Representing polynomial of CONNECTIVITY

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Abstract

We show that the coefficients of the representing polynomial of any monotone Boolean function are the values of a Möbius function of an atomistic lattice related to this function. Using this we determine the representing polynomial of any Boolean function corresponding to a directed acyclic graph connectivity problem. Only monomials corresponding to unions of paths have non-zero coefficients which are \((-1)^D\) where \(D\) is an easily computable function of the graph corresponding to the monomial (it is the number of plane regions in the case of planar graphs). We estimate the number of monomials with non-zero coefficients for the two-dimensional grid connectivity problem as being between \(\Omega(1.641^{2n^2})\) and \(O(1.654^{2n^2})\).

1 Introduction

In this paper we study the representing polynomials of the Boolean functions corresponding to certain graph connectivity problems. Every Boolean function \(f : \{0, 1\}^n \rightarrow \{0, 1\}\) can be expressed as a real multilinear polynomial in a unique way. We will refer to it as the representing polynomial of the Boolean function. Such polynomial is often used to estimate various complexity measures of a Boolean function, or to construct an algorithm related to it. Some fine examples of such applications can be found in the textbook [ODo14].

The representing polynomials are often studied in the Fourier basis in which the bits 0 and 1 are replaced by values 1 and \(-1\) respectively, in particular the cited textbook mostly deals with these polynomials in the Fourier basis. However, we will concentrate on these polynomials in the standard, \(\{0, 1\}\) basis, using approach similar to the one employed by Beniamini and Nisan in the recent papers [BN21; Ben20] involving lattices, their Möbius functions and convex polytopes.

The Boolean functions we mainly deal with in this paper are the ones corresponding to the directed acyclic graph connectivity (DAG-Connectivity) problems. In such problem, the input bits correspond to the edges of a given directed acyclic graph, denoting presence (1) or absence (0) of the edge, and the Boolean function is equal to 1 iff there is a path (consisting of present edges) from a starting vertex (source) to a final vertex (sink). Some examples of such problems with interesting connections are the one where the graph is the Boolean hypercube (which is related to the Travelling Salesman Problem, see, e.g., [Amb+19]), and the one where the graph is two-dimensional grid (which is related to the Edit Distance problem, see, e.g., [Amb+20]).

DAG-Connectivity Boolean functions are monotone. In Section 3 we show that the coefficients of the representing polynomial of any monotone Boolean function are the values of the

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Möbius function (with minus sign) of the poset of its unions of prime implicants (Theorem 3). We also characterize the posets obtainable as such unions of prime implicants: they are exactly the finite atomistic lattices (Proposition 1 and Theorem 2).

Then, relating a directed acyclic graph $G$ to a flow polytope and using results from [Hil03], we compute the corresponding Möbius function obtaining formula for the representing polynomial (Section 4):

$$p_G(x) = \sum_{H \in U(P_G) \setminus \{\emptyset\}} (-1)^{D(H)} \prod_{i \in H} x_i$$

where the summation variable $H$ ranges over non-empty unions of paths from source to sink, and $D(H)$ is an easily computable function of $H$ (in the case of a planar graph it is essentially the number of regions in which $H$ divides the plane).

Since the number of monomials with non-zero coefficients is related to the communication complexity of the Boolean function (see, e.g., Section 4.4.1 in [BN21]), we estimate it in the case when $G$ is a two-dimensional grid (Section 5). In Section 2 we provide some preliminaries, and Section 6 contains the conclusion.

## 2 Posets, their Möbius functions, and convex polytopes

Let $O = \langle S, \leq_O \rangle$ be a poset over $S$ with order relation $\leq_O$. An antichain is any subset of $S$ consisting of mutually incomparable elements (under the order relation $\leq_O$). Consider the set inclusion poset $O = \langle 2^S, \subset \rangle$, and let $A$ be some antichain of this poset. Then let $U(A)$ be the induced poset of unions: $U(A) = \langle \{ \bigcup_{b \in B} b \mid B \subseteq A \}, \subset \rangle$.

For every poset $O$ there exists a unique function called the Möbius function $\mu : S \times S \to \mathbb{R}$, with the following two properties:

1) For all $s \in S$: $\mu(s, s) = 1$, and

2) For all $u, v \in S$ such that $u \leq_O v$:

$$\sum_{s : u \leq_O s \leq_O v} \mu(u, s) = 0.$$

Let $H = \{ x \in \mathbb{R}^n \mid ax = b \}$ denote a hyperplane and let $H^+ = \{ x \in \mathbb{R}^n \mid ax \geq b \}$ denote one of the halfspaces whose boundary is $H$. An intersection of halfspaces $B = H_1^+ \cap H_2^+ \cap \ldots \cap H_m^+$ that is bounded is called a convex polytope. If $H^+ \cap B = B$, then the intersection $H \cap B$ is called a face of $B$. Using the duality of linear programming one can show that

**Lemma 1.** A non-empty $F \subset \mathbb{R}^n$ is a face of $H_1^+ \cap H_2^+ \cap \ldots \cap H_m^+$ if and only if $F = \bigcap_{i \in M} H_i^+$ for some $M \subseteq [m]$. [Sch86].

In other words, a face corresponds to a system where some inequality signs are replaced by equality signs. This representation may be non-unique. Denote by $F(B)$ the set of faces of polytope $B$. Then $\langle F(B), \subset \rangle$ is a poset that is a lattice called the face lattice of polytope $B$. For a polytope $B$ let the dimension of the face $\dim B$ be the largest $n$ such that there exist $v_1, v_2, \ldots, v_{n+1} \in B$ such that $v_1 - v_{n+1}, v_2 - v_{n+1}, \ldots, v_n - v_{n+1}$ are linearly independent. Let $\dim \emptyset = -1$. Then
**Theorem 1** (Euler’s relation, [Grü03; Bro83]). For any two faces $F_1, F_2 \in \mathcal{F}(B)$, such that $F_1 \subseteq F_2$:

$$\sum_{F_1 \subseteq F \subseteq F_2} (-1)^{\dim F} = 0.$$

Let $\mu_B$ denote the Möbius function of the face lattice of polytope $B$.

**Corollary 1.**

$$\mu_B(F_1, F_2) = (-1)^{\dim F_2 - \dim F_1}.$$  

**Proof.** Since the Möbius function is unique, it is sufficient to verify that $\mu_B$ as defined here satisfies the properties 1) and 2). Clearly, $\mu_B(F, F) = 1$. For $F_1 \subseteq F_2$:

$$\sum_{F_1 \subseteq F \subseteq F_2} \mu_B(F_1, F) = \sum_{F_1 \subseteq F \subseteq F_2} (-1)^{\dim F - \dim F_1} = (-1)^{\dim F_1} \sum_{F_1 \subseteq F \subseteq F_2} (-1)^{\dim F} = 0. \quad \square$$

### 3 Representing monotone functions

Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be a Boolean function on $n$ variables. For two $n$-bit strings $x$ and $y$ we will denote by $x_i$ the $i$-th bit of the string, and say that $x \leq y$ if for all $i$: $x_i \leq y_i$. We say that a Boolean function is monotone if for all $x, y \in \{0,1\}^n$: $x \leq y \Rightarrow f(x) \leq f(y)$. We say that a set $I \subseteq [n]$ is a prime implicant of $f$ if the property $(\forall i \in I: x_i = 1) \Rightarrow f(x) = 1$ holds for $I$, but fails for every subset $I' \subset I$. Every monotone Boolean function has a unique representation in terms of the set of its prime implicants denoted by $P_f$. Since $P_f$ is an antichain in the inclusion poset of the subsets of $[n]$, we can consider its subposet $U(P_f)$.

**Proposition 1.** The poset $\langle U(P_f), \subset \rangle$ is an atomistic lattice.

**Proof.** Since $\langle U(P_f), \subset \rangle$ is a subposet of the lattice $\langle 2^{[n]}, \subseteq \rangle$ containing all the original joins (unions) of the elements of $U(P_f)$, it is an upper semilattice. Moreover, it is a lattice, since $U(P_f)$ contains $\emptyset$ as the least element, and any finite bounded upper semilattice is a lattice (see e.g. Proposition 3.3.1 in [Sta12]). Since the elements of $P_f$ are incomparable, and $U(P_f)$ contains beside them only their unions (including the empty one), $P_f$ is the set of atoms of $U(P_f)$, and every element of $U(P_f)$ is expressible as their join (union), i.e. the lattice $\langle U(P_f), \subset \rangle$ is atomistic. \(\square\)

**Theorem 2.** Every finite atomistic lattice is isomorphic to $\langle U(P_f), \subset \rangle$ for some monotone Boolean function $f$.

**Proof.** Let $\langle L, \leq_L \rangle$ be a finite atomistic lattice, and let $A = \{a_1, a_2, \ldots, a_m \} \subset L$ be its set of atoms.

We will build the corresponding Boolean function $f$ as follows. We will pick an appropriate $n$ (number of the input bits) and injective mapping $u : [n] \rightarrow 2^{[m]}$ so that prime implicants $P_f = \{b_1, b_2, \ldots, b_m\}$ defined by $b_j = \{k \in [n] \mid j \in u(k)\}$ would generate $U(P_f)$ which is isomorphic to $L$ (in this isomorphism each prime implicant $b_j$ will correspond to the atom $a_j$). In such construction it is irrelevant which element of $[n]$ is mapped by $u$ to which element of the image $Im(u)$, only the set $Im(u)$ itself is important.

Notice that, given $k \in [n]$ and $S \subseteq [m]$, $k$ belongs to the union $\bigcup_{j \in S} b_j$ iff $S \cap u(k) \neq \emptyset$. Thus the presence of such $k$ (or, rather, $u(k)$) ensures that any union $\bigcup_{j \in T} b_j$ with $T \cap u(k) = \emptyset$ is distinct from any union $\bigcup_{j \in S} b_j$ with $S \cap u(k) \neq \emptyset$.  

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Initially we let $\text{Im}(u) = 2^m$ (later we will remove some of its elements). Since $L$ is atomistic, any of its elements is expressible as a union of its atoms. However, some unions of atoms may be equal between themselves. Suppose an equality $\bigcup_{j \in S} a_j = \bigcup_{j \in T} a_j$ holds for some distinct $S, T \subseteq [m]$. To achieve that a corresponding equality holds for $b_j$, we remove from $\text{Im}(u)$ all $u(k)$ which make these unions distinct, that is all $u(k)$ such that $S \cap u(k) = \emptyset \neq T \cap u(k)$ or $S \cup u(k) = \emptyset = T \cap u(k)$. We decrease $n$ and remap $u$ from the new interval $[n]$ accordingly.

We perform this operation for every equality which holds between unions of atoms $a_j$. Thus we ensure that the corresponding equality holds also for the unions of $b_j$. We claim that $\text{Im}(u)$ that remains after these operations is what we need. To show that, it remains to prove that we haven’t introduced any undesired equality among unions of $b_j$ by removing too many $u(k)$’s.

Suppose the contrary: that for some $S, T \subseteq [m]$ we have $\bigcup_{j \in S} a_j = \bigcup_{j \in T} a_j$, but $\bigcup_{j \in S} b_j = \bigcup_{j \in T} b_j$. Here among different $S$ that give the same union $\bigcup_{j \in S} a_j$, let us use the maximum $S$ (i. e. such that no superset of $S$ gives the same union), similarly let us use the maximum $T$. Since an equality among unions of $a_j$ implies an equality among the corresponding unions of $b_j$, $\bigcup_{j \in S} b_j = \bigcup_{j \in T} b_j$ holds also for the new $S$ and $T$, if we replaced any of them.

At least one of the sets $S \cap T$ and $T \setminus S$ is nonempty; WLOG suppose that $S \cap T = \emptyset$. The equality $\bigcup_{j \in S} b_j = \bigcup_{j \in T} b_j$ means that, among others, we have removed $[m] \setminus T$ from $\text{Im}(u)$ because otherwise it would make these unions distinct: $S \cap ([m] \setminus T) = S \setminus T = \emptyset = T \cap ([m] \setminus T)$. Let the equality which caused the removal of $[m] \setminus T$ be $\bigcup_{j \in S'} a_j = \bigcup_{j \in T} a_j$ where $S' \cap ([m] \setminus T) = \emptyset = T' \cap ([m] \setminus T)$. Then $T' \subseteq T \subseteq S' \cup T$ and $\bigcup_{j \in T} a_j = \bigcup_{j \in (T \setminus T') \cup T'} a_j = \bigcup_{j \in (T \setminus T') \cup S'} a_j$. Note that in an atomistic lattice, if $\bigcup_{j \in S} a_j = \bigcup_{j \in Y} a_j$, then $\bigcup_{j \in X} a_j = \bigcup_{j \in Y} a_j = \bigcup_{j \in X \cup Y} a_j$, thus the last equality implies $\bigcup_{j \in T} a_j = \bigcup_{j \in T \cup S'} a_j$ which contradicts the maximality of $T$.

Thus no undesired equality was introduced, and $L$ is isomorphic to $U(P_f)$ by mapping that for each $S \subseteq [m]$ maps $\bigcup_{j \in S} a_j$ to $\bigcup_{j \in S} b_j$.

Let $p : \mathbb{R}^n \to \mathbb{R}$ be a multilinear polynomial. We can write $p$ as a linear combination of monomials:

$$p(x_1, x_2, \ldots, x_n) = \sum_{S \subseteq [n]} \alpha_S \prod_{i \in S} x_i.$$  

We say that a real multilinear polynomial $p$ represents a Boolean function $f$ if $p(x) = f(x)$ for all $x \in \{0, 1\}^n$. Every Boolean function has a unique polynomial $p$ that represents it. The poset $U(P_f)$ and its associated Möbius function $\mu_{U(P_f)}$ have a crucial role in determining the coefficients $\alpha_S$:

**Theorem 3.** Let $f$ be a monotone Boolean function not identical to 1. Then its representing polynomial is

$$p_f(x_1, x_2, \ldots, x_n) = \sum_{S \in U(P_f) \setminus \{\emptyset\}} -\mu_{U(P_f)}(\emptyset, S) \prod_{i \in S} x_i.$$  

**Proof.** We will show that the polynomial $p_f$ equals $f$ on all $x \in \{0, 1\}^n$. Clearly, when $f(x) = 0$ then for each $S \in U(P_f) \setminus \{\emptyset\}$ there exists an $i$ such that $x_i = 0$, and so $p_f(x) = 0$. If $f(x) = 1$, then let $S$ be such that $x_i = 1 \iff i \in S$. Let $S' = \cup \{I \in P_f, I \subseteq S\}$. $S'$ cannot be empty since $f(x) = 1$. Then

$$p_f(x) = \sum_{T \in U(P_f) : T \subseteq S'} \alpha_T = \sum_{T \in U(P_f) : T \subseteq S'} -\mu_{U(P_f)}(\emptyset, T) + \mu_{U(P_f)}(\emptyset, T) = 1. \quad \square$$
4 Möbius function of DAG-Connectivity

Let $G = \langle V, E \rangle$ be a directed acyclic graph. For our purposes we can fix $V$ and associate $G$ with the set of its arcs $E$. Thus we will write $H \subseteq G$ when $H$ is a subgraph of $G$, etc. Let $S(G) \subseteq V$ be the set of vertices with no incoming arcs – sources, and let $T(G) \subseteq V$ be vertices with no outgoing arcs – sinks.

**Definition 1.** In DAG-Connectivity$_G$, we are given a subgraph $H$ of some fixed graph $G$ as a list of bits defining its arcs $x_e = \begin{cases} 1 & \text{if } e \in H \\ 0 & \text{if } e \notin H \end{cases}$, and our task is to determine if there is a non-empty path from a source to a sink.

Let $s: E \to V$ and $t: E \to V$ denote the source and target of an arc. If we allow multigraphs, i.e., graphs which can have multiple arcs $e_1, \ldots, e_m$ with the same sources and targets $s(e_1) = \ldots = s(e_m)$, $t(e_1) = \ldots = t(e_m)$, then for our purposes we can assume that there is exactly one source and one sink, because we can merge all sources into one (and similarly – all sinks) without affecting connectivity. Henceforth, we will denote the unique source as $s$ and the unique sink as $t$.

Given a DAG $G$, let $P_G$ be the finite set of paths connecting sources and sinks. Note that $P_G$ has to be an antichain. The following two theorems are a special case of Theorem 3.2 from [Hil03].

We give our proofs for completeness.

**Theorem 4.** For all directed acyclic graphs $G$ the poset $U(P_G)$ is isomorphic to a face lattice of a convex polytope.

**Proof.** For a graph $G$ and its subgraph $H \subseteq G$, let $F(H)$ be the set of unit flows from $s$ to $t$, i.e., $F(H)$ consists of all vectors in $\mathbb{R}^{\vert G \vert}$

$$F(H) = \{(f_e)_{e \in G} \}_{f \in \mathbb{R}^{\vert G \vert}},$$

such that the following additional constraints are satisfied:

1. Flow is non-negative: $\forall e \in G : f_e \geq 0$;
2. The total flow is 1: $\sum_{e \in H} f_e = 1$;
3. Flow conservation: $\forall v \notin \{s, t\} : \sum_{e \in H} f_e = \sum_{e \in H} f_e$.
4. Flow is restricted to the subgraph: $\forall e \notin H : f_e = 0$;

Next, we show that $F: U(P_G) \to 2_{\mathbb{R}^{\vert G \vert}}$ is indeed the bijection we sought. First, $F(H)$ shares almost all constraints with $F(G)$ except equalities (4). By Lemma 4 the faces of $F(G)$ correspond to the system of $F(G)$ where some inequalities are replaced by inequalities, i.e., $f_e = 0$ for some $H' \subseteq G$. Thus, $F(H)$ is a face of $F(G)$ for any $H \in U(P_G)$ since the inequalities $f_e \geq 0$ are replaced by equalities $f_e = 0$ for $e \notin H$. On the other hand, for a subgraph $H'$ there exists a $H \in U(P_G)$ such that $H = \bigcup \{p \in P_G|p \subseteq H'\}$. But subgraphs $H$ and $H'$ correspond to the same face; since all $e \in H \setminus H'$ have no path in $H'$ containing them, the flow on these arcs must be zero: $f_e = 0$ for $e \in H' \setminus H$. By construction the inclusion property is obviously preserved, since the flow is more constrained on a subgraph. \qed
Next we give a simple formula for computing the dimension of a face of the flow polytope. Let 
\( D(H) = |H| - |\{ s(e) | e \in H \} \cup \{ t(e) | e \in H \} \setminus \{ s, t \} | - 1 \).

**Theorem 5.** If \( H \in U(P_G) \), then the dimension of the corresponding face is
\[
\dim F(H) = D(H).
\]

**Proof.** First, the lemma is clearly true for empty graph and a single path from \( P_G \).

Denote by \( A \subset B \), if \( A, B \in U(P_G), A \subset B \) and \( \exists C \in U(P_G) : A \subset C \subset B \). Let us show that for all \( H, H' \in U(P_G) \) such that \( H \subset H' \): \( \Delta := H' \setminus H \) is an ear of \( H' \), i.e., a path \( (v_0, v_1, \ldots, v_k) \) whose internal vertices \( v_1, v_2, \ldots, v_{k-1} \) have no other adjacent arcs in \( H' \). Hence \( D(H') = D(H) + 1 \).

Let us prove by contradiction assuming that \( \Delta \) is not an ear. Clearly, there exists some subset \( \delta \subseteq \Delta \) that is an ear of \( \delta \cup H \); one can start a path with any arc of \( \Delta \) and extend the path by traveling backwards and forwards along arcs in \( \Delta \) until a vertex with an adjacent arc in \( H \) is encountered. Consider the initial vertex \( v_{init} \) of \( \Delta \). There must be a path \( \alpha \subseteq H \) [potentially empty] from \( s \) to \( v_{init} \). If \( v_{init} \) has no incoming arcs in \( H \) then \( v_{init} = s \). If \( v_{init} \) has an incoming edge in \( H \) then \( \alpha \) is a path from \( s \) to \( v_{init} \). Symmetrically reasoning we obtain a path \( \omega \) from the final vertex to \( t \).

Clearly, \( \alpha \cup \delta \cup \omega \in P_G \) and so \( \delta \cup H \in U(P_G) \). Thus \( H \subset H \cup \delta \subset H \cup \Delta = H' \) — a contradiction.

![Ear decomposition](image)

Each \( H \in U(P_G) \) has an “ear decomposition” described as \( \emptyset = H_{-1} \subset H_0 \subset H_1 \subset \ldots \subset H_{D(H)-1} \subset H = H_D \). In particular, let the path \( \alpha \cup \delta \cup \omega \in P_G \) added from \( H_{k-1} \) to \( H_k \) be denoted by \( p_k \). Then \( H_k = H_{k-1} \cup p_k \).

Let \( 1(p) = (f_e)_{e \in G} \) denote the unit flow along path \( p \): \( f_e = \begin{cases} 1 & \text{if } e \in p \\ 0 & \text{if } e \notin p \end{cases} \). Any flow satisfying all but the non-negativity constraints (i.e., satisfying (2-4)) for \( H \) is an affine combination of \( \{ 1(p_i) | i \in \{ 0, 1, \ldots, D(H) \} \} \). We can establish this by induction on \( D(H) \). This is obviously the case for \( D(H) = -1 \) and \( D(H) = 0 \). Assuming that it is true for \( D(H) = k - 1, k > 0 \), let \( f \) be the flow vector. Note that the flow \( f_e \) for arcs \( e \) in the ear \( e \in H_{D(H)} \setminus H_{D(H)-1} \) must be equal. Consider the flow
\[ f' = f - f_e(1(p_k) - 1(p_{k-1})). \]

\( f' \) satisfies constraints (2-4) for \( H_{D(H)-1} \) and so by inductive assumption \( f' \) is in the affine span of \( \{ 1(p_i) | i \in \{ 0, 1, \ldots, D(H) - 1 \} \} \). We conclude that \( \dim F(H) \leq |\{ p_0, p_1, \ldots, p_{D(H)} \}| - 1 = D(H) \).

\( \{ 1(p_i) - 1(p_0) | i \in \{ 1, \ldots, D(H) \} \} \) are linearly independent because each \( 1(p_i) - 1(p_0) \) is outside the linear span of \( \{ 1(p_i) - 1(p_0) | i \in \{ 1, \ldots, i - 1 \} \} \). Obviously, \( \{ 1(p_i) | i \in \{ 0, 1, \ldots, D(H) \} \} \) belong to \( F(H) \). Therefore \( \dim F(H) \geq D(H) \).

\( \square \)
Corollary 2. The unique multilinear polynomial representing $\text{DAG-CONNECTIVITY}_G$ is:

$$p_G(x) = \sum_{H \in U(P_G) \setminus \{\emptyset\}} (-1)^{D(H)} \prod_{i \in H} x_i$$

Proof. By Theorem 3 we have:

$$p_G(x) = \sum_{H \in U(P_G) \setminus \{\emptyset\}} -\mu_{U(P_G)}(\emptyset, H) \prod_{i \in H} x_i.$$  

By Theorem 4, the poset $U(P_G)$ is isomorphic to the face lattice of a polytope, and by the Corollary of Euler’s relation:

$$p_G(x) = \sum_{H \in U(P_G) \setminus \{\emptyset\}} (-1)^{\dim F(H) - \dim \emptyset} \prod_{i \in H} x_i = \sum_{H \in U(P_G) \setminus \{\emptyset\}} (-1)^{\dim F(H)} \prod_{i \in H} x_i.$$  

Finally, by Theorem 5 we conclude that

$$p_G(x) = \sum_{H \in U(P_G) \setminus \{\emptyset\}} (-1)^{D(H)} \prod_{i \in H} x_i. \quad \blacksquare$$

Corollary 3. The degree of $p_G(x)$ is maximal: $\deg p_G(x) = |G|$.

Proof. Since the whole graph $G$ is also a union of paths: $G \in U(P_G)$, the coefficient at its monomial is $(-1)^{D(G)}$, i.e. not zero. \quad \blacksquare

5 Size of $U(P_G)$ for 2D grids

Let $G_n$ be a directed grid: the vertices of this graph are labeled by $\{0, 1, \ldots, n\}^2$ and there is an arc from vertex $(i_1, j_1)$ to $(i_2, j_2)$ iff $i_1 = i_2$ and $j_2 = j_1 + 1$ or $j_1 = j_2$ and $i_2 = i_1 + 1$.

Theorem 6. $\Omega \left(1.641^{2n^2}\right) \leq |U(P_{G_n})| \leq O \left(1.654^{2n^2}\right)$

Proof. For both bounds we use the spectral method on appropriately constructed matrices.

For a vertex that is not a sink or source we will say that a vertex is valid, if it has an incoming arc if it has an outgoing arc. Clearly, a subset of arcs is a union of paths iff all the vertices are valid. To show the lower bound we count only the unions of paths where every vertex at distance $2di$ has a path going through it (for integers $i$ – the depth and $d$ – the width, an integer constant).
We will call the arcs between these vertices layers; and we exploit the fact that arcs from different layers can be selected for the union independently. To count the number of arc arrangements per layer we split each layer into strips. We say that a strip is valid, if the interior vertices are valid and top an bottom vertices have an incoming and outgoing arc respectively. Now we construct a directed graph whose vertices are the valid strips and an arc between two strips \( u \) and \( v \) exists iff \( v \) can follow \( u \) in a layer, that is, the vertices shared between the two strips are valid. Let \( S_d \) be its adjacency matrix. Then \( S_d^{m} [u,v] \) equals the number of ways to arrange \( m + 1 \) strips consecutively starting with strip \( u \) and ending with \( v \). Tallying all the ways over all the layers we obtain

\[
\left( \prod_{i=0}^{n} \langle 1 | S_d^{2di} | 1 \rangle \right)^2.
\]

For the purposes of this lower bound we can disregard the possible ways to arrange arcs outside the layers (near the borders of the grid) and instead assume all the outside arcs are present.

The matrix \( S_d \) is non-negative and irreducible (consult [HJ13, Chapter 8] for definitions and theorems used in this part). In fact already \( S_d^2 \) is a positive matrix, since we can place a strip with all arcs between any two strips to make a valid sequence of 3 strips. Furthermore, \( S_d \) is a primitive matrix because it has positive values on the diagonal, i.e., the diagonal entry corresponding to the strip with all arcs is 1. \( S_d \) has a unique eigenvalue \( \lambda \) such that \( |\lambda| = \rho(S_d) \) and the corresponding right and left eigenvectors \( |r \rangle, |l \rangle \) consist of non-negative entries only. Thus by [HJ13, Theorem 8.5.1]

\[
\lim_{n \to \infty} \left( \frac{S_d}{\lambda} \right)^n = |r \rangle \langle l |.
\]

which implies \( \lim_{n \to \infty} \langle 1 | \left( \frac{S_d}{\lambda} \right)^n | 1 \rangle = \Omega(1) \) and so \( \langle 1 | S_d^n | 1 \rangle = \Omega(\lambda^n) \). Plugging this back into \( [1] \)
we obtain

\[
\left(\prod_{i=0}^{\frac{n}{d}} \langle 1 \mid S_{d}^{2di} \mid 1 \rangle\right)^{2} \geq \left(\prod_{i=0}^{\frac{n}{d}} c\lambda^{2di}\right)^{2} \geq c^{O(n)} \lambda^{2\frac{n^{2}}{2d}}.
\]

Having computed \(\lambda = \rho(S_d)\) for \(d\) up to 5, we obtain the lower bound.

For the upper bound the approach is slightly different. We will divide the grid into layers that are diagonal parallel to the direction between source and sink.

The vertices on the borders of a layer will not be required to have a path or even be valid. Similarly as before we will divide the layer into strips, but now we will define \(S'_d\) as a \(2^d \times 2^d\) matrix whose rows and columns are indexed by strings \(\{0, 1\}^d\) which determine whether a vertex has a path through it or not for vertices on, respectively, the left and right border of a strip (or a sequence of strips). \(S'_d[x, y]\) will be the number of strips with left vertex assignment of \(x\) and right vertex assignment of \(y\). \(S'^m_d[x, y]\) now counts the number of sequences of \(m\) strips starting with assignment \(x\) and ending with \(y\), with only valid vertices in between. Similarly as in the lower bound, the ways to choose arcs within all the strips is

\[
\left(\prod_{i=0}^{\frac{n}{d}} \langle 1 \mid S'_{d}^{2di} \mid 1 \rangle\right)^{2} \leq \left(\prod_{i=0}^{\frac{n}{2^d-1}} \| S'_{d}^{2di} \|\right)^{2} \leq \left(\prod_{i=0}^{\frac{n}{d}} \| S'_{d} \|^{2di}\right)^{2} = \| S'_d \|^{\frac{n}{2^d} + n}. \tag{2}
\]

There are \(O(n)\) arcs outside the strips which contribute a multiplier of at most \(2^{O(n)}\). Having computed \(\| S'_d \|\) for \(d\) up to 15, we obtain the upper bound.

We conjecture that the base of the exponent tends to \(\sqrt{1 + \sqrt{3}} \approx 1.653\).
6 Conclusion

Like Beniamini and Nisan in [BN21], we studied also the dual of the function: 

\[ f^*(x_1, x_2, \ldots, x_n) = \neg f(\neg x_1, \neg x_2, \ldots, \neg x_n). \]

Let \( \text{DAG-Connectivity}^*(x_1, x_2, \ldots, x_n) = \sum_{S \subseteq [n]} \alpha_S^* \prod_{i \in S} x_i. \) Prime implicants of \( \text{DAG-Connectivity}^* \) are minimal cuts. Even though we did not manage to prove it for arbitrary graphs, we found that for the graphs we analyzed the following properties are true:

* \( \alpha_S^* \in \{-1, 0, 1\} \);
* The sets \( S \) corresponding to non-zero \( \alpha_S^* \) together with empty set constitute an Eulerian lattice with the usual subset inclusion relation. However, unlike for \( \text{DAG-Connectivity} \) the elements of this lattice are not all unions of minimal cuts, but a subset of them. In particular, a slight generalization of Lemma 4.6 from [BN21] to any function with the union of prime implicants \( U(P_f) \) being an Eulerian lattice holds. However, it is not a sufficient criterion for \( \text{DAG-Connectivity} \).

The following topics could be of interest for future research:

* What other useful classes of monotone Boolean functions, besides Bipartite Perfect Matching [BN21] and \( \text{DAG-Connectivity} \), have simple Möbius functions?
* The lattices of Bipartite Perfect Matching and \( \text{DAG-Connectivity} \) are Eulerian implying the simplicity of the formula for the representing polynomial. Is there a simple (good, useful) characterization of the monotone Boolean functions with Eulerian lattices?
* Can the representing polynomial of some \( \text{DAG-Connectivity} \) problem be used to improve its complexity estimations? For instance, the current quantum query complexity estimations for the \( n \times n \) two-dimensional grid still have gap between \( \Omega(n^{1.5}) \) and \( O(n^2) \) [Amb+20]. Since the quantum query complexity is lower bounded by the degree of approximate polynomial (divided by 2), one of the questions is: can the representing polynomial be used to obtain lower bounds exceeding \( \Omega(n^{1.5}) \) for the degree of approximate polynomial?

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