DISCRETENESS AND HOMOGENEITY OF THE TOPOLOGICAL FUNDAMENTAL GROUP

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Abstract. For a locally path connected topological space, the topological fundamental group is discrete if and only if the space is semilocally simply-connected. While functoriality of the topological fundamental group for arbitrary topological spaces remains an open question, the topological fundamental group is always a homogeneous space.

1. Introduction

The concept of a natural topology for the fundamental group appears to have originated with Hurewicz [8] in 1935. It received further attention by Dugundji [2] in 1950 and by Biss [1], Fabel [3, 4, 5, 6], and others more recently. The purpose of this note is to prove the following folklore theorem.

Theorem 1. Let $X$ be a locally path connected topological space. The topological fundamental group $\pi_1^{\top}(X)$ is discrete if and only if $X$ is semilocally simply-connected.

Theorem 5.1 of [1] is Theorem 1 without the hypothesis of local path connectedness. However a counterexample of Fabel [4] shows that this stronger result is false. Fabel [4] also proves a weaker version of Theorem 1 assuming that $X$ is locally path connected and a metric space. In this note we remove the metric hypothesis.

Our proof proceeds from first topological principles making no use of rigid covering fibrations [1] nor even of classical covering spaces. We make no use of the functoriality of the topological fundamental group, a property which was also a main result in [1, Cor. 3.4] but in fact is unproven [6, pp. 188–189]. Beware that the misstep in the proof of [1, Prop. 3.1], namely the assumption that the product of quotient maps is a quotient map, is repeated in [7, Thm. 2.1].

In general the homeomorphism type of the topological fundamental group depends on a choice of basepoint. We say that $\pi_1^{\top}(X)$ is discrete without reference to basepoint provided $\pi_1^{\top}(X, x)$ is discrete for each $x \in X$. If $x$ and $y$ are connected by a path in $X$, then $\pi_1^{\top}(X, x)$ and $\pi_1^{\top}(X, y)$ are homeomorphic. This fact was proved in [1, Prop. 3.2] and a detailed proof is in Section 4 below for completeness. Theorem 1 now immediately implies the following.

Corollary. Let $X$ be a path connected and locally path connected topological space. The topological fundamental group $\pi_1^{\top}(X, x)$ is discrete for some $x \in X$ if and only if $X$ is semilocally simply-connected.
As mentioned above it is open whether $\pi_{1}^{\text{top}}$ is a functor from the category of pointed topological spaces to the category of topological groups. The unsettled question is whether multiplication

$$\pi_{1}^{\text{top}}(X, x) \times \pi_{1}^{\text{top}}(X, x) \xrightarrow{\mu} \pi_{1}^{\text{top}}(X, x)$$

\(([f], [g]) \mapsto [f] \cdot [g]\)

is continuous. By Theorem 1 if $X$ is locally path connected and semilocally simply-connected, then $\pi_{1}^{\text{top}}(X, x)$, and hence $\pi_{1}^{\text{top}}(X, x) \times \pi_{1}^{\text{top}}(X, x)$, is discrete and so $\mu$ is trivially continuous. Continuity of $\mu$ in general remains an interesting question.

Lemma 4 below shows that if $(X, x)$ is an arbitrary pointed topological space, then left and right multiplication by any fixed element in $\pi_{1}^{\text{top}}(X, x)$ are continuous self maps of $\pi_{1}^{\text{top}}(X, x)$. Therefore $\pi_{1}^{\text{top}}(X, x)$ acts on itself by left and right translation as a group of self homeomorphisms. Clearly these actions are both transitive. Thus we obtain the following result.

**Theorem 2.** If $(X, x)$ is a pointed topological space, then $\pi_{1}^{\text{top}}(X, x)$ is a homogeneous space.

This note is organized as follows. Section 2 contains definitions and conventions, Section 3 proves two lemmas and Theorem 1, Section 4 addresses change of basepoint, and Section 5 shows left and right translation are homeomorphisms.

### 2. Definitions and Conventions

By convention, neighborhoods are open. Unless stated otherwise, homomorphisms are inclusion induced.

Let $X$ be a topological space and $x \in X$. A neighborhood $U$ of $x$ is *relatively inessential* (in $X$) provided $\pi_{1}(U, x) \to \pi_{1}(X, x)$ is trivial. $X$ is *semilocally simply-connected* at $x$ provided there exists a relatively inessential neighborhood $U$ of $x$. $X$ is *semilocally simply-connected* provided it is so at each $x \in X$. A neighborhood $U$ of $x$ is *strongly relatively inessential* (in $X$) provided $\pi_{1}(U, y) \to \pi_{1}(X, y)$ is trivial for every $y \in U$.

The fundamental group is a functor from the category of pointed topological spaces to the category of groups. Consequently if $A$ and $B$ are any subsets of $X$ such that $x \in A \subset B \subset X$ and $\pi_{1}(B, x) \to \pi_{1}(X, x)$ is trivial, then $\pi_{1}(A, x) \to \pi_{1}(X, x)$ is trivial as well. This observation justifies the convention that neighborhoods are open.

If $X$ is locally path connected and semilocally simply-connected, then each $x \in X$ has a path connected relatively inessential neighborhood $U$. Such a $U$ is necessarily a strongly relatively inessential neighborhood of $x$ as the reader may verify (see for instance [9, Ex. 5 p. 330]).

Let $(X, x)$ be a pointed topological space and let $I = [0, 1] \subset \mathbb{R}$. The space

$$C_{x}(X) = \{f : (I, \partial I) \to (X, x) \mid f \text{ is continuous}\}$$
is endowed with the compact-open topology. The function
\[ C_x(X) \xrightarrow{q} \pi_1(X, x) \]
\[ f \mapsto [f] \]
is surjective so \( \pi_1(X, x) \) inherits the quotient topology and one writes \( \pi_{1}^{\text{top}}(X, x) \) for the resulting topological fundamental group. Let \( e_x \in C_x(X) \) denote the constant map. If \( f \in C_x(X) \), then \( f^{-1} \) denotes the path defined by \( f^{-1}(t) = f(1-t) \).

3. Proof of Theorem 1

We prove two lemmas and then Theorem 1.

**Lemma 1.** Let \((X, x)\) be a pointed topological space. If \( \{[e_x]\} \) is open in \( \pi_{1}^{\text{top}}(X, x) \), then \( x \) has a relatively inessential neighborhood in \( X \).

**Proof.** The quotient map \( q \) is continuous and \( \{[e_x]\} \subset \pi_{1}^{\text{top}}(X, x) \) is open, so \( q^{-1}([e_x]) = [e_x] \) is open in \( C_x(X) \). Therefore \( e_x \) has a basic open neighborhood
\[
(1) \quad e_x \in V = \bigcap_{n=1}^{N} V(K_n, U_n) \subset [e_x] \subset C_x(X)
\]
where each \( K_n \subset I \) is compact, each \( U_n \subset X \) is open, and each \( V(K_n, U_n) \) is a subbasic open set for the compact-open topology on \( C_x(X) \). We will show that
\[
U = \bigcap_{n=1}^{N} U_n
\]
is a relatively inessential neighborhood of \( x \) in \( X \). Clearly \( U \) is open in \( X \) and, by (1), \( x \in U \). Finally, let \( f : (I, \partial I) \to (U, x) \). For each \( 1 \leq n \leq N \) we have
\[
f(K_n) \subset U \subset U_n.
\]
Thus \( f \in [e_x] \) by (1) and so \( [f] = [e_x] \) is trivial in \( \pi_1(X, x) \). \( \square \)

**Lemma 2.** Let \((X, x)\) be a pointed topological space and let \( f \in C_x(X) \). If \( X \) is locally path connected and semilocally simply-connected, then \( \{[f]\} \) is open in \( \pi_{1}^{\text{top}}(X, x) \).

**Proof.** As \( q \) is a quotient map, we must show that \( q^{-1}([f]) = [f] \) is open in \( C_x(X) \). So let \( g \in [f] \). For each \( t \in I \) let \( U_t \) be a path connected relatively inessential neighborhood of \( g(t) \) in \( X \). The sets \( g^{-1}(U_t), t \in I \), form an open cover of \( I \). Let \( \lambda > 0 \) be a Lebesgue number for this cover. Choose \( N \in \mathbb{N} \) so that \( 1/N < \lambda \). For each \( 1 \leq n \leq N \) let
\[
I_n = \left[ \frac{n-1}{N}, \frac{n}{N} \right] \subset I.
\]
Reindex the \( U_t \)'s so that
\[
g(I_n) \subset U_n \quad \text{for each} \quad 1 \leq n \leq N.
\]
The \( U_n \)'s are not necessarily distinct, nor does the proof require this condition. For each \( 1 \leq n \leq N \) let \( W_n \) denote the path component of \( U_n \cap U_{n+1} \) containing \( g(n/N) \), so
\[
(2) \quad g \left( \frac{n}{N} \right) \in W_n \subset (U_n \cap U_{n+1}) \subset X.
\]
Consider the basic open set
\[(3) \quad V = \left( \bigcap_{n=1}^{N} V(I_n, U_n) \right) \cap \left( \bigcap_{n=1}^{N-1} V \left( \left\{ \frac{n}{N} \right\}, W_n \right) \right) \subset C_x(X).\]

By construction, \(g \in V\). It remains to show that \(V \subset [f]\). So let \(h \in V\). As \([g] = [f]\), it suffices to show that \([h] = [g]\).

By (3) we have
\[(4) \quad h \left( \frac{n}{N} \right) \in W_n \quad \text{for each } 1 \leq n \leq N.\]

For each \(1 \leq n \leq N - 1\) let \(\gamma_n : I \to W_n\) be a continuous path such that
\[
\gamma_n(0) = h \left( \frac{n}{N} \right) \quad \text{and} \quad \gamma_n(1) = g \left( \frac{n}{N} \right),
\]
which exists by (2) and (3). Let \(\gamma_0 = e_x\) and \(\gamma_N = e_x\). For each \(1 \leq n \leq N\) define
\[
\begin{align*}
I & \quad \xrightarrow{s_n} I_n \\
\frac{t}{N} t + \frac{n-1}{N} & \quad \xrightarrow{\gamma_n^{-1}} W_n
\end{align*}
\]
and let
\[
\begin{align*}
g_n &= g \circ s_n \quad \text{and} \\
h_n &= h \circ s_n.
\end{align*}
\]

So \(g_n\) and \(h_n\) are affine reparameterizations of \(g|_{I_n}\) and \(h|_{I_n}\) respectively. For each \(1 \leq n \leq N\)
\[
\delta_n = g_n \ast \gamma_n^{-1} \ast h_n^{-1} \ast \gamma_{n-1}
\]
is a loop in \(U_n\) based at \(g_n(0)\) (see Figure 1). As \(U_n\) is a strongly relatively inessential neighborhood, \([\delta_n] = 1 \in \pi_1(X, g_n(0))\). Therefore \(g_n\) and \(\gamma_{n-1}^{-1} \ast h_n \ast \gamma_n\) are path

\[\text{Figure 1. Loop } \delta_n = g_n \ast \gamma_n^{-1} \ast h_n^{-1} \ast \gamma_{n-1} \text{ in } U_n \text{ based at } g_n(0).\]
homotopic. In \( \pi_1(X, x) \) we have
\[
[h] = [h_1 * h_2 * \cdots * h_N]
= [\gamma_0^{-1} * \gamma_1 * \gamma_1^{-1} * h_2 * \gamma_2 * \cdots * \gamma_N^{-1} * h_N]
= [g_1 * g_2 * \cdots * g_N]
= [g]
\]
proving the lemma. \( \square \)

In the previous proof, the second collection of subbasic open sets in (3) are essential. Figure 2 shows two loops \( g \) and \( h \) based at \( x \) in the annulus \( X = S^1 \times I \). All conditions in the proof are satisfied except \( g(1/N) \) and \( h(1/N) \) fail to lie in the same connected component of \( U_1 \cap U_2 \). Clearly \( g \) and \( h \) are not homotopic loops.

**Proof of Theorem 4.** First assume \( \pi_1^{\text{top}}(X) \) is discrete and let \( x \in X \). By definition \( \pi_1^{\text{top}}(X, x) \) is discrete and so \( \{[e_x]\} \) is open in \( \pi_1^{\text{top}}(X, x) \). By Lemma 1 \( x \) has a relatively inessential neighborhood in \( X \). The choice of \( x \in X \) was arbitrary and so \( X \) is semilocally simply-connected.

Next assume \( X \) is semilocally simply-connected and let \( x \in X \). Points in \( \pi_1^{\text{top}}(X, x) \) are open by Lemma 2 and so \( \pi_1^{\text{top}}(X, x) \) is discrete. The choice of \( x \in X \) was arbitrary and so \( \pi_1^{\text{top}}(X) \) is discrete. \( \square \)

4. **Basepoint change**

**Lemma 3.** Let \( X \) be a topological space and \( x, y \in X \). If \( x \) and \( y \) lie in the same path component of \( X \), then \( \pi_1^{\text{top}}(X, x) \) and \( \pi_1^{\text{top}}(X, y) \) are homeomorphic.

**Proof.** Let \( \gamma : I \to X \) be a continuous path with \( \gamma(0) = y \) and \( \gamma(1) = x \). Define the function
\[
\begin{align*}
C_y(X) & \xrightarrow{\gamma} C_x(X) \\
f & \mapsto (\gamma^{-1} * f) * \gamma
\end{align*}
\]
First we show that $\Gamma$ is continuous. Let $I_1 = [0, 1/4]$, $I_2 = [1/4, 1/2]$, and $I_3 = [1/2, 1]$. Define the affine homeomorphisms

\[
\begin{align*}
I_1 &\xrightarrow{s_1} I \\
t &\xrightarrow{4t} 4t \\
I_2 &\xrightarrow{s_2} I \\
t &\xrightarrow{4t - 1} 4t - 1 \\
I_3 &\xrightarrow{s_3} I \\
t &\xrightarrow{2t - 1} 2t - 1
\end{align*}
\]

and note that

\[
\begin{align*}
I &\xrightarrow{\Gamma(f)} X \\
t &\xrightarrow{\gamma^{-1} \circ s_1(t)} 0 \leq t \leq \frac{1}{4} \\
t &\xrightarrow{f \circ s_2(t)} \frac{1}{4} \leq t \leq \frac{1}{2} \\
t &\xrightarrow{\gamma \circ s_3(t)} \frac{1}{2} \leq t \leq 1
\end{align*}
\]

Consider an arbitrary subbasic open set $V = V(K, U) \subset C_x(X)$. Observe that $\Gamma(f) \in V$ if and only if

(5) $\gamma^{-1} \circ s_1(K \cap I_1) \subset U,$

(6) $f \circ s_2(K \cap I_2) \subset U,$ and

(7) $\gamma \circ s_3(K \cap I_3) \subset U.$

Define the subbasic open set $V' = V(s_2(K \cap I_2), U) \subset C_y(X)$. Observe that $f \in V'$ if and only if (6) holds. As conditions (5) and (7) are independent of $f$, either $\Gamma^{-1}(V) = \emptyset$ or $\Gamma^{-1}(V) = V'$. Thus $\Gamma$ is continuous. Next consider the diagram

\[
\begin{array}{ccc}
C_y(X) & \xrightarrow{\Gamma} & C_x(X) \\
\downarrow q_y & & \downarrow q_x \\
\pi^\text{top}_1(X, y) & \xrightarrow{\pi(\Gamma)} & \pi^\text{top}_1(X, x)
\end{array}
\]

The composition $q_x \circ \Gamma$ is constant on each fiber of $q_y$ so there is a unique set function making the diagram commute, namely $\pi(\Gamma) : [f] \mapsto [\Gamma(f)]$. As $q_y$ is a quotient map, the universal property of quotient maps \cite[Thm. 11.1 p. 139]{9} implies that $\pi(\Gamma)$ is continuous. It is well known that $\pi(\Gamma)$ is a bijection \cite[Thm. 2.1 p. 327]{9}. Repeating the above argument with the roles of $x$ and $y$ interchanged and the roles of $\gamma$ and $\gamma^{-1}$ interchanged, we see that $\pi(\Gamma)^{-1}$ is continuous. Thus $\pi(\Gamma)$ is a homeomorphism as desired. \hfill \square

5. Translation

**Lemma 4.** Let $(X, x)$ be a pointed topological space. If $[f] \in \pi^\text{top}_1(X, x)$, then left and right translation by $[f]$ are self homeomorphisms of $\pi^\text{top}_1(X, x)$.

**Proof.** Fix $[f] \in \pi^\text{top}_1(X, x)$ and consider left translation by $[f]$ on $\pi^\text{top}_1(X, x)$

\[
\begin{array}{ccc}
\pi^\text{top}_1(X, x) & \xrightarrow{L_1[f]} & \pi^\text{top}_1(X, x) \\
\downarrow [g] & & \downarrow [f \cdot [g]]
\end{array}
\]
Plainly $L_{[f]}$ is a bijection of sets. Consider the commutative diagram

$$
\begin{array}{ccc}
C_x(X) & \xrightarrow{L_f} & C_x(X) \\
\downarrow{q} & & \downarrow{q} \\
\pi_1^{\text{top}}(X,x) & \xrightarrow{L_{[f]}} & \pi_1^{\text{top}}(X,x)
\end{array}
$$

where $L_f$ is defined by

$$
\begin{array}{ccc}
C_x(X) & \xrightarrow{L_f} & C_x(X) \\
g & \xrightarrow{f \ast g} & f \ast g
\end{array}
$$

First we show $L_f$ is continuous. Let $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$. Define the affine homeomorphisms

$$
\begin{array}{ccc}
I_1 & \xrightarrow{s_1} & I \\
t & \xrightarrow{2t} & 2t
\end{array}
\quad
\begin{array}{ccc}
I_2 & \xrightarrow{s_2} & I \\
t & \xrightarrow{2t - 1} & 2t - 1
\end{array}
$$

and note that

$$
\begin{array}{ccc}
I & \xrightarrow{f \ast g} & X \\
t & \xrightarrow{f \circ s_1(t)} & f \circ s_1(t) \\
& 0 \leq t \leq \frac{1}{2} & 0 \leq t \leq \frac{1}{2}
\end{array}
\quad
\begin{array}{ccc}
I & \xrightarrow{g \circ s_2(t)} & g \circ s_2(t) \\
& \frac{1}{2} \leq t \leq 1 & \frac{1}{2} \leq t \leq 1
\end{array}
$$

Consider an arbitrary subbasic open set

$$
V = V(K, U) \subset C_x(X).
$$

Observe that $f \ast g \in V$ if and only if

$$
\begin{align*}
(9) & \quad f \circ s_1(K \cap I_1) \subset U \\
(10) & \quad g \circ s_2(K \cap I_2) \subset U.
\end{align*}
$$

Define the subbasic open set

$$
V' = V(s_2(K \cap I_2), U) \subset C_x(X).
$$

Observe that $g \in V'$ if and only if (10) holds. As condition (9) is independent of $g$, either $L_f^{-1}(V) = \emptyset$ or $L_f^{-1}(V) = V'$. Thus $L_f$ is continuous. The composition $q \circ L_f$ is constant on each fiber of the quotient map $q$ and (8) commutes, so the universal property of quotient maps [9, Thm. 11.1 p. 139] implies that $L_{[f]}$ is continuous.

Applying the previous argument to $f^{-1}$ we get $L_{[f]}^{-1} = L_{[f^{-1}]}$ is continuous and $L_{[f]}$ is a homeomorphism. The proof for right translation is almost identical. □

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