ON THE LAST NONZERO DIGITS OF $n!$ IN A GIVEN BASE

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Abstract. In this paper we study the sequence of strings of $k$ last nonzero digits of $n!$ in a given base $b$. We determine for which $b$ this sequence is automatic and show how to generate it using a uniform morphism. We also compute how often each possible string of $k$ digits appears as the $k$ last nonzero digits of $n!$.

1. Introduction

The sequence $\{\ell_b(n!}\}_{n \geq 0}$ of the last nonzero digit in the base-$b$ expansion of $n!$ has been an object of interest of several authors. For example, Dresden [7] showed that the number

$$\sum_{n=0}^{\infty} \frac{\ell_{10}(n!)}{10^n}$$

is transcendental. Deshouillers and Ruzsa [6] investigated the case $b = 12$ and computed that for $a \in \{1, \ldots, 11\}$ the asymptotic density of the set

$$\{n \geq 0 : \ell_{12}(n!) = a\}$$

is 1/2 if $a \in \{4, 8\}$ and 0 otherwise. Recall that by the asymptotic density of a set $A \subset \mathbb{N}$ we mean the limit

$$\lim_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n},$$

if it exists. The sequence $\{\ell_{12}(n!}\}_{n \geq 0}$ was further studied by Deshouillers [3, 4]. Deshouillers and Luca [5] proved that

$$|\{0 \leq n \leq x : n! \text{ is a sum of three squares}\}| = \frac{7}{8}x + O(x^{2/3}).$$

This can also be interpreted as a result concerning the last two nonzero digits of $n!$ in base 4. Indeed, by Legendre’s three-square theorem, $m$ is not a sum of three squares if and only if the string of two last nonzero digits of $m$ in base 4 is either 13 or 33. All of these results rely on showing that the respective sequences are automatic or coincide with an automatic sequence on a set of asymptotic density 1, as in the case $b = 12$. We recall the notion of automaticity in Section 3.

Motivated by these results, we will consider a general problem concerning the $k$ last nonzero digits of $n!$ in base $b$, where $k \geq 1$ is a fixed integer. For

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a positive integer \( m \) let \( \ell_{b,k}(m) \) be the number whose base-\( b \) expansion is given by \( k \) last nonzero digits of \( m \). We can add leading zeros if necessary. If \( k = 1 \), then for simplicity we will use the previous notation \( \ell_b(m) \). For a positive integer \( c \) (not necessarily prime) denote by \( \nu_c(m) \) the largest integer \( l \) such that \( c^l \) divides \( m \), and by \( m \mod c \) the residue of \( m \) modulo \( c \) lying in the set \( \{0, 1, \ldots, c - 1\} \). Then we have

\[
\ell_{b,k}(m) = \frac{m}{b^{\nu_b(m)}} \mod b^k.
\]

Obviously, \( \ell_{b,k}(m) \) is not divisible by \( b \) and satisfies \( 1 \leq \ell_{b,k}(m) \leq b^k - 1 \).

Our first aim is to study the automaticity of the sequence \( \{\ell_{b,k}(n!)\}_{n \geq 0} \) and find (in the automatic case) its explicit description in terms of uniform morphisms. Moreover, we would like to determine how often each \( a \in \{1, \ldots, b^k - 1\} \) appears in this sequence. More precisely, we are interested in computing the asymptotic density (if it exists) of the set

\[
\{ n \geq 0 : \ell_{b,k}(n!) = a \},
\]

which we will also call the asymptotic frequency of \( a \) in \( \{\ell_{b,k}(n!)\}_{n \geq 0} \) and denote by \( f_{b,k}(a) \). We could also rephrase the latter question and ask how often a fixed string of digits \( a_k \cdots a_1 \), with \( a_i \in \{0, \ldots, b - 1\} \) and \( a_1 \neq 0 \), appears as \( k \) last nonzero digits of \( n! \). In Section \( \ref{sec:main-results} \) we answer these questions in a general situation.

## 2. Main results

Consider the prime factorization of \( b \):

\[
b = p_1^{l_1} \cdots p_r^{l_r},
\]

where \( p_1, \ldots, p_r \) are distinct primes and \( l_1, \ldots, l_r \) are positive integers. If \( r \geq 2 \), then we can additionally renumber the primes so that the following technical conditions are satisfied:

(1) \[
l_1(p_1 - 1) \geq l_2(p_2 - 1) \geq \cdots \geq l_r(p_r - 1)
\]

and

(2) \[
p_1 = \max\{p_i : l_i(p_i - 1) = l_1(p_1 - 1)\}.
\]

It was recently proved by Lipka \( \cite{Lipka} \) that \( \{\ell_b(n!)\}_{n \geq 0} \) is automatic if and only if \( b = p_1^{l_1} \) or \( l_1(p_1 - 1) > l_2(p_2 - 1) \). Our first result shows that a similar claim remains true for the sequence \( \{\ell_{b,k}(n!)\}_{n \geq 0} \) describing the \( k \) last nonzero digits. Moreover, we show that if \( l_1(p_1 - 1) = l_2(p_2 - 1) \), then the sequence \( \{\ell_{b,k}(n!)\}_{n \geq 0} \) is “almost” automatic.

**Theorem 1.** If either \( b = p_1^{l_1} \) or \( l_1(p_1 - 1) > l_2(p_2 - 1) \), then the sequence \( \{\ell_{b,k}(n!)\}_{n \geq 0} \) is \( p_1 \)-automatic. If \( l_1(p_1 - 1) = l_2(p_2 - 1) \), then the sequence \( \{\ell_{b,k}(n!)\}_{n \geq 0} \) is not \( m \)-automatic for any \( m \). However, it coincides with a \( p_1 \)-automatic sequence on a set of asymptotic density 1.

At the end of Section \( \ref{sec:main-results} \) we give a fairly effective way to generate \( \{\ell_{b,k}(n!)\}_{n \geq 0} \) (or the corresponding \( p_1 \)-automatic sequence) using uniform morphisms.
The following theorem gives for each number \( a \in \{1, \ldots, b^k - 1\} \) the frequency \( f_{b,k}(a) \) of \( a \) in \( \{\ell_{b,k}(n!)\}_{n \geq 0} \). It turns out that these frequencies remain the same after restricting to subsequences along arithmetic progressions.

**Theorem 2.** Let \( a \in \{1, \ldots, b^k - 1\} \) be such that \( b \nmid a \). We have

\[
f_{b,k}(a) = \frac{1}{(p_1 - 1)p_1^{l_1(a) - k(l_1 + 1)}}
\]

if \( (b/p_1)^k \mid a \), otherwise \( f_{b,k}(a) = 0 \). Moreover, for any integers \( c \geq 1 \) and \( d \geq 0 \), the number \( a \) appears in the subsequence \( \{\ell_{b,k}((cn + d)!)\}_{n \geq 0} \) with asymptotic frequency \( f_{b,k}(a) \), regardless of \( c \) and \( d \).

Note that for \( b = 12, k = 1 \) we obtain the result of Deshouillers and Ruzsa [6], whereas for \( b = 4, k = 2 \) we get a slightly weaker version of the result of Deshouillers and Luca [5]. We can easily deduce a corollary concerning a special case of \( b \), for which all the nonzero frequencies are equal.

**Corollary 1.** Assume that \( l_1 = 1 \) and let \( a \in \{1, \ldots, b^k - 1\} \) be such that \( b \nmid a \). We have

\[
f_{b,k}(a) = \frac{1}{(p_1 - 1)p_1^{l_1(a) - 1}}
\]

if \( (b/p_1)^k \mid a \), otherwise \( f_{b,k}(a) = 0 \)

In the following example we examine the last nonzero digit of \( n! \) with \( b = 720 \).

**Example 1.** Consider the sequence \( \{\ell_{720}(n!)\}_{n \geq 0} \) of the last nonzero digit of \( n! \) in base \( b = 720 = 5 \cdot 3^2 \cdot 2^4 \).

We label the primes \( p_1 = 5, p_2 = 3, p_3 = 2 \) and the exponents \( l_1 = 1, l_2 = 2, l_3 = 4 \) accordingly, so that conditions (1) and (2) are satisfied. In fact, we have the equality

\[
l_1(p_1 - 1) = l_2(p_2 - 1) = l_3(p_3 - 1).
\]

By Theorem 2 the sequence \( \{\ell_{720}(n!)\}_{n \geq 0} \) is not automatic, however it coincides with a 5-automatic sequence on a set of asymptotic density 1. A computation similar as in Proposition 4 below shows that we can define such a 5-automatic sequence by \( \beta(0) = 576 \) and the recurrence relation

\[
\beta(5n + j) = \beta(n) \cdot j! \mod 720
\]

for all \( n \geq 0 \) and \( j = 0, 1, 2, 3, 4 \). The terms \( \beta(n) \) take values in the alphabet \( \Sigma = \{144, 288, 432, 576\} \). Define a 5-uniform morphism \( \varphi : \Sigma^* \to \Sigma^* \) by

\[
\varphi(x) = x_0x_1x_2x_3x_4,
\]

where \( x_j = x \cdot j! \mod 720 \). We can explicitly write

\[
\varphi(144) = 144 144 288 144 576, \quad \varphi(288) = 288 288 576 288 432,
\]

\[
\varphi(432) = 432 432 144 432 288, \quad \varphi(576) = 576 576 432 576 144.
\]
This morphism has exactly 4 fixed points. By (3) it is clear that \( \{ \beta(n) \}_{n \geq 0} \) is the fixed point of \( \varphi \) starting with 576, so it is a pure morphic sequence (does not require a coding).

We have \( \beta(n) = \ell_{720}(n!) \) on a set of asymptotic density 1, and hence the frequency of each symbol in both sequences is the same. In particular, if \( a \not\in \Sigma \), then \( f_{720}(a) = 0 \). To compute the frequencies of \( a \in \Sigma \), consider the \( 4 \times 4 \) incidence matrix \( M \) associated with \( \varphi \), which is of the form:

\[
M = \begin{bmatrix}
3 & 0 & 1 & 1 \\
1 & 3 & 1 & 0 \\
0 & 1 & 3 & 1 \\
1 & 1 & 0 & 3
\end{bmatrix},
\]

where the rows and columns correspond to the elements of \( \Sigma \) arranged in ascending order. All the entries of \( M^2 \) are positive, and therefore \( \varphi \) is primitive. The matrix \( \frac{1}{4} M \) is row-stochastic, thus by the discussion in Section 3 we have \( f_{720}(a) = 1/4 \) for \( a \in \Sigma \).

In general, bases \( b \) such that \( \{ \ell_{b,k}(n!) \}_{n \geq 0} \) is not automatic and the nonzero frequencies \( f_{b,k}(a) \) are not all equal, require considerably larger alphabets \( \Sigma \) (at least when using the approach presented in Section 4). One can check that the method of Proposition 4 applied to \( b = 144, k = 1 \) produces the "smallest" such example (in terms of \(|\Sigma|\)) with \(|\Sigma| = 48 \).

3. Preliminaries

We recall the definition of automatic sequences in terms of uniform morphisms. For a more detailed description see Chapters 4–6. Let \( \Sigma \) and \( \Delta \) be finite alphabets and denote by \( \Sigma^* \) and \( \Delta^* \) the sets of finite words created from letters in \( \Sigma \) and \( \Delta \), respectively, together with the empty word \( \varepsilon \). We call a map \( \varphi : \Sigma^* \to \Delta^* \) a morphism if \( \varphi(xy) = \varphi(x)\varphi(y) \) for all \( x, y \in \Sigma^* \). Clearly, a morphism is uniquely determined by its values on single letters in \( \Sigma \). We say that a morphism is \( l \)-uniform for an integer \( l \geq 1 \) if \( |\varphi(x)| = l \) for all \( x \in \Sigma \), where \( |y| \) denotes the length of a word \( y \). A 1-uniform morphism is called a coding. If \( \Sigma = \Delta \) then we denote by \( \varphi^i \) the \( i \)-th iterate of \( \varphi \) (with \( \varphi^0 \) being the identity morphism on \( \Sigma^* \)). A morphism \( \varphi \) is said to be prolongable on \( x \in \Sigma \) if \( \varphi(x) = xy \) for some \( y \in \Sigma^* \) and \( \varphi^i(y) \neq \varepsilon \) for all \( i \geq 0 \). If \( \varphi \) is prolongable on \( x \) then the sequence of words \( x, \varphi(x), \varphi^2(x), \ldots \) converges to the infinite word

\[
\varphi^\omega(x) = xy \varphi(y) \varphi^2(y) \cdots
\]

in the sense that each \( \varphi^i(x) \) is a prefix of \( \varphi^\omega(x) \) for \( i \geq 0 \). We can naturally extend \( \varphi \) to infinite words over \( \Sigma \). Then one can check that \( \varphi^\omega(x) \) is a fixed point of \( \varphi \), that is \( \varphi(\varphi^\omega(x)) = \varphi^\omega(x) \). Moreover, it is the unique fixed point of \( \varphi \) starting with \( x \). A sequence \( \{ \alpha(n) \}_{n \geq 0} \) with values in \( \Delta \), treated as an infinite word, is called a morphic sequence if \( \{ \alpha(n) \}_{n \geq 0} = \tau(\varphi^\omega(x)) \) for some morphism \( \varphi \) prolongable on \( x \) and a coding \( \tau : \Sigma \to \Delta \). We call \( \{ \alpha(n) \}_{n \geq 0} \) an \( l \)-automatic sequence if we can choose the morphism \( \varphi \) to be \( l \)-uniform. To prove that a sequence \( \{ \beta(n) \}_{n \geq 0} \) is a fixed point of an
l-uniform morphism \( \varphi \) it is necessary and sufficient that for all \( n \geq 0 \) we have

\[ \varphi(\beta(n)) = \beta(ln)\beta(ln+1)\cdots\beta((l+1)n - 1). \]

The asymptotic frequency of a letter in an automatic sequence does not always exist (see \cite{1} Example 8.1.2). To guarantee its existence it is enough to assume that the morphism \( \varphi \) generating the sequence is primitive, that is, there exists an integer \( i \geq 1 \) such that for all \( x, y \in \Sigma \) the letter \( y \) appears in \( \varphi^i(x) \). Below we give an overview of a general method of finding the frequencies of symbols in \( \varphi^\omega(x) \), where \( \varphi \) is prolongable on \( x \in \Sigma \). Once these frequencies are known, it is straightforward, given a coding \( \tau : \Sigma^* \rightarrow \Delta^* \), to compute the frequency of each \( a \in \Delta \) in the infinite word \( \tau(\varphi^\omega(x)) \). For more details and examples see \cite{1} Chapter 8.

Let \( \Sigma = \{a_1, \ldots, a_d\} \). For any word \( w \in \Sigma^* \) denote by \( |w|_{a_i} \) the number of occurrences of \( a_i \) in \( w \). We associate with the morphism \( \varphi \) the \( d \times d \) incidence matrix \( M = [m_{i,j}]_{1 \leq i, j \leq d} \), where \( m_{i,j} = |\varphi(a_j)|_{a_i} \). One can show that for any word \( w \in \Sigma^* \) and integer \( n \geq 1 \) we have

\[
\begin{bmatrix}
|\varphi^n(w)|_{a_1} \\
\vdots \\
|\varphi^n(w)|_{a_d}
\end{bmatrix} = M^n 
\begin{bmatrix}
|w|_{a_1} \\
\vdots \\
|w|_{a_d}
\end{bmatrix}.
\]

The task of finding frequencies of letters in \( \varphi^\omega(x) \) essentially boils down to studying the behavior of \( M^n \) as \( n \) tends to infinity. A nonnegative square matrix \( D \) is called primitive if there exists an integer \( n \geq 1 \) such that \( D^n \) has all entries positive. It is easy to see that \( \varphi \) is primitive if and only if its incidence matrix \( M \) is primitive. By \cite{1} Theorem 8.4.7, if \( \varphi \) is a primitive morphism, then the frequencies of all letters in \( \varphi^\omega(x) \) exist and are nonzero. Moreover, the vector of frequencies is the positive normalized eigenvector associated with the Perron–Frobenius eigenvalue of the incidence matrix \( M \) (where the \( i \)-th entry of the vector corresponds to \( a_i \)). If additionally \( M \) is a row-stochastic matrix multiplied by \( C > 0 \), then its Perron–Frobenius eigenvalue is equal to \( C \) and the frequencies of \( a_1, \ldots, a_d \) are all equal to \( 1/d \).

4. Proofs

Let us write the prime factorization of \( b \):

\[ b = p_1^{l_1} \cdots p_r^{l_r} \]

and denote \( q_i = p_i^{l_i} \) for \( i = 1, \ldots, r \). To begin with, in Lemmas \cite{1} and \cite{2} we show that if \( b \) has at least two prime factors, then with the choice of \( p_i \) satisfying \cite{1} and \cite{2}, the value \( \ell_{b,k}(n!) \) depends only on \( \ell_{q_1,k}(n!) \) and \( \nu_{q_i}(n!) \) for almost all \( n \geq 0 \).

Lemma 1. Let \( b \) have at least two distinct prime factors and let \( m \geq 1 \) be an integer. Assume that

\[ \nu_{q_1}(m) + k \leq \nu_{q_i}(m) \]

(4)
for all \( i = 2, \ldots, r \). Then
\[
\ell_{b,k}(m) = \ell_{q_1,k}(m)q^k l_{q_1}(m) - k \mod b^k,
\]
where \( q = b/q_1 \) and \( t \) is the multiplicative inverse of \( q \) modulo \( q_1^k \).

**Proof.** We have
\[
m = b^{\nu_b(m)}(b^k l + \ell_{b,k}(m)),
\]
where \( l \) is some nonnegative integer. By the assumption \([4]\) we have
\[
0 \leq \nu_{q_i}(b^k l + \ell_{b,k}(m)) \leq \nu_{q_i}(b^k l + \ell_{b,k}(m)) - k,
\]
for each \( i = 2, \ldots, r \), which implies that \( \nu_{q_i}(\ell_{b,k}(m)) \geq k \). Therefore, we obtain
\[
(7) \quad \ell_{b,k}(m) \equiv 0 \pmod{q^k}.
\]
Now let
\[
m = q_1^{\nu_{q_1}(m)}(q_1^k j + \ell_{q_1,k}(m)),
\]
where \( j \) is some nonnegative integer. Again, by \([4]\) we have \( \nu_b(m) = \nu_{q_1}(m) \). Comparing \([6]\) and \([8]\) we get
\[
q^{\nu_{q_1}(m)} \ell_{b,k}(m) \equiv \ell_{q_1,k}(m) \pmod{q_1^k},
\]
or, equivalently,
\[
(9) \quad \ell_{b,k}(m) \equiv \ell_{q_1,k}(m) \pmod{q_1^k}.
\]
By applying the Chinese Remainder Theorem to the system of congruences \([7]\) and \([9]\), we finally obtain the result. \( \square \)

Assume that the primes \( p_1, \ldots, p_r \) are numbered in such a way that conditions \([1]\) and \([2]\) are satisfied. Recall that for \( p \) prime and \( n \geq 0 \) we have Legendre’s formula:
\[
\nu_p(n!) = \frac{n - s_p(n)}{p - 1},
\]
where \( s_p(n) \) denotes the sum of digits of the base-\( p \) expansion of \( n \). The following lemma shows that for \( m = n! \) the assumption of Lemma \([1]\) is satisfied on a set of \( n \) of asymptotic density 1.

**Lemma 2.** If \( b \) has at least two distinct prime factors, then the set
\[
S_{b,k} = \{ n \geq 0 : \nu_{q_i}(n!) \geq \nu_{q_1}(n!) + k \text{ for } i = 2, \ldots, r \}
\]
has asymptotic density 1. If furthermore \( l_1(p_1 - 1) > l_2(p_2 - 1) \), then \( S_{b,k} \) contains all but a finite number of nonnegative integers.

**Proof.** First, consider the case \( l_1(p_1 - 1) > l_2(p_2 - 1) \). Fix any \( 2 \leq i \leq r \). By Legendre’s formula we obtain
\[
\lim_{n \to \infty} \frac{\nu_{q_i}(n!)}{\nu_{q_1}(n!)} = \frac{l_1(p_1 - 1)}{l_i(p_i - 1)} > 1,
\]
and therefore all \( n \) sufficiently large are in \( S_{b,k} \).
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Now consider the case $l_1(p_i - 1) = l_i(p_i - 1)$ for $i = 2, \ldots, s$. By Lemma 3 there exists $\delta_i > 0$ such that the set of nonnegative integers $n$ satisfying

$$s_{p_i}(n) \leq s_{p_i}(n) - \delta_i \log n$$

has asymptotic density 1. For such $n$ we obtain

$$\nu_{q_i}(n!) = \left\lfloor \frac{n - s_{p_i}(n)}{l_i(p_i - 1)} \right\rfloor \geq \left\lfloor \frac{n - s_{p_i}(n) + \delta_i \log n}{l_i(p_i - 1)} \right\rfloor \geq \nu_{q_i}(n!) + \left\lfloor \frac{\delta_i \log n}{l_1(p_1 - 1)} \right\rfloor.$$

If we additionally take $n$ sufficiently large, then $n \in S_{b,k}$. \hfill \Box

When $b$ is a prime power we put $S_{b,k} = \mathbb{N}$. Then the formula (5) is satisfied with $m = n!$ for any base $b \geq 2$ and all $n \in S_{b,k}$. We can further factorize the right side of (5) into simpler terms. It is easy to check that

$$\ell_{q_i,k}(m) = \ell_{p_i,k}(m)p_1^{\nu_{p_1}(m) \mod l_1 \mod q_i^k}.$$

If we additionally denote by $s$ the multiplicative order of $q$ modulo $q_i^k$, then the contribution of $\nu_{q_i}(m)$ to (5) depends only on its residue modulo $s$. Using these observations, we can rewrite (5) with $m = n!$ as

$$\ell_{b,k}(n!) = \ell_{p_1,k}(n!)p_1^{\nu_{p_1}(n!) \mod l_1 \mod q_i^k} q_i^{k l_1 + (\nu_{p_1}(n!)/l_1 \mod s) \mod b^k},$$

whenever $n \in S_{b,k}$.

We will now consider the family of sequences

$$\{\alpha_{p,k,u,v}(n)\}_{n \geq 0} = \{(\ell_{p,k}(n!), \nu_p(n!) \mod u, n \mod v)\}_{n \geq 0},$$

where $p$ is a prime number, $k, u \geq 1$ are integers and $v$ is a positive integer divisible by $\text{lcm}(p^{k-1}, u, 2)$. In the particular case $p = 2, k = 2$ we additionally require that $v$ is divisible by 4. The motivation for this approach is that for $p = p_1$, $k = l_1 k, u = l_1 s$ and $n \in S_{b,k}$ we can express $\ell_{b,k}(n!)$ by a coding of $\alpha_{p,k',u,v}(n)$, as seen in (10). The term $n \mod v$ does not appear in the coding itself, however it plays a vital role in the recurrence relations describing the terms $\alpha_{p,k,u,v}(n)$. The advantage of considering general $u$ and $v$, instead of immediately focusing on $\{\ell_{b,k}(n!}\}_{n \geq 0}$, is that it allows to study this sequence along any arithmetic progression.

Below we briefly outline the structure of our reasoning. We first prove in Proposition 1 that the sequence $\{\alpha_{p,k,u,v}(n)\}_{n \geq 0}$, treated as an infinite word, is a fixed point of a $p$-uniform morphism. Next, in Proposition 2 we show that this morphism is primitive. Proposition 3 uses the method described in Section 3 to prove that each of the possible values of $\alpha_{p,k,u,v}(n)$ appears with equal frequency. Combined with the relation between $\alpha_{p,k',u,v}(n)$ and $\ell_{b,k}(n!)$, this finally leads to the results of Section 2.

We start with deriving recurrence relations for $\alpha_{p,k,u,v}(n)$. For a fixed integer $c \geq 2$ we will use the notation

$$n_c! = \begin{cases} 1 & \text{if } n = 0, \\ \prod_{1 \leq m \leq n, \gcd(c,m) = 1} m & \text{if } n > 0. \end{cases}$$
This function is sometimes called the Gauss factorial (see [2] for some of its interesting properties). We begin with an auxiliary lemma.

Lemma 3. We have
\[(p^k)_p! \equiv \begin{cases} 1 \pmod{p^k} & \text{if } p = 2 \text{ and } k \neq 2, \\ -1 \pmod{p^k} & \text{otherwise.} \end{cases} \]

The proof is straightforward and relies on the fact that the product of all elements in a finite abelian group is equal to the product of the elements of order two. The following lemma gives a recurrence relation satisfied by the terms \(\ell_{p,k}(n!)\).

Lemma 4. For all \(n \geq 0\) and \(i = 0, \ldots, p - 1\) we have
\[
\ell_{p,k}((pn + i)!) = \ell_{p,k}(n!)(pn + i)_p! \pmod{p^k}.
\]

Proof. Obviously, (11) is true for \(n = 0\) and \(i = 0\). By induction on \(n\) we can compute that
\[
\ell_{p,k}((p(n + 1)!) \equiv \ell_{p,k}((pn)!)_p \ell_{p,k}(pn + 1) \prod_{i=1}^{p-1}(pn + i) \\
\equiv \ell_{p,k}(n!)(pn + 1)! \prod_{i=1}^{p-1}(pn + i) \\
\equiv \ell_{p,k}((n + 1)!)(pn + 1)! \pmod{p^k}.
\]
This ends the proof for \(i = 0\). Since none of the numbers \(pn + 1, \ldots, pn + p - 1\) is divisible by \(p\), the result for \(i > 0\) follows. 

We are now ready to show that the sequence \(\{\alpha_{p,k,u,v}(n)\}_{n \geq 0}\) is a fixed point of a \(p\)-uniform morphism. The terms \(\alpha_{p,k,u,v}(n)\) take values in the alphabet
\[
\Lambda_{p,k,u,v} = (\mathbb{Z}/p^k\mathbb{Z})^\times \times (\mathbb{Z}/u\mathbb{Z})^+ \times \{0, 1, \ldots, v - 1\},
\]
where \((\mathbb{Z}/p^k\mathbb{Z})^\times\) and \((\mathbb{Z}/u\mathbb{Z})^+\) denote the multiplicative group modulo \(p^k\) and the additive group modulo \(u\), respectively. It is convenient to treat the first two coordinates of \((x, y, z) \in \Lambda_{p,k,u,v}\) as elements of a group, as will be seen in the proof of Lemma 7. Define a \(p\)-uniform morphism \(\psi_{p,k,u,v} : \Lambda_{p,k,u,v}^* \to \Lambda_{p,k,u,v}^*\) as follows:
\[
\psi_{p,k,u,v}(x, y, z) = (x_0, y_0, z_0)(x_1, y_1, z_1) \cdots (x_{p-1}, y_{p-1}, z_{p-1}),
\]
where
\[
x_i = x(pz + i)_p! \pmod{p^k}, \\
y_i = y + z \pmod{u}, \\
z_i = pz + i \pmod{v}.
\]
We have the following:

Proposition 1. The sequence \(\{\alpha_{p,k,u,v}(n)\}_{n \geq 0}\) is the fixed point of \(\psi_{p,k,u,v}\) starting with \((1, 0, 0)\).
Proof. Take any \( n \geq 0 \) and let
\[
(x, y, z) = (\ell_{p,k}(n!), \nu_p(n!)) \mod u, n \mod v).
\]
We can write \( n = mv + z \) for some integer \( m \geq 0 \). Fix any \( 0 \leq i \leq p - 1 \).
By Lemmas 3 and 4 we have
\[
\ell_{p,k}((pn + i)!) \equiv \ell_{p,k}(n!)(pn + i)_p! \equiv x(pvm + pz + i)_p!
\equiv x((p^k)_p!)^{mv/p^{k-1}}(pz + i)_p! \equiv x_i \pmod{p^k}.
\]
Note that in the case \( p = 2, k = 2 \) we have to use the assumption that \( 4 | v \)
because \( 4! \equiv -1 \pmod{4} \). Furthermore, we have
\[
\nu_p((pn + i)! = \nu_p((pn)!)) = \frac{pn - sp_p((pn))}{p - 1} = \nu_p(n!) + n,
\]
and therefore \( \nu_p((pn + i)! \equiv y_i \pmod{u} \).
Obviously, \( pz + i \equiv z_i \pmod{v} \), which completes the proof. \( \square \)

Observe that if \((x', y', z')\) appears on the \( i \)-th position in \( \psi_{p,k,u,v}^m(x,y,z) \)
for some integer \( m \geq 1 \), then \((x''', y'' + y''', z')\) appears on the \( i \)-th position
in \( \psi_{p,k,u,v}^m(x'''', y + y'''', z) \). We will use this property a number of times. Our
aim is now to prove that each symbol from \( \Lambda_{p,k,u,v} \) appears in the sequence
\( \{\alpha_{p,k,u,v}(n)\}_{n \geq 0} \) with the same frequency. First, we give some auxiliary
lemmas.

**Lemma 5.** For any two symbols \((x,y,z),(x',y',z')\) \( \in \Lambda_{p,k,u,v} \) denote
\[
(x, y, z)R(x', y', z')
\]
if there exists an integer \( m \geq 0 \) such that \((x', y', z')\) appears in \( \psi_{p,k,u,v}^m(x,y,z) \).
Then \( R \) is an equivalence relation on \( \Lambda_{p,k,u,v} \).

**Proof.** The relation \( R \) is obviously transitive. We will now prove that it is symmetric.
Fix any \((x,y,z),(x',y',z')\) \( \in \Lambda_{p,k,u,v} \) such that \((x, y, z)R(x', y', z') \).
For any integer \( m \geq 0 \) the third coordinate of \( \psi_{p,k,u,v}^m(x', y', z') \) forms a
finite sequence of \( p^m \) consecutive integers taken modulo \( v \), so we have
\[
(x', y', z')R(xx'', y + y'', z) \text{ for some } (x'', y'') \in (\mathbb{Z}/p^k\mathbb{Z})^x \times (\mathbb{Z}/u\mathbb{Z})^y.
\]
If \((x'', y'') = (1, 0)\), then we are done. Otherwise, transitivity gives \((x, y, z)R(xx'', y + y'', z) \) and by using our earlier observation one can show inductively that
\[
(xx'', y + y'', z)R(x(x'')^n, y + ny'', z) \text{ for all } n \geq 2.
\]
If we choose \( n \) to be the order of \((x'', y'')\) in \((\mathbb{Z}/p^k\mathbb{Z})^x \times (\mathbb{Z}/u\mathbb{Z})^y\),
then by transitivity we get \((x', y', z')R(x, y, z) \). Hence, the relation \( R \) is an equivalence relation,
since each element of \( \Lambda_{p,k,u,v} \) is in relation with at least one element. \( \square \)

In other words, Lemma 5 says that the automaton associated with \( \psi_{p,k,u,v} \)
is a disjoint union of its strongly connected components. To prove the primitivity of \( \psi_{p,k,u,v} \),
we need to show that all the elements of \( \Lambda_{p,k,u,v} \) are related through \( R \).
The following lemma displays a useful identity concerning the terms \( \ell_{p,k}(n!) \).
Lemma 6. For any integers \(n,m \geq 0\) not divisible by \(p\), and integers \(s \geq 0, t \geq k + \lfloor \log_p m \rfloor\) we have
\[
\ell_{p,k}(p^{s+t}n - p^s m!) \equiv (p^s m!) \ell_{p,k}((p^s m)!)) \equiv (-1)^{p^m - 1} \ell_{p,k}((p^{s+t}n)!)) \pmod{p^k}.
\]

Proof. We can write
\[
\ell_{p,k}(p^{s+t}n)! \equiv \ell_{p,k}(p^{s+t}n - p^s m)!n \prod_{j=1}^{p^s m - 1} \ell_{p,k}(n p^{s+t} - j) \pmod{p^k}.
\]
The condition \(t \geq k + \lfloor \log_p m \rfloor\) guarantees that for each \(j = 1, \ldots, p^s m - 1\) we have
\[
\ell_{p,k}(n p^{s+t} - j) \equiv -\ell_{p,k}(j) \pmod{p^k}.
\]
Hence, multiplying both sides of (13) by \(m\), we obtain the desired result. \(\Box\)

In the following lemma we prove that \(\alpha_{p,k,u,v}(n)\) takes on all the possible values.

Lemma 7. Each symbol \((x,y,z)\in \Lambda_{p,k,u,v}\) appears in \(\psi_{p,k,u,v}(1,0,0)\).

Proof. Consider the set
\[
H = \{(x,y) \in (\mathbb{Z}/p^k \mathbb{Z})^\times \times (\mathbb{Z}/u \mathbb{Z})^+ : (x,y,0) \text{ appears in } \psi_{p,k,u,v}(1,0,0)\}.
\]
By Proposition [12] we have
\[
H = \{(\ell_{p,k}((vn)!), \nu_p((vn)!) \pmod{u}) : n \geq 0\}.
\]
First, we will show that \(H\) is a subgroup of \((\mathbb{Z}/p^k \mathbb{Z})^\times \times (\mathbb{Z}/u \mathbb{Z})^+\). Obviously, \((1,0) \in H\). Now if \((x,y), (x',y') \in H\) then \((1,0)R(x,y,0)\) and \((1,0)R(x',y',0)\). By a reasoning similar as in Lemma 5 we obtain \((x,y,0)R(x^{-1},-y,0)\) and therefore \((1,0,0)R(x^{-1},-y,0)\). This means that \((x',y',0)R(x'^{-1},y'-y,0)\), thus \((x'x^{-1},y'-y) \in H\).

The next step is to show that in fact \(H = (\mathbb{Z}/p^k \mathbb{Z})^\times \times (\mathbb{Z}/u \mathbb{Z})^+\). To do this we will prove that a set of generators of \((\mathbb{Z}/p^k \mathbb{Z})^\times \times (\mathbb{Z}/u \mathbb{Z})^+\) appears in \(H\). First, denote by \(H_1\) the projection of \(H\) on the first coordinate, which is a subgroup of \((\mathbb{Z}/p^k \mathbb{Z})^\times\). Consider the congruence (12) with \(s = \nu_p(v), m = v/p^s, t \geq k + \lfloor \log_p m \rfloor\), and let \(n\) run over the set \(\{xm : x \in (\mathbb{Z}/p^k \mathbb{Z})^\times\}\). Then \(pm\) is even and (12) takes the form
\[
\ell_{p,k}((p^t x v - v)!)) \ell_{p,k}(v!))x \equiv \ell_{p,k}((p^t x v)!)) \pmod{p^k}.
\]
We have \(\ell_{p,k}((p^t x v - v)!)), \ell_{p,k}(v!)), \ell_{p,k}((p^t x v)!)) \in H_1\), and therefore \(p^k - x \in H_1\) as well. It follows that \(H_1 = (\mathbb{Z}/p^k \mathbb{Z})^\times\).

Now fix \(x = 1\) in (14). By Lemmas 3 and 4 we get
\[
\ell_{p,k}((p^t v)!)) \equiv \ell_{p,k}(v!)) \prod_{i=1}^{t} (p^i v)_{p^i}! \equiv \ell_{p,k}(v!)) \pmod{p^k}.
\]
Here we used the assumption that \(v\) is divisible by \(\text{lcm}(p^{k-1},2)\). The congruence (14) further simplifies to the form
\[
\ell_{p,k}((v(p^t - 1))!)) \equiv -1 \pmod{p^k}.
for any integer $t \geq k + \lfloor \log_p m \rfloor$. We have
\[
\nu_p((v(p^{t+1} - 1))) - \nu_p((v(p^t - 1))) = \frac{p^t v(p - 1) - s_p(m(p^{t+1} - 1)) + s_p(m(p^t - 1))}{p - 1} = p^t v - 1,
\]
since for $t \geq 1 + \lfloor \log_p m \rfloor$ the base-$p$ expansion of $mp^{t+1} - m$ has exactly one more digit (equal to $p - 1$) than $mp^t - m$, with other digits unchanged. If we denote $y = \nu_p((v(p^t - 1)))$, then it follows that $(p^k - 1, y), (p^k - 1, y - 1) \in H$, and hence $(1, 1) \in H$. Combined with the earlier reasoning, this means that $H = (\mathbb{Z}/p^k\mathbb{Z})^\times \times (\mathbb{Z}/u\mathbb{Z})^\times$, as $H$ contains a set of generators of the latter group.

Now take any $(x, y, z) \in \Lambda_{p,k,u,v}$. Directly from the definition of the morphism $\psi_{p,k,u,v}$, there exist some $(x', y') \in (\mathbb{Z}/p^k\mathbb{Z})^\times \times (\mathbb{Z}/u\mathbb{Z})^\times$ such that $(x', y', 0) R(x, y, z)$ and the result follows. □

We are now ready to prove primitivity of $\psi_{p,k,u,v}$.

**Proposition 2.** The morphism $\psi_{p,k,u,v}$ is primitive.

**Proof.** By Lemma 7 we have $(1, 0, 0) R(x, y, z)$ for each $(x, y, z) \in \Lambda_{p,k,u,v}$, and thus all the elements of $\Lambda_{p,k,u,v}$ are related. In consequence, the incidence matrix $M = [m_{i,j}]_{1 \leq i,j \leq d}$ associated with $\psi_{p,k,u,v}$ is irreducible, i.e., for each $i, j$ there exists a positive integer $n$ such that $m_{i,j}^{(n)} > 0$, where $M^n = [m_{i,j}^{(n)}]_{1 \leq i,j \leq d}$. However, $M$ has a nonzero diagonal element corresponding to $(1, 0, 0) \in \Lambda_{p,k,u,v}$, so by a well-known fact it is primitive. □

Proposition 2 implies that the frequencies of all symbols $(x, y, z) \in \Lambda_{p,k,u,v}$ in the sequence $\{\alpha_{p,k,u,v}(n)\}_{n \geq 0}$ are positive. In the following proposition we show that they are in fact all equal.

**Proposition 3.** Each element $(x, y, z) \in \Lambda_{p,k,u,v}$ appears in $\{\alpha_{p,k,u,v}(n)\}_{n \geq 0}$ with frequency
\[
\frac{1}{p^{k-1}(p - 1)uv}.
\]
Equivalently, for any integers $c \geq 1$ and $0 \leq d \leq c - 1$ each element $(x, y) \in (\mathbb{Z}/p^k\mathbb{Z})^\times \times (\mathbb{Z}/u\mathbb{Z})^\times$ appears in $\{\ell_{p,k}((cn + d)!), \nu_p((cn + d)!) \mod u\}_{n \geq 0}$ with frequency
\[
\frac{1}{p^{k-1}(p - 1)u}.
\]

**Proof.** By the discussion in Section 3 and Proposition 2 it suffices to prove that the matrix $\frac{1}{p} M$ is row-stochastic. In other words, we need to show that each fixed $(x', y', z') \in \Lambda_{p,k,u,v}$ appears exactly $p$ times in the words $\psi_{p,k,u,v}(x, y, z)$ with $(x, y, z) \in \Lambda_{p,k,u,v}$.

If $(x', y', z')$ appears in $\psi_{p,k,u,v}(x, y, z)$, then $(x' x'', y' + y'', z')$ appears the same number of times in $\psi_{p,k,u,v}(x a'', y + y'', z)$ for any $(x'', y'') \in (\mathbb{Z}/p^k\mathbb{Z})^\times \times (\mathbb{Z}/u\mathbb{Z})^\times$. Therefore, for each fixed $z'$ the number of occurrences of $(x', y', z')$ in the words $\psi_{p,k,u,v}(x, y, z)$ with $(x, y, z) \in \Lambda_{p,k,u,v}$ does not depend on $x'$ and $y'$. 


Now we will show that it does not depend on $z'$ either. The symbol $(x', y', z')$ appears on the $i$-th position in $\psi_{p, k, u, v}(x, y, z)$ for some $(x, y, z) \in \Lambda_{p, k, u, v}$ if and only if the congruence
\begin{equation}
{pz + i \equiv z'} \pmod{v},
\end{equation}
is satisfied. Regardless of $z'$, the congruence (15) has exactly $p$ solutions $(z, i) \in \{0, 1, \ldots, v - 1\} \times \{0, 1, \ldots, p - 1\}$, which completes the proof. □

According to Proposition 3, we could informally say that for a random integer $n \geq 0$ the three coordinates of $\alpha_{p, k, u, v}(n)$ behave like independent uniformly distributed random variables.

We will now proceed to prove the results of Section 2. As in the beginning of this section, write
\[ b = p_1^{l_1} \cdots p_r^{l_r} \]
and assume that the conditions (1) and (2) are satisfied. Until the end of this section let $q_1 = p_1^{l_1}, q = b/q_1$ and denote by $t$ and $s$ the multiplicative inverse and the order of $q$ modulo $q_1^k$, respectively. We assign specific values to the parameters $p, k', u, v$ of the sequence $\{\alpha_{p, k', u, v}(n)\}_{n \geq 0}$. Put $u = l_1s$ and
\[ v = \begin{cases} \lcm(p_1^{l_1-1}, 2, u) & \text{if } b^k \neq 4, \\ 4 & \text{if either } b = 2, k = 2 \text{ or } b = 4, k = 1. \end{cases} \]

To appropriately describe $\{\ell_{b, k}(n!)\}_{n \geq 0}$, define
\begin{align*}
\Sigma_{b, k} & = \Lambda_{p_1 l_1 k, u, v}, \\
\varphi_{b, k} & = \psi_{p_1 l_1 k, u, v}.
\end{align*}

The terms $\ell_{b, k}(n!)$ take values in the alphabet
\[ \Delta_{b, k} = \{1 \leq a \leq b^k : b \nmid a\}. \]

We also define a coding $\tau_{b, k} : \Sigma_{b, k}^* \to \Delta_{b, k}^*$ in the following way:
\[ \tau_{b, k}(x, y, z) = xp_1^{(y \mod l_1)}qk^{s+([y/l_1] \mod s)} \mod b^k \]
and denote
\[ \{\beta_{b, k}(n)\}_{n \geq 0} = \tau_{b, k}(\varphi_{b, k}(1, 0, 0)). \]

The terms $\beta_{b, k}(n)$ take values in the alphabet
\[ \tilde{\Delta}_{b, k} = \{a \in \Delta_{b, k} : q^k|a\}. \]

The formula (10) shows that $\beta_{b, k}(n) = \ell_{b, k}(n!)$ for $n \in S_{b, k}$. We are now ready to prove Theorem 1.

**Proof of Theorem 1.** The sequence $\{\beta_{b, k}(n)\}_{n \geq 0}$ is $p_1$-automatic, which immediately gives the desired result when $b$ is a prime power. If $b$ has multiple prime factors and $l_1(p_1 - 1) > l_2(p_2 - 1)$, then Lemma 2 guarantees that $\beta_{b, k}(n) = \ell_{b, k}(n!)$ for all but a finite number of nonnegative integers. However, an $l$-automatic sequence remains $l$-automatic after changing a finite number of terms. Non-automaticity of $\{\ell_{b, k}(n!)\}_{n \geq 0}$ in the case $l_1(p_1 - 1) = l_2(p_2 - 1)$ follows immediately from the result of Lipka [8], since $\ell_{b}(n!)$ is a coding of $\ell_{b, k}(n!)$. □
In the case when \( \{\ell_{b,k}(n!}\}_{n \geq 0} \) is automatic, we use the description of \( \{\beta_{b,k}(n)\}_{n \geq 0} \) in order to express the former sequence as the image under a coding of a fixed point of a uniform morphism. Indeed, we can add finitely many extra letters to \( \Sigma_{b,k} \) to handle the terms \( \ell_{b,k}(n!) \neq \beta_{b,k}(n) \) and modify \( \varphi_{b,k} \) and \( \tau_{b,k} \) accordingly. In the following proof of Theorem 2 we rely on directly studying \( \beta_{b,k}(n) \) instead of \( \ell_{b,k}(n!) \).

**Proof of Theorem 2** The frequency of each \( a \in \Delta_{b,k} \) is the same in \( \{\ell_{b,k}(n!}\}_{n \geq 0} \) and \( \{\beta_{b,k}(n)\}_{n \geq 0} \), as these sequences agree on a set of \( n \) of density 1, thus we can focus on the latter sequence. Obviously, if \( a \notin \Delta_{b,k} \), then this frequency equals 0.

The value of \( \tau_{b,k}(x,y,z) \) does not depend on \( z \), so we can define an alphabet \( \Sigma'_{b,k} \) as the first two components of \( \Sigma_{b,k} \) and a coding \( \tau'_{b,k}(x,y) = \tau_{b,k}(x,y,z) \). In the case when \( q^k | a \), the equality \( \tau'_{b,k}(x,y) = a \) is equivalent to the system of congruences

\[
y \equiv \nu_{p_1}(a) \pmod{l_1},
\]

\[
xt \equiv a \pmod{q^k},
\]

which has exactly \( sp_1^{\nu_{p_1}(a)} \) solutions \( (x,y) \in \Sigma'_{b,k} \). The alphabet \( \Sigma'_{b,k} \) has exactly \( p_1^{l_1k-1}(p_1 - 1)/l_1s \) elements, therefore by Proposition 3 the symbol \( a \) appears in \( \{\beta_{b,k}(n)\}_{n \geq 0} \) and its subsequences \( \{\beta_{b,k}(cn+d)\}_{n \geq 0} \) with asymptotic frequency

\[
f_{b,k}(a) = \frac{1}{(p_1 - 1)/l_1} \frac{p_1^{\nu_{p_1}(a) - kl_1 + 1}}{
which ends the proof. \( \square \)

It is in fact possible to generate the sequence \( \{\beta_{b,k}(n)\}_{n \geq 0} \) using an alphabet smaller (in most cases) than \( \Sigma_{b,k} \). Let

\[
\widetilde{\Sigma}_{b,k} = \widetilde{\Delta}_{b,k} \times \{0, \ldots, v - 1\}
\]

and define a \( p_1 \)-uniform morphism \( \widetilde{\varphi}_{b,k} : \widetilde{\Sigma}_{b,k} \rightarrow \widetilde{\Sigma}_{b,k} \) by

\[
\widetilde{\varphi}_{b,k}(x,z) = (x_0, z_0) (x_1, z_1) \cdots (x_{p-1}, z_{p-1}),
\]

where for \( j = 0, 1, \ldots, p_1 - 1 \), we put

\[
x_j = xp_1^{(z \mod l_1) + \varepsilon(x,z)} \mod s(p_1z + j)p_1! \mod b^k,
\]

\[
z_j = p_1z + j \mod v,
\]

and

\[
\varepsilon(x,z) = \begin{cases} 
0 & \text{if } \nu_{p_1}(x) \mod l_1 + z \mod l_1 < l_1, \\
1 & \text{otherwise}.
\end{cases}
\]

In particular, if \( l_1 = 1 \), then \( x_j \) takes a simpler form

\[
x_j = xt^{\varepsilon(z) - 1} \mod b^k.
\]

Let \( \widetilde{\tau}_{b,k} : \widetilde{\Sigma}_{b,k} \rightarrow \widetilde{\Delta}_{b,k} \) be a coding defined by \( \widetilde{\tau}_{b,k}(x,y) = x \). The following proposition shows that the sequence \( \{\beta_{b,k}(n)\}_{n \geq 0} \) can also be described in terms of \( \widetilde{\varphi}_{b,k} \) and \( \widetilde{\tau}_{b,k} \).
Proposition 4. We have \( \{\beta_{b,k}(n)\}_{n \geq 0} = \tilde{\tau}_{b,k}(\tilde{\varphi}_{b,k}(q^k t^k, 0)) \).

Proof. It suffices to show that 
\[
\tilde{\varphi}_{b,k}(q^k t^k, 0) = \{ (\beta_{b,k}(n), n \mod v) \}_{n \geq 0}.
\]

By definition \( \beta_{b,k}(0) = q^k t^k \). Let \( n \geq 0 \) and denote \( (x, z) = (\beta_{b,k}(n), n \mod v) \). Using the recurrence relations in Lemma 4 and Proposition 1, we get for each \( j = 0, 1, \ldots, p_1 - 1 \):
\[
\beta_{b,k}(p_1 n + j) \equiv \ell_{p_1, j, k}((p_1 n + j)!|p_1^{q_{p_1}(p_1 n + j)!} \mod t_1 q^k t^{k + [(q_{p_1}(n! + n) / t_1] \mod s}) \mod t_1 q^k t^{k + [(q_{p_1}(n! + n) / t_1] \mod s}) \equiv \beta_{b,k}(n)(p_1 z + j), \quad p_1^x \mod t_1 t^{k l (z / t_1 + \varepsilon(x, z)) \mod s}
\]
\[
\equiv x_j \quad (\text{mod } b^k).
\]
The result follows immediately. \( \square \)

Observe that 
\[
|\tilde{\Sigma}_{b,k}| = q_1^{k - 1}(q_1 - 1)v \leq q_1^{k - 1}(q_1 - q_1 / p_1)uv = |\Sigma_{b,k}|
\]
with equality if and only if \( u = 1 \) or, equivalently, \( l_1 = s = 1 \). In other words, \( \tilde{\Sigma}_{b,k} \) contains roughly \( u \) times less elements than \( \Sigma_{b,k} \).

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