THE SOLUTION OF THE MILNE PROBLEM FOR MAGNETIZED ATMOSPHERE

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Abstract

The numerical solution of the Milne problem for semi-infinite plane-parallel magnetized electron atmosphere is obtained. It is assumed that magnetic field is directed along the normal to the atmosphere. The angular dependence, the polarization degree and positional angle of outgoing radiation are presented in the tables for various values of the Faraday rotation parameter and the degree of absorption $q=0$, 0.2 and 0.4. We assume that magnetic field $B \leq 10^6$ G when one can neglect the effects of circular dichroism and take into account only the Faraday rotation effect.
1 Introduction

When a magnetic field is present in hot electron atmospheres and circumstellar shells, in accretion disks near quasars, and the nuclei of active galaxies, radiation is subject to the Faraday rotation of its plane of polarization. The angle of rotation $\psi$ is related to the parameters of the medium and the path length $l$ over which the light travels by [1]:

$$\psi = \frac{1}{2} \delta \tau_T \cos \Theta,$$

where $\tau_T = N_e l \sigma_T$ is the Thomson optical thickness of the optical path, $\sigma_T = (8\pi/3)r_e^2 \approx 6.65 \cdot 10^{-25} \text{cm}^2$ is the Thomson cross section, $r_e = e^2/m_e c^2 \approx 2.82 \cdot 10^{-13} \text{cm}$ is the classical radius of electron, $N_e$ is the electron number density, and $\Theta$ is the angle between the direction $n$ of the beam of light and the magnetic field $B$. The plane of polarization undergoes a right-handed rotation for $\Theta < 90^\circ$ and the opposite for $\Theta > 90^\circ$ if one views along the propagation direction of the light.

The parameter $\delta$ is equal to the angle of rotation of the polarization plane on a path $\tau_T = 2$ along the magnetic field and is given by

$$\delta = \frac{3}{4\pi} \cdot \frac{\lambda}{r_e} \cdot \frac{\omega_B}{\omega} \approx 0.8\lambda^2(\mu m)B(G).$$

Here, $\lambda = 2\pi c/\omega$ is the wavelength of the radiation, $\omega = 2\pi \nu$ is the angular frequency of the light, $\omega_B = |e|B/m_e c$ is the electron cyclotron frequency and $\omega_B/\omega \approx 0.93 \cdot 10^{-8}\lambda(\mu m)B(G)$.

In the general case of elliptically polarized light, the Faraday rotation describes the rotation of the polarization ellipse as the light passes through a magnetized medium.

For atmospheres with $\omega_B/\omega \ll 1$, right and left circularly polarized electromagnetic waves propagate in the medium independently at phase velocities corresponding to the refractive indices $n_r$ and $n_l$. A linearly polarized wave can be represented as a sum of right and left polarized coherent waves and the difference in the phase velocities of these waves leads to a rotation of the polarization plane ($\psi = 0.5(\omega/c)(n_l - n_r)$). For $\omega_B/\omega \ll 1$ the scattering cross sections for all the waves equal to the Thomson cross section $\sigma_T$. We shall examine only this case here. For optical wavelengths ($\lambda \approx 0.5\mu m$), this means $B \leq 10^6 G$.

The Faraday rotation leads to depolarization of the radiation since photons arriving from different optical depths have exposed different rotations of their planes of polarization.

We are considering the so-called Milne problem, i.e., multiple scattering of light in a semi-infinite plane-parallel atmosphere where the sources of unpolarized radiation are located at very high optical depth from the surface of the atmosphere. Besides scattering of the light on electrons, we shall also consider the intrinsic absorption of the light, the degree of which we denote by $q(q = \sigma_a/(\sigma_a + \sigma_T)$, where
\( \sigma_a \) is the intrinsic absorption cross section. The Milne problem corresponds to the conditions in stellar atmospheres, as well as the passage of light through optically thick circumstellar shells and accretion disks.

Ambartsumyan’s invariance principle \([2]\) is usually used in solving the Milne problem, that is, the angular distribution and polarization of emerging radiation are independent of the adding (or removal) of any layer to a semi-infinite atmosphere. Here the intensity and polarization of the emerging radiation are expressed in terms of so-called H-functions, which satisfy a nonlinear integral equation in just angular variables. Chandrasekhar \([3]\) proposed an extremely efficient method for solving this equation, the ”fork” method, in which the successive approximations represent the H-function with excess and deficiency. For magnetized atmospheres the invariance principle has been used \([1,4,5]\) to obtain systems of nonlinear equations for tensor H-functions. Unfortunately, there is no efficient method for solving these extremely cumbersome systems of equations. For the most interesting case of a conservative atmosphere \((q = 0)\), the iteration procedure converges very poorly. Even in the simplest case of a magnetic field perpendicular to the surface, one must solve a system of six nonlinear equations.

For large values of the Faraday rotation parameter, \( \delta \gg 1 \), however, the H-function technique can be used to obtain simple asymptotic solutions of the Milne problem and of problems with power law and exponential distributions of the sources in magnetized atmospheres \([5]\). These solutions are suitable for arbitrary inclinations of the magnetic field to the outward normal \( N \) to the atmosphere. Numerical solutions of these problems have been obtained \([7,8]\) for the case of zero magnetic field.

For \( \delta \leq 1 \) and arbitrary inclination of magnetic field, all the methods of solution lead to extremely cumbersome formulas and calculations.

For a magnetic field \( B \), directed along the normal \( N \) to the surface of the atmosphere the calculations are much simpler, since the problem has axial symmetry. In \([9]\) the Milne problem for \( B \parallel N \) has been solved by Monte Carlo method and in \([10]\) by Feautrier method. However, the results of these calculations, which were given in the form of graphs, are sometimes rather different. In this paper the Milne problem for \( B \parallel N \) is solved by the classical Chandrasekhar method \([3]\) using gaussian quadrature formulas to reduce the integro-differential transport equation to a system of linear differential equations. By increasing the order of the gaussian quadrature and comparing the results, one can estimate the accuracy of the solutions that are obtained. We have attained accuracies to the first four significant figures. This is the most accurate solution of the Milne problem to date. In some cases our results differ by 10-15\% from the less accurate results of \([10]\). It is extremely important that the results obtained here can be used to estimate the accuracy of the simple asymptotic formulas in \([6]\).
2 Solution of the Milne problem

We shall consider the Milne problem for the case when the magnetic field \( B \) is directed along the outward normal \( N \) to the semi-infinite atmosphere. In the absence of a magnetic field the problem reduces (see, [3]) to a system of coupled equations for the intensity \( I(\tau, \mu) \) and the Stokes polarization parameter \( Q(\tau, \mu) \) (either for the intensities of the radiation polarized in the \((nN)\) - plane or in the perpendicular plane, \( I_x(\tau, \mu) \) and \( I_y(\tau, \mu) \), respectively: \( I = I_x + I_y, \ Q = I_x - I_y \)). Here, \( \mu = \cos \theta \) is the cosine of the angle between the propagation direction \( n \) of the light and the normal \( N \), \( \tau \) is the optical thickness, including absorption, taken from the surface into the interior of the medium. As usual, we have chosen the \( x \) axis of the observer’s system lying in the \((nN)\) plane where \( n \) is the direction to the telescope. Remind that the classical Chandrasekhar solution for the emerging radiation \((\tau = 0)\) gives an elongation of the angular distribution of \( J(\mu = 1) \equiv I(\mu = 1)/I(\mu = 0) \) = 3.06 and a peak value of the degree of polarization for \( \mu = 0 \) of 11.71%, while the oscillations of the electric field vector of the radiation for all \( \mu \) are perpendicular to \((nN)\)-plane, i.e. \( Q(\mu) < 0 \).

The presence of a magnetic field leads to the appearance of the Stokes parameter \( U(\tau, \mu) = I_{x'} - I_{y'} \), where \( x' \) and \( y' \) are the coordinate axes turned in the positive (right hand) direction by 45° from the basic \( x \) and \( y \) axes. The parameter \( U \) means that the plane of polarization is no longer perpendicular to the plane \((nN)\). Remind that the angle of inclination \( \chi \) of this plane with respect to the perpendicular to the \((nN)\) is given by \( \tan(2\chi) = U/Q \). The azimuthal symmetry of this case, \( B \parallel N \), means that \( U \) does not contribute to the scattering on electrons, that is, the integral term in the transport equation coincides with the case \( B = 0 \). In this situation \( U \) is completely determined by the Faraday rotation process on leaving the atmosphere. This means that the angle \( \chi \) is taken in the right hand sense from the plane of the oscillations without a magnetic field relative to the line of sight at the telescope with a magnetic field directed outward from the medium. For a magnetic field directed towards the interior of the atmosphere, \( \chi \) is taken in the opposite direction.

The system of equations for \( I(\tau, \mu), Q(\tau, \mu) \) and \( U(\tau, \mu) \) according to the general formulas of [1,4] is

\[
\frac{\mu}{d\tau}I(\tau, \mu) = I(\tau, \mu) - \frac{3}{16}(1 - q) \int_{-1}^{1} d\mu' \left\{ [(3 - \mu'^2) + \mu'^2(3\mu'^2 - 1)]I(\tau, \mu') + (1 - 3\mu'^2)(1 - \mu'^2)Q(\tau, \mu') \right\}, \tag{3}
\]

\[
\frac{\mu}{d\tau}Q(\tau, \mu) = Q(\tau, \mu) + (1 - q)\delta \mu U(\tau, \mu) - \frac{3}{16}(1 - q)(1 - \mu^2) \int_{-1}^{1} d\mu' \left[ (1 - 3\mu'^2)I(\tau, \mu') + 3(1 - \mu'^2)Q(\tau, \mu') \right], \tag{4}
\]
\[ \mu \frac{d}{d\tau} U(\tau, \mu) = U(\tau, \mu) - (1 - q)\delta \mu Q(\tau, \mu). \]  

Here, \( q \) is the degree of true absorption of the light, \( \tau \) is total optical thickness including absorption, and for \( B || N \), \( \cos \Theta = \mu \).

The boundary conditions for the system (3)-(5) are the usual ones: \( I(0, -\mu) = 0, Q(0, -\mu) = 0 \) and \( U(0, -\mu) = 0 \), that is, there is no radiation incident from outside. In addition, it is assumed that none of the Stokes parameters have exponentially increasing terms for \( \tau \to \infty \). Following Chandrasekhar’s method [3], we use gaussian quadratures to replace the integral terms with sums where the parameters are taken at discrete points \( \mu_i \):

\[ I_i = I(\tau, \mu_i), \quad Q_i = Q(\tau, \mu_i) \quad \text{and} \quad U_i = U(\tau, \mu_i). \]

The points \( \mu_i \) are the roots of the Legendre polynomial \( P_2^n(\mu) \). The number \( n \) determines the order of the gaussian quadrature formula. The system of integro-differential equations (3)-(5) is thereby converted into a system of ordinary differential equations:

\[ \mu_i \frac{d}{d\tau} I_i = I_i - \frac{3}{16} (1 - q) \sum_{j=\pm 1}^{\pm n} a_j \left\{ [(3 - \mu_i^2) + \mu_i^2 (3\mu_j^2 - 1)] I_j + (1 - 3\mu_j^2)(1 - \mu_j^2)Q_j \right\}, \]  

\[ \mu_i \frac{d}{d\tau} Q_i = Q_i + (1 - q)\delta \mu_i U_i - \frac{3}{16} (1 - q)(1 - \mu_i^2) \sum_{j=\pm 1}^{\pm n} a_j \left[ (1 - 3\mu_j^2)I_j + 3(1 - \mu_j^2)Q_j \right], \]  

\[ \mu_i \frac{d}{d\tau} U_i = U_i - (1 - q)\delta \mu_i Q_i. \]  

Here, \( \mu_i \) are the roots of the Legendre polynomial \( P_2^m(\mu) = 0 \), \( \mu_{-i} = -\mu_i \), and \( a_i \) are the known weights of the gaussian quadrature formula, with \( a_{-i} = a_i \).

We seek a solution of the system (6)-(8) of the form

\[ I_i = g_i \exp(-k\tau), \quad Q_i = h_i \exp(-k\tau), \quad U_i = f_i \exp(-k\tau). \]  

Substituting (9) in (6)-(8) yields the formulas

\[ f_i = \frac{(1 - q)\delta \mu_i}{1 + k\mu_i} h_i, \]

\[ g_i = \frac{\beta - \alpha \mu_i^2}{1 + k\mu_i}, \]

\[ h_i = \alpha \frac{(1 - \mu_i^2)(1 + k\mu_i)}{(1 + k\mu_i)^2 + [(1 - q)\delta \mu_i]^2}. \]  

A homogeneous system of algebraic equations is derived from (6) and (7) for finding the numbers \( \alpha \) and \( \beta \). The condition for this system to be soluble, namely
that the determinant equal to zero, yields the characteristic equation for finding the eigenvalues \( k \). We can only determine the ratio \( \alpha/\beta \) from the homogeneous second order system, so that one of \( \alpha \) or \( \beta \) remains unknown. This number is found using the condition that the radiation flux leaving the atmosphere \( F \) is given.

As a result, we have obtained the angular distribution

\[
J(\mu) = I(0, \mu)/I(0, 0),
\]

the degree of polarization

\[
p(\mu) = \frac{\sqrt{Q^2(0, \mu) + U^2(0, \mu)}}{I(0, \mu)},
\]

and the angle of inclination \( \chi(\mu) \) of the electric field oscillations of the radiation relative to a plane perpendicular to the plane \( (nN) \),

\[
\tan(2\chi) = \frac{U(0, \mu)}{Q(0, \mu)}.
\]

These quantities are listed in Tables 1–6 for a various values of parameters \( \delta \) and \( q \).

3 Conclusion

We now analyze these results briefly and offer a qualitative explanation of them. First of all, it is evident that the polarization of the radiation \( p(\mu) \) becomes ever more peaked as \( \delta \) increases, with a maximum at \( \mu = 0 \), i.e., in a direction perpendicular to the magnetic field. This is a manifestation of the depolarization of the radiation owing to Faraday’s rotation, since the radiation emerging from the medium consists of the fluxes of light undergoing different amounts of the Faraday rotation of their planes of polarization. At the same time, as \( \delta \) increases, there is an increase in the angle \( \chi \) by which the polarization plane of the emerging radiation turns relative to the plane of polarization in the absence of a magnetic field, i.e., relative perpendicular to the \((nN)\) plane. In the limit \( \delta \rightarrow \infty \), the angle \( \chi \rightarrow 45^\circ \). This behaviour of the turning angle \( \chi \) can be explained qualitatively as follows: the emerging radiation mainly passes from the outer layer of the atmosphere with \( \tau/\mu \approx 1 \). According to (5), the Stokes parameter \( U(\mu) \) acquires the value \( -\sim Q(\mu)(1 - q)\delta\mu\tau/\mu \), which leads to a ratio \( U(\mu)/Q(\mu) \sim (1 - q)\delta\mu \). Thus, the turning angle \( \chi \) is given by

\[
\tan(2\chi) \sim (1 - q)\delta\mu.
\]

For \( (1 - q)\delta\mu \gg 1 \) the positional angle \( \chi \) actually tends to the limit \( 45^\circ \). The Faraday rotation is determined solely by the presence of free electrons along the path of radiation, i.e., by the Thomson optical thickness \( \tau_T = (1 - q)\tau \). For \( q \rightarrow 1 \), the outer layer of the atmosphere with \( \tau \approx 1 \) contains too few electrons \( (\tau_T \rightarrow 0) \) for the Faraday rotation to influence the distribution of the polarization plane of
emerging radiation. In this case, we can neglect the parameter \( U(\tau, \mu) \) and the system (3)-(5) transforms into the ordinary equations of radiative transfer without magnetic field. These qualitative arguments and estimates are general in nature and not associated with the specifics of the Milne problem. Thus, for arbitrary strongly absorbing atmospheres \((q \to 1)\) the Faraday rotation is unimportant. The Milne problem is not interesting for highly absorbing atmospheres, since the distribution of thermal radiation sources, which is proportional to the distribution of absorbing particles, plays a dominant role. It is known that radiation emerging from highly absorbing layers is essentially unpolarized.

Polarization causes little change in the distribution of the emerging radiation, even in the absence of a magnetic field. Thus, the Milne problem with polarization taken into account (3) and (4) yields an elongation \( J(0) = 3.06 \), while solving (3) with the \( Q(\tau, \mu) \) term omitted (the equation just for the intensity with the Rayleigh phase function) gives \( J(0) = 3.02 \). That is, the angular distributions are essentially the same (see, Table 6).

The Faraday rotation causes depolarization of the radiation for all the directions except perpendicular to the magnetic field. Thus, with the increasing \( \delta \) the contribution of the polarization terms to the formation of the angular distribution becomes ever smaller. Equation (3) is transformed into a separate equation for just the intensity with the Rayleigh phase function. Our tables show the gradual approach of the angular distribution to this limiting form (see columns 3-5 of Table 6) as the Faraday rotation parameter \( \delta \) increases.

The contribution of the polarization terms \( Q \) and \( U \) to the overall polarization of the radiation emerging from the atmosphere is much greater than their influence on the formation of the angular distribution. Thus, the calculated degrees of polarization using known intensities of the radiation are 9.37% instead of 11.71%. This means that the difference 11.71%\( -9.37\% = 2.34\% \) (or 20% of the total polarization) is created by the polarization terms. As can be seen from the Tables, the Faraday rotation substantially reduces the polarization and, for \( \delta \gg 1 \) the contribution of the polarization terms to the degree of polarization itself goes to zero. Here the intensity of radiation, itself, differs somewhat from the case of zero magnetic field and is determined by the scalar transport equation with the Rayleigh phase function (we are comparing the second and third columns of Table 6). Thus, at the polarization maximum \((\mu = 0)\) we obtain a slightly lower value of 9.14% instead of 9.37%. Our values of \( p(0) \) approach precisely this limit as \( \delta \to \infty \) (see, Tables 1-3).

The simple asymptotic formulas in [6] for a number of standard problems in radiation transport theory correspond to the approximation in which the radiation intensity is determined by the transport equation with the Rayleigh phase function, while the polarization is treated as the result of single scattering of a known radiation flux and its transformation by the Faraday rotation. They give somewhat overestimated polarization values. A comparison of the calculations using these formulas with the exact solutions obtained here shows that for \( q = 0 \), with \( \delta = 10 \) the asymptotic formulas yield polarization values with an relative error of \( \approx 10\% \). For
δ = 5 the error exceeds ≈ 20%.

These simple formulas have the major advantage of providing an analytical description of the polarization for an arbitrary arrangement of the magnetic field in the atmosphere.

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Table 1: The degree of polarization $p(\mu)\%$, the positional angle of the polarization $\chi^\circ$, and angular distribution $J(\mu)$ of emerging radiation for $q = 0$.

| $\mu$ | $\delta = 0$ | 1 | 2 | 3 |
|-------|-------------|---|---|---|
|       | $p$ | $\chi$ | $J$ | $p$ | $\chi$ | $J$ | $p$ | $\chi$ | $J$ |
| 0     | 11.71 | 0 | 1 | 11.56 | 0 | 1 | 11.26 | 0 | 1 |
| 0.05  | 8.997 | 0 | 1.1433 | 8.865 | 1.29 | 1.1437 | 8.497 | 2.537 | 1.1441 | 8.141 | 3.748 | 1.1442 |
| 0.10  | 7.467 | 0 | 1.2627 | 7.268 | 2.40 | 1.2635 | 6.842 | 4.650 | 1.2640 | 6.425 | 6.760 | 1.2642 |
| 0.15  | 6.323 | 0 | 1.3742 | 6.098 | 3.38 | 1.3752 | 5.633 | 6.439 | 1.3758 | 5.183 | 9.213 | 1.3760 |
| 0.20  | 5.430 | 0 | 1.4815 | 5.176 | 4.25 | 1.4827 | 4.693 | 7.962 | 1.4833 | 4.234 | 11.22 | 1.4832 |
| 0.25  | 4.682 | 0 | 1.5862 | 4.424 | 5.03 | 1.5875 | 3.939 | 9.267 | 1.5880 | 3.491 | 12.88 | 1.5877 |
| 0.30  | 4.052 | 0 | 1.6891 | 3.795 | 5.72 | 1.6905 | 3.322 | 10.39 | 1.6907 | 2.896 | 14.26 | 1.6902 |
| 0.35  | 3.511 | 0 | 1.7907 | 3.259 | 6.35 | 1.7920 | 2.809 | 11.37 | 1.7920 | 2.414 | 15.43 | 1.7912 |
| 0.40  | 3.040 | 0 | 1.8912 | 2.798 | 6.91 | 1.8926 | 2.377 | 12.23 | 1.8923 | 2.016 | 16.41 | 1.8911 |
| 0.45  | 2.625 | 0 | 1.9910 | 2.397 | 7.42 | 1.9923 | 2.008 | 12.98 | 1.9916 | 1.684 | 17.26 | 1.9901 |
| 0.50  | 2.257 | 0 | 2.0901 | 2.045 | 7.89 | 2.0913 | 1.692 | 13.65 | 2.0903 | 1.404 | 18.00 | 2.0885 |
| 0.55  | 1.927 | 0 | 2.1887 | 1.733 | 8.31 | 2.1898 | 1.417 | 14.24 | 2.1884 | 1.166 | 18.64 | 2.1862 |
| 0.60  | 1.630 | 0 | 2.2869 | 1.455 | 8.70 | 2.2879 | 1.178 | 14.78 | 2.2859 | 0.960 | 19.20 | 2.2834 |
| 0.65  | 1.360 | 0 | 2.3847 | 1.206 | 9.06 | 2.3855 | 0.967 | 15.26 | 2.3831 | 0.783 | 19.70 | 2.3802 |
| 0.70  | 1.115 | 0 | 2.4822 | 0.982 | 9.39 | 2.4829 | 0.780 | 15.69 | 2.4800 | 0.627 | 20.14 | 2.4766 |
| 0.75  | 0.890 | 0 | 2.5795 | 0.779 | 9.70 | 2.5799 | 0.613 | 16.08 | 2.5765 | 0.490 | 20.53 | 2.5728 |
| 0.80  | 0.683 | 0 | 2.6765 | 0.595 | 9.98 | 2.6767 | 0.465 | 16.44 | 2.6728 | 0.369 | 20.89 | 2.6687 |
| 0.85  | 0.493 | 0 | 2.7733 | 0.426 | 10.24 | 2.7733 | 0.331 | 16.77 | 2.7688 | 0.262 | 21.21 | 2.7643 |
| 0.90  | 0.316 | 0 | 2.8700 | 0.272 | 10.49 | 2.8697 | 0.210 | 17.07 | 2.8646 | 0.165 | 21.51 | 2.8597 |
| 0.95  | 0.152 | 0 | 2.9665 | 0.131 | 10.72 | 2.9659 | 0.100 | 17.35 | 2.9603 | 0.078 | 21.77 | 2.9550 |
| 1     | 0     | 0 | 3.0628 | 0 | 10.95 | 3.0620 | 0 | 17.60 | 3.0558 | 0 | 22.02 | 3.0500 |
Table 2: The degree of polarization $p(\mu)\%$, the positional angle of the polarization $\chi^\circ$, and the angular distribution $J(\mu)$ of emerging radiation for $q = 0$.

| $\mu$ | $\delta = 4$ | $\delta = 5$ | $\delta = 6$ | $\delta = 7$ |
|-------|---------------|---------------|---------------|---------------|
|       | $p$  | $\chi$ | $J$    | $p$  | $\chi$ | $J$  | $p$  | $\chi$ | $J$  |
| 0     | 10.75 | 0     | 1     | 10.56 | 0     | 1     | 10.40 | 0     | 1     |
| 0.05  | 7.802 | 4.913 | 1.1446 | 7.550 | 6.078 | 1.1447 | 7.336 | 7.218 | 1.1446 |
| 0.10  | 6.020 | 8.724 | 1.2646 | 5.712 | 10.64 | 1.2646 | 5.434 | 12.43 | 1.2644 |
| 0.15  | 4.751 | 11.71 | 1.3762 | 4.418 | 14.06 | 1.3761 | 4.120 | 16.17 | 1.3758 |
| 0.20  | 3.805 | 14.06 | 1.4834 | 3.477 | 16.64 | 1.4831 | 3.190 | 18.88 | 1.4826 |
| 0.25  | 3.083 | 15.94 | 1.5876 | 2.776 | 18.62 | 1.5872 | 2.515 | 20.89 | 1.5866 |
| 0.30  | 2.520 | 17.45 | 1.6899 | 2.243 | 20.17 | 1.6893 | 2.012 | 22.42 | 1.6885 |
| 0.35  | 2.074 | 18.69 | 1.7907 | 1.828 | 21.41 | 1.7899 | 1.627 | 23.62 | 1.7890 |
| 0.40  | 1.713 | 19.71 | 1.8903 | 1.499 | 22.41 | 1.8894 | 1.326 | 24.57 | 1.8883 |
| 0.45  | 1.418 | 20.57 | 1.9891 | 1.232 | 23.24 | 1.9880 | 1.085 | 25.35 | 1.9876 |
| 0.50  | 1.173 | 21.30 | 2.0871 | 1.014 | 23.94 | 2.0858 | 0.890 | 26.00 | 2.0844 |
| 0.55  | 0.968 | 21.93 | 2.1846 | 0.832 | 24.53 | 2.1831 | 0.728 | 26.55 | 2.1815 |
| 0.60  | 0.793 | 22.48 | 2.2815 | 0.679 | 25.03 | 2.2799 | 0.592 | 27.01 | 2.2781 |
| 0.65  | 0.643 | 22.95 | 2.3780 | 0.549 | 25.47 | 2.3762 | 0.478 | 27.41 | 2.3743 |
| 0.70  | 0.513 | 23.37 | 2.4742 | 0.437 | 25.85 | 2.4722 | 0.379 | 27.76 | 2.4701 |
| 0.75  | 0.399 | 23.74 | 2.5700 | 0.339 | 26.19 | 2.5679 | 0.294 | 28.06 | 2.5657 |
| 0.80  | 0.300 | 24.07 | 2.6656 | 0.254 | 26.49 | 2.6633 | 0.220 | 28.33 | 2.6609 |
| 0.85  | 0.212 | 24.37 | 2.7610 | 0.179 | 26.76 | 2.7585 | 0.155 | 28.57 | 2.7559 |
| 0.90  | 0.133 | 24.64 | 2.8561 | 0.113 | 27.00 | 2.8534 | 0.097 | 28.79 | 2.8507 |
| 0.95  | 0.063 | 24.88 | 2.9511 | 0.053 | 27.22 | 2.9482 | 0.046 | 28.98 | 2.9454 |
| 1     | 0     | 25.10 | 3.0459 | 0     | 27.41 | 3.0428 | 0     | 29.16 | 3.0398 |

| $\mu$ | $\delta = 7$ |
|-------|---------------|
| 1     | 0     | 30.53 | 3.0375 |
Table 3: The degree of polarization $p(\mu)\%$, the positional angle of the polarization $\chi^\circ$, and the angular distribution $J(\mu)$ of emerging radiation for $q = 0$.

| $\mu$ | $\delta = 8$ |  |  | $\delta = 9$ |  |  | $\delta = 10$ |  |  | $\delta = 100$ |  |  |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
|       | $p$  | $\chi$ | $J$ | $p$  | $\chi$ | $J$ | $p$  | $\chi$ | $J$ | $p$  | $\chi$ | $J$ |
| 0     | 10.17 | 0 | 1 | 10.08 | 0 | 1 | 10.01 | 0 | 1 | 9.173 | 0 | 1 |
| 0.05  | 6.959 | 9.413 | 1.1444 | 6.787 | 10.46 | 1.1444 | 6.622 | 11.48 | 1.1444 | 1.540 | 37.72 | 1.1441 |
| 0.10  | 4.931 | 15.67 | 1.2641 | 4.702 | 17.11 | 1.2640 | 4.487 | 18.46 | 1.2639 | 0.706 | 40.44 | 1.2631 |
| 0.15  | 3.601 | 19.76 | 1.3753 | 3.377 | 21.28 | 1.3751 | 3.174 | 22.63 | 1.3749 | 0.430 | 41.37 | 1.3734 |
| 0.20  | 2.713 | 22.51 | 1.4819 | 2.516 | 23.99 | 1.4816 | 2.342 | 25.29 | 1.4814 | 0.294 | 41.83 | 1.4792 |
| 0.25  | 2.097 | 24.45 | 1.5856 | 1.931 | 25.86 | 1.5853 | 1.786 | 27.08 | 1.5850 | 0.215 | 42.11 | 1.5821 |
| 0.30  | 1.654 | 25.87 | 1.6874 | 1.515 | 27.21 | 1.6869 | 1.396 | 28.36 | 1.6866 | 0.164 | 42.29 | 1.6830 |
| 0.35  | 1.324 | 26.95 | 1.7876 | 1.208 | 28.23 | 1.7871 | 1.109 | 29.32 | 1.7876 | 0.128 | 42.42 | 1.7824 |
| 0.40  | 1.070 | 27.79 | 1.8867 | 0.974 | 29.02 | 1.8861 | 0.892 | 30.06 | 1.8856 | 0.102 | 42.52 | 1.8807 |
| 0.45  | 0.870 | 28.47 | 1.9849 | 0.790 | 29.65 | 1.9843 | 0.723 | 30.65 | 1.9837 | 0.082 | 42.60 | 1.9781 |
| 0.50  | 0.710 | 29.02 | 2.0824 | 0.643 | 30.16 | 2.0817 | 0.588 | 31.13 | 2.0810 | 0.066 | 42.66 | 2.0747 |
| 0.55  | 0.578 | 29.49 | 2.1793 | 0.523 | 30.59 | 2.1785 | 0.478 | 31.53 | 2.1778 | 0.053 | 42.71 | 2.1708 |
| 0.60  | 0.469 | 29.88 | 2.2757 | 0.424 | 30.95 | 2.2748 | 0.387 | 31.86 | 2.2740 | 0.043 | 42.75 | 2.2664 |
| 0.65  | 0.377 | 30.21 | 2.3716 | 0.341 | 31.26 | 2.3707 | 0.310 | 32.14 | 2.3699 | 0.034 | 42.78 | 2.3615 |
| 0.70  | 0.299 | 30.50 | 2.4673 | 0.270 | 31.52 | 2.4662 | 0.246 | 32.39 | 2.4653 | 0.027 | 42.81 | 2.4564 |
| 0.75  | 0.231 | 30.75 | 2.5625 | 0.209 | 31.76 | 2.5614 | 0.190 | 32.60 | 2.5605 | 0.021 | 42.83 | 2.5509 |
| 0.80  | 0.173 | 30.98 | 2.6576 | 0.156 | 31.96 | 2.6564 | 0.142 | 32.79 | 2.6554 | 0.015 | 42.86 | 2.6451 |
| 0.85  | 0.121 | 31.17 | 2.7524 | 0.109 | 32.14 | 2.7511 | 0.099 | 32.95 | 2.7500 | 0.011 | 42.88 | 2.7391 |
| 0.90  | 0.076 | 31.35 | 2.8470 | 0.068 | 32.30 | 2.8456 | 0.062 | 33.10 | 2.8445 | 0.007 | 42.89 | 2.8329 |
| 0.95  | 0.036 | 31.51 | 2.9414 | 0.032 | 32.44 | 2.9399 | 0.029 | 33.33 | 2.9387 | 0.003 | 42.91 | 2.9265 |
| 1     | 0     | 31.65 | 3.0356 | 32.57 | 3.0341 | 33.35 | 3.0328 | 42.92 | 3.0200 |  |  |  |
Table 4: The degree of polarization $p(\mu)\%$, the positional angle of the polarization $\chi^\circ$, and the angular distribution $J(\mu)$ of emerging radiation for $q = 0.2$.

| $\mu$ | $\delta = 1$ | 5 | 10 | 50 |
|-------|--------------|---|----|----|
|       | $p$  | $\chi$ | $J$ | $p$  | $\chi$ | $J$ | $p$  | $\chi$ | $J$ | $p$  | $\chi$ | $J$ |
| 0     | 0    | 25.05  | 0   | 1   | 20.12 | 0   | 1   | 18.96 | 0   | 1   | 17.65 | 0   |
| 0.05  | 0    | 22.95  | 1.188| 1.1240| 17.73 | 5.774| 1.1248| 15.67 | 11.01 | 1.1251| 7.022 | 31.24 | 1.1270|
| 0.10  | 0    | 21.53  | 2.475| 1.2337| 15.44 | 11.42| 1.2348| 11.95 | 19.69 | 1.2357| 3.491 | 37.32 | 1.2386|
| 0.15  | 0    | 20.28  | 3.868| 1.3424| 13.09 | 16.56| 1.3436| 8.777 | 25.61 | 1.3450| 2.171 | 39.53 | 1.3480|
| 0.20  | 0    | 19.07  | 5.368| 1.4542| 10.85 | 20.96| 1.4551| 6.650 | 29.56 | 1.4567| 1.499 | 40.67 | 1.4594|
| 0.25  | 0    | 17.85  | 6.973| 1.5715| 8.866 | 24.61| 1.5716| 5.068 | 32.27 | 1.5732| 1.095 | 41.35 | 1.5752|
| 0.30  | 0    | 16.59  | 8.681| 1.6963| 7.190 | 27.57| 1.6949| 3.922 | 34.20 | 1.6962| 0.828 | 41.81 | 1.6975|
| 0.35  | 0    | 15.27  | 10.49| 1.8305| 5.805 | 29.97| 1.8269| 3.070 | 35.64 | 1.8277| 0.639 | 42.14 | 1.8279|
| 0.40  | 0    | 13.91  | 12.38| 1.9761| 4.671 | 31.92| 1.9696| 2.420 | 36.75 | 1.9694| 0.499 | 42.38 | 1.9784|
| 0.45  | 0    | 12.50  | 14.35| 2.1355| 3.745 | 33.53| 2.1249| 1.913 | 37.62 | 2.1236| 0.393 | 42.58 | 2.1211|
| 0.50  | 0    | 11.06  | 16.38| 2.3112| 2.986 | 34.86| 2.2955| 1.512 | 38.33 | 2.2928| 0.309 | 42.73 | 2.2885|
| 0.55  | 0    | 9.605  | 18.47| 2.5065| 2.363 | 35.98| 2.4843| 1.189 | 38.91 | 2.4798| 0.243 | 42.86 | 2.4734|
| 0.60  | 0    | 8.171  | 20.58| 2.7253| 1.850 | 36.93| 2.6951| 0.927 | 39.40 | 2.6883| 0.189 | 42.96 | 2.6795|
| 0.65  | 0    | 6.779  | 22.69| 2.9723| 1.426 | 37.75| 2.9323| 0.713 | 39.81 | 2.9229| 0.145 | 43.05 | 2.9111|
| 0.70  | 0    | 5.456  | 24.80| 3.2537| 1.077 | 38.45| 3.2017| 0.538 | 40.16 | 3.1891| 0.109 | 43.13 | 3.1738|
| 0.75  | 0    | 4.226  | 26.87| 3.5773| 0.790 | 39.06| 3.5108| 0.394 | 40.47 | 3.4943| 0.080 | 43.19 | 3.4750|
| 0.80  | 0    | 3.108  | 28.89| 3.9534| 0.555 | 39.60| 3.8694| 0.277 | 40.74 | 3.8483| 0.056 | 43.25 | 3.8240|
| 0.85  | 0    | 2.120  | 30.84| 4.3958| 0.364 | 40.07| 4.2907| 0.182 | 40.98 | 4.2640| 0.037 | 43.30 | 4.2339|
| 0.90  | 0    | 1.270  | 32.72| 4.9235| 0.211 | 40.50| 4.7928| 0.105 | 41.19 | 4.7594| 0.021 | 43.35 | 4.7224|
| 0.95  | 0    | 0.563  | 34.51| 5.5636| 0.091 | 40.88| 5.4018| 0.046 | 41.38 | 5.3603| 0.009 | 43.39 | 5.3147|
| 1     | 0    | 36.21  | 6.3553| 0   | 41.22 | 6.1556| 0   | 41.55 | 6.1043| 0   | 43.42 | 6.0485|
Table 5: The degree of polarization $p(\mu)\%$, the positional angle of the polarization $\chi^\circ$, and the angular distribution $J(\mu)$ of emerging radiation for $q = 0.4$.

| $\mu$ | $\delta = 1$ | $5$ | $10$ | $50$ |
|-------|-------------|-----|------|-----|
|       | $p$ | $\chi$ | $J$ | $p$ | $\chi$ | $J$ | $p$ | $\chi$ | $J$ | $p$ | $\chi$ | $J$ |
| 0     | 39.89 | 0 | 1 | 33.20 | 0 | 1 | 31.45 | 0 | 1 | 29.49 | 0 | 1 |
| 0.05  | 38.30 | 0.91 | 1.1055 | 31.28 | 4.499 | 1.1062 | 28.50 | 8.75 | 1.1066 | 15.00 | 28.51 | 1.1087 |
| 0.10  | 37.00 | 1.924 | 1.2046 | 28.93 | 9.228 | 1.2054 | 23.90 | 16.74 | 1.2063 | 7.819 | 36.17 | 1.2105 |
| 0.15  | 35.64 | 3.043 | 1.3081 | 25.98 | 13.92 | 1.3085 | 19.02 | 23.09 | 1.3100 | 4.902 | 39.11 | 1.3150 |
| 0.20  | 34.13 | 4.276 | 1.4198 | 22.65 | 18.33 | 1.4195 | 14.82 | 27.81 | 1.4213 | 3.379 | 40.63 | 1.4265 |
| 0.25  | 32.44 | 5.633 | 1.5431 | 19.24 | 22.32 | 1.5413 | 11.49 | 31.29 | 1.5432 | 2.454 | 41.55 | 1.5479 |
| 0.30  | 30.55 | 7.124 | 1.6810 | 16.00 | 25.81 | 1.6769 | 8.933 | 33.88 | 1.6785 | 1.838 | 42.17 | 1.6824 |
| 0.35  | 28.46 | 8.762 | 1.8370 | 13.09 | 28.81 | 1.8297 | 6.958 | 35.85 | 1.8306 | 1.401 | 42.61 | 1.8332 |
| 0.40  | 26.16 | 10.56 | 2.0157 | 10.56 | 31.36 | 2.0038 | 5.421 | 37.39 | 2.0036 | 1.077 | 42.95 | 2.0045 |
| 0.45  | 23.68 | 12.52 | 2.2223 | 8.407 | 33.53 | 2.2046 | 4.213 | 38.62 | 2.2027 | 0.830 | 43.21 | 2.2013 |
| 0.50  | 21.04 | 14.65 | 2.4643 | 6.604 | 35.38 | 2.4387 | 3.254 | 39.61 | 2.4346 | 0.638 | 43.41 | 2.4302 |
| 0.55  | 18.27 | 16.94 | 2.7509 | 5.110 | 36.95 | 2.7155 | 2.489 | 40.44 | 2.7084 | 0.486 | 43.58 | 2.7002 |
| 0.60  | 15.45 | 19.40 | 3.0956 | 3.883 | 38.31 | 3.0474 | 1.876 | 41.13 | 3.0364 | 0.366 | 43.72 | 3.0236 |
| 0.65  | 12.63 | 22.00 | 3.5168 | 2.883 | 39.48 | 3.4525 | 1.386 | 41.72 | 3.4365 | 0.270 | 43.84 | 3.4177 |
| 0.70  | 9.908 | 24.71 | 4.0422 | 2.078 | 40.50 | 3.9573 | 0.995 | 42.23 | 3.9349 | 0.194 | 43.95 | 3.9085 |
| 0.75  | 7.377 | 27.49 | 4.7137 | 1.438 | 41.39 | 4.6022 | 0.687 | 42.67 | 4.5718 | 0.134 | 44.04 | 4.5359 |
| 0.80  | 5.123 | 30.29 | 5.5996 | 0.940 | 42.18 | 5.4541 | 0.449 | 43.05 | 5.4132 | 0.088 | 44.12 | 5.3652 |
| 0.85  | 3.221 | 33.06 | 6.8179 | 0.562 | 42.88 | 6.6284 | 0.268 | 43.39 | 6.5742 | 0.052 | 44.18 | 6.5107 |
| 0.90  | 1.723 | 35.75 | 8.5928 | 0.289 | 43.50 | 8.3468 | 0.138 | 43.69 | 8.2758 | 0.027 | 44.25 | 8.1930 |
| 0.95  | 0.650 | 38.33 | 11.407 | 0.106 | 44.06 | 11.094 | 0.050 | 43.97 | 11.004 | 0.010 | 44.30 | 10.899 |
| 1     | 0     | 40.76 | 16.531 | 0     | 44.56 | 16.170 | 0     | 44.21 | 16.075 | 0     | 44.35 | 15.962 |
Table 6: Some solutions of the Milne problem without a magnetic field. The first two columns give the Chandrasekhar values of the degree of polarization $p(\mu)\%$ and the angular distribution $J(\mu)$ of emerging radiation for $q = 0$. The third, fourth and fifth columns describe the angular distribution of the radiation obtained by solving the equation just for the intensity with the Rayleigh phase function for $q = 0, 0.2,$ and $0.4$, respectively. The last columns are our solutions of the Milne problem for $q = 0.2,$ and $0.4$.

| $\mu$ | 0 | 0.2 | 0.4 | 0.2 | 0.4 |
|-------|---|-----|-----|-----|-----|
|       | $p$ | $J$ | $J$ | $J$ | $J$ |
| 0     | 11.71 | 1 | 1 | 1 | 28.63 | 1 | 44.54 | 1 |
| 0.05  | 8.979 | 1.1460 | 1.1469 | 1.1301 | 1.1122 | 26.49 | 1.1236 | 42.93 | 1.1052 |
| 0.10  | 7.448 | 1.2644 | 1.2647 | 1.2407 | 1.2133 | 25.04 | 1.2333 | 41.59 | 1.2045 |
| 0.15  | 6.311 | 1.3755 | 1.3746 | 1.3496 | 1.3173 | 23.80 | 1.3424 | 40.21 | 1.3085 |
| 0.20  | 5.410 | 1.4826 | 1.4801 | 1.4606 | 1.4284 | 22.62 | 1.4549 | 38.72 | 1.4212 |
| 0.25  | 4.667 | 1.5871 | 1.5828 | 1.5761 | 1.5495 | 21.47 | 1.5732 | 37.09 | 1.5459 |
| 0.30  | 4.041 | 1.6898 | 1.6835 | 1.6981 | 1.6836 | 20.30 | 1.6994 | 35.30 | 1.6858 |
| 0.35  | 3.502 | 1.7913 | 1.7829 | 1.8282 | 1.8340 | 19.11 | 1.8355 | 33.37 | 1.8445 |
| 0.40  | 3.033 | 1.8918 | 1.8810 | 1.9684 | 2.0047 | 17.88 | 1.9836 | 31.29 | 2.0265 |
| 0.45  | 2.619 | 1.9915 | 1.9783 | 2.1208 | 2.2009 | 16.61 | 2.1459 | 29.08 | 2.2375 |
| 0.50  | 2.252 | 2.0906 | 2.0773 | 2.2878 | 2.4290 | 15.29 | 2.3254 | 26.75 | 2.4849 |
| 0.55  | 1.923 | 2.1892 | 2.1709 | 2.4723 | 2.6981 | 13.93 | 2.5252 | 24.32 | 2.7783 |
| 0.60  | 1.627 | 2.2873 | 2.2665 | 2.6778 | 3.0202 | 12.52 | 2.7493 | 21.79 | 3.1315 |
| 0.65  | 1.358 | 2.3851 | 2.3616 | 2.9088 | 3.4128 | 11.08 | 3.0029 | 19.19 | 3.5635 |
| 0.70  | 1.112 | 2.4826 | 2.4564 | 3.1709 | 3.9017 | 9.588 | 3.2920 | 16.52 | 4.1023 |
| 0.75  | 0.888 | 2.5798 | 2.5508 | 3.4712 | 4.5267 | 8.062 | 3.6249 | 13.81 | 4.7909 |
| 0.80  | 0.682 | 2.6768 | 2.6450 | 3.8193 | 5.3529 | 6.503 | 4.0122 | 11.07 | 5.6988 |
| 0.85  | 0.492 | 2.7736 | 2.7389 | 4.2281 | 6.4945 | 4.913 | 4.4680 | 8.302 | 6.9457 |
| 0.90  | 0.316 | 2.8703 | 2.8327 | 4.7151 | 8.1718 | 3.297 | 5.0119 | 5.528 | 8.7577 |
| 0.95  | 0.152 | 2.9667 | 2.9263 | 5.3059 | 10.872 | 1.658 | 5.6714 | 2.758 | 11.619 |
| 1     | 0     | 3.0631 | 3.0197 | 6.0376 | 15.932 | 0     | 6.4868 | 0     | 16.786 |