The general subclass of the extended double-Kerr solution describing the exterior field of two spinning sources in gravitational equilibrium is presented in the physical parametrization involving individual masses and angular momenta of the constituents. The analytical formulae relating equilibrium states in the double-Kerr and double-Reissner-Nordström configurations are also obtained.

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I. INTRODUCTION

Although the general equilibrium problem for the extended double-Kerr (EDK) solution was solved more than a decade ago [1,2], the physical parametrization of the 4-parameter family of equilibrium configurations in terms of the Komar quantities [4] has not yet been obtained up to date. This can be explained by numerous technical difficulties that one has to overcome for being able to express all the “canonical” parameters of the EDK solution and various associated constant quantities in terms of the physical parameters. Recently, nonetheless, we have succeeded in finding the desired reparametrization for a 3-parameter equilibrium configuration [5] that describes a Schwarzschild black hole levitating in the field of a superextreme Kerr source, and have studied physical effects in that binary system. To reach a more ambitious goal, in the present paper we will reparametrize the entire 4-parameter family of the EDK equilibrium configurations in terms of the Komar physical quantities. This will be done with the aid of two sets of the inversion formulae involving parameters of the solution and individual physical characteristics of the constituents.

In the next section we briefly review the EDK equilibrium configurations and present the first set of inversion formulae. The Komar individual characteristics of the constituents and their relation to the “canonical” parameters of the EDK solution are discussed in section 3. The general 4-parameter family of equilibrium binary systems determined by the EDK solution is reparametrized in physical parameters in section 4; here, in particular, we obtain a physical representation for the Dietz-Hoenselaers (DH) solution [6] describing two identical corotating superextreme Kerr sources in gravitational equilibrium. In section 5 we give a simple new proof of the absence of balance between two Kerr black holes, and also derive formal analytical formulae relating the equilibrium states in the EDK and double-Reissner-Nordström [6] solutions. The paper ends with concluding remarks in section 6.

II. SOLUTION OF THE EDK EQUILIBRIUM PROBLEM IN “CANONICAL” PARAMETERS AND THE FIRST SET OF INVERSION FORMULAE

The main advantage of the EDK solution over the non-extended one originally obtained by Kramer and Neugebauer for two black-hole constituents [8] consists in a remarkable possibility of its use for solving in a unified manner the equilibrium problem for any combination of two Kerr sources – black holes or superextreme objects. Such a possibility becomes feasible due to the presence in the EDK solution of the parameters \(\alpha\), which can assume arbitrary real values or occur in complex conjugate pairs. A pair of two real \(\alpha\) then naturally determines an underextreme Kerr constituent (a black hole if its mass is positive), while a complex conjugate pair defines a superextreme constituent (the four main types of binary configurations are shown in Fig. 1).

The equilibrium configurations in the EDK solution are defined by an Ernst complex potential \(\mathcal{E}\) [8] of the following form [2]:

\[
\mathcal{E} = \frac{\Lambda + \Gamma}{\Lambda - \Gamma}, \quad \Lambda = \sum_{1 \leq i < j \leq 4} \lambda_{ij} r_i r_j, \quad \Gamma = \sum_{i=1}^{4} \gamma_i r_i,
\]

\[
\lambda_{ij} = (-1)^{i+j}(\alpha_i - \alpha_j)(\alpha_i - \alpha_{j'})X_i X_{j'}, \quad (i', j' \neq i, j; \; i' < j')
\]

\[
\gamma_i = (-1)^i(\alpha_{i'} - \alpha_{j'})(\alpha_{i'} - \alpha_{k'})X_i, \quad (i', j', k' \neq i; \; i' < j' < k')
\]

\[
r_i = \sqrt{\rho^2 + (z - \alpha_i)^2},
\]

where the parameters \(\alpha_i, \; i = 1, 2, 3, 4\), as was already mentioned, occur as arbitrary real constants or complex
conjugate pairs, and \( X_i \) are given by the formulae

\[
X_1 = \varphi \frac{\epsilon_1 \omega_1 - \varphi}{1 - \epsilon_1 \omega_1 \varphi}, \quad X_2 = \varphi \frac{1 - \epsilon_1 \omega_1 \varphi}{\epsilon_1 \omega_1 - \varphi}, \quad X_3 = -\varphi \frac{1 + i \epsilon_4 \omega_4 \varphi}{i \epsilon_4 \omega_4 - \varphi}, \quad X_4 = \varphi \frac{i \epsilon_4 \omega_4 - \varphi}{1 + i \epsilon_4 \omega_4 \varphi}.
\]

\[
\omega_1 = \sqrt{\frac{(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4)}{(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_4)}}, \quad \omega_4 = \sqrt{\frac{(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_4)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)}}.
\]

(2)

the complex constant \( \varphi \) being subject to the constraint \( |\varphi|^2 \equiv \varphi \bar{\varphi} = 1 \) (a bar over a symbol means complex conjugation), while \( \epsilon_1 = \pm 1 \) and \( \epsilon_4 = \pm 1 \).

The potential \( \mathcal{E} \) defined by (1)-(2) is an exact solution of the Ernst equation \( 4 \) constructed with the aid of Sibgatullin’s method \( 10, 11 \), and the entire metric associated with this potential has the form \( 1, 12 \)

\[
ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2,
\]

\[
f = \frac{\Lambda \Lambda - \Gamma \Gamma}{(\Lambda - \Gamma)(\Lambda + \Gamma)}, \quad e^{2\gamma} = \frac{\Lambda \Lambda - \Gamma \Gamma}{\lambda_0 \lambda_0 r_1 r_2 r_3 r_4}, \quad \omega = 2\text{Im}(\sigma_0) - \frac{2\text{Im}[G(\bar{\Lambda} - \bar{\Gamma})]}{\Lambda \Lambda - \Gamma \Gamma},
\]

\[
G = \varepsilon \Gamma + \sum_{1 \leq i < j \leq 4} (\alpha_i + \alpha_j) \lambda_{ij} r_i r_j - \frac{4}{3} \sum_{i=1}^{4} (\alpha_i \varphi' + \alpha_{i'} \alpha_{k'} \gamma_{i'} r_i),
\]

\[
\lambda_0 = \sum_{1 \leq i < j \leq 4} \lambda_{ij}, \quad \gamma_0 = \sum_{i=1}^{4} \gamma_i, \quad \sigma_0 = \frac{1}{\lambda_0} [\gamma_0 + \sum_{1 \leq i < j \leq 4} (\alpha_i + \alpha_j) \lambda_{ij}].
\]

(3)

Mention that the Weyl-Papapetrou cylindrical coordinates \( \rho \) and \( z \) enter into the potential \( \mathcal{E} \) from (1) and into the metric coefficients \( f, \gamma, \omega \) from (3) only through the functions \( r_i \).

Formulae (1)-(3) represent a “canonical” form of the solution describing equilibrium configurations in the EDK spacetime. In order to rewrite them in physical parameters, we find it helpful first to express the parameters \( \alpha_i \) in terms of the quantities \( \omega_1 \) and \( \omega_4 \). For this purpose we introduce two additional constants, \( z_0 \) and \( s \), defined as

\[
z_0 \equiv \frac{1}{4}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \quad s \equiv \frac{1}{2}(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4),
\]

(4)

the constant \( z_0 \) permitting one to make an appropriate choice of the origin of coordinates on the symmetry axis, and \( s \) being the relative coordinate distance between the centers of the two constituents.
The sets of $\alpha$s describing each type of the binary system in Fig. 1 are the following (the notation is obvious):

$$
A_{BB} = \{ \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4 \},
$$
$$
A_{BS} = \{ \alpha_1 > \alpha_2 > \text{Re}(\alpha_3) = \text{Re}(\alpha_4), \text{Im}(\alpha_3) < 0, \alpha_4 = \bar{\alpha}_3 \},
$$
$$
A_{SB} = \{ \text{Re}(\alpha_1) = \text{Re}(\alpha_2) > \alpha_3 > \alpha_4, \text{Im}(\alpha_1) < 0, \alpha_2 = \bar{\alpha}_1 \},
$$
$$
A_{SS} = \{ \text{Re}(\alpha_1) = \text{Re}(\alpha_2) > \text{Re}(\alpha_3) = \text{Re}(\alpha_4), \text{Im}(\alpha_1) < 0, \text{Im}(\alpha_3) < 0, \alpha_2 = \bar{\alpha}_1, \alpha_4 = \bar{\alpha}_3 \}.
$$

(5)

The proposed change of parametrization is going to transform the above sets into the new ones, namely,

$$
A_{BB} \rightarrow \Omega_{(0,0)},
$$
$$
A_{BS} \rightarrow \Omega_{(0,-1)} \cup \Omega_{(0,+1)},
$$
$$
A_{SB} \rightarrow \Omega_{(-1,0)} \cup \Omega_{(+1,0)},
$$
$$
A_{SS} \rightarrow \Omega_{(-1,-1)} \cup \Omega_{(+1,-1)} \cup \Omega_{(-1,+1)},
$$

(6)

where

$$
\Omega_{(0,0)} = \{ \omega_1 > 1, \omega_4 > 1 \},
$$
$$
\Omega_{(0,-1)} = \{ \omega_1 > 1, \omega_4 \omega_4 = 1, \text{Im}(\omega_4) < 0, \text{Re}(\omega_4) > 0 \},
$$
$$
\Omega_{(0,+1)} = \{ \omega_1 > 1, \omega_4 \omega_4 = 1, \text{Im}(\omega_4) > 0, 1/\omega_1 > \text{Re}(\omega_4) \geq 0 \},
$$
$$
\Omega_{(-1,0)} = \{ \omega_4 > 1, \omega_1 \omega_4 = 1, \text{Im}(\omega_4) < 0, \text{Re}(\omega_1) > 0 \},
$$
$$
\Omega_{(+1,0)} = \{ \omega_4 > 1, \omega_1 \omega_4 = 1, \text{Im}(\omega_4) > 0, 1/\omega_4 > \text{Re}(\omega_1) \geq 0 \},
$$
$$
\Omega_{(-1,-1)} = \{ \omega_1 \omega_4 = 1, \text{Im}(\omega_4) < 0, \text{Re}(\omega_1) < 0, \text{Re}(\omega_4) > 0, \text{Im}(\omega_4) > 0, \text{Re}(\omega_1) > 0, \text{Re}(\omega_4) \geq 0 \},
$$
$$
\Omega_{(+1,-1)} = \{ \omega_1 \omega_4 = 1, \text{Im}(\omega_4) < 0, \text{Re}(\omega_4) > 0, \text{Re}(\omega_1) \geq 0 \},
$$
$$
\Omega_{(-1,+1)} = \{ \omega_1 \omega_4 = 1, \text{Im}(\omega_4) < 0, \text{Re}(\omega_1) > 0, \text{Re}(\omega_4) \geq 0 \},
$$

(7)

and also

$$
-\infty < z_0 < +\infty, \quad s > 0
$$

(8)

for all $\Omega$s. Note that the subindexes in $\Omega$s have been designed in such a way that they provide one with the information about the presence of a black-hole constituent (0) and the sign of the imaginary part of $\omega_1$ or $\omega_4$.

The inverse parameter change, i.e. the one that maps $\Omega$s into the original $A$’s, can be described by means of the following bi-valued relations ($\delta = \pm 1$):

$$
\alpha_1 = z_0 + \frac{s}{2} + s \frac{\delta \omega_4 (\omega_1^2 - 1)}{(\omega_1 + \delta \omega_4)(1 + \delta \omega_1 \omega_4)},
$$
$$
\alpha_2 = z_0 + \frac{s}{2} - s \frac{\delta \omega_4 (\omega_1^2 - 1)}{(\omega_1 + \delta \omega_4)(1 + \delta \omega_1 \omega_4)},
$$
$$
\alpha_3 = z_0 - \frac{s}{2} + s \frac{\omega_1 (\omega_1^2 - 1)}{(\omega_1 + \delta \omega_4)(1 + \delta \omega_1 \omega_4)},
$$
$$
\alpha_4 = z_0 - \frac{s}{2} - s \frac{\omega_1 (\omega_1^2 - 1)}{(\omega_1 + \delta \omega_4)(1 + \delta \omega_1 \omega_4)}.
$$

(9)

It is of course understood that for each $\Omega$ one has to use only one of the two branches in the above formulae, and the criterion of choosing the appropriate branch is very simple: if one of the two subindexes of $\Omega$ is equal to +1 (the imaginary part of any of the two $\omega$s is positive) then one has to use the branch $\delta = -1$, if not – then the branch $\delta = +1$.

We now turn to the consideration of the physical Komar quantities associated with the EDK solution.

### III. Komar Masses and Angular Momenta. The Second Set of Inversion Formulae

Explicit analytical formulae for the physical masses and angular momenta of the balancing constituents in the EDK solution were obtained in the paper [2]. The Komar masses $m_u$ and $m_d$ (the subindexes “u” and “d” are abbreviations
from “up” and “down”, referring to the location of the upper and lower constituents on the symmetry axis) are given by the formulae

\[ m_u = -s \frac{C(C_1 - C)}{CC_1 + SC_4 - 1 + \epsilon \delta CS}, \]
\[ m_d = -s \frac{S(C_4 - S)}{CC_1 + SC_4 - 1 + \epsilon \delta CS}, \]  
(10)

while Komar angular momenta \( j_u \) and \( j_d \) are defined by the expressions

\[ a_u \equiv \frac{j_u}{m_u} = s \frac{\epsilon \delta C[(C - \epsilon \delta S)C_1 - 1 + \epsilon \delta CS]}{(C_1 + \epsilon \delta C_4)(CC_1 + SC_4 - 1 + \epsilon \delta CS)}, \]
\[ a_d \equiv \frac{j_d}{m_d} = s \frac{S[(S - \epsilon \delta C)C_4 - 1 + \epsilon \delta CS]}{(C_1 + \epsilon \delta C_4)(CC_1 + SC_4 - 1 + \epsilon \delta CS)}, \]  
(11)

where the new constants \( C, S, C_1, C_4 \) and \( \epsilon \) are introduced via the relations

\[ \varphi \equiv C + iS, \quad C_1 \equiv \frac{1}{2} \epsilon \delta \left( \frac{1}{\omega_1} + \frac{1}{\omega_1} \right), \quad C_4 \equiv \frac{1}{2} \epsilon \delta \left( \frac{1}{\omega_4} + \frac{1}{\omega_4} \right), \quad \epsilon \equiv \epsilon_1 \epsilon_4. \]  
(12)

The above Komar quantities (10) and (11) constitute a set of four parameters with a clear physical meaning. Then a question arises, whether these quantities can be used for parametrizing the equilibrium solution? Remarkably, the answer is yes, and the best practical way to do this is by means of the following inversion formulae:

\[ C_1 = C - \epsilon \delta \frac{m_u}{M + s}, \]
\[ C_4 = S - \epsilon \delta \frac{m_d}{M + s}, \]
\[ C = \kappa \frac{M + s + \epsilon \delta a_u}{\sqrt{(M + s + \epsilon \delta a_u)^2 + (M + s + \epsilon \delta a_d)^2}}, \]
\[ S = \kappa \frac{\epsilon \delta (M + s + \epsilon \delta a_d)}{\sqrt{(M + s + \epsilon \delta a_u)^2 + (M + s + \epsilon \delta a_d)^2}}, \]  
(13)

where \( \kappa = \pm 1 \), while \( s \) satisfies the quadratic equation

\[ s^2 + [2M + \epsilon \delta (a_u + a_d)]s + M^2 + \epsilon \delta J = 0, \quad M \equiv m_u + m_d, \quad J \equiv j_u + j_d. \]  
(14)

Note that Eq. (14), after rewriting it in the form

\[ \epsilon \delta (M + s)^2 + s(a_u + a_d) + J = 0, \]  
(15)

can be immediately recognized as the equilibrium law for two arbitrary Kerr constituents originally derived in our paper [3].

Mention that the \( \kappa \) sign, to be congruent with all our previous conventions, has to be chosen in such a way that \( C > 0 \). It is also clear that, since \( s > 0 \), the admissible values of the masses and angular momenta are those that correspond to at least one positive \( s \) in Eq. (14).

Therefore, the set \((z_0, m_u, m_d, j_u, j_d)\) can be used for parametrizing the equilibrium class of the EDK solution. Apparently, the constant \( z_0 \) can be always fixed at some specific value, for instance if one wants to bring the origin of coordinates into the center of mass or into some other point related to a concrete binary configuration that might look attractive from the physical standpoint.

### IV. PHYSICAL PARAMETRIZATION OF \( \alpha_i \) AND \( X_i \). THE METRIC FUNCTIONS

In order to rewrite the complex potential (9) and corresponding metric (8) in the physical parameters, it is necessary to find the reparametrized form of the quantities \( \alpha_i \) and \( X_i \). As it follows from (9), the constants \( \alpha_i \) can be written in the form

\[ \alpha_1 = z_0 + \frac{s}{2} + \sigma_u, \quad \alpha_2 = z_0 + \frac{s}{2} - \sigma_u, \quad \alpha_3 = z_0 - \frac{s}{2} + \sigma_d, \quad \alpha_4 = z_0 - \frac{s}{2} - \sigma_d, \]  
(16)
where

\[ \sigma_u = s \frac{\delta \omega_1 (\omega_1^2 - 1)}{(\omega_1 + \delta \omega_1) (1 + \delta \omega_1 \omega_1)}, \quad \sigma_d = s \frac{\omega_1 (\omega_1^2 - 1)}{(\omega_1 + \delta \omega_1) (1 + \delta \omega_1 \omega_1)}. \]

The desired “physical” form of \( \sigma_u \) and \( \sigma_d \) is then obtainable with the aid of formulae \([12, 14]\), yielding after tedious but straightforward algebraic manipulations the following final expressions:

\[ \sigma_u = \sqrt{m_u^2 - a_u^2 + m_u a_u \frac{a_u (M + m_u + 2s) - 2m_u [a_u + \varepsilon (M + s)]}{(M + s)^2}}, \]

\[ \sigma_d = \sqrt{m_d^2 - a_d^2 + m_d a_d \frac{a_d (M + m_d + 2s) - 2m_d [a_d + \varepsilon (M + s)]}{(M + s)^2}}, \]

where \( \varepsilon \equiv \epsilon \delta \). The above \( \sigma_u \) and \( \sigma_d \) differ considerably from \( \sigma = \sqrt{m^2 - \alpha^2} \) of a single Kerr source \([13]\) due to interaction of the constituents. It is worth mentioning that in the case of the real-valued \( \alpha \), say \( \alpha_1 \) and \( \alpha_2 \), the corresponding \( \sigma_u^2 > 0 \); however, if \( \alpha_2 = \alpha_1 \) then \( \sigma_u^2 < 0 \) and one must use the convention \( \sigma_u = -i \sqrt{-\sigma_u^2} \) if one wants to pass to a positive definite radicand in \([15]\). In Fig. 2 we have shown two reparametrized equilibrium configurations for which the origin of coordinates is chosen at the center of the lower constituent \((z_0 = s/2)\).

![FIG. 2: Physical reparametrization of the equilibrium configurations (a) and (b) from Fig. 1.](image)

In a similar manner, by using \([12, 14]\), it is possible to rewrite formulae \([2]\) in terms of the Komar quantities; below we give the resulting reparametrized form of \( X_i \):

\[ X_1 = \frac{(M + s + \varepsilon a_d) (M + s - i \varepsilon m_u) + i \varepsilon (M + s) \sigma_u}{(M + s + \varepsilon a_d) (M + s + i \varepsilon m_u) - i \varepsilon (M + s) \sigma_u}, \]

\[ X_2 = \frac{(M + s + \varepsilon a_d) (M + s - i \varepsilon m_u) - i \varepsilon (M + s) \sigma_u}{(M + s + \varepsilon a_d) (M + s + i \varepsilon m_u) + i \varepsilon (M + s) \sigma_u}, \]

\[ X_3 = \frac{(M + s + \varepsilon a_u) (M + s - i \varepsilon m_d) + i \varepsilon (M + s) \sigma_d}{(M + s + \varepsilon a_u) (M + s + i \varepsilon m_d) - i \varepsilon (M + s) \sigma_d}, \]

\[ X_4 = \frac{(M + s + \varepsilon a_u) (M + s + i \varepsilon m_d) - i \varepsilon (M + s) \sigma_d}{(M + s + \varepsilon a_u) (M + s - i \varepsilon m_d) + i \varepsilon (M + s) \sigma_d}. \]

\[ (19) \]
An alternative way of writing $X_i$ which may be advantageous for some calculations is this:

$$X_1 = \frac{1}{m_u \Delta} [(M + s)(m_u + i \varepsilon u) - \varepsilon (sa_u - m_u a_d)],$$

$$X_2 = \frac{1}{m_u \Delta} [(M + s)(m_u - i \varepsilon u) - \varepsilon (sa_u - m_u a_d)],$$

$$X_3 = \frac{i}{m_d \Delta} [(M + s)(\varepsilon m_d + i \sigma_d) - sa_d + m_d a_u],$$

$$X_4 = \frac{i}{m_d \Delta} [(M + s)(\varepsilon m_d - i \sigma_d) - sa_d + m_d a_u],$$

$$\Delta = -(M + s + \varepsilon a_u) + i[\varepsilon (M + s) + a_d].$$

(20)

Now we are able to write down the reparametrized complex potential [14-2]: its new simple representation is the following:

$$\mathcal{E} = E_\pm / E_+, \quad E_\mp = [s^2 - (\sigma_u + \sigma_d)^2](X_1 r_1 - X_2 r_2) + 2 \sigma_u)(X_3 r_3 - X_4 r_4) + 2 \sigma_d)$$

$$\Gamma = 2 \sigma_u [(s + \sigma_u)^2 - \sigma_d^2] X_2 r_2 - [(s - \sigma_u)^2 - \sigma_d^2] X_1 r_1$$

$$\alpha_1 = \frac{3}{2} + \sigma_u, \quad \alpha_2 = \frac{3}{2} - \sigma_u, \quad \alpha_3 = -\frac{3}{2} + \sigma_d, \quad \alpha_4 = -\frac{3}{2} - \sigma_d,$$

(21)

where $\alpha_u$, $\alpha_d$ and $X_i$ are determined by [13] and [15] or [20], and where we have set $z_0 = 0$ in the expressions of $\alpha_i$. The above potential $\mathcal{E}$ can be also written in the form

$$\mathcal{E} = \frac{\Lambda + \Gamma}{\Lambda - \Gamma},$$

\[\Lambda = [s^2 - (\sigma_u + \sigma_d)^2](X_1 r_1 - X_2 r_2)(X_3 r_3 - X_4 r_4) - 4 \sigma_u \sigma_d (X_2 r_2 - X_3 r_3)(X_1 r_1 - X_4 r_4),\]

\[\Gamma = 2 \sigma_u [(s + \sigma_u)^2 - \sigma_d^2] X_2 r_2 - [(s - \sigma_u)^2 - \sigma_d^2] X_1 r_1\]

$$+ 2 \sigma_u [(s - \sigma_d)^2 - \sigma_u^2] X_4 r_4 - [(s + \sigma_d)^2 - \sigma_u^2] X_3 r_3,$$

(22)

and below we will use the functions $\Lambda$ and $\Gamma$ for presenting the reparametrized coefficients $f$, $\gamma$ and $\omega$ in the metric [13]:

\[f = \frac{\Lambda \Lambda - \Gamma \Gamma}{(\Lambda - \Gamma)(\Lambda + \Gamma)} e^{2\gamma} = \frac{\Lambda \Lambda - \Gamma \Gamma}{K_{0} r_1 r_2 r_3 r_4}, \quad \omega = \omega_0 = \frac{2 \text{Im}[G(\Lambda - \Gamma)]}{\Lambda \Lambda - \Gamma \Gamma},\]

\[G = z \Gamma + 4 s \alpha_u \sigma_d [(X_3 r_3 + \alpha_3)(X_4 r_4 + \alpha_4) - (X_1 r_1 + \alpha_1)(X_2 r_2 + \alpha_2)]\]

$$+ (\sigma_u + \sigma_d)[s^2 - (\sigma_u - \sigma_d)^2][(X_1 r_1 + \alpha_1)(X_3 r_3 + \alpha_3) - (X_2 r_2 + \alpha_2)(X_4 r_4 + \alpha_4)]$$

$$+ (\sigma_u - \sigma_d)[s^2 - (\sigma_u + \sigma_d)^2][(X_2 r_2 + \alpha_2)(X_3 r_3 + \alpha_3) - (X_1 r_1 + \alpha_1)(X_4 r_4 + \alpha_4)],$$

\[K_0 = \frac{64}{m_1 m_2^2} s^2 |\sigma_u|^2 |\sigma_d|^2 (M + s)^2, \quad \omega_0 = -2 \varepsilon (M + s).\]

(23)

Therefore, we have obtained a physical representation for the general family of equilibrium configurations in the EDK solution. Its interesting particular case which we would like to mention in conclusion of this section is the DH configuration for two balancing identical cootating superextreme Kerr particles [6] possessing an additional symmetry with respect to the equatorial plane [14, 15]. For this specific two-body system $m_u = m_d = m$, $a_u = a_d = a$, $\sigma_u = \sigma_d = \sigma$, and it is convenient to solve Eq. (14) for $a$, yielding ($\delta = +1$)

$$a = -\frac{\varepsilon(s + 2m)^2}{2(s + m)},$$

(24)

which means that $m$ and $s$ are chosen as arbitrary parameters of the solution. Then we readily obtain for $X_i$ the expressions

$$X_1 = \frac{s + 2i \varepsilon m + \varepsilon \mu}{s + (2 + i \varepsilon m - \varepsilon \mu)}, \quad X_2 = \frac{i \varepsilon[s + (2 + i \varepsilon m - \varepsilon \mu)]}{s + (2 - i \varepsilon m + \varepsilon \mu)},$$

$$X_3 = \frac{i \varepsilon[s + (2 - i \varepsilon m + \varepsilon \mu)]}{s + (2 + i \varepsilon m - \varepsilon \mu)}, \quad X_4 = \frac{s + (2 + i \varepsilon m - \varepsilon \mu)}{s + (2 - i \varepsilon m + \varepsilon \mu)}.$$

(25)
while $\sigma$ becomes a pure imaginary quantity (since $m > 0$, $s > 0$) whose explicit form is the following:

$$\sigma = -\frac{i\mu s}{2(s + m)}, \quad \mu \equiv \sqrt{s^2 + 6ms + 7m^2}.$$  \hspace{1cm} (26)$$

For $\alpha_i$ and $r_i$ in the equatorially symmetric case one has

$$\begin{align*}
\alpha_1 &= -\alpha_4 = \frac{s}{2} + \sigma, \quad \alpha_2 = -\alpha_3 = \frac{s}{2} - \sigma, \\
r_1 &= \sqrt{\rho^2 + (z - \alpha_1)^2}, \quad r_2 = \sqrt{\rho^2 + (z - \alpha_2)^2}, \quad r_3 = \sqrt{\rho^2 + (z + \alpha_2)^2}, \quad r_4 = \sqrt{\rho^2 + (z + \alpha_1)^2},
\end{align*}$$

\hspace{1cm} (27)

and the potential $\mathcal{E}$ of the DH equilibrium configuration, after the substitutions into formulae (22) and subsequent simplifications, finally takes the form

$$\mathcal{E} = \frac{\Lambda + \Gamma}{\Lambda - \Gamma},$$

$$\Lambda = (s^2 - 4\rho^2)(\mu_2 - \mu_3)(\mu_3 - \mu_4) - 4\rho^2(\mu_2 - i\epsilon\mu_3)(i\epsilon\mu_4 + \mu_3 - \mu_4),$$

$$\Gamma = 2ms\sigma[(1 - i\epsilon)(s - 2\sigma)(\mu_3 - \mu_4 + i\epsilon\mu_3) - (1 + i\epsilon)(s + 2\sigma)(\mu_2 - i\epsilon\mu_3)],$$

\hspace{1cm} (28)

whereas the corresponding metric functions $f$, $\gamma$ and $\omega$ can be written as

$$f = \frac{\Lambda \bar{\Lambda} - \Gamma \bar{\Gamma}}{(\Lambda - \Gamma)(\bar{\Lambda} - \bar{\Gamma})}, \quad e^{2\gamma} = \frac{\Lambda \bar{\Lambda} - \Gamma \bar{\Gamma}}{K_0 r_1 r_2 r_3 r_4}, \quad \omega = \omega_0 - 2\text{Im}[G(\Lambda \bar{\Gamma} - \bar{\Lambda} \Gamma)],$$

$$G = z\Gamma + s\sigma(2s\mu_2^2 r_4 - \mu_2^2 r_1 r_3) - 8iems\sigma(r_1 r_2 + r_3 r_4) + (1 + i\epsilon)m(s^2 - 4\sigma^2)$$

$$\times [\mu_+(r_3 + i\epsilon r_1) - \mu_-(r_3 + i\epsilon r_2)],$$

$$K_0 = 256s^2\sigma^2(s + 2m)^2, \quad \omega_0 = -2\epsilon(s + 2m), \quad \mu_\pm \equiv s + 3m \mp \epsilon m.$$  \hspace{1cm} (29)

To consider a particular DH configuration, one only needs to choose the values of $m$ and $s$, and find from (24) the corresponding value of $a$ at which the balance occurs. Formulae (26)-(29) will then describe the spacetime for that parameter choice.

\hspace{1cm} V. DISCUSSION

Although the general formulae worked out in the previous section are applicable to all four types of the two-Kerr configurations from Fig. 1, the equilibrium states with $m_a > 0$, $m_d > 0$ are only possible for the systems (b), (c) and (d) containing at least one superextreme component. Various particular equilibrium configurations between a black-hole and a superextreme constituents, or between two unequal superextreme constituents were considered in the paper [1], and recently we have shown [5] that balance can be achieved even between a Schwarzschild black hole and a Kerr superextreme object. The absence of the equilibrium between two underextreme Kerr constituents with positive Komar masses (the systems (a) in Fig. 1) was strictly proved in our paper [2], and the non-existence proof was later extended to the case of two extreme Kerr constituents [10], thus ruling out the two-black-hole equilibrium states in the EDK solution. In this respect it is worth noting that the criterion according to which an underextreme or extreme constituent of the double-Kerr solution is a black hole if it possesses positive Komar mass was originally employed by Hoenselaers [17]. However, this simple and very sound criterion has been recently objected in a series of papers by Neugebauer and Hennig [18–21] whose only argument against it is a purely numerical result by Ansorg and Petroff [22] on a regular black hole with negative mass in the presence of surrounding matter which the former authors interpret as a “convincing counterexample” to the general belief about the unphysical character of negative Komar mass. In our opinion, it is not quite legitimate to use numerical results as arguments against the analytical studies, especially when the two approaches treat different physical models. In this relation it may be remarked that the recent paper [23] questions the correctness of the Ansorg and Petroff’s claim on the existence of regular black-hole spacetimes with negative Komar masses and points out possible sources of error in their numerical analysis. On the other hand, the use by Neugebauer and Hennig of a specific black-hole inequality between the angular momentum and horizon area [24] as an alternative to the positive Komar mass condition, is not free of shortages too and thus can hardly be viewed as an enhancement of the known non-existence proof [2]. Indeed, the inequality [24] was originally derived for a single black hole, so that its applicability to each underextreme constituent in a two-body configuration actually needs a justification which would take into account the interaction effects. Moreover, the aforementioned inequality requires the regularity of spacetime as prerequisite to its use in a concrete problem; however, the analytical
Theorem I.

connection between Eqs. (15) and (34) is described by the following two theorems.

systems only one of the inequalities (32) has to be satisfied.

\[ \omega_d \gamma_d, \gamma_d \] constant values of the functions \( \omega \) and \( \gamma \) on the respective horizons, are able to provide us with a simple demonstration that the individual Komar masses \( m_u \) and \( m_d \) cannot simultaneously take on positive values in such configurations. Taking into account that \( \delta = +1 \) in the \( (a) \)-type equilibrium states, one can arrive at the following final expressions for \( A_u \) and \( A_d \):

\[
\begin{align*}
A_u &= -\frac{4\pi m_u(s + m_d)(M + s + \epsilon a_d) - \sigma_u(M + s)}{s(M + s)(M + s + \epsilon a_d)}, \\
A_d &= -\frac{4\pi m_d(s + m_u)(M + s + \epsilon a_u) - \sigma_d(M + s)}{s(M + s)(M + s + \epsilon a_u)},
\end{align*}
\]

whence it follows immediately that in order the masses of the black-hole constituents and areas of the horizons could take positive values simultaneously, the following two conditions must be satisfied:

\[ M + s + \epsilon a_d < 0, \quad M + s + \epsilon a_u < 0. \quad (32) \]

However, after rewriting the equilibrium condition (14) in the form (\( \delta = +1 \))

\[ s(M + s) - (m_u + s)(M + s + \epsilon a_u) - (m_d + s)(M + s + \epsilon a_d) = 0, \quad (33) \]

we see that, under the suppositions made, the inequalities (32) convert the left-hand side of (33) into a strictly positive quantity, which signifies the absence of equilibrium configurations of two Kerr black holes. Note, however, that in the systems (\( b \)) and (\( c \)) the black-hole component has the horizon area defined by one of the expressions (31), with \( \epsilon \) substituted by \( \epsilon \delta \), so that the balance condition (33) may have physically meaningful solutions because in such systems only one of the inequalities (32) has to be satisfied.

It would certainly be of interest to briefly discuss a formal interrelation existing between the equilibrium configurations of the EDK solution and analogous configurations of the double-Reissner-Nordström (DRN) solution [7, 26]. While the former configurations are defined by the condition (15), the latter equilibrium states of two electrically charged Reissner-Nordström sources [27, 28] are defined by the balance condition

\[ m_u m_d - \left( q_u + \frac{m_u q_d - m_d q_u}{m_u + m_d + s} \right) \left( q_d + \frac{m_d q_u - m_u q_d}{m_u + m_d + s} \right) = 0, \quad (34) \]

(the reader is referred to [7, 27] for the details of its derivation), where \( m_u \) and \( m_d \) are Komar masses of the upper and lower constituents, \( q_u \) and \( q_d \) are the corresponding charges, while \( s \) is the relative coordinate distance. The connection between Eqs. (15) and (34) is described by the following two theorems.

**Theorem I.** If \( m_u, m_d, a_u, a_d, s \) is an equilibrium configuration of the EDK solution, then the substitution

\[
\begin{align*}
a_u &= \frac{\epsilon \delta q_u (m_u q_d - m_d q_u)}{m_u m_d - q_u q_d}, \\
a_d &= \frac{-\epsilon \delta q_u (m_u q_d - m_d q_u)}{m_u m_d - q_u q_d}, \\
m_u q_d - m_d q_d \neq 0,
\end{align*}
\]

into Eq. (15) defines an equilibrium configuration of the DRN solution.

**Theorem II.** Given an equilibrium configuration of the DRN solution, \( m_u, m_d, q_u, q_d, s \), the substitution

\[
\begin{align*}
q_u^2 &= -\frac{m_u m_d \Delta_0^2}{\Delta_0}, \\
q_d^2 &= -\frac{m_d m_u \Delta_0^2}{\Delta_0}, \\
q_u q_d &= \frac{m_u m_d a_u a_d}{\Delta_0},
\end{align*}
\]

with \( \Delta_0 \equiv a_u a_d + \epsilon \delta (m_u a_d + m_d a_u) \neq 0, \quad \Delta_0 m_u m_d < 0 \), converts Eq. (34) into condition (15).

The proof of these theorems is straightforward and consists in the substitution of (35) and (36) into Eqs. (15) and (34), respectively.
VI. CONCLUSION

We hope that the physical representation of the general family of equilibrium configurations of two Kerr sources obtained in the present paper will make this family more accessible for concrete applications and will simplify the analysis of particular cases which exhibit interesting physical properties. Although two Kerr black holes cannot be in the gravitational equilibrium, this fact does not diminish the importance of the EDK solution because there are other physically meaningful equilibrium configurations it offers – those between a black hole and a superextreme source, and between two superextreme Kerr constituents, both types of the configurations permitting their components to have exclusively positive Komar masses. It is probably worth remarking that for many years the superextreme solutions had been largely underestimated compared to the black-hole ones in spite of the theoretical evidence that they can arise from the gravitational collapse [29, 30], or can open new horizons for the gravitational experiment (an important prediction made four decades ago by Penrose [31]). It seems that the recent paper of Jacobson and Sotiriou [32] on destroying black holes with test bodies establishes an interesting physical bridge between the two types of exact solutions, and we expect that the binary equilibrium configurations described by the EDK solution will be able to shed additional light on the physical interaction of black holes and superextreme sources.

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