Werner-like States and Strategic Form of Quantum Games

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Abstract

We quantize prisoners dilemma, chicken game and battle of sexes to explore the effect of quantization on their strategic form. The games start with Werner-like state as an initial state. We show that for the measurement in entangled basis the strategic forms of these games remain unaffected by quantization. On the other hand when measurement is performed in product basis then these games could not retain their strategic forms.
In game theoretic situations two or more rational players compete to maximize their payoffs by suitable choice of available strategies \([1, 2]\). The set of strategies from which unilateral deviation of any player reduces his/ her payoff is called the Nash Equilibrium (NE) of the game \([3]\). In its normal form a game is represented by a payoff matrix. In order to obtain the strategic form of a game certain constraints are imposed on the elements of its payoff matrix. Prisoner dilemma game (PD), for example, is the story of interrogation of two arrested suspects, Alice and Bob, who have allegedly committed a crime together. Each of the prisoners have to decide whether to confess the crime (to defect \(D\)) or to deny the crime (to cooperate \(C\)) without any communication between them. According to payoff matrix \([1]\), if both players receive \(R\) and \(U\) for mutual cooperation and mutual defection respectively; and a cooperator and defector engaged in a contest against each other receive \(S\) and \(T\) respectively; then the strategic form of PD demands that \(T > R > U > S\) \([4, 5]\). Due these constraints PD takes a form where the rational reasoning forces each player to defect. As a result \(DD\) appears as a NE of the game with a small payoff \(U\) for each player. This NE is not Pareto optimal because the players could have obtained better payoff \(R\) by playing \(C\). This is referred to as the dilemma of this game.

Chicken game (CG) is another interesting example in this regard. It depicts a situation in which two players drive their cars straight towards each other. The first to swerve to avoid the collision (to cooperate \(C\)) is the loser (chicken) and the one who keeps on driving straight (to defect \(D\)) is the winner. By assigning \(R\) and \(U\) to mutual cooperation and defection respectively; \(S\) and \(T\) to a cooperator and a defector against each other then the strategic form of CG requires the constraints on the elements of payoff matrix as \(T > R > S > U\), see payoff matrix \([1]\). Certainly if both players cooperate they can avoid a crash and none of them will be winner. If one of them steers away (defects \(D\)) he will be loser but will survive but the opponent will receive the entire honor. If they crash then the cost of both of them will be higher than the cost of being chicken and the payoff will be lower \([4]\). There is no dominant strategy and \(CD, DC\) are two NE in this game. The dilemma of this game is that \(CC\) which is Pareto optimal is not a NE.

The payoff matrix for the Battle of Sexes (BoS) game is of the form \([2]\). In the usual exposition of this game the players Alice and Bob are trying to decide a place to spend Saturday evening. Alice wants to attend Opera while Bob is interested in watching TV at home and both would prefer to spend the evening together. If \(O\) and \(T\) represent Opera
and TV respectively and both players receive $\alpha$ and $\beta$ for playing $O$ and $T$ respectively. They obtain $\sigma$ for strategy pairs $(O,T)$ and $(T,O)$. The constraint imposed on the element of this game is $\alpha > \beta > \sigma$. There exist two NE $(O,O)$ and $(T,T)$ in the classical form of the game. In absence of any communication between Alice and Bob, there exists a dilemma as NE $(O,O)$ suits Alice whereas Bob prefers $(T,T)$. As a result both players could end up with worst payoff $\sigma$ in case they play mismatched strategies.

The analysis of games in quantum domain helped in resolving such dilemmas. One of the elegant and foremost step in this direction was by Eisert et al [6] to remove dilemma in PD. In this quantization scheme the strategy space of the players is a two parameter set of $2 \times 2$ unitary operators. Starting with maximally entangled initial quantum state the authors showed that for a suitable quantum strategy the dilemma disappears from the game. The quantum strategy pair $Q \otimes Q$ appears as a NE with payoffs $R$ for both players and is Pareto optimal. They also pointed out that the quantum strategy $Q$ always wins over all classical strategies. Eisert et al [7] also showed that $Q \otimes Q$ is a unique NE in CG and is Pareto optimal. An experimental demonstration of this quantization scheme for PD has been achieved on a two qubit nuclear magnetic resonance (NMR) computer with full range of entanglement parameter $\gamma$ ranging from $0$ to $\frac{\pi}{2}$. It is interesting to note that these results are in good agreement with theory. Such a type of demonstration has also been proposed on the optical computer [10]. Some other interesting issues that have been analyzed using this quantization scheme are, the proof of quantum Nash equilibrium theorem [11], evolutionarily stable strategies (ESS) [13], quantum verses classical player [14–16], the difference between classical and quantum correlations [17–19] and the model of decoherence in the quantum games [23, 24]. Eisert et al scheme can easily be implemented to all kinds of $2 \times 2$ games. A possible classification of $2 \times 2$ games has also been given by Huertas-Rosero [25]. Later on, Marinatto and Weber [8] introduced another interesting and simple scheme for the quantization of non-zero sum games. They gave Hilbert structure to the strategic spaces of the players. They also used the maximally entangled initial state and allowed the players to play their tactics by applying probabilistic choices of unitary operators. Applying their scheme to Battle of Sexes game they found the strategy for which both the players have equal payoffs. Marinatto and Weber quantization scheme gave very interesting results while investigating evolutionarily stable strategies (ESS) [13, 20, 37] and in the analysis of repeated games [21] etc.
In our earlier work we quantized PD and CG to explore the role of quantum discord in quantum games [38]. To establish this connection we use Werner-like state as an initial state of the game. We showed that the dilemma in both PD and CG can be resolved by separable states with non-zero quantum discord. Recently we find that the strategic form of quantized PD depends upon entanglement of initial quantum state as well as on the type of measurement basis (entangled or product) [45]. For both type of measurements there exist respective cutoff values of entanglement of initial quantum state up to which strategic form of game remains intact. Beyond these cutoffs the quantized PD behaves like chicken game up to another cutoff value. Here we show if a quantum game starts with Werner-like state as an initial quantum state then the strategic form of quantized game remains intact. Quantizing PD, CG and BoS we show that when measurement is performed in entangled basis the strategic form of quantized games is unaffected. However when the measurement is performed in product basis then PD, CG and BoS lose their strategic form after quantization. This result high lights the fact that despite being nonlocal, for certain range of parameter $p$, when shared between two parties then Werner states behave as a powerful resource in comparison to classical randomness [42].

\[
\begin{align*}
\text{Bob} & \\
& \\
& \\
& \\
& \begin{array}{c} C \\
D \end{array} \\
C & \begin{array}{c} (R,R) \\
(S,T) \end{array} \\
D & \begin{array}{c} (T,S) \\
(U,U) \end{array}
\end{align*}
\]

(1)

Matrix 1: The constraints for PD are $T > R > U > S$
and for $T > R > S > U$ for CG.
\begin{align*}
\text{Bob} & \\
O & T \\
\text{Alice} & \begin{bmatrix} (\alpha, \beta) & (\sigma, \sigma) \\ (\sigma, \sigma) & (\beta, \alpha) \end{bmatrix}
\end{align*}

(2)

Matrix 2: For BoS it is required that \( \alpha > \beta > \sigma. \)

Before investigating the role of Werner like states in quantum games we present a brief introduction to these states following Refs. [33–36]. Werner states are linear combination of a maximally entangled and a maximally mixed state. Their entanglement and nonlocality depends upon a parameter \( p \) with values lying in the range \( 0 \leq p \leq 1. \) For \( 0 < p \leq \frac{1}{3} \) they are separable, for \( \frac{1}{3} < p \leq \frac{1}{\sqrt{2}} \) entangled but not nonlocal and for the range \( \frac{1}{\sqrt{2}} < p < 1 \) they become inseparable and nonlocal [33]. This behavior is in contrast with their well known separability at \( p \leq \frac{1}{3}. \)

A two-qubit Werner like state is of the form

\[ \rho_{in} = p |\phi^+\rangle \langle \phi^+| + \frac{(1-p)}{4} I \otimes I \]

(3)

where \( |\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \) is standard Bell state.

Next we quantize a game with a general payoff matrix given by

\begin{align*}
\text{Bob} & \\
B_1 & B_1 \\
\text{Alice} & \begin{bmatrix} \langle 00 | & \langle 01 | & \langle 10 | & \langle 11 | \\ \langle 00 | & \langle 01 | & \langle 10 | & \langle 11 | \end{bmatrix}
\end{align*}

(4)

using Werner like state (3) as an initial quantum state. The strategy of each of the players is represented by the unitary operator \( U_i \) given as

\[ U_i = \cos \frac{\theta_i}{2} R_i + \sin \frac{\theta_i}{2} C_i, \]

(5)

where \( i = 1 \) or 2 and \( R_i, C_i \) are the unitary operators defined as

\begin{align*}
R_i |0\rangle &= e^{i\phi_i} |0\rangle, \quad R_i |1\rangle = e^{-i\phi_i} |1\rangle, \\
C_i |0\rangle &= - |1\rangle, \quad C_i |1\rangle = |0\rangle.
\end{align*}

(6)
Here we restrict our treatment to two parameter set of strategies for mathematical simplicity in accordance with Ref. [6]. After the application of the strategies, the initial state given by Eq. (3) transforms into

$$\rho_f = (U_1 \otimes U_2) \rho_{in} (U_1 \otimes U_2)^\dagger. \quad (7)$$

The payoff operators for Alice and Bob are

$$P^A = s^A_{00} P_{00} + s^A_{11} P_{11} + s^A_{01} P_{01} + s^A_{10} P_{10},$$

$$P^B = s^B_{00} P_{00} + s^B_{11} P_{11} + s^B_{01} P_{01} + s^B_{10} P_{10}, \quad (8)$$

where

$$P_{00} = |\psi_{00}\rangle \langle \psi_{00}|, \quad |\psi_{00}\rangle = \cos \frac{\delta}{2} |00\rangle + i \sin \frac{\delta}{2} |11\rangle, \quad (9a)$$

$$P_{11} = |\psi_{11}\rangle \langle \psi_{11}|, \quad |\psi_{11}\rangle = \cos \frac{\delta}{2} |11\rangle + i \sin \frac{\delta}{2} |00\rangle, \quad (9b)$$

$$P_{10} = |\psi_{10}\rangle \langle \psi_{10}|, \quad |\psi_{10}\rangle = \cos \frac{\delta}{2} |10\rangle - i \sin \frac{\delta}{2} |01\rangle, \quad (9c)$$

$$P_{01} = |\psi_{01}\rangle \langle \psi_{01}|, \quad |\psi_{01}\rangle = \cos \frac{\delta}{2} |01\rangle - i \sin \frac{\delta}{2} |10\rangle, \quad (9d)$$

with $\delta \in [0, \frac{\pi}{2}]$ being the entanglement of the measurement basis. Above payoff operators reduce to that of Eisert’s scheme for $\delta$ equal to $\gamma$, which represents the entanglement of the initial state [6]. For $\delta = 0$ above operators transform into that of Marinatto and Weber’s scheme [8]. The payoffs for the players are calculated as

$$s_j(\theta_1, \phi_1, \theta_2, \phi_2) = \text{Tr}(P^A \rho_f),$$

$$s_j(\theta_1, \phi_1, \theta_2, \phi_2) = \text{Tr}(P^B \rho_f), \quad (10)$$

where Tr represents the trace of a matrix. Using Eqs. (7), (8) and (10) the payoffs for players $j = A, B$ are obtained as

$$s_j(\theta_1, \phi_1, \theta_2, \phi_2) = s^j_{00} \text{Tr}(P_{00}\rho_f) + s^j_{01} \text{Tr}(P_{01}\rho_f) + s^j_{10} \text{Tr}(P_{10}\rho_f) + s^j_{11} \text{Tr}(P_{11}\rho_f) \quad (11)$$
where we have defined

\[
\text{Tr}(P_{00}\rho_f) = p \left[ \frac{1}{2} \sin^2 (\phi_1 + \phi_2) \sin \delta \right] \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \\
\frac{(\sin \delta - 1)}{2} \left\{ \cos^2 \frac{\theta_1}{2} + \cos^2 \frac{\theta_2}{2} - \frac{1}{2} \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) \right\} - \\
\frac{\sin \delta}{2} + 1 + \frac{p}{4}
\]

(12a)

\[
\text{Tr}(P_{01}\rho_f) = p \left[ \frac{1 + \cos 2\phi_1 \sin \delta}{2} \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \frac{1 - \cos 2\phi_2 \sin \delta}{2} \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \\
\frac{(-1 + \sin \delta) \sin \phi_1 \cos \phi_2 - (1 + \sin \delta) \cos \phi_1 \sin \phi_2}{4} \sin \theta_1 \sin \theta_2 \right] + 1 - \frac{p}{4}
\]

(13)

\[
\text{Tr}(P_{10}\rho_f) = p \left[ \frac{1 - \cos 2\phi_1 \sin \delta}{2} \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \frac{1 + \cos 2\phi_2 \sin \delta}{2} \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} - \\
\frac{(1 + \sin \delta) \sin \phi_1 \cos \phi_2 + (1 - \sin \delta) \cos \phi_1 \sin \phi_2}{4} \sin \theta_1 \sin \theta_2 \right] + 1 - \frac{p}{4}
\]

(14)

\[
\text{Tr}(P_{11}\rho_f) = p \left[ \left\{ \frac{1 - \cos^2 (\phi_1 + \phi_2) \sin \delta}{2} \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \\
\frac{(\sin \delta + 1)}{2} \left\{ \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \frac{1}{2} \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) \right\} \right\} + \\
\frac{1 - p}{4}
\]

(15)

In the framework of our generalized quantization scheme [43] measurement can be performed either using entangled basis ($\delta = \frac{\pi}{2}$) or product basis ($\delta = 0$). Next we discuss both these cases one by one.
Case 1: Measurement in entangled basis

For the measurement in entangled basis with the help of Eq. (11) the payoffs for players become

\[
\begin{align*}
\mathcal{S}_j(\theta_1, \phi_1, \theta_2, \phi_2) &= p \left[ \mathcal{S}_{00}^j \left( \cos^2 (\phi_1 + \phi_2) \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right) + \right. \\
&\quad \left. \mathcal{S}_{01}^j \left( \cos \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \sin \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 + \right. \\
&\quad \left. \mathcal{S}_{10}^j \left( \sin \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \cos \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 + \right. \\
&\quad \left. \mathcal{S}_{11}^j \left( \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin (\phi_1 + \phi_2) + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)^2 \right] \\
&\quad + \left( 1 - p \right) \frac{9}{4} \left( \mathcal{S}_{00}^j + \mathcal{S}_{01}^j + \mathcal{S}_{10}^j + \mathcal{S}_{11}^j \right)
\end{align*}
\]

For PD with payoff matrix elements \( \mathcal{A}_{00}^A = \mathcal{B}_{00}^B = 3, \mathcal{A}_{01}^A = \mathcal{B}_{10}^B = 0, \mathcal{A}_{10}^A = \mathcal{B}_{01}^B = 5 \) and \( \mathcal{A}_{11}^A = \mathcal{B}_{11}^B = 1 \) the above equation reduces to

\[
\begin{align*}
\mathcal{A}_j(\theta_1, \phi_1, \theta_2, \phi_2) &= p \left[ 3 \left( \cos^2 (\phi_1 + \phi_2) \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right) + \right. \\
&\quad \left. 5 \left( \sin \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \cos \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 + \right. \\
&\quad \left. \left( \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin (\phi_1 + \phi_2) + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)^2 \right] \\
&\quad + \frac{9}{4} (1 - p)
\end{align*}
\]

\[
\begin{align*}
\mathcal{B}_j(\theta_1, \phi_1, \theta_2, \phi_2) &= p \left[ 3 \left( \cos^2 (\phi_1 + \phi_2) \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right) + \right. \\
&\quad \left. 5 \left( \cos \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \sin \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 + \right. \\
&\quad \left. \left( \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin (\phi_1 + \phi_2) + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)^2 \right] \\
&\quad + \frac{9}{4} (1 - p)
\end{align*}
\]

For \( p = 1 \) these results reduce to that of Eisert et al. \( \text{[6]} \) and the dilemma in game is resolved for players strategies \( U(\theta_1, \phi_1, \theta_2, \phi_2) = U(0, \frac{\pi}{2}, 0, \frac{\pi}{2}) = Q \) with \( \mathcal{A}_0(0, \frac{\pi}{2}, 0, \frac{\pi}{2}) = \mathcal{B}_0(0, \frac{\pi}{2}, 0, \frac{\pi}{2}) = (3, 3) \). Next we investigate whether the strategy \( Q \) is NE for \( p \neq 1 \). Then
the NE conditions

$$A_0\left(0, \frac{\pi}{2}, 0, \frac{\pi}{2}\right) - A_1\left(\theta_1, \phi_1, 0, \frac{\pi}{2}\right) \geq 0$$

$$B_0\left(0, \frac{\pi}{2}, 0, \frac{\pi}{2}\right) - B_1\left(0, \frac{\pi}{2}, \theta_2, \phi_2\right) \geq 0$$

(19)

give

$$p \left(3 \sin^2 \frac{\theta_1}{2} + 2 \cos^2 \frac{\theta_1}{2} \cos \phi_1\right) \geq 0.$$  

(20)

The above inequality is satisfied for all values of \( p \geq 0 \) showing that the strategy pair \((Q, Q)\) continues to be NE for all values of \( p > 0 \). It shows that although state \((3)\) is not entangled for \( p \leq \frac{1}{3} \) yet when shared between two players it is proved to be a better resource compared to classical randomness. On the other hand at \( p = 0 \) when the initial state becomes maximally mixed the payoffs become \( \frac{9}{4} \) irrespective of players strategies.

Now we investigate whether the quantized PD with \( Q \otimes Q \) as NE has the strategic form like that of PD. Using Eqs. (17, 18) the elements of payoff matrix of quantized PD are

$$R = \frac{3}{4}p + \frac{9}{4}, \; S = \frac{9}{4} - \frac{9}{4}p, \; T = \frac{11}{4}p + \frac{9}{4}, \; U = \frac{9}{4} - \frac{5}{4}p.$$  

(21)

It is easy to see that these payoff elements obey the constraints \( T > R > U > S \) for all values of \( p > 0 \). Therefore we conclude that the strategic form of PD remains unaffected by quantization if it starts with Werner-like state as an initial quantum state. The payoff elements (21) are shown in figure (1) which shows that the payoffs elements obey the constraints required by PD.

For CG with payoff matrix elements \( A_{00} = B_{00} = 3, A_{01} = B_{10} = 1, A_{10} = B_{01} = 4 \) and \( A_{11} = B_{11} = 0 \) the payoffs given in Eq. (16) become

$$A_1(\theta_1, \phi_1, \theta_2, \phi_2) = p \left[ 3 \left( \cos^2 (\phi_1 + \phi_2) \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right) + \left( \cos \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \sin \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 + 4 \left( \sin \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \cos \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 \right] + 2(1 - p)$$  

(22)
FIG. 1: The payoff elements $R, S, T$ and $U$ versus $p$. It shows that the constraints required to maintain the strategic form of PD are satisfied for all values of $p > 0$.

\[
B(\theta_1, \phi_1, \theta_2, \phi_2) = p \left[ 3 \left( \cos^2 (\phi_1 + \phi_2) \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right) + \left( \sin \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \cos \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 + 4 \left( \cos \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \sin \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 \right] + 2 (1 - p) \]  
(23)

With the help of Eqs. (19) the strategy pair $U(\theta_1, \phi_1, \theta_2, \phi_2) = U(0, \pi/2, 0, \pi/2)$ will be NE of this game if

\[
p \left[ 2 + \cos^2 \frac{\theta_1}{2} (3 \cos^2 \phi_1 - 2) \right] \geq 0. \]  
(24)

The above condition is satisfied for all values of $p \geq 0$. It means that dilemma can be resolved in CG when the players share the state (3) with $p > 0$. Furthermore it can be checked by Eqs. (22, 23) that for $p = 0$ the payoffs of the players become 2, independent of players decisions.

Now we investigate the strategic form of quantized CG with $Q \otimes Q$ as a NE. Using Eqs. (22, 23) the elements of payoff matrix of quantized CG become

\[
R = p + 2, \ S = 2 - p, \ T = 2p + 2, \ U = 2 - 2p. \]  
(25)

It is evident that these payoff elements obey the constraints $T > R > S > U$ required for a game to behave like CG for all values of $p > 0$. We plot these payoff elements in figure.
which shows that the strategic form of CG is not affected by quantization when it starts with an initial state of the form of Werner-like states.

FIG. 2: The payoff elements $R, S, T$ and $U$ versus $p$. It shows that the constraints required to maintain the strategic form of CG are satisfied for all values of $p > 0$.

For BoS using payoff matrix elements $\alpha = 2, \beta = 1$ and $\gamma = 0$ the payoffs given in Eq. (16) become

$$
A(\theta_1, \phi_1, \theta_2, \phi_2) = p \left[ 2 \left( \cos^2(\phi_1 + \phi_2) \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right) + \left( \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin(\phi_1 + \phi_2) + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)^2 \right] + \frac{3(1-p)}{4}$$

(26)

$$
B(\theta_1, \phi_1, \theta_2, \phi_2) = p \left[ \left( \cos^2(\phi_1 + \phi_2) \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right) + 2 \left( \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin(\phi_1 + \phi_2) + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)^2 \right] + \frac{3(1-p)}{4}

(27)

With the help of the above payoffs we see that

$$
\alpha = \frac{5}{4}p + \frac{3}{4}, \beta = \frac{1}{4}p + \frac{3}{4}, \sigma = \frac{3}{4} - \frac{3}{4}p

(28)

These payoff elements obey the constraints required by a game to behave like BoS for all values of $p > 0$. We plot the payoff elements in figure (??) below. It is clear that after
quantization the strategic form of BoS remain unaffected if it starts with a Werner-like initial quantum state.

![Graph showing payoffs α, β and γ versus p.](image)

FIG. 3: The payoff elements α, β and γ versus p. It shows that the constraints required to maintain the strategic form of BoS are satisfied for all values of \( p > 0 \).

It shows that for quantized versions of PD, CG and BoS the strategic forms of the games remain unaffected if the initial quantum state is Werner-like state.

**Case 2: Measurement in product basis**

For the measurement performed in product basis (i.e. for \( \delta = 0 \) in Eqs. (9a to 9d) ) the Eq. (11) reduces to

\[
\mathcal{J}(\theta_1, \phi_1, \theta_2, \phi_2) = \frac{p}{2} \left[ (\mathcal{J}_{00}^j + \mathcal{J}_{11}^j) \left\{ \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \right. \\
+ \frac{1}{2} \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) \right\} + (\mathcal{J}_{01}^j + \mathcal{J}_{10}^j) \left\{ \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \\
+ \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} - \frac{1}{2} \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) \right\} \right]
+ (1 - p) \left( \frac{1}{4} (\mathcal{J}_{00}^j + \mathcal{J}_{01}^j + \mathcal{J}_{10}^j + \mathcal{J}_{11}^j) \right)
\]  

(29)

For \( p > 0 \) the above payoffs remain equivalent to the payoffs obtained by Marinatto and Weber’s quantization scheme where the players also have the option to manipulate the phase \( \phi \) of the given qubit [8, 44]. However at \( p = 0 \) when the quantum discord disappears the payoffs given by Eq. (29) become average value of the entries of payoff matrix (4).

For PD with payoff matrix elements \( \mathcal{J}_{00}^A = \mathcal{J}_{00}^B = 3, \mathcal{J}_{01}^A = \mathcal{J}_{10}^B = 0, \mathcal{J}_{10}^A = \mathcal{J}_{01}^B = 5 \) and
$A_{11} = B_{11} = 1$ the payoff given in Eq. (29) become
\[ R = U = \frac{9}{4} - \frac{p}{4}, \quad S = T = \frac{p}{4} + \frac{9}{4}. \] (30)

For CG with payoff matrix elements $A_{00} = B_{00} = 3, A_{01} = B_{10} = 1, A_{10} = B_{01} = 4$ and $A_{11} = B_{11} = 0$ using Eq. (29) we get
\[ R = U = 2 - \frac{p}{2}, \quad S = T = \frac{p}{2} + 2. \] (31)

Similarly for BoS with payoff matrix elements $\alpha = 2, \beta = 1$ and $\gamma = 0$ the elements of the quantized payoff matrix become
\[ \alpha = \beta = \frac{3}{4}p + \frac{3}{4}, \gamma = \frac{3}{4} - \frac{3}{4}p. \] (32)

For all three games it is easy to check that when the measurement is performed in product basis then the strategic form of the game never remains the same.

In summary we quantized PD, CG and BoS taking Werner-like state as an initial quantum state to explore the strategic form of their quantized versions. We performed measurements in entangled and product basis. For the measurement in entangled basis we showed that the strategic form of quantized PD, CG and BoS remains intact for all values of $p > 0$. This highlights the fact that despite being nonlocal, for certain range of parameter $p$, when shared between two parties these states are a powerful resource in comparison to classical randomness [42]. On the other hand when measurement is performed in product basis then the strategic form of all the three games does not remain intact.

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