LOGARITHMIC SCHRÖDINGER EQUATIONS IN INFINITE DIMENSIONS

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Abstract. We study the logarithmic Schrödinger equation with finite range potential on $\mathbb{R}^d$. Through a ground-state representation, we associate and construct a global Gibbs measure and show that it satisfies a logarithmic Sobolev inequality. We find estimates on the solutions in arbitrary dimension and prove the existence of weak solutions to the infinite-dimensional Cauchy problem.

1. Introduction

Consider the logarithmic Schrödinger equation (LSE) given by

$$i\partial_t \psi = H\psi \equiv -\Delta \psi + V\psi + \lambda \log \frac{|\psi|^2}{\int |\psi|^2 dx}$$

with $\lambda \in \mathbb{R}$ and where $\Delta$ denotes the Laplacian in $\mathbb{R}^n$. It has the following property of statistical independence for non-interacting systems. Suppose the potential can be written as $V(x) = V_1(x_1) + V_2(x_2)$, where $V_j$ depends on coordinates $x_j$, $j = 1, 2$, such that $\{x_1\} \cap \{x_2\} = \emptyset$ and $x = (x_1, x_2) \in \mathbb{R}^n$. Let

$$H_j \psi_j \equiv -\Delta_j \psi_j + V_j \psi_j + \lambda \psi_j \log \frac{|\psi_j|^2}{\int |\psi_j|^2 dx_j}$$

with $\Delta_j \equiv \Delta_{\{x_j\}}$. Then, for an initial condition $\psi(x, 0) = \psi_1(x_1, 0)\psi_2(x_2, 0)$, the function $\psi = \psi_1\psi_2$ solves the equation (1.1)

$$i\partial_t \psi = H\psi = (\psi_2 H_1 \psi_1 + \psi_1 H_2 \psi_2).$$

Thus the probability density $|\psi|^2$ of a composite system, consisting of two non-interacting subsystem, is described by the product of probability densities $|\psi_j|^2$, $j = 1, 2$, describing both systems separately. Non-interacting systems remain statistically independent. Furthermore, for the solutions of the above equations the normalisation $\int |\psi|^2 dx = 1$ is independent of time and we have the following physical property of additivity of energy

$$\langle \psi, H\psi \rangle = \sum_j \langle \psi_j, H_j \psi_j \rangle.$$

To study the asymptotic stability of large interacting systems we investigate an analog of the ground state representation. To this end, for the equation (1.1) we would like to consider the solution $\psi \equiv \varphi e^{-\frac{1}{\lambda}U}$ with some $U : \mathbb{R}^n \to \mathbb{R}$ independent of

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time. By choosing
\[ -\frac{1}{2} \Delta U + \frac{1}{4} |\nabla U|^2 + \lambda U = V \] (1.2)
we can rewrite our equation as follows
\[ i\partial_t \varphi = -\mathcal{L} \varphi + \lambda \varphi \log \frac{\varphi^2}{\mu |\varphi|^2} \] (1.3)
with the linear operator \( \mathcal{L} \equiv \Delta - \nabla U \cdot \nabla \) and the measure \( \mu(\varphi) \equiv \int \varphi e^{-U} d\mathbf{x} \).

Later on we examine (1.3) in finite and infinite dimensional spaces, therefore it will be vital that the formal Gibbs measure \( d\mu \equiv e^{-U} d\mathbf{x} \) can be well defined as a probability measure. In particular, to discuss the equation in an infinite dimensional setting, where the Lebesgue measure is not well defined, it will be necessary to have a reference probability measure. For this we will need to add some technical conditions. First of all we note that any potential energy function \( V \) is required to have an additive structure, which includes a local external potential as well as a multi-particle interaction. Since the relation between \( V \) and \( U \) is nonlinear, even an additive structure of \( U \) may not necessarily provide a stable \( V \). However we notice that this can be achieved if \( U \) itself is finite range. This provides us with an interesting family of systems we will consider in this paper; however more general cases should be discussed elsewhere.

The LSE plays an important role in the class of nonlinear Schrödinger equations. Indeed, the equations with logarithmic nonlinearity admit a wide range of applications, for example, in quantum mechanics [1, 2], quantum optics [3], modelling magma transport [4], nuclear physics [5], transport and diffusion phenomena [6], information theory [7, 8], quantum gravity [9], theory of Bose–Einstein condensation [10] and others.

Over a long period, including a number of very recent publications, the LSE was extensively studied in mathematical literature. In particular, many results on existence and smoothness of solutions to different forms of the LSE have been provided (see [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]). These works concern finite-dimensional configuration space, which rules out the systems with infinitely many particles. This paper aims to explore such systems by introducing an infinite-dimensional setup for the LSE in the context of statistical mechanics.

In Section 2 we introduce the infinite-dimensional configuration space \( \Omega \equiv \mathbb{R}^{Z_d} \) and finite-range potential \( U \). We present a construction of the associated global Gibbs measure \( \mu \) and the operator \( \mathcal{L} \) corresponding to the Dirichlet form in \( L^2(\mu) \). In Section 3 we show that under a set of well-chosen assumptions the measure \( \mu \) satisfies a logarithmic Sobolev inequality.

**Theorem 1.1.** The global Gibbs measure \( \mu \) satisfies a log-Sobolev inequality on \( \Omega = \mathbb{R}^{Z_d} \) (under suitable growth conditions on \( U \)), that is, for all \( \varphi \in H^1(\mu) \)
\[ \text{Ent}_\mu \left( |\varphi|^2 \right) \equiv \mu \left( |\varphi|^2 \log \frac{|\varphi|^2}{\mu |\varphi|^2} \right) \leq c \| \nabla \varphi \|_{L^2(\mu)}^2 \] (1.4)
with some constant \( c \in (0, \infty) \) independent of the function \( \varphi \).
In particular, it follows from (1.4) that the right hand side of the LSE (1.3) defined by addition of two unbounded operators can be well defined, uniformly in dimension, for an appropriate choice of $\lambda$. Although the general idea of our proof follows those of [25], [26], [27], [28], [29], [30] we need some care due to non-compactness of the space $\Omega = \mathbb{R}^{2d}$ and the unboundedness of the interaction potential. The full statement, after the constructions in Section 2 is given in Theorem 3.1.

In Section 4 we show that our strategy can be used to recover nice, direct results on the analyticity of the theory, providing an alternative point of view to the one presented in [26].

Following the construction of $\mu$ and proof of LSI, we will investigate various estimates for the solutions of LSE in Section 5. We summarise these in the statement below.

**Theorem 1.2.** Assume that $U$ satisfies the same conditions in Theorem 1.1. Let $\varphi$ be the solution to (1.3) in arbitrary dimension with initial data $f \in H^1(\mu)$ dependent on a finite number of variables, i.e. $f$ is a function of $\{x_k\}_{k \in \Lambda}$, where $\Lambda \subset \mathbb{Z}^d$. Then, for $\lambda > 0$, it follows that

$$\|\nabla \varphi(\cdot, t)\|_{L^2(\mu)}^2 \leq e^{2\lambda t} \|\nabla f\|_{L^2(\mu)}^2 \quad \text{and} \quad \text{Ent}_{\mu}(\|\varphi(\cdot, t)\|^2) \leq (\lambda^{-1} + c) \|\nabla f\|_{L^2(\mu)}^2$$

for all $t > 0$, where $c$ is the constant in (1.4). Furthermore, for any $j \in \mathbb{Z}^d$

$$\|\nabla(x_j)\varphi(\cdot, t)\|_{L^2(\mu)}^2 \leq e^{-N_j} \|\nabla f\|_{L^2(\mu)}^2$$

holds for all $t \leq \epsilon N_j$, where $\epsilon = \epsilon(\lambda, U) > 0$ and $N_j$ is proportional to the distance between $j$ and $\Lambda$.

The last bound tells us that, in finite time, the LSE can only introduce small dependence on variables far away from the dependence of the initial data. This property is analogous to the one discussed in the semigroup case (see e.g. [26], [29]) and is vital for the construction of the solution of the infinite dimensional LSE. For the full statements of these bounds, see theorems 5.1, 5.3 and 5.4 in Section 5.

In Section 6 we study the existence of weak solutions to the infinite-dimensional LSE. We restrict to the one-dimensional lattice and introduce the following result.

**Theorem 1.3.** Let $U$ be a finite-range potential with bounded multi-spin interaction (see Section 2) and consider initial data $f \in H^1(\mu)$ dependent on finitely many variables. Then there exists a weak solution to the infinite-dimensional LSE problem, i.e. there exists $\varphi \in H^1(\mu)$ such that

$$\mu (g i \partial_t \varphi) = \mu \left( \nabla g \cdot \nabla \varphi + g\lambda \varphi \log \frac{\|\varphi\|^2}{\mu \|\varphi\|^2} \right)$$

$$\varphi|_{t=0} = f$$

for all smooth compactly supported function $g : \mathbb{R}^2 \to \mathbb{C}$ that depends on finitely many variables.
For the proof we construct a sequence of finite sets \( \Lambda_n \subset \mathbb{Z} \) such that, as \( \Lambda_n \) invades \( \mathbb{Z} \), the solution \( \varphi_{\Lambda_n} \) to the finite-dimensional LSE on \( \mathbb{R}^{\Lambda_n} \) converges to a weak solution \( \varphi \) of the infinite-dimensional LSE on \( \mathbb{R}^\mathbb{Z} \). In higher dimensions the same argument can be used, but more care has to be taken with the growth of the interaction (see Remark 6.3). The full statement of the result is given by Theorem 6.1.

We conclude in Section 7 by discussing the construction of solitary solutions in finite and infinite dimensions.

2. Infinite dimensional setup

2.1. Configuration space. Consider our physical space being modelled by a lattice \( \mathbb{Z}^d \) (\( d \in \mathbb{N}^+ \)), endowed with the \( l^\infty \) distance \( |i - j| \equiv \max_{1 \leq n \leq d} |i_n - j_n| \) for all \( i, j \) in \( \mathbb{Z}^d \). For simplicity, we define

\[
\text{dist}(\Lambda, \Lambda') \equiv \inf_{i \in \Lambda, j \in \Lambda'} |i - j|,
\]

\[
\text{diam}(\Lambda) \equiv \sup_{i,j \in \Lambda} |i - j| \quad \forall \, \Lambda, \Lambda' \subset \mathbb{Z}^d.
\]

We assign each \( j \in \mathbb{Z}^d \) a random variable \( x_j \) with values in \( \mathbb{R} \). Then the configuration of our system is given by \( x = (x_j)_{j \in \mathbb{Z}^d} \), living in the infinite-dimensional space \( \Omega \equiv \mathbb{R}^{\mathbb{Z}^d} \).

Denoting by \( \Lambda \subset \mathbb{Z}^d \) any compact subset \( \Lambda \) of \( \mathbb{Z}^d \), we define the projection \( \pi_{\Lambda} : \Omega \to \mathbb{R}^\Lambda \) by

\[
\pi_{\Lambda}(x) = (x_j)_{j \in \Lambda} \equiv x_\Lambda.
\]

The \( \sigma \)-algebra \( \Sigma \) on \( \Omega \) is the smallest \( \sigma \)-algebra that makes all projections measurable. Similarly, any sub-\( \sigma \)-algebra \( \Sigma_{\Lambda} \) on \( \mathbb{R}^\Lambda \) is the smallest \( \sigma \)-algebra for which \( \pi_{\{j\}} \) is measurable for all \( j \in \Lambda \).

We say that a function \( f \) on \( \Omega \) is localised in \( \Lambda \) if it is \( \Sigma_{\Lambda} \)-measurable (or equivalently, \( f \) only depends on coordinates in \( \mathbb{R}^\Lambda \)). Let \( \Lambda_f \subset \mathbb{Z}^d \) be the smallest set in which \( f \) is localised, we call \( f \) a local or cylinder function if \( \Lambda_f \) is compact. Denote by \( \mathfrak{F}(\Omega) \) the set of all local and bounded functions \( f : \Omega \to \mathbb{C} \).

2.2. Interactions and Gibbs measures. The potential on \( \Omega \) will be given by a family \( \{J_X\}_{X \subset \subset \mathbb{Z}^d} \) of differentiable local functions with finite range \( R \in \mathbb{N}^+ \). That is for each \( X \subset \subset \mathbb{Z}^d \), \( J_X : \Omega \to \mathbb{R} \) is localised in \( X \), twice differentiable (as a function on \( \mathbb{R}^X \)) and \( J_X \equiv 0 \) if \( \text{diam}(X) > R \). The potential energy \( U_\Lambda \) in any finite volume \( \Lambda \subset \subset \mathbb{Z}^d \) is well-defined by

\[
U_\Lambda(x) = \sum_{\Lambda \cap X \neq \emptyset} J_X(x).
\]

Example 2.1 (Bilinear Potential). Consider the family of potentials

\[
J_X(x) = \begin{cases} 
C_{ii}x_i^2 & \text{if } X = \{i\} \\
C_{ij}x_ix_j & \text{if } X = \{i, j\} \\
0 & \text{otherwise}
\end{cases}
\]

where \( C_{ij} \in \mathbb{R} \) uniformly bounded and \( C_{ij} = 0 \) if \( |i - j| > R \). The associated \( U_\Lambda \), the bilinear interaction in \( \Lambda \), follows from the above.
Note that in general $U_\Lambda$ may depend on $x_j$ for some $j \notin \Lambda$. Such dependence outside $\Lambda$ shall be fixed as boundary conditions, i.e. $x_\Lambda^c = \omega_\Lambda^c$ for some constant $\omega \in \Omega$. For this reason, we can write

$$U_\Lambda(x) = U_\Lambda^\omega(x) \equiv U_\Lambda(x|x_\Lambda^c = \omega_\Lambda^c).$$

In other words, $U_\Lambda^\omega$ represents the energy of configuration $x_\Lambda$ in the finite sub-system $\mathbb{R}_\Lambda$, with all variables outside this system being fixed by $\omega$. The (local) Gibbs measure on $\mathbb{R}_\Lambda$ with boundary conditions $\omega$ is, by definition, the probability measure $E_\Lambda^\omega$ satisfying

$$dE_\Lambda^\omega = \frac{e^{-U_\Lambda^\omega}}{\int_{\mathbb{R}_\Lambda} e^{-U_\Lambda^\omega}} dx_\Lambda$$

where $dx_\Lambda$ is the product Lebesgue measure on $\mathbb{R}_\Lambda$. For $E_\Lambda^\omega$ to be well-defined in all finite volume $\Lambda$, we assume

$$0 < \int_{\mathbb{R}_\Lambda} e^{-U_\Lambda^\omega} dx_\Lambda < \infty \quad \forall \Lambda \subset \subset \mathbb{Z}^d \quad \forall \omega \in \Omega \quad (2.1)$$

and hereinafter we only consider $U_\Lambda$ that satisfies the condition (2.1).

The quantity $dE_\Lambda^\omega$ has all the properties of a regular conditional probability of the system $\mathbb{R}_\Lambda$ being in configuration $x_\Lambda$, given the boundary $\omega_\Lambda^c$ outside the system. Frequently it will be convenient to identify $E_\Lambda^\omega$ as an operator defined by

$$E_\Lambda^\omega f \equiv \int f(x_\Lambda|x_\Lambda^c = \omega_\Lambda^c) dE_\Lambda^\omega$$

on measurable functions $f : \Omega \to \mathbb{C}$. To ease the notation, we will write $E_\Lambda = E_\Lambda^\omega$ when the role of $\omega$ is insignificant, and we call the family $\{E_\Lambda\}_{\Lambda \subset \subset \mathbb{Z}^d}$ a local specification since it specifies the probability density in every local system $\mathbb{R}_\Lambda$. In the following lemma we recall some useful properties of the local specification.

**Lemma 2.2** ([31]). Let $\{E_\Lambda\}_{\Lambda \subset \subset \mathbb{Z}^d}$ be the local specification defined as above. Then the following properties are satisfied for all boundary conditions:

(i) Normalisation: $E_\Lambda(1) = 1$.

(ii) Additivity: Let $\{f_n \geq 0\}$ be a sequence in $\mathfrak{F}(\Omega)$, then

$$\sum_n E_\Lambda f_n = E_\Lambda \left( \sum_n f_n \right).$$

(iii) Locality: $E_\Lambda f$ is $\Sigma_\Lambda^c$-measurable for $f \in \mathfrak{F}(\Omega)$, and if $f$ is $\Sigma_\Lambda^c$-measurable then $E_\Lambda f = f$.

(iv) Compatibility: If $\Lambda_1 \subset \Lambda_2$ then

$$E_{\Lambda_2} E_{\Lambda_1} = E_{\Lambda_2}.$$

Using the local specification, we can now define a global Gibbs measure $\mu$ on the whole space $\Omega$. Given any boundary condition $\omega \in \Omega$, the probability $\mu$ on $\Omega$ conditional to $\omega$ should match the probability kernel $E_\Lambda^\omega$ on $\mathbb{R}_\Lambda$. Hence, we require

$$\mu(f | \{x_\Lambda^c = \omega_\Lambda^c\}) = E_\Lambda^\omega f \quad (2.2)$$
for all \( \Lambda \subset \subset \mathbb{Z}^d \), \( \omega \in \mathbb{R}^\Lambda \) and \( f \in \mathcal{F}(\Omega) \). The statement (2.2) is equivalent to

\[
\mu \mathcal{E}_\Lambda f = \mu f, \tag{2.3}
\]

known as the DLR equation introduced by Dobrushin, Lanford, and Ruelle [32].

Consequently, any solution to (2.3) shall be called a (global) Gibbs measure associated to the local specification \( \{ \mathcal{E}_\Lambda \}_{\Lambda \subset \subset \mathbb{Z}^d} \). Later we formulate conditions that guarantee the existence and uniqueness of the global Gibbs measure \( \mu \). For a general discussion of the solutions to the DLR equation see e.g. [31], [32], [33], [34], [35].

2.3. **Differential operators.** For simplicity, we shall denote by \( \nabla_j \equiv \partial/\partial x_j \) and \( \Delta_j \equiv (\partial/\partial x_j)^2 \) the derivatives on \( \mathbb{R}^U \). For any differentiable function \( f : \Omega \to \mathbb{C} \) and \( \Lambda \subset \subset \mathbb{Z}^d \), we define the gradient \( \nabla_\Lambda f \equiv (\nabla_j f)_{j \in \Lambda} \) and \( \nabla f \equiv (\nabla_j f)_{j \in \mathbb{Z}^d} \), with the Laplacian \( \Delta_\Lambda \equiv \sum_{j \in \Lambda} \Delta_j \) and \( \Delta \equiv \sum_{j \in \mathbb{Z}^d} \Delta_j \). The dot product is naturally defined by

\[
\nabla f \cdot \nabla g \equiv \sum_{j \in \mathbb{Z}^d} (\nabla_j f)(\nabla_j g), \quad \nabla_\Lambda f \cdot \nabla_\Lambda g \equiv \sum_{j \in \Lambda} (\nabla_j f)(\nabla_j g),
\]

inducing the \( L^2 \) length \( |\nabla f|^2 \equiv \nabla f \cdot \nabla f \) and \( |\nabla_\Lambda f|^2 \equiv \nabla_\Lambda f \cdot \nabla_\Lambda f \). Consider \( f \in \mathcal{F}(\Omega) \) that are twice differentiable, then the linear operator

\[
\mathcal{L}_\Lambda f = \Delta_\Lambda f - \nabla_\Lambda U_\Lambda \cdot \nabla_\Lambda f
\]

is well-defined for all \( \Lambda \subset \subset \mathbb{Z}^d \).

2.4. **Properties of Gibbs measures.** We recall some well-known properties and coercive inequalities relating to the operator \( \mathcal{L}_\Lambda = \Delta_\Lambda - \nabla U_\Lambda \cdot \nabla_\Lambda \) and the associated Gibbs measures \( \mathcal{E}_\Lambda \) and \( \mu \) that we constructed in the previous subsections. First, we show that the Gibbs measures are invariant and reversible with respect to \( \mathcal{L}_\Lambda \).

**Lemma 2.3.** For any \( \Lambda \subset \subset \mathbb{Z}^d \), \( \mu \) and \( \mathcal{E}_\Lambda \) are invariant with respect to \( \mathcal{L}_\Lambda \), i.e. for all \( \omega \in \Omega \) it holds

\[
\mu(\mathcal{L}_\Lambda f) = \mathcal{E}_\Lambda^\omega(\mathcal{L}_\Lambda f) = 0.
\]

**Proof.** Let \( j \in \Lambda \) and we have

\[
\mathcal{E}_\Lambda(\Delta_j f) = \frac{1}{Z_\Lambda} \int (\Delta_j f)e^{-U_\Lambda}dx_\Lambda = \frac{1}{Z_\Lambda} \int \nabla_j f \cdot \nabla_j \left( e^{-U_\Lambda} \right) dx_\Lambda = \mathcal{E}_\Lambda(\nabla_j U_\Lambda \cdot \nabla_j f)
\]

due to integration by parts. Then \( \mathcal{E}_\Lambda(\mathcal{L}_\Lambda f) = 0 \) and by the DLR equation one obtains the same property for \( \mu \). \( \square \)

**Lemma 2.4.** For any \( \Lambda \subset \subset \mathbb{Z}^d \), \( \mu \) and \( \mathcal{E}_\Lambda \) are reversible for \( \mathcal{L}_\Lambda \), i.e. for all \( \omega \in \Omega \) it holds

\[
\mu(f \mathcal{L}_\Lambda g) = \mu(g \mathcal{L}_\Lambda f), \quad \mathcal{E}_\Lambda^\omega(f \mathcal{L}_\Lambda g) = \mathcal{E}_\Lambda^\omega(g \mathcal{L}_\Lambda f).
\]

**Proof.** For any \( j \in \Lambda \) we have

\[
\int (f \Delta_j g - g \Delta_j f)e^{-U_\Lambda}dx_\Lambda = \int \nabla_j (f \nabla_j g - g \nabla_j f)e^{-U_\Lambda}dx_\Lambda = -\int (f \nabla_j g - g \nabla_j f) \nabla_j \left( e^{-U_\Lambda} \right) dx_\Lambda
\]
Lemma 2.7. \[ \text{If } E \text{ is enough, then by the LSI for } c \text{ gap inequality with coefficient } c \text{ same coefficient } c \]

Let

\[ \text{Proof.} \]

Remark 2.6. \[ f \text{ functions. To see that, let } \omega \text{ for all boundary conditions } E \text{ using inequality } |\nabla| \]

which gives \( \mathbb{E}_A(f \mathcal{L}_A g) = \mathbb{E}_A(g \mathcal{L}_A f) \). The DLR equation leads to the same reversibility of \( \mu \). \[ \square \]

Definition 2.5. We say \( \mathbb{E}_A \) satisfies the spectral gap inequality (SGI) with coefficient \( c_{SG} \in (0, \infty) \) if

\[ \mathbb{E}_A^\ast |f - \mathbb{E}_A^\ast f|^2 \leq \mathbb{E}_A^\ast (f^\ast (-\mathcal{L}_A) f) = c_{SG} \mathbb{E}_A^\ast |\nabla_A f|^2 \]

and \( \mathbb{E}_A \) satisfies the logarithmic Sobolev inequality (LSI) with coefficient \( c_{LS} \in (0, \infty) \)

\[ \text{Ent}_A(|f|^2) \equiv \mathbb{E}_A^\ast \left( |f|^2 \log \frac{|f|^2}{\mathbb{E}_A^\ast |f|^2} \right) \leq c_{LS} \mathbb{E}_A^\ast |\nabla_A f|^2 \]

for all boundary conditions \( \omega \in \Omega \).

Remark 2.8. The above property and proof is due to Rothaus [36]. Another idea (due to S.G.Bokov and F.Götze), is to notice that right hand side of LSI is invariant with
respect to a shift by a constant and that one has

\[
\frac{1}{2} \mathbb{E}_\Lambda |f - \mathbb{E}_\Lambda f|^2 \leq \sup_{a \in \mathbb{R}} \mathbb{E}_\Lambda \left( |f|^2 \log \frac{|f|^2}{\mathbb{E}_\Lambda |f|^2} \right).
\]

For an interesting discussion when \( c_{SG} = \frac{1}{2} c_{LS} \) is or is not true see \[37\].

Finally, we state an important, well-known, property of log-Sobolev inequalities.

**Lemma 2.9.** If \( \mu_1, \mu_2 \) are two probability measures on two non-interacting systems \( \Omega_1, \Omega_2 \) satisfying LSI with coefficient \( c_{LS} \) (or SGI with coefficient \( c_{SG} \)), then their tensorisation \( \mu_1 \otimes \mu_2 \) on the product space \( \Omega_1 \times \Omega_2 \) also satisfies a LSI with coefficient \( c_{LS} \) (or an SGI with coefficient \( c_{SG} \)).

The proof can be found in Theorem 2.5 and Theorem 4.4 in \[25\].

3. Logarithmic Sobolev Inequalities for Global Gibbs Measure

Hereafter, we make the assumption that the global Gibbs measure, \( \mu \) exists, which we justify with Remark 3.6 below. Following the construction in Section 2, we present sufficient conditions on the potential \( U \) such that the associated measure \( \mu \) satisfies a logarithmic Sobolev inequality. We now restate Theorem 1.1 in its full form.

**Theorem 3.1.** Suppose \( U_\Lambda \) satisfies the condition (2.1) introduced in Section 2 and \( \|
abla_i \nabla_j U_\Lambda \|_\infty \leq A \) for all \( \Lambda \subset \subset \mathbb{Z}^d \), \( i, j \in \mathbb{Z}^d \) with \( i \neq j \) where \( A > 0 \) is independent of \( i, j, \Lambda \). If the finite-volume log-Sobolev inequality

\[
\mathbb{E}_\Lambda \left( |f|^2 \log \frac{|f|^2}{\mathbb{E}_\Lambda |f|^2} \right) \leq \hat{c} \mathbb{E}_\Lambda |\nabla_\Lambda f|^2
\]

holds for all \( \Lambda \subset \subset \mathbb{Z}^d \) and some \( \hat{c} > 0 \) independent of \( \Lambda \), then the global Gibbs measure \( \mu \) associated to \( \{ \mathbb{E}_\Lambda \} \) is unique and satisfies the log-Sobolev inequality

\[
\mu \left( |f|^2 \log \frac{|f|^2}{\mu |f|^2} \right) \leq c \mu |\nabla f|^2
\]

for all \( f \in H^1(\mu) \) and some constant \( c \in (0, \infty) \) independent of \( f \).

**Remark 3.2.** A sufficient condition for the volume-uniform LSI (3.1) is that

\[
\inf_{x \in \Omega} \Delta_i U_\Lambda(x) \geq B
\]

for all \( \Lambda \subset \subset \mathbb{Z}^d \), \( i \in \Lambda \) and some \( B > 0 \) independent of \( i \) and \( \Lambda \). This is a consequence of the Bakry-Emery criterion \[38\], which states that if

\[
\Gamma_2(f, f) \geq b \Gamma_1(f, f)
\]

for some \( b > 0 \) independent of \( f \), where

\[
\Gamma_1(f, g) \equiv \mathcal{L}_\Lambda(fg) - f \mathcal{L}_\Lambda g - g \mathcal{L}_\Lambda f,
\]

\[
\Gamma_2(f, g) \equiv \mathcal{L}_\Lambda(\Gamma_1(f, g)) - \Gamma_1(f, \mathcal{L}_\Lambda g) - \Gamma_1(g, \mathcal{L}_\Lambda f),
\]

then (3.1) holds with \( c = 2/b \). By direct calculation, (3.4) is satisfied due to (3.3). In other words, one has a volume-uniform LSI if the potential is strongly convex at all sites with the same strong convexity constant.

For application of the above result, we return to the examples of the bilinear potential and its perturbation by bounded local interactions.

**Example 3.3.** Consider the bilinear potential

\[
U_{\Lambda}(x) = \sum_{\{i,j\} \cap \Lambda \neq \emptyset} C_{ij} x_i x_j
\]

where \( C^- \leq C_{ii} \leq C^+ \) and \(|C_{ij}| \leq C^+\) for some \( 0 < C^- \leq C^+ \), and \( C_{ij} = 0 \) if \(|i-j| > R\). Then \( U_{\Lambda} \) satisfies all conditions in Theorem 3.1 by Remark 3.2.

**Example 3.4.** Consider the same bilinear potential but with bounded perturbations

\[
U_{\Lambda}(x) = \sum_{\{i,j\} \cap \Lambda \neq \emptyset} C_{ij} x_i x_j + \varepsilon \sum_{X \subset \subset \mathbb{R}^d} W_X(x_{X})
\]

where \( \varepsilon \in \mathbb{R} \), \( W_X \in C^2(\mathbb{R}^X) \) has uniformly in \( X \) bounded derivatives of order \( n \),\( \varepsilon \) small if \(|\varepsilon|\) all conditions in Theorem 3.1 are satisfied.

As a final example we introduce the restriction to the interactions with no boundary conditions. Later on we will use this extensively to study existence results for the LSE.

**Example 3.5.** Theorem 3.1 holds true when the boundary conditions are removed. That is, suppose \( U \) satisfies the conditions of Theorem 3.1 then we denote by

\[
U^o_{\Lambda}(x) \equiv \sum_{X \subseteq \Lambda} J_X(x)
\]

the restriction to \( J_X \) strictly localised inside \( \Lambda \). Then, we denote by \( E^o_{\Lambda} \) the local Gibbs measure corresponding to \( U^o_{\Lambda} \), i.e. \( dE^o_{\Lambda} = (e^{-U^o_{\Lambda}}dx_{\Lambda})(fe^{-U^o_{\Lambda}}dx_{\Lambda})^{-1} \), and by \( \mu^o \) the global Gibbs measure associated to \( \{E^o_{\Lambda}\} \) through the DLR equation. Then Theorem 3.1 holds with \( (U_{\Lambda}, E_{\Lambda}, \mu) \) replaced by \( (U^o_{\Lambda}, E^o_{\Lambda}, \mu^o) \).

**Remark 3.6 (Existence).** We remark that by the compactness argument in [31], the existence of \( \mu \) in our setting can always be obtained by restricting the configuration space \( \Omega \), to the space of slowly increasing sequences (tempered sequences) \( S_{\Omega} \equiv \{ x \in \Omega : \exists N > 0 \text{ such that } \sup_j |x_j|/|j|^N < \infty \} \), provided the multi-spin interaction satisfies suitable growth conditions. The restriction to the subspace \( S_{\Omega} \) can be justified by the fact that \( S_{\Omega} \) is dense in \( \Omega \), i.e. \( \mu(S_{\Omega}) = 1 \) (see [31 Proposition A.1]), and that one can obtain compactness argument on such space.

### 3.1. Strategy to prove Theorem 3.1

The proof of Theorem 3.1 follows the ideas of [25], [26], [27], [28], [29], [30] with some modifications for the non-compactness of the space \( \Omega \equiv \mathbb{R}^{2d} \) and the unboundedness of the potential \( U \). The strategy is to
construct a probability measure $\Pi$ on $\Omega$ that satisfies a LSI and where $\Pi^n \to \mu$ as $n \to \infty$. Then one can obtain the $\mu$-LSI by using the $\Pi$-LSI repeatedly on a telescoping series of entropy terms associated to the sequence $\{\Pi^n\}_{n \in \mathbb{N}}$.

Here we present a construction of such $\Pi$, the idea is to define $\Pi$ as an infinite tensorisation of local Gibbs measures on equally sized cubes that partition the whole lattice. To this end, for some parameter $L \in \mathbb{N}$ and the interaction range $R \in \mathbb{N}$, we define the base cube $X_0 \equiv [0, 2(L + R)]^d \cap \mathbb{Z}^d$ and denote by $X_k \equiv k + X_0$ the translation of $X_0$ by the vector $k \in \mathbb{Z}^d$. For any $s \in \mathbb{N}$ let $v_s = (v_s^{(1)}, \ldots, v_s^{(d)}) \in \{0, 1\}^d$ be the binary representation of $s$, i.e. $s = \sum_{n=1}^d v_s^{(n)}2^{n-1}$. We define the following collection of cubes

$$\Gamma_s = \bigcup_{k \in T_s} X_k$$

for $s = 0, 1, \ldots, 2^d - 1$ and the translation set $T_s \equiv \{k \in \mathbb{Z}^d : k \in 2(L + 2R)\mathbb{Z}^d + (L + 2R)v_s\}$. By such construction, each $\Gamma_s$ contains disjoint cubes of shape $X_0$, equally distributed in $\mathbb{Z}^d$, and separated by a distance greater than $2R$. In addition, as $s$ goes over $\{0, 1, \ldots, 2^d - 1\}$ the vector $v_s$ will cover all directions in $\{0, 1\}^d$, hence the union $\bigcup_{s=0,1,\ldots,2^d-1} \Gamma_s = \mathbb{Z}^d$ covers the whole lattice. Let $E_{X_k}$ be the local Gibbs measure on the cube $X_k$ and we set $E_s \equiv \bigotimes_{X_k \subset \Gamma_s} E_{X_k}$. Then we define the measure $\Pi$ by

$$\Pi^\omega \equiv E_{\Gamma_{2^d-1}}E_{\Gamma_{2^d-2}} \cdots E_1 E_0$$

with boundary conditions $x(\Gamma_{2^d-1})^c = \omega(\Gamma_{2^d-1})^c$. The measure $\Pi$ satisfies the following four properties.

**Proposition 3.7.** Let $\Pi$ be as constructed above, then the following hold.

(i) The DLR equation holds:

$$\mu \Pi f = \mu f.$$

(ii) There exists $\bar{c} > 0$ such that

$$\mu \left( |\Pi|^2 \log |f|^2 \right) - \Pi |f|^2 \log \Pi |f|^2 \leq \bar{c} \mu |\nabla f|^2.$$

(iii) There exists $\gamma \in (0, 1)$ such that

$$\mu |\nabla (\Pi |f|^2)^{\frac{1}{2}}|^2 \leq \gamma \mu |\nabla f|^2.$$

(iv) The sequence $\Pi^n$ converges to $\mu$ almost surely, i.e.

$$\lim_{n \to \infty} \Pi^n(f) = \mu(f) \quad \mu \text{- a.s.,}$$

and the limit defines a unique global Gibbs measure $\mu$ associated to $\{E_\Lambda\}$.

Before proving Proposition 3.7 we first demonstrate how to derive the $\mu$-LSI (3.2) by using $\Pi$. 

Proof of Theorem 3.1. By (i), we have
\[ \mu \left( |f|^2 \log \frac{|f|^2}{\mu f^2} \right) = \sum_{n=0}^{N-1} \mu \left( \Pi \left( |f_n|^2 \log |f_n|^2 \right) - \Pi |f_n|^2 \log \Pi |f_n|^2 \right) + \mu \left( |f_N|^2 \log \frac{|f_N|^2}{\mu |f|^2} \right) \]
then by (ii), it follows
\[ \mu \left( |f|^2 \log \frac{|f|^2}{\mu f^2} \right) \leq \bar{c} \sum_{n=0}^{N-1} \mu |\nabla f_n|^2 + \mu \left( |f_N|^2 \log \frac{|f_N|^2}{\mu |f|^2} \right) . \]
Since \( \gamma \in (0, 1) \), applying (iii) gives
\[ \mu \left( |f|^2 \log \frac{|f|^2}{\mu f^2} \right) \leq \frac{\bar{c}}{1-\gamma} \mu |\nabla f|^2 + \mu \left( |f_N|^2 \log \frac{|f_N|^2}{\mu |f|^2} \right) . \]
Then by (iv) and the monotone convergence theorem, the last term follows
\[ \mu \left( |f_N|^2 \log \frac{|f_N|^2}{\mu |f|^2} \right) \to 0 \]
as \( N \to \infty \), which gives the desired LSI for \( \mu \) with coefficient \( c \equiv \bar{c}/(1-\gamma) \).

Remark 3.8. The choice of \( \Pi \) is not unique. In the proof Proposition 3.7 the base cube \( X_0 \) can in fact take any shape as long as it is optimised under a set of conditions. Our choice of homogeneous cube partition is just to provide an easy presentation of the formulae.

3.2. Proof of Proposition 3.7. Here we present a proof of conditions (i) to (iv). Condition (i) easily follows from the DLR equation (2.3) for each local Gibbs measure \( \mathbb{E}_{X_k} \) in the cube setup. To verify conditions (ii) to (iv), we use the following lemma.

Lemma 3.9. Suppose all conditions in Theorem 3.1 are satisfied, then for any cube \( X_k \) and \( j \notin X_k \) the ‘sweeping out’ inequalities
\[ \mu \left| \nabla_j (\mathbb{E}_{X_k} f) \right|^2 \leq \sum_{i \in \mathbb{Z}^d} \alpha_{ji} \mu \left| \nabla_i f \right|^2 \]  
(3.6)
and
\[ \mu \left| \nabla_j (\mathbb{E}_{X_k} |f|^2)^{\frac{1}{2}} \right|^2 \leq \sum_{i \in \mathbb{Z}^d} \alpha_{ji} \mu \left| \nabla_i f \right|^2 \]  
(3.7)
hold with \( 0 \leq \alpha_{ji} \leq D |X_0| e^{-M|j-i|} \) for some positive constants \( D, M \) independent of \( j \) and \( X_k \).

Assuming Lemma 3.9 is true, we now prove condition (ii). Denote by \( f_{-1} \equiv f \) and \( f_s \equiv (\mathbb{E}_{X_0} \cdots \mathbb{E}_{X_{s-1}} |f|^2)^{1/2} \), one has the telescopic representation
\[ \mu \left( \Pi (|f|^2 \log |f|^2) - \Pi |f|^2 \log \Pi |f|^2 \right) = \sum_{s=0}^{2^d-1} \mu \mathbb{E}_s \left( f_{s-1}^2 \log \frac{f_{s-1}^2}{\mathbb{E}_s f_{s-1}^2} \right) \]
y by the DLR equation. For any \( s \), each local measure \( \mathbb{E}_{X_k} \) with \( X_k \subset \Gamma_s \) satisfies a LSI with coefficient \( \bar{c} \) by (3.1), their tensorisation \( \mathbb{E}_s \) therefore satisfies the same LSI by
Lemma 2.4, leading to
\[ \mu \left( \Pi \left( |f|^2 \log |f|^2 \right) - \Pi |f|^2 \log |f|^2 \right) \leq \tilde{c} \sum_{s=1}^{2^{d-1}} \mu |\nabla \Gamma_s f_{s-1}|^2. \]  
(3.8)

We now present a simple study of the quantity \( \nabla \Gamma_s f_{s-1} \). For any \( j \in \Gamma_s \), if \( j \in \Gamma_{s-1} \) then \( \nabla_j f_{s-1} = 0 \). If \( j \notin \Gamma_{s-1} \), there exists only one cube \( X_{k(j)} \subset \Gamma_{s-1} \) such that \( \text{dist}(j, X_{k(j)}) \leq R \). In the second case, we have
\[ \nabla_j f_{s-1} = \nabla_j \left( \mathbb{E}_{\Gamma_{s-1} \setminus X_{k(j)} \mathbb{E}_{X_{k(j)}} f_{s-2}^2 \right)^{\frac{1}{2}} \leq \mathbb{E}_{\Gamma_{s-1} \setminus X_{k(j)}} \left| \nabla_j (\mathbb{E}_{X_{k(j)}} f_{s-2}^2)^{\frac{1}{2}} \right|^2 \]
by Cauchy-Schwartz inequality. Integrating this with \( \mu \) and using the sweeping out inequality (3.7) for \( \Lambda = X(j) \), it follows
\[ \mu \left| \nabla_j f_{s-1} \right|^2 \leq \sum_{i \in \mathbb{Z}^d \setminus \Gamma_{s-2}} \alpha_{ji} \mu \left| \nabla_i f_{s-2} \right|^2, \]
where we have \( i \subset \mathbb{Z}^d \setminus \Gamma_{s-2} \) because \( \Gamma_{s-2} = (\mathbb{E}_{\Gamma_{s-2}} |f_{s-2}|^2)^{1/2} \) has no dependence in \( \Gamma_{s-2} \). Now that \( \nabla_j f_{s-1} \) is bounded by \( \nabla_i f_{s-2} \), the same estimate can be used recursively until we reach the gradients on \( f \), i.e.
\[ \mu \left| \nabla_j f_{s-1} \right|^2 \leq \sum_{i \in \mathbb{Z}^d} \eta_{ji} \eta_{ji} \mu \left| \nabla_i f \right|^2, \]  
(3.9)

where
\[ \eta_{ji} \equiv \sum_{i_1 \in \mathbb{Z}^d \setminus \Gamma_{s-2}} \sum_{i_2 \in \mathbb{Z}^d \setminus \Gamma_{s-3}} \cdots \sum_{i_{s-1} \in \mathbb{Z}^d \setminus \Gamma_0} \left( \alpha_{ji_1} \alpha_{ji_2} \cdots \alpha_{ji_{s-1}} \right). \]  
(3.10)

Summing (3.9) over all \( j \in \Gamma_s \), the constant before \( \mu \left| \nabla_i f \right|^2 \) is therefore
\[ \sum_{j \in \Gamma_s} \eta_{ji} \leq \left( \sup_{i \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \alpha_{ki} \right) s \leq (|X_0|)^s \left( \sum_{k \in \mathbb{Z}^d} e^{-M|k|} \right)^s \leq (|X_0| C_d)^s \]  
(3.11)
for some finite constant \( C_d = \sum_k e^{-M|k|} \) that only depends on \( M \) and the lattice dimension \( d \). Taking this back into (3.8), one gets
\[ \mu \left( \Pi \left( |f|^2 \log |f|^2 \right) - \Pi |f|^2 \log |f|^2 \right) \leq \tilde{c} \mu \left| \nabla f \right|^2 \]  
(3.12)
with \( \tilde{c} \equiv \tilde{c} (1 + (|X_0| C_d) + \ldots + (|X_0| C_d)^{2^d}) \), which concludes the proof of (ii).

To verify (iii), we use the same estimate as in (3.9) with \( s = 2^d \), which gives
\[ \mu \left| \nabla (\Pi |f|^2)^{\frac{1}{2}} \right|^2 \leq \mu \left| \nabla_j f_{2^{d-1}} \right|^2 \leq \sum_{i \in \mathbb{Z}^d} \eta_{ji} \eta_{ji} \mu \left| \nabla_i f \right|^2. \]  
(3.13)

By the cube construction (3.5), the intersection \( X_k \cap X_{k'} \) between any two overlapping cubes \( X_k \) and \( X_{k'} \) is a (hyper)rectangle whose shortest edge has a length of \( L \). Therefore, along each path \( \{j, i_1, i_2, \ldots, i_{2^d-1}, i\} \) in the sum (3.10) with \( s = 2^d \), there exists at least one pair of adjacent sites \( (i_n, i_{n+1}) \) such that \( |i_n - i_{n+1}| \geq L/2 \). Incorporate
this property into the estimate (3.11), one gets
\[
\mu |\nabla (\Pi f)^2|^2 \leq (D|X_0|C_d)^{2d}e^{-L/2}\mu |\nabla f|^2.
\]
(3.14)
Since \(|X_0|\) grows polynomially with \(L\), one can make \(L\) sufficiently large so that \(\gamma \equiv (D|X_0|C_d)^{2d}e^{-L/2} < 1\) and this concludes (iii).

To verify (iv), we redefine \(f_{-1} \equiv f\) and \(f_s \equiv E_sE_{s-1} \cdots E_0(f)\). By the argument for (3.8), each \(E_s\) satisfies a LSI (and thus a SGI by Lemma 2.7) with coefficient \(\bar{c}\). Hence, one can bound the II-variance by
\[
\mu |f - \Pi f|^2 \leq 2^{2d} \sum_{s=0}^{2d-1} \mu E_s|f_{s-1} - E_s f_{s-1}|^2 \leq 2^{2d} \bar{c} \sum_{s=0}^{2d-1} \mu |\nabla f, f_{s-1}|^2.
\]
Comparing this with (3.8), the situation is essentially the same and therefore we can follow the steps from (3.8) until (3.12) to conclude that
\[
\mu \Pi |f - \Pi f|^2 \leq 2^{2d} \bar{c} \mu |\nabla f|^2
\]
for the same \(\bar{c}\) in (3.12). Similarly, the argument of (3.13) and (3.14) leads to the gradient bound
\[
\mu |\nabla (\Pi f)|^2 \leq \beta \mu |\nabla f|^2
\]
for some \(\beta \in (0, 1)\). Using the above two estimates with \(f\) replaced by \(\Pi^n f\), we obtain that
\[
\mu |\Pi^n f - \Pi^{n+1} f|^2 \leq 2^{2d} \bar{c} \mu |\nabla (\Pi^n f)|^2 \leq 2^{2d} \beta^n \bar{c} \mu |\nabla f|^2.
\]
Since \(\beta \in (0, 1)\) and \(\mu |\nabla f|^2 < \infty\) for \(f \in H^1(\mu)\), we can use Borel-Cantelli lemma to show that the sequence \(\{\Pi^n f\}_n\) converges \(\mu\)-almost surely to some function \(F_\lim : \Omega \rightarrow \mathbb{C}\) (for the technical details, see [39, Lemma 5.4.6]). Similarly, one can prove that \(|\nabla (\Pi^n f)|\) converges to zero \(\mu\)-a.s., meaning that the limit \(F_\lim\) must be a constant function. Therefore, we have
\[
\lim_{n \rightarrow \infty} \Pi^n f = F_\lim = \mu(F_\lim) = \lim_{n \rightarrow \infty} \mu(\Pi^n f) = \mu(f) \quad \mu\text{-a.s.}
\]
where the limit can be taken out of \(\mu\) by dominated convergence theorem (since \(f\) is bounded), and the last equality follows from the DLR equation (i).

3.3 Proof of Lemma 3.9. We will first prove (3.7) and then (3.6) follows similarly. Let \(\Lambda\) be any cube \(X_k\) in our setup (thus \(\text{diam}(\Lambda) > 2R\)). For any \(j \notin \Lambda\) let \(i \in \Lambda\) be such that \(|j - i| > R\) and we denote by \(F \equiv (\mathbb{E}_{\Lambda}|f|^2)^{1/2}\). By the compatibility property \(\mathbb{E}_\Lambda = \mathbb{E}_\Lambda \mathbb{E}_{\Lambda \setminus i}\) in Lemma 2.2, we have
\[
|\nabla_j (\mathbb{E}_\Lambda |f|^2)^{1/2}|^2 = |\nabla_j (\mathbb{E}_\Lambda F^2)^{1/2}|^2 \leq \frac{|\mathbb{E}_\Lambda (\nabla_j F^2)|^2}{4 \mathbb{E}_\Lambda F^2} + \frac{|\mathbb{E}_\Lambda (F^2 - \nabla_j U_\Lambda)|^2}{4 \mathbb{E}_\Lambda F^2}
\]
(3.15)
with the covariance \(\mathbb{E}_\Lambda(f; g) \equiv \mathbb{E}_\Lambda(fg) - (\mathbb{E}_\Lambda f)(\mathbb{E}_\Lambda g)\). Let \(I_1\) and \(I_2\) be the first and second term on the right hand side of (3.15), then it simply follows \(I_1 \leq \mathbb{E}_\Lambda |\nabla_j F|^2\) by
Cauchy-Schwartz inequality. For $I_2$, since $F$ has no dependence in $\Lambda \setminus i$ the covariance follows $\mathbb{E}_A(F^2; -\nabla_j U_\Lambda) = \mathbb{E}_A(F^2; U)$ with $U \equiv \mathbb{E}_{A\setminus i}(-\nabla_j U_\Lambda)$. To continue the calculation, we define $\tilde{\mathbb{E}}_A \equiv \mathbb{E}_A(d\tilde{x}_\Lambda)$ as an isomorphic copy of $\mathbb{E}_A(d\tilde{x}_\Lambda)$ on a variable $\tilde{x}_\Lambda$ independent of $x_\Lambda$, and similarly $\tilde{F} \equiv F(\tilde{x}_\Lambda)$, $\tilde{U} \equiv U(\tilde{x}_\Lambda)$, while all configurations outside $\Lambda$ are fixed by the same boundary conditions $\omega_{\Lambda'}$. Now we can rewrite the covariance as

$$|\mathbb{E}_A(F^2; U)|^2 = \frac{1}{4} \mathbb{E}_A(\tilde{F}^2(\tilde{U} - \tilde{U}))^2 \leq \frac{1}{4} \mathbb{E}_A(F^2)\mathbb{E}_A((F - \tilde{F})^2(\tilde{U} - \tilde{U}))^2 \leq \left( \mathbb{E}_A F^2 \right) \tilde{\mathbb{E}}_A \left( (F - \tilde{F})^2(\tilde{U} - \tilde{U})^2 \right).$$

(3.16)

Combining $I_1$ and $I_2$, we obtain the bound

$$|\nabla_j(\mathbb{E}_A|f|^2)^{\frac{1}{2}}|^2 \leq \mathbb{E}_A|\nabla_j F|^2 + \frac{1}{4}\mathbb{E}_A\tilde{\mathbb{E}}_A \left( (F - \tilde{F})^2(\tilde{U} - \tilde{U})^2 \right).$$

(3.17)

Note that $U$ only has one free variable on $x_i$ since all configurations are integrated by $\mathbb{E}_{A\setminus i}$ inside $\Lambda \setminus i$ and fixed by $\omega$ outside $\Lambda$. Therefore, we can write $U(x_\Lambda) = U(x_i)$ and estimate the difference

$$|U(x_i) - U(\tilde{x}_i)| = \left| \int_{\tilde{x}_i}^{x_i} \nabla_i U(\tilde{x}_i) \, d\tilde{x}_i \right| \leq |x_i - \tilde{x}_i| \|\nabla_i U\|_\infty = |x_i - \tilde{x}_i| \left\| \mathbb{E}_{A\setminus i}(\nabla_i U_{A\setminus i}; \nabla_j U_{A\setminus i}) \right\|_\infty$$

where the last step follows from the fact $\nabla_j U_\Lambda = \nabla_j U_{A\setminus i}$ since $|i - j| > R$.

To bound the above covariance, we use the following result adapted from [10] Theorem 2.4 and Remark 2.6.

**Theorem 3.10.** If $\|\nabla_i \nabla_j U_\Lambda\|_\infty \leq Ae^{-|j-i|}$ for some $A > 0$ independent of $i, j, \Lambda$, then the volume-uniform LSI (3.1) is equivalent to the following mixing condition

$$|\mathbb{E}_A(f; g)|^2 \leq C|\Lambda_f| |\Lambda_f| e^{-M\mathrm{dist}(\Lambda_f, \Lambda_g)} \left( \mathbb{E}_A|\nabla_f|^2 \right) \left( \mathbb{E}_A|\nabla g|^2 \right)$$

(3.18)

for all $f, g \in H^1(\mu)$ localised in $\Lambda_f, \Lambda_g$ respectively and $C, M > 0$ independent of $\Lambda, f, g$.

Note that the condition on $\nabla_i \nabla_j U_\Lambda$ is trivially satisfied by the condition in Theorem 3.1 and the fact that the interaction is of finite range. Therefore, one can apply (3.18) to obtain the following covariance bound

$$\left\| \mathbb{E}_{A\setminus i}(\nabla_i U_{A\setminus i}; \nabla_j U_{A\setminus i}) \right\|_\infty^2 \leq CA^4 e^{2MR(2R + 1)^d} e^{-M|j-i|}.$$

Collecting every piece together in (3.11), we have

$$I_2 \leq \frac{1}{4} CA^4 e^{2MR(2R + 1)^d} e^{-M|j-i|} \mathbb{E}_A \tilde{\mathbb{E}}_A \left( (F - \tilde{F})^2(x_i - \tilde{x}_i)^2 \right).$$
For simplicity let \( \hat{F} \equiv F - \tilde{F} \), \( \hat{x}_i \equiv x_i - \tilde{x}_i \) and \( \hat{E}_\Lambda \equiv E_\Lambda \tilde{E}_\Lambda \), one has the following relative entropy inequality

\[
\hat{E}_\Lambda \left( \frac{\hat{F}^2}{\hat{x}_i^2} \right) \leq \frac{1}{\varepsilon} \hat{E}_\Lambda \left( \frac{\hat{F}^2}{E_\Lambda F^2} \right) + \frac{1}{\varepsilon} \left( \log e^{\hat{E}_\Lambda \hat{x}_i^2} \right) \hat{E}_\Lambda \hat{F}^2
\]

(3.19) for any \( \varepsilon > 0 \). By the uniform LSI (3.1) and Lemma 2.9 the double measure \( \hat{E}_\Lambda \) satisfies a LSI with coefficient \( \hat{c} \) and thus the entropy term is bounded by \( \hat{c} \hat{E}_\Lambda (|\nabla \hat{F}|^2 + |\nabla_\Lambda \hat{F}|^2) = 2\hat{c} \hat{E}_\Lambda |\nabla_i F|^2 \), where only \( \nabla_i \) remains since \( F \) has no dependence in \( \Lambda \setminus i \). By Lemma 2.7 the measure \( \hat{E}_\Lambda \) also satisfies an SGI with coefficient \( \hat{c}/2 \), hence

\[
\hat{E}_\Lambda \hat{F}^2 = 2\hat{E}_\Lambda (F - E_\Lambda F)^2 \leq \hat{c} \hat{E}_\Lambda |\nabla_\Lambda F|^2 = \hat{c} \hat{E}_\Lambda |\nabla_i F|^2.
\]

For the log-term on \( \hat{E}_\Lambda \hat{x}_i^2 \), we use the following lemma adapted from [41, Theorem 4.5].

**Lemma 3.11.** Let \( \nu \) be a probability measure on \( \mathbb{R}^n \) that satisfies a LSI with coefficient \( \hat{c} > 0 \). Let \( g : \mathbb{R}^n \to \mathbb{R} \) be any differentiable function such that \( |\nabla_{\mathbb{R}^n} g|^2 \leq a g \) for some \( a > 0 \), then for all \( \varepsilon \in (0, \frac{1}{a\hat{c}}) \) the following bound holds

\[
\log \nu(e^g) \leq 2\varepsilon \nu(g).
\]

Note that \( |\nabla_\Lambda \hat{x}_i^2|^2 + |\nabla_\Lambda \hat{x}_i^2|^2 = 8\hat{x}_i^2 \) and therefore by setting \( \varepsilon < \frac{1}{8\hat{c}} \) one can use Lemma 3.11 to show that

\[
\log e^{\hat{E}_\Lambda \hat{x}_i^2} \leq 2\varepsilon \hat{E}_\Lambda \hat{x}_i^2 = 4\varepsilon \hat{E}_\Lambda (x_i - E_\Lambda (x_i))^2 \leq 2\varepsilon \hat{c} \hat{E}_\Lambda |\nabla_\Lambda x_i|^2 = 2\varepsilon \hat{c}.
\]

Taking this back into (3.19), (3.16), (3.15) and integrating with \( \mu \), one gets

\[
\mu |\nabla_j (E_\Lambda |f|^2)^{1/2}|^2 \leq \mu |\nabla_j F|^2 + C_R e^{-M|j-i|} \mu |\nabla_i F|^2
\]

(3.20)

with \( C_R \equiv \max \{ 1, \hat{c} C A^4 e^{2M R(2 R + 1)|d(\varepsilon^{-1} + \hat{c})} \} \) independent of \( i, j, \Lambda \).

Recall that \( F = (E_{\Lambda \setminus i} |f|^2)^{1/2} \), hence we can define \( F_2 \equiv (E_{\Lambda \setminus \{i_1, i_2\}} |f|^2)^{1/2} \) with \( |j - i_2| > R \) and use (3.20) again to bound \( \nabla_j F \) by \( \nabla_j F_2 \) and \( \nabla_{i_2} F_2 \). With this idea, we may construct a sequence \( \{i_n\} \) in \( \Lambda \) such that \( |j - i_n| \) is decreasing and the sequence stops at \( i_N \) when \( |j - i_{N+1}| \leq R \). Then (3.20) can be applied inductively to bound each \( \nabla_j F_n \) where \( F_n \equiv (E_{\Lambda \setminus \{i_1, i_2, \ldots, i_n\}} |f|^2)^{1/2} \). To deal with the second term \( \nabla_{i_n} F_n \) at each iteration, we shall use the following bound

\[
\mu |\nabla_i (E_{\Lambda \setminus i} |f|^2)^{1/2}|^2 \leq \mu |\nabla_i f|^2 + \tilde{C}_R \mu |\nabla_{\Lambda \setminus i} f|^2,
\]

(3.21)

which holds for all \( \Lambda' \subset \Lambda, i \notin \Lambda' \) and some \( \tilde{C}_R \geq 1 \) depending only on \( \hat{c}, A, R, d \). (3.21) can be deduced by the similar strategy as for (3.20) while keeping \( f \) instead of \( F \) in the beginning. Despite losing the exponential factor in this case, we can use (3.21) to terminate the iteration by directly getting the gradient bound on \( f \).

With (3.20) and (3.21), we now carry on the iteration as follows. At each step \( n \), (3.20) generates two gradient terms \( \nabla_j F_n \) and \( \nabla_{i_n} F_n \). For the \( \nabla_j F_n \) term, we continue the iteration by applying (3.20) again which generates \( \nabla_j F_{n+1} \) and \( \nabla_{i_{n+1}} F_{n+1} \). For the \( \nabla_{i_n} F_n \) term, we terminate the iteration by applying (3.21) which generates...
The final output of this procedure is therefore $E$ with additional conditions on the structure of systems using the idea of Section 3. Sobolev inequality). Below we describe a way one can obtain a nalyticity for infinite (which by the celebrated result of Len Gross is an equivalent condition to logarithmic finite speed of propagation of information for Markov semigroups and hypercontractivity boundary conditions) Logarithmic Sobolev inequality. The proof relied on use of freezing inequality (3.7).

Suppose we replace the map given by regular conditional expectation $\mathbb{E}_\Lambda$ by a map

$$\mathbb{E}_{\Lambda, \varepsilon}(f) = \frac{\mathbb{E}_\Lambda(e^{-\varepsilon V_\Lambda} f)}{\mathbb{E}_\Lambda(e^{-\varepsilon V_\Lambda})},$$

with complex parameter $\varepsilon$ and an interaction $V_\Lambda$, such that $\mathbb{E}_\Lambda(e^{-\varepsilon V_\Lambda}) \neq 0$. In particular this holds for sufficiently small $|\varepsilon|$ if $V$ is a bounded measurable function. But, with additional conditions on the structure of $\mathbb{E}_{\Lambda, \varepsilon}$, it can also be achieved for unbounded interactions $V$. For example if the density of $\mathbb{E}_\Lambda$ is $e^{-U_\Lambda}/\int e^{-U_\Lambda}$ and $V_\Lambda - U_\Lambda$ is a bounded function. We remark that for the map $\mathbb{E}_{\Lambda, \varepsilon}$ one can show the following
relation
\[ \mu \left| \nabla_j \mathbb{E}_{\Lambda,\varepsilon}(f) \right| \leq C_\varepsilon \mu \left| \nabla_j f \right| + \sum_{i \in \Lambda} C_\varepsilon \alpha_{ij} \mu \left| \nabla_i f \right| \]
for any sufficiently smooth complex function \( f \) with some real constant \( C_\varepsilon \rightarrow \varepsilon \rightarrow 0 \).

With this property, defining a map \( \Pi_\varepsilon \) corresponding to the family \( \{ \mathbb{E}_{\Lambda,\varepsilon} \} \), one can show for small \( |\varepsilon| \) the uniform convergence of \( \Pi_\varepsilon^n \) for any localised Lipschitz function \( f \). By our construction \( \Pi_\varepsilon^n f \) is a sequence of analytic in \( \varepsilon \) functions. Hence by the uniform convergence the limiting function
\[ \mu_\varepsilon f \equiv \lim_{n \rightarrow \infty} \Pi_\varepsilon^n f \]
is also an analytic function of the complex parameter \( \varepsilon \). This suggest that the spectral gap is a real analytic function of \( \varepsilon \) in a corresponding small neighbourhood of the original theory for which the first order sweeping out relations hold.

5. Bounds for Solutions of LSE

We now return to the logarithmic Schrödinger equation (LSE) problem in finite volume \( \Lambda \subset \subset \mathbb{Z}^d \), given by

\[ i \partial_t \varphi = -\mathcal{L}_\Lambda \varphi + \lambda \varphi \log \frac{|\varphi|^2}{\mu |\varphi|^2} \quad \text{(LSE}_\Lambda) \]
where \( \mu \) is the infinite-volume Gibbs measure, constructed in Section 2 and \( \mathcal{L}_\Lambda \equiv \Delta_{\Lambda} - \nabla_{\Lambda} U_\Lambda \cdot \nabla_{\Lambda} \). We shall consider initial data \( f \in H^1(\mu) \) localised in some \( \Lambda_f \subset \Lambda \) so that all initial particles are covered by the potential. Due to the finite-range interactions, the solution \( \varphi \) is localised in \( \{ j : \text{dist}(j, \Lambda) \leq R \} \) for some \( R > 0 \), therefore the problem (LSE\(_\Lambda\)) lives in finite dimensions and is well-defined. In this section we find various estimates uniformly in \( \Lambda \). In the next section we show that these are crucial for constructing solutions to the infinite-volume LSE as \( \Lambda \rightarrow \mathbb{Z}^d \).

Denote by \( |\cdot|_\mu \equiv \sqrt{\mu |\cdot|^2} \) the \( L^2(\mu) \) norm, we first show that the mass and energy of (LSE\(_\Lambda\)) are conserved over time.

**Theorem 5.1.** For any \( \Lambda \subset \subset \mathbb{Z}^d \), let \( \varphi(\cdot, t) \) be the solution of (LSE\(_\Lambda\)) with initial data \( f \in H^1(\mu) \). Then for all time \( t \geq 0 \) the following conservation of mass property holds

\[ \| \varphi(\cdot, t) \|^2_\mu = \| f \|^2_\mu. \]

If \( U_\Lambda \) is twice differentiable, the following conservation of energy property holds

\[ \| \nabla_\Lambda \varphi(\cdot, t) \|^2_\mu + \lambda \text{Ent}_\mu |\varphi(\cdot, t)|^2 = \| \nabla_\Lambda f \|^2_\mu + \lambda \text{Ent}_\mu |f|^2 \quad (5.1) \]

where \( \text{Ent}_\mu F \equiv F \log(F/\mu F) \) denotes the \( \mu \)-entropy.

**Proof.** The mass conservation easily follows from

\[ \partial_t \mu |\varphi|^2 = 2 \text{Re}(\mu (\varphi^* \partial_t \varphi)) = -2 \text{Im}(\mu (\varphi^* \mathcal{L}_\Lambda \varphi)) = 0 \]
using \((\text{LSE}_\Lambda)\) and the reversibility of \(\mu\) for \(\mathcal{L}_\Lambda\), due to Lemma 2.4. To obtain the energy conservation, we will similarly show that
\[
\partial_t \left( \mu |\nabla_\Lambda \varphi|^2 + \lambda \text{Ent}_\mu |\varphi|^2 \right) = 0. \tag{5.2}
\]
In what follows we consider regularised solution \(e^{-\varepsilon \mathcal{L}_\Lambda} \varphi\) of \((\text{LSE}_\Lambda)\) and pass to the limit with \(\varepsilon\) to 0 at the end. Keeping this in mind, below we provide the stream of essential steps omitting explicit reference to \(\varepsilon\). First we note that for any \(j \in \mathbb{Z}^d\) the time evolution of \(|\nabla_j \varphi|^2\) follows
\[
\partial_t |\nabla_j \varphi|^2 = 2 \text{Im} \left( - \nabla_j (\mathcal{L}_\Lambda \varphi) \cdot \nabla_j \varphi^* + \lambda \nabla_j \left( \varphi \log \frac{|\varphi|^2}{\mu|\varphi|^2} \right) \cdot \nabla_j \varphi^* \right). \tag{5.3}
\]
Rewrite the first product as
\[
\nabla_j (\mathcal{L}_\Lambda \varphi) \cdot \nabla_j \varphi^* = \left[ \nabla_j, \mathcal{L}_\Lambda \right] \varphi \cdot \nabla_j \varphi^* + \left( \nabla_j \varphi^* \right) \mathcal{L}_\Lambda (\nabla_j \varphi)
\]
where the commutator follows \(\left[ \nabla_j, \mathcal{L}_\Lambda \right] = - \sum_{k \in \Lambda} \nabla_{jk} U_\Lambda \cdot \nabla_k\) with \(\nabla_{jk} \equiv \nabla_j \nabla_k\). Inserting this into (5.3) and integrating with \(\mu\), one gets
\[
\partial_t \mu |\nabla_j \varphi|^2 = \text{Im} \left( 2\mu \sum_{k \in \Lambda} \nabla_{jk} U_\Lambda \cdot \nabla_k \varphi \cdot \nabla_j \varphi^* - \text{Im} \left( 2\mu \left( (\nabla_j \varphi^*) \mathcal{L}_\Lambda (\nabla_j \varphi) \right) \right) \right. \tag{5.4}
\]
\[
+ \text{Im} \left( 2\lambda \mu \left( \nabla_j \left( \varphi \log \frac{|\varphi|^2}{\mu|\varphi|^2} \right) \cdot \nabla_j \varphi^* \right) \right).
\]
By summing over \(j \in \Lambda\), the first term in (5.4) vanishes due to symmetry and the second term also vanishes by the reversibility of \(\mu\). Therefore we only have the last term being summed over, which gives
\[
\partial_t \mu |\nabla_\Lambda \varphi|^2 = \text{Im} \left( 2\lambda \mu \left( \nabla_\Lambda (\varphi \log |\varphi|^2) \cdot \nabla_\Lambda \varphi^* \right) \right). \tag{5.6}
\]
For the entropy term, using mollification argument with the invariance of the \(L^2\) property we have
\[
\frac{\partial}{\partial t} \text{Ent}_\mu |\varphi|^2 = - \text{Im} \left( 2\mu \left( \varphi^* \log |\varphi|^2 \mathcal{L}_\Lambda \varphi \right) \right) - \text{Im} \left( 2\mu \left( \varphi^* \mathcal{L}_\Lambda \varphi + \lambda (1 + \log |\varphi|^2) \text{Ent}_\mu |\varphi|^2 \right) \right).
\]
The second term vanishes again due to the reversibility of \(\mu\) and therefore we get
\[
\frac{\partial}{\partial t} \text{Ent}_\mu |\varphi|^2 = - \text{Im} \left( 2\mu \left( \varphi^* \log |\varphi|^2 \mathcal{L}_\Lambda \varphi \right) \right) = \text{Im} \left( 2\mu \left( \nabla_\Lambda (\varphi^* \log |\varphi|^2) \cdot \nabla_\Lambda \varphi \right) \right) \tag{5.7}
\]
using integration by parts. By noticing that the measure in (5.7) is the complex conjugate of the measure in (5.6), one easily verifies (5.2) and hence the energy conversation property.

\[\square\]

Remark 5.2. The mass conservation property holds true with any measure \(\mu_\Lambda\) and the operator \(\mathcal{L}_\Lambda\) corresponding to the Dirichlet form \(\mu_\Lambda(f \mathcal{L}_\Lambda g) = -\mu_\Lambda(\nabla_\Lambda f \cdot \nabla_\Lambda g)\).
With similar steps one can construct time-dependent bounds for the gradient and the entropy. In particular, we show that these bounds can be uniform in time if $\mu$ satisfies a log-Sobolev inequality (e.g. it satisfies Theorem 3.2).

**Theorem 5.3.** For any $\Lambda \subset \subset \mathbb{Z}^d$ let $\varphi$ be the solution of (LSE$_\Lambda$) with localised initial data $f \in H^1(\mu)$. If $U_\Lambda$ is twice differentiable, then for all $t \geq 0$ one has the gradient bound
\[
\|\nabla_\Lambda \varphi(\cdot, t)\|_\mu^2 \leq e^{2|\lambda|t} \|\nabla f\|_\mu^2
\] (5.8) and the entropy bound
\[
\text{Ent}_\mu[\varphi(\cdot, t)]^2 \leq \text{Ent}_\mu[f]^2 + \frac{1}{|\Lambda|} (e^{2|\lambda|t} - 1) \|\nabla f\|_\mu^2.
\] (5.9)

If $\lambda > 0$ and $\mu$ satisfies a log-Sobolev inequality with coefficient $c > 0$, one has the time-uniform bounds
\[
\|\nabla_\Lambda \varphi(\cdot, t)\|_\mu^2 \leq (1 + \lambda c) \|\nabla f\|_\mu^2 \quad \text{and} \quad \text{Ent}_\mu[\varphi(\cdot, t)]^2 \leq (1/\lambda + c) \|\nabla f\|_\mu^2.
\] (5.10)

**Proof.** Following the proof of Theorem (5.1), the estimate (5.6) leads to
\[
\partial_t \mu|\nabla_\Lambda \varphi|^2 \leq 2|\lambda| \sum_{j \in \Lambda} \mu \left| \text{Im} \left( \varphi^* \nabla_j \varphi \right) \right| \left( \nabla_j \log |\varphi|^2 \right) \leq 2|\lambda| \sum_{j \in \Lambda} \mu |\nabla_j \varphi|^2.
\]
Using Grönwall’s lemma one easily obtains the gradient bound (5.8). For the entropy bound, a follow-up calculation of (5.7) shows that
\[
\frac{\partial}{\partial t} \text{Ent}_\mu[\varphi]^2 = -2 \sum_{j \in \Lambda} \text{Im} \left( \mu \left( \varphi^* \nabla_j \varphi \right) \right) \leq 2\mu|\nabla_\Lambda \varphi|^2 \leq 2 e^{2|\lambda|t} \mu |\nabla f|^2,
\]
where the last inequality is by (5.8) that we just verified. Integrating with respect to $t$ and one obtains the entropy bound (5.9). The uniform bounds (5.10) easily follow from the log-Sobolev inequality of $\mu$ and the energy conservation property (5.1). \(\square\)

With the gradient bound of $\nabla_\Lambda \varphi$ inside $\Lambda$, we now proceed to estimate $\nabla_j \varphi$ for all $j \in \mathbb{Z}^d$. Due to the finite-range interactions, one can show that the norm of $\nabla_j \varphi$ decays exponentially as $j$ moves away from the initial region $\Lambda_f$. This property corresponds to the finite speed propagation of information, which is presented in the following theorem.

**Theorem 5.4.** For any $\Lambda \subset \subset \mathbb{Z}^d$ let $\varphi$ be the solution of (LSE$_\Lambda$) with initial data $f \in H^1(\mu)$ localised in some $\Lambda_f \subset \subset \mathbb{Z}^d$. Suppose the potential $U_\Lambda$ satisfies $\|\nabla_j \nabla_k U_\Lambda\|_\infty \leq A$ for all $\Lambda \subset \subset \mathbb{Z}^d$, $j, k \in \mathbb{Z}^d$ with $j \neq k$ and some $A > 0$ independent of $i, j, \Lambda$. Then the following estimate holds
\[
\|\nabla_j \varphi(\cdot, t)\|_\mu^2 \leq e^{-N_j t} \|\nabla f\|_\mu^2
\]
for any $j \in \mathbb{Z}^d$ and all time $t \leq \epsilon N_j$ where $\epsilon^{-1} \equiv 9(2|\lambda| + A)(2R + 1)^2d$ is the propagation speed and $N_j$ is the distance $N_j \equiv \left[ \text{dist}(j, \Lambda_f)R^{-1} \right]$. 


Proof. Using the calculation in (5.4), (5.5) and (5.6), we have for any \( j \in \mathbb{Z}^d \) the estimate
\[
\partial_t (\mu |\nabla \varphi|^2) \leq 2|\lambda| \mu |\nabla \varphi|^2 + \sum_{k \in \Lambda \setminus \{j\}} \| \nabla_{jk} U_\Lambda \|_{\infty} (\mu |\nabla_k \varphi|^2 + \mu |\nabla_j \varphi|^2).
\] (5.11)

Assuming the uniform bound on \( |\nabla_{jk} U_\Lambda| \) holds, one gets the following differential inequality
\[
\partial_t (\mu |\nabla_j \varphi|^2) \leq \sum_{k \in \mathbb{Z}^d} A_{jk} \mu |\nabla_k \varphi|^2
\] (5.12)
where \( A_{jj} = 2|\lambda| + A(2R + 1)^d \), \( A_{jk} = A \) if \( 0 < |j - k| \leq R \) and \( A_{jk} = 0 \) otherwise. We will use this inequality to get an improved estimate on \( \| \nabla_j \varphi \|_\mu \) when the initial data \( f \) is a local function. First we remark that as a consequence of (5.8) and (5.11) it follows
\[
\| \nabla_{jk} U_\Lambda \|_{\infty} \equiv \sup_{k \in \mathbb{Z}^d} \sup_{t \in [0,t]} \| \nabla_k \varphi (\cdot, s) \|^2 \leq a e^{e t} \mu |\nabla f|^2
\] (5.13)
holds for any finite time \( t \geq 0 \). Let \( f \in H_1 (\mu) \) be localised in some finite volume \( \Lambda_f \subset \mathbb{Z}^d \). Taking into account that the interaction is of finite range \( R \) we will show that within a fixed time period \( \| \nabla_j \varphi \|_\mu \) decays exponentially with \( \text{dist}(j, \Lambda_f) \). To this end, let \( N_j \equiv \left[ \text{dist}(j, \Lambda_f) R^{-1} \right] \) and consider the integral inequality
\[
\mu |\nabla_j \varphi (\cdot, t)|^2 \leq \mu |\nabla_j f|^2 + \sum_{k \in \mathbb{Z}^d} A_{jk} \int_0^t dt_1 \mu |\nabla_k \varphi (\cdot, t_1)|^2
\] (5.14)
deduced from (5.12). Note that \( \nabla_j f \) is zero if \( j \notin \Lambda_f \), thus we can iterate (5.14) for \( M > N_j \) times until we get a nonzero derivative of \( f \). This yields
\[
\mu |\nabla_j \varphi (\cdot, t)|^2 \leq \sum_{n=N_j}^M \frac{t^n}{n!} \sum_{k_1, \ldots, k_n \in \mathbb{Z}^d} (A_{j k_1} \cdots A_{k_n k_1}) \mu |\nabla_{k_1 \cdots k_n} f|^2
\]
\[
+ \sum_{k_1, \ldots, k_{M+1} \in \mathbb{Z}^d} (A_{j k_1} \cdots A_{k_M k_{M+1}}) \int_0^t dt_1 \cdots \int_0^{t_{M+1}} dt_{M+1} \mu |\nabla_{k_{M+1}} \varphi (\cdot, t_{M+1})|^2.
\]
Denoting by \( A_* = A_{jj} (2R + 1)^d \), the above sum is then bounded by
\[
\left( \sum_{n=N_j}^M \frac{(A_* t)^n}{n!} \mu |\nabla f|^2 \right) + \frac{(A_* t)^M}{M!} \bar{\varphi}_t
\]
with \( \bar{\varphi}_t \) defined in (5.13). Since \( \bar{\varphi}_t < \infty \) for all finite time \( t \), sending \( M \to \infty \) leads to
\[
\mu |\nabla_j \varphi (\cdot, t)|^2 \leq \sum_{n=N_j}^\infty \frac{(A_* t)^n}{n!} \mu |\nabla f|^2 \leq \frac{(A_* t)^{N_j}}{N_j!} \mu |\nabla f|^2,
\]
where by Stirling’s formula \(n! \geq n^ne^{-n}\) one has
\[
\frac{(A,t)^{N_j}}{N_j!}e^{A,t} \leq \exp \left( N_j \left( \log(A_*) + 1 - \log \frac{N_j}{t} + A_* \frac{t}{N_j} \right) \right).
\]
For all \(t \leq \frac{N_j}{2A}\) it easily follows \(\log(A_*) + 1 - \log(N_j/t) + A_(t/N_j) \leq -1\), which concludes the proof of Theorem 5.4. \(\square\)

We note that where \(U_\Lambda\) is given by the bilinear or perturbed bilinear potentials, detailed in Section 3, we have the above estimates in Theorem 5.1, 5.3, 5.4. Recall that \(U_\Lambda\) is the potential for the ground state representation of the original solution \(\psi \equiv \varphi e^{-\frac{t}{4}U_\Lambda}\). For consistency, in the following remark we relate \(U_\Lambda\) back to the original potential \(V_\Lambda\) by the differential equation (1.2), namely \(V_\Lambda = -\frac{1}{2}\Delta U_\Lambda + \frac{1}{4} |\nabla U_\Lambda|^2 + \lambda U_\Lambda\).

**Remark 5.5.** Consider the bilinear example of \(U_\Lambda\) with conditions given in Example 3.3. Assuming \(C_{jk} = -C_{kj}\) for simplicity we have
\[
V_\Lambda(x) = D_\Lambda + \sum_{j,k \in \Lambda_R} D_{jk}x_jx_k,
\] (5.15)
where \(\Lambda_R \equiv \{j : \text{dist}(j,\Lambda) \leq R\}\), \(D_\Lambda \equiv -\sum_{j \in \Lambda} C_{jj}\), and \(D_{jk} \equiv \lambda C_{jk} 1_{j \in \Lambda} \cap k \in \Lambda\) + \(\sum_{l \in \Lambda} C_{jl} C_{lk}\). With the conditions on \(C_{jk}\), it follows that \(D_{jk}\) is uniformly bounded and of finite range \(2R\). In case \(\lambda\) is positive, the quadratic coefficient \(D_{jj}\) is uniformly bounded below by some positive constant. If \(\lambda\) is negative and large in absolute value (e.g. \(\lambda < -\sup_j C_{jj}\) when \(C_{ij}\) is diagonal), the quadratic terms become negative and delicate issues may arise. For the latter case, we note that LSE with negative bilinear \(V\) part were recently considered in [15] as an interesting mathematics situation where non-unique positive stationary solutions could exist, with each one generating a continuous family of solitary waves. We remark that large negative values of \(\lambda\) correspond in our setup to the loss of log-Sobolev inequality under which one has uniqueness and strict positivity of the ground state.

For the perturbed bilinear case in Example 3.4, denote by \(Y_\Lambda \equiv \sum_{\chi \subset \subset \mathbb{Z}^d : X \cap \Lambda \neq \emptyset} W_\chi\) and assume \(C_{jk}\) is symmetric for simplicity, then in this case \(V_\Lambda\) takes the quadratic form
\[
V_\Lambda(x) = D_\Lambda + \sum_{k \in \Lambda_R} D_kx_k + \sum_{j,k \in \Lambda_R} D_{jk}x_jx_k,
\]
where \(\Lambda_R\) and \(D_{jk}\) are defined as in (5.15) with the coefficient \(D_k \equiv \varepsilon \sum_{j \in \Lambda} C_{jk} (\nabla_j Y_j)\) and the bounded perturbation
\[
D_\Lambda = \lambda \varepsilon Y_\Lambda + \sum_{j \in \Lambda} \left( -C_{jj} + \frac{\varepsilon^2}{4} |\nabla_j Y_j|^2 - \frac{\varepsilon}{2} \Delta_j Y_j \right).
\]
It is easy to check that \(V_\Lambda\) has uniformly bounded coefficients \(D_k\), \(D_{jk}\) and is of finite range \(2R\).

Finally, for any \(\Lambda \subset \subset \mathbb{Z}^d\) consider the setting with removed boundary conditions, \((U_\Lambda^\circ, \mathbb{E}_\Lambda^\circ)\) defined in Example 3.5, where \(U_\Lambda^\circ \equiv \sum_{\chi \subset \Lambda} J_\chi\) is the potential strictly inside
\( \Lambda \) and \( \mathbb{E}^\Lambda \) is the local Gibbs measure corresponding to \( U^\Lambda \). The LSE in \( \Lambda \) with no boundary conditions is given by

\[
 i \partial_t \varphi_\Lambda = -\mathcal{L}^\Lambda \varphi_\Lambda + \lambda \varphi_\Lambda \log \frac{\varphi_\Lambda^2}{\mathbb{E}^\Lambda \varphi_\Lambda^2} \quad (\text{LSE}^\Lambda)
\]

with \( \mathcal{L}^\Lambda \equiv \Delta_\Lambda - \nabla_\Lambda U^\Lambda \cdot \nabla_\Lambda \) and initial condition \( f \) localised in \( \Lambda_f \subset \Lambda \). For this setup, we have the following results.

**Corollary 5.6.** If the \( \mu \)-estimates in Theorem 5.1, 5.3, 5.4 hold for \( \text{LSE}^\Lambda \), they remain true for \( \text{LSE}^\circ \Lambda \) with \( \mu \) replaced by \( \mathbb{E}^\circ \Lambda \). That is, for the same conditions on \( U^\circ \), one has the conservation property

\[
 \mathbb{E}^\circ \Lambda |\nabla \varphi_\Lambda|^2 = \mathbb{E}^\circ \Lambda |f|^2
\]

the gradient and entropy bounds

\[
 \mathbb{E}^\circ \Lambda |\nabla \varphi_\Lambda|^2 \leq e^{2|\lambda|t} \mathbb{E}^\circ \Lambda |f|^2 \mu \quad \text{and} \quad \text{Ent}_{\mathbb{E}^\circ \Lambda} |\varphi_\Lambda|^2 \leq \text{Ent}_{\mathbb{E}^\circ \Lambda} |f|^2 + |\lambda|^{-1}(e^{2|\lambda|t} - 1)\mathbb{E}^\circ \Lambda |\nabla f|^2,
\]

and the finite-speed propagation property

\[
 \mathbb{E}^\circ \Lambda |\nabla_j \varphi_\Lambda|^2 \leq e^{-N_j} \mathbb{E}^\circ \Lambda |\nabla f|^2 \quad (5.16)
\]

for all \( j \in \mathbb{Z}^d \) and time \( t \leq \varepsilon N_j \), where \( \varepsilon \) and \( N_j \) are defined as in Theorem 5.4.

Note that in the above theorem we have uniform estimates for the total gradient \( \nabla \varphi_\Lambda \) instead of \( \nabla_\Lambda \varphi_\Lambda \) since the solution \( \varphi_\Lambda \) of \( \text{LSE}^\Lambda \) is localised entirely in \( \Lambda \) due to non-existence of boundary conditions. In the next section, we will use the finite-volume problem \( \text{LSE}^\circ \Lambda \) with the estimates in Corollary 5.6 to construct solutions of the infinite-volume LSE problem.

### 6. Existence of Weak Solutions in Infinite Volume

In this section we establish the existence of weak solutions to the infinite-volume LSE by finite-dimensional approximations. Namely, we will construct a sequence \( \{\Lambda_n\} \), increasing in volume, such that the sequence of solutions \( \varphi_{\Lambda_n} \) of \( \text{LSE}^\circ_{\Lambda_n} \) is compact in \( H^1(\mu) \). Then we choose a suitable convergent subsequence whose limit defines a weak infinite-volume solution.

We consider \( \text{LSE}^\circ \) since it lives entirely inside \( \mathbb{R}^\Lambda \), which is technically simpler and for which we already have existence and uniqueness of the solutions. When studying uniqueness of the infinite-volume solution, it is therefore convenient to compare the solution obtained from \( \text{LSE}^\circ \) with the one from other setups such as \( \text{LSE}^\Lambda \).

For simplicity, we shall hereafter work on the space \( \Omega \equiv \mathbb{R}^\mathbb{Z} \) and assume the multi-particle interaction is bounded, i.e. the potentials \( J_X \) have bounded derivatives of order \( n = 0, 1, 2 \) uniformly in \( X \) for all \( |X| \geq 2 \). For the higher dimensional case see Remark 6.3. Moreover, we assume that the measures \( \mathbb{E}_{\Lambda} \) and \( \mathbb{E}_{\circ} \) associated to \( \{J_X\} \).
satisfy a log-Sobolev inequality with coefficient uniform in $\Lambda$. With these conditions, we are ready to introduce the existence theorem.

**Theorem 6.1.** Let $\Omega = \mathbb{R}^2$ and suppose $\{ J_X \}$ satisfies the above conditions. For any $\lambda \in \mathbb{R}$ and $f \in H^1(\mu)$, if the Cauchy problem (LSE$^\lambda$) with initial data $f$ and coupling constant $\lambda$ admits at least one solution for all $\Lambda \subset \subset \mathbb{Z}^d$, then there exists a weak solution to the infinite-volume LSE problem with initial data $f$ and coupling constant $\lambda$, i.e. there exists $\varphi \in H^1(\mu)$ such that

$$
\mu(g \partial_t \varphi) = \mu \left( \nabla g \cdot \nabla \varphi + g \lambda \varphi \log \frac{|\varphi|^2}{|\mu|} \right) 
$$

$$
\varphi(t = 0) = f 
$$

for all smooth and compactly supported local functions $g : \Omega \to \mathbb{C}$.

**Proof of Theorem 6.1.** Let $f \in H^1(\mu)$ be localised in $\Lambda_f \subset [-L, L]$ for some $L \geq R$, where $R$ is the interaction range. Consider the sequence $\Lambda_n \equiv [-8nL, 8nL] \cap \mathbb{Z}$ and suppose (LSE$^\lambda$) with initial condition $f$ admits at least one solution $\varphi_n \equiv \varphi_{\Lambda_n}$ for all $n \in \mathbb{N}$. We will prove the following convergence of the norms

$$
\lim_{n \to \infty} \mu|\varphi_n|^2 = \mu|f|^2. 
$$

(6.1)

Denote by $\rho_n \equiv d\mathbb{E}_{\Lambda_n}/d\mathbb{E}_{\varphi_n}^\lambda$ the density ratio, we can rewrite $\mu|\varphi_n|^2 = \mu \mathbb{E}_{\lambda} |\varphi_n|^2 = \mu \mathbb{E}_{\lambda}^\varphi (|\varphi_n|^2 \rho_n)$. With the notation $\nu_n \equiv \mu \mathbb{E}_{\lambda}^\varphi$ and a notation $\nu(g; h) \equiv \nu(gh) - \nu(g)\nu(h)$ for the $\nu$-covariance, we then have

$$
\mu|\varphi_n|^2 = \nu_n(|\varphi_n|^2; \rho_n) + \left( \nu_n(|\varphi_n|^2) \right) \left( \nu_n(\rho_n) \right) = \nu_n(|\varphi_n|^2; \rho_n) + \nu_n|f|^2 
$$

(6.2)

since $\mathbb{E}_{\lambda}^\varphi(\rho_n) = 1$ and $\mathbb{E}_{\lambda}^\varphi|\varphi_n|^2 = \mathbb{E}_{\lambda}^\varphi|f|^2$ by the mass conservation property in Corollary 5.6. Using (6.2) for the norms of $\varphi_n$ and $\varphi_{n+1}$, one has the following estimate

$$
|\mu|\varphi_{n+1}|^2 - \mu|\varphi_n|^2| \leq \nu_n(|\varphi_n|^2; \rho_n) + \nu_{n+1}(|\varphi_{n+1}|^2; \rho_{n+1}) + \nu_n|f|^2 - \nu_n|f|^2. 
$$

(6.3)

To bound the first covariance, let $\Gamma = (\{-12nL, -4nL\} \cup \{4nL, 12nL\}) \cap \mathbb{Z}$ and we rewrite the covariance by

$$
\nu_n(|\varphi_n|^2; \rho_n) = \nu_n(|\varphi_n|^2 - \mathbb{E}_\Gamma|\varphi_n|^2; \rho_n) + \nu_n(\mathbb{E}_\Gamma|\varphi_n|^2; \rho_n). 
$$

(6.4)

To proceed, we first note that if the lattice is one-dimensional and $J_X$ is uniformly bounded for $|X| \geq 2$ then $\rho_n$ is bounded uniformly in $n$ by $1/|B_1| \leq \rho_n \leq B_1$, where

$$
B_1 = \exp \left( 4 \sup_{k \in \mathbb{Z}} \left( \sum_{x \in k \cap |X| \geq 2} \|J_X\|_\infty \right) \right) 
$$

(6.5)

is finite since the interaction is of finite range. Hence, for any function $g \geq 0$ one has $\mu(g) \leq B_1 \nu_n(g)$ and $\nu_n(g) \leq B_1 \mu(g)$. With this density bound, the first covariance in (6.4) can be estimated by

$$
\nu_n(|\varphi_n|^2 - \mathbb{E}_\Gamma|\varphi_n|^2; \rho_n) \leq 2B_1 |\varphi_n|^2 - \mathbb{E}_\Gamma|\varphi_n|^2 \leq 2B_1 \mu \mathbb{E}_\Gamma \mathbb{E}_\Gamma \|\varphi_n|^2 - |\varphi_n|^2\|. 
$$

(6.6)
Applying the same argument as in (3.16) on the double measure $\mathbb{E}_\Gamma \tilde{\mathbb{E}}_\Gamma$, it follows

$$E(6.6) \leq 4B_1\mu \left( \mathbb{E}_\Gamma |\varphi_n|^2 \right)^{1/2} \left( \mathbb{E}_\Gamma |\varphi_n - \mathbb{E}_\Gamma \varphi_n|^2 \right)^{1/2} \leq 4(\frac{B_1}{2})^2 \left( \mathbb{E}_\Gamma^2 |f|^2 \right)^{1/2} \mu \left( \mathbb{E}_\Gamma |\nabla \varphi_n|^2 \right)^{1/2},$$

where we used the mass conservation for $\mathbb{E}_\Gamma$ and the spectral gap inequality for $\mathbb{E}_\Gamma$. Using the finite-speed propagation property (5.16) for $\mathbb{E}_\Gamma |\nabla \varphi_n|^2$, one has the estimate

$$E(6.6) \leq D_1|\Lambda_n|e^{-\frac{\epsilon}{8}\text{diam}(\Lambda_n)} \left( \mu |f|^2 + \mu |\nabla f|^2 \right)$$

with $D_1 \equiv 4(\frac{B_1}{2})^5$. By the construction of $\Gamma$ and $\Lambda_n$, we have dist($\Gamma, \Lambda_f$) $\geq \frac{1}{8}\text{diam}(\Lambda_n)$ and therefore there exists $\epsilon_1 > 0$ independent of $n$ such that

$$\left| \mu_n \left( |\varphi_n|^2 - \mathbb{E}_\Gamma |\varphi_n|^2, \rho_n \right) \right| \leq D_1|\Lambda_n|e^{-\epsilon_1\text{diam}(\Lambda_n)} \left( \mu |f|^2 + \mu |\nabla f|^2 \right).$$

(6.7)

Here we note that due to the finite-speed propagation property (5.16), the estimate (6.7) only holds in the time range $t \leq C\text{diam}(\Lambda_n)$ for some $C > 0$ independent of $n$. As $n \to \infty$ in the sequence, this time range will eventually extend to $[0, \infty)$ and the above estimate becomes global in time.

To bound the second covariance in (6.4), we use the mixing condition (3.18) for the measure $\mathbb{E}_\Lambda$, which holds true since $\mathbb{E}_\Lambda$ satisfies a uniform LSI and $\nabla_j \mathbb{U}_\lambda$ is bounded due to the bounded multi-particle interaction. Therefore, there exist $C_{\text{mix}}, M > 0$ independent of $n$ such that

$$E_\Lambda(\mathbb{E}_\Gamma |\varphi_n|^2; \rho_n) \leq C_{\text{mix}}|\Lambda_f| |\Lambda_{\rho_n}| e^{-M \text{dist}(\Gamma, \Lambda_{\rho_n})} \left( \mathbb{E}_\Lambda |\nabla \Lambda_n \mathbb{E}_\Gamma |\varphi_n|^2 \right)^{1/2} \left( \mathbb{E}_\Lambda |\nabla \Lambda_n \rho_n|^2 \right)^{1/2}$$

where $\Lambda_f, \Lambda_{\rho_n}$ are the sets where $\mathbb{E}_\Gamma |\varphi_n|^2$ and $\rho_n$ are localised, respectively. First, we note that both $|\Lambda_f|$ and $|\Lambda_{\rho_n}|$ are smaller than $|\Lambda_n|$. Moreover, since dist($\Gamma, \Lambda_{\rho_n}$) $\geq \frac{1}{8}\text{diam}(\Lambda_n)$ there exists $\epsilon_2 > 0$ independent of $n$ such that the exponential term is bounded by $e^{-\epsilon_2\text{diam}(\Lambda_n)}$. To estimate the $\nabla \Lambda_n \rho_n$ term, we use the following bound

$$\|\nabla \Lambda_n \rho_n\|_{1/\infty} \leq 2B_1 \sup_{k \in \mathbb{Z}} \left( \sum_{|X| \leq k, |X| \geq 2} \|\nabla J_X\|_{1/\infty} \right) \equiv B_2$$

where $B_2$ is finite since $J_X$ is of finite range and has uniformly bounded derivatives for $|X| \geq 2$. For the gradient $\nabla \Lambda_n \mathbb{E}_\Gamma |\varphi_n|^2$, direct calculation gives

$$\nabla_j \mathbb{E}_\Gamma |\varphi_n|^2 = \mathbb{E}_\Gamma \left( \nabla_j |\varphi_n|^2 \right) - \mathbb{E}_\Gamma \left( |\varphi_n|^2 \nabla_j \mathbb{U}_\lambda \right)$$

for any $j \notin \Gamma$. The first term is simply bounded by $\mathbb{E}_\Gamma |\varphi_n|^2 + \mathbb{E}_\Gamma |\nabla_j \varphi_n|^2$ due to Young’s inequality. For the covariance, an application of the relative entropy inequality (similar as (5.19)) and Lemma 3.11 gives

$$E_\Gamma \left( |\varphi_n|^2; \nabla \mathbb{U}_\lambda \right) \leq C_{\text{RE}} |\Lambda_n| \left( \mathbb{E}_\Gamma |\varphi_n|^2 + \mathbb{E}_\Gamma |\nabla_j \varphi_n|^2 \right)$$

for some $C_{\text{RE}} > 0$ independent of $n$. Combining all the estimates, we have

$$\left| \mu \mathbb{E}_\Lambda(\mathbb{E}_\Gamma |\varphi_n|^2; \rho_n) \right| \leq B_2 C_{\text{mix}}(1 + C_{\text{RE}})|\Lambda_n|^4 e^{-\epsilon_2\text{diam}(\Lambda_n)} \left( \mu |\varphi_n|^2 + \mu |\nabla \varphi_n|^2 \right).$$
Using the mass conservation for $\mathbb{E}_{\Lambda_n} |\varphi_n|^2$ and the gradient bound for $\mathbb{E}_{\Lambda_n} \nabla \varphi_n$, it follows
\[ |\mu \mathbb{E}_{\Lambda_n} (\mathbb{E}_n |\varphi_n|^2; \rho_n) | \leq D_2 |\Lambda_n| 4e^{-\varepsilon_2 \text{diam}(\Lambda_n)} \left( \mu |f|^2 + e^{2|\lambda| t} \mu |
abla f|^2 \right) \]
with $D_2 \equiv B_1^2 B_2 C_{\text{mix}} (1 + C_{RE})$. Taking (6.7) and (6.8) into (6.4), we arrive at
\[ |\nu_n (|\varphi_n|^2; \rho_n) | \leq D_{12} |\Lambda_n| e^{-\varepsilon_1 \text{diam}(\Lambda_n)} \left( \mu |f|^2 + e^{2|\lambda| t} \mu |
abla f|^2 \right) \]
with $D_{12} \equiv \max\{D_1, D_2\}$ and $\varepsilon_{12} \equiv \min \{\varepsilon_1, \varepsilon_2\}$. This concludes the estimate of the first term in (6.3). The second term in (6.3) follows similarly and therefore we will finish the proof of (6.1) by estimating the third term in (6.3).

To this end, for any $\theta \in [0, 1]$ we define the interpolating potential
\[ U^\theta_2 \equiv U^\theta_{\Lambda_n} \rightleftarrows \sum_{k \in \Lambda_n+1 \backslash \Lambda_n} \theta J_X \]
and the corresponding local Gibbs measure $d\mathbb{E}_\theta \equiv \left( \int e^{-U^\theta_2} d\lambda_{\Lambda_n+1} \right)^{-1} e^{-U^\theta_2} d\lambda_{\Lambda_n+1}$. Then one has the following integral representation
\[ \mathbb{E}_\theta |f|^2 - \mathbb{E}_{\Lambda_n} |f|^2 = \int_0^1 d\theta d\theta \mathbb{E}_\theta |f|^2, \]
where the $\theta$-derivative follows $d\theta \mathbb{E}_\theta |f|^2 = -\mathbb{E}_\theta \left( |f|^2; d\theta U^\theta \right)$. When the lattice is one-dimensional, the size $|\Lambda_{n+1} \backslash \Lambda_n|$ is independent of $n$ and therefore the density $d\mathbb{E}_{\Lambda_n+1} / d\mathbb{E}_\theta$ is bounded uniformly in $n$ and $\theta$. Then by the arguments of [13, Lemma 5.1], if $\mathbb{E}_{\Lambda_n+1}$ satisfies a uniform LSI the measure $\mathbb{E}_\theta$ also satisfies a LSI uniformly in $n$ and $\theta$. Thus, Theorem 3.10 holds for $\mathbb{E}_\theta$ and one has the mixing condition
\[ \left| \mathbb{E}_\theta \left( |f|^2; d\theta U^\theta \right) \right| \leq C_{\text{mix}} |\Lambda_\theta| |\Lambda_f| e^{-M' \text{dist}(\Lambda_\theta, \Lambda_f)} \left( \mathbb{E}_\theta |\nabla \lambda_{n+1} | f|^2 |^2 \right)^{1/2} \left( \mathbb{E}_\theta |\nabla \lambda_{n+1} \left( \frac{d\theta U^\theta}{\theta} \right) |^2 \right)^{1/2} \]
for some $C_{\text{mix}}, M' > 0$ independent of $n$ and $\theta$, where $\Lambda_\theta$ is the set where $d\theta U^\theta$ is localised in. By the construction of $\Lambda_n$, we have $|\Lambda_\theta| |\Lambda_f| \leq |\Lambda_n|^2$ and $\text{dist}(\Lambda_\theta, \Lambda_f) \geq \frac{1}{4} \text{diam}(\Lambda_n)$. Moreover, the gradient on $d\theta U^\theta$ is bounded by
\[ \left\| \nabla \lambda_{n+1} \left( \frac{d\theta U^\theta}{\theta} \right) \right\|_{\infty} \leq \sup_{n \in \mathbb{N}} \left( \sum_{|\Lambda_n| \neq 0} \| \nabla J_X \|_{\infty} \right) \equiv B_3. \]
Finally, by noticing that $\mathbb{E}_\theta (g) \leq B_1 \mu (g)$ for all $\theta \in [0, 1]$ and $g \geq 0$, one gets the estimate
\[ |\nu_{n+1} | f |^2 - |\nu_n| f |^2 | \leq D_3 |\Lambda_n| 2 e^{-\varepsilon_3 \text{dist}(\Lambda_n)} \left( \mu |f|^2 + \mu |\nabla f|^2 \right) \]
with $D_3 \equiv B_1 B_3 C_{\text{mix}}$ and $\varepsilon_3 \equiv M'$. Gathering (6.9) and (6.10) in (6.3), we have the bound
\[ |\mu |\varphi_{n+1} |^2 - |\mu| \varphi_n |^2 | \leq D |\Lambda_n| 4e^{-\varepsilon \text{diam}(\Lambda_n)} \left( \mu |f|^2 + e^{2|\lambda| t} \mu |\nabla f|^2 \right) \]
for some $D, \varepsilon > 0$ independent of $n$. Since the series $\sum_n |\Lambda_n| 4e^{-\varepsilon \text{diam}(\Lambda_n)}$ converges, the real sequence $\mu |\varphi_n |^2$ is Cauchy and thus convergent to some $C_{\lim} \geq 0$. Using the same argument with triangle inequality, one can show that $|\mu |\varphi_n (t_1) |^2 - \mu |\varphi_n (t_2) |^2 |$ tends
to zero for any time \( t_1, t_2 \). Hence, the limit \( C_{\text{lim}} \) is independent of time and therefore identical to \( \mu |f|^2 \).

Using consideration of Section 5 and in particular Theorem 5.1 and 5.3 in the situations described by (5.1) and (5.9) we have a compactness of the set of the solutions of \( \text{LSE}_\Lambda^{\infty} \) associated to the sequence \( \Lambda_n \) invading \( \mathbb{Z} \). In case when entropy term is bounded we get that the sequence of solutions \( \varphi_n \) is weakly compact and since the corresponding sequence of the norms \( \mu |\varphi_n|^2 \) is convergent, by general principles this implies that in fact our sequence converges strongly in \( L^2(\mu) \). Hence by general principles there exists a subsequence \( \varphi_{n_m}, m \in \mathbb{N} \), which converges \( \mu \)-a.e. to some function \( \varphi \in L^2(\mu) \). Hence we can conclude that \( \varphi \) satisfies the infinite-volume LSE in the ultra weak sense, i.e. for any smooth and compactly supported local functions \( g : \Omega \to \mathbb{C} \) one has

\[
\mu (g \partial_t \varphi) = \mu \left( (-\mathcal{L}g)\varphi + g\lambda \varphi \log \frac{|\varphi|^2}{\mu|\varphi|^2} \right)
\]

where \( \mathcal{L} \equiv \lim_{\Lambda \to \mathbb{Z}} \mathcal{L}_\Lambda \) is well-defined for all \( g \). Since the theorems in Section 5 provide us with uniform bound in the corresponding \( H^1(\mu) \) space we could choose a suitable weakly convergent subsequence (for which also the nonlinear term would converge as we have also uniform bound for entropy). The limit of such the sequence would satisfy the following weak form of the equation

\[
\mu (g \partial_t \varphi) = \mu \left( \nabla g \cdot \nabla \varphi + g\lambda \varphi \log \frac{|\varphi|^2}{\mu|\varphi|^2} \right).
\]

It is possible that following the ideas of [45] one could develop a technique for higher order estimates which would allow us to get a strong convergence of approximating sequence corresponding to a smooth bounded local initial data and reasonable interactions (including Gaussian perturbed by smooth short range multi-particle interactions).

In the case of the bilinear interaction, we give an explicit characterisation of the bounded multi-particle interaction condition.

**Example 6.2.** For any \( X \subset \subset \mathbb{Z} \), consider the following perturbed quadratic potential

\[
J_X(x) = \begin{cases} 
C_j x_j^2 & \text{if } X = \{j\} \\
\varepsilon W_X(x) & \text{if } |X| \geq 2
\end{cases}
\]

where \( \varepsilon \in \mathbb{R}, W_X \in C^2(\mathbb{R}^X) \) has uniformly in \( X \) bounded derivatives of order \( n = 0, 1, 2 \) and \( W_X \equiv 0 \) if \( \text{diam}(X) > R \). Suppose \( C^- \leq C_j \leq C^+ \) for some \( 0 < C^- \leq C^+ \), then for sufficiently small \( |\varepsilon| \) the measures \( \mathbb{E}_\Lambda \) and \( \mathbb{E}_\Lambda^0 \) both satisfy a uniform LSI by Remark 3.2. Moreover, by [46, Proposition 1.3], the Cauchy problem \( \text{LSE}_\Lambda^{\infty} \) associated to the potential \( \{J_X\} \) admits a unique solution for all \( \lambda \in \mathbb{R} \) and localised initial condition \( f \in H^1(\mu) \). Therefore, all conditions in Theorem 6.1 are satisfied
and there exists a solution of the infinite-volume LSE associated to the perturbed quadratic potential on \( \mathbb{R}^\mathbb{Z} \).

**Remark 6.3.** The restriction to \( d = 1 \) is required for the finite quantity \( B_1 \), since the interaction-range, \( R \), extension of the finite cubes contain a fixed number of points. In higher dimensions, the size of the extensions grow as the cubes get larger. This could be offset with further assumptions on the potential.

### 7. Solitons

In finite dimensions the solitons for the logarithmic Schrödinger equation, introduced originally in [47], were discussed in number of works see e.g. [11], [12], [48], [3], [49], [50], [51], [52], [53], [54], [55], [6], [56], [57], [58], [59], [60], [61], [62], [63]. In \( \mathbb{R}^n \) one can try construct general solutions to the following LSE

\[
i\partial_t \psi = -\Delta \psi + V\psi + \lambda \psi \log |\psi|^2
\]

by performing separation of variables so that \( \psi(t, x) = e^{-iEt} \varphi(x) \), where \( \varphi : \mathbb{R}^n \to \mathbb{R} \) is the stationary state satisfying the eigen-equation

\[
-\Delta \psi + V\psi + \lambda \psi \log |\psi|^2 = E \psi \tag{7.1}
\]

with energy eigenvalue \( E \in \mathbb{R} \). For the sake of simplicity, we shall first consider \( V = 0 \) and try to solve

\[
-\Delta \psi + \lambda \psi \log |\psi|^2 = E \psi. \tag{7.2}
\]

Inspired by the results in [1], we take the soliton-like ansatz \( \psi = \exp(-\frac{a}{2}|x|^2) \) into (7.2) and find that \( a = -\lambda \) provides a solution with \( E = -\lambda n \). Letting \( \psi_0 = \exp(-\frac{1}{2}|x|^2) \), one can readily see that for any nonzero \( b \in \mathbb{R} \), \( \psi_b \equiv b\psi_0 \) is also a solution to (7.2) with eigenvalue \( E_b = -\lambda (n - \log b^2) \). In order \( \psi \) to be normalizable, we require \( \lambda < 0 \) and the normalised solution is \( \psi = (-2\pi/\lambda)\frac{1}{2} \exp(-\frac{1}{4}|x|^2) \) with eigenvalue \( E = -\lambda n (1 - \log(-2\pi/\lambda)) \). Next, we investigate (7.1) with a quadratic potential \( V = \sum_{j=1}^n a_j x_j^2 \) where \( a_j > 0 \), and this time we use a more general ansatz \( \psi = \exp(-U(x)) \) to get

\[
\Delta U - |\nabla U|^2 - 2\lambda U = E - \sum_{j=1}^n a_j x_j^2.
\]

Choosing \( U = \sum_{j=1}^n \frac{1}{2} c_j x_j^2 \), with \( c_j = \frac{1}{2} (\sqrt{4a_j + \lambda^2} - \lambda) > 0 \), one gets a normalised solution \( \psi = (2\pi)^{n/2} \prod c_j \exp(-\sum_j c_j x_j^2) \) with eigenvalue \( E = \lambda n \log(2\pi) + \sum_j (c_j - \lambda \log c_j) \).

Use of Galilean covariance provides us with a soliton type solution [48]. However for more general potential this transformation requires suitable adaptation of the potential.

In the case of composite systems consisting of a number of elementary entities (atoms, molecules, cells, etc) one would also like to model states where the wave function changes in time, with concentration to a given part of the system.
In infinite dimensional systems with nontrivial interaction the typical situation is more complicated. Under log-Sobolev inequality, (thanks to Lemma 2.7), assuming \( \lambda > -\frac{\varepsilon}{c} \) with some \( \varepsilon \in [0, 1) \), we have the following analog of spectral gap property

\[
\mu \left( -\varphi^* \mathcal{L} \varphi + \lambda |\varphi|^2 \log \frac{|\varphi|^2}{\mu |\varphi|^2} \right) \geq \frac{2(1 - \varepsilon)}{c} \left( \frac{1}{c} + \lambda \right) \mu |\varphi|^2.
\]

However, in general the structure of spectrum is not discrete. In the best case what one encounters (for example in statistical mechanics see e.g. [63], [64] or QFT see e.g. [65] and references there in) are separated windows of continuous spectrum. Thus the classical notion of soliton may not make sense. However it is possible that states with soliton behaviour on some large timescale could be reasonable. It would be interesting to develop such theory (with potential applications e.g. to quantum computing).

References

[1] I. Bialynicki-Birula and J. Mycielski, “Nonlinear wave mechanics,” Annals of Physics, vol. 100, no. 1-2, pp. 62–93, 1976.
[2] I. Białynicki-Birula and J. Mycielski, “Uncertainty relations for information entropy in wave mechanics,” Communications in Mathematical Physics, vol. 44, no. 2, pp. 129–132, 1975.
[3] H. Buljan, A. Šiber, M. Soljačić, T. Schwartz, M. Segev, and D. N. Christodoulides, “Incoherent white light solitons in logarithmically saturable noninstantaneous nonlinear media,” Physical Review E, vol. 68, no. 3, p. 036607, 2003.
[4] S. De Martino, M. Falanga, C. Godano, and G. Lauro, “Logarithmic schrödinger-like equation as a model for magma transport,” EPL (Europhysics Letters), vol. 63, no. 3, p. 472, 2003.
[5] E. F. Hefter, “Application of the nonlinear schrödinger equation with a logarithmic inhomogeneous term to nuclear physics,” Physical Review A, vol. 32, no. 2, p. 1201, 1985.
[6] T. Hansson, D. Anderson, and M. Lisak, “Propagation of partially coherent solitons in saturable logarithmic media: A comparative analysis,” Physical Review A, vol. 80, no. 3, p. 033819, 2009.
[7] J. D. Brasher, “Nonlinear wave mechanics, information theory, and thermodynamics,” International journal of theoretical physics, vol. 30, no. 7, pp. 979–984, 1991.
[8] K. Yasue, “Quantum mechanics of nonconservative systems,” Annals of Physics, vol. 114, no. 1, pp. 479–496, 1978.
[9] K. G. Zloshchastiev, “Logarithmic nonlinearity in theories of quantum gravity: Origin of time and observational consequences,” Gravitation and Cosmology, vol. 16, pp. 288–297, Oct 2010.
[10] A. V. Avdeenkov and K. G. Zloshchastiev, “Quantum bose liquids with logarithmic nonlinearity: Self-sustainability and emergence of spatial extent,” Journal of Physics B: Atomic, Molecular and Optical Physics, vol. 44, no. 19, p. 195303, 2011.
[11] A. H. Ardila, “Existence and stability of standing waves for nonlinear fractional schrödinger equation with logarithmic nonlinearity,” Nonlinear Analysis, vol. 155, pp. 52–64, 2017.
[12] A. H. Ardila, “Orbital stability of gausson solutions to logarithmic schrödinger equations,” Electron. J. Diff. Eqns., vol. 2016 (335), pp. 1–9, 2016.
[13] W. Bao, R. Carles, C. Su, and Q. Tang, “Error estimates of a regularized finite difference method for the logarithmic schrödinger equation,” SIAM Journal on Numerical Analysis, vol. 57, no. 2, pp. 657–680, 2019.
[14] R. Carles and I. Gallagher, “Universal dynamics for the defocusing logarithmic schrödinger equation,” Duke Mathematical Journal, vol. 167, no. 9, pp. 1761–1801, 2018.
[15] R. Carles and C. Su, “Nonuniqueness and nonlinear instability of gaussons under repulsive harmonic potential,” arXiv preprint arXiv:2107.10024, 2021.
[16] T. Cazenave, “Stable solutions of the logarithmic Schrödinger equation,” Nonlinear Analysis: Theory, Methods & Applications, vol. 7, no. 10, pp. 1127–1140, 1983.
[17] T. Cazenave and A. Haraux, “Équations d’évolution avec non linéarité logarithmique,” in Annales de la Faculté des sciences de Toulouse: Mathématiques, vol. 2, pp. 21–51, 1980.
[18] T. Cazenave, Semilinear Schrödinger Equations, vol. 10. American Mathematical Soc., 2003.
[19] P. d’Avenia, E. Montefusco, and M. Squassina, “On the logarithmic Schrödinger equation,” Communications in Contemporary Mathematics, vol. 16, no. 02, p. 1350032, 2014.
[20] G. Ferriere, “The focusing logarithmic Schrödinger equation: Analysis of breathers and nonlinear superposition,” Discrete & Continuous Dynamical Systems, vol. 40, no. 11, pp. 6247–6274, 2020.
[21] G. Ferriere, “Convergence rate in wasserstein distance and semiclassical limit for the defocusing logarithmic Schrödinger equation,” Analysis & PDE, vol. 14, no. 2, pp. 617–666, 2021.
[22] P. Guerrero, J. López, and J. Nieto, “Global h1 solvability of the 3d logarithmic Schrödinger equation,” Nonlinear Analysis: Real World Applications, vol. 11, no. 1, pp. 79–87, 2010.
[23] H.-M. Nguyen and M. Squassina, “Logarithmic Sobolev inequality revisited,” Comptes Rendus Mathematique, vol. 355, no. 4, pp. 447–451, 2017.
[24] W. C. Troy, “Uniqueness of positive ground state solutions of the logarithmic Schrödinger equation,” Archive for Rational Mechanics and Analysis, vol. 222, no. 3, pp. 1581–1600, 2016.
[25] A. Guionnet and B. Zegarlinski, “Lectures on logarithmic Sobolev inequalities,” in Séminaire de probabilités XXXVI, pp. 1–134, Springer, 2003.
[26] D. W. Stroock and B. Zegarlinski, “The logarithmic Sobolev inequality for continuous spin systems on a lattice,” Journal of Functional Analysis, vol. 104, no. 2, pp. 299–326, 1992.
[27] B. Zegarlinski, “Log-sobolev inequalities for infinite one dimensional lattice systems,” Communications in Mathematical Physics, vol. 133, pp. 147–162, Sep 1990.
[28] B. Zegarlinski, “Dobrushin uniqueness theorem and logarithmic Sobolev inequalities,” Journal of Functional Analysis, vol. 105, no. 1, pp. 77–111, 1992.
[29] B. Zegarlinski, “The strong decay to equilibrium for the stochastic dynamics of unbounded spin systems on a lattice,” Communications in Mathematical Physics, vol. 175, pp. 401–432, Jan 1996.
[30] D. W. Stroock and B. Zegarlinski, “The logarithmic Sobolev inequality for discrete spin systems on a lattice,” Communications in Mathematical Physics, vol. 149, no. 1, pp. 175–193, 1992.
[31] J. Béllissard and R. Høegh-Krohn, “Compactness and the maximal gibbs state for random gibbs fields on a lattice,” Communications in Mathematical Physics, vol. 84, no. 3, pp. 297–327, 1982.
[32] C. Preston, “Random fields and specifications,” in Random Fields, pp. 11–32, Springer, 1976.
[33] H.-O. Georgii, “Gibbs measures and phase transitions,” de Gruyter Studies in Mathematics, vol. 9, 1988.
[34] Y. G. Sinai, Theory of phase transitions: rigorous results. Pergamon, 1982.
[35] S. Friedli and Y. Velenik, Statistical mechanics of lattice systems: a concrete mathematical introduction. Cambridge University Press, 2017.
[36] O. S. Rothaus, “Logarithmic Sobolev inequalities and the spectrum of Schrödinger operators,” Journal of Functional Analysis, vol. 42, no. 1, pp. 110–120, 1981.
[37] O. S. Rothaus, “Lower bounds for eigenvalues of regular sturm-liouville operators and the logarithmic Sobolev inequality,” Duke Mathematical Journal, vol. 45, no. 2, pp. 351–362, 1978.
[38] D. Bakry and M. Émery, “Diffusions hypercontractives,” in Séminaire de probabilités XIX 1983/84, pp. 177–206, Springer, 1985.
[39] J. D. Inglis, Coercive inequalities for generators of Hörmander type. PhD thesis, Department of Mathematics, Imperial College London, 2010.
[40] C. Henderson and G. Menz, “Equivalence of a mixing condition and the lsi in spin systems with infinite range interaction,” Stochastic Processes and their Applications, vol. 126, no. 10, pp. 2877–2912, 2016.
[41] W. Hebisch and B. Zegarlinski, “Coercive inequalities on metric measure spaces,” Journal of Functional Analysis, vol. 258, no. 3, pp. 814–851, 2010.
[42] D. W. Stroock and B. Zegarlinski, “The equivalence of the logarithmic sobolev inequality and the Dobrushin-Shlosman mixing condition,” Communications in Mathematical Physics, vol. 144, no. 2, pp. 303–323, 1992.
[43] H. Ye, J. Gao, and Y. Ding, “A generalized Gronwall inequality and its application to a fractional differential equation,” Journal of Mathematical Analysis and Applications, vol. 328, no. 2, pp. 1075–1081, 2007.
[44] R. Holley and D. Stroock, “Logarithmic sobolev inequalities and stochastic ising models,” Journal of Statistical Physics, vol. 46, no. 5, pp. 1159–1194, 1987.
[45] V. Kontis, M. Ottobre, and B. Zegarlinski, “Long-and short-time behaviour of hypocoercive-type operators in infinite dimensions: An analytic approach,” Infinite Dimensional Analysis, Quantum Probability and Related Topics, vol. 20, no. 03, p. 1750015, 2017.
[46] R. Carles and G. Ferriere, “Logarithmic schrödinger equation with quadratic potential,” Nonlinearity, vol. 34, no. 12, p. 8283, 2021.
[47] I. Bialynicki-Birula and J. Mycielski, “Gaussons: solitons of the logarithmic schrodinger equation,” Physica Scripta, vol. 20, no. 3-4, p. 539, 1979.
[48] H. R. Brown and P. R. Holland, “The galilean covariance of quantum mechanics in the case of external fields,” American Journal of Physics, vol. 67, no. 3, pp. 204–214, 1999.
[49] R. Côte and X. Friederich, “On smoothness and uniqueness of multi-solitons of the non-linear schrödinger equations,” Communications in Partial Differential Equations, vol. 46, no. 12, pp. 2325–2385, 2021.
[50] R. Côte, “On the soliton resolution for equivariant wave maps to the sphere,” Communications on Pure and Applied Mathematics, vol. 68, no. 11, pp. 1946–2004, 2015.
[51] R. Côte and S. Le Coz, “High-speed excited multi-solitons in nonlinear schrödinger equations,” Journal de mathématiques pures et appliquées, vol. 96, no. 2, pp. 135–166, 2011.
[52] R. Côte, Y. Martel, and F. Merle, “Construction of multi-soliton solutions for the $l^2$-supercritical gkdv and nls equations,” Revista Matematica Iberoamericana, vol. 27, no. 1, pp. 273–302, 2011.
[53] W. Eckhaus and P. Schuur, “The emergence of solitons of the korteweg-de vries equation from arbitrary initial conditions,” Mathematical Methods in the Applied Sciences, vol. 5, no. 1, pp. 97–116, 1983.
[54] G. Ferriere, “Existence of multi-solitons for the focusing logarithmic non-linear schrödinger equation,” in Annales de l’Institut Henri Poincaré C, Analyse non linéaire, vol. 38, pp. 841–875, Elsevier, 2021.
[55] M. Grillakis, J. Shatah, and W. Strauss, “Stability theory of solitary waves in the presence of symmetry, i,” Journal of functional analysis, vol. 74, no. 1, pp. 160–197, 1987.
[56] E. Hernandez and B. Remaud, “General properties of gausson-conserving descriptions of quantal damped motion,” Physica A: Statistical Mechanics and its Applications, vol. 105, no. 1-2, pp. 130–146, 1981.
[57] W. Królikowski, D. Edmundson, and O. Bang, “Unified model for partially coherent solitons in logarithmically nonlinear media,” Physical Review E, vol. 61, no. 3, p. 3122, 2000.
[58] S. Le Coz, “Standing waves in nonlinear schrödinger equations,” Analytical and numerical aspects of partial differential equations, pp. 151–192, 2009.
[59] Y. Martel and F. Merle, “Multi solitary waves for nonlinear schrödinger equations,” Annales de l’Institut Henri Poincaré C, vol. 23, no. 6, pp. 849–864, 2006.
[60] Y. Martel, F. Merle, and T.-P. Tsai, “Stability and asymptotic stability for subcritical gkdv equations,” *Communications in mathematical physics*, vol. 231, no. 2, pp. 347–373, 2002.

[61] Y. Martel, F. Merle, and T.-P. Tsai, “Stability in h1 of the sum of k solitary waves for some nonlinear schrödinger equations,” *Duke Mathematical Journal*, vol. 133, no. 3, pp. 405–466, 2006.

[62] P. C. Schuur, *Asymptotic analysis of soliton problems: an inverse scattering approach*, vol. 1232. Springer, 2006.

[63] C. Zhang and X. Zhang, “Bound states for logarithmic schrödinger equations with potentials unbounded below,” *Calculus of Variations and Partial Differential Equations*, vol. 59, p. 23, Jan 2020.

[64] R. A. Minlos and A. Trishch, “The complete spectral decomposition of a generator of glauber dynamics for the one-dimensional ising model,” *Russian Mathematical Surveys*, vol. 49, no. 6, p. 210, 1994.

[65] J. Glimm and A. Jaffe, *Quantum physics: a functional integral point of view*. Springer, 1987.

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