Toy models for D. H. Lehmer’s conjecture II

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Abstract.
In the previous paper, we studied the “Toy models for D. H. Lehmer’s conjecture”. Namely, we showed that the $m$-th Fourier coefficient of the weighted theta series of the $\mathbb{Z}^2$-lattice and the $A_2$-lattice does not vanish, when the shell of norm $m$ of those lattices is not the empty set. In other words, the spherical 4 (resp. 6)-design does not exist among the nonempty shells in the $\mathbb{Z}^2$-lattice (resp. $A_2$-lattice).

This paper is the sequel to the previous paper. We take 2-dimensional lattices associated to the algebraic integers of imaginary quadratic fields whose class number is either 1 or 2, except for $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, then, show that the $m$-th Fourier coefficient of the weighted theta series of those lattices does not vanish, when the shell of norm $m$ of those lattices is not the empty set. Equivalently, we show that the corresponding spherical 2-design does not exist among the nonempty shells in those lattices.

Key Words and Phrases. weighted theta series, spherical $t$-design, modular forms, lattices, Hecke operator.

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1 Introduction

The concept of spherical $t$-design is due to Delsarte-Goethals-Seidel [7]. For a positive integer $t$, a finite nonempty subset $X$ of the unit sphere $S^{n-1} = \{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \}$ is called a spherical $t$-design on $S^{n-1}$ if the following condition is satisfied:

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x) d\sigma(x),$$

for all polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree not exceeding $t$. Here, the righthand side means the surface integral on the sphere, and $|S^{n-1}|$ denotes the surface volume of the sphere $S^{n-1}$. The meaning of spherical $t$-design is that the average value of the integral of any polynomial of degree up to $t$ on the sphere is replaced by the average value at a finite set on the sphere. A finite subset $X$ in $S^{n-1}(r)$, the sphere of radius $r$ centered at the origin, is also called a spherical $t$-design if $\frac{1}{r}X$ is a spherical $t$-design on the unit sphere $S^{n-1}$.

We denote by $\text{Harm}_j(\mathbb{R}^n)$ the set of homogeneous harmonic polynomials of degree $j$ on $\mathbb{R}^n$. It is well known that $X$ is a spherical $t$-design if and only if the condition

$$\sum_{x \in X} P(x) = 0$$

holds for all $P \in \text{Harm}_j(\mathbb{R}^n)$ with $1 \leq j \leq t$. If the set $X$ is antipodal, that is $-X = X$, and $j$ is odd, then the above condition is fulfilled automatically. So we reformulate the condition of spherical $t$-design on the antipodal set as follows:

**Proposition 1.1.** A nonempty finite antipodal subset $X \subset S^{n-1}$ is a spherical $2s + 1$-design if the condition

$$\sum_{x \in X} P(x) = 0$$

holds for all $P \in \text{Harm}_{2j}(\mathbb{R}^n)$ with $2 \leq 2j \leq 2s$. 

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It is known [7] that there is a natural lower bound (Fisher type inequality) for the size of a spherical $t$-design in $S^{n-1}$. Namely, if $X$ is a spherical $t$-design in $S^{n-1}$, then

$$|X| \geq \left( n - 1 + \left\lfloor \frac{t}{2} \right\rfloor \right) + \left( n + \left\lfloor \frac{t}{2} \right\rfloor - 2 \right) \left\lfloor \frac{t}{2} \right\rfloor - 1$$

if $t$ is even, and

$$|X| \geq 2 \left( n - 1 + \left\lfloor \frac{t}{2} \right\rfloor \right) \left\lfloor \frac{t}{2} \right\rfloor$$

if $t$ is odd.

A lattice in $\mathbb{R}^n$ is a subset $\Lambda \subset \mathbb{R}^n$ with the property that there exists a basis $\{v_1, \cdots, v_n\}$ of $\mathbb{R}^n$ such that $\Lambda = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_n$, i.e., $\Lambda$ consists of all integral linear combinations of the vectors $v_1, \cdots, v_n$. The dual lattice $\Lambda$ is the lattice

$$\Lambda^\ast := \{ y \in \mathbb{R}^n \mid (y, x) \in \mathbb{Z}, \text{ for all } x \in \Lambda \},$$

where $(x, y)$ is the standard Euclidean inner product. The lattice $\Lambda$ is called integral if $(x, y) \in \mathbb{Z}$ for all $x, y \in \Lambda$. An integral lattice is called even if $(x, x) \in 2\mathbb{Z}$ for all $x \in \Lambda$, and it is odd otherwise. An integral lattice is called unimodular if $\Lambda^\ast = \Lambda$. For a lattice $\Lambda$ and a positive real number $m > 0$, the shell of norm $m$ of $\Lambda$ is defined by

$$\Lambda_m := \{ x \in \Lambda \mid (x, x) = m \} = \Lambda \cap S^{n-1}(m).$$

Let $\mathbb{H} := \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$ be the upper half-plane.

**Definition 1.1.** Let $\Lambda$ be the lattice of $\mathbb{R}^n$. Then for a polynomial $P$, the function

$$\Theta_{\Lambda, P}(z) := \sum_{x \in \Lambda} P(x)e^{iz(x,x)}$$

is called the theta series of $\Lambda$ weighted by $P$.

**Remark 1.1** (See Hecke [8], Schoeneberg [18, 19]).
(i) When $P = 1$, we get the classical theta series

$$\Theta_{\Lambda}(z) = \Theta_{\Lambda, 1}(z) = \sum_{m \geq 0} |\Lambda_m| q^m, \text{ where } q = e^{\pi i z}.$$
(ii) The weighted theta series can be written as

\[ \Theta_{\Lambda, P}(z) = \sum_{x \in \Lambda} P(x) e^{i\pi z(x, x)} = \sum_{m \geq 0} a_m^{(P)} q^m, \]

where \( a_m^{(P)} := \sum_{x \in \Lambda_m} P(x). \)

These weighted theta series have been used efficiently for the study of spherical designs which are the nonempty shells of Euclidean lattices. (See [22, 23, 5, 15, 6]. See also [2].)

**Lemma 1.1** (cf. [22, 23, 15, Lemma 5]). Let \( \Lambda \) be an integral lattice in \( \mathbb{R}^n \). Then, for \( m > 0 \), the non-empty shell \( \Lambda_m \) is a spherical \( t \)-design if and only if

\[ a_m^{(P)} = 0 \]

for all \( P \in \text{Harm}_{2j}(\mathbb{R}^n) \) with \( 1 \leq 2j \leq t \), where \( a_m^{(P)} \) are the Fourier coefficients of the weighted theta series

\[ \Theta_{\Lambda, P}(z) = \sum_{m \geq 0} a_m^{(P)} q^m. \]

The theta series of \( \Lambda \) weighted by \( P \) is a modular form for some subgroup of \( SL_2(\mathbb{R}) \). We recall the definition of the modular forms.

**Definition 1.2.** Let \( \Gamma \subset SL_2(\mathbb{R}) \) be a Fuchsian group of the first kind and let \( \chi \) be a character of \( \Gamma \). A holomorphic function \( f : \mathbb{H} \to \mathbb{C} \) is called a modular form of weight \( k \) for \( \Gamma \) with respect to \( \chi \), if the following conditions are satisfied:

(i) \( f \left( \frac{az + b}{cz + d} \right) = \left( \frac{cz + d}{\chi(\sigma)} \right)^k f(z) \) for all \( \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \)

(ii) \( f(z) \) is holomorphic at every cusp of \( \Gamma \).

If \( f(z) \) has period \( N \), then \( f(z) \) has a Fourier expansion at infinity, [10]:

\[ f(z) = \sum_{m=0}^{\infty} a_m q_N^m, \quad q_N = e^{2\pi i z/N}. \]
We remark that for \( m < 0 \), \( a_m = 0 \), by the condition (ii). A modular form with constant term \( a_0 = 0 \), is called a cusp form. We denote by \( M_k(\Gamma, \chi) \) (resp. \( S_k(\Gamma, \chi) \)) the space of modular forms (resp. cusp forms) with respect to \( \Gamma \) with the character \( \chi \). When \( f \) is the normalized eigenform of Hecke operators, p.163, [10], the Fourier coefficients satisfy the following relations:

**Lemma 1.2** (cf. [10, Proposition 32, 37, 40, Exercise 2, p.164]). Let \( f(z) = \sum_{m \geq 1} a(m)q^m \in S_k(\Gamma, \chi) \). If \( f(z) \) is the normalized eigenform of Hecke operators, then the Fourier coefficients of \( f(z) \) satisfy the following relations:

\[
  a(mn) = a(m)a(n) \quad \text{if} \quad (m, n) = 1
\]

\[
  a(p^{\alpha+1}) = a(p)a(p^{\alpha}) - \chi(p)p^{k-1}a(p^{\alpha-1}) \quad \text{if} \quad p \text{ is a prime.}
\]

We set \( f(z) = \sum_{m \geq 1} a(m)q^m \in S_k(\Gamma, \chi) \). When \( \dim S_k(\Gamma, \chi) = 1 \) and \( a(1) = 1 \), then \( f(z) \) is the normalized eigenform of Hecke operators, [10]. So, the coefficients of \( f(z) \) have the relations as mentioned in Lemma 1.2. It is known that

\[
  |a(p)| < 2p^{(k-1)/2}
\]

for all primes \( p \). Note that this is the Ramanujan conjecture and its generalization, called the Ramanujan-Petersson conjecture for cusp forms which are eigenforms of the Hecke operators. These conjectures were proved by Deligne as a consequence of his proof of the Weil conjectures, [10, page 164], [9]. Moreover, for a prime \( p \) with \( \chi(p) = 1 \) the following equation holds, [11].

\[
  a(p^{\alpha}) = p^{(k-1){\alpha}/2} \frac{\sin(\alpha + 1)\theta_p}{\sin \theta_p},
\]

where \( 2 \cos \theta_p = a(p)p^{-(k-1)/2} \).

It is well known that the theta series of \( \Lambda \subset \mathbb{R}^n \) weighted by harmonic polynomial \( P \in \text{Harm}_j(\mathbb{R}^n) \) is a modular form of weight \( n/2 + j \) for some subgroup \( \Gamma \subset SL_2(\mathbb{R}) \). In particular, when \( \text{deg}(P) \geq 1 \), the theta series of \( \Lambda \) weighted by \( P \) is a cusp form.

For example, we consider the even unimodular lattice \( \Lambda \). Then the theta series of \( \Lambda \) weighted by harmonic polynomial \( P, \Theta_{\Lambda,P}(z) \), is a modular form with respect to \( SL_2(\mathbb{Z}) \).
Example 1.1. Let $\Lambda$ be the $E_8$-lattice. This is an even unimodular lattice of $\mathbb{R}^8$, generated by the $E_8$ root system. The theta series is as follows:

$$\Theta_{\Lambda}(z) = E_4(z) = 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^{2m}$$

$$= 1 + 240q^2 + 2160q^4 + 6720q^6 + 17520q^8 + \cdots,$$

where $\sigma_3(m)$ is a divisor function $\sigma_3(m) = \sum_{d|m} d^3$.

For $j = 2, 4$ and $6$, the theta series of $\Lambda$ weighted by $P \in \text{Harm}_j(\mathbb{R}^8)$ is a weight 6, 8 and 10 cusp form with respect to $SL_2(\mathbb{Z})$. However, it is well known that for $k = 6, 8$ and $10$, $\dim S_k(SL_2(\mathbb{Z})) = 0$, that is, $\Theta_{\Lambda, P}(z) = 0$. Then by Lemma 1.1 all the nonempty shells of $E_8$-lattice are spherical 6-design.

For $j = 8$, the theta series of $\Lambda$ weighted by $P$ is a weight 12 cusp form with respect to $SL_2(\mathbb{Z})$. Such a cusp form is uniquely determined up to constant, i.e., it is Ramanujan’s delta function:

$$\Delta_{24}(z) = q^2 \prod_{m \geq 1} (1 - q^{2m})^{24} = \sum_{m \geq 1} \tau(m) q^{2m}.$$

The following proposition is due to Venkov, de la Harpe and Pache [5, 6, 15, 22].

Proposition 1.2 (cf. [15]). Let the notation be the same as above. Then the following are equivalent:

(i) $\tau(m) = 0$.

(ii) $(\Lambda)_{2m}$ is an 8-design.

It is a famous conjecture of Lehmer that $\tau(m) \neq 0$. So, Proposition 1.2 gives a reformulation of Lehmer’s conjecture. Lehmer proved in [11] the following theorem.

Theorem 1.1 (cf. [11]). Let $m_0$ be the least value of $m$ for which $\tau(m) = 0$. Then $m_0$ is a prime if it is finite.

There are many attempts to study Lehmer’s conjecture ([11] 20), but it is difficult to prove and it is still open.
Recently, however, we showed the “Toy models for D. H. Lehmer’s conjecture” [3]. We take the two cases $\mathbb{Z}^2$-lattice and $A_2$-lattice. Then, we consider the analogue of Lehmer’s conjecture corresponding to the theta series weighted by some harmonic polynomial $P$. Namely, we show that the $m$-th coefficient of the weighted theta series of $\mathbb{Z}^2$-lattice does not vanish when the shell of norm $m$ of those lattices is not an empty set. Or equivalently, we show the following result.

**Theorem 1.2** (cf. [3]). The nonempty shells in $\mathbb{Z}^2$-lattice (resp. $A_2$-lattice) are not spherical 4-designs (resp. 6-designs).

This paper is sequel to the previous paper [3]. In this paper, we take some lattices related to the imaginary quadratic fields. Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field, and let $\mathcal{O}_K$ be its ring of algebraic integers. Let $\text{Cl}_K$ be the ideal classes. In this paper, we only consider the cases $|\text{Cl}_K| = 1$ and $|\text{Cl}_K| = 2$ except for Section 6. So, when we consider the cases $|\text{Cl}_K| = 1$ and $|\text{Cl}_K| = 2$, we denote by $\mathfrak{o}$ (resp. $\mathfrak{a}$) the principal (resp. nonprincipal) ideal class.

We denote by $d_K$ the discriminant of $K$:

$$d_K = \begin{cases} -4d & \text{if } -d \equiv 2, 3 \pmod{4}, \\ -d & \text{if } -d \equiv 1 \pmod{4}. \end{cases}$$

**Theorem 1.3** (cf. [24, page 87]). Let $d$ be a positive square-free integer, and let $K = \mathbb{Q}(\sqrt{-d})$. Then

$$\mathcal{O}_K = \begin{cases} \mathbb{Z} + \mathbb{Z} \sqrt{-d} & \text{if } -d \equiv 2, 3 \pmod{4}, \\ \mathbb{Z} + \mathbb{Z} \frac{-1 + \sqrt{-d}}{2} & \text{if } -d \equiv 1 \pmod{4}. \end{cases}$$

Therefore, we consider $\mathcal{O}_K$ to be the lattice in $\mathbb{R}^2$ with the basis

$$\begin{cases} (1, 0), (1, \sqrt{-d}) & \text{if } -d \equiv 2, 3 \pmod{4}, \\ (1, 0), \left( -\frac{1}{2}, \frac{\sqrt{-d}}{2} \right) & \text{if } -d \equiv 1 \pmod{4}, \end{cases}$$

denoted by $L_\mathfrak{o}$.

Generally, it is well-known that there exists one-to-one correspondence between the set of reduced quadratic forms $f(x, y)$ with a fundamental discriminant $d_K < 0$ and the set of fractional ideal classes of the unique quadratic
field $\mathbb{Q}(\sqrt{-d})$ \cite{24}, page 94]. Namely, For a fractional ideal $A = \mathbb{Z}\alpha + \mathbb{Z}\beta$, we obtain the quadratic form $ax^2 + bxy + cy^2$, where $a = \alpha\alpha'/N(A)$, $b = (\alpha\beta' + \alpha'\beta)/N(A)$ and $c = \beta\beta'/N(A)$. Conversely, for a quadratic form $ax^2 + bxy + cy^2$, we obtain the fractional ideal $\mathbb{Z} + \mathbb{Z}(b + \sqrt{d_{K}})/2a$. We remark that $N(A)$ is a norm of $A$ and $\alpha'$ is a complex conjugate of $\alpha$.

Here, we define the automorphism group of $f(x, y)$ as follows:

$$U_f = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) \mid f(\alpha x + \beta y, \gamma x + \delta y) = f(x, y) \right\}.$$ 

Then, for $n \geq 1$, the number of the nonequivalent solutions of $f(x, y) = n$ under the action of $U_f$ is equal to the number of the integral ideals of norm $n$ \cite{24}.

**Theorem 1.4** (cf. \cite{24}, page 63). Let $f(x, y)$ be the reduced quadratic form with a fundamental discriminant $D < 0$ and $U_f$ be the automorphism group of $f(x, y)$. Then

$$\sharp U_f = \begin{cases} 6 & \text{if } D = -3, \\ 4 & \text{if } D = -4, \\ 2 & \text{if } D < -4. \end{cases}$$

These classical results are due to Gauss, Dirichlet, etc. Let $a$ be an ideal class and $f_a(x, y)$ be the reduced quadratic form corresponding to $a$. Moreover, let $L_a$ be the lattice corresponding to $f(x, y)$. We denote by $N(A)$ the norm of an ideal $A$. Then, using Theorem 1.4, we have

$$\sum_{x \in L_a} q^{(x,x)} = 1 + \sharp U_f \sum_{n=1}^{\infty} \sharp \{ A \mid A \text{ is an integral ideal of } a, N(A) = n \} q^n.$$ 

When $|\text{Cl}_K| = 2$, we give the generators of $L_a$ in Appendix. Here, we remark that when $K = \mathbb{Q}(\sqrt{-1})$ (resp. $K = \mathbb{Q}(\sqrt{-3})$), $L_o$ is $\mathbb{Z}^2$-lattice (resp. $A_2$-lattice). We studied the spherical designs of shells of those lattices in the previous paper \cite{3}.

In this paper, we take the imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$, with $d \neq 1$ and $d \neq 3$. Then, we consider the analogue of Lehmer’s conjecture corresponding to its theta series weighted by some harmonic polynomial $P$. Here, we consider the following problem that whether the nonempty shells of $L_o$ and $L_a$ are spherical 2-designs (hence 3-designs) or not.
In Section 4, we study the case that the class number is 1. We show that the $m$-th coefficient of the weighted theta series of $L_\alpha$-lattice does not vanish when the shell of norm $m$ of those lattices is not an empty set. Or equivalently, we show the following result:

**Theorem 1.5.** Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field whose class number is 1 and $d \neq 1, 3$ i.e., $d$ is in the following set: $\{2, 7, 11, 19, 43, 67, 163\}$. Then, the nonempty shells in $L_\alpha$ are not spherical $2$-designs.

Similarly, in Section 5, we study the case that the class number is 2 and show the following result:

**Theorem 1.6.** Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field whose class number is 2 i.e., $d$ is in the following set: $\{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\}$. Then, the nonempty shells in $L_\alpha$ and $L_\alpha$ are not spherical $2$-designs.

In Section 6, we consider the case that the class number is 3 and study the property of Hecke characters. In Section 7, we give some concluding remarks and state a conjecture for the future study.

# 2 Preliminaries

In this section, we review the theory of imaginary quadratic fields.

**Theorem 2.1** (cf. [1, page 104, Proposition 5.16]). *We can classify the prime ideals of a quadratic field as follows:*

1. If $p$ is an odd prime and $(d_K/p) = 1$ (resp. $d_K \equiv 1 \pmod{8}$) then $(p) = P\overline{P}$ (resp. $(2) = P\overline{P}$), where $P$ and $\overline{P}$ are prime ideals with $P \neq \overline{P}$, $N(P) = N(\overline{P}) = p$ (resp. $N(P) = 2$).

2. If $p$ is an odd prime and $(d_K/p) = -1$ (resp. $d_K \equiv 5 \pmod{8}$) then $(p) = P$ (resp. $(2) = P$), where $P$ is a prime ideal with $N(P) = p^2$ (resp. $N(P) = 4$).

3. If $p \mid d_k$ then $(p) = P^2$, where $P$ is a prime ideal with $N(P) = p$. 
Lemma 2.1. Let $I$ be an integral ideal of $K$. For $n \in \mathbb{N}$, if $N(I) = n$ and $I$ is a principal ideal, namely, $I \in \mathfrak{o}$ then there exist $a, b \in \mathbb{Z}$ such that for $-d \equiv 2, 3 \pmod{4}$

$$n = a^2 + db^2,$$

for $-d \equiv 1 \pmod{4}$

$$n = a^2 + db^2 \quad \text{or} \quad n = \frac{a^2 + db^2}{4}.$$

If $|\text{Cl}_K| = 2$, $N(I) = n$ and $I$ is a nonprincipal ideal, namely, $I \in \mathfrak{a}$ and assume that $m$ is one of the norm of nonprincipal ideals then there exist $a, b \in \mathbb{Z}$ such that for $-d \equiv 2, 3 \pmod{4}$

$$mn = a^2 + db^2,$$

for $-d \equiv 1 \pmod{4}$

$$mn = a^2 + db^2 \quad \text{or} \quad mn = \frac{a^2 + db^2}{4}.$$

Proof. We assume that $|\text{Cl}_K| = 1$. For $-d \equiv 2, 3 \pmod{4}$, we can write $I = (a + b\sqrt{-d})$, then $N(I) = a^2 + db^2$. For $-d \equiv 1 \pmod{4}$, we can write $I = (a + b\sqrt{-d})$ or $I = ((a + b\sqrt{-d})/2)$, then $N(I) = a^2 + db^2$ or $N(I) = (a^2 + db^2)/4$.

Here, we assume that $|\text{Cl}_K| = 2$. Let $J$ be the nonprincipal ideal of $K$ whose norm is $m$. If $I$ is a nonprincipal ideal then, $JI$ is a principal ideal of $K$. Therefore, for $-d \equiv 2, 3 \pmod{4}$, we can write $JI = (a + b\sqrt{-d})$, then $N(JI) = a^2 + db^2$. Hence, $mn = a^2 + db^2$. For $-d \equiv 1 \pmod{4}$, we can write $JI = (a + b\sqrt{-d})$ or $JI = ((a + b\sqrt{-d})/2)$, then $N(JI) = a^2 + db^2$ or $N(JI) = (a^2 + db^2)/4$. Hence, $mn = a^2 + db^2$ or $mn = (a^2 + db^2)/4$. $\square$

Proposition 2.1. Let $F(m)$ be the number of the integral ideals of norm $m$ of $K$. Let $p$ be a prime number. Then, if $p \neq 2$

$$F(p^e) = \begin{cases} e + 1 & \text{if } (d_K/p) = 1, \\ (1 + (-1)^e)/2 & \text{if } (d_K/p) = -1, \\ 1 & \text{if } p \mid d_K, \end{cases}$$

if $p = 2$

$$F(2^e) = \begin{cases} e + 1 & \text{if } d_K \equiv 1 \pmod{8}, \\ (1 + (-1)^e)/2 & \text{if } d_K \equiv 5 \pmod{8}, \\ 1 & \text{if } 2 \mid d_K. \end{cases}$$
Proof. When \((d_K/p) = 1\) i.e., \((p) = P\overline{P}\) and \(P \neq \overline{P}\), since \(P\) and \(\overline{P}\) are the only integral ideals of norm \(p\), we have \(F(p) = 2\). Moreover, the integral ideals of norm \(p^e\) are as follows: \(P^e, P^{e-1}\overline{P}, \ldots, (\overline{P})^e\). So, we have \(F(p^e) = e + 1\). The other cases can be proved similarly.\(\square\)

3 Hecke characters and Theta series

In this section, we introduce the Hecke character and discuss the relationships between the Hecke character and the weighted theta series of the lattices \(L_o\) and \(L_a\). Then, we show that for \(|\text{Cl}_K| = 1\) and \(P_1 = (x^2 - y^2)/2\), the weighted theta series \(\Theta_{L_o,P_1}\) is a normalized Hecke eigenform. For \(|\text{Cl}_K| = 2\) and \(P_2 = x^2 - y^2\), a certain sum of the two weighted theta series \(c_1\Theta_{L_o,P_2} + c_2\Theta_{L_a,P_2}\) is a normalized Hecke eigenform. Later, we give the explicit values of \(c_1\) and \(c_2\).

A Hecke character \(\phi\) of weight \(k \geq 2\) with modulus \(\Lambda\) is defined in the following way. Let \(\Lambda\) be a nontrivial ideal in \(\mathcal{O}_K\) and let \(I(\Lambda)\) denote the group of fractional ideals prime to \(\Lambda\). A Hecke character \(\phi\) with modulus \(\Lambda\) is a homomorphism

\[
\phi : I(\Lambda) \to \mathbb{C}^\times
\]

such that for each \(\alpha \in K^\times\) with \(\alpha \equiv 1 \pmod{\Lambda}\) we have

\[
(7) \quad \phi(\alpha \mathcal{O}_K) = \alpha^{k-1}.
\]

Let \(\omega_\phi\) be the Dirichlet character with the property that

\[
\omega_\phi(n) := \phi((n))/n^{k-1}
\]

for every integer \(n\) coprime to \(\Lambda\).

Theorem 3.1 (cf. [14, page 9], [13, page 183]). Let the notation be the same as above, and define \(\Psi_{K,\Lambda}(z)\) by

\[
(8) \quad \Psi_{K,\Lambda}(z) := \sum_A \phi(A)q^{N(A)} = \sum_{n=1}^{\infty} a(n)q^n,
\]

where the sum is over the integral ideals \(A\) that are prime to \(\Lambda\) and \(N(A)\) is the norm of the ideal \(A\). Then \(\Psi_{K,\Lambda}(z)\) is a cusp form in \(S_k(\Gamma_0(d_K \cdot N(\Lambda)), (\frac{d_K}{d}) \omega_\phi)\).
We remark that function (8) is a normalized Hecke eigenform [11, 21]. Moreover, if the class number of $K$ is $h$ then the character as given in (7) will have $h$ extensions to nonprincipal ideals. Namely, the function (8) has $h$ choices, so we denote by $\Psi_{K,\Lambda}^{(1)}(z), \ldots, \Psi_{K,\Lambda}^{(h)}(z)$ each functions (see [16]).

**Example 3.1.**

(i) $d = 2$.

We calculate $\Psi_{K,\Lambda}(z) = \sum_{m \geq 1} a(m)q^m$, where $\Lambda = (1)$ and the weight of the Hecke character is 3. We remark that $|\Cl_K| = 1$. By the definitions (7) and (8), we have $a(1) = 1^2 = 1$, $a(2) = \sqrt{-2}^2 = -2$, $a(3) = (-1 + \sqrt{-2})^2 + (-1 - \sqrt{-2})^2 = 2$, $a(4) = 2^2$, $\ldots$. Thus, we obtain

$$\Psi_{K,\Lambda}^{(1)}(z) = q - 2q^2 - 2q^3 + 4q^4 + 4q^6 - 8q^8 - 5q^9 + \cdots.$$ 

(ii) $d = 5$.

We calculate $\Psi_{K,\Lambda}(z) = \sum_{m \geq 1} a(m)q^m$, where $\Lambda = (1)$ and the weight of the Hecke character is 3. We remark that $|\Cl_K| = 2$. When $A$ of norm $m$ is a nonprincipal ideal, $A^2$ is a principal ideal, so, $\phi(A^2)$ is computable by the definition (7). For example, $\phi((2, 1 + \sqrt{-5})^2) = \phi((2)) = 4$, so, we can assume that $\phi((2, 1 + \sqrt{-5})) = 2$, i.e., $a(2) = 2$. Then, since $(2, 1 + \sqrt{-5})(3, 1 + \sqrt{-5}) = (1 - \sqrt{-5})$ and $(2, 1 + \sqrt{-5})(3, 1 - \sqrt{-5}) = (-1 - \sqrt{-5})$, we have $a(3) = ((1 + \sqrt{-5})^2 + (1 - \sqrt{-5})^2)/2 = -4$, $a(4) = 2^2$, $\ldots$. Thus, we obtain

$$\Psi_{K,\Lambda}^{(1)}(z) = q + 2q^2 - 4q^3 + 4q^4 - 5q^5 - 8q^6 + 4q^7 + 8q^8 + 7q^9 + \cdots.$$ 

| Table 1: Integral ideals of small norm of $d = 2$ and $d = 5$ |
|---|---|
| $N(A)$ | $A$: ideal |
| 1 | (1) |
| 2 | (\sqrt{-2}) |
| 3 | (-1 + \sqrt{-2}) |
| 4 | (2) |
| 5 | (\sqrt{-5}) |
| 6 | (1 - \sqrt{-5}) |

$\phi(A^2)$ is computable by the definition (7). For example, $\phi((2, 1 + \sqrt{-5})^2) = \phi((2)) = 4$, so, we can assume that $\phi((2, 1 + \sqrt{-5})) = 2$, i.e., $a(2) = 2$. Then, since $(2, 1 + \sqrt{-5})(3, 1 + \sqrt{-5}) = (1 - \sqrt{-5})$ and $(2, 1 + \sqrt{-5})(3, 1 - \sqrt{-5}) = (-1 - \sqrt{-5})$, we have $a(3) = ((1 + \sqrt{-5})^2 + (1 - \sqrt{-5})^2)/2 = -4$, $a(4) = 2^2$, $\ldots$. Thus, we obtain

$$\Psi_{K,\Lambda}^{(1)}(z) = q + 2q^2 - 4q^3 + 4q^4 - 5q^5 - 8q^6 + 4q^7 + 8q^8 + 7q^9 + \cdots.$$
On the other hand, we assume that $\phi((2, 1 + \sqrt{-5})) = -2$, i.e., $a(2) = -2$. Then, we have

$$\Psi_{K, \Lambda}^{(2)}(z) = q - 2q^2 + 4q^3 + 4q^4 - 5q^5 - 8q^6 - 4q^7 - 8q^8 + 7q^9 + \cdots.$$ 

Here, we discuss the relationships between the Hecke character and the weighted theta series of the lattices $L_o$ and $L_a$. First, we quote the following theorem:

**Theorem 3.2** (cf. [13, page 192]). Let $L$ be an integral lattice with the Gram matrix $A$ and $N$ be the natural number such that the elements of $NA^{-1}$ are rational integers. Let the character $\chi(d)$ be

$$\chi(d) = \left(\frac{(-1)^{r/2} \det L}{d}\right).$$

Then, for $P \in \text{Harm}_2(\mathbb{R}^2)$,

(1) $\Theta_{L, P} \in M_3(\Gamma_0(4N), \chi)$.

(2) If all the diagonal elements of $A$ are even, then $\Theta_{L, P} \in M_3(\Gamma_0(2N), \chi)$.

(3) If all the diagonal elements of $A$ and $NA^{-1}$ are even, then $\Theta_{L, P} \in M_3(\Gamma_0(N), \chi)$.

Then, we obtain the following lemmas:

**Lemma 3.1.** Let $K$ be an imaginary quadratic field whose class number is 1 and $L_o$ be the lattice corresponding to the principal ideal class $\mathfrak{o}$. Let $\phi$ be the Hecke character of weight 3 with modulus $\Lambda$. Assume that $\Lambda = (1)$ and $P_1 = (x^2 - y^2)/2 \in \text{Harm}_2(\mathbb{R}^2)$. Then, $\Psi_{K, \Lambda}(q) = \Theta_{L_o, P_1}(q)$.

**Lemma 3.2.** Let $K$ be an imaginary quadratic field whose class number is 2 and $L_o$ (resp. $L_a$) be the lattice corresponding to the principal ideal class $\mathfrak{o}$ (resp. nonprincipal ideal class $\mathfrak{a}$). Let $\phi$ be the Hecke character of weight 3 with modulus $\Lambda$. Assume that $\Lambda = (1)$ and $P_2 = x^2 - y^2 \in \text{Harm}_2(\mathbb{R}^2)$. Then, $\Psi_{K, \Lambda}(q) = c_1 \Theta_{L_o, P_2}(q) + c_2 \Theta_{L_a, P_2}(q)$, where $c_1$ and $c_2$ are given as in table 2.

**Proof of Lemmas 3.1 and 3.2.** First, we assume that the lattices are integral lattices, if not we multiple the Gram matrix of $L$ by 2.
Table 2: Coefficients, $c_1$ and $c_2$

| $-d$ | $-5$ | $-6$ | $-10$ | $-13$ | $-15$ | $-22$ | $-35$ | $-37$ | $-51$ |
|-----|-----|-----|------|------|------|------|------|------|------|
| $c_1$ | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 |
| $c_2$ | 1/2 | 1/2 | 1/2 | 1/2 | 2 | 1/2 | 3 | 1/2 | 1/2 |

| $-d$ | $-58$ | $-91$ | $-115$ | $-123$ | $-187$ | $-235$ | $-267$ | $-403$ | $-427$ |
|-----|-----|-----|------|------|------|------|------|------|------|
| $c_1$ | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 |
| $c_2$ | 1/2 | 5/3 | 1/2 | 1/2 | 7/3 | 1/2 | 11/9 | 1/2 |

Because of the Theorems 3.1 and 3.2, $\Psi_{K, \Lambda}(q)$, $\Theta_{L_0, P}(q)$ and $\Theta_{L_0, P}(q)$ with $P = P_1$, $P_2$ are modular forms of the same group $\Gamma$. Therefore, we calculate the basis of the space of modular forms of group $\Gamma$ and check $\Psi_{K, \Lambda}(q) = \Theta_{L_0, P_1}(q)$ and $\Psi_{K, \Lambda}(q) = c_1\Theta_{L_0, P_2}(q) + c_2\Theta_{L_0, P_2}(q)$ explicitly (using “Sage”, Mathematics Software [17]).

**Corollary 3.1.** Let the notation be the same as above. If $|\text{Cl}_K| = 1$ then $\Theta_{L_1, P_1}(q)$ is a normalized Hecke eigenform. If $|\text{Cl}_K| = 2$ then $c_1\Theta_{L_1, P_2}(q) + c_2\Theta_{L_2, P_2}(q)$ is a normalized Hecke eigenform.

**Proof.** The function (8) is a normalized Hecke eigenform [11][21].

Finally, we give the following proposition, which is an analogue of Theorem 1.1 and the crucial part of the proof of Theorems 1.5 and 1.6.

**Proposition 3.1.** Assume that $\sum_{m \geq 1} a(m)q^m$ is a normalized Hecke eigenform of $S_3(\Gamma, \chi)$ and the coefficients $a(m)$ are rational integers. Moreover, let $p$ be the prime such that $\chi(p) = 1$. Let $\alpha_0$ be the least value of $\alpha$ for which $a(p^\alpha) = 0$. If $a(p) \neq \pm p$ then $\alpha_0 = 1$ if it is finite.

**Proof.** Assume the contrary, that is, $\alpha_0 > 1$, so that $a(p) \neq 0$. By the equation (5),

$$a(p^{\alpha_0}) = 0 = p^{\alpha_0} \frac{\sin(\alpha_0 + 1)\theta_p}{\sin \theta_p}.$$  

This shows that $\theta_p$ is a real number of the form $\theta_p = \pi k/(1 + \alpha_0)$, where $k$ is an integer. Now the number

$$z = 2 \cos \theta_p = a(p)p^{-1},$$

(9)
being twice the cosine of a rational multiple of \(2\pi\), is an algebraic integer. On the other hand \(z\) is a root of the equation

\[(10) \quad pz - a(p) = 0.\]

Hence \(z\) is a rational integer. By (4) and (9), we have \(|z| \leq 1\). Therefore \(z = \pm 1\) and the equation (10) becomes \(a(p) = \pm p\). By assumption, this is a contradiction. \(\square\)

\section{The case of \(|\text{Cl}_K| = 1\)}

Let \(K := \mathbb{Q}(\sqrt{-d})\) be an imaginary quadratic field. If the class number of \(K\) is 1 then \(d\) is in the following set \(\{1, 2, 3, 7, 11, 19, 43, 67, 163\}\). In particular, we only consider the cases where \(d\) is in the set: \(\{2, 7, 11, 19, 43, 67, 163\}\) since the cases \(d = 1\) and \(d = 3\) are considered in \([3]\).

In this section, we assume that \(a(m)\) and \(b(m)\) are the coefficients of the following functions:

\[
\Theta_{L_0}(q) = \sum_{m \geq 0} a(m)q^m, \quad \Theta_{L_0,P_1}(q) = \sum_{m \geq 1} b(m)q^m,
\]

where \(P_1 = (x^2 - y^2)/2 \in \text{Harm}_2(\mathbb{R}^2)\).

\textbf{Lemma 4.1.} Let \(d\) be one of the elements in \(\{2, 7, 11, 19, 43, 67, 163\}\). We set \(a'(m) = a(m)/2\) for all \(m\). Then,

\[
a'(p^e) = \begin{cases} 
  e + 1 & \text{if } (d_K/p) = 1, \\
  (1 + (-1)^e)/2 & \text{if } (d_K/p) = -1, \\
  1 & \text{if } p \mid d_K.
\end{cases}
\]

\textit{Proof.} Because of the equation (9), \(a'(m)\) is the number of integral ideals of \(K\) of norm \(m\). Therefore, it can be proved by Proposition 2.1. \(\square\)

\textbf{Lemma 4.2.} Let \(p\) be a prime number such that \((d_K/p) = 1\). Then, \(b(p) \neq 0\). Moreover, if \(p \neq d\) then \(b(p) \neq \pm p\).

\textit{Proof.} We remark that by Corollary 3.1 \(\Theta_{L_0,P_1}(q) = \Psi_{K,A}(q)\). So, the numbers \(b(m)\) are the coefficients of \(\Psi_{K,A}(q)\).
First, we assume that \( d \neq 2 \) i.e., \( -d \equiv 1 \pmod{4} \) and \( \mathcal{O}_K = \mathbb{Z} + \mathbb{Z}(1 + \sqrt{-d})/2 \). If \( N((a + b\sqrt{-d})) \) is equal to \( p \) then by Lemma 2.1,
\[
p = a^2 + db^2.
\]
Because of the definition of \( \Psi_{\mathcal{O}_K}(q) \),
\[
b(p) = (a + b\sqrt{-d})^2 + (a - b\sqrt{-d})^2 = 2(a^2 - db^2).
\]
If \( b(p) = 0 \) then \( a^2 = db^2 \). This is a contradiction. Assume that \( b(p) = \pm p \).
Then,
\[
2(a^2 - db^2) = \pm(a^2 + db^2),
\]
that is, \( a^2 = 3db^2 \) or \( 3a^2 = db^2 \). This is a contradiction.

Next, we assume that \( d = 2 \) i.e., \( -d \equiv 2 \pmod{4} \) and \( \mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\sqrt{-2} \). If \( N((a + b\sqrt{-2})) \) is equal to \( p \) then by Lemma 2.1,
\[
p = a^2 + 2b^2.
\]
Because of the definition of \( \Psi_{\mathcal{O}_K}(q) \),
\[
b(p) = (a + b\sqrt{-2})^2 + (a - b\sqrt{-2})^2 = 2(a^2 - 2b^2).
\]
If \( b(p) = 0 \) then \( a^2 = 2b^2 \). This is a contradiction. Assume that \( b(p) = \pm p \).
Then,
\[
2(a^2 - 2b^2) = \pm(a^2 + 2b^2),
\]
that is, \( a^2 = 6b^2 \) or \( 3a^2 = 2b^2 \). This is a contradiction.
Proof of Theorem 1.5. We will show that \( b(m) \neq 0 \) when \((L_0)_m \neq \emptyset\).

By Theorem 3.1, \( \Theta_{L_0,P_1} \) is a normalized Hecke eigenform. So, we assume that \( m \) is a power of prime, if not we could apply the equation (2). We will divide our considerations into the following three cases.

(i) Case \( m = p^\alpha \) and \( p \mid d_K \):
   By \( a(m) = 2 \) and the inequality (11), the shells \((L_0)_m\) are not spherical 2-designs. Hence, \( b(m) \neq 0 \).

(ii) Case \( m = p^\alpha \) and \((d_K/p) = -1\):
   By Lemma 4.1,
   \[
   a(p^n) = \begin{cases} 
   0 & \text{if } n \text{ is odd}, \\
   2 & \text{if } n \text{ is even}.
   \end{cases}
   \]
   By \( a(m) = 2 \) and the inequality (11), when \( n \) is even, the shells \((L_0)_m\) are not spherical 2-designs. Hence, \( b(m) \neq 0 \).

(iii) Case \( m = p^\alpha \) and \((d_K/p) = 1\):
   By Proposition 3.1 and Lemma 4.2, we have \( b(m) \neq 0 \). This completes the proof of Theorem 1.5. \( \square \)

5 The case of \( |\text{Cl}_K| = 2 \)

Let \( K := \mathbb{Q}(\sqrt{-d}) \) be an imaginary quadratic field. In this section, we assume that the class number of \( K \) is 2. So, we consider that \( d \) is in the following set: \( \{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\} \). We denote by \( \mathfrak{o} \) (resp. \( \mathfrak{a} \)) the principal (resp. nonprincipal) ideal class.

In this section, we also assume that \( a(m) \) and \( b(m) \) are the coefficients of the following functions:

\[
\Theta_{L_0}(q) + \Theta_{L_a}(q) = \sum_{m \geq 0} a(m)q^m,
\]

\[
c_1 \Theta_{L_0,P_2}(q) + c_2 \Theta_{L_a,P_2}(q) = \sum_{m \geq 1} b(m)q^m,
\]

where \( c_1 \) and \( c_2 \) are defined in Lemma 3.2.
Lemma 5.1. Set \( l_1 := \{N(O) \mid x \in L_o\} \) and \( l_2 := \{N(A) \mid A \in a\} \). Then, \( l_1 \cap l_2 = \emptyset \). Therefore, the set \( L_o \cap L_a \) consists of the origin.

Proof. Let \( p \) be the prime number such that \( (d_K/p) = 1 \). Then there exist prime ideals \( P \) and \( P' \) such that \( (p) = PP' \) and \( N(P) = N(P') = p \). Since the class number is 2, we have \( P \) and \( P' \) or \( P \) and \( P' \) are in \( a \). If \( P \) and \( P' \) are in \( a \) we denote by \( p_i \) such a prime. If \( P \) and \( P' \) are in \( a \) we denote by \( p'_i \) such a prime.

Let \( p \) be the prime number such that \( (d_K/p) = -1 \). Then \( (p) \) is the prime ideal and \( N((p)) = p^2 \). We denote by \( q_i \) such a prime.

Let \( p \) be the prime number such that \( p \mid d_K \). Then there exists a prime ideal \( P \) such that \( (p) = P^2 \) and \( N(P) = p \). Since the class number is 2, we have \( P \in o \) or \( P \in a \). If \( P \in o \) we denote by \( r_i \) such a prime. If \( P \in a \) we denote by \( r'_i \) such a prime.

We take the element \( n \in l_1 \cap l_2 \) and perform a prime factorization, \( n = p_1 \cdots p'_1 \cdots q_1 \cdots q_i \cdots r_1 \cdots r'_1 \cdots \). Then, \( p_1, \ldots, q_1, \ldots, r_1, \ldots \) correspond to principal ideals. So, if the number of the primes \( p' \) and \( r' \) is even then \( n \in l_1 \) and if the number of the primes \( p' \) and \( r' \) is odd then \( n \in l_2 \). This completes the proof of Lemma 5.1.

\[ a'(p^e) = \begin{cases} e + 1 & \text{if } (d_K/p) = 1, \\ (1 + (-1)^e)/2 & \text{if } (d_K/p) = -1, \\ 1 & \text{if } p \mid d_K. \end{cases} \]

Proof. Because of the equation \( \Theta \), \( a'(m) \) is the number of integral ideals of \( K \) of norm \( m \). Therefore, it can be proved by Proposition 2.1.

Lemma 5.3. Let \( p \) be a prime number such that \( (d_K/p) = 1 \). Then, \( b(p) \neq 0 \). Moreover, if \( p \neq d \) then \( b(p) \neq \pm p \).

Proof. We remark that by Corollary 3.1 \( c_1 \Theta_{L_a,P_2}(q) + c_2 \Theta_{L_a,P_2}(q) = \Psi_{K,A}(q) \). So, the numbers \( b(m) \) are the coefficients of \( \Psi_{K,A}(q) \).

We set \( N(J) = p \). When \( J \) is a principal ideal, it can be proved by the similar method in Lemma 4.2. So, we assume that \( J \) is nonprincipal.

We list the smallest prime number \( m \) such that \( m \mid d_K \) and \( m \in \{N(I) \mid I \in a\} \), and the values \( b(m) \) are in Table 3. First, we assume that \(-d \equiv 2 \) or
Table 3: Values of $m$ and $b(m)$

| $-d$ | $-5$ | $-6$ | $-10$ | $-13$ | $-15$ | $-22$ | $-35$ | $-37$ | $-51$ |
|------|------|------|------|------|------|------|------|------|------|
| $m$  | 2    | 2    | 2    | 3    | 2    | 5    | 2    | 3    |      |
| $b(m)$ | 2    | 2    | 2    | $-3$ | 2    | $-5$ | 2    | 3    |      |

| $-d$ | $-58$ | $-91$ | $-115$ | $-123$ | $-187$ | $-235$ | $-267$ | $-403$ | $-427$ |
|------|------|------|------|------|------|------|------|------|------|
| $m$  | 2    | 7    | 5    | 3    | 11   | 5    | 3    | 13   | 7    |
| $b(m)$ | 2    | $-7$ | $-5$ | 3    | $-11$ | 5    | 3    | $-13$ | 7    |

3 (mod 4). If $N(J)$ is equal to $p$ then by Lemma 2.1

$$mp = a^2 + db^2.$$ 

Because of the definition of $\Psi_{K,\Lambda}(q)$,

$$b(mp) = (a + b\sqrt{-d})^2 + (a - b\sqrt{-d})^2 = 2(a^2 - db^2).$$

Since $b(mp) = b(m)b(p)$ and the value of $b(m)$ in Table 3, we have $b(p) = a^2 - db^2$. If $b(p) = 0$ then $a^2 = db^2$. This is a contradiction. Assume that $b(p) = \pm p$. Then,

$$a^2 - db^2 = \pm \frac{a^2 + db^2}{2},$$

that is, $a^2 = 3db^2$ or $3a^2 = db^2$. This is a contradiction.

Next, we assume that $-d \equiv 1$ (mod 4). If $N(J)$ is equal to $p$ then by Lemma 2.1 there exist $a, b \in \mathbb{Z}$ such that

$$mp = a^2 + db^2 \text{ or } mp = \frac{a^2 + db^2}{4}.$$ 

Because of the definition of $\Psi_{K,\Lambda}(q)$,

$$b(mp) = (a + b\sqrt{-d})^2 + (a - b\sqrt{-d})^2 = 2(a^2 - db^2).$$

or

$$b(mp) = \left(\frac{a + b\sqrt{-d}}{2}\right)^2 + \left(\frac{a - b\sqrt{-d}}{2}\right)^2 = a^2 - db^2.$$ 

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Since $b(mp) = b(m)b(p)$ and the value of $b(m)$ in Table 3, we have $b(p) = 2/b(m) \times (a^2 - db^2)$ or $b(p) = 1/b(m) \times (a^2 - db^2)/2$. If $b(p) = 0$ then $a^2 = db^2$. This is a contradiction. Assume that $b(p) = \pm p$. Then,

$$\frac{2(a^2 - db^2)}{b(m)} = \pm \frac{a^2 + db^2}{m},$$

or

$$\frac{a^2 - db^2}{2b(m)} = \pm \frac{a^2 + db^2}{4m},$$

that is, $a^2 = 3db^2$ or $3a^2 = db^2$ since $m = \pm b(m)$ for $-d \equiv 1 \pmod{4}$. This is a contradiction.

Proof of Theorem 1.6. Because of Lemma 5.1 it is enough to show that $b(m) \neq 0$ when $(L_\sigma)_m \neq \emptyset$ or $(L_\lambda)_m \neq \emptyset$.

By Theorem 3.1 $c_1 \Theta_{L_\sigma, p_1} + c_2 \Theta_{L_\lambda, p_2}$ is a normalized Hecke eigenform. So, We assume that $m$ is a power of prime, if not we could apply the equation (2). We will divide into the three cases.

(i) Case $m = p^\alpha$ and $p \mid d_K$:

By $a(m) = 2$ and (1), the shells $(L)_m$ are not spherical 2-designs. Hence, $b(m) \neq 0$.

(ii) Case $m = p^\alpha$ and $(d_K/p) = -1$:

By Lemma (4.1),

$$a(p^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

By $a(m) = 2$ and (1), when $n$ is even, the shells $(L)_m$ are not spherical 2-designs. Hence, $b(m) \neq 0$.

(iii) Case $m = p^\alpha$ and $(d_K/p) = 1$:

By Proposition 3.1 and Lemma 5.3 $b(m) \neq 0$. This completes the proof of Theorem 1.6. □
6 The case of $|\text{Cl}_K| = 3$

In the previous sections, we studied the cases of class number $h = |\text{Cl}_K|$ is either 1 or 2. However, it seems that the situation is somewhat different for the cases of class numbers $h \geq 3$. In this section, we discuss briefly how it is different, by considering the case of $d = 23$ ($h = 3$).

We first remark that one reason of success for the cases $h = 1$ and $h = 2$ is that the coefficients $a(m)$ of the Hecke eigenform $\Psi_{K,A}$ are all integers. Therefore, by the formula (10) $z = a(p)/p$ is a rational number (and since it is an algebraic integer), and so it must be a rational integer. It seems that this situation is no more true in general for the cases of $h \geq 3$. We will give more details information, concentrating the special (and typical) case of $d = 23$.

We denoted by $\mathfrak{o}, \mathfrak{a}_1$ and $\mathfrak{a}_2$ the ideal classes. The corresponding quadratic forms are $x^2 + xy + 6y^2$, $2x^2 - xy + 3y^2$ and $2x^2 + xy + 3y^2$, namely, $L_\mathfrak{o} = \langle (1, 0), (1/2, \sqrt{3}/2) \rangle$, $L_{\mathfrak{a}_1} = \langle (2, 0), (1/2, \sqrt{3}/2) \rangle$ and $L_{\mathfrak{a}_2} = \langle (2, 0), (-1/2, \sqrt{3}/2) \rangle$, respectively. We give the weighed theta series of those ideal lattices. We set $P_1 = x^2 - y^2$ and $P_2 = xy$ in this section.

$$
\Theta_{L_\mathfrak{o}} = 1 + 2q + 2q^4 + 4q^6 + 4q^8 + 2q^9 + 4q^{12} + 2q^{16} + 4q^{18} + 2q^{23} + 4q^{24} + 2q^{25} + 4q^{26} + 4q^{27} + 4q^{32} + 6q^{36} + 4q^{39} + 8q^{48} + 2q^{49} + 4q^{52} + 4q^{54} + 4q^{58} + 4q^{59} + 4q^{62} + 6q^{64} + 8q^{72} + 4q^{78} + 2q^{81} + 4q^{82} + 4q^{87} + 2q^{92} + 4q^{93} + 4q^{94} + 8q^{96} + 2q^{100} + O[q]^{101}
$$

$$
\frac{1}{2} \times \Theta_{L_\mathfrak{o}, P_1} = q + 4q^4 - 11q^6 - 7q^8 + 9q^9 + q^{12} + 16q^{16} + 13q^{18} - 23q^{23} - 44q^{24} + 25q^{25} + 29q^{26} - 38q^{27} - 28q^{32} + 85q^{36} - 14q^{39} + 77q^{48} + 49q^{49} - 103q^{52} - 99q^{54} - 91q^{58} + 26q^{59} + 101q^{62} - 15q^{64} - 11q^{72} + 133q^{78} + 81q^{81} - 43q^{82} + 82q^{87} - 92q^{92} - 182q^{93} - 19q^{94} - 7q^{96} + 100q^{100} + O[q]^{101}
$$

$$
\Theta_{L_{\mathfrak{a}_1}, P_2} = 0
$$

$$
\Theta_{L_{\mathfrak{a}_1}, P_1} = 8q^2 - 11q^3 - 7q^4 + q^6 + 32q^8 + 13q^9 - 88q^{12} + 29q^{13} - 56q^{16} + 121q^{18} + 81q^{24} - 103q^{26} - 99q^{27} + 91q^{29} + 101q^{31} + 49q^{32} + 41q^{36} + 133q^{39} - 43q^{41} - 18q^{46} - 19q^{47} - 183q^{48} + 200q^{50} + 232q^{52} - 295q^{54} + 209q^{58} + 41q^{62} - 224q^{64} + 253q^{69} + 77q^{71} + 393q^{72} - 283q^{73} - 275q^{75} - 375q^{78} + 418q^{81} - 247q^{82} - 227q^{87} + 161q^{92} - 203q^{93} + 353q^{94} + 616q^{96} +
$$
392q^{98} - 175q^{100} + O[q]^{101}

\frac{4}{\sqrt{23}} \times \Theta_{L_1, \nu_2} = q^3 - 3q^4 + 5q^6 - 7q^9 + 9q^{13} - 11q^{18} + 13q^{24} - 3q^{26} + 9q^{27} - 15q^{29} - 15q^{31} + 21q^{32} - 27q^{36} + 17q^{39} + 33q^{41} - 39q^{47} - 19q^{48} + 45q^{54} + 31q^{58} - 51q^{62} - 23q^{69} + 57q^{71} + 5q^{72} - 15q^{73} + 25q^{75} - 35q^{78} - 38q^{81} + 45q^{82} - 55q^{87} + 69q^{92} + 65q^{93} - 27q^{94} - 75q^{100} + O[q]^{101}

\Theta_{L_2, \nu_1} = 1 + 2q^2 + 2q^3 + 2q^4 + 2q^6 + 2q^8 + 4q^{12} + 2q^{13} + 4q^{16} + 4q^{18} + 6q^{24} + 2q^{26} + 2q^{27} + 2q^{29} + 2q^{31} + 4q^{32} + 6q^{36} + 2q^{39} + 2q^{41} + 2q^{46} + 2q^{47} + 6q^{48} + 2q^{50} + 4q^{52} + 6q^{54} + 2q^{58} + 2q^{62} + 4q^{64} + 2q^{69} + 2q^{71} + 8q^{72} + 2q^{73} + 2q^{75} + 6q^{78} + 4q^{81} + 2q^{82} + 2q^{87} + 2q^{92} + 2q^{93} + 2q^{94} + 8q^{96} + 2q^{98} + 2q^{100} + O[q]^{101}

2 \times \Theta_{L_2, \nu_1} = 8q^2 - 11q^3 - 7q^4 + q^6 + 32q^8 + 13q^9 - 88q^{12} + 29q^{13} - 56q^{16} + 121q^{18} + 81q^{24} - 103q^{26} - 99q^{27} - 91q^{29} + 101q^{31} + 49q^{32} + 41q^{36} + 133q^{39} - 43q^{41} - 184q^{46} - 19q^{47} - 183q^{48} + 200q^{50} + 232q^{52} - 295q^{54} + 209q^{58} + 41q^{62} - 224q^{63} + 253q^{69} + 77q^{71} + 393q^{72} - 283q^{73} - 275q^{75} - 375q^{78} + 418q^{81} - 247q^{82} - 227q^{87} + 161q^{92} - 203q^{93} + 353q^{94} + 616q^{96} + 392q^{98} - 175q^{100} + O[q]^{101}

We calculate the Hecke character of weight 3 and modulus (1), i.e, we calculate \( \Psi_{K, \Lambda} = \sum_{m \geq 1} a(m)q^m \), where \( \Lambda = (1) \) and \( k = 3 \). When \( A \) of norm \( m \) is a nonprincipal ideal, \( A^3 \) is a principal ideal. Then we set \( \phi(A^3) = \phi(A)^3 \). For example, \( (2, -1/2 + \sqrt{-23}/2)^3 = (-3/2 - \sqrt{-23}/2) \). Because of

\[
\phi \left( \left( \frac{-3 - \sqrt{-23}}{2} \right) \right) = \left( \frac{-3 - \sqrt{-23}}{2} \right)^2 = \frac{-7 + 3\sqrt{-23}}{2},
\]

\( \phi((2, -1/2 + \sqrt{-23}/2)) \) is one of the roots of

\[
(11) \quad x^3 - \left( \frac{-7 + 3\sqrt{-23}}{2} \right) = 0.
\]

We denote by \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) the roots of equation (11), namely, \( \alpha_1 \sim -1.86272 + 0.728188i, \alpha_2 \sim 0.300733 - 1.97726i \) and \( \alpha_3 \sim 1.56199 + 1.24907i \), respectively. Then, \( \phi((2, -1/2 + \sqrt{-23}/2)) \) is one of \( \alpha_1, \alpha_2 \) or \( \alpha_3 \). (Actually there are three different Hecke characters in this case.) First let us set \( \phi((2, -1/2 + \sqrt{-23}/2)) = \alpha_1 \). By the equation \( (2, -1/2 + \sqrt{-23}/2) \)
\(\sqrt{-23}/2 = (2),\)

\[\phi\left(\left(2, \frac{-1 + \sqrt{-23}}{2}\right)\right) \times \phi\left(\left(2, \frac{1 + \sqrt{-23}}{2}\right)\right) = \phi((2)).\]

We get

\[\alpha_1 \times \phi\left(\left(2, \frac{1 + \sqrt{-23}}{2}\right)\right) = 4,\]

hence, \(\phi((2, 1/2 + \sqrt{-23}/2)) = 4/\alpha_1.\) So,

\[a(2) = \phi\left(\left(2, \frac{-1 + \sqrt{-23}}{2}\right)\right) + \phi\left(\left(2, \frac{1 + \sqrt{-23}}{2}\right)\right) = \alpha_1 + 4/\alpha_1.\]

By the equation \((2, -1/2 + \sqrt{-23}/2) \times (3, 1/2 - \sqrt{-23}/2) = (1/2 - \sqrt{-23}/2),\)

\[\phi\left(\left(2, \frac{-1 + \sqrt{-23}}{2}\right)\right) \times \phi\left(\left(3, \frac{1 - \sqrt{-23}}{2}\right)\right) = \phi\left(\left(\frac{1 - \sqrt{-23}}{2}\right)\right).\]

We get

\[\alpha_1 \times \phi\left(\left(3, \frac{1 - \sqrt{-23}}{2}\right)\right) = \left(\frac{1 - \sqrt{-23}}{2}\right)^2 = \frac{-11 - \sqrt{-23}}{2},\]

hence, \(\phi((3, 1/2 - \sqrt{-23}/2)) = (-11 - \sqrt{-23})/2 \times 1/\alpha_1.\) Similarly, \(\phi((3, -1/2 - \sqrt{-23}/2)) = (-11 + \sqrt{-23})/2 \times \alpha_1/\alpha_1^2 + 4).\) So,

\[a(3) = \phi\left(\left(3, \frac{1 - \sqrt{-23}}{2}\right)\right) + \phi\left(\left(3, \frac{-1 - \sqrt{-23}}{2}\right)\right)\]

\[= \frac{-11 - \sqrt{-23}}{2} \times \frac{1}{\alpha_1} + \frac{-11 + \sqrt{-23}}{2} \times \frac{\alpha_1}{\alpha_1^2 + 4}.\]

We recall \(\alpha_1 \sim -1.86272 + 0.728188i.\) Then, we obtain

\[\Psi_{K,\Lambda}^{(i)} = q - 3.72545q^2 + 4.24943q^3 + \cdots.\]

Actually, it is possible to continue this calculation, but we need the information on the basis of all the ideals, which is rather complicated. So, we determine the Hecke eigenforms \(\Psi_{K,\Lambda}^{(i)}\) by a different method. By computer calculation (using “Sage” [17]), we know that the space of the modular forms
of weight 3 where $\Psi_{K,\Lambda}$ belongs is of dimension 3. We can calculate the basis of this modular form explicitly, and the three basis elements are of the form:

\begin{align*}
q + 4q^4 - 11q^6 - 7q^8 + 9q^9 + \cdots, \\
q^2 - 5q^4 + 7q^6 + 4q^8 - 8q^9 + \cdots, \\
q^3 - 3q^4 + 5q^6 - 7q^9 + \cdots.
\end{align*}

On the other hand, because of Theorems 3.1 and 3.2, $\Theta_{L_0,P_1}$, $\Theta_{L_{a_1},P_1}$ and $\Theta_{L_{a_2},P_2}$ are in the same space of Hecke eigenforms $\Psi_{K,\Lambda}^{(i)}$. Therefore, comparing the first three coefficients of the following equation:

$$
\Psi_{K,\Lambda}^{(i)}(q) = \frac{1}{2} \Theta_{L_0,P}(q) + a_2 \Theta_{L_{a_1},P}(q) + b_2 \sqrt{23} \Theta_{L_{a_2},P}(q),
$$

we can find numbers $a$ and $b$ as follows:

$$(a,b) = \begin{cases}
(A_1, B_2), \\
(A_2, B_1), \\
(A_3, B_3),
\end{cases}
$$

where $A_1, A_2$ and $A_3$ are the elements defined by

$$
\{x \mid 512x^3 - 96x + 7 = 0\} = \{A_1 = -0.465681, A_2 = 0.0751832, A_2 = 0.390498\},
$$

respectively, and $B_1, B_2$ and $B_3$ are the elements defined by

$$
\{x \mid 512x^3 - 2208x + 1587 = 0\} = \{B_1 = -2.37065, B_2 = 0.873067, B_3 = 1.49759\},
$$

respectively.

In this way, we can calculate the Hecke eigenforms $\Psi_{K,\Lambda}^{(i)}$. Namely,

$$
\Psi_{K,\Lambda}^{(1)} = q - 3.72545q^2 + 4.24943q^3 + 9.87897q^4 - 15.831q^5 - 21.9018q^6 + 9.05761q^7 + 41.9799q^8 - 21.3624q^9 + 42.0781q^{10} - 33.7437q^{11} - 23q^{12} - 93.07q^{13} + 25q^{14} + 79.5844q^{15} + 0.244826q^{16} + 55.473q^{17} - 33.9378q^{18} - 69.1528q^{19} + 89.4799q^{20} - 90.7777q^{21} - 8.78692q^{22} + 85.6853q^{23} + 42.8975q^{24} + 178.808q^{25} + 49q^{26} - 93.1362q^{27} + O[q^{28}],
$$

$$
\Psi_{K,\Lambda}^{(2)} = q + 0.601466q^2 + 1.54364q^3 - 3.63824q^4 + 0.928445q^5 - 4.59414q^6 - 6.61718q^7 - 5.61612q^8 + 23.5162q^9 + 11.7897q^{10} - 3.98001q^{11} - 23q^{12} - 7.90168q^{13} + 25q^{14} + 14.1442q^{15} -
$$
24.1073q^{27} - 42.4015q^{29} - 27.9663q^{31} + 25.4677q^{32} + 24.0749q^{36} + 36.3005q^{39} + 74.9986q^{41} - 13.8337q^{46} - 93.8839q^{47} + 18.1991q^{48} + 49q^{49} + 15.0366q^{50} + O[q]^{51}.

Ψ_{K,A}^{(3)} = q + 3.12398q^2 - 5.79306q^3 + 5.75927q^4 - 18.0974q^6 + 5.49593q^8 + 24.5596q^9 - 33.3638q^{12} - 2.15383q^{13} - 5.86788q^{16} + 76.7237q^{18} - 23q^{21} - 31.8383q^{24} + 25q^{25} - 6.72853q^{26} - 90.1376q^{27} - 13.0715q^{29} + 61.9041q^{31} - 40.3149q^{32} + 141.445q^{36} + 12.4773q^{39} - 66.2117q^{41} - 71.8516q^{46} + 50.9864q^{47} + 33.993q^{48} + 49q^{49} + 78.0996q^{50} + O[q]^{51}.

The coefficients $a(m)$ for this case are far from integers. In fact they are not elements in a cyclotomic number field in general. So, it seems difficult to use the Hecke eigenforms obtained this way to apply for the case of the class number 3 or more in general. Some new additional ideas will be needed to treat the case of $d = 23$ or more generally the cases of class numbers $h \geq 3$. We have included the presentation of the results (although they are not conclusive) for $d = 23$, hoping that it might help the reader for the future study on this topic.

**Remark 6.1.** We remark that the coefficients of $Ψ_{K,A}^{(i)}$ in above calculator results are not exact values but approximate values.

| $N(A)$ | $A$: ideal          | $N(A)$ | $A$: ideal          |
|--------|---------------------|--------|---------------------|
| 1      | (1)                 | 6      | (1/2 - $\sqrt{-23}$/2) |
|        |                     |        | (6, 5/2 + $\sqrt{-23}$/2) |
| 2      | (2, -1/2 + $\sqrt{-23}$/2) |        | (6, 7/2 + $\sqrt{-23}$/2) |
|        | (2, 1/2 + $\sqrt{-23}$/2) |        | (1/2 + $\sqrt{-23}$/2) |
| 3      | (3/2 + $\sqrt{-23}$/2) | 7      | -                   |
|        | (3, 1/2 - $\sqrt{-23}$/2) |        |                     |
| 4      | (9, 5/2 + $\sqrt{-23}$/2) | 8      | (-3/2 - $\sqrt{-23}$/2) |
|        | (4, 3/2 + $\sqrt{-23}$/2) |        | (4, -1 + $\sqrt{-23}$/2) |
|        | (2)                 |        | (-3/2 + $\sqrt{-23}$/2) |
|        | (4, 5/2 + $\sqrt{-23}$/2) | 9      | (9, 11/2 + $\sqrt{-23}$/2) |
|        |                     |        | (3)                 |
|        |                     | 10     | (9, 7/2 + $\sqrt{-23}$/2) |

Table 4: Integral ideals of small norm of $d = 23$
7 Concluding Remarks

(1) In this paper, we use the mathematics software “Sage” [17]. In particular, The results in Tables 1 and 2 are computed by “Sage” using the command “K.ideals_of_bdd_norm()”. We remark that this command does not always give a $\mathbb{Z}$-basis of ideal. We must make sure the command “(ideal).basis()”.

(2) In Appendix C, we list theta series of lattices obtained from $\mathbb{Q}(\sqrt{-5})$. The other cases are listed in one of the author’s websites [12].

(3) In the previous paper [3], we studied the spherical designs in the nonempty shells of the $\mathbb{Z}^2$-lattice and $A_2$-lattice. The results state that any shells in the $\mathbb{Z}^2$-lattice (resp. $A_2$-lattice) are spherical 2-design (resp. 4-design). However, the nonempty shells in the $\mathbb{Z}^2$-lattice (resp. $A_2$-lattice) are not spherical 4-design (resp. 6-design). It is interesting to note that no spherical 6-design among the nonempty shells of any Euclidean lattice of 2-dimension is known. It is an interesting open problem to prove or disprove whether these exist any 6-design which is a shell of a Euclidean lattice of 2-dimension.

Responding to the authors’ request, Junichi Shigezumi performed computer calculations to determine whether there are spherical $t$-designs for bigger $t$, in the 2- and 3-dimensional cases. His calculation shows that among the nonempty shells of integral lattices in 2-dimension (with relatively small discriminant and small norms), there are only 4-designs. That is, no 6-designs were found. (So far, all examples of such 4-designs are the union of vertices of regular 6-gons, although they are the nonempty shells of many different lattices). In the 3-dimensional case, all examples obtained are only 2-designs. No 4-designs which are shells of a lattice were found. It is an interesting open problem whether this is true in general for the dimensions 2 and 3. Moreover, it is interesting to note that no spherical 12-design among the nonempty shells of any Euclidean lattice (of any dimension) is known. It is also an interesting open problem to prove or disprove whether these exist any 12-design which is a shell of a Euclidean lattice.

Finally, we state the following conjecture for the 2-dimensional lattices.

**Conjecture 7.1.** Let $L$ be the Euclidean lattice of 2-dimension, whose quadratic form is $ax^2 + bxy + cy^2$. 

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(i) Assume that $b^2 - 4ac = \text{(Integer)}^2 \times (-3)$. Then, all the nonempty shells of $L$ are not spherical 6-designs and some of the nonempty shells of $L$ are spherical 4-designs. Moreover, if all the nonempty shells of $L$ are spherical 4-designs then $b^2 - 4ac = -3$, that is, $A_2$-lattice.

(ii) Assume that $b^2 - 4ac = \text{(Integer)}^2 \times (-4)$. Then, all the nonempty shells of $L$ are not spherical 4-designs and some of the nonempty shells of $L$ are spherical 2-designs. Moreover, if all the nonempty shells of $L$ are spherical 2-designs then $b^2 - 4ac = -4$, that is, $\mathbb{Z}^2$-lattice.

(iii) Otherwise, all the nonempty shells of $L$ are not spherical 2-designs.

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A The case of $|\text{Cl}_K| = 1$

| $-d$ | $-d \pmod{4}$ | $d_K$ | $L_\alpha$ |
|------|----------------|-------|-----------|
| $-1$ | 3              | $-2^2$ | $[1, \sqrt{-1}]$ |
| $-2$ | 2              | $-2^3$ | $[1, \sqrt{-2}]$ |
| $-3$ | 1              | $-3$   | $[1, (1 + \sqrt{-3})/2]$ |
| $-7$ | 1              | $-7$   | $[1, (1 + \sqrt{-7})/2]$ |
| $-11$ | 1            | $-11$ | $[1, (1 + \sqrt{-11})/2]$ |
| $-19$ | 1            | $-19$ | $[1, (1 + \sqrt{-19})/2]$ |
| $-43$ | 1            | $-43$ | $[1, (1 + \sqrt{-43})/2]$ |
| $-67$ | 1            | $-67$ | $[1, (1 + \sqrt{-67})/2]$ |
| $-163$ | 1          | $-163$ | $[1, (1 + \sqrt{-163})/2]$ |
B  The case of $|\text{Cl}_K| = 2$

Table 6: $|\text{Cl}_K| = 2$

| $-d$ | $-d \pmod{4}$ | $d_K$  | $L_a$    | $L_b$   |
|------|----------------|--------|----------|---------|
| -5   | 3              | $-2^2 \times 5$ | $[1, \sqrt{-5}]$ | $[2, 1 + \sqrt{-5}]$ |
| -6   | 2              | $-2^2 \times 3$ | $[1, \sqrt{-6}]$ | $[2, \sqrt{-6}]$ |
| -10  | 2              | $-2^3 \times 5$ | $[1, \sqrt{-10}]$ | $[2, \sqrt{-10}]$ |
| -13  | 3              | $-2^2 \times 13$ | $[1, \sqrt{-13}]$ | $[2, 1 + \sqrt{-13}]$ |
| -15  | 1              | $-3 \times 5$ | $[1, (1 + \sqrt{-15})/2]$ | $[2, (1 + \sqrt{-15})/2]$ |
| -22  | 2              | $-2^3 \times 11$ | $[1, \sqrt{-22}]$ | $[2, \sqrt{-22}]$ |
| -35  | 1              | $-5 \times 7$ | $[1, (1 + \sqrt{-35})/2]$ | $[3, (1 + \sqrt{-35})/2]$ |
| -37  | 3              | $-2^2 \times 37$ | $[1, \sqrt{-37}]$ | $[2, 1 + \sqrt{-37}]$ |
| -51  | 1              | $-3 \times 17$ | $[1, (1 + \sqrt{-51})/2]$ | $[3, (3 + \sqrt{-51})/2]$ |
| -58  | 2              | $-2^2 \times 29$ | $[1, \sqrt{-58}]$ | $[2, \sqrt{-58}]$ |
| -91  | 1              | $-7 \times 13$ | $[1, (1 + \sqrt{-91})/2]$ | $[5, (3 + \sqrt{-91})/2]$ |
| -115 | 1              | $-5 \times 23$ | $[1, (1 + \sqrt{-115})/2]$ | $[5, (5 + \sqrt{-115})/2]$ |
| -123 | 1              | $-3 \times 41$ | $[1, (1 + \sqrt{-123})/2]$ | $[3, (3 + \sqrt{-123})/2]$ |
| -187 | 1              | $-11 \times 17$ | $[1, (1 + \sqrt{-187})/2]$ | $[7, (3 + \sqrt{-187})/2]$ |
| -235 | 1              | $-5 \times 47$ | $[1, (1 + \sqrt{-235})/2]$ | $[5, (5 + \sqrt{-235})/2]$ |
| -267 | 1              | $-3 \times 89$ | $[1, (1 + \sqrt{-267})/2]$ | $[3, (3 + \sqrt{-267})/2]$ |
| -403 | 1              | $-13 \times 31$ | $[1, (1 + \sqrt{-403})/2]$ | $[11, (9 + \sqrt{-403})/2]$ |
| -427 | 1              | $-7 \times 61$ | $[1, (1 + \sqrt{-427})/2]$ | $[7, (7 + \sqrt{-427})/2]$ |
C \ \text{Theta series of } L_0 \ \text{and } L_\alpha \ \text{of } \mathbb{Q}(\sqrt{-5})

\Theta_{L_\sigma} = 1 + 2q + 2q^2 + 2q^3 + 4q^4 + 6q^5 + 6q^6 + 4q^7 + 4q^8 + 2q^9 + 2q^{10} + q^{10} + q^{12} + q^{13} + q^{14} + q^{15} + q^{16} + q^{18} + q^{20} + q^{22} + q^{23} + q^{24} + q^{25} + q^{26} + q^{27} + q^{28} + q^{29} + q^{30} + q^{31} + q^{32} + q^{33} + q^{34} + q^{35} + q^{36} + q^{37} + q^{38} + q^{39} + q^{40} + q^{41} + q^{42} + q^{43} + q^{44} + q^{45} + q^{46} + q^{47} + q^{48} + q^{49} + q^{50} + O(q^{51})

\Theta_{L_\alpha} = 1 + 2q^2 + 2q^3 + 4q^4 + 6q^5 + 6q^6 + 4q^7 + 4q^8 + 2q^9 + 2q^{10} + q^{10} + q^{12} + q^{13} + q^{14} + q^{15} + q^{16} + q^{18} + q^{20} + q^{22} + q^{23} + q^{24} + q^{25} + q^{26} + q^{27} + q^{28} + q^{29} + q^{30} + q^{31} + q^{32} + q^{33} + q^{34} + q^{35} + q^{36} + q^{37} + q^{38} + q^{39} + q^{40} + q^{41} + q^{42} + q^{43} + q^{44} + q^{45} + q^{46} + q^{47} + q^{48} + q^{49} + q^{50} + O(q^{51})

\Theta_{L_\sigma, \alpha} = q + 4q^4 - 5q^5 - 8q^6 - 7q^7 + 8q^8 + 16q^9 + 16q^{10} - 20q^{11} + 32q^{12} + 25q^{13} + 25q^{14} + 20q^{15} + 30q^{16} + 28q^{17} + 62q^{18} - 35q^{19} - 35q^{20} - 35q^{21} - 35q^{22} - 35q^{23} - 35q^{24} - 35q^{25} - 35q^{26} - 35q^{27} - 35q^{28} - 35q^{29} - 35q^{30} - 35q^{31} - 35q^{32} - 35q^{33} - 35q^{34} - 35q^{35} - 35q^{36} - 35q^{37} - 35q^{38} - 35q^{39} - 35q^{40} - 35q^{41} - 35q^{42} - 35q^{43} - 35q^{44} - 35q^{45} - 35q^{46} - 35q^{47} - 35q^{48} - 35q^{49} - 35q^{50} - O(q^{51})
\[ \Psi^{(1)}_{K,q}(z) = q + 2q^2 - 4q^3 + 4q^4 - 5q^5 - 8q^6 + 4q^7 + 8q^8 + 7q^9 - 10q^{10} - 16q^{12} - 8q^{14} + 20q^{15} + 16q^{16} + 14q^{18} - 20q^{20} - 16q^{21} - 4q^{23} - 32q^{24} + 25q^{25} + 8q^{27} + 16q^{28} - 22q^{29} + 40q^{30} + 32q^{32} - 20q^{35} + 28q^{36} - 40q^{40} + 62q^{41} - 32q^{42} + 76q^{43} - 35q^{45} - 88q^{46} + 4q^{47} - 64q^{48} - 33q^{49} + 50q^{50} + 16q^{51} + 32q^{54} - 44q^{58} + 80q^{60} - 5q^{61} + 25q^{63} + 16q^{64} - 11q^{66} + 17q^{69} - 40q^{70} + 56q^{72} - 100q^{75} - 80q^{80} - 95q^{81} + 124q^{92} + 76q^{93} - 64q^{94} + 152q^{96} + 86q^{97} - 142q^{98} - 70q^{99} - 176q^{100} + 32q^{94} - 128q^{96} - 66q^{98} + 100q^{100} + 122q^{101} - 44q^{103} + 80q^{105} + 124q^{107} + 32q^{108} + 38q^{109} + 64q^{112} + 22q^{115} - 8q^{116} + 16q^{120} + 121q^{121} + 116q^{122} - 248q^{123} - 125q^{125} + 56q^{126} - 236q^{127} + 128q^{128} - 304q^{129} - 232q^{130} + 40q^{135} + 352q^{138} - 80q^{140} - 16q^{141} + 112q^{144} + 110q^{145} + 132q^{147} + 278q^{149} + 200q^{150} - 160q^{160} + 170q^{161} - 190q^{162} - 164q^{163} + 248q^{164} + 152q^{166} + 244q^{167} - 128q^{168} + 169q^{169} + 304q^{172} + 170q^{174} + 100q^{175} - 25q^{178} - 140q^{180} - 358q^{181} + 232q^{183} - 352q^{184} + 16q^{188} + 32q^{189} - 256q^{192} - 132q^{196} + 200q^{200} + 464q^{201} + 244q^{202} - 88q^{203} - 310q^{205} - 88q^{206} - 308q^{207} + 160q^{210} + 248q^{214} - 380q^{215} + 64q^{216} + 76q^{218} + 436q^{223} + 128q^{224} + 175q^{225} - 356q^{227} - 262q^{229} + 440q^{230} - 176q^{232} - 20q^{235} + 320q^{240} + 302q^{241} + 242q^{242} + 308q^{243} - 232q^{244} + 165q^{245} - 496q^{246} - 304q^{249} - 250q^{250} + 112q^{252} - 472q^{254} + 256q^{256} - 608q^{258} + 154q^{261} - 28q^{263} - 568q^{267} - 64q^{268} + 38q^{269} - 80q^{270} + 704q^{276} - 160q^{280} - 248q^{281} - 32q^{282} + 316q^{283} + 284q^{287} + 224q^{288} + 289q^{289} + 220q^{290} + 264q^{294} + 556q^{298} - 400q^{300} + 304q^{301} - 488q^{303} + 290q^{305} - 596q^{307} + 176q^{309} - 140q^{315} - 320q^{320} - 496q^{321} - 352q^{322} - 380q^{324} - 328q^{326} - 152q^{327} + 496q^{328} + 16q^{329} + 304q^{332} + 488q^{334} + 580q^{335} - 256q^{336} + 338q^{338} + 328q^{339} - 352q^{342} - 152q^{347} + 496q^{348} + 289q^{349} + 256q^{352} + 496q^{358} + 604q^{382} + 704q^{384} + 176q^{401} - 488q^{404} + 475q^{405} - 176q^{406} - 802q^{409} - 620q^{410} + 176q^{412} - 616q^{414} - 380q^{415} + 320q^{420} - 77q^{421} + 28q^{423} - 232q^{424} + 496q^{428} - 760q^{430} + 128q^{432} + 440q^{435} + 152q^{436} - 231q^{441} + 796q^{443} + 710q^{445} + 872q^{446} - 1112q^{447} + 256q^{448} + 398q^{449} + 350q^{450} - 712q^{454} - 524q^{458} + 880q^{460} + 764q^{463} - 352q^{464} + 124q^{467} - 64q^{469} - 40q^{470} + 640q^{480} + 604q^{482} + 704q^{483} + 48q^{484} + 616q^{486} + 484q^{487} - 464q^{488} - 656q^{489} + 330q^{490} - 992q^{492} - 608q^{498} + 500q^{500} + O(q)^{501}
\]

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