Radio emission of the Crab and Crab-like pulsars

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ABSTRACT

The pulsar radio emission is commonly associated with the plasma outflow in the open field line tube; then a pencil beam is emitted along the pulsar magnetic axis. Observations suggest that there is an additional radio emission mechanism specific for pulsars with high magnetic field at the light cylinder. These pulsars are known to be strong sources of non-thermal high energy radiation, which could be attributed to reconnection in the current sheet separating, just beyond the light cylinder, the oppositely directed magnetic fields. Pulsars with the highest magnetic field at the light cylinder (> 100 kG) exhibit also radio pulses in phase with the high energy pulses. Moreover, giant radio pulses are observed in these pulsars. I argue that the reconnection process that produces high energy emission could also be responsible for the radio emission. Namely, coalescence of magnetic islands in the sheet produces magnetic perturbations that propagate away in the form of electro-magnetic nano-shots. I estimate the parameters of this emission and show that they are compatible with observations.

Key words: magnetic reconnection – (magnetohydrodynamics) MHD – plasmas – radiation mechanisms: non-thermal – (stars:) pulsars: general

1 INTRODUCTION

Pulsars with large magnetic fields at the light cylinder differ significantly from ordinary pulsars. First of all, non-thermal high-energy emission is observed only from pulsars with the field at the light cylinder of about or larger than 1 kG (see, e.g., fig. 2 in the review by Venter et al. 2018). Moreover, the radio properties of many of them differ significantly from the properties of ordinary pulsars. The Crab pulsar is a good example. It exhibits two well separated main peaks of radio emission. Taking into account that the pulsar rotation axis is inclined to the line of sight by 60°, as follows from the X-ray image of the nebula (e.g., Ng & Romani 2004), this is incompatible with the standard presumption that the pulsar radio emission forms a pencil beam generated in the polar cap outflow and therefore directed along the magnetic axis. In the standard picture, two-peaked profiles could be formed either in orthogonal or in aligned rotators (in the last case, the double peak is formed by a hollow cone beam). Moreover, if pulses of ordinary pulsars generally widen with decreasing frequency (e.g., Graham-Smith 2003), the Crab main pulses do not exhibit any significant frequency evolution. One more specific feature of the Crab radio emission is giant pulses. The Crab twin in the LMC, PSR B0540-69, exhibits similar properties (e.g., Johnston et al. 2004). All this evidences for an emission mechanism different from that responsible for the "standard" pulsar emission.

It seems that the same specific emission mechanism operates also in many recycled (millisecond) pulsars because properties of their radio emission (Kramer et al. 1998) resemble those of the Crab. First of all, the abundance of double pulses in recycled pulsars is incompatible with the standard explanation of double-peaked pulsars as orthogonal rotators or nearly aligned rotators with a hollow cone beam. Moreover, there is no or very small frequency development of the pulses. In millisecond pulsars, the beam width is less than what is predicted by the canonical pulsar model and in some cases is even less than the size of the polar cap. The millisecond pulsars are also known as gamma-ray sources and in some of them, giant pulses are observed. One sees a similarity between the Crab-like and the recycled pulsars; therefore there should be a special radio emission mechanism associated with the high-energy emission region of pulsars. Of course the "standard" radio emission mechanism could also operate in these pulsars so that, depending on the orientation, the observer sees radiation either from the polar tube outflow, or from the site where the high energy emission is generated, or from both. For example in the Crab pulsar, the "standard" mechanism may be responsible for the precursor radio emission (e.g., Graham-Smith 2003).

The common feature of all these pulsars is a large magnetic field at the light cylinder. The model successfully explaining the high energy emission of pulsars places the emission source to the current sheet separating, just beyond the light cylinder, the oppositely directed magnetic fields (Lyubarskii 1998; Bai & Spitkovsky 2010; Arka & Dubus 2011).
The magnetic reconnection heats the particles in the sheet and if the magnetic field in the light cylinder zone is high enough, the synchrotron emission of the particles forms a powerful high-energy fan beam. Inasmuch as such a beam rotates together with the magnetosphere, the observer generally sees a double-peaked light curve.

In this paper, I propose a mechanism for the radio emission from a reconnecting current sheet in pulsars. The reconnection process occurs via formation and coalescence of magnetic islands/pinches within the current sheet (e.g., Kagan et al. 2013). Two islands coalescing with the velocity of the order of the speed of light perturb the magnetic field in the vicinity of the coalescence point thus producing magnetohydrodynamic (MHD) waves propagating away. I show that fast magneto-sonic (fms) waves escape from the magnetosphere in the form of radio waves. In particular, the coalescence of large islands could be responsible for giant nano-shots observed in the Crab pulsar (Hankins et al. 2002; Eilek & Hankins 2010).

The paper is organized as follows. In sect. 2, I present the emission mechanism and estimate the properties of nano-shots produced by a coalescence event. In sect. 3, I consider non-linear interactions between waves generated by many coalescence events and estimate the radio luminosity provided by the proposed mechanism. Discussion and conclusions are presented in sect. 4.

2 RADIO EMISSION FROM MAGNETIC ISLANDS COALESCING IN THE PULSAR CURRENT SHEET

Coalescence of two magnetic islands produces a magnetic perturbation in the vicinity of the coalescence region. Therefore MHD waves are generated around the reconnecting current sheet. There are generally three MHD waves, the Alfven wave and two magnetosonic waves. In the relativistic case, their phase velocities are presented, e.g., by Appl & Camenzind (1988). In the simplest case of a cold plasma, only the Alfven and the fast magnetosonic (fms) waves remain; their phase velocities are reduced to

\[ v_A = c \sqrt{\frac{\sigma}{1 + \sigma}} \cos \theta; \]
\[ v_{fms} = c \sqrt{\frac{\sigma}{1 + \sigma}}. \]

(1)
(2)

where \( \theta \) is the angle between the magnetic field and the direction of propagation, \( \sigma = B^2 / 4\pi pc^2 \) the magnetization parameter, \( B \) the background magnetic field, \( p \) the plasma density. The group velocity of the Alfven waves is directed along the magnetic field lines therefore, they do not transfer the energy away from the current sheet. Fast magnetosonic waves do propagate across the magnetic field lines therefore any coalescence event produces a quasi-spherical fms pulse of the duration \( \sim a / c \), where \( a \) is the transverse size of the island.

In order to demonstrate that this wave escapes from the system in the form of a vacuum electromagnetic wave, let us consider what happens to this wave when it propagates towards smaller plasma densities. In the harmonic wave with the frequency \( \omega \) and the wave vector \( k \), the electric current is found from Maxwell’s equations as

\[ j = \frac{i}{4\pi} \left( \frac{k(k \cdot E) - k^2 E}{\omega} + \omega E \right), \]

(3)

where \( E \) is the electric field of the wave. In the fms wave, the electric field is perpendicular both to the background magnetic field and to the direction of propagation therefore \( k \cdot E = 0 \). Then the ratio of the conductivity to the displacement current in the fms wave is presented, with the aid of eq. (2), as

\[ \frac{j}{i\omega E} = \frac{1}{4\pi(1 + \sigma^2)}. \]

(4)

One sees that when the ratio of the plasma to the magnetic energy density goes to zero, \( \sigma \to \infty \), the conductivity current vanishes. Therefore the wave smoothly transforms to a vacuum electro-magnetic wave when the plasma density goes to zero.

More generally, one can abandon the MHD approximation and consider the wave in the scope of the two-fluid hydrodynamics. For the electron-positron plasma, the dispersion relation for waves polarized perpendicularly both to the background magnetic field and to the direction of propagation is found as (e.g., Melrose 1997)

\[ \omega^2 = k^2 v^2 - \frac{2\omega_p^2 \omega}{\omega_B^2 - \omega^2}. \]

(5)

where \( \omega_p = \sqrt{4\pi e^2 n / m} \) is the plasma frequency, \( \omega_B = eB / mc \) the Larmor frequency, \( n \) the electron density, \( e \) and \( m \) the electron charge and mass, correspondingly. Taking into account that \( \sigma = \omega_p^2 / (2\omega_B^2) \), one sees that this equation is reduced to eq. (2) in the limit \( \omega, \omega_p \ll \omega_B \).

Numerical simulations of pulsar magnetospheres (Philippov et al. 2013) show that the plasma fills the magnetosphere and the wind inhomogeneously such that empty regions remain that propagate outwards. When the fms wave enters these regions, it becomes truly vacuum electro-magnetic wave. If the wave remains within the plasma, it eventually meets the cyclotron resonance where it could be absorbed, at least partially. The optical depth for the cyclotron absorption depends on the plasma parameters in this region (Blandford & Scharlemann 1976; Mikhailovskii et al. 1982; Lyubarskii & Petrova 1993; Luo & Melrose 2001; Petrova 2002; Fussell et al. 2002), which lies well outside the light cylinder in pulsars with a high magnetic field at the light cylinder. The question of how the radio emission passes through the resonance region is common for all pulsar radiation models; it is not addressed here. For our purpose, it is enough to notice that even if the radio emission is effectively absorbed by the plasma in the resonance layer, some radiation still could escape due to the mentioned above empty regions.

Let us now consider parameters of the emitted pulses. The pulse width in the frame moving with the plasma within the current sheet is of the order of the transverse size of magnetic islands, \( a \). In the lab frame, the duration of the pulse is

\[ \tau \sim \frac{a}{c\Gamma}. \]

(6)

where \( \Gamma \) is the Lorentz factor of the plasma flow.
within the sheet. According to the available models (e.g., Timokhin & Arons 2013), the magnetospheric plasma moves outwards along the magnetic field lines with Lorentz factors \( \sim 100 - 1000 \). When the plasma enters, via the reconnection process, into the current sheet, it is decelerated because the magnetic field lines now cross the sheet giving rise to the decelerating \( j \times B \) force. Therefore one can expect that \( \Gamma \) is less than the Lorentz factor of the plasma flow in the magnetosphere. It was found by Lyubarskii & Spitkovsky 2014) that at \( \Gamma \approx 10 \), parameters of the high energy emission from the current sheet are compatible with the observed parameters. Therefore \( \Gamma \) will be normalized as \( \Gamma = 10 \Gamma_1 \).

The size of magnetic islands scales with the width of the current sheet, \( \Delta \). In pulsars with high magnetic fields at the light cylinder, the width of the sheet is determined by the balance between the dissipative heating of the plasma in the sheet and the synchrotron cooling. According to the available rough estimates (Lyubarskii 1996; Uzdensky & Spitkovsky 2014),

\[
\Delta \sim r_e^{-1/2} \left( \frac{c}{\omega_B} \right)^{3/2} = 1.3 B_6^{-3/2} \text{m},
\]

where \( r_e \) is the classical electron radius, \( B = 10^6 B_6 \) G the magnetic field at the light cylinder.

Numerical simulations of the reconnection process show (Petropoulou et al. 2016, 2018) that the size of islands spans a wide range. The large islands are ten or even more times larger than the width of the sheet therefore, \( a \) will be normalized by 10 m, \( a = 10^8 a_3 \) cm. Now the observed duration of the pulse, eq. (6), is estimated as

\[
\tau \sim 3 a_3^{3/2} \Gamma_1 \text{ns}.
\]

The corresponding frequency is \( f \sim \tau^{-1} \sim 3 \cdot 10^8 \) Hz. In principle, one can expect that after the coalescence of two large islands, the newly born island oscillates therefore one can write, more generally, \( f \tau \approx \text{few} \). Recall that the observed nano-shots exhibit \( f \tau \approx 10 \) (Hankins et al. 2003; Eilek & Hankins 2016) however, such fine details could not be captured by the presented rough model.

The amplitude of the magnetic perturbation produced by the coalescence of the large islands is comparable with the strength of the background field therefore the total energy of the pulse may be estimated as the energy density of the background field multiplied by the volume of the island. Taking into account that the islands are in fact current ropes elongated in the direction of the current, one can take the length of the rope, \( l = \zeta a \), where \( \zeta > 1 \); then the energy of the pulse in the frame moving with the plasma in the current sheet is

\[
E' \sim \frac{B^2}{8\pi} l a^3 = \frac{\zeta B^2}{8\pi} a^3.
\]

In the lab frame, \( E = E' \Gamma \). This energy is emitted within the solid angle \( \sim \pi \Gamma^{-2} \) during the time \( \tau \). The spectral flux detected at the distance \( D = 2D_2 \) kpc is

\[
S \sim \frac{E \Gamma^2}{\pi f \tau D^2} \approx \frac{\zeta^3 B^3 a^3}{8\pi D^2} = 350 \frac{\zeta^3 B_6^3 a_3}{D_2^2} \text{Jy},
\]

where \( \zeta = 10 \zeta_1 \). One sees that the estimated temporary spectral flux are compatible with the observed parameters of giant nano-shots from the Crab pulsar (Hankins et al. 2003; Eilek & Hankins 2016).

The above estimates assume that the pulses freely escape and reach the observer. However, pulses from different coalescence events generally intersect each other above the current sheet. This would lead to non-linear interaction, which is considered in the next section.

### 3 NON-LINEAR INTERACTIONS OF FMS WAVES

An fms pulse produced by a coalescence events could pass through pulses produced in coalescence events throughout the sheet. Therefore the non-linear interaction between the pulses should be generally taken into account. The simplest is the interaction of three waves (e.g., Tsytovich 1970) satisfying the resonance conditions

\[
\omega_1 + \omega_2 = \omega; \quad k_1 + k_2 = k.
\]

Let us consider this process in the force-free limit because outside the currents sheet, the magnetic field energy significantly exceeds the plasma energy density.

In this case, the fms and the Alfven waves have the dispersion relations (see, e.g., Appendix)

\[
\omega = kc
\]

and

\[
\omega = kc |\cos \theta|, \quad \theta = \text{angle between the wave vector and the background magnetic field}. \quad \text{These fms could not satisfy the resonance conditions} \quad (11) \quad \text{however, an fms wave could decay into another fms wave and an Alfven wave or into two Alfven waves.}
\]

When considering non-linear processes, one can conveniently use the wave amplitudes, \( a_k \), defined such that

\[
n_k = |a_k|^2
\]

is the number density of quanta with the wave vector \( k \), the wave energy density being

\[
\varepsilon_k = \omega_k n_k.
\]

The electric and magnetic fields are equal in the force-free fms and Alfven waves (see, e.g., Appendix). Therefore for a harmonic wave,

\[
E = E_k \exp(i k \cdot r - i \omega_k t) + \text{c.c.},
\]

the average wave energy density is just

\[
\varepsilon_k = \frac{\overline{E^2}}{4\pi} = \frac{|E_k|^2}{2\pi}.
\]

Then

\[
a_k = \frac{E_k}{\sqrt{2\pi \omega_k}}
\]

More exactly, three fms waves could satisfy the conditions (11), if they are aligned. But it is shown in Appendix, that even in this case, the non-linear interaction vanishes because in the force-free case, fms waves do not excite either currents or charge in the plasma.

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Due to the three-wave interaction, the amplitudes of waves satisfying the resonance condition \( (11) \) vary according to equations

\[
\frac{\partial a_{k_1}}{\partial t} = V_{1k_2} a_{k_2} \quad (19)
\]

\[
\frac{\partial a_{k_2}}{\partial t} = V_{2k_1} a_{k_1} \quad (20)
\]

\[
\frac{\partial a_k}{\partial t} = V_3 a_{k_1} a_{k_2} \quad (21)
\]

In the three-wave process, annihilation of a quantum \( k \) results in creation of one quantum \( k_1 \) and one quantum \( k_2 \), so that \( n_k + n_{k_1} = \text{const} \) and \( n_k + n_{k_2} = \text{const} \) (the Manley-Rowe relations); this implies

\[
V_3 = -V_2^* = -V_1^* \equiv V_{k_1k_2k}.
\]

The matrix elements, \( V_{k_1k_2k} \), for the interaction of an fms wave with two fms waves or with an Alfven wave and an fms wave are calculated in Appendix.

The decay time of a monochromatic fms wave \( (\omega, k) \) could be estimated assuming that initially \( a_{k_1} \), \( a_{k_2} \ll a_k \). Then eliminating, e.g., \( a_{k_2} \) from eqs. \((19)\) and \((20)\), one gets

\[
\frac{\partial^2 a_k}{\partial t^2} = \left| V_{k_1k_2k} a_k \right|^2 a_k.
\]

One sees that \( a_k \) (and also \( a_{k_2} \)) grow exponentially with the characteristic time

\[
\tau = \left( \left| V_{k_1k_2k} a_k \right|^2 \right)^{-1/2}.
\]

This means that at the time scale of the order of a few \( \tau \), the waves \( k_1 \) and \( k_2 \) take a significant fraction of the initial energy. Therefore the amplitude of the initial wave, \( a_k \), decreases at the same time-scale. Substituting the estimate \((A.33)\) for the matrix element, one gets an estimate for the decay time

\[
\tau \sim \left( \frac{E_k}{B_0^2} \right)^{1/3} \quad (25)
\]

In eqs. \((19)\) and \((21)\), the waves are assumed to be monochromatic with fixed phases. In a more realistic case of wide spectra and random phases, one presents the electric field as a superposition of harmonics,

\[
E = \int \left[ E_k \exp(i k \cdot r - i \omega t) + c.c. \right] dk,
\]

such that the wave energy density is

\[
\mathcal{E} = \int \mathcal{E}_k dk,
\]

where the energy of each harmonic, \( \mathcal{E}_k \), is given by eq. \((17)\).

In this case, the wave field is described by the number density of quanta defined by eq. \((15)\), the evolution being governed by the kinetic equations

\[
\frac{\partial n_k}{\partial t} = \int W_{k_1k_2k}(n_{k_1} n_{k_2} - n_k n_k) \delta(\omega_k - \omega_{k_1} - \omega_{k_2}) dk_1 dk_2,
\]

where

\[
W_{k_1k_2k} = 2\pi |V_{k_1k_2k}|^2.
\]

The characteristic interaction time may be now estimated as

\[
\tau \sim \left( \frac{W_{k_1k_2k} n_k k^2}{B_0^2} \right)^{-1} \sim \left( \frac{\mathcal{E}}{B_0^2} \right)^{1/3} \quad (30)
\]

where \( U_0 = B_0^2/8\pi \) the energy density of the background field; \( \mathcal{E} = \omega_n n_k k^3 \) the energy density of the waves.

One sees that the interaction of fms waves inevitably produces Alfven waves. The Alfven waves evolve into a cascade transferring the energy to small scales where they eventually decay and heat the plasma \( (\text{Thompson & Blaes, 1998).} \)

Therefore if the time scale \((30)\) is smaller than the characteristic time of the system \( (\text{e.g., the escape time from the system} \), most of the initial wave energy is eventually converted to heat.

If the reconnection in the current sheet just beyond the light cylinder proceeds continuously generating many fms pulses from local coalescing events, the interaction between the pulses would transfer their energy to the plasma until the characteristic wave interaction time \((30)\) becomes equal to the escape time, \( \tau \sim R_L/c = \Omega^{-1} \). Then the radiation energy density is \( \mathcal{E} \sim (\Omega/c) U_0 \). Therefore the total luminosity \( (L) \) is estimated\( \hspace{1em} (31) \)

\[
L \sim \mathcal{E} c R_L^2 \sim \frac{U_0 R_L c^2}{\omega}.
\]

Taking into account that the pulsar spin-down power is estimated as \( L_{sd} \sim U_0 c R_L^2 \), one can present the luminosity in the form

\[
L = \frac{\Omega}{P} L_{sd} = \frac{L_{sd}}{P f},
\]

where \( P = 2\pi/\Omega \) is the pulsar rotation period, \( f \) the radio frequency. Substituting the Crab rotational period and \( f = 100 \text{ MHz} \) \((\text{the Crab pulsar spectrum is very steep so that most of the energy is emitted at low frequencies})\), one gets \( L/L_{sd} \sim 3 \cdot 10^{-7} \), which is roughly compatible with the observed Crab radio luminosity \( L \approx 7 \cdot 10^{31} \text{ erg/s} \) \((\text{e.g., Malov et al., 1994}).

4 DISCUSSION AND CONCLUSIONS

In this paper, I consider the radio emission generated by coalescence of magnetic islands in a reconnecting pulsar current sheet just beyond the light cylinder. A coalescence event produces a short fms pulse that is smoothly converted into an electro-magnetic wave when propagates towards the decreasing plasma density. The duration of the pulses, and therefore the effective emission wavelength, depends on the size of the islands, which scale with the width of the current sheet. The last is determined by the balance between the dissipative heating and synchrotron cooling. According to the estimate \((17)\), the width of the sheet rapidly grows with decreasing of the magnetic field strength, therefore the proposed mechanism works only in pulsars with a high enough magnetic field at the light cylinder.

\[2\] The non-linear interactions in the zero electric field frame are estimated in the zero electric field frame; it is not affected by plasma moving along the magnetic field lines. Just beyond the light cylinder, the magnetospheric electric and magnetic fields are of the same order but not too close to each other so that the drift velocity, \( cE/B \), is only mildly relativistic. Therefore to within factors of the order of unity, the parameters in the zero electric field frame and in the lab frame are the same.
Already for the Vela pulsar parameters, one gets the sheet width of about 100 m. Therefore most of the emission is expected to be in an extremely low frequency band. Taking into account that islands of smaller sizes are presented in the current sheet, one cannot exclude that in this case, the emission from the current sheet still could be observed in the decameter range. But this emission could hardly be observed from pulsars with larger periods. Similar considerations show that this emission could be observed from millisecond pulsars with the highest values of the magnetic field at the light cylinder exceeds 100 kG (Johnson et al. 2014; Ng et al. 2014). The giant pulses are also observed only from millisecond pulsars with B > 100 kG (Bilous et al. 2015). Therefore the presented model is consistent with observations.

It was shown in this paper that at the Crab pulsar parameters, the coalescence of large magnetic islands produces nano-shots with the energy and duration compatible with the observed giant pulses. When many nano-shots are continuously produced in the current sheet, non-linear interactions between them transform most of the energy into heat, the average luminosity being determined by the condition that the system (in our case, the near zone of the pulsar wind with the size of the order of the light cylinder radius) is marginally transparent for non-linear interactions. The Crab radio luminosity estimated from these considerations is compatible with that observed.

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APPENDIX. THREE-WAVE INTERACTIONS IN THE FORCE-FREE MHD

The general theory of the nonlinear wave interactions in the force-free MHD has been developed by Thompson & Blaes (1998). Here I calculate straightforwardly the interaction rates.

The force-free MHD equations are written as

\[ \rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B} = 0; \quad (A.1) \]
\[ \mathbf{E} \cdot \mathbf{B} = 0. \quad (A.2) \]

They should be complemented by Maxwell’s equations

\[ \nabla \cdot \mathbf{E} = 4\pi \rho; \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}; \quad (A.3) \]
\[ \nabla \cdot \mathbf{B} = 0; \quad \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}. \quad (A.4) \]

Let the background magnetic field be directed along z-axis, \( \mathbf{B}_0 = B_0 \hat{z} \). Assuming that perturbations are small, one can solve eqs. (A.1) and (A.2) perturbatively, \( \mathbf{E} = \mathbf{E}^{(1)} + \mathbf{E}^{(2)} + \ldots; \quad \mathbf{B} = \mathbf{B}^{(1)} + \mathbf{B}^{(2)} + \ldots. \)
Substituting this solution into Maxwell’s equations, one gets
\begin{equation}
\mathbf{j}^{(1)} \times \mathbf{\hat{z}} = 0; \quad (A.5)
\end{equation}
\begin{equation}
E_z^{(1)} = 0. \quad (A.6)
\end{equation}
Making use of Maxwell’s equations, one expresses the current, \( j^{(1)} \), via the fields; then eq. (A.5) is written, for a harmonic wave, as
\begin{equation}
[(\omega^2 - k^2c^2)E_k^{(1)} + c^2(\mathbf{k} \cdot E_k^{(1)})\mathbf{k}] \times \mathbf{\hat{z}} = 0. \quad (A.7)
\end{equation}
The set of equations (A.6) and (A.7) has two solutions.

The first solution is polarized perpendicularly to the \( \mathbf{kB}_0 \) plane so that one can write the electric field of the wave as
\begin{equation}
E_k^{(1)} = \frac{\mathbf{\hat{z}} \times \mathbf{k}}{k \sin \theta} E_k^{(1)}; \quad (A.8)
\end{equation}
where \( \theta \) is the angle between the wave vector and the background magnetic field. Then eq. (A.7) yields
\begin{equation}
\omega = kc. \quad (A.9)
\end{equation}
Substituting this solution into Maxwell’s equations, one gets
\begin{equation}
B_k^{(1)} = \frac{1}{k} \mathbf{k} \times E_k^{(1)} = \frac{\mathbf{\hat{z}} - \cos \theta k}{k \sin \theta} E_k^{(1)}; \quad (A.10)
\end{equation}
\begin{equation}
\rho^{(1)} = j^{(1)} = 0. \quad (A.11)
\end{equation}
This is the fms wave.

The second solution is polarized in the \( \mathbf{kB}_0 \) plane, so that one writes
\begin{equation}
E_k^{(1)} = \frac{\mathbf{k} - \mathbf{\hat{z}} \cos \theta}{k \sin \theta} E_k^{(1)}; \quad (A.12)
\end{equation}
In this case,
\begin{equation}
\omega = kc\cos \theta; \quad (A.13)
\end{equation}
\begin{equation}
B_k^{(1)} = \frac{(\mathbf{k} \cdot \mathbf{\hat{z}})}{\omega k \sin \theta} \mathbf{\hat{z}} \times \mathbf{k} E_k^{(1)}; \quad (A.14)
\end{equation}
\begin{equation}
\rho_k^{(1)} = \frac{i k \sin \theta E_k^{(1)}}{4\pi}, \quad j_k^{(1)} = \text{sgn}(\cos \theta) c \rho_k^{(1)} \mathbf{\hat{z}}. \quad (A.15)
\end{equation}
This is the Alfven wave.

In the second order, eqs. (A.1) and (A.2) are written as
\begin{equation}
\rho^{(1)}E^{(1)} + \frac{1}{c} j^{(2)} \times \mathbf{E}^{(1)} + j^{(2)} \times \mathbf{\hat{z}} B_0 = 0 \quad (A.16)
\end{equation}
\begin{equation}
\mathbf{E}^{(1)} + \mathbf{B}^{(1)} + B_\phi \mathbf{E}^{(2)} \mathbf{\hat{z}} = 0. \quad (A.17)
\end{equation}
In this order, the wave amplitudes slowly vary with time therefore Maxwell’s equations should be presented, in Fourier components, as
\begin{equation}
i \omega B_k^{(2)} - \frac{\partial B_k^{(1)}}{\partial t} = i k \times E_k^{(2)}; \quad (A.18)
\end{equation}
\begin{equation}
\frac{4\pi}{c} j_k^{(2)} = i k \times B_k^{(2)} + i \omega E_k^{(2)} - \frac{\partial E_k^{(1)}}{\partial t} \quad (A.19)
\end{equation}
\begin{equation}
= \frac{1}{i \omega} \left\{ \left( \omega^2 - k^2 \right) E_k^{(2)} + (\mathbf{k} \cdot E_k^{(2)}) \mathbf{k} \right\} + \frac{1}{i \omega^2} \left\{ \left( \frac{\partial E_k^{(1)}}{\partial t} \right) \mathbf{k} - (\omega^2 + k^2) \frac{\partial E_k^{(1)}}{\partial t} \right\}. \quad (A.20)
\end{equation}
The Fourier transform of eq. (A.10) yields
\begin{equation}
\sum_{k^\prime} \left( \rho_k^{(1)} E_{k-k^\prime}^{(1)} + j_k^{(1)} \times B_{k-k^\prime}^{(1)} \right) + B_0 j_k^{(2)} \times \mathbf{\hat{z}} = 0. \quad (A.21)
\end{equation}
For waves satisfying the resonance condition (11), it is reduced to
\begin{equation}
\rho_k^{(1)} E_{k-k}^{(1)} + j_k^{(1)} \times B_{k-k}^{(1)} + j_{k^1} \times B_{k^1-k}^{(1)} = B_0 \mathbf{\hat{z}} \times j_k^{(2)}. \quad (A.22)
\end{equation}
Now let us consider specific cases.

Interaction of three fms waves. In fms waves, the current and charge density vanish in the first approximation, therefore for three fms waves, eq. (A.10) is reduced to
\begin{equation}
\mathbf{j}^{(2)} \times \mathbf{\hat{z}} = 0. \quad (A.23)
\end{equation}
The non-linear current (A.20) for the fms wave is found, by applying eqs. (A.18) and (A.19), as
\begin{equation}
\frac{4\pi}{c} j_k^{(2)} = i \left( \frac{k \cdot E_k^{(2)}}{k} \right) k - 2 \frac{\partial E_k^{(1)}}{\partial t} \mathbf{\hat{z}} \times k. \quad (A.24)
\end{equation}
Substituting this expression into eq. (A.23) and making a dot product of this equation with \( k \) in order to kill the term with \( E_k^{(2)} \) yields \( \frac{\partial E_k^{(1)}}{\partial t} = 0 \) therefore three fms waves do not interact.

Decay of an fms wave into an fms and an Alfven waves. Taking into account eqs. (A.11) and (A.15), one writes eq. (A.22) as
\begin{equation}
\rho_k^{(1)} \left( E_k^{(1)} + \text{sgn}(\cos \theta) \mathbf{\hat{z}} \times B_k^{(1)} \right) = B_0 \mathbf{\hat{z}} \times j_k^{(2)}; \quad (A.25)
\end{equation}
where \( \mathbf{k}_1 \) is for the Alfven wave and \( \mathbf{k}_2 \) for the fms wave. Making a dot product of this equation with \( k \) and applying eqs. (A.5, A.10, A.12, A.15, and A.24) yields
\begin{equation}
\frac{\partial E_k^{(1)}}{\partial t} = -i \left( \frac{\mathbf{k} \cdot \mathbf{E}_k^{(1)}}{2} - \frac{\partial E_k^{(1)}}{\partial t} \mathbf{k} \cdot \mathbf{E}_k^{(1)} \right). \quad (A.26)
\end{equation}
This equation is reduced to the form of (24) by transforming, according to eq. (15), the wave amplitudes from \( E_k \) to \( \mathbf{k} \); then one gets the expression for the matrix coefficient
\begin{equation}
\phi_{k_1k_2k} = \left( \frac{\pi}{2} \right)^{1/2} \left( \frac{k_1 c}{2k_2} \right)^{3/2} \sin \theta \left[ -1 - \text{sgn}(\cos \theta_1) \text{sgn}(\cos \theta_2) \right] \frac{(k_2 \times k_1) \cdot \mathbf{\hat{z}}}{B_0 \cos \theta_1 \cos \theta_2}. \quad (A.28)
\end{equation}
Decay of an fms wave into two Alfven waves. Now eq. (A.22) is written, with account of eq. (15), as
\begin{equation}
\rho_k^{(1)} \left( E_k^{(1)} + \text{sgn}(\cos \theta_1) \text{sgn}(\cos \theta_2) \right) = -i \left( \frac{\mathbf{k} \cdot \mathbf{E}_k^{(1)}}{2} \right) \left( \frac{k_1 c}{2k_2} \right)^{3/2} \sin \theta \left[ -1 - \text{sgn}(\cos \theta_1) \text{sgn}(\cos \theta_2) \right] \frac{(k_2 \times k_1) \cdot \mathbf{\hat{z}}}{B_0 \cos \theta_1 \cos \theta_2}. \quad (A.29)
\end{equation}
where both \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) are for the Alfven waves. One sees that the decay is possible only if the Alfven waves propagate in the opposite directions with respect to the background magnetic field,
\begin{equation}
\cos \theta_1 \cos \theta_2 < 0. \quad (A.30)
\end{equation}
This is a partial case of the general fact that Alfven waves propagating in the same direction do not interact with each other. Let us assume that this condition is satisfied; then making a dot product of eq. (A.29) with \( k \) and applying eqs. (A.13, A.15, and A.24), one gets
\begin{equation}
\frac{\partial E_k^{(1)}}{\partial t} = i \left( \frac{c}{B_0} \left( k_1 \sin \theta_1 \cos \phi_2 + k_2 \sin \theta_2 \cos \phi_1 \right) (E_k^{(1)} \cdot E_k^{(1)}) \right). \quad (A.31)
\end{equation}
Here I used the coordinate system such that

\[ k_y = 0 \] so that

\[ k = k \left( \sin \theta, 0, \cos \theta \right), \quad k_i = k_i \left( \sin \theta_i, \cos \phi_i, \sin \phi_i, \cos \theta_i \right). \]

Transforming, according to eq. [18], the wave amplitudes from \( E_k \) to \( a_k \), yields the expression for the matrix coefficient

\[
V_{S \rightarrow A+A}^{k_1 k_2 k} = i \left( \frac{2 \pi \omega_1 \omega_2 c}{k} \right)^{1/2} \frac{k_1 \sin \theta_1 \cos \phi_2 + k_2 \sin \theta_2 \cos \phi_1}{B_0}.
\]  \hspace{1cm} (A.32)

One sees from the resonance conditions and the dispersion relations, that the frequencies of all three interacting waves are generally of the same order and the angles are generally of the order of unity. Then one gets a rough estimate for the matrix coefficients

\[
|V_{S \rightarrow A+A}^{k_1 k_2 k}| \sim |V_{S \rightarrow S+A}^{k_1 k_2 k}| \sim \frac{\omega^{3/2}}{B_0}.
\]  \hspace{1cm} (A.33)