Stochastic processes on non-Archimedean spaces. III. Stochastic processes on totally disconnected topological groups.

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Abstract

Stochastic processes on totally disconnected topological groups are investigated. In particular they are considered for diffeomorphism groups and loop groups of manifolds on non-Archimedean Banach spaces. Theorems about a quasi-invariance and a pseudo-differentiability of transition measures are proved. Transition measures are used for the construction of strongly continuous representations including irreducible of these groups. In addition stochastic processes on general Banach-Lie groups, loop monoids, loop spaces and path spaces of manifolds on Banach spaces over non-Archimedean local fields also are investigated.

1 Introduction.

This part is the continuation of the previous two [13, 16], where stochastic processes on Banach spaces over local fields and stochastic antiderivational equations on them were investigated. This part is devoted to stochastic

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processes on a totally disconnected topological group which is complete, separable and ultrametrizable. In particular stochastic processes on diffeomorphism groups and loop groups of manifolds on Banach spaces over a local field are considered. These groups were defined and investigated in previous articles of the author [17, 19, 20, 23, 21]. These groups are non-locally compact and for them the Campbell-Hausdorff formula is not valid (in an open local subgroup). In this article topological groups satisfying locally the Campbell-Hausdorff formula also are considered.

Finite-dimensional Lie groups satisfy locally the Campbell-Hausdorff formula. This is guaranteed, if to impose on a locally compact topological Hausdorff group $G$ two conditions: it is a $C^\infty$-manifold and the following mapping $(f, g) \mapsto f \circ g^{-1}$ from $G \times G$ into $G$ is of class $C^\infty$. But for infinite-dimensional $G$ the Campbell-Hausdorff formula does not follow from these conditions. Frequently topological Hausdorff groups satisfying these two conditions also are called Lie groups, though they can not have all properties of finite-dimensional Lie groups, so that the Lie algebras for them do not play the same role as in the finite-dimensional case and therefore Lie algebras are not so helpful. If $G$ is a Lie group and its tangent space $T_eG$ is a Banach space, then it is called a Banach-Lie group, sometimes it is undermined, that they satisfy the Campbell-Hausdorff formula locally for a Banach-Lie algebra $T_eG$. In some papers the Lie group terminology undermines, that it is finite-dimensional. It is worthwhile to call Lie groups satisfying the Campbell-Hausdorff formula locally (in an open local subgroup) by Lie groups in the narrow sense; in the contrary case to call them by Lie groups in the broad sense.

In this article also theorems about a quasi-invariance and a pseudo-differentiability of transition measures on the totally disconnected topological group $G$ relative to the dense subgroup $G'$ are proved. For measures on Banach spaces over locally compact non-Archimedean fields their quasi-invariance and pseudo-differentiability were investigated in [18] (see also [21, 23] for diffeomorphism and loop groups, but for measures not related with stochastic processes). In each concrete case of $G$ it is necessary to construct a stochastic process and $G'$. Below path spaces, loop spaces, loop monoids, loop groups and diffeomorphism groups are considered not only for finite-dimensional, but also for infinite-dimensional manifolds.

In particular, loop and diffeomorphism groups are important for the development of the representation theory of non-locally compact groups. Their
representation theory has many differences with the traditional representation theory of locally compact groups and finite-dimensional Lie groups, because non-locally compact groups have not $C^*$-algebras associated with the Haar measures and they have not underlying Lie algebras and relations between representations of groups and underlying algebras (see also [22]).

In view of the A. Weil theorem if a topological Hausdorff group $G$ has a quasi-invariant measure relative to the entire $G$, then $G$ is locally compact. Since loop groups $(L^M N)_{\xi}$ are not locally compact, they can not have quasi-invariant measures relative to the entire group, but only relative to proper subgroups $G'$ which can be chosen dense in $(L^M N)_{\xi}$, where an index $\xi$ indicates on a class of smoothness. The same is true for diffeomorphism groups.

In this article classes of smoothness of the type $C^n$ by Schikhof are used. We recall shortly their definition. Let $K$ be a local field, that is, a finite algebraic extension of the $p$-adic field $Q_p$ for the corresponding prime number $p$ [27, 29, 30, 32]. For $b \in \mathbb{R}$, $0 < b < 1$, we consider the following mapping:

$$j_b(\zeta) := p^{b \times \text{ord}_p(\zeta)} \in \Lambda_p$$

for $\zeta \neq 0$, $j_b(0) := 0$, such that $j_b(*) : K \rightarrow \Lambda_p$, where $K \subset C_p$, $C_p$ denotes the field of complex numbers with the non-Archimedean valuation extending that of $Q_p$, $p^{-\text{ord}_p(\zeta)} := |\zeta|_K$, $\Lambda_p$ is a spherically complete field with a valuation group $\{|x| : 0 \neq x \in \Lambda_p\} = (0, \infty) \subset \mathbb{R}$ such that $C_p \subset \Lambda_p$ [1, 28, 30, 32]. Then we denote $j_1(x) := x$ for each $x \in K$. Let us consider Banach spaces $X$ and $Y$ over $K$. Suppose $F : U \rightarrow Y$ is a mapping, where $U \subset X$ is an open bounded subset. The mapping $F$ is called differentiable if for each $\zeta \in K$, $x + \zeta h \in U$ with $x + \zeta h \in U$ there exists a differential such that

$$(1) \quad DF(x, h) := dF(x + \zeta h)/d\zeta |_{\zeta=0} := \lim_{\zeta \to 0}\{F(x + \zeta h) - F(x)\}/\zeta$$

and $DF(x, h)$ is linear by $h$, that is, $DF(x, h) =: F'(x)h$, where $F'(x)$ is a bounded linear operator (a derivative). Let

$$(2) \quad \Phi^1 F(x; h; \zeta) := \{F(x + \zeta h) - F(x)\}/\zeta$$

be a partial difference quotient of order 1 for each $x + \zeta h \in U$, $\zeta h \neq 0$. If $\Phi^1 F(x; h; \zeta)$ has a bounded continuous extension $\Phi^1 F$ onto $U \times V \times S$,
where $U$ and $V$ are open neighbourhoods of $x$ and $0$ in $X$, $U + V \subset U$, $S = B(K, 0, 1)$, then

$$
\|\Phi^1 F(x; h; \zeta)\| := \sup_{x \in U, h \in V, \zeta \in S} \|\Phi^1 F(x; h; \zeta)\|_Y < \infty
$$

and $\Phi^1 F(x; h; 0) = F'(x)h$. Such $F$ is called continuously differentiable on $U$. The space of such $F$ is denoted $C(1, U \to Y)$. Let

$$
\Phi^b F(x; h; \zeta) := (F(x + \zeta h) - F(x))/j_b(\zeta) \in Y_{\Lambda_p}
$$

be partial difference quotients of order $b$ for $0 < b < 1$, $x + \zeta h \in U$, $\zeta h \neq 0$, $\Phi^0 F := F$, where $Y_{\Lambda_p}$ is a Banach space obtained from $Y$ by extension of a scalar field from $K$ to $\Lambda_p$. By induction using Formulas $(1 - 4)$ we define partial difference quotients of orders $n + 1$ and $n + b$:

$$
\Phi^{n+1} F(x; h_1, \ldots, h_{n+1}; \zeta_1, \ldots, \zeta_{n+1}) := \frac{\Phi^n F(x + \zeta_{n+1}h_{n+1}; h_1, \ldots, h_n; \zeta_1, \ldots, \zeta_n) - \Phi^n F(x; h_1, \ldots, h_n; \zeta_1, \ldots, \zeta_n)}{\zeta_{n+1}}
$$

and derivatives $F^{(n)} = (F^{(n-1)})'$. Then $C(t, U \to Y)$ is a space of functions $F : U \to Y$ for which there exist bounded continuous extensions $\Phi^v F$ for each $x$ and $x + \zeta_i h_i \in U$ and each $0 \leq v \leq t$, such that each derivative $F^{(k)}(x) : X^k \to Y$ is a continuous $k$-linear operator for each $x \in U$ and $0 < k \leq [t]$, where $0 \leq t < \infty$, $h_i \in V$ and $\zeta_i \in S$, $[t] = n \leq t$ and $\{t\} = b$ are the integral and the fractional parts of $t = n + b$ respectively. The norm in the Banach space $C(t, U \to Y)$ is the following:

$$
\|F\|_{C(t, U \to Y)} := \sup_{x, x + \zeta_i h_i \in U; h_i \in V; \zeta_i \in S; i=1,\ldots,s=\lfloor v \rfloor + \text{sign}(v): 0 \leq v \leq t} \|\Phi^v F(x; h_1, \ldots, h_s; \zeta_1, \ldots, \zeta_s)\|_{Y_{\Lambda_p}}
$$

where $0 \leq t \in \mathbb{R}$, $\text{sign}(y) = -1$ for $y < 0$, $\text{sign}(y) = 0$ for $y = 0$ and $\text{sign}(y) = 1$ for $y > 0$.

It is necessary to note that there are quite another groups with the same name loop groups, but they are infinite-dimensional Banach-Lie groups of
mappings \( f : M \to H \) into a finite-dimensional Lie group \( H \) with the pointwise group multiplication of mappings with values in \( H \). The loop groups considered here are geometric loop groups.

On the other hand, representation theory of non-locally compact groups is little developed apart from the case of locally compact groups. For locally compact groups theory of induced representations is well developed due to works of Frobenius, Mackey, etc. (see [10] and references therein). But for non-locally compact groups it is very little known. In particular geometric loop and diffeomorphism groups have important applications in modern physical theories (see [23, 21] and references therein).

Then measures are used for the study of associated unitary representations of dense subgroups \( G' \).

2 Stochastic antiderivational equations and measures on totally disconnected topological groups.

To avoid misunderstandings we first remind our definitions from [20, 23, 21].

2.1. Definitions and Notes. 1. Let \( X \) be a Banach space over a local field \( K \). Suppose \( M \) is an analytic manifold modelled on \( X \) with an atlas \( At(M) \) consisting of disjoint clopen charts \( (U_j, \phi_j), j \in \Lambda_M, \Lambda_M \subset \mathbb{N} \). That is, \( U_j \) and \( \phi_j(U_j) \) are clopen in \( M \) and \( X \) respectively, \( \phi_j : U_j \to \phi_j(U_j) \) are homeomorphisms, \( \phi_j(U_j) \) are bounded in \( X \). Let \( X = c_0(\alpha, K) \), where

\[
(1) \quad c_0(\alpha, K) := \{ x = (x^i : i \in \alpha) | x^i \in K, \text{ and for each } \epsilon > 0 \text{ the set } \}
\]
\[
(\{ i : |x^i| > \epsilon \} \text{ is finite } \} \text{ with }\]
\[
(2) \quad ||x|| := \sup_{i} |x^i| < \infty
\]

and the standard orthonormal base \( (e_i : i \in \alpha) \) [24], \( \alpha \) is considered as an ordinal due to the Kuratowski-Zorn lemma, \( \alpha \geq 1 \). Its cardinality is called a dimension \( \text{card}(\alpha) =: \text{dim}_K c_0(\alpha, K) \) over \( K \).

Then \( C(t, M \to Y) \) for \( M \) with a finite atlas \( At(M) \), \( \text{card}(\Lambda_M) < \aleph_0 \), denotes a Banach space of functions \( f : M \to Y \) with an ultranorm

\[
(3) \quad ||f||_t = \sup_{j \in \Lambda_M} ||f|_{U_j}||_{C(t, U_j \to Y)} < \infty,
\]
where \( Y := \mathcal{C}_0(\beta, K) \) is the Banach space over \( K \), \( 0 \leq t \in \mathbb{R} \), their restrictions \( f|_{U_j} \) are in \( \mathcal{C}(t, U_j \to Y) \) for each \( j, \beta \geq 1 \).

2.1.2. Let \( X, Y \) and \( M \) be the same as in §2.1.1 for a local field \( K \). We denote by \( \mathcal{C}_0(t, M \to Y) \) a completion of a subspace of cylindrical functions restrictions of which on each chart \( f|_{U_i} \) are finite \( K \)-linear combinations of functions \( \{ \overline{Q}_{m}(x_m)q_i|_{U_i} : i \in \beta, m \} \) relative to the following norm:

\[
\|f\|_{\mathcal{C}_0(t,M \to Y)} := \sup_{i,m,l} |a(m, f|_{U_i})| J_l(t, m),
\]

where multipliers \( J_l(t, m) \) are defined as follows:

\[
J_l(t, m) := \|\overline{Q}_{m}|_{U_l}\|_{\mathcal{C}(l,\phi_l(U_l)\cap K^n \to K)}.
\]

2.1.3.a. Let \( N \) be an analytic manifold modelled on \( Y \) with an atlas

\[
\text{At}(N) = \{(V_k, \psi_k) : k \in \Lambda_N\},
\]

such that \( \psi_k : V_k \to \psi_k(V_k) \subset Y \) are homeomorphisms, \( \text{card}(\Lambda_N) \leq \aleph_0 \) and \( \theta : M \to N \) be a \( C(t') \)-mapping, also \( \text{card}(\Lambda_M) < \aleph_0 \), where \( V_k \) are clopen in \( N \), \( t' \geq \max(1, t) \) is the index of a class of smoothness, that is, for each admissible \((i, j)\):

\[
(2) \quad \theta_{i,j} \in C(t', U_{i,j} \to Y)
\]

with * empty or an index * taking value 0 respectively,

\[
(3) \quad \theta_{i,j} := \psi_i \circ \theta|_{U_{i,j}},
\]

where \( U_{i,j} := [U_j \cap \theta^{-1}(V_i)] \) are non-void clopen subsets. We denote by \( C^0(\xi, M \to N) \) for \( \xi = t \) with \( 0 \leq t \leq \infty \) a space of mappings \( f : M \to N \) such that

\[
(4) \quad f_{i,j} - \theta_{i,j} \in C(\xi, U_{i,j} \to Y).
\]

In view of Formulas (1 - 4) we supply it with an ultrametric

\[
(5) \quad \rho^\xi_{\xi}(f, g) = \sup_{i,j} \|f_{i,j} - g_{i,j}\|_{C(\xi,U_{i,j} \to Y)}
\]

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for each $0 \leq \xi < \infty$.

2.1.3.b. Let $M$ and $N$ be two analytic manifolds with finite atlases, $\dim_K M = n \in \mathbb{N}$, $\theta_{i,j} \in C(\infty, U_j \to Y)$ for each $i, j$.

We denote by $C^0_\theta((t, s), M \to N)$ a completion of a locally $K$-convex space

(1) $\{ f \in C^0_\theta(t + sn, M \to N) : \rho_0^{(t,s)}(f, \theta) < \infty$ and for each $\epsilon > 0$ a set $\{(k, m) : \sum_{i,j} a(m, f^k_{i,j} - \theta^k_{i,j}) |J((t, s), m)| > \epsilon\}$ is finite $\}$

relative to an ultrametric

(2) $\rho_0^{(t,s)}(f, g) := \sup_{i,j,m,k} |a(m, f^k_{i,j} - g^k_{i,j})| J((t, s), m),$

where $s \in \mathbb{N}_0, 0 \leq t < \infty$, $|J_{\xi}(\xi, \theta) := c^0(\xi, (U_{\xi} \to Y)) \cap \hom(M),$ $\text{Diff}(\xi, \theta) := c^0(\xi, (U_{\xi} \to M)) \cap \hom(M),$ for each integer $\gamma$ such that $1 \leq \gamma \leq s$ and for each $v \in \{[t] + \gamma n, t + \gamma n\}$.

2.1.4. For infinite atlases we use the traditional procedure of inductive limits of spaces. For $M$ with the infinite atlas, $\text{card}(\Lambda_M) = \aleph_0$, and the Banach space $Y$ over $K$ we denote by $C^0_\theta(\xi, M \to Y)$ for $\xi = t$ with $0 \leq t \leq \infty$ or for $\xi = (t, s)$ a locally $K$-convex space, which is the strict inductive limit

(1) $C^0_\theta(\xi, M \to Y) := \text{str} - \text{ind}\{C^0_\theta(\xi, (U^E \to Y)), \pi^E, \Sigma\},$

where $E \in \Sigma, \Sigma$ is the family of all finite subsets of $\Lambda_M$ directed by the inclusion $E < F$ if $E \subset F$, $U^E := \bigcup_{j \in E} U_j$ (see also §2.4 [24]).

For mappings from one manifold into another $f : M \to N$ we therefore get the corresponding uniform spaces. They are denoted by $C^0_\theta(\xi, M \to N)$.

We introduce notations

(2) $G_i(\xi, M) := C^0_\theta(\xi, M \to M) \cap \hom(M),$ $D_i f(\xi, M) = C^0(\xi, M \to M) \cap \hom(M),$ that are called groups of diffeomorphisms (and homeomorphisms for $0 \leq t < 1$ and $s = 0$), $\theta = id$, $id(x) = x$ for each $x \in M$, where $\hom(M) := \{ f :
\( f \in C(0, M \to M) \), \( f \) is bijective, \( f(M) = M \), \( f \) and \( f^{-1} \in C(0, M \to M) \) denotes the usual homeomorphism group. For \( s = 0 \) we may omit it from the notation, which is always accomplished for \( M \) infinite-dimensional over \( K \).

2.2. Notes. Henceforth, ultrametrizable separable complete manifolds \( \bar{M} \) and \( N \) are considered. Since a large inductive dimension \( Ind(\bar{M}) = 0 \) (see Theorem 7.3.3 \([9]\)) hence \( \bar{M} \) has not boundaries in the usual sense. Therefore,

\[
(1) \quad At(\bar{M}) = \{(\bar{U}_j, \bar{\phi}_j) : \ j \in \Lambda_{\bar{M}}\}
\]

has a refinement \( At'(\bar{M}) \) which is countable and its charts \((\bar{U}'_j, \bar{\phi}'_j)\) are clopen and disjoint and homeomorphic with the corresponding balls \( B(X, y_j, \bar{r}'_j) \), where

\[
(2) \quad \bar{\phi}'_j : \bar{U}'_j \to B(X, y'_j, \bar{r}'_j) \text{ for each } j \in \Lambda'_{\bar{M}}
\]

are homeomorphisms (see \([9, 24]\)). For \( \bar{M} \) we fix such \( At'(\bar{M}) \).

We define topologies of groups \( G_i(\xi, \bar{M}) \) and locally \( K \)-convex spaces \( C^0(\xi, \bar{M} \to Y) \) relative to \( At'(\bar{M}) \), where \( Y \) is the Banach space over \( K \).

Therefore, we suppose also that \( \bar{M} \) and \( N \) are clopen subsets of the Banach spaces \( X \) and \( Y \) respectively. Up to the isomorphism of loop semigroups (see below their definition) we can suppose that \( s_0 = 0 \in \bar{M} \) and \( y_0 = 0 \in N \).

For \( M = \bar{M} \setminus \{0\} \) let \( At(M) \) consists of charts \((U_j, \phi_j)\), \( j \in \Lambda_M \), while \( At'(M) \) consists of charts \((U'_j, \phi'_j)\), \( j \in \Lambda'_M \), where due to Formulas (1, 2) we define

\[
(3) \quad U_1 = \bar{U}_1 \setminus \{0\}, \phi_1 = \bar{\phi}_1|_{U_1}; \ U_j = \bar{U}_j \text{ and } \phi_j = \bar{\phi}_j \text{ for each } j > 1,
\]

\[
0 \in \bar{U}_1, \ \Lambda_M = \Lambda_{\bar{M}}, \ U'_1 = \bar{U}'_1 \setminus \{0\}, \ \phi'_1 = \bar{\phi}'_1|_{U'_1}, \ U'_j = \bar{U}'_j \text{ and } \phi'_j = \bar{\phi}'_j \text{ for each } j > 1, \ j \in \Lambda'_M = \Lambda'_{\bar{M}}, \ \bar{U}'_1 \ni 0.
\]

2.3. Definitions and Notes. 1. Let the spaces be the same as in §2.1.4 (see Formulas 2.1.4.(1-3)) with the atlas of \( M \) defined by Conditions 2.2.(3). Then we consider their subspaces of mappings preserving marked points:

\[
(1) \quad C^0_\theta(\xi, (M, s_0) \to (N, y_0)) := \{ f \in C^0_\theta(\xi, \bar{M} \to N) : \lim_{|G_1|+\ldots+|G_k| \to 0} \bar{\Phi}^v(f - \theta)(s_0; h_1, \ldots, h_k; \zeta_1, \ldots, \zeta_k) = 0 \text{ for each } v \in \{0, 1, \ldots, [t], t\}, \ k = [v] + sign\{v\},
\]

where
where for $s > 0$ and $\xi = (t, s)$ in addition Condition 2.1.3.b.(4) is satisfied for each $1 \leq \gamma \leq s$ and for each $v \in \{[t] + n\gamma, t + n\gamma\}$, and the following subgroup:

$$\text{(2) } G_0(\xi, M) := \{f \in G_0(\xi, \bar{M}) : f(s_0) = s_0\}$$

of the diffeomorphism group, where $s \in N_0$ for $\dim K M < \aleph_0$ and $s = 0$ for $\dim K M = \aleph_0$.

With the help of them we define the following equivalence relations $K_\xi$: $f K_\xi g$ if and only if there exist sequences

$$\{\psi_n \in G_0(\xi, M) : n \in N\},$$

$$\{f_n \in C_0^g(\xi, M \to N) : n \in N\} \text{ and }$$

$$\{g_n \in C_0^g(\xi, M \to N) : n \in N\} \text{ such that }$$

$$\text{(3) } f_n(x) = g_n(\psi_n(x)) \text{ for each } x \in M \text{ and } \lim_{n \to \infty} f_n = f \text{ and } \lim_{n \to \infty} g_n = g.$$

Due to Condition (3) these equivalence classes are closed, since $(g(\psi(x))' = g'(\psi(x))\psi'(x), g(s_0) = s_0, g'(s_0) = 0$ for $t + s \geq 1$. We denote them by $< f >_{K_\xi}$. Then for $g < f >_{K_\xi}$ we write $gK_\xi f$ also. The quotient space $C_0^g(\xi, (M, s_0) \to (N, y_0))/K_\xi$ we denote by $\Omega_\xi(M, N)$, where $\theta(M) = \{y_0\}$.

2.3.2. Let as usually $A \lor B := A \times \{b_0\} \cup \{a_0\} \times B \subset A \times B$ be the wedge product of pointed spaces $(A, a_0)$ and $(B, b_0)$, where $A$ and $B$ are topological spaces with marked points $a_0 \in A$ and $b_0 \in B$. Then the composition $g \circ f$ of two elements $f, g \in C_0^g(\xi, (M, s_0) \to (N, y_0))$ is defined on the domain $M \lor \bar{M} \setminus \{s_0 \times s_0\} =: M \lor \bar{M}$.

Let $M = \bar{M} \setminus \{0\}$ be as in §2.2. We fix an infinite atlas $\bar{M}'(M) := \{(\bar{U}'_j, \phi'_j) : j \in N\}$ such that $\phi'_j : \bar{U}'_j \to B(X, y'_j, r'_j)$ are homeomorphisms,

$$\lim_{k \to \infty} r'_j(k) = 0 \text{ and } \lim_{k \to \infty} y'_j(k) = 0$$

for an infinite sequence $\{j(k) \in N : k \in N\}$ such that $\bar{cl}_M[\bigcup_{k=1}^\infty \bar{U}'_{j(k)}]$ is a clopen neighbourhood of 0 in $\bar{M}$, where $\bar{cl}_A A$ denotes the closure of a subset $A$ in $\bar{M}$. In $M \lor \bar{M}$ we choose the following atlas $\bar{M}'(M \lor \bar{M}) = \{(\bar{W}_l, \xi_l) : l \in N\}$ such that $\xi_l : \bar{W}_l \to B(X, z_l, a_l)$ are homeomorphisms,

$$\lim_{k \to \infty} a_l(k) = 0 \text{ and } \lim_{k \to \infty} z_l(k) = 0.$$
for an infinite sequence \( \{ l(k) \in \mathbb{N} : k \in \mathbb{N} \} \) such that \( cl_{\text{cl} \mathbb{M} \vee \text{cl} \bar{M}}[\bigcup_{k=1}^{\infty} W_{l(k)}] \) is a clopen neighbourhood of \( 0 \times 0 \) in \( \mathbb{M} \vee \bar{M} \) and
\[
\text{card}(\mathbb{N} \setminus \{ l(k) : l \in \mathbb{N} \}) = \text{card}(\mathbb{N} \setminus \{ j(k) : k \in \mathbb{N} \}).
\]

Then we fix a \( C(\infty) \)-diffeomorphisms \( \chi : \mathbb{M} \vee M \to M \) such that
\[
(1) \quad \chi(W_{l(k)}) = \tilde{U}'_{j(k)} \quad \text{for each } k \in \mathbb{N}
\]
\[
(2) \quad \chi(W_l) = \tilde{U}'_{j(l)} \quad \text{for each } l \in (\mathbb{N} \setminus \{ l(k) : k \in \mathbb{N} \}),
\]
where
\[
(3) \quad \kappa : (\mathbb{N} \setminus \{ l(k) : k \in \mathbb{N} \}) \to (\mathbb{N} \setminus \{ j(k) : k \in \mathbb{N} \})
\]
is a bijective mapping for which
\[
(4) \quad p_{-1} \leq a_{l(k)}/r'_{j(k)} \leq p \quad \text{and} \quad p_{-1} \leq a_{l}/r'_{\kappa(l)} \leq p.
\]

This induces the continuous injective homomorphism
\[
(5) \quad \chi^* : C^d_0(\xi, (M \vee M, s_0 \times s_0) \to (N, y_0)) \to C^d_0(\xi, (M, s_0) \to (N, y_0))
\]
such that
\[
(6) \quad \chi^*(g \vee f)(x) = (g \vee f)(\chi^{-1}(x))
\]
for each \( x \in M \), where \( (g \vee f)(y) = f(y) \) for \( y \in M_2 \) and \( (g \vee f)(y) = g(y) \) for \( y \in M_1, M_1 \vee M_2 = M \vee M, M_i = M \) for \( i = 1, 2 \). Therefore
\[
(7) \quad g \circ f := \chi^*(g \vee f)
\]
may be considered as defined on \( M \) also, that is, to \( g \circ f \) there corresponds the unique element in \( C^d_0(\xi, (M, s_0) \to (N, y_0)) \).

**2.3.3.** The composition in \( \Omega_\xi(M, N) \) is defined due to the following inclusion \( g \circ f \in C^d_0(\xi, (M, s_0) \to (N, y_0)) \) (see Formulas 2.3.2.(1-7)) and then using the equivalence relations \( K_\xi \) (see Condition 2.3.1.(3)).

It is shown below that \( \Omega_\xi(M, N) \) is the monoid, which we call the loop monoid.

**2.4. Note.** For each chart \( (V_i, \psi_i) \) of \( At(N) \) (see Equality 2.1.3.a.(1)) there are local normal coordinates \( y = (y^j : j \in \beta) \in B(Y, a_i, r_i), Y = c_0(\beta, K) \). Moreover, \( TV_i = V_i \times Y \), consequently, \( TN \) has the disjoint atlas \( At(TN) = \{ (V_i \times X, \psi_i \times I) : i \in \Lambda_N \} \), where \( I_Y : Y \to Y \) is the unit mapping, \( \Lambda_N \subset \mathbb{N} \), \( TN \) is the target vector bundle over \( N \).
Suppose $V$ is an analytic vector field on $N$ (that is, by definition $V|_{V_i}$ are analytic for each chart and $V \circ \psi_i^{-1}$ has the natural extension from $\psi_i(V_i)$ on the balls $B(X, a_i, r_i)$). Then by analogy with the classical case we can define the following mapping

$$\bar{\exp}_y(zV) = y + zV(y)$$

for which

$$\partial^2 \bar{\exp}_y(zV(y))/\partial z^2 = 0$$

(this is the analog of the geodesic), where $|V(y)||_Y|z| \leq r_i$ for $y \in V_i$ and $\psi_i(y)$ is also denoted by $y, z \in K, V(y) \in Y$. Moreover, there exists a refinement $At''(N) = \{(V''_i, \psi''_i) : i \in \Lambda'' M\}$ of $At(N)$. This $At''(N)$ is embedded into $At(N)$ by charts such that it is also disjoint and analytic and $\psi''_i(V''_i)$ are $K$-convex in $Y$. The latter means that $\lambda x + (1 - \lambda)y \in \psi''_i(V''_i)$ for each $x, y \in \psi''_i(V''_i)$ and each $\lambda \in B(K, 0, 1)$. Evidently, we can consider $\bar{\exp}_y$ injective on $V''_i, y \in V''_i$. The atlas $At''(N)$ can be chosen such that

$$\bar{\exp}_y|_{V''_i} : V''_i \times B(Y, 0, \tilde{r}_i) \to V''_i$$

is the analytic homeomorphism for each $i \in \Lambda'' M$, where $\infty > \tilde{r}_i > 0, y \in V''_i,$

$$\bar{\exp}_y : (\{y\} \times B(Y, 0, \tilde{r}_i)) \to V''_i$$

is the isomorphism. Therefore, $\bar{\exp}$ is the locally analytic mapping, $\bar{\exp} : \tilde{T}N \to N$, where $\tilde{T}N$ is the corresponding neighbourhood of $N$ in $TN$.

Then

1. $T_f C^{\theta}_*(\xi, M \to N) = \{g \in C^{(\theta, 0)}_{\ast}(\xi, M \to TN) : \pi_N \circ g = f\},$

consequently,

2. $C^{\theta}_{\ast}(\xi, M \to TN) = \bigcup_{f \in C^{\theta}_{\ast}(\xi, M \to N)} T_f C^{\theta}_{\ast}(\xi, M \to N) = TC^{\theta}_{\ast}(\xi, M \to N),$

where $\pi_N : TN \to N$ is the natural projection, $* = 0$ or $* = \emptyset$ ($\emptyset$ is omitted).

Therefore, the following mapping

3. $\omega_{\bar{\exp}} : T_f C^{\theta}_{\ast}(\xi, M \to N) \to C^{\theta}_{\ast}(\xi, M \to N)$

is defined by the formula given below

4. $\omega_{\bar{\exp}}(g(x)) = \bar{\exp}_{f(x)} \circ g(x),$
that gives charts on $C^0(\xi, M \to N)$ induced by charts on $C^0(\xi, \tilde{M} \to T\tilde{N})$.

2.5. Definition and Note. In view of Equalities 2.4.(1,2) the space $C^0(\xi, \tilde{M} \to N)$ is isomorphic with $C^0(\xi, (M, s_0) \to (N, y_0)) \times N^\xi$, where $y_0 = 0$ is the marked point of $N$. Here

$$(1) \quad N^\xi := N \otimes (\bigotimes_{j=1}^{d} \tilde{L}_\xi(X^j \to Y)) \text{ for } t \in N_0 \text{ with } t + s > 0;$$

$$(2) \quad N^\xi = N \text{ for } t + s = 0;$$

$$(3) \quad N^\xi = N \otimes (\bigotimes_{j=1}^{d} \tilde{L}_\xi(X^j \to Y)) \otimes C^0_0(0, M^k \to Y_\lambda) \text{ for } t \in R \setminus N,$$

where $N^\xi$ is with the product topology, $d = [t]$ for $\xi = t$, $d = [t] + n\alpha$ for $\xi = (t, s)$ with $\alpha = \dim_k M < \aleph_0$, when $s > 0$, $k = d + \text{sign}\{t\}$, $Y_\lambda := c_0(\beta, \lambda)$, $\lambda$ is the least subfield of $A_p$ such that $\lambda \supset K \cup j(t)(K)$ (see §2.1 [23]). Then $\tilde{L}_\xi(X^j \to Y)$ denotes the Banach space of continuous $j$-linear operators $f_j : X^j \to Y$ with

$$(4) \quad \|f_j\|_{\tilde{L}_\xi(X^j \to Y)} := \sup_{i,m} \|f_j^i\|_m \text{ and}$$

$$(5) \quad \lim_{i+m+k \to \infty} \|f_j^i\|_m = 0, \text{ where}$$

$$(6) \quad \|f_j^i\|_m := \sup_{0 \neq h \in K, l=1,\ldots,j} \|f_j^i(h_{1}, \ldots, h_{j})\|_Y J'(\xi, m)/(\|h_1\|_X \ldots \|h_j\|_X),$$

$K^k := sp_K(e_1, \ldots, e_k) \hookrightarrow X$ is a $K$-linear span of the standard basic vectors, $m = (m_1, \ldots, m_k)$, $|m| = m_1 + \ldots + m_k$, $k \in N$; $h_1 = \ldots = h_{m_1}, \ldots, h_{m_{k-1}+1} = \ldots = h_{m_k}$ for $s = 0$; in addition Condition 2.1.3.b.(4) is satisfied for each $0 < \gamma \leq s$, when $s > 0$; $f = (f_0, f_1, \ldots, f_j, \ldots) \in N^\xi$, $\sum_i f_j^i q_i = f_j$, $f_j^i : X^j \to K$,

$J'(\xi, m) := |\partial^m \tilde{Q}_m(x)|_{m=0}|_K$

(see §2.2 [23] and Equations 2.1.2.(1-5), 2.1.3.b.(1-3)).

2.6. Definitions. A function $f : K \to C$ is called pseudo-differentiable of order $b$, if there exists the following integral:

$$(1) \quad PD(b, f(x)) := \int_{K} \left[ (f(x) - f(y)) \times g(x, y, b) \right] v(dy),$$

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where \(g(x, y, b) := |x - y|^{-1-b}\) with the nonnegative Haar measure \(v\) and \(b \in C\) (see also \(\S\) 2.1 [23]). We introduce the following notation \(PD_c(b, f(x))\) for such integral by \(B(K, 0, 1)\) instead of the entire \(K\).

2.7. Definitions. Let \(G\) be a topological Hausdorff semigroup and \((M, R)\) be a space \(M\) of measures on \((G, Bf(G))\) with values in \(R\), where \(Bf(G)\) denotes the Borel \(\sigma\)-algebra of \(G\). Let also \(G'\) and \(G''\) be dense subsemigroups in \(G\) such that \(G'' \subset G'\) and a topology \(T\) on \(M\) is compatible with \(G'\), that is, \(\mu \mapsto \mu_h\) is the homomorphism of \((M, R)\) into itself for each \(h \in G'\), where \(\mu_h(A) := \mu(h \circ A)\) for each \(A \in Bf(G)\). Let \(T\) be the topology of convergence for each \(E \in Bf(G)\). If \(\mu \in (M, R)\) and \(\mu_h \sim \mu\) are the equivalent measures for each \(h \in G'\) then \(\mu\) is called quasi-invariant on \(G\) relative to \(G'\). We shall consider \(\mu\) with the continuous quasi-invariance factor

\[(1) \rho_\mu(h, g) := \mu_h(dg)/\mu(dg).\]

If \(G\) is a group, then we use the traditional definition of \(\mu_h\) such that \(\mu_h(A) := \mu(h^{-1} \circ A)\).

Let \(S(r, f) = g(r, f)\) be a curve on the subsemigroup \(G''\), such that \(S(0, f) = f\) and there exists \(\partial S(r, f)/\partial r \in TG''\) and \(\partial S(r, f)/\partial r|_{r=0} =: A_f \in T_f G''\), where \(r \in B(K, 0, R), \infty > R \geq 1\). Then a measure \(\mu\) on \(G\) is called pseudo-differentiable of order \(b\) relative to \(S\) if there exists \(PD_c(b, S(r, \mu)(B))\) by \(r \in B(K, 0, 1)\) for each \(B \in Bf(G)\), where \(S(r, \mu)(B) := \mu(S(-r, B))\) for each \(B \in Bf(G)\). A measure \(\mu\) is called pseudo-differentiable of order \(b\) if there exists a dense subsemigroup \(G''\) of \(G\) such that \(\mu\) is pseudo-differentiable of order \(b\) for each curve \(S(r, f)\) on \(G''\) described above, where \(b \in C\).

Naturally Definitions 2.7 have generalizations, when \(G\) is a topological manifold on which a topological group (or a semigroup) \(G'\) acts continuously from the left \(G' \times G \ni (g, x) \mapsto gx \in G\).

2.8. Note. Now let us describe dense loop submonoids which are necessary for the investigation of quasi-invariant measures on the entire monoid. For finite \(At(M)\) and \(\xi = (t, s)\) let \(C_0^{\theta}(\xi, M \to Y)\) be a subspace of \(C_0^g(\xi, M \to Y)\) consisting of mappings \(f\) for which

\[(1) \|f - \theta\|_{C_0^{\theta}(\xi, M \to Y)} := \sup_{i, m, j} |a(m, f^i|U_j)| \cdot k \cdot J_j(\xi, m) p_{k(i, m)} < \infty\]

and

\[(2) \lim_{i + |m| + \text{Ord}(m) \to \infty} \sup_j |a(m, f^i|U_j)| \cdot k \cdot J_j(\xi, m) p_{k(i, m)} = 0,\]

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where \( k(i, m) := c' \times i + c \times (|m| + \text{Ord}(m)) \), \( c' \) and \( c \) are non-negative constants, \( |m| := \sum_i m_i \),

\[
\text{Ord}(m) := \max\{i : m_i > 0 \text{ and } m_l = 0 \text{ for each } l > i\}
\]

(see also Formulas 2.1.2.(2) and 2.1.3.b.(3)).

For finite-dimensional \( M \) over \( K \) this space is isomorphic with \( C_0^{\theta}(\xi, M \to Y) \), where \( k'(i, m) = c' \times i + c \times |m| \). For finite-dimensional \( Y \) over \( K \) the space \( C_0^{\theta}(\xi, M \to Y) \) is isomorphic with \( C_0^{\theta}(\xi, M \to Y) \), where \( k''(i, m) = c \times (|m| + \text{Ord}(m)) \). For \( c' = c = 0 \) this space coincides with \( C_0^{\theta}(\xi, M \to Y) \) and we omit \( \{k\} \).

Then as in §2.3 we define spaces \( C_0^{\theta}(\xi, (M, s_0) \to (N, 0)) \), groups

\[
G^{\{k\}}(\xi, M) := C_0^{\theta}(\xi, M \to M) \cap \text{Hom}(M),
\]

\[
G_0^{\{k\}}(\xi, M) := \{\psi \in G^{\{k\}}(\xi, M) : \psi(s_0) = s_0\}
\]

and the equivalence relation \( K_{\xi,\{k\}} \) in it for each \( M \) and \( N \) from §2.1 and §2.2. Therefore,

\[
G' := \Omega_{\xi}^{\{k\}}(M, N) =: C_0^{\theta}(\xi, (M, s_0) \to (N, 0))/K_{\xi,\{k\}}
\]

is the dense submonoid in \( \Omega_{\xi}(M, N) \).

2.9. **Note and Definition.** For a commutative monoid \( \Omega_{\xi}(M, N) \) with the unity and the cancellation property (see [23]) there exists a commutative group \( L_{\xi}(M, N) \) equal to the Grothendieck group. This group is the quotient group \( F/B \), where \( F \) is a free Abelian group generated by \( \Omega_{\xi}(M, N) \) and \( B \) is a closed subgroup of \( F \) generated by elements \( [f + g] - [f] - [g] \), \( f \) and \( g \in \Omega_{\xi}(M, N) \), \( [f] \) denotes an element of \( F \) corresponding to \( f \). The natural mapping

\[
(1) \gamma : \Omega_{\xi}(M, N) \to L_{\xi}(M, N)
\]

is injective. We supply \( F \) with a topology inherited from the Tychonoff product topology of \( \Omega_{\xi}(M, N)^\mathbb{Z} \), where each element \( z \) of \( F \) is

\[
(2) \quad z = \sum_f n_{f,z}[f],
\]

\( n_{f,z} \in \mathbb{Z} \) for each \( f \in \Omega_{\xi}(M, N) \),

\[
(3) \quad \sum_f |n_{f,z}| < \infty.
\]
In particular $[nf] - n[f] \in \mathcal{B}$, where $1f = f$, $nf = f \circ (n-1)f$ for each $1 < n \in \mathbb{N}$, $f + g := f \circ g$. We call $L_\xi(M,N)$ the loop group.

2.10. **Note.** Let $\Omega^{(k)}(M,N)$ be the loop submonoid as in §2.8 such that $c > 0$ and $c' > 0$. Then it generates the loop group $G' := L^{(k)}_\xi(M,N)$ as in §2.9 such that $G'$ is the dense subgroup in $G = L_\xi(M,N)$.

2.11. **Remarks.** Let $M$ be a manifold on the Banach space $X$ with an atlas $At(M)$ consisting of disjunctive charts $(U_j, \phi_j)$, $j \in \Lambda$, $\Lambda \subset \mathbb{N}$, where $U_j$ and $\phi_j(U_j)$, are clopen in $M$ and $X$ respectively, $\phi_j : U_j \to \phi_j(U_j)$ is a homeomorphism, also $\phi_j(U_j) = B(x,x_j,r_j)$ is a ball in $X$ with a radius $0 < r_j < \infty$ for each $j$.

For $\Lambda = \omega_0$ we define a Banach space

$$\tilde{C}_s(t,M \to X) := \{f|_{U_j} \in C_s(t,U_j \to X), \|f\|_{C_s(t,M \to X)} := \sup_{j \in \Lambda}(\|f|_{U_j}\|_{C_s(t,U_j \to X)})$$

/ $\min(1,r_j) < \infty$ and $(\|f|_{U_j}\|_{C_s(t,U_j \to X)}/\min(1,r_j)) \to 0$ while $j \to \infty$,}

where $0 \leq t < \infty$, $* = 0$ for spaces $C_0(t,U \to X)$, $* = \emptyset$ or simply is omitted for $C(t,U \to X)$. For the finite atlas $At(M)$ the spaces $\tilde{C}_s(t,U \to X)$ and $C_s(t,U \to X)$ are linearly topologically isomorphic. By $C^*_s(t,M \to \hat{M})$ for $0 \leq t \leq \infty$ is denoted the following space of functions $f : M \to \hat{M}$ such that $(\phi_i - \theta_i) \in C_s(t,M \to X)$ for each $i \in \Lambda$ and $f_i = \psi_i \circ f$, $\theta_i = \psi_i \circ \theta$. We introduce the following group

$$G(t,M) := \tilde{C}^{id}_0(t,M \to M) \cap \text{Hom}(M),$$

which is called the diffeomorphism group (and the homeomorphism group for $0 \leq t < 1$), where $\text{Hom}(M)$ is the group of continuous homeomorphisms.

Each function $f \in C_0(t,M \to X)$ has the following decomposition:

$$f(x)|_{U_j} = \sum_{(i \in \mathbb{N}, n \in \mathbb{N})} f^i(n;x)|_{U_j} e_i \tilde{z}(n), \text{ and } \{e_i \tilde{z}(n)(\tilde{Q}_m(x)|_{U_j}) : i, n, \text{Ord}(m) = n, j \} \text{ is the orthogonal basis, moreover,}$$

$$f_n(x)|_{U_j} := \sum_i f^i(n;x)|_{U_j} e_i \in C_0(t,U_j \to X), \text{ where}$$

$$X_{\tilde{z}(n)} := \{f_n(x) : f_n|_{U_j} \in C_0(t,U_j \to X)\}$$
is the Banach space with the norm induced from $C_0(t, M \to X)$ such that

$$f^i(n; x)|_{U_j} := \sum_{(\text{Ord } m=n, m=(m(1), ..., m(n)), m(j) \in \mathbb{N}_0)} a(m, f^i|_{U_j}) Q_m(x)|_{U_j},$$

where $Q_m(x)|_{U_j} = 0$ for $x \in M \setminus U_j$.

For the manifold $M$ we fix a subsequence $\{M_n : n \in \mathbb{N}_0\}$ of submanifolds in $M$ such that $M_n \hookrightarrow M_{n+1} \hookrightarrow \ldots$ for each $n$, $\dim_K M_n = \beta(n) \in \mathbb{N}$ for each $n \in \mathbb{N}_0$, $\bigcup_n M_n$ is dense in $M$, where $\beta(n) < \beta(n+1)$ for each $n$ and there exists $n_0 \in \mathbb{N}$ with $\beta(n) = n$ for each $n > n_0$.

We take the following subgroup

$$G' := \{f \in G(t, M) : (f^i(n; x) - i^0(n; x)) =: g^i(n; x) \in C_0(t_n, M_n \to K) \text{ and } |a(m; g^i(n; x)|_{U_j})| J_j(t_n, m) \leq c(f)p^{v'(m,j,i)}\},$$

where $c(f) > 0$ is a constant, $v'(m, j, i) = -c'i - c'n - c''j$, $n = \text{Ord}(m)$, $c' = \text{const} > 0$ and $c'' = \text{const} \geq 0$, $c'' > 0$ for $\Lambda = \omega_0$, $t_n = t + s(n)$ for $0 \leq t < \infty$, $s(n) > n$ for each $n$ and $\lim_{n \to \infty} s(n)/n =: \zeta > 1$. Then there exists the following ultrametric in $G'$:

$$d(f, id) = \sup_{m,n,j} \{|a(m; g^i(n; x)|_{U_j})| J_j(t_n, m)p^{-v'(m,j,i)}\}.$$

### 2.12. Note.

At first it is necessary to prove theorems about the quasi-invariance and the pseudo-differentiability of transition measures of stochastic processes on Banach spaces over local fields. We consider two types of measures on $c_0(\omega_0, K)$. The first is the $q$-Gaussian measure

$$\mu = \mu_{J, \gamma, q} := \bigotimes_{j=1}^\infty \mu_j(dx^j), \text{ where } \mu_j(dx^j) = C_{\gamma_j}^{-\gamma, \gamma_j, q} f_{\gamma_j}^{-\gamma, \gamma_j, q} v(dx^j)$$

(see §2 [13]). The characteristic functional of the $q$-Gaussian measure is positive definite, hence $\mu$ is nonnegative (see also §2.6 [15]). The second is specified below and is the particular case of measures considered in §4.3 [14].

Let $w$ be the real-valued nonnegative Haar measure on $K$ with $w(B(K, 0, 1)) = 1$. We consider the following measure $\mu$ on $c_0(\omega_0, K)$

(i) $\mu(dx) = \bigotimes_{j=1}^\infty \mu_j(dx^j)$, where $x \in c_0(\omega_0, K)$, $x = (x^j : j \in \omega_0)$, $x^j \in K$, $x = \sum_j x^j e_j$, $e_j$ is the standard orthonormal base in $c_0(\omega_0, K)$. 

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Let now on the Banach space \( c_0 := c_0(\omega_0, K) \) there is given an operator \( J \in L_1(c_0) \) such that \( Je_i = v_i e_i \) with \( v_i \neq 0 \) for each \( i \). We consider a measure \( \nu_i(dx) := f_i(x)w(dx) \) on \( K \), where \( f_i : K \to [0, 1] \) is a function belonging to the space \( L^1(K, w, R) \) such that \( f_i(x) = f(x/v_i) + h_i(x/v_i) \), where \( f \) is a locally constant positive function, \( f(x) = \sum_{j=1}^{\infty} C_jCh_{B_j}(x) \), \( B_j := B(K, x_j, r_j) \) is a ball in \( K \), \( Ch_V \) is the characteristic function of a subset \( V \) in \( K \), that is, \( Ch_V(x) = 1 \) for each \( x \in V \), \( Ch_V(x) = 0 \) for each \( x \in K \backslash V \), \( x_1 := 0 \), \( r_1 := 1 \), \( \inf_j r_j = 1 \), \( \{ B_j : j \} \) is the disjoint covering of \( K \), \( 1 \geq C_j > 0 \), \( \lim_{|x| \to \infty} f(x) = 0 \), \( h_i \in L^1(K, w, R) \) such that \( \text{ess} w - \sup_{x \in K} |h_i(x)/f(x)| = \delta_i < 1 \), \( \sum_i \delta_i < \infty \) and \( \nu_i(K) = 1 \). Then \( \nu_i(S) > 0 \) for each open subset \( S \) in \( K \). There exists a \( \sigma \)-additive product measure

\[(ii) \quad \mu_j(dx) := \prod_{i=1}^{\infty} \mu_i(dx^i) \text{ on the } \sigma\text{-algebra of Borel subsets of } c_0 \text{, since the}
\]
Borel \( \sigma \)-algebras defined for the weak topology of \( c_0 \) and for the norm topology of \( c_0 \) coincide, where \( \mu_i(dx^i) := \nu(dx^i/v_i) \).

Let \( A : c_0 \to c_0 \) be a linear topological isomorphism, that is, \( A \) and \( A^{-1} \in L(c_0) \), then for a measure \( \mu \in c_0 \) there exists in image \( \mu_A(S) := \mu(A^{-1}S) \) for each Borel subset \( S \) in \( c_0 \). In view of Proposition 2.12.2 \[15\] \( L_q(c_0) \) is the ideal in \( L(c_0) \). This produces new \( q \)-Gaussian measures \( (\mu_{J,\gamma,q})_A =: \mu_{AJ,\gamma,q} \) and measures of the second type \( (\mu_J)_A =: \mu_{AJ} \). In view of \( \S 2.9 \) \[13\] each injective linear operator \( S \in L_q(c_0) \) with \( E(c_0) \) dense in \( c_0 \) can be presented in the form \( S = AJ \). Hence for each such \( S \) there exists the \( \sigma \)-additive measure \( \mu_{S,\gamma,q} \) and \( \mu_S \). These measures are induced by the corresponding cylinder measures \( \mu_{\gamma,\gamma,q} \) or \( \mu_{\gamma} \) on \( K^{\otimes_0} \), where \( I \) is the unit operator, since \( c_0 \) in the weak topology is isomorphic with \( K^{\otimes_0} \). Here the algebra \( U \) of cylindrical subsets is generated by subsets \( \pi_V^{-1}(A) \), where \( A \) is a Borel subset in \( K^n \), \( card(V) = n < \aleph_0 \), \( V \subset N \), \( \pi_V : K^{\otimes_0} \to \prod_{i \in V} K_i \) is the natural projection.

On the space \( C_0^0(T, H) = C_0^0(T, K) \otimes H \) let \( S = S_1 \otimes S_2 \) and \( \gamma = \gamma^1 \otimes \gamma^2 \), where \( S_1 \) is a linear operator on \( C_0^0(T, K) \) and \( S_2 \) is a linear operator on \( H \), \( \gamma^1 \in C_0^0(T, K) \), \( \gamma^2 \in H \) such that the measure \( \mu_{S,\gamma,q} \) is the product of measures \( \mu_{S_1,\gamma^1,q} \) on \( C_0^0(T, K) \) and \( \mu_{S_2,\gamma^2,q} \) on \( H \), analogously \( \mu_S \) is the product of measures \( \mu_{S_1} \) on \( C_0^0(T, K) \) and \( \mu_{S_2} \) on \( H \). With the help of such measures on the space \( C_0^0(T, H) \) the stochastic process \( w(t, \omega) \) is defined as in \( \S 3.4.2 \) and 4.3 \[13\] and \( \S 3.2 \) \[15\].

2.13. Let \( Y \) be a Banach space over the local field \( K \) and \( V \) be a neighbourhood of zero in \( Y \). Consider either the measure \( \mu_{S,\gamma,q} \) or \( \mu_S \) outlined in \( \S 2.12 \). Suppose that in stochastic antiderivational equations 3.4.(i) and
3.5.(i) [\[10\]] mappings \(a\) and \(E\) be dependent on the parameter \(y \in V\), that is, \(a = a(t, \omega, \xi, y)\) and \(E = E(t, \omega, \xi, y)\); moreover, \(a_{k,l} = a_{k,l}(t, \xi, y)\) for each \(k\) and \(l\) in the latter equation, the condition 3.4.1(LLC) [\[10\]] is satisfied for each \(0 < r < \infty\) with the constant \(K_r\) independent from \(y \in V\) for each \(y \in V\).

Evidently, Equation 3.4.(i) is the particular case of 3.5.(i), when in the latter equation the corresponding \(a_{0,1}\) and \(a_{1,0}\) are chosen with all others \(a_{k,l} = 0\) (when \(k + l \neq 1\)). Let also

(i) \(a, E\) and \(a_{k,l}\) be of class \(C^1\) by \(y \in V\) such that
\[
a \in C^1(V, L^2(\Omega, F, \lambda; C^0(B_R, L^q(\Omega, F, \lambda; C^0(B_R, H)))))
\]
and
\[
E \in C^1(V, L^2(\Omega, F, \lambda; C^0(B_R, L(L^q(\Omega, F, \lambda; C^0(B_R, H)))))
\]
\[
a_{m-l,l} \in C^1(V, C^0(B_R, L^q(\Omega, F, \lambda; C^0(B_R, H)), 0, R_2), L_m(H^{\otimes m}; H))\) (continuous and bounded on its domain) for each \(n, l, 0 < R_2 < \infty\) and
\[
\lim_{n \to \infty} \sup_{0 \leq t \leq n} \|a_{n-l,l}\|_{C^1(V, C^0(B_R, \times B(L^q(\Omega, F, \lambda; C^0(B_R, H)), 0, R_2), L_n(H^{\otimes n}, H)))} = 0
\]
for each \(0 < R_1 \leq R < \infty\) or each \(0 < R_1 < R\) when \(R = \infty\), for each \(0 < R_2 < \infty\);

(ii) \(ker(E(t, \omega, \xi, y)) = 0\) for each \(t, \xi\) and \(y\), also for \(\lambda\)-almost every \(\omega\);

(iii) \(a(t, \omega, \xi, y)\) and \(\partial a(t, \omega, \xi, y)/\partial y \in X_{0,s}(H) := \{z : S^{-1}z \in H_s\}\)
and \(\partial E(t, \omega, \xi, y)/\partial y \in L_r(H)\) for \(\lambda\)-almost all \(\omega\) and each \(t, \xi, y, \) where \(H_s := \{z : z \in H; \sum_{j=1}^{\infty} |z_j|^s < \infty\}\) for each \(0 < s < \infty\), \(H_\infty := H\), with \(s = r = q\) for \(\mu_{s,\gamma,q}\); \(s = \infty\) and \(r = 0\) for the measure of the second type \(\mu_S, z_j\) are the coordinates of the vector \(z\) in the standard base in \(H\); in addition for Equation 3.5.(i)

(iv) \(\partial a(t, \omega, \xi, y)/\partial y \in L_{k+l,r}(H^{\otimes(k+l)}; H)\) for each \(l\) and each \(k\) with either \(r = q\) or \(r = 0\) correspondingly. The following theorem states the quasi-invariance of the transition measure \(\mu^{F_{t,u}}(\{\omega : \xi(t_0, \omega, y) = 0, \xi(t, \omega, y) \in A\}) = \mu_y(A)\) where \(F_{t,u}(\xi) := \xi(t, \omega, y) - \xi(u, \omega, y)\).

**Theorem.** Let either Conditions (i – iii) or (i – iv) be satisfied, then the transition measure \(P_y(A)\) of the stochastic process \(\xi(t, \omega, y)\) being the solution of Equation either 3.4.(i) or 3.5.(i) [\[10\]] and depending on the parameter \(y \in V\) is quasi-invariant relative to the corresponding \(\lambda\)-invariant measure \(U(y_2, y; \xi(t, \omega, y)) := \xi(t, \omega, y_2)\) for each \(y\) and \(y_2 \in V\).

**Proof.** The Kakutani theorem (see II.4.1 [\[1\] \]) states, whether \(\prod_{k=1}^{\infty} \alpha_k\) converges to a positive number or diverges to zero, the measure \(\mu\) is absolutely continuous or orthogonal with respect to \(\nu\), correspondingly, where \(\alpha_k := \int_{X_k} (p_k(x_k))^{1/2} \nu_k(dx_k), \mu_k\) is absolutely continuous relative to \(\nu_k, \mu = \otimes k \mu_k, \nu = \otimes k \nu_k, \mu_k\) and \(\nu_k\) are probability measures on measurable spaces \(X_k\) for each \(k \in N\), \(p_k(x) := \mu_k(dx)/\nu_k(dx)\). In the first case \(\prod_k p_k(x_k)\) converges
in the mean to $\mu(dx)/\nu(dx)$. In the considered here case let $X_k = K$ for
each $k \in \mathbb{N}$. Let $\mu_k(dx) = C_f(x - y)v(dx)$, $\nu_k(dx) = C_f(x)v(dx)$, where
$v$ is the non-negative Haar measure on $k$. $f$ is a positive function such
that $f \in L^1(K,v,R)$, $C = \text{const} > 0$ such that $\nu(K) = 1$. Then $p_k(x) =
f(x-y)/f(x)$ and $\alpha_k = \int_{k}(f(x-y)f(x))^{1/2}v(dx)$. For the $g$-Gaussian measure
$f(x) = f_K\exp(-\beta|x|^q)\chi_y(x)\chi_{1}(z)\nu(dx)$ (see [29] and §7 [31]). If $|xy| \leq 1$, then
$\chi_{1}(y) = 1$. Therefore, there is a constant $C_1 > 0$ independent from
$\beta$ and $\gamma$ such that $|f(z - y) - f(z)| \leq |f(z)|(1 + C_1\exp(-\beta r^{-q}))$ for each
$y$ with $|y| < r$, where $\beta r^{-q} > 1$, since due to Cauchy-Schwarz-Bunyakovskii
inequality
\[\int_{|x|>1/r} \exp(-\beta|x|^q)\chi_{1}(z)\chi_{1}(z)\chi_{1}(z)\nu(dx) \leq \int_{|x|>1/r} \exp(-\beta|x|^q)\chi_{1}(z)\chi_{1}(z)\chi_{1}(z)\nu(dx)|g(y, z) \leq |f(z)||g(y, z),\]
where $g(y, z) := \int_{|x|>1/r} \exp(-\beta|x|^q)\chi_{1}(z)\chi_{1}(z)\chi_{1}(z)\nu(dx)$. Let $|y_j/v_j| = r_j < 1$ for each $j > j_0$, then $|\alpha_j - 1| \leq C\exp(-\beta r_j^{-q})$ for each $j > j_0$,
where $C = \text{const} > 0$. In view of Proposition 2.12.2 [13] and the Kakutani theorem $\mu_{S,\gamma,q}^z$ is equivalent to $\mu_{S,\gamma,q}$ for each $z \in X_0,q(C_0^0(T, H))$, where
$\mu_{S,\gamma,q}^z(A) := \mu(A - z)$ for each Borel subset $A$ in $C_0^0(T, H)$, that is, $\mu_{S,\gamma,q}$ is quasi-invariant relative to shifts $z \in X_0,q(C_0^0(T, H))$.

For the measure $\mu_{\gamma}$ and $|y| < 1/|v|$ there is the equality $f((x - y)/v) =
f(x/v)$ for each $x \in K$ and $0 \neq v \in K$. In view of the definition of $f_k$
there is the equality $p_k(x) = f_k(x - y_k)/f_k(x) = [f((x - y_k)/v_k)/f(x/v_k)]/[1 + h_k((x - y_k)/v_k)/f((x - y_k)/v_k)]/[1]
+ h_k(x/v_k)/f(x/v_k)]$. If $|y_k/v_k| \leq 1$, then $f((x - y_k)/v_k)/f(x/v_k) = 1$ for each $x \in K$. From the conditions imposed on
$h_k$ and $f$ and the Kakutani theorem and Proposition 2.12.2 [13] it follows,
that $\mu_{S,\gamma,q}$ is quasi-invariant relative to shifts $z \in X_{0,q}(C_0^0(T, H))$.

The quasi-invariance factor $\rho(z, x) := \mu_{S,L}(dx)/\mu(dx)$ is Borel measurable
as follows from the construction of $\mu$ and the Kakutani theorem and the
Lebesgue theorem about majorized convergence (see §2.4.9 [3]), since this is true for each its one-dimensional projection. The Banach theorem states: if $G$
is a topological group and $A \subset G$ is a Borel measurable set of second category,
then $A \circ A^{-1}$ is a neighbourhood of unit (see §5.5 [14]). The quasi-invariance
factor satisfies the cocycle condition: $\rho(z + h, x) = \rho(z, x - h)\rho(h, x)$ for
each $z$ and $h \in X_{0,q}(C_0^0(T, H))$ and each $x \in C_0^0(T, H)$. Therefore, in view
of the Lusin theorem (see §2.3.5 [4]) \( \rho(z, x) := \frac{\mu(z)}{\mu(x)} \) is such that 

\( \mu(W_L) = 1 \) for each finite-dimensional subspace \( L \) in \( X_{0,s}(C^0_0(T, H)) \), where 

either \( \mu = \mu_{s,n,q} \) or \( \mu = \mu_S \), \( W_L := \{ x : \rho(z, x) \) is defined and continues by 

\( z \in L \} \).

In view of the preceding consideration \( \lim_{n\to\infty} \rho(P_n z, x) = \rho(z, x) \) for 

\( \mu \)-almost all \( x \in C^0_0(T, H) \), moreover, this convergence is uniform by \( z \) in each 

ball \( B(L, 0, c) \) for each finite-dimensional subspace \( L \) in \( X_{0,s}(C^0_0(T, H)) \), 

where \( P_n \) is a projection on a subspace \( sp(e_1, ..., e_n) = K^n \), where \( \{ e_j : j \} \) is the 

orthonormal base in \( X_{0,s}(C^0_0(T, H)) \). Evidently, \( X_{0,s}(C^0_0(T, H)) \) is dense 

in \( C^0_0(T, H) \).

Stochastic antiderivational Equation 3.4.(i) [16] is the particular case of 

3.5.(i). Therefore, it is sufficient to consider the latter equation. Below it is 

shown that the one-parameter family of solutions \( \xi(t, \omega, y) \) is of class \( C^1 \) by 

\( y \in V \). Let \( X_0(t, y) = x(y), \ldots, \)

\[
X_n(t, y) = x(y) + \sum_{m=1}^{\infty} \sum_{l=0}^{m} (P_{m-l} x_n(u(y), y) - a_{m-l}(u, X_{n-1}(u, y), y)) \circ (I^\otimes b \otimes a^{(m-l)} \otimes E^\otimes l)] y_{=t},
\]

correspondingly, 

\[
X_{n+1}(t, y) - X_n(t, y) = \sum_{m=1}^{\infty} \sum_{l=0}^{m} (P_{m-l} x_n(u(y), y) - a_{m-l}(u, X_{n-1}(u, y), y)) \circ (I^\otimes b \otimes a^{(m-l)} \otimes E^\otimes l)] y_{=t},
\]

where \( t_j = \sigma_j(t) \) for each \( j = 0, 1, 2, \ldots \), for the shortening of the notation \( X_n, \)

\( x \) and \( a_{l,k} \) are written without the argument \( \omega, a \) and \( E \) are written without 

their variables. Then

\[
M \sup_{u,y} \| \hat{P}_{b+l} - a_{m-l}(u, X_{n-1}(u, y), y) \| (B_{R_1} \times B(L^\otimes 0, R_2) \times V) \circ (I^\otimes b \otimes a^{(m-l)} \otimes E^\otimes l)] y_{=t})^g \leq K(M \| \hat{P}_{b+l} - a_{m-l}(u, X_{n-1}(u, y), y) \|^g) (M \sup_{u,y} \| a \|_{m-l}) (M \sup_{u,y} \| E \|^l),
\]

20
where \( X_n \in C_0^0(B_R, H) \) for each \( \omega, y \in V \) and for each \( n, K \) is the same constant as in §3.4, \( 1 \leq g < \infty \). On the other hand,

\[
X_1(t, y) = x(t, y) + \sum_{m+b=1}^{\infty} \sum_{l=0}^{m} (\hat{P}_{u+b-l, w(u, \omega)}[a_{m-l+b, l}(u, x(u, y)) \circ (\hat{I}^{\otimes b} \otimes a^{\otimes (m-l)} \otimes E^{\otimes l})])_{u=t},
\]

consequently,

\[
\|X_1(t, y) - X_0(t, y)\|^g \leq \sup_{m,l,b} (\|\hat{P}_{u+b-m-l, w(u, \omega)}[a_{m-l+b, l}(u, x(u, y)) \circ (\hat{I}^{\otimes b} \otimes a^{\otimes (m-l)} \otimes E^{\otimes l})])_{u=t}\|^g.
\]

Due to Condition \((ii)\) for each \( \epsilon > 0 \) and \( 0 < R_2 < \infty \) there exists \( B_\epsilon \subset B_R \) such that

\[
K \sup_{m,l,b} (\|\hat{P}_{u+b-m-l, w(u, \omega)}[a_{m-l+b, l}(u, x(u, y)) \circ (\hat{I}^{\otimes b} \otimes a^{\otimes (m-l)} \otimes E^{\otimes l})])_{u=t}\|^g
\]

\( =: c < 1 \). On the other hand, the partial difference quotient has the continuous extension \( \hat{\Phi}^1 \) of \( a_{t,k} \), \( a \) and \( E \) themselves, where \( y \in V, h \in Y, \xi \in K \) such that \( y + \xi h \in V \), since analogous to \( (X_{n+1} - X_n) \) estimates are true for \( \hat{\Phi}^1(X_{n+1} - X_n) \). Therefore, there exists the unique solution on each \( B_\epsilon \) and it is of class \( C^1 \) by \( y \in V \), since \( \sup_{u,y} \max(\|X_1(u, y) - X_0(u, y)\|_{L^0(\Omega, H)}, \|\hat{\Phi}^1_1(X_1(u, y) - X_0(u, y))\|_{L^0(\Omega, H)} < \infty \) and \( \lim_{t \to \infty} C = 0 \) for each \( C > 0 \), hence there exists \( \lim_{n \to \infty} X_n(t, y) = X(t, y) = \xi(t, \omega, y)_{|B_\epsilon} \), where \( C := M \sup_{u \in B_\epsilon, y \in V} \max(\|X_1(u, y) - X_0(u, y)\|_{L^0(\Omega, H)}, \|\hat{\Phi}^1_1(X_1(u, y) - X_0(u, y))\|_{L^0(\Omega, H)}) \leq (c + 1)K < \infty \), here \( B_\epsilon \) is an arbitrary ball of radius \( \epsilon \) in \( B_R \), \( t \in B_\epsilon \). Therefore, \( \xi(t, \omega, y) \in C^1(V, L^q(\Omega, F, \lambda; C^0(B_R, H))). \)

From Proposition 3.11 [16] it follows, that the multiplicative operator functional \( T(t, v; \omega; y) \) is of class \( C^1 \) by the parameter \( y \in V \) such that \( \xi(t, \omega, y) = T(t, v; \omega; y)\xi(v, \omega, y) \) for each \( t, v \in T \).

Due to the existence and uniqueness of the solution \( \xi(t, \omega, y) \) for each \( y \in V \), there exists the operator \( U(y_2, y; \xi(t, \omega, y)) := \xi(t, \omega, y_2) \), that may be nonlinear by \( \xi \). The variation of the family of solutions \( \{\xi(t, \omega, y) : y\} \) corresponds to the differential \( D_y \xi(t, \omega, y) \). Since \( \xi(t, \omega, y) \) is of class \( C^1 \) by \( y \), then \( U(y_2, y; \xi(t, \omega, y)) \) is of class \( C^1 \) by \( y \) and \( y_2 \). The operator \( U(y_2, y; \ast) \) has the inverse, since \( U(y, y_2; U(y_2, y; \xi(t, \omega, y))) = \xi(t, \omega, y) \) for each \( y_2 \) and
\[ y \in V, \ t \in T \text{ and } \omega \in \Omega. \text{ Therefore, } U^{-1}(y_2, y;*) \text{ is also of class } C^1 \text{ by } y_2 \text{ and } y. \text{ In view of Conditions } (iii, iv) \text{ and } \xi(t, \omega, y_2) - \xi(t, \omega, y) \in X_{0,s}(H).\]

On the other hand, either \( \mu_{S,\gamma,q} \) or \( \mu_S \) is quasi-invariant relative to shifts \( z \in X_{0,q}(C^0_0(T, H)) \) and \( S = S_1 \otimes S_2 \), consequently, the transition measure \( P_y \) is quasi-invariant relative to shifts \( z \in X_{0,s}(H) \). In view of Conditions \( (ii - iv) \) \( \partial U(y_2, y; \eta)/\partial \eta - I \in L_\tau(H) \) for each \( y_2 \) and \( y \in V \), where \( \eta \in \{ \xi(t, \omega, y) : y \} \), either \( r = q \) or \( r = 0 \) respectively. Since \( \mu_S(C^0_0(T, H)) = 1 \), then \( P_y(H) = 1 \), hence \( U(y_2, y;*) \) is defined \( P_y \) - almost everywhere on \( H \) for each \( y_2 \) and \( y \in V \). Therefore, there exists \( n \) such that for each \( j > n \) the mappings \( V(j; x) := x + P_j(U^{-1}(x) - x) \) and \( U(j; x) := x + P_j(U(x) - x) \) are invertible and \( \lim_j | \det U'_x(j; x) | = | \det U'_x(x) | \) and \( \lim_j | \det V'_x(j; x) | = 1/| \det U'_x(x) | \), where \( U(x) := U(y_2, y; x), y_2 \) and \( y \in V \).

In view of Theorem 3.28 \[ \] for each \( y_2 \) and \( y \in V \) the transition measures \( P_{y_2} \) and \( P_y \) are equivalent.

**2.14. Theorem.** Let Conditions 2.13. \( (i - iv) \) be satisfied and let \( \phi \) be a \( C^1 \)-diffeomorphism of a subset \( V \) clopen in \( K \) onto the unit ball \( B(K, 0, 1) \). Then

1. the transition measure \( P_y \) corresponding to \( \mu_{S,\gamma,q} \) is pseudo-differentiable by the parameter \( y = \phi(z) \) of order \( b \in C \) for each \( \text{Re}(b) \geq 0 \), where \( z \in V \);
2. \( P_y \) corresponding to \( \mu_S \) with \( h_k \) such that \( \sum_k \delta_k < \infty \), where \( \delta_k := \sup_{x \in B(K,0,1) | |PD_x(b, h_k(x))|} \) is pseudo-differentiable by the parameter \( y = \phi(z) \) of order \( b \geq 0 \), moreover, \( P_y \) is pseudo-differentiable for each \( b \in C \), when each \( f_k \) is locally constant, that is, \( h_k = 0 \) for each \( k \in N \).

**Proof.** Up to a constant multiplier the operator \( PD_x(b, h(x)) \) of \S 2.7 coincides with the pseudo-differential operator \( D^b(h(x)Ch_{B(K,0,1)}(x)) \) from \S 9 \[ \] where \( Ch_A \) is the characteristic function of the subset \( A \) in \( K \). If \( \psi \in L^2(K, w, C) \) and \( b > 0 \), then due to the Cauchy-Schwarz-Bunyakowski inequality there exists \( \int_{K \setminus B_{K,\infty}} |\psi(x) - \psi(y)||x - y|^{-1-b}w(dy) \), where \( w \) is the Haar nonnegative measure on \( K \). Then \( \int (D^b(h(x))) = |x|^b F[h(x)] \), where \( F(h)(x) := \int_{K} h(y)\chi_1((x,y))w(dx) \) is the Fourier transform \[ \] (see also \S 3.6 \[ \]). In view of Theorem 7.4 \[ \] the Fourier transform \( f \mapsto F[f] \) is the bijective continuous isomorphism of \( L^2(K, w, C) \) onto itself such that \( f(x) = \lim_{r \to \infty} \int_{B(0,r)} F[f](y)\chi_1(-(y,x))w(dy) \) and \( (f,g) = (F[f], F[g]) \) for each \( f,g \in L^2(K, w, C) \). If \( \psi(x) = C \exp(-\beta|x|\gamma)h(x) \), then there exists \( D^b\psi(x) \) for each \( b > 0 \). In accordance with Example 4.3.9 \( \int_{K} \chi_1(x)w(dx) = 0 \) for each \( \gamma \neq 0 \). In view of Example 4.3.10 \[ \] \( \int_{Q_{\mu}} |x|^q\chi_1(yx)w(dx) = \)
\[ 1 - p^q \] \[ 1 - p^{-n(q+1)} \] |y|^{-n(q+1)} \] for each \( q \in \mathbb{C} \) with Re(\( q \)) > 0 and \( n \in \{1, 2, 3, \ldots \} \).

If \( f \) is a locally constant function as in §2.13, then \( PD_c(b, f) \) exists for each \( b \in \mathbb{C} \). On the other hand, \( PD_c(b, f + h_k) = PD_c(b, f) + PD_c(b, h_k) \).

Let \( g \) be a continuously differentiable function \( g : \mathbb{R} \to \mathbb{R} \) such that \( \|g\|_{C^1(\mathbb{R}, \mathbb{R})} := \sup_x |g(x)| + \sup_x |g'(x)| < \infty \), that is \( g \in C^1_b(\mathbb{R}, \mathbb{R}) \). If for \( f : K \to \mathbb{R} \) and \( x \in K \) there exists \( [f(x) - f(y)]|x - y|^{-1-b} \in L^1(K, w, \mathbb{C}) \) as the function by \( y \in K \), then \( f(K)[g \circ f(x) - g \circ f(y)]|x - y|^{-1-b}w(dy) = \int_{S(f, x)} [g \circ f(x) - g \circ f(y)][f(x) - f(y)]^{-1}[f(x) - f(y)]|x - y|^{-1-b}w(dy) \), where \( S(f, x) := \{ y : y \in K, f(x) \neq f(y) \} \), consequently, there exists \( PD(b, g \circ f)(x) \).

If instead of \( g \) there exists \( h \in C^1(\mathbb{K}, \mathbb{K}) \) such that \( \|h\|_{C^1(\mathbb{K}, \mathbb{K})} := \max(\sup_x |h(x)|, \sup_{x,y} |\Phi^1 h(x; 1; y)|) < \infty \), that is \( h \in C^1_b(\mathbb{K}, \mathbb{K}) \), then \( f(K)[f \circ h(x) - f \circ h(y)]|x - y|^{-1-b}w(dy) = \int_{S(h, x)} [f \circ h(x) - f \circ h(y)]|h(x) - h(y)|^{-1-b}w(dy) \), hence there exists \( PD(b, f \circ h)(x) \). Analogous two statements are true for the operator \( PD_c \) instead of \( PD \).

In view of Equation 9.1(5) \([11]\) \( D^\alpha D^\beta \psi = D^\beta D^\alpha \psi = D^{\alpha+\beta} \psi \) for each \( \alpha \neq -1, \beta \neq -1 \) and \( \alpha + \beta \neq -1 \) for each \( \psi \in \mathbb{D}' \) such that there exists \( D^\alpha \psi \), \( D^\beta \psi \) and \( D^{\alpha+\beta} \psi \), where \( \mathbb{D}' \) is the topologically dual space to the space \( \mathbb{D} \) of locally constant functions \( \phi : K \to \mathbb{R} \). On the other hand, \( \mathbb{D} \) is dense in \( \mathbb{D}' \) in the weak topology (see §6 [31]). Evidently, \( L^2 \cap \mathbb{D} \) is dense in \( L^1(K, w, \mathbb{R}) \) also. The characteristic functional of the Gaussian measure belongs to \( \mathbb{D}' \) and is locally constant on \( K \setminus \{0\} \). Due to §§7.2 and 7.3 the Fourier transform is the linear topological isomorphism of \( \mathbb{D} \) on \( \mathbb{D}' \) and of \( \mathbb{D}' \) on \( \mathbb{D} \). Then \( \mu_{\gamma, q}^g(dx)/w(dx) \in L^1(K, w, \mathbb{R}) \cap \mathbb{D}' \) for each \( g \in C^0_0(T, H)^* \).

In view of Theorem 4.3 \([38]\) and using the Kakutani theorem as in §2.13 we get the statements of this theorem, since the quasi-invariance factor \( P_y(dx)/P_u(dx) \) is pseudo-differentiable as the function by \( y \) of order \( b \) for each fixed \( u \in B(K, 0, 1) \).

2.15. Theorem. Let \( G \) be either a loop group or a diffeomorphism group defined as in §§2.9 and 2.11, then there exists a stochastic process \( \xi(t, \omega) \) on \( G \) which induces a quasi-invariant transition measure \( P \) on \( G \) relative to \( G' \) and \( P \) is pseudo-differentiable of order \( b \) for each \( b \in \mathbb{C} \) such that \( \text{Re}(b) \geq 0 \) relative to \( G' \), where a dense subgroup \( G' \) is given in §§2.10 and 2.11.

Proof. These topological groups also have structures of \( C^\infty \)-manifolds, which are infinite-dimensional over the local field \( K \), but they do not satisfy
the Campbell-Hausdorff formula in any open local subgroup \[21, 23\]. Their manifold structures and actions of \( G' \) on \( G \) will be sufficient for the construction of desired measures. These separable Polish groups have embeddings as clopen subsets into the corresponding tangent Banach spaces \( Y' \) and \( Y \) in accordance with \[24\] and \( \S \S 2.1 - 2.11 \), where \( Y' \) is the dense subspace of \( Y \). As usually \( TG = \bigcup_{x \in G} T_x G \) and \( T_x G = (x, Y) \).

Let \( G \) be a complete separable relative to its metric \( \rho \) \( C^\infty \)-manifold on a Banach space \( Y \) over \( K \) such that it has an embedding into \( Y \) as the clopen subset. Let \( \tau_G : TG \to G \) be a tangent bundle on \( G \). It is trivial, since \( TG = G \times Y \) for the considered here case. Let \( \theta : Z_G \to G \) be a trivial bundle on \( G \) with the fibre \( Z \) such that \( Z_G = Z \times G \), then \( L_{1.2}(\theta, \tau_G) \) be an operator bundle with a fibre \( L_{1.2}(Z, Y) \) (see \( \S 2.13 \) \[13\]). Let \( \Pi := \tau_G \oplus L_{1.2}(\theta, \tau_G) \) be a Whitney sum of bundles \( \tau \) and \( L_{1.2}(\theta, \tau_G) \).

Since \( G \) is clopen in \( Y \), the valuation group of \( K \) is discrete in \( (0, \infty) \), then it has a clopen disjoint covering by balls \( B(Y, x_j, r_j) \). That is, the atlas \( \text{At}(G) \) of \( G \) has a refinement \( \text{At}'(G) \) being a disjoint atlas.

On \( Y \) consider the measure \( \mu_{S,\gamma,q} \) or \( \mu_S \) as in \( \S 2.12 \). Then in view of Theorems 4.3 \[15\] and 2.2 \[16\] there exists the stochastic process \( w(t, \omega) \) corresponding to \( \mu_{S,\gamma,q} \) or \( \mu_S \) (see also Definitions 4.2 \[14\] and 3.2 \[16\]). Suppose that \( f \) and \( h_k \) for each \( k \in \mathbb{N} \) defining the measure \( \mu_S \) satisfy the Conditions of \( \S 2.12 \) and of Theorem 2.14.

Now let \( G \) be a loop or a diffeomorphism group of the corresponding manifolds over the field \( K \). Consider for \( G \) a field \( U \) with a principal part \( (a_\eta, E_\eta) \), where \( a_\eta \in T_\eta G \) and \( E_\eta \in L_{1.2}(H, T_\eta G) \) and \( \ker(E_\eta) = \{0\} \), \( \theta : H_G \to G \) is a trivial bundle with a Banach fiber \( H \) and \( H_G := G \times H \), \( L_{1.2}(\theta, \tau_\eta) \) is an operator bundle with a fibre \( L_{1.2}(H, T_\eta G) \) such that \( (a_\eta, E_\eta) \) satisfies Conditions of Theorem 3.4 \[16\]. For Equation 3.5.(i) \[16\] we take additionally \( (a_{l,k})_\eta \) for each \( l, k \) satisfying conditions of Theorem 3.5 \[19\]. To satisfy conditions of quasi-invariance and pseudo-differentiability of transition measures theorems we choose \( a_\eta, E_\eta \) and \( (a_{k,l})_\eta \) of class \( C^1 \) and satisfying Conditions 2.13.(iii,iv) by \( \eta := \eta \in G' := V \) for each \( l, k \).

We can take initially \( \mu_{1,s,q} \) or \( \mu_T \) a cylindrical measure on a Banach space \( X' \) such that \( T_\eta G' \subset X' \subset T_\eta G \). If \( A_\eta \) is the \( L_q \)-operator or the \( L_1 \)-operator with \( \ker(A_\eta) = \{0\} \), then \( A_\eta \) gives the \( \sigma \)-additive measure \( \mu_{A_\eta, A_{1,q}, A_{z,q}} \) or \( \mu_{A_\eta} \) in the completion \( X'_{1,\eta} \) of \( X' \) with respect to the norm \( \|x\|_1 := \|A_\eta x\| \) (see also \( \S 2.12 \)).

There exists the solution \( \xi(t, \omega, \eta) = \xi_\eta(t, \omega) \) of stochastic antiderivational
Consider left shifts $L_h : G \to G$ such that $L_h \eta := h \circ \eta$. Let us take $a_e \in T_e G', A_e \in L_{1,q}(T_e G', T_e G)$ or $A_e \in L_{1,1}(T_e G', T_e G)$ respectively, $(a_{k,l})_\eta \in L_{k+l}(T_e G)^{\otimes (k+l)}; T_e G)$ for each $k$ and each $l$, where $H, T e G'$ and $T_e G$ in their own norm uniformities are isomorphic with $\mathcal{C}_0(\omega_0, K)$. Then we put $a_x = (DL_x) a_e$ and $A_x = (DL_x) \circ A_e$ for each $x \in G$, hence $a_x \in T_e G$ and $A_x \in L_{1,s}(H_x, (DL_x)T_e G)$, where $(DL_x)T_e G = T_x G$ and $T_e G' \subset T_x G$, $H_x := (DL_x)T_e G'$, $s = q$ or $s = 1$. Operators $L_h$ are (strongly) $C^\infty$-differentiable diffeomorphisms of $G$ such that $D_h L_h : T_\eta G \to T_{h \eta} G$ is correctly defined, since $D_h L_h = h_*$ is the differential of $h$. In view of the choice of $G'$ in $G$ each partial difference quotient $\tilde{\Phi}^n L_h(x_1, ..., x_n; \zeta_1, ..., \zeta_n)$ is of class $C^0$ and $D^n L_h$ is of class $L_{n+1,s}(TG'^n \times G', TG)$ for each vector fields $X_1, ..., X_n$ on $G'$, $\zeta_1, ..., \zeta_n \in K$ with $\zeta_j p_2(X_j) + h \in G'$ and $h \in G'$, since for each $0 \leq l \in \mathbb{Z}$ the embedding of $T^l G'$ into $T^l G$ is the product of two operators of the $L_q$-class or the embedding is of the $L_1$-class, where $T^0 G := G$, $X = (x, X_x) \in T_x G$, $x \in G'$, $X_x \in Y'$, $p_1(X) = x$, $p_2(X) = X_x$. Take a dense subgroup $G'$ from §2.10 or §2.11 correspondingly and consider left shifts $L_h$ for $h \in G'$.

The considered here groups $G$ are separable, hence the minimal $\sigma$-algebra generated by cylindrical subalgebras $f^{-1}(B_n)$, $n=1,2,...$, coincides with the $\sigma$-algebra $B$ of Borel subsets of $G$, where $f : G \to K^n$ are continuous functions, $B_n$ is the Borel $\sigma$-algebra of $K^n$. Moreover, $G$ is the topological Radon space (see Theorem 1.1.2 and Proposition 1.1.7 [5]). Let

$$P(t_0, \psi, t, W) := P(\{ \omega : \xi(t_0, \omega) = \psi, \xi(t, \omega) \in W \})$$

be the transition probability of the stochastic process $\xi$ for $t \in T$, which is defined on a $\sigma$-algebra $B$ of Borel subsets in $G$, $W \in B$, since each measure $\mu_{A_n, A_{n+1}, q}$ is defined on the $\sigma$-algebra of Borel subsets of $T_{\eta} G$ (see above). On the other hand, $T(t, \tau, \omega)g x = g T(t, \tau, \omega) x$ is the stochastic evolution family of operators for each $\tau \neq t \in T$. There exists the transition measure $P(t_0, \psi, t, W)$ such that it is a $\sigma$-additive quasi-invariant pseudo-differentiable of order $b$ relative to the action of $G'$ by the left shifts $L_h$ on $\mu$ measure on $G$, for example, $t_0 = 0$ and $\psi = \varepsilon$ with the fixed $t_0 \in T$ (see Definitions 2.6 and 2.7).
2.16. Note. In §2.15 $G'$ is on the Banach space $Y'$ and $G$ on the Banach space $Y$ over $K$ such that $G'$ and $G$ are complete relative to their uniformities $U_{G'}$ and $U_G$. There are inclusions $TG' = G' \times Y' \subset G \times Y' \subset G \times Y = TG$. The completion of $TG'$ relative to the uniformity $U_G \times U_Y$ produces the uniform space $G \times Y'$. Therefore, each $U_G \times U_Y$-uniformly continuous vector field $X = (x, X_x)$ on $G'$ has the unique extension on $G$ such that $X_x \in Y'$ for each $x \in G$ (see §8.3 [3]), where $U_G|_{G'} \subset U_{G'}$. Thus the $U_G \times U_{Y'}$-extension on $G$ and it provides the 1-parameter group $\rho : K \times G \to G$ of $C^\infty$-diffeomorphisms of $G$ generated by a $U_G \times U_{Y'}$-$C^\infty$-vector field $X_\rho$ on $G'$ [17, 21], that is, $(\partial \rho(v, x)/\partial v)|_{v=0} = X_\rho(x)$ for each $x \in G$, where $v \in K$, $X_\rho(x) \in G \times Y'$. In view of §2.15 the transition measure $P$ is quasi-invariant and pseudo-differentiable of order $b$ relative to the 1-parameter group $\rho$.

This approach is also applicable to the case of two Polish manifolds $G'$ and $G$ of class $C^\infty$ on $Y'$ and $Y$ over $K$. The quasi-invariance and pseudo-differentiability of the measure $P$ on $G$ relative to the 1-parameter group $\rho$ (by the definition) means such properties of $P$ relative to the $U_G \times U_{Y'}$-$C^\infty$-vector field $X$ on $G'$.

Evidently, considering different $(a, E)$ and $\{a_{k,l} : k, l\}$ we see that there exist $c = card(R)$ nonequivalent stochastic (in particular, Wiener) processes on $G$ and $c$ orthogonal quasi-invariant pseudo-differentiable of order $b \in C$ with $Re(b) > 0$ measures on $G$ relative to $G'$.

If $M$ is compact, then in the case of the diffeomorphism group its dense subgroup $G'$ can be chosen such that $G' \supset Diff(t', M)$ for $dim_K M = n \in N$ and $t' = t + s$ for $0 \leq t \in R$, $s > nv$, $v = dim_{Q_p}(K)$. Analogously can be considered the manifold $M \subset B(K^n, 0, r)$ and the group $G := Diff(an_r, M)$ of analytic diffeomorphisms $f : M \to M$ having analytic extensions on $B(K, 0, r)$ with the corresponding norm topology, where $r > 0$ and $r < \infty$. Then there exists the stochastic process $\xi$ on $T_c G$ such that it generates the transition measure $P$ on $T_c G$, its restriction on the clopen subset $G$ embedded into $T_c G$ produces the quasi-invariant and pseudo-differentiable of each order $b \in C$ with $Re(b) \geq 0$ measure $P_G$ relative to the dense subgroup $G' := Diff(an_R, M)$ for $R > r > 0$, since the embedding $T_c G'$ into $T_c G$ is of class $L_1$ (see also §2.17).

2.17. Theorem. Let $G$ be a separable Banach-Lie group over a local field $K$. Then there exists a probability quasi-invariant and pseudo-differentiable of each order $b \in C$ with $Re(b) > 0$ transition measure $P$ on $G$ relative
to a dense subgroup \( G' \) such that \( P \) is associated with a non-Archimedean stochastic process.

**Proof.** We consider two cases: (I) \( G \) satisfies locally the Campbell-Hausdorff formula; (II) \( G \) does not satisfy it in any neighbourhood of \( e \) in \( G \). The first case permits to describe \( G' \) more concretely. There exists the embedding of \( G \) into \( T_eG \) as the clopen subgroup, since \( G \) is the Polish group. The second case can be considered quite analogously to \( \S 2.15 \), where the dense subgroup \( G' \) can be characterized by the condition that the embedding of \( T_eG' \) into \( T_eG \) is \( \theta = \theta_1 \theta_2 \) with \( \theta_1 \) and \( \theta_2 \) of class \( L_q \) or \( \theta \) of class \( L_1 \), where \( s = q \) or \( s = 1 \) for stochastic processes associated either with \( \mu_{S,\gamma,q} \) or \( \mu_S \) respectively.

It remains to consider the first case. For \( G \) there exists a Banach-Lie algebra \( \mathfrak{g} \) and the exponential mapping \( \exp : V \to U \), where \( V \) is a neighbourhood of 0 in \( \mathfrak{g} \) and \( U \) is a neighbourhood of \( e \) in \( G \) such that \( \exp(V) = U \), where \( \exp(X + Y) = \exp(X)\exp(Y) \) for commuting elements \( X \) and \( Y \) of \( \mathfrak{g} \), that is, \([X,Y] = 0\), \( \exp(X)Y\exp(-X) = \exp(ad\,X)Y \), \( \exp(\lambda X) = \sum_{j=0}^{\infty} \lambda^j X^j/j! \), \( V = B(\mathfrak{g},0,r) \) is a ball of radius \( 0 < r < \infty \) in \( \mathfrak{g} \), \( \lambda \in \mathbb{K} \), \( \lambda X \in V \), \( \mathfrak{g} = T_eG \). The radii of convergence of the exponential and Hausdorff series corresponding to \( \log(\exp(X)\exp(Y)) \) are positive such that for each \( 0 < R < p^{1/(1-p)} \) to a ball \( B(\mathfrak{g},0,R) \) there corresponds a clopen subgroup \( G_1 \) supplied with the Hausdorff function (see \( \S II.6 \) and \( \S II.8 \)). Therefore, the exponential mapping \( \exp \) supplies \( G \) with the structure of the analytic manifold over \( \mathbb{K} \). By \( \text{At}(G) = \{(U_j, \phi_j) : j \in \mathbb{N}\} \) is denoted the analytic atlas of \( \mathbb{N} \), that is \( \phi_j : U_j \to V_j \) are diffeomorphisms of \( U_j \) onto \( V_j \), where \( U_j \) and \( V_j \) are clopen in \( G \) and in \( \mathfrak{g} \) respectively, connecting mappings \( \phi_j \circ \phi_i^{-1} \) are analytic on \( \phi_i(U_i \cap U_j) \subset \mathfrak{g} \). Therefore, the exponential mapping provides \( G \) with the covariant derivation \( \nabla \) and a bilinear tensor \( \Gamma \) such that \( \nabla_X Y = \lambda X \nabla_X Y = -\Lambda T(X,Y)/2 \), where the left-invariant derivation on \( G \) is defined by \( \lambda X \nabla_X Y = 0 \) for an arbitrary left-invariant vector field \( \hat{Y} \) and all vector fields \( X \) on \( G \), a vector field \( \hat{Y} \) is called left-invariant if \( TL_g \hat{Y}(h) = \hat{Y}(gh) \), \( L_gh := gh \) for each \( g, h \in G \), \( TL_g \) is the tangent mapping of \( L_g, \nabla_X Y = DY_u.X_u + \Gamma_u(X_u,Y_u) \). For such \( \nabla \) the torsion tensor is zero (see \( \S 1.7 \), \( \S 12 \) and \( \S 14.7 \)). It defines the rigid analytic geometry and the corresponding atlas on \( G \). Nevertheless \( \text{At}(G) \) has the refinement \( \text{At}'(G) \) such that charts of \( \text{At}'(G) \) compose the disjoint covering of \( G \).

Let \( a_x \) be an analytic vector field and \( A_x \) be a analytic operator field on \( G \) such that \( A_x \) is an injective compact operator of class \( L_s \) for each
\( x \in G \), since \( g \) is of separable type over a spherically complete field \( K \) and hence isomorphic with \( c_0(\omega_0, K) \) (see Chapter 3 in [29]), where \( s = q \) or \( s = 1 \). Let \( w_x(t, \omega) \) be a non-Archimedean stochastic (or, in particular, Wiener) process in \( T_x G \) such that \( a_x t + A_x w_x(t) \in T_x G \), since the space \( C_0^d \) is isomorphic with \( c_0 \). For a ball \( B_R := B(K, 0, R) \) in \( K \) for \( 0 < R < \infty \) let \( B(K, t_j, r) \) be a disjoint paving for sufficiently small \( 0 < r < \infty \) for which \( \xi^q(t) = \exp_{\xi^q, k} \{ a_{\xi^q, k} (t - t_k) + A_{\xi^q, k} [w_{\xi^q, k} (t) - w_{\xi^q, k} (t_k)] \} \) is defined, where \( \xi^q = \xi^q(t_k) \) for \( k = 0, 1, ..., n, \xi^q(0) = x \), \( q \) denotes the partition of \( B_R \) into \( B(K, t_j, r) \). Then there exists the process \( \xi = \lim_q \xi^q(t) \) which is by our definition a solution of the following stochastic equation:

\[
(i) \quad d\xi(t, \omega) = \exp_{\xi(t, \omega)} \{ a_{\xi(t, \omega)} dt + A_{\xi(t, \omega)} dw(t, \omega) \}
\]

for \( t \in B_{R} \). A function \( f(t, x) \) such that \( f(t, \xi) := \ln_{\xi(t, \omega)} \xi(t, \omega) \) satisfies the condition of Theorem 4.8 [15] on the corresponding domain \( W \), where \( (t, x) \in W \subset T \times H \). In view of Theorem 4.8 [14] after coordinate mapping of a chart \( (U, \phi) \) this equation takes the following form on \( g \):

\[
(ii) \quad \phi(\xi(t, \omega)) = \phi(\xi(t_0, \omega)) + (\hat{P}_{\phi} a_{\xi(u)}^\phi)|_{u=t} + (\hat{P}_{\phi} a_{\xi(u)}^\phi)|_{u=t} + \sum_{m=2}^\infty (m!)^{-1} \sum_{l=0}^m \left( m \atop l \right) (\hat{P}_{u^{m-l}} a_{\xi(u)}^\phi)|_{u=t} + (\hat{P}_{u^{m-l}} a_{\xi(u)}^\phi)|_{u=t},
\]

where \( E = A_{\xi(u, \omega)} \), \( a^\phi = (\partial \phi_x / \partial x) a_x \), \( A^\phi_x = (\partial \phi_x / \partial x) A_x (\partial \phi_x^{-1} / \partial x) \), since \( h^\phi = (\partial g^\phi / \partial x) f^\phi + \Gamma^\phi_{\phi(x)} (f^\phi, g^\phi) \) for \( h = \nabla f g \), \( f^\phi = (\partial \phi / \partial x) f \), \( g^\phi = (\partial \phi / \partial x) g \), \( h^\phi = (\partial \phi / \partial x) h \), \( \Gamma^\phi \) is a bilinear operator of Christoffel in \( g \), which has the transformation property \( D(\psi \circ \phi^{-1}) \). \( \Gamma^\phi_{\phi(x)} = D^2(\psi \circ \phi^{-1}) \). \( \Gamma^\phi_{\phi(x)} \circ (D(\psi \circ \phi^{-1}) \times D(\psi \circ \phi^{-1})) \) such that \( \nabla_X Y^\phi = DY^\phi \). \( X^\phi + \Gamma^\phi_{\phi(x)} (X^\phi, Y^\phi) \), \( \Gamma^\phi_{\phi(x)} \) denotes \( \Gamma \) for the chart \( (U, \phi) \), \( \psi \) corresponds to another chart \( (V, \psi) \) such that \( U \cap V \neq \emptyset \), \( f, g \) and \( h \) are vector fields, since \( [\partial (\psi \circ \phi^{-1}) / \partial t] = 0 \), that is Corollary 4.7 [15] is applicable instead of Theorem 4.8 [15] because \( f \) corresponds to \( (\psi \circ \phi^{-1}) \) (see §1.5 [14] and [3]). Since \( a_x \) and \( A_x \) are analytic, then \( a \) and \( E \) satisfy conditions of Theorem 3.4 [16].

The function \( \Gamma \) is analytic on the corresponding domain. On the other hand, \( g \) is isomorphic with \( c_0(\omega_0, K) \) as the Banach space. If \( Z \) is the center of \( g \), then \( ad : g/Z \rightarrow gl(c_0(\omega_0, K)) \) is the injective representation, where \( gl(c_0) \)
denotes the general linear algebra on $c_0$, $ad(x)y := [x,y]$ for each $x,y \in g$. Since $Z$ is commutative it also has injective representation in $gl(c_0)$, consequently, $g$ has embedding into $gl(c_0(\omega_0),K)$, since $c_0 \oplus c_0$ is isomorphic with $c_0$. Therefore, each $x \in g$ can be written in the form $x = \sum x_{i,j}X_{i,j}$, where $\{X_{i,j} : i,j \in N\}$ is the orthonormal basis of $g$ as the Banach space, $x_{i,j} \in K$, $\lim_{i+j \to \infty} x_{i,j} = 0$, consequently, $g$ has an embedding into $L_0(c_0(\omega_0),K)$. Then $\Gamma$ can be written in local coordinates $x_{s(i,j)} := x_{i,j}$, where $s : N^2 \to N$ is a bijection for which $\lim_{i+j \to \infty} s(i,j) = \infty$, $X_{i,j} := q_s(i,j)$, since $(x^n)^{i,k} = \sum x_{i,j}x_{l,m}^x...x_{r,u}^x$, when $x_{i,j} = (\delta_{a,j} \delta_{b,k} : a,b \in N)$, $\psi \circ \phi^{-1}(x) = \sum a_m a_n a_m x_m$ with $a_m^s \in K$ and $\lim s+|m| + \text{Ord}(m) \to \infty a_m^s = 0$, since $\exp(x)$ has a radius of convergence $0 < \tilde{r} = p^{-1}$ for $\text{char}(K) = 0$ (see Theorem 25.6 [30]), where $m = (m_1,...,m_k)$, $k = \text{Ord}(m)$, $0 \leq m_1 \in Z,...,0 \leq m_k \in Z$, $0 < m_k \in Z$, $0 \leq k \in Z$. Evidently, there exists $0 < r \leq \infty$ such that the series for $\psi \circ \phi^{-1}$ converges in $B(c_0,0,r)$ for $V \cup U \neq \emptyset$. Hence each $a_{m-l,t} := (\partial^{n-2} \Gamma_{\phi(x)}/\partial^{m-2}x)/m!$ for $m \geq 2$ and $a_{1,0} = a_{0,0} = (\partial \phi/\partial x)$ satisfies Condition 3.5.(ii) [16]. Due to Theorem 3.5 [16] there exists the unique solution of Equation 3.5.(i). Consider $G'$ corresponding to $g'$ such that the embedding $\theta$ of $g'$ into $g$ is of class $L_1$ for $\mu_S$ or $\theta = \theta_1 \theta_2$, where $\theta_1$ and $\theta_2$ are of class $L_q$ for $\mu_{S_\tau_{\gamma,q}}$

Let $T \in L_1(g)$ or $T = T_1T_2$ with $T_1$ and $T_2 \in L_s(g)$, where $s = 1$ or $s = q$. Consider $h_1 := T(g)$, $h_2 := sp_K([h_1,g] \cup h_1)$ and by induction $h_{n+1} := sp_K([h_n,g] \cup h_n)$, then $h_{n+1} \supset h_n$ and $h_n$ is the subalgebra in $g$ for each $n \in N$. In view of Proposition 2.12.2 the space $L_s(g)$ is the ideal in $L(g)$. Therefore, $h := \cup h_n$ is the ideal in $g$ due to the anticommutativity and the Jacobi identity. Since $K$ is spherically complete there exists $h_{n+1} \supset h_n = : t_{n+1}$ for each $n \in N$ and $t_1 := h_1$ such that $t_n$ is the $K$-linear subspace of $g$ ([29]). By $c_0(g,\{t_n : n\}) = : y$ is denoted the completion in $g$ of vectors $z$ such that $z = \sum z_n$ with $z_n \in t_n$ for each $n$ and $\lim_{n \to \infty} z_n = 0$. Evidently $y$ is the parabolic ideal in $g$ such that $h \subset y$, since $g$ is infinite dimensional over $K$. Then the embedding $\theta$ of $y$ into $g$ is either of class $L_1$ or $\theta = \theta_1 \theta_2$ such that $\theta_1$ and $\theta_2$ belong to $L_q$.

Due to this let $a$ and $A$ be such that $[a_{x,y}] \subset y$ and $[A_{x},ad(y)] \subset ad(y)$ for each $x \in G$, where $ad(x)y := [x,y]$ for each $x,y \in g$, that is, $ad(x) \in L(g)$. If $g \in y \cap V$, then $\exp(ad(g)) - I$ is either of the class $L_1$ or the product of two operators each of which is of the class $L_q$.

There exists a countable family $(g_j,W_j) : j \in N)$ of elements $g_j \in G \setminus W$ for each $j > 1$ and clopen subsets $e \in W_j \subset W$ such that $g_1 = e$, $W_1 =
$W$ and $\{g_j W_j : j\}$ is a locally finite covering of $G$, since $G$ is separable and ultrametrizable (see §5.3 [8]). If $P$ is a quasi-invariant and pseudo-differentiable of order $b$ measure on a clopen subgroup $W$ relative to a dense subgroup $W'$, then $P(S) := (\sum_j P((g_j^{-1} S) \cap W_j) 2^{-j})(\sum_j P(W_j) 2^{-j})^{-1}$ for each Borel subset $S$ in $G$ is quasi-invariant and pseudo-differentiable of order $b$ measure on $G$ relative to the dense subgroup $G' := \bigcup_j g_j(W_j \cap W')$. The group $G$ is totally disconnected and is left-invariantly ultrametrizable (see §8 and Theorem 5.5 [13] and §6.2 [8]), consequently, in each neighbourhood of $e$ there exists a clopen subgroup in $G$. Then conditions of Theorems 2.13 and 2.14 are satisfied. Therefore, analogously to §2.15 there are $S$, $\gamma$ and the stochastic process corresponding to $\mu_{S,\gamma,q}$ or $\mu_S$ such that the transition measure $P$ is quasi-invariant and pseudo-differentiable relative to $G'$.

2.18. Theorem 2.15 gives the subgroup $G'$ concretely for the given group $G$, but Theorem 2.17 describes concretely $G'$ only for the case of $G$ satisfying the Campbell-Hausdorff formula. For a Banach-Lie group not satisfying locally the Campbell-Hausdorff formula Theorem 2.17 gives only the existence of $G'$. These transition measures $P =: \nu$ on $G$ induce strongly continuous unitary regular representations of $G'$ given by the following formula: $T^n \nu f(g) := (\nu^h(dg)/\nu(dg))^{1/2} f(h^{-1} g)$ for $f \in L^2(G,\nu,\mathbb{C}) =: H$, $T^n \nu \in U(H)$, $U(H)$ denotes the unitary group of the Hilbert space $H$. For the strong continuity of $T^n \nu$ the continuity of the mapping $G' \ni h \mapsto \rho_{\nu}(h,g) \in L^1(G,\nu,\mathbb{C})$ and that $\nu$ is the Borel measure are sufficient, where $g \in G$, since $G$ is the Polish space and hence the Radon space (see Theorem I.1.2 [5]). On the other hand, the continuity of $\rho_{\nu}(h,g)$ by $h$ from the Polish group $G'$ into $L^1(G,\nu,\mathbb{C})$ follows from $\rho_{\nu}(h,g) \in L^1(G,\nu,\mathbb{C})$ for each $h \in G'$ and that $G'$ is the topological subgroup of $G$ (see [11]).

Then analogously to §2.15 there can be constructed quasi-invariant and pseudo-differentiable measures on the manifold $M$ relative to the action of the diffeomorphism group $G_M$ such that $G' \subset G_M$. Then Poisson measures on configuration spaces associated with either $G$ or $M$ can be constructed [20]. There exists the stochastic process corresponding to $\mu_{S,\gamma,q}$ with the certain choice of $a$, $E$ and $a_{k,l}$ such that the regular representation is irreducible, for the stochastic process corresponding to $\mu_S$ it can be taken the family of $\{f_k : k\}$ and $a$, $E$ and $a_{k,l}$ such that the regular representation is irreducible.

More generally it is possible to consider instead of the group $G$ a Polish topological space $X$ on which $G'$ acts jointly continuously: $\phi : (G' \times X) \ni
\((h, x) \mapsto hx =: \phi(h, x) \in X, \phi(e, x) = x\) for each \(x \in X\), \(\phi(v, \phi(h, x)) = \phi(vh, x)\) for each \(v\) and \(h \in G'\) and each \(x \in X\). If \(\phi\) is the Borel function, then it is jointly continuous \([11]\). From §II.3.2 [23] (see also [21, 25, 26]) there is the following result.

**Theorem.** Let \(X\) be a Polish topological space with a \(\sigma\)-additive \(\sigma\)-finite nonnegative nonzero ergodic Borel measure \(\nu\) quasi-invariant relative to a Polish topological group \(G'\) acting on \(X\) by the Borel function \(\phi\). If 

(i) \(sp_C\{\psi| \psi(g) := (\nu^h(dg)/\nu(dg))^{1/2}, h \in G'\}\) is dense in \(H\) and 

(ii) for each \(f_{1,j}\) and \(f_{2,j}\) in \(H\), \(j = 1, ..., n, n \in \mathbb{N}\) and each \(\epsilon > 0\) there exists \(h \in G'\) such that \(|(T_h f_{1,j}, f_{2,j})| \leq \epsilon|(f_{1,j}, f_{2,j})|, \) when \(|(f_{1,j}, f_{2,j})| > 0\).

Then the regular representation \(T: G' \rightarrow U(H)\) is irreducible.

There can be used pseudo-differentiable measures of order \(l\) either for each \(l \in \mathbb{N}\) or \(-l \in \mathbb{N}\), that is used for the verification of Condition (i). Transition measures corresponding to stochastic processes that are quasi-invariant and pseudo-differentiable of each order \(b \in \mathbb{C}\) with \(Re(b) \leq 0\) can be constructed analogously starting with the corresponding measures \(\mu_S\). To satisfy the conditions of this theorem, for example, in §2.15 it can be taken \(a = 0\), \(E\) nondegenerate independent from \(t\) and each \(a_{k,l} = 0\) besides \(a_{0,1} = 1\); in §2.17 it can be taken \(a = 0\), \(E\) nondegenerate independent from \(t\) and \(a_{k,l}\) defined by the exponential mapping for \(G\).

In view of Proposition II.1 [28] for the separable Hilbert space \(H\) the unitary group endowed with the strong operator topology \(U(H)_s\) is the Polish group. Let \(U(H)_n\) be the unitary group with the metric induced by the operator norm. In view of the Pickrell’s theorem (see §II.2 [28]): if \(\pi: U(H)_n \rightarrow U(V)_s\) is a continuous representation of \(U(H)_n\) on the separable Hilbert space \(V\), then \(\pi\) is also continuous as a homomorphism from \(U(H)_s\) into \(U(V)_s\). Therefore, if \(T: G' \rightarrow U(H)_s\) is a continuous representation, then there are new representations \(\pi \circ T: G' \rightarrow U(V)_s\). On the other hand, the unitary representation theory of \(U(H)_n\) is the same as that of \(U_\infty(H) := U(H) \cap (1 + L_0(H))\), since the group \(U_\infty(H)\) is dense in \(U(H)_s\).

Two theorems about induced representations of the dense subgroups \(G'\) were proved in [24], which are also applicable to the considered here cases.
3 Stochastic antiderivational equations and measures on a loop monoid and a path space.

3.1. Theorem. On the monoid $G = \Omega_{\xi}(M,N)$ from §2.3.1 and each $b_0 \in \mathbb{C}$ with $\text{Re}(b_0) \geq 0$ there exists a stochastic process $\eta(t,\omega)$ on $G$ such that the transition measure $P$ is quasi-invariant and pseudo-differentiable of each order $b \in \mathbb{C}$ with $\text{Re}(b) \geq \text{Re}(b_0)$ relative to the dense submonoid $G' := \Omega^{(k)}_{\xi}(M,N)$ from §2.8 (with $c > 0$ and $c' > 0$).

Proof. In view of Lemma I.2.17 it is sufficient to consider the case of $M$ with the finite atlas $At(M)$. The rest of the proof is quite analogous to that of Theorem 2.15 using the definitions of the quasi-invariance and the pseudo-differentiability for semigroups from §2.7.

3.2. Definition and Note. In view of §2.5 each space $N$ has the additive group structure, when $N = B(Y,0,R)$, $0 < R \leq \infty$.

Therefore, the factorization by the equivalence relation $K_{\xi} \times \text{id}$ produce the monoid of paths $C^0_{\xi}(\xi, M \to N)/ (K_{\xi} \times \text{id}) =: S_{\xi}(M,N)$ in which compositions are defined not for all elements, where $y_1 \text{id} y_2$ if and only if $y_1 = y_2 \in N$. There exists a composition $f_1 f_2 = (g_1 g_2, y)$ if and only if $y_1 = y_2 = y$, where $f_i = (g_i, y_i), g_i \in \Omega_{\xi}(M,N)$ and $y_i \in N$, $i \in \{1,2\}$. The latter semigroup has elements $e_y$ such that $f = e_y \circ f = f \circ e_y$ for each $f$, when their composition is defined, where $y \in N$, $f = (g, y), g \in \Omega_{\xi}(M,N)$, $e_y = (e, y)$. If $N$ is a monoid, then $S_{\xi}(M,N)$ can be supplied with the structure of a direct product of two monoids. Therefore, $P_{\xi}(M,N) := L_{\xi}(M,N) \times N$ is called the path group.

3.3. Theorem. On the path group $G = P_{\xi}(M,N)$ from §3.2, when $N = B(Y,0,R)$ and $N_{\xi}$ is supplied with the additive group structure, and each $b_0 \in \mathbb{C}$ with $\text{Re}(b_0) \geq 0$ there exists a stochastic process $\eta(t,\omega)$ for which a transition measure $P$ is quasi-invariant and pseudo-differentiable of each order $b \in \mathbb{C}$ with $\text{Re}(b) \geq \text{Re}(b_0)$ relative to a dense subgroup $G'$.

Proof. Since $P_{\xi}(M,N) = L_{\xi}(M,N) \times N_{\xi}$, it is sufficient to construct two stochastic processes on $L_{\xi}(M,N)$ and $N_{\xi}$ and to consider transition measures for them. In view of Theorems 2.15 and 2.17 the desired processes and transition measures for them exist.

3.4. Definition. Let the topology of $\Omega_{\xi}(M,N)$ be defined relative to countable $At(M)$. If $F$ is the free Abelian group corresponding to $\Omega_{\xi}(M,N)$, then there exists a set $\tilde{W}$ generated by formal finite linear combinations over
of elements from $C^0_0(\xi, (M, s_0) \rightarrow (N, y_0))$ and a continuous extension $K_\xi$ of $K_\xi$ onto $W_\xi(M, N)$ and a subset $\bar{B}$ of $\tilde{W}$ generated by elements $[f + g] - [f] - [g]$ such that $W_\xi(M, N)/K_\xi$ is isomorphic with $L_\xi(M, N)$, where

$$W_\xi(M, N) := \tilde{W}/\bar{B},$$

$f$ and $g \in C^0_0(\xi, (M, s_0) \rightarrow (N, y_0))$, $[f]$ is an element in $\tilde{W}$ corresponding to $f$, $\tilde{W}$ is in a topology inherited from the space $C^0_0(\xi, (M, s_0) \rightarrow (N, y_0))^\mathbb{Z}$ in the Tychonoff product topology. We call $W_\xi(M, N)$ an $O$-group. Clearly the composition in $C^0_0(\xi, (M, s_0) \rightarrow (N, y_0))$ induces the composition in $W_\xi(M, N)$. Then $W_\xi(M, N)$ is not the algebraic group, but associative compositions are defined for its elements due to the homomorphism $\chi^*$ given by Formulas 2.3.2.(5,6), hence $W_\xi(M, N)$ is the monoid without the unit element.

Let $\mu_h(A) := \mu(h \circ A)$ for each $A \in Bf(W_\xi(M, N))$ and $h \in W_\xi(M, N)$, then as in §2.7 we get the definition of quasi-invariant and pseudo-differentiable measures.

Let now $G' := W_\xi^{[k]}(M, N)$ be generated by $C^0_0(\xi, (M, s_0) \rightarrow (N, 0))$ as in §2.8, then it is the dense $O$-subgroup in $W_\xi(M, N)$, where $c > 0$ and $c' > 0$.

**3.5. Theorem.** Let $G := W_\xi(M, N)$ be the $O$-group as in §3.4, $At(M)$ be finite and $b_0 \in C$ with $Re(b_0) \geq 0$. Then there exists a stochastic process $\eta(t, \omega)$ on $G$ for which the transition measure $P$ is quasi-invariant and pseudo-differentiable of each order $b \in C$ with $Re(b) \geq Re(b_0)$ on $G$ relative to a dense $O$-subgroup $G'$.

**Proof.** The uniform space $C^0_0(\xi, M \rightarrow N)$ has the embedding as the clopen subset into $C^0_0(\xi, M \rightarrow Y)$. Here we can take $a \in TG'$ and $A \in L_{1, s}(\theta, \tau)$ without relations with $DL_h$, where $s = q$ or $s = 1$ respectively. Then repeating the major parts of the proof of §2.15 without $L_h$ and so more simply, but using actions of vectors fields of $TG'$ by $\rho_X$ on $G$ we get the statement of this theorem, since $(DX\rho_X)Y$ and $[(\nabla_X)^n(DX\rho_X)]Y$ are products of two operators of of class $L_{n+2,q}((TG')^{n+2}, TG)$ and also of class $L_{n+2,1}((TG')^{n+2}, TG)$ for each $C^\infty$-vector fields $X$ and $Y$ on $G'$ and each $n \in \mathbb{N}$. In view of §2.16 there exists a stochastic process $\eta(t, \omega)$ for which the transition measure $P$ is quasi-invariant and pseudo-differentiable relative to each 1-parameter diffeomorphism group of $G'$ associated with a $U_G \times U_{Y'}$-$C^\infty$-vector field on $G'$. 

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