COMBINATORICS OF INTEGER PARTITIONS WITH PRESCRIBED PERIMETER

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Abstract. We prove that the number of even parts and the number of times that parts are repeated have the same distribution over integer partitions with a fixed perimeter. This refines Straub’s analog of Euler’s Odd-Distinct partition theorem. We generalize the two concerned statistics to these of the part-difference less than $d$ and the parts not congruent to 1 modulo $d + 1$ and prove a distribution inequality, that has a similar flavor as Alder’s ex-conjecture, over partitions with a prescribed perimeter. Both of our results are proved analytically and combinatorially.

1. Introduction

Integer partitions [1] play important roles in combinatorics, number theory and other related mathematical branches. For a positive integer $n$, a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of weakly decreasing positive integers such that $\sum_i \lambda_i = n$ is called a partition of $n$. The $\lambda_i$’s are called the parts of $\lambda$ and $k$ is the number of parts that we denote $\ell(\lambda)$. The perimeter of $\lambda$ is defined to be

$$\Gamma(\lambda) = \lambda_1 + \ell(\lambda) - 1$$

and the minimal part-difference of $\lambda$ is

$$\min\{\lambda_i - \lambda_{i+1} : 1 \leq i < \ell(\lambda)\}.$$ 

One of the most famous results in partition theory is Euler’s Odd-Distinct partition theorem, which asserts that the number of partitions of $n$ with odd parts is equal to the number with distinct parts. Two beautiful bijective proofs (see [4, pp. 63-65]) of Euler’s partition theorem with different refinements were constructed respectively by Sylvester and Glaisher. Recently, Straub [5, Theorem 1.4] considered the set $\mathcal{H}_n$ of partitions with perimeter $n$, rather than the set of all partitions of $n$, and proved the following analog of Euler’s partition theorem.

Theorem 1 (Straub). There are as many partitions in $\mathcal{H}_n$ with odd parts as with distinct parts.

For a fixed positive integer $d$, Alder’s ex-conjecture (see [1, Sec. 4.3] and [2]) states that there are not more partitions of $n$ into parts congruent to $\pm 1$ modulo $d + 3$ than into parts with minimal part-difference at least $d$. Indeed, Alder’s ex-conjecture is Euler’s partition theorem (resp. Rogers–Ramanujan identity) for $d = 1$ (resp. $d = 2$). Inspired by Alder’s ex-conjecture, Fu and Tang [2, Theorem 2.15] proved the following $d$-extension of Straub’s result.

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Theorem 2 (Fu–Tang). There are as many partitions in \( H_n \) with parts congruent to 1 modulo \( d+1 \) as with minimal part-difference at least \( d \).

The purpose of this paper is to study further the combinatorics of integer partitions with a prescribed perimeter after the aforementioned works by Straub and Fu–Tang.

We are interested in two natural statistics on partitions. Define for a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \) the statistics
\[
\text{rep}(\lambda) = |\{1 \leq i \leq \ell(\lambda) - 1 : \lambda_i = \lambda_{i+1}\}| \quad \text{and} \quad \text{even}(\lambda) = |\{1 \leq i \leq \ell(\lambda) : \lambda_i \text{ is even}\}|
\]
called the number of repeated parts and the number of even parts, respectively. Our first result refines Theorem 1 by the above two statistics.

Theorem 3. For any positive integer \( n \), we have
\[
\sum_{\lambda \in H_n} t^{\text{rep}(\lambda)} = \sum_{\lambda \in H_n} t^{\text{even}(\lambda)}.
\]
See Fig. 1 for an example of (1.1) for \( n = 5 \), which shows
\[
\sum_{\lambda \in H_5} t^{\text{rep}(\lambda)} = 5 + 5t + 4t^2 + t^3 + t^4 = \sum_{\lambda \in H_5} t^{\text{even}(\lambda)}.
\]
Setting \( t = 0 \) in (1.1) recovers Theorem 1. We will investigate the combinatorics of Theorem 3 by providing three different proofs:

1. a recurrence relation proof with connection to extraordinary subsets studied by Grimaldi [3];
2. a generating function proof with some related consequences;
3. a bijective proof that admits an unexpected extension to general \( d \) for our next result.

Our next result can be considered as an extension of Theorem 2, as will be seen (see Theorem 19) when exploiting the \( d \)-extension of our bijective proof of Theorem 3.

Theorem 4. Fix integers \( d \geq 1 \) and \( n \geq 1 \). The total number of indices \( i \) with \( 1 \leq i < \ell(\lambda) \) satisfying \( \lambda_i - \lambda_{i+1} < d \) in all partitions \( \lambda \) in \( H_n \) is not less than the total number of parts not congruent to 1 modulo \( d+1 \) in all partitions in \( H_n \).
Again, three different proofs for Theorem 4 will be provided: a generating function proof, an injective proof and a bijective proof.

The proofs of Theorems 3 and 4 are given in Sections 2 and 3, respectively. This paper is concluded with further remarks in Section 4.

2. Three proofs of Theorem 3

2.1. A recurrence relation proof. Let \( A(n, k) \) (resp. \( B(n, k) \)) be the number of partitions \( \lambda \) in \( \mathcal{H}_n \) with \( \text{rep}(\lambda) = k \) (resp. \( \text{even}(\lambda) = k \)). In the following two lemmas, we prove that \( A(n, k) \) and \( B(n, k) \) share the same (binomial-like) recurrence relation, which proves Theorem 3.

Lemma 5. The number \( A(n, k) \) satisfies the recurrence relation

\[
(2.1) \quad A(n, k) = A(n - 1, k) + A(n - 1, k - 1) + A(n - 2, k) - A(n - 2, k - 1)
\]

for \( n \geq 2 \) with the initial value \( A(1, 0) = 1 \).

Proof. Suppose that \( n \geq 2 \) and \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \) is a partition in \( \mathcal{H}_n \) with \( \text{rep}(\lambda) = k \). To obtain the recursion formula for \( A(n, k) \), we consider three cases:

(i) \( \lambda_1 = \lambda_2 \). In this case, by deleting the largest part of \( \lambda \), we obtain \( \lambda' = (\lambda_2, \lambda_3, \ldots) \). Then \( \lambda' \in \mathcal{H}_{n-1} \) and \( \text{rep}(\lambda') = k - 1 \). Furthermore, \( \lambda' \) can be any of the partitions satisfying \( \lambda' \in \mathcal{H}_{n-1} \) and \( \text{rep}(\lambda') = k - 1 \). Therefore, the number of partitions \( \lambda \in \mathcal{H}_n \) satisfying \( \lambda_1 = \lambda_2 \) and \( \text{rep}(\lambda) = k \) is equal to \( A(n - 1, k - 1) \).

(ii) \( \lambda_1 = \lambda_2 + 1 \). In this case, again by deleting the largest part of \( \lambda \), we obtain \( \lambda' = (\lambda_2, \lambda_3, \ldots) \). It is clear that \( \lambda' \in \mathcal{H}_{n-2} \) and \( \text{rep}(\lambda') = k \). Furthermore, \( \lambda' \) can be any of the partitions satisfying \( \lambda' \in \mathcal{H}_{n-2} \) and \( \text{rep}(\lambda') = k \). Therefore, the number of partitions \( \lambda \in \mathcal{H}_n \) satisfying \( \lambda_1 = \lambda_2 + 1 \) and \( \text{rep}(\lambda) = k \) is equal to \( A(n - 2, k) \).

(iii) \( \lambda_1 \geq \lambda_2 + 2 \). In this case, by removing one box in the first row of the Young diagram of \( \lambda \), we obtain \( \lambda' = (\lambda_1 - 1, \lambda_2, \lambda_3, \ldots) \). It is clear that \( \lambda' \in \mathcal{H}_{n-1} \), \( \text{rep}(\lambda') = k \) and \( \lambda'_1 \geq \lambda'_2 + 1 \). By (i), the number of partitions \( \lambda' \) satisfying \( \lambda' \in \mathcal{H}_{n-1} \), \( \text{rep}(\lambda') = k \) and \( \lambda'_1 = \lambda'_2 \) is equal to \( A(n - 2, k - 1) \). Thus, the number of partitions \( \lambda' \) satisfying \( \lambda' \in \mathcal{H}_{n-1} \), \( \text{rep}(\lambda') = k \) and \( \lambda'_1 \geq \lambda'_2 + 1 \) is equal to \( A(n - 1, k - 1) - A(n - 2, k - 1) \). Finally, this implies that the number of partitions \( \lambda \in \mathcal{H}_n \) satisfying \( \lambda_1 \geq \lambda_2 + 2 \) and \( \text{rep}(\lambda) = k \) is equal to \( A(n - 1, k) - A(n - 2, k - 1) \).

Combining the above three cases gives (2.1). \( \square \)

Lemma 6. The number \( B(n, k) \) satisfies the recurrence relation

\[
(2.2) \quad B(n, k) = B(n - 1, k) + B(n - 1, k - 1) + B(n - 2, k) - B(n - 2, k - 1)
\]

for \( n \geq 2 \) with the initial value \( B(1, 0) = 1 \).

Proof. Suppose that \( n \geq 2 \) and \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \) is a partition in \( \mathcal{H}_n \) with \( \text{even}(\lambda) = k \). Let \( B_o(n, k) \) (resp. \( B_e(n, k) \)) be the number of partitions \( \lambda \in \mathcal{H}_n \) with \( \lambda_1 \) odd (resp. even) and \( \text{even}(\lambda) = k \). To obtain the recursion formula for \( B(n, k) \) is more involved and we again consider three cases:

(i) \( \lambda_1 = \lambda_2 \). In this case, consider the deletion of the largest part of \( \lambda \), we see that the number of partitions \( \lambda \in \mathcal{H}_n \) satisfying \( \lambda_1 \) odd (resp. even) and \( \text{even}(\lambda) = k \) is equal to \( B_o(n - 1, k) \) (resp. \( B_e(n - 1, k - 1) \)).
(ii) $\lambda_1 = \lambda_2 + 1$. In this case, again by considering the deletion of the largest part of $\lambda$, we see that the number of partitions $\lambda \in \mathcal{H}_n$ satisfying $\lambda_1$ odd (resp. even), $\lambda_1 = \lambda_2 + 1$ and even$(\lambda) = k$ is equal to $B_e(n-2,k)$ (resp. $B_o(n-2,k-1)$).

(iii) $\lambda_1 \geq \lambda_2 + 2$. In this case, by removing two boxes in the first row of the Young diagram of $\lambda$, we obtain $\lambda' = (\lambda_1 - 2, \lambda_2, \lambda_3, \ldots)$. It is clear that $\lambda' \in \mathcal{H}_{n-2}$ and even$(\lambda') = k$. Furthermore, $\lambda'$ can be any of the partitions satisfying $\lambda' \in \mathcal{H}_{n-2}$ and even$(\lambda') = k$. Therefore, the number of partitions $\lambda \in \mathcal{H}_n$ satisfying $\lambda_1$ odd (resp. even), $\lambda_1 \geq \lambda_2 + 2$ and even$(\lambda) = k$ is equal to $B_o(n-2,k)$ (resp. $B_e(n-2,k)$).

Combining the above three cases together, we derive that

\begin{equation}
B(n, k) = B(n-2, k) + B_o(n-1, k) + B_o(n-2, k-1) + B_e(n-1, k-1) + B_e(n-2, k).
\end{equation}

On the other hand, by deleting the first part of $\lambda$ when $\lambda_1 = \lambda_2$ or by removing one box in the first row of the Young diagram of $\lambda$ when $\lambda_1 \geq \lambda_2 + 1$, we see that

\begin{equation}
B_o(n-1, k-1) = B_o(n-2, k-1) + B_e(n-2, k)
\end{equation}

and

\begin{equation}
B_e(n-1, k) = B_e(n-2, k-1) + B_o(n-2, k-1) = B(n-2, k-1).
\end{equation}

Now (2.3), (2.4) and (2.5) together results in (2.2), as desired.

For positive integers $n$ and $k$ with $1 \leq k \leq n$, a subset $S$ of $[n] := \{1, 2, \ldots, n\}$ is called a $k$-extraordinary subset if $|S|$ equals the $k$-th smallest element of $S$. For example, all the 2-extraordinary subsets of $[5]$ are

\{1, 2\}, \ {1, 3, 4}, \ {1, 3, 5}, \ {2, 3, 4}, \ {2, 3, 5}.

Let $C(n, k)$ be the number of $k$-extraordinary subsets of $[n]$. In [3], Grimaldi showed that

$$C(n, k) = C(n-1, k) + \sum_{i=1}^{k} C(n-k-2+i, i)$$

for $n \geq 2$ and the initial value $C(1, 1) = 1$. It follows from the above recursion that

$$C(n, k) = C(n-1, k) + C(n-2, k) + \sum_{i=1}^{k-1} C(n-k-2+i, i)$$

$$= C(n-1, k) + C(n-2, k) + C(n-1, k-1) - C(n-2, k-1).$$

Comparing with the recursions for $A(n, k)$ and $B(n, k)$ derived in the above two lemmas (and their initial values) yields the following result.

**Proposition 7.** For $n \geq 1$ and $0 \leq k \leq n-1$, we have

$$A(n, k) = B(n, k) = C(n, k+1).$$
2.2. A generating function proof. We will compute the generating function for the joint distribution of \((\text{rep}, \text{even})\) on \(\mathcal{H}_n\). For a partition \(\lambda\), let \(\text{dist}(\lambda)\) be the number of distinct parts of \(\lambda\). Then, \(\text{dist}(\lambda) = \ell(\lambda) - \text{rep}(\lambda)\). A partition \(\lambda\) can also be represented as \(\lambda = 1^{m_1}2^{m_2} \cdots\), where \(m_i\) is the number of parts equating \(i\) of \(\lambda\). Consider the weight function

\[\text{wt}(\lambda) = p^{\text{dist}(\lambda)} q^{\text{even}(\lambda)} x_1^{m_1} x_2^{m_2} \cdots.\]

Then,

\[1 + \sum_{\lambda \in \mathcal{P}} \text{wt}(\lambda) = \left(1 + \frac{px_1}{1 - x_1}\right) \left(1 + \frac{pqx_2}{1 - qx_2}\right) \left(1 + \frac{px_3}{1 - x_3}\right) \left(1 + \frac{pqx_4}{1 - qx_4}\right) \cdots,\]

where \(\mathcal{P}\) denotes the set of all integer partitions. Let \(\mathcal{P}_n\) be the set of partitions \(\lambda = (\lambda_1, \lambda_2, \ldots)\) such that \(\lambda_1 = n\). Then,

\[\sum_{\lambda \in \mathcal{P}_{2n}} \text{wt}(\lambda) = \left(1 + \frac{px_1}{1 - x_1}\right) \left(1 + \frac{pqx_2}{1 - qx_2}\right) \cdots \left(1 + \frac{px_{2n-1}}{1 - x_{2n-1}}\right) \left(\frac{pqx_{2n}}{1 - qx_{2n}}\right)\]

and

\[\sum_{\lambda \in \mathcal{P}_{2n+1}} \text{wt}(\lambda) = \left(1 + \frac{px_1}{1 - x_1}\right) \left(1 + \frac{pqx_2}{1 - qx_2}\right) \cdots \left(1 + \frac{pqx_{2n}}{1 - qx_{2n}}\right) \left(\frac{px_{2n+1}}{1 - x_{2n+1}}\right).\]

It follows from (2.7) and (2.8) that

\[
\sum_{\lambda \in \mathcal{P}} p^{\text{dist}(\lambda)} q^{\text{even}(\lambda)} x^{\ell(\lambda)} y^{\lambda_1}
= \frac{pxy}{1 - x} + \left(\frac{pqx}{1 - qx} + \frac{pxy(1 - qx + pqx)}{(1 - x)(1 - qx)}\right) \sum_{n \geq 1} \left(1 + \frac{px}{1 - x}\right)^n \left(1 + \frac{pqx}{1 - qx}\right)^{n-1} y^{2n}
= \frac{pxy}{1 - x} + \left(\frac{pqx}{1 - qx} + \frac{pxy(1 - qx + pqx)}{(1 - x)(1 - qx)}\right) \left(1 + \frac{px}{1 - x}\right) y^2
\cdot \frac{1 - \left(1 + \frac{px}{1 - x}\right) \left(1 + \frac{pqx}{1 - qx}\right)}{1 - \left(1 + \frac{px}{1 - x}\right) \left(1 + \frac{pqx}{1 - qx}\right)} y^2.
\]

Therefore, we have

**Theorem 8.** The generating function for partitions by size of first parts, number of parts and the pair \((\text{dist}, \text{even})\) is

\[\sum_{\lambda \in \mathcal{P}} p^{\text{dist}(\lambda)} q^{\text{even}(\lambda)} x^{\ell(\lambda)} y^{\lambda_1} = \frac{pxy}{1 - x} + \left(\frac{pqx}{1 - qx} + \frac{pxy(1 - qx + pqx)}{(1 - x)(1 - qx)}\right) \left(1 + \frac{px}{1 - x}\right) y^2 \frac{1 - \left(1 + \frac{px}{1 - x}\right) \left(1 + \frac{pqx}{1 - qx}\right)}{1 - \left(1 + \frac{px}{1 - x}\right) \left(1 + \frac{pqx}{1 - qx}\right)} y^2.\]

Consequently \((p \leftarrow p^{-1}, x \leftarrow px, y \leftarrow x)\),

\[
\sum_{\lambda \in \mathcal{P}} p^{\text{rep}(\lambda)} q^{\text{even}(\lambda)} x^{\Gamma(\lambda)} = \frac{x(1 - (p - 1)q(x^2 + x))}{1 - p(1 + q)x - (1 - p^2q)x^2 - (1 - p)(1 + q)x^3 - (p - 1)^2qx^4}.
\]
In particular,
\begin{equation}
\sum_{\lambda \in \mathcal{P}} p^{\text{rep}(\lambda)} x^{\Gamma(\lambda)} = \sum_{\lambda \in \mathcal{P}} p^{\text{even}(\lambda)} x^{\Gamma(\lambda)} = \frac{x}{1 - (1 + p)x - (1 - p)x^2}.
\end{equation}

Remark 9. Dividing both sides of (2.9) by $x$ and then substituting $p \leftarrow p^{-1}, q \leftarrow 1, x \leftarrow -px, y \leftarrow x$ yields
\begin{equation}
\sum_{\lambda \in \mathcal{P}} (-1)^{\ell(\lambda)} p^{\text{rep}(\lambda)} x^{\Gamma(\lambda)} = \frac{x}{(p - 1)(x^2 - x) - 1}.
\end{equation}
Thus, if we denote
\begin{equation}
h_n(p) = \sum_{\Gamma(\lambda) = n} (-1)^{\ell(\lambda)} p^{\text{rep}(\lambda)} = \sum_{\Gamma(\lambda) = n, \ell(\lambda) \text{ even}} p^{\text{rep}(\lambda)} - \sum_{\Gamma(\lambda) = n, \ell(\lambda) \text{ odd}} p^{\text{rep}(\lambda)},
\end{equation}
then $\sum_{n \geq 1} h_n(p)x^n = \frac{x}{(p-1)(x^2-x)-1}$, which is equivalent to the recurrence relation
\begin{equation}
h_n(p) = (1 - p)(h_{n-1}(p) - h_{n-2}(p))
\end{equation}
for $n \geq 3$ with initial values $h_1(p) = -1$ and $h_2(p) = p - 1$. In particular, we have
\begin{equation}
h_n(0) = \begin{cases} (-1)^m, & \text{if } n = (6m - 3 \pm 1)/2; \\ 0, & \text{otherwise.} \end{cases}
\end{equation}

This was first proved by Fu and Tang [2, Theorem 1.4], which is an analog to Euler’s pentagonal number theorem (see [1, Sec. 3.5]). Two other special evaluations of $h_n(p)$ deserve to be mentioned:

- We have $h_n(1) = 0$, which means that there are as many partitions in $\mathcal{H}_n$ with odd number of parts as with even number of parts. It is an interesting exercise to construct an involution proof of this simple fact.
- $|h_n(2)|$ is equal to the $n$-th Fibonacci number $F_n$, which satisfies $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ and initial values $F_1 = F_2 = 1$.

Let $a_o$ (resp. $a_e$) be the total number of odd (resp. even) parts in all partitions in $\mathcal{H}_n$. It can be easily deduced from (2.9) the following closed formulae for $a_o$ and $a_e$.

Corollary 10. For all positive integer $n \geq 2$, we have
\begin{equation}
a_o = (n + 2) \cdot 2^{n-3} \quad \text{and} \quad a_e = n \cdot 2^{n-3}.
\end{equation}
The above result means that the average difference between the number of odd parts and the number of even parts in all partitions with perimeter $n$ is equal to $1/2$.

2.3. A bijective proof. This subsection is devoted to a bijective proof of the equidistribution on $\mathcal{H}_n$:
\begin{equation}
\sum_{\lambda \in \mathcal{H}_n} t^{\text{rep}(\lambda)} = \sum_{\lambda \in \mathcal{H}_n} t^{\text{even}(\lambda)}.
\end{equation}

Theorem 11. There exists a recursively defined bijection $\phi : \mathcal{H}_n \rightarrow \mathcal{H}_n$ satisfying
\begin{equation}
\text{rep}(\lambda) = \text{even}(\phi(\lambda))
\end{equation}
for every $\lambda \in \mathcal{H}_n$. 

Figure 2. The 01-sequence of $\lambda = (6, 3, 3, 1)$ is $w(\lambda) = 0100110001$.

It is convenient to use the following representation of partitions as 01-sequences.

**Lemma 12.** There exists a natural bijection $\lambda \mapsto w(\lambda) = w_0w_1w_2\cdots w_n$ between partitions in $H(n)$ and the set $\Lambda_n$ of 01-sequences such that $w_0 = 0$, $w_n = 1$. Furthermore,

\[
\ell(\lambda) = |\{1 \leq i \leq n : w_i = 1\}|
\]

\[
\text{rep}(\lambda) = |\{1 \leq i \leq n : w_i = w_{i-1} = 1\}|
\]

and

\[
\text{even}(\lambda) = |\{1 \leq i \leq n : w_i = 1 \text{ and } \#\{0 \leq j < i : w_j = 0\} \text{ is even}\}|.
\]

**Proof.** Visually, the 01-sequence representation $w(\lambda)$ of a partition $\lambda \in H_n$ can be obtained as follows. For the edges in the boundary of the Young diagram of $\lambda$, starting at the left bottom and ending to the right top, we label the vertical (resp. horizontal) edges with 1 (resp. 0); see Fig. 2 for an example. In this way, we get a 01-sequence $w(\lambda) \in \Lambda_n$. The three desired properties are obvious from this correspondence. □

**Proof of Theorem 11.** For each $w \in \Lambda_n$, denote

\[
\widetilde{\text{rep}}(w) = |\{1 \leq i \leq n : w_i = w_{i-1} = 1\}|
\]

\[
\widetilde{\text{even}}(w) = |\{1 \leq i \leq n : w_i = 1 \text{ and } \#\{0 \leq j < i : w_j = 0\} \text{ is even}\}|.
\]

In view of Lemma 12, we aim to define recursively a bijection $\tilde{\phi} : \Lambda_n \rightarrow \Lambda_n$ satisfying

(2.13)

\[
\tilde{\text{rep}}(w) = \widetilde{\text{even}}(\tilde{\phi}(w))
\]

for each $w \in \Lambda_n$ and then set $\phi = w^{-1} \circ \tilde{\phi} \circ w$. Set $\tilde{\phi}(\emptyset) = \emptyset$ and $\tilde{\phi}(01) = 01$. For each $w \in \Lambda_n$ ($n \geq 2$), we distinguish the following two cases:

- If $w = 00w_2\cdots w_n$, then $\tilde{\phi}(w)$ is obtained from $\tilde{\phi}(00w_2\cdots w_n)$ by inserting 1 immediately after the initial 0.

- If $w = 01w_2\cdots w_n$, then suppose $w_m = 0$ is the second 0 in $w$ if exists, otherwise set $m = n + 1$. Define $\tilde{\phi}(w)$ to be the concatenation $00w_2\cdots w_{m-1}\tilde{\phi}(w_m\cdots w_n)$.

For example, we have

| 001 $\mapsto$ 011 $\mapsto$ 001; 0101 $\mapsto$ 0001 $\mapsto$ 0111 $\mapsto$ 0011 $\mapsto$ 0101; |
| 00001 $\mapsto$ 01111 $\mapsto$ 00111 $\mapsto$ 01011 $\mapsto$ 0001, 01001 $\mapsto$ 00011 $\mapsto$ 01101 $\mapsto$ 00101 $\mapsto$ 01001 |
under the mapping \( \tilde{\phi} \). It is routine to check by induction on \( n \) that \( \tilde{\phi} \) is well-defined and satisfies (2.13). To see that \( \tilde{\phi} \) is a bijection, we define its inverse explicitly. Given \( w \in \Lambda_n \) \((n \geq 2)\), we consider the following two cases:

- If \( w = 01w_2 \cdots w_n \), then \( \tilde{\phi}^{-1}(w) \) is obtained from \( \tilde{\phi}(0w_2 \cdots w_n) \) by inserting 0 immediately after the initial 0.
- If \( w = 00w_2 \cdots w_n \), then suppose \( w_m = 0 \) is the third 0 in \( w \) if exists, otherwise set \( m = n + 1 \). Define \( \tilde{\phi}^{-1}(w) \) to be the concatenation \( 01w_2 \cdots w_{m-1} \tilde{\phi}^{-1}(w_m \cdots w_n) \).

It is easy to check by induction on \( n \) that \( \tilde{\phi} \) and \( \tilde{\phi}^{-1} \) are inverse to each other. \( \square \)

3. Three proofs of Theorem 4

For a partition \( \lambda \), let

\[
\text{mod}_d(\lambda) := |\{i : \lambda_i \equiv 1(\text{mod } d + 1)\}| \quad \text{and} \quad \text{mod}'_d(\lambda) := \ell(\lambda) - \text{mod}_d(\lambda).
\]

Then, \( \text{mod}'_d(\lambda) \) is \( d \)-generalization of \( \text{even}(\lambda) \) as \( \text{mod}_1(\lambda) = \text{even}(\lambda) \). Introduce the \( d \)-generalization of \( \text{rep}(\lambda) \) by

\[
\text{dif}_d(\lambda) := |\{i : 1 \leq i < \ell(\lambda), \lambda_i - \lambda_{i+1} < d\}|.
\]

It is clear that Theorem 4 is equivalent to

\[
(3.1) \quad \sum_{\lambda \in \mathcal{H}_n} \text{dif}_d(\lambda) \geq \sum_{\lambda \in \mathcal{H}_n} \text{mod}'_d(\lambda).
\]

3.1. A generating function proof. First we compute the generating function for partitions by the perimeter and the statistic \( \text{mod}_d \).

**Lemma 13.** We have

\[
(3.2) \quad \sum_{\lambda \in \mathcal{P}} t^{\text{mod}_d(\lambda)} x^{\ell(\lambda)} y^{\lambda_1} = t x y (1 - x)^d + \frac{x y}{1 - y - 1}(y^{d+1} - y(1 - x)^d)}{(1 - t x)(1 - x)^d - y^{d+1}}.
\]

In particular,

\[
(3.3) \quad \sum_{\lambda \in \mathcal{P}} t^{\text{mod}'_d(\lambda)} d^{\Gamma(\lambda)} = \frac{x(t x^{d+1} + (1 - t x)^d(x - 1))}{(1 - x - t x)((1 - t x)^d(x - 1) + x^{d+1})}.
\]

**Proof.** Since every integer partition \( \lambda \) can be written as \( \lambda = 1^{m_1}2^{m_2} \cdots \), we have

\[
\sum_{\lambda \in \mathcal{P}} t^{m_d(\lambda)} x^{\ell(\lambda)} y^{\lambda_1} = t x \sum_{n \equiv 1(\text{mod } d + 1)} y^n \left( \frac{1}{1 - t x} \right)^{[n/(d+1)]} \left( \frac{1}{1 - x} \right)^{n-[n/(d+1)]},
\]

\[+ x \sum_{n \not\equiv 1(\text{mod } d + 1)} y^n \left( \frac{1}{1 - t x} \right)^{[n/(d+1)]} \left( \frac{1}{1 - x} \right)^{n-[n/(d+1)]},
\]

\[= t x \sum_{k \geq 0} \left( \frac{y}{1 - t x} \right)^{k+1} \left( \frac{y}{1 - x} \right)^{dk},
\]

\[+ x \sum_{i=1}^d \sum_{k \geq 0} \left( \frac{y}{1 - t x} \right)^{k+1} \left( \frac{y}{1 - x} \right)^{dk+i},
\]
which is simplified to (3.2).

Next we compute the generating function for partitions by the perimeter and the statistic $\text{dif}_d$.

Lemma 14. We have

\begin{equation}
\sum_{\lambda \in P} t^{\text{dif}_d(\lambda)} x^{\ell(\lambda)} y_{\lambda_1} = \frac{x y}{1 - y - tx(1 - y^d) - xy^d}.
\end{equation}

In particular,

\begin{equation}
\sum_{\lambda \in P} t^{\text{dif}_d(\lambda)} x^{\Gamma(\lambda)} = \frac{x}{1 - x - tx(1 - x^d) - x^{d+1}}.
\end{equation}

Proof. Notice that in the 01-sequence $w(\lambda)$ of $\lambda$, each maximal segment $00 \cdots 01$ ($k$ 0’s followed by one 1) after the leftmost 1 contributes a $txy^k$ in the following generating function if $0 \leq k \leq d - 1$; and contributes a $xy^k$ if $k \geq d$. Therefore, we have

\[
\sum_{\lambda \in P} t^{\text{dif}_d(\lambda)} x^{\ell(\lambda)} y_{\lambda_1} = \frac{x y}{1 - y} \left(1 + \left(\frac{tx(1 - y^d)}{1 - y} + \frac{xy^d}{1 - y}\right) + \left(\frac{tx(1 - y^d)}{1 - y} + \frac{xy^d}{1 - y}\right)^2 + \cdots \right)
\]

\[
= \frac{x y}{1 - y - tx(1 - y^d) - xy^d},
\]

as desired. \qed

Take the derivative with respect to $t$ and then set $t = 1$ in (3.3) gives

\begin{equation}
\sum_{n \geq 1} x^n \sum_{\lambda \in H_n} \text{mod}_d'(\lambda) = \frac{x(1 - x)(x^{d+1} - x(1 - x)^d)}{(1 - 2x)^2(x^{d+1} - (1 - x)^{d+1})}.
\end{equation}

On the other hand, the same operation on (3.5) yields

\begin{equation}
\sum_{n \geq 1} x^n \sum_{\lambda \in H_n} \text{dif}_d(\lambda) = \frac{x^2(1 - x^d)}{(1 - 2x)^2}.
\end{equation}

To finish the proof of Theorem 4 (or equivalently, inequality (3.1)), it remains to show that

\begin{equation}
\frac{x^2(1 - x^d)}{(1 - 2x)^2} - \frac{x(1 - x)(x^{d+1} - x(1 - x)^d)}{(1 - 2x)^2(x^{d+1} - (1 - x)^{d+1})} = \frac{x^{d+2}(1 - 2x + x^{d+1} - (1 - x)^{d+1})}{(1 - 2x)^2((1 - x)^{d+1} - x^{d+1})}
\end{equation}

has nonnegative coefficients for each $d \geq 1$. Since

\[
\frac{1 - 2x + x^{d+1} - (1 - x)^{d+1}}{(1 - 2x)^2((1 - x)^{d+1} - x^{d+1})} = \frac{1}{1 - 2x} \left(\frac{1}{(1 - x)^{d+1} - x^{d+1}} - \frac{1}{1 - 2x}\right),
\]

Theorem 4 then follows from the following interesting positivity result.

Theorem 15. The rational function

\[
\frac{1}{(1 - x)^{d+1} - x^{d+1}} - \frac{1}{1 - 2x} =: \Delta_d(x)
\]

has nonnegative coefficients for each $d \geq 1$.\]
Note that $\Delta_0(x) = \Delta_1(x) = 0$ and

$$
\Delta_2(x) = \sum_{n \geq 1} x^n \sum_{i \geq 0} \left( \frac{n}{3i+1} \right) = x + 2x^2 + 3x^3 + 5x^4 + 10x^5 + 21x^6 + \cdots,
$$

$$
\Delta_3(x) = \sum_{n \geq 1} x^n \sum_{i \geq 0} 2 \left( \frac{n+1}{4i+2} \right) = 2x + 6x^2 + 12x^3 + 20x^4 + 32x^5 + 56x^6 + \cdots,
$$

$$
\Delta_4(x) = \sum_{n \geq 1} x^n \sum_{i \geq 0} \left( 3 \left( \frac{n+2}{5i+3} \right) - \left( \frac{n}{5i+2} \right) \right) = 3x + 11x^2 + 27x^3 + 54x^4 + 95x^5 + 156x^6 + \cdots.
$$

For two polynomials $f(x), g(x) \in \mathbb{Z}[x]$, we write

$$
f(x) \geq_x g(x) \iff f(x) - g(x) \text{ has nonnegative coefficients}.
$$

To prove Theorem 15, we need the following auxiliary lemma.

**Lemma 16.** For $d \geq 1$, we have

$$
\frac{x}{(1-x)^d} \geq_x \frac{x^d}{(1-x)^d}.
$$

**Proof.** This follows from the binomial theorem

$$
(1-x)^{-d} = \sum_{k \geq 0} \binom{d+k-1}{k} x^k
$$

and the monotonicity $\binom{d+k}{k+1} \geq \binom{d+k-1}{k}$ for $k \geq 0$. \hfill \Box

We can now prove Theorem 15.

**Proof of Theorem 15.** By Lemma 16 and by induction on $d$, we have

$$
\frac{1}{(1-x)^d} - \frac{1}{(1-x)^d} \geq_x \frac{1}{(1-x)^d} - \frac{1}{(1-x)^d} \left( 1 + \frac{x}{1-x} + \left( \frac{x}{1-x} \right)^2 + \cdots + \left( \frac{x}{1-x} \right)^{d-1} \right)
$$

$$
\geq_x \frac{1}{(1-x)^{d-1}} - \frac{1}{1-x} \left( 1 + \frac{x}{1-x} + \left( \frac{x}{1-x} \right)^2 + \cdots + \left( \frac{x}{1-x} \right)^{d-2} \right)
$$

$$
= \frac{1}{1-x} \left( 1 + \frac{x}{1-x} + \left( \frac{x}{1-x} \right)^2 + \cdots + \left( \frac{x}{1-x} \right)^{d-2} \right)
$$

$$
\geq_x 0
$$

for each $d \geq 2$. It then follows from the above assertion that

$$
\Delta_{d-1}(x) = \frac{1}{(1-x)^d - x^d} - \frac{1}{1-2x}
$$

$$
= \frac{1}{(1-x)^d} \left( 1 + \left( \frac{x}{1-x} \right)^d + \left( \frac{x}{1-x} \right)^{2d} + \left( \frac{x}{1-x} \right)^{3d} + \cdots \right)
$$

$$
- \frac{1}{1-x} \left( 1 + \frac{x}{1-x} + \left( \frac{x}{1-x} \right)^2 + \left( \frac{x}{1-x} \right)^3 + \cdots \right)
$$
Proof. \[ \left( \sum_{k \geq 0} \left( \frac{x}{1-x} \right)^{kd} \right) \left( \frac{1}{(1-x)^d} - \frac{1}{1-x} \left( 1 + \frac{x}{1-x} + \left( \frac{x}{1-x} \right)^2 + \cdots + \left( \frac{x}{1-x} \right)^{d-1} \right) \right) \geq_x 0, \]

which completes the proof of the theorem. \[ \square \]

3.2. An injective proof. This section is motivated by finding an interpretation for $\sum_{\lambda \in \mathcal{H}_n} \text{dif}_d(\lambda) - \sum_{\lambda \in \mathcal{H}_n} \text{mod}_d(\lambda)$. A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda^*_i, \cdots)$ whose $i$-th part, $1 \leq i \leq \ell(\lambda)$, receives a star is called a labeled partition. It is convenient to represent such a labeled partition by $\lambda, i$. Let us consider the two sets of labeled partitions:

\[ \mathcal{D}_{n,d} := \{(\lambda, i) : \lambda \in \mathcal{H}_n, 0 \leq \lambda_{i-1} - \lambda_i < d \} \quad \text{and} \quad \mathcal{M}_{n,d} := \{(\lambda, i) : \lambda \in \mathcal{H}_n, \lambda_i \not\equiv 1(\text{mod } d + 1) \}. \]

For example,

\[ \mathcal{D}_{4,2} = \{32^*, 33^*, 211^*, 21^*1, 221^*, 22^*1, 222^*, 22^*2, 1111^*, 111^*1, 11^*11\} \]

and

\[ \mathcal{M}_{4,2} = \{3^*1, 32^*, 3^*2, 33^*, 3^*3, 2^*11, 221^*, 2^*21, 222^*, 22^*2, 2^*22\}. \]

It is clear that Theorem 4 is equivalent to $|\mathcal{D}_{n,d}| \geq |\mathcal{M}_{n,d}|$ for $n, d \geq 1$. The purpose of this section is to provide an injective proof of Theorem 4 that leads to a partition interpretation of $|\mathcal{D}_{n,d}| - |\mathcal{M}_{n,d}|$.

**Theorem 17.** For any fixed $n, d \geq 1$, there exists an injection $\xi$ from $\mathcal{M}_{n,d}$ to $\mathcal{D}_{n,d}$ such that

\[ (3.9) \quad \mathcal{D}_{n,d} \setminus \xi(\mathcal{M}_{n,d}) = \{(\lambda, i) \in \mathcal{D}_{n,d} : \lambda_{i-1} \equiv 1(\text{mod } d + 1), \lambda_i \not\equiv 1(\text{mod } d + 1) \}. \]

**Proof.** For a labeled partition $(\lambda, i) \in \mathcal{M}_{n,d}$, we define $\xi(\lambda, i) \in \mathcal{D}_{n,d}$ according to the following two cases. Suppose that $\lambda_i = l(d+1) + k$ for some $2 \leq k \leq d + 1$ and $l \geq 0$. Note that $0 \leq k - 2 < d$ and we use the convention $\lambda_{l+1}(\lambda) = 0$.

- If $\lambda_i - \lambda_{i+1} \leq k - 2$, then set $\xi(\lambda, i) = (\lambda', i + 1)$, where $\lambda' = \lambda_1' \lambda_2' \cdots \lambda'_{\ell(\lambda)+1}$ is a partition with one more part than $\lambda$ defined as

\[ \lambda'_j = \begin{cases} \lambda_j - 1, & \text{if } j \leq i; \\ l(d+1) + 1, & \text{if } j = i + 1; \\ \lambda_{j-1}, & \text{if } i + 1 < j \leq \ell(\lambda) + 1. \end{cases} \]

Since $\lambda'_{i+1} = l(d+1) + k - 1 - l(d+1) + 1 = k - 2$, we see $\xi(\lambda, i) \in \mathcal{D}_{n,d}$. In this case, $\lambda_{i+1} \equiv 1(\text{mod } d + 1)$.

- If $\lambda_i - \lambda_{i+1} > k - 2$, then set $\xi(\lambda, i) = (\lambda', i + 1)$, where $\lambda' = \lambda_1' \lambda_2' \cdots \lambda'_{\ell(\lambda)+1}$ is a partition with one more part than $\lambda$ defined as

\[ \lambda'_j = \begin{cases} \lambda_j - 1, & \text{if } j \leq i; \\ l(d+1) + 1, & \text{if } j = i + 1; \\ \lambda_{j-1}, & \text{if } i + 1 < j \leq \ell(\lambda) + 1. \end{cases} \]

Since $\lambda'_{i+1} = l(d+1) + k - 1 - l(d+1) + 1 = k - 2$, we see $\xi(\lambda, i) \in \mathcal{D}_{n,d}$.

For example, if $(\lambda, 3) = (14, 13, 11^*, 10, 5, 2) \in \mathcal{M}_{19,5}$ then $\xi(\lambda, 3) = (14, 13, 11, 10^*, 5, 2) \in \mathcal{D}_{19,5}$ is constructed in the first case, and if $(\lambda, 3) = (14, 13, 11^*, 5, 5, 2) \in \mathcal{M}_{19,5}$ then $\xi(\lambda, 3) = (13, 12, 10^*, 5, 5, 2) \in \mathcal{D}_{19,5}$ is constructed in the second case.

To see that the mapping $\xi$ is an injection, observe that for any $(\lambda, i + 1) \in \mathcal{D}_{n,d}$:
\begin{itemize}
\item $(\lambda, i + 1)$ is the image under $\xi$ from the first case above if $\lambda_{i+1} \not\equiv 1 (\text{mod } d + 1)$ and $\lambda_i \not\equiv 1 (\text{mod } d + 1)$;
\item $(\lambda, i + 1)$ is the image under $\xi$ from the second case above if $\lambda_{i+1} \equiv 1 (\text{mod } d + 1)$;
\item $(\lambda, i + 1)$ is not an image under $\xi$ if $\lambda_{i+1} \not\equiv 1 (\text{mod } d + 1)$ but $\lambda_i \equiv 1 (\text{mod } d + 1)$.
\end{itemize}

Since the above two cases of $\xi$ are reversible, $\xi$ is an injection and (3.9) holds. \hfill $\square$

\textbf{Example 18.} As an example of (3.9), we see that the number $|\mathcal{D}_{6,2}| - |\mathcal{M}_{6,2}| = 4$ counts the labeled partitions $(4, 3^*, 1)$, $(4, 3^*, 2)$, $(4, 4, 3^*)$ and $(4, 3^*, 3)$.

3.3. A bijective proof. The following stronger version of Theorem 4 was originally suggested by numerical computations.

\textbf{Theorem 19.} Fix $d \geq 1$. There exists a bijection $\phi_d : \mathcal{H}_n \rightarrow \mathcal{H}_n$ such that

\[ \text{dif}_d(\lambda) \geq \text{mod}'(\phi_d(\lambda)) \]

for every $\lambda \in \mathcal{H}_n$. Moreover, $\text{dif}_d(\lambda) = 0$ if and only if $\text{mod}'(\phi_d(\lambda)) = 0$.

\textbf{Remark 20.} Theorem 19 is a common generalization of Theorems 2 and 4. At the beginning, we attempted to prove Theorem 19 by using generating function but failed. To find such a proof remains an interesting open problem.

The construction of $\phi_d$ is a $d$-extension of $\phi$ defined in Theorem 11.

\textbf{Proof of Theorem 19.} Under the correspondence $\lambda \mapsto w = w(\lambda)$ in Lemma 12, the two statistics $\text{dif}_d(\lambda)$ and $\text{mod}'(\lambda)$ are transformed to

\[ \tilde{\text{dif}}_d(w) = |\{i \in [n-1] : w_i = w_{i+k} = 1 \text{ and } w_{i+1} = \cdots = w_{i+k-1} = 0 \text{ for some } k \in [d]\}|, \]

\[ \tilde{\text{mod}}'(w) = |\{2 \leq i \leq n : w_i = 1 \text{ and } \#\{0 \leq j < i : w_j = 0\} \not\equiv 1 (\text{mod } d + 1)\}|. \]

So it is sufficient to define recursively a bijection $\tilde{\phi}_d : \Lambda_n \rightarrow \Lambda_n$ such that

\[ (3.10) \quad \tilde{\text{dif}}_d(w) \geq \tilde{\text{mod}}'(\tilde{\phi}_d(w)) \]

for each $w \in \Lambda_n$ and then set $\phi_d = w^{-1} \circ \tilde{\phi}_d \circ w$.

Set $\tilde{\phi}_d(0) = 0$ and $\tilde{\phi}_d(01) = 01$. For each $w \in \Lambda_n$ ($n \geq 2$), we distinguish the following two cases:

- If $w = 00w_2 \cdots w_n$, then $\tilde{\phi}_d(w)$ is obtained from $\tilde{\phi}_d(0w_2 \cdots w_n)$ by inserting 1 immediately after the initial 0.
- If $w = 01w_2 \cdots w_n$, then suppose $w_m = 0$ is the $(d+1)$-th 0 in $w$ if exists, otherwise set $m = n + 1$. Define $\tilde{\phi}_d(w)$ to be the concatenation $00w_2 \cdots w_{m-1} \tilde{\phi}_d(w_{m} \cdots w_n)$.

For example, we have

\begin{itemize}
\item $001 \mapsto 011 \mapsto 001$; \hspace{.5cm} $0101 \mapsto 0001 \mapsto 0111 \mapsto 0011 \mapsto 0101$;
\item $00001 \mapsto 01111 \mapsto 00111 \mapsto 01011 \mapsto 00011 \mapsto 01101 \mapsto 00101 \mapsto 01001 \mapsto 00001$.
\end{itemize}

under the mapping $\tilde{\phi}_2$. We need to verify that $\tilde{\phi}_d$ satisfies (3.10) by induction on $n$ depending on two cases of $\tilde{\phi}_d$:

- Since inserting 1 after the initial 0 of $\tilde{\phi}_d(0w_2 \cdots w_n)$ dose not change $\tilde{\text{mod}}'$, we have

\[ \tilde{\text{dif}}_d(w) = \tilde{\text{dif}}_d(0w_2 \cdots w_n) \geq \tilde{\text{mod}}'(\tilde{\phi}_d(0w_2 \cdots w_n)) = \tilde{\text{mod}}'(\tilde{\phi}_d(w)). \]
The proof of the theorem is now complete. □

To see that \( \tilde{\phi}_d \) is a bijection, we construct its inverse \( \tilde{\phi}_d^{-1} \) explicitly. Given \( w \in \Lambda_n \) \((n \geq 2)\), we consider the following two cases:

- If \( w = 01w_2 \cdots w_n \), then \( \tilde{\phi}_d^{-1}(w) \) is constructed from \( \tilde{\phi}_d^{-1}(0w_2 \cdots w_n) \) by inserting 0 immediately after the initial 0.
- If \( w = 00w_2 \cdots w_n \), then suppose \( w_m = 0 \) is the \((d+2)\)-th 0 in \( w \) if exists, otherwise set \( m = n+1 \). Define \( \tilde{\phi}_d^{-1}(w) \) to be the concatenation \( 01w_2 \cdots w_m-1 \tilde{\phi}_d^{-1}(w_m \cdots w_n) \).

It can be checked routinely that \( \tilde{\phi}_d \) and \( \tilde{\phi}_d^{-1} \) are inverse to each other, which proves that \( \tilde{\phi}_d \) is a bijection.

Finally, we need to verify that whenever \( \text{mod}_d'(w) = 0 \) then \( \text{dif}_d(\tilde{\phi}_d^{-1}(w)) = 0 \) by induction on \( n \) according to the two cases of \( \tilde{\phi}_d^{-1} \):

- In the first case, we have
  \[ 0 = \text{mod}_d'(w) = \text{mod}_d'(0w_2 \cdots w_n) = \text{dif}_d(\tilde{\phi}_d^{-1}(0w_2 \cdots w_n)) = \text{dif}_d(\tilde{\phi}_d^{-1}(w)). \]
- In the second case, since \( \text{mod}_d'(w) = 0 \), we must have \( m = d + 2 \) and the prefix \( 00w_2 \cdots w_m \) of \( w \) are all 0’s, which leads to
  \[ \text{dif}_d(\tilde{\phi}_d^{-1}(w)) = \tilde{\phi}_d^{-1}(w_m \cdots w_n) = 0. \]

The proof of the theorem is now complete.

4. Further remarks

We will conclude this paper with the following three remarks.

1. Note that Eq. (3.7) is equivalent to the following closed formula
   \[ \sum_{\lambda \in \mathcal{H}_n} \text{dif}_d(\lambda) = (n - 1) \cdot 2^{n-2} - (n - d - 1) \cdot 2^{n-d-2}, \]
   which can also be deduced directly as follows. By the 01-sequence representation of partitions, we have
   \[ \sum_{\lambda \in \mathcal{H}_n} \text{dif}_d(\lambda) = \sum_{\lambda \in \Lambda_n} \tilde{\text{dif}}_d(\lambda) \]
   \[ = \sum_{j=1}^{d} \sum_{w \in \Lambda_n} \#\{1 \leq i \leq n-j : w_i = w_{i+j} = 0, w_{i+1} = w_{i+2} = \ldots = w_{i+j-1} = 1\} \]
Figure 3. Partitions in $H_5$ (represented as Young diagrams) with their pair of statistics $(\text{rep}^*, \text{even}^*)$ on the top.

\[
\begin{align*}
&= \sum_{j=1}^{d} \sum_{i=1}^{n-j} \left| \{ w \in \Lambda_n : w_i = w_{i+j} = 0, \ w_{i+1} = w_{i+2} = \ldots = w_{i+j-1} = 1 \} \right| \\
&= \sum_{j=1}^{d} \left( (n - j - 1)2^{n-j-2} + 2^{n-j-1} \right) \\
&= (n - 1) \cdot 2^{n-2} - (n - d - 1) \cdot 2^{n-d-2}.
\end{align*}
\]

(2) In Theorem 8, we have computed the generating function for the joint distribution of $(\text{rep}, \text{even})$ over $H_n$, which is a rational formal power series. The pair $(\text{dif}_d, \text{mod}'_d)$ is a $d$-extension of $(\text{rep}, \text{even})$, however, in Lemmas 13 and 14 we have to deal separably with the distribution of $\text{dif}_d$ and $\text{mod}'_d$ on $H_n$. It remains an open problem to compute the generating function for the joint distribution of $(\text{dif}_d, \text{mod}'_d)$ over $H_n$ for general $d$.

(3) In [6], Wilf proved via the Principle of Inclusion-Exclusion [4, Sec. 2.1] an interesting refinement of Euler’s Odd-Distinct partition theorem by using two valued version of our (position) statistics “rep” and “even”:

\[
\text{rep}^*(\lambda) = |\{ \lambda_i : \lambda_i = \lambda_{i+1} \}| \quad \text{and} \quad \text{even}^*(\lambda) = |\{ \lambda_i : \lambda_i \text{ is even} \}|,
\]

called the number of repeated part sizes and the number of even part sizes, respectively. Namely, he proved the equidistribution

\[
\sum_{\lambda \in \text{Par}_n} t^{\text{rep}^*(\lambda)} = \sum_{\lambda \in \text{Par}_n} t^{\text{even}^*(\lambda)},
\]

where $\text{Par}_n$ is the set of all partitions of $n$. Interestingly, this equidistribution holds true when replacing $\text{Par}_n$ by $H_n$:

\[
\sum_{\lambda \in \text{H}_n} t^{\text{rep}^*(\lambda)} = \sum_{\lambda \in \text{H}_n} t^{\text{even}^*(\lambda)},
\]

which is a valued version of Theorem 3. See Fig. 3 for an example of (4.2) for $n = 5$. The approach of Wilf via Inclusion-Exclusion doesn’t seem to work for (4.2). However, the bijection $\phi$ constructed in Theorem 11 also proves (4.2). Is there any unified generalization of both (4.1) and (4.2)?
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