon, n_j^\varepsilon) \\
& + P_\delta(A_{\rho^\varepsilon, n^\varepsilon} \cdot \nabla \rho_j^\varepsilon, B_{\rho^\varepsilon, n^\varepsilon} \cdot \nabla n_j^\varepsilon) - P_\delta(A_{\rho^\delta, n^\delta} \cdot \nabla \rho_j^\delta, B_{\rho^\delta, n^\delta} \cdot \nabla n_j^\delta),
\end{aligned}
$$

(1.17)

where $(A_{\rho, n}, B_{\rho, n})$ is defined by

$$(A_{\rho, n}, B_{\rho, n}) := \begin{cases} 
\left( \frac{\rho}{\rho + n}, \frac{n}{\rho + n} \right), & \text{if } \rho + n > 0, \\
(0, 0), & \text{if } \rho + n = 0.
\end{cases}
$$

(1.18)

We can prove that for some domain $Q \subset \mathbb{T}^{2d} \times (0, T)$ with $|\mathbb{T}^{2d} \times (0, T)/Q|$ arbitrarily small, the first three terms on the right-hand side of (1.17) on $Q$ tend to 0 as $h \to 0$ uniformly with respect to $\delta$, and the last term associated with $P_\delta(A_{\rho, n} \cdot \nabla \theta, B_{\rho, n} \cdot \nabla \theta)$ in one variable $\theta$ can be estimated by the arguments as used in [10] with some modifications.

Meanwhile, to deal with the effective viscous flux part, we make use of the structures of the momentum equation and the commutator estimates of the Riesz operator, which is different from the previous analysis [10] and may be applicable to other related models in fluid dynamics.

The rest of this paper is organized as follows. The global existence of weak solutions to the approximate system with two small parameters $\varepsilon \in (0, 1)$ and $\delta \in (0, 1)$ is obtained in Section 2. In Section 3, we pass to the limit as $\varepsilon \to 0$ by the compactness theorem of Lions and Feireisl. In Section 4, we prove the strong convergence of two densities by deriving the quantitative regularity estimate of their sum and show the convergence of the approximate sequence to an expected weak solution as $\delta \to 0$. In Appendix A, we give some technical lemmas which are used in this paper.

## 2 The Faedo-Galerkin approximation

We are ready to solve the following approximate system for $\varepsilon \in (0, 1)$ and $\delta \in (0, 1)$:

$$
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= \varepsilon \Delta \rho, \\
\partial_t n + \text{div}(n u) &= \varepsilon \Delta n, \\
\partial_t ((\rho + n)u) + \text{div}((\rho + n)u \otimes u) + \nabla P_\delta(\rho, n) + \varepsilon \nabla u \cdot \nabla (\rho + n) \\
&= \mu \Delta u + (\mu + \lambda) \nabla \text{div} u, & x \in \mathbb{T}^d, & t > 0
\end{align*}
$$

(2.1)

with the initial data

$$
(\rho, n, u)(x, 0) = (\rho_{0, \delta}, n_{0, \delta}, u_{0, \delta})(x), & x \in \mathbb{T}^d,
$$

(2.2)

where $P_\delta(\rho, n)$ is the artificial pressure

$$
P_\delta(\rho, n) := 1_{\rho + n \leq \delta} P(\rho, n) + \delta(\rho + n)^{p_0}, & \quad p_0 > \gamma + \tilde{\gamma} + \alpha + \tilde{\alpha} + 1,
$$

(2.3)

and $(\rho_{0, \delta}, n_{0, \delta}, u_{0, \delta})$ is the regularized initial data

$$
(\rho_{0, \delta}, n_{0, \delta}, u_{0, \delta})(x) := \left( \rho_0 * \delta_\varepsilon + \rho_0 * \delta_\varepsilon + \delta_\varepsilon, \frac{n_0}{\sqrt{\rho_0 + n_0}} * \delta_\varepsilon + \delta_\varepsilon + \frac{n_0}{\sqrt{\rho_0 + n_0}} * \delta_\varepsilon + \delta_\varepsilon \right)(x).
$$

(2.4)
Here, \(1_{s \geq k}\) is a cut-off function satisfying
\[
1_{s \leq k} := \begin{cases} 
1, & \text{if } 0 \leq s \leq k, \\
\text{smooth, if } k \leq s \leq 2k, \\
0, & \text{if } s \geq 2k,
\end{cases}
\]
and \(j_\delta\) is a function such that
\[
\begin{align*}
&j_\delta \in C^\infty(\mathbb{T}^d), \quad \|j_\delta\|_{L^1} = 1, \quad 0 \leq j_\delta \leq \delta^{-\frac{d}{m_0}}, \\
&j_\delta * f \to f \quad \text{in } L^p(\mathbb{T}^d) \quad \text{as } \delta \to 0, \quad \forall f \in L^p(\mathbb{T}^d), \quad p \in [1, \infty).
\end{align*}
\]

It is easy to verify under the assumptions of either Theorem 1.2 or Theorem 1.6 that \((\rho_0, \delta, n_0, \delta, (\rho_0, \delta + n_0, \delta)u_0, \delta)\) converges to \((\rho_0, u_0, n_0)\) strongly in
\[
L^\gamma(\mathbb{T}^d) \times L^n(\mathbb{T}^d) \times L^{\frac{2m_0}{m_0-\gamma}}(\mathbb{T}^d)
\]
and satisfies
\[
\begin{align*}
&\frac{\|\rho_0, \delta\|_{L^\gamma}}{\|\rho_0\|_{L^\gamma}}, \quad \frac{\|n_0, \delta\|_{L^n}}{\|n_0\|_{L^n}}, \quad \sqrt{\frac{\|\rho_0, \delta + n_0, \delta u_0, \delta\|_{L^2}}{\|\rho_0, \delta\|_{L^2}}}, \\
&0 < \delta \leq \rho_0, \delta(x), n_0, \delta(x) \leq C\delta^{-\frac{1}{m_0}} \frac{1}{c_{+ \delta}} \leq \frac{n_0, \delta(x)}{\rho_0, \delta(x)} \leq c_{+ \delta}, \quad x \in \mathbb{T}^d, \quad c_{+ \delta} := C\delta^{-\frac{1}{m_0}} > 1,
\end{align*}
\]
Without loss of generalization, we can assume
\[
P(\rho, n) \in C^2(\mathbb{R}_+ \times \mathbb{R}_+), \quad |\partial_{nn}P(\rho, n)| \leq C(1 + \rho^{m_0-3} + n^{m_0-3}).
\]
Indeed, if this is not the case, one may approximate \(P(\rho, n)\) by a regular sequence since (2.7) is only used to derive the estimate (2.15) but has no effect on the limit process as \(\delta \to 0\) in Section 4. We can compute a constant \(c_\delta \to \infty\) as \(\delta \to 0\) such that both \(\partial_t P_\delta(\rho, n) > 0\) and \(\partial_n P_\delta(\rho, n) > 0\) hold for any \((\rho, n) \in \mathbb{R}_+ \times \mathbb{R}_+\) satisfying \(\rho + n \geq c_\delta > 0\), and then set
\[
\begin{align*}
P_{1, \delta}(\rho, n) := & P_\delta(\rho, n) + C_11_{\rho + n \leq C_\delta}(\rho + n)^{\gamma + \alpha} + \rho + n, \\
P_{2, \delta}(\rho, n) := & C_11_{\rho + n \leq C_\delta}(\rho + n)^{\gamma + \alpha} + \rho + n
\end{align*}
\]
with \(C_1 > 1\) a sufficiently large constant. Then the pressure \(P_\delta(\rho, n)\) can be re-written as
\[
P_\delta(\rho, n) = P_{1, \delta}(\rho, n) - P_{2, \delta}(\rho, n).
\]
Note that \(P_{1, \delta}(\rho, n) \in C^2(\mathbb{R}_+ \times \mathbb{R}_+)\) is monotonically increasing with respect to both \(\rho \geq 0\) and \(n \geq 0\), and \(P_{2, \delta}(\rho, n) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)\) satisfies \(P_{2, \delta}(\rho, n) > 0\) and \(P_{2, \delta}(\rho, n) = 0\) for \(\rho + n \geq 2C_\delta\), so we can employ the classical Lions-Feireisl approach [23, 37] to solve the approximate system (2.1).

Let \(\psi \in C^\infty(\mathbb{T}^d) (\ell = 1, 2, \ldots)\) be an orthonormal basis in \(L^2\). Denote the finite-dimensional space \(X_\ell \subset L^2\) and its projector \(P_\ell : L^2 \to X_\ell\) by
\[
X_\ell := \text{span}\{\psi_j\}_{j=1}^\ell, \quad P_\ell f := \sum_{i=1}^\ell \psi_i \int_{\mathbb{T}^d} \psi_i f dx, \quad \forall f \in X_\ell.
\]

**Proposition 2.1.** Let \(\rho_0 > \gamma + \tilde{\gamma} + \alpha + \tilde{\alpha} + 1, \delta \in (0, 1), \varepsilon \in (0, 1)\) and an integer \(\ell \geq 1\). Then under the assumptions of either Theorem 1.2 or Theorem 1.6, there exists a unique global regular solution \((\rho_\ell, n_\ell, u_\ell)\) for (2.1) with the initial data \((\rho_{0\ell}, n_{0\ell}, P_{0\ell})\). In addition, it holds for any \(T > 0\) that
\[
\begin{align*}
&\|(\rho_\ell, n_\ell)\|_{L^\infty(0, T; L^\infty)} + \varepsilon \frac{1}{C_\delta} \|(\nabla \rho_\ell, \nabla n_\ell)\|_{L^2(0, T; L^2)} \leq C_\delta, \\
&\frac{1}{C_\delta} \|(\rho_\ell, n_\ell)\|_{L^\infty(0, T; L^\infty)} + \|u_\ell\|_{L^2(0, T; H^1)} \leq C_\delta, \\
&0 \leq \frac{1}{c_{+ \delta}} \rho_\ell(x, t) \leq n_\ell(x, t) \leq c_{+ \delta} \rho_\ell(x, t), \quad \text{a.e. } (x, t) \in \mathbb{T}^d \times (0, T), \\
&0 \leq \rho_\ell(x, t) \leq n_\ell(x, t) \leq \tau \rho_\ell(x, t), \quad \text{if } \rho_\ell(x) \leq n_0(x) \leq \tau \rho_0(x), \quad \text{a.e. } (x, t) \in \mathbb{T}^d \times (0, T),
\end{align*}
\]
where $C_\delta > 0$ is a constant independent of $\varepsilon$ and $\ell$, and $c_{*,\delta} > 1$ is the constant given by (2.6)$_2$.

Furthermore, we have

$$\int_{\mathbb{T}_t} \left[ \frac{1}{2} (\rho_t + n_\ell)|u_\ell|^2 + G_\delta(\rho_t, n_\ell) \right] dx + \int_0^t \int_{\mathbb{T}_t} [\mu|\nabla u_\ell|^2 + (\mu + \lambda)(\text{div} u_\ell)^2] dx dt$$

$$\leq \int_{\mathbb{T}_t} \left[ \frac{1}{2} (\rho_0, n_\ell)|u_0, n_\ell|^2 + G_\delta(\rho_0, n_\ell, n, \delta) \right] dx + C_\delta \varepsilon,$$

where $G_\delta(\rho, n)$ is defined by

$$G_\delta(\rho, n) := \begin{cases} \rho \int_1^\rho P\left( \frac{s}{\rho}, \frac{n}{\rho} \right) s^{-2} ds, & \text{under the assumptions of Theorem 1.2}, \\ \rho \int_0^\rho P\left( \frac{s}{\rho}, \frac{n}{\rho} \right) s^{-2} ds, & \text{under the assumptions of Theorem 1.6}. \end{cases}$$

Proof. By the Faedo-Galerkin approximation and fixed point arguments [24, 26, 42], there is a time $T_\ell \in (0, T]$ such that the system (2.1) with the initial data $(\rho_0, n, 0, \rho_0, n)$ can be solved uniquely on $[0, T_\ell]$. The details are omitted here.

To show $T_\ell = T$, we need to establish the a priori estimates (2.10) on $\mathbb{T}^d \times (0, T_\ell)$ uniformly in $\ell$. Note that $u_\ell$ is regular since all the Sobolev norms in the finite-dimensional space $X_\ell$ are equivalent. By the comparison principle for (2.1)$_1$–(2.1)$_2$, we have

$$0 \leq \frac{1}{c_{*,\delta}} \rho_t(x, t) \leq n_\ell(x, t) \leq c_{*,\delta} \rho_t(x, t), \quad (x, t) \in \mathbb{T}^d \times (0, T_\ell).$$

(2.13)

Multiplying (1.1)$_1$, (1.1)$_2$ and (1.1)$_3$ by $\partial_{\rho_t} G_\delta(\rho_t, n_\ell)$, $\partial_{n_\ell} G_\delta(\rho_t, n_\ell)$ and $u_\ell$, respectively, and using the fact that

$$\rho \partial_{\rho_t} G_\delta(\rho, n) + n \partial_{n_\ell} G_\delta(\rho, n) - P_\delta(\rho, n) = 0,$$

we show for $t \in [0, T_\ell]$ that

$$\frac{d}{dt} \int_{\mathbb{T}_t} \left( \frac{1}{2} (\rho_t + n_\ell)|u_\ell|^2 + G_\delta(\rho_t, n_\ell) \right) dx$$

$$+ \int_{\mathbb{T}_t} (\mu|\nabla u_\ell|^2 + (\mu + \lambda)(\text{div} u_\ell)^2 + \delta \rho_0 \rho_0^\delta |\nabla n_\ell|^2 + \delta \rho_0 n_\ell^\delta |\nabla n_\ell|^2) dx$$

$$= -\varepsilon \int_{\mathbb{T}_t} (\partial_{\rho_t}^2 G_\delta(\rho_t, n_\ell)|\nabla n_\ell|^2) dx + 2 \rho_\ell^\delta G_\delta(\rho_t, n_\ell)|\nabla n_\ell|^2 + 2 \rho_\ell^\delta G_\delta(\rho_t, n_\ell)|\nabla n_\ell|^2.$$

(2.14)

It follows from (1.13), (2.3) and (2.13) that

$$\begin{cases} G_\delta(\rho_t, n_\ell) \geq \frac{\delta}{p_0 + 1} \rho_0 - C_\delta, \\ (|\partial_{\rho_t}^2 G_\delta| + |\partial_{\rho_t n_\ell} G_\delta| + |\partial_{n_\ell n_\ell} G_\delta|)(\rho_t, n_\ell) \leq C_\delta(\rho_t + n_\ell)^{p_0 - 2} + C_\delta. \end{cases}$$

(2.15)

By (2.15)$_2$ and Young’s inequality, the right-hand side of (2.14) can be estimated as

$$-\varepsilon \int_{\mathbb{T}_t} (\partial_{\rho_t}^2 G_\delta(\rho_t, n_\ell)|\nabla n_\ell|^2 + 2 \rho_\ell n_\ell^\delta G_\delta(\rho_t, n_\ell)|\nabla n_\ell| + 2 \rho_\ell n_\ell^\delta G_\delta(\rho_t, n_\ell)|\nabla n_\ell|^2) dx$$

$$\leq \frac{\varepsilon \delta p_0}{2} \int_{\mathbb{T}_t} (\rho_\ell^\delta |\nabla n_\ell|^2 + n_\ell^\delta |\nabla n_\ell|^2) dx + C_\delta \varepsilon \int_{\mathbb{T}_t} (|\nabla n_\ell|^2 + |\nabla n_\ell|^2) dx + C_\delta \varepsilon.$$

(2.16)

To control the second term on the right-hand side of (2.16), we deduce from (2.1)$_1$–(2.1)$_2$ that

$$C_\delta \frac{d}{dt} \int_{\mathbb{T}_t} (|\rho_t|^2 + |n_\ell|^2) dx + 2 C_\delta \varepsilon \int_{\mathbb{T}_t} (|\nabla n_\ell|^2) dx$$

$$= -2 C_\delta \int_{\mathbb{T}_t} (\rho_t + n_\ell) \text{div} u_\ell dx$$
Adding (2.14) and (2.16)–(2.17) together and making use of (1.3), (2.6), (2.15) and the Grönwall inequality, we obtain
\[
\sup_{t \in [0,T]} \int_{\Omega} \left[ \left( \rho + n_\varepsilon \right) u_\varepsilon \right] + n_\varepsilon^p \rho_\varepsilon^p + n_\varepsilon^p \nabla \rho_\varepsilon \nabla \nabla \rho_\varepsilon = C_\delta,
\]
which together with [24, Lemma 3.2] leads to
\[
\|u_\varepsilon\|_{L^2(0,T;L^2)} \leq C_\delta.
\]
By the Sobolev inequality, it also holds that
\[
\|\rho_\varepsilon + n_\varepsilon\|_{L^{p^0+1}(0,T;L^{p^0+1})} = \|\rho_\varepsilon + n_\varepsilon\|_{L^{p^0+1}(0,T;L^{p^0+1})} \leq \|\rho_\varepsilon + n_\varepsilon\|_{L^{1}(0,T;L^{1})} + C\|\nabla \rho_\varepsilon + n_\varepsilon\|_{L^{2}(0,T;L^{2})}.
\]
The combination of (1.3), (2.13) and (2.18)–(2.20) gives rise to \(T_\varepsilon = T\) and (2.10). By (2.14), (2.16) and (2.18), one can show (2.11). The proof of Proposition 2.1 is completed.

With the help of Proposition 2.1 and standard arguments (see [41,44]), we have the following global existence of weak solutions to the IVP (2.1)—(2.2).

**Proposition 2.2.** Let \(p_0 > \gamma + \gamma + \alpha + \bar{\alpha} + 1, \delta \in (0,1)\) and \(\varepsilon \in (0,1)\). Then under the assumptions of either Theorem 1.2 or Theorem 1.6, there exists a global weak solution \((\rho_\varepsilon, n_\varepsilon, (\rho_\varepsilon + n_\varepsilon)u_\varepsilon)\) to the IVP (2.1)—(2.2) such that for any \(T > 0\),
\[
\begin{align*}
\|\rho_\varepsilon + n_\varepsilon\|_{L^\infty(0,T;L^{p_0})} + \varepsilon \|\nabla \rho_\varepsilon, \nabla n_\varepsilon\|_{L^2(0,T;L^2)} & \leq C_\delta, \\
\sqrt{\rho_\varepsilon + n_\varepsilon} u_\varepsilon & \leq c_\delta \rho_\varepsilon \rho_\varepsilon, \quad \text{a.e. } (x,t) \in \Omega \times (0,T), \\
\|\rho_\varepsilon + n_\varepsilon\|_{L^p(0,T;L^p)} & \leq C_\delta,
\end{align*}
\]
where \(C_\delta > 0\) is a constant independent of \(\varepsilon\), and \(c_\delta > 1\) is the constant given by (2.6).

Furthermore, we have
\[
\int_{\Omega} \left[ \rho_\varepsilon + n_\varepsilon \right] + n_\varepsilon \|u_\varepsilon\|^2 + G_\varepsilon(\rho_\varepsilon, n_\varepsilon) \, dx + \int_{0}^{T} \int_{\Omega} \mu \|\nabla u_\varepsilon\|^2 + (\rho + \lambda)(\nabla \rho_\varepsilon)^2 \, dx \, dt \\
\leq \int_{\Omega} \left[ \rho_0 + n_0 \right] + n_0 \|u_0\|^2 + G_0(\rho_0, n_0) \, dx + C_\delta \varepsilon, \quad \text{a.e. } t \in (0,T)
\]
with \(G_\varepsilon(\rho, n)\) defined by (2.12).

### 3 Vanishing artificial viscosity

In this section, we aim to study the following IVP:
\[
\begin{cases}
\partial_t \rho + \text{div}(\rho \mu) = 0, \\
\partial_t n + \text{div}(n \mu) = 0, \\
\partial_t ((\rho + n)u) + \text{div}((\rho + n)u \otimes u) + \nabla P_\delta(\rho, n) = \mu \Delta u + (\mu + \lambda) \nabla \text{div}u, & x \in \Omega, \quad t > 0
\end{cases}
\]
with the initial data

\[(\rho, n, u)(x, 0) = (\rho_{0, \delta}, n_{0, \delta}, u_{0, \delta})(x), \quad x \in \mathbb{T}^d, \quad (3.2)\]

where \(P_{\delta}(\rho, n)\) and \((\rho_{0, \delta}, n_{0, \delta}, u_{0, \delta})\) are given by (2.3) and (2.4), respectively.

First, by (2.21) and standard compactness arguments, we have the following convergence of weak solutions to the IVP (3.1)–(3.2).

**Lemma 3.1.** Let \(T > 0, \rho_0 > \gamma + \bar{\gamma} + \alpha + \bar{\alpha} + 1\) and \((\rho_\varepsilon, n_\varepsilon, (\rho_\varepsilon + n_\varepsilon)u_\varepsilon)\) be the weak solution to the IVP (2.1)–(2.2) for \(\varepsilon \in (0, 1)\) given by Proposition 2.2. Then under the assumptions of either Theorem 1.2 or Theorem 1.6, there is a limit \((\rho, n, (\rho + n)u)\) such that as \(\varepsilon \to 0\), it holds up to a subsequence (still denoted by \((\rho_\varepsilon, n_\varepsilon, (\rho_\varepsilon + n_\varepsilon)u_\varepsilon)\)) that

\[
\begin{align*}
\left\{(\rho_\varepsilon, n_\varepsilon) &\to (\rho, n) \quad \text{in } L^{p_0+1}(0, T; L^{p_0+1}(\mathbb{T}^d)) \times L^{p_0+1}(0, T; L^{p_0+1}(\mathbb{T}^d)), \\
\varepsilon(\nabla \rho_\varepsilon, \nabla n_\varepsilon) &\to 0 \quad \text{in } L^2(0, T; L^2(\mathbb{T}^d)), \\
\varepsilon_\varepsilon &\to u \quad \text{in } L^2(0, T; H^1(\mathbb{T}^d)), \\
(\rho_\varepsilon, n_\varepsilon) &\to (\rho, n) \quad \text{in } C([0, T]; L^{p_0}(\mathbb{T}^d)) \times C([0, T]; L^{p_0}(\mathbb{T}^d)), \\
(\rho_\varepsilon, n_\varepsilon) &\to (\rho, n) \quad \text{in } C([0, T]; H^{-1}(\mathbb{T}^d)) \times C([0, T]; H^{-1}(\mathbb{T}^d)), \\
(\rho_\varepsilon + n_\varepsilon)u_\varepsilon &\to (\rho + n)u \quad \text{in } C([0, T]; L^{p_0}(\mathbb{T}^d)) \cap C([0, T]; H^{-1}(\mathbb{T}^d)).
\end{align*}
\]

Before showing the strong convergence of two densities, we need to have the following lemma.

**Lemma 3.2.** Let \(T > 0, (\rho_\varepsilon, n_\varepsilon, (\rho_\varepsilon + n_\varepsilon)u_\varepsilon)\) be the weak solution to the IVP (2.1)–(2.2) for \(\varepsilon \in (0, 1)\) given by Proposition 2.2, and \((\rho, n, (\rho + n)u)\) be the limit obtained by Lemma 3.1. Then under the assumptions of either Theorem 1.2 or Theorem 1.6, it holds that

\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^d} A_{\rho_\varepsilon, n_\varepsilon} - A_{\rho, n} \rho_\varepsilon \, dx \, dt = 0, \quad p \in [1, \infty),
\]

\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^d} B_{\rho_\varepsilon, n_\varepsilon} - B_{\rho, n} \rho_\varepsilon \, dx \, dt = 0, \quad p \in [1, \infty),
\]

where \((A_{\rho, n}, B_{\rho, n})\) is defined by (1.18).

Furthermore, as \(\varepsilon \to 0\), we have

\[
(\rho_\varepsilon, n_\varepsilon) \to (\rho, n) \quad \text{in } L^1(0, T; L^1(\mathbb{T}^d)) \times L^1(0, T; L^1(\mathbb{T}^d))
\]

\[
\Leftrightarrow \rho_\varepsilon + n_\varepsilon \to \rho + n \quad \text{in } L^1(0, T; L^1(\mathbb{T}^d)).
\]

**Proof.** The proof of (3.4) can be found in [44]. Define \(F_\sigma(\rho, n) := \frac{\sigma^2}{\rho + n + \sigma}\) for any \(\sigma > 0\) so that it holds that

\[
\begin{align*}
\left|\rho_\varepsilon \rho_\sigma F_\sigma(\rho_\varepsilon, n_\varepsilon) + n_\varepsilon \rho_\sigma F_\sigma(\rho_\varepsilon, n_\varepsilon) - F_\sigma(\rho_\varepsilon, n_\varepsilon)\right| &= \frac{\sigma \rho^2}{(\rho + n + \sigma)^2} \leq \sigma, \\
\rho_\varepsilon \rho_\sigma F_\sigma(\rho_\varepsilon, n_\varepsilon) &\cdot \nabla \rho_\varepsilon \cdot \nabla n_\varepsilon + \rho_\varepsilon \rho_\sigma F_\sigma(\rho_\varepsilon, n_\varepsilon) \nabla \rho_\varepsilon \cdot \nabla n_\varepsilon \leq 0,
\end{align*}
\]

\[
\left|\rho_\varepsilon \rho_\sigma F_\sigma(\rho_\varepsilon, n_\varepsilon)\right| + |\rho_\varepsilon \rho_\sigma F_\sigma(\rho_\varepsilon, n_\varepsilon)| \leq 4.
\]

Let \(\eta_\sigma \in C_c^\infty(\mathbb{T}^d)\) for any \(\sigma > 0\) be the Friedrichs mollifier. Applying \(\eta_\sigma * \) to both sides of (2.1)–(2.1)$_2$, we obtain

\[
\begin{align*}
\partial_t \eta_\sigma * \rho_\varepsilon + \text{div}(\eta_\sigma * (\rho_\varepsilon * u_\varepsilon)) &= \varepsilon \Delta(\eta_\sigma * \rho_\varepsilon) + r_1_\sigma, \\
\partial_t \eta_\sigma * \rho_\varepsilon + \text{div}(\eta_\sigma * \rho_\varepsilon u_\varepsilon) &= \varepsilon \Delta(\eta_\sigma * \rho_\varepsilon) + r_1_\sigma, \\
\partial_t \eta_\sigma * n_\varepsilon + \text{div}(\eta_\sigma * n_\varepsilon u_\varepsilon) &= \varepsilon \Delta(\eta_\sigma * n_\varepsilon) + r_2_\sigma, \\
\partial_t \eta_\sigma * n_\varepsilon + \text{div}(\eta_\sigma * n_\varepsilon u_\varepsilon) &= \varepsilon \Delta(\eta_\sigma * n_\varepsilon) + r_2_\sigma.
\end{align*}
\]

By the commutator estimates of Friedrichs mollifier (see [17, 36]), the terms \(r_1_\sigma\) and \(r_2_\sigma\) converge to 0 strongly in \(L^1(0, T; L^1(\mathbb{T}^d))\). Multiplying (3.7)$_1$ and (3.7)$_1$ by \(\partial_t \eta_\sigma * \rho_\varepsilon, \eta_\sigma * \rho_\varepsilon, \eta_\sigma * n_\varepsilon\) and \(\partial_t \eta_\sigma * n_\varepsilon \rho_\varepsilon * u_\varepsilon \)

\[\rho_{\varepsilon, \eta_{\sigma} * n_{\varepsilon}}, \text{ respectively, adding the resulted equations together, and then making use of (3.6), we have}\]
\[
\partial_t F_{\sigma}(\eta_{\sigma} * \rho_{\varepsilon}, \eta_{\sigma} * n_{\varepsilon}) + \text{div}(F_{\sigma}(\eta_{\sigma} * \rho_{\varepsilon}, \eta_{\sigma} * n_{\varepsilon})u_{\varepsilon}) \\
\leq \sigma|\text{div}u_{\varepsilon}| + \varepsilon \Delta F_{\sigma}(\eta_{\sigma} * \rho_{\varepsilon}, \eta_{\sigma} * n_{\varepsilon}) + 4(|r_{1,\sigma}| + |r_{2,\sigma}|),
\]

which gives rise to
\[
\int_0^T \int_{\mathbb{T}^d} F_{\sigma}(\eta_{\sigma} * \rho_{\varepsilon}, \eta_{\sigma} * n_{\varepsilon}) \, dx \, dt \\
\leq T \int_{\mathbb{T}^d} F_{\sigma}(\eta_{\sigma} * \rho_{0,\delta}, \eta_{\sigma} * n_{0,\delta}) \, dx + \sigma T^2 \|\text{div}u_{\varepsilon}\|_{L^2(0,T;L^2)} + 4T\|\sigma\|_{L^1(0,T;L^1)}.
\]  
(3.8)

By the dominated convergence theorem, we take the limit as \(\sigma \to 0\) in (3.8) to derive
\[
\int_0^T \int_{\mathbb{T}^d} \frac{\rho^2}{\rho + n} \, dx \, dt \leq T \int_{\mathbb{T}^d} \frac{\rho^2_{0,\delta}}{\rho_{0,\delta} + n_{0,\delta}} \, dx.
\]  
(3.9)

Similarly, one has
\[
\int_0^T \int_{\mathbb{T}^d} \frac{\rho^2}{\rho + n} \, dx \, dt = T \int_{\mathbb{T}^d} \frac{\rho^2_{0,\delta}}{\rho_{0,\delta} + n_{0,\delta}} \, dx.
\]  
(3.10)

By using (3.3), (3.9)–(3.10) and
\[
\begin{aligned}
\rho &= (\rho + n)A_{\rho,n}, \quad \frac{\rho^2}{\rho + n} = \rho A_{\rho,n} = (\rho + n)A^2_{\rho,n}, \\
n &= (\rho + n)B_{\rho,n}, \quad \frac{n^2}{\rho + n} = n B_{\rho,n} = (\rho + n)B^2_{\rho,n},
\end{aligned}
\]  
(3.11)

we obtain
\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^d} (\rho_{\varepsilon} + n_{\varepsilon})A_{\rho_{\varepsilon},n_{\varepsilon}} - A_{\rho,n}^2 \, dx \, dt \\
= \lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^d} \left( \frac{\rho^2_{\varepsilon}}{\rho_{\varepsilon} + n_{\varepsilon}} - 2\rho_{\varepsilon}A_{\rho,n} + (\rho_{\varepsilon} + n_{\varepsilon})A^2_{\rho,n} \right) \, dx \, dt \\
\leq \int_0^T \int_{\mathbb{T}^d} \left( \frac{\rho^2}{\rho + n} - 2\rho A_{\rho,n} + (\rho + n)A^2_{\rho,n} \right) \, dx \, dt = 0,
\]

which together with Hölder’s inequality and the fact that
\[
0 \leq A_{\rho,n}, B_{\rho,n} \leq 1
\]  
(3.12)

leads to (3.4)1. Similarly, one can verify (3.4)2. Then, we are ready to prove (3.5). Obviously, as \(\varepsilon \to 0\), the strong convergence of both \(\rho_{\varepsilon}\) and \(n_{\varepsilon}\) in \(L^1(0,T;L^1(\mathbb{T}^d))\) implies the strong convergence of \(\rho_{\varepsilon} + n_{\varepsilon}\) in \(L^1(0,T;L^1(\mathbb{T}^d))\). Assume \(\rho_{\varepsilon} + n_{\varepsilon} \to \rho + n\) in \(L^1(0,T;L^1(\mathbb{T}^d))\) as \(\varepsilon \to 0\). By virtue of (3.4)1 for \(p = 1\) and (3.11)–(3.12), we have
\[
\lim_{\varepsilon \to 0} \|\rho_{\varepsilon} - \rho\|_{L^1(0,T;L^1)} \\
= \lim_{\varepsilon \to 0} \|\rho_{\varepsilon} + n_{\varepsilon}\|_{L^1(0,T;L^1)} - (\rho_{\varepsilon} + n_{\varepsilon})A_{\rho,n} + (\rho_{\varepsilon} + n_{\varepsilon})A_{\rho,n} - (\rho + n)A_{\rho,n} \|_{L^1(0,T;L^1)} \\
\leq \lim_{\varepsilon \to 0} \|\rho_{\varepsilon} - (\rho_{\varepsilon} + n_{\varepsilon})A_{\rho,n}\|_{L^1(0,T;L^1)} + \|\rho_{\varepsilon} + n_{\varepsilon} - \rho - n\|_{L^1(0,T;L^1)} = 0,
\]

where we recall \(\rho = (\rho + n)A_{\rho,n}\). Similarly, one can show that \(n_{\varepsilon}\) converges to \(n\) strongly in \(L^1(0,T;L^1(\mathbb{T}^d))\) as \(\varepsilon \to 0\). The proof of Lemma 3.2 is completed. \[\square\]
We consider the cut-off function
\[ T_k(s) := \begin{cases} 
  s, & \text{if } 0 \leq s \leq k, \\
  \text{smooth and concave,} & \text{if } k \leq s \leq 3k, \\
  2k, & \text{if } s \geq 3k,
\end{cases} \quad (3.13) \]
which was introduced in [26] and the references therein. As in [26,37], we can prove the following property of the effect viscous flux.

**Lemma 3.3.** Let \( T > 0, \delta \in (0,1), (\rho_\varepsilon, n_\varepsilon, (\rho_\varepsilon + n_\varepsilon)u_\varepsilon) \) be the weak solution to the IVP (3.1)-(3.2) for \( \varepsilon \in (0,1) \) given by Proposition 2.2, and \((\rho, u, (\rho + n)u)\) be the limit obtained by Lemma 3.1. Then under the assumptions of either Theorem 1.2 or Theorem 1.6, for any \( k > 0 \), it holds up to a subsequence (still denoted by \((\rho_\varepsilon, n_\varepsilon, (\rho_\varepsilon + n_\varepsilon)u_\varepsilon)\)) that
\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^d} (P_k(\rho_\varepsilon, n_\varepsilon) - (2\mu + \lambda) \text{div} u_\varepsilon) T_k(\rho_\varepsilon + n_\varepsilon) dx dt = \int_0^T \int_{\mathbb{T}^d} (P_k(\rho, n) - (2\mu + \lambda) \text{div} u) T_k(\rho + n) dx dt,
\]
where \( T \) denotes the weak limit of \( f_\varepsilon \) as \( \varepsilon \to 0 \).

Finally, we prove the strong convergence of two densities.

**Lemma 3.4.** Let \( T > 0, p_0 > \gamma + \tilde{\gamma} + \alpha + \tilde{\alpha} + 1, (\rho_\varepsilon, n_\varepsilon, (\rho_\varepsilon + n_\varepsilon)u_\varepsilon) \) be the weak solution to the IVP (2.1)-(2.2) for \( \varepsilon \in (0,1) \) given by Proposition 2.2, and \((\rho, n, (\rho + n)u)\) be the limit obtained by Lemma 3.1. Then under the assumptions of either Theorem 1.2 or Theorem 1.6, as \( \varepsilon \to 0 \), it holds up to a subsequence (still denoted by \((\rho_\varepsilon, n_\varepsilon)\)) that
\[
(\rho_\varepsilon, n_\varepsilon) \to (\rho, n) \quad \text{in } L^p(0,T;L^p(\mathbb{T}^d)) \times L^p(0,T;L^p(\mathbb{T}^d)), \quad p \in [1,p_0+1).
\]

**Proof.** Let \( \vartheta_\varepsilon := \rho_\varepsilon + n_\varepsilon, \vartheta := \rho + n \) and \( b_\sigma(s) := (s+\sigma) \log(s+\sigma) \) for \( s \geq 0 \) and \( \sigma > 0 \). By (3.7), one has
\[
\partial_t (\eta_\sigma * \vartheta_\varepsilon) + \text{div}(\eta_\sigma * \vartheta_\varepsilon) u_\varepsilon = \varepsilon \Delta (\eta_\sigma * \vartheta_\varepsilon) + r_{1,\sigma} + r_{2,\sigma},
\]
where \( \eta_\sigma \in C_c^\infty(\mathbb{T}^d) \) for \( \sigma > 0 \) is the Friedrichs mollifier, and \( r_{i,\sigma} \) \( (i = 1, 2) \) are given by (3.7). Multiplying (3.16) by \( b'(\eta_\sigma * \vartheta_\varepsilon) \), we get
\[
\partial_t b(\eta_\sigma * \vartheta_\varepsilon) + \text{div}(b(\eta_\sigma * \vartheta_\varepsilon) u_\varepsilon) + b'(\eta_\sigma * \vartheta_\varepsilon) \eta_\sigma * \vartheta_\varepsilon - b(\eta_\sigma * \vartheta_\varepsilon) \eta_\sigma * \vartheta_\varepsilon \leq \varepsilon \Delta (b(\eta_\sigma * \vartheta_\varepsilon)) + b'(\eta_\sigma * \vartheta_\varepsilon) r_{1,\sigma} + r_{2,\sigma}.
\]
We integrate (3.17) over \( \mathbb{T}^d \times [0,t] \) for \( t \in [0,T] \) and take the limit as \( \sigma \to 0 \) to derive
\[
\int_{\mathbb{T}^d} \partial_\vartheta \log \vartheta dx \leq \int_{\mathbb{T}^d} (\rho_0, \delta + n_0, \delta) \log(\rho_0, \delta + n_0, \delta) dx - \int_0^t \int_{\mathbb{T}^d} \vartheta_\varepsilon \text{div} u_\varepsilon dx dt, \quad \text{a.e. } t \in (0,T).
\]
Similarly, one has
\[
\int_{\mathbb{T}^d} \partial \log \vartheta dx \leq \int_{\mathbb{T}^d} (\rho_0, \delta + n_0, \delta) \log(\rho_0, \delta + n_0, \delta) dx - \int_0^t \int_{\mathbb{T}^d} \partial \text{div} u dx dt, \quad \text{a.e. } t \in (0,T).
\]
Let \( \mathcal{I} \) denote the weak limit of \( f_\varepsilon \) as \( \varepsilon \to 0 \), and \( P_{1,\delta}(\rho, n), P_{2,\delta}(\rho, n), (A_{\rho,n}, B_{\rho,n}) \) and \( T_k(f) \) be defined by (2.8)1, (2.8)2, (1.18) and (3.13), respectively. It can be shown by (2.9), (3.11), (3.14) and (3.18)-(3.19) that
\[
\int_{\mathbb{T}^d} (\vartheta \log \vartheta - \vartheta \log \vartheta) dx \leq \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{T}^d} (\vartheta \text{div} u - \vartheta_\varepsilon \text{div} u_\varepsilon) dx dt = \sum_{i=1}^4 I_i^1, \quad \text{a.e. } t \in (0,T),
\]
where $I_i^4$ ($i = 1, 2, 3, 4$) are given by

$$
I_1^4 := \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{T}^d} \left[ (\vartheta - T_k(\vartheta^*)) \nabla \cdot (\vartheta_\varepsilon - T_k(\vartheta_\varepsilon)) \right] dx dt,
$$

$$
I_2^4 := \frac{1}{2\mu + \lambda} \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{T}^d} \left[ P_3(\rho_\varepsilon, \varepsilon_\varepsilon) - P_3(\rho_{p,n}, \varepsilon_\varepsilon) \right] \left( \frac{1}{T_k(\vartheta_\varepsilon)} - T_k(\vartheta_\varepsilon) \right) dx dt,
$$

$$
I_3^4 := \frac{1}{2\mu + \lambda} \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{T}^d} \left[ P_3(\rho_\varepsilon, \varepsilon_\varepsilon) - P_3(\rho_{p,n}, \varepsilon_\varepsilon) \right] \left( \frac{1}{T_k(\vartheta_\varepsilon)} - T_k(\vartheta_\varepsilon) \right) dx dt,
$$

$$
I_4^4 := \frac{1}{2\mu + \lambda} \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{T}^d} \left[ P_3(\rho_\varepsilon, \varepsilon_\varepsilon) - P_3(\rho_{p,n}, \varepsilon_\varepsilon) \right] \left( \frac{1}{T_k(\vartheta_\varepsilon)} - T_k(\vartheta_\varepsilon) \right) dx dt.
$$

It follows from (2.21), (3.3) and the lower semi-continuity of the weak limit $\vartheta = T_k(\vartheta^*)$ that

$$
I_1^4 \leq \sup_{\varepsilon \to 0} \left( \left\| \vartheta - T_k(\vartheta^*) \right\|_{L^2(0,T;L^2)} \right) \left( \left\| \nabla \vartheta \right\|_{L^2(0,T;L^2)} \right)
$$

$$
\leq C_\delta \sup_{\varepsilon \to 0} \left( \left\| \vartheta_\varepsilon - T_k(\vartheta_\varepsilon) \right\|_{L^2(0,T;L^2)} \right)
$$

$$
\leq \frac{C_\delta \| \vartheta \|_{L^{p_1}(0,T;L^{p_1})}}{k^{p_1-1}} \leq \frac{C_\delta}{k^{p_1-1}}.
$$

For any $x_1, x_2, y_1, y_2 \geq 0$ and $F(x, y) \in C^1(\mathbb{R}_+ \times \mathbb{R}_+)$, we have

$$
F(x_1, y_1) - F(x_2, y_2) = (x_1 - x_2) \int_0^1 \partial_x F(x_2 + \theta(x_1 - x_2), y_2 + \theta(y_1 - y_2)) d\theta
$$

$$
+ (y_1 - y_2) \int_0^1 \partial_y F(x_2 + \theta(x_1 - x_2), y_2 + \theta(y_1 - y_2)) d\theta,
$$

which together with (1.4), (2.3), (2.21) and (3.4) yields

$$
I_1^3 \leq C k \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{T}^d} \left( \vartheta_\varepsilon + 1 \right)^{p_0-1} \vartheta_\varepsilon^{1-p_0} \vartheta_\varepsilon^{1-p_0} \left( |A_{\rho_\varepsilon, \varepsilon_\varepsilon} - A_{\rho_{p,n}}| + |B_{\rho_\varepsilon, \varepsilon_\varepsilon} - B_{\rho_{p,n}}| \right) dx dt
$$

$$
\leq C k \sup_{\varepsilon \to 0} \left( \left( \int_0^t \int_{\mathbb{T}^d} \left( \vartheta_\varepsilon + 1 \right)^{p_0+1} dx dt \right) \right)
$$

$$
\times \lim_{\varepsilon \to 0} \left( \left( \int_0^t \int_{\mathbb{T}^d} \vartheta_\varepsilon \left( |A_{\rho_\varepsilon, \varepsilon_\varepsilon} - A_{\rho_{p,n}}|^{p_0} + |B_{\rho_\varepsilon, \varepsilon_\varepsilon} - B_{\rho_{p,n}}|^{p_0} \right) dx dt \right) \right)^{1/p_0} = 0.
$$

Noticing that both $T_k(s)$ and $P_{3,4}(A_{\rho_{p,n}, s}, B_{\rho_{p,n}, s})$ are monotonically increasing with respect to $s \geq 0$, we apply [25, Theorem 10.19] to have

$$
I_3^1 \leq 0.
$$

Substituting the above estimates of $I_i^1$ ($i = 1, 2, 3$) into (3.20) and taking the limit as $k \to \infty$, we deduce

$$
\int_{\mathbb{T}^d} (\vartheta \log \vartheta - \theta \log \theta) dx dt \leq \frac{1}{2\mu + \lambda} \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{T}^d} \left[ P_2(\rho_\varepsilon, \varepsilon_\varepsilon) - P_2(\rho_{p,n}, \varepsilon_\varepsilon) \right] \left( \vartheta_\varepsilon - \vartheta \right) dx dt.
$$

Since $P_2(sA_{\rho_{p,n}, sB_{\rho_{p,n}}})$ has compact support in $\{s \in \mathbb{R}_+ \mid s \leq 2C_\delta\}$ due to (2.8), there is a constant $C_{s, \delta} > 0$ such that both $C_\delta s \log s - s P_2(A_{\rho_{p,n}, sB_{p,n}, s})$ and $C_\delta s \log s + P_2(A_{\rho_{p,n}, sB_{p,n}s})$ are strictly convex for any $s \geq 0$, and therefore we get

$$
\lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{T}^d} \left[ P_2(\rho_\varepsilon, \varepsilon_\varepsilon) - P_2(\rho_{p,n}, \varepsilon_\varepsilon) \right] \vartheta_\varepsilon dx dt
$$

$$
\leq C_{s, \delta} \int_0^t \int_{\mathbb{T}^d} (\vartheta \log \vartheta - \theta \log \theta) dx dt.
$$
and
\[
\lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{T}^d} [P_{2,\delta}(\rho, n) - P_{2,\delta}(A_{\rho, n} \varphi, B_{\rho, n} \varphi)] \, dx \, dt
\]
\[
\leq \lim_{\varepsilon \to 0} \int_{\{(x, \tau)\in \mathbb{T}^d \times [0, T] \mid \varphi \leq 2C_{c_\delta}\}} [P_{2,\delta}(A_{\rho, n} \varphi, B_{\rho, n} \varphi)] \, dx \, dt
\]
\[
\leq C_{\ast} \lim_{\varepsilon \to 0} \int_{\{(x, \tau)\in \mathbb{T}^d \times [0, T] \mid \varphi \leq 2C_{c_\delta}\}} (\varphi \log \varphi - \varphi \log \varphi) \, dx \, dt
\]
\[
\leq 2C_{\ast} C_{c_\delta} \int_0^t \int_{\mathbb{T}^d} (\varphi \log \varphi - \varphi \log \varphi) \, dx \, dt,
\]
where one used (3.11), the fact that \(P_{2,\delta}(sA_{\rho, n}, sB_{\rho, n}) \geq 0\), the compact support of \(P_{2,\delta}(sA_{\rho, n}, sB_{\rho, n})\) and the convexity of \(C_{\ast} s \log s - P_{2,\delta}(sA_{\rho, n}, sB_{\rho, n})\). By (3.24)–(3.26), we have
\[
\int_{\mathbb{T}^d} (\varphi \log \varphi - \varphi \log \varphi) \, dx \leq \frac{C_{\ast} C_{c_\delta} + 1}{2\mu + \lambda} \int_0^t \int_{\mathbb{T}^d} (\varphi \log \varphi - \varphi \log \varphi) \, dx \, dt.
\]
The combination of (2.21), (3.5), (3.27), the Grönwall inequality and the convexity of \(s \log s\) gives rise to (3.15). The proof of Lemma 3.4 is completed.

With the help of Proposition 2.2 and Lemmas 3.1–3.4, we have the global existence of weak solutions to the IVP (3.1)–(3.2).

**Proposition 3.5.** Let \(p_0 > \gamma + \tilde{\gamma} + \alpha + \tilde{\alpha} + 1\) and \(\delta \in (0, 1)\). Then under the assumptions of either Theorem 1.2 or Theorem 1.6, there exists a global weak solution \((\rho_\delta, n_\delta, (\rho_\delta + n_\delta) u_\delta)\) to the IVP (3.1)–(3.2) satisfying for any \(T > 0\) that
\[
\begin{align*}
\|\rho_\delta\|_{L^\infty(0,T; L^\gamma)} + \|n_\delta\|_{L^\infty(0,T; L^\alpha)} + \delta \frac{C}{2\mu + \lambda} \|\rho_\delta, n_\delta\|_{L^\infty(0,T; L^\mu)} & \leq C, \\
\|\sqrt{\rho_\delta} + n_\delta u_\delta\|_{L^\infty(0,T; L^2)} + \|u_\delta\|_{L^2(0,T; H^1)} & \leq C, \\
\rho_\delta(x, t) & \geq 0, \quad n_\delta(x, t) \geq 0, \quad a.e. \quad (x, t) \in \mathbb{T}^d \times (0, T), \\
\varphi_\delta(x, t) & \leq n_\delta(x, t) \leq \varphi_\delta(x, t), \quad if \quad \varphi_\delta(x) \leq n_\delta(x) \leq \varphi_\delta(x), \quad a.e. \quad (x, t) \in \mathbb{T}^d \times (0, T),
\end{align*}
\]
where \(C > 0\) is a constant independent of \(\delta\).

Furthermore, it holds that
\[
\int_{\mathbb{T}^d} \left[ \frac{1}{2} (\rho_\delta + n_\delta) |u_\delta|^2 + G_\delta(\rho_\delta, n_\delta) \right] dx + \int_0^t \int_{\mathbb{T}^d} \left[ ||\nabla u_\delta||^2 + (\mu + \lambda) (\text{div} u_\delta)^2 \right] dx \, dt
\]
\[
\leq \int_{\mathbb{T}^d} \left[ \frac{1}{2} (\rho_\delta + n_\delta) |u_\delta|^2 + G_\delta(\rho_\delta, n_\delta) \right] dx, \quad a.e. \quad t \in (0, T),
\]
where \(G_\delta(\rho, n)\) is given by (2.12).

4 Vanishing artificial pressure
4.1 Higher integrability of densities

In this section, we prove Theorems 1.2 and 1.6 by taking the limit in (3.1) as \(\delta \to 0\). Since the uniform estimates (3.28) are not enough to show the convergence of \(P_\delta(\rho_\delta, n_\delta)\) to \(P(\rho, n)\) as \(\delta \to 0\), we need the higher integrability of \(\rho_\delta\) and \(n_\delta\) uniformly in \(\delta\) as follows.

**Lemma 4.1.** Let \(T > 0\), \(p_0 > \gamma + \tilde{\gamma} + \alpha + \tilde{\alpha} + 1\), \((\rho_\delta, n_\delta, (\rho_\delta + n_\delta) u_\delta)\) be the weak solution to the IVP (3.1)–(3.2) for \(\delta \in (0, 1)\) given by Proposition 3.5, and \(\varphi_\delta := \rho_\delta + n_\delta\). Then under the assumptions of Theorem 1.2, it holds that
\[
\int_0^T \int_{\mathbb{T}^d} (\varphi_\delta^{\max(\gamma, \alpha)} + \delta \varphi_\delta^{\alpha + \theta}) \, dx \, dt \leq C
\]
with $C > 0$ a constant independent of $\delta$ and

$$\theta := \frac{2}{d} \max\{\gamma, \alpha\} - 1 > 0.$$  

In addition, under the assumptions of Theorem 1.6, we have

$$\int_0^T \int_{\mathbb{T}^d} \left( \rho_b^{\gamma+\theta_1} + n_b^{\gamma+\theta_2} + \delta \rho^\gamma \partial_\nu + \delta \rho^\nu \partial_\gamma \right) dx dt \leq C$$  

(4.2)

with

$$\theta_1 := \frac{2}{d} \gamma - \frac{\gamma}{\min\{\gamma, \alpha\}} > 0, \quad \theta_2 := \frac{2}{d} \alpha - \frac{\alpha}{\min\{\gamma, \alpha\}} > 0.$$  

Proof. We are ready to show (4.2), and (4.1) can be proved by similar arguments. Denote by $F_\delta$ the effect viscous flux as

$$F_\delta := F_0(\rho, n) - \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}^d} F_0(\rho, n) dx - (2\mu + \lambda) \text{div} u.$$  

(4.3)

One deduces from the equation (3.1) that

$$F_\delta = (-\Delta)^{-1} \text{div}[\partial_t(\partial_\delta u) + \text{div}(\partial_\delta u \otimes u)]$$  

(4.4)

where the operator $(-\Delta)^{-1} : W^{k-2,p} (\mathbb{T}^d) \to W^{k,p} (\mathbb{T}^d)$ for $k \in \mathbb{R}$ and $p \in (1, \infty)$ is defined by

$$(-\Delta)^{-1} f = g.$$  

(4.5)

Here, $g$ is the solution to the problem $-\Delta g = f,$ $\int_{\mathbb{T}^d} f = 0.$ It can be verified by the Sobolev inequality and $L^p(\mathbb{T}^d)-L^p(\mathbb{T}^d)$ boundedness of the operator $\nabla (-\Delta)^{-1} \text{div}$ that

$$\|(-\Delta)^{-1} \text{div} f\|_{L^p(\mathbb{T}^d)} \leq C \|\nabla (-\Delta)^{-1} \text{div} f\|_{L^p(\mathbb{T}^d)} \leq C \|f\|_{L^p}, \quad \forall f \in L^p, \quad p \in (1,d).$$  

(4.6)

By (3.1) and (4.4), we obtain

$$\int_0^T \int_{\mathbb{T}^d} \rho_\delta^{\theta_1} P_\delta(\rho, n) dx dt = \sum_{i=1}^4 I_i^2.$$  

(4.7)

with

$$I_1 := \int_0^T \int_{\mathbb{T}^d} \rho_\delta^{\theta_1} \left( \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}^d} P_0(\rho, n) dx + (2\mu + \lambda) \text{div} u \right) dx dt,$$

$$I_2 := \int_0^T \int_{\mathbb{T}^d} \rho_\delta^{\theta_1} (-\Delta)^{-1} \text{div}(\partial_\delta u) dx dt,$$

$$I_3 := (\theta - 1) \int_0^T \int_{\mathbb{T}^d} \rho_\delta^{\theta_1} \text{div}(\partial_\delta u) dx dt,$$

$$I_4 := \sum_{i,j=1}^d \int_0^T \int_{\mathbb{T}^d} \rho_\delta^{\theta_1} \left[ R_i R_j (\partial_\delta u_i u_j) - u_i^2 R_i R_j (\partial_\delta u_j) \right] dx dt, \quad R_i := (-\Delta)^{-1} \partial_i, \quad i = 1, \ldots, d.$$  

First, one deduces from (3.28) and $2\theta_1 < \gamma + \theta_1$ that

$$I_1^2 \leq T \|\rho_\delta^{\theta_1}\|_{L^\infty(0,T;L^1)} \|P_0(\rho, n)\|_{L^\infty(0,T;L^1)} + (2\mu + \lambda) T^{\frac{d}{2}} \|\rho_\delta^{\theta_1}\|_{L^2(0,T;L^2)} \|\text{div} u\|_{L^2(0,T;L^2)}$$

$$\leq C \left( 1 + \frac{1}{\varepsilon_0} \right) + \varepsilon_0 \int_0^T \int_{\mathbb{T}^d} \rho_\delta^{\gamma+\theta_1} dx dt$$

with the constant $\varepsilon_0 > 0$ to be chosen later. By (3.28), one has

$$\|\partial_\delta\|_{L^\infty(0,T;L^\infty)} + \|\partial_\delta u\|_{L^\infty(0,T;L^{2\gamma_0})} \leq C,$$

(4.8)
where we used the convenient notation $\gamma_0 := \min\{\gamma, \alpha\}$. For the term $I_2^1$, making use of (2.6), (3.28), (4.6), (4.8) and $\frac{1}{\gamma} + 1 - \frac{\alpha}{\gamma} > \frac{2n+1}{2n\gamma}$, we have

$$
I_2^1 \leq \|\rho_0^\delta\|_{L^{\infty}(0,T;L^{\frac{2m+3}{m}})} \|(-\Delta)^{-1}\text{div}(\theta_0 u_\delta)\|_{L^{\infty}(0,T;L^{\frac{2m}{2m-3}})} + \|\rho_0\|_{L^1} \|(-\Delta)^{-1}\text{div}((\rho_0, n_\delta) u_\delta)\|_{L^{\infty}(0,T;L^{\frac{2m}{2m-3}})} \\
\leq C\|\rho_0^\delta\|_{L^{\infty}(0,T;L^2)} \|\theta_0 u_\delta\|_{L^{\infty}(0,T;L^{\frac{2m}{m}})} + C\|\rho_0\|_{L^1} \|((\rho_0, n_\delta) u_\delta\|_{L^{\infty}(0,T;L^{\frac{2m}{m}})} \leq C.
$$

As in [37], we analyze the terms $I_i^2$ ($i = 3, 4$) for two cases of $d \geq 3$ and $d = 2$, respectively.

**Case 1.** $d \geq 3$.

Since it holds that $\gamma_0 + \frac{1}{\gamma} + d - \frac{2}{d} = 1$, we obtain by (3.28), (4.8), the Sobolev embedding $H^1(\mathbb{T}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{T}^d)$ and the boundedness of the Riesz operator that

$$
I_3^2 \leq C\|\rho_0^\delta\|_{L^{\infty}(0,T;L^{\frac{2m}{m}})} \|\theta_0\|_{L^{\infty}(0,T;L^{\gamma_0})} \|u_\delta\|_{L^2(0,T;L^{\frac{2m}{m}})} \leq C.
$$

Similarly, the term $I_2^2$ for $d \geq 4$ can be estimated by

$$
I_2^2 \leq C\|\rho_0^\delta\|_{L^{\infty}(0,T;L^{\frac{2m}{m}})} \|\theta_0\|_{L^{\infty}(0,T;L^{\gamma_0})} \|u_\delta\|_{L^2(0,T;L^{\frac{2m}{m}})} \leq C.
$$

For $d = 3$, it follows from (3.28), (4.6), (4.8) and the Sobolev embedding $H^1(\mathbb{T}^3) \hookrightarrow L^6(\mathbb{T}^3)$ that

$$
I_3^2 \leq C\|\rho_0^\delta\|_{L^{\frac{2m}{m+3}}(0,T;L^{\frac{2m}{m+3}})} \|\theta_0\|_{L^{\frac{2m}{m+3}}(0,T;L^{\gamma_0})} \|u_\delta\|_{L^2(0,T;L^{\frac{2m}{m}})} \leq C\left(1 + \frac{1}{\varepsilon_0}\right) + \varepsilon_0 \int_0^T \int_{\mathbb{T}^3} \rho_3^{\gamma+\delta} \text{d}x\text{d}t.
$$

**Case 2.** $d = 2$.

One can show by (3.28) and (4.6) that

$$
I_3^2 \leq C\|\rho_0^\delta\|_{L^{\frac{2m}{m+4}}(0,T;L^{\frac{2m}{m+4}})} \|\theta_0\|_{L^{\frac{2m}{m+4}}(0,T;L^{\gamma_0})} \|u_\delta\|_{L^2(0,T;L^{\frac{2m}{m}})} \leq C\left(1 + \frac{1}{\varepsilon_0}\right) + \varepsilon_0 \int_0^T \int_{\mathbb{T}^2} (\rho_3^{\gamma+\delta} + u_3^{\alpha+\delta}) \text{d}x\text{d}t.
$$

In addition, one deduces by the critical Sobolev embedding $H^1(\mathbb{T}^2) \hookrightarrow BMO(\mathbb{T}^2)$ for $d = 2$ and the commutator estimate (A.11) that

$$
I_2^2 \leq \sum_{i,j=1}^2 \|\rho_0^\delta\|_{L^{\frac{2m}{m+4}}(0,T;L^{\frac{2m}{m+4}})} \|\mathcal{R}_i\mathcal{R}_j (\theta_0 u_\delta)\|_{L^{\frac{2m}{m+4}}(0,T;L^{\frac{2m}{m+4}})} \leq C\|\rho_0^\delta\|_{L^{\frac{2m}{m+4}}(0,T;L^{\frac{2m}{m+4}})} \|\theta_0\|_{L^2(0,T;BMO)} \|u_\delta\|_{L^2(0,T;H^1)} \leq C\left(1 + \frac{1}{\varepsilon_0}\right) + \varepsilon_0 \int_0^T \int_{\mathbb{T}^2} (\rho_3^{\gamma+\delta} + u_3^{\alpha+\delta}) \text{d}x\text{d}t.
$$

Substituting the above estimates of $I_i^2$ ($i = 1, 2, 3, 4$) into (4.7), we obtain

$$
\int_0^T \int_{\mathbb{T}^d} \rho_3^\delta P_3(\rho_3, n_3) \text{d}x\text{d}t \leq C \left(1 + \frac{1}{\varepsilon_0}\right) + 3\varepsilon_0 \int_0^T \int_{\mathbb{T}^d} (\rho_3^{\gamma+\delta} + u_3^{\alpha+\delta}) \text{d}x\text{d}t. \quad (4.9)
$$
Similarly, one can show
\[
\int_0^T \int_{\mathbb{R}^d} n_\delta \eta \beta_\delta \rho \, dx \, dt \leq C \left( 1 + \frac{1}{\varepsilon_0} \right) + \frac{3\varepsilon_0}{T} \int_0^T (\rho^\beta \eta + n_\delta \eta^2) \, dx \, dt. \tag{4.10}
\]
By (1.4) and (2.3), it holds that
\[
(\rho^\beta + n_\delta \eta^2) P_\delta(\rho, n_\delta) \geq (\rho^\beta + n_\delta \eta^2) (P(\rho_\delta, n_\delta) + \delta(\rho_\delta + n_\delta) \rho_\delta - 1)_{\delta} = \frac{1}{2C_H} (\rho^\beta + n_\delta \eta^2 + \delta(\rho_\delta + n_\delta)(\rho_\delta + n_\delta)^2) - C. \tag{4.11}
\]
Combining (4.9)–(4.11) and choosing \( \varepsilon_0 = \frac{1}{2C_H} \), we derive (4.2). The proof of Lemma 4.1 is completed.

By (3.28) and (4.11), we have the convergence of weak solutions to the IVP (3.1)–(3.2).

**Lemma 4.2.** Let \( T > 0, \rho_0 > \gamma + \gamma + \alpha + \alpha + 1, \) and \((\rho_\delta, n_\delta, (\rho_\delta + n_\delta) u_\delta)\) be the weak solution to the IVP (3.1)–(3.2) for \( \delta \in (0, 1) \) given by Proposition 3.5. Then under the assumptions of either Theorem 1.2 or Theorem 1.6, there is a limit \((\rho, n, (\rho + n) u)\) such that as \( \delta \to 0 \), it holds up to a subsequence (still denoted by \((\rho_\delta, n_\delta, (\rho_\delta + n_\delta) u_\delta)\)) that

\[
\begin{aligned}
(\rho_\delta, n_\delta) &\to (\rho, n) \quad \text{in } L^\infty(0, T; L^\gamma(\mathbb{T}^d)) \times L^\infty(0, T; L^\alpha(\mathbb{T}^d)), \\
\delta(\rho_\delta + n_\delta) u_\delta &\to 0 \quad \text{in } L^1(0, T; L^1(\mathbb{T}^d)), \\
u_\delta &\to u \quad \text{in } L^2(0, T; H^1(\mathbb{T}^d)), \\
(\rho_\delta, n_\delta) &\to (\rho, n) \quad \text{in } C([0, T]; L^\gamma_{\text{weak}}(\mathbb{T}^d)) \times C([0, T]; L^\gamma_{\text{weak}}(\mathbb{T}^d)), \\
(\rho_\delta, n_\delta) &\to (\rho, n) \quad \text{in } C([0, T]; \tilde{H}^{-1}(\mathbb{T}^d)) \times C([0, T]; \tilde{H}^{-1}(\mathbb{T}^d)), \\
(\rho_\delta + n_\delta) u_\delta &\to (\rho + n) u \quad \text{in } C([0, T]; L^2_{\text{weak}}(\mathbb{T}^d)) \cap C([0, T]; \tilde{H}^{-1}(\mathbb{T}^d)).
\end{aligned} \tag{4.12}
\]

**4.2 Strong convergence of densities**

By similar arguments as used in Lemma 3.2, the strong convergence of two densities \( \rho_\delta \) and \( n_\delta \) and the strong convergence of their sum \( \rho_\delta + n_\delta \) are equivalent.

**Lemma 4.3.** Let \( T > 0, (\rho_\delta, n_\delta, (\rho_\delta + n_\delta) u_\delta) \) be the weak solution to the IVP (3.1)–(3.2) for \( \delta \in (0, 1) \) given by Proposition 3.5, and \((\rho, n, (\rho + n) u)\) be the limit obtained by Lemma 4.2. Then under the assumptions of either Theorem 1.2 or Theorem 1.6, as \( \delta \to 0 \), it holds that

\[
\begin{aligned}
(\rho_\delta, n_\delta - (\rho_\delta + n_\delta) A_{\rho, n}) &\to 0 \quad \text{in } L^1(0, T; L^1(\mathbb{T}^d)), \\
(\rho_\delta + n_\delta) u_\delta - (\rho_\delta + n_\delta) B_{\rho, n} &\to 0 \quad \text{in } L^1(0, T; L^1(\mathbb{T}^d)),
\end{aligned} \tag{4.13}
\]

where \((A_{\rho, n}, B_{\rho, n})\) is defined by (1.18).

Furthermore, as \( \delta \to 0 \), we have

\[
(\rho_\delta, n_\delta) \to (\rho, n) \quad \text{in } L^1(0, T; L^1(\mathbb{T}^d)) \times L^1(0, T; L^1(\mathbb{T}^d))
\]

\[
\Leftrightarrow \rho_\delta + n_\delta \to \rho + n \quad \text{in } L^1(0, T; L^1(\mathbb{T}^d)). \tag{4.14}
\]

As in [2,10,11], introduce the symmetric periodic kernel

\[
K_h(x) := \begin{cases} 
1 & \text{if } 0 \leq |x| \leq \frac{1}{2}, \\
\frac{1}{(h + |x|)^d}, & \text{if } \frac{1}{2} \leq |x| \leq \frac{2}{3}, \\
\text{smooth,} & \text{if } \frac{2}{3} \leq |x| \leq \frac{3}{4}, \\
\text{smooth and independent of } h, & \text{if } \frac{3}{4} \leq |x| \leq 1. 
\end{cases} \tag{4.15}
\]
It is easy to show for a constant $C > 1$ independent of $h$ and a suitably small constant $h_0 \in (0, 1)$ that

$$|x| |\nabla \mathcal{K}_h(x)| \leq C \mathcal{K}_h(x), \quad \frac{1}{C} |\log h| \leq \|\mathcal{K}_h\|_{L^1} \leq C |\log h|, \quad h \in (0, h_0). \quad (4.16)$$

Define the functional

$$L_{h,p}(f) := \int_{\mathbb{T}^d} \mathcal{K}_h(x - y)|\Lambda[f]|^p dx dy,$$  \quad (4.17)

where we use the convenient notations

$$\Lambda[f] := f^x - f^y, \quad f^x := f(x, t), \quad \mathcal{K}_h := \frac{K_h}{\|K_h\|_{L^1}}.$$

By virtue of Lemma A.3 below, to derive the strong convergence of $\vartheta_\delta := \rho_\delta + n_\delta$, we need

$$\lim h \to 0 \lim \sup_{\delta \to 0} \ess \sup_{t \in (0, T]} L_{h,1}(\vartheta_\delta) = 0.$$  \quad (4.18)

It is shown in Lemma 4.4 below that the above estimate is the same thing as

$$\lim h \to 0 \lim \sup_{\delta \to 0} \ess \sup_{t \in (0, T]} \int_{\mathbb{T}^d} \mathcal{K}_h(x - y)\chi(\Lambda[\vartheta_\delta])(w_3^x + w_3^y) dx dy = 0,$$

where $w_3$ is given by (4.21) below, and $\chi$ is the function

$$\chi(s) := \begin{cases} |s|^2, & \text{if } |s| \leq 1, \\ \text{smooth, if } 1 \leq |s| \leq 2, \\ |s|, & \text{if } |s| \geq 2, \end{cases} \quad (4.19)$$

which satisfies

$$\begin{cases} \left|\chi(s) - \frac{1}{2} \chi'(s) s\right| \leq \frac{1}{2} \chi'(s)s, & 0 \leq \chi'(s)s \leq C \chi(s) \leq C |s|, \quad s \in \mathbb{R}, \\ \chi(s) \geq \frac{1}{C} |s|, & |s| \geq 1. \end{cases} \quad (4.20)$$

**Lemma 4.4.** Let $T > 0$, $h \in (0, h_0)$, $(\rho_\delta, n_\delta, (\rho_\delta + n_\delta)u_\delta)$ be the weak solution to the IVP (3.1)–(3.2) for $\delta \in (0, 1)$ given by Proposition 3.5, $\vartheta_\delta := \rho_\delta + n_\delta$, and $w_3$ be the solution to the auxiliary problem

$$\begin{align*}
\partial_t w_3 + u_\delta \cdot \nabla w_3 + \lambda_0 \xi \delta w_3 &= 0, \quad x \in \mathbb{T}^d, \quad t \in (0, T], \\
\xi \delta := & \vartheta_\delta [\text{div} u_\delta] + |\text{div} u_\delta| + M |\nabla u_\delta| + \rho_\delta^* + \rho_\delta^2 + n_\delta^2 + n_\delta^3 + 1, \\
w_3(x, 0) &= e^{-\lambda_0(\rho_\delta + n_\delta)}(x), \quad x \in \mathbb{T}^d, \quad (4.21)
\end{align*}$$

where $\lambda_0 \geq 1$ is a constant to be chosen, and $M$ is the localized maximal operator defined by (A.9). Then under the assumptions of either Theorem 1.2 or Theorem 1.6, we have

$$\ess \sup_{t \in [0, T]} \int_{\mathbb{T}^d} \vartheta_\delta \log (1 + |\log w_3|) dx \leq C(1 + \lambda_0), \quad (4.22)$$

and for $\sigma_* > 0$ to be chosen later,

$$\ess \sup_{t \in [0, T]} (L_{h,1}(\vartheta_\delta))^2 \leq \frac{C(1 + \lambda_0)}{\log (1 + |\log \sigma_*|)} + \frac{1}{\sigma_*} \ess \sup_{t \in [0, T]} \int_{\mathbb{T}^d} \mathcal{K}_h(x - y)\chi(\Lambda[\vartheta_\delta])(w_3^x + w_3^y) dx dy, \quad (4.23)$$

where $L_{h,1}(f)$, $\Lambda[f]$, $f^x$, $\mathcal{K}_h$ and $\chi$ are given by (4.17), (4.18), (4.18)$_1$, (4.18)$_2$, (4.18)$_3$ and (4.19), respectively, and $C > 0$ is a constant independent of $\delta$, $h$ and $\sigma_*$. 

Proof. First, it follows from the maximal principle for the transport equation (4.21) that

$$0 \leq w_\delta \leq 1,$$

(4.24)

which leads to $|\log w_\delta| = -\log w_\delta$. Since it holds that

$$\partial_t \vartheta_\delta + u_\delta \cdot \nabla \vartheta_\delta + \vartheta_\delta \text{div} u_\delta = 0 \quad \text{in } D'(\mathbb{T}^d \times (0, T)),$$

(4.25)

one derives by (4.21) and the argument of renormalized solutions as in (3.16)–(3.18) that

$$\partial_t e^{-\lambda_0 \vartheta_\delta} + u_\delta \cdot \nabla e^{-\lambda_0 \vartheta_\delta} + \lambda_0 \Xi_{\vartheta_\delta} e^{-\lambda_0 \vartheta_\delta} = \lambda_0 (\Xi_{\vartheta_\delta} + \vartheta_\delta \text{div} u_\delta) e^{-\lambda_0 \vartheta_\delta} \geq 0 \quad \text{in } D'(\mathbb{T}^d \times (0, T)).$$

(4.26)

From the comparison principle for (4.21) and (4.26), we have (4.22).

Next, by (4.22) and the argument of renormalized solutions (formally multiplying (4.21) by $-\vartheta_\delta (1 + |\log w_\delta|) w_\delta^{-1}$), we obtain

$$\partial_t (\vartheta_\delta \log (1 + |\log w_\delta|)) + \text{div}(u_\delta \vartheta_\delta \log (1 + |\log w_\delta|))$$

$$= \frac{\lambda_0 \vartheta_\delta \Xi_{\vartheta_\delta}}{1 + |\log w_\delta|} \leq \frac{\lambda_0 \vartheta_\delta \Xi_{\vartheta_\delta}}{1 + \lambda_0 \vartheta_\delta} \leq \Xi_{\vartheta_\delta} \quad \text{in } D'(\mathbb{T}^d \times (0, T)).$$

(4.27)

Since we have

$$\int_{\mathbb{T}^d} \vartheta_\delta \log (1 + |\log w_\delta|) |_{t=0} dx \leq C \lambda_0,$$

and the term $\Xi_{\vartheta_\delta}$ on the right-hand side of (4.27) is uniformly bounded in $L^1(0, T; L^1(\mathbb{T}^d))$ due to (1.12) and (1.15), (2.28) and (4.1)–(4.2), one shows (4.22) after integrating (4.27) over $\mathbb{T}^d \times [0, t]$.

Finally, one has by (4.19) and (4.20) that

$$L_{h, 1}(\vartheta_\delta) = \left( \int_{\{|x, y| \leq 2^d |\Lambda[\vartheta_\delta]| \leq 1\}} \mathcal{K}_h(x - y) |\Lambda[\vartheta_\delta]| dx dy + \int_{\{|x, y| \geq 2^d |\Lambda[\vartheta_\delta]| \geq 1\}} \mathcal{K}_h(x - y) |\Lambda[\vartheta_\delta]| dx dy \right) \leq C \left( \int_{\mathbb{T}^d} \mathcal{K}_h(x - y) \chi(|\Lambda[\vartheta_\delta]|) dx dy \right)^{\frac{1}{2}}.$$

(4.28)

Owing to (4.19), Young’s inequality of convolution type and $w_x^\delta + w_y^\delta \geq \sigma_\ast$ for $w_x^\delta \geq \sigma_\ast$ or $w_y^\delta \geq \sigma_\ast$, we obtain

$$\int_{\mathbb{T}^d} \mathcal{K}_h(x - y) \chi(|\Lambda[\vartheta_\delta]|) dx dy$$

$$\leq \left( \int_{\{|x, y| \leq 2^d |\Lambda[\vartheta_\delta]| \leq 1\}} \mathcal{K}_h(x - y) \chi(|\Lambda[\vartheta_\delta]|) dx dy + \int_{\{|x, y| \geq 2^d |\Lambda[\vartheta_\delta]| \geq 1\}} \mathcal{K}_h(x - y) \chi(|\Lambda[\vartheta_\delta]|) dx dy \right) \leq C \left( \int_{\mathbb{T}^d} \mathcal{K}_h(x - y) \chi(|\Lambda[\vartheta_\delta]|) (w_x^\delta + w_y^\delta) dx dy, \right.$$

which together with (4.22) and (4.28) leads to (4.23). \qed

With the help of the continuity equation (4.25), we have the quantitative regularity estimate with weight.

**Lemma 4.5.** Let $T > 0$, $h \in (0, h_0)$, $\lambda_0 \geq 1$, $(\rho_3, \nu_3, (\rho_3 + \nu_3) u_3)$ be the weak solution to the IVP (3.1)–(3.2) for $\delta \in (0, 1)$ given by Proposition 3.5, $\vartheta_\delta := \rho_3 + \nu_3$, and $w_\delta$ be the solution to the IVP (4.21). Then under the assumptions of either Theorem 1.2 or Theorem 1.6, it holds for $t \in [0, T]$ that

$$\int_{\mathbb{T}^d} \mathcal{K}_h(x - y) \chi(|\Lambda[\vartheta_\delta]|) (w_x^\delta + w_y^\delta) dx dy$$

$$\leq L_{h, 1}(\rho_0, \nu_0, \nu_0) + \frac{C}{|\log h|^2} + (C - 2\lambda_0) \int_0^t \int_{\mathbb{T}^d} \mathcal{K}_h(x - y) \chi(|\Lambda[\vartheta_\delta]|) \Xi_{\vartheta_\delta} w_x^\delta dx dy d\tau.$$
By virtue of (4.21), (4.30) and the argument of renormalized solutions as in (3.1), one can show
\[\int_{\mathbb{T}^d} \mathcal{K}_h(x-y)\chi'(\Lambda[\vartheta_3]) (\vartheta_x^y + \vartheta_y^y) \, dx \, dy \]
where
\[\mathcal{K}_h(x-y) = \int_0^1 f^x(T) \, d\lambda_h(x-y) \chi \Lambda[\vartheta_3] \vartheta_x^y + \vartheta_y^y \]
and
\[\chi \Lambda[\vartheta_3] \vartheta_x^y + \vartheta_y^y \in \mathcal{D}'(\mathbb{T}^d \times [0, T]).\]  
(4.30)

By virtue of (4.21), (4.30) and the argument of renormalized solutions as in (3.16)–(3.18) (formally multiplying (4.30) by \(\mathcal{K}(x-y)\chi'(\Lambda[\vartheta_3])(\vartheta_x^y + \vartheta_y^y)\)) and integrating the resulted equation over \(\mathbb{T}^d \times [0, t]\), one can show
\[\int_{\mathbb{T}^d} \mathcal{K}_h(x-y)\chi'(\Lambda[\vartheta_3]) (w_x^y + w_y^y) \, dx \, dy \]
where \(I^3_i\) (\(i = 1, 2, 3, 4\)) are given by
\[I^3_1 := 2 \int_0^t \int_{\mathbb{T}^d} \nabla \mathcal{K}_h(x-y) \cdot \Lambda[u_3] \chi' \Lambda[\vartheta_3] \vartheta_x^y \, dx \, dy \, dt,
I^3_2 := 2 \int_0^t \int_{\mathbb{T}^d} \mathcal{K}_h(x-y)\chi'(\Lambda[\vartheta_3]) (\vartheta_x^y + \vartheta_y^y \cdot \nabla_x \vartheta_x^y) \, dx \, dy \, dt,
I^3_3 := \int_0^t \int_{\mathbb{T}^d} \mathcal{K}_h(x-y)\chi'(\Lambda[\vartheta_3]) (\vartheta_x^y + \vartheta_y^y \cdot \nabla_x \vartheta_x^y) \, dx \, dy \, dt,
I^3_4 := -\int_0^t \int_{\mathbb{T}^d} \mathcal{K}_h(x-y)\chi'(\Lambda[\vartheta_3]) (\vartheta_x^y + \vartheta_y^y \cdot \nabla_x \vartheta_x^y) \, dx \, dy \, dt.

By (4.16), (4.20), (4.22), (A.6) and (A.10), it holds that
\[I^3_1 \leq C \int_0^t \int_{\mathbb{T}^d} \mathcal{K}_h(x-y) (D_xu) (\vartheta_x^y + \vartheta_y^y) \chi' \Lambda[\vartheta_3] \vartheta_x^y \, dx \, dy \, dt,
\[\leq C \int_0^t \int_{\mathbb{T}^d} \mathcal{K}_h(x-y) (|D_xu| u_y^y) \chi' \Lambda[\vartheta_3] \vartheta_x^y \, dx \, dy \, dt,
\[\leq C \int_0^t \|\vartheta_3(\tau)\|_{L^2} \int_{\mathbb{T}^d} |\mathcal{K}_h(x) ||D_xu^y| - D_xu_y(\tau)| \, dx \, d\tau,
\[+ C \int_0^t \int_{\mathbb{T}^d} \mathcal{K}_h(x-y)\chi' \Lambda[\vartheta_3] \vartheta_x^y \, dx \, dy \, dt,
\[\leq C \frac{\|\vartheta_3\|_{L^2}}{\sqrt{|\log h|}} \cdot \frac{\|u_y\|_{L^2(0,T;H^1)}}{\|u_y\|_{L^2(0,T;H^1)}} + C \int_0^t \int_{\mathbb{T}^d} \mathcal{K}_h(x-y)\chi' \Lambda[\vartheta_3] \vartheta_x^y \, dx \, dy \, dt,
\]
where $D_{x[1]}f$ and $Mf$ are defined by (A.7) and (A.9), respectively. For $I_2^3$, it follows from (4.21) that

$$I_2^3 = -2\lambda_0 \int_0^t \int_{Q^{2a}} \mathcal{K}_h(x-y)\chi(\Lambda[\vartheta_S])\Xi^\delta_w \omega^2 dxdydt.$$ 

In addition, by (4.20)1 and the fact that

$$\begin{align*}
(2\chi(\Lambda[\vartheta_S]) - \chi'(\Lambda[\vartheta_S])\Lambda[\vartheta_S])(\partial_x \omega^2 + \partial_y \omega^2) + (\theta^2 + \theta_y^2)\chi'(\Lambda[\vartheta_S])\Lambda[\div \omega] \\
= 2(2\chi(\Lambda[\vartheta_S]) - \chi'(\Lambda[\vartheta_S])\Lambda[\vartheta_S])\partial_x \omega^2 - 2(\chi'(\Lambda[\vartheta_S]))\theta^2 + \chi(\Lambda[\vartheta_S])\Lambda[\div \omega],
\end{align*}$$

we infer

$$I_2^3 + I_2^4 \leq C \int_0^t \int_{Q^{2a}} \mathcal{K}_h(x-y)\chi(\Lambda[\vartheta_S])\div \omega^2 \omega^2 dxdydt$$

$$- 2\int_0^t \int_{Q^{2a}} \mathcal{K}_h(x-y)\Lambda[\div \omega](\chi'(\Lambda[\vartheta_S]))\theta^2 + \chi(\Lambda[\vartheta_S])\omega^2 dxdydt.$$

One obtains (4.29) after substituting the above estimates of $I_2^3$ ($i = 1, 2, 3, 4$) into (4.31).

Next, the truncated part of $\Lambda[\div \omega]$ can be decomposed into the pressure flux part.

**Lemma 4.6.** Let $T > 0$, $p_0 > \gamma + \alpha + \alpha + 1$, $h \in (0, h_0)$, $(\rho_0, n_0, (\rho_0 + n_0)u_0)$ be the weak solution to the IVP (3.1)–(3.2) for $\delta \in (0, 1)$ given by Proposition 3.5, $\vartheta_S := \rho_0 + n_0$, and $\omega_0$ be the solution to the IVP (4.21). Then under the assumptions of either Theorem 1.2 or Theorem 1.6, it holds for $k \geq 1$ and $t \in [0, T]$ that

$$- \int_0^t \int_{Q^{2a}} \mathcal{K}_h(x-y)\Lambda[\div \omega]^2(\chi'(\Lambda[\vartheta_S]))\theta^2 + \chi(\Lambda[\vartheta_S]))(1 - \theta^2 \chi^1 \chi^2 \chi^3 \chi^4) \omega^2 dxdydt$$

$$\leq C \int_0^t \int_{Q^{2a}} \mathcal{K}_h(x-y)\Lambda[\div \omega]^2\chi(\Lambda[\vartheta_S])\theta^2 + \chi(\Lambda[\vartheta_S]),$$

with

$$\begin{align*}
\Psi_{s,k} := (\chi'(\Lambda[\vartheta_S]))\theta^2 + \chi(\Lambda[\vartheta_S]))1_{\theta^2 < k} \chi^1 \chi^2 \chi^3 \chi^4 \theta^2, 
\end{align*}$$

where $P_S$, $F_S$, $L_{s,1}(f)$, $\Lambda[f]$, $f^\delta$, $\mathcal{K}_h$ and $\chi$ are given by (2.3), (4.3), (4.17), (4.18), (4.18)1, (4.18)2, (4.18)3 and (4.19), respectively, and $C > 0$ is a constant independent of $\delta$, $h$, $k$ and $\lambda_0$.

**Proof.** By (3.28), (4.1)–(4.2), (4.19)1 and (4.22)1, we have

$$- \int_0^t \int_{Q^{2a}} \mathcal{K}_h(x-y)\Lambda[\div \omega]^2(\chi'(\Lambda[\vartheta_S]))\theta^2 + \chi(\Lambda[\vartheta_S]))(1 - \theta^2 \chi^1 \chi^2 \chi^3 \chi^4) \omega^2 dxdydt$$

$$\leq C \int_0^t \int_{Q^{2a}} \mathcal{K}_h(x-y)[\div \omega^2 + |\div \omega|](\theta^2 + 1) dxdydt$$

$$\leq C \int_0^t \int_{Q^{2a}} \mathcal{K}_h(x-y)[\div \omega^2 + |\div \omega|](\theta^2 + 1) dxdydt$$

$$\leq C \frac{k}{\lambda_0^2} \|\div \omega\|_{L^2(0,T,L^2)} \|\vartheta_S\|_{L^2(0,T,L^2)} \leq C \frac{k}{\lambda_0^2},$$

where one used the facts that $|\chi'(\Lambda[\vartheta_S])|\theta^2 \leq C(\theta^2 + 1)$ if $\lambda(\vartheta_S) \leq 2$, $\chi'(\Lambda[\vartheta_S])\theta^2 + \chi(\Lambda[\vartheta_S]) = \theta^2 \sign(\Lambda[\vartheta_S])$ for $|\lambda(\vartheta_S)| \geq 2$, $\theta^2 + \theta^2 \geq k$ for either $\theta^2 \geq k$ or $\theta^2 \geq k$, and $\vartheta_S \omega \leq \vartheta_S e^{-\lambda_0 \vartheta_S} \leq \frac{1}{\lambda_0} \leq 1$.

By (4.3) and (4.34), (4.32) holds. \qed
We are ready to estimate the key pressure part.

**Lemma 4.7.** Let $T > 0, p_0 > \gamma + \bar{\gamma} + \alpha + \bar{\alpha} + 1, h \in (0, h_0), (\rho_\delta, n_\delta, (\rho_\delta + n_\delta)u_\delta)$ be the weak solution to the IVP (3.1)–(3.2) for $\delta \in (0, 1)$ given by Proposition 3.5, $\theta_\delta := \rho_\delta + n_\delta$, and $w_\delta$ be a solution to the IVP (4.21). Then for any $\zeta > 0$, there is a constant $\delta_1(\zeta) \in (0, 1)$ such that it holds for $\delta \in (0, \delta_1(\zeta))$, $k \geq 1$ and $t \in [0, T]$ that

$$-\int_0^t \int_{\mathbb{T}^d} \nabla_{x,y} \cdot (\rho_\delta \nabla_x \phi_\delta) \Psi_{\delta,k} = C k^{p_0} \delta^\alpha + \int_0^t \left( L_{h,1}(A_{\rho,n}) + L_{h,1}(B_{\rho,n}) \right) dt$$

$$+ C \int_0^t \int_{\mathbb{T}^d} \nabla_{x,y} \cdot (\rho_\delta \nabla_x \phi_\delta) \Psi_{\delta,k} dxdydt, \quad (4.35)$$

where $(A_{\rho,n}, B_{\rho,n}), L_{h,1}(f)$, $\Lambda[f]$, $f^\ast, \nabla_{x,y}$ and $\Psi_{\delta,k}$ are given by (1.18), (4.17), (4.18), (1.18), (1.18) and (3.8) respectively, and $C > 0$ is a constant independent of $\delta, h, k$ and $\lambda_0$.

**Proof.** To begin with, as $\delta \to 0$, we deduce by (4.13) that

$$\left\{ \begin{array}{ll}
\rho_\delta - A_{\rho,n} \theta_\delta \to 0 & \text{a.e. in } \mathbb{T}^d \times (0, T),
\rho_\delta - B_{\rho,n} \theta_\delta \to 0 & \text{a.e. in } \mathbb{T}^d \times (0, T),
\end{array} \right.$$
It follows from (1.4), (3.11)–(3.12), (3.21), (4.22) and (4.36) that
\[
I_4^1 \leq C \int_{\mathbb{Q}_{\zeta,t}} \mathcal{K}_h(x - y) [((\rho_0^2 + A_{\rho,n} \delta_0^2)\delta^{-1} + \delta(\rho_0^2 + A_{\rho,n} \delta_0^2)) P_{\rho,n}^{-1} + 1]|\rho_0^2 - A_{\rho,n} \delta_0^2| \\
+ (n_0^2 + B_{\rho,n} \delta_0^2))\delta^{-1} + \delta(n_0^2 + B_{\rho,n} \delta_0^2)) P_{\rho,n}^{-1} + 1)]|n_0^2 - B_{\rho,n} \delta_0^2|||\delta_0^2 + \delta_0^2)|1_{\delta_0^2 < h}dx dy d\tau
\leq CK^p_0 \zeta.
\]

Similarly, one has
\[
I_2^1 \leq CK^p_0 \zeta.
\]

For the term \(I_3^1\), we have by (1.4), (3.12), (3.21) and (4.22) that
\[
I_3^1 \leq CK^p_0 (L_{h,1}(A_{\rho,n}) + L_{h,1}(B_{\rho,n})).
\]

Additionally, it is easy to prove
\[
I_4^1 + I_2^1 \leq CK^p_0 \delta^\beta.
\]

The key term \(I_0^1\) can be controlled by the damping term in (4.21) with respect to \(x\), so we need to overcome the difficulties caused by the possible dependence of \(I_0^1\) on \(y\). To this end, in the spirit of [10], we analyze \(I_0^1\) in the following three cases:

**Case 1.** \((P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2)) - P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2)) \Lambda[\delta] > 0.\)

Note that if \(P(\rho, n)\) is monotonically increasing in both \(\rho\) and \(n\), then all the cases reduce to Case 1. In this case, since \(P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2) - P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2), \Lambda[\delta]\) and \(\chi'(\Lambda[\delta])\) have the same sign, it holds by (4.20) that
\[
- (P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2) - P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2)) \Lambda[\delta] < 0 \text{ and } \Lambda[\delta] < 0 \text{ and } \Lambda[\delta] < 0.
\]

We make use of (1.4), (3.12), (3.21) and (4.36) to obtain
\[
\begin{align*}
- (P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2) - P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2)) \Lambda[\delta] & < 0 \text{ and } \Lambda[\delta] < 0 \text{ and } \Lambda[\delta] < 0.
\end{align*}
\]

**Case 2.** \((P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2) - P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2)) \Lambda[\delta] < 0 \text{ and } \Lambda[\delta] < 2 \delta_0^2.\)

We make use of (1.4), (3.12), (3.21) and (4.36) to obtain
\[
\begin{align*}
- (P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2) - P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2)) \Lambda[\delta] & < 0 \text{ and } \Lambda[\delta] < 2 \delta_0^2.
\end{align*}
\]

**Case 3.** \((P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2) - P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2)) \Lambda[\delta] < 0 \text{ and } \Lambda[\delta] > 2 \delta_0^2.\)

In this case, one has
\[
\left\{ \begin{array}{l}
|\Lambda[\delta]| = \delta_0^2 - \delta_0^2 > \delta_0^2 \geq 0, \quad \chi'(\Lambda[\delta]) < 0, \quad \Lambda[\delta] = |\Lambda[\delta]| + \delta_0^2 < 2|\Lambda[\delta]|,

P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2) > P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2),
\end{array} \right.
\]

which gives rise to
\[
\begin{align*}
& - (P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2) - P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2)) \Lambda[\delta] \\
& < (P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2) - P(A_{\rho,n} \delta_0^2, B_{\rho,n} \delta_0^2)) (\chi'(\Lambda[\delta]) |\Lambda[\delta]| - \chi(\Lambda[\delta])) \Lambda[\delta] \\
& \leq C(\rho_0^2 + \zeta) \gamma + (n_0^2 + \zeta)^2 + 1 + 1) \chi(\Lambda[\delta]) \Lambda[\delta], \quad (x, y, \tau) \in \mathbb{Q}_{\zeta,t}.
\end{align*}
\]
Combining the above three cases together, we get

\[ I_6^4 \leq C \int_0^1 \int_{\mathbb{T}^2d} \mathcal{K}_h(x - y)((\rho_\delta^5)^\gamma + (\rho_\delta^5)^\tilde{\gamma} + (n_\delta^5)^\alpha + (n_\delta^5)^\tilde{\alpha} + 1)\chi(\Lambda[\partial_\delta])u_h^\delta dx dy dr. \]

Substituting the above estimates of $I_6^4$ ($i = 1, \ldots, 6$) into (4.37), we derive (4.35). The proof of Lemma 4.7 is completed.

**Remark 4.8.** Let $\Pi(\rho)$ be the general pressure laws for compressible Navier-Stokes equations given in [10]. For $P(\rho, n) = \Pi(\rho + n)$, one can repeat the same arguments as in [10] to estimate the pressure part (4.35). However, even for $P(\rho, n) = \Pi(\rho) + \Pi(n)$, we have to reduce the two variables $\rho_\delta$ and $n_\delta$ into one variable $\partial_\delta$ by applying the decomposition (1.17).

Finally, different from the previous work by Bresch and Jabin [10], we make use of the structures of the equations (3.1) and the commutator estimates (A.11)–(A.12) of the Riesz operator $\mathcal{R}_i = (-\Delta)^{-1}\partial_i$ to control the effect viscous flux part.

**Lemma 4.9.** Let $T > 0$, $p_0 > \gamma + \tilde{\gamma} + \alpha + \tilde{\alpha} + 1$, $h \in (0, h_0)$, $(\rho_\delta, n_\delta, (\rho_\delta + n_\delta)u_\delta)$ be the weak solution to the IVP (3.1)–(3.2) for $\delta \in (0, 1)$ given by Proposition 3.5, $\partial_\delta := \rho_\delta + n_\delta$, and $u_\delta$ be the solution to the IVP (4.21). Then under the assumptions of either Theorem 1.2 or Theorem 1.6, it holds for $k \geq 1$ and $t \in [0, T]$ that

\[ \int_0^t \int_{\mathbb{T}^2d} \mathcal{K}_h(x - y)\Lambda[F_{\delta,k}]\Psi_{\delta,k} dx dy dr \leq \frac{C\lambda_0 k_{p_0}}{|\log h|^{\frac{1}{2}}\min\{\frac{1}{2}, \frac{2\gamma_0 - 2}{\gamma_0 - 1}\}}, \quad \gamma_0 := \min\{\gamma, \alpha\}, \quad (4.38) \]

where $F_{\delta}$, $\Lambda[f]$, $f^\ast$, $\mathcal{K}_h$ and $\Psi_{\delta,k}$ are given by (4.3), (4.18)1, (4.18)2, (4.18)3 and (4.39), respectively, and $C > 0$ is a constant independent of $\delta$, $k$ and $\lambda_0$.

**Proof.** By the argument of renormalized solutions to (4.21)1 and (4.25), one has

\[ \partial_\delta \Psi_{\delta,k} + \text{div}_x(\Psi_{\delta,k}u_\delta^k) + \text{div}_y(\Psi_{\delta,k}u_\delta^y) = R_{\delta,k} \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^{2d} \times (0, T)) \quad (4.39) \]

with

\[ R_{\delta,k} := (\Psi_{\delta,k} - \partial_\delta^2 \Psi_{\delta,k}) \text{div}_x u_\delta^x + (\Psi_{\delta,k} - \partial_\delta^2 \Psi_{\delta,k}) \text{div}_y u_\delta^y - \lambda_0 \partial_\delta^2 \Psi_{\delta,k} \Xi^\delta w_\delta^5. \]

By (4.21)2 and (4.22)1, it is easy to verify

\[ \left\{ \begin{array}{l}
\|\Psi_{\delta,k}\|_{L^\infty(0, T; L^1(\mathbb{T}^{2d}))} \leq C k, \\
\|R_{\delta,k}\|_{L^\infty(0, T; L^1(\mathbb{T}^{2d}))} \leq C\lambda_0 k_{p_0}(\|\text{div}_x u_\delta^x\| + \|\text{div}_y u_\delta^y\| + M(\|\nabla_x u_\delta^x\| + 1),
\end{array} \right. \quad (4.40) \]

where the constant $C > 0$ is independent of $\delta$, $k$ and $\lambda_0$, and the localized maximal operator $M : L^p(\mathbb{T}^d) \rightarrow L^p(\mathbb{T}^d)$ for any $p \in [1, \infty)$ is defined by (A.7). From (4.4), we have

\[ \Lambda[F_{\delta}] = \partial_\delta \Lambda[(-\Delta)^{-1}\text{div}(\partial_\delta u_\delta)] + \Lambda[(-\Delta)^{-1}\text{div}(\partial_\delta u_\delta \otimes u_\delta)]. \quad (4.41) \]

One deduces by (4.39), (4.41) and integration by parts that

\[ \int_0^t \int_{\mathbb{T}^2d} \mathcal{K}_h(x - y)\Lambda[F_{\delta,k}]\Psi_{\delta,k} dx dy dr = \sum_{i=1}^4 I_5^i, \quad (4.42) \]

where $I_5^i$ ($i = 1, 2, 3, 4$) are given by

\[ I_1^5 := \int_{\mathbb{T}^2d} \mathcal{K}(x - y)\Lambda[(-\Delta)^{-1}\text{div}(\partial_\delta u_\delta)]\Psi_{\delta,k} dx dy \bigg|_0^t, \quad I_2^5 := -\int_0^t \int_{\mathbb{T}^2d} \mathcal{K}_h(x - y)\Lambda[(-\Delta)^{-1}\text{div}(\partial_\delta u_\delta)]R_{\delta,k} dx dy dr, \]

\[ I_3^5 := \int_0^t \int_{\mathbb{T}^2d} \nabla \mathcal{K}(x - y) \cdot \Lambda[u_\delta] \Lambda[(-\Delta)^{-1}\text{div}(\partial_\delta u_\delta)]\Psi_{\delta,k} dx dy dr, \quad I_4^5 := \sum_{i,j=1}^d \int_0^t \int_{\mathbb{T}^2d} \mathcal{K}_h(x - y)\Lambda[R_i R_j((\partial_\delta u_\delta^i)u_\delta^j - u_\delta^i R_i R_j(\partial_\delta u_\delta^j))]\Psi_{\delta,k} dx dy dr. \]
First, we estimate the term $I_1^5$. It follows from (4.8), (A.3) and the boundedness of the operator $\nabla(-\Delta)^{-1}\text{div}$ that

$$
L_{h,p}((-\Delta)^{-1}\text{div}(\partial_3 u_3)) 
\leq C\|((-\Delta)^{-1}\text{div}(\partial_3 u_3))\|_{L^p} + C\|\nabla(-\Delta)^{-1}\text{div}(\partial_3 u_3)\|_{L^{2q}} \left(\frac{1}{\|\log h\|^\frac{2}{2q}} + \frac{1}{\|\log h\|^p\min\left\{0, \frac{1}{2} - \frac{2}{2q}\right\}}\right), \quad p \in \left[1, \frac{2q(\gamma_0 + 1) - 2\gamma_0}{d(\gamma_0 + 1) - 2\gamma_0}\right],
$$

which together with (2.6), (4.8) and (4.40) implies

$$
I_1^5 \leq Ck \text{ess sup}_{t \in [0,T]} L_{h,1}((-\Delta)^{-1}\text{div}(\partial_3 u_3)) + Ck L_{h,1}((-\Delta)^{-1}\text{div}(\rho_{0,\delta} + n_{0,\delta}) u_{0,\delta}) 
\leq \frac{Ck}{\|\log h\|^\frac{2}{2q}}.
$$

To control the term $I_2^5$, we obtain by (3.28), (4.8), (4.40) and (4.43) that

$$
I_2^5 \leq Ck^p \lambda_0 \int_0^t \int_{\mathbb{T}^d} K_{h}(x - y)|\nabla(-\Delta)^{-1}\text{div}(\partial_3 u_3)|||\text{div}_x u_3^x|| + ||\text{div}_y u_3^y|| + M||\nabla_x u_3^x|| + M||\nabla_y u_3^y|| + M\text{ess sup}_{t \in [0,T]} L_{h,2}((-\Delta)^{-1}\text{div}(\partial_3 u_3))^{\frac{1}{2}} + \text{ess sup}_{t \in [0,T]} L_{h,1}((-\Delta)^{-1}\text{div}(\partial_3 u_3))^{\frac{1}{2}} 
\leq \frac{Ck^p \lambda_0}{\|\log h\|^\min\left\{\frac{1}{4}, \frac{2q - 2}{2q}\right\}}.
$$

Similar to the estimate of $I_1^5$ in (4.31), since $(-\Delta)^{-1}\text{div}(\partial_3 u_3)$ is uniformly bounded in $L^2(0,T; L^2(\mathbb{T}^d))$ due to $\gamma_0 > \frac{1}{2}$, one can prove

$$
I_3^5 \leq Ck \int_0^t \int_{\mathbb{T}^d} K_{h}(x - y)|\nabla(-\Delta)^{-1}\text{div}(\partial_3 u_3)||M||\nabla_x u_3^x||\text{dxdy} + \frac{Ck}{\|\log h\|^\frac{2}{2q}}.
$$

By (3.28), (4.8) and (4.43), we have

$$
\int_0^t \int_{\mathbb{T}^d} K_{h}(x - y)|\nabla\Lambda(-\Delta)^{-1}\text{div}(\partial_3 u_3)||M||\nabla_x u_3^x||\text{dxdy}
\leq C\|M||\nabla\Lambda u_3||_{L^2(0,T; L^2)}\text{ess sup}_{t \in [0,T]} L_{h,2}((-\Delta)^{-1}\text{div}(\partial_3 u_3))^{\frac{1}{2}}
\leq \frac{C}{\|\log h\|^\min\left\{\frac{1}{4}, \frac{2q - 2}{2q}\right\}}.
$$

We obtain from (4.44)–(4.45) that

$$
I_4^5 \leq \frac{Ck}{\|\log h\|^\min\left\{\frac{1}{4}, \frac{2q - 2}{2q}\right\}}.
$$

Finally, we are ready to estimate the last term $I_5^5$. Inspired by [37], we make use of the commutator estimate (A.12) to deduce

$$
\|\nabla(R_i R_j (\partial_3 u_{3}^x u_3^y) - u_3^x R_i R_j (\partial_3 u_3^y)) (t)||_{L^p} \leq C\|\nabla u_3^x(t)||_{L^2}||\partial_3 u_3^y(t)||_{L^q}, \quad \text{a.e. } t \in (0,T)
$$

where the constants $p \in (1, \infty)$ and $q \in (2, \infty)$ satisfy

$$
\frac{1}{2} + \frac{1}{q} = \frac{1}{p} < 1.
$$
Note that one may not apply (4.46) directly for \( \gamma_0 \leq d \). For example, \( \partial_\delta u_3 \) is uniformly bounded in \( L^2(0, T; L^{6/5+d}(\mathbb{T}^d)) \) for \( d = 3 \), but we have

\[
\frac{1}{2} + \frac{6 + \gamma_0}{6\gamma_0} \geq 1
\]

provided that \( \gamma_0 \leq 3 \). To overcome this difficulty, we truncate \( \partial_\delta \) by using \( 1 = 1_{\partial_\delta \leq L} + 1_{\partial_\delta \geq L} \) for \( L \geq 1 \) leading to \( I_1^6 = I_1^0 + I_1^6 \) with

\[
I_1^0 := \sum_{i,j=1}^d \int_0^t \int_{\mathbb{T}^d} \mathcal{R}_h(x-y)A[\mathcal{R}_i \mathcal{R}_j(\partial_\delta 1_{\partial_\delta \leq L} u_3^i u_3^j) - u_3^i \mathcal{R}_i \mathcal{R}_j(\partial_\delta 1_{\partial_\delta \leq L} u_3^j)]\psi_\delta k dx dy dt,
\]

\[
I_1^6 := \sum_{i,j=1}^d \int_0^t \int_{\mathbb{T}^d} \mathcal{R}_h(x-y)A[\mathcal{R}_i \mathcal{R}_j(\partial_\delta 1_{\partial_\delta \geq L} u_3^i u_3^j) - u_3^i \mathcal{R}_i \mathcal{R}_j(\partial_\delta 1_{\partial_\delta \geq L} u_3^j)]\psi_\delta k dx dy dt.
\]

The terms \( I_1^0 \) and \( I_1^6 \) can be estimated in the cases of \( d \geq 3 \) and \( d = 2 \) as follows:

**Case 1.** \( d \geq 3 \).

We show after using (4.40)_1, (A.3), (A.12) and \( H^1(\mathbb{T}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{T}^d) \) that

\[
I_1^0 \leq Ck \sum_{i,j=1}^d \int_0^t \int_{\mathbb{T}^d} L_{i,j}^{\frac{1}{2}}(\mathcal{R}_i \mathcal{R}_j(\partial_\delta 1_{\partial_\delta \leq L} u_3^i u_3^j) - u_3^i \mathcal{R}_i \mathcal{R}_j(1_{\partial_\delta \leq L} \partial_\delta u_3^j)) dt
\]

\[
\leq Ck \left( \frac{1}{|\log h|^\frac{d}{2}} + \frac{1}{|\log h|^2} \sum_{i,j=1}^d \left\| \nabla [\mathcal{R}_i \mathcal{R}_j(\partial_\delta 1_{\partial_\delta \leq L} u_3^i u_3^j) - u_3^i \mathcal{R}_i \mathcal{R}_j(1_{\partial_\delta \leq L} \partial_\delta u_3^j)] \right\|_{L^1(0,T; L^{\frac{2d}{d-2}})} \right)
\]

\[
\leq \frac{Ck}{|\log h|^\frac{d}{2}} \left( 1 + \| \nabla u_3 \|_{L^2(0,T; L^2)} \| \partial_\delta 1_{\partial_\delta \leq L} u_3 \|_{L^2(0,T; L^{\frac{2d}{d-2}})} \right)
\]

\[
\leq Ck L |\log h|^\frac{d}{2}.
\]

For the term \( I_1^6 \), it holds that

\[
I_1^6 \leq Ck \sum_{i,j=1}^d \left( \| \mathcal{R}_i \mathcal{R}_j(\partial_\delta 1_{\partial_\delta \geq L} u_3^i u_3^j) \|_{L^1(0,T; L^{p_2})} + \| u_3^i \mathcal{R}_i \mathcal{R}_j(\partial_\delta 1_{\partial_\delta \geq L} u_3^j) \|_{L^1(0,T; L^{p_2})} \right)
\]

\[
\leq Ck \| \partial_\delta 1_{\partial_\delta \geq L} \|_{L^\infty(0,T; L^{p_2})} \| u_3^i \|_{L^2(0,T; L^{\frac{2d}{d-2}})}
\]

\[
\leq Ck \left( \frac{20}{L^\infty(0,T; L^{p_2})} \right) \| u_3^i \|_{L^2(0,T; H^1)}^2 \leq \frac{Ck}{L^\infty(0,T; L^{p_2})} \leq \frac{Ck}{L^\infty(0,T; L^{p_2})} \leq \frac{Ck}{L^\infty(0,T; L^{p_2})}
\]

with the constants \( p_2 \in (1, \infty) \) and \( p_3 \in \left( \frac{d}{2}, \gamma_0 \right) \) satisfying

\[
\frac{1}{p_2} + \frac{d-2}{d} = \frac{1}{p_2}, \quad \gamma_0 - p_3 = \frac{1}{d} \left( \gamma_0 - \frac{d}{2} \right).
\]

**Case 2.** \( d = 2 \).

By (4.40)_1, (A.3), (A.12) and \( H^1(\mathbb{T}^2) \hookrightarrow L^p(\mathbb{T}^2) \) for any \( p \in [1, \infty) \), the term \( I_1^0 \) can be estimated by

\[
I_1^0 \leq Ck \sum_{i,j=1}^2 \int_0^t \int_{\mathbb{T}^2} L_{i,j}^{\frac{1}{2}}(\mathcal{R}_i \mathcal{R}_j(\partial_\delta 1_{\partial_\delta \leq L} u_3^i u_3^j) - u_3^i \mathcal{R}_i \mathcal{R}_j(1_{\partial_\delta \leq L} \partial_\delta u_3^j)) dt
\]

\[
\leq Ck \left( \frac{1}{|\log h|^\frac{d}{2}} + \frac{1}{|\log h|^2} \sum_{i,j=1}^2 \left\| \nabla [\mathcal{R}_i \mathcal{R}_j(\partial_\delta 1_{\partial_\delta \leq L} u_3^i u_3^j) - u_3^i \mathcal{R}_i \mathcal{R}_j(1_{\partial_\delta \leq L} \partial_\delta u_3^j)] \right\|_{L^1(0,T; L^{\frac{2d}{d-2}})} \right)
\]

\[
\leq \frac{Ck}{|\log h|^\frac{d}{2}} \left( 1 + \| \nabla u_3 \|_{L^2(0,T; L^2)} \| \partial_\delta 1_{\partial_\delta \leq L} u_3 \|_{L^2(0,T; L^2)} \right)
\]

\[
\leq Ck(1 + L \| u_3 \|_{L^2(0,T; H^1)}) \leq \frac{CkL}{|\log h|^\frac{d}{2}}.
\]
For the term $I_5^0$, we make use of (4.40), (A.3), (A.11) and $H^1(\mathbb{T}^2) \hookrightarrow BMO(\mathbb{T}^2) \hookrightarrow L^p(\mathbb{T}^2)$ for any $p \in [1, \infty)$ to obtain

\[
I_5^0 \leq C k \sum_{i,j=1}^2 \|R_i R_j (\partial_\delta \chi_{\delta \leq L} u_i^j u_j^i) - u_i^j R_i R_j (\theta_\delta \chi_{\delta \leq L} \theta_\delta u_i^j)\|_{L^1(0,T;L^{2\gamma_0})} \\
\leq C k \|u_i^j\|_{L^2(0,T;BMO)} \|\partial_\delta \chi_{\delta \leq L} u_i^j\|_{L^2(0,T;L^{2\gamma_0})} \\
\leq C k \|\theta_\delta \chi_{\delta \leq L} u_i^j\|_{L^2(0,T;H^1)} \|\partial_\delta \chi_{\delta \leq L} u_i^j\|_{L^\infty(0,T;L^{2\gamma_0})} \\
\leq \frac{C k}{L^{\frac{2\gamma_0}{1-\delta}}} \|u_i^j\|_{L^2(0,T,H^1)} \|\theta_\delta \chi_{\delta \leq L} u_i^j\|_{L^2(0,T;L^{2\gamma_0})} \leq \frac{C k}{L^{\frac{2\gamma_0}{1-\delta}}}.
\]

Combining the above two cases about the estimates of $I_5^0$ ($i = 1, 2$), we derive

\[
I_5^0 \leq C k \left( \frac{L}{|\log h|^{\frac{1}{2}}} + \frac{1}{L^{\frac{1}{2}}(\gamma_0 - \frac{2}{\gamma})} \right), \quad d \geq 2.
\]  

(4.47)

Hence one can choose $L = |\log h|^{-\frac{1}{2} + \frac{2}{\gamma}}$ in (4.47) to get

\[
I_5^0 \leq \frac{C k}{|\log h|^{\frac{1}{2} \min\{\frac{2\gamma_0}{1-\delta}, \frac{\gamma_0 - \frac{2}{\gamma}}{\gamma}\}}.
\]

Substituting the above estimates of $I_5^0$ ($i = 1, 2, 3, 4$) into (4.42), we prove (4.38). The proof of Lemma 4.9 is completed. \qed

**Remark 4.10.** It should be noted that Lemma 4.9 about the compactness of effect viscous flux indeed holds for any adiabatic constants $\gamma, \alpha > \frac{2}{\gamma}$.

**Proofs of Theorems 1.2 and 1.6.** Let $T > 0$, $(\rho_0, n_0, (\rho_0 + n_0) u_0)$ be the weak solution to the IVP (3.1)-(3.2) for $\delta \in (0, 1)$ given by Proposition 3.5, $\theta_\delta := \rho_0 + n_0, w_0$ be the solution to the IVP (4.21), and $(\rho, u, (\rho + n) u)$ be the limit obtained by Lemma 4.2. Then with the results from Lemma 4.5 to Lemma 4.9, for any $\zeta > 0$, there exists a constant $\delta_1(\zeta) \in (0, 1)$ such that for $\delta \in (0, \delta_1(\zeta))$, $h \in (0, h_0)$, $k \geq 1$ and $\lambda_0 \geq 1$, we have

\[
\int_{\mathbb{T}^2} \mathcal{K}_h(x - y) \chi(\Lambda[\theta_\delta])(w_0^2 + w_0^8) dxdy \\
\leq \frac{C k}{k} + C\lambda_0 k^p h_1(\delta, \zeta, h) + (C - 2\lambda_0) \int_0^T \int_{\mathbb{T}^2} \mathcal{K}_h(x - y) |\Lambda[\theta_\delta]| \mathbb{E}_x u_0^2 dxdy d\tau,
\]

where $C > 0$ is a constant independent of $\delta, h, \zeta$ and $\lambda_0$, and $h_1(\delta, \zeta, h) \in (0, 1)$ is given by

\[
h_1(\delta, \zeta, h) := L_{h,1}(\rho_0, \delta + n_0, \delta) + \frac{1}{|\log h|^{\frac{1}{2} \min\{\frac{2\gamma_0}{1-\delta}, \frac{\gamma_0 - \frac{2}{\gamma}}{\gamma}\}}} \\
+ \delta^\beta + \zeta + \int_0^T (L_{h,1}(\rho, \rho + n) + L_{h,1}(\rho, \rho + n)) dt, \quad \gamma_0 := \min\{\gamma, \alpha\}.
\]

Choosing

\[
\lambda_0 = \frac{C k}{2} + 1, \quad k = h_1(\delta, \zeta, h)^{-\frac{1}{\gamma_0 + \tau}},
\]

we have

\[
\int_{\mathbb{T}^2} \mathcal{K}_h(x - y) \chi(\Lambda[\theta_\delta])(w_0^2 + w_0^8) dxdy \leq C h_1(\delta, \zeta, h)^{-\frac{1}{\gamma_0 + \tau}},
\]

which together with (4.23) implies for any $\sigma_* > 0$ that

\[
(L_{h,1}(\theta_\delta))^2 \leq \frac{C}{\log \left(1 + |\log \sigma_*|\right)} + \frac{C h_1(\delta, \zeta, h)^{-\frac{1}{\gamma_0 + \tau}}}{\sigma_*}.
\]
Thus, one can choose \( \sigma_* = h_1(\delta, \zeta, h) \frac{1}{\log(1 + |\log h_1(\delta, \zeta, h)|)} \) to obtain

\[
(L_{h,1}(\vartheta_\delta))^2 \leq \frac{C}{\log (1 + |\log h_1(\delta, \zeta, h)|)} + h_1(\delta, \zeta, h) \frac{1}{\log(1 + |\log h_1(\delta, \zeta, h)|)}.
\]

Since it follows that

\[
\lim_{h \to 0} \limsup_{\delta \to 0} h_1(\delta, \zeta, h) = \zeta
\]

for any small \( \zeta > 0 \), we gain

\[
\lim_{h \to 0} \limsup_{\delta \to 0} \sup_{t \in [0,T]} L_{h,1}(\vartheta_\delta) = 0.
\]  
(4.48)

Due to (4.14), (4.48), Lemma A.3 below and the fact that \( \partial_t \vartheta_\delta \) is uniformly bounded in

\[
L^\infty(0, T; W^{-1, \frac{2m}{m+1}}(\mathbb{T}^d)),
\]

it holds up to a subsequence (still denoted by \( (\rho_\delta, n_\delta) \)) that

\[
(\rho_\delta, n_\delta) \to (\rho, n) \quad \text{in} \quad L^1(0, T; L^1(\mathbb{T}^d)) \times L^1(0, T; L^1(\mathbb{T}^d)) \quad \text{as} \quad \delta \to 0,
\]  
(4.49)

which together with Lemma 4.1 and the Egorov theorem gives rise to

\[
P_\delta(\rho_\delta, n_\delta) \to P(\rho, n) \quad \text{in} \quad L^1(0, T; L^1(\mathbb{T}^d)) \quad \text{as} \quad \delta \to 0.
\]  
(4.50)

By (4.12) and (4.50), one can show that \( (\rho, n, (\rho + n)a) \) satisfies the properties (1)–(3) in Definition 1.1.

Finally, we show the energy inequality (1.9) under the assumptions of Theorem 1.2, and the case of Theorem 1.6 can be proved similarly. For any \( L \geq 1 \) and the smooth function \( j_\delta \) satisfying (2.5), it is easy to verify for \( (\rho_{0,\delta}, n_{0,\delta}) := ((\rho_0 1_{\rho_0 \leq L}) * j_\delta + \delta, (n_0 1_{n_0 \leq L}) * j_\delta + \delta) \) that

\[
\left| \int_{\mathbb{T}^d} G(\rho_{0,\delta}, n_{0,\delta}) dx \right| \to 0 \quad \text{as} \quad \delta \to 0,
\]  
(4.51)

and for \( (\rho_{0,\delta}, n_{0,\delta}) := ((\rho_0 1_{\rho_0 \geq L}) * j_\delta + \delta, (n_0 1_{n_0 \geq L}) * j_\delta + \delta) \) that

\[
\left| \int_{\mathbb{T}^d} G(\rho_{0,\delta}, n_{0,\delta}) dx \right| \\
\leq \int_{\mathbb{T}^d} \rho_{0,\delta}^2 \int_{\mathbb{T}^d} G_1(\rho_0 1_{\rho_0 \leq L} + \delta, (n_0 1_{n_0 \leq L}) * j_\delta + \delta) dx \\\n\leq \int_{\mathbb{T}^d} \rho_{0,\delta}^2 \left( \int_{\mathbb{T}^d} G_1(\rho_0 1_{\rho_0 \geq L} + \delta, (n_0 1_{n_0 \geq L}) * j_\delta + \delta) dx \right) \\\n\leq C \int_{\mathbb{T}^d} (\rho_0 1_{\rho_0 \geq L}) + (n_0 1_{n_0 \leq L}) \to 0 \quad \text{as} \quad L \to \infty \quad \text{uniformly in} \ \delta,
\]  
(4.52)

where one used (1.4), (1.10)–(1.11) and Young’s inequality for convolutions. By (2.6), we also infer

\[
\left| \int_{\mathbb{T}^d} \rho_0 \int_{\mathbb{T}^d} \left( 1_{s + \frac{n_0 \delta}{\rho_0 \delta} \leq P\left( s, \frac{n_0 \delta}{\rho_0 \delta} s \right) + \delta \left( s + \frac{n_0 \delta}{\rho_0 \delta} s \right) \frac{n_0}{\rho_0 \delta} s \right) s^{-2} dx \right| \to 0 \quad \text{as} \quad \delta \to 0.
\]  
(4.53)

By (4.51)–(4.53), it follows that

\[
\int_{\mathbb{T}^d} G(\rho_0, n_0) dx = \lim_{\delta \to 0} \int_{\mathbb{T}^d} G_\delta(\rho_0, \delta, n_0, \delta) dx.
\]  
(4.54)

By (4.1)–(4.2), (4.49) and a similar argument of truncation, one gets

\[
\int_0^T \psi \int_{\mathbb{T}^d} G(\rho, n) dx dt = \lim_{\delta \to 0} \int_0^T \psi \int_{\mathbb{T}^d} G_\delta(\rho_\delta, n_\delta) dx dt, \quad \forall \psi \in C^\infty_c(0, T).
\]  
(4.55)
By (2.22), (3.29), (4.12), (4.54)–(4.55) and the lower semi-continuity of weak limits, the solution \((\rho, u, (\rho + n)u)\) to the IVP (1.1)–(1.2) satisfies the energy inequality (1.9) for a.e. \(t \in (0, T)\). Theorems 1.2 and 1.6 are proved.

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Appendix A

In this appendix, we present some technical lemmas which are useful to our analysis.

Lemma A.1. Let \( p \in [1, \infty), \varepsilon > 0, \) and \( f_{\varepsilon} \) be uniformly bounded in \( L^p(\mathbb{T}^d) \). Then for any \( h \in (0, h_0) \), it holds that

\[
\sup_{|z| < h} \int_{\mathbb{T}^d} |f_{\varepsilon}(x + z) - f_{\varepsilon}(x)|^p dx \leq C \left( \frac{\|f_{\varepsilon}\|_{L^p}^p}{\log h} + L_{h,p}(f_{\varepsilon}) \right), \tag{A.1}
\]

\[
L_{h,p}(f_{\varepsilon}) \leq C \left( \frac{\|f_{\varepsilon}\|_{L^p}^p}{\log h} + \sup_{|z| < \frac{h}{2}} \int_{\mathbb{T}^d} |f_{\varepsilon}(x + z) - f_{\varepsilon}(x)|^p dx \right), \tag{A.2}
\]

where \( L_{h,p}(f) \) is defined by (4.17), and \( C > 0 \) is a constant independent of \( h \) and \( \varepsilon \).

Furthermore, if \( f_{\varepsilon} \) is uniformly bounded in \( W^{1,q}(\mathbb{T}^d) \) for \( q \in [1, d] \), then we have

\[
L_{h,p}(f_{\varepsilon}) \leq C \left( \frac{\|f_{\varepsilon}\|_{L^p}^p}{\log h} + \frac{\|\nabla f_{\varepsilon}\|_{L^q}^q}{\log h^{pd\min(0, \frac{1}{p} - \frac{1}{q})}} \right), \quad p \in \left[ 1, \frac{qd}{d-q} \right]. \tag{A.3}
\]

Proof. First, for any \( |z| < h \) and \( \sigma \in (0, 1) \), it can be shown by (4.16) and the fact that \( \|\nabla K_\sigma\|_{L^\infty} \leq \frac{C}{\sigma^{d+\tau}} \) that

\[
\int_{\mathbb{T}^d} |f_{\varepsilon}(x + z) - f_{\varepsilon}(x)|^p dx \leq C \int_{\mathbb{T}^d} \frac{K_\sigma}{|K_\sigma|_{L^1}} * f_{\varepsilon}(x + z) - K_\sigma * f_{\varepsilon}(x) |^p dx + C \int_{\mathbb{T}^d} f_{\varepsilon}(x) - K_\sigma * f_{\varepsilon}(x) |^p dx
\]

\[
\leq \frac{C}{|K_\sigma|_{L^1}} \int_{\mathbb{T}^d} \left( K_\sigma(x + z - y) - K_\sigma(x - y) \right) f_{\varepsilon}(y) dy |^p dx
\]

\[
+ \frac{C}{|K_\sigma|_{L^1}} \int_{\mathbb{T}^d} K_\sigma(x - y) f_{\varepsilon}(x) - f_{\varepsilon}(y) dy |^p dx
\]

\[
\leq \frac{C|z|^p}{\log \sigma^p} \|\nabla K_\sigma\|_{L^\infty} \|f_{\varepsilon}\|_{L^1}^p + \frac{C}{\log \sigma} \int_{\mathbb{T}^d} K_\sigma(x - y) |f_{\varepsilon}(x) - f_{\varepsilon}(y)|^p dy
\]

\[
\leq \frac{C h^p |f_{\varepsilon}|_{L^p}^p}{\log \sigma^p} + L_{\sigma,p}(f_{\varepsilon}).
\]

One may choose \( \sigma = h^{\frac{1}{d+\tau}} \) in the above inequality to prove (A.1).

Next, we show (A.2). It follows that

\[
L_{h,p}(f_{\varepsilon}) = \sum_{i=1}^{3} I_i^7,
\]

where \( I_i^7 \) \((i = 1, 2, 3)\) are given by

\[
I_1^7 := \int_{|x-y| > \frac{1}{2} h} \mathbb{K}_h(x-y)|f_{\varepsilon}(x) - f_{\varepsilon}(y)|^p dy,
\]

\[
I_2^7 := \int_{\frac{1}{4} h < |x-y| \leq \frac{1}{2} h} \mathbb{K}_h(x-y)|f_{\varepsilon}(x) - f_{\varepsilon}(y)|^p dy,
\]

\[
I_3^7 := \int_{0 \leq |x-y| < \frac{1}{4} h} \mathbb{K}_h(x-y)|f_{\varepsilon}(x) - f_{\varepsilon}(y)|^p dy.
\]
Due to (4.16) and the fact that $K_h(x) \leq C$ for any $|x| \geq \frac{1}{2}$, one can bound $I_1^2$ by

$$I_1^2 \leq C \frac{\|f_h\|^p_{L_p}}{\log h}.$$

For the term $I_2^2$, we have

$$I_2^2 \leq C \sum_{n=0}^{\log \log h - 1} \frac{1}{\log h} \int_{\frac{1}{2}e^{n-1} \leq |z| \leq \frac{1}{2}e^n} e^{\nu d} \int_{T^d} |f_\varepsilon(x) - f_\varepsilon(x-z)|^p dx dz$$

$$\leq C \frac{\|f_h\|^p_{L_p}}{\log h} \sum_{n=0}^{\infty} e^{-\frac{1}{2}n} \leq C \frac{\|f_h\|^p_{L_p}}{\log h}$$

where we used the facts that $K_h(z) \leq \frac{C}{n^2}$ for any $|z| > 0$ and $\log h \leq \frac{1}{2}e^{n+1}$ for any integer $0 \leq n \leq \log \log h - 1$. For the term $I_3^2$, it is easy to obtain

$$I_3^2 = \int_{|z| < \frac{1}{4\log h}} K_h(z) dz \int_{T^d} |f_\varepsilon(x) - f_\varepsilon(x-z)|^p dx \leq \sup_{|z| < \frac{1}{4\log h}} \int_{T^d} |f_\varepsilon(x+z) - f_\varepsilon(x)|^p dx.$$

Combining the above estimates of $I_i^2$ $(i = 1, 2, 3)$, we derive (A.2).

Moreover, for any $|z| < h$, due to the Sobolev inequality and the fact that

$$f_\varepsilon(x+z) - f_\varepsilon(x) = \int_0^1 z \cdot \nabla f_\varepsilon(x + \omega z) d\omega,$$

it holds that

$$\left( \int_{T^d} |f_\varepsilon(x+z) - f_\varepsilon(x)|^p dx \right)^{\frac{1}{p}} \leq \left( \int_{T^d} |f_\varepsilon(x+z) - f_\varepsilon(x)|^p dx \right)^{\frac{1}{p}} \leq |z| \|\nabla f_\varepsilon\|_{L^q}, \quad p \in [1, q]$$

and

$$\left( \int_{T^d} |f_\varepsilon(x+z) - f_\varepsilon(x)|^p dx \right)^{\frac{1}{p}} \leq \left( \int_{T^d} |f_\varepsilon(x+z) - f_\varepsilon(x)|^q dx \right)^{\frac{1}{q}} \left( \int_0^1 \int_{T^d} |\nabla f_\varepsilon(x + \omega z)|^q d\omega dx \right)^{\frac{1}{q}}$$

$$\leq C|z|^{-\theta} \left\| f_\varepsilon - \frac{1}{|T^d|} \int_{T^d} f dx \right\|_{L^q} \|\nabla f_\varepsilon\|_{L^q}^{1-\theta}$$

$$\leq C|z|^{-\theta} \|\nabla f_\varepsilon\|_{L^q}, \quad p \in \left( q, \frac{qd}{d-q} \right),$$

where the constant $\theta \in (0, 1)$ satisfies

$$\frac{1}{p} = \frac{d-q}{qd} \theta + \frac{1}{q}(1-\theta).$$

Thus, (A.3) follows from (A.2) and the above two estimates. The proof of Lemma A.1 is completed. \[\square\]

The following compactness criterion is a consequence of Lemma A.1 and the Riesz-Fréchet-Kolmogorov criterion (see [13, p. 111]).

**Lemma A.2.** Let $\varepsilon > 0$, and $f_\varepsilon$ be uniformly bounded in $L^p(T^d)$ for $p \in [1, \infty)$. Then the sequence $f_\varepsilon$ is strongly compact in $L^p(T^d)$ if and only if

$$\lim_{h \to 0} \limsup_{\varepsilon \to 0} L_{h,\varepsilon}(f_\varepsilon) = 0,$$

(A.4)

where $L_{h,\varepsilon}(f_\varepsilon)$ is defined by (4.17).
For the compactness in time and space, we have the following lemma of Lions-Aubin type.

**Lemma A.3.** Let \( T > 0, \varepsilon > 0, f_\varepsilon \) be uniformly bounded in \( L^p(0,T; L^p(\mathbb{T}^d)) \) for \( p \in [1, \infty) \), and \( \partial_t f_\varepsilon \) be uniformly bounded in \( L^q(0,T; W^{-m,1}(\mathbb{T}^d)) \) for \( q > 1 \) and \( m \geq 0 \). Then \( f_\varepsilon \) is strongly compact in \( L^p(0,T; L^p(\mathbb{T}^d)) \) if and only if

\[
\lim_{h \to 0} \limsup_{\varepsilon \to 0} \int_0^T L_{h,p}(f_\varepsilon) dt = 0, \tag{A.5}
\]

where \( L_{h,p}(f_\varepsilon) \) is defined by (4.17).

**Proof.** For any \( |z| < h < \frac{T}{2} \), it holds that

\[
\|f_\varepsilon(x + z, t + h) - f_\varepsilon(x, t)\|_{L^p(0, \frac{T}{3}; L^p)} = \sum_{i=1}^{3} I_i^h
\]

with

\[
I_1^h := \|f_\varepsilon(x + z, t + h) - f_\varepsilon(x, t + h)\|_{L^p(0, \frac{T}{3}; L^p)},
I_2^h := \|f_\varepsilon \cdot \eta_\delta(x, t + h) - f_\varepsilon \cdot \eta_\delta(x, t)\|_{L^p(0, \frac{T}{3}; L^p)},
I_3^h := \|f_\varepsilon(x, t + h) - f_\varepsilon \cdot \eta_\delta(x, t + h)\|_{L^p(0, \frac{T}{3}; L^p)} + \|f_\varepsilon(x, t) - f_\varepsilon \cdot \eta_\delta(x, t)\|_{L^p(0, \frac{T}{3}; L^p)},
\]

where \( \eta_\delta \in C_0^\infty(\mathbb{T}^d) \) for \( \delta \in (0, 1) \) is the Friedrichs mollifier. It can be verified by (A.1) and (A.5) that

\[
\lim_{h \to 0} \limsup_{\varepsilon \to 0} I_1^h = 0,
\lim_{h \to 0} \limsup_{\varepsilon \to 0} I_2^h \leq \lim_{h \to 0} \limsup_{\varepsilon \to 0} h^{1-\frac{d}{p}} \|\partial_t f_\varepsilon \cdot \eta_\delta\|_{L^p(0,T; L^p)} = 0, \quad \delta \in (0, 1),
\lim_{\delta \to 0} \limsup_{h \to 0} \limsup_{\varepsilon \to 0} I_3^h \leq 2 \lim_{\delta \to 0} \limsup_{h \to 0} \limsup_{\varepsilon \to 0} \|f_\varepsilon(x + z, t) - f_\varepsilon(x, t)\|_{L^p(0,T; L^p)} = 0.
\]

By the above estimates and the Riesz-Fréchet-Kolmogorov criterion, \( f_\varepsilon \) is strongly compact in \( L^p(0, \frac{T}{2}; L^p(\mathbb{T}^d)) \). Repeating the same arguments for \( \tilde{f}_\varepsilon(x, t) := f_\varepsilon(x, T - t) \), we derive the compactness of \( f_\varepsilon \) in \( L^p(\frac{T}{2}, T; L^p(\mathbb{T}^d)) \).

The next lemma is useful to Lemmas 4.6 and 4.9.

**Lemma A.4** (See [2,11]). For any \( f \in H^1(\mathbb{T}^d) \), it holds that

\[
|f(x) - f(y)| \leq C|x - y|(D_{|x-y|}|f(x) + D_{|x-y|}|f(y)), \tag{A.6}
\]

where \( C > 0 \) is a constant depending only on \( d \), and \( D_r f(x) \) is denoted by

\[
D_r f(x) := \frac{1}{r} \int_{|z| \leq r} \frac{|
abla f(x + z)|}{|z|^{d-1}} dz, \tag{A.7}
\]

which satisfies

\[
D_r f(x) \leq CM|\nabla f|(x). \tag{A.8}
\]

Here the localized maximal operator \( M : L^p(\mathbb{T}^d) \to L^p(\mathbb{T}^d) \) for any \( p \in [1, \infty) \) is defined by

\[
M f(x) := \sup_{r \in (0,1] \frac{1}{|B(0,r)|} \int_{B(0,r)} f(x + z) dz. \tag{A.9}
\]

In addition, we have

\[
\int_{\mathbb{T}^d} K_h(z) \|D_{|z|} f - D_{|z|} f(\cdot + z)\|_{L^2} dz \leq C\|f\|_{H^1}|\log h|^\frac{1}{4}, \tag{A.10}
\]

where the kernel \( K_h \) is given by (4.15).
The following lemma about the commutator estimates of the Riesz operator $R_i := (-\Delta)^{-\frac{1}{2}} \partial_i$ was introduced by Coifman et al. [15] and Coifman and Meyer [14], and applied to the global existence of the weak solution to isentropic compressible Navier-Stokes equations by Lions [37].

**Lemma A.5** (See [14, 15]). For $p \in (1, \infty)$, it holds that

$$
\| f R_i R_j g - R_i R_j (fg) \|_{L^p} \leq C \| f \|_{BMO} \| g \|_{L^p}, \quad f \in BMO(T^d), \quad g \in L^p(T^d),
$$

(A.11)

where $BMO(T^d)$ denotes the bounded mean oscillation space, and $C > 0$ is a constant depending only on $p$ and $d$.

Moreover, for $q_i \in (1, \infty)$ ($i = 1, 2, 3$) satisfying $\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}$, it holds that

$$
\| \nabla [ f R_i R_j g - R_i R_j (fg) ] \|_{L^{q_1}} \leq C \| \nabla f \|_{L^{q_1}} \| g \|_{L^{q_3}}, \quad f \in W^{1,q_2}(T^d), \quad g \in L^{q_3}(T^d),
$$

(A.12)

where $C > 0$ is a constant depending only on $q_i$ ($i = 1, 2$) and $d$. 