Limit theorems for eigenvectors of the normalized Laplacian for random graphs

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Abstract

We prove a central limit theorem for the components of the eigenvectors corresponding to the $d$ largest eigenvalues of the normalized Laplacian matrix of a finite dimensional random dot product graph. As a corollary, we show that for stochastic blockmodel graphs, the rows of the spectral embedding of the normalized Laplacian converge to multivariate normals and furthermore the mean and the covariance matrix of each row are functions of the associated vertex’s block membership. Together with prior results for the eigenvectors of the adjacency matrix, we then compare, via the Chernoff information between multivariate normal distributions, how the choice of embedding method impacts subsequent inference. We demonstrate that neither embedding method dominates with respect to the inference task of recovering the latent block assignments.

1 Introduction

Statistical inference on graphs is a burgeoning field of research in machine learning and statistics, with numerous applications to social network, neuroscience, etc. Many statistical inference procedures for graphs involve a preprocessing step of finding a representation of the vertices as points in some low-dimensional Euclidean space. This representation is usually given by the truncated eigen-decomposition of the adjacency matrix or related matrices such as the combinatorial Laplacian or the normalized Laplacian. For example, given a point cloud lying in some purported low-dimensional manifold in a high-dimensional ambient space, many manifold learning or non-linear dimension reduction algorithms such as Laplacian eigenmaps [8] and diffusion maps [15] use the eigenvectors of the normalized Laplacian constructed from a neighborhood graph of the points as a low-dimensional Euclidean representation of the point cloud before performing inference such as clustering or classification. Spectral clustering
algorithms such as the normalized cuts algorithm [35] proceed by embedding a
graph into a low-dimensional Euclidean space followed by running $K$-means on
the embedding to obtain a partitioning of the vertices. Some network compar-
ison procedures embed the graphs and then compute a kernel-based distance
measure between the resulting point clouds [3, 41].

The choice of the matrix used in the embedding step and its effect on subsequent
inference is, however, rarely addressed in the literature. In a recent pioneering
work, the authors of [6] addressed this issue by analyzing, in the context of
stochastic blockmodel graphs where the subsequent inference task is the recovery
of the block assignments, a metric given by the average distance between the
vertices of a block and its cluster centroid for the spectral embedding of the
adjacency matrix and the normalized Laplacian matrix. The metric is then
used as a surrogate measure for the performance of the subsequent inference
task, i.e., the metric is a surrogate measure for the error rate in recovering
the vertices to block assignments. The stochastic blockmodel [20] is a popular
generative model for random graphs with latent community structure and many
results are known regarding consistent recovery of the block assignments; see
for example [7, 13, 23, 27, 28, 30, 34, 36, 39] and the references therein.

It was shown in [6] that for two-block stochastic blockmodels, for a large regime
of parameters the normalized Laplacian spectral embedding reduces the within-
block variance (occasionally by a factor of four) while preserving the between-
block variance, as compared to that of the adjacency spectral embedding. This
suggests that for a large region of the parameters space for two-block stochastic
blockmodels, the spectral embedding of the Laplacian is to be preferred over
that of the adjacency matrix for subsequent inference. However, we observed
that the metric in [6] is intrinsically tied to the use of $K$-means as the clustering
procedure, i.e., a smaller value of the metric for the Laplacian spectral embed-
ding as compared to that for the adjacency spectral embedding only implies that
clustering the Laplacian spectral embedding using $K$-means is possibly better
than clustering the adjacency spectral embedding using $K$-means.

Motivated by the above observation, one main goal of this paper is to propose
a metric that is independent of any specific clustering procedure, i.e., a metric
that characterizes the minimum error achievable by any clustering procedure
that uses only the spectral embedding, for the recovery of block assignments
in stochastic blockmodel graphs. We achieve this by establishing distributional
limit results for the eigenvectors corresponding to the few largest eigenvalues of
the adjacency or Laplacian matrix and then characterizing, through the notion
of statistical information, the distributional differences between the blocks for
either embedding method. Roughly speaking, smaller statistical information
implies less information to discriminate between the blocks of the stochastic
blockmodel.

More specifically, the limit result in [4] states that, for stochastic blockmodel
graphs, conditional on the block assignments the scaled eigenvectors correspond-
ning to the few largest eigenvalues of the adjacency matrix converge to a multi-
variate normal (see e.g., Theorem 2.2) as the number of vertices increases. Furthermore, the associated covariance matrix is not necessarily spherical and hence $K$-means clustering for the adjacency spectral embedding does not always yield minimum error for recovering the block assignment. Analogous limit results (see e.g., Theorem 3.2) for the eigenvectors of the normalized Laplacian matrix then facilitate comparison between the two embedding methods via the classical notion of Chernoff information [11]. The Chernoff information is a supremum of the Chernoff $\alpha$-divergences for $\alpha \in (0, 1)$ and characterizes the error rate of the Bayes decision rule in hypothesis testing; the Chernoff $\alpha$-divergence is an example of a $f$-divergence [1, 16] and it satisfies the information processing lemma and is invariant with respect to invertible transformations [24].

Our paper is thus structured as follows. We recall in Section 2 the definition of random dot product graphs, stochastic blockmodel graphs, and spectral embedding of the adjacency and Laplacian matrices. We then state in Section 2.1 several limit results for the eigenvectors of the adjacency spectral embedding. These results are generalizations of results from [4, 40]. The main technical contribution of this paper, namely analogous limit results for the eigenvectors of the Laplacian spectral embedding, are then given in Section 3. We then discuss the implications of these limit results in Section 4; in particular Section 4.3 characterizes, via the notion of Chernoff statistical information, the large-sample optimal error rate of spectral clustering procedures. We demonstrate that neither embedding method dominates for the inference task of recovering block assignments in stochastic blockmodels. We conclude the paper with some brief remarks on potential extensions of the results presented herein. Proofs of stated results are given in the appendix.

2 Background and Setting

We first recall the notion of a random dot product graph [31].

**Definition 1.** Let $F$ be a distribution on a set $X \subseteq \mathbb{R}^d$ satisfying $x^\top y \in [0, 1]$ for all $x, y \in X$. We say $(X, A) \sim \text{RDPG}(F)$ with sparsity factor $\rho_n \leq 1$ if the following hold. Let $X_1, \ldots, X_n \sim F$ be independent random variables and define

$$X = [X_1 \mid \cdots \mid X_n]^\top \in \mathbb{R}^{n \times d} \text{ and } P = \rho_n XX^\top \in [0, 1]^{n \times n}. \quad (2.1)$$

The $X_i$ are the latent positions for the random graph, i.e., we do not observe $X$, rather we observe only the matrix $A$. The matrix $A \in \{0, 1\}^{n \times n}$ is defined to be symmetric with all zeroes on the diagonal such that for all $i < j$, conditioned on $X_i, X_j$ the $A_{ij}$ are independent and

$$A_{ij} \sim \text{Bernoulli}(\rho_n X_i^\top X_j), \quad (2.2)$$

namely,

$$P[A \mid X] = \prod_{i<j} (\rho_n X_i^\top X_j)^{A_{ij}} (1 - \rho_n X_i^\top X_j)^{(1 - A_{ij})}. \quad (2.3)$$
Remark. We note that non-identifiability is an intrinsic property of random dot product graphs. More specifically, if \((X, A) \sim \text{RDPG}(F)\) where \(F\) is a distribution on \(\mathbb{R}^d\), then for any orthogonal transformation \(U\), \((Y, B) \sim \text{RDPG}(F \circ U)\) is identically distributed to \((X, A)\); we write \(F \circ U\) to denote the distribution of \(Y = UX\) whenever \(X \sim F\). Furthermore, there also exists a distribution \(F'\) on \(\mathbb{R}^d\) with \(d' > d\) such that \((Y, B) \sim \text{RDPG}(F')\) is identically distributed to \((X, A)\). Non-identifiability due to orthogonal transformations cannot be avoided given the observed \(A\). We avoid the other source of non-identifiability by assuming throughout this paper that if \((X, A) \sim \text{RDPG}(F)\) then \(F\) is non-degenerate, i.e., \(\mathbb{E}[XX^\top]\) is of full rank.

As an example of random dot product graphs, we could take \(X\) to be the unit simplex in \(\mathbb{R}^d\) and let \(F\) be a mixture of Dirichlet distributions or logistic-normal distribution. Random dot product graphs are a specific example of latent position graphs or inhomogeneous random graphs \([8, 19]\), in which each vertex is associated with a latent position \(\kappa\). A random dot product graph on \(n\) vertices is also, when viewed as an induced subgraph of an infinite graph, an exchangeable random graph \([17]\). Random dot product graphs are related to stochastic block model graphs \([20]\) and degree-corrected stochastic block model graphs \([21]\); for example, a stochastic blockmodel graph on \(K\) blocks with a positive semidefinite block probability matrix \(B\) corresponds to a random dot product graph where \(F\) is a mixture of \(K\) point masses.

For a given matrix \(M\) with non-negative entries, denote by \(L(M)\) the normalized Laplacian of \(M\) defined as

\[
L(M) = (\text{diag}(M1))^{-1/2}M(\text{diag}(M1))^{-1/2}
\]

where, given \(z = (z_1, \ldots, z_n) \in \mathbb{R}^n\), \(\text{diag}(z)\) is the \(n \times n\) diagonal matrix whose diagonal entries are the \(z_i\)'s. Our definition of the normalized Laplacian is slightly different from that often found in the literature, e.g., in \([14, 35]\) the normalized Laplacian is \(I - L(M)\). For the purpose of this paper, namely the notion of the Laplacian spectral embedding via the eigenvalues and eigenvectors of the normalized Laplacian, these two definitions of the normalized Laplacian are equivalent. We shall henceforth refer to \(L(M)\) as the Laplacian of \(M\), in contrast to the *combinatorial* Laplacian \(\text{diag}(M1) - M\) of \(M\). See \([29]\) for a survey of the combinatorial Laplacian and its connection to graph theory.

**Definition 2** (Adjacency and Laplacian spectral embedding). Let \(A\) be a \(n \times n\) adjacency matrix. Suppose the eigendecomposition of \(A\) is given by \(A = \sum_{i=1}^{n} \lambda_i u_i u_i^\top\) where \(|\lambda_1| \geq |\lambda_2| \geq \ldots\) are the eigenvalues and \(u_1, u_2, \ldots, u_n\) are the corresponding orthonormal eigenvectors. Given a positive integer \(d \leq n\), denote by \(S_A = \text{diag}(|\lambda_1|, \ldots, |\lambda_d|)\) the diagonal matrix whose diagonal entries are the \(|\lambda_1|, \ldots, |\lambda_d|\), and denote by \(U_A\) the \(n \times d\) matrix whose columns are
the corresponding eigenvectors $u_1, \ldots, u_d$. The adjacency spectral embedding (ASE) of $A$ into $\mathbb{R}^d$ is then the $n \times d$ matrix $\hat{X} = \hat{U}_A \hat{S}_A^{1/2}$. Similarly, let $L(A)$ denote the normalized Laplacian of $A$ and suppose the eigendecomposition of $L(A)$ is given by $L(A) = \sum_{i=1}^{n} \lambda_i \hat{u}_i \hat{u}_i^\top$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ are the eigenvalues and $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n$ are the corresponding orthonormal eigenvectors.

Then given a positive integer $d \leq n$, denote by $\hat{S}_A = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$ the diagonal matrix whose diagonal entries are the $\lambda_1, \ldots, \lambda_d$ and denote by $\hat{U}_A$ the $n \times d$ matrix whose columns are the eigenvectors $\hat{u}_1, \ldots, \hat{u}_d$. The Laplacian spectral embedding of $A$ into $\mathbb{R}^d$ is then the $n \times d$ matrix $\hat{X} = \hat{U}_A \hat{S}_A^{1/2}$.

**Remark.** Let $(X, A) \sim \text{RDPG}(F)$ with sparsity factor $\rho_n$ and suppose that the $d \times d$ matrix $E[XX^\top]$ is of full-rank where $X \sim F$. The $n \times d$ matrix $\hat{X}$, the adjacency spectral embedding $\hat{X}$ of $A$ into $\mathbb{R}^d$, can then be viewed as a consistent estimate of $\rho_n^{1/2}X$. See [38] for a comprehensive overview of the consistency results and their implications for subsequent inference. On the other hand, as $L(cM) = L(M)$ for any constant $c > 0$, the $n \times d$ matrix $\hat{X}$ – the normalized Laplacian embedding of $A$ into $\mathbb{R}^d$ – can be viewed as a consistent estimate of $(\rho_n \operatorname{diag}(XX^\top 1))^{-1/2} \rho_n^{1/2}X$ which does not depend on the sparsity factor $\rho_n$. This is in contrast to the adjacency spectral embedding. For previous consistency results of $\hat{X}$ as an estimator for $X$ in various random graphs models, the reader is referred to [33, 34, 42] among others. However, to the best of our knowledge, Theorem 3.2 – namely the distributional convergence of $\hat{X}$ to a mixture of multivariate normals in the context of random dot product graphs and stochastic blockmodel graphs – had not been established prior to this paper.

Finally, we remark that $X$ and $\hat{X}$ are estimating quantities that, while closely related – $X$ and $(\operatorname{diag}(XX^\top 1))^{-1/2} \rho_n^{1/2}X$ are one-to-one transformations of each other – are in essence distinct “parametrizations” of random dot product graphs. It is therefore not entirely straightforward to facilitate a direct comparison of the “efficiency” of $X$ and $\hat{X}$ as estimators. This thus motivates our consideration of the $f$-divergences between the multivariate normals since the family of $f$-divergences satisfy the information processing lemma and are invariant with respect to invertible transformations.

**Remark.** For simplicity we shall assume henceforth that either $\rho_n = 1$ for all $n$, or that $\rho_n \to 0$ with $n\rho_n = \omega(\log^4 n)$. We note that for our purpose, namely the distributional limit results in Section 2.1 and Section 3, the assumption that $\rho_n = 1$ for all $n$ is equivalent to the assumption that there exists a constant $c > 0$ such that $\rho_n \to c$. The assumption that $n\rho_n = \omega(\log^4 n)$ is so that we can apply the concentration inequalities from [25] to show concentration, in spectral norm, of $A$ and $L(A)$ around $\rho_n XX^\top$ and $L(XX^\top)$, respectively.

### 2.1 Limit results for the adjacency spectral embedding

We now recall several limit results for $\hat{X} - X$. These results are restatements of earlier results from [4] and [40]. Theorem 2.2 as stated below is a slight
generalization of Theorem 1 in [4]; the result in [4] assumed a more restrictive
distinct eigenvalues assumption for the matrix \(E[XX^\top]\) where \(X \sim F\). We shall
assume throughout this paper that \(d\), the rank of \(E[XX^\top]\) where \(X \sim F\), is
fixed and known a priori.

**Remark.** For ease of exposition, many of the bounds in this paper are said to
hold “with high probability”. We say that a random variable \(\xi \in \mathbb{R}\) is \(O_p(f(n))\)
if, for any positive constant \(c > 0\) there exists a \(n_0\) and a constant \(C > 0\) (both
of which possibly depend on \(c\)) such that for all \(n \geq n_0\), \(|\xi| \leq Cf(n)\) with
probability at least \(1 - n^{-c}\); in addition, we say that a random variable \(\xi \in \mathbb{R}\)
is \(o_p(f(n))\) if for any positive constant \(c > 0\) and any \(\epsilon > 0\) there exists a \(n_0\)
such that for all \(n \geq n_0\), \(|\xi| \leq \epsilon f(n)\) with probability at least \(1 - n^{-c}\).
Similarly, when \(\xi\) is a random vector in \(\mathbb{R}^d\) or a random matrix in \(\mathbb{R}^{d_1 \times d_2}\), \(\xi = O_p(f(n))\)
or \(\xi = o_p(f(n))\) if \(||\xi|| = O_p(f(n))\) or \(||\xi|| = o_p(f(n))\), respectively. Here \(||x||\)
denotes the Euclidean norm of \(x\) when \(x\) is a vector and the spectral norm
of \(x\) when \(x\) is a matrix. We write \(\xi = \zeta + O_p(f(n))\) or \(\xi = \zeta + o_p(f(n))\) if
\(\xi - \zeta = O_p(f(n))\) or \(\xi - \zeta = o_p(f(n))\), respectively.

**Theorem 2.1.** Let \((X_n, A_n) \sim \text{RDPG}(F)\) with sparsity factor \(\rho_n\). Then there
exists a \(d \times d\) orthogonal matrix \(W_n\) and a \(n \times d\) matrix \(R_n\) such that
\[
\hat{X}_n W_n - \rho_n^{1/2} X_n = \rho_n^{-1/2} (A_n - P_n)X_n (X_n^\top X_n)^{-1} + R_n.
\] (2.5)

Furthermore, \(||R|| = O_p((\rho_n n)^{-1/2})\). Let \(\mu_F = E[X_1]\) and \(\Delta = E[X_1 X_1^\top]\). If
\(\rho_n = 1\) for all \(n\), then there exists a sequence of orthogonal matrices \(W_n\) such that
\[
\|X_n W_n - X_n\|_F^2 \overset{\text{a.s.}}{\to} \text{tr} \Delta^{-1} \left( E[X_1 X_1^\top (X_1^\top \mu_F - X_1^\top \Delta X_1)] \right) \Delta^{-1}.
\] (2.6)

If, however, \(\rho_n \to 0\) and \(n \rho_n = \omega(\log n)\), then
\[
\|\hat{X}_n W_n - \rho_n^{1/2} X_n\|_F^2 \overset{\text{a.s.}}{\to} \text{tr} \Delta^{-1} \left( E[X_1 X_1^\top (X_1^\top \mu_F)] \right) \Delta^{-1}.
\] (2.7)

**Theorem 2.2.** Assume the setting and notations of Theorem 2.1. Denote by \(\hat{X}_i\) the \(i\)-th row of \(\hat{X}_n\). Let \(\Phi(z, \Sigma)\) denote the cumulative distribution function
for the multivariate normal, with mean zero and covariance matrix \(\Sigma\), evaluated
at \(z\). Also denote by \(\Sigma(x)\) the matrix
\[
\Sigma(x) = \Delta^{-1} E[X_1 X_1^\top (x^\top X_1 - x^\top X_1 x)] \Delta^{-1}.
\]

If \(\rho_n = 1\) for all \(n\), then there exists a sequence of orthogonal matrices \(W_n\) such that
for each fixed index \(i\) and any \(z \in \mathbb{R}^d\),
\[
P\left\{ \sqrt{n}(W_n \hat{X}_i - X_i) \leq z \right\} \overset{\text{d}}{\to} \int \Phi(z, \Sigma(x)) dF(x)
\] (2.8)

That is, the sequence \(\sqrt{n}(W_n \hat{X}_i - X_i)\) converges in distribution to a mixture
of multivariate normals. We denote this mixture by \(\mathcal{N}(0, \tilde{\Sigma}(X_i))\). If, however,
\( \rho_n \to 0 \) and \( n\rho_n = \omega(\log^4 n) \) then there exists a sequence of orthogonal matrices \( W_n \) such that

\[
P\left\{ \sqrt{n}(W_n\hat{X}_i - \rho_n^{1/2}X_i) \leq z \right\} \xrightarrow{d} \int \Phi(z, \Sigma_{o(1)}(x))dF(x) \quad (2.9)
\]

where \( \Sigma_{o(1)}(x) = \Delta^{-1}E[X_1X_1^\top x^\top X_1]\Delta^{-1}. \)

An important corollary of Theorem 2.2 is the following result for when \( F \) is a mixture of \( K \) point masses, i.e., \( (X, A) \sim \text{RDPG}(F) \) is a \( K \)-block stochastic blockmodel graph. Then for any fixed index \( i \), the event that \( X_i \) is assigned to block \( k \in \{1, 2, \ldots, K\} \) has non-zero probability and hence one can conditioned on the block assignment of \( X_i \) to show that the conditional distribution of \( \sqrt{n}(W_n\hat{X}_i - X_i) \) converges to a multivariate normal. This is in contrast to the unconditional distribution being a mixture of multivariate normals as in Eq. (2.8) and Eq. (2.9).

**Corollary 2.3.** Assume the setting and notations of Theorem 2.1 and let

\[
F = \sum_{k=1}^{K} \pi_k \delta_{\nu_k}, \quad \pi_1, \ldots, \pi_K > 0, \sum \pi_k = 1
\]

be a mixture of \( K \) point masses in \( \mathbb{R}^d \) where \( \delta_{\nu_k} \) is the Dirac delta measure at \( \nu_k \). Then if \( \rho_n \equiv 1 \), there exists a sequence of orthogonal matrices \( W_n \) such that for any fixed index \( i \),

\[
P\left\{ \sqrt{n}(W_n\hat{X}_i - X_i) \leq z \mid X_i = \nu_k \right\} \xrightarrow{d} \mathcal{N}(0, \Sigma_k) \quad (2.10)
\]

where \( \Sigma_k = \Sigma(\nu_k) \) is as defined in Eq. (2.8). If \( \rho_n \to 0 \) and \( n\rho_n = \omega(\log^4(n)) \) as \( n \to \infty \), then the sequence of orthogonal matrices \( W_n \) satisfies

\[
P\left\{ \sqrt{n}(W_n\hat{X}_i - \rho_n^{1/2}X_i) \leq z \mid X_i = \nu_k \right\} \xrightarrow{d} \mathcal{N}(0, \Sigma_{o(1),k}) \quad (2.11)
\]

where \( \Sigma_{o(1),k} = \Sigma_{o(1)}(\nu_k) \) is as defined in Eq. (2.9).

### 3 Limit results for Laplacian spectral embedding

We now present the main technical results of this paper, namely analogues of the limit results in Section 2.1 for the Laplacian spectral embedding.

**Theorem 3.1.** Let \( (A_n, X_n) \sim \text{RDPG}(F) \) for \( n \geq 1 \) be a sequence of random dot product graphs with sparsity factors \( (\rho_n)_{n \geq 1} \). Denote by \( D_n \) and \( T_n \) the \( n \times n \) diagonal matrices \( \text{diag}(A_n) \) and \( \text{diag}(\rho_n X_n X_n^\top 1) \), respectively, i.e., the diagonal entries of \( D_n \) are the vertex degrees of \( A_n \) and the diagonal entries of \( T_n \) are
the expected vertex degrees. Let \( \tilde{X}_n = \rho_n^{1/2}T_n^{-1/2}X_n = \text{diag}(X_nX_n^\top)\) where the expectation in Eq. (3.4) is taken with respect to \( X_1 \) and \( X_2 \) being i.i.d. drawn according to \( F \). Equivalently,

\[
\begin{align*}
\frac{n}{F} \tilde{X}_n W_n - \tilde{X}_n \xrightarrow{a.s.} & \ tr \left[ g(X_1, X_2) X_1^\top X_2 - X_1^\top X_2 X_2^\top X_1 \right].
\end{align*}
\]

If \( \rho_n \to 0 \) and \( n \rho_n = o(\log^2 n) \) then the sequence \( (W_n)_{n \geq 1} \) satisfies

\[
\frac{n}{F} \tilde{X}_n W_n - \tilde{X}_n \xrightarrow{a.s.} \ tr \left[ \frac{\tilde{X}_n X_1 X_1^\top X_2}{X_1^\top X_2} - \frac{3}{4} \right].
\]
That is, the sequence \( n(\mathbf{W}_n X_i - \mathbf{X}_i / \sqrt{\sum_j \mathbf{X}_j^\top \mathbf{X}_j}) \) converges in distribution to a mixture of multivariate normals. We denote this mixture by \( \mathcal{N}(0, \tilde{\Sigma}(X_i)) \). If \( \rho_n \to 0 \) and \( n\rho_n = \omega(\log^3 n) \) then there exists a sequence of orthogonal matrices \( \mathbf{W}_n \) such that

\[
\mathbb{P}\left\{ n\rho_n^{1/2}(\mathbf{W}_n X_i - \mathbf{X}_i / \sqrt{\sum_j \mathbf{X}_j^\top \mathbf{X}_j}) \leq z \right\} \xrightarrow{d} \int \Phi(z, \tilde{\Sigma}(x)) dF(x). \tag{3.8}
\]

where \( \tilde{\Sigma}(x) \) is defined by

\[
\tilde{\Sigma}(x) = \mathbb{E}\left[ \left( \frac{\Delta^{-1} \mathbf{X}_i}{\mathbf{X}_i^\top \mu} - \frac{x}{2x^\top \mu} \right) \left( \frac{\Delta^{-1} \mathbf{X}_i}{\mathbf{X}_i^\top \mu} - \frac{x^\top \mathbf{X}_1}{x^\top \mu} \right) \right]. \tag{3.9}
\]

The proofs of Theorem 3.1 and Theorem 3.2 are given in Section B. We end this section by stating the conditional distribution of \( n\rho_n (\mathbf{W}_n X_i - \mathbf{X}_i / \sqrt{\sum_j \mathbf{X}_j^\top \mathbf{X}_j}) \) when \((\mathbf{X}, \mathbf{A}) \sim \text{RDPG}(F)\) is a \( K \)-block stochastic blockmodel graph.

**Corollary 3.3.** Assume the setting and notations of Theorem 3.1 and let

\[
F = \sum_{k=1}^K \pi_k \delta_{\nu_k}, \quad \pi_1, \ldots, \pi_K > 0, \sum_k \pi_k = 1
\]

be a mixture of \( K \) point masses in \( \mathbb{R}^d \). Then if \( \rho_n \equiv 1 \), there exists a sequence of orthogonal matrices \( \mathbf{W}_n \) such that for any fixed index \( i \),

\[
\mathbb{P}\left\{ n\rho_n^{1/2}(\mathbf{W}_n X_i - \mathbf{X}_i / \sqrt{\sum_j \mathbf{X}_j^\top \mathbf{X}_j}) \leq z \mid X_i = \nu_k \right\} \xrightarrow{d} \mathcal{N}(0, \tilde{\Sigma}_k) \tag{3.10}
\]

where \( \tilde{\Sigma}_k = \tilde{\Sigma}(\nu_k) \) is as defined in Eq. (3.6) and \( n_k \) for \( k \in \{1, 2, \ldots, K\} \) denote the number of vertices in \( \mathbf{A}_n \) that are assigned to block \( k \). If instead \( \rho_n \to 0 \) and \( n\rho_n = \omega(\log^3(n)) \) as \( n \to \infty \) then the sequence of orthogonal matrices \( \mathbf{W}_n \) satisfies

\[
\mathbb{P}\left\{ n\rho_n^{1/2}(\mathbf{W}_n X_i - \mathbf{X}_i / \sqrt{\sum_j \mathbf{X}_j^\top \mathbf{X}_j}) \leq z \mid X_i = \nu_k \right\} \xrightarrow{d} \mathcal{N}(0, \tilde{\Sigma}_{o(1), k}) \tag{3.11}
\]

where \( \tilde{\Sigma}_{o(1), k} = \tilde{\Sigma}_{o(1)}(\nu_k) \) is as defined in Eq. (3.9).

**Remark.** As a special case of Corollary 3.3, we have that if \( \mathbf{A} \) is an Erdős-Rényi graph on \( n \) vertices with edge probability \( p^2 \) – which corresponds to a random dot product graph where the latent positions are identically \( p \) – then for each fixed index \( i \), the normalized Laplacian embedding satisfies

\[
n(\hat{\mathbf{X}}_i - \frac{1}{\sqrt{n}}) \xrightarrow{d} \mathcal{N}(0, \frac{1-p^2}{4p^2}),
\]

while the adjacency spectral embedding satisfies

\[
\sqrt{n}(\hat{\mathbf{X}}_i - p) \xrightarrow{d} \mathcal{N}(0, 1 - p^2).
\]
As another example, if $A$ is a stochastic blockmodel graph with block probabilities matrix $B = \begin{bmatrix} p^2 & pq \\ pq & q^2 \end{bmatrix}$ and block assignment probabilities $(\pi, 1 - \pi)$ – which corresponds to a random dot product graph where the latent positions are either $p$ with probability $\pi$ or $q$ with probability $1 - \pi$ – then for each fixed index $i$, the normalized Laplacian embedding satisfies

\[ n(\tilde{X}_i - \frac{p}{\sqrt{n_1 p^2 + n_2 pq}}) \xrightarrow{d} \mathcal{N}(0, \frac{\pi p(1-p^2) + (1-\pi)q(1-q^2)}{4(p^2 + (1-\pi)q^2)}) \] if $X_i = p$,  
\[ n(\tilde{X}_i - \frac{q}{\sqrt{n_1 pq + n_2 q^2}}) \xrightarrow{d} \mathcal{N}(0, \frac{\pi p(1-pq) + (1-\pi)q(1-q^2)}{4(p^2 + (1-\pi)q^2)}) \] if $X_i = q$.  

(3.12)  
(3.13)

where $n_1$ and $n_2 = n - n_1$ are the number of vertices of $A$ with latent positions $p$ and $q$. The adjacency spectral embedding meanwhile satisfies

\[ \sqrt{n}(\tilde{X}_i - p) \xrightarrow{d} \mathcal{N}(0, \frac{\pi p^3(1-p^2)+q^4(1-q^2)}{(p^2 + (1-\pi)q^2)^2}) \] if $X_i = p$,  
\[ \sqrt{n}(\tilde{X}_i - q) \xrightarrow{d} \mathcal{N}(0, \frac{\pi p^3 q(1-pq)+q^4(1-q^2)}{(p^2 + (1-\pi)q^2)^2}) \] if $X_i = q$.  

(3.14)  
(3.15)

Remark. We note that the quantity $n_k$ appears in Eq. (3.7) and Eq. (3.8). Replacing $n_k$ by $n\pi_k$ in Eq. (3.7) and Eq. (3.8) is, however, not straightforward. For example, for the two-block stochastic blockmodel considered in Eq. (3.12), letting $\zeta = \frac{np}{{n_1 p^2 + n_2 pq}} = \frac{np}{{n_1 p^2 + n(1-\pi)pq}}$ we have

\[ \zeta = \frac{np}{n_1 p^2 + n_1 p^2 + n_2 pq} = \frac{np}{n_1 p^2 + n_2 pq} = \frac{np}{n_2 pq} = \frac{np}{n_2 pq - n_1 p^2 - n_2 pq} = \frac{np(n_1 p^2 + n_2 pq - n_1 p^2 - n_2 pq)}{(n_1 p^2 + n_2 pq - n_1 p^2 - n_2 pq)(n_1 p^2 + n_2 pq)} = \frac{np(n_1 p^2 + n_2 pq - n_1 p^2 - n_2 pq)}{(n_1 p^2 + n_2 pq - n_1 p^2 - n_2 pq)(n_1 p^2 + n_2 pq)}.
\]

By the strong law of large numbers and Slutsky’s theorem, we have

\[ \frac{n^{3/2}}{(\sqrt{n_1 p^2 + n(1-\pi)pq} + \sqrt{n_1 p^2 + n_2 pq})} \xrightarrow{a.s.} \frac{1}{2(p^2 + pq)} \rightarrow \mathcal{N}(0, \frac{n\pi - n_1}{\sqrt{n_1 p^2 + n(1-\pi)pq}}) \]

We note that, as the $n_k$ are assumed to be random variables, i.e., we are not conditioning on the block sizes, by the central limit theorem we have

\[ \frac{1}{\sqrt{n_1 p^2 + n(1-\pi)pq}} \xrightarrow{d} \mathcal{N}(0, \pi(1-\pi)). \]

Therefore, by Slutsky’s theorem, we have

\[ \zeta = \frac{np}{\sqrt{n_1 p^2 + n_2 pq}} = \frac{np}{\sqrt{n_1 p^2 + n(1-\pi)pq}} \xrightarrow{d} \mathcal{N}(0, \frac{p(1-p^2)}{4(p^2 + pq)^2}). \]

To replace $n_k$ by $n\pi_k$ in Eq. (3.7) and Eq. (3.8), we thus need to include the random term $\zeta$. While we surmise that Eq. (3.7) and Eq. (3.8) can be adapt to account for this randomness in $n_k$, we shall not do so in this paper.
3.1 Proofs sketch for Theorem 3.1 and Theorem 3.2

We present in this subsection a sketch of the main ideas in the proofs of Theorem 3.1 and Theorem 3.2; the detailed proofs are given in Section B of the appendix. We start with the motivation behind Eq. (3.1). Given $\tilde{X}_n$, the entries of the right hand side of Eq. (3.1), except for the term $\mathbf{R}_n$, can be expressed explicitly in terms of linear combinations of the entries $a_{ij} - p_{ij}$ of $\mathbf{A}_n - \mathbf{P}_n$.

This is in contrast with the left hand side of Eq. (3.1) which depends on the quantities $\mathbf{U}_\mathbf{A}$ and $\mathbf{S}_\mathbf{A}$ (recall Definition 2); since the quantities $\mathbf{U}_\mathbf{A}$ and $\mathbf{S}_\mathbf{A}$ cannot be express explicitly in terms of the entries of $\mathbf{A}_n$ and $\mathbf{P}_n$, we conclude that the right hand side of Eq. (3.1) is simpler to analyze. From Eq. (3.1), the squared Frobenius norm $n\rho_n\|\mathbf{X}_n\mathbf{W}_n - \tilde{\mathbf{X}}_n\|_F^2$ is

$$n\rho_n\|\mathbf{T}_n^{-1/2}(\mathbf{A}_n - \mathbf{P}_n)\mathbf{T}_n^{-1/2}\tilde{\mathbf{X}}_n(\tilde{\mathbf{X}}_n^\top\tilde{\mathbf{X}}_n)^{-1} + \frac{1}{2}(\mathbf{I} - \mathbf{D}_n\mathbf{T}_n^{-1})\tilde{\mathbf{X}}_n\|_F^2 + O_P((n\rho_n)^{-1/2}).$$

Then conditional on $\mathbf{P}_n$, the above expression is, up to the term of order $O_P((n\rho_n)^{-1/2})$, a function of the independent random variables $\{a_{ij} - p_{ij}\}_{i < j}$. We can then apply concentration inequalities such as those in [9] to show that the squared Frobenius norm $n\rho_n\|\mathbf{X}_n\mathbf{W}_n - \tilde{\mathbf{X}}_n\|_F^2$ is, conditional on $\mathbf{P}_n$, concentrated around its expectation. Here the expectation is taken with respect to the random entries of $\mathbf{A}_n$. Eq. (3.4) and Eq. (3.5) then follows by direct evaluation of this expectation, for the case when $\rho_n \equiv 1$ and for when $\rho_n \rightarrow 0$, respectively.

Once Eq. (3.1) is established, we can derive Theorem 3.2 as follows. Let $\xi_i$ denotes the $i$-th row of $n\rho_n^{1/2}(\mathbf{W}_n\tilde{\mathbf{X}}_n - \tilde{\mathbf{X}}_n)$ and let $r_i$ denotes the $i$-th row of $\mathbf{R}_n$. Eq. (3.1) then implies

$$\xi_i = (\tilde{\mathbf{X}}_n^\top\tilde{\mathbf{X}}_n)^{-1}\frac{n\rho_n^{1/2}}{\sqrt{t_i}} \left( \sum_j a_{ij} - p_{ij} \tilde{x}_j \right) + \frac{n\rho_n^{1/2}(t_i - d_i)}{2t_i}\tilde{x}_i + n\rho_n^{1/2}r_i$$

$$= (\tilde{\mathbf{X}}_n^\top\tilde{\mathbf{X}}_n)^{-1}\frac{n\rho_n^{1/2}}{\sqrt{t_i}} \left( \sum_j \frac{\sqrt{n\rho_n}(a_{ij} - p_{ij})X_j}{t_j} - \frac{n\rho_n X_i}{2t_i^{3/2}} \sum_j (a_{ij} - p_{ij}) + n\rho_n^{1/2}r_i \right)$$

$$= \frac{n\rho_n^{1/2}}{\sqrt{t_i}} \sum_j \frac{(a_{ij} - p_{ij})}{\sqrt{n\rho_n}} \left( \frac{(\tilde{\mathbf{X}}_n^\top\tilde{\mathbf{X}}_n)^{-1}X_j}{t_j/(n\rho_n)} - \frac{X_i}{2t_i/(n\rho_n)} \right) + n\rho_n^{1/2}r_i$$

We then show that $n\rho_n^{1/2}r_i \overset{d}{\rightarrow} 0$. Indeed, there are $n$ rows in $\mathbf{R}_n$ and $\|\mathbf{R}_n\|_F = O((n\rho_n)^{-1})$; hence, on average, for each index $i$, $\|r_i\|^2 = O_P(n^{-3}\rho_n^{-2})$. Furthermore, $t_i/(n\rho_n) = \sum_j X_i^\top X_j/n \xrightarrow{a.s.} X_i^\top\mu$ as $n \rightarrow \infty$. Finally, $\tilde{\mathbf{X}}_n^\top\tilde{\mathbf{X}}_n = \sum_i (X_iX_i^\top/(\sum_j X_j^\top X_j))$ which, as we show in Section B, converges to $\tilde{\Delta} = \sum_i X_iX_i^\top/\sum_j X_j^\top X_j$. 

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E \left[ \frac{X_i X_i^\top}{X_i^\top \mu} \right] as n \to \infty. We therefore have, after additional manipulations, that

\[ \xi_i = \frac{\sqrt{n} \rho_n}{\sqrt{t_i}} \sum_j (a_{ij} - \rho_n) \left( \frac{\Delta^{-1} X_j}{X_j^\top} - \frac{X_i}{2X_i^\top} \right) + o_p(1). \]

Then conditioning on \( X_i = x \), the above expression for \( \xi_i \) is roughly a sum of independent and identically distributed mean 0 random variables. The multivariate central limit theorem can then be applied to the above expression for \( \xi_i \), thereby yielding Theorem 3.2.

We now sketch the derivation of Eq. (3.1). For simplicity, we ignore the subscript \( n \) in the matrices \( A_n, X_n, P_n \) and related matrices. First, consider the following expression.

\[ \bar{U}_A \bar{S}_A^{1/2} - \bar{U}_P \bar{S}_P^{1/2} \bar{P}_p \bar{U}_A = \mathcal{L}(A) \bar{U}_A \bar{S}_A^{-1/2} - \mathcal{L}(P) \bar{U}_P \bar{S}_P^{-1/2} \bar{P}_p \bar{U}_A \]

\[ = \mathcal{L}(A) \bar{U}_A \bar{U}_A^\top \bar{S}_A^{1/2} - \mathcal{L}(P) \bar{U}_P \bar{S}_P^{-1/2} \bar{P}_p \bar{U}_A \]

Now \( \mathcal{L}(A) \) is “concentrated” around \( \mathcal{L}(P) \), i.e., \( \| \mathcal{L}(A) - \mathcal{L}(P) \| = O_p((n \rho_n)^{-1/2}) \) (see Theorem 2 in [25]). Since \( \| \mathcal{L}(P) \| = \Theta(1) \) and the non-zero eigenvalues of \( \mathcal{L}(P) \) are all of order \( \Theta(1) \), this implies, by the Davis-Kahan theorem, that the eigenspace spanned by the largest eigenvalues of \( \mathcal{L}(A) \) is “close” to that spanned by the \( d \) largest eigenvalues of \( \mathcal{L}(P) \). More precisely, \( \bar{U}_A \bar{U}_A^\top = \bar{U}_P \bar{U}_P^\top + O_p((n \rho_n)^{-1/2}) \) and

\[ \bar{U}_A \bar{S}_A^{1/2} - \bar{U}_P \bar{S}_P^{1/2} \bar{P}_p \bar{U}_A = \mathcal{L}(A) \bar{U}_P \bar{U}_P^\top \bar{U}_A \bar{S}_A^{1/2} - \mathcal{L}(P) \bar{U}_P \bar{S}_P^{-1/2} \bar{P}_p \bar{U}_A + O_p((n \rho_n)^{-1}). \]

We then consider the terms \( \bar{S}_P^{-1/2} \bar{P}_p \bar{U}_A \bar{S}_A^{-1/2} \). Since \( \bar{U}_P \) and \( \bar{U}_A \) both have orthonormal columns, \( \bar{U}_A \bar{U}_A^\top = \bar{U}_P \bar{U}_P^\top + O_p((n \rho_n)^{-1/2}) \) implies that there exists an orthogonal matrix \( W^* \) such that \( \bar{U}_P \bar{U}_A = W^* + O_p((n \rho_n)^{-1}) \) (see Proposition B.2). Furthermore, \( W^* \) satisfies an important property, namely that \( \bar{W}^* \bar{S}_P^{-1/2} \bar{S}_A^{-1/2} = O_p((n \rho_n)^{-1}) \). We can thus juxtapose \( \bar{U}_P \bar{U}_A \) and \( \bar{S}_A^{-1/2} \) in the above expression and replace \( \bar{U}_P \bar{U}_A \) by the orthogonal matrix \( W^* \), thereby yielding

\[ \bar{U}_A \bar{S}_A^{1/2} - \bar{U}_P \bar{S}_P^{1/2} W^* = (\mathcal{L}(A) - \mathcal{L}(P)) \bar{U}_P \bar{S}_P^{-1/2} W^* + O_p((n \rho_n)^{-1}). \]

As \( \bar{X} \bar{X}^\top = \mathcal{L}(P) = \bar{U}_P \bar{S}_P^{1/2} \bar{P}_p \bar{U}^\top \), we have \( \bar{X} = \bar{U}_P \bar{S}_P \bar{W} \) for some orthogonal matrix \( \bar{W} \). Therefore,

\[ \bar{U}_A \bar{S}_A^{1/2} - \bar{X} \bar{W}^\top W^* = (\mathcal{L}(A) - \mathcal{L}(P)) \bar{U}_P \bar{S}_P^{-1/2} W^* + O_p((n \rho_n)^{-1}) \]

\[ = (\mathcal{L}(A) - \mathcal{L}(P)) \bar{U}_P \bar{S}_P^{1/2} \bar{W} \bar{W}^\top \bar{S}_P^{-1/2} W^* + O_p((n \rho_n)^{-1}) \]

\[ = (\mathcal{L}(A) - \mathcal{L}(P)) \bar{X} (\bar{X} \bar{X})^{-1} \bar{W}^\top W^* + O_p((n \rho_n)^{-1}). \]
Equivalently,

\[
\tilde{U}_A \tilde{S}_A^{1/2} (W^*)^\top \tilde{W} - \tilde{X} = (L(A) - L(P)) \tilde{X}(\tilde{X}^\top \tilde{X})^{-1} + O_P((n \rho_n)^{-1}). \tag{3.16}
\]

The right hand side of Eq. (3.16) can be written explicitly in terms of the entries of \( A \). However, since \( L(A) = D^{-1/2} A D^{-1/2} \), the entries of the right hand side of Eq. (3.16) are not linear/affine combinations of the entries of \( A \). Nevertheless, by a Taylor-series expansion of the entries of \( D^{-1/2} \), we have \( D^{-1/2} = T^{-1/2} + \frac{1}{2} T^{-3/2}(T-D) + O_P((n \rho_n)^{-3/2}) \). Substituting this into Eq. (3.16) followed by further simplifications yield Eq. (3.1).

\section{Subsequent Inference}

In this section we demonstrate how the results of Section 2.1 and Section 3 provide insights into subsequent inference. We first consider graphs generated according to a stochastic blockmodel with parameters

\[
B = \begin{bmatrix} 0.42 & 0.42 \\ 0.42 & 0.5 \end{bmatrix}; \quad \text{and} \quad \pi = (0.6, 0.4). \tag{4.1}
\]

We sample an adjacency matrix \( A \) for graphs on \( n \) vertices from the above model for various choices of \( n \). For each adjacency matrix \( A \), we compute the normalized Laplacian embedding of \( A \). Figure 1 presents examples of the scatter plots for these embeddings for \( n = 1000, 2000 \) and 4000. The points in the scatter plots are colored according to the block membership of the corresponding vertices in the blockmodel. For each block, we also plot the ellipses showing the empirical (dashed lines) and theoretical (solid lines) 95% level curves for the distribution of \( \tilde{X}_i \). The theoretical level curves are as specified in Theorem 3.2.

![Figure 1](image1.png)

Figure 1: Plot of the estimated latent positions in a two-block stochastic blockmodel graph on \( n \) vertices. The points are colored according to the block membership of the corresponding vertices. Dashed ellipses give the 95% level curves for the empirical distributions. Solid ellipses give the 95% theoretical level curves for the distributions as specified by Theorem 3.2.

We next investigate the implication of the multivariate normal distribution from Theorem 3.2 on subsequent inference. Spectral clustering refers to a large class
of techniques used in partitioning data points into clusters that proceed by first performing a truncated eigendecomposition of a similarity matrix between the data points to obtain a low-dimensional Euclidean representation of these data points followed by clustering of the data points in this low-dimensional representation; see [26] for a comprehensive introduction. The normalized cuts algorithm of [35] is a popular and widely-used instance of spectral clustering where the similarity matrix is a normalized Laplacian matrix and clustering is done using the $K$-means algorithm.

![Figure 2: Comparison of clustering error rates for Gaussian mixture model (GMM) clustering, $K$-means clustering, linear classifier, and Bayes-optimal classifier. The error rate for each $n \in \{1000, 1250, 1500, \ldots, 4000\}$ was obtained by averaging 100 Monte Carlo iterations and are plotted on a log scale. The plot indicates that the assumption of a mixture of multivariate normals can yield significant improvement in the accuracy of the spectral clustering procedure.](image)

It was shown in [34] that the normalized cuts algorithm, i.e., the normalized Laplacian embedding followed by $K$-means, is consistent for estimating the block memberships of stochastic blockmodels graphs. The result of Corollary 3.3, however, suggests that $K$-means clustering is suboptimal unless the covariance matrices of the estimated latent positions for the blocks are spherical. We illustrate this by generating sequences of stochastic blockmodel graphs on $n$ vertices with parameters as given in Eq. (4.1) where $n \in \{1000, 1250, 1500, \ldots, 4000\}$. For each graph, we embed its normalized Laplacian matrix into $\mathbb{R}^2$ and cluster the embedded vertices via either $K$-means or the MCLUST Gaussian mixture model-based clustering algorithm [18]. We then measure the error rate of the clustering solution. The error rates, averaged over 100 replicates of the experiment, are presented on log-scale in Figure 2. We see that the Gaussian mixture model-based clustering does yield significant improvement over $K$-means clustering. For further comparison, we plot the Bayes-optimal error rate and that of a linear classifier which assigns an embedded point to the closest theoretical centroid. The error rate of the linear classifier is computed under the assump-
tion that the rows of the Laplacian spectral embedding are indeed multivariate normal with known covariance matrices and centered around the centroid of the respective blocks; this error rate serves as a lower-bound for that of K-means clustering.

4.1 Comparison of ASE and LSE via within-class covariances

We now discuss a comparison of the use of adjacency spectral embedding and Laplacian spectral embedding for subsequent inference. We consider as our subsequent inference task the problem of recovering the block assignments in stochastic blockmodel graphs. Our first metric of comparison is the notion of within-block variance for each block of the stochastic blockmodel, following the work of [6]. We partially extend the results of [6] for two-block stochastic blockmodels to $K$-block stochastic blockmodels with positive semidefinite block probability matrices. However, while the collection of within-block variances is a meaningful surrogate for the performance of our subsequent inference task, we argue that it is not the “right” metric as it captures only the trace of the block-conditional covariance matrices and not the form of the block-conditional covariance matrices. That is to say, the use of the within-block variances as a surrogate measure is similar to the oracle $K$-means lower bound in Figure 2. A more appropriate surrogate is the collection of pairwise Chernoff informations between the block-conditional multivariate normals, which behave similarly to the oracle Bayes lower bound in Figure 2. The discussion of Chernoff information is postponed to the next subsection.

**Definition 3** (Within-block variances). Let $(X, A) \sim \text{RDPG}(F)$ with sparsity factor $\rho_n$ where $F = \sum_k \pi_k \delta_{\nu_k}$ is a mixture of $K$ point masses at $\nu_1, \nu_2, \ldots, \nu_K \in \mathbb{R}^d$ and $\delta_{\nu_k}$ denotes the Dirac delta function. Given $A$, let $C_k$ for $k \in \{1, 2, \ldots, K\}$ denote the set of vertices of $A$ assigned to block $k$. Recall the definitions of $U_A$ and $\tilde{U}_A$ in Definition 2, i.e., $U_A$ and $\tilde{U}_A$ are the $n \times d$ matrices containing the $d$ largest eigenvectors of the adjacency matrix and the Laplacian matrix, respectively. For any index $i$, let $U_A(i, :)$ and $\tilde{U}_A(i, :)$ denote the $i$-th row of $U_A$ and $\tilde{U}_A(i, :)$, respectively. Then for any $k, l \in \{1, 2, \ldots, K\}$, the ASE variance between block $k$ and block $l$ is defined as

\[
\hat{d}_{kl} = \hat{d}_{kl}(A) = \frac{1}{|C_k|} \sum_{i \in C_k} \| U_A(i, :) - \hat{\mu}_l \|^2; \quad \hat{\mu}_l = \frac{1}{|C_l|} \sum_{j \in C_l} U_A(j, :).
\] (4.2)

Similarly, the LSE variance between block $k$ and block $l$ is

\[
\tilde{d}_{kl} = \tilde{d}_{kl}(A) = \frac{1}{|C_k|} \sum_{i \in C_k} \| \tilde{U}_A(i, :) - \tilde{\mu}_l \|^2; \quad \tilde{\mu}_l = \frac{1}{|C_l|} \sum_{j \in C_l} \tilde{U}_A(j, :).
\] (4.3)

When $k = l$, $\hat{d}_{kk}$ and $\tilde{d}_{kk}$ are referred to as the ASE within-block variance for block $k$ and the LSE within-block variance for block $k$, respectively.
We then have the following large-sample limit results for $\hat{d}_{kl}$ and $\tilde{d}_{kl}$. Their proofs are similar to those of Theorem 2.1 and Theorem 3.1 and therefore will be omitted. Nevertheless, we verify in Section C of the appendix that Theorem 4.1 and Theorem 4.2 are indeed generalizations of Theorem 3.1 and Theorem 3.2 from [6]. We emphasize that neither Theorem 4.1 nor Theorem 4.2 assume distinct eigenvalues of the matrix $XX^\top$ or $L(XX^\top)$; distinct eigenvalues is a necessary assumption used in the proofs of Theorem 3.1 and Theorem 3.2 in [6] (see Section 8 of the cited paper).

**Theorem 4.1.** Assume the setting and notations of Theorem 2.1 and suppose furthermore that $F = \sum_k \pi_k \delta_{\nu_k}$ is a mixture of $K$ distinct point masses at $\nu_1, \nu_2, \ldots, \nu_K \in \mathbb{R}^d$. Let $\bar{U}_{P_n}$ denote the $n \times d$ matrix whose columns are the orthonormal eigenvectors corresponding to the non-zero eigenvalues of the matrix $P_n = \rho_n X_n X_n^\top$. For any $k \in \{1, 2, \ldots, K\}$, let $S_k$ be the $n \times n$ diagonal matrix with diagonal entries $(s_k(1), s_k(2), \ldots, s_k(n))$ such that $s_k(i) = 1$ if $X_i = \nu_k$ and $s_k(i) = 0$ otherwise. We then have, for any $k \in \{1, 2, \ldots, K\}$

\[
\begin{align*}
  n^2 \hat{d}_{kk} &= \frac{n^2}{|C_k|} \| S_k (\bar{U}_{A_n} W_n - \bar{U}_{P_n}) \|_F^2 + o_P(1) \\
  &= \frac{n^2}{|C_k|} \| S_k (A_n - P_n) X_n (X_n^\top X_n)^{-3/2} \|_F^2 + o_P(1).
\end{align*}
\]

Therefore, if $\rho_n \equiv 1$, then for any $k \in \{1, 2, \ldots, K\}$

\[
\begin{align*}
  n^2 \hat{d}_{kk} &\xrightarrow{a.s.} \text{tr} \Delta^{-3} \mathbb{E}[X_1 X_1^\top (\nu_k^\top X_1 - \nu_k^\top X_1 X_1^\top \nu_k)] \\
  \text{as } n \to \infty.
\end{align*}
\]

If, however, $\rho_n \to 0$ and $n \rho_n = \omega(\log^k(n))$, then

\[
\begin{align*}
  n^2 \hat{d}_{kk} &\xrightarrow{a.s.} \text{tr} \Delta^{-3} \mathbb{E}[X_1 X_1^\top \nu_k^\top X_1] \\
  \text{as } n \to \infty.
\end{align*}
\]

For the $\tilde{d}_{kk}$, we have the following result.

**Theorem 4.2.** Assume the setting and notations of Theorem 3.1 and suppose furthermore that $F = \sum_k \pi_k \delta_{\nu_k}$ is a mixture of $K$ distinct point masses at $\nu_1, \nu_2, \ldots, \nu_K \in \mathbb{R}^d$. Let $\bar{U}_{P_n}$ denote the $n \times d$ matrix whose columns are the orthonormal eigenvectors corresponding to the non-zero eigenvalues of the matrix $L(P_n) = \mathcal{L}(\rho_n X_n X_n^\top) = \mathcal{L}(X_n X_n^\top)$. For any $k \in \{1, 2, \ldots, K\}$, let $S_k$ be the $n \times n$ diagonal matrix with diagonal entries $(s_k(1), s_k(2), \ldots, s_k(n))$ such that $s_k(i) = 1$ if $X_i = \nu_k$ and $s_k(i) = 0$ otherwise. We then have, for any $k \in \{1, 2, \ldots, K\}$

\[
\begin{align*}
  n^2 \tilde{d}_{kk} &= \frac{n^2}{|C_k|} \| S_k (\bar{U}_{A_n} W_n - \bar{U}_{P_n}) \|_F^2 + o_P(1) \\
  &= \frac{n^2}{|C_k|} \| S_k M_1 (X_n X_n^\top)^{-3/2} + \frac{1}{2} S_k M_2 (X_n X_n^\top)^{-1/2} \|_F^2 + o_P(1)
\end{align*}
\]

\[
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\]
where $M_1$ and $M_2$ are defined as

\begin{align}
M_1 &= T_n^{-1/2}(A_n - P_n)T_n^{-1/2}\bar{X}_n \\
M_2 &= T_n^{-1/2}(T_n - D_n)T_n^{-1/2}\bar{X}_n.
\end{align}

Therefore, if $\rho_n \equiv 1$, then for any $k \in \{1, 2, \ldots, K\}$

\begin{align}
n^2\tilde{d}_{kk} &\xrightarrow{a.s.} \operatorname{tr} \tilde{\Delta}^{-3}E\left[\left(\frac{X_1}{X_1^\top \mu} - \frac{\tilde{\Delta} \nu_k}{2\nu_k^\top \mu}\right)\left(\frac{X_1^\top}{X_1^\top \mu} - \frac{\nu_k^\top \tilde{\Delta}}{2\nu_k^{\top} \mu}\right)\right] \\
&\quad \times \left(\frac{\nu_k^\top X_1 - \nu_k^\top X_1 X_1^\top \nu_k}{\nu_k^\top \mu}\right)
\end{align}

as $n \to \infty$. If, however, $\rho_n \to 0$ and $n\rho_n = \omega(\log^4(n))$, then

\begin{align}
n^2\tilde{d}_{kk} &\xrightarrow{a.s.} \operatorname{tr} \tilde{\Delta}^{-3}E\left[\left(\frac{X_1}{X_1^\top \mu} - \frac{\tilde{\Delta} \nu_k}{2\nu_k^\top \mu}\right)\left(\frac{X_1^\top}{X_1^\top \mu} - \frac{\nu_k^\top \tilde{\Delta}}{2\nu_k^{\top} \mu}\right)\right] \\
&\quad \times \left(\frac{\nu_k^\top X_1}{\nu_k^\top \mu}\right)
\end{align}

as $n \to \infty$.

**Remark.** We note that the $\tilde{d}_{kl}$ and $\bar{d}_{kl}$ are defined in terms of $U_A$ and $\tilde{U}_A$ and not in terms of $X = U_A S_A^{1/2}$ and $\bar{X} = \tilde{U}_A \tilde{S}_A^{1/2}$. This is because $||S_A^{1/2}|| > ||\tilde{S}_A^{1/2}||$. In addition, as we alluded to previously, the $\tilde{d}_{kl}$ and $\bar{d}_{kl}$ do not explicitly take into account the structure of the block-conditional covariance matrices; instead they measure only the average Euclidean distance of a point to its block-conditional cluster centroid – this coincides with taking the trace of the covariance matrices. Therefore, the $\tilde{d}_{kl}$ and $\bar{d}_{kl}$ serve as a surrogate only for the performance of the $K$-means $\circ$ ASE and $K$-means $\circ$ LSE procedures for recovering block assignments. As Figure 2 illustrates, the $K$-means $\circ$ ASE and $K$-means $\circ$ LSE procedures do not yield the optimal error rate for the inference task at hand. That is to say, the within-block variances cannot be used to compare the ASE and LSE for subsequent inference in a way that is independent of the clustering procedure used. Roughly speaking, what we want is to be able to compare, for a given stochastic blockmodel graph $A$, the large-sample error rate of $\text{inf}_{T', T} T \circ$ ASE versus the large-sample error rate of $\text{inf}_{T', T'} T' \circ$ LSE; here $T$ and $T'$ range over all possible transformations and clusterings procedure. This comparison is facilitated by the limit results of Corollary 2.3 and Corollary 3.3 and the notion of the Chernoff information.

### 4.2 Chernoff Information

Let $F_0$ and $F_1$ be two absolutely continuous multivariate distributions in $\Omega = \mathbb{R}^d$ with density functions $f_0$ and $f_1$, respectively. Suppose that $Y_1, Y_2, \ldots, Y_m$ are independent and identically distributed random variables, with $Y_i$ distributed either $F_0$ or $F_1$. We are interested in testing the simple null hypothesis $H_0: F = F_0$ against the simple alternative hypothesis $H_1: F = F_1$. A test $T$ can be viewed as a sequence of mappings $T_m: \Omega^m \Rightarrow \{0, 1\}$ such that given $Y_1 = y_1, Y_2 = y_2, \ldots, Y_m = y_m$, the test rejects $H_0$ in favor of $H_1$ if $T_m(y_1, y_2, \ldots, y_m) = 1$; similarly, the test favors $H_0$ if $T_m(y_1, y_2, \ldots, y_m) = 0$.
The Neyman-Pearson lemma states that, given $Y_1 = y_1, Y_2 = y_2, \ldots, Y_m = y_m$ and a threshold $\eta_m \in \mathbb{R}$, the likelihood ratio test which rejects $\mathbb{H}_0$ in favor of $\mathbb{H}_1$ whenever

$$\left( \sum_{i=1}^{m} \log f_0(y_i) - \sum_{i=1}^{m} \log f_1(y_i) \right) \leq \eta_m$$

is the most powerful test at significance level $\alpha_m = \alpha(\eta_m)$, i.e., the likelihood ratio test minimizes the type-II error $\beta_m$ subject to the constraint that the type-I error is at most $\alpha_m$.

Assuming that $\pi \in (0, 1)$ is a prior probability that $\mathbb{H}_0$ is true. Then, for a given $\alpha^*_m \in (0, 1)$, let $\beta_m^* = \beta_m^*(\alpha^*_m)$ be the type-II error associated with the likelihood ratio test when the type-I error is at most $\alpha^*_m$. The quantity $\inf_{\alpha^*_m \in (0, 1)} \pi \alpha^*_m + (1 - \pi)\beta^*_m$ is then the Bayes risk in deciding between $\mathbb{H}_0$ and $\mathbb{H}_1$ given the $m$ independent random variables $Y_1, Y_2, \ldots, Y_m$. A classical result of Chernoff [11, 12] states that the Bayes risk is intrinsically linked to a quantity known as the Chernoff information. More specifically, let $C(F_0, F_1)$ be the quantity

$$C(F_0, F_1) = -\log \left[ \inf_{t \in (0,1)} \int_{\mathbb{R}^d} f_0^t(x) f_1^{1-t}(x) dx \right]$$

$$= \sup_{t \in (0,1)} \left[ -\log \int_{\mathbb{R}^d} f_0^t(x) f_1^{1-t}(x) dx \right]. \quad (4.12)$$

Then we have

$$\lim_{m \to \infty} \frac{1}{m} \inf_{\alpha^*_m \in (0, 1)} \log(\pi \alpha^*_m + (1 - \pi)\beta^*_m) = -C(F_0, F_1). \quad (4.13)$$

Thus $C(F_0, F_1)$, the Chernoff information between $F_0$ and $F_1$, is the exponential rate at which the Bayes error $\inf_{\alpha^*_m \in (0, 1)} \pi \alpha^*_m + (1 - \pi)\beta^*_m$ decreases as $m \to \infty$; we note that the Chernoff information is independent of $\pi$. We also define, for a given $t \in (0, 1)$ the Chernoff divergence $C_t(F_0, F_1)$ between $F_0$ and $F_1$ by

$$C_t(F_0, F_1) = -\log \int_{\mathbb{R}^d} f_0^t(x) f_1^{1-t}(x) dx.$$

The Chernoff divergence is an example of a $f$-divergence as defined in [1, 16]. When $t = 1/2$, $C_t(F_0, F_1)$ is the Bhattacharyya distance between $F_0$ and $F_1$. As we mentioned previously, any $f$-divergence satisfies the information processing lemma and is invariant with respect to invertible transformations [24]. Thus any $f$-divergence such as the Kullback-Liebler divergence can also be used to compare the two embedding methods. We chose the Chernoff information mainly because of its explicit relationship with the Bayes risk.

The result of Eq. (4.13) can be extended to $K + 1 \geq 2$ hypotheses. Let $F_0, F_1, \ldots, F_K$ be distributions on $\mathbb{R}^d$ and suppose that $Y_1, Y_2, \ldots, Y_m$ are independent and identically distributed random variables with $Y_i$ distributed
\( F \in \{ F_0, F_1, \ldots, F_K \} \). We are thus interested in determining the distribution of the \( Y_i \) among the \( K + 1 \) hypothesis \( \mathbb{H}_0 \): \( F = F_0, \ldots, \mathbb{H}_K \): \( F = F_K \). Suppose also that hypothesis \( \mathbb{H}_k \) has a priori probability \( \pi_k \). Then for any decision rule \( \delta \), the risk of \( \delta \) is \( r(\delta) = \sum_k \pi_k \sum_{l \neq k} \alpha_{lk}(\delta) \) where \( \alpha_{lk}(\delta) \) is the probability of accepting hypothesis \( \mathbb{H}_k \) when hypothesis \( \mathbb{H}_l \) is true. Then we have \([22]\)

\[
\inf_{\delta} \lim_{m \to \infty} \frac{r(\delta)}{m} = -\min_{k \neq l} C(F_k, F_l). \tag{4.14}
\]

where the infimum is over all decision rules \( \delta \). That is to say, for any \( \delta \), \( r(\delta) \) decreases to 0 as \( m \to \infty \) at a rate no faster than \( \exp(-m \min_{k \neq l} C(F_k, F_l)) \).

It was also shown in \([22]\) that the Maximum A Posteriori decision rule achieves this rate.

For this paper, we are interested in computing the Chernoff information \( C(F_0, F_1) \) when \( F_0 \) and \( F_1 \) are multivariate normals. Suppose \( F_0 = \mathcal{N}(\mu_0, \Sigma_0) \) and \( F_1 = \mathcal{N}(\mu_1, \Sigma_1) \); then, denoting by \( \Sigma_t = t\Sigma_0 + (1 - t)\Sigma_1 \), we have

\[
C(F_0, F_1) = \sup_{t \in (0,1)} \left( \frac{t(1 - t)}{2} (\mu_1 - \mu_2)\Sigma_t^{-1}(\mu_1 - \mu_2) + \frac{1}{2} \log \frac{|\Sigma_t|}{|\Sigma_0|^{1/t}|\Sigma_1|^{1-(1/t)}} \right). \tag{4.15}
\]

### 4.3 Comparison of ASE and LSE via Chernoff information

We now employ the limit results of Corollary 2.3 and Corollary 3.3 to compare the performance of the Laplacian spectral embedding and the adjacency spectral embedding for subsequent inference. Our subsequent inference task is once again the problem of recovering the block assignments in stochastic blockmodel graphs; furthermore, we are interested in estimating the large-sample optimal error rate for recovering the underlying block assignments after the spectral embedding step is carried out. The discussion in Section 4.2 indicates that an appropriate measure for the large-sample optimal error rate for spectral clustering using adjacency or Laplacian spectral embedding is in terms of the minimum of the pairwise Chernoff informations between the multivariate normal distributions as specified in Corollary 2.3 or Corollary 3.3. More specifically, let \( \mathbf{B} \in [0, 1]^{K \times K} \) and \( \pi \in \mathbb{R}^K \) be the matrix of block probabilities and the vector of block assignment probabilities for a \( K \)-block stochastic blockmodel. We shall assume that \( \mathbf{B} \) is positive semidefinite. Then given an \( n \) vertex instantiation of the SBM graph with parameters \( (\pi, \mathbf{B}) \), for sufficiently large \( n \), the large-sample optimal error rate for recovering the block assignments when adjacency spectral embedding is used as the initial embedding step can be characterized by the quantity \( \rho_A = \rho_A(n) \) defined by

\[
\rho_A = \min_{k \neq l} \sup_{t \in (0,1)} \frac{1}{2} \log \frac{|\Sigma_{kl}(t)|}{|\Sigma_k|^1|\Sigma_l|^{1-t}} + \frac{nt(1-t)}{2}(\nu_k - \nu_l)^\top \Sigma_{kl}^{-1}(t)(\nu_k - \nu_l) \tag{4.15}
\]

where \( \Sigma_{kl}(t) = t\Sigma_k + (1-t)\Sigma_l \). We recall Eq. (4.14), in particular the fact that as \( \rho_A \) increases, the large-sample optimal error rate decreases. Similarly, the
large-sample optimal error rate when Laplacian spectral embedding is used as the pre-processing step can be characterized by the quantity $\rho_L = \rho_L(n)$ defined by

$$
\rho_L = \min_{k \neq l} \sup_{t \in (0, 1)} \frac{1}{2} \log \frac{\| \Sigma_{kl}(t) \|}{\| \Sigma_k^{1/2} \|^{1-t} \| \Sigma_l^{1/2} \|^{1-t}} + \frac{nt(1-t)}{2} (\tilde{\nu}_k - \tilde{\nu}_l)^\top \tilde{\Sigma}_{kl}^{-1}(t) (\tilde{\nu}_k - \tilde{\nu}_l) \tag{4.16}
$$

where $\tilde{\Sigma}_{kl}(t) = t\Sigma_k + (1-t)\Sigma_l$ and $\tilde{\nu}_k = \nu_k / (\sum_k \pi_k \nu_k \nu_k^\top)^{1/2}$. We emphasize that we have made the simplifying assumption that $n_k = n\pi_k$ in our expression for $\tilde{\nu}_k$ in Eq. (4.16). This is for ease of comparison between $\rho_A$ and $\rho_L$ in our subsequent discussion.

We thus propose to use the ratio $\rho_A / \rho_L$ as a measure of the relative large-sample performance of the adjacency spectral embedding as compared to the Laplacian spectral embedding for subsequent inference, at least in the context of stochastic blockmodel graphs. That is to say, for given parameters $\pi$ and $B$, if $\rho_A / \rho_L > 1$ then adjacency spectral embedding is to be preferred over Laplacian spectral embedding when $n$, the number of vertices in the graph, is sufficiently large; similarly, if $\rho_A / \rho_L < 1$ then Laplacian spectral embedding is to be preferred over adjacency spectral embedding.

**Remark.** We note that if the block-conditional covariance matrices $\Sigma_k$ are all non-singular, then for sufficiently large $n$, the term log $\frac{\| \Sigma_{kl}(t) \|}{\| \Sigma_k^{1/2} \|^{1-t} \| \Sigma_l^{1/2} \|^{1-t}}$ in the definition of $\rho_A$ is negligible; similarly, the term log $\frac{\| \Sigma_{kl}(t) \|}{\| \Sigma_k^{1/2} \|^{1-t} \| \Sigma_l^{1/2} \|^{1-t}}$ in the definition of $\rho_L$ is also negligible. However, on occasion, some of the block-conditional covariance matrices $\Sigma_k$ are singular. As an example, we consider a completely associative two-block stochastic blockmodel with $B = \begin{bmatrix} p & 0 \\ 0 & q^2 \end{bmatrix}$ and $\pi = (\pi_1, \pi_2)$. Then the block-conditional covariance matrices are

$$
\Sigma_1 = (1 - p^2) \begin{bmatrix} \pi_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}; \quad \Sigma_2 = (1 - p^2) \begin{bmatrix} 0 & 0 \\ 0 & \pi_2^{-1} \end{bmatrix}
$$

$$
\overline{\Sigma}_1 = \frac{(1 - p^2)}{4p^2} \begin{bmatrix} \pi_1^{-2} & 0 \\ 0 & 0 \end{bmatrix}; \quad \overline{\Sigma}_2 = \frac{(1 - p^2)}{4p^2} \begin{bmatrix} 0 & 0 \\ 0 & \pi_2^{-2} \end{bmatrix},
$$

and $\rho_A = \rho_L = \infty$. Therefore, ASE and LSE are equivalent with respect to the subsequent inference task. In contrast, [6] showed that the within-block variances for ASE are four times larger than that of the within-block variances of LSE, while the between-block variances for ASE and LSE are the same. We conclude that the within-block variances measure fails to capture the fact that the block-conditional covariance matrices $\Sigma_1$ and $\Sigma_2$ are singular but in different subspaces, and similarly $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ are also singular but in different subspaces, and thus if we had used the within-block variances measure as a surrogate, we would have been misled into believing that LSE is preferable to ASE for this particular subsequent inference task. Indeed, had we ignored the terms log $\frac{\| \Sigma_{kl}(t) \|}{\| \Sigma_k^{1/2} \|^{1-t} \| \Sigma_l^{1/2} \|^{1-t}}$ in the definitions of $\rho_A$ and $\rho_L$, we would have
Figure 3: The ratio $\rho_A/\rho_L$ displayed for various values of $p \in [0.2, 0.8]$ and $r = q - p \in [-0.15, 0.15]$. The labeled lines are the contour lines for $\rho_A/\rho_L$.

come to the similar conclusion that $\rho_L = \frac{2p^2}{1-p^2} \max\{\pi_1, \pi_2\} = 4\rho_A$ for sufficiently large $n$.

As an illustration of the ratio $\rho_A/\rho_L$, we first consider the collection of 2-block stochastic blockmodels where $B = \begin{bmatrix} p^2 & pq \\ pq & q^2 \end{bmatrix}$ for $p, q \in (0, 1)$ and $\pi = (\pi_1, \pi_2)$ with $\pi_1 + \pi_2 = 1$. Then for sufficiently large $n$, $\rho_A$ is approximately

$$\rho_A \approx \sup_{t \in (0, 1)} \frac{nt(1-t)}{2} \frac{\pi_1 p^2 + \pi_2 q^2}{(p-q)^2(t\sigma_1^2 + (1-t)\sigma_2^2)^{-1}}$$

where $\sigma_1$ and $\sigma_2$ are as specified in Eq. (3.14) and Eq. (3.15), respectively. Simple calculations yield

$$\rho_A \approx \frac{n(p-q)^2(\pi_1 p^2 + \pi_2 q^2)}{2(\sqrt{\pi_1 p^4(1-p^2)} + \pi_2 pq^3(1-pq) + \sqrt{\pi_1 p^3q(1-pq)} + \pi_2 q^4(1-q^2))^2}$$

for sufficiently large $n$. Similarly, denoting by $\tilde{\sigma}_1^2$ and $\tilde{\sigma}_2^2$ the variances specified in Eq. (3.12) and Eq. (3.13), we have

$$\rho_L \approx \sup_{t \in (0, 1)} \frac{nt(1-t)}{2} \frac{\pi_1 p^2 + \pi_2 q^2}{(p - q)^2(t\tilde{\sigma}_1^2 + (1-t)\tilde{\sigma}_2^2)^{-1}}$$

$$\approx \frac{2n(\sqrt{\pi_1 p}(1-p^2) + \pi_2 q(1-pq))}{(\sqrt{\pi_1 p(1-p^2)} + \pi_2 q(1-pq))^2} \frac{(t\tilde{\sigma}_1^2 + (1-t)\tilde{\sigma}_2^2)^{-1}}{t\sigma_1^2 + (1-t)\sigma_2^2}$$

for sufficiently large $n$. Fixing $\pi = (0.6, 0.4)$, we computed the ratio $\rho_A/\rho_L$ for a range of $p$ and $q$ values, with $p \in [0.2, 0.8]$ and $q = p + r$ where $r \in [-0.15, 0.15]$. The results are plotted in Figure 3. The $y$-axis of Figure 3 denotes the values of $p$ and the $x$-axis are the values of $r$.

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We also generate instances of a stochastic blockmodel graph on 200 vertices with parameters $p = 0.75$ and $q = 0.6$. For each graph we measure the error rate of the spectral embedding followed by the Gaussian mixture-model based clustering procedure in recovering the block assignments. The error rate for the GMM $\circ$ ASE procedure, averaged over 1000 Monte Carlo replicates, is $0.079$ with a standard error of $6.6 \times 10^{-4}$; meanwhile the error rate for the GMM $\circ$ LSE procedure, also averaged over 1000 Monte Carlo replicates, is $0.083$ with a standard error of $7.2 \times 10^{-6}$. The difference in the mean error rate is statistically significant at $\alpha = 0.001$. Conversely, when $p = 0.2$ and $q = 0.3$ and the graphs are on 400 vertices, the mean error rate, over 1000 Monte Carlo replicates, for the GMM $\circ$ ASE procedure is $0.161$ while the mean error rate for the GMM $\circ$ LSE procedure is $0.151$ and this difference is also statistically significant at $\alpha = 0.001$.

We next consider the collection of stochastic blockmodels with parameters $p$ and $q$ where

$$B = \begin{bmatrix} p & q & q \\ q & p & q \\ q & q & p \end{bmatrix}, \quad p, q \in (0, 1), \quad \pi = (0.8, 0.1, 0.1).$$

First we compute the ratio $\rho_A/\rho_L$ for $p \in [0.3, 0.9]$ and $r = q - p$ with $r \in [-0.2, -0.01]$. The results are plotted in Figure 4, with the $y$-axis of Figure 4 being the values of $p$ and the $x$-axis being the values of $r$. We then generate instances of a stochastic blockmodel graph on 800 vertices with $p = 0.9$ and $q = 0.72$ and estimate the error rate of the GMM $\circ$ ASE and the GMM $\circ$ LSE procedures in recovering the block assignments. The GMM $\circ$ ASE and GMM $\circ$ LSE error rates, averaged over 1000 Monte Carlo replicates, are $0.29$ and $0.38$, respectively. For these choice of parameters, $\rho_A/\rho_L \approx 1.01$. We also generate instances of a stochastic blockmodel graph on 1600 vertices with $p = 0.34$ and $q = 0.15$. The ratio $\rho_A/\rho_L$ in this case is $\approx 0.98$; the GMM $\circ$ ASE and GMM $\circ$ LSE error rates, averaged over 1000 Monte Carlo replicates, are $0.18$ and $0.06$, respectively.

5 Summary and Conclusions

We shown in this paper several limit results for the eigenvectors corresponding to the largest eigenvalues of the normalized Laplacian matrix of random graphs. In particular, we show that for stochastic blockmodel graphs, conditioned on the block assignments, each row of the Laplacian spectral embedding converges to a multivariate normal distribution. We then discuss the relationship between spectral embeddings of the adjacency and normalized Laplacian matrices and subsequent inference. When the subsequent inference task is the problem of clustering the vertices of a graph, we show that the Chernoff information between the multivariate normals approximation of the embedding is a suitable measure for the large-sample optimal error rate, i.e., it characterizes the minimum error
The ratio $\rho_A/\rho_L$ displayed for various values of $p \in [0.2, 0.8]$ and $r = q - p \in [-0.2, -0.01]$ for the 3-block stochastic blockmodel of Eq. (4.17). The labeled lines are the contour lines for $\rho_A/\rho_L$.

rate achievable by any clustering procedure that operates only on the spectral embedding. As a result, we are able to theoretically compare the use of spectral embedding of the adjacency matrix versus that of the normalized Laplacian for subsequent inference, thereby refining and extending the pioneering work of [6].

We now mention several potential extensions of this work. The normalized Laplacian considered in this paper is just one example of possible normalization. In particular, given $\tau > 0$ one can define the $\tau$-regularized normalized Laplacian $\mathcal{L}_\tau$ via $\mathcal{L}_\tau(A) = (D + \tau I)^{-1/2} A (D + \tau I)^{-1/2}$ or $\mathcal{L}_\tau(A) = (D + \tau I)^{-1/2} (A + \tau 11^T) (D + \tau I)^{-1/2}$[2, 10, 33]. It had been shown that regularization is particularly useful for spectral clustering in sparse graphs. It will thus be of interest to derive limit results for the eigenvectors of $\mathcal{L}_\tau(A)$ analogous to those in this paper; such results can potentially allow one to choose the regularization parameter $\tau$.

The limit results in this paper are for the spectral embedding of $(X, A) \sim \text{RPDG}(F)$ into $\mathbb{R}^d$ when $d$, the rank of the matrix $\mathbb{E}[XX^T]$ where $X \sim F$, is fixed and known. Similar results can be derived when the spectral embedding of $A$ is into $\mathbb{R}^{d'}$ where $d' < d$. Limit results for spectral embedding of the adjacency matrix or Laplacian matrix into $\mathbb{R}^d$ when $d' > d$ is, to the best of our knowledge, an open problem. A related inquiry is limit results for spectral embedding into $\mathbb{R}^{d'}$ when $d' < d$ but $d$ varies with $n$ and is not fixed, such as when the graph arises from a latent position model where the link function, viewed as an integral operator, has infinite rank. Since new results on stochastic blockmodels indicate that they can be regarded as a universal approximation
to latent positions model graphs or graphons of exchangeable random graphs [43, 44], limit results for the adjacency and Laplacian spectral embedding will be useful in further understanding of this approximation property.

Finally, the Chernoff information used in this paper is a measure of the effect of spectral embedding on subsequent inference for a single graph. Recently, however, there has been interests in two-sample inference for graphs, e.g., network comparisons or two-sample hypothesis testing for graphs [3, 40, 41]. As an example, given two distributions $F$ and $G$, the problem of testing whether $F = G$ given two random dot product graphs $A \sim \text{RDPG}(F)$ and $B \sim \text{RDPG}(G)$ was considered in [41]; the proposed test statistic is a kernel-based distance measure between the spectral embedding $\hat{X}$ of $A$ and $\hat{Y}$ of $B$. Determining a measure that characterizes the effect of spectral embedding for two-sample graphs inference problems, akin to how the Chernoff information characterize the effect of spectral embedding for single graph inference, is of significant interest.

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A Proof of Theorem 2.1 and Theorem 2.2

We first present a sketch of the proof of Theorem 2.1, noting that the main arguments are given in [40]. We also note that similar, albeit more involved, arguments are used in the proof of Theorem 3.1. Since the proof of Theorem 3.1 will be presented in much greater detail in Section B, to avoid repetitions, we chose to omit the details in the current proof. Nevertheless, we emphasize that the statements of the results in [40] are slightly different from how they are stated in the current paper; these differences stem mainly from how sparseness in the graphs is incorporated. More specifically [40] considered a sequence of random dot product graphs where for each $n$, the matrix of latent positions $X_n$ are fixed but unknown (see Definition 1 in [40]) and furthermore, there need not exist any relationship between $X_n$ and $X_{n'}$ for $n \neq n'$. Sparseness of the graphs is thus implicit (see for example the condition on the minimum vertex’s degree in Assumption 1 in [40]). The current paper, however, assumes that the rows of $X_n$ are independently sampled according to a distribution $F$. As such, sparseness needs to be made explicit through the sparsity factor $\rho_n$.

Remark. For ease of exposition, henceforth we shall on many occasions remove the subscript $n$ from the matrices $X_n, X_n, A_n, P_n$ and other related matrices such as $U_{A_n}, U_{P_n}$, etc. The subsequent statements are thus to be interpreted as holding for sufficient large $n$. Since we are concerned with limit results, this should lead to minimal confusion.

We first note that Eq. (2.5) follows from Theorem A.5 in [40]. More specifically, if $(X, A) \sim \text{RDPG}(F)$ with sparsity factor $\rho_n$, then Theorem A.5 in [40] yields

$$\|X - \rho_n^{1/2}XW\|_F = \|(A - P)U_PS_P^{-1/2}\|_F + O_P((n\rho_n)^{-1/2}).$$

Since $P = \rho_nXX^\top$, we have $U_PS_P^{-1/2}W = \rho_n^{1/2}X$ for some orthogonal matrix $W$. Therefore,

$$\|(A - P)U_PS_P^{-1/2}\|_F = \|(A - P)U_PS_P^{1/2}WW^\top S_P^{-1/2}W\|_F$$

$$= \|(A - P)\rho_n^{1/2}X(\rho_nX^\top X)^{-1}\|_F$$

$$= \rho_n^{-1/2}\|(A - P)XX^\top X^{-1}\|_F.$$  

Eq. (2.5) is thus established. We now show Eq. (2.6) and Eq. (2.7). We shall use the convention that, unless stated otherwise, expectation of a random variable dependent on $A$ is taken with respect to $A$ conditional on $P$. Let $\zeta = \rho_n\|(A - P)U_PS_P^{-1/2}\|_F^2$. Then, conditional on $P$, $\zeta$ is a linear function of the independent random variables $\{a_{ij} - p_{ij}\}_{i<j}$. Lemma A.5 in [40] shows that $\zeta$ is tightly concentrated around its expectation $E[\zeta]$. We then have

$$E[\zeta] = E[\|(A - P)U_PS_P^{-1/2}\|_F^2]$$

$$= \rho_n^{-1}E[\|(A - P)XX^\top X^{-1}\|_F^2]$$

$$= \text{tr} n(X^\top X)^{-1} \left( n^{-2}\rho_n^{-1}X^\top E[(A - P)^2] X \right) n(X^\top X)^{-1}.$$
Now, the $ij$-th entry of $(A - P)^2$ is of the form $\sum_k (a_{ik} - p_{ik})(a_{kj} - p_{kj})$. As the upper diagonal entries of $A$ are independent conditional on $P$, we have

$$E\left[ \sum_k (a_{ik} - p_{ik})(a_{kj} - p_{kj}) \right] = \begin{cases} 0 & \text{if } i \neq j \\ \sum_{k \neq i} p_{kj}(1 - p_{kj}) & \text{if } i = j \end{cases}$$

By the strong law of large numbers, $n^{-1}X^T X_n = n^{-1} \sum_i X_i X_i^\top$ converges to $\Delta = E[X_1 X_1^\top]$ almost surely as $n \to \infty$. Hence $n(X^\top X)^{-1}$ converges to $\Delta^{-1}$ almost surely. In addition,

$$n^{-2} \rho_n^{-1} X^\top E[(A - P)^2] X = n^{-2} \rho_n^{-1} \sum_{i=1}^n \sum_{j \neq i} X_i X_i^\top p_{ik}(1 - p_{ik})$$

$$= n^{-2} \rho_n^{-1} \sum_{i=1}^n \sum_{k \neq i} X_i X_i^\top (\rho_n X_i^\top X_k - \rho_n^2 (X_i^\top X_k)^2)$$

$$= n^{-2} \sum_{i=1}^n \sum_{k \neq i} X_i X_i^\top (X_i^\top X_k - \rho_n X_i^\top X_k X_k^\top X_i)$$

If $\rho_n = 1$ for all $n$, the above term converges to $E[X_1 X_1^\top (X_1^\top \mu_F - X_1^\top \Delta_F X_1)]$ almost surely. When $\rho_n \to 0$, the above term converges to $E[X_1 X_1^\top X_1^\top \mu_F]$ almost surely. Eq. (2.6) and Eq. (2.7) is thus established.

We now sketch the proof of Theorem 2.2. We emphasize that Theorem 2.2 is a generalization of the corresponding result in [4, 38], the generalization being that Theorem 2.2 does not assume distinct eigenvalues of the matrix $E[XX^\top]$ where $X \sim F$; distinct eigenvalues is a necessary assumption for the proof given in [4, 38].

Let $a_{ij}$ and $p_{ij}$ denote the $ij$-th entry of $A$ and $P$. From Eq. (2.5), by exchangeability of the collection $\{W_n X_j - \rho_n^{1/2} X_j\}_{j=1}^n$, for any fixed index $i$ we have

$$\sqrt{n}(W_n X_i - \rho_n^{1/2} X_i) = \sqrt{n} \rho_n^{-1/2} (X^\top X)^{-1} \sum_{j \neq i} (a_{ij} - p_{ij}) X_j + o_P(1)$$

$$= \rho_n^{-1/2} (n^{-1} X^\top X)^{-1} \sum_{j \neq i} (a_{ij} - p_{ij}) X_j + o_P(1)$$

$$= (n^{-1} X^\top X)^{-1} \sum_{j \neq i} (a_{ij} - \rho_n X_i^\top X_j) X_j + o_P(1).$$

Now conditional on $X_i$, the quantity $\sum_{j \neq i} (a_{ij} - \rho_n X_i^\top X_j) X_j$ is a sum of independent and identically distributed mean 0 random variables. Thus by the multivariate central limit theorem, conditioning on $X_i = x$ yields

$$\sum_{j \neq i} \frac{(a_{ij} - \rho_n X_i^\top X_j) X_j}{\sqrt{n} \rho_n} \xrightarrow{d} \mathcal{N}(0, E[X_1 X_1^\top (x^\top X_1 - \rho_n x^\top X_1 X_1^\top x)]).$$
Furthermore, since \( n^{-1}X^\top X = n^{-1} \sum X_i X_i^\top \overset{a.s.}{\to} \Delta \) as \( n \to \infty \), we have by Slutsky’s theorem that

\[
\sqrt{n}(W_n \tilde{X}_i - \rho_n^{1/2} X_i) \overset{d}{\to} N(0, \Delta^{-1} E[X_1 X_1^\top (x^\top X_1 - \rho_n x^\top X_1 X_1^\top x)] \Delta^{-1}),
\]

thereby establishing Theorem 2.2.

## B Proof of Theorem 3.1 and Theorem 3.2

For ease of exposition, we present in Section B.1 a proof of Theorem 3.2, assuming Eq. (3.1) in Theorem 3.1 holds. We next derive, in Section B.2, Eq. (3.1) in Theorem 3.1. We then show, in Section B.4 that the Frobenius norms in Eq. (3.4) and Eq. (3.5) are tightly concentrated around their expectations. We complete the proof of Theorem 3.1 by computing these expectations explicitly when \( \rho_n \equiv 1 \) and when \( \rho_n \to 0 \).

### B.1 Proof of Theorem 3.2

Recall that we suppress the dependency on \( n \) in the subscripts of the matrices \( A_n, X_n, P_n \) and other related matrices. In addition, recall that \( \bar{X} = \rho_n^{1/2} T^{-1/2} X = \text{diag}(XX^\top)^{-1/2} X \). Eq. (3.1) from Theorem 3.1 then implies

\[
\bar{X} W - \bar{X} = T^{-1/2}(A - P) T^{-1/2} \bar{X} (\bar{X}^\top \bar{X})^{-1} + \frac{1}{2} T^{-1}(T - D) \bar{X} + R
\]

for some orthogonal matrix \( W \) and \( n \times d \) matrix \( R \) with \( \| R \|_F = O((n \rho_n)^{-1}) \).

For a fixed index \( i \), let \( \zeta_i \) denote the \( i \)-th row of \( n^{1/2} \bar{X} W - \bar{X} \). Also let \( r_i \) denote the \( i \)-th row of \( R \). Now exchangeability of the \( \{ \bar{X}_j \}_{j=1}^n \) implies exchangeability of the \( \{ \bar{X}_i \}_{i=1}^n \) and exchangeability of the \( \{ \bar{X}_j \}_{j=1}^n \). This also implies exchangeability of the \( \{ \zeta_j \}_{j=1}^n \) and thus exchangeability of the \( \{ r_j \}_{j=1}^n \). Now, for any fixed index \( i \), by exchangeability of the \( \{ r_j \}_{j=1}^n \), we have

\[
n^2 \rho_n E[\| r_i \|^2] = n^2 \rho_n \frac{1}{n} E[\sum_j \| r_j \|^2] = n \rho_n E[\| R \|_F^2]
\]

Now, with probability at least \( 1 - n^{-3} \), \( \| R \|_F \leq C_0(n \rho_n)^{-1} \) for some constant \( C_0 \). In addition, \( \| R \|_F \leq n \) almost surely. Thus \( E[\| R \|_F^2] \leq C_0^2 (n \rho_n)^{-2} (1 - n^{-3}) + n \times n^{-3} = O((n \rho_n)^{-2}) \). Therefore \( n^2 \rho_n E[\| r_i \|^2] = O((n \rho_n)^{-1}) \). Since \( n \rho_n = \omega(\log^4(n)) \), we therefore have \( n^2 \rho_n E[\| r_i \|^2] \to 0 \) as \( n \to \infty \), i.e., \( n \rho_n^{1/2} r_i \overset{d}{\to} 0 \) as \( n \to \infty \).

Let \( a_{ij} \) and \( p_{ij} \) denote the \( ij \)-th entry of \( A \) and \( P \), respectively. The above
reasoning implies that for a fixed index $i$, $\zeta_i$ is of the form

$$\zeta_i = (\mathbf{X}^T \mathbf{X})^{-1} \frac{n \rho_n^{1/2}}{\sqrt{t_i}} \left( \sum_j \frac{a_{ij} - p_{ij}}{\sqrt{t_j}} \tilde{X}_j \right) + \frac{n \rho_n^{1/2}}{2t_i} (t_i - d_i) \tilde{X}_i + o_p(1)$$

$$= (\mathbf{X}^T \mathbf{X})^{-1} \frac{n \rho_n}{\sqrt{t_i}} \left( \sum_{j \neq i} \frac{\sqrt{n \rho_n} (a_{ij} - p_{ij}) X_j}{t_j} \right) - \frac{(n \rho_n)^{3/2}}{2t_i^{3/2}} \sum_{j \neq i} \frac{(a_{ij} - p_{ij})}{\sqrt{n \rho_n}} + o_p(1).$$

We first note that $\mathbf{X}^T \mathbf{X} = \mathbf{X} \text{diag}(\mathbf{XX}^T)^{-1} \mathbf{X}$ converges almost surely to $\tilde{\mathbf{X}}$ as $n \to \infty$. This can be seen as follows. Denoting $\mu = \mathbb{E}[X_1]$, we have

$$\tilde{\mathbf{X}}^T \tilde{\mathbf{X}} = \sum_{i=1}^n \frac{X_i X_i^T}{\sum_j X_i^T X_j}$$

$$= \left( \sum_{i=1}^n \frac{X_i X_i^T}{\sum_j X_i^T X_j} \mu \right) + \sum_{i=1}^n \frac{X_i X_i^T}{\sum_j X_i^T X_j} \left( \frac{1}{\sum_j X_i^T X_j} - \frac{1}{\sum_j X_i^T X_j} \mu \right)$$

$$= \left( \sum_{i=1}^n \frac{X_i X_i^T}{\sum_j X_i^T X_j} \mu \right) + \sum_{i=1}^n \frac{X_i X_i^T}{\sum_j X_i^T X_j} \left( \frac{\mu - \sum_j X_i^T X_j}{\sum_j X_i^T X_j} \right).$$

Now, for any index $i$, let $c_i = [(n X_i^T \mu - \sum_j X_i^T X_j)/(\sum_j X_i^T X_j)]$. Then by Hoeffding’s inequality, $c_i = O_p(n^{-1/2})$. As $X_i X_i^T$ is positive semidefinite for each index $i$, we thus have

$$-c_i \frac{X_i X_i^T}{\sum_j X_i^T X_j} \preceq \frac{X_i X_i^T}{\sum_j X_i^T X_j} \left( \frac{\mu - \sum_j X_i^T X_j}{\sum_j X_i^T X_j} \right) \preceq c_i \frac{X_i X_i^T}{\sum_j X_i^T X_j}.$$

where $\preceq$ denotes the positive semidefinite ordering of matrices. Hence

$$-\left( \sup_{j \in [n]} c_j \right) \sum_i \frac{X_i X_i^T}{\sum_j X_i^T X_j} \preceq \sum_i \frac{X_i X_i^T}{\sum_j X_i^T X_j} \left( \frac{\mu - \sum_j X_i^T X_j}{\sum_j X_i^T X_j} \right) \preceq \left( \sup_{j \in [n]} c_j \right) \sum_i \frac{X_i X_i^T}{\sum_j X_i^T X_j}.$$

We then have by a union bound that $\sup_{i \in [n]} c_i = O_p(\sqrt{n^{-1} \log n})$ and hence $\sup_{i \in [n]} c_i \xrightarrow{a.s.} 0$ as $n \to \infty$. In addition, by the strong law of large numbers

$$\sum_i \frac{X_i X_i^T}{n X_i^T \mu} \xrightarrow{a.s.} \mathbb{E} \left[ \frac{X_1 X_1^T}{X_1^T \mu} \right] \quad \text{(B.1)}$$

as $n \to \infty$. Thus,

$$\sum_i \frac{X_i X_i^T}{n X_i^T \mu} \left( \frac{\mu - \sum_j X_i^T X_j}{\sum_j X_i^T X_j} \right) \xrightarrow{a.s.} 0.$$
as $n \to \infty$. We thus conclude that
\[
\bar{X}^\top \bar{X} = \left( \frac{1}{n} \sum_{i=1}^{n} X_i X_i^\top \right) + \frac{1}{n} \sum_{i=1}^{n} X_i X_i^\top \mu - \sum_j X_j^\top X_j \quad \text{as} \quad n \to \infty.
\]

Therefore $(\bar{X}^\top \bar{X})^{-1}$ converges almost surely to $\Delta^{-1}$ as $n \to \infty$. In addition, $t_i/(n\rho_n) \to X_i^\top \mu$ as $n \to \infty$ and hence $\sqrt{n\rho_n/t_i} \to (X_i^\top \mu)^{-1/2}$ as $n \to \infty$. We next consider the term
\[
\sum_{j \neq i} \sqrt{n\rho_n}(a_{ij} - p_{ij})X_j = \sum_{j \neq i} (a_{ij} - p_{ij})X_j \quad \text{as} \quad n \to \infty.
\]

The second sum on the right hand side of the above display is, conditioned on $\mathbf{P}$, a sum of mean 0 random variables. Hoeffding’s inequality implies that the event
\[
\left\| \sum_{j \neq i} \frac{(a_{ij} - p_{ij})X_j n\rho_n X_j^\top \mu - t_j}{\sqrt{n\rho_n X_j^\top \mu}} \right\| \geq s
\]
occurs with probability at most
\[
2 \exp \left( -Cs^2 \sum_{j \neq i} \left\| X_j \right\|^2 \frac{1}{2(n\rho_n X_j^\top \mu - t_j)^2} \right)
\]
for some constant $C > 0$. Therefore,
\[
\sum_{j \neq i} \frac{(a_{ij} - p_{ij})X_j n\rho_n X_j^\top \mu - t_j}{\sqrt{n\rho_n X_j^\top \mu}} \quad \text{as} \quad n \to \infty.
\]

as $n \to \infty$. We thus have
\[
\zeta_i = (\bar{X}^\top \bar{X})^{-1} \frac{\sqrt{n\rho_n}}{\sqrt{t_i}} \left( \sum_{j \neq i} \frac{(a_{ij} - p_{ij})X_j}{\sqrt{n\rho_n X_j^\top \mu}} \right) = \frac{n\rho_n \sqrt{n\rho_n} X_i}{2t_i \sqrt{t_i}} \sum_{j \neq i} \frac{(a_{ij} - p_{ij})}{\sqrt{n\rho_n}} + o_P(1).
\]

We now show that
\[
\frac{n\rho_n \sqrt{n\rho_n} X_i}{2t_i \sqrt{t_i}} \sum_{j \neq i} \frac{(a_{ij} - p_{ij})}{\sqrt{n\rho_n}} = \frac{\sqrt{n\rho_n} (\bar{X}^\top \bar{X})^{-1} X_i}{2 \sqrt{t_i} \mu} \sum_{j \neq i} \frac{(a_{ij} - p_{ij})}{\sqrt{n\rho_n}} + o_P(1).
\]

This can be done as follows. We first consider the term
\[
\frac{n\rho_n \sqrt{n\rho_n} X_i}{2t_i \sqrt{t_i}} \sum_{j \neq i} \frac{(a_{ij} - p_{ij})}{\sqrt{n\rho_n}} = \frac{\sqrt{n\rho_n}}{2 \sqrt{t_i}} \left( \sum_{j \neq i} \frac{(a_{ij} - p_{ij})}{\sqrt{n\rho_n}} \right) \left( \frac{X_i}{X_i^\top \mu} + \frac{n\rho_n X_i}{t_i} - \frac{X_i}{X_i^\top \mu} \right)
\]

Once again, conditional on $\mathbf{P}$,
\[
\frac{\sqrt{n\rho_n}}{2 \sqrt{t_i}} \left( \sum_{j \neq i} \frac{(a_{ij} - p_{ij})}{\sqrt{n\rho_n}} \right) \left( \frac{n\rho_n X_i}{t_i} - \frac{X_i}{X_i^\top \mu} \right)
\]

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is a sum of mean 0 random variable. Hence, by Hoeffding’s inequality, we also have that
\[
\frac{\sqrt{n\rho_n}}{2\sqrt{t_i}} \left( \sum_{j \neq i} \frac{(a_{ij} - p_{ij})}{\sqrt{n\rho_n}} \right) \left( \frac{n\rho_n X_i}{t_i} - \frac{X_i}{X_i^\top \mu} \right) \xrightarrow{a.s.} 0
\]
as \(n \to \infty\). We thus have
\[
\frac{n\rho_n \sqrt{n\rho_n} X_i}{2\sqrt{t_i}} \sum_{j \neq i} \frac{(a_{ij} - p_{ij})}{\sqrt{n\rho_n}} = \frac{\sqrt{n\rho_n}}{2\sqrt{t_i}} \sum_{j \neq i} \frac{(a_{ij} - p_{ij}) X_i}{\sqrt{n\rho_n} X_i^\top \mu} + o_p(1). \tag{B.5}
\]
We next write
\[
\frac{\sqrt{n\rho_n}}{2\sqrt{t_i}} \sum_{j \neq i} \frac{(a_{ij} - p_{ij}) X_i}{\sqrt{n\rho_n} X_i^\top \mu} = \frac{\sqrt{n\rho_n} (\vec{X}^\top \vec{X})^{-1}}{2\sqrt{t_i}} \sum_{j \neq i} \frac{(a_{ij} - p_{ij}) (\vec{X}^\top \vec{X} - \vec{\Delta}) X_i}{\sqrt{n\rho_n} X_i^\top \mu}.
\]
We again evoke Hoeffding’s inequality conditionally on \(P\) to conclude that
\[
\frac{\sqrt{n\rho_n} (\vec{X}^\top \vec{X})^{-1}}{2\sqrt{t_i}} \sum_{j \neq i} \frac{(a_{ij} - p_{ij}) (\vec{X}^\top \vec{X} - \vec{\Delta}) X_i}{\sqrt{n\rho_n} X_i^\top \mu} \xrightarrow{a.s.} 0 \tag{B.6}
\]
as \(n \to \infty\). Eq. (B.4) then follows from Eq. (B.5) and Eq. (B.6).

Combining Eq. (B.3) and Eq. (B.4), we arrive at
\[
\zeta_i = (\vec{X}^\top \vec{X})^{-1} \frac{\sqrt{n\rho_n}}{\sqrt{t_i}} \left( \sum_{j \neq i} \frac{(a_{ij} - p_{ij})}{\sqrt{n\rho_n}} \left( \frac{X_j}{X_j^\top \mu} - \frac{\vec{\Delta} X_i}{2X_i^\top \mu} \right) \right) + o_p(1)
\]
\[
= (\vec{X}^\top \vec{X})^{-1} \frac{\sqrt{n\rho_n}}{\sqrt{t_i}} \left( \sum_{j \neq i} \frac{(a_{ij} - p_{ij})}{\sqrt{n\rho_n}} \left( \frac{X_j}{X_j^\top \mu} - \frac{\vec{\Delta} X_i}{2X_i^\top \mu} \right) \right) + o_p(1).
\]
Now, for each fixed index \(i\), conditioning on \(X_i = x\), the quantity
\[
\frac{1}{\sqrt{n\rho_n}} \sum_{j \neq i} (a_{ij} - p_{n} X_i^\top X_j) \left( \frac{X_j}{X_j^\top \mu} - \frac{\vec{\Delta} X_i}{2X_i^\top \mu} \right) \tag{B.7}
\]
is a sum of independent and identically distributed mean 0 random variables. Therefore, by the multivariate central limit theorem, we have that conditional on \(X_i = x\), the term in Eq. (B.7) converges in distribution to
\[
\mathcal{N}(0, \mathbb{E} \left[ \left( \frac{X_j}{X_j^\top \mu} - \frac{\vec{\Delta} x}{2x^\top \mu} \right) \left( \frac{x^\top X_j - p_n x^\top X_j x_{ij}^\top x}{x^\top \mu} - \frac{\vec{\Delta} x}{2x^\top \mu} \right) \right]^\top )
\]
Finally, recall that \((\vec{X}^\top \vec{X})^{-1}\) and \(\sqrt{n\rho_n/t_i}\) converge almost surely to \(\vec{\Delta}^{-1}\) and \((X_i^\top \mu)^{-1/2}\) as \(n \to \infty\). Therefore, by Slutsky’s theorem, conditional on \(X_i = x\), \(\zeta_i = n\rho_n^{1/2} (W \tilde{X}_i - \tilde{X}_i)\) converges in distribution to
\[
\mathcal{N}(0, \mathbb{E} \left[ \left( \frac{\vec{\Delta}^{-1} X_j}{X_j^\top \mu} - \frac{x}{2x^\top \mu} \right) \left( \frac{x^\top X_j - p_n x^\top X_j x_{ij}^\top x}{x^\top \mu} - \frac{\vec{\Delta} x}{2x^\top \mu} \right) \right]^\top )
\]as desired.
B.2 Proof of Eq. (3.1)

We start with a concentration inequality for the spectral norm of $A - P$ and $\mathcal{L}(A) - \mathcal{L}(P)$ in the case when $A$ is an edge-independent inhomogenous random graph.

**Lemma B.1 ([25, 32]).** Let $A \sim \text{Bernoulli}(P)$, i.e., $A$ is a symmetric matrix whose upper triangular entries are independent Bernoulli random variables with $P[a_{ij} = 1] = p_{ij}$. Let $\Delta = \max_i \sum_{j \neq i} p_{ij}$ and $\delta = \min_i \sum_{j \neq i} p_{ij}$ denote the maximum and minimum row sums of $P$. Suppose $\delta$ satisfies $\delta \gg \log^4(n)$. Then

$$\|A - P\| = O_P(\sqrt{\Delta}),$$

$$\|\mathcal{L}(A) - \mathcal{L}(P)\| = O_P(\delta^{-1/2}).$$

When $P = \rho_n X X^\top$ then $\delta$ and $\Delta$ are both of order $\Theta(n \rho_n)$. Furthermore, the non-zero eigenvalues of $P$ are all of order $\Theta(n \rho_n)$ while the non-zero eigenvalues of $\mathcal{L}(P)$ are all of order $\Theta(1)$. In light of Lemma B.1, for our subsequent derivation, we shall assume that $\rho_n = \omega((\log^k(n))$ for some positive integer $k \geq 4$.

Lemma B.1 implies the following proposition.

**Proposition B.2.** Let $(A, X) \sim \text{RDPG}(F)$ with sparsity factor $\rho_n$. Let $W_1 \Sigma W_2^\top$ be the singular value decomposition of $\tilde{U}_A^\top P \tilde{U}_A$. Then

$$\|\tilde{U}_P^\top \tilde{U}_A - W_1 W_2^\top\|_F = O_P((n \rho_n)^{-1}).$$

**Proof.** Let $\sigma_1, \sigma_2, \ldots, \sigma_d$ denote the singular values of $\tilde{U}_P^\top \tilde{U}_A$ (the diagonal entries of $\Sigma$). Then $\sigma_i = \cos(\theta_i)$ where the $\theta_i$ are the principal angles between the subspaces spanned by $\tilde{U}_A$ and $U_P$. Furthermore, by the Davis-Kahan sin(Θ) theorem (see e.g., Theorem 3.6 in [37]),

$$\|\tilde{U}_A^\top \tilde{U}_A - \tilde{U}_P^\top \tilde{U}_P\| = \max_i |\sin(\theta_i)| \leq \frac{\|\mathcal{L}(A) - \mathcal{L}(P)\|}{\lambda_d(\mathcal{L}(P))} = O_P((n \rho_n)^{-1/2}).$$

Here $\lambda_d(\mathcal{L}(P))$ denotes the $d$ largest eigenvalue of $\mathcal{L}(P)$. We thus have

$$\|\tilde{U}_P^\top \tilde{U}_A - W_1 W_2^\top\|_F = \|\Sigma - I\|_F = \left(\sum_{i=1}^d (1 - \sigma_i)^2\right)^{1/2} \leq \sum_{i=1}^d (1 - \sigma_i^2) = \sum_{i=1}^d \sin^2(\theta_i).$$

Therefore $\|\tilde{U}_P^\top \tilde{U}_A - W_1 W_2^\top\|_F = O_P((n \rho_n)^{-1})$ as desired.

From now on, we shall denote by $W^*$ the orthogonal matrix $W_1 W_2^\top$ as defined in the above proposition. Next, we state the following lemma.
Lemma B.3. Let \((A, X) \sim \text{RDPG}(F)\) with sparsity factor \(\rho_n\). Then
\[
\begin{align*}
n\rho_n\|\tilde{U}_p^T\tilde{U}_A\tilde{S}_A - \tilde{S}_p\tilde{U}_p^T\tilde{U}_A\| &= O_p(1), \quad (B.8) \\
n\rho_n\|\tilde{U}_p^T\tilde{U}_A\tilde{S}_A^{1/2} - \tilde{S}_p^{1/2}\tilde{U}_p^T\tilde{U}_A\| &= O_p(1), \quad (B.9) \\
n\rho_n\|\tilde{U}_p^T\tilde{U}_A\tilde{S}_A^{-1/2} - \tilde{S}_p^{-1/2}\tilde{U}_p^T\tilde{U}_A\| &= O_p(1). \quad (B.10)
\end{align*}
\]

In proving Lemma B.3, we need the following technical result. Lemma B.3 and Lemma B.4 are the key technical lemmas of this paper. Roughly speaking, Lemma B.3 along with Proposition B.2 allows us to interchange the order of the orthogonal transformation \(W^*\) with the diagonal scaling matrices \(S_A\) or \(S_A\); Lemma B.4 simplifies various expressions involving \(A, D, \mathcal{L}(A)\) and \(\tilde{U}_A\).

Lemma B.4. Let \((A, X) \sim \text{RDPG}(F)\) with sparsity factor \(\rho_n\). Then the following holds simultaneously
\[
\begin{align*}
D^{-1/2} - T^{-1/2} &= \frac{1}{2}T^{-3/2}(T - D) + O_p((n\rho_n)^{-3/2}), \quad (B.11) \\
\mathcal{L}(A) &= T^{-1/2}(A - P)T^{-1/2} + D^{-1/2}PD^{-1/2} + O_p((n\rho_n)^{-1}), \quad (B.12) \\
D^{-1/2}PD^{-1/2} - \mathcal{L}(P) &= \frac{1}{2}T^{-3/2}(T - D)PD^{-1/2} \\
&+ \frac{1}{2}T^{-1/2}PT^{-3/2}(T - D) + O_p((n\rho_n)^{-1}). \quad (B.13)
\end{align*}
\]

We continue with the proof of Eq. (3.1). Let \(\Pi = \tilde{U}_p\tilde{U}_p^T\) and \(\Pi^\perp = I - \Pi\). Proposition B.2 and Lemma B.3 then yield
\[
\begin{align*}
\tilde{U}_A\tilde{S}_A^{1/2} - \tilde{U}_pS_p^{1/2}W^* &= \tilde{U}_A\tilde{S}_A^{1/2} - \tilde{U}_pS_p^{1/2}\tilde{U}_p^T\tilde{U}_A + O_p((n\rho_n)^{-1}) \\
&= \tilde{U}_A\tilde{S}_A^{1/2} - \tilde{U}_p\tilde{U}_p^T\tilde{U}_A\tilde{S}_A^{1/2} + O_p((n\rho_n)^{-1}) \\
&= \Pi^\perp\mathcal{L}(A)\tilde{U}_A\tilde{S}_A^{-1/2} + O_p((n\rho_n)^{-1}).
\end{align*}
\]

Since \(\mathcal{L}(P) = \tilde{U}_p\tilde{S}_p\tilde{U}_p^T\), \(\Pi^\perp\mathcal{L}(P) = 0\) and hence
\[
\begin{align*}
\tilde{U}_A\tilde{S}_A^{1/2} - \tilde{U}_pS_p^{1/2}W^* &= \Pi^\perp(\mathcal{L}(A) - \mathcal{L}(P))\tilde{U}_A\tilde{S}_A^{-1/2} + O_p((n\rho_n)^{-1}). \quad (B.18)
\end{align*}
\]

In addition,
\[
\begin{align*}
(\mathcal{L}(A) - \mathcal{L}(P))\tilde{U}_A\tilde{S}_A^{-1/2} &= (\mathcal{L}(A) - \mathcal{L}(P))\Pi^\perp\tilde{U}_A\tilde{S}_A^{-1/2} + (\mathcal{L}(A) - \mathcal{L}(P))\Pi\tilde{U}_A\tilde{S}_A^{-1/2} \\
&= O_p((n\rho_n)^{-1}) + (\mathcal{L}(A) - \mathcal{L}(P))\Pi\tilde{U}_A\tilde{S}_A^{-1/2},
\end{align*}
\]

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where we bound \((\mathcal{L}(A) - \mathcal{L}(P))\Pi U_A S_A^{-1/2}\) using Eq. (B.14) and the submultiplicativity of the spectral norm. Eq. (B.18) then implies

\[
\tilde{U}_A S_A^{1/2} - \tilde{U}_P S_P^{1/2}W^* = \Pi (\mathcal{L}(A) - \mathcal{L}(P)) \tilde{U}_A S_A^{-1/2} + O_p((n\rho_n)^{-1})
\]

\[
= \Pi (\mathcal{L}(A) - \mathcal{L}(P)) \Pi \tilde{U}_A S_A^{-1/2} + O_p((n\rho_n)^{-1}).
\]

(B.19)

By Eq. (B.17) and sub-multiplicativity of the Frobenius norm, we also have

\[
\Pi (\mathcal{L}(A) - \mathcal{L}(P)) \Pi \tilde{U}_A S_A^{-1/2} = (\mathcal{L}(A) - \mathcal{L}(P)) \Pi \tilde{U}_A S_A^{-1/2} - \Pi(\mathcal{L}(A) - \mathcal{L}(P)) \Pi \tilde{U}_A S_A^{-1/2}
\]

\[
= (\mathcal{L}(A) - \mathcal{L}(P)) \Pi \tilde{U}_A S_A^{-1/2} + O_p((n\rho_n)^{-1}).
\]

Eq. (B.19) then becomes

\[
\tilde{U}_A S_A^{1/2} - \tilde{U}_P S_P^{1/2}W^* = \Pi (\mathcal{L}(A) - \mathcal{L}(P)) \Pi \tilde{U}_A S_A^{-1/2} + O_p((n\rho_n)^{-1})
\]

\[
= (\mathcal{L}(A) - \mathcal{L}(P)) \Pi \tilde{U}_A S_A^{-1/2} + O_p((n\rho_n)^{-1})
\]

\[
= (\mathcal{L}(A) - \mathcal{L}(P)) \tilde{U}_P S_p^{1/2}W^* + O_p((n\rho_n)^{-1}).
\]

(B.20)

Recall from Eq. (B.12) the decomposition

\[
\mathcal{L}(A) = T^{-1/2}(A - P)T^{1/2} + D^{-1/2}P D^{-1/2} + O_p((n\rho_n)^{-1}).
\]

Therefore, from Eq. (B.20), we have

\[
\tilde{U}_A S_A^{1/2} - \tilde{U}_P S_P^{1/2}W^* = O_p((n\rho_n)^{-1}) + T^{-1/2}(A - P)T^{1/2}\tilde{U}_P S_P^{-1/2}W^*
\]

\[
+ D^{-1/2}P D^{-1/2} - T^{-1/2}P T^{-1/2}\tilde{U}_P S_P^{-1/2}W^*.
\]

(B.21)

We next recall from Eq. (B.13) the decomposition

\[
D^{-1/2}P D^{-1/2} - T^{-1/2}P T^{-1/2} = \frac{1}{2}T^{1/2}(T - D)P D^{-1/2} + \frac{1}{2}T^{-1/2}P T^{-1/2}(T - D)
\]

\[
+ O_p((n\rho_n)^{-3/2}).
\]

In addition, we recall from Eq. (B.15) that

\[
T^{-1/2}P T^{-1/2}(T - D)\tilde{U}_P = O_p((n\rho_n)^{-1}).
\]

Eq. (B.21) therefore reduces to

\[
\tilde{U}_A S_A^{1/2} - \tilde{U}_P S_P^{1/2}W^* = O_p((n\rho_n)^{-1}) + T^{-1/2}(A - P)T^{1/2}\tilde{U}_P S_P^{-1/2}W^*
\]

\[
+ \frac{1}{2}T^{-3/2}(T - D)P D^{-1/2}\tilde{U}_P S_P^{-1/2}W^*.
\]

(B.22)
Now
\[
T^{-3/2}(T - D)PD^{-1/2} = T^{-3/2}(T - D)P(D^{-1/2} + T^{-1/2})
\]
\[
= T^{-3/2}(T - D)PT^{-1/2} + O_p((n\rho_n)^{-1}),
\]
and thus Eq. (B.22) further simplifies to
\[
\tilde{U}_A \tilde{S}_A^{1/2} - \tilde{U}_p \tilde{S}_p^{1/2} W^* = O_p((n\rho_n)^{-1}) + T^{-1/2}(A - P)T^{-1/2} \tilde{U}_p \tilde{S}_p^{1/2} W^*
\]
\[
+ \frac{1}{2} T^{-3/2}(T - D)PT^{-1/2} \tilde{U}_p \tilde{S}_p^{1/2} W^*.
\]

(B.23)

Since \(T\) and \(D\) are diagonal matrices, we note that
\[
T^{-3/2}(T - D)PT^{-1/2} \tilde{U}_p \tilde{S}_p^{-1/2} W^* = T^{-1}(T - D)T^{-1/2}PT^{-1/2} \tilde{U}_p \tilde{S}_p^{-1/2} W^*
\]
\[
= T^{-1}(T - D)\mathcal{L}(P) \tilde{U}_p \tilde{S}_p^{-1/2} W^*
\]
\[
= T^{-1}(T - D) \tilde{U}_p \tilde{S}_p^{-1/2} W^*
\]
\[
= T^{-1}(T - D) \tilde{U}_p \tilde{S}_p^{1/2} W^*.
\]

We therefore arrive at
\[
\tilde{U}_A \tilde{S}_A^{1/2} - \tilde{U}_p \tilde{S}_p^{1/2} W^* = O_p((n\rho_n)^{-1}) + T^{-1/2}(A - P)T^{-1/2} \tilde{U}_p \tilde{S}_p^{-1/2} W^*
\]
\[
+ \frac{1}{2} T^{-1}(T - D) \tilde{U}_p \tilde{S}_p^{1/2} W^*.
\]

(B.24)

To conclude the proof of Eq. (3.1), we recall that \(\tilde{X} = L(P) = \tilde{U}_p \tilde{S}_p \tilde{U}_p^T\); hence \(\tilde{X} = \tilde{U}_p \tilde{S}_p^{1/2} \tilde{W}\) for some orthogonal matrix \(\tilde{W}\). Therefore
\[
\tilde{U}_p \tilde{S}_p^{1/2} W^* = \tilde{U}_p \tilde{S}_p^{1/2} \tilde{W} \tilde{W}^T W^* = \tilde{X} \tilde{W}^T W^*
\]
\[
\tilde{U}_p \tilde{S}_p^{-1/2} W^* = \tilde{U}_p \tilde{S}_p^{1/2} \tilde{W} \tilde{W}^T \tilde{S}_p^{-1} \tilde{W} \tilde{W}^T W^* = \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \tilde{W}^T W^*.
\]

Substituting the above equations into Eq. (B.24) yields
\[
\tilde{U}_A \tilde{S}_A^{1/2} - \tilde{X} \tilde{W}^T W^* = O_p((n\rho_n)^{-1}) + T^{-1/2}(A - P)T^{-1/2} \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \tilde{W}^T W^*
\]
\[
+ \frac{1}{2} T^{-1}(T - D) \tilde{X} \tilde{W}^T W^*.
\]

Equivalently,
\[
\tilde{U}_A \tilde{S}_A^{1/2} (W^*)^T \tilde{W} - \tilde{X} = O_p((n\rho_n)^{-1}) + T^{-1/2}(A - P)T^{-1/2} \tilde{X} (\tilde{X}^T \tilde{X})^{-1}
\]
\[
+ \frac{1}{2} T^{-1}(T - D) \tilde{X}.
\]

Eq. (3.1) is thereby established.
B.3 Proof of Lemma B.3 and Lemma B.4

We first present the proof of Lemma B.4. We recall the notations $D = \text{diag}(A1)$ and $T = \text{diag}(P1)$. Denote by $d_i$ and $t_i$ the $i$-th diagonal elements of $D$ and $T$. The $i$-th diagonal element of $D^{-1/2} - T^{-1/2}$ can be written as

$$\frac{1}{\sqrt{d_i}} = \frac{t_i - d_i}{(\sqrt{d_i} + \sqrt{t_i})\sqrt{d_i}\sqrt{t_i}} = \frac{t_i - d_i}{2t_i^{3/2}} + \frac{(t_i - d_i)(1)}{d_i\sqrt{t_i} + t_i\sqrt{d_i}} = \frac{t_i - d_i}{2t_i^{3/2}} + \frac{t_i\sqrt{t_i} - \sqrt{d_i}(t_i - d_i)\sqrt{t_i}}{2t_i^{3/2}(d_i\sqrt{t_i} + t_i\sqrt{d_i})}.$$

We have, by Chernoff’s bound, that $|t_i - d_i| = O(\sqrt{n\rho_n})$ for any given index $i$, and hence $|\sqrt{t_i} - \sqrt{d_i}| = O(1)$. Therefore,

$$(t_i - d_i)\frac{t_i\sqrt{t_i} - \sqrt{d_i}(t_i - d_i)\sqrt{t_i}}{2t_i^{3/2}(d_i\sqrt{t_i} + t_i\sqrt{d_i})} = O(\sqrt{n\rho_n})\frac{O(n\rho_n)}{O(n\rho_n)^3} = O((n\rho_n)^{-3/2}).$$

Upon taking an union bound over all indices $i = 1, 2, \ldots, n$, we have

$$D^{-1/2} - T^{-1/2} = \frac{1}{2}T^{-3/2}(T - D) + O((n\rho_n)^{-3/2} \log n). \quad (B.25)$$

Eq. (B.11) is thereby established. Eq. (B.13) follows directly from Eq. (B.11) and the definition of $\mathcal{L}(P) = T^{-1/2}PT^{-1/2}$. We next show Eq. (B.12). Consider the following decomposition of $\mathcal{L}(A)$

$$\mathcal{L}(A) = D^{-1/2}(A - P)D^{-1/2} + D^{-1/2}PD^{-1/2} = T^{-1/2}(A - P)T^{-1/2} + T^{-1/2}(A - P)(D^{-1/2} - T^{-1/2}) + (D^{-1/2} - T^{-1/2})(A - P)D^{-1/2} + D^{-1/2}PD^{-1/2}.$$

By Lemma B.1, we have

$$\|(A - P)T^{-1/2}\| \leq \|A - P\| \times \|T^{-1/2}\| = O(1). \quad (B.26)$$

Similarly, Lemma B.1 and Chernoff bound yield

$$\|(A - P)D^{-1/2}\| \leq \|(A - P)\| \times \|D^{-1/2}\| = O(1). \quad (B.27)$$

Combining Eq. (B.25) and Eq. (B.27), we have

$$\|(D^{-1/2} - T^{-1/2})(A - P)D^{-1/2}\| \leq \|(T^{-3/2}(D - T))\|/2 + O((n\rho_n)^{-3/2})) \times O(1) = O((n\rho_n)^{-1}).$$

Similarly, Eq. (B.25) and Eq. (B.26) implies

$$\|T^{-1/2}(A - P)(D^{-1/2} - T^{-1/2})\| = O((n\rho_n)^{-1}).$$
We thus have

$$\mathcal{L}(A) = T^{-1/2}(A - P)T^{-1/2} + D^{-1/2}PD^{-1/2} + O_\varphi((n\rho_n)^{-1}).$$  \hspace{1cm} (B.28)$$

Eq. (B.12) is thereby established.

We next derive Eq. (B.15) through Eq. (B.17). From Eq. (B.28), we have

$$\tilde{U}_p^\top(\mathcal{L}(A) - \mathcal{L}(P))\tilde{U}_p = \tilde{U}_p^\top T^{-1/2}(A - P)T^{-1/2}\tilde{U}_p$$

$$+ \tilde{U}_p^\top(D^{-1/2}PD^{-1/2} - T^{-1/2}PT^{-1/2})\tilde{U}_p \hspace{1cm} (B.29)$$

We first bound the spectral norm of $\tilde{U}_p^\top T^{-1/2}(A - P)T^{-1/2}\tilde{U}_p$. Let $\tilde{u}_i$ be the $i$-th column of $\tilde{U}_pT^{-1/2}$; the $ij$-th entry of $\tilde{U}_p^\top T^{-1/2}(A - P)T^{-1/2}\tilde{U}_p$ is then of the form

$$\tilde{u}_i^\top(A - P)\tilde{u}_j = \sum_{k<l} 2\tilde{u}_{ik}(a_{kl} - p_{kl})\tilde{u}_{jl} + \sum_k \tilde{u}_{ik}p_{kk}\tilde{u}_{jk}$$

where $\tilde{u}_{ik}$ is the $k$-th element of the vector $\tilde{u}_i$. We note that

$$|\sum_k \tilde{u}_{ik}p_{kk}\tilde{u}_{jk}| \leq \rho_n\|\tilde{u}_i\| \times \|\tilde{u}_j\| \leq \rho_n\delta_n^{-1} = O_\varphi(\rho_n(n\rho_n)^{-1}).$$

In addition, $\sum_{k<l} 2\tilde{u}_{ik}(a_{kl} - p_{kl})\tilde{u}_{jl}$ is, conditioned on $P$, a sum of mean 0 random variables. Hoeffding’s inequality then implies

$$P \left[ \left| \sum_{k<l} 2\tilde{u}_{ik}(a_{kl} - p_{kl})\tilde{u}_{lj} \right| \geq t \right] \leq \exp\left(-\frac{t^2}{2(\sum_{k<l} \tilde{u}_{ik}^2 \tilde{u}_{lj}^2)}\right)$$

$$\leq \exp\left(-\frac{t^2}{2 \sum_k \sum_j \tilde{u}_{ik}^2 \tilde{u}_{jl}^2}\right)$$

$$\leq \exp\left(-\frac{t^2}{2\delta_n^{-2}}\right).$$

Hence $\tilde{u}_i^\top(A - P)\tilde{u}_j = O_\varphi(\delta_n^{-1})$. As $\tilde{U}_pT^{-1/2}(A - P)T^{-1/2}\tilde{U}_p$ is a $d \times d$ matrix, a union bound then implies

$$\tilde{U}_p^\top T^{-1/2}(A - P)T^{-1/2}\tilde{U}_p = O_\varphi(\delta^{-1}) = O_\varphi((n\rho_n)^{-1}).$$  \hspace{1cm} (B.30)$$

We next bound the spectral norm of $\tilde{U}_p^\top(D^{-1/2}PD^{-1/2} - T^{-1/2}PT^{-1/2})\tilde{U}_p$. Let $\zeta_{ij}$ denote the $ij$-th entry of $\tilde{U}_p^\top(D^{-1/2}PD^{-1/2} - T^{-1/2}PT^{-1/2})\tilde{U}_p$. From Eq. (B.25), we have

$$\zeta_{ij} = \tilde{u}_i^\top \left( (D^{-1/2} - T^{-1/2})PD^{-1/2} + T^{-1/2}P(D^{-1/2} - T^{-1/2}) \right)\tilde{u}_j$$

$$= \frac{1}{2} \tilde{u}_i^\top \left( T^{-3/2}(T - D)PD^{-1/2} + T^{-1/2}PT^{-3/2}(T - D) \right)\tilde{u}_j + O_\varphi((n\rho_n)^{-3/2}).$$

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Now let $\zeta_{ij}^{(1)}$ and $\zeta_{ij}^{(2)}$ denote the quantities

$$\zeta_{ij}^{(1)} = \frac{1}{2} u_i^T T^{-3/2} (T - D) P D^{-1/2} u_j,$$

$$\zeta_{ij}^{(2)} = \frac{1}{2} u_i^T T^{-1/2} P T^{-3/2} (T - D) u_j.$$

Because $P = \rho_n X X^T$, we have

$$\zeta_{ij}^{(1)} \leq \frac{1}{2} \|\rho_n^{1/2} u_i^T T^{-3/2} (T - D) X\| \times \|\rho_n^{1/2} X^T D^{-1/2} u_j\|,$$

$$\zeta_{ij}^{(2)} \leq \frac{1}{2} \|\rho_n^{1/2} u_i^T T^{-1/2} X\| \times \|\rho_n^{1/2} X^T T^{-3/2} (T - D) u_j\|.$$

For $k \in \{1, 2, \ldots, d\}$, let $x_k$ denote the $k$-th column of $X$. Furthermore, for $l \in \{1, 2, \ldots, n\}$, let $x_{kl}$ denote the $l$-th entry of $x_k$ – equivalently the $k$-th entry of $X_l$ (recall that $X = [X_1 | \cdots | X_n]^T$). Also let $\tilde{u}_{kl}$ denotes the $l$-th entry of $\tilde{u}_k$. Then $\rho_n^{-1/2} u_i^T T^{-3/2} (T - D) X$ is a vector in $\mathbb{R}^d$ whose $k$-th element is of the form

$$\rho_n^{1/2} \tilde{u}_i^T T^{-3/2} (D - T) x_k = \rho_n^{1/2} \sum_l \frac{\tilde{u}_i l}{t_{il}^{3/2}} (d_l - t_l) x_{kl}$$

$$= \rho_n^{1/2} \sum_l \sum_m \frac{\tilde{u}_i l}{t_{il}^{3/2}} (a_{lm} - p_{lm}) x_{kl}$$

$$= 2 \rho_n^{1/2} \sum_{l < m} \frac{\tilde{u}_i l}{t_{il}^{3/2}} (a_{lm} - p_{lm}) x_{kl} + \rho_n^{1/2} \sum_l \frac{\tilde{u}_i l}{t_{il}^{3/2}} p_l x_{kl}$$

Conditioned on $P$, the above is a sum of mean 0 random variables and a term of order $O((n \rho_n)^{-3/2})$. Hoeffding’s inequality then yields

$$\mathbb{P} \left[ \left| 2 \rho_n^{1/2} \sum_{l < m} \frac{\tilde{u}_i l}{t_{il}^{3/2}} (a_{lm} - p_{lm}) x_{kl} \right| \geq s \right] \leq 2 \exp \left( -\frac{s^2}{2 \rho_n \sum_{l < m} t_{il}^{-3} \tilde{u}_i l^2 x_{kl}^2} \right)$$

$$\leq 2 \exp \left( -\frac{s^2}{2 n \rho_n \sum_{l < m} t_{il}^{-3} \tilde{u}_i l^2 x_{kl}^2} \right)$$

$$\leq 2 \exp \left( -\frac{s^2}{2 n \rho_n \delta_n^{-3} \sum_l \tilde{u}_i l} \right)$$

$$\leq 2 \exp \left( -\frac{s^2 \delta_n^3}{2 n \rho_n \|\tilde{u}_i\|^2} \right)$$

$$\leq 2 \exp \left( -\frac{s^2 \delta_n^3}{2 n \rho_n} \right)$$

where we used the fact that $x_{kl}^2 \leq 1$ for all indices $k$ and $l$ (as $(A, X) \sim RDPG(F)$). We thus have

$$\rho_n^{1/2} \tilde{u}_i^T T^{-3/2} (D - T) x_k = O_P ((n \rho_n)^{-1}) \quad \text{(B.31)}$$
A union bound over the $d$ entries of $\rho_n^{1/2} \tilde{u}_i^\top T^{-3/2} (T - D) X$ along with the bound $\|\rho_n^{1/2} T^{-1/2} X\| = \Theta_p(1)$ yield that $\zeta_{ij}^{(1)} = \Theta_p((n \rho_n)^{-1})$. An identical argument also yield that $\zeta_{ij}^{(2)} = \Theta_p((n \rho_n)^{-1})$. Therefore, $\zeta_{ij} = O((n \rho_n)^{-1})$. A union bound over the indices $i, j \in \{1, 2, \ldots, d\}$ also implies

$$T^{-1/2} P T^{-3/2} (T - D) \tilde{U}_P = \Theta_p((n \rho_n)^{-1}), \quad (B.32)$$

$$\tilde{U}_P^\top T^{-3/2} (T - D) P D^{-1/2} = \Theta_p((n \rho_n)^{-1}), \quad (B.33)$$

$$\|\tilde{U}_P^\top (D^{-1/2} P D^{-1/2} - T^{-1/2} P T^{-1/2}) \tilde{U}_P\| = \Theta_p((n \rho_n)^{-1}). \quad (B.34)$$

We thus derive Eq. (B.15) and Eq. (B.16). Eq. (B.17) follows from Eq. (B.29), Eq. (B.30) and Eq. (B.34). Lemma B.4 is thereby established.

Lemma B.3 now follows directly from Lemma B.4. Indeed, by Eq. (B.14) and Eq. (B.17), we have

$$\tilde{U}_P \tilde{U}_A S_A - \tilde{S}_P \tilde{U}_P \tilde{U}_A = \tilde{U}_P L(A) \tilde{U}_A - \tilde{U}_P L(P) \tilde{U}_A$$

$$= \tilde{U}_P^\top (L(A) - L(P))(R + \tilde{U}_P \tilde{U}_P^\top \tilde{U}_A)$$

$$= \Theta_p((n \rho_n)^{-1}) + \tilde{U}_P (L(A) - L(P)) \tilde{U}_P \tilde{U}_P^\top \tilde{U}_A$$

$$= \Theta_p((n \rho_n)^{-1}). \quad (B.35)$$

Eq. (B.8) is thereby established. We now establish Eq. (B.9), noting that the same argument applies also to Eq. (B.10). For $i, j \in \{1, 2, \ldots, d\}$, let $r_{ij}$ denote the $ij$-th entry of $\tilde{U}_P \tilde{U}_A$. Also, for $i \in \{1, 2, \ldots, d\}$, let $\lambda_i(A)$ and $\lambda_j(P)$ denote the $i$-th eigenvalue of $L(A)$ and $L(P)$, respectively. Then the $ij$-th entry of $\tilde{U}_P \tilde{U}_A S_A - \tilde{S}_P \tilde{U}_P \tilde{U}_A$ is of the form

$$r_{ij} (\tilde{\lambda}_j^{1/2}(A) - \tilde{\lambda}_i^{1/2}(P)) = \frac{r_{ij} (\lambda_j(A) - \lambda_i(P))}{\lambda_j^{1/2}(A) + \lambda_i^{1/2}(P)}.$$ 

Since $\tilde{\lambda}_i(A) = \Theta_p(1)$ and $\tilde{\lambda}_j(P) = \Theta_p(1)$, the previous expression and Eq. (B.35) yield

$$r_{ij} (\tilde{\lambda}_j^{1/2}(A) - \tilde{\lambda}_i^{1/2}(P)) = \Theta_p((n \rho_n)^{-1}).$$

A union bound over $i, j$ then implies Eq. (B.9).

**B.4 Proof of Eq. (3.4) and Eq. (3.5)**

Recall Eq. (3.1), i.e., with $\zeta = (\tilde{X} W - \tilde{X})$, we have

$$\|\zeta\|_F = \|T^{-1/2} (A - P) T^{-3/2} (X - \tilde{X}) \| + \frac{1}{2} T^{-1/2} (T - \tilde{T}) \tilde{X} \|_F + \Theta_p((n \rho_n)^{-1}).$$

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The above implies,
\[ \|z\|^2_F = \|T^{-1/2}(A - P)T^{-1/2}\tilde{X}(\tilde{X}^\top \tilde{X})^{-1}\|^2_F + \frac{1}{\gamma} \|T^{-1}(T - D)\tilde{X}\|^2_F \\
+ \text{tr} \tilde{X}^\top T^{-1}(T - D)T^{-1/2}(A - P)T^{-1/2}\tilde{X}(\tilde{X}^\top \tilde{X})^{-1} + O_p((n\rho_n)^{-3/2}). \]

We show Eq. (3.4) and Eq. (3.5) by analyzing each term in the right hand side of the above display. In particular, we shall show that these terms are concentrated around their expected values; evaluation of these expected values, in the limit as \( n \to \infty \), yield Eq. (3.4) and Eq. (3.5).

We first consider the term \( Z = \|T^{-1/2}(A - P)T^{-1/2}\tilde{X}(\tilde{X}^\top \tilde{X})^{-1}\|^2_F \). We note that conditional on \( P \), \( Z \) is a function of the \( n(n-1)/2 \) independent random variables \( \{a_{ij}\}_{i<j} \). It is therefore expected that \( Z \) will be concentrated around its expectation \( \mathbb{E}[Z] \) where the expectation is taken with respect to \( A \), conditional on \( P \). We verify this below.

Let \( A' = (a'_{ij}) \) be an independent copy of \( A \), i.e., the upper triangular entries of \( A' \) are independent Bernoulli random variables with mean parameters \( \{p_{ij}\}_{i<j} \). Let \( A^{(ij)} \) be the matrix obtained by replacing the \( (i,j) \) and \( (j,i) \) entries of \( A \) by \( a'_{ij} \) and let \( Z^{(ij)} = \|T^{-1/2}(A^{(ij)} - P)T^{-1/2}\tilde{X}(\tilde{X}^\top \tilde{X})^{-1}\|^2_F \). We show concentration of \( Z \) around \( \mathbb{E}[Z] \) using the following concentration inequality from [9, Theorem 5 and Theorem 6].

**Theorem B.5.** Assume that there exists positive constants \( a \) and \( b \) such that
\[ \sum_{i<j} (Z - Z^{(ij)})^2 \leq aZ + b. \]

Then for all \( t > 0 \),
\[ \mathbb{P}[Z - \mathbb{E}[Z] \geq t] \leq \exp\left(\frac{-t^2}{4a\mathbb{E}[Z] + 4b + 2at}\right), \tag{B.36} \]
\[ \mathbb{P}[Z - \mathbb{E}[Z] \leq -t] \leq \exp\left(\frac{-t^2}{4a\mathbb{E}[Z]}\right). \tag{B.37} \]

We now bound \( \sum_{i<j} (Z - Z^{(ij)})^2 \). For notational convenience, we denote the \( i \)-th row of \( \tilde{X}(\tilde{X}^\top \tilde{X})^{-1} \) by \( \zeta_i \) and the \( i \)-th row of \( T^{-1/2}(A - P)T^{-1/2}\tilde{X}(\tilde{X}^\top \tilde{X})^{-1} \) by \( \xi_i \). We shall also denote the inner product between vectors in Euclidean space by \( \langle \cdot, \cdot \rangle \). For each \( i \), \( \xi_i = \sum_{j=1}^n \frac{a_{ij} - p_{ij}}{\sqrt{t_{ij}}} \zeta_j \) and hence
\[ Z = \sum_{k=1}^n \xi_k^2 = \sum_{k=1}^n \sum_{\ell=1}^n \sum_{\ell'=1}^n \frac{(ak_{\ell} - p_{k\ell})(ak_{\ell'} - p_{k\ell'})}{t_k \sqrt{t_\ell t_{\ell'}}} \langle \zeta_\ell, \zeta_{\ell'} \rangle. \]

Now \( A \) and \( A^{(ij)} \) differs possibly only in the \((i,j)\) and \((j,i)\) entries; furthermore, the \( \{t_i\} \) do not depend on the entries of \( A \) and \( A^{(ij)} \). We thus have, upon
Thus, since \( a_{ij} = a_{i'j} \) and \( \mathbf{Z} \) denote a generic constant, not depending on \( \mathbf{Z} \), in the above display and could change from line to line. In the above derivation, we have used the fact that \( C_0 \sqrt{n} \leq \| \mathbf{X} \| \leq \sqrt{n} \) for some constant \( C_0 > 0 \) and \( \| \mathbf{T} \| \geq \delta \geq C_1 n \rho_n \) for some constant \( C_1 > 0 \).
We then have, by Theorem B.5, that for all \( t > 0 \),
\[
\mathbb{P}[Z - \mathbb{E}[Z] > t] \leq \exp\left(-\frac{-Ct^2}{(n\rho_n)^{-2}\mathbb{E}[Z] + 2(n\rho_n)^{-2}t}\right) \tag{B.38}
\]
\[
\mathbb{P}[Z - \mathbb{E}[Z] > -t] \leq \exp\left(-\frac{-Ct^2}{(n\rho_n)^{-2}\mathbb{E}[Z]}\right). \tag{B.39}
\]
In addition, it is straightforward to see that \( \mathbb{E}[Z] \leq C_3(n\rho_n)^{-1} \), for some constant \( C_3 > 0 \); here the expectation is taken with respect to \( \mathbf{A} \) conditional on \( \mathbf{P} \). We therefore have that there exists a constant \( C > 0 \) such that \( t = C(n\rho_n)^{-3/2} \log^{1/2} n \) yield
\[
Z = \mathbb{E}[Z] + O_P((n\rho_n)^{-3/2} \log^{1/2} n)
\]
\[
= \mathbb{E}\|\mathbf{T}^{-1/2}(\mathbf{A} - \mathbf{P})\mathbf{T}^{-1/2}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\|_F^2 + O_P((n\rho_n)^{-3/2} \log^{1/2} n) \tag{B.40}
\]
We now evaluate \( \mathbb{E}[Z] \). We have
\[
\mathbb{E}[Z] = \mathbb{E}\left[\text{tr} (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{T}^{-1/2}(\mathbf{A} - \mathbf{P})\mathbf{T}^{-1}(\mathbf{A} - \mathbf{P})\mathbf{T}^{-1/2}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\right]
\]
\[
= \text{tr} (\mathbf{X}^\top\mathbf{X})^{-1}\mathbb{E}[\mathbf{X}]^{-1/2}(\mathbf{A} - \mathbf{P})\mathbf{T}^{-1/2}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}. \tag{B.41}
\]
We note that \( \mathbf{T}^{-1/2}(\mathbf{A} - \mathbf{P})\mathbf{T}^{-1}(\mathbf{A} - \mathbf{P})\mathbf{T}^{-1/2} \) is a \( n \times n \) matrix whose \( ij \)-th entry \( \xi_{ij} \) is of the form
\[
\xi_{ij} := \sum_k t_i^{-1} t_j^{-1/2} (a_{ik} - p_{ik})(a_{kj} - p_{kj})
\]
and hence
\[
\mathbb{E}[\xi_{ij}] = \begin{cases} 0 & \text{if } i \neq j \\ \sum_k t_i^{-1} t_j^{-1} p_{ik}(1 - p_{ik}) & \text{if } i = j \end{cases}
\]
We shall denote by \( \mathbf{\bar{M}} \) the diagonal matrix \( (\mathbb{E}[\xi_{ij}]) \) as given above. Then
\[
n\rho_n\mathbb{E}[Z] = n\rho_n \text{tr} (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{\bar{M}}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}
\]
We first recall from Eq. (B.2) that \( \mathbf{X}^\top\mathbf{X} \xrightarrow{a.s.} \mathbf{\bar{X}} \) and \( (\mathbf{X}^\top\mathbf{X})^{-1} \xrightarrow{a.s.} \mathbf{\bar{X}}^{-1} \) as \( n \to \infty \).
We next consider \( n\rho_n\mathbf{X}^\top\mathbf{\bar{M}}\mathbf{X} \). Let \( \bar{m}_i \) denote the \( i \)-th diagonal element of \( \mathbf{\bar{M}} \). We have
\[
n\rho_n\mathbf{X}^\top\mathbf{\bar{M}}\mathbf{X} = n\rho_n \sum_i \frac{\rho_n X_i \mu^\top \bar{m}_i}{t_i}
\]
\[
= n\rho_n \sum_i \frac{\rho_n X_i \mu^\top \bar{m}_i}{\rho_n \sum_j X_i^\top X_j}
\]
\[
= \sum_i \frac{X_i \mu^\top \bar{m}_i}{nX_i^\top \mu} + \sum_i \frac{X_i \mu^\top \bar{m}_i}{nX_i^\top \mu} \left( \frac{nX_i^\top \mu - \sum_j X_i^\top X_j}{\sum_j X_i^\top X_j} \right), \tag{44}
\]
Similar to our derivation of Eq. (B.2), we have
\[-(\sup_{j \in [n]} n \rho_n \bar{m}_j c_j) \leq \sum_i X_i X_i^\top n \rho_n \bar{m}_i \leq (\sup_{j \in [n]} n \rho_n \bar{m}_j c_j) \sum_i X_i X_i^\top \]

In addition, for each index $i$,
\[n \rho_n \bar{m}_i = n \rho_n \sum_k t_i^{-1} t_k^{-1} p_{ik}(1 - p_{ik}) = O_p(1)\]
and hence $\sup_{i \in [n]} n \rho_n \bar{m}_i \xrightarrow{a.s.} 0$ as $n \to \infty$. Therefore
\[
\sum_i X_i X_i^\top n \rho_n \bar{m}_i \xrightarrow{a.s.} 0 \tag{B.41}
\]
as $n \to \infty$. We thus only need to consider
\[
\sum_i X_i X_i^\top n \rho_n \bar{m}_i = \sum_i \sum_k \frac{n \rho_n X_i X_i^\top p_{ik}(1-p_{ik})}{(X_i^\top \mu)\mu_{ik}}
\]
\[
= \sum_i \sum_k \frac{n \rho_n X_i X_i^\top p_{ik}(1-p_{ik})}{(X_i^\top \mu)\mu_{ik}} - \sum_k \frac{n \rho_n X_k X_k^\top}{(X_k^\top \mu)\mu_{k}}
\]
\[
= \sum_i \sum_k \frac{n \rho_n X_i (X_i^\top X_k - p_{ik} X_i^\top X_k X_k^\top X_i)}{n^2 (X_i^\top \mu)^2 (X_k^\top \mu)}
\]
\[
+ \sum_i \sum_k \frac{n \rho_n X_i (X_i^\top X_k - p_{ik} X_i^\top X_k X_k^\top X_i)}{n^2 (X_i^\top \mu)^2 (X_k^\top \mu)} \xrightarrow{a.s.} 0
\]
as $n \to \infty$. An analogous argument to that used in deriving Eq. (B.41) yield
\[
\sum_i \sum_k \frac{n \rho_n X_i (X_i^\top X_k - p_{ik} X_i^\top X_k X_k^\top X_i)}{n^2 (X_i^\top \mu)^2 (X_k^\top \mu)} \xrightarrow{a.s.} 0
\]
as $n \to \infty$. It thus remains to evaluate
\[
\sum_i \sum_k \frac{n \rho_n X_i (X_k X_i^\top - p_{ik} X_i^\top X_k X_k^\top X_i)}{n^2 (X_i^\top \mu)^2 (X_k^\top \mu)}
\]
The strong law of large numbers implies
\[
\sum_i \sum_k \frac{n \rho_n X_i X_i^\top X_k X_k^\top}{n^2 (X_i^\top \mu)^2 (X_k^\top \mu)} \xrightarrow{a.s.} \mathbb{E} \left[ \frac{X_1 X_1^\top X_1^\top \bar{\mu}}{(X_1^\top \mu)^2} \right]
\]
\[
\rho_n \sum_i \sum_k \frac{n \rho_n X_i X_i^\top X_k X_k^\top}{n^2 (X_i^\top \mu)^2 (X_k^\top \mu)} \rightarrow \rho_n \mathbb{E} \left[ \frac{X_1 X_1^\top X_1^\top \Delta X_1}{(X_1^\top \mu)^2} \right].
\]
We invoke Slutsky’s theorem and conclude that
\[
\begin{align*}
    n\rho_n Z &= n\rho_n \|T^{-1/2}(A - P)T^{-1/2}\overline{X}(\overline{X}^\top \overline{X})^{-1}\|^2_F \\
    &= n\rho_n \text{tr} (\overline{X}^\top \overline{X})^{-1} \overline{X}^\top \overline{X} (\overline{X}^\top \overline{X})^{-1} + O_P((n\rho_n)^{-1/2} \log^{1/2} n) \\
    &\to \text{tr} \Delta^{-1} \mathbb{E} \left[ \frac{X_1 X_1^\top (X_1^\top \mu_{ij} - \rho_n X_1^\top \Delta X_1)}{(X_1^\top \mu)^2} \right] \Delta^{-1}.
\end{align*}
\]

We next bound \( Z := \|(T - D)T^{-1}\overline{X}\|^2_F \). \( Z \) is again a function of the \( n(n - 1)/2 \) independent random variables \( \{a_{ij}\}_{i<j} \). Let \( Z^{(ij)} = \|(T - D^{(ij)})T^{-1}\overline{X}\| \) where \( D^{(ij)} \) is the diagonal matrix whose diagonal entries are the degrees of \( A^{(ij)} \); we recall that \( A^{(ij)} \) is obtained by replacing the \((i, j)\) and \((j, i)\) entries of \( A \) with an independent copy \( a'_{ij} \) of \( a_{ij} \). We now bound \( \sum_{i<j}(Z - Z^{(ij)})^2 \). Let \( \overline{X}_i \) denote the \( i \)-th row of \( \overline{X} \). Then
\[
Z = \sum_i \frac{(t_k - d_k)^2}{t_k^2} \|\overline{X}_k\|^2,
\]
and hence (with \( d_k^{(ij)} \) denoting the degree of vertex \( k \) in \( A^{(ij)} \))
\[
Z - Z^{(ij)} = \sum_i ((t_k - d_k)^2 - (t_k - d_k^{(ij)})^2) \|\overline{X}_k\|^2 \\
= \sum_i (d_k^{(ij)} - d_k)(2t_k - d_k - d_k^{(ij)}) \|\overline{X}_k\|^2 \\
= (a'_{ij} - a_{ij}) \left((2t_i - 2d_i + a_{ij} - a'_{ij}) \|\overline{X}_i\|^2 + (2t_j - 2d_j + a_{ij} - a'_{ij}) \|\overline{X}_j\|^2 \right).
\]
Using the fact that \((b + c)^2 \leq 2b^2 + 2c^2\) and that \( a_{ij} = a_{ji}, a'_{ij} = a'_{ji} \) we have
\[
(Z - Z^{(ij)})^2 \leq 2(a'_{ij} - a_{ij})^2(2t_i - 2d_i + a_{ij} - a'_{ij}) \frac{\|\overline{X}_i\|^4}{t_i^4} + 2(a'_{ji} - a_{ji})^2(2t_j - 2d_j + a_{ij} - a'_{ij}) \frac{\|\overline{X}_j\|^4}{t_j^4},
\]
from which we derive
\[
\sum_{i<j} (Z - Z^{(ij)})^2 \leq \sum_{i=1}^n \sum_{j=1}^n (a'_{ij} - a_{ij})^2(16(t_i - d_i)^2 + 4) \frac{\|\overline{X}_i\|^4}{t_i^4} \\
\leq \sum_{i=1}^n \sum_{j=1}^n (16(t_i - d_i)^2 + 4) \frac{\|\overline{X}_i\|^4}{t_i^4} \\
\leq \sum_{i=1}^n \sum_{j=1}^n (16(t_i - d_i)^2 + 4) \frac{\|\overline{X}_i\|^4}{t_i^4} \frac{(\rho_n t_i^{-1}) \|X_i\|^2}{t_i^2} \\
\leq C \sum_{i=1}^n \sum_{j=1}^n (16(t_i - d_i)^2 + 4) \frac{\|\overline{X}_i\|^2}{t_i^2} - n^{-3} \rho_n^{-2} \\
\leq C_1(n\rho_n)^{-2} Z + C_2(n\rho_n)^{-4} \leq C_3(n\rho_n)^{-2} Z
\]
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for some constants $C_1, C_2, C_3 > 0$. Once again, we apply Theorem B.5 to conclude

$$
P[Z - \mathbb{E}[Z] > t] \leq \exp\left(-\frac{Ct^2}{(n\rho_n)^{-2}\mathbb{E}[Z] + 2(n\rho_n)^{-2}t}\right) \quad \text{(B.43)}$$

$$
P[Z - \mathbb{E}[Z] > -t] \leq \exp\left(-\frac{Ct^2}{(n\rho_n)^{-2}\mathbb{E}[Z]}\right). \quad \text{(B.44)}$$

In addition, $\mathbb{E}[Z] = \mathbb{E}[\sum_k (t_k - d_k)^2 t_k^{-2}\|X_k\|^2] \leq C(n\rho_n)^{-1}$ for some constant $C > 0$; here the expectation is taken with respect to $D$ conditional on $P$. We thus conclude

$$
Z = \| (T - D)T^{-1}X \|_F^2
= \mathbb{E}[\| (T - D)T^{-1}X \|_F^2] + O_p((n\rho_n)^{-3/2}\log^{1/2}(n)). \quad \text{(B.45)}
$$

We now evaluate $n\rho_n\mathbb{E}[\|1/2 (T - D)T^{-1}X \|_F^2] = n\rho_n\frac{1}{4}\text{tr}\tilde{X}^\top(T - D)^2T^{-1}\mathbb{E}[\| (T - D)T^{-1}X\|_F^2].$

Now $T^{-1}\mathbb{E}[\| (T - D)^2\|]T^{-1}$ is a diagonal matrix whose $i$-th diagonal entry is of the form $t_i^{-2}\sum_j p_{ij}(1 - p_{ij})$. Hence,

$$
n\rho_n\tilde{X}^\top T^{-1}\mathbb{E}[\| (T - D)^2\|]T^{-1}\tilde{X} = n\rho_n^2 \sum_i t_i^{-3}X_iX_i^\top \sum_j p_{ij}(1 - p_{ij})
= \sum_i n\rho_n^2 X_iX_i^\top \sum_j \frac{1}{(n\rho_n X_i^\top X_i)^3} \rho_n X_i^\top X_j(1 - \rho_n X_i^\top X_j) + o_p(1)
= \sum_i n^{-1} \frac{X_iX_i^\top}{(X_i^\top X_i)^3} \sum_j n^{-1} X_i^\top X_j(1 - \rho_n X_i^\top X_j) + o_p(1)
= \sum_i n^{-1} \frac{X_iX_i^\top}{(X_i^\top X_i)^3} \sum_j n^{-1} X_i^\top X_j(1 - \rho_n X_i^\top X_i) + o_p(1)
$$

We therefore have

$$
n\rho_n\mathbb{E}[\|1/2 (T - D)X \|_F^2] \xrightarrow{a.s.} \frac{1}{4}\text{tr}\left(\mathbb{E}\left[\frac{X_iX_i^\top}{(X_i^\top X_i)^3}\left(1 - \frac{\rho_n X_i^\top \Delta X_i}{(X_i^\top X_i)^3}\right)\right]\right) \quad \text{(B.46)}$$

as $n \to \infty$.

Finally we consider $Z := n\rho_n\text{tr} \tilde{X}^\top T^{-1}(T - D)T^{-1/2}(A - P)T^{-1/2}\tilde{X}(\tilde{X}^\top \tilde{X})^{-1}$. A similar, albeit slightly more tedious, argument to that used in deriving Eq. (B.40) and Eq. (B.45) yields

$$
Z = \text{tr} \tilde{X}^\top T^{-1}(T - D)T^{-1/2}(A - P)T^{-1/2}\tilde{X}(\tilde{X}^\top \tilde{X})^{-1}
= \text{tr} \mathbb{E}\left[\tilde{X}^\top T^{-1}(T - D)T^{-1/2}(A - P)T^{-1/2}\tilde{X}(\tilde{X}^\top \tilde{X})^{-1}\right] + O_p((n\rho_n)^{-3/2}\log^{1/2}(n)).
$$
We now evaluate $E[Z]$. We have

$$E[Z] = \text{tr} \tilde{X}^T T^{-3/2} E[(T - D)(A - P)] T^{-1/2} \tilde{X} (\tilde{X}^T \tilde{X})^{-1}$$

Now the $ij$-th entry of $E[(A - P)(T - D)]$ is of the form

$$E[(a_{ij} - p_{ij})(t_j - d_j)] = E[(a_{ij} - p_{ij}) \sum_k (p_{jk} - a_{jk})]$$

$$= \sum_k E[(a_{ij} - p_{ij})(p_{kj} - a_{kj})] = -p_{ij}(1 - p_{ij}),$$

and hence, with $\circ$ denoting the Hadamard product of matrices,

$$n_\rho E[Z] = -n_\rho \text{tr} \tilde{X}^T T^{-3/2} (P - P \circ P) T^{-1/2} \tilde{X} (\tilde{X}^T \tilde{X})^{-1}$$

$$= -n_\rho \text{tr} \tilde{X}^T T^{-1} (T^{-1/2} PT^{-1/2} - T^{-1/2}(P \circ P)T^{-1/2}) \tilde{X} (\tilde{X}^T \tilde{X})^{-1}$$

$$= -n_\rho \text{tr} \tilde{X}^T T^{-1} \tilde{X} + n_\rho \text{tr} \tilde{X}^T T^{-3/2}(P \circ P)T^{-1/2} \tilde{X} (\tilde{X}^T \tilde{X})^{-1}.$$

(B.47)

We first consider the term $n_\rho \text{tr} \tilde{X}^T T^{-1} \tilde{X}$. We have

$$n_\rho \text{tr} \tilde{X}^T T^{-1} \tilde{X} = n_\rho \sum_i \frac{\rho_n X_i^T X_i}{t_i^2} = -\frac{1}{n} \sum_i X_i^T X_i \text{tr} \left[ \frac{X_1 X_1^T}{(X_1^T \mu)^2} \right] + o_p(1),$$

and hence

$$-n_\rho \text{tr} \tilde{X}^T T^{-1} \tilde{X} \xrightarrow{a.s.} -\text{tr} E \left[ \frac{X_1 X_1^T}{(X_1^T \mu)^2} \right].$$

(B.48)

Finally, we consider the term $n_\rho \text{tr} \tilde{X}^T T^{-3/2}(P \circ P) T^{-1/2} \tilde{X}$. We recall that $(\tilde{X}^T \tilde{X})^{-1} \xrightarrow{a.s.} \Delta^{-1}$ as $n \to \infty$. In addition,

$$n_\rho \text{tr} \tilde{X}^T T^{-3/2}(P \circ P) T^{-1/2} \tilde{X} = n_\rho^2 \tilde{X}^T T^{-2}(P \circ P) T^{-1} \tilde{X}$$

$$= n_\rho^2 \sum_i \sum_j \frac{p_{ij}^2}{t_i^4} X_i X_j^T$$

$$= n_\rho^2 \sum_i \sum_j \frac{p_{ij}^2}{(n \rho_n)^3 (X_i^T \mu)^2 X_j^T \mu} X_i X_j^T + o_p(1)$$

$$= n_\rho^2 \sum_i \sum_j \frac{\rho_n^2 (X_i^T X_j)^2}{(n \rho_n)^3 (X_i^T \mu)^2 X_j^T \mu} X_i X_j^T + o_p(1)$$

$$= \rho_n \sum_i \sum_j \frac{1}{n} \frac{X_i^T X_j X_j^T X_i}{(X_i^T \mu)^2 X_j^T \mu} X_i X_j^T + o_p(1).$$

We thus conclude

$$n_\rho \text{tr} \tilde{X}^T T^{-3/2}(P \circ P) T^{-1/2} \tilde{X} (\tilde{X}^T \tilde{X})^{-1} \xrightarrow{a.s.} \rho_n \text{tr} E \left[ \frac{X_1^T X_2 X_2^T X_1}{(X_1^T \mu)^2 X_2^T \mu} X_1 X_1^T \right] \Delta^{-1}$$

(B.49)
where the expectation is taken with respect to $X_1, X_2$ being i.i.d drawn from $F$. Combining Eq. (B.48) and Eq. (B.49) yield

$$
np \rho_n \text{tr} \left[ \bar{X}^\top T^{-1}(T - D)T^{-1/2}(A - P)T^{-1/2}\bar{X} \bar{X}^\top \right] \xrightarrow{a.s.} \rho_n \text{tr} \left[ \frac{X_1^\top X_2X_2^\top X_1 - X_1X_1^\top}{(X_1^\top \mu)(X_2^\top \mu)} \right] \Delta - \text{tr} \left[ \frac{X_1X_1^\top}{(X_1^\top \mu)^2} \right]. \quad (B.50)
$$

Eq. (3.4) and Eq. (3.5) then follows directly from Eq. (B.42), Eq. (B.46) and Eq. (B.50).

## C Within-block variances

We now verify that Theorem 4.1 and Theorem 4.2 are indeed generalizations of Theorem 3.1 and Theorem 3.2 from [6]. Suppose that $K = d$, i.e., that $B$ is invertible. Then denoting by $\nu$ the $d \times d$ matrix $\nu = [\nu_1 | \nu_2 | \cdots | \nu_d]$, we have that $\nu$ is also invertible and that $B = \nu^\top \nu$ and $\Delta = \nu \text{diag}(\nu) \nu^\top$. Let $z_k = (\nu_k^\top \nu_1 - \nu_k^\top \nu_2 X_1^\top X_1 \nu_k)$ and $\bar{z}_k = (\nu_k^\top \nu_1 - \nu_k^\top \nu_d X_1^\top X_1 \nu_k)$. Then

$$
\text{tr} \left[ \nu \text{diag}(\nu) \text{diag}(z_k) \nu^\top \right] = \nu \left( \text{diag}(\pi) \text{diag}(z_k) \right) \nu^\top.
$$

Then Eq. (4.5) in Theorem 4.1 simplifies to

$$
n^2 \hat{d}_{kk} \xrightarrow{a.s.} \text{tr} \Delta^{-3} \text{tr} \left[ X_1X_1^\top (\nu_k^\top X_1 - \nu_k^\top X_1 \nu_k) \right] = \text{tr} \left( \nu \text{diag}(\pi) \nu^\top \right)^{-3} \nu \left( \text{diag}(\pi) \text{diag}(z_k) \right) \nu^\top \quad (C.1)
$$

$$
= \text{tr} \left( (\nu^\top)^{-1} \text{diag}(\pi)^{-1} \nu^{-1} \text{diag}(\pi) \text{diag}(z_k) \nu^{-1} \nu^\top \right)
$$

$$
= \text{tr} \left( \text{diag}(\pi)^{-1} B^{-1} \right) \text{diag}(z_k)
$$

$$
= \text{tr} \left( \text{diag}(\pi)^{-1/2} B^{-1} \text{diag}(\pi)^{-1/2} \right) \text{diag}(z_k)
$$

$$
= \sum_l \sum_{l'} \nu_k^\top \nu_l (1 - \nu_k^\top \nu_l) \frac{(B_{ll'}^{-1})^2}{\pi_l \pi_{l'}}
$$

$$
= \sum_l \sum_{l'} \left( B_{ll'}^{-1} \right) \frac{(B_{ll'}^{-1})^2}{\pi_l \pi_{l'}}
$$

where $B_{ll'}^{-1}$ is the $ll'$-th entry of $B^{-1}$. We note that the above expression for $\hat{d}_{kk}$ can be written purely in terms of the entries of $B$ and $\pi$ without the need to find the $\nu_1, \nu_2, \ldots, \nu_d$ explicitly.

We compare Eq. (C.1) with Theorem 3.1 in [6]. Let $A$ be sampled from a stochastic blockmodel with parameters $B = [A_n \ B_n \ C_n \ D_n] \text{ and } \pi = (\pi_1, \pi_2)$ with $\alpha_n, \beta_n \neq \gamma_n$. In [6], it is assume that the number of vertices assigned to block 1
and block 2 are $n\pi_1$ and $n\pi_2$, respectively. For ease of exposition and without loss of generality, suppose that the row indices of $A$ are such that the first $n\pi_1$ rows correspond to vertices assigned to block 1 and the last $n\pi_2 = n - n\pi_1$ rows correspond to vertices assigned to block 2. Let $v_1$ and $v_2$ denote the eigenvectors corresponding to the largest and second largest eigenvector of $P = ZBZ^\top$ where $Z$ is a $n \times 2$ matrix whose $i$-th row is $(1, 0)$ for $i = 1, 2, \ldots, n\pi_1$ and is $(0, 1)$ for $i = n\pi_1 + 1, n\pi_1 + 2, \ldots, n$. We then have that $v_1 = (x_1, x_1, \ldots, x_1, y_1, y_1, \ldots, y_1)$ for some $x_1, y_1$, i.e., the first $n\pi_1$ elements of $v_1$ are $x_1$ and the remaining $n\pi_2$ elements are $y_1$. Similarly, we have $v_2 = (x_2, x_2, \ldots, x_2, y_2, y_2, \ldots, y_2)$ for some $x_2, y_2$. Then Eq. (3.1) in [6] states that (the notation $a_n \sim b_n$ in [6] means $a_n/b_n = 1 + o_p(1)$)

$$
\hat{d}_{11} \sim \left(\left(\frac{x_1^2}{\lambda_1} + \frac{x_2^2}{\lambda_2}\right)n\pi_1\alpha_n(1 - \alpha_n) + \left(\frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2}\right)n\pi_2\gamma_n(1 - \gamma_n)\right) \quad (C.2)
$$

where $\lambda_1$ and $\lambda_2$ are the largest and second largest eigenvalues of $P$. We can rewrite Eq. (C.2) as

$$
\hat{d}_{11} \sim \text{tr} (P^\dagger)^2 \text{diag}((\alpha_n(1 - \alpha_n), \gamma_n(1 - \gamma_n), \ldots)) \quad (C.3)
$$

where $P^\dagger$ is the Moore-Penrose pseudo-inverse of $P$ and the first $n\pi_1$ entries of the diagonal matrix $\text{diag}(\alpha_n(1 - \alpha_n), \gamma_n(1 - \gamma_n), \ldots)$ are $\alpha_n(1 - \alpha_n)$ while the remaining $n\pi_2$ diagonal entries are $\gamma_n(1 - \gamma_n)$. As $Z$ is of full-column rank, we have $Z^\dagger = (Z^\top Z)^{-1}Z^\top = \text{diag}(1/(n\pi_1), 1/(n\pi_2))Z^\top$. Furthermore, $B$ is invertible and hence

$$
P^\dagger = (ZBZ^\top)^\dagger = (Z^\top)^\dagger B^{-1}Z^\dagger = n^{-2}Z \text{diag}(\pi)^{-1}B^{-1}\text{diag}(\pi)^{-1}Z^\top.
$$

Therefore,

$$(P^\dagger)^2 = n^{-3}Z \text{diag}(\pi)^{-1}B^{-1}\text{diag}(\pi)^{-1}B^{-1}\text{diag}(\pi)^{-1}Z^\top$$

and hence

$$n^2\hat{d}_{11} \sim n^2 \text{tr} (P^\dagger)^2 \text{diag}((\alpha_n(1 - \alpha_n), \gamma_n(1 - \gamma_n), \ldots))$$

$$\sim n^{-1} \text{tr} Z(\text{diag}(\pi)^{-1}B^{-1})^2\text{diag}(\pi)^{-1}Z^\top \text{diag}(\alpha_n(1 - \alpha_n), \gamma_n(1 - \gamma_n), \ldots))$$

$$\sim \text{tr} (\text{diag}(\pi)^{-1}B^{-1})^2 \text{diag}(\alpha_n(1 - \alpha_n), \gamma_n(1 - \gamma_n))$$

which is a special case of Eq. (C.1). Theorem 4.1 is thus an extension of Theorem 3.1 in [6] to general $K$-block stochastic blockmodels, provided that the block probability matrix is positive semidefinite.

We now consider $n^2\hat{d}_{kk}$. When $B$ is invertible, Eq. (4.10) in Theorem 4.2 can be simplified in a manner similar to the derivation of Eq. (C.1). Let $\mu = (\mu_1, \mu_2, \ldots, \mu_d)$ where $\mu_k = \nu_k^\top \mu$. Then $\Delta = \nu(\text{diag}(\pi)\text{diag}(\mu)^{-1})\nu^\top$. The right
hand side of Eq. (4.10) can be decompose as $\zeta_1 - \zeta_2 + \zeta_3$ with $\zeta_1$ given by

$$
\zeta_1 = \text{tr} \tilde{\Delta}^{-3} \mathbb{E} \left[ \frac{X_1 X_1^T}{(X_1^T \mu)^2} \nu_k^T X_1 - \nu_k^T X_1 X_1^T \nu_k \right] \\
= \frac{1}{\mu_k} \text{tr} \tilde{\Delta}^{-3} \nu^T (\text{diag}(\pi)\text{diag}(\mu)^{-2}\text{diag}(z_k)) \nu^T \\
= \frac{1}{\nu_k^T \mu} \text{tr} (\text{diag}(\pi)^{-1}\text{diag}(\mu)\nu^{-1}(\nu^T)^{-1})^2\text{diag}(\mu)^{-1}\text{diag}(z_k) \\
= \frac{1}{\mu_k} \text{tr} (\text{diag}(\pi)^{-1}\text{diag}(\mu)B^{-1})^2\text{diag}(\mu)^{-1}\text{diag}(z_k) \\
= \frac{1}{\mu_k} \text{tr} (\text{diag}(\pi)^{-1/2}\text{diag}(\mu)^{1/2}B^{-1}\text{diag}(\mu)^{1/2}\text{diag}(\pi)^{-1/2})^2\text{diag}(\mu)^{-1}\text{diag}(z_k) \\
= \sum_i \sum_{\nu} \frac{(B^{l}_{i\nu})^2 \nu_k \nu_l (1 - \nu_k^T \nu_l)}{\pi_l \pi_{\nu} \mu_k} = \sum_i \sum_{\nu} \frac{B_{kl}(1 - B_{kl})(B^{l}_{i\nu})^2 \mu_l}{\pi_l \pi_{\nu} \mu_k}.
\]

Let $e_k$ denote the vector whose $i$-th element is 1 if $i = k$ and 0 otherwise. For $\zeta_2$, we have

$$
\zeta_2 = \text{tr} \tilde{\Delta}^{-2} \mathbb{E} \left[ \frac{X_1 X_1^T}{X_1^T \mu} \nu_k^T X_1 - \nu_k^T X_1 X_1^T \nu_k \right] \\
= \frac{1}{\mu_k^2} \text{tr} \tilde{\Delta}^{-2} \nu^T (\text{diag}(\pi)\text{diag}(\mu)^{-1}\text{diag}(z_k)) 1 \nu^T \\
= \frac{1}{\mu_k^2} \text{tr} \text{diag}(\pi)^{-1}\text{diag}(\mu)\nu^{-1}(\nu^T)^{-1}\text{diag}(z_k) 1 e^T_k \\
= \frac{1}{\mu_k^2} \text{tr} \text{diag}(\pi)^{-1}\text{diag}(\mu)B^{-1}\text{diag}(z_k) 1 e^T_k \\
= \frac{1}{\pi_k \mu_k} \sum_i \nu_k^T \nu_l (1 - \nu_k^T \nu_l) B^{-1}_{kl} = \frac{1}{\pi_k \mu_k} \sum_i B_{kl}(1 - B_{kl}) B^{-1}_{kl}.
\]

Finally for $\zeta_3$ we have

$$
\zeta_3 = \frac{1}{4\mu_k^2} \text{tr} \tilde{\Delta}^{-1} \nu_k \nu_k \mathbb{E}[\nu_k^T X_1 - \nu_k^T X_1 X_1^T \nu_k] \\
= \frac{\mathbb{E}[\nu_k^T X_1 - \nu_k^T X_1 X_1^T \nu_k]}{4\mu_k^2} \text{tr} \nu_k^T \tilde{\Delta}^{-1} \nu_k \\
= \frac{\mathbb{E}[\nu_k^T X_1 - \nu_k^T X_1 X_1^T \nu_k]}{4\mu_k^2} \text{tr} \nu_k^T (\nu^T)^{-1}(\text{diag}(\pi)^{-1}\text{diag}(\mu))\nu^{-1} \nu_k \\
= \frac{\mathbb{E}[\nu_k^T X_1 - \nu_k^T X_1 X_1^T \nu_k]}{4\mu_k^2} \frac{\mu_k}{\pi_k} = \sum_i \frac{\pi_l B_{il}(1 - B_{il})}{4\pi_k \mu_k^2}.
\]

As $\mu_k = \sum_i \pi_i \nu_k^T \nu_l = \sum_i \pi_i B_{il}$, $\zeta_1$, $\zeta_2$ and $\zeta_3$ can also be written purely in terms of the entries of $B$ and $\pi$.
For the two-block stochastic blockmodel, Eq. (3.3) in [6] states that
\[ n^2 \tilde{d}_{11} \sim \frac{\alpha_n (1 - \alpha_n)}{\mu_1^2} \left( \frac{1}{4} + \frac{\pi_2 \gamma_n}{\mu_1 \lambda_2^2} \right) + \frac{\gamma_n (1 - \gamma_n)}{\mu_2} \left( \frac{\pi_2}{4 \pi_1} + \frac{\pi_1 \alpha_n}{\mu_2 \lambda_2^2} \right) \]  
(C.4)

where \( \lambda_2 = \pi_1 \pi_2 (\alpha_n \beta_n - \gamma_n^2) / (\mu_1 \mu_2) \) is the second largest eigenvalue of \( L(P) \) (c.f. Lemma 6.1 in [6]). Verifying that \( \zeta_1 - \zeta_2 + \zeta_3 \) does indeed yield Eq. (C.4) for the two-block stochastic blockmodel is a straightforward computation. We omit the details. Theorem 4.2 is thus an extension of Theorem 3.2 in [6] for general \( K \)-blocks stochastic blockmodels whenever the matrix of block probabilities is positive semidefinite.