A PARAMETRIX FOR QUANTUM GRAVITY?

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In the sixties, DeWitt discovered that the advanced and retarded Green functions of the wave operator on metric perturbations in the de Donder gauge make it possible to define classical Poisson brackets on the space of functionals that are invariant under the action of the full diffeomorphism group of spacetime. He therefore tried to exploit this property to define invariant commutators for the quantized gravitational field, but the operator counterpart of such classical Poisson brackets turned out to be a hard task. On the other hand, in the mathematical literature, it is by now clear that, rather than inverting exactly an hyperbolic (or elliptic) operator, it is more convenient to build a quasi-inverse, i.e. an inverse operator up to an operator of lower order which plays the role of regularizing operator. This approximate inverse, the parametrix, which is, strictly, a distribution, makes it possible to solve inhomogeneous hyperbolic (or elliptic) equations. We here suggest that such a construction might be exploited in canonical quantum gravity provided one understands what is the counterpart of classical smoothing operators in the quantization procedure. We begin with the simplest case, i.e. fundamental solution and parametrix for the linear, scalar wave operator; the next step are tensor wave equations, again for linear theory, e.g. Maxwell theory in curved spacetime. Last, the nonlinear Einstein equations are studied, relying upon the well-established Choquet-Bruhat construction, according to which the fifth derivatives of solutions of a nonlinear hyperbolic system solve a linear hyperbolic system. The latter is solved by means of Kirchhoff-type formulas, while the former fifth-order equations can be solved by means of well-established parametrix techniques for elliptic operators. But then the metric components that solve the vacuum Einstein equations can be obtained by convolution of such a parametrix with Kirchhoff-type formulas. Some basic functional equations for the parametrix are also obtained, that help in studying classical and quantum version of the Jacobi identity.

Keywords: quantum gravity, Peierls bracket, parametrix

1. Introduction

The Hamiltonian road to quantization has played a key role, over the last century, in the development of quantum mechanics [1], quantum field theory in flat spacetime [2], including quantum Yang-Mills [3] and the particle physics standard model, as well as in the formulation of canonical quantum gravity [4,5,6,7,8,9,10]. The main drawback of the Hamiltonian formulation, despite its beautiful and powerful applications to the classical Cauchy problem of general relativity [11,12,13], lies in the loss of the full diffeomorphism group of four-dimensional spacetime, with the associated undoing of the unification of space and time into the spacetime manifold.
(see, however, the valuable work in Refs. [14,15] on the way to circumvent this problem).

Indeed, at classical level, the tools of global differential geometry make it possible to obtain a spacetime covariant formulation of the constraint equations [16], which turn out to be linear [17] on a bigger space, the space of multimomenta. Even earlier, at quantum level, the work of Peierls [18] and DeWitt [19] made it possible to define a Poisson bracket on the space of all field functionals that remain invariant under the action of the infinite-dimensional Lie (pseudo-)group of the theory (for gravity, this is the group of spacetime diffeomorphisms). If \( A \) and \( B \) are any two such functionals of the field variables \( \varphi^i(x) \), their classical Peierls bracket reads as

\[
(A, B) \equiv A_i \tilde{G}^{ij} B_j = \int d^4 x \int d^4 z \frac{\delta A}{\delta \varphi^i} \tilde{G}^{ij}(x, z) \frac{\delta B}{\delta \varphi^j}(z),
\]

where \( \tilde{G}^{ij} \), the supercommutator function [20,21], is the difference between advanced and retarded Green functions for the invertible operator \( F_{ij} \) acting on fields:

\[
\tilde{G}^{ij} \equiv G^{+ij} - G^{-ij} = -\tilde{G}^{ji}.
\]

The advanced and retarded Green functions are both left- and right-inverses of \( F_{ij} \), i.e.

\[
G^{\pm ij} F_{jk'} = -\delta^i_{k'} = -\delta^i_{k'} \delta(x, x'), \quad F_{ij} G^{\pm jk'} = -\delta^j_{k'} = -\delta^j_{k'} \delta(x, x').
\]

In the framework of Ref. [19], the state of the art on the application of such ideas to Einstein’s gravity was as follows.

(i) The absolute invariants (also called observables or gauge-invariant functionals) \( A \) and \( B \) satisfy, by definition, the conditions

\[
\nabla_\nu \delta T = 0, \quad T = A, B,
\]

which are a particular case of the gauge-invariance equation \( Q_\alpha T = 0 \) of Appendix A (see (A5)), with \( T = A, B \), because the infinitesimal diffeomorphism

\[
\delta g_{\mu\nu} = \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu
\]

is a particular case of the general equation \( \delta \varphi^i = Q^i_\alpha \delta \xi^\alpha \) of Appendix A (see (A3)).

(ii) The infinitesimal variation suffered from \( A \), say, reads as

\[
\delta \pm A = \int \frac{\delta A}{\delta g_{\mu\nu}} \delta \pm g_{\mu\nu} A^4 x.
\]

Since the spacetime metric is not an invariant, its variations \( \delta \pm g_{\mu\nu} \) are determined only up to a coordinate transformation (see (1.6)), while Eqs. (1.5) guarantee that the advanced and retarded variations \( \delta \pm A \) do not suffer from redundancies.
(iii) If $S$ is the Einstein-Hilbert action, the associated Euler-Lagrange equations are (hereafter $g \equiv -\det g_{\mu\nu}$)

$$0 = \frac{\delta S}{\delta g_{\mu\nu}} = G^\mu{}_{\nu} = \sqrt{g} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right).$$

(1.8)

If the action functional undergoes the change

$$S \rightarrow S + \epsilon B,$$  

(1.9)

with constant parameter $\epsilon$, the new Euler-Lagrange equations read as

$$\delta \pm G_{\mu\nu} = -\epsilon \frac{\delta B}{\delta g_{\mu\nu}}.$$  

(1.10)

The general solution of Eq. (1.10) is obtained by adding (1.6) to a particular solution determined by appropriate boundary and supplementary conditions. A convenient form of supplementary (or gauge-fixing) condition is [19]

$$\left( g^{\mu\sigma} g^{\nu\tau} - \frac{1}{2} g^{\mu\nu} g^{\sigma\tau} \right) \nabla_\nu \delta \pm g_{\sigma\tau} = 0.$$  

(1.11)

If Eq. (1.11) is fulfilled, the Euler-Lagrange equations (1.10) take the form

$$\sqrt{g} \left[ \left( g^{\mu\sigma} g^{\nu\tau} - \frac{1}{2} g^{\mu\nu} g^{\sigma\tau} \right) g^{\lambda\chi} \nabla_\lambda \nabla_\rho - 2 R^{\mu\sigma\nu\tau} \right] \delta \pm g_{\sigma\tau} = -2 \epsilon \frac{\delta B}{\delta g_{\mu\nu}}.$$  

(1.12)

One can then solve for the infinitesimal variation of the spacetime metric in the form

$$\delta \pm g_{\mu\nu} = \epsilon \int G^\pm_{\mu\nu\alpha\beta}(x,z) \frac{\delta B}{\delta g_{\alpha\beta}} d^4z,$$  

(1.13)

where $G^\pm_{\mu\nu\alpha\beta}$ are the advanced and retarded Green functions for Eq. (1.12); they satisfy

$$g^{\sigma\tau} \nabla_\tau \nabla_\sigma G^\pm_{\mu\nu\alpha\beta} - 2 R^{\sigma\tau} G^\pm_{\mu\nu\alpha\beta} = - \left( U_{\mu\alpha} U_{\nu\beta} + U_{\mu\beta} U_{\nu\alpha} - g_{\mu\nu} g_{\alpha\beta} \right) g^{-4}(x) \delta^{(4)}(x,z) g^{-4}(z).$$  

(1.14)

With the notation of Ref. [19], the indices $\alpha, \beta$ refer to the point $z$ while the indices $\mu, \nu$ refer to the point $x$. Parallel displacement along the geodesic between $x$ and $z$ is performed by the bivector $U_{\mu\alpha}$. The four-dimensional Dirac delta is a scalar density at both $x$ and $z$. By construction, the Green functions $G^\pm_{\mu\nu\alpha\beta}$ are bi-tensors (see the beginning of Sec. III), by virtue of their transformation properties at two different spacetime points.

(iv) In canonical quantum gravity, inspired by the Dirac map in quantum mechanics [22], according to which the commutator of position and momentum operators is the imaginary unit times their classical Poisson bracket, DeWitt considered the commutator defined by

$$[A, B] \equiv i \int d^4x \int d^4z \frac{\delta A}{\delta g_{\mu\nu}(x)} \bar{G}_{\mu\nu\alpha\beta}(x,z) \frac{\delta B}{\delta g_{\alpha\beta}(z)}(z),$$  

(1.15)
where ˜\(G_{\mu\nu\alpha\beta}\) is the supercommutator for quantum Einstein’s gravity, i.e.
\[
\tilde{G}_{\mu\nu\alpha\beta} \equiv G_{\mu\nu\alpha\beta}^+ - G_{\mu\nu\alpha\beta}^-,
\]
and boldface characters are here used to stress that we deal with the operator counterpart of classical Green functions. The work in Ref. [19] pointed out that, if it were possible to ignore the noncommutativity of factors on the right-hand side of Eq. (1.15), this (formal) commutator can be shown to satisfy all identities of a quantum Poisson bracket, and hence the question of a consistent quantum theory of gravity was reduced (but not solved!) to finding an appropriate definition of operator propagator.

We are here initiating a research program aimed at showing that one can get pretty close to fulfilling such a task after a careful consideration of well-known properties of classical hyperbolic equations. For this purpose, Sec. II introduces fundamental solution and parametrix of a scalar wave operator; tensor wave equations are considered in Sec. III, with emphasis on Maxwell theory in curved spacetime; nonlinear wave equations are studied in Sec. IV, by focusing on the Choquet-Bruhat method for solving Cauchy’s problem for the Einstein equations. Sec. V proves the existence of the parametrix necessary to solve the classical Einstein equations, by using a well-known technique for proving the existence of a parametrix for elliptic (rather than hyperbolic!) equations. Section VI studies the role of the parametrix in the Jacobi identity for the Peierls bracket, Sec. VII obtains novel functional equations for the parametrix, while Sec. VIII discusses the remainder in the supercommutator function and presents our concluding remarks. Relevant background material is described in the appendices: DeWitt’s notation for gauge theories, spacetime geometry for the wave equation, characteristic conoid and Kirchhoff formulas for the linear hyperbolic system associated to the nonlinear Einstein equations.

2. Fundamental solution and parametrix of a scalar wave operator

Let us consider, for simplicity, a scalar differential operator \(P\) with \(C^\infty\) coefficients
\[
Pu = \Box u + \langle a, \nabla u \rangle + bu = g^{\mu\nu} \nabla_\mu \nabla_\nu u + a^\mu \nabla_\mu u + bu,
\]
that is defined on a connected open set \(\Omega\) of four-dimensional spacetime. A fundamental solution of \(P\) is a distribution \((G_q(p), \phi(p))\) which is a function of the spacetime point \(q\) such that
\[
P G_q - \delta_q = 0. \tag{2.2}
\]
This means that
\[
(P G_q, \phi) = \phi(q), \quad \phi \in C^\infty_0(\Omega). \tag{2.3}
\]
We note that the fundamental solution is not uniquely defined, unlike the case of Green’s functions [24], for each of which there is a specific boundary condition. A \(C^\infty\) parametrix of \(P\) is instead a distribution \(\pi_q\) such that
\[
P \pi_q - \delta_q = \omega \in C^\infty(\Omega). \tag{2.4}
\]
This concept is of interest because, although a fundamental solution is a useful tool for solving linear partial differential equations, some of the problems in which it plays a role can be handled better by means of functions possessing a singularity that is not annihilated but merely smoothed out by the differential operator under investigation. The smoothing might even be so weak that the singularity is actually augmented but acquires a less rapid growth than was to be anticipated from the order of the differential operator [25].

If one studies the scalar wave equation on Minkowski spacetime in the coordinates \((ct, x^1, x^2, x^3)\), with \(x^i \equiv (x^1, x^2, x^3)\), the solution of the Cauchy problem

\[
\Box \phi = 0, \quad \phi(t = 0, x) = u_0(x), \quad \frac{\partial \phi}{\partial t}(t = 0, x) = u_1(x)
\]  

(2.5)
can be expressed in the form

\[
\phi(x, t) = \sum_{j=0}^{1} E_j(t)u_j(x),
\]

(2.6)
where, on denoting by \(\hat{u}_j(x)\) the Fourier transforms of Cauchy data, the operators \(E_j(t)\) act \([26]\) in such a way that (here \(\xi \equiv (\xi_1, \xi_2, \xi_3)\))

\[
E_j(t)u_j(x) = \sum_{k=1}^{2} (2\pi)^{-3} \int e^{i\varphi_k(x,t,\xi)} \alpha_{jk}(x,t,\xi)\hat{u}_j(\xi)d^3\xi + R_j(t)u_j(x),
\]

(2.7)
where the \(\varphi_k\) are real-valued phase functions which satisfy the initial condition

\[
\varphi_k(t = 0, x, \xi) = \langle \xi, x \rangle = x \cdot \xi = \sum_{s=1}^{3} x^s \xi_s,
\]

(2.8)
and \(R_j(t)\) is a regularizing operator (see our earlier comments) which smoothes out the singularities acted upon by it \([26]\). Thus, the amplitudes \(\alpha_{jk}\) and phases \(\varphi_k\) make it possible to build the parametrix for the Cauchy problem. In our analysis we are going to need the generalization of this construction to the tensor wave equation, bearing in mind that Eq. (1.14) is the tensor version of Eq. (2.2).

Meanwhile, let us try to develop a heuristic argument on the relation between a fundamental solution \(G\) and a \(C^\infty\) parametrix \(\pi\) of a partial differential operator \(P\) as in (2.1). By omitting, for simplicity of notation, the subscript \(q\), if we consider the split

\[
G = \pi + Y,
\]

(2.9)
we find

\[
P\tilde{G} = P\pi + PY = \delta = \delta + R + PY = -P^{-1}R.
\]

(2.10)
Thus, a fundamental solution of \(P\) differs from its \(C^\infty\) parametrix by minus the inverse of \(P\) applied to the regularizing term \(R \in C^\infty(\Omega)\). In the simplest case, i.e.
operators on the real line, we would therefore solve the inhomogeneous equation
\[ P \chi = \psi \]
in the form
\[ \chi(x) = \int_{y_0}^{y} \pi(x, y) \psi(y) dy - \int_{y_0}^{y} \left( \int_{z_0}^{y} G(x, z) R(z) dz \right) \psi(y) dy. \] (2.11)

Now, to the extent that the parametrix leads to a good approximate inverse of \( P \), the integral
\[ \int_{y_0}^{y} \pi(x, y) \psi(y) dy \]
on the right-hand side of (2.11) provides a good approximation of
\[ \int_{y_0}^{y} G(x, y) \psi(y) dy. \]

Similarly, in the tensor wave equations which are the ultimate goal of our investigation, with the associated Peierls bracket, we are going to re-express (1.1) as the sum of the parametrix and remainder contribution, respectively.

The difference between fundamental solution and parametrix is further elucidated, with the notation summarized in Appendix B, by sections 4.2 and 4.3 of Ref. [23], which prove that there exist fundamental solutions of the operator \( P \) having the form (see (B1)-(B6))
\[ G_{\pm} = \frac{1}{2\pi} \left( U \delta_{\pm}(\Gamma) + V H_{\pm}(\Gamma) \right), \] (2.12)
as well as parametrices reading as
\[ \pi_{\pm} = \frac{1}{2\pi} \left( U \delta_{\pm}(\Gamma) + \bar{V} H_{\pm}(\Gamma) \right), \] (2.13)
where \( U(p, q) \) is a smooth function belonging to \( C^\infty(\Omega \times \Omega) \), expressible by
\[ U(x) = \left| \frac{g(0)}{g(x)} \right|^\frac{1}{2} \exp \left( -\frac{1}{2} \int a_\mu(x) x^\mu dr \right), \] (2.14)
the \( a_\mu \) being the covariant components of the vector \( a^\mu \) in (2.1), while \( V \) solves the characteristic initial value problem
\[ PV = 0, \quad V = V_0 \text{ on } C(q) \] (2.15)
and is real-analytic, hence expressible as
\[ V = \sum_{l=0}^{\infty} V_l \frac{\Gamma^l}{l!} \] (2.16)

*Following Ref. [23], we use local coordinates that are normal at \( q \), for which \( g_{\mu\nu}(x) x^\nu = g_{\mu\nu}(0) x^\nu \), while the world function \( \Gamma \) takes the form \( \Gamma = g_{\mu\nu}(0) x^\mu x^\nu \).*
the $V_i$ being elements of $C^\infty(\Omega \times \Omega)$ computable in the form

\[
V_0(x, y) = -\frac{1}{4} U(x, y) \int_0^1 \left. \frac{P U}{U} \right|_{x=z(s)} dS,
\]

\[
V_i(x, y) = -\frac{1}{4} U(x, y) \int_0^1 \left. \frac{PV_{i-1}}{U} \right|_{x=z(s)} s^i dS, \tag{2.17}
\]

while

\[
\bar{V} = V_0 + \sum_{l=1}^{\infty} V_l \Gamma_l \sigma(k_l \Gamma), \tag{2.18}
\]

where $\sigma \in C_0^\infty(\mathbb{R})$ and takes values

\[
\sigma(t) = 1 \text{ if } |t| \leq \frac{1}{2}, \quad \sigma(t) = 0 \text{ if } |t| \geq 1, \tag{2.19}
\]

the $\{k_l\}$ being a sequence of positive numbers, strictly increasing and tending to infinity. By virtue of (2.12)-(2.18), one finds (cf. (2.10))

\[
G_\pm q - \pi_\pm = Y_\pm = \frac{1}{2\pi} (V - \bar{V}) H_\pm(\Gamma), \tag{2.20}
\]

where

\[
V - \bar{V} = \sum_{l=1}^{\infty} V_l \Gamma_l \left(1 - \sigma(k_l \Gamma)\right). \tag{2.21}
\]

Note that the parametrix is not determined uniquely. If $F \in C^\infty(\Omega \times \Omega)$ and $F \sim 0$ when the world function $\Gamma \to 0$, then $\bar{V} + F = V_0$ on $C(q)$, the null cone of $q$, and $P(\bar{V} + F) \sim 0$ when $\Gamma \to 0$. Hence one can replace $\bar{V}$ with $\bar{V} + F$ in the formulas (2.13), to obtain another pair of parametrices of $P$ in $\Omega$. If $\Omega$ is a causal domain (see appendix B), the operator $P$ in (2.1) has a fundamental solution $G^+_q$ in $\Omega$, such that

\[
P G^+_q = \delta_q, \quad \text{supp} G^+_q \subset J^+(q). \tag{2.22}
\]

It is of the form

\[
G^+_q = \frac{1}{2\pi} \left( U \delta_+(\Gamma) + V^+ \right), \tag{2.23}
\]

with $U = U(p, q)$ defined as in (2.14), whereas, given the set

\[
\Delta^+ \equiv \{(p, q) : (p, q) \in \Omega \times \Omega, \; p \in J^+(q)\}, \tag{2.24}
\]

the function $V^+(p, q)$ is of class $C^\infty$ on $\Delta^+$ and has support contained in $\Delta^+$. When the world function $\Gamma(p, q) \to 0$ for $(p, q) \in \Delta^+$, the Hadamard series on the right-hand side of (2.16) is an asymptotic expansion for $V^+$. In analogous way, $P$ has also a fundamental solution $G^-_q$ in $\Omega$ such that

\[
P G^-_q = \delta_q, \quad \text{supp} G^-_q \subset J^-(q), \tag{2.25}
\]

Note that the parametrix is not determined uniquely. If $F \in C^\infty(\Omega \times \Omega)$ and $F \sim 0$ when the world function $\Gamma \to 0$, then $\bar{V} + F = V_0$ on $C(q)$, the null cone of $q$, and $P(\bar{V} + F) \sim 0$ when $\Gamma \to 0$. Hence one can replace $\bar{V}$ with $\bar{V} + F$ in the formulas (2.13), to obtain another pair of parametrices of $P$ in $\Omega$. If $\Omega$ is a causal domain (see appendix B), the operator $P$ in (2.1) has a fundamental solution $G^+_q$ in $\Omega$, such that

\[
P G^+_q = \delta_q, \quad \text{supp} G^+_q \subset J^+(q). \tag{2.22}
\]

It is of the form

\[
G^+_q = \frac{1}{2\pi} \left( U \delta_+(\Gamma) + V^+ \right), \tag{2.23}
\]

with $U = U(p, q)$ defined as in (2.14), whereas, given the set

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\[
P G^-_q = \delta_q, \quad \text{supp} G^-_q \subset J^-(q), \tag{2.25}
\]
which is of the form
\[ G^q_\Gamma = \frac{1}{2\pi} \left( U\delta_\Gamma + V^- \right), \tag{2.26} \]
where \( V^- \) has properties similar to those of \( V^+ \), with past and future interchanged.

We know that there are infinitely many \( C^\infty \) parametrices of \( P \) supported in the causal future of \( q \), and the previous theorem shows that one can construct a fundamental solution \( G^q_\Gamma \) from any one of these. However, another theorem ensures that there is in fact only one forward fundamental solution \[23\], which is precisely the advanced Green function needed in the construction of the Peierls bracket. Thus, Eq. (2.23) is the definitive form of the forward fundamental solution of \( P \) in \( \Omega \). It may be viewed as the field of a point source at \( q \). It consists of a singular part \( U\delta_\Gamma \) which is a measure, supported on the future null cone of \( q \), and of a regular part \( V^+ \), which is a function. The function \( p \to V^+(p,q) \) has its support contained in \( J^+(q) \), and lies in \( C^\infty(J^+(q)) \). It is said to be the tail term of the fundamental solution; it distinguishes the curved spacetime case from the Minkowskian case, where \( G^q_\Gamma \) is sharp, its support being the future null cone of \( q \).

3. Tensor wave equations: Maxwell theory in curved spacetime

The tensor analogue of the differential operator \( P \) in (2.1) can be written, by introducing tensor multi-indices \( I = (i_1,\ldots,i_m) \) and \( J = (j_1,\ldots,j_m) \), in either of the forms \[23\]
\[ (Pu)_I = \nabla_{j} \nabla^{j} u_I + a^{IJ}_{I} \nabla_{j} u_{J} + b_{I}^{J} u_{J}, \tag{3.1} \]
or
\[ (Pu)^I = \nabla_{j} \nabla^{j} u^{I} + a^{IJ}_{I} \nabla_{j} u^{J} + b_{I}^{J} u^{J}. \tag{3.2} \]

The class of \( C^\infty \) tensor fields of rank \( m \) defined on a causal domain \( \Omega_0 \) is a vector space over the complex numbers, denoted by \( E^m(\Omega_0) \). The subspace of \( E^m(\Omega_0) \) consisting of fields with compact support is denoted by \( D^m(\Omega_0) \). A tensor distribution \( u \in D^m(\Omega_0) \) is then a continuous linear map \( D^m(\Omega_0) \to C \). Moreover, if \( p' \) is a point of \( \Omega_0 \), a tensor-valued distribution \( T \) is a continuous linear map \( \xi \to (T,\xi) \) of \( D^m(\Omega_0) \) into the finite-dimensional vector space of tensors of rank \( m \) at \( p' \).

A bitensor of type \((0,m)\) at \( p \) and of type \((m',0)\) at \( p' \) is a multilinear form on the product of \( m \) copies of the cotangent space \( T^*M_p \) and of \( m' \) copies of the tangent space \( TM_{p'} \). Bitensor indices can be raised and lowered by means of the appropriate components of the metric tensor at \( p \) and at \( p' \), respectively.

The Dirac distribution of rank \( m \), denoted by \( \delta^{(m)}_{p'}(p) \), maps \( \xi \in D^m(\Omega_0) \) to \( \xi(p') \), according to
\[ \left( \delta^{(m)}_{p'}(p),\xi(p) \right) = \xi(p'). \tag{3.3} \]
A fundamental solution of the differential operator (3.1) or (3.2) is a tensor-valued distribution $G_{p'}(p)$ such that
\[ PG_{p'}(p) = \delta^{(m)}_{p'}(p). \] (3.4)

There exist two basic fundamental solutions $G^{+}_{p'}(p)$ and $G^{-}_{p'}(p)$ whose supports are contained in $J^{+}(p')$ and in $J^{-}(p')$, respectively.

The transport bitensor (see $U_{\mu\alpha}$ in (1.14)) in a geodesically convex domain $\Omega$ is a bitensor field $m\tau(p,p')$ of rank $m$ at both $p'$ and $p$ which satisfies the differential equations (hereafter the world function $\Gamma = \Gamma(p,p')$, and the covariant derivatives $\nabla_{j}$ and $\nabla^{j}$ act at $p$)
\[ \nabla^{j}\Gamma \nabla_{j} \left( m\tau_{I}' \right) = 0, \] (3.5)
jointly with the initial conditions
\[ \left( m\tau_{I}' \right) \bigg|_{p=p'} = \delta_{I_{1}}^{i_{1}}...\delta_{I_{m}}^{i_{m}}. \] (3.6)

We shall also need the biscalar $\kappa(p,p')$, which solves the differential equation [23]
\[ 2(\nabla\Gamma, \nabla\kappa) + (\Box\Gamma - 8)\kappa = 0, \] (3.7)
with initial condition
\[ \kappa(p', p') = 1, \] (3.8)
and is given, in local coordinates, by [27,28]
\[ \kappa(x,y) = \sqrt{|\frac{\det \frac{\partial\tau_{I}}{\partial x}\frac{\partial\tau_{I}}{\partial y}}{4|g(x)g(y)|^{*}}|}. \] (3.9)

Both $m\tau$ and $\kappa$ are symmetric functions of their arguments. We can now state the tensor analogue of the theorem at the end of Sec. II for the forward fundamental solution [23].

**Theorem** In a causal domain $\Omega_0$, the tensor differential operator
\[ (Pu)_{I} = \nabla_{j}\nabla^{j}u_{I} + b_{I}^{j}u_{j} \] (3.10)
has a fundamental solution $G^{+}_{p'}(p)$ in $\Omega_0$ such that
\[ PG^{+}_{p'}(p) = \delta^{(m)}_{p'}(p), \text{ supp}G^{+}_{p'}(p) \subset J^{+}(p'), \] (3.11)
which is of the form
\[ G^{+}_{p'}(p) = \kappa(p,p')_{m\tau}(p,p')\delta_{+}(\Gamma) + V^{+}(p,p'), \] (3.12)
where $V^{+}(p,p')$ is a bitensor field that vanishes if $p \notin J^{+}(p')$ and is of class $C^{\infty}$ on the closed set
\[ \{(p',p) : (p',p) \in \Omega_0 \times \Omega_0, p \in J^{+}(p') \}. \] (3.13)
Now we remark that vacuum Maxwell theory in the absence of charges and currents is ruled precisely by a tensor differential operator of the kind (3.10), because, upon imposing the Lorenz \[29\] supplementary (or gauge-fixing) condition, i.e.
\[
\nabla_\mu A_\mu = 0, \tag{3.14}
\]
the potential \(A_\mu\) obeys the wave equation
\[
P_\mu^\nu A_\nu = 0, \tag{3.15}
\]
where
\[
P_\mu^\nu \equiv -\delta_\mu^\nu \Box + R_\mu^\nu, \tag{3.16}
\]
having denoted by \(\Box\) the d’Alembert operator \(g^{\rho\sigma} \nabla_\rho \nabla_\sigma\), and by \(R_\mu^\nu\) the Ricci tensor of the spacetime manifold \((M, g)\) endowed with a Levi-Civita connection \(\nabla\).

4. Nonlinear wave equation: Einstein’s equations

So far, we have met the concepts of fundamental solution and parametrix for linear partial differential equations, of either scalar or tensorial nature. But how to define fundamental solutions for the nonlinear partial differential equations provided by Einstein’s theory of gravitation? Indeed, the vacuum Einstein equations in four spacetime dimensions are equivalent to Ricci-flatness, i.e. \(R_{\alpha\beta} = 0\), where, upon defining
\[
F^\lambda = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^\lambda_\mu \right), \tag{4.1}
\]
one has the split
\[
R_{\alpha\beta} = -L_{\alpha\beta} - N_{\alpha\beta}, \tag{4.2}
\]
having defined \[11\]
\[
L_{\alpha\beta} = \frac{1}{2} \left( g_{\alpha\mu} \partial_\beta g_{\mu\alpha} + g_{\beta\mu} \partial_\alpha g_{\mu\beta} \right) F^\mu, \tag{4.3}
\]
\[
N_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} \frac{\partial^2 g_{\alpha\beta}}{\partial x^\lambda \partial x^\mu} + H_{\alpha\beta}, \tag{4.4}
\]
where \(H_{\alpha\beta}\) is a polynomial of covariant and contravariant metric components, and of their first partial derivatives. Equations (4.1)-(4.3) suggest considering the supplementary condition \(F^\lambda = 0\), so that the vacuum Einstein equations read eventually as
\[
g^{\lambda\mu} \frac{\partial^2 g_{\alpha\beta}}{\partial x^\lambda \partial x^\mu} + 2H_{\alpha\beta} = 0. \tag{4.5}
\]
Now if we set
\[
g^{\lambda\mu} = A^{\lambda\mu}, \quad g_{\alpha\beta} = W_s, \quad 2H_{\alpha\beta} = f_s, \tag{4.6}
\]
we realize that our gauge-fixed vacuum Einstein equations take the standard form of quasilinear hyperbolic systems [11]

$$A^\lambda_\mu \frac{\partial^2 W_s}{\partial x^\lambda \partial x^\mu} + f_s = 0. \quad (4.7)$$

Suppose now, following again Ref. [11], that in a spacetime domain $D$, centred at the point $M$ with coordinates $(x^i, 0)$, and defined by

$$\left| x^i - \bar{x}^i \right| \leq d, \quad \left| x^0 \right| \leq \varepsilon, \quad (4.8)$$

and for values of the unknown functions $W_s$ and their first derivatives satisfying

$$\left| W_s - W_s(M) \right| \leq l, \quad \left| \frac{\partial W_s}{\partial x^\alpha} - \frac{\partial W_s}{\partial x^\alpha}(M) \right| \leq l, \quad (4.9)$$

the coefficients $A^\lambda_\mu$ and $f_s$ possess partial derivatives with respect to their arguments up to the fifth order. One can now show that, by differentiating 5 times the Eqs. (4.7) with respect to the variables $x^\alpha$, one obtains a linear system of second-order partial differential equations

$$A^\lambda_\mu \frac{\partial^2 u_s}{\partial x^\lambda \partial x^\mu} + B_r^\lambda \frac{\partial u_r}{\partial x^\lambda} + f_s = 0, \quad (4.10)$$

where the associated quadratic form $A^\lambda_\mu X_\lambda X_\mu$ is of normal hyperbolic type, i.e.

$$A^{00} > 0, \quad \sum_{i,j=1}^{3} A^{ij} X_i X_j < 0. \quad (4.11)$$

In the following calculations, it is useful to define

$$\frac{\partial W_s}{\partial x^\alpha} = W_{s\alpha}, \quad \frac{\partial^2 W_s}{\partial x^\alpha \partial x^\beta} = W_{s\alpha\beta}, \quad \ldots, \quad (4.12)$$

until we denote by $U_S$ the partial derivatives of fifth order of $W_s$, i.e.

$$\frac{\partial^5 W_s}{\partial x^\alpha \partial x^\beta \partial x^\gamma \partial x^\delta \partial x^\varepsilon} = W_{s\alpha\beta\gamma\delta\varepsilon} = U_S. \quad (4.13)$$

Differentiation of Eqs. (4.7) with respect to any variable $x^\alpha$ whatsoever leads to $n$ equations having the form

$$A^\lambda_\mu \frac{\partial^2 W_{s\alpha}}{\partial x^\lambda \partial x^\mu} + \left[ \frac{\partial A^\lambda_\mu}{\partial x^\alpha} + \frac{\partial A^\lambda_\mu}{\partial W_{r\nu}} W_{r\alpha} + \frac{\partial A^\lambda_\mu}{\partial W_{r\nu}} \frac{\partial W_{r\nu}}{\partial x^\alpha} \right] \frac{\partial W_{s\mu}}{\partial x^\lambda}$$

$$+ \frac{\partial f_s}{\partial x^\alpha} + \frac{\partial f_s}{\partial W_{r\nu}} W_{r\alpha} + \frac{\partial f_s}{\partial W_{r\nu}} \frac{\partial W_{r\nu}}{\partial x^\alpha} = 0. \quad (4.14)$$
By repeating four times this process, the following system of \( N \) equations is obtained:

\[
A^\lambda \mu \frac{\partial^2 W_{s\alpha\beta\gamma\delta\epsilon}}{\partial x^\alpha \partial x^\beta} + \left[ \frac{\partial A^\lambda \mu}{\partial x^\alpha} \frac{\partial A^\lambda \mu}{\partial W_r} W_r \alpha \mu \right] \frac{\partial}{\partial x^\lambda} W_{s\beta\gamma\delta\mu} \\
+ \frac{\partial A^\lambda \mu}{\partial x^\alpha} \frac{\partial A^\lambda \mu}{\partial W_r} W_r \alpha \mu + \frac{\partial A^\lambda \mu}{\partial W_r} W_r \beta \gamma \delta \mu + \ldots \\
+ \frac{\partial A^\lambda \mu}{\partial x^\alpha} \frac{\partial A^\lambda \mu}{\partial W_r} W_r \alpha \mu + \frac{\partial A^\lambda \mu}{\partial W_r} W_r \gamma \delta \mu + \ldots \\
+ \frac{\partial A^\lambda \mu}{\partial W_r} \frac{\partial W_{r\alpha\beta\gamma\delta}}{\partial x^\alpha} + \frac{\partial f_s}{\partial W_r} \frac{\partial W_{r\alpha\beta\gamma\delta}}{\partial x^\alpha} + F_S = 0,
\]

(4.15)

where \( F_S \) is a function of the variables \( x^\alpha \), the unknown functions \( W_s \) and their partial derivatives up to the fifth order included, but not of their higher-order derivatives. By virtue of Eq. (4.15), the fifth derivatives \( U_S \) of the functions \( W_s \) satisfy, within the domain \( D \) and under the assumptions previously stated, a system of \( N \) equations

\[
A^\lambda \mu \frac{\partial^2 U_S}{\partial x^\lambda \partial x^\mu} + B^T \lambda \Sigma \frac{\partial U_T}{\partial x^\lambda} + F_S = 0,
\]

(4.16)

which is precisely of the linear type studied in (4.10), because

\[
A^\lambda \mu = A^\mu (x^\alpha, W_s, W_{sa}), \quad B^T \lambda \Sigma = B^T \lambda (x^\alpha, W_s, W_{sa}, W_{sa\beta}), \\
F_S = F_S(x^\alpha, W_s, (W_s, ..., U_s)).
\]

(4.17)

By virtue of the method described in appendix C, we know that Eqs. (4.16) can be solved explicitly through Kirchhoff-type formulas as in Eq. (C33). Thus, by virtue of (4.13), if we can find a fundamental solution, or at least a parametrix \( \pi \) of the fifth-order linear equation

\[
\frac{\partial^5}{\partial x^\alpha \partial x^\beta \partial x^\gamma \partial x^\delta \partial x^\epsilon} W_s = U_S(x^\mu),
\]

(4.18)

\( U_S \) being given by the integral formula (C33), we will know, at least in principle, the classical Peierls bracket for gravity, which is in turn necessary to arrive at the desired definition of quantum Peierls bracket (1.15) for gravity. We recall that the \( W_s \) functions in Eq. (4.18) are the covariant components of the metric tensor of the spacetime manifold. They can be expressed, from Eq. (4.18) and its parametrix, through the convolution formula

\[
W_s = \int \pi(x^\alpha, \{y^\beta\}) U_S(\{y^\beta\}) d\mu_y + \text{remainder}.
\]

(4.19)

5. Existence of the parametrix

We can point out that, for every choice \( (\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}, \overline{\epsilon}) \) of the indices \( (\alpha, \beta, \gamma, \delta, \epsilon) \), we can set

\[
x^\overline{\alpha} = x^1, \ x^\overline{\beta} = x^2, \ x^\overline{\gamma} = x^3, \ x^\overline{\delta} = x^4, \ x^\overline{\epsilon} = x^5,
\]

where \( F_S \) is a function of the variables \( x^\alpha \), the unknown functions \( W_s \) and their partial derivatives up to the fifth order included, but not of their higher-order derivatives. By virtue of Eq. (4.15), the fifth derivatives \( U_S \) of the functions \( W_s \) satisfy, within the domain \( D \) and under the assumptions previously stated, a system of \( N \) equations

\[
A^\lambda \mu \frac{\partial^2 U_S}{\partial x^\lambda \partial x^\mu} + B^T \lambda \Sigma \frac{\partial U_T}{\partial x^\lambda} + F_S = 0,
\]

(4.16)

which is precisely of the linear type studied in (4.10), because

\[
A^\lambda \mu = A^\mu (x^\alpha, W_s, W_{sa}), \quad B^T \lambda \Sigma = B^T \lambda (x^\alpha, W_s, W_{sa}, W_{sa\beta}), \\
F_S = F_S[x^\alpha, W_s, (W_s, ..., U_s)].
\]

(4.17)

By virtue of the method described in appendix C, we know that Eqs. (4.16) can be solved explicitly through Kirchhoff-type formulas as in Eq. (C33). Thus, by virtue of (4.13), if we can find a fundamental solution, or at least a parametrix \( \pi \) of the fifth-order linear equation

\[
\frac{\partial^5}{\partial x^\alpha \partial x^\beta \partial x^\gamma \partial x^\delta \partial x^\epsilon} W_s = U_S(x^\mu),
\]

(4.18)

\( U_S \) being given by the integral formula (C33), we will know, at least in principle, the classical Peierls bracket for gravity, which is in turn necessary to arrive at the desired definition of quantum Peierls bracket (1.15) for gravity. We recall that the \( W_s \) functions in Eq. (4.18) are the covariant components of the metric tensor of the spacetime manifold. They can be expressed, from Eq. (4.18) and its parametrix, through the convolution formula

\[
W_s = \int \pi(x^\alpha, \{y^\beta\}) U_S(\{y^\beta\}) d\mu_y + \text{remainder}.
\]

(4.19)
and hence the operator on the left-hand side of Eq. (4.18) may be viewed as a constant-coefficient fifth-order elliptic operator on $\mathbb{R}^5$. At this stage, we can exploit the following theorem [30]:

**Theorem.** Every elliptic operator $P(D)$ with constant coefficients has a parametrix which is a $C^\infty$ function in $\mathbb{R}^n - \{0\}$.

**Proof.** The symbol of $P(D)$ can be written as

$$ P(\xi) = P_m(\xi) + P_{m-1}(\xi) + ... + P_0, \quad (5.1) $$

where the term $P_j$ is homogeneous of degree $j$, and $P_m(\xi) \neq 0$ when $\xi \neq 0$. Then there exists a constant $c$ for which

$$ |P_m(\xi)| \geq c > 0 \quad (5.2) $$

when $|\xi| = \sqrt{(\xi_1)^2 + ... + (\xi_n)^2} = 1$, so that the homogeneity provides the minorization

$$ |P_m(\xi)| = |\xi|^m P_m \left( \frac{\xi}{|\xi|} \right) \geq c|\xi|^m, \quad \xi \in \mathbb{R}^n. \quad (5.3) $$

By virtue of (5.3), one can write for some constants $C$ and $R$

$$ |P(\xi)| \geq |P_m(\xi)| - |P_{m-1}(\xi)| - ... \geq c|\xi|^m - C\left(|\xi|^{m-1} + ... + 1\right) \geq c\frac{|\xi|^m}{2}, \quad (5.4) $$

provided that $\xi \in \mathbb{R}^n$ has $|\xi| \geq R$. Since a derivative of order $k$ of $\frac{1}{P(\xi)}$ is of the form $\frac{Q(\xi)}{P(\xi)^{k+1}}$, with $Q$ of degree $\leq (m-1)k$, as one can prove by induction, one finds that, when $|\xi| > R$,

$$ \left| \xi^\beta D^\alpha \left( \frac{1}{P(\xi)} \right) \right| \leq C_{\alpha\beta} |\xi|^{\beta - |\alpha| - m}. \quad (5.5) $$

Choose now $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in the subset of $\mathbb{R}^n$ given by $\{\xi : |\xi| < R\}$. This ensures that $\frac{1 - \chi(\xi)}{P(\xi)}$ is a bounded $C^\infty$ function, hence it can be viewed as the Fourier transform of a distribution $\pi$. Hence one has

$$ P(D)\pi = \delta + \omega, \quad (5.6) $$

where the Fourier transform of $\omega$ is $-\chi$. Hence $\omega$ itself is an element of the Schwartz space, and it follows from (5.5) that $D^\beta x^\alpha \pi$ is continuous when $|\beta| - |\alpha| - m < -n$. Q.E.D.

As is stated in Ref. [30], the error term $\omega$ obtained in such a proof has an analytic extension to $C^n$. Moreover, the parametrix $\pi$ admits an analytic extension to a conic neighborhood of $\mathbb{R}^n - \{0\}$. 
Inspired by the split (2.9), let us first re-express the classical advanced and retarded Green functions mentioned in the Introduction in the form

$$G_+^{ij} = \pi_+^{ij} + U_+^{ij},$$  \hspace{1cm} (6.1)

$$G_-^{ij} = \pi_-^{ij} + U_-^{ij},$$  \hspace{1cm} (6.2)

where $\pi^{ij}$ is the parametrix and $U^{ij}$ is the remainder term. Moreover, inspired by the definition (1.2), let us introduce

$$\tilde{\pi}^{ij} \equiv \pi_+^{ij} - \pi_-^{ij},$$

$$\tilde{U}^{ij} \equiv U_+^{ij} - U_-^{ij},$$

$$\tilde{G}^{ij} = \tilde{\pi}^{ij} + \tilde{U}^{ij}.$$  \hspace{1cm} (6.3)

The lack of uniqueness of the parametrix (see comments after (2.21)) can be exploited to ensure that $\tilde{\pi}^{ij}$ has the same antisymmetry property of the supercommutator function $\tilde{G}^{ij}$. In other words, given the parametrices $\pi_+^{ij}$, suppose that

$$\tilde{\pi}^{ij} = \tilde{\pi}^{ij} + \tilde{\pi}^{ij} = \tilde{\pi}^{ij} + \tilde{\pi}^{ij},$$  \hspace{1cm} (6.4)

where $\tilde{\pi}^{ij} \not= 0$ and $\tilde{\pi}^{ij} \not= 0$. Now we can add to $\tilde{\pi}^{ij}$ a function $\tilde{f}^{ij} \in C^\infty(\Omega \times \Omega)$, tending to 0 when $\Gamma \to 0$, such that

$$\tilde{\pi}^{ij} = \tilde{\pi}^{ij} + \tilde{f}^{ij} \implies \tilde{\pi}^{ij} = \tilde{\pi}^{ij} + \tilde{f}^{ij},$$  \hspace{1cm} (6.5)

and hence it is enough to choose

$$\tilde{f}^{ij} = -\tilde{\pi}^{ij}$$  \hspace{1cm} (6.6)

to achieve the desired skew-symmetry of $\tilde{\pi}^{ij}$. The remainder term $\tilde{U}^{ij}$ is then skew-symmetric as well, by virtue of (6.3).

In the proof that the classical Peierls bracket is a Poisson bracket on the space of gauge-invariant functionals, a crucial role is played by the verification of the Jacobi identity [31] obtained from the sum

$$J(A, B, C) \equiv (A, (B, C)) + (B, (C, A)) + (C, (A, B))$$

$$= A_i \tilde{G}^{il} \left( B_j \tilde{G}^{jk} C_k + B_j \tilde{G}^{jl} C_l \right) + B_j \tilde{G}^{il} \left( C_k \tilde{G}^{ki} A_i + C_k \tilde{G}^{kl} A_l \right)$$

$$+ C_k \tilde{G}^{kl} \left( A_i \tilde{G}^{lj} B_j + A_i \tilde{G}^{lj} B_j \right)$$

$$+ \left[ A_i \tilde{G}^{il} B_j \tilde{G}^{jk} C_k + B_j \tilde{G}^{il} C_k \tilde{G}^{kj} A_i + C_k \tilde{G}^{kl} A_i \tilde{G}^{lj} B_j \right],$$  \hspace{1cm} (6.7)

where $J(A, B, C)$ is found to vanish classically, as shown in detail in Ref. [31] and, first, in Ref. [32]. By virtue of the split (6.3), the Jacobi identity (6.7) can be written as the sum of three terms, i.e.

$$J(A, B, C) = J_{\pi}(A, B, C) + J_{\tilde{G}}(A, B, C) + J_{\tilde{G}}(A, B, C),$$  \hspace{1cm} (6.8)

\[\text{Strictly speaking, we should write } G^{ij'}, \gamma^{ij'}, R^{ij'}, \text{ as in (1.3) and (1.4), to stress that the indices } i, j \text{ refer to different spacetime points, but we refrain ourselves from doing so for simplicity of notation.}\]
where \( \tilde{J} \tilde{\pi} \) (respectively, \( \tilde{J} \tilde{U} \)) can be obtained from (6.7) upon replacing everywhere \( \tilde{G}^{ij} \) (respectively, \( \tilde{U}^{ij} \)), whereas \( \tilde{J} \tilde{\pi} \tilde{U} \) denotes the sum of all mixed terms, e.g.

\[
A, i \tilde{\pi} B, j C, k + B, j \tilde{U} k C, ij + \ldots .
\]

Let us here focus on the term \( \tilde{J} \tilde{\pi} \) in the classical Jacobi identity, because, within our scheme, it is equal to the classical part of the quantum Jacobi identity. By virtue of (6.3) and (6.7), we find

\[
\tilde{J} \tilde{\pi} = A, i \tilde{\pi} B, j C, k + B, j \tilde{\pi} k C, ij + A, i \tilde{\pi} k C, jl (\tilde{\pi} ij + \tilde{\pi} kl) + \ldots ,
\]

(6.9)

The skew-symmetry of \( \tilde{\pi} \), jointly with commutation of functional derivatives:

\[
T, i l = T, l i \text{ for all } T = A, B, C,
\]

implies that the first three terms on the right-hand side of (6.9) vanish. For example, one finds \[32,31\]

\[
A, i \tilde{\pi} B, j C, k + B, j \tilde{\pi} k C, ij = A, l \tilde{\pi} B, j C, k + B, j \tilde{\pi} k C, lj = -A, i \tilde{\pi} B, j C, k + B, j \tilde{\pi} k C, ij = 0,
\]

(6.10)

and an identical procedure can be applied to the terms containing the second functional derivatives \( B, jl \) and \( C, kl \) in (6.9).

7. Functional equations for the parametrix

The last term on the right-hand side of (6.9) requires new calculations because it contains functional derivatives of \( \tilde{\pi} \). These can be dealt with after taking infinitesimal variations of an equation like (2.4). With the DeWitt notation used here, the defining equation for the parametrix reads as

\[
P_{ij} \tilde{\pi}^{k} = -\delta_{i}^{k} + \omega_{i}^{k},
\]

(7.1)

where the signs on the right-hand side have been chosen so as to agree with Eq. (1.4) when \( \omega_{\pm} = 0 \) (which means that the parametrix is actually a fundamental solution or even a Green function). The operator \( P_{ij} \) is obtained, in classical theory, as \[32,31\]

\[
P_{ij} = S_{ij} + h_{ik} Q_{\alpha}^{k} t^{\alpha \beta} h_{\beta j} Q_{\beta}^{l},
\]

(7.2)

where \( h_{ij} \) is a local and symmetric matrix which is taken to transform like \( S_{ij} \) under group transformations, and \( t^{\alpha \beta} \) is a local, nonsingular, symmetric matrix which transforms according to the adjoint representation of the infinite-dimensional invariance group. Such matrices act as metrics that raise and lower indices of the generators \( Q_{\alpha}^{i} \) according to the rules \[32\]

\[
Q_{\alpha}^{i} \equiv h_{ij} Q_{\alpha}^{j}, \quad Q_{i}^{\alpha} \equiv t^{\alpha \beta} Q_{i}^{\beta}.
\]

(7.3)
If we study the transformation properties of Eq. (A3) under the infinitesimal gauge transformations (A3), we find

\[ P\pi_\pm = -I + \omega_\pm \implies (\delta P)\pi_\pm + P(\delta \pi_\pm) = \delta \omega_\pm. \quad (7.4) \]

Now we act with \( \pi_\pm \) on both sides of Eq. (7.4), and we find, by virtue of (7.1), the relation

\[ (I - \omega_\pm)\delta \pi_\pm = \pi_\pm (\delta \pi_\pm) - \pi_\pm \delta \omega_\pm. \quad (7.5) \]

At this stage we revert to condensed-index notation, hence writing, from (7.5),

\[ \left( \delta^j_i - \omega^i_{j,\pm} \right) \pi_{\pm,\alpha}^{jk} = \pi_{\pm,\alpha}^{ia} \pi_{\pm,\beta}^{bk} - \pi_{\pm,\alpha}^{ia} \omega_{j,\pm}^{bk,\alpha}, \quad (7.6) \]

At this stage, we are going to re-express in (7.7) the terms \( Q_{\pm,\alpha}^{bk} \) and \( \pi_{\pm,\alpha}^{ia} \). For this purpose, inspired by the method adopted in Ref. [32], we begin by noticing that, for background fields that solve the classical field equations, one has (see Eq. (A7) and set \( S_\gamma = 0 \) therein)

\[ P_{ik} Q_{\pm,\alpha}^{bk} = Q_{\pm,\beta}^{i,\rho} \hat{P}_{\rho,\alpha}, \quad (7.8) \]

where the operator \( \hat{P}_{\rho,\alpha} \) is defined by [32]

\[ \hat{P}_{\rho,\alpha} \equiv Q_{\rho,\beta}^{k} Q_{\alpha}^{k}. \quad (7.9) \]

The associated parametrix \( \pi_{\pm,\alpha}^{\alpha,\beta} \) is here defined, by analogy with (7.1), in the form

\[ \hat{P}_{\rho,\alpha} \pi_{\pm,\alpha}^{\alpha,\beta} = -\delta_{\rho,\alpha} + \omega_{\pm,\alpha}^{\beta}. \quad (7.10) \]

Now we act first with \( \pi_{\pm,\alpha}^{\alpha,\beta} \) on \( P_{ik} Q_{\pm,\alpha}^{bk} \), finding, in light of (7.1), the functional equation

\[ Q_{\pm,\alpha}^{i,\alpha} = \omega_{j,\pm}^{i,\alpha} Q_{\pm,\beta}^{k,\alpha} - \pi_{\pm,\alpha}^{i,\alpha} Q_{\pm,\beta}^{k,\alpha} \hat{P}_{\rho,\alpha}. \quad (7.11) \]

Furthermore, we act upon Eq. (7.11) with the parametrix \( \pi_{\pm,\alpha}^{\alpha,\beta} \) and find, by virtue of (7.10), the functional equation resulting from (7.11), i.e.

\[ \pi_{\pm,\alpha}^{i,\alpha} Q_{\pm,\beta}^{k,\alpha} = Q_{\pm,\alpha}^{i,\alpha} \pi_{\pm,\alpha}^{\alpha,\beta} + \pi_{\pm,\alpha}^{i,\alpha} Q_{\pm,\beta}^{k,\alpha} \omega_{\pm,\rho}^{\beta} - \omega_{\pm,\rho}^{i,\alpha} Q_{\pm,\beta}^{k,\alpha} \pi_{\pm,\alpha}^{\alpha,\beta}. \quad (7.12) \]

The left-hand side of (7.12) is exactly of the type \( \pi_{\pm,\alpha}^{i,\alpha} Q_{\pm,\beta}^{k,\alpha} \) occurring in Eq. (7.7). If we were dealing with Green functions, Eq. (7.12) would reduce to the familiar relation [32]

\[ G_{\pm,\alpha}^{i,\beta} Q_{\pm,\beta}^{k,\alpha} = Q_{\pm,\alpha}^{i,\alpha} G_{\pm,\alpha}^{\alpha,\beta}. \quad (7.13) \]

Interestingly, on going from Green functions to parametrices, a basic functional equation like (7.13) receives terms linear in \( \omega_{\pm,\rho}^{i,\alpha} \) and \( \omega_{\pm,\rho}^{\alpha,\beta} \). As a second step, we act with \( \pi_{\pm,\alpha}^{\alpha,\beta} \) from the left on Eq. (7.8), finding that

\[ Q_{\pm,\alpha}^{i,\alpha} = Q_{\pm,\alpha}^{i,\alpha} \omega_{\pm,\rho}^{\alpha,\beta} - \pi_{\pm,\alpha}^{i,\alpha} P_{ik} Q_{\pm,\beta}^{k,\alpha}. \quad (7.14) \]
which implies

$$Q^\alpha_{ij} = \pi^\alpha \pi^j - \pi^\beta Q^j.$$  \hspace{1cm} (7.15)

This is the transposed of Eq. (7.12), and generalizes the more familiar functional equation for Green functions [32]

$$Q^\alpha_{ij} = G^{\pm \alpha \beta} Q^j.$$  \hspace{1cm} (7.16)

By virtue of the group invariance property satisfied by all physical observables (set $T = A$ or $B$ or $C$ in Eq. (A5)), the terms like $Q^a_{ij} \pi^\alpha_{ij}$ and $\pi^\beta Q^j$ on the right-hand side of (7.12) and (7.15) give vanishing contribution to $J_\pi(A, B, C)$. One is therefore left with the contributions involving third functional derivatives of the action $S$, plus terms linear in $\omega^i_j$ and $\omega^\alpha_{ij}$, hereafter denoted by $O(\omega^i_j, \omega^\alpha_{ij})$. Bearing in mind that $S_{abc} = S_{acb} = S_{bac}$ = ...one can relabel indices summed over, finding eventually (cf. Ref. [31])

$$J_\pi(A, B, C) = A_i B_j C_k \left[(\pi^i_k - \pi^i_j)(\pi^j_k \pi^k_j - \pi^j_k \omega^k_j) + (\pi^j_k - \pi^j_i)(\pi^k_i \pi^i_k - \pi^k_i \omega^i_k) + (\pi^k_j - \pi^k_i)(\pi^i_j \pi^j_i - \pi^i_j \omega^i_j)\right] S_{abc} = 0,$$  \hspace{1cm} (7.17)

where we have assumed the nontrivial properties

$$\pi^i_+ = \pi^i_-, \pi^i_- = \pi^i_+.$$  \hspace{1cm} (7.18)

The sum in (7.17) vanishes because it involves six pairs of double products of parametrixes with opposite signs, i.e.

$$\pi^i_+ \pi^j_- \pi^k_+ \pi^l_- - \pi^i_- \pi^j_+ \pi^k_- \pi^l_+ + \pi^i_- \pi^j_- \pi^k_+ \pi^l_+ - \pi^i_+ \pi^j_+ \pi^k_- \pi^l_- + \pi^i_- \pi^j_+ \pi^k_- \pi^l_+ - \pi^i_+ \pi^j_- \pi^k_- \pi^l_+,$$

$$\pi^i_- \pi^j_+ \pi^k_- \pi^l_+ - \pi^i_+ \pi^j_- \pi^k_+ \pi^l_- + \pi^i_+ \pi^j_- \pi^k_- \pi^l_+ - \pi^i_- \pi^j_+ \pi^k_+ \pi^l_- + \pi^i_+ \pi^j_+ \pi^k_- \pi^l_- - \pi^i_- \pi^j_- \pi^k_+ \pi^l_+ = 0.$$  \hspace{1cm} (7.19)

8. Remainder term in the supercommutator function and concluding remarks

Along the years, Peierls brackets have been considered also in the modern literature to define Poisson brackets on the space of histories [33,34], to define the Feynman functional integral [35], to reconsider the covariant form of Hamiltonian dynamics [36], to study Poisson brackets for fermionic fields in de Sitter space [37], and even to investigate the perturbative construction of models of algebraic quantum field theory [38]. The material in our Secs. II and III is necessary to achieve gradually the transition towards nonlinear hyperbolic equations, but at that early stage the
parametrix is neither compelling nor more useful than the standard canonical approaches. However, since the Green functions of hyperbolic operators play such a key role in defining and evaluating Peierls brackets, while for nonlinear hyperbolic equations not even the fundamental solution is defined, even nowadays, our approach comes into play. We have exploited the work in Ref. [11] to argue that, since one knows how to solve linear hyperbolic systems through integral equations, and how to obtain the solution of nonlinear hyperbolic systems from the solution of linear ones, one can obtain good computational recipes for approximate evaluation of Peierls brackets, even though the exact Peierls bracket for gravity, obtainable in principle from the fully covariant formalism, remains inaccessible.

More precisely, in the quantum commutator (1.15) for gravity, the factors on the right-hand side do not commute. This means that we have to study (6.8) when $\tilde{G}^{ij}$ is replaced by its unknown operator version, here denoted by boldface characters, i.e. $\mathbf{G}^{ij}$, for which

$$
\tilde{G}^{ik} T_{kj} \neq T_{kj} \tilde{G}^{jk} \forall T = A, B, C, \quad (8.1)
$$

$$
\mathbf{G}^{il} T_{jl} \neq T_{jl} \mathbf{G}^{il} \forall T = A, B, C. \quad (8.2)
$$

At this stage, our idea is to generalize the splits (6.1), (6.2) and the definitions (6.3), assuming that the operator Green’s function $G^{ij}$ has a classical part given by the parametrix $\pi^{ij}$ and a quantum part given by the as yet unknown operator $U^{ij}$, the latter being responsible for the lack of commutativity in (8.1) and (8.2). Hence we write

$$
G^{ij} = \pi^{ij} + U^{ij} \implies \mathbf{G}^{ij} = \frac{\pi^{ij}}{\pi} + \mathbf{U}^{ij}. \quad (8.3)
$$

It is this operator equation that should be inserted into the operator version of $J_{\pi U}^{(A, B, C)}$ and $J_{\mathbf{U}}^{(A, B, C)}$ in (6.8), and this is the technical challenge ahead of us in the years to come.

In our paper we have developed ideas and have exploited well-established properties of hyperbolic and elliptic partial differential equations to obtain a novel perspective on the commutators of canonical quantum gravity. We have shown that the fully diff-invariant classical Peierls bracket among diff-invariant functionals in general relativity can be approximated by means of a parametrix of a fifth-order elliptic operator, convoluted with functions solving Kirchhoff-type formulas. The procedure is more systematic than the Eq. (1.14) leading to the advanced and retarded Green functions, which was advocated but not solved in Ref. [19]. Although the parametrix of a partial differential operator is not unique, this apparent drawback can be exploited in order to evaluate with increasing accuracy the desired classical Peierls bracket. After all, the best we can do in science is to evaluate with high accuracy the functional relations we are interested in. We have assumed that the operator Green functions of canonical quantum gravity consist of a classical part given by the above parametrix, plus a quantum part that is responsible of technical complications but should be kept under control. Our original work in
Secs from V to VII puts on firm ground the existence of the desired parametrix for gravity, and derives the novel functional equations (7.6), (7.7), (7.12) and (7.15) that are a concrete step towards applying parametrices in canonical quantization of field theories possessing an infinite-dimensional invariance group.

From the point of view of general formalism, another interesting issue is whether a relation exists with the work in Ref. [39], where the author has studied a formulation of the ordering problems of general relativity inspired by the Grönewald-Van Hove theorem. This work of Segre establishes a negative result, i.e., the lack of a suitable quantization map and of a suitable extension of such a map to general relativity. We hope to understand better the issue in future, so as to investigate the possible implications (if any) for the quantum part $U^{ij}$ of the Green function in our Eq. (8.3).

If it were possible to achieve our goals, a novel perspective on the longstanding problems of (canonical) quantum gravity would emerge.

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Appendix A. DeWitt notation and gauge theories

The DeWitt notation [32] makes it possible to use matrix-like notation for the otherwise rather lengthy functional equations encountered in field theory. It does not solve, by itself, the open problems of theoretical physics, but makes it possible to see easily what features are shared by seemingly very different field theories.

The field variables are denoted by $\varphi^i$, e.g. the electromagnetic potential $A_\mu^i$, or the Yang-Mills potential $A_\mu^i$ or the spacetime metric $g_{\mu\nu}$. For each functional $T$ of the fields, its functional derivatives are denoted by

$$\frac{\delta T}{\delta \varphi^i} = T_{,i} = T_1,$$

$$\frac{\delta^2 T}{\delta \varphi^i(x) \delta \varphi^j(x')} = T_{,ij} = T_2,$$

and so on. Since the Latin indices used here for fields carry information about the spacetime point where the field is evaluated, an expression like $T_{,ij} U^{ij}$ is meant to be

$$\sum_j \int \frac{\delta^2 T}{\delta \varphi^i(x) \delta \varphi^j(x')} U^{ij}(x') dx',$$

for $x'$.
 Unlike tensor calculus, where the Einstein convention $T_{\mu\nu}U^\nu$ in $n$-dimensional spacetime is a short-hand notation for a sum without integration, i.e. $\sum_{\nu=0}^{n-1} T_{\mu\nu}(x)U^\nu(x)$.

In the gauge theories of theoretical physics, the action functional remains unchanged in value under certain continuous changes in the dynamical variables, which are confined to finite but otherwise arbitrary spacetime domains. The set of all such changes constitutes a transformation group. The abstract group of which the transformation group forms a representation is the \textit{invariance group} of the system \cite{32}. Since the finite domain over which the changes in the dynamical variables differ from zero is arbitrary, the invariance group is necessarily infinite-dimensional, i.e. it is a pseudo-group \cite{32}. Under an infinitesimal group transformation the changes may be expressed in the general form

$$\delta \varphi^i = \int Q^i_{\alpha'} \delta \xi^\alpha' \, dx' = Q^i_{\alpha} \delta \xi^\alpha,$$ \hfill (A.3)

where, for local theories, the $Q^i_{\alpha'}$ are linear combinations of Dirac’s delta and its derivatives, while the $\delta \xi^\alpha$ are arbitrary infinitesimal differentiable functions of the spacetime points, known as group parameters, and vanish outside the arbitrary domain under consideration \cite{32}.

When Eq. (A3) holds, the infinitesimal variation of any functional $T$ of the fields is expressed by

$$\delta T = T_{,i} \delta \varphi^i = T_{,i} Q^i_{\alpha} \delta \xi^\alpha.$$ \hfill (A.4)

In particular, if $T$ is gauge-invariant, its variation $\delta T$ in (A4) vanishes, and one finds

$$T_{,i} Q^i_{\alpha} = 0 \implies Q_{\alpha} T = 0,$$ \hfill (A.5)

where $Q_{\alpha}$ are vector fields on the space of field configurations, i.e.

$$Q_{\alpha} \equiv Q^i_{\alpha} \frac{\delta}{\delta \varphi^i}.$$ \hfill (A.6)

Note also that, if $T$ coincides with the action functional $S$ of a gauge theory, functional differentiation of Eq. (A5) yields

$$S_{,ij} Q^i_{\alpha} + S_{,i} Q^i_{\alpha,j} = 0.$$ \hfill (A.7)

Thus, restriction of Eq. (A7) to the dynamical subspace, where the Euler-Lagrange equations $S_{,i} = 0$ hold, tells us that the operator $S_{,ij}$ is not invertible, because it then possesses nonvanishing eigenvectors belonging to the zero eigenvalue. Thus, one has to resort to the imposition of suitable supplementary conditions whenever the field theory is a gauge theory (see, for example, Eq. (1.11) taken from Ref. \cite{19}). For Einstein’s general relativity, the explicit form of $S_{,ij}$ was first obtained by Levi-Civita \cite{40} and then, in a more accessible reference, by DeWitt \cite{32}, but it is remarkable that, before any detailed calculation, the lack of Green functions or fundamental solutions of $S_{,ij}$ for any gauge theory is already clear from first principles.
Appendix B. Spacetime geometry for the wave equation

A connected open set $D$ of the spacetime manifold $(M, g)$ is said to be a geodesically convex domain if any two points $q$ and $p$ in $D$ are joined by a unique geodesic in $D$. In a geodesically convex domain $\Omega$, let the local coordinates of two points $q$ and $p$ be $x$ and $y$, respectively. If $\rho : \tau \rightarrow z(\tau), \tau \in [0, 1]$, is a parametrized $C^2$ curve joining $q$ and $p$ in $\Omega$, the arc-length of $\rho$ is defined by

$$s \equiv \int_0^1 \sqrt{g_{\mu \nu}(z(\tau))} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} d\tau. \quad (B.1)$$

When $\rho$ is the unique geodesic joining $q$ and $p$ in $\Omega$, the arc-length $s$ is said to be the geodesic distance of $q$ and $p$, and the square of the geodesic distance is $\Gamma(p, q)$:

$$\Gamma(p, q) \equiv s^2, \quad (B.2)$$

and is written as $\Gamma(x, y)$ in local coordinates. In the literature on general relativity, $\Gamma(p, q)$ is called the world function. It was first introduced by Hadamard [41] in the analysis of the Cauchy problem for linear partial differential equations, then by Ruse [42] and Synge [43], and eventually by DeWitt and Brehme [44] in the investigation of electromagnetic Green functions on curved spacetime.

In a geodesically convex domain $\Omega$, the future (resp. past) null semi-cone $C^+(q)$ (resp. $C^-(q)$) is the set of all points $p$ of $\Omega$ such that there exists a future-directed (resp. past-directed) null geodesic from $q$ to $p$. The future (resp. past) domain of dependence $D^+(q)$ (resp. $D^-(q)$) is the set of all points $p \in \Omega$ that can be reached along future-directed (resp. past-directed) timelike geodesics from $q$. One has therefore

$$C^+(q) = \partial D^+(q), \quad C^-(q) = \partial D^-(q). \quad (B.3)$$

The closure of $D^+(q)$ (resp. $D^-(q)$) is instead called the future (resp. past) emission of $q$:

$$J^+(q) \equiv \overline{D^+(q)}, \quad J^-(q) \equiv \overline{D^-(q)}. \quad (B.4)$$

Such sets are also called causal future ($J^+$) and causal past ($J^-$) of $q$ [45].

In Eqs. (2.12) and (2.13), we have

$$H_\pm(\Gamma) \equiv 1 \text{ if } p \in J^\pm(q), \quad 0 \text{ if } p \notin J^\pm(q). \quad (B.5)$$

Moreover,

$$U\delta_\pm(\Gamma) \equiv U(p, q)\delta_\pm(\Gamma(p, q)) \quad (B.6)$$

are distributions that act on test functions $\phi(p) \in C^\infty_0(\Omega)$.

A connected open set $\Omega$ is called a causal domain if

(i) there is a geodesically convex domain $\Omega_0$ such that $\Omega \subset \Omega_0$, and

(ii) $\forall p, q \in \Omega$, the set $J^+(q) \cap J^-(p)$ is either a compact subset of $\Omega$, or otherwise is empty.
Appendix C. Characteristic conoid and Kirchhoff formulas

For systems of linear second-order partial differential equations, the theory of characteristics is developed starting from the equations \[ E_s = \sum_{r=1}^{m} \sum_{\lambda,\mu=0}^{n} E^\lambda_\lambda^\mu_{sr} \frac{\partial^2 \varphi_r}{\partial x^\lambda \partial x^\mu} + \Phi_s = \sum_{r=1}^{m} E_{s0}^0 \frac{\partial^2 \varphi_r}{\partial (x^0)^2} + ... = 0, \]
which are soluble with respect to \( \frac{\partial^2 \varphi_r}{\partial (x_0)^2} \) (this is the normality condition) if \( \Omega \equiv \det E_{s0}^0 \neq 0 \). We then ask ourselves under which conditions is the normality character preserved if, to the independent variables \( x^0, x^1, ..., x^n \), we apply the transformation

\[ (x^0, x^1, ..., x^n) \rightarrow (z, z^1, ..., z^n) \]
so that the hyperplane \( x^0 = a^0 \) is transformed into an hypersurface having equation

\[ z(x^0, x^1, ..., x^n) = z^0, \]
starting from which one can determine, at least in a neighborhood, the \( \varphi_r \) functions. For this purpose, let us consider the covector (or covariant vector or one-form) variables

\[ \xi_\mu \equiv \frac{\partial z}{\partial x^\mu}, \]
from which we get \[ \frac{\partial \varphi_r}{\partial x^\mu} = \frac{\partial \varphi_r}{\partial z} \xi_\mu + \sum_{\lambda=1}^{n} \frac{\partial \varphi_r}{\partial z^\lambda} \frac{\partial z^\lambda}{\partial x^\mu} \xi_\mu + ..., \]

\[ \frac{\partial^2 \varphi_r}{\partial x^\lambda \partial x^\mu} = \frac{\partial^2 \varphi_r}{\partial z^2} \xi_\lambda \xi_\mu + .... \]

Hence the original system (C1) gets transformed into

\[ \sum_{r=1}^{m} \frac{\partial^2 \varphi_r}{\partial z^2} \sum_{\lambda,\mu=0}^{n} E^\lambda_\lambda^\mu_{s} \xi_\lambda \xi_\mu + ... = 0, \quad s = 1, 2, ..., m. \]

Thus, upon considering the principal (or leading) symbol of the operator in Eqs. (C1), i.e. the matrix of polynomials

\[ \omega_{sr} \equiv \sum_{\lambda,\mu=0}^{n} E^\lambda_\lambda^\mu_{s} \xi_\lambda \xi_\mu, \]
also called the characteristic polynomial, the normality condition of (C1) is preserved if \( \Omega \equiv \det \omega_{sr} \neq 0 \). If instead \( \Omega \) vanishes, it is no longer possible to apply (whatever the value of \( z^0 \)) the Cauchy theorem starting from the hypersurfaces \( z = z^0 \). The equation

\[ \Omega = \det \omega_{sr} = 0 \]
is the equation defining the characteristics manifolds [46].

In particular, the characteristic surfaces of the system (4.10) are three-dimensional manifolds (null hypersurfaces [23]) of four-dimensional spacetime with coordinates $x^\alpha$ and solve the differential system

$$F = \sum_{\lambda,\mu=0}^{3} A^{\lambda\mu} y_\lambda y_\mu = 0,$$

(C.9)

$$\sum_{\lambda=0}^{3} y_\lambda dx^\lambda = 0.$$  

(C.10)

The four quantities $y_\lambda$ denote a system of directional parameters of the normal vector field. Let us take this system, which is only defined up to a proportionality factor, in such a way that $y_0 = 1$, and let us set $y_i = \xi_i$. The desired surfaces are therefore a solution of

$$F = A^{00} + 2 \sum_{i=1}^{3} A^{i0} \xi_i + \sum_{i,j=1}^{3} A^{ij} \xi_i \xi_j = 0,$$  

(C.11)

$$dx^0 + \sum_{i=1}^{3} \xi_i dx^i = 0.$$  

(C.12)

The characteristics of this differential system, which are bicharacteristics (i.e. null geodesics [23]) of the Eqs. (4.10), satisfy the following differential equations:

$$\frac{dx^i}{\left(A^{i0} + \sum_{j=1}^{3} A^{ij} \xi_j\right)} = \frac{dx^0}{\left(A^{00} + \sum_{i=1}^{3} A^{i0} \xi_i\right)} = -\frac{1}{2} \left(\frac{\partial F}{\partial x^i} - \xi_i \frac{\partial F}{\partial x^0}\right) = d\lambda_1,$$  

(C.13)

$\lambda_1$ being an auxiliary parameter. The characteristic conoid $\Sigma_0$ with vertex $M_0(x_0^0)$ is the characteristic surface generated from the bicharacteristics passing through $M_0$. Such a bicharacteristic solves the system of integral equations [11]

$$x^i = x_i^0 + \int_0^{\lambda_1} T^i d\lambda_1, \quad T^i \equiv A^{i0} + A^{ij} \xi_j,$$  

(C.14)

$$x^0 = (x_0^0) + \int_0^{\lambda_1} T^0 d\lambda_1, \quad T^0 \equiv A^{00} + \sum_{i=1}^{3} A^{i0} \xi_i,$$  

(C.15)

$$\xi_i = \xi_i^0 + \int_0^{\lambda_1} R_i d\lambda_1, \quad R_i \equiv -\frac{1}{2} \left(\frac{\partial F}{\partial x^i} - \xi_i \frac{\partial F}{\partial x^0}\right),$$  

(C.16)

where the $\xi_i^0$ satisfy the relation (C3), i.e.

$$A_0^{00} + 2 \sum_{i=1}^{3} A_0^{i0} \xi_i^0 + \sum_{i,j=1}^{3} A_0^{ij} \xi_i^0 \xi_j^0 = 0,$$  

(C.17)
being the value of $A^{\lambda \mu}$ at the vertex $M_0$ of the conoid $\Sigma_0$. One can assume the following values:

$$A^{00}_0 = 1, \quad A^{i0}_0 = 0, \quad A^{ij}_0 = -\delta^{ij}, \quad (C.18)$$

so that $(C17)$ reduces to

$$\sum_{i=1}^{3} (\xi^i_0)^2 = 1. \quad (C.19)$$

Besides the parameter $\lambda_1$ which defines the position of a point on a given bicharacteristic, we will need two more parameters $\lambda_2$ and $\lambda_3$ which vary with the bicharacteristic under investigation and are given by [11]

$$\xi^0_1 = (\sin \lambda_2)(\cos \lambda_3), \quad \xi^0_2 = (\sin \lambda_2)(\sin \lambda_3), \quad \xi^0_3 = \cos \lambda_2. \quad (C.20)$$

Moreover, following again Ref. [11], for any function $\varphi$ of the spacetime coordinates $x^\alpha$, we denote by $[\varphi]$ its restriction to the characteristic conoid, i.e.

$$[\varphi] \equiv \varphi(x^\alpha)|_{\Sigma_0}, \quad (C.21)$$

and we consider the second-order operator

$$M(\varphi) \equiv \sum_{i,j=1}^{3} [A^{ij}] \frac{\partial^2 \varphi}{\partial x^i \partial x^j}, \quad (C.22)$$

and its (formal) adjoint

$$\overline{M}(\psi) \equiv \sum_{i,j=1}^{3} \frac{\partial^2 ([A^{ij}]\psi)}{\partial x^i \partial x^j}. \quad (C.23)$$

At this stage, the work in Ref. [11] took linear combinations of Eqs. (4.10), with left-hand side (with free index $r$) denoted by $E_r$, the coefficients of linear combination being some a priori unknown functions $\sigma^r_s$. Remarkably, such functions were found to solve a set of equations and turned out to obey the factorization property

$$\sigma^r_s = \theta \omega^r_s, \quad (C.24)$$

where

$$\omega^r_s = \int_{0}^{\lambda_1} \left( \sum_{t=1}^{3} Q^r_t \omega^t_s + Q \omega^s_r \right) d\lambda_1 + \delta^r_s, \quad (C.25)$$

having defined

$$Q^r_t \equiv \frac{1}{2} \left( [B^r_t] + [B^t_r] \xi_t \right), \quad (C.26)$$

$$Q \equiv -\frac{1}{2} \sum_{i,j=1}^{3} \left( \xi_i \frac{\partial}{\partial x^i} [A^{ij}] - \frac{1}{2} \sum_{t=1}^{3} \left( \frac{\partial}{\partial x^t} [A^{i0}] \right) \right), \quad (C.27)$$
while, on denoting by

$$ J = \frac{D(x^1, x^2, x^3)}{D(\lambda_1, \lambda_2, \lambda_3)} $$

(C.28)

the Jacobian of the change of variables \( x^i = x^i(\lambda_j) \) on the conoid \( \Sigma_0 \), one finds \[11\]

$$ \theta = \frac{\sqrt{|\sin \lambda_2|}}{\sqrt{|J|}}, \quad \lim_{\lambda_1 \to 0} \theta \lambda_1 = 1. \quad \text{(C.29)} $$

The work in Ref. \[11\] arrived at Kirchhoff formulas for the solution of Eqs. (4.10) where the integrand is built from the functions

$$ E_s^i = \sum_{j,r=1}^3 \left\{ [A^{ij}] \sigma^r_s \frac{\partial u_r}{\partial x^j} - [u_r] \frac{\partial}{\partial x^j} ( [A^{ij}] \sigma^r_s ) + [B^r_i][u_s] \sigma^r_s \right\} + 2\sigma^r_s \left\{ [A^{ij}] \xi_j + [A^{00}] \right\} \left( \frac{\partial u_r}{\partial x^j} \right), $$

(C.30)

$$ L_s^r = \mathcal{M}(\sigma^r_s) - \sum_{i=1}^3 \frac{\partial}{\partial x^i} ( [B^r_i][\sigma^r_s] ), \quad \text{(C.31)} $$

and also from the parameter \( T \) resulting from the following geometric considerations.

The surfaces upon which we perform integration are surfaces \( x^0 = \text{constant} \) traced on the characteristic conoid \( \Sigma_0 \). Thus, in light of (C10), they fulfill the differential relation \( p_i dx^i = 0 \). To compute the surface element \( dS \) we re-express the volume element, originally written as

$$ dV = \prod_{i=1}^3 dx^i = \prod_{i=1}^3 d\lambda_i, $$

by exploiting the surfaces \( x^0 = \text{constant} \) and the bicharacteristics (where only \( \lambda_1 \) is varying), i.e. \[11\]

$$ dV = (\cos \nu) \sqrt{|T|} \ dl_1 \ dS, $$

(C.32)

where \( \sqrt{|T|} \ dl_1 \) is the length element of the bicharacteristic, and \( \nu \) is the angle of the bicharacteristic with the normal to the surface \( S \) at the point considered.

Having defined all the concepts we need, we can now state the theorem proved in Ref. \[11\] for linear hyperbolic systems.

**Theorem.** Let the linear hyperbolic system (4.10) be given, satisfying the following assumptions:

(i) At the point \( M_0 \), the conditions (C18) hold.

(ii) The functions \( A^{\lambda \mu} \) and \( B^{r \lambda}_s \) have partial derivatives with respect to \( x^0 \) of order 4 and 2, respectively, continuous and bounded in a domain

$$ D : |x^i - \bar{x}^i| \leq d, \quad |x^0| \leq \varepsilon. $$
The functions \( f_r \) are continuous and bounded.

(iii) The fourth partial derivatives of \( A^{\lambda \mu} \) and the second partial derivatives of \( B^\nu_s \) fulfill Lipschitz conditions.

Then every continuous, bounded solution of the Eqs. (4.10) with continuous and bounded first derivatives within \( D \) satisfies the Kirchhoff integral relations

\[
4\pi u_s(x_j) = \sum_{r=1}^{3} \left[ \int_0^2 \int_0^\pi \int_0^\pi \left( [u_r] L^r_s + \sigma^r_s [f_r] \right) \frac{J}{T^0} \, dx^0 \, d\lambda_2 \, d\lambda_3 \right.
\]
\[
+ \int_0^{2\pi} \int_0^\pi \left\{ \frac{E^r_s J_{\xi r}}{T^0} \right\}_{x^0=0} \, d\lambda_2 \, d\lambda_3 \right],
\]

(C.33)

if the coordinates \( x_0^0 \) of \( M_0 \) fulfill majorizations of the form

\[
| (x_0)^0 | \leq \varepsilon_0, \quad | x_0^i - \mathbf{\pi} | \leq d,
\]

(C.34)

which define a domain \( D_0 \subset D \).

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