NEAR EQUALITY IN THE RIESZ-SOBOLEV INEQUALITY

MICHAEL CHRIST

Abstract. The Riesz-Sobolev inequality provides a sharp upper bound for a trilinear expression involving convolution of indicator functions of sets. Equality is known to hold only for indicator functions of appropriately situated intervals. We characterize ordered triples of subsets of \( \mathbb{R}^1 \) that nearly realize equality, with quantitative bounds of power law form with the optimal exponent.

1. Introduction

Denote by \(|A|\) the Lebesgue measure of a set \( A \subset \mathbb{R}^d \), and by \( 1_A \) the indicator function of \( A \). By \( \langle f, g \rangle \) we will mean \( \int_{\mathbb{R}^d} fg \, dm \) where \( m \) is Lebesgue measure. Let \( f^* \) denote the radially symmetric nonincreasing rearrangement of \( f \), whose definition is reviewed below.

Let \( f, g, h : \mathbb{R}^d \to [0, \infty) \) be nonnegative Lebesgue measurable functions such that \( \{x : f(x) > t\} < \infty \) for all \( t > 0 \), and likewise for \( g, h \). The inequality of Riesz and Sobolev \([14],[16]\) states that

\[
\langle f \ast g, h \rangle \leq \langle f^* \ast g^*, h^* \rangle.
\]

Of particular importance is the special case of indicator functions of measurable sets with finite Lebesgue measures, for which the inequality becomes

\[
\langle 1_A \ast 1_B, 1_C \rangle \leq \langle 1_{A^*} \ast 1_{B^*}, 1_{C^*} \rangle
\]

where \( A^*, B^*, C^* \) are balls centered at 0 whose measures equal the measures of \( A, B, C \) respectively. The right-hand side is of course a function of \( \{|A|, |B|, |C|\} \) alone. This foundational inequality for indicator functions of sets directly implies the formally more general version \([14],[16]\) for nonnegative functions. The multiplicity function \( (1_A \ast 1_B)(x) \) is a continuum measurement of the number of ways in which \( x \) is represented in the form \( a + b \) with \( (a, b) \in A \times B \); \( \langle 1_A \ast 1_B, 1_C \rangle \) represents the total number of ways in which elements of \( C \) can be so represented.

Inverse theorems that characterize cases of equality in inequalities \([1,2]\) and/or \([1,1]\) have been a useful tool in the analysis of extremizers of other inequalities. For instance, one element of Lieb’s characterization \([11]\) of extremizers of the Hardy-Littlewood-Sobolev inequality was the fact that if \( h = h^* \), and if \( h^* \) is positive and strictly decreasing along rays emanating from the origin, then equality holds in \([1,1]\) only if \( f = f^* \) and \( g = g^* \) up to suitable translations. See for instance Theorem 3.9 in \([12]\). More recently, this author in \([2]\) used a sharper inverse theorem of Burchard \([1]\) concerning \([1,2]\) to determine all extremizers of an inequality for the Radon transform. The one-dimensional case of Burchard’s theorem states that if \( (A, B, C) \) is an ordered triple of subsets of \( \mathbb{R}^1 \) whose measures satisfy a natural admissibility condition introduced below, then equality holds in \([1,2]\) only if \( A, B, C \) coincide with intervals, up to null sets. Equality also forces the centers
of these intervals to satisfy \( a + b = c \). These conditions together are also sufficient for equality.

An associated question is what properties \((A, B, C)\) must have if the condition of exact equality in (1.2) is relaxed to near equality. If \(|\langle A \ast B, C \rangle|\) is nearly maximal among such expressions for all ordered triples with \(|A|, |B|, |C|\) specified, must \((A, B, C)\) be close to some extremizing triple? In what metric must it be close? How close? One seeks a compactness principle, modulo the action of a noncompact symmetry group. This paper is one of a series devoted to these stability questions, for functionals and inequalities that are governed by the Abelian group structure of Euclidean space and have the group of all affine automorphisms as an underlying symmetry group. One of our principal tools, an additive combinatorial inverse theorem, was originally developed in the context of finite sets and discrete groups, but has proved effective in the continuum as well.

A weak theorem describing triples realizing near equality in (1.2) was established in [3]. This seems to have been the first usage of the inverse theorem in the context of such analytic inequalities for Euclidean space. It served as the central element of a characterization [4] of those functions which nearly extremize Young’s convolution inequality in \( \mathbb{R}^d \), for arbitrary \( d \). Notwithstanding its adequacy for that application, this inverse theorem suffers from severe limitations: It is only applicable if one is given two sets \( C_1, C_2 \) such that both triples \((A, B, C_i)\) achieve near equality; \(|C_2|\) must be very nearly equal to \(3|C_1|\); and its conclusion is of “little \( o \)” form.

In this paper we establish a more definitive result for \( d = 1 \), with natural hypotheses and a quantitative conclusion of power law type in which the principal exponent is the best possible. However, the analysis developed here exploits structural aspects of the one-dimensional case which do not extend to higher dimensions. We plan to address the dimensional limitation in subsequent work, by combining the one-dimensional result with other arguments, rather than by extending the one-dimensional method of proof. Some progress on related problems in higher dimensions was made by this author in [4], [5], [6] and by Figalli and Jerison [8]. The latter authors have obtained a power law type bound, in an analogous result for the Brunn-Minkowski inequality.

2. Main Theorem

Let \((A_1, A_2, A_3)\) be an ordered triple of measurable subsets of \( \mathbb{R}^1 \) with finite, positive Lebesgue measures. No inverse theorem is possible without a natural hypothesis, called admissibility by Burchard [1]. \((A_1, A_2, A_3)\) is said to be admissible if \(|A_i| + |A_j| \geq |A_k|\) for every permutation \((i, j, k)\) of \((1, 2, 3)\), and to be strictly admissible if \(|A_i| + |A_j| > |A_k|\) for every permutation. These definitions are independent of the ordering. Formulation of our main theorem requires the following more quantitative version of strict admissibility.

**Definition 2.1.** Let \( \eta \in (0, 1] \). Let \( A_j \) be measurable subsets of \( \mathbb{R}^1 \) satisfying \( 0 < |A_j| < \infty \). The ordered triple \((A_1, A_2, A_3)\) is \( \eta \)-strictly admissible if

\[
|A_i| + |A_j| \geq |A_k| + \eta \max(|A_1|, |A_2|, |A_3|)
\]

for every permutation \((i, j, k)\) of \((1, 2, 3)\).

An immediate consequence of \( \eta \)-strict admissibility is mutual comparability of the measures of the sets in question:

\[
\min_m |A_m| \geq \eta \max_n |A_n|.
\]

This will be proved below.
the left-hand side of the inequality, \( A, B, C \) satisfying the hypotheses of Theorem 2.1, \( (2.6) \) There exists a constant \( K < \infty \) for which the following holds. Let \( \eta \in (0, 1] \). Let \( (A, B, C) \) be an \( \eta \)-strictly admissible ordered triple of measurable subsets of \( \mathbb{R}^1 \) with finite, positive Lebesgue measures. If
\[
(1_A \ast 1_B, 1_C) \geq (1_{A^*} \ast 1_{B^*}, 1_{C^*}) - \varepsilon \max(|A|, |B|, |C|)^2
\]
and if \( \varepsilon \leq K^{-1}\eta^4 \) then there exist intervals \( I, J, L \subset \mathbb{R} \) such that
\[
|A \triangle I| \leq K\eta^{-1}\varepsilon^{1/2} \max(|A|, |B|, |C|)
\]
and the corresponding upper bounds hold for \( |J \triangle B| \) and \( |L \triangle C| \). The centers \( a, b, c \) of \( I, J, L \) satisfy
\[
|a + b - c| \leq K\eta^{-2}\varepsilon^{1/4} \max(|A|, |B|, |C|).
\]
The exponent \( \frac{1}{2} \) in (2.3) is optimal. Indeed, if \( A, B \) are intervals centered at 0, if \( C = [-\gamma, \gamma] \cup [\gamma + \delta, \gamma + 2\delta] \), and if \( (A, B, [-\gamma, \gamma]) \) is strictly admissible then for sufficiently small \( \delta \), (2.3) holds with \( \varepsilon \simeq \delta^2 \).

The Riesz-Sobolev inequality can be viewed as a statement about additive combinatorics. The quantity \( (1_A \ast 1_B, 1_C) \) is interpreted as the number of ordered pairs \( (a, b) \in A \times B \) for which the sum \( a + b \) lies in \( C \).

Theorem 2.1 can be interpreted as a sharpening of the Riesz-Sobolev inequality, in the following way. The infimum in the following inequality is taken over all bounded intervals \( I \subset \mathbb{R} \).

**Theorem 2.2.** There exists a constant \( c_0 > 0 \) such that for any \( \eta \in (0, 1] \) and any sets \( A, B, C \) satisfying the hypotheses of Theorem 2.1,
\[
(1_A \ast 1_B, 1_C) \leq (1_{A^*} \ast 1_{B^*}, 1_{C^*}) - c_0\eta^2 \inf_I |A \triangle I|^2.
\]
The hypotheses are symmetric in \( (A, B, C) \) in a natural way, so in the second term on the left-hand side of the inequality, \( A \) can equally be replaced by \( B \) or by \( C \).

The Riesz-Sobolev inequality is very closely related to another one, the KPRGT inequality. In Theorem 13.1 we formulate an analogous inverse result for the KPRGT inequality, and deduce it as a corollary of Theorem 2.1.

The author thanks Marcos Charalambides and Ed Scerbo for proofreading and for valuable suggestions which have improved the exposition, and Terence Tao for calling his attention to the KPRGT inequality.

### 3. Outline and notations

The leading idea in the proof, as in [3], is to relate near equality in the Riesz-Sobolev inequality to near equality in the Brunn-Minkowski inequality, for which a characterization is already available. The essential difference between the two situations is that for Brunn-Minkowski, one is given that \( a + b \in C \) for every ordered pair \( (a, b) \in A \times B \), whereas for Riesz-Sobolev it is given that \( a + b \in C \) for a subset of \( A \times B \) whose complement has measure comparable to that of \( A \times B \), even in cases of exact equality.

The superlevel sets
\[
S_{A,B}(t) = \{ x \in \mathbb{R}^1 : (1_A \ast 1_B)(x) > t \}
\]
play a central role in the analysis. There are multiple steps, organized as follows although not in this order.

\[ \triangle \]
(1) If an ordered triple \((A, B, C)\) nearly attains equality in the Riesz-Sobolev inequality, then \(C\) nearly coincides with the superlevel set \(S_{A,B}(\alpha)\) for a certain parameter \(\alpha\) which depends only on \(|A|, |B|, |C|\). Moreover, the ordered triple \((A, B, S_{A,B}(\alpha))\) also nearly attains equality, so that \((A, B, C)\) can be replaced by \((A, B, S_{A,B}(\alpha))\).

(2) Superlevel sets associated to convolutions of indicator functions of arbitrary sets satisfy an additive inclusion relation: the difference set \(S_{A,B}(\alpha) - S_{A,B}(\beta)\) is contained in \(S_{A,-A}(\alpha + \beta - |B|)\).

(3) An inverse theorem associated to the one-dimensional Brunn-Minkowski inequality asserts that if \(|A + B|\) is nearly equal to \(|A| + |B|\), then \(A, B\) nearly coincide with intervals. Thus in order to show that \(S_{A,B}(\alpha)\) and hence \(C\) are nearly equal to intervals, it suffices to show that the measure of the difference set \(S_{A,B}(\alpha) - S_{A,B}(\alpha)\) is only slightly greater than twice the measure of \(S_{A,B}(\alpha)\). By the inclusion relation, this in turn would follow from the same upper bound for \(|S_{A,-A}(2\alpha - |B|)|\).

(4) The Riesz-Sobolev inequality is equivalent to another inequality, which we call the (sharpened) KPRGT inequality. Whereas the Riesz-Sobolev upper bound is expressed in terms of \(|A|, |B|, |S_{A,B}(\tau)|\), the KPRGT bound is expressed in terms of \(|A|, |B|, \tau\). Therefore it is potentially possible to study whether such a triple of sets nearly extremizes the KPRGT inequality, without knowing \(|S_{A,B}(\tau)|\).

(5) If \((A, B, S_{A,B}(\alpha))\) nearly realizes equality in the Riesz-Sobolev inequality, then the ordered triple \((A, -A, S_{A,-A}(2\alpha - |B|))\) nearly achieves equality in the KPRGT inequality — but our argument for this implication applies only under the excruciatingly restrictive extra hypothesis that \(|A| = |B|\). This step uses the inclusion relation involving differences of superlevel sets, and the Brunn-Minkowski inequality to obtain lower bounds for measures of these differences.

(6) Whenever \((A, B, S_{A,B}(\tau))\) nearly extremizes the KPRGT inequality, a nearly tight bound must hold for \(|S_{A,B}(\tau)|\). In the present context, this is the desired upper bound for \(|S_{A,-A}(2\alpha - |B|)|\).

(7) By the inverse theorem, \(S_{A,B}(\alpha)\) and hence \(C\) nearly coincide with an interval, concluding the proof (for \(C\)) when \(|A| = |B|\).

(8) An alteration procedure makes it possible to replace sets with certain subsets, without sacrificing the hypothesis of near equality in the Riesz-Sobolev inequality. This is used to replace \(A, B\) by subsets with equal measures, making the special case treated above applicable.

(9) The alteration procedure is sufficiently flexible to give rise to a rich family of subsets of \(A\). Near coincidence of all subsets in such a family with intervals is shown to imply near coincidence of \(A\) itself with a larger interval.

Our exposition includes largely self-contained proofs of the Riesz-Sobolev, KPRGT, and sharpened KPRGT inequalities, of the additive combinatorial inverse theorem (in the relevant continuum version) of Freıman that is a keystone of the analysis, and of the equivalence of two formulations of the Riesz-Sobolev inequality.

Symmetric nonincreasing rearrangements are defined as follows. Let \(|S|\) denote the Lebesgue measure of \(S \subset \mathbb{R}^d\). If \(S\) is Lebesgue measurable and \(0 < |S| < \infty\), then \(S^*\) denotes the open ball \(B\) centered at \(0 \in \mathbb{R}^d\) which satisfies \(|B| = |S|\). If \(f: \mathbb{R}^d \to [0, \infty)\) is a Lebesgue measurable function for which \(|\{x : f(x) > t\}|\) is finite for any \(t > 0\), then \(f^*\) is defined to be the unique radially symmetric function such that \(r \mapsto f^*(rx)\) is a nonincreasing function of \(r > 0\) for each \(0 \neq x \in \mathbb{R}^d\), \(|\{x : f^*(x) > t\}| = |\{x : f(x) > t\}|\) for all \(t > 0\), and \(r \mapsto f^*(rx)\) is right continuous for each \(0 \neq x\).
To prove the claim made above, let \((A, B, C)\) be an \(\eta\)-strictly admissible ordered triple and assume without loss of generality that \(|A| \geq |B| \geq |C|\). The inequality to be proved is then that \(|C| \geq \eta|A|\). It is given that \(|B| + |C| \geq |A| + \eta|A|\), so \(|C| \geq \eta|A| + (|A| - |B|) \geq \eta|A|\). \(\square\)

4. THE RIESZ-SOBOLEV INEQUALITY RECAST

To any Lebesgue measurable sets \(A, B \subset \mathbb{R}^1\) with finite Lebesgue measures are associated the superlevel sets

\[
S_{A,B}(t) = \{x : (1_A * 1_B)(x) > t\}.
\]

These sets are open since \(1_A * 1_B\) is a continuous function.

For any two bounded intervals \(I, J\), centered at 0

\[
(1_I * 1_J)(x) = \begin{cases} 
\min(|I|, |J|) & \text{if } 2|x| \leq |I| - |J| \\
\frac{|I|}{2} + \frac{|J|}{2} - |x| & \text{if } |I| - |J| \leq 2|x| \leq |I| + |J| \\
0 & \text{if } 2|x| \geq |I| + |J|.
\end{cases}
\]

Thus

\[
|S_{I,J}(t)| = \begin{cases} 
0 & \text{if } t \geq \min(|I|, |J|) \\
|I| + |J| - 2t & \text{if } 0 \leq t < \min(|I|, |J|).
\end{cases}
\]

The Riesz-Sobolev inequality for \(\mathbb{R}^1\) states that

\[
\int_E (1_A * 1_B) \leq \int_{E'} (1_A * 1_B)
\]

for any sets \(A, B, E \subset \mathbb{R}^1\) with finite Lebesgue measures. A proof is sketched in [10]. The right-hand side can be expressed in terms of \(|A|, |B|, |E|\) using the formulas above.

The following notation will be used throughout the discussion.

**Definition 4.1.** For any Lebesgue measurable sets \(A, B, C \subset \mathbb{R}^d\) with finite Lebesgue measures,

\[
\mathcal{D}(A, B, C) = \langle 1_{A^*} * 1_{B^*}, 1_{C^*} \rangle - \langle 1_A * 1_B, 1_C \rangle
\]

The Riesz-Sobolev inequality states that \(\mathcal{D}(A, B, C) \geq 0\) for all \((A, B, C)\).

If

\[
|A| - |B| \leq |E| \leq |A| + |B|
\]

and if \(\sigma \in [0, \min(|A|, |B|)]\) is defined by

\[
|E| = |A| + |B| - 2\sigma
\]

then

\[
\langle 1_{A^*} * 1_{B^*}, 1_{E^*} \rangle = |A| \cdot |B| - \sigma^2
\]

and the Riesz-Sobolev inequality becomes

\[
\langle 1_A * 1_B, 1_C \rangle \leq |A| \cdot |B| - \sigma^2 = |A| \cdot |B| - \frac{1}{4}(|A| + |B| - |E|)^2.
\]

For \(|E| > |A| + |B|\) the Riesz-Sobolev inequality for \(\mathbb{R}^1\) states the trivial upper bound \(\int_E (1_A * 1_B) \leq |A| \cdot |B|\). For \(|E| < |A| + |B|\), it gives the also trivial upper bound \(|E|\min(|A|, |B|)\).
The identity
\begin{equation}
\int_{S_{A,B}(\tau)} (1_A \ast 1_B) = \tau |S_{A,B}(\tau)| + \int_{\tau}^{\infty} |S_{A,B}(t)| \, dt
\end{equation}
will be useful throughout our analysis. It allows one to express the $\mathbb{R}^1$ Riesz-Sobolev inequality, for the special case when $C$ is a superlevel set of $1_A \ast 1_B$, in the form

**Lemma 4.1.** Let $A, B \subset \mathbb{R}^1$ be Lebesgue measurable sets with finite, positive measures. Let $\tau \in [0, \min(|A|, |B|)]$. Define $\sigma$ by
\begin{equation}
|S_{A,B}(\tau)| = |A| + |B| - 2\sigma.
\end{equation}
If
\begin{equation}
|A| - |B| \leq |S_{A,B}(\tau)| \leq |A| + |B|
\end{equation}
then
\begin{equation}
\tau |S_{A,B}(\tau)| + \int_{\tau}^{\infty} |S_{A,B}(t)| \, dt \leq |A| \cdot |B| - \sigma^2
\end{equation}

The assumption (4.11) is equivalent to $\sigma \in [0, \min(|A|, |B|)]$. We will show in §14 how the Riesz-Sobolev inequality for general sets $C$ can in turn be deduced from this lemma.

5. APPROXIMATION BY SUPERLEVEL SETS

If $A, B$ are given then in order to maximize $\langle 1_A \ast 1_B, 1_C \rangle$ over all sets $C$ of specified measure, $C$ should be chosen to be a superlevel set of that measure, provided such a superlevel set exists. The purpose of this section is to show that if $(A, B, C)$ is a nearly extremizing ordered triple, then $C$ must nearly coincide with some superlevel set $S_{A,B}(t)$, and moreover $|S_{A,B}(t)|$ must be nearly equal to $|A| + |B| - 2t$. This was shown in [3], but we give more precise bounds here.

**Lemma 5.1.** Let $A, B, E \subset \mathbb{R}^1$ be Lebesgue measurable sets of finite, positive measures. Suppose that
\begin{equation}
|A| - |B| + 2D(A, B, E)^{1/2} < |E| < |A| + |B| - 2D(A, B, E)^{1/2}.
\end{equation}
Define $\tau$ by $|E| = |A| + |B| - 2\tau$. Then
\begin{align}
|S_{A,B}(\tau) \triangle E| &\leq 4D(A, B, E)^{1/2} \\
| |S_{A,B}(\tau)| - (|A| + |B| - 2\tau) | &\leq 4D(A, B, E)^{1/2}.
\end{align}

**Proof of Lemma 5.1.** To simplify notation write $S = S_{A,B}(\tau)$, $\lambda = |A| - |B|$, and $D = D(A, B, E)$. By definition of $D(A, B, E)$,
\begin{equation}
\langle 1_A \ast 1_B, 1_E \rangle = |A| \cdot |B| - \frac{1}{4}(|A| + |B| - |E|)^2 - D.
\end{equation}
Since $|E| > \lambda + 2D^{1/2}$, if $|E \setminus S|$ were strictly greater than $2D^{1/2}$ then there would exist a measurable set $T$ satisfying $E \cap S \subset T \subset E$ with $|T| \geq \lambda$ and $|E \setminus T| > 2D^{1/2}$. Indeed, if $|E \cap S| \geq \lambda$ choose $T = E \cap S$. Otherwise choose any measurable set $T$ satisfying $|T| = \lambda$ with $E \cap S \subset T \subset E$. 

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Because $|T| \geq \lambda = |A| - |B|$, 
\[ \langle 1_A \ast 1_B, 1_E \rangle = \langle 1_A \ast 1_B, 1_T \rangle + \langle 1_A \ast 1_B, 1_{E \setminus T} \rangle \]
\[ \leq |A| \cdot |B| - \frac{1}{4}((|A| + |B| - |T|)^2 + \tau |E \setminus T|) \]
\[ = |A| \cdot |B| - \frac{1}{4}(2\tau + |E \setminus T|)^2 + \tau |E \setminus T| \]
\[ = |A| \cdot |B| - \tau^2 - \frac{1}{4}|E \setminus T|^2 \]
\[ = |A| \cdot |B| - \frac{1}{4}(|A| + |B| - |E|)^2 - \frac{1}{4}|E \setminus T|^2. \]

Thus $|E \setminus T| \leq 2D^{1/2}$, which is a contradiction.

To establish an upper bound for $|S \setminus E|$, consider any set $T$ satisfying $E \subset T \subset E \cup S$ with $|T| \leq |A| + |B|$. Since $1_A \ast 1_B > \tau$ at each point of $T \setminus E$,
\[ \langle 1_A \ast 1_B, 1_T \rangle \geq \langle 1_A \ast 1_B, 1_E \rangle + \tau |T \setminus E| \]
\[ = |A| \cdot |B| - \frac{1}{4}(|A| + |B| - |E|)^2 - D + \tau |T \setminus E|. \]

Bearing in mind that $|T \setminus E| = |T| - |E|$,
\[ |A| \cdot |B| - \frac{1}{4}(|A| + |B| - |E|)^2 + \tau |T \setminus E| \]
\[ = |A| \cdot |B| - \frac{1}{4}(|A| + |B| - |E|)^2 + \frac{1}{2}(|A| + |B| - |E|)(|T| - |E|) \]
\[ = |A| \cdot |B| - \frac{1}{4}(|A| + |B| - |T|)^2 - \frac{1}{2}(|A| + |B| - |T|)(|T| - |E|) - \frac{1}{4}(|T| - |E|)^2 \]
\[ = |A| \cdot |B| - \frac{1}{4}(|A| + |B| - |T|)^2 + \frac{1}{2}(|T| - |E|)^2 \]
\[ = |A| \cdot |B| - \frac{1}{4}(|A| + |B| - |T|)^2 + \frac{1}{4}|T \setminus E|^2. \]

Thus
\[ \langle 1_A \ast 1_B, 1_T \rangle \geq |A| \cdot |B| - \frac{1}{4}(|A| + |B| - |T|)^2 + \frac{1}{4}|T \setminus E|^2 - D. \]

Since the Riesz-Sobolev inequality guarantees that $\langle 1_A \ast 1_B, 1_T \rangle$ cannot exceed the quantity $|A| \cdot |B| - \frac{1}{4}(|A| + |B| - |T|)^2$, it follows that
\[ \frac{1}{4}|T \setminus E|^2 \leq D. \]

If $x \leq y$ are numbers, $A$ is a measurable set, and $|B| \leq x$ for every measurable subset $B \subset A$ satisfying $|B| \leq y$, then $|A| \leq x$. From this principle and the preceding inequality, since $|E| \leq |A| + |B| - 2D^{1/2}$ it now follows that $|S \setminus E| \leq 2D^{1/2}$.

Summing the bounds for $|E \setminus S|$ and $|S \setminus E|$ demonstrates that $S = S_{A,B}(\tau)$ satisfies $|S \Delta E| \leq 4D^{1/2}$. Since $\tau$ is defined so that $|E| = |A| + |B| - 2\tau$, 
\[ |S_{A,B}(\tau) - (|A| + |B| - 2\tau)| = |S_{A,B}(\tau) - |E|| \leq |S_{A,B}(\tau) \setminus E| \leq 4D^{1/2}, \]
which is the final conclusion [5.3] of the lemma.

\[ \square \]

Lemma 5.1 can be reinterpreted as a refinement of the $\mathbb{R}^1$ Riesz-Sobolev inequality.

**Lemma 5.2.** Under the hypotheses of Lemma 5.1, defining $\tau_E$ by the relation
\[ (5.4) \quad |E| = |A| + |B| - 2\tau_E, \]

one has
\[ (5.5) \quad \langle 1_A \ast 1_B, 1_E \rangle + \frac{1}{16} \left( |S_{A,B}(\tau_E)| - (|A| + |B| - 2\tau_E) \right)^2 \leq \langle 1_A \ast 1_B, 1_E \rangle. \]
It follows directly from the upper bound for $|S_{A,B}(\tau) \triangle E|$ that the triple $(A, B, S_{A,B}(\tau))$ nearly attains equality in the Riesz-Sobolev inequality, with a discrepancy majorized by a constant multiple of $\mathcal{D}(A, B, E)^{1/2} \max(|A|, |B|, |E|)$. The next lemma gives a better bound, which will be essential in the attainment of the optimal exponent in Theorem 2.1.

**Lemma 5.3.** Let $(A, B, E)$ be an ordered triple of sets satisfying the hypotheses of Lemma 5.1. Define $\tau$ by $|E| = |A| + |B| - 2\tau$. Then the superlevel set $S_{A,B}(\tau)$ satisfies

$$\mathcal{D}(A, B, S_{A,B}(\tau)) \leq \mathcal{D}(A, B, E).\tag{6.6}$$

**Proof.** Write $\mathcal{D} = \mathcal{D}(A, B, E)$. One has

$$||A| - |B|| < |S_{A,B}(\tau)| < |A| + |B|$$

since it was shown above that $|E \setminus S_{A,B}(\tau)| \leq 2\mathcal{D}^{1/2}$, and likewise for $|S_{A,B}(\tau) \setminus E|$.

The convolution $1_A * 1_B$ satisfies $(1_A * 1_B)(x) \geq \tau$ for $x \in S_{A,B}(\tau) \setminus E$, and $\leq \tau$ for $x \in E \setminus S_{A,B}(\tau)$. Thus

$$\langle 1_A * 1_B, 1_{S_{A,B}(\tau)} \rangle \geq \langle 1_A * 1_B, 1_E \rangle + \tau|S_{A,B}(\tau) \setminus E| - \tau|E \setminus S_{A,B}(\tau)|$$

$$= \langle 1_A * 1_B, 1_E \rangle + \tau(|S_{A,B}(\tau)| - |E|)$$

$$= |A| \cdot |B| - \tau^2 - \mathcal{D} + \tau(|S_{A,B}(\tau)| - |E|).$$

Defining $\sigma$ by $|S_{A,B}(\tau)| = |A| + |B| - 2\sigma$, this can be rewritten

$$\langle 1_A * 1_B, 1_{S_{A,B}(\tau)} \rangle \geq |A| \cdot |B| - \sigma^2 + \sigma^2 - \tau^2 - \mathcal{D} + \tau(|S_{A,B}(\tau)| - |E|)$$

$$= |A| \cdot |B| - \sigma^2 + \sigma^2 - \tau^2 - \mathcal{D} + \tau(-2\sigma + 2\tau)$$

$$= |A| \cdot |B| - \sigma^2 - \mathcal{D} + (\sigma - \tau)^2$$

$$\geq |A| \cdot |B| - \sigma^2 - \mathcal{D}$$

$$= \langle 1_A, 1_B, 1_{S_{A,B}(\tau)} \rangle - \mathcal{D}.$$

□

### 6. The KPRGT Inequality

The Riesz-Sobolev inequality has a close relative, which we will call the KPRGT inequality. Important contributions to its theory were made by Kemperman [10], Pollard [13], Ruzsa [15], and Green and Ruzsa [9]; the form most directly relevant to our discussion was established by Tao [18]. The KPRGT inequality, in the version of [18], states that for any compact connected Abelian group $G$ equipped with a translation-invariant Borel probability measure $\mu$, for any Borel sets $A, B \subseteq G$ and any $\tau \in [0, \min(\mu(A), \mu(B))]$,

$$\int_G \min \left( (1_A * 1_B)(x), \tau \right) d\mu(x) \geq \tau \min \left( \mu(A) + \mu(B) - \tau, 1 \right).\tag{6.1}$$

This has as a corollary the KPRGT inequality for $\mathbb{R}^d$:

$$\int_{\mathbb{R}^d} \min \left( (1_A * 1_B)(x), \tau \right) dx \geq \tau(|A| + |B| - \tau) \tag{6.2}$$

provided $0 \leq \tau \leq \min(|A|, |B|)$. To deduce (6.2) from (6.1), first consider bounded sets apply (6.1) to their images under the quotient map $\mathbb{R}^d \to \mathbb{R}^d/\mathbb{Z}^d$ defined by $x \mapsto \varepsilon x$ modulo $\mathbb{Z}^d$, for sufficiently small $\varepsilon > 0$. The case of unbounded sets with finite Lebesgue measures follows by a limiting argument. The details are omitted, since an alternative proof will be provided below.
Since $\int_{\mathbb{R}^d} (1_A \ast 1_B) \, dx = |A| \cdot |B|$, and
\[
\int_{\mathbb{R}^d} f = \tau |\{ f > \tau \}| + \int_{f > \tau} f + \int_{f < \tau} (f - \tau) = \int_{\mathbb{R}^d} \min(f, \tau) + \int_{f > \tau} (f - \tau)
\]
for any nonnegative measurable function $f$ and any $\tau \geq 0$, (6.2) is equivalent for $0 \leq \tau \leq \min(|A|, |B|)$ to
\[
\int_{S_{A,B}(\tau)} ((1_A \ast 1_B)(x) - \tau) \, dx \leq |A| \cdot |B| - \tau(|A| + |B| - \tau) = (|A| - \tau)(|B| - \tau),
\]
which can also be equivalently written as
\[
\int_{\tau}^\infty |S_{A,B}(t)| \, dt \leq (|A| - \tau)(|B| - \tau) \quad \text{for} \quad 0 \leq \tau \leq \min(|A|, |B|)
\]
for $A, B \subset \mathbb{R}^d$.

This is not a sharp inequality for $d > 1$, and we restrict the discussion henceforth to $d = 1$. We will refer to (6.4) as the KPRGT inequality. It should be compared with the Riesz-Sobolev inequality (4.12), which provides an upper bound for the sum of the left-hand side of (6.4) plus $\tau|S_{A,B}(\tau)|$.

Both the KPRGT (6.4) and Riesz-Sobolev (4.12) inequalities give integrated upper bounds for the measures of superlevel sets, rather than any bound for any individual superlevel set. Yet our present goal is a tight bound for $|S_{A,-A}(2\alpha - B)|$ when $S_{A,B}(\alpha)$ is as in Lemma 5.1.

7. Connection between the Riesz-Sobolev and KPRGT inequalities

The connection between these two inequalities can be expressed succinctly using the following variant of $D(A, B, S_{A,B}(\tau))$.

**Definition 7.1.** For any sets $A, B$ and any real number $\tau \in [0, \min(|A|, |B|)]$, the deficit $D'(A, B, \tau)$ is
\[
D'(A, B, \tau) = (|A| - \tau)(|B| - \tau) - \int_{\tau}^\infty |S_{A,B}(t)| \, dt.
\]

The KPRGT inequality asserts simply that $D'(A, B, \tau) \geq 0$ for $0 \leq \tau \leq \min(|A|, |B|)$. The two quantities $D'(A, B, \tau)$ and $D(A, B, S_{A,B}(\tau))$ are related by the following identities.

**Lemma 7.1.** Let $A, B \subset \mathbb{R}^d$ be measurable sets with finite, positive Lebesgue measures. Let $\tau \in [0, \min(|A|, |B|)]$, and suppose that
\[
||A| - |B|| \leq |S_{A,B}(\tau)| \leq |A| + |B|.
\]
Then
\[
D'(A, B, \tau) = D(A, B, S_{A,B}(\tau)) + (\sigma - \tau)^2
\]
where $|S_{A,B}(\tau)| = |A| + |B| - 2\sigma$.

If $|S_{A,B}(\tau)| \geq |A| + |B|$ then
\[
D'(A, B, \tau) = D(A, B, S_{A,B}(\tau)) + (|A| - |B|)(\tau - |A| - |B|) + \tau^2.
\]

If $|S_{A,B}(\tau)| \leq \min(|A|, |B|)$ then
\[
D'(A, B, \tau) = D(A, B, S_{A,B}(\tau)) + (|B| - \tau)(|A| - |S| - \tau).
\]

The expressions $\tau(|S_{A,B}(\tau) - |A| - |B|)$ in (7.4) and $(|B| - \tau)(|A| - |S| - \tau)$ in (7.5) are nonnegative, under the indicated assumptions about $|S_{A,B}(\tau)|$. 

Proof. Suppose that (7.2) holds. Recall that under this assumption, \( \mathcal{D} = \mathcal{D}(A, B, S_{A,B}(\tau)) \) can be expressed as

\[
\mathcal{D} = |A| \cdot |B| - \sigma^2 - \int_{S_{A,B}(\tau)} (1_A \ast 1_B).
\]

By (1.1),

\[
\int_\tau^\infty |S_{A,B}(t)| \, dt = \int_{S_{A,B}(\tau)} (1_A \ast 1_B) - \tau |S_{A,B}(\tau)|
\]

\[
= |A| \cdot |B| - \sigma^2 - \mathcal{D} - \tau(|A| + |B| - 2\sigma)
\]

\[
= (|A| - \tau)(|B| - \tau) - (\tau - \sigma)^2 - \mathcal{D}.
\]

Next let \( S = S_{A,B}(\tau) \) and suppose that \( |S_{A,B}(\tau)| > |A| + |B| \). Then \( 1_A \ast 1_B \), \( 1_S \ast 1_B \) = \( |A| \cdot |B|, \) so \( \mathcal{D}(A, B, S_{A,B}(\tau)) = |A| \cdot |B| - \int_S 1_A \ast 1_B \). Therefore

\[
\mathcal{D}'(A, B, \tau) = (|A| - \tau)(|B| - \tau) - \int_\tau^\infty |S_{A,B}(t)| \, dt
\]

\[
= (|A| - \tau)(|B| - \tau) - \int_S (1_A \ast 1_B) + \tau |S|
\]

\[
= (|A| - \tau)(|B| - \tau) - |A| \cdot |B| + \mathcal{D}(A, B, S) + \tau |S|
\]

\[
= \tau(\tau - |A| - |B|) + \tau |S| + \mathcal{D}(A, B, S)
\]

\[
= \tau(|S| - |A| - |B|) + \tau^2 + \mathcal{D}(A, B, S).
\]

If \( |S| < \min(|A|, |B|) \) suppose without loss of generality that \( |A| \geq |B| \). Then \( 1_A \ast 1_B, 1_S \ast 1_B \) = \( |B| \cdot |S|, \) so \( \mathcal{D}(A, B, S_{A,B}(\tau)) = |B| \cdot |S| - \int_S 1_A \ast 1_B \). Therefore as above,

\[
\mathcal{D}'(A, B, \tau) = (|A| - \tau)(|B| - \tau) - \int_\tau^\infty |S_{A,B}(t)| \, dt
\]

\[
= (|A| - \tau)(|B| - \tau) - |S| \cdot |B| + \mathcal{D}(A, B, S) + \tau |S|
\]

\[
= \mathcal{D}(A, B, S) + (|B| - \tau)(|A| - |S| - \tau).
\]

In particular, \( 0 \leq \mathcal{D}(A, B, S_{A,B}(\tau)) \leq \mathcal{D}'(A, B, \tau) \) in all cases. Conversely, in the main case (7.2), an inequality in the reverse direction holds provided that \( |\sigma - \tau| \) can be suitably controlled. Thus for superlevel sets, near equality in the KPRGT inequality implies near equality in the Riesz-Sobolev inequality, while the reverse holds if a suitable upper bound is valid for \( |\sigma - \tau| \).

The KPRGT inequality is sharp, in the sense that equality holds whenever \( A, B \) are intervals and \( 0 \leq \tau \leq \min(|A|, |B|) \). Nonetheless, a yet sharper inequality is implicit in the identities of Lemma 7.1.

**Corollary 7.2.** Let \( A, B \subset \mathbb{R}^1 \) be measurable sets with finite, positive Lebesgue measures. Let \( \tau \in [0, \min(|A|, |B|)] \), and suppose that \( |A| - |B| \leq |S_{A,B}(\tau)| \leq |A| + |B| \). Define \( \sigma \) by \( |S_{A,B}(\tau)| = |A| + |B| - 2\sigma \). Then

\[
(7.6) \quad \mathcal{D}'(A, B, \tau) \geq (\sigma - \tau)^2 = \frac{1}{\tau}(|A| + |B| - |S_{A,B}(\tau)|)^2
\]

if \( |A| - |B| \leq |S_{A,B}(\tau)| \leq |A| + |B| \) while

\[
(7.7) \quad \mathcal{D}'(A, B, \tau) \geq \begin{cases} 
\tau(|S| - |A| - |B|) + \tau^2 & \text{if } |S_{A,B}(\tau)| \geq |A| + |B| \\
(|B| - \tau)(|A| - |S| - \tau) & \text{if } |S_{A,B}(\tau)| \leq |A| - |B| \nonumber
\end{cases}
\]
The first conclusion can be equivalently restated as

\[(7.8) \quad \int_{\tau}^{\infty} |S_{A,B}(t)| \, dt + (\sigma - \tau)^2 \leq (|A| - \tau)(|B| - \tau)\]

provided that \(|A| - |B| \leq |S_{A,B}(\tau)| \leq |A| + |B|\), with corresponding restatements of the second and third conclusions.

**Proof.** According to the Riesz-Sobolev inequality, \(D(A, B, S_{A,B}(\tau)) \geq 0\). Lemma 7.1 thus gives all three conclusions. \(\square\)

Alternatively, (7.8) can be deduced from the KPRGT inequality (6.4) itself by application of (6.4) to \(\int_{\tau}^{\infty} |S_{A,B}(t)| \, dt\), and comparison of this integral with \(\int_{\tau}^{\infty} |S_{A,B}(t)| \, dt\), in analogy with the proof of Lemma 5.1.

We have shown via the identity (7.3) that the sharpened KPRGT inequality (7.8) for \(\mathbb{R}^1\) is equivalent to the Riesz-Sobolev inequality for \(\mathbb{R}^1\), specialized to superlevel sets; in particular, this provides an independent proof of the KPRGT inequality for \(\mathbb{R}^1\). Inequality (7.8) is strictly sharper than (6.4) unless \(\sigma = \tau\), that is, unless \(|S_{A,B}(\tau)| = |A| + |B| - 2\tau\). Inequality (7.8) asserts in particular that near inequality in the (unsharpened) KPRGT inequality in the form (6.4), can only hold if \(|S_{A,B}(\tau)|\) is nearly equal to \(|A| + |B| - 2\tau\).

A situation will arise below in which it will not be known that \(|S_{A,B}(\tau)| \leq |A| + |B|\). The following version of the KPRGT inequality (implicit in (7.4)) will be useful in that situation.

**Lemma 7.3.** Suppose that \(A, B \subset \mathbb{R}^1\) are measurable sets with positive, finite Lebesgue measures. Suppose that \(0 \leq \tau < \max(|A|, |B|)\). If \(|S_{A,B}(\tau)| \geq |A| + |B|\) then

\[(7.9) \quad \int_{\tau}^{\infty} |S_{A,B}(t)| \, dt \leq |A| \cdot |B| - \tau|S_{A,B}(\tau)|.\]

**Proof.** Since

\[
\tau|S_{A,B}(\tau)| + \int_{\tau}^{\infty} |S_{A,B}(t)| \, dt = \langle 1_A \ast 1_B, 1_{S_{A,B}(\tau)} \rangle \leq |A| \cdot |B|,
\]

the integral satisfies

\[
\int_{\tau}^{\infty} |S_{A,B}(t)| \, dt = \tau|S_{A,B}(\tau)| + \int_{\tau}^{\infty} |S_{A,B}(t)| \, dt - \tau|S_{A,B}(\tau)|
\]

\[
= |A| \cdot |B| - \tau|S_{A,B}(\tau)|.
\]

\(\square\)

8. An additive relation between superlevel sets

Recall the definition of superlevel sets: For \(U, V \subset \mathbb{R}^1\),

\[(8.1) \quad S_{U,V}(\alpha) = \{x : (1_U \ast 1_V)(x) > \alpha\}.\]

Define \(A - x = \{a - x : a \in A\}\) and \(-B = \{-b : b \in B\}\).

**Lemma 8.1.**

\[(8.2) \quad S_{U,V}(\alpha) = \{x : \|1_{U-x} - 1_{-V}\|_1 < |U| + |V| - 2\alpha\}.\]
Proof. This is a direct consequence of the elementary identities
\[(1_A * 1_B)(x) = |(A - x) \cap (-B)|\]
\[(8.3)\]
\[|(A - x) \triangle (-B)| + 2|(A - x) \cap (-B)| = |A| + |B|\]
\[(8.4)\]
\[\|1_{A-x} - 1_{-B}\|_1 = |(A - x) \triangle (-B)|\] \hspace{1cm} (8.5)

\[\square\]

Lemma 8.2. Let \(U, V \subset \mathbb{R}^d\) be measurable sets with finite Lebesgue measures. Let \(\alpha_1, \alpha_2 > 0\). Then
\[S_{U,V}(\alpha_1) - S_{U,V}(\alpha_2) \subset S_{U,-U}(\alpha_1 + \alpha_2 - |V|).\]

Proof. Because \(1_U * 1_V\) and \(1_U * 1_{-U}\) are continuous, the associated superlevel sets are open and there is no ambiguity in asserting that an individual point belongs to such a set. Let \(x_i \in S_{U,V}(\alpha_i)\) for \(i = 1, 2\). By Lemma 8.1,
\[\|1_{U-x_i} - 1_V\|_1 < |U| + |V| - 2\alpha_i.\]
By the triangle inequality,
\[\|1_{U-x_1} - 1_{U-x_2}\|_1 < 2|U| + 2|V| - 2\alpha_1 - 2\alpha_2.\]
This is equivalent to
\[\|1_{U-x} - 1_U\|_1 < 2|U| + 2|V| - 2\alpha_1 - 2\alpha_2\]
where \(x = x_1 - x_2\). By Lemma 8.1 again, this is in turn equivalent to \(x \in S_{U,-U}(\beta)\) where
\[|U| + |-U| - 2\beta = 2|U| + 2|V| - 2\alpha_1 - 2\alpha_2,\]
that is, \(\beta = \alpha_1 + \alpha_2 - |V|\). \hspace{1cm} (8.6)

\[\square\]

9. Analysis of the case \(|A| = |B|\)

We arrive at the heart of the proof of the main theorem. As in [3], we will rely on the following characterization of near equality in the Brunn-Minkowski inequality.

Theorem 9.1. Let \(A, B \subset \mathbb{R}^1\) be nonempty Borel sets satisfying
\[|A + B| < |A| + |B| + \min(|A|, |B|).\]
Then
\[\text{diameter}(A) \leq |A + B| - |B|.\]
\[(9.1)\]
\[(9.2)\]

This is Proposition 3.1 of [5]; it is simply a continuum analogue of a theorem of Frei\"man concerning finite sumsets of \(\mathbb{Z}\). It can be deduced from the finite version by an approximation and limiting argument. A proof is included in §15.

A technical point is that while the sum of two Borel sets is Lebesgue measurable, the sum of two Lebesgue measurable sets need not be so. In this paper, this theorem will be applied only to superlevel sets, which are open. In that case the sum set is also open, so its measurability is elementary.

In this section we show that if an ordered triple \((A, B, C)\) of subsets of \(\mathbb{R}^1\) nearly attains equality in the Riesz-Sobolev inequality, and if \(|A| = |B|\), then under certain auxiliary hypotheses, \(C\) is nearly equal to an interval.
Lemma 9.2. Let \((A, B, C)\) be an \(\eta\)-strictly admissible ordered triple of Lebesgue measurable subsets of \(\mathbb{R}\) having positive, finite Lebesgue measures. Suppose that
\[
|A| = |B| \quad (9.3)
\]
\[
|A| - |C| \geq 4D(A, B, C)^{1/2} \quad (9.4)
\]
\[
\mathcal{D}(A, B, C)^{1/2} < \frac{1}{2\eta}|A|. \quad (9.5)
\]
Then there exists an interval \(I \subset \mathbb{R}\) such that
\[
|C \triangle I| < 14D(A, B, C)^{1/2}. \quad (9.6)
\]
Both hypotheses \((9.3)\) and \((9.4)\) are unnatural from the perspective of our main theorem, and will eventually be circumvented in \((11)\). The main step in the proof of Lemma 9.2 will be:

Lemma 9.3. Let \((A, B, C)\) be an \(\eta\)-strictly admissible ordered triple of subsets of \(\mathbb{R}\) having positive, finite Lebesgue measures. Let \(\mathcal{D} = \mathcal{D}(A, B, C)\). Suppose that \(|A| = |B|\), \(\mathcal{D}^{1/2} < \frac{1}{2\eta}|A|\), and \(|C| \leq |A| - 4D^{1/2}\). Then
\[
|S_{A,-A}(\gamma) - 2|C| \leq 8\mathcal{D}^{1/2}. \quad (9.7)
\]

Proof. The Riesz-Sobolev inequality guarantees that \(\mathcal{D} \geq 0\), while there is the trivial upper bound
\[
|S_{A,-A}(\gamma) - 2|C| \leq \max(|A|, |B|, |C|)^2 = |A|^2. \quad (9.8)
\]

Define \(\beta\) by
\[
C = |A| + |B| - 2\beta = 2|A| - 2\beta
\]
so that \(\beta = |A| - \frac{1}{2}|C|\) and \(\gamma = |A| - |C| = 2\beta - |A|\).

By the \(\eta\)-strict admissibility hypothesis, \(\gamma \geq \eta|A| < 2D^{1/2}\). On the other hand, \(|C| \leq |A| \leq |A| + |A| - 2D^{1/2}\) since \(2D^{1/2} < \eta|A| \leq |A|\). Therefore the triple \((A, B, C)\) satisfies the hypothesis of Lemma 5.3. We conclude that \(S_{A,B}^{\beta}\) satisfies
\[
C \setminus I \leq 2D^{1/2}. \quad (9.9)
\]

By Lemma 5.3
\[
\mathcal{D}(A, B, \beta) \leq \mathcal{D}. \quad (9.10)
\]
\[
|S_{A,B}^{\beta}| \geq |C - 2D^{1/2}| > 0, \text{ while } |S_{A,B}^{\beta}| \leq |C| + 2D^{1/2} \leq |A| + 2D^{1/2} \leq 2|A| = |A| + |B|. \quad (9.11)
\]
Corollary 7.2 therefore applies. The quantity denoted by \((\sigma - \tau)^2\) in Corollary 7.2 is here \(\frac{1}{4}(|S_{A,B}^{\beta}| - |C|)^2 \leq \mathcal{D}\), so the Corollary gives
\[
\mathcal{D}'(A, B, \beta) = \mathcal{D}(A, B, \beta) + \frac{1}{4}|S_{A,B}^{\beta}| - |C| \leq \mathcal{D} + \mathcal{D} = 2\mathcal{D}.
\]
Since \(|B| = |A|\),
\[
\mathcal{D}'(A, B, \beta) = |A| \cdot |B| = 2|A| + |B| = |A|^2 - \int_\beta^\infty |S_{A,B}^{\beta}| \, dt = (|A| - \beta)^2 - \int_\beta^\infty |S_{A,B}^{\beta}| \, dt.
\]
Thus we have established near equality in the KPRGT inequality:
\[
\int_\beta^\infty |S_{A,B}^{\beta}| \, dt \geq (|A| - \beta)^2 - 2\mathcal{D}.
\]
This can be equivalently written as
\[
\mathcal{D}'(A, B, \beta) \leq 2\mathcal{D}. \quad (9.12)
\]
Up to this point, the analysis has been rather formal, involving manipulations of expressions involving integrals of measures of superlevel sets but using very little about the definitions of those superlevel sets. We now introduce the underlying additive structure of the Riesz-Sobolev and KPRGT inequalities through the relation
\[(9.13) \quad S_{A,B}(t) - S_{A,B}(t) \subset S_{A,-A}(2t - |B|).\]
This holds for any \(t > S\) the Riesz-Sobolev and KPRGT inequalities through the relation definitions of those superlevel sets. We now introduce the underlying additive structure of
\[14 \text{MICHAEL CHRIST}\]
If \(R\) by the Brunn-Minkowski inequality for \(\gamma\)
targeted to obtain
\[(9.17) \quad \int_\gamma |S_{A,-A}(\alpha)| d\alpha \geq 2 \int_\gamma |S_{A,B}(\frac{1}{2}(\alpha + |A|))| d\alpha.
= 4 \int_\beta |S_{A,B}(t)| dt
\geq 4(|A| - \beta)^2 - 8D
= 4|A|^2 - 2\beta|A| + \beta^2) - 8D
= 4|A|^2 - 4(\gamma + |A|)|A| \gamma + |A|)^2 - 8D
= (|A| - \gamma)^2 - 8D.\]
That is,
\[(9.14) \quad |S_{A,B}(t) - S_{A,B}(t)| \geq 2|S_{A,B}(t)|\]
by the Brunn-Minkowski inequality for \(\mathbb{R}^1\). Since \(|B| = |A|\),
\[(9.15) \quad |S_{A,-A}(2t - |A|)| = |S_{A,-A}(2t - |B|)| \geq |S_{A,B}(t) - S_{A,B}(t)| \geq 2|S_{A,B}(t)|.
Equivocally, if \(0 < \alpha \leq |A|\) then
\[(9.16) \quad |S_{A,-A}(\alpha)| \geq 2|S_{A,B}(\frac{1}{2}(\alpha + |A|))|\).
As \(\alpha\) varies over \([\gamma, \infty]\), \(\frac{1}{2}(\alpha + |A|)\) varies over \([\beta, \infty]\). Consequently \((9.16)\) can be integgated to obtain
\[\int_\gamma |S_{A,-A}(\alpha)| d\alpha \geq 2 \int_\gamma |S_{A,B}(\frac{1}{2}(\alpha + |A|))| d\alpha.
That is,
\[D'(A,-A,\gamma) \leq 8D.\]

If \(|S_{A,-A}(\gamma)| \leq 2|A|\) then define \(\sigma\) by \(|S_{A,-A}(\gamma)| = |A| + |A| - 2\sigma\). By Corollary \(7.2\)
\((\sigma - \gamma)^2 \leq D'(A,-A,\gamma) \leq 8D\). Inserting the definition of \(\sigma\), this inequality becomes
\[(9.18) \quad \left| |S_{A,-A}(\gamma)| - (2|A| - 2\gamma) \right| \leq 2(8D)^{1/2}.\]
Since \(\gamma\) was defined to be \(|A| - |C|\), this is the desired upper bound for the quantity \(|S_{A,-A}(\gamma)| - 2|C|\).
It is not possible to have \(|S_{A,-A}(\gamma)| > 2|A|\). Indeed, by Lemma \(7.3\)
\((|A| - \gamma)^2 - 8D \leq |A|^2 - \gamma|S_{A,-A}(\gamma)|.\)
If \(|S_{A,-A}(\gamma)| > 2|A|\) it follows that \((|A| - \gamma)^2 - 8D < |A|^2 - 2\gamma|A|\), so \(\gamma^2 < 8D\). Since
\(\gamma^2 = (|A| - |C|)^2\) by its definition, this contradicts the hypothesis that \(|C| \leq |A| - 4D^{1/2}\). \(\square\)

**Remark 9.1.** This proof implicitly produces an upper bound for
\[\int_\beta \left| |S_{A,B}(t) - S_{A,B}(t)| - 2|S_{A,B}(t)| \right| dt.\]
Since $S_{A,B}(t)$ is empty for all $t \geq |A| = |B|$, this in concert with Chebyshev’s inequality provides an upper bound for $(|A| - \beta)^{-1} |S_{A,B}(t) - S_{A,B}(t)| - 2|S_{A,B}(t)|$ for most values of $t$. Therefore Theorem 9.1 could be applied to conclude that $S_{A,B}(t)$ nearly coincides with an interval, for most $t$. We will argue slightly differently below to show this for the specific parameter $t = \beta$, which is the value directly relevant to the proof of Theorem 2.1.

**Remark 9.2.** The assumption $|A| = |B|$ is essential in this argument. If $|A| > |B|$, then the lower bound (9.11) changes form. Following the resulting changes in the ensuing steps, one arrives at a modified lower bound for $\int_{\infty}^{\gamma} |S_{A,A}(\alpha)| \, d\alpha$ in which the main term becomes $4(|A| - |B| - \beta |A| - \beta |B| + \beta^2)$. Whenever $|B| < |A|$, this quantity is strictly less than the desired $(|A| - \gamma)^2$, which represents equality in the KPRGT inequality for $1_A * 1_{-A}$.

**Remark 9.3.** Because the aim is to attain an upper bound for $|S_{A,A}(\gamma)|$, it is natural to execute the reasoning above in the framework of near equality for the KPRGT inequality, rather than for the Riesz-Sobolev inequality. The latter, in its form (4.12), is only relevant if the measure of the superlevel set in question is known; but it is precisely this measure which we seek here to control. Thus the close connection between the two inequalities is an essential element of our reasoning.

**Proof of Lemma 9.2.** Write $D = D(A, B, C)$. Define $\beta, \gamma$ as above. Since $S_{A,B}(\beta) - S_{A,B}(\beta) \subset S_{A,A}(\gamma)$,

(9.19) 

$|S_{A,B}(\beta) - S_{A,B}(\beta)| \leq |S_{A,A}(\gamma)| \leq 2|C| + 8D^{1/2}$

by (9.7). On the other hand, $2|S_{A,B}(\beta)| \geq 2|C| - 4D^{1/2}$ by (9.9). Thus

$|S_{A,B}(\beta) - S_{A,B}(\beta)| \leq 2|S_{A,B}(\beta)| + 12D^{1/2}$.

Since

$|S_{A,B}(\beta)| \geq |C| - 2D^{1/2} \geq \eta |A| - 2D^{1/2} \geq \frac{1}{2} \eta \max(|A|, |B|, |C|)$,

$12D^{1/2} < |S_{A,B}(\beta)|$ and thus $|S_{A,B}(\beta) - S_{A,B}(\beta)| < 3|S_{A,B}(\beta)|$. Therefore Theorem 9.1 applies, and certifies that $S_{A,B}(\beta)$ is contained in some interval $I$ which satisfies $|I| \leq |S_{A,B}(\beta)| + 12D^{1/2}$.

It has already been noted that $|C \triangle S_{A,B}(\beta)| < 2D^{1/2}$. Therefore

$|C \triangle I| < 14D^{1/2}$. 

\square

## 10. Truncations

Next we review and generalize a device used by Burchard [11] and related to work of F. Riesz [14], which makes it possible to modify the measures of the sets $A, B, C$ without sacrificing the hypothesis of near equality in the Riesz-Sobolev inequality. This device will be used to remove the undesirable hypotheses (9.3) and (9.4) from Lemma 9.2. As developed here, this device involves two free parameters $\eta, \eta'$. The works [11] and later [7] exploited only the restricted case $\eta = \eta'$, but the generalization to distinct parameters $\eta, \eta'$ will be quite useful in [11].

**Definition 10.1.** Let $S \subset \mathbb{R}$ be a Lebesgue measurable set with finite, positive measure. Let $\eta, \eta' > 0$, and assume that $\eta + \eta' < |S|$. The truncation $S_{\eta,\eta'}$ of $S$ is

(10.1) 

$S_{\eta,\eta'} = S \cap [a, b]$
Lemma 10.1. For any sets $A, B, C$ and any $\eta, \eta' \geq 0$ such that $\eta + \eta' < \min(|A|, |B|)$,
\begin{equation}
(10.3)
(1_A \ast 1_B, 1_C) \leq (1_{A_{\eta, \eta'}} \ast 1_{B_{\eta', \eta}}, 1_C) + (\eta + \eta')|C|.
\end{equation}

A point to note is that $B_{\eta', \eta}$ appears, rather than $B_{\eta, \eta'}$. In the special case $\eta = \eta'$, this lemma appears in the paper of Burchard \[1\], and appears to be rooted in work of F. Riesz \[14\].

Proof. Consider any $x \in \mathbb{R}$ and set $\tilde{B} = x - B$. Then
\begin{equation}
(10.4)
x - B_{\eta', \eta} = (x - B)_{\eta, \eta'} = \tilde{B}_{\eta, \eta'}
\end{equation}
(note that $\tilde{B}_{\eta', \eta}$ means $(\tilde{B})_{\eta, \eta'}$ and that $\tilde{B}_{\eta', \eta}$ appears, rather than $\tilde{B}_{\eta, \eta'}$) and
\begin{equation}
(10.5)
(1_{A_{\eta, \eta'}} \ast 1_{B_{\eta', \eta}})(x) = |A_{\eta, \eta'} \cap (x + (-B)_{\eta, \eta'})| = |A_{\eta, \eta'} \cap \tilde{B}_{\eta, \eta'}|,
\end{equation}
while
\begin{equation}
(1_A \ast 1_B)(x) = |A \cap (x - B)| = |A \cap \tilde{B}|.
\end{equation}

Let $a < a'$ and $b < b' \in \mathbb{R}$ satisfy
\begin{equation}
A_{\eta, \eta'} = A \cap [a, a']
\end{equation}
\begin{equation}
\tilde{B}_{\eta, \eta'} = B \cap [b, b']
\end{equation}
with $a, b$ minimal and $a', b'$ maximal. There are four possible cases to be analyzed, depending on which of $a, b$ is larger, and which of $a', b'$ is larger. If for instance $a \leq b$ and $a' \leq b'$ then
\begin{equation}
(10.6)
A \cap \tilde{B} = [A_{\eta, \eta'} \cap \tilde{B}_{\eta, \eta'}] \cup [(A \cap \tilde{B} \cap (-\infty, b)] \cup [(A \cap \tilde{B} \cap (a', \infty))
\end{equation}
\begin{equation}
\subset [A_{\eta, \eta'} \cap \tilde{B}_{\eta, \eta'}] \cup [\tilde{B} \cap (-\infty, b)] \cup [A \cap (a', \infty)]
\end{equation}
\begin{equation}
\subset A_{\eta, \eta'} \cap \tilde{B}_{\eta, \eta'} + \eta + \eta'.
\end{equation}
The other three cases are analyzed in the same way, with the same result (10.5).

Thus we have shown that for every $x \in \mathbb{R}$,
\begin{equation}
(10.7)
(1_A \ast 1_B)(x) \leq (1_{A_{\eta, \eta'}} \ast 1_{B_{\eta', \eta}})(x) + \eta + \eta'
\end{equation}
Integrate both sides with respect to $x \in C$ to conclude the proof. \hfill \Box

Lemma 10.2. Let $\eta, \eta' > 0$. For any intervals $I, J, K \subset \mathbb{R}$ centered at 0 and satisfying $|I| > \eta + \eta'$, $|J| > \eta + \eta'$, and $|K| \leq |I| + |J|$,
\begin{equation}
(10.8)
\langle 1_I \ast 1_J, 1_K \rangle = \langle 1_{(I_{\eta, \eta'})^*} \ast 1_{(J_{\eta', \eta})^*}, 1_K \rangle + (\eta + \eta')|K|.
\end{equation}

The verification is a straightforward calculation. This statement also appears in \[1\], in the case $\eta = \eta'$.

Corollary 10.3. Let $\eta, \eta', \mathcal{D} \geq 0$. Let $A, B, C$ satisfy $|A| > \eta + \eta'$, $|B| > \eta + \eta'$, and $|C| \leq |A| + |B|$. If
\begin{equation}
(10.9)
\langle 1_A \ast 1_B, 1_C \rangle = \langle 1_A^{*}, 1_B^{*}, 1_C^{*} \rangle - \mathcal{D}
\end{equation}
then
\begin{equation}
(10.10)
\langle 1_{A_{\eta, \eta'}} \ast 1_{B_{\eta', \eta}}, 1_C \rangle \geq \langle 1_{(A_{\eta, \eta'})^*} \ast 1_{(B_{\eta', \eta})^*}, 1_C^* \rangle - \mathcal{D}.
\end{equation}
Proof. By Lemmas 10.1 and 10.2

\[
\langle 1_{A_{\eta,\eta'}} \ast 1_{B_{\eta',\eta}}, 1_C \rangle \geq \langle 1_A \ast 1_B, 1_C \rangle - (\eta + \eta')|C|
\]

\[
= \langle 1_A \ast 1_B, 1_{C*} \rangle - D - (\eta + \eta')|C|
\]

\[
= \langle (1_{A_{\eta,\eta'}}) \ast 1_{(B_{\eta',\eta})^*}, 1_{C*} \rangle + (\eta + \eta')|C| - D - (\eta + \eta')|C|.
\]

We pause to indicate how the Riesz-Sobolev inequality can be proved using truncations. Let measurable sets \(A, B, C \subset \mathbb{R}^1\) with positive, finite Lebesgue measures be given. If \((A, B, C)\) is not strictly admissible then we may suppose without loss of generality that \(|C| \geq |A| + |B|\). Then \(\langle 1_A \ast 1_B, 1_C \rangle \leq \|1_A \ast 1_B\|_{L^1} = |A| \cdot |B|\) while \(\langle 1_{A*} \ast 1_{B*}, 1_{C*} \rangle = |A| \cdot |B|\) since \(A* + B* \subset C^*\). If \((A, B, C)\) is strictly admissible, choose \(\rho^* > 0\) so that \((|A| - \rho^*) + (|B| - \rho^*) = |C|\). Then \(\rho^* < \min(|A|, |B|)\); for instance, the inequality \(\rho^* < |A|\) is equivalent by a bit of algebra to \(|B| < |A| + |C|\), which holds by strict admissibility. Set \(\rho = \frac{1}{2}\rho^*\). Then

\[
\langle 1_A \ast 1_B, 1_C \rangle \leq \langle 1_{A_{\rho,\rho}} \ast 1_{B_{\rho,\rho}}, 1_C \rangle + \rho|C|
\]

\[
\leq |A_{\rho,\rho}| \cdot |B_{\rho,\rho}| + \rho|C|
\]

\[
= (|A| - \rho^*)(|B| - \rho^*) + \rho|C|
\]

\[
= (|A*| - \rho^*)(|B*| - \rho^*) + \rho|C^*|
\]

\[
= \langle 1_{A*} \ast 1_{B*}, 1_{C*} \rangle + \rho|C^*|
\]

\[
= \langle 1_{A*} \ast 1_{B*}, 1_{C*} \rangle.
\]

\[
□
\]

Lemma 10.4. There exists an absolute constant \(K < \infty\) with the following property. Let \(A \subset \mathbb{R}\) be a Lebesgue measurable set of positive, finite measure. Let \(\varepsilon > 0\) and \(0 < \lambda < 1\). Suppose that for any \(\rho, \rho' \geq 0\) satisfying \(\rho + \rho' = (1 - \lambda)|A|\), there exists an interval \(I \subset \mathbb{R}\) such that

\[
|A_{\rho,\rho'} \triangle I| \leq \varepsilon|A|.
\]

Then there exists an interval \(I \subset \mathbb{R}\) such that

\[
|A \triangle I| \leq K\lambda^{-1}\varepsilon|A|.
\]

If \(\lambda\) is small then \(|A_{\rho,\rho'}| = \lambda|A|\) is small relative to \(A\), so the hypothesis becomes effectively weaker, leading to the lost power \(\eta^{-1}\) in the bound.

Proof. Without loss of generality we may suppose that \(\lambda = N^{-1}\) for some \(N \in \mathbb{N}\), and then that \(\varepsilon < (4N)^{-1}\).

For \(j \in \{0, 1, 2, \cdots, 2N\}\) set \(A_j = A_{\rho,\rho}\) where \(\rho = \frac{j}{2N}|A|\) and

\[
\rho' = (1 - N^{-1})|A| - \rho = \frac{2N - 2 - j}{2N}|A|.
\]

Then \(|A_j| = N^{-1}|A|\) for each index \(j\), and \(|A_j \cap A_{j+1}| = (2N)^{-1}|A|\) for \(0 \leq j < 2N\). There exists an interval \(I_j\) satisfying \(|A_j \triangle I_j| \leq \varepsilon|A|\).

For any index \(j\),

\[
|I_j| - N^{-1}|A| = |I_j| - |A_j| \leq |A_j \triangle I_j| \leq \varepsilon|A|,
\]

so

\[
|I_j| \geq (N^{-1} - \varepsilon)|A|.
\]
On the other hand, for any \( j \in \{0, 1, 2, \ldots, 2N - 1\} \),
\[
|I_j \triangle I_{j+1}| \leq |I_j \triangle A_j| + |A_j \triangle A_{j+1}| + |A_{j+1} \triangle I_{j+1}|
\leq \varepsilon|A| + N^{-1}|A| + \varepsilon|A|,
\]
The assumption that \( \varepsilon < \frac{1}{2}N^{-1} \) is equivalent to \((N^{-1} + 2\varepsilon) < 2(N^{-1} - \varepsilon)\). Therefore \( |I_j \triangle I_{j+1}| < |I_j| + |I_{j+1}|\). This forces the two intervals \( I_j, I_{j+1} \) to intersect.

Since \( I_j \) are intervals and \( I_j \) intersects \( I_{j+1} \) for every \( j < 2N \), \( I = \cup_{j=0}^{2N} I_j \) is an interval. Since \( A \setminus I = \bigcup_j A_j \setminus I_j \subset \bigcup_j (A_j \setminus I_j) \),
\[
|A \setminus I| \leq \sum_j |A_j \setminus I_j| \leq 2N\varepsilon|A|.
\]

In the same way,
\[
|I \setminus A| \leq \sum_j |I_j \setminus A_j| \leq 2N\varepsilon|A|,
\]
so
\[
|A \triangle I| \leq 4N\varepsilon|A| \leq K\lambda^{-1}\varepsilon|A|.
\]

\( \square \)

11. The general case

We now use truncations to complete the proof of Theorem 2.1. Let \((A, B, C)\) be an \( \eta \)-strictly admissible ordered triple of Lebesgue measurable subsets of \( \mathbb{R}^1 \) with finite, positive measures.

The analysis is broken into cases, depending on the relative sizes of \(|A|, |B|, |C|\).

**Lemma 11.1.** If
\[
(11.1) \quad \mathcal{D}(A, B, C)^{1/2} \leq \frac{1}{18}\eta^2 \max(|A|, |B|, |C|)
\]
and
\[
(11.2) \quad |A| > \max(|B|, |C|) - \frac{1}{4}\eta \max(|A|, |B|, |C|)
\]
then there exists an interval \( I \) satisfying
\[
(11.3) \quad |A \triangle I| \leq K\eta^{-1}\mathcal{D}(A, B, C)^{1/2}.
\]

**Proof.** Let \( \mathcal{D} = \mathcal{D}(A, B, C) \). Choose \( \delta > 0 \) so that
\[
(11.4) \quad 4\mathcal{D}^{1/2} \leq \delta \leq \frac{1}{8}\eta \max(|A|, |B|, |C|).
\]
Define the nonnegative quantities
\[
\sigma^* = |A| - |B| + \delta, \quad \rho^* = |A| - |C| + \delta.
\]
For any nonnegative parameters \( \rho, \rho', \sigma, \sigma' \) satisfying \( \rho + \rho' = \rho^* \) and \( \sigma + \sigma' = \sigma^* \), define
\[
A = A_{\rho+\sigma, \rho'+\sigma'}, \quad B = B_{\rho', \rho}, \quad C = C_{\sigma, \sigma'}.
\]
We next verify that \((A, B, C)\) satisfies the hypotheses of Lemma 9.2
\[
|A| = |A| - \rho^* - \sigma^*
\]
\[
= |B| + |C| - |A| - 2\delta
\]
\[
\geq \eta \max(|A|, |B|, |C|) - 2\delta
\]
\[
\geq \frac{1}{8}\eta \max(|A|, |B|, |C|)
\]
since \( \delta \leq \frac{1}{2}\max(|A|, |B|, |C|) \). Therefore \( A, B, C \) all have positive Lebesgue measures, \( |B| = |C| = |B| + |C| - |A| - \delta, \) and \( |A| = |B| - \delta. \) Moreover,

\[
|A| + |B| - |C| = |A| \geq \frac{1}{2}\max(|A|, |B|, |C|) \geq \frac{1}{2}\max(|A|, |B|, |C|),
\]

so the triple \( (A, B, C) \) is \( \frac{1}{2}\eta \)-strictly admissible. By two consecutive applications of Corollary 10.3,

\[
\langle 1_A * 1_B, 1_C \rangle \geq \langle 1_{A*} * 1_{B*}, 1_{C*} \rangle - D.
\]

Since \( |C| - |A| = \delta \geq 4D^{1/2}, \) \( (B, C, A) \) satisfies (10.3). Finally (10.4) holds with \( \eta \) replaced by \( \frac{1}{2}\eta \) because

\[
|B| = |B| + |C| - |A| - \delta \geq \eta \max(|A|, |B|, |C|) - \delta \geq \frac{1}{2}\eta \max(|A|, |B|, |C|),
\]

so that \( D^{1/2} \leq \frac{1}{2}\eta |B| \). Thus \( (B, C, A) \) satisfies the hypotheses of Lemma 11.2 with \( \eta \) replaced by \( \frac{1}{2}\eta \). We conclude that there exists an interval \( I \) such that

\[
|A \triangle I| \leq 14D^{1/2}.
\]

This has been proved for \( A = A_{\rho+\sigma,\rho'+\sigma'} \) whenever \( \rho + \rho' = \rho^* \) and \( \sigma + \sigma' = \sigma^* \). By Lemma 10.2 this implies that there exists an interval \( I \) satisfying \( |A \triangle I| \leq K\lambda^{-1}D^{1/2} \) where \( K < \infty \) is an absolute constant and

\[
\lambda = 1 - \frac{\rho^* + \sigma^*}{|A|}
\]

\[
= \frac{|B| + |C| - |A| - 2\delta}{|A|}
\]

\[
\geq \eta - 2\delta|A|^{-1}
\]

\[
\geq \eta - \frac{2}{\eta}|A|^{-1} \max(|A|, |B|, |C|).
\]

The hypothesis (11.1) guarantees that \( |A| \geq \frac{1}{2}\max(|A|, |B|, |C|) \) and therefore \( \lambda \geq \frac{1}{2}\eta \). Thus \( |A \triangle I| \leq K\eta^{-1}D^{1/2} \). \( \square \)

Next consider an arbitrary \( \eta \)-strictly admissible triple \( (A, B, C) \). Continue to simplify notation by writing \( D = D(A, B, C) \). By permuting these sets and invoking the identities

\[
\langle 1_A * 1_B, 1_C \rangle = \langle 1_A * 1_{-C}, 1_{-B} \rangle
\]

\[
\langle 1_A * 1_B, 1_C \rangle = \langle 1_B * 1_A, 1_C \rangle
\]

we may suppose that \( |A| \geq |B| \geq |C| \). Then \( |A| \geq \max(|B|, |C|) \geq \max(|B|, |C|) - \frac{1}{2}\eta \max(|A|, |B|, |C|) \), so Lemma 11.1 can be applied to conclude that \( A \) has suitably small symmetric difference with some interval. If \( |B| \geq |A| - \delta, \) then the same applies also to \( B \); likewise for \( C \) if \( |C| \geq |A| - \delta \). \( \square \)

Continuing to assume that \( |A| \geq |B| \geq |C| \), consider the case in which \( |B| < |A| - \frac{1}{4}\eta |A| \), so that Lemma 11.1 can no longer be applied directly to \( B \). Consider the triple \( (\tilde{A}, \tilde{B}, \tilde{C}) \) with \( \tilde{B} = A_{\rho,\rho}, \tilde{A} = B, \) and \( \tilde{C} = C_{\rho,\rho'} \) where \( \rho, \rho' \) are nonnegative and \( \rho + \rho' = \rho^* = |A| - |B|. \) Then \( |\tilde{A}| = |\tilde{B}| > |\tilde{C}| \), and

\[
|\tilde{A}| - |\tilde{C}| = |B| - (|C| - (|A| - |B|)) = |A| - |C| \geq \frac{1}{4}\eta |A|^2 \geq \frac{1}{4}\eta \max(|\tilde{A}|, |\tilde{B}|, |\tilde{C}|).
\]

Moreover

\[
|\tilde{B}| + |\tilde{C}| - |\tilde{A}| = |\tilde{C}| = |C| - \rho^* = |C| + |B| - |A| \geq \eta |A| \geq \eta |\tilde{A}|,
\]
so \((\tilde{A}, \tilde{B}, \tilde{C})\) is \(\eta\)-strictly admissible. And

\[
\max(|\tilde{A}|, |\tilde{B}|, |\tilde{C}|) = |\tilde{A}| = |B| \geq \eta \max(|A|, |B|, |C|);
\]

there is a loss of a factor of \(\eta\) here in comparison with \(\max(|A|, |B|, |C|)\).

The discrepancy \(\tilde{D} = (1_{\tilde{A}} \ast 1_{\tilde{B}}, 1_{\tilde{C}}) - (1_{\tilde{A}} \ast 1_{\tilde{B}}, 1_{\tilde{C}})\) satisfies \(\tilde{D} \leq \tilde{D}\). Assuming that \(D^{1/2} \leq K^{-1} \eta^4 |A|\) for a sufficiently large constant \(K\), it follows that

\[
D^{1/2} \leq K^{-1} \eta^2 |\tilde{A}|.
\]

Since \(|\tilde{A}| - |\tilde{C}| \geq \frac{1}{4} \eta \max(|\tilde{A}|, |\tilde{B}|, |\tilde{C}|)\), Lemma \(\ref{11.1}\) can be applied to \((\tilde{A}, \tilde{B}, \tilde{C})\) to conclude that there exists an interval \(J\) satisfying

\[
|\tilde{A} \triangle J| \leq K\eta^{-1} D^{1/2}.
\]

\(\tilde{A}\) was defined to equal \(B\), so this is the desired conclusion for \(B\).

We have shown thus far that in all cases, both \(A, B\) nearly coincide with intervals, in the desired sense. Moreover, the same conclusion holds for \(C\) if \(|C| \geq |A| - \frac{1}{4} \eta |A|\). It remains only to analyze \(C\), under the assumption that \(|C| < |A| - \frac{1}{4} \eta |A|\). Let \(\rho^* = |A| - |B|\). Consider \((\tilde{A}, \tilde{B}, \tilde{C}) = (A_{\rho, \rho'}, B, C_{\rho, \rho'})\) where \(\rho, \rho'\) are nonnegative and \(\rho + \rho' = \rho^*\). Then

\[
|\tilde{C}| = |C| - (|A| - |B|) = |B| + |C| - |A| \geq \eta |A| \geq \eta |\tilde{A}|.
\]

Moreover

\[
|\tilde{A}| - |\tilde{C}| = |A| - |C| \geq \frac{1}{4} \eta |A| \geq \frac{1}{4} \eta |\tilde{A}|.
\]

Therefore if \(D^{1/2} \leq K^{-1} \eta^4 \max(|A|, |B|, |C|)\) for sufficiently large \(K\) then Lemma \(\ref{11.1}\) can be applied once more to give the desired conclusion for \(C\). This concludes the proof of Theorem \(\ref{2.1}\). \(\square\)

12. CENTERS OF INTERVALS

Let \(A, B, C, \eta, \varepsilon\) be as in the statement of Theorem \(\ref{2.1}\). The theorem states that there exist intervals \(I, J, L\) that satisfy its first conclusion \(\ref{2.4}\), with the additional property \(\ref{2.5}\) that their centers \(a, b, c\) satisfy \(|a + b - c| \leq K \eta^{-2} \varepsilon^{1/4} \max(|A|, |B|, |C|)\). We will show that \(\ref{2.5}\) holds for the centers of any intervals \(I, J, L\) that satisfy \(\ref{2.4}\), that is, \(|A \triangle I| \leq K \eta^{-1} \varepsilon^{1/2} \max(|A|, |B|, |C|)\) and likewise for \(J, L\) respectively relative to \(B, C\). Indeed, denoting by \(K\) a constant whose value is allowed to change from one occurrence to the next,

\[
\langle 1_I \ast 1_J, 1_L \rangle \geq \langle 1_{A \cap I} \ast 1_{B \cap J}, 1_{C \cap L} \rangle \geq \langle 1_A \ast 1_B, 1_C \rangle - K \max(|A|, |B|, |C|) \max(|A \setminus I|, |B \setminus J|, |C \setminus L|) \geq \langle 1_A \ast 1_B, 1_C \rangle - K \eta^{-1} \varepsilon^{1/2} \max(|A|, |B|, |C|)^2 \geq \langle 1_A \ast 1_B, 1_C \rangle - K \eta^{-1} \varepsilon^{1/2} \max(|A|, |B|, |C|)^2 - \varepsilon \max(|A|, |B|, |C|)^2 \geq \langle 1_A \ast 1_B, 1_C \rangle - K \eta^{-1} \varepsilon^{1/2} \max(|A|, |B|, |C|)^2 \geq \langle 1_J \ast 1_L, 1_I \rangle - K \eta^{-1} \varepsilon^{1/2} \max(|A|, |B|, |C|)^2 - K \max(|A|, |B|, |C|) \max(|I|, |J|, |L|) \geq \langle 1_J \ast 1_L, 1_I \rangle - K \eta^{-1} \varepsilon^{1/2} \max(|A|, |B|, |C|)^2 \geq \langle 1_I \ast 1_J, 1_L \rangle - K \eta^{-1} \varepsilon^{1/2} \max(|A|, |B|, |C|)^2.
\]
Theorem 13.1. For any \( A, B \) measures satisfying \( S(13.4) \)

Under the hypotheses of Theorem 13.1, \( \tau \) is replaced by \( \eta \).

Thus we have reduced matters to the situation in which \( (I, J, L) \) satisfies

\[
\Delta \leq \frac{1}{2} \eta \max(|K|, |J|, |L|) - K\eta^{-1/2} \varepsilon^{1/2} \max(|A|, |B|, |C|)
\]

provided that \( \varepsilon \) is sufficiently small, since it is assumed that \( \eta = O(\eta^2) \). Therefore \( (I, J, L) \) is \( \frac{1}{2} \eta \)-strictly admissible.

Thus we have reduced matters to the situation in which \( (I, J, L) \) is replaced by \( (I, J, L) \), \( \eta \) is replaced by \( \frac{1}{2} \eta \), and \( \varepsilon \) is replaced by \( K\eta^{-1/2} \varepsilon^{1/2} \). In this situation it is elementary that \( |a + b - c| \leq K\eta^{-1/2} \varepsilon / 4 \).

13. Near equality in the KPRGT inequality

The following is an analogue for the KPRGT inequality of our main theorem.

**Theorem 13.1.** For any \( \eta \in (0, \frac{1}{2}] \) there exists an absolute constant \( K < \infty \) for which the following holds. Let \( A, B \) be measurable subsets of \( \mathbb{R}^1 \) with finite, positive Lebesgue measures satisfying

\[
\min(|A|, |B|) \geq \eta (1 - \eta)^{-1} \max(|A|, |B|).
\]

Let

\[
\tau \in [\eta \max(|A|, |B|), (1 - \eta) \min(|A|, |B|)].
\]

If the deficit

\[
\mathcal{D}' = \mathcal{D}'(A, B, \tau) = \left[ |A| \cdot |B| - \tau(|A| + |B|) + \tau^2 \right] - \int_\tau^\infty |S_{A,B}(t)| \, dt
\]

satisfies

\[
\mathcal{D}' < \min(\tau^2, (|B| - \tau)^2, K^{-1}\eta^8 \max(|A|, |B|)^2)
\]

then there exist intervals \( I, J \) satisfying \( |A \triangle I| \leq K \sqrt{D'} \) and \( |B \triangle J| \leq K \sqrt{D'} \).

We will deduce this from Theorem 2.1. A preliminary step is to control the measure of \( S_{A,B}(\tau) \).

**Lemma 13.2.** Under the hypotheses of Theorem 13.1,

\[
|A| - |B| \leq |S_{A,B}(\tau)| \leq |A| + |B|.
\]

**Proof.** Write \( S = S_{A,B}(\tau) \). Assume without loss of generality that \( |A| \geq |B| \). If \( |S| < |A| - |B| \) then

\[
\int_\tau^\infty |S_{A,B}(t)| \, dt = \langle 1_A * 1_B, 1_S \rangle - \tau |S| \leq |B| \cdot |S| - \tau |S|.
\]

Now

\[
|B| \cdot |S| - \tau |S| - |A| \cdot |B| + \tau(|A| + |B|) - \tau^2 = (|B| - \tau)(|S| - (|A| - |B|)) - (|B| - \tau)^2.
\]

Therefore

\[
\mathcal{D}' \geq (|B| - \tau)(|A| - |B| - |S|) + (|B| - \tau)^2.
\]
Since $|B| > (1 - \eta)|B| = (1 - \eta) \min(|A|, |B|) \geq \tau$ and $|A| > |B| + |S|$, this contradicts the hypothesis that $\mathcal{D}' < ((|B| - \tau)^2$.

If on the other hand $|S| > |A| + |B|$ then

$$\int_\tau^{\infty} |S_{A,B}(t)| \, dt \leq |A| \cdot |B| - \tau |S|$$

so

$$\mathcal{D}' \geq |A| \cdot |B| - \tau (|A| + |B|) + \tau^2 - \left(|A| \cdot |B| - \tau |S|\right) = \tau (|S| - |A| - |B|) + \tau^2,$$

contradicting the assumption that $\mathcal{D}' < \tau^2$. \hfill \Box

Proof of Theorem 13.1. Define $\sigma$ by $|S_{A,B}(\tau)| = |A| + |B| - 2\sigma$. Lemma 13.2 asserts that $(A, B, S_{A,B}(\tau))$ satisfies the hypothesis of Corollary 7.2. Therefore

$$\langle 1_A \ast 1_B, 1_{S_{A,B}(\tau)} \rangle - \langle 1_A \ast 1_B, 1_{S_{A,B}(\tau)} \rangle \leq \mathcal{D}'$$

and

$$\left| |S_{A,B}(\tau)| - (|A| + |B| - 2\tau) \right| = 2|\sigma - \tau| \leq 2\mathcal{D}'^{1/2}.$$

Simple calculations using the hypotheses 13.1 and $\mathcal{D}' \leq \frac{1}{2} \eta^2 \max(|A|, |B|)^2$ show that

$$|S_{A,B}(\tau)| \leq (1 + 2\eta)(|A| + |B|) \leq (2 + 4\eta) \max(|A|, |B|)$$

and that the ordered triple $(A, B, S_{A,B}(\tau))$ is strictly $\gamma$-admissible, where

$$\gamma = 2\eta^2 - 2\mathcal{D}'^{1/2} \max(|A|, |B|)^{-1} \geq \eta^2.$$

Theorem 2.1 applies and yields the desired conclusion, provided that

$$\max(|A|, |B|)^{-2} \mathcal{D}' \leq K^{-1}\gamma^4,$$

that is, $\mathcal{D}' \leq K^{-1}\gamma^4 \max(|A|, |B|)^2$. The hypothesis $\mathcal{D}' \leq K^{-1}\gamma^4 \max(|A|, |B|)^2$ ensures this. \hfill \Box

14. Equivalence of two formulations

As was shown in Lemma 4.12, the Riesz-Sobolev inequality implies the upper bound

$$\int_{S_{A,B}(\tau)} 1_A \ast 1_B \leq |A| \cdot |B| - \sigma^2$$

where $|S_{A,B}(\tau)| = |A| + |B| - 2\sigma$, provided that $|S_{A,B}(\tau)|$ lies in $[\max(|A|, |B|) - \min(|A|, |B|), |A| + |B|]$. The Riesz-Sobolev inequality concerns general sets, rather than only superlevel sets, but is nonetheless a consequence of this one by formal reasoning. For the sake of completeness we indicate here a proof. In conjunction with Corollary 7.2 this provides a proof that the KPRGT inequality for $\mathbb{R}^1$ implies the Riesz-Sobolev inequality for $\mathbb{R}^1$.

Recall the notation $\sigma(t) = \sigma$ where $t = |A| + |B| - 2\sigma$. It suffices to consider $\int_E 1_A \ast 1_B$ when $0 < |E| < |A| + |B|$, for sets $E \subset \{x : (1_A \ast 1_B)(x) > 0\}$. Among all sets $E$ of a given measure $|E|$, $\int_E 1_A \ast 1_B$ is maximized when $E$ is a superlevel set $S_{A,B}(\tau)$, where $\tau$ is chosen so that $|S_{A,B}(\tau)| = |E|$, provided that such a value of $\tau$ exists. In this case, 4.12 for this value of $\tau$ implies that $\int_E 1_A \ast 1_B \leq |A| \cdot |B| - \sigma(|E|)^2$.

Consider next the case in which $0 < |E| < |A| + |B|$, but no such parameter $\tau$ exists. By the Brunn-Minkowski inequality, $\{x : (1_A \ast 1_B)(x) > 0\}$ has measure $\geq |A| + |B| > |E|$. $\tau \mapsto |S_{A,B}(\tau)|$ is a nonincreasing upper semicontinuous function. Therefore there exists $\tau$ such that $|S_{A,B}(\tau)| < |E|$, but $\mathcal{S} = \{x : (1_A \ast 1_B)(x) = \tau\}$ satisfies $|\mathcal{S}| + |S_{A,B}(\tau)| \geq |E|$. Write $\mathcal{S} = S_{A,B}(\tau)$. Choose a measurable set $\mathcal{S}^1 \subset \{x : (1_A \ast 1_B)(x) = \tau\}$ Then
\[ \int_{E^\dagger} 1_A * 1_B \leq \int_{E^\dagger} 1_A * 1_B \] for any subset \( E^\dagger \) of \( S \cup S \) satisfying \( |E^\dagger| = |E| \) and \( E^\dagger \supset S \). So it suffices to bound \( \int_{E^\dagger} 1_A * 1_B \) for such sets.

Let \( T = S \cup S \) if \( |S| + |S| \leq |A| + |B| \), and if \( |S| + |S| > |A| + |B| \) let \( T \) be a measurable set satisfying \( S \subset T \subset S \cup S \) with \( |T| = |A| + |B| \).

\[
\int_{E^\dagger} 1_A * 1_B = \int_T 1_A * 1_B - \tau(|T|) \leq |A| \cdot |B| - \sigma(|T|)^2 - \tau(|T| - |E|).
\]

By the same reasoning, since \( S \subset E^\dagger \) and \( 1_A * 1_B \equiv \tau \) on \( E^\dagger \setminus S \),

\[
\int_{E^\dagger} 1_A * 1_B \leq |A| \cdot |B| - \sigma(|S|)^2 + \tau(|E| - |S|),
\]

so it suffices to verify that

\[
(14.1) \quad \min \left( -\sigma(|T|)^2 - \tau|T|, -\sigma(|S|)^2 - \tau|S| \right) \leq -\sigma(|E|)^2 - \tau|E|.
\]

Set \( a = |S|, b = |T|, \) and \( x = |E| \). For \( t \in \mathbb{R} \) define

\[
h(t) = -\sigma(t)^2 - \tau t = -\frac{1}{4}(|A| + |B| - t)^2 - \tau t.
\]

In these terms, (14.1) becomes

\[
(14.2) \quad \min (h(a), h(b)) \leq h(x).
\]

Since \( a \leq x \leq b \), and since \( h \) is concave, (14.2) holds.

15. Proof of Theorem 9.1

Theorem 9.1, a continuum version of a theorem of Freiman concerned with finite sets, was proved in [3]. We repeat the proof here, again for the sake of completeness.

**Lemma 15.1.** Let \( A, B \subset \mathbb{R} \) be compact sets. Let \( c \in \mathbb{R} \) and \( I = (-\infty, c] \). If \( B \cap I \neq \emptyset \) then

\[
|A + (B \cap I)| - |A| - |B \cap I| \leq |A + B| - |A| - |B|.
\]

**Proof.** Since \( B \) is closed, we may suppose without loss of generality that \( c \in B \). By independently translating \( A, B \) we may assume that \( c = 0 \) and that the maximal element of \( A \) is equal to 0. Then \( A + (B \cap I) \subset (-\infty, 0] \), while \( B \setminus I = \{0 \} + (B \setminus I) \subset (0, \infty) \). Since \( A + B \supset \{0 \} + (B \setminus I), A + B \) contains both \( A + (B \cap I) \) and \( \{0 \} + (B \setminus I), \) which have at most the single point 0 in common. Therefore

\[
|A + B| \geq |A + (B \cap I)| + |B \setminus I|,
\]

and consequently

\[
|A + (B \cap I)| - |A| - |B \cap I| \leq |A + B| - |B \setminus I| - |A| - |B \cap I| = |A + B| - |A| - |B|.
\]

\( \square \)

It suffices to prove Theorem 9.1 for compact sets. Indeed, let \( A, B \) be measurable sets satisfying the hypothesis. It suffices to prove that any compact subset \( K \subset A \) has diameter \( \leq |A + B| - |B| \). Let \( \varepsilon > 0 \) be arbitrary. Choose compact sets \( \tilde{K} \) satisfying \( K \subset \tilde{K} \subset A \) and \( L \subset B \) such that \( |A| - |\tilde{K}| \) and \( |B| - |L| \) are sufficiently small to ensure that \( |\tilde{K}| + |L| + \min(|\tilde{K}|, |L|) > |A + B| \) and \( |A + B| - |L| < |A + B| - |B| + \varepsilon \). Then \( |\tilde{K} + L| - |L| \leq |A + B| - |L| < |A + B| - |B| + \varepsilon \) and

\[
|\tilde{K} + L| < |\tilde{K}| + |L| + \min(|\tilde{K}|, |L|).
\]
Therefore if the conclusion of Theorem 9.1 is assumed to hold for compact sets, it follows that for any \( \varepsilon > 0 \),
\[
\text{diameter}(K) \leq \text{diameter}(\tilde{K}) < |A + B| - |B| + \varepsilon.
\]

**Proof of Theorem 9.1.** Let \( A, B \) be arbitrary compact sets that satisfy \(|A + B| < |A| + |B| + \min(|A|, |B|)\). Consider first the case in which \( \text{diameter}(A) \geq \text{diameter}(B) \). By scaling we may assume without loss of generality that \( \text{diameter}(A) = 1 \). By independently translating \( A, B \) we may assume that \( 0 = \min(A) \) and \( 1 = \max(A) \). Thus \( A + B \subset [0, 2] \).

Let \( \pi : \mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z} \) be the quotient map. Since \( A, B \) have diameters \( \leq 1 \), \( |\pi(A)| = |A| \) and \( |\pi(B)| = |B| \). Since \( 0 \in A, B \subset A + B \) and therefore \( \pi(B) \subset \pi(A + B) \).

We claim that
\[
(15.1) \quad |A + B| \geq |\pi(A + B)| + |B|.
\]
Indeed, there exists at least one measurable set \( S \subset A + B \) satisfying both
\[
\begin{align*}
\pi(S) &= \pi(A + B) \setminus \pi(B) \\
|S| &= |\pi(A + B) \setminus \pi(B)|.
\end{align*}
\]
Choose such a set \( S \).

Then \( A + B \) contains the three sets \( B, \{1\} + B, \) and \( S \). Since \( \pi(B) = \pi(\{1\} + B) \), and since \( \pi(S) \) is disjoint from \( \pi(B) \), \( S \) is disjoint from \( B \cup (\{1\} + B) \). The two sets \( B \subset [0, 1] \) and \( \{1\} + B \subset [1, 2] \) can have at most one element in common since \( \text{diameter}(B) \leq 1 \).

Therefore
\[
|A + B| \geq |B| + |\{1\} + B| + |S| \\
= 2|B| + |S| \\
= 2|\pi(B)| + |\pi(A + B) \setminus \pi(B)| \\
= 2|\pi(B)| + |\pi(A + B)| - |\pi(B)| \\
= |\pi(A + B)| + |\pi(B)| \\
= |\pi(A + B)| + |B|,
\]
as claimed.

A theorem of Kemperman [10] (see also [15] and [18] for an alternative proof) states that for any Borel subsets of \( \mathbb{T} \), \( |A + B| \geq \min(|A| + |B|, 1) \). Apply this to \( \pi(A), \pi(B) \), noting that \( \pi(A) + \pi(B) = \pi(A + B) \), to conclude that
\[
(15.2) \quad |\pi(A + B)| \geq \min(|\pi(A)| + |\pi(B)|, 1) = \min(|A| + |B|, 1).
\]

If \(|A| + |B| \leq 1 \) then \(|\pi(A + B)| \geq |A| + |B| \) and therefore by (15.1), \(|A + B| \geq |A| + |B| + \min(|A|, |B|)\), contradicting the hypothesis \(|A + B| < |A| + |B| + \min(|A|, |B|)\). Therefore \(|A| + |B| \geq 1 \).

By (15.2), this forces \(|\pi(A + B)| \geq 1 \).

By (15.1), \(|A + B| \geq 1 + |B| \). By the normalization \( 1 = \text{diameter}(A) \), this is equivalent to
\[
\text{diameter}(A) = 1 \leq |A + B| - |B|,
\]
as was to be proved.

There remains the case in which \( \text{diameter}(A) < \text{diameter}(B) \). Applying the case already treated with the roles of \( A, B \) reversed then gives \( \text{diameter}(B) \leq |A + B| - |A| \), so by transitivity \( \text{diameter}(A) < |A + B| - |A| \). If \(|A| \geq |B| \), this gives a stronger bound than required. In particular, we have established the desired inequality whenever \(|A| = |B| \), regardless of which set has the larger diameter.
If $|B| > |A|$, choose $I = (-\infty, c]$ so that $|B \cap I| = |A|$. By Lemma 15.1,

$$|A + (B \cap I)| \leq |A| + |B \cap I| + (|A + B| - |A| - |B|).$$

Therefore

$$|A + (B \cap I)| < |A| + |B \cap I| + \min(|A|, |B|)$$

$$= |A| + |B \cap I| + \min(|A|, |B \cap I|),$$

so we can conclude from what is shown above that

$$\text{diameter}(A) \leq |A + (B \cap I)| - |B \cap I|$$

$$= \left( |A + (B \cap I)| - |A| - |B \cap I| \right) + |A|$$

$$\leq \left( |A + B| - |A| - |B| \right) + |A|$$

$$= |A + B| - |B|.$$

□

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Michael Christ, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA
E-mail address: mchrist@math.berkeley.edu