Robust Remanufacturing Planning with Parameter Uncertainty

Zhicheng Zhu\textsuperscript{1}, Yisha Xiang\textsuperscript{1}, Ming Zhao\textsuperscript{2}, and Yue Shi\textsuperscript{1}

\textsuperscript{1}Department of Industrial, Manufacturing, and Systems Engineering, Texas Tech University, Lubbock, TX, USA
\textsuperscript{2}Department of Business Administration, University of Delaware, Newark, DE, USA

\textbf{Abstract}

We consider the problem of remanufacturing planning in the presence of statistical estimation errors. Determining the optimal remanufacturing timing, first and foremost, requires modeling of the state transitions of a system. The estimation of these probabilities, however, often suffers from data inadequacy and is far from accurate, resulting in serious degradation in performance. To mitigate the impacts of the uncertainty in transition probabilities, we develop a novel data-driven modeling framework for remanufacturing planning in which decision makers can remain robust with respect to statistical estimation errors. We model the remanufacturing planning problem as a robust Markov decision process, and construct ambiguity sets that contain the true transition probability distributions with high confidence. We further establish structural properties of optimal robust policies and insights for remanufacturing planning. A computational study on the NASA turbofan engine shows that our data-driven decision framework consistently yields better worst-case performances and higher reliability of the performance guarantee.

\textbf{Keywords:} remanufacturing planning, robust Markov decision process, control-limit policy, data-driven solutions
1 Introduction

The manufacturing industry is a major consumer of materials and energy and imposes a significant impact on environment. Manufacturing activities are responsible for approximately 31% of the United States’ total energy usage and 19% of the world’s greenhouse gas emissions within the industrial sector [Diaz et al., 2010; Faludi et al., 2015]. Sustainable manufacturing with improved environmental performance has drawn a great attention from governments, companies and scientific communities. In the past decade, remanufacturing has emerged as one of the critical elements for developing a sustainable manufacturing industry [Ijomah et al., 2007]. Comparing to manufacturing a new product, remanufacturing can reduce up to 80% of energy consumption and carbon dioxide emissions [Sutherland et al., 2008], and 40-65% of manufacturing costs [Shi and Min, 2014; Ford and Despeisse, 2016]. The proceedings of the G7 Alliance on Resource Efficiency noted remanufacturing as one of the most important drivers for getting to a closed-loop economy and improving resource efficiency [EPA, 2016]. The growing remanufacturing industry has become an important economic force in the United States with a well-developed market estimated at $53 billion per year [Lund and Hauser, 2012]. Many products, especially the ones with long lifespans such as turbine blades and automobile parts (e.g., engines, water pumps), are now routinely remanufactured at the near-end of their life cycles and returned to service [Dulman and Gupta, 2018; Seitz, 2007].

Remanufacturing is an industrial process whereby used or broken-down products (or components), referred to as “cores”, are restored to a like-new condition with an extended lifetime [Östlin et al., 2009]. During this process, the cores pass through a number of operations including inspection, dismantling, part reprocessing, repair, replacement and reassembly. The performance of the remanufactured cores is expected to meet the desired product standards similar to the original product. Remanufacturing is distinctly different from related activities, such as reuse, repair, and recycling [Skrainka, 2012; Lund and Hauser, 2012]. A product is reused when it has not completed its life-cycle and the user decides to stop its use, and a consumer sector is willing to accept it in its current use state, perhaps to its original purpose. Repair fixes what is broken or worn with no attempt to fully restore the product to a like-new condition or to a new life. Recycling recovers materials at the end of the product life, returning them into the use stream. Recycling a complex product (e.g., car), however, can result in a loss of up to 95 percent of the value-added content (e.g., labor,
energy) (Giuntini and Gaudette, 2001), in addition to the loss of all functionality of the original product. By reclaiming the material content and retaining the embodied energy and labors used for manufacturing the original product, remanufacturing is a more efficient means of resource recirculation than these related activities.

Current remanufacturing practices mainly consider end-of-life products. Such a reactive approach has a number of drawbacks (Sutherland et al., 2008; Song et al., 2015). First, for many products, remanufacturability is significantly reduced due to no adequate techniques to remanufacture retired products. Second, remanufacturing costs often increase significantly towards the end of a product’s life cycle. Third, substantial negative environmental effects are usually generated because much higher energy and material consumption is required for remanufacturing due to heavy damage at the end of a product’s life. Several studies have shown that in some cases, remanufacturing actually consumes more energy than manufacturing a new product (Chandler, 2011; Gutowski et al., 2011). It is paramount that decision makers make informed decisions on the timing of remanufacturing and ensure it is conducted when it is worth the effort.

The robustness of the remanufacturing planning decisions, however, can be threatened by data inadequacy. Since the optimal decision involves suggesting the optimal action, such as no intervention, remanufacturing, or scrapping, at different states (conditions), it is first and foremost to estimate the transition dynamics of a component. The estimation often faces a great challenge because of limited field data. The situation may be further exacerbated as the field data typically contain a large amount of noises and incorrect information. This data inadequacy poses a critical question to decision makers: How does uncertainty in model parameters translate into uncertainty in the performance of interest? The decision makers must assess whether any observed nominal improvement in the environmental and economic effects resulted from remanufacturing at certain states is likely to be a true improvement, suggesting remanufacturing in those states, or conversely, a consequence of the parameter uncertainties due to statistical estimation errors, favoring remanufacturing when it causes negative effects.

Several studies attempted to address the impacts of the uncertainty in model parameters. It has been shown that when the parameter uncertainty—the deviation of the model parameters from the true ones, is largely ignored, the optimal policy can lead to serious degradation in performance, because the optimal
policy in sequential decision making is quite sensitive to parameter uncertainty, particularly perturbations in the transition probability (Mannor et al., 2007; Iyengar, 2005; Xu et al., 2012). Therefore, with a tacit understanding that the state transition dynamics of a component has to be estimated from historical data in practice, remanufacturing planning is confronted with the external uncertainty due to the deviation of the estimates from their true values in addition to the internal uncertainty due to the stochastic nature of a component’s condition evolution. To address both internal and external variations and prescribe robust remanufacturing policies, we formulate the remanufacturing planning problem as a robust Markov decision process (RMDP). We investigate the structural properties of the robust remanufacturing policies in the presence of statistical estimation errors. We further conduct a computational study based on the operational dataset of the turbofan engine operated by NASA (Frederick et al., 2007). The main contribution of this paper is threefold.

- Provide a novel data-driven modeling framework for remanufacturing planning in which decision makers can remain robust with respect to statistical estimation errors in transition dynamics.

- Establish structural properties of optimal robust policies and provide insights for decision making in remanufacturing planning in the presence of parameter uncertainties. Our findings also make contributions to the robust Markov decision process literature in which few papers have focused on characterizing the structure of the optimal robust policy.

- Conduct computational studies to demonstrate the optimal robust policies, investigate the out-of-sample performance of the resulting optimal robust remanufacturing policies, and derive data-driven solutions to improve the out-of-sample performance.

The remainder of this paper is organized as follows. Section 2 reviews relevant literature on remanufacturing planning and sequential decision making with parameter uncertainty. In Section 3, we formulate the remanufacturing planning problem as a RMDP. Section 4 establishes conditions to ensure the optimal robust policies are of control-limit type. In Section 5, we present a computational study using simulated operational data of NASA’s turbofan engines. Section 6 concludes this study and suggests future research directions.
2 Literature Review

Our study is related to two streams of the literature: remanufacturing planning and sequential decision-making with parameter uncertainty.

2.1 Remanufacturing Planning

The management of remanufacturing production and control activities greatly differs from management activities in traditional manufacturing. Production planning and control activities are more complex for remanufacturing firms due to complicating characteristics such as the uncertain timing and quantity of returns (Guide Jr, 2000). The majority of literature in remanufacturing operational management focus on production planning and control activities, such as reverse logistic management, balancing returns and demands, and inventory control. Van Der Laan et al. (1999) consider production planning and inventory control in systems where manufacturing and remanufacturing operations occur simultaneously. Savaskan et al. (2004) identify and model three close-loop supply chains to address the problem of choosing the appropriate reverse channel structure for the collection of used products. Galbreth and Blackburn (2010) study the optimal core acquisition quantity problem subject to the uncertainty of core conditions. We refer the readers to a review (Govindan et al., 2015) and the papers therein for more research works relevant to reverse logistics and closed-loop supply chain.

Remanufacturing planning, while being recognized, has received little attentions. Existing works on remanufacturing timing decisions often either ignore both types of uncertainties in transition dynamics of a remanufacturing system or only focus on the internal variation. For example, Song et al. (2015) determine remanufacturing timing based on a deterministic degradation process charaterized by residual strength factors. Wang et al. (2016) recommend remanufacturing based on online monitoring: Products are remanufactured when it reaches the limit condition beyond which the product is no longer remanufacturable. External variation is largely ignored, and hence, remanufacturing could be blindly suggested even if it might lead to increased negative environmental or economic impacts, resulting in the robustness of remanufacturing planning decisions in question.

Remanufacturing planning bears a close resemblance to maintenance planning which aims to determine
the optimal timing of preventive maintenance. In this paper, we use Markov models for remanufacturing planning; the most relevant works in maintenance optimization literature are the ones that model maintenance problems using a Markov decision process (e.g., Kurt and Kharoufeh (2010), Elwany et al. (2011), Kim and Makis (2013)). Most maintenance optimization models that are formulated as a Markov decision process (MDP), however, assume that the cost parameters and the transition kernel are known, and hence, cannot provide satisfactory out-of-sample performances when future realizations deviate from the predicted ones. One of the few papers that consider ambiguity in transition probabilities is by Kim (2016). In his paper, Kim (2016) considers a failing system whose underlying state is unobservable and accounts uncertainties in both posterior distributions and transition probabilities. The optimization of an ambiguous partially observable MDP (APOMDP) is generally challenging even for small state spaces. For remanufacturing planning, the state space would become large, rendering the optimization of the resulting APOMDP computationally prohibitive.

### 2.2 Sequential Decision-Making with Parameter Uncertainty

Conventional MDP assumes that the transition probabilities and rewards can be estimated confidently and focuses solely on the uncertainty stemming from the stochastic nature of MDPs. In practice, however, estimating the true transition probabilities is difficult, if not impossible, because of the limited data availability or inevitable statistical errors or both. When parameters deviate from the true ones and such uncertainty is ignored, the optimal policy can lead to serious inferior performance, because the optimal policy in sequential decision making is quite sensitive to parameter uncertainty, particularly deviations in the transition probability (Mannor et al. 2007; Iyengar 2005; Xu et al. 2012).

Early works on the MDPs with parameter uncertainties, including Silver (1963); Satia and Lave Jr (1973); White III and El-Deib (1986) and White III and Eldeib (1994), formulate the uncertainty in either a game-theoretic or Bayesian approach. In the game-theoretic formulation, it is assumed that the uncertainty about the transition probabilities is encoded by describing the set of all transition probability rows. Hence, when the decision maker makes a decision for a given state, the nature selects a transition probability row from the set to minimize the reward. Satia and Lave Jr (1973) use the game-theoretic formulation to model...
the transition uncertainty in MDP and proposed a policy iteration procedure to solve the problem. White III and Eldeib (1994) further develop a modified policy iteration-based algorithm for the MDP with imprecise transition probabilities. The Bayesian approach, first introduced by Silver (1963), assumes a known priori probability distribution of each transition probability row. Hence, the transition probabilities can be updated along the Bellman’s equations. Dirichlet priors are a common choice of modeling the uncertainty in transition probabilities (Delage and Mannor, 2010).

Most of the early contributions, however, do not concern the construction of ambiguity sets. Inspired by the data-driven approaches, recent RMDP works (Iyengar, 2005; Nilim and El Ghaoui, 2005; Wiesemann et al., 2013) have developed various methods to construct the uncertainty set of transition probabilities that contain the true transition probabilities with high confidence. Many statistical methods, such as likelihood constraints, deviation-type constraints and distance metrics (e.g., Wasserstein ball, φ−divergence balls), have been applied to construct an uncertainty set of transition probabilities with historical samples (Iyengar, 2005; Nilim and El Ghaoui, 2005; Wiesemann et al., 2013). Reformulation of RMDPs with different types of ambiguity sets and the corresponding tractability have also been studied in the literature. Compared to the theoretical orientation of these works, our present work focuses more narrowly on developing methods for a specific problem class, establishing structural properties of optimal robust policies, and providing executable insights.

3 Robust Remanufacturing Planning Problem

Consider remanufacturing planning of a single-unit system that degrades during its operation. Because we focus on single-unit systems, the words system and component are used interchangeably throughout the paper. The component is inspected at equally spaced discrete time epochs \( T = \{0, 1, \ldots \} \). Let \((\mathcal{S}, \mathcal{K})\) be the state space, where \( \mathcal{S} = \{0, 1, 2, \ldots, S\} \) represents the set of conditions and \( \mathcal{K} = \{0, 1, \ldots\} \) represents the set of cumulative numbers of completed remanufacturing activities. Note that a larger state denotes a worse condition and the worst state \( S \) is an absorbing state, meaning the systems stays there if there is no intervention. At each epoch, a decision maker observes the state of the component and then chooses an action from the set \( \mathcal{A} = \{0, 1, 2\} \), where 0 means continuing operation to the next observation time, 1 means
remanufacturing, which takes one period, and 2 means scraping the component.

The most important objective of remanufacturing is to minimize the negative environmental impacts while sustaining profitable growth. The direct environmental impacts of a manufactured/remanufactured system are often measured by greenhouse gas emissions (e.g., CO₂, CH₄, N₂O, etc.) using life cycle assessment (LCA). LCA is a technique that compiles an inventory of relevant energy consumption and material inputs and environmental releases, and then evaluates the potential environmental impacts associated with identified inputs and releases \cite{Curran2011}. Such an evaluation can be done using published databases (e.g., EcoScan) or commercial softwares (e.g., GaBi) if the system is complicated. There are several types of LCA. For instance, cradle-to-grave is the assessment of a full product life cycle from resource extraction (“cradle”) to use phase and disposal phase (“grave”), cradle-to-gate is an assessment of a partial product life cycle from resource extraction to the factory gate (i.e., before it transported to the consumer), and gate-to-gate, also a partial LCA, evaluates the eco-burden of a manufacturing facility. In this paper, rather than model different types of greenhouse gas emissions and formulate a multi-objective problem, we model the environmental impacts using carbon cost, which is determined by the amount of carbon emissions and the carbon price. As carbon trading increasingly recognized as one of the most effective approaches to incentivising companies to become environmental friendly \cite{Abdallah2012} and more carbon trading systems established around the world, remanufacturing planning models that consider the carbon costs will become more relevant and applicable.

To model the profit of a remanufacturing system, we assume that during each decision period, the decision maker receives a gain \( g(s, k) \) (e.g., production revenue) if operation is not interrupted and incurs some environmental costs \( e(s, k) \). The reward of keeping operation in one period is thus denoted by \( r(s, k) = g(s, k) - e(s, k) \). If the decision is to remanufacture the component, a remanufacturing cost \( c_r \), which comprises the manufacturing and carbon costs, is incurred. If the system is scrapped, a salvage value \( c_s \) is received. We assume that \( c_r \) and \( c_s \) are the same regardless of a component’s condition.

Although remanufacturing is supposed to restore a component to like-new conditions, each remanufacturing process typically makes the component less resistant to deterioration. Hence, a component’s deterioration process depends on the cumulative number of completed remanufacturing activities, which is modeled as fol-
allows. For a system that has been remanufactured $k$ times, let $P = [p(s'|s,k)]_{s,s' \in S, k \in K}$ denote the transition probability matrix when the decision is to wait. Note that remanufacturing brings the system status to an as-good-as-new condition, but increments the cumulative number of remanufacturings by one. We assume that the deterioration process is irreversible, and hence $P$ is an upper triangular matrix. Due to limited data availability and statistical estimation errors, the transition probability of a remanufacturing system is fundamentally unknown. To mitigate the effects of uncertain transition probabilities, we assume that the true transition kernel is contained in an ambiguity set.

Next, we present an important assumption regarding the ambiguity set, which ensures deterministic and Markovian policies (Iyengar, 2005).

**Assumption 1 (Rectangularity).** An RMDP problem has a rectangular ambiguity set if the ambiguity set has the form $\mathcal{U} = \bigotimes_{s \in S, k \in K} \mathcal{U}_{sk}$ where $\bigotimes$ stands for the Cartesian product, and $\mathcal{U}_{sk}$ is the projection of $\mathcal{U}$ onto the parameters of state $(s,k)$.

The implications of the rectangularity assumption is often interpreted in an adversarial setting (Iyengar, 2005; Nilim and El Ghaoui, 2005): The decision maker first chooses a policy $\pi$. Then an adversary observes $\pi$, and chooses a distribution that minimizes the reward. In this context, rectangularity is a form of an independence assumption: The choice of a particular distribution for a given state $(s,k)$ does not limit the choices of the adversary of other states. There are two possible models for transition matrix uncertainty. In the first model, referred to as the stationary uncertainity model, the transition matrices chosen by the adversary depending on the policy once and for all, and remain fixed thereafter. In the second model, referred to as the time-varying uncertainty model, the transition matrices can vary arbitrarily with time, within their prescribed bounds. It has been shown in Nilim and El Ghaoui (2005) that for a finite horizon problem with a discounted cost function, the gap between the optimal value of the stationary uncertainty problem and that of its time-varying counterpart goes to zero as the horizon length goes to infinity. In this paper, we consider the stationary worst-case distribution, that is, the choices of $p(\cdot|s,k)$ are the same every time the state $(s,k)$ is encountered. Note that there is no ambiguity in transitions in the period during which remanufacturing is conducted, since remanufacturing takes one period and there is no transition in that period. Because the optimal robust policies of the remanufacturing planning are Markovian and deterministic
under the rectangularity assumption, we have the robust remanufacturing planning optimization model in
the following recursive form:

\[ V(s, k) = \sup_{a \in A} w(s, k; a), \quad \text{(RRmPO)} \]

where

\[ w(s, k; a) = \begin{cases} 
\inf_{\mathbf{p} \in U} r(s, k) + \beta \sum_{s' \in S} \mathbf{p}(s'|s, k)V(s', k), & a = 0, \\
-c_r + \beta V(0, k + 1), & a = 1, \\
c_s, & a = 2.
\end{cases} \]

3.1 Construction of Ambiguity Sets

As mentioned in the introduction, the motivation for the robust methodology is the presence of the
statistical errors associated with estimating the transition probabilities using historical data. A natural
choice for the ambiguity set is the confidence regions associated with density estimation. We thus use the
Kullback-Leibler (KL) distance to construct the ambiguity set around the empirical transition probabilities.
Bootstrap resampling, a common non-parametric method to address the uncertainty, is also used to construct
confidence intervals of estimators.

An important property of the ambiguity set constructed using these two methods is that it cannot pop a
scenario while allows scenario suppression. A scenario that never occurs in the nominal problem cannot have
a positive probability (or, “pop”) in the ambiguous problems. Such a property ensured that the assumption
of no-self-improving in system’s deterioration is held in the ambiguous problem. More importantly, the
ambiguity sets constructed using these two methods are convex, and hence leads to a computational tractable
problem. In addition, the KL distance and bootstrapping are already being used in statistics, making them
attractive to deal with data directly.

KL Distance. In a data-driven setting, the empirical distribution typically serves as the nominal
distribution. For notational convenience, we drop the notations of \( k \) and \( a \). Let \( \hat{\mathbf{p}}_s \) be the maximum
likelihood estimator of \( \mathbf{p}_s \) given a state \( s \in \mathcal{S} \):

\[ \hat{p}_s(s') = \frac{n(s'|s)}{\sum_{s' \in \mathcal{S}} n(s'|s)}. \quad (1) \]
where \( n(s'|s) \) is the number of transitions from state \( s \) to \( s' \). The KL distance between \( \hat{p}_s \) and \( p_s \) is defined as

\[
D(p_s||\hat{p}_s) = \sum_{s' \in S} p_s(s') \log \left( \frac{p_s(s')}{\hat{p}_s(s')} \right).
\]  

(2)

It is obvious that \( D(p_s||\hat{p}_s) \geq 0 \) with equality holds when \( p_s = \hat{p}_s \). Given a state \( s \in S \), the KL-distance-based ambiguity set is given by

\[
\mathcal{U}_s = \left\{ p_s: D(p_s||\hat{p}_s) \leq \theta, \sum_{s' \in S} p_s(s') = 1, p_s(s') \in [0,1], s' \in S \right\}.
\]  

(3)

Let \( N_s = \sum_{s' \in S} n(s'|s) \). It has been shown that the normalized estimated KL-distance \( 2N_s D(p_s||\hat{p}_s) \) asymptotically follows a \( \chi^2_{|S|-1} \) distribution (see more details in [Ben-Tal et al. (2013)]). We thus have the following (approximate) \((1-\alpha)\)-confidence set around \( \hat{p}_s \)

\[
\mathcal{U}_s = \left\{ p_s: D(p_s||\hat{p}_s) \leq \theta \right\}
\]  

(4)

where \( \theta = \chi^2_{|S|-1,1-\alpha}/(2N_s) \).

**Bootstrap Resampling.** Let \( D \) be realizations of the Markov chain \( \{S_n; n \geq 0\} \) with transition matrix \( P \) and \( \hat{P} \) be the maximum likelihood estimator of \( P \) based on the observed data \( D \). For a large number of statistics of interest, the bootstrap distribution approximates the sampling distribution (i.e., asymptotic normality of \( \sqrt{N}(\hat{P} - P) \) where \( \hat{P} \) is the bootstrap estimator of \( P \)). From the bootstrap distribution, one can assess the uncertainty of each probability in the transition matrix and construct the confidence interval for each probability. The bootstrap distribution can be calculated by direct theoretical calculation, which draws samples with replacement, or Monte Carlo approximation. The reader is referred to [Efron and Tibshirani (1994)] for more detailed bootstrapping procedures. Note that the ambiguity set constructed by bootstrap resampling has the same form of the ambiguity set constructed using the interval matrix method in [Nilim and El Ghaoui (2005)]. Therefore, we also refer to the method of constructing ambiguity sets using bootstrap sampling as the interval matrix method hereinafter since it better describes the form of the ambiguity sets.
4 Structure of the Optimal Robust Policy

In this section, we investigate the structural properties of the optimal robust remanufacturing policies. We will focus our attention on control-limit policies. We establish sufficient conditions that ensure the existence of monotonically control-limit policies. The optimality of such structured policies is important because they are appealing to decision makers and enables efficient computation and are easy to implement. Our analysis will make significant use of the notion of the stochastic dominance, which helps establish stochastic dominance relationships for transition behaviors. Below, we define some stochastic order concepts that are used in our analysis.

Definition 1.

(a) A transition probability matrix $P = [p(i|j)]_{i,j=0,1,...,n}$ is said to be IFR (increasing failure rate) if $\sum_{i=m}^{n} p(i|j)$ is non-decreasing in $j$ for all $m = 0, 1, ..., n$.

(b) For two transition probability matrices $P_1 = [p_1(i|j)]_{i,j=0,1,...,n}$ and $P_2 = [p_2(i|j)]_{i,j=0,1,...,n}$, we say $P_1 \succeq P_2$, if $\sum_{i=m}^{n} p_1(i|j) \geq \sum_{i=m}^{n} p_2(i|j)$ for all $j, m = 0, 1, ..., n$.

Assumption 2. Let $\hat{P}(\cdot|\cdot, k)$ denote the nominal transition probability matrix for a system that has been remanufactured $k$ times,

(a) $\hat{P}(\cdot|\cdot, k)$ is IFR for all $k \in K$.

(b) $\hat{P}(\cdot|\cdot, k + 1) \succeq \hat{P}(\cdot|\cdot, k)$ for all $k \in K$.

Assumption [2]a) implies that, given the cumulative number of completed remanufacturing activities $k$, the system in a worse state at the current epoch is more likely than the other to be found in a worse condition at the next epoch. Assumption [2]b) imposes a first-order stochastic dominance relationship among the system’s deterioration matrices corresponding to different remanufacturing histories. More explicitly, given two systems with the same condition but different remanufacturing histories, the system with a larger $k$ is more likely to get worse than the other during operation. Additional assumption is made regarding the operational gains, environmental costs, and the salvage value.

Assumption 3.
(a) The operational gain $g(s,k)$ is non-increasing in $s \in S$ and $k \in K$, and the carbon cost $c(s,k)$ is non-decreasing in $s \in S$ and $k \in K$;

(b) The reward at state $S$, the salvage value $c_s$ and the discount factor $\beta$ satisfy the following condition:

$$\frac{r(S,0)}{1-\beta} < c_s.$$

Assumption 3(a) implies that as the number of completed remanufacturing activities increases and its condition worsens, the gain decreases and the carbon cost increases. For example, an engine in a worse state usually incurs higher maintenance costs and consumes more gasoline or electricity. Assumption 3(b) ensures that the decision of no intervention (i.e., $a = 0$) is excluded when a system is at the worst state for all $k \in K$ because it is not practical that the system stays in the worst condition $S$ for an infinitely long time. This unrealistic scenario is eliminated by assuming that the total expected reward from doing nothing at state $(S,0)$, computed as $\sum_{t=0}^{\infty} \beta^t r(S,0) = \frac{r(S,0)}{1-\beta}$, is less than the salvage value. Since $r(S,0) \geq r(S,k)$ for all $k > 0$, the condition also eliminates the no-intervention option for state $(S,k)$ for all $k > 0$.

### 4.1 Remanufacturing Planning with KL-Distance-Based Ambiguity Sets

In this section, we consider Model (RRmPO) with KL-distance-based ambiguity sets. We provide reformulations and establish conditions that ensure control-limit type policies. We first reduce the bi-level problem (RRmPO) to a single-level problem by applying the Lagrangian dual theory, and then investigate the structure of the robust value function, which is necessary for establishing control-limit robust remanufacturing planning policies.

**Proposition 1.** For Model (RRmPO) with KL-distance-based ambiguity sets, $w(s,k;0)$ can be reformulated as

$$w(s,k;0) = \sup_{\mu > 0} r(s,k) - \mu \log \left( \sum_{s' \in S} \hat{p}(s'|s,k) \exp \left( \frac{-\beta V(s',k)}{\mu} \right) \right) - \mu \theta,$$

and the worst-case distribution is

$$p^*(s'|s,k) = \frac{\hat{p}(s'|s,k) \exp \left( \frac{-\beta V(s',k)}{\mu_{sk}^*} \right)}{\sum_{s' \in S} \hat{p}(s'|s,k) \exp \left( \frac{-\beta V(s',k)}{\mu_{sk}^*} \right)},$$

where

$$\mu_{sk}^* = \frac{V(s',k)}{\beta \theta}$$

for all $s \in S$, $k \in K$. 

where $\mu_{s,k}^*$ is the optimal solution of the dual problem \([5]\) given $s$ and $k$.

**Proof.** See Appendix A.1.

**Proposition 2.** For Model \([\text{RRmPO}]\) with ambiguity sets constructed using the KL distance, the value function $V(s,k)$ is non-increasing in $s \in S$ and $k \in K$.

**Proof.** See Appendix A.2.

Next, we establish conditions that ensure control-limit robust policy structures, that is, the remanufacturing decisions are of control-limit type with respect to the condition of the system and the cumulative number of completed remanufacturing activities.

**Theorem 1.** For Model \([\text{RRmPO}]\) with KL-distance-based ambiguity sets, there exists a cumulative number of completed remanufacturing activities $k^* \in K$, and an operational state $s_k \in S$ such that for $k < k^*$

$$a(s,k) = \begin{cases} 0 & \text{if } s < s_k, \\ 1 & \text{if } s \geq s_k, \end{cases}$$

and for $k \geq k^*$

$$a(s,k) = \begin{cases} 0 & \text{if } s < s_k, \\ 2 & \text{if } s \geq s_k. \end{cases}$$

**Proof.** See Appendix A.3.

Theorem 1 shows that for a system that has $k < k^*$, the optimal decision is either wait until the next period or remanufacture, and the system is remanufactured when the condition is equal to or exceeds a limit. When the cumulative number of completed remanufacturing activities reaches the threshold $k^*$, the optimal decision is either wait until the next period or scrap and there exists a scrapping threshold. This implies that despite the cost savings and environmental benefits that make remanufacturing appealing, remanufacturing activity is not recommended after being conducted certain number of times. Note that $k^* = 0$ is a special case that remanufacturing is not optimal for all $k \in K$. Let $\zeta_{\text{rm}}(k)$ denote the control limit $s_k$ for $k < K^*$ and $\zeta_{\text{scrap}}(k)$ denote the control limit $s_k$ for $k \geq K^*$. We further examine the structure of $\zeta_{\text{rm}}(k)$ and $\zeta_{\text{scrap}}(k)$ in the next theorem.
Theorem 2. Consider Model \(\text{(RRmPO)}\) with the KL-distance-based ambiguity sets. Then, the following holds:

(a) If \(\frac{\beta r(0,0)}{1-\beta} - \beta c_s \leq r(s,k) - r(s,k+1)\), \(\zeta_{rm}(k)\) is non-increasing in \(k, k < k^*\).

(b) \(\zeta_{scrap}(k)\) is non-increasing in \(k, k \geq k^*\).

Proof. See Appendix A.4.

The condition in Theorem 2(a) is restrictive. We will show that most violations do not change the monotone structure of \(\zeta_{rm}(k)\) and \(\zeta_{scrap}(k)\) in Section 5.4.1 through some computational studies.

4.1.1 Solution Methodology for KL Distance Model

Model \(\text{(RRmPO)}\) can be efficiently solved using robust value iteration as described below:

**Algorithm 1 Robust Value Iteration**

1: Initialization:
   \[ V(s,k), a^*(s,k) \leftarrow 0, V(s,k) \leftarrow M, \forall (s,k) \in S \times K, \epsilon > 0 \]
2: while \( ||V - \bar{V}|| \geq \frac{(1-\beta)\epsilon}{4\beta} \) do
3: \( V \leftarrow \bar{V} \)
4: for \((s,k) \in S \times K\) do
5: \( \bar{V}(s,k) \leftarrow \max_{a \in A} w(s,k; a) \)
6: \( a^*(s,k) \leftarrow \arg \max_{a \in A} w(s,k; a) \)
7: end for
8: end while
9: return \( \bar{V}, a^* \)

To solve the inner problem \(w(s,k; 0)\) in step 5, we can either employ a numerical search for the dual problem in Equation (5), or solve the primal problem as the following conic program:

\[
w(s,k; 0) = r(s,k) + \min_{p(s',k) \in \mathcal{U}_k} \sum_{s' \in S} p(s'|s,k)V(j,k)
\]

s.t. \( p(s'|s,k) \geq 0, s' \in S, \sum_{s' \in S} p(s'|s,k) = 1 \)

\[
\sum_{s' \in S} p(s'|s,k) \log \left( \frac{p(s'|s,k)}{\bar{p}(s'|s,k)} \right) \leq \theta,
\]

(7)
where constraint (7) can be represented by the following exponential cone:

\[
\sum_{s' \in S} z(s') = \theta,
\]

\[
（\hat{p}(s'|s,k), p(s'|s,k), -z(s')） \in K_{\text{exp}}, s' \in S,
\]

where \(K_{\text{exp}} = \{(x_1, x_2, x_3) : x_1 \geq x_2 e^{x_3/x_2}, x_2 > 0\}\) is the exponential cone in \(\mathbb{R}^3\).

### 4.2 Remanufacturing Planning with Interval-Matrix-Based Ambiguity Sets

In this section, we establish conditions that guarantee structural properties of the optimal robust policies of Model (RRmPO) with ambiguity sets constructed using the interval matrix model. The interval matrix model describes the uncertainty on the rows of the transition matrices in the form \(U_{sk} = \{p(\cdot|s,k) : p(s'|s,k) \leq p(s'|s,k) \leq \bar{p}(s'|s,k), \sum_{s' \in S} p(s'|s,k) = 1\}\). Note that since components in each row vector of a transition matrix are constrained by \(p(\cdot|s,k)\) ^T 1 = 1, more effective lower and upper bounds of the transition probability of each state \((s,k) \in S \times K\) can be obtained. The effective upper bounds are

\[
\min \{\bar{p}(s'|s,k), 1 - \sum_{s'' \in S \setminus \{s'\}} p(s''|s)\}, \forall s' \in S.
\]

and the effective lower bounds are

\[
\max \{p(s'|s,k), 1 - \sum_{s'' \in S \setminus \{s'\}} \bar{p}(s''|s,k)\}, \forall s' \in S,
\]

The upper and lower bounds hereinafter are referred to the bounds defined in Equations (8) and (9). We first establish conditions that ensure the value function is monotone and the worst-case transition matrices are IFR, and then examine the structures of the robust optimal remanufacturing planning policies.

**Proposition 3.** Consider Model (RRmPO) with the ambiguity set constructed using the interval matrix model.
(a) If the lower and upper bounds of all states \((s, k) \in \mathcal{S} \times \mathcal{K}\) satisfy the following conditions,

\[
\sum_{s''=0}^{i} p(s''|s, k) \geq \sum_{s''=0}^{i} p(s''|s', k), \forall i \in \mathcal{S}, k \in \mathcal{K}, \text{if } s' \geq s,
\]

(10)

\[
\sum_{s''=1}^{S} \bar{p}(s''|s, k) \leq \sum_{s''=1}^{S} \bar{p}(s''|s', k), \forall i \in \mathcal{S}, k \in \mathcal{K}, \text{if } s' \geq s
\]

(11)

then the value function \(V(s, k)\) is non-increasing in \(s\) for all \(k \in \mathcal{K}\).

(b) If the lower and upper bounds of all states \((s, k) \in \mathcal{S} \times \mathcal{K}\) satisfy conditions (10) and (11), and the following conditions,

\[
\sum_{s''=0}^{i} p(s''|s, k) \geq \sum_{s''=0}^{i} p(s''|s', k'), \forall i \in \mathcal{S}, s \in \mathcal{S}, \text{if } k' \geq k
\]

(12)

\[
\sum_{s''=1}^{S} \bar{p}(s''|s, k) \leq \sum_{s''=1}^{S} \bar{p}(s''|s', k'), \forall i \in \mathcal{S}, s \in \mathcal{S}, \text{if } k' \geq k
\]

(13)

then the value function \(V(s, k)\) is non-increasing for all \(k \in \mathcal{K}\).

**Proof.** See Appendix A.5. \qed

Proposition 3 provides a set of conditions sufficient to guarantee the structure of the value function and the worst-case transition matrices, which are critical to the existence of control-limit robust optimal policies. These conditions are generally not restrictive, and are expected to be roughly satisfied. Conditions (10) and (11) ensure that the worst case transition matrices are IFR, and conditions (12) and (13) guarantee that the worst transition matrix \(P^*(\cdot|\cdot, k + 1)\) dominates the worst transition matrix \(P^*(\cdot|\cdot, k)\) for all \(k \in \mathcal{K}\).

In the proof of Proposition 3, we also obtain the transition behaviors of worst-case transition probability matrices when the decision is to wait, which are summarized in the following corollaries.

**Corollary 1.** For Model (RmPO) with the ambiguity set constructed using the interval matrix model, if the lower and upper bounds of all states \((s, k) \in \mathcal{S} \times \mathcal{K}\) satisfy conditions (10) and (11), then the worst-case transition matrices are:

\[
p^*(s'|s, k) = \begin{cases} 
\bar{p}(s'|s, k), & s' > \delta_{sk}, \\
p(s'|s, k), & s' < \delta_{sk}, \\
1 - \sum_{s'=1}^{\delta_{sk}-1} p(s'|s, k) - \sum_{s'=\delta_{sk}+1}^{S} \bar{p}(s'|s, k), & s' = \delta_{sk}, 
\end{cases}
\]

(14)

where \(\delta_{sk} = \min \{\delta \in \mathcal{S} : \sum_{s'=0}^{\delta} p(s'|s, k) + \sum_{s'=\delta+1}^{S} \bar{p}(s'|s, k) \leq 1\}\) for all \(s\) and \(k\).
Proof. See Appendix A.6.

Let $P^*(\cdot|\cdot, k)$ denote the worst-case transition matrices in Corollary 1. Corollary 2 summarizes the structure of $P^*(\cdot|\cdot, k)$.

**Corollary 2.** $P^*(\cdot|\cdot, k)$ is IFR for all $k \in \mathcal{K}$; if $k_1 > k_2$ ($k_1, k_2 \in \mathcal{K}$), $P^*(\cdot|\cdot, k_1) \succeq P^*(\cdot|\cdot, k_2)$.

Next we show that similar to optimal robust remanufacturing policies for model with KL-distance-based ambiguity sets, the optimal robust policies for model with interval-matrix-based ambiguity sets exhibit the same control-limit structure with respect to $s$ and $k$ under some conditions.

**Theorem 3.** For Model \textit{[RRmPO]} with the ambiguity set constructed by the interval matrix model, if the lower and upper bounds of all states $(s, k) \in \mathcal{S} \times \mathcal{K}$ satisfy conditions (10) to (13), then there exist a cumulative number of remanufacturings $k^* \in \mathcal{K}$ and an operational state $s_k \in \mathcal{S}$ such that for $k < k^*$

$$a(s, k) = \begin{cases} 0 & \text{if } s < s_k, \\ 1 & \text{if } s \geq s_k, \end{cases}$$

and for $k \geq k^*$

$$a(s, k) = \begin{cases} 0 & \text{if } s < s_k, \\ 2 & \text{if } s \geq s_k. \end{cases}$$

**Proof.** See Appendix A.7.

**Theorem 4.** For Model \textit{[RRmPO]} with the ambiguity set constructed using the interval matrix model, if the lower and upper bounds of all states $(s, k) \in \mathcal{S} \times \mathcal{K}$ satisfy conditions (10) to (13), then the following holds:

(a) If $\frac{\beta r(0,0) - \beta c_s}{1 - \beta} \leq r(s, k) - r(s, k + 1)$, is satisfied, $\zeta_{\text{rm}}(k)$ is non-increasing in $k, k < k^*$.

(b) $\zeta_{\text{scrap}}(k)$ is non-increasing in $k, k \geq k^*$.

**Proof.** See Appendix A.8.

### 4.2.1 Solution Methodology for Interval Matrix Model

Model \textit{[RRmPO]} with the ambiguity set constructed using the interval matrix model can be efficiently solved by the robust value iteration algorithm in Section 4.1.1 where the dual problem of the inner problem...
\( w(s, k; 0) \) is given by:

\[
w(s, k; 0) = \max_{\lambda} \left\{ r(s, k) + \sum_{s' \in S} \beta V(s', k) \bar{p}(s'|s, k) + \lambda \sum_{s' \in S} (1 - \bar{p}(s'|s, k)) \right. \\
\left. + \sum_{s' \in S} (\beta V(s', k) - \lambda)^+ (\bar{p}(s'|s, k) - \bar{p}(s'|s, k)) \right\},
\]

where \((\beta V(s', k) - \lambda)^+ = \max\{\beta V(s', k) - \lambda, 0\}\). Since this dual problem is piecewise linear on \( \lambda \) with break points \( \beta V(s', k) \), \( \forall s' \in S \), the optimality is obtained at one of the break points. Suppose \( \lambda^* = \beta V(s'', k) \) for \( s'' \in S \). Then, the complementary slackness suggests \( p^*(s'|s, k) = \bar{p}(s'|s, k) \) if \( V(s', k) > V(s'', k) \), \( p^*(s'|s, k) = \bar{p}(s'|s, k) \) if \( V(s', k) < V(s'', k) \), and \( p^*(s''|s, k) = 1 - \sum_{s' \in S \setminus \{s''\}} p^*(s'|s, k) \). Corollary \([\text{I}]\) follows when \( V(s', k) \) is non-increasing in \( s' \in S \) for any given \( k \).

### 4.3 Sensitivity Analysis

Consider two problem instances \( (\Lambda_1 \text{ and } \Lambda_2) \) with ambiguity sets constructed using the KL distance or the interval matrix model. Assume that these two problem instances satisfy the following: (1) reward functions are the same, (2) the ambiguity set of problem \( \Lambda_1 \) is contained in its counterpart of problem \( \Lambda_2 \) (i.e. \( \mathcal{U}^1_{sk} \subseteq \mathcal{U}^2_{sk} \)), and (3) all conditions that are needed to ensure the control-limit structure of the optimal robust policies are satisfied. Theorem 5 addresses the relationship between the optimal robust policies of these two systems. Let \( \zeta^{i}_{\text{rm}}(k) \) and \( \zeta^{i}_{\text{scrap}}(k) \) denote the remanufacturing and scrap limits of problem \( \Lambda_i \), \( i = 1, 2 \), given \( k \), respectively. Let \( k^*_i \) be the same threshold defined in Theorems 1 and 3. We show that given \( k \), the remanufacturing threshold in problem \( \Lambda_2 \), \( \zeta^2_{\text{rm}}(k) \) is higher than or the same as its counterpart in problem \( \Lambda_2 \). In contrast, problem \( \Lambda_2 \) has a higher scrap control limit given \( k \). This shows that a decision maker needs to be more conservative about initiating a remanufacturing process in anticipation of more transition uncertainties and that decision makers should consider scrapping early to receive the terminal rewards, to hedge against uncertainties in future operational gains. We summarize our results in the following theorem.

**Theorem 5.** Let \( \Lambda_1 \) and \( \Lambda_2 \) be two problem instances defined in this section. Then, the following holds.

(a) for \( k < k^*_1 \), \( \zeta^1_{\text{rm}}(k) \leq \zeta^2_{\text{rm}}(k) \) if \( p^*_1(\tilde{s}|\tilde{s}, k) = 0 \), where \( \tilde{s} = \zeta^1_{\text{rm}}(k) - 1 \).

(b) for \( k \geq k^*_1 \), \( \zeta^1_{\text{scrap}}(k) \geq \zeta^2_{\text{scrap}}(k) \)

(c) \( k^*_1 \geq k^*_2 \).
Suppose that a decision maker has problem $\Lambda_1$ implemented and is aware of its optimal robust policy. Theorem 5 offers valuable insights on the optimal policy if the decision maker decides to be more conservative by considering a larger ambiguity set. It is worth noting that if the system does not allow self-transition, i.e., $\hat{p}(s|s,k)$ for all $s \in S, k \in K$, then Theorem 5 holds.

5 Computational Study

5.1 System Model Description

Real-world turbofan engine operating data acquired from sensors are used to demonstrate our robust remanufacturing planning model. Procuring actual turbofan engine system fault progression data is typically time consuming and expensive. Hence, we use the data simulated using the Commercial Modular Aero-Propulsion System Simulation (C-MAPSS) software (Frederick et al., 2007) developed at NASA to demonstrate our robust remanufacturing planning model and examine the performance of the optimal robust remanufacturing policies. The C-MAPSS engine is a 90,000 lb thrust class turbofan engine and has five rotating components: fan, low pressure compressor (LPC), high pressure compressor (HPC), high pressure turbine (HPT), and low pressure turbine (LPT). The engine diagram in Figure 1 shows the main elements of the engine model (Saxena et al., 2008).

![Simplified diagram of engine simulated in C-MAPSS](image)

Figure 1: Simplified diagram of engine simulated in C-MAPSS (Saxena et al., 2008)

The overall simulation is implemented in the MATLAB and Simulink environment, providing flexible interaction with the software user. C-MAPSS offers 14 inputs and can produce several outputs for analysis. The system inputs include fuel flow, deviation from scheduled variable stator vanes angle, deviation from scheduled variable bleed valve position, and a set of 13 health parameters (e.g., Fan efficiency modifier, LPC
flow modifier, HPC pressure-ratio modifier) that consist of flow, efficiency, and pressure-ratio modifiers for the fan, LPC, and HPC, and flow and efficiency modifiers for the HPT and LPT. Advanced users can readily modify and customize the model to their specific requirements to simulate the deterioration in any of the engine’s five rotating components. The outputs include various sensor response surfaces and operability margins. The total of 21 outputs are summarized in Table 1.

Table 1: C-MAPSS outputs (Saxena et al., 2008)

| Symbol  | Description                 | Units |
|---------|-----------------------------|-------|
| T2      | Total temperature at fan inlet | °R    |
| T24     | Total temperature at LPC outlet | °R  |
| T30     | Total temperature at HPC outlet | °R  |
| T50     | Total temperature at LPT outlet | °R  |
| P2      | Pressure at fan inlet       | psia  |
| P15     | Total pressure in bypass-duct | psia |
| P30     | Total pressure at HPC outlet | psia  |
| Nf      | Physical fan speed          | rpm   |
| Nc      | Physical core speed         | rpm   |
| epr     | Engine pressure ratio (P50/P2) | -     |
| Ps30    | Static pressure at HPC outlet | psia |
| phi     | Ratio of fuel flow to Ps30   | pps/psi |
| NRf     | Corrected fan speed         | rpm   |
| NRc     | Corrected core speed        | rpm   |
| BPR     | Bypass Ratio                | -     |
| farB    | Burner fuel-air ratio       | -     |
| htBleed | Bleed Enthalpy              | -     |
| Nf_dmd  | Demanded fan speed          | rpm   |
| PCNR_dmd| Demanded corrected fan speed | rpm |
| W31     | HPT coolant bleed           | lbm/s |
| W32     | LPT coolant bleed           | lbm/s |

5.2 Dataset Description

We consider the data pertaining to a single failure mode and a single operating condition. The dataset considered in this work consists of 100 units which are run to failure. Note that end-of-life can be subjectively determined as a function of operational thresholds that can be measured; these thresholds depend on user specifications to determine safe operational limits. For illustration purposes, we arbitrarily choose four features and plot the time series of these features for a randomly selected unit and all units (Figure 2).

From Figure 2 we can see that the data contains a lot of noises. Various sources can contribute to noises,
Figure 2: Illustrations of raw sensor data sequences. (a)-(d), time series of the selected features of unit 6. (e)-(h), time series of the selected features of all units. Solid lines are the time series of the unit that has the most maximum (yellow line) and minimum (red line) points.

and the main sources of noise are manufacturing and assembly variations, process noise, and measurement noise to name a few important ones (Saxena et al., 2008). Due to the large amount of noises and limited real-world operational data available, there often exists a high level of uncertainties in transition probabilities of the turbofan engines, and operators and manufacturers are in great need of robust remanufacturing planning.

5.3 Parameter Estimation

It is typically desirable to reduce the dimensionality of the data and reconstruct them from a lower dimensional samples. We therefore use the principal component analysis method to compress the high-dimensional sensor outputs and use the first principle component that accounts for the largest variability of data as the system health indicator. We further discretize the obtained health indicator into 7 intervals, representing 7 condition states, as recommended by Moghaddass and Zuo (2014). The $k$-means method is used to partition health indicators. The nominal transition probability is estimated using the maximum likelihood method, i.e., $\hat{p}(s'|s) = \frac{\sum_{i=1}^{m} n_i(s'|s)}{\sum_{i=1}^{m} \sum_{s' \in S} n_i(s'|s)}$, where $n_i(s'|s)$ is the number of transitions from state $s$ to $s'$ for unit $i$, and $m$ is the total number of units in a sample. We construct the ambiguity sets as described in Section 3.1. Note that when constructing interval-matrix-based ambiguity sets, 30 bootstrap samples are used. Based on discussions with researchers and field engineers, an engine typically lose about 7% useful
life each time it is remanufactured. We modify the nominal transition probability matrix obtained for new
turbonfan engines (i.e., $k = 0$) to reflect such a loss for $k > 0$. For all the following experiments, the
nominal transitional probability matrices satisfy Assumption 2 and the lower and upper bounds of transition
probabilities satisfy conditions [10] to [13].

5.4 Experiments

Next, we demonstrate the structure of the optimal robust remanufacturing policy and examine the out-
of-sample performance of the optimal robust policies. The following cost data is used for all experiments in
this section: $g(s, k) = 4 - 0.25s - 0.25k, e(s, k) = 1 + 0.25s + 0.25k, c_r = 2$, and $c_s = 0.5$. The discount factor
$\beta$ is 0.9 for all following experiments.

5.4.1 Policy Structures

We have established conditions to ensure control-limit policies for Model (RRmPO) with ambiguity sets
constructed using the KL distance or the interval matrix method. For illustration purposes, we show the
structure of optimal robust policies for Model (RRmPO) with KL-distance-based ambiguity sets.

As Figure 3 shows, the remanufacturing policies exhibit control-limit structure. We can also see that
as $\theta$ increases, the remanufacturing threshold $\zeta_{rm}(k)$ increases and $k^*$ decreases (i.e., the scrap is performed
earlier). This implies that when parameter uncertainty is large, a decision maker needs to be cautious
about remanufacturing used products and to consider scrapping at an earlier stage. This is because (1)
the remanufacturing cost may not be offset by the subsequent operational gains due to large parameter
uncertainties and (2) securing the fixed salvage value better hedges against uncertainties in future gains.

![Figure 3: Optimal robust policies for different $\theta$s](image)

(a) $\theta = 0$  (b) $\theta = 0.5$  (c) $\theta = 1$
As stated earlier, the condition of Theorem 2(a), which is the same as the condition of Theorem 4 is restrictive and difficult to satisfy. We further examine whether the optimal robust policies are still of control-limit type when this condition is violated. We test a total of 5000 instances and the generation of the test instances is described in Appendix B.1. Out of the 3060 test instances that violate the condition of Theorem 2(a), only 209 (i.e., approximately 6.8%) instances violate the monotone structure. Therefore, we believe that a control-limit policy with respect to $k$ can be obtained in most practical cases even when the condition that guarantees it is violated.

5.4.2 Impact of the Parameter Uncertainty

We first conduct experiments to investigate the impact of the parameter uncertainty on the out-of-sample performance. The radius $\theta$ determines the size of the KL-distance-based ambiguity set and the confidence level $\alpha$ determines the size of the ambiguity set constructed using bootstrap resampling. For notational convenience, we use $\psi$ to denote the hyperparameter that controls the size of the ambiguity set.

We use a training dataset $\mathcal{N}$. The optimal robust policies of Model (RRmPO) with ambiguity sets constructed under different hyperparameter values using the training dataset, $\pi_{\mathcal{N}}(\psi)$, are then implemented in a test dataset $\mathcal{M}$ to assess the out-of-sample performance. We examine two performance measurements: the average reward and the reliability of performance guarantees. The average reward is defined as $\bar{\nu}_{\mathcal{N}}(\psi) = \frac{\sum_{i \in \mathcal{M}} \nu_i(\pi_{\mathcal{N}}(\psi))}{|\mathcal{M}|}$, where $\nu_i(\pi_{\mathcal{N}}(\psi))$ is the expected reward of robust policy $\pi_{\mathcal{N}}(\psi)$ for test sample $i \in \mathcal{M}$ when the system is brand new ($s = 0, k = 0$). The reliability is defined as the probability of the event $\bar{\nu}_{\mathcal{N}}(\psi) \geq V_{\mathcal{N}}(\psi)$, where $V_{\mathcal{N}}(\psi)$ is the in-sample value of $V(0, 0)$ under $\psi$.

Figure 4 depicts the experiment results when the sizes of training dataset is 5 ($|\mathcal{N}| = 5$) and size of the test dataset is 50 ($|\mathcal{M}| = 50$). From Figure 4(a), we observe that the average reward of the robust policy is slightly higher than that of the nominal policy when $\theta$ is not too large. As $\theta$ keeps increasing, the average reward of the robust policy deteriorates because the robust policy is too conservative. The empirical reliability visualized in Figure 4(b) is in general non-decreasing in $\theta$, and the reliability of the performance guarantee under the robust approach is much higher than that under the nominal approach.

We also find that the out-of-sample average reward using a robust approach is better as long as the reliability
of the performance guarantee is noticeably smaller than 1 and deteriorates when it is close to 1. Figure 4(c) and (d) present the out-of-sample performance and the reliability of Model (RRmPO) with the interval-matrix-based ambiguity sets, respectively. Similar patterns are observed. Results of this experiment provide an empirical justification of adopting a robust remanufacturing approach, especially when the size of the dataset is small.

![Figure 4](image)

Figure 4: Out-of-sample reward $\bar{\nu}_N(\psi)$ and reliability $\Pr\{\bar{\nu}_N(\psi) \geq V_N(\psi)\}$ as a function of $\psi$. (a)-(b) KL-distance-based ambiguity set. (c)-(d) Interval-matrix-model-based ambiguity set.

### 5.4.3 Remanufacturing Planning Driven by Out-of-Sample Performance

From the previous experiment on the impact of the parameter uncertainty, it is shown that different hyperparameter $\psi$ values may lead to robust remanufacturing policies with different out-of-sample performance $\bar{\nu}_N(\psi)$. It is desired to select a $\psi$ that maximizes the average award $\bar{\nu}_N(\psi)$. This, however, requires the true transition probability that is not precisely known. We select the optimal $\psi$ via validation using the training data. Specifically, we randomly select 60% of the training dataset $N$ for training and the remaining 40% of the training data is used for validation. Using newly formed training dataset to construct the ambiguity
sets, solve Model \((\text{RRmPO})\) for a finite number of candidate hyperparameter \(\psi\). Use the validation dataset to evaluate the out-of-sample performance of \(\pi_N(\psi)\), select the optimal \(\psi^*\) as the one that maximizes \(\bar{\nu}_N(\psi)\) of the validation set, and report \(\pi_N(\psi^*)\) as the data-driven solution.

Figure 5(a) shows the mean value of the out-of-sample performance \(\bar{\nu}_N(\psi^*)\) as a function of the sample size \(|N|\). We also observe that both out-of-sample and in-sample performances exhibit asymptotic consistency. Figure 5(b) shows the mean of the reliability of the guaranteed performance under different sample sizes. We can see that the robust policy significantly outperforms the nominal one, particularly when the training data is scarce. As more data become available, the optimal robust policy converges to the nominal policy, and so does the performance of the robust policy. Figure 5(c) reports in-sample estimate \(\bar{V}_N(\psi)\). We can see that the nominal approach is over-optimistic while the robust approaches act on the cautious side.

5.4.4 Remanufacturing Planning Driven by Reliability

From the previous experiment, we can see that there exists some trade-off between the out-of-sample performance and the reliability of the performance guarantee; reliability may be sacrificed if the optimal
Figure 6: Out-of-sample reward $\bar{\nu}_N(\psi, \gamma)$, reliability $\Pr\{\bar{\nu}_N(\psi, \gamma) \geq V_N(\psi, \gamma)\}$, and in-sample reward $V_N(\psi, \gamma)$ as a function of $|N|$ with a reliability guarantee of $\gamma$. (a)-(c) KL-distance-based ambiguity set. (d)-(f) Interval-matrix-model-based ambiguity set.

hyperparameter $\psi$ is selected by only maximizing the out-of-sample performance. In the next experiment, we consider an alternative objective that chooses a hyperparameter $\psi$ that results in satisfactory out-of-sample performance while ensuring a prescribed reliability level. We use the method described in Esfahani and Kuhn (2018) to find the smallest $\psi$ for which a desired reliability level ($\gamma$) is guaranteed. An estimator of $\psi_\gamma$ is constructed via bootstrapping the training data as follows. Construct $q$ bootstrap samples (with replacement) from the original training dataset. Solve Model (RRmPO) for a finite number of hyperparameters for each bootstrap sample ($N_q$) and obtain the optimal value $V_{N_q}(\psi_\gamma)$, and then estimate the out-of-sample performance for the corresponding validation dataset. Set $\psi_\gamma$ to the smallest $\psi$ that leads to a reliability level of $\gamma$, that is, the out-of-sample performance $\bar{\nu}_{N_q}(\psi_\gamma)$ for the validation sets exceeds the in-sample performance $V_{N_q}(\psi, \gamma)$ in at least $\gamma \times q$ different bootstrap samples. Resolve Model (RRmPO) for $\psi_\gamma$ and obtain the data-driven remanufacturing policies using the original training dataset $N$. The ambiguity sets constructed in this method calibrate their size to guarantee the desired reliability level $\gamma$.

Figure 6 depicts the result when $q = 30$ and $\gamma = 0.7$. From Figures 6(b) and 6(e), we can see that the empirical reliabilities obtained from the robust approach are close to the desired reliability target and exceed
the prescribed target in many cases.

6 Conclusion and Future Work

In this paper, we consider the problem of remanufacturing planning in the presence of parameter uncertainty. We formulate the problem as a robust Markov decision process in which the true transition probability is unknown but lies in an ambiguity set with high confidence. Two statistical methods are used to construct the ambiguity set: the KL distance, and bootstrap resampling. We investigate the structure of the optimal robust policies and establish conditions to ensure the policies are of control-limit type. We also establish sufficient conditions for some of the intuitive results seen in our computational study. In particular, we derive the general decision insights for two systems—one system’s ambiguity set is contained in the other’s; we show that when large uncertainty in transition dynamics presents, the decision maker needs to be cautious about remanufacturing a product and should consider early scrapping to hedge against future uncertainties. We demonstrate the structure of the optimal robust policies via a computational study using the simulated operational data of the turbofan engine operated by NASA, investigate the out-of-sample performance, and derive the data-driven solutions to improve the out-of-sample performance.

Future extensions of this work will focus on investigating optimal production planning and inventory control policies for remanufacturing that build on this work. Moreover, at each decision epoch, decision makers make new observation about the system, and an important question that arises is that how the information that becomes available in the decision process can be leveraged to resolve some ambiguity, so that the optimal robust policies are not overly conservative. In addition, an implicit assumption made in this paper is that the states of a system are directly observable (i.e., the sensor data reveal the underlying state of the system with certainty). In practice, many systems are not directly observable and the states have to be inferred from signals collected. Future work will investigate the partially observable Markov decision process with parameter uncertainty.
References

Abdallah, T., Farhat, A., Diabat, A., and Kennedy, S. (2012). Green supply chains with carbon trading and environmental sourcing: Formulation and life cycle assessment. *Applied Mathematical Modelling*, 36(9):4271–4285.

Ben-Tal, A., Den Hertog, D., De Waegenaere, A., Meelenberg, B., and Rennen, G. (2013). Robust solutions of optimization problems affected by uncertain probabilities. *Management Science*, 59(2):341–357.

Chandler, D. L. (2011). When is it worth remanufacturing?

Curran, M. A. (2011). Scientific applications international corporation (saic). *Life cycle assessment: principles and practice*, dostupno na: http://www.epa.gov/nrmrl/lcaccess/pdfs/600r06060.pdf, 10.

Delage, E. and Mannor, S. (2010). Percentile optimization for markov decision processes with parameter uncertainty. *Operations research*, 58(1):203–213.

Diaz, N., Choi, S., Helu, M., Chen, Y., Jayanathan, S., Yasui, Y., Kong, D., Pavanaskar, S., and Dornfeld, D. (2010). Machine tool design and operation strategies for green manufacturing.

Dulman, M. T. and Gupta, S. M. (2018). Maintenance and remanufacturing strategy: using sensors to predict the status of wind turbines. *Journal of Remanufacturing*, 8(3):131–152.

Efron, B. and Tibshirani, R. J. (1994). *An introduction to the bootstrap*. CRC press.

Elwany, A. H., Gebraeel, N. Z., and Maillart, L. M. (2011). Structured replacement policies for components with complex degradation processes and dedicated sensors. *Operations research*, 59(3):684–695.

EPA, U. (2016). G7 alliance on resource efficiency: U.s.-hosted workshop on the use of life cycle concepts in supply chain management to achieve resource efficiency.

Esfahani, P. M. and Kuhn, D. (2018). Data-driven distributionally robust optimization using the wasserstein metric: Performance guarantees and tractable reformulations. *Mathematical Programming*, 171(1-2):115–166.
Faludi, J., Bayley, C., Bhogal, S., and Iribarne, M. (2015). Comparing environmental impacts of additive manufacturing vs traditional machining via life-cycle assessment. Rapid Prototyping Journal.

Ford, S. and Despeisse, M. (2016). Additive manufacturing and sustainability: an exploratory study of the advantages and challenges. Journal of Cleaner Production, 137:1573–1587.

Frederick, D. K., DeCastro, J. A., and Litt, J. S. (2007). User’s guide for the commercial modular aero-propulsion system simulation (e-mapss).

Galbreth, M. R. and Blackburn, J. D. (2010). Optimal acquisition quantities in remanufacturing with condition uncertainty. Production and Operations Management, 19(1):61–69.

Giuntini, R. and Gaudette, K. (2001). Remanufacturing, the next great opportunity for improving us productivity. Business Horizons.

Govindan, K., Soleimani, H., and Kannan, D. (2015). Reverse logistics and closed-loop supply chain: A comprehensive review to explore the future. European journal of operational research, 240(3):603–626.

Guide Jr, V. D. R. (2000). Production planning and control for remanufacturing: industry practice and research needs. Journal of operations Management, 18(4):467–483.

Gutowski, T. G., Sahni, S., Boustani, A., and Graves, S. C. (2011). Remanufacturing and energy savings. Environmental science & technology, 45(10):4540–4547.

Ijomah, W. L., McMahon, C. A., Hammond, G. P., and Newman, S. T. (2007). Development of design for remanufacturing guidelines to support sustainable manufacturing. Robotics and Computer-Integrated Manufacturing, 23(6):712–719.

Iyengar, G. N. (2005). Robust dynamic programming. Mathematics of Operations Research, 30(2):257–280.

Kim, M. J. (2016). Robust control of partially observable failing systems. Operations Research, 64(4):999–1014.

Kim, M. J. and Makis, V. (2013). Joint optimization of sampling and control of partially observable failing systems. Operations Research, 61(3):777–790.
Kurt, M. and Kharoufeh, J. P. (2010). Optimally maintaining a markovian deteriorating system with limited imperfect repairs. *European Journal of Operational Research*, 205(2):368–380.

Lund, R. T. and Hauser, W. (2012). The database of remanufacturers. *Boston University [www.reman.org/Papers/Reman_Database_Lund.pdf]*.

Mannor, S., Simester, D., Sun, P., and Tsitsiklis, J. N. (2007). Bias and variance approximation in value function estimates. *Management Science*, 53(2):308–322.

Moghaddass, R. and Zuo, M. J. (2014). An integrated framework for online diagnostic and prognostic health monitoring using a multistate deterioration process. *Reliability Engineering & System Safety*, 124:92–104.

Nilim, A. and El Ghaoui, L. (2005). Robust control of markov decision processes with uncertain transition matrices. *Operations Research*, 53(5):780–798.

Östlin, J., Sundin, E., and Björkman, M. (2009). Product life-cycle implications for remanufacturing strategies. *Journal of Cleaner Production*, 17(11):999–1009.

Puterman, M. L. (2014). *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons.

Satia, J. K. and Lave Jr, R. E. (1973). Markovian decision processes with uncertain transition probabilities. *Operations Research*, 21(3):728–740.

Savaskan, R. C., Bhattacharya, S., and Van Wassenhove, L. N. (2004). Closed-loop supply chain models with product remanufacturing. *Management science*, 50(2):239–252.

Saxena, A., Goebel, K., Simon, D., and Eklund, N. (2008). Damage propagation modeling for aircraft engine run-to-failure simulation. In *2008 international conference on prognostics and health management*, pages 1–9. IEEE.

Seitz, M. A. (2007). A critical assessment of motives for product recovery: the case of engine remanufacturing. *Journal of Cleaner Production*, 15(11-12):1147–1157.
Shi, W. and Min, K. J. (2014). Product remanufacturing and replacement decisions under operations and maintenance cost uncertainties. *The Engineering Economist, 59*(2):154–174.

Silver, E. A. (1963). Markovian decision processes with uncertain transition probabilities or rewards. Technical report, MASSACHUSETTS INST OF TECH CAMBRIDGE OPERATIONS RESEARCH CENTER.

Skrainka, M. R. S. (2012). *Analysis of the environmental impact on remanufacturing wind turbines*. Rochester Institute of Technology.

Song, S., Liu, M., Ke, Q., and Huang, H. (2015). Proactive remanufacturing timing determination method based on residual strength. *International Journal of Production Research, 53*(17):5193–5206.

Sutherland, J. W., Adler, D. P., Haapala, K. R., and Kumar, V. (2008). A comparison of manufacturing and remanufacturing energy intensities with application to diesel engine production. *CIRP annals, 57*(1):5–8.

Van Der Laan, E., Salomon, M., Dekker, R., and Van Wassenhove, L. (1999). Inventory control in hybrid systems with remanufacturing. *Management science, 45*(5):733–747.

Wang, Y., Hu, J., Ke, Q., and Song, S. (2016). Decision-making in proactive remanufacturing based on online monitoring. *Procedia CIRP, 48*:176–181.

White III, C. C. and El-Deib, H. K. (1986). Parameter imprecision in finite state, finite action dynamic programs. *Operations Research, 34*(1):120–129.

White III, C. C. and Eldeib, H. K. (1994). Markov decision processes with imprecise transition probabilities. *Operations Research, 42*(4):739–749.

Wiesemann, W., Kuhn, D., and Rustem, B. (2013). Robust markov decision processes. *Mathematics of Operations Research, 38*(1):153–183.

Xu, H., Caramanis, C., and Mannor, S. (2012). A distributional interpretation of robust optimization. *Mathematics of Operations Research, 37*(1):95–110.
Appendix

A.1 Proof of Proposition 1

The value function defined in (RRmPO) involves solving an inner problem for any given $s \in S$ and $k \in K$ as follows

$$w(s, k; 0) = \min_{r(s, k)} r(s, k) + \beta \sum_{s' \in S} p(s'|s, k) V(s', k)$$

s.t. $\sum_{s' \in S} p(s'|s, k) = 1$, $\sum_{s' \in S} \log \left( \frac{p(s'|s, k)}{\hat{p}(s'|s, k)} \right) p(s'|s, k) \leq \theta$

$$p(s'|s, k) \geq 0, s' \in S. \quad (A.1)$$

The Lagrangian dual problem of (A.1) is

$$\max_{\lambda \text{ free}, \mu \geq 0} L(\lambda, \mu) \text{ s.t. } L(\lambda, \mu) = \min_{p(\cdot|s, k) \geq 0} L(\lambda, \mu, p(\cdot|s, k))$$

where the Lagrangian function is

$$L(\lambda, \mu, p(\cdot|s, k)) = r(s, k) + \beta \sum_{s' \in S} p(s'|s, k) V(s', k) + \lambda \left( 1 - \sum_{s' \in S} p(s'|s, k) \right)$$

$$+ \mu \left( \sum_{s' \in S} p(s'|s, k) \log \left( \frac{p(s'|s, k)}{\hat{p}(s'|s, k)} \right) - \theta \right)$$

$$= r(s, k) + \lambda - \mu \theta + \sum_{s' \in S} \left( \beta V(s', k) - \lambda + \mu \log \left( \frac{p(s'|s, k)}{\hat{p}(s'|s, k)} \right) \right) p(s'|s, k).$$

The strong duality holds because $\hat{p}(\cdot|s, k)$ is a strictly feasible solution to the problem (A.1) and the Slater condition holds. The first order conditions of the Lagrangian function give

$$\frac{\partial L(\lambda, \mu, p(\cdot|s, k))}{\partial p(s'|s, k)} = \beta V(s', k) - \lambda + \mu \log \left( \frac{p(s'|s, k)}{\hat{p}(s'|s, k)} \right) + \mu = 0, \forall s' \in S$$

$$\Rightarrow \quad p(s'|s, k) = \hat{p}(s'|s, k) \exp \left( -\frac{\beta V(s', k) - \lambda - \mu}{\mu} \right), \forall s' \in S. \quad (A.2)$$

By substituting (A.2) into the Lagrangian function, the dual problem becomes

$$\max_{\lambda \text{ free}, \mu \geq 0} L(\lambda, \mu) = r(s, k) + \lambda - \mu \theta - \exp \left( \frac{\lambda - \mu}{\mu} \right) \mu \sum_{s' \in S} \hat{p}(s'|s, k) \exp \left( -\frac{\beta V(s', k)}{\mu} \right).$$
Again, the first order conditions give

$$\frac{\partial L(\lambda, \mu)}{\partial \lambda} = 1 - \exp \left( \frac{\lambda - \mu}{\mu} \right) \sum_{s' \in S} \hat{p}(s'|s, k) \exp \left( \frac{-\beta V(s', k)}{\mu} \right) = 0$$

$$\Rightarrow \lambda = -\mu \log \left( \sum_{s' \in S} \hat{p}(s'|s, k) \exp \left( \frac{-\beta V(s', k)}{\mu} \right) \right) + \mu. \quad (A.3)$$

The dual problem can be rewritten as

$$\max_{\mu \geq 0} L(\mu) = r(s, k) - \mu \log \left( \sum_{s' \in S} \hat{p}(s'|s, k) \exp \left( \frac{-\beta V(s', k)}{\mu} \right) \right) - \mu \theta.$$ 

By combining (A.2) and (A.3), we have the worst-case transitional probabilities as

$$p^*(s'|s, k) = \frac{\hat{p}(s'|s, k) \exp \left( -\beta V(s', k)/\mu_{sk}^* \right)}{\sum_{s'' \in S} \hat{p}(s''|s, k) \exp \left( -\beta V(s'', k)/\mu_{sk}^* \right)}, \quad \forall s' \in S,$$

where $\mu_{sk}^*$ is the optimal solution of the dual problem with given $s$ and $k$.

**A.2 Proof of Proposition 2**

Let $V^n(s, k) = \max_{a \in A} w^n(s, k; a)$ denote the value function at the $n$th iteration of the robust value iteration algorithm in Section 4.1.1. We will show that $V^n(s, k)$ is non-increasing in $s \in \mathcal{S}$ and $k \in \mathcal{K}$ for any integer $n \geq 0$ by induction. Then, the theorem follows because the robust value iteration algorithm converges to an optimal policy.

We set the initial value as $V^0(s, k) = 0$ for all $s \in \mathcal{S}$ and $k \in \mathcal{K}$. First, we show that $V(s, k)$ is non-increasing in $s \in \mathcal{S}$ for all $k \in \mathcal{K}$. Because $V^0(s, k) = 0$ for all $s \in \mathcal{S}$, the induction holds at the initial iteration. Assume that $V^n(s, k)$ is non-increasing in $s \in \mathcal{S}$ for $n = 1, \ldots, m-1$. Let $s', s \in \mathcal{S}$ with $s' > s$ and $\mu_{sk}^*$ be the optimal solution of the dual problem [5] defined in Theorem [1] for any given state $(s, k) \in \mathcal{S} \times \mathcal{K}$. We consider two cases at iteration $m$. If $a = 0$, we have
\[ w^m(s', k; 0) = \max_{\mu > 0} r(s', k) - \mu \log \left( \sum_{s'' \in S} \hat{p}(s''|s', k) \exp \left( \frac{-\beta V^{m-1}_m(s'', k)}{\mu} \right) \right) - \mu \theta \]

\[ = r(s', k) - \mu^*_{s', k} \log \left( \sum_{s'' \in S} \hat{p}(s''|s', k) \exp \left( \frac{-\beta V^{m-1}_m(s'', k)}{\mu^*_{s', k}} \right) \right) - \mu^*_{s', k} \theta \]

\[ \leq r(s, k) - \mu^*_{s, k} \log \left( \sum_{s'' \in S} \hat{p}(s''|s, k) \exp \left( \frac{-\beta V^{m-1}_m(s'', k)}{\mu^*_{s, k}} \right) \right) - \mu^*_{s, k} \theta \] \quad (A.4)

\[ \leq r(s, k) - \mu^*_{s, k} \log \left( \sum_{s'' \in S} \hat{p}(s''|s, k) \exp \left( \frac{-\beta V^{m-1}_m(s'', k)}{\mu^*_{s, k}} \right) \right) - \mu^*_{s, k} \theta \] \quad (A.5)

\[ \leq \max_{\mu > 0} r(s, k) - \mu \log \left( \sum_{s'' \in S} \hat{p}(s''|s, k) \exp \left( \frac{-\beta V^{m-1}_m(s'', k)}{\mu} \right) \right) - \mu \theta \]

\[ = w^m(s, k; 0) \]

The inequality (A.4) holds because \( r(s', k) \leq r(s, k) \). The inequality (A.5) follows Lemma 4.7.2 in [Puterman (2014)] because \( P(\cdot|\cdot, k) \) is IFR and \( V^{m-1}_m(s, k) \) is non-increasing in \( s \) given \( k \) by the induction hypothesis.

If \( a = 1 \), we have \( w^m(s, k; 1) = w^m(s', k; 1) = -c_r + \beta V^{m-1}_m(0, k+1) \). Therefore, \( w^m(s, k; 1) \) is non-increasing in \( s \) given \( k \). Similarly, since \( w^m(s, k; 2) = w^m(s', k; 2) = c_s \), \( w^m(s, k; 2) \) is also non-increasing in \( s \) given \( k \). Since \( V^m(s, k) = \max_{a \in A} w^m(s, k; a) \geq \max_{a \in A} w^m(s', k; a) \), the induction hypothesis holds at iteration \( m \).

Next, we show that \( V(s, k) \) is non-increasing in \( k \in \mathcal{K} \) for all \( s \in \mathcal{S} \). Because \( V^0(s, k) = 0 \) for all \( k \in \mathcal{K} \), the induction holds at the initial iteration. Assume for any given \( s \in \mathcal{S} \), \( V^n(s, k) \) is non-increasing in \( k \in \mathcal{K} \) for \( n = 0, \ldots, m - 1 \). We consider two cases at iteration \( m \). If \( a = 0 \), we have
The inequality (A.6) holds because \( r(s, k + 1) \leq r(s, k) \) and \( V^{m-1}(s, k + 1) \leq V^{m-1}(s, k) \) by the induction hypothesis. The Inequality (A.7) follows Lemma 4.7.2 in Puterman (2014) because \( \mathbf{P}(\cdot | s, k + 1) \preceq \mathbf{P}(\cdot | s, k) \) by Assumption 2(b) and \( V^{m-1}(s, k) \) is non-increasing in \( s \in \mathcal{S} \).

If \( a = 1 \), we have \( w^m(s, k; 1) = -c_r + \beta V^{m-1}(0, k + 1) \geq -c_r + \beta V^{m-1}(0, k + 2) = w^m(s, k + 1; 1) \).

Therefore, \( w^m(s, k; 1) \) is non-increasing in \( k \) for all \( s \in \mathcal{S} \). Similarly, since \( w^m(s, k; 2) = w^m(s, k + 1; 2) = c_a, \)

\( w^m(s, k; 2) \) is also non-increasing in \( k \) for all \( s \in \mathcal{S} \). Since \( V^m(s, k) = \max_{a \in \mathcal{A}} w^m(s, k; a) \geq \max_{a \in \mathcal{A}} w^m(s, k + 1; a) = V^m(s, k + 1) \), the induction hypothesis holds at iteration \( m \).

**A.3 Proof of Theorem 1**

We first show that the optimal policy is of control-limit type for all \( k \in \mathcal{K} \). Let \( s' > s \). We consider two cases: (i) If \( a^*(s, k) = 1 \), then \( V(s, k) = w(s, k; 1) = -c_r + \beta V(0, k + 1) = w(s', k; 1) \leq V(s', k) \).

Because \( V(s, k) \) is non-increasing in \( s \) for all \( k \in \mathcal{K}, V(s, k) \geq V(s', k) \). Thus, we have \( V(s', k) = w(s', k; 1) \) and \( a^*(s', k) = 1 \). (ii) If \( a^*(s, k) = 2 \), then \( V(s, k) = w(s, k; 2) = c_a = w(s', k; 2) \), and by Theorem 2 \( V(s, k) \geq V(s', k) \), we have \( V(s', k) = w(s', k; 2) \) and \( a^*(s', k) = 2 \).

Next, we show the existence of the threshold \( k^* \). This is equivalent to show that if \( a^*(s, k) = 2 \) for some \( k \), then \( a^*(s, k + 1) = 2 \). Since \( V(s, k) = w(s, k; 2) = c_a = w(s, k + 1; 2) \leq V(s, k + 1) \) and \( V(s, k) \geq V(s, k + 1) \),
we have $V(s, k + 1) = w(s, k + 1; 2)$ and hence $a^*(s, k + 1) = 2$.

### A.4 Proof of Theorem 2

We first prove that $\zeta_{rm}(k)$ is non-increasing in $k$, $\forall k \in \{0, \ldots, k^* - 1\}$. This is equivalent to show that $a^*(s, k + 1) = 1$ if $a^*(s, k) = 1 \forall k \in \{0, \ldots, k^* - 2\}$. We prove this by contradiction. Suppose $a^*(s, k) = 1$ but $a^*(s, k + 1) = 0$ for some $s \in S$ and $k \in \{0, \ldots, k^* - 2\}$. Then, we have $w(s, k; 1) \geq w(s, k; 0)$, $w(s, k + 1; 1) < w(s, k + 1; 0)$ and hence,

$$w(s, k; 1) - w(s, k + 1; 1) > w(s, k; 0) - w(s, k + 1; 0).$$  \hspace{1cm} (A.8)

The right hand side (RHS) of Equation (A.8) be rewritten as

$$\text{RHS} = r(s, k) + \max_{\mu > 0} \left( -\mu \log \left( \sum_{s' \in S} \hat{p}(s'|s, k) \exp \left( \frac{-\beta V(s', k)}{\mu} \right) \right) - \mu \theta \right)$$

$$- r(s, k + 1) - \max_{\mu > 0} \left( -\mu \log \left( \sum_{s' \in S} \hat{p}(s'|s, k + 1) \exp \left( \frac{-\beta V(s', k + 1)}{\mu} \right) \right) - \mu \theta \right)$$

$$\geq r(s, k) + \left( -\mu^*_{s,k+1} \log \left( \sum_{s' \in S} \hat{p}(s'|s, k) \exp \left( \frac{-\beta V(s', k)}{\mu^*_{s,k+1}} \right) \right) - \mu^*_{s,k+1} \theta \right)$$

$$- r(s, k + 1) - \left( -\mu^*_{s,k+1} \log \left( \sum_{s' \in S} \hat{p}(s'|s, k + 1) \exp \left( \frac{-\beta V(s', k + 1)}{\mu^*_{s,k+1}} \right) \right) - \mu^*_{s,k+1} \theta \right)$$

$$\geq r(s, k) - r(s, k + 1),$$  \hspace{1cm} (A.9)

where inequality (A.9) follows Lemma 4.7.2 in Puterman (2014) because $V(s, k)$ is non-increasing in $k \in K$ and $\hat{p}(|\cdot|, k + 1) \succeq \hat{p}(|\cdot|, k)$ in Assumption 2. And the left hand side (LHS) of Equation (A.8) be rewritten as

$$\text{LHS} = -c_r + \beta V(0, k + 1) + c_r - \beta V(0, k + 2) \leq \beta V(0, k + 1) - \beta c_s \leq \frac{\beta r(0, 0)}{1 - \beta} - \beta c_s,$$  \hspace{1cm} (A.10)

where the first inequality holds because $V(0, k + 2) \geq w(s, k + 2; 2) = c_s$, and the second inequality holds because $V(0, k + 1) \leq \sum_{t=0}^{\infty} \beta^t r(0, 0) = r(0, 0)/(1 - \beta)$. By (A.9) and (A.10), we have $\beta r(0, 0)/(1 - \beta) - \beta c_s \geq r(s, k) - r(s, k + 1)$, which violates condition in Theorem 2(a) and implies that $a^*(s, k + 1) = 1$ if $a^*(s, k) = 1$.

It is straightforward that $\zeta_{scrap}(k)$ is non-increasing in $k \in K$ because $a^*(s, k + 1) = 2$ if $a^*(s, k) = 2$ as shown in the proof of Theorem 1.
A.5 Proof of Proposition 3

Before proving our main results, we first present a lemma that identifies the worst distribution of the following problem:

$$\begin{align*}
\min_{p(\cdot|s)} & \sum_{s' \in S} p(s'|s)\nu(s') \\
\text{s.t.} & \sum_{s' \in S} p(s'|s) = 1 \\
& \underline{p}(s'|s) \leq p(s'|s) \leq \bar{p}(s'|s), \forall s' \in S
\end{align*}$$

(A.11)

with given $s \in S$, where $p(\cdot|s)$ and $\bar{p}(\cdot|s)$ are effective lower and upper bounds defined by equations (8) and (9).

**Lemma A.1.** If $\nu(s)$ is non-increasing in $s \in S$, then the optimal solution of the problem (A.11) (i.e., the worst distribution) $p^*(\cdot|s)$ is as follows:

$$p^*(s'|s) = \begin{cases} 
\bar{p}(s'|s), & s' > \delta_s \\
p(s'|s), & s' < \delta_s, \\
1 - \sum_{s''=0}^{s''=s+1} \bar{p}(s''|s), & s' = \delta_s,
\end{cases}$$

(A.12)

where $\delta_s = \min \{ \delta \in S : \sum_{s'=0}^{s} p(s'|s) + \sum_{s''=s+1}^{s} \bar{p}(s''|s) \leq 1 \}$.

**Proof.** We prove Lemma A.1 by introducing a contradiction. Suppose $p'(\cdot|s)$ is an optimal solution and there exists an $i > \delta_s$ such that $p'(i|s) < p^*(i|s) = \bar{p}(i|s)$. This implies that there exists an $j \leq \delta_s$ such that $p'(j|s) > p^*(j|s)$. Let $\Delta p = \min \{ p^*(i|s) - p'(i|s), p^*(j|s) - p'(j|s) \}$. We construct a new distribution $p''(s'|s)$ such that $p''(s'|s) = p'(s'|s)$ for $s' \in S \setminus \{i, j\}$, $p''(i|s) = p'(i|s) + \Delta p$, and $p''(j|s) = p'(j|s) - \Delta p$. It is easy to verify that $p''(\cdot|s)$ is feasible to problem (A.11). Because $\nu(i) \leq \nu(j)$, we have

$$\begin{align*}
\sum_{s'=0}^{S} p''(s'|s)\nu(s') &= \sum_{s' \in S \setminus \{i, j\}} p''(s'|s)\nu(s') + p''(i|s)\nu(i) + p''(j|s)\nu(j) \\
&= \sum_{s' \in S \setminus \{i, j\}} p'(s'|s)\nu(s') + (p'(i|s) + \Delta p)\nu(i) + (p'(j|s) - \Delta p)\nu(j) \\
&= \sum_{s'=0}^{S} p'(s'|s)\nu(s') + \Delta p(\nu(i) - \nu(j)) \leq \sum_{s'=0}^{S} p'(s'|s)\nu(s').
\end{align*}$$

This means there exists a feasible solution $p''(\cdot|s)$ that is no worse than $p'(\cdot|s)$. Thus there exists a contradiction. $\square$
We now prove the monotonicity of the value function. We first show that \( p^*(-| \cdot, k) \) is IFR for any given \( k \in K \) if \( p(s'|s, k) \) and \( \bar{p}(s'|s, k) \) satisfy conditions (10) and (11) for all \((s, k) \in \mathcal{S} \times K\). Let \( s' \geq s \) and \( \delta \), \( s \) be the same \( \delta \), defined in Lemma 3.1. If \( i \leq \delta \), we have

\[
\sum_{s''=i}^S p^*(s'', s', k) = 1 - \sum_{s''=0}^{i-1} p(s'', s', k) \geq 1 - \sum_{s''=0}^{i-1} p(s'', s, k) \geq 1 - \sum_{s''=0}^{i-1} p^*(s'', s, k) = \sum_{s''=i}^S p^*(s'', s, k),
\]

where the first inequality is a result of condition (10). If \( i > \delta \), we have

\[
\sum_{s''=i}^S p^*(s'', s', k) = \sum_{s''=i}^S \bar{p}(s'', s', k) \geq \sum_{s''=i}^S \bar{p}(s'', s, k) \geq \sum_{s''=i}^S p^*(s'', s, k),
\]

where the first inequality is a result of condition (11). Therefore, the result follows. We can similarly show that \( p^*(-| \cdot, k + 1) \geq p^*(-| \cdot, k) \) for all \( k \in K \) if conditions (12) and (13) for all \((s, k) \in \mathcal{S} \times K \) are satisfied.

Having established the structure properties of transition matrices, we next prove part (a) regarding the monotonicity of \( V(s, k) \) with respect to \( s \) for all \( k \in K \).

The proof is based on the robust value iteration algorithm in [Iyengar (2005)]. Let \( V^n(s, k) \) be the value function of the state \((s, k)\) at the end of iteration \( n \). We show that \( V^n(s, k) \) is non-increasing in \( s \) for all \( k \in K \) in every iteration \( n \) and therefore \( V(s, k) \) is non-increasing in \( s \) for all \( k \in K \) as the algorithm converges.

We prove this by induction. Let the initial values in the robust value iteration algorithm be \( V^0(s, k) = 0 \) for all \( s \in \mathcal{S} \) and \( k \in K \), then the induction hypothesis holds at the initial iteration. Assume \( V^n(s, k) \) is non-increasing in \( s \) for \( k \in K \) for \( n = 1, \ldots, m - 1 \). Let \( s', s \in \mathcal{S} \) with \( s' > s \). We first show that \( p^*(-| \cdot, k) \) is IFR for any given \( k \in K \). At iteration \( m \), if \( a = 0 \),

\[
w^m(s, k; 0) = r(s, k) + \beta \sum_{s'' \in \mathcal{S}} p^*(s''|s, k)V^{m-1}(s'', k) \geq r(s', k) + \beta \sum_{s'' \in \mathcal{S}} p^*(s''|s, k)V^{m-1}(s'', k)
\]

\[
\geq r(s', k) + \beta \sum_{s'' \in \mathcal{S}} p^*(s''|s', k)V^{m-1}(s'', k)
\]

\[
= w^m(s', k; 0),
\]

where inequality (A.13) follows Lemma 4.7.2 in [Puterman (2014)] because \( V^{m-1}(s, k) \) is non-increasing in \( s \) by the induction hypothesis and \( p^*(-| \cdot, k) \) is IFR given \( k \).

If \( a = 1 \), \( w^m(s', k; 1) = -c_r + \beta V^{m-1}(1, k + 1) = w^m(s, k; 1) \). Therefore, \( w^m(s, k; 1) \) is non-increasing in
s for all \( k \in \mathcal{K} \). We can similarly prove that \( w^m(s, k; 2) \) is also non-increasing in \( s \) for all \( k \in \mathcal{K} \).

Because \( V^m(s, k) = \max_{a \in \mathcal{A}} w^m(s, k; a) \geq \max_{a \in \mathcal{A}} w^m(s', k; a) = V^m(s', k) \). Therefore, the induction hypothesis holds at iteration \( m \).

The proof of part (b) is similar to that of part (a), and is omitted.

A.6 Proof of Corollary 1

Corollary 1 is a direct result of Lemma A.1 and the proof is omitted.

A.7 Proof of Corollary 2

Corollary 2 has already been proved in Theorem 3 and the proof is omitted.

A.8 Proof of Theorem 3

The proof is similar to the proof of Theorem 1.

A.9 Proof of Theorem 4

The proof is similar to the proof of Theorem 2.

A.10 Proof of Theorem 5

Suppose we solve the two problems simultaneously using the robust value iteration algorithm. We first show that starting with a value of 0 for all states in both problems, at the end of each iteration of the algorithm, the value function of \( \Lambda_1 \) will be greater than or equal to the value function of \( \Lambda_2 \) for each state. Let \( V_1^n(s, k) \) be the value function of the state \((s, k) \in S \times \mathcal{K}\) of problem \( \Lambda_1 \) at the end of iteration \( n \). Let \( \mathcal{U}_{sk}^i \), \( p_i(\cdot|s, k) \), and \( p_i^*(\cdot|s, k) \) denote the ambiguity set, transition probability, and the worst transition probability for state \((s, k) \in S \times \mathcal{K}\) of problem \( \Lambda_i \), respectively.

We prove this by induction. Since \( V_1^0(s, k) = V_2^0 = 0 \) for \((s, k) \in S \times \mathcal{K}\), the induction holds at the initial iteration. Now, assume that \( V_1^n(s, k) \geq V_2^n(s, k), (s, k) \in S \times \mathcal{K}, \) for \( n = 1, \ldots, m - 1 \). Then we want to show that \( V_1^m(s, k) \geq V_2^m(s, k), (s, k) \in S \times \mathcal{K} \). At iteration \( m \), if \( a = 0 \), we have
\[ w_1^m(s, k; 0) = \min_{p_1(\cdot | s, k) \in \mathcal{U}^1_{sk}} r(s, k) + \beta \sum_{s' \in \mathcal{S}} p_1(s' | s, k) V^{m-1}_1(s', k) \]
\[ \geq \min_{p_1(\cdot | s, k) \in \mathcal{U}^1_{sk}} r(s, k) + \beta \sum_{s' \in \mathcal{S}} p_1(s' | s, k) V^{m-1}_2(s', k) \tag{A.14} \]
\[ \geq \min_{p_2(\cdot | s, k) \in \mathcal{U}^2_{sk}} r(s, k) + \beta \sum_{s' \in \mathcal{S}} p_2(s' | s, k) V^{m-1}_2(s', k) \tag{A.15} \]
\[ = w_2^m(s, k; 0), \tag{A.16} \]

where inequality \([A.14]\) follows the induction hypothesis and the inequality in \([A.15]\) follows \(\mathcal{U}^1_{sk} \subseteq \mathcal{U}^2_{sk}\).

If \(a = 1\), \(w_1^m(s, k; 1) = -c_r + V^{m-1}_1(s, k) \geq -c_r + V^{m-1}_2(s, k) = w_2^m(s, k; 1)\), since \(V^{m-1}_1(s, k) \geq V^{m-1}_2(s, k)\) for \((s, k) \in \mathcal{S} \times \mathcal{K}\) by the induction assumption. If \(a = 2\), \(w_1^m(s, k; 2) = w_2^m(s, k; 2) = c_s\).

Since \(V^m_1(s, k) = \max_{a \in \mathcal{A}} w_1^m(s, k; a) \geq \max_{a \in \mathcal{A}} w_2^m(s, k; a) = V^m_2(s, k)\) for all \((s, k) \in \mathcal{S} \times \mathcal{K}\), the induction hypothesis holds at iteration \(m\). Because the value function of \(\Lambda_1\) is always greater than or equal to that of \(\Lambda_2\) at each iteration of the value-iteration algorithm, the optimal value function of \(\Lambda_1\) is greater than or equal to that of \(\Lambda_2\).

Next, we prove part (a) that \(\zeta^1_{rm}(k) \leq \zeta^2_{rm}(k)\) for \(k < k^*_1\), where \(\tilde{s} = \zeta^1_{rm}(k) - 1\) and \(k^*_1\) is the threshold defined in Theorems \([2]\) and \([4]\) for \(\Lambda_1\). This is equivalent to show \(a^*_2(\tilde{s}, k) = 0\) if \(a^*_1(\tilde{s}, k) = 0\) and \(\tilde{p}(\tilde{s} | \tilde{s}, k) = 0\) for all \(k < k^*_1\). We prove this by introducing a contradiction. Suppose \(a^*_2(\tilde{s}, k) = 1\) when \(a^*_1(\tilde{s}, k) = 0\). Then we have \(w_1(\tilde{s}, k; 0) \geq w_1(\tilde{s}, k; 1)\) and \(w_2(\tilde{s}, k; 1) \geq w_2(\tilde{s}, k; 0)\). Thus, we have

\[ w_1(\tilde{s}, k; 0) - w_2(\tilde{s}, k; 0) \geq w_1(\tilde{s}, k; 1) - w_2(\tilde{s}, k; 1). \tag{A.17} \]
The left-hand-side (LHS) of (A.17) can be rewritten as

\[
\text{LHS} = r(\tilde{s}, k) + \kappa \sum_{s' \in \mathcal{S}} p_1^*(s' | \tilde{s}, k) V_1(s', k) - r(\tilde{s}, k) - \kappa \sum_{s' \in \mathcal{S}} p_2^*(s' | \tilde{s}, k) V_2(s', k)
\]

\[
= \beta p_1^*(\tilde{s} | \tilde{s}, k) V_1(\tilde{s}, k) + \beta \sum_{s' \geq \tilde{s} + 1} p_1^*(s' | \tilde{s}, k) V_1(s', k) - \beta \sum_{s' \geq \tilde{s} + 1} p_2^*(s' | \tilde{s}, k) V_2(s', k)
\]

\[= \beta \left( w_1(\tilde{s}, k; 1) - w_2(\tilde{s}, k; 1) \right), \quad (A.20) \]

\[= \beta \left( w_1(\tilde{s}, k; 1) - w_2(\tilde{s}, k; 1) \right) - \beta w_2(\tilde{s}, k; 1) \quad (A.19) \]

The equality (A.18) is obtained by simply rearranging terms. The equality (A.19) holds because \( a^*_2(s', k) = 1 \) and \( a^*_2(s', k) = 1 \) for all \( s' \geq \tilde{s} \) following Theorem 1. The equality (A.20) holds because \( p^*_1(\tilde{s} | \tilde{s}, k) = 0 \).

By (A.17) and (A.20), we must have \( \beta \left( w_1(\tilde{s}, k; 1) - w_2(\tilde{s}, k; 1) \right) \geq w_1(\tilde{s}, k, 1) - w_2(\tilde{s}, k, 1) \). However, the inequality does not hold since \( \beta < 1 \) and leads to a contradiction. Therefore, we have \( a^*_2(\tilde{s}, k) = 0 \) if \( a^*_1(\tilde{s}, k) = 0 \) and \( p^*_1(\tilde{s} | \tilde{s}, k) = 0 \).

Next, we prove part (b) that \( \xi^*_1(\tilde{s}, k) \geq \xi^*_2(\tilde{s}, k) \) for \( k \geq k^*_1 \). This is equivalent to show \( a^*_2(s, k) = 2 \) if \( a^*_1(s, k) = 2 \). Because \( V_1(s, k) = w_1(s, k, 2) = c_s = w_2(s, k, 2) \leq V_2(s, k) \) and \( V_1(s, k) \geq V_2(s, k) \) by the induction, we have \( V_2(s, k) = w_2(s, k, 2) \) and thus \( a^*_2(s, k) = 2 \).

Since \( a^*_2(s, k) = 2 \) if \( a^*_1(s, k) = 2 \) for \( k \geq k^*_1 \), part (c) follows.

**B.1 Experiment Parameters**

The following table provides the experiment parameters used in the experiment that examines the existence of control limit policies when the condition of Theorem 2(a) is violated. Note that for easy parameter control, we redefine the reward as \( r(s, k) = a_0 - a_1 k - a_2 s \). Parameter values are drawn from their respective uniform distributions.

|      | \( a_0 \) | \( a_1 \) | \( a_2 \) | \( c_r \) | \( c_s \) | \( \theta \) | \( \beta \) |
|------|----------|----------|----------|----------|----------|----------|----------|
|      | \( U(10, 50) \) | \( U(1, 15) \) | \( U(1, 15) \) | \( U(0, 10) \) | \( U(0, 10) \) | \( U(0, 2) \) | \( U(0.01, 0.99) \) |