1. Introduction

A $V_{12}$ Fano threefold is a smooth Fano threefold $X$ of index 1 with $\text{Pic } X = \mathbb{Z}$ and $(-K_X)^3 = 12$, see [Is, IP]. Let $X$ be a $V_{12}$ threefold. It was shown by Mukai [Mu] that $X$ admits an embedding into a connected component $\mathbf{LGr}_+(V)$ of the Lagrangian Grassmannian $\mathbf{LGr}(V)$ of Lagrangian (5-dimensional) subspaces in a vector space $V = \mathbb{C}^{10}$ with respect to a nondegenerate quadratic form $Q$, and moreover, $X = \mathbf{LGr}_+(V) \cap \mathbb{P}^8$. Let $\mathcal{O}_X$ be the structure sheaf and let $\mathcal{U}_+$ denote the restriction to $X$ of the tautological (5-dimensional) subbundle from $\mathbf{LGr}_+(V) \subset \mathbf{Gr}(5, V)$. Then it is easy to show that $(\mathcal{U}_+, \mathcal{O}_X)$ is an exceptional pair in the bounded derived category of coherent sheaves on $X$, $\mathcal{D}^b(X)$. Therefore, triangulated subcategory $(\mathcal{U}_+, \mathcal{O}_X)$ generated by the pair is admissible and there exists a semiorthogonal decomposition $\mathcal{D}^b(X) = (\mathcal{U}_+, \mathcal{O}_X, \mathcal{A}_X)$, where $\mathcal{A}_X = \perp (\mathcal{U}_+, \mathcal{O}_X) \subset \mathcal{D}^b(X)$ is the orthogonal subcategory. The main result of this note is an equivalence $\mathcal{A}_X \cong \mathbf{D}^b(C^\vee)$, where $C^\vee$ is a curve of genus 7.

The curve $C^\vee$ arising in this way in fact is nothing but the orthogonal section of the Lagrangian Grassmannian considered by Iliev and Markushevich [IM1]. Recall that the components $\mathbf{LGr}_+(V)$ and $\mathbf{LGr}_-(V)$ of the Lagrangian Grassmannian $\mathbf{LGr}(V)$ lie in the dual projective spaces $\mathbb{P}(S^+V)$ and $\mathbb{P}(S^-V)$ respectively, where $S^\pm V$ are the spinor (16-dimensional) representations of the corresponding spinor group $\mathbf{Spin}(V)$. So, with any linear subspace $\mathbb{P}^8 \subset \mathbb{P}(S^+V)$ one can associate its orthogonal $(\mathbb{P}^8)^\perp = \mathbb{P}^6 \subset \mathbb{P}(S^-V)$ and consider following [IM1] the orthogonal section $C^\vee := \mathbf{LGr}_-(V) \cap \mathbb{P}^6$, which can be shown to be a smooth genus 7 curve, whenever $X$ is smooth.

Further, Iliev and Markushevich explained in [IM1] the intrinsic meaning of the curve $C^\vee$ associated to the threefold $X$. They have shown that it is isomorphic to the moduli space of stable rank 2 vector bundles on $S$ with $c_1 = 1$, $c_2 = 5$. Considering a universal bundle $\mathcal{E}_1$ on $X \times C^\vee$ we obtain the corresponding kernel functor $\Phi_{\mathcal{E}_1} : \mathbf{D}^b(C^\vee) \rightarrow \mathcal{D}^b(X)$. It follows from [BO] and [IM1] that $\Phi_{\mathcal{E}_1}$ is fully faithful. Moreover, it can be shown that its image is contained in the orthogonal subcategory $\mathcal{A}_X = \perp (\mathcal{U}_+, \mathcal{O}_X) \subset \mathcal{D}^b(X)$. Thus, it remains to check that $\Phi_{\mathcal{E}_1} : \mathbf{D}^b(C^\vee) \rightarrow \mathcal{A}_X$ is essentially surjective.

To prove the surjectivity of the functor $\Phi_{\mathcal{E}_1}$ we use the following approach. Take arbitrary smooth hyperplane section $X \supset S := X \cap \mathbb{P}^7 = \mathbf{LGr}_+(V) \cap \mathbb{P}^7$ and consider the orthogonal section $S^\vee = \mathbf{LGr}_-(V) \cap (\mathbb{P}^7)^\perp$. Then both $S$ and $S^\vee$ are $K3$ surfaces, moreover $S^\vee$ is smooth and $C^\vee$ is a hyperplane section of $S^\vee$. Iliev and Markushevich have shown in [IM1] that the moduli space of stable rank 2 vector bundles on $S$ with $c_1 = 1$, $c_2 = 5$ is isomorphic to $S^\vee$, so we can again consider a universal bundle $\mathcal{E}_2$ on $S \times S^\vee$ and the corresponding kernel functor $\Phi_{\mathcal{E}_2} : \mathbf{D}^b(S^\vee) \rightarrow \mathbf{D}^b(S)$. Again it follows from [BO] and [IM1] that $\Phi_{\mathcal{E}_2}$ is fully faithful, hence an equivalence by [Br]. Further, it is clear that we have an isomorphism $\mathcal{E}_{1\mid S \times C^\vee} \cong \mathcal{E}_{2\mid S \times C^\vee}$, hence the composition of $\Phi_{\mathcal{E}_1}$ with pushforward from $C^\vee$ to $S^\vee$ coincides with the composition of $\Phi_{\mathcal{E}_2}$ with restriction from $X$ to $S$: $\alpha^\ast \circ \Phi_{\mathcal{E}_1} \cong \Phi_{\mathcal{E}_2} \circ \beta_\ast$, where $\alpha : S \rightarrow X$ and $\beta : C^\vee \rightarrow S^\vee$ are the embeddings. The crucial observation however is that the bundles $\mathcal{E}_1$ on $X \times C^\vee$ and $\mathcal{E}_2$ on $S \times S^\vee$ can be glued on $X \times S^\vee$ so that the kernel functor $\mathbf{D}^b(X) \rightarrow \mathbf{D}^b(S^\vee)$ corresponding to the glueing vanishes on the subcategory $\mathcal{A}_X$. In other words, we have $(\beta_\ast \circ \Phi_{\mathcal{E}_1})_{\mid \mathcal{A}_X} \cong (\Phi_{\mathcal{E}_2} \circ \alpha^\ast)_{\mid \mathcal{A}_X}$, where $\Phi_{\mathcal{E}_1}$ and $\Phi_{\mathcal{E}_2}$.
are the left adjoint functors. Now the proof goes as follows. Take an object \( F \in \mathcal{A}_X \), orthogonal to the image of \( \Phi_{\xi_1} \). Then \( \Phi_{\xi_1}^*(F) = 0 \). Hence \( (\Phi_{\xi_2}^* \circ \alpha^*)(F) = (\beta_* \circ \Phi_{\xi_1}^*)(F) = 0 \). But \( \Phi_{\xi_2} \) is an equivalence, hence \( \Phi_{\xi_2}^* \) is an equivalence, hence \( \alpha^*(F) = 0 \). Since these arguments apply to any smooth hyperplane section \( S \subset X \), it follows that the restriction of such \( F \) to any smooth hyperplane section is zero, but this immediately implies that \( F = 0 \).

Having in mind the semiorthogonal decomposition \( \mathcal{D}^b(X) = \langle \mathcal{U}_+, O_X, \mathcal{D}^b(C^\vee) \rangle \) one can informally say that the nontrivial part of the derived category \( \mathcal{D}^b(X) \) is described by the curve \( C^\vee \). Therefore, the curve \( C^\vee \) should appear in all geometrical questions related to \( X \). As a demonstration of this phenomenon we show that the Fano surface of conics on \( X \) is isomorphic to the symmetric square of \( C^\vee \). This fact was known to Iliev and Markushevich, see [IM2], however we decided to include our proof into the paper for two reasons: it demonstrates very well how the above semiorthogonal decomposition can be used, and, moreover, the same approach allows to investigate any other moduli space on \( X \).

The paper is organised as follows. In section 2 we recall briefly results of [IM1]. In section 3 we give an explicit description of universal bundles on \( X \times C^\vee \) and \( S \times S^\vee \) and of their glueing on \( X \times S^\vee \). In section 4 we give necessary cohomological computations. In section 5 we consider the derived categories and prove the equivalence \( \mathcal{A}_X \cong \mathcal{D}^b(C^\vee) \). Finally, in section 6 we investigate conics on \( X \) and prove that the Fano surface \( F_X \) is isomorphic to \( S^2C^\vee \).

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2. Preliminaries

Fix a vector space \( V = \mathbb{C}^{10} \) and a quadratic nondegenerate form \( Q \) on \( V \). Let \( S^+V, S^-V \) denote the spinor (16-dimensional) representations of the spinor group \( \text{Spin}(Q) \). Recall that the spaces \( S^\pm V \) coincide with the (duals of the) spaces of global sections of the ample generators of the Picard group of connected components \( L\text{Gr}_\pm(V) \) of the Lagrangian Grassmanian of \( V \) with respect to \( Q \). In particular, we have canonical embeddings \( L\text{Gr}_\pm(V) \to \mathbb{P}(S^\pm V) \).

Choose a pair of subspaces \( A_8 \subset A_9 \subset S^+V \), \( \dim A_i = i \), and consider the intersections

\[
S = L\text{Gr}_+(V) \cap \mathbb{P}(A_8) \subset \mathbb{P}(S^+V),
X = L\text{Gr}_+(V) \cap \mathbb{P}(A_9) \subset \mathbb{P}(S^+V). \tag{1}
\]

It is easy to see that if \( X \) is smooth then \( X \) is a \( V_{12} \) threefold, and if \( S \) is smooth then \( S \) is a polarized \( K3 \) surface of degree 12.

**Theorem 2.1** ([Mu]). If \( X \) is a \( V_{12} \) Fano threefold, and \( S \subset X \) is its smooth \( K3 \) surface section, then there exists a pair of subspaces \( A_8 \subset A_9 \subset S^+V \), such that \( S \) and \( X \) are obtained by (1).

Recall that the spinor representations \( S^-V \) and \( S^+V \) are canonically dual to each other, and denote by \( B_7 \subset B_8 \subset S^-V \) the orthogonal subspaces,

\[
B_i = A_{16-i}^\perp \subset S^+V^* \cong S^-V,
\]

and consider the dual pair

\[
C^\vee = L\text{Gr}_-(V) \cap \mathbb{P}(B_7) \subset \mathbb{P}(S^-V),
S^\vee = L\text{Gr}_-(V) \cap \mathbb{P}(B_8) \subset \mathbb{P}(S^-V). \tag{2}
\]
Again, it is easy to see that if $S'$ is smooth then $S'$ is a polarized $K3$ surface of degree 12, and if $C'$ is smooth then $C'$ is a canonically embedded curve of genus 7.

We denote by $H_X$, $L_X$, and $P_X$ the classes of a hyperplane section, of a line, and of a point in $H^*(X, \mathbb{Z})$. The same notation is used for varieties $S$, $S'$ and $C'$. For example, $P_{S'} \in H^1(S', \mathbb{Z})$ stands for the class of a point on $S'$.

Let $\mathcal{U}_+, \mathcal{U}_-$ denote the tautological subbundles on $L\text{Gr}_+(V) \subset \text{Gr}(5, V)$, $L\text{Gr}_-(V) \subset \text{Gr}(5, V)$ respectively, and by $\mathcal{U}_{+x}, \mathcal{U}_{-y}$ their fibers at points $x \in L\text{Gr}_+(V)$, $y \in L\text{Gr}_-(V)$ respectively.

Recall the relation between the canonical duality of $L\text{Gr}_+(V)$ and the intersection of subspaces.

**Lemma 2.2 ([IM1]).** Let $x \in L\text{Gr}_+(V) \subset \mathbb{P}(S^+V)$, $y \in L\text{Gr}_-(V) \subset \mathbb{P}(S^+V)$ and denote by $\langle -, - \rangle$ the duality pairing on $S^+V \times S^+V$. Then

\[
\begin{align*}
\langle x, y \rangle & \neq 0 \iff \mathcal{U}_{+x} \cap \mathcal{U}_{-y} = 0, \\
\langle x, y \rangle & = 0 \implies \dim (\mathcal{U}_{+x} \cap \mathcal{U}_{-y}) \geq 2.
\end{align*}
\]

It follows that for any $(x, y) \in X \times C'$ or $(x, y) \in S \times S'$ we have $\dim (\mathcal{U}_{+x} \cap \mathcal{U}_{-y}) \geq 2$.

**Lemma 2.3 ([IM1]).** We have the following equivalences:

(i) $C'$ is smooth $\iff X$ is smooth $\iff$ for all $x \in X$, $y \in C'$ we have $\dim (\mathcal{U}_{+x} \cap \mathcal{U}_{-y}) = 2$;

(ii) $S'$ is smooth $\iff S$ is smooth $\iff$ for all $x \in S$, $y \in S'$ we have $\dim (\mathcal{U}_{+x} \cap \mathcal{U}_{-y}) = 2$.

The following theorem reveals the intrinsic meaning of the curve $C'$ and of the surface $S'$ in terms of $X$ and $S$ respectively.

**Theorem 2.4 ([IM1]).** (i) The curve $C'$ is the fine moduli space of stable rank 2 vector bundles $E$ on $X$ with $c_1(E) = H_X$, $c_2(E) = 5L_X$. If $E_y$, $E_y'$ are the bundles on $X$ corresponding to points $y, y' \in C'$, then

\[
\text{Ext}^p(E_y, E_y') = \begin{cases} 
\mathbb{C}, & \text{for } p = 0, 1 \text{ and } y = y' \\
0, & \text{otherwise}
\end{cases}
\]

(ii) The surface $S'$ is the fine moduli space of stable rank 2 vector bundles $E$ on $S$ with $c_1(E) = H_S$, $c_2(E) = 5P_S$. If $E_y$, $E_y'$ are the bundles on $S$ corresponding to points $y, y' \in S'$, then

\[
\text{Ext}^p(E_y, E_y') = \begin{cases} 
\mathbb{C}, & \text{for } p = 0, 2 \text{ and } y = y' \\
\mathbb{C}^2, & \text{for } p = 1 \text{ and } y = y' \\
0, & \text{otherwise}
\end{cases}
\]

### 3. The Universal Bundles

Consider one of the following two products

\[
either W_1 = X \times C', \\
or W_2 = S \times S'.
\]

Denote by $\mathcal{U}_+$ and $\mathcal{U}_-$ the pullbacks of the tautological subbundles on $L\text{Gr}_+(V)$ and $L\text{Gr}_-(V)$ to $W_i \subset L\text{Gr}_+(V) \times L\text{Gr}_-(V)$, and consider the following natural composition of morphisms of vector bundles on $W_i$

\[
\xi_i : \mathcal{U}_- \to V \otimes \mathcal{O}_{W_i} \cong V^* \otimes \mathcal{O}_{W_i} \to \mathcal{U}_+^*.
\]

**Lemma 3.1.** If $X$ (resp. $S$) is smooth then the rank of $\xi_1$ (resp. $\xi_2$) equals 3 at every point of $W_1$ (resp. $W_2$).

**Proof:** Since the kernel of the natural projection $V^* \otimes \mathcal{O}_{L\text{Gr}_+(V)} \to \mathcal{U}_+^*$ equals $\mathcal{U}_+$, it suffices to show that for all points $(x, y) \in W_i \subset L\text{Gr}_+(V) \times L\text{Gr}_-(V)$ we have $\dim (\mathcal{U}_{+x} \cap \mathcal{U}_{-y}) = 2$ which follows from lemma 2.3. \(\square\)
Lemma 3.2. We have \( \text{Ker} \xi_i \cong (\text{Coker} \xi_i)^* \).

Proof: We have the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & U_+ & \longrightarrow & V \otimes \mathcal{O}_{W_i} & \longrightarrow & U_+^* & \longrightarrow & 0 \\
& & \downarrow \xi_i & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_+^* & \longrightarrow & U_+^* & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\]

Note that the middle vertical arrow is surjective and its kernel is \( U_+ \). Hence, the long exact sequence of kernels and cokernels gives \( 0 \to \text{Ker} \xi_i \to U_+ \to U_+^* \to \text{Coker} \xi_i \to 0 \). Moreover, it is clear that the map \( U_+ \to U_+^* \) in this sequence coincides with the dual map \( \xi_i^* \). It follows immediately that \( \text{Ker} \xi_i \cong \text{Ker} \xi_i^* \). On the other hand, it is clear that \( \text{Ker} \xi_i^* \cong (\text{Coker} \xi_i)^* \). \( \square \)

Let \( E_i \) denote the cokernel of \( \xi_i \) on \( W_i \). It follows that \( E_i \) is a rank 2 vector bundle on \( W_i \) and we have an exact sequence

\[
0 \to E_i^* \to U_+ \xrightarrow{\xi_i} U_+^* \to E_i \to 0.
\] (3)

Dualizing, we obtain another sequence

\[
0 \to E_i^* \to U_+^* \xrightarrow{\xi_i^*} U_+ \to E_i \to 0.
\] (4)

Lemma 3.3. The Chern classes of bundles \( E_i \) are given by the following formulas

\[
\begin{align*}
\text{c}_1(E_1) &= H_X + H_{C^\vee}, & \text{c}_2(E_1) &= \frac{L}{12} H_X H_{C^\vee} + 5L_X + \eta, \\
\text{c}_1(E_2) &= H_S + H_{S^\vee}, & \text{c}_2(E_2) &= \frac{L}{12} H_S H_{S^\vee} + 5P_S + 5P_{S^\vee},
\end{align*}
\]

with \( \eta \in \left( H^3(X, \mathbb{C}) \otimes H^1(C^\vee, \mathbb{C}) \right) \cap H^4(X \times C^\vee, \mathbb{Z}) \).

Proof: It follows from (3) that

\[
\text{ch}(U_+^*) - \text{ch}(U_+) = \text{ch}(E_i) - \text{ch}(E_i^*) = 2\text{ch}_1(E_i) + 2\text{ch}_3(E_i).
\]

This allows to compute

\[
\begin{align*}
\text{ch}_1(E_1) &= H_X + H_{C^\vee}, & \text{ch}_2(E_1) &= -\frac{1}{2} P_X, \\
\text{ch}_1(E_2) &= H_S + H_{S^\vee}, & \text{ch}_2(E_2) &= -\frac{1}{2} P_S - \frac{1}{2} P_{S^\vee}.
\end{align*}
\]

Further, it is clear that \( \text{c}_1(E_i) = \text{ch}_1(E_i) \), and by Künneth formula we have

\[
\begin{align*}
\text{c}_2(E_i) &= a_1 H_X H_{C^\vee} + b_1 L_X + \eta, & \text{c}_2(E_2) &= a_2 H_S H_{S^\vee} + b_2 P_S + c_2 P_{S^\vee},
\end{align*}
\]

for some \( a_1, a_2, b_1, b_2, c_2 \in \mathbb{Q} \), \( \eta \in \left( H^3(X, \mathbb{C}) \otimes H^1(C^\vee, \mathbb{C}) \right) \cap H^4(X \times C^\vee, \mathbb{Z}) \). Further, since the correspondence \( S \leftrightarrow S^\vee \) is symmetric, it is clear that \( c_2 = b_2 \). Finally, \( a_1 \) and \( b_1 \) can be found from the equality \( 3c_1(E_i)c_2(E_i) = \text{ch}_1(E_i)^3 - 6\text{ch}_3(E_i) \). \( \square \)

Remark 3.4. Using the Riemann–Roch formula on \( X \times C^\vee \) one can compute \( \eta^2 = 14 \).

Corollary 3.5. The bundle \( E_1 \) (resp. \( E_2 \)) is a universal family of rank 2 vector bundles with \( c_1 = H_X, c_2 = 5L_X \) on \( X \) (resp. with \( c_1 = H_S, c_2 = 5P_S \) on \( S \)).

Proof: For every \( y \in C^\vee \) (resp. \( y \in S^\vee \)) we denote by \( E_{1y} \) the fiber of \( E_1 \) over \( X \times y \) and by \( E_{2y} \) the fiber of \( E_2 \) over \( S \times y \). It will be shown in lemmas 4.3 and 4.5 below that all bundles \( E_{1y} \) on \( X \) for \( y \in C^\vee \) and all bundles \( E_{2y} \) on \( S \) for \( y \in S^\vee \) are stable, hence there exist morphisms

\[
f_1 : C^\vee \to \mathcal{M}_X(2, H_X, 5L_X), \\
f_2 : S^\vee \to \mathcal{M}_S(2, H_S, 5P_S),
\]
Hence they are isomorphisms, and the bundles $E_{\mu}$ bundles on $C$ respectively,

The first sequence is exact by lemma 3.7 and definition of $\tilde{\mu}$. We have exact sequence on $C^\vee$ and $S^\vee$ are the projections, and $L_1$ and $L_2$ are line bundles on $C^\vee$ and $S^\vee$ respectively.

It is easy to see that the maps $f_1$ and $f_2$ coincide with the maps $\rho$ constructed in [IM1], section 4. Hence they are isomorphisms, and the bundles $E_1$ and $E_2$ are universal.

Let $\alpha : S \to X$ and $\beta : C^\vee \to S^\vee$ denote the embeddings and put $\lambda_1 = \alpha \times \text{id}_{C^\vee}$, $\lambda_2 = \text{id}_S \times \beta$, $\mu_1 = \text{id}_X \times \beta$, $\mu_2 = \alpha \times \text{id}_{S^\vee}$, $\nu = \alpha \times \beta$. Then we have a commutative diagram

$$
\begin{array}{ccc}
S \times C^\vee & \overset{\lambda_1}{\longrightarrow} & X \times C^\vee \\
\uparrow & & \uparrow \\
S \times S^\vee & \overset{\lambda_2}{\longrightarrow} & X \times S^\vee \\
\downarrow & & \downarrow \\
X \times S^\vee & \overset{\nu}{\longrightarrow} & X \times S^\vee
\end{array}
$$

Lemma 3.6. We have canonical isomorphism $\lambda_1^*E_1 = \lambda_2^*E_2$.

Proof: The claim is clear since $\lambda_1^*E_1$ is the cokernel of $\lambda_1^*\xi_i$, and $\lambda_1^*\xi_1 = \lambda_2^*\xi_2$ by definition of $\xi_i$. □

We denote the bundle $\lambda_1^*E_1 = \lambda_2^*E_2$ on $S \times C^\vee$ by $E$.

Consider the product $\tilde{W} = X \times S^\vee$ and the composition

$$
\xi : U_i \to V \otimes O_{\tilde{W}} \cong V^* \otimes O_{\tilde{W}} \to U_i^*.
$$

It is clear that

$$
\mu_i^*\xi = \xi_i. \quad (6)
$$

Lemma 3.7. The rank of $\tilde{\xi}$ equals 5 at $X \times S^\vee \setminus (\mu_1(X \times C^\vee) \cup \mu_2(S \times S^\vee))$.

Proof: Follows from lemma 2.2. □

Let $\tilde{E}$ denote the cokernel of $\tilde{\xi}$.

Lemma 3.8. We have exact sequences on $X \times S^\vee$

$$
0 \to U^- \overset{\xi}{\longrightarrow} U_i^* \to \tilde{E} \to 0,
$$

$$
0 \to \tilde{E} \to \mu_1^*E_1 \oplus \mu_2^*E_2 \to \nu_*E \to 0.
$$

Proof: The first sequence is exact by lemma 3.7 and definition of $\tilde{E}$. To verify exactness of the second sequence we note that $\mu_i^*E_i = E_i$ by (6) and (3), and the canonical surjective maps $\tilde{E} \to \mu_i^*\tilde{E} = \mu_i^*E_i$ glue to a surjective map $\tilde{E} \to \text{Ker}(\mu_1^*E_1 \oplus \mu_2^*E_2 \to \nu_*E)$. On the other hand, it is easy to check that the Chern characters of $\tilde{E}$ and $\text{Ker}(\mu_1^*E_1 \oplus \mu_2^*E_2 \to \nu_*E)$ coincide, hence $\tilde{E} \cong \text{Ker}(\mu_1^*E_1 \oplus \mu_2^*E_2 \to \nu_*E)$ and we are done. □

Corollary 3.9. We have exact sequence on $X \times S^\vee$

$$
0 \to \mu_1^*E_1 \otimes O(-H_X) \to \tilde{E} \to \mu_2^*E_2 \to 0.
$$
4. Cohomological computations

Lemma 4.1. The pair $(U_+, O_X)$ in $D^b(X)$ is exceptional. In other words,

\[ \text{Ext}^k(U_+, U_+) = H^k(X, U_+^* \otimes U_+) = \text{Ext}^k(O_X, O_X) = H^k(X, O_X) = \begin{cases} \mathbb{C}, & \text{for } k = 0 \\ 0, & \text{for } k \neq 0 \end{cases} \]

\[ H^*(X, U_+) = 0. \]

Proof: Recall that $X$ is a complete intersection $X = \mathbb{P}(A_9) \cap \text{LGr}_+(V) \subset \mathbb{P}(S^+ V)$ and $S^+ V/A_9 = B_7^\ast$. Hence $X \subset \text{LGr}_+(V)$ is the zero locus of a section of the vector bundle $B_7^\ast \otimes \mathcal{O}_{\text{LGr}_+(V)}(H_{\text{LGr}_+(V)})$. Therefore, the Koszul complex $\Lambda^*(B_7^\ast \otimes \mathcal{O}_{\text{LGr}_+(V)}(H_{\text{LGr}_+(V)}))$ is a resolution of the structure sheaf $O_X$ on $\text{LGr}_+(V)$. In other words, we have an exact sequence

\[ 0 \to \Lambda^7(B_7^\ast \otimes \mathcal{O}_{\text{LGr}_+(V)}(H_{\text{LGr}_+(V)})) \to \ldots \to \Lambda^1(B_7^\ast \otimes \mathcal{O}_{\text{LGr}_+(V)}(H_{\text{LGr}_+(V)})) \to \mathcal{O}_{\text{LGr}_+(V)} \to O_X \to 0. \]

Tensoring it by $U_+$ and $U_+^* \otimes U_+$ we see that it suffices to compute $H^*(\text{LGr}_+(V), F(-kH_{\text{LGr}_+(V)}))$ for $F = \mathcal{O}_{\text{LGr}_+(V)}$, $F = U_+$ and $F = U_+^* \otimes U_+$ and $0 \leq k \leq 7$. These cohomologies are computed by Borel–Bott–Weil Theorem [D], since all the bundles under the question are the pushforwards of equivariant line bundles on the flag variety of the spinor group $\text{Spin}(V)$. \hfill \Box

Since the canonical class of $X$ equals $-H_X$, the Serre duality on $X$ gives

Corollary 4.2. We have

\[ H^*(X, U_+^*(-H_X)) = 0, \quad H^k(X, U_+ \otimes U_+^*(-H_X)) = \begin{cases} \mathbb{C}, & \text{for } k = 3 \\ 0, & \text{for } k \neq 3 \end{cases}. \]

Lemma 4.3. For any $y \in C^\vee$ we have $H^p(X, \mathcal{E}_{1y}(-H_X)) = H^p(X, \mathcal{E}_{1y} \otimes U_+^*(-H_X)) = 0$. In particular, $\mathcal{E}_{1y}$ is stable.

Proof: Recall that by definition $C^\vee = \text{LGr}_-(V) \cap \mathbb{P}(B_7)$, and $X = \text{LGr}_+(V) \cap \mathbb{P}(A_9)$ with $A_9 = B_7^\perp$. Choose a hyperplane $\mathbb{P}(B_9) \subset \mathbb{P}(B_7)$ such that $\mathbb{P}(B_9)$ intersects $C^\vee$ transversally and doesn’t contain $y$. Take $A_{10} = B_6^\perp$ and consider $\hat{X} = \text{LGr}_+(V) \cap \mathbb{P}(A_{10})$. Then the arguments of lemma 2.3 show that $\hat{X}$ is a smooth Fano fourfold of index 2 containing $X$ as a hyperplane section. Moreover, the arguments similar to that of lemma 3.8 show that the composition of morphisms on $\hat{X}$

\[ \hat{\xi} : U_{-y} \otimes O_{\hat{X}} \to V \otimes O_{\hat{X}} \to U_+^* \]

is injective and its cokernel is isomorphic to the pushforward of $\mathcal{E}_{1y}$ via the embedding $i : X \to \hat{X}$. In other words, we have the following exact sequence on $\hat{X}$:

\[ 0 \to U_{-y} \otimes O_{\hat{X}} \to U_+^* \to i_* \mathcal{E}_{1y} \to 0, \quad (7) \]

On the other hand, using Borel–Bott–Weil Theorem and the Koszul resolution of $\hat{X} \subset \text{LGr}_+(V)$ along the lines of lemma 4.1 one can compute

\[ H^*(\hat{X}, U_+^* \otimes U_+^*(-H_{\hat{X}})) = H^*(\hat{X}, U_+^*(-H_{\hat{X}})) = H^*(\hat{X}, O_{\hat{X}}(-H_{\hat{X}})) = 0 \]

and the claim follows from the cohomology sequences of (7) twisted by $O_{\hat{X}}(-H_{\hat{X}})$ and $U_+^*(-H_{\hat{X}})$ respectively, since

\[ H^*(X, \mathcal{E}_{1y}(-H_X)) = H^*(\hat{X}, i_* \mathcal{E}_{1y}(-H_{\hat{X}})), \quad H^*(X, \mathcal{E}_{1y} \otimes U_+^*(-H_X)) = H^*(\hat{X}, i_* \mathcal{E}_{1y} \otimes U_+^*(-H_{\hat{X}})). \]

\hfill \Box

Lemma 4.4. For any $y \in C^\vee$ we have $H^1(X, \mathcal{E}_{1y}(-2H_X)) = 0$. 

Proof: Restricting exact sequence (3) to \( X = X \times \{ y \} \subset X \times C^\vee \), twisting it by \( \mathcal{O}_X(-H_X) \) and taking into account lemma 3.3 we obtain exact sequence
\[
0 \to \mathcal{E}_{1y}(-2H_X) \to U_{-y} \otimes \mathcal{O}_X(-H_X) \to U'_y(-H_X) \to \mathcal{E}_{1y}(-H_X) \to 0.
\]
It follows from corollary 4.2 and lemma 4.3 that \( H^1(X, \mathcal{E}_{1y}(-2H_X)) = U_{-y} \otimes H^1(X, \mathcal{O}_X(-H_X)) \), but using Serre duality we have \( H^1(X, \mathcal{O}_X(-H_X)) = H^2(X, \mathcal{O}_X)^* = 0 \) by lemma 4.1.

Lemma 4.5. For any \( y \in S^\vee \) we have \( H^0(S, \mathcal{E}_{2y}(-H_S)) = 0 \). In particular, \( \mathcal{E}_{2y} \) is stable.

Proof: For \( y \in C^\vee \) we have \( \mathcal{E}_{2y} = \mathcal{E}_{1y}|_S \), hence the claim follows from exact sequence
\[
H^0(X, \mathcal{E}_{1y}(-H_X)) \to H^0(S, \mathcal{E}_{2y}(-H_S)) \to H^1(X, \mathcal{E}_{1y}(-2H_X)),
\]
since the first term vanishes by lemma 4.3, and the third term vanishes by lemma 4.4.

Now we note that while \( S \) (and hence \( S^\vee \)) is fixed we can take for \( C^\vee \) any smooth hyperplane section of \( S^\vee \), consider the corresponding smooth \( X \supset S \), and repeat the above arguments in this situation. Since any point \( y \in S^\vee \) lies on a smooth hyperplane section, these arguments prove the claim for all \( y \in S^\vee \).

Corollary 4.6. For any \( y \in C^\vee \) we have \( H^*(X, \mathcal{E}_{1y} \otimes U_+(-H_X)) = 0 \).

Proof: Tensor exact sequence \( 0 \to U_+ \to V \otimes \mathcal{O}_X \to U'_+ \to 0 \) with \( \mathcal{E}_{1y}(-H_X) \) and consider the cohomology sequence.

5. Derived categories

Consider the kernel functors taking \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) for kernels:
\[
\Phi_1 : \mathcal{D}^b(C^\vee) \to \mathcal{D}^b(X), \quad \Phi_2 : \mathcal{D}^b(S^\vee) \to \mathcal{D}^b(S), \quad \Phi_i(-) = Rp_{i*}(Lq_i^*(-) \otimes \mathcal{E}_i),
\]
where \( p_i \) and \( q_i \) are the projections onto the first and the second factors:

\[
\begin{array}{ccc}
X \times C^\vee & \xrightarrow{\alpha} & C^\vee \\
\downarrow p_1 & \quad & \downarrow q_1 \\
S \times S^\vee & \xrightarrow{\beta} & S^\vee
\end{array}
\]

Theorem 5.1. The functor \( \Phi_i \) is fully faithful.

Proof: According to the result of Bondal and Orlov [BO] it suffices to check that for the structure sheaves of any two points \( y_1, y_2 \in C^\vee \) (resp. \( y_1, y_2 \in S^\vee \)) and all \( p \in \mathbb{Z} \) we have
\[
\text{Ext}^p(\Phi_i(\mathcal{O}_{y_1}), \Phi_i(\mathcal{O}_{y_2})) = \text{Ext}^p(\mathcal{O}_{y_1}, \mathcal{O}_{y_2}).
\]
But clearly \( \Phi_i(\mathcal{O}_{y_k}) = \mathcal{E}_{iy_k} \) and it remains to apply corollary 3.5 and theorem 2.4.

Corollary 5.2. The functor \( \Phi_2 : \mathcal{D}^b(S^\vee) \to \mathcal{D}^b(S) \) is an equivalence.

Proof: Any fully faithful functor between the derived categories of K3 surfaces is an equivalence, see [Br].

Consider the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{E}_1} & C^\vee \\
S & \xrightarrow{\mathcal{E}_2} & S^\vee
\end{array}
\]
where the dotted line connecting two varieties means that we consider the corresponding kernel on their product. This diagram induces a diagram of functors

\[
\begin{array}{ccc}
\mathcal{D}^b(X) & \xrightarrow{\Phi_1} & \mathcal{D}^b(C^\vee) \\
\alpha^* & & \beta_* \\
\mathcal{D}^b(S) & \xrightarrow{\Phi_2} & \mathcal{D}^b(S^\vee)
\end{array}
\]

which is commutative by lemma 3.6, since the functor \(\alpha^* \circ \Phi_1\) is given by the kernel \(\lambda^* \mathcal{E}_1\), and the functor \(\Phi_2 \circ \beta_*\) is given by the kernel \(\lambda^*_2 \mathcal{E}_2\).

Let \(\Phi_1^* : \mathcal{D}^b(X) \to \mathcal{D}^b(C^\vee)\) and \(\Phi_2^* : \mathcal{D}^b(S) \to \mathcal{D}^b(S^\vee)\) denote the left adjoint functors. The standard computation shows that these functors are given by the kernels

\[
\mathcal{E}_1^*(-H_X)[3] = \mathcal{E}_1(-2H_X - H_{C^\vee})[3] \quad \text{on } X \times C^\vee, \quad \text{and} \quad \mathcal{E}_2^*[2] = \mathcal{E}_2(-H_S - H_{S^\vee})[2] \quad \text{on } S \times S^\vee
\]

respectively. Consider the following diagram

\[
\begin{array}{ccc}
\mathcal{D}^b(X) & \xrightarrow{\Phi_1^*} & \mathcal{D}^b(C^\vee) \\
\alpha^* & & \beta_* \\
\mathcal{D}^b(S) & \xrightarrow{\Phi_2^*} & \mathcal{D}^b(S^\vee)
\end{array}
\]

This diagram is no longer commutative, however, the following proposition shows that it becomes commutative if one replaces \(\mathcal{D}^b(X)\) by its subcategory \(\perp \langle \mathcal{U}_+, \mathcal{O}_X \rangle \subset \mathcal{D}^b(X)\).

**Proposition 5.3.** The functors \(\beta_* \circ \Phi_1^*\) and \(\Phi_2^* \circ \alpha^* : \mathcal{D}^b(X) \to \mathcal{D}^b(S^\vee)\) are isomorphic on the subcategory \(\perp \langle \mathcal{U}_+, \mathcal{O}_X \rangle \subset \mathcal{D}^b(X)\).

**Proof:** It is clear that the functors \(\beta_* \circ \Phi_1^*\) and \(\Phi_2^* \circ \alpha^*\) are given by the kernels \(\mu_1 \mathcal{E}_1(-2H_X - H_{S^\vee})[3]\) and \(\mu_2 \mathcal{E}_2(-H_X - H_{S^\vee})[2]\) on \(X \times S^\vee\) respectively. Considering the helix of the exact sequence of corollary 3.9 twisted by \(\mathcal{O}(-H_X - H_{S^\vee})\) we see that there exists a distinguished triangle

\[
\mu_2 \mathcal{E}_2(-H_X - H_{S^\vee})[2] \to \mu_1 \mathcal{E}_1(-2H_X - H_{S^\vee})[3] \to \tilde{\mathcal{E}}(-H_X - H_{S^\vee})[3].
\]

It remains to show that a kernel functor \(\mathcal{D}^b(X) \to \mathcal{D}^b(S^\vee)\) given by the kernel \(\tilde{\mathcal{E}}(-H_X - H_{S^\vee})\) vanishes on the triangulated subcategory \(\perp \langle \mathcal{U}_+, \mathcal{O}_X \rangle \subset \mathcal{D}^b(X)\).

Note that lemma 3.8 implies that \(\tilde{\mathcal{E}}(-H_X - H_{S^\vee})\) is isomorphic to a cone of the morphism \(\tilde{\xi} : \mathcal{U}_-(-H_X - H_{S^\vee}) \to \mathcal{U}_+(-H_X - H_{S^\vee})\) on \(X \times S^\vee\), so it suffices to check that the kernel functors \(\mathcal{D}^b(X) \to \mathcal{D}^b(S^\vee)\) given by the kernels \(\mathcal{U}_-(-H_X - H_{S^\vee})\) and \(\mathcal{U}_+(-H_X - H_{S^\vee})\) on \(X \times S^\vee\) vanish on the triangulated subcategory \(\perp \langle \mathcal{U}_+, \mathcal{O}_X \rangle \subset \mathcal{D}^b(X)\). Let \(\tilde{p} : X \times S^\vee \to X\) and \(\tilde{q} : X \times S^\vee \to S^\vee\) denote the projections. The straightforward computation using the projection formula and the Serre duality on \(X\) shows that for any object \(F \in \mathcal{D}^b(X)\) we have

\[
\Phi_{\mathcal{U}_-(-H_X - H_{S^\vee})}(F) = R\tilde{q}_*(L\tilde{p}^*(F) \otimes \mathcal{U}_-(-H_X - H_{S^\vee})) = R\Gamma(X, F(-H_X)) \otimes \mathcal{U}_-(-H_{S^\vee}) = R\text{Hom}(F, \mathcal{O}_X)^* \otimes \mathcal{U}_-(-H_{S^\vee}),
\]

\[
\Phi_{\mathcal{U}_+(-H_X - H_{S^\vee})}(F) = R\tilde{q}_*(L\tilde{p}^*(F) \otimes \mathcal{U}_+(-H_X - H_{S^\vee})) = R\Gamma(X, F \otimes \mathcal{U}_+(-H_X)) \otimes \mathcal{O}_{S^\vee}(-H_{S^\vee}) = R\text{Hom}(F, \mathcal{U}_+)^* \otimes \mathcal{O}_{S^\vee}(-H_{S^\vee}).
\]

In particular, the above kernel functors vanish for all objects \(F \in \perp \langle \mathcal{U}_+, \mathcal{O}_X \rangle \subset \mathcal{D}^b(X)\) and we are done. \(\square\)
Theorem 5.4. We have a semiorthogonal decomposition
\[ \mathcal{D}(X) = \langle \mathcal{U}_+, \mathcal{O}_X, \Phi_1(\mathcal{D}(C_\vee)) \rangle. \] (8)

Proof: It is clear that \( \mathcal{O}_X \) is an exceptional bundle, and \( \mathcal{U}_+ \) is an exceptional bundle by lemma 4.1. Now, let us verify the semiorthogonality. Indeed,
\[ \text{Ext}^\bullet(\mathcal{O}_X, \mathcal{U}_+) = H^\bullet(X, \mathcal{U}_+). \]
by lemma 4.1. Moreover, denoting by \( \Phi \) by lemma 4.3 and corollary 4.6. Hence \( \Phi \) by lemma 4.3 and corollary 4.6. Hence \( \Phi(0) = 0 \), since \( \{\mathcal{O}_y\}_{y \in C_\vee} \) is a spanning class (see [Br]) in \( \mathcal{D}(C_\vee) \), hence
\[ \text{Ext}^\bullet(\Phi^1(0), F) = \text{Ext}^\bullet(\Phi^1(\mathcal{O}_y), F) = \text{Ext}^\bullet(\mathcal{E}_y, F) = H^p(X, \mathcal{E}_y \otimes F) = H^p(X, \mathcal{E}_y \otimes F(\mathcal{H}_X)) = 0 \]
by lemma 4.3 and corollary 4.6. Hence \( \Phi(F) = 0 \), since \( \{\mathcal{O}_y\}_{y \in C_\vee} \) is a spanning class (see [Br]) in \( \mathcal{D}(C_\vee) \), hence
\[ \text{Ext}^\bullet(\Phi^1(\mathcal{G}), F) = \text{Ext}^\bullet(\mathcal{G}, \Phi^1(\mathcal{F})) = \text{Ext}^\bullet(\mathcal{G}, 0) = 0 \]
for all \( \mathcal{G} \in \mathcal{D}(C_\vee) \).

It remains to check that \( \mathcal{D}(X) \) is generated by \( \mathcal{U}_+, \mathcal{O}_X \), and \( \Phi_1(\mathcal{D}(C_\vee)) \) as a triangulated category. Indeed, assume that \( F \in \mathcal{D}(X) \), \( \mathcal{O}_X \), and \( \Phi_1(\mathcal{D}(C_\vee)) \) we have \( \Phi(F) = 0 \). On the other hand, since \( F \in \mathcal{D}(X) \), \( \mathcal{O}_X \) we have by proposition 5.3
\[ \Phi^1 \circ \alpha(F) = \beta \circ \Phi^1(F) = 0. \]
But \( \Phi^2 \) is an equivalence by corollary 5.2, hence \( \Phi^2 \) is an equivalence, hence \( \alpha^*(F) = 0 \).

Now we note, that while \( X \) (and hence \( C_\vee \) is fixed, we can take for \( S \) any smooth hyperplane section of \( X \). Then the above arguments imply that for any \( F \in \mathcal{D}(X) \), \( \mathcal{O}_X \), \( \Phi_1(\mathcal{D}(C_\vee)) \) and \( \mathcal{D}(X) \) its restriction to any smooth hyperplane section is isomorphic to zero. Thus the proof is finished by the following lemma.

Lemma 5.5. If \( X \) is a smooth algebraic variety and \( F \) is a complex of coherent sheaves on \( X \) which restriction to every smooth hyperplane section of \( X \) is acyclic, then \( F \) is acyclic.

Proof: Assume that \( F \) is not acyclic and let \( k \) be the maximal integer such that \( H^k(F) \neq 0 \). Let \( x \in X \) be a point in the support of the sheaf \( H^k(F) \). Choose a smooth hyperplane section \( j : S \times X \) passing through \( x \). Since the restriction functor \( j^* \) is right-exact it is clear that \( H^k(Lj^*F) \neq 0 \), a contradiction.

6. Application: the Fano surface of conics

Let \( F_X \) denote the Fano surface of conics (rational curves of degree 2) on \( X \).

Lemma 6.1. If \( R \subset X \) is a conic then \( \mathcal{U}_+[R] \cong \mathcal{O}_R \oplus \mathcal{O}_R(-1) \oplus 4 \).

Proof: Since \( \mathcal{U}_+ \) is a subbundle of the trivial vector bundle \( V \otimes \mathcal{O}_X \), and since we have \( r(\mathcal{U}_+[R]) = 5 \), \( \text{deg}(r(\mathcal{U}_+[R])) = -4 \) we have \( \mathcal{U}_+[R] \cong \bigoplus_{j=0}^5 \mathcal{O}_R(-u_j) \), where \( u_j \geq 0 \) and \( \sum u_j = 4 \). Thus it suffices to check that \( \dim H^0(R, \mathcal{U}_+[R]) = 1 \). Actually, \( \dim H^0(R, \mathcal{U}_+[R]) \geq 1 \) follows from above, so it remains to show that \( \dim H^0(R, \mathcal{U}_+[R]) \geq 2 \) is impossible.

Indeed, assume \( \dim H^0(R, \mathcal{U}_+[R]) \geq 2 \). Choose a 2-dimensional subspace \( U \subset H^0(R, \mathcal{U}_+[R]) \subset H^0(R, V \otimes \mathcal{O}_R) = V \) and consider \( V' = U^\perp/U \). Then \( \text{LGr}_+(V') \subset \text{LGr}_+(V) \), and it is clear that \( R \subset X' := \text{LGr}_+(V') \cap X \). Since \( X \) is a plane section of \( \text{LGr}_+(V) \), therefore \( X' \) is a plane section of \( \text{LGr}_+(V') \). But \( V' = \mathbb{C}^6 \), hence \( \text{LGr}_+(V') \cong \mathbb{P}^3 \). But a plane section of \( \mathbb{P}^3 \) containing a conic contains a plane \( \mathbb{P}^2 \), hence \( X' \) contains \( \mathbb{P}^2 \), hence \( X \) contains \( \mathbb{P}^2 \) which contradicts Lefschetz theorem for \( X \).}

\[ \square \]
Lemma 6.2. We have $\bigcap_{y \in C'} U_{-y} = 0$.

Proof: Assume that $0 \neq v \subset \bigcap_{y \in C'} U_{-y}$ and consider $V'' = v/\mathbb{C}v$. Then $C' \subset \text{LGr}_-(V'') \subset \text{LGr}_-(V)$. Moreover, since $C'$ is a plane section of $\text{LGr}_-(V)$, hence $C'$ is a plane section of $\text{LGr}_-(V'')$. Further, $V'' = \mathbb{C}^8$, hence $\text{LGr}_-(V'')$ is a quadric, and a curve which is a plane section of a quadric is a line or a conic. But $C'$ is neither. \hfill \Box

Theorem 6.3. We have $F_X \cong S^2 C'$.

Proof: Let $R \subset X$ be a conic and consider a decomposition of its structure sheaf with respect to the semiorthogonal decomposition

\[ \mathcal{D}^b(X) = \langle \mathcal{O}_X, \mathcal{U}, \Phi_1(\mathcal{D}^b(C')) \rangle, \]

obtained from the decomposition (8) by mutating $\mathcal{U}$ through $\mathcal{O}_X$. To this end we compute

\[ \text{Ext}^p(\mathcal{O}_R, \mathcal{O}_X) = H^{3-p}(X, \omega_X \otimes \mathcal{O}_R)^* = H^{3-p}(R, \omega_R)^* = \begin{cases} \mathbb{C}, & \text{if } p = 2 \\ 0, & \text{otherwise} \end{cases} \]

\[ \text{Ext}^p(\mathcal{O}_R, \mathcal{U}) = \text{Ext}^{3-p}(\mathcal{U}, \omega_X \otimes \mathcal{O}_R)^* = \text{Ext}^{3-p}(\mathcal{U}_+^R, \omega_R)^* = H^{3-p}(R, \mathcal{U}_+^R \otimes \omega_R)^* = H^{p-2}(R, \mathcal{U}_+^R) = \begin{cases} \mathbb{C}, & \text{if } p = 2 \\ 0, & \text{otherwise} \end{cases} \]

by lemma 6.1. Hence the decomposition gives the following exact sequence

\[ 0 \to \mathcal{O}_X \to \mathcal{U}_+ \to \Phi_1(\mathcal{O}_R) \to \mathcal{O}_R \to 0, \quad (9) \]

where

\[ \Phi_1 : \mathcal{D}^b(X) \to \mathcal{D}^b(C'), \quad \Phi_1(-) = Rq_{1*}(Lp_{1*}(-) \otimes \mathcal{E}_1^*(H_{C'}))[1], \]

is the right adjoint to $\Phi_1$ functor.

Lemma 6.4. $\Phi_1(\mathcal{O}_R)$ is a pure sheaf.

Proof: In order to understand $\Phi_1(\mathcal{O}_R) = Rq_{1*}(Lp_{1*}(\mathcal{O}_R) \otimes \mathcal{E}_1^*(H_{C'}))[1] \in \mathcal{D}^b(C')$, we investigate $H^*(X, \mathcal{E}_1^* \otimes \mathcal{O}_R) = H^*(R, \mathcal{E}_1^*[R])$ for all $y \in C'$. The sheaf $\mathcal{E}_1^*$ by (3) is a subsheaf of the trivial vector bundle $\mathcal{U}_- \otimes \mathcal{O}_X$, therefore $H^0(R, \mathcal{E}_1^*[R]) \subset \mathcal{U}_- \subset \mathcal{V}$. On the other hand, by (4) we have $\mathcal{E}_1^* = \mathcal{E}_1(-H_X)$, is a subsheaf of $\mathcal{U}_+$, hence $H^0(R, \mathcal{E}_1^*[R]) \subset H^0(R, \mathcal{U}_+^R) = \mathbb{C} \subset \mathcal{V}$. Therefore, if $H^0(R, \mathcal{E}_1^*[R]) \not= 0$ for all $y \in C'$ then $\bigcap_{y \in C'} \mathcal{U}_- \not= 0$ which is false by lemma 6.2. Thus for generic $y \in C'$ we have $H^0(R, \mathcal{E}_1^*[R]) = 0$, hence $R^0q_{1*}(Lp_{1*}(\mathcal{O}_R) \otimes \mathcal{E}_1^*(H_{C'})) = 0$. On the other hand, since $R$ is 1-dimensional we have $R^kq_{1*}(Lp_{1*}(\mathcal{O}_R) \otimes \mathcal{E}_1^*(H_{C'})) = 0$ for $k \not= 0, 1$. Hence $\Phi_1(\mathcal{O}_R)$ is a pure sheaf. \hfill \Box

Corollary 6.5. $\Phi_1(\mathcal{O}_R)$ is an artinian sheaf of length 2 on $C'$.

Proof: Computation of the Chern character of $\Phi_1(\mathcal{O}_R)$ via the Grothendieck–Riemann–Roch. \hfill \Box

It follows from above that $\Phi_1(\mathcal{O}_R)$ is either the structure sheaf of a length 2 subscheme in $C'$, or $\Phi_1(\mathcal{O}_R) = \mathcal{O}_y \otimes \mathcal{O}_y$ for some $y \in C'$. We claim that the second never happens. To this end we need the following

Lemma 6.6. We have $\Phi_1^*(\mathcal{U}_+) = \mathcal{O}_{C'}$.\hfill \Box
Proof: It is clear that
\[ \Phi_1^*(U_+^*) = Rq_{1*}(Lp_1^*(U_+^*) \otimes \mathcal{E}_1^*(-H_X))[3] = Rq_{1*}(\mathcal{E}_1 \otimes U_+^*(-2H_X - H_C^\vee))[3]. \]
On the other hand, tensoring (4) with \( U_+^*(-H_X) \) we obtain exact sequence
\[ 0 \to \mathcal{E}_1 \otimes U_+^*(-2H_X - H_C^\vee) \to U_+ \otimes U_+^*(-H_X) \to U_+^* \otimes U_+^*(-H_X) \to \mathcal{E}_1 \otimes U_+^*(-H_X) \to 0. \]
Lemma 4.3 implies that \( R^*q_{1*}(\mathcal{E}_1 \otimes U_+^*(-H_X)) = 0 \) and corollary 4.2 implies that
\[ R^*q_{1*}(U_+^* \otimes U_+^*(-H_X)) = 0, \quad R^k q_{1*}(U_+ \otimes U_+^*(-H_X)) = \begin{cases} \mathcal{O}_{C^\vee}, & \text{for } k = 3 \\ 0, & \text{for } k \neq 3 \end{cases} \]
and the claim follows from the spectral sequence. \( \square \)

**Lemma 6.7.** \( \Phi_1^*(\mathcal{O}_R) \neq \mathcal{O}_y \oplus \mathcal{O}_y \).

**Proof:** If the above would be true then the decomposition (9) would take form
\[ 0 \to \mathcal{O}_X \to U_+^* \to \mathcal{E}_{1y} \oplus \mathcal{E}_{1y} \to \mathcal{O}_R \to 0. \]
On the other hand, it follows from lemma 6.6 that
\[ \text{Hom}(U_+^*, \mathcal{E}_{1y}) = \text{Hom}(U_+^*, \Phi_1(\mathcal{O}_y)) = \text{Hom}(\Phi_1^*(U_+^*), \mathcal{O}_y) = \text{Hom}(\mathcal{O}_{C^\vee}, \mathcal{O}_y) = \mathbb{C}, \]
hence the map \( U_+^* \to \mathcal{E}_{1y} \oplus \mathcal{E}_{1y} \) must have rank 2 and the above sequence is impossible. \( \square \)

**Corollary 6.8.** \( \Phi_1^*(\mathcal{O}_R) \) is the structure sheaf of a length 2 subscheme in \( C^\vee \).

Thus the functor \( \Phi_1^* \) induces a map \( F_X \to S^2C^\vee \).

Vice versa, if \( Z \) is a length 2 subscheme in \( C^\vee \) then
\[ \text{Hom}(\mathcal{O}_{C^\vee}, \mathcal{O}_Z) = \text{Hom}(\Phi_1^*(U_+^*), \mathcal{O}_Z) = \text{Hom}(U_+^*, \Phi_1(\mathcal{O}_Z)). \]
Therefore, the canonical projection \( \mathcal{O}_{C^\vee} \to \mathcal{O}_Z \) induces canonical morphism \( f : U_+^* \to \Phi_1(\mathcal{O}_Z) \).

Its kernel, being a rank 1 reflexive sheaf with \( c_1 = 0 \), must be isomorphic to \( \mathcal{O}_X \), and it is easy to show that its cokernel is the structure sheaf of a conic. Therefore, the map \( \Phi_1^* : F_X \to S^2C^\vee \) is an isomorphism. \( \square \)

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