An analytical solution to the problem of stability loss of the rod under the combined action of concentrated force and dead weight

Yu Krutii1, S Bekshaev1 and V Osadchyi1

1 Odesa State Academy of Civil Engineering and Architecture, 4 Didrihson Street, 65029, Odesa, Ukraine

Email: yurii.krutii@gmail.com

Abstract. The exact solution of the differential equation of buckling of the prismatic column under the combined action of a longitudinal concentrated end force and its own weight is constructed. Formulas are written for the state parameters of the rod, expressed in terms of the initial parameters. For six cases of ideal boundary conditions, finite analytical formulas are obtained that establish the dependence of the critical force on the weight of the rod.

1. Introduction

The rods are a widespread model in the design of structural elements in machine engineering, shipbuilding, aircraft, tool engineering and other fields of technology. In the form of a rod model can be considered not only a separate structural element, but also the structure as a whole. For example, the columns of frame structures, steel supports, antennas of various designs, industrial high-rise structures in the form of chimneys, water towers, etc. can be represented as the rods. The theory of rod calculation for stability has been studied in detail in the works [1-3] and many others.

The dead weight of short vertical rods is usually neglected, as it is considered small in comparison with the applied load. However, it is extremely necessary to take into consideration its own weight when calculating the stability of high-rise and massive framed structures. A number of publications have been devoted to solving the problem of the loss of rod stability under the action of its own weight, among which we single out the works [4, 5].

Despite the relevance and the undoubted practical significance, up to this day, the exact solution to the problem of stability loss of the column under the simultaneous action of concentrated force and dead weight has not been found. The mathematical model of such a problem will be a differential equation with variable coefficients. It is quite clear that its exact solution has the information of a qualitative nature and forms the most complete picture of the physical phenomenon of stability loss. However, the variability of the coefficients of the equation introduces significant mathematical difficulties in the procedure for constructing such a solution. It can probably explain the lack of the exact solution to this problem in the scientific literature. In this work, these difficulties were overcome. The results described below are obtained on the basis of the exact solution of the differential stability equation.

For example, figure 1 shows the general diagram of a hinged rod, and figure 2 shows the forces that arise in its element. However, all the formulas, obtained below, will be valid under any boundary conditions.
Figure 1. Design scheme of the column.

Figure 2. The internal forces acting upon the element of a column.

List of symbols:

- $EI$ – the flexural rigidity of a road;
- $N$ – constant axial load;
- $q$ – the variable axial (compressive) force, where $q$ – weight per unit length of beam;
- $y(x)$ – the displacement of the axis point of the column with coordinate $x$ (deflection);
- $\varphi(x)$ – the angular displacement;
- $M(x)$ – the bending moment;
- $Q(x)$ – the transverse force.

The differential equation of rod stability in our case has the form

$$EI y'''(x) + ((N + qx)y'(x))' = 0. \tag{1}$$

The stress-strain state of the rod is completely described by displacements $y(x)$, $\varphi(x)$ and internal forces $M(x)$, $Q(x)$. We will briefly name them as rod state parameters. As you know [1-3], they are interconnected by formulas:

$$\varphi(x) = y'(x); \tag{2}$$
$$M(x) = -EI \varphi'(x); \tag{3}$$
$$Q(x) = -EI \varphi''(x) - (N + qx)\varphi(x). \tag{4}$$

The purpose of the work is to establish the dependence of the critical force on the dead weight of the rod in an analytical form. To achieve the goal, the following tasks are formulated and solved:
- find the exact solution to the equation (1);
- get the formulas for the state parameters of the rod;
- get the final calculation formulas for various boundary conditions.

2. Materials and methods

Earlier in the work [6], the method of direct integration of differential equations with continuous variable coefficients has been developed. Using this method, the exact solutions for a number of differential equations of statics and dynamics were also constructed there. The essence of the proposed
method can also be considered, for example, according to the publications [7–11], where a number of mechanics problems of a deformable solid body is solved with its help.

In this publication, at the initial stage, using the direct integration method, it was possible to determine the form of the equation solution (1). It turned out that the solution should be sought in the form of double power series with unknown coefficients. Due to this, the problem was reduced only to the determination of these coefficients.

3. Exact solution of the differential stability equation and formulas for the rod state parameters

Integrating the equation (1) and carrying out the replacement (2), we obtain

\[ EI\varphi''(x) + (N + qx)\varphi(x) = -Q(0), \tag{5} \]

or

\[ l^2\varphi''(x) + \left(K + \frac{x}{l}\right)\varphi(x) = -\frac{l^2}{EI}Q(0), \tag{6} \]

where

\[ K = \frac{Nl^2}{EI}, \tag{7} \]

\[ S = \frac{ql^3}{EI} \tag{8} \]

– dimensionless parameters. In addition to the equation (6), we will also consider an equivalent system of differential equations

\[ \frac{d\Phi(x)}{dx} = H(x)\Phi(x) + h, \tag{9} \]

where

\[ \Phi(x) = \begin{pmatrix} \varphi(x) \\ M(x) \end{pmatrix} = \begin{pmatrix} \varphi(x) \\ -EI\varphi'(x) \end{pmatrix}, \tag{10} \]

\[ H(x) = \begin{pmatrix} 0 & -1/EI \\ \frac{EI}{l^2}\left(K + \frac{x}{l}\right) & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 \\ Q(0) \end{pmatrix}. \]

We identify the fundamental solutions of the homogeneous equation by \( Z_n(x) \) \( (n=1,2) \), corresponding to (6), and the particular solutions of the non-homogeneous equation by \( Z_3(x) \) (6), for which we take the representation

\[ Z_n(x) = -\frac{l^2}{EI}Q(0)Z_n(x). \tag{11} \]

Then the functions \( Z_n(x) \) \( (n=1, 2, 3) \) must satisfy the equations:

\[ l^2Z_n''(x) + \left[K + \frac{x}{l}\right]Z_n(x) = 0 \quad (n = 1, 2); \tag{12} \]

\[ l^2Z_3''(x) + \left[K + \frac{x}{l}\right]Z_3(x) = 1. \tag{13} \]

We add the following boundary conditions to these equations:
We will look the functions $Z_n(x)$ \((n=1, 2, 3)\) in a dimensionless format in the form of double power series

$$Z_n(x) = \left(\frac{x}{l}\right)^{n-1} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{n,k,j} (-K)^k (-S)^j \left(\frac{x}{l}\right)^{2k+3j},$$

or

$$Z_n(x) = \left(\frac{x}{l}\right)^{n-1} c_{n,0,0} + \sum_{j=1}^{\infty} c_{n,0,j} (-S)^j \left(\frac{x}{l}\right)^{3j} + \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} c_{n,k,j} (-K)^k \left(\frac{x}{l}\right)^{2k}$$

$$+ \sum_{k=1}^{\infty} \sum_{j=2}^{\infty} c_{n,k,j} (-K)^k (-S)^j \left(\frac{x}{l}\right)^{2k+3j)},$$

where $c_{n,k,j}$ \((n=1, 2, 3)\) \((k=0, 1, 2, \ldots)\) \((j=0, 1, 2, \ldots)\) – dimensionless coefficients, that have to be determined. While we assume that the series (16) uniformly converge and the operation of their double differentiation is possible.

Due to this choice, as the analysis of formula (17) shows, the condition (15) is certainly carried out, and the condition (14) will be satisfied if we put

$$c_{1,0,0} = c_{2,0,0} = 1.$$

Inserting (16) into the equations (12), (13), we have

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_{n,k,j} c_{n,k,j} (-K)^k (-S)^j \left(\frac{x}{l}\right)^{n+2k+3j-3} = \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} c_{n,k,j} (-K)^k \left(\frac{x}{l}\right)^{2k} \left(\frac{x}{l}\right)^{2k+3j}$$

$$- \sum_{j=2}^{\infty} \sum_{k=0}^{\infty} c_{n,k,j} (-K)^k (-S)^j \left(\frac{x}{l}\right)^{2k+3j-1} = \delta_n (n=1, 2, 3),$$

where

$$f_{n,k,j} = (n+2k+3j-2)(n+2k+3j-1), \quad \delta_1 = \delta_2 = 0, \quad \delta_3 = 1.$$ (19)

In the second sum, we shift the index $k$ by one, so, replace $k$ with $k-1$. We perform the same operation in the third sum with the index $j$. As a result, we get

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_{n,k,j} c_{n,k,j} (-K)^k (-S)^j \left(\frac{x}{l}\right)^{n+2k+3j-3} = \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} c_{n,k-1,j} (-K)^k (-S)^j \left(\frac{x}{l}\right)^{n+2k+1j-3}$$

$$- \sum_{j=2}^{\infty} \sum_{k=0}^{\infty} c_{n,k,j-1} (-K)^k (-S)^j \left(\frac{x}{l}\right)^{n+2k+3j-3} = \delta_n (n=1, 2, 3).$$

Further, after the generations, we come to the necessity of fulfilling the equality

$$f_{n,0,0} c_{n,0,0} \left(\frac{x}{l}\right)^{n-3} + \sum_{k=1}^{\infty} (f_{n,k,0} c_{n,k,0} - c_{n,k-1,0}) (-K)^k \left(\frac{x}{l}\right)^{n+2k-3} + \sum_{j=2}^{\infty} (f_{n,0,j} c_{n,0,j} - c_{n,0,j-1}) (-S)^j \left(\frac{x}{l}\right)^{n+3j-3}$$
\[
+ \sum_{k=1}^{n} \sum_{j=1}^{3} \left( f_{n,k,j} c_{n,k,j} - c_{n,k+1,j} - c_{n,k,j-1} \right) \left( -K \right)^{k} \left( -S \right)^{j} \left( \frac{x}{l} \right)^{n+2k+3j-3} = \delta_{n} \quad (n = 1, 2, 3). 
\]

Hence, taking into account the equalities (18), (19), we obtain the following set of recurrence formulas for determining the desired dimensionless coefficients:

\[
c_{n,0,0} = \frac{1}{(n-1)!} \quad (n = 1, 2, 3) ; 
\]

(20)

\[
c_{n,k,0} = \frac{c_{n,k-1,0}}{(n+2k-2)(n+2k-1)} \quad (k = 1, 2, 3, \ldots) ; 
\]

(21)

\[
c_{n,0,j} = \frac{c_{n,0,j-1}}{(n+3j-2)(n+3j-1)} \quad (j = 1, 2, 3, \ldots) ; 
\]

(22)

\[
c_{n,k,j} = \frac{c_{n,k-1,j} + c_{n,k,j-1}}{(n+2k+3j-2)(n+2k+3j-1)} \quad (k = 1, 2, 3, \ldots)(j = 1, 2, 3, \ldots). 
\]

(23)

Thus the coefficients of the series (16), (17) can be considered known.

Note the fact that the formulas (21), (22) can also be written in explicit form:

\[
c_{n,k,0} = \frac{1}{(n+2k-1)!} \quad (k = 1, 2, 3, \ldots) ; 
\]

(24)

\[
c_{n,0,j} = \frac{n(n+3)...(n+3j-3)}{(n+3j-1)!} \quad (j = 1, 2, 3, \ldots). 
\]

(25)

The proof of the series (16) convergence is based on the following estimates

\[
c_{n,k,j} \leq \frac{3^{j} j!}{(2k)!} \quad (n = 1, 2, 3)(k = 0, 1, 2\ldots)(j = 0, 1, 2\ldots), 
\]

(26)

which can be obtained by method of induction, based on the formulas (20), (24), (25), (23). Taking into consideration (26), we have

\[
\left| Z_{n}(x) \right| \leq \left( \frac{x}{l} \right)^{n-1} \sum_{k=0}^{\infty} \sum_{j=0}^{3} c_{n,k,j} K^{k} S^{j} \left( \frac{x}{l} \right)^{2k+3j} \leq \left( \frac{x}{l} \right)^{n-1} \left[ \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left( \sqrt{K} \frac{x}{l} \right)^{2k} \right] \left[ \sum_{j=0}^{\infty} \frac{3^{j} j!}{(3j)!} \left( \frac{3S x}{l} \right)^{3j} \right] \]

\[
= \left( \frac{x}{l} \right)^{n-1} \cosh \sqrt{K} \frac{x}{l} \sum_{j=0}^{\infty} \frac{3^{j} j!}{(3j)!} \left( \frac{3S x}{l} \right)^{3j} . 
\]

Moreover, for the radius of series convergence in powers of the variable \((\sqrt{S} x/l)^{3}\), according to the D’Alembert criterion, we obtain

\[
R = \lim_{j \to \infty} (3j + 1)(3j + 2) = \infty . 
\]

Thus, it is proved that the series (16) converge uniformly. The convergence of series, as a result of single and double differentiation, is proved in a similar way (16).

Therefore, we can state that two solutions \(Z_{n}(x)\) \((n=1,2)\) of the homogeneous equation, corresponding one (6), as well as a particular solution \(Z^{*}(x)\) of the non-homogeneous equation (6), were found.

Each of the solutions \(Z_{n}(x)\) \((n=1,2)\) generates a vector according to the formula (10)
\[ \Phi_n(x) = \begin{pmatrix} Z_n(x) \\ -EI Z'_n(x) \end{pmatrix} \quad (n = 1, 2) \]

solving a homogeneous system of differential equations corresponding to (9). Then the matrix

\[ \Omega(x) = \begin{pmatrix} Z_1(x) & Z_2(x) \\ -EI Z'_1(x) & -EI Z'_2(x) \end{pmatrix}, \]

developed with these vectors, will also satisfy a homogeneous system. Moreover, taking into account (14), we find

\[ \Omega(0) = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{EI}{l} \end{pmatrix}, \quad \left[ \Omega(0) \right] = -\frac{EI}{l} \neq 0. \]

Therefore, the matrix (27) is the fundamental matrix [12] for the system (9).

Multiplying \( \Omega(x) \) on the right by a constant matrix \( \Omega^{-1}(0) \), we obtain a new fundamental matrix \( \Lambda(x) = \Omega(x)\Omega^{-1}(0) \) for which

\[ \Lambda(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

As it is known [12], a fundamental matrix, possessing the characteristic (28), is determined definitely and is called a matrizant. In this case, the general solution of the homogeneous system, corresponding to (9), is written in the form \( \Phi(x) = \Lambda(x)\Phi(0) \).

Forming the vector using (11)

\[ \Phi_*(x) = -Q(0)\int_0^x \left( \begin{array}{c} Z_1(x) \\ -EI Z'_1(x) \end{array} \right), \]

we obtain a particular solution of the non-homogeneous system (9) that satisfies the condition (15). This is easily verified by substitution.

Therefore, the general solution of the system of differential equations (9) can be written as follows

\[ \Phi(x) = \Lambda(x)\Phi(0) + \Phi_*(x). \]

Hence, for the rotation angle and bending moment, we will have:

\[ \varphi(x) = \varphi(0)Z_1(x) - M(0)\frac{l}{EI}Z_2(x) - Q(0)\frac{l^2}{EI}Z_1(x); \]

\[ M(x) = -\varphi(0)EI Z'_1(x) + M(0)lZ'_2(x) + Q(0)l^2Z'_1(x). \]

Next, integrating the equality (2), we obtain the formula for displacement

\[ y(x) = y(0) + \int_0^x \varphi(x)dx, \]

or

\[ y(x) = y(0) + \varphi(0)\int_0^x Z_1(x)dx - M(0)\int_0^x Z_2(x)dx - Q(0)\int_0^x lZ'_1(x)dx. \]

We rewrite the formulas (29)-(31) in a different form by means of the following dimensionless functions
\[ X_n(x) = \frac{1}{l} \int_0^l Z_n(x) dx, \quad \bar{X}_n(x) = lX_n(x), \quad \bar{X}_n'(x) = lX_n'(x) \quad (n = 1, 2, 3). \] (32)

As a result, we obtain the final formulas for the rod state parameters expressed in terms of the initial parameters:

\[ y(x) = y(0) + \varphi(0) \int_0^x X_1(x) dx - M(0) \frac{l^2}{EI} X_2(x) - Q(0) \frac{l^3}{EI} X_3(x); \] (33)
\[ \varphi(x) = \varphi(0) \bar{X}_1(x) - M(0) \frac{l}{EI} \bar{X}_2(x) - Q(0) \frac{l^2}{EI} \bar{X}_3(x); \] (34)
\[ M(x) = -\varphi(0) \frac{EI}{l} \bar{X}_1(x) + M(0) \bar{X}_2(x) + Q(0) l \bar{X}_3(x); \] (35)
\[ Q(x) = Q(0). \] (36)

where

\[ X_n(x) = \left( \frac{x}{l} \right)^n \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{c_{n,k,j}}{n + 2k + 3j} (-K)^k (-S)^j \left( \frac{x}{l} \right)^{2k+3j} (n = 1, 2, 3), \] (37)
\[ \bar{X}_n(x) = Z_n(x) \quad (n = 1, 2, 3), \] (38)
\[ \bar{X}_n'(x) = \left( \frac{x}{l} \right)^{n-2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (n + 2k + 3j - 1)c_{n,k,j} (-K)^k (-S)^j \left( \frac{x}{l} \right)^{2k+3j} \quad (n = 1, 2, 3). \] (39)

We note that the formula for the transverse force (36) follows directly from the equalities (4), (5).

The presence of the formulas (33)-(39), (17), (20)-(23) allows us to investigate the rod stability in the case of the combined action of a concentrated force and dead weight under any boundary conditions.

4. Analytical presentation for longitudinal force and rod weight

Directly from the equalities (7), (8) we find:

\[ N = K \frac{EI}{l^2}; \] (40)
\[ ql = S \frac{EI}{l^2}. \] (41)

These formulas establish a direct analytical dependence of the longitudinal force \( N \) and the weight of the rod \( ql \) on other parameters of the mechanical system that is being considered.

These formulas are well known in stability theory [1-3]. There will be the formula of the form (40), studying the stability of the rod under the action of concentrated force without taking its own weight into account. There will be the formula of the form (41) when the stability of the rod loaded only with its own weight is considered. Moreover, the critical values of dimensionless parameters \( K \) and \( S \) depend only on the boundary conditions for the rod with constant stiffness.

As a matter of fact, these formulas are shown to remain correct even in the case of the combined action of a concentrated force and rod own weight. However, in our case, the critical value of parameter \( K \), in addition to the boundary conditions, will also depend on the weight of the rod. The critical value of parameter \( S \) will not depend only on the boundary conditions, but also on the applied axial force.

5. Analytical dependence of the critical force on the weight of the rod

We establish the analytical dependence of the critical force on the weight of the rod for six ideal boundary conditions listed in table 1.
Characteristic equations, corresponding to different boundary conditions, admit a uniform notation

\[ \eta_0(S) - \eta_1(S)K + \eta_2(S)K^2 - \eta_3(S)K^3 + ... = 0, \]  

(42)

where \( \eta_k(S) (k=0,1,2,\ldots) \) – dimensionless coefficients. The left side of this equation will always be an absolutely convergent series.

Based on the formulas (37)-(39), (16) we can write:

\[ X_x(l) = \sum_{k=0}^{\infty} \gamma_{n,k}(S)(-K)^k; \]  

(43)

\[ \tilde{X}_x(l) = \sum_{k=0}^{\infty} \tilde{\gamma}_{n,k}(S)(-K)^k; \]  

(44)

\[ \tilde{X}_x(l) = \sum_{k=0}^{\infty} \tilde{\gamma}_{n,k}(S)(-K)^k, \]  

(45)

where

\[ \gamma_{n,k}(S) = \sum_{j=0}^{\infty} \frac{c_{n,k,j}}{n+2k+3j}(-S)^j, \]  

(46)

\[ \tilde{\gamma}_{n,k}(S) = \sum_{j=0}^{\infty} c_{n,k,j}(-S)^j, \]  

(47)

\[ \tilde{\gamma}_{n,k}(S) = \sum_{j=0}^{\infty} (n+2k+3j-1)c_{n,k,j}(-S)^j. \]  

(48)

**Table 1.** Types of rod ends fixings, corresponding boundary conditions and characteristic equations.

| Case | Type of ends fixing | Boundary condition | Characteristic equation |
|------|---------------------|--------------------|------------------------|
| x = 0 | x = l               | y(0) = 0; M(0) = 0; y(l) = 0; M(l) = 0. | \( X_x(l)\tilde{X}_x(l) - \tilde{X}_x(l)X_x(l) = 0 \) |
| 1    | Pinned              | y(0) = 0; M(0) = 0; y(l) = 0; M(l) = 0. |                        |
| 2    | Free                | M(0) = 0; Q(0) = 0; y(l) = 0; \( \varphi(l) = 0 \). | \( \tilde{X}_x(l) = 0 \) |
| 3    | Clamped             | M(0) = 0; Q(0) = 0; y(l) = 0; \( \varphi(l) = 0 \). |                        |
| 4    | Pinned              | y(0) = 0; M(0) = 0; y(l) = 0; \( \varphi(l) = 0 \). | \( X_x(l)\tilde{X}_x(l) - \tilde{X}_x(l)X_x(l) = 0 \) |
| 5    | Sliding restraint   | \( \varphi(0) = 0; Q(0) = 0 \); y(l) = 0; \( \varphi(l) = 0 \). | \( \tilde{X}_x(l) = 0 \) |
| 6    | Sliding restraint   | \( \varphi(0) = 0; Q(0) = 0 \); y(l) = 0; M(l) = 0. | \( \tilde{X}_x(l) = 0 \) |

We suggest to consider the case of a hinge supported rod in detail:

\[ y(0) = 0; M(0) = 0; y(l) = 0; M(l) = 0. \]
In this case, the initial parameters $y(0), M(0)$ are known in advance. Realizing with the help of (33), (35) the conditions at the end $x=l$, as for $\phi(0), Q(0)$ we will have the system:

\[
\begin{aligned}
&I X_\ell(l) \phi(0) - \frac{I^3}{EI} X_\ell(l) Q(0) = 0; \\
&\frac{EI}{I} \ddot{X}_\ell(l) \phi(0) + I \ddot{X}_s(l) Q(0) = 0.
\end{aligned}
\]

Writing down the condition for the solvability of the system, we obtain the characteristic equation

\[
X_\ell(l) \ddot{X}_\ell(l) - \ddot{X}_s(l) X_s(l) = 0. \tag{49}
\]

The characteristic equations for all the boundary conditions are given in table 1.

Based on the representations (43), (45) and multiplying and subtracting the series on the left side of the equation (49), we have it in the form (42), where

\[
\eta_k(S) = \sum_{i=0}^{k} \left( \gamma_{i,i}(S) \ddot{\gamma}_{3,i-k,i}(S) - \ddot{\gamma}_{3,i}(S) \gamma_{3,i-k,i}(S) \right) (k = 0, 1, 2, \ldots).
\]

Transforming the last expression taking into account (46), (48), we obtain

\[
\eta_k(S) = \sum_{i=0}^{k} \sum_{j=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{2(k-i) + 3(j-\nu) + 2}{2i + 3\nu + 1} \frac{2i + 3\nu}{2(k-i) + 3(j-\nu) + 3} c_{i,j-\nu} c_{3,k-i,j-\nu}(-S)^i.
\]

Thus, according to the formula (8), calculating the value of the parameter $S$ for a specified rod and finding the corresponding smallest positive root of the equation (42), we will have a pair of significance points $(S, K)$ at which the rod will lose stability. In this case, the critical force value can be found by the formula (40).

It is easy to set the range of values for the dimensionless parameters $K$ and $S$. As it is known [1-3], for a weightless rod (case $S = 0$), the critical force is solved by the formula (40), where $K = \pi^2$. It is quite clear that the calculated value of the critical force for a weighted rod (case $S \neq 0$) should be less in comparison with a weightless rod. Hence, we conclude that in our case $0 \leq K \leq \pi^2$.

It is also known [1-5] that in the absence of axial force (case $K = 0$), the critical weight of the rod is determined by the formula (41), moreover, for a hinge supported rod $S = 18.5687$. If the axial load is applied (case $K \neq 0$), the calculated value of the critical rod weight should be less compared with the case when the axial force is absent. In other words, for our case the values $0 \leq S \leq 18.5687$ are relevant.

In figure 3 there are the calculating results of the parameter $K$, corresponding to some given values of the parameter $S$ from the specified interval in increments of 1.85687. Considering the parameter $K$ as a function of a variable $S$, as well as having many values of this function that correspond to the values of an explanatory variable $0 \leq S \leq 18.5687$, we approximate the function $K$ by a polynomial. In this case, the degree of the approximating polynomial is chosen from the condition that the determination coefficient is equal to 1. As a result, we obtain

\[
K = 9.8700 - 0.5004 S - 0.0017 S^2.
\]

The value 9.8700 differs from the number $\pi^2$ only by an amount 0.0004. After the replacement, we get an approximate formula

\[
K = \pi^2 - 0.5004 S - 0.0017 S^2. \tag{50}
\]

The graph of the found dependence is shown in Figure 3.

Thus, a multiplicity of value pairs $(S, K)$, at which the rod loses the stability, forms the line of critical values, shown in figure 3.
Substituting the value (50) into the formula (40) instead of the parameter $K$ and taking into account (8), we finally obtain

$$N_{cr} = \pi^2 \frac{EI}{l^2} - 0.5004(ql) - 0.0017(ql)^2 \frac{l^2}{EI}. \quad (51)$$

This formula establishes the dependence of the critical force $N_{cr}$ on the dead weight of the rod $ql$. It is important to note that when considering a weightless rod (case $q = 0$), from the formula (51) we obtain the well-known Euler formula.

| $S$      | $K$      |
|----------|----------|
| 0        | $\pi^2$  |
| 1.8569   | 8.9352   |
| 3.7137   | 7.9890   |
| 5.5706   | 7.0310   |
| 7.4275   | 6.0613   |
| 9.2844   | 5.0798   |
| 11.1412  | 4.0866   |
| 12.9981  | 3.0819   |
| 14.8550  | 2.0659   |
| 16.7118  | 1.0385   |
| 18.5687  | 0.0000   |

**Figure 3.** Calculating results and corresponding critical line for the case 1 from table 1.

Similar calculations were carried out for other cases of boundary conditions. The formulas for calculating the coefficients of the characteristic equations in the form (42) for all six cases are presented in table 2. The final formulas for the critical forces are presented in table 3.

**Table 2.** Formulas for calculating the coefficients of the characteristic equations.

| Case | Coefficients of characteristic equations ($k = 0,1,2,...$) |
|------|----------------------------------------------------------|
| 1    | $\eta_k(S) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \frac{2(k+i)+3(j+i)+2}{2i+3j+1} - \frac{2i+3j}{2(k+i)+3(j+i)+3} \right) c_{i,j} c_{i+k-i,j} (-S)^i$ |
| 2    | $\eta_k(S) = \sum_{j=0}^{\infty} c_{1,k,j} (-S)^j$ |
| 3    | $\eta_k(S) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \frac{1}{2i+3j+2} - \frac{1}{2(k+i)+3(j+i)+3} \right) c_{i,j} c_{i+k-i,j} (-S)^i$ |
| 4    | $\eta_k(S) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \frac{1}{2i+3j+1} - \frac{1}{2(k+i)+3(j+i)+3} \right) c_{i,j} c_{i+k-i,j} (-S)^i$ |
| 5    | $\eta_k(S) = \sum_{j=0}^{\infty} c_{2,k,j} (-S)^j$ |
| 6    | $\eta_k(S) = \sum_{j=0}^{\infty} (2k+3j+1) c_{2,k,j} (-S)^j$ |
Table 3. Critical Forces for various cases of boundary conditions.

| Case | Formulas for Critical Force |
|------|-----------------------------|
| 1    | $N_{cr} = \pi^2 \frac{EI}{l^2} - 0.5004(ql) - 0.0017(ql)^2\frac{l^2}{EI}$ |
| 2    | $N_{cr} = \frac{\pi^2}{4} \frac{EI}{l^2} - 0.2968(ql) - 0.0023(ql)^2\frac{l^2}{EI}$ |
| 3    | $N_{cr} = 4\pi^2 \frac{EI}{l^2} - 0.5003(ql) - 0.0004(ql)^2\frac{l^2}{EI}$ |
| 4    | $N_{cr} = -\frac{\pi^2}{0.699^2} \frac{EI}{l^2} - 0.3442(ql) - 0.0008(ql)^2\frac{l^2}{EI}$ |
| 5    | $N_{cr} = \frac{\pi^2}{4} \frac{EI}{l^2} - 0.5001(ql) - 0.0011(ql)^2\frac{l^2}{EI}$ |
| 6    | $N_{cr} = \frac{\pi^2}{4} \frac{EI}{l^2} - 0.7028(ql) - 0.0020(ql)^2\frac{l^2}{EI}$ |

6. Results and discussion
Whenever the study of a physical phenomenon is reduced to a differential equation, the key question is constructing its exact (analytical) solution. However, the researchers often face a well-known mathematical problem, which is the lack of a universal method for integrating differential equations with the variable coefficients. Probably, this can explain the predominant use of approximate methods. This completely refers to the problem of loss of rod stability under the combined action of a concentrated force and dead weight. In this paper, these difficulties were overcome. As a result, the following results were obtained:
1. The exact solution of the differential stability equation is constructed;
2. Formulas for displacements and internal forces in the rod are obtained;
3. For six cases of the boundary conditions in an analytical form, the formulas for the critical force are obtained.

In fact, the solution of the original problem is reduced only to the implementation of the given boundary conditions and the solution of the obtained characteristic equation.

7. Conclusions
The problem of the loss of rod stability under the combined action of a concentrated force and dead weight is solved. Existence of exact formulas for the rod state parameters is a real alternative to using approximate methods for solving this class of problems. It is quite clear that based on the exact solution of the differential equation, a more established picture of the stability loss can be obtained in comparison with the approximate methods. It is the exact solution that carries the qualitative information and forms the most complete picture of the physical phenomenon that is being studied.

References
[1] Timoshenko S P and Gere J M 1961 Theory of Elastic Stability (New York: McGraw-Hill) 541
[2] Vol’mir A S 1967 Ustoychivost’ deformiraiemymkh sistem (Moscow: Nauka) 984
[3] Bazant Z P and Cedolin L 1991 Stability of Structures (New York: Oxford University Press) 1008
[4] Chai Y H and Wang C M 2006 An application of differential transformation to stability analysis of heavy columns Int. J. Structural Stability and Dynamics 06 (03) pp 317–332
[5] Krutii Yu and Vandynskyi V 2019 Exact solution of buckling problem of the column loaded by self-weight *IOP Conference Series: Mat. Sci. and Eng.* 708 (1) 012062

[6] Krutii Yu S 2016. *Rozrobka metodu rozv’yazannya zadach stiykosti i kolyvan’ deformivnykh system zi zminnymy neperervnymy parametramy* Sc. D. diss. Lutsk National Technical University.

[7] Krutii Yu S 2018 Construction of a solution of the problem of stability of a bar with arbitrary continuous parameters *J. Mat. Sci.* 231 pp 665-677

[8] Shvab’yuk V I,Krutii Yu S and Sur’yaninov M G 2016 Investigation of the Free Vibrations of Bar Elements with Variable Parameters Using the Direct Integration Method *Strength of Materials* 48(3) pp 384-393

[9] Krutii Yu 2015 Analysis of longitudinal oscillations for systems with continuous variable parameters using force integration method *Technical journal* 9(4) pp 420-425

[10] Krutii Yu, Surianinov M and Vandynskyi V 2018 Development of the method for calculation of cantilever construction's oscillations taking into account own weight *Eastern-European J. of Enterprise Technologies* 3(7) pp 13-19

[11] Krutii Yu, Surianinov M and Vandynskyi V 2019 Analytic Formulas for the Cantilever Structures’ Natural Frequencies with Taking into Account the Dead Weight *Materials Science Forum* 968 pp 450-459.

[12] Gantmacher F R 1959 *The Theory of Matrices* (Chelsea, New York: American Mathematical Soc.) 374