SINGULAR CURVES OVER A FINITE FIELD AND WITH MANY POINTS

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Abstract. Recently Fukasawa, Homma and Kim introduced and studied certain projective singular curves over \( \mathbb{F}_q \) with many extremal properties. Here we extend their definition to more general non-rational curves.

1. Introduction

Fix a prime \( p \) and a \( p \)-power \( q \). Recently S. Fukasawa, M. Homma and S. J. Kim introduced a family of singular rational curves defined over \( \mathbb{F}_q \), with many singular points over \( \mathbb{F}_q \) and, conjecturally, some extremal properties. In this paper we discuss a similar type of curves, discuss their extremal properties and, in some cases, show that they are, more or less, the curves introduced in [5]. The zeta-function \( Z_Y(t) \) of a singular curve \( Y \) is explicitly given in terms of the Frobenius on a “topological” invariant \( H^1_{\text{ét}}(Y, \mathbb{Q}_\ell) \) ([2], [1], p. 2, [8], [12]). Hence \( Z_Y(t) \) does not detect the finer invariants of the singular points of \( Y \) (it does not distinguish between unibranch points defined over the same extension of \( \mathbb{F}_q \); in particular it does not distinguish between a smooth point and a cusp). Using gluing of points of the normalization with the same residue field we may define a “minimal” singular curve with prescribed normalization and prescribed zeta-function.

Let \( Y \) be a geometrically integral projective curve defined over \( \mathbb{F}_q \). Let \( u : C \to Y \) denote the normalization. Since any finite field is perfect, \( C \) and \( u \) are defined over \( \mathbb{F}_q \). Hence for every integer \( n \geq 1 \) we have \( u(C(\mathbb{F}_q^n)) \subseteq Y(\mathbb{F}_q^n) \) and for each \( P \in Y(\mathbb{F}_q^n) \) the scheme \( u^{-1}(P) \) is defined over \( \mathbb{F}_q^n \). Hence the finite set \( u^{-1}(P)_{\text{red}} \) is defined over \( \mathbb{F}_q \) (but of course if \( \sharp(u^{-1}(P)_{\text{red}}) > 1 \) the points of \( u^{-1}(P)_{\text{red}} \) may only be defined over a larger extension of \( \mathbb{F}_q \)). We are interested in properties of the set \( Y(\mathbb{F}_q) \) knowing \( C \). A. Weil’s study of the zeta-function of smooth projective curves was extended to the case of singular curves ([2]). We will use the very useful and self-contained treatment given by Y. Aubry and M. Perret ([1]). There are infinitely many curves \( Y' \) defined over \( \mathbb{F}_q \), with \( C \) as their normalization and with the same zeta-function (see Examples 1, 2 and Lemma 2). However, given \( Y \), there is one natural such curve if we prescribe also the sets \( u^{-1}(P)_{\text{red}} \) as subsets of \( C(\mathbb{F}_q) \). Let \( w_s : Y_s \to Y \) be the seminormalization of \( Y \) ([2], [1]). We recall that \( Y_s \) is an integral projective curve with \( C \) as its normalization and that \( u = w_s \circ u_s \), where \( u_s : C \to Y_s \) is the normalization map. Over an algebraically closed field the one-dimensional seminormal singularities with embedding dimension \( n \geq 1 \) are exactly the singularities formally isomorphic to the local ring at the origin of the union of the coordinate axis in \( \mathbb{A}^n \). Even over a finite field the curve we will study...
is defined in this way, i.e. the curves $C_{[q,n]}$, $n \geq 2$, defined below are obtained in the same way, i.e. the gluing process introduced by C. Traverso ([9]) gives always a seminormal curve and if the base field is algebraically closed, then all seminormal curve singularities are obtained in this way (over an algebraically closed base field a more general construction is given in [7], p. 70). We call axial singularities the curve singularities obtained in this way. Hence by definition we say that $(Y,P)$ is an axial singularity with embedding dimension $n$ if and only if over $\overline{F}_q$ it is formally isomorphic at $P$ to the germ at 0 of the union of of the $n$ axis of $\mathbb{A}^n$. An axial singularity of embedding dimension $n > 2$ is not Gorenstein. An axial singularity of embedding dimension 2 is an ordinary double point except that over a non-algebraically closed base field, say $\mathbb{F}_q$, we need to distinguish if the two branches of $Y$ at $P$ (or the two lines of its tangent cone) are defined over $\mathbb{F}_q$ or not (in the latter case each of them is defined over $\mathbb{F}_{q^2}$). Similarly, for an axial singularity $(Y,P)$ of embedding dimension $t \geq 2$ defined over $\mathbb{F}_q$, the $t$ lines of the tangent cone $(P,Y)$ are defined over $\mathbb{F}_{q^t}$ and their union is defined over $\mathbb{F}_q$. In the examples we are interested in, none of these lines will be defined over a field $\mathbb{F}_{q^e}$ with $e < t$. If $P \in Y$ is a singular point, then we may associate a non-negative integer $p_a(Y,P)$ (usually called the arithmetic genus of the singularity or the drop of genus the singular point $P$) such that $p_a(Y) = p_a(C) + \sum_{P \in \text{Sing}(Y)} p_a(Y,P)$. When $Y$ is an axial singularity with embedding dimension $n$, then $p_a(Y,P) = n - 1$.

We fix $q$, $C$ and an integer $n \geq 2$. We construct a singular curve $C_{[q,n]}$ with $C$ as its normalization and $\sharp(C_{[q,n]}(\overline{F}_q))$ very large in the following way. Fix an integer $t$ such that $2 \leq t \leq n$. For each $P \in C(\overline{F}_q) \setminus C(\overline{F}_{q^{t-1}})$ the orbit of the Frobenius $F_q$ has order $t$, say $\{P, \ldots, F_q^{t-1}(P)\}$. Let $C_{[q,n]}$ be the only curve obtained by gluing each of these orbits (for all possible $t \leq n$) (see Remark 4). By construction $C_{[q,n]}$ is a seminormal curve defined over $\mathbb{F}_q$, each singular points of $C_{[q,n]}$ is defined over $\mathbb{F}_q$ and $\sharp(C_{[q,n]}(\overline{F}_q)) = N_1 + \sum_{i=2}^n (N_i - N_{i-1})/i$, where $N_i := \sharp(C(\overline{F}_{q^i}))$. Fix $P \in C_{[q,n]}$ with embedding dimension $t \geq 2$. The Frobenius $F_q$ acts on the local ring $C_{C_{[q,n]},P}$ and hence on the $t$ branches of $C_{[q,n]}$ at $P$ (i.e. the $t$ smooth branches through 0 of the tangent cone of $C_{[q,n]}$ at $P$). Since $P$ is an axial singularity, the action of Frobenius is the restriction to $u^{-1}(P)$ of the action of the Frobenius $F_q : C(\overline{F}_q) \rightarrow C(\overline{F}_q)$. Hence this action is cyclic, i.e. it has a unique orbit. Thus if $O = u(P)$ with $P \in C(\overline{F}_q) \setminus C(\overline{F}_{q^{t-1}})$, then $p_a(C_{[q,n]}) = t - 1$ and none of the $t$ branches of $C_{[q,n]}$ at $u(O)$ is defined over a proper subfield of $\mathbb{F}_{q^t}$. See Propositions 2, 3, Question 1 and Remark 2 for the relations between $\mathbb{F}_{q^t}$ and the curves $B$ and $B_n$ studied in [5].

2. THE CURVES $C_{[q,n]}$ AND THEIR MAXIMALITY PROPERTIES

Let $u : C \rightarrow Y$ denote the normalization map. We often write $u^{-1}(P)$ instead of $u^{-1}(P)_{\text{red}}$. Set $\Delta_Y := \sharp(u^{-1}(\text{Sing}(Y)(\overline{F}_q))) - \sharp(\text{Sing}(Y)(\overline{F}_q))$. The zeta-function $Z_Y(t)$ of $Y$ is the product of the zeta-function $Z_C(t)$ of $C$ and a degree $\Delta_Y$ polynomials whose inverse roots are roots of unity ([2], [1], Theorem 2.1 and Corollary 2.4). Let $\omega_i$, $1 \leq i \leq 2g$, be the inverse roots of numerator of $Z_C(t)$ and $\beta_j$, $1 \leq j \leq \Delta_Y$ the inverse roots of the polynomial $Z_Y(t)/Z_C(t)$. For every integer $n \geq 1$ we have

$$\#(Y(\mathbb{F}_{q^n})) = q^n + 1 - \sum_{i=1}^{2g} \omega_i^n - \sum_{j=1}^{\Delta_Y} \beta_j^n$$
Proof. We have \( z(C(F_q^n)) = q^n + 1 - \sum_{i=1}^{2g} \omega_i \). Recall that \( |\beta_j| = 1 \) for all \( j \). Hence among all curves with fixed normalization \( C \) and with fixed \( \Delta_Y \) the integer \( z(Y(F_q^n)) \) is maximal (resp. minimal) for a curve with \( \beta_j = -1 \) for all \( j \) (resp. \( \beta_j = 1 \)) for all \( j \), if any such curve exists. If \( \beta_j = -1 \) for all \( j \), then for all odd (resp. even) positive integers \( n \) among all curves with fixed normalization \( C \) and with fixed \( \Delta_Y \) the value \( z(Y(F_q^n)) \) is maximal (resp. minimal). If \( \beta_j = 1 \) for all \( j \), then for every \( n > 0 \) the value \( z(Y(F_q^n)) \) is minimal among all curves with fixed normalization \( C \) and with fixed \( \Delta_Y \).

**Lemma 1.** Let \( Y \) be a geometrically integral projective curve and \( u : C \to Y \) its normalization. The degree \( \Delta_Y \) polynomial \( Z_Y(t)/Z_C(t) \) has all its inverse roots equal to \(-1\) if and only if for each \( P \in \text{Sing}(Y) \) either \( z(u^{-1}(P)) = 1 \) or \( P \in Y(F_q^n) \) and \( u^{-1}(P) \) is formed by two points of \( C(F_q^n) \) (in the latter case these two points are exchanged by the Frobenius and they are in \( C(F_q^n) \setminus C(F_q) \)).

Proof. For each \( Q \in C(F_q^n) \) let \( d_Q \) be the minimal integer \( x > 0 \) such that \( Q \in C(F_q^{nx}) \). The explicit form of the polynomial \( Z_Y(t)/Z_C(t) \) is given in [1, Theorem 2.1]. \( Z_Y(t)/Z_C(t) \) is a product of polynomials, each of them associated to a different singular point of \( Y \). Hence it is sufficient to consider separately the contribution of each singular point of \( Y \). Fix \( P \in \text{Sing}(Y) \) and call \( Z_P(t) \) the associated polynomial. Let \( d_P \) be the minimal integer \( t \geq 1 \) such that \( P \in Y(F_q^{nt}) \). We have \( (1 - t^{d_P})Z_P(t) = \prod_{Q \in u^{-1}(P)}(1 - t^{d_Q}). \) Since \( Y \) is defined over \( F_q \), the orbit of \( P \) by the Frobenius of \( Y \) has order \( d_P \). For any point \( P' \neq P \) in this orbit, say \( F_q^n(P) \) for some \( x \in \{1, \ldots, d_P - 1\} \) we have \( u^{-1}(P') = F_q^n(u^{-1}(P)) \). Hence \( d_Q \geq d_P \) for each \( Q \in u^{-1}(P) \) and for each \( Q \in u^{-1}(P') \).

First assume \( z(u^{-1}(P)) = 1 \). The only point, \( Q \), of \( u^{-1}(P) \) is defined over \( F_q^{d_P} \). Since \( d_{u(Q)} \leq d_Q \), we get \( d_Q = d_P \). We easily get that \( z(u^{-1}(P)) = 1 \) if and only if the constant 1 is the factor of \( Z_Y(t)/Z_C(t) \) associated to the orbit of \( P \). Hence from now on we assume \( \alpha := z(u^{-1}(P)) \geq 2 \).

If \( d_Q > d_P \) for some \( Q \in u^{-1}(P) \) and either \( d_P \geq 2 \) or \( d_Q \geq 3 \), then we get that \( (1 - t^{d_P})Z_P(t) \) has a root of order \( > \max(d_P, 2) \) and hence \( Z_P(t) \) has a root \( \neq -1 \). Now assume \( d_P \geq 2 \) and \( d_Q = d_P \) for all \( Q \in u^{-1}(P) \). We get \( Z_P(t) = (1 - t^{d_P})^\alpha \). Since we assumed \( \alpha \geq 2 \), even in this case \( Z_P(t) \) has a root \( \neq -1 \).

Now assume \( d_P = 1 \). It remains to analyze the case \( d_Q \in \{1, 2\} \) for any \( Q \in u^{-1}(P) \). If \( d_Q = 1 \) for at least one \( Q \in u^{-1}(P) \), then \( Z_P(1) = 0 \). If \( d_Q = 2 \) for all \( Q \in u^{-1}(P) \), then \( Z_P = (1 + t)(1 - t^2)^\alpha - 1 \) has \( \alpha - 1 \) roots equal to 1.

**Remark 1.** Fix \( q, g, C \) with genus \( g \) and an integer \( n \geq 2 \). Set \( N_i := z(C(F_q^n)) \). We have \( z(C_{[q,n]}(F_q^n)) = N_1 + \sum_{i=2}^{n} (N_i - N_{i-1})/i , \) i.e.

\[
(2) \quad z(C_{[q,n]}(F_q^n)) = \sum_{i=1}^{n-1} \frac{N_i}{i+1} + N_n/n
\]

Now assume that \( g > 0 \) and that \( q \) is a square. If \( C \) is a minimal curve for \( F_q \), then it is a minimal curve for each \( F_q^n \), \( i \geq 2 \) (use that \( C \) is minimal if and only if \( Z_C(t) = \frac{1 - t^{2g}}{(1 - t)^g} \)). Hence for fixed \( q \) and \( n \) the integer \( z(C_{[q,n]}(F_q^n)) \) is minimal varying \( C \) among all smooth curves of genus \( g \) if and only if \( C \) is a minimal curve. Now we assume \( n \geq 3 \) and generalize the construction of the curve \( C_{[q,n]} \). Fix a positive integer \( s \leq n - 2 \) and integers \( t_1, \ldots, t_s \) such that \( 2 \leq t_1 < \cdots < t_s \leq n \).
Let \( C_{[q,t_1,\ldots,t_s]} \) be the curve obtained from \( C \) gluing only the \( F_q \)-orbits of the points of \( C(\mathbb{F}_q) \setminus C(\mathbb{F}_q^{-1}) \). We get

\[
\sharp(C_{[q,t_1,\ldots,t_s]}(\mathbb{F}_q)) = N_1 + \sum_{t \in \{t_1,\ldots,t_s\}} (N_t - N_{t-1})/t.
\]

Recall that \( N_1 \) is maximal if and only if each \( N_t \) with \( t \) odd is maximal and each \( N_t \) with \( t \) even is minimal \((\text{[H0]}, \text{[5]}, \text{Theorem 10.1})\). Hence if all \( t_i \) are odd, then \( \sharp(C_{[q,t_1,\ldots,t_s]}) \) is maximal if and only if \( C \) is a maximal curve.

In the case \( q = 0 \) we get the following result.

**Proposition 1.** For every integer \( n \geq 2 \) we have \( \sharp(\mathbb{P}^1_{[q,n]}) = q + 1 + \sum_{t=2}^n (q^t - q^{t-1})/(t-1)/t \).

**Proposition 2.** The curve \( \mathbb{P}^1_{[q,2]} \) is isomorphic over \( \mathbb{F}_q \) to the plane curve \( B \subset \mathbb{P}^2 \) defined in \([4]\) and \([5]\).

**Proof.** Let \( u : \mathbb{P}^1 \to \mathbb{P}^1_{[q,2]} \) denote the normalization map. The normalization map \( \Phi : \mathbb{P}^1 \to B \) is unramified, because the composition of it with the inclusion \( B \to \mathbb{P}^2 \) is unramified \((\text{part (i) of } \text{[5]}, \text{Theorem 2.2})\). By \([5]\), part (iii) of Theorem 2.2, \( B \) is a degree \( q + 1 \) plane curve with \( (q^2 - q)/2 \) singular points and \( \Phi(P) = \Phi(Q) \) with \( P \neq Q \) if and only if \( u(P) = u(Q) \). Hence the universal property of the seminormalization gives the existence of a morphism \( \psi : \mathbb{P}^1_{[q,2]} \to B \) such that \( \psi \) is a bijection. Since \( p_n(\mathbb{P}^1_{[q,2]}) = (q^2 - q)/2 \), we have \( p_n(\mathbb{P}^1_{[q,2]}) = p_n(B) \). Hence \( \psi \) is an isomorphism.

**Proposition 3.** Fix an integer \( n \geq 3 \). Then \( \mathbb{P}^1_{[q,n]} \) is the seminormalization of the curve \( B_n \subset \mathbb{P}^n \), defined in \([5]\), \( \S 6 \), and there is a birational morphism \( \psi_{q,n} : \mathbb{P}^1_{[q,n]} \to B_n \) defined over \( \mathbb{F}_q \) such that \( \psi_{q,n} : \mathbb{P}^1_{[q,n]}(K) \to B_n(K) \) is bijective for every field \( K \supseteq \mathbb{F}_q \).

**Proof.** Let \( u : \mathbb{P}^1 \to \mathbb{P}^1_{[q,n]} \) and \( \Phi_n : \mathbb{P}^1 \to B_n \) denote the normalization maps. By \([5]\), part (ii) of Theorem 6.4, each point \( P \in \text{Sing}(B_n) \) corresponds to an integer \( t \in \{2, \ldots, n\} \) and an orbit of the Frobenius on \( \mathbb{P}^1(\mathbb{F}_q') \setminus \mathbb{P}^1(\mathbb{F}_q^{-1}) \). Hence the definition of \( \mathbb{P}^1_{[q,n]} \) and the universal property of the seminormalization gives a birational morphism \( \psi_{q,n} : \mathbb{P}^1_{[q,n]} \to B_n \) defined over \( \mathbb{F}_q \) such that \( \psi_{q,n} : \mathbb{P}^1_{[q,n]}(K) \to B_n(K) \) is bijective for every field \( K \supseteq \mathbb{F}_q \).

**Question 1.** We guess that \( \psi_{q,n} \) is an isomorphism.

**Remark 2.** Fix a prime power \( q \) and the integer \( n \geq 3 \). Let \( \Phi_n : \mathbb{P}^1 \to B_n \) denote the normalization map. By \([5]\), part (i) of Theorem 6.4, \( \Phi_n \) is unramified \((\text{this is a necessary condition for being } \psi_{q,n} \text{ an isomorphism})\). The following conditions are equivalent:

\begin{enumerate}
  \item the morphism \( \psi_{q,n} \) is an isomorphism;
  \item \( p_n(B_n) = p_n(\mathbb{P}^1_{[q,n]}) \);
  \item for each \( P \in \text{Sing}(B_n) \), say with \( P = \Phi_n(Q) \) and \( Q \in \mathbb{P}^1(\mathbb{F}_q') \setminus \mathbb{P}^1(\mathbb{F}_q^{-1}) \), the singularity \( (B_n, P) \) has arithmetic genus \( t - 1 \);
  \item for each \( P \in \text{Sing}(B_n) \), say with \( P = \Phi_n(Q) \) and \( Q \in \mathbb{P}^1(\mathbb{F}_q') \setminus \mathbb{P}^1(\mathbb{F}_q^{-1}) \), the tangent cone \( C(P, B_n) \subset \mathbb{P}^n \) is formed by \( t \) lines through \( P \) spanning a \( t \)-dimensional linear subspace.
\end{enumerate}
Example 1. Fix a geometrically integral projective curve $Y$ of $P$ and assume $\triangle$ would have $v$:

Remark 3. unramified, we have the fibers of $\nu$ need to add some conditions on the curve questions raised in [5], Remark 2.5. Examples 1, 2 and Lemma 2 show that we $\#$ and $\delta$ equality holds only if each singular point of $Y$ be the degree of the polynomial $\sharp$.

Proposition 4. Let $C$ be a smooth and geometrically irreducible projective curve defined over $\mathbb{F}_q$. Set $\delta := \sharp(C(\mathbb{F}_q^2)) - \sharp(C(\mathbb{F}_q ))$. Let $Y$ a projective curve defined over $\mathbb{F}_q$ with $C$ as its normalization. We have $\sharp(Y(\mathbb{F}_q )) \geq \sharp(C(\mathbb{F}_q )) + \delta$ and $p_a(Y) \leq g + \delta$ if and only if $Y$ is isomorphic to $C_{[q,2]}$ over $\mathbb{F}_q$.

Proof. The “ if ” part is true, because $p_a(C_{[q,2]}) = g + \delta$ and $\sharp(C_{[q,2]}(\mathbb{F}_q )) = g + \delta$. Assume $\sharp(Y(\mathbb{F}_q )) \geq \sharp(C(\mathbb{F}_q )) + \delta$ and $p_a(Y) \leq g + \delta$. Let $u : C \rightarrow Y$ be the normalization map. The morphism $u$ is defined over $\mathbb{F}_q$, i.e. over a field on which $Y$ is defined, because any finite field is perfect. We have $\sharp(\text{Sing}(Y)) \leq p_a(Y) - \delta$ and equality holds only if each singular point of $Y$ is formally isomorphic over $\mathbb{F}_q$ either to a node or an ordinary cusp. Set $\delta_Y := \sharp(u^{-1}(\text{Sing}(Y)(\mathbb{F}_q )) - \sharp(\text{Sing}(Y)(\mathbb{F}_q ))$. The polynomial $Z_Y(t)/Z_C(t)$ has degree $\delta_Y$ and $\sharp(Y(\mathbb{F}_q )) \leq \sharp(C(\mathbb{F}_q )) + \delta_Y$ and equality holds if and only if each inverse root of $Z_Y(t)/Z_C(t)$ is equal to $-1$. Since $\delta_Y \leq p_a(Y) - g$, we get $\delta_Y = \delta$ and $p_a(Y) = g + \delta$. Since $p_a(Y) = g + \delta$ and $\sharp(\text{Sing}(Y)(\mathbb{F}_q )) \geq \delta$, we get $\text{Sing}(Y)(\mathbb{F}_q ) = \text{Sing}(Y)(\mathbb{F}_q )$, $\sharp(\text{Sing}(Y)(\mathbb{F}_q )) = \delta$ and that for each $P \in \text{Sing}(Y)(\mathbb{F}_q )$ the set $u^{-1}(P)$ is formed by two points of $C(\mathbb{F}_q ) \setminus C(\mathbb{F}_q )$ exchanged by the Frobenius (Lemma 1). Since $p_a(Y) = g + \sharp(\text{Sing}(Y)(\mathbb{F}_q ))$ and $\sharp(u^{-1}(P)) \geq 2$ for each $P \in \text{Sing}(Y)(\mathbb{F}_q )$, we also get that $Y$ is nodal. Hence $Y$ is seminormal. The structure of the fibers of $u^{-1}(P)$, $P \in \text{Sing}(Y)(\mathbb{F}_q )$, gives $Y = C_{[q,2]}$.

Proposition 5. Let $Y$ be a geometrical integral projective curve defined over $\mathbb{F}_q$ and with only seminormal singularity. Let $u : C \rightarrow Y$ be the normalization map. Let $\delta_Y$ be the degree of the polynomial $Z_Y(t)/Z_C(t)$. Assume $2\delta_Y \leq \sharp(C(\mathbb{F}_q^2)) - \sharp(C(\mathbb{F}_q ))$.

We have $\sharp(Y(\mathbb{F}_q )) \leq \sharp(C(\mathbb{F}_q ))$ and equality holds if and only if $Y$ is isomorphic to $C_{[q,2]}$ over $\mathbb{F}_q$.

Proof. We have $\sharp(Y(\mathbb{F}_q )) \leq \sharp(C(\mathbb{F}_q )) + \delta_Y$ and equality holds if and only if each inverse root of $Z_Y(t)/Z_C(t)$ is equal to $-1$. Hence we may assume $2\delta_Y = \sharp(C(\mathbb{F}_q^2)) - \sharp(C(\mathbb{F}_q ))$. Since $Y$ has only seminormal singularities, $u$ is unramified. Since $u$ is unramified, we have $\sharp(u^{-1}(P)) \geq 2$ for all $P \in \text{Sing}(Y)$. Hence Lemma 1 gives that the fibers of $u$ are the fibers of the normalization map $C \rightarrow C_{[q,2]}$. Since $Y$ and $C_{[q,2]}$ are seminormal, we get that they are isomorphic. They are isomorphic over $\mathbb{F}_q$, because $u$ is defined over $\mathbb{F}_q$ and the seminormalization is defined over $\mathbb{F}_q$.

Remark 3. In the case $C \cong \mathbb{P}^1$ Propositions 4 and 5 are partial answers to a question raised in [4], Remark 2.5. Examples 1, 2 and Lemma 2 show that we need to add some conditions on the curve $Y$, not only to fix the normalization $\mathbb{P}^1$ and assume $\delta_Y \leq (q^2 - q)/2$.

Example 1. Fix a geometrically integral projective curve $A$ defined over $\mathbb{F}_q$ and $P \in A(\mathbb{F}_q )$. Now we define a geometrically integral projective curve $Y$ defined over $\mathbb{F}_q$ and a morphism $v : A \rightarrow Y$ defined over $\mathbb{F}_q$, such $v \setminus A \setminus \{P\}$ is an isomorphism onto $Y \setminus v(P)$, but $v$ is not an isomorphism. Notice that for each such pair $(Y,v)$ we would have $p_a(Y) > p_a(A)$ and that for every integer $t \geq 1$ induces a bijection $A(\mathbb{F}_q^t) \rightarrow Y(\mathbb{F}_q^t)$. To define $Y$ and $v$ it is sufficient to define them in a neighborhood of $P$ in $A$ and the glue to it the identity map $A \setminus \{P\} \rightarrow A \setminus \{P\}$. Fix an embedding $j : A \hookrightarrow \mathbb{P}^r$, $r \geq 3$, and take a projection of $j(A)$ into $\mathbb{P}^2$ from an $(r-3)$-dimensional
linear subspace not containing $j(P)$, but intersecting the Zariski tangent space of $j(A)$ at $j(P)$.

**Example 2.** Fix a geometrically integral projective curve $A$ defined over $\mathbb{F}_q$ and any point $P \in A_{\text{reg}}(\mathbb{F}_q)$. Let $t$ be the minimal integer $t \geq 1$ such that $P \in A(\mathbb{F}_q^t)$. We assume $t \geq 2$, because the case $t = 1$ is covered by Example 1. Hence the orbit of $P$ by the action of the Frobenius $F_q$ has order $t$ (it is $\{P, F_q(P), \ldots, F_{q}^{t-1}(P)\}$). Let $Y$ denote the only curve and $\nu: A \to Y$ the only morphism obtained in the following way. We fix a bijection of sets $\nu: A \to Y$ and use it to define a topology on the set $Y$. Now we define $Y$ as a ringed space. On $Y \setminus \nu(\{P, F_q(P), \ldots, F_{q}^{t-1}(P)\})$ we assume that $\nu$ is an isomorphism of local ringed spaces. For each $Q \in \{P, F_q(P), \ldots, F_{q}^{t-1}(P)\}$ we impose that $\mathcal{O}_{Y,Q}$ is the local ring of a unibranch singular point and that $\nu$ is the normalization map (it may be done using the method of Example 1 with $Q$ instead of $P$ and $\mathbb{F}_q^t$ instead of $\mathbb{F}_q$). We need to do the construction simultaneously over all $Q \in \{P, F_q(P), \ldots, F_{q}^{t-1}(P)\}$ and in such a way that the morphism is defined over $\mathbb{F}_q$. As in Example 1 it is sufficient to define $\nu U$, where $U$ is a neighborhood of $\{P, F_q(P), \ldots, F_{q}^{t-1}(P)\}$. There is an embedding $j: A \to \mathbb{P}^r$, $r \geq t + 2$, such that the $t$ lines $T_{q,j}(j(A), Q \in \{P, F_q(P), \ldots, F_{q}^{t-1}(P)\}$, are linearly independent. Since $j$ is defined over $\mathbb{F}_q$ the Frobenius $F_q$ of $\mathbb{F}_q^t$ acts on $j(A)$ and on the tangent developable of $A$. Since $j(P)$ is defined over $\mathbb{F}_q$. Hence $T_{j(P),j(A)}(\mathbb{F}_q^t) \setminus j(P)$ has $(d^t - 1)/q - 1 - 1$ elements. Fix any $O \in T_{j(P),j(A)}(\mathbb{F}_q^t) \setminus j(P)$. For each $x \in \{1, t - 1\}$, $F_{q,x}(O) \in T_{j(F_{q,x}(P),j(A)}(\mathbb{F}_q^t) \setminus j(F_{q,x}(P))$. Since the $t$ tangent lines are linearly independent, the linear space $E := \langle O, F_{q,x}(O), \ldots, F_{q,1}^{t-1}(O) \rangle$ has dimension $t - 1$. Since $E$ is $F_q$-invariant, it is defined over $\mathbb{F}_q$. Let $\pi : \mathbb{P}^r \setminus E \to \mathbb{P}^{r-t}$ denote the linear projection from $E$. Since $E$ is defined over $\mathbb{F}_q$, $\pi$ is defined over $\mathbb{F}_q$. Hence the integral projective curve $T := \pi(j(A) \setminus E \cap j(A)) \subseteq \mathbb{P}^{r-t}$ is defined over $\mathbb{F}_q$. Since the $t$ tangent lines are linearly independent and $O \neq j(P)$, we have $E \cap j(P) = \emptyset$. Hence $E \cap j(U) = \emptyset$ for a sufficiently small neighborhood $U$ of $\{P, F_q(P), \ldots, F_{q}^{t-1}(P)\}$. Assume for the moment that $\pi[j(A) \setminus j(A) \cap E$ is birational onto its image. Since $\pi[j(A) \setminus j(A) \cap E$ is birational onto its image, it is separable. Hence only finitely many points of $j(A_{\text{reg}})$ have a tangent line intersecting $E$. Restricting if necessary $U \subseteq A_{\text{reg}}$ we may assume that for no other point $Q \in j(U)(\mathbb{F}_q)$ the Zariski tangent space $T_{j(Q),j(A)}$ intersects $E$. Since $\pi[j(A) \setminus j(A) \cap E$ is birational onto its image, it is generically injective. Hence restricting $U \subseteq A_{\text{reg}}$ we may assume that $\pi[j(U)$ is injective and an isomorphism outside $\{j(P), j(F_q(P)), \ldots, j(F_{q}^{t-1}(P))\}$. At these points the curve $T$ has a cusp, but perhaps not an ordinary cusp, i.e., it is a unibranch singular point. Hence to conclude the example it is sufficient to find $j$ such that $\pi[j(A) \setminus j(A) \cap E$ is birational onto its image. We take as $j$ is a linearly normal embedding of degree $d > \max\{2p_a(A) - 2, p_a(A) + t\}$. Since $d > \max\{2p_a(A) - 2, p_a(A) + t\}$, Riemann-Roch gives $r = d - p_a(A)$. Assume that $\pi[j(A) \setminus j(A) \cap E$ is not birational onto its image and call $x \geq 2$ its degree. Thus $\deg(T) \leq d/x \leq d/2$. Since $j(A)$ spans $\mathbb{P}^r$, $T$ spans $\mathbb{P}^{r-t}$. Hence $\deg(T) \geq r - t = d - p_a(A) - t$. Hence $(d - p_a(A) - t) \geq 2(d - p_a(A) - t)$, contradicting our assumption $d > p_a(A) + t$.

**Lemma 2.** Fix an integer $y > 0$. Let $A$ be a geometrically integral projective curve defined over $\mathbb{F}_q$. Assume $A_{\text{reg}}(\mathbb{F}_q) \neq \emptyset$ and fix $P \in A_{\text{reg}}(\mathbb{F}_q)$. Then there are a
geometrically integral projective curve $Y$ and a morphism $u : A \to Y$ defined over $\mathbb{F}_q$ such that $u$ induces an isomorphism of $A \setminus \{P\}$ onto $Y \setminus u(P)$ and $p_a(Y) = p_a(A) + y$.

Proof. Let $m$ be the maximal ideal of the local ring $O_{A,P}$. By assumption $O_{A,P}/m \cong \mathbb{F}_q$ and $\mathbb{F}_q \cdot 1 \subset O_{A,P}$. Hence the $\mathbb{F}_q$-vector space $O_{A,P}$ is the direct sum of its subspaces $\mathbb{F}_q \cdot 1$ and $m$. Set $O_{Y,u(P)} := \mathbb{F}_q \cdot 1 + m^{y+1} \subset O_{A,P}$. It is easy to check that $O_{Y,u(P)}$ is a local ring with $m^{y+1}$ as its maximal ideal. Since $P \in A_{reg}$, $O_{A,P}$ is a DVR. Hence $m/m^{y+1}$ is a $\mathbb{F}_q$-vector space of dimension $y$. We take $Y$ the same topological space as $Y$, but with $O_{Y,u(P)}$ at the point $u(P)$ associated to $P$ instead of $O_{A,P}$. With this definition of $u$ we have $\dim_{\mathbb{F}_q} (u_*(O_A)/O_Y) = y$. Hence $p_a(Y) = p_a(A) + y$. 

\[ \square \]

Remark 4. Fix $q$, $C$ and an integer $n \geq 2$. Here we explain one way to check the existence of the curve $C_{[q,n]}$. We obtain $C_{[q,n]}$ in finitely many steps each of them similar to the one described in Example 2. We use $z$ steps, where $z$ is the number of orbits of $F_q$ in $C(\mathbb{F}_{q^n}) \setminus C(\mathbb{F}_q)$. At each of the steps we glue together one of these orbit. We do not need any notion of gluing, except that set-theoretically in each step one of these orbits is sent to a single point and for all other points the map is an isomorphism. Fix $Q \in C(\mathbb{F}_{q^n}) \setminus C(\mathbb{F}_q)$ and assume $Q \in C(\mathbb{F}_{q^n}) \setminus C(\mathbb{F}_{q^{n-1}})$. Hence $\{Q,F_q(Q),\ldots,F_q^{n-1}(Q)\}$ is the orbit of $Q$ for the action of $F_q$. Call $A$ the geometrically integral curve arising in the steps at which we want to glue this orbit. Hence there is a geometrically integral projective $A$ curve defined over $\mathbb{F}_q$ with $C$ as its normalization (call $u : C \to A$) and $u(Q) \in A_{reg}$ (in the previous steps if any) the maps where isomorphism at each point of $\{Q,F_q(Q),\ldots,F_q^{n-1}(Q)\}$. Set $P := u(Q)$. Since $u$ is defined over $\mathbb{F}_q$ and $u$ is an isomorphism in a neighborhood of $u^{-1}(\{P,F_q(P),\ldots,F_q^{n-1}(P)\})$, we have $\{P,F_q(P),\ldots,F_q^{n-1}(P)\} \subset A_{reg}$ and these $t$ points are distinct. Hence $P \in A_{reg}(\mathbb{F}_{q^n}) \setminus A_{reg}(\mathbb{F}_{q^{n-1}})$. As in Example 2 we get several curves $Y$ and morphism $v : A \to Y$ defined over $\mathbb{F}_q$, sending $\{P,F_q(P),\ldots,F_q^{n-1}(P)\}$ to a single point, $O$, of $Y$ and induces an isomorphism of $A \setminus \{P,F_q(P),\ldots,F_q^{n-1}(P)\}$ onto $Y \setminus \{O\}$. Let $A_1$ be the seminormalization of $Y$ in $A$. Then we use $A_1$ instead of $A$. After $z$ steps we get $C_{[q,n]}$. To get $C_{[q,n]}$ we get an existence property for the seminormalization. The result does not depend from the order of the gluing. Hence $C_{[q,n]}$ depends only from $q$, $C$ and $n$. Hence the curves $\mathbb{P}^1_{[q,n]}$ depends only from $q$ and $n$.

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