Thermodynamical Bethe Ansatz analysis in an
SU(2)×U(1) symmetric σ-model

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Abstract

Four different types of free energies are computed by both thermodynamical Bethe Ansatz (TBA) techniques and by weak coupling perturbation theory in an integrable one-parameter deformation of the O(4) principal chiral σ-model (with SU(2)×U(1) symmetry). The model exhibits both ‘fermionic’ and ‘bosonic’ type free energies and in all cases the perturbative and the TBA results are in perfect agreement, strongly supporting the correctness of the proposed $S$ matrix. The mass gap is also computed in terms of the Λ parameters of the modified minimal subtraction scheme and a lattice regularized version of the model.

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1. Introduction

We shall study the following one-parameter family of 1+1 dimensional σ-models described by the Lagrangian

\[ \mathcal{L}_0 = -\frac{1}{2\lambda} \left( L^1_\mu L^1_\mu + L^2_\mu L^2_\mu + (1 + g) L^3_\mu L^3_\mu \right), \tag{1.1} \]

where

\[ \frac{1}{2} L^a_\mu \sigma^a = G^{-1} \partial_\mu G, \quad G \in SU(2), \tag{1.2} \]

g is a (real) parameter and \( \sigma^a \) stand for the standard Pauli matrices. The Lagrangian (1.1) can be interpreted as a deformation of the SU(2) × SU(2) (or O(4)) symmetric nonlinear σ-model (NLS) by the parameter \( g \). It possesses an SU(2) \( L \times U(1)_R \) (global) symmetry. We shall assume in the following that the deformation parameter, \( g \), satisfies

\[ -1 \leq g \leq 0, \tag{1.3} \]

because with \( g \) being in this range the model, (1.1), is asymptotically free (AF) (see Appendix D). At the classical level the deformed model, (1.1), interpolates between the SU(2) × SU(2) (\( g = 0 \)) and O(3) (\( g = -1 \)) NLS models. It has already appeared in the Bethe Ansatz approach of Polyakov and Wiegmann to solve the SU(2) × SU(2) and O(3) NLS \([1, 2]\). The deformed model (1.1) is known to possess a Lax-pair and is believed to be integrable both at the classical and at the quantum level \([3]\). Its \( S \)-matrix has been proposed long time ago \([1, 3]\) and at present a considerable amount of evidence corroborates the belief that the deformed SU(2) × U(1) symmetric σ-model is quantum integrable indeed \([4, 5, 6]\). Its spectrum is thought to contain two massive doublets whose scattering is described by the tensor product of an SU(2) × U(1) symmetric solution of the bootstrap \( S \)-matrix equations:

\[ S(\theta) = S^{(\infty)}(\theta) \otimes S^{(p)}(\theta), \tag{1.4} \]

where \( S^{(p)}(\theta) \) is the Sine-Gordon (SG) \( S \)-matrix depending on the parameter \( p \). In Eq. (1.4) the limit \( p \to \infty \) corresponds to an SU(2) × SU(2) symmetric \( S \)-matrix.

In this paper we carry out a rather thorough consistency check on the proposed \( S \)-matrix of the one parametric deformation of the O(4) σ-model, (1.1). The basic idea is to investigate the system at large particle density when both perturbation theory in the coupling \( \lambda \) and a calculation based on the \( S \)-matrix of the massive physical particles apply. (For a review, see Ref. \([4]\).) To achieve this one introduces a chemical potential, \( h \), coupled to a Noether charge of the corresponding symmetry, and considers the ground state of the modified Hamiltonian

\[ H = H_0 - hQ. \tag{1.5} \]

The Legendre transform of the ground state energy density or (zero temperature) free energy of the system must be the form:

\[ \mathcal{F} \equiv \delta f(h) \equiv f(h) - f(0) = -h^2 F_0(h/m, Q), \tag{1.6} \]
where $m$ is the mass of the particles. (Eq. (1.6) follows simply from dimensional analysis.) The Thermodynamical Bethe Ansatz (TBA) method based on the $S$-matrix leads to a set of integral equations, which can be solved, for large chemical potential $h$, in the form of an asymptotic series expansion in $h/m$ using a modified Wiener-Hopf technique [8].

On the other hand in an asymptotically free theory the result of a perturbative computation of $F_0$ (which is of course renormalization group invariant) is an asymptotic series in the running coupling, $\bar{\lambda}(h/\Lambda) \propto [\ln(h/\Lambda)]^{-1}$, where the $\Lambda$ parameter (of dimension mass) is the usual renormalization group invariant combination of the ultraviolet cutoff and the bare coupling, $\lambda$.

Since $F$ is a physical quantity, the results obtained by TBA method and by perturbation theory should agree. Thus comparison of the two results provides a rather stringent consistency check on the $S$-matrix (and also on the self-consistency of the hypothesis used in the course of the calculation), moreover one can also extract the exact $m/\Lambda$ ratio, a rather important non-perturbative parameter of the theory which can then be also measured e.g. by lattice simulations. This comparison, following the pioneering work in Refs. [9, 10], has since been applied to many bosonic, fermionic and supersymmetric integrable models.

In Ref. [6] a two-parameter family of integrable models has been studied by similar techniques. The models were mainly treated as perturbed conformal field theories allowing for a proof of integrability at the quantum level. The SU(2)×U(1) symmetric $S$-matrix (1.4) is a limiting case, ($p_1 \to \infty$, $p_2 = p$), of the $S$-matrix of Ref. [4]. In the present paper, however, we put the emphasis on the $\sigma$-model representation suitable for traditional weak-coupling perturbation theory. Although the model (1.1) corresponds to a limiting case of the two-parameter family of Ref. [4], one cannot read off directly the asymptotic expansions for the ground state energies from Ref. [3]. This is due to the fact that both in perturbation theory and using the TBA only the asymptotic series are computable. The coefficients of these series diverge in the $p_1 \to \infty$ limit, thus one cannot directly recover the coefficients of the limiting case from those valid for finite values of the parameters. For this reason we had to perform the TBA analysis directly in the $p \to \infty$ limit.

As there are two conserved Noether charges in the model (1.1), $Q_L, Q_R$, there are three types of free energies corresponding to ground states either coupled to $Q_L$ or to $Q_R$ or to both of them. In fact when the ground state is coupled to $Q_L$, one has to distinguish between two different cases namely $p > 1$ and $p < 1$ as then bound states (breathers) appear in the spectrum. We have calculated all the (four) different types of (zero temperature) free energies both by the TBA method and by perturbation theory in the modified minimal substraction scheme (MS) and extracted the corresponding (four) $m/\Lambda$ ratios. Three of the kernels of the integral equation determining the ground state energies are simply the $p_1 \to \infty$ limits of the corresponding ones in [4] whereas in the $p < 1$ case with bound states we have
computed the corresponding kernel directly using the bootstrap fusion method. We have found complete consistency for all four cases, in particular all of the four $m/\Lambda$ ratios agree, the result being:

$$\frac{m}{\Lambda_{\text{MS}}} = 2^{3-\frac{p}{2}} e^{\frac{p}{2} - 1} \frac{\Gamma(1 + \frac{p}{2})}{\pi p}, \quad (1.7)$$

where $m$ is the mass of the SU(2) doublets. The $p \to 0$ limit (corresponding to the running deformation parameter $\tilde{g}(h) \to -1$ in (1.1)) is expected to give the O(3) NLS model (and a decoupled free field) [2]. Our results are in perfect agreement with his expectation, as the previously calculated $m_1/\Lambda_{\text{MS}} = 8/e$ ratio in the O(3) NLS model [9] (where now $m_1$ denotes the mass of the triplet in the O(3) model) is reproduced in the $p \to 0$ limit. This can be immediately seen from Eq. (1.7) using the fact that in the $p \to 0$ limit $m_1 \to \pi p m$.

We have also proposed a lattice action for the deformed O(4) NLS model and calculated the ratio of the $\Lambda$-parameters corresponding to the two regularization schemes:

$$\frac{\Lambda_{\text{MS}}}{\Lambda_L} = \sqrt{32e} \pi. \quad (1.8)$$

The plan of the paper is as follows: in Section 2 the calculation of the free energies by the TBA method is outlined, in Section 3 the perturbative computation is presented in some detail and the $m/\Lambda$ ratios are also given there. Section 4 is devoted to the perturbative calculation of the free energy using lattice regularization leading to the $\Lambda_{\text{MS}}/\Lambda_L$ ratio. To make the paper rather self contained we included four Appendices with some details of the computations. Appendix A contains some useful formulae to obtain the free energy by solving the pertinent integral equation. In Appendix B the scattering phase shift of the lowest mass bound state is calculated. Appendix C contains some details of the perturbative calculations, while Appendix D is devoted to renormalization group considerations.

### 2. Calculation of the free energies from the integral equation

As the model (1.1) admits $G = SU(2)_L \times U(1)_R$ as its (global) symmetry group, one can introduce two chemical potentials coupled to the two Noether charges, $Q_L, Q_R$, corresponding to the two U(1) (L and R) subgroups of $G$. The modified Hamiltonian then reads

$$H = H_0 - h_L Q_L - h_R Q_R. \quad (2.1)$$

For definiteness we chose $h_L, h_R \geq 0$ and normalize the charges of the four particles as

$$(1, 1), \quad (-1, 1), \quad (1, -1), \quad (-1, -1). \quad (2.2)$$

We distinguish between the three different types of finite density ground states depending on $h_L, h_R$:
1. \( h_L, h_R > 0 \), (DIAG)

2. \( h_L = 0, h_R = h > 0 \), (RIGHT)

3. \( h_L = h > 0, h_R = 0 \). (LEFT)

In the following we shall analyse these three cases.

2.1 The diagonal current

We recall that our strategy will be to calculate the ground state energy density of the system at large particle density (and at zero temperature), that is we consider the system with \( h_L/m, h_R/m \gg 1 \) (\( m \) is the mass of the particles). Then it is clear that particles of charge (1,1) condense into the vacuum. It is less clear what other kind of particles (necessarily with a smaller charge/mass ratio) will appear in the vacuum state. We shall assume that the vacuum consist of only particles of charge (1,1). This seemingly radical assumption has apparently worked in all analogous examples studied so far, and it greatly simplifies the solution of TBA equations.

The scattering phase of the (1,1) type particle, \( \delta(\theta) \), can be read off from Eq. (B.11) in Appendix B:

\[
\delta(\theta) = \delta_{\infty}(\theta) + \delta_p(\theta) .
\] (2.3)

When the ground state is assumed to contain only particles of charge (1,1) the calculation of the free energy using the TBA method reduces to the solution of a single integral equation (A.1) where the kernel \( K \) is given by the logarithmic derivative of the relevant \( S \)-matrix element,

\[
K(\theta) = \frac{1}{2\pi i} d \ln S(\theta) .
\] (2.4)

In the present case \( S(\theta) = \exp \{i\delta(\theta)\} \) where scattering phase, \( \delta(\theta) \), is given by Eq. (2.3). We note that the effective chemical potential, \( h \), in Eq. (A.1) is just \( h = h_L + h_R \). From Eqs. (2.3, B.8) the Fourier transform of the relevant kernel is easily found to be:

\[
1 - \hat{K}_D(\omega) = \frac{\tanh \frac{\pi \omega}{2}}{2 \sinh \frac{\pi p \omega}{2}} e^{\frac{\pi p |\omega|}{2}} ,
\] (2.5)

with \( p > 0 \). The result of the splitting of \( 1 - \hat{K}_D(\omega) = [G_+(\omega)G_-(\omega)]^{-1} \) where \( G_+(\omega) \) resp. \( G_-(\omega) \) are analytic functions in the upper resp. lower half plane can be written as

\[
G_+(\omega) = \sqrt{2p} \frac{\Gamma^2(1 - \frac{i\omega}{2})}{\Gamma(1 - i\omega) \Gamma(1 - \frac{ip\omega}{2})} e^{\frac{ip\omega}{2} (1 - \ln(-\frac{ip\omega}{2}))} e^{-i\omega \ln 2} ,
\] (2.6)

and \( G_-(\omega) = G_+(-\omega) \).
For the present (fermionic type) kernel \( (2.3) \) the constants in Eqs. \( (A.5-A.6) \) of Appendix A are given as

\[
\tilde{a} = -\frac{p}{2}, \quad \tilde{b} = \frac{p}{2} \left( \ln \frac{p}{2} + 1 - \Gamma'(1) \right) + \ln 2, \quad (2.7)
\]

\[
\tilde{k} = \sqrt{2p}, \quad G_+(i) = \frac{\pi}{2} \frac{\sqrt{2p}}{\Gamma(1 + \frac{p}{2})} e^{-\frac{\pi}{4}(1-\ln \frac{p}{2})}.
\]

Then the expression for free energy \( (A.3) \) finally reads:

\[
\delta f_D(h) = -\frac{h^2}{\pi} p \left[ 1 - \frac{p}{2t} - \frac{p^2 \ln t}{4 t^2} + \frac{\tilde{A}}{t^2} + O\left(\frac{\ln t}{t^3}\right) \right], \quad (2.8)
\]

where \( t = \ln(h/m) \) and the constant \( \tilde{A} \), which enters in our final result for the \( m/\Lambda \) ratio is given as

\[
\tilde{A} = \frac{p}{2} \left[ \ln \Gamma(1 + \frac{p}{2}) + \frac{p}{2} \left( \frac{3}{2} - \ln 2 \right) - 1 + 3 \ln 2 - \ln \pi \right]. \quad (2.9)
\]

We remark here that from the classical term in Eq. \( (2.3) \), \( -h^2 p/\pi \), one can immediately read off the level of an underlying ultraviolet (UV) current algebra, \( k \), as \( k = p \) \([11]\).

2.2 The right current

In analogy to the assumption made in the diagonal case we now expect the ground state to consist of a mixture of particles of the same charge with respect to \( U(1)_R \), i.e. those of charge (1,1) and (-1,1) with equal densities. In Ref. \([3]\) the result of the diagonalisation of the pertinent coupled TBA system has been given and the relevant kernel, \( \hat{K}_R(\omega) \) turns out to be the \( p_1 \to \infty \) limit in Eq. (57) of Ref. \([3]\). From this one finds (somewhat surprisingly) \( \hat{K}_R(\omega) = \hat{K}_D(\omega) \), implying that \( \delta f_R(h) = \delta f_D(h) \), where we have denoted the chemical potential for the right current as \( h_R = h \). So the free energies are precisely the same as functions of the effective chemical potentials in these two apparently rather different cases. We do not have an explanation of this fact, we note, however, that as long as \( p_1 < \infty \) \( \hat{K}_R(\omega) \neq \hat{K}_D(\omega) \).

2.3 The left current

Analogously to the RIGHT case the ground state is expected to consist of an equal density mixture of particles of the same charge, this time with respect to \( U(1)_L \), i.e. those of charge (1,1) and (1,-1). Using again the result of Ref. \([3]\) the Fourier transform of the relevant (bosonic type) kernel is found to be:

\[
1 - \hat{K}_L(\omega) = \frac{\tanh \frac{\pi |\omega|}{2}}{2 \cosh \frac{\pi |\omega|}{2}} e^{\frac{\pi |\omega|}{2}}, \quad (2.10)
\]
where \( x = p - 1 \) and \( x \) is assumed to be positive. The functions appearing in the decomposition of \( 1 - \hat{K}_L(\omega) \) are now given as

\[
G_+(\omega) = \frac{2\sqrt{\pi}}{\sqrt{-i\omega}} \frac{\Gamma\left(1 - \frac{i\omega}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{i\omega}{2}\right)\Gamma\left(\frac{1}{2} - \frac{i\omega}{2}\right)} e^{\frac{i\omega}{2}(1-\ln(-\frac{i\omega}{2}))},
\]

(2.11)

and \( G_-(\omega) = G_+(\omega) \). For the kernel (2.10), the constants in Eq. (A.8) are not difficult to calculate and we find:

\[
a = -\frac{x}{2}, \quad b = -\frac{1}{2} [x \ln x + x(\ln 2 - 1 - \Gamma'(1)) + 2 \ln 2],
\]

\[
k = \frac{2}{\sqrt{\pi}}, \quad G_+(i) = \frac{\pi}{\Gamma\left(\frac{1}{2} - \frac{i\omega}{2}\right)} e^{\frac{-x}{2}(1-\ln \frac{x}{2})}.
\]

(2.12)

So the asymptotic series of the free energy density can be simply written as

\[
\delta f_L(h) = -\frac{h^2}{\pi} \left( t + (1 - \frac{p}{2}) \ln t + A + \ldots \right),
\]

(2.13)

where the constant \( A \), which enters to our final result for the ratio \( m/\Lambda \) is found to be

\[
A = \frac{5}{2} \ln 2 - \ln x - 1 + \ln \Gamma \left(1 + \frac{x}{2}\right) + \frac{x}{2}(1 - \ln 2).
\]

(2.14)

At this point we would like to remark that although the \( p \to \infty \) limit in the \( S \)-matrix is smooth (c.f. Eq. (B.8)) this is not the case in the final results for the free energy, i.e. in Eqs. (2.8), (2.13). This is not so surprising since Eqs. (2.8), (2.13) have been obtained as asymptotic expansions valid for \( h/m \to \infty \) for fixed \( p \), and the lack of a smooth \( p \to \infty \) limit of the coefficients shows that the \( h/m \to \infty \) and the \( p \to \infty \) limits cannot be simply interchanged in these asymptotic expansions.

Next it might be worth pointing out the following (somewhat puzzling) fact. In Eq. (2.8) the ratio of the coefficients of the \( \ln t/t^2 \) term, \(-p^2/4\), and the \(-1/t \) term, \( p/2 \), is \( r_0 = -p/2 \), while in Eq. (2.13) the ratio of the coefficients of the \( \ln t \) term, \( 1 - p/2 \), and that of the \( t \) term is \( r_L = 1 - p/2 \). This appears to be a serious problem since on general grounds in a theory with a coupling, \( \lambda \), the perturbative result for a fermionic type free energy is expected to be of the form

\[
\delta f_{\text{Fer}}(h) = -\frac{h^2f_0}{\pi} \left( 1 - f_1 \frac{\bar{\lambda}(h)}{2\pi} + \mathcal{O}(\bar{\lambda}^2) \right),
\]

(2.15)

while in the same theory for a bosonic case it is expected to be given by

\[
\delta f_{\text{Bos}}(h) = -\frac{h^2b_0}{\pi} \left( \frac{1}{\lambda(h)} + \text{const.} + \mathcal{O}(\bar{\lambda}) \right),
\]

(2.16)

where \( f_0, f_1, b_0 \) are constants and \( \bar{\lambda}(h) \) is the running coupling. Since from the renormalization group

\[
\frac{1}{\lambda(h)} = \beta_0 \ln \frac{h}{\Lambda} + \frac{\beta_1}{\beta_0} \ln \ln \frac{h}{\Lambda} + \mathcal{O}\left(\frac{1}{h/\Lambda}\right),
\]

(2.17)
one would expect \( r_{\text{Fer}} = r_{\text{Bos}} = \beta_1/\beta_0^2 \). As in the present case \( r_D \neq r_L \), it seems impossible to match Eq. (2.8) and Eq. (2.13) simultaneously with the corresponding perturbative expansion. Fortunately this turns out to be only an apparent paradox which is resolved, however, in a not entirely obvious way (see Section 3).

Finally we note that the leading term of \( \delta f_L(h)/h^2 \) diverges as \( \ln(h/m) \) which can be interpreted that either the underlying UV current algebra does not exist or that if it could be defined its level is divergent.

2.4 The left current for \( p < 1 \)

In the previous subsection it has been assumed that the parameter, \( x = p - 1 > 0 \). Although in the \( S \)-matrix any \( p \geq 0 \) is allowed in Eq. (2.10) the range of \( p \) is, however, restricted to \( p > 1 \). This restriction follows from the fact that in Eq. (2.10) \( \lim_{\omega \to \infty} \hat{K}_L(\omega) = 1 \) for \( 0 \leq p < 1 \), that is for \( p < 1 \) \( \hat{K}_L(\omega) \) ceases to be the Fourier transform of a function. Then clearly one cannot just simply analytically continue in \( p \) the result for the free energy, (2.13), for \( p < 1 \).

In order to clarify what happens when \( p \) becomes smaller than one, we recall that then in the Sine-Gordon (SG) theory bound states (breathers) appear in the spectrum (for \( p < 1 \) one enters simply the attractive regime). The SG mass spectrum is given by the well known formula:

\[
m_r = 2m \sin \frac{\pi pr}{2}, \quad r = 1, 2 \ldots < \frac{1}{p},
\]

where \( m \) denotes the mass of the SG kinks. As the \( S \) matrix of the present model, Eq. (1.4), contains the SG \( S \) matrix as a factor the spectrum then changes accordingly, i.e. breathers also appear (with appropriate multiplicity) in the deformed NLS model. (We remark that \( m \) is also the mass of the SU(2) doublets, of course).

The appearance of these new particles implies that as soon as \( p \) becomes smaller than 1, the vacuum changes as then it becomes energetically favourable for the \( r = 1 \) charge \((2,0)\) breather to condense. This charge \((2,0)\) breather can be interpreted as the bound state of the \((1,1)\) and \((1,-1)\) particles (kinks). In fact as this particle has the highest charge/mass ratio, we can again assume (as in Subsection 2.1) that the true vacuum consists only of the condensate of the \( r = 1 \) charge \((2,0)\) breather. Then one needs the scattering phase, \( \delta^{(2,0)}(\theta) \), of the \((2,0)\) breathers in order to compute \( \delta f_L(h) \) for the \( p < 1 \) case.

The calculation of \( \delta^{(2,0)}(\theta) \) amounts to a simple application of the well known bootstrap-fusion method [13]. In Appendix B we summarized this method and calculated the breather-breather phase shift, which is given by Eq. (B.16). Using Eq. (B.18) the Fourier transform of the pertinent kernel is found to be

\[
1 - \hat{K}_B(\omega) = 2 \tanh \frac{\pi |\omega|}{2} \cosh \frac{\pi |\omega|x}{2} e^{-\frac{\pi |\omega|x}{2}},
\]

(2.19)
where $|x| = |p - 1| = 1 - p$. The functions appearing in the decomposition of $1 - \hat{K}_B(\omega)$ are now given as

$$G_+(\omega) = \frac{1}{\sqrt{-i\omega}} \frac{\Gamma(1 - \frac{i|\omega|}{2}) \Gamma(\frac{1}{2} - \frac{i|\omega|}{2})}{\Gamma(\frac{1}{2} - \frac{i|\omega|}{2})} e^{-\frac{i|\omega|}{2} (1 - \ln(-\frac{i|\omega|}{2}))},$$

and again $G_-(\omega) = G_+(-\omega)$.

For the kernel (2.19) the constants in (A.8) are found to be:

$$a = \frac{|x|^2}{2}, \quad b = \frac{1}{2} \left| |x| \ln |x| + |x| (\ln 2 - 1 - \Gamma'(1)) - 2 \ln 2 \right|,$$

$$k = \frac{1}{\sqrt{\pi}}, \quad G_+(i) = \frac{1}{2} \Gamma \left(1 + \frac{|x|}{2}\right) e^{\frac{|x|^2}{2} (1 - \ln \frac{|x|}{2})}.$$

So the final result for the asymptotic expansion of the free energy density for the $r = 1$ charge (2,0) breather condensate can be written as

$$\delta f_B(h_B) = -\frac{h_B^2}{4\pi} \left(\tau + (1 - \frac{p}{2}) \ln \tau + A_B + \ldots\right),$$

where $\tau = \ln h_B/m_1$ and the constant $A_B$ is given as

$$A_B = 3 \ln 2 - \frac{3}{2} - \ln \pi + \ln \Gamma \left(\frac{p}{2}\right) + \frac{p}{2} (1 - \ln 2) + \ln \left(\sin \frac{\pi p}{2}\right).$$

It is now easy to verify that expressing the breather mass, $m_1$, through the kink mass, $m$, using the SG spectrum formula Eq. (2.18) and taking into account that the charge of the breather is twice that of the kink, i.e. $h_B = 2h$, the free energy density of the breather condensate, $\delta f_B(h_B, m_1)$, becomes identical to that of the charge $(1, \pm 1)$ kink mixture, $\delta f_L(h, m)$ analytically continued for negative values of $x$. This is very reassuring as the result for $\delta f_L(h, \Lambda)$ from perturbation theory turns out to be continuous for the full range $0 \leq p < \infty$, which is certainly to be expected.

**2.5 The O(3) model as the $p \to 0$ limit.**

In this last subsection we discuss the $p \to 0$ limit and recapitulate some arguments advanced a long time ago by Wiegmann [2] that the SU(2)$_L \times$U(1)$_R$ invariant theory (1.1) in this limit becomes equivalent to the O(3) $\sigma$-model and a decoupled free massive particle. First of all from Eq. (2.18) one sees that $m/m_1 \to \infty$ as $p \to 0$ i.e. the kinks ‘disappear’ from the spectrum. Furthermore the higher breathers with $r > 1$ also ‘disappear’ from the spectrum, but for a different reason. The binding energy of the $r > 1$ breathers (which can be considered as bound states of the $r = 1$ ones) tends to zero giving rise to zero energy bound states. Therefore the limiting spectrum consists of 4 massive particles, transforming as $(3L + 1L, 1R)$ under SU(2)$_L \times$U(1)$_R$. As already claimed in Ref. [2] and confirmed in Ref. [5] in the $p \to 0$ limit the $S$-matrix of these remaining $r = 1$ kinks becomes effectively $S_{O(3)} \otimes 1$ i.e.
the tensor product of the (minimal) $S$ matrix of the O(3) model proposed by the Zamolodchikov brothers [17] with the $S$-matrix of a free boson, consistently with this somewhat heuristic argument. As this is a highly non-trivial limit it is clearly desirable to perform as many quantitative checks on it as possible. In the following we add yet another piece of evidence corroborating the arguments of Wiegmann, by showing that the $p \to 0$ limit of the $m_1/\Lambda_{\text{MS}}$ ratio agrees precisely with the one calculated directly in the O(3) model [9].

3. Perturbative Calculations

In this section we calculate the ground state energy of the system in the presence of external fields in perturbation theory. As in the previous section, we shall consider fields coupled to the Noether charges corresponding to the $U(1)_L \times U(1)_R$ transformation

$$\delta G = i \epsilon_L \sigma^3 G - i \epsilon_R G \sigma^3.$$

(3.1)

To calculate the ground state energy one has to gauge the symmetries in the Euclidean field theory formulation and consider the case of constant, imaginary gauge fields [9]. The corresponding covariant derivative $D_\mu$ is given by

$$D_2 G = \partial_2 G + h_L \sigma^3 G - h_R G \sigma^3, \quad D_1 G = \partial_1 G.$$

(3.2)

Putting (3.2) into (1.1) we write the gauged Lagrangian as:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2,$$

(3.3)

where $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the terms linear and quadratic in the external fields, respectively.

To actually calculate the free energy in the presence of the external field, we take the master formula

$$\mathcal{F}(h) = \delta f(h) = f(h) - f(0),$$

(3.4)

where

$$e^{-\int d^n x f(h)} = \int \mathcal{D}G \exp\left\{-\int d^n x \left(\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2\right)\right\},$$

(3.5)

and expand it in powers of the (bare) coupling $\lambda_0$. From now on the bare lagrangian parameters (couplings) will be denoted as $\lambda_0$ and $g_0$. We have also written the volume element as $d^n x$, where $n = 2 - \epsilon$ to indicate that we are going to use dimensional regularization in the perturbative calculations.

In perturbation theory the action has to be expanded around a (stable) solution of the classical equations of motion. Thus our next task is to find elements of $SU(2)$, $G_0$, that correspond to (local) minima of the quadratic part of the gauged Lagrangian,

$$\mathcal{L}_2 = -\frac{2h_L^2}{\lambda_0} (1 + g_0 z^2) - \frac{2h_R^2}{\lambda_0} (1 + g_0) + \frac{4h_L h_R (1 + g_0) z}{\lambda_0},$$

(3.6)
where
\[ z = \frac{1}{2} \text{Tr} \left\{ \sigma^3 G^{-1} \sigma^3 G \right\}, \quad (3.7) \]
which plays the role of the potential energy. In this paper we shall consider the following two solutions. The first one, called ‘bosonic’, (BOS) is given as:
\[ G_0 = \frac{1}{\sqrt{2}} (1 + i \sigma^2), \quad h_R = 0. \quad (3.8) \]
The other one, which shall be referred to as ‘fermionic’ (FER) is written as
\[ G_0 = i \sigma^2, \quad (3.9) \]
where we also require that \( h_L \) and \( h_R \) satisfy the inequality
\[ \Delta = 4 \left[ h_L^2 g_0 + h_L h_R (1 + g_0) \right] \geq 0, \quad (3.10) \]
ensuring the non-negativity of the mass term.

The terms ‘bosonic’ (resp. ‘fermionic’) are used here by anticipating that they actually correspond to the bosonic (resp. fermionic) type free energies discussed in the previous section. The BOS-type solution will be used to calculate the free energy corresponding to the pure \( U(1)_L \) charge (\( h_R = 0 \), LEFT) case, while the FER-type solution is appropriate to calculate the free energy for both the pure \( U(1)_R \) charge (\( h_L = 0 \), RIGHT) and the ‘diagonal’ (DIAG) case, where both \( h_L \) and \( h_R \) are different from zero.

We now start to analyse the BOS case first. The leading, \( \mathcal{O}(\lambda_0^{-1}) \), term in perturbation theory is given by the potential energy, \( \mathcal{L}_2 \), at its minimum, which is immediately seen to be \( (z_0 = 0, h_R = 0) \)
\[ \mathcal{F}^{(-1)} = \mathcal{L}^{(-1)} = -\frac{2h_L^2}{\lambda_0}. \quad (3.11) \]

Then we turn to the calculation of the next, 1-loop, term in perturbation theory. The calculation is elementary but for completeness we give some details in Appendix C. The general 1-loop contribution is given by Eq. (C.15), which is a rather complicated function of three parameters. It simplifies somewhat in the two cases we are interested in. In the present, BOS, case after using (C.12), we obtain
\[ \mathcal{F}^{(0)} = \frac{4h_L^2}{n} \int \frac{d^n p}{(2\pi)^n} \frac{(1 + g_0) p_0^2 - g_0 p_2^2}{(p^2)^2 - 4g_0 h_L^2 p_0^2 + 4(1 + g_0) h_L^2 p_2^2}. \quad (3.12) \]
Because of the explicit dependence of the integrand on the second component of the momentum we have to modify the usual rules of dimensional regularization by the following. In all Feynman integrals we perform the one-dimensional integral over
\[ p_2 \text{ first, followed by the integration over the remaining momentum components, continued to } n - 1 \text{ dimensions, i.e. in the present paper} \]
\[
\int \frac{d^n p}{(2\pi)^n} := \int \frac{d^{n-1} p}{(2\pi)^{n-1}} \int_{-\infty}^{\infty} \frac{dp_2}{(2\pi)}. \quad (3.13)
\]

It is possible to calculate the integral (3.12) exactly \[16\]. For our purposes, however, it is sufficient to calculate its divergent part (the part that becomes singular in the \( \epsilon \rightarrow 0 \) limit) and to show that \( \mathcal{F}^{(0)} \) must be of the form
\[
\mathcal{F}^{(0)} = h^n_L \left\{ \frac{1 - g_0}{2\pi\epsilon} + W(g_0) \right\}, \quad (3.14)
\]
where \( W(g_0) \) is a (complicated) finite function of its argument. Fortunately all we shall need is the value of \( W(g_0) \) at the special point \( g_0 = -1 \). For this very special \( g_0 \) (3.12) simplifies enormously:
\[
\mathcal{F}^{(0)} \bigg|_{g_0=-1} = \frac{4h_L^2}{n} \int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2 + 4h_L^2}. \quad (3.15)
\]
Calculating (3.15) and comparing the result to (3.14) yields
\[
W(-1) = \frac{1}{2\pi} \left[ 1 + \Gamma'(1) + \ln \pi \right]. \quad (3.16)
\]
Having obtained the bare 1-loop result, we now turn to renormalization and renormalization group (RG) improvements. The methods are well-known but we summarized the details of the calculation in Appendix D to make this paper relatively self-contained.

The RG improved perturbative result gives the asymptotic expansion of the free energy for large values of the external fields. Writing the sum of the classical contribution (3.11) and the one-loop correction (3.14) and using definitions (D.2), (D.3) we obtain the renormalized form
\[
\mathcal{F}(h) = -\frac{2h^2}{\lambda} - \frac{(1-g)h^2}{4\pi} \ln \frac{h^2}{\mu^2} + h^2 W(g) + \mathcal{O}(\lambda), \quad (3.17)
\]
where \( h = h_L \). Expressing the result in terms of the single effective coupling, \( \lambda_{\text{eff}} \), defined by Eq. (D.29) we get
\[
\mathcal{F}(h) = -2h^2 \left\{ \frac{1}{\lambda_{\text{eff}}} + \frac{1}{2\pi} \left( \ln 2 - \frac{1}{2} \right) + \cdots \right\}. \quad (3.18)
\]
At this point we remark that we have identified the RG invariant parameter defined in terms of the running couplings \( \bar{g}(t) \) and \( \bar{\lambda}(t) \) by Eq. (D.21) with the ‘deformation
parameter, \( p \) appearing in the \( S \) matrix (1.4). Finally expanding (3.18) with the help of (D.32) the free energy in the BOS case can be expressed as

\[
\mathcal{F}(h) = -\frac{h^2}{\pi} \left\{ s + \left( 1 - \frac{p}{2} \right) \ln s + \left( \ln 2 - \frac{1}{2} \right) + \cdots \right\} .
\] (3.19)

Next we turn to the computation in the FER case. The leading (classical) term of the free energy is then

\[
\mathcal{F}(-1) = \mathcal{L}(-1) = -\frac{2(1 + g_0)}{\lambda_0} (h_L + h_R)^2 .
\] (3.20)

Note that \( \mathcal{F}(-1) \) for the diagonal charge depends only on the sum \( h_L + h_R \). This is consistent with the results of the TBA analysis found in Section 2 for the DIAG case. In the present (FER) case (C.15) becomes

\[
\mathcal{F}(0) = 2 \int d^np \left( 2\pi \right)^n \Re \frac{\Delta - i\omega p_2}{p^2 + \Delta - 2i\omega p_2} ,
\] (3.21)

where

\[
\omega = h_L(1 - g_0) - h_R(1 + g_0) .
\] (3.22)

Evaluating Eq. (3.21) using the modified dimensional scheme (c.f. (3.13)) yields:

\[
\mathcal{F}(0) = (1 + g_0)^n (h_L + h_R)^n \left\{ \frac{1}{2\pi\epsilon} + \frac{1}{4\pi} \left[ 1 + \Gamma'(1) + \ln 4\pi \right] \right\} .
\] (3.23)

It is gratifying to find that the one loop term (3.23) also depends only on the sum \( h_L + h_R \), just like the classical one (3.21). This is not an accident. It is in fact absolutely essential that \( \mathcal{F}(h_L, h_R) \) be a function of \( h_L + h_R \) in order to be able to match the perturbative result with the one found by the TBA method for the DIAG case in Subsection 2.1. As shown in Appendix C, in the FER case the free energy does depend only on the sum \( h_L + h_R \) to all orders in perturbation theory. Moreover it is easy to see that the free energy in the FER case must be of the form

\[
\mathcal{F}(h) = -\frac{2h^2(1 + g_0)}{\lambda_0} + \frac{M^2}{\lambda_0} \sum_{L=1}^{\infty} F_L(g_0) \left( \frac{\lambda_0}{M^\epsilon} \right)^L .
\] (3.24)

Where

\[
M = (1 + g_0)(h_L + h_R) .
\] (3.25)

and the summation is over the number of loops, \( L \). (3.24) is nothing but dimensional analysis based on the fact that the only dimensionful parameter available is \( M \), which serves as an infrared cutoff in our perturbative computations. The second form
illustrates the fact that all higher order terms are proportional to \((1 + g_0)^2\), which fact plays an important role in our calculations.

Adding now the classical and the one-loop terms, (3.20) and (3.23) and expressing the result in terms of renormalized quantities we obtain:

\[
\mathcal{F}(h) = -2h^2 \left\{ \frac{1 + \frac{g}{\lambda}}{\lambda} + \frac{(1 + \frac{g}{\lambda})^2}{8\pi} \left[ \frac{1}{\mu^2} \ln \left( \frac{(1 + \frac{g}{\lambda})^2 h^2}{\mu^2} \right) - 1 - 2\gamma \right] + O(\lambda) \right\}, \tag{3.27}
\]

where \(h = h_L + h_R\) and the constant \(\gamma\) is defined as

\[
\gamma = \frac{1}{2} \Gamma'(1) + \ln \sqrt{4\pi}. \tag{3.28}
\]

As we need the expansion of \(\mathcal{F}(h)\) up to \(O(\lambda_{\text{eff}}^2)\) in the effective coupling in this case, this would necessitate a three-loop computation at first sight. In fact we obtain all the \(O(\lambda_{\text{eff}}^2)\) terms from the one loop result (3.27) alone! This ‘mini miracle’ is due to the \((1 + \bar{g})^2\) factor in front of the one-loop term together with the fact that \((1 + \bar{g})^2 = O(\lambda_{\text{eff}}^2)\), implying that the one-loop term is already \(O(\lambda_{\text{eff}}^2)\). Moreover as it follows from Eq. (3.25) the contribution of the higher order terms will be at least \(O(\lambda_{\text{eff}}^3)\) hence they can be safely ignored. Putting then everything together we find

\[
\mathcal{F}(h) = -\frac{ph^2}{\pi} \left\{ 1 - \frac{p}{4s} \lambda_{\text{eff}} + \frac{p^2}{32\pi^2} \lambda_{\text{eff}}^2 + \frac{p\lambda_{\text{eff}}^2}{8\pi^2} \left[ \ln p + \ln \left( \frac{\lambda_{\text{eff}}^2}{\pi^2} \right) \right] + O(\lambda_{\text{eff}}^3) \right\}. \tag{3.29}
\]

Note the non-analytic contribution, \(\ln \lambda_{\text{eff}}\), in the last term of (3.29). This comes ultimately from the \(g_0\)-dependence in (3.24) of \(M\), which plays the role of the infrared cutoff in our calculations. The presence of this non-analytic term in \(\lambda_{\text{eff}}\) explains why the coefficient of the \(\ln s\) term differs for the BOS and the FER cases. Indeed using Eq. (D.32) we find that the large \(s\) expansion in the FER case is finally given as

\[
\mathcal{F}(h) = -\frac{ph^2}{\pi} \left\{ 1 - \frac{p}{2s} - \frac{p^2}{4s^2} \ln s + \frac{p}{2s^2} \left[ \ln p + \frac{p}{4} \right] + \cdots \right\}. \tag{3.30}
\]

Having calculated the asymptotic form of the free energy both in perturbation theory and with the TBA method, by comparing them (with \(t = s - \ln(m/\Lambda_{\text{MS}})\)) we obtain the relation between the mass of the doublet particles, \(m\), and \(\Lambda_{\text{MS}}\). In both the BOS resp. FER case the comparison of Eqs. (2.13) and (3.19) resp. (2.8) and (3.30) leads to the result (1.7) as already announced in the Introduction.

We emphasize that it is already a very nontrivial check on the overall consistency of our assumptions that all the expansion coefficients, for both the BOS and the FER cases, agree. A further, very stringent consistency check is that the \(m/\Lambda\) ratio obtained in the BOS case is exactly the same as the one obtained in the FER case. Finally by comparing the free energy of the breather condensate, computed from the TBA method, Eq. (2.22), with the perturbative one, Eq. (3.19), one finds

\[
\frac{m_1}{\Lambda_{\text{MS}}} = 2^{3 - \frac{d}{2}} e^{\frac{d-1}{2}} \Gamma(1 + \frac{p}{2}) \frac{2\sin \frac{\pi p}{2}}{\pi^2}. \tag{3.31}
\]
By recalling that the mass of the lowest lying \(r = 1\) breather is \(m_1 = 2m \sin \frac{\pi p}{2}\) one immediately obtains from Eq. (3.31) the \(m/\Lambda\) ratio in Eq. (1.7). This is clearly a further nontrivial check on our result and on the mutual consistency of the hypotheses made in the course of the calculation.

The ratio (1.7) diverges in the \(p \to \infty\) limit. This illustrates the fact, discussed in Subsection 2.3, that it is not possible to recover the asymptotic expansion of the ground state energy of the \(O(4)\) model from the generic case in this limit. Furthermore \(m/\Lambda_{\text{MS}}\) also diverges as \(p \to 0\) consistently with the Wiegmann scenario as discussed in Subsection 2.5, while

\[
\lim_{p \to 0} \frac{m_1}{\Lambda_{\text{MS}}} = \frac{8}{\epsilon},
\]

which exactly reproduces the \(m/\Lambda\) ratio of the \(O(3)\) model first computed in Ref. [9]. In our view this beautiful agreement certainly yields some more quantitative support as to the correctness of the ‘conventional wisdom’.

4. Lattice regularization

In this section we propose a lattice action for the deformed principal model and calculate the free energy in this lattice version of the model. This is then used to calculate the ratio of the \(\Lambda\)-parameters of the dimensionally regulated theory and the lattice version.

First we rewrite the Lagrangian (1.1) in the form:

\[
\mathcal{L}_0 = (1 + g)\mathcal{L}^{O(4)} - g\mathcal{L}^{O(3)},
\]

exhibiting that it interpolates between the \(O(3)\) and \(O(4)\) nonlinear \(\sigma\)-models. In fact

\[
\mathcal{L}^{O(4)} = \frac{1}{\lambda} \text{tr}\{\partial_\mu G^{-1}\partial_\mu G\},
\]

is the familiar action of the SU(2) principal chiral (or O(4)) nonlinear \(\sigma\)-model and

\[
\mathcal{L}^{O(3)} = \frac{1}{\lambda} \partial_\mu \Sigma^a \partial_\mu \Sigma^a,
\]

is the Lagrangian of an \(O(3)\) symmetric nonlinear \(\sigma\)-model with a composite \(O(3)\) field

\[
\Sigma^a = \frac{1}{2} \text{tr}\{G^{-1} \sigma^a G \sigma^3\}.
\]

Motivated by this form of the Lagrangian we take as our lattice action the following linear combination.

\[
S_{\text{Lattice}} = (1 + \tilde{g}_0) S^{O(4)} - \tilde{g}_0 S^{O(3)}.
\]
Here
\[ S^{O(4)} = \frac{1}{\tilde{\lambda}_0} \sum_{x,\mu} \text{tr} \left\{ 2 - G^{-1}(x)G(x + \hat{\mu}) - G^{-1}(x + \hat{\mu})G(x) \right\}, \tag{4.6} \]
is the standard O(4) lattice action in terms of the SU(2) principal model variable \( G(x) \) associated to the lattice site \( x \). (As usual, the lattice spacing \( a \) is taken to be unity and \( \hat{\mu} \) denotes the unit vector in the \( \mu \) direction. Moreover, we have denoted the bare lattice couplings by \( \tilde{\lambda}_0 \) and \( \tilde{g}_0 \) to distinguish them from their counterparts in the dimensional scheme.) Similarly
\[ S^{O(3)} = \frac{1}{\tilde{\lambda}_0} \sum_{x,\mu} \left\{ 1 - \Sigma^a(x)\Sigma^a(x + \hat{\mu}) \right\} \tag{4.7} \]
is the standard O(3) lattice action in terms of the composite field (4.4).

For simplicity we shall consider here the free energy associated to the \( U(1)_L \) charge only. This corresponds to the BOS case of Section 3 (i.e. \( h_L = h \), \( h_R = 0 \)). An appropriate gauging of the Lagrangian (4.5) and taking imaginary gauge fields leads to:
\[ S_{\text{gauged}} = \frac{1 + \tilde{g}_0}{\tilde{\lambda}_0} \sum_{x,\mu} \text{tr} \left\{ 2 - G^{-1}(x)e^{-h_\mu \sigma^3}G(x + \hat{\mu}) - G^{-1}(x + \hat{\mu})e^{h_\mu \sigma^3}G(x) \right\} \]
\[ + \frac{\tilde{g}_0}{2\tilde{\lambda}_0} \sum_{x,\mu} \text{tr} \left\{ G(x)\sigma^3G^{-1}(x)e^{-h_\mu \sigma^3}G(x + \hat{\mu})e^{h_\mu \sigma^3} - 1 \right\}, \tag{4.8} \]
with \( h_\mu = h_\delta \mu^2 \).

The perturbative calculation of the free energy on the lattice is now completely analogous to the one in the dimensional scheme. The leading contribution is given again by the minimum of the gauged action for constant fields. (4.8) also takes its minimum at the group element (3.8) and is given by
\[ F^{(-1)}_L = -4(1 + \tilde{g}_0) \frac{1}{\tilde{\lambda}_0a^2} \left( \cosh ha - 1 \right) + \frac{\tilde{g}_0}{\tilde{\lambda}_0a^2} \left( \cosh 2ha - 1 \right), \tag{4.9} \]
where the \( a \)-dependence has been reconstructed. In the continuum limit, \( a \to 0 \), (4.9) reproduces (3.11).

The 1-loop contribution is half the logarithmic determinant of the operator corresponding to the quadratic term in the expansion of the lattice action around the group element (3.8). We have not tried to calculate the 1-loop determinant for generic \( \tilde{g}_0 \), because for our purposes it is sufficient to show that it must be of the form
\[ F^{(0)}_L = -h^2 \left\{ \frac{1 - \tilde{g}_0}{2\pi} \ln ha + W_L(\tilde{g}_0) + O(a^2) \right\}. \tag{4.10} \]
\( W_L(\tilde{g}_0) \) similarly to its dimensional counterpart, is a complicated function of its argument but all we are going to need is its value at the \( O(3) \) point \( \tilde{g}_0 = -1 \), where it is relatively easy to evaluate. We find

\[
W_L(-1) = -\frac{1}{2\pi} - \frac{3}{2}\pi \ln 2 - \frac{5}{8},
\]

(4.11)

We now proceed, as before, from the bare results (4.9) and (4.10) via the renormalized ones to the RG improved 1-loop formula

\[
\mathcal{F}_L(h) = -\frac{h^2}{\pi} \left\{ \ln \frac{h}{\Lambda_L} + \left( 1 - \frac{p}{2} \right) \ln \ln \frac{h}{\Lambda_L} - \frac{1}{2} - \frac{3}{2}\pi \ln 2 - \frac{5\pi}{8} \right\} + \ldots ,
\]

(4.12)

where we have used the results of Appendix D. Comparing (3.19) and (4.12) gives the ratio of the \( \Lambda \)-parameters of the two regularization schemes:

\[
\frac{\Lambda_{\text{MS}}}{\Lambda_L} = \sqrt{32 e^{\frac{5\pi}{8}}}. \]

(4.13)

Note that the ratio (4.13) is independent of the RG-invariant parameter \( p \), but is nevertheless different from the corresponding ratio of the standard \( O(3) \) model, which is given by a formula similar to (4.13), but with an exponent that is \( \pi/2 \) instead of \( 5\pi/8 \). This means that our lattice regularization is different from the standard one in the \( O(3) \) limit. A possible explanation for this might have been that the difference is due to the compositeness of our \( O(3) \) field (4.4). But by a simple computation we find the standard \( \Lambda_L \) for \( \tilde{g}_0 = -1 \) if the first term in (4.5) is simply omitted. To clarify this apparent paradox we note that the difference is due to the following term in the quadratic piece of the gauged lattice action.

\[
S_\Phi = \frac{4(1 + \tilde{g}_0)}{\lambda_0} \sum_{x,\mu} \cosh h_\mu \Phi(x) \left[ \Phi(x) - \Phi(x + \hat{\mu}) \right].
\]

(4.14)

(4.14) is, of course, absent if \( \tilde{g}_0 = -1 \). This is the case in the standard lattice version of the \( O(3) \) model. If, however, \( \tilde{g}_0 \neq -1 \), then the coupling dependence can be scaled out of (4.14). Close to the continuum limit, after rescaling, (4.14) corresponds to the continuum Lagrangian

\[
\mathcal{L}_\phi = \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{1}{4} h^2 a^2 (\partial_2 \phi)^2 + \mathcal{O}(h^4 a^4),
\]

(4.15)

from which one would naively conclude that this field is decoupled from the external field \( h \) in the continuum limit. This is, however, not correct since the second term in (4.15) does in fact contribute to the continuum free energy, because of quadratic divergences in lattice perturbation theory. This contribution is \( h^2/8 \) leading to the modified exponent in (4.13). We expect to find the same modified \( \Lambda_L/\Lambda_{\text{MS}} \) ratio by comparing the 4-point function computed in the two regularization schemes.
For completeness we also computed the $\Lambda_{\text{MS}}/\Lambda_L$ ratio in the $O(4)$ symmetric case $g = 0$. In this special case the function defined in (3.14) is relatively easy to calculate. In the dimensional regularization scheme

$$W(0) = \frac{1 + 2\gamma}{4\pi},$$

whereas in the analogous calculation for lattice regularization we find

$$W_L(0) = \frac{1}{4\pi} \left[ 1 + \ln 32 + \frac{\pi}{2} \right].$$

Using (4.16) and (4.17) we obtain the $\Lambda_{\text{MS}}/\Lambda_L$ ratio

$$\frac{\Lambda_{\text{MS}}}{\Lambda_L} = \sqrt{32} e^{\frac{\pi}{4}},$$

in agreement with the well-known result. We note that what is new in our calculation is that we considered the external field coupled to $Q_L$ here instead of the diagonal charge considered previously. The $\Lambda_{\text{MS}}/\Lambda_L$ ratio is the same of course, as it should.

We think that it would be extremely interesting to perform Monte Carlo simulations using the lattice action (4.5). This would provide us with an independent way of determining the ratio of the mass of the physical particles and the lattice $\Lambda$ parameter and also with an interesting example of a lattice field theory model with more than one relevant couplings.

5. Summary and Conclusions

In this paper we have computed the ground state energies of the deformed principal model in the presence of external fields for all three possible orientations of the external field. We have found a complete consistency between perturbation theory and the results from the TBA method based on the proposed S-matrix of the model. In all three cases the leading and the first few subleading coefficients of the large field expansion for the ground state energy are identical once we calculate the $m/\Lambda$ ratio using one of the expansions. This beautiful agreement is absolutely nontrivial since it can only be achieved after identifying an unusual non-analytic term in the perturbative expansion for the fermionic case. Such non-analytic terms can only occur in perturbation theory in the presence of several couplings. To our knowledge our example is the first one where the results of the TBA program is compared to the results of asymptotically free perturbation theory with several coupling constants.

Actually the presence of the running deformation parameter, $\hat{g}$, made the computations much easier as compared to a purely fermionic model. In a fermionic model, generically a three-loop calculation is necessary to obtain the $m/\Lambda$ ratio, whereas in the deformed model here, it has been sufficient to calculate only to one-loop order! This nice simplification is due to the fact that all higher terms are proportional to $(\hat{g} + 1)^2$, and since for large fields $\hat{g} \to -1$, they are effectively supressed.
We have also outlined a proof to all orders of perturbation theory that the free energies for the two apparently different fermionic cases (DIAG and RIGHT) actually have the same form as functions of the effective chemical potentials. We have no explanation for this fact but it is yet another very nontrivial consistency check since the TBA integral equations are identical for the two cases.

It is also quite remarkable that the agreement between perturbation theory and the TBA method is valid for all values of the RG invariant parameter, \( p = 2\pi(1 + \bar{g})/\bar{\lambda} \), including the attractive regime \( 0 < p < 1 \), where the bootstrap S-matrix indicates the existence of a complex bound state structure in the spectrum of the model. We have found that the structure of the ground state is changing with a decreasing \( p \) in the sense that it is always the lowest lying breather bound state that condenses to the vacuum. The final formula for the ground state energy is, however, analytic for the whole range of \( 0 < p < \infty \) and agrees with the results of PT. Finally our results are also in perfect agreement with the Wiegmann scenario for the \( p \to 0 \) limit since \( \lim_{p \to 0} m_1/\Lambda = \frac{8}{e} \), \( (m_1 \text{ being the mass of the lowest lying breather}) \), thus we reproduce the \( m/\Lambda \) ratio computed previously in the O(3) model. What we found here lends also additional support to the empirical rule that only the particle with the highest charge/mass ratio condenses in the ground state. This rule was found to be correct in all cases considered so far \cite{7} and proved to be correct also for the deformed model in the attractive regime. In view of the complexity of the spectrum of physical excitations in the attractive regime it would be extremely interesting to find an independent explanation of this phenomenon.

We have also proposed a simple lattice version of the \( SU(2) \times U(1) \) symmetric model and computed the \( \Lambda_{\text{MS}}/\Lambda_L \) ratio where \( \Lambda_L \) denotes the lattice lambda parameter.

An interesting question that requires further study is the application of the TBA program to calculate thermodynamical quantities for finite temperature and comparing them to the results of perturbative calculations.

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Appendix A. Solution of the TBA integral equation

In the Appendix of Ref. \cite{10} the solution of the pertinent integral equation determining the free energy has been obtained for a generic kernel by applying the generalized Wiener–Hopf technique. The integral equation is given as follows:

\[
\epsilon(\theta) - \int_{-B}^{B} d\theta' K(\theta - \theta')\epsilon(\theta') = h - m \cosh \theta , \tag{A.1}
\]

together with the boundary condition \( \epsilon(\pm B) = 0 \). In terms of the solution of
Eq. (A.1) the free energy is
\[ \delta f(h) = -\frac{m}{2\pi} \int_{-B}^{B} d\theta \cosh(\theta) \epsilon(\theta). \]  

(A.2)

We do not go into the details of the solution technique of Eq. (A.1) as they can be found in the literature (e.g. Ref. [10]). Here we just present the results in a somewhat simplified form which could prove convenient for future applications. The crucial step in solving Eq. (A.1) by the generalized Wiener-Hopf method is to split the Fourier transform of the kernel in Eq. (A.1), \( \hat{K}(\omega) \), as
\[ 1 - \hat{K}(\omega) = \left[ G_+(\omega)G_-(\omega) \right]^{-1}, \]  

(A.3)

where \( G_+(\omega) \) resp. \( G_-(\omega) \) is analytic in the upper resp. lower half plane. When the kernel is of the ‘fermionic’ type, i.e. \( \hat{K}(\omega) \neq 1 \), one can assume the following expansion of \( G_+ (i\xi) \) around \( \xi = 0 \) for \( \xi > 0 \):
\[ G_+ (i\xi) = \tilde{k} \exp \{-a_0 \xi \ln \xi \} \left( 1 + \tilde{a} \xi + O(\xi^2) \right). \]  

(A.4)

Then in the limit of high density \( h/m \gg 1 \), the asymptotic series for \( \delta f(h) \) can be obtained by a simple iterative technique. The final result, Eq. (A.34) of Ref. [10], can be conveniently recast in the following form:
\[ \delta f(h) = -\frac{h^2}{4} \frac{k^2}{2\pi} \left\{ 1 + \tilde{a} \ln \left( \frac{2\tilde{k}}{G_+(i\xi)} \right) + \frac{\tilde{A}}{t^2} + \ldots \right\}, \]  

(A.5)

where \( t = \ln(h/m) \) and the constant \( \tilde{A} \) is given as
\[ \tilde{A} = \tilde{a} \left( 1 - \tilde{b} - \ln \frac{2\tilde{k}}{G_+(i\xi)} \right) + \tilde{a}^2 \left( \frac{3}{2} + \Gamma'(1) - \ln 2 \right). \]  

(A.6)

For ‘bosonic’ type kernels, \( \hat{K}(0) = 1 \) so the splitting functions \( G_+(\omega), G_-(\omega) \) are singular at \( \omega = 0 \). The solution of the integral equation (A.1) is then more complicated than in the previous (FER-type) case as the corresponding Neumann series is not uniformly convergent and the solution cannot be obtained by simple iteration. The solution of Eq. (A.1) for a generic BOS-type kernel in the limit \( t = \ln(h/m) \to \infty \) has been given in Ref. [12] up to order of \( \ln t/t \). According to the results of Ref. [12] for \( G_+(i\xi) \) admitting the following expansion around \( \xi = 0 \) (\( \xi > 0 \))
\[ G_+ (i\xi) = \frac{k}{\sqrt{\xi}} \exp \{-a_0 \xi \ln \xi \} \left( 1 - b_0 \xi + O(\xi^2) \right), \]  

(A.7)

the asymptotic series for \( t \gg 1 \) of the Legendre transform of ground state energy density can be expressed in the form
\[ \delta f(h) = -\frac{h^2}{4} k^2 \left( t + (a + \frac{1}{2}) \ln t + A + \ldots \right), \]  

(A.8)
where the constant $A$ is given by

$$A = \ln \left( \frac{\sqrt{2\pi} ke^{-b}}{G_+(i)} \right) - 1 + a(\ln 8 - 1 - \Gamma'(1)). \quad (A.9)$$

**Appendix B. Bootstrap fusion**

In this appendix we calculate the phase shifts corresponding to the scattering of two bound state particles of charge $(2,0)$ and of lowest mass $m_1 = 2m \sin \frac{\pi}{2}$ in terms of the scattering phases of the “elementary” particles of charge $(1,1)$ and $(1,-1)$. The procedure is called bootstrap fusion and being an important part of the whole bootstrap program is discussed at length in the literature [13, 18, 20]. Therefore we just apply the general framework to the special case of our interest, but even this could prove to be useful for other models.

We shall assume that the S-matrix describing the scattering of the “elementary” particles of mass $m$ can be decomposed as

$$S_{\beta'\alpha'}(\theta) = \sum_e \varphi_e^{\beta'\alpha'} S_e(\theta) \varphi_e^{\alpha\beta}, \quad (B.1)$$

where the orthogonal projectors satisfy

$$\sum_e \varphi_e^{\alpha\beta} \varphi_e^{\alpha'\beta'} = \delta_{\alpha\alpha'} \delta_{\beta\beta'}, \quad \sum_{\alpha,\beta} \varphi_{\alpha\beta} \varphi_{\alpha'\beta'} = \delta_{e' e}^{\alpha' \alpha} \delta_{e' e}^{\beta' \beta}. \quad (B.2)$$

The scattering phases may have poles in the physical sheet $0 \leq \text{Im} \theta \leq \pi$ of the form

$$S_e(\theta) \approx \frac{i R_e}{\theta - i u_b}. \quad (B.3)$$

If the pole in (B.3) is associated with a bound state of mass

$$m_b = 2m \cos \frac{u_b}{2}, \quad (B.4)$$

then one can use the bootstrap fusion procedure to calculate the S-matrix of the bound states from the S-matrix of their constituents. We assume further that the bound state, $b$, occurs only in one of the channels (which we shall label $e = b$) and that the scattering of $b$ is diagonal with all other (elementary or composite) particles from the spectrum of the model. Then the scattering of $b$ and $X$ (which can be one of the constituents or some composite particle) is given by

$$S_{bX}(\theta) = \sum_{\alpha,\beta, X, \tilde{\alpha}, \tilde{\beta}} \varphi_b^{\alpha \beta} S_{\alpha X}(\theta + \frac{i u_b}{2}) S_{\tilde{\alpha} \tilde{\beta} X}(\theta - \frac{i u_b}{2}) \varphi_{b \tilde{\alpha} \tilde{\beta}}. \quad (B.5)$$

\[2\] Bound states are not the only possibility to explain the presence of these, or higher order, poles. See [18, 20].
In the case of the Sine-Gordon model the particles are indexed by their U(1) charges (which can be 1 or -1) and the four scattering channels are labeled \((0, 0), (\bar{0}, \bar{0}), (\pm, \pm)\).

The projectors are

\[
\begin{align*}
\varphi_{\alpha\beta}^{0\beta} &= \varphi_{\alpha\beta}^{0} = \delta_1^\alpha \delta_1^\beta, \\
\varphi_{\alpha\beta}^{\bar{0}\beta} &= \varphi_{\alpha\beta}^{\bar{0}} = \delta_{-1}^\alpha \delta_{-1}^\beta, \\
\varphi_{\alpha\beta}^{\alpha\beta} &= \varphi_{\alpha\beta}^{\pm} = \frac{1}{\sqrt{2}} (\delta_1^\alpha \delta_{-1}^\beta \pm \delta_{-1}^\alpha \delta_1^\beta).
\end{align*}
\]  

The soliton-soliton and antisoliton-antisoliton scattering phases are

\[
S_0^{(p)}(\theta) = S_{\bar{0}}^{(p)}(\theta) = -e^{i\delta_p(\theta)}, \quad (B.7)
\]

where

\[
\delta_p(\theta) = \int_{-\infty}^{\infty} d\omega \frac{\sin(\omega \theta)}{\omega} K_p(\omega), \quad \hat{K}_p(\omega) = \frac{\sinh \left( \frac{\pi \omega (p-1)}{2} \right)}{2 \cosh \left( \frac{\pi \omega}{2} \right) \sinh \left( \frac{\pi \omega p}{2} \right)}. \quad (B.8)
\]

The other two eigenvalues are

\[
S_+^{(p)}(\theta) = -\frac{\sinh \left( \frac{\theta + i\pi}{2p} \right)}{\sinh \left( \frac{\theta - i\pi}{2p} \right)} S_0^{(p)}(\theta), \quad (B.9)
\]

and

\[
S_-^{(p)}(\theta) = -\frac{\cosh \left( \frac{\theta + i\pi}{2p} \right)}{\cosh \left( \frac{\theta - i\pi}{2p} \right)} S_0^{(p)}(\theta). \quad (B.10)
\]

The lowest breather has mass \(m_b = 2m \sin \frac{\pi}{2p} m_1\) corresponding to \(u_b = \pi(1-p) = u_1\) and it is present in the channel \(e = -\) only.

Now we return to the deformed principal model. The S-matrix of the model, Eq. (1.4), is a tensor product of two copies of the SG S-matrix discussed above with \(p = \infty\) in the first factor. Using the \((Q_L, Q_R)\) labeling of particles, (1.4) actually means

\[
S_{(\beta_{L}^\prime, \beta_{R}^\prime)}^{(\alpha_{L}^\prime, \alpha_{R}^\prime)}(\theta) = S_{_{\alpha_{L}^\prime, \alpha_{R}^\prime}}^{(\infty)}(\theta) S_{_{\beta_{L}^\prime, \beta_{R}^\prime}}^{(p)}(\theta). \quad (B.11)
\]

Using a tensor product labeling \(E = (e_L; e_R)\) also for the scattering channels, the projectors can also be written in the product form

\[
\Phi_{\alpha_{L}^\prime, \alpha_{R}^\prime}(\theta) = \varphi_{\alpha_{L}^\prime, \alpha_{R}^\prime}^{e_L} \varphi_{\alpha_{L}^\prime, \alpha_{R}^\prime}^{e_R}, \quad (B.12)
\]

and similarly for \(\Phi_{E}\). Finally

\[
S_{(e_{L}, e_{R})}(\theta) = S_{e_{L}}^{(\infty)}(\theta) S_{e_{R}}^{(p)}(\theta). \quad (B.13)
\]

We are interested in the bound state \(B = (0; -)\), which is composed of the elementary particles \((1,1)\) and \((1,-1)\) and has lowest mass \(m_1\) in this charge \((2,0)\).
Applying the bootstrap fusion formula (B.5) to calculate the scattering of B and one of its constituents \((1, \gamma_R)\) gives

\[
S_{B(1, \gamma_R)}(\theta) = e^{i\delta(\theta + i\frac{\pi}{2})} e^{i\delta(\theta - i\frac{\pi}{2})} S_1^{(p)}(\theta),
\]

where

\[
S_1^{(p)}(\theta) = \frac{\sinh \theta + i \sin \frac{\pi(1+p)}{2}}{\sinh \theta - i \sin \frac{\pi(1+p)}{2}},
\]

is the analogous soliton-breather scattering in the SG model. We can interpret (B.14) as the product of this SG scattering coming from the second factor in (1.4) multiplied by the diagonal soliton-soliton scattering coming from the first one.

Having obtained the bound state-particle scattering, we now apply (B.5) once more to find the \(B - B\) scattering phase, \(\delta(2, 0)(\theta)\). Using (B.14) in (B.5) we find

\[
S_{BB}(\theta) = e^{i\delta(2, 0)(\theta)} = e^{2i\delta(\theta)} e^{i\delta(\theta + iu_1)} e^{i\delta(\theta - iu_1)} S_{11}^{(p)}(\theta),
\]

where

\[
S_{11}^{(p)}(\theta) = \frac{\sinh \theta + i \sin p\pi}{\sinh \theta - i \sin p\pi}, \quad u_1 = \pi(1 - p) = \pi |x|,
\]

is the breather-breather scattering phase in the SG model. The interpretation of (B.16) is completely analogous to that of (B.14). Finally for later convenience we also give here the Fourier representation of \(S_{11}^{(p)}(\theta)\):

\[
S_{11}^{(p)}(\theta) = -\exp \left\{ -i \int_{-\infty}^{\infty} d\omega \frac{\sin \omega \theta \cosh \left(\frac{1}{2} - p\right)\pi \omega}{\omega \cosh \frac{1}{2} \pi \omega} \right\}.
\]

**Appendix C. Perturbative calculations**

We chose the following parametrization for the group valued field \(G\):

\[
G = G_0 \frac{1}{\sqrt{1 + |\Psi|^2}} \begin{pmatrix} 1 & -\Psi^* \\ \Psi & 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}.
\]

Here \(\Psi\) is a complex scalar field, \(\Phi\) is a real scalar field and \(G_0 \in SU(2)\) is an appropriate constant group element given by (3.8) or (3.9).

Before we can expand the Lagrangian in powers of the coupling constant \(\lambda_0\) we have to rescale our fields as follows.

\[
\Psi = \sqrt{\frac{\lambda_0}{2}} \psi, \quad \Phi = \frac{1}{2} \sqrt{\frac{\lambda_0}{1 + g_0}} \phi.
\]

In terms of the rescaled fields we have

\[
\mathcal{L}_0 = \frac{\partial_{\mu} \psi \partial^\mu \psi^*}{(1 + \frac{\lambda_0}{2} |\psi|^2)^2} + \frac{1}{2} \partial_{\mu} \phi \partial^\mu \phi - \frac{i}{2} \sqrt{\lambda_0(1 + g_0)} \partial_{\mu} \phi A_\mu - \frac{(1 + g_0)\lambda_0}{8} A_\mu A_\mu,
\]

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where
\[ A_\mu = \frac{\psi \partial_\mu \psi^* - \psi^* \partial_\mu \psi}{1 + \frac{\lambda_0}{2} |\psi|^2}. \] (C.4)

Due to the fact that the two cases BOS and FER correspond to expansions around different points, \( L_1 \) and \( L_2 \) are different in terms of the rescaled fields. For the BOS case we find
\[
L_1 = i \sqrt{2(1 + g_0) h_L} \frac{\psi + \psi^*}{1 + \frac{\lambda_0}{2} |\psi|^2} \partial_2 \phi + h_L \sqrt{\frac{\lambda_0}{2} \frac{|\psi|^2}{1 + \frac{\lambda_0}{2} |\psi|^2}} (\partial_2 \psi - \partial_2 \psi^*) + g_0 h_L \frac{\psi + \psi^*}{1 + \frac{\lambda_0}{2} |\psi|^2} A_2, \] (C.5)

and
\[
L_2 = -\frac{2 h^2_L}{\lambda_0} - g_0 h^2_L \frac{(\psi + \psi^*)^2}{(1 + \frac{\lambda_0}{2} |\psi|^2)^2}. \] (C.6)

whereas in the FER case
\[
L_1 = 2 i h_L \sqrt{\lambda_0(1 + g_0)} \partial_2 \phi \frac{|\psi|^2}{1 + \frac{\lambda_0}{2} |\psi|^2} + \left[ h_L (1-g_0) - h_R (1+g_0) + \frac{g_0 \lambda_0 h_L |\psi|^2}{1 + \frac{\lambda_0}{2} |\psi|^2} \right] A_2, \] (C.7)

and
\[
L_2 = -\frac{2 (h_L + h_R)^2 (1 + g_0)}{\lambda_0} + 4 h^2_L g_0 \frac{|\psi|^2}{(1 + \frac{\lambda_0}{2} |\psi|^2)^2} + 4 h_L h_R (1 + g_0) \frac{|\psi|^2}{1 + \frac{\lambda_0}{2} |\psi|^2}. \] (C.8)

The leading, \( \mathcal{O}(\lambda_0^{-1}) \), terms of the free energy are (3.11) and (3.20). The next terms are of the order \( \mathcal{O}(1) \) and are given by half the logarithmic determinant of the operator corresponding to the ‘free’ Lagrangian, that is the piece quadratic in the fields \( \psi \) and \( \phi \). This ‘free’ Lagrangian can be written in the BOS case as
\[
\mathcal{L}^{(0)} = \partial_\mu \psi^* \partial_\mu \psi + \frac{1}{2} \partial_\mu \phi \partial_\mu \phi - g_0 h^2_L (\psi + \psi^*)^2 + i \sqrt{2(1 + g_0)} h_L (\psi + \psi^*) \partial_2 \phi, \] (C.9)

and for the FER case as:
\[
\mathcal{L}^{(0)} = \partial_\mu \psi^* \partial_\mu \psi + \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \Delta |\psi|^2 + \omega (\psi \partial_2 \psi^* - \psi^* \partial_2 \psi). \] (C.10)

In both cases, only two fields are really coupled to the external fields. These are the real part of \( \psi \) and \( \phi \) in the BOS case and the real and imaginary parts of \( \psi \) in the FER case. This allows us to write \( \mathcal{L}^{(0)} \) as an effective Lagrangian of two bosons \( \phi_1, \phi_2 \) in both cases:
\[
\mathcal{L}^{(0)}_{\text{eff}} = \frac{1}{2} \left[ \partial_\mu \phi_1 \partial_\mu \phi_1 + \partial_\mu \phi_2 \partial_\mu \phi_2 \right] + \frac{m_1^2}{2} \phi_1^2 + \frac{m_2^2}{2} \phi_2^2 + i \nu \left( \phi_2 \partial_2 \phi_1 - \phi_1 \partial_2 \phi_2 \right). \] (C.11)
where
\[ m_1^2 = 0, \quad m_2^2 = -4g_0h_L^2, \quad \nu = \sqrt{1 + g_0 h_L}, \]  
\tag{C.12}
and
\[ m_1^2 = m_2^2 = \Delta, \quad \nu = \omega, \]  
\tag{C.13}
for the BOS and FER cases, respectively. The one-loop contribution is half the logarithmic determinant of the linear operator corresponding to \( L_{\text{eff}}^{(0)} \):
\[ f^{(0)} = \frac{1}{2} \int \frac{d^n p}{(2\pi)^n} \ln \left[ (p^2 + m_1^2) (p^2 + m_2^2) + 4\nu^2 p_2^2 \right]. \]  
\tag{C.14}
Taking the derivative of (C.14) with respect to the external fields and then integrating it again from zero, combined with dimensional analysis in \( n \) dimensions finally gives
\[ F^{(0)} = \delta f^{(0)} = \frac{1}{n} \int \frac{d^n p}{(2\pi)^n} \frac{(m_1^2 + m_2^2) p^2 + 2m_1^2 m_2^2 + 4\nu^2 p_2^2}{(p^2 + m_1^2) (p^2 + m_2^2) + 4\nu^2 p_2^2}. \]  
\tag{C.15}

We have seen in Section 3 that the 1-loop result in the FER case, (3.23), is a function of the sum \( h_L + h_R \) only, similarly to the classical term (3.20). The free energy calculated by the TBA method also has this property. It is a very non-trivial fact that this behaviour can be reproduced by the perturbative calculation. In fact, we will now show that in the diagonal case the free energy depends only on the sum \( h_L + h_R \) to all orders of perturbation theory.

To show this, we introduce the shifted derivatives
\[ \tilde{\partial}_2 \psi = \partial_2 \psi + \omega \psi, \quad \text{and} \quad \tilde{\partial}_2 \psi^* = \partial_2 \psi^* - \omega \psi^*. \]  
\tag{C.16}
Using this notation, the gauged Lagrangian can be compactly written as
\[ \mathcal{L} = \tilde{\mathcal{L}}_0 - \frac{2(h_L + h_R)^2(1 + g_0)}{\lambda_0} + iM \sqrt{\lambda_0 (1 + g_0)} \frac{|\psi|^2}{1 + \frac{\lambda_0}{2} |\psi|^2} \frac{\partial_2 \phi}{\partial \phi} \]  
\[ + \frac{\lambda_0 g_0 M}{2} \frac{|\psi|^2 (\psi \tilde{\partial}_2 \psi^* - \psi^* \tilde{\partial}_2 \psi)}{\left( 1 + \frac{\lambda_0}{2} |\psi|^2 \right)^2} + \frac{M^2 |\psi|^2}{\left( 1 + \frac{\lambda_0}{2} |\psi|^2 \right)^2} \left\{ 1 + \frac{\lambda_0 (1 - g_0) |\psi|^2}{2} \right\}, \]  
\tag{C.17}
where \( \tilde{\mathcal{L}}_0 \) stands for the original Lagrangian, (C.3), with all \( \partial_2 \psi \) and \( \partial_2 \psi^* \) substituted by the shifted derivatives (C.16). The advantage of using the form of the Lagrangian, (C.17), is that the explicit dependence on the external fields \( h_L, h_R \) occurs only through the combination \( M = (1 + g_0)(h_L + h_R) \). In addition to this, there is still an implicit dependence on the external fields through \( \omega \) hidden in the shifted derivatives (C.16). This disappears, however, since the Lagrangian (C.17) is in fact equivalent to another one, denoted by \( \mathcal{L}_M \), differing from \( \mathcal{L} \) only by replacing all shifted derivatives, \( \tilde{\partial}_2 \), by ordinary ones.
The fact that $\mathcal{L}$ and $\mathcal{L}_M$ are completely equivalent can be best illustrated in the case of the $\mathcal{O}(1)$ contribution to the free energy. The relevant piece of the Lagrangian is (C.10), which can also be written as

$$\mathcal{L}^{(0)} = \tilde{\partial}_\mu \psi^* \tilde{\partial}_\mu \psi + \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + M^2 |\psi|^2. \quad (C.18)$$

The substitution $\tilde{\partial}_2 \to \partial_2$ corresponds to the shift $p_2 \to p_2 + i\omega$ in momentum space. More precisely, when integrating over $p_2$ in Eq. (3.21) the above shift corresponds to deforming the integration contour parallel to the real axis. This deformation is allowed since the singularities of the integrand at

$$p_2 = i\omega \pm i\sqrt{p_1^2 + M^2}, \quad (C.19)$$

lie outside the region between the real axis and the shifted contour (guaranteed by the inequality (3.10)). After shifting the integration variable Eq. (3.21) takes the simple form

$$\mathcal{F}^{(0)} = \frac{2}{n} \int \frac{d^n p}{(2\pi)^n} \frac{M^2}{p^2 + M^2}, \quad (C.20)$$

which is easily evaluated and yields (3.23). Now Eq. (C.20) can be derived from the simplified Lagrangian

$$\mathcal{L}_M^{(0)} = \partial_\mu \psi^* \partial_\mu \psi + \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + M^2 |\psi|^2, \quad (C.21)$$

exhibiting the one-loop equivalence between $\mathcal{L}$ and $\mathcal{L}_M$.

It is easy to generalize this argument to show that there is a one-to-one correspondence between the Feynman diagrams generated by $\mathcal{L}$ and $\mathcal{L}_M$ and that the perturbative results are identical for these two models, diagram by diagram, to all orders of perturbation theory. This equivalence follows from the fact that the shift $p_2 \to p_2 + i\omega$ can be performed simultaneously in all momentum variables corresponding to the lines of a multi-loop Feynman diagram. In this way we have shown that all dependence on the external fields comes from the $M$-dependence of $\mathcal{L}_M$. This means, in particular, that the free energy in the diagonal case takes the form (3.24, 3.25) and it is a function of the sum $h = h_L + h_R$ only, which is exactly the same as the free energy for the $U(1)_R$ case (as a function of $h_R$).

**Appendix D. Renormalization Group Equations**

Due to its SU(2)$_L \times U(1)_R$ symmetry, the model (1.1) is renormalizable in perturbation theory. (Of course, in Eq. (1.1) actually the bare coupling $\lambda_0$ and the bare $g$-parameter $g_0$ occur.) Using dimensional regularization, the relation between the bare parameters, $(\lambda_0, g_0)$, and the renormalized ones, $(\lambda, g)$, is written as:

$$\lambda_0 = \mu^\epsilon \lambda Z_\lambda, \quad \text{and} \quad 1 + g_0 = (1 + g) Z_g. \quad (D.1)$$
In Eq. (D.1) the dimensionful parameter, $\mu$, is introduced as usual, to carry the mass dimension of $\lambda_0$ (which is dimensionless in 2 dimensions). The renormalization constants contain only pole terms in the variable $\epsilon = 2 - n$:

$$Z_{\lambda} = 1 + \sum_{r=1}^{\infty} \frac{y_r(\lambda, g)}{\epsilon^r}, \quad \text{(D.2)}$$

$$Z_g = 1 + \sum_{r=1}^{\infty} \frac{w_r(\lambda, g)}{\epsilon^r}. \quad \text{(D.3)}$$

The residues of the poles, $y_r$ and $w_r$, can be calculated in perturbation theory:

$$y_r(\lambda, g), w_r(\lambda, g) = \mathcal{O}(\lambda^r). \quad \text{(D.4)}$$

It is clear from (D.1) that if $g = -1$, then also $g_0 = -1$. This is a consequence of the fact that at this value of the deformation parameter the model (1.1) reduces to the $O(3)$ nonlinear $\sigma$-model (plus a free boson), which is obvious from the parametrization used in Section 4. An other special value of the deformation coupling is $g = 0$ implying $g_0 = 0$ and $w_r(\lambda, 0) = 0$ in Eq. (D.3). In this case the deformation actually vanishes and we have the SU(2)$_L \times$SU(2)$_R$ (or $O(4)$) symmetric principal $\sigma$-model. In a sense the deformed model (1.1) interpolates between the $O(3)$ and $O(4)$ nonlinear $\sigma$-models.

Physical quantities depend on the renormalized couplings $\lambda$, $g$ and the dimensionful parameter $\mu$ in such a way that the action of the renormalization group (RG) operator

$$\mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta_{\lambda}(\lambda, g) \frac{\partial}{\partial \lambda} + \beta_g(\lambda, g) \frac{\partial}{\partial g}, \quad \text{(D.5)}$$

vanishes on them. Here the two $\beta$-functions are obtained from the residues of the first order poles in (D.2) and (D.3) as

$$\beta_{\lambda} = \lambda^2 \frac{\partial y_1}{\partial \lambda}, \quad \text{and} \quad \beta_g = (1 + g)\lambda \frac{\partial w_1}{\partial \lambda}. \quad \text{(D.6)}$$

For later use we note that in the FER case the free energy, Eq. (3.25), can be written as

$$\mathcal{F}(h) = -2h^2 \left\{ \frac{1 + g_0}{\lambda_0} - \frac{(1 + g_0)^2}{2\lambda_0} S(g_0, \lambda_0[h(1 + g_0)]^{-\epsilon}) \right\}, \quad \text{(D.7)}$$

where $S$ stands for the infinite sum in (3.25). From (D.7) we can conclude that at $g = -1$ $Z_{\lambda} = Z_g$ and in particular,

$$w_1(\lambda, -1) = y_1(\lambda, -1). \quad \text{(D.8)}$$
In both BOS and FER cases, the free energy takes the form
\[ F(h) = -h^2 F_0(\lambda, g, \mu, h), \] (D.9)
where the function \( F_0 \) is RG invariant: \( \mathcal{D}F_0 = 0 \). As we are interested in the asymptotic expansion of the free energy for large external fields we write
\[ h = h_0 e^t, \] (D.10)
where \( h_0 \) is a fixed, finite value of the external field and \( t \to \infty \). Standard RG considerations then show that
\[ F(h) = -h^2 F_0(\tilde{\lambda}(t), \tilde{g}(t), \mu, h_0), \] (D.11)
where the running coupling \( \tilde{\lambda} \) and the running deformation parameter \( \tilde{g} \) satisfy the following set of differential equations and initial conditions.
\[
\frac{d\tilde{\lambda}}{dt} = \beta_\lambda(\tilde{\lambda}, \tilde{g}), \quad \tilde{\lambda}(0) = \lambda, \tag{D.12}
\]
\[
\frac{d\tilde{g}}{dt} = \beta_g(\tilde{\lambda}, \tilde{g}), \quad \tilde{g}(0) = g. \tag{D.13}
\]

In the deformed principal model the \( \beta \)-functions are known to be of the form \[ \text{[14, 15]} \]
\[
\beta_\lambda = -\frac{\lambda^2}{4\pi} \left\{ 1 - g + \lambda p_2(g) + \lambda^2 p_3(g) + \cdots \right\}, \tag{D.14}
\]
\[
\beta_g = \frac{\lambda g(1 + g)}{2\pi} \left\{ 1 + \lambda q_1(g) + \lambda^2 q_2(g) + \cdots \right\}, \tag{D.15}
\]
where the two-loop \( \beta \)-function coefficients are given by \[ \text{[14, 15]} \]
\[ p_2(g) = \frac{1 - 2g + 5g^2}{8\pi}, \quad \quad \text{and} \quad \quad q_1(g) = \frac{1 - g}{4\pi}. \tag{D.16} \]
The three-loop coefficients \( p_3(g), \ q_2(g) \) are not known at present, but fortunately we will not need their explicit form beyond the combination
\[ u = 2q_2(-1) - p_3(-1). \tag{D.17} \]

We are interested in solutions of (D.12) and (D.13) with asymptotically free (AF) behaviour, that is \( \tilde{\lambda} \to 0 \) together with \( \tilde{g} \to g_1 \) as \( t \to \infty \), where \( g_1 \) is some constant. Analyzing Eqs. (D.14), (D.15), we find three different AF solutions:

1. \( g_1 = 0 \quad g \equiv g_0 \equiv \tilde{g} \equiv 0 \), \tag{D.18}
2. \( g_1 = -1 \quad g \equiv g_0 \equiv \tilde{g} \equiv -1 \), \tag{D.19}
3. \( g_1 = -1 \quad -1 < \tilde{g} < 0 \). \tag{D.20}
Solutions (D.18) resp. (D.19) correspond to the O(4) resp. the O(3) NLS model, while the solution (D.20) corresponds to the ‘generic’ case of deformed principal model. In what follows we shall concentrate on the generic case. We introduce the RG-invariant combination of the two couplings:

\[ p = 2\pi \lim_{t \to \infty} \frac{1 + \bar{g}(t)}{\lambda(t)}, \quad (D.21) \]

which we identify with the parameter \( p \) in the \( S \) matrix (1.4). A RG-invariant quantity can almost be treated just as a numerical parameter. We can express the running deformation coupling, \( \bar{g} \), in terms of the running coupling, \( \bar{\lambda} \) and the RG-invariant \( p \) as

\[ \bar{g} = \Gamma(\bar{\lambda}, p) \quad (D.22) \]

and define an effective \( \beta \)-function for \( \lambda \) as

\[ \beta_{\text{eff}}(\lambda, p) = \beta_\lambda(\lambda, \Gamma(\lambda, p)) \quad (D.23) \]

Using the perturbative expressions (D.14) and (D.15) together with (D.16) and (D.17) we have

\[ \Gamma(\lambda, p) = -1 + \frac{p}{2\pi} \lambda + p^{(2)} \lambda^2 + p^{(3)} \lambda^3 + \cdots, \quad (D.24) \]

where

\[ p^{(2)} = -\frac{p^2}{8\pi^2}, \quad \text{and} \quad p^{(3)} = \frac{p^2(p + 2)}{64\pi^3} + \frac{pu}{8\pi}. \quad (D.25) \]

The unknown three-loop coefficients contribute only to \( u \) defined in Eq. (D.17). It is not difficult to show using Eq. (D.8) that

\[ 2q_2(-1) = p_3(-1), \quad (D.26) \]

implying \( u = 0 \).

Using the perturbative form (D.24) in the definition (D.23) we have

\[ \beta_{\text{eff}}(\lambda, p) = -\frac{\lambda^2}{2\pi} + \frac{p - 2}{8\pi^2} \lambda^3 + \cdots, \quad (D.27) \]

which allows the definition of the RG-invariant \( \Lambda \)-parameter in the \( \overline{\text{MS}} \) scheme the usual way:

\[ \Lambda_{\overline{\text{MS}}} = \mu e^{-\frac{2\pi}{\pi} \left( \frac{\lambda}{2\pi} \right)^{\frac{p}{2}-1}} e^{\gamma \left( 1 + O(\lambda) \right)}. \quad (D.28) \]

The next step in the RG analysis is the introduction of the effective coupling \( \lambda_{\text{eff}}(h) \) defined by the transcendental equation

\[ \frac{2\pi}{\lambda_{\text{eff}}} + \left( \frac{p}{2} - 1 \right) \ln \frac{2\pi}{\lambda_{\text{eff}}} = \ln \frac{h}{\Lambda_{\overline{\text{MS}}}}. \quad (D.29) \]
The advantage of using the effective charge is that it is a function of the physical quantity
\[ s = \ln \frac{h}{\Lambda_{\text{MS}}} \] (D.30)
only and moreover the running coupling can be expressed in terms of \( \lambda_{\text{eff}}(h) \) perturbatively (in the sense that it is an infinite power series):
\[ \bar{\lambda} = \lambda_{\text{eff}} + \frac{1}{2\pi} \left( \ln \frac{h_0}{\mu} - \gamma \right) \lambda_{\text{eff}}^2 + \cdots . \] (D.31)
The asymptotic expansion (for large \( s \)) of the effective coupling (containing terms \( \propto \ln s \)) can be written as
\[ \lambda_{\text{eff}} = \frac{2\pi}{s} + \frac{\pi(p-2)}{s^2} \ln s + \cdots . \] (D.32)

So far we have concentrated on the generic case (D.20). The O(3) symmetric limit, case (D.19), \( (p \to 0) \) corresponds to putting simply \( p = 0 \) in Eq. (D.32). The O(4) limit, (D.18), however, cannot be obtained from the generic case although it corresponds formally to the \( p \to \infty \) limit. (See the discussion in Subsection 2.3.)

For lattice regularization some of the above formulae are modified. First of all, instead of (D.1) and (D.2-D.3) the relation between the bare parameters \( (\tilde{\lambda}_0, \tilde{g}_0) \) and the renormalized ones \( (\tilde{\lambda}, \tilde{g}) \) are given by
\[ \tilde{\lambda}_0 = \tilde{\lambda} \tilde{Z}_{\lambda} , \quad \text{and} \quad 1 + \tilde{g}_0 = (1 + \bar{g}) \bar{Z}_g , \] (D.33)
where
\[ \tilde{Z}_{\lambda} = 1 + \sum_{r=1}^{\infty} \tilde{y}_r(\tilde{\lambda}, \bar{g}) (\ln \tilde{\mu}a)^r , \] (D.34)
\[ \tilde{Z}_g = 1 + \sum_{r=1}^{\infty} \tilde{w}_r(\tilde{\lambda}, \bar{g}) (\ln \tilde{\mu}a)^r . \] (D.35)

Here the parameter \( \tilde{\mu} \) is the lattice analogue of the \( \mu \)-variable of the dimensional scheme and naturally it is \( \tilde{\mu} \) that appears in the lattice version of (D.5). The lattice \( \beta \)-functions are given by
\[ \tilde{\beta}_\lambda = -\tilde{\lambda} \tilde{y}_1(\tilde{\lambda}, \bar{g}) , \quad \text{and} \quad \tilde{\beta}_g = -(1 + \bar{g}) \tilde{w}_1(\tilde{\lambda}, \bar{g}) . \] (D.36)

The rest of the RG analysis is identical to the one in the dimensional scheme. Although the higher order \( \beta \)-functions are different, the main features of the RG trajectories (D.18-D.20) remain the same and also the first two coefficients of \( \beta_{\text{eff}}(\tilde{\lambda}, p) \) are unchanged. The lattice version of the \( \Lambda \)-parameter is given by
\[ \Lambda_L = \tilde{\mu} e^{-\frac{2\pi}{\bar{\lambda}} \left( \frac{\bar{\lambda}}{2\pi} \right)^{(\frac{p}{2}-1)}} \left( 1 + \mathcal{O}(\tilde{\lambda}) \right) . \] (D.37)
References

[1] A. Polyakov, P.B. Wiegmann, Phys. Lett. 131B (1983) 121
[2] P.B. Wiegmann, Phys. Lett. 152B (1985) 209.
[3] I.V. Cherednik Sov. J. Nucl. Phys., 33 (1981) 144
[4] H.M. Babujian, A.M. Tsvelick, Nucl. Phys. B265 (1986) 24.
[5] A. Kirillov, N.Yu. Reshetikhin in Proc. of the Paris-Meudon Colloquium, String Theory, Quantum Cosmology and Quantum Gravity, Integrable and Conformal Invariant Theories, (1986), eds. N. Sanchez, H. de Vega, (World Scientific, Singapore).
[6] V.A. Fateev, Nucl. Phys. B473[FS] (1996) 509.
[7] J.M. Evans and T.J. Hollowood, Nucl. Phys. Proc. Suppl. 45A (1996) 130;
[8] G. Japaridze, A. Nersesyan and P. Wiegmann, Nucl. Phys. B230 (1984) 511.
[9] P. Hasenfratz, M. Maggiore and F. Niedermayer, Phys. Lett. 245B (1990) 522; ibid 529.
[10] P. Forgács, F. Niedermayer and P. Weisz, Nucl. Phys. B367 (1991) 123.
[11] P. Fendley and K. Intriligator, Phys. Lett. 319B (1993) 132;
[12] J. Balog, S. Naik, F. Niedermayer and P. Weisz, Phys. Rev. Lett. 69 (1992) 873.
[13] M. Karowski, Nucl. Phys. B153 (1979) 244.
[14] P. Azaria, P. Lecheminant and D. Mouhanna, Nucl. Phys. B455 (1995) 648;
[15] J. Balog, P. Forgács, Z. Horváth and L. Palla, Nucl. Phys. Proc. Suppl. 49B (1996) 16;
[16] L. Karp and L. Palla, to be published
[17] A.B. Zamolodchikov and A.B. Zamolodchikov, Ann. Phys. (NY) 120 (1979) 253
[18] S. Coleman and H.J. Thun, Comm. Math. Phys. 61 (1978) 31
[19] H. Babujian, A. Fring, M. Karowski and A. Zapletal, Nucl. Phys. B538 (1999) 535
[20] G. Mussardo, Phys. Rep. 218 (1992) 215