INFINITE LOOP SPACES AND POSITIVE SCALAR CURVATURE

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Abstract. We study the homotopy type of the space of metrics of positive scalar curvature on high-dimensional compact spin manifolds. Hitchin used the fact that there are no harmonic spinors on a manifold with positive scalar curvature to construct a secondary index map from the space of positive scalar metrics to a suitable space from the real $K$-theory spectrum. Our main results concern the nontriviality of this map. We prove that for $2n \geq 6$, the natural $KO$-orientation from the infinite loop space of the Madsen–Tillmann–Weiss spectrum factors (up to homotopy) through the space of metrics of positive scalar curvature on any $2n$-dimensional spin manifold. For manifolds of odd dimension $2n + 1 \geq 7$, we prove the existence of a similar factorisation.

When combined with computational methods from homotopy theory, these results have strong implications. For example, the secondary index map is surjective on all rational homotopy groups. We also present more refined calculations concerning integral homotopy groups.

To prove our results we use three major sets of technical tools and results. The first set of tools comes from Riemannian geometry: we use a parameterised version of the Gromov–Lawson surgery technique which allows us to apply homotopy-theoretic techniques to spaces of metrics of positive scalar curvature. Secondly, we relate Hitchin’s secondary index to several other index-theoretical results, such as the Atiyah–Singer family index theorem, the additivity theorem for indices on noncompact manifolds and the spectral-flow index theorem. Finally, we use the results and tools developed recently in the study of moduli spaces of manifolds and cobordism categories. The key new ingredient we use in this paper is the high-dimensional analogue of the Madsen–Weiss theorem, proven by Galatius and the third named author.

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References
1. Introduction

1.1. Statement of results. Among the several curvature conditions one can put on a Riemannian metric, the condition of positive scalar curvature (hereafter, \( \text{psc} \)) has the richest connection to topology, and in particular to cobordism theory. This strong link is provided by two fundamental facts.

The first comes from index theory. Let \( \mathfrak{D} \) be the Atiyah–Singer Dirac operator on a Riemannian spin manifold \((W, g)\). The scalar curvature appears in the remainder term in the Weitzenböck formula (or, more appropriately, Schrödinger–Lichnerowicz formula, cf. [42] and [33]). This forces the Dirac operator \( \mathfrak{D} \) to be invertible provided that \( g \) has positive scalar curvature, and hence forces the index of \( \mathfrak{D} \) to vanish. The Atiyah–Singer index theorem in turn equates the index of \( \mathfrak{D} \) with the \( \hat{A} \)-invariant of \( W \), an element in \( \text{KO}^{-d}(\ast) \), which can be defined in homotopy-theoretic terms through the Pontrjagin–Thom construction. Thus if \( W \) has a psc metric, then \( \hat{A}(W) = 0 \).

The second fundamental fact, due to Gromov and Lawson [24], is that if a manifold \( W \) with a psc metric is altered by a suitable surgery to a manifold \( W' \), then \( W' \) again carries a psc metric. These results interact extremely well provided the manifolds are spin, simply connected and of dimension at least five. Under these circumstances, the question of whether \( W \) admits a psc metric depends only on the spin cobordism class of \( W \), which reduces it to a problem in stable homotopy theory. Stolz [45] managed to solve this problem, and thereby showed that such manifolds admit a psc metric precisely if their \( \hat{A} \)-invariant vanishes. Much work has been undertaken to relax these three hypotheses; see [42] and [43] for surveys.

Rather than the existence question, we are interested in understanding the homotopy type of the space \( \mathcal{R}^+(W) \) of all psc metrics on a manifold \( W \). If \( W \) has boundary \( \partial W \), we consider the space \( \mathcal{R}^+(W)_h \) of all psc metrics on \( W \) which are of the form \( dt^2 + h \) near the boundary, for a fixed psc metric \( h \) on \( \partial W \).

To state our main result, we recall Hitchin’s definition of a secondary index-theoretic invariant for psc metrics [27], which we shall call the index difference. Ignoring some technical details for now, the definition is as follows. For a closed spin \( d \)-manifold \( W \) choose a basepoint psc metric \( g_0 \in \mathcal{R}^+(W) \), so for another psc metric \( g \) we can form the path of metrics \( g_t = (1 - t) \cdot g + t \cdot g_0 \) for \( t \in [0, 1] \). There is an associated path of Dirac operators in the space \( \text{Fred}^d \) of \( \text{Cl}^d \)-linear Fredholm operators on Hilbert space, and it starts and ends in the subspace of invertible operators, which is contractible. As the space \( \text{Fred}^d \) represents \( KO^{-d}(-) \), we obtain an element indiff\(_{g_0}(g) \in KO^{-d}([0, 1] \cdot \{0, 1\}) = KO^{-d-1}(\ast) = KO_{d+1}(\ast) \). This construction generalises to manifolds with boundary and to families, and gives a well-defined homotopy class of maps

\[
\text{indiff}_{g_0} : \mathcal{R}^+(W)_h \longrightarrow \Omega^{\infty + d+1}KO
\]

to the infinite loop space which represents real \( K \)-theory. The computational consequences of our main result concern the induced map on homotopy groups,

\[
A_k(W, g_0) : \pi_k(\mathcal{R}^+(W)_h, g_0) \longrightarrow KO_{k+d+1}(\ast) = \begin{cases} \mathbb{Z} & k + d + 1 \equiv 0 \pmod{4} \\ \mathbb{Z}/2 & k + d + 1 \equiv 1, 2 \pmod{8} \\ 0 & \text{else} \end{cases}
\]

and we state it first.
**Theorem A.** Let \( W \) be a spin manifold of dimension \( d \geq 6 \), and fix \( h \in \mathcal{R}^+(\partial W) \) and \( g_0 \in \mathcal{R}^+(W)_h \). If \( k = 4s - d - 1 \geq 0 \) then the map

\[
A_k(W, g_0) \otimes Q : \pi_k(\mathcal{R}^+(W)_h, g_0) \otimes Q \to KO_{4s}(*) \otimes Q = Q
\]

is surjective. If \( e = 1, 2 \) and \( k = 8s + e - d - 1 \), then the map

\[
A_k(W, g_0) : \pi_k(\mathcal{R}^+(W)_h, g_0) \to KO_{8s+e}(*) = \mathbb{Z}/2
\]

is surjective. In other words, the map \( A_k(W, g_0) \) is nontrivial if \( k \geq 0, d \geq 6 \) and the target is nontrivial.

Theorem A supersedes, to our knowledge, all previous results in the literature concerning the nontriviality of the maps \( A_k(W, g_0) \). These are

(i) Hitchin [27] showed that \( A_k(S^d, g_0) \) is surjective for \( k = 0, 1 \) and \( k+d+1 \equiv 1, 2 \) (mod 8), where \( g_0 \) denotes the round metric on the euclidean sphere.

(ii) Gromov–Lawson [23, Theorem 4.47] showed that \( A_k(S^d, g_0) \) is surjective for \( k = 0 \) and \( d+1 \equiv 0 \) (mod 4). See also Carr [11, Theorem 4] for more details.

(iii) Hanke–Schick–Steimle [25] showed that \( A_k(W, g_0) \otimes Q \) is surjective for \( k \) small compared to \( d \).

(iv) Crowley–Schick [14] showed that \( A_{8s+1-d}(W, g_0) \) is surjective, for \( d \geq 7 \).

Our main result does not involve homotopy groups but rather the construction of maps \( \rho : X \to \mathcal{R}^+(W)_h \) from certain infinite loop spaces \( X \), and the identification of the composition \( indiff_{g_0} \circ \rho \) with a well-known infinite loop map. It is then an easy task to deduce Theorem A from this construction, using standard techniques and results from homotopy theory.

Let us turn to this space-level version of our result, which is stated as Theorems B and C below. To formulate it, we must recall the definition of Madsen–Tillmann–Weiss spectra. Let \( \text{Gr}^\text{Spin}_{d,n} := \frac{\text{Spin}(n)}{\text{Spin}(d) \times \text{Spin}(n-d)} \) denote the spin Grassmannian, which carries a tautological bundle

\[
V_{d,n} := \text{Spin}(n) \times_{\text{Spin}(d)} \mathbb{R}^d \subset \text{Gr}^\text{Spin}_{d,n} \times \mathbb{R}^n
\]

having \((n-d)\)-dimensional orthogonal complement \( V_{d,n}^\perp \). There are structure maps \( \Sigma \text{Th}(V_{d,n}^\perp) \to \text{Th}(V_{d,n}^\perp) \) between the Thom spaces of these vector bundles, forming a spectrum (in the sense of stable homotopy theory) which we denote \( \text{MTSpin}(d) \).

This spectrum has associated infinite loop spaces

\[
\Omega^{\infty+1} \text{MTSpin}(d) := \colim_{n \to \infty} \Omega^{n+1} \text{Th}(V_{d,n}^\perp),
\]

where the colimit is formed using the adjoints of the structure maps.

The parametrised Pontrjagin–Thom construction associates to any smooth bundle \( \pi : E \to B \) of compact \( d \)-dimensional spin manifolds a natural map

\[
\alpha_E : B \to \Omega^{\infty} \text{MTSpin}(d),
\]

which encodes many invariants of smooth fibre bundles. For example, there is a map of infinite loop spaces \( \alpha^d : \Omega^{\infty} \text{MTSpin}(d) \to \Omega^{\infty+d} \text{KO} \) such that the composition \( \alpha^d \circ \alpha_E : B \to \Omega^{\infty+d} \text{KO} \) represents the family index of the Dirac operators on the fibres of \( \pi \). The collision of notation with the \( \alpha \)-invariant mentioned earlier is intended: There is a map of spectra \( \text{MTSpin}(d) \to \Sigma^{-d} \text{MSpin} \) into the desuspension of the classical spin Thom spectrum, and the classical \( \alpha \)-invariant is induced by a spectrum map \( \alpha : \text{MSpin} \to \text{KO} \) constructed by Atiyah–Bott–Shapiro. Our
map $\mathcal{A}_d$ is the composition of these maps (or rather the infinite loop map induced by the composition).

**Definition 1.1.1.** Maps $f_0, f_1 : X \to Y$ are called weakly homotopic [3] if for each finite CW complex $K$ and each $g : K \to X$, the maps $f_0 \circ g$ and $f_1 \circ g$ are homotopic.

The technical core of the present paper is the following result.

**Theorem B.** Let $W$ be a spin manifold of dimension $2n \geq 6$. Fix $h \in R^+(\partial W)$ and $g_0 \in R^+(W)_h$. Then there is a map $\rho : \Omega^{\infty+1}MTSpin(2n) \to R^+(W)_h$ such that the composition

$$\Omega^{\infty+1}MTSpin(2n) \xrightarrow{\rho} R^+(W)_h \xrightarrow{\text{indiff}_{0}} \Omega^{\infty+2n+1}KO$$

is weakly homotopic to $\Omega_{\mathcal{A}_{2n}}$, the loop map of $\mathcal{A}_{2n}$.

For odd dimensions, we have a result that looks very similar; we state it separately as it is deduced from Theorem B, and the proofs are quite different.

**Theorem C.** Let $W$ be a spin manifold of dimension $2n+1 \geq 7$. Fix $h \in R^+(\partial W)$ and $g_0 \in R^+(W)_h$. Then there is a map $\rho : \Omega^{\infty+2}MTSpin(2n) \to R^+(W)_h$ such that the composition

$$\Omega^{\infty+2}MTSpin(2n) \xrightarrow{\rho} R^+(W)_h \xrightarrow{\text{indiff}_{0}} \Omega^{\infty+2n+2}KO$$

is weakly homotopic to $\Omega^2_{\mathcal{A}_{2n}}$, the double loop map of $\mathcal{A}_{2n}$.

In both cases the map $\rho$ depends on the choice of the metric $g_0$.

1.2. **Further computations.** Theorem [A] is a consequence of Theorems [B] and [C] and relatively easy computations in stable homotopy theory. However, the geometric form of Theorems [B] and [C] gives an interpretation in terms of spaces of psc metrics to more difficult and interesting stable homotopy theory computations as well. We state the following more detailed computations for even-dimensional manifolds: they have odd-dimensional analogues too, which we leave to the reader to deduce from the results of Section 5.

The first concerns the surjectivity of the map on homotopy groups induced by the index difference, without any localisation.

**Theorem D.** Let $W$ be a spin manifold of dimension $2n \geq 6$. Fix $h \in R^+(\partial W)$ and $g_0 \in R^+(W)_h$. Then the map

$$A_k(W, g_0) : \pi_k(R^+(W)_h, g_0) \to KO_{k+2n+1}(*)$$

is surjective for $0 \leq k \leq 2n - 1$.

One application of this theorem is as follows. Let $B^8$ be a spin manifold such that $\mathcal{A}(B) \in KO_8(*)$ is the Bott class (such a manifold is sometimes called a “Bott manifold”). By the work of Joyce [29, §6] there is a Bott manifold which admits a metric $g_B$ with holonomy group $Spin(7)$. Then $g_B$ must be Ricci-flat and hence scalar-flat. For any closed spin $d$-manifold $W$, cartesian product with $(B, g_B)$ thus defines a direct system

$$R^+(W) \to R^+(W \times B) \to R^+(W \times B \times B) \to \cdots$$

and as $\mathcal{A}(B)$ is the Bott class there is an induced map from the direct limit

$$\text{indiff}_h[B^{-1}] : \Omega^+(W)[B^{-1}] := \text{hocolim}_{k \to \infty} \Omega^+(W \times B^k) \to \Omega^{\infty+d+1}KO.$$
It then follows from Theorem \[\text{[1]}\] (or its odd-dimensional analogue) that this map is surjective on all homotopy groups.

Secondly, working away from the prime 2 we are able to use work of Madsen–Schlichtkrull [38] to obtain an upper bound on the index of the image of the index difference map on homotopy groups.

**Theorem E.** Let \( W \) be a spin manifold of dimension \( 2n \geq 6 \). Fix \( h \in \mathcal{R}^+(\partial W) \) and \( g_0 \in \mathcal{R}^+(W)_h \). Then the image of the map

\[
A_{4m-2n-1}(W, g_0)[\frac{1}{2}] : \pi_{4m-2n-1}(\mathcal{R}^+(W)_h, g_0)[\frac{1}{2}] \to KO_{4m}(\ast)[\frac{1}{2}]
\]

has finite index, dividing

\[
A(m, n) := \gcd \left\{ \prod_{i=1}^n (2^{2m_i} - 1) \cdot \text{Num} \left( \frac{B_{m_i}}{2m_i} \right) \middle| \sum_{i=1}^n m_i = m \right\}
\]

(where we adopt the convention that \((2^{2m-1} - 1) \cdot \text{Num}(\frac{B_{m}}{2m}) = 1 \) when \( m = 0 \)).

While the numbers \( A(m, 1) = (2^{2m-1} - 1) \cdot \text{Num}(\frac{B_{m}}{2m}) \) are complicated, computer calculation shows that \( A(m, 2) = 1 \) for at least \( m \leq 2001 \). Hence \( A(m, 2\ell) = 1 \) for \( m \leq 2001 \cdot \ell \), and so for \( W \) a spin manifold of dimension \( 4\ell \geq 6 \) the map

\[
A_k(W, g_0)[\frac{1}{2}] : \pi_k(\mathcal{R}^+(W)_h, g_0)[\frac{1}{2}] \to KO_{k+4\ell+1}(\ast)[\frac{1}{2}]
\]

is surjective for \( k < 2000 \cdot 4\ell \). (One can deduce similar ranges for manifolds whose dimensions have other residues modulo 4, cf. Section 5.4.)

**Remark 1.2.1.** It would be extremely interesting to compute the numbers \( A(m, 2) \) for a significantly larger range of \( m \), and even more interesting to establish whether (or not) they are always equal to 1.

**Remark 1.2.2.** In Section 5.4.2 we show that the estimate in Theorem \[\text{[E]}\] is approximately sharp: the quotient of \( A(m, n) \) by the index of the image of the map

\[
\pi_{4m-2n}(\text{MTSpin}(2n))[\frac{1}{2}] \to \pi_{4m-2n-1}(\mathcal{R}^+(W)_h, g_0)[\frac{1}{2}] \to KO_{4m}(\ast)[\frac{1}{2}]
\]

has all prime factors \( p \) relatively small, in the sense that \( p \leq 2m + 2 \). Thus any prime \( q \) dividing \( A(m, n) \) with \( q > 2m + 2 \) must in fact divide the index of the image of this composition.

Thirdly, we study the \( p \)-local homotopy type of the spaces \( \mathcal{R}^+(S^d) \) of positive scalar curvature metrics on spheres. In [54], Walsh has shown that \( \mathcal{R}^+(S^d) \) admits the structure of an \( H \)-space (we will give our own perspective on this in Section 4.1.1), so for any prime \( p \) we may form the localisation \( \mathcal{R}^+_p(S^d) \) of the identity component of this \( H \)-space, that is, the component of the round metric. This may be constructed, for example, as the mapping telescope of the \( r \)th power maps on this \( H \)-space, over all \( r \) coprime to \( p \).

**Theorem F.** Let \( d \geq 6 \) and \( p \) be an odd prime. Then there is a map

\[
f : (\Omega^\infty + d + 1 \text{KO})_p \to \mathcal{R}^+_p(S^d)_p
\]

such that \( (\text{inndiff}_{g_0})_p \circ f \) induces multiplication by \((2^{2n-1} - 1) \cdot \text{Num}(B_n/2n)\) times a \( p \)-local unit on \( \pi_{4n-d-1} \).
Thus if \( p \) is an odd regular prime (i.e. is not a factor of \( \text{Num}(B_n) \) for any \( n \)) and in addition does not divide any number of the form \( (2^{2n-1} - 1) \), then there is a splitting
\[
\mathcal{R}^+_o(S^d)_{(p)} \simeq (\Omega_0^{\infty+d+1}KO)_{(p)} \times F_{(p)},
\]
where \( F \) is the homotopy fibre of \( \text{inddiff}_{g_k} \). In particular, for such primes the map induced by \( \text{inddiff}_{g_k} \) on \( F_p \)-cohomology is injective.

At the prime 2 we are not able to obtain such a strong splitting result, but we can still establish enough information to obtain the cohomological implication.

**Theorem G.** For \( d \geq 6 \), the map \( \text{inndiff}_{g_k} : \mathcal{R}^+(S^d) \to \Omega^{\infty+d+1}KO \) is injective on \( F_2 \)-cohomology.

### 1.3. Outline of the proofs of the main results.

We now turn to a brief outline of the proofs of Theorems 13 and 14 starting with the even case. The driving force behind Theorem 13 (besides the surgery technique for \( \text{psc} \) metrics and index theory) is the high-dimensional analogue of the Madsen–Weiss theorem, proved by Galatius and the third named author [21, 20]. The input is the construction of a \( 2n \)-dimensional spin cobordism \( W : \emptyset \to S^{2n-1} \) satisfying certain properties, including having an \( n \)-connected structure map to \( B\text{Spin} \); manifolds of this flavour were first considered by Kreck [33]. Then, writing
\[
K := (\{0,1\} \times S^{2n-1}) \# \Phi(S^n): S^{2n-1} \to S^{2n-1}
\]
and \( W_k \) for the composition \( W \cup kK \) of \( W \) and \( k \) copies of the self-cobordism \( K \), it follows from the results of [21, 20] that there is a map
\[
BDiff(W_{\infty}) := \hocolim_{k \to \infty} BDiff_\partial(W_k) \to \Omega_0^\infty MT\text{Spin}(2n),
\]
(1.3.1)
induced by a parametrised Pontrjagin–Thom construction, which is acyclic. Recall that a map \( f : X \to Y \) of spaces is called acyclic if for each \( y \in Y \), the homotopy fibre \( h_{\text{fib}}(f) \) has the singular homology of a point. In particular, the map (1.3.1) may be identified with the Quillen plus construction of \( B\text{Diff}_\partial(W_{\infty}) \).

The diffeomorphism group \( \text{Diff}_\partial(W_k) \) acts on the space \( \mathcal{R}^+(K)_{g_k} \), by pulling metrics back, and we form the Borel construction
\[
EDiff_\partial(W_k) \times_{\text{Diff}_\partial(W_k)} \mathcal{R}^+(K)_{g_k} \to B\text{Diff}_\partial(W_k).
\]
By picking a metric \( h \in \mathcal{R}^+(K)_{g_k} \), we get stabilisation maps between the Borel constructions for different values of \( k \). The first key point is that the cobordism invariance theorem of Chernysh [12] and Walsh [53] allows us to choose \( h \) so that the map induced by gluing, \( \mathcal{R}^+(W_k)_{g_k} \to \mathcal{R}^+(W_{k+1})_{g_k} \), is a homotopy equivalence. Therefore, in the colimit \( k \to \infty \), we get a fibration over \( B\text{Diff}_\partial(W_{\infty}) \) whose fibre is homotopy equivalent to \( \mathcal{R}^+(W)_{g_k} \).

The second key point of the proof is that we prove that this fibration is induced from a fibration over the target space of the map (1.3.1). This argument is by obstruction theory. The main input is that for each pair of diffeomorphisms \( f_0, f_1 \in \text{Diff}_\partial(W_k) \), the automorphisms \( f_0^*, f_1^* : \mathcal{R}^+(W_k)_{g_k} \to \mathcal{R}^+(W_k)_{g_k} \) commute up to homotopy. This is again derived from the cobordism invariance theorem, along with a formal argument of Eckmann–Hilton type.

Once these arguments are in place, the map \( \rho \) is defined to be the fibre transport map of the fibration over \( \Omega_0^\infty MT\text{Spin}(2n) \), based at some \( g_0 \in \mathcal{R}^+(W)_{g_k} \). To show that the composition is (weakly homotopic to) the map \( \Omega_\mathcal{R}^+(W)_{g_k} \), we use Bunke’s additivity theorem for the index [10] and, needless to say, the Atiyah–Singer family
index theorem. At this point, one arrives at the statement of Theorem B but with the specific psc manifold \((W,q_0)\). To generalise to an arbitrary manifold, we use the cobordism invariance theorem to deduce Theorem B for a certain psc metric on \(D^{2n}\), and then the additivity theorem (which implies a cut and paste invariance of the index difference) to obtain the result for all \(d\)-dimensional spin manifolds.

The deduction of Theorem C from Theorem B is much easier and by a rather short index-theoretic argument. The point here is that for closed \(W\), there is an alternative description of the index difference map, due to Gromov and Lawson [24], in terms of a boundary value problem of Atiyah–Patodi–Singer type on the cylinder \(W \times [0,1]\). The proof of Theorem C relates the index difference for \((2n+1)\)-dimensional manifolds (Hitchin’s definition) with the index difference for \(2n\)-dimensional manifolds (Gromov–Lawson’s definition). In order to make the argument conclusive we need to know that both definitions agree, and this follows from a family version of the spectral-flow-index theorem which was proved by the second named author [16].

Remark 1.3.2. The argument for the deduction of Theorem C from Theorem B also yields that \(\text{Im}(A_{k+1}(\partial W,h)) \subset \text{Im}(A_k(W,g))\) when \(g \in \mathcal{R}^+(W)_h\). This allows one to prove Theorem A by induction on the dimension, starting with \(d = 6\). Along the same lines, if Theorem B is established for \(2n = 6\), we get for all \(d \geq 6\) a factorisation

\[
\Omega^{d-5}_\mathbf{s} : \Omega^{\infty+d-5}_{\mathbf{MTSpin}(6)} \xrightarrow{\partial} \mathcal{R}^+(W^d)_h \xrightarrow{\text{indiff}} \Omega^{\infty+d+1}_\mathbf{KO},
\]

which suffices for some computational applications.

In addition, the proof of Theorem B in the special case \(2n = 6\) enjoys several simplifications, the principal one being that the results of [21] suffice, and the more difficult results of [20] are not required. Consequently, we have given (in Section 4.3.1) a separate proof of this special case.

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2. Spaces of metrics of positive scalar curvature

We begin this chapter by precisely defining the spaces which we shall study, in Section 2.1. In the other sections, we survey results on spaces of positive scalar curvature metrics which we shall need later. Section 2.2 provides technical but mostly elementary results, for later reference. The most important result, to be discussed in Section 2.3.3, is the cobordism invariance theorem of Chernysh and Walsh and the hasty reader can jump directly to that section.

2.1. Definitions. Let \(W : M_0 \leadsto M_1\) be a cobordism between closed manifolds. For us, a cobordism will always be a morphism in the cobordism category in the sense of [22] §2.1. In particular, the boundary of \(W\) is collared: for some \(a_0 < c_0 < c_1 < a_1 \in \mathbb{R}\), there is an embedding \(b = b_0 \sqcup b_1 : [a_0, c_0] \times M_0 \sqcup [c_1, a_1] \times M_1 \to W\) which induces the canonical identification of \(\{a_i\} \times M_i\) with \(M_i\). Let \(\Gamma(W; \text{Sym}^2(TW))\) be the space of smooth symmetric \((0,2)\)-tensor fields on \(W\). This
is a Fréchet topological vector space; the topology is generated by the maximum norms $\|\nabla^k u\|_{C^0}$, where $u \in \Gamma(W; \text{Sym}^2(TW))$, and the gradients are taken with respect to any fixed reference metric on $W$.

For $\epsilon_i > 0$ small enough, we write $\mathcal{R}(W)^{\epsilon_0, \epsilon_1} \subset \Gamma(W; \text{Sym}^2(TW))$ for the subspace of all Riemannian metrics $g$ on $W$ for which there exist Riemannian metrics $h_i$ on $M_i$ such that $b_i^*(g) = h_i + dt^2$ on the collars $[a_0, a_0 + \epsilon_0] \times M_0$ and $[a_1 - \epsilon_1, a_1] \times M_1$. This is a convex subspace and hence contractible. The subspace $\mathcal{R}^+(W)^{\epsilon_0, \epsilon_1} \subset \mathcal{R}(W)^{\epsilon_0, \epsilon_1}$ of metrics with positive scalar curvature is open, and we define

$$\mathcal{R}^+(W) := \lim_{\epsilon_0, \epsilon_1 \to 0} \mathcal{R}^+(W)^{\epsilon_0, \epsilon_1}.$$  

(We will show in Lemma 2.2.1 that this colimit is weakly homotopy equivalent to any of its finite stages.)

Any metric $g$ on $W$ induces a metric $h$ on $\partial W$ by restriction, and if $g$ has positive scalar curvature, then so does $h$. This yields a continuous map

$$\text{res} : \mathcal{R}^+(W) \longrightarrow \mathcal{R}^+(M_0) \times \mathcal{R}^+(M_1).$$

For a pair $(h_0, h_1) \in \mathcal{R}^+(M_0) \times \mathcal{R}^+(M_1)$ we write

$$\mathcal{R}^+(W)_{h_0, h_1} := \text{res}^{-1}(h_0, h_1) \subset \mathcal{R}^+(W).$$

In plain language: this is the space of all positive scalar curvature metrics whose restriction to a small collar around $M_i$ coincides with $h_i + dt^2$. If one of the manifolds $M_i$ is empty then we write $\mathcal{R}^+(W)_{h}$, where $h$ is the boundary metric.

Let $W : M_0 \simeq M_1$ and $W' : M_1 \simeq M_2$ be two cobordisms and $W \cup W' = W \cup_{M_1} W'$ be their composition. Let $h_i \in \mathcal{R}^+(M_i)$, $i = 0, 1, 2$, be given. Then there is a map

$$\mu : \mathcal{R}^+(W)_{h_0, h_1} \times \mathcal{R}^+(W')_{h_1, h_2} \rightarrow \mathcal{R}^+(W \cup W')_{h_0, h_2},$$

where the metric $\mu(g, g')$ is defined to agree with $g$ on $W$ and with $g'$ on $W'$. If we fix $g' \in \mathcal{R}^+(W')_{h_1, h_2}$, then we obtain a map

$$\mu_{g'} : \mathcal{R}^+(W)_{h_0, h_1} \longrightarrow \mathcal{R}^+(W \cup W')_{h_0, h_2}$$

$$g \mapsto \mu(g, g')$$

by gluing in the metric $g'$. Sometimes we abbreviate $g \cup g' := \mu(g, g')$.

2.2. Some basic constructions with psc metrics.

2.2.1. Collar stretching.

**Lemma 2.2.1.** For $0 < \delta_i \leq \epsilon_i < c$, the inclusion $\mathcal{R}^+(W)^{\epsilon_0, \epsilon_1}_{h_0, h_1} \hookrightarrow \mathcal{R}^+(W)^{\delta_0, \delta_1}_{h_0, h_1}$ is a homotopy equivalence.

**Proof.** For typographical simplicity, we assume that $a_0 = 0$, $M_1 = \emptyset$ and write $(\epsilon, \delta, c, h) := (\epsilon_0, \delta_0, c_0, h_0)$. Let $H_s : \mathbb{R} \to \mathbb{R}$, $s \in [0, 1]$, be an isotopy such that

1. $H_0 \equiv \text{id}$,
2. $H_s \equiv \text{id}$ near $[c, \infty)$,
3. $(H_s)' = 1$ near $\delta$,
4. $H_1(\delta) = \epsilon$ and
5. $H_s \leq H_u$ for $s \leq u$. 


This induces an isotopy of embeddings, also denoted $H_s$, of the collar to itself, and by condition (ii) also of $W$ into itself. Define a homotopy $F_s : \mathcal{R}^+(W)^h \to \mathcal{R}^+(W)^h$ by the formula

$$F_s(g) := \begin{cases} dt^2 + h & \text{on } [0, H_s(\delta)], \\ (H_s)^* g & \text{elsewhere.} \end{cases}$$

By construction, $F_s$ maps the subspace $\mathcal{R}^+(W)^h$ to itself, $F_0$ is the identity and $F_1$ maps $\mathcal{R}^+(W)^h$ into $\mathcal{R}^+(W)^h$. This proves that $F_1$ is a two-sided homotopy inverse of the inclusion, as claimed. \qed

Lemma 2.2.1 has the following immediate consequence.

Corollary 2.2.2.

(i) The inclusion $\mathcal{R}^+(W)_{h_0,h_1} \to \mathcal{R}^+(W)_{h_0,h_1}$ is a weak homotopy equivalence.

(ii) For $a_1 > a_2$ and $h_1 \in \mathcal{R}^+(M_1)$, let $g = dt^2 + h_1 \in \mathcal{R}^+(a_1,a_2) \times M_1)$. The gluing map $\mu_1 : \mathcal{R}^+(W)_{h_0,h_1} \to \mathcal{R}^+(W \cup ([a_1,a_2] \times M_1))_{h_0,h_1}$ is a weak homotopy equivalence.

2.2.2. The trace of an isotopy. Let $M$ be a closed manifold and let $\mathcal{P}R^+(M)$ be the space of all smooth paths in $\mathcal{R}^+(M)$. An alternative name for a path $(h : t \mapsto h_t) \in \mathcal{P}R^+(M)$ is an isotopy of psc metrics. We would like to consider the metric $g = dt^2 + h_t$ on $[0,1] \times M$ (the trace of the isotopy $h$), and we would like to have a map

$$F : \mathcal{P}R^+(M) \to \mathcal{R}^+(0,1] \times M)$$

given by $h \mapsto dt^2 + h_t$. However the metric $dt^2 + h_t$ has in general neither positive scalar curvature nor a product form near the boundary $\{0,1\} \times M$. This can be fixed, using the following result of Gajer [18, p. 184].

Lemma 2.2.4. Let $M$ be a closed manifold and $h : [0,1] \to \mathcal{R}^+(M)$ be a smooth path. Then there is a constant $\Lambda = \Lambda(h) > 0$, which depends on the path $h$ and its first two derivatives (in both directions) such that whenever $f : \mathbb{R} \to [0,1]$ is a smooth function with $|f'|, |f''| \leq \Lambda$, the metric $dt^2 + h_{f(t)}$ on $\mathbb{R} \times M$ has positive scalar curvature.

The constant $\Lambda(h)$ can be chosen so that it has the property stated in the Lemma for all paths close to $h$, and furthermore we can assume that $\Lambda \leq 1$. Using that $\mathcal{R}^+(M)$ is a metric space and hence paracompact, there is a continuous function $\Lambda : \mathcal{P}R^+(M) \to (0,1]$, such that $\Lambda(h)$ has this property for the path $h$. Once and for all, pick $C > 0$ and a smooth $f_1 : \mathbb{R} \to [0,1]$, with $f_1(t) = 0$ near $(-\infty,0]$ and $f_1(t) = 1$ near $[C,\infty)$ and $|f_1'|, |f_1''| \leq 1$. Let $f_\Lambda(t) := f_1(\Lambda t)$, so that $|f_\Lambda'|, |f_\Lambda''| \leq \Lambda$. Furthermore, pick an isotopy $(H_s)_{s>0}$ of diffeomorphisms of $\mathbb{R}$, such that $H_s = id$ near $(-\infty,0]$ and $H_s(x) = s + x - 1$ for $x$ near $[1,\infty)$. By taking products with $id_M$, we get an isotopy of $\mathbb{R} \times M$, also denoted $H_s$. The true definition of the map $F$ is then

$$F(h) := H_s^{f_\Lambda} (dt^2 + h_{f_\Lambda(h)}).$$

By construction, the metric $F(h)$ lies in $\mathcal{R}^+([0,1] \times M)_{h_0,h_1}$, where $h_i = h(i)$ for $i = 0, 1$, and so $F$ restricts to a map

$$F : \mathcal{P}R_{h_0,h_1} \to \mathcal{R}^+([0,1] \times M)_{h_0,h_1}.$$
interval. Careful choice of \( f_1 \) also allows us to map into the space \( \mathcal{R}^+([0,1] \times M)^a \), for some small \( a > 0 \). As it stands, \( F \) depends on various choices, but the space of these choices is contractible, whence the homotopy class of \( F \) is well-defined.

2.2.3. The quasifibration theorem. One application of this construction is a result of Chernysh \[13\] Theorem 1.1, which we shall use once in later chapters. Let \( W \) be a manifold with collared boundary \( M \) and \( \text{res} : \mathcal{R}^+(W)^c \to \mathcal{R}^+(M) \) be the restriction map. For \( h \in \mathcal{R}^+(M) \), the geometric fibre \( \text{res}^{-1}(h) \) is the space \( \mathcal{R}^+(W)^c_h \) while the homotopy fibre \( \text{hofib}_h(\text{res}) \) is the space of pairs \((g,p)\), with \( g \in \mathcal{R}^+(W)^c \) and \( p \) a continuous path in \( \mathcal{R}^+(M) \) from \( \text{res}(g) \) to \( h \). Inside the homotopy fibre, we have the space \((\text{hofib}_h(\text{res}))(\text{C}^\infty)\), which is defined by the condition that \( p \) has to be a smooth path. The inclusion \( i : (\text{hofib}_h(\text{res}))(\text{C}^\infty) \to \text{hofib}_h(\text{res}) \) is a homotopy equivalence \[13\] Lemma 2.3. Using the map \( F \), Chernysh constructs a map

\[
S' : (\text{hofib}_h(\text{res}))(\text{C}^\infty) \longrightarrow \mathcal{R}^+(W)^c_h
\]
as follows: pick an embedding \( j : W \to W \) onto the complement of a small collar, then the metric \( S'(g,p) \) is defined to be \((j^{-1})^* g \) on the image of \( j \) and \( F(p) \) on the small collar. Chernysh proves that \( S \) is a two-sided homotopy inverse to the fibre inclusion \( \mathcal{R}^+(W)^c_h \to (\text{hofib}_h(\text{res}))(\text{C}^\infty) \) \[13\]Lemma 2.2. By inverting \( i \) and passing to the colimit \( \epsilon \to 0 \), we obtain a homotopy class of map \( S : \text{hofib}_h(\text{res}) \to \mathcal{R}^+(W)^c_h \).

Theorem 2.2.7. (Chernysh \[13\])

(i) The map \( S \) is a two-sided homotopy inverse to the fibre inclusion \( \mathcal{R}^+(W)^c_h \to \text{hofib}_h(\text{res}) \).

(ii) In particular, the restriction map \( \text{res} : \mathcal{R}^+(W) \to \mathcal{R}^+(M) \) is a quasifibration.

2.2.4. Gluing of traces. Another application of the trace map is the proof of the following result.

Lemma 2.2.8. Let \( M = \partial W \), and \( h_0, h_1 \in \mathcal{R}^+(M) \) be isotopic psc metrics. Then there exists a \( g \in \mathcal{R}^+([0,1] \times M)_{h_0,h_1} \) such that the gluing map

\[
\mu_g : \mathcal{R}^+(W)_{h_0} \longrightarrow \mathcal{R}^+(W \cup ([0,1] \times M))_{h_1}
\]
is a weak homotopy equivalence.

Proof. Let \( h_t \) be a smooth isotopy from \( h_0 \) to \( h_1 \), which is assumed to be constant near 0 and 1. Let \( h_t := h_{1-t} \) the opposite isotopy. The concatenation \( h \ast h \) is a point in the space of smooth paths \([0,2] \to \mathcal{R}^+(M)\) starting and ending at \( h_0 \), and this path is nullhomotopic. In the construction of the trace map \( F \), we make little adjustments: first, the range of the map \( f \) is \([0,2] \), and we assume that \( f(C/2) = 1 \). The trace of the concatenated isotopy \( h \ast h \) is a point \( F(h \ast h) \in \mathcal{R}^+([0,2] \times M) \). Since the isotopy \( h \) is constant near 0, 1 and because of the condition on \( f \), we can write \( F(h \ast h) \) as the composition of a metric \( F(h) \) on \([0,1] \times M \) with a metric \( F(h) \) on \([0,1] \times M \). As \( h \ast h \) is nullhomotopic, the metric \( F(h \ast h) \) is isotopic to a cylinder, and hence the map

\[
\mu_{F(h \ast h)} = \mu_{F(h)} \circ \mu_{F(h)} : \mathcal{R}^+(W)_{h_0} \longrightarrow \mathcal{R}^+(W \cup [0,1] \times M)_{h_1} \longrightarrow \mathcal{R}^+(W \cup [0,2] \times M)_{h_0}
\]
is a weak homotopy equivalence, by Corollary 2.2.2. Therefore \( \mu_{F(h)} \) has a right homotopy inverse and \( \mu_{F(h)} \) has a left homotopy inverse. As \( \mu_{F(h)} \) is of the same form as \( \mu_{F(h)} \), \( \mu_{F(h)} \) has a left homotopy inverse as well and is therefore a weak homotopy equivalence. Thus \( \mu_{F(h)} \) is a one-sided homotopy inverse to a weak homotopy equivalence, so is a weak homotopy equivalence itself. \( \square \)
2.3. The cobordism theorem.

2.3.1. Standard metrics. On the disc \( D^d \), fix a collar of its boundary \( S^{d-1} \subset D^d \) by the formula

\[
b : S^{d-1} \times (-1,0] \to D^d \quad (v,t) \mapsto (1 + t) \cdot v.
\]

We assume that disks are always equipped with this collar.

On the sphere \( S^d \), let \( g_0^d = g_0 \in \mathcal{R}^+(S^d) \) be the ordinary metric of the Euclidean sphere of radius 1 (which has positive scalar curvature as long as \( d \geq 2 \)), and let \( g_{\text{hemi}} \) be the metric on \( D^d \) which comes from identifying \( D^d \subset \mathbb{R}^d \) with the lower hemisphere of \( S^d \subset \mathbb{R}^{d+1} \) via

\[
D^d \to S^d \quad x \mapsto (x, -\sqrt{1 - |x|^2}).
\]

We say that a rotation-invariant psc-metric \( g \) on \( D^d \) is a torpedo metric if

(i) \( b^*(g) \) agrees with the product metric \( g_0^{d-1} + dt^2 \) near \( S^{d-1} \times \{0\} \),

(ii) \( g \) agrees with \( g_{\text{hemi}} \) near the origin.

We fix a torpedo metric \( g_{\text{tor}}^d \) on \( D^d \) once and for all (for each \( d \geq 3 \)). In [52 §2.3], it is proved that \( g_{\text{tor}}^d \) can be chosen to have the following extra property: the metric on \( S^d \) obtained by gluing together two copies of \( g_{\text{tor}}^d \) on the upper and lower hemispheres is isotopic to \( g_0^d \). Such a metric on \( S^d \) will be called a double torpedo metric and denoted by \( g_{\text{d-tor}}^d \).

2.3.2. Spaces of metrics which are standard near a submanifold. Let \( W \) be a compact manifold of dimension \( d \) with boundary \( M \), equipped with a collar \( b : M \times (-1,0] \to W \). Let \( X \) be a \((k-1)\)-dimensional manifold and \( \phi : X^{k-1} \times D^{d-k+1} \to W^d \) be an embedding, and first suppose that \( \partial X \) is empty and \( \phi \) and \( b \) are disjoint. Let \( g_X \in \mathcal{R}(X) \) be a Riemannian metric, not necessarily of positive scalar curvature. However, we assume that the metric \( g_X + g_{\text{tor}}^{d-k+1} \) on \( X \times D^{d-k+1} \) has positive scalar curvature (this is the case for example if \( g_X \) has non-negative scalar curvature). Fix \( h \in \mathcal{R}^+(M) \) and let

\[
\mathcal{R}^+(W; \phi, g_X)_h \subset \mathcal{R}^+(W)_h
\]

be the subspace of those metrics \( g \) such that \( \phi^* g = g_X + g_{\text{tor}}^{d-k+1} \). We call this the space of psc metrics on \( W \) which are standard near \( X \). One of the main ingredients of the proof of Theorem 2.3.1 is the following result, due to Chernysh [12]. A different proof was later given by Walsh [53].

**Theorem 2.3.1.** (Chernysh, Walsh) If \( d - k + 1 \geq 3 \), then the inclusion map

\[
\mathcal{R}^+(W; \phi, g_X)_h \to \mathcal{R}^+(W)_h
\]

is weak homotopy equivalence.

Both authors state the result when the manifold \( W \) is closed. However, the deformations of the metrics appearing in the proof take place in a given tubular neighbourhood of \( X \), and therefore the global structure of \( W \) does not play a role. The predecessor of Theorem 2.3.1 is the famous surgery theorem of Gromov and Lawson [23]: that if \( \mathcal{R}^+(W)_h \) is nonempty, then \( \mathcal{R}^+(W; \phi, g_X)_h \) is nonempty. One might state this by saying that the inclusion map is \((-1)\)-connected. Gajer [18]
showed that the inclusion map is 0-connected, i.e. that each psc metric on $W$ is isotopic to one which is standard near $X$.

We also need a version of this result when $X$ has nonempty boundary. In that case, we require that the embedding $\phi$ is neat. In other words, $\phi^{-1}(\partial W) = \partial X \times D^{d-k+1}$, and $X$ is equipped with a collar $c : \partial X \times (-1,0] \to X$ such that the restricted embedding $\partial \phi : \partial X \times D^{d-k+1} \to \partial W$ satisfies $\phi(c(y,s),z) = b(\partial \phi(y,z),s)$ for all $(y,s,z) \in \partial X \times (-1,0] \times D^{d-k+1}$.

We fix a Riemannian metric $g_X$ on $X$ which is a product near its boundary (with respect to the collar $c$), and let $h_{\partial X}$ be the restriction of $g_X$ to $\partial X$. Neither $g_X$ nor $h_{\partial X}$ need to have positive scalar curvature, but we again insist that $g_X + g_{\text{tor}}^{d-k+1}$ does have positive scalar curvature. For $h \in R^+(M; \partial \phi, h_{\partial X})$ a psc metric on $M$ which is standard near $\phi(\partial X \times D^{d-k+1})$, we again let

$$\mathcal{R}^+(W; \phi, g_X) \subset \mathcal{R}^+(W)$$

be the subspace of psc metrics $g$ such that $\phi^* g = g_X + g_{\text{tor}}^{d-k+1}$. This can be viewed as the space of psc metrics that are standard near the manifold $X$.

Chernysh [13] Theorem 1.3 proved the following extension of Theorem 2.3.1: It follows from Theorem 2.3.1 and Theorem 2.2.4.

**Theorem 2.3.2.** The statement of Theorem 2.3.1 holds for $\partial W \neq \emptyset$ too.

**2.3.3. Bordism invariance of the space of psc metrics.** The original application of Theorem 2.3.1 was to show that for a closed, simply connected, spin manifold $W$ of dimension at least 5, the homotopy type of $\mathcal{R}^+(W)$ only depends on the spin cobordism class of $W$.

**Theorem 2.3.3.** (Chernysh, Walsh) Let $W : M_0 \sim M_1$ be a compact $d$-dimensional cobordism, $\phi : S^{k-1} \times D^{d-k+1} \to \text{int} W$ be an embedding, and $W'$ be the result of surgery along $\phi$. Fix $h_i \in \mathcal{R}^+(M_i)$. If $2 \leq k-1 \leq d-3$ then there is a weak homotopy equivalence

$$SE_\phi : \mathcal{R}^+(W)h_0,h_1 \simeq \mathcal{R}^+(W')h_0,h_1.$$  

Furthermore, the surgery datum $\phi$ determines a preferred homotopy class of $SE_\phi$.

The map $SE_\phi$ is called the surgery equivalence induced by the surgery datum $\phi$.

**Proof.** Since the surgery is in the interior of $W$, the boundary of $W$ and the metric on $\partial W$ are not affected. So we may, for typographical simplicity, assume that $W$ is closed. Let $W^o := W \setminus \phi(S^{k-1} \times \text{int}(D^{d-k+1}))$, a manifold with boundary $S^{k-1} \times S^{d-k}$, and let

$$W' = W^o \cup_{S^{k-1} \times D^{d-k+1}} (D^k \times S^{d-k})$$

be the result of doing a surgery on $\phi$ to $W$. There is a canonical embedding $\phi' : D^k \times S^{d-k} \to W'$, and if we do surgery on $\phi'$, we recover $W$. Note that the restriction of the psc metric $g_{\text{tor}}^{k-1} + g_{\text{tor}}^{d-k+1}$ on $S^{k-1} \times D^{d-k+1}$ to the boundary $S^{k-1} \times S^{d-k}$ is $g_{\text{tor}}^{k-1} + g_{\text{tor}}^{d-k}$, by the definition of a torpedo metric. Similarly, the restriction of the psc metric $g_{\text{tor}}^k + g_{\text{tor}}^{d-k}$ on $D^k \times S^{d-k}$ to the boundary is $g_{\text{tor}}^{k-1} + g_{\text{tor}}^{d-k}$.
Therefore we get maps
\[
\mathcal{R}^+(W; \phi, g^{k-1}_0) \cong \mathcal{R}^+(W'; \phi', g^{d-k}_0) \\
\Downarrow \iota_0 \hspace{1cm} \Downarrow \iota_1 \\
\mathcal{R}^+(W) \hspace{1cm} \mathcal{R}^+(W')
\]
(2.3.4)

By Theorem 2.3.1, the map \(\iota_0\) (\(\iota_1\), respectively) is a weak equivalence if \(d-k+1 \geq 3\) (if \(k \geq 3\), respectively).

The cobordism invariance of the space \(\mathcal{R}^+(W)\) for closed, simply connected, spin manifolds of dimension at least five follows by the same use of Smale’s handle cancellation technique as in [23].

2.3.4. \textit{Existence of stabilizing metrics.} We use Theorem 2.3.1 to deduce the existence of psc metrics \(g\) on certain cobordisms \(W\) such that \(\mu_g\) is a weak equivalence.

\textbf{Lemma 2.3.5.} Let \(M^{d-1} = \partial W\) be simply connected and spin, and \(K : M \sim M\) be a cobordism which is simply connected and spin, and is in turn spin cobordant to \([0, 1] \times M\) relative to its boundary. Suppose \(d \geq 5\). Then for any boundary condition \(h \in \mathcal{R}^+(M)\) there is a \(g \in \mathcal{R}^+(K)_{h, h}\) such that

\[
\mu_g : \mathcal{R}^+(W)_{h} \rightarrow \mathcal{R}^+(W \cup_M K)_h
\]
is a weak homotopy equivalence.

\textbf{Proof.} By assumption there is a relative spin cobordism \(V^{d+1}\) from \(K\) to \([0, 1] \times M\). Since \(\dim(V) \geq 6\), by doing surgery in the interior of \(V\) we can achieve that \(V\) is 2-connected. In particular, the inclusions \([0, 1] \times M \rightarrow V\) and \(K \rightarrow V\) are both 2-connected maps. As the manifolds \(K\), \([0, 1] \times M\), and \(V\) are all simply connected, we can apply Smale’s handle cancellation technique to \(V\); the result is that we can assume that the cobordism \(V\) is obtained by attaching handles of index \(3 \leq k \leq d-2\) to the interior of either of its boundaries. The result we need is a step in the proof of the \(h\)-cobordism theorem and stated explicitly as [22, Theorem VIII.4.1], except that we consider a cobordism of manifolds with boundary. But \(V\) is a relative cobordism, with boundary decomposed as \(([0, 1] \times M) \cup K \cup (\partial K \times [0, 1])\) (the last part is constant). The proof of loc. cit. applies verbatim to this case.

Let \(\phi : S^{k-1} \times D^{d-k+1} \rightarrow K\) be a piece of surgery data in the interior of \(K\) such that surgery along it yields a manifold \(K'\) (this corresponds to a surgery of index \(k\)). Similarly to the proof of Theorem 2.3.3 there is a commutative diagram

\[
\begin{align*}
\mathcal{R}^+(W)_{h} \times \mathcal{R}^+(K)_{h, h} \rightarrow \mathcal{R}^+(W \cup K)_{h} \\
\Downarrow \hspace{1cm} \Downarrow \\
\mathcal{R}^+(W)_{h} \times \mathcal{R}^+(K; \phi, g^{k-1}_0)_{h, h} \rightarrow \mathcal{R}^+(W \cup K; \phi, g^{k-1}_0)_{h} \\
\Downarrow \hspace{1cm} \Downarrow \\
\mathcal{R}^+(W)_{h} \times \mathcal{R}^+(K')_{h, h} \rightarrow \mathcal{R}^+(W \cup K')_{h}
\end{align*}
\]

where, if \(2 \leq k - 1 \leq d - 3\), all the vertical maps are weak homotopy equivalences. Thus gluing on a metric \(g \in \mathcal{R}^+(K)_{h, h}\) induces a weak homotopy equivalence if and only if gluing on the corresponding metric \(g' \in \mathcal{R}^+(K')_{h, h}\) does.
By Theorem 2.3.3 the cobordism $V$ induces a surgery equivalence
$$\mathcal{R}^+(K)_{h,h} \simeq \mathcal{R}^+([0, 1] \times M)_{h,h}.$$ Gluing on $([0, 1] \times M, dt^2 + h)$ induces a weak homotopy equivalence, by Corollary 2.2.2 so if we let $g \in \mathcal{R}^+(K)_{h,h}$ be in a path component corresponding to that of $h + dt^2$ under the surgery equivalence then gluing on $(K, g)$ also induces a weak homotopy equivalence, as required.  

3. The secondary index invariant

This chapter contains the index theoretic arguments that go into the proof of our main results. We begin by stating our framework for $K$-theory in Section 3.1. Then we recall the basic properties of the Dirac operator on a spin manifold and on bundles of spin manifolds, including those with noncompact fibres, in Section 3.2. These analytical results allow the definition of the secondary index invariant, $\text{inddiff}$, to be presented in Section 3.3.

Conceptually simple as the definition of $\text{inddiff}$ is, it seems to be impossible to compute directly. The purpose of the rest of this chapter is to provide computational tools. One computational strategy is to use the additivity property of the index, which results in a cut-and-paste property for the index difference. This is done in Section 3.4. The other computational strategy is to relate the secondary index to a primary index. In Section 3.5, we describe the abstract setting necessary to carry out such a comparison. This is then applied in two different situations. The first is the passage from even to odd dimensions, in other words the derivation of Theorem C from Theorem B, which is carried out in Section 3.6. The second situation in which we apply the general comparison pattern is when we compute the index difference by a family index in the classical sense. This will be essential for the proof of Theorem B and is done in Section 3.8. The classical indices can be computed using the Atiyah–Singer index theorem for families of Clifford-linear differential operators, which we first discuss in Section 3.7. Also in Section 3.8, the index theorem is interpreted in homotopy-theoretic terms, and there, another key player of this paper enters the stage: the Madsen–Tillmann–Weiss spectra.

3.1. Real $K$-theory. The homotopy theorists’ definition of real $K$-theory is in terms of the periodic $K$-theory spectrum $KO$. By definition, the $KO$-groups of a space pair $(X, Y)$ are given by
$$KO^k(X, Y) := [\Omega^\infty^{-k}KO, *].$$ The specific choice of a model for $KO$ is irrelevant, as long as one considers only spaces having the homotopy type of CW complexes. For index theoretic arguments, we use the Fredholm model, which we now briefly describe. Our model is a variant of a classical result by Atiyah–Singer [7] and Karoubi [29]. More details and further references can be found in [10, §2]. We begin by recalling some subtleties concerning Hilbert bundles and their maps.

3.1.1. Hilbert bundles. Let $X$ be a space (usually paracompact and Hausdorff) and let $H \to X$ be a real or complex Hilbert bundle. For us, a Hilbert bundle will always have separable fibres and the unitary group with the compact-open topology as a structure group. An operator $F : H_0 \to H_1$ is a fibre-preserving, fibrewise linear continuous map. It is determined by a family $(F_x)_{x \in X}$ of bounded operators $F_x : (H_0)_x \to (H_1)_x$. Some care is necessary when defining properties of the
operator $F$ by properties of the individual operators $F_x$. We will recall the basic facts and refer the reader to [16] for a more detailed discussion. An operator $F$ is adjointable if the family of adjoints $(F_x^*)$ also is an operator. The algebra of adjointable operators on $H$ is denoted by $\text{Lin}_X(H)$. There is a notion of a compact operator which is due to Dixmier–Douady [15], §22; the set of compact operators is denoted $\text{Kom}_X(H)$ and is a $*$-ideal in $\text{Lin}_X(H)$. A Fredholm family is an element in $\text{Lin}_X(H)$ which is invertible modulo $\text{Kom}_X(H)$. The reader is warned that being compact (or Fredholm) is a stronger condition on an operator $F$ than just saying that all $F_x$ are compact (or Fredholm). For our purposes, the following fact is all we need to know. We say that $F$ is locally norm-continuous if each point $x \in X$ admits a neighborhood $U$ and a trivialisation of $H|_U$ such that in this trivialisation, $F$ is given by a continuous map $U \to \text{Lin}(H_x)$ (with the norm topology in the target). If $F$ is locally norm-continuous and each $F_x$ is compact (or invertible, or Fredholm), then $F$ is compact (or invertible, or Fredholm).

3.1.2. Clifford bundles. Let $V, W \to X$ be two Riemannian vector bundles. A Cl$(V^+ \oplus W^-)$-Hilbert bundle over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a triple $(H, \iota, c)$, where $H \to X$ is a $\mathbb{K}$-Hilbert bundle (always assumed to have separable fibres), $\iota$ is a $\mathbb{Z}/2$-grading (i.e. a self-adjoint involution) on $H$ and $c : V \oplus W \to \text{End}(H)$ is a bundle map such that

$$c(v, w)\iota + \iota c(v, w) = 0$$

$$c(v, w)^* = c(-v, w)$$

$$c(v, w)c(v', w') + c(v', w)c(v, w) = -2(\langle v, v' \rangle - \langle w, w' \rangle)$$

hold for all $v \in V$, $w \in W$. If $V = X \times \mathbb{R}^p$ and $W = X \times \mathbb{R}^q$, we abbreviate the term “Cl$(X \times \mathbb{R}^p)^+ \oplus (X \times \mathbb{R}^q)^-$-bundle” to “Cl$p,q$-Hilbert bundle”.

The opposite Cl$(V^+ \oplus W^-)$-Hilbert bundle has the same underlying Hilbert bundle, but the Clifford multiplication and grading are replaced by $-c$ and $-\iota$. We often denote Cl$p,q$-Hilbert bundles simply by $H$, and the opposite one by $H^{op}$. Instead of “finite-dimensional Cl$(V^+ \oplus W^-)$-Hilbert bundle”, we shall say Cl$(V^+ \oplus W^-)$-module. A Cl$p,q$-Fredholm family is a Fredholm family such that $F_1 = -\iota F$ and $Fc(v) = c(v)F$ for all $v \in \mathbb{R}^{p+q} := (\mathbb{R}^p)^+ \oplus (\mathbb{R}^q)^-$ (note that the convention on $\mathbb{R}^{p+q}$ differs from that in [4], we think it is easier to memorise).

3.1.3. $K$-theory. We denote the product of pairs of spaces by $(X, Y) \times (A, B) := (X \times A, X \times B \cup Y \times A)$. A $(p, q)$-cycle on $X$ is a tuple $(H, \iota, c, F)$, consisting of a Cl$p,q$-Hilbert bundle and a Cl$p,q$-Fredholm family. If $Y \subseteq X$ is a subspace, then a relative $(p, q)$-cycle is a $(p, q)$-cycle $(H, \iota, cF)$ with the additional property that the operator $F|_Y$ is invertible. Clearly $(p, q)$-cycles can be pulled back along continuous maps, and there is an obvious notion of isomorphism of $(p, q)$-cycles and of direct sum of finitely many $(p, q)$-cycles. A concordance of relative $(p, q)$-cycles $(H_i, \iota_i, c_i, F_i)$, $i = 0, 1$ on $(X, Y)$ consists of a relative $(p, q)$-cycle $(H, \iota, c, F)$ on $(X, Y) \times [0, 1]$ and isomorphisms $(H, \iota, c, F)|_{X \times \{i\}} \cong (H_i, \iota_i, c_i, F_i)$. A $(p, q)$-cycle $(H, \iota, c, F)$ is acyclic if $F$ is invertible. Often, we abbreviate $(H, \iota, c, F)$ to $(H, F)$ if there is no risk of confusion. Occasionally, we write $x \mapsto (H_x, F_x)$ to describe a $(p, q)$-cycle on $X$.

**Definition 3.1.1.** Let $X$ be a paracompact Hausdorff space and $Y \subseteq X$ be a closed subspace. The group $K^{op,q}(X, Y)$ is the quotient of the abelian monoid
of concordance classes of relative \((p,q)\)-cycles, divided by the submonoid of those concordance classes which contain acyclic \((p,q)\)-cycles.

The monoid so obtained is in fact a group, and the additive inverse is given by, [16] Lemma 2.19],

\[-[H, \iota, c, F] = [H, -\iota, -c, F].\quad (3.1.2)\]

In [16], the following variant of a classical result by Atiyah–Singer [7] and Karoubi [29] is proved. Recall that a \(\text{Cl}^d\)-Hilbert space is \textit{ample} if it contains any finite-dimensional irreducible \(\text{Cl}^d\)-Hilbert space with infinite multiplicity.

**Theorem 3.1.3.** For each compact Hausdorff pair \((X,Y)\), there is a natural isomorphism \(KO^{p,q}(X,Y) \cong KO^{-p}(X,Y)\). The functor \(KO^{p,q}\) is representable by a space pair \((F^{p,q}, D^{p,q})\), where \(F^{p,q}\) is the space of \(\text{Cl}^d\)-Fredholm operators on an ample \(\text{Cl}^d\)-Hilbert space and \(D^{p,q}\) is the subspace of invertible operators, which is contractible. More precisely, there is a natural bijection

\[
[(X,Y), (F^{p,q}, D^{p,q})] \cong KO^{p,q}(X,Y).\quad (3.1.4)
\]

We remark that the topology on \(F^{p,q}\) is different from the norm topology. Theorem 3.1.3 implies that \((F^{p,q}, D^{p,q})\) is weakly homotopy equivalent to \(\Omega^{\infty+p-q}KO, \ast\). For any class \(b \in KO^{p,q}(X,Y)\), we call the resulting homotopy class of maps \((X,Y) \to (\Omega^{\infty+p-q}KO, \ast)\) the homotopy-theoretic realisation of \(b\). Henceforth, we denote \(KO^{-p}(X,Y) := KO^{0,0}(X,Y)\). We let \(\langle 1, \partial 1 \rangle := \langle [-1, 1], \{ -1, 1 \} \rangle\) and use the following notation:

\[
\Omega KO^{-p}(X,Y) := KO^{-p}(\langle X, Y \rangle \times \langle 1, \partial 1 \rangle).
\]

Bott periodicity in this setting states that the map

\[
KO^{-p}(X,Y) \to \Omega KO^{-p-1}(X,Y)\quad (3.1.5)
\]

\[
(H,F) \mapsto ((x,s) \mapsto (H_x, F_x + st_x(e_1))),\quad (3.1.6)
\]

is an isomorphism of abelian groups. In particular, we get an isomorphism

\[
KO^{-p}(X,Y) \to \Omega^p KO^{p}(X,Y)
\]

which realise the isomorphism stated in Theorem 3.1.3. From [16] Theorem A.6 and [16] Proposition A.13, we deduce the following statement.

**Proposition 3.1.7.** Let \(U\) be an ample \(\text{Cl}^d\)-Hilbert space and \(J\) be an invertible \(\text{Cl}^d\)-operator on \(U\). If \((H_0, F_0)\) and \((H_1, F_1)\) are two \((p,q)\)-cycles on \(X\), then they determine the same element of \(KO^{p,q}(X,Y)\) if and only if \((H_0, F_0) \oplus (U, J)\) and \((H_1, F_1) \oplus (U, J)\) are concordant.

### 3.2. Generalities on Dirac operators.

#### 3.2.1. The spin package.

A basic reference for spin vector bundles and associated constructions is [34] §II.7. Let \(V \to X\) be a real vector bundle of rank \(d\). A \textit{topological spin structure} is a reduction of the structure group of \(V\) to the group \(\tilde{\text{GL}}_d^+(\mathbb{R})\), the connected 2-fold covering group of the group \(\text{GL}_d^+(\mathbb{R})\) of matrices of positive determinant. In the presence of a Riemannian metric, a topological spin structure induces a reduction of the structure group to Spin\((d)\), which is the familiar notion of a spin structure. However, for the purpose of index theory, it is more convenient to define a spin structure on a Riemannian vector bundle \(V \to X\) of rank \(d\) as a fibrewise irreducible real \(\text{Cl}(V \oplus \mathbb{R}^{0,d})\)-module \(\Phi_V\). The
opposite spin structure $\mathfrak{S}_V^\text{op}$ has the same underlying vector bundle as $\mathfrak{S}_V$ and the same Clifford multiplication, but the grading is inverted (note that this is not the opposite bundle in the sense of the previous section). If $V = TM$ is the tangent bundle of a Riemannian manifold $M^d$ with metric $g$, we denote spin structures typically by $\mathfrak{S}_M$. There is a canonical connection $\nabla$ on $\mathfrak{S}_M$ derived from the Levi-Civita connection on $M$. The Dirac operator $\mathfrak{D} = \mathfrak{D}_g$ acts on sections of $\mathfrak{S}_M$ and is defined as the composition

$$\Gamma(M; \mathfrak{S}_M) \xrightarrow{\nabla} \Gamma(M; TM \otimes \mathfrak{S}_M) \xrightarrow{c} \Gamma(M; \mathfrak{S}_M)$$

(it would be more appropriate to call $\mathfrak{D}$ the Atiyah–Singer Dirac operator, but no other Dirac operator will occur in this paper). This is a linear formally self-adjoint elliptic differential operator of order 1, which anticommutes with the grading and Clifford multiplication by $\mathbb{R}^{0,d}$. We can change the $\mathrm{Cl}^{0,d}$-multiplication on $\mathfrak{S}_M$ to a $\mathrm{Cl}^{d,0}$-multiplication by replacing $c(v)$ by $uc(v)$. With this new structure, $\mathfrak{D}$ becomes $\mathrm{Cl}^{d,0}$-linear. Passing to the opposite spin structure leaves the operator $\mathfrak{D}$ unchanged, but changes the sign of the grading and of the $\mathrm{Cl}^{d,0}$-multiplication.

The relevance of the Dirac operator to scalar curvature stems from the well-known Schrödinger–Lichnerowicz formula (also known as Lichnerowicz–Weitzenböck formula) [44, 35], or [34, Theorem II.8.8]:

$$\mathfrak{D}^2 = \nabla^* \nabla + \frac{1}{4} \text{scal}(g). \quad (3.2.1)$$

### 3.2.2. The family case.

We need to study the Dirac operator for families of manifolds and also for nonclosed manifolds. Let $X$ be a paracompact Hausdorff space. We study bundles $\pi : E \rightarrow X$ of possibly noncompact manifolds with $d$-dimensional fibres. The fibres of $\pi$ are denoted $E_x := \pi^{-1}(x)$. The vertical tangent bundle is $T_v E \rightarrow E$ and we always assume implicitly that a (topological) spin structure on $T_v E$ is fixed (then of course the fibres are spin manifolds). A fibrewise Riemannian metric $(g_x)_{x \in X}$ on $E$ then gives rise to the spinor bundle $\mathfrak{S}_E \rightarrow E$, a $\mathrm{Cl}(T_v E)^+ \oplus \mathbb{R}^{0,d}$-module. The restriction of the spinor bundle to the fibre over $x$ is denoted $\mathfrak{S}_x \rightarrow E_x$, and the Dirac operator on $E_x$ is denoted $\mathfrak{D}_x$. When we consider bundles of noncompact manifolds, they are always required to have a simple structure outside a compact set.

**Definition 3.2.2.** Let $\pi : E \rightarrow X$ be a bundle of noncompact $d$-dimensional spin manifolds. Let $t : E \rightarrow \mathbb{R}$ be a fibrewise smooth function such that $(\pi, t) : E \rightarrow X \times \mathbb{R}$ is proper. Let $a_0 < a_1 : X \rightarrow [\pm \infty, \infty]$ be continuous functions. Abusing notation, we denote $X \times (a_0, a_1) = \{(x, t) \mid a_0(x) < t < a_1(x)\}$ and $E_{(a_0, a_1)} := (\pi, t)^{-1}(X \times (a_0, a_1))$. We say that $E$ is cylindrical over $(a_0, a_1)$ if there is a $(d-1)$-dimensional spin manifold $M$ such that $E_{(a_0, a_1)} \cong X \times (a_0, a_1) \times M$ (over $X \times (a_0, a_1)$) as spin manifold bundles. The bundle is said to have cylindrical ends if there are functions $a_-, a_+ : X \rightarrow \mathbb{R}$ such that $E$ is cylindrical over $(-\infty, a_-)$ and $(a_+, \infty)$. A fibrewise Riemannian metric $g = (g_x)_{x \in X}$ on $E$ is called cylindrical over $(a_0, a_1)$ if $g_x = dt^2 + h_x$ for some metric $h_x$ on $M$, over $(a_0, a_1)$. We always consider Riemannian metrics which are cylindrical over the ends. We say that a bundle with cylindrical ends and metrics has positive scalar curvature at infinity if there is a function $\epsilon : X \rightarrow (0, \infty)$, such that the metrics on the ends of $E_x$ have scalar curvature $\geq \epsilon(x)$. 
Any bundle of closed manifolds, equipped with an arbitrary function \( t \), tautologically has cylindrical ends and positive scalar curvature at infinity. Note that the fibres of a bundle with cylindrical ends are automatically complete in the sense of Riemannian geometry. Fibre bundles \( \pi : E \to X \) of spin manifolds with boundary can be fit into the above framework by a construction called elongation.

**Definition 3.2.3.** Let \( \pi : E \to X \) be a bundle of compact manifolds with collared boundary, and assume that the boundary bundle is trivialised: \( \partial E = X \times \partial N \) as a spin bundle. Let \( g = (g_x)_{x \in X} \) be a fibrewise metric on \( E \) so that \( g_x \) is of the form \( dt^2 + h_x \) near the boundary, for psc metrics \( h_x \in \mathbb{R}^+ (N) \). The elongation of \((E, g)\) is the bundle \( \hat{E} = E \cup_{\partial E} (X \times [0, \infty) \times N) \), with the metric \((dt^2 + h_x)\) on the added cylinders. The elongation has cylindrical ends and positive scalar curvature at infinity.

### 3.2.3. Analysis of Dirac operators

Let \( E \to X \) be a spin manifold bundle, equipped with a Riemannian metric \( g \), and with cylindrical ends. Let \( L^2(E; \mathfrak{S})_x \) be the Hilbert space of \( L^2 \)-sections of the spinor bundle \( \mathfrak{S}_x \to E_x \). These Hilbert spaces assemble to a \( \text{Cl}^{d, 0} \)-Hilbert bundle on \( X \). From now on, we make the crucial assumption that the scalar curvature of \( g \) is positive at infinity. It is known that the Dirac operator \( \mathfrak{D}_x \) is an essentially self-adjoint (unbounded) Fredholm operator on the Hilbert space \( L^2(E_x; \mathfrak{S}_x) \) (the maximal domain is the Sobolev space \( W^1(E_x; \mathfrak{S}_x) \)). The bounded transform

\[
F_x := \frac{\mathfrak{D}_x}{(1 + \mathfrak{D}_x^2)^{1/2}}
\]

is a bounded \( \text{Cl}^{d, 0} \)-Fredholm operator. The family of operators \( (F_x)_{x \in X} \) is a \( \text{Cl}^{d, 0} \)-Fredholm family over \( X \), which is moreover locally norm-continuous in the sense of Section 3.1. These standard facts are proven in detail in [16] §3.1. If moreover for each \( y \in Y \subset X \), the metric \( g_y \) has positive scalar curvature (on all of \( E_y \), not only at the ends), then the operator \( \mathfrak{D}_y \) has trivial kernel, by the Bochner method using the Schrödinger–Lichnerowicz formula [3.2.1], cf. [34] II Corollary 8.9. Consequently, the operator \( F_y \) is invertible for all \( y \in Y \).

**Definition 3.2.4.** Let \( \pi : E \to X \) be a bundle of Riemannian spin manifolds with cylindrical ends and positive scalar curvature at infinity, with metric \( g \). We denote by \( \text{Dir}(E, g) \) the \((d, 0)\)-cycle \( x \mapsto (L^2(E_x, \mathfrak{S}_{E_x}), F_x) \) and by \( \text{ind}(E, g) \) its class in \( KO^{-d}(X) \). If it is understood that the metric \( g_y \) has positive scalar curvature for all \( y \in Y \subset X \), we use the same symbol to denote the class in the relative \( K \)-group \( KO^{-d}(X, Y) \).

When we meet a bundle with boundary, by the symbols \( \text{Dir}(E, g) \) and \( \text{ind}(E, g) \) we always mean the index of the elongated manifolds. Alternatively, one could use the Dirac operators on \( E \) with Atiyah–Patodi–Singer boundary conditions [6].

**Lemma 3.2.5.** Let \( \pi : E \to X \) be a spin manifold bundle with cylindrical ends and let \( g_0, g_1 \) be fibrewise metrics which both have positive scalar curvature at infinity. Assume that \( g_0 \) and \( g_1 \) agree on the ends. Let \( Y \subset X \) and assume that \( g_0 \) and \( g_1 \) agree over \( Y \) and have positive scalar curvature there. Then \( \text{ind}(E, g_0) = \text{ind}(E, g_1) \in KO^{-d}(X, Y) \).
Proof. This follows from the homotopy invariance of the Fredholm index or by considering the metric \((1 - t)g_0 + tg_1\) on the product bundle \(E \times I \to X \times I\), which gives a concordance of cycles.

In particular, if the fibres of \(\pi\) are closed, then \(\text{ind}(E, g)\) does not depend on \(g\) at all. This observation justifies the notation \(\text{ind}(E)\) for closed bundles and \(\text{ind}(E, g)\) when \(E\) is a bundle with cylindrical ends, and the psc metric \(g\) is only defined on the ends. In that case, \(g = dt^2 + h\), and we could also write \(\text{ind}(E, h)\), emphasizing the role of \(h\) as a boundary condition. Finally, we remark that if we pass to the opposite bundle \(E^{op} \to X\) (with the opposite spin structure), then

\[
\text{ind}(E^{op}, g) = -\text{ind}(E, g).
\] (3.2.6)

This follows from the definition of the opposite spin structure given in subsection 3.2.1 and from the formula for the additive inverse of a \(K\)-theory class, given after Definition 3.1.1.

3.3. The secondary index invariants. We now define the secondary index invariant, the index difference. There are two definitions of this invariant which we will consider. The first one is due to Hitchin [27] and we take it as the main definition.

3.3.1. Hitchin’s definition. Let \(W\) be a manifold with collared boundary \(M\), and let \(h \in \mathcal{R}^+(M)\). On the space \(I \times \mathcal{R}^+(W)_h \times \mathcal{R}^+(W)_h\), we consider the elongation of the trivial bundle with fibre \(W\), and introduce the following fibrewise Riemannian metric \(g\): on the fibre over \((t, g_0, g_1)\) it is \(\frac{1}{1 + t}g_0 + \frac{1}{1 + t}g_1\) (since both metrics agree on \(M\), this has positive scalar curvature at infinity). For \(t = \pm 1\), this metric has positive scalar curvature and thus applying the results from the previous section, we get an element

\[
\text{indiff} := \text{ind}(I \times \mathcal{R}^+(W)_h \times \mathcal{R}^+(W)_h, g) \in \Omega KO^{-d}((\mathcal{R}^+(W)_h \times \mathcal{R}^+(W)_h, \Delta))
\] (3.3.1)

(where \(\Delta\) is the diagonal) the path space version of the index difference.

Remark 3.3.2. Slightly imprecisely, one can phrase this construction by saying that each pair \((g_0, g_1)\) of psc metrics defines a path \(t \mapsto D_{\frac{1}{1 - t}g_0 + \frac{1}{1 - t}g_1}\) in \(F^{d,0}\), which for \(t = \pm 1\) is invertible, and hence in the contractible subspace \(D^{d,0}\). The space of all paths \(\gamma : (I, \{\pm 1\}) \to (F^{d,0}, D^{d,0})\) is homotopy equivalent to the loop space \(\Omega F^{d,0}\). This is conceptually clear, but since the spinor bundle depends on the underlying metric and thus the operators \(D_{\frac{1}{1 - t}g_0 + \frac{1}{1 - t}g_1}\) do not act on the same Hilbert space, does not strictly make sense.

Note that it is important that the metrics are fixed on \(M\), since otherwise the metrics \(\frac{1}{1 - t}g_0 + \frac{1}{1 - t}g_1\) might not have positive scalar curvature at infinity. In fact, the class \((3.3.1)\) does not have an extension to all of \(\mathcal{R}^+(W)\), and our argument in Section 3.6 shows this fact.

If we keep a basepoint \(g \in \mathcal{R}^+(W)_h\) fixed, we get an element

\[
\text{indiff}_g \in \Omega KO^{-d}((\mathcal{R}^+(W)_h, g))
\]

by fixing the first variable. Using Theorem 3.3.3, we can represent this element in a unique way by a homotopy class of pointed maps

\[
\text{indiff}_g : (\mathcal{R}^+(W)_h, g) \to (\Omega^{\infty + d + 1}KO, \ast).
\]
3.3.2. Gromov–Lawson’s definition. The second version of the index difference is due to Gromov and Lawson [24], and we will use it as a computational tool. Let \( W \) be a \( d \)-dimensional closed manifold and consider the trivial bundle over \( \mathcal{R}^+(W) \times \mathcal{R}^+(W) \) with fibre \( \mathbb{R} \times W \). Choose a smooth function \( \varphi : \mathbb{R} \to [0, 1] \) that is equal to 0 on \((-\infty, 0]\) and equal to 1 on \([1, \infty)\). Equip the fibre over \((g_0, g_1)\) with the metric \( h_{(g_0, g_1)} := dt^2 + (1 - \varphi(t))g_0 + \varphi(t)g_1 \), which has positive scalar curvature at infinity and so gives an element

\[
\text{indiff}_{GL} := \text{ind}(\mathcal{R}^+(W) \times \mathcal{R}^+(W) \times \mathbb{R} \times W, h) \in KO^{-d-1}(\mathcal{R}^+(W) \times \mathcal{R}^+(W), \Delta)
\] (3.3.3)

(the degree shift appears since \( \mathbb{R} \times M \) has dimension \( d + 1 \)). We do not try to define this version of the index difference for manifolds with boundary. Again, we obtain \( \text{indiff}_{GL}^g \in KO^{-d-1}(\mathcal{R}^+(M), g) \) by fixing a basepoint. The homotopy-theoretic realisations of both index differences are pointed maps

\[
\text{indiff}_g, \text{indiff}_{GL}^g : (\mathcal{R}^+(M), g) \to (\Omega^{\infty + d+1}KO, \ast).
\] (3.3.4)

The following theorem answers the obvious question whether both definitions of the index difference yield the same answer. It will be the main ingredient for the derivation of Theorem C from Theorem B.

**Theorem 3.3.5.** (Spectral flow-index theorem [16]) For each closed spin manifold \( M \), the two definitions of the secondary index invariant agree, i.e. the maps (3.3.4) are weakly homotopic (cf. Definition [1, 7]).

The name of this result is because it relies on a generalisation of the classical spectral flow-index theorem [41] to the Clifford-linear family case, also proved in [16].

3.4. The additivity theorem. An efficient tool to compute the secondary index invariant is the additivity theorem for the index of operators on noncompact manifolds. There are several versions of this result in the literature, but the version that is most useful for our purposes is due to Bunke [10].

**Assumptions 3.4.1.** Let \( X \) be a paracompact Hausdorff space. Let \( E \to X \) and \( E' \to X \) be two Riemannian spin manifold bundles of fibre dimension \( d \) with metrics \( g \) and \( g' \) and with cylindrical ends, such that \( E \) and \( E' \) have positive scalar curvature at infinity. Assume that there exist functions \( a_0 < a_1 : X \to \mathbb{R} \) such that \( E \) and \( E' \) are cylindrical over \( X \times (a_0, a_1) \) and agree there: \( E_{(a_0, a_1)} = E'_{(a_0, a_1)} \)

(more precisely, we mean that there exists a spin-preserving isometry of bundles of closed manifolds over \( X \times (a_0, a_1) \)). Assume that the scalar curvature on \( E_{(a_0, a_1)} \) is positive. Let

\[
E_0 = E_{(-\infty, a_1)}; \quad E_1 = E_{(a_0, \infty)}; \quad E_2 = E'_{(-\infty, a_1)}; \quad E_3 = E'_{(a_0, \infty)}
\]

and define \( E_{ij} := E_i \cup E_j \) for \( (i, j) \in \{(01), (23), (03), (21)\} \). Note that \( E = E_{01} \) and \( E' = E_{23} \). These are bundles of spin manifolds with cylindrical ends, having positive scalar curvature at infinity.
Theorem 3.4.2. (Additivity theorem) Under Assumptions 3.4.1, we have
\[ \text{ind}(E_{01}) + \text{ind}(E_{23}) + \text{ind}(E_{03}^{op}) + \text{ind}(E_{21}^{op}) = 0 \in KO^{-d}(X). \]

Furthermore, if both bundles \( E_{01} \) and \( E_{23} \) have positive scalar curvature over the closed subspace \( Y \subset X \), then the above equation holds in \( KO^{-d}(X, Y) \).

If \( X = \ast \), this is due to Bunke [10], and the case of arbitrary \( X \) and \( Y = \emptyset \) is straightforward from his argument. For the case \( Y \neq \emptyset \), we need to give an additional argument, and this forces us to go into some details of Bunke’s proof. Furthermore, the setup used by Bunke is slightly different from ours (cf. §1.1 loc.cit.), and so we decided to sketch the full proof here.

3.4.2. The proof of the additivity theorem. Let \( H_{ij} := L^2(E_{ij}; \mathcal{F}) \) be the Hilbert bundle on \( X \) associated with the bundle \( E_{ij} \). The sum of the indices showing up in Theorem 3.4.2 is represented by the tuple \((H, \iota, c, F)\); the graded \( Cl^{d,0}\)-bundle is \( H := H_{01} \oplus H_{23} \oplus H_{03} \oplus H_{21} \), with Clifford action, involution, and operator given by

\[
\begin{align*}
c &:= \begin{pmatrix} c_{01} & -c_{03} & c_{23} & -c_{21} \\ -c_{03} & c_{23} & -c_{01} & c_{21} \\ -c_{21} & c_{23} & -c_{03} & c_{01} \\ c_{21} & -c_{23} & c_{03} & -c_{01} \end{pmatrix} \\
\iota &:= \begin{pmatrix} \iota_{01} & -\iota_{03} \\ \iota_{03} & -\iota_{01} \\ -\iota_{23} & \iota_{21} \\ \iota_{21} & -\iota_{23} \end{pmatrix} \\
F &:= \frac{D}{(1 + D^2)^{1/2}}.
\end{align*}
\]

Pick smooth functions \( \lambda_0, \mu_0 : \mathbb{R} \to [0, 1] \) with \( \text{supp}(\lambda_0) \subset [0, \infty), \text{supp}(\mu_0) \subset (-\infty, 1] \) and \( \mu_0^2 + \lambda_0^2 = 1 \); we can choose them to have \( |\lambda_0|, |\mu_0| \leq 2 \). We define functions \( \mu, \lambda : X \times \mathbb{R} \to [0, 1] \) by \( \mu(x, t) = \mu_0(\frac{t - a_0(x)}{a_1(x) - a_0(x)}) \) and \( \lambda(x, t) = \lambda_0(\frac{t - a_0(x)}{a_1(x) - a_0(x)}) \).

The formula
\[
J_0 := \begin{pmatrix} \mu & -\lambda & -\mu & -\lambda \\ \lambda & \mu & -\lambda & -\mu \end{pmatrix}
\]
defines an operator on \( H \) (the interpretation should be clear: for example, multiplying a spinor over \( E_{01} \) by \( \mu \) gives a spinor with support in \( E_0 \), and we can transplant it to \( E_{03} \)). Then \( J := J_{0t} \) is an odd, \( Cl^{d,0}\)-linear involution.

Bunke proves that the anticommutator \( FJ + JF \) is compact (stated and proved as Lemma 3.4.7 below), whence

\[
s \mapsto \cos(s)F + \sin(s)J
\]
defines a homotopy from \( F \) to \( J \); since \( J \) is invertible, the element \([H, \iota, c, F] \in KO^{0,d}(X)\) is zero, which proves Theorem 3.4.2 for \( Y = \emptyset \). Assume that the scalar curvature is positive over \( Y \subset X \), so that \( F \) is invertible over \( Y \). If the homotopy \( FJ + JF \) would go through invertible operators over \( Y \), the proof of Theorem 3.4.2 would be complete. However, \( s \mapsto \cos(s)F + \sin(s)J \) is not in general invertible if \( F \) is, and so we have to adjust this homotopy.
Lemma 3.4.4. There exists $C > 0$ with the following property. Let $x \in X$ and $\ell_x := a_1(x) - a_0(x)$ be the length of the straight cylinder in the middle. Then
\[ \|F_x J_x + J_x F_x\| \leq \frac{C}{\ell_x}. \]

Proof. We ease notation by dropping the index $x$. First, compute the anticommutator
\[ DJ + JD = \begin{pmatrix} -\mu' & -\lambda' \\ \mu' & \lambda' \end{pmatrix} e_1, \]
where $e$ denotes Clifford multiplication by the unit vector field $\partial_t$ on $E$. The functions $\mu$ and $\lambda$ have been chosen so that $|\mu'|, |\lambda'| \leq 2/\ell$. Thus $DJ + JD$ is bounded, and $\|DJ + JD\| \leq \frac{2\sqrt{2}}{\ell}$. We write $Z(t) = (1 + D^2 + t^2)^{-1}$ and use the absolutely convergent integral representation (taken from [10])
\[ F = \frac{D}{(1 + D^2)^{1/2}} = \frac{2}{\pi} \int_0^\infty Z(t)Ddt. \]
Hence
\[ FJ + JF = \frac{2}{\pi} \int_0^\infty (Z(t)DJ + JZ(t)D)dt. \tag{3.4.5} \]
Write $P = DJ + JD$ and note that $Z(t)$ and $D$ commute. Thus $ZDJ + JZD = ZP + [J, Z]D$. Moreover, $[J, Z] = Z[Z^{-1}, J]Z$ and $[Z^{-1}, J] = DP - PD$. Altogether, this shows that the integrand can be written as
\[ Z(t)DJ + JZ(t)D = Z(t)P - Z(t)PD^2Z(t) + Z(t)DPZ(t)D. \tag{3.4.6} \]
We have the following estimates:
\[ \|Z(t)\| \leq \frac{1}{1 + t^2}, \]
\[ \|D^2Z(t)\| \leq 1, \]
\[ \|Z(t)D\| \leq \frac{1}{2\sqrt{1 + t^2}}. \]
The first two are clear, and the third follows from $\sup_x \frac{x}{1 + x^2 + t^2} = \frac{1}{2\sqrt{1 + t^2}}$. Therefore, the norm of the operator (3.4.6) is bounded by
\[ \frac{2\sqrt{2}}{\ell} \left( \frac{1}{1 + t^2} + \frac{1}{1 + t^2} + \frac{1}{4(1 + t^2)} \right) = \frac{9\sqrt{2}}{2\ell} \frac{1}{1 + t^2}, \]
and so $\|FJ + JF\| \leq \frac{C}{\ell}$, with $C = \frac{2\sqrt{2}}{\ell}$.

Lemma 3.4.7. ([10]) The anticommutator $FJ + JF$ is compact.

Proof. We use the integral formula (3.4.3), and revert to using the index $x$. The integrand is $(Z_x = Z_x(x))$
\[ Z_x P_x - (Z_x P_x)(D_x^2 Z_x) + Z_x[D_x, P_x]Z_x D_x - (Z_x P_x)(D_x^2 Z_x) \]
(note that $D_x$ and $Z_x$ commute). The operator $Z_x P_x$ is compactly supported (since $P_x$ is) and a pseudo-differential operator of order $-2$, hence compact by Rellich’s theorem. As $D_x^2 Z_x$ is bounded, the first two and the last two are finite. The operator $[D_x, P_x]$ is of order $1$ and has compact support. Therefore $Z_x[D_x, P_x]Z_x D_x$ is a pseudo-differential operator of order $-2$ with compact support, hence compact by Rellich’s theorem. Since the integral (3.4.6) converges absolutely,
By the Schrödinger–Lichnerowicz formula we have
\[ W \text{ as above and let } E \]
and see the discussion in Section 3.1.1. However, by [19, Proposition 3.7], the family \( x \mapsto F_x \) is a Fredholm family, and the proof in loc. cit. shows that \( F \) is locally norm-continuous. But \( J \) is also locally norm-continuous, and so is \( FJ \). Therefore, by the remark in Section 3.1.1, compactness of \( FJ \) follows.

**Proof of Theorem 3.4.2.** Let \( \kappa : Y \to (0, \infty) \) be a lower bound for the scalar curvature of \( E \) and \( E' \), i.e. a function such that for \( y \in Y \)
\[ \kappa(y) \leq \text{scal}(g_y), \text{scal}(g'_y). \]
By the Schrödinger–Lichnerowicz formula we have \( D^2 \geq \frac{\kappa}{4} \). Consider the operator homotopy \( H(s) = \cos(s)F + \sin(s)J \). By Lemma 3.4.4
\[ H(s)^2 = \cos(s)^2F^2 + \sin(s)^2 + \cos(s)\sin(s)(FJ + JF) \geq \cos(s)^2 \frac{\kappa/4}{1 + \kappa/4} + \sin(s)^2 - \frac{C}{2\ell}, \]
we see that \( H(s) \) is invertible for all \( s \in [0, \pi/2] \), provided that
\[ \ell > \frac{C(1 + \kappa/4)}{2\kappa}. \tag{3.4.8} \]
with \( C \) being the constant from Lemma 3.4.4. This proves Theorem 3.4.2 under the additional assumption that the length \( \ell_x \) of the straight cylinder in the middle is long enough to satisfy (3.4.8).

The key observation to treat the general case is now that stretching the cylinder \((a_0, a_1) \) to arbitrarily big length \( \ell \) does not affect the lower bound \( \kappa \) for the scalar curvature (the metric is a product metric on the cylinder, and stretching the cylinder in the \( \mathbb{R} \)-direction does not change the scalar curvature). Here is how the stretching is done. We will simultaneously change the function to \( \mathbb{R} \) and the metric on the common piece \( E(a_0, a_1) = E'(a_0, a_1) \). Let \( b : X \times \mathbb{R} \to [0, \infty) \) be a function (the restriction to \( x \in \mathbb{R} \) should be smooth) with support in \( X \times (a_0, a_1) \). We change the metric \( g = g' \) on \( E(a_0, a_1) = E'(a_0, a_1) \) by adding \( b(x, t)^2 dt^2 \). The straight cylinder now has length \( \int_{a_0}^{a_1} \sqrt{1 + b^2} \right d\tau \). We change the projection maps \( E, E' \to \mathbb{R} \) to \((z \in E, E') \) to
\[ \ell(z) := t(z) + \int_{-\infty}^{t(z)} \sqrt{1 + b^2(\pi(z), \tau)} d\tau. \]
Let \( a'_1 := a_0 + \int_{a_0}^{a_1} \sqrt{1 + b^2(\pi(z), \tau)} d\tau \); with the new metric and the new projection functions, the two bundles are cylindrical over a \( X \times (a_0, a'_1) \). Clearly, the new family with the stretched cylinder is concordant to the original one. If the function \( b \) is picked such that for \( y \in Y \), the inequality
\[ \int_{a_0}^{a_1} \sqrt{1 + b^2(\pi(z), \tau)} d\tau \geq \frac{2C(1 + \kappa(\pi(z))/4)}{2\kappa(\pi(z))} \]
holds, then the homotopy (3.4.3) is through invertible operators over \( Y \). \( \square \)

### 3.4.3. A more useful formulation of the additivity theorem.
We reformulate the additivity theorem slightly, to a form better adapted to our needs. Let \((X, Y) \) be as above and let \( W : M_0 \to M_1 \), \( W' : M_1 \to M_2 \) be \( d \)-dimensional spin cobordisms. Let \( E \to X \) (\( E' \to X \) resp.) be a \( W \)-bundle (\( W' \)-bundle, resp.) with trivialized boundary and a spin structure. Let \( h_i \in R^+(M_i) \) be psc metrics and let \( g \) (\( g' \), resp.) be a fibrewise Riemannian metric on \( W \) (\( W' \), resp.) which coincides over the
Proof. Assume that over \( Y \), the metrics \( g \) and \( g' \) have positive scalar curvature.

**Corollary 3.4.9.** Under the above assumptions, the following holds:

\[
\text{ind}(E, g) + \text{ind}(E', g') = \text{ind}(E \cup X \times M_i, E', g \cup g') \in KO^{-d}(X, Y).
\]

**Proof.** Let \( E_0 := X \times (-\infty, 0] \times M_0 \cup X \times M_0 \), \( E_1 := X \times [1, \infty) \times M_1 \), \( E_2 := X \times (-\infty, 1] \times M_1 \) and \( E_3 := E' \cup X \times M_2 [2, \infty) \times M_2 \), with the metrics extended by cylindrical metrics. By Theorem 3.4.2

\[
\text{ind}(E_{01}) + \text{ind}(E_{23}) + \text{ind}(E_{03}^{\text{op}}) + \text{ind}(E_{21}^{\text{op}}) = 0 \in KO^{-d}(X, Y).
\]

The manifold bundle \( E_{21} \) has positive scalar curvature and thus \( \text{ind}(E_{21}^{\text{op}}) = 0. \) On the other hand, \( \text{ind}(E_{03}^{\text{op}}) = -\text{ind}(E_{03}) \) by Remark 3.2.6. But tracing through the definitions shows that \( \text{ind}(E_{01}) = \text{ind}(E,g) \), \( \text{ind}(E_{23}) = \text{ind}(E', g') \) and \( \text{ind}(E_{03}) = \text{ind}(E \cup X \times M_1, E', g \cup g') \). This completes the proof. \( \square \)

### 3.4.4 Additivity property of the index difference.

Here we show what the additivity theorem yields for the index difference.

**Theorem 3.4.10.** Let \( M_0 \xrightarrow{V} M_1 \xrightarrow{W} M_2 \) be spin cobordisms, \( h_i \in \mathcal{R}^+(M_i) \) be boundary conditions, and \( g \in \mathcal{R}^+(V)_{h_0,h_1} \) and \( m \in \mathcal{R}^+(W)_{h_1,h_2} \) be psc metrics satisfying these boundary conditions. Then the diagram

\[
\begin{array}{ccc}
\mathcal{R}^+(V)_{h_0,h_1} & \xrightarrow{\mu_m} & \mathcal{R}^+(V \cup W)_{h_0,h_2} \\
\text{indiff}_g & \downarrow & \text{indiff}_{g,m} \\
& \Omega^{\infty+d+1}\text{KO} & \\
\end{array}
\]

is homotopy commutative.

**Proof.** Write \( X := \mathcal{R}^+(V)_{h_0,h_1} \). On the space \( X \times \mathbb{I} \), we consider the following trivial fibre bundles equipped with Riemannian metrics:

(i) \( E_{01} \) has fibre \((-\infty, 0] \times M_0 \cup V \cup [1, \infty) \times M_1\); the metric over \((x,t)\) is \( \frac{1-t}{2}g + \frac{t}{1-t}x \), extended to the cylinders by a product.

(ii) \( E_{23} \) has fibre \((-\infty, 1] \times M_1 \cup W \cup [2, \infty) \times M_2\); the metric over \((x,t)\) is \( m \), extended to the cylinders by a product.

These coincide over \([1] \times M_1\). As in Section 3.4, these bundles are partitioned into \( E_i \), \( i = 0, \ldots, 3 \), and \( E_1, E_2, \) and \( E_3 \) have positive scalar curvature. By Corollary 3.4.9 we get a concordance between \( \text{Dir}(E_{01}) \oplus (U, I) \) and \( \text{Dir}(E_{03}) \oplus (U, I) \), so \( \text{ind}(E_{01}) = \text{ind}(E_{03}) \in \Omega KO^{-d}(X) \). But by construction, \( \text{ind}(E_{01}) = \text{indiff}_g \) and \( \text{ind}(E_{03}) = \text{indiff}_{g,m} \), which concludes the proof. \( \square \)

**Remark 3.4.11.** Let \( W^d \) be closed and \( \phi : S^{k-1} \times D^{d-k+1} \rightarrow W \) be an embedding, with \( 3 \leq k \leq d-2 \). We use the notations of the proof of Theorem 2.3.3. There are maps

\[
\mu_{g_{k-1}+g_{d-k+1}} : \mathcal{R}^+(W^g_{k-1}+g_{d-k}) \rightarrow \mathcal{R}^+(W)
\]

and

\[
\mu_{g_{d-k}+g_{d-k}} : \mathcal{R}^+(W^g_{d-k}) \rightarrow \mathcal{R}^+(W'),
\]

and the surgery equivalence map \( SE_{\phi} \) is the composition of a homotopy inverse of the first with the second map. Hence, the index difference satisfies an appropriate cobordism invariance. We leave the precise formulation to the interested reader, as we will not use this fact.
3.4.5. Propagating a detection theorem. We spell out a consequence of Theorem 3.4.10 which is important for the global structure of this paper. If one manages to prove a detection theorem for the index difference for a certain single spin cobordism \( W^d \) (satisfying the conditions listed below), then the same detection theorem follows for all spin manifolds of dimension \( d \).

**Proposition 3.4.12.** Let \( W : \emptyset \to S^{d-1} \) be a simply connected spin cobordism of dimension \( d \geq 6 \), which is spin cobordant to \( D^d \) relative to its boundary. Let \( g \in \mathcal{R}^+(W)_{g_0} \) be in a path component which corresponds to that of \( g_{\text{tor}}^d \in \mathcal{R}^+(D^d)_{g_0} \) under a surgery equivalence \( \mathcal{R}^+(W)_{g_0} \simeq \mathcal{R}^+(D^d)_{g_0} \).

Let \( W' \) be an arbitrary \( d \)-dimensional spin cobordism with boundary \( M' \) and let \( h' \in \mathcal{R}^+(M') \) and \( g' \in \mathcal{R}^+(W')_{h'} \). Let \( X \) be a CW complex, \( \hat{a} : X \to \Omega^{\infty+d+1}KO \) be a map and let a factorisation

\[
X \xrightarrow{\rho'} \mathcal{R}^+(W')_{h'} \xrightarrow{\text{indiff}} \Omega^{\infty+d+1}KO
\]
of \( \hat{a} \) up to homotopy be given. Then there exists another factorisation

\[
X \xrightarrow{\rho} \mathcal{R}^+(W)_{g_0} \xrightarrow{\text{indiff}_{\text{tor}}} \Omega^{\infty+d+1}KO
\]
of \( \hat{a} \) up to homotopy.

**Proof.** Let \( W_0 = W \setminus \text{int}(D^d) \) be \( W \) with an open disc in the interior removed. As \( W \) is simply connected and spin cobordant to \( D^d \) relative to its boundary, it follows that \( W_0 \) is simply connected and spin cobordant to \( [0,1] \times S^{d-1} \) relative to its boundary. Thus by Lemma 3.5 there is a metric \( g_0 \in \mathcal{R}^+(W_0)_{g_0} \) such that the map

\[
\mu_{g_0} : \mathcal{R}^+(D^d)_{g_0} \to \mathcal{R}^+(D^d \cup W_0)_{g_0}
\]

is a weak homotopy equivalence, and \( g^d_{\text{tor}} \cup g_0 \) is isotopic to \( g \). As \( X \) is a CW-complex, there is a map \( \rho' : X \to \mathcal{R}^+(D^d)_{g_0} \) so that \( \mu_{g_0} \circ \rho' \simeq \rho \), and using Theorem 3.4.10 the composition

\[
X \xrightarrow{\rho'} \mathcal{R}^+(D^d)_{g_0} \xrightarrow{\text{indiff}_{\text{tor}}} \Omega^{\infty+d+1}KO
\]
is homotopic to \( \hat{a} \).

Let \( W' \) be a spin cobordism with a psc metric \( g' \) as in the assumption of the proposition. We can change \( g' \) by an isotopy supported in a small disc in the interior so that \( g' \) agrees with \( g^d_{\text{tor}} \) in that disc. This does not change the homotopy class of the index difference, and we can write \( W' = D^d \cup W_0' \) and \( g' = g^d_{\text{tor}} \cup g_1 \). The desired factorisation is

\[
X \xrightarrow{\rho'} \mathcal{R}^+(D^d)_{g_0} \xrightarrow{\mu_{g_1}} \mathcal{R}^+(W')_{g'} \xrightarrow{\text{indiff}_{h'}} \Omega^{\infty+d+1}KO
\]

and the composition is homotopic to \( \hat{a} \), again by Theorem 3.4.10. \( \square \)

3.5. The relative index construction in an abstract setting. For the proof of Theorems B and C we need a precise tool to express secondary indices in terms of a primary index. Our tool is the relative index construction.

Write \( I = [0,1] \). Let \( f : (X, x_0) \to (Y, y_0) \) be a map of pointed spaces (in practice, \( f \) will be a fibration of some kind). Recall the notions of mapping cylinder \( \text{Cyl}(f) := (X \times I) \coprod Y \sim, (x, 1) \sim f(x) \) and homotopy fibre \( \text{hofib}_y(f) = \{(x, c) \in X \times Y | f(x) = c(0), c(1) = y\} \). There is a natural map \( \epsilon_{y_0} : f^{-1}(y_0) \to \text{hofib}_{y_0}(f), x \mapsto (x, c_{y_0}) \), where \( c_{y_0} \) denotes the constant path at \( y_0 \). Let \( (x_0, c_{y_0}) \in \text{hofib}_{y_0}(f) \)
and \( x_0 \in f^{-1}(y_0) \) be the basepoints, so that \( \epsilon_{y_0} \) is a pointed map. There is a natural map
\[
\eta_{y_0} : (I, \{ \pm 1 \}) \times (\text{hofib}_{y_0}(f), *) \to (\text{Cyl}(f), X \cup \{x_0\} \times I)
\]
defined by
\[
(t, x, c) \mapsto \begin{cases} (x, 1 + t) & t \leq 0 \\ c(t) & t \geq 0 \end{cases}
\]
and the composition \( \eta_{y_0} \circ \epsilon_{y_0} \) is homotopic to the map \( \iota_{y_0} : I \times f^{-1}(y_0) \to \text{Cyl}(f) ; (t, x) \mapsto (x, \frac{1+2t}{2}) \) (as maps of pairs). Moreover, there is the fibre transport map from the loop space of \( Y \) based at \( y_0 \) to the homotopy fibre (it is a based map):
\[
\tau : \Omega_{y_0} Y \to \text{hofib}_{y_0}(f)
\]
\[
c \mapsto (x_0, c).
\]
We write \( KO_{p,q}(f) := KO_{p,q}(\text{Cyl}(f), X \cup \{x_0\} \times I \cup \{y_0\}) \). There is an induced map
\[
\text{trg} : KO_{p,q}(f) \to \Omega KO_{p,q}(\text{hofib}_{y_0}(f), *)
\]
\[
\alpha \mapsto \eta_{y_0}^* \alpha,
\]
called the transgression. (Often, we consider the composition \( \epsilon_{y_0}^* \circ \text{trg} \) and call this composition transgression as well.) The inclusion \( i : Y \to \text{Cyl}(f) \) induces
\[
\text{bas} : KO_{p,q}(f) \to KO_{p,q}(Y, y_0)
\]
\[
\alpha \mapsto i^* \alpha.
\]
We call \( \text{bas}(\alpha) \) the base class of the relative class \( \alpha \). Finally, there is a map, the loop map \( \Omega : KO_{p,q}(Y, y_0) \to \Omega KO_{p,q}(\Omega_{y_0} Y) \), given by pulling back by the evaluation map \( I \times \Omega_{y_0} Y \to Y \).

**Lemma 3.5.1.** Let \( \alpha \in KO_{p,q}(f) \) be given. Then
\[
\tau^* \text{trg}(\alpha) = \Omega \text{bas}(\alpha) \in \Omega KO_{p,q}(\Omega_{y_0} Y, *)
\]

**Proof.** The proof is easier to understand than the statement. The diagram
\[
(\mathbb{I}, \{ \pm 1 \}) \times (\Omega_{y_0} Y, *) \xrightarrow{ev} (Y, y_0) \xrightarrow{\tau} (\mathbb{I}, \{ \pm 1 \}) \times (\text{hofib}_{y_0}(f), *) \xrightarrow{\eta_{y_0}} (\text{Cyl}(f), X \cup \{x_0\} \times I)
\]
is homotopy commutative (in space pairs), and hence the associated diagram in \( K \)-theory is commutative. By tracing through the definitions, this commutativity is expressed by the formula in the statement of the lemma. \( \square \)

This lemma has the following homotopy-theoretic interpretation. Recall that we write relative \( K \)-theory classes as homotopy classes of maps, via (3.1.4).

**Corollary 3.5.2.** Let \( \alpha \in KO_{p,q}(f) \) be a relative \( K \)-theory class. Then the diagram
\[
\Omega_{y_0} Y \xrightarrow{\tau} \text{hofib}_{y_0}(f) \xrightarrow{\text{trg}(\alpha)} \Omega^\infty + p - q \text{KO} \xrightarrow{\Omega \text{bas}(\alpha)} \Omega^\infty + p - q + 1 \text{KO}
\]
is homotopy commutative.
We will use this to translate index-theoretic results into homotopy-theoretic conclusions.

### 3.6. Increasing the dimension

The first application of the generalities from Section 3.3 is the derivation of Theorem C from Theorem B (the true ingredients are Theorems 3.3.5, 2.2.7, and the additivity theorem).

**Theorem 3.6.1.** Let $W^d$ be a compact spin manifold with boundary $M$. Let $h_0 \in \mathcal{R}^+(M)$ and $g_0 \in \mathcal{R}^+(W)_{h_0}$. Then there exists a homotopy class of maps $T : \Omega_{h_0} \mathcal{R}^+(M) \to \mathcal{R}^+(W)_{h_0}$ such that the diagram

\[
\begin{array}{ccc}
\Omega_{h_0} \mathcal{R}^+(M) & \xrightarrow{T} & \mathcal{R}^+(W)_{h_0} \\
\ominus\text{inndiff}_{h_0} & \searrow & \ominus\text{inndiff}_{g_0}
\end{array}
\]

is weakly homotopy commutative.

Before we embark on the proof of Theorem 3.6.1 we show how it accomplishes the promised goal.

*Proof of Theorem C, assuming Theorems B and 3.6.1.* In the situation of Theorem 3.6.1 put $W = D^{2n+1}$ and $g_0 := \sigma_{tor}$. Then $h_0 = g_0 \in \mathcal{R}^+(S^{2n})$. Consider the diagram

\[
\begin{array}{ccc}
\Omega_{h_0}^\infty \mathcal{M}_{Spin}(2n) & \xrightarrow{-\Omega_\rho} & \Omega_{h_0} \mathcal{R}^+(S^{2n}) \\
\ominus\text{inndiff}_{h_0} & \searrow & \ominus\text{inndiff}_{g_0}
\end{array}
\]

The map $\Omega_\rho$ is provided by Theorem B, the composition $(-\Omega \text{inndiff}_{h_0}) \circ (\Omega_\rho) \simeq \Omega(\text{inndiff}_{h_0} \circ \rho)$ is weakly homotopic to $\Omega^2 \omega_{S^{2n}}$, also by Theorem B. The triangle is weakly homotopy commutative by Theorem 3.6.1 and $T \circ (-\Omega_\rho)$ is the map whose existence is claimed by Theorem C.

The rest of this section is devoted to the proof of Theorem 3.6.1. Consider the restriction map $\text{res} : \mathcal{R}^+(W) \to \mathcal{R}^+(M)$. We will construct and analyse a relative class $\beta \in KO^{d,0}(\text{res})$.

Metrics of positive scalar curvature on $M$ can be extended to (arbitrary) metrics on $W$, by the following procedure. Pick a cut-off function $a : W \to [0, 1]$ which is supported in the collar around $M$ and is equal to 1 near the boundary. For any $h \in \mathcal{R}^+(M)$, let $\sigma(h) := a(h + dt^2) + (1 - a)g_0$. The result is an extension map $\sigma : \mathcal{R}^+(M) \to \mathcal{R}(W)$ (to $\mathcal{R}(W)$, not to $\mathcal{R}^+(W)$) such that $\sigma(h_0) = g_0$ and such that $\sigma(h)$ restricted to a collar is equal to $h + dt^2$.

Consider the trivial fibre bundle with fibre $W$ over the mapping cylinder $\text{Cyl}(\text{res})$ of the restriction map, and define the following fibrewise Riemannian metric $m$ on it: over the point $h \in \mathcal{R}^+(M) \subset \text{Cyl}(\text{res})$ take the metric $\sigma(h)$, and over the point $(g, t) \in \mathcal{R}^+(W) \times [0, 1]$ take the metric $m_{(g, t)} := t\sigma(\text{res}(g)) + (1 - t)g$. By the construction of $\sigma$,

(i) the metric $m$ is psc when restricted to the boundary bundle $\text{Cyl}(\text{res}) \times M$,
(ii) $m_{g, 0} = g$ for $g \in \mathcal{R}^+(W)$ and
(iii) $m_{g_0, t} = t\sigma(\text{res}(g_0)) + (1 - t)g_0 = g_0$, for $t \in [0, 1]$. 
Therefore, $m$ has positive scalar curvature over $(\mathcal{R}^+(W) \times \{0\}) \cup (\{g_0\} \times I) \subset \text{Cyl}(\text{res})$. We define the class

$$\beta = \text{ind}(W, m) \in KO^{d, 0}(\text{Cyl}(\text{res}), (\mathcal{R}^+(W) \times \{0\}) \cup (\{g_0\} \times I)).$$

The only choice made in defining the class $\beta$ was the section $\sigma$, but this is a convex choice and does not affect $\beta$. However, $\beta$ might depend on $g_0$.

**Proposition 3.6.2.**

(i) Let $\epsilon_{h_0} : \mathcal{R}^+(W)_{\text{ho}} \to \text{hofib}_{h_0}(\text{res})$ be the fibre comparison map. Then $\epsilon_{h_0}^*(\text{trg}(\beta)) : \mathcal{R}^+(W)_{h_0} \to \Omega_{\infty + d + 1}^d KO$ is homotopic to $-\text{inddiff}_{g_0}$.

(ii) $\text{bas}(\beta) : \mathcal{R}^+(M) \to \Omega_{\infty + d} KO$ is weakly homotopic to $\text{inddiff}_{h_0}$.

**Proof.** The first part is straightforward. The class $\epsilon_{h_0}^*(\text{trg}(\beta))$ is represented by the cycle $\epsilon_{h_0}^*\beta$, and $\epsilon_{h_0} : I \times \mathcal{R}^+(W)_{h_0} \to \text{Cyl}(\text{res})$ is the map $(t, g) \mapsto (g, \frac{1 + t}{2})$. So $\epsilon_{h_0}^*\beta$ is the cycle

$$(t, g) \mapsto \frac{1 + t}{2} - \sigma(\text{res}(g)) + \frac{1 - t}{2} - \sigma(h_0) = \frac{1 + t}{2} \sigma(h_0) + \frac{1 - t}{2} \sigma(h_0),$$

because $g \in \mathcal{R}^+(W)_{h_0}$ and $\sigma(h_0) = g_0$, and this represents minus the index difference.

The second part is deeper and uses the additivity theorem and the spectral flow theorem. The base class $\text{bas}(\beta)$ lies in $KO^{d, 0}(\mathcal{R}^+(M))$, and by tracing through the definitions, we find that

$$\text{bas}(\beta) = \text{ind}(W, \sigma(h)),$$

the notation indicates that $\text{bas}(\beta)$ is represented by the cycle whose value at $h \in \mathcal{R}^+(M)$ is the Dirac operator of the metric $\sigma(h)$ on the manifold $W$. For two metrics $h_0, h_1 \in \mathcal{R}^+(M)$, denote by $[h_0, h_1]$ the metric $ds^2 + \varphi(s)h_1 + (1 - \varphi(s))h_0$ on $M \times \mathbb{R}$, where $\varphi : \mathbb{R} \to [0, 1]$ is a smooth function that is 0 near $(-\infty, 0]$ and 1 near $[1, \infty)$. Now we calculate, using the additivity theorem (Corollary 3.4.9), that

$$\text{ind}(W, \sigma(h)) + \text{ind}(M \times \mathbb{R}, [h, h_0]) + \text{ind}(W^{\text{op}}, g_0) = \text{ind}(W \cup M \times [0, 1] \cup W^{\text{op}}, \sigma(h) \cup [h, h_0] \cup g_0) = 0.$$

To see that the index is zero, note first that the manifold $W \cup (M \times [0, 1]) \cup W^{\text{op}}$ is closed and therefore the index on the right hand side does not depend on the choice of the metric. Moreover $g_0 \cup [h_0, h_0] \cup g_0$ is a psc metric on $W \cup (M \times [0, 1]) \cup W^{\text{op}}$, and therefore the right-hand side is zero. On the other hand, the contribution $\text{ind}(W^{\text{op}}, g_0)$ on the left hand side is zero by Bochner’s vanishing argument with the Lichnerowicz–Schrödinger formula, and therefore

$$\text{ind}(W, \sigma(h)) = -\text{ind}(M \times \mathbb{R}, [h, h_0]) = \text{ind}(M \times \mathbb{R}, [h_0, h]).$$

The last equality is true by the additivity theorem and by Lemma 3.2.5. But, by Theorem 3.3.3 the right hand side is equal to $\text{inddiff}_{h_0} \in KO^{d - 1}(\mathcal{R}^+(M))$. □

**Proof of Theorem 3.6.1** Consider the diagram:

$$\begin{array}{ccc}
\mathcal{R}^+(W)_{h_0} & \xrightarrow{\epsilon_{h_0}} & \text{hofib}_{h_0}(\text{res}) \\
\downarrow_{\text{inddiff}_{g_0}} & & \downarrow_{\text{trg}(\beta)} \\
\Omega_{\infty + d + 1} KO & \xrightarrow{\text{inddiff}_{h_0}} & \mathcal{R}^+(M)
\end{array}$$
The left triangle is homotopy commutative by Proposition 3.6.2[1], and the right triangle is homotopy commutative by Proposition 3.6.2[1] and Corollary 3.5.2. The final ingredient is the quasifibration theorem (Theorem 2.2.7): to get the map $T$, take the homotopy inverse of $\epsilon_{ho}$ provided by Theorem 2.2.7. \hfill \Box

Remark 3.6.4. The homotopy inverse to $\epsilon_{ho}$ is given by an explicit construction. This allows a concrete description of the map $T$: one takes a (smooth) curve $p \in \Omega_{ho}\mathcal{R}^+(M)$, forms the trace $F(p) \in \mathcal{R}^+(\{0,1\} \times M)_{ho,ho}$ and glues this trace to the metric $g_0 \in \mathcal{R}^+(W)_{ho}$. We leave the precise formulation to the reader.

3.7. The Atiyah–Singer index theorem. In the case when the spin manifold bundle $\pi : E \to X$ has closed $d$-dimensional fibres, the index $\text{ind}(E) \in K\mathcal{O}^{-d}(X)$ can be expressed in homotopy-theoretic terms using the Clifford-linear version of the Atiyah–Singer family index theorem. We recall the result here.

3.7.1. $KR$-theory and the Atiyah–Bott–Shapiro construction. The formulation of the index theorem which we shall use is in terms of Atiyah’s $KR$-theory [1]. Let $Y$ be a locally compact space with an involution $\tau$ (a “Real space” in the terminology of [4]). Classes in the \textit{compactly supported} Real $K$-theory $KR_c(Y)$ are represented by triples $(E,\iota,f)$, with $E \to Y$ a finite-dimensional “Real vector bundle”, $\iota$ a $\mathbb{Z}/2$-grading on $E$ and $f : E \to E$ an odd, self-adjoint bundle map which is an isomorphism outside a compact set. Both $\iota$ and $f$ are required to be compatible with the Real structure. Out of the set of concordance classes of triples $[E,\iota,f]$, one defines $KR_c(Y)$ by a procedure completely analogous to that of Definition 3.1.1. No danger of confusion arises by writing $KR(Y) = KR_c(Y)$.

Recall the Atiyah–Bott–Shapiro construction [3]: let $X$ be a compact space, $V,W \to X$ be Riemannian vector bundles and $p : V \oplus W \to X$ be their sum. Let $V^+ \oplus W^-$ be the total space of $V \oplus W$, equipped with the involution $(x,y) \mapsto (x,-y)$. Let $E$ be a real $\text{Cl}(V^+ \oplus W^-)$-module. Then the triple $(p^*E,\iota,\gamma)$ with $\gamma_{v,w} := \iota(c(v) + ic(w))$ represents a class $\text{abs}(E) \in KR(V^+ \oplus W^-)$, the Atiyah–Bott–Shapiro class of $E$.

If $V \to X$ is a rank $d$ spin vector bundle with spin structure $\mathfrak{S}_V$, then the $KO$-theory Thom class of $V$ is defined to be

$$\lambda_V := \text{abs}(\mathfrak{S}_V) \in KR(V^+ \oplus \mathbb{R}^{0,d}) \cong KO^d(V) \cong KO^{-d}(\text{Th}(V)),$$

where the first isomorphism is proved in [1] §2-3 (and the second one is a standard property of compactly supported $K$-theory).

3.7.2. The Clifford-linear index theorem. Let $\pi : E \to X$ be a spin manifold bundle with closed $d$-dimensional fibres over a compact base space. Assume that $E \subset X \times \mathbb{R}^n$ and that $\pi$ is the projection to the second factor. Let $N_vE$ be the vertical normal bundle, so that $T_vE \oplus N_vE = E \times \mathbb{R}^n$ and let $U \subset X \times \mathbb{R}^n$ be a tubular neighborhood of $E$. The spin structure on $T_vE$ induces a spin structure on $N_vE$. The open inclusion $N_vE \cong U \subset X \times \mathbb{R}^n$ induces a map

$$\psi : KR(N_vE^+ \oplus \mathbb{R}^{0,n}) \to KR(X \times \mathbb{R}^{n,n}) \cong KR(X);$$

the last is the $(1,1)$-periodicity isomorphism [3] Theorem 2.3]. The Clifford linear index theorem in is

\textbf{Theorem 3.7.2.} With the notation introduced above, $\text{ind}(E) = \psi(\lambda_{N_vE})$. 

This result is due to Hitchin [27, Proposition 4.2] and is of course a consequence of the Atiyah–Singer index theorem for families of real elliptic operators [8]. However, Hitchin’s explanation of the argument (as well as the treatment in [34, §16]) leaves out a critical detail. For sake of completeness, we discuss this detail.

Recall that $\mathcal{D}_{T,E}$ has an action $c$ of the Clifford algebra $\mathbb{C}l^{d,0}$. The Dirac operator $\mathbb{D}_x$ on $E_x$ anticommutes with the self-adjoint bundle endomorphism $ic(v)$, for each $(x,v) \in X \times \mathbb{R}^{d,0}$. Using the isomorphism of Theorem 3.13, the family of Dirac operators defines a family

$$R_{x,v} := \frac{\mathbb{D}_x}{(1 + \mathbb{D}_x^2)^{1/2}} + tv$$

over $X \times \mathbb{R}^d$ which is invertible unless $v = 0$ and so gives an element in the group $KO^0(X \times D^d, X \times S^{d-1})$. Hitchin uses the real family index theorem [8] to compute the index of the family $(D_{x,v})_{(x,v) \in X \times \mathbb{R}^{d,0}}$. However, the family index theorem as proven in [8] only compares elements in absolute $K$-theory, not in relative $K$-theory, and it is not completely evident how the topological index of the family $(D_{x,v})$ (as an element in relative $K$-theory) is defined. Therefore, a slight extension of the real family index theorem is necessary.

**Theorem 3.7.3.** Let $\pi : E \subseteq X \times \mathbb{R}^n \to X$ be a closed manifold bundle over a compact base. Let $Q : \Gamma(E;V_0) \to \Gamma(E;V_1)$ be a family of real elliptic pseudo-differential operators of order $0$. Assume that $Y \subseteq X$ is a closed subspace such that for $y \in Y$, the operator $Q_y$ is a bundle isomorphism. The symbol class $\text{smb}(D)$ is then an element in $KR(T_\pi(E_{X-Y}^-))$ and the family index $\text{ind}(Q) \in KO(X,Y) = KO(X-Y)$ can be computed as the image of $\text{smb}(D)$ under the composition

$$KR(T_\pi(E_{X-Y}^-)) \to KR(T_\pi(E_{X-Y}^-) \oplus N_v(E_{X-Y}^-) \oplus N_v(E_{X-Y}^+)) = KO^n(N_v,E_{X-Y}) \to KO^n((X-Y) \times \mathbb{R}^n) \cong KO(X-Y)$$

of the Thom isomorphism, the map induced by the inclusion and Bott periodicity.

**Proof.** For $Y = \emptyset$, this is precisely the Atiyah–Singer family index theorem [8]. In the general case, form $Z = X \cup_Y X$, $E' = E \cup_{E|_0} E$. Since $Q$ is a bundle isomorphism over $Y$, we can form the clutching $V_0' = V_0 \cup_Y V_1$ and $V_1' = V_1 \cup_Y V_0$ over $E'$ and extend the family $Q$ by the identity over the second copy $X_2$ of $X$. We obtain a new family $P$, which coincides with $Q$ over the first copy $X_1$ of $X$ and is a bundle isomorphism over $X_2$. Therefore, under the map $KO(X,Y) \cong KO(Z,X_2) \to KO(Z)$, the topological (resp. analytical) index of $Q$ is mapped to the topological (resp. analytical) index of $P$. By the index theorem, the topological and analytical indices of $P$ agree. Finally, as the inclusion $X_2 \to Z$ splits, the map $KO(Z,X_2) \to KO(Z)$ is injective, and the indices agree in $KO(X,Y) = KO(Z,X_2)$.

**Proof of Theorem 3.7.2** We have to compute the family index of the family $R_{x,v} := \frac{\mathbb{D}_x}{(1 + \mathbb{D}_x^2)^{1/2}} + tv$ over $X \times D^d$, as an element in $KO(X \times (D^d, S^{d-1}))$. Consider the homotopy $(t \in [0,1])$

$$R_{t,x,v} := \sqrt{1 - t^2|v|^2} \frac{\mathbb{D}_x}{(1 + \mathbb{D}_x^2)^{1/2}} + tv$$
of families of order 0 pseudo-differential operators. Because

\[ R^2_{t,x,v} = (1 - t^2|v|^2) \frac{\partial_x^2}{1 + \partial_x^2} + |v|^2, \]

we find that \( R_{t,x,v} \) is invertible for \( |v| = 1 \). If \( \xi \in T_v E \) has norm 1, then

\[ \text{smbr}_{R^2_{t,x,v}}(\xi) = (1 - t^2|v|^2) \frac{|\xi|^2}{1 + |\xi|^2} + |v|^2 \]

and this is invertible for all \( x, v, \) and \( t \), so the family \( R_{t,x,v} \) is elliptic. Thus we can replace the original family \( R_{x,v} = R_{0,x,v} \) by \( R_{1,x,v} \), which over \( X \times S^{d-1} \) is just the family \( \varphi \) of bundle automorphisms. Thus the relative family index theorem (Theorem 3.7.3) applies. The computation of the topological index of this family is done in the proof of [27, Proposition 4.2].

3.8. **Fibre bundles, Madsen–Tillmann–Weiss spectra, and index theory.** We can reformulate the Atiyah–Singer index theorem in terms of the Madsen–Tillmann–Weiss spectrum \( \text{MTSpin}(d) \). The basic reference for these spectra is [22] and the connection to the index theorem was pointed out in [17]. We refer to these papers for more details. Later we present the variation for manifolds with boundary, and finally relate the index difference to the family index.

3.8.1. **Madsen–Tillmann–Weiss spectra.** Let \( \gamma_d \rightarrow B\text{Spin}(d) \) be the universal spin vector bundle. By definition, the Madsen–Tillmann-Weiss spectrum \( \text{MTSpin}(d) \) is the Thom spectrum of the additive inverse of \( \gamma_d \), which may be described concretely as follows. For \( n \in \mathbb{N} \), we let

\[ \text{Gr}_{d,n}^{\text{Spin}} := \frac{\text{Spin}(n)}{\text{Spin}(d) \times \text{Spin}(n - d)} \]

be the spin Grassmannian manifold. There are tautological bundles

\[ V_{d,n} = \text{Spin}(n) \times \text{Spin}(d) \mathbb{R}^d \subset \text{Gr}_{d,n}^{\text{Spin}} \times \mathbb{R}^n \quad \text{and} \quad V_{d,n}^{\perp} \subset \text{Gr}_{d,n}^{\text{Spin}} \times \mathbb{R}^n. \]

By stabilizing with respect to \( n \), one obtains structure maps

\[ \Sigma \text{Th}(V_{d,n}^{\perp}) \rightarrow \text{Th}(V_{d,n+1}^{\perp}), \]

and by definition the sequence of these spaces form the spectrum \( \text{MTSpin}(d) \). Adjoining these structure maps, we obtain

\[ \Omega^\infty \text{Th}(V_{d,n}^{\perp}) \rightarrow \Omega^{n+1} \text{Th}(V_{d,n+1}^{\perp}), \]

and by definition \( \Omega^\infty \text{MTSpin}(d) \) is the colimit over these maps. Similarly the space \( \Omega^{\infty+k} \text{MTSpin}(d) \) is the colimit of the \( \Omega^{n+k} \text{Th}(V_{d,n}^{\perp}) \), for \( k \in \mathbb{Z} \). There are maps \( V_{d-1,n-1}^{\perp} \rightarrow V_{d,n}^{\perp} \) which are compatible with the structure maps of the spectra and give a spectrum map \( \text{MTSpin}(d-1) \rightarrow \Sigma \text{MTSpin}(d) \), which on infinite loop spaces induces a map \( \Omega^\infty \text{MTSpin}(d-1) \rightarrow \Omega^{\infty-1} \text{MTSpin}(d) \).

3.8.2. **Spin fibre bundles and the Pontrjagin–Thom construction.** A bundle \( \pi : E \subset X \times \mathbb{R}^n \rightarrow X \) of \( d \)-dimensional closed spin manifolds has a Pontrjagin–Thom map

\[ \alpha_E : X \rightarrow \Omega^\omega \text{Th}(V_{d,n}^{\perp}) \rightarrow \Omega^\infty \text{MTSpin}(d) \]

whose homotopy class does not depend on the embedding of \( E \) into \( X \times \mathbb{R}^n \). We also need a version for manifolds with boundary. Let \( W^d \) be a manifold with boundary \( M \), and consider fibre bundles \( E \rightarrow X \) with structure group \( \text{Diff}_\partial(W) \),
that \((E, \partial E)\) is constant over connected, and let \(\pi : E \to X\) be a smooth fibre bundle with fibre \(W\) and trivialised boundary bundle, \(\partial E \cong X \times M\). For each spin structure on \(W\) there is a unique spin structure on \(T_xE\) which is isomorphic to the given one on the fibre and which is constant over \(\partial E\).

**Lemma 3.8.1.** Let \(W\) be a manifold with boundary \(M\) such that \((W, M)\) is 1-connected, and let \(\pi : E \to X\) be a smooth fibre bundle with fibre \(W\) and trivialised boundary bundle, \(\partial E \cong X \times M\). For each spin structure on \(W\) there is a unique spin structure on \(T_xE\) which is isomorphic to the given one on the fibre and which is constant over \(\partial E\).

**Proof.** The Leray–Serre spectral sequence for the fibration pair \((E, \partial E) \to X\) proves that \((E, \partial E)\) is homologically 1-connected and that the fibre inclusion \((W, M) \to (E, \partial E)\) induces an injection \(H^2(E, \partial E; \mathbb{Z}/2) \to H^2(W, M; \mathbb{Z}/2)\). The obstruction to extending the spin structure on \(\partial E\) to all of \(E\) lies in \(H^2(E, \partial E; \mathbb{Z}/2)\) and goes to zero in \(H^2(W, M; \mathbb{Z}/2)\) since \(W\) is assumed to be spin and the spin structure on \(M\) is assumed to extend over \(W\). Thus the obstruction is trivial, which shows the existence of the spin structure. Uniqueness follows from \(H^1(E, \partial E; \mathbb{Z}/2) = 0\). \(\square\)

A closed spin manifold \(M^{d-1}\) determines a point \([M] \in \Omega^{\infty-1}\text{MTSpin}(d)\), namely the image under

\[
\Omega^\infty\text{MTSpin}(d-1) \to \Omega^{\infty-1}\text{MTSpin}(d)
\]

of the point in \(\Omega^\infty\text{MTSpin}(d-1)\) determined by the Pontrjagin–Thom map of the trivial bundle \(M \to *\). Of course, \([M]\) is not unique, but depends on an embedding of \(M\) into \(\mathbb{R}^\infty\) and a tubular neighbourhood, which is a contractible choice.

If \(W\) is a \(d\)-dimensional manifold with boundary \(M\), and \(\pi : E \to X\) is a smooth family of manifolds with fibre \(W\) equipped with a trivialisation \(\partial E \cong X \times M\) of the boundary and a fibrewise spin structure which is constant along the boundary, then there is a Pontrjagin–Thom map

\[
\alpha_E : X \to \Omega_{\emptyset, [M]} \Omega^{\infty-1}\text{MTSpin}(d)
\]

to the space of paths in \(\Omega^{\infty-1}\text{MTSpin}(d)\) from the basepoint \(\emptyset\) to \([M]\). A spin nullbordism \(V : M \to \emptyset\) determines a path from \([M]\) to \(\emptyset\) and thus a homotopy equivalence

\[
\theta_V : \Omega_{\emptyset, [M]} \Omega^{\infty-1}\text{MTSpin}(d) \simeq \Omega^\infty\text{MTSpin}(d)
\]

and we put \(\alpha_{E,V} := \theta_V \circ \alpha_E : X \to \Omega^\infty\text{MTSpin}(d)\). For two nullbordisms \(V_0\) and \(V_1\), the maps \(\alpha_{E,V_0}\) and \(\alpha_{E,V_1}\) differ by loop addition with the constant map \(\alpha_{V_0 \cup V_1}\). We will typically take \(V = W^{op}\).

3.8.3. **The index theorem and homotopy theory.** The vector bundles \(V_{d,n}^\perp \to \text{Gr}_{d,n}^{\text{Spin}}\) have spin structures, so have \(\text{KO}\)-theory Thom classes \(\lambda_{V_{d,n}^\perp} \in \text{KO}^{n-d}(\text{Th}(V_{d,n}^\perp))\).

These fit together to a spectrum \(\text{KO}\)-theory class \(\lambda_{-d} \in \text{KO}^{-d}(\text{MTSpin}(d))\), alias a spectrum map

\[
\lambda_{-d} : \text{MTSpin}(d) \to \Sigma^{-d}\text{KO}.
\]

The infinite loop map of \(\lambda_{-d}\) is denoted \(\nu_d := \Omega^\infty \lambda_{-d} : \text{MTSpin}(d) \to \Omega^{\infty+d}\text{KO}\).

With these definitions, we arrive at the following version of the index theorem.
Theorem 3.8.2. Let $\pi : E \to X$ be a bundle of closed $d$-dimensional spin manifolds. Then the maps

$$\text{ind}(E, \Omega^\infty \lambda_{-d}) \circ \alpha_E : X \to \Omega^{\infty + d}KO$$

are weakly homotopic.

The translation of Theorem 3.7.2 into Theorem 3.8.2 is exactly parallel to the translation of the complex family index theorem described in [17].

For manifold bundles with nonempty boundary, we need a psc metric $h \in \mathcal{R}^+(M)$ to be able to talk about $\text{ind}(E, h) := \text{ind}(E, dt^2 + h) \in KO^{-d}(X)$. To express the index in this situation in terms of homotopy theory, an additional hypothesis on $h$ is needed.

Theorem 3.8.3. Let $\pi : E \to X$ be a bundle of $d$-dimensional spin manifolds, with trivialised boundary $X \times M$. Let $h \in \mathcal{R}^+(M)$ and let $V : M \to \emptyset$ be a nullbordism which carries a psc metric $g \in \mathcal{R}^+(V)$. Then the maps

$$\text{ind}(E, h, \Omega^\infty \lambda_{-d}) \circ \alpha_{E,V} : X \to \Omega^{\infty + d}KO$$

are weakly homotopic.

Proof. By the additivity theorem (Corollary 3.4.9), $\text{ind}(E, h) = \text{ind}(E \cup_{\partial E} (X \times V))$. The result follows in a straightforward manner from Theorem 3.8.2, by the definition of $\alpha_{E,V}$ as $\alpha_{E \cup_{\partial E} (X \times V)}$. \qed

In most cases of interest to us, we will be able to take $V = W^{op}$, where $W$ is a fibre of $E$.

3.8.4. Spin fibre bundles and the index difference. We now show how to fit the index difference into this context. Let $W$ be a $d$-dimensional manifold with boundary $M$, and let $\pi : E \to X$ be a smooth fibre bundle with structure group $\text{Diff}_0(W)$ and fibre $W$, and with underlying $\text{Diff}_0(W)$-principal bundle $Q \to X$. We assume that $X$ is paracompact. By Lemma 3.8.1, there is a spin structure on the vertical tangent bundle $T_{\pi}E \to E$, which is constant along $\partial E = X \times M \subset E$. Let $h_0 \in \mathcal{R}^+(M)$ be fixed, and consider the induced fibre bundle

$$p : Q \times_{\text{Diff}_0(W)} \mathcal{R}^+(W)_{h_0} \to X.$$ 

Choose a basepoint $x_0 \in X$ and identify $\pi^{-1}(x_0)$ with $W$. Then $p^{-1}(x_0)$ may be identified with $\mathcal{R}^+(W)_{h_0}$ and we also choose a basepoint $g_0 \in p^{-1}(x_0)$.

We will now introduce an element $\beta \in KO^{d,0}(p)$, depending only on the bundle $\pi$ and the metrics $g_0$ and $h_0$. To begin the construction of $p$, choose a fibrewise Riemannian metric $k$ on the fibre bundle $\pi : E \to X$, such that

(i) on $\pi^{-1}(x_0) = W$, the metric $k$ is equal to $g_0$,

(ii) near $\partial E$, $k$ has a product structure and the restriction to $\partial E$ is equal to $h_0$.

It is easy to produce such a metric using a partition of unity, which of course will typically not have positive scalar curvature. Now let $\tilde{E} \to \text{Cyl}(p)$ be the pullback of the bundle $\pi$ along the natural map $\text{Cyl}(p) \to X$. Observe that a point in $Q \times_{\text{Diff}_0(W)} \mathcal{R}^+(W)_{h_0}$ is a pair $(x, g)$, where $x \in X$ and $g$ is a psc metric on $\pi^{-1}(x)$ which is equal to $h_0$ on the boundary. The bundle $\tilde{E}$ has the following fibrewise metric: over a point $x \in X \subset \text{Cyl}(p)$, we take the metric $k_x$, and over a point $(x, g, t) \in Q \times_{\text{Diff}_0(W)} \mathcal{R}^+(W)_{h_0} \times [0, 1]$, we take the metric $(1 - t)g + tk_x$. This metric restricts to $h_0$ on $\partial \tilde{E}$ and it has psc if $t = 0$ and over $x_0 \in X \subset \text{Cyl}(p)$.
Since $E$ and hence $\hat{E}$ is spin, there is a Dirac operator for this metric, so a well-defined element $\beta \in KO^{d,0}(p)$ (defined using the elongation of the bundle $\hat{E}$). The following properties of this construction are immediately verified.

**Proposition 3.8.4.**

(i) The base class of $\beta$ is the usual family index of $E$, with the metric $k$: that is $\text{bas}(\beta) = \text{ind}(E, k) \in KO^d(X)$.

(ii) The transgression of $\beta$ to $p^{-1}(x_0) = \mathcal{R}^+(W)_{h_0}$ is the index difference class $\text{inddiff}_{g_0} \in \Omega KO^{d,0}(\mathcal{R}^+(W)_{h_0})$.

**3.8.5. Informal remarks.** Using Corollary 3.5.2 we obtain a homotopy commutative diagram

$$
\Omega_x \xrightarrow{\tau} \mathcal{R}^+(W)_{h_0} \xrightarrow{\sim} \text{hofib}_{x_0}(p) \xrightarrow{\text{inddiff}_{g_0}} \Omega^{\infty+d+1} KO.
$$

By itself, the power of this construction is quite limited. For example, in the universal case $X = BDiff(W)$ the fibre transport map can be identified, up to homotopy, with the orbit map $Diff(W) \to \mathcal{R}^+(W)_{h_0}$ at $g_0$. Thus the above diagram translates to

$$
\xymatrix{ Diff(W) \ar[d]^\tau \ar[r] & \mathcal{R}^+(W)_{h_0} \ar[r]^{\text{inddiff}_{g_0}} & \Omega^{\infty+d+1} KO. }
$$

which expresses a conceptually pleasing property of the index difference, but without knowledge about the homotopy type of $Diff(W, \partial W)$, it cannot be used to explore the homotopy type of $\mathcal{R}^+(W)_{h_0}$. Hitchin [27, §4.4] used this diagram and deep results about $\pi_*(Diff(S^n))$ for the proof of his detection result. One of the pillars of the proof of Theorem B is a less direct application of Proposition 3.8.4.

4. PROOF OF THE MAIN RESULTS

In this section, we will prove Theorem B. Before beginning the proof in earnest, we will establish a result (Theorem 4.1.1) which will be a fundamental tool in the proof, but is also of independent interest. For expositional purposes, we have structured the proof of Theorem B into three parts. In Section 4.2., the main constructive argument of the paper is carried out, stated as Theorem 4.2.4. This theorem assumes the existence of a space $X$ which can be approximated homologically by $\text{hocolim}_k BDiff(W_k)$, where $W_k$ is a certain sequence of spin cobordisms. Out of this data, we produce a map $\Omega X \to \mathcal{R}^+(W_k)$. Sections 4.3 and 4.4. provide the data assumed for Theorem 4.2.4. In Section 4.3, we give the general framework and finish the proof in the case $n = 3$ (which is done by directly quoting [21]). Section 4.4 deals with the general case, which is more difficult and uses [20].

In order to avoid some homotopy-theoretic irritations later, regarding weak and strong homotopy equivalences, in this section we shall occasionally replace the spaces we are interested in by CW-complexes, as follows. For a manifold $V$ with
collared boundary $M$ and a psc metric $h$ on $M$, we let

$$\text{Diff}_\beta(V) := |\text{Sing}_v \text{Diff}_\beta(V)|$$
$$\mathcal{R}^+(V)_h := |\text{Sing}_v \mathcal{R}^+(V)_h|$$

be the geometric realisations of the associated singular simplicial sets. There is an action of $\text{Diff}_\beta(V)$ on $\mathcal{R}^+(V)_h$, induced by the action of $\text{Diff}_\beta(V)$ on $\mathcal{R}^+(V)_h$.

### 4.1. Action of the diffeomorphism group on the space of psc metrics

The right action of $\text{Diff}_\beta(V)$ on $\mathcal{R}^+(V)_h$ induces a homomorphism

$$\Gamma(V) := \pi_0(\text{Diff}_\beta(V)) \to \pi_0(\text{Aut}(\mathcal{R}^+(V)_h)), \quad (4.1.1)$$

to the group of homotopy classes of self homotopy equivalences of the space $\mathcal{R}^+(V)_h$.

#### Theorem 4.1.2

Let $V : S^{d-1} \to \emptyset$ be a simply connected spin bordism of dimension $d \geq 6$, which is in turn spin cobordant to $D^d : S^{d-1} \to \emptyset$ relative to its boundary. Then for $h = g_0^{d-1}$ the image of the homomorphism $(4.1.1)$ is an abelian group.

The conclusion can also be expressed by saying that the action in the homotopy category of $\Gamma(V)$ on $\mathcal{R}^+(V)_{g_0^{d-1}}$ is through an abelian group.

The proof of this theorem will be an application of the cobordism invariance of spaces of psc metrics (Theorem 2.3.1) as well as a formal argument of Eckmann-Hilton flavour. In order to apply this formal argument, we must first endow certain spaces of psc metrics with multiplicative structures.

#### 4.1.1. An $H$-space structure

Let $W : M \to N$ be a $d$-dimensional cobordism, $h_0 \in \mathcal{R}^+(M)$ and $h_1 \in \mathcal{R}^+(N)$. We will define the structure of an $H$-space on $\mathcal{R}^+(M \times [0,1])_{h_0,h_1}$ and a left action of this $H$-space on $\mathcal{R}^+(W)_{h_0,h_1}$. Rather than doing this directly it is convenient to pass to a homotopy equivalent model where we can construct a strictly associative multiplication.

#### Definition 4.1.3

Let

$$\mathcal{R}^+(M \times (-\infty,0])_{h_0,h_0} := \text{colim}_{b \to -\infty} \mathcal{R}^+(M \times [b,0])_{h_0,h_0},$$

where the colimit is formed by extending by the product metric $h_0 + dt^2$. This is the space of all psc metrics $g$ on $M \times (-\infty,0]$ which are of the form $h_0 + dt^2$ near $M \times 0$ and near $M \times (-\infty, b]$ for some $b \leq 0$.

In a similar way, let

$$\mathcal{R}^+(M \times (-\infty,0]) \cup_M W)_{h_0,h_1} := \text{colim}_{b \to -\infty} \mathcal{R}^+(M \times [b,0]) \cup W)_{h_0,h_1}$$

be the space of all psc metrics on the one-sided elongation of $W$ which are equal to $h_0 + dt^2$ near $M \times (-\infty,b]$ for some $b \leq 0$ and equal to $h_1 + dt^2$ near $N$.

#### Definition 4.1.4

Let

$$\mathfrak{R}^+(M \times (-\infty,0], h_0) \subset \mathcal{R}^+(M \times (-\infty,0])_{h_0,h_0} \times (-\infty,0)$$

be the subspace of pairs $(g, a)$ such that $g$ coincides with $h_0 + dt^2$ near $M \times (-\infty, a]$. We define a composition law

$$\rho((g, a), (g', a')) := (g \cdot g', a + a'),$$

where the metric $g \cdot g'$ agrees with $t_{-a}^- g$ on $M \times (-\infty, a']$ and with $g'$ on $M \times [a', \infty)$, and we write $t_b$ for the translation map $(x, s) \mapsto (x, s + b)$ on $M \times \mathbb{R}$. 
In a similar way, let 
\[ \mathcal{R}^+(W, h_0, h_1) \subset \mathcal{R}((M \times (-\infty, 0]) \cup_M W) \times (-\infty, 0] \]
be the subspace of pairs \((g, a)\) such that \(g\) coincides with \(h_0 + dt^2\) near \((M \times (-\infty, a])\), and with \(h_1\) along the other boundary \(N\). We define a left action by
\[ \nu : \mathcal{R}^+(M \times (-\infty, 0], h_0) \times \mathcal{R}^+(W, h_0, h_1) \to \mathcal{R}^+(W, h_0, h_1) \]
\[ ((g, a), (g', a')) \mapsto (g \circ g', a + a') \]
where \(g \circ g'\) agrees with \(g'\) on \((M \times [a', 0]) \cup_M W\) and with \(t^*_{a}g\) on \((M \times (-\infty, a']\).

The following lemma is immediate from the formulae for \(\rho\) and \(\nu\).

**Lemma 4.1.5.** The multiplication \(\rho\) is strictly associative and homotopy unital. The left action \(\nu\) is strictly associative and homotopy unital.

There is an inclusion
\[ i : \mathcal{R}^+(M \times [-1, 0])_{h_0, h_0} \to \mathcal{R}^+(M \times (-\infty, 0], h_0) \quad (4.1.6) \]
given by \(g \mapsto (\bar{g}, -1)\), where \(\bar{g}\) denotes the metric on \(M \times \mathbb{R}\) which is \(g\) on \(M \times [-1, 0]\) and \(h + dt^2\) outside of it. Similarly, there is an inclusion
\[ i : \mathcal{R}^+(W)_{h_0, h_1} \to \mathcal{R}^+(W, h_0, h_1) \quad (4.1.7) \]
given by \(g \mapsto (\bar{g}, 0)\), where \(\bar{g}\) denotes the metric on \((M \times (-\infty, 0]) \cup_M W\) given by \(g\) on \(W\) and \(h_0 + dt^2\) on \(M \times (-\infty, 0]\).

We decompose the sphere \(S^{d-1}\) as \(S^{d-1} = U_+ \cup U_-\), where \(U_+\) are discs and \(U_+ \cap U_- \cong S^{d-2} \times [0, 1]\), and choose a double torpedo metric \(h\) on \(S^{d-1}\) so that \(h|_{U_\pm} = g_{\text{tor}}^{d-1}\), and \(h|_{U_+ \cap U_-}\) is a product metric. For each choice of sign \(\pm\), we let
\[ \mathcal{R}^+(S^{d-1} \times (-\infty, 0], h_0)_{\pm} \subset \mathcal{R}^+(S^{d-1} \times (-\infty, 0], h_0) \]
denote the subspace of those metrics which are of the form \(g_{\text{tor}}^{d-1} + dt^2\) on \(U_\pm \times \mathbb{R}\). One could call \(\mathcal{R}^+(S^{d-1} \times (-\infty, 0], h_\pm)\) the space of metrics having support in \(U_\pm \times \mathbb{R}\) (where the “support” of a metric is the region in which it differs from the standard metric \(h_0 + dt^2\)). If we let \(\phi_{\pm} : D^{d-1} \times [0, 1] \hookrightarrow S^{d-1} \times [0, 1]\) be the embedding obtained by identifying \(U_\pm\) with \(D^{d-1}\), then the inclusion (4.1.6) restricts to an inclusion
\[ \mathcal{R}^+(S^{d-1} \times [-1, 0]; \phi_{\pm}, g_{\text{tor}}^{d-1})_{h_0} \to \mathcal{R}^+(S^{d-1} \times (-\infty, 0], h_0)_{\pm}. \quad (4.1.8) \]

**Lemma 4.1.9.** The inclusions (4.1.6), (4.1.7) and (4.1.8) are weak homotopy equivalences.

**Proof.** We give the argument for (4.1.6); it is completely analogous for (4.1.7), and for (4.1.8) we explain the necessary minor modification at the end of the proof. Consider the composition
\[ \mathcal{R}^+(M \times [-1, 0])_{h_0, h_0} \to \mathcal{R}^+(M \times (-\infty, 0], h_0) \to \mathcal{R}^+(M \times (-\infty, 0], h_0) \]
where the map \(F\) forgets the \(\mathbb{R}\)-coordinate. The composition is the inclusion of \(\mathcal{R}^+(M \times [-1, 0])_{h_0, h_0}\) into the colimit and by Lemma (2.2.1) it is a weak homotopy equivalence, so it is enough to prove that the map \(F\) is a weak equivalence. Note that the fibres of \(F\) are nonempty intervals in \(\mathbb{R}\) and hence convex. On the other hand, for each \(b > 0\), there is a cross section of \(F\) defined on the set \(R_b := \mathcal{R}^+(M \times [-b, 0])_{h_0, h_0}\) (take the constant value \(b\) in the \(\mathbb{R}\)-coordinate). Therefore, \(F|_{F^{-1}(R_b)} : F^{-1}(R_b) \to R_b\) is a homotopy equivalence. By a well-known property of the colimit topology
as well as unit and commutativity homotopies. Let $G$

Proof of Theorem 4.1.2. To see that the analogue of the composition above is a weak equivalence in the case (4.1.8), one simply has to observe that the homotopies used for the proof in the case (4.1.6) restrict to homotopies of the corresponding subspaces for (4.1.8). □

The essential property of this construction is that $\rho$ is, rather surprisingly, a homotopy commutative multiplication when $N$ is a sphere and $h_0$ is a double torpedo metric. This may be thought of as having a similar flavour to the recent theorem of Walsh [54] showing that $\mathcal{R}^+(S^d)$ has an action of the little $d$-discs operad.

**Proposition 4.1.10.** The multiplication $\rho$ on $\mathcal{R}^+(S^{d-1} \times (-\infty, 0], g_{\text{dtor}}^{d-1})$ is weakly homotopy commutative.

**Proof.** It is enough to prove the following two claims:

(i) Elements of $\mathcal{R}^+(S^{d-1} \times (-\infty, 0], g_{\text{dtor}}^{d-1})_+$ commute up to homotopy with elements of $\mathcal{R}^+(S^{d-1} \times (-\infty, 0], g_{\text{dtor}}^{d-1})_-$, in a natural way, i.e. the diagram

\[
\begin{array}{ccc}
\mathcal{R}(S^{d-1} \times (-\infty, 0], g_{\text{dtor}}^{d-1})_+ & \xrightarrow{\rho} & \mathcal{R}(S^{d-1} \times (-\infty, 0], g_{\text{dtor}}^{d-1})_-\\
\downarrow & & \downarrow \\
\mathcal{R}(S^{d-1} \times (-\infty, 0], g_{\text{dtor}}^{d-1})_- & \xrightarrow{\rho} & \mathcal{R}(S^{d-1} \times (-\infty, 0], g_{\text{dtor}}^{d-1})_+
\end{array}
\]

commutes up to homotopy.

(ii) The inclusions $\mathcal{R}^+(S^{d-1} \times (-\infty, 0], g_{\text{dtor}}^{d-1})_\pm \to \mathcal{R}^+(S^{d-1} \times (-\infty, 0], g_{\text{dtor}}^{d-1})$ are weak homotopy equivalences.

For the claim (i), note that for $(g_\pm, a_\pm) \in \mathcal{R}^+(S^{d-1} \times (-\infty, 1], g_{\text{dtor}}^{d-1})_\pm$ the formula

\[
s \mapsto \begin{cases} 
t^*_s a_- g_+ & \text{on } U_- \\
t^*_s (1-s)a_+ g_- & \text{on } U_+
\end{cases}
\]

defines a path in $\mathcal{R}(S^{d-1} \times (-\infty, 0], g_{\text{dtor}}^{d-1})$ (as the translated metric $t^*_s g_\pm$ coincides with $g_{\text{dtor}}^{d-1} + dt^2$ when restricted to $U_\pm \times \mathbb{R}$). This goes from $\rho((g_-, a_-), (g_+, a_+))$ to $\rho((g_+, a_+), (g_-, a_-))$, and it clearly depends continuously on $(g_\pm, a_\pm)$ so yields the homotopy needed for the first claim.

By Theorem 2.3.2 the inclusions

\[
\mathcal{R}^+(S^{d-1} \times [-1, 0]; \phi_{\pm}, g_{\text{dtor}}^{d-1})_\phi^{d-1} \to \mathcal{R}^+(S^{d-1} \times [-1, 0]; g_{\text{dtor}}^{d-1})
\]

are weak homotopy equivalences, and the second claim follows by Lemma 4.1.9. □

4.1.2. Proof of Theorem 4.1.3 We begin with the Eckmann–Hilton style argument.

**Proposition 4.1.11.** Let $Z$ be a homotopy (unital, associative, and commutative) $H$-space with multiplication $\rho$, and $X$ be a space with a homotopy (unital, associative) left $Z$-action $\nu$. In particular, we have

\[
\rho(\rho(-, -), -) \simeq \rho(-, \rho(-, -)) \quad \nu(\rho(-, -), -) \simeq \nu(-, \nu(-, -)),
\]
as well as unit and commutativity homotopies. Let $G$ be a group acting on $X$ on the right via $(x, g) \mapsto x \cdot g$, and suppose that the $Z$- and $G$-actions commute in the
sense that \( \nu(-, -) \simeq \nu(-, -) \cdot - \). Suppose further that there is an \( x_0 \in X \) so that the orbit map

\[
\begin{array}{c}
Z \\ z \mapsto \nu(z, x_0)
\end{array} \xrightarrow{\sim} X
\]

is a homotopy equivalence. Then the homomorphism \( \pi_0(G) \to \pi_0(\text{Aut}(X)) \) has abelian image.

Proof. For a point \( y \) in a space \( Y \), we denote by \([y] \in \pi_0(Y)\) the path component it belongs to. Consider the maps

\[
G \xrightarrow{g \mapsto x_0 \cdot g} X \leftarrow X \xrightarrow{z \mapsto \nu(z, x_0)} Z,
\]

where the rightmost map is a homotopy equivalence by assumption. For each \( g \in G \), there exists a point \( z_g \in Z \) so that \([x_0 \cdot g] = [\nu(z_g, x_0)]\). We claim that that the self-maps of \( X \) given by \( x \mapsto \nu(z_g, x) \) and \( x \mapsto x \cdot g \) are homotopic. To see this, consider the diagram

\[
\begin{array}{c}
Z \\ z \mapsto \rho(z, z_g)
\end{array} \xrightarrow{\sim} X \leftarrow X \xrightarrow{z \mapsto \nu(z, x_0)} Z
\]

Going over the top gives \( \nu(-, x_0) \cdot g \), which is homotopic to \( \nu(-, x_0 \cdot g) \) and so to \( \nu(-, \nu(z_g, x_0)) \) and hence to \( \nu(\rho(-, z_g), x_0) \), which is the map given by going along the bottom. Thus the square commutes up to homotopy. The horizontal maps are homotopy equivalences by assumption, and the right-hand map is a homeomorphism, so \( \rho(-, z_g) \) is also a homotopy equivalence. If the right-hand map in (4.1.12) is replaced by \( x \mapsto \nu(z_g, x) \) then the square also commutes up to homotopy (going over the top is the map \( \nu(z_g, \nu(-, x_0)) \) which is homotopic to \( \nu(\rho(z_g, -), x_0) \) and so to \( \nu(\rho(-, z_g), x_0) \) by homotopy commutativity of \( \rho \)). So as the remaining maps are all homotopy equivalences we find that \( x \mapsto \nu(z_g, x) \) and \( x \mapsto x \cdot g \) are homotopic.

Let \( f \in G \) be another element. We have constructed homotopies

\[
(-) \cdot f \cdot g \simeq \nu(z_f, -) \cdot g \simeq \nu(z_g, \nu(z_f, -)) \simeq \nu(\rho(z_g, z_f), -),
\]

and as \( \rho \) is homotopy commutative this is homotopic to \( \nu(\rho(z_f, z_g), -) \), and so by reversing the analogous chain of homotopies to \( (-) \cdot g \cdot f \). This proves that the translation maps by \( f \cdot g \) and \( g \cdot f \) are homotopic, as claimed.

We now proceed with the proof of Theorem 4.1.2. In the notation introduced in the last section, the pair

\[
\begin{align*}
Z &:= |\text{Sing}_{\bullet} \mathcal{R}^+(S^{d-1} \times (-\infty, 0], g_{\text{drt}}^{d-1})| \\
X &:= |\text{Sing}_{\bullet} \mathcal{R}^+(V, g_{\text{drt}}^{d-1})|
\end{align*}
\]

(4.1.13)

gives a monoid \( Z \) and a space \( X \) on which \( Z \) acts on the left. Furthermore, the topological group

\[
G := |\text{Sing}_{\bullet} \text{Diff}(V)|
\]

(4.1.14)

acts on \( X \), by the evident action of \( \text{Diff}(V) \) on \( \mathcal{R}^+(V, g_{\text{drt}}^{d-1}) \). The \( G \)- and \( Z \)-actions on \( X \) strictly commute with each other, and by Proposition 4.1.10 the monoid \( Z \) is homotopy-commutative.
Lemma 4.1.15. Let $V : S^{d-1} \sim \emptyset$ be a spin cobordism satisfying the hypotheses of Theorem 4.1.2. Then there is an $x_0 \in \mathcal{R}^+(V, g^{d-1}_{dtor})$ such that

$$\nu(-, x_0) : \mathcal{R}^+(S^{d-1} \times (-\infty, 0], g^{d-1}_{dtor}) \longrightarrow \mathcal{R}^+(V, g^{d-1}_{dtor})$$

is a weak homotopy equivalence. Hence the induced map $\nu(-, x_0) : Z \to X$ is a homotopy equivalence.

Proof. The first step is to reduce the problem to the case $V = D^d$. For a manifold $W$ with boundary $S^{d-1}$ and an embedding $\phi : S^{k-1} \times D^{d-k+1} \hookrightarrow W$ in the interior, let

$$\mathcal{R}^+(W, g^{d-1}_{dtor}, \phi, g^{k-1}_o) \subset \mathcal{R}^+(W, g^{d-1}_{dtor})$$

be the subspace of those metrics on $(S^{d-1} \times (-\infty, 0]) \cup S_{d-1}$, $W$ which pull back under $\phi$ to $g^{k-1}_o + g^{d-k+1}_{dtor}$. This is a sub-$\mathcal{R}^+(S^{d-1} \times (-\infty, 0], g^{d-1}_{dtor})$-space, and by combining Lemma 4.1.9 and Theorem 2.3.1 it follows that the inclusion is a weak homotopy equivalence.

As in the proofs of Theorem 4.1.2 and Lemma 2.3.5, if $\phi : S^{k-1} \times D^{d-k+1} \hookrightarrow W$ is a piece of surgery data in the interior of index 3, and $W'$ is the manifold obtained by performing this surgery, which contains a dual surgery datum $\phi' : S^{d-k} \times D^k \hookrightarrow W'$, then there is a zig-zag

$$\mathcal{R}^+(W, g^{d-1}_{dtor}, \phi, g^{k-1}_o) \cong \mathcal{R}^+(W', g^{d-1}_{dtor}, \phi', g^{d-k}_o)$$

$$\mathcal{R}^+(W, g^{d-1}_{dtor}) \quad \mathcal{R}^+(W', g^{d-1}_{dtor})$$

of maps which are both maps of $\mathcal{R}^+(S^{d-1} \times (-\infty, 0], g^{d-1}_{dtor})$-spaces and weak homotopy equivalences.

Returning to the situation at hand, by assumption there is a spin cobordism from $V$ to $D^d$ relative to the common boundary of these two manifolds. As $V$ is simply connected and spin, this may be realised by a sequence of surgeries on the interior of $V$ of indices $2 \leq k \leq d - 3$, and so by the above there is a zig-zag of weak homotopy equivalences of left $\mathcal{R}^+(S^{d-1} \times (-\infty, 0], g^{d-1}_{dtor})$-spaces between $\mathcal{R}^+(V, g^{d-1}_{dtor})$ and $\mathcal{R}^+(D^d, g^{d-1}_{dtor})$. Thus it is enough to show that $\mathcal{R}^+(D^d, g^{d-1}_{dtor})$ has the property claimed in the lemma.

To see this, we let $E^d$ be a copy of $D^d$ and choose a decomposition $D^d = (S^{d-1} \times [0, 1]) \cup S_{d-1} E^d$. We let $g_{iso} \in \mathcal{R}^+(S^{d-1} \times [0, 1], g^{d-1}_{dtor}, g^{d-1}_o)$ be the metric produced by Lemma 2.2.8 out of an isotopy from $g^{d-1}_{dtor}$ to $g^{d-1}_o$ and we let $x_0 = \mu(g_{iso}, g^d_{tor}) \in \mathcal{R}^+(D^d, g^{d-1}_{dtor})$ (where $g^d_{tor}$ is a torpedo metric on $E^d$). Consider the diagram

$$\begin{array}{ccc}
\mathcal{R}^+(S^{d-1} \times [-1, 0], g^{d-1}_{dtor}, g^{d-1}_o) & \xrightarrow{a = -1} & \mathcal{R}^+(S^{d-1} \times (-\infty, 0], g^{d-1}_{dtor}) \\
\mu_{g_{iso}} & & \nu(-, x_0) \\
\mathcal{R}^+(S^{d-1} \times [-1, 1], g^{d-1}_{dtor}, g^{d-1}_o) & \xrightarrow{\mu g^d_{tor}} & \mathcal{R}^+(S^{d-1} \times (-\infty, 1] \cup E^d, g^{d-1}_{dtor}) \\
\mathcal{R}^+(S^{d-1} \times [-1, 1] \cup E^d, g^{d-1}_{dtor}, g^{d-1}_o) & \xrightarrow{a = -1} & \mathcal{R}^+(S^{d-1} \times (-\infty, 1] \cup E^d, g^{d-1}_{dtor})
\end{array}$$
which clearly commutes. The two left vertical maps are weak equivalences: \( \mu_{g_{\text{tor}}} \) by Lemma 2.2.8 and \( \partial \) by Theorem 2.3.1. The two horizontal maps are weak equivalences by Lemma 4.1.9. Thus the right vertical map is a weak equivalence, which shows that \( x_0 \) indeed has the claimed property.

**Proof of Theorem 4.1.4** Let \( X, Z \) and \( G \) be as in (4.1.13) and (4.1.14). By Proposition 4.1.10 and Lemma 4.1.15, the triple \((X, Z, G)\) satisfies the assumptions of Proposition 4.1.11. Therefore, the action map \( \Gamma(V) \to \pi_0(\text{Aut}(\text{Sing}_\bullet \mathcal{R}^+(V, g_{\text{tor}}^{-1}))) \) has abelian image. The standard inclusion

[\[ \mathcal{R}^+(V)_{g_{\text{tor}}^{-1}} \to \mathcal{R}^+(V, g_{\text{tor}}^{-1}) \]]

is \( \text{Diff}_\partial(V) \)-equivariant and a weak homotopy equivalence by Lemma 4.1.9. Thus after taking geometric realisations of the associated singular simplicial sets, we find that \( \Gamma(V) \to \pi_0(\text{Aut}(\mathcal{R}^+(V)_{g_{\text{tor}}^{-1}})) \) has abelian image. This is almost the statement of Theorem 4.1.2, except that the boundary condition is \( g_{\text{tor}}^{-1} \), not \( g_{\text{tor}}^{-1} \).

Now the metric \( g_{\text{tor}}^{-1} \in \mathcal{R}^+(S^{d-1}) \) is isotopic to \( g_{\text{tor}}^{-1} \), and gluing on the trace of an isotopy gives a map

[\[ \mathcal{R}^+(V)_{g_{\text{tor}}^{-1}} \to \mathcal{R}^+([0, 1] \times S^{2n-1}) \cup V)_{g_{\text{tor}}^{-1}} \cong \mathcal{R}^+(V)_{g_{\text{tor}}^{-1}} \]

which is a homotopy equivalence by Lemma 2.2.8. This map is also equivariant in the homotopy category with respect to the homomorphism

[\[ \Gamma(V) \to \Gamma(([0, 1] \times S^{2n-1}) \cup V) \cong \Gamma(V), \]

which is an isomorphism by collar stretching. This finishes the proof. \( \square \)

4.2. **Constructing maps into spaces of psc metrics.**

**Definitions and statement of the main construction theorem.** In this section we shall carry out the main constructive step for the proof of Theorem 4.1.4. Let \( 2n \geq 6 \) and we shall suppose that

\[ W : \emptyset \simeq S^{2n-1} \]

is a simply connected spin cobordism, such that \( W \) is spin cobordant to \( D^{2n} \) relative to its boundary. Let

\[ K := ([0, 1] \times S^{2n-1}) \# (S^n \times S^n) : S^{2n-1} \simeq S^{2n-1}. \]

For \( i = 0, 1, 2, \ldots \) let \( K|_i := S^{2n-1} \) and \( K|_{i+1} : K|_i \simeq K|_{i+1} \) be a copy of \( K \). Also, consider \( W \) as a cobordism to \( K|_0 \). Then we write

\[ W_k := W \cup K|_{0,k} := W \cup \bigcup_{i=0}^{k-1} K|_{i,i+1} : \emptyset \simeq K|_k \]

for the composition of \( W \) and \( k \) copies of \( K \), so \( W_0 = W \). Define the group \( D_k := \text{Diff}(W_k, K|_k) \), and write \( B_k := BD_k \) for the classifying space of this group and \( \pi_k : E_k := ED_k \times p_k W_k \to B_k \) for the universal bundle. There is a homomorphism \( D_k \to D_{k+1} \) given by extending diffeomorphisms over \( K|_{k+1} \) by the identity, and this induces a map \( \lambda_k : B_k \to B_{k+1} \) on classifying spaces. Let \( \text{hocolim}_k B_k \) denote the mapping telescope.

Before we state the main result of this section, let us collect the important consequences of our previous work.
Proposition 4.2.1.

(i) There is a surgery equivalence \( R^+(W_0)_{g_2} \simeq R^+(D^{2n})_{g_2} \), and so in particular \( R^+(W_0)_{g_2} \) is non-empty. Thus we may choose an \( h_{-1} \in R^+(W_0)_{g_2} \) which lies in the component of \( g_2 \), under the surgery equivalence.

(ii) There are metrics \( h_i \in R^+(K[0,i+1])_{g_2} \) so that the gluing maps

\[ \mu_i : R^+(W_i)_{g_2} \to R^+(W_{i+1})_{g_2} \]

are weak homotopy equivalences. Let

\[ m_i := h_{-1} \cup h_0 \cup h_1 \cup \cdots \cup h_{i-1} \in R^+(W_i)_{g_2}. \]

(iii) The action homomorphism \( \Gamma(W_k) \to \pi_0(\text{Aut}(R^+(W_k)_{g_2})) \) has abelian image.

Proof. This is straightforward from the previous work: Because \( W_0 = W \) is spin cobordant to \( D^{2n} \) relative to its boundary by assumption, the first part follows from Theorem 2.3.3. Because the manifold \( K[0,i+1] = ([0,1] \times S^{2n-1}) \# (S^n \times S^n) \) is cobordant to a cylinder relative to its boundary, the second part follows from Lemma 2.3.4. The third assertion follows from Theorem 4.1.2 again using the assumption that \( W_0 \) is spin cobordant to \( D^{2n} \) relative to its boundary. \( \square \)

By Lemma 3.8.1 there is a unique fibrewise spin structure on each bundle \( E_k \). Thus there is the family of Dirac operators on the fibre bundles \( \pi_k \). We now list further hypotheses on the manifold \( W \), which will allow us to carry out an obstruction-theoretic argument. We will later make particular choices of \( W \) which fulfil these assumptions.

Assumptions 4.2.2.

(i) There is a space \( X \) with \( \pi_1(X) \) abelian and an acyclic map \( \Psi : B_\infty \to X \).

(ii) There is a class \( \hat{a} \in KO^{-2n}(X) \) such that \( \Psi^*(\hat{a}) \) restricts to the family index

\[ \text{ind}(E_k, g_2^{-1}) \in KO^{-2n}(B_k) \],

for all \( k \), up to phantom maps.

Remark 4.2.3. Recall that a map \( f : X \to Y \) of spaces is called acyclic if for each \( y \in Y \) the homotopy fibre \( \text{hofib}_y(f) \) has the singular homology of a point. This is equivalent to \( f \) inducing an isomorphism on homology for every system of local coefficients on \( Y \); if \( Y \) is not simply-connected then it is stronger than merely being a homology equivalence.

Recall that a map \( f : X \to Z \) to a pointed space is called phantom if it is weakly homotopic to the constant map to the basepoint. Then maps \( f_0, f_1 : X \to Z \) to a loop space are said to agree up to phantom maps if their difference \( f_0 \cdot f_1^{-1} \) is phantom.

We will write \( R_k := |\text{Sing}_* R^+(W_k)_{g_2}| \) and \( X := |\text{Sing}_* X| \). The rest of this section is devoted to the proof of the following result.

Theorem 4.2.4. If the spin cobordism \( W : \emptyset \to S^{2n-1} \) is such that \( W \) is simply connected and is spin cobordant to \( D^{2n} \) relative to its boundary, and Assumptions 4.2.2 hold, then there is a map \( \rho : \Omega X \to R_0 \) such that the composition with \( \text{indiff}_{m_0} \) agrees with \( \Omega \hat{a} \), up to phantom maps.

In the following sections we will show how a manifold \( W \) satisfying these hypotheses can be constructed, but in the rest of this section we will prove Theorem 4.2.4 and so suppose that the hypotheses of this theorem hold.
Setting the stage for the obstruction argument. Theorem 4.2.4 is proved by an obstruction-theoretic argument, which needs some preliminary constructions. We introduce the abbreviations \( R_k := R^+(W_k)_{g_k}^{2n-1}, T_k := ED_k \times_D R_k, \) write \( p_k : T_k \to B_k \) for the projection map and write \( \mu_k := \mu_{h_k} : R_k \to R_{k+1} \) for the gluing maps defined by \( h_k. \) The map \( \mu_k \) is \( D_k \)-equivariant (by construction), so there is an induced map between the Borel constructions

\[
\begin{array}{ccc}
R_k & \xrightarrow{\mu_k} & R_{k+1} \\
\downarrow & & \downarrow \\
T_k & \xrightarrow{\nu_k} & T_{k+1} \\
\downarrow & & \downarrow \\
B_k & \xrightarrow{\lambda_k} & B_{k+1}.
\end{array}
\]

By Proposition 4.2.1 [1], the top map is a weak homotopy equivalence, so the lower square is weakly homotopy cartesian. Using the (unique) spin structure that each fibre bundle \( \pi_k : E_k \to B_k \) has (cf. Lemma 3.8.1), the construction of Section 3.8 gives relative \( KO \)–classes \( \beta_k \in KO^{-d}(p_k) \). Let \( \text{hocolim}_k p_k : \text{hocolim}_k T_k \to \text{hocolim}_k B_k \) be the induced map on homotopy colimits (alias mapping telescopes).

**Proposition 4.2.5.** There is a relative \( KO \)–class \( \beta_\infty \in KO^{-d}(\text{hocolim}_k p_k), \) such that the restriction to \( KO^{-d}(p_k) \) is equal to \( \beta_k. \)

**Proof.** The map

\[
KO^{-d}(\text{hocolim}_k p_k) \to \varprojlim KO^{-d}(p_k)
\]

is surjective by Milnor’s \( \text{lim}^1 \) sequence. The classes \( \beta_k \in KO^{-d}(p_k) \) give a consistent collection by Corollary 3.4.9, and so there exists a \( \beta_\infty \in KO^{-d}(\text{hocolim}_k p_k) \) restricting to them. \( \square \)

Let us move to the simplicial world, in which we will carry out the proof of Theorem 4.2.4. The simplicial group \( \text{Sing}_* D_k \) (obtained by taking singular simplices) acts on the simplicial set \( \text{Sing}_* R_k. \) We denote the geometric realisations by \( D_k := |\text{Sing}_* D_k| \) and \( R_k := |\text{Sing}_* R_k|, \) and also write \( B_k := BD_k. \) Let \( D_\infty \) and \( R_\infty \) be the colimits of the induced maps

\[
\lambda_k : D_k \to D_{k+1} \quad \text{and} \quad \mu_k : R_k \to R_{k+1},
\]

which, as these maps are cellular inclusions, are also homotopy colimits. Let \( B_\infty := BD_\infty \) and \( \Psi : B_\infty \to X := |\text{Sing}_* X| \) be a map in the homotopy class induced by

\[
B_\infty = \text{colim}_{k \to \infty} B_k \xrightarrow{\sim} \text{hocolim}_k B_k \to |\text{Sing}_* (B_\infty)| \xrightarrow{\text{Sing}_* \Psi} |\text{Sing}_* (X)| =: X.
\]

Let

\[
p_k : T_k := ED_k \times_D R_k \to B_k
\]

be the induced fibre bundles, for \( k \in \mathbb{N} \cup \{\infty\}. \) There is a commutative diagram

\[
\begin{array}{ccc}
T_\infty & \xrightarrow{\sim} & \text{hocolim}_k T_k \\
\downarrow & & \downarrow \\
B_\infty & \xrightarrow{\sim} & \text{hocolim}_k B_k
\end{array}
\]

and \( \Psi : \text{hocolim}_k p_k \to \text{hocolim}_k p_k \)
As the four left spaces in the diagram are all CW-complexes, there exists a class $\beta_\infty \in KO^{-2n}(p_\infty)$ whose pullback to $KO^{-2n}(\text{holim}_k p_k)$ coincides with the pullback of the class $\beta_\infty$ constructed in Proposition 4.2.5 to $\text{Cyl}(\text{holim}_k p_k)$. The square

$$
\begin{array}{ccc}
T_k & \longrightarrow & T_\infty \\
| \quad & & | \\
B_k & \longrightarrow & B_\infty
\end{array}
$$

is homotopy cartesian, as both maps are Serre fibrations, the map $R_k \rightarrow R_\infty$ on fibres is a homotopy equivalence by Proposition 4.2.1 (ii), and all the spaces have the homotopy type of CW complexes. The following property of the class $\beta_\infty$ is clear from the construction and Proposition 4.2.5.

**Lemma 4.2.7.** The pullback of $\beta_\infty$ to each $\text{Cyl}(p_k)$ agrees with the pullback of $\beta_k$ along the weak homotopy equivalence $\text{Cyl}(p_k) \rightarrow \text{Cyl}(p_k)$.

*The obstruction argument.* The topological group $D_\infty$ acts on $R_\infty$, and the map $p_\infty : T_\infty \rightarrow B_\infty$ is the associated Borel construction. In particular, there is an associated homomorphism

$$D_\infty \rightarrow \text{Aut}(R_\infty)$$

of topological monoids, which on classifying spaces gives a map

$$B_\infty \rightarrow B\text{Aut}(R_\infty).$$

**Lemma 4.2.8.** The monodromy map $\pi_1(B_\infty) \rightarrow \pi_0(\text{Aut}(R_\infty))$ has abelian image.

*Proof.* Because a group is commutative if all its finitely generated subgroups are, and because the diagram (4.2.6) is homotopy cartesian, it is enough to prove that the monodromy map $\pi_1(B_k) \rightarrow \pi_0(\text{Aut}(R_k))$ has abelian image for each $k$. But $\pi_1(B_k) = \pi_0(D_k) = \Gamma(W_k)$, and under this identification, the monodromy map becomes the homomorphism $\Gamma(W_k) \rightarrow \pi_0(\text{Aut}(R_k))$ induced from the action, and we have shown that this acts through an abelian group in Lemma 4.2.1. \qed

**Proposition 4.2.9.** The exists a commutative and homotopy cartesian square

$$
\begin{array}{ccc}
T_\infty & \longrightarrow & T^+_\infty \\
| \quad & & | \\
B_\infty & \longrightarrow & X
\end{array}
$$

Moreover, there is a unique class $\beta^+_\infty \in KO^{-2n}(p^+_\infty)$ which restricts to $\beta_\infty \in KO^{-2n}(p_\infty)$.

*Proof.* First, we invoke May’s general classification theory for fibrations [37]. The result (loc. cit. Theorem 9.2) is that there is a universal fibration $E \rightarrow B\text{Aut}(R_\infty)$ with fibre $R_\infty$ over the classifying space of the topological monoid $\text{Aut}(R_\infty)$ and a homotopy cartesian diagram

$$
\begin{array}{ccc}
T_\infty & \longrightarrow & E \\
| \quad & & | \\
B_\infty & \longrightarrow & B\text{Aut}(R_\infty).
\end{array}
$$
By Assumption 4.2.2 (i) there is an acyclic map $Ψ : B_∞ \to X$ to a space with abelian fundamental group. We claim that the obstruction problem

\[
\begin{array}{ccc}
B_∞ & \xrightarrow{Ψ} & X \\
\downarrow{h} & & \downarrow{g} \\
B \text{ Aut}(R_∞) & \end{array}
\]

can be solved, up to homotopy. Since $Ψ$ is acyclic, by the universal property of acyclic maps [26, Proposition 3.1] it is sufficient to prove that $\ker(π_1(Ψ)) \subset \ker(π_1(h))$. The group $\ker(π_1(Ψ))$ is perfect, but by Lemma 4.2.8 the group $π_1(h)$ has abelian image, and so $π_1(h)(\ker(π_1(Ψ)))$ is trivial; in other words, $\ker(π_1(Ψ)) \subset \ker(π_1(h))$.

Therefore, we have constructed a factorisation

\[
h : B_∞ \xrightarrow{Ψ} X \xrightarrow{g} B \text{ Aut}(R_∞)
\]

up to homotopy, but without loss of generality we can assume that $B_∞ \to X$ is a cofibration and so achieve that the above factorisation holds strictly. Let us denote by $p_∞ : T_∞^+ \to X$ the fibration obtained by pulling $E$ back along the map $g$, so there is an induced commutative square

\[
\begin{array}{ccc}
T_∞ & \xrightarrow{Ψ'} & T_∞^+ \\
\downarrow{p_∞} & & \downarrow{p_∞^+} \\
B_∞ & \xrightarrow{Ψ} & X.
\end{array}
\]

By construction, the square is homotopy cartesian. Therefore, since $Ψ$ is acyclic, so is $Ψ'$. Thus the class $β_∞ \in KO^{-d}(p_∞)$ from Lemma 4.2.7 extends to a unique class $β_∞^+ \in KO^{-d}(p_∞^+)$. □

Now we define the map $ρ_∞ : ΩX \to R_∞$ as the fibre transport of the fibration $p_∞^+$. Since $R_0 \to R_∞$ is a homotopy equivalence, we can lift $ρ_∞$ to a map $ρ : ΩX \to R_0$. It remains to check that $ρ$ has the property stated in Theorem 4.2.4. We apply the relative index construction (i.e. Corollary 3.5.2) to the class $β_∞$ and obtain a homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega X & \xrightarrow{ρ} & \Omega^{∞+2n+1}KO \\
\downarrow{ρ_∞} & & \downarrow{\text{trg}(β_∞^+)} \\
R_0 & \xrightarrow{\cong} & R_∞
\end{array}
\]

The composition $R_0 \to R_∞ \xrightarrow{\text{trg}(β_∞^+)} Ω^{∞+2n+1}KO$ is homotopic to (the pullback to $R_0$ of) the index difference with respect to the psc metric $m_0 \in R^+(W_0)_{R_0}^{2n-1}$, by construction, Lemma 4.2.7 and Proposition 3.8.3. Recall that in Assumption 4.2.2 (ii) we have chosen a map $â : X \to Ω^{∞+2n}KO$ which restricts to the family index on each $B_k$.

**Proposition 4.2.10.**

(i) The classes $\text{bas}(β_∞^+)$ and $â$ in $[X, Ω^{∞+2n}KO]$ agree up to phantom maps.
(ii) The classes $\Omega \text{bas}(\beta^+_k)$ and $\Omega \hat{a}$ in $[X, \Omega^{\infty+2n+1}\text{KO}]$ agree up to phantom maps.

The proof of Theorem 4.2.4 will be completed by the second statement of Proposition 4.2.10. For the proof, we need a general result about the relation between phantom maps into loop spaces and homology equivalences.

**Lemma 4.2.11.** Let $f : X \to Y$ be a homology equivalence and $h : Y \to \Omega Z$ be a map to a loop space. If $h \circ f$ is a phantom, then so is $h$.

**Proof.** Without loss of generality, we can assume that $X$ and $Y$ are connected CW complexes and that $h$ and $f$ are pointed maps. It is also enough to prove that $h$ becomes nullhomotopic when composed with pointed maps with finite CW source. First we show that if $k : X \to \Omega Z$ is a phantom, then so is its adjoint $k^{ad} : \Sigma X \to Z$ under the loop/suspension adjunction. Let $l : F \to \Sigma X$ be any map from a finite CW complex, and we wish to show that $k^{ad} \circ l$ is nullhomotopic. As any finite subcomplex of $\Sigma X$ is contained in the suspension $\Sigma L$ of a finite $L \subset X$, we can write $l = (\Sigma i) \circ j$, where $i : L \to X$ is the inclusion and $j : F \to \Sigma L$ some map. But then $k^{ad} \circ l = k^{ad} \circ \Sigma i \circ j = (k \circ i)^{ad} \circ j$. Since $k$ is a phantom and $L$ is finite, $k \circ i$ and hence the adjoint $(k \circ i)^{ad}$ is nullhomotopic.

Now return to the notation of the lemma. Let $F$ be a finite complex and $g : F \to Y$ be a map; we wish to show that $h \circ g$ is nullhomotopic. It is enough to prove that $(h \circ g)^{ad} = h^{ad} \circ (\Sigma g) : \Sigma F \to \Sigma Y \to Z$ is nullhomotopic. Now as $f$ was assumed to be a homology equivalence of connected CW complexes, $\Sigma f : \Sigma X \to \Sigma Y$ is a homotopy equivalence. Thus there exists a map $m : \Sigma F \to \Sigma X$ with $(\Sigma f) \circ m \simeq \Sigma g$. Hence

$$(h \circ g)^{ad} = h^{ad} \circ (\Sigma g) \simeq h^{ad} \circ (\Sigma f) \circ m = (h \circ f)^{ad} \circ m.$$
4.3. Starting the proof of Theorem \textbf{B} \footnote{\textbf{Proposition 4.3.3.} The maps \(\text{ind}(E_k, g_0^{d-1}) : B_k \rightarrow \Omega^\infty(\mathrm{KO})\) and \(B_k \rightarrow B_\infty \xrightarrow{\alpha} \Omega^\infty_{\infty(\mathrm{MTSpin}(2n))\Omega^\infty(\mathrm{KO})}\), agree up to phantom maps.}

Consider a sequence of spin cobordisms

\[
\emptyset \xrightarrow{W} K_{[0]} \xrightarrow{K_{[1]}} K_{[2]} \xrightarrow{K_{[3]}} \cdots \tag{4.3.1}
\]

as in the previous section. The associated fibre bundles \(E_k : E_k \rightarrow B_k\) admit unique spin structures—which can be taken to be compatible—by Lemma \textbf{3.8.1}. Hence we obtain a map

\[
\hocolim_{k \to \infty} B_k \rightarrow \hocolim_{k \to \infty} \Omega_0[K_{[k]})\Omega^\infty(\mathrm{MTSpin}(2n))
\]

on homotopy colimits. Each of the maps

\[
\Omega_0[K_{[k]})\Omega^\infty(\mathrm{MTSpin}(2n)) \rightarrow \Omega_0[K_{[k+1]})\Omega^\infty(\mathrm{MTSpin}(2n)),
\]

which concatenates a path with the path obtained from the Pontrjagin–Thom construction applied to \(K_{[k,k+1]}\), is a homotopy equivalence, and as \(K_{[-1]} = \emptyset\) we obtain a map

\[
\alpha_\infty : B_\infty := \hocolim_{k \to \infty} B_k \rightarrow \Omega^\infty_{\infty(\mathrm{MTSpin}(2n))}
\]

well-defined up to homotopy.

Suppose for now that the map \(\alpha_\infty\) is acyclic. Then it satisfies Assumption \textbf{1.2.2}, and for Assumption \textbf{1.2.2} we take the class

\[
\Omega^\infty(\lambda_{-2}) \in K0^{-2n}(\Omega^\infty_{\infty(\mathrm{MTSpin}(2n))})
\]

represented by the infinite loop map of the \(K\)-theory Thom class of \(\mathrm{MTSpin}(2n)\).

Thus in order to finish the proof of Theorem \textbf{B} in dimension \(2n\) we must produce a spin cobordism \(W : \emptyset \rightarrow S^{2n-1}\) such that the following three conditions are satisfied

\(i\) \(W\) is 1-connected,
(ii) $W$ is spin cobordant to $D^{2n}$ relative to its boundary,

(iii) the associated map $\alpha_\infty$ is acyclic.

The main result of [21] gives criteria on $W$ for the map $\alpha_\infty$ to be a homology isomorphism, as long as $2n \geq 6$. This result is enough to prove Theorem B in the case $2n = 6$, and we explain it first.

4.3.1. **Finishing the proof of of Theorem B: the 6-dimensional case.** In this case we let $W = D^6$, which clearly satisfies the first two conditions. The manifold $W_k$ is then the $k$-fold connected sum $\#^k(S^3 \times S^3)$, minus a disc. To establish the acyclicity of the map $\alpha_\infty$ in this case, we use the following theorem.

**Theorem 4.3.4** (Galatius, Randal-Williams). The map

$$\alpha_\infty : \text{hocolim}_{k \to \infty} \text{BDiff}(\#^k(S^3 \times S^3) \setminus D^6, S^5) \to \Omega^\infty_0 \text{MTSpin}(6)$$

is a homology equivalence.

This is a consequence of the general theorem [21, Theorem 1.8], and is explicitly discussed in [21, §1.5]. Having a homology equivalence is not quite enough for the obstruction theory in the previous subsection, where we needed the map to be acyclic. However, we also have the following.

**Proposition 4.3.6.** The space $\Omega^\infty_0 \text{MTSpin}(6)$ is simply connected.

This follows from recent calculations of Galatius and the third named author [19, Lemma 5.7]. We have been informed by Bökstedt that his calculations with Dupont and Svane [9] can be used to give an alternative proof. It follows from this proposition that the map (4.3.5) is actually acyclic (as the target is simply connected, so there are no local coefficient systems to check), which finishes the proof of Theorem B for $d = 6$.

4.4. **Finishing the proof of Theorem B.** For $2n > 6$ the main result of [21] does not suffice to prove Theorem B. Instead, we have to use the more difficult results of [20]. There are three ways in which the case $2n > 6$ is more difficult to handle than the case $2n = 6$. Firstly, the infinite loop space $\Omega^\infty_0 \text{MTSpin}(2n)$ is not necessarily simply connected, so it is not automatic that the homology equivalences coming from [21] are acyclic. Secondly, the manifold obtained by the countable composition of the cobordisms $K = ([0,1] \times S^{2n-1}) \#(S^n \times S^n)$ does not form a “universal spin-end” in the sense of [21, Definition 1.7] unless $2n = 6$, and so the results of [21] do not apply. Thirdly, even if the results of [21] did apply to this stabilisation, we must show that there is a spin cobordism $W : \emptyset \to S^{2n-1}$ whose structure map $\ell : W \to B\text{Spin}(2n)$ is $n$-connected and which in addition satisfies the conditions given in Section 4.3.

The first two of these difficulties can be avoided by appealing instead to the results of [20], which build on those of [21]. These results upgrade those of [21] to always give acyclic maps instead of merely homology equivalences, and to allow more general stabilisations than by “universal ends”. The third difficulty must be confronted directly, and we shall do so shortly. First, let us state the version of the result of [20] which we shall use, and show how to extract it from [20].

**Theorem 4.4.1** (Galatius, Randal-Williams). Let $W : \emptyset \to S^{2n-1}$ be a spin cobordism such that the structure map $\ell : W \to B\text{Spin}(2n)$ is $n$-connected. Then the
map

$$\alpha_\infty : \operatorname{hocolim}_{k \to \infty} B \operatorname{Diff}(W_k, K|_k) \to \Omega_0^\infty \operatorname{MTSpin}(2n)$$

is acyclic.

**Proof.** We adopt the notation of [20]. The manifold $W_k$ is obtained from $W$ by attaching $n$-handles and higher, so $\ell : W_k \to B \operatorname{Spin}(2n)$ is also $n$-connected. Thus the canonical tangential structure associated to $W_k$ is that of a spin structure, $\theta : B \operatorname{Spin}(2n) \to BO(2n)$. Choose a $\theta$-structure $\hat{\ell}_{K|_k} : TW_k|_{K|_k} \to \theta^* \gamma_{2n}$ on the boundary of $W_k$ which extends to $W_k$. Elementary obstruction theory shows that the space $\operatorname{Bun}_\theta(W_k; \hat{\ell}_{K|_k})$ of $n$-connected $\theta$-structures on $W_k$ is contractible, as the pair $(W_k, K|_k)$ is 1-connected and the map $\theta$ is 2-connected. Thus there is a homotopy equivalence

$$\mathcal{M}^\theta(W_k; \hat{\ell}_{K|_k}) \sim \to B \operatorname{Diff}(W_k, K|_k).$$

The theorem will thus follow if the map

$$\alpha_k : B \operatorname{Diff}(W_k, K|_k) \simeq \mathcal{M}^\theta(W_k; \hat{\ell}_{K|_k}) \longrightarrow \Omega^\infty_{[W_k, \hat{\ell}_{K|_k}]} \operatorname{MTSpin}(2n)$$

is acyclic in a range of degrees which tends to $\infty$ with $k$, but this is precisely an application of [20, Theorem 1.5] □

It remains to produce a spin cobordism $W : \emptyset \rightsquigarrow S^{2n-1}$ such that

- (i) $W$ is spin cobordant to $D^{2n}$ relative to its boundary,
- (ii) the structure map $\ell : W \to B \operatorname{Spin}(2n)$ is $n$-connected.

(This second condition implies that $W$ is simply connected, as $B \operatorname{Spin}(2n)$ is.) We shall do this as follows. Suppose first that we may find a $2n$-dimensional spin cobordism $V : \emptyset \rightsquigarrow P$ such that

- (i) the pair $(V, P)$ is $(n-1)$-connected,
- (ii) the structure map $\ell : V \to B \operatorname{Spin}(2n)$ is $n$-connected.

Then the double $D(V) := V \cup_P \overline{V}$ of $V$ may be presented as the boundary of the manifold obtained by smoothing the corners of $V \times [0, 1]$. The manifold $V \times [0, 1]$ has a spin structure coming from that of $V$, and so $D(V)$ has a spin structure which agrees with the given one on $V$ and which bounds. Furthermore, by homotopy excision the pair $(D(V), V)$ is $(n-1)$-connected, and $\ell : V \to B \operatorname{Spin}(2n)$ is $n$-connected, so $\ell : D(V) \to B \operatorname{Spin}(2n)$ is also $n$-connected. We may thus take $W := D(V) \setminus \operatorname{int}(D^{2n})$; this still has $n$-connected structure map, and is spin cobordant to $D^{2n}$ because $D(V)$ is spin nullbordant. This produces $W$, given $V$.

The technique for the construction of the cobordism $V$ is quite general, and is not specific to the tangential structure $\theta : B \operatorname{Spin}(2n) \to BO(2n)$. Since the special case of spin structures is no simpler than the general case we prove a general version of the result.

**Proposition 4.4.2.** Let $n \geq 3$ and $\theta : B \to BO(2n)$ be a tangential structure such that $B$ satisfies Wall’s finiteness condition $F_n$ (cf. [49]). Then there is a $\theta$-cobordism $V : \emptyset \rightsquigarrow P$ such that

- (i) the pair $(V, P)$ is $(n-1)$-connected,
- (ii) the structure map $\ell : V \to B$ is $n$-connected.
Let us make a remark about the finiteness condition. If a $V$ in the statement of the proposition exists, then (being a compact smooth manifold) it is homotopy equivalent to a finite CW complex. By Poincaré–Lefschetz duality with any local coefficients $A$ on $V$ we have

$$H^i(V; A) \cong H_{2n-i}(V, P; A \otimes \mathcal{O}_V)$$

and as the pair $(V, P)$ is $(n-1)$-connected we have $H^i(V; A) = 0$ for all $i > n$ and it follows from [50 Corollary 5.1] that $V$ is homotopy equivalent to a finite CW complex having cells of dimension at most $n$. By attaching cells to $V$ we may obtain a homotopy equivalence $\ell' : V' \tilde{\to} B$ from a CW complex with finite $n$-skeleton (cf. [49 Lemma 1.2]). Thus by [49 Theorem A] the space $B$ satisfies the finiteness condition $F_n$. Thus this is certainly a necessary condition.

**Proof.** We may assume that $B$ is connected, and otherwise work component by component. Let $f : X \to B$ be an $n$-connected map from a finite complex $X$ which exists by [49 Theorem A]. Classical surgery below the middle dimension, namely [51 Theorem 1.4], produces a $2n$-dimensional manifold with boundary $(V, P)$ and a factorisation of its Gauss map

$$V \xrightarrow{\ell} X \xrightarrow{f} B \xrightarrow{\theta} BO(2n)$$

such that $\ell : V \to X$ is $n$-connected and $\ell|_P : P \to X$ is $(n-1)$-connected. It follows that the pair $(V, P)$ is $(n-1)$-connected and that $f \circ \ell : V \to B$ is $n$-connected. \qed

Clearly $B\text{Spin}(2n)$ is of type $F_n$, so this proposition provides the required cobordism $V$. This finishes the proof of Theorem \textbf{B}.

5. Computational results

In this section, we will derive the computational consequences of Theorems \textbf{B} and \textbf{C}. To do so, we will study the effect of the maps

$$\Omega^{\infty+1} \lambda_{-2n} : \Omega_0^{\infty+1} \text{MTSpin}(2n) \to \Omega^{\infty+2n+1}\text{KO}$$
$$\Omega^{\infty+2} \lambda_{-2n} : \Omega_0^{\infty+2} \text{MTSpin}(2n) \to \Omega^{\infty+2n+2}\text{KO}$$

on homotopy and homology, and in particular their images; Theorems \textbf{B} and \textbf{C} show that these maps factor through $R^+(S^{2n})$ and $R^+(S^{2n+1})$ respectively, so the images of these maps are contained in the the images of the respective secondary index maps. This section is almost entirely homotopy-theoretic, and except for Theorem 5.4.11 we shall not mention spaces of psc metrics any further.

Recall that $\text{MTSpin}(d)$ is the Thom spectrum $\text{Th}(-\gamma_d)$ of the additive inverse of the universal vector bundle $\gamma_d \to B\text{Spin}(d)$. For any virtual spin vector bundle $V \to X$ of rank $r \in \mathbb{Z}$, we denote the $KO$-theoretic Thom class by $\lambda_V \in KO^r(\text{Th}(V)) = [\text{Th}(V), \Sigma^r \text{KO}]$. Note that there is a unique lift of $\lambda_V$ to $\Sigma^r \text{ko}$, the appropriate suspension of the connective $KO$-spectrum, which we denote by the same symbol. In the special case $V = -\gamma_d$, we denote the Thom class by $\lambda_{-d} \in [\text{MTSpin}(d), \Sigma^{-d}(\text{ko})]$. We are interested in the groups

$$J_{d,k} := \text{Im} ((\lambda_{-d})_* : \pi_k(\text{MTSpin}(d)) \to \pi_{d+k}(\text{ko})).$$
5.1. Multiplicative structure of Madsen–Tillmann–Weiss spectra. The spectrum $\text{ko}$ has a ring structure, and the algebraic structure of $\pi_*(\text{ko})$ is well-known, due to Bott periodicity. There are elements $\eta \in \pi_4(\text{ko})$, $\kappa \in \pi_4(\text{ko})$ and $\beta \in \pi_5(\text{ko})$, such that

$$\pi_*(\text{ko}) = \mathbb{Z}\langle \eta, \kappa, \beta \rangle/(2\eta, \eta^3, \kappa^2 - 4\beta, \kappa \eta).$$

(5.1.1)

Even though $\text{MTSpin}(d)$ is not itself a ring spectrum, there is a useful product structure available to us as the collection $\{\text{MTSpin}(d)\}_{d \geq 0}$ form what one might call a graded ring spectrum. Namely, there are maps

$$\mu : \text{MTSpin}(d) \times \text{MTSpin}(e) \to \text{MTSpin}(d + e)$$

which come from the bundle maps $\gamma_d \times \gamma_e \to \gamma_{d+e}$ which cover the Whitney sum maps $B\text{Spin}(d) \times B\text{Spin}(e) \to B\text{Spin}(d + e)$. The usual multiplicative property of Thom classes translates into the statement that the diagram

$$\text{MTSpin}(d) \times \text{MTSpin}(e) \xrightarrow{\mu} \text{MTSpin}(d + e)$$

$$\Sigma^{-d}\text{ko} \times \Sigma^{-e}\text{ko} \xrightarrow{\Sigma^{-d} \times \Sigma^{-e}} \Sigma^{-(d+e)}\text{ko}$$

(where the bottom horizontal map is the ring spectrum structure map) commutes up to homotopy. On the level of homotopy groups, this commutativity means that for $a \in \pi_k(\text{MTSpin}(d))$, $b \in \pi_l(\text{MTSpin}(e))$, we have

$$(\lambda_{-d} \times \lambda_{-e})_* (\mu(a, b)) = (\lambda_{-d})_*(a) \cdot (\lambda_{-e})_* (b) \in \pi_{d+e+k+l}(\text{ko}).$$

(5.1.2)

In order to write down elements in $\pi_* (\text{MTSpin}(d))$, the interpretation of this homotopy group in terms of Pontrjagin–Thom theory is useful.

**Theorem 5.1.3.** The group $\pi_k (\text{MTSpin}(d))$ is isomorphic to the cobordism group of triples $(M, \mathcal{V}, \phi)$, where $M$ is a closed $(k + d)$-manifold, $\mathcal{V} \to M$ a spin vector bundle of rank $d$ and $\phi : \mathcal{V} \oplus c_1^\mathcal{V} \cong TM$ a stable isomorphism of vector bundles.

This is just a special case of the classical Pontrjagin–Thom theorem, see e.g. [46, Chapter II]. There are homomorphisms

$$\pi_{k+1}(\text{MTSpin}(d-1)) \to \pi_k(\text{MTSpin}(d)) \to \Omega^\text{Spin}_{d+k}$$

(5.1.4)

where the symbol $\Omega^\text{Spin}_{d+k}$ denotes the ordinary spin cobordism group of $(d + k)$-manifolds. The first homomorphism sends $[M, \mathcal{V}, \phi] \to [M, \mathcal{V} \oplus \mathbb{R}, \phi]$, and the second forgets $\mathcal{V}$ and $\phi$ (but keeps the spin structure on $M$ that is induced by them). The homomorphism $\pi_k(\text{MTSpin}(d)) \to \Omega^\text{Spin}_{d+k}$ is surjective for $k \leq 0$ and bijective for $k < 0$. The image of $\pi_k(\text{MTSpin}(d)) \to \Omega^\text{Spin}_{d+k}$ (for $k > 0$) is the group of all cobordism classes which contain manifolds whose stable tangent bundle splits off an $k$-dimensional trivial summand. Any $d$-dimensional spin manifold $M$ defines an element $[M, TM, id] \in \pi_0(\text{MTSpin}(d))$, but this construction does not descend to a homomorphism $\Omega^\text{Spin}_d \to \pi_0(\text{MTSpin}(d))$.

The product has a pleasant description in terms of manifolds: if $[M_i, V_i, \phi_i] \in \pi_{k_i}(\text{MTSpin}(d_i))$, $i = 0, 1$, then

$$[M_0 \times M_1, V_0 \times V_1, \phi_0 \times \phi_1] = \mu([M_0, V_0, \phi_0], [M_1, V_1, \phi_1]) \in \pi_{k_0 + k_1}(\text{MTSpin}(d_0 + d_1)).$$

It is a consequence of the Atiyah–Singer index theorem that

$$(\lambda_{-d})_* ([M, \mathcal{V}, \phi]) = \text{ind}(\mathcal{F}_M) \in KO^{-d-k} = \pi_{k+d}(\text{ko})$$
for \([M, V, \phi] \in \pi_k(\text{MTSpin}(d))\). From now on, we will denote this invariant by the classical notation \(\hat{A}(M)\). For \(k + d \equiv 0 \pmod{4}\), the value of \(\hat{A}(M)\) can be computed in terms of characteristic classes by the formula

\[
\hat{A}(M) = \begin{cases} 
\langle \hat{A}(TM), [M] \rangle \cdot \beta^r & \text{if } d + k = 8r, \\
\frac{1}{2} \langle \hat{A}(TM), [M] \rangle \cdot \beta^r \kappa & \text{if } d + k = 8r + 4.
\end{cases}
\]  

(5.1.5)

For each \(d \geq 0\) there is a class \(e_d := [\ast, \mathbb{R}^d, \text{id}] \in \pi_{-d}(\text{MTSpin}(d))\), which is a generator for the group \(\pi_{-d}(\text{MTSpin}(d)) \cong \mathbb{Z}\). These classes clearly satisfy \(\mu(e_d, e_e) = e_{d+e}\) and \(\hat{A}(e_d) = 1\). Moreover, \(e_0\) is a unit for the multiplication \(\mu\). Multiplication by \(e_1\) defines a map

\[ S^{-1} \wedge \text{MTSpin}(d) \longrightarrow \text{MTSpin}(1) \wedge \text{MTSpin}(d) \longrightarrow \text{MTSpin}(d+1), \]  

(5.1.6)

which coincides with the analogous map in [22, §3] and which on homotopy groups induces the first map in (5.1.4). The composition

\[ S^{-d} e_d \rightarrow \text{MTSpin}(d) \xrightarrow{\lambda_{-d}} \Sigma^{-d} \text{ko} \]

is the \(d\)th desuspension of the unit map of the ring spectrum \(\text{ko}\). To sum up, we obtain a homotopy commutative diagram:

\[
\begin{array}{ccc}
S^0 & \xrightarrow{e_0} & \text{MTSpin}(0) \\
& e_1 \searrow & \lambda_{-0} \nearrow \eta_1 \\
& & \Sigma \text{MTSpin}(1) \xrightarrow{\lambda_{-1}} \Sigma \text{MTSpin}(2) \\
& & \eta_2 \\
& & \Sigma^2 \text{MTSpin}(2) \\
& & \vdots
\end{array}
\]

(5.1.7)

From (5.1.2) and (5.1.7), we obtain

**Corollary 5.1.8.** There are inclusions \(J_{d,k} \supset J_{d-1,k+1} \) and \(J_{d,k} J_{e,l} \subset J_{d+e,k+l}\).

5.2. **Proof of Theorem [A]** In this section we shall provide the homotopy theoretic calculations which, when combined with Theorem [B] and [C] establish Theorem [A].

We first investigate the effect of the maps \(\lambda_{-d}\) on rational homotopy groups.

**Theorem 5.2.1.** For each \(d \geq 2\) and \(d + k \equiv 0 \pmod{4}\), the map

\[ (\lambda_{-d})_* : \pi_k(\text{MTSpin}(d)) \otimes \mathbb{Q} \longrightarrow \pi_{k+d}(\text{ko}) \otimes \mathbb{Q} \cong \mathbb{Q} \]

is surjective.

**Proof.** The proof is a standard calculation with characteristic classes, but we present the details as they will be used later on.

Let \(\pi : V \rightarrow X\) be a complex vector bundle of rank \(n\) whose underlying real bundle has a spin structure. The spin structure determines a Thom class \(\lambda_V \in KO^{2n}(\text{Th}(V))\), which for this proof we shall write as \(\lambda_V^{\text{Spin}}\). On the other hand, the
complex structure determines a Thom class \( \lambda^C \in K^{-2n}(\text{Th}(V)) \). The groups \( SO(2), \) Spin(2), and \( U(1) \) are all isomorphic, but Spin(2) \( \to SO(2) \) is a double cover. Identifying all these groups with \( U(1) \), it follows that a spin structure on a complex line bundle is precisely a complex square root. In particular, the spin structure on \( V \) determines a square root \( \det(V)^{1/2} \) of the complex determinant line bundle of \( V \).

The relation between the Thom classes \( \lambda^\text{spin}_L \) and \( \lambda^C_L \) under the complexification map \( c : KO \to K \) is given by the following formula\(^1\) cf. [34, (D.16)]:

\[
c(\lambda^\text{spin}_L) = \lambda^C_L \cdot \pi^* \det(V)^{-1/2} \in K^{-2n}(\text{Th}(V)).
\]

If \( V \oplus V^\perp \cong \epsilon^2 \), we obtain, using that \( \det(V) \otimes \det(W) = \det(V \oplus W) \), the formula

\[
c(\lambda^\text{spin}_{-V}) = \lambda^C_{-V} \cdot \pi^* \det(V)^{1/2} \in K^{-2n}(\text{Th}(V^\perp)).
\]

This relation is preserved under stabilisation, and therefore we get an equation in the \( K \)-theory of the Thom spectrum

\[
c(\lambda^\text{spin}_L) = \lambda^C_{-L} \cdot \pi^* \det(V)^{1/2} \in K^{-2n}(\text{Th}(-V)).
\]

After these generalities let us begin the proof of the theorem. By Corollary 5.1.8 it is enough to consider the case \( d = 2 \), and as Spin(2) can be identified with \( U(1) \) we may identify \( B\text{Spin}(2) \) with \( CP^\infty \). Under this identification the universal rank 2 spin bundle is identified with \( L^2 \), the (realification of the) tensor square of the universal complex line bundle over \( CP^\infty \).

Specialising the above general theory to this case, we obtain

\[
c(\lambda^\text{spin}_{L^2}) = \lambda^C_{-L^2} \cdot \pi^* L \in K^{-2}(\text{MTSpin}(2)).
\]

The relation between the Chern character and the \( K \)-theory Thom class is\(^2\)

\[
\text{ch}(\lambda^C_{-L^2}) = \frac{c_1(L^2)}{1 - e^{c_1(L^2)}} \cdot u_{-2},
\]

where \( u_{-2} \in H^{-2}((\text{MTSpin}(2)); \mathbb{Q}) \) is the cohomological Thom class. If we define \( x := c_1(L) \), which generates \( H^2(\mathbb{CP}^\infty; \mathbb{Z}) \), then we obtain

\[
\text{ch}(c(\lambda^\text{spin}_{L^2})) = \frac{2x}{1 - e^{2x}} \cdot u_{-2} = -\frac{x}{\sinh(x)} \cdot u_{-2} \in H^*((\text{MTSpin}(2)); \mathbb{Q}),
\]

and following [38] Appendix B the identity \( \frac{1}{\sinh(x)} = \frac{1}{\tanh(x)} - \frac{1}{\tanh(2x)} \) yields

\[
\frac{x}{\sinh(x)} = 1 + \sum_{m=1} \frac{(-1)^m (2^{2m} - 2) B_m}{(2m)!} x^{2m},
\]

where \( B_m \) is the \( m \)th Bernoulli number (our notation for Bernoulli numbers also follows [38], Appendix B). We hence obtain the formula

\[
\text{ch}(c(\lambda^\text{spin}_{L^2})) = -\left( 1 + \sum_{m=1} (-1)^m \frac{(2^{2m} - 2) B_m}{(2m)!} x^{2m} \right) \cdot u_{-2}. \tag{5.2.3}
\]

---

\(^1\)It is important here to adopt the correct convention for \( K \)-theory Thom classes of complex vector bundles: one should take the convention used in [34] Theorem C.8, which is characterised by the identity \( \langle \lambda^C_L \rangle^2 = (1 - L) \cdot \lambda^C_L \in K^0(\text{Th}(L)) \) when \( L \to \mathbb{CP}^\infty \) is the universal line bundle.

\(^2\)This is obtained by taking Chern characters of \( \langle \lambda^C_L \rangle^2 = (1 - L^2) \cdot \lambda^C_L \), simplifying, and using multiplicativity.
The point of the proof is now that \( \frac{(2^m - 2)B_m}{(2m)!} \neq 0 \) for all \( m > 0 \). More precisely, consider the diagram

\[
\begin{array}{ccc}
\pi_{4m-2}(\text{MTSpin}(2)) \otimes \mathbb{Q} & \xrightarrow{(\lambda_{-2})_*} & \pi_{4m}(\text{ko}) \otimes \mathbb{Q} \\
\downarrow h & & \downarrow h \\
H_{4m-2}(\text{MTSpin}(2); \mathbb{Q}) & \xrightarrow{(\lambda_{-2})_*} & H_{4m}(\text{ko}; \mathbb{Q}) \\
\cong & & (ph_{4m}, -) \\
H_{4m}(B\text{Spin}(2); \mathbb{Q}) & \xrightarrow{e} & \mathbb{Q}.
\end{array}
\]

The upper vertical maps are the Hurewicz homomorphisms, which are isomorphisms by Serre’s finiteness theorem; the left bottom vertical arrow is the inverse Thom isomorphism, and the right bottom vertical map is the evaluation against the \( m \)-th component of the Pontrjagin character. The upper square is commutative, and the above calculation shows that the lower square commutes if \( e \) is the evaluation against \((-1)^{m+1}\frac{(2^m - 2)B_m}{(2m)!}x^{2m} \neq 0 \in H_{4m}(B\text{Spin}(2); \mathbb{Q}) \). As this class is nonzero, the lower horizontal map is onto, and so is the upper horizontal map, as claimed. \( \square \)

We now describe the effect of the maps \( \lambda_{-d} \) onto the \( \mathbb{Z}/2 \) summands.

**Theorem 5.2.4.** For each \( d \geq 0 \) and \( 0 \leq i \equiv 1, 2 \pmod{8} \), the map

\[
(\lambda_{-d})_* : \pi_i(\Sigma^d\text{MTSpin}(d)) \to \pi_i(\text{ko}) \cong \mathbb{Z}/2
\]

is surjective.

**Proof.** That the unit map \( S^0 \to \text{ko} \) hits all 2-torsion follows from the work of Adams on the \( J \)-homomorphism [2, Theorem 1.2]. But the \( d \)-fold desuspension of the unit map is \( \lambda_{-d} \circ e_d \), by the remarks before diagram (5.1.7). \( \square \)

### 5.3. Integral surjectivity.

The following implies Theorem [3].

**Theorem 5.3.1.** The homomorphism \( (\lambda_{-d})_* : \pi_k(\text{MTSpin}(d)) \to \pi_{d+k}(\text{ko}) \) is surjective for all \( k \leq d \).

**Proof.** If \( k + d \equiv 1 \) or 2 \pmod{8} \), the statement follows from Theorem [5.2.4]. Using the multiplicative structure (5.1.8) and (5.1.1), it will be enough to create elements \( \mathfrak{t} \in \pi_2(\text{MTSpin}(2)) \) with \( \mathcal{A}(\mathfrak{t}) = \kappa \) and \( b \in \pi_4(\text{MTSpin}(4)) \) with \( \mathcal{A}(b) = \beta \). For both cases, we use Theorem [6.1.3] and both elements will be given by a (2m - 1)-connected 4m-manifold, \( m = 1, 2 \), with the desired value of \( \mathcal{A}(M) \), plus a spin vector bundle \( V \to M \) of rank \( 2m \), and a stable isomorphism \( V \oplus \varepsilon_{2m}^3 \cong TM \). We will write \( u_M \in H^4(M) \) for the generator with \( \langle u_M, [M] \rangle = 1 \). Recall the formulae for the low-dimensional \( A \)-classes and the Hirzebruch classes

\[
\begin{align*}
\hat{A}_1 &= -\frac{1}{2^1 \cdot 3} p_1 \\
\hat{A}_2 &= \frac{1}{2^2 \cdot 3^2 \cdot 5} (-4p_2 + 7p_1^2) \\
L_1 &= \frac{1}{3} p_1 \\
L_2 &= \frac{1}{3^2 \cdot 5} (7p_2 - p_1^2).
\end{align*}
\]

We first construct the element \( \mathfrak{t} \in \pi_2(\text{MTSpin}(2)) \). Let \( K \) be a K3 surface, which is a simply connected spin manifold. It is well-known that the intersection form of \( K \) is \( q = 2(-E_8) \oplus 3H \), the direct sum of two times the negative \( E_8 \)-form
and three hyperbolic summands. The signature of this form is $-16$, which by Hirzebruch’s signature theorem means that $p_1(TK) = -48u_K$. Therefore, $\tilde{A}(TK) = 2u_K$ and so $\delta f(K) = \kappa$ by (6.1.3), as required. We claim that there exists a complex line bundle $L \to K$ such that $p_1(L \otimes \mathbb{C}) = p_1(TK)$. Let $a \in H^2(K)$ and $L_a$ be the line bundle with $c_1(L_a) = a$. Since

$$p_1(L_a \otimes \mathbb{C}) = c_1(L_a \otimes \mathbb{C})^2 = 4q(a) \cdot u_K,$$

we have to pick $a$ such that $q(a) = -12$. It is easy to see that a quadratic form which contains a hyperbolic summand represents any even number, and therefore such an $a$ exists. We now claim that $TK$ and $L \otimes \mathbb{C} \oplus \epsilon_R^2$ are stably isomorphic. To see this, we must show that the triangle in the following diagram commutes.

$$\begin{array}{ccc}
BSpin(2) & \longrightarrow & K(\mathbb{Z}, 4) \\
\downarrow & & \downarrow \\
L_a \otimes \mathbb{C} & \longrightarrow & BSpin \longrightarrow \frac{p_1}{2} \\
TK & \longrightarrow & BSpin \\
\end{array}$$

As the map $p_1/2 : BSpin \longrightarrow K(\mathbb{Z}, 4)$ is 7-connected, it is enough to show that $p_1(TK) = p_1(L_a \otimes \mathbb{C} \oplus \epsilon_R^2) \in H^4(K; \mathbb{Z})$, but we have arranged for this to be true. Hence there is a stable isomorphism $\phi : L_a \otimes \mathbb{C} \oplus \epsilon_R^2 \cong TK$ and the element $t = [K, L_a, \phi] \in \pi_2(M_{Spin}(2))$ has the desired properties.

In the 8-dimensional case, the proof is similar, but more difficult (with the exception that every vector bundle on a 3-connected 8-manifold is spin, so we do not have to take care of this condition). Let $P$ be the 8-dimensional $E_8$-plumbing manifold and consider $\sharp^{28}P$, the boundary connected sum of 28 copies of $P$. This is a parallelisable manifold, with signature $2^5 \cdot 7$, and the boundary $\partial(\sharp^{28}P)$ is diffeomorphic to $S^7$, by the calculation of Kervaire–Milnor [30] that the group of homotopy 7-spheres is a cyclic group of order 28. The manifold $M := \sharp^{28}N \cup_{S^7} D^8$ is parallelisable away from a point, and therefore $p_1(TM) = 0$. The Hirzebruch signature theorem and (6.1.2) shows that

$$p_2(TM) = 2^5 \cdot 3^2 \cdot 5u_M$$

and $\tilde{A}(TM) = u_M$, so that $\delta g(M) = \beta$, by (6.1.3). For $r \geq 0$ let $K_r := \sharp^r(S^4 \times S^4)$ and $M_r := M \sharp K_r$. This is still parallelisable away from a point, whence $p_1(TM_r) = 0$, and since $M_r$ is cobordant to $M$, we still have $\delta g(M_r) = \beta$. We now claim that for $r = 12$, we can find a 4-dimensional vector bundle $V_r \to K_r$ so that the connected sum of $V_r$ with the trivial bundle over $M$ yields a vector bundle over $M_r$ with the same Pontrjagin classes as $TM_r$. Consider the exact sequence

$$KO^{-1}(M_r^{(4)}) \rightarrow \tilde{KO}^0(S^8) \rightarrow KO^0(M_r) \rightarrow KO^0(M_r^{(4)}) \rightarrow \tilde{KO}^1(S^8) = 0$$

coming from the cofibre sequence $M_r^{(4)} \to M_r \to S^8$. The group $KO^{-1}(M_r^{(4)})$ is torsion, as $M_r^{(4)}$ is a bouquet of 4-spheres. Since $\tilde{KO}^0(S^8) \cong \mathbb{Z}$, the map $\delta$ is zero. Because $KO^0(M_r^{(4)})$ is free abelian, it follows that $KO^0(M_r)$ is torsion-free, and this implies that the Pontrjagin character $\phi : KO^0(M_r) \to H^*(M_r; \mathbb{Q})$ is injective. Therefore, a stable vector bundle on $M_r$ is determined by its Pontrjagin classes. Thus if we are able to find a 4-dimensional vector bundle $V_r \to K_r$ such that $c_2(V_r) \to M \sharp K_r = M_r$ has the same Pontrjagin classes as $TM_r$, then it will be stably isomorphic to $TM_r$. 
Isomorphism classes of 4-dimensional stably trivial vector bundles on $K_r - \ast \simeq \sqrt[r]{S^4 \vee S^4}$ are in bijection with $2H^4(K_r) \cong (\mathbb{Z}^2)^r$, the group of cohomology classes that are divisible by 2. The bijection is given by the Euler class. For each such vector bundle, there is an obstruction in

$$\pi_7(B\text{Spin}(4)) \cong \pi_6(S^3 \times S^3) \cong \mathbb{Z}/12 \oplus \mathbb{Z}/12$$

against extending the vector bundle over $K_r$ (the isomorphism $\pi_6(S^3) \cong \mathbb{Z}/12$ is classical, see [39, p. 186]). If we take $r = 12$ and the same vector bundle on each copy of $S^4 \vee S^4$, then the obstruction is zero, and the vector bundle can be extended. In other words, for each $a \in 2H^4(S^4 \times S^4)$, we obtain a 4-dimensional vector bundle $V_a$ on $K_{12} = \mathbb{Z}/12(S^4 \times S^4)$ with Euler class

$$(a, \ldots, a) \in H^4(K_{12}; \mathbb{Z}) \cong H^4(K_r; \mathbb{Z}).$$

When restricted to the 4-skeleton, the bundle $V_a$ is stably trivial and so $p_1(V_a) = 0$. Now we take the connected sum of $V_a$ with the trivial vector bundle on $M$ and get a vector bundle $W_{12} \to M_{12} = M_{12}K_{12}$, with $p_1(W_{12}) = 0$ and Euler class

$$e(W_{12}) = (a, \ldots, a, 0) \in H^4(K_{12}) \cong H^4(M) = H^4(M_{12}).$$

Let $q_0$ be the intersection form on $H^4(S^4 \times S^4)$, and compute

$$p_2(W_{12}) = e(W_{12})^2 = q(a, \ldots, a, 0)u_{M_{12}} = 12q_0(a)u_{M_{12}}.$$ 

In order to achieve that $p_2(W_{12}) = p_2(TM_{12}) = 2^5 \cdot 3^2 \cdot 5u_{M_{12}}$, we have to find an even $a$ so that $q_0(a) = 2^3 \cdot 3 \cdot 5$. As in the four-dimensional case, we can find an even $a$ with $q_0(a) = 8s$, for each $s \in \mathbb{Z}$, and picking $s = 15$ finishes the proof. □

5.3.1. Homological conclusions. We can use Theorem 5.3.1 to obtain results on the image of $H_*(\Omega_{\infty+1}MT\text{Spin}(d); \mathbb{F}) \to H_*(\Omega_{\infty+d+1}\text{ko}; \mathbb{F})$ when $\mathbb{F}$ is a field. When $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_2$, the result is particularly nice. For $\mathbb{F} = \mathbb{Q}$ and $d \geq 2$, we find that

$$H_*(\Omega_{\infty+1}MT\text{Spin}(d); \mathbb{Q}) \to H_*(\Omega_{\infty+d+1}\text{ko}; \mathbb{Q})$$

is surjective, using Theorem 5.2.1. For $\mathbb{F} = \mathbb{F}_2$, we have the following result, proving Theorem 5.3.1.

**Proposition 5.3.3.** For $n \geq 0$, the Thom class maps

$$\Omega^{\infty+1}\lambda_{-2n} : \Omega^{\infty+1}MT\text{Spin}(2n) \to \Omega^{\infty+2n+1}\text{ko}$$
$$\Omega^{\infty+2}\lambda_{-2n} : \Omega^{\infty+2}MT\text{Spin}(2n) \to \Omega^{\infty+2n+2}\text{ko}$$

are surjective on $\mathbb{F}_2$-homology.

**Proof.** Since $\Omega^{\infty+1}\lambda_{-2n}$ is an infinite loop map, its image in $\mathbb{F}_2$-homology contains the algebra over the Dyer–Lashof algebra generated by the Hurewicz image of $\lambda_{-2n}$. We require some information about $H_*(\Omega^{\infty+1}\text{ko}; \mathbb{F}_2) = H_*(\Omega^k(\mathbb{Z} \times BO); \mathbb{F}_2)$ as algebras over the Dyer–Lashof algebra. These are monogenic: they are generated as an algebra over the Dyer–Lashof algebra by a single class. When $k \equiv 0, 1, 2, 4 \mod 8$, so $\Omega^k(\mathbb{Z} \times BO)$ is disconnected, the class $[1] \in H_0(\Omega^k(\mathbb{Z} \times BO); \mathbb{F}_2)$ of the path component corresponding to a generator of $\pi_k(\mathbb{Z} \times BO)$ generates all the homology, and for the remaining $k$ the unique nontrivial class in the lowest nonvanishing reduced homology group generates all the homology. These claims follow from the calculations of Kochman [51] and Priddy [40], a pleasant reference for which is [59]. On the other hand, in Theorems 5.2.4 and 5.3.1, we have shown that these classes are contained in the image of $\Omega^{\infty+j}\lambda_{-2n}$, $j = 1, 2$. □
5.4. Away from the prime 2. Away from the prime 2, we can considerably improve the surjectivity result Theorem 5.3.3 by applying work of Madsen and Schlichtkrull [36]. All spaces that occur in the sequel are infinite loop spaces; in particular they are simple in the sense of homotopy theory, and so they admit localisations at primes in the sense of Sullivan [47].

Theorem 5.4.1. Let \( p \) be an odd prime. There is a loop map \( f : (\Omega_0^{\infty+2}KO)(p) \to \Omega_0^{\infty+2}MTSpin(2)(p) \) such that the composition

\[
(\Omega_0^{\infty+2}KO)(p) \xrightarrow{f} \Omega_0^{\infty}MTSpin(2)(p) \xrightarrow{\Omega^\infty_L - 2} (\Omega_0^{\infty+2}KO)(p)
\]
on \( \pi_{4m-2}(-) \) is multiplication by a \( p \)-local unit times \( (2^{2m-1} - 1) \cdot \text{Num}(\frac{m^2}{2m}) \).

Proof. Let us write \( \Omega_0^2(Z \times BO) \) for \( \Omega_0^{\infty+2}KO \). We will deduce this from work of Madsen–Schlichtkrull [36]. Their work allows us to construct the following homotopy commutative diagram, where all infinite loop spaces are implicitly localised at \( p \), and \( k \in \mathbb{N} \) is chosen such that its residue class generates \((\mathbb{Z}/p^2)^\times\).

Here \( MTSO(2) = \text{Th}(-\gamma_2) \) is the Thom spectrum of minus the tautological bundle \( \gamma_2 \to BSO(2) \); it receives a spectrum map from \( MTSpin(2) \) coming from the map \( BSpin(2) \to BSO(2) \). Identifying \( BSO(2) \) with \( \mathbb{CP}^\infty \), and \( \gamma_2 \) with the tautological complex line bundle \( L \to \mathbb{CP}^\infty \), the map \( \omega \) is the “inclusion” map \( \text{Th}(\!-\!L) \to \text{Th}(\!-\!L \oplus L) = \Sigma^\infty \mathbb{CP}^\infty \). The map \( \rho \) is obtained by looping the complexification maps twice, and \( \psi^k \) is the \( k \)th Adams operation. The diagram is constructed as follows.

(i) The square \( \square \) is constructed in [36], and the only property we require of it, apart from its existence, is the fact that \( \Omega s \circ (L - 1) \simeq \text{inc} - [1] \), cf. proof of Lemma 7.5 in [36].

(ii) The map \( f \) is defined so as to make the square \( \square \) commute. This uses the fact that the lower map in \( \square \) is an equivalence, as we are working at an odd prime so \( BSpin(2) \to BSO(2) \) is a \( p \)-local equivalence.

(iii) In the square \( \square \) the left hand map is the infinite loop map of \( \lambda_{-2} \), the \( KO \)-theory Thom class of \( MTSpin(2) \).

(iv) In the square \( \square \) the right hand map is the infinite loop map corresponding to the class \( r(t) \cdot \lambda^c_{-L} \in K_0(p)(MTSO(2)) \) in the \( p \)-local \( K \)-theory of \( MTSO(2) \), where \( \lambda^c_{-L} \in K_0(p)(MTSO(2)) \) is the Thom class, and

\[
r(t) = \sqrt{1 + t} \in K_0(p)(BSO(2)) = \mathbb{Z}_p[[t]] \quad t = L - 1
\]
is the formal power series expansion of $\sqrt{1+t}$, whose coefficients lie in $\mathbb{Z}[\frac{1}{2}]$ so are $p$-local integers for any odd prime $p$, so this defines an element in $p$-local $K$-theory. Under the map $BS\text{Spin}(2) \to BSO(2)$ the line bundle $L$ pulls back to $L^\otimes 2$, and so $r(t)$ pulls back to $L$.

(v) The commutativity of square (i) is another way of expressing formula (5.2.2) from the proof of Theorem 5.2.1.

We wish to compute the effect of the composition $\Omega^\infty \lambda_{-2} \circ f$ on $\pi_{4m-2}$, where it must be multiplication by some $p$-local integer $A_m \in \mathbb{Z}_{(p)}$. As the complexification map

$$\rho : \pi_{4m-2}(\Omega^2_0(\mathbb{R} \times BO)) \longrightarrow \pi_{4m-2}(BU)$$

is an isomorphism, the effect of the composition $\Omega^\infty (r(t) \cdot \chi_L^c) \circ \Omega \tilde{s}$ on $\pi_{4m-2}$ must also be multiplication by $A_m$. Classes in $\pi_{4m-2}(BU)$ are detected faithfully by their evaluations against the Chern character class $\text{ch}$.

Thirdly, we wish to compute $(\Omega^\infty \lambda_{-2}) \circ \text{ch}_{2m-1}$, and $\Omega^\infty (r(t) \cdot \chi_L^c) \circ \Omega \tilde{s}$ is a loop map so respects primitives and must therefore send $\text{ch}_{2m-1}$ to $A_m \cdot \text{ch}_{2m-1}$. Thus we may compute in rational cohomology.

Let us identify $BSO(2)$ with $\mathbb{CP}^\infty$, so the tautological bundle is given by the universal complex line bundle $L$. Write $x := c_1(L) \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$. Firstly

$$(\Omega^\infty (r(t) \cdot \chi_L^c))^* \text{ch}_{2m-1} = (-1)^{m+1} \frac{(2^m - 2) \cdot B_m}{(2m)!} \sigma^* (x^{2m} \cdot u_{-2}),$$

where $\sigma^* : H^*(\text{MTSO}(2)) \to H^*(\Omega^\infty \text{MTSO}(2))$ denotes the cohomology suspension. This comes from formula (5.4.3) of the proof of Theorem 5.2.1. As $\omega^*(x^t) = x^{t+1} \cdot u_{-2}$, we may write the above equation as

$$(\Omega^\infty (r(t) \cdot \chi_L^c))^* \text{ch}_{2m-1} = (-1)^{m+1} \frac{(2^m - 2) \cdot B_m}{(2m)!} \omega^*(\Omega^\infty \omega)^* (\sigma^* (x^{2m-1})).$$

Secondly, we wish to compute $(\Omega \tilde{s})^* (\sigma^* (x^{2m-1}))$. The class $\sigma^* (x^{2m-1})$ is primitive and $\Omega \tilde{s}$ is a loop map, so this class will again be primitive and so a multiple of $\text{ch}_{2m-1}$. To determine which multiple, we may pull it back further to $\mathbb{CP}^\infty$, where

$$(\Omega \tilde{s} \circ (L - 1))^* (\sigma^* (x^{2m-1})) = (\text{inc} - [1])^* (\sigma^* (x^{2m-1})) = x^{2m-1}.$$ 

As $(L - 1)^* \text{ch}_{2m-1} = x^{2m-1} / (2m - 1)!$, we find that

$$(\Omega \tilde{s})^* (\sigma^* (x^{2m-1})) = (2m - 1)! \cdot \text{ch}_{2m-1}.$$ 

Thirdly, we wish to compute $(\Omega^2 (\psi^k - 1))^* \text{ch}_{2m-1}$. Again, this will be primitive and we may find which multiple of $\text{ch}_{2m-1}$ it is by pulling back further to $\mathbb{CP}^\infty$. The map $\Omega^2 (\psi^k - 1) \circ (L - 1)$ classifies the virtual vector bundle $kL^k - L - (k - 1)$, which may be seen by the formula

$$(\psi^k - 1)(\beta \otimes (L - 1)) = \beta \otimes (kL^k - L - (k - 1)) \in K^0(S^2 \wedge \mathbb{CP}^\infty),$$

where $\beta \in K^0(S^2)$ is the Bott element. Thus it pulls $\text{ch}_{2m-1}$ back to $(k^{2m-1} - 1) / (2m - 1)! \cdot x^{2m-1}$, and so

$$(\Omega^2 (\psi^k - 1))^* \text{ch}_{2m-1} = (k^{2m-1} - 1) \cdot \text{ch}_{2m-1}.$$ 

Fourthly, the commutativity of (ii) along with (5.4.3), (5.4.4), and (5.4.4) gives

$$(\Omega^\infty (r(t) \cdot \chi_L^c) \circ \Omega \tilde{s})^* \text{ch}_{2m-1} = (-1)^{m+1} \frac{(2^m - 2) \cdot B_m}{2m} \cdot (k^{2m-1} - 1) \cdot \text{ch}_{2m-1}.$$
and so
\[ A_m = (2^{m-1} - 1) \cdot \text{Num}(\frac{B_m}{2m}) \cdot \left( (-1)^{m+1} \cdot 2 \cdot \frac{k^{2m} - 1}{\text{Den}(\frac{B_m}{2m})} \right). \]

Finally, it is well-known that $\frac{(k^{2m} - 1)}{\text{Den}(\frac{B_m}{2m})}$ is a $p$-local unit whenever $p$ is an odd prime and $k$ generates $(\mathbb{Z}/p^2)\times$. This follows from Lemma 2.12 and Theorem 2.6 of [1], together with Von Staudt’s Theorem. Hence $A_m$ is $(2^{m-1} - 1) \cdot \text{Num}(\frac{B_m}{2m})$ times a $p$-local unit, as claimed.

There are two types of consequences of Theorem 5.4.1. The first concerns the index of the groups $J_{d,k}[\frac{1}{2}] \subset \pi_{d+k}(\text{ko})[\frac{1}{2}]$, and the other is a splitting result for the index difference. We begin with the homotopy group statements. One derives from Theorem 5.4.1:

Corollary 5.4.5. The subgroup $J_{2,4m-2}[\frac{1}{2}] \subset \pi_{4m}(\text{ko})[\frac{1}{2}]$ has finite index which divides $(2^{m-1} - 1) \cdot \text{Num}(\frac{B_m}{2m})$.

Next, we adopt the convention that $(2^{m-1} - 1) \cdot \text{Num}(\frac{B_m}{2m}) = 1$ for $m = 0$ and set
\[ A(m,n) := \gcd \left\{ \prod_{i=1}^{n} (2^{m_i} - 1) \cdot \text{Num} \left( \frac{B_{m_i}}{2m_i} \right) \right\}. \]

Using products and Corollary 5.4.5, we find that $J_{2n,4m-2n}[\frac{1}{2}] \subset \pi_{4m}(\text{ko})[\frac{1}{2}]$ has finite index dividing $A(m,n)$. Using the maps 5.1.0 we arrive at the following conclusion, which proves Theorem 5.

Corollary 5.4.7. For each $n, m, q \geq 0$, $J_{2n+q,4m-2n-q}[\frac{1}{2}] \subset \pi_{4m}(\text{ko})[\frac{1}{2}]$ has finite index dividing $A(m,n)$.

The strength of this result can be demonstrated by some concrete calculations. Recall that $\text{Num}(\frac{B_m}{2m}) = \pm 1$ for $m \in \{1, 2, 3, 4, 5, 7\}$ and $\text{Num}(\frac{B_6}{2}) = -691$, so any prime $p$ dividing $A(m, 2)$ in particular divides each of
\[
1 \cdot (2^{m-1} - 1) \cdot \text{Num}(\frac{B_m}{2m})
\]
\[
1 \cdot (2^{m-3} - 1) \cdot \text{Num}(\frac{B_m}{2(m-1)})
\]
\[
7 \cdot (2^{m-5} - 1) \cdot \text{Num}(\frac{B_m}{2(m-2)})
\]
\[
31 \cdot (2^{m-7} - 1) \cdot \text{Num}(\frac{B_m}{2(m-3)})
\]
\[
127 \cdot (2^{m-9} - 1) \cdot \text{Num}(\frac{B_m}{2(m-4)})
\]
\[
7 \cdot 73 \cdot (2^{m-11} - 1) \cdot \text{Num}(\frac{B_m}{2(m-5)})
\]
\[
691 \cdot 23 \cdot 89 \cdot (2^{m-13} - 1) \cdot \text{Num}(\frac{B_m}{2(m-6)})
\]
\[
8191 \cdot (2^{m-15} - 1) \cdot \text{Num}(\frac{B_m}{2(m-7)})
\]

Computer calculation shows that for $m \leq 2001$ the first three conditions already imply that $A(m, 2) = 1$, and so $J_{4m-4}[\frac{1}{2}] = \pi_{4m}(\text{ko})[\frac{1}{2}]$, in this range. Therefore, further taking products, we see that
\[
J_{4t+4k-2}[\frac{1}{2}] = \pi_{4t+4k}(\text{ko})[\frac{1}{2}]
\]
for $q \geq 0$ and $k \leq 2000 \cdot \ell$. 

5.4.1. A homotopy splitting. The final of our computational results is the following splitting theorem. To state it, we introduce a condition on prime numbers.

**Definition 5.4.8.** Recall that an odd prime number is called regular if it does not divide any of the numbers \( \text{Num}(\frac{B_m}{2m}) \). We say that an odd prime is very regular if in addition it does not divide any of the numbers \( (2^{2m-1} - 1) \).

**Remark 5.4.9.** The usual definition of a regular prime \( p \) is one which does not divide \( \text{Num}(B_m) \) for any \( 2m \leq p - 3 \). This is equivalent to not dividing \( \text{Num}(\frac{B_m}{2m}) \) for any \( 2m \leq p - 3 \), and is also equivalent to not dividing \( \text{Num}(\frac{B_m}{2m}) \) for any \( m \): If such a \( p \) divided \( \text{Num}(\frac{B_m}{2m}) \) for some \( 2n > p - 3 \) then we cannot have \( p - 1 \mid 2n \) (as then \( p \mid \text{Den}(\frac{B_m}{2m}) \) by von Staudt’s theorem) so we must have \( p - 1 \mid 2n \). Thus we may find \( 0 \neq 2n \equiv 2m \mod (p - 1) \) with \( 2m \leq p - 3 \). Kummer’s congruence (of \( p \)-integers) \( \frac{B_m}{2m} \equiv \frac{B_m}{2m} \mod p \) then contradicts \( p \) being regular in the usual sense.

**Remark 5.4.10.** By Fermat’s little theorem, if a prime \( p \) divides \( (2^{2m-1} - 1) \) for some \( m \), then it also divides \( (2^{2m'-1} - 1) \) for some \( m' \leq \frac{p-1}{2} \). Hence it is easy to check the condition of not dividing any \( 2^{2m-1} - 1 \), and for example of the regular primes less than \( 100 \) the primes 7, 23, 31, 47, 71, 73, 79 and 89 are not very regular, and the remaining primes 3, 5, 11, 13, 17, 19, 29, 41, 43, 53, 61, 63, 83, and 97 are very regular.

The space \( R^+(S^d) \) is an \( H \)-space, and the component \( R_0^+(S^d) \) of the round metric is a grouplike \( H \)-space. Therefore, it is a simple space in the sense of homotopy theory and admits a \( p \)-localisation \( R_0^+(S^d)(p) \) for each prime number \( p \).

**Theorem 5.4.11.** For \( d \geq 6 \) and each odd very regular prime, there is a weak homotopy equivalence

\[
R_0^+(S^d)(p) \simeq \Omega_0^{\infty+d+1} \text{ko}(p) \times F(p)
\]

where \( F \) is the homotopy fibre of the index difference map

\[
\text{indiff}_{gd} : R_0^+(S^d) \to \Omega_0^{\infty+d+1} \text{ko}.
\]

**Proof.** We consider the composition

\[
\Omega_0^{\infty+d+1} \text{ko}(p) \xrightarrow{\Omega_0^{\infty+d-1} \text{MTSpin}(2)(p)} \Omega_0^{\infty+d-5} \text{MTSpin}(6)(p) \xrightarrow{\text{MTSpin}(6)} \Omega_0^{\infty+d+1} \text{ko}(p).
\]

The first map is from Theorem 5.4.1 and the second is the map \( \text{MTSpin}(4) \). The third map is the map from Theorem 3.6.1 which gives the left triangle of the (weakly) homotopy commutative diagram

\[
\Omega_0, R^+(S^{m-1}) \xrightarrow{T} R^+(D^m) \xrightarrow{\mu_{m,n}} R^+(S^n) \xrightarrow{\text{indiff}_{D^m}} \Omega_0^{\infty+m+1} \text{ko}.
\]

The (homotopy) commutativity of the right triangle follows from the additivity theorem, more precisely Theorem 3.4.10. The \((d-6)\)-fold iteration of maps along
the top of this diagram (and \( p \)-localisation) yields the map \( L \) in \((5.4.12)\); the commutativity up to homotopy of this diagram shows that \( \text{inndiff}_g \circ L \) is homotopic to \( \Omega^{d-6} \text{indiff}_g \). By Theorem \(1\) the composition
\[
\Omega_0^{\infty+d-5} \text{MTSpin}(6)_p \longrightarrow \Omega^{d-6} \mathcal{R}_0^+ (S^d)(p) \xrightarrow{L} \mathcal{R}_0^+ (S^d)(p) \xrightarrow{\text{inndiff}} \Omega_0^{\infty+d+1} \text{ko}(p)
\]
is the same as \( \Omega^{\infty+d-5} \lambda_{-6} \), and so precomposing with the map
\[
\Omega_0^{\infty+d-1} \text{MTSpin}(2)_p \longrightarrow \Omega_0^{\infty+d-5} \text{MTSpin}(6)_p
\]
gives \( \Omega^{\infty+d+1} \lambda_{-2} \). Therefore, by Theorem \((5.4.1)\) the composition \((5.4.12)\) induces multiplication by a \( p \)-local unit times \((2^{2m-1}-1) \cdot \text{Num}(\frac{B_\mathbb{Z}}{\mathbb{Z}})\), which proves Theorem \(2\). If \( p \) is in addition very regular then the composition \((5.4.12)\) is a weak homotopy equivalence, so composing all but the last map gives a composition
\[
\Omega_0^{\infty+d+1} \text{ko}(p) \xrightarrow{\theta} \mathcal{R}_0^+ (S^d)(p) \xrightarrow{\text{inndiff}} \Omega_0^{\infty+d+1} \text{ko}(p)
\]
which is a weak homotopy equivalence.

Let \( j : F \to \mathcal{R}_0^+ (S^d) \) be the inclusion of the homotopy fibre of the map \( \text{inndiff}_g \). Implicitly localising all spaces at \( p \), it follows that the map
\[
\pi_*(\Omega^{\infty+d+1}_0 \text{ko}) \oplus \pi_*(F) \xrightarrow{\delta \oplus j_*} \pi_*(\mathcal{R}_0^+ (S^d)) \oplus \pi_*(\mathcal{R}_0^+ (S^d)) \xrightarrow{\delta} \pi_*(\mathcal{R}_0^+ (S^d))
\]
is an isomorphism. But \( \mathcal{R}_0^+ (S^d)(p) \) is an \( H \)-space, and therefore, by the classical Eckmann–Hilton lemma, addition on \( \pi_*(\mathcal{R}_0^+ (S^d)(p)) \) is induced by the \( H \)-space multiplication \( \mu \) on \( \mathcal{R}_0^+ (S^d)(p) \). Therefore, the map
\[
\mu \circ (\theta \times j) : \Omega_0^{\infty+d+1}_0 \text{ko}(p) \times F(p) \longrightarrow \mathcal{R}_0^+ (S^d)(p)
\]
is a weak homotopy equivalence. \( \square \)

5.4.2. Sharpness. Recall that we write \( c : \text{KO} \to \text{K} \) for the complexification map. The identity (cf. \( \text{[34]} \) Proposition 12.5])
\[
\text{ch}(c(\lambda_{-2n})) = (-1)^n \hat{A}(\gamma_{2n}) \cdot u_{-2n} \in H^*(\text{MTSpin}(2n); \mathbb{Q})
\]
yields the commutative diagram
\[
\begin{array}{c}
\pi_{4m-2n}(\text{MTSpin}(2n))[\frac{1}{2}] \\
\bigg| \\
H_{4m-2n}(\text{MTSpin}(2n); \mathbb{Z}[\frac{1}{2}])
\end{array} \xrightarrow{\chi_{2m}} \begin{array}{c}
\pi_{4m-2n}(\Sigma^{-2n} \text{ko})[\frac{1}{2}] \\
\bigg| \\
H_{4m}(\text{BSpin}(2n); \mathbb{Z}[\frac{1}{2}])
\end{array} \xrightarrow{(-1)^n \hat{A}_{4m}} \mathbb{Q}.
\]
The map \( \prod_{i=1}^n \text{BSpin}(2) \to \text{BSpin}(2) \) induces a surjection on homology with \( \mathbb{Z}[\frac{1}{2}] \)-coefficients (as with these coefficients \( \text{BSpin}(k) \to \text{BO}(k) \) is an isomorphism), and so the image of
\[
\hat{A}_{4m} : H_{4m}(\text{BSpin}(2n); \mathbb{Z}[\frac{1}{2}]) \longrightarrow \mathbb{Q}
\]
is the same as the image of
\[
\hat{A}_{4m} : \left( \bigotimes_{i=1}^n H_* (\text{BSpin}(2); \mathbb{Z}[\frac{1}{2}]) \right)_{4m} \longrightarrow \mathbb{Q}.
\]
Identify \( \text{BSpin}(2) \) with \( \mathbb{C}P^{\infty} \), so that \( L^{\otimes 2} \) is the universal spin rank 2 bundle, and let \( x = c_1(L) \). Write \( x_i \) for the pullback of \( x \) to the \( i \)th factor of \( \prod_{i=1}^n \text{BSpin}(2) \).
Then the \( \hat{A} \) class of the sum of universal bundles over \( \prod_{i=1}^{n} B \text{Spin}(2) \) is, by multiplicativity,
\[
\prod_{i=1}^{n} \frac{x_i}{\sinh(x_i)},
\]
and if we let \( A_\ell := (-1)^{\ell} \frac{(2^{2\ell-2})^{B_\ell}}{(2\ell)!} \) be the coefficient of \( x^{2\ell} \) in \( \frac{x}{\sinh(x)} \) then the image of \( A_{4m} \) is precisely the \( \mathbb{Z}[\frac{1}{2}] \)-linear span
\[
\left\langle \prod_{i=1}^{n} A_{m_i} \left| \sum_{i=1}^{n} m_i = m \right. \right\rangle \subset \mathbb{Q}.
\]

For a prime \( p \) such that \( 4m + 1 < 2p - 3 \) the map \( h(p) \) is an isomorphism by the Atiyah–Hirzebruch spectral sequence
\[
E^2_{s,t} = H_s(M \text{Spin}(2n); \pi_t(S^0)_{(p)}) \Rightarrow \pi_{s+t}(M \text{Spin}(2n))_{(p)},
\]
as \( H_*(M \text{Spin}(2n); \mathbb{Z}_{(p)}) \) is torsion-free, and the first \( p \)-torsion in the stable homotopy groups of spheres is in degree \( 2p - 3 \). In this case we find that each \( A_\ell \) with \( \ell \leq m \) is a \( p \)-integer (\( 2\ell \leq 2m < p \) shows that \( (2\ell)! \) is a \( p \)-local unit, and von Staudt’s theorem shows that \( \text{Den}(\frac{B_\ell}{2}) \) is a \( p \)-local unit) and so if we write \( j_{d,k} \) for the index of \( J_{d,k} \) in \( KO_{d+k} \) then its \( p \)-adic valuation is
\[
v_p(j_{2n,4m-2n}) = \gcd \sum_{m_i=m} v_p \left( \prod_{i=1}^{n} (2^{2m_i-1} - 1) \cdot \text{Num}(B_{m_i}/2m_i) \right).
\]

Thus, in terms of the constants \( A(m, n) \) defined in \( \S 5.4.6 \), there is an identity
\[
j_{2n,4m-2n} = A(m, n) \cdot \frac{A}{B}
\]
where (by Corollary \( \S 5.4.7 \)) \( A \) is a power of 2, and \( B \) is an integer all of whose prime factors \( p \) satisfy \( p \leq 2m + 2 \).

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