UNIFORM BOUNDS FOR INVARIANT SUBSPACE PERTURBATIONS
ANIL DAMLE* AND YUEKAI SUN†

Abstract. For a fixed matrix $A$ and perturbation $E$ we develop purely deterministic bounds on how invariant subspaces of $A$ and $A + E$ can differ when measured by a suitable “row-wise” metric rather than via traditional norms such as two or Frobenius. Understanding perturbations of invariant subspaces with respect to such metrics is becoming increasingly important across a wide variety of applications and therefore necessitates new theoretical developments. Under minimal assumptions we develop new bounds on subspace perturbations under the two-to-infinity matrix norm and show in what settings these row-wise differences in the invariant subspaces can be significantly smaller than the two or Frobenius norm differences. We also demonstrate that the constitutive pieces of our bounds are necessary absent additional assumptions and therefore our results provide a natural starting point for further analysis of specific problems.

Key words. Invariant subspace perturbation theory

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1. Introduction. Given a matrix $A$, it is natural to try and “understand” properties of $A$ after it has been perturbed in some way. Understanding can take many forms, as can the way we chose to perturb $A$. In this work we are primarily concerned with additive perturbations and understanding how spectral properties of the matrix change—i.e., given a perturbation $E$ how do certain invariant subspaces of $A + E$ relate to those of $A$. This is a long standing question and one that has received extensive attention over the years. This includes the well-known Davis-Kahan theorem [9], work by Wedin [24], and more general and extensive perturbation theory; see, e.g., [21] for an overview.

Nevertheless, many problems of growing interest in mathematics, statistics, and computer science require new variants of such theory. Most notably, this manifests as modifications to the metrics we use to assess the similarity of invariant subspaces of $A$ and $A + E$. Concretely, whereas traditional theory is often interested in classical notions of subspace distance measured by spectral or Frobenius norms, we will be interested in row-wise$^1$ (or closely related) measures of error. In Section 2 we will formally outline the specifics of these metrics, contrast them to traditionally considered metrics, and provide additional preliminary material relevant to our work.

The impetus for these new types of bounds is often, though not exclusively, problems arising in statistics and computer science such as matrix completion [4, 16], principal component analysis [3, 18], robust factor analysis [11], spectral clustering [1, 2, 8, 17, 19, 23], and more. In these settings $A$ will often represent some model and a given instance of the model $\tilde{A}$ can be thought of as a (random) perturbation to this baseline, i.e. $\tilde{A} = A + E$. Many models $A$ have highly structured and meaningful invariant subspaces whose properties form the basis for a wide variety of algorithmic development and analysis of the underlying problem. Therefore, given $\tilde{A}$ we would like to understand if that structure can still be reliably leveraged. For many of these applications traditional measures of distance do not necessarily provide

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*Department of Computer Science, Cornell University, Ithaca, NY (damle@cornell.edu).
†Department of Statistics, University of Michigan, Ann Arbor, MI (yuekai@umich.edu).

$^1$As elaborated on later, we must somewhat carefully define what it means for two subspaces to be close row-wise.
adequate control over changes to the invariant subspaces.

A simple, concrete illustration of the types of bounds we will develop is encapsulated in the following scenario. Given a rank-$k$ symmetric matrix $A$, an $n \times k$ matrix $V$ with orthonormal columns representing the subspace associated with the non-zero eigenvalues, and a perturbation $E$, when is the dominant invariant subspace of $A + E$ closer to $V$ row-wise than may be expected based on the smallest non-zero eigenvalue of $A$ and the spectral norm of $E$? As it turns out, in situations where $E$ and $V$ have relatively uniform row norms (i.e., they are incoherent [4]) we may expect significantly better bounds than what is captured by traditional subspace perturbation theory. We will formalize these results in Section 3 where we also provide proofs and investigate the behavior of our bounds. To complement our theoretical developments, Section 4 provides several numerical examples illustrating our bounds in appropriate scenarios.

Given the potential usefulness of such bounds and the extent of relevant applications, this area has received significant attention over the past several years [1, 5, 10, 12]. Our main contributions are three fold:

1. We develop new deterministic row-wise perturbation bounds for orthonormal bases of invariant subspaces. Often, existing work entangles deterministic aspects of the perturbation analysis with the subsequent analysis of it in specific random settings. Nevertheless, our deterministic bounds are easily amenable to further analysis in the probabilistic settings.

2. We show that our bounds are sharp by constructing adversarial perturbations that saturate the bounds.

3. Our perturbation bounds apply under more general conditions than preceding results and we argue that our assumptions are minimal in certain respects by considering specific examples.

While some aspects of our bounds are broadly in alignment with prior work, as noted above others are new, rely on less restrictive assumptions, and are more directly interpretable. We will draw specific comparisons to existing results parallel to the development of our bounds in Section 3.

2. Preliminaries.

2.1. Matrices. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric (not necessarily positive-definite) matrix. We arrange its eigenvalues in descending order
\[ \lambda_1 \geq \cdots \geq \lambda_n, \]
and denote its eigen-decomposition as
\[ A = V_1 \Lambda_1 V_1^T + V_2 \Lambda_2 V_2^T, \]
where $\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_r)$ and $\Lambda_2 = \text{diag}(\lambda_{r+1}, \ldots, \lambda_n)$ and $V_1 \in \mathbb{R}^{n \times r}$ and $V_2 \in \mathbb{R}^{n \times (n-r)}$ are matrices whose columns are the associated eigenvectors.\(^2\) Note that for our later results it is important that $\Lambda_2$ explicitly include any zero eigenvalues of $A$ as they must be incorporated into our measure of how close $\Lambda_1$ and $\Lambda_2$ are. Because we are interested in algebraic orderings of the eigenvalues we use dominant refer to the algebraically largest eigenvalues (in contrast to the largest in magnitude).

Remark 1. Note that the restriction to the $r$ algebraically largest eigenvalues is not essential. Our results will only depend on spectral separation and therefore may be easily adapted to any isolated collection of $r$ eigenvalues. However, such an adaptation

\(^2\)In the case of repeated eigenvalues any orthonormal basis for the associated eigenspace suffices.
introduces notational overhead without adding anything fundamentally new. Similarly, with appropriate adaptation these results are applicable to magnitude based ordering of the eigenvalues—though the ordering itself may be more sensitive to perturbations than the associated subspaces. Therefore, to streamline the exposition we present everything for the \( r \) algebraically largest eigenvalues.

Now, let \( \hat{A} = A + E \) represent a perturbation of \( A \) by a symmetric matrix \( E \) and let \( \hat{V}_1 \in \mathbb{R}^{n \times r} \) be a matrix with orthonormal columns whose range is the \( r \)-dimensional invariant subspace of \( \hat{A} \) associated with the algebraically largest eigenvalues. For the moment we will assume \( \hat{\lambda}_r > \hat{\lambda}_{r+1} \) so this notation is well defined, later assumptions we make will ensure this property. The primary contributions of this paper are centered around relating \( V_1 \) and \( \hat{V}_1 \).

We will often be interested in projections of matrices onto the invariant subspaces associated with \( A \) (represented by \( V_1 \) and \( V_2 \)). Therefore, for any matrix \( B \in \mathbb{R}^{n \times n} \) define \( B_{i,j} \) as \( V_i^T B V_j \) with \( i, j \in \{1, 2\} \). Lastly, for any matrix \( B \in \mathbb{R}^{n \times n} \) we define the Sylvester operator \( S_B : \mathbb{R}^{n \times r} \to \mathbb{R}^{n \times r} \) as

\[
S_B : Z \to ZB_{1,1} - B_{2,2}Z.
\]

Note that we have embedded \( V_1 \) and \( V_2 \) directly into the definition of this Sylvester operator for convenience.

2.2. Norms. Throughout this paper, we let \( \| \cdot \|_1, \| \cdot \|_\infty, \) and \( \| \cdot \|_2 \) denote the standard \( \ell_p \) vector norms and their associated induced matrix norms. Similarly, we let \( \| \cdot \|_F \) denote the Frobenius norm. In this work we will also be concerned with the two-to-infinity induced matrix norm. Specifically, we denote this norm by \( \| \cdot \|_{2, \infty} \) and note that for an \( n \times k \) matrix \( B \) it can be defined as the maximum \( \ell_2 \) norm of rows of \( B \), i.e.

\[
\| B \|_{2, \infty} = \max_{i=1, \ldots, n} \| B_{i, \cdot} \|_2.
\]

We outline a few easily verified properties of \( \| \cdot \|_{2, \infty} \) that will be useful later:

UNITARY INVARIANCE FROM THE RIGHT: For any orthogonal \( Z \) of appropriate size

\[
\| BZ \|_{2, \infty} = \| B \|_{2, \infty}.
\]

In other words, the norm is invariant under orthogonal transforms from the right (though, notably, not from the left). This property follows immediately from the unitary invariance of the two-norm.

SUB-MULTIPlicative RELATIONS: Relevant sub-multiplicative relationships are

\[
\| B_1 B_2 \|_{2, \infty} \leq \| B_1 \|_{2, \infty} \| B_2 \|_2 \quad \text{and} \quad \| B_1 B_2 \|_{2, \infty} \leq \| B_1 \|_\infty \| B_2 \|_{2, \infty}.
\]

These inequalities follow from the definition of induced matrix norms.

2.3. Subspace distances. Importantly, given our fairly loose assumptions on the eigenvalues of \( A \) we cannot always talk about convergence to individual eigenvectors. Instead, we consider the distances between invariant subspaces. Given two \( n \times k \) matrices with orthonormal columns \( W \) and \( \hat{W} \) distance between the subspaces spanned by \( W \) and \( \hat{W} \) is

\[
\text{dist}(W, \hat{W}) = \| WW^T - \hat{W} \hat{W}^T \|_2.
\]

Equivalent definitions include (see, e.g., [13]):
**Complementary subspaces:** Let $W_2 \in \mathbb{R}^{n \times n - k}$ be an orthonormal basis for the orthogonal complement of the subspace spanned by $W$, then

$$\text{dist}(W, \tilde{W}) \equiv \|W_2^T \tilde{W}\|_2.$$ 

**Sine-$\Theta$ distance:** Let $\Theta(W, \tilde{W})$ be a diagonal matrix containing the principle angles between $W$ and $\tilde{W}$, then

$$\text{dist}(W, \tilde{W}) \equiv \|\sin \Theta(W, \tilde{W})\|_2.$$ 

The bounds we develop in Section 3 will focus on a slightly different measure between $W$ and $\tilde{W}$. Specifically, we will be concerned with the row-wise error metric

$$\min\{\|\tilde{W}U - W\|_{2,\infty} : U \in O^k\},$$

where the minimization over orthogonal matrices ensures the metric is appropriate for subspaces (as opposed to relying on a specific choice of basis). Conceptually, this measure is closely related to the so-called orthogonal Procrustes problem

$$\min\{\|\tilde{W}U - W\|_F : U \in O^k\},$$

which is well-studied and has a known solution easily computable via the SVD of $\tilde{W}^T W$ (see, e.g., [13]). Notably, the entry-wise definitions of (2.1) and (2.2) immediately show it is plausible that (2.1) can be considerably (by a factor of $1/\sqrt{n}$) smaller than (2.2)—using $\|\cdot\|_{2,\infty}$ allows us to understand how well the error is distributed over the rows.

**2.4. Separation of matrices.** An important concept for our work is the separation of matrices in different norms. Specifically, for any two matrices $B \in \mathbb{R}^{\ell \times \ell}$ and $C \in \mathbb{R}^{m \times m}$ and norm $\| \cdot \|_*$ on $\mathbb{R}^{m \times \ell}$, define

$$\text{sep}_*(B, C) = \inf\{\|ZB - CZ\|_* : \|Z\|_* = 1\}.$$ 

Perhaps the most pervasive use of $\text{sep}$ is in traditional invariant subspace perturbation theory. In fact

$$\text{sep}_F(B, C) = \min_{\lambda \in \Lambda(B), \mu \in \Lambda(C)} |\lambda - \mu|,$$

thereby recovering the oft used notion of an eigengap (see, e.g., [21]).

Importantly, and in alignment with our algebraic ordering of eigenvalues above, $\text{sep}$ is shift invariant in any norm, i.e.

$$\text{sep}_*(B + \xi I, C + \xi I) = \text{sep}_*(B, C)$$

for any $\xi \in \mathbb{R}$. Furthermore, in any unitarily invariant norm $\text{sep}$ is relatively insensitive to perturbations of small spectral norm. In other words (see Proposition 2.1 of [15])

$$\text{sep}_2(B + E_B, C + E_C) \geq \text{sep}_2(B, C) - \|E_B\|_2 - \|E_C\|_2.$$ 

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3 The optimal value of this minimization problem is also closely related to the aforementioned notions of subspace distance and principle angles between $W$ and $\tilde{W}$.

4 This specific form of $\text{sep}$ differs slightly notationally, though not mathematically, from the standard way it is written for the two or Frobenius norm. Since we will ultimately be dealing with norms where $\|B\|_* \neq \|B^T\|_*$, this definition is required for consistency.
and
\[ \text{sep}_F(B + E_B, C + E_C) \geq \text{sep}_F(B, C) - \|E_B\|_2 - \|E_C\|_2 \]
for appropriately sized matrices \( E_B \) and \( E_C \). Lastly, given diagonal matrices it is possible to control \( \text{sep} \) in a variety of norms necessary for our work. These bounds are captured collectively in Lemma 2.1 (the results about \( \text{sep}_2 \) and \( \text{sep}_F \) are well known).

**Lemma 2.1.** Let \( D_1 \in \mathbb{R}^{\ell \times \ell} \) and \( D_2 \in \mathbb{R}^{m \times m} \) be diagonal matrices and assume that \( \lambda_{\min}(D_1) \geq \lambda_{\max}(D_2) \), then
\[ \text{sep}_2(D_1, D_2) = \text{sep}_F(D_1, D_2) = \text{sep}_{2,\infty}(D_1, D_2) = \lambda_{\min}(D_1) - \lambda_{\max}(D_2). \]

**Proof.** We defer the proof to Appendix A.1.

In addition to the above “canonical” definition of separation, some of our bounds requires we introduce a slight variant of \( \text{sep} \). In particular, we will occasionally consider the separation measured only over matrices in a linear subspace. More specifically, let \( W \in \mathbb{R}^{n \times k} \) be an orthonormal basis for a \( k \)-dimensional linear subspace and define
\[ \text{sep}_{*,W}(B, C) = \inf\{\|ZB - CZ\|_* : Z \in \text{ran} W, \|Z\|_* = 1\}. \]

It is immediate that \( \text{sep}_{*,W}(B, C) \geq \text{sep}_i(B, C) \) for any \( W \) and therefore, as will become evident, consideration of this restricted version of \( \text{sep} \) can only improve our bounds. For us, the key use of this restricted separation quantity will be when \( C = WD_2W^T \) for some diagonal matrix \( D_2 \).

In anticipation of its use later, we prove some results about this restricted version of \( \text{sep} \) analogous to our earlier statements. First, we generalize the notion of \( \text{sep} \) being shift invariant in Lemma 2.2.

**Lemma 2.2.** Consider \( B \in \mathbb{R}^{\ell \times \ell} \) and \( C \in \mathbb{R}^{m \times m} \), and let \( W \in \mathbb{R}^{n \times m} \) with \( n \geq m \) be a matrix with orthonormal columns. Then,
\[ \text{sep}_{*,W}(B + \xi I, WCW^T + \xi WW^T) = \text{sep}_{*,W}(B, WCW^T) \]
for any \( \xi \in \mathbb{R} \).

**Proof.** The proof follows immediately from the fact that for any \( Z \in \text{ran} W \), \( \xi WW^T Z = \xi Z \). 

Perhaps more importantly, and as illustrated in Lemma 2.3 in any unitarily invariant norm we can relate this restricted version of \( \text{sep} \) directly to \( \text{sep}(B, C) \).

**Lemma 2.3.** Consider \( B \in \mathbb{R}^{\ell \times \ell} \) and \( C \in \mathbb{R}^{m \times m} \), and let \( W \in \mathbb{R}^{n \times m} \) with \( n \geq m \) be a matrix with orthonormal columns. Then, for any unitarily invariant norm \( \| \cdot \|_* \),
\[ \text{sep}_{*,W}(B, WCW^T) = \text{sep}_*(B, C) \]
for any \( \xi \in \mathbb{R} \).

**Proof.** We first rewrite the infimum over \( \text{ran} W \) in terms of coefficients of \( Z \) in the orthonormal basis \( W \) as
\[ \text{sep}_{*,W}(B, C) = \inf\{\|WXB - WCW^TX\|_* : \|X\|_* = 1\}, \]
where we have used the fact that \( \| \cdot \|_* \) is unitarily invariant to say that \( \|WX\|_* = \|X\|_* \). Using that \( W^TW = I \) and unitary invariance once more concludes the proof.

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5 One consequence of this restriction is that it will allow us to eliminate any artificial requirement that \( A_1 \) be separated from zero when \( A \) is not low-rank.
We now use these results to control a restricted version of \( \text{sep}_{2,\infty} \) (a quantity that will be of particular importance to us and is not unitarily invariant) in terms of more traditional and directly interpretable quantities. These results are encapsulated in Lemma 2.4 and we stress that they are worse case bounds that may be far from achieved in practice or provably loose in specific cases.\(^6\) Nevertheless, the fact that the \( 2,\infty \) norm is not unitarily invariant from the left significantly changes the landscape of possible outcomes.

**Lemma 2.4.** Consider \( B \in \mathbb{R}^{\ell \times \ell} \) and \( C \in \mathbb{R}^{m \times m} \), and let \( W \in \mathbb{R}^{n \times m} \) with \( n \geq m \) be a matrix with orthonormal columns. Then,

\[
\text{sep}_{(2,\infty),W}(B,WCW^T) \geq \max \left\{ \frac{1}{\sqrt{n}}, \beta_W \right\} \text{sep}_F(B,C),
\]

where

\[
\beta_W = \inf \left\{ \frac{\|WX\|_{2,\infty}}{\|W\|_{2,\infty}} : \|X\|_F = 1 \right\}.
\]

**Proof.** We prove two lower bounds on \( \text{sep}_{(2,\infty),W}(B,WCW^T) \) that always hold and then maximize over them. First, observe that for any \( Z \in \text{ran} \ W \) there exists an \( X \) such that \( Z = WX \) in which case

\[
\frac{\|ZB - WCW^T Z\|_{2,\infty}}{\|Z\|_{2,\infty}} = \frac{\|WXB - WCX\|_{2,\infty}}{\|WX\|_{2,\infty}} \geq \frac{\|W(XB - CX)\|_F}{\sqrt{n}\|X\|_F} \geq \frac{\text{sep}_F(B,C)}{\sqrt{n}},
\]

where we have used that \( \frac{1}{\sqrt{n}}\|A\|_F \leq \|A\|_{2,\infty} \leq \|A\|_F \) for any \( A \in \mathbb{R}^{n \times m} \).

Once again using that there exists an \( X \) such that \( Z = UX \) we see that

\[
\frac{\|ZB - WCW^T Z\|_{2,\infty}}{\|Z\|_{2,\infty}} = \frac{\|WXB - WCX\|_{2,\infty}}{\|WX\|_{2,\infty}} \geq \frac{\|W(XB - CX)\|_{2,\infty}}{\|W\|_{2,\infty}\|X\|_F} \geq \frac{\|WT\|_{2,\infty}}{\|W\|_{2,\infty}},
\]

where \( T = (XB - CX)/\|X\|_F \). Since \( \|T\|_F \geq \text{sep}_F(B,C) \) we take an infimum of the lower bound over \( X \) to conclude the proof. \( \square \)

Lastly, in Lemma 2.5 we provide a direct bound on \( \text{sep}_{2,\infty} \) in terms of traditional matrix norms. In some situations, this may provide the most direct control over \( \text{sep}_{2,\infty} \) while in others it may be vacuous and one must resort to Lemma 2.4 instead.

**Lemma 2.5.** Consider \( B \in \mathbb{R}^{\ell \times \ell} \) and \( C \in \mathbb{R}^{m \times m} \), and let \( W \in \mathbb{R}^{n \times m} \) with \( n \geq m \) be a matrix with orthonormal columns. Then,

\[
\text{sep}_{(2,\infty),W}(B,WCW^T) \geq \sigma_{\min}(B) - \|WCW^T\|_{\infty}.
\]

\(^6\)Concrete examples being when \( C = 0 \) or \( W = I \) and \( B \) and \( C \) are diagonal (see Lemma 2.1) in which case \( \text{sep}_F = \text{sep}_2 = \text{sep}_{2,\infty} \).
3. Main result. Given the notation and concepts from Section 2 we may now proceed to present our core results bounding \( \| \cdot \|_{2,\infty} \) changes in invariant subspaces of symmetric matrices \( A \) under symmetric perturbation.

**Theorem 3.1.** Let \( \text{gap} = \min \{ \text{sep}_2(A_1, A_2), \text{sep}_{(2,\infty)}(A_1, V_2 A_2 V_2^T) \} \). If \( \| E \|_2 \leq \frac{\text{gap}}{5} \) then

\[
\min \{ \| \hat{V}_1 U - V_1 \|_{2,\infty} : U \in \mathbf{O}^r \} \leq 8 \| V_1 \|_{2,\infty} \left( \frac{\| E \|_2}{\text{sep}_2(A_1, A_2)} \right)^2 \\
+ 2 \| V_2 E_{2,1} \|_{2,\infty} \text{gap} + 4 \| V_2 V_T E \|_{2,\infty} \| E \|_2 \text{gap} \times \text{sep}_2(A_1, A_2),
\]

where \( \hat{V}_1 \) is any matrix with orthonormal columns whose range is the dominant \( r \)-dimensional invariant subspace of \( \hat{A} \).

First, we briefly remark on the conditions and implications of Theorem 3.1. The condition \( \| E \|_2 \leq \frac{\text{gap}}{5} \) is standard in the literature; it ensures the two parts of \( \sigma(A) \) corresponding to the \( r \)-largest eigenvalues and the \( n - r \) smallest eigenvalues of \( A \) are disjoint.\(^7\) The first term on the right hand side is also expected, it looks like a traditional Davis-Kahan bound reduced by factors of the incoherence of \( V_1 \) and \( \| E \|_2 \). The second term captures how (in)coherent \( V_2 V_T^T E V_1 \) is, a term we often expect to be well controlled. Lastly, the third term is controlled by the incoherence of \( E \) itself (relative to its spectral norm).\(^8\)

As we will see later, often it is the case that either the second or third term dominates the upper bound. Though, as we will discuss later, we believe that the third term can be more sharply controlled in many random scenarios. In addition, in Section 3.3 we will argue that the presence of \( \text{sep}_{(2,\infty)}(A_1, V_2 A_2 V_2^T) \) is essential, though Lemma 2.4 provides some control over it via more interpretable quantities.

**Remark 2.** Of particular note, when \( A \) is rank \( r \) and therefore \( \Lambda_2 = 0 \) Theorem 3.1 simplifies significantly since under the additional assumption that \( \lambda_r \geq 0 \) we have that

\[
\text{gap} = \text{sep}_2(A_1, A_2) = \text{sep}_{(2,\infty)}(A_1, V_2 A_2 V_2^T) = \lambda_r.
\]

Prior to embarking on a proof of the main result, we present two natural corollaries of Theorem 3.1. Corollary 3.2 characterizes the error when rather than a minimum over \( U \in \mathbf{O}^r \) we use a fixed \( U \). Corollary 3.3 simplifies our result in the case where the infinity norm of \( E \) is sufficiently bounded relative to the spectral gap and the incoherence of \( V_1 \).

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\(^7\)Technically, the constant in the denominator just needs to be bigger than 4.

\(^8\)Note that if we define \( \mu = \sqrt{7} \| V_1 \|_{2,\infty} \), it is possible to further simplify the bound by observing that \( \| V_2 V_T^T E \|_{2,\infty} \leq \| V_2 V_T^T \|_{\infty} \| E \|_{2,\infty} \leq (1 + \mu^2) \| E \|_{2,\infty} \).
**Corollary 3.2.** Let \( \text{gap} = \min\{\text{sep}_2(A_1, A_2), \text{sep}_{(2, \infty)}(A_1, V_1, V_2^T)\} \). If \( \|E\|_2 \leq \frac{\text{gap}}{5} \) then

\[
\|\hat{V}_1 \hat{U} - V_1\|_{2, \infty} \leq 8 \|V_1\|_{2, \infty} \left( \frac{\|E\|_2}{\text{sep}_2(A_1, A_2)} \right)^2 + 2 \frac{\|V_2E_{2,1}\|_{2, \infty}}{\text{gap}} + 4 \frac{\|V_2V_2^T E\|_{2, \infty}}{\text{gap} \times \text{sep}_2(A_1, A_2)} \|E\|_2,
\]

where \( \hat{V}_1 \) is any matrix with orthonormal columns whose range is the dominant \( r \)-dimensional invariant subspace of \( \hat{A} \) and \( \hat{U} \) solves the orthogonal Procrustes problem

\[
\min\{\|\hat{V}_1 U - V_1\|_{F} : U \in O^r\}.
\]

**Remark 3.** Corollary 3.2 is particularly useful in circumstances where it is possible to estimate \( \hat{U} \) given only \( \hat{V}_1 \) and some structural assumptions about \( V_1 \). Algorithms based around this paradigm have been developed for spectral clustering [8] and localization of basis functions in Kohn-Sham Density Functional Theory [6, 7].

**Corollary 3.3.** Let \( \text{gap} = \min\{\text{sep}_2(A_1, A_2), \text{sep}_{(2, \infty)}(A_1, V_1, V_2^T)\} \) and \( \mu = \sqrt{n}\|V_1\|_{2, \infty} \). If \( \|E\|_2 \leq \frac{\text{gap}}{10} \) and \( \|E\|_{\infty} \leq \text{gap}/4(1 + \mu^2) \) then

\[
\min\{\|\hat{V}_1 U - V_1\|_{2, \infty} : U \in O^r\} \leq 8 \|V_1\|_{2, \infty} \left( \frac{\|E\|_2}{\text{sep}_2(A_1, A_2)} \right)^2 + 4 \frac{\|V_2E_{2,1}\|_{2, \infty}}{\text{gap}} \|E\|_2,
\]

where \( \hat{V}_1 \) is any matrix with orthonormal columns whose range is the dominant \( r \)-dimensional invariant subspace of \( \hat{A} \).

### 3.1. Related work

The most closely related results to our own are the two-to-infinity bounds in [5], though other results exist for single eigenvectors [10] and for similar, though distinct, measures of subspace perturbations [1]. The results in [5] concern orthonormal bases of the singular subspaces of (possibly non-symmetric) matrices. However, when specialized to orthonormal bases of the invariant subspaces of symmetric matrices our results lead to sharper bounds. In [5] the authors establish a general decomposition of \( \hat{V}_1 \hat{U} - V_1 \), where \( \hat{U} \) solves the orthogonal Procrustes problem

\[
\min\{\|\hat{V}_1 U - V_1\|_{F} : U \in O^r\},
\]

and deduce general bounds on \( \|\hat{V}_1 \hat{U} - V_1\|_{2, \infty} \) through repeated use of the triangle inequality.

The first bound in [5] (Theorem 3.7) is most similar to our main result, albeit proved in a significantly different manner. In the case where \( A \) has rank \( r \) their first bound is comparable to our main result. However, if \( A \) is not low-rank our result implies faster convergence rates. In this case, \( \|V_2V_2^T AV_2V_2^T\|_{2, \infty} \) is non-zero, and the right side of their bound is dominated by the term \( \|\sin \Theta(V_1, V_1)\|_2 \). If \( E \) is a matrix of iid \( \mathcal{N}(0, \frac{1}{n^2}) \) random scalars, their bound implies \( \|\hat{V}_1 \hat{U} - V_1\|_{2, \infty} \) vanishes at the rate \( \hat{O}(1/\sqrt{n}) \). On the other hand, our bound shows that \( \|\hat{V}_1 \hat{U} - V_1\|_{2, \infty} \) vanishes at the faster rate \( \hat{O}(1/n) \)—an observation illustrated in Section 4.

In [1] the authors develop similar results to Corollary 3.2 as corollaries to their main results. Specifically, their final expressions in Theorem 2.1 and Corollary 2.1 are row-wise perturbations bounds on orthonormal bases of invariant subspaces. However, this is not the main focus of their work, so these results are generally looser than
our bounds. Furthermore, in contrast to our results their bounds are not purely determin- 
istic; they rely on probabilistic assumptions on the error matrix $E$ and additional assumptions on $A$ itself.

3.2. Proof the main result. At a high level, the proof has two parts. In the first part, we develop a specific characterization of $\hat{V}_1$ parametrized by a matrix $\hat{X}$ that is a root of a quadratic matrix equation. In the second part, we show that $\|\hat{V}_1 - V_1\|_{2,\infty}$ is small under our stated assumptions.

Part 1: Our starting point is the bound

$$\min\{\|\hat{V}_1 U - V_1\|_{2,\infty} : U \in O^r\} \leq \|\hat{V}_1 U - V_1\|_{2,\infty},$$

where $\hat{U}$ is the solution the orthogonal Procrustes problem:

$$\min\{\|\hat{V}_1 U - V_1\|_F : U \in O^r\}.$$  

Notably, the solution to this problem is well known and computable given $\hat{V}_1$ and $V_1$, which will prove useful in our numerical experiments. More pertinent to our needs at the moment, this means that $\hat{V}_1 \hat{U}$ is the closest matrix with orthonormal columns to $V_1$ in Frobenius norm whose range is the dominant $r$-dimensional invariant subspace of $\hat{A}$.

We start by constructing a matrix with orthonormal columns $\hat{V}_1$ whose range is the dominant $r$-dimensional invariant subspace of $\hat{A}$ and a matrix $\hat{V}_2$ characterizing the orthogonal complement of $\hat{V}_1$. Specifically, define

$$(3.3) \quad \hat{V}_1 = (V_1 + V_2 \hat{X}) (I_r + \hat{X}^T \hat{X})^{-\frac{1}{2}},$$

and

$$(3.4) \quad \hat{V}_2 = (V_2 - V_1 \hat{X}^T) (I_r + \hat{X} \hat{X}^T)^{-\frac{1}{2}}$$

for some $\hat{X} \in \mathbb{R}^{(n-r) \times r}$.

Remark 4. A clean derivation of this characterization is to start with the general formula for an arbitrary invariant subspace $\hat{V}_1 = V_1 H + V_2 X$ for some appropriately sized matrices $H$ and $X$. Requiring that $\hat{V}_1^T \hat{V}_1 = I$ ensures that $H$ is non-singular as long as $\|X\|_2 < 1$, which is guaranteed by our assumptions. If we additionally enforce that the solution to $\min\{\|\hat{V}_1 U - V_1\|_F : U \in O^r\}$ is the identity we conclude that $\hat{V}_1^T V_1 = H$ must be positive definite. Letting $\hat{X} = X H^{-1}$ the condition that $\hat{V}_1$ has orthonormal columns shows that

$$H^2 + H \hat{X}^T \hat{X} H = I.$$  

Multiplying on the left and right by $H^{-1}$ we conclude that $H^{-2} = I + \hat{X}^T \hat{X}$ and arrive at (3.3).

It is not hard to check that $\hat{V}_1$ and $\hat{V}_2$ have orthonormal columns and their ranges are complementary subspaces of $\mathbb{R}^n$. Thus $\text{ran}\hat{V}_1$ is an invariant subspace of $\hat{A}$ if and only if

$$(3.5) \quad 0 = \hat{V}_2^T \hat{A} \hat{V}_1 = -\hat{A}_{2,1} + \hat{X} \hat{A}_{1,1} - \hat{A}_{2,2} \hat{X} + \hat{X} \hat{A}_{1,2} \hat{X}.$$  

In other words, $\hat{X}$ is a root of the map $F : \mathbb{R}^{(n-r) \times r} \to \mathbb{R}^{(n-r) \times r}$ defined as

$$F : X \to -\hat{A}_{2,1} + X \hat{A}_{1,1} - \hat{A}_{2,2} X + X \hat{A}_{1,2} X.$$
We find a root of $F$ by appealing to a Newton-type method (for root-finding). Starting at $X_0 = 0$, we construct the sequence

$$X_{t+1} \leftarrow X_t - S^{-1}_A(F(X_t)).$$

To characterize the limit of $(X_t)$ we appeal to the Newton-Kantorovich theorem (Theorem B.1). We remark that this construction is similar but not identical to that in [20, §3].

**Lemma 3.4.** As long as $\|E\|_2 \leq \frac{\text{sep}_2(A_1, A_2)}{4}$, $(X_t)$ converges to $\hat{X}$ such that $\hat{X}$ satisfies (3.5) and $\|\hat{X}\|_2 \leq \frac{4\|E\|_2}{\text{sep}_2(A_1, A_2)}$.

**Proof.** We defer the proof to Appendix B.1. 

Since $\hat{X}$ satisfies (3.5), $\text{ran}\hat{V}_1$ is an invariant subspace of $\hat{A}$. It remains to show that $\text{ran}\hat{V}_1$ is the dominant $r$-dimensional invariant subspace of $\hat{A}$. We block-diagonalize $\hat{A}$ to obtain

$$\begin{bmatrix} \hat{V}_1 & \hat{V}_2 \end{bmatrix}^T \hat{A} \begin{bmatrix} \hat{V}_1 & \hat{V}_2 \end{bmatrix} = \begin{bmatrix} \hat{V}_1^T \hat{A} \hat{V}_1 & \hat{V}_1^T \hat{A} \hat{V}_2 \\ \hat{V}_2^T \hat{A} \hat{V}_1 & \hat{V}_2^T \hat{A} \hat{V}_2 \end{bmatrix}.$$ 

The first diagonal block is

$$\hat{V}_1^T \hat{A} \hat{V}_1 = (I_r + \hat{X}^T \hat{X})^{-\frac{1}{2}}(V_1 + V_2 \hat{X})^T \hat{A}(V_1 + V_2 \hat{X})(I_r + \hat{X}^T \hat{X})^{-\frac{1}{2}}$$

$$= (I_r + \hat{X}^T \hat{X})^{-\frac{1}{2}}(\hat{A}_{1,1} + \hat{A}_{1,2} \hat{X} + \hat{X}^T \hat{A}_{2,1} + \hat{X}^T \hat{A}_{2,2}) (I_r + \hat{X}^T \hat{X})^{-\frac{1}{2}}.$$ 

Recalling $\hat{X}$ satisfies (3.5), we have

$$\hat{X}^T \hat{A}_{2,1} + \hat{X}^T \hat{A}_{2,2} \hat{X} = \hat{X}^T \hat{X} \hat{A}_{1,1} + \hat{X}^T \hat{X} \hat{A}_{1,2} \hat{X}.$$ 

Plugging this expression into the right side of the preceding display, we obtain

$$\hat{V}_1^T \hat{A} \hat{V}_1 = (I + \hat{X}^T \hat{X})^{-\frac{1}{2}}(\hat{A}_{1,1} + \hat{A}_{1,2} \hat{X} + \hat{X}^T \hat{X} \hat{A}_{1,1} + \hat{X}^T \hat{X} \hat{A}_{1,2} \hat{X})(I + \hat{X}^T \hat{X})^{-\frac{1}{2}}$$

$$= (I + \hat{X}^T \hat{X})^{-\frac{1}{2}}(I + \hat{X}^T \hat{X})(I_{1,1} + \hat{A}_{1,2} \hat{X})(I + \hat{X}^T \hat{X})^{-\frac{1}{2}}$$

$$= (I + \hat{X}^T \hat{X})^{\frac{1}{2}}(I_{1,1} + \hat{A}_{1,2} \hat{X})(I + \hat{X}^T \hat{X})^{-\frac{1}{2}}.$$ 

In other words, the first diagonal block is similar to $\hat{A}_{1,1} + \hat{A}_{1,2} \hat{X}$. This implies

$$\sigma(\hat{V}_1^T \hat{A} \hat{V}_1) = \sigma(\hat{A}_{1,1} + \hat{A}_{1,2} \hat{X})$$

$$= \sigma(\hat{A}) + \sigma(\hat{E}_{1,1} + \hat{E}_{1,2} \hat{X})$$

$$\subset \sigma(\hat{A}) + (\|E_{1,1}\|_2 + \|E_{1,2}\|_2 \|\hat{X}\|_2)[-1, 1]$$

(Bauer-Fike theorem)

$$\subset \sigma(\hat{A}) + (\|E\|_2 + \frac{4\|E\|_2}{\text{sep}_2(A_1, A_2)})[-1, 1]$$

$$\subset \sigma(\hat{A}) + 2\|E\|_2[-1, 1]$$

Similarly, it is possible to show that the second diagonal block is similar to $\hat{A}_{2,2} - \hat{X} \hat{A}_{1,2}$ and

$$\sigma(\hat{V}_2^T \hat{A} \hat{V}_2) \subset \sigma(\hat{A}_{2,2}) + 2\|E\|_2[-1, 1].$$

Recalling $\|E\|_2 \leq \frac{\text{sep}_2(A_1, A_2)}{\text{sep}_2(A_1, A_2)}$, we have

$$\min\{\lambda_1 : \lambda_1 \in \sigma(\hat{V}_1^T \hat{A} \hat{V}_1)\} \geq \lambda_r - 2\|E\|_2$$

$$> \lambda_{r+1} + 2\|E\|_2$$

$$> \max\{\lambda_2 : \lambda_2 \in \sigma(\hat{V}_2^T \hat{A} \hat{V}_2)\},$$
Lemma 3.4. we can use the Taylor expansion (valid for |x| < 1) to conclude
\[
\begin{align*}
\hat{V}_1^T V_1 &= (I + \hat{X}^T \hat{X})^{-\frac{1}{2}} (V_1 + V_2 \hat{X})^T V_1 \\
&= (I + \hat{X}^T \hat{X})^{-\frac{1}{2}} (V_1^T V_1 + \hat{X} V_2^T V_1) \\
&= (I + \hat{X}^T \hat{X})^{-\frac{1}{2}},
\end{align*}
\]
that polar factor is the identity and as desired \( \hat{V}_1 \) is the closest matrix to \( V_1 \) in Frobenius distance among all matrices of the form \( \hat{V}_1 U \), where \( U \in O^r \). Note that this is exactly the set of matrices with orthonormal columns whose range is the dominant \( r \)-dimensional invariant subspace of \( \hat{A} \).

Part 2: For the remainder of the proof \( \hat{V}_1 \) is as defined in (3.3) and we proceed to explicitly bound
\[
\|\hat{V}_1 - V_1\|_{2,\infty}.
\]
We start by decomposing the error \( \hat{V}_1 - V_1 \) into its components in \( \text{ran}V_1 \) and \( \text{ran}V_1^\perp \) as
\[
\hat{V}_1 - V_1 = V_1 V_1^T (\hat{V}_1 - V_1) + V_2 V_2^T (\hat{V}_1 - V_1) = V_1 V_1^T \hat{V}_1 - V_1 + V_2 V_2^T \hat{V}_1.
\]
Recalling that \( V_1^T \hat{V}_1 = (I + \hat{X}^T \hat{X})^{-\frac{1}{2}} \) (from part 1) we observe that
\[
V_2^T \hat{V}_1 = V_2^T (V_1 + V_2 \hat{X}) (I + \hat{X}^T \hat{X})^{-\frac{1}{2}} = \hat{X} (I + \hat{X}^T \hat{X})^{-\frac{1}{2}}
\]
and deduce
\[
(3.7) \quad \hat{V}_1 - V_1 = V_1 ((I_r + \hat{X}^T \hat{X})^{-\frac{1}{2}} - I_r) + V_2 \hat{X} (I + \hat{X}^T \hat{X})^{-\frac{1}{2}}.
\]
We will now address each part of this decomposition of the error separately.

The \((2, \infty)\)-norm of the first term on the right side of (3.7) is at most
\[
\|V_1 ((I_r + \hat{X}^T \hat{X})^{-\frac{1}{2}} - I_r)\|_{2,\infty} \leq \|V_1\|_{2,\infty} \|(I_r + \hat{X}^T \hat{X})^{-\frac{1}{2}} - I_r\|_2.
\]
Since
\[
\|(I_r + \hat{X}^T \hat{X})^{-\frac{1}{2}} - I_r\|_2 = \left| 1 - (1 + \|\hat{X}\|_2^2)^{-1/2} \right|,
\]
we can use the Taylor expansion (valid for \(|x| < 1\))
\[
(1 + x)^{-1/2} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{\prod_{\ell=1}^{k} (\ell - 1/2)}{k!} x^k
\]
to conclude\footnote{The terms in the Taylor expansion alternate and decay in magnitude, and \(\|\hat{X}\|_2 \leq 1/2\) by Lemma 3.4.} that
\[
\|(I_r + \hat{X}^T \hat{X})^{-\frac{1}{2}} - I_r\|_2 \leq \frac{1}{2} \|\hat{X}\|_2^2.
\]
By Lemma 3.4, \( \| \hat{X} \|_2 \leq \frac{4\|E\|_2}{\text{sep}_2(A_1, A_2)} \), which implies that
\[
\| V_1 ((I_r + \hat{X}^T \hat{X})^{-\frac{1}{2}} - I_r) \|_{2,\infty} \leq 8 \| V_1 \|_{2,\infty} \left( \frac{\| E \|_2}{\text{sep}_2(A_1, A_2)} \right)^2.
\]

At this point we have control over the first term in (3.7) and since the \( (2, \infty) \)-norm of the second term on the right side of (3.7) is at most \( \| V_2 \hat{X} (I_r + \hat{X}^T \hat{X})^{-\frac{1}{2}} \|_{2,\infty} \leq \| V_2 \hat{X} \|_{2,\infty} \| (I_r + \hat{X}^T \hat{X})^{-\frac{1}{2}} \|_2 \leq \| V_2 \hat{X} \|_{2,\infty} \)
we have that
\[
\| \hat{V}_1 - V_1 \|_{2,\infty} \leq 8 \| V_1 \|_{2,\infty} \left( \frac{\| E \|_2}{\text{sep}_2(A_1, A_2)} \right)^2 + \| V_2 \hat{X} \|_{2,\infty}.
\] (3.8)

For the remainder of the proof we focus on bounding \( \| V_2 \hat{X} \|_{2,\infty} \). At first glance, we are tempted to appeal to the compatibility of \( \| \cdot \|_{2,\infty} \) and \( \| \cdot \|_2 \) to obtain
\[
\| V_2 \hat{X} \|_{2,\infty} \leq \| V_2 \|_{2,\infty} \| \hat{X} \|_2 \leq \| V_2 \|_{2,\infty} \frac{2\| E \|_2}{\text{sep}_2(A_1, A_2)}.
\]
Unfortunately, this bound is generally inadequate because \( \| V_2 \|_{2,\infty} \) may be much larger than \( \| V_1 \|_{2,\infty} \). Instead, we must study \( \| V_2 \hat{X} \|_{2,\infty} \) directly. To start, observe that \( V_2 \hat{X} \) satisfies
\[
0 = -V_2 \hat{A}_{2,1} + V_2 \hat{X} \hat{A}_{1,1} - V_2 \hat{A}_{2,2} V_2^T V_2 \hat{X} + V_2 \hat{X} \hat{A}_{1,2} V_2^T V_2 \hat{X}.
\] (3.9)
In other words, \( V_2 \hat{X} \) is a root of the map \( G: \mathbb{R}^{n \times r} \rightarrow \mathbb{R}^{n \times r} \) defined as
\[
G: Y \rightarrow -V_2 \hat{A}_{2,1} + Y \hat{A}_{1,1} - V_2 \hat{A}_{2,2} V_2^T Y + Y \hat{A}_{1,2} V_2^T Y.
\]
Letting \( \hat{Y} = V_2 \hat{X} \), we rearrange (3.9) to obtain
\[
\hat{Y} \hat{A}_{1,1} - V_2 A_2 V_2^T \hat{Y} = -V_2 \hat{A}_{2,1} + \hat{Y} \hat{A}_{1,2} V_2^T \hat{Y} + V_2 E_{2,2} V_2^T \hat{Y}
= -V_2 E_{2,1} + \hat{Y} E_{1,2} V_2^T \hat{Y} + V_2 E_{2,2} V_2^T \hat{Y}.
\]
Taking norms, we have that
\[
\| \hat{Y} \hat{A}_{1,1} - V_2 A_2 V_2^T \hat{Y} \|_{2,\infty}
\leq \| V_2 E_{2,1} \|_{2,\infty} + \| \hat{Y} E_{1,2} V_2^T \hat{Y} \|_{2,\infty} + \| V_2 E_{2,2} V_2^T \hat{Y} \|_{2,\infty}
\leq \| V_2 E_{2,1} \|_{2,\infty} + \| \hat{Y} \|_{2,\infty} \| \hat{Y} E_{1,2} V_2^T \hat{Y} \|_{2,\infty} + \| V_2 E_{2,2} V_2^T \hat{Y} \|_{2,\infty}
\leq \| V_2 E_{2,1} \|_{2,\infty} + \| \hat{Y} \|_{2,\infty} \| V_2 E_{1,2} V_2^T \hat{Y} \|_{2,\infty} + \| V_2 E_{2,2} V_2^T \hat{Y} \|_{2,\infty},
\]
where we appealed to \( \hat{Y} \in \text{ran} V_2 \). Next, by Lemma 3.4
\[
\| \hat{Y} \hat{A}_{1,1} - V_2 A_2 V_2^T \hat{Y} \|_{2,\infty}
\leq \| V_2 E_{2,1} \|_{2,\infty} + \| \hat{Y} \|_{2,\infty} \frac{2\| E \|_2}{\text{gap}} + \| V_2 E_{2,2} V_2^T \hat{Y} \|_{2,\infty}.
\]
Observe that the left side is at least
\[
\| \tilde{Y} A_{1,1} - V_2 A_2 V_2^T \tilde{Y} \|_{2,\infty} \geq \| \tilde{Y} A_1 - V_2 A_2 V_2^T \tilde{Y} \|_{2,\infty} \\
\geq \operatorname{sep}_{2(\infty),V_3}(A_1, V_2 A_2 V_2^T) \| \tilde{Y} \|_{2,\infty} - \| \tilde{Y} \|_{2,\infty} \| E_{1,1} \|_2 \\
\geq \operatorname{sep}_{2(\infty),V_3}(A_1, V_2 A_2 V_2^T) \| \tilde{Y} \|_{2,\infty} - \| \tilde{Y} \|_{2,\infty} \| E \|_2 \\
\geq \frac{3}{4} \operatorname{gap} \| \tilde{Y} \|_{2,\infty},
\]
and, therefore,
\[
\left( \frac{3}{4} \operatorname{gap} - \frac{2\|E\|^2_2}{\operatorname{gap}} \right) \| \tilde{Y} \|_{2,\infty} \leq \| V_2 E_{2,1} \|_{2,\infty} + \| V_2 V_2^T \tilde{Y} \|_{2,\infty}.
\]
Since \( 2\|E\|_{2}/\operatorname{gap} \leq 1 \) and \( \|E\|_{2} \leq \operatorname{gap}/4 \) by assumption we have that
\[
(3.10) \quad \| \tilde{Y} \|_{2,\infty} \leq \frac{2\|V_2 E_{2,1}\|_{2,\infty}}{\operatorname{gap}} + \frac{2\|V_2 V_2^T \tilde{Y}\|_{2,\infty}}{\operatorname{gap}}.
\]

Prior to concluding the proof, we summarize our results up to this point in Lemma 3.5. We partly pause to highlight a natural launching point for problem specific analysis, particularly in settings where it is possible to control \( \| V_2 V_2^T \tilde{Y} \|_{2,\infty} \) in a tighter manner than suggested by our worst case bounds that follow.

**Lemma 3.5.** Let \( \operatorname{gap} = \min \{ \operatorname{sep}_{2}(A_1, A_2), \operatorname{sep}_{2(\infty),V_3}(A_1, V_2 A_2 V_2^T) \} \). If \( \|E\|_{2} \leq \frac{\operatorname{gap}}{4} \) then
\[
\| \tilde{V}_1 \tilde{U} - V_1 \|_{2,\infty} \leq 8\|V_1\|_{2,\infty} \left( \frac{\|E\|_2}{\operatorname{sep}_{2}(A_1, A_2)} \right)^2 + 2\|V_2 V_2^T \tilde{Y} V_1\|_{2,\infty} + \|V_2 V_2^T \|_{2,\infty} \| \tilde{Y} \|_{2,\infty},
\]
where \( \tilde{V}_1 \) is a matrix with orthonormal columns whose range is the dominant \( r \)-dimensional invariant subspace of \( \tilde{A} \), \( \tilde{U} \) solves the orthogonal Procrustes problem
\[
\min \{ \| \tilde{V}_1 U - V_1 \|_{F} : U \in \mathbb{O}^r \},
\]
and \( \tilde{Y} \in \text{ran} V_2 \) is a root of
\[
G : Y \rightarrow -V_2 \tilde{A}_{2,1} + Y \tilde{A}_{1,1} - V_2 \tilde{A}_{2,2} V_2^T Y + Y \tilde{A}_{1,2} V_2^T Y.
\]

Moving forward, Lemma 3.5 immediately implies that
\[
\| \tilde{Y} \|_{2,\infty} \leq 2\|V_2 E_{2,1}\|_{2,\infty} + 4\|V_2 V_2^T \tilde{Y} V_1\|_{2,\infty} + \|V_2 V_2^T \tilde{Y}\|_{2,\infty},
\]
where we have used the sub-multiplicative relationships
\[
\|V_2 V_2^T \tilde{Y} V_1\|_{2,\infty} \leq \|V_2 V_2^T \tilde{Y}\|_{2,\infty},
\]
in conjunction with Lemma 3.4 to bound \( \| \tilde{Y} \|_{2} \). This concludes the proof of Theorem 3.1 and Corollary 3.2.

We now briefly retrace our steps to prove Corollary 3.3. In particular, returning to Lemma 3.5 we can instead conclude that
\[
\| \tilde{Y} \|_{2,\infty} \left( 1 - \frac{2(1 + \mu^2)}{\operatorname{gap}} \| \tilde{Y} \|_{2,\infty} \right) \leq \frac{2\|V_2 E_{2,1}\|_{2,\infty}}{\operatorname{gap}},
\]
where we have used the observation that
\[
\|V_2 V_T^T\|_{\infty} = \|I_n - V_1 V_T^T\|_{\infty} \\
\leq 1 + \max \{ \sum_{j=1}^{n} |v_i^T v_j| : i \in [n] \} \\
\leq 1 + n\|V_1\|_{2,\infty}^2 \\
\leq 1 + \mu^2.
\]
Now, since \( \|E\hat{Y}\|_{2,\infty}/\|\hat{Y}\|_{2,\infty} \leq \|E\|_{\infty} \) if we further assume that \( \|E\|_{\infty} \leq \text{gap}/4(1 + \mu^2) \) we get that
\[
\left( 1 - \frac{2(1 + \mu^2)}{\text{gap}} \|E\hat{Y}\|_{2,\infty} \right) \geq \frac{1}{2}.
\]
Therefore,
\[
\|\hat{Y}\|_{2,\infty} \leq 4 \frac{\|V_2 E,1\|_{2,\infty}}{\text{gap}},
\]
which concludes the proof of Corollary 3.3.

3.3. Observations and implications. We now discuss several aspects of our bounds in greater detail. In particular, we first construct specific examples that show any of the 3 terms in the bound of Theorem 3.1 may tightly control the error and therefore are all necessary. We then argue why \( \text{sep}_{(2,\infty),V_2}(\Lambda_1, V_2 \Lambda_2 V_T^T) \) should be directly included in our bounds by showing that in the worst case it may be considerably smaller than \( \text{sep}_{F}(\Lambda_1, \Lambda_2) \). Lastly, we discuss the use of our bound in certain probabilistic scenarios and highlight how our bounds can facilitate further analysis of those situations.

3.3.1. When the upper bound is tight. The first term of our upper bound represents the projection of the error onto \( V_1 \) while the latter two terms arise from the projection onto \( V_2 \). Therefore, we focus on the latter piece to understand if both the terms are necessary and examine our potentially loose use of the triangle inequality and sub-multiplicative bounds in the proof. To accomplish this, we construct a specific example and examine the behavior of our bound.

We bounded the projection of the error onto \( \text{ran}V_2 \) as
\[
\|V_2 V_T^T (\hat{V}_1 - V_1)\|_{2,\infty} \leq \frac{2\|V_2 E,1\|_{2,\infty}}{\text{gap}} + \frac{4\|V_2 V_T^T E\|_{2,\infty}\|E\|_{2,\infty}}{\text{gap} \times \text{sep}_{2}(\Lambda_1, \Lambda_2)}.
\]
While the first term on the right hand side of (3.11) is a natural part of our bound given the quadratic form (3.9), the second term arose from the sub-multiplicative bound
\[
\|V_2 V_T^T E\hat{Y}\|_{2,\infty} \leq \|V_2 V_T^T E\|_{2,\infty}\|\hat{Y}\|_2 \leq \frac{2}{\text{sep}_{2}(\Lambda_1, \Lambda_2)}\|V_2 V_T^T E\|_{2,\infty}\|E\|_2.
\]
Nevertheless, both terms are necessary—there are perturbations that saturate each part of the bound.

To show this, we build an example that demonstrates two clear regimes—one where the first term of (3.11) controls the error tightly and one where the second term does. We accomplish this by picking \( E \) such that for the resulting \( \hat{Y} \)
\[
\|E\hat{Y}\|_{2,\infty} \approx \|V_2 V_T^T E\|_{2,\infty} \frac{\|E\|_2}{\text{gap}}.
\]
To keep thing simple, we consider the \( r = 1 \) case with \( \lambda_1 = 1 \) and \( \lambda_2, \ldots, \lambda_n = 0 \), which implies \( \text{gap} = 1 \). We then let \( V_1 = 1/\sqrt{n} \) and observe that if \( E_{1,2} = 0 \) and \( E_{1,1} = 0 \) then \( \hat{Y} \) satisfies
\[
(I - V_2 E_{2,2} V_2^T) \hat{Y} = V_2 E_{2,1}.
\]
The core insight in our construction is that we can now \( V_2 E_{2,2} V_2^T \) and \( V_2 E_{2,1} \) carefully to accomplish our goal. This is because we can essentially determine \( \hat{Y} \) (in fact, to first order it looks like \( V_2 E_{2,1} \) if the norm of \( E \) is sufficiently small).

Now, let
\[
V_2 E_{2,2} V_2^T = V_2 V_2^T (e_1 1^T \pm \sqrt{n} + 1 \pm e_i^T) V_2 V_2^T
\]
In this case there exists a \( y \) with \( y_1 = 1 \) and \( y_2, \ldots, y_n = \mathcal{O}(1/\sqrt{n}) \) such that \( (I - V_2 E_{2,2} V_2^T) y = \mathcal{O}(1/\sqrt{n}) \) entry-wise. Setting \( V_2 E_{2,1} \) to be proportional to \( V_2 V_2^T (I - V_2 E_{2,2} V_2^T) y \) lets us deterministically construct an \( E \) where \( \hat{Y} \) essentially saturates the sub-multiplicative bound.\(^{10}\) The preceding construction yields a purely deterministic counter-example illustrating in Figure 1b that either part of (3.11) can be dominant. Similarly, Figure 1a shows, as expected, that our bound on the projection of the error onto \( V_1 \) tightly captures the asymptotic behavior.

3.3.2. Inclusion of norm specific separation. In the proof of Theorem 3.1 \( \text{sep}_{\nu_2}(\lambda_1, V_2 V_2^T) \) arises somewhat naturally. Nevertheless, ideally one would be able to generically relate it tightly to traditional notions of an eigengap. Unfortunately, the lower bound provided in Lemma 2.4 is essentially tight. To show this we explicitly construct an example that achieves (to within a small constant) the lower bound \( \text{sep}_{\nu_2}(\lambda_1, V_2 V_2^T) \geq \text{sep}_{\nu_2}(\lambda_1, \lambda_2)/\sqrt{n} \).

Assume \( n \) is even and let \( 1 \) be the vector of all ones and \( (1 \pm)_i = -1 \) if \( i > n/2 \) and 1 otherwise. Now, define \( v_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T / \sqrt{n} \) and \( v_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T / \sqrt{n} \) and consider the \( n+1 \times n+1 \) matrix
\[
A = 2 * v_1 v_1^T + [e_1 \quad v_2] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} [e_1 \quad v_2]^T
\]
In our framework this corresponds to setting \( \Lambda_1 = 2 \), \( \Lambda_2 = \text{diag}(1, -1, 0, \ldots, 0) \in \mathbb{R}^{n \times n} \), and letting \( V_2 \) to be any matrix with orthonormal columns spanning the orthogonal complement of \( v_1 \) such that
\[
[e_1 \quad v_2] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} [e_1 \quad v_2]^T = V_2 \Lambda_2 V_2^T.
\]
In this case, by picking the vector \( q = e_1 + 2 v_2 \) (which is in the range of \( V_2 \) and satisfies \( \|q\|_{2,\infty} = 1 \) so long as \( n \geq 4 \)) we see that
\[
\|q \Lambda_1 - V_2 \Lambda_2 V_2^T q\|_{2,\infty} = \|2q - 2e_1 - v_2\|_{2,\infty}
\]
\[
= \|3v_2\|_{2,\infty}
\]
\[
= \frac{3}{\sqrt{n}}.
\]
\(^{10}\)Practically one can make \( E \) symmetric by setting \( E_{1,2} \) appropriately without destroying the counterexample and scale \( E \) by \( n^{-1/3} \) so that we expect convergence in \( n \). Details are available in the online materials referred to in the numerical experiments section. Choices of scaling constants in individual parts of \( E \) control where the crossover point occurs between the two bounds.
Fig. 1: Asymptotic behavior of $\min_{\tilde{U} = \pm 1} \| \tilde{V}_1 - V_1 \tilde{U} \|_{2,\infty}$ split into the component of the error in $\text{ran} V_1$ and $\text{ran} V_2$. This example shows that either part of the upper bound in Theorem 3.1 associated with the error projected onto $\text{ran} V_2$ can tightly control the rate of decay. Similarly, our control over the projection of the error onto $\text{ran} V_1$ matches the observed rate for this example. Note that, as indicated by the legend, we have not included constants (they would appear to be slightly loose in this case) and therefore the dotted lines technically represent decay rates rather than upper bounds.

3.3.3. Probabilistic settings. While the two preceding sections illuminate why various terms in our constructed bounds are necessary, one may expect that in random settings these terms can be controlled more effectively and the expected behavior may be far from the worst case. In particular, if we consider $E = \sigma Z$ where $Z$ is symmetric and $Z_{i,j}$ are iid $N(0,1)$ random variables we have by standard properties of Gaussian random matrices (see, e.g., [22])

$$\mathbb{P}(\|E\|_2 > 3\sigma \sqrt{n}) \lesssim e^{-\frac{n}{2}}.$$ 

Furthermore, using the fact that $V_1$ incoherent and $E$ is independent from $V_1$ allows us to assert (again via standard properties of Gaussian random matrices and a union bound [22]) that

$$\|E_{2,1}\|_{2,\infty} \leq (1 + \mu^2)\|EV_1\|_{2,\infty} \lesssim \sigma \sqrt{\log n}$$

with high probability. Ideally, these results would directly imply that in such a setting

$$\min\{\|\tilde{V}_1 U - V_1\|_{2,\infty} : U \in O^*\} \lesssim \sigma \sqrt{\log n}$$
with high probability. Unfortunately, this does not follow directly from Theorem 3.1 as for certain values of $\sigma$ the error bound is dominated by the term $\|E\|_{2,\infty} \|E\|_2 \lesssim \sigma^2 n$.

Figure 1 shows that this is not an artifact of our analysis; it is possible to construct examples that saturate the error bound. However, these examples are adversarial. In particular, the independence among the entries of $Z$ permits more direct control of $\|E\tilde{Y}\|_{2,\infty}$ and [1] appeal to a leave-one-out technique to achieve such direct control. Based on our analysis, we conjecture that their column-wise independence condition may be relaxed to independence among the entries of $Z$, which permits more direct control of $\|E\tilde{Y}\|_{2,\infty}$.

4. Numerical simulations. We now provide numerical simulations to illustrate the effectiveness of our bounds and elaborate on a key difference between them and prior work. We consider two settings, one where $A$ is low-rank and one where $A$ is not low-rank and $\|V_2A_2V_2^T\|_{2,\infty}$ is constant with respect to $n$. In all these experiments, and as before, we let $1$ be the vector of all ones and $(1_\pm)_i = -1$ be $1$ in the first half of the entries and $-1$ in the second half. We also let $E = \sigma Z$ where $Z$ is a symmetric matrix whose entries are iid $N(0,1)$. Code to generate these plots (and Figure 1) is available at https://github.com/asdamle/rowwise-perturbation.

4.1. $A$ is low-rank. Assume $n$ is even, let

\[
V_1 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1_\pm \end{bmatrix},
\]

and consider $A = V_1V_1^T$. In this setting, gap $= 1$ and if $\sigma = 1/n$ Corollary 3.2 shows that $\|\tilde{V}_1\tilde{U} - V_1\|_{2,\infty} \lesssim_P \frac{\sqrt{\log n}}{n}$, where the term controlling the rate with respect to $n$ is $\|V_2V_2^TEV_1\|_{2,\infty}$. Increasing $n$ we compute $\tilde{V}_1$ and the solution to the orthogonal Procrustes problem so we can measure $\|\tilde{V}_1\tilde{U} - V_1\|_{2,\infty}$. Figure 2a clearly shows the expected behavior for both the traditional subspace distance and our bound on $\|\cdot\|_{2,\infty}$.

Perhaps more interestingly, we also consider the case where $E = (1/n^{3/4})Z$. In this case, our deterministic upper bounds predict $\|\tilde{V}_1\tilde{U} - V_1\|_{2,\infty} \lesssim_P \mathcal{O}\left(\frac{1}{n}\right)$. However, as expected in this setting, the bound used to control $\|E\tilde{Y}\|_{2,\infty}$ is loose and Figure 2b shows that the error acts as if $E$ and $\tilde{Y}$ were independent (though they are decidedly not) yielding an observed convergence rate of $\sqrt{\frac{\log n}{n^{1/4}}}$.

4.2. $A$ is not low-rank. Now, we consider the case where $A$ itself is no longer low-rank. Now let

\[
V_1 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1_\pm \end{bmatrix},
\]

$v_2 = e_1 - e_2 \in \mathbb{R}^{n \times 1}$, and

\[
A = 4V_1V_1^T + v_2v_2^T.
\]

Notably, $A$ is no longer low-rank and the component of $A$ orthogonal to $V_1$ is coherent, of significant relative magnitude, and does not decay with $n$. Nevertheless, our results immediately imply that the asymptotic behavior of $\|\tilde{V}_1\tilde{U} - V_1\|_{2,\infty}$ should match that of the low-rank case.\(^{11}\) In contrast, this behavior is not accurately predicted by the upper bounds given in [5]. Using the same experimental set up as before, Figures 3a and 3b clearly illustrate the asymptotic behavior we expect—mirroring that of Figures 2a and 2b respectively.

\(^{11}\)Here $\text{sep}_{2,\infty}(A_1, V_2A_2V_2^T) \geq 2$ as a consequence of Lemma 2.5
5. Extensions of our bounds for non-normal matrices. While we have constructed our bounds for symmetric matrices $A$ subject to arbitrary additive perturbations, they directly hold for normal matrices $A$. Furthermore, they may extended in several directions and we briefly articulate how such extensions are readily obtained following the same proof strategy used for Theorem 3.1.

5.1. Schur form subspaces. Our results can be directly extended to Schur form for non-normal matrices. We now let

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} T_{1,1} & T_{1,2} \\ 0 & T_{2,2} \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix}^*$$

where $U_1 \in \mathbb{C}^{n \times r}$ and $U_2 \in \mathbb{C}^{n-r \times r}$ have orthonormal columns, and $T_{1,1}$ and $T_{2,2}$ are upper triangular.

Adding one additional assumption about the norm of $T_{1,2}$ our results extend via Theorem 5.1 to Schur form. For simplicity we have been loose with our assumptions on $\|E\|_2$ and $\|T_{1,2}\|_2$ and small improvements to the necessary constants are possible. Importantly, rather then $T_{1,2}$ showing up in the upper bounds it shows up in the assumptions—thereby controlling the matrices for which this result is valid.

**Theorem 5.1.** Let $\text{gap} = \min\{\text{sep}_2(T_{1,1}, T_{2,2}), \text{sep}_{(2,\infty)}(T_{1,1}, U_2 T_{2,2} U_2^*)\}$. If $\|E\|_2 \leq \frac{\text{gap}}{10}$ and $\|T_{1,2}\|_2 \leq \frac{\text{gap}}{10}$ then there exists a matrix $\hat{Y} \in \mathbb{C}^{n-r \times r}$ such that

$$\hat{U}_1 = (U_1 + U_2 \hat{Y})(I + \hat{Y}^* \hat{Y})^{-1/2}$$
forms an invariant subspace for $\hat{A}$ satisfying

$$\min\{\|\hat{U}Q - U_1\|_{2,\infty} : Q \in O^r\} \leq 8\|U_1\|_{2,\infty} \left(\frac{\|E\|_2}{\text{sep}_2(T_{1,1}, T_{2,2})}\right)^2$$

$$+ 2 \frac{\|U_2^* E U_1\|_{2,\infty}}{\text{gap}} + 4 \frac{\|U_2^* E\|_{2,\infty} \|E\|_2}{\text{gap} \times \text{sep}_2(T_{1,1}, T_{2,2})}.$$  

Proof. Note that our additional assumptions ensure that $\|\hat{A}_{1,2}\|_2 \leq \text{gap}/5$. Therefore, the result follows from the same argument as Lemma 3.4 and Theorem 3.1 where we simply use $\hat{A}_{1,2}$ in place of $E_{1,2}$ and $T_{1,1}$ and $T_{2,2}$ in lieu of $\Lambda_1$ and $\Lambda_2$. 

5.2. Singular vectors and subspaces. More generally, and perhaps of more interest for non-normal matrices, analogous questions about subspace perturbations can be posed for singular subspaces. While omitted here, we believe the proof strategy employed in this work can be extended to develop similar bounds for pairs of singular subspaces. This assertion is based on the quadratic forms given in [20] for singular subspaces of $A + E$, though we leave such developments for future work.

6. Conclusions. Throughout this manuscript we have developed bounds on $\|\hat{V}_1 - V_1 \hat{U}\|_{2,\infty}$ that are characterized by easily interpretable quantities (such as $\|V_1\|_{2,\infty}$) and rely on minimal assumptions. By additionally demonstrating that various aspects of our bounds are “essential” when allowing for arbitrary symmetric $A$ and $E$ we clearly show where the limits are for this problem absent additional assumptions. Nevertheless, this effort also provides a natural launching point for further
analysis, as it points to the key assumptions that have to (or may) be made to further understand the behavior of $\|\hat{V}_1 - V_1 \hat{U}\|_{2,\infty}$ in specific settings. One concrete example of this is the random setting explored in Section 4, where more refined control of $\|EY\|_{2,\infty}$ is possible. Lastly, there are several ways in which our bounds show commonly made assumptions in prior work (such as incoherence of $A$ or certain assumptions on $V_2 A_2 V_2^T$) are unnecessary. The consequence of this is that our bounds are sharper in certain situations. Collectively, we believe that these qualities make our bounds useful and interpretable across a broad range of applications.

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Appendix A. Proofs on properties of separation.

A.1. Proof of Lemma 2.1. First, we observe that because $\text{sep}$ is shift invariant it suffices to prove the result for non-negative $D_1$ and $D_2$. Therefore, we assume $D_1$ and $D_2$ have non-negative entries for the remainder of this proof. We now prove lower bounds for all three variants of $\text{sep}$.

2-norm. For any $Z$ we let $U_Z \Sigma_Z V_T^Z$ denote its reduced SVD and note that since $\|Z\|_2 = 1$ we have that $\sigma_1 = 1$. Now we observe that

\[
\|ZD_1 - D_2 Z\|_2 \geq \|ZD_1\|_2 - \|D_2 Z\|_2 \\
\geq \lambda_{\text{min}}(D_1) - \|D_2\|_2 \\
\geq \lambda_{\text{min}}(D_1) - \lambda_{\text{max}}(D_2)
\]

where we have used that $\|XD_1\|_2 = \|\Sigma_Z V_T^Z D_1\|_2$.

Frobenius norm. For any $Z$ with unit Frobenius norm observe that

\[
\|ZD_1 - D_2 Z\|_F \geq \|ZD_1\|_F - \|D_2 Z\|_F \\
\geq \lambda_{\text{min}}(D_1) - \|D_2\|_2 \\
\geq \lambda_{\text{min}}(D_1) - \lambda_{\text{max}}(D_2).
\]

2,\infty-norm. For any $Z$ with $\|Z\|_{2,\infty} = 1$ there exists an index $k$ such that $\|e_k^T Z\|_2 = 1$ where $e_k \in \mathbb{R}^m$ is a canonical basis vector. Now, observe that

\[
\|ZD_1 - D_2 Z\|_{2,\infty} \geq \|ZD_1\|_{2,\infty} - \|D_2 Z\|_{2,\infty} \\
\geq \|e_k^T ZD_1\|_2 - \|D_2 Z\|_{2,\infty} \\
\geq \lambda_{\text{min}}(D_1) - \lambda_{\text{max}}(D_2),
\]

where the last inequality follows because $D_2$ represents a row scaling of $Z$.

Finally, let $j$ denote the column in which $\lambda_{\text{min}}(D_1)$ arises and let $i$ denote the column in which $\lambda_{\text{max}}(D_2)$ arises. Now, observe that

\[
e_i e_j^T D_1 - D_2 e_i e_j^T = (\lambda_{\text{min}}(D_1) - \lambda_{\text{max}}(D_2)) e_i e_j^T,
\]
where \( e_i \in \mathbb{R}^m \) and \( e_j \in \mathbb{R}^\ell \) are canonical basis vectors. Since
\[
\|e_i e_j^T\|_2 = \|e_i e_j^T\|_F = \|e_i e_j^T\|_{2,\infty} = 1
\]
we achieve the aforementioned lower bounds in all cases and thereby conclude the proof.

Appendix B. The Newton-Kantorovich theorem. This is the version of the Newton-Kantorovich theorem we appeal to. It appears in [14, pp 536].

**Theorem B.1.** Let \( X, Y \) be Banach spaces and \( F : X \rightarrow Y \) be twice-continuously (Fréchet) differentiable in a neighborhood of \( U \) of \( x \in X \). Assume there is a linear map \( J : X \rightarrow Y \) such that \( S^{-1}_A \) is bounded and satisfies
1. \( \|J^{-1}(F(x))\| \leq \eta \),
2. \( \|J^{-1} \circ \partial F(x) - I\| \leq \delta \),
3. \( \|J^{-1} \circ \partial^2 F(y)\| \leq K \) for all \( y \in U \).

If \( \delta < 1 \) and \( h \in \left( \frac{\eta}{1 - \delta} \right)^2 < \frac{1}{2} \), then the sequence \((x_t)\) defined recursively as
\[
\begin{align*}
x_0 & = x \\
x_{t+1} & = x_t - J^{-1}(F(x_t))
\end{align*}
\]
converges to \( \bar{x} \in X \) such that \( F(\bar{x}) = 0 \) and
\[
\|\bar{x} - x\| \leq \frac{2\eta}{(1 - \delta)(1 + \sqrt{1 - 2h})}.
\]

**B.1. Proof of Lemma 3.4.** We start by evaluating the derivatives of \( F \):
\[
\begin{align*}
\partial F(0) : X & \rightarrow X \hat{A}_{1,1} - \hat{A}_{2,2}X, \\
\partial^2 F(X) : X_1, X_2 & \rightarrow X_1 \hat{A}_{1,2}X_2.
\end{align*}
\]
Recognizing \( \|S^{-1}_A\|_2 = \frac{1}{\text{sep}_2(\hat{A}_{1,1}, \hat{A}_{2,2})} \) and \( F(0) = E_{2,1} \), we have
\[
\|S^{-1}_A(F(0))\|_2 \leq \frac{\|E_{2,1}\|_2}{\text{sep}_2(\hat{A}_{1,1}, \hat{A}_{2,2})},
\]
so the first condition of Theorem B.1 is satisfied by \( \eta = \frac{\|E_{2,1}\|_2}{\text{sep}_2(\hat{A}_{1,1}, \hat{A}_{2,2})} \). We recognize \( S_A = \partial F(0) \), so the second condition of Theorem B.1 is satisfied by \( \delta = 0 \). Finally, we have
\[
\|S^{-1}_A(\partial^2 F(0)(X_1, X_2))\|_2 \leq \frac{\|X_1 \hat{A}_{1,2}X_2\|_2}{\text{sep}_2(\hat{A}_{1,1}, \hat{A}_{2,2})} \leq \frac{\|X_1\|_2 \|E_{1,2}\|_2 \|X_2\|_2}{\text{sep}_2(\hat{A}_{1,1}, \hat{A}_{2,2})},
\]
so the third condition of Theorem B.1 is satisfied by \( K = \frac{\|E_{1,2}\|_2}{\text{sep}_2(\hat{A}_{1,1}, \hat{A}_{2,2})} \). From Proposition 2.1 of [15] we then have that
\[
\text{sep}_2(\hat{A}_{1,1}, \hat{A}_{2,2}) \geq \text{sep}_2(\Lambda_1 + E_{1,1}, \Lambda_2 + E_{2,2})
\]
\[
= \text{sep}_2(\Lambda_1, \Lambda_2) + \|E_{1,1}\|_2 - \|E_{2,2}\|_2
\]
\[
\geq \text{sep}_2(\Lambda_1, \Lambda_2) + 2\|E\|_2.
\]
We combine this bound on \( \text{sep}_2(\tilde{A}_{11}, \tilde{A}_{22}) \) with the condition \( \|E\|_2 \leq \frac{\text{sep}_2(\Lambda_1, \Lambda_2)}{4} \) to obtain

\[
\eta = \frac{\|E_{2,1}\|_2}{\text{sep}_2(\tilde{A}_{11}, \tilde{A}_{22})} \leq \frac{\|E_{2,1}\|_2}{\text{sep}_2(\Lambda_1, \Lambda_2) - 2\|E\|_2} \leq \frac{2\|E_{2,1}\|_2}{\text{sep}_2(\Lambda_1, \Lambda_2)},
\]

\[
h = \frac{\eta K}{(1 - \delta)^2} = \frac{\|E_{2,1}\|_2\|E_{1,2}\|_2}{\text{sep}_2(\tilde{A}_{11}, \tilde{A}_{22})^2} \leq \frac{\|E\|_2}{(\text{sep}_2(\Lambda_1, \Lambda_2) - 2\|E\|_2)^2} \leq \frac{1}{4} < \frac{1}{2},
\]

so the NK theorem implies \( F \) has a root \( \tilde{X} \) such that

\[
\|\tilde{X}\|_2 \leq \frac{2\eta}{(1 - \delta)(1 + \sqrt{1 - 2h})} < 2\eta \leq \frac{4\|E_{2,1}\|_2}{\text{sep}_2(\Lambda_1, \Lambda_2)}
\]

as claimed.

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