Almost all of the nontrivial zeros of the Riemann zeta-function are on the critical line

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\begin{align*}
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\end{align*}

Abstract

Applying Littlewood’s lemma in connection to Riemann’s Hypothesis and exploiting the symmetry of Riemann’s \( \xi \) function we show that almost all nontrivial Riemann’s Zeta zeros are on the critical line.

1 Introduction

In his only paper devoted to the number theory published in 1859 \[24\] (it was also included as an appendix in \[9\]) Bernhard Riemann continued analytically the series

\[ \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it, \quad \sigma > 1 \]  

(1)

to the complex plane with exception of \( s = 1 \), where the above series is a harmonic divergent series. He has done it using the integral

\[ \zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^s \, dz}{e^z - 1} , \]  

(2)

where the contour \( C \) is

\[ \text{C} \]

The definition of \((-z)^s\) is \((-z)^s = e^{s \log(-z)}\), where the definition of \( \log(-z) \) conforms to the usual definition of \( \log(z) \) for \( z \) not on the negative real axis as the branch which is real for positive real \( z \), thus \((-z)^s\) is not defined on the positive real axis,
see [9, p.10]. Appearing in (2) the gamma function $\Gamma(z)$ has many representations, we present the Weierstrass product:

$$\Gamma(z) = \frac{e^{-Cz}}{z} \prod_{k=1}^{\infty} \frac{e^{z/k}}{(1 + \frac{z}{k})}.$$  \hspace{1cm} (3)

Here $C$ is the Euler–Mascheroni constant

$$C = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log(n) \right) = 0.577216 \ldots \hspace{1cm} (4)$$

From (3) it is seen that $\Gamma(z)$ is defined for all complex numbers $z$, except $z = -n$ for integer $n > 0$, where are the simple poles of $\Gamma(z)$. The most popular definition of the gamma function given by the integral $\Gamma(z) = \int_{0}^{\infty} e^{-t} t^{z-1} dt$ is valid only for $\Re[z] > 0$. Recently perhaps over 100 representations of $\zeta(s)$ are known, for review of the integral and series representations see [22].

The function $\zeta(s)$ has two kinds of zeros: trivial zeros at $s = -2n, \ n = 1, 2, 3, \ldots$ and nontrivial zeros in the critical strip $0 < \Re[s] < 1$. In [24] Riemann made the assumption, now called the Riemann Hypothesis (RH for short in following), that all nontrivial zeros $\rho_n$ lie on the critical line $\Re[s] = \frac{1}{2}$: $\rho_n = \frac{1}{2} + i\gamma_n$. Contemporary the above requirement is augmented by the demand that all nontrivial zeros are simple. Riemann has shown that $\zeta(s)$ fulfills the functional identity:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s), \quad \text{for } s \in \mathbb{C} \setminus \{0, 1\}. \hspace{1cm} (5)$$

The above form of the functional equation is explicitly symmetrical with respect to the line $\Re(s) = 1/2$: the change $s \to 1-s$ on both sides of (5) shows that the values of the combination of functions $\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ are the same at points $s$ and $s-1$. Thus it is convenient to introduce the function

$$\xi(s) = \frac{1}{2} s(s-1) \Gamma\left(\frac{s}{2}\right) \zeta(s). \hspace{1cm} (6)$$

Then the functional identity takes the simple form:

$$\xi(1-s) = \xi(s) \hspace{1cm} (7)$$

The fact that $\zeta(s) \neq 0$ for $\Re(s) > 1$ and the form of the functional identity entails that nontrivial zeros $\rho_n = \beta_n + i\gamma_n$ are located in the critical strip:

$$0 \leq \Re[\rho_n] = \beta_n \leq 1.$$
1 − ρ_n = 1 − β_n − iγ_n and 1 − ̅ρ_n = 1 − β_n + iγ_n are also zeros: they are located symmetrically around the straight line \( \Re [s] = \frac{1}{2} \) and the axis \( t = 0 \), see Fig. 1.

The classical (from XX century) references on the RH are [30], [9], [15], [19]. In the XXI century there appeared two monographs about the zeta function: [4] and [5].

There was a lot of attempts to prove RH and the common opinion was that it is true. However let us notice that there were famous mathematicians: J. E. Littlewood [6, p.345], [12, p. 390], P. Turan and A.M. Turing [3, p.1209] M. Huxley [8, p. 357] believing that the RH is not true, see also the paper “On some reasons for doubting the Riemann hypothesis” [16] (reprinted in [4, p.137]) written by A. Ivić. In Karatsuba’s talk [18] at 1:01:10 he mentions that Atle Selberg had serious doubts whether RH is true or not. New arguments against RH can be found in [2]. When J. Derbyshire asked A. Odlyzko about his opinion on the validity of RH he

\[ s = \sigma + it \]

**Figure 1:** The location of zeros of the Riemann \( \zeta(s) \) function.

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1 We thank A. Kourbatov for bringing this fact to our attention.
replied “Either it’s true, or else it isn’t” [8, p. 357–358]. There were some attempts to prove RH using the physical methods, see [25] or [32].

In [29, p. 81] we read: “Hilbert said that if he could rise from the dead in 200 years, his first thought would not be to ask what social or technological progress there had been, but what had been discovered about the zeros of the zeta function \( \zeta \) because that is not only the most interesting unanswered mathematical question, but the most interesting of all questions...”

2 Zeta’s zeros on the critical line

In 1914 G.H. Hardy [13] (reprinted in [4]) proved first result in favor of the RH: there are infinitely many zeros of \( \zeta(s) \) on the critical line \( \frac{1}{2} + i\gamma_n \) with imaginary part \( 0 < \gamma_n < T \). In 1921 Hardy and Littlewood in [14] proved that \( N_0(T) > \text{const} \cdot T \) for large \( T \). Let \( N(T) \) denotes the number of zeta zeros \( \rho_n = \beta + i\gamma_n \) in the critical strip up to \( T \), i.e. the number of zeta zeros in the rectangle \( 0 < \beta < 1, \ 0 < \gamma_n < T \). In 1942 A. Selberg in [27] proved that \( N_0(T) > \text{const} \cdot T \log \log \log(T) \) for large \( T \) and in [26] he improved it to \( N_0(T) > \text{const} \cdot T \log(T) \) for large \( T \) (these two papers are reprinted in [28]). Norman Levinson proved [20] that more than one-third of zeros of Riemann’s \( \zeta(s) \) are on critical line \( N_0(T) > N(T)/3 \) by relating the zeros of the zeta function to those of its derivative. Later Levinson [21] improved the proportion of zeros on the critical line to 0.3474. Brian Conrey in 1989 improved this further to two-fifths (precisely 40.77 %) [7]. Next S. Feng proved that at least 41.28% of the zeros of the Riemann zeta function are on the critical line [10]. The present record seems to belong to K. Pratt et. al. who proved that at least \( \frac{5}{12} = 0.41666 \ldots \) of the zeros of the Riemann zeta function are on the critical line [23].

In this paper we are going to apply the Littlewood’s Lemma to show that almost all zeta zeros are on the critical line:

**Theorem:** Almost all zeros \( \rho_n = \beta_n + i\gamma_n \) of the \( \zeta(s) \) function have \( \beta_n = \frac{1}{2} \). Here by “almost all” we mean, that

\[
\frac{N_0(T)}{N(T)} = 1 + \mathcal{O}\left(\frac{1}{T \log(T)}\right),
\]

where \( T \) is not an imaginary part of the nontrivial zeta’s zero.

3 Proof of the Theorem

We will use the Littlewood’s Lemma (see e.g. [17, Chap.21]):
**Littlewood’s Lemma:** Let $F(s)$ be holomorphic function inside rectangle $D$ with sides parallel to axes not vanishing on the boundary $\partial D$ and let $\text{dist}(\rho)$ denotes the distance of the zero $\rho$ of $F(s)$, i.e. $F(\rho) = 0$, to the left side of $D$. Then

$$\sum_{\rho \in D} \text{dist}(\rho) = \frac{1}{2\pi} \oint_{\partial D} \log(F(s)) ds, \quad s = \sigma + it$$

(9)

For the function $F(s)$ we will substitute the Riemann’s $\xi(s)$ function defined in (6) which has the same zeros as $\zeta(s)$:

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

(10)

In addition to functional equation (7) it fulfils:

$$\overline{\xi(s)} = \xi(\overline{s}).$$

(11)

We have

$$\log(\xi(s)) = \log |\xi(s)| + i \arg(\xi(s)).$$

(12)

Let $0 < \alpha < \frac{1}{2}$ be a real number. We consider the rectangle $D(\alpha, T)$ with vertices $A = (1 - \alpha, iT), B = (\alpha, iT), C = (\alpha, -iT), D = (1 - \alpha, -iT)$, see Fig.2 and we will look for zeros of $\xi(s)$ which are the same as zeros of $\zeta(s)$. We have in this case (see .eq(21) in [17]):

$$\sum_{\rho \in D(\alpha)} \text{dist}(\rho) = \frac{1}{2\pi} \int_{-T}^{T} \left( \log(|\xi(\alpha + it)|) - \log(|\xi(1 - \alpha + it)|) \right) dt$$

$$+ \frac{1}{2\pi} \int_{\alpha}^{1-\alpha} \left( \arg(\xi(\sigma + iT)) - \arg(\xi(\sigma - iT)) \right) d\sigma,$$

(13)

where the argument is defined by continuous variation starting with any fixed value at a chosen point.

From functional identity (7) and (11) we have

$$|\xi(\alpha + it)| = |\xi(1 - \alpha - it)| = |\xi(1 - \alpha + it)|$$

(14)

thus the first integral in (13) vanishes. Next, we know that

$$\arg(\xi(\sigma - iT)) = - \arg(\xi(\sigma + iT))$$

(15)

and it follows

$$\sum_{\rho \in D(\alpha)} \text{dist}(\rho) = \frac{1}{\pi} \int_{\alpha}^{1-\alpha} \arg(\xi(\sigma + iT)) d\sigma$$

(16)

First we will calculate rhs of (16). We have:

$$\arg(\xi(s)) = \arg\left(\frac{1}{2} s(s-1)\right) + \arg\left(\pi^{-s/2}\right) + \arg\left(\Gamma\left(\frac{s}{2}\right)\right) + \arg(\zeta(s))$$

(17)
From the proviso in (20) we have to restrict $s$ to principal branch, thus argument for $s = \sigma + iT$ is close to $\pi/2$ because $\arg(s)$ is a little bit less than $\pi/2$ and $\arg(s - 1)$ is a little bit more than $\pi/2$ thus we write

$$\arg(s(s - 1)) = \pi + o(1).$$

Hence we have

$$\int_{\alpha}^{1-\alpha} \arg \left( \frac{1}{2} s(s - 1) \right) d\sigma = (1 - 2\alpha) (\pi + o(1)).$$

From Stirling formula, see e.g. [1, eq. 6.1.37 and eq. (6.1.41)] we have

$$\log(\Gamma(z)) = \left( z - \frac{1}{2} \right) \log(z) - z + \frac{1}{2} \log(2\pi) + \frac{1}{12z} + \ldots, \quad \arg(z) < \pi$$

and in our case:

$$\log \left( \Gamma \left( \frac{s}{2} \right) \right) = \left( \frac{\sigma}{2} - \frac{1}{2} + \frac{iT}{2} \right) \left( \log \left( \frac{\sigma^2}{4} + \frac{T^2}{4} \right) \right)^{\frac{1}{2}} + i \arg \left( \frac{\sigma}{2} + \frac{iT}{2} \right)$$

$$- \left( \frac{\sigma}{2} + \frac{iT}{2} \right) + \frac{1}{2} \log(2\pi) + o(1)$$

Hence

$$\arg \left( \Gamma \left( \frac{s}{2} \right) \right) = \text{Im} \left( \log \left( \Gamma \left( \frac{s}{2} \right) \right) \right) =$$

$$\frac{T}{4} \log \left( \frac{\sigma^2}{4} + \frac{T^2}{4} \right) - \frac{T}{2} + \left( \frac{\sigma}{2} - \frac{1}{2} \right) \arg \left( \frac{\sigma}{2} + \frac{iT}{2} \right)$$

For the last term we can write for very large $T$

$$\left( \frac{\sigma}{2} - \frac{1}{2} \right) \arg \left( \frac{\sigma}{2} + \frac{iT}{2} \right) = \left( \frac{\sigma}{2} - \frac{1}{2} \right) \left( \frac{\pi}{2} + o(1) \right)$$

For very large $T$ we can skip terms with $\alpha$ under logarithms:

$$T \log \left( \frac{\sigma^2}{4} + \frac{T^2}{4} \right) = T \log \left( \frac{T^2}{4} \left( 1 + \frac{\sigma^2}{T^2} \right) \right) =$$

$$2T \log \left( \frac{T}{2} \right) + T \log \left( 1 + \frac{\sigma^2}{T^2} \right) = 2T \log \left( \frac{T}{2} \right) + O \left( \frac{1}{T} \right).$$

Further we have

$$\arg(\pi^{-(\sigma+iT)/2}) = -\frac{T}{2} \log(\pi)$$

Together from above equations we have:

$$\arg(\xi(\sigma + iT)) = \frac{T}{2} \log \left( \frac{\sigma^2}{4} + \frac{T^2}{4} \right) + C(\sigma) - \frac{T}{2} (1 + \log(\pi)) + \arg(\zeta(\sigma + iT)) + o(1).$$
Above $C(\sigma)$ absorbs constants:

$$C(\sigma) = \pi + \frac{\pi}{2} \left( \frac{\sigma}{2} - \frac{1}{2} \right) + o(1) = \frac{3\pi}{4} + \frac{\pi}{4} \sigma + o(1) \quad (27)$$

Figure 2. Rectangle $\mathcal{D}(\alpha, T)$, in red critical line is plotted.

Sides $AB$ and $CD$ have length $1 - 2\alpha$ while sides $CB$ and $DA$ have length $2T$.

Integrating by parts we obtain

$$\int \log \left( \frac{\sigma^2 + T^2}{4} \right) d\sigma = \sigma \log \left( \frac{\sigma^2 + T^2}{4} \right) - 2\sigma + 2T \arctan \left( \frac{\sigma}{T} \right) + \text{constant} \quad (28)$$
and it gives
\begin{equation}
\int_{\alpha}^{1-\alpha} \arg(\xi(\sigma + iT))d\sigma = (1 - \alpha) \frac{T}{4} \left( \log \left( \frac{(1 - \alpha)^2}{4} + \frac{T^2}{4} \right) - 2 \right) - \alpha \frac{T}{4} \left( \log \left( \frac{\alpha^2}{4} + \frac{T^2}{4} \right) - 2 \right) + \frac{T^2}{2} \left( \arctan \left( \frac{1 - \alpha}{T} \right) - \arctan \left( \frac{\alpha}{T} \right) \right)
- (1 - 2\alpha) \frac{T}{2} (1 + \log(\pi)) + C_2 + \int_{\alpha}^{1-\alpha} \arg(\zeta(\sigma + iT))d\sigma + \mathcal{O}(1) + C_2
\end{equation}

where
\begin{equation}
C_2 = \int_{\alpha}^{1-\alpha} C(\sigma)d\sigma = (1 - 2\alpha) \left( \frac{7\pi}{8} + o(1) \right)
\end{equation}

Using the Maclaurin series expansion of \( \arctan(x) \) we find that for large \( T \)
\begin{equation}
T \left( \arctan \left( \frac{1 - \alpha}{T} \right) - \arctan \left( \frac{\alpha}{T} \right) \right) = 1 - 2\alpha + \mathcal{O} \left( \frac{1}{T} \right).
\end{equation}

Adding (19) and (25) gives us finally that the rhs of (16) is equal to
\begin{equation}
\frac{1}{\pi} \int_{\alpha}^{1-\alpha} \left\{ \arg \left( \frac{1}{2}s(s - 1) \right) + \arg \left( \pi^{-s/2} \right) + \arg \left( \Gamma \left( \frac{s}{2} \right) \right) + \arg(\zeta(s)) \right\}d\sigma
= (1 - 2\alpha) \left( \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} + \frac{C_2}{\pi} + o(1) \right) + \frac{1}{\pi} \int_{\alpha}^{1-\alpha} \arg(\zeta(s))d\sigma.
\end{equation}

We give estimate for the integral of the argument of \( \zeta(s) \). The mean value theorem for definite integrals: Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Then there exists \( c \in (a, b) \) such that
\begin{equation}
\int_{a}^{b} f(x) \, dx = f(c)(b - a).
\end{equation}

Applying it to \( \arg(\zeta(\sigma + iT)) \), when \( T \) is not equal to imaginary part of nontrivial zero (\( \arg(\zeta(s)) \) is not defined for \( s \) equal to zero of zeta) we have:
\begin{equation}
\int_{\alpha}^{1-\alpha} \arg(\zeta(\sigma + iT))d\sigma = (1 - 2\alpha) \arg(\zeta(\sigma_0 + iT))
\end{equation}

for some \( \sigma_0 \in (\alpha, 1 - \alpha) \). Letting \( \alpha \to \frac{1}{2} \) we have
\begin{equation}
\lim_{\alpha \to \frac{1}{2}} \frac{1}{1 - 2\alpha} \int_{\alpha}^{1-\alpha} \arg(\zeta(\sigma + iT))d\sigma = \arg \left( \zeta \left( \frac{1}{2} + iT \right) \right)
\end{equation}

Thus finally we obtain for \( \alpha \) close to \( \frac{1}{2} \)
\begin{equation}
\frac{1}{\pi} \int_{\alpha}^{1-\alpha} \arg(\xi(\sigma + iT))d\sigma =
\end{equation}
\[
= (1 - 2\alpha) \left\{ \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} + \frac{7}{8} + \frac{1}{\pi} \arg \left( \zeta \left( \frac{1}{2} + iT \right) \right) \right\} + o(1). \tag{36}
\]

From the Riemann–von Mangoldt formula we have that the number of zeta zeros \(N(T)\) with positive imaginary parts \(< T\) (and real part inside critical strip) is (see eq.(2.3.6) in [1])

\[
N(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} + \frac{7}{8} + \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + iT \right) + \mathcal{O}(T^{-1}). \tag{37}
\]

The term 7/8 was already known to von Mangoldt, see [31, p.10 eq.(8)]. Comparing (36) and (37) we obtain

\[
\frac{1}{\pi} \int_{\alpha}^{1-\alpha} \arg(\xi(\sigma + iT))d\sigma = (1 - 2\alpha) (N(T) + o(1)) \tag{38}
\]

Now we will focus on the lhs of (16). This needs information about specific location of zeros. Because there are only finite number of zeta zeros inside rectangle \(ABCD\), we can choose \(\alpha\) so close to \(\frac{1}{2}\) that zeros off critical line be outside rectangle \(D(\alpha,T)\), i.e. our symmetric rectangle closely hugs the critical line that inside are only zeros on \(\Re(s) = \frac{1}{2}\). Then we have

\[
\sum_{\rho \in D(\alpha,T)} \text{dist}(\rho) = \left( \frac{1}{2} - \alpha \right) 2N_0(T). \tag{39}
\]

Comparing (38) and (39) we can cancel \(1 - 2\alpha \neq 0\) on both sides what gives:

\[
N_0(T) = N(T) + o(1) \tag{40}
\]

and we obtain (8). \(\square\)

4 Comments

The von Mangoldt formula (37) was obtained from the Argument Principle. This principle gives the total number of zeros in some region and it needs the information on the change of the argument. The Littlewood lemma is more powerful: it involves specific locations of zeros and the integrals of the argument.

Our method is based on the symmetry relations satisfied by the \(\xi(s)\) function (in relation to Littlewood’s lemma), which drastically simplify the calculations. The other two major ideas in our work are, first, the use of the mean value theorems for integrals, (and second) in combination with that limit process when the rectangle hugs the critical line together with the behavior of various terms in our calculations.
under this process. So there are about three fundamental ideas that we use in our work. We rely more on symmetry and the consequences of a limit process, rather than the computational techniques involving smoothing functions (mollifiers), like Conrey and Levinson employ. When we apply Littlewood’s lemma, we know that there exists an analytic branch of $\log(\xi)$ on the simply connected domain represented by our rectangle without the horizontal branch cuts connecting the left side of our rectangle with $\zeta$ zeros inside our rectangle. For any branch of the logarithm, in other words for any branch of the argument, the conclusions of our calculations are not significantly affected.

Because in (8) in the limit $T \to \infty$ there will be equality $N_0(T) = N(T)$ we propose

**Conjecture:** There exists such a constant $Y > 0$ that if $\beta + i\gamma$ is a zero of $\zeta(s)$ and $|\gamma| > Y$ then $\beta = \frac{1}{2}$.

If RH is true then the above Conjecture is empty, i.e. $Y = 0$. In the paper [11] I.J. Good and R.F. Churchhouse gave the arguments that RH is true with probability 1. Above we have showed that almost 100% of the $\zeta(s)$ are on the critical line.

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