Filtration and canonical completeness for continuous modal $\mu$-calculi

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joint work with

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The modal $\mu$-calculus
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\[(\text{ML}) \quad \varphi ::= p \mid \neg p \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid \Diamond \varphi \mid \Box \varphi, \quad p \in P\]
The modal $\mu$-calculus

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Given a formula $\varphi \in ML$ and a variable $x \in P$, we may regard $x$ as a free variable of $\varphi$. For every Kripke model $S = (S, R, V)$, this induces a function:

$$\varphi^S_x : \mathcal{P}(S) \rightarrow \mathcal{P}(S) \text{ given by } \varphi^S_x(A) := \llbracket \varphi \rrbracket^S_{x \mapsto A}$$
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Observation

If $x$ occurs only positively in $\varphi$, then $\varphi^S_x$ is monotone and so, by the Knaster-Tarski theorem, it has both a least and a greatest fixpoint.
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\[(\mu ML) \quad \varphi ::= p \mid \neg p \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid \Diamond \varphi \mid \Box \varphi \mid \mu x \psi \mid \nu x \psi,\]

where $p \in P$ and $x$ occurs only positively in $\psi$. 
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(ML) $\varphi ::= p \mid \neg p \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid \diamond \varphi \mid \Box \varphi, \ p \in P$

Given a formula $\varphi \in \text{ML}$ and a variable $x \in P$, we may regard $x$ as a free variable of $\varphi$. For every Kripke model $S = (S, R, V)$, this induces a function:

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where $p \in P$ and $x$ occurs only positively in $\psi$.

$$\llbracket \mu x \varphi \rrbracket^S_x := \text{LFP}(\varphi^S_x) \quad \llbracket \nu x \varphi \rrbracket^S_x := \text{GFP}(\varphi^S_x)$$
Evaluation game

The evaluation game $E(\xi, S)$ takes positions in $S_f(\xi) \times S$ and has the following ownership function and admissible moves.

Position | Player | Admissible moves
--- | --- | ---
$(\phi_1 \lor \phi_2, s)$ | $\exists \{ (\phi_1, s), (\phi_2, s) \}$ | $(\phi_1 \land \phi_2, s)$ | $\forall \{ (\phi_1, s), (\phi_2, s) \}$ | $(\Box \phi, s)$ | $\exists \{ (\phi, t) : s R t \}$ | $(\Box \phi, s)$ | $\forall \{ (\phi, t) : s R t \}$

$(\eta x. \delta x, s)$ with $x \in BV(\xi) - \{ (\delta x, s) \}$

$(p, s)$ with $p \in FV(\xi)$ and $s \in V(p)$

$\forall \emptyset (\neg p, s)$ with $p \in FV(\xi)$ and $s \in V(p)$

$\exists \emptyset (p, s)$ with $p \in FV(\xi)$ and $s \in V(p)$

$\forall \emptyset (\neg p, s)$ with $p \in FV(\xi)$ and $s \in V(p)$

An infinite match is won by $\exists (\forall)$ if the 'most important' fixpoint variable reached infinitely often is a $\nu$-variable (a $\mu$-variable).

Example: $\mu x \Box x$ is true at a state $s_0$ iff there is no infinite path starting at $s_0$.

$(\mu x \Box x, s_0) \rightarrow (\Box x, s_0) \rightarrow (x, s_1) \rightarrow (\Box x, s_1) \rightarrow (x, s_2) \rightarrow \cdots$
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| $(\varphi_1 \lor \varphi_2, s)$ | $\exists$ | $\{(\varphi_1, s), (\varphi_2, s)\}$ |
| $(\varphi_1 \land \varphi_2, s)$ | $\forall$ | $\{(\varphi_1, s), (\varphi_2, s)\}$ |
| $(\Diamond \varphi, s)$          | $\exists$ | $\{(\varphi, t) : sRt\}$      |
| $(\Box \varphi, s)$             | $\forall$ | $\{(\varphi, t) : sRt\}$      |
| $(\eta x . \delta x, s)$        | $\exists$ | $\{(\delta x, s)\}$           |
| $(x, s)$ with $x \in BV(\xi)$   | $\forall$ | $\emptyset$                    |
| $(p, s)$ with $p \in FV(\xi)$ and $s \in V(p)$ | $\exists$ | $\emptyset$                    |
| $(-p, s)$ with $p \in FV(\xi)$ and $s \in V(p)$ | $\forall$ | $\emptyset$                    |
| $(p, s)$ with $p \in FV(\xi)$ and $s \notin V(p)$ | $\exists$ | $\emptyset$                    |
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| $(\varphi_1 \land \varphi_2, s)$           | $\forall$ | $\{((\varphi_1, s), (\varphi_2, s))\}$ |
| $(\square \varphi, s)$                      | $\exists$ | $\{((\varphi, t) : sRt)\}$ |
| $(\square \varphi, s)$                      | $\forall$ | $\{((\varphi, t) : sRt)\}$ |
| $(\eta x. \delta_x, s)$                    | $\exists$ | $\{(\delta_x, s)\}$ |
| $(x, s)$ with $x \in BV(\xi)$              | $\forall$ | $\emptyset$ |
| $(p, s)$ with $p \in FV(\xi)$ and $s \in V(p)$ | $\forall$ | $\emptyset$ |
| $(\neg p, s)$ with $p \in FV(\xi)$ and $s \in V(p)$ | $\exists$ | $\emptyset$ |
| $(p, s)$ with $p \in FV(\xi)$ and $s \notin V(p)$ | $\exists$ | $\emptyset$ |
| $(\neg p, s)$ with $p \in FV(\xi)$ and $s \notin V(p)$ | $\forall$ | $\emptyset$ |

An infinite match is won by $\exists (\forall)$ if the ‘most important’ fixpoint variable reached infinitely often is a $\nu$-variable (a $\mu$-variable).

Example: $\mu x \square x$ is true at a state $s_0$ iff there is no infinite path starting at $s_0$.

$$(\mu x \square x, s_0) \rightarrow (\square x, s_0) \rightarrow (x, s_1) \rightarrow (\square x, s_1) \rightarrow (x, s_2) \rightarrow \cdots$$
Motivation of the paper

The modal $\mu$-calculus is highly expressive, yet retains many of the desirable properties of basic modal logic, e.g. bisimulation invariance and the finite model property.

However, two important methods fail: (i) filtration and (ii) canonical models.

Both of these methods are well-known to work for PDL.

Question

Can we do better? That is, is there a natural fragment of $\mu$ML that subsumes PDL and to which the methods of filtration and canonical models can be applied?

Our answer (very roughly)

Yes, namely the continuous modal $\mu$-calculus.
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Yes, namely the continuous modal $\mu$-calculus.
Filtration

Let $S = (S, R, V)$ be a Kripke model and let $\Sigma$ be a finite and closed set of formulas. Let $\sim_S^\Sigma$ be the equivalence relation given by:

$s \sim_S^\Sigma s' \iff s, s' \in \\left[ [\phi] \right]_S \ \iff \ s' \sim_S^\Sigma t'$

for all $\phi \in \Sigma$. And define $S := S/\sim_S^\Sigma$.

Pick any relation $R \subseteq S \times S$ such that $R_{\min} \subseteq R \subseteq R_{\max}$, where

$R_{\min} := \{ (s, t) : \text{there are } s', t' \sim_S^\Sigma s, t \text{ such that } Rs' t' \} \ \quad \quad \quad R_{\max} := \{ (s, t) : \text{for all } \Box \phi \in \Sigma; \text{ if } s \models \Box \phi, \text{ then } t \models \phi \}.$

Finally, let $V(p) := \{ s : s \models p \}$ for every $p \in \Sigma \cap P$. Then the model $S := (S, R, V)$ is called a filtration of $S$ through $\Sigma$. 
Filtration

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Let $\mathcal{S} = (S, R, V)$ be a Kripke model and let $\Sigma$ be a finite and closed set of formulas.

Let $\sim_{\Sigma}^{\mathcal{S}}$ be the equivalence relation given by:

$$s \sim_{\Sigma}^{\mathcal{S}} s' \text{ if and only if } s \in [\varphi]^{\mathcal{S}} \iff s' \in [\varphi]^{\mathcal{S}} \text{ for all } \varphi \in \Sigma.$$ 

and define $\overline{\mathcal{S}} := \mathcal{S}/\sim_{\Sigma}^{\mathcal{S}}$. 

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Let $S = (S, R, V)$ be a Kripke model and let $\Sigma$ be a finite and closed set of formulas.

Let $\sim^S_\Sigma$ be the equivalence relation given by:

$$s \sim^S_\Sigma s' \text{ if and only if } s \models [\varphi]^S \iff s' \models [\varphi]^S \text{ for all } \varphi \in \Sigma.$$ 

and define $\overline{S} := S/\sim^S_\Sigma$.

Pick any relation $\overline{R} \subseteq \overline{S} \times \overline{S}$ such that $R^\text{min} \subseteq \overline{R} \subseteq R^\text{max}$, where

$$R^\text{min} := \{(\overline{s}, \overline{t}) : \text{there are } s' \sim^S_\Sigma s \text{ and } t' \sim^S_\Sigma t \text{ such that } Rs't'}\},$$

$$R^\text{max} := \{(\overline{s}, \overline{t}) : \text{for all } \Box \varphi \in \Sigma; \text{ if } s \models \Box \varphi, \text{ then } t \models \varphi}\}.$$
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Filtration

Let $S = (S, R, V)$ be a Kripke model and let $\Sigma$ be a finite and closed set of formulas.

Let $\sim^S_\Sigma$ be the equivalence relation given by:

$$s \sim^S_\Sigma s' \text{ if and only if } s \in \downarrow \varphi \iff s' \in \downarrow \varphi \text{ for all } \varphi \in \Sigma.$$ 

and define $\overline{S} := S/\sim^S_\Sigma$.

Pick any relation $\overline{R} \subseteq \overline{S} \times \overline{S}$ such that $R^{\text{min}} \subseteq \overline{R} \subseteq R^{\text{max}}$, where

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Finally, let $\overline{V}(p) := \{\overline{s} : s \models p\}$ for every $p \in \Sigma \cap P$.

Then the model $\overline{S} := (\overline{S}, \overline{R}, \overline{V})$ is called a filtration of $S$ through $\Sigma$. 
Filtration (ii)

The Filtration Theorem holds for a modal language $\mathcal{D}$ if for any finite and closed set $\Sigma$ of $\mathcal{D}$-formulas and any filtration $S$ through $\Sigma$ we have:

\[ s \in [\[ \phi \]\]_S \iff s \in [\[ \phi \]\]_{S'} \]

for every $\phi \in \Sigma$.

The Filtration Theorem holds for ML, for PDL, but not for $\mu$-ML:

Consider the formula $\phi := \mu x \Box x$ and the model $S := (N, <, V)$:

\[ 0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \]

with transitive arrows.
The Filtration Theorem holds for a modal language $D$ if for any finite and closed set $\Sigma$ of $D$-formulas and any filtration $\bar{S}$ of $S$ through $\Sigma$ we have:

$$\bar{s} \in \llbracket \varphi \rrbracket^{\bar{S}} \iff s \in \llbracket \varphi \rrbracket^{S}$$

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Consider the formula $\varphi := \mu x \Box x$ and the model $S := (\mathbb{N}, <, V)$:

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$+$ transitive arrows
The continuous modal $\mu$-calculus

Idea: restrict the use of the least and greatest fixpoint operators to (formulas that induce) functions that are Scott continuous, rather than merely monotone.

Fontaine (2008) proves the following syntactic characterisation:

$\phi ::= x \mid \alpha \mid \phi \lor \phi \mid \phi \land \phi \mid \Box \phi \mid \mu y \phi'$

where $x \in X$, $y \in P$, $\alpha \in \mu_{\text{c}}_\text{ML} \text{X-free}$, and $\phi' \in \text{Con}_X \cup \{y\}$ ($\mu_{\text{c}}_\text{ML}$).

Roughly: under a $\mu$ we disallow $\Box$ and $\nu$ and, dually, under a $\nu$ we disallow $\Box$ and $\mu$. 
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Roughly: under a $\mu$ we disallow $\Box$ and $\nu$ and, dually, under a $\nu$ we disallow $\Diamond$ and $\mu$. 
Properties of \( \mu_c \text{ML} \)

Constructive: fixpoints are reached after at most \( \omega \) iterations.

Strictly more expressive than PDL.

Properties of the evaluation game played with \( \mu_c \text{ML} \)-formulas:

1. A match progresses at most finitely often from a position \((s, \eta x. \delta)\) to a position \((t, \eta y. \theta)\).

2. A match progresses at most finitely often from a position \((s, \mu x. \delta)\) to a position \((t, \Box \psi)\).
Properties of $\mu_{c}ML$

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Properties of $\mu_cML$

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Properties of the evaluation game played with $\mu_c$ML-formulas:

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2. A match progresses at most finitely often from a position $(s, \mu x.\delta)$ to a position $(t, \Box \psi)$. 
Theorem (Filtration Theorem for $\mu_c$ML)

For any finite and closed set $\Sigma$ of $\mu_c$ML-formulas and any filtration $\overline{S}$ of $S$ through $\Sigma$ it holds that:

$$\overline{s} \in \llbracket \varphi \rrbracket ^{\overline{S}} \iff s \in \llbracket \varphi \rrbracket ^{S}$$

for every $\varphi \in \Sigma$. 

Proof sketch.

Suppose $\exists$ has a winning strategy $f$ for $G$ at $(\varphi, s)$; we must show that she has a winning strategy for $G$ at $(\varphi, s)$. We play a shadow match, copying in $G$ the moves suggested to $\exists$ by the strategy $f$ in $G$, and simulating in $G$ the moves played by $\forall$ in $G$. Note: at each position $(s, \Box \varphi)$ we must reset the shadow match. However, if the obtained strategy would be losing for $\exists$ this reset could happen only finitely often, contradicting the assumption that $f$ is winning.
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Other results

Definition
A class of models $\mathcal{M}$ is said to admit filtration with respect to a language $D$ if for every model $S$ in $\mathcal{M}$ and every finite closed set of $D$-formulas $\Sigma$, the class $\mathcal{M}$ contains a filtration of $S$ through $\Sigma$. A class of frames $\mathcal{F}$ is said to admit filtration if the class of models $\{(S, R, V) : (S, R) \in \mathcal{F}\}$ does.
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Lemma
For any logic $L$, the class $\text{Mod}(L)$ admits filtration wrt $\mu^cML$ iff it admits filtration wrt $\mu^cML$. 
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Lemma
For any logic $L$, the class $\text{Mod}(L)$ admits filtration wrt $\text{ML}$ iff it admits filtration wrt $\mu_c\text{ML}$.

Corollary (Finite Model Property)
Let $L$ be a logic such that $\text{Mod}(L)$ admits filtration with respect to $\text{ML}$, and let $\phi$ be a formula of the continuous $\mu$-calculus. Then $\phi$ is valid in every $L$-model if and only if $\phi$ is valid in every finite $L$-model.
Other results

Definition
A class of models $\mathcal{M}$ is said to admit filtration with respect to a language $D$ if for every model $S$ in $\mathcal{M}$ and every finite closed set of $D$-formulas $\Sigma$, the class $\mathcal{M}$ contains a filtration of $S$ through $\Sigma$. A class of frames $\mathcal{F}$ is said to admit filtration if the class of models $\{(S, R, V) : (S, R) \in \mathcal{F}\}$ does.

Lemma
For any logic $L$, the class $\text{Mod}(L)$ admits filtration wrt $ML$ iff it admits filtration wrt $\mu_cML$.

Corollary (Finite Model Property)
Let $L$ be a logic such that $\text{Mod}(L)$ admits filtration with respect to $ML$, and let $\phi$ be a formula of the continuous $\mu$-calculus. Then $\phi$ is valid in every $L$-model if and only if $\phi$ is valid in every finite $L$-model.

For example: $\mu_cML$ has the FMP over symmetric models.
Other results (ii)

Theorem

Let $L$ be a canonical logic in the basic modal language such $Fr(L)$ admits filtration. Then $\mu_c$-$L$ is sound and complete with respect to $Fr(L)$.

For example: $L = KB, K4, S4, S5, \ldots$

The last two results generalise results for PDL in Kikot, Shapirovsky & Zolin (AiML 2020).
Other results (ii)

Theorem

Let \( L \) be a canonical logic in the basic modal language such \( \text{Fr}(L) \) admits filtration. Then \( \mu_c\text{-}L \) is sound and complete with respect to \( \text{Fr}(L) \).

For example: \( L = \text{KB}, \text{K4}, S4, S5, \ldots \)

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**Theorem**

Let $L$ be a canonical logic in the basic modal language such $Fr(L)$ admits filtration. Then $\mu_c-L$ is sound and complete with respect to $Fr(L)$.

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The last two results generalise results for PDL in Kikot, Shapirovsky & Zolin (AiML 2020).
Future work
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▶ Relation to constructiveness.
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- Is $\mu_c$ML somehow a maximal ‘natural’ fragment of $\mu$ML to which filtration is applicable?
Future work

- Relation to constructiveness.
- Is $\mu_c\text{ML}$ somehow a maximal ‘natural’ fragment of $\mu\text{ML}$ to which filtration is applicable?
- Can the currently separate proofs of the Filtration Theorem and canonical completeness be unified by taking a filtration of some canonical model (as with PDL).
Thank you