FINITE MASS SELF-SIMILAR BLOWING-UP SOLUTIONS OF A CHEMOTAXIS SYSTEM WITH NON-LINEAR DIFFUSION

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Abstract. For a specific choice of the diffusion, the parabolic-elliptic Patlak-Keller-Segel system with non-linear diffusion (also referred to as the quasi-linear Smoluchowski-Poisson equation) exhibits an interesting threshold phenomenon: there is a critical mass $M_c > 0$ such that all the solutions with initial data of mass smaller or equal to $M_c$ exist globally while the solution blows up in finite time for a large class of initial data with mass greater than $M_c$. Unlike in space dimension 2, finite mass self-similar blowing-up solutions are shown to exist in space dimension $d \geq 3$.

1. Introduction

In space dimension $d = 2$, the parabolic-elliptic Patlak-Keller-Segel (PKS) system is a simplified model which describes the collective motion of cells in the following situation: cells diffuse in space and emit a chemical signal, the chemo-attractant, which results in the cells attracting each other. If $\rho$ denotes the density of cells and $c$ the concentration of the chemo-attractant, the PKS system reads $[13, 19]$

$$\begin{cases}
\partial_t \rho(t, x) = \text{div} [\nabla \rho(t, x) - \rho(t, x) \nabla c(t, x)], \\
c(t, x) = (E_2 * \rho)(t, x), \\
E_2(x) = -\frac{1}{2\pi} \ln |x|,
\end{cases}$$

$t, x \in [0, \infty) \times \mathbb{R}^2$.

This model may be seen as an elementary brick to understand the aggregation of cells in mathematical biology as it exhibits the following interesting and biologically relevant feature: there is a critical mass above which the density of cells is expected to concentrate near isolated points after a finite time, a property which is related to the formation of fruiting bodies in the slime mold Dictyostelium discoideum. Such a phenomenon does not take place if the density of cells is too low. More precisely, given a non-negative integrable initial condition $\rho_0$ with finite second moment, the system (1) has a unique maximal classical solution $(\rho, c)$ defined on some maximal time interval $[0, T), \; T \in (0, \infty]$. Its first component $\rho$ is non-negative and the mass of $\rho$ (that is, its $L^1$-norm) remains constant through time evolution

$$\|\rho(t)\|_1 = M := \|\rho_0\|_1, \quad t \in [0, T).$$

It is well-known that, if $M < 8\pi$, the solution to (1) exists globally in time while it blows up in finite time if $M > 8\pi$, see $[3, 6, 11, 12]$ and the references therein.

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More recently, it was shown that there is global existence as well for the critical mass \( M = 8\pi \), the blowup occurring in infinite time with a profile being a Dirac mass of mass \( 8\pi \) \[1\]. When the mass \( M \) is above \( 8\pi \), the shape of the finite time blowup is not self-similar according to asymptotic expansions computed in \[5, 15\] (see also \[10\] for a related problem in a bounded domain). In addition, there is no integrable and radially symmetric blowing-up self-similar solution to \[11, 18, \text{Theorem 8}\].

In space dimension \( d \geq 3 \), the system \( \eqref{PKS} \) seems to be less relevant from the biological point of view as blowup may occur whatever the value of \( M \) \[9, 17\]. This means that the diffusion is too weak to balance the aggregation resulting from the chemotactic term. It is however well-known that one can enhance the effect of diffusion to prevent crowding by considering a diffusion of porous medium type which increases the diffusion of the cells when their density \( \rho \) is large. This is the generalised version of the Patlak-Keller-Segel model considered in, e.g., \[2, 4, 22, 23, 24\]:

\[
\begin{aligned}
&\partial_t \rho(t, x) = \text{div} \left( \rho^m(t, x) \nabla c(t, x) \right), \\
&c(t, x) = \left( E_d \ast \rho \right)(t, x),
\end{aligned}
\]  
\( (t, x) \in [0, \infty) \times \mathbb{R}^d, \)

where \( m > 1 \), \( c_d := 1/((d - 2) \, \sigma_d) \), and \( \sigma_d := 2 \pi^{d/2}/\Gamma(d/2) \) denotes the surface area of the sphere \( S^{d-1} \) of \( \mathbb{R}^d \). The system \( \eqref{porous} \) also arises in astrophysics \[4\] (being then referred to as the generalised Smoluchowski-Poisson equation), and \( \rho \) and \( c \) denote the density of particles and the gravitational potential, respectively.

For \( \eqref{porous} \), it turns out that there is only one critical exponent of the non-linear diffusion, namely \( m_d := 2(d - 1)/d \), such that the mass plays a similar role to that in \( \eqref{PKS} \). Indeed, if \( m > m_d \) the diffusion enhancement is too strong and the solutions always exist globally in time whereas if \( m < m_d \) the diffusion is not strong enough to compensate the aggregation term and there are solutions blowing up in finite time whatever the value of the mass \[22, 23\]. The relevant diffusion is thus achieved in the case when \( m = m_d \). In this case, it was proved in \[2\] that there is a unique threshold mass \( M_c > 0 \) with the following properties: if the mass \( M = \| \rho_0 \|_1 \) of the initial condition \( \rho_0 \) is less or equal to \( M_c \), then the corresponding solution to \( \eqref{porous} \) exists globally in time, whereas given any \( M > M_c \) there are initial data \( \rho_0 \) with mass \( M \) such that the corresponding solution blows up in finite time. Thus, for the peculiar choice \( m = m_d \) and \( d \geq 3 \), the system \( \eqref{porous} \) exhibits the same qualitative behaviour as the PKS system \( \eqref{PKS} \) in space dimension 2. Still, there is a fundamental difference as the latter has no fast-decaying stationary solution with mass \( 8\pi \) while the former has a two-parameter family of non-negative, integrable, and compactly supported stationary solutions with mass \( M_c \) for each \( d \geq 3 \) \[2, \text{Section 3}\].

It is then tempting to figure out whether this striking difference extends above the critical mass \( M_c \) and this leads us to investigate the existence of blowing-up (or backward) self-similar solutions with finite mass. More precisely, since mass remains unchanged throughout time evolution, we look for solutions \((\rho, c)\) to \( \eqref{porous} \) with \( m = m_d \) and \( d \geq 3 \) of the form

\[
\begin{aligned}
&\rho(t, x) = \frac{1}{s(t)^d} \, \Phi \left( \frac{x}{s(t)} \right) \quad \text{and} \quad c(t, x) = \frac{1}{s(t)^{d-2}} \, \Psi \left( \frac{x}{s(t)} \right),
\end{aligned}
\]
for \((t, x) \in [0, T) \times \mathbb{R}^d\) with \(s(t) := [d(T - t)]^{1/d}\), the time \(T\) being an arbitrary positive real number. Note that \(s(t)\) converges to zero as \(t\) increases to the blowup time \(T\).

Our main result is then the following:

**Theorem 1** (Existence of finite mass self-similar blowing-up solutions). There exists \(M_2 \in (M_c, \infty)\) such that, for any \(M\) in \((M_c, M_2]\), there exists at least a non-negative self-similar blowing-up solution \((\rho_M, c_M)\) to (2) of the form (3) with a radially symmetric, compactly supported, and non-increasing profile \(\Phi_M\) satisfying 
\[\|\rho_M(t)\|_1 = \|\Phi_M\|_1 = M\] for \(t \in [0, T)\) and \(\|\rho_M(t)\|_\infty \to \infty\) as \(t \to T\).

As a consequence of Theorem 1, we realize that non-negative, integrable, and radially symmetric self-similar blowing-up solutions to (2) with a non-increasing profile only exist below a threshold mass. Another by-product of our analysis is the existence of non-negative and non-integrable self-similar blowing-up solutions to (2), see Proposition 8 below.

### 2. Blowing-up self-similar profiles

From now on,

\[d \geq 3\] and \(m = m_d = \frac{2(d - 1)}{d}\),

and we look for a solution \((\rho, c)\) to (2) of the form

\[
\rho(t, x) = \frac{1}{s(t)^d} \Phi \left( \frac{x}{s(t)} \right) \quad \text{and} \quad c(t, x) = \frac{1}{s(t)^{d-2}} \Psi \left( \frac{x}{s(t)} \right)
\]

with \(s(t) = [d(T - t)]^{1/d}\) and \((t, x) \in [0, T) \times \mathbb{R}^d\) for some given \(T > 0\). We further assume that \(\Phi\) enjoys the following properties:

- \(\Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\) is radially symmetric and non-negative,
- \(\Phi^{m-1} \in W^{1,\infty}(\mathbb{R}^d)\).

Inserting the ansatz in (2) gives that \((\Phi, \Psi)\) solves

\[
\begin{cases}
\text{div} (\nabla [\Phi^m(y)]) - \Phi(y) \nabla \Psi(y) - \Phi(y) y = 0, \\
\Psi(y) = (E_d \ast \Phi)(y),
\end{cases}
\]

for \(y \in \mathbb{R}^d\). Since \(\Psi = E_d \ast \Phi\), the radial symmetry of \(\Phi\) ensures that of \(\Psi\) and, introducing the profiles \((\varphi, \psi)\) of \((\Phi, \Psi)\)

\[
\Phi(y) = \varphi(|y|), \quad \Psi(y) = \psi(|y|), \quad y \in \mathbb{R}^d.
\]

By [14, Theorem 9.7, Formula (5)], we have

\[
\psi(r) = \frac{1}{(d - 2)r^{d-2}} \int_0^r \varphi(s) s^{d-1} \, ds + \frac{1}{d - 2} \int_r^\infty \varphi(s) s \, ds
\]

for \(r \geq 0\). We can also write the equation for \(\varphi\) as

\[
\partial_r \left( r^{d-1} \varphi(r) \partial_r J(r) \right) = 0 \quad \text{with} \quad J(r) := \frac{2(d - 1)}{d - 2} \varphi^{(d-2)/d}(r) - \psi(r) - \frac{r^2}{2}
\]

\[\Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\] is radially symmetric and non-negative,
for \( r \in (0, \infty) \). Since we are looking for an integrable profile, we formally conclude that
\[
\partial_r J(r) = 0 \quad \text{for} \quad r \in \mathcal{P}_\phi := \{ s \in (0, \infty) : \phi(s) > 0 \}.
\]
In particular, \( J \) is constant on any connected component of \( \mathcal{P}_\phi \). But, if \( C \) is a connected component of \( \mathcal{P}_\phi \), we have either
\[
C = (0, R_s) \quad \text{for some} \quad R_s \in (0, \infty],
\]
or
\[
C = (R_i, R_s) \quad \text{for some} \quad R_i \in (0, \infty) \quad \text{and} \quad R_s \in (0, \infty].
\]

**Remark 2.** If we additionally assume that the profile \( \phi \) is non-increasing then \( \mathcal{P}_\phi \) has only one connected component which is necessarily of the form (9).

Now, take a connected component \( C \) of \( \mathcal{P}_\phi \). It follows from (8) that there is \( \mu \in \mathbb{R} \) such that
\[
J(r) = \frac{2(d-1)}{d-2} \phi^{(d-2)/d}(r) - \psi(r) - \frac{r^2}{2} = -\mu \quad \text{for} \quad r \in C.
\]
Owing to the assumed integrability of \( \Phi \), the function \( r \mapsto r^{d-1} \phi(r) \) belongs to \( L^1(0, \infty) \) and it follows from (6) that the function \( r \mapsto r^{d-2} \psi(r) \) is bounded in \( C \). Therefore (10) only complies with the integrability of \( \Phi \) if \( R_s < \infty \) which implies the boundedness of \( C \). Introducing
\[
\Xi := \phi^{(d-2)/d}
\]
and taking the Laplacian of both sides of (10) yield that \( \Xi \) is a positive solution to
\[
- \frac{d^2 \Xi}{dr^2}(r) - \frac{d-1}{r} \frac{d \Xi}{dr}(r) = \frac{d-2}{2(d-1)} \left( \Xi(r)^{d/(d-2)} - d \right) \quad \text{in} \quad C,
\]
with either
\[
\partial_r \Xi(0) = \Xi(R_s) = 0 \quad \text{if} \quad C = (0, R_s)
\]
or
\[
\Xi(R_i) = \Xi(R_s) = 0 \quad \text{if} \quad C = (R_i, R_s).
\]
A final change of scale, namely
\[
\eta(r) := \frac{1}{\lambda_d} \Xi \left( \frac{r}{\mu_d} \right), \quad \lambda_d := d^{(d-2)/d}, \quad \mu_d := d^{1/d} \left( \frac{d-2}{2(d-1)} \right)^{1/2},
\]
leads us to the following boundary-value problem for \( \eta \): either
\[
\begin{cases}
\frac{d^2 \eta}{dr^2}(r) + \frac{d-1}{r} \frac{d \eta}{dr}(r) + \eta(r)^{d/(d-2)} - 1 = 0, & r \in (0, \mu_d R_s), \\
\frac{d \eta}{dr}(0) = 0, & \eta(\mu_d R_s) = 0,
\end{cases}
\]
or
\[
\begin{cases}
\frac{d^2 \eta}{dr^2}(r) + \frac{d-1}{r} \frac{d \eta}{dr}(r) + \eta(r)^{d/(d-2)} - 1 = 0, & r \in (\mu_d R_i, \mu_d R_s), \\
\eta(\mu_d R_i) = 0, & \eta(\mu_d R_s) = 0.
\end{cases}
\]
We have thus reduced our study to one or several boundary-value problems (depending on the number of connected components of $P_\varphi$) for a nonlinear second order differential equation. The purpose of the next section is then a precise study of this ordinary differential equation.

However, before going on, let us point out that (11) is not equivalent to (10). Indeed, since

$$
\partial_r J(r) = \frac{2(d-1)}{d-2} \partial_r \Xi(r) + \frac{1}{r^{d-1}} \int_0^r \Xi(s)^{d/(d-2)} s^{d-1} \, ds - r, \quad r \in \mathcal{C},
$$

by (7), the fact that $\Xi$ is a solution to (11) only guarantees that $\partial_r (r^{d-1} \partial_r J(r)) = 0$ for $r \in \mathcal{C}$. Consequently, there are constants $C_1$ and $C_2$ such that

$$
\partial_r J(r) = -\frac{(d-2)C_1}{r^{d-1}}, \quad J(r) = \frac{C_1}{r^{d-2}} + C_2, \quad r \in \mathcal{C},
$$

from which (10) follows only if $C_1 = 0$. On the one hand, if $\mathcal{C} = (R_i, R_s)$ with $0 < R_i < R_s$, it is yet unclear whether the boundary conditions (13) might imply this property. On the other hand, if $\mathcal{C} = (0, R_s)$, the boundary conditions (12) ensure that $\partial_r J(0) = 0$ and thus $C_1 = 0$. We shall only deal with this case in the remaining of this paper and thus focus on the non-increasing profiles $\varphi$.

3. An auxiliary ordinary differential equation

For $a \in \mathbb{R}$, let $u(\cdot, a) \in \mathcal{C}^1([0, r_{\max}(a))]$ denote the maximal solution to the Cauchy problem

$$
\begin{cases}
\quad u''(r, a) + \frac{d-1}{r} u'(r, a) + |u(r, a)|^{p-1} u(r, a) - 1 = 0, & r \in [0, r_{\max}(a)), \\
\quad u(0, a) = a, \quad u'(0, a) = 0,
\end{cases}
$$

with $r_{\max}(a) \in (0, \infty]$ and $p = d/(d-2)$.

Clearly, if $a = 1$ then $u(\cdot, 1) \equiv 1$ is a stationary solution and $r_{\max}(1) = \infty$. We first show that $u(\cdot, a)$ is global for all $a \in \mathbb{R}$ and oscillates around the value 1 if $a \neq 1$.

**Lemma 3.** For each $a \in \mathbb{R} \setminus \{1\}$, $r_{\max}(a) = \infty$, and the solution $u(\cdot, a)$ to (16) is an oscillatory function in $(0, \infty)$. More precisely,

- if $a > 1$, there is an increasing sequence $(r_i(a))_{i \geq 0}$ of real numbers such that $r_0(a) = 0$,

$$
\begin{cases}
\quad u'(r_i(a), a) = 0, \quad (-1)^i u'(r, a) < 0 \quad \text{for} \quad r \in (r_i(a), r_{i+1}(a)), \\
\quad u(r_{2i}(a), a) > u(r_{2i+2}(a), a) > 1 > u(r_{2i+3}(a), a) > u(r_{2i+1}(a), a)
\end{cases}
$$

for $i \geq 0$,

- if $a < 1$, there is an increasing sequence $(r_i(a))_{i \geq 1}$ of real numbers such that $r_1(a) = 0$

$$
\begin{cases}
\quad u'(r_i(a), a) = 0, \quad (-1)^i u'(r, a) < 0 \quad \text{for} \quad r \in (r_i(a), r_{i+1}(a)), \\
\quad u(r_{2i}(a), a) > u(r_{2i+2}(a), a) > 1 > u(r_{2i+1}(a), a) > u(r_{2i-1}(a), a)
\end{cases}
$$

for $i \geq 1$. 

These properties are illustrated in Figure 1. Notice that, for $a = 7$, $u(., 7)$ vanishes at a finite $r$ and thus provides a solution to (14).

**Figure 1.** Various oscillating behaviours of $u(., a)$ for $a \in \{0.2, 1, 3, 7\}$.

**Proof of Lemma 3.** For any $r \in [0, r_{\text{max}}(a))$ consider the functional
\[
E(r, a) := \frac{|u'(r, a)|^2}{2} + \frac{|u(r, a)|^{p+1}}{p+1} - u(r, a).
\]
By (16), for all $r \in [0, r_{\text{max}}(a))$
\[
\frac{dE}{dr}(r, a) = -\frac{d-1}{r} |u'(r, a)|^2 \leq 0,
\]
Obviously $E(r, a) \geq -p/(p+1)$. Owing to (18), $E(r, a) \in [-p/(p+1), E(0, a)]$ for $r \in [0, r_{\text{max}}(a))$ which prevents $u(., a)$ of becoming unbounded at a finite value of $r$, thereby implying that $r_{\text{max}}(a) = \infty$. We next argue using Sturm’s oscillations theorem as in [16, Lemma 9], to establish the oscillatory behaviour of $u(., a)$ for $a \neq 1$.

According to (14), we are interested in finding solutions to the initial value problem (16) which are positive and vanish at a finite value of $r$. We thus focus on the case $a > 0$ and investigate the positivity properties of $u(., a)$.

**Lemma 4.** There is a constant $a_c > 1$ such that
- if $a \in (0, a_c)$, then $u(r, a) > 0$ for all $r \geq 0$,
• if $a = a_c$, then there is $R(a_c) > 0$ such that
  \[
  \begin{cases}
  u(R(a_c), a_c) = 0 \\
  u'(R(a_c), a_c) = 0 \\
  u(r, a_c) > 0 \quad \text{for } r \in [0, R(a_c)) \,.
  \end{cases}
  \]

• if $a \in (a_c, \infty)$, then there is $R(a) > 0$ such that
  \[
  \begin{cases}
  u(R(a), a) = 0 \\
  u'(R(a), a) < 0 \\
  u(r, a) > 0 \quad \text{for } r \in [0, R(a)) \,.
  \end{cases}
  \]

These three possibilities are drawn in Figure 2.

\begin{figure}[ht]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Behaviour of $u(., a)$ for $a > a_c$, $a = a_c$ and $a < a_c$.}
\end{figure}

\textit{Proof of Lemma 4.} For $a > 0$, we define
\[ R(a) := \inf\{ R > 0 : u(r, a) > 0 \quad \text{for} \quad r \in [0, R) \} \,.
\]
Notice that the positivity of $a$ and the continuity of $u(., a)$ guarantee that $R(a) > 0$.

We consider the sets
\[
\begin{align*}
\mathcal{P} & := \{ a > 0 : R(a) = \infty \} \\
\mathcal{N} & := \{ a > 0 : R(a) < \infty \text{ and } u'(R(a), a) < 0 \} \\
\mathcal{N}_0 & := \{ a > 0 : R(a) < \infty \text{ and } u'(R(a), a) = 0 \} 
\end{align*}
\]
Clearly, $\mathcal{P} \cup \mathcal{N} \cup \mathcal{N}_0 = (0, \infty)$ and $1 \in \mathcal{P}$. Actually, if $a \in (0, (p + 1)^{1/p})$, then $E(0, a) < 0$ and the monotonicity \((18)\) of $E$ entails that $E(r, a) < 0$ for all $r \geq 0$. But, if $R(a) < \infty$, it readily follows from the definition \((17)\) of the functional $E$ that $E(R(a), a) \geq 0$ whence a contradiction. Therefore, $R(a) = \infty$ for any $a \in (0, (p + 1)^{1/p})$ so that

\[
(0, (p + 1)^{1/p}) \subset \mathcal{P}.
\]

Consider now $a \in \mathcal{N}_0$. Then $U(x) := u(|x|, a)$ is a radial positive solution to the homogeneous Dirichlet-Neumann free boundary problem $\Delta U + U^p - 1 = 0$ in $B(0, R(a))$ with $U = \partial_r U = 0$ on $\partial B(0, R(a))$. According to \cite[Theorem 3 (ii)], there is only one value of $a$ for which this solution has a positive radial solution and it is unique. Consequently, there is a unique $a_c > 0$ such that $\mathcal{N}_0 = \{a_c\}$.

Consider next $a \in \mathcal{N} \cup \mathcal{N}_0$ and recall that $a > 1$ by \((19)\). Following \cite[Lemma 11]{16}, let us assume for contradiction that there is $\rho \in (0, R(a))$ such that $u'(\rho, a) = 0$. Either $\rho \leq 1$ and we infer from the definition, the monotonicity of $E$, see \((17)\)-\((18)\), and the definition of $R(a)$ that $0 > E(\rho, a) \geq E(R(a), a) \geq 0$ which is a contradiction. Or $\rho > 1$ and the oscillating behaviour of the solutions implies, using the notation of Lemma 3 that $\rho \geq r_2(a)$. This implies that $r_1(a) < R(a)$. Then $u(r_1(a), a) \in (0, 1)$ and using again \((17)\), \((18)\), and the definition of $R(a)$, we conclude that $0 > E(r_1(a), a) \geq E(R(a), a) \geq 0$, hence a contradiction. Therefore,

\[
u'(r, a) < 0 \quad \text{for} \quad r \in (0, R(a)) \quad \text{if} \quad a \in \mathcal{N} \cup \mathcal{N}_0.
\]

Let us now prove that $\mathcal{P}$ and $\mathcal{N}$ are open subsets of $(0, \infty)$. We first consider $a \in \mathcal{N}$: by \((20)\) there are $\rho > R(a)$ and $\epsilon > 0$ such that $u(\rho, a) < 0$ and $u'(r, a) < -2\epsilon$ for $r \in (0, \rho)$. By continuous dependence, there is $\delta \in (0, a)$ such that $u(\rho, b) < 0$ and $u'(r, b) < -\epsilon$ for $r \in (0, \rho)$ and $b \in (a - \delta, a + \delta)$. Since $u(0, b) = b > 0$, we readily deduce that, for each $b \in (a - \delta, a + \delta)$, we have $R(b) \in (0, \rho)$ with $u'(R(b), b) < -\epsilon < 0$. Consequently, $(a - \delta, a + \delta)$ and $\mathcal{N}$ is open in $(0, \infty)$. Consider next $a \in \mathcal{P}$, $a > 1$. By Lemma 3 and \((17)\), we have $u(r, a) \geq u(r_1(a), a) \in (0, 1)$ for $r \in [0, r_1(a)]$ and $E(r_1(a), a) < 0$. By continuous dependence, there is $\delta > a$ such that $u(r, b) \geq u(r_1(a), a)/2 > 0$ for $r \in [0, r_1(a)]$, $u(r_1(a), b) \in (0, 1)$, and $E(r_1(a), b) < 0$ for $b \in (a - \delta, a + \delta)$. Assume now for contradiction that there is $b \in (a - \delta, a + \delta)$ such that $R(b) < \infty$. Owing to \((17)\), \((18)\), and the definition of $R(b)$, we obtain $0 > E(r_1(a), b) > E(R(b), b) \geq 0$ and a contradiction. Consequently, $(a - \delta, a + \delta) \subset \mathcal{P}$ and $\mathcal{P}$ is also open in $(0, \infty)$.

We finally argue as in \cite[Lemma 15]{16} to show that there is $A > 0$ such that $(A, \infty) \subset \mathcal{N}$.

Since $\mathcal{P}$ and $\mathcal{N}$ are open subsets of $(0, \infty)$, $\mathcal{N}_0 = \{a_c\}$, $(0, (p + 1)^{1/p}) \subset \mathcal{P}$, and $(A, \infty) \subset \mathcal{N}$, we readily conclude that $\mathcal{P} = (0, a_c)$ and $\mathcal{N} = (a_c, \infty)$.

We next study the properties of the map $a \mapsto R(a)$. An efficient tool for that purpose is the variation of $u(., a)$ with respect to $a$ defined by

$$
\psi(r, a) := \frac{\partial u}{\partial a}(r, a), \quad (r, a) \in [0, \infty) \times (0, \infty),
$$
which solves the second order linear differential equation
\[
\vartheta''(r,a) + \frac{d-1}{r} \vartheta'(r,a) + p u(r,a)^{p-1} \vartheta(r,a) = 0, \quad r \in [0, \infty),
\]
(21)
\[
\vartheta(0,a) = 1, \quad \vartheta'(0,a) = 0.
\]
We argue as in [7, 25] to prove the following lemma.

Lemma 5. If \( a > a_c \), there is a unique \( z(a) \in (0, R(a)) \) such that
\[
\begin{cases}
\vartheta(r,a) > 0 & \text{for } r \in [0, z(a)), \\
\vartheta(z(a),a) = 0 \\
\vartheta(r,a) < 0 & \text{for } r \in (z(a), R(a)].
\end{cases}
\]
In addition, \( u(z(a),a) > 1 \) and the ratio \( \vartheta(.,a)/u(.,a) \) is a decreasing function of \( r \) on \( (0,R(a)) \).

Proof of Lemma 5. Since the proof follows rather closely that of [25] and [7, Lemma 2.1], we sketch it briefly for the sake of completeness. Fix \( a > a_c \) and set \( u = u(.,a) \) and \( \vartheta = \vartheta(.,a) \) to simplify notations. We first argue as in [16, Lemma 17] to show that \( \vartheta \) vanishes at least once in the interval \( (0, z_1(a)) \), where \( z_1(a) \) denotes the unique zero in \( (0, R(a)) \) of \( u - 1 \). Indeed, (16) also reads
\[
(u(r) - 1)'' + \frac{d-1}{r} (u(r) - 1)' + \frac{u(r)^p - 1}{u(r) - 1} (u(r) - 1) = 0, \quad r \in [0, \infty)
\]
and \( (u(r)^p - 1)/(u(r) - 1) \leq p u(r)^{p-1} \) for \( r \in [0, z_1(a)) \). It then follows from Sturm’s comparison theorem that \( \vartheta \) vanishes at least once in the interval \( (0, z_1(a)) \). Let \( z \in (0, z_1(a)) \) denote the first zero of \( \vartheta \).

We now aim at showing that \( \vartheta \) cannot vanish once more in the interval \( (z, R(a)) \). To this end, we define
\[
\xi(r) := r^{d-1} \left[ u'(r) \vartheta(r) - u(r) \vartheta'(r) \right] = -r^{d-1} u(r)^2 \left( \frac{\vartheta}{u'} \right)'(r), \quad r \in [0, R(a)),
\]
which encodes the monotonicity of \( \vartheta/u \). It follows from (16) and (21) that
\[
\xi'(r) = r^{d-1} \left[ (p-1) u'(r) + 1 \right] \vartheta(r), \quad r \in [0, R(a)).
\]
Clearly, \( \xi'(r) > 0 \) for \( r \in (0, z) \) and \( \xi(0) = 0 \), so that \( \xi(r) > 0 \) for \( r \in (0, z] \). Assume now for contradiction that there is \( \vartheta \in (z, R(a)) \) such that
\[
\xi(r) > 0 \quad \text{for } r \in (0, \vartheta) \quad \text{and} \quad \xi(\vartheta) = 0.
\]
Observing that \( \vartheta'(z) < 0 \), we realize that, if \( \vartheta(\vartheta) \geq 0 \), there is \( \sigma \in (z, \vartheta) \) such that \( \vartheta(r) < 0 \) for \( r \in (z, \sigma) \) and \( \vartheta(\sigma) = 0 \). In that case, \( \vartheta'(\sigma) \geq 0 \) and thus
\[
\xi(\sigma) = -\sigma^{d-1} u(\sigma) \vartheta'(\sigma) \leq 0,
\]
leading us to a contradiction. Consequently,
\[
\vartheta(\vartheta) < 0.
\]
We next introduce the functions
\[ T(r) := \frac{2 (u(r)^p - 1)}{(p - 1) u(r)^p + 1} \xi(r) - \zeta(r), \]
\[ \zeta(r) := r^d \left[ u'(r) \vartheta'(r) + (u(r)^p - 1) \vartheta(r) \right] + (d - 2) r^{d-1} u'(r) \vartheta(r), \]
for \( r \in [0, R(a)) \) and use (16), (21), and (22) to obtain
\[ \zeta'(r) = 2 r^{d-1} \left( u(r)^p - 1 \right) \vartheta(r), \]
\[ T'(r) = 2 p^2 \frac{u(r)^{p-1}}{[(p - 1) u(r)^p + 1]^2} u'(r) \xi(r), \]
for \( r \in [0, R(a)) \). Integrating (24) over \((0, \varrho)\) and using the negativity of \( u' \) and the positivity of \( \xi \) on this interval give
\[ \zeta(\varrho) = -T(\varrho) > 0. \]
Since \( \xi(\varrho) = 0 \), we have \( u(\varrho) \vartheta'(\varrho) = u'(\varrho) \vartheta(\varrho) \) and we have
\[ \zeta(\varrho) = Q(\varrho) \frac{\vartheta(\varrho)}{u(\varrho)}, \]
where
\[ Q(r) := r^d \left[ u'(r)^2 + u(r)^{p+1} - u(r) \right] + (d - 2) r^{d-1} u(r) u'(r), \quad r \in [0, R(a)). \]
It then follows from (23), (25), and the positivity of \( u \) that
\[ Q(\varrho) < 0. \]
Finally, define
\[ P(r) := r^d \left( u'(r)^2 + 2 \frac{u(r)^{p+1}}{p+1} - 2 u(r) \right) + (d - 2) r^{d-1} u(r) u'(r) \]
for \( r \in [0, R(a)) \). On the one hand, we notice that
\[ P(r) = Q(r) - u(r) - \frac{p - 1}{p + 1} u(r)^{p+1} < Q(r), \quad r \in [0, R(a)). \]
On the other hand, we deduce from (16) and (18) that
\[ P'(r) = r^{d-1} u(r) \left( \frac{d - 2}{d - 1} u(r)^p - (d + 2) \right), \quad r \in [0, R(a)). \]
At this point, we realize that we have necessarily \( a > (d + 2)(d - 1)/(d - 2) \) and that there is \( s \in (0, R(a)) \) such that \( P'(r) > 0 \) if \( r < (0, s) \) and \( P'(r) < 0 \) if \( r \in (s, R(a)) \). Since \( P(0) = 0 \) and \( P(R(a)) > 0 \), we conclude that \( P(\varrho) > 0 \) and then \( Q(\varrho) > 0 \) by (27). But this contradicts (26). We have thus established that \( \xi \) is positive in \((0, R(a))\) from which Lemma 4 follows.

We are now in a position to state and prove some properties of the map \( a \mapsto R(a) \).

**Proposition 6.** The map \( a \mapsto R(a) \) is a decreasing function on \((a_c, \infty)\) and there is \( z_1 > 0 \) such that
\[ \lim_{a \searrow a_c} R(a) = R(a_c) \quad \text{and} \quad \lim_{a \to \infty} a^{(p-1)/2} R(a) = z_1. \]
The monotonicity of \( a \mapsto R(a) \) is shown in Figure 3. According to numerical simulations, the function \( a \mapsto a^{(p-1)/2} R(a) \) also seems to be a decreasing function of \( a \in [a_c, \infty) \), see Figure 3.

**Figure 3.** Monotonicity of the radius \( R \) and \( a \mapsto a^{(p-1)/2} R(a) \) \((d = 3)\).

**Proof of Proposition 6.** By Lemma 4, \( u'(R(a), a) < 0 \) for all \( a \in (a_c, \infty) \) and the implicit function theorem warrants that \( R \in C^1((a_c, \infty)) \) with

\[
\frac{dR}{da}(a) = -\frac{\vartheta(R(a), a)}{w'(R(a), a)}.
\]

Since \( \vartheta(R(a), a) < 0 \) by Lemma 5, the previous formula implies the strict monotonicity of \( a \mapsto R(a) \). We next define

\[
R_l := \sup_{a \in (a_c, \infty)} R(a) \in (0, \infty] .
\]

If \( R_l > R(a_c) \), there is \( \varrho \in (R(a_c), R_l) \) such that \( u(\varrho, a_c) > 0 \) by Lemmata 3 and 4. Then, there is \( \delta > 0 \) such that \( R(a) > \varrho \) for \( a \in (a_c, a_c + \delta) \). It then follows from the continuous dependence of \( u(\cdot, a) \) with respect to \( a \) and the monotonicity of \( u(\cdot, a) \) with respect to \( r \) that

\[
0 = u(R(a_c), a_c) = \lim_{a \searrow a_c} u(R(a), a) \geq \lim_{a \searrow a_c} u(\varrho, a) = u(\varrho, a_c) > 0 ,
\]

and a contradiction. Therefore, \( R_l \leq R(a_c) \) is finite and we have

\[
u(R_l, a_c) = \lim_{a \searrow a_c} u(R(a), a) = 0 ,
\]

from which we conclude that \( R_l = R(a_c) \).

Finally, define

\[
v(r, a) := \frac{1}{a} \left( \frac{r}{a^{(p-1)/2}}, a \right) , \quad (r, a) \in [0, \infty) \times (0, \infty).
\]
Owing to (16), \( v(., a) \) solves
\[
\begin{cases}
  v''(r, a) + \frac{d-1}{r} v'(r, a) + |v(r, a)|^{p-1} v(r, a) - a^{-p} = 0, & r \in [0, \infty), \\
  v(0, a) = 1, & v'(0, a) = 0.
\end{cases}
\]
In addition,
\[
(30) \quad v(r, a) > 0 \quad \text{for} \quad r \in \left(0, a^{(p-1)/2} R(a)\right)
\]
for \( a > a_c \) by Lemma 4. Since \( a^{-p} \rightarrow 0 \) as \( a \rightarrow \infty \), we have
\[
(31) \quad \limsup_{a \rightarrow \infty} \sup_{r \in [0, \varrho]} |v(r, a) - w(r)| = 0 \quad \text{for all} \quad \varrho > 0,
\]
where \( w \) denotes the unique solution to
\[
(32) \quad \begin{cases}
  w''(r) + \frac{d-1}{r} w'(r) + |w(r)|^{p-1} w(r) = 0, & r \in [0, \infty), \\
  w(0) = 1, & w'(0) = 0.
\end{cases}
\]
By (3), there is \( z_1 > 0 \) such that
\[
(33) \quad w(r) > 0 \quad \text{and} \quad w'(r) < 0 \quad \text{for} \quad r \in [0, z_1), \quad w(z_1) = 0, \quad w'(z_1) < 0.
\]
Owing to (33), there is \( \delta > 0 \) such that \( w(r) < 0 \) for \( r \in (z_1, z_1 + \delta) \). It then follows from (31) that, given \( r \in (z_1, z_1 + \delta) \), \( v(r, a) < 0 \) for \( a \) large enough (depending on \( r \)), whence \( a^{(p-1)/2} R(a) \leq r \) for \( a \) large enough by (30). Letting \( r \rightarrow z_1 \) guarantees that
\[
\limsup_{a \rightarrow \infty} a^{(p-1)/2} R(a) \leq z_1.
\]
Next, if \( \varrho \in (0, \gamma) \), we have \( w(r) > w(\varrho) > 0 \) for \( r \in [0, \varrho] \) and we infer from (31) that \( v(r, a) > w(\varrho)/2 > 0 \) for \( r \in [0, \varrho] \) and \( a \) large enough. Consequently, \( \varrho < a^{(p-1)/2} R(a) \) for \( a \) large enough, from which we conclude that
\[
\liminf_{a \rightarrow \infty} a^{(p-1)/2} R(a) \geq z_1.
\]
Combining the above two inequalities completes the proof of Proposition 6. \( \square \)

The above information allow us to estimate from above and from below a specific integral of \( u(., a) \).

**Proposition 7.** For \( a \in [a_c, \infty) \), we define
\[
\mathcal{M}(a) := d |B(0, 1)| \int_0^{R(a)} u(r, a)^p r^{d-1} \, dr.
\]
Recalling that \( w \) is the solution to (32) and \( z_1 \) is its first positive zero, we have
\[
(34) \quad \lim_{a \rightarrow \infty} \mathcal{M}(a) = \mathcal{M}_c := d |B(0, 1)| \int_0^{z_1} w(r)^p r^{d-1} \, dr,
\]
\[
(35) \quad \mathcal{M}_2 := \sup_{a \in [a_c, \infty)} \mathcal{M}(a) < \infty.
\]
Proof of Proposition 7. Let $a \geq a_c$. Since $u(0, a) = a$, it follows from the monotonicity of $u(., a)$ that
\[
\mathcal{M}(a) \leq d |B(0, 1)| \int_0^{R(a)} a^p r^{d-1} dr = |B(0, 1)| \left( a^{(p-1)/2} R(a) \right)^d.
\]
The upper bound (35) is then a straightforward consequence of (28) and the above inequality.

Next, recalling that $v(., a)$ is defined by (29), we have
\[
\mathcal{M}(a) = d |B(0, 1)| \int_0^{a a^{(p-1)/2} R(a)} v(r, a)^p r^{d-1} dr,
\]
and we infer from (28) and (31) that (34) holds true. □

4. Proof of Theorem 1

Thanks to the analysis done in the previous sections, we are now in a position to construct self-similar blowing-up solutions to (2) having either finite or infinite mass.

Proposition 8. Given $a > 0$ and $T > 0$, define
\[
\varphi(r) := \lambda_d^d/(d-2) u(\mu_d r, a)^d/(d-2) \quad \text{for} \quad r \in [0, \infty) \quad \text{if} \quad a \in (0, a_c),
\]
and
\[
\varphi(r) := \begin{cases} 
\lambda_d^d/(d-2) u(\mu_d r, a)^d/(d-2) & \text{for} \quad r \in [0, R(a)/\mu_d] \\
0 & \text{for} \quad r \geq R(a)/\mu_d
\end{cases} \quad \text{if} \quad a \in [a_c, \infty).
\]

Define next $v$, $\Phi$, and $\Psi$ by (1) and (5), respectively. Then the functions $(\rho, c)$ defined by (1) in $(0, T) \times \mathbb{R}^d$ with $s(t) = [d(T - t)]^{1/d}$ is a non-negative self-similar blowing-up solution to (2) with finite mass if $a \geq a_c$ and infinite mass if $a \in (0, a_c)$.

The proof of Proposition 8 readily follows from the analysis performed in Sections 2 and 3. As for Theorem 1 it is a straightforward consequence of Proposition 8, the threshold values $M_c$ and $M_2$ being given by
\[
M_c := d^{1/d} \left( \frac{2(d-1)}{d-2} \right)^{(d-1)/2} \mathcal{M}_c \quad \text{and} \quad M_2 := d^{1/d} \left( \frac{2(d-1)}{d-2} \right)^{(d-1)/2} \mathcal{M}_2.
\]

5. Discussion

We have proved the existence of non-negative, integrable, and radially symmetric self-similar blowing-up solutions for (2). The profile $\varphi$ of these self-similar solutions is compactly supported and non-increasing, and the mass of the corresponding self-similar solution ranges in the bounded interval $(M_c, M_2]$, the threshold mass $M_c$ corresponding to the onset of blowup found in [2]. Our analysis thus reveals the existence of a second threshold value $M_2 > M_c$ of the mass above which no radially symmetric and non-increasing self-similar blowing-up solution exist. The meaning of this second threshold value for the mass is yet unclear. It is worth mentioning at this point that a related situation was uncovered for the critical unstable thin-film equation
\[
\partial_t u = -\partial_x \left( u^n \partial_x^3 u + u^{n+2} \partial_x u \right), \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]
in [21] for \( n \in (0, 3/2) \). It is likely that, given \( M \in (M_1, M_2) \), there is only a unique radially symmetric and non-increasing self-similar blowing-up solution with mass \( M \) and Figure 4 provides some numerical evidence of this fact. Besides this uniqueness question, the question of stability of these blowing-up solutions is also of interest.

![Figure 4](image-url)

**Figure 4.** Monotonicity of the mass \( a \mapsto M(a) \).

Another challenging question is the existence (or non-existence) of integrable profiles \( \varphi \) with a non-connected positivity set as discussed in Section 2. Figure 5 provides numerical evidence that, if \( a > a_c \) is large enough, \( u(\cdot, a) \) may have several zeroes and each positive “hump” actually corresponds to a solution of (15) for suitable values of \( R_i \) and \( R_s \). Whether the additional constraint (16) may be satisfied does not seem to be clear.

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