C*-TENSOR CATEGORIES AND FREE PRODUCT BIMODULES

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Abstract. A C*-tensor category with simple unit object is realized by von Neumann algebra bimodules of finite Jones index if and only if it is rigid.

Introduction

Given a subfactor $N \subset M$ of finite Jones index, the associated sequence of higher relative commutants $\{N_i \cap M_j\}_{j \geq 0}$ with $M_0 = M$ contains rich information on the relative position of the inclusion and has been a good source of combinatorial structures behind subfactors.

An axiomatization of higher relative commutants is performed by S. Popa in the form of so-called standard lattices, which is based on a generalization of his preceding result on free product construction of irreducible subfactors of arbitrarily given indices. On the other hand, F. Radulescu invented a method to construct free product subfactors from commuting squares satisfying some strong conditions on non-degeneracy and connectedness.

Under the background of these results on subfactors, we shall present here an abstract characterization of C*-tensor categories which can be realized by bimodules of finite Jones index.

To do this, we need to impose some structure of rigidity in C*-tensor categories: we assume Frobenius duality together with a systematic choice of dual objects in C*-tensor categories. The notion of Frobenius duality is formulated in [21] as an abstraction from the Ocneanu's Frobenius reciprocity in bimodules of finite Jones index (see [4], Chapter 9), which produces promptly all the combinatorial features in subfactor theory such as commuting squares, Markov traces, Perron-Frobenius eigenvectors and so on ([24]).

Frobenius duality, together with the associated cyclic tensor products, is utilized in our previous paper [3] to realize amenable C*-tensor categories as bimodules over the AFD II$_1$-factor, where random walks on the fusion algebra (the Grothendieck ring of a tensor category) is coupled with the structure of tensor categories. The constructed bimodules are referred to as random walk bimodules in what follows.

Note here that random walk bimodules are based on non-factor von Neumann algebras and far from being irreducible generally. (The keypoint in [3] is that amenability is enough to prove the irreducibility.)

Our main result in this paper is that the random walk construction is combined with the Radulescu’s method to produce bimodules over amalgamated free product

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factors so that it gives a fully faithful realization of a C*-tensor category with Frobenius duality as bimodules of finite Jones index.

Since a C*-tensor category with Frobenius duality is characterized abstractly as a rigid C*-tensor category with simple unit object ([25]), the above result gives the following more satisfactory characterization: A C*-tensor category with simple unit object is realized as that of bimodules of finite Jones index over a factor if and only if it is rigid.

1. Preliminaries

In this paper, we shall work with C*-tensor categories, which we may assume to be strict without loss of generality by coherence theorem ([12, Theorem 7.2.1]).

A conjugation in a C*-tensor category $C$ is, by definition, a conjugate-linear monoidal C*-functor, $X \mapsto X^*$, $\text{Hom}(X,Y) \ni f \mapsto f^* \in \text{Hom}(X,Y)$ with the accompanied conjugate multiplicativity $\{c_{X,Y} : X \otimes Y \to X \otimes Y\}$ and a natural equivalence $\{d_X : X \to X^*\}$ satisfying $d_X = d_{X^*}$. The object $X^*$ is often denoted by $X^{**}$ with the associated contravariant functor defined by $t_f = f^{**} = f^{*}$ for a morphism $f$.

Definition 1.1. Let $C, C'$ be C*-tensor categories with conjugations. A monoidal C*-functor $F : C \to C'$ with multiplicativity $m = \{m_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)\}$ is said to be *-monoidal if there is a natural family $\{s_V : F(V^*) \to F(V)^*\}$ of unitaries in $C'$ satisfying

$$
\begin{align*}
F(W^*) \otimes F(V^*) & \xrightarrow{s \otimes s} F(W)^* \otimes F(V)^* \\
F(W^* \otimes V^*) & \xrightarrow{F(c)} F((V \otimes W)^*) \\
& \xrightarrow{t_m} F(V \otimes W)^*
\end{align*}
$$

Definition 1.2. Let $F, G : C \to C'$ be monoidal functors. A natural C*-transformation $\{\varphi_V : F(V) \to G(V)\}$ is called a *-monoidal transformation if it is monoidal (multiplicative) and satisfies

$$
\begin{align*}
F((V^*)^*) & \xrightarrow{s} F(V^*) \\
F(V) & \xrightarrow{d^{-1}} F(V^{**}) \\
& \xrightarrow{t_s}
\end{align*}
$$

and

$$
\begin{align*}
F(V^*) & \xrightarrow{s^F} F(V)^* \\
& \xrightarrow{t_{\varphi_V}}
\end{align*}
$$

Definition 1.3. A *-monoidal transformation is called a *-monoidal equivalence if $\varphi_V$ is an isomorphism for each $V$. 

$$
\begin{align*}
G(V^*) & \xrightarrow{s^G} G(V)^* \\
& \xrightarrow{t_{\varphi_V}}
\end{align*}
$$
**Definition 1.3.** A $^\ast$-monoidal functor $F : \mathcal{C} \to \mathcal{C}'$ is called a $^\ast$-monoidal isomorphism if there are a $^\ast$-monoidal functor $G : \mathcal{C}' \to \mathcal{C}$ and two $^\ast$-monoidal equivalences

$$\varphi : FG \cong \text{id}_{\mathcal{C}'} \quad \psi : GF \cong \text{id}_{\mathcal{C}}.$$  

Two $C^\ast$-tensor categories $\mathcal{C}, \mathcal{C}'$ with conjugations are said to be isomorphic if there is a $^\ast$-monoidal isomorphism $F : \mathcal{C} \to \mathcal{C}'$.

A conjugation is said to be strict if $\{c_{X,Y}\}$ and $\{d_X\}$ are identities. The coherence theorem can be extended to tensor categories with conjugations and we can safely restrict ourselves to strict conjugations (cf. [21]).

**Definition 1.4.** A Frobenius duality in a (strict) $C^\ast$-tensor category $\mathcal{C}$ is a (strict) conjugation together with a family of self-conjugate morphisms $\{\epsilon_X = \tau_X : X \otimes X^* \to I\}_{X \in \text{Objects}}$ satisfying the following conditions ($I$ being the unit object in $\mathcal{C}$).

(i) (Multiplicativity)

$$X \otimes Y \otimes Y^* \otimes X^* \xrightarrow{\epsilon_X \otimes Y} I$$

(ii) (Naturality) For a morphism $f : X \to Y$ in $\mathcal{C}$,

$$X \otimes Y^* \xrightarrow{f \otimes 1} Y \otimes Y^*$$

(iii) (Faithfulness) The map

$$\text{Hom}(X, Y) \ni f \mapsto \epsilon_Y \circ (f \otimes 1) \in \text{Hom}(X \otimes Y^*, I)$$

is injective for $X, Y \in \text{Object}(\mathcal{C})$.

(iv) (Neutrality) For a morphism $f \in \text{End}(X)$, we have

$$\epsilon_X(f \otimes 1) = \epsilon_X(1 \otimes f) \epsilon_{X^*}.$$  

**Example 1.5.** Let $N$ be a $\Pi_1$-factor and consider the tensor category $\mathcal{C}$ of $N$-$N$ bimodules of finite index. Then, together with the obvious operation of conjugation, $\mathcal{C}$ admits a canonical Frobenius duality $\{\epsilon_X\}$ defined by

$$\epsilon_X : X \otimes N X^* = L^2(\text{End}(X_N)) \ni x(\tau \circ E)^{1/2} \mapsto [X]^{1/4} E(x)^{-1/2}$$

for an irreducible $X$ ($\tau$ is the normalized trace on $N$, $E : \text{End}(X_N) \to N$ is the conditional expectation and $[X]$ is the Jones index for the inclusion $N \subset \text{End}(X_N)$) and then by

$$\epsilon_X = \sum_{i=1}^n \epsilon_X(T_i \otimes T_i)$$

for a general $X$, where $X \cong \bigoplus_{i=1}^n X_i$ with $\{T_i : X \to X_i\}$ an orthogonal family of coisometries.
Definition 1.6. An object $X$ in a C*-tensor category $\mathcal{C}$ is said to be rigid if we can find an object $Y$ and morphisms $\epsilon : X \otimes Y \to I$, $\delta : I \to Y \otimes X$ such that

$$X \xrightarrow{1 \otimes \delta} X \otimes Y \otimes X \xrightarrow{\epsilon \otimes 1} X,$$

$$Y \xrightarrow{\delta \otimes 1} Y \otimes X \otimes Y \xrightarrow{1 \otimes \epsilon} Y$$

are identities.

An object $Y$ in the definition of rigidity turns out to be unique up to isomorphism and is referred to as a dual object of $X$ (see [10] for example).

A C*-tensor tensor category $\mathcal{C}$ is rigid if any object in $\mathcal{C}$ is rigid.

It is easy to see that a C*-tensor category is rigid if it admits a Frobenius duality. Conversely, we have the following by [25].

Theorem 1.7. In a rigid C*-tensor category $\mathcal{C}$ with simple unit object, there exists a Frobenius duality and any Frobenius duality in $\mathcal{C}$ is unique up to unitary isomorphisms.

The next result is due to Longo and Roberts ([11, Lemma 3.2]). We here present an independent structural proof.

Proposition 1.8 (Longo-Roberts). Let $\mathcal{C}$ be a C*-tensor category with simple unit object. Then for any rigid object $X$ in $\mathcal{C}$, $\text{End}(X)$ is finite-dimensional.

In particular, the tensor category $\mathcal{C}$ is semisimple, i.e., any object $X$ in $\mathcal{C}$ is isomorphic to a direct sum of finitely many simple objects (by adding subobjects to $\mathcal{C}$ if necessary).

Proof. Let $\epsilon : X^* \otimes X \to I$, $\delta : I \to X \otimes X^*$ be a rigidity pair and $F : \text{End}(X) \to \text{Hom}(I, X \otimes X^*)$ be the associated Frobenius transform: $F(f) = (f \otimes 1)\delta$ and $F^{-1}(g) = (1 \otimes \epsilon)(g \otimes 1)$. Note here that $\text{Hom}(I, X \otimes X^*)$ is a Hilbert space because of the simplicity of the unit object $I$.

From the inequalities

$$\|F(f)\|^2 = \delta^*(f^* f \otimes 1)\delta \leq \|f\|^2 \|\delta\|^2,$$

$$\|F^{-1}(g)\|^2 = (1 \otimes \epsilon)(gg^* \otimes 1)(1 \otimes \epsilon^*) \leq \|g\|^2 \|\epsilon\|^2,$$

the C*-algebra $\text{End}(X)$ is continuously isomorphic to the Hilbert space $\text{Hom}(I, X \otimes X^*)$ with the bounded inverse, whence $\text{End}(X)$ is reflexive as a Banach space, proving $\dim(\text{End}(X)) < +\infty$.

Proposition 1.9. Let $N$ be a factor. An $N$-$N$ bimodule $X$ has finite Jones index if and only if $X$ is rigid in the tensor category $\mathcal{C}$ of $N$-$N$ bimodules.

Proof. If $X$ has the finite Jones index, it is rigid as a consequence of the existence of Frobenius duality.

Conversely, let $X$ be a rigid object in $\mathcal{C}$. Then by Longo-Roberts' finite-dimensionality, we may assume that $X$ is simple. In that case, the finiteness of Jones index follows from the non-triviality of $\text{Hom}(N L^2(N) N, N X \otimes N X_N^*)$ and $\text{Hom}(N L^2(N) N, N X^* \otimes N X_N)$ as pointed out in [11, Lemma 10].

In what follows, we shall work with a rigid C*-tensor category $\mathcal{C}$ with simple unit object (Frobenius duality being implicitly assumed by Theorem 1.7). Let $S$ be the set of equivalence classes of simple objects in $\mathcal{C}$. By the semisimplicity of $\mathcal{C}$, the free vector space $\mathbb{C}[S]$ over the set $S$ admits the algebra structure as a Grothendieck ring. Moreover, the existence of dual objects in $\mathcal{C}$ allows us to define the *-operation...
in $\mathbb{C}[S]$ by $[X]^* = [X^*]$. The *-algebra $\mathbb{C}[S]$ is, by definition, the **fusion algebra** of $\mathcal{C}$.

Now we review the construction and some of its basic properties introduced in \cite{6}. By choosing a representative set of simple objects in $\mathcal{C}$, we regard $S$ as a set of simple objects in $\mathcal{C}$.

Let $\mu$ be a probability measure on $S$ which supports the whole set $S$. To introduce the random walk construction of bimodules, we choose an orthogonal family $\{e_s\}_{s \in S}$ of projections in the AFD $II_1$-factor $R$ such that

$$\omega(e_s) = \frac{\mu(s)}{d(s)},$$

where $\omega$ denotes the normalized trace of $R$.

For a finite sequence $x = (x_n, \ldots, x_1) \in S^n$, we define the projection $e_x \in R^{\otimes n}$ by $e_x = e_{x_n} \otimes \cdots \otimes e_{x_1}$.

We denote the space $e_x R^{\otimes n} e_y$ simply by $x R_y$ and keep to use $\omega$ to denote the normalized trace on $R^{\otimes n}$.

In the following, the conventional symbol $\otimes$ for tensor products is often omitted to simplify notations.

Now, given an object $X$ in $\mathcal{C}$, we define an increasing sequence of finite von Neumann algebras $\{A_n(X)\}_{n \geq 0}$ and an increasing sequence of Hilbert spaces $\{X_n\}_{n \geq 0}$ by

$$A_n(X) = \bigoplus_{x,y \in S^n} \left[ x_n \ldots x_1 X \right] \otimes x R_y,$$

$$X_n = \bigoplus_{x,y \in S^n} \left[ x_n \ldots x_1 X \right] \otimes L^2(x R_y),$$

with the imbeddings $A_n(X) \to A_{n+1}(X)$ and $X_n \to X_{n+1}$ defined by

$$\sigma \otimes a \mapsto \sum_{s \in S} (1_s \otimes \sigma) \otimes (e_s \otimes a),$$

$$\xi \otimes a \omega^{1/2} \mapsto \sum_{s \in S} (1_s \otimes \xi) \otimes (e_s \otimes a) \omega^{1/2}.$$

If we introduce a faithful tracial functional $\tau^X_n$ on $A_n(X)$ by

$$\tau^X_n(\sigma \otimes a) = \delta_{x,y}(\sigma) \omega(a) \quad \text{with} \quad \langle \sigma \rangle 1_I = \epsilon_{x,X}(\sigma \otimes 1) \epsilon^*_x X,$$

then it turns out that the family $\{\tau^X_n\}$ is compatible with the inclusion $A_n(X) \to A_{n+1}(X)$, whence it defines the trace $\tau_X$ on the inductive limit $\bigcup_{n \geq 0} A_n(X)$.

The standard Hilbert space $L^2(A_n(X))$ of $A_n(X)$ is naturally identified with $(XX^*)_n$ by

$$(\sigma \otimes a) \tau^X_n \leftrightarrow \tilde{\sigma} \otimes a \omega^{1/2},$$

where $\tilde{\sigma} \in \left[ x_n \ldots x_1 XX^* \right]$ is the Frobenius transform of $\sigma$.

If we write $A_n$ to stand for $A_n(I)$ with the trace $\tau$ denoted by $\tau$, the Hilbert space $X_n$ is an $A_n$-$A_n$ bimodule by

$$(\sigma \otimes a)(\xi \otimes x \omega^{1/2})(\sigma' \otimes a') = (\sigma \otimes 1_X)(\xi \sigma' \otimes x a a' \omega^{1/2}),$$

which is referred to as a **random walk bimodule**.
Lemma 1.10. Every algebraic vector in $X_n$ is $\tau_n$-bounded and the associated $A_n$-valued inner product is given by the formula

$$A_n[\xi \otimes a \omega^{1/2}, \eta \otimes b \omega^{1/2}] = (\xi \eta^*)_X \otimes ab^* \in A_n,$$

where $\xi \in \frac{xX}{x'}, \eta \in \frac{yX}{y'}, a \in xR_{x'}, b \in yR_{y'}$ and

$$(\xi \eta^*)_X = (1 \otimes \epsilon_X)(\xi \eta^* \otimes 1_{X^*})(1 \otimes \epsilon_X^*) \in \frac{x}{y}$$

denotes the partial trace of $\xi \eta^*$.

2. Free Product Bimodules

Among finite von Neumann algebras constructed by random walks, we here concentrate on the lowest inclusion $A_1(X) \subset A_2(X)$ (higher inclusions can be used as well). To simplify the notation, we denote these as $A(X)$ and $B(X)$ with the convention $A = A(I)$ and $B = B(I)$.

Recall that all these are isomorphic to a direct sum of countably many AFD $\mathbb{II}_1$-factors.

We also use the notation $A_XA$ and $B_XB$ to stand for random walk bimodules $X_1$ and $X_2$ respectively. (The notation in fact indicates the fact that $A_XA$ is an $A$-$A$ bimodule.)

Lemma 2.1. The inclusion $A \subset B$ is connected, i.e.,

$$Z(A) \cap Z(B) = \mathbb{C}1.$$

Proof. Recall that

$$A = \bigoplus_{s,t \in S} \left[ \begin{smallmatrix} s \\ t \end{smallmatrix} \right] \otimes sR_t = \bigoplus_{s \in S} \left[ \begin{smallmatrix} s \\ \bullet \end{smallmatrix} \right] \otimes sR_s \cong \bigoplus_{s \in S} sR_s$$

is imbedded into

$$B = \bigoplus_{s,t \in S^2} \left[ \begin{smallmatrix} s \\ t \end{smallmatrix} \right] \otimes sR_t$$

by

$$1_s \otimes x \mapsto \sum_{t \in S} (1_t \otimes 1_s) \otimes (e_t \otimes x).$$

Thus a central element $\sum_s f(s)1_s \otimes e_s$ in $A$ is mapped into

$$\sum_s f(s)1_s \otimes e_s = \sum_u \sum_s \sum_{\xi: u \to st} f(s) \frac{d(u)}{(\xi|\xi)} \xi \xi^* \otimes e_{st}.$$
in $B$, then we have
\[ f(s) \sum_{\xi: u \to s \in B} d(u) \xi \xi^* = g(u) \sum_{\xi: u \to s \in B} d(u) \xi \xi^* \]
for any $s$, $t$ and $u \in S$ ($e_{st} \neq 0$ as $\mu$ supports the whole $S$), i.e.,
\[ f(s) = g(u) \quad \text{for any } s, u \in S \]
because $\begin{bmatrix} st \\ u \end{bmatrix} \neq \{0\}$ for some $t \in S$. \qed

**Lemma 2.2.** For $s, t, x, y$ and $z \in S$,
\[ xyRs z(e_s \otimes zR_t) = xyR st. \]

**Proof.** Consider the case $\mu(z)/d(z) \leq \mu(t)/d(t)$. By the assumption $\mu(z) > 0$ for any $z \in S$, we can find a finite family of partial isometries $\{u_i\}_{0 \leq i \leq n}$ in $R$ ($n$ can be chosen as the positive integer satisfying $n \leq \mu(t)d(z)/d(t)\mu(z) < n + 1$) such that
\[ e_t = \sum_i u_i^* e_z u_i. \]
Then $u_i e_t \in zR_t$ and we have
\[
xyRs z(e_s \otimes zR_t) \supset \sum_i xyRs z(e_s \otimes u_i e_t) \\
\supset \sum_i e_{xy}(R \otimes R)(e_s \otimes u_i^* e_z u_i e_t) \\
= xyR st.
\]
The reverse inclusion is trivial. \qed

**Lemma 2.3.** Both of $A_X A_B$ and $B_A X_A$ are dense in $B X_B$.

**Proof.** We will only show the density of $B A X_A$. Let $\xi \in B X_B$ be orthogonal to $B A X_A$ and express it as
\[ \xi = \bigoplus_{s, t \in S^2} \xi_i(s, t) \otimes \alpha_i(s, t) \]
with $\{\alpha_i(s, t)\}_{i \geq 1}$ an orthonormal basis in $L^2(sR_t)$ and $\xi_i(s, t) \in \text{Hom}(t, sX)$.

Given any $u \in S$, choose $\sigma \in \text{Hom}(t_2 u, s)$, $\eta \in \text{Hom}(t_1, uX)$, $a \in sR_{t_2 u}$ and $b \in uR_{t_1}$ arbitrarily. Then $\xi$ is orthogonal to the vector
\[
(\sigma \otimes 1_X)(1_{t_2} \otimes \eta) \otimes a(e_{t_2} \otimes b)\omega^{1/2}
\]
by assumption, i.e.,
\[ \sum_i (\xi_i(s, t)\langle \sigma \otimes 1_X)(1_{t_2} \otimes \eta)\rangle (\alpha_i(s, t)\langle a(e_{t_2} \otimes b)\rangle\omega^{1/2}) = 0. \]
Since the set $\{a(e_{t_2} \otimes b)\omega^{1/2}\}$ is total in $L^2(sR_t)$ by Lemma 2.2, we can simplify the orthogonality condition to
\[
(\xi_i(s, t)\langle \sigma \otimes 1_X)(1_{t_2} \otimes \eta)\rangle = 0
\]
for any $i \geq 1$. 

On the other hand, Frobenius isomorphisms
\[
\bigoplus_{u \in S} \begin{bmatrix} s \\ t_2 u \end{bmatrix} \otimes uX \underset{\cong}{=} \bigoplus_{u \in S} \begin{bmatrix} t_2 s \\ t_1 u \end{bmatrix} \otimes u \begin{bmatrix} t_1 X^* \\ t \end{bmatrix},
\]
show that \{ (σ ⊗ 1X)(1τ ⊗ ω) \} is total in Hom(t, sX).
Consequently we have \( ξ_i(t, s) = 0 \) for any \( i \geq 1 \) and any \( s, t \in S^2 \).

**Lemma 2.4.** Given an object \( X \) of \( C \), the following formulas define unitary maps
\( \text{L}^2(B) \otimes_A X_A \to \text{B}X_B \) and \( \text{A}X \otimes_A \text{L}^2(B) \to \text{B}X_B \):
\[
(σ \otimes b\omega^{1/2}) \otimes_A (ξ \otimes x\omega^{1/2}) \mapsto \sum_{u \in S} (σ \otimes 1X)(1_u \otimes ξ) \otimes b(e_u \otimes x)\omega^{1/2},
\]
\[
(ξ \otimes x\omega^{1/2}) \otimes_A (σ \otimes b\omega^{1/2}) \mapsto \sum_{u \in S} (1_u \otimes ξ)σ \otimes (e_u \otimes x)b\omega^{1/2},
\]
where
\[
σ \in \begin{bmatrix} s_2 s_1 \\ t_2 t_1 \end{bmatrix}, \quad b \in s_2 s_1 R_{t_2 t_1} \quad \text{and} \quad ξ \in \begin{bmatrix} sX \\ t \end{bmatrix}, \quad x \in sR_t
\]
with \( s, t, s_1, t_1 \in S \).

**Proof.** By the formula of \( A \)-valued inner product in \( \text{A}X \) (Lemma 1.10),
\[
\| (σ \otimes b\omega^{1/2}) \otimes_A (ξ \otimes x\omega^{1/2}) \|^2 = \sum_{u} (σ \otimes b\omega^{1/2}) (σ \otimes b\omega^{1/2})(1_u \otimes <ξ^*>,X \otimes (e_u \otimes xx^*))
\]
\[
= \sum_{u} (σ^*σ \otimes 1X)(1_u \otimes ξ^*)(b^*b(e_u \otimes xx^*))
\]
\[
= \| \sum_{u} (σ \otimes 1X)(1_u \otimes ξ) \otimes b(e_u \otimes x)\omega^{1/2} \|^2,
\]
i.e., the correspondance
\[
(σ \otimes b\omega^{1/2}) \otimes_A (ξ \otimes x\omega^{1/2}) \mapsto \sum_{u \in S} (σ \otimes 1X)(1_u \otimes ξ) \otimes b(e_u \otimes x)\omega^{1/2}
\]
gives a well-defined isometry \( \text{L}^2(B) \otimes_A X_A \to \text{B}X_B \), which is in fact a unitary by Lemma 2.3.

The unitary maps in the above lemma are clearly functorial in the variable \( X \) and, composing these unitary maps, we obtain a natural family of unitary maps
\( \text{A}L^2(B) \otimes_A X_A \to \text{A}X \otimes_A \text{L}^2(B)_A \),
which intertwines the \( A \)-actions and is referred to as shift isomorphisms.

**Corollary 2.5.** The shift isomorphism transfers the subspace \( \text{L}^2(B)^\circ \otimes_A X_A \) onto the subspace \( \text{A}X \otimes_A \text{L}^2(B)^\circ \). Here \( \text{L}^2(B)^\circ \) denotes the orthogonal complement of \( \text{L}^2(A) \) in \( \text{L}^2(B) \) (\( \text{L}^2(A) \) being imbedded into \( \text{L}^2(B) \) by the trace-preserving conditional expectation).

**Proof.** From the definition of unitaries, it is immediate to check that both of \( τ_B^{1/2} \otimes_A τ_A^{1/2} \xi \) and \( ξ \otimes A_{1/2} τ_B^{1/2} \) are mapped to \( ξ \in \text{A}X_A \). Thus the unitaries turn out to be the obvious identification
\[
\text{L}^2(A) \otimes_A X_A = \text{A}X_A = \text{A}X \otimes_A \text{L}^2(A)
\]
on the subspaces $L^2(A) \otimes_A X_A$ and $A X \otimes_A L^2(A)$. Since $L^2(B)^{\circ} \otimes_A X_A$ and $A X \otimes_A L^2(B)^{\circ}$ are orthogonal complements of these, the assertion holds.

We are now ready to introduce a free product construction of bimodules. Given a continuous von Neumann algebra $Q$ with a faithful normalized trace $\tau_Q$, we set

$$N = N(Q) = (Q \otimes A) *_A B,$$

which is a $\mathrm{II}_1$-factor by the connectedness of the inclusion $A \subset B$ (see [17, §1] or Lemma 3.8 below).

Given an object $X$ in $\mathcal{C}$, we shall construct an $N-N$ bimodule in the following way. To define the base Hilbert space of a free-product bimodule, we start with the Hilbert space $A X_A$ and take $A$-tensor products with $A$-bimodules

$$L^2(Q)^{\circ} \otimes L^2(A) = L^2(A) \otimes L^2(Q)^{\circ}$$

alternately so that

(i) tensor products of the following two types are not allowed to appear,

$$(L^2(Q)^{\circ} \otimes L^2(A)) \otimes_A X_A \otimes_A (L^2(Q)^{\circ} \otimes L^2(A)), \quad L^2(B)^{\circ} \otimes_A X_A \otimes_A L^2(B)^{\circ},$$

(ii) $L^2(Q)^{\circ}$ commutes with $L^2(A)A$ and $A X_A$,

(iii) if the tensor component of the form $L^2(B)^{\circ} \otimes_A X$ appears, it is identified with $X \otimes_A L^2(B)^{\circ}$ by the shift isomorphism.

Note that tensor components can be lined up sequentially for the identification in (iii) and hence there arise no coherence problems here.

The direct sum of resulting Hilbert spaces is denoted by $F(X)$, which we shall make into an $N-N$ bimodule.

By the above requirements for identification, the position of $A X_A$ can be freely moved inside summands. For example, if we move the component $A X_A$ to the right end in each summand, we have the following expression for $F(X)$:

$$F(X) = A X_A \oplus (L^2(Q)^{\circ} \otimes L^2(A)) \otimes_A X_A \oplus L^2(B)^{\circ} \otimes_A X_A$$

$$\oplus L^2(B)^{\circ} \otimes_A (L^2(Q) \otimes L^2(A)) \otimes_A A X_A$$

$$\oplus (L^2(Q)^{\circ} \otimes L^2(A)) \otimes_A L^2(B)^{\circ} \otimes_A X_A$$

$$\oplus (L^2(Q)^{\circ} \otimes L^2(A)) \otimes_A L^2(B)^{\circ} \otimes_A (L^2(Q) \otimes L^2(A)) \otimes_A A X_A$$

$$\oplus L^2(B)^{\circ} \otimes_A (L^2(Q) \otimes L^2(A)) \otimes_A L^2(B)^{\circ} \otimes_A X_A$$

$$\oplus \ldots,$$

which is further reduced to

$$F(X) = A X_A + L^2(Q)^{\circ} \otimes_A X_A + L^2(B)^{\circ} \otimes_A X_A$$

$$+ L^2(B)^{\circ} \otimes_A L^2(Q)^{\circ} \otimes A X_A$$

$$+ L^2(Q)^{\circ} \otimes L^2(B)^{\circ} \otimes_A X_A$$

$$+ L^2(Q)^{\circ} \otimes L^2(B)^{\circ} \otimes_A L^2(Q)^{\circ} \otimes A X_A$$

$$+ L^2(B)^{\circ} \otimes A L^2(Q)^{\circ} \otimes A L^2(B)^{\circ} \otimes A X_A$$

$$+ \ldots$$

if we use the obvious identification $L^2(A) \otimes_A V = V = V \otimes_A L^2(A)$ for an $A-A$ bimodule $V$. 

Let \( \mathcal{N} \) be the dense \(*\)-subalgebra of \( \mathcal{N} = (Q \otimes A) * A \) algebraically generated by \( Q \cup B \). If we rearrange \( F(X) \) so that the first factors in summands are coupled to \( L^2(Q) = \mathbb{C} \oplus L^2(Q)^\circ \), then we obtain the expression

\[
F(X) = L^2(Q) \otimes_A X_A + L^2(Q) \otimes (B) \otimes_A L^2(Q)^\circ \otimes_A X_A + \ldots,
\]
on which \( Q \) acts from left by multiplication. Let us denote this by \( \lambda_Q \).

On the other hand, if we consider the rearrangement to form \( L^2(B) = L^2(A) \oplus L^2(B)^\circ \) at the left ends of summands, we get the expression like

\[
F(X) = L^2(B) \otimes_A X_A + L^2(B) \otimes_A L^2(Q)^\circ \otimes_A X_A + \ldots
\]

with the left representation of \( B \) denoted by \( \lambda_B \). It is then immediate to check the commutativity of \( \lambda_Q(Q) \) and \( \lambda_B(A) \), whence we have a \(*\)-representation \( \lambda \) of \( \mathcal{N} \) on \( F(X) \) so that \( \lambda|Q = \lambda_Q \) and \( \lambda|B = \lambda_B \).

Similarly, we can define an antirepresentation \( \rho \) of \( \mathcal{N} \) on \( F(X) \) by right multiplication after rearrangements of summands at the right ends. It then turns out that \( \lambda \) and \( \rho \) commute by straightforward computations and the Hilbert space \( F(X) \) becomes an \( \mathcal{N} \)-\( \mathcal{N} \) module.

Note that by the way of constructions, the \( \mathcal{N} \)-\( \mathcal{N} \) action is described according to the free product prescription (cf. [20]).

Lemma 2.6. Let \( K \) be an object in \( C \) and \( V \) be an \( A(K) \)-\( A(K) \) module. Then we can define \( A(K) \)-\( A(K) \) linear unitary maps \( L^2(B)^\circ \otimes_A V \to L^2(B(K))\circ \otimes_A V \) and \( V \otimes A L^2(B)^\circ \to V \otimes A(K) L^2(B(K))\circ \) so that

\[
(\sigma \otimes b)_{1/2}^{\tau_B} \otimes_{\tau_A}^{\tau_{-1/2}} v \mapsto (\sigma \otimes b)_{1/2}^{\tau_K} \otimes_{\tau_A}^{\tau_{-1/2}} v,
\]

\[
v \otimes_{\tau_A}^{\tau_{-1/2}} (\sigma \otimes b)_{1/2}^{\tau_B} \mapsto v \otimes_{\tau_A}^{\tau_{-1/2}} (\sigma \otimes b)_{1/2}^{\tau_K},
\]

where \( \sigma \in \text{Hom}(t, s) \) and \( b \in s R_t \) for \( s, t \in S^2 \).

Proof. By Lemma 2.4

\[
V \otimes_A L^2(B(K))\circ \cong V \otimes_A L^2(A(K)) \otimes_A L^2(B)^\circ \cong V \otimes A L^2(B)^\circ,
\]

where an element \( v \otimes_{\tau_{-1/2}} (\sigma \otimes b)_{1/2}^{\tau} \) in the right end is transferred as

\[
v \otimes_{\tau_{-1/2}} (\sigma \otimes b)_{1/2}^{\tau_B} \leftrightarrow v \otimes_{\tau_{-1/2}} (\sigma \otimes b)_{1/2}^{\tau_K} \leftrightarrow v \otimes_{\tau_{-1/2}} (\sigma \otimes b).\]

Remark. A vector \( (\sigma \otimes b)^{1/2} \tau_K = (\sigma \otimes b)^{1/2} \otimes b \) in \( L^2(B(K)) \) corresponds to the vector \( (\sigma \otimes 1_K) \otimes b \) in \( B K K_B^* \).

Lemma 2.7. The obvious imbedding \( \mathcal{N} \to (Q \otimes A(K)) * A(K) \) is extended to an injective normal \(*\)-homomorphism of \( N \).
Proof. Let $K$ be an object in $\mathcal{C}$. Then we have the following commuting squares

$$
\begin{array}{ccc}
Q \otimes A(K) & \longrightarrow & A(K) \\
\downarrow & & \downarrow \\
Q \otimes A & \longrightarrow & A
\end{array}
\quad
\begin{array}{ccc}
A(K) & \longleftarrow & B(K) \\
\downarrow & & \downarrow \\
A & \longleftarrow & B
\end{array}
$$

and then we can apply the imbedding theorem of amalgamated free products (see Lemma 6.1 in [2] for example) to find that $N = (Q \otimes A) *_A B$ is a von Neumann subalgebra of the amalgamated free product $(Q \otimes A(K)) *_{A(K)} B(K)$. \hfill \Box

**Lemma 2.8.** The Hilbert space $F(K \otimes K^*)$ is naturally isometrically isomorphic to $L^2((Q \otimes A(K)) *_{A(K)} B(K))$ with the $\mathcal{N}$-$\mathcal{N}$ action on $F(K \otimes K^*)$ identified with the left and right action of $\mathcal{N}$ by multiplication when $N$ is imbedded into $(Q \otimes A(K)) *_{A(K)} B(K)$ by the previous lemma.

Proof. Let $M = (Q \otimes A(K)) *_{A(K)} B(K)$. Then we have

$$L^2(M) = L^2(A(K)) \oplus L^2(Q) \otimes L^2(A(K)) \oplus L^2(B(K))^\circ \oplus \ldots,$$

where the summation is taken over alternate $A(K)$-tensor products of $L^2(Q) \otimes L^2(A(K))$ and $L^2(B(K))^\circ$.

If we apply the isomorphism in Lemma 2.4 from outside in each summand (note that $L^2(Q) \otimes L^2(A(K))$ commutes with $L^2(A(K))$ and does not touch on the $A(K)$-action), we end up with a realization of a summand in $F(KK^*)$ with the tensor component $\Lambda KK^*_{A}$ at a prescribed position: two realizations with $\Lambda KK^*_{A}$ at adjacent positions correspond to the choices of isomorphisms $L^2(B(K))^\circ \cong L^2(B) \otimes_A L^2(A(K))$ or $L^2(B(K))^\circ \cong L^2(A(K)) \otimes_A L^2(B)$ at the last stage of reductions.

There are two ways of ambiguity other than the ones related by shift isomorphisms as indicated by the following diagram

$$
\begin{array}{ccc}
L^2(B(K))^\circ \otimes_{A(K)} L^2(B(K))^\circ & \longrightarrow & L^2(B)^\circ \otimes_A L^2(A(K)) \otimes_{A(K)} L^2(B(K))^\circ \\
\downarrow & & \downarrow \\
L^2(B(K))^\circ \otimes_{A(K)} L^2(A(K)) \otimes_A L^2(B)^\circ & \longrightarrow & L^2(B)^\circ \otimes_A L^2(B(K))^\circ
\end{array}
$$

This is however nothing but two ways of reductions in the right hand side of

$$
L^2(B(K))^\circ \otimes_{A(K)} L^2(B(K))^\circ \cong (L^2(B)^\circ \otimes_A L^2(A(K))) \otimes_{A(K)} (L^2(A(K)) \otimes_A L^2(B))^\circ
$$

and the commutativity of the diagram is eventually reduced to the associativity of multiplication in the algebra $A(K)$.

It is now straightforward to check that the $\mathcal{N}$-$\mathcal{N}$ action on $F(KK^*)$ corresponds to the left and right multiplications of $N$ on $L^2(M)$. \hfill \Box

**Proposition 2.9.** The $\mathcal{N}$-$\mathcal{N}$ action on $F(X)$ is extended to the $N$-$N$ action by weak continuity: the Hilbert space $F(X)$ is furnished with the structure of an $N$-$N$ bimodule.
Proof. Let $K$ be such that $X \subset KK^*$ ($K = I \oplus X$ for example). Then $F(X)$ is realized as an $\mathcal{N}\mathcal{N}$ invariant closed subspace of $F(KK^*)$. Since the $\mathcal{N}\mathcal{N}$ action is continuously extended to the $\mathcal{N}\mathcal{N}$ action on $F(KK^*)$ by previous lemmas, the same holds on $F(X)$. 

It is also clear that a morphism $f : X \to Y$ in $\mathcal{C}$ induces an $\mathcal{N}\mathcal{N}$-intertwiner $F(f) : F(X) \to F(Y)$ so that

$$F(f)(\xi \otimes x\omega^{1/2}) = (1_s \otimes f)\xi \otimes x\omega^{1/2}$$

for $\xi \in \text{Hom}(t, sX)$ and $x \in sR_t$ with $s, t \in S$.

The correspondence $X \mapsto F(X)$, together with $f \mapsto F(f)$, gives a C*-functor from $\mathcal{C}$ into the C*-tensor category of $\mathcal{N}\mathcal{N}$ bimodules, which is referred to as a free product functor in what follows.

**Lemma 2.10.** The free product functor is faithful, i.e.,

$$F : \text{Hom}(X, Y) \to \text{Hom}(F(X), F(Y))$$

is injective for any objects $X, Y$ in $\mathcal{C}$.

Proof. Since $F$ is a C*-functor, it suffices to consider the case $X = Y$. From the above definition of $F(f)$, given $s \in S$ and $0 \neq x \in 1R_s$, the subspace $1X_s \otimes x\omega^{1/2}$ of $F(X)$ is invariant under the action of $F(\text{End}(X))$ and is equivalent to the obvious irreducible representation of $\text{End}(X)$ on the vector space $X_s$. Since the family $\{\text{Hom}(s, X)\}_{s \in \text{supp}(X)}$ is a complete set of irreducible representations of $\text{End}(X)$, we conclude that $f \mapsto F(f)$ is a faithful representation of $\text{End}(X)$.

3. Free Product Functors

In this section, we shall reveal the monoidal structure of free product functors towards our realization theorem.

The following formula on operator-valued inner products is frequently used to identify relative tensor products.

**Lemma 3.1.** Each element in $\mathcal{N}X_A$ is right $\mathcal{N}$- and left $\mathcal{N}$-bounded at the same time and the associated operator valued-inner products (relative to the normalized trace of $\mathcal{N}$) is given by

$$[\xi, \eta]_{\mathcal{N}} = [\xi, \eta]_A \quad \text{and} \quad N[\xi, \eta] = A[\xi, \eta].$$

Therefore, for $a, b \in \mathcal{N}$, we have

$$[a\xi, \eta]_{\mathcal{N}} = a^*[\xi, \eta]_A b, \quad N[a\xi, b\eta] = a_A[\xi, \eta]b^*.$$

Proof. Let $\mathcal{N} \subset \mathcal{N}$ be the algebraic free product subalgebra. For $x \in \mathcal{N}$ of the form

$$x = aqv_1q_1b_2q_2 \ldots,$$

with $a \in A, q \in Q, q_1 \in Q^\circ$ and $b_2 \in B^\circ$ (here $Q^\circ = \ker \tau_Q$ and $B^\circ = \ker E^A_B$), we have

$$\xi x = \xi a \otimes q_1t_1^{1/2} \otimes_A b_1 t^{1/2} \otimes_A \ldots$$
we show the multiplicativity of free product functors.

Lemma 3.2. Let $X$ and $Y$ be objects in $\mathcal{C}$. Then we can define an $N$-$N$ linear unitary map $m_{X,Y} : F(X) \otimes_N F(Y) \to F(X \otimes Y)$ by the formula

$$m_{X,Y}((\xi \otimes x \omega^{1/2}) \otimes_N (\eta \otimes y \omega^{1/2})) = (\xi \otimes 1_Y)\eta \otimes xy \omega^{1/2},$$

where $\xi \in \left[ sX \atop s' \right], \eta \in \left[ tY \atop t' \right], x \in sR_s'$, and $y \in tR_{t'}$ with $s$, $s'$, $t$ and $t'$ in $S$.

Proof. Set $\zeta = (\xi \otimes 1_Y)\eta \in \left[ sXY \atop t' \right]$. Then, from the operator-valued inner product formula,

$$\| (\xi \otimes x \omega^{1/2}) \otimes_N (\eta \otimes y \omega^{1/2}) \|^2 = (\eta \otimes y \omega^{1/2})[\xi \otimes x \omega^{1/2}, \xi \otimes x \omega^{1/2}]_A (\eta \otimes y \omega^{1/2})
= (\eta \otimes y \omega^{1/2})[\zeta \otimes x \omega^{1/2}, (\eta \otimes y \omega^{1/2})]
= (\zeta \otimes xy \omega^{1/2})[\zeta \otimes xy \omega^{1/2}].$$

Thus we can define an isometry of $A X \otimes_N Y_A$ to $F(XY)$ by the same formula as $m_{X,Y}$, which is obviously $A$-$A$ linear and has the range $AXY_A$.

Now, for $a, b \in N$, writing $a(\xi \otimes x \omega^{1/2}) = \alpha \otimes_A (\xi \otimes x \omega^{1/2})$ and $(\eta \otimes y \omega^{1/2}) b = (\eta \otimes y \omega^{1/2}) \otimes_A \beta$ with $\alpha, \beta$ elements in alternate $A$-tensor products of $L^2(Q)^0$ and $L^2(B)^0$, we have

$$\| a(\xi \otimes x \omega^{1/2}) \otimes_N (\eta \otimes y \omega^{1/2}) \|^2
= \| \alpha \otimes_A (\xi \otimes x \omega^{1/2}) \otimes_N (\eta \otimes y \omega^{1/2}) \otimes_A \beta \|^2
= (\xi \otimes x \omega^{1/2}) \otimes_N (\eta \otimes y \omega^{1/2}) \otimes_N (\eta \otimes y \omega^{1/2})_A[\beta, \beta]
= (\zeta \otimes xy \omega^{1/2})[\alpha, \alpha]_A (\zeta \otimes xy \omega^{1/2})_A[\beta, \beta].$$

(here we apply the isometry just defined)

$$= \| \alpha \otimes_A (\zeta \otimes xy \omega^{1/2}) \otimes_A \beta \|^2
= \| a(\zeta \otimes xy \omega^{1/2}) b \|^2.$$

The unitary map $m_{X,Y}$ is apparently natural in variables $X, Y$.

Lemma 3.3. The natural family $\{m_{X,Y}\}$ is associative:

$$m_{X,Y,Z}(m_{X,Y} \otimes 1_{F(Z)}) = m_{X,Y,Z}(1_{F(X)} \otimes m_{Y,Z})$$

for objects $X, Y$ and $Z$ in $\mathcal{C}$. 

---

$(\otimes_A$ is with respect to the canonical trace $\tau$ on $A$) and hence

$$\|\xi x\|^2 = (\tau^{1/2} \otimes q_{1/2} \otimes b_1 \tau^{1/2} \otimes q_{1/2} \otimes \ldots) [\xi, \xi]_{A \tau^{1/2} \otimes A \tau^{1/2} \otimes A \tau^{1/2} \otimes \ldots}
= (\tau^{1/2} x)[\xi, \xi]_{A \tau^{1/2} \otimes b_1 \tau^{1/2} \otimes q_{1/2} \otimes \ldots}
= (\tau^{1/2} x)[\xi, \xi]_{A \tau^{1/2} \otimes x}$$
shows that $[\xi, \xi]_N = [\xi, \xi]_A \in A \subset N$. 

We now show the multiplicativity of free product functors.
Proof. We first show the associativity on the subspace $AX_A \otimes_N AY_A \otimes_N AZ_A$, which is checked by

$$m_{XY,Z}(m_{X,Y}((\xi \otimes x_{1/2}^N (\eta \otimes y_{1/2}^N)) \otimes_N (\zeta \otimes z_{1/2}^N))$$

$$= m_{XY,Z}((\xi \otimes 1_y)\eta \otimes xy_{1/2}^N (\zeta \otimes z_{1/2}^N))$$

$$= (\xi \otimes 1_Y)(\eta \otimes 1_Z)\zeta \otimes xy_{1/2}^N$$

$$= m_{X,Y,Z}((\xi \otimes x_{1/2}^N) \otimes_N m_{Y,Z}((\eta \otimes y_{1/2}^N) \otimes_N (\zeta \otimes z_{1/2}^N))).$$

Now the associativity is extended to the whole space as follows: By the density of $NAX_A$ and $AX_AN$ in $F(X)$, we see that

$$\overline{N(AX_A \otimes_N AY_A \otimes_N AZ_A)N} = F(X) \otimes_N AY_A \otimes_N AZ_AN$$

$$= F(X) \otimes_N NAY_A \otimes_N AZAN$$

$$= F(X) \otimes N F(Y) \otimes N F(Z).$$

Since $m_{X,Y}$’s are $N$-$N$ linear, the above density ensures the overall validity of the desired associativity.

So far, we have checked that the free product functor $F$ is monoidal with multiplicity given by the family $\{m_{X,Y}\}$.

We next show that the monoidal functor $F$ preserves conjugations.

Lemma 3.4. The natural isomorphism $AX_A^* \rightarrow (AX_A)^*$ is extended to a unitary map $s_X : F(X^*) \rightarrow F(X)^*$ so that it intertwines $N$-$N$ actions.

Proof. Recall that the $A-A$ linear unitary map $AX_A^* \rightarrow (AX_A)^*$ is defined by

$$\xi^* \otimes x^*_{1/2} \mapsto (\xi \otimes x_{1/2}), \quad \xi \in \begin{bmatrix} x^* \\ t \end{bmatrix}, \ x \in sR_t,$$

where $\xi^* \in \begin{bmatrix} tX^* \\ s \end{bmatrix}$ denotes the Frobenius transform of $\xi^* \in \begin{bmatrix} t \\ sX \end{bmatrix}$.

Let $a, b \in N$ and write $a(\xi \otimes x_{1/2}^N)b = \alpha \otimes_A (\xi \otimes x_{1/2}^N) \otimes_A \beta$ as before. Then we have

$$\|a(\xi \otimes x_{1/2}^N)b\|^2 = \|\alpha \otimes_A (\xi \otimes x_{1/2}^N) \otimes_A \beta\|^2$$

$$= ((\xi \otimes x_{1/2}^N)_{1/2}^A[\alpha,\alpha]_A(\xi \otimes x_{1/2}^N)_{1/2}^A[\beta,\beta])$$

$$= (\xi^* \otimes x^*_{1/2} \omega_{1/2}^A[\alpha,\alpha]_A(\xi^* \otimes x^*_{1/2} \omega_{1/2}^A)_{1/2}^A[\beta,\beta])$$

$$= \|\alpha \otimes_A (\xi^* \otimes x^*_{1/2})_{1/2}^A[\beta,\beta]\|^2$$

$$= \|a(\xi^* \otimes x^*_{1/2})_{1/2}^A\|^2.$$

Thus

$$a(\xi^* \otimes x^*_{1/2}^N)b \mapsto a(\xi \otimes x_{1/2}^N)b$$

defines a unitary map $s_X : F(X^*) \rightarrow F(X)^*$.

Lemma 3.5. The family $\{s_X\}$ is natural in $X$: The diagram

$$\begin{array}{ccc}
F(X^*) & \xrightarrow{s_X} & F(X)^* \\
\downarrow{F(f)} & & \uparrow{tF(f)} \\
F(Y^*) & \xrightarrow{s_Y} & F(Y)^*
\end{array}$$
commutes for $f : X \to Y$.

Proof. By the naturality of Frobenius duality, we have $(1 \otimes t^f)\eta^* = ((1 \otimes f^*)\eta)^*$, which is utilized in the following way:

$$s_XF(t^f)(\eta^* \otimes y^*\omega^{1/2}) = s_X((1 \otimes t^f)\eta^* \otimes y^*\omega^{1/2})$$
$$= s_X(((1 \otimes f^*)\eta) \otimes y\omega^{1/2})^*$$
$$= (F(f^*)(\eta \otimes y\omega^{1/2}))^*$$
$$= (F(f)^*(\eta \otimes y\omega^{1/2}))^*$$
$$= t^f(f(\eta \otimes y\omega^{1/2}))^*.$$

\[ \square \]

Lemma 3.6. The family $\{s_X\}$ is multiplicative:

$$F(Y^*) \otimes_N F(X^*) \xrightarrow{s_Y \otimes s_X} F(Y)^* \otimes_N F(X)^*$$

Proof. By $t^m_{X,Y} = m_{X,Y}$, we shall verify the equivalent relation $m_{X,Y}(s_Y \otimes s_X) = s_{XY}m_{Y,X^*}$, which is checked on the $N$-cyclic subspace in the following way:

$$m_{X,Y}(s_Y \otimes s_X)((\eta^* \otimes y\omega^{1/2}) \otimes_N (\xi^* \otimes x\omega^{1/2}))$$
$$= m_{X,Y}((\eta \otimes y\omega^{1/2})^* \otimes_N (\xi \otimes x\omega^{1/2})^*)$$
$$= m_{X,Y}((\xi \otimes x\omega^{1/2}) \otimes_N (\eta \otimes y\omega^{1/2}))^*$$
$$= (m_{X,Y}((\xi \otimes x\omega^{1/2}) \otimes_N (\eta \otimes y\omega^{1/2})))^*$$
$$= ((\xi \otimes 1_Y)\eta \otimes xy\omega^{1/2})^*,$$

which coincides with

$$s_{XY}m_{Y,X^*}((\eta^* \otimes y\omega^{1/2}) \otimes_N (\xi^* \otimes x\omega^{1/2}))$$

if we use

$$(\eta^* \otimes 1_X)\xi^* = ((\xi \otimes 1_Y)\eta)^*,$$

which is an easy consequence of Frobenius transforms.

\[ \square \]

Lemma 3.7. The family $\{s_X\}$ is compatible with duality: Two unitary maps $s_{X^*} : F(X^{**}) \to F(X^*)^*$ and $t^ss_{X} : F(X)^{**} \to F(X)^*$ are the same if we apply the identification $F(X^{**}) = F(X) = F(X)^{**}$.

Proof. We compute as

$$\overline{s_{X^*}}s_{X^*}((\xi \otimes x\omega^{1/2})^* = \overline{s_{X}}(\xi^* \otimes x^*\omega^{1/2})^* = (s_X(\xi^* \otimes x^*\omega^{1/2}))^*$$
$$= (\xi \otimes x\omega^{1/2})^{**} = \xi \otimes x\omega^{1/2},$$

which implies $\overline{s_{X^*}}s_{X^*} = 1_{F(X)}$ by the density argument and we are done as $t^s_{X^1} = \overline{s_X}$.  \[ \square \]
We can now conclude that the free product functor is *-monoidal with the accompanied isomorphisms given by \( \{ s_X \} \).

From the properties of the free product functor established so far, we find that, for each object \( K \) in \( \mathcal{C} \), \( F(K)^* \) is a dual object of \( F(K) \) in the \( C^* \)-tensor category of \( N-N \) bimodules. Since \( N \) is a factor, the unit object \( L^2(N) \) is irreducible and hence the Longo-Roberts finite-dimensionality theorem can be applied to show that \( \text{End}(F(K)) \) is finite-dimensional and at the same time each \( F(K) \) has finite Jones index by the rigidity characterization of index-finiteness (Proposition 1.9). Moreover we know the following weaker version of Frobenius reciprocity

\[
\text{End}(F(K)) \cong \text{Hom}(L^2(N), F(K) \otimes_N F(K)^*).
\]

By the natural identification

\[
F(K) \otimes_N F(K)^* \cong F(KK^*) \cong L^2((Q \otimes A(K)) *_{A(K)} B(K)),
\]

the above fact implies that \( \text{End}(F(K)) \) is isomorphic to the set of \( N \)-central vectors in \( L^2((Q \otimes A(K)) *_{A(K)} B(K)) \), which is further isomorphic to the relative commutant

\[
N' \cap ((Q \otimes A(K)) *_{A(K)} B(K)).
\]

Now the lemma below shows that this is in fact isomorphic to \( \text{End}(K) \), whence

\[
\text{End}(K) \ni f \mapsto F(f) \in \text{End}(F(K))
\]

is a surjective isomorphism for any \( K \) and then reducing it to off-diagonal corners, the functor \( F \) is found to be fully faithful:

\[
F : \text{Hom}(X, Y) \to \text{Hom}(F(X), F(Y))
\]

is a surjective isomorphism for any objects \( X, Y \) in \( \mathcal{C} \).

**Lemma 3.8.** We have

\[
N' \cap ((Q \otimes A(K)) *_{A(K)} B(K)) = \text{End}(K).
\]

**Proof.** By the ergodicity property of free products (Theorem 4.1 in [4]), we have

\[
Q' \cap ((Q \otimes A(K)) *_{A(K)} B(K)) = Q \otimes A(K)
\]

and hence

\[
N' \cap ((Q \otimes A(K)) *_{A(K)} B(K)) = B' \cap A(K).
\]

Taking into account the commutativity with elements in \( B \) of the form \( 1_s \otimes x \) with \( s \in S^2 \) and \( x \in sR_s \), we first reduce an element in \( B' \cap A(K) \) to the form

\[
\sigma = \bigoplus_{s \in S} \sigma_s \otimes e_s \quad \text{with} \quad \sigma_s \in \left[ \begin{array}{c} sK \\ sK \end{array} \right].
\]

Now the commutativity with the Jones projection \( d(s)^{-1} \epsilon_s^* \epsilon_s^* \in \left[ \begin{array}{c} s^* s \\ s^* s \end{array} \right] \) reveals that \( \sigma_s \) is further restricted to

\[
\sigma_s = 1_s \otimes f_s \quad \text{with} \quad f_s \in \text{End}(K).
\]

To see the independence of \( f_s \) on \( s \in S \), let \( s, t \in S \) and \( \rho = \epsilon_s^* \epsilon_t \in \left[ \begin{array}{c} s^* s \\ t^* t \end{array} \right] \). Then the commutativity of \( \sigma \) with \( \rho \otimes s_{s^*} R_{t^*} \subset B \) means

\[
(\rho \otimes f_s) \otimes x = (\rho \otimes f_t) \otimes x
\]
for \( x \in s^*Rt^* \). Since \( s^*Rt^* \neq \{0\} \) and \( \rho \neq 0 \), this implies \( f_s = f_t \) for any \( s, t \in S \).

Thus, letting \( f = f_s \in \text{End}(K) \), we see that \( \sigma = f \) belongs to \( \text{End}(K) \). \( \square \)

**Theorem 3.9.** Let \( \mathcal{C} \) be a \( C^* \)-tensor category with Frobenius duality. Then the free product functor \( F \) gives a fully faithful realization of \( \mathcal{C} \) as that of \( N-N \) bimodules of finite index. Moreover, \( F \) preserves Frobenius dualities as well as conjugations.

**Proof.** Non-trivial is the equality \( F(\epsilon_X) = \epsilon_{F(X)} \). By the additivity of \( \{\epsilon_X\} \) and \( \{\epsilon_{F(X)}\} \), we may restrict ourselves to the case of a simple \( X \), which is checked by showing the positivity of \( F(\epsilon_X) \):

\[
F(\epsilon_X)((\xi \otimes x\omega^{1/2}) \otimes_N (\xi \otimes x\omega^{1/2}))^* = F(\epsilon_X)((\xi \otimes 1_X^*)\xi^* \otimes xx^*\omega^{1/2}) \\
= (1 \otimes \epsilon_X)(\xi \otimes 1_X^*)\xi^* \otimes xx^*\omega^{1/2} \\
= (\xi^*)X \otimes xx^*\omega^{1/2},
\]

which obviously belongs to the positive cone of \( L^2(N) \). \( \square \)

4. Comments

If we apply our main theorem to the Tannaka dual of a compact quantum group \( G \) (see [22] for the discussion of Frobenius duality), the resulting bimodule realization can be interpreted as giving a kind of coaction of \( G \) on the finite factor \( N \) (the reference von Neumann algebra of bimodules), which is minimal because the realization is fully faithful ([23]). The crossed product algebra \( M \) is then a factor and the “dual” action of \( G \) on \( M \) is minimal in the sense that \( M' \cap (M \rtimes G) = \mathbb{C}1 \) by Takesaki’s duality (see [3] §8 for details).

Thus we can recover the result due to Y. Ueda [18]. Note that his construction is more direct than ours: the minimal action is described as the free product of a faithful action and a trivial action, whereas subfactors corresponding to our bimodule realization are constructed according to the method of A. Wassermann ([19]). In this sense, our construction is a kind of reverse process of Ueda’s although it is not clear at present whether they really give the same subfactors.

If we apply our realization to the bicategory generated by a single object \( X \), then we obtain a realization of the associated standard lattice as higher relative commutants of the subfactor \( N \subset \text{End}(X_N) \), recovering the result due to S. Popa. It is therefore interesting to make clear the relationship between the Popa’s realization and ours. Roughly speaking, our free product factors are amplified in its ingredients by the AFD II\(_1\)-factor compared with the Popa’s subfactors but no clues to explicit connections presently.

Finally we remark here that our realization theorem cannot be obtained by just reformulating the Popa’s result: Firstly, although it is straightforward to produce standard lattices from tensor categories (with Frobenius duality), the reverse implication is highly non-trivial and is not established yet (cf. [8]). Secondly, standard lattices should correspond to singly generated tensor categories and there would be no good way to approximate tensor categories by their subcategories especially in the realization problem (a kind of cohomological adjustments are needed among approximating realizations, which seem hopeless).

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