Remote Time Manipulation

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Harnessing the flow of proper time of arbitrary external systems over which we exert little or no control has been a recurring theme in both science and science-fiction. Unfortunately, all relativistic schemes to achieve this effect beyond mere time dilation are utterly unrealistic. In this work, we find that there exist non-relativistic scattering experiments which, if successful, freeze out, speed up or even reverse the free dynamics of any ensemble of quantum systems present in the scattering region. This “time warping” effect is universal, i.e., it is independent of the particular interaction between the scattering particles and the target systems, or the (possibly non-Hermitian) Hamiltonian governing the evolution of the latter. The protocols require careful preparation of the probes which are scattered, and success is heralded by projective measurements of these probes at the conclusion of the experiment. We fully characterize the possible time translations which we can effect on $n$ target systems through a scattering protocol of fixed duration; the core result is that time can be freely distributed between the systems, and reversed at a small cost. For high $n$, our protocols allow one to quickly send a single system to its far future or past.

Since the advent of the special theory of relativity, our modern understanding of time in Physics has been “that which clocks measure” [1]. With this conception in mind, the relativistic effect by which two clocks lose synchrony if one of them is translated through space is commonly regarded as a “time dilation”. What makes relativistic time dilation extraordinary is the fact that, in order to make one of the clocks tick more slowly, we do not need to be granted control of its inner workings: it suffices to push it. The ability to manipulate the proper time of a physical system in such a high level way, a phenomenon known as time warp, has inspired numerous works of science fiction (see, e.g., [2]).

There have been several proposals to achieve command of the proper time of arbitrary external systems, all of them based on special or general relativity. Most of them rely on the existence of natural “time machines” [3, 4]. More realistic schemes, like the time translator of Aharonov et al. [5], while theoretically feasible, would operate under an astronomically small probability of success.

To address this issue, in this paper we propose a class of non-relativistic scattering experiments with the property to warp the time of any number of physical systems placed within the scattering region. In these experiments, a number of particles are produced, let to propagate freely and subsequently measured after some time $T'$. Depending on the outcome of this measurement, the experiment is regarded as either a “success” or a “failure”. If there is nothing in the scattering region, the experiment always fails. However, if the scattering region holds $n$ identical quantum systems of a given dimensionality $d$ and the experiment succeeds, then each system $i$ will leap to the quantum state it would have had if it had been evolving unperturbed for time $T_i \neq T'$, where $T_i$ can be negative. The experiment does not rely on any knowledge on the Hamiltonian of the target systems or their interaction with the scattered particles. Since they effect a high-level manipulation of the proper time of each system in the scattering region, such prepare-and-measure protocols can be regarded as a non-relativistic form of time warp.

We find that, in this scenario, evolution time behaves as a material resource, like gold, in the sense that it cannot be created, but it can be transferred for free between identical systems. Hence, with a scattering experiment of duration $T'$ we can transfer all the evolution time accumulated by the $n$ systems to a single system, “fast-forwarding” the latter $nT'$ time units to its future. Time can also be inverted, at a cost $d - 1$. Combining the two approaches, we can invert such an aggregated time, thus projecting that same system $nT'/d$ time units to its past. By taking higher values of $n$, we can make these time warping effects increasingly dramatic.

For $n = 1$ our results resonate with those of [6], where one of the authors showed that a similar experimental setup allows making a single uncontrolled system leap to the state it had $T$ time units before the experiment started. In this regard, our present work shows that a single system can be rewinded to its past much faster than the protocols introduced in [6] allowed. Note that there exist other methods to invert an unknown unitary [7–10], but they demand the ability to effect controlled quantum operations on the target system.

Before we proceed to present our results, a note on scope is in order. In this work, we will be mainly concerned with the theoretical feasibility of the scattering experiments referred to above. That is, we will be content with devising time warping protocols with a non-zero probability of success, leaving for future work legitimate concerns such as reasonably high success rates or simple experimental implementations. For the simplest scenarios we will nonetheless present some examples of scattering experiments whose implementation is within the reach of state-of-the-art or near-future quantum technologies.
The model

We consider a scenario where the experimental setup consists in two parts: a controlled lab, where we can prepare any quantum state and conduct any quantum operation; and a scattering region. The latter contains \( n \) identical physical target systems of Hilbert space dimension \( d \) at separate locations, see Fig. [1]. We assume that they remain in the same place during the course of the experiment. The initial (internal) quantum state of the \( n \) systems is unknown; for simplicity, we will take it pure and denote it by \( |\psi_{1,...,n}\rangle \).

If left unperturbed, each of these systems will independently evolve according to a (unknown) time-independent Hamiltonian \( H_0 \). That is, after time \( T \) the state of the \( n \) systems will evolve to \( e^{-i\sum_{k=1}^{n} H_0(k)T} |\psi_{1,...,n}\rangle \). To incorporate decay processes in this framework, we allow \( H_0 \) to be non-Hermitian.

In order to influence systems \( k = 1,...,n \), we can prepare a particle in the controlled lab and let it propagate within the scattering region. While in the scattering region, these particles or probes interact with each system \( k \) via the non-Hermitian Hamiltonian \( H_I(k)\bar{q}_k \), where \( \bar{r} \) (\( \bar{q}_k \)) denotes the probe’s (system \( k ' s \) position. The joint state of systems \( 1,...,n \) and \( P \) thus evolves as

\[
\frac{\partial}{\partial t} |\psi_{1,...,n,P}\rangle = H_P + \sum_{k=1}^{n} H_0^{(k)} + H_I^{(k,P)} |\psi_{1,...,n,P}\rangle,
\]

where \( H_P \) denotes the free Hamiltonian of the probe. For technical reasons, \( H_I \) is assumed a bounded operator; otherwise \( H_I, H_0 \) and \( H_P \) are arbitrary and unknown.

We will allow multiple probes at a time within the scattering region. Such probes could, in principle, interact with each other as well as with the uncontrolled systems. Nonetheless, we will demand yet another condition on our experimental setup, that we will call the targeting assumption. Namely, we postulate that it is possible to prepare a probe in such a way that it interacts with a single uncontrolled system (its target) and nothing else. Such a probe will either return to the lab within a given time \( \Delta t \) and through a given channel or else be absorbed by the environment or lost in free space. Moreover, if several probes with different targets are prepared simultaneously, then the evolution of each probe and its target will be independent and identical among the different pairs of probe-uncontrolled system. Meanwhile, those uncontrolled systems without a targeting probe will keep evolving through \( H_0 \).

Note that the targeting assumption can be justified in many experimental setups where the uncontrolled systems are sufficiently separated in space. Throughout the text, we will further assume that \( \Delta t \) can be taken arbitrarily small.

In our protocols, we will allow each probe to enter the scattering region and return to the controlled lab a number of times. While in the lab, we are allowed to interact with the probe in any controlled way. We can, e.g., entangle it with a quantum memory inside the lab, or with any other probe that happens to be within the lab at the same time. Note that, except for the presence of several uncontrolled systems/probes in the scattering region, this is very similar to the scenario considered in \([4]\).

The scattering protocol ends at time \( t = T' \), when we conduct a dichotomic heralding measurement over the quantum registers present in the lab. If the outcome is “success”, we expect the state of systems \( 1,...,n \) to be

\[
|\psi_{1,...,n}'\rangle = U(T_1,...,T_n) |\psi_{1,...,n}\rangle,
\]

with \( U(T_1,...,T_n) = \bigotimes_{k=1}^{n} e^{-iH_0T_k} \).

We will be mainly interested in whether such a time-warping experiment is possible, disregarding its actual probability of success: we just demand that the latter is non-zero for generic \( H_0, H_I, H_P \). Note, e.g., that, if the probes do not interact with the uncontrolled systems at all, then Eq.\([2]\) can only hold if \( T_1 = T_2 = ... = T_n = T' \).

Fundamental restrictions of time warping

In the Supplementary Material, we prove that, in the above scenario, a scattering experiment of duration \( T' \) leading to Eq.\([2]\) can only be possible if
\[
\sum_{i: T_i > 0} T_i + \sum_{i: T_i < 0} |T_i|(d-1) \leq n T'.
\] (3)

Let us analyze the significance of this equation. For \( n = 1 \), it implies that \( T_1 \in [-\frac{2 T'}{d-1}, T'] \). This means that, in principle, we could invert the evolution of the uncontrolled system in the scattering region. Scattering protocols to achieve this “resetting effect” were already provided in [6]. However, they required an experiment of duration \( T' = O(d^2 |T_1|) \), as opposed to \( T' = (d-1)|T_1| \).

The latter bound is consistent with the work of [10], where the authors prove that, in order to invert a unitary transformation, the uncontrolled system can be inverted at a cost \( (d - 1)\). However, they required an experiment of duration \( T' = O(d) \).

We will first study the case \( n = 1 \). We start by dividing the evolution of the system into two parts: a) evolution of the probe, and b) evolution of the rest of the system. The crucial question is whether any rate \( nT' \) is compatible with Eq. (3) can be achieved in practice. We will see next that this is the case asymptotically. That is, \( \sum_{i: T_i > 0} T_i + \sum_{i: T_i < 0} |T_i|(d-1) \) can be achieved in time \( T' + \epsilon \), where \( \epsilon \) can be made arbitrarily small.

### The protocols

The protocols we will study are:

1. **Controlled set**: We start by dividing the probe's Hilbert space into the factors \( \mathcal{H}_p, \mathcal{H}_r \), where \( \mathcal{H}_p \) denotes the probe's internal degree of freedom; and \( \mathcal{H}_r = L^2(\mathbb{R}^3) \) is used to model the position of the probe's center of mass. Let \( \{|i\}^d_p \) be an orthonormal basis for \( \mathcal{H}_p \), and let \( R \) denote a qubit register within the lab.

2. **Scattering protocol**: We will prepare the probe in a superposition of states, one inside and another one outside the lab, controlled by the qubit register \( R \). The utility is that, we prepare our first probe in the state \( \frac{1}{\sqrt{2}}(|\varphi\rangle_R + |\varphi\rangle_R) \), where \( |\varphi\rangle \) is the state, allowed to the probing target assumption, that allows the probe to interact with system 1 or else be absorbed. |\varphi\rangle \) is some bounded state within the lab.

3. **The world line marked by state \( |1\rangle_R \) hence propagates through the scattering region, interacting with system 1, initially in state \( |\psi\rangle_1 \), until it re-enters the lab after time \( \Delta t \). When the probe re-enters the lab, its spatial degree of freedom is projected onto the state \( |\varphi\rangle_1 \), which we subsequently transform to the state |\varphi\rangle_1 \). The final (unnormalized) joint state of system 1 and the lab is thus

\[
\frac{1}{\sqrt{2}} \left( \sum_{j=1}^d W_j |\psi\rangle_1 |j\rangle_R + W_0 |\psi\rangle_1 |2\rangle_R |1\rangle_R \right) \otimes |\varphi\rangle_1.
\] (4)

where the \( d \times d \) matrices \( W_0, \ldots, W_d \) are given by

\[
W_0 = e^{-i H_0 \Delta t},
\]

\[
W_j = \langle j |_R \langle \varphi_1 |_R e^{-i (H_0 + H_{pr}) \Delta t} |\varphi_1 \rangle_\mathcal{P}_R, \quad j = 1, \ldots, d.
\] (5)

For the state |\varphi\rangle_1 \) of the probe's center of mass does not play a role once the probe returns to the lab, in the following we will omit it. Similarly, the states of the register and the internal degree of freedom of the probe, can be combined into a single label \( j \): the final state of the joint system can thus be rewritten as

\[
\frac{1}{d} \sum_{j=0}^d W_j |\psi\rangle_1 |j\rangle_1.
\]

Now, suppose that we sequentially send \( m - 1 \) more probes in this fashion. At time \( T' = m \Delta t \), we post-select the lab's degree of freedom to the pure state \( \sum_{j=1}^d g_j |j_1, \ldots, j_m\rangle \). Then the final state of system 1 would be

\[
\sum_{j_1, \ldots, j_m} g_{j_1, \ldots, j_m} W_{j_1} \ldots W_{j_m} |\psi\rangle_1.
\] (6)

Here \( G(W) = \sum \sum g_W \) is an instance of a homogeneous matrix polynomial of degree \( m \). Conversely, for any homogeneous polynomial \( G(W) \) of degree \( m \), we can use the above scheme to make system 1 leap to a state proportional to \( G(W) |\psi\rangle_1 \). Note that there exist other procedures which achieve this same effect. In fact, as we shall see later, some of those have a higher probability of success, or are experimentally preferable to the scheme presented above. Notice as well that, in principle, the index \( j \) of \( W_j \) only varies over \( 0, \ldots, d_p \). However, by sending \( D \) probes one after the other, we get access to matrices of the form \( W_0, \ldots, W_L \). Hence, for any \( L \), any matrix polynomial of the form \( G(W_0, \ldots, W_L) \) can be interpreted as a scattering protocol just by re-scaling \( \Delta t \) to \( D \Delta t \), for \( D \) high enough. This is so even if the physical probes have no internal degrees of freedom at all (\( d_p = 1 \)).

Since the non-commuting variable \( W_0 = e^{-i H_0 \Delta t} \) will play a special role, from now on we will name it \( V \), and...
denote any polynomial of the form \( G(W) \) as \( G(V, \overline{W}) \), with \( \overline{W} = (W_1, ..., W_d) \).

The problem of identifying scattering experiments with the property to reset system 1 to time \( T_1 < 0 \) is thus mapped to that of finding matrix polynomials \( G(V, \overline{W}) \) such that \( G(V, \overline{W}) \propto V^{-s} \), with \( s\Delta t = |T_1| \). To do so, we will borrow an old concept from the theory of matrix algebras.

A central matrix polynomial for dimension \( d \) is a matrix polynomial \( G(X_1, ..., X_d) \) with no constant term with the property that, evaluated on \( d \times d \) matrices, is always proportional to the identity matrix. That is, for any \( A_1, ..., A_L \in M_d \),

\[
G(A_1, ..., A_L) = g(A_1, ..., A_L)I_d, \quad (7)
\]

where the scalar \( g(A_1, ..., A_L) \) is non-zero for some \( d \times d \) matrices \( A_1, ..., A_L \). In \([14]\), it is shown that, for any \( d \), there exists a homogeneous central matrix polynomial \( F_d(X, Y_1, ..., Y_d) \) for dimension \( d \) with degree \( d^2 \) and linear in the variables \( Y_1, ..., Y_d \).

Now, consider the matrix polynomial \( F_d(V, W_1 V^*, ..., W_d V^*) \). This is a central matrix polynomial of the form \( R_d(V, \overline{W})V^* \), where \( R_d \) is a homogeneous polynomial of degree \( d^2 + s(d-1) \). It follows that, when evaluated on \( d \times d \) matrices, \( R_d(V, \overline{W}) \propto V^{-s} \). That is, \( R_d \) corresponds to a resetting protocol of duration \( T' = [d^2 + s(d-1)]\Delta t \) that evolves the uncontrollable system by time \( T_1 = -s\Delta t \). By shortening the duration \( \Delta t \) of the targeting interactions and increasing \( s \) correspondingly, we can make the quotient \( \frac{T'}{T_1} \) as close as we want to \( d-1 \) while keeping \( T \) constant.

One can achieve any value of \( T_1 \) within the interval \( (\frac{T'}{T_1}, T') \) allowed by Eq.\((3)\) just by applying the previous resetting protocol for time \( \alpha T' \), with \( 0 \leq \alpha \leq 1 \) and then letting the system evolve naturally for time \( (1-\alpha)T' \).

Let us now move to the case where \( n \) uncontrollable systems are present in the scattering region. Invoking again the targeting assumption, we can, by playing with superpositions of probes inside and outside the lab, effect over systems 1, ..., \( n \) transformations of the form

\[
|\psi_{1,...,n}'\rangle = G(V, \overline{W})|\psi_{1,...,n}\rangle, \quad (8)
\]

where this time \( G(V, \overline{W}) \) is what we will call a homogeneous tensor matrix polynomial of degree \( m \) in each factor, i.e., a polynomial combining both dot and tensor products, with \( m-1 \) dot products on each of the \( n \) factors. More formally, \( G(X_1, ..., X_L) \) is a tensor matrix polynomial of degree \( m \) in each factor if it can be written as

\[
G(X_1, ..., X_L) = \sum_j g_j X_{j11} ... X_{jm1} \otimes ... \otimes X_{j1m} ... X_{jm}, \quad (9)
\]

for some complex coefficients \( \{g_j\}_J \). Here the sum in \( J \) is taken over \( \{1, ..., L\}^{\times mn} \).

In the Supplementary Material, we show that, for any \( d \), there exists an interesting generalization of central polynomials. These are are permutation polynomials, or tensor polynomial matrices, which evaluated on \( d \times d \) matrices, return a matrix proportional to a fixed linear combination of permutation matrices. In particular, for any permutation \( \pi \) of \( n \) elements there exists a tensor matrix polynomial \( \Omega_d^\pi(X_1, ..., X_d) \) with the property that, when evaluated on \( d \times d \) matrices \( A_1, ..., A_L \),

\[
\Omega_d^\pi(A_1, ..., A_L) \propto \mathcal{P}_\pi. \quad (10)
\]

Here \( \{\mathcal{P}_\pi : \pi \in S_n\} \) denote the permutation operators, defined via \( \mathcal{P}_\pi|i_1\rangle \ldots |i_n\rangle = |i_{\pi(1)}\rangle \ldots |i_{\pi(n)}\rangle \).

This has an immediate consequence for our time-warp protocols, for it allows us to transfer evolution time from some systems into others. For instance, suppose that we wish to transfer evolution time from all systems to system \( j \). If we denote \( V \) acting on factor \( k \) by \( V_k \), then one can verify that the tensor polynomial

\[
E_d^j(V, \overline{W}) = V_j \prod_{k \neq j} \Omega_d^{(j,k)}(\overline{W})V_k \Omega_d^{(j,k)}(\overline{W}), \quad (11)
\]

is homogeneous of degree \( s + O(1) \) in each factor. On the other hand, \( E_d^j(V, \overline{W}) \propto V_j^m \). The corresponding scattering protocol thus evokes system \( j \) by \( T_j = n\Delta t \) in \( T' \approx s\Delta t \) time units, while keeping the rest of the systems frozen. As before, by taking the limit \( \Delta t \to 0 \) while keeping \( s\Delta t \) constant, we can transfer \( n\Delta t \) time units to system \( j \) by means of a protocol of duration \( T' \).

To both transfer and invert time one can use the protocol associated to the homogeneous tensor polynomial

\[
D_d^j(V, \overline{W}) \equiv E_d^j \left( R_d(V, \overline{W}), \overline{W}\right). \quad (12)
\]

By applying the scattering protocols associated to \( D_d^j(V, \overline{W}) \) and \( E_d^k(V, \overline{W}) \) in sequence, we can achieve in time \( T' + O(\Delta t) \) any configuration \( T_1, ..., T_n \) compatible with Eq.\((3)\).

**Examples**

For most interactions \( \overline{W}, V \), the general protocols presented above exhibit a very low probability of success. Therefore, in the following lines we provide a couple of scattering protocols for qubit systems for which the success rate is reasonably high.

Consider the case where we place a single qubit in the scattering area, i.e., \( n = 1, d = 2 \). A famous central polynomial for qubits is \( F(A, B) = [A, B]^2 \) (this can be verified, e.g., by direct parametrization of each \( 2 \times 2 \) matrix \( A, B \)). As we did in the derivation of \( R_d(V, \overline{W}) \), we will use \( F(A, B) \) as a template to devise a new protocol for quantum resetting. Take \( A = V, B = W_1 V^* \). Then we have that
We prepare a photon in the state $\frac{1}{\sqrt{2}}(|p_1⟩-|p_2⟩)\otimes|H⟩$. A photon in path $p_1$ would re-enter the controlled lab just as a photon in path $p_2$ would enter the scattering region. The photon’s path degree of freedom is post-selected to the state $\frac{1}{\sqrt{2}}(|p_1⟩+|p_2⟩)$. Finally, we measure its polarization in the basis $\{|H⟩, |V⟩\}$.

$$I \propto F(W_1V^*, V) = g(V, W_1)V^*, \quad (13)$$

with $g(V, W_1) = [W_1, V]V^*[W_1, V]$. This implies that $g(V, W_1) \propto V^{-s}$, provided that $V, W_1$ are $2 \times 2$ matrices.

To implement this protocol with linear optics, one can use photons as probes and their polarization as the internal degree of freedom identifying horizontal (vertical) polarization with $|g⟩$ ($|v⟩$). To verify the identity, evolution time behaves like a material resource, that happens to correspond to any linear superposition of permutation operators. One consequence of this is that we can devise scattering protocols which permute the quantum states of the uncontrolled systems. We believe that this result is interesting on its own right.

Although at the end of the paper we provided simple instances of scattering protocols with a reasonably high (average) probability of success, our general constructions most likely represent very improbable processes. Future research on time manipulation should focus on devising simple, robust and probable time warp scattering experiments.

Conclusion

In this paper we have characterized how one can probabilistically warp the evolution time of an ensemble of uncontrolled systems of known dimensionality by means of scattering experiments. We have seen that, in such scenarios, evolution time behaves like a material resource, in the sense that it can be transferred and wasted, but not created. It can also be inverted, at a cost, via an irreversible process.

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A. Limits on resetting for $n = 1$

In our scattering scenario, by playing with the preparation and processing of the probes, the most general transformation which we can effect on the initial state $|\psi_1\rangle$ of system 1 is of the form:

$$|\psi_1\rangle\langle\psi_1| \rightarrow \sum_j A_j |\psi_1\rangle\langle\psi_1| A_j^*, \quad (A1)$$

where each $A_j$ is a linear combination of operators of the form

$$\Pi_1 e^{-iH_0 t_1} \Pi_2 e^{-iH_0 t_2} \cdots, \quad (A2)$$

with $\sum t_i = T'$.

The operators $\Pi_k^j$ depend on the Hamiltonians $H_1, H_P$ and the way we process the probes inside the controlled lab.

To see why this is the case, note that, when a probe or collection thereof interacts with the uncontrolled system from time $t = 0$ to $t = t_f$, the Hamiltonian guiding the evolution of the joint system is:

$$H_0 + \tilde{H}_P + \tilde{H}_I, \quad (A3)$$

where $\tilde{H}_P, \tilde{H}_I$ are, respectively, $\sum_{k=1}^{N} H_P^{(k)}$ and $\sum_{j=1}^{N} H_I^{(k)}$. If $H_I$ is bounded and the number of probes $N$ is finite, then we can use the Dyson series to model the evolution of the joint system as $|\psi\rangle \rightarrow \tilde{U}(t_f, 0)|\psi\rangle$, with $\tilde{U}(t_f, 0)$ given by

$$\tilde{U}(t_f, 0) = \sum_{j=0}^{\infty} (-i)^j \int \text{d}t_1 \cdots \text{d}t_j e^{-\frac{i}{\hbar}(\tilde{H}_P + H_0)(t_f - t_j)} \tilde{H}_I e^{-\frac{i}{\hbar}(\tilde{H}_P + H_0)(t_{j-1} - t_j)} \cdots e^{-\frac{i}{\hbar}(\tilde{H}_P + H_0)(t_{j-1} - t_j)} \tilde{H}_I e^{-\frac{i}{\hbar}(\tilde{H}_P + H_0) t_j}. \quad (A4)$$

While the probes are in the lab for time $\tau$, the system evolves via the operator $e^{-iH_0 \tau}$, and whatever post-selection we might be applying to the probes will just affect system 1 through the bipartite terms $H_I$ in the expression above. Taking into account that $e^{-i(\tilde{H}_P + H_0) t} = e^{-i\tilde{H}_P t} e^{-iH_0 t}$, the form of Eq. (A2) is thus justified.

We are interested in situations where

$$\sum_j A_j |\psi_{1,\ldots,n}\rangle\langle\psi_{1,\ldots,n}| A_j^* \propto e^{-iH_0 T_1} |\psi_{1,\ldots,n}\rangle\langle\psi_{1,\ldots,n}| e^{iH_0 T_1}, \quad (A5)$$

with $T_1 = -T < 0$. By convexity, it follows that the above equation can hold for some non-zero proportionality scalar iff there exists an operator $A$ (namely, any non-zero $A_i$), admitting a decomposition in terms of linear combinations of operators of the form of Eq. (A2), such that

$$A \propto e^{iH_0 T}, \quad (A6)$$

for all Hamiltonians $H_0, H_I, H_C$.

Suppose then that there exists such an operator, and let $H_0$ be any generic Hamiltonian such that the right hand side of Eq. (A6) does not vanish. Since $H_0$ is a generic operator, it must admit a Jordan decomposition of the form $H_0 = B \sum j=1^{d} \alpha_j |j\rangle\langle j| B^{-1}$, with $\{\alpha_k\}_{k=1}^{d} \subset \mathbb{C}$, for some invertible $d \times d$ matrix $B$. Since $A = f(\tilde{\alpha}) e^{iH_0 T}$ for some scalar $f(\tilde{\alpha})$, we have that

$$\tilde{A}_{jj} = f(\tilde{\alpha}) e^{i\alpha_j T}, \quad (A7)$$

for $j = 1, \ldots, d$. Here $\tilde{A}$ denotes $B^{-1} A B$. On the other hand, because of Eq. (A2), we have that

$$\tilde{A}_{jj} = \sum_{\tilde{\alpha} \in T} \tilde{c}(\tilde{\alpha}) e^{-i\tilde{\alpha} \cdot \tilde{t}}, \quad (A8)$$

where $T = \{\tilde{t} \in \mathbb{R}^d, t_i \geq 0, \sum t_i = T'\}$.

It follows that $f(\tilde{\alpha})$ admits the decompositions

$$f(\tilde{\alpha}) = \sum_{\tilde{\alpha} \in T_{j}} \tilde{c}(\tilde{\alpha}) e^{-i\tilde{\alpha} \cdot \tilde{t}}, \quad (A9)$$

for $j = 1, \ldots, d$. Here $T_j = \{\tilde{t} \in \mathbb{R}^d, t_i \geq T_{\delta_j}, t_i = T' + T_j\}$.

Now, express the vector $\tilde{\alpha} = \tilde{\beta} + i\tilde{\gamma}$ in terms of its real and imaginary parts $\tilde{\beta}, \tilde{\gamma}$. Fixing $\tilde{\gamma}$, we have that the above expressions depend on $\tilde{\beta}$ as

$$f(\tilde{\alpha}) = \sum_{\tilde{\alpha} \in T_{j}} \tilde{c}(\tilde{\alpha}) e^{-i\tilde{\alpha} \cdot \tilde{t}}. \quad (A10)$$

The exponentials $e^{-i\tilde{\beta} \cdot \tilde{t}}$ are linearly independent functions of $\tilde{\beta}$. This implies that $\tilde{c}(\tilde{\alpha}, \tilde{\gamma})$ must vanish if, for some $i, \tilde{t} \notin T_i$. This leads to the expression

$$f(\tilde{\alpha}) = \sum_{\tilde{t} \notin \bigcup_{i} T_{j}} \tilde{c}(\tilde{\alpha}, \tilde{\gamma}) e^{-i\tilde{\beta} \cdot \tilde{t}}. \quad (A11)$$

Any $\tilde{t} \notin \bigcap_{i} T_{j}$ satisfies $t_i \geq T_i, \sum t_i = T + T'$. Combining these two inequalities, we conclude that, for all $\tilde{\gamma}$ for which $f(\tilde{\alpha})$ does not vanish, $dT \leq \sum_{i=1}^{d} t_i = T + T'$. That is,
Note that the argument above does not invoke at any point the uncontrollability of system 1: it holds even if we know the form of the operators \( \{\Pi_i\}_i \) in Eq. (A2). In fact, it holds if we further know the similarity transformation that diagonalizes \( H_0 \).

### B. General limits for \( n \) systems

Suppose that, through a scattering experiment of duration \( T' \), we were able to induce a transformation of the type \( A \propto U(T_1, ..., T_n) \), for some times \( T_1, ..., T_n \). Let us assume, w.l.o.g., that \( T_1, ..., T_k < 0 \) and \( T_{k+1}, ..., T_n \geq 0 \).

Now, let \( B \) be the similarity transformation that diagonalizes \( H_0 \), i.e., \( B^{-1}H_0B = \sum_i \alpha_i |i\rangle\langle i| \), and consider the operator

\[
\tilde{A} = \prod_{j=1}^{d-1} (B^{\otimes n} \Gamma^{j}(B^{-1})^{\otimes n}) A (B^{\otimes n} \Gamma^{-j}(B^{-1})^{\otimes n}), \tag{B1}
\]

where \( \Gamma = I^{\otimes k} \otimes \tilde{\Gamma}^{\otimes n-k} \), with \( \tilde{\Gamma} = \sum_{i=1}^{d} |i\rangle \langle i+1| \), where \( i+1 \) equals \( i+1 \) for \( i < d \); and 1, otherwise. Noting that for any \( d \times d \) diagonal matrix \( \bar{Y} \), \( \prod_{j=1}^{d-1} \bar{Y}^{T^{j-1}} = \det(Y)^{d-1} \), we have that \( \tilde{A} \propto U((d-1)T_1, ..., (d-1)T_k, -T_{k+1}, ..., -T_n) \).

Finally, define the linear map \( \Lambda : M_d^{\otimes n} \to M_d \) by

\[
\Lambda(X) = \sum_{i_1, \ldots, i_{n-1}} (I \otimes |i_1\rangle \otimes \ldots \otimes |i_{n-1}\rangle) X (|i_1\rangle \otimes \ldots \otimes |i_{n-1}\rangle \otimes I). \tag{B2}
\]

It can be seen that \( \Lambda(X_1 \otimes \ldots \otimes X_n) = X_1 \otimes \ldots \otimes X_n \); it follows that \( \Lambda(\tilde{A}) \propto U((d-1)\sum_{i=1}^{k} T_i - \sum_{i=k+1}^{n} T_i) \), i.e., \( \Lambda(\tilde{A}) \) is a resetting transformation. Clearly, \( \Lambda(\tilde{A}) \) can be expressed as linear combinations of product operators of the form Eq. (A2) for \( n = 1 \), with the particularity that \( \sum_{i} t_i = (d-1)nT' \). By Eq. (A12) we have, then, that the total resetting time \( (d-1)\sum_{i=1}^{k} |T_i| + \sum_{i=k+1}^{n} |T_i| \) is upper bounded by \( nT' \). That is,

\[
\sum_{i=1}^{k} |T_i|(d-1) + \sum_{i>k} |T_i| \leq nT'. \tag{B3}
\]

### C. Controlled operations in scattering scenarios

Given \( n \) identical uncontrolled systems of dimension \( d \), one can, by sending probes, effect controlled operations of the form \( \sum_{\pi \in S_n} c_{\pi} P_{\pi} \), where \( P_{\pi} \) is a permutation operator. We will next prove the result for \( n = 2 \) and sketch how to prove it for general \( n \).

This construction requires a lot of matrix variables; let us start with \( X_1, \ldots, X_m \). Note that any linear combination \( C \) of the matrices \( X_i \otimes X_i \) is such that \( P_{\{1,2\}} \cdot C \cdot P_{\{1,2\}} = C \). This implies that \( C = C_S \otimes C_A \), where \( S \) and \( A \) denote, respectively, the symmetric and antisymmetric subspaces of \( \mathbb{C}^d \otimes \mathbb{C}^d \), with dimensions \( d(d+1)/2 \) and \( d(d-1)/2 \).

Notice that any central polynomial of dimension \( D \) vanishes when evaluated with matrices of dimension \( D' < D \). This is so because we can embed any \( D' \times D' \) matrix \( M \) into \( D \times D \) matrices as \( M \oplus 0_{D' - D'} \). If we evaluate the considered central polynomial with such \( D \times D \) matrices, the result should be proportional to the identity. At the same time, it must be of the form \( M' \oplus 0_{D' - D'} \). It follows that the proportionality constant is zero.

Now, let \( G(Z_1, \ldots, Z_m) \) be a central polynomial for dimension \( d(d+1)/2 \), and let \( C_1(X) = \ldots, C_m(X) \) be linear combinations of \( X_1 \otimes X_1, \ldots, X_m \otimes X_m \) such that, for some \( d \times d \) matrices \( X_1, \ldots, X_m \), \( G(C_1(X), \ldots, C_m(X)) \neq 0 \). Since the dimensionality of the symmetric (antisymmetric) space is equal to (smaller than) \( 0 \), the previous observations we have that \( G(X) \equiv G(C_1(X), \ldots, C_m(X)) \) must be proportional to \( I_S \otimes A \) when evaluated with \( d \times d \) matrices. That is, the polynomial \( G(X) \) is generically nonzero and proportional to the symmetric projector \( \Pi_S \).

Given another set of matrix variables \( \{Y_i\}_i \), consider the polynomials \( P_{ij}^m(Y) \equiv Y_i \otimes \bar{Y}_j - \bar{Y}_j \otimes Y_i \). Any linear combination of \( C \) of those satisfies \( P_{\{1,2\}} \cdot C \cdot P_{\{1,2\}} = -C \). This implies that

\[
C = \begin{pmatrix} 0 & C_1 \\ C_2 & 0_A \end{pmatrix}. \tag{C1}
\]

Therefore, the image of the polynomials \( \tilde{P}_{ijkl} \equiv P_{ij} \tilde{G} \tilde{P}_{kl} \), when evaluated on \( d \times d \) matrices, are matrices of the form \( 0_S \otimes \bullet_A \). In turn, any matrix of the form \( 0_S \otimes \bullet_A \) can be expressed as a linear combination of \( P_{ij}(Y) \), for certain \( Y \)'s. Now, choose a central matrix polynomial \( H(Z_1, \ldots, Z_m) \) for dimension \( d(d-1)/2 \), and choose linear combinations \( \tilde{C}_1, \ldots, \tilde{C}_{m'} \) of the polynomials \( \tilde{P}_{ijkl}(X, Y) \) such that \( \tilde{H}(X, Y) \equiv H(\tilde{C}_1(X, Y), \ldots, \tilde{C}_{m'}(X, Y)) \neq 0 \), for some \( X, Y \). It follows that \( \tilde{H}(X, Y) \) is generically nonzero, and proportional to the antisymmetric projector \( \Pi_A \).

The problem is that the proportionality scalars in \( \tilde{G} \) and \( \tilde{H} \) may differ. To fix this, choose a central matrix polynomial \( F(Z_1, \ldots, Z_p) \) for dimension \( d^2 \), and let \( Q_1 = T_1 \otimes T_2, Q_2 = T_3 \otimes T_4, \ldots, Q_p = T_{2p-1} \otimes T_{2p} \), where \( T_1, \ldots, T_{2p} \) are new matrix variables. Then we have that

\[
\bar{F} = F \left( (\tilde{G} + \tilde{H})Q_1(\tilde{G} + \tilde{H}), \ldots, (\tilde{G} + \tilde{H})Q_p(\tilde{G} + \tilde{H}) \right). \tag{C2}
\]
is a central polynomial. We can express it as

\[ \tilde{F} = \tilde{G}F^3\tilde{G} + \tilde{G}F^2\tilde{H} + \tilde{H}F^3\tilde{G} + \tilde{H}F^2\tilde{H}, \]

(C3)

for some polynomials \( F^1, F^2, F^3, F^4 \).

Let \( f(X,Y,T) \) be the scalar function satisfying \( \tilde{F}(X,Y,T) = f(X,Y,T)\lambda^2 \). Then we have that

\[
\begin{align*}
  f(X,Y,T)\Pi_S &= \Pi_S \tilde{F}(X,Y,T)\Pi_S = \tilde{G}F^3\tilde{G}, \\
  f(X,Y,T)\Pi_A &= \Pi_A \tilde{F}(X,Y,T)\Pi_A = \tilde{H}F^2\tilde{H}.
\end{align*}
\]

(C4)

The two polynomials we are looking for are thus \( \tilde{S} \equiv GF^3\tilde{G}, A \equiv H^2\tilde{H} \). Since \( \Pi_S + \Pi_A = \Pi_\lambda \), we combine the polynomials \( \tilde{S}, \tilde{G} \), we can induce any linear combination of the two permutation operators on \( n = 2 \) systems.

There is, though, a subtlety. The way they were constructed, \( \tilde{S}, \tilde{A} \) are not homogeneous: the reason is that \( \tilde{G}, \tilde{H} \) have different degree. This can be fixed, e.g., by redefining them as \( \tilde{G} \equiv GJG, \tilde{H} \equiv \tilde{H}JH \), where \( J \) are central polynomials for dimension \( d^2 \) on new variables such that \( \text{deg}(G) + \text{deg}(JG) = \text{deg}(H) + \text{deg}(JH) \). Such polynomials always exist: indeed, for all dimensions there exists a central matrix polynomial of the form \( J(U_1, \ldots, U_q) \), linear in \( U_1 \). Its degree can be increased by an arbitrary amount \( k \) via the transformation \( J(U_1, \ldots, U_q) \rightarrow J(U_1^{k+1}, \ldots, U_q) \).

It remains to show that one can extend this construction for general \( n \). This follows from the fact that any permutation of \( n \) parties can be expressed as the product of \( n - 1 \) gates which are either permutations of the form \( \pi \) or identities. Define the tensor polynomial \( P^{ij} \) as \( P^{ij} = \tilde{S} + (-1)^{\delta_{ij}}\tilde{A}^{ij} \otimes \bigotimes_{k \neq \ell,\pi} C_k, \) where \( C \) is a central polynomial of the same degree as \( \tilde{S}, \tilde{A} \). Then,

\[
P^{ij} = \lambda^{\pi}, \text{ for } i = j; \quad \lambda P_{(i,j)}, \text{ otherwise.}
\]

(C5)

Now, given a permutation \( \pi \in S_n \), define the sequence of permutations \( \pi_k = (k, \pi_{k-1}(k)) \circ \pi_{k-1}, \) with \( \pi_0 = \pi \). It follows that

\[
\prod_{k=1}^{n-1} P^{k,\pi_{k-1}(k)} = \lambda^{n-1} P_{\pi}.
\]

D. Invariant tensor matrix polynomials

In the previous section we showed how to construct a tensor matrix polynomial \( G(X_1, \ldots, X_L) \) such that it was proportional to a permutation operator between the identical systems. As the proportionality constant is the same for all permutations, this allows arbitrary linear combinations of permutations to be realised independently of \( \{X_i\} \). We now show that these are the only invariant tensor matrix polynomials.

Let us consider a general such polynomial \( G(X_1, \ldots, X_L) \propto M \). We require that it remain proportional to the same \( M \) for different variables, which includes the unitary rotation \( G(UX_1U^\dagger, \ldots, UX_LU^\dagger) \) for \( U \in SU(d) \). From the tensor product structure of \( G \) this implies

\[
U^{\otimes n} M(U^\dagger)^{\otimes n} = \lambda_U M,
\]

(D1)

where \( \lambda_U \) is a constant of proportionality and the equation must hold for all \( U \). Vectorising this equation by applying the transformation

\[
M = \sum_{i,j} m_{ij} |i\rangle\langle j| \rightarrow \sum_{i,j} m_{ij} |i\rangle\langle j| = |M\rangle,
\]

(D2)

leads to

\[
U^{\otimes n} \otimes (U^*)^{\otimes n} |M\rangle = \lambda_U |M\rangle.
\]

(D3)

From the Peter-Weyl theorem of representation theory \cite{15}, we decompose the group action into a direct sum of irreducible representations,

\[
U^{\otimes n} \otimes (U^*)^{\otimes n} |M\rangle = \bigoplus_{i} r_{d_i}^{k_i} |\lambda_{d_i} \rangle,
\]

(D4)

where \( r_{d_i} \in SU(d_i) \) and \( k_i \) is the multiplicity. The equality in Eq.(D3) must hold over all tensor blocks independently which implies

\[
r_{d_i} |m_{d_i}^{\ell} \rangle = \lambda_{U} |m_{d_i}^{j} \rangle \quad \forall i, j, U.
\]

(D6)

For \( d_i > 1 \) this requires that \( \lambda_U = 0 \) as there is no nontrivial eigenvector for every matrix in \( SU(d_i) \). However, for \( d_i = 1 \) then \( r_{d_i} = 1 \), and therefore \( \lambda_U = 1 \) for all \( |m_{d_i}^{\ell} \rangle \). Hence, the only \( |M\rangle \) which can satisfy Eq.(D3) are those which lie in the span of \( |m_{d_i}^{j} \rangle \), and they do so with \( \lambda_U = 1 \). This means that \( M \) lies in the span of the trivial representation of the conjugate action of \( U^{\otimes n} \). By the Schur-Weyl duality, this is exactly the same space as the permutations \( S_n \).

E. Finding SWAP polynomials numerically

The method used to find matrix polynomials for the SWAP operation between two systems is similar to the one described in \cite{6}. We want to find tensor matrix polynomials for \( n = 2 \) systems, where the operators acting on the two systems are unknown but identical. We write a general tensor matrix polynomial as

\[
P(A_1, \ldots, A_{d}) = \sum_{i,j} p_{i,j} A_{i_1} A_{i_2} \ldots A_{i_k} \otimes A_{j_1} A_{j_2} \ldots A_{j_k},
\]

(E1)

where \( k \) is the degree of the polynomial in each system, the \( A \)'s are \( d \times d \) matrices, \( d \) is the dimension of the probes (or the different possible paths they can take),
and \( \bar{r}, \bar{j} = \{0, 1, \ldots, d_p\}^k \). The \( p_{r,j} \) can be thought of as the coefficient in the vector space of polynomials, where a vector is expressed as \( \sum r_j p_{r,j} |r, j\rangle \).

The aim is to characterise the size of the subspace of this vector space which produces SWAP polynomials. To do this we take random \( A_i \)'s and two vectors \(|R\rangle, |L\rangle\) which \( A_i \) acts on, and construct the polynomial with coefficients

\[
p_{\bar{r}, \bar{j}} = \langle L | A_{i_1} A_{i_2} \ldots A_{i_k} \otimes A_{j_1} A_{j_2} \ldots A_{j_k} | R \rangle
\]

repeating this with different random sets of matrices (from the dimension of the space, at most \( d_p^2 \) are required) and finding a basis for such \( p_{\bar{r}, \bar{j}} \) characterises the space \( V^\perp \). By construction, a vector from the orthonormal complement \( V \) has the property that

\[
\sum_{\bar{r}, \bar{j}, \bar{p}, \bar{q}} \langle \bar{r}, \bar{j} | p_{\bar{r}, \bar{j}} \langle L | A_{p_1} \ldots A_{p_k} \otimes A_{q_1} \ldots A_{q_k} | R \rangle | \bar{p}, \bar{q} \rangle = 0,
\]

\[
\sum_{\bar{r}, \bar{j}} p_{\bar{r}, \bar{j}} \langle L | A_{i_1} \ldots A_{i_k} \otimes A_{j_1} A_{j_2} \ldots A_{j_k} | R \rangle = 0
\]

for almost all choice of matrices \( A \). In order for this to generate SWAP polynomials between two qubits (\( d_s = 2 \)) we construct \( V \) using each of \(|R\rangle = \{|00\rangle, |01\rangle, |11\rangle\}\), and \(|L\rangle = |10\rangle\). The only possible matrix polynomials which satisfy these constraints are the SWAP and the 0 operator. The 0 operators can themselves be found by including \(|10\rangle\) in the set of \(|R\rangle\). Taking the orthogonal complement of the latter in the former results in isolating the SWAP polynomials. For qubit probes, \( d_p = 2 \), this set is empty for \( k < 5 \) and has dimension 3 for \( k = 5 \). From this three independent SWAP polynomials can be easily extracted.

F. Searching for simple protocols for time transfer

The transfer of evolution time between two identical uncontrolled systems does not require us to play with linear superpositions of probe paths. Indeed, consider a scenario where we send each probe to interact with its corresponding uncontrolled system \( s \) times. Every time we send the probe to the scattering region, we copy its state to a register in the lab and set the state of the probe to \(|0\rangle\). At the end of this procedure, the joint state between the uncontrolled systems and the lab memories will be

\[
\sum_{\bar{r}, \bar{j}} W_{\bar{r}} \otimes W_{\bar{j}} |\psi_{1,2}\rangle |\bar{i}, M_1 \rangle |\bar{j}, M_2 \rangle.
\]

Here \( \bar{r}, \bar{j} \in \{0, \ldots, d_P - 1\}^s \), and \( W_r \equiv W_{r_1,0} \ldots W_{r_s,0} \).

Next, we let system 2 evolve for \( m \) iterations, after which we will send the same probe \( s \) times in the same way. Meanwhile, we do the opposite with system 1: that is, we send the same probe \( s \) times and then we let system 1 evolve alone for \( m \) iterations. The final state of the system will be

\[
\sum_{\bar{r}, \bar{j}, \bar{k}, \bar{l}} V^m W_k W_{\bar{r}} \otimes W_{\bar{j}} V^m W_{\bar{j}} |\psi_{1,2}\rangle |\bar{i}, M_1 \rangle |\bar{j}, M_2 \rangle |\bar{k}, M'_1 \rangle |\bar{l}, M'_2 \rangle.
\]

Now, let \(|\phi\rangle = \sum_{\bar{r}, \bar{j}} \phi_{\bar{r}, \bar{j}} |\bar{i}, \bar{j}\rangle\) be a normalized vector, such that

\[
\sum_{\bar{r}, \bar{j}} \phi_{\bar{r}, \bar{j}} W_{\bar{r}} \otimes W_{\bar{j}} \propto \mathcal{P}_{(1,2)}.
\]

It follows that, if we post-select the lab registers \( M_1, M_2 \) and \( M'_1, M'_2 \) in the state \(|\phi\rangle\), then the final state of the uncontrolled systems will be \((V^m \otimes I) |\psi_{1,2}\rangle\). That is, we will have made system 1 evolve by \( T = 2m\Delta t \) in time \((m + 2s)\Delta t\). Similarly, if a set of orthonormal vectors \( \{|\phi^\alpha\rangle\}_\alpha \) satisfy Eq. (F3), then a protocol to transfer evolution time to system 1 from system 2 would be given by following the previous steps and then post-selecting the registers to one of the states \( \{|\phi^\alpha\rangle \otimes |\phi^\beta\rangle\}_{\alpha,\beta} \). For a fixed interaction \( W \), the probability of success of such a protocol is

\[
\left( \sum_{\alpha} |\lambda^\alpha(W)|^2 \right)^2,
\]

where

\[
\lambda^\alpha(W) \equiv \langle 0 | \otimes \sum_{\bar{r}, \bar{j}} \phi_{\bar{r}, \bar{j}}^\alpha W_{\bar{r}} \otimes W_{\bar{j}} |0 \rangle \otimes \langle 0 |^2
\]

Using the three SWAP polynomials found in App. E for the \( \{|\phi^\alpha\rangle\} \) and averaging over the \( W \) according to the Haar measure, we get an average success probability of \( 6.9 \pm 0.6 \times 10^{-5} \). While this may be too small to be of practical importance, it is still large enough to be detectable experimentally. Furthermore, it may be possible to boost this probability significantly by increasing the number of probes used to implement the SWAP.