The Microscopic Spectral Density of the QCD Dirac Operator

P.H. Damgaard\textsuperscript{1}, J.C. Osborn\textsuperscript{2}, D. Toublan\textsuperscript{2}, J.J.M. Verbaarschot\textsuperscript{2}

\textsuperscript{1}Niels Bohr Institute, Blegdamsvej 17, Copenhagen, Denmark
\textsuperscript{2}Department of Physics and Astronomy, SUNY, Stony Brook, New York 11794

Abstract

We derive the microscopic spectral density of the Dirac operator in $SU(N_c \geq 3)$ Yang-Mills theory coupled to $N_f$ fermions in the fundamental representation. An essential technical ingredient is an exact rewriting of this density in terms of integrations over the super Riemannian manifold $Gl(N_f + 1|1)$. The result agrees exactly with earlier calculations based on Random Matrix Theory.

\textit{PACS:} 11.30.Rd, 12.39.Fe, 12.38.Lg, 71.30.+h

\textit{Keywords:} QCD Dirac operator; Chiral random matrix theory; Partially quenched chiral perturbation theory; super-Riemannian manifolds; Microscopic spectral density; Valence quark mass dependence
1. Introduction

For the understanding of chiral symmetry breaking (and restoration) in QCD and other QCD-like theories with massless or near-massless quarks it is essential to know the distribution of the small eigenvalues of the Dirac operator. In the conventional thermodynamic limit where the space-time volume $V$ is taken to infinity prior to taking quark masses to zero, the accumulation of eigenvalues $\lambda$ near $\lambda \sim 0$ may or may not lead to a non-vanishing density of the eigenvalues at the origin, $\rho(0)/V$. According to the Banks-Casher relation \[ \Sigma_0 \equiv \langle \bar{\psi} \psi \rangle = \pi \rho(0)/V, \] this spectral density is an order parameter for chiral symmetry breaking. In all that follows we only consider the case where the chiral condensate does not vanish in the chiral limit.

The full spectral density $\rho(\lambda)$ is prohibitively difficult to compute, sensitive as it is to physics at all scales from the infrared up to the ultraviolet cut-off. Because of its relation to the chiral condensate, even just computing $\rho(0)$ is tantamount to understanding in all detail the dynamics that underlies chiral symmetry breaking in QCD. Moreover, since $\rho(0)$ involves an energy scale in the deepest infrared, only non-perturbative techniques such as lattice regularizations can have a hope of computing this single number from first principles. The fact that only low-momentum modes of the Dirac spectrum are of relevance here nevertheless opens up the interesting possibility of studying the problem in a new setting, using the low-momentum representation of the QCD partition function. This effective theory is a chiral Lagrangian. Its equivalence with the conventional representation of QCD in terms of microscopic degrees of freedom (quarks and gluons) becomes exact in precisely the limit of interest here: zero (or almost-zero) momentum. There is only one input parameter in this approach: the chiral condensate $\Sigma_0$, and hence the value of spectral density evaluated at the origin, $\rho(0)$. For computational purposes it is convenient to impose a condition which ensures that only exact zero-momentum modes dominate the euclidean QCD partition function. This can be achieved by restricting the (large) space-time volume $V$ to obey the inequalities

\[
\frac{1}{\Lambda_{\text{QCD}}} \ll V^{1/4} \ll \frac{1}{m_\pi},
\]

where $m_\pi$ is the mass of the Goldstone bosons associated with the lightest quark mass. The first inequality is required to separate the Goldstone modes from the other hadronic excitations with a mass scale of $\Lambda_{\text{QCD}}$ or higher. When $V \to \infty$ the second inequality gives an unphysical limit in which the pion is ultra-light, and never fits into the space-time volume. However, a wealth of information about the Dirac operator spectrum can be computed \textit{exactly} in this limit.

\footnote{The domain of validity of the zero momentum mode approximation is determined by the pion decay constant $F$.}
Once it is assured that only zero-momentum modes contribute to the effective QCD partition function, its evaluation becomes remarkably simple. The space-time integration of the effective Lagrangian density yields just an overall volume factor, and the partition function becomes identical to a zero-dimensional group integral over the coset determined by the pattern of chiral symmetry breaking. As was noted by Leutwyler and Smilga [4], it is highly advantageous to consider these partition functions in sectors corresponding to definite topological charge $\nu$ (in order to avoid absolute value signs, we will take $\nu$ positive or zero from now on). Consider $SU(N_c \geq 3)$ gauge theories with $N_f$ fermions in the fundamental representation. If chiral symmetry breaks according to $SU_L(N_f) \times SU_R(N_f) \rightarrow SU_V(N_f)$, the effective partition function of the zero momentum modes takes on the form [3, 4]

$$Z_{GLS}^{N_f, \nu}(U) = \int U(N_f) dU \det(U)^{\nu} \exp \left\{ V \Sigma_0 \Re \text{Tr}[MU^\dagger] \right\} \quad (2)$$

in a sector of topological charge $\nu$, and with fermion mass matrix $M$. Note that the effective partition function only depends on masses $m_i$ and four-volume $V$ in the scaling combination of $\mu_i \equiv m_i V \Sigma_0$. A few years ago it was realized that this effective partition function itself has an entirely different representation in terms of large-$N$ Random Matrix Theory [5, 6]. The precise relationship is as follows.

Define a “chiral” random matrix partition function with the same global symmetries as the field theory partition function by [4, 7]

$$Z_{N_f, \nu}^\beta(m_1, \cdots, m_{N_f}) = \int DW \prod_{f=1}^{N_f} \det(D + m_f) e^{-N^\beta \text{Tr}(W^\dagger W)}, \quad (3)$$

where

$$D = \left( \begin{array}{cc} 0 & iW \\ iW^\dagger & 0 \end{array} \right), \quad (4)$$

and $W$ is a $n \times m$ matrix with $\nu = |n - m|$ and $N = n + m$. The Random Matrix Theory potential $V(W^\dagger W)$ is not restricted by any symmetry conditions, but it can be shown that all relevant results in the proper limit do not depend on this potential once one imposes that the matrix spectral density at the origin $\rho(0)$ be non-vanishing [8, 9, 10, 11, 12, 13]. The equivalent of the topological charge $\nu$ in the effective partition function (4) is taken to be fixed. In the large-$N$ limit we thus have $n = N/2$. The matrix elements of $W$ are either real ($\beta = 1$, chiral Gaussian Orthogonal Ensemble (chGOE)), complex ($\beta = 2$, chiral Gaussian Unitary Ensemble (chGUE)), or quaternion real ($\beta = 4$, chiral Gaussian Symplectic Ensemble (chGSE)) [7]. For Yang-Mills theory with three or more colors and quarks in the fundamental representation the matrix elements of the Dirac operator are complex, and we have $\beta = 2$. It can be demonstrated that in the microscopic domain [8]
the random matrix partition function for any potential \(V(W^\dagger W)\) can be mapped \textit{exactly} onto the effective finite volume partition function (2). This was shown in ref. [5, 6] for a Gaussian potential, and, because the random matrix partition function does not depend on \(V(W^\dagger W)\) in this limit [14], it holds for any potential. For more discussion of the random matrix partition function we refer to [15].

Because of the spontaneous breaking of chiral symmetry, the smallest eigenvalues of the Dirac operator are spaced as \(1/\rho(0) = \pi/\Sigma_0 V\). In order to study the behavior of the smallest eigenvalues in the approach to the thermodynamic limit, it is natural to rescale the eigenvalues according to \(u = \lambda V \Sigma_0\) and to introduce the microscopic limit of the spectral density [5]

\[
\rho_s(u) = \lim_{V \to \infty} \frac{1}{V \Sigma_0} \rho \left( \frac{u}{V \Sigma_0} \right).
\]

(5)

For broken chiral symmetry this results in a nontrivial limiting function.

To understand the significance of the identification between the chiral Lagrangian (2) and the Random Matrix Theory (3) in the microscopic domain, it helps to view these partition functions \(Z_\nu[\mu_i]\) as generating functions for the chiral condensates \(\langle \bar{\psi}_i \psi_i \rangle\). Conventionally they are defined by taking the zero-mass limit in the end. However, it is useful to focus instead on the mass-dependent chiral condensate, defined in the obvious way by

\[
\Sigma_i(\mu_1, \ldots, \mu_{N_f}) \equiv \frac{1}{V} \frac{\partial}{\partial \mu_i} \log Z_\nu[\mu_1, \ldots, \mu_{N_f}] .
\]

(6)

As is evident from the spectral representation of the condensate, this quantity carries much information about the microscopic spectral density of the Dirac operator:

\[
\Sigma_i(\mu_1, \ldots, \mu_{N_f}) = \Sigma_0 \int du \frac{\rho_s(u; \mu_1, \ldots, \mu_{N_f})}{iu + \mu_i}
\]

(7)

The issue at hand is whether this relation can be uniquely inverted to provide the microscopic spectral density \(\rho_s(u; \mu_1, \ldots, \mu_{N_f})\) in terms of the mass-dependent chiral condensate \(\Sigma_i(\mu_1, \ldots, \mu_{N_f})\). While the relation (7) resembles the Stieltjes transform of \(\rho_s(u; \mu_1, \ldots, \mu_{N_f})\) (which under suitable convergence criteria has a unique inverse), the \(\mu_i\)-dependence of the microscopic spectral density ruins this identification. This problem was recently solved in a paper by three of us [16], where it was noted that the introduction of an additional fermion species into the theory can be used to provide the needed unique inverse of a relation of the form (7). Clearly what is needed is that the microscopic spectral density becomes insensitive to the addition of this additional species. The solution is to simultaneously introduce yet another quark species, but this time of opposite statistics [16]. In the original field theory formulation this corresponds to a Euclidean partition
function of the form

\[ Z_{pq}^\nu = \left( \prod_{f=1}^{N_f} m_f^\nu \right)^\nu \int [dA]^\nu \frac{\det(iD - m_{v1})}{\det(iD - m_{v2})} \prod_{f=1}^{N_f} \det(iD - m_f) e^{-S_{YM}[A]} . \]  

(8)

When \( m_{v1} = m_{v2} \) this partition function simply coincides with the original one. However, it is now also the generator of a mass-dependent chiral condensate for the additional (say, fermionic) quark species, i.e.,

\[ \Sigma(m_v; m_1, \ldots, m_{N_f}) = \frac{1}{V} \frac{\partial}{\partial m_{v1}} \bigg|_{m_{v1}=m_{v2}=m_v} \log Z_{pq}^\nu . \]  

(9)

In terms of the spectral density the valence quark mass dependence of the chiral condensate can be rewritten as [17, 18, 19]

\[ \Sigma(m_v; m_1, \ldots, m_{N_f}) = \frac{1}{V} \sum_k \langle \frac{1}{i\lambda_k + m_v} \rangle = \frac{1}{V} \int d\lambda \rho(\lambda; m_1, \ldots, m_{N_f}) \frac{i\lambda + m_v}{i\lambda + m_v}. \]  

(10)

Here, \( \langle \cdots \rangle \) denotes an average with respect to the distribution of the eigenvalues. Notice that the spectral density in this equation is for \( m_{v1} = m_{v2} \) and coincides with the original QCD one. Contrary to (9), the relation (10) can then be inverted to give \( \rho(\lambda; m_1, \ldots, m_{N_f}) \). As mentioned in [16], the spectral density follows from the discontinuity across the imaginary axis,

\[ \text{Disc}_{m_v=i\lambda} \Sigma(m_v) = \lim_{\epsilon \to 0} \Sigma(i\lambda + \epsilon) - \Sigma(i\lambda - \epsilon) = 2\pi \sum_k \langle \delta(\lambda + \lambda_k) \rangle = 2\pi \rho(\lambda), \]  

(11)

where we have suppressed the dependence on the sea-quark masses.

The formulation of the spectral density as a derivative of the partition function (8) provides a natural explanation for the result that the microscopic spectral density can be related to the usual finite volume partition function with two additional flavors [20, 16].

There is a close analogy to the quenching prescription of lattice gauge theory, but it is worthwhile stressing that, since we eventually restrict ourselves to equal masses \( m_{v1} = m_{v2} \), there are no approximations involved in introducing two additional quark species of opposite statistics in this manner.

The effective Lagrangian corresponding to the partition function (8) with two additional quark species is determined by its underlying global (super-)symmetry structure. Its precise form is very close to the usual one, except for the fact that the group manifold of the Goldstone modes is that of a supergroup. The main object of this paper is the calculation of the valence quark mass dependence of the chiral condensate from this effective theory. Below we will focus on the domain (1) where the nonzero momentum modes
factorize from the effective partition function. This super-symmetric effective Lagrangian has been considered in a different context by Bernard and Golterman [21, 22], who, by analogy with lattice gauge theory, call it the partially quenched chiral Lagrangian. The terms “sea quarks” for the original physical fermions and “valence quarks” for the additional (fermionic) species have been borrowed from lattice gauge theory as well. In perturbation theory the cancellation of the associated determinants in the path integral corresponds to the omission of loops with this fermion. We stress again that in the present context this is not an approximation, but precisely what is required for the associated spectral density to be equal to the QCD spectral density. The quenched version of this effective Lagrangian was recently derived from a two-sublattice random flux model [23] using the flavor-color transformation introduced by Zirnbauer [24].

Below we will find that the microscopic spectral density obtained from the extreme infrared sector of (8) agrees with the result [7, 25] derived from chiral Random Matrix Theory. One can wonder how exact results can be obtained this way. The answer is simple. In the microscopic scaling regime (1) the effective QCD Lagrangian is entirely free of dynamics. The only assumption is that the theory supports spontaneous breaking of chiral symmetry. With this knowledge alone the microscopic limit of the QCD partition function can be written in terms of a zero-dimensional group integral over the Goldstone manifold. This is both true for the usual chiral Lagrangian and for the chiral Lagrangian corresponding to a partition function with valence quarks. The latter results in the exact distribution of the eigenvalues of the Dirac operator in the microscopic scaling limit. This computable and universal end of the Dirac operator spectrum is, as was to be expected, entirely independent of the detailed QCD dynamics.

The organization of this paper is as follows. In section 2 we introduce the effective theory corresponding to QCD with additional quark species of different statistics and discuss the geometry of the Goldstone manifold. Some of the pitfalls associated with the evaluation of super-integrals are discussed in section 3. In section 4 we calculate the measure using an explicit representation of the super-unitary group. In section 5 we evaluate the valence quark mass dependence of the chiral condensate in the sector of topological charge $\nu$, and derive directly from this the microscopic spectral density $\rho_s(\lambda)$, which is found to agree exactly with earlier results based on Random Matrix Theory. A general expression for arbitrary number of flavors and topological charge is obtained in section 6. This expression is evaluated in section 7 for an arbitrary number of massless quarks in the sector of zero topological charge and for one sea quark in a sector of topological charge $\nu$. Concluding remarks are made in section 8. Additional technical details of the calculations can be found in appendices A and B.
2. Low Energy limit of QCD

If chiral invariance is spontaneously broken in QCD, the low energy limit of the theory is dominated by the Goldstone modes associated with this spontaneous symmetry breaking. The basic flavor symmetry of the QCD action with \( N_f \) (physical) sea quarks and \( N_v \) valence quarks is

\[
Gl_L(N_f + N_v) \otimes Gl_R(N_f + N_v) \text{.} \tag{12}
\]

The reason for this bigger symmetry group is that a priori we do not relate the integration variables in the partition function by complex conjugation. This symmetry group is not necessarily a symmetry of the QCD partition function. The symmetry transformations may violate the convergence of the integrals. There are no problems for the Grassmann integration. However, the integrations of the bosonic quark fields are only convergent if the fields are related by complex conjugation. For this reason a \( U_A(N_v) \) transformation on the bosonic quark fields is not a symmetry of the partition function. However, a \( Gl(V) \)/\( U(N_v) \) axial transformation is consistent with convergence requirements.

After spontaneous breaking of the chiral symmetry according to

\[
Sl_L(N_f + N_v) \otimes Sl_R(N_f + N_v) \rightarrow Sl_V(N_f + N_v) \text{,} \tag{13}
\]

the symmetry of the QCD partition function is reduced to \( Sl_V(N_f + N_v) \otimes Gl_V(1) \) where \( \otimes \) denotes the semi-direct product. Notice that an axial \( Gl(1) \) subgroup of the group (12) is broken explicitly by the anomaly.

The Goldstone manifold corresponding to the symmetry breaking pattern (13) is based on the symmetric superspace \( Sl_A(N_f + N_v) \). In our effective partition function the terms that break the axial symmetry will be included explicitly resulting in an integration manifold given by \( Sl_A(N_f + N_v) \otimes Gl_A(1) \). However, this manifold is not a super-Riemannian manifold and is not suitable as an integration domain for the low energy partition function. As an integration domain we choose a maximum Riemannian submanifold of \( Gl_A(N_f + N_v) \). This results in a fermion-fermion block given by the compact domain \( SU_A(N_f + N_v) \), whereas the boson-boson block is restricted to the non-compact domain \( Gl(N_v)/U(N_v) \). Because of the super-trace, this compact/non-compact structure is required for obtaining a positive definite quadratic form for the mass term and the kinetic term of our low energy effective partition function [26]. For a detailed mathematical discussion of this construction we refer to a paper by Zirnbauer [27].

In this paper we restrict ourselves to the case of just one valence quark, \( N_v = 1 \), which is all that is needed to derive the microscopic spectral density of the QCD Dirac operator. We will denote our integration manifold by \( Gl(N_f + 1) \). The fields are parametrized by

\[
U = \exp(i\sqrt{2}\Phi/F) \text{,} \tag{14}
\]
where $\Phi$ is a $(N_f + 2) \times (N_f + 2)$ superfield:

$$
\Phi = \begin{pmatrix}
\phi & \bar{\chi} \\
\chi & i\tilde{\phi}
\end{pmatrix}
$$

and $F$ is the pion decay constant. Here, $\phi$ is a $(N_f + 1) \times (N_f + 1)$ Hermitean matrix containing the ordinary mesons made of quarks and antiquarks. The factor $i$ in the field $\tilde{\phi}$ provides us with a parametrization of $Gl(1)/U(1)$. It represents a meson made out of two ghost quarks. Finally, $\chi$ and $\bar{\chi}$ (not related by complex conjugation) represent fermionic mesons consisting of a ghost-quark and an ordinary anti-quark. They are the Goldstone fermions associated with the spontaneous breaking of the chiral supersymmetry (13). We choose a diagonal mass matrix with $N_f$ sea quark masses resulting in the $(N_f + 2) \times (N_f + 2)$ quark-mass matrix

$$
\hat{M} = \text{diag}(m_1, \ldots, m_{N_f}, m_v, m_v - J).
$$

The source $J$ acts as to lift the degeneracy between the valence quark and its superpartner. We will eventually set $J = 0$.

The axial symmetry of the QCD partition function in the sector of topological charge $\nu$ is broken explicitly by a mismatch in the number of left-handed and right-handed modes. Under a global axial transformation $U_A$, the integrand in (8) is multiplied by a factor $\text{Sdet}^\nu(U_A)$. Taking into account the explicit breaking by the mass term as well, the QCD partition function in the range (1) reduces to the effective low energy partition function (16)

$$
Z^\nu_{N_f}(\hat{M}) = \int_{U \in Gl(N_f + 1|1)} dU \text{Sdet}^\nu(U)e^{\frac{1}{2}\text{Str}(\hat{M}U + \hat{M}U^{-1})}.
$$

As discussed before, the integration manifold is given by the maximum Riemannian submanifold for the symmetric superspace $Gl(N_f + 1|1)$. For a definition of the super-determinant, Sdet, and the super-trace, Str, we refer to the book by Efetov (28).

A consistency check of this effective partition follows by comparing (17) with (2), which implies

$$
Z^\nu_{N_f}(\hat{M}) \big|_{J=0} = Z^\text{GLS}_{N_f,\nu}(\hat{M})
$$

(18)

While this identity may appear quite trivial, only a careful analysis of the superintegral involved on the left hand side of this equation will ensure that it is fulfilled. This issue is connected with the appearance of Efetov-Wegner terms, which we will discuss next.

3. A Simple Example of a Superintegral
In this section we remind the reader of some of the problems that occur in integrations over a supermanifold. In spite of the fact that Grassmann integrations are always convergent, the actual integration over a supermanifold can be quite an involved task. Even an apparently innocent change of variables may result in a subtle paradox [28, 29]. To illustrate this fact, let us look at an example given by Zirnbauer [30]. Consider the supermatrix

\[ A = \begin{pmatrix} a & \alpha \\ \beta & ib \end{pmatrix}, \]  

and the following Gaussian integral

\[ \mathcal{I} = \int da \, db \, d\alpha \, d\beta \, e^{-\frac{1}{2} \text{Str} A^2} = \int da \, db \, d\alpha \, d\beta \, e^{-\frac{1}{2} \left( a^2 + b^2 \right) - a\beta}. \]  

By explicit integration or by Wegner’s theorem [29], one knows that \( \mathcal{I} = 1 \). After Zirnbauer, let us change variables according to

\[ A \rightarrow A' = \begin{pmatrix} s - \sigma \tau (s - it) \\ \sigma (s - it) \end{pmatrix} - \tau (s - it) \begin{pmatrix} s \sigma (s - it) \\ -\tau (s - it) \end{pmatrix}. \]  

The Berezinian of this transformation is easily computed to be:

\[ \left| \frac{\partial (a, b, \alpha, \beta)}{\partial (s, t, \sigma, \tau)} \right| = \frac{1}{(s - it)^2}. \]  

Therefore the integral (20) becomes

\[ \mathcal{I} = \int ds \, dt \, d\sigma \, d\tau \, e^{-\frac{1}{2} \text{Str} A'^2} = \int ds \, dt \, d\sigma \, d\tau \, e^{-\frac{1}{2} (s^2 + t^2)} \frac{1}{(s - it)^2}. \]  

The change of variables (21) has removed all the Grassmann variables from the integrand. One might naively conclude that \( \mathcal{I} = 0 \), in flagrant contradiction with the result we derived before with the original set of variables! However, this conclusion is wrong: The Berezinian is singular at \( s = t = 0 \), and the integral over \( s \) and \( t \) in (23) is therefore not defined.

An even more puzzling substitution is to keep the fermionic variables \( \alpha, \beta \) of (19), while using the eigenvalues of \( A \) as bosonic integration variables [31]. They are easily computed to be:

\[ \begin{align*}
\tilde{a} &= a + \frac{a \beta}{a - ib} \\
\tilde{b} &= ib + \frac{\beta a}{a - ib}.
\end{align*} \]  

The Berezinian of this change of coordinates is equal to one, and the integral (20) therefore becomes

\[ \mathcal{I} = \int d\tilde{a} \, d\tilde{b} \, d\alpha \, d\beta \, e^{-\frac{1}{2} \left( \tilde{a}^2 + \tilde{b}^2 \right)}. \]
Again the Grassmann variables have disappeared from the integrand, and one could naively conclude that $\mathcal{I} = 0$. This is not true. The substitution above generates boundary contributions, the so-called Efetov-Wegner terms [28, 29]. In order to compensate for the nilpotent terms in the integration domain of the superintegral, the measure has to be modified accordingly. The correct measure, including the Efetov-Wegner term, is known in this example [31].

This short discussion exemplifies the subtleties involved in the computation of a super-integral. In a superintegral, one has to be suspicious about any change of variables. When bosonic coordinates are shifted by nilpotent terms, the path of integration may contain even functions of the Grassmann variables. This can result in the notorious Efetov-Wegner terms in the measure [28, 29, 27, 32, 33, 34, 35]. This is not an academic problem. In our case, if the integration of the $Gl(N_f + 1|1)$ partition function is performed via a supersymmetric generalization of the Itzykson-Zuber integral [36, 37, 38], Efetov-Wegner terms appear in the measure in a way very similar to the second change of variables in the example above. However, as is evident from the example, the possible appearance of Efetov-Wegner terms depends on the choice of parametrization. Below we introduce a parametrization without such anomalous terms for the observables under consideration. For example we will find that the partition function is properly normalized without the need for anomalous terms.

4. Parametrization of $Gl(N_f + 1|1)$ and calculation of the measure

In order to construct an explicit parametrization for $Gl(N_f + 1|1)$ we remind the reader of a parametrization of the Riemannian superspace associated with $U(1|1)$ which appeared earlier in the literature [39],

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \exp \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}. \tag{26}$$

The usefulness of this parametrization lies in the factorization into an ordinary and a Grassmannian factor. As was discussed in previous section the Goldstone manifold is a Riemannian superspace that requires a parametrization in terms of compact and non-compact variables [10, 27, 10]. More specifically, instead of the parametrization in (26), the boson-boson block is given by $Gl(1)/U(1)$, obtained by the replacement $\exp i\phi \rightarrow \exp s$. The generalization to $Gl(N_f + 1|1)$ is immediate:

$$U = \begin{pmatrix} U_n & 0 \\ \frac{\partial}{\partial r} & e^s \end{pmatrix} \exp \begin{pmatrix} 0 & \cdots & 0 & \alpha_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_n \\ \beta_1 & \cdots & \beta_n & 0 \end{pmatrix}, \tag{27}$$
where \( \vec{0} \) is a null-vector of length \( n \equiv N_f + 1 \), \( U_n \) is a unitary matrix and the \( \{ \alpha_i \} \) and \( \{ \beta_i \} \) are Grassmann variables. If this product is written as

\[
U = U_o U_g,
\]

we find that

\[
\delta U' \equiv U_o^{-1} dUU^{-1} U_o = U_o^{-1} dU_o + dU_g U_g^{-1}.
\]

Since the Jacobian of a similarity transformation is unity, the invariant measure of the super-Riemannian manifold is given by

\[
d[U] = B(U_o, U_g) DU_n d\bar{s} \prod_{k=1}^{n} d\beta_k d\alpha_k,
\]

where \( DU_n \) is the invariant measure of \( U(N_f + 1) \), and the Berezinian is given by

\[
B = \text{Sdet} \frac{\delta U'}{\delta U_o \delta s \delta \alpha_1 \cdots \delta \beta_n}.
\]

The \( n^2 \) differentials \( \delta U_n \) are defined by \( \delta U_n \equiv U_n^{-1} dU_n \). From the definition of the superdeterminant we find that the Berezinian factorizes as follows

\[
B = \frac{B_1}{B_2},
\]

with

\[
B_1 = \text{Sdet} \frac{\delta U'}{\delta U_o \delta s \delta \alpha_{1n+1} \cdots \delta \alpha_{2n+1} \delta \alpha_{3n+1} \cdots \delta \alpha_{n+1n}}
\]

and

\[
B_2 = \det \frac{\delta U_{1n+1} \cdots \delta U_{n+1n}}{\delta \alpha_1 \cdots \delta \alpha_n \delta \beta_1 \cdots \delta \beta_n}.
\]

Here, \( \delta U_g \equiv dU_g U_g^{-1} \). By inspection one finds that the matrix in \( B_1 \) contains a block of zeros. One then trivially verifies that \( B_1 = 1 \). This implies the factorization of the measure in an ordinary unitary invariant measure and a Grassmannian factor.

The calculation of the second determinant proceeds as follows. We first rewrite \( U_g \) as

\[
U_g = 1 + \frac{\cosh x - 1}{x^2} G^2 + \frac{\sinh x}{x} G,
\]

where \( G \) denotes the exponent in \( U_g = \exp G \) and

\[
x^2 = \sum_{k=1}^{N_f+1} \beta_k \alpha_k.
\]
Here and below, $\sinh x/x$ and $\cosh x$ are meant as a formal power series in $x^2$. The expression for $U_g$ can be easily derived by means of the identity $G^3 = x^2G$. After calculating $\delta U_g = dU_gU_g^{-1}$ one obtains the following explicit expression for $B_2$,

$$B_2 = \det \left[ \frac{\sinh x}{x} \left( \begin{array}{cc} \delta_{kl} + a \beta_k \alpha_l & b \beta_k \beta_l \\ -b \alpha_k \alpha_l & \delta_{kl} - a \alpha_k \beta_l \end{array} \right) \right],$$

(37)

where

$$a = -\frac{1}{2x \sinh x} + \frac{1}{2x^2} - \frac{\cosh x - 1}{2x^2},$$

$$b = -\frac{1}{2x \sinh x} + \frac{1}{2x^2} + \frac{\cosh x - 1}{2x^2}.$$  

(38)

A general expression for the determinant with the structure of (37) is given in Appendix A. The final result for $B$ turns out to be remarkably simple

$$B = \cosh x \left( \frac{x}{\sinh x} \right)^{2N_f+3}.$$

(39)

For $N_f = 0$ one immediately finds that $B = 1$ and, for $N_f = 1$, the result is $B = 1 + \frac{1}{3}(\alpha_1 \beta_1 + \alpha_2 \beta_2)$.

Instead of (27) one could use an alternative parametrization defined by

$$U = V \Lambda V^{-1},$$

(40)

where $\Lambda$ is a diagonal matrix with matrix elements $\Lambda_{kk} \exp i \theta_k$, $k = 1, \ldots, N_f + 1$ and $\Lambda_{N_f+2N_f+2} = \exp s$. The matrix $V$ is a (compact) super-unitary matrix. The Berezinian from the transformation from the $U$-variables to the $V$ and $\Lambda$ variables is given by

$$\prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^2 \prod_k (e^s - e^{i\theta_k})(e^{-s} - e^{-i\theta_k}).$$

(41)

This parametrization is particularly useful if the integrals are calculated by means of a supersymmetric generalization of the Itzykson-Zuber integral [36, 37, 38]. However, in this case the Efetov-Wegner terms may be nonvanishing and must be carefully analyzed.

5. The Microscopic Spectral Density in the Quenched Limit

The valence quark mass dependence of the chiral condensate for $N_f = 0$ and $\nu = 0$ was calculated in [11] using the parametrization (11). It is straightforward to generalize this computation to arbitrary topological charge. As a warm-up exercise for the calculation for arbitrary $N_f$, we first calculate $\Sigma(m_v)$ in a sector of topological charge $\nu$ using the parametrization (27) for $N_f = 0$.  

12
For $N_f = 0$ the parametrization (27) yields a measure that is simply flat (see eq. (39)). In this genuinely quenched limit, the zero mode partition function (17) can be written as

$$Z_{N_f=0}(m_v, m_v - J) = \int \frac{d\theta}{2\pi} ds d\beta d\alpha e^{\nu(i\theta-s)} \times \exp \left[ \Sigma_0 V \text{Str} \begin{pmatrix} m_v & 0 \\ 0 & m_v - J \end{pmatrix} \left( \frac{1 + \frac{1}{2} \alpha \beta}{\beta(e^s - e^{-\theta})/2} \left( \cos \theta + \cosh s \right) \right) \right].$$

(42)

The normalization of this partition function, $Z_{N_f=0}(m_v, m_v)$, for $m_v \neq 0$ follows by simply expanding the Grassmann variables. Using the Wronskian identity for the modified Bessel functions,

$$K_{\nu}(x) I_{\nu+1}(x) + I_{\nu}(x) K_{\nu+1}(x) = \frac{1}{x},$$

(43)

we find $Z_{N_f=0}(m_v, m_v) = 1$. This is in agreement with Wegner’s theorem for super-unitary invariant integrals [29]. Moreover, in our context it has an immediate interpretation in terms of eq. (18), whose right hand side in this $N_f = 0$ case simply reduces to a mass-independent constant (which conveniently can be chosen as unity, as done here).

After differentiation with respect to $J$ and a trivial integration over the Grassmann variables, the valence quark mass dependence of the chiral condensate is found to be

$$\Sigma(m_v) = \frac{\Sigma_0}{2} \int \frac{d\theta}{2\pi} ds e^{\mu_v (\cos \theta - \cosh s)} e^{\nu(i\theta-s)} \left[ \cosh s (\mu_v \cos \theta + \mu_v \cosh s - 1) \right],$$

(44)

where $\mu_v = m_v V \Sigma_0$. This integral is easily computed, it can be expressed in terms of Bessel functions. Using standard recursion relations and the Wronskian identity (43) the result we get is:

$$\frac{\Sigma(m_v)}{\Sigma_0} = \mu_v \left[ I_{\nu}(\mu_v) K_{\nu}(\mu_v) + I_{\nu+1}(\mu_v) K_{\nu+1}(\mu_v) \right] + \frac{\nu}{\mu_v}.$$  

(45)

The last term corresponds to the number of zero modes of the Dirac operator. It is remarkable that this term, which has such a simple interpretation in the framework of the original QCD partition function (where it comes from the explicit factor $m^\nu$ in eq. (8)), is reproduced by the supersymmetric chiral Lagrangian (which does not contain such an explicit factor $m^\nu$). This phenomenon is general, and occurs in the framework of the usual chiral Lagrangian as well [4, 41].

The result (45) coincides exactly with the valence quark mass dependence of the chiral condensate obtained from Random Matrix Theory by integrating the microscopic spectral density according to (10) [18]. What is most important in the present context is that this relation can be inverted. Using eq. (11) we immediately find

$$\rho_s(u) = \frac{u}{2} \left[ J_{\nu}(u)^2 - J_{\nu+1}(u) J_{\nu-1}(u) \right].$$

(46)
This is the microscopic spectral density of the Dirac operator in QCD for \( N_f = 0 \) in a sector of arbitrary topological charge \( \nu \). It agrees exactly with the result obtained earlier by means of chiral Random Matrix Theory \([7]\). Here, it has been derived directly from the effective finite-volume partition function of QCD in the microscopic scaling regime.

The results derived in this section can be tested by means of lattice QCD simulations. Results obtained with staggered fermions for three colors convincingly show that the valence quark mass dependence of the chiral condensate \([18]\) and the microscopic spectral density \([42]\) are given by the above expressions for \( \nu = 0 \). Apparently, effects of the topological charge become only visible in simulations close to continuum limit. For completeness we also mention that lattice QCD results for the microscopic spectral density for \( N_c = 2 \) with staggered fermions is given by similar universal expressions as well \([43]\).

### 6. The Microscopic Spectral Density for QCD with \( N_f \) Flavors

Let us now turn to the general case of \( N_f \) flavors and topological charge \( \nu \). The (partially) supersymmetric effective partition function is given in eq. \((17)\). It will turn out that the measure of this partition function is normalized such that \( Z_{N_f}^\nu (\hat{\mathcal{M}}) = N_f! \) for \( J = 0 \) and zero sea quark masses. In general, it must satisfy the identity \((18)\) up to a normalization factor.

We compute the valence quark mass dependence of the chiral condensate by

\[
\Sigma(m_v) = \frac{1}{V} \frac{\partial}{\partial J} \log Z_{N_f}^\nu (\hat{\mathcal{M}}) \bigg|_{J=0}.
\]  

Using the parameterization \((27)\) with \( U = U_o U_g \), we can rewrite the partition function as

\[
Z_{N_f}^\nu (\hat{\mathcal{M}}) = \int d[U] \text{Sdet}^\nu (U) \exp \left\{ \frac{\Sigma_0 V}{2} \text{Str}[U_g \hat{\mathcal{M}} U_o + \hat{\mathcal{M}} U_g^{-1} U_o^{-1}] \right\}.
\]  

Separating out the fermion-fermion (\( FF \)) and the boson-boson (\( BB \)) blocks we have in the exponential

\[
Z_{N_f}^\nu (\hat{\mathcal{M}}) = \int d[U] \text{Sdet}^\nu (U) \exp \left\{ \Sigma_0 V \left( \frac{1}{2} \text{Tr}[U_g^{FF} \hat{\mathcal{M}}_n U_n + \hat{\mathcal{M}}_n U_g^{FF} U_n^{-1}] - \hat{\mathcal{M}}^{BB} U_g^{BB} \cosh s \right) \right\},
\]  

where the \( FF \) superscript stands for the upper left \( (N_f + 1) \times (N_f + 1) \) block and \( BB \) represents the lower right element. The \( FF \) blocks of \( \hat{\mathcal{M}} \) and \( U \) are denoted by \( \hat{\mathcal{M}}_n \) and \( U_n \), respectively. We have also used the property that \( U_g \) and \( U_g^{-1} \), have the same \( FF \) blocks and \( BB \) blocks which is easily seen from the definition of \( U_g \) and its expansion in powers of the Grassmann elements.
Using that $S\det^\nu(U) = \exp(-\nu s)\det^\nu(U_n)$ the $s$ integral can be done immediately resulting in a modified Bessel function with argument $\hat{\hat{M}}_{BB}U_{BB} = \mu_J \cosh x$ where $\mu_J$ is defined as $\mu_J \equiv (m_v - J)\Sigma_0$. The next step in evaluating the partition function is to expand the exponent in terms of the Grassmann variables in $U^g_{FF} = 1 + G^2(\cosh x - 1)/x^2$ (see eq. (35)). Exploiting the rotational invariance of the scalar product $x^2$, the exponent can be expressed in terms of the eigenvalues, $\Lambda_k$ of the matrix

$$A = \Sigma_0 V(\hat{\hat{M}}_nU_n + U_n^{-1}\hat{\hat{M}}_n)/2. \quad (50)$$

This allows us to make the following replacement in the integrand of (49):

$$\exp\left\{\text{Tr} AU_{g}^{FF}\right\} \rightarrow \exp\left\{\sum_k \Lambda_k \left(1 + \frac{\cosh x - 1}{x^2} \alpha_k \beta_k\right)\right\}. \quad (51)$$

Writing the exponent of the Grassmann variables as a product and once more using the symmetry of the Grassmann variables in the integral we find that an even more symmetric integrand is obtained by making the replacement

$$\exp\left\{\frac{\cosh x - 1}{x^2} \sum_k \alpha_k \beta_k \Lambda_k\right\} \rightarrow \sum_{l=0}^{n} \left(1 - \cosh x\right)^l \frac{(n - l)!}{n!} \sum_{1 \leq k_1 < \cdots < k_l \leq n} \Lambda_{k_1} \cdots \Lambda_{k_l}$$

$$= \sum_{l=0}^{n} \left(1 - \cosh x\right)^l \frac{(n - l)!}{n!} \frac{(\partial y)^l}{l!} \bigg|_{y=0} \exp \left\{\text{Tr} \log(1 + yA)\right\}. \quad (52)$$

The partition function can thus be written as

$$Z_{N_f}^\nu(\hat{\hat{M}}) = 2 \int \prod_k d\beta_k d\alpha_k \cosh x \left(\frac{x}{\sinh x}\right)^{2n+1} K_{\nu}(\mu_J \cosh x) \sum_{l=0}^{n} \left(1 - \cosh x\right)^l \frac{(n - l)!}{n!} \Omega_l(\hat{\hat{M}}_n), \quad (53)$$

where

$$\Omega_l(\hat{\hat{M}}_n) = \int dU_n \det U_n \nu \frac{(\partial y)^l}{l!} \bigg|_{y=0} \det(1 + yA) \exp \left\{\text{Tr} A\right\}. \quad (54)$$

In (53) we have included our result for the Berezinian (39) which factorizes into an ordinary part an a Grassmannian part.

Up to a factor $(-1)^n n!$, performing the final Grassmann integrals is equivalent to finding the coefficient of $x^{2n}$ in the remaining integrand. Using the residue theorem the latter may be expressed as a contour integral around the pole at $x = 0$ of the quotient of the integrand and $x^{2n+1}$. We may then make the substitution $z = \sinh x$ to simplify the integrand resulting in

$$Z_{N_f}^\nu(\hat{\hat{M}}) = 2 \int \frac{dz}{2\pi i} \frac{(-1)^n n!}{z^{2n+1}} K_{\nu}(\mu_J \sqrt{1 + z^2}) \sum_{l=0}^{n} \left(1 - \sqrt{1 + z^2}\right)^l \frac{(n - l)!}{n!} \Omega_l(\hat{\hat{M}}_n). \quad (55)$$
The term containing $K_\nu$ can be expanded using the product theorem for Bessel functions which can be written as

$$K_\nu(\mu_J \sqrt{1 + z^2}) = (1 + z^2)^\nu/2 \sum_{k=0}^{\infty} \frac{1}{k!} K_{k+\nu}(\mu_J) \left[ \frac{-z^2}{2\mu_J} \right]^k. \quad (56)$$

The other factor in the integrand can also be expanded in a series in $z^2$. After multiplying the two out and extracting the coefficient of $z^{2n}$ we are left with the final result for the partition function

$$Z_{N_f}^\nu(\hat{M}) = 2(-1)^n \sum_{k=0}^{n} \sum_{l=0}^{\nu} \sum_{s=0}^{\nu} (-1)^s \binom{l}{s} \binom{\frac{s+\nu}{2}}{n-k} \frac{1}{k!} \left( \frac{-\mu_J}{2} \right)^k K_{k+\nu}(\mu_J)(n-l)!\Omega_l(\hat{M}_n). \quad (57)$$

For $\nu = 0$ this result can be further simplified by using the combinatorial identity

$$\sum_{s=0}^{l} (-1)^s \binom{l}{s} \binom{s/2}{p} = \left( -\frac{1}{4} \right)^p l2^p \frac{(2p-l-1)!}{(p-l)!p!} \quad (58)$$

for $0 \leq l \leq p$ and it gives zero otherwise. Here we take the right hand side of this expression to be one when $p = l = 0$.

In general, the integrals over the unitary group give a fairly complicated expression. However, the calculation greatly simplifies for an arbitrary number of massless quarks and zero topological charge. Of course, the calculation is relatively simple for the case of $N_f = 1$, even with the quark masses included and arbitrary topological charge.

7. Some Special Cases

While the above formulas are quite involved in general, there are some important special cases in which they can be reduced to simple expressions in terms of elementary functions. Consider first the case of one physical quark of mass $m_1$, i.e. $N_f = 1$. The expression (57) then reduces to

$$Z_{N_f=1}^\nu(\hat{M}) = \frac{1}{2} K_\nu(\mu_J) [\nu(\nu - 2)\Omega_0 - (\nu - \frac{1}{2})\Omega_1 + \Omega_2] - \frac{1}{2} \mu_J K_{1+\nu}(\mu_J) [2\nu\Omega_0 - \Omega_1] + \frac{1}{2} \mu_J^2 K_{2+\nu}(\mu_J) \Omega_0. \quad (59)$$

Using the $\Omega_l$ as computed in appendix B, we find that the normalization of the partition function is given by

$$Z_{N_f=1}^\nu(\hat{M}) \big|_{J=0} = I_\nu(m_1). \quad (60)$$
This result was to be expected. It coincides with the finite volume partition function for \( N_f = 1 \) as derived in [4] (without valence quarks) and is just a special case of (13). The valence quark mass dependence of the chiral condensate follows by differentiating \( \log Z^\nu_{N_f}(\hat{M}) \) with respect to the source term \( J \). This results in

\[
\frac{\Sigma(m_v; m_1)}{\Sigma_0} = \mu_v \left[ I_{\nu+1}(\mu_v)K_{\nu+1}(\mu_v) + I_{\nu+2}(\mu_v)K_{\nu}(\mu_v) \right] + \frac{\nu}{\mu_v} + 2\mu_1 \frac{K_{\nu}(\mu_v)\mu_v I_{\nu}(\mu_v)I_{\nu+1}(\mu_1) - \mu_1 I_{\nu}(\mu_1)I_{\nu+1}(\mu_v)}{\mu_v^2 - \mu_1^2}.
\]

This can easily be compared with results computed from Random Matrix Theory [44, 45] by integrating the microscopic spectral density obtained there according to eq. (10). However, what is important here is not the precise form of this mass dependent chiral condensate, but again the fact that the relation (10) can now be inverted to yield the microscopic spectral density itself. Using eq. (11) we find

\[
\rho_s(u; m_1) = \frac{u}{2} \left[ J_{\nu+1}(u)^2 - J_{\nu+2}(u)J_{\nu}(u) \right] + \mu_1 \frac{J_{\nu}(u)\mu_v I_{\nu}(\mu_v)I_{\nu+1}(u) - u I_{\nu+1}(\mu_1)J_{\nu}(u)}{(u^2 + \mu_1^2)}.
\]

This result agrees exactly with the answer obtained from Random Matrix Theory [44, 45].

One can of course continue by considering a larger number of massive flavors. But the computations rapidly get rather involved, and we have not succeeded in finding a simple compact expression for the mass dependent chiral condensate for the general case of \( N_f \) massive flavors in a sector of arbitrary topological charge \( \nu \). There is, however, one important case which can be computed rather easily: that of an arbitrary number of massless quarks in a sector of zero topological charge. In this case the calculation greatly simplifies since then \( \Omega_l = 0 \) for \( l \geq 3 \). This is readily seen from its definition in terms of the eigenvalues of \( A \) and the observation that for massless sea-quarks only two eigenvalues of \( A \) are nonzero. In this case the expression (57) reduces to

\[
Z^\nu_{N_f}(\hat{M}) = 2 \left( \frac{\mu_l}{2} \right)^n K_n(\mu_j)\Omega_0 + \left( \frac{\mu_l}{2} \right)^{n-1} K_{n-1}(\mu_j)\Omega_1,
\]

where we have used the identity (see appendix B)

\[
(n - 1)\Omega_1 + 2\Omega_2 = 0,
\]

which is valid for the case of zero sea quark masses and topological charge equal to zero. From the explicit expressions in Appendix B for \( \Omega_0 \) and \( \Omega_1 \) we obtain our final result for the partition function

\[
Z^\nu_{N_f}(\hat{M}) = (n - 1)! \left[ \mu^n_j K_n(\mu_j)\frac{I_{n-1}(m_v)}{m_{v-1}^n} + \mu^{n-1}_j K_{n-1}(\mu_j)\frac{I_n(m_v)}{m_{v-2}^n} \right].
\]
Next, from the Wronskian identity (43) we immediately find that $Z(\hat{\mathcal{M}}) = (n-1)! = N_f!$ for $J = 0$. The valence quark mass dependence of the chiral condensate follows from differentiation with respect to $J$, resulting in

$$\frac{\Sigma(m_v)}{\Sigma_0} = \mu_v \left[ I_{N_f}(\mu_v) K_{N_f}(\mu_v) + I_{N_f+1}(\mu_v) K_{N_f-1}(\mu_v) \right]. \quad (66)$$

This expression agrees with the result obtained previously from Random Matrix Theory [18]. Again our purpose here is just the opposite: we can now derive directly from field theory the microscopic spectral density of the Dirac operator from the discontinuity of $\Sigma(m_v)$ as given in eq. (66). The result is

$$\rho_s(u) = \frac{u}{2} \left[ J_{N_f}(u)^2 - J_{N_f+1}(u)J_{N_f-1}(u) \right]. \quad (67)$$

This is the celebrated result obtained first using Random Matrix Theory [25]. Indeed, this taken together with the earlier results constitute an analytical proof that the smallest eigenvalues of the QCD Dirac operator are correlated according to a Random Matrix Theory whose form is dictated by the global symmetries of the QCD Dirac operator.

8. Conclusions

In the limit of vanishing light quark masses the infrared sector of the QCD partition function is dominated by the Goldstone modes associated with the assumed spontaneous breaking of chiral symmetry. However, the usual chiral Lagrangian does not allow us to access the Dirac spectrum. For that reason one has to extend the partition function with one valence quark and its superpartner [16]. The chiral Lagrangian of this partition function is based on the super-group $Gl(N_f + 1|1) \times Gl(N_f + 1|1)$. In agreement with a supersymmetric generalization of the Vafa-Witten theorem [46] and the maximum breaking of chiral symmetry, the symmetry is broken to the diagonal subgroup $Gl(N_f + 1|1)$. As dictated by the convergence of the integrals the integration manifold is restricted to a super-Riemannian symmetric submanifold. It is characterized by a symbiosis between compact and non-compact degrees of freedom. Remarkably, precisely this balance between compact and non-compact variables in the effective Lagrangian is what leads to the correct cut structure in the complex valence quark mass plane. The supersymmetric extension of the effective chiral Lagrangian has thus passed a highly non-trivial self-consistency test. Moreover, in the present context this super-extension of the chiral Lagrangian for QCD is not used to study artifacts of (partially) quenched numerical simulations, but rather to derive physical results in standard QCD with dynamical fermions.

In a previous paper [16], three of us have shown that the microscopic spectral density can be obtained from a partition function with compact degrees of freedom only, i.e. by
replacing $Gl(1)/U(1) \to U(1)$. The integrals over the supergroups could be performed conveniently by means of a supersymmetric generalization of the Itzykson-Zuber integral. However, a similar approach applied to the direct calculation of the valence quark mass dependence of the chiral condensate only worked for the quenched approximation, i.e. for the $Gl(1|1)$ partition function, but failed in the case of a nonzero number of sea quarks. The general reason for such failure is well understood. The measure contains anomalous terms which have to be included. The appearance of these so-called Efetov-Wegner terms has its origin in the fact that the integration contour contains nilpotent terms, which after expansion in a power series, result in total derivatives. Typically, such terms contribute in the neighborhood of singularities.

In this work we have followed a different approach. We have directly calculated the $Gl(N_f + 1|1)$ partition function without relying on super-symmetric Itzykson-Zuber integrals. Technically, this calculation was possible because we found a parametrization without anomalous contributions for the observables under consideration.

In conclusion, we have shown analytically that the microscopic distribution of eigenvalues of the QCD Dirac operator can be computed directly from a supersymmetric extension of the effective finite-volume QCD partition function. The results agree exactly with the original computations which were based on chiral Random Matrix Theory. It is quite remarkable that what could appear as a forbiddingly difficult field-theory computation of the microscopic Dirac operator spectrum was first performed on the basis of universality arguments and Random Matrix Theory. What has now been proven is that these results are exact and indeed can be derived directly from field theory, without recourse to Random Matrix Theory. The underlying reason should be clear: Random Matrix Theories with the global symmetries of the QCD partition function can be reduced to the finite volume partition function that has been studied in this paper.

Acknowledgements

This work was partially supported by the US DOE grant DE-FG-88ER40388. The work of P.H.D. was supported in part by EU TMR grant ERBFMRXCT97-0122, and the work of D.T. was supported by Schweizerischer Nationalfonds. T. Guhr, A. Schäfer, A. Smilga, M. Stephanov, R. Szabo, H.A. Weidenmüller and T. Wettig are acknowledged for useful discussions. We particularly benefitted from discussions with A. Altland and M. Zirnbauer on the super-unitary measure.
Appendix A. Calculation of a determinant

In this appendix we show that

\[
\det\left( \begin{array}{cc} \delta_{kl} + a\beta_k\alpha_l & b\beta_k\beta_l \\ -b\alpha_k\alpha_l & \delta_{kl} - a\alpha_k\beta_l \end{array} \right) = (1 - 2ax^2 + (a^2 - b^2)x^4)^{-1}. \tag{68}
\]

Here, \(\alpha_k \) and \(\beta_k \) are Grassmann variables and \(a\) and \(b\) are scalar functions. We use the notations that \(\alpha_k\alpha_l\) represents an \(n \times n\) matrix with entries \(\alpha_k\alpha_l\) and \(x^2\) is defined by

\[
x^2 = \sum_{k=1}^{n} \beta_k\alpha_k. \tag{69}
\]

The proof is as follows. If we introduce the matrix

\[
A = \left( \begin{array}{cc} a\beta_k\alpha_l & b\beta_k\beta_l \\ -b\alpha_k\alpha_l & -a\alpha_k\beta_l \end{array} \right) \tag{70}
\]

the determinant is calculated by the relation

\[
\det(1 + A) = \sum_{k=1}^{\infty} \exp\left[ \frac{(-1)^{k+1}}{k} \text{Tr}A^k \right]. \tag{71}
\]

One can easily show that the powers of \(A\) have the same structure as the matrix \(A\). Therefore, and

\[
A^p = \left( \begin{array}{cc} a_p\beta_k\alpha_l & b_p\beta_k\beta_l \\ -b_p\alpha_k\alpha_l & -a_p\alpha_k\beta_l \end{array} \right). \tag{72}
\]

It is straightforward to derive a recursion relation for the coefficients \(a_p\) and \(b_p\)

\[
a_{p+1} = -ax^2a_p - bx^2b_p, \tag{73}
\]

\[
b_{p+1} = -bx^2a_p - ax^2b_p, \tag{74}
\]

with \(a_1 = 1\) and \(b_1 = b\). The solution of these recursion relations is given by

\[
a_p = \frac{1}{2}(a + b)^p(-x^2)^{(p-1)} + \frac{1}{2}(a - b)^p(-x^2)^{(p-1)}
\]

\[
b_p = \frac{1}{2}(a + b)^p(-x^2)^{(p-1)} - \frac{1}{2}(a - b)^p(-x^2)^{(p-1)}. \tag{75}
\]

By resumming the power series into a logarithm and taking the trace of the matrices, one easily recovers the expression for the determinant.
Appendix B. Some integrals over Unitary groups

In this appendix we calculate the integrals

\[ \Omega_l(\hat{M}) = \int dU \det U \nu \left( \frac{\partial y}{l!} \right) |_{y=0} \det(1 + yA) \exp \{ \text{Tr}A \} , \]  

(76)

where

\[ A = \Sigma_0 V(\hat{M}U + U^{-1}\hat{M})/2, \]  

(77)

and the integral is over the unitary group \( U(n) \). We evaluate these integrals for \( l = 0, 1 \) and \( 2 \). These integrals are of the form

\[ \int d[U] f(U, U^\dagger) \det(U)^\nu \exp \left\{ \frac{1}{2} \text{Tr}[M(U + U^\dagger)] \right\} \equiv \langle f(U, U^\dagger) \rangle \]  

(78)

with \( M \) a diagonal matrix and \( U \) ranging over the group \( U(n) \). To evaluate these integrals we use the following result \[^{17}\]

\[ W(J, J^\dagger) = \int d[U] \det(U)^\nu \exp \left\{ \text{Tr}[JU + U^\dagger J^\dagger] \right\} = \prod_{k=1}^{n-1} 2^k k! \left( \frac{\det J^\dagger}{\det J} \right)^2 \frac{\det_{ij} [z]^j_{i-1+\nu}(z_i)}{\Delta(z^2_i)} \]  

(79)

where \( z_i = 2\sqrt{\lambda_i} \) with the \( \lambda_i \) being the eigenvalues of the matrix \( J^\dagger J \) and \( \Delta(z^2_i) \) is the Vandermonde determinant \( \prod_{k\neq l}(z_k^2 - z_l^2) \). From the invariance of the measure and the fact that an arbitrary complex matrix can be diagonalized by two unitary matrices resulting in a matrix with positive diagonal elements, it follows immediately that the integral factorizes into a function of the eigenvalues of \( J^\dagger J \) and a ratio of determinants of \( J^\dagger \) and \( J \). This result can be proven naturally \[^{19}\] by means of a generalization of the Itzykson-Zuber integral to arbitrary complex matrices. We can now rewrite the original integral as

\[ \langle f(U, U^\dagger) \rangle = f \left( \frac{\partial}{\partial J} \frac{T}{T}, \frac{\partial}{\partial J^\dagger} \right) W(J, J^\dagger) \bigg|_{J=J^\dagger=M/2}. \]  

(80)

Since \( W \) contains the eigenvalues \( \lambda \), we will need to change the derivatives with respect to the sources \( J \) and \( J^\dagger \) into derivatives on \( \lambda \) by the chain rule

\[ \frac{\partial}{\partial J_{ij}} = \sum_{a=1}^{n} \frac{\partial \lambda_a}{\partial J_{ij}} \frac{\partial}{\partial \lambda_a}. \]  

(81)

The derivative of \( \lambda \) with respect to the sources can then be obtained from the identities

\[ \sum_{p=1}^{n} \chi^r_p = \text{Tr}(J^\dagger J)^r \]  

(82)
for \( r = 1, \ldots, n \). By applying derivatives with respect to the sources \( J \) and \( J^\dagger \) on both sides of the equation we get a system of equations which we can then solve for the required derivative of \( \lambda \).

As an example we calculate the integrals corresponding to \( \Omega_1 \) and \( \Omega_2 \) as defined in (70) which are used in the text. Our goal is to express them in terms of \( \Omega_0 = \langle 1 \rangle \) and its derivatives with respect to the masses. We can then use the explicit expression for \( \Omega_0 \) for the two cases considered in the text to obtain an expression for these integrals. We consider the case with \( N_f = 1 \) sea quark with mass \( m_1 \), one valence quark with mass \( m_v (n = 2) \) for topological charge \( \nu \) and the case with \( N_f \) massless flavors, one valence quark with mass \( m_v (n = N_f + 1) \) for zero topological charge (\( \nu = 0 \)). In these cases, the values of \( \Omega_0 \) are given by

\[
\Omega_0 = 2 \frac{m_v I_{1+\nu}(m_v) I_{\nu}(m_1) - m_1 I_{1+\nu}(m_1) I_{\nu}(m_v)}{m_v^2 - m_1^2},
\]

and

\[
\Omega_0 = 2^{N_f} N_f! \frac{I_{N_f}(m_v)}{m_v^{N_f}},
\]

respectively. For simplicity we absorb the factors of \( \Sigma_0 V \) into the masses in this appendix. We emphasize that in this appendix the sea quark masses and the valence quark mass enter on the same footing. We nevertheless make this distinction since it allows us to make contact with the formulas in the main text.

First we calculate

\[
\Omega_1 = \langle Tr A \rangle = \frac{1}{2} \sum_i m_i \left( \frac{\partial}{\partial J_{ii}} + \frac{\partial}{\partial J^\dagger_{ii}} \right) W(J, J^\dagger) \bigg|_{J=J^\dagger=M/2}.
\]

One can check that the derivatives acting on the term (\( \det J^\dagger / \det J \))\(^{\nu/2} \) gives no contribution to the final result. Since this calculation only requires diagonal sources, it is easy to see that the derivatives acting on the remaining term gives

\[
\Omega_1 = \sum_i m_i \frac{\partial}{\partial m_i} \Omega_0.
\]

For the two cases above, for \( N_f = 1 \) and one valence quark (\( n = 2 \)) in the sector of topological charge \( \nu \) we obtain

\[
\Omega_1 = 2 I_{\nu}(m_v) I_{\nu}(m_1) - 2 \Omega_0,
\]

and for \( N_f \) massless flavors and one valence quark (\( n = N_f + 1 \)) with zero topological charge (\( \nu = 0 \)) we find

\[
\Omega_1 = 2^{N_f} N_f! \frac{I_{N_f+1}(m_v)}{m_v^{N_f-1}}.
\]
The next integral,

$$\Omega_2 = \frac{1}{2} \left\langle (\text{Tr}A)^2 - \text{Tr}A^2 \right\rangle,$$  \hspace{1cm} (89)

includes off diagonal sources and thus becomes more complicated. The first term is similar to the previous example and can be written

$$\left\langle (\text{Tr}A)^2 \right\rangle = \sum_{i,j} m_i m_j \frac{\partial}{\partial m_i} \frac{\partial}{\partial m_j} \Omega_0.$$  \hspace{1cm} (90)

Here again the term \((\det J^\dagger / \det J)^{\nu/2}\) in \(W\) did not contribute. The second term is

$$\left\langle \text{Tr}A^2 \right\rangle = \frac{1}{4} \left\langle \text{Tr}[2M^2 + (MU)^2 + (U^\dagger M)^2] \right\rangle.$$  \hspace{1cm} (91)

We thus need to calculate \(\left\langle \text{Tr}MUMU \right\rangle\) and its conjugate. In order to evaluate these integrals we write

$$\left\langle \text{Tr}(MU)^2 \right\rangle = \sum_{i,j} m_i m_j \frac{\partial}{\partial J_{ij}} \frac{\partial}{\partial J_{ji}} W(J, J^\dagger) \bigg|_{J=J^\dagger=M/2},$$  \hspace{1cm} (92)

and similarly for its conjugate. Now the term \((\det J^\dagger / \det J)^{\nu/2}\) does make a contribution. Summing the contribution for this case and its conjugate gives \(2n\nu^2\Omega_0\). The remainder of these integrals can again be evaluated using the chain rule

$$\frac{\partial}{\partial J_{ij}} \frac{\partial}{\partial J_{ji}} = \sum_a \frac{\partial^2 \lambda_a}{\partial J_{ij} \partial J_{ji}} \frac{\partial}{\partial \lambda_a} + \sum_{a,b} \frac{\partial \lambda_a}{\partial J_{ij}} \frac{\partial \lambda_b}{\partial J_{ji}} \frac{\partial^2}{\partial \lambda_a \partial \lambda_b}.$$  \hspace{1cm} (93)

Applying the two derivatives to the identities \[(82)\] and then solving the resulting system of equations, we find (for \(i \neq j\))

$$\frac{\partial \lambda_a}{\partial J_{ij} \partial J_{ji}} = \frac{m_i m_j}{m_i^2 - m_j^2},$$  \hspace{1cm} (94)

when \(i = a\), \(i\) and \(j\) are reversed when \(j = a\), and it is zero otherwise. Putting this all together we end up with

$$\Omega_2 = \frac{1}{2} \sum_{i,j} m_i m_j \frac{\partial^2 \Omega_0}{\partial m_i \partial m_j} + \frac{1}{4} \sum_i \left( m_i \frac{\partial \Omega_0}{\partial m_i} - m_i^2 \frac{\partial^2 \Omega_0}{\partial m_i^2} - m_i^2 \Omega_0 \right) - \frac{1}{4} n\nu^2\Omega_0$$

$$- \frac{1}{2} \sum_{i \neq j} \frac{m_i m_j}{m_i^2 - m_j^2} \left( m_j \frac{\partial \Omega_0}{\partial m_i} - m_i \frac{\partial \Omega_0}{\partial m_j} \right).$$  \hspace{1cm} (95)

This can be further simplified by using the identity \[(48)\]

$$\sum_i \left[ (2n - 1)m_i \frac{\partial \Omega_0}{\partial m_i} + m_i^2 \frac{\partial^2 \Omega_0}{\partial m_i^2} - m_i^2 \Omega_0 \right] + 2 \sum_{i \neq j} \frac{m_i m_j}{m_i^2 - m_j^2} \left( m_j \frac{\partial \Omega_0}{\partial m_i} - m_i \frac{\partial \Omega_0}{\partial m_j} \right) = n\nu^2\Omega_0.$$  \hspace{1cm} (96)
As a final result we obtain
\[
\Omega_2 = -\frac{1}{2}(n - 1) \sum_i^n m_i \frac{\partial \Omega_0}{\partial m_i} + \frac{1}{2} \sum_{i,j}^n m_i m_j \frac{\partial^2 \Omega_0}{\partial m_i \partial m_j} - \sum_{i \neq j}^n \frac{m_i m_j}{m_i^2 - m_j^2} \left( m_j \frac{\partial \Omega_0}{\partial m_i} - m_i \frac{\partial \Omega_0}{\partial m_j} \right).
\]
(97)

For \( N_f = 1 \) with sea quark mass \( m_1 \) and valence quark mass \( m_v \) the expression \( \Omega_2 \) becomes for arbitrary \( \nu \)
\[\Omega_2 = -\frac{1}{2} \Omega_1 - m_v^2 \Omega_0 - \nu^2 \Omega_0 + \nu \Omega_1 + 2 \nu \Omega_0 + 2 m_v I_{1+\nu}(m_v) I_{\nu}(m_v), \]
(98)
while for \( N_f \) massless flavors and one valence quark we get for \( \nu = 0 \)
\[\Omega_2 = -\frac{1}{2} N_f \Omega_1. \]
(99)

References

[1] T. Banks and A. Casher, Nucl. Phys. B169 (1980) 103.

[2] J.C. Osborn and J.J.M. Verbaarschot, Phys. Rev. Lett. 81 (1998) 268.

[3] J. Gasser and H. Leutwyler, Phys. Lett. 188B (1987) 477; Nucl. Phys. B307 (1988) 763.

[4] H. Leutwyler and A. Smilga, Phys. Rev. D46 (1992) 5607.

[5] E.V. Shuryak and J.J.M. Verbaarschot, Nucl. Phys. A560 (1993) 306.

[6] M.A. Halasz and J.J.M. Verbaarschot, Phys. Rev. D52 (1995) 2563.

[7] J. Verbaarschot, Phys. Rev. Lett. 72 (1994) 2531; Phys. Lett. B329 (1994) 351; Nucl. Phys. B426[FS] (1994) 559.

[8] G. Akemann, P. Damgaard, U. Magnea and S. Nishigaki, Nucl. Phys. B 487[FS] (1997) 721.

[9] E. Brézin, S. Hikami and A. Zee, Nucl. Phys. B464 (1996) 411.

[10] A.D. Jackson, M.K. Sener and J.J.M. Verbaarschot, Nucl. Phys. B479 (1996) 707.

[11] T. Guhr and T. Wettig, Nucl. Phys. B506 (1997) 589.

[12] A.D. Jackson, M.K. Sener and J.J.M. Verbaarschot, Nucl. Phys. B506 (1997) 612.

[13] M.K. Sener and J.J.M. Verbaarschot, Phys. Rev. Lett. 81 (1998) 248.
[14] P.H. Damgaard, hep-th/9807026.

[15] J.J.M. Verbaarschot, Lectures given at NATO Advanced Study Institute on Confinement, Duality and Nonperturbative Aspects of QCD, Cambridge, 1997, hep-th/9710114.

[16] J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, hep-th/9806110, accepted for publication in Nucl. Phys. B.

[17] S. Chandrasekharan, Nucl. Phys. Proc. Suppl. 42 (1995) 475; S. Chandrasekharan and N. Christ, Nucl. Phys. Proc. Suppl. 42 (1996) 527; N. Christ, Nucl. Phys. B (Proc. Suppl.) 53 (1997) 253.

[18] J.J.M. Verbaarschot, Phys. Lett. B368 (1996) 137.

[19] J.J.M. Verbaarschot, in Nonperturbative Approaches to Quantum Chromodynamics, D. Diakonov, ed., Gatchina 1995.

[20] P.H. Damgaard, Phys. Lett. B424 (1998) 322; G. Akemann and P.H. Damgaard, Phys. Lett. 432 (1998) 390; G. Akemann and P.H. Damgaard, Nucl. Phys. B528 (1998) 411.

[21] A. Morel, J. Physique 48 (1987) 1111.

[22] C. Bernard and M. Golterman, Phys. Rev. D46 (1992) 853, Phys. Rev. D49 (1994) 486; M. Golterman, hep-lat/9411005, Chiral Perturbation Theory and the quenched approximation of QCD.

[23] B. Simons and A. Altland, cond-mat/9811134.

[24] M.R. Zirnbauer, J. Math. Phys. 38 (1997) 2007; J. Phys. A 29 (1996) 7113.

[25] J. Verbaarschot and I. Zahed, Phys. Rev. Lett. 70, 3852 (1993).

[26] H. Leutwyler, Ann. Phys. 235 (1994) 165.

[27] M. Zirnbauer, J. Math. Phys. 37 (1996) 4986.

[28] K.B. Efetov, Supersymmetry in disorder and chaos, Cambridge University Press, (1997); K.B. Efetov, Adv. Phys. 32 (1983) 53.

[29] F. Wegner, private communication, 1983.

[30] M.R. Zirnbauer, Nucl. Phys. B 265 [FS15] (1986) 375.

[31] T. Guhr, Nucl. Phys. A560 (1993) 223.
[32] M. J. Rothstein, Trans. Am. Math. Soc. 299, 387 (1987).

[33] J. Alfaro and L.F. Urrutia, hep-th/9810130.

[34] M. R. Zirnbauer and F. D. M. Haldane, Phys. Rev. B52 (1995) 8729.

[35] Z. Pluhár and H.A. Wiedenmüller, cond-mat/9809276.

[36] T. Guhr, J. Math. Phys. 32 (1991) 336.

[37] J. Alfaro, R. Medina and L.F. Urrutia, J. Math. Phys. 36 (1995) 3085.

[38] T. Guhr and T. Wettig, J. Math. Phys. 37 (1996) 6395.

[39] T. Guhr, J. Math. Phys. 34 (1993) 2523; ibid. 34 (1993) 2541.

[40] A.V. Andreev, B.D. Simons, and N. Taniguchi, Nucl. Phys. B432 [FS] (1994) 487.

[41] P.H. Damgaard, Phys. Lett. B425 (1998) 151.

[42] P.H. Damgaard, U.M. Heller and A. Krasnitz, hep-lat/9810060. M. Göckeler, H. Hehl, P.E.L. Rakow, A. Schäfer and T. Wettig, hep-lat/9811018.

[43] M.E. Berbenni-Bitsch, S. Meyer, A. Schäfer, J.J.M. Verbaarschot and T. Wettig, Phys. Rev. Lett. 80 (1998) 1146; M.E. Berbenni-Bitsch, S. Meyer and T. Wettig, Phys. Rev. D58 (1998) 071502.

[44] P.H. Damgaard and S.M. Nishigaki, Nucl. Phys. B518 (1998) 495.

[45] T. Wilke, T. Guhr and T. Wettig, Phys. Rev. D57 (1998) 6486; B. Seif, T. Wettig and T. Guhr, hep-th/9811044.

[46] C. Vafa and E. Witten, Nucl. Phys. B234 (1984) 173.

[47] R. Brower, P. Rossi and C-I. Tan, Nucl.Phys. B190 [FS3] (1981) 699.

[48] R. Brower and M. Nauenberg, Nucl.Phys. B180 [FS2] (1981) 221.

[49] A.D. Jackson, M.K. Sener and J.J.M. Verbaarschot, Phys. Lett. B 387 (1996) 355.