GENERAL ISSUES IN THE EVOLUTION OF FERMION MASSES AND MIXINGS

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ABSTRACT

General issues in the renormalization group evolution of fermion masses and mixings is discussed. An effective fixed point in the top quark Yukawa coupling can strongly constrain its value at the electroweak scale. Predictions following from Yukawa coupling unification are affected by threshold corrections at the grand unified scale. The Landau pole translates into an upper limit on the strong gauge coupling $\alpha_3(M_Z)$. Given the hierarchy in the fermion sector, the evolution of the Cabbibo-Kobayashi-Maskawa matrix can be expressed in terms of a single scaling parameter $S$. Using this scaling factor and analogous scaling factors for the quark and lepton masses, we outline a systematic strategy that readily yields electroweak predictions for any GUT scale texture.

1. Introduction

The additional symmetry in grand unified theories (GUTs) can be used to reduce the number of arbitrary parameters in the standard model. Gauge coupling unification eliminates one of these free parameters. Yukawa coupling unification can potentially provide a much more expansive reduction. In this case, symmetries at the GUT scale can provide relations between the 13 parameters of the flavor sector (9 fermion masses and 4 parameters that characterize the mixing in the Cabbibo-Kobayashi-Maskawa (CKM) matrix). From another point of view, the low-energy measurements of fermion masses and mixings can provide a window into the symmetries at the GUT scale. In the following we will concentrate on a few general topics that are relevant to the evolution of fermion masses and mixings.

2. Fixed Points

Fixed point solutions could apply for a wide range of top quark Yukawa couplings arising in a more fundamental theory. One can obtain a simple estimate of the location of the fixed point by setting the one-loop top quark Yukawa renormalization group equation (RGE) in the minimal supersymmetric standard model (MSSM) to zero,

$$\frac{d\lambda_t}{dt} = \frac{\lambda_t}{16\pi^2} \left( - \sum c_i g_i^2 + 6\lambda_t^2 + \lambda_b^2 \right) = 0.$$  (1)

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with $c_1 = 13/15$, $c_2 = 3$, $c_3 = 16/3$. This is only accurate to about 10% in practice because the gauge couplings are themselves evolving. A careful analysis of the two-loop RGEs in the MSSM using experimental input for the gauge couplings yields an effective fixed point of $\lambda^\text{fp} \simeq 1.1$ near the electroweak scale $\mu = M_Z$ as shown in Figure 1. Top quark Yukawa couplings exceeding the fixed point value at the GUT scale evolve rapidly to the fixed point, while the approach from below is more gradual.

The prediction for the $m_b/m_\tau$ ratio provides motivation for the fixed point solution. This behaviour can be understood immediately from the one-loop RGE in the MSSM for $R_{b/\tau} \equiv \lambda_b/\lambda_\tau$,

$$\frac{dR_{b/\tau}}{dt} = \frac{R_{b/\tau}}{16\pi^2} \left( - \sum d_i g_i^2 + \lambda_t^2 + 3\lambda_b^2 - 3\lambda_\tau^2 \right), \quad (2)$$

with $d_1 = -4/3$, $d_2 = 0$, $d_3 = 16/3$. If the $b$-quark is sufficiently light and $R_{b/\tau} = 1$ at the GUT scale, large Yukawa couplings are required to counteract the “overshoot” from the gauge coupling contributions from Eq. (2). Here we take as inputs $m_\tau = 1.784$ GeV and the running mass $m_b(m_b) = 4.25$ GeV. In the standard model the effective fixed point solution implies that the top quark is heavy $m_t > 200$ GeV. In the MSSM the Yukawa coupling must be large ($\simeq 1$) at the electroweak scale, implying a linear correlation between $m_t$ and $\sin \beta$ (neglecting contributions from $\lambda_b$ and $\lambda_\tau$ which have a significant effect only for very large $\tan \beta$),

$$m_t(m_t) = \frac{\lambda_t^\text{fp} v \sin \beta}{\sqrt{2}} = \frac{\lambda_t^\text{fp} v}{\sqrt{2}} \frac{\tan \beta}{\sqrt{1 + \tan^2 \beta}}, \quad (3)$$

where $v = 246$ GeV and $\tan \beta$ is the ratio of the vevs of the two Higgs doublets in the

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Figure 1: The top quark Yukawa coupling evolves rapidly to the effective fixed point value from above. The constraint $d\lambda_t/dt = 0$ varies with scale because the gauge couplings are evolving.
MSSM. As $\alpha_3(\mu)$ is increased, $\lambda_t(\mu)$ must be correspondingly increased to preserve the $m_b/m_\tau$ prediction. Hence for larger input $\alpha_3(M_Z)$, the solutions tend to display more strongly the fixed point character. The fixed point does not require that $\tan\beta$ be small, but allows for large $\tan\beta$ if $m_t$ is sufficiently large. There is an intermediate region of $\tan\beta$ in which the effects of $\lambda_b$ and $\lambda_t$ are negligible in the RG evolution, but for which Eq. (3) is valid. However if $m_t^{\text{pole}}$ is below 160 GeV, the fixed point gives $\tan\beta < 2$ with interesting consequences for Higgs boson phenomenology\(^7\).

Another interesting result is the observation\(^3\) that the observed $m_b/m_\tau$ ratio can be obtained if the masses of all three members of the heavy generation are determined by fixed points (without necessarily assuming the GUT scale unification constraint $\lambda_b = \lambda_t$). This solution requires that $\lambda_t$, $\lambda_b$ and $\lambda_\tau$ be large, and therefore that $\tan\beta$ be large. In some minimal models large $\tan\beta$ will cause a violation of proton decay constraints.

3. Threshold Corrections at the GUT Scale

Figure 2 shows the effects of taking threshold corrections to the GUT scale unification constraint $\lambda_b(M_G) = \lambda_t(M_G)$ for two different values of $\alpha_3(M_Z)$. The top quark mass plotted is the running mass $m_t(m_t)$. For $\alpha_3(M_Z) = 0.11$, the top quark Yukawa coupling can be pushed below its fixed point for threshold corrections as large as 20\%, and the solution of the RGEs is not sufficiently close to the fixed point solution to provide a constraint in the $m_t - \tan\beta$ plane. For $\alpha_3(M_Z) = 0.12$ the fixed point solution is useful even for large threshold corrections, since the solutions display a stronger fixed point nature.

Threshold corrections to the GUT scale unification constraint $\lambda_b = \lambda_t$ generally are larger if the top quark Yukawa is large at the GUT scale; however, Figure 2 indicates that these GUT threshold corrections become less important in determining the relation between $m_t$ and $\tan\beta$ for a large top quark Yukawa coupling. It is precisely the fixed point nature of $\lambda_t$ that make $m_t,\tan\beta$ solutions insensitive to even large GUT threshold corrections. One also expects threshold corrections to the other Yukawa coupling unification conditions, including those involving the CKM mixing angles.

4. Landau Pole

The two-loop part of the RGE’s are known for the Yukawa couplings and for the mixing angles in the MSSM. Comparing the two-loop to the one-loop can give a quantitative estimate of the proximity of the Landau pole. Any criteria one might define as the breakdown is admittedly subjective. We adopt one in which the two-loop contribution to the evolution be less than $\frac{1}{4}$ of the one-loop contribution over the entire range of the Yukawa coupling evolution. Since the top Yukawa coupling is rising toward the Landau pole as one evolves upward in scale, this condition is restrictive at the highest scales. The Landau pole indicates that there is an upper limit\(^4,6\) on the value of the strong coupling $\alpha_3(M_Z) \leq 0.125$. 

\[^7\] It is precisely the fixed point nature of $\lambda_t$ that make $m_t,\tan\beta$ solutions insensitive to even large GUT threshold corrections.
Figure 2: The effect of threshold corrections on the Yukawa coupling unification condition $\lambda_b(M_G) = \lambda_\tau(M_G)$ with $m_b = 4.25$ GeV for $\alpha_3(M_Z) = 0.11$ and 0.12. The corrections have a more pronounced effect for smaller values of $\alpha_3(M_Z)$ for which the solutions are closer to the fixed point. The Landau pole provides a constraint on corrections with $\lambda_b(M_G) > \lambda_\tau(M_G)$. 
Figure 1 also shows that for $\lambda_b(M_G) > \lambda_\tau(M_G)$, the top quark Yukawa coupling is pushed up against the Landau pole. This potentially can give new constraints on the size of GUT scale threshold corrections.

5. Universal Evolution of the CKM Matrix

The one-loop evolution equations for the Yukawa coupling matrices are

$$\frac{d U}{dt} = \frac{1}{16\pi^2} \left( x_u I + y_u U U^\dagger + a_d D D^\dagger \right) U ,$$

$$\frac{d D}{dt} = \frac{1}{16\pi^2} \left( x_d I + y_d D D^\dagger + a_u U U^\dagger \right) D ,$$

where the coefficients $x_i$, $y_i$, $a_i$ depend upon the particle content of the theory and are functions of the dimensionless gauge and Yukawa couplings, i.e. $a_i = a_i(g_1^2, g_2^2, g_3^2, Tr[U U^\dagger], Tr[D D^\dagger], Tr[E E^\dagger])$ and Higgs quartic couplings. When there is a hierarchy of masses in the Yukawa matrices, the evolution of the quark masses and CKM mixing angles is given as a simple scaling. The hierarchy required is the following: Light generations with small Yukawa couplings (i.e. $<< 1$), and a heavy third generation. With such a hierarchy it will be these heavy Yukawas along with the gauge couplings that are important in the CKM evolution. Mixing between the heavy and light generations must be small, which occurs naturally for a hierarchy in the Yukawa matrices. The only terms in Eqs. (4) and (5) that contribute to the running of the CKM matrix are the ones involving $a_u$ and $a_d$. The CKM mixing angles scale as

$$\frac{d W_1}{dt} = -\frac{W_1}{8\pi^2} \left( a_d \lambda_t^2 + a_u \lambda_b^2 \right) ,$$

where $W_1 = |V_{cb}|^2, |V_{ub}|^2, |V_{ts}|^2, |V_{td}|^2$, the CP-violation parameter $J$ and

$$\frac{d W_2}{dt} = 0 ,$$

where $W_2 = |V_{us}|^2, |V_{ud}|^2, |V_{cb}|^2, |V_{ub}|^2, |V_{ts}|^2, |V_{td}|^2$. The two-loop versions of Eqs. (4)-(7) can be found in Ref. [14]. The solution of Eq. (6) is

$$W_1(M_G) = W_1(\mu) S(\mu) ,$$

where $S$ is a scaling factor defined by

$$S(\mu) = \exp \left\{ -\frac{1}{8\pi^2} \int_{\mu}^{M_G} \left( a_d \lambda_t^2 + a_u \lambda_b^2 \right) d \ln \mu' \right\} .$$

The lightest two generations do not affect the evolution, and one does not need the mixing between the first two generations to be small for the universal scaling described above to occur. This makes the scaling universality an especially good approximation since the Cabbibo angle is the largest of the quark mixings. Any amount of mixing between light generations is allowed, which is intuitively the case
since they have a negligible impact on the evolution. The scaling behavior can be demonstrated to all orders in perturbation theory\textsuperscript{14}.

Corollaries to the universality of the scaling of the CKM matrix are the following:

- The ratios $|V_{ub}/V_{cb}|$, $|V_{td}/V_{ts}|$ do not scale.

- The CP-violation parameter $J$ has the same scaling factor $S$. There is a simple way to understand the scaling of $J$ in terms of the mixing angles. The unitarity relation

$$V_{11}V_{12}^* + V_{21}V_{22}^* + V_{31}V_{32}^* = 0,$$  \hspace{1cm} (10)

can be represented by the triangle in Figure 3a. Since $|V_{11}V_{12}^*| \approx |V_{21}V_{22}^*| >> |V_{31}V_{32}^*|$, this particular unitarity triangle is very slim. Only the short side scales to leading order in the approximation, as shown in Figure 3b. Since the CP-violation parameter $J$ is twice the area of any unitarity triangle, it must scale with the same factor $S$. Of course the other sides of the triangle must change a very small amount to preserve unitarity

$$V_{11}V_{12}^* + V_{21}V_{22}^* + SV_{31}V_{32}^* = 0,$$  \hspace{1cm} (11)

but these changes are subleading in the hierarchy approximation. A similar argument exists for the other unitarity triangles. Equivalent ways to think about the scaling is in terms of the Wolfenstein parameterization\textsuperscript{18} or the DHR parameterization\textsuperscript{17,19}. In the former case the scaling manifests itself as the running of only $A$, and the nonevolution of $\lambda$, $\rho$, and $\eta$. In the latter case, the mixing angles $s_1$ and $s_2$ do not scale, while $s_3$ scales with the factor $S^{1/2}$.

- If the mixing between two generations is exactly zero, then it must be zero at all scales. This is true even if there is no hierarchy.

Figure 3: Scaling of a unitarity triangle.
A general texture analysis can be performed by diagonalizing mass matrices at the scale where they are simple (i.e. at the GUT scale where the zero structure is defined). The largest corrections from subleading terms in the hierarchy will come at this stage (they can be as large as $\mathcal{O}(\lambda^2) \approx 5\%$). The contribution of the subleading terms to the RGEs that are neglected in the hierarchy approximation is much smaller and can be neglected entirely.

A crucial point to be emphasized here is that the mass matrices themselves contain more information than can be observed. To compare predictions with experiment it is only necessary to evolve the observables, i.e. the masses and mixings. The zeroes disappear from the mass matrices as the low energy theory does not respect the discrete (or otherwise) symmetries that gave rise to them, but the evolution of the observables is particularly simple given that the hierarchy exists.

A practical, systematic strategy to generate the electroweak predictions of various GUT textures is the following:

1. There are scaling quantities for the heavy and light Yukawa couplings and for the CKM matrix that depend on the values of the heavy Yukawa couplings and the gauge couplings. For any particular choice of these couplings at the electroweak scale there are particular solutions for the scaling parameters that can be calculated using the RGEs. For example, after fixing the gauge couplings at the electroweak scale, contours of the scaling factors can be obtained in the $m_t, \tan\beta$ plane.

2. For any given texture find the diagonal Yukawa couplings, the CKM matrix, and the parameter $J$ at the “texture” or GUT scale. One can retain the contributions of the subleading terms in the diagonalization to any degree of accuracy. These contributions (if desired) can be obtained analytically (e.g. Ref. [20]) or numerically if necessary.

3. The evolution of the observables calculated in step (2) can now be evolved to the electroweak scale by multiplying by the scaling factors calculated in step (1).

4. Step (2) can be repeated for a different texture to obtain a different set of boundary conditions. The scaling factors from step (1) are obtained from the evolution equations alone and need not be recalculated. The evolution for the new texture is obtained by simply multiplying by the scaling factors in step (1).

A more sophisticated algorithm is needed to evolve the Yukawa matrices as a whole. This extra work is unnecessary, however. Only the observable components need to be evolved, and this can be done without even making the hierarchy approximation. However, then the evolution is only approximately described by scaling (although the approximation is quite good). The complete two-loop evolution equations for the mixing angles and the quark masses are known for any theory in which the RGEs for the Yukawa matrices are known\textsuperscript{14}. Each entry in the Yukawa matrices is not known after evolution, but the observable combinations are known and this is the full information needed to compare with experiment.

6. Conclusions

Low energy observables in the flavor sector can provide a powerful probe
of the GUT scale symmetries. The most predictive models provide an enormous reduction in the number of arbitrary parameters. Even if these models are eventually in contradiction with improvements in experimental data, we are confident that the renormalization group scaling of low energy observables will continue to be a valuable tool in the search for higher symmetries.

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