One-dimensional Dirac electrons on the surface of weak topological insulators

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We show that a class of weak three-dimensional topological insulators feature one-dimensional Dirac electrons on their surfaces. Their hallmark is a line-like energy dispersion along certain directions of the surface Brillouin zone. These one-dimensional Dirac line degeneracies are topologically protected by a symmetry that we refer to as in-plane time-reversal invariance. We show how this invariance leads to Dirac lines in the surface spectrum of stacked Kane-Mele systems and more general models for weak three-dimensional topological insulators.

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The field of topological properties of matter is one of the most active areas in condensed matter physics [1–5]. In particular, topological insulators (TIs) have attracted a lot of attention in recent years due to their unique physical properties and potential applications [6, 7]. In contrast to conventional insulators, TIs feature topologically robust gapless edge or surface states with a linear Dirac dispersion. Most interestingly, these so-called topological surface states are a direct physical consequence of the nontrivial topology of the bulk band structure and protected by time-reversal symmetry.

The number of inequivalent classes of TIs depends on the dimension of the systems under consideration [8–9]. In two dimensions, there is only one class which is characterized by a single topological invariant. Experimental realizations are, for instance, HgTe quantum well structures [10–11]. In three dimensions, TIs are described by a quadruple of topological invariants and their variety is, therefore, much larger. One then distinguishes between weak and strong TIs. The latter possess an odd number of Dirac-like surface states at the time-reversal invariant (TRI) momenta of the surface Brillouin zone (BZ). For this reason, they are stable against generic perturbations preserving time-reversal symmetry. On the contrary, weak TIs are unstable against translational-symmetry breaking perturbations, such as charge density waves, due to the presence of an even number of Dirac-like degeneracies in the surface band structure. Furthermore, they are topologically equivalent to a stack of layers of two-dimensional (2D) TIs. Both classes have been identified experimentally, including, for instance, the strong TIs Bi1−xSbx [12–13], Bi2Se3 [14], Bi2Te3 [14], as well as the recently found weak TI Bi14Rh2I9 [15].

The topologically protected surface Kramers doublets found in three-dimensional (3D) TIs are usually referred to as Dirac points [16], since in their vicinity the dispersion of the surface states resembles that of a 2D Dirac electron (see Fig. 1(b)). However, a recent experimental work by Gibson et al. [17] suggests that 3D TIs can also exhibit Dirac-like line degeneracies on their surfaces corresponding to effectively one-dimensional (1D) Dirac electrons. By angle-resolved photoemission spectroscopy they established the presence of highly anisotropic surface states on the (113) surface of Ru3Sn2, which show an almost line-like dispersion along certain high-symmetry directions.

Thus far, 1D Dirac states have not been investigated in the context of 3D topological insulators. The first question that arises is whether and under what circumstances topological insulators can exhibit Dirac-like line degeneracies on their surface, and, secondly, whether they are protected by any kind of symmetry. In this Letter, we establish that Dirac lines naturally occur in weak topological insulators during the transition from effectively two dimensions to three dimensions. Furthermore, these lines are topologically protected by a symmetry that we refer to as in-plane time-reversal invariance.

Below we first introduce and motivate in-plane time reversal invariance and show how it leads to topologically protected Dirac lines on the surface of weak TIs. We will then indicate why protected 1D Dirac surface states are not possible in strong TIs and demonstrate explicitly how Dirac lines arise due to in-plane time-reversal invariance in certain weak TI model Hamiltonians, namely stacked Kane-Mele systems [18–19].

**In-plane time-reversal invariance** — Let us consider a stack of arbitrary but identical 2D TIs where the stacking direction is, without loss of generality, the z direction. Furthermore, let the system be finite in a direction perpendicular to the stacking direction, e.g. the x direction. Generically, each of the 2D TI ribbons will have topologically protected edge states inside the bulk energy gap with an edge Kramers doublet at a TRI point [19, 20]. Let us first inspect the case where the layers are completely decoupled and the band structure of the system does not disperse in the kx direction. Thus, the Kramers doublet of the 1D BZ of the layers become perfectly flat line degeneracies, more precisely Dirac lines, in the surface BZ of the stack. With this stacking procedure we have constructed a weak topological insulator with Z2 indices (τ2; τ1ν2τ3) = (0; 001) [8, 9], and with Dirac lines instead of Dirac points on its surfaces. This is not sur-
prprising, since the stack of 2D TIs is still an effectively 2D system. Generally, one would expect the Dirac lines to break up and leave only Dirac points at TRI momenta once an arbitrary time-reversal interlayer coupling is introduced. This, in turn, leads to a typical surface band dispersion of a weak TI with an even number of surface Dirac cones \(8\). However, in the following we will show that this is not necessarily the case and that the presence of in-plane time-reversal invariance leads to topologically protected Dirac lines even in the full 3D system with coupled layers.

In-plane time-reversal invariance is the 2D analogue of the conventional time-reversal symmetry with respect to a specific plane \(\Pi\) (see Fig. 1(a)). The corresponding antiunitary operator \(T_{\Pi}\) acts similarly to its 3D analogue except that the momentum component perpendicular to the plane \(\Pi\) remains unchanged. Without loss of generality, we choose the plane to be \((001)\). Then, the in-plane time-reversal operator for spin-\(\frac{1}{2}\) particles reads

\[
T_{(001)} := i(1 \otimes s^y)K \text{ with } k_x, k_y, k_z \rightarrow -k_x, -k_y, k_z. \quad (1)
\]

Here, \(s^y\) acts on the spin part of the system, where, again without loss of generality, we have chosen \(z\) as the quantization axis of spin and the Pauli matrices as a spin representation. The identity matrix \(1\) is in the space spanned by additional degrees of freedom, such as orbital or sublattice. We further denote the matrix part of the operator, without the action on momentum, by \(T_{(001)}\). Note that \(T_{(001)}\) has the same operator structure as the conventional time-reversal operator \(T\). Obviously, for a 2D system \(T_{(001)}\) and \(T\) are identical. If we deal with a weak TI with weak indices \(\nu = (\nu_1, \nu_2, \nu_3)\), we denote the in-plane time-reversal operator associated with the \((\nu_1, \nu_2, \nu_3)\) plane by \(T_{\nu}\).

Since both types of time-reversal operators appear to be very similar, one expects the presence of an analogue of Kramers theorem for the in-plane time-reversal symmetric system. To establish this, we recall that a time-reversal symmetric system satisfies \(TH(k_x, k_y, k_z)T^{-1} = H(-k_x, -k_y, -k_z)\), i.e. the Bloch Hamiltonian commutes with \(T\) at TRI points where \((k_x, k_y, k_z) = (-k_x, -k_y, k_z)\). For particles with half-integer spin, this implies Kramers theorem \(8\). For an in-plane time-reversal symmetric system, we have instead

\[
T_{(001)}H(k_x, k_y, k_z)T_{(001)}^{-1} = H(-k_x, -k_y, k_z). \quad (2)
\]

Hence, for half-integer particles the energy spectrum must be at least two-fold degenerate along in-plane time-reversal invariant lines satisfying \((k_x, k_y, k_z) = (-k_x, -k_y, k_z)\). This thus gives rise to Kramers lines along which Kramers-like degeneracies are guaranteed and topologically protected by \(T_{(001)}\) invariance. Since this condition includes also the 8 TRI points in the BZ, the Kramers lines always connect certain Kramers points in the BZ.

By the bulk-edge correspondence, one generically finds topologically protected Dirac lines on surfaces perpendicular to the stacking direction. The Dirac lines resemble the dispersion of a 1D Dirac particle in a 2D space: along one direction, it shows the typical linear Dirac dispersion whereas the dispersion is line-like in the perpendicular direction. This is illustrated in Fig. 1(c). If \(T_{\nu}\) is broken without breaking conventional time-reversal symmetry, only degeneracies at the Kramers points in the BZ are still topologically protected. Therefore, each Dirac line splits and the associated 1D Dirac particle decays into two 2D Dirac particles.

Precisely for this reason, topologically protected Dirac lines cannot exist in strong TIs. Consider a strong TI with one Dirac point and without any in-plane time-reversal symmetry. It is always possible to reduce the number of Dirac points of a strong TI to one by introducing suitable translational symmetry breaking and surface potential terms. Now choose an arbitrary plane \((klm)\) and establish the corresponding \(T_{klm}\) symmetry. Before that, the single Dirac point of the strong TI must be connected to the bulk continuum along any line that will later become a Kramers line once in-plane time-reversal symmetry has been established. Otherwise, there would be a second Dirac point at the opposite TRI momentum and the system would not be a strong TI. In the process of establishing \(T_{klm}\) symmetry, the “arms” of the Dirac point, therefore, pull down the bulk bands to which they are attached and thereby close the bulk energy gap. Thus the system cannot be an insulator and either becomes a semimetal or a metal. For this reason, the presence of in-plane time-reversal symmetry, which is essential for
the appearance of Dirac lines, is not reconcilable with a strong TI. An explicit demonstration of this behavior for a specific model Hamiltonian is given in the supplemental material [21].

There is another appealing point of view which illustrates the connection between in-plane time-reversal symmetry and Dirac lines in weak TIs. For this, we consider a weak TI with weak indices (001) such that the associated stacking direction is the z direction. Furthermore, let us treat \( k_z \) in the corresponding Hamiltonian \( H(\mathbf{k}_x, \mathbf{k}_y, k_z) \) as a parameter. If now in-plane time-reversal symmetry \( T_{001} \) is present in the weak TI, we have a 2D quantum spin-Hall insulator (QSHI) for any value of the parameter \( k_z \) (see Fig. 1(a)). This can be easily seen. First, this is clearly the case for \( k_z = 0 \) and \( k_z = \pi \) since the system obeys conventional time-reversal symmetry and is topologically non-trivial. However, as we move along \( k_z \), the 2D systems always preserve time-reversal symmetry in the 2D sense due to the presence of in-plane time-reversal symmetry in the full 3D system. Moreover, the bulk gap does not close. Therefore, we cannot have a topological phase transition and the collection of 2D systems stay in the QSHI phase for all \( k_z \). This also implies that any such 2D QSHI will have topologically protected (spin-filtered) edge states that will form edge Kramers doublets at the TRI momenta of the 1D BZ. When we now move along \( k_z \), the edge Kramers doublets cannot be broken, which implies the existence of Dirac lines along the in-plane TRI lines in surface BZ of the weak TI.

**Stacked Kane-Mele models** — Having established the general consequences of in-plane time-reversal invariance, we apply these notions explicitly to stacked Kane-Mele systems. The Kane-Mele model [18, 19] is known to be a realization of a 2D TI in certain parameter ranges. It comprises a 2D nearest-neighbor tight-binding model on a honeycomb lattice with additional TRI-invariant spin-orbit interaction terms, where \( T \) is the *conventional* time-reversal operator (see Ref. [5]). The model is described by the following Hamiltonian [18, 19]

\[
\mathcal{H}_{\text{KM}} = t \sum_{\langle \mathbf{l}, \mathbf{j} \rangle, \sigma} c_{\mathbf{l} \sigma}^\dagger c_{\mathbf{j} \sigma} + i\lambda_{\text{SO}} \sum_{\langle \mathbf{l}, \mathbf{j} \rangle, \sigma^l, \sigma^r} \nu_{l} c_{\mathbf{l} \sigma^l}^\dagger \xi_{\sigma^l}^{x} c_{\mathbf{j} \sigma^r} + i\lambda_{\text{R}} \sum_{\langle \mathbf{l}, \mathbf{j} \rangle, \sigma^l, \sigma^r} \xi_{\sigma^l}^{z} c_{\mathbf{l} \sigma^l}^\dagger c_{\mathbf{j} \sigma^r},
\]

where the notations and coefficients are as in Ref. [19]. We note that the second term describes in-plane \( z \rightarrow -z \) symmetric spin-orbit coupling (SOC), the third term represents in-plane Rashba SOC, and the third term is a staggered sublattice potential which breaks inversion symmetry in the plane. As usual, we denote the two interpenetrating hexagonal sublattices of the honeycomb lattice as \( A \) and \( B \) and follow the conventions of Ref. [19].

From this 2D model we build up a 3D system by stacking the Kane-Mele layers along the \( z \) direction. This is done in such a way that corresponding lattice points of the same sublattice but in different layers lie on top of each other (AA stacking). In order to couple the layers, we introduce a nearest-neighbor interlayer hopping term and an interlayer SOC term, leading us to the model Hamiltonian

\[
\mathcal{H} = \sum_{\mathbf{l}} \mathcal{H}_{\text{KM}, \mathbf{l}} + \tau \sum_{\langle \mathbf{l}, \mathbf{l}' \rangle} \sum_{\sigma^l, \sigma^r} c_{\mathbf{l} \sigma^l}^\dagger c_{\mathbf{l}' \sigma^r}^\dagger \xi_{\sigma^l}^{x} c_{\mathbf{l}' \sigma^r} + i\lambda_{\text{SO}, \perp} \sum_{\langle \mathbf{l}, \mathbf{l}' \rangle} \mu_{l'} c_{\mathbf{l} \sigma^l}^\dagger \xi_{\sigma^l}^{z} c_{\mathbf{l}' \sigma^r}^\dagger,
\]

where \( \tau \) indexes the layers, and \( \mu_{l'} = \pm 1 \) for \( \mathbf{l} \gtrless \mathbf{l}' \). We note that after a Fourier transformation, the corresponding Bloch Hamiltonian \( H(\mathbf{k}) \) is a \( 4 \times 4 \) matrix which can be expanded in terms of Dirac matrices and their commutators similar to Ref. [19]. Furthermore, these matrices can be written as Kronecker products of Pauli matrices \( \sigma^i \) in sublattice space and Pauli matrices \( s^i \) in spin space. In this way, the interlayer terms of the Hamiltonian become

\[
\mathcal{H}_{l l'}(\mathbf{k}) = 2\tau \cos k_z (\mathbf{1} \otimes \mathbf{1}),
\]

\[
\mathcal{H}_{\text{SO}, \perp}(\mathbf{k}) = -2\lambda_{\text{SO}, \perp} \sin k_z (\mathbf{1} \otimes s^z),
\]

where the interlayer distance has been set to unity. The other terms of the Hamiltonian can be found in Ref. [19]. The relevant in-plane time-reversal operator for this model is \( T_{001} = i(\mathbf{1} \otimes s^y)K \) with \( (k_x, k_y, k_z) \rightarrow (-k_x, -k_y, k_z) \).

In particular, we are interested in surface states which will be studied for a slab of thickness \( W \) with (010) surfaces, where \( W \) is measured in the number of unit cells. The surface cuts out so-called zigzag edges from each Kane-Mele layer. In other words, we will investigate a stack of infinitely many Kane-Mele layers with zigzag termination. The corresponding Bloch Hamiltonian \( H^{(010)}(k_x, k_z) \) of the slab is a \( 4W \times 4W \) matrix whose energies are obtained by exact diagonalization.

In Fig. 2, the band structure of the (010) Kane-Mele slab along high-symmetry lines of the surface BZ is shown.
the surface BZ for the zigzag termination. Therefore, a

Along the surface band emerging from the bulk. The bands meet in the light of in-plane time-reversal symmetry. With the presence of in-plane time-reversal invariance. One might actually view such an in-plane time-reversal invariant weak TI as a collection of 2D QSHIs in momentum space, where the momentum component perpendicular to the surface described by the weak indices serves as a parameter. Topologically protected 1D Dirac electrons cannot appear on the surface of strong TIs. Experimentally, the surface Dirac lines connecting two time-reversal invariant points in an in-plane time-reversal invariant weak TI can in principle be detected by angle-resolved photoemission spectroscopy.

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Appendix A: Stacked Kane-Mele model with mass term

In the main part of this work, we ignored the staggered sublattice potential term in $H_{KM}$. However, it is well-known that such a mass term can result in a transition from a topological to a trivial insulator in the 2D Kane-Mele model by closing and reopening the bulk energy gap [18,19]. What happens to the Dirac line in the stacked system, if we increase the mass? First of all, it is easy to check that the mass term preserves the relevant in-plane time-reversal symmetry. Therefore, the Dirac line cannot be destroyed in the process. How can the surface states then be trivial in the trivial sector? The key is the closing of the bulk energy gap. Before, the surface states are connected to both the upper and the lower bulk continuum. However, the process of closing and reopening the gap allows them to merge only with the upper or the lower bulk continuum, respectively. In this way, the Dirac line remains intact but the surface states do no longer traverse the bulk energy gap and are, therefore, topologically trivial (see Fig. 4).

FIG. 4. (color online). Energy bands for the stacked Kane-Mele model with mass term.
Appendix B: Cubic Liu-Qi-Zhang model

The cubic Liu-Qi-Zhang model \[16\] is derived from a model introduced by Zhang et al. \[13\], which has been successfully used to describe the Bi$_2$Se$_3$ family of strong topological insulators. It is a 3D nearest-neighbor tight-binding model on a simple cubic lattice with two orbital and two spin degrees of freedom per site. The corresponding Hamiltonian in momentum representation is \[16\]

$$
\mathcal{H} = \sum_{\mathbf{k}} \sum_{a,b=1}^{2} \sum_{\sigma\sigma'} H_{a\sigma,b\sigma'}(\mathbf{k}) d_{k\sigma a}^\dagger d_{k\sigma' b},
$$

with the $4 \times 4$ Bloch Hamiltonian \[16\]

$$
H(\mathbf{k}) = [M_0 + 6B - 2B \sum_{i=1}^{3} \cos k_i] \Gamma_5 + A \sum_{i=1}^{3} \Gamma_i \sin k_i.
$$

Here, $\Gamma_j$ denotes the Dirac matrices of Ref.\[16\] where also the introduced parameters and notations are explained. We note that the Dirac matrices are Kronecker products of Pauli matrices $s^i$ in spin space and Pauli matrices $\sigma^i$ in orbital space. The coordinate system used is aligned with the edges of the cubic unit cell and we write $\mathbf{k} = (k_1, k_2, k_3) \equiv (k_x, k_y, k_z)$. The model describes a trivial insulator for $M_0 > 0$ and $M_0 < -12$, a strong TI with $\mathbb{Z}_2$ indices $(1;000)$ or $(1;111)$ for $0 > M_0 > -4B$ or $-8B > M_0 > -12B$, and a weak TI with $\mathbb{Z}_2$ indices $(0;111)$ for $-4B > M_0 > -8B$ \[16\], where in all cases we have $A = B$. In particular, we focus on the weak TI phase $(0;111)$. For this set-up, surface states are studied for a slab of thickness $W$ with $(001)$ surfaces. The corresponding Bloch Hamiltonian, we have to work with, is $4W \times 4W$ and the band structures in the slab geometry are obtained by exact diagonalization.

Let us first check the weak TI phase for in-plane time-reversal invariance with respect to the $yz$ plane. The corresponding operator is

$$
\mathcal{T}_{(100)} = i (s^y \otimes 1) K \text{ with } k_y, k_z \rightarrow -k_y, -k_z.
$$

It is easy to verify that this symmetry is broken due to the $A \sin k_x \Gamma_1$ term in Eq. (7). In Fig. 5 we show what happens to the band structure of the systems in a $(001)$ slab geometry, if we tune the symmetry-breaking term to zero. Obviously, the two-fold degeneracy for the surface states along $\Gamma X$ ($YM$ is not shown) is restored. However, along this line the surface states were connected to the bulk before carrying out the deformation. For this reason, they pull the bulk bands down, resulting in a closing of the bulk energy gap. Hence, the system undergoes a semimetal transition, if we reestablish the in-plane time-reversal symmetry with respect to the $yz$ plane. Besides, this would have happened also for the $xz$ and the $xy$ plane (not shown).

Nevertheless, by analogy with the stacked Kane-Mele model we expect the weak TI with $(0;111)$ to develop Dirac lines without a semimetal transition, if we restore the in-plane time-reversal symmetry with respect to the $(111)$ plane – the plane described by the weak indices.

![FIG. 5. (color online). Energy bands for the cubic Liu-Qi-Zhang model: bands of a (001) slab of width $W = 40$ are shown along high symmetry lines of the surface BZ associated with the original cubic unit cell. The model parameters are $M_0 = -5.0$, $A = B = 1.0$ for panels (a), (c), and $M_0 = -2.0$, $A = B = 1.0$ for panels (b), (d). We show the band structures for broken (upper panels) and restored (lower panels) in-plane time-reversal symmetry with respect to the $yz$ plane. Surface states are highlighted in red.](image)

![FIG. 6. (color online). Alternative description of the lattice in the cubic Liu-Qi-Zhang model: the three inequivalent layers corresponding to different elements $A, B, C$ of the new hexagonal unit cell are illustrated. Within the layers, the hexagonal structure of the basal plane is clearly visible. Moreover, the new primitive lattice vectors $a_1, a_2, c$ and the shift vector $\Delta$ are shown.](image)
For convenience, we choose a different unit cell with a different coordinate system attached to it. Since a weak TI with weak indices (111) is topologically equivalent to a stack of 2D TIs stacked along the (111) direction, we are going to construct the new unit cell in this light.

The Z axis of the new coordinate system points along the (111) direction of the original coordinate system. We further want to align the c axis of the new unit cell with the new Z axis. It turns out that the cubic lattice is best described, in this way, by a hexagonal unit cell with a basis comprising three elements. The primitive lattice vectors of the new unit cell are $a_1 = a(1,−1,0)$, $a_2 = a(0,1,−1)$, $c = a(1,1,1)$ with respect to the original coordinate system, and $a_1 = \sqrt{2}a(1,0,0)$, $a_2 = \sqrt{2}a(−1,\sqrt{3},0)$, $c = a(0,0,1)$ with respect to the new rotated coordinates. The elements of the basis lie in different planes, where corresponding adjacent points are relatively shifted by a vector $\Delta = 1/3(−a_1 + a_2 + c)$.

After a Fourier transformation of (6) to real space, the model parameters can be translated to the new coordinate system. Another Fourier transformation back to momentum space then yields a new Bloch Hamiltonian $\tilde{H}(k)$ which is a 12 × 12 matrix due to the additional sublattice degrees of freedom $m, m'$. Here, $m, m'$ can assume the values $A, B, C$. As in Ref. 16, the spin parts and the orbital parts can be expanded in terms of $\Gamma$ matrices. In addition, the sublattice part can be expanded in $3 \times 3$ Gell-Mann matrices $\lambda_i$ and the corresponding unit matrix denoted by $I$. The components of the crystal momentum $k$ are now $k_x, k_y, k_z$ with respect to the rotated coordinates. With this, the explicit structure of the Bloch Hamiltonian is

$$\tilde{H} = \tilde{H}_0 \otimes \Gamma_5 + \frac{A}{2} \sum_{i=1}^{3} \tilde{H}_i \otimes \Gamma_i,$$

with

$$\tilde{H}_0 = (M_0 + 6B)I - B(2\cos \tilde{X} + \cos \tilde{Y})(\lambda_1 + \lambda_6) + B\sin \tilde{Y}(\lambda_2 + \lambda_7) - 2B\cos \tilde{X}\cos \tilde{Y}\cos \tilde{Z}\lambda_4 - B\cos \tilde{Z}\lambda_5 + 2B\cos \tilde{X}\sin \tilde{Y}\cos \tilde{Z}\lambda_5 + B\sin \tilde{Z}\lambda_5$$

$$\tilde{H}_1 = \sin \tilde{X}(\lambda_1 + \lambda_6) - \cos \tilde{X}(\lambda_2 + \lambda_7) + \cos(\tilde{X} + \tilde{Y})\cos \tilde{Z}\lambda_5 + \sin(\tilde{X} + \tilde{Y})\cos \tilde{Z}\lambda_4 - \sin(\tilde{X} + \tilde{Y})\sin \tilde{Z}\lambda_5 + \cos(\tilde{X} + \tilde{Y})\sin \tilde{Z}\lambda_4$$

where we have used the notations $\tilde{X} = k_Xa/\sqrt{2}$, $\tilde{Y} = k_Ya\sqrt{3}/2$, and $\tilde{Z} = k_Za\sqrt{3}$. Here, $a$ denotes the lattice constant of the original cubic unit cell. The underlined terms break in-plane time-reversal symmetry with respect to the $XY$ plane.

In the following, everything is expressed in terms of the new coordinate system. The weak indices are now (001). Therefore, we are particularly interested in the in-plane time-reversal symmetry with respect to the (001) plane. The associated operator reads

$$T_{(001)} = i(I \otimes s^t \otimes 1)K$$

with $k_X, k_Y \rightarrow −k_X, −k_Y$ (14)

where $I$ denotes the $3 \times 3$ unit matrix in sublattice space associated with the three-component basis of the lattice.

We will now study surface states for a slab of thickness $W$ with (010) surfaces. For this, we have to work with a $12W \times 12W$ Bloch Hamiltonian $\tilde{H}^{(010)}(k_X, k_Z)$. The band structures in the slab geometry are again obtained by exact diagonalization.

Let us first have a look at the case, where this symmetry is broken. In Fig. 7(a), we show the band structure in a (010) slab geometry along high symmetry lines of

![Figure 7](image-url)
the surface BZ associated with the new unit cell. We find a Dirac point at $A$, another one at $\Gamma$, and two line degeneracies along $M_xL_x$. However, these line degeneracies come in pairs and could be easily pushed out of the bulk energy gap. For this reason, they are trivial surface bands. This is also reflected in the fact that in-plane time-reversal symmetry is broken and therefore, they are not topologically protected.

In Eq. (13), all terms of the Bloch Hamiltonian that break in-plane time-reversal symmetry are underlined. We choose to tune all $k_Z$ dependent terms to zero except the $\cos k_Z \lambda_5 \otimes \Gamma_3$ term, which preserves in-plane TR symmetry. This is possible without closing the bulk energy gap, so the system stays in the weak TI phase. The effect is shown in Fig. 7(c). We observe that the trivial line degeneracies along $M_xL_x$ are pushed out of the bulk energy gap. Along the other trivial direction $A\Gamma$, a Dirac line forms, which is now topologically protected by in-plane time-reversal symmetry. This is in perfect agreement with the previous observations for the stacked Kane-Mele model.

Out of curiosity, we also ask what happens for the strong TI phase, if we establish in-plane time-reversal symmetry. This is shown in Fig. 5(b), (d) for $T_{(100)}$ symmetry with respect to the original coordinate system, and in Fig. 7(b), (d) for $T_{(001)}$ symmetry with respect to the rotated coordinate system. Without the symmetry, in both cases we find one Dirac point at the $\Gamma$ point of the surface BZ. However, once we restore the considered in-plane time-reversal symmetry, the bulk energy gap closes. As already pointed out in the main part of this Letter, this is due to the connection of the surface states to the bulk continuum. It causes the bulk bands to be pulled down along $\Gamma X$ or $\Gamma A$, respectively. Therefore, it is not possible for strong TIs to have topologically protected Dirac lines.