MULTIVARIABLE $(\varphi, \Gamma)$-MODULES AND LOCALLY ANALYTIC VECTORS

by

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Abstract. — Let $K$ be a finite extension of $\mathbb{Q}_p$ and let $G_K = \text{Gal}(\overline{\mathbb{Q}}_p/K)$. There is a very useful classification of $p$-adic representations of $G_K$ in terms of cyclotomic $(\varphi, \Gamma)$-modules (cyclotomic means that $\Gamma = \text{Gal}(K_{\infty}/K)$ where $K_{\infty}$ is the cyclotomic extension of $K$). One particularly convenient feature of the cyclotomic theory is the fact that any $(\varphi, \Gamma)$-module is overconvergent.

Questions pertaining to the $p$-adic local Langlands correspondence lead us to ask for a generalization of the theory of $(\varphi, \Gamma)$-modules, with the cyclotomic extension replaced by an infinitely ramified $p$-adic Lie extension $K_{\infty}/K$. It is not clear what shape such a generalization should have in general. Even in the case where we have such a generalization, namely the case of a Lubin-Tate extension, most $(\varphi, \Gamma)$-modules fail to be overconvergent.

In this paper, we develop an approach that gives a solution to both problems at the same time, by considering the locally analytic vectors for the action of $\Gamma$ inside some big modules defined using Fontaine's rings of periods. We show that, in the cyclotomic case, we recover the usual overconvergent $(\varphi, \Gamma)$-modules. In the Lubin-Tate case, we can prove, as an application of our theory, a folklore conjecture in the field stating that $(\varphi, \Gamma)$-modules attached to $F$-analytic representations are overconvergent.

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Introduction

Let $K$ be a finite extension of $\mathbb{Q}_p$ and let $G_K = \text{Gal}(\overline{\mathbb{Q}}_p/K)$. The basic idea of $p$-adic Hodge theory is to construct an intermediate extension $K \subset K_\infty \subset \overline{\mathbb{Q}}_p$ such that $K_\infty/K$ is simple enough, but still contains most of the ramification of $\overline{\mathbb{Q}}_p/K$ (we say that $K_\infty/K$ is deeply ramified, see [CG96]). This is for example the case if $K_\infty/K$ is an infinitely ramified $p$-adic Lie extension ([Sen72] and [CG96]). The usual choice for $K_\infty/K$ is the cyclotomic extension. One important application of this idea is Fontaine’s construction [Fon90] of cyclotomic $(\varphi, \Gamma)$-modules. By a theorem of Cherbonnier and Colmez [CC98], these cyclotomic $(\varphi, \Gamma)$-modules are always overconvergent; this is a fundamental result which allows us to relate Fontaine’s $(\varphi, \Gamma)$-modules and classical $p$-adic Hodge theory. The resulting $(\varphi, \Gamma)$-modules give rise to free modules of finite rank over the Robba ring. If $V$ is a $p$-adic representation of $G_K$, the cyclotomic $(\varphi, \Gamma)$-module $D_{\text{rig}}^\dagger(V)$ over the Robba ring attached to $V$ can be constructed in the following way. Let $K_\infty$ be the cyclotomic extension of $K$, let $H_K = \text{Gal}(\overline{\mathbb{Q}}_p/K_\infty)$ and let $\Gamma_K = \text{Gal}(K_\infty/K)$. Let $\mathcal{B}_{\text{rig}}^\dagger$ be one of the big rings of $p$-adic periods [Ber02], let $\tilde{\mathcal{B}}_{\text{rig}, K}^\dagger = (\mathcal{B}_{\text{rig}}^\dagger)^{H_K}$ and let $\tilde{D}_{\text{rig}}^\dagger(V) = (\tilde{\mathcal{B}}_{\text{rig}}^\dagger \otimes_{\mathbb{Q}_p} V)^{H_K}$. By étale descent, we have $\mathcal{B}_{\text{rig}}^\dagger \otimes_{\mathcal{B}_{\text{rig}, K}^\dagger} \tilde{D}_{\text{rig}}^\dagger(V) = \mathcal{B}_{\text{rig}}^\dagger \otimes_{\mathbb{Q}_p} V$.

We then use an analogue of Tate’s normalized traces to descend from $\tilde{D}_{\text{rig}}^\dagger(V)$ to a module $D_{\text{rig}}^\dagger(V)$ over the Robba ring $\mathcal{B}_{\text{rig}, K}^\dagger$: this is the basic idea of the Colmez-Sen-Tate method [BC08]. However, the space $\tilde{D}_{\text{rig}}^\dagger(V)$ is also a $p$-adic Banach representation of $\Gamma_K$, and it is easy to see that $D_{\text{rig}}^\dagger(V)$ consists of some vectors of $\tilde{D}_{\text{rig}}^\dagger(V)$ that are locally analytic for the action of $\Gamma_K$ (more precisely: pro-analytic, denoted by $^\text{pa}$) so that $D_{\text{rig}}^\dagger(V) \subset \tilde{D}_{\text{rig}}^\dagger(V)$. Moreover, by theorem 7.4, we have $\tilde{D}_{\text{rig}}^\dagger(V)^\text{pa} = \cup_{n \geq 0} \varphi^{-n}(D_{\text{rig}}^\dagger(V))$.

The construction of the $p$-adic local Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$ (see for instance [Bre10], [Col10] and [Ber11]) uses these cyclotomic $(\varphi, \Gamma)$-modules in an essential way. In order to extend this correspondence to $\text{GL}_2(F)$, where $F$ is a finite extension of $\mathbb{Q}_p$, it seems necessary to have at our disposal a theory of $(\varphi, \Gamma)$-modules for which $\Gamma = \text{Gal}(K_\infty/K)$ where $F \subset K$ and $K_\infty$ is generated by the torsion points of a Lubin-Tate group attached to $F$. Generalizing the theory of $(\varphi, \Gamma)$-modules to higher dimensional $p$-adic Lie groups $\Gamma$ is a difficult problem, which is raised in the introduction to [Fon90]. It does not seem to be always possible; for example, the main result of [Ber14] implies that under a reasonable additional assumption, $\Gamma$ needs to be abelian for the theory to
work. If \( K_\infty / K \) is a Lubin-Tate extension as above, then the theory does extend \([KR09]\) but the resulting \( (\varphi, \Gamma) \)-modules are usually not overconvergent \([FX13]\). Our solution to the problems of extending the theory and of the lack of overconvergence is to construct \( (\varphi, \Gamma) \)-modules with coefficients in some rings of pro-analytic vectors, which is a straightforward generalization of the above observation that \( \tilde{D}_{\text{rig}}^\dagger (V)^{pa} = \cup_{n \geq 0} \varphi^{-n} (\tilde{D}_{\text{rig}}^\dagger (V)) \) in the cyclotomic case. Our main result in this direction is the following (see theorem 8.1 and the rest of the article for notation).

**Theorem A.** — If \( K_\infty \) contains a subextension \( L_\infty \), cut out by some unramified twist of the cyclotomic character, then \( \tilde{D}_{\text{rig},K}^\dagger (V)^{pa} = (\tilde{B}_{\text{rig},K}^\dagger)^{pa} \otimes_{\tilde{B}_{\text{rig},L}^\dagger} D_{\text{rig},L}(V) \), so that \( \tilde{D}_{\text{rig},K}^\dagger (V)^{pa} \) is a free \( (\tilde{B}_{\text{rig},K}^\dagger)^{pa} \)-module of rank \( \dim(V) \), stable under \( \varphi_q \) and \( \Gamma_K \).

This theorem allows us to construct \( (\varphi, \Gamma) \)-modules over some rings of pro-analytic vectors such as \( (\tilde{B}_{\text{rig},K}^\dagger)^{pa} \). It would be interesting to determine the precise structure of these rings. When \( K_\infty / K \) is generated by the torsion points of a Lubin-Tate group attached to a Galois extension \( F/\mathbb{Q}_p \) contained in \( K \), so that \( \Gamma_K \) is an open subgroup of \( \mathcal{O}_F^\times \), we show (see \S 8) that \( (\tilde{B}_{\text{rig,K}}^\dagger)^{pa} \) contains as a dense subset the ring \( \varphi^{-\infty}(\mathcal{R}) \), where \( \mathcal{R} \) is a Robba ring in \([F: \mathbb{Q}_p]\) variables. This is why we call \( (\varphi, \Gamma) \)-modules over \( (\tilde{B}_{\text{rig},K}^\dagger)^{pa} \) multivariable \( (\varphi, \Gamma) \)-modules.

We then give an application of theorem A to the overconvergence of some Lubin-Tate \( (\varphi, \Gamma) \)-modules. We first compute the pro-\( F \)-analytic vectors \( (\tilde{B}_{\text{rig},K}^\dagger)^{F,pa} \) of \( \tilde{B}_{\text{rig},K}^\dagger \) when \( K_\infty \) is generated by the torsion points of a Lubin-Tate group attached to \( F \). The following is theorem 11.6 where \( \tilde{B}_{\text{rig},K}^\dagger \) is the Robba ring in one “Lubin-Tate” variable.

**Theorem B.** — We have \( (\tilde{B}_{\text{rig},K}^\dagger)^{F,pa} = \cup_{n \geq 0} \varphi^{-n} (\tilde{B}_{\text{rig},K}^\dagger) \).

We also determine enough of the structure of the ring \( (\tilde{B}_{\text{rig},K}^\dagger)^{pa} \) in the Lubin-Tate setting (theorem 5.3) to be able to prove a monodromy theorem concerning the descent from \( (\tilde{B}_{\text{rig},K}^\dagger)^{pa} \) to \( (\tilde{B}_{\text{rig},K}^\dagger)^{F,pa} \). We refer to theorem 6.1 for a precise statement. These results suggest the possibility of constructing some Lubin-Tate \( (\varphi, \Gamma) \)-modules over \( B_{\text{rig},K}^\dagger \) by descending \( \tilde{D}_{\text{rig},K}(V)^{pa} \) to a module over \( (\tilde{B}_{\text{rig},K}^\dagger)^{F,pa} \), which is done by solving \( p \)-adic analogues of the Cauchy-Riemann equations. Recall now that if \( F \) is a finite extension of \( \mathbb{Q}_p \) contained in \( K \) and if \( V \) is an \( F \)-linear representation of \( G_K \), we say that \( V \) is \( F \)-analytic if \( C_p \otimes_F V \) is the trivial semilinear \( C_p \)-representation for all non-trivial embeddings \( \tau: F \rightarrow \overline{\mathbb{Q}}_p \). This definition is the natural generalization of Kisin and Ren’s notion of \( L \)-crystalline representations (§3.3.7 of \([KR09]\)) and it also appears in the study of vector bundles over Fargues and Fontaine’s curve (remark 16.28 of \([FF12]\)).
Recall that using Fontaine’s classical theory, we can attach some “Lubin-Tate \((\varphi, \Gamma_K)\)-modules” over the two-dimensional local field \(\mathcal{B}_K\) to all representations of \(G_K\) ([Fon90] and [KR09]). Using our monodromy theorem, we prove the following result.

**Theorem C.** — The Lubin-Tate \((\varphi, \Gamma_K)\)-modules of \(F\)-analytic representations are overconvergent.

Theorem C was previously known for \(F = Q_p\) (Cherbonnier and Colmez [CC98]), for crystalline representations of \(G_K\) (Kisin and Ren [KR09]), as well as for some reducible representations (Fourquaux and Xie [FX13]).

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1. Lubin-Tate extensions

Throughout this paper, \(F\) is a finite Galois extension of \(Q_p\) with ring of integers \(\mathcal{O}_F\), uniformizer \(\pi_F\) and residue field \(k_F\). Let \(q = p^h\) be the cardinality of \(k_F\) and let \(F_0 = W(k_F)[1/p]\). Let \(e\) be the ramification index of \(F\), so that \(eh = [F : Q_p]\). Let \(\sigma\) denote the absolute Frobenius map on \(F_0\). Let \(E\) denote the set of embeddings of \(F\) in \(\overline{Q}_p\) so that \(E = \text{Gal}(F/Q_p)\). If \(\tau \in E\), then there exists \(n(\tau) \in \mathbb{Z}/h\mathbb{Z}\) such that \(\tau = [x \mapsto x^p]^{n(\tau)}\) on \(k_F\). Let \(W = W(F^{\text{unr}}/Q_p)\) be the Weil group of \(F^{\text{unr}}/Q_p\). If \(w \in W\), then the pair \((w|_F \in E, n(w) \in \mathbb{Z})\) determines \(w\), and \(n(w|_F) \equiv n(w) \mod h\).

Let \(\text{LT}\) be a Lubin-Tate formal \(\mathcal{O}_F\)-module attached to \(\pi_F\). If \(a \in \mathcal{O}_F\), let \([a](T)\) denote the power series that gives the multiplication-by-\(a\) map on LT. We fix a local coordinate \(T\) on LT such that \([\pi_F](T) = T^q + \pi_FT\). Let \(F_n = F(\text{LT}[\pi_F^n])\) and let \(F_\infty = \cup_{n \geq 1} F_n\). Let \(H_F = \text{Gal}(\overline{Q}_p/F_\infty)\) and \(\Gamma_F = \text{Gal}(F_\infty/F)\). By Lubin-Tate theory (see [LT65]), \(\Gamma_F\) is isomorphic to \(\mathcal{O}_F^\times\) via the Lubin-Tate character \(\chi_F : \Gamma_F \to \mathcal{O}_F^\times\). There exists an unramified character \(\eta_F : G_F \to \mathbb{Z}_p^\times\) such that \(N_{F/Q_p}(\chi_F) = \eta_F\chi_{\text{cyc}}\).

If \(K\) is a finite extension of \(F\), let \(K_n = KF_n\) and \(K_\infty = KF_\infty\) and \(\Gamma_K = \text{Gal}(K_\infty/K)\). Let \(\Gamma_n = \text{Gal}(K_\infty/K_n)\) so that \(\Gamma_n = \{g \in \Gamma_K \text{ such that } \chi_F(g) \in 1 + \pi_K^h\mathcal{O}_F\}\). Let \(u_0 = 0\) and for each \(n \geq 1\), let \(u_n \in \overline{Q}_p\) be such that \([\pi_F](u_n) = u_{n-1}\), with \(u_1 \neq 0\). We have \(\text{val}_p(u_n) = 1/q^{n-1}(q-1)e\) if \(n \geq 1\) and \(F_n = F(u_n)\). Let \(Q_k(T)\) be the minimal polynomial
of $u_k$ over $F$. We have $Q_0(T) = T$, $Q_1(T) = [\pi_F](T)/T$ and $Q_{k+1}(T) = Q_k([\pi_F](T))$ if $k \geq 1$. Let $\log_{LT}(T) \in F[T]$ denote the Lubin-Tate logarithm map, which converges on the open unit disk and satisfies $\log_{LT}(\alpha(T)) = \alpha \cdot \log_{LT}(T)$ if $\alpha \in O_F$. Note that $\log_{LT}(T) = T \cdot \prod_{k \geq 1} Q_k(T)/\pi_F$. Let $\exp_{LT}(T)$ denote the inverse of $\log_{LT}(T)$.

2. Locally analytic and pro-analytic vectors

Let $G$ be a $p$-adic Lie group (in this paper, $G$ is most of the time an open subgroup of $O_F^\times$) and let $W$ be a Banach representation of $G$. The space of locally analytic vectors of $W$ is defined in §7 of [ST03]. Here we follow the construction given in the monograph [Eme11]. Let $H$ be an open subgroup of $G$ such that there exist coordinates $c_1, \ldots, c_d : H \to \mathbb{Z}_p$ giving rise to an analytic isomorphism $c : H \to \mathbb{Z}_p^d$. If $w \in W$, we say that $w$ is an $H$-analytic vector if there exists a sequence $\{w_k\}_{k \in \mathbb{N}^d}$ with $w_k \to 0$ in $W$, such that $g(w) = \sum_{k \in \mathbb{N}^d} c(g)^k w_k$ for all $g \in H$. Let $W^{H,\text{an}}$ denote the space of $H$-analytic vectors. This space injects into $C^{\text{an}}(H,W)$ and we endow it with the induced topology, so that $W^{H,\text{an}}$ is a Banach space. We say that a vector $w \in W$ is locally analytic if there exists an open subgroup $H$ as above such that $w \in W^{H,\text{an}}$. Let $W^{la}$ denote the space of such vectors. We have $W^{la} = \cup_H W^{H,\text{an}}$ where $H$ runs through a sequence of open subgroups of $G$. We endow $W^{la}$ with the inductive limit topology, so that $W^{la}$ is a Banach space. In the sequel, we use the following results.

**Lemma 2.1.** — If $W$ is a ring, such that $\|xy\| \leq \|x\| \cdot \|y\|$ if $x, y \in W$, then $W^{H,\text{an}}$ is a ring and $\|xy\|_H \leq \|x\|_H \cdot \|y\|_H$ if $x, y \in W^{H,\text{an}}$.

**Proof.** — This is a straightforward computation, cf. §2.1 of [BC14].

**Proposition 2.2.** — Let $W$ and $B$ be two Banach representations of $G$. If $B$ is a ring and if $W$ is a free $B$-module of finite rank, having a basis $w_1, \ldots, w_d$ such that $g \mapsto \text{Mat}(g)$ is a locally analytic map $G \to \text{GL}_d(B)$, then $W^{la} = \bigoplus_{j=1}^d B^{la} \cdot w_j$.

**Proof.** — This is proved in §2.1 of [BC14], but we recall the proof for the convenience of the reader. It is clear that $\bigoplus_{i=1}^d B^{la} \cdot w_i \subset W^{la}$, so we show the reverse inclusion. If $w \in W$, then we can write $w = \sum_{i=1}^d b_i w_i$. Let $f_i : W \to B$ be the map $w \mapsto b_i$. Write $\text{Mat}(g) = (m_{i,j}(g))_{i,j}$. If $g \in G$, then $g(w) = \sum_{i,j=1}^d g(b_i)m_{i,j}(g)w_j$. If $w \in W^{la}$, then $g \mapsto f_j(g(w)) = \sum_{i=1}^d g(b_i)m_{i,j}(g)$ is a locally analytic map $G \to B$. If $\text{Mat}(g)^{-1} = (m_{i,j}(g))_{i,j}$, then $g(b_i) = \sum_{j=1}^d f_j(g(w)) m_{i,j}(g)$ so that $b_i \in B^{la}$. 


Let $W$ be a Fréchet space, whose topology is defined by a sequence $\{p_i\}_{i \geq 1}$ of seminorms. Let $W_i$ denote the Hausdorff completion of $W$ for $p_i$, so that $W = \lim_{i \to 1} W_i$.

**Definition 2.3.** — If $W = \lim_{i \to 1} W_i$ is a Fréchet representation of $G$, then a vector $w \in W$ is pro-analytic if its image $\pi_i(w)$ in $W_i$ is a locally analytic vector for all $i$. We denote by $W^{\text{pa}}$ the set of such vectors.

We extend the definition of $W^{\text{la}}$ and $W^{\text{pa}}$ to the cases when $W$ is an LB space and an LF space respectively. Note that if $W$ is an LB space, then $W^{\text{la}} = W^{\text{pa}}$. If $W$ is an LF space, then $W^{\text{la}} \subset W^{\text{pa}}$ but $W^{\text{pa}}$ will generally be bigger.

**Proposition 2.4.** — Let $W$ and $B$ be two Fréchet representations of $G$. If $B$ is a ring and if $W$ is a free $B$-module of finite rank, having a basis $w_1, \ldots, w_d$ such that $g \mapsto \text{Mat}(g)$ is a pro-analytic map $G \to \text{GL}_d(B)$, then $W^{\text{pa}} = \bigoplus_{j=1}^d B^{\text{pa}} \cdot w_j$.

**Proof.** — If $w \in W$, then one can write $w = \sum_{j=1}^d b_j w_j$ with $b_j \in B$. If $w \in W^{\text{pa}}$ and $i \geq 1$, then $\pi_i(b_j) \in B_i^{\text{la}}$ for all $i$ by proposition 2.2, so that $b_j \in B^{\text{pa}}$. □

The map $\ell : g \mapsto \log_{p_0} \chi_F(g)$ gives an $F$-analytic isomorphism between $\Gamma_n$ and $\pi_n^F \mathcal{O}_F$ for $n \gg 0$. If $W$ is an $F$-linear Banach representation of $\Gamma_K$ and $n \gg 0$, we say that an element $w \in W$ is $F$-analytic on $\Gamma_n$ if there exists a sequence $\{w_k\}_{k \geq 1}$ of elements of $W$ with $\pi_n^{w_k} w_k \to 0$ such that $g(x) = \sum_{k \geq 1} \ell(g)^k w_k$ for all $g \in \Gamma_n$. Let $W^{\Gamma_n,\text{an},F^{\text{la}}}$ denote the space of such elements. Let $W^{\Gamma_n,\text{an},F^{\text{la}}} = \bigcap_{n \geq 1} W^{\Gamma_n,\text{an},F^{\text{la}}}$. A short computation shows that $W^{\Gamma_n,\text{an},F^{\text{la}}} = W^{\Gamma_n,\text{an}} \cap W^{\text{Fla}}$. Recall the following simple result (§2.1 of [BC14]).

**Lemma 2.5.** — If $w \in W^{\text{la}}$, then $\|w\|_{\Gamma_n} = \|w\|$ for $m \gg 0$.

If $\tau \in E$, we have the “derivative in the direction $\tau$”, which is an element $\nabla \tau \in F \otimes \text{Lie}(\Gamma_F)$. It can be constructed in the following way (after §3.1 of [D13]). If $W$ is an $F$-linear Banach representation of $\Gamma_K$ and if $w \in W^{\text{la}}$, then there exists $m \gg 0$ and elements $\{w_k\}_{k \in \mathbb{N}^E}$ such that if $g \in \Gamma_m$, then $g(w) = \sum_{k \in \mathbb{N}^E} \ell(g)^k w_k$, where $\ell(g)^k = \prod_{\tau \in E} \tau \circ \ell(g)^k \tau$. We then set $\nabla \tau (w) = w_{\Lambda \tau}$, where $1_\tau$ is the $E$-uple whose entries are 0 except the $\tau$-th one which is 1. If $k \in \mathbb{N}^E$, and if we set $\nabla^k (w) = \prod_{\tau \in E} \nabla^k \tau (w)$, then $w_k = \nabla^k (w)/k!$.

**Lemma 2.6.** — Let $X, Y$ be $F$-representations of $\Gamma_n$, $\tau \in E$, and $f : X \to Y$ a $\Gamma_n$-equivariant map such that $f(ax) = \tau^{-1}(a)f(x)$. If $x \in X^{\text{pa}}$, then $\nabla_{\text{Id}}(f(x)) = f(\nabla \tau (x))$. 
3. Rings of $p$-adic periods

In this § , we recall the definition of a number of rings of $p$-adic periods. These definitions can be found in [Fon90, Fon94] and [Ber02], but we also use the “Lubin-Tate” generalization given for instance in §§8,9 of [Col02]. Let $\tilde{E}^+ = \{(x_0, x_1, \ldots) , \text{ with } x_n \in \mathcal{O}_{C_p}/\pi_F \text{ and } x_{n+1}^q = x_n \text{ for all } n \geq 0\}$. This ring is endowed with the valuation $\text{val}_{\tilde{E}}(\cdot)$ defined by $\text{val}_{\tilde{E}}(x) = \lim_{n \to +\infty} q^n \text{val}_\pi(\hat{x}_n)$ where $\hat{x}_n \in \mathcal{O}_{C_p}$ lifts $x_n$. The ring $\tilde{E}^+$ is complete for $\text{val}_{\tilde{E}}(\cdot)$. If the $\{u_n\}_{n \geq 0}$ are as in §11, then $\pi = (\overline{\pi}_0, \overline{\pi}_1, \ldots) \in \tilde{E}^+$ and $\text{val}_\pi(\pi) = q/(q-1)e$. Let $\tilde{E}$ be the fraction field of $\tilde{E}^+$.

Let $W_F(\cdot)$ denote the functor $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} W(\cdot)$ of $F$-Witt vectors. Let $\tilde{A}^+ = W_F(\tilde{E}^+)$ and let $\tilde{B}^+ = \tilde{A}^+[1/\pi_F]$. These rings are preserved by the Frobenius map $\varphi_q = \text{Id} \otimes \varphi^h$. Every element of $\tilde{B}^+[1/\{\overline{\pi}\}]$ can be written as $\sum_{k \gg -\infty} \pi_F^k[x_k]$ where $\{x_k\}_{k \in \mathbb{Z}}$ is a bounded sequence of $\tilde{E}$. If $r \geq 0$, define a valuation $V(\cdot, r)$ on $\tilde{B}^+[1/\{\overline{\pi}\}]$ by

$$V(x, r) = \inf_{k \in \mathbb{Z}} \left( \frac{k}{e} + \frac{p-1}{pr} \text{val}_{\tilde{E}}(x_k) \right) \text{ if } x = \sum_{k \gg -\infty} \pi_F^k[x_k].$$

This valuation is normalized as in §2 of [Ber02]. The valuation defined in §3 of [Ber13] is normalized differently (sorry), it is $pr/(p-1)$ times this one. If $I$ is a closed subinterval of $[0; +\infty[$, let $V(x, I) = \inf_{r \in I} V(x, r)$. The ring $\tilde{B}^I$ is defined to be the completion of $\tilde{B}^+[1/\{\overline{\pi}\}]$ for the valuation $V(\cdot, I)$ if $0 \notin I$ and if $I = [0; r]$, then $\tilde{B}^I$ is the completion of $\tilde{B}^+$ for $V(\cdot, I)$. When $F = Q_p$, the ring $\tilde{B}^I$ is the same as the one denoted by $\tilde{B}_I$ in §2.1 of [Ber02]. Let $\tilde{A}^I$ be the ring of integers of $\tilde{B}^I$ for $V(\cdot, I)$.

If $k \geq 1$, let $r_k = p^{h-1}(p-1)$. The map $\theta \circ \varphi_q^{-k} : \tilde{A}^+ \to \mathcal{O}_{C_p}$ extends by continuity to $\tilde{A}^I$ provided that $r_k \in I$ and then $\theta \circ \varphi_q^{-k}(\tilde{A}^I) \subset \mathcal{O}_{C_p}$. By §9.2 of [Col02], there exists $u \in \tilde{A}^+$, whose image in $\tilde{A}^+$ is $\overline{\pi}$, and such that $\varphi_q^{-1}(u) = \overline{\pi}F([u])$ if $g \in \Gamma_F$. For $k \geq 0$, let $Q_k = Q_k(u) \in \tilde{A}^+$. The kernel of $\theta : \tilde{A}^+ \to \mathcal{O}_{C_p}$ is generated by $\varphi_q^{-1}(Q_1)$ (see proposition 8.3 of [Col02]), so that $\varphi_q^{-1}(Q_1)/([\overline{\pi}F] - \pi_F)$ is a unit of $\tilde{A}^+$ and therefore, $Q_k/([\overline{\pi}F] - \pi_F)$ is a unit of $\tilde{A}^+$ for all $k \geq 1$.

Lemma 3.1. — If $y \in \tilde{A}^{[0; r_k]}$, then there exists a sequence $\{a_i\}_{i \geq 0}$ of elements of $\tilde{A}^+$, converging $p$-adically to 0, such that $y = \sum_{i \geq 0} a_i \cdot (Q_k/\pi_F)^i$.

Proof. — See §2.1 of [Ber02] for $F = Q_p$, the proof for other $F$ being similar. □

Lemma 3.2. — Let $r = r_\ell$ and $s = r_k$, with $1 \leq \ell \leq k$.

1. $\theta \circ \varphi_q^{-k}(\tilde{A}^{[r;s]}) = \mathcal{O}_{C_p}$ and $\ker(\theta \circ \varphi_q^{-k} : \tilde{A}^{[r;s]} \to \mathcal{O}_{C_p}) = (Q_k/\pi_F) \cdot \tilde{A}^{[r;s]}$;
2. $\pi_F\tilde{A}^{[r;s]} \cap (Q_k/\pi_F) \cdot \tilde{A}^{[r;s]} = Q_k \cdot \tilde{A}^{[r;s]}$;
3. $\pi_F\tilde{A}^{[r;s]} \cap \tilde{A}^{[0;s]} = \pi_F\tilde{A}^{[0;s]}$. 
Proof. — Item (1) follows from the straightforward generalization of §2.2 of [Ber02] from $\mathbb{Q}_p$ to $F$ (note that proposition 2.11 of ibid. is only correct if the element $[\hat{p}]/p - 1$ actually belongs to $\hat{\mathbb{A}}^\dagger$) and the fact that $Q_k/([\hat{\pi}_F]^{q_k} - \pi_F)$ is a unit of $\hat{\mathbb{A}}^\dagger$. If $x \in \hat{\mathbb{A}}^{[r; s]}$ and $\pi_F x \in \ker(\theta \circ \varphi_q^{-k})$, then $x \in \ker(\theta \circ \varphi_q^{-k})$ and this together with (1) implies (2). Finally, if $x \in \hat{\mathbb{A}}^{[r; s]}$ is such that $\pi_F x \in \hat{\mathbb{A}}^{[0; s]}$, then $x \in \hat{\mathbb{B}}^{[0; s]}$ and $V(x, s) \geq V(x, [r; s]) \geq 0$, so that $x \in \hat{\mathbb{A}}^{[0; s]}$ and $\pi_F x \in \pi_F \hat{\mathbb{A}}^{[0; s]}$. □

Proposition 3.3. — If $y \in \hat{\mathbb{A}}^{[0; s]} + \pi_F \cdot \hat{\mathbb{A}}^{[r; s]}$ and if $\{y_i\}_{i \geq 0}$ is a sequence of elements of $\hat{\mathbb{A}}^\dagger$ such that $y - \sum_{j=0}^{i-1} y_i \cdot (Q_k/\pi_F)^i$ belongs to $\ker(\theta)^j$ for all $j \geq 1$, then there exists $j \geq 1$ such that $y - \sum_{i=0}^{j-1} y_i \cdot (Q_k/\pi_F)^i \in \pi_F \cdot \hat{\mathbb{A}}^{[r; s]}$.

Proof. — By lemma 3.1 there exist $j \geq 1$ and $a_0, \ldots, a_{j-1}$ of $\hat{\mathbb{A}}^\dagger$ such that

\[
\begin{align*}
(A) \quad & y - \left( a_0 + a_1 \cdot (Q_k/\pi_F) + \cdots + a_{j-1} \cdot (Q_k/\pi_F)^{j-1} \right) \\
& \in \pi_F \hat{\mathbb{A}}^{[r; s]}
\end{align*}
\]

We have $a_0, y_0 \in \hat{\mathbb{A}}^\dagger$ and $\theta \circ \varphi_q^{-k}(y_0 - a_0) \in \pi_F \mathcal{O}_C$ by the above, so there exists $c_0, d_0 \in \hat{\mathbb{A}}^\dagger$ such that $a_0 = y_0 + Q_k c_0 + \pi_F d_0$. In particular, (A) holds if we replace $a_0$ by $y_0$. Assume now that $f \leq j - 1$ is such that (A) holds if we replace $a_i$ by $y_i$ for $i \leq f - 1$. The element

\[
\begin{align*}
& \left( a_0 + a_1 \cdot (Q_k/\pi_F) + \cdots + a_{j-1} \cdot (Q_k/\pi_F)^{j-1} \right) \\
& \quad - \left( y_0 + y_1 \cdot (Q_k/\pi_F) + \cdots + y_{j-1} \cdot (Q_k/\pi_F)^{j-1} \right)
\end{align*}
\]

belongs to $\pi_F \hat{\mathbb{A}}^{[r; s]} + (Q_k/\pi_F)^f \hat{\mathbb{A}}^{[r; s]}$. If $a_i = y_i$ for $i \leq f - 1$, then the element

\[
\begin{align*}
& \left( a_f + a_{f+1} \cdot (Q_k/\pi_F) + \cdots + a_{j-1} \cdot (Q_k/\pi_F)^{j-1-f} \right) \\
& \quad - \left( y_f + y_{f+1} \cdot (Q_k/\pi_F) + \cdots + y_{j-1} \cdot (Q_k/\pi_F)^{j-1-f} \right)
\end{align*}
\]

belongs to $\pi_F \hat{\mathbb{A}}^{[r; s]} + (Q_k/\pi_F)^{j-f} \hat{\mathbb{A}}^{[r; s]}$ since $\pi_F \hat{\mathbb{A}}^{[r; s]} \cap (Q_k/\pi_F)^{j-f} \hat{\mathbb{A}}^{[r; s]} = \pi_F (Q_k/\pi_F)^{j-f} \hat{\mathbb{A}}^{[r; s]}$ by applying repeatedly (2) of lemma 3.2. We have $a_f, y_f \in \hat{\mathbb{A}}^\dagger$ and the above implies that $\theta \circ \varphi_q^{-k}(y_f - a_f) \in \pi_F \mathcal{O}_C$. There exist therefore $c_f, d_f \in \hat{\mathbb{A}}^\dagger$ such that $a_f = y_f + Q_k c_f + \pi_F d_f$ which shows that (A) holds if we also replace $a_f$ by $y_f$. This shows by induction on $f$ that $y - (y_0 + y_1 \cdot (Q_k/\pi_F) + \cdots + y_{j-1} \cdot (Q_k/\pi_F)^{j-1})$ belongs to $\pi_F \hat{\mathbb{A}}^{[r; s]}$, which proves the proposition. □

Lemma 3.4. — If $r > 1$, then $u/|u|$ is a unit of $\hat{\mathbb{A}}^{1/r}$.

Proof. — We have $u = |u| + \sum_{k \geq 1} \pi_F^k [v_k]$ with $v_k \in \hat{\mathbb{E}}^\dagger$ and the lemma follows from the fact that if $s \geq r > 1 \geq (p - 1)/p \cdot q/(q - 1)$, then $V(\pi_F/|u|, s) > 0$. □

If $\rho > 0$, then let $\rho' = \rho \cdot e \cdot p/(p - 1) \cdot (q - 1)/q$. Lemma 3.3 and the fact that $\text{val}_F(u) = q/(q - 1)e$ imply that if $r > 1$, then $V(u', r) = i/r'$ for $i \in \mathbb{Z}$ (compare with proposition 3.1 of [Ber13], bearing in mind that our normalization of $V(\cdot, r)$ is different).
Let $I$ be either a subinterval of $]1; +\infty[$ or such that $0 \in I$, and let $f(Y) = \sum_{k \in \mathbb{Z}} a_k Y^k$ be a power series with $a_k \in F$ and such that $\text{val}_p(a_k) + k/\rho \to +\infty$ when $|k| \to +\infty$ for all $\rho \in I$. The series $f(u)$ then converges in $\widetilde{B}^I$ and we let $B^I_F$ denote the set of $f(u)$ where $f(Y)$ is as above. It is a subring of $B^I_F = (\widetilde{B}^I)^{\mathbb{F}_F}$, which is stable under the action of $\Gamma_F$. The Frobenius map gives rise to a map $\varphi_q : B^I_F \to B^I_{F^q}$. If $m \geq 0$, then $\varphi_q^{-m}(B^m_{F^q}) \subset B^I_F$ and we let $B^I_{F,m} = \varphi_q^{-m}(B^m_{F^q})$ so that $B^I_{F,m} \subset B^I_{F,m+1}$ for all $m \geq 0$. For example, if $t_F = \log_{|L|}(u)$ then $t_F \in B^{|r|+\infty}_F$, and $\varphi_q(t_F) = \pi_F t_F$ and $g(t_F) = \chi_F(g)t_F$ for $g \in G_F$.

Let $B^r_{\text{rig},F}$ denote the ring $B^r_{F[\mathbb{Q}]}$. This is a subring of $B^r_{\tilde{F}_{\text{rig}}}$ for all $s \geq r$. Let $B^r_{\text{rig},F}$ denote the set of $f(u) \in B^r_{\text{rig},F}$ such that in addition $\{a_k\}_{k \in \mathbb{Z}}$ is a bounded sequence. Let $B^I_F = \cup_{r \geq 0} B^I_{r,F}$. This a henselian field (cf. §2 of [Mat95]), whose residue field $E_F$ is isomorphic to $E_q((u))$. Let $K$ be a finite extension of $F$. By the theory of the field of norms (see [FW79a], [FW79b] and [Win83]), there corresponds to $K/F$ a separable extension $E_K/E_F$, of degree $[K_\infty : F_\infty]$. Since $B^I_F$ is a henselian field, there exists a finite unramified extension $E_K/B^I_F$ of degree $[K_\infty : F_\infty]$ whose residue field is $E_K$ (cf. §3 of [Mat95]). There exists therefore $r(K) > 0$ and elements $x_1, \ldots, x_c$ in $B^I_{K,\{r(K)\}}$ such that $B^I_K = \oplus_{i=1}^c B^I_{r(K)} x_i$ for all $s \geq r(K)$. Let $B^I_{K,s}$ denote the completion of $B^I_{K,s}$ for $V(\cdot, I)$ where $r(K) \leq \min(I)$, so that $B^I_K = \oplus_{i=1}^c B^I_{r(K)} \cdot x_i$. Let $B^I_{K,m} = \varphi_q^{-m}(B^m_{K^q})$ and $B^I_{K,\infty} = \cup_{m \geq 0} B^I_{K,m}$ so that $B^I_{K,m} \subset \tilde{B}^I_K = (\tilde{B}^I)^{\mathbb{F}_K}$.

Let $B^r_{\text{rig},K}$ denote the Fréchet completion of $B^r_{K}$ for the valuations $\{V(\cdot, \{r; s\})\}_{s \geq r}$. Let $B^r_{\text{rig},K,m} = \varphi_q^{-m}(B^r_{\text{rig},K,m})$ and $B^r_{\text{rig},K,\infty} = \cup_{m \geq 0} B^r_{\text{rig},K,m}$. We have $B^r_{\text{rig},K,\infty} \subset \tilde{B}^r_{\text{rig},K}$ for all $s \geq r$. Let $\tilde{B}^r_{\text{rig},K}$ denote the Fréchet completion of $\tilde{B}^r_{\text{rig},K}$ for the valuations $\{V(\cdot, \{r; s\})\}_{s \geq r}$; $B^r_{\text{rig},K}$ is a subring of $\tilde{B}^r_{\text{rig},K}$ for all $s \geq r$. Let $\tilde{B}^r_{\text{rig},F} = \cup_{r \geq 0} \tilde{B}^r_{\text{rig},K}$ and $\tilde{B}^r_{\text{rig},F} = (\tilde{B}^r_{\text{rig},K})^{\mathbb{F}_K}$ and $\tilde{B}^r_{\text{rig},F} = (\tilde{B}^r_{\text{rig},K})^{\mathbb{F}_K}$. Note that $\tilde{B}^r_{\text{rig},K}$ contains $B^r_{\text{rig},K}$.

4. Locally $F$-analytic vectors of $\tilde{B}^I_{\text{rig},K}$

In this §, we compute the pro-$F$-analytic vectors of $\tilde{B}^I_{\text{rig},K}$. Recall that if $n \geq 1$, then we set $r_n = p^{nh-1}(p-1)$. From now on, let $r = r_\ell$ and $s = r_k$, with $\ell \leq k$. Let $I = \{r; s\}$ with $r = r_\ell$ and $s = r_k$.

**Proposition 4.1.** — If $f(Y) \in \mathcal{O}_F[Y]$, then $\varphi_q^{-m}(f(u)) \in (\tilde{B}^I_{F})^{\mathbb{F}_F}$.

**Proof.** — By lemma [21] it is enough to show that $\varphi_q^{-m}(u) \in (\tilde{B}^I_{F})^{\mathbb{F}_F}$. By Lubin-Tate theory, there exists a family $\{c_n(T)\}_{n \geq 0}$ of elements of $F[T]$ such that $[a(T)] = \sum_{n \geq 0} c_n(a) \cdot T^n$ if $a \in \mathcal{O}_F$. The polynomials $c_n(T)$ are of degree at most $n$ and $c_n(\mathcal{O}_F) \subset \mathcal{O}_F$. Let $\{g_n(T)\}_{n \geq 0}$ denote the family of polynomials constructed in §1.8 of [DS09]. Since $c_n(\mathcal{O}_F) \subset \mathcal{O}_F$ and the family $\{g_n(T)\}_{n \geq 0}$ is a Mahler basis (§1.2 of ibid), there are
elements $b_{n,i} \in \mathcal{O}_F$ such that $c_n(T) = \sum_{i=0}^{n} b_{n,i} g_n(T)$. If $n \geq 0$, let $n_0 + n_1 q + \cdots + n_{m-1} q^{m-1}$ denote the representation of $n$ in base $q$. Let $h = k + m$, let

$$w_{n,h} = \sum_{i=h}^{m-1} n_i \frac{q^{i-h} - 1}{q - 1}.$$ 

By proposition 4.2 of ibid (see also §10 of [Ami64]), the elements $\{\pi_{F_i}^{w_{n,h}} g_n\}_{n \geq 0}$ form a Banach basis of the Banach space $\mathcal{L}(\mathcal{O}_F)$ of functions on $\mathcal{O}_F$ that are analytic on closed disks of radius $|\pi_F|^h$. Let $\|\cdot\|_s$ denote the norm on $\hat{B}$ given by $\|x\| = p^{-V(x,s)}$. In order to prove the proposition, it is enough to show that $\|g_n\|_{\mathcal{L}(\mathcal{O}_F)} \cdot \|\varphi_q^{-m}(u)^n\|_s \to 0$ as $n \to +\infty$. We have

$$w_{n,h} = \sum_{i=h}^{m-1} n_i \frac{q^{i-h} - 1}{q - 1} \leq \sum_{i=h}^{m-1} n_i \frac{q^{i-h}}{q - 1} \leq n \cdot \frac{1}{q^{h(q-1)}}.$$ 

On the other hand, $\|\varphi_q^{-m}(u)^n\|_{r_k} = \|u^n\|_{r_k+m} = |\pi_F|^{n/h(q-1)}$. This implies that

$$\|g_n\|_{\mathcal{L}(\mathcal{O}_F)} \cdot \|\varphi_q^{-m}(u)^n\|_s \leq |\pi_F|^n \left(\frac{1}{q^{n/h(q-1)} - q^{n/(q-1)}}\right),$$

so that $\|g_n\|_{\mathcal{L}(\mathcal{O}_F)} \cdot \|\varphi_q^{-m}(u)^n\|_s \to 0$ as $n \to +\infty$. \hfill \Box

**Remark 4.2.** — In a previous version of this paper, proposition 4.1 was proved under the assumption that the ramification index of $F$ was at most $p-1$, by bounding the norm of $\nabla^i(f)/i!$ as $i \to +\infty$. I am grateful to P. Colmez for suggesting the above proof.

Let $m_0 \geq 0$ be such that $t_F$ and $t_F/Q_k \in (\hat{B}_F)^{\Gamma_{m_0} \cdot \mathcal{F}_{\Lambda}}$.

**Lemma 4.3.** — If $m \geq m_0$, $a \in \hat{B}_F$ and $Q_k \cdot a \in (\hat{B}_F)^{\Gamma_m \cdot \mathcal{F}_{\Lambda}}$, then $a \in (\hat{B}_F)^{\Gamma_m \cdot \mathcal{F}_{\Lambda}}$.

**Proof.** — Write $a = 1/t_F \cdot t_F/Q_k \cdot Q_k a$. The lemma follows from the facts that $g(1/t_F) = \chi_F(g)^{-1} \cdot (1/t_F)$ and that $t_F/Q_k$ is $F$-analytic on $\Gamma_m$, and lemma 2.1 \hfill \Box

**Theorem 4.4.** — If $I = [r_k; r_k]$ with $\ell \leq k$, then $(\hat{B}_F)^{\mathcal{F}_{\Lambda}} = \hat{B}_{K,\infty}^{F_{\Lambda}}$.

**Proof.** — We first prove the theorem for $K = F$. The action of $\Gamma_F$ on $\hat{B}_{F,m}^{F_{\Lambda}}$ is locally $F$-analytic, so that $\hat{B}_{F,\infty} \subset (\hat{B}_F)^{F_{\Lambda}}$, and we now prove the reverse inclusion. Take $x \in (\hat{B}_F^{[r:s]})^{F_{\Lambda}} \cap \hat{A}^{[r:s]}$.

Since $x \in (\hat{B}_F^{[r:s]})^{F_{\Lambda}}$, there exists $m \geq m_0$ such that $x \in (\hat{B}_F^{[r:s]})^{\Gamma_{m+k} \cdot \mathcal{F}_{\Lambda}}$. If $d = q^{f-1}(q-1)$, then $\hat{A}^{[r:s]} = \hat{A}^{[0:s]} \{\pi_F/d\}$. So that for all $n \geq 1$, there exists $k_n \geq 0$ such that $(u^d/\pi_F)^{k_n} \cdot x \in \hat{A}^{[0:s]} + \pi_F^{k_n} \hat{A}^{[r:s]}$. If $x_n = (u^d/\pi_F)^{k_n} \cdot x$, then $x_n \in (\hat{B}_F^{[r:s]})^{\Gamma_{m+k} \cdot \mathcal{F}_{\Lambda}}$, so that $\theta \circ \varphi_q^{-k}(x_n) \in \mathcal{O}_{F_{\infty}}^{\Gamma_{m+k} \cdot \mathcal{F}_{\Lambda}}$.

By §4.1 of [BC14], $\hat{F}_{\infty} = F_{\infty}$ and therefore, $\mathcal{O}_{F_{\infty}}^{\Gamma_{m+k} \cdot \mathcal{F}_{\Lambda}} = \mathcal{O}_{F_{m+k}}$. 


There exists \( y_{n,0} \in \mathcal{O}_F[\varphi_q^{-m}(u)] \) such that \( \theta \circ \varphi_q^{-k}(y_{n,0}) = \theta \circ \varphi_q^{-k}(y_{n,0}) \). By (11) of lemma 3.2 and lemma 4.3 there exists \( x_{n,1} \in (\widetilde{B}_F^{[r:s]})^{\Gamma_{m+k,an,F,la}} \cap \mathcal{A}^{[r:s]} \) such that \( x_n - y_{n,0} = (Q_k/\pi_F) \cdot x_{n,1} \). Applying this procedure inductively gives us a sequence \( \{y_{n,i}\}_{i \geq 0} \) of elements of \( \mathcal{O}_F[\varphi_q^{-m}(u)] \) such that for all \( j \geq 1 \), we have

\[
x_n - (y_{n,0} + y_{n,1} \cdot (Q_k/\pi_F) + \cdots + y_{n,j-1} \cdot (Q_k/\pi_F)^{-1}) \in \ker(\theta)^j.
\]

Proposition 3.3 shows that there exists \( j \gg 0 \) such that

\[
x_n - (y_{n,0} + y_{n,1} \cdot (Q_k/\pi_F) + \cdots + y_{n,j-1} \cdot (Q_k/\pi_F)^{-1}) \in \pi_F \mathcal{A}^{[r:s]},
\]

and therefore belongs to \( \pi_F(\mathcal{A}^{[0:s]} + \pi_F^{-1} \mathcal{A}^{[r:s]}) \), since \( \pi_F \mathcal{A}^{[r:s]} = \pi_F \mathcal{A}^{[0:s]} \) by (3) of lemma 3.2. Write \( x_n - (y_{n,0} + y_{n,1} \cdot (Q_k/\pi_F) + \cdots + y_{n,j-1} \cdot (Q_k/\pi_F)^{-1}) - \pi_F x_n \) with \( x_n \in \mathcal{A}^{[0:s]} + \pi_F^{-1} \mathcal{A}^{[r:s]} \). By proposition 4.1 we have \( x_n \in (\widetilde{B}_F^{[r:s]})^{\Gamma_{m+k,an,F,la}} \). Applying to \( x_n \) the same procedure which we have applied to \( x_n \), and proceeding inductively, allows us to find some \( j \gg 0 \) and some elements \( \{y_{n,i}\}_{i \leq j} \) of \( \mathcal{O}_F[\varphi_q^{-m}(u)] \) such that if

\[
y_n = y_{n,0} + y_{n,1} \cdot (Q_k/\pi_F) + \cdots + y_{n,j} \cdot (Q_k/\pi_F)^{-1},
\]

then \( y_n - x_n \in \pi_F \mathcal{A}^{[r:s]} \). If \( z_n = (\pi_F/\pi)^{k_0} y_n \), then \( z_n - x = (\pi_F/\pi)^{k_0} (y_n - x_n) \in \pi_F \mathcal{A}^{[r:s]} \) so that \( \{z_n\}_{n \geq 1} \) converges \( \pi_F \)-adically to \( x \), and \( z_n \in \mathcal{A}^{[r:s]} \) so that \( x \in \mathcal{A}^{[r:s]} \). This proves the theorem when \( K = F \).

We now consider the case when \( K \) is a finite extension of \( F \). We first prove that \( \mathcal{B}^{[r]}_{K,\infty} \subset (\mathcal{B}^{[r]}_F)^{F,la} \). Since \( \mathcal{B}^{[r]}_F = \oplus_{i=1}^n \mathcal{B}^{[r]}_x \), at the end of §3 each element of \( \mathcal{B}^{[r]}_{K,\infty} \) is integral over \( \mathcal{B}^{[r]}_{F,\infty} \). Take \( x \) in \( \mathcal{B}^{[r]}_{K,\infty} \) and let \( P(T) \in \mathcal{B}^{[r]}_{F,\infty}[T] \) denote its minimal polynomial over \( \mathcal{B}^{[r]}_{F,\infty} \). If \( g \in \Gamma_K \) is close enough to 1, then \( (gP)(gx) = 0 \) and the coefficients of \( gP \) are analytic functions in \( \ell(g) \). We also have \( P'(x) \neq 0 \), so that \( x \) is locally \( F \)-analytic by the implicit function theorem for analytic functions (which follows from the inverse function theorem given on page 73 of [Ser06]). Note that if \( P(x) = 0 \) and \( D \in \text{Lie}(\Gamma_K) \), then \( (DP)(x) + P'(x)D(x) = 0 \), which gives us an explicit way to compute the derivatives of \( x \). This proves the first inclusion.

We have \( \mathcal{B}^{[r]}_K = \oplus_{i=1}^n \mathcal{B}^{[r]}_F \cdot x_i \), and the reverse inclusion now follows from proposition 2.2 which implies that \( (\mathcal{B}^{[r]}_K)^{F,la} = \oplus_{i=1}^n (\mathcal{B}^{[r]}_F)^{F,la} \cdot x_i \), and the case \( K = F \). \( \square \)

Lemma 4.5. — Let \( r \geq \max(r(K), (p-1)e/p) \). If \( x \in \mathcal{B}^{[r:s]}_K \) and \( \varphi^n(x) \in \mathcal{B}^{[m,s;q.m]}_K \) for some \( t \geq s \), then \( x \in \mathcal{B}^{[r:s]}_K \).

Proof. — Let \( \psi_q : \mathcal{B}^{[r:s]}_F \rightarrow \mathcal{B}^{[r:s/q]}_F \) be the map constructed for \( r > (p-1)e/p \) in §2 of [PX13]. It satisfies \( V(\psi_q(x), [r/q, s/q]) \geq V(x, [r, s]) - h \) and \( \psi_q(\varphi_q(x)) = x \). Recall that if \( x_1, \ldots, x_e \) is a basis of \( \mathcal{B}^{[r]}_K \) over \( \mathcal{B}^{[r]}_F \), then \( \mathcal{B}^{[r:s]}_K = \oplus_{i=1}^e \mathcal{B}^{[r:s]}_F \cdot x_i \). We can assume that
Let \( x_i = \varphi_q(y_i) \) with \( y_i \in B^{1,r(K)}_K \) (cf. §III.2 of [CC98]). We then extend \( \psi_q \) to \( B^{r;\alpha}_K \) by the formula \( \psi_q(\sum_{i=1}^n \lambda_i \varphi_q(y_i)) = \sum_{i=1}^n \psi_q(\lambda_i) y_i. \)

If \( x \in B^{r;\alpha}_K \) and \( \varphi^m(x) \in B^{[m^r;\alpha_{m^n}] q}_K, \) then \( x = \psi_q(\varphi^m(x)) \) and \( \psi_q(\varphi^m(x)) \in B^{r;\alpha}_K. \)

\[ \square \]

**Theorem 4.6.** — We have \( \tilde{B}^{1,r}_{\rig, K} = B^{1,r}_{\rig, K, \infty}. \)

*Proof.* — If \( x \in (\tilde{B}^{1,r}_{\rig, K})^{F,\pa} \), then theorem 4.4 implies that for each \( s \geq r \), the image of \( x \) in \( B^{[r,s]}_K \) lies in \( B^{[r,s]}_{K, m} \) for some \( m = m(s) \). We have \( \varphi_q^m(x) \in B^{[m^r;\alpha_{m^n}] q}_K \) and lemma 4.5 implies that \( m(s) \) is independent of \( s \gg 0 \). The theorem then follows from the fact that \( B^{1,r}_{\rig, K, m} = \lim_{s \to \infty} B^{[r,s]}_K. \)

\[ \square \]

5. Rings of locally analytic periods

We now prove that the elements of \( (\tilde{B}^{1,r}_K)^{\la} \) can be written as power series with coefficients in \( (\tilde{B}^{1,r}_K)^{\la} \). Let \( K \) be a finite extension of \( F \) and let \( K_\infty = K F_\infty \) as above. If \( \tau \in E \) and \( f(Y) = \sum_{k \in Z} a_k \tau^k \) with \( a_k \in F \), let \( f^\tau(Y) = \sum_{k \in Z} \tau(a_k) Y^k \). For \( \tau \in E \), let \( \tilde{n}(\tau) \) be the lift of \( n(\tau) \in Z/hZ \) belonging to \( \{0, \ldots, h-1\} \).

Let \( y_r = (\tau \otimes \varphi^{n(\tau)})(u) \in \tilde{A}^+= \mathcal{O}_F \otimes_{\mathcal{O}_{Y_0}} W(\tilde{E}^+). \) The element \( y_r \) satisfies \( g_r(y_r) = [\chi_F(g)]^\tau(y_r) \) and \( \varphi(y_r) = [\pi_F]^\tau(y_r) = t_\tau \cdot Y^h + y_r^2. \) Let \( t_\tau = (\tau \otimes \varphi^{n(\tau)})(t_F) = \log_{1/t_F}(y_r). \)

Recall that \( W = W(F^{unr}/Q_p) \). If \( g \in W \) and \( p^{n(g)} - 1 \in I \) then we have a map \( t_g : \tilde{B}^I \to B^{1,r}_\dir \) given by \( x \mapsto (g|_F^1 \otimes \varphi^{-n(g)})(x). \)

**Lemma 5.1.** — If \( g \in W \) and \( p^{n(g)} - 1 \in I \), with \( g|_F = \tau \) and \( n(g) = \tilde{n}(\tau) = kh \), then \( \ker(\theta \circ t_g : \tilde{B}^I \to \mathcal{C}_p) = \mathcal{Q}_p^I(y_r). \)

*Proof.* — This follows from the definitions and (1) of lemma 3.2

Let \( \nabla_\tau \) be the derivative in the direction of \( \tau \). If \( f(Y) \in \mathcal{R}(Y) \), then \( \nabla_\tau f(y_r) = t_\tau \cdot v_\tau \cdot df/dY(y_r) \) where \( v_\tau = \partial(T \otimes_{\mathcal{L}T} U)/\partial U \) is a unit (see §2.1 of [KR09]). Let \( \partial_\tau = t_\tau^{-1}v_\tau^{-1}\nabla_\tau \) so that \( \partial_\tau f(y_r) = df/dY(y_r) \) (this notation is slightly incompatible with that of [3]). Note that \( \partial_\tau \circ \partial_\nu = \partial_\nu \circ \partial_\tau \) if \( \tau, \nu \in E \).

**Lemma 5.2.** — We have \( \partial_\tau((\tilde{B}^{1,r}_{\rig, K})^{\pa}) \subset (\tilde{B}^{1,r}_{\rig, K})^{\pa}. \)

*Proof.* — Take \( x \in (\tilde{B}^{1,r}_{\rig, K})^{\pa} \) and take \( n = hm + \tilde{n}(\tau) \) with \( m \) such that \( r_n \geq r \). Let \( g \in W \) be such that \( g|_F = \tau \) and \( n(g) = n \). We have \( \theta \circ t_g(x) \in \tilde{K}^{\la}_\infty \). Corollary 4.3 of [BC14] implies that \( \nabla_\Id = 0 \) on \( \tilde{K}^{\la}_\infty \) and therefore that \( \theta \circ t_g(\nabla_\tau(x)) = 0 \) by lemma 2.6.

By lemma 5.1, this implies that \( \nabla_\tau(x) \) is divisible by \( Q^r_m(y_r) \) for all \( m \) such that \( r_n \geq r \). Since \( t_\tau = y_r \cdot \prod_{m \geq 1} Q^r_m(y_r)/\tau(\pi_F), \) this implies the lemma.

\[ \square \]
Lemma 5.3. — If \( x \in \hat{A}_F \), \( I \) is a closed interval, and \( n \geq 1 \), there exists \( \ell \geq 0 \) and \( x_n \in \mathcal{O}_F[\varphi_\ell^{-\ell}(u)] \) such that \( x - x_n \in p^n \hat{A}_F \).

Proof. — Let \( k \geq 1 \) be such that \( u^k \in p^n \hat{A}_F \). By corollary 4.3.4 of \cite{Win83}, the ring \( \bigcup_{m \geq 0} \varphi_m^{\pm m}(F_q[u]) \) is \( u \)-adically dense in \( \mathbb{E}_F \). By successive approximations, we find \( \ell \geq 0 \) and \( x_n \in \mathcal{O}_F[\varphi_\ell^{-\ell}(u)] \) such that \( x - x_n \in p^n \hat{A}_F + u^k \hat{A}_F \) so that \( x - x_n \in p^n \hat{A}_F \).

For \( n \geq 1 \) and \( I \) a closed interval, let \( y_{\tau,n} \in \mathcal{O}_F[\varphi_\ell^{-\ell}(u)] \) be as in lemma 5.3 so that \( y_\tau - y_{\tau,n} \in p^n \hat{A}_F \). Let \( E_0 = E \setminus \{1d\} \). If \( k \in \mathbb{N}^E_0 \), let \( |k| = \sum_{\tau \in E_0} k_\tau \) and let \( k! = \prod_{\tau \in E_0} k_\tau \) and let \( 1 \), be the tuple whose entries are 0 except the \( \tau \)-th one which is 1. Let \( (y - y_n)^k = \prod_{\tau \in E_0} (y_\tau - y_{\tau,n})^{k_\tau} \) and \( \partial^k = \prod_{\tau \in E_0} \partial_{\tau}^k \). We have

\[
\partial_{\tau}(y - y_n)^k = \begin{cases} 0 & \text{if } k_\tau = 0, \\ k_\tau (y - y_n)^{k - 1} & \text{if } k_\tau \geq 1. 
\end{cases}
\]

By lemma 2.25, there exists \( m \geq 1 \) such that \( y_\tau - y_{\tau,n} \in (B_1^I)_{\Gamma_m} \) and \( \|y_\tau - y_{\tau,n}\|_{\Gamma_m} \leq p^{-n} \) for all \( \tau \in E_0 \). Let \( \{x_1\}_{\tau \in \mathbb{N}^E_0} \) be a sequence of elements of \( (B_1^I)_{\Gamma_m} \) such that \( \|p^{\ell}x_1\|_{\Gamma_m} \rightarrow 0 \) as \( |i| \rightarrow +\infty \). The series \( \sum_{i \in \mathbb{N}^E_0} x_1(y - y_n)^i \) then converges in \( (B_1^I)_{\Gamma_m} \).

Theorem 5.4. — If \( x \in (B_1^I)_{\la} \) and \( n_0 \geq 0 \), then there exists \( m, n \geq 1 \) and a sequence \( \{x_1\}_{\tau \in \mathbb{N}^E_0} \) of \( (B_1^I)_{\la,\Gamma_m} \) such that \( \|p^{(n-n_0)|i|}x_1\|_{\Gamma_m} \rightarrow 0 \) and \( x = \sum_{i \in \mathbb{N}^E_0} x_1(y - y_n)^i \).

Proof. — The maps \( \partial_{\tau} : (B_1^I)_{\Gamma_m} \rightarrow (B_1^I)_{\Gamma_m} \) are continuous and hence there exists \( m, n \geq 1 \) such that \( x \in (B_1^I)_{\Gamma_m} \) and \( \|\partial_{\tau} x\|_{\Gamma_m} \leq p^{(n-n_0)|i|} \|x\|_{\Gamma_m} \) for all \( k \in \mathbb{N}^E_0 \). If \( i \in \mathbb{N}^E_0 \), let

\[
x_1 = \frac{1}{i!} \sum_{k \in \mathbb{N}^E_0} (-1)^i k! (y - y_n)^k \partial^k x_1(\tau) (x).
\]

The series above converges in \( (B_1^I)_{\Gamma_m} \) to an element \( x_1 \) such that \( \partial_{\tau}(x_1) = 0 \) for all \( \tau \in E_0 \), so that \( x_1 \in (B_1^I)_{\la,\Gamma_m} \). In addition, \( \|x_1\|_{\Gamma_m} \leq p^{(n-n_0)|i|} \|x\|_{\Gamma_m} \) so that \( \|p^{(n-n_0)|i|}x_1\|_{\Gamma_m} \rightarrow 0 \), the series \( \sum_{i \in \mathbb{N}^E_0} x_1(y - y_n)^i \) converges, and its limit is \( x \).

Corollary 5.5. — If \( F \neq \mathbb{Q}_p \) and \( \tau \in E \), then \( \partial_{\tau} : (B_1^I)_{\la} \rightarrow (B_1^I)_{\la} \) is onto.

Proof. — Suppose that \( \tau \neq Id \), and write \( x = \sum_{i \in \mathbb{N}^E_0} x_1(y - y_n)^i \) as in theorem 5.4 with \( n_0 = 1 \). Since \( \partial_{\tau}(x_1(y - y_n)^i) = x_1 i_\tau(y - y_n)^{i-1} + \frac{x_1}{i_\tau + 1} (y - y_n)^{i+1} \). The series converges because \( \|x_1\|_{\Gamma_m} \leq p^{(n-1)|i|} \|x\|_{\Gamma_m} \). If \( \tau = Id \), one may use the fact that the embeddings play a symmetric role.
Remark 5.6. — Corollary 5.5 is false if $F = \mathbb{Q}_p$. Note also that if $x = f(y_\tau)$ with $f(Y) = \sum_k x_k Y^k \in \mathcal{R}^I(Y)$, then the series above for $\partial_\tau^{-1}(x)$ does not converge to $\sum_k x_k y_\tau^{k+1}/(k+1)$ since that series is not defined unless $x_{-1} = 0$, and even then does not converge in $\mathcal{R}^I(y_\tau)$ in general.

6. A multivariable monodromy theorem

In this §, we explain how to descend certain $(\tilde{\mathcal{B}}_{\text{rig},K}^I)^{\text{pa}}$-modules to $(\tilde{\mathcal{B}}_{\text{rig},K}^I)^{F,\text{pa}}$. Let $M$ be a free $(\tilde{\mathcal{B}}_{\text{rig},K}^I)^{\text{pa}}$-module, endowed with a bijective Frobenius map $\varphi_q : M \to M$ and with a compatible pro-analytic action of $\Gamma_K$, such that $\nabla_\tau(M) \subset t_\tau \cdot M$ for all $\tau \in E_0$. Write $\partial_\tau = v_\tau^{-1}t_\tau^{-1}\nabla_\tau$ so that $\partial_\tau(M) \subset M$ if $\tau \in E_0$. Let

$$\text{Sol}(M) = \{ x \in M, \text{ such that } \partial_\tau(x) = 0 \text{ for all } \tau \in E_0 \}$$

so that $\text{Sol}(M)$ is a $(\tilde{\mathcal{B}}_{\text{rig},K}^I)^{F,\text{pa}}$-module stable under $\Gamma_K$, and such that $\varphi_q : \text{Sol}(M) \to \text{Sol}(M)$ is a bijection. Our monodromy theorem is the following result.

Theorem 6.1. — If $M$ is a free $(\tilde{\mathcal{B}}_{\text{rig},K}^I)^{\text{pa}}$-module with a bijective Frobenius map $\varphi_q$ and a compatible pro-analytic action of $\Gamma_K$, such that $\partial_\tau(M) \subset M$ for all $\tau \in E_0$, then $\text{Sol}(M)$ is a free $(\tilde{\mathcal{B}}_{\text{rig},K}^I)^{F,\text{pa}}$-module, and $M = (\tilde{\mathcal{B}}_{\text{rig},K}^I)^{\text{pa}} \otimes (\tilde{\mathcal{B}}_{\text{rig},K}^I)^{F,\text{pa}} \text{Sol}(M)$.

Remark 6.2. — The usual monodromy conjecture asks for solutions after possibly performing a finite extension $L/K$ and adjoining a logarithm. In this case:

1. there is no need to perform a finite extension $L/K$ since by an analogue of proposition 1.3.2 of [Ber08b], a $(\tilde{\mathcal{B}}_{\text{rig},L}^I)^{F,\text{pa}}$-module with an action of $\text{Gal}(L_\infty/K)$ descends to a $(\tilde{\mathcal{B}}_{\text{rig},K}^I)^{F,\text{pa}}$-module with an action of $\text{Gal}(K_\infty/K)$. In the classical case, the coefficients are too small to be able to perform this descent.

2. there is no need to adjoin a log since the maps $\partial_\tau : (\tilde{\mathcal{B}}_K^I)^{\text{la}} \to (\tilde{\mathcal{B}}_K^I)^{\text{la}}$ are onto.

Proof of theorem 6.1. — Let $r \geq 0$ be such that $M$ and all its structures are defined over $(\tilde{\mathcal{B}}_{\text{rig},K}^I)^{\text{pa}}$, and let $I \subset [r; +\infty[$ be a closed interval, such that $I \cap qI \neq \emptyset$. Let $m_1, \ldots, m_d$ be a basis of $M$, and let $M^I = \oplus_{i=1}^d (\tilde{\mathcal{B}}_K^I)^{\text{la}} \cdot m_i$. Let $D_\tau = \text{Mat}(\partial_\tau)$ for $\tau \in E_0$. We first prove that $\text{Sol}(M^I)$ is a free $\mathcal{B}_{K_\infty}^I$-module of rank $d$, such that $M = (\tilde{\mathcal{B}}_K^I)^{\text{la}} \otimes \mathcal{B}_{K_\infty}^I \text{Sol}(M^I)$. This amounts to finding a matrix $H \in \text{GL}_d((\tilde{\mathcal{B}}_K^I)^{\text{la}})$ such that $\partial_\tau(H) + D_\tau H = 0$ for all $\tau \in E_0$. If $k \in \mathbb{N}^{E_0}$, let $H_k = \text{Mat}(\partial_\tau^k)$. If $n$ is large enough, then

$$H = \sum_{k \in \mathbb{N}^{E_0}} (-1)^{|k|} H_k \frac{(y - y_n)^k}{k!}$$
converges in $M_d((\tilde{B}_K^k)^{la})$ to a solution of the equations $\partial_r(H) + D_rH = 0$ for $\tau \in E_0$. If in addition $n > 0$, then $\|H_k \cdot (y - y_n)k/k!\| < 1$ if $|k| \geq 1$ so that $H \in \text{GL}_d((\tilde{B}_K^k)^{la})$.

This proves that $\text{Sol}(M^f)$ is a free $\tilde{B}_K^{f,\infty}$-module of rank $d$ such that $M = (\tilde{B}_K^k)^{la} \otimes \tilde{B}_K^{f,\infty}$ is the matrix of $\varphi$ over $B$ for all $\varphi$ in our case (see the remark preceding theorem 3.2.16 in [Wen03] and the state of the art concerning projective limits of such spaces seems to be insufficient in our case (see the remark preceding theorem 3.2.16 in [Wen03]). We use instead the Frobenius map to show that we can remain at a “finite level”, that is work with modules over $\tilde{B}_K^{k,n}$ for a fixed $n$.

Let $m_1, \ldots, m_d$ be a basis of $\text{Sol}(M^f)$. The Frobenius map $\varphi_q$ gives rise to bijections $\varphi_q^k : \text{Sol}(M^f) \rightarrow \text{Sol}(M^{p,f})$ for all $k \geq 0$. Let $J = I \cap qI$ and let $P \in \text{GL}_d((\tilde{B}_K^k)^{la})$ be the matrix of $\varphi_q(m_1), \ldots, \varphi_q(m_d)$ in the basis $m_1, \ldots, m_d$. We have $P \in \text{GL}_d((\tilde{B}_K^k)^{F,la})$ because $\partial_r(m_i) = 0$ and $\partial_r(\varphi_q(m_i)) = 0$ for all $\tau \in E_0$ and $1 \leq i \leq d$. By theorem 4.3, there exists therefore some $n \geq 0$ such that $P \in \text{GL}_d(\tilde{B}_K^k)$. For $k \geq 0$, let $I_k = q^kI$ and $J_k = I_k \cap I_{k+1}$ and $E_k = \oplus_{i=1}^d \tilde{B}_K^{k,n} \cdot \varphi_q^k(m_i)$. The fact that $P \in \text{GL}_d(\tilde{B}_K^{k,n})$ implies that $\varphi_q^k(P) \in \text{GL}_d(\tilde{B}_K^{k,n})$ and hence

$$\tilde{B}_K^{J_k} \otimes_{\tilde{B}_K^{J_k,n}} E_k = \tilde{B}_K^{J_k} \otimes_{\tilde{B}_K^{J_k+1,n}} E_{k+1}$$

for all $k \geq 0$. The collection $\{E_k\}_{k \geq 0}$ therefore forms a vector bundle over $\tilde{B}_K^{[r;+\infty]}$ for $r = \min(I)$. By theorem 2.8.4 of [Ked05] (see also §3 of [ST03]), there exists elements $n_1, \ldots, n_d$ of $\cap_{k \geq 0} E_k \subset M$ such that $E_k = \oplus_{i=1}^d \tilde{B}_K^{k,n_i}$ for all $k \geq 0$. These elements give a basis of $\text{Sol}(M)$ over $(\tilde{B}_K^k)^{F,pa}$, which is also a basis of $M$ over $(\tilde{B}_K^k)^{pa}$, and this proves the theorem.

\[ \square \]

7. Lubin-Tate $(\varphi, \Gamma)$-modules

We now review the construction of Lubin-Tate $(\varphi, \Gamma)$-modules. If $K$ is a finite extension of $F$, let $B_K$ be the $p$-adic completion of the field $B_K^\dagger$ defined in §3 and let $A_K$ denote the ring of integers of $B_K$ for $\text{val}_p(\cdot)$. A $(\varphi_q, \Gamma_K)$-module over $B_K$ is a finite dimensional $B_K$-vector space $D$, along with a semilinear Frobenius map $\varphi_q$ and a compatible action of $\Gamma_K$. We say that $D$ is étale if $D = B_K \otimes_{A_K} D_0$ where $D_0$ is a $(\varphi_q, \Gamma_K)$-module over $A_K$.

Let $B$ be the $p$-adic completion of $\cup_{K/F} B_K$. By specializing the constructions of [Fon90], Kisin and Ren prove the following theorem in their paper (theorem 1.6 of [KR09]).
Theorem 7.1. — The functors $V \mapsto (B \otimes_F V)^{\overline{pK}}$ and $D \mapsto (B \otimes B_K D)^{\varphi_{q=1}}$ give rise to mutually inverse equivalences of categories between the category of $F$-linear representations of $G_K$ and the category of étale $(\varphi_q, \Gamma_K)$-modules over $B_K$.

We say that a $(\varphi_q, \Gamma_K)$-module $D$ is overconvergent if there exists a basis of $D$ in which the matrices of $\varphi_q$ and of all $g \in \Gamma_K$ have entries in $B^\dagger_K$. This basis then generates a $B^\dagger_K$-vector space $D^\dagger$ which is canonically attached to $D$. The main result of [CC98] states that if $F = Q_p$, then every étale $(\varphi_q, \Gamma_K)$-module over $B_K$ is overconvergent (the proof is given for $\pi_F = p$, but it is easy to see that it works for any uniformizer). If $F \neq Q_p$, then some simple examples (cf. [FX13]) show that this is no longer the case.

We say that an $F$-linear representation of $G_K$ is $F$-analytic if $C_p \otimes_F V$ is the trivial $C_p$-semilinear representation of $G_K$ for all embeddings $\tau \neq \text{Id} \in \text{Gal}(F/Q_p)$. This definition is the natural generalization of Kisin and Ren’s $L$-crystalline representations (§3.3.7 of [KR09]). See also remark 16.28 of [FF12]. Kisin and Ren then go on to show that if $K \subset F_\infty$, and if $V$ is a crystalline $F$-analytic representation of $G_K$, then the $(\varphi_q, \Gamma_K)$-module attached to $V$ is overconvergent (see §3.3 of [KR09]).

If $D$ is a $(\varphi_q, \Gamma_K)$-module over $B^\dagger_{ec,K}$, and if $g \in \Gamma_K$ is close enough to 1, then by standard arguments (see §2.1 of [KR09]), the series $\log(g) = \log(1 + (g - 1))$ gives rise to a differential operator $\nabla_g : D \to D$. The map $\text{Lie}_F \Gamma_F \to \text{End}(D)$ arising from $v \mapsto \nabla_{\exp(v)}$ is $Q_p$-linear, and we say that $D$ is $F$-analytic if this map is $F$-linear (see §2.1 of [KR09] and §1.3 of [FX13]). This is equivalent to the requirement that the elements of $D$ be pro-$F$-analytic vectors for the action of $\Gamma_K$. The following is theorem 4.2 of [Ber13].

Theorem 7.2. — If $F/Q_p$ is unramified, if $K \subset F_\infty$ and if $V$ is an overconvergent $F$-representation of $G_K$, then $B^\dagger_{ec,K} \otimes B^\dagger_K D^\dagger(V)$ is $F$-analytic if and only if $V$ is $F$-analytic.

In §9 we prove the theorem below. Note that it was previously known for $F = Q_p$ by the main result of [CC98], for crystalline representations by §3 of [KR09] and for reducible (or even trianguline) 2-dimensional representations by theorem 0.3 of [FX13].

Theorem 7.3. — If $V$ is $F$-analytic, then it is overconvergent.

We now assume that $K$ is a finite extension of $Q_p$ and that $L_\infty/K$ is the extension of $K$ attached to $\eta_{\chi_{\text{cyc}}}$ where $\eta$ is an unramified character of $G_F$. When $\eta = 1$, $L_\infty$ is the cyclotomic extension of $K$ and the Cherbonnier-Colmez theorem (see [CC98]) says that there is an equivalence of categories between étale $(\varphi, \Gamma)$-modules over $B^\dagger_L$ and $F$-representations of $G_K$. If $\eta$ is not the trivial character, then there is still such an equivalence of categories, where $B^\dagger_p$ is a field of power series with coefficients in $F$ and
in one variable $X_\eta$ and $B^\dagger_L$ is the corresponding extension. This can be seen in at least two ways.

1. One can redo the whole proof of the Cherbonnier-Colmez theorem for $L_\infty/K$, and this works because $\text{Gal}(L_\infty/K)$ is an open subgroup of $\mathbb{Z}_p^\times$;
2. One can use the fact that $L_\infty \cdot \mathbb{Q}_p^{\text{unr}} = K(\mu_{p^{10^7}}) \cdot \mathbb{Q}_p^{\text{unr}}$, apply the classical Cherbonnier-Colmez theorem, and then descend from $L_\infty \cdot \mathbb{Q}_p^{\text{unr}}$ to $L_\infty$, which poses no problem since that extension is unramified.

The variable $X_\eta$ is then an element of $\mathcal{O}_{\mathcal{F}_{\text{unr}}}[[X]]$, of the form $z_1X + \cdots$ with $z_1 \in \mathcal{O}_{\mathcal{F}_{\text{unr}}}^\times$.

Let $V$ be a $\mathbb{Q}_p$-linear representation of $G_K$. By the above generalization of the Cherbonnier-Colmez theorem, $V$ is overconvergent, so that we can attach to $V$ the $B^\dagger_L$-vector space $D^r_V = \bigcup_{r > 0} D^r_{L(r)}$. Let $D^v_{L(v)}$ and $D^r_{\varphi(L)}$ denote the various completions of $D^r_{L(v)}$. Let

$$ \check{D}^v_{L(v)} = (B^{[r:s]} \otimes \mathbb{Q}_p)_{H_L} \text{ and } \check{D}^r_{\varphi(L)} = (B^{[r:s]} \otimes \mathbb{Q}_p)_{H_L}. $$

The Cherbonnier-Colmez theorem implies that $\check{D}^v_{L(v)} = B^{[r,s]}_{\varphi(L)} D_{\varphi(L)}$ and that $\check{D}^r_{\varphi(L)} = B^{[r:s]}_{\varphi(L)} D_{\varphi(L)}$.

**Theorem 7.4.** — We have

1. $\check{D}^v_{L(v)} = B^{[r,s]}_{L(v)} \otimes B^{[r,s]}_{\varphi(L)} D_{L(v)}$;
2. $\check{D}^r_{\varphi(L)} = B^{[r,s]}_{\varphi(L)} \otimes B^{[r,s]}_{\varphi(L)} D_{\varphi(L)}$.

**Proof.** — We have $\check{D}^v_{L(v)} = B^{[r,s]}_{L(v)} \otimes B^{[r,s]}_{\varphi(L)} D_{L(v)}$, and (1) now follows from theorem 4.34 and from the fact that the elements of $D_{\varphi(L)}$ are locally analytic (see §2.1 of [KR09]). Likewise, (2) follows from theorem 4.36 and proposition 2.3 and from the fact that the elements of $D_{\varphi(L)}$ are pro-analytic.

**8. Multivariable $(\varphi, \Gamma)$-modules**

We now explain how to construct some $(\varphi, \Gamma)$-modules over the ring $(B^{\dagger}_{\varphi(K)})$. Let $L_\infty$ be as in §7 and let $K_\infty/K$ be a $p$-adic Lie extension, such that $L_\infty \subset K_\infty$. Let $\Gamma_K = \text{Gal}(K_\infty/K)$. Let $H_K = \text{Gal}(\overline{\mathbb{Q}}_p/K_\infty)$, let $V$ be a $p$-adic representation of $G_K$ of dimension $d$, and let

$$ \check{D}^v_K = (B^{[r,s]} \otimes \mathbb{Q}_p)_{H_K} \text{ and } \check{D}^r_{\varphi(L)} = (B^{[r,s]} \otimes \mathbb{Q}_p)_{H_K}. $$

These two spaces are topological representations of $\Gamma_K$.

**Theorem 8.1.** — We have
1. \( \tilde{D}_K^{[r,s]}(V)^{\text{la}} = (\tilde{B}_K^{[r,s])^{\text{la}}} \otimes_{\tilde{B}_L^{[r,s]}} D_L^{[r,s]}(V); \)
2. \( \tilde{D}_{\text{rig},K}^{[r,s]}(V)^{\text{pa}} = (\tilde{B}_{\text{rig},K}^{[r,s])^{\text{pa}}} \otimes_{\tilde{B}_{\text{rig},L}^{[r,s]}} D_{\text{rig},L}^{[r,s]}(V). \)

**Proof.** — We have \( \tilde{B}^{[r,s]} \otimes_{\mathbb{Q}_p} V = \tilde{B}^{[r,s]} \otimes_{\tilde{B}_L^{[r,s]}} D_L^{[r,s]}(V), \) so that \( \tilde{D}_K^{[r,s]}(V) = \tilde{B}_K^{[r,s]} \otimes_{\tilde{B}_L^{[r,s]}} D_L^{[r,s]}(V), \) and item (1) follows from proposition 2.2. Item (2) is proved similarly. \( \Box \)

Let \( \tilde{D}_{\text{rig},K}(V)^{\text{pa}} = \cup_{r>0} \tilde{D}_{\text{rig},K}^{[r,s]}(V)^{\text{pa}}. \) Theorem 5.1 implies that \( \tilde{D}_{\text{rig},K}^{[r,s]}(V)^{\text{pa}} \) is a free \( (\tilde{B}_{\text{rig},K}^{[r,s]})^{\text{pa}} \)-module of rank \( \dim(V) \) stable under \( \varphi_q \) and \( \Gamma_K. \) We propose this module as a first candidate for a \((\varphi_q, \Gamma_K)\)-module in the case \( \Gamma_K = \text{Gal}(\mathbb{K}_\infty/\mathbb{K}). \) One can then attempt to construct some multivariable \((\varphi, \Gamma)\)-modules by descending from \( (\tilde{B}_{\text{rig},K}^{[r,s]})^{\text{pa}} \) to certain nicer rings of power series. For example, if \( F \) is unramified over \( \mathbb{Q}_p \) and \( \pi_F = p \) and \( K = F \) and \( K_\infty \) is generated by the torsion points of \( \text{LT}, \) then by theorem A of \([\text{Ked13}]) \) one can descend \( \tilde{D}_{\text{rig},K}(V)^{\text{pa}} \) to a reflexive coadmissible module on the ring \( \mathcal{R}^{[0;+\infty]}(Y_0, \ldots, Y_{h-1}) \) of functions on the \( h \)-dimensional open unit disk. Note that the cyclotomic element \( X = \varepsilon-1 \) belongs to \( (\tilde{B}_{\text{rig},K}^{[r,s]})^{\text{pa}}, \) but it is not in the image of \( \cup_{n>0} \varphi^{-n}_q \mathcal{R}(Y_0, \ldots, Y_{h-1}) \) where \( \mathcal{R}(Y_0, \ldots, Y_{h-1}) \) denotes the “Robba ring in \( h \) variables” (defined in \([\text{Ber13}]) \). Therefore, descending to smaller subrings of \( (\tilde{B}_{\text{rig},K}^{[r,s]})^{\text{pa}} \) may be quite complicated. In general, it will be useful to answer the following.

**Question 8.2.** — What is the structure of the ring \( (\tilde{B}_{\text{rig},K}^{[r,s]})^{\text{pa}} \)?

Finally, we mention that definition 7.8 and conjecture 7.9 of \([\text{Ked13}]) \) discuss some necessary and sufficient conditions for certain elements of \( \tilde{B}_{\text{rig},K}^{[r,s]} \) to be locally analytic.

### 9. Overconvergence of \( F \)-analytic representations

We now give the proof of conjecture 7.3 using the construction of multivariable \((\varphi, \Gamma)\)-modules and the monodromy theorem.

**Theorem 9.1.** — The Lubin-Tate \((\varphi_q, \Gamma_K)\)-modules of \( F \)-analytic representations are overconvergent.

Let \( V \) be an \( F \)-linear representation of \( G_K \) and let \( \tilde{D}_{\text{rig},K}^{[r,s]}(V) = (\tilde{B}_{\text{rig},K}^{[r,s]} \otimes_F V)^{H_K}. \) Since \( K_\infty \) contains \( L_\infty, \) the \( \tilde{B}_{\text{rig},K}^{[r,s]} \)-module \( \tilde{D}_{\text{rig},K}^{[r,s]}(V) \) is free of rank \( d = \dim(V) \) and there is an isomorphism compatible with \( G_K \) and \( \varphi_q \)

\[
\tilde{B}_{\text{rig}}^{[r,s]} \otimes_{\tilde{B}_{\text{rig},K}^{[r,s]}} \tilde{D}_{\text{rig},K}^{[r,s]}(V) = \tilde{B}_{\text{rig}}^{[r,s]} \otimes_F V.
\]
Lemma 9.2. — If $V$ is an $F$-representation of $G_K$ that is $C_p$-admissible at $\tau \in E$, then

$$\nabla_\tau (\tilde{D}_{\text{rig},K}^\dagger (V))^{pa} \subset t_\tau \cdot \tilde{D}_{\text{rig},K}^\dagger (V)^{pa}.$$  

Proof. — Take $n = hm + \tilde{n}(\tau)$ with $m$ such that $r_n \geq r$ and let $g \in W$ be such that $g|_F = \tau$ and $n(g) = n$. Let $e_1, \ldots, e_d$ be a basis of $(C_p \otimes_F V)^{G_K}$ over $K$, so that it is also a basis of $(C_p \otimes_F V)^{H_K}$ over $\tilde{K}_\infty$. If $g \in (C_p \otimes_F V)^{H_K}$ is $Q_p$-analytic, then we can write $y = \sum_{i=1}^d y_i e_i$ and by lemma 2.2 we have $y_i \in \tilde{K}_\infty$. Corollary 4.3 of [BCT14] implies that $\nabla_{id} = 0$ on $(C_p \otimes_F V)^{H_K}$ and therefore that if $x \in \tilde{D}_{\text{rig},K}^\dagger (V)^{pa}$, then $\theta \circ t_g (\nabla_\tau (x)) = 0$ by lemma 2.6. Lemma 5.1 implies that if $x \in \tilde{D}_{\text{rig},K}^\dagger (V)^{pa}$, then $\nabla_\tau (x)$ is divisible by $Q_m(y_r)$ for all $m$ such that $r_n \geq r$. Since $t_\tau = y_\tau \cdot \prod_{m \geq 1} Q_m(y_r)/\tau(\pi_F)$, this implies the lemma. \qed

Proof of theorem 9.1. — Let $V$ be an $F$-representation of $G_K$ that is $F$-analytic and let $M = \tilde{D}_{\text{rig},K}^\dagger (V)^{pa}$. By theorem 8.1 $M$ is a free $(\tilde{B}_{\text{rig},K})^{pa}$-module stable under $\Gamma_K$ and $\varphi_q$. Lemma 9.2 implies that $M$ is stable under the differential operators $\{\partial_\tau\}_{\tau \in E \setminus \{id\}}$. By theorem 6.4, $\text{Sol}(M)$ is a free $(\tilde{B}_{\text{rig},K})^{F,pa}$-module of rank $d$ such that there is an isomorphism compatible with $G_K$ and $\varphi_q$

$$\tilde{B}_{\text{rig}} \otimes^{(\tilde{B}_{\text{rig},K})^{F,pa}} \text{Sol}(M) = \tilde{B}_{\text{rig}} \otimes_F V.$$  

By theorem 4.6 we have $(\tilde{B}_{\text{rig},K})^{F,pa} = \tilde{B}_{\text{rig},K,\infty}$. This implies that there exists $n \geq 0$, and a basis $s_1, \ldots, s_d$ of $\text{Sol}(M)$ such that $\text{Mat}(\varphi_q^n) \in \text{GL}_d(\tilde{B}_{\text{rig},K,n})$ as well as $\text{Mat}(g) \in \text{GL}_d(\tilde{B}_{\text{rig},K,n})$ for all $g \in \Gamma_F$. If we set $D_{\text{rig}} = \bigotimes_{i=1}^d B_{\text{rig},K}^\dagger \cdot \varphi_q^n(s_i)$, then $D_{\text{rig}}$ is a $(\varphi_q, \Gamma_K)$-module over $B_{\text{rig},K}$ such that Sol$(M) = (\tilde{B}_{\text{rig},K})^{pa} \otimes^{B_{\text{rig},K}^\dagger} D_{\text{rig}}$. The module $D_{\text{rig}}$ is uniquely determined by this condition: if there are two and if $X$ denotes the change of basis matrix and $P_1, P_2$ the matrices of $\varphi_q$, then $X \in \text{GL}_d(B_{\text{rig},K,n})$ for some $n \gg 0$, and the equation $X = P_2^{-1} \varphi(X) P_1$ implies that $X \in \text{GL}_d(B_{\text{rig},K})$.

The isomorphism $\tilde{B}_{\text{rig}} \otimes^{B_{\text{rig},K}} D_{\text{rig}} = \tilde{B}_{\text{rig}} \otimes_F V$ implies that $D_{\text{rig}}$ is pure of slope 0 (see [Ked05]). By theorem 6.3.3 of [Ked05], there is an étale $(\varphi_q, \Gamma_K)$-module $D^\dagger$ over $B_{\text{rig},K}$ such that $D_{\text{rig}} = \tilde{B}_{\text{rig},K} \otimes^{B_{\text{rig},K}} D^\dagger$.

Since $D^\dagger$ is étale, there exists an $F$-representation $W$ of $G_K$ such that $\tilde{B}_{\text{rig}} \otimes^{B_{\text{rig},K}} D^\dagger = \tilde{B}_{\text{rig}} \otimes_F W$. Taking $\varphi_q$-invariants in $\tilde{B}_{\text{rig}} \otimes_F W = \tilde{B}_{\text{rig}} \otimes_F V$ shows that $W = V$. This proves theorem 9.1 for $V$, with $D^\dagger(V) = D^\dagger$. \qed

Remark 9.3. — The same proof shows that theorem 9.1 extends to an equivalence of categories between the category of $F$-analytic $B$-pairs (see [Ber08a]) and the category of all Lubin-Tate $(\varphi_q, \Gamma_K)$-modules over $B_{\text{rig},K}$. These $F$-analytic $B$-pairs appear as Galois equivariant vector bundles on Fargues and Fontaine’s curve (§16 of [FF12]).
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