First and second order necessary optimality conditions for stochastic distributed systems

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Abstract. An optimal control problem of stochastic system, which that dynamics is described by hyperbolic type stochastic partial differential equations with a multipoint type quality criterion, is considered. A stochastic analog of the Euler equation is obtained and singular controls in the classical sense are investigated.

1. Introduction

It is known that optimal control problems with Goursat–Darboux systems [1-10], etc., have been sufficiently studied by now, and various necessary optimality conditions have been obtained. However, the above works don’t take into account the impact on the control object of random disturbances present in real control systems. For this reason, studies of control problems for Goursat–Darboux stochastic systems, the behavior of which cannot be described with complete certainty, are of particular interest. A number of issues related to the derivation of optimality conditions in control problems described by stochastic Goursat–Darboux systems are studied in [11–15] and others.

In the proposed work, some ideas from [7–9] are used to solve one optimal control problem of Goursat–Darboux stochastic system. First, the necessary first-order optimality conditions were established in the form of a stochastic analog of the Euler equation, then the cases of singular controls in the classical sense were considered, and various type second-order necessary optimality conditions for singular controls were obtained [17, 7].

2. Problem statement

Let \( (\Omega, \mathcal{F}, P) \) be some probability space and on it a non-decreasing flow of \( \sigma \)-algebra \( \{\mathcal{F}_{tx}, (t,x) \in D = T \times X = [t_0, t_1] \times [x_0, x_1]\} \). \( \mathcal{R}(D) \) is a space of measurable in \((t,x,\omega)\) and \( \mathcal{F}_t \) are consistent processes \( z: D \times \Omega \to \mathbb{R}^n \), such that

\[
E \int_{t_0}^{t_1} \int_{x_0}^{x_1} \|z(t,x)\|^2 \, dt \, dx < +\infty,
\]

where \( E \) is sign of expectation.

Consider in domain \( D \) controlled system of stochastic partial differential equation

\[
\frac{\partial^2 z(t,x)}{\partial t \partial x} = f(t,x,z,z_t,z_x,u) + g(t,x,z) \frac{\partial^2 W(t,x)}{\partial t \partial x}, (t, x) \in D,
\]

(1)
with boundary conditions Goursat type

\[ z(t_0, x) = a(x), x \in X, \]

\[ z(t, x_0) = b(t), t \in T, \quad a(x_0) = b(t_0). \]

Here \( f(t, x, p, u) \) is a given \( n \) - dimensional vector function, continuous in set of variables together with partial derivatives with respect to \((p, u)\) up to second order, inclusive where \( p = (z, z_t, z_x) \); \( g(t, x, z) \) is a given \( n \) - dimensional vector function, continuous in the set of variables together with partial derivatives in \( z \) up to second order inclusive; \( a(x), b(t) \) are edge \( n \) -dimensional vector functions defined on \( X \) and \( T \), respectively, satisfy the Lipschitz condition;

\[
\frac{\partial^2 W(t,x)}{\partial t \partial x} - n - \text{dimensional two-parameter independent “white noise” on plane.}
\]

As admissible controls, we choose measurable bounded functions \( u(t,x) \), provided

\[ u(t,x) \in U, (t,x) \in D, \]

where \( U \) is given non-empty, bounded, open set from \( R^r_u(u(t,x) \in L_\infty(D, U)) \).

Assumed that each admissible control \( u(t,x) \) correspond with probability 1 unique solution \( z(t,x) \) of problem (1)–(2) in sense [11, 16].

On the set of such solutions, we define multipoint quality functional

\[ S(u) = E\{\varphi(z(T_1, X_1), z(T_2, X_2), ..., z(T_k, X_k))\}, \tag{4} \]

with the condition that \( \varphi(a_1, a_2, ..., a_k) \) is a given, twice continuously differentiable scalar function, and

\[ (T_i, X_i), i = 1, k, t_0 < T_1 < T_2 < ... < T_k \leq t_1; x_0 < X_1 < X_2 < ... < X_k \leq x_1 \]

are the given points.

Consider the following optimal control problem: the controls \( u(t, x) \) must be chosen so that the quality functional (4) takes the smallest possible value.

Aim of our study is to obtain a stochastic analog of the Euler equation and to study the singular controls in the classical sense for considered problem (1)–(4).

3. The increment formula in terms first and second variations of the quality functional

Let \( u(t, x) \) and \( \bar{u}(t, x) \) be admissible controls, and \( z(t, x) \) and \( \bar{z}(t, x) = z(t, x) + \Delta z(t, x) \) be the corresponding solutions of system (1) – (2).

For convenience of further presentation, we introduce the following notation:

\[ H(t, x, p(t, x), u(t, x), \psi(t, x)) = \psi'(t, x)f\left(t, x, p(t, x), u(t, x)\right), \]
\[ H_p(t, x) = H_p(t, x, p(t, x), u(t, x), \psi(t, x)), \]
\[ H_{pp}[t, x] = H_{pp}(t, x, p(t, x), u(t, x), \psi(t, x)), \]
\[ f_p[t, x] = f_p(t, x, p(t, x), u(t, x)), g_x[t, x] = g_x(t, x, z(t, x)). \]

Here \( \psi(t, x) \in L_\infty(D, R^n) \) is a vector-function of conjugate variables that are a solution of the following two-dimensional Volterra type stochastic integral equation (adjoint system):

\[
\psi(t, x) = -\sum_{i=1}^k a_i(t, x) \frac{\partial \varphi(z(T_1, X_1), z(T_2, X_2), ..., z(T_k, X_k))}{\partial a_i} + \\
\int_t^1 \int_x^x H_z[\tau, s] d\tau d\tau + \int_t^1 \int_x H_z[z(t, x)] d\tau + \int_x^x H_z[z(t, s)] d\tau + \int_t^1 \int_x^x g_x[z(t, s)] d\tau d\tau,
\]

where \( a_i(t, x), i = 1, k \) is characteristic function of domain \([t_0, T_1] \times [x_0, X_1]\).

Using the Taylor formula, taking into account the accepted notation, the increment of functional (4) corresponding to the controls \( u(t, x), \bar{u}(t, x) \) can be reduced to the form
\[ \Delta S(u) = S(\bar{u}) - S(u) = E \left\{ - \int_{t_0}^{t_1} \int_{x_0}^{x_1} H_u'[t,x] \Delta u(t,x) \, dx \, dt + \frac{1}{2} \sum_{i,j=1}^{k} \Delta z'(T_i,X_i) \frac{\partial^2 \varphi(z(T_i,X_i),z(T_j,X_j),...,z(T_k,X_k))}{\partial a_i \partial a_j} \Delta z(T_i,X_i) - \int_{t_0}^{t_1} \int_{x_0}^{x_1} \Delta p'(t,x) H_p[p,t,x] \, dx \, dt - 2 \int_{t_0}^{t_1} \int_{x_0}^{x_1} \Delta u'(t,x) H_u[p,t,x] \, dx \, dt - \int_{t_0}^{t_1} \int_{x_0}^{x_1} \Delta u'(t,x) H_u[u,t,x] \Delta u(t,x) \, dx \, dt \right\} + \eta_1(\Delta u(t,x)), \]

where by definition
\[ \eta_1(\Delta u(t,x)) = E \left\{ o_1 \left( \sum_{i=1}^{k} \| \Delta z(T_i,X_i) \|^2 \right)^2 \right\} + \int_{t_0}^{t_1} \int_{x_0}^{x_1} o_2 (\| \Delta u(t,x) \| + \| \Delta p(t,x) \|)^2 \, dx \, dt. \]

Here the values \( o_i(\cdot), i = 1,2, \) are determined, respectively, from the expansions
\[ \varphi(\bar{z}(T_1,X_1),\bar{z}(T_2,X_2),...,\bar{z}(T_k,X_k)) - \varphi(z(T_1,X_1),z(T_2,X_2),...,z(T_k,X_k)) = \sum_{i=1}^{k} \frac{\partial \varphi(z(T_i,X_i),z(T_2,X_2),...,z(T_k,X_k))}{\partial a_i} \Delta z(T_i,X_i) + \frac{1}{2} \sum_{i,j=1}^{k} \Delta z'(T_i,X_i) \frac{\partial^2 \varphi(z(T_1,X_1),z(T_2,X_2),...,z(T_k,X_k))}{\partial a_i \partial a_j} \Delta z(T_i,X_i) \]

Hence, by analogous reasoning from [7-10], for optimality an admissible control \( u(t,x) \) it is necessary that for all \( \delta u(t,x) \in L_\infty(D,U) \) the first variation (in the classical sense) of the quality criterion is equal to zero, and the second variation was non-negative:
\[ \delta^2 S(u;\delta u) = E \left\{ - \int_{t_0}^{t_1} \int_{x_0}^{x_1} H_u'[t,x] \delta u(t,x) \, dx \, dt, \right\} \]

\[ \delta^2 S(u;\delta u) = E \left\{ \sum_{i=1}^{k} \delta z'(T_i,X_i) \frac{\partial^2 \varphi(z(T_1,X_1),z(T_2,X_2),...,z(T_k,X_k))}{\partial a_i \partial a_j} \Delta z(T_i,X_i) - \int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta p'(t,x) H_p[p,t,x] \, dx \, dt - 2 \int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u'(t,x) H_u[p,t,x] \, dx \, dt - \int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u'(t,x) H_u[u,t,x] \Delta u(t,x) \, dx \, dt \right\} \geq 0. \]

Here by definition \( \delta p(t,x) = \left( \delta z(t,x), \delta z_x(t,x), \delta z_x(t,x) \right)', \) and \( \delta z(t,x) \) is the variation of the state vector of process, this is the solution of stochastic equation in variation
\[ \frac{\partial^2 \delta z(t,x)}{\partial t \partial x} = f'_z[t,x] \delta z(t,x) + f'_u'[t,x] \delta u(t,x) + g'_z[t,x] \delta z(t,x) - \frac{\partial^2 W(t,x)}{\partial t \partial x}, \]

4. Necessary optimality conditions
From identity (5), since \( \delta u(t,x) \in L_\infty(D,R^r) \) is arbitrary, implies the identity
\[ EH'_u[\theta, \xi] = 0. \]  

(8)

In other words, it is proved

**Theorem 1 (analog of the Euler equation).** For optimality of the admissible control \( u(t, x) \) in problem (1)–(4), it is necessary that equality (8) be satisfied for all \( (\theta, \xi) \in [t_0, t_1] \times [x_0, x_1] \) — Lebesgue point of control \( u(t, x) \).

Optimality condition (8) is a stochastic analog of the Euler equation for the problem under consideration and is a necessary first-order optimality condition.

Further, following the papers [17, 7], the admissible controls \( u(t, x) \) satisfying the Euler equation (8) will be called the classical extremal of problem (1)–(4).

Therefore, the following is true.

**Theorem 2.** For the optimality of the classical extremal \( u(t, x) \) in problem (1)–(4), it is necessary that inequality (6) be satisfied for all \( \delta u(t, x) \in R^n \).

As can be seen, inequality (6) is an implicit second order necessary optimality condition. For this reason, let us consider the derivation of the necessary second-order optimality conditions, explicitly expressed directly through the parameters of problem (1)–(4).

In this way, we essentially need the following

**Lemma.** The solution to equation (7) with probability 1, everywhere in the domain \( D \) admits the representation

\[
\delta z(t, x) = \int_{t_0}^{t} \int_{\tau}^{x} R(t, x; \tau, s) f_u[\tau, s] \delta u(\tau, s) ds d\tau, \tag{9}
\]

where \( R(t, x; \tau, s) \) — measurable and bounded \( (n \times n) \) — Riemann matrix is the solution of the stochastic integral equation Volterra:

\[
R(t, x; \tau, s) = I + \int_{\tau}^{t} \int_{\tau}^{x} R(t, x; \tau, \beta) f_z[\tau, \beta] d\alpha d\beta + \int_{\tau}^{t} R(t, x; \alpha, s) f_{zz}[\alpha, s] d\alpha + \int_{\tau}^{t} \int_{\tau}^{x} R(t, x; \tau, \beta) f_z[\tau, \beta] d\alpha d\beta + \int_{\tau}^{t} \int_{\tau}^{x} R(t, x; \alpha, \beta) g_{zz}[\alpha, \beta] \frac{\partial^2 W(\alpha, \beta)}{\partial \alpha \partial \beta} d\alpha d\beta,
\]

where \( I \) — \( (n \times n) \) is identity matrix.

The lemma is proved according to the scheme from [18].

Also from representation (9) for the partial derivatives of the solution \( \delta z(t, x) \) with respect to \( t, x \) we have:

\[
\begin{align*}
\delta z_t(t, x) &= \int_{t_0}^{t} \int_{\tau}^{x} R(t, x; t, s) f_u[\tau, s] \delta u(t, s) ds + D_1(t, x), \tag{10} \\
\delta z_x(t, x) &= \int_{t_0}^{t} \int_{\tau}^{x} R(t, x; \tau, s) f_u[\tau, s] \delta u(\tau, s) d\tau + D_2(t, x), \tag{11}
\end{align*}
\]

where by definition

\[
\begin{align*}
D_1(t, x) &= \int_{t_0}^{t} \int_{\tau}^{x} R_t(t, x; \tau, s) f_u[\tau, s] ds d\tau, \\
D_2(t, x) &= \int_{t_0}^{t} \int_{\tau}^{x} R_x(t, x; \tau, s) f_u[\tau, s] ds d\tau.
\end{align*}
\]

Using formulas (10), (11), according to the formula Dirichlet [19] for permutations of integration redistributions, we have
\[
\int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u'(t, x) H_{t x z} [t, x] \delta z_t (t, x) dx dt = \int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u'(t, x) H_{t x z} [t, x] D_1 (t, x) dt dx +
\]
\[
\int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u(t, s) H_{t x z} [t, x] R(t, s; t, x) ds \int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u(t, x) H_{t x z} [t, x] \delta z_t (t, x) dx dt,
\]

(12)

\[
\int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u'(t, x) H_{t x z} [t, x] \delta z_t (t, x) dx dt = \int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u'(t, x) H_{t x z} [t, x] D_2 (t, x) dt dx +
\]
\[
\int_{t_0}^{t_1} \int_{x_0}^{x_1} \int_{t}^{t_1} \delta u(\tau, x) H_{t x z} [\tau, x] R(\tau, x; t, x) d\tau \int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u(t, x) H_{t x z} [\tau, x] \delta z_t (\tau, x) dx dt.
\]

(13)

Let us introduce into consideration the matrix functions, which are a stochastic analog of the matrices function introduced in [7-9], etc.

\[
K(x; \tau, s) = \int_{\max (\tau, s)}^{t_1} R(t, x; \tau, x) H_{z x x} [t, x] R(t, x; s, x) dt,
\]

\[
M(t; \tau, s) = \int_{\max (\tau, s)}^{x_1} R(t, x; t, \tau) H_{z t z} [t, x] R(t, x; t, s) dx.
\]

Hence, taking into account the matrix functions \(K(x; \tau, s)\), and \(M(t; \tau, s)\) we make sure that

\[
\int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta z(t, x) H_{z t z} [t, x] \delta z_t (t, x) dx dt + \int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta z_x (t, x) H_{z t z} [t, x] \delta z_x (t, x) dx dt =
\]
\[
\int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u(\tau, x) f_{u[t, \tau]} M(t; \tau, s) f_{u[t, s]} (x) ds d\tau dt +
\]
\[
\int_{t_0}^{t_1} \int_{x_0}^{x_1} \int_{t}^{t_1} \delta u'(\tau, x) f_{u[t, \tau]} K(x; \tau, s) f_{u[s, x]} ds d\tau dt + D_3 (t, x),
\]

(14)

where by definition

\[
D_3 (t, x) = \int_{t_0}^{t_1} \int_{x_0}^{x_1} \left( \int_{t_0}^{t_1} \int_{x_0}^{x_1} \int_{t}^{t_1} R(t, x; t, s) f_{u[t, s]} (x) \delta u(t, s) ds \right) H_{z t z} [t, x] D_1 (t, x) +
\]
\[
D_1 (t, x) H_{z t z} [t, x] \delta z_t (t, x) + D_2 (t, x) H_{z t z} [t, x] \delta z_x (t, x) +
\]
\[
\left( \int_{t_0}^{t_1} \int_{x_0}^{x_1} \int_{t}^{t_1} R(t, x; t, s) f_{u[t, s]} (x) \delta u(t, s) d\tau \right) H_{z t z} [t, x] D_2 (t, x) \right)^{'} dx dt.
\]

Then, taking into account relation (12) – (14), the second variation \(\delta^2 S(u; \delta u)\), defined by expression (6), can be replaced by the following form:
\[
\delta^2 S(u; \delta u) = \left\{ - \int_{t_0}^{t_1} \int_{x_0}^{x_1} \int_{t_0}^{t_1} \delta u'(\tau, x) f_{u}[\tau, x] K(x; \tau, s) f_{\delta u}[s, t] \delta u(s, t) ds dt d\tau - \right. \\
\left. \int_{t_0}^{t_1} \int_{x_0}^{x_1} \int_{t_0}^{t_1} \delta u'(t, \tau) f_{u}[t, \tau] M(t; \tau, s) f_{\delta u}[s, t] \delta u(s, t) ds dt d\tau d\tau \right. \\
2 \int_{t_0}^{t_1} \int_{x_0}^{x_1} \int_{t}^{t_1} \delta u'(t, \tau) H_{u\delta u}[\tau, \tau] R(\tau, \tau; \tau, x) d\tau + \right. \\
\left. \int_{t_0}^{t_1} \int_{x_0}^{x_1} \int_{t_0}^{t_1} \delta u'(t, \tau) H_{u\delta u}[\tau, \tau] R(\tau, \tau; \tau, x) d\tau d\tau \right. \\
\left. \int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u'(t, x) H_{u\delta u}[t, x] \delta u(t, x) dt d\tau \right. \\
\left. + D_4(t, x), \right. \\
\text{where by definition} \\
D_4(t, x) = E \left\{ \sum_{i,j=1}^{k} \delta z'(T_i, X_i) \frac{\partial^2 \varphi(z(T_i, X_i), z(T_j, X_j), \ldots, z(T_k, X_k))}{\partial a_i \partial a_j} \delta z(T_i, X_i) - \right. \\
\int_{t_0}^{t_1} \int_{x_0}^{x_1} \left[ \delta z'(t, x) H_{z\delta z}[t, x] \delta z(t, x) + \delta z'(t, x) H_{zz}[t, x] \delta z(t, x) + \delta z'_x(t, x) H_{zz,x}[t, x] \delta z_x(t, x) + \delta z'_x(t, x) H_{z\delta z,x}[t, x] \delta z_x(t, x) \right] dt d\tau \\
\left. + 2 \int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u'(t, x) H_{u\delta u}[t, x] \delta z(t, x) + \right. \\
\left. \delta u'(t, x) H_{u\delta u}[t, x] D_1(t, x) + H_{u\delta u}[t, x] D_2(t, x) \right] dt d\tau - D_3(t, x). \right.
\]

Let \( L_\infty([a, b], U) \) be the set of \( r \)-dimensional measurable and bounded vector functions \( l(\cdot) \) with values from \( U \). Determining in turn a special variation of the optimal control \( u(t, x) \) respectively, by the formula

\[
\delta u_\varepsilon(t, x) = \left\{ \begin{array}{ll}
l_1(t, x) & \text{if } (t, x) \in D(\varepsilon) = T \times [\xi, \xi + \varepsilon), \\
0 & \text{if } (t, x) \in D \setminus D(\varepsilon), \\
\end{array} \right.
\]

where \( \varepsilon > 0 \) is a small enough number \( \xi \in [x_0, x_1], l_1(t) \in L_\infty([t_0, t_1], R^r) \), and \( \theta \in [t_0, t_1], l_2(t) \in L_\infty([x_0, x_1], R^r) \) then from inequality (6) it follows that along optimal process \( u(t, x), z(t, x) \) the following inequalities hold:

\[
E \left\{ \int_{t_0}^{t_1} \int_{t_0}^{t_1} l'_1(\tau) f_{u}[\tau, \xi] K(\xi, \tau, s) f_{\delta u}[s, \xi] l_1(s) ds d\tau + \int_{x_0}^{x_1} l'_1(\tau) H_{u\delta u}[\tau, \xi] l_1(t) dt + \right. \\
2 \int_{t_0}^{t_1} \int_{t}^{t_1} l'_1(\tau) H_{u\delta u}[\tau, \xi] R(\tau, \xi; \tau, \xi) d\tau d\tau \right. \\
\left. \int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u'(t, \tau) H_{u\delta u}[\tau, \tau] R(\tau, \tau; \tau, \tau) d\tau d\tau \right. \\
\left. \int_{t_0}^{t_1} \int_{x_0}^{x_1} \delta u'(t, x) H_{u\delta u}[t, x] \delta u(t, x) dt d\tau \right. \\
\left. + D_4(t, x), \right. \\
\left. + D_4(t, x) \right\} \leq 0,
\]

for all \( \xi \in [x_0, x_1], l_1(t) \in L_\infty([t_0, t_1], R^r), \).
$$E \left\{ \int_{x_0}^{x_1} l_1^2(\tau) f_u[\theta, \tau] M(\theta, \tau, s) f_u[\theta, s] l_2(s) ds d\tau + \int_{x_0}^{x_1} l_2^2(x) H_{uu}[\theta, x] l_2(x) dx \right\}$$

$$2 \int_{x_0}^{x_1} \left[ \int_{x}^{x_1} l_2^2(\tau) H_{uuz}[\theta, \tau] R(\theta, \tau; \theta, x) d\tau \right] f_u[\theta, x] l_2(x) dx \leq 0,$$

(17)

for all $\theta \in [t_0, t_1], l_2(x) \in L_\infty([x_0, x_1], R^r)$.

Thus, it is proved

**Theorem 3.** (second order necessary optimality condition). For the optimality of the classical extremal $u(t, x)$ in problem (1)–(4), it is necessary that inequalities (16), (17) hold.

**Consequence.** (analog of the Legendre–Clebsch condition). For the optimality of the classical extremal $u(t, x)$ in problem (1)–(4), it is necessary that the inequality

$$Ev'H_{uu}[\theta, \xi]v \leq 0,$$

were satisfied for all $v \in R^r,(\theta, \xi) \in [t_0, t_1] \times [x_0, x_1]$.

At the end, we consider the case of degeneration of an analogue of the Legendre–Clebsch condition.

**Definition [17, 7-10].** An admissible control $u(t, x)$ is called a singular, in the classical sense, control if for all $(\theta, \xi) \in [t_0, t_1] \times [x_0, x_1]$

$$EH_{uu}[\theta, \xi] = 0, EH_{uu}[\theta, \xi] = 0.$$

From theorem 3 implies

**Theorem 4.** For the optimality of a singular control in the classical sense $u(t, x)$ in problem (1)–(4), it is necessary that the following relations hold:

$$E \left\{ \int_{t_0}^{t_1} \int_{t_0}^{t_1} l_1^1(\tau) f_u[\tau, \xi] K(\xi, \tau, s) f_u[s, \xi] l_1(s) ds d\tau + \right.$$

$$2 \int_{t_0}^{t_1} \left[ \int_{t}^{t_1} l_1^1(\tau) H_{uuz}[\tau, \xi] R(\tau, \xi; t, \xi) d\tau \right] f_u[t, \xi] l_1(t) dt \right\} \leq 0,$$

for all $\xi \in [x_0, x_1], l_1(t) \in L_\infty([t_0, t_1], R^r),$

$$E \left\{ \int_{x_0}^{x_1} \int_{x_0}^{x_1} l_2^2(\tau) f_u[\theta, \tau] M(\theta, \tau, s) f_u[\theta, s] l_2(s) ds d\tau + \right.$$

$$2 \int_{x_0}^{x_1} \left[ \int_{x}^{x_1} l_2^2(\tau) H_{uuz}[\theta, \tau] R(\theta, \tau; \theta, x) d\tau \right] f_u[\theta, x] l_2(x) dx \right\} \leq 0,$$

for all $\theta \in [t_0, t_1], l_2(x) \in L_\infty([x_0, x_1], R^r)$.

5. Conclusion

Applying the method of increment, and at the same time, taking into account the properties of the stochastic integral, a second-order representation for the increment of the quality functional is obtained. On the basis of this representation, first-order necessary optimality condition is proved in the form of an analog of the Euler equation. Then, with the help of a series of special variations, the second-order necessary optimality conditions are proved.

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