A machine learning based control of complex systems

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Abstract

In this work, inspired in the symbolic dynamic of chaotic systems and using machine learning techniques, a control strategy for complex systems is designed. Unlike the usual methodologies based on modeling, where the control signal is obtained from an approximation of the dynamic rule, here the strategy rest upon an approach of a function, that from the current state of the system, give the necessary perturbation to bring the system closer to a homoclinic orbit that naturally goes to the target.

The proposed methodology is data-driven or can be developed in a based-model context and is illustrated with computer simulations of chaotic systems given by discrete maps, ordinary differential equations and coupled maps networks.

Results shows the usefulness of the design of control techniques based on machine learning and numerical approach of homoclinic orbits.

1 Introduction

There are experimental and/or numerical evidence of the chaotic behavior in areas such as: population dynamics[1], epidemiology[2], ecosystems[3], cardiac rhythm[4], neurological systems[5], optoelectronic systems[6], chemical reactions[7], economic systems[8], communication systems[9] and a lot of more situations in physics, chemistry, biology and engineering[10]. Added to this, that behavior can be useful in areas such as secure communication[11], where chaotic systems offer an alternative for the development of new technologies.

As can be seen in most of the before mentioned cases, it is desirable to have strategies that allow the evolution of the system to be regulated: the population of animal communities can be controlled to prevent extinctions or the spread of diseases, for obvious reasons the epidemic diseases must be controlled, the heart rate must be controlled in the case of arrhythmia, brain activity must be controlled in case of epilepsy or finally some chemical, physical or computational systems can be controlled for the benefit of all. This represents sufficient motivation to devote effort to the design of models and/or control schemes for such systems.

From a general perspective these systems can be modeled as \( s_t = f_t^r(s_0) \), where \( r \) is a set of parameters, \( t \in \mathbb{R} \) or \( t \in \mathbb{N} \) represents the time, \( s \in S \) the state of the system and \( f \), a function that regulates the evolution of the system; so that, the control over them can only be achieved applying disturbances of some of these components.

When systems like the previously mentioned have chaotic behavior: on the one hand they have sensitivity to the initial conditions, which makes them difficult to predict in the long term, but on the other it offers the opportunity to change their behavior radically using small perturbations, this is, with a low cost and without producing important alterations in the system.
The seminal article on this topic presents a method for the control of chaotic systems, known as OGY method\cite{OGY}, this is the archetype of the methods that belong to the class of those that disturb the parameter. As evidence of it importance, from the point of view of basic research or technology of the finding of Ott, Grebogi and York\cite{OGY}, where the term controlling chaos was coined, there are many other methods for stabilizing chaotic systems that have been suggested afterwards, see for instance \cite{Lewis} and references therein.

Another method almost as celebrated, as the previous one, is a feedback method proposed by Pyragas et. al. \cite{Pyragas} belonging to the methods that disturb $f$. In this method, the stabilization of unstable periodic orbits of a chaotic system is achieved by combined feedback with the use of a specially designed external oscillator. Neither of the two methods require an a priory analytical knowledge of the dynamics system and may be applicable in experimental situations.

Among the strategies mentioned above, and many others, there are a significant amount of control strategies based on linear approximations of the dynamics around the target. When is it so, it is necessary that the linear approach is a good representation of the system or equivalently that the trajectory of the system is close enough to the target.

In this work, unlike those that model the dynamics to design the control function, here the control function is directly obtained by using Gaussian Processes Regression method\cite{GP} which is training using as examples a set of states, that are close to a points of a homoclinic orbit. The article is organized as follow: in Section II a symbolic dynamic inspired but learning machine based approach to control of chaotic dynamic is presented. Here, model-based and data-driven versions of the method are presented. Section III is devoted to presented some examples of the performance of the methodology and finally in Section IV we give some concluding remarks.

2 A learning machine approach to control chaos

Colloquially, a control problem can be paraphrased as a sequence of three steps: (i) the identification of the control’s object, (ii) the selection of the control’s target and (iii) the design of a control strategy.

2.1 Control object and target

We will consider chaotic systems in one or more dimensions, continuous or discrete and represented by discrete maps, ordinary differential equations or coupled maps networks. However, in order to show the ideas behind our control problem in a simple form, we will start considering as control object a chaotic, one-dimensional and uni-modal map $s_{t+1} = f_r(s_t)$, for later in the results section use as control objects the rest of previously listed systems.

The targets, in all examples, are given by an unstable fixed-point solutions ($s^*$) of the dynamical system, although the control strategy can be adapted to the case of other types of periodic orbits.

2.2 Model-based approach of the control strategy

Our approach to the control strategy propose to the estimate a small perturbation $u_t$, to the actual state $s_t$, in such a way that in the $(t+1)$-th iteration of $f$ the orbit come close to some point $s_0$, whose evolution has the form $s_0, s_1, \ldots, s_{k-1}, s^* + \delta$, where $\delta$ is a small value. This is, we should to estimate a perturbation $u_t$, such that, if

$$s_{t+1} = f_r(s_t) + u_t,$$

then $|s_{t+1} - s_0| < |s_t - s_0|$, with $s^* + \delta = f^k_r(s_0)$. 


2.2.1 Target improve

In some sense, the idea behind of the methodology is like to the OGY method, it consists in bringing the system closer, not immediately, to the control’s target but to points of orbits that naturally takes it as close as possible to target. Thus, if we are able to estimate these new points and include it in the target, we expect that the control scheme to be more effective. The orbits that fulfill that condition, in the case of the fixed points of chaotic systems, are the homoclinic orbits. This orbits joins a saddle equilibrium point to itself[16], and offer a number of potential targets equal to the points in this orbit. However, we must to note that here, unlike the OGY method, we do not disturb the parameters of the system or use explicitly a linear approximation of the system.

Our control strategy start by determining the set of states \( \{s^i_0\}_{t=1}^q \). Clearly, these states can be estimated if we iterate the inverse function of \( f \), from \( s^* + \delta \), this is, \( s_0 = f_r^{-k}(s^* + \delta) \), but this can no be done simply because the inverse of a nonlinear function is multi-valued.

On the other hand, it is well known[17] that the phase space evolution of an uni-modal discrete map can be translated, in biunivocal way, into a binary representation by using a partition of the state space in two regions, labeled each of them by one symbol (\( \sigma \)), 0 or 1. The symbolic representation is obtained replacing \( s_t \) by the label associated to the element of the partition which belongs \( s_t \rightarrow \sigma_t \). With this new information \( f_r^{-1} \), can be written as as:

\[
f_r^{-1}(s_t, \Sigma) = \begin{cases} f_{r,0}^{-1}(s_t) & if \sigma_t = 0 \\ f_{r,1}^{-1}(s_t) & if \sigma_t = 1 \\ \end{cases}
\]

where \( f_{r,0} \) and \( f_{r,1} \) are the monotone branch at the left or right from the maximum of \( f \), respectively. In this form, every state \( s_t \), can be represented by a sequence of symbols \( \Sigma = \{\sigma_{t+i}\}_{i=0}^l \), given by the symbolic chain generated from the initial condition \( s_t \). Here \( l \) give the accuracy with which \( s_t \) is represented. In particular, the states from which the system reaches a neighborhood of the fixed point are given by:

\[
s_0 = f_{r,\sigma_1}^{-1} \circ \cdots \circ f_{r,\sigma_l}^{-1} \circ f_{r,\sigma_l}^{-1}(s^* + \delta)
\]

Due to the contraction of \( f_{r,\sigma_k}^{-1} \) and for \( l \) large enough, the iteration of (3) converges approximately to the real number \( s_0 \) independently of the value of \( \delta \), if the sequence of symbols \( \sigma_t \) is appropriate, i. e. if these sequences are of the form

\[
\Sigma_i = \sigma_1, \sigma_2, \cdots, \sigma_k, \sigma^*, \cdots, \sigma^* \quad (l-k \text{ times})
\]

and are large enough.

It form a set of \( 2^k \) possible binary sequences. In general not all of them are admissible (see for example[17], Sec. 2.5.5), but the existence of some sequences of states associated with the admissible chains is guaranteed by the presence of homoclinic orbits in the attractor of \( f \). Hence this sequences approximating segments of homoclinic orbits, will be used to calculate the \( 2^k \) initial conditions, \( s^i_0 = f_r^{-l}(s_R, \Sigma_i) \) with \( s_R \) a random real number in the adequate domain, which places the system in a stable orbit towards the target \( s^* \). Finally, the addition of the set \( s_0 \)'s to the target, clearly will contributes to decrease the convergence time from the system’s current state to the control’s target.

2.2.2 Control strategy

In order to estimate the sequence \( u_k \) we start generating, from \( q \) strings like [4], \( q \) initial conditions \( S_0 = \{s^1_0, s^2_0, \cdots, s^q_0\} \), from which the system reach a neighborhood of \( s^* \) in \( l \) iterations.
With the before estimate states, $M$ pairs $(s_t, u_t)$ are generated using,
\[
u_t = s_{t+1} - N(S_0, s_{t+1}),
\]
from the elements of an typical orbit \(\{s_t\}_{t=1}^{N}\) of the system. Here, \(N(S_0, s_{t+1})\) is a function that pick out, the element of \(S_0\) nearest to \(s_{t+1}\) is defined as:
\[
N(S_0, s_t) = \arg \min_{s_0} \{|s_0^i - s_{t+1}|\}.
\]
These pairs will later serve as training data for a machine learning scheme that estimates \(u_t = u(s_t)\), for any \(s_t\) in the domain of \(f\). In this way the control problem is turned into a regression problem, that we propose can be to solved using artificial intelligence techniques.

As one, among many alternatives to approximate \(u\), in this work we use a powerful strategy from machine learning known as Gaussian Process Regression\[15\]. Here we will understand these processes according to their usual definition: a set of indexed random variables, such that every finite subset of those random variables has a multivariate normal distribution. The distribution of a \(GP\) is the joint distribution of all those random variables, and as such, it is a distribution over functions with a continuous domain.

These process are specified by its \textit{mean function} \(m(s)\) and \textit{covariance function} \(k(s, s_i)\) and are a natural generalization of the Gaussian distribution whose mean and covariance are represented by a vector and a matrix, respectively. We will represent this process, with the usual notation, as:
\[
u \sim GP(m(s), k(s, s'))
\]
and we will read this, as: \textit{the function u is distributed as a GP with prior mean function m(x) and covariance function k(s, s')}. If we generate a random vector from this distribution, \(u \sim N(m(s), k(s, s'))\), this vector will have the function values \(u(s)\) as coordinates indexed by \(s\).

In this work, the strategy of control depend on the methodology to approximate \(u\) but we believe that any methodology used, to interpolate the response of the system to states that are not present in the their training set, can solve the control problem.

### 2.3 Model-free approach of the control strategy

Since it is not always possible to have the symbolic dynamics of the system, this section shows how it is possible to implement our idea using data from a typical orbit of the system. Thus a data-driven version of the strategy is easily designed if we replace the symbolic dynamics knowledge with an algorithm that identifies, in the time series, a set of sequences of states converging to \(s^*\).

At this point, we could use a numerical method of detection of homoclinic orbits as in references [19, 20, 21] and find the points which we will include in the target. However, we will opt for a simplier strategy: we start identifying, in the time series, the set of the nearest states to \(s^*\), \(\{s_{m1}, s_{m2}, \ldots, s_{mq}\}\), so that \(s_0^i = s_{m_i-1}\), with \(i = 1, \cdots, q\) and \(l\) is an integer identifying the \(l\)-th predecessor of \(s_{m_i}\). Thus the set \(S_0 = \{s_0^i\}_{i=1}^q\) is constructed and given that set the control strategy is the same that in previous section.

In the following section the performance of the methodology is shown. In the case of the logistic map we will present both approximations (mode-based and model-free versions), in the rest it is not so simple to obtain the symbolic representation so we will only show results of the model-free approach.
3 Results

In order to show the performance of the proposed scheme, we use as examples: the Logistic map, the Henon map, the Lorenz system and a network of coupled discrete maps. These represent one, two, three and \( n \)-dimensional systems with discrete time and continuous time.

3.1 Logistic map

In this case, \( f(s_t) = 4s_t(1 - s_t) \) has an unstable fixed point at \( s^* = 3/4 \). The inverse function of \( f \) can be written using the symbolic dynamic:

\[
    f^{-1}_4(s_t, \Sigma) = \begin{cases} 
    \frac{1}{2} + \frac{\sqrt{1 - s_t}}{2} & \text{if } \sigma_t = 0 \\
    \frac{1}{2} - \frac{\sqrt{1 - s_t}}{2} & \text{if } \sigma_t = 1 
\end{cases}, \tag{8}
\]

The controlled system can be written as \( s_{t+1} = 4s_t(1 - s_t) + u_t \), with \( u_t = u(x_t) \). In this case, since the symbolic dynamic is known, the model-based and data-driven control schemes are showed. Figures 1 and 2, shows the results of the control scheme application using 500 data points with \( l = 6 \), \( k = 4 \) and \( q = 16 \) in the case of model based control and \( l = 6 \) and \( q = 16 \) in the case of data-driven control.

![Figure 1: Model based control approach of Logistic map. Left: Black points represents the training set of the Gaussian process. Gray dashed line represents the predicted values of the function \( u(s_t) \) and gray shadow represents the confidence interval of the predictions, given by two times the standard deviation of the errors. Right: Evolution of the controlled system. The inset shows the applied perturbations. In both cases, the results shown were averaged for 100 initial random conditions.](image)

It is worth to note that in general, if all possible sequences of symbolic states are considered as training data generation, the quality of the performance of the strategy could be affected, because this number of pairs does not necessarily coincide with the admissible ones. Now, if the dynamics are available, it is possible to determine the admissible symbolic sequences using the strategy proposed in [17]. The study of this aspect of the problem is not the subject of this article, but will be addressed in future works.

3.2 Henon map

The Henon map is used as canonical example of bi-dimensional chaotic system and given by a difference equation system. It has an unstable fixed point at \( (x^*, y^*) = (0.6313, 0.1894) \) and the controlled system is given by:

\[
    \begin{align*}
        x_{t+1} &= 1 - a(x_t^2 + y_t) + u_t, \\
        y_{t+1} &= bx_t.
    \end{align*} \tag{9}
\]
Figure 2: Data-driven control approach of Logistic map. Left: Black points represents the training set of the Gaussian process. Gray dashed line represents the predicted values of the function $u(s_t)$ and gray shadow represents the confidence interval of the predictions, given as in the model-based case. Right: Evolution of the controlled system. The inset shows the applied perturbations. In both cases, the results shown were averaged for 100 initial random conditions.

Here $u_t = u(x_t, y_t)$ and $10^3$ data points are used to train the Gaussian Process with parameters, $l = 6$, $k = 3$ and $q = 64$.

Figure 3: Henon map. Evolution of the controlled system. The inset shows the applied perturbations. In both cases, the results shown were averaged for 100 initial random conditions.

### 3.3 Lorenz system

The Lorenz system is the typical example of the continuous chaotic dynamical system represented by the set of ordinary differential equations. It has an unstable fixed point at $(8.4553, 8.4553, 27)$. The controlled version in this case is:

\[
\begin{align*}
\dot{x}(t) &= \sigma(y(t) - x(t)), \\
\dot{y}(t) &= -x(t)z(t) + rx(t) - y(t) + u_t, \\
\dot{z}(t) &= x(t)y(t) - bz(t).
\end{align*}
\]

Here $u_t = u(x(t), y(t), z(t))$ and $8 \times 10^3$ data points are used to train the Gaussian Process with parameters, $l = 8$, $k = 4$ and $q = 128$.

### 3.4 A coupled maps network

The systems composed of parts that have some degree of autonomy and interact, are frequent in innumerable situations and in areas from basic to applied science, so they must be an obligatory example of control object. Here, without loss of generality we use the logistic map, $f(s_t) = \ldots$
rs_t^i(1 - s_t^i), as the nonlinear local dynamic to construct a generic complex dynamical network consisting of $M$ identical nodes,

$$s_{t+1}^i = f(s_t^i) - \frac{\epsilon}{k_i} \sum_{j=1}^{M} L_{ij} f(s_t^j),$$

(10)

where $L = (L_{ij})_{i,j=1}^{M}$ is the usual (symmetric) Laplacian matrix with diagonal entries $L_{ii} = k_i$ and $k_i$ the out degree of the node $i$. Thus, the network (10) can be represented as

$$s_{t+1} = F(s_t) = f(s_t) - \epsilon \ C \ f(s_t),$$

(11)

where the bolt font are vectors, $C = (L_{ij}/k_i)^N_{i,j=1}$ is an asymmetric matrix with real eigenvalues and $\epsilon$ a real parameter. Thus, the network can be represented as $s_{t+1} = Af(s_t)$, with $A = I - \epsilon \ C$. This system have a unstable and homogeneous fixed point given by $s^* = (s_1^*, s_2^*, \ldots, s_N^*)$. In our case $M = 4$, $\epsilon = 0.05$ and

$$A = \begin{pmatrix}
0.95 & 0.025 & 0.025 & 0. \\
0.025 & 0.95 & 0. & 0.025 \\
0.025 & 0. & 0.95 & 0.025 \\
0. & 0.025 & 0.025 & 0.95 \\
\end{pmatrix}.$$  

(12)

Although the symbolic dynamics of this network can be constructed in a way similar to the one-dimensional case, here we will only present results for the data-dependent control strategy. The following figure shows evolution of the controlled system a a function of the time.

In this example, the controlled system is written as:

$$s_{t+1}^i = f(s_t^i) - \frac{\epsilon}{k_i} \sum_{j=1}^{M} L_{ij} f(s_t^j) + u_t^i,$$

(13)

where $u_t^i = u^i(s_t)$.

Figures 5 and 6 shows the structure of the network used as example in this article and the evolution of the controlled network, respectively. The control functions $u_t^i$ was generated by training the Gaussian Process with $10^5$ data points with parameters $l = 4$, $k = 3$ and $q = 4 \times 10^3$.

In spite of the results shown in Figure 6, refer to a network with the topology of the network in Figure 5, the performance of the control scheme is similar for many networks with the same number of units. Numerical experiments suggest that, as this number increases the amount of data needed to obtain good approximations of $u$ quickly grows so it would be necessary, in the case of large networks, to combine the control strategy with a numerical methodology to
Figure 5: Coupled chaotic maps.

Figure 6: Coupled maps network. Evolution of the controlled system. Continuous black, dashed black, continuous gray and dashed gray lines, shows the evolution of the four nodes. The inset shows the applied perturbations.

approximate homoclinic orbits. This is out of the scope of this work and it is currently under investigation.

4 Final remarks

A methodology for control of complex systems based on the estimation of the control function, from the observation of the system and using a machine learning technique, was presented. Although in this particular case we have used the regression method based on Gaussian processes, the methodology allows us to use the regression technique that best suits the particular problem.

Despite the existence of other control methods that use artificial intelligence strategies, the proposed method is novel in the sense that it directly approximates the control signal and does so with very little computational effort.

Finally, results shows that the use machine learning techniques and numerical approximation of homoclinic orbits can be useful in the design of control techniques of complex systems.
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