Phase-slip avalanches in the superflow of $^4$He through arrays of nanopores

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Recent experiments by Sato et al. have explored the dynamics of $^4$He superflow through an array of nanopores. These experiments have found that, as the temperature is lowered, phase-slippage in the pores changes its character, from synchronous to asynchronous. Inspired by these experiments, we construct a model to address the characteristics of phase-slippage in superflow through nanopore arrays. We focus on the low-temperature regime, in which the current-phase relation for a single pore is linear, and thermal fluctuations may be neglected. Our model incorporates two basic ingredients: (1) each pore has its own random value of critical velocity (due, e.g., to atomic-scale imperfections); and (2) an effective inter-pore coupling, mediated through the bulk superfluid. The inter-pore coupling tends to cause neighbours of a pore that has already phase-slipped also to phase-slip; this process may cascade, creating an avalanche of synchronously slipping phases. As the temperature is lowered, the distribution of critical velocities is expected to effectively broaden, owing to the reduction in the superfluid healing length, leading to a loss of synchronicity in phase-slippage. Furthermore, we find that competition between the strength of the disorder in the critical velocities and the strength of the inter-pore interaction leads to a phase transition between non-avalanching and avalanching regimes of phase-slippage.

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a measurable “whistle” at the Josephson frequency [26]. We remark that, in the setting of multi-link superconducting devices, it has been shown that the irreversible regime can be utilised for magnetic-field and related phase-sensitive measurements [27].

In this Paper, we address the issue of the transition from synchronous to asynchronous phase-slip dynamics of an array of nanopores connecting a pair of superfluid reservoirs. The main ingredients of our description are pores that have random, temperature-dependent critical velocities, along with an effective inter-pore coupling mediated via superflow in the reservoirs. We develop a model that incorporates these ingredients, and analyse it via both a mean-field approximation and exact numerical analysis for arrays consisting of a relatively small number of pores. Thus, we identify two effects: (a) strong disorder washes out the synchronicity of phase slips, which leads to the loss of the “whistle”; and (b) if the disorder is sufficiently weak, the phase-slip dynamics undergoes a disorder-driven phase transition between avalanching and non-avalanching regimes (see Fig. 2). This enables us to obtain the current-phase relation, as well as the amplitude of the current oscillations at the Josephson frequency corresponding to a fixed chemical-potential difference. We believe that this model and our analysis of it captures the essential physics taking place in the Berkeley group’s experiments [1].

Disparate physical systems featuring competition between quenched disorder and interactions, such as sliding tectonic plates [28], random-field magnets [29], and solids with disorder-pinned charge-density waves (CDWs) [30, 31], are well known to show phenomena analogous to the avalanching-to-non-avalanching transition described here. The model and analysis that we employ are similar to those that have been applied in the aforementioned settings [32].

Basic model—The system we wish to describe consists of two reservoirs of superfluid 4He, separated by a rigid barrier, embedded in which is an array of pores, as shown in Fig. 1. We shall specialise to the case of an $N \times N$ array of pores, each having radius $r_0$, situated at the sites of a square lattice of lattice parameter $\ell$. It is straightforward to extend our analysis to other array geometries. It is convenient to regard the system as comprising three components: two are bulk components [i.e., the left (L) and right (R) reservoirs]; the third consists of the superfluid inside the pores. We describe the state of the bulk helium in terms of the superfluid order-parameter phase fields $\phi^L, \phi^R$. In doing this we are neglecting effects of amplitude excitations of the order parameter, including vortices. In contrast, within the pores we retain both amplitude and phase degrees of freedom. We imagine controlling the system by specifying the two phases, $\phi^L, \phi^R$, to which the phases in the left and right reservoirs tend, far from the pore array. We believe that this level of description allows us to capture the following important elements: (a) pores that exhibit narrow-wire-like meta-stable states, these states being connected by phases slips; and (b) interactions mediated through the bulk superfluid in the two reservoirs, which couple pairs of pores to one another and also couple the pores to the control phases, $\phi^L, \phi^R$.

To summarise, we describe the bulk superfluid helium reservoirs by the phase-only Hamiltonians

$$H^{L/R} = \frac{K_s}{2} \int_{L/R} d^3r \left| \nabla \phi^{L/R}(r) \right|^2,$$

where $K_s \equiv \hbar^2 n_s/m$ is the superfluid stiffness, in which

![FIG. 1: Schematic of the model system.](image1)

**FIG. 1:** Schematic of the model system. Left: the location of the pore array on the membrane is indicated by the black region, and the phases on hemispheres at infinity are labelled $\phi^L$ and $\phi^R$. Centre: slice through the membrane, with pores being represented by breaks in the membrane (white). Right: boundary conditions on hemispheres at the openings of the $i$-th pore.

![FIG. 2: Phase diagram showing avalanching and non-avalanching regimes of the phase-slip dynamics, as a function of the effective pore strength $J/2\pi r_0$ and disorder strength $w$. The diagram was computed via our mean field theory, in the large-array limit, with top-hat critical phase twist (or, equivalently, the velocity) distribution of width $w$. Inset: The fraction of pores that have phase slipped, as a function of the control-parameter $(\phi^L - \phi^R)/2\pi$. Comparison between numerics for a $25 \times 25$ array of pores (points) and our mean-field theory (lines) with Gaussian distribution of critical velocities.](image2)
$n_s$ is the superfluid number density and $m$ is the mass of a $^4$He atom. To account for phase-slippage processes within a pore, which arise from vortex lines crossing the pore, we shall use a modified phase-only model that keeps track of the number of phase slips. Therefore, we take the energy of the superfluid inside the $i^{th}$ pore to be

$$H_i = \frac{K_s}{2} J \left( \phi_i^L - \phi_i^R - 2\pi n_i \right)^2,$$  

(2)

in which $J \equiv \pi r_0^2/d$ accounts for the geometry of the pore, where $d$ is of the order of the membrane thickness, and $\phi_i^L/R$ is the phase of the order parameter in the vicinity of the left/right side of the $i^{th}$ pore. Furthermore, $n_i$ counts the net number of phase slips that would occur in the $i^{th}$ pore if the system were to progress from a reference state in which the phase were uniform throughout. This description of the pores must be supplemented by specifying the critical velocities at which phase slips occur, as we shall discuss in detail below.

To reduce the model to one involving only the phase-differences across the pores and the phases at infinity, we minimise the energy $H$ in the reservoirs, which forces $\chi^L/R(r)$ to obey the Laplace equation. Focusing on the left reservoir, the appropriate boundary conditions are (see Fig. 1): (a) the phase on the hemisphere at infinity is $\Phi^L$; (b) no current flows through the membrane surface between the pores, i.e. $\nabla \cdot \chi^L = 0$ there; and (c) the phase on the hemisphere of radius $r_0$ centred at the opening of the $i^{th}$ pore is specified to be $\phi_i^L$. The choice of surface for the last of the boundary conditions is made for convenience, as it simplifies the resulting mixed boundary value problem whilst enforcing the physical condition of continuity of the phase in the vicinity of the pore opening.

To solve this mixed boundary value problem we appeal to its electrostatics analogy, in which the phase and the superfluid stiffness $K_s$ respectively play the roles of the scalar potential and the permittivity $\varepsilon_0$. To compute the energy at the minimum in terms of the boundary data we apply the divergence theorem to Eq. (1) to obtain the energy in terms of a surface integral:

$$H^L = \frac{K_s}{2} \int_{\partial L} dS \cdot \chi^L \nabla \chi^L - \frac{K_s}{2} \int_{L} d^3\tau \chi^L \nabla^2 \chi^L,$$  

(3)

where the volume (i.e. last) term vanishes because at the minimum-energy configuration $\chi^L$ obeys $\nabla^2 \chi^L = 0$. To simplify the evaluation of the surface integral we make use of the fact that the energy is invariant under global shifts of the potential. Thus, we lower all potentials by $\Phi^L$, which eliminates the contribution from the (left) surface at infinity. What remains are the contributions from the membrane: those from between the pores give zero, owing to the zero-flux boundary condition; those from those from the hemispherical surfaces covering the pores give

$$H^L = \frac{1}{4} \sum_i (\phi_i^L - \Phi^L) q_i,$$  

(4)

where, by assumption, the phase $\chi^L$ takes the constant values $\phi_i^L$ on the hemispherical surfaces, and the remaining surface integrals have been replaced (via the analogue of Gauss’s law) by half of the (analogue of) the charge enclosed. For pore $i$ this charge is $q_i \equiv K_s \int dS \cdot \nabla \chi^L$, where the integral is taken over the spherical surface that completes the hemispherical one. Reflecting the problem across the membrane’s left surface, we see that we are looking for half of the electrostatic energy of a set of suitably charged metal spheres at potentials $\phi_i^L - \Phi^L$, and what remains is to determine the charges $q_i$. To do this, we consider the $i^{th}$ sphere: in the limit $\ell \gg r_0$ (i.e. ignoring di- and higher-order charge-multipoles), the potential on its surface obeys

$$K_s \left( \phi_i^L - \Phi^L \right) = \sum_j C_{ij}^{-1} q_j,$$  

(5a)

$$C_{ij}^{-1} = \frac{\delta_{ij}}{4\pi r_0} + \frac{1 - \delta_{ij}}{4\pi |r_{ij}|},$$  

(5b)

where $|r_{ij}|$ is the distance between the $i^{th}$ and $j^{th}$ pores (sphere centres), and $C_{ij}$ is the analogue of the capacitance matrix. By solving Eq. (5a) for the $q_i$’s and eliminating them from Eq. (1), we arrive at the combined energy of the left and right superfluid reservoirs and the superfluid in the pores:

$$E = \frac{K_s}{2} \sum_{ij} (\phi_i^L - \Phi^L) C_{ij} (\phi_j^L - \Phi^L) + \sum_i H_i,$$  

(6)

where we have, without loss of generality, restricted our attention to spatially anti-symmetric states, for which $\Phi^R = -\Phi^L$ and $\phi_i^L = -\phi_i^R$.

We complete the description of our model by specifying the single-pore dynamics, and thus the mechanism by which energy is dissipated in the pores. The superfluid velocity $v_i$ in a pore of thickness $d$ is defined by the phases at the pore openings: $v_i = h\nabla \phi_i/m \approx h (2\phi_i^L - 2\pi n_i)/m$. Correspondingly the current through the pore is given by

$$I_i = \frac{K_s J}{h} (2\phi_i^L - 2\pi n_i).$$  

(7)

When the velocity through the $i^{th}$ pore exceeds its critical value $v_{c,i}$ (or, equivalently, $\phi_i^L - \pi n_i$ exceeds $\phi_{c,i}$), a vortex line nucleates and moves across the pore, which decreases the phase-difference across the pore by $2\pi$. Thus, immediately after a phase slip, the velocity through the $i^{th}$ pore is decreased by $\Delta v \equiv 2\pi h/m d$, a quantum of superfluid velocity drop for a pore having fixed phases at the openings, i.e. $v_i \to v_i - \Delta v$. However, after a very short time, controlled by the speed of sound, the system balances the superflow in the bulk reservoirs and through the various pores. Therefore, after this relaxation process is complete, the actual drop in the superfluid velocity through the $i^{th}$ pore is always less than $\Delta v$.
the left hemispherical surface at infinity through the \( i \)th pore to the right hemispherical surface at infinity drops by \( 2\pi \), whilst the phase-difference along a path through any other pore remains unaffected. In our model, we implement this kind of phase-slip event by sending \( n_i \) to \( n_i + 1 \) (assuming all flow is to the left) and finding a new set of values for all of the \( \phi_i \)'s by minimising the total energy, Eq. (9).

Not too near \( T_\lambda \) the experimentally-observed temperature-dependence of the critical velocity for superflow through a pore is approximately linear \([33, 34]\):

\[
v_c(T) = v_c(0)(1 - T/T_0), \quad \text{where } T_0 \approx 2.5 K.
\]

This \( T \)-dependence is most closely reproduced by the thermal nucleation of half-ring vortices mechanism \([35]\). However, the linear \( T \)-dependence does not hold in the temperature regime that we are primarily interested in; instead, the critical velocity is proportional to the superfluid stiffness there \([24, 25, 36]\):

\[
\xi(T) \approx \xi_0(1 - T/T_\lambda)^{-2/3} \text{ is the temperature-dependent healing length.}
\]

Extrinsic effects are known to reduce critical velocities from the intrinsic values discussed above. We hypothesise that in the Berkeley group’s experiments the extrinsic effects originate in atomic-scale roughness of the pore walls, and play a pivotal role in generating critical-velocity variability amongst the pores. This variability is expected to be temperature dependent because only roughness on length-scales longer than \( \xi(T) \) can substantially perturb the order parameter and thus influence the critical velocities. Hence, at higher temperatures the impact of surface roughness is expected to be weaker and, correspondingly, the distribution of critical velocities is expected to be narrower. Thus, lowering the temperature has the important effective consequence of increasing the effective disorder \([37]\).

**Implications of the model** — We shall work at fixed (negative) difference \( \Delta \mu \) in the chemical potential between the reservoirs, so that the control parameter \( \Phi \) evolves linearly in time, according to the Josephson-Anderson relation

\[
\Phi(t) = -\Phi(t) = -\frac{\Delta \mu}{2\hbar} t.
\]

As \( \Phi - \Phi_R \) grows, so do the superfluid velocities through the various pores, punctuated at regular intervals by velocity drops associated with the phase-slip processes. As the total energy of the state is periodic in \( \Phi \) with period \( \pi \), the total current through the array must be a periodic function of time with the period given by the Josephson frequency \( \omega_J = \Delta \mu / \hbar \). Due to the randomness of the critical velocities amongst the pores, the velocities in the various pores do not reach their critical values simultaneously. Therefore, the pores having the smaller critical velocities (i.e. the weaker pores) tend to slip first. If the distribution of critical velocities is sufficiently narrow, the array may, as we demonstrate below, suffer an avalanche.

By an avalanche we mean that when the weaker pores slip, superflow through the neighbouring pores that have yet to slip increases, due to the inter-pore interaction, and this drives them to their own \( v_c \)'s, causing a cascade of phase slips in which an appreciable fraction of pores in the array slip. Experimentally, avalanches are reflected in a periodic series of sharp drops in the total current through the array of pores as a function of time. Time-traces of the total current in the avalanching and non-avalanching regimes are contrasted in Fig. 3.

For arrays having a small number of pores, the quasi-static state of (mechanical) equilibrium may be numerically determined, as the control parameter \( \Phi^L - \Phi^R \) evolves parametrically. As a consequence of the long-range nature of the inter-pore couplings, the array dynamics is well approximated by a mean-field theory. We shall describe a numerical approach first, and then the mean-field theory.

The numerics take as input: \( J \), the \( v_c \)'s, and the (non-inverted) matrix \( C_{ij}^{-1} \) [see Eq. (5)], which itself depends on \( r_0, l, \) and \( N \). At each time-step, \( \Phi^L \) is incremented, and the new \( \phi^L \)'s are obtained. If the superflow through any pore is found to now exceed its critical velocity, that pore phase-slips (i.e. its value of \( n_i \) is incremented by plus unity); next, the various \( \phi^L_i \)'s are recomputed using the new set of \( n_i \)'s, and the program goes back to recheck if any other pore now exceeds its critical velocity. This continues until no new phase-slips are found to occur, at which point the control parameter is incremented and the procedure is repeated.

Next, we construct a mean-field theory. We assert that

![FIG. 3: Traces of total current through an array of pores as a function of time, at various disorder strengths.](image-url)
the control phase-difference is monotonically increasing in time, so that phase differences $\phi^L_i - \phi^R_i$ are always increasing and the superflow in the pores undergoes only $n_i$-increasing phase slips. Then we may proceed by selecting an arbitrary pore $i$, and minimising the energy $\Theta$ with respect to $\phi^L_i$; by replacing the $\phi^L_{ji}$'s by the mean-field value $\langle \phi^L \rangle$ we obtain an equation for the phase at the $i$th pore:

$$\phi^L_i (C + 2J) - B \langle \phi^L \rangle = A \Phi^L + 2\pi J n_i, \quad (10)$$

where $A \equiv \sum_i C_{ii}$, $B \equiv C_{ii} - A$, and $C \equiv C_{ij}$ (The necessary inversion of $C_{ij}^{-1}$ can readily be accomplished either analytically, by transforming to Fourier space, or numerically.) By averaging the left- and right-hand sides of this equation over sites, we arrive at a self-consistency condition on $\langle \phi^L \rangle$. Next, by assuming self-averaging with respect to the disorder in the critical velocities, we may replace the average over sites by an average over disorder. This procedure amounts to replacing, in the above condition, $\phi^L_i$ by $\langle \phi^L \rangle$ and $n_i$ by

$$\langle n \rangle (\langle \phi^L \rangle) = \sum_k k \int_0^\infty d\phi_c Q(\phi_c) \left[ \Theta \left( \phi_c - \frac{A \Phi^L + B \langle \phi^L \rangle - \pi k C}{C + 2J} \right) - \Theta \left( \phi_c - \frac{A \Phi^L + B \langle \phi^L \rangle - \pi (k-1) C}{C + 2J} \right) \right], \quad (11)$$

where $Q(\phi_c)$ is the probability distribution of half-critical phase twists in the pores $\phi_c \equiv d\nu_c / \hbar$. The averaged form of Eq. (11) can be solved graphically, by plotting the left- and right-hand sides as functions of $\langle \phi^L \rangle$; see Fig. 4. It is evident from this graphical approach that whenever the maximum slope of the right-hand side, $2\pi J B \langle n \rangle / \langle \phi^L \rangle$, fails to exceed the slope of the left-hand side, $(A + 2J) \langle \phi^L \rangle$, the self-consistency condition yields a unique solution for the average phase $\langle \phi^L \rangle$, and that this phase evolves continuously with the (increasing) control phase $\Phi^L$. This corresponds to the non-avalanching regime. By contrast, whenever the maximum slope of the right-hand side does exceed the slope of the left-hand side, the self-consistency condition no longer yields a unique solution for $\langle \phi^L \rangle$. Instead, as the control-phase increases, the continuous evolution of $\langle \phi^L \rangle$ is punctuated by jumps, which occur when pairs of solutions merge and disappear. These jumps reflect avalanching behaviour, and we refer to this as the avalanching regime.

One can use this mean-field theory to construct a phase diagram that demarcates avalanching and non-avalanching regimes, for any choice of disorder distribution $Q$. For the case of a top-hat distribution of critical phase-twists, a simple inequality defines the avalanching regime:

$$w \leq w_c \equiv \frac{2\pi J B}{(A + 2J)(C + 2J)}, \quad (12)$$

where $w$ is the width of the top hat, and $w_c$ is its critical value; such a phase boundary is exemplified in Fig. 2. In experiments, one can explore this phase diagram by tuning the temperature, which, as we have hypothesised above, effectively tunes the strength of the disorder.

Thermal fluctuations of the phases wash out the disorder-driven phase transition if they exceed the width of the disorder distribution. To avoid this, the temperature has to be smaller than the energy cost of winding the phase of a single pore by the critical disorder width, i.e. $k_B T \lesssim K_s T C w_c^2 / 2$. We estimate that this inequality is satisfied provided that $T$ is not too close to the $\lambda$-point, i.e. $T_\lambda - T \gtrsim 3 \text{ mK}$ in the setting of Ref. [1]. Furthermore, we have assumed that the current-phase relation is linear, which is true provided that $T_\lambda - T \gtrsim 5 \text{ mK}$, as was measured for a similar setup in Ref. [14].

To test the results of the mean-field theory, we have compared them to results from a numerical investigation performed on a finite, periodic lattice [33]. For various widths of the disorder distribution (which we have taken to be Gaussian), the fractions of pores that have phase-slipped, as a function of control-parameter, are shown in the inset of Fig. 2. As predicted by the mean-field theory, avalanches occur when the distribution of critical velocities is narrow but not when it is broad. At a critical strength of the disorder, which separates these two regimes, the discontinuity in the mean-field fraction of slipped pores just vanishes. Moreover, at this critical
disorder there are always values of the control phase at which the response of the system diverges. The mean-field and numerical results appear to agree with one another rather well, as one can see from Fig. 2, at least in the vicinity of the critical point.

The two main results of our Paper are summarised in Fig. 5. The blue curve shows the dependence of the amplitude of the current oscillations as a function of temperature, using the disorder model described in the text. Dashed line: expected amplitude in the absence of disorder.

Comparison with experiments—In their experiments, the Berkeley group have measured the amplitude of the “whistle” (i.e. the amplitude of the current oscillations ΔI) as a function of temperature at a fixed chemical potential difference. The experiments found that at T<sub>λ</sub> there are no oscillations of the current. As the temperature is lowered below T<sub>λ</sub>, ΔI first increases, and then gradually decreases. To obtain the behaviour of ΔI as a function of T we extend the model for the critical velocity, Eq. [8], to include disorder, by assuming that

\[
v_{c,i}(T) \approx \frac{\hbar}{m(\xi(T) + x_i)},
\]

where x<sub>i</sub> is the temperature-independent characteristic scale of the surface roughness in the pore, which we take to have a Gaussian distribution. For T<sub>λ</sub> − T > 5 mK, we can compare the results of our modelling to those of the experiments. The general features are reproduced: the initial increase in ΔI is associated with an increase in the superfluid fraction; the gradual decrease at lower temperatures is due to the loss of synchronicity amongst the pores, which is caused by the effective increase in the strength of disorder. To demonstrate these features, we plot the amplitude of current oscillations as a function of temperature; see the inset in Fig. 5 for which we have set δx = 4 nm.

We also note that the general features of the current-vs.-time traces, Fig. 3, are similar to those of the type III experiments described in Ref. 1. In both, as the temperature is lowered (i.e. the disorder is increased), the avalanche gradually disappears, and then so do the oscillations in the current.

Concluding remarks—Motivated by recent experiments performed by the Berkeley group on superflow through nanopore arrays, we have developed a model to describe phase-slip dynamics of such systems. The main features of our model are effective inter-pore couplings, mediated through the bulk superfluid, as well as randomness in the critical velocities of the pores, the latter being effectively controlled through the temperature.

Within our model, we find that the competition between (a) site-disorder in the critical velocities and (b) effective inter-pore coupling leads to the emergence of rich collective dynamics, including a transition between avalanching and non-avalanching regimes of the phase-slip dynamics. We identify a line of critical disorder-strengths in the phase diagram, at which there is a divergent susceptibility, in the sense that near to this line small changes in the control parameter can lead to large changes in the fraction of phase-slipped pores.

Our model reproduces the key physical features of the Berkeley group’s experiments, including a high-temperature synchronous regime, a low-temperature asynchronous regime, and a transition between the two. We therefore feel that the model captures the essential physics explored in these experiments.

In a forthcoming paper we shall extend our approach in order to address the transition from the Josephson regime to the avalanching phase-slippage regime described here, by including thermal fluctuations together with a generalised description of pore energies that is valid in both regimes.

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[32] In particular, one can make evident the kinship between the present superfluid model and the elastic model for CDW depinning (without topological excitations) by exchanging control via the chemical-potential difference between the reservoirs for control via the supercurrent, which then plays the role of the CDW driving force.

[33] Beecken, B. P. & Zimmermann, W. Jr., Variation of the critical order-parameter phase difference with temperature from 0.4 to 1.9 K in the flow of superfluid $^3$He through a tiny orifice. *Phys. Rev. B* 35, 1630-1635 (1987).

[34] Avenel, O., Ihas, G. G. & Varoquaux, E. The nucleation of vortices in superfluid $^4$He: answers and questions. *J. Low Temp. Phys.* 93, 1031-1057 (1993).

[35] Volovik G. E. Quantum-mechanical formation of vortices in a superfluid. ZhETF Pis. Red. 15, 116-120 (1972) [JETP Lett. 15, 81-83 (1972)].

[36] Hess, G. B. Critical velocities in superfluid helium flow through 10 µm-diameter pinholes. *Phys. Rev. Lett.* 27, 977-979 (1971).

[37] Similar ideas have been suggested by Y. Sato and co-workers (private communication (2006)).

[38] $A = 0.02 \mu m; B = 0.16 \mu m; C = 0.18 \mu m; J = 0.05 \mu m.$
We wish to eliminate effects arising from edges of the array of pores. We do this by measuring distances between pairs of sites, allowing paths to wrap around the edges and picking the shortest one. This renders the problem translationally invariant. In particular, $A$, $B$, and $C_{ii}$ are then independent of any pore index.

The Gaussian distribution is suitably cut-off and renormalized to avoid negative or very large critical velocities.