Multidesigns for the graph pair formed by the 6-cycle and 3-prism

Yizhe Gao, Dan Roberts
Illinois Wesleyan University, Bloomington, IL, USA
ygao@iwu.edu, drobert1@iwu.edu

Abstract

Given two graphs $G$ and $H$, a $(G, H)$-multidecomposition of $K_n$ is a partition of the edges of $K_n$ into copies of $G$ and $H$ such that at least one copy of each is used. We give necessary and sufficient conditions for the existence of $(C_6, \overline{C}_6)$-multidecomposition of $K_n$ where $C_6$ denotes a cycle of length 6 and $\overline{C}_6$ denotes the complement of $C_6$. We also characterize the cardinalities of leaves and paddings of maximum $(C_6, \overline{C}_6)$-multipackings and minimum $(C_6, \overline{C}_6)$-multicoverings, respectively.

Keywords: graph pair, decomposition, multidecomposition, packing, covering, cycle, prism
Mathematics Subject Classification: 05C51, 05C70
DOI: 10.5614/ejgta.2020.8.1.10

1. Introduction

Let $G$ and $H$ be graphs. Denote the vertex set of $G$ by $V(G)$ and the edge set of $G$ by $E(G)$. A $G$-decomposition of $H$ is a partition of $E(H)$ into a set of edge-disjoint subgraphs of $H$ each of which is isomorphic to $G$. Graph decompositions have been extensively studied. This is particularly true for the case where $H \cong K_n$, see [2] for a recent survey. A $G$-decomposition of $K_n$ is sometimes referred to as a $G$-design of order $n$. As an extension of a graph decomposition we can permit more than one graph, up to isomorphism, to appear in the partition. A $(G, H)$-multidecomposition of $K_n$ is a partition of $E(K_n)$ into a set of edge-disjoint subgraphs each of...
which is isomorphic to either $G$ or $H$, and at least one copy of $G$ and one copy of $H$ are elements of the partition. When a $(G, H)$-multidecomposition of $K_n$ does not exist, we would like to know how “close” we can get. More specifically, define a $(G, H)$-multipacking of $K_n$ to be a collection of edge-disjoint subgraphs of $K_n$ each of which is isomorphic to either $G$ or $H$ such that at least one copy of each is present. The set of edges in $K_n$ that are not used as copies of either $G$ or $H$ in the $(G, H)$-multipacking is called the leave of the $(G, H)$-multipacking. Similarly, define a $(G, H)$-multicovering of $K_n$ to be a partition of the multiset of edges formed by $E(K_n)$ where some edges may be repeated into edge-disjoint copies of $G$ and $H$ such that at least one copy of each is present. The multisets of repeated edges is called the padding. A $(G, H)$-multipacking is called maximum if its leave is of minimum cardinality, and a $(G, H)$-multicovering is called minimum if its padding is of minimum cardinality. The term multidesign is used to encompass multidecompositions, multipackings, and multicoverings.

A natural way to form a pair of graphs is to use a graph and its complement. To this end, we have the following definition which first appeared in [1]. Let $G$ and $H$ be edge-disjoint, non-isomorphic, spanning subgraphs of $K_n$ each with no isolated vertices. We call $(G, H)$ a graph pair of order $n$ if $E(G) \cup E(H) = E(K_n)$. For example, the only graph pair of order 4 is $(C_4, E_2)$, where $E_2$ denotes the graph consisting of two disjoint edges. Furthermore, there are exactly 5 graph pairs of order 5. In this paper we are interested in the graph pair formed by a 6-cycle, denoted $C_6$, and the complement of a 6-cycle, denoted $\overline{C}_6$.

Necessary and sufficient conditions for multidecompositions of complete graphs into all graph pairs of orders 4 and 5 were characterized in [1]. They also characterized the cardinalities of leaves and paddings of multipackings and multicoverings for the same graph pairs. We advance those results by solving the same problems for a graph pair of order 6, namely $(C_6, \overline{C}_6)$. Note that $\overline{C}_6$ is sometimes referred to as the 3-prism, but we used the former notation for brevity. We first address multidecompositions, then multipackings and multicoverings. Our main results are stated in the following three theorems.

**Theorem 1.1.** The complete graph $K_n$ admits a $(C_6, \overline{C}_6)$-multidecomposition of $K_n$ if and only if $n \equiv 0, 1 \pmod{3}$ with $n \geq 6$, except $n \in \{7, 9, 10\}$.

**Theorem 1.2.** For each $n \equiv 2 \pmod{3}$ with $n \geq 8$, a maximum $(C_6, \overline{C}_6)$-multipacking of $K_n$ has a leave of cardinality 1. Furthermore, a maximum $(C_6, \overline{C}_6)$-multipacking of $K_7$ has a leave of cardinality 6, and a maximum $(C_6, \overline{C}_6)$-multipacking of either $K_9$ or $K_{10}$ has a leave of cardinality 3.

**Theorem 1.3.** For each $n \equiv 2 \pmod{3}$ with $n \geq 8$, a minimum $(C_6, \overline{C}_6)$-multicovering of $K_n$ has a padding of cardinality 2. Furthermore, a minimum $(C_6, \overline{C}_6)$-multicovering of $K_7$ has a padding of cardinality 6, and a minimum $(C_6, \overline{C}_6)$-multicovering of either $K_9$ or $K_{10}$ has a padding of cardinality 2.

Let $G$ and $H$ be vertex-disjoint graphs. The join of $G$ and $H$, denoted $G \vee H$, is defined to be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{u, v\} : u \in V(G), v \in V(H)$. We use the shorthand notation $\bigvee_{i=1}^t G_i$ to denote $G_1 \vee G_2 \vee \cdots \vee G_t$, and when $G_i \cong G$ for all $1 \leq i \leq t$ we write $\bigvee_{i=1}^t G$. For example, $K_{12} \cong \bigvee_{i=1}^4 K_3$. 

www.ejgta.org
For notational convenience, let \((a, b, c, d, e, f)\) denote the copy of \(C_6\) with vertex set \{a, b, c, d, e, f\} and edge set \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{e, f\}, \{a, f\}\}, as seen in Figure 1. Let \([a, b, c; d, e, f]\) denote the copy of \(C_6\) with vertex set \{a, b, c, d, e, f\} and edge set
\[\{\{a, b\}, \{b, c\}, \{a, c\}, \{d, e\}, \{e, f\}, \{d, f\}, \{a, d\}, \{b, e\}, \{c, f\}\}.

![Figure 1. Labeled copies of \(C_6\) and \(\overline{C_6}\), denoted by \((a, b, c, d, e, f)\) and \([a, b, c; d, e, f]\), respectively.](image)

Next, we state some known results on graph decompositions that will help us prove our main result. Sotteau’s theorem gives necessary and sufficient conditions for complete bipartite graphs (denoted by \(K_{m,n}\) when the partite sets have cardinalities \(m\) and \(n\)) to decompose into even cycles of fixed length. Here we state the result only for cycle length 6.

**Theorem 1.4** (Sotteau [5]). A \(C_6\)-decomposition of \(K_{m,n}\) exists if and only if \(m \geq 4\), \(n \geq 4\), \(m\) and \(n\) are both even, and 6 divides \(mn\).

Another celebrated result in the field of graph decompositions is that the necessary conditions for a \(C_k\)-decomposition of \(K_n\) are also sufficient. Here we state the result only for \(k = 6\).

**Theorem 1.5** (Šajna [4]). Let \(n\) be a positive integer. A \(C_6\)-decomposition of \(K_n\) exists if and only if \(n \equiv 1, 9 \pmod{12}\).

The necessary and sufficient conditions for a \(\overline{C_6}\)-decomposition of \(K_n\) are also known, and stated in the following theorem.

**Theorem 1.6** (Kang et al. [3]). Let \(n\) be a positive integer. A \(\overline{C_6}\)-decomposition of \(K_n\) exists if and only if \(n \equiv 1 \pmod{9}\).

2. Multidecompositions

We first establish the necessary conditions for a \((C_6, \overline{C_6})\)-multidecomposition of \(K_n\).

**Lemma 2.1.** If a \((C_6, \overline{C_6})\)-multidecomposition of \(K_n\) exists, then
1. \(n \geq 6\), and
2. \(n \equiv 0, 1 \pmod{3}\).

135
Proof. Assume that a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_n\) exists. It is clear that condition (1) holds. Considering that the edges of \(K_n\) are partitioned into subgraphs isomorphic to \(C_6\) and \(\overline{C}_6\), we have that there exist positive integers \(x\) and \(y\) such that \(\binom{n}{2} = 6x + 9y\). Hence, 3 divides \(\binom{n}{2}\), which implies \(n \equiv 0, 1 \pmod{3}\), and condition (2) follows. \(
\)

2.1. Small examples of multidecompositions

In this section we present various non-existence and existence results for \((C_6, \overline{C}_6)\)-multidecompositions of small orders. The existence results will help with our general constructions.

2.1.1. Non-existence results

The necessary conditions for the existence of a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_n\) fail to be sufficient in exactly three cases, namely \(n = 7, 9, 10\). We will now establish the non-existence of \((C_6, \overline{C}_6)\)-multidecompositions of \(K_n\) for these cases.

Lemma 2.2. A \((C_6, \overline{C}_6)\)-multidecomposition of \(K_7\) does not exist.

Proof. Assume the existence of a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_7\), call it \(\mathcal{G}\). There must exist positive integers \(x\) and \(y\) such that \(\binom{7}{2} = 21 = 6x + 9y\). The only solution to this equation is \((x, y) = (2, 1)\); therefore, \(\mathcal{G}\) must contain exactly one copy of \(\overline{C}_6\). However, upon examining the degree of each vertex contained in the single copy of \(\overline{C}_6\) we see that there must exist a non-negative integer \(p\) such that \(6 = 2p + 3\). This is a contradiction. Thus, a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_7\) cannot exist. \(
\)

Lemma 2.3. A \((C_6, \overline{C}_6)\)-multidecomposition of \(K_9\) does not exist.

Proof. Assume the existence of a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_9\), call it \(\mathcal{G}\). There must exist positive integers \(x\) and \(y\) such that \(\binom{9}{2} = 36 = 6x + 9y\). The only solution to this equation is \((x, y) = (3, 2)\); therefore, \(\mathcal{G}\) must contain exactly two copies of \(\overline{C}_6\).

Turning to the degrees of the vertices in \(K_9\), we have that there must exist positive integers \(p\) and \(q\) such that \(8 = 2p + 3q\). The only possibilities are \((p, q) \in \{(4, 0), (1, 2)\}\). Note that \(K_6\) does not contain two edge-disjoint copies of \(\overline{C}_6\). Since \(\mathcal{G}\) contains exactly two copies of \(\overline{C}_6\), there must exist at least one vertex \(a \in V(K_9)\) that is contained in exactly one copy of \(\overline{C}_6\). However, this contradicts the fact that vertex \(a\) must be contained in either 0 or 2 copies of \(\overline{C}_6\). Thus, a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_9\) cannot exist. \(
\)

Lemma 2.4. A \((C_6, \overline{C}_6)\)-multidecomposition of \(K_{10}\) does not exist.

Proof. Assume the existence of a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_{10}\), call it \(\mathcal{G}\). There must exist positive integers \(x\) and \(y\) such that \(\binom{10}{2} = 45 = 6x + 9y\). Thus, \((x, y) \in \{(6, 1), (3, 3)\}\); therefore, \(\mathcal{G}\) must contain at least one copy of \(\overline{C}_6\). However, if \(\mathcal{G}\) consists of exactly one copy of \(\overline{C}_6\), then the vertices of \(K_{10}\) which are not included in this copy would have odd degrees remaining after the removal of the copy of \(\overline{C}_6\). Thus, the case where \((x, y) = (6, 1)\) is impossible.

Upon examining the degree of each vertex in \(K_{10}\), we see that there must exist positive integers \(p\) and \(q\) such that \(9 = 2p + 3q\). The only solutions to this equation are \((p, q) \in \{(3, 1), (0, 3)\}\). From the above argument, we know that \(\mathcal{G}\) contains exactly 3 copies of \(\overline{C}_6\), say \(A, B, \) and \(C\). Let
Example 2.1.2. Existence results

We now present some multidecompositions of small orders that will be useful for our general recursive constructions.

**Example 1.** $K_{13}$ admits a $(C_6, \overline{C}_6)$-multidecomposition.

Let $V(K_{13}) = \{1, 2, \ldots, 13\}$. The following is a $(C_6, \overline{C}_6)$-multidecomposition of $K_{13}$.

\[
\left\{ [1, 2, 3; 7, 9, 8], [1, 4, 5; 9, 12, 10], [3, 4, 6; 7, 11, 10], [2, 5, 6; 8, 12, 11] \right\} \\
\cup \left\{ (13, 1, 6, 8, 5, 11), (13, 2, 4, 7, 6, 12), (13, 3, 5, 9, 4, 10), (13, 7, 12, 3, 9, 6), \\
(13, 8, 10, 2, 7, 5), (13, 9, 11, 1, 8, 4), (1, 10, 3, 11, 2, 12) \right\}
\]

**Example 2.** $K_{15}$ admits a $(C_6, \overline{C}_6)$-multidecomposition.

Let $V(K_{15}) = \{1, 2, \ldots, 15\}$. The following is a $(C_6, \overline{C}_6)$-multidecomposition of $K_{15}$.

\[
\left\{ [1, 5, 10; 6, 8, 12], [4, 8, 13; 9, 11, 15], [7, 11, 1; 12, 14, 3], [10, 14, 4; 15, 2, 6], \\
[13, 2, 7; 3, 5, 9] \right\} \\
\cup \left\{ (1, 12, 11, 13, 5, 15), (4, 15, 14, 1, 8, 3), (7, 3, 2, 4, 11, 6), (10, 6, 5, 7, 14, 9), \\
(13, 9, 8, 10, 2, 12), (1, 2, 11, 3, 6, 13), (4, 5, 14, 6, 9, 1), (7, 8, 2, 9, 12, 4), \\
(10, 11, 5, 12, 15, 7), (13, 14, 8, 15, 3, 10) \right\}
\]

**Example 3.** $K_{19}$ admits a $(C_6, \overline{C}_6)$-multidecomposition.

Let $V(K_{19}) = \{1, 2, \ldots, 19\}$. The following is a $(C_6, \overline{C}_6)$-multidecomposition of $K_{19}$.

\[
\left\{ [2, 11, 14; 17, 4, 18], [3, 12, 15; 18, 5, 19], [4, 13, 16; 19, 6, 11], [5, 14, 17; 11, 7, 12], \\
[6, 15, 18; 12, 8, 13], [7, 16, 19; 13, 9, 14], [8, 17, 11; 14, 10, 15], [9, 18, 12; 15, 2, 16], \\
[10, 19, 13; 16, 3, 17] \right\} \\
\cup \left\{ (2, 12, 14, 3, 11, 1), (3, 13, 15, 4, 12, 1), (4, 14, 16, 5, 13, 1), (5, 15, 17, 6, 14, 1), \\
(6, 16, 18, 7, 15, 1), (7, 17, 19, 8, 16, 1), (8, 18, 11, 9, 17, 1), (9, 19, 12, 10, 18, 1), \\
(10, 11, 13, 2, 19, 1), (2, 3, 10, 4, 9, 5), (2, 6, 8, 7, 3, 4), (2, 7, 4, 5, 3, 8), \\
(2, 10, 8, 4, 6, 9), (3, 6, 10, 5, 7, 9), (5, 6, 7, 10, 9, 8) \right\}
\]
2.2. General constructions for multidecompositions

Lemma 2.5. If \( n \equiv 0 \pmod{6} \) with \( n \geq 6 \), then \( K_n \) admits a \((C_6, \overline{C}_6)\)-multidecomposition.

Proof. Let \( n = 6x \) for some integer \( x \geq 1 \). Note that \( K_{6x} \cong \bigvee_{i=1}^x K_6 \). On each copy of \( K_6 \) place a \((C_6, \overline{C}_6)\)-multidecomposition of \( K_6 \). The remaining edges form edge-disjoint copies of \( K_{6,6} \), which admits a \( C_6 \)-decomposition by Theorem 1.4. Thus, we obtain the desired \((C_6, \overline{C}_6)\)-multidecomposition of \( K_n \).

Lemma 2.6. If \( n \equiv 1 \pmod{6} \) with \( n \geq 13 \), then \( K_n \) admits a \((C_6, \overline{C}_6)\)-multidecomposition.

Proof. Let \( n = 6x + 1 \) for some integer \( x \geq 2 \). The proof breaks into two cases.

Case 1: \( x = 2k \) for some integer \( k \geq 1 \). Notice that \( K_{12k+1} \cong K_1 \lor \bigvee_{i=1}^k K_{12} \). Each of the \( k \) copies of \( K_{13} \) formed by \( K_1 \lor K_{12} \) admits a \((C_6, \overline{C}_6)\)-multidecomposition by Example 1. The remaining edges form edge-disjoint copies of \( K_{12,12} \), which admits a \( C_6 \)-decomposition by Theorem 1.4. Thus, we obtain the desired \((C_6, \overline{C}_6)\)-multidecomposition of \( K_n \).

Case 2: \( x = 2k + 1 \) for some integer \( k \geq 2 \). Notice that \( K_{12k+7} \cong K_1 \lor K_6 \lor \bigvee_{i=1}^k K_{12} \). The single copy of \( K_{19} \) formed by \( K_1 \lor K_6 \lor K_{12} \) admits a \((C_6, \overline{C}_6)\)-multidecomposition by Example 3. The remaining \( k-1 \) copies of \( K_{13} \) formed by \( K_1 \lor K_{12} \) each admit a \((C_6, \overline{C}_6)\)-multidecomposition by Example 1. The remaining edges form edge-disjoint copies of either \( K_{6,12} \) or \( K_{12,12} \). Both of these graphs admit \( C_6 \)-decompositions by Theorem 1.4. Thus, we obtain the desired \((C_6, \overline{C}_6)\)-multidecomposition of \( K_n \).

Lemma 2.7. If \( n \equiv 3 \pmod{6} \) with \( n \geq 15 \), then \( K_n \) admits a \((C_6, \overline{C}_6)\)-multidecomposition.

Proof. Let \( n = 6x + 3 \) for some integer \( x \geq 2 \). The proof breaks into two cases.

Case 1: \( x = 2k \) for some integer \( k \geq 1 \). Notice that \( K_{12k+3} \cong K_1 \lor K_{14} \lor \bigvee_{i=1}^{k-1} K_{12} \). The remainder of the proof is similar to the proof of Case 1 of Lemma 2.6 where the ingredients required are \( C_6 \)-decompositions of \( K_{12,12} \), and \( K_{12,14} \), as well as \((C_6, \overline{C}_6)\)-multidecompositions of \( K_{13} \) and \( K_{15} \). Note that both of these graphs admit \( C_6 \)-decompositions by Theorem 1.4, \( K_{8,12} \), and \( K_{12,12} \), as well as a \((C_6, \overline{C}_6)\)-multidecomposition of \( K_{13} \).

Case 2: \( x = 2k + 1 \) for some integer \( k \geq 1 \). Notice that \( K_{12k+9} \cong K_1 \lor K_8 \lor \bigvee_{i=1}^k K_{12} \). The remainder of the proof is similar to the proof of Case 2 of Lemma 2.6 where the ingredients required are \( C_6 \)-decompositions of \( K_9 \) (which exists by Theorem 1.5), \( K_{8,12} \), and \( K_{12,12} \), as well as a \((C_6, \overline{C}_6)\)-multidecomposition of \( K_{13} \).

Lemma 2.8. If \( n \equiv 4 \pmod{6} \) with \( n \geq 16 \), then \( K_n \) admits a \((C_6, \overline{C}_6)\)-multidecomposition.

Proof. Let \( n = 6x + 4 \) where \( x \geq 2 \) is an integer. Note that \( K_{6x+4} \cong K_{10} \lor \bigvee_{i=1}^{x-1} K_6 \). The remainder of the proof is similar to the proof of Case 2 of Lemma 2.6 where the ingredients required are \( C_6 \)-decompositions of \( K_{6,6} \) and \( K_{6,10} \), a \( \overline{C}_6 \)-decomposition of \( K_{10} \) (which exists by Theorem 1.6), as well as a \((C_6, \overline{C}_6)\)-multidecomposition of \( K_6 \).

Combining Lemmas 2.5, 2.6, 2.7, and 2.8, we have proven Theorem 1.1.
3. Maximum Multipackings

Now we turn our attention to \((C_6, \overline{C}_6)\)-multipackings in the cases where \((C_6, \overline{C}_6)\)-multidecompositions do not exist.

3.1. Small examples of maximum multipackings

Example 4. A maximum \((C_6, \overline{C}_6)\)-multipacking of \(K_7\) has a leave of cardinality 6.

Note that the number of edges used in a \((C_6, \overline{C}_6)\)-multipacking of any graph must be a multiple of 3, since \(\gcd(6, 9) = 3\). Since no \((C_6, \overline{C}_6)\)-multidecomposition of \(K_7\) exists the next possibility is a leave of cardinality 3. However, the equation \(18 = 6x + 9y\) has no positive integer solutions. Thus, the minimum possible cardinality of a leave is 6. Let \(V(K_7) = \{1, ..., 7\}\). The following is a \((C_6, \overline{C}_6)\)-multipacking of \(K_7\), with leave \(\{1, 7\}, \{2, 7\}, \{3, 7\}, \{4, 7\}, \{5, 7\}, \{6, 7\}\).

\[
\{[1, 3, 5; 4, 6, 2], (1, 2, 3, 4, 5, 6)\}
\]

Example 5. A maximum \((C_6, \overline{C}_6)\)-multipacking of \(K_8\) has a leave of cardinality 1.

Let \(V(K_8) = \{1, ..., 8\}\). The following is a \((C_6, \overline{C}_6)\)-multipacking of \(K_8\), with leave \(\{3, 6\}\).

\[
\{[2, 5, 7; 4, 1, 8], (1, 2, 3, 4, 5, 6), (1, 3, 5, 6, 7, 9), (3, 8, 2, 6, 4, 7)\}
\]

Example 6. A maximum \((C_6, \overline{C}_6)\)-multipacking of \(K_9\) has a leave of cardinality 3.

Let \(V(K_9) = \{1, ..., 9\}\). The following is a \((C_6, \overline{C}_6)\)-multipacking of \(K_9\), with leave \(\{2, 4\}, \{2, 9\}, \{4, 9\}\).

\[
\{[1, 2, 3; 6, 5, 4], [1, 4, 7; 9, 8, 3], [2, 6, 8; 7, 9, 5], (1, 5, 3, 6, 7, 8)\}
\]

Example 7. A maximum \((C_6, \overline{C}_6)\)-multipacking of \(K_{10}\) has a leave of cardinality 3.

A \((C_6, \overline{C}_6)\)-multipacking of \(K_{10}\) with a leave of cardinality 3 can be obtained by starting with a \(\overline{C}_6\)-decomposition of \(K_{10}\). Then remove three vertex-disjoint edges from one copy of \(\overline{C}_6\), forming a \(C_6\). This gives us the desired \((C_6, \overline{C}_6)\)-multipacking of \(K_{10}\) where the three removed edges form the leave.

Example 8. A maximum \((C_6, \overline{C}_6)\)-multipacking of \(K_{11}\) has a leave of cardinality 1.

Let \(V(K_{11}) = \{1, ..., 11\}\). The following is a \((C_6, \overline{C}_6)\)-multipacking of \(K_{11}\), with leave \(\{1, 2\}\).

\[
\{[1, 7, 10; 9, 6, 3], [1, 5, 6; 4, 10, 2], [2, 5, 7; 11, 8, 4], [1, 3, 11; 8, 2, 9]\}
\cup\{(3, 4, 9, 10, 6, 8), (4, 5, 9, 7, 11, 6), (3, 5, 11, 10, 8, 7)\}
\]

Example 9. A maximum \((C_6, \overline{C}_6)\)-multipacking of \(K_{17}\) has a leave of cardinality 1.
Let \( V(K_{17}) = \{1, \ldots, 17\} \). The following is a \((C_6, \overline{C_6})\)-multipacking of \( K_{17} \), with leave \{1, 10\}:
\[
\{[2, 3, 5; 7, 8, 1], [3, 6, 4; 9, 8, 10], [2, 4, 9; 6, 5, 7]\}
\cup\{[2, 12, 5, 10, 11, 14], [2, 10, 17, 4, 13, 11], [4, 7, 13, 14, 5, 15], [4, 11, 15, 8, 16, 12],
(1, 15, 14, 16, 5, 17), (3, 12, 11, 17, 6, 15), (1, 2, 16, 7, 14, 4), (2, 13, 5, 8, 14, 17),
(7, 15, 10, 13, 9, 17), (1, 13, 6, 9, 11, 16), (1, 9, 12, 7, 3, 11), (3, 10, 12, 8, 4, 16),
(3, 13, 16, 6, 12, 14), (2, 8, 13, 17, 12, 15), (6, 10, 16, 15, 9, 14), (5, 9, 16, 17, 8, 11),
(1, 3, 17, 15, 13, 12), (1, 6, 11, 7, 10, 14)\}\n
### 3.2. General Constructions of maximum multipackings

**Lemma 3.1.** If \( n \equiv 2 \pmod{6} \) with \( n \geq 14 \), then \( K_n \) admits a \((C_6, \overline{C_6})\)-multipacking with leave cardinality 1.

*Proof.* Let \( n = 6x + 2 \) for some integer \( x \geq 2 \). Notice that \( K_{6x+2} \cong K_2 \lor (\lor_{i=1}^{x} K_6) \). Let \( \{u, v\} = V(K_2) \). Each of the \( x \) copies of \( K_6 \) formed by \( K_2 \lor K_6 \) admit a \((C_6, \overline{C_6})\)-multipacking with leave cardinality 1 by Example 5. Note that we can always choose the leave edge to be \( \{u, v\} \) in each of these multipackings. The remaining edges form edge disjoint copies of \( K_6,6 \), each of which admits a \( C_6 \)-decomposition by Theorem 1.4. Thus, we obtain the desired \((C_6, \overline{C_6})\)-multipacking of \( K_n \). \( \square \)

**Lemma 3.2.** If \( n \equiv 5 \pmod{6} \) with \( n \geq 11 \), then \( K_n \) admits a \((C_6, \overline{C_6})\)-multipacking with leave cardinality 1.

*Proof.* Let \( n = 6x + 5 \) for some integer \( x \geq 1 \).

**Case 1:** \( x = 2k \) for some integer \( k \geq 1 \). Notice that \( K_{12k+5} \cong K_1 \lor K_{16} \lor (\lor_{i=1}^{k-1} K_{12}) \). Each of the \( k-1 \) copies of \( K_{13} \) formed by \( K_1 \lor K_{12} \) admit a \((C_6, \overline{C_6})\)-multidecomposition by Example 1. The copy of \( K_{17} \) formed by \( K_1 \lor K_{16} \) admits a \((C_6, \overline{C_6})\)-multipacking with leave of cardinality 1 by Example 9. The remaining edges form edge disjoint copies of \( K_{12,12} \) or \( K_{12,16} \), each of which admits a \( C_6 \)-decomposition by Theorem 1.4. Thus, we obtain the desired \((C_6, \overline{C_6})\)-multipacking of \( K_n \).

**Case 2:** \( x = 2k + 1 \) for some integer \( k \geq 1 \). Notice that \( K_{12k+11} \cong K_1 \lor K_{10} \lor (\lor_{i=1}^{k} K_{12}) \). On each of the \( k \) copies of \( K_{13} \) formed by \( K_1 \lor K_{12} \) admit a \((C_6, \overline{C_6})\)-multidecomposition by Example 1. The copy of \( K_{11} \) formed by \( K_1 \lor K_{10} \) admits a \((C_6, \overline{C_6})\)-multipacking with leave of cardinality 1 by Example 8. The remaining edges form edge disjoint copies of \( K_{12,12} \) or \( K_{10,12} \), each of which admits a \( C_6 \)-decomposition by Theorem 1.4. Thus, we obtain the desired \((C_6, \overline{C_6})\)-multipacking of \( K_n \). \( \square \)

Combining Lemmas 3.1 and 3.2 along with Examples 4, 6, and 7 we have proven Theorem 1.2.

### 4. Minimum Multicoverings

Now we turn our attention to minimum \((C_6, \overline{C_6})\)-multicoverings in the cases where \((C_6, \overline{C_6})\)-multidecompositions do not exist.
4.1. Small examples of minimum multicovertings

**Example 10.** A minimum $(C_6, \overline{C}_6)$-multicovering of $K_7$ has a padding of cardinality 6.

We first rule out the possibility of a minimum $(C_6, \overline{C}_6)$-multicovering of $K_7$ with a padding of cardinality 3. The only positive integer solution to the equation $24 = 6x + 9y$ is $(x, y) = (1, 2)$. In such a covering there would be one vertex left out of one of the copies of $\overline{C}_6$. It would be impossible to use all edges at this vertex with the remaining copies of $C_6$ and $\overline{C}_6$. Thus, the best possible cardinality of a padding is 6. Let $V(K_7) = \{1, ..., 7\}$. The following is a minimum $(C_6, \overline{C}_6)$-multicovering of $K_7$, with padding of $\{\{1, 2\}, \{1, 5\}, \{1, 6\}, \{3, 6\}, \{4, 5\}, \{5, 6\}\}$.

$\{[1, 2, 3; 6, 5, 4], (1, 4, 7, 6, 3, 5), (1, 6, 2, 4, 5, 7), (1, 2, 7, 3, 6, 5)\}$

**Example 11.** A minimum $(C_6, \overline{C}_6)$-multicovering of $K_8$ has a padding of cardinality 2.

Let $V(K_8) = \{1, ..., 8\}$. The following is a minimum $(C_6, \overline{C}_6)$-multicovering of $K_8$, with padding of $\{\{1, 8\}, \{3, 5\}\}$.

$\{[1, 2, 8; 4, 3, 5], [1, 5, 6; 3, 7, 8], (1, 7, 2, 6, 4, 8), (2, 4, 7, 6, 3, 5)\}$

**Example 12.** A minimum $(C_6, \overline{C}_6)$-multicovering of $K_9$ has a padding of cardinality 3.

A $(C_6, \overline{C}_6)$-multicovering of $K_9$ with a padding of cardinality 3 can be obtained by starting with a $C_6$-decomposition of $K_9$ which exists by Theorem 1.5. One of the copies of $C_6$ contained in this decomposition can be transformed into a copy of $\overline{C}_6$ by carefully adding 3 edges. This gives us the desired $(C_6, \overline{C}_6)$-multicovering of $K_9$ where the three added edges form the padding.

**Example 13.** A minimum $(C_6, \overline{C}_6)$-multicovering of $K_{10}$ has a padding of cardinality 3.

A $(C_6, \overline{C}_6)$-multicovering of $K_{10}$ with a padding of cardinality 3 can be obtained by starting with a $\overline{C}_6$-decomposition of $K_{10}$. One copy of $\overline{C}_6$ can be transformed into two copies of $C_6$ by carefully adding three edges. This gives us the desired $(C_6, \overline{C}_6)$-multicovering of $K_{10}$ where the three added edges form the padding.

**Example 14.** A minimum $(C_6, \overline{C}_6)$-multicovering of $K_{11}$ has a padding of cardinality 2.

Let $V(K_{11}) = \{1, ..., 11\}$. The following is a minimum $(C_6, \overline{C}_6)$-multicovering of $K_{11}$, with padding of $\{\{3, 4\}, \{8, 11\}\}$.

$\{[1, 2, 11; 6, 5, 7], [1, 3, 5; 10, 2, 9], [4, 6, 10; 7, 9, 8]\}$

$\cup \{[3, 4, 5, 8, 11, 6], (1, 8, 2, 7, 3, 9), (2, 4, 9, 11, 8, 6), (1, 4, 3, 11, 10, 7),$

$\{3, 8, 4, 11, 5, 10\}\}$

**Example 15.** A minimum $(C_6, \overline{C}_6)$-multicovering of $K_{17}$ has a padding of cardinality 2.

Let $V(K_{17}) = \{1, ..., 17\}$. Apply Theorem 1.5 and let $B_1$ be a $C_6$-decomposition on the copy of $K_9$ formed by the subgraph induced by the vertices $\{9, \ldots, 17\}$. Apply Theorem 1.4 and let $B_2$ be a $C_6$-decomposition of the copy of $K_{6,8}$ formed by the subgraph of $K_{17}$ with vertex bipartition
Multidesigns for the graph pair formed by the 6-cycle and 3-prism

The pair (A, B) where $A = \{1, \ldots, 8\}$ and $B = \{12, \ldots, 17\}$. The following is a minimum $(C_6, \overline{C}_6)$-multicovering of $K_{17}$, with padding of $\{(3, 5), (7, 8)\}$.

$$B_1 \cup B_2 \cup \{1, 2, 3, 6, 5, 4\}, [1, 4, 8; 7, 2, 6]$$

$$\cup \{(1, 5, 7, 8, 3, 9), (1, 10, 3, 7, 4, 11), (2, 8, 7, 11, 6, 9)\}$$

$$\cup \{(5, 11, 8, 9, 7, 10), (3, 5, 9, 4, 10, 6), (2, 11, 3, 5, 8, 10)\}$$

4.2. General constructions of minimum multicoverings

**Lemma 4.1.** If $n \equiv 2 \pmod{6}$ with $n \geq 8$, then $K_n$ admits a minimum $(C_6, \overline{C}_6)$-multicovering with a padding of cardinality 2.

**Proof.** Let $n = 6x + 2$ for some integer $x \geq 1$. Notice that $K_{6x+2} \cong K_8 \lor \bigvee_{i=1}^{x-1} K_6$. Each of the $x - 1$ copies of $K_6$ admit a $(C_6, \overline{C}_6)$-multidecomposition by Lemma 2.5. The copy of $K_8$ admits a $(C_6, \overline{C}_6)$-multicovering with a padding of cardinality 2 by Example 11. The remaining edges form edge disjoint copies of $K_{6,6}$ or $K_{6,8}$, each of which admit a $C_6$-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multicovering of $K_n$. □

**Lemma 4.2.** If $n \equiv 5 \pmod{6}$ with $n \geq 11$, then $K_n$ admits a minimum $(C_6, \overline{C}_6)$-multicovering with a padding of cardinality 2.

**Proof.** Let $n = 6x + 5$ for some integer $x \geq 1$. The proof breaks into two cases.

**Case 1:** $x = 2k$ for some integer $k \geq 1$. Notice that $K_{12k+5} \cong K_1 \lor K_4 \lor \bigvee_{i=1}^{k} K_{12}$. One copy of $K_{17}$ is formed by $K_1 \lor K_4 \lor K_{12}$, and admits a $(C_6, \overline{C}_6)$-multicovering with a padding of cardinality 2 by Example 15. The $k - 1$ copies of $K_{13}$ formed by $K_1 \lor K_{12}$ admit a $(C_6, \overline{C}_6)$-multidecomposition by Example 1. The remaining edges form edge disjoint copies of $K_{12,12}$ or $K_{4,12}$, each of which admits a $C_6$-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multicovering of $K_n$.

**Case 2:** $x = 2k + 1$ for some integer $k \geq 1$. Notice that $K_{12k+11} \cong K_1 \lor K_4 \lor K_6 \lor \bigvee_{i=1}^{k} K_{12}$. One copy of $K_{11}$ is formed by $K_1 \lor K_4 \lor K_6$, and admits a $(C_6, \overline{C}_6)$-multicovering with a padding of cardinality 2 by Example 14. The $k$ copies of $K_{13}$ formed by $K_1 \lor K_{12}$ admit a $(C_6, \overline{C}_6)$-multidecomposition by Example 1. The remaining edges form edge disjoint copies of $K_{12,12}$, $K_{4,12}$, or $K_{6,12}$, each of which admits a $C_6$-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multicovering of $K_n$. □

Combining Lemmas 4.1 and 4.2, we have proven Theorem 1.3.

5. Conclusion

The cardinalities of the leaves of maximum $(C_6, \overline{C}_6)$-multipackings and paddings of minimum $(C_6, \overline{C}_6)$-multicoverings of $K_n$ have been characterized. However, the achievable structures of these leaves and paddings are still yet to be characterized. This leads to the following open question.

**Open Problem 1.** For each positive integer $n$, characterize all possible graphs (multigraphs) which are leaves (paddings) of a $(C_6, \overline{C}_6)$-multipacking (multicovering) of $K_n$.
Furthermore, it would be of interest to know when a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_n\) exists with \(p\) copies of \(C_6\) and \(q\) copies of \(\overline{C}_6\) where \((p, q)\) is any solution to the equation \(6p + 9q = \binom{n}{2}\). This leads to the following open problem.

**Open Problem 2.** Let \(p, q\) and \(n\) be positive integers for which \(6p + 9q = \binom{n}{2}\). Determine whether a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_n\) exists with \(p\) copies of \(C_6\) and \(q\) copies of \(\overline{C}_6\).

**Acknowledgement**

We would like to thank Mark Liffiton and Wenting Zhao for finding \((C_6, \overline{C}_6)\)-multidecompositions of \(K_{11}\) and \(K_{17}\) using the MiniCard solver. MiniCard source code is available at https://github.com/liffiton/minicard.

**References**

[1] A. Abueida and M. Daven, Multidesigns for Graph-Pairs of Order 4 and 5, *Graphs and Combinatorics* (2003) 19, 433–447.

[2] P. Adams, D. Bryant, and M. Buchanan, A Survey on the Existence of G-Designs, *J. Combin. Designs* **16** (2008), 373–410.

[3] Q. Kang, H. Zhao, and C. Ma, Graph designs for nine graphs with six vertices and nine edges, *Ars Combin.* **88** (2008), 379–395.

[4] M. Šajna, Cycle decompositions III: Complete graphs and fixed length cycles, *J. Combin. Des.* **10** (2002) 1, 27–78.

[5] D. Sotteau, Decomposition of \(K_{m,n} (K_{m,n}^*)\) into Cycles (Circuits) of Length \(2k\), *J. Combin. Theory Ser. B* **30** 1981, 75–81.