The Minimum $S$-Divergence Estimator under Continuous Models: The Basu-Lindsay Approach

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Abstract

Robust inference based on the minimization of statistical divergences has proved to be a useful alternative to the classical maximum likelihood based techniques. Recently Ghosh et al. (2013) proposed a general class of divergence measures for robust statistical inference, named the $S$-Divergence Family. Ghosh (2014) discussed its asymptotic properties for the discrete model of densities. In the present paper, we develop the asymptotic properties of the proposed minimum $S$-Divergence estimators under continuous models. Here we use the Basu-Lindsay approach (1994) of smoothing the model densities that, unlike previous approaches, avoids much of the complications of the kernel bandwidth selection. Illustrations are presented to support the performance of the resulting estimators both in terms of efficiency and robustness through extensive simulation studies and real data examples.

1 Introduction

In case of parametric statistical inference, the usefulness and properties of various density based minimum divergence estimators have been extensively studied in the recent literature; see Basu et al. (2011). The key in density-based minimum divergence estimation is the quantification of the discrepancy between the parametric model and the sample data through a suitable density based divergence. Then one obtains the estimate of the unknown parameter of interest by minimizing this divergence as a function of the unknown parameter. The class of density-based divergences that may be useful in this approach includes the Pearson’s chi-square (Pearson, 1900), the Kullback-Leibler divergence (Kullback and Leibler, 1951), the $\phi$-divergence or disparity family (Csiszár, 1963) including the Hellinger distance, the Bregman divergence (Bregman, 1967), the Burbea-Rao divergence (Burbea and Rao, 1982), Cressie-Read family of Power divergence (Cressie and Read, 1984), the density power divergence (Basu et al., 1998) etc. The power divergence family is a subclass of the class of $\phi$-divergences while the density power divergence family is a subclass of Bregman divergences.

The main advantage of minimum distance methods in parametric estimation is the robustness property which some of them inherently possess. In the past statisticians believed that the goals of robustness and efficiency were conflicting and could not be achieved simultaneously. However, several density-based minimum distance estimators have been demonstrated to have strong robustness properties along with full asymptotic efficiency [e.g., Beran (1977), Tamura and Boos (1986), Simpson (1987, 1989) and Lindsay (1994)]. Several of the minimum distance estimators based on
the Cressie-Read family of power divergences are examples of such estimators. However, standard estimation techniques under continuous models using the power divergence family require the use of nonparametric smoothing for the construction of the data generating density and hence inherit all the complications of the kernel density estimation process and bandwidth selection. Later, Basu et al. (1998) developed a rich class of density-based divergence measures named the density power divergences that produce robust estimates without using nonparametric smoothing but with a small loss in efficiency. Both the density power divergence and the power divergence families have similarities in their outlier downweighting philosophy. Although these two families have different forms (except for the Kullback-Leibler divergence which is the only divergence common to both families), Patra et al. (2013) provided an useful connection between these two families which allows us to develop either family from the other through specific motivations.

Combining the forms of the power divergence and the density power divergences, Ghosh et al. (2013) developed a two parameter family of density-based divergences, called the “$S$-Divergence”, that connects each member of the Cressie-Read family of power divergences smoothly to the $L_2$-divergence at the other end. Through various numerical examples, these authors illustrated that several of the minimum divergence estimators within the $S$-divergence family also have strong robustness properties and are often competitive with the classical estimators in terms of efficiency. Ghosh (2014) has provided the asymptotic distribution of the corresponding minimum $S$-divergence estimators under discrete model families. In the discrete case there is a natural nonparametric density estimator of the true unknown density; this is simply the vector of the sample relative frequencies, the construction of which requires no additional artifacts like the kernel function or the bandwidth. However, in the case of continuous models it is not so simple and one needs to use the kernel density estimate for the same purpose.

In this present article, we will develop the theoretical properties of the minimum $S$-divergence estimators under the set up for a continuous model. The first approach under the similar set-up with kernel density estimators was presented by Beran (1977) in the context of minimum Hellinger distance estimators. But, his approach contains all the complications of the kernel estimator along with the issue of bandwidth selection. For simplicity, we will use the later approach of Basu and Lindsay (1994) that helps us to at least partially avoid the problem with kernel bandwidth selection. In this approach we replace ordinary model densities with the smoothed model densities to obtain an estimator slightly different than the minimum $S$-divergence estimator; we will refer to this estimator as the minimum $S^*$-divergence estimator. We will prove the consistency and asymptotic normality of the minimum $S^*$-divergence estimator under appropriate assumptions. Interestingly, we will see that the asymptotic distributions of the minimum $S^*$-divergence estimators are also independent of the parameter $\lambda$ in the definition of $S$-divergence, as in the distribution of the minimum $S$-divergence estimators in the discrete models.

The rest of the paper is organized as follows. We start with a brief description of the $S$-divergence family and the corresponding minimum $S$-divergence estimators in Section 2. Then we will describe the two different approaches of the minimum divergence estimator under continuous models, Beran’s approach and Basu-Lindsay approach, in Section 3. The minimum $S^*$-divergence estimators will be introduced in that section and details for the particular case of $\lambda = 0$ will be presented in Section 4. Sections 5 and 6 will describe the influence function analysis and the asymptotic properties of the general minimum $S^*$-divergence estimators. All the results will be supported by suitable simulation studies in Section 7 and some interesting real data examples in Section 8. We will introduce the concept of the $\alpha$-transparent kernels in Section 9. We will end the paper with a small discussion on the choice of tuning parameters in Section 10.
2 The $S$-Divergence Family and its Estimating Equation

The $S$-divergence family of divergences (Ghosh et al. 2013) is defined in terms of two parameters $\alpha \geq 0$ and $\lambda \in \mathbb{R}$ as

$$S(\alpha, \lambda)(g, f) = \frac{1}{1 + \lambda(1 - \alpha)} \int \left[ \left( f^{1+\alpha} - g^{1+\alpha} \right) - \frac{(1 + \alpha)}{(\alpha - \lambda(1 - \alpha))} g^{1+\lambda(1-\alpha)} \left( f^{\alpha-\lambda(1-\alpha)} - g^{\alpha-\lambda(1-\alpha)} \right) \right]$$

$$= \frac{1}{A} \int f^{1+\alpha} - \frac{1 + \alpha}{AB} \int f^B g^A + \frac{1}{B} \int g^{1+\alpha},$$

with $A = 1 + \lambda(1 - \alpha)$ and $B = \alpha - \lambda(1 - \alpha)$. Note that, $A + B = 1 + \alpha$. If $A = 0$ then the corresponding $S$-divergence measures are defined as the continuous limit of (1) as $A \to 0$ and are given by

$$S(\alpha, \lambda; A=0)(g, f) = \lim_{A \to 0} S(\alpha, \lambda)(g, f) = \int f^{1+\alpha} \log \left( \frac{f}{g} \right) - \int \frac{(f^{1+\alpha} - g^{1+\alpha})}{1 + \alpha}.$$ 

(2)

Similarly, for $B = 0$ then the corresponding $S$-divergence measures are defined as

$$S(\alpha, \lambda; B=0)(g, f) = \lim_{B \to 0} S(\alpha, \lambda)(g, f) = \int g^{1+\alpha} \log \left( \frac{g}{f} \right) - \int \frac{(g^{1+\alpha} - f^{1+\alpha})}{1 + \alpha}.$$ 

(3)

Note that for $\alpha = 0$, this family reduces to the Cressie-Read family with parameter $\lambda$ and for $\alpha = 1$, it gives the $L_2$ divergence (independently of $\lambda$). Again for $\lambda = 0$, the expression in (1) generates the Density Power divergence measure having parameter $\alpha$. Ghosh et. al. (2013) showed that the $S$-divergence family defined in (1), (2) and (3) indeed represents a family of genuine statistical divergence measures in the sense that $S(\alpha, \lambda)(g, f) \geq 0$ for all densities $g, f$ defined with respect to a common measure and for all $\lambda \in \mathbb{R}$, $\alpha \geq 0$, and it equals zero if and only if the argument densities are identically equal.

Now, let us assume that we have a sample from the true unknown density $g$ (having $G$ as the corresponding cumulative distribution function), and we want to model it by the parametric family \{f_\theta : \theta \in \Theta \subseteq \mathbb{R}^p\} of densities. The minimum $S$-divergence estimator of the unknown parameter $\theta$ is obtained by minimizing the divergence $S(\alpha, \lambda)(\hat{g}, f_\theta)$ with respect to $\theta \in \Theta$, where $\hat{g}$ is some nonparametric estimate of the true density $g$ based on the observed data. For the discrete model, the vector of relative frequencies may be taken as $\hat{g}$, but for continuous models we need to use kernel density estimation or a similar technique to obtain $\hat{g}$. The corresponding estimating equation for the minimum $S$-divergence estimator is given by

$$\int f^{1+\alpha}_\theta u_\theta - \int f^B \hat{g}^A u_\theta = 0$$

(4)

or,

$$\int K(\delta(x)) f^{1+\alpha}_\theta(x) u_\theta(x) = 0,$$

(5)

where $\delta(x) = \frac{\hat{g}(x)}{f_\theta(x)} - 1$, $K(\delta) = \frac{(\delta+1)^{A-1}}{A}$, $u_\theta(x) = \frac{\partial}{\partial \theta} \log f_\theta(x)$, the maximum likelihood score function.
Let \( T_{\alpha,\lambda} \) denotes the minimum \( S \)-divergence functional at the distribution \( G \) defined by the relation
\[
S_{(\alpha,\lambda)}(g, f_{T_{\alpha,\lambda}(G)}) = \min_{\theta \in \Theta} S_{(\alpha,\lambda)}(g, f_{\theta}),
\]
whenever the minimum exists. One can easily derive the influence function of this functional by taking a derivative of the estimating equation (5) under a mixture contaminated density with respect to a mixture contamination. Then, the influence function of this minimum \( S \)-divergence functional is given by
\[
IF(y; T_{\alpha,\lambda}, G) = J^{-1} \left[ Au_\theta(y) f_B(y) g^{A-1} - \xi \right],
\]
where
\[
\xi = \xi(\theta) = A \int u_\theta f^B g^A, \tag{7}
\]
\[
J = J(\theta) = A \int u_\theta u_\theta^T f_\theta^{1+\alpha} + \int (i_\theta - B u_\theta u_\theta^T) (g^A - f_\theta^A) f_\theta^B, \tag{8}
\]
and \( i_\theta(x) = -\nabla [u_\theta(x)] \) with \( \nabla \) representing the gradient with respect to \( \theta \). However, if the true density belongs to the model family, i.e., \( g = f_\theta \), the influence function simplifies to
\[
IF(y; T_{\alpha,\lambda}, F_\theta) = \left( \int u_\theta u_\theta^T f_\theta^{1+\alpha} \right)^{-1} \left[ u_\theta(y) f_\theta^\alpha (y) - \int u_\theta f_\theta^{1+\alpha} \right]. \tag{9}
\]
Interestingly the influence function at the model depends only on the parameter \( \alpha \) and not on \( \lambda \).

## 3 The Minimum \( S \)-Divergence Estimator under Continuous Models: Different Approaches

Let us now consider the estimation under the continuous models by minimizing the \( S \)-divergence. Let \( G \) denotes the class of all probability distributions having densities with respect to the Lebesgue measure. We will assume that the true, data generating distribution \( G \) and the model family \( F = \{F_\theta : \theta \in \Theta \subseteq \mathbb{R}^p\} \) belong to \( G \); also \( G \) and \( F_\theta \) have densities \( g \) and \( f_\theta \) with respect to the Lebesgue measure.

Let \( X_1, \ldots, X_n \) be a random sample of size \( n \) from the true distribution \( G \) which is modeled by \( F \), and we want to estimate the unknown model parameter \( \theta \). As in the case of discrete models, the minimum \( S \)-divergence estimator (MSDE) of the unknown parameter \( \theta \) is to be obtained by choosing the model density \( f_\theta \) which gives the closest fit to the data with respect to the \( S \)-divergence measure. However, unlike the discrete case, this gives rise to an immediate challenge; the data are discrete, but the model is continuous, so that there is an obvious incompatibility of measures in constructing a distance between the two. We cannot simply use relative frequencies to represent a nonparametric density estimate of the true data generating distribution in this case.

In this context, Beran (1977) suggested the construction of a continuous density estimate using some appropriate nonparametric density estimation method such as kernel density estimation. Let us assume that
\[
g_n^\alpha(x) = \frac{1}{n} \sum_{i=1}^n W(x, X_i, h_n) = \int W(x, y, h_n) dG_n(y) \tag{10}
\]
represents a nonparametric kernel density estimator where \( W(x, y, h_n) \) is a smooth kernel function with bandwidth \( h_n \) and \( G_n \) is the empirical distribution function as obtained from the data. Usually the kernel is chosen as a symmetric location-scale density with scale \( h_n \), i.e.,

\[
W(x, X_i, h_n) = \frac{1}{h_n} w\left(\frac{x - X_i}{h_n}\right),
\]

where \( w(\cdot) \) is a symmetric nonnegative density function. According to Beran’s approach, we can now estimate \( \theta \) by minimizing the \( S \)-divergence measure between \( g_n^* \) and \( f_\theta \). Under suitable differentiability assumptions, the estimating equation is then given by (5) with \( \hat{g} \) replaced by the kernel estimator \( g_n^* \). Although the rest of the estimation and the asymptotic properties of the estimator are similar to the discrete case, the inclusion of the kernel density estimation process leads to substantial difficulties in practice. The theoretical derivation of the asymptotic normality of the minimum \( S \)-divergence estimators and the description of their other asymptotic properties are far more complex in this case. In particular, the choice of the sequence of kernels (or, more precisely, the sequence of smoothing parameters) becomes critical and also various complicated conditions have to be imposed for the consistency of the kernel density estimator; these complications are unavoidable for Beran’s approach.

The approach taken by Basu and Lindsay (1994) differs from the Beran’s approach in that it proposes that the model be convoluted with the same kernel as well. Suppose \( f_\theta^* \) represents the kernel integrated “smoothed” version of the model defined as

\[
f_\theta^*(x) = \int W(x, y) dF_\theta(y). \tag{12}
\]

In the Basu-Lindsay approach of model smoothing, we consider the \( S \)-divergence measure between \( g_n^* \) and \( f_\theta^* \), and minimize it over \( \theta \) to obtain the corresponding minimum \( S \)-divergence estimator. The procedure may be intuitively justified as follows. The intent here is to minimize a measure of discrepancy between the data and the model. To make the data continuous, one has to import an artificial kernel. However, one needs to ensure — through the imposition of suitable conditions on the kernel function and the smoothing parameter — that the additional smoothing effect due to the kernel vanishes asymptotically. In the Basu-Lindsay approach, we convolute the model with the same kernel used on the data. In a sense, this compensates for the bias due to the imposition of the kernel on the data by imposing the same bias on the model. It is therefore expected that the kernel will play a less important role in the estimation procedure than it plays in Beran’s approach, particularly in small samples. As we will see later in this section, one gets consistent estimators of the parameter \( \theta \) even when the smoothing parameter is held fixed as the sample size increases to infinity.

In this work, we will consider only the simpler Basu-Lindsay approach for estimation under the continuous model based on \( S \)-divergence. We need consider the corresponding smoothed version of the likelihood score function

\[
\tilde{u}_\theta(x) = \nabla \log f_\theta^*(x);
\]

we will denote its \( j \)th entry by \( \tilde{u}_{j\theta}(x) \). Further, for the second derivative matrix \( \nabla^2 \tilde{u}_\theta(x) = \nabla^2 \log f_\theta^*(x) \), we denote its \((j, k)\)th element by \( \tilde{u}_{j,k\theta}(x) \). Now, let us consider \( S_{(\alpha, \lambda)}(g_n^*, f_\theta^*) \) and minimize this measure of discrepancy between the model and data with respect to the parameter.
A routine differentiation shows that the corresponding estimating equation is given by

\[
\int (f_\theta^*)^B (g_n^*)^A \tilde{u}_\theta - \int (f_\theta^*)^{1+\alpha} \tilde{u}_\theta = 0,
\]

or,

\[
\int K(\delta_n^*) (f_\theta^*)^{1+\alpha} \tilde{u}_\theta = 0,
\]

with \( \delta_n^*(x) = \frac{g_n^*(x)}{f_\theta^*(x)} - 1 \). We will denote the estimator obtained by this approach as minimum \( S^* \)-divergence estimator (MSDE*) which is, in general, not the same as the estimator obtained by minimizing the \( S \)-divergence measure \( S_{(\alpha,\lambda)}(g_n^*, f_\theta) \). The main reason behind this inequality is the substitution of the model family by its kernel smoothed version.

4 The Minimum \( S^* \)-Divergence Estimator: Special Case (\( \lambda = 0 \))

The density power divergence measure, originally proposed by Basu et al. (1998), can also be obtained from the \( S \)-divergence divergence family by putting \( \lambda = 0 \) and has the form

\[
d_\alpha(g, f_\theta) = S_{(\alpha,0)}(g, f_\theta) = \int f_\theta^{1+\alpha} - \frac{1+\alpha}{\alpha} \int f_\theta^\alpha g + \frac{1}{\alpha} \int g^{1+\alpha}, \quad \alpha > 0.
\]

This particular family of divergences has the special property that we can find the minimum DPD estimator without using kernel density estimators even under continuous models. In fact, we can write the minimum DPD estimating equation in terms of the empirical distribution function \( G_n \) based on the observed data as

\[
\int f_\theta^\alpha u_\theta dG_n - \int f_\theta^{1+\alpha} u_\theta = 0,
\]

or,

\[
\frac{1}{n} \sum_{i=1}^{n} f_\theta^\alpha(X_i) u_\theta(X_i) - E_\theta[f_\theta^\alpha(X) u_\theta(X)] = 0.
\]

Note that the above estimating equation is an unbiased estimating equation at the model and we can minimize it to get the minimum DPD estimator; there is no need for kernel smoothing. The asymptotic properties of the minimum DPD estimators are well-studied in the literature and are valid under continuous models also.

Now, suppose we apply the approach of smoothed density (Basu and Lindsay, 1994) to find the corresponding MSDE* for this special case. Let us denote the MSDE* under this particular case of \( \lambda = 0 \) by the minimum DPD* estimator (MDPDE*). The corresponding estimating equation is given by

\[
\int (f_\theta^*)^a g_n^* \tilde{u}_\theta - \int (f_\theta^*)^{1+\alpha} \tilde{u}_\theta = 0.
\]
However, here we can further simplify this estimating equation by noting that
\[
\int g_n^*(x)f_\theta^*(x)^{\alpha \tilde{u}_\theta}(x)dx = \int \left[ \int W(x, y, h)dG_n(y) \right] f_\theta^*(x)^{\alpha \tilde{u}_\theta}(x)dx \\
= \int \left[ \int f_\theta^*(x)^{\alpha \tilde{u}_\theta}(x)W(x, y, h)dx \right] dG_n(y) \\
\quad \text{(By Fubini’s theorem)} \\
= \int u_\theta^*(y)dG_n(y) \\
= \frac{1}{n} \sum_{i=1}^{n} u_\theta^{*}(X_i). 
\]
Here, we have used the notation
\[
u_\theta^{\alpha *}(y) = \int \tilde{u}_\theta(x)\{f_\theta^*(x)\}^{\alpha}W(x, y, h)dx, \tag{18}
\]
in the spirit of Basu and Lindsay (1994). Further, taking expectation with respect to the density \( f_\theta \),
\[
E_\theta[u_\theta^*(Y)] = \int \left[ \int f_\theta^*(x)^{\alpha \tilde{u}_\theta}(x)W(x, y, h)dx \right] f_\theta(y)dy \\
= \int f_\theta^*(x)^{\alpha \tilde{u}_\theta}(x) \left[ \int W(x, y, h)f_\theta(y)dy \right] dx \\
= \int (f_\theta^*(x)^{\alpha \tilde{u}_\theta}(x)) f_\theta^*(x)dx \\
= \int f_\theta^*(x)^{1+\alpha \tilde{u}_\theta}(x). 
\]
Thus, the estimating equation of the minimum DPD* estimator becomes
\[
\frac{1}{n} \sum_{i=1}^{n} u_\theta^{*}(X_i) - E_\theta[u_\theta^*(Y)] = 0, \tag{19}
\]
which is clearly an unbiased estimating equation. Therefore the standard asymptotic results for unbiased estimating equations also hold for the minimum DPD* estimators. Under certain restrictions on the kernel function, the results of Section 9 will show that these two estimating equations (16) and (19) become the same, producing identical estimators.

5 Influence Function of the Minimum \( S^* \)-Divergence Estimator

Consider the general case of the minimum \( S^* \)-divergence estimators and the kernel smoothed version \( g^* \) of the true density \( g \) defined by
\[
g^*(x) = \int W(x, y, h)dG(y). \tag{20}
\]
Then the minimum $S^\ast$-divergence estimator functional $T^\ast_{(\alpha, \lambda)}(G)$ is defined by the relation

$$S_{(\alpha, \lambda)}(g^\ast, f^\ast_{T^\ast_{(\alpha, \lambda)}}(G)) = \min_{\theta \in \Theta} S_{(\alpha, \lambda)}(g^\ast, f^\ast_\theta),$$

provided such a minimum exists. Then the influence function of $T^\ast_{(\alpha, \lambda)}$ can be derived from the corresponding estimating equation (which is the same as Equation [14] with $g^\ast_n$ replaced by $g^\ast$).

Let us now consider the contaminated distribution $G_\epsilon(x) = (1 - \epsilon)G(x) + \epsilon \wedge_y (x)$ with $\wedge_y$ being the degenerate distribution at the contamination point $y$. Suppose $g_\epsilon$ denotes the corresponding contaminated density, and $g^\ast_\epsilon$ denotes the associated smoothed density. Note that $g^\ast_\epsilon = (1 - \epsilon)g^\ast + \epsilon W(x, y, h)$. Then, by definition, the minimum $S^\ast$-divergence estimator functional $T(G_\epsilon)$ satisfies

$$\int K(\delta_\epsilon(x)) f^\ast_\theta(x)^{1+\alpha} \tilde{u}_\theta(x) dx = 0, \quad (21)$$

where $\delta_\epsilon(x) = g^\ast_\epsilon(x)/f^\ast_\theta(x) - 1$. Then the influence function of the MSDE’s at the distribution $G$ can be computed by taking a derivative of both sides of (21) and is presented in the following theorem.

**Theorem 5.1** The Influence Function for the minimum $S^\ast$-divergence estimator functional $T^\ast_{(\alpha, \lambda)}$ at the distribution $G$ is given by

$$IF(y; G, T^\ast_{(\alpha, \lambda)}) = [J^\ast_g]^{-1} N^\ast_g(y) \quad (22)$$

where

$$N^\ast_g(y) = N^\ast_{(\alpha, \lambda)}(y; g) = A \left[ \int (f^\ast_{\theta_0}(x)) B(g^\ast(x)) A^{-1} \tilde{u}_{\theta_0}(x) W(x, y, h) dx - \int (f^\ast_{\theta_0}) B(g^\ast) A^{-1} \tilde{u}_{\theta_0} \right], \quad (23)$$

and

$$J^\ast_g = J^\ast_{(\alpha, \lambda)}(g) = A \int (f^\ast_{\theta_0})^{1+\alpha} \tilde{u}_{\theta_0} \tilde{w}^T_{\theta_0} + \int \tilde{i}_{\theta_0} - B \tilde{u}_{\theta_0} \tilde{w}^T_{\theta_0} (g^\ast)^A - (f^\ast_{\theta_0})^A (f^\ast_{\theta_0})^B, \quad (24)$$

with $\theta^g = T^\ast_{(\alpha, \lambda)}(G)$ being the best fitting parameter under $G$ and $\tilde{i}_{\theta_0}(x) = -\nabla [\tilde{u}_{\theta_0}(x)]$.

**Corollary 5.2** When the true density $g$ belongs to the model family $\{f_\theta : \theta \in \Theta\}$, i.e., $g = f_\theta$ for some $\theta \in \Theta$, then the Influence Function for the Minimum $S^\ast$-divergence Estimator functional $T^\ast_{(\alpha, \lambda)}$ at the distribution $G = F_\theta$ becomes

$$IF(y; F_\theta, T^\ast_{(\alpha, \lambda)}) = [J^\ast(\theta)]^{-1} \left\{ u^\ast_\theta(y) - E_\theta[u^\ast_\theta(X)] \right\}, \quad (25)$$

where

$$J^\ast = J^\ast_{(\alpha, \lambda)}(f_\theta) = E_\theta[u^\ast_\theta(X)^2], \quad (26)$$
with
\[ u_\theta^{2\alpha}\ast(y) = \int \tilde{u}_\theta(x)\tilde{u}_\theta(x)^T \{ f_\theta^*(x) \}^\alpha W(x, y, h) dx. \] (27)

Note that the Influence Function at the model distribution turns out to be independent of the parameter \( \lambda \), just as it was for the minimum \( S \)-divergence estimator (see Equation 19). □

Remark 5.1 It is interesting to note that, the matrix \( J^\ast \) defined in 26 can also be written as
\[ J^\ast = E_\theta[u_\theta^{2\alpha}\ast(X)] = E_\theta[-\nabla u_\theta^{\alpha}\ast(X)]. \] (28)

To see this, note that
\[ \nabla u_\theta^{\alpha}\ast(X) = \int \nabla \tilde{u}_\theta(x)\{ f_\theta^*(x) \}^\alpha W(x, X, h) dx + \alpha u_\theta^{2\alpha}\ast(X), \]

and
\[ E_\theta \left[ \int \nabla \tilde{u}_\theta(x)\{ f_\theta^*(x) \}^\alpha W(x, X, h) dx \right] = \int \left[ \int \nabla \tilde{u}_\theta(x)\{ f_\theta^*(x) \}^\alpha W(x, y, h) dx \right] f_\theta(y) dy \]
\[ = \int \nabla \tilde{u}_\theta(x)\{ f_\theta^*(x) \}^\alpha \left[ \int f_\theta(y) W(x, y, h) dy \right] dx \]
\[ = \int \nabla \tilde{u}_\theta(x)\{ f_\theta^*(x) \}^{1+\alpha} dx \]
\[ = -(1 + \alpha) \int \tilde{u}_\theta(x)\tilde{u}_\theta(x)^T \{ f_\theta^*(x) \}^{1+\alpha} dx, \] (29)
where the last step follows using integration by parts. A similar argument shows that
\[ J^\ast = E_\theta[u_\theta^{2\alpha}\ast(X)] = \int \tilde{u}_\theta(x)\tilde{u}_\theta(x)^T \{ f_\theta^*(x) \}^{1+\alpha} dx. \] (30)

Combining all these, we get
\[ E_\theta [\nabla u_\theta^{\alpha}\ast(X)] = E_\theta \left[ \int \nabla \tilde{u}_\theta(x)\{ f_\theta^*(x) \}^\alpha W(x, X, h) dx \right] + \alpha E_\theta \left[ u_\theta^{2\alpha}\ast(X) \right], \]
or,
\[ E_\theta [\nabla u_\theta^{\alpha}\ast(X)] = -(1 + \alpha) J^\ast + \alpha J^\ast = -J^\ast. \]
This proves (28). □

6 Asymptotic Distribution of the Minimum \( S^\ast \)-Divergence Estimator

Now, we will derive the asymptotic properties of the minimum \( S^\ast \)-divergence estimator and prove its asymptotic equivalence with the minimum \( S \)-divergence estimators under some assumptions on the kernel used. Consider the parametric set-up of the continuous model \( \{ f_\theta : \theta \in \Theta \} \) with corresponding smoothed model \( f_\theta^* \) as defined in equation (12). We have \( n \) independent and identically
distributed observations \(X_1, \ldots, X_n\) from the true density \(g\) and the corresponding smoothed density \(g^\ast\) is as defined in equation (20). Along with all the notations of the previous sections, we will further assume some more conditions to give a rigorous proof of the consistency and asymptotic normality of the minimum \(S^\ast\)-divergence estimator. The assumptions will be based on the following general definitions about the properties of the model densities.

**Definition 6.1** The parametric model \(\mathcal{F}\) is said to be identifiable, if for any \(\theta_1, \theta_2 \in \Theta\), \(\theta_1 \neq \theta_2\) implies \(f_{\theta_1}(x) \neq f_{\theta_2}(x)\) on a set of positive dominating measure, where \(f_{\theta}\) represents the density function of \(F_{\theta}\).

**Definition 6.2** Let the parametric densities \(\{f_\theta : \theta \in \Theta\}\) have common support \(K^\ast\), independent of \(\theta\). Then the true density \(g\) is said to be compatible with the family \(\{f_\theta\}\) if \(K^\ast\) is also the support of \(g\).

**Definition 6.3** The kernel integrated family of distributions \(f_{\theta}^\ast(x)\) defined in equation (12) is called smooth if the conditions of Lehmann (1983, p. 409, p. 429) are satisfied with \(f_{\theta}^\ast(x)\) is place of \(f_{\theta}(x)\).

Then the assumptions required for proving the asymptotic results are as follows:

(SB1) The model family \(\mathcal{F}\) is identifiable in the sense of Definition 6.1.

(SB2) The probability density functions \(f_\theta\) of the model distribution have common support so that the set \(\mathcal{X} = \{x : f_\theta(x) > 0\}\) is independent of \(\theta\). Also the true distribution \(g\) is compatible with the model family in the sense of Definition 6.2.

(SB3) The kernel integrated family of distributions \(f_{\theta}^\ast(x)\) is smooth in the sense of Definition 6.3.

(SB4) The matrix \(J^\ast(\theta^\ast)\) is positive definite.

(SB5) The quantities \(\int (g_{\theta}^\ast)^{1/2}(x)(f_{\theta}^\ast(x))^\alpha|u_{j\theta}(x)|dx\), 
\(\int (g_{\theta}^\ast)^{1/2}(x)(f_{\theta}^\ast(x))^\alpha |u_{j\theta}(x)||u_{k\theta}(x)|dx\) and \(\int (g_{\theta}^\ast)^{1/2}(x)(f_{\theta}^\ast(x))^\alpha |u_{j\theta}(x)||u_{k\theta}(x)|dx\) are bounded for all \(j,k,l\) and for all \(\theta \in \omega\), an open neighborhood of the best fitting parameter \(\theta^\ast\).

(SB6) For almost all \(x\), there exists functions \(M_{jkl}(x), M_{jkl}(x), M_{jkl}(x)\) that dominate, in absolute value,
\[ [(f_{\theta}^\ast(x))^\alpha u_{j\theta}(x)] \quad [(f_{\theta}^\ast(x))^\alpha u_{j\theta}(x) u_{k\theta}(x)] \quad \text{and} \quad [(f_{\theta}^\ast(x))^\alpha u_{j\theta}(x) u_{k\theta}(x) u_{l\theta}(x)] \]
respectively for all \(j,k,l\) and are uniformly bounded in expectation with respect to \(g^\ast\) and \(f_{\theta}^\ast\) for all \(\theta \in \omega\).

(SB7) The function \(\left(\frac{g^\ast(x)}{f_{\theta}^\ast(x)}\right)^{A-1}\) is uniformly bounded (by, say, \(C\)) for all \(\theta \in \omega\).

To prove the desired asymptotic results for the minimum \(S^\ast\)-divergence estimator, we will, from now on, assume that the conditions [SB1] [SB7] hold. Also, as in the case of asymptotic
distribution of the MSDE under discrete models, here also we will first prove some important Lemmas. Let us define $\eta_n(x) = \sqrt{n}(\sqrt{\delta_n^*} - \sqrt{\delta_g^*})^2$ with

$$\delta_g^* = \frac{g^*(x)}{f^*_g(x)} - 1.$$

**Lemma 6.1** Provided it exists, $\text{Var}_g(g_n^*(x)) = \frac{\nu(x)}{n}$, where $\nu(x)$ is defined as

$$\nu(x) = \int W^2(x, y, h)g(y)dy - [g^*(x)]^2.$$  \hfill (31)

For the proof of this Lemma see Basu and Lindsay (1994). Next we need to assume that the boundedness of the kernel function $W(x, y, h)$. So, from now on, let

$$W(x, y, h) \leq M(h) < \infty,$$

where $M(h)$ depends on $h$, but not on $x$ or $y$. Then, we have

$$\nu(x) \leq \int W^2(x, y, h)g(y)dy \leq M(h) \int W(x, y, h)g(y)dy = M(h)g^*(x).$$  \hfill (32)

The next three Lemmas follow from Basu and Lindsay (1994) and are stated here without proof.

**Lemma 6.2** Under the above mentioned set-up and notations,

$$n^{1/4} \left[ (g_n^*(x))^{1/2} - g^*(x)^{1/2} \right] \to 0,$$

with probability 1, provided $\nu(x) < \infty$. \hfill $\square$

**Lemma 6.3** For any $k \in [0, 2]$, we have

1. $E[\eta_n^*(x)^k] \leq n^{\frac{k}{2}} E[|\delta_n^*(x) - \delta_g^*(x)|^k] \leq \left[ \frac{\nu(x)}{(f^*_g(x))^2} \right]^k$.

2. $E[|\delta_n^*(x) - \delta_g^*(x)|] \leq \frac{\sqrt{\nu(x)}}{f^*_g(x)}.$

\hfill $\square$

**Lemma 6.4** $E[(\eta_n^*(x))^k] \to 0$, as $n \to \infty$, for $k \in [0, 2)$.

\hfill $\square$
Next, let us define,
\[
    a_n^*(x) = K(\delta_n^*(x)) - K(\delta_g^*(x)),
\]
\[
b_n^*(x) = (\delta_n^*(x) - \delta_g^*(x))K'(\delta_g^*(x))
\]
and
\[
    \tau_n^*(x) = \sqrt{n}|a_n^*(x) - b_n^*(x)|.
\]

We will need the limiting distributions of
\[
    S_{1n}^* = \sqrt{n} \sum_x a_n^*(x)(f_\theta^*(x))^{1+\alpha} \tilde{u}_\theta(x), \quad \text{and} \quad
    S_{2n}^* = \sqrt{n} \sum_x b_n^*(x)(f_\theta^*(x))^{1+\alpha} \tilde{u}_\theta(x).
\]

**Lemma 6.5** Assume condition \(\text{(SB5)}\) Then
\[
    E|S_{1n}^* - S_{2n}^*| \to 0, \quad \text{as} \quad n \to \infty,
\]
and hence \(S_{1n}^* - S_{2n}^* \xrightarrow{P} 0, \quad \text{as} \quad n \to \infty.\)

**Proof:** By Lemma 25 of Lindsay (1994), there exists some positive constant \(\beta\) such that
\[
    \tau_n^*(x) \leq \beta \sqrt{n}(\sqrt{\delta_n^*} - \sqrt{\delta_g^*})^2 = \beta \eta_n^*(x).
\]

Also, by Lemma 6.3 and equation (32),
\[
    E[\tau_n^*(x)] \leq \beta \frac{M^{1/2}(h)(g^*(x))^{1/2}}{f_\theta^*(x)}.
\]

By Lemma 6.4,
\[
    E[\tau_n^*(x)] = \beta E[\eta_n^*(x)] \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus we get,
\[
    E|S_{1n}^* - S_{2n}^*| \leq \sum_x E[\tau_n^*(x)(f_\theta^*(x))^{1+\alpha} |\tilde{u}_\theta(x)|]
\]
\[
    \leq \beta M^{1/2}(h) \sum_x (g^*(x))^{1/2}(f_\theta^*(x))^{\alpha} |\tilde{u}_\theta(x)|
\]
\[
    < \infty, \quad \text{(by assumption SB5)}.
\]

So, by the Dominated Convergence Theorem (DCT),
\[
    E|S_{1n}^* - S_{2n}^*| \to 0 \quad \text{as} \quad n \to \infty.
\]
Hence, by Markov inequality,
\[
    S_{1n}^* - S_{2n}^* \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty.
\]

**Lemma 6.6** Suppose,
\[
    V_g^* = V_{(a,\lambda)}^* (g) = Var \left[ \int W(x, X, h)K'(\delta_g^*(x))(f_\theta^*(x))^{\alpha} \tilde{u}_\theta(x)dx \right]
\]

is finite. Then under the true distribution \(g\),
\[
    S_{1n}^* \xrightarrow{P} N(0, V_g^*)
\]

□
Further, if we let $h$ be the bandwidth, the asymptotic variance matrix is given by

$$V_h = \frac{1}{n} \sum_{i=1}^{n} \left[ W(x, X_i, h) - E\{ W(x, X_i, h) \} \right] K'(\delta_\theta(x))(f_\theta^*(x))^{\alpha}\tilde{u}_\theta(x)dx,$$

This completes the proof of the Lemma.

Now, using the above lemmas, we can prove the main theorem of this section regarding the consistency and asymptotic normality of the Minimum $S^*$-divergence Estimator.

**Theorem 6.7** Under Assumptions [(SB1),(SB7)], there exists a consistent sequence $\theta_n^*$ of roots to the Minimum $S^*$-divergence estimating equation (14).

Also, the asymptotic distribution of $\sqrt{n}(\theta_n^* - \theta)$ is $p$-dimensional normal with mean 0 and variance $[J_g^*]^{-1}V_g^*[J_g^*]^{-1}$, where $J_g^*$ and $V_g^*$ are as defined in equation (24) and (33).

**Proof:** The proof is similar to that of the minimum $S$-divergence estimators under discrete models (Ghosh, 2014) with slight modifications based on Lemmas 6.1 to 6.6.

**Corollary 6.8** When the true density $g$ belongs to the model family $\{f_\theta : \theta \in \Theta\}$, i.e., $g = f_\theta$, the asymptotic distribution of $\sqrt{n}(\theta_n^* - \theta)$ is Normal with mean zero and variance-covariance matrix $[J_g^*]^{-1}V_g^*[J_g^*]^{-1}$, where $V_g^* = V_g^*(f_\theta) = V_\theta[u_\theta^*(X)]$ and $J_g^*$ is defined in equation (26). Note that this asymptotic distribution turns out to be independent of the parameter $\lambda$. (This independence is in-line with that of the minimum $S$-divergence estimator for discrete models derived in Ghosh, 2014.)

**Remark 6.1** It is to be noted that the result of Theorem 6.7 is a fixed bandwidth $(h)$ results. Further, if we let $h \to 0$, then we can easily show that under appropriate regularity conditions the asymptotic variance matrix $[J_g^*]^{-1}V_g^*[J_g^*]^{-1}$ at the model density converges element-wise to matrix

$$\left( E_\theta[u_\theta(X)u_\theta(X)^T f_\theta^*(X)] \right)^{-1} V_\theta[u_\theta(X)f_\theta^*(X)] \left( E_\theta[u_\theta(X)u_\theta(X)^T f_\theta^*(X)] \right)^{-1}.$$

Interestingly this matrix is the same as the asymptotic variance of the minimum $S$-divergence estimators at the discrete model, as obtained in Ghosh (2014).

In the same spirit, we also expect that under the additional condition $h \to 0$ along with the assumptions of Theorem 6.7, the asymptotic distribution of $\sqrt{n}(\theta_n^* - \theta)$ will be $p$-variate normal with mean vector 0 and variance given by expression (34), which corresponds to the discrete case, although we do not have a fully rigorous proof at this point.
As an illustration of this expectation, we compute the asymptotic variance $[J^*]^{-1}V^*[J^*]^{-1}$ of the minimum $S^*$-divergence estimators at the model for $N(\theta, \sigma^2)$ model distribution with several values of the fixed bandwidth $h$ converging to zero and compare these values with the asymptotic variance given in expression (34). We consider the Gaussian kernel with $W(x,y,h)$ being the $N(y,h^2)$ density at $x$ and different values of $\alpha \geq 0$. Then, a simple calculation shows that the smoothed density $f_\theta^*(x)$ is the normal $N(\theta, \sigma^2 + h^2)$ density and the corresponding asymptotic variance at the model given in Corollary 6.8 simplifies to

$$[J^*]^{-1}V^*[J^*]^{-1} = \left[\frac{(1 + \alpha)^2(\sigma^2 + h)^2}{((1 + \alpha)h^2 + \sigma^2)((1 + \alpha)h^2 + (1 + 2\alpha)\sigma^2)}\right]^{3/2} \sigma^2 = \zeta_{\alpha,h} \sigma^2.$$  

Interestingly, in this case, we have $\zeta_{0,h} = 1$ so that the value of $[J^*]^{-1}V^*[J^*]^{-1}$ at $\alpha = 0$ further simplifies to $\sigma^2$ which is independent of $h$ and also the same as the corresponding value of expression (34); we will examine this special property towards the end of this paper in Section 9. For all other $\alpha > 0$, the value of the expression (34) equals $\zeta_\alpha \sigma^2$ with $\zeta_\alpha = (1 + \alpha)^3(1 + 2\alpha)^{-3/2}$ and clearly

$$\zeta_{\alpha,h} \to \zeta_\alpha, \quad \text{as} \quad h \to 0.$$  

The expression $\zeta_\alpha \sigma^2$ is the asymptotic variance of the MDPDE of $\mu$. This gives some substance to our description in the previous paragraphs relating to the asymptotic variance of the MSDE* under the condition $h \to 0$.

7 Simulation Studies: Normal Model

We will now explore the performance of the minimum $S^*$-divergence estimator through a detailed simulation study. For simplicity, we will consider the model density $f_\theta$ to be the normal density with unknown mean $\mu$ and unknown variance $\sigma^2$ so that the parameter of interest is $\theta = (\mu, \sigma)$ and the parameter space is $\Theta = \mathbb{R} \times [0, \infty)$. We will simulate sample data of size $n = 50$ from a normal distribution with mean 0 and variance 3 and compute the minimum $S^*$-divergence estimators based on the sample drawn. We compute the empirical bias and MSE for the minimum $S^*$-divergence estimator over 1000 replications. For computation of the minimum $S^*$-divergence estimator under the Basu-Lindsay approach, we will use the Gaussian kernel and, following the normal reference rule (Scott, 2001), use the bandwidth

$$h_n = 1.06\sigma n^{-1/5},$$

with $\sigma$ being the standard deviation. In this expression of the bandwidth, we will replace $\sigma$ by its robust estimate

$$\hat{\sigma} = \frac{\text{Median}_i |X_i - \text{Median}_j X_j|}{0.6745}.$$  

Also, we need to form the smoothed model density as described in previous sections. Here our model is the $N(\mu, \sigma^2)$ density and we choose the Gaussian kernel with $W(x,y,h)$ being the $N(y,h^2)$ density at $x$. We have already observed that the smoothed density $f_\theta^*(x)$ is the normal $N(\mu, \sigma^2 + h^2)$ density in this case.

First we will consider the case of pure data without any contamination. So, we generate samples from pure $N(0,3)$ density; the computed MSE of the mean and variance parameter are reported in Tables 1 and 2. Clearly, as $\alpha$ increases from 0 to 1, the MSE increases slightly and so there is a
Table 1: MSEs of the location parameter $\mu$ without any contamination

| $\lambda$ | $\alpha = 0$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.8$ | $\alpha = 1$ |
|-----------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| -1        | 0.22         | 0.17         | 0.17         | 0.18         | 0.19         | 0.21         | 0.22         |
| -0.7      | 0.16         | 0.17         | 0.17         | 0.18         | 0.19         | 0.21         | 0.22         |
| -0.5      | 0.16         | 0.17         | 0.17         | 0.18         | 0.19         | 0.21         | 0.22         |
| -0.3      | 0.16         | 0.17         | 0.17         | 0.18         | 0.19         | 0.21         | 0.22         |
| 0         | 0.16         | 0.16         | 0.17         | 0.18         | 0.19         | 0.21         | 0.22         |
| 0.5       | 0.16         | 0.16         | 0.17         | 0.18         | 0.19         | 0.21         | 0.22         |
| 1         | 0.17         | 0.17         | 0.17         | 0.18         | 0.18         | 0.21         | 0.22         |
| 1.5       | 0.17         | 0.17         | 0.17         | 0.18         | 0.18         | 0.21         | 0.22         |
| 2         | 0.17         | 0.17         | 0.17         | 0.18         | 0.18         | 0.21         | 0.22         |

Table 2: MSEs of the scale parameter $\sigma$ without any contamination

| $\lambda$ | $\alpha = 0$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.8$ | $\alpha = 1$ |
|-----------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| -1        | 0.16         | 0.11         | 0.11         | 0.11         | 0.12         | 0.13         | 0.14         |
| -0.7      | 0.11         | 0.10         | 0.11         | 0.11         | 0.12         | 0.13         | 0.14         |
| -0.5      | 0.10         | 0.10         | 0.10         | 0.11         | 0.12         | 0.13         | 0.14         |
| -0.3      | 0.10         | 0.10         | 0.10         | 0.11         | 0.11         | 0.13         | 0.14         |
| 0         | 0.09         | 0.09         | 0.10         | 0.11         | 0.11         | 0.13         | 0.14         |
| 0.5       | 0.10         | 0.09         | 0.10         | 0.10         | 0.11         | 0.13         | 0.14         |
| 1         | 0.10         | 0.10         | 0.10         | 0.10         | 0.11         | 0.13         | 0.14         |
| 1.5       | 0.12         | 0.11         | 0.10         | 0.10         | 0.11         | 0.13         | 0.14         |
| 2         | 0.13         | 0.12         | 0.11         | 0.10         | 0.10         | 0.13         | 0.14         |

loss in efficiency of the minimum $S^*$-divergence estimators with increasing $\alpha$. However, this loss in efficiency is quite small and does not limit its application in pure data.

The main reason of using the minimum $S^*$-divergence estimator is that it gives the option of several robust estimators at different but high levels of efficiency. From the influence function analysis of Section 5, it is suggested that the robustness of these estimators should increase with increasing $\alpha$. To study the robustness under several types of contaminations, we next consider data from the $N(0, 3)$ distribution but with 5%, 10% or 20% contamination from different distributions as follows:

(i) mean shifted density – $N(15, 3)$

(ii) larger variance – $N(0, 10)$

(iii) smaller variance – $N(0, 1)$

(iv) symmetric contamination with heavy tail – $t$-density with 1 degrees of freedom.

(v) Non-symmetric contamination with heavy tail – $\chi^2$-density with 10 degrees of freedom.

For brevity, we will only present the MSE under some interesting cases of contaminations in Figures 1 to 8. If we consider contamination by the mean shifted density (case (i)), then the variance of the overall distribution also gets affected and the MLE of both $\mu$ and $\sigma$ gets affected even under
a small contamination of 5%. The minimum $S^*$-divergence estimator with positive $\lambda$ and small $\alpha$ perform worse than the MLE under such contaminations; but that corresponding estimators at $\lambda < 0$ perform quite robustly ignoring the effect of contamination. Even under the heavy 20% contamination in case (i), the members of the minimum $S^*$-divergence family with negative $\lambda$ and
Figure 3: Bias and MSE of the estimator of location parameter $\mu$ with different contamination proportion $\epsilon$ for case (ii)

Figure 4: Bias and MSE of the estimator of scale parameter $\sigma$ with different contamination proportion $\epsilon$ for case (ii)

moderately large $\alpha$ successfully ignore the contamination to give the robust estimates of the location parameter $\mu$.

If we consider the contamination by a density with larger variance (as in case (ii)), it should ideally affect the scale parameter only – not the location parameter. The minimum $S^*$-divergence
estimators with negative $\lambda$ and larger $\alpha$ give us quite robust estimates of the scale parameter $\sigma$ even with 20% contamination in this case; also the MSE of the location parameter $\mu$ remains almost equal to that of the MLE since they are not significantly affected by the contamination in variance. But the minimum $S^*$-divergence estimators of $\mu$ with $\lambda > 0$ and small $\alpha$ close to zero are seen to
Figure 7: Bias and MSE of the estimator of location parameter $\mu$ with different contamination proportion $\epsilon$ for case (v)

Figure 8: Bias and MSE of the estimator of scale parameter $\sigma$ with different contamination proportion $\epsilon$ for case (v)

be affected by such contamination also implying their inferiority compared to the MLE. Under the contamination with a symmetric heavy-tail density (case (iv)), the performance of the minimum $S^*$-divergence estimators is exactly similar to the case of large variance contamination (case (ii)); and their performance under the non-symmetric contamination (case (v)) is again similar to that of
the mean-shifted contamination (case (i)). In case (iii) where the true distribution is contaminated by a distribution with smaller variance but the same mean, we did not get a consistent pattern to make specific recommendations about which choices of the tuning parameter would lead to better estimators in terms of their stability under contamination. As such we have not reported the findings for this case.

Therefore, the overall performance of the minimum $S^*$-divergence estimators can be characterized into two groups; just as in the case of minimum $S$-divergence estimators. One group consists of the estimators corresponding to $\lambda < 0$ and moderate $\alpha$ close to 0.5 generating highly robust estimator with comparable efficiencies with respect to the maximum likelihood estimators. The second group consists of those with $\lambda > 0$ and small $\alpha$ close to zero which perform even worse than the maximum likelihood estimator in terms of the robustness under any type of contamination (except case (iii)). The robustness of the minimum $S^*$-divergence estimators are, thus, dependent on both the parameters $\lambda$ and $\alpha$.

8 Real Data Examples

8.1 Short’s Data

In this example we consider Short’s data for the determination of the parallax of the sun, the angle subtended by the earth’s radius, as if viewed and measured from the surface of the sun. From this angle and available knowledge of the physical dimensions of the earth, one can easily calculate the mean distance of the earth to the sun. The raw observations are presented in Data Set 2 of Stigler (1977).

The data were previously analyzed by many authors including Basu et al. (2011). Observing the pattern of the data, one might model it using a normal density with some mean $\mu$ and variance $\sigma^2$. It was observed that the data contain a large outlier at 5.76 which severely affects the maximum likelihood estimator. The maximum likelihood estimates of location ($\mu$) and scale ($\sigma$) are 8.378 and

![Figure 9: Normal density fits to Short’s data (solid line corresponds to $\alpha = 0.5$ and $\lambda = -0.5$, dotted line corresponds to kernel density, dashed line corresponds to MLE)](image-url)
0.846, respectively. But, removing the large outlier at 5.76, the maximum likelihood estimates of the location and scale become 8.541 and 0.552. So, there is a clear need of using a suitable robust technique to estimate the parameters based on this data set.

Table 3: Estimates of the location parameter $\mu$ for Short’s data.

| $\lambda$ | $\alpha = 0$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.8$ | $\alpha = 1$ |
|-----------|---------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $-1$      | 8.38          | 8.39           | 8.39           | 8.40           | 8.40           | 8.40           | 8.41           |
| $-0.7$    | 8.40          | 8.39           | 8.39           | 8.40           | 8.40           | 8.40           | 8.41           |
| $-0.5$    | 8.45          | 8.42           | 8.40           | 8.40           | 8.40           | 8.40           | 8.41           |
| $-0.3$    | 8.49          | 8.47           | 8.42           | 8.41           | 8.40           | 8.40           | 8.41           |
| $0$       | 8.38          | 8.43           | 8.46           | 8.43           | 8.41           | 8.40           | 8.41           |
| $0.5$     | 8.26          | 8.30           | 8.39           | 8.44           | 8.45           | 8.41           | 8.41           |
| $1$       | 8.21          | 8.24           | 8.32           | 8.36           | 8.42           | 8.41           | 8.41           |
| $1.5$     | 8.17          | 8.20           | 8.26           | 8.31           | 8.36           | 8.41           | 8.41           |
| $2$       | 8.14          | 8.16           | 8.22           | 8.26           | 8.31           | 8.42           | 8.41           |

Table 4: Estimates of the scale parameter $\sigma$ for Short’s data.

| $\lambda$ | $\alpha = 0$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.8$ | $\alpha = 1$ |
|-----------|---------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $-1$      | 0.32          | 0.33           | 0.33           | 0.33           | 0.33           | 0.34           | 0.35           |
| $-0.7$    | 0.36          | 0.35           | 0.34           | 0.34           | 0.34           | 0.35           | 0.35           |
| $-0.5$    | 0.45          | 0.41           | 0.36           | 0.35           | 0.35           | 0.35           | 0.35           |
| $-0.3$    | 0.57          | 0.51           | 0.41           | 0.38           | 0.36           | 0.35           | 0.35           |
| $0$       | 0.85          | 0.76           | 0.51           | 0.44           | 0.39           | 0.35           | 0.35           |
| $0.5$     | 0.97          | 0.93           | 0.79           | 0.65           | 0.49           | 0.36           | 0.35           |
| $1$       | 1.01          | 0.98           | 0.89           | 0.81           | 0.69           | 0.37           | 0.35           |
| $1.5$     | 1.03          | 1.01           | 0.94           | 0.88           | 0.80           | 0.38           | 0.35           |
| $2$       | 1.04          | 1.02           | 0.97           | 0.92           | 0.86           | 0.39           | 0.35           |

We will apply the proposed minimum $S^*$-divergence estimators here with a normal kernel $W(x, y, h)$ as discussed in the previous sections. The bandwidth will be chosen from the normal reference rule as in the case of our simulation studies, described in Section 7. In Tables 3 and 4 we provide the minimum $S^*$-divergence estimates of the location and scale parameters under several values of $\alpha$ and $\lambda$. We can clearly see that many of the $S^*$-divergence estimators are not affected by the outliers. This includes the $S^*$-divergence with negative $\lambda$ with small $\alpha$ and the same with positive $\lambda$ but large positive $\alpha$. However the $S^*$-divergence estimators with $\lambda > 0$ and small positive $\alpha$ are very sensitive with respect to the outlier like the MLE (which corresponds to $\alpha = 0$ and $\lambda = 0$). Thus the performance of the minimum $S^*$-divergence estimators for the continuous model is also exactly similar to that of the minimum $S$-divergence estimators under the discrete model. There is a clear triangular region, in the lower left hand corner of Table 4, where the estimators are adversely affected by the outlier, in some cases severely. This is consistent with our findings of Section 7.
8.2 Newcomb’s Data

This example involves Newcomb’s light speed data (Stigler, 1977, Table 5). The data set shows a nice unimodal structure, and the normal model would have provided an excellent fit to the data if the two large outliers were not there. The data set was previously analyzed by many researchers who had shown the need of suitable robust estimation methods for fitting a normal model because the usual maximum likelihood estimator is seen to be highly affected by the presence of the two outliers in the data set. So, we will compute the robust minimum $S^*$-divergence estimators for the present data, with the kernel and the corresponding bandwidth being the same as in the previous example. The estimates obtained for different members of the $S^*$-divergence family are presented in Tables 5 and 6.

Table 5: Estimates of the location parameter $\mu$ for Newcomb’s data.

| $\lambda$ | $\alpha = 0$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.8$ | $\alpha = 1$ |
|-----------|---------------|----------------|---------------|---------------|---------------|---------------|---------------|
| -1        | 27.72         | 27.69          | 27.63         | 27.60         | 27.56         | 27.47         | 27.42         |
| -0.7      | 27.73         | 27.70          | 27.63         | 27.60         | 27.57         | 27.47         | 27.42         |
| -0.5      | 27.73         | 27.70          | 27.64         | 27.60         | 27.57         | 27.47         | 27.42         |
| -0.3      | 27.72         | 27.70          | 27.64         | 27.61         | 27.57         | 27.47         | 27.42         |
| 0         | 26.21         | 27.58          | 27.64         | 27.61         | 27.57         | 27.47         | 27.42         |
| 0.5       | 22.69         | 23.80          | 26.47         | 27.57         | 27.48         | 27.42         | 27.42         |
| 1         | 21.39         | 22.23          | 24.25         | 25.52         | 27.48         | 27.42         | 27.42         |
| 1.5       | 20.62         | 21.29          | 22.97         | 24.06         | 25.41         | 27.48         | 27.42         |
| 2         | 20.12         | 20.66          | 22.09         | 23.05         | 24.25         | 27.48         | 27.42         |

Table 6: Estimates of the scale parameter $\sigma$ for Newcomb’s data.

| $\lambda$ | $\alpha = 0$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.8$ | $\alpha = 1$ |
|-----------|---------------|----------------|---------------|---------------|---------------|---------------|---------------|
| -1        | 4.94          | 4.97           | 4.98          | 4.97          | 4.95          | 4.90          | 4.87          |
| -0.7      | 4.99          | 5.00           | 4.99          | 4.98          | 4.96          | 4.90          | 4.87          |
| -0.5      | 5.02          | 5.02           | 5.00          | 4.98          | 4.96          | 4.90          | 4.87          |
| -0.3      | 5.10          | 5.06           | 5.01          | 4.99          | 4.97          | 4.90          | 4.87          |
| 0         | 10.66         | 5.48           | 5.05          | 5.01          | 4.98          | 4.90          | 4.87          |
| 0.5       | 18.63         | 17.10          | 10.76         | 5.19          | 5.01          | 4.91          | 4.87          |
| 1         | 20.48         | 19.53          | 16.49         | 13.69         | 7.02          | 4.91          | 4.87          |
| 1.5       | 21.36         | 20.65          | 18.51         | 16.75         | 13.89         | 4.92          | 4.87          |
| 2         | 21.88         | 21.30          | 19.61         | 18.29         | 16.29         | 4.92          | 4.87          |

Many of the minimum $S^*$-divergence estimates automatically discount these large observations, unlike the maximum likelihood estimate (corresponding to $\alpha = 0$ and $\lambda = 0$ in the $S^*$-divergence measure). Once again the minimum $S^*$-divergence estimates for all $\alpha$ with $\lambda < 0$ and for large $\alpha$ close to one with $\lambda \geq 0$ are remarkably close to each other and robust with respect to the large outliers. Once again the region of instability is in the lower left hand corner of the tables.
9 A Suggestion for the Kernels: $\alpha$-Transparent Kernel

In this paper, we have developed the properties of the minimum $S^*$-divergence estimators under the continuous models following the approach of Basu and Lindsay (1994). The work of Basu and Lindsay (1994) was in the context of minimum disparity estimators, where they have also derived some conditions on the kernel density so that the minimum disparity estimators obtained by the smoothed model approach under continuous models have similar asymptotic distribution as that of the original minimum disparity estimators without any kernel smoothing under discrete models and hence they are fully efficient. They have termed those special kernels as the “Transparent kernel”, although there are only few examples of them. In this section, we will derive similar conditions on the kernel so that the asymptotic distribution of the proposed minimum $S^*$-divergence estimators under continuous models (as derived in previous sections) have the same form as that of the minimum $S$-divergence estimators under discrete models (as derived in Ghosh, 2014).

Let us again start with the special case corresponding to $\lambda = 0$ which gives the well-known density power divergence; in this case we can compare the results of the corresponding smoothed model based estimators with the existing results of the minimum density power divergence estimators. In particular, comparing the estimating equation (19) with the MDPDE estimating equation (16), we can find the required conditions on the kernel function under which the MDPDE* (the MSDE* for $\lambda = 0$) coincides with the MDPDE. The result is presented in the following Lemma.

Lemma 9.1 Suppose the kernel function $W(x, y, h)$ used in the smoothing densities is such that

$$u^*_\theta(y) = Mf^*_\theta(y)u_\theta(y) + L$$

for a $p$-vector $L$ depending only on $\alpha$ and $h$, and a $p \times p$ nonsingular matrix $M$ depending on $\theta$, $\alpha$ and $h$ where for each components $\theta_j$ of theta ($j = 1, \ldots, p$), we have

either “\( \int u_\theta f_\theta^{\alpha+\lambda} = 0 \)”, or “the $j$-th column of $M$ is independent of $\theta$”.

Then the estimating equation for the Minimum DPD* Estimator is the same as that for the Minimum DPD Estimator and hence the two estimators are indeed equal.

□

Further, just as in the case of disparities, the condition (35) imposed on the kernel function also ensure that the asymptotic variance of the MSDE*s will be equal to that of the MSDEs under discrete model, beyond this special case of $\lambda = 0$ also. We will justify this condition further by comparing the influence functions and asymptotic distributions of the two types of estimators in the following two corollaries.

Corollary 9.2 Suppose the true density $g$ belongs to the model family \{\( f_\theta : \theta \in \Theta \), i.e., $g = f_\theta$ for some $\theta \in \Theta$. Also assume that the kernel function $W(x, y, h)$ used in the smoothing satisfies the condition (35) for some nonsingular matrix $M$ and vector $L$ as in Lemma 9.1 along with Condition (36). Then the Influence Function for the minimum $S^*$-divergence estimator functional $T^*_{(\alpha, \lambda)}$ at the distribution $G = F_\theta$ becomes the same as that of the minimum $S$-divergence estimator at $G = F_\theta$, given in Equation (9).
**Proof:** In view of Corollary 5.2 and Equation (28), we only need to show that, Condition (35) along with (36) implies

\[ E[−∇u^*_θ(X)] = M \int u_θ(x)u_θ(x)^T f_θ^{1+α}(x)dx. \]  

(37)

Then, it follows that

\[ IF(y; F_θ, T^*_θ) = [J^*(θ)]^{-1} \{u^*_θ(y) - E[\theta^*_θ(X)]\} \]

\[ = [M \int u_θ(x)u_θ(x)^T f_θ^{1+α}(x)dx]^{-1} \]

\[ \times \{Mu_θ(y)f_θ^α(y) + L - E[Mu_θ(X)f_θ^α(X) + L]\} \]

\[ = [\int u_θ(x)u_θ(x)^T f_θ^{1+α}(x)dx]^{-1} \{u_θ(y)f_θ^α(y) - E[\theta^*_θ(X)f_θ^α(X)]\}, \]

which is the same as given in (9); it proves the corollary.

Now, to prove (37), let us take the derivative with respect to \(θ\) on both sides of the condition (35) to get

\[ ∇u^*_θ(x) = M∇u_θ(x)f_θ^α(x) + αMu_θ(x)u_θ(x)^T f_θ^α(x) \]

\[ + [(∇_1M)u_θ(x) (∇_2M)u_θ(x) \cdots (∇_nM)u_θ(x)] f_θ^α(x), \]  

(38)

where \(∇_j\) represents the derivative with respect to the \(j\)-th component of \(θ\). Taking expectation with respect to \(f_θ\) in both sides of (38), we have

\[ E[−∇u^*_θ(X)] = M \int ∇u_θ(x)f_θ^{1+α}(x)dx + αM \int u_θ(x)u_θ(x)^T f_θ^{1+α}(x)dx, \]  

(39)

since the expectation of the third term in (38) is zero by Condition (36). Also, integrating the first integral in (39) by parts, we get

\[ \int ∇u_θ(x)f_θ^{1+α}(x)dx = -(1 + α) \int u_θ(x)u_θ(x)^T f_θ^{1+α}(x)dx. \]  

(40)

Combining (39) and (40) we get the desired result (37), completing the proof. \(□\)

**Corollary 9.3** Suppose the true density \(g\) belongs to the model family \(\{f_θ : θ ∈ Θ\}\), i.e., \(g = f_θ\) for some \(θ ∈ Θ\). Also assume that the kernel function \(W(x, y, h)\) used in the smoothing satisfies the condition (35) for some nonsingular matrix \(M\) and vector \(L\) as in Lemma 9.1 along with Condition (36). Then the asymptotic distribution for the minimum \(S^\ast\)-divergence estimator \(θ_n^\ast\) is normal with mean zero and variance-covariance matrix as given by the expression (34). \(□\)

Thus we have seen that if the kernel function satisfies Equation (35) then the minimum \(S^\ast\)-divergence estimator and the minimum \(S\)-divergence estimator have the same influence function at the model distribution. Thus, under this assumption, all the first order asymptotic properties of the MSDE\(^\ast\)'s should be same as that of the MSDEs under discrete model. In particular, we have also seen such equivalence in terms of their asymptotic distribution. We will refer to kernel function satisfying Equation (35) as the \(α\)-transparent kernel for the model \(f_θ\); at \(α = 0\) this notion coincides with that of the “transparent kernel” as defined in Basu and Lindsay (1994).
Definition 9.1 Consider the parametric model $F = \{ F_\theta : \theta \in \Theta \subseteq \mathbb{R}^p \}$. Let $u_\alpha^*(x)$ be as defined in Equation (18). Then a kernel function $W(x,y,h)$ will be called a $\alpha$-transparent kernel for the above family of models if it satisfies the Condition (35) for some nonsingular matrix $M$ and vector $L$ as in Lemma 9.1 along with Condition (36).

For the mean parameter of the normal model, the Gaussian kernel provides one example of an $\alpha$-transparent kernel at $\alpha = 0$; in this case the asymptotic variance of the MSDE* becomes independent of the bandwidth $h$ as seen in Remark 6.1. Although the calculations are not provided here, the same is true for the normal variance. In general, however, the $\alpha$-transparent kernel is a theoretical construct and we do not have other examples of $\alpha$-transparent kernels at this point. If one does exist, the problem under consideration will have simple solutions. However, even if one does not exist, we expect that letting $h \to 0$ will allow the asymptotic variance of the MSDE* to stabilize around the expression in equation (34), as discussed in Section 6.

10 Concluding Remarks : Choice of the Tuning parameters

In this work we have developed the theoretical properties of the minimum $S$-divergence estimators under the general framework of the continuous models. The $S$-divergence family provides a large collection of divergences with different properties; it includes several divergences where the corresponding estimators have near optimal efficiency properties and strong robustness properties, and provides a general framework to explore the properties of the minimum divergence estimators based on this vast class of divergence measures through two tuning parameter $\alpha \geq 0$ and $\lambda \in \mathbb{R}$. Ghosh et al. (2013) explored the robustness performances of this general class of minimum divergence estimators through the influence function and breakdown point analysis; Ghosh (2014) proved the asymptotic properties of these estimators under the discrete models and linked the theoretical properties with the empirical findings and robustness presented in Ghosh et al. (2013). Several other applications of this family of $S$-divergence measures are under the lens of many recent researches; see for example, Ghosh, Maji and Basu (2013) and Ghosh, Basu and Pardo (2014).

In this context, it is useful to have a general theoretical results for the minimum $S$-divergence estimators beyond the discrete models, since there are many real life problems that can not be modeled by the discrete distributions. The present paper fill up this gap by providing a general asymptotic theory for the minimum $S$-divergence estimators under continuous model families. To avoid the complications of kernel bandwidth selection, we here considered the Basu-Lindsay (1994) approach of smoothed densities to define minimum $S^*$-divergence estimator from minimum $S$-divergence estimators along with their equivalence under suitable assumptions and discussed its practical implications. All the theoretical results derived have been well-supported by extensive simulation study and two popular real data examples.

Combining all these asymptotic properties and empirical findings presented above, a clear trade-off between efficiency and robustness is observed between different members of the $S$-divergence family. This trade-off solely depends on the tuning parameters $\alpha$ and $\lambda$ defining the divergence measure and is exactly similar to that observed for the minimum $S$-divergence estimators in discrete cases (Ghosh et al., 2013; Ghosh, 2014). In particular, efficiency decreases and robustness increases as $\alpha$ increases. However the first order efficiency and influence function are independent of the other tuning parameter $\lambda$ theoretically, though the numerical illustrations suggest that the robustness of the minimum $S$-divergence estimators or minimum $S^*$-divergence estimators depend clearly on $\lambda$ – the estimators with $\lambda < 0$ and $\lambda > 0$ with larger values of $\alpha$ are robust than the other members of the
family; this is consistent with the higher order influence analysis of Ghosh et al. (2013). Therefore, in this present case of continuous model also we can choose an optimum range of values of the tuning parameters following the logic discussed in Ghosh (2014) for the discrete case. Combining the asymptotic and empirical findings, it was suggested that low values of $\alpha \in [0.1, 0.25]$ coupled with moderately large negative values of $\lambda \in [-0.3, -0.5]$ be considered as appropriate choices of the tuning parameter to obtain highly robust estimators with only an insignificant loss of efficiency. We also stick with these choice of tuning parameters for the continuous models and the minimum $S^*$-divergence estimators considered here as empirical suggestions. However, this issue of an optimum tuning parameter needs further theoretical justification for both discrete and continuous cases that we are hoping to pursue in our future research.

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