ON THE O-MINIMAL LS-CATEGORY

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ABSTRACT. We introduce the o-minimal LS-category of definable sets in o-minimal expansions of ordered fields and we establish a relation with the semialgebraic and the classical one. We also study the o-minimal LS-category of definable groups. Along the way, we show that two definably connected definably compact definable groups $G$ and $H$ are definable homotopy equivalent if and only if $L(G)$ and $L(H)$ are homotopy equivalent, where $L$ is the functor which associates to each definable group its corresponding Lie group via Pillay’s conjecture.

1 Introduction

This paper is a contribution to the development of o-minimal topology, in particular to o-minimal homotopy. This development has found a successful approach through homology and cohomology theories. Recently, in [2] the author and M. Otero also provide a homotopy theory to the o-minimal setting (see Fact 2.2).

We introduce the o-minimal Lusternik-Schnirelmann category (in short o-minimal LS-category) of definable sets in an o-minimal expansion $\mathcal{R}$ of a real closed field. The classical one was originally introduced to provide a lower bound on the number of critical points for any smooth function on a manifold, then it became an important subject in algebraic topology. The LS-category of a topological space $X$, denoted by $\text{cat}(X)$, is the least integer $m$ such that $X$ has an open cover of $m + 1$ elements with each of them contractible to a point in $X$. The o-minimal LS-category of a definable set $X$, denoted by $\text{cat}(X)^{\mathcal{R}}$, is defined in the obvious way (see Definition 3.4).

In Section 3, in analogy with the homotopy results in [2], we prove the following comparison result.

**Theorem 1.1.** Let $X$ be a semialgebraic set defined without parameters. Then $\text{cat}(X)^{\mathcal{R}} = \text{cat}(X(\mathbb{R}))$.

On the other hand, recall that given a definably compact $d$-dimensional definable group $G$, the work of several authors (e.g. A. Berarducci, E. Hrushovski, Y. Peterzil, A. Pillay, M. Otero and others) in the positively

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resolution of Pillay’s conjecture has shown that there exist a smallest type-
definable subgroup $G^{00}$ of $G$ with bounded index such that $L(G) := G/G^{00}$
with the logic topology is a compact $d$-dimensional Lie group (see \cite{6,12,15}).
Moreover, if $G$ is definably connected then $L(G)$ is connected. Recall that
a subset of $L(G)$ is closed in the logic topology if and only if its preimage
by the projection $\pi : G \to L(G)$ is a type-definable subset of $G$. With the
logic topology $\pi : G \to L(G)$ is a continuous epimorphism.

The main motivation to study the o-minimal LS-category is establish a
topological analogy between a definably compact definably connected group
$G$ and the connected compact Lie group $L(G)$ associated to it. In this direc-
tion, it has been proved by A. Berarducci in \cite{3} that the cohomology groups
of $G$ are isomorphic to those of $L(G)$. Moreover, using the development of
o-minimal homotopy mentioned above, A. Berarducci, M. Maminio and M.
Otero prove in \cite{5} that the homotopy groups of $G$ and $L(G)$ are isomorphic.
The aim of Section 4 is to prove that $\text{cat}(G)^R$ and $\text{cat}(L(G))$ are equal (see
Corollary \ref{cor:cat-equality}). To do this we establish the following stronger result.

**Theorem 1.2.** Let $G$ be a definably connected definably compact definable
group whose underlying set is a semialgebraic set defined without parameters.
Then $G(R)$ is homotopy equivalent to $L(G)$.

As a consequence of Theorem \ref{thm:main-theorem} we get the following.

**Corollary 1.3.** Let $G$ and $H$ be definably connected definably compact de-
finable groups. Then $G$ and $H$ are definable homotopy equivalent if and only
if $L(G)$ and $L(H)$ are homotopy equivalent.

We point out that the preprint \cite{4} by A. Berarducci and M. Mamino has
recently appeared with similar results to Theorem \ref{thm:main-theorem} and Corollary \ref{cor:main-corollary}.
However, the points of view of each paper are different. Here, we obtain the
results via a transfer approach. In \cite{4}, the development goes through a new
homotopic study of the projection map $\pi : G \to L(G)$.

The results of Section 4 of this paper are part of the author’s Ph.D.
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## 2 Notation and preliminaries

For the rest of the paper we fix an o-minimal expansion $\mathcal{R}$ of a real closed
field $R$. We shall denote by $\mathcal{R}_{sa}$ the field structure of the real closed field $R$.
We always take *definable* to mean definable in $\mathcal{R}$ with parameters, except
otherwise stated. We take the order topology on $R$ and the product topology
on $R^n$ for $n > 1$. All definable maps are assumed to be continuous. If a set $X$ is definable with parameters in some structure $\mathcal{M}$ we denote by $X(M)$ the realization of $X$ in $\mathcal{M}$.

Given a definable set $S$ and some definable subsets $S_1, \ldots, S_l$ of $S$ we say that $(K, \phi)$ is a triangulation in $R^n$ of $S$ partitioning $S_1, \ldots, S_l$ if $K$ is a simplicial complex formed by a finite number of (open) simplices in $R^n$ and $\phi : |K| \to S$ is a definable homeomorphism, with $|K| = \bigcup_{\sigma \in K} \sigma \subset R^n$ the realization of $K$, such that each $S_i$ is the union of the images by $\phi$ of some simplices of $K$. We say that a simplicial complex $K'$ is a subdivision of a simplicial complex $K$ if each simplex of $K'$ is contained in a simplex of $K$ and each simplex of $K$ equals the union of finitely many simplices of $K'$. We will use the standard notion of barycentric subdivision of a simplicial complex (see [10, Ch.8, §1.8]).

Now, let us collect some results needed in the sequel. We start with a refinement of the triangulation theorem.

**Fact 2.1** (Normal triangulation theorem). [1, Thm.1.4] Let $K$ be a simplicial complex and let $S_1, \ldots, S_l$ be definable subsets of its realization $|K|$. Then, there is a subdivision $K'$ of $K$ and a definable homeomorphism $\phi' : |K'| \to |K|$ such that

(i) $(K', \phi')$ partitions all $S_1, \ldots, S_l$ and each $\sigma \in K$,

(ii) for every $\tau \in K'$ and $\sigma \in K$, if $\tau \subset \sigma$ then $\phi'(\tau) \subset \sigma$.

We say that $(K', \phi')$ is a normal triangulation of $|K|$ partitioning the subsets $S_1, \ldots, S_l$.

Using the notation above, it follows from property (ii) that $\phi'$ is definably homotopic to $\text{id}_{|K|}$ (see the proof of [1, Thm.1.1]). For this reason the normal triangulation theorem is a key tool to prove the following results concerning o-minimal homotopy already mentioned in the introduction. Let $X$ and $Y$ be semialgebraic sets in $R_{sa}$ and let $A$ and $B$ be semialgebraic subsets of $X$ and $Y$ respectively. The o-minimal homotopy set $[(X, A), (Y, B)]^R$ is the collection of definable maps $f : X \to Y$ with $f(A) \subset B$ modulo definable homotopy mapping $A$ in $B$ (see [2, §3]).

**Fact 2.2.** With the notation above, if $A$ is relatively closed in $X$ then,

(a) [2, Cor.3.3] the map $[(X, A), (Y, B)]^{R_{sa}} \to [(X, A), (Y, B)]^R : [f] \mapsto [f]$ is a bijection,

(b) [3, Ch.III, Thm.4.1] if $S$ a real closed field extension of $R$ then the map $[(X, A), (Y, B)]^{R_{sa}} \to [(X(S), A(S)), (Y(S), B(S))]^{S_{sa}} : [f] \mapsto [f(S)]$ is a bijection,

(c) [3, Ch.III, Thm.5.1] if $R = \mathbb{R}$ then $[(X, A), (Y, B)]^{R_{sa}} \to [(X, A), (Y, B)] : [f] \mapsto [f]$ is a bijection, where $[(X, A), (Y, B)]$ is the classical homotopy set.

Moreover, if the homotopy sets under consideration are the homotopy groups of a semialgebraic pointed set $(X, x)$, i.e., $\pi_n(X, x)^R = [(I^n, \partial I^n), (X, x)]^R$. 

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then the bijections are isomorphisms (see [2, §4]). We shall omit the superscript \( R \) if it is clear from the context.

We finish with the following interesting application of Fact 2.2 to the study of definable groups which will be used in Section 4.

**Fact 2.3.** [5, Thm.3.7] Let \( G \) be a definably connected definably compact group. Then \( \pi_n(G) \cong \pi_n(L(G)) \) for all \( n \geq 1 \).

### 3 o-Minimal LS-category of definable sets

In this section we introduce the Lusternik-Schnirelmann category (in short LS-category) for definable sets. We apply the results of o-minimal homotopy from [2] and the normal triangulation theorem to prove some comparison theorems concerning the LS-category. For a general reference on the classical Lusternik-Schnirelmann category see [8].

**Definition 3.1.** Let \( X \) be a definable set. We say that a definable subset \( A \) of \( X \) is definably categorical in \( X \) if \( A \) is definably contractible to a point in \( X \) (not necessarily definably contractible in itself). We say that a definable cover \( \{V_i\}_{i=1}^m \) of \( X \) is a definable categorical cover of \( X \) if each \( V_i \) is definably categorical in \( X \).

**Fact 3.2.** [8, Lem.1.29] Let \( X \) and \( Y \) be definable sets. Let \( f : X \to Y \) and \( g : Y \to X \) be definable maps such that \( g \circ f \sim id_X \). Then \( f^{-1}(V) \) is a definable categorical subset of \( X \) for each definable categorical subset \( V \) of \( Y \).

**Proof.** Let \( F : X \times I \to X \) be a definable homotopy from \( id_X \) to \( g \circ f \). Let \( y_0 \in Y \) and let \( H : V \times I \to Y \) be a definable homotopy such that \( H(y,0) = y \) for all \( y \in V \) and \( H(y,1) = y_0 \) for all \( y \in V \). Denote by \( U = f^{-1}(V) \) and consider the definable map \( G : U \times I \to X \) defined by

\[
G(x,t) = \begin{cases} 
F(x,2t) & \text{for all } (x,t) \in U \times [0,\frac{1}{2}], \\
g(H(f(x),2t-1)) & \text{for all } (x,t) \in U \times [\frac{1}{2},1]. 
\end{cases}
\]

Note that \( G(x,0) = x \) and \( G(x,1) = g(x_0) \) for all \( x \in U \), i.e., \( U \) is a definable categorical subset of \( X \).

Every definable set \( X \) has a definable categorical open cover. Indeed, by the triangulation theorem and Fact 3.2 we can assume that \( X = |K| \) for a simplicial complex \( K \). We denote by \( K' \) the first barycentric subdivision of \( K \). By [9, Prop III.1.6], the open definable subset \( \text{St}_{K'}(v) \) of \( |K| \) is definably categorical in \( |K| \) for each \( v \in \text{Vert}(K) \cap |K| \). Therefore, \( \{\text{St}_{K'}(v) : v \in \text{Vert}(K) \cap |K|\} \) is a finite definable categorical open cover of \( |K| \). Alternatively, every definable set \( X \) is a finite union of open cells (which are already definably contractible).
Definition 3.3. The o-minimal LS-category of a definable set $X$, denoted by $\text{cat}(X)^R$, is the least integer $m$ such that $X$ has a definable categorical open cover of $m + 1$ elements.

For example, by definition we have that a definable set $X$ is definably contractible if and only if $\text{cat}(X)^R = 0$. Now, we prove that the o-minimal LS-category is homotopy invariant.

Fact 3.4. \cite[Lem.1.30]{s} Let $X$ and $Y$ be definable homotopy equivalent definable sets. Then $\text{cat}(X)^R = \text{cat}(Y)^R$.

Proof. Let $f : X \to Y$ and $g : Y \to X$ be definable maps such that $f \circ g \sim \text{id}_Y$ and $g \circ f \sim \text{id}_X$. We show that $\text{cat}(Y)^R \leq \text{cat}(X)^R$, the other inequality by symmetry. Let $\{U_i\}_{i=1}^{m+1}$ be a definable categorical open cover of $X$. By Fact 3.2, $g^{-1}(U_i)$ is definably categorical in $Y$ for each $i = 1, \ldots, m + 1$. Hence $\{g^{-1}(U_i)\}_{i=1}^{m+1}$ is a categorical open cover of $Y$.

Theorem 3.5. Let $X \subset R^n$ be a semialgebraic set. Then $\text{cat}(X)^{R_{sa}} = \text{cat}(X)^R$.

Proof. Clearly, $\text{cat}(X)^{R_{sa}} \geq \text{cat}(X)^R$. We show that $\text{cat}(X)^{R_{sa}} \leq \text{cat}(X)^R$. By the Triangulation theorem we can assume that $X = |K|$ for some simplicial complex $K$. Let $m = \text{cat}(X)^R$ and let $U_1, \ldots, U_{m+1}$ be a definable categorical open cover of $|K|$. By the normal triangulation theorem \cite[2.1]{s} there is a subdivision $K'$ of $K$ and a definable homeomorphism $\phi : |K'| \to |K|$ such that $(K', \phi)$ partitions $U_1, \ldots, U_{m+1}$. Note that since $K'$ is a subdivision of $K$ we have $|K'| = |K|$ (this is the reason why we use the normal triangulation theorem instead of the standard one). Therefore the open subset $V_i := \phi^{-1}(U_i)$ is the realization of a subcomplex of $K'$ and hence a semialgebraic subset of $|K'|$ for each $i = 1, \ldots, m + 1$. Since $\phi$ is a definable homeomorphism, $V_i$ is a definable categorical subset of $|K|$ for all $i = 1, \ldots, m + 1$ (see Fact 3.2). Now, by Fact 2.2 \cite[(a)]{s} if a semialgebraic subset $A$ of a semialgebraic set $B$ is definably contractible in $B$, then it is semialgebraically contractible in $B$. Hence, $V_i$ is a semialgebraic categorical subset of $|K|$ for all $i = 1, \ldots, m + 1$. Then $\{V_i\}_{i=1}^{m+1}$ is a semialgebraic categorical open cover of $|K|$ and hence $\text{cat}(X)^{R_{sa}} \leq m$.

Theorem 3.6. Let $X \subset R^n$ be a semialgebraic set. Let $S$ be a real closed field extension of $R$. Then $\text{cat}(X)^{R_{sa}} = \text{cat}(X(S))^{S_{sa}}$.

Proof. It is immediate that $\text{cat}(X)^{R_{sa}} \geq \text{cat}(X(S))^{S_{sa}}$. For, given a semialgebraic categorical open cover $\{U_i\}_{i=1}^{m+1}$ of $X$, $\{U_i(S)\}_{i=1}^{m+1}$ is clearly a semialgebraic categorical open cover of $X(S)$. We show that $\text{cat}(X)^{R_{sa}} \leq \text{cat}(X(S))^{S_{sa}}$. Let $\{V_i\}_{i=1}^{m+1}$ be a semialgebraic categorical open cover of $X(S)$. Let $H_i : V_i \times I \to X(S)$
be a semialgebraic homotopy such that \( H_i(x,0) = x \) for all \( x \in V_i \) and \( H_i(x,1) = x_i \in X(S) \) for all \( x \in V_i \) for each \( i = 1, \ldots, m+1 \). Without loss of generality we can assume that \( x_i \in R^n \). Indeed, let \( x'_i \in X(S) \cap R^n \) be a point which lies in the same semialgebraic connected component of \( x_i \). Consider a semialgebraic curve \( \alpha_i : I \to X(S) \) such that \( \alpha_i(0) = x_i \) and \( \alpha_i(1) = x'_i \). Then, we can replace \( H_i \) by the semialgebraic homotopy \( H'_i : V_i \times I \to X(S) \) with \( H'_i(x,t) = H_i(x,2t) \) for all \( (x,t) \in V_i \times [0,\frac{1}{2}] \) and \( H'_i(x,t) = \alpha_i(2t - 1) \) for all \( (x,t) \in V_i \times [\frac{1}{2},1] \).

On the other hand, by the triangulation theorem and Fact 3.4 we can assume that \( X(S) = |K| \) for a simplicial complex \( K \) in \( S \) whose vertices lie in \( R \) as well as the \( V_i \)'s are realizations of subcomplexes of \( K \) semialgebraically categorical in \( |K| \) (possibly with parameters from \( S \)). Furthermore, by Fact 2.2(b) the \( V_i \)'s are also semialgebraically categorical in \( |K| \) with parameters from \( R \) and therefore \( \{V_i(R)\}_{i=1}^{m+1} \) is a semialgebraic categorical open cover of \( X \). Hence, \( \text{cat}(X)^{\text{sa}} \leq \text{cat}(X(S))^{\text{sa}} \), as required.

The following fact allows us to work with closed simplicial complexes in the proof of Theorem 3.8.

**Fact 3.7.** [11 Ch.III, Prop.1.6, 1.8] Let \( K \) be the first barycentric subdivision of some simplicial complex. Let \( \text{co}(K) \) be the closed simplicial subcomplex of \( K \) consisting in all simplexes of \( K \) whose faces are also simplexes of \( K \). Then there is a semialgebraic retraction \( r : |K| \to |\text{co}(K)| \) such that \( (1-t)x + t \cdot r(x) \in |K| \) for all \( (x,t) \in |K| \times I \) and hence \( H : |K| \times I \to |K| : (x,t) \mapsto (1-t)x + t \cdot r(x) \) is a canonical semialgebraic strong deformation retraction.

**Theorem 3.8.** Let \( X \subset \mathbb{R}^n \) be a semialgebraic set. Then \( \text{cat}(X)^{\text{sa}} = \text{cat}(X) \), where \( \text{cat}(X) \) denotes the classical LS-category of \( X \).

**Proof.** Clearly, \( \text{cat}(X)^{\text{sa}} \geq \text{cat}(X) \). We show that \( \text{cat}(X)^{\text{sa}} \leq \text{cat}(X) \). We can assume that \( X = |K| \) for some simplicial complex \( K \). Moreover, since strong deformation retracts are homotopy equivalences, by Fact 3.7 and Fact 3.4 we can assume that \( K \) is closed. Let \( \{U_i\}_{i=1}^{m+1} \) be a categorical open cover of \( |K| \). We will construct a semialgebraic categorical open cover \( \{V_i\}_{i=1}^{m+1} \) of \( |K| \). Firstly, by the shrinking lemma we can assume that each \( U_i \) is contractible in \( |K| \). Furthermore, by the Lebesgue's number lemma we can also assume that for each \( \sigma \in K \) there is \( i \in \{1, \ldots, m+1\} \) such that \( \sigma \subset U_i \). We define \( \mathcal{F}_i := \{\sigma \in K : \sigma \subset U_i\} \) and

\[
A_i := \bigcup_{\sigma \in \mathcal{F}_i} \sigma
\]

for each \( i = 1, \ldots, m+1 \). Note that (i) \( |K| = A_1 \cup \cdots \cup A_{m+1} \) and (ii) each \( \overline{A}_i \) is contractible in \( |K| \). On the other hand, each \( \overline{A}_i \) is a semialgebraic strong deformation retract of the open semialgebraic set \( V_i := \text{St}_{|K^r|}(\overline{A}_i) \),
where $K'$ is the first barycentric subdivision of $K$ (see [9, Prop III.1.6]). Therefore, by (ii), $V_i$ is (not necessarily semialgebraically) contractible in $|K|$. Now, by Fact 2.2(c) if a semialgebraic subset $A$ of a semialgebraic set $B$ is contractible in $B$, then it is semialgebraically contractible in $B$. Hence, each $V_i$ is semialgebraically contractible in $|K|$ and hence, by (i), $\{V_i\}_{i=1}^{m+1}$ is a semialgebraic categorical open cover of $|K|$. We deduce that $\text{cat}(X)^{\mathbb{R}\text{sa}} \leq \text{cat}(X)$, as required. 

**Corollary 3.9.** The o-minimal LS-category is invariant under elementary extensions and o-minimal expansions.

**Proof.** This follows from Theorem 3.5 and Theorem 3.6.

**Proof of Theorem 1.1.** We denote by $\mathbb{Q}$ the real algebraic numbers. It follows from Corollary 3.9 and Theorem 3.8 that $\text{cat}(|K|)^{\mathbb{R}\text{sa}} = \text{cat}(X)^{\mathbb{R}\text{sa}} = \text{cat}(X^{\mathbb{R}}) = \text{cat}(X)$. We deduce that $\text{cat}(X)^{\mathbb{R}\text{sa}} \leq \text{cat}(X)$, as required. 

**Corollary 3.10.** Let $X$ be a definably connected definable set. Then

$$\text{cat}(X)^{\mathbb{R}} \leq \dim(X).$$

**Proof.** By the Triangulation theorem, Fact 3.7 and 3.4, we can assume that $X = |K|$ for a closed simplicial complex $K$ whose vertices lie in the real algebraic numbers $\mathbb{Q}$. Now, it follows from Theorem 1.1 that $\text{cat}(|K|)^{\mathbb{R}} = \text{cat}(X^{\mathbb{R}})$. By the classical version of Corollary 3.10 (see [8, Thm. 1.7]),

$$\text{cat}(|K|)^{\mathbb{R}} \leq \dim(|K|)^{\text{top}},$$

where $\dim(|K|)^{\text{top}}$ denotes the covering dimension of $|K|$. On the other hand, since $K$ is a simplicial complex, $\dim(|K|)^{\text{top}}$ is exactly the dimension of $K$ as a simplicial complex, i.e., $\dim(|K|)^{\text{top}} = \dim(|K|)$, the latter being also the o-minimal dimension, as required.

**Corollary 3.11.** Let $X$ be a definable set and let $n \geq 1$ such that $\pi_r(X)^{\mathbb{R}} = 0$ for all $r = 0, \ldots, n-1$. Then $\text{cat}(X)^{\mathbb{R}} \leq \dim(X)/n$.

**Proof.** Similar to the proof of Corollary 3.10 using the corresponding classical statement to that of Corollary 3.11 (see [8, Thm. 1.50]).

**Corollary 3.12.** Let $X$ be a definable set. Let $\text{cuplength}_Q(X)^{\mathbb{R}}$ be the least integer $k$ such that all $(k+1)$-fold cup products vanish in the reduced cohomology $\widetilde{H}^*(X; \mathbb{Q})^{\mathbb{R}}$. Then $\text{cat}(X)^{\mathbb{R}} \geq \text{cuplength}_Q(X)^{\mathbb{R}}$.

**Proof.** As before, this follows from Theorem 1.1, the o-minimal cohomological theory developed in [11] and the corresponding classical statement to that of Corollary 3.12 (see [8, Thm. 1.50]).
4 Homotopy types of definable groups

In this section we assume that $\mathcal{R}$ is sufficiently saturated. In accordance with the notation used in the introduction, we shall denote by $\mathbb{L}$ the functor

$$\mathbb{L}: \{\text{Definably compact definable groups}\} \to \{\text{Compact Lie groups}\}$$

$$G \mapsto G/G^0.$$

which maps definable homomorphisms to continuous homomorphisms (see [3, Thm.5.2]). We collect here some properties of the functor $\mathbb{L}$ studied in [3] which will be used in the sequel without mention.

**Fact 4.1.** Let $G$ and $H$ be definably compact definable groups and let $\pi : G \to \mathbb{L}(G)$ be the projection.

(i) The functor $\mathbb{L}$ is exact.

(ii) If $H$ is a definable subgroup of $G$ then $\pi(H) = \mathbb{L}(H)$.

(iii) $\mathbb{L}(G^0) = \mathbb{L}(G)^0$.

(iv) $\mathbb{L}(G \times H) = \mathbb{L}(G) \times \mathbb{L}(H)$.

**Proof.** (i) and (ii) can be found in [3, Thm.5.2] and [3, Thm.4.4] respectively. (iii) follows from (i) and (iii) follows from [3, Cor.4.7].

As we pointed out in the introduction, the aim of this section is to prove Theorem 1.2. To do this we first prove the following. Recall that given a definable group $G$ the commutator subgroup $G' := [G, G]$ might not be definable. However, if $G$ is definably compact definably connected then $G'$ is a definably connected definable subgroup of $G$ (see below and [13, Cor.6.4]).

**Theorem 4.2.** Let $G$ be a definably connected definably compact definable group. Then $Z(G)^0 \times G'$ is definable homotopy equivalent to $G$.

The latter is motivated by the following classical result concerning compact Lie groups proved by A. Borel. We include the proof for completeness.

**Fact 4.3.** [7, Prop.3.1] Let $G$ be a compact, connected Lie group. Then $G$ is homeomorphic to the topological direct product of its commutator subgroup $G'$ and the connected component $Z(G)^0$ of its center.

**Proof.** Firstly, note that $G = Z(G)^0 G'$ and $Z(G)^0 \cap G'$ is finite. Let $n$ be the dimension of $Z(G)^0$. Consider a connected $(n - 1)$-dimensional Lie subgroup $Z_1$ of the torus $Z(G)^0$. Then, the projection $G \to G/Z_1 G'$ is a principal bundle with a circle as its base and with the connected Lie group $Z_1 G'$ as its fiber. On the other hand, the equivalence classes of principal bundles over an sphere $S^n$ with an arcwise connected group $Z_1 G'$ as its fiber is in 1-1 correspondence with $\pi_{n-1}(Z_1 G')$ (see [16, Cor.18.6]). Hence, since $\pi_0(Z_1 G') = 0$, the bundle $G \to G/Z_1 G'$ is equivalent to the trivial one, so that $G$ is homeomorphic to $G/Z_1 G' \times Z_1 G'$. By recurrence we get that $G$ is homeomorphic to $Z(G)^0 \times G'$.

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Remark 4.4. The proof of Fact 4.3 does not apply in the o-minimal setting. For, even though we could prove an o-minimal version of the bundle classification used above, there are definably connected definably compact abelian definable groups without any proper infinite definable subgroup (see [14, §5]). Anyway, we conjecture that every definably connected definably compact definable group $G$ is definably homeomorphic to $Z(G)^0 \times G'$.

On the other hand, to prove Theorem 4.2 we will use the following structural result about definably compact groups recently proved by E. Hrushovski, A. Pillay and Y. Peterzil. Recall that a definable group $G$ is semisimple if and only if it has no infinite abelian normal (definable) subgroup.

Fact 4.5. [13, Cor.6.4] Let $G$ be a definably connected definably compact group. Then $G' := [G, G]$ is definable, definably connected and semisimple. Moreover, the map $p : Z(G)^0 \times G' \to G : (g, h) \mapsto gh$ is a homomorphism with finite kernel.

The following is an immediate consequence of Fact 4.5.

Lemma 4.6. Let $G$ be a definably connected, definably compact definable group. Then $L(G') = L(G)'$ and $L(Z(G)^0) = Z(L(G))^0$.

Proof. The projection $\pi : G \to L(G)$ is a surjective homomorphism. Hence, $\pi(G') = \pi(G)'$. Since $\pi(G') = L(G')$, we deduce that $L(G') = L(G)'$. Now, we show that $L(Z(G)^0) = Z(L(G))^0$. Since $\pi$ is a surjective homomorphism, $L(Z(G)) = \pi(Z(G)) < Z(\pi(G)) = Z(L(G))$. Then,

$$L(Z(G)^0) = L(Z(G))^0 < Z(L(G))^0.$$  

Hence, it is enough to prove that $\dim(L(Z(G)^0)) = \dim(Z(L(G))^0)$. By Fact 4.5, $\dim(G) = \dim(Z(G)^0 \times G') = \dim(Z(G)^0) + \dim(G')$ and hence we have $\dim(L(G)) = \dim(L(Z(G)^0)) + \dim(L(G'))$. On the other hand, it follows from Fact 4.3 that $\dim(L(G)) = \dim(Z(L(G))^0) + \dim(L(G'))$. Since $\dim(L(G')) = \dim(L(G'))$, we deduce that $\dim(L(Z(G)^0)) = \dim(Z(L(G))^0)$, as required. \hfill $\square$

Corollary 4.7. Let $G$ be a definably connected, definably compact definable group. Then $\pi_n(G) \cong \pi_n(Z(G)^0 \times G')$ for all $n \geq 1$.

Proof. By Lemma 4.6, Fact 2.3 and Fact 4.3 we have the following isomorphisms

$$\pi_n(G) \cong \pi_n(L(G)) \cong \pi_n(Z(L(G))^0) \times \pi_n(L(G')) \cong \pi_n(Z(L(G))^0) \times \pi_n(L(G')) \cong \pi_n(Z(G)^0) \times \pi_n(Z(G')) \cong \pi_n(Z(G)^0) \times \pi_n(G')$$

as required. \hfill $\square$
Note that the definable homomorphism $p : Z(G)^0 \times G' \to G : (g, h) \mapsto gh$ from Fact 4.5 is a definable homotopy equivalence if and only if $p$ is an isomorphism. Indeed, if $p$ is a definable homotopy equivalence then $p_* : \pi_1(Z(G)^0 \times G') \to \pi_1(G')$ is an isomorphism. On the other hand, since $p$ is a surjective homomorphism with finite kernel we have that $p$ is a definable covering (see [11, Prop.2.11]). Hence, it follows from [11, Prop.2.9] that the cardinality of $p^{-1}(e)$ equals the one of $\pi_1(G')/p_*\pi_1(Z(G)^0 \times G'))$, so that $p^{-1}(e)$ is trivial. Then, $p$ is an isomorphism, as required.

Even though $p$ above is not necessarily a definable homotopy equivalence, we now prove that $Z(G)^0 \times G'$ and $G$ are actually definable homotopy equivalent.

Proof of Theorem 4.2. Let $d = \dim(Z(G)^0)$. Note that by Lemma 4.6 we also have $d = \dim(Z(L(G))^0)$. It suffices to prove that $G$ is definable homotopy equivalent to

$$T_R^d \times G',$$

where $T_R^d$ is the $d$-dimensional torus defined as the subset $[0, 1)^d$ of $R^d$ with the sum operation modulo 1. Indeed, $Z(G)^0$ is definable homotopy equivalent to $T_R^d$ (see [5, Cor.4.4]).

Firstly, we show that $\pi_1(G) \cong Z^d \times \text{Tor}(\pi_1(G))$. For, it follows from the proof of Corollary 4.7 that $\pi_1(G) \cong \pi_1(Z(L(G)^0)) \times \pi_1(L(G)^0) \cong Z^d \times \pi_1(L(G)^0)$. Moreover, $L(G)^0$ is a semisimple compact Lie group and hence $\pi_1(L(G)^0)$ is finite, so that $\pi_1(L(G)^0) \cong \text{Tor}(\pi_1(G))$, as required. In particular, since $\pi_1(G^0) \cong \pi_1(L(G)^0)$ (see Fact 2.2), we have proved that $\pi_1(G^0)$ and $\text{Tor}(\pi_1(G))$ are isomorphic finite groups.

Now, take $\gamma_1, \ldots, \gamma_d : I \to G$ definable curves such that

$$[\gamma_1] + \text{Tor}(\pi_1(G)), \ldots, [\gamma_d] + \text{Tor}(\pi_1(G)),$$

freely generate the group $\pi_1(G)/\text{Tor}(\pi_1(G)) \cong Z^d$. Consider the definable map,

$$f : T_R^d \times G' \to G : (t_1, \ldots, t_d, g) \mapsto \gamma_1(t_1) \cdots \gamma_d(t_d)g.$$

We show that $f$ is a definable homotopy equivalence. By the o-minimal Whitehead theorem (see [2, Thm.5.6]) it suffices to prove that the homomorphism $f_* : \pi_n(T_R^d \times G') \to \pi_n(G)$ is actually an isomorphism for all $n \geq 1$. Consider the definable maps $i : G' \to G : g \mapsto g$ and $j : T_R^d \to G : (t_1, \ldots, t_d) \mapsto \gamma_1(t_1) \cdots \gamma_d(t_d)$. Since $G$ is a definable group, we can regard $f_*$ as the homomorphism

$$f_* : \pi_n(T_R^d) \times \pi_n(G') \to \pi_n(G) : (x, y) \mapsto j_*(x) + i_*(y).$$

Claim: The homomorphism $i_* : \pi_n(G') \to \pi_n(G)$ is an isomorphism for every $n \geq 2$ and injective for $n = 1$. 


Granted the claim and since \( \pi_n(\mathbb{T}_R^d) = \pi_n(\mathbb{T}_R^d) = 0 \) for all \( n \geq 2 \) (see Fact 2.2), we deduce that \( f_* = i_* \) is an isomorphism for all \( n \geq 2 \). To finish the proof we have to show that \( f_* \) is an isomorphism for \( n = 1 \). Firstly, note that by definition of the \( \gamma \)'s we have that the \( \gamma \)'s are \( \mathbb{Z} \)-linear independent and hence \( j_* \) is injective. In particular, \( j_*(\pi_1(\mathbb{T}_R^d)) \) is torsion free. On the other hand, it follows from the claim that \( i_* \) is injective. Hence, \( i_*(\pi_1(G')) \) is a finite subgroup of \( \pi_1(G) \) with the cardinality of \( \mathrm{Tor}(\pi_1(G)) \), so that \( i_*(\pi_1(G')) = \mathrm{Tor}(\pi_1(G)) \). Therefore, \( j_*(\pi_1(\mathbb{T}_R^d)) \cap i_*(\pi_1(G')) \) is trivial. We deduce that \( f_* \) is injective. Finally, by definition of the definable curves \( \gamma \)'s we have that \( \mathrm{Im}(f_*)/\mathrm{Tor}(\pi_1(G)) = \pi_1(G)/\mathrm{Tor}(\pi_1(G)) \) and hence \( \mathrm{Im}(f_*) = \pi_1(G) \), as required.

**Proof of the Claim.** By Fact 4.5 the kernel of \( p : Z(G)^0 \times G' \to G : (h, g) \mapsto h \cdot g \) is finite and therefore \( p \) is a definable covering (see [11, Prop.2.11]). Then \( p \) is a definable fibration (see [12, Thm.4.10]) and hence, by the definable fibration property (see [12, Cor.4.11]), we have that \( p_* \) is an isomorphism for all \( n \geq 2 \) and injective for \( n = 1 \). Since \( Z(G)^0 \) is a definably compact abelian definable group, \( \pi_1(Z(G)^0) = 0 \) for all \( n \geq 2 \) (see [5, Cor.3.3]). We deduce that \( i_* = p_* \) is an isomorphism for all \( n \geq 2 \). On the other hand, the inclusion \( k : G' \to Z(G)^0 \times G' \) induces and injective map \( k_* : \pi_1(G') \to \pi_1(Z(G)^0 \times G') \) and hence \( i_* = p_* \circ k_* \) is injective for \( n = 1 \).

As a consequence of Theorem 4.2 we obtain the following general result (compare with the discussion preceding the proof of Theorem 4.2 above).

**Corollary 4.8.** Let \( H \) and \( G \) be definably connected definably compact definable groups such that \( \pi_1(G) \cong \pi_1(H) \). If there exists a surjective homomorphism \( p : G \to H \) with finite kernel then \( G \) and \( H \) are definable homotopy equivalent.

**Proof.** Firstly, it follows from Corollary 4.4 that \( \pi_1(G) \cong \mathbb{Z}^n \times \pi_1(G') \) and \( \pi_1(H) \cong \mathbb{Z}^m \times \pi_1(H') \), where \( n = \dim(Z(G)^0) \) and \( m = \dim(Z(H)^0) \). Moreover, since \( G' \) and \( H' \) are definably compact and semisimple, both \( \pi_1(G') \) and \( \pi_1(H') \) are finite. Now, since \( \pi_1(G) \cong \pi_1(H) \), we deduce that \( n = m \) and the cardinality of \( \pi_1(G') \) and \( \pi_1(H') \) are equal.

By the proof of Theorem 4.2 we have that \( G \) and \( H \) are definably homotopy equivalent to \( \mathbb{T}^n \times G' \) and \( \mathbb{T}^m \times H' \) respectively. Since \( n = m \), it is enough to prove that \( G' \) and \( H' \) are definable homotopy equivalent. Actually, we show that \( p|_{G'} : G' \to H' \) is an isomorphism. Since \( p|_{G'} \) is a surjective homomorphism with finite kernel we have that \( p|_{G'} \) is a definable covering homomorphism. In particular, \( (p|_{G'})_* : \pi_1(G') \to \pi_1(H') \) is injective (see [11, Cor.2.8]). Moreover, since \( \pi_1(G') \) and \( \pi_1(H') \) are finite groups with the same cardinality, we deduce that \( (p|_{G'})_*(\pi_1(G')) = \pi_1(H') \). On the other hand, the cardinality of \( (p|_{G'})^{-1}(e) \) equals the one of \( \pi_1(H')/(p|_{G'})_*(\pi_1(G')) \), so that \( p|_{G'} \) is an isomorphism.

We are now ready to prove Theorem 4.2.
Proof of the Theorem 1.2. By Fact 4.5, $G'$ is a definably connected definably compact semisimple definable group and hence, by a result of M. Edmundo, G. Jones and N. Peatfield, $G'$ has a very good reduction, i.e., there is a semialgebraic group $H$ defined without parameters such that $G$ is definably isomorphic to $H$ (see, e.g., [13, Thm.4.4]). By Lemma 4.6 we have that $d := \dim (Z(G)^0) = \dim (Z(L(G))^0)$. We show that both $G(\mathbb{R})$ and $L(G)$ are homotopy equivalent to

$$
\mathbb{T}_d^d \times H(\mathbb{R}),
$$

where $\mathbb{T}_d^d$ is the $d$-dimensional torus defined as the subset $[0, 1]^d$ of $\mathbb{R}^d$ with the sum operation modulo 1. First, we prove that $L(G)$ is homotopy equivalent to $\mathbb{T}_d^d \times H(\mathbb{R})$. For, by Lemma 4.6 and [3, Thm.1.6] we have $L(G)' = L(G') \cong H(\mathbb{R})$ and hence $Z(L(G))^0 \times L(G)'$ is isomorphic to $\mathbb{T}_d^d \times H(\mathbb{R})$. Then, by Fact 4.3, $L(G)$ is homotopy equivalent (actually homeomorphic) to $\mathbb{T}_d^d \times H(\mathbb{R})$. On the other hand, by the proof of Theorem 4.2, $G$ is definable homotopy equivalent to $\mathbb{T}_d^d \times H$. Now, since both $G$ and $\mathbb{T}_d^d \times H$ are semialgebraic sets defined without parameters, it follows from Fact 2.2 that $G$ is semialgebraic homotopy equivalent to $\mathbb{T}_d^d \times H$ without parameters. In particular, $G(\mathbb{R})$ is semialgebraic homotopy equivalent to $\mathbb{T}_d^d \times H(\mathbb{R})$ without parameters, as required.

Proof of Corollary 1.3. By the triangulation theorem we can assume that the underlying sets of both $G$ and $H$ are semialgebraic without parameters. By Theorem 1.2, $L(G)$ and $L(H)$ are homotopy equivalent to $G(\mathbb{R})$ and $H(\mathbb{R})$ respectively. Now, if $G$ and $H$ are definable homotopy equivalent then $G$ and $H$ are semialgebraic homotopy equivalent without parameters (see Fact 2.2). Hence $G(\mathbb{R})$ and $H(\mathbb{R})$ are (semialgebraic) homotopy equivalent (without parameters), so that $L(G)$ and $L(H)$ are homotopy equivalent. On the other hand, if $L(G)$ and $L(H)$ are homotopy equivalent then $G(\mathbb{R})$ and $H(\mathbb{R})$ are homotopy equivalent. Hence, by Fact 2.2(c) we have that $G(\mathbb{R})$ and $H(\mathbb{R})$ are semialgebraic homotopy equivalent without parameters, so that $G$ and $H$ are semialgebraic homotopy equivalent (without parameters).

Remark 4.9. Theorem 1.2 allows us to extend the functor $L$ to definable maps up to homotopy. That is, given two definably connected definably compact definable groups $G$ and $H$, consider the o-minimal homotopy set $[G, H]$ and the homotopy set $[L(G), L(H)]$ (see the definition after Fact 2.1). We define a map $\bar{L} : [G, H] \to [L(G), L(H)]$ as follows. Let $f : G \to H$ be a definable map. We can assume that the underlying sets of both $G$ and $H$ are semialgebraic without parameters. Then, by Fact 2.2 the map $f$ is definably homotopic to a semialgebraic map $g : G \to H$ defined without parameters. On the other hand, by Theorem 1.2 there are $\phi_G : L(G) \to G(\mathbb{R})$ and $\psi_H : H(\mathbb{R}) \to L(H)$ definable homotopy equivalences. Finally, we define $\bar{L}([f]) := [\psi_H \circ g(\mathbb{R}) \circ \phi_G]$. Note that $\bar{L}$ is well-defined since it depends neither on the choice of $\phi_G$, $\psi_H$ and $g$ nor the representant of $[f]$. 

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Corollary 4.10. Let $G$ be a definably connected definably compact definable group. Then $\text{cat}(G)^R = \text{cat}(L(G))$.

Proof. By the triangulation theorem we can assume that the underlying set of $G$ is semialgebraic without parameters. By Theorem 4.2, $G(\mathbb{R})$ is homotopy equivalent to $L(G)$ and hence $\text{cat}(G(\mathbb{R})) = \text{cat}(L(G))$. Then, it follows from Theorem 4.4 that $\text{cat}(G)^R = \text{cat}(G(\mathbb{R})) = \text{cat}(L(G))$, as required.

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