Dynamics of an imprecise stochastic Holling II one-predator two-prey system with jumps

Fei Sun*

School of Mathematics and Computational Science, Wuyi University, Jiangmen 529020, China

Abstract
Groups in ecology are often affected by sudden environmental perturbations. Parameters of stochastic models are often imprecise due to various uncertainties. In this paper, we formulate a stochastic Holling II one-predator two-prey system with jumps and interval parameters. Firstly, we prove the existence and uniqueness of the positive solution. Moreover, the sufficient conditions for the extinction and persistence in the mean of the solution are obtained.

Keywords: Holling II predator-prey model, imprecise, jumps, persistence and extinction

1. Introduction
In ecology and mathematical ecology, the study of interrelationship between species has become one of the main topics. And there have been growing interests on the dynamical behavior of the population species living in groups, such as Holling type I, II, and III functional response. For a better review of Holling II functional response and its extension, see [1]-[9] as well as there references.

However, the sudden environmental perturbations may bring substantial social and economic losses. For example, the recent COVID-19 has a serious impact on the world. It is more realistic to study the population dynamics with imprecise parameters. Panja et al. [14] studied a cholera epidemic model with imprecise numbers and discussed the stability condition of equilibrium points of the system. Das and Pal [15] analyzed the stability of the system and solved the optimal control problem by introducing an imprecise SIR model. Other studies on imprecise parameters include those of [10]-[13], and the references therein.

The main focus of this paper is dynamics of an imprecise stochastic Holling II one-predator two-prey model with jumps. To this end, we first introduce the imprecise stochastic Holling II one-predator two-prey model. With the help of Lyapunov functions, we prove the existence and uniqueness of the positive solution. Further, the sufficient conditions for the extinction and persistence in the mean of the solution are obtained.

The remainder of this paper is organized as follows. In Sect. 2, we introduce the basic models. In Sect. 3, the unique global positive solution of the system is proved. The sufficient conditions for the extinction and persistence in the mean of the solution are derived in Sect. 4.

2. Imprecise stochastic Holling II one-predator two-prey system
In this section, we introduce the imprecise stochastic system. Let \( x_i(t) \) \( (i = 1, 2) \) and \( y(t) \) denote the population sizes of prey species and the population size of predator species at time \( t \), respectively. Then a

*Corresponding author
Email address: sunfei@whu.edu.cn (fsun.sci@outlook.com) (Fei Sun)
stochastic Holling II one-predator two-prey system takes the following form [18].

\[
\begin{align*}
\begin{aligned}
    dx_1(t) &= x_1(t)(r_1 - a_{11}x_1(t) - a_{12}x_2(t) - \frac{a_{13}y(t)}{1 + x_1(t)})dt + \sigma_1x_1(t)dB_1(t) + \int \gamma c_1(u)x_1(t^-)N(dt, du), \\
    dx_2(t) &= x_2(t)(r_2 - a_{21}x_1(t) - a_{22}x_2(t) - \frac{a_{23}y(t)}{1 + x_2(t)})dt + \sigma_2x_2(t)dB_2(t) + \int \gamma c_2(u)x_2(t^-)N(dt, du), \\
    dy(t) &= y(t)[-r_3 - a_{33}y(t) + \frac{a_{31}x_1(t)}{1 + x_1(t)} + \frac{a_{32}x_2(t)}{1 + x_2(t)}]dt + \sigma_3y(t)dB_3(t) + \int \gamma c_3(u)y(t^-)N(dt, du),
\end{aligned}
\end{align*}
\]

where \(x_1(t^-), x_2(t^-)\) and \(y(t^-)\) are the left limits of \(x_1(t), x_2(t)\) and \(y(t)\), respectively. \(r_i > 0\) \((i = 1, 2, 3)\) are the intrinsic growth rates or death rate, \(a_{ij}\) \((i = 1, 2, 3)\) stand for the intraspecies interaction, \(a_{ij}\) \((i \neq j)\) represent the effect of species \(j\) upon the growth rate of species \(i\). \(B_i(t), \ (i = 1, 2, 3)\) are mutually independent Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, \mathcal{F}_{t \geq 0}, \mathbb{P})\). \(\sigma_i^2\) represent the intensities of \(B_i(t)\). Let \(\lambda\) be the characteristic measure of \(N\) which is defined on a finite measurable subset \(\mathbb{Y}\) of \((0, +\infty)\) with \(\lambda(\mathbb{Y}) < \infty\). Define the compensated random measure by \(\tilde{N}(dt, du) = N(dt, du)\lambda(du)dt\).

Before we state the imprecise stochastic Holling II one-predator two-prey system, definitions of Interval-valued function should recalled (Pal [15]).

**Definition 2.1.** (Interval number) An interval number \(A\) is represented by closed interval \([g^l, g^u]\) and defined by \(A = [g^l, g^u] = \{x | g^l \leq x \leq g^u, x \in \mathbb{R}\}\), where \(\mathbb{R}\) is the set of real numbers and \(g^l, g^u\) are the lower and upper limits of the interval numbers, respectively. The interval number \([g, g]\) represents a real number \(g\). The arithmetic operations for any two interval numbers \(A = [g^l, g^u] \) and \(B = [h^l, h^u]\) are as follows:

- **Addition:** \(A + B = [g^l + h^l, g^u + h^u]\).
- **Subtraction:** \(A - B = [g^l - h^u, g^u - h^l]\).
- **Scalar multiplication:** \(\alpha A = [\alpha g^l, \alpha g^u]\), where \(\alpha\) is a positive real number.
- **Multiplication:** \(AB = [g^l h^l, g^u h^u]\).
- **Division:** \(A/B = [g^l/h^l, g^u/h^u]\).

**Definition 2.2.** (Interval-valued function) Let \(g > 0, h > 0\). If the interval is of the from \([g, h]\), the interval-valued function is take as \(f(k) = g^{(1-k)}h^k\) for \(k \in [0, 1]\).

Let \(\hat{r}_i, \hat{a}_{ij}, \hat{\sigma}_i\) represent the interval numbers of \(r_i, a_{ij}, \sigma_i\) \((i, j = 1, 2, 3)\), respectively. The system (2.1) with imprecise parameters becomes:

\[
\begin{align*}
\begin{aligned}
    dx_1(t) &= \hat{r}_1 - \hat{a}_{11}x_1(t) - \hat{a}_{12}x_2(t) - \frac{\hat{a}_{13}y(t)}{1 + x_1(t)}]dt + \hat{\sigma}_1x_1(t)dB_1(t) + \int \gamma c_1(u)x_1(t^-)N(dt, du), \\
    dx_2(t) &= \hat{r}_2 - \hat{a}_{21}x_1(t) - \hat{a}_{22}x_2(t) - \frac{\hat{a}_{23}y(t)}{1 + x_2(t)}]dt + \hat{\sigma}_2x_2(t)dB_2(t) + \int \gamma c_2(u)x_2(t^-)N(dt, du), \\
    dy(t) &= \hat{y}(t)[-\hat{r}_3 - \hat{a}_{33}y(t) + \frac{\hat{a}_{31}x_1(t)}{1 + x_1(t)} + \frac{\hat{a}_{32}x_2(t)}{1 + x_2(t)}]dt + \hat{\sigma}_3y(t)dB_3(t) + \int \gamma c_3(u)y(t^-)N(dt, du),
\end{aligned}
\end{align*}
\]

where \(\hat{r}_i = [r_i^l, r_i^u], \hat{a}_{ij} = [a_{ij}^l, a_{ij}^u], \hat{\sigma}_i = [\sigma_i^l, \sigma_i^u]\) \((i, j = 1, 2, 3)\).

According to the Theorem 1 in Pal et al. [10] and considering the interval-valued function \(f(p) = (f^l)^{1-p}(f^u)^p\) for interval \(\hat{f} = [f^l, f^u]\) for \(p \in [0, 1]\), we can prove that system (2.2) is equivalent to the following system:

\[
\begin{align*}
\begin{aligned}
    dx_1(t) &= x_1(t)([r_1^l]^{1-p}(r_1^u)^p) - (a_{11}^l)^{1-p}(a_{11}^u)^p x_1(t) - (a_{12}^l)^{1-p}(a_{12}^u)^p x_2(t) - \frac{(a_{13}^l)^{1-p}(a_{13}^u)^p y(t)}{1 + x_1(t)}]dt + \sigma_1^l x_1(t)dB_1(t) + \int \gamma^l c_1(u)x_1(t^-)N(dt, du), \\
    dx_2(t) &= x_2(t)([r_2^l]^{1-p}(r_2^u)^p) - (a_{21}^l)^{1-p}(a_{21}^u)^p x_1(t) - (a_{22}^l)^{1-p}(a_{22}^u)^p x_2(t) - \frac{(a_{23}^l)^{1-p}(a_{23}^u)^p y(t)}{1 + x_2(t)}]dt + \sigma_2^l x_2(t)dB_2(t) + \int \gamma^l c_2(u)x_2(t^-)N(dt, du), \\
    dy(t) &= y(t)[-([r_3^l]^{1-p}(r_3^u)^p) - (a_{31}^l)^{1-p}(a_{31}^u)^p y(t) + (a_{31}^l)^{1-p}(a_{31}^u)^p x_1(t) + \frac{(a_{32}^l)^{1-p}(a_{32}^u)^p x_2(t)}{1 + x_1(t)} + \frac{(a_{33}^l)^{1-p}(a_{33}^u)^p y(t)}{1 + x_2(t)}]dt + \sigma_3^l y(t)dB_3(t) + \int \gamma^l c_3(u)y(t^-)N(dt, du),
\end{aligned}
\end{align*}
\]
for $p \in [0, 1]$.

Throughout this paper, let

$$b_i = (r^i_1)^{1-p}(r^i_2)^p - \frac{((\sigma^i_1)^{1-p}(\sigma^i_2)^p)^2}{2} + \int_Y \ln(1 + c_i(u)) \lambda(du), \quad i = 1, 2.$$ 

$$b_3 = - (r^3_1)^{1-p}(r^3_2)^p - \frac{((\sigma^3_1)^{1-p}(\sigma^3_2)^p)^2}{2} + \int_Y \ln(1 + c_3(u)) \lambda(du).$$

$$\langle f(t) \rangle = t^{-1} \int_0^t f(s)ds, \quad \langle f(t) \rangle^{*} = \limsup_{t \to \infty} t^{-1} \int_0^t f(s)ds, \quad \langle f(t) \rangle_{\ast} = \liminf_{t \to \infty} t^{-1} \int_0^t f(s)ds.$$

3. Existence and uniqueness of positive solution of system (2.3)

For convenience in the following investigation, we require that

(H1) $\int_Y (|c_i(u)| \lor |c_i(u)|^2) \lambda(du) \leq \infty.$

(H2) $\int_Y (|\ln(1 + c_1(u))| \lor |\ln(1 + c_2(u))|^2) \lambda(du) \leq \infty$ \quad $i = 1, 2, 3.$

The following theorem will prove that system (2.3) admits a unique global positive solution.

Theorem 3.1. Let Assumptions (H1) and (H2) hold. Then for any given initial value $(x_1(0), x_2(0), y(0)) \in \mathbb{R}_+^3$, system (2.3) has a unique solution $(x_1(t), x_2(t), y(t)) \in \mathbb{R}_+^3$ for all $t \geq 0$ almost surely (a.s.).

Proof. Since $t \geq 0$, by system (2.3), we can construct the following system

$$
\begin{align*}
\frac{du_1(t)}{dt} &= \left( b_1 - (a_{11}^u)^{1-p}(a_{11}^u)^p e^{u_1(t)} - (a_{12}^u)^{1-p}(a_{12}^u)^p e^{u_2(t)} - \frac{(a_{13}^u)^{1-p}(a_{13}^u)^p e^{u_3(t)}}{1 + e^{u_3(t)}} \right) dt \\
&\quad + (\sigma_{11}^u)^{1-p}(\sigma_{11}^u)^p dB_1(t) + \int_Y \ln(1 + c_1(u)) N(dt, du), \\
\frac{du_2(t)}{dt} &= \left( b_2 - (a_{21}^u)^{1-p}(a_{21}^u)^p e^{u_1(t)} - (a_{22}^u)^{1-p}(a_{22}^u)^p e^{u_2(t)} - \frac{(a_{23}^u)^{1-p}(a_{23}^u)^p e^{u_3(t)}}{1 + e^{u_2(t)}} \right) dt \\
&\quad + (\sigma_{21}^u)^{1-p}(\sigma_{21}^u)^p dB_2(t) + \int_Y \ln(1 + c_2(u)) N(dt, du), \\
\frac{dv(t)}{dt} &= \left( b_3 - (a_{31}^v)^{1-p}(a_{31}^v)^p e^{v(t)} + \frac{(a_{32}^v)^{1-p}(a_{32}^v)^p e^{u_3(t)}}{1 + e^{u_1(t)}} + \frac{(a_{33}^v)^{1-p}(a_{33}^v)^p e^{u_3(t)}}{1 + e^{u_2(t)}} \right) dt \\
&\quad + (\sigma_{31}^v)^{1-p}(\sigma_{31}^v)^p dB_3(t) + \int_Y \ln(1 + c_3(u)) N(dt, du),
\end{align*}
$$

Because the coefficients of this system are local Lipschitz continuous (Mao [16]), for any given initial value $(u_1(0), u_2(0), v(0)) = (\ln x_1(0), \ln x_2(0), \ln y(0)) \in \mathbb{R}_+^3$, there is a unique local solution $(u_1(t), u_2(t), v(t))$ on $t \in [0, \tau)$, where $\tau$ is the explosion time (see Mao [16]). Hence, system (2.3) admits unique positive local solution $(x_1(t), x_2(t), y(t)) = (e^{u_1(t)}, e^{u_2(t)}, e^{v(t)})$. The proof of global existence of this local solution to system (2.3) is rather standard. Therefore, we omit the proof here.

By Theorem 3.1, system (2.3) admits a unique global positive solution. Next, we will show that the solution of system (2.3) is stochastically bounded.

Theorem 3.2. Let Assumptions (H1) and (H2) hold. Then for any given initial value $(x_1(0), x_2(0), y(0)) \in \mathbb{R}_+^3$ and $k > 0$, the solution of system (2.3) satisfies

$$\limsup_{t \to \infty} \mathbb{E}(x_1^k(t) + x_2^k(t) + y^k(t)) \leq K,$$

where $K$ is a generic positive constant.

Proof. The Itô’s formula (Situ [13]) shows that

$$\mathbb{E}(e^{t}(x_1^k(t) + x_2^k(t) + y^k(t))) = x_1^k(0) + x_2^k(0) + y^k(0) + \mathbb{E} \int_0^t e^s F(s)ds,$$
Lemma 4.1.

Thus, which implies

\[ F = - (a_{11}^l)^{1-p}(a_{11}^u)^{p}kx_1^{k+1} + \left( 1 + k(r_1^l)^{1-p}(r_1^u)^{p} \right) + \frac{k(k-1)}{2}((\sigma_1^l)^{1-p}(\sigma_1^u)^{p})^2 + \int_Y ((1 + c_1(u))^k - 1) \lambda(du) x_1^k \]

\[ - (a_{12}^l)^{1-p}(a_{12}^u)^{p}kx_1^{k+1} - \frac{(a_{13}^l)^{1-p}(a_{13}^u)^{p}kx_1^{k} y}{1 + x_1} \]

\[ - (a_{22}^l)^{1-p}(a_{22}^u)^{p}kx_2^{k+1} + \left( 1 + k(r_2^l)^{1-p}(r_2^u)^{p} \right) + \frac{k(k-1)}{2}((\sigma_2^l)^{1-p}(\sigma_2^u)^{p})^2 + \int_Y ((1 + c_2(u))^k - 1) \lambda(du) x_2^k \]

\[ - (a_{21}^l)^{1-p}(a_{21}^u)^{p}kx_1x_2^k - \frac{(a_{23}^l)^{1-p}(a_{23}^u)^{p}kx_2^{k} y}{1 + x_2} \]

\[ - (a_{33}^l)^{1-p}(a_{33}^u)^{p}ky^{k+1} + \left( 1 - k(r_3^l)^{1-p}(r_3^u)^{p} \right) + \frac{k(k-1)}{2}((\sigma_3^l)^{1-p}(\sigma_3^u)^{p})^2 + \int_Y ((1 + c_3(u))^k - 1) \lambda(du) y^k \]

\[ + \frac{(a_{31}^l)^{1-p}(a_{31}^u)^{p}kx_1y^{k}}{1 + x_1} + \frac{(a_{32}^l)^{1-p}(a_{32}^u)^{p}kx_2y^{k}}{1 + x_2} \]

\[ \leq - (a_{11}^l)^{1-p}(a_{11}^u)^{p}kx_1^{k+1} + \left( 1 + k(r_1^l)^{1-p}(r_1^u)^{p} \right) + \frac{k(k-1)}{2}((\sigma_1^l)^{1-p}(\sigma_1^u)^{p})^2 + \int_Y ((1 + c_1(u))^k - 1) \lambda(du) x_1^k \]

\[ - (a_{22}^l)^{1-p}(a_{22}^u)^{p}kx_2^{k+1} + \left( 1 + k(r_2^l)^{1-p}(r_2^u)^{p} \right) + \frac{k(k-1)}{2}((\sigma_2^l)^{1-p}(\sigma_2^u)^{p})^2 + \int_Y ((1 + c_2(u))^k - 1) \lambda(du) x_2^k \]

\[ - (a_{33}^l)^{1-p}(a_{33}^u)^{p}ky^{k+1} + \left( 1 - k(r_3^l)^{1-p}(r_3^u)^{p} \right) + (a_{31}^l)^{1-p}(a_{31}^u)^{p} k + (a_{32}^l)^{1-p}(a_{32}^u)^{p} k \]

\[ + \frac{k(k-1)}{2}((\sigma_3^l)^{1-p}(\sigma_3^u)^{p})^2 + \int_Y ((1 + c_3(u))^k - 1) \lambda(du) y^k \]

\[ \leq K. \]

Thus,

\[ \mathbb{E}(e^t(x_1^{k}(t) + x_2^{k}(t) + y^{k}(t))) \leq x_1^k(0) + x_2^k(0) + y^k(0) + Ke^t, \]

which implies

\[ \limsup_{t \to \infty} \mathbb{E}(x_1^{k}(t) + x_2^{k}(t) + y^{k}(t)) \leq K. \]

This completes the proof. \( \square \)

4. Extinction

When studying mathematical ecology, two of the most interesting issues are persistence and extinction. In this section, we discuss the extinction of populations in system (2.3) and leave its persistence to the next section.

Definition 4.1.

(1) \( x(t) \) is said to be extinct if \( \lim_{t \to \infty} x(t) = 0 \) a.s.

(2) \( x(t) \) is said to be strongly persistent in the mean if \( \liminf_{t \to \infty} t^{-1} \int_0^t x(s)ds > 0 \) a.s.

Before we state the main results of this section, several lemmas (Zhang et al. [18]) should be recalled without the proofs and relevant explanations.

Lemma 4.1. For any given initial value \((x_1(0), x_2(0), y(0)) \in \mathbb{R}^3_+\), the solution \((x_1(t), x_2(t), y(t))\) of system (2.3) satisfies

\[ \limsup_{t \to \infty} \frac{\ln x_i(t)}{t} \leq 0, \quad \limsup_{t \to \infty} \frac{\ln y(t)}{t} \leq 0 \quad a.s. \quad i = 1, 2. \]

Lemma 4.2. Assume

\[ dx(t) = x(t)((r_1)^{1-p}(a^u)^{p} - (a_1)^{1-p}(a_1)^{p}x(t))dt + (a_1)^{1-p}(a_1)^{p}x(t)dB(t) + \int_\mathbb{Y} c(u)x(t^-)N(dt, du). \]
If \((a^t)^{1-p}(a^n)^p > 0\) and \((r^t)^{1-p}(r^n)^p - \frac{((\sigma^t)^{1-p}(\sigma^n)^p)^2}{2} + \int_Y \ln(1 + c(u))\lambda(du) \geq 0\), we have

\[
\lim_{t \to \infty} \frac{\ln x(t)}{t} = 0 \text{ a.s.}
\]

and

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t x(s) ds = \frac{(a^t)^{1-p}(r^n)^p - \frac{((\sigma^t)^{1-p}(\sigma^n)^p)^2}{2} + \int_Y \ln(1 + c(u))\lambda(du)}{(a^t)^{1-p}(a^n)^p} \text{ a.s.}
\]

We now establish sufficient conditions for extinction of populations in system (2.3).

**Theorem 4.1.** If \(b_1 > 0\), \(b_2 > 0\) and \(b_3 < 0\), all the populations in system (2.3) go to extinction.

**Proof.** The Itôs formula of system (2.3) yields

\[
d\ln x_1(t) = \left( b_1 - (a^t_{11})^{1-p}(a^n_{11})^p x_1(t) - (a^t_{12})^{1-p}(a^n_{12})^p x_2(t) - \frac{(a^t_{13})^{1-p}(a^n_{13})^p y(t)}{1 + x_1(t)} \right) dt
\]

\[
+ (\sigma^t_{11})^{1-p}(\sigma^n_{11})^p dB_1(t) + \int_Y \ln(1 + c_1(u))\tilde{\lambda}(dt, du)
\]

(4.1) from 0 to \(t\) and then dividing by \(t\) on both sides, we obtain

\[
\frac{\ln(x_1(t)/x_1(0))}{t} = b_1 - (a^t_{11})^{1-p}(a^n_{11})^p x_1(t) - (a^t_{12})^{1-p}(a^n_{12})^p x_2(t) - (a^t_{13})^{1-p}(a^n_{13})^p \left(\frac{y(t)}{1 + x_1(t)}\right)
\]

\[
+ \frac{M_1(t)}{t} + \frac{\tilde{M}_1(t)}{t}.
\]

(4.2)

Similarly, we have

\[
\frac{\ln(x_2(t)/x_2(0))}{t} = b_2 - (a^t_{21})^{1-p}(a^n_{21})^p x_1(t) - (a^t_{22})^{1-p}(a^n_{22})^p x_2(t) - (a^t_{23})^{1-p}(a^n_{23})^p \left(\frac{y(t)}{1 + x_2(t)}\right)
\]

\[
+ \frac{M_2(t)}{t} + \frac{\tilde{M}_2(t)}{t},
\]

and

\[
\frac{\ln(y(t)/y(0))}{t} = b_3 - (a^t_{31})^{1-p}(a^n_{31})^p y(t) + (a^t_{31})^{1-p}(a^n_{31})^p \left(\frac{x_1(t)}{1 + x_1(t)}\right) + (a^t_{32})^{1-p}(a^n_{32})^p \left(\frac{x_2(t)}{1 + x_2(t)}\right)
\]

\[
+ \frac{M_3(t)}{t} + \frac{\tilde{M}_3(t)}{t}.
\]

(4.3)

(4.4)

where \(M_i(t) := \int_0^t (\sigma^t_{i1})^{1-p}(\sigma^n_{i1})^p dB_i(s)\) and \(\tilde{M}_i(t) := \int_0^t \int_Y [\ln(1 + c_i(u))\hat{\lambda}(ds, du), i = 1, 2, 3\) are all martingale terms. Thus, by strong law of large numbers, we have

\[
\lim_{t \to \infty} \frac{M_i(t)}{t} = 0 \text{ a.s. and } \lim_{t \to \infty} \frac{\tilde{M}_i(t)}{t} = 0 \text{ a.s.}
\]

(4.5)

Thus, (4.2), (4.3) and (4.5) yields

\[
\limsup_{t \to \infty} \frac{\ln x_i(t)}{t} \leq b_i \text{ a.s. } i = 1, 2.
\]

Which means \(\lim_{t \to \infty} x_i(t) = 0\text{ a.s. } i = 1, 2\text{ when } b_1 < 0\text{ and } b_2 < 0\). This together with (4.4) and (4.5) implies

\[
\limsup_{t \to \infty} \frac{\ln y(t)}{t} \leq b_3 \text{ a.s.}
\]

Therefore, \(b_3 < 0\) implies \(\lim_{t \to \infty} y(t) = 0\text{ a.s.}\) This completes the proof. 

\[\Box\]
Theorem 4.2. If \( b_1 > 0, b_2 < 0 \) and \((a_{31}^1)^{1-p}(a_{31}^0)^p < 0\), \( x_2(t) \) and \( y(t) \) are extinct and
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t x_1(s)ds = \frac{b_1}{(a_{11}^1)^{1-p}(a_{11}^0)^p} \ a.s.
\]

**Proof.** \( b_2 < 0 \) together with (4.3) and (4.5) yields
\[
\limsup_{t \to \infty} \frac{\ln x_2(t)}{t} \leq b_2 < 0 \ a.s.,
\]
which means \( \lim_{t \to \infty} x_2(t) = 0 \). Combining this with (4.4) and (4.5), we know that
\[
\limsup_{t \to \infty} \frac{\ln y(t)}{t} \leq b_3 + (a_{31}^1)^{1-p}(a_{31}^0)^p < 0 \ a.s.,
\]
which also implies \( \lim_{t \to \infty} y(t) = 0 \). It is easy to check that, for any \( 0 < \epsilon < \frac{b_1}{2} \), there exist a positive constant \( t_0 \) and a set \( \Omega_\epsilon \) such that \( P(\Omega_\epsilon) \geq 1 - \epsilon \), and for \( t \geq t_0 \) we get \((a_{12}^1)^{1-p}(a_{12}^0)^p x_2 < \epsilon, (a_{13}^1)^{1-p}(a_{13}^0)^p y < \epsilon \). Thus, for any \( \omega \in \Omega_\epsilon \),
\[
dx_1(t) \leq x_1(t)[(r_{11}^1)^{1-p}(r_{11}^0)^p - (a_{11}^1)^{1-p}(a_{11}^0)^p x_1(t)]dt + (\sigma_{11}^1)^{1-p}(\sigma_{11}^0)^p x_1(t)dB_1(t) + \int_y c_1(u)x_1(t^-)N(dt, du),
\]
and
\[
dx_1(t) \geq x_1(t)[(r_{11}^1)^{1-p}(r_{11}^0)^p - 2\epsilon - (a_{11}^1)^{1-p}(a_{11}^0)^p x_1(t)]dt + (\sigma_{11}^1)^{1-p}(\sigma_{11}^0)^p x_1(t)dB_1(t) + \int_y c_1(u)x_1(t^-)N(dt, du).
\]
By Lemma 4.2 and stochastic comparison theorem, for \( b_1 > 0 \),
\[
b_1 - 2\epsilon \leq \liminf_{t \to \infty} \frac{1}{t} \int_0^t x_1(s)ds \leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t x_1(s)ds \leq \frac{b_1}{(a_{11}^1)^{1-p}(a_{11}^0)^p} \ a.s.
\]
Thus, we have
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t x_1(s)ds = \frac{b_1}{(a_{11}^1)^{1-p}(a_{11}^0)^p} \ a.s.,
\]
when \( \epsilon \to 0 \). This completes the proof.

Next, we establish sufficient conditions for persistence in the mean of system (2.3). Before that, we need to consider several stochastic differential equations with jumps.

\[
d\phi_1(t) = \phi_1(t)[(r_{11}^1)^{1-p}(r_{11}^0)^p - (a_{11}^1)^{1-p}(a_{11}^0)^p \phi_1(t)]dt + (\sigma_{11}^1)^{1-p}(\sigma_{11}^0)^p \phi_1(t)dB_1(t) + \int_y c_1(u)\phi_1(t^-)N(dt, du),
\]
\[
d\phi_2(t) = \phi_2(t)[(r_{22}^1)^{1-p}(r_{22}^0)^p - (a_{22}^1)^{1-p}(a_{22}^0)^p \phi_2(t)]dt + (\sigma_{22}^1)^{1-p}(\sigma_{22}^0)^p \phi_2(t)dB_2(t) + \int_y c_2(u)\phi_2(t^-)N(dt, du),
\]
\[
d\phi_3(t) = \phi_3(t)[-(r_{33}^1)^{1-p}(r_{33}^0)^p - (a_{33}^1)^{1-p}(a_{33}^0)^p \phi_3(t)]dt + (\sigma_{33}^1)^{1-p}(\sigma_{33}^0)^p \phi_3(t)dB_3(t) + \int_y c_3(u)\phi_3(t^-)N(dt, du),
\]
\[
d\phi_4(t) = \phi_4(t)[-(r_{44}^1)^{1-p}(r_{44}^0)^p + (a_{44}^1)^{1-p}(a_{44}^0)^p]dt + (\sigma_{44}^1)^{1-p}(\sigma_{44}^0)^p \phi_4(t)dB_4(t) + \int_y c_4(u)\phi_4(t^-)N(dt, du).
\]
Thus, we know from stochastic comparison theorem that
\[
x_1(t) \leq \phi_1(t), \quad x_2(t) \leq \phi_2(t), \quad \phi_3(t) \leq y(t) \leq \phi_4(t).
\]
**Theorem 4.3.** If $b_3 > 0$, population $y(t)$ of system (2.3) satisfies
\[
\frac{b_3}{(a_{13}^{l})^{1-p}(a_{33}^{u})^{p}} \leq \liminf_{t \to \infty} \frac{1}{t} \int_{0}^{t} y(s)ds \leq \limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} y(s)ds \leq \frac{b_3 + (a_{31}^{l})^{1-p}(a_{31}^{u})^{p} + (a_{32}^{l})^{1-p}(a_{32}^{u})^{p}}{(a_{33}^{l})^{1-p}(a_{33}^{u})^{p}} \quad \text{a.s.}
\]
Moreover, if
\[
b_1 > \max\{0, ((a_{12}^{l})^{1-p}(a_{12}^{u})^{p})^{b_2}(a_{22}^{u})^{p}((a_{13}^{l})^{1-p}(a_{13}^{u})^{p})\}
\]
and
\[
b_2 > \max\{0, ((a_{21}^{l})^{1-p}(a_{21}^{u})^{p})^{b_1}(a_{11}^{u})^{p}((a_{13}^{l})^{1-p}(a_{13}^{u})^{p})\}
\]
we have
\[
\frac{1}{(a_{11}^{l})^{1-p}(a_{11}^{u})^{p}}(b_1 - ((a_{12}^{l})^{1-p}(a_{12}^{u})^{p})^{b_2}(a_{22}^{u})^{p}((a_{13}^{l})^{1-p}(a_{13}^{u})^{p}) - ((a_{13}^{l})^{1-p}(a_{13}^{u})^{p})^{b_3} + (a_{31}^{l})^{1-p}(a_{31}^{u})^{p} + (a_{32}^{l})^{1-p}(a_{32}^{u})^{p})
\]
\[
\leq \liminf_{t \to \infty} \frac{1}{t} \int_{0}^{t} x_1(s)ds \leq \limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} x_1(s)ds \leq \frac{b_1}{(a_{11}^{l})^{1-p}(a_{11}^{u})^{p}} \quad \text{a.s.,}
\]
and
\[
\frac{1}{(a_{22}^{l})^{1-p}(a_{22}^{u})^{p}}(b_2 - ((a_{21}^{l})^{1-p}(a_{21}^{u})^{p})^{b_1}(a_{11}^{u})^{p}((a_{13}^{l})^{1-p}(a_{13}^{u})^{p}) - ((a_{13}^{l})^{1-p}(a_{13}^{u})^{p})^{b_3} + (a_{31}^{l})^{1-p}(a_{31}^{u})^{p} + (a_{32}^{l})^{1-p}(a_{32}^{u})^{p})
\]
\[
\leq \liminf_{t \to \infty} \frac{1}{t} \int_{0}^{t} x_2(s)ds \leq \limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} x_2(s)ds \leq \frac{b_2}{(a_{22}^{l})^{1-p}(a_{22}^{u})^{p}} \quad \text{a.s.,}
\]
which means all the populations in system (2.3) are strongly persistent in the mean.

**Proof.** By
\[
\phi_3(t) \leq y(t) \leq \phi_4(t),
\]
we know from Lemma 4.2 that if $b_3 > 0$,
\[
\frac{b_3}{(a_{13}^{l})^{1-p}(a_{33}^{u})^{p}} \leq \liminf_{t \to \infty} \frac{1}{t} \int_{0}^{t} y(s)ds \leq \limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} y(s)ds \leq \frac{b_3 + (a_{31}^{l})^{1-p}(a_{31}^{u})^{p} + (a_{32}^{l})^{1-p}(a_{32}^{u})^{p}}{(a_{33}^{l})^{1-p}(a_{33}^{u})^{p}} \quad \text{a.s.}
\]
By Lemma 4.2 we still know that
\[
\limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} x_1(s)ds \leq \limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} \phi_1(s)ds = \frac{b_1}{(a_{11}^{l})^{1-p}(a_{11}^{u})^{p}} \quad \text{a.s.,}
\]
and
\[
\limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} x_2(s)ds \leq \limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} \phi_2(s)ds = \frac{b_2}{(a_{22}^{l})^{1-p}(a_{22}^{u})^{p}} \quad \text{a.s.}
\]
These together with Lemma 4.3 and 4.2 imply
\[
(a_{11}^{l})^{1-p}(a_{11}^{u})^{p}\liminf_{t \to \infty} \frac{1}{t} \int_{0}^{t} x_1(s)ds \geq \liminf_{t \to \infty} \left\{ -\frac{\ln(x_1(t)/x_1(0))}{t} + b_1 - (a_{12}^{l})^{1-p}(a_{12}^{u})^{p}\frac{1}{t} \int_{0}^{t} y(s)ds \right\}
\]
\[
- (a_{13}^{l})^{1-p}(a_{13}^{u})^{p}\frac{1}{t} \int_{0}^{t} y(s)ds + \frac{M_1(t)}{t} + \frac{\bar{M}_1(t)}{t}
\]
\[
\geq b_1 - \limsup_{t \to \infty} \frac{\ln(x_1(t))}{t} - (a_{12}^{l})^{1-p}(a_{12}^{u})^{p}\limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} \phi_2(s)ds
\]
\[
- (a_{13}^{l})^{1-p}(a_{13}^{u})^{p}\limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} y(s)ds
\]
\[
\geq b_1 - (a_{12}^{l})^{1-p}(a_{12}^{u})^{p}\frac{b_2}{(a_{22}^{l})^{1-p}(a_{22}^{u})^{p}}
\]
\[
- (a_{13}^{l})^{1-p}(a_{13}^{u})^{p}\frac{b_3 + (a_{31}^{l})^{1-p}(a_{31}^{u})^{p} + (a_{32}^{l})^{1-p}(a_{32}^{u})^{p}}{(a_{33}^{l})^{1-p}(a_{33}^{u})^{p}} \quad \text{a.s.}
\]
Similarly, we also have
\[
(a_{22})^{1-p}(a_{22}^p)\lim_{t \to \infty} \frac{1}{t} \int_0^t x_2(s)ds \geq b_2 - (a_{21})^{1-p}(a_{21}^p)\frac{b_1 (a_{11})^{1-p}(a_{11}^p)}{(a_{11})^{1-p}(a_{11}^p) b_3 + (a_{31})^{1-p}(a_{31}^p) + (a_{32})^{1-p}(a_{32}^p)} a.s.
\]

This completes the proof. \(\square\)

References

[1] H.I. Freedman, Deterministic Mathematical Models in Population Ecology, Marcel Dekker, New York, 1980.
[2] E. Beretta, Y. Kuang, Global analysis in some delayed ratio-dependent predator-prey systems, Nonlinear Anal. 32 (1998) 381-408.
[3] F. Berezovskaya, G. Karev, R. Arditi, Parametric analysis of the ratio-dependent predator-prey model, J. Math. Biol. 43 (2001) 221-246.
[4] M. Fan, K. Wang, Periodicity in a delayed ratio-dependent predator-prey system, J. Math. Anal. Appl. 362 (2001) 179-190.
[5] S.B. Hsu, T.W. Hwang, Y. Kuang, Global analysis of the Michaelis-Menten ratio-dependent predator-prey system, J. Math. Biol. 42 (2001) 489-506.
[6] Y. Kuang, E. Beretta, Global qualitative analysis of a ratio-dependent predator-prey system, J. Math. Biol. 36 (1998) 389-406.
[7] R. Xu, L. Chen, Persistence and stability for a two-species ratio-dependent predator-prey system with time delay in a two-patch environment, Comput. Math. Appl. 40 (2000) 577-588.
[8] Y. Pei, Y. Yang, C. Li, Dynamics of an impulsive control system which prey species share a common predator, Chaos Solitons Fractals 41 (2009) 2429-2436.
[9] R. Xu, L. Chen, Persistence and global stability for n-species ratio-dependent predator-prey system with time delays, J. Math. Anal. Appl. 275 (2002) 27-43.
[10] D. Pal, G.S. Mahaptra, G.P. Samanta, Optimal harvesting of prey-predator system with interval biological parameters: a bioeconomic model, Math. Biosci. 241 (2013) 181187.
[11] Q. Wang, Z. Liu, X. Zhang, R.A. Cheke, Incorporating prey refuge into a predator-prey system with imprecise parameter estimates, Comput. Appl. Math. 36 (2017) 10671084.
[12] S. Sharma, G.P. Samanta, Optimal harvesting of a two species competition model with imprecise biological parameters, Nonlinear Dynam. 77 (2014) 11011119.
[13] D. Kiouach, Y. Sabbar, Ergodic Stationary Distribution of a Stochastic Hepatitis B Epidemic Model with Interval-Valued Parameters and Compensated Poisson Process, Comput. Math. Method. M. (2020), https://doi.org/10.1155/2020/9676501
[14] P. Panja, S.K. Mondal, J. Chattopadhyay, Dynamical study in fuzzy threshold dynamics of a cholera epidemic model, Fuzzy Inf. Eng. 9 (2017) 381401.
[15] A. Das, M. Pal, A mathematical study of an imprecise SIR epidemic model with treatment control, J. Appl. Math. Comput. (2017) http://dx.doi.org/10.1007/s12190-017-1083-6.
[16] X.R. Mao, Stochastic Differential Equations and Applications, second ed., Horwood, Chichester, UK, 2007.
[17] R. Situ, Theory of Stochastic Differential Equations with Jumps and Applications, Springer, 2005.
[18] X. Zhang , W. Li, M. Liu, K. Wang, Dynamics of a stochastic Holling II one-predator two-prey system with jumps, Phys. A 421 (2015) 571582.