On Friendly Index Set of Graphs

M. Teffilia and J. Devaraj

1Department of Mathematics, Nesamony Memorial Christian College, Marthandam,Tamilnadu, India
2Department of Mathematics, Nesamony Memorial Christian College, Marthandam,Tamilnadu, India

*Corresponding author: teffiliafranklin@gmail.com
Available online at: www.isroset.org
Accepted 18/Aug/2018, Online 30/Aug/2018

Abstract—A function f from V(G) to \{0,1\} where for each edge xy , \(f^*(xy) = (f(x) + f(y)) mod 2\), let \(v_i(f)\) is the number of vertices v with \(f(v) = i\) and \(e_i(f)\) is the number of edges e with \(f^*(e) = i\) is called friendly if \(|v_i(f) - v_j(f)| \leq 1\). The friendly index set of a graph G is FI(G) = \{|e_i(f) - e_j(f)| \text{ where f runs over all friendly labelings of G}\}. In this paper we find the friendly index set of the umbrella graph, Spl(K_{1,n}), Globe graph ,P_2+mK_1 and union of a path and a star sharing a vertex in common.

Keywords— Friendly labeling, Friendly index set, Umbrella graph, Spl(K_{1,n}), Globe graph

I. INTRODUCTION

For all terminology and notations in graph theory we follow harary[3]. Unless mentioned or otherwise a graph in this paper shall mean a simple finite graph without isolated vertices. Let G be a graph with vertex set V(G) and edge set E(G). A labeling \(f: V(G) \rightarrow Z\) induces an edge labeling \(f^*: E(G) \rightarrow Z\) defined by \(f^*(xy) = f(x) + f(y)\) for each edge xyE(G). For i\in Z, let \(v_i(f) = \text{card}\{v\in V(G): f(v) = i\}\) and \(e_i(f) = \text{card}\{e\in E(G): f^*(e) = i\}\). A labeling f of a graph G is said to be friendly if \(|v_i(f) - v_j(f)| \leq 1\). The friendly index set of the graph G, FI(G) is defined as \{|e_i(f) - e_j(f)| : the vertex labeling f is friendly\}. Lee and Ng [3] define the friendly index set of graphs. This is a generalization of graph cordiality.

This paper consists of four sections. Section I is the introduction part of friendly index set. Section II contains preliminaries and notations. Section III contains main results and illustrations and section IV contains conclusion part.

II. PRELIMINARIES AND NOTATIONS

Definition 2.1[4] : A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions.

Definition 2.2 [10] : A fan graph \(f_n\) obtained by joining all vertices of a path \(P_n\) to further vertex, called the centre. Thus \(f_n\) contains \(n+1\) vertices \(c_1,v_1,v_2, ..., v_n\) and (2n-1) edges say \(cv_i, 1 \leq i \leq n\) and \(v_iv_{i+1}, 1 \leq i \leq n-1\)

Definition 2.3 [2] : Star \(K_{1,n}\) is the graph with one vertex of degree n called apex and n vertices of degree one.

Definition 2.4[7] : A Umbrella graph \(U(m,n)\) is the graph obtained by joining a path \(P_n\) with the central vertex of a fan \(f_m\).

Definition 2.5 [5] : For each vertex v of a graph G take a new vertex v. Join v to all the vertices of G adjacent to v. The graph spl(G) thus obtained is called splitting graph of G.

Definition 2.6 [6] : A globe is a graph obtained from two isolated vertex are joined by n paths of length two. It is denoted by Gl(n)

III. MAIN RESULTS

Theorem 3.1 : The Umbrella graph \(U(m,n)\) where n = 2 has the friendly index set \{0,2,4,...,m+1\} if m is odd, m \geq 9 and \{0,2,4,...,m\} if m is even, m \geq 4

Proof:
Let G = U(m,2)

The vertex set of U(m,2) is \(V(G) = \{x_1,x_2,...,x_m,y_1,y_2\}\)

The edge set of U(m,2) is \(E(G) = \{(x_i,x_{i+1}), 1 \leq i \leq m-1\} \cup \{(y_1,y_2)\}\)

Then \(|V(G)| = m+2, |E(G)| = 2m\)

Case (i) : m is odd
First label alternatively with 0’s and 1’s starting with 0 in x₁’s and label y₁ by 1 and y₂ by 0. Then v(0) - v(1) = 1 and |e(0)| = m+1.

Next interchange only y₁ and y₂. Then v(0) - v(1) = 1 and |e(0)| = m-1.

Next label x₁ by ‘0’ and next two vertices by ‘1’ and following next two vertices by ‘0’ and then alternatively by 0’s and 1’s starting with ‘1’ and label y₁ by 1 and y₂ by 0. Then |v(0) - v(1)| = 1 and |e(0)| = m-3.

Then interchange only y₁ and y₂. Again |v(0) - v(1)| = 1 and |e(0)| = m-5.

Next label x₁ by 0 and next three vertices by 1 and following next three vertices by 0 and then alternatively by 0’s and 1’s starting with 1. Then label y₁ by 1 and y₂ by 0. Then |v(0) - v(1)| = 1 and |e(0)| = m-9.

Continuously proceeding like this ie, label x₁ by 0 and next continuously label up to \( \frac{m-3}{2} \) vertices by 1 and \( \frac{m-3}{2} \) vertices by 0 and then alternatively 0’s and 1’s starting with 1.

At the last step we get the vertex is labeled friendly and |e(0)| = m.

∴ The friendly index set is \{0,2,4,...,m\}

Case (ii) : m is even

Fix y₁ by 1 and y₂ by 0.

Label the vertices x₁, x₂, ..., xₘ alternatively by 0’s and 1’s starting with ‘0’. Therefore the vertex labeling is friendly and |e(0)| = m.

Keep x₁ and x₂ remain unchanged and then label all the other vertices ie, x₃,x₄,...,xₘ by their complement. Then the vertex labeling is friendly and |e(0)| = m-2.

Then keep x₁, x₂, x₃, x₄ remain unchanged and then label all the other vertices ie, x₅,x₆,...,xₘ by its complement. Again the vertex labeling is friendly and |e(0)| = m-4.

Continue this process to get the friendly index set \{2,4,...,m\}

Atlast change the labels xₘ and y₁ to its complement and we get the vertex is friendly and |e(0)| = m.

∴ The friendly index set is \{0,2,4,...,m\}.

Illustration 3.1 :

FI(U(9,2)) = \{0,2,4,6,8,10\}

FI(U(8,2)) = \{0,2,4,6,8\}

FI(U(8,2)) = \{0,2,4,6,8\}

FI(U(8,2)) = \{0,2,4,6,8\}
Theorem 3.2:
The friendly index set of $\text{spl}(k_{1,n})$ is $\{0,2,4,...,n\}$, if $n$ is even and $\{1,3,...,n\}$, if $n$ is odd.

Proof:
Let $v_1, v_2, ..., v_n$ be the pendant vertices, $v$ be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, ..., u_n$ are the added vertices corresponding to $v, v_1, v_2, ..., v_n$ in $\text{spl}(k_{1,n})$.

Let $V(G) = \{v, v_1, v_2, ..., v_n, u, u_1, u_2, ..., u_n\}$

$E(G) = \{vv_i/1 \leq i \leq n\} \cup \{uu_i/1 \leq i \leq n\} \cup \{uv_i/1 \leq i \leq n\}$

Thus $\text{spl}(k_{1,n})$ has $2n+2$ vertices and $3n$ edges.

Case (i)
When $n$ is even
Fix $u = 0$ and $v = 1$

First label the vertices $v_i$'s as $(0,0,...,0)$ and $u_i$'s as $(1,1,...,1)$ in $\text{spl}(k_{1,n})$. Then the vertex labeling is friendly and $|e(0) - e(1)| = n$.

In next step decrease one vertex labeling ie, '0' from $v_i$'s and label with its complement ie, '1'. Also decrease one vertex labeling, i.e. '1' from $u_i$'s and label with its complement ie, '0'. Again the vertex labeling is friendly and $|e(0) - e(1)| = n-2$.

The vertices remain unchanged and continue the above step up to $\left(\frac{n}{2}\right)$ vertices having the label '0' in $v_i$'s and $\left(\frac{n}{2}\right)$ vertices having the label '1' in $u_i$'s.

∴ The friendly index set is $\{0,2,4,...,n\}$

Case (ii)
When $n$ is odd
Fix $u = 0$ and $v = 1$

First label the vertices $v_i$'s as $(0,0,...,0)$ and $u_i$'s as $(1,1,...,1)$ in $\text{spl}(k_{1,n})$. Then the vertex labeling is friendly and $|e(0) - e(1)| = n$.

In the next step decrease one vertex labeling i.e., '0' from $v_i$'s and label with its complement i.e., '1'. Also decrease one vertex labeling i.e., '1' from $u_i$'s and label with its complement i.e., '0'. Again the vertex labeling is friendly and $|e(0) - e(1)| = n-2$.

All changed vertices remain unchanged and continue this process up to $\left[\frac{n}{2}\right]$ vertices having the label '1' in $v_i$'s and $\left[\frac{n}{2}\right]$ vertices having the label '0' in $u_i$'s.

∴ The friendly index set is $\{1,3,...,n\}$

Illustration 3.2
When $n$ is even
$FI(\text{Spl}(k_{1,n})) = \{0,2,4\}$

When $n$ is odd
$FI(\text{Spl}(k_{1,n})) = \{0\}$
Theorem 3.3:
\[ \text{FI}(\text{Gl}(n)) = \{0, 4\}, \text{if } n \text{ is even and } \text{FI}(\text{Gl}(n)) = \{0, 2, 6\}, \text{if } n \text{ is odd} \]

Proof:
Let \((x_1, x_2, \ldots, x_n)\) be the vertices of \(n\) paths of length two. Let \(u, v\) be the vertices which are joined by those \(n\) paths.

It has \(n+2\) vertices and 2n edges.

Case (i): When \(n\) is even

Label \(u\) by 0 and \(v\) by 1 and label \((\frac{n}{2})\) vertices of \(x_i\)'s by 0 and other \((\frac{n}{2})\) vertices by 1. Then \(v(0) - v(1) = 0\). Therefore \(|e(0) - e(1)| = 0\)

Next label \(u\) and \(v\) by 0 and label \((\frac{n}{2})\) vertices of \(x_i\)'s by 0 and \((\frac{n}{2}) + 1\) vertices of \(x_i\)'s by 1. Again \(v(0) - v(1) = 0\). Then \(|e(0) - e(1)| = 4\)

Then the friendly index set is \(\{0, 4\}\)

Case (ii): When \(n\) is odd

Label \(u\) and \(v\) by 0 and label \((\frac{n}{2})\) vertices of \(x_i\)'s by 0 and the remaining vertices by 1. Then the vertex labeling is friendly with \(v(0) - v(1) = 1\). Therefore \(|e(0) - e(1)| = 2\).

Next label \(u\) by 0 and \(v\) by 1 and label \((\frac{n}{2}) + 1\) vertices of \(x_i\)'s by 0 and the remaining vertices by 1. Then \(v(0) - v(1) = 1\). Therefore \(|e(0) - e(1)| = 0\).

Next label \(u\) and \(v\) by 1 and label \((\frac{n}{2}) + 2\) vertices of \(x_i\)'s by 0 and the remaining vertices by 1. Again \(v(0) - v(1) = 1\) and \(|e(0) - e(1)| = 6\).

Then the friendly index set is \(\{0, 2, 6\}\)

Illustration 3.3

\[ \text{FI}(\text{Gl}(4)) = \{0, 4\} \]
Theorem 3.4:
FI (G(5)) = {0, 2, 6}

Next consider an odd m.
Label $v_1$ and $v_2$ as 0 and $y_{\left\lfloor \frac{m}{2} \right\rfloor + 1}, \ldots, y_m$ as 1 and all the remaining vertices as 0. This labeling is also friendly with $v(1) - v(0) = 1$ Then $|e(0) - e(1)| = 5$

Illustration 3.4
FI (P_{2} + 6K_{1}) = \{1, 3\}

FI (P_{2} + 5K_{1}) = \{1, 5\}

Proof:
Consider a path $P_2$ with two vertices $v_1, v_2$. Let $y_1, y_2, \ldots, y_m$ be the m isolated vertices. Join $v_1, v_2$ with $y_i, 1 \leq i \leq m$. The graph obtained is $P_2 + mk_i$. The vertex set of $G$ is $V(G) = \{v_1, v_2, y_1, y_2, \ldots, y_m\}$. The edge set of $G$ is

$$E(G) = \{(v_1, v_2)\} \cup \{(v_1, y_i) | 1 \leq i \leq m\} \cup \{(v_2, y_i) | 1 \leq i \leq m\}$$

Then $|V(G)| = 2 + m$ and $|E(G)| = 2m + 1$

First consider an even m
Label $v_1$ and $v_2$ as 0 and $y_{\left\lfloor \frac{m}{2} \right\rfloor + 1}, \ldots, y_m$ as 1 and the remaining vertices as 0. Then the vertex labeling is friendly with $v(1) - v(0) = 0$ and $|e(1) - e(0)| = 3$.

Then label $v_1$ as 0 and $v_2$ as 1 and label the vertices of $y_i$ as alternatively 0 and 1. This vertex labeling is also friendly with $v(1) - v(0) = 0$ Then $|e(0) - e(1)| = 1$.

The friendly index set is {1, 3}. 
Theorem 3.5:
Union of a path and a star sharing a vertex in common has the friendly index set \{ 1,3,5,\ldots,2n-5 \}

Proof
Let the \( n \) vertices of path \( P_n \) be \( u_1, u_2, \ldots, u_n \). Let the \( n \) spokes of star \( S_n \) be \( v_1, v_2, \ldots, v_n \) and \( u_1 \) be the centre vertex of the star. Identify \( u_1 \) and \( v_1 \). Let the graph so obtained is \( G \).

Clearly \( G \) has \( 2n \) vertices and \( 2n-1 \) edges.

First label \( (u_1, u_2, u_3, \ldots, u_n) = (1,0,0,\ldots,0) \) and \( (v_1, v_2, \ldots, v_n) = (0,1,1,\ldots,1) \) The vertex labeling is friendly and \(| e(0) - e(1) | = 2n-5 \)

Now keep all the vertex remain unchanged. Next change the vertex \( u_2 \) and \( v_2 \) to its complement. This labeling is friendly. Then \(| e(0) - e(1) | = 2n-3 \)

Now keep all the vertex remains unchanged and change the vertex \( u_3 \) and \( v_3 \) to its complement. This labeling is vertex friendly. Then \(| e(0) - e(1) | = 2n-1 \)

Continue this process upto changing the vertex \( u_{n-2} \) and \( v_{n-2} \) to its complement.

Then the friendly index set is \{ 1,3,5,\ldots,2n-5 \}

Illustration 3.5:

IV. CONCLUSION
Labeling of graphs in graph theory is one of the interesting research topic in the recent days. Here we have find new results of friendly index sets of five graphs related to labeling of graphs. Similar work can be carried out for other graphs also.

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