COMPLEMENTING MAPS, CONTINUATION
AND GLOBAL BIFURCATION

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ABSTRACT. We state, and indicate some of the consequences of, a theorem whose sole assumption is the nonvanishing of the Leray-Schauder degree of a compact vector field, and whose conclusions yield multidimensional existence, continuation and bifurcation results.

Complementing maps and the Theorem. Let $X$ be a Banach space, $m$ be a positive integer, and $O \subseteq \mathbb{R}^m \times X$ be open. Suppose $f: O \rightarrow X$ is an $m$-parameter compact vector field; i.e. $f(\lambda, x) = x - F(\lambda, x)$, for $(\lambda, x) \in O$, where $F$ is continuous and maps bounded sets into relatively compact sets. A continuous map $g: O \rightarrow \mathbb{R}^m$, which maps bounded sets into bounded sets, will be called a complement for $f: O \rightarrow X$ provided that the Leray-Schauder degree, $\deg((g, f), O, 0)$, is defined and nonzero: $(g, f)((\lambda, x)) = (g(\lambda, x), f(\lambda, x))$, for $(\lambda, x) \in O$, and since $O$ is not assumed to be bounded, “defined” means $(g, f)^{-1}(0)$ is compact.

By cohomology we will mean Čech cohomology with integral coefficients. By dimension of a topological space we mean the Čech-Lebesgue covering dimension, and if $p \in A$, the space $A$ will be said to have dimension at least $m$ at $p$ provided that each neighborhood, in $A$, of $p$ has dimension at least $m$.

THEOREM. Let $X$ be a Banach space, $m$ be a positive integer, and $O \subseteq \mathbb{R}^m \times X$ be open. Suppose that $f: O \rightarrow X$ is complemented by $g: O \rightarrow \mathbb{R}^m$. Then there exists a closed connected subset, $C$, of $f^{-1}(0)$, whose dimension at each point is at least $m$, and (*) whenever $K$ is a compact subset of $C$, with $g^{-1}(0) \cap C \subseteq K$, the map of pairs $g: (C, C - K) \rightarrow (\mathbb{R}^m, \mathbb{R}^m - 0)$ induces a nontrivial map in the $m$th cohomology group. In particular, $C \cap g^{-1}(0) \neq \emptyset$ and either $C$ is unbounded or $C \cap \partial O \neq \emptyset$. In the case when $f$ and $g$ are defined on $\overline{O}$ with $f^{-1}(0) \cap g^{-1}(0) \cap \partial O = \emptyset$, $C$ also has the following properties: if $C$ is bounded, then $\dim(C \cap \partial O) \geq m - 1$, when $m > 1$, and $C \cap \partial O$ has at least two points, when $m = 1$; if $g: f^{-1}(0) \cap \overline{O} \rightarrow \mathbb{R}^m$ is proper and $\dim(C \cap \partial O) < m - 1$, then $g(C) = \mathbb{R}^m$.

SKELETON OF THE PROOF. Since $\deg((g, f), O, 0) \neq 0$, by using the cup-product in cohomology, it follows that whenever $K$ is compact and $g^{-1}(0) \subseteq K \subseteq f^{-1}(0)$ the map $g: (f^{-1}(0), f^{-1}(0) - K) \rightarrow (\mathbb{R}^m, \mathbb{R}^m - 0)$ is cohomologically nontrivial. Passing to the limit over all such $K$'s we obtain a nontrivial class, $\xi$, in the $m$th Čech cohomology group with compact supports of $f^{-1}(0)$. The continuity of Čech theory enables us to choose a set, $C$, which is minimal.
among the closed subsets of $f^{-1}(0)$ to which $\xi$ restricts nontrivially. This $C$ has the properties claimed. \qed

**Some consequences of the Theorem.** In what follows, $f: O \subseteq \mathbb{R}^m \times X \rightarrow X$ is an $m$-parameter compact vector field.

1. **Continuation under global hypotheses.** Let $\lambda_0 \in \mathbb{R}^m$ and let $f_{\lambda_0}$ be the section of $f$ over the slice $O_{\lambda_0}$. One shows that if $\operatorname{deg}(f_{\lambda_0}, O_{\lambda_0}, 0) \neq 0$ then $f: O \rightarrow X$ is complemented by $g: O \rightarrow \mathbb{R}^m$ defined by $g(\lambda, x) = \lambda - \lambda_0$. Thus the theorem furnishes a description of how $f^{-1}(0)$ emanates from $O_{\lambda_0}$. This is a multidimensional refinement of the Leray-Schauder continuation principle (see [4, 6 and 7]).

2. **Continuation under local hypotheses.** Let $(\lambda_0, x_0) \in O$ and suppose that the map $x \rightarrow f(\lambda_0, x)$ has a Fréchet derivative, $L$, at $x = x_0$. Assume $L \in L(X, X)$ is invertible. Then, letting $U = O - \{(\lambda_0, x) | x \neq x_0, f(\lambda_0, x) = 0\}$, one shows that $f: U \rightarrow X$ is complemented by $g: U \rightarrow \mathbb{R}^m$ defined by $g(\lambda, x) = \lambda - \lambda_0$. Thus, there is an $m$-dimensional connected subset, $C$, of $f^{-1}(0) \cap U$, which contains $(\lambda_0, x_0)$, and which is either unbounded or $C \cap \partial O \cup \{(\lambda_0, x) | x \neq x_0, f(\lambda_0, x) = 0\} \neq \emptyset$. Another global version of the implicit function theorem was obtained in [3].

3. **Nonlinear perturbation of linear Fredholm operators.** Let $\Omega \subseteq \mathbb{R}^2$ be simply connected, open and bounded, with $\partial \Omega$ a smooth closed curve. Suppose $\tau: \partial \Omega \rightarrow S^1$ is smooth and such that the winding number of $\tau: \partial \Omega \rightarrow S^1$ equals $-k < 0$. Given $\phi, \psi: \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ we consider the following nonlinear Riemann-Hilbert problem: find $u, v: \Omega \rightarrow \mathbb{R}$ such that, if $r = (\tau_1, \tau_2)$

\[
\begin{align*}
(i) \quad \begin{cases}
u_x - v_y = \phi(x, y, u, v), \\
v_x + u_y = \psi(x, y, u, v)
\end{cases} 
\text{in } \Omega,
(ii) \quad u \tau_1 - v \tau_2 = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(R-H)

Let $\alpha \in (0, 1)$ be such that $\psi$ and $\phi$ lie in $C^{1+\alpha}(\Omega \times A, \mathbb{R})$ for each bounded subset $A$ of $\mathbb{R}^2$ ($C^{1+\alpha}$ denotes the usual Schauder space). Under the assumption that $\psi(x, y, 0, 0) = \phi(x, y, 0, 0) = 0$, for each $(x, y) \in \Omega$, it follows that for each $r > 0$, $\{(u, v) \in C^{1+\alpha}(\Omega, \mathbb{R}^2) | (u, v) \text{ solves } \text{(R-H)}, ||(u, v)||_{1+\alpha} = r\}$ has dimension at least $2k$.

Let $W = \{(u, v) \in C^{1+\alpha}(\Omega, \mathbb{R}^2) | (u, v) \text{ satisfies (ii)}\}$, and let $L: W \rightarrow C^\alpha(\Omega, \mathbb{R}^2)$ be the linear operator defined by the left-hand side of (i). Choose $z_1, \ldots, z_k$ in $\Omega$ and define $g: W \rightarrow \mathbb{R}^{2k+1}$ by

\[
g((u, v)) = (u(z_1), v(z_1), \ldots, u(z_k), v(z_k), \int_{\partial \Omega} [\tau_1 v + \tau_2 u] ds).
\]

Letting $X = g^{-1}(0)$, the linear theory (see [10]) implies $L: X \rightarrow C^\alpha(\Omega, \mathbb{R}^2)$ has an inverse, $T$, and $W = V \oplus X$, with $\dim(V) = 2k + 1$.

If we rewrite (R-H) as $f((u, v)) = T(L - H)((u, v)) = 0$, one shows that $f: V \oplus X \rightarrow X$ is complemented by $g$ on each ball about the origin in $W$, and so we can apply the Theorem.

4. **Global bifurcation.** For simplicity, we assume $O = \mathbb{R}^m \times X$. We assume $\mathbb{R}^m \times \{0\} \subseteq f^{-1}(0)$, and call $\mathbb{R}^m \times \{0\}$ the trivial solutions of $f$. Suppose $\alpha, \beta \in \mathbb{R}^m$ are such that $(\alpha, 0)$ and $(\beta, 0)$ are not bifurcation points of $f^{-1}(0)$ and that
ind(\(f_\alpha,0\)) \neq \text{ind}(f_\beta,0), \) where “ind” denotes the Leray-Schauder index. Then, if \( \Gamma \) is any open curve (i.e. homeomorphic image of \( \mathbb{R} \)) in \( \mathbb{R}^m \times \{0\} \) which passes through \((\alpha,0)\) and \((\beta,0)\), there exists a connected set, \( C \), of nontrivial zeros of \( f \), whose dimension at each point is at least \( m \), which intersects the segment, \((\alpha,0), (\beta,0)\), of \( \Gamma \), determined by \((\alpha,0)\) and \((\beta,0)\), and either \( C \) is unbounded or \( C \) intersects \( \Gamma - \{(\alpha,0), (\beta,0)\} \).

When \( \alpha = 0 \), \( \beta = (1,0,\ldots) \) and \( \Gamma \) is the line through \( \alpha \) and \( \beta \) the proof runs as follows. Choose \( r > 0 \) such that \( f(\lambda, x) \neq 0 \) when \( 0 < ||x|| < r \) and either \( |\lambda| \leq 3r \) or \( |\lambda - \beta| \leq 3r \). Let \( h: \mathbb{R} \to [0,r] \) be continuous, vanish outside of \([-r, 1+r]\), and equal \( r \) on \([r, 1-r]\). Then define \( g: \mathbb{R}^m \times X \to \mathbb{R}^m \) by

\[
g(\lambda_1,\ldots,\lambda_m) = (||x||^2 - (h(\lambda_1))^2, \lambda_2,\ldots,\lambda_m).
\]

One shows that if \( U = \mathbb{R}^m \times \{X - \{0\}\} \), then \( \text{deg}((g,f),U,0) = \text{ind}(f_\beta,0) - \text{ind}(f_\alpha,0) \), and so \( g \) complements \( f \) on \( U \). So we extract the subset, \( C \), of \( f^{-1}(0) \cap U \), having the properties in the conclusion of the Theorem. Conclusion (\( \ast \)) implies our assertions regarding \( C \cap \Gamma \).

This bifurcation result yields the principle abstract global bifurcation results of [9 and 1]. J. Ize (see [8]) has given a proof of the bifurcation theorem in [9] using a map similar to the above \( g \).

**Remark.** In the definition of complementing map if one replaces the Leray-Schauder degree by the Browder-Petryshyn degree for \( A \)-proper mappings (see [5]) the Theorem still holds. We believe that approximation results similar to those used in [2] will also yield the Theorem when \( F \) is assumed to be condensing.

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