Classification of two-orbit varieties

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1. Introduction

Our base field is the field $\mathbb{C}$ of complex numbers. We study normal complete algebraic varieties $X$ endowed with an action of a connected reductive algebraic group $G$ admitting a dense orbit $\Omega$. The stabilizer $H$ of a point of the dense orbit will be called generic stabilizer. Clearly, $H$ is parabolic if and only if $X = \Omega$.

If $H$ is connected, BOREL showed in [3] that the homogeneous space $G/H$ has at most two ends, i.e. the complement $X \setminus \Omega$ has at most two connected components. When $X \setminus \Omega$ is disconnected, the generic stabilizer contains a maximal unipotent subgroup $U$ of $G$. This property results from the connectedness (proved in [8, 11]) of the $U$-fixed point subset of $X$. In this situation, the varieties $X$ share other peculiarities which were clarified by AHIEZER in [4] where he studied and classified these objects.

This article deals with the left situation – $X \setminus \Omega$ is connected –, as we assume this complementary to be $G$-homogeneous. Actually, this case appears as the most natural and “simplest” one to consider after the case of projective homogeneous varieties (Grassmannian, flag varieties, etc.). We call naturally such varieties $X$ two-orbit varieties. These varieties have been intensively studied before. AHIEZER in [4] gave a classification of the pairs $(G, H)$ such that $\Omega = G/H$ admits a compactification $X$ by one homogeneous divisor. This list was independently obtained by HUCKLEBERRY and SNOW [12] in the more general context of kählerian varieties. Later, BRION gave in [11] a purely algebraic approach to these results using the general theory of embeddings developed by Luna and VUST (see [14]). Finally,
Feldmüller classified in [10] all pairs \((G, H)\) giving rise to two-orbit varieties whose closed orbit is of codimension 2.

The aim of this paper is to give the complete classification of all two-orbit varieties and to prove Luna’s conjecture, namely that all two-orbit varieties are spherical, i.e. admit a dense orbit of a Borel subgroup \(B\) of \(G\). In particular, spherical varieties have only finitely many \(B\)-orbits. We refer the reader to [14, 5] for an introduction to this subject.

To obtain the classification, we use two main steps. First of all, we show that defining a two-orbit variety as being normal is not a major restriction since the normalization map for a complete variety with two orbits is bijective (see section 2). Secondly, we give only an explicit list of all cuspidal two-orbit varieties from which we obtain all the two-orbit varieties by a parabolic induction procedure explicitly described in section 2.

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Note added in proof. Alexander Smirnov found similar results while I was submitting this article.

2. Notation and main results

Let \(G\) be a connected reductive algebraic group. We fix a Borel subgroup \(B\) and a maximal torus \(T\) in \(B\). Let \(\Phi\) be the root system of \(G\), we denote \(\Phi^+ \subset \Phi\) the set of positive roots relative to \(B\) and \(\Delta \subset \Phi^+\) the set of simple roots corresponding to \(B\). In case \(G\) is simple, we enumerate the simple roots as in [4] and when it is more convenient, we use the short notation of [loc. cit.] for the description of positive roots. For example, if \(G\) is of type \(F_4\), \(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\) stands for the root \(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\).

The subset \(\langle \gamma_1, \ldots, \gamma_r \rangle\) of \(\Phi\) will refer to the set of roots obtained as \(Q\)-linear combinations of the given roots \(\gamma_1, \ldots, \gamma_r\).

For \(\alpha\) a root of \(G\), we have the corresponding root space \(g_\alpha \subset g = \text{Lie } G\). We let \(Y_\alpha\) be any element of \(g_\alpha \setminus \{0\}\), then \(\mathbb{C}Y_\alpha = g_\alpha\). The 0-th root space of \(g\) is \(\text{Lie } T = t\).

Throughout this article, \(X\) will denote a connected normal complete algebraic variety endowed with an action of \(G\) such that \(X\) has two \(G\)-orbits. Such a variety is naturally called a two-orbit \(G\)-variety. If \(G \cdot x\) corresponds to the dense orbit and \(G \cdot y\) to the closed one, we have \(X = G \cdot x \cup G \cdot y = \text{Cl}(G \cdot x)\) (the closure of \(G \cdot x\)). In other words, a two-orbit \(G\)-variety can be regarded as a complete two-orbit embedding of the homogeneous space \(G/H\) where \(H\) is a generic stabilizer.

Actually, a two-orbit variety is even projective. For, according to Sumihiro (see [16]), the closed orbit admits a \(G\)-stable quasi-projective neighbourhood – which has to be all of \(X\). As a weak converse, we state: a connected projective \(G\)-variety with two \(G\)-orbits (say \(X' = G \cdot x' \cup G \cdot y'\)) is bijective to its normalization,
which is, in particular, a two-orbit $G$-variety. Let us sketch the proof. Consider the $G$-equivariant morphism $\pi : X' \to X'$ given by the normalization of $X'$. First of all, $\pi$ being birational, $X'$ has a dense $G$-orbit isomorphic to $G \cdot x'$. Secondly, due to the finiteness of $\pi$ together with the connectedness of a parabolic subgroup, we get that $\pi^{-1}(G \cdot y')$ consists of a finite union of $G$-orbits all bijective to $G \cdot y'$. Therefore, all these orbits are projective. Finally, we prove, by contradiction, that actually $\pi^{-1}(G \cdot y')$ is $G$-homogeneous. Suppose that $\pi^{-1}(G \cdot y')$ consists of more than one (projective) $G$-orbit. By \( \tilde{\pi} \), $X' \setminus G \cdot x'$ will have two connected components. Thus, (see the corresponding argument given in the introduction), the generic stabilizer of a point of $X'$, and hence of $X'$, will contain a maximal unipotent subgroup of $G$, say $U$ subgroup of $B$. Consequently, we will end up with two $B$-fixed points in $X'$ - a situation which can not occur.

Similarly, by considering the $G$-equivariant finite morphism given by the projection $G/H \to G/H^\circ$, one can prove that: if $G/H$ has a complete two-orbit embedding, then so does $G/H^\circ$. The converse is obviously true. More precisely, if $Z$ is a complete two-orbit embedding of $G/H^\circ$, $G/H$ inherits a complete two-orbit embedding given as the quotient of $Z$ by $H/H^\circ$. As a consequence, we will restrict our study to two-orbit varieties with connected generic stabilizers, without loss of generality.

Before stating the main results on the two-orbit varieties, we need to make one more remark. Blowing down a given two-orbit variety along its closed orbit may produce a new two-orbit variety. We will illustrate and clarify this natural geometrical construction via the next example.

Let $G$ be a simple group of rank 2 and $\lambda$ a dominant weight such that $\langle \lambda, \alpha_i^\vee \rangle > 1$ and $\langle \lambda, \alpha_2^\vee \rangle = 0$, $\langle \cdot, \cdot \rangle$ being the Killing form of $G$ and $\alpha_i^\vee$ the coroot associated to $\alpha_i$. We consider the irreducible $G$-module $V(\lambda)$ associated to $\lambda$ and the weight vector $v_{\lambda - \alpha_1} \in V(\lambda)$ of weight $\lambda - \alpha_1$. Take $P \supset B$ the parabolic subgroup associated to $\alpha_1$. Then $\text{Cl}(P \cdot [v_{\lambda - \alpha_1}]) \subseteq \mathbb{P}(V(\lambda))$ is a $P$-variety with two $P$-orbits and $\text{Cl}(G \cdot [v_{\lambda - \alpha_1}]) = G \cdot \text{Cl}(P \cdot [v_{\lambda - \alpha_1}])$ is a two-orbit $G$-variety. We have an obvious $G$-equivariant morphism $G \times_P \text{Cl}(P \cdot [v_{\lambda - \alpha_1}]) \to \text{Cl}(G \cdot [v_{\lambda - \alpha_1}])$. This morphism is only isomorphic on the open $G$-orbits, the morphism on the closed $G$-orbits being $G/B \to G/Q$ where $Q \supset B$ is the parabolic subgroup associated to $\alpha_2$.

This example is very instructive in the sense that it gives the procedure to obtain all two-orbit varieties and motivates the following notion inspired by Luna’s work \cite{Luna}. In the statement of this definition (and in the rest of the text), the terminology two-orbit $P$-variety for $P$ a parabolic subgroup of $G$ will refer, by abuse of language, to a complete normal $P$-variety, with two $P$-orbits, on which the unipotent radical of $P$ acts trivially.

Definition. A two-orbit $G$-variety $X$ is obtained by parabolic induction from a pair $(P, Y)$ if $P \subseteq G$ is a parabolic subgroup and $Y$ a two-orbit $P$-variety such that

(i) the radical of $P$ acts trivially on $Y$;
(ii) there exists a $P$-equivariant injective morphism $\varphi : Y \to X$ inducing a birational morphism $G \times_p Y \to X$.

A two-orbit variety is cuspidal if it can not be obtained by parabolic induction.

**Proposition 2.1.**

(i) The geometry of a two-orbit variety obtained by parabolic induction from a pair $(P, Y)$ is completely determined by its closed orbit and the geometry of $Y$.

(ii) Each (non cuspidal) two-orbit variety is obtained by parabolic induction from a unique pair $(P, Z)$ such that $Z$ is cuspidal.

Once this result is proved (see section 4), we can concentrate only on cuspidal two-orbit varieties. We get their description as follows. We start to show which groups can occur as generic stabilizers (see sections 5 and 6). This is done mainly with combinatorial methods. Thus, afterwards, it is just a matter of computations, using Luna-Vust Theory (see [14]), to describe the corresponding complete two-orbit embeddings. We list the pairs in this section but the embeddings only in the appendix to avoid too much notation in this part of the text. Recall that we consider only the case of connected generic stabilizers $H$, which is not restricted at all as explained in the beginning of this section.

**Theorem 2.2.**

The cuspidal two-orbit varieties are obtained as embeddings of the homogeneous spaces $G/H$ with $(G, H)$ in Table 1 and Table 2. In particular, $G$ is simple or equal to $SL_2 \times SL_2$, in this situation. Moreover, each such $G/H$ has exactly one complete two-orbit embedding described in the appendix.

**Corollary 2.3.** Two-orbit varieties are spherical.

**Proof.** As the property of being spherical is not destroyed by parabolic induction, it suffices to check that the homogeneous spaces $G/H$ obtained from Table 1 and 2 are indeed spherical. It appears that these embeddings either have been already classified by [14] or are wonderful (thus spherical by [14]) of rank 2 (see [14]). $\square$

**3. General properties**

First of all, note that we can assume the group $G$, acting on $X$, to be semisimple. Indeed, the radical $R(G)$ of $G$ acts trivially on $X$. For, if there exists a generic element of $X$ not fixed by $R(G)$, we will obtain at least two $R(G)$-fixed points in the closure of its $R(G)$-orbit. But it is a well-know fact, that these fixed points belong to different connected components of the $R(G)$-fixed point set in $X$ – which is connected in our situation since it is equal to the closed $G$-orbit of $X$, according to
Table 1. Pairs of type I

\[ A_n \supset \mathfrak{gl}_n \]
\[ B_n \supset \mathfrak{so}_{2n} \]
\[ B_n \supset t \bigoplus \alpha \in \Psi g_{\alpha}, \quad \text{with } \Psi = \langle \pm \alpha_1, \ldots, \pm \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n \rangle \]
\[ B_n \text{ or } C_n \supset t \bigoplus \alpha \in \Psi g_{\alpha}, \quad \text{with } \Psi = \langle \pm \alpha_2, \ldots, \pm \alpha_{n-1}, \alpha_n, \varepsilon_1 + \varepsilon_n \rangle \]
\[ C_n \supset \mathfrak{sp}_2 \times \mathfrak{sp}_{2n-2} \]
\[ C_n \supset t \bigoplus \alpha \in \Psi g_{\alpha}, \quad \text{with } \Psi = \langle \pm \alpha_2, \ldots, \pm \alpha_n, 2\varepsilon_1 \rangle \]
\[ F_4 \supset \text{spin}_9 \]
\[ F_4 \supset t \bigoplus \alpha \in \Psi g_{\alpha}, \quad \text{with } \Psi = \langle \pm \alpha_2, \alpha_3, \pm \alpha_4, 120 \rangle \]
\[ G_2 \supset t \bigoplus \alpha \in \Psi g_{\alpha}, \quad \text{with } \Psi = \langle \alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 \rangle \]
\[ G_2 \supset t \bigoplus \alpha \in \Psi g_{\alpha}, \quad \text{with } \Psi = \langle \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 \rangle \]
\[ G_2 \supset t \bigoplus \alpha \in \Psi g_{\alpha}, \quad \text{with } \Psi = \langle \pm \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 \rangle \]
\[ G_2 \supset \mathfrak{sl}_3 \]

Table 2. Pairs of type II

\[ A_1 \times A_1 \supset \mathfrak{sl}_2 \quad \text{(diagonally embedded)} \]
\[ A_2, B_2 \text{ or } G_2 \supset \text{Lie}(\ker(\alpha_1 - \alpha_2)) \oplus C(Y_{\alpha_1} + Y_{\alpha_2}) \bigoplus_{\alpha \in \Phi^+ \setminus \{\alpha_1, \alpha_2\}} g_{\alpha} \]
\[ B_3 \supset g_2 \]
\[ C_3 \supset \mathfrak{sl}_2 \oplus C \oplus C(Y_{-\alpha_2} + Y_{\varepsilon_1 + \varepsilon_2}) \bigoplus_{\alpha \in \Phi^+ \setminus \{\alpha_2, \varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_3, 2\varepsilon_2\}} g_{\alpha} \]
\[ \text{with } t \cap \mathfrak{sl}_2 \oplus C = \text{Lie}(\ker(\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3)) \]
\[ D_n \supset \mathfrak{so}_{2n-1} \]
\[ F_4 \supset g_2 \oplus C \oplus C(Y_{-\alpha_4} + Y_{1232}) \bigoplus_{\alpha \in \{1121, 1221, 0121, \alpha_3\}} g_{\alpha} \]
\[ \text{with } t \cap g_2 \oplus C = \text{Lie}(\ker(\alpha_4 + 1232)) \]
Lemma 3.1. Let $H$ be a subgroup of $G$ and $T'$ be a torus of $G$. If $G'$ is the identity-component of the centralizer in $G$ of $T'$, then the connected components of $(G/H)^{T'}$, the $T'$-fixed points of $G/H$, are exactly given by its $G'$-orbits.

Proof. We prove this standard statement on the corresponding tangent spaces. 

We will start with the case of the two-orbit $SL_2$-varieties. We work out this case separately not only because it is simple but also because it will be used quite often later on.

Proposition 3.2. The two-orbit $SL_2$-varieties are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ with the obvious $SL_2$-actions. The respective generic stabilizers are the maximal torus $T$ and its normalizer.

Proof. The a priori possible dimensions for a two-orbit $SL_2$-variety (that is, the possible dimensions for a $SL_2$-orbit) are 0, 1, 2 or 3.

Using the fact that projective $SL_2$-orbits are 0- or 1-dimensional (and vice versa) and that 3-dimensional $SL_2$-orbits are affine, we prove that a two-orbit $SL_2$-variety can only be 2-dimensional. Therefore, it has at least three points fixed by $T$ a maximal torus of $SL_2$ (see [3, Theorem 25.2]) Since a projective $SL_2$-orbit has only two $T$-fixed points, a generic stabilizer $H$ must contain this torus $T$. Finally, we get that $H/T$ is finite because $\dim H = 1 = \dim T$ so $H = T$ or $H = N_G(T)$. The respective two-orbit $SL_2$-varieties are clearly $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. Note that the action of $SL_2$ on $\mathbb{P}^2$ is induced by $SO_3$. 

From now on, the group $G$ is of rank greater than 2. Let us start by studying locally the two-orbit $G$-variety $X$. Consider $y$ the $B$-fixed point of $X$ and $P$ its stabilizer in $G$. The group $P$ is parabolic and $P^uL$ will be its Levi decomposition such that $T \subset L$. By [5, Theorem 1.4], there exists an affine $L$-stable subvariety $Z$ of $X$ such that $\dim Z \geq 1$ and $Z \cap G \cdot y = \{y\}$. Using the fact that $T$-orbits are affine, it is easy to see that an affine $T$-variety containing fixed points also contains 1-dimensional orbits if the action is not trivial. Therefore, there is a generic element $x$ of $X$ such that $\text{codim}_T T_x \leq 1$. In other words, we have just obtained:

Proposition 3.3. The ranks of the algebraic group $G$ and of the generic stabilizer differ at most by 1.

Definition. The two-orbit varieties such that the rank of $G$ is equal to (resp. different from) the rank of $H$ are said of type I (resp. of type II).

Suppose that the generic element $x$ is such that the torus $T^o_x$ is a maximal torus of $G_x$. Let $L'$ be the centralizer in $G$ of $T^o_x$. Then, by the above proposition, $L'$ is equal to $T$ or is a Levi subgroup of semisimple rank 1. We are going to prove that the latter possibility can not occur i.e.
Proposition 3.4. The maximal tori of the generic stabilizer are regular tori of the group $G$.

Proof. We proceed by contradiction; suppose that $L' \neq T$ and consider the $L'$-variety $Cl(L' \cdot x)$. Since $T^0_x \neq T$, $Cl(L' \cdot x)$ is actually a $SL_2$-variety but not a two-orbit $SL_2$-variety (see Proposition 3.2). More precisely, $Cl(L' \cdot x) \setminus L' \cdot x$ is a finite union of closed $L'$-orbits of dimension $\leq 1$ all contained in $G \cdot y$ because of Lemma 3.1. Therefore (see also the proof of Proposition 3.2), the $L'$-variety $Cl(L' \cdot x)$ is of dimension 2.

Let $B_{L'}$ be the Borel subgroup of $L'$ equal to $B \cap L'$ and $U_{L'}$ be its unipotent radical. There exists an element, say $x'$, in $L' \cdot x$ fixed by $U_{L'}$ (see the introduction). Moreover, $T^0_x$ fixes obviously $x'$ so, by Proposition 3.2, $T^0_x = T^0_{x'}$. Thus, $Cl(L' \cdot x') = L \cdot Cl(B_{L'} \cdot x') = L' \cdot Cl(T \cdot x')$. If $y_1$ and $y_2$ denote the $T$-fixed points in $Cl(T \cdot x')$ then $Cl(L' \cdot x) = L' \cdot x \cup L' \cdot y_1 \cup L' \cdot y_2$. The variety $Cl(L' \cdot x)$, being 2-dimensional, has at least three $T$-fixed points, (see [13, Theorem 25.2]). Therefore, $y_1$ and $y_2$ can not be simultaneously fixed by $L'$. Suppose that $L' \cdot y_1 \neq y_1$ and consider the element, say $y'$, satisfying $Cl(T \cdot y') = L' \cdot y_1$. Then we get:

$$T^0_{y'} = T^0_{x'}, \quad \dim(T \cdot x') = \dim(T \cdot y') = 1 \quad \text{and} \quad y_1 \in Cl(T \cdot x') \cap Cl(T \cdot y').$$

It results from the lemma below that $y_2 \in L \cdot y_1$ which is incompatible with the fact that $Cl(L' \cdot x)$ is not a two-orbit variety. Thus the assumption $L' \neq T$ was absurd – which proves Proposition 3.4. \hfill \Box

Lemma 3.5. Let $x$ and $z$ be two elements of a projective $G$-variety such that

$$\dim T \cdot x = \dim T \cdot z = 1 \quad \text{and} \quad T^0_x = T^0_z.$$  

There are two or four $T$-fixed points in $Cl(T \cdot x) \cup Cl(T \cdot z)$ assuming that they are all contained in one same closed $G$-orbit.

Proof. By embedding the given variety $G$-equivariantly into the projective space of a representation, we can write the elements $x$ and $z$ as: $x = [v_1 + \ldots + v_r]$ and $z = [v_1 + \ldots + v_s]$. By assumption, the weights of the support of $x$ (resp. of the support of $z$) sit on a same affine line $D_x$ (resp. $D_z$) and moreover,

$$\ker(\mu_1 - \mu_r)^0 = T^0_x = T^0_z = \ker(\nu_1 - \nu_s)^0.$$  

From these equalities, we can deduce that the lines $D_x$ and $D_z$ are strictly parallel or equal. The first situation clearly yields four $T$-fixed points in $Cl(T \cdot x) \cup Cl(T \cdot z)$ and the second one only two $T$-fixed points (use the fact that the Weyl group acts transitively on the $T$-fixed points of a projective orbit). \hfill \Box

As a consequence of Proposition 3.4, we have:

Corollary 3.6. Consider two elements $x$ and $z$ of the dense $G$-orbit of a two-orbit variety such that $T^0_x$ (resp. $T^0_z$) is a maximal torus of $G_x$ (resp. $G_z$). Then $z \in N_G(T) \cdot x$; in particular, if $x$ and $z$ are $B$-conjugated they are actually $T$-conjugated.
4. Parabolic induction

This section is the main step to the classification of two-orbit varieties. It consists essentially in proving that the problem of classifying the two-orbit varieties can be reduced to a subclass of two-orbit varieties called \textit{cuspidal}, i.e. the two-orbit varieties which can not be obtained by parabolic induction (see section 2).

4.1. Statements

Proposition 4.1. If $X$ is obtained by parabolic induction from a pair $(P, Y)$, then it is completely determined by its closed orbit and the variety $Y$.

Proof. This statement is a direct consequence of the fact that the morphism $G \times_P \ Y \to X$ is a blowing down of the closed $G$-orbit of $X$ and that the variety $X$ is completely determined, according to a result of Luna and Vust (see theorem 8.3 of [14]), by its $G$-stable prime divisors and by the $B^-$-stable prime divisors of its dense $G$-orbit, whose closure in $X$ contains the closed $G$-orbit of $X$. \hfill \Box

We can put an order $\leq$ on the set of induction pairs of $X$, defined naturally, for two pairs $(P_1, Y_1)$ and $(P_2, Y_2)$, by:

$$(P_1, Y_1) \leq (P_2, Y_2) \quad \text{if } P_1 \subseteq P_2 \text{ and } \varphi_1(Y_1) \subseteq \varphi_2(Y_2).$$

Theorem 4.2. The set of induction pairs of a two-orbit variety (endowed with the order $\leq$) has an unique minimal element.

4.2. Proof of Theorem 4.2

In order to prove Theorem 4.2, we are constructing explicitly, in this section, the minimal induction pair.

Let us consider an induction pair $(P, Y)$ and two elements $x$ and $z$ in the dense $G$-orbit of $X$ such that $B \cdot x$ and $B \cdot z$ are closed in $G \cdot x$. These elements can be chosen such that $T^*_x$ (resp. $T^*_z$) is a maximal torus of $G_x$ (resp. $G_z$). Then, by Corollary 3.6, they are conjugated by an element $n \in N_{G}(T)$. Because of the choice made on $B \cdot x$ and on $B \cdot z$, one can show by a short computation that $n$ is in fact an element of $P$. We have got:

Lemma 4.3. If $X$ is obtained from a pair $(P, Y)$ by parabolic induction then the $B$-orbits which are closed in the dense $G$-orbit are in a same $P$-orbit of $X$. In particular, they are contained in $\varphi(Y)$.

Let $x$ be as above, denote $G_x$ by $H$. Let $P_1$ be the parabolic subgroup of $G$ generated by $B$, $H$ and the elements $n$ of $N_{G}(T)$ such that $Bn \cdot x$ is closed in $G \cdot x$. By construction, for all pairs $(P, Y)$, we have: $P_1 \subseteq P$ and $X_1 = \text{Cl}(P_1 \cdot x) \subset \varphi(Y)$. However, $X_1$ may not be a two-orbit $P_1$-variety. More precisely, we have:
$X_1 \cap G \cdot x = P_1 \cdot x$ but an analogous equality may not hold for $X_1 \cap G \cdot y$ ($y$ a $T$-fixed element of the closed orbit in $X_1$). So, instead of $P_1$, we have to consider $P_H$ the parabolic subgroup of $G$ generated by $P_1$ and the elements $w \in W$ such that $w \cdot y \in X_1$. Thus, $X_H = \text{Cl}(P_H \cdot x) = \text{Cl}(P_H \cdot X_1)$ is a two-orbit $P_H$-variety with $P_H \subset P$ and $X_H \subset \varphi(Y)$.

Finally, let us consider $\tilde{X}_H$ the normalized variety of $X_H$. The pair $(P_H, \tilde{X}_H)$ is the required element.

**Proposition 4.4.** The pair $(P_H, \tilde{X}_H)$ is an induction pair; this is the minimal element for the set of induction pairs of $X$.

We already know that the variety $\tilde{X}_H$ is a two-orbit $P_H$-variety. So we are left only to prove that the radical of $P_H$ acts trivially on $\tilde{X}_H$. For this, we use the following

**Proposition 4.5.** Let $Y$ be a two-orbit $Q$-variety with $Q$ a parabolic subgroup of $G$ containing $B$. Consider an element $z$ of its dense $Q$-orbit such that the torus $T^o_z$ is a maximal torus of $Q_z$. Suppose that the radical of $Q$ doesn’t act trivially on $Y$. Then the orbit $L \cdot z$ is complete for $L$ a Levi subgroup of $Q$ containing the torus $T$.

**Proof of Proposition 4.4.** According to Proposition 4.3 applied to $\tilde{X}_H$ and $P_H$, if we prove that $L \cdot z$ is not complete, Proposition 4.4 will follow. If $T^o_z \neq T$, that $L \cdot z$ is not complete is obvious. Consider then the other case: $T^o_z = T$. If $\alpha$ is a simple root of $P_H$ such that $U_{\alpha} \not\subset G_z$ then (because of the assumption made on $B \cdot z$) $s_{\alpha} B s_{\alpha} \cap G_z$ contains a Borel subgroup of $G_z$. In other words, $s_{\alpha} B s_{\alpha}$ is closed in $P_H \cdot z$ and by definition of $P_H$, it means that $SL_2(\alpha) \subset L$. To conclude that $L \cdot z$ is not complete, we need only to observe that $\text{Cl}(L \cdot z)$ contains a $T$-fixed point of the closed $G$-orbit (because $\text{Cl}(U_{\alpha} \cdot z)$ does by Corollary 3.6). \(\square\)

**Proof of Proposition 4.4.** First of all, note that the radical of $Q$ acts trivially on the closed $Q$-orbit of $Y$. So the proposition relies only on the dense $Q$-orbit.

We choose an element $z$ verifying the hypotheses of the proposition and such that $B \cdot z$ is closed in $Q \cdot z$.

Let us start with the case: $T^o_z \neq T$. Thus $L \cdot z$ is not complete. Let $\beta \in \Phi^+$ be such that $U_\beta \not\subset Q_z$. Such a root exists otherwise $Y$ will have two $B$-fixed points given by the $T$-fixed points of $\text{Cl}(T \cdot z)$. Consider the 2-dimensional variety $X^0_\beta = \text{Cl}(U_\beta T \cdot z)$ and denote by $T^o_{X^0_\beta} (\subset \ker \beta)$ its generic stabilizer in $T$. To get our result, we are going to prove that $T^o_{X^0_\beta}$ contains the identity-component $Z(L)^o$ of the center of $L$.

Let $y_1$ and $y_2$ be the $T$-fixed points of $\text{Cl}(T \cdot z)$. If $y_i$ ($i = 1, 2$) is not fixed by $U_\beta$, we denote by $y'_i$ the other $T$-fixed point in $\text{Cl}(U_\beta \cdot y_i)$; otherwise, we set $y'_i = y_i$. The points $y'_1$ and $y'_2$ are distinct; this follows easily from Lemma 3.3 and from the fact that $T^o_{X^0_\beta}$ is regular.

The variety $X^o_{\beta^o}$, being connected and containing two distinct elements, $y_1$ and $y_2$, is 1-dimensional.
To construct the elements $z_i$ and $\rho_i$, it suffices to consider the convex hull of the support of $X_\beta$; the elements $z_i$ (resp. $\rho_i$) have as support the edges (resp. vertices) of this polytope.

Because of the claim, we now have at hand at least two elements $u$ and $v$ simultaneously in the closed $Q$-orbit and in $X_\beta$ such that $\dim(T \cdot u) = 1 = \dim(T \cdot v)$ and $\Cl(T \cdot u) \cap \Cl(T \cdot v) = 1$. Therefore, $(T_u \cap T_v)^\circ$ (of codimension 2) is equal to $T_{X_\beta}^\circ$. But since $Z(L)$ acts trivially on the closed $Q$-orbit, we end up with the required inclusion: $Z_L \subset T_{X_\beta}^\circ$.

Assume now that $T_{X_\beta}^\circ = T$ and that $L \cdot z$ is not closed. Therefore, there exists at least one positive root, say $\alpha$, in the root system of $(L, L)$ such that $U_\alpha \not\subset Q_z$. Recall that we want to prove that the radical of $Q$ acts trivially on $Y$. To do so, we proceed by contradiction: suppose there exists $\beta \in \Phi^+$, $\beta \neq \alpha$ such that $U_\beta \subset Q^\circ$ and $U_\beta \not\subset Q_z$. Then, the variety $X_{\beta, \alpha} = \Cl(U_\beta U_\alpha \cdot z)$ is 2-dimensional and contains a dense $T$-orbit; denote by $T_{X_{\beta, \alpha}}$ its generic stabilizer in $T$.

If $y_1$ and $y_2$ are the $T$-fixed points (distinct from $z$) of $\Cl(U_\alpha \cdot z)$ and of $\Cl(U_\beta \cdot z)$ respectively, then necessarily $y_1$ and $y_2$ are distinct (as they do not have the same support). If $U_\beta \cdot y_1 \neq y_1$, we denote by $y_1'$ the other $T$-fixed point of $\Cl(U_\beta \cdot y_1)$; otherwise, we set $y_1' = y_1$. Then the variety $X_{\beta, \alpha}^{U_\beta}$ is 1-dimensional, since it contains the distinct points $y_1'$ and $y_2$. But, $y_1$ and $y_2$ must be the only $T$-fixed points of the closed $Q$-orbit of $Y$ in $X_{\beta, \alpha}$ otherwise with the same arguments used in the first case ($T_{X_\beta}^\circ \neq T$), we will have: $Z(L)^\circ \subset T_{X_\beta}$. So, we can conclude that $U_\beta$ must fix $y_1'$ and that there exists an element $u$ in the closed $Q$-orbit of $Y$ such that: $X_{\beta, \alpha}^{U_\beta} = \Cl(T \cdot u)$.

By considering the variety $\Cl(U_\alpha \cdot y_1)$, we get $y_3$, the other $T$-fixed point in it. For the same reasons as before, the points $y_1$ and $y_3$ are distinct.

To conclude, we have constructed two 1-dimensional subvarieties of the closed $Q$-orbit, $X_{\beta, \alpha}$ and $\Cl(U_\alpha \cdot y_1)$ such that $y_1$ is their only common $T$-fixed point. According to Lemma 3.3, this yields the contradiction: $(T_u \cap \ker \alpha)^\circ = T_{X_\beta}^\circ$.

5. Two-orbit varieties of type I

In this section, the two-orbit $G$-variety $X = \Cl(G \cdot x) = G \cdot x \cup G \cdot y$ is of type I, i.e. $T^\circ = T$. Recall (see section 3) that a two-orbit variety is projective. So we can embed $X$ in $P(V)$ with $V$ a finite $G$-module. The elements $x$ and $y$ of $X$ can be written as: $x = [v_\mu]$ and $y = [v_\lambda]$ with $v_\mu$ and $v_\lambda$ weight vectors of $V$. We can choose $\lambda$ to be dominant.

Let $\beta \in \Phi^+$ be such that $U_\beta \not\subset G_x$ (such a root exists otherwise $B \subset G_x$) and consider $z = [v_\mu + v_{\mu + \beta} + \cdots + v_{\mu + k_\beta}] \in U_\beta \cdot x$. With a judicious choice of $x$, (for instance $\mu$ dominant), one shows easily that $[v_{\mu + k_\beta}] \in G \cdot y$. So finally, we can
choose \( x \) such that \( \lambda = \mu + k\beta \) (take a \( W \)-conjugate of the previous \( x \)). Define the support of a root (denoted \( \text{supp} \)) as the set of simple roots really involved in its writing.

**Proposition 5.1.** If \( X \) is cuspidal then \( G \) is simple and \( \text{supp} \beta = \Delta \).

**Proof.** The first assertion comes from the construction of the minimal induction pair (see section 5.2). More precisely, as \( X \) is cuspidal, \( G \) must be generated by the parabolic subgroup \( P_1 \) and the elements \( w \in W \) such that \( w \cdot [v_\lambda] \in X_1 \). Recall that \( P_1 \) is spanned by \( H, B \) and the elements \( n \in N_G(T) \) such that \( Bn \cdot x \) is closed in \( G \cdot x \) and \( X_1 = \text{Cl}(P_1 \cdot x) \).

If \( G = G_1 \times \cdots \times G_r \) with \( G_i \) simple and \( \Delta = \Delta_1 \times \cdots \times \Delta_r \) with \( \Delta_i \) associated to \( G_i \), we are going to show that \( H = H_1 \times \cdots \times H_r \) with \( H_i = G_i \) for all \( i \neq i_0 \) and \( \text{supp} \beta \subset \Delta_{i_0} \).

First of all, note that if \( \alpha \notin \text{supp} \beta \), then \( U_\alpha \subset G_x \). Indeed, \( \mu + \ell \alpha \) (\( \ell > 0 \)) is not of shape \( \lambda - \sum_{\gamma \in \Delta} n_\gamma \cdot \gamma \) (weights of \( V \)) since \( \lambda = \mu + k\beta \) (see the choice of \( x \) made above). Moreover, if \( \alpha \in \Delta \setminus \Delta_{i_0} \) then with the above description of \( G \), we must have \( U_{-\alpha} \subset G_x \). Thus: \( U_{\pm \alpha} \subset G_x \), for all \( \alpha \in \Delta \setminus \Delta_{i_0} \). The acting group \( G \) can then be assumed to be simple.

To obtain the second assertion, consider the parabolic subgroup \( P \) associated to \( \text{supp} \beta \) and the \( P \)-variety \( Z = X \cap \mathbb{P} \left( \bigoplus_{\nu} V_\nu \right) \) for \( \nu = \lambda - \sum n_\alpha \alpha \) with \( \alpha \in \text{supp} \beta \) and \( n_\alpha \geq 0 \). Then, if \( \text{supp} \beta \neq \Delta \) \( (P, Z) \) is an induction pair of \( X \). \( \square \)

From now on, \( G \) will be simple and \( x \) will satisfy the above conditions as well as the two following ones. If \( \lambda = \mu + k\beta \) then \( k \) is minimal and moreover if \( [v_\nu] \) is another \( T \)-fixed generic element satisfying the same assumptions as \( x \) then \( \mu > \mu' \) for \( \mu \) and \( \mu' \) comparable.

Consider a second positive root, say \( \gamma \), with \( \gamma \neq \beta \) and \( U_\gamma \not\subset G_x \) (there exists at least one such a root which is simple). Denote by \( L \) the Levi subgroup associated to \( \beta \) and \( \gamma \), i.e. \( L \) is the centralizer in \( G \) of \( (\ker \beta \cap \ker \gamma)^0 \). Then we have:

**Lemma 5.2.** The variety \( \text{Cl}(L \cdot x) \) is a cuspidal two-orbit \( L \)-variety.

**Proof.** From Lemma 5.1 we have \( L \cdot x = \text{Cl}(L \cdot x) \cap G \cdot x \) and \( \text{Cl}(L \cdot x) \setminus L \cdot x \) consists of a finite union of complete \( L \)-orbits. Thus, if \( \text{Cl}(L \cdot x) \) is not a two-orbit \( L \)-variety, we will have \( (L \cdot x)^{U_L} \neq \emptyset \) for \( U_L \) an unipotent maximal subgroup of \( L \) (again the same argument as in the introduction). It will follow that \( \text{Cl}(L \cdot x) = L \cdot x \). But this equality cannot hold since \( [v_\lambda] \in \text{Cl}(L \cdot x) \setminus L \cdot x \).

As we can not find any proper parabolic subgroup \( P \) such that \( P^u \subset L_x \subset P \) (because \( U_{\pm \beta}, U_{\pm \gamma} \not\subset G_x \)), \( L \cdot x \) must be cuspidal. \( \square \)

By this procedure, we have constructed two-orbit varieties for some subgroups \( L \) of \( G \) of semisimple rank 2. Thus, once we know what the two-orbit varieties are, for the simple groups of rank 2, we will know \( L_x \) (for all \( L \)'s) and then \( G_x \).
Let us start with determining the cuspidal two-orbit varieties in the rank 2 case. For this, we need two technical lemmas. Let \( \Lambda \) be the convex hull of the support of the variety \( \text{Cl}(U_7 U_\beta \cdot x) \). We have the following picture and notation in the weight lattice \( \mathcal{X} \) of \( G \).

**Lemma 5.3.** The extremal points \( v_i \) of \( \Lambda \) are \( W \)-conjugated to \( \lambda \) and the \( \beta_i \)'s are some roots of \( \Phi \).

*Proof.* First of all, note that the points \([v_{\nu_i}]\) and the elements \([v_i]\) whose support is \([\nu_i, \nu_{i+1}] \cap \mathcal{X}\) sit in \( \text{Cl}(U_7 U_\beta \cdot x) \). If \( \Lambda \) has more than three extremal points then the cardinality of \( \text{supp}(v_i) \) is smaller than \( \text{supp} x \)'s one. So by the minimality assumption made on \( x \), we must have \([v_i] \in G \cdot y \) and thus \([v_{\nu_i}] \in G \cdot y \). If \( \Lambda \) has three extremal points, the assumed maximality of the weight \( \mu \) \((x = [v_{\mu}] \)) forces \( \nu_1 \) to be in \( W \cdot \lambda \).

The second assertion of the lemma follows from the fact that \( T^0_{[v_i]} = \ker(\nu_i - \nu_{i+1})^\circ \) is singular. \( \square \)

**Lemma 5.4.** Let \( \delta \) be a positive root such that \([v_{s_{\gamma'}(\lambda)}] \in \text{Cl}(U_\delta \cdot x) \), for \( \gamma' \in \Delta \). Suppose that: if there exists \( r \geq 0 \) such that \( \mu + r\gamma' \) is extremal as weight of \( V \), we must have \( \mu + r\gamma' = s_\alpha(\lambda) \) for \( \alpha \in \Delta \). Then, \( U_{\gamma'} \subset G_x \) if \( (\mu, \gamma') \geq 0 \) and \( U_{-\gamma'} \subset G_x \) if \( (\mu, \gamma') \leq 0 \).

*Proof.* If there is no \( r > 0 \) such that \( \mu + r\gamma' \) is extremal then by Lemma 5.3, we must have \( U_{\gamma'} \subset G_x \) if \((\mu, \gamma') \geq 0 \). If \((\mu, \gamma') \leq 0 \) and \( U_{-\gamma'} \not\subset G_x \) then \( \text{Cl}(U_{-\gamma'} \cdot x) \) contains a \( T \)-fixed point of \( G \cdot y \) and so does \( \text{Cl}(U_{\gamma'} \cdot x) \) – which is absurd.

Suppose now \( \mu + r\gamma' \) is extremal and that \((\mu, \gamma') \geq 0 \). Let \( x' = \varepsilon_\delta(1) \cdot x \in U_\delta \cdot x \), \( \varepsilon_\delta \) being the natural map associated to \( \delta \) from \( \mathbb{C} \) to \( U_\delta \). Then by assumption \( x' = [v_{\mu} + \cdots + v_{s_{\gamma'}(\lambda)}] \). The study of \( \text{supp}(\varepsilon_\beta(1) \cdot x') \) gives raise to a \( j > 0 \) such that

\[
-\frac{Y^{n_j}}{n_j!} v_{\mu + j\delta} = s Y_{\gamma'} \cdot v_{s_{\gamma'}(\lambda)} \quad \text{for } s \in \mathbb{C} \setminus \{0\} \text{ and } n_j \geq 0. \tag{1}
\]

Recall (see section 2) that \( Y_\beta \) denotes an element of \( g_\beta \setminus \{0\} \).

Furthermore, consider the support of the variety \( \text{Cl}(T \cdot \exp(Y_{\beta} + sY_{\gamma'} \cdot x')) \). The points \( s_{\alpha}(\lambda), s_{\gamma'}(\lambda) \) are extremal points of this support but \( \lambda \) is not; otherwise, we will have a gap in this support since the weight \( \gamma' + s_{\gamma'}(\lambda) \) is missing by equality (1). It implies that \([s_{\alpha}(\lambda), s_{\gamma'}(\lambda)]\) must be an edge of this support – which contradicts Lemma 5.3 since \( s_{\alpha}(\lambda) - s_{\gamma'}(\lambda) \) is not a root, \( \alpha \) and \( \gamma' \) being simple.

If \((\mu, \gamma') \leq 0 \), we can go back to the positive case with the element \( s_{\gamma'} \cdot x \). \( \square \)

Let us show how we can apply these two lemmas to get the two-orbit varieties in the rank 2 case, through the following
Example. Suppose $G$ of type $B_2$. Then there are two possibilities for $\beta$: $\beta = \alpha_1 + \alpha_2$ or $\beta = \alpha_1 + 2\alpha_2$. Let us compute $g_x$ in case $\beta = \alpha_1 + \alpha_2$.

If $U_{\alpha_i} \not\subset G_x$ then $\alpha_2$ verifies the conditions of Lemma 5.4 with $(\mu, \alpha_2^\vee) \geq 0$. Therefore $U_{\alpha_2} \subset G_x$. Let us show that $U_{\alpha_1 + 2\alpha_2} \subset G_x$. If there does not exist $r > 0$ such that $\mu + r(\alpha_1 + 2\alpha_2)$ is extremal then this inclusion is given by Lemma 5.3. Otherwise, we will have $\mu + r\alpha = s_1(\lambda)$. A simple computation leads to: $2(\lambda, \alpha_1^\vee) = (\lambda, \alpha_2^\vee)$. But in this latter case, $\mu$ does not satisfy the good conditions; in particular, $\lambda$ is not minimal. We have obtained the pair: $(SO_5, g_x = t \oplus g_{\alpha_2} \oplus g_{\alpha_1 + 2\alpha_2})$.

If $U_{\alpha_1} \subset G_x$ then $U_{\alpha_2} \not\subset G_x$ and we have $(\mu, \pm\alpha_1) = 0$. Thus by Lemma 5.4, we get $U_{\pm\alpha_1} \subset G_x$. It follows from Lemma 5.3 that $U_{\alpha_1 + 2\alpha_2} \subset G_x$. Therefore, we have obtained the pairs: $(SO_5, \mathfrak{g}_x = t \oplus g_{\pm\alpha_1} \oplus g_{\alpha_1 + 2\alpha_2})$.

By the procedure given in this example, we get the two-orbit varieties for the rank 2 case. In other words, we have a description of $\Phi(G_x)$ the root system of $G_x$.

Lemma 5.5. Let $\gamma \in \Phi^+$, we have the following:

(i) if $(\gamma, \beta)$ is of type $A_1 \times A_1$ then $\pm \gamma \in \Phi(G_x)$;
(ii) if $(\gamma, \beta)$ is of type $A_2$ then $\pm \gamma \in \Phi(G_x)$ or $\pm s_\gamma(\beta) \in \Phi(G_x)$;
(iii) if $(\gamma, \beta)$ is of type $B_2$ with $\beta = \varepsilon_1 + \varepsilon_j$ then $\Phi(G_x) \cap (\gamma, \beta) = \{\varepsilon_1 - \varepsilon_j, \varepsilon_1\}$;
(iv) if $(\gamma, \beta)$ is of type $B_2$ with $\beta = \varepsilon_1$ then $\Phi(G_x) \cap (\gamma, \beta) \supset \{\varepsilon_1 + \varepsilon_j, \pm(\varepsilon_1 - \varepsilon_j)\}$ or $\Phi(G_x) \cap (\gamma, \beta) = \{\varepsilon_1 + \varepsilon_j, \varepsilon_j\}$.

By applying this statement to $\beta$, a positive root of maximal support and to any other positive root, we get Table 1 just by computations. Let us work out two examples to understand how it works.

Example. Suppose $G$ of type $A_n$. Then $\beta = \alpha_1 + \cdots + \alpha_n$. Since $(\alpha_i, \beta)$ is of type $A_1 \times A_1$, by Lemma 5.5(i), we have $\pm \alpha_i \in \Phi(G_x)$ for $2 \leq i \leq n - 1$. Moreover, Lemma 5.5(ii) yields: $\pm \alpha_1 \in \Phi(G_x)$ or $\pm \alpha_n \in \Phi(G_x)$. This gives the pair $(SL_{n+1}, \mathfrak{g}_{\ln})$.

Suppose $G$ of type $B_n$ and $\beta = \varepsilon_1$. Applying Lemma 5.5(i) and Lemma 5.2(ii) respectively, we get $\{\pm \varepsilon_2, \ldots, \pm \alpha_n\} \subset \Phi(G_x)$ and respectively $\{\varepsilon_1 + \varepsilon_i : i \geq 2\} \subset \Phi(G_x)$ with $\pm (\varepsilon_1 - \varepsilon_i) \in \Phi(G_x)$ or $\varepsilon_i \in \Phi(G_x)$ for all $i \geq 2$. If $\pm \alpha_i \notin \Phi(G_x)$ then $\{\pm \alpha_1, \ldots, \pm \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n\} \subset \Phi(G_x)$. Therefore, we obtain the two pairs $(SO_{2n+1}, \mathfrak{g}_{\ln})$ and $(SO_{2n+1}, \mathfrak{g}_x = \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha \oplus t)$ where $\Psi = \{\pm \alpha_1, \ldots, \pm \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n\}$. If $\pm \alpha_1 \notin \Phi(G_x)$, we get the pair $(SO_{2n+1}, \mathfrak{g}_x = \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha \oplus t)$, with $\Psi = \{\pm \alpha_2, \ldots, \pm \alpha_{n-1}, \alpha_n, \varepsilon_1 + \varepsilon_n\}$.

6. Two-orbit varieties of type II

In this section, the two-orbit $G$-variety $X = \text{Cl}(G \cdot x)$ is of type II, that is, by definition, $T_x^G \neq T$. We embed $X$ in $\mathbb{P}(V)$ with $V$ a finite $G$-module as in the previous section. Then the generic element $x$ can be written as $[v_{\lambda_0} + \cdots + v_{\lambda_{r+1}}]$.
with the \( \lambda_i \)'s sitting on a same affine line, say \( D_x \) (because of Proposition 3.3), we order the weights \( \lambda_i \) of \( \supp x \) in such way that: \( \Cl(T \cdot x) = T \cdot x \cup \{ \nu_{\lambda_0}, \nu_{\lambda_{r+1}} \} \).

As elements of \( G \cdot y, [\nu_{\lambda_0}] \) and \( [\nu_{\lambda_{r+1}}] \) are \( W \)-conjugate, i.e. there exists \( w \in W \) such that \( \lambda_0 = w(\lambda_{r+1}) \). We choose \( x \) such that \( \lambda_0 = \lambda \) and satisfying the following condition of minimality: if \( x' = [\nu_{\lambda} + \cdots + v_{w'(\lambda)}] \) is another generic element then \( w < w' \) for \( w \) and \( w' \) comparable.

Let \( \alpha \) be a simple root such that \( s_\alpha w < w \) and \( U_\alpha \not\subset G_x \). Take for instance \( \alpha \) such that \( w(\lambda), \alpha \rangle < 0 \). Consider the variety \( X_\alpha = \Cl(TU_\alpha \cdot x) \) and in particular, the convex hull \( \Lambda \) of its support pictured below in the weight lattice \( \mathcal{X} \) of \( G \).

Similarly as for Lemma 5.3, we get:

**Lemma 6.1.** The elements \( y_i \in X_\alpha \) whose support sits on the line \( D_i \) belong to the closed \( G \)-orbit of \( X \). The directions of the affine lines \( D_i \) are given by roots \( \beta_i \).

In particular, the extremal points \( \nu_i \) are \( W \)-conjugated and \( \supp y_i = [\nu_i, \nu_{i+1}] \cap \mathcal{X} \).

The roots \( \beta_i \) span a root system of rank 2; let \( \{ \alpha, \beta \} \) be a basis of this root system.

**Corollary 6.2.** \( w = s_\alpha s_\beta \).

**Proof.** We know that \( s_\alpha w < w \) and that \( \lambda - w(\lambda) \) can not be, up to a scalar, a root because \( T^\circ_x = \ker(\lambda - w(\lambda))^\circ \) is regular (see Proposition 3.4). So we can assume \( \langle \alpha, \beta \rangle \) to be of type \( G_2 \) with \( w = (s_\alpha s_\beta)^2 \) – the case \( w = w_0 \) being easily ruled out.

Consider the convex hull \( \Lambda \) (see the corresponding picture). Since we have: \( \supp y_i = [\nu_i, \nu_{i+1}] \cap \mathcal{X} \), there exists a weight \( \nu \in D_x \) such that \( \lambda - \beta_i = \nu + k\alpha, \) \( k \geq 0 \). In other words, if \( D_{\alpha, \lambda - \beta_i} \) denotes the line of direction \( \alpha \) passing through \( \lambda - \beta_i \), we must have:

\[
D_x \cap D_{\alpha, \lambda - \beta_i} \cap \mathcal{X} \neq \emptyset. \tag{2}
\]

If \( w = (s_\alpha s_\beta)^2 \) with \( \langle \alpha, \beta \rangle \) of type \( G_2 \) then \( \beta_i = \beta \) or \( \beta = \beta + \alpha \). But (2) forces \( (\lambda, \alpha)/(\langle \lambda, \alpha \rangle + (\lambda, \beta)) \) to be an integer – which can not occur because \( (\lambda, \alpha) \cdot (\lambda, \beta) \neq 0 \) since \( T^\circ_x \) is regular. \( \square \)

Using this corollary and the same arguments given in the proof of Proposition 5.4, we get:

**Proposition 6.3.** If \( X \) is cuspidal of type II then \( G \) is simple or of type \( A_1 \times A_1 \).

Furthermore, \( \supp \beta \cup \{ \alpha \} = \Delta \).

From now on \( G \) will be assumed to be simple or of type \( A_1 \times A_1 \). Let \( L \) be the Levi subgroup associated to \( \alpha \) and \( \beta \) and \( l \) be its Lie algebra.

**Proposition 6.4.** (i) If \( \langle \alpha, \beta \rangle \) is of type \( A_1 \times A_1 \) then \( I_x = t' + \mathbb{C}(Y_{-\alpha} + Y_\beta) + \mathbb{C}(Y_\alpha + Y_{-\beta}) \).
(ii) Otherwise, \( l_x = l' \oplus \mathbb{C}(Y_{-\alpha} + Y_{s_{\alpha}(\beta)}) \oplus \bigoplus_{\gamma \in \Phi^+ \cap \{\alpha, \beta\}} \mathfrak{g}_\gamma. \)

Here, \( l' \) is the kernel of \( \alpha + s_{\alpha}(\beta) \) considered as element of the dual \( \mathfrak{t}^*. \)

**Proof.** Set \( \delta = s_{\alpha}(\beta), (s_{\alpha}s_{\beta}(\lambda), \delta^\vee) < 0. \) Thus \( Y_\delta \notin \mathfrak{g}_{v_{m\omega}} \) and \( Y_\delta \notin \mathfrak{g}_x. \)

In order to get: \( Y_\delta + Y_{-\alpha} \in \mathfrak{g}_x, \) we are going to prove \((\lambda, \alpha^\vee) = (\lambda, \beta^\vee).\) (3)

Set \( \lambda = m\omega_{\alpha} + n\omega_{\beta} (m, n > 0) \) and consider the variety \( \text{Cl}(TU_{\alpha} \cdot x). \) The arguments used in the proof of the previous corollary give a weight \( \lambda_i \in \text{supp} \ x \) such that \( Y_{\lambda_i} \cdot v_{\lambda_i} \) is of weight \( \lambda - \beta \) (for \( r > 0 \)). Translating the latter in terms of equations, we get \( \lambda_i = \lambda - \alpha_i/m\delta \) and \( n/m \in \mathbb{N}. \) Similarly, considering the variety \( \text{Cl}(U_{\alpha} \cdot x), \) we get \( m/n \in \mathbb{N} \) thus \( m = n \) and also

\[
Y_{-\alpha} \cdot v_{\lambda_i} = -qY_\delta \cdot v_{\lambda_i}, \quad q \in \mathbb{C}^*. \quad (4)
\]

Finally, let \( Z_t \) be the variety \( \text{Cl}(T \exp t(Y_\delta + qY_{-\alpha}) \cdot x), t \in \mathbb{C}. \) Its support is entirely contained in the triangle of vertices \( \lambda, s_{\alpha}(\lambda) \) and \( w(\lambda). \) But there is a gap in this support: the weight \( \lambda - \alpha \) is missing because of \([4].\) This implies that \( \exp t(Y_\delta + qY_{-\alpha}) \in G_x. \)

To show that \( U_{\gamma} \subset G_x \) for all \( \gamma \in \Phi \setminus \{\alpha, \delta\}, \) we proceed by contradiction and as before, we will find a gap in the support of \( \text{Cl}(U_{\alpha} \cdot x). \)

To conclude, we have to note that if \( \langle \alpha, \beta \rangle \) is not of type \( A_1 \times A_1, \) then \( Y_{\alpha} + Y_{-s_{\alpha}(\beta)} \notin \mathfrak{g}_x. \) The first assertion is obtained just by symmetry. \( \Box \)

As a consequence of the previous proof, we have

**Corollary 6.5.** (i) \( (\lambda, \alpha^\vee) = (\lambda, \beta^\vee); \)

(ii) \( T_x^\circ = \ker (\alpha + s_{\alpha}(\beta))^\circ; \)

(iii) \( N_G(H)/H \) is finite;

(iv) \( \text{Cl}(L \cdot x) \) is a cuspidal two-orbit \( L \)-variety.

The main thing to do, in order to get the cuspidal two-orbit varieties of type II, is to give the list of the possible \((G, \alpha, \beta)\) where \( \alpha \) and \( \beta \) are the positive roots considered previously. If \( G \) is of rank \( 2, \) it is done already by Proposition [1,3].

So the acting group \( G \) will be definitely of rank greater than 2. Sum up the properties of \( \alpha \) and \( \beta: \)

1. \( \alpha \in \Delta, (w(\lambda), \alpha) \leq 0 \) and \( U_\alpha \not\subset G_x; \)
2. \( \lambda - w(\lambda) \in \langle \alpha, \beta \rangle; \)
3. \( \{\alpha, \beta\} \) basis of \( \langle \alpha, \beta \rangle; \)
4. \( \text{supp} \beta \cup \{\alpha\} = \Delta; \)
5. \( w = s_{\alpha}s_{\beta}; \)
6. \( (\lambda, \alpha^\vee) = (\lambda, \beta^\vee). \)
It will appear quickly that there are very few roots satisfying all these conditions mainly because of the two following statements.

**Remark.** Let $\gamma_1$, $\gamma_2$, and $\gamma_3$ be three roots of $\Phi$ spanning a root system $\Psi$ of rank 3. Suppose $\Psi$ verifies the property:

$$\eta = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3 \in \Psi \implies \eta - n_3\gamma_3 \in \Phi \text{ (up to a scalar).}$$

Then, $\langle \gamma_1, \gamma_2 \rangle \cap \langle \gamma_3, \eta \rangle$ is generated by a root or is equal to \{0\}, if $\eta$ is a root of $\Phi$.

Let us consider a simple root $\delta$ such that $(\beta, \delta) > 0$ (then $\delta \in \text{supp} \beta$). Set $\gamma = s_\alpha(\delta)$. Then

$$(w(\lambda), \gamma) \leq 0. \quad (5)$$

**Lemma 6.6.** If $\alpha$, $\beta$, and $\gamma$ satisfy the property given in the remark with $\gamma = \gamma_3$ then $(w(\lambda), \gamma) = 0$.

**Proof.** If $(w(\lambda), \gamma) \neq 0$ then according to (5), $(w(\lambda), \gamma) < 0$. And therefore, there exists $\eta \in \Phi$ such that $\lambda - w(\lambda) \in \langle \gamma, \eta \rangle$ (argue similarly as we did to get the root $\beta$). It implies: $\lambda - w(\lambda) \in \langle \alpha, \beta \rangle \cap \langle \gamma, \eta \rangle$. But this is impossible because of the remark and the fact that $T^\circ_\gamma$ is regular (see Proposition 3.4). \hfill \square

Start with $G$ classical and suppose: $(\alpha, \delta) = 0$. If $\text{supp} \beta = \Delta$ then $\alpha$, $\beta$, and $\delta$ satisfy the conditions of Lemma 6.6. Therefore we get: $(\lambda, \delta) = (\lambda, \beta)$. But this equality is incompatible with $(\lambda, \alpha'') = (\lambda, \beta'')$. So assume that $\text{supp} \beta = \Delta \setminus \{\alpha\}$. Then we have the following possibilities for $(G, \alpha, \beta)$:

- $(A_n, \alpha_i(i = 1, n), \alpha - \alpha_i)$;
- $(G, \alpha_n, \varepsilon_1 - \varepsilon_n)$ for $G = B_n, C_n$;
- $(G, \alpha_1, \varepsilon_2 + \varepsilon_n)$ for $G = B_n, C_n$;
- $(G, \alpha_1, \varepsilon_2 + \varepsilon_j(2 < j < n))$ for $G = B_n, C_n, D_n$;
- $(D_n, \alpha_{n-1}, \varepsilon_1 + \varepsilon_n)$;
- $(D_n, \alpha_n, \varepsilon_1 - \varepsilon_n)$.

Applying Lemma 6.6, we end up again with a contradiction. Thus necessarily, $(\alpha, \delta) < 0$. The possible triples $(G, \alpha, \beta)$ are now:

- $(A_3, \alpha_2, \tilde{\alpha})$;
- $(G, \alpha_2, \varepsilon_1 + \varepsilon_3)$ for $G = B_n, C_n$ or $D_n$;
- $(B_3, \alpha_3, \tilde{\alpha})$;
- $(B_n, \alpha_1, \varepsilon_2)$;
- $(C_n, \alpha_2, \tilde{\alpha})$;
- $(C_n, \alpha_1, 2\varepsilon_2)$. 

Claim. In all these cases, \( (w(\lambda), \gamma) \neq 0 \).

But because of Lemma 6.6, we may also have:

- \((A_3, \alpha_2, \hat{\alpha})\);
- \((C_3, \alpha_2, \hat{\alpha})\);
- \((B_3, \alpha_3, \hat{\alpha})\);
- \((C_n, \alpha_2, \hat{\alpha})\);
- \((D_n, \alpha_1, \hat{\alpha})\).

The fourth triple is ruled out just by considering \((w(\lambda), \alpha_1)\). The other ones give raise to some pairs of Table 2.

For the exceptional case, we proceed similarly and we obtain the left pairs of Table 2. This ends the proof of the main theorem: the classification of two-orbit varieties.

Appendix

To make Table 3 and Table 4 readable to the reader, we will need the following notation.

We shall recall, at first, that, in these tables, the two-orbit varieties are cuspidal with connected generic stabilizers. All other two-orbit varieties are obtained (see section 2) either by parabolic induction or as the quotient of a cuspidal two-orbit variety by \( H/H^\circ \) for \( H^\circ \) the corresponding connected generic stabilizer given in Tables 1 and 2.

In the first column, we have listed the type of the group \( G \) acting on the two-orbit variety designed in the second column. When the action is obvious, we shall not state it precisely.

Once we have the acting group, we fix a Borel subgroup \( B \) and a maximal torus in it and use the standard notation (see [4]) to denote \( \alpha_i \) the simple roots, \( \omega_i \) the fundamental weight corresponding to \( \alpha_i \), \( s_i \) the simple reflection of the Weyl group associated to \( \alpha_i \) and \( P_i \) the maximal parabolic subgroup attached to \( \alpha_i \).

The Grassmannian of \( m \)-planes in \( \mathbb{C}^n \) is denoted by \( \text{Gr}(n; m) \). For instance, \( \text{Gr}(n; 1) \) is just \( \mathbb{P}^n \) and \( \text{Gr}(n; n - 1) \) is \( \mathbb{P}^n \).

We have a natural action of \( SO_n \) on the quadric \( Q(n) = \{ [z_0 : z_1 : \ldots : z_n] \in \mathbb{P}^n : z_0^2 = \sum z_i^2 \} \) given by \( g \cdot [z_0 : z'] = [z_0 : g \cdot z'] \), \( g \in SO_n \) and \( z' \in \mathbb{C}^n \).

As usual, \( \Sigma_k \) denotes the Hirzebruch surface: the 2-dimensional smooth complete torus embedding corresponding to the integer \( k \). In particular, when \( k = 1 \), the normalization of \( \Sigma_k \) is just \( \mathbb{P}^2 \).
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### Table 3. Two-orbit varieties of type I

| Group | Description                        |
|-------|------------------------------------|
| $A_n$ | $\mathbb{P}^n \times \mathbb{P}^n$ |
| $B_n$ | $Q(2n+1)$                           |
| $B_n$ | $\{ (\ell, P) \in \mathbb{P}^{2n+1} \times \text{Gr}(2n+1; n) : \ell \subset P, \ P \text{ totally isotropic} \}$ |
| $B_n$ or $C_n$ | $G/P_1 \times G/P_n$ |
| $C_n$ | $\text{Gr}(2n; 2)$               |
| $C_n$ | $\{ (\ell, P) \in \mathbb{P}^{2n} \times \text{Gr}(2n; 2) : \ell \subset P \}$ |
| $F_4$ | $E_6/P_6$                           |
| $F_4$ | $\text{Cl}(G \cdot x) \subset \mathbb{P}(V(\lambda) \otimes V(\omega_4))$ with $x = [v_\mu], \ \mu = \lambda - \beta, \ \lambda = \omega_1 + \omega_4$ and $\beta = 1111$ |
| $G_2$ | $G \times_B \Sigma_3$             |
| $G_2$ | $G \times_B \Sigma_2$             |
| $G_2$ | $G_2 \times P_1 \mathbb{P}^2$ compactification of $P_1/H = \mathbb{C}^2$ |
| $G_2$ | $Q(7)$ action induced by $SO_7$'s |

### Table 4. Two-orbit varieties of type II

| Group | Description                        |
|-------|------------------------------------|
| $A_1 \times A_1$ | $\mathbb{P}^3$ compactification of $SL_2$ |
| $A_2$ | $\{ (A, z) \in SL_3 \times \mathbb{P}^2 : A \text{ nilpotent and } Az = 0 \}$ |
| $B_2$ or $G_2$ | $\text{Cl}(G \cdot [v_\lambda + v_w(\lambda)]) \subset \mathbb{P}(V(\lambda))$ with $\lambda = \omega_1 + \omega_2, \ w = s_1s_2$ |
| $B_3$ | $Q(8)$ action induced by $SO_9$'s |
| $C_3$ | $\{ z = [\sum_{i,j} v_i \wedge v_j] \in \mathbb{P}(\wedge^2 \mathbb{C}^6) : \sum_{i,j} \omega(v_i, v_j) = 0 \}$ |
| $D_n$ | $Q(2n)$                             |
| $F_4$ | $\text{Cl}(G \cdot x) \subset \mathbb{P}(V(\lambda))$ with $x = [v_\lambda + v_w(\lambda)], \ \lambda = \omega_4, \ w = s_\alpha s_\beta, \ \alpha = \alpha_4$ and $\beta = 1231$ |