Segment representation of a subclass of co-planar graphs

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Abstract. A graph is said to be a segment graph if its vertices can be mapped to line segments in the plane such that two vertices have an edge between them if and only if their corresponding line segments intersect. Kratochvíl and Kuběna [2] asked the question of whether the complements of planar graphs are segment graphs. We show here that the complements of all partial 2-trees are segment graphs.

1 Introduction

Given a family of sets $\mathcal{F}$, a simple, undirected graph $G(V,E)$ is said to be an “intersection graph of sets from $\mathcal{F}$" if there exists a function $f : V(G) \to \mathcal{F}$ such that for $u, v \in V(G)$ such that $u \neq v$, $uv \in E(G) \iff f(u) \cap f(v) \neq \emptyset$. We let $IG(\mathcal{F}) = \{ G \mid G \text{ is an intersection graph of sets from } \mathcal{F} \}$. When $\mathcal{F}$ is a collection of geometric objects, $IG(\mathcal{F})$ is said to be a class of “geometric intersection graphs”. Some well-known classes of geometric intersection graphs are:

$\text{INT} = IG(\{\text{all intervals on the real line}\})$ ("Interval graphs")

$\text{STRING} = IG(\{\text{all simple curves in the plane}\})$ ("String graphs")

$\text{CONV} = IG(\{\text{all convex arc-connected regions in the plane}\})$

$\text{SEG} = IG(\{\text{all straight line segments in the plane}\})$ ("Segment graphs")

Clearly, $\text{SEG} \subseteq \text{CONV} \subseteq \text{STRING}$. The last inclusion follows from the fact that string graphs are exactly the intersection graphs of arc-connected regions in the plane. It was shown in [2] that the complement of every planar graph is in $\text{CONV}$. In the same paper, the authors pose the question of whether the complement of every planar graph is in $\text{SEG}$. A positive answer to this question would imply that the $\text{MAXCLIQUE}$ problem for segment graphs is NP-complete, thus resolving a long standing open problem raised by Kratochvíl and Nešetřil in 1990 [3]. It is worth noting that the question of whether every planar graph is in $\text{SEG}$, known as Scheinerman’s conjecture, was resolved by Chalopin and Gonçalves [1] who showed that every planar graph is indeed the intersection graph of line segments in the plane.

Partial 2-trees are a subclass of planar graphs that includes series-parallel graphs and outer-planar graphs as proper subclasses. In this paper, we show that the complement of every partial 2-tree is in $\text{SEG}$. 
2 Definitions

All the graphs that we consider shall be finite, simple and undirected. We denote the vertex set of a graph \(G\) by \(V(G)\) and its edge set by \(E(G)\). Given a graph \(G\) and \(X \subseteq V(G)\), we denote the subgraph induced by \(V(G) \setminus X\) in \(G\) as \(G - X\). The complement of a graph \(G\), denoted as \(\overline{G}\), is the graph with \(V(\overline{G}) = V(G)\) and \(E(\overline{G}) = \{uv \mid u \neq v\text{ and } uv \not\in E(G)\}\).

2.1 Partial 2-trees

A \(2\)-tree is defined as follows:

1. A single edge is a \(2\)-tree.
2. If \(G\) is a \(2\)-tree, then the graph \(G'\) with \(V(G') = V(G) \cup \{v\}\) and \(E(G') = E(G) \cup \{vx, vy\}\) where \(xy \in E(G)\) is a \(2\)-tree.

A partial \(2\)-tree is any spanning subgraph of a \(2\)-tree.

2.2 Segments, rays and compatible segment representations

Let \(a, b \in \mathbb{R}^2\) be two points in the plane. Then:

- A segment with end-points \(a\) and \(b\), denoted as \(ab\), is the set \(\{a + \rho(b - a) \mid \rho \in [0, 1]\}\). Any point on a segment that is not one of its end-points is said to be an interior point of the segment. For the purposes of this paper, we shall assume that every segment has a non-zero length—i.e., the end-points of a segment may not coincide.
- A ray starting at a point \(a\) and passing through a point \(b\) is the set \(\{a + \rho(b - a) \mid \rho \in [0, \infty)\}\).
  A ray has a single end-point, which is its starting point.

Given \(l_1\) and \(l_2\), where each could be a segment or a ray, they are said to cross each other if \(l_1 \cap l_2\) consists exactly of a single point that is not an end-point of either \(l_1\) or \(l_2\). If \(l_1 \cap l_2 = \emptyset\), then they are said to be disjoint.

Given a segment graph \(G\), there exists a function \(f : V(G) \rightarrow \mathcal{R}\) such that \(\forall u, v \in V(G), u \neq v, f(u) \cap f(v) \neq \emptyset \Leftrightarrow uv \in E(G)\) where \(\mathcal{R}\) is a collection of segments. We say that \(\mathcal{R}\) is a “segment representation” of \(G\).

**Definition 1.** Let \(G\) be a partial \(2\)-tree and let \(G_T\) be a \(2\)-tree of which \(G\) is a spanning subgraph. Let \(\mathcal{R}\) be a segment representation of \(G\) with segments \(\{s_u \mid u \in V(G)\}\). \(\mathcal{R}\) is said to be a segment representation of \(G\) that is “compatible with \(G_T\)” if for each \(uv \in E(G_T)\), a ray \(r_{uv}\) can be drawn in \(\mathcal{R}\) such that the collection of these rays satisfies the following properties:

1. \(r_{uv}\) starts from an interior point on one of \(s_u\) or \(s_v\) and passes through an end-point of the other and meets no other points of either \(s_u\) or \(s_v\).
2. \(r_{uv}\) crosses every segment other than \(s_u\) and \(s_v\), and
3. \(r_{uv}\) crosses every ray \(r_{xy}\) where \(xy \in E(G_T) \setminus \{uv\}\).

The rays \(r_{uv}\) where \(uv \in E(G_T)\) shall be called “special rays”.
3 The construction

Theorem 1. If $G$ is a partial 2-tree which is a spanning subgraph of a 2-tree $G_T$, then $\overline{G}$ has a segment representation that is compatible with $G_T$.

Proof. We shall prove this by induction on $|V(G)|$. There is nothing to prove for $|V(G)| < 2$. If $|V(G)| = 2$, then clearly, $\overline{G}$ has a segment representation compatible with $G_T$ as shown in Figure 1. Consider a partial 2-tree $G$ with $|V(G)| > 2$. By definition of a 2-tree, there exists a degree 2 vertex $v$ in $G_T$ with neighbours $x$ and $y$ such that $xy \in E(G_T)$ and $G_T - \{v\}$ is also a 2-tree. Let $G'$ be $G - \{v\}$. $G'$ is also a partial 2-tree as it is a spanning subgraph of the 2-tree $G_T - \{v\}$. For ease of notation, we shall denote the 2-tree $G_T - \{v\}$ by $G'_T$. By our induction hypothesis, there is a segment representation for $G'$ that is compatible with $G'_T$. We shall show how we can extend this to a segment representation for $\overline{G}$ that is compatible with $G_T$, thereby completing the proof.

Let $R'$ be a segment representation of $\overline{G'}$ which is compatible with $G'_T$ and let $s_u$ denote the segment corresponding to a vertex $u \in V(G')$. We shall add a new segment $s_v$ to $R'$ so that we get the required segment representation $R$ for $\overline{G}$.

Since $R'$ is compatible with $G'_T$, there is a special ray $r_{xy}$ in $R'$ which we shall assume without loss of generality starts from an interior point of $s_x$ and passes through an end-point of $s_y$. Let $p$ be the starting point of the ray $r_{xy}$ on $s_x$ (refer Figure 2). $l_1$ and $l_2$ are two parallel rays starting from points $p_1$ and $p_2$ on $s_x$ on either side of $p$ and parallel to $r_{xy}$ such that they cross every segment and special ray that $r_{xy}$ crosses. By our definition of crossing, the ray $r_{xy}$ meets every segment and special ray that it crosses at a point that is an end-point of neither of them. Therefore, we can always choose rays $l_1$ and $l_2$ distinct from $r_{xy}$ as long as $s_x$ is not a point or parallel to $r_{xy}$. Note that by our definition of $r_{xy}$, $s_x$ cannot lie along $r_{xy}$ and $s_x$ can also not be a point which ensures that the two rays $l_1$ and $l_2$ distinct from $r_{xy}$ can be obtained. Of all the crossing points of segments and special rays on $l_1$, let $q_1$ be the farthest from $p_1$. Let $q_2$ be the similarly defined point on $l_2$. Clearly, any segment or special ray that meets the segment $p_1p_2$ and crosses $q_1q_2$ crosses every segment and special ray that $r_{xy}$ crosses.

The segment $s_v$ is placed in the following way in $R'$ to obtain $R$:

Case 1. $yv \in E(G)$. We place $s_v$ as shown in Figure 3. As $yv \notin E(G)$, the segment $s_v$ is disjoint from the segment $s_y$. Let us first consider the case when $xv \notin E(G)$. In $\overline{G}$, $v$ is adjacent to all vertex $v$ in $G_T$ with neighbours $x$ and $y$ such that $xy \in E(G_T)$ and $G_T - \{v\}$ is also a 2-tree.
Fig. 2. Starting point of construction

Fig. 3. The case when $yv \in E(G)$

Fig. 4. The case when $yv \notin E(G)$
the vertices in $V(G) \setminus \{v, y\}$. This requirement is satisfied, as since $s_v$ meets $p_1p_2$ and crosses $q_1q_2$, it crosses all the segments that cross $r_{xy}$ and meets $s_x$ too. Moreover, $s_v$ also crosses all the special rays in $R'$ including $r_{xy}$. We will show that we can now draw two rays $r_{vy}$ and $r_{xv}$, so that they, together with the special rays in $R'$, form the collection of special rays $R'$. The rays $r_{vy}$ and $r_{xv}$ can be drawn as shown in the figure so that they cross $s_x$ and $s_y$ respectively. In addition, both of them cross each other and $r_{xy}$. Since they also meet $p_1p_2$ and cross $q_1q_2$, each of them crosses all the special rays and segments that cross $r_{xy}$. Thus $R$ is a segment representation of $\overline{G}$ that is compatible with $G_T$. Note that we can draw the segment $s_v$ and the rays $r_{vy}$ and $r_{xv}$ in this way as long as the segments $s_x$ and $s_y$ have non-zero length and $s_y$ does not lie along the ray $r_{xy}$. But our definitions of segments and special rays ensure that these pathological situations do not occur. Now, if $xv \in E(G)$, then $s_v$ can be slightly shortened at the end $p_3$ so that it becomes disjoint from $s_x$ without affecting any of the other arguments so that we still obtain the segment representation $R$ for $\overline{G}$.

Case 2. $yv \not\in E(G)$. We place $s_v$, $r_{xv}$ and $r_{vy}$ as shown in Figure 4. The rest of the argument is similar to that of Case 1.

We thus have a segment representation of $\overline{G}$ which is compatible with $G_T$. This completes the proof.

Corollary 1. If $G$ is any partial 2-tree, then $\overline{G}$ is a segment graph.

Corollary 2. The complements of series-parallel graphs and outerplanar graphs are segment graphs.

Proof. Series-parallel graphs and outerplanar graphs are subclasses of partial 2-trees.

References

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