BOUNDS ON THE NORMS OF MAXIMAL OPERATORS ON WEYL SUMS

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Abstract. We obtain new estimates on the maximal operator applied to the Weyl sums. We also consider the quadratic case (that is, Gauss sums) in more details. In wide ranges of parameters our estimates are optimal and match lower bounds. Our approach is based on a combination of ideas of Baker (2021) and Chen and Shparlinski (2020).

1. Introduction

1.1. Set-up and motivation. Given a family \( \varphi = (\varphi_1, \ldots, \varphi_d) \in \mathbb{Z}[T]^d \) of \( d \) distinct nonconstant polynomials, a positive integer \( k \leq d \) and a real positive parameter \( \rho \), we consider the \( L^\rho \)-norms of the so called maximal operator

\[
M_{k,\rho}(\varphi, N) = \left\| \sup_{y \in \mathbb{T}^{d-k}} |S_\varphi(x, y; N)| \right\|_{L^\rho(\mathbb{T}_k)}
\]

\[
= \left( \int_{\mathbb{T}_k} \sup_{y \in \mathbb{T}^{d-k}} |S_\varphi(x, y; N)|^\rho \, dx \right)^{1/\rho}
\]

on the Weyl sums

\[
S_\varphi(x, y; N) = \sum_{n=1}^N e\left( \sum_{j=1}^k x_j \varphi_j(n) + \sum_{j=1}^{d-k} y_j \varphi_{k+j}(n) \right),
\]

where \( e(z) = \exp(2\pi iz) \) with two groups coefficients

\( x = (x_1, \ldots, x_k) \in \mathbb{T}_k \) and \( y = (y_1, \ldots, y_{d-k}) \in \mathbb{T}_{d-k} \),

where

\( \mathbb{T}_\nu = [0, 1]^\nu \)

is the \( \nu \)-dimensional unit cube.

Such bounds, as well as bounds on \( \sup_{y \in \mathbb{T}^{d-k}} |S_\varphi(x, y; N)| \) which hold for almost all \( x \in \mathbb{T}_k \) have recently been considered in a number of papers.
of works [2, 3, 6–9, 11, 12, 14, 20, 26]. Results of this kind add to our understanding of Weyl sums. Besides the interest to these results is ignited by applications outside of number theory, see [1–3, 8, 20, 23].

Here, to exhibit our idea in the clearest possible form, we consider

\[ \{ \varphi_1(T), \ldots, \varphi_d(T) \} = \{ T, \ldots, T^d \}. \]

We emphasise that in (1.1) we request the equality of sets rather than of sequences. Thus (1.1) means that

\[ \varphi_i(T) = T^{\pi(i)}, \quad i = 1, \ldots, d, \]

for some permutation \( \pi \in S_d \).

We also note that most of our results depend on the following parameters

\[ \tau_k(\varphi) = \sum_{j=1}^{k} \deg \varphi_j \quad \text{and} \quad \sigma_k(\varphi) = \sum_{j=k+1}^{d} \deg \varphi_j. \]

It is also convenient to define

\[ s(d) = \frac{d(d + 1)}{2}. \]

In the case \( k = 0 \) we trivially have \( M_{0, \rho}(\varphi, N) = N \). We also observe that the case of \( k = d \) corresponds to the Vinogradov Mean Value Theorem, which has recently been obtained in an optimal form by Bourgain, Demeter and Guth [10] and Wooley [27, 28]. Hence we are mostly interested in the case \( 1 \leq k < d \).

1.2. Notation. Throughout the paper, the notation \( U = O(V), \ U \ll V \) and \( V \gg U \) are equivalent to \( |U| \leq cV \) for some positive constant \( c \), which throughout the paper may depend on the degree \( d \) and occasionally on the small real positive parameter \( \varepsilon \) and the arbitrary real parameter \( t \).

For any quantity \( V > 1 \) we write \( U = V^{o(1)} \) (as \( V \to \infty \)) to indicate a function of \( V \) which satisfies \( V^{-\varepsilon} \ll |U| \ll V^{\varepsilon} \) for any \( \varepsilon > 0 \), provided \( V \) is large enough. One additional advantage of using \( V^{o(1)} \) is that it absorbs \( \log V \) and other similar quantities without changing the whole expression.

1.3. Previous results. Here we give a brief outline of previously known upper and lower bounds on \( M_{k, \rho}(\varphi, N) \). We recall our definitions (1.2) and (1.3).
The first result in this direction has been given by Chen and Shparlinski [11, Theorem 2.1], which implies that for any real positive $\rho \leq 2s(d) + d - k$,

\begin{equation}
M_{k,\rho}(\varphi, N) \leq N^\mu(k,\varphi) + o(1),
\end{equation}

where

$$\mu(k,\varphi) = \frac{s(d) + \sigma_k(\varphi) + d - k}{2s(d) + d - k} = 1 - \frac{\tau_k(\varphi)}{2s(d) + d - k}.$$

It is also shown in [11, Theorem 2.1] that in the special cases of the functions

$$\varphi_{2,1}(T) = (T^2, T)$$

we have

\begin{equation}
N^{1/2} \leq M_{1,2}(\varphi_{2,1}, N) \ll (N \log N)^{1/2},
\end{equation}

while for

$$\varphi_{3,2,1}(T) = (T^3, T^2, T)$$

we have

$$2^{1/4}N^{1/2} + O(N^{-1/2}) \leq M_{1,4}(\varphi_{3,2,1}, N) \ll N^{3/4}(\log N)^{1/4}.$$

Barron [7] has recently obtained the following estimate

\begin{equation}
M_{1,4}(\varphi_{1,2}, N) \leq N^{3/4 + o(1)}
\end{equation}

for

$$\varphi_{1,2}(T) = (T, T^2).$$

Using a different approach, Baker [6] has refined (1.6) with matching upper and lower bounds:

\begin{equation}
N^{a(\rho)}(\log N)^{b(\rho)} \ll M_{1,\rho}(\varphi_{1,2}, N) \ll N^{a(\rho)}(\log N)^{b(\rho)},
\end{equation}

where

$$a(\rho) = \begin{cases} 3/4 & \text{for } 1 \leq \rho \leq 4, \\ 1 - 1/\rho & \text{for } \rho > 4, \end{cases} \quad b(\rho) = \begin{cases} 1/4 & \text{for } \rho = 4, \\ 0 & \text{for } \rho \geq 1, \rho \neq 4. \end{cases}$$

Here we also consider the quadratic case, that is, the case of Gauss sums, in more detail. It is convenient to introduce the notations

$$G(x, y; N) = \sum_{n=1}^{N} e\left(xn + yn^2\right)$$
and

\[ K_\rho(N) = \left\| \sup_{y \in T} |G(x, y; N)| \right\|_{L^p(T)} = \left( \int_T \left| \sup_{y \in T} |G(x, y; N)|^p \right| dx \right)^{1/p}, \]

\[ L_\rho(N) = \left\| \sup_{x \in T} |G(x, y; N)| \right\|_{L^p(T)} = \left( \int_T \left| \sup_{x \in T} |G(x, y; N)|^p \right| dy \right)^{1/p}, \]

where

\[ T = T_1 = [0, 1]. \]

In particular, the bound (1.5) can now be written as

\[ N^{1/2} \leq L_2(N) \ll (N \log N)^{1/2} \]

and the bound (1.7) as

\[ N^{a(\rho)}(\log N)^{b(\rho)} \ll K_\rho(N) \ll N^{a(\rho)}(\log N)^{b(\rho)}. \]

We also consider norms of maximal operators along other straight lines.

1.4. New bounds. Here we combine the ideas from [6] and [11] and obtain new bounds which improve (1.4).

For large \( \rho \) we have the following upper and lower bounds of the same order of magnitude.

**Theorem 1.1.** Suppose that \( \varphi \in \mathbb{Z}[T]^d \) satisfies (1.1). For any real positive \( \rho \geq 2s(d) + d - k \) we have

\[ N^{1 - \tau_k(\varphi)/\rho} \ll M_{k, \rho}(a, \varphi, N) \leq N^{1 - \tau_k(\varphi)/\rho + o(1)}, \quad N \to \infty. \]

Note that by the convexity (that is, by the Hölder inequality), Theorem 1.1, taken with \( \rho = 2s(d) + d - k \), implies (1.4).

Our next result gives better bounds for small values of \( d \), namely for \( 3 \leq d \leq 6 \), and some choices of other parameters.

**Theorem 1.2.** Suppose that \( d \geq 3 \) and \( \varphi \in \mathbb{Z}[T]^d \) satisfies (1.1). For any real \( \rho > 0 \) we have

\[ M_{k, \rho}(a, \varphi, N) \leq N^{1 - 2^{-d} + o(1)} + N^{1 - \tau_k(\varphi)/\rho + o(1)}, \quad N \to \infty. \]

Elementary computation shows that Theorem 1.2 provides a better bound than (1.4) in the range \( \rho < 2^{d-1}\tau_k(\varphi) \) provided

\[ 2^{d-1}\tau_k(\varphi) < 2s(d) + d - k. \]

For large \( d \) this condition is never satisfied, however for each \( d \in \{3, 4, 5, 6\} \) we give examples of parameters when Theorem 1.2 improves (1.4).

For each \( 1 \leq k < d \) denote

\[ \varphi(T)|_k = \{\varphi_1(T), \ldots, \varphi_k(T)\}. \]
We now list all the possible choices of \(d, k\) and \(\varphi(T)|_k\) as in (1.1) such that Theorem 1.2 gives better bounds than (1.4).

For \(d = 3\), the condition (1.8) becomes \(\tau_k(\varphi) < (15 - k)/4\). Thus for \(k = 1\) it is sufficient to have \(\tau_1(\varphi) < 7/2\), which holds for all the possible choices that \(\varphi(T)|_1 \in \{T, T^2, T^3\}\). For \(k = 2\), it is sufficient to have \(\tau_2(\varphi) < 13/4\). Since \(\tau_2(\varphi) \neq 1, 2\), we obtain that \(\tau_2(\varphi) = 3\). Thus we have only one choice for \(\varphi(T)|_2\), that is \(\varphi(T)|_2 = \{T, T^2\}\).

For \(d = 4\), the condition (1.8) becomes \(\tau_k(\varphi) < (24 - k)/8\). Thus for \(k = 1\) it is sufficient to have \(\tau_k(\varphi) = 1\), which implies that \(\varphi(T)|_1 \in \{T, T^2\}\). For \(k = 2\), then it is sufficient to have \(\tau_2(\varphi) = 1\). But by our definition of \(\varphi\) we can not have \(\tau_2(\varphi) = 1\).

For \(d = 5\), the condition (1.8) becomes \(\tau_k(\varphi) < (35 - k)/16\). For \(k = 1\) it is sufficient to have \(\tau_k(\varphi) = 1\) which implies that \(\varphi(T)|_1 = T\). However, for other choices of \(k\) Theorem 1.2 does not yield a new bound.

We remark that for any \(d \geq 7\) and \(1 \leq k < d\), and any \(\varphi(T)|_k\) the bound (1.4) gives a better upper bound than Theorem 1.2.

For the maximal operators on Gauss sums we have the following result.

**Theorem 1.3.** For any real \(\rho > 0\) we have

\[
L_\rho(N) \leq N^{1/2 + o(1)} + N^{1 - 2/\rho + o(1)}, \quad N \to \infty.
\]

Note that if \(\rho \geq 4\) then the second term in the bound of Theorem 1.3 dominates and it becomes a special case of the optimal upper bound of Theorem 1.1. We also note that for any \(y \in \mathbb{T}\) we have

\[
\int_{\mathbb{T}} \left| \sum_{n=1}^{N} e(xn + yn^2) \right|^2 dx = N,
\]

thus \(\sup_{x \in \mathbb{T}} |G(x, y)| \geq N^{1/2}\), and hence

\[
\int_{\mathbb{T}} \sup_{x \in \mathbb{T}} |G(x, y)|^\rho dy \geq N^{\rho/2}.
\]

Therefore, we conclude that Theorem 1.3 is optimal for any \(\rho > 0\).

From Theorem 1.3 we derive the following bounds for the mean values of short sums of Gauss sums, which improves the bounds [11, Corollary 2.2] for this setting. For \(M \in \mathbb{Z}\), we consider Gauss sums over
short intervals, that is,

\[ G(x, y; M, N) = \sum_{n=M+1}^{M+N} e(xn + yn^2). \]

Elementary arithmetic shows that

\[ |G(x, y; M, N)| = |G(x + 2My, y; N)|, \]

and hence

\[ \sup_{M \in \mathbb{Z}} |G(x, y; M, N)| \leq \sup_{u \in T} |G(u, y; N)|. \]

**Corollary 1.4.** Using Theorem 1.3 and above notation, for any \( \rho > 0 \) we have

\[
\int_{T^2} \sup_{M \in \mathbb{Z}} |G(x, y; M, N)|^\rho dx dy \leq N^{\rho/2+o(1)} + N^{1-2/\rho+o(1)}.
\]

For any \( t \in \mathbb{R} \) the projection \( \pi_t : \mathbb{R}^2 \to \mathbb{R} \) is defined as

\[ \pi_t(x, y) = x + ty. \]

Let

\[ P_{t, \rho}(N) = \left( \int_{\mathbb{R}} \sup_{(x, y) \in \pi_t^{-1}(z) \cap T^2} |G(x, y; N)|^\rho \, dz \right)^{1/\rho}. \]

A similar quantity for much more general sums has been treated in [11, Theorem 2.3]. In the special case of Gauss sums we obtain a stronger result.

**Theorem 1.5.** For any \( t \in \mathbb{R} \) and any \( \rho > 0 \) we have

\[ P_{t, \rho}(N) \leq N^{5/6+o(1)} + N^{1-1/\rho+o(1)}, \quad N \to \infty. \]

We remark that Theorem 1.5 improves the bound \( N^{6/7+o(1)} \) for \( \rho \leq 7 \) which follows from the general estimate of [11, Theorem 2.3].

For a rational number \( t \) we have the following better bounds.

**Theorem 1.6.** For any \( t \in \mathbb{Q} \) and any \( \rho > 0 \) we have

\[ P_{t, \rho}(N) \leq N^{3/4+o(1)} + N^{1-1/\rho+o(1)}, \quad N \to \infty. \]

We observe that for \( t = 0 \) we have

\[ P_{0, \rho}(N) = M_{1, \rho}(\varphi_{1, 2}; N) = K_{\rho}(N). \]

Hence Theorem 1.6 implies a slightly less precise version of the upper bound in (1.7).
2. Bounds of Weyl sums

2.1. Approximations of Weyl sums. We recall a result of Vaughan [25, Theorem 7.2] approximating general Weyl sums

\[ S_d(u; N) = \sum_{n=1}^{N} e(u_1n + \ldots + u_d n^d) \]

by complete rational sums.

**Lemma 2.1.** Suppose that for integers \( q, r_1, \ldots, r_d \) with \( \gcd(q, r_1, \ldots, r_d) = 1 \) we have

\[ \left| u_j - \frac{r_j}{q} \right| \leq \xi_j, \quad j = 1, \ldots, d, \]

for some \( \xi_1, \ldots, \xi_d \in \mathbb{R} \). Then

\[ S_d(u; N) = q^{-1}S_d \left( q^{-1}r; q \right) l(\xi) + \Delta, \]

where \( r = (r_1, \ldots, r_d) \), \( \xi = (\xi_1, \ldots, \xi_d) \),

\[ l(\xi) = \int_{0}^{N} e \left( \xi_1 z + \ldots + \xi_d z^d \right) dz \]

and \( \Delta \) satisfies the bound

\[ \Delta \ll q \left( 1 + |\xi_1|N + \ldots + |\xi_d|N^d \right). \]

To apply Lemma 2.1 we need to recall well-known bounds for the complete rational sum \( S_d(q^{-1}r; q) \) and the oscillating integral \( l(\xi) \).

For \( l(\xi) \) by [25, Theorem 7.3] we have

\[ l(\xi) \ll N \min_{j=1,\ldots,d} \{ 1, \xi_j^{-1/d}N^{-j/d} \}. \]

Hence, we now concentrate on the sums \( S_d(q^{-1}r; q) \).

2.2. Bounds of complete rational sums. We denote

\[ S_{d,q}(b) = S_d \left( q^{-1}b; q \right) = \sum_{n=1}^{q} e_q(b_1n + \ldots + b_d n^d), \]

where \( e_q(k) = \exp(2\pi k/q) \).

If \( p \) is a prime number, then we have the classical Weil bound, see, for example, [22, Theorem 5.38].

**Lemma 2.2.** For a prime \( p \), and \( b \in \mathbb{Z}^d \) with

\[ \gcd(p, b_1, \ldots, b_d) = 1, \]

we have \( S_{d,p}(b) \leq (d - 1)\sqrt{p} \).
If \( q = p^m \) for some prime number \( p \) and integer \( m \geq 2 \) then we have the following bound, see, for example, [16, Equation (2.5)].

**Lemma 2.3.** For a prime \( p \), an integer \( m \geq 1 \) and \( b \in \mathbb{Z}^d \) with 
\[
\gcd(p, b_1, \ldots, b_d) = 1,
\]
we have \( S_{d,p^m}(b) \leq (d-1)p^{m-1} \).

The following estimate (see, for example, [18,24]) is a slight improvement of the classical bound which contains an additional factor \( q^{o(1)} \), see [25, Theorem 7.1]. It makes our calculation slightly less cluttered (but is not necessary for our final result).

**Lemma 2.4.** For an integer \( q \geq 1 \) and \( b \in \mathbb{Z}^d \) with 
\[
\gcd(q, b_1, \ldots, b_d) = 1,
\]
we have 
\[
S_{d,q}(b) \ll q^{1-1/d}.
\]

An integer number \( n \) is called
- \( r \)-th power free if any prime number \( p \mid n \) satisfies \( p^r \nmid n \);
- \( r \)-th power full if any prime number \( p \mid n \) satisfies \( p^r \mid n \).

We note that 1 is both \( r \)-th power free and \( r \)-th power full for any \( r \in \mathbb{N} \).

Our main tool is the following bound on \( |S_{d,q}(b)| \).

**Lemma 2.5.** Write an integer \( q \geq 1 \) as \( q = q_2 \cdots q_d \) with \( \gcd(q_i, q_j) = 1 \), \( 2 \leq i < j \leq d \), such that
- \( q_2 \geq 1 \) is cube free,
- \( q_i \) is \( i \)-th power full but \( (i+1) \)-th power free when \( 3 \leq i \leq d-1 \),
- \( q_d \) is \( d \)-th power full.

For \( b \in \mathbb{Z}^d \) with 
\[
\gcd(q, b_1, \ldots, b_d) = 1,
\]
we have 
\[
|S_{d,q}(b)| \leq \prod_{i=2}^{d} q_i^{1-1/i} q^{o(1)}.
\]

**Proof.** We factor \( q \) as in a product of distinct primes and prime squares as
\[
q = p_1^{m_1} \cdots p_s^{m_s},
\]
It is well known that the function \( S_{d,q}(b) \) is “multiplicative”, this follows by applying an argument similar to [21, Equation (12.21)] or [25,
Lemma 2.10]. It means that for any vector $b = (e_{1,j}, \ldots, e_{d,j})$ such that
\[ \gcd(p_j, e_{1,j}, \ldots, e_{d,j}) = 1, \]
for $i = 1, \ldots, s$, and
\[ S_{d,q}(b) = \prod_{j=1}^{s} S_{d,p_j^{m_j}}(e_j). \]
Applying Lemmas 2.2, 2.3 and 2.4, we obtain
\[ S_{d,p_j^{m_j}}(e_j) \leq (d-1)p_j^{-1} \prod_{i=1}^{d} q_i^{1-1/i}. \]
We now form
- $q_2$ as the product of powers $p_j^{m_j}$ with $m_j = 1, 2,$
- $q_i$, $3 \leq i \leq d-1$, as the product of powers $p_j^{m_j}$ with $m_j = i$,
- $q_d$, as the product of powers $p_j^{m_j}$ with $m_j \geq d$.
and
\[ |S_{d,q}(b)| \leq (d-1)^s \prod_{i=2}^{d} q_i^{1-1/i}. \]
Since obviously
\[ s! \leq \prod_{i=1}^{s} p_i \leq q, \]
we see that $(d-1)^s = q^{o(1)}$, which finishes the proof. \qed

2.3. Structure of large Weyl sums. The following combination of results of Baker [4, Theorem 3] and [5, Theorem 4] describes the structure of large Weyl sums.

Lemma 2.6. We fix $d \geq 2$, some $\varepsilon > 0$, and suppose that for a real
\[ A > N^{1-1/D+\varepsilon}, \]
where
\[ D = \min\{2^{d-1}, 2d(d-1)\}, \]
we have $|S_d(u; N)| \geq A$. Then there exist integers $q, r_1, \ldots, r_d$ such that
\[ 1 \leq q \leq (NA^{-1})^d N^{o(1)}, \quad \gcd(q, r_1, \ldots, r_d) = 1, \]
and
\[ \left| u_j - \frac{r_j}{q} \right| \leq q^{-1} (NA^{-1})^d N^{-j+o(1)}, \quad j = 1, \ldots, d. \]
We now use Lemma 2.6 to get a slightly more precise statement.
Lemma 2.7. We fix $d \geq 3$, some $\varepsilon > 0$, and suppose that for a real $A > N^{1-1/D+\varepsilon}$, where

$$D = \min\{2^{d-1}, 2d(d-1)\},$$

we have $|S_d(u; N)| \geq A$. Then there exist positive integers $q_2 \ldots q_d$ with $\gcd(q_i, q_j) = 1$, $2 \leq i < j \leq d$, such that

1. $q_2$ is cube free,
2. $q_i$ is $i$-th power full but $(i+1)$-th power free when $3 \leq i \leq d-1$,
3. $q_d$ is $d$-th power full,

and

$$\prod_{i=2}^{d} q_i^{\beta_j} \leq N^{1+o(1)} A^{-1}$$

and integers $b_1, \ldots, b_d$ with

$$\gcd(q_2 \ldots q_d, b_1, \ldots, b_d) = 1$$

such that

$$|u_j - \frac{b_j}{q_2 \ldots q_d}| \leq (NA^{-1})^d N^{-j+o(1)} \prod_{i=2}^{d} q_i^{-d/i}, \quad j = 1, \ldots, d.$$ 

Proof. By Lemma 2.6, there exist integers $q, r_1, \ldots, r_d$ such that

$$1 \leq q \leq (NA^{-1})^d N^{o(1)}, \quad \gcd(q, r_1, \ldots, r_d) = 1,$$

and

$$\beta_j = \left|u_j - \frac{r_j}{q}\right| \leq q^{-1} (NA^{-1})^d N^{-j+o(1)}, \quad j = 1, \ldots, d.$$ 

By Lemma 2.1 we have

$$S_d(u; N) = q^{-1} S_{d,q}(b) l(\beta) + \Delta,$$

where $S_{d,q}(b)$ is given by (2.2), and with

$$\Delta \ll q + (NA^{-1})^d N^{o(1)} \leq (NA^{-1})^d N^{o(1)}.$$ 

By the condition $A \geq N^{1-1/D+\varepsilon}$ and $d \geq 3$ we see from (2.4) that $|\Delta| \leq A/2$ provided that $N$ is large enough. Thus, it follows from (2.3) and the triangle inequality that

$$A/2 \leq |S_d(u; N)| - |\Delta| \leq q^{-1} |S_{d,q}(b)| |l(\beta)|.$$
We factorise \( q = q_2 \ldots q_d \) as in Lemma 2.5. Hence, by Lemma 2.5 we have
\[
|S_{d,q}(b)| \leq \prod_{i=2}^{d} q_i^{1-1/i} q^{o(1)}.
\]
Thus, recalling the bound (2.1), we derive from (2.5)
\[
A \leq N^{1+o(1)} \prod_{i=2}^{d} q_i^{1-1/i} \min_{j=1,\ldots,d} \{1, \beta_j^{-1/d} N^{-j/d}\}
\]
\[
= N^{1+o(1)} \prod_{i=2}^{d} q_i^{-1/i} \min_{j=1,\ldots,d} \{1, \beta_j^{-1/d} N^{-j/d}\}.
\]
In particular,
\[
A \leq N^{1+o(1)} \prod_{i=2}^{d} q_i^{-1/i},
\]
which implies the desired restriction on \( q_2, \ldots, q_d \).
Furthermore, for each \( j = 1, \ldots, d \), we have
\[
\beta_j \leq (NA^{-1})^d N^{-j+o(1)} \prod_{i=2}^{d} q_i^{-d/i},
\]
which finishes the proof. \( \square \)

2.4. Frequency of large Weyl sums. Let \( \lambda \) denote the Lebesgue measure on \( T_k \) (for an appropriate \( k \)). Let \( \mathbf{x} \in T_k \) we define \( y(\mathbf{x}) \) by
\[
|S_\mathbf{\varphi}(\mathbf{x}, y(\mathbf{x}); N)| = \sup_{y \in T_{d-k}} |S_\mathbf{\varphi}(\mathbf{x}, y; N)|
\]
(if there are several choices we fix any, for example, the lexicographically smallest).
For any \( A > 0 \) denote
\[
\lambda_{\varphi,k}(A; N) = \lambda \left( \{ \mathbf{x} \in T_k : |S_\mathbf{\varphi}(\mathbf{x}, y(\mathbf{x}); N)| \geq A \} \right).
\]
We start with recalling the bound
\[
\lambda_{\varphi,k}(A; N) \leq N^{s(d)+1+k} A^{-\tau_k(\varphi)+o(1)} A^{2s(d)-d+k}
\]
from [11, Lemma 3.2], which using (1.2) and (1.3) we write as follows.

Lemma 2.8. Let \( A \) be real number with \( 1 \leq A \leq N \). Then
\[
\lambda_{\varphi,k}(A; N) \leq N^{2s(d)+k-\gamma_k(\varphi)+o(1)} A^{-2s(d)-d+k}.
\]
We now show that sometimes we have better bounds. For any integer \( i \geq 2 \) it is convenient to denote 
\[
\mathcal{F}_i = \{ n \in \mathbb{N} : n \text{ is } i\text{-th power full} \}\quad \text{and} \quad \mathcal{F}_i(x) = \mathcal{F}_i \cap [1, x].
\]
The classical result of Erdős and Szekeres [19] gives an asymptotic formula for the cardinality of \( \mathcal{F}_i(x) \) which we present here in a very relaxed form as the upper bound
\[
(2.6) \quad \# \mathcal{F}_i(x) \ll x^{1/i}. 
\]

**Lemma 2.9.** Suppose that \( d \geq 3 \) and 
\[
A > N^{1-1/D+\varepsilon}
\]
for some fixed \( \varepsilon > 0 \), where 
\[
D = \min\{2^{d-1}, 2d(d-1)\}.
\]
Then we have 
\[
\lambda_{\varphi, k}(A; N) \leq N^{dk+1-\tau_k(\varphi)+o(1)} A^{-dk-1}.
\]

**Proof.** Denote 
\[
Q = (NA^{-1})^d.
\]
We also fix some \( \eta > 0 \). Let \( \mathcal{U}_{q_2, \ldots, q_d} \) be set of vectors \( \mathbf{u} \in \mathbb{T}_d \) with components satisfying the inequalities of Lemma 2.7, that is, 
\[
\mathcal{U}_{q_2, \ldots, q_d} = \left\{ \mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{T}_d : \right. \left. \left| u_j - \frac{b_j}{q_2 \cdots q_d} \right| \leq cQ N^{-\deg \varphi_j+\eta} \prod_{i=2}^{d} q_i^{-d/i}, \ j = 1, \ldots, k \right\}
\]
with some constant \( c > 0 \), which depends only on \( d \) and \( \eta \). Clearly, 
\[
(2.7) \quad \lambda(\mathcal{U}_{q_2, \ldots, q_d}) \ll (q_2 \cdots q_d)^k Q^k N^{-\tau_k(\varphi)+k\eta} \prod_{i=2}^{d} q_i^{-dk/i}.
\]
By Lemma 2.7 we obtain 
\[
(2.8) \quad \{ \mathbf{x} \in \mathbb{T}_k : |S_{\varphi}(\mathbf{x}, y(\mathbf{x}); N)| \geq A \} \subseteq \bigcup_{(q_2, \ldots, q_d) \in \Omega} \mathcal{U}_{q_2, \ldots, q_d},
\]
where, slightly relaxing the conditions of Lemma 2.7, for any \( \eta > 0 \) we can take 
\[
\Omega = \left\{ (q_2, \ldots, q_d) \in \mathbb{N}^{d-1} : q_i \in \mathcal{F}_i, \ 3 \leq i \leq d, \ \prod_{i=2}^{d} q_i^{1/i} \leq CQ^{1/d} N^{\eta} \right\}
\]
for some constant \( C > 0 \), which depends only on \( d \) and \( \eta \).
For $Z \geq 1$, we write $z \sim Z$ to denote that $Z/2 < z \leq Z$. We now fix some real numbers $Q_2, \ldots, Q_d$ and consider the measure $U(Q_2, \ldots, Q_d)$ of the contribution to the right hand side of (2.8) from $(q_2, \ldots, q_d) \in \Omega$ with $q_i \sim Q_i$, $2 \leq i \leq d$.

Covering $\Omega$ by $O((\log N)^d)$ dyadic boxes, we see that from (2.8) that

$$
\lambda_{\varphi,k}(A; N) \ll \max \left\{ U(Q_2, \ldots, Q_d) : Q_2, \ldots, Q_d \geq 1, \prod_{i=2}^{d} Q_i^{1/i} \leq CQ^{1/d}N^{\eta} \right\} (\log N)^d.
$$

Thus, using (2.7), we obtain

$$
U(Q_2, \ldots, Q_d) \leq \sum_{(q_2, \ldots, q_d) \in \Omega \atop q_i \sim Q_i, 2 \leq i \leq d} \lambda(\mathcal{U}_{q_2, \ldots, q_d})
\ll Q^k N^{-\tau_k(\varphi) + k\eta} \sum_{(q_2, \ldots, q_d) \in \Omega \atop q_i \sim Q_i, 2 \leq i \leq d} \prod_{i=2}^{d} q_i^{k-dk/i}.
$$

Recalling the definition of $\Omega$, we see that

$$
U(Q_2, \ldots, Q_d) \ll Q^k N^{-\tau_k(\varphi) + k\eta}
\sum_{q_2 \sim Q_2} q_2^{k-dk/2} \prod_{i=3}^{d} \sum_{q_i \sim Q_i} q_i^{k-dk/i}.
$$

(2.10)

Applying the bound (2.6), we derive from (2.10) that

$$
U(Q_2, \ldots, Q_d) \ll Q^k N^{-\tau_k(\varphi) + k\eta} \prod_{i=2}^{d} Q_i^{\alpha_i},
$$

(2.11)

where

$$
\alpha_2 = k - dk/2 + 1 \quad \text{and} \quad \alpha_i = k - (dk - 1)/i, \quad i = 3, \ldots, d.
$$

Observe that for $i = 2, \ldots, d$, we have

$$
\alpha_i \leq 1/i,
$$

(2.12)

which for $i \geq 3$ is obvious from

$$
\alpha_i = k - (dk - 1)/i = 1/i - (d/i - 1)k
$$

and for $i = 2$ from

$$
\alpha_2 = 1 - (d/2 - 1)k
$$
and \( d \geq 3 \).

Since \( Q_i \geq 1 \), using (2.12), under the condition on \( Q_2, \ldots, Q_d \) in (2.9) we derive

\[
\prod_{i=2}^{d} Q_i^{a_i} \leq \prod_{i=2}^{d} Q_i^{1/i} \ll Q^{1/d} N^\eta,
\]

which is achieved for the choice \( Q_2 = \ldots = Q_{d-1} = 1 \) and \( Q_d \sim Q N^{d \eta} \).

Combining (2.13) with (2.9) and (2.11), we obtain

\[
\lambda_{p, k}(A; N) \leq (NA^{-1})^{dk+1} N^{-\eta_k(p)+o(1)}
\]

(since \( \eta > 0 \) is arbitrary). This finishes the proof. \( \square \)

2.5. Quadratic Weyl sums. In this case we have the following analogue of Lemma 2.9.

**Lemma 2.10.** For any \( A \geq N^{1/2+\varepsilon} \) with some fixed \( \varepsilon > 0 \) we have

\[
\lambda \left( \left\{ y \in T : \sup_{x \in T} |G(x, y; N)| \geq A \right\} \right) \leq N^{2+o(1)} A^{-4}.
\]

**Proof.** We see from Lemma 2.6, that there is some \( Q = (NA^{-1})^2 N^{o(1)} \) such that if \( |G(x, y; N)| \geq A \) for some \( x \in T \), then for some \( q \leq Q \) the coefficients belong to one of at most \( q \) intervals corresponding to \( 1 \leq a_2 \leq q \) and the length of each interval is \( O(QN^{-2}q^{-1}) \). Therefore for some constant \( c > 0 \) depending only on \( \varepsilon \), we have

\[
\lambda \left( \left\{ y \in T : \sup_{x \in T} |G(x, y; N)| \geq A \right\} \right) \ll \sum_{1 \leq q \leq Q} q \frac{Q}{qN^2}
\]

\[
= QN^{-2} \sum_{1 \leq q \leq Q} 1 \leq Q^2 N^{-2} = N^{2+o(1)} A^{-4},
\]

which concludes the proof. \( \square \)

We fix some \( \varepsilon > 0 \). Denote \( Q = (NA^{-1})^2 N^\varepsilon \). Suppose that

\[
|G(x, y; N)| \geq A \geq N^{1/2+\varepsilon}.
\]

Then by Lemma 2.6 there exist \( q \leq Q \), \( (a_1, a_2) \in [q]^2 \) and constant \( c > 0 \) which depends on \( \varepsilon \) only such that

\[
(x, y) \in \mathcal{R}_{q, a_1, a_2},
\]

where the box

\[
\mathcal{R}_{q, a_1, a_2} = \left[ \frac{a_1}{q} - c \frac{Q}{qN}, \frac{a_1}{q} + c \frac{Q}{qN} \right] \times \left[ \frac{a_2}{q} - c \frac{Q}{qN^2}, \frac{a_2}{q} + c \frac{Q}{qN^2} \right].
\]
In what follows, for $m \in \mathbb{N}$ it is convenient to denote
$$[m] = \{0, 1, \ldots, m - 1\}.$$  

We note that the implied constant below may depend on the parameter $t$.

**Lemma 2.11.** For any $t \in \mathbb{R}$ and any $A \geq N^{1/2+\varepsilon}$ we have
$$\lambda\left(\left\{z \in \mathbb{R} : \sup_{(x,y) \in \pi_t^{-1}(z) \cap T_2} |G(x,y); N)| \geq A\right\}\right) \leq N^{5+o(1)} A^{-6}.$$  

**Proof.** Applying Lemma 2.6 and elementary geometry, we have
$$\left\{z \in \mathbb{R} : \sup_{(x,y) \in \pi_t^{-1}(z) \cap T_2} |G(x,y); N)| \geq A\right\} \subseteq \pi_t \left(\bigcup_{q \leq Q} \bigcup \mathcal{R}_{q,a_1,a_2}\right)$$  
$$= \bigcup_{q \leq Q} \pi_t \left(\bigcup (a_1,a_2) \in [q]^2 \mathcal{R}_{q,a_1,a_2}\right).$$  

Observe that for any integer $q \geq 1$ we have
$$\bigcup (a_1,a_2) \in [q]^2 \mathcal{R}_{q,a_1,a_2} = \{(a_1/q, a_2/q) : (a_1, a_2) \in [q]^2\} + \mathcal{R}_{q,0,0},$$  
where the summation symbol ‘+’ means the arithmetic (or Minkowski) sum, that is, for sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$, 
$$\mathcal{A} + \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}.$$  

Thus
$$\pi_t \left(\bigcup (a_1,a_2) \in [q]^2 \mathcal{R}_{q,a_1,a_2}\right) = \pi_t(q^{-1}([q] \times [q])) + \pi_t(\mathcal{R}_{q,0,0})$$  
$$= q^{-1}([q] + t[q]) + \pi_t(\mathcal{R}_{q,0,0}),$$  
where
$$[q] + t[q] = \{a_1 + ta_2 : (a_1, a_2) \in [q]^2\},$$  
and for a set $\mathcal{S} \subseteq \mathbb{R}$ and a scalar $\alpha \in \mathbb{R}$, we denote $\alpha \mathcal{S} = \{\alpha s : s \in \mathcal{S}\}$.  

Note that
$$\#([q] + t[q]) \ll q^2.$$  

(2.15)
It follows that
\[
\lambda \left( \pi_t \left( \bigcup_{(a_1, a_2) \in [q]^2} \mathcal{R}_{q, a_1, a_2} \right) \right) \\
\leq \# \left( q^{-1} ([q] + t[q]) \right) \lambda (\pi_t(\mathcal{R}_{q, 0, 0})) \ll qQ/N.
\]

Thus we see from (2.14) that
\[
\lambda \left( \left\{ z \in \mathbb{R} : \sup_{(x, y) \in \pi_t^{-1}(z) \cap T_2} |G(x, y); N| \geq A \right\} \right) \\
\ll Q^2/N = N^{5+o(1)} A^{-6},
\]
which finishes the proof.

We now turn our attention to the case \( t \in \mathbb{Q} \), and note that for this setting the bound (2.15) can be replaced by by \( q \) (the smallest possible upper bound). The geometric meaning of this is that there are overlaps of the projection \( \pi_t \) when \( t \in \mathbb{Q} \). We record this as the following elementary sumset estimate.

**Lemma 2.12.** For \( m \in \mathbb{N} \) denote \([m] = \{0, 1, 2, \ldots, m\}\). For \( t \in \mathbb{Q} \) and \( n \in \mathbb{N} \) we have
\[
\# ([n] + t[n]) \ll n.
\]

**Proof.** Suppose that \( t = a/b \) with \( a, b \in \mathbb{Z} \) and \( b \neq 0 \). Then we have
\[
\# ([n] + t[n]) = \# ([b[n] + a[n]] \leq (a + b)n + 1,
\]
which finishes the proof.

**Lemma 2.13.** For any \( t \in \mathbb{Q} \) and any \( A \geq N^{1/2+\varepsilon} \) we have
\[
\lambda \left( \left\{ z \in \mathbb{R} : \sup_{(x, y) \in \pi_t^{-1}(z)} |G(x, y); N| \geq A \right\} \right) \ll N^{3+o(1)} A^{-4}.
\]

**Proof.** Applying the similar arguments as in the proof of Lemma 2.11, but using the bound
\[
\# ([n] + t[n]) \ll q
\]
of Lemma 2.12, instead of (2.15), we obtain the desired bound.

3. **Proofs of main results**

3.1. **Preliminaries.** We recall the following rather elementary but useful general result, which is a slightly modified version of [11, Lemma 4.1].
Lemma 3.1. Let $\mathcal{X}$ be a metric space and $\nu$ be a Radon measure on $\mathcal{X}$ with $\nu(\mathcal{X}) < \infty$. Let $M \leq N$ be two positive numbers and $F : \mathcal{X} \to [0, N]$ be a function such that for any $M \leq A \leq N$,

$$\nu(\{x \in \mathcal{X} : F(x) \geq A\}) \leq N^a A^{-b}.$$ 

Then for any $\rho > 0$,

$$\int_{\mathcal{X}} F(x)^\rho d\nu(x) \ll \nu(\mathcal{X}) M^\rho + N^a M^{\rho-b} \log N + N^{a+b-b} \log N.$$

Proof. Taking a dyadic decomposition, we obtain

$$\{x \in T_k : M < F(x) \leq N\} = \bigcup_{i=1}^I \{x \in T_k : M 2^{i-1} < F(x) \leq 2^i M\},$$

where $I$ is the integer number such that $2^{I-1} M \leq N < 2^I M$. Thus we obtain

$$\int_{\mathcal{X}} F(x)^\rho d\nu(x) = \int_{\{x : F(x) \leq M\}} F(x)^\rho d\nu(x) + \int_{\{x : M < F(x) \leq N\}} F(x)^\rho d\nu(x)$$

$$\leq M^\rho + \sum_{i=1}^I (2^i M)^\rho \nu(\{x \in T_k : M 2^{i-1} < F(x) \leq 2^i M\})$$

$$\ll M^\rho + N^a M^{\rho-b} \sum_{i=1}^I 2^{i(\rho-b)}.$$ 

Considering the cases $\rho \leq b$ or $\rho > b$ separately, we obtain the desired bounds. \qed

3.2. Proof of Theorem 1.1. Let $a = s(d) + \sigma_k + d - k$, $b = 2s(d) + d - k$ and $M = N^{a/b}$. Applying Lemmas 2.8 and 3.1 we obtain

$$\int_{T_k} |S_\varphi(x, y(x); N)|^\rho dx \leq N^{\rho a/(b+o(1))} + N^{\rho a-b+o(1)} \leq N^{\rho a-b+o(1)}.$$ 

The second inequality holds since $\rho \geq b$. By our notation, we have

$$\rho + a - b = \rho - \tau_k(\varphi),$$ 

which gives the desired upper bound.

For the lower bound we note that for an appropriate constant $c > 0$, which depends only on $d$, if

$$0 \leq x_i \leq c N^{-\deg \varphi}, \quad i = 1, \ldots, k,$$
and \( y = 0 \) we have \(|S_\varphi(x, 0; N)| \gg N\). Hence,
\[
\int_{T_k} \sup_{y \in \mathbb{T}_{d-k}} |S_\varphi(x, y; N)|^\rho \, dx \\
\geq \int_0^{c_N^{-\deg \varphi_1}} \ldots \int_0^{c_N^{-\deg \varphi_k}} \sup_{y \in \mathbb{T}_{d-k}} |S_\varphi(x, y; N)|^\rho \, dx_1 \ldots dx_k \\
\geq \int_0^{c_N^{-\deg \varphi_1}} \ldots \int_0^{c_N^{-\deg \varphi_k}} |S_\varphi(x, 0; N)|^\rho \gg N^{\rho - \tau_k(\varphi)},
\]
which concludes the proof.

3.3. Proof of Theorem 1.2. Let fix some \( \varepsilon > 0 \) and set
\[
a = dk + 1 - \tau_k(\varphi), \quad b = dk + 1, \quad M = N^{1 - 1/2^{d-1} + \varepsilon}.
\]
We assume that \( \varepsilon > 0 \) is sufficiently small, so that \( M < N \).

Applying Lemmas 2.9 and 3.1 for
\[
I = \int_{T_k} |S_\varphi(x, y(x); N)|^\rho \, dx
\]
we obtain
\[
I \ll M^\rho + N^{\alpha + o(1)} M^{\rho - b} + N^{\rho + a - b + o(1)}.
\]

**Case 1.** Suppose that \( 0 < \rho \leq b \). Then \( M^{\rho - b} \gg N^{\rho + a - b} \) and we see that
\[
I \ll M^\rho + N^{\alpha + o(1)} M^{\rho - b}.
\]
Taking the values \( M, a, b \) and using the fact that for any \( d \geq 3 \),
\[
\tau_k(\varphi) \geq \frac{dk + 1}{2^{d-1}},
\]
we obtain \( M^\rho \gg N^a M^{\rho - b} \), and hence
\[
I \ll M^\rho = N^{\rho(1 - 1/2^{d-1} + \varepsilon)}.
\]

**Case 2.** Suppose that \( \rho > b \). Then \( N^a M^{\rho - b} < N^{\rho + a - b} \) and we see that
\[
I \ll M^\rho + N^{\rho + a - b + o(1)}.
\]
(i) If \( b < \rho \leq \tau_k(\varphi) 2^{d-1} \) then \( M^\rho \gg N^{\rho + a - b} \), and hence
\[
I \ll N^{\rho(1 - 1/2^{d-1} + \varepsilon + o(1))}.
\]
(ii) If \( \rho \geq \tau_k(\varphi) 2^{d-1} \) then we have
\[
I \ll N^{\rho + a - b + \varepsilon + o(1)} = N^{\rho - \tau_k(\varphi) + \varepsilon + o(1)}.
\]
Putting all cases together, since \( \varepsilon > 0 \) is arbitrary, we obtain the desired result.
3.4. **Proof of Theorem 1.3.** Let \( a = 2 + \varepsilon, \) \( b = 4 \) and \( M = N^{1/2+\varepsilon} \). By Lemma 2.10 and Lemma 3.1 we obtain
\[
\int_T \sup_{x \in T} |G(x, y; N)| dy \ll N^{\rho/2+\varepsilon} + N^{\rho-2+\varepsilon},
\]
since \( \varepsilon > 0 \) is arbitrary, which finishes the proof.

3.5. **Proof of Theorem 1.5.** Let \( a = 5 + \varepsilon, \) \( b = 6 \) and \( M \geq N^{1/2+\varepsilon} \) be some parameter which is to be determined later. By Lemmas 2.11 and 3.1 we obtain
\[
\int_{\mathbb{R}} \sup_{(x, y) \in \pi_2^{-1}(z) \cap T_2} |G(x, y; N)|^\rho dz \ll M^\rho + N^5 M^{\rho-6} + N^{\rho-1}.
\]
Hence, taking \( M = N^{5/6} \), since \( \varepsilon > 0 \) is arbitrary, we obtain the desired bound.

3.6. **Proof of Theorem 1.6.** Taking a similar argument as in the proof of Theorem 1.5 and using Lemma 2.13 instead of Lemma 2.11, with \( M = N^{3/4} \) we obtain the desired bound.

4. **Comments**

Perhaps one can use the description of large sums in Section 2.3 to get new estimates for the mean values
\[
\int_\Gamma |S_d(u; N)|^\rho d\mu(u) \quad \text{and} \quad \int_\mathbb{R} |S_d(u; N)|^\rho du
\]
for a smooth surface \( \Gamma \) with an attached Radon measure \( \mu \) on \( \Gamma \) and for a box \( \mathfrak{B} = [\xi_1, \xi_1+\delta] \times \ldots \times [\xi_d, \xi_d+\delta] \) for a small \( \delta \). Several results for such average values can be found in [13, 15, 17]. We hope that our approach can improve them in some ranges.

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