A Hopf’s Lemma and the Boundary Regularity for the Fractional P-Laplacian

Lingyu Jin * Yan Li †

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Abstract

We begin the paper with a Hopf’s lemma for a fractional p-Laplacian problem on a half-space. Specifically speaking, we show that the derivative of the solution along the outward normal vector is strictly positive on the boundary of the half-space. Next we show that positive solutions to a fractional p-Laplacian equation possess certain Hölder continuity up to the boundary.

Key words Fractional p-Laplacian; Dirichlet problem; Hopf’s lemma; Boundary regularity

1 Introduction

The strong maximum principle of Eberhard Hopf, often known as the Hopf’s lemma[1], is a classical and fundamental result of the theory of second order elliptic partial differential equations. Its main idea is that if a function satisfies a second order partial differential inequality of a certain kind in a domain of \( \mathbb{R}^n \) and attains a maximum in the domain then the function is constant. The Hopf’s lemma has been generalized to describe the behavior of the solution to an elliptic problem as it approaches a point on the boundary where its maximum is attained.

*Partially supported by .
†Corresponding author.
Its history can be first traced back to the maximum principle for harmonic functions. In the past decade this lemma has been generalized as the strong maximum principle for singular quasi-linear elliptic differential inequalities ([5]). For a while it was thought that the Hopf’s maximum principle applies only to linear differential operators. In the later sections of his original paper, however, Hopf considered a more general situation which permits certain nonlinear operators and, in some cases, leads to uniqueness statements in the Dirichlet problem for the mean curvature operator and the Monge-Ampère equation.

In the first part of this paper, we prove a Hopf’s lemma for a nonlinear non-local pseudo-differential operator – the fractional p-Laplacian. Nonlocal fractional operators, in particular the fractional Laplacian, have gained a lot of popularity among researchers working in a variety of fields. For instance, the fractional Laplacian has been utilized to model the dynamics in the Hamiltonian chaos in astrophysics (see [6]), random motions such as the Brownian motion and the Poisson process in physics (see [7] and [8]), the jump process in finance and probability (see [9]) as well as the the acoustic wave in mechanics. In the diffusion process, it has been used to derive heat kernel estimates for a large class of symmetric jump-type processes (see [10], [11]). The fractional Laplacian has also been applied to the study of the game theory, image processing, Lévy processes, optimization and so on. Readers who are interested in the application of the fractional Laplacians can refer to [12], [13], [14] and the references therein.

The interest in the fractional operators continues to grow strong in this decade. More and more beautiful results, whose counterparts are powerful tools in elliptic PDE analysis, have been proved in the fractional setting. The generalization, however, is no small feat due to the essential difference in how the fractional operators and the traditional differential operators are defined. In light of this, let’s take a look at the fractional p-Laplacian. Let \( s \in (0, 1) \) and \( p > 1 \). The fractional p-Laplacian is defined as

\[
(-\Delta)^s_p u(x) = C_{n,s,p} \lim_{\varepsilon \to 0} \int_{R^n \setminus B_{\varepsilon}(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+ps}} \, dy
\]

\[
= C_{n,s,p} \text{PV} \int_{R^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+ps}} \, dy, \quad (1)
\]

where PV stands for the Cauchy principal value. To ensure that the integral
in (1) is well defined, we assume that
\[ u \in C^{1,1}_{loc}(\Omega) \cap L^{sp}(\mathbb{R}^n) \]
with
\[ L^{sp} = \{ u \in L^1_{loc}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|u(x)|^{p-1}}{1 + |x|^{n+sp}} dx < \infty \} . \]

When \( s = 1 \) in Eq.(1), it is the p-Laplacian. When \( p = 2 \), Eq.(1) becomes the nonlocal fractional Laplacian \((-\Delta)^s\). A quick observation of the integral domain \( \mathbb{R}^n \) easily points to a characteristic shared among such integro-differential operators. Different from the traditional differential operators, such as the Laplace operator, these are not locally defined. To give an example of what’s new in nonlocal problems compared with local ones, we consider the Dirichelet and the Neumann problem on a bounded domain \( \Omega \subset \mathbb{R}^n \). To study the Laplacian problems, we require of information of solutions on the boundary \( \partial \Omega \). But this is not enough for the fractional problems, which demand knowledge of solutions on both \( \partial \Omega \) and \( \mathbb{R}^n \setminus \bar{\Omega} \). This raises a natural discussion in how to install appropriate boundary conditions in different cases so that the solutions can be extended in a way that preserve proper regularity in the whole space \( \mathbb{R}^n \). The challenge is especially true in computer-based simulations given that there is a limited amount of data we can gather over time. On top of this, when \( p \neq 2 \), the complexity increases because nonlinearity appears in the numerator.

In this paper, we are interested in the fractional p-Laplacian problems with Dirichlet boundary conditions. Our first main result is a Hopf’s lemma in a half-space. So far there are a few interesting results in the fractional settings on the Hopf’s maximum principle.

In [2], Caffarelli et al. quoted a generalized Hopf’s lemma for the smooth solution to a harmonic fractional equation on a smooth domain \( \Omega \subset \mathbb{R}^n \). Either by the Harnack inequality or the Riesz potential, they claimed it true that if there is a point \( X_0 \in \partial \Omega \) for which \( v(X_0) = 0 \), then there exists \( \lambda > 0 \) such that \( v(x) \geq \lambda ((x - X_0) \cdot \nu(X_0))^\alpha \), where \( \nu(X_0) \) is the inner normal to \( \partial \Omega \) at \( X_0 \).

In [17], Greco and Servadei considered
\[ (-\Delta)^s u(x) \geq c(x) u(x), \quad x \in \Omega. \]
Assuming that \( c(x) \leq 0 \) in bounded domain \( \Omega \), they derived that
\[ \inf_{x \to \partial \Omega} \frac{u(x)}{(\text{dist}(x, \partial \Omega))^s} > 0, \quad \text{as } x \to \partial \Omega. \quad (2) \]
Quite recently, Chen and Li [3] proved a Hopf’s lemma in terms of the boundary derivative for anti-symmetric functions on a half space through an elementary yet rather delicate analysis.

Lemma 1.1 (Chen-Li) Assume that \( w \in C^3_{\text{loc}}(\bar{\Sigma}) \), \( \lim_{x \to \partial \Sigma} \frac{1}{\text{dist}(x, \partial \Sigma)^2} = o(1) \), and

\[
\begin{align*}
(-\Delta)^s w(x) + c(x)w(x) &= 0, \quad \text{in } \Sigma, \\
w(x) &> 0, \quad \text{in } \Sigma, \\
w(x^\lambda) &= -w(x), \quad \text{in } \Sigma.
\end{align*}
\]

Then

\[
\frac{\partial w}{\partial \nu}(x) < 0, \quad x \in \partial \Sigma.
\]

In [16], Pezzo and Quass considered a fractional p-Laplacian problem on a bounded domain \( \Omega \) satisfying the interior ball condition:

\[
(-\Delta)^p u = c(x)|u|^{p-2}u, \quad x \in \Omega.
\]

(3)

Under certain assumptions on \( c(x) \), they obtained a similar result to that in [2] for the weak super-solution of (3). Following the spirit in [3], we present a Hopf’s lemma for \((-\Delta)^p\) via the boundary derivative.

Let

\[
T_\lambda = \{ x \in \mathbb{R}^n \mid x_1 = \lambda, \text{ for some } \lambda \in \mathbb{R} \}
\]

be the moving planes,

\[
\Sigma_\lambda = \{ x \in \mathbb{R}^n \mid x_1 > \lambda \}
\]

be the region to the right of the plane \( T_\lambda \),

\[
x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)
\]

be the reflection of \( x \) about \( T_\lambda \) and

\[
w_\lambda(x) = u_\lambda(x) - u(x).
\]

Theorem 1 For \( p \geq 3 \), assume that \( u \in C^3_{\text{loc}}(\Sigma) \cap L_{sp} \) and satisfies

\[
\begin{align*}
(-\Delta)^p u_\lambda(x) - (-\Delta)^p u(x) + c(x)w(x) &= 0, \quad \text{in } \Sigma, \\
w(x) &> 0, \quad \text{in } \Sigma, \\
w(x^\lambda) &= -w(x), \quad \text{in } \Sigma.
\end{align*}
\]

(4)
Let $\nu$ be the outward normal vector on $\partial \Sigma$. If
\[
\lim_{x \to \partial \Sigma} c(x) = o\left(\frac{1}{\text{dist}(x, \partial \Sigma)^2}\right),
\]
then
\[
\frac{\partial w}{\partial \nu}(x) < 0, \quad x \in \partial \Sigma.
\]

Following this we present our second main result—a Lipschitz boundary regularity for the fractional $p$-Laplacian.

In [20], Bogdan derived a boundary Harnack inequality for nonnegative solutions for a harmonic fractional problem with Dirichlet condition. Other boundary regularity for fractional equations were obtained by Caffarelli et al. in [2] for a homogeneous fractional heat equation, and by Kim and Lee in [21] for a free boundary problem for the fractional Laplacian. In both papers the authors proved that the limit of $u(x) / \text{dist}(x, \partial \Omega)$ exists point-wise on the boundary.

In a recent paper by Ros-Oton and Serra [15], the authors considered
\[
\begin{cases}
(-\Delta)^s u = g, & x \in \Omega, \\
u = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]
Under the assumption that $g \in L^\infty(\Omega)$ for a bounded $\Omega$, they proved that the solution is $C^s(\mathbb{R}^n)$ and $\frac{u(x)}{\text{dist}(x, \partial \Omega)}$ is $C^\alpha$ up to $\partial \Omega$ through a Krylov boundary Harnack inequality.

Later, in [19] Chen et al. proved a similar result for the classical solutions through a good plain argument. The closest result to ours was obtained by Iannizzotto et al. in [23]. There the authors proved $C^\alpha$ regularity, $\alpha \in (0, s]$, up to the boundary for the weak solutions of a fractional $p$-Laplacian problem. Their proof was carried out in the spirit of Krylov's approach to boundary regularity and was quite complicated.

Inspired the work in [19] and [23], we apply some of the ideas in [19] on the following equation,
\[
\begin{cases}
(-\Delta)^p u(x) = f(x), & x \in \Omega, \\
u \equiv 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]

**Theorem 2** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with exterior tangent spheres on the boundary, $s \in (0, 1)$ and $p > 2$. Assume that $\|f\|_{L^\infty(\Omega)} < \infty$ and
\(u(x) \in L^p\). If \(u\) is a solutions of (7), then there exists some \(\nu \in (0, s)\) such that for \(x\) close to the boundary
\[
|u(x)| \leq c[\text{dist}(x, \partial \Sigma)]^\nu, \quad x \in \Sigma. \tag{8}
\]

Remark 1 For \(p = 2\), \(\nu\) can be up to \(s\) (see [15]).

For convenience’s sake, we let
\[
|u(x)|^{p-2}u(x) =: [u(x)]^{p-1}.
\]

Throughout the paper, we denote positive constants by \(c, C_i\) whose values may vary from line to line.

2 A Hopf’s Lemma

In this section, we prove Theorem 1. For simplicity, in this section, we write \(w_\lambda\) as \(w\) and \(\Sigma_\lambda\) as \(\Sigma\).

Proof. To prove (6), we develop a contradictive argument. Suppose there exists some \(\tilde{x}\in \partial \Sigma\) such that (6) is not true, then
\[
\frac{\partial w}{\partial \nu}(\tilde{x}) = 0. \tag{9}
\]
Without loss of generality, let \(\lambda = 0\) and \(\tilde{x}\) be the origin. Let the ray from \(\tilde{x}\) in the direction of \(-\nu\) be the \(x_1\) axis. By (9) and the anti-symmetry of \(w\), we know that
\[
\frac{\partial^2 w}{\partial x_1^2}(0) = 0.
\]
For some \(\bar{x} = (\bar{x}, 0') \in \mathbb{R}^n\) and close to the origin, by the Taylor expansion, we obtain
\[
w(\bar{x}) = w(0) + Dw(0) \cdot \bar{x} + \bar{x} \cdot D^2 w(0) \cdot \bar{x}^T + O(|\bar{x}|^3) = O(|\bar{x}|^3). \tag{10}
\]
For simplicity’s sake, let
\[
\delta = |\bar{x}_1| = \text{dist}(\bar{x}, T_0). \tag{11}
\]
Then we have \(w(\bar{x}) = O(\delta^3)\), and
\[
|Dw(\bar{x})| = O(\delta^2), |D^2 w(\bar{x})| = O(\delta). \tag{12}
\]
For $\bar{x}$ sufficiently close to the origin, i.e. $\delta$ sufficiently small, it’s trivial that
\begin{equation}
    c(\bar{x})w(\bar{x}) = o(1)\delta. \tag{13}
\end{equation}

Using the estimate we have on $w_\lambda$ and its derivatives, we can prove that for $\delta$ small and some $c_1 > 0$, it holds that
\begin{equation}
    (-\Delta)^s_p u_\lambda(\bar{x}) - (-\Delta)^s_p u(\bar{x}) \leq -\frac{c_1}{4} \delta. \tag{14}
\end{equation}

We postpone the proof of (14) for the moment. Combining (13) and (14), we arrive at
\[ (-\Delta)^s_p u_\lambda(\bar{x}) - (-\Delta)^s_p u(\bar{x}) + c(\bar{x})w(\bar{x}) < 0. \]

This contradicts to (4) and thus proves the theorem.

Now we prove (14). Recall that $y^\lambda = y^0 = (-y_1, y')$. By (1), we have
\begin{align*}
    (-\Delta)^s_p u_\lambda(\bar{x}) - (-\Delta)^s_p u(\bar{x}) &= C_{n,s,p} \text{PV} \int_{R^n} \frac{(u_\lambda(\bar{x}) - u_\lambda(y))^{p-1} - (u(\bar{x}) - u(y))^{p-1}}{|\bar{x} - y|^{n+ps}} dy \\
    &= C_{n,s,p} \text{PV} \int_{\Sigma} \frac{(u_\lambda(\bar{x}) - u_\lambda(y))^{p-1} - (u(\bar{x}) - u(y))^{p-1}}{|\bar{x} - y|^{n+ps}} dy \\
    &\quad + C_{n,s,p} \int_{R^n/\Sigma} \frac{(u_\lambda(\bar{x}) - u_\lambda(y))^{p-1} - (u(\bar{x}) - u(y))^{p-1}}{|x - y|^{n+ps}} dy \\
    &= C_{n,s,p} \text{PV} \int_{\Sigma} \frac{(u_\lambda(\bar{x}) - u_\lambda(y))^{p-1} - (u(\bar{x}) - u(y))^{p-1}}{|\bar{x} - y|^{n+ps}} dy \\
    &\quad + C_{n,s,p} \int_{\Sigma} \frac{(u_\lambda(\bar{x}) - u(y))^{p-1} - (u(\bar{x}) - u_\lambda(y))^{p-1}}{|\bar{x} - y^0|^{n+ps}} dy \\
    &= C_{n,s,p} \text{PV} \int_{\Sigma} \left(1 - \frac{1}{|\bar{x} - y|^{n+ps}} + \frac{1}{|\bar{x} - y^0|^{n+ps}} \right) dy \\
    &\quad + C_{n,s,p} \int_{\Sigma} \frac{(u_\lambda(\bar{x}) - u_\lambda(y))^{p-1} - (u(\bar{x}) - u(y))^{p-1}}{|\bar{x} - y^0|^{n+ps}} dy \\
    &\quad + (u_\lambda(\bar{x}) - u(y))^{p-1} - (u(\bar{x}) - u_\lambda(y))^{p-1} dy \\
    &=: C_{n,s,p} \text{PV} \int_{\Sigma} I dy + C_{n,s,p} \int_{\Sigma} II dy. \tag{15}
\end{align*}
We first take care of $\int_{\Sigma} II \, dy$ for later.  
Let $R_o > 0$ be a given positive number. Then

$$ \Sigma = (\Sigma \cap B_{R_o}(\bar{x})) \cup (\Sigma \cap B_{R_o}^c(\bar{x})). $$

For $y \in \Sigma \cap B_{R_o}(\bar{x})$, by the mean value theorem we have

$$ (u_\lambda(\bar{x}) - u_\lambda(y))^{p-1} - (u(\bar{x}) - u(y))^{p-1} + (u_\lambda(\bar{x}) - u(y))^{p-1} - (u(\bar{x}) - u_\lambda(y))^{p-1} $$

$$ = (p-1)(|\xi_1|^{p-2} + |\xi_2|^{p-2}) w_\lambda(\bar{x}) $$

$$ \leq cw_\lambda |\bar{x} - y^0|^{p-2}, \quad (16)$$

with $\xi_1$ between $u_\lambda(\bar{x}) - u_\lambda(y)$ and $u(\bar{x}) - u_\lambda(y)$, $\xi_2$ between $u_\lambda(\bar{x}) - u(y)$ and $u(\bar{x}) - u(y)$. The last inequality holds because, under the assumption $w(y) > 0$ for $y \in \Sigma$, we have

$$ |\xi_1| \leq \max\{|u_\lambda(\bar{x}) - u_\lambda(y)|, |u(\bar{x}) - u_\lambda(y)|\} $$

$$ < \max\{|u_\lambda(\bar{x}) - u_\lambda(y)|, |u(\bar{x}) - u(y)|\} $$

$$ \leq c \max\{|\bar{x} - y^0|, |\bar{x} - y|\} = c|\bar{x} - y^0|. $$

Similarly, one can show that

$$ |\xi_2| < c \max\{|\bar{x} - y^0|, |\bar{x} - y|\} = c|\bar{x} - y^0|. $$

From (16), for $\delta$ sufficiently small we deduce that

$$ \int_{\Sigma \cap B_{R_o}(\bar{x})} |II| \, dy = \int_{\Sigma \cap B_{R_o}(\bar{x})} II \, dy $$

$$ \leq c \int_{\Sigma \cap B_{R_o}(\bar{x})} \frac{w(\bar{x})}{|\bar{x} - y^0|^{n+ps-p+2}} \, dy $$

$$ \leq cw(\bar{x}) \int_{B_{2R_o}(\bar{x}) \setminus B_\delta(\bar{x})} \frac{1}{|\bar{x} - y^0|^{n+ps-p+2}} \, dy $$

$$ \leq c_1 \max\{\delta^{1+p-ps}, \delta^2\}. \quad (17)$$
For $y \in \Sigma \cap B^c_{R_0}(\bar{x})$, using $u \in L^p$ and the Hölder inequality we have

$$
\int_{\Sigma \cap B^c_{R_0}(\bar{x})} II \, dy \\
\leq c w(\bar{x}) \int_{\Sigma \cap B^c_{R_0}(\bar{x})} \frac{|u(\bar{x})|^{p-2} + |u(\lambda(\bar{x}))|^{p-2} + |u(y)|^{p-2} + |u(\lambda(y))|^{p-2}}{|\bar{x} - y^0|^{n+ps}} \, dy \\
\leq c w(\bar{x}) \left[ C \int_{|y| \geq R/2} \frac{1}{(1 + |y|)^{n+ps}} \, dy + 2 \int_{|y| \geq R/2} \frac{|u(y)|^{p-2}}{(1 + |y|)^{n+ps}} \, dy \right] \\
\leq c w(\bar{x}) \left( C + 2 \left( \int_{|y| \geq R/2} \frac{|u(y)|^{p-1}}{(1 + |y|)^{n+ps}} \, dy \right)^{\frac{p}{p-1}} \left( \int_{|y| \geq R/2} \frac{1}{(1 + |y|)^{n+ps}} \, dy \right)^{\frac{1}{p-1}} \right) \\
\leq c w(\bar{x}) \\
\leq c \delta^3.
$$

Together with [17], it shows that for $\delta$ small we have

$$
\int_{\Sigma} I I \, dy \leq c \max\{\delta^{1+p-ps}, \delta^2\}. \tag{18}
$$

Next we estimate $\int_{\Sigma} I \, dy$.

For some $R >> 1$ large, let $B^+_R(0) = \{ x \in B_R(0) \mid x_1 > 0 \}$. To take care of the possible singularities, we divide $B^+_R(0)$ into five subregions( see Fig.2) defined as below.

- $D_1 = \{ x \mid 1 \leq x_1 \leq 2, |x'| \leq 1 \}$,
- $D_2 = \{ x \in B_R(0) \mid x_1 \geq \eta \}$,
- $D_3 = \{ x \mid 0 \leq x_1 \leq 2\delta, |x'| < \delta \}$,
- $D_4 = \{ x \mid 0 \leq x_1 \leq \eta, |x'| < \eta, x \notin D_3 \}$,
- $D_5 = \{ x \in B_R(0) \mid 0 \leq x_1 \leq \eta, |x'| > \eta \}$.

We estimate the integral in each region accordingly. Later, we will discuss the requirements that $R$ and $\eta$ must satisfy. Roughly speaking, we need to take $R$ sufficiently large and $\eta > \delta$ sufficiently small.

We start with $D_1$. By the mean value theorem we have

$$
\frac{1}{|\bar{x} - y|^{n+ps}} - \frac{1}{|\bar{x} - y^0|^{n+ps}} \\
= (-\frac{n + ps}{2}) \frac{1}{|\xi_3|^{\frac{n+ps}{2}+1}} \left( |\bar{x} - y|^{2} - |\bar{x} - y^0|^{2} \right) \\
= \frac{n + ps}{2} \frac{1}{|\xi_3|^{\frac{n+ps}{2}+1}} 4\bar{x}_1 y_1 \tag{19}
$$
Figure 1: Subregions

with
\[ |\bar{x} - y|^2 \leq \xi_3 \leq |\bar{x} - y^0|^2, \]

and
\[
(u_\lambda(\bar{x}) - u_\lambda(y))^{p-1} - (u(\bar{x}) - u(y))^{p-1} = (p - 1)|\xi_4|^{p-2}[w(\bar{x}) - w(y)],
\] (20)

where \(\xi_4\) is between \(u_\lambda(\bar{x}) - u_\lambda(y)\) and \(u(\bar{x}) - u(y)\).

Since \(w(x) > 0\) in \(\Sigma\) and \(w(0) = 0\), for \(y \in D_1\) and \(\bar{x}\) sufficiently close to the origin, it is trivial that
\[
w(\bar{x}) - w_\lambda(y) < -c < 0.
\] (21)

Hence
\[
u_\lambda(\bar{x}) - u_\lambda(y) < u(\bar{x}) - u(y).
\]

Together with (20), it shows that
\[
\xi_4 \neq 0, \quad y \in \Sigma.
\]

Therefore there exists some \(c\) such that
\[
|\xi_4| \geq c > 0.
\]

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Combine this result with (19) and (20), it gives

\[
\int_{D_1} \left( [u_\lambda(\bar{x}) - u_\lambda(y)]^{p-1} - [u(\bar{x}) - u(y)]^{p-1} \right) \left( \frac{1}{|\bar{x} - y|^{n+ps}} - \frac{1}{|\bar{x} - y^0|^{n+ps}} \right) dy
\]

\[
\leq c \int_{D_1} |\xi|^2 [w(\bar{x}) - w(y)] \frac{\bar{x}_1 y_1}{|\bar{x} - y|^{n+ps+2}} dy
\]

\[
\leq - c_1 \delta
\]  

(22)

We estimate the integral on \(D_2\). Later, in the proof for \(D_4\) and \(D_5\), we will discuss the ranges of \(\eta\) and \(R\) respectively. For now, we assume both \(R\) and \(\eta\) have already been selected and fixed. Then it’s obvious that

\[
w_\lambda(x) - w_\lambda(y) \leq 0, \quad y \in \Omega_{R,\eta}, \text{as } \delta \to 0.
\]

Thus

\[
\int_{D_2} \left( [u_\lambda(\bar{x}) - u_\lambda(y)]^{p-1} - [u(\bar{x}) - u(y)]^{p-1} \right) \left( \frac{1}{|\bar{x} - y|^{n+ps}} - \frac{1}{|\bar{x} - y^0|^{n+ps}} \right) dy
\]

\[
\leq \int_{D_1} \left( u_\lambda(\bar{x}) - u_\lambda(y) \right)^{p-1} - (u(\bar{x}) - u(y))^{p-1} \left( \frac{1}{|\bar{x} - y|^{n+ps}} - \frac{1}{|\bar{x} - y^0|^{n+ps}} \right) dy
\]

\[
\leq - c_1 \delta
\]

(23)

On \(D_3\), we separate the integrand \(I\) into two pieces. On one hand, by Taylor expansion, we have

\[
\left| \int_{D_3} [u_\lambda(\bar{x}) - u_\lambda(y)]^{p-1} - [u(\bar{x}) - u(y)]^{p-1} dy \right|
\]

\[
\leq c \int_{D_3} \frac{|\xi|^2}{|\bar{x} - y^0|^{n+ps}} |w(\bar{x}) - w(y)| dy
\]

\[
\leq c \int_{D_3} \frac{|\bar{x} - y|^{p-2}}{|\bar{x} - y^0|^{n+ps}} \left( |Dw(\bar{x}) \cdot (\bar{x} - y)| + |(\bar{x} - y) \cdot D^2w(\bar{x}) \cdot (\bar{x} - y)^T| + |O(|\bar{x} - y|^3)| \right) dy
\]

\[
\leq c \delta^2 \int_{D_3} \frac{1}{|\bar{x} - y^0|^{n+ps-1+p}} dy
\]

\[
\leq c \max\{\delta^2, \delta^{1+p-ps}\}.
\]

(24)

The last inequality is true as \(\delta \to 0\).
On the other hand, for \( \xi_5 \) between \( u(x) - u(y) \) and \( Du(x) \cdot (x - y) \), we have

\[
\int_{D_3} \frac{[u(\bar{x}) - u(\bar{y})]^{p-1}}{|\bar{x} - \bar{y}|^{n+ps}} dy = 0,
\]

as a result of the symmetry of \( D_3 \) with respect to \( \bar{x} \). Similarly, for \( \xi_6 \) between \( u(x) - u(y) \) and \( Du(x) \cdot (x - y) \), we have

\[
\int_{D_3} \frac{[u(\bar{x}) - u(\bar{y})]^{p-1}}{|\bar{x} - \bar{y}|^{n+ps}} dy = 0,
\]

and

\[
\int_{D_3} \frac{|Du(\bar{x}) \cdot (\bar{x} - y)|^{p-1} + (p - 1)|\xi_p|^{p-2}[(\bar{x} - y) \cdot D^2u(\bar{x}) \cdot (\bar{x} - y) + O(|\bar{x} - y|^3)] dy
\]

We obtain the second to last equation from the fact that

\[
\int_{D_3} \frac{|Du(\bar{x}) \cdot (\bar{x} - y)|^{p-1} + (p - 1)|\xi_p|^{p-2}[(\bar{x} - y) \cdot D^2u(\bar{x}) \cdot (\bar{x} - y) + O(|\bar{x} - y|^3)] dy
\]

as a result of the symmetry of \( D_3 \) with respect to \( \bar{x} \). Similarly, for \( \xi_6 \) between \( u(x) - u(y) \) and \( Du(x) \cdot (x - y) \), we have

\[
\int_{D_3} \frac{|Du(\bar{x}) \cdot (\bar{x} - y)|^{p-1} + (p - 1)|\xi_p|^{p-2}[(\bar{x} - y) \cdot D^2u(\bar{x}) \cdot (\bar{x} - y) + O(|\bar{x} - y|^3)] dy
\]

as a result of the symmetry of \( D_3 \) with respect to \( \bar{x} \). Similarly, for \( \xi_6 \) between \( u(x) - u(y) \) and \( Du(x) \cdot (x - y) \), we have

\[
\int_{D_3} \frac{|Du(\bar{x}) \cdot (\bar{x} - y)|^{p-1} + (p - 1)|\xi_p|^{p-2}[(\bar{x} - y) \cdot D^2u(\bar{x}) \cdot (\bar{x} - y) + O(|\bar{x} - y|^3)] dy
\]

as a result of the symmetry of \( D_3 \) with respect to \( \bar{x} \). Similarly, for \( \xi_6 \) between \( u(x) - u(y) \) and \( Du(x) \cdot (x - y) \), we have

\[
\int_{D_3} \frac{|Du(\bar{x}) \cdot (\bar{x} - y)|^{p-1} + (p - 1)|\xi_p|^{p-2}[(\bar{x} - y) \cdot D^2u(\bar{x}) \cdot (\bar{x} - y) + O(|\bar{x} - y|^3)] dy
\]
Combining (24) with (25) it gives
\[ | \int_{D_3} I dy | \leq c \max \{ \delta^2, \delta^{1+p-p_s} \}. \] (26)

Below we deal with \( D_4 \).

\[ \left| \int_{D_4} \left( [u_{\lambda}(\bar{x}) - u_{\lambda}(y)]^{p-1} - [u(\bar{x}) - u(y)]^{p-1} \right) \left( \frac{1}{|\bar{x} - y|^{n+p_s}} - \frac{1}{|\bar{x} - y^0|^{n+p_s}} \right) dy \right| \leq c \int_{D_4} |\xi_4|^{p-2} \left| w(\bar{x}) - w(y) \right| \frac{x_1 y_1}{|\bar{x} - y^0|^{n+p_s+2}} dy \]

\[ \leq c \int_{D_4} |\bar{x} - y|^{p-2} \left( |Dw(\bar{x}) \cdot (\bar{x} - y)| + |(\bar{x} - y) \cdot D^2 w(\bar{x}) \cdot (\bar{x} - y)^T| \right) + O(|\bar{x} - y|^3) \left( \frac{\delta}{|\bar{x} - y|^{n+p_s+1}} \right) dy \]

\[ \leq c\delta \int_{B_{2\eta}(\bar{x}) \setminus B_4(\bar{x})} \frac{1}{|\bar{x} - y|^{n+p_s-p}} dy \]

\[ = c\delta \frac{(2\eta)^{p-p_s - \delta_p - p}}{p - p_s} \]

\[ \leq \frac{c_1}{8} \delta. \] (27)

The last inequality is true when \( \eta \) is sufficiently small.

On \( D_5 \), we have

\[ \left| \int_{D_5} \left( [u_{\lambda}(\bar{x}) - u_{\lambda}(y)]^{p-1} - [u(\bar{x}) - u(y)]^{p-1} \right) \left( \frac{1}{|\bar{x} - y|^{n+p_s}} - \frac{1}{|\bar{x} - y^0|^{n+p_s}} \right) dy \right| \leq c \int_{D_5} |\xi_4|^{p-2} \left| w(\bar{x}) - w(y) \right| \frac{x_1 y_1}{|\bar{x} - y^0|^{n+p_s+2}} dy \]

\[ \leq c \int_{D_5} |\bar{x} - y|^{p-2} \left( |Dw(\bar{x}) \cdot (\bar{x} - y)| + |(\bar{x} - y) \cdot D^2 w(\bar{x}) \cdot (\bar{x} - y)^T| \right) + O(|\bar{x} - y|^3) \left( \frac{\delta \eta}{|\bar{x} - y|^{n+p_s+2}} \right) dy \]

\[ \leq c\delta \eta \int_{B_{2R}(\bar{x}) \setminus B_{\eta}(\bar{x})} \frac{1}{|\bar{x} - y|^{n+p_s-p+1}} dy \]

\[ = c\delta \eta \frac{(2R)^{3+p-p_s} - (\eta)^{3+p-p_s}}{3 + p - p_s} \]

\[ \leq \frac{c_1}{8} \delta. \] (28)
The validity of the last inequality results from $\eta$ being sufficiently small for $R$ fixed.

Gathering the estimates on $D_i$, $i = 1, 2, 3, 4, 5$, that is, (22), (23), (26), (27) and (28), it shows that for $\eta$ sufficiently small,

$$\int_{B_R^+(0)} I \, dy \leq -c_1\delta. \quad (29)$$

What remains to do is the integral on $\Sigma \setminus B_R^+(0)$.

$$\left| \int_{\Sigma \setminus B_R^+(0)} \left( [u_{\lambda}(\bar{x}) - u_{\lambda}(y)]^{p-1} - [u(\bar{x}) - u(y)]^{p-1} \right) \left( \frac{1}{|\bar{x} - y|^{n+ps}} - \frac{1}{|\bar{x} - y_0|^{n+ps}} \right) dy \right|$$

$$\leq c\delta \int_{\Sigma \setminus B_R^+(0)} \frac{|u(\bar{x})|^{p-1} + |u_{\lambda}(\bar{x})|^{p-1} + |u_{\lambda}(y)|^{p-1} + |u(y)|^{p-1}}{|\bar{x} - y|^{n+ps+1}} dy$$

$$\leq c\delta \int_{\Sigma \setminus B_R^+(0)} \frac{|u(\bar{x})|^{p-1} + |u_{\lambda}(\bar{x})|^{p-1}}{|\bar{x} - y|^{n+ps+1}} dy + c\delta \int_{\partial^n} \frac{|u(y)|^{p-1}}{(1 + |y|)^{n+ps}} dy$$

$$\leq \frac{c\delta}{R^{1+ps}} + \frac{c\delta}{R}$$

$$\leq \frac{c_1}{8}\delta. \quad (30)$$

The last inequality holds when $R$ is sufficiently large. Together with (29), it gives

$$\int_{\Sigma} I \, dy \leq -\frac{c_1\delta}{2}. \quad (31)$$

Combining this with (18), for $\delta$ sufficiently small, we conclude that

$$(-\Delta)_p^s u_{\lambda}(\bar{x}) - (-\Delta)_p^s u(\bar{x}) \leq -\frac{c_1\delta}{4}.$$

This proves (14) and completes the proof of the theorem.

3 Boundary Regularity

In this section we prove Theorem 2. Here the analysis of regularity up to the boundary is based on the existence of some super-solution, sometimes referred to as the barrier function in boundary regularity analysis, to the fractional $p$-Laplacian equation. To construct the barrier function, we begin with an equation in $R_1^+ := \{x \in R \mid x > 0\}$, whose solution is known explicitly.
Lemma 3.1 For $0 < \nu < s$,

$$(-\Delta)^s_p(x^\nu) = C\nu x^{(p-1)\nu-ps}, \ x \in \mathbb{R}^+,$$

with

$$C\nu = \int_{-\infty}^{+\infty} \frac{(1 - z_+^\nu)^{p-1}}{|1 - z|^{1+ps}} dz > 0.$$

Proof. Since $x > 0$, we have $x_+ = x$, and

$$(-\Delta)^s_p(x^\nu) = \int_{\mathbb{R}} \frac{(x^\nu - y^\nu)}{|x - y|^{1+ps}} dy$$

$$= \int_{-\infty}^{+\infty} \frac{x^{(p-1)\nu}(1 - z_+^\nu)^{p-1}}{x^{1+ps}|1 - z|^{1+ps}} x dz \ (y = xz)$$

$$= x^{(p-1)\nu-ps} \int_{-\infty}^{+\infty} \frac{(1 - z_+^\nu)^{p-1}}{|1 - z|^{1+ps}} dz$$

$$= C\nu x^{(p-1)\nu-ps},$$

with $C\nu = \int_{-\infty}^{+\infty} \frac{(1 - z_+^\nu)^{p-1}}{|1 - z|^{1+ps}} dz$. Then

$$C\nu = \int_{0}^{+\infty} \frac{(1 - z_+^\nu)^{p-1}}{|1 - z|^{1+ps}} dz + \int_{-\infty}^{0} \frac{(1 - z_+^\nu)^{p-1}}{|1 - z|^{1+ps}} dz$$

$$= \int_{0}^{+\infty} \frac{(1 - z_+^\nu)^{p-1}}{|1 - z|^{1+ps}} dz + \frac{1}{ps}.$$

(34)

For $0 < \nu \leq \frac{ps-1}{p-1}$,

$$\int_{0}^{+\infty} \frac{(1 - z_+^\nu)^{p-1}}{|1 - z|^{1+ps}} dz$$

$$= \int_{0}^{1} (1 - z^\nu)^{p-1} dz + \int_{1}^{+\infty} \frac{(1 - z^\nu)^{p-1}}{|1 - z|^{1+ps}} dz$$

$$= \int_{0}^{1} \frac{(1 - z^\nu)^{p-1}}{|1 - z|^{1+ps}} dz + \int_{0}^{1} \frac{(1 - w^\nu)^{p-1}}{|1 - w|^{1+ps}} \frac{1}{w^2} dw$$

$$= \int_{0}^{1} \frac{(1 - z^\nu)^{p-1}}{|1 - z|^{1+ps}} dz + \int_{0}^{1} \frac{(1 - w^\nu)^{p-1}}{|1 - w|^{1+ps}} w^{1+ps-np-1} dw$$

$$= \int_{0}^{1} \frac{(1 - z^\nu)^{p-1}}{|1 - z|^{1+ps}} (1 - z^{ps-np-1}) dz \geq 0.$$
Together with (34) it implies that
\[ C_\nu > 0, \quad \text{for} \ 0 < \nu < \frac{ps - 1}{p - 1}. \] (35)

To continue, we need Lemma 3.1 in \cite{23} which states that
\[ (-\Delta)^s_p(x^s_+) \big|_{x=1} = 0, \quad x \in R. \] (36)

Then for \( \frac{ps - 1}{p - 1} < \nu < s \), it follows that
\[
C_\nu = \int_{\infty}^{\infty} \frac{(1 - z^\nu)^{p-1} - (1 - z^s)^{p-1}}{|1 - z|^{1+ps}} dz
= \int_{0}^{\infty} \frac{(1 - z^\nu)^{p-1} - (1 - z^s)^{p-1}}{|1 - z|^{1+ps}} dz
= \int_{1}^{\infty} \frac{(1 - z^{-\nu})^{p-1} - (1 - z^{-s})^{p-1}}{|1 - z^{-1}|^{1+ps}} \frac{1}{z^2} dz + \int_{0}^{1} \frac{(1 - z^\nu)^{p-1} - (1 - z^s)^{p-1}}{|1 - z|^{1+ps}} dz
= \int_{0}^{1} \frac{(1 - z^\nu)^{p-1} - (1 - z^s)^{p-1}}{|1 - z|^{1+ps}} dz
+ \int_{0}^{1} \frac{(1 - z^s)^{p-1} - (1 - z^s)^{p-1}}{|1 - z|^{1+ps}} dz
= \int_{0}^{1} \frac{(1 - z^\nu)^{p-1} - (1 - z^s)^{p-1}}{|1 - z|^{1+ps}} (1 - z^{ps-1-(p-1)\nu}) dz
+ \int_{0}^{1} \frac{(1 - z^s)^{p-1}}{|1 - z|^{1+ps}} (z^{s-1} - z^{ps-1-(p-1)\nu}) dz > 0.
\]

Together with (35), we conclude that
\[ C_\nu > 0 \quad \text{for} \ \nu \in (0, s). \]

Next we generalize Lemma 3.1 to \( n \)-dimensions. Let \( R^+_n := \{ x \in R^n \mid x_n > 0 \} \).

**Corollary 3.1** For \( 0 < \nu < s \),
\[ (-\Delta)^s_p(x_n)^\nu = C_{\nu,n}(x_n)^{(p-1)\nu - ps}, \quad \text{for} \ x \in R^+_n, \] (37)
\[ \text{with} \]
\[ C_{\nu,n} = C_\nu \int_{0}^{\infty} \frac{t^{n-2}}{(1 + t^2)^{\frac{2+ps}{2}}} dt > 0. \]
Proof. Let \( x = (x', x_n) \in \mathbb{R}^n \), \( r = |x' - y'| \) and \( \tau = |x_n - y_n| \). From Lemma 3.1 we have

\[
(-\Delta)^s p (x_n)^\nu = \int_{\mathbb{R}^n} \frac{((x_n)^\nu - (y_n)^\nu)^{p-1}}{|x - y|^{n + p}} dy
\]

\[
= \int_{\mathbb{R}^n} \frac{((x_n)^\nu - (y_n)^\nu)^{p-1}}{(x_n - y_n)^2 + (x' - y')^2 \frac{n + p s}{2}} dy
\]

\[
= \int_{-\infty}^{+\infty} \frac{(x_n)^\nu - (y_n)^\nu)^{p-1}}{|x_n - y_n|^{n + p s}} dy \int_0^\infty \frac{w_n - 2^n - 2}{\tau^2 + r^2 \frac{n + p s}{2}} dr dn
\]

\[
= \int_{-\infty}^{+\infty} \frac{(x_n)^\nu - (y_n)^\nu)^{p-1}}{|x_n - y_n|^{n + p s}} dy \int_0^\infty \frac{w_n - 2^n - 2}{1 + t^2 \frac{n + p s}{2}} dt (r = \tau t)
\]

\[
= (x_n)^{\nu(p-1) - ps} C \int_0^\infty \frac{w_n - 2^n - 2}{1 + t^2 \frac{n + p s}{2}} dt
\]

\[
:= (x_n)^{\nu(p-1) - ps} C_{\nu,n}.
\]

Now we are ready to construct the barrier function.

Lemma 3.2 Let \( \phi(x) = (|x|^2 - 1)^\nu \) in \( \mathbb{R}^n \) with \( \nu \in (0, s) \). Then there exists some \( \epsilon > 0 \) small and \( C_0 > 0 \) such that

\[
(-\Delta)^s p \phi(x) \geq C_0 (|x| - 1)^{\nu(p-1) - ps}, \quad x \in B_{1+\epsilon}(0) \backslash B_1(0). \tag{38}
\]

Proof. To prove the lemma, we argue by contradiction. Suppose (38) is not true, then there exists a sequence \( \{x^k\} \in B_1(0) \) so that \( |x^k| \to 1 \) and

\[
(-\Delta)^s p \phi(x^k)(|x^k| - 1)^{ps - \nu(p-1)} \to 0, \quad k \to \infty. \tag{39}
\]

Without loss of generality, let \( x^k = (0, 1 + d_k) \). Then

\[
d_k = |x^k| - 1 \to 0, \quad k \to \infty.
\]

Here we use an equivalent form of (1) via the difference quotient

\[
(-\Delta)^s_p \phi(x^k) = \frac{C_{n,s,p}}{2} \int_{\mathbb{R}^n} \frac{[\phi(x^k) - \phi(x^k + y)]^{p-1} + [\phi(x^k) - \phi(x^k - y)]^{p-1}}{|y|^{n + ps}} dy.
\]
Then by Lemma 3.1, we have

\[
\left(\left|x^k - 1\right|^{p \cdot \nu(p-1)}(-\Delta)^{s}\phi(x^k)ight)
= \frac{d_k^{ps - \nu(p-1)}}{2} C_{n,s,p} \int_{\mathbb{R}^n} \frac{[\phi(x^k) - \phi(x^k + y)]^{p-1} + [\phi(x^k) - \phi(x^k - y)]^{p-1}}{|y|^{n+ps}} dy
\]

\[
= C_{n,s,p} d_k^{ps - \nu(p-1)} \left( \int_{\mathbb{R}^n} \frac{\left(\left|x^k \right|^2 - 1\right)_+^{\nu} - \left(\left|x^k + y \right|^2 - 1\right)_+^{\nu}}{|y|^{n+ps}}^{p-1} dy \right)
\]

\[
= C_{n,s,p} d_k^{ps - \nu(p-1)} \left( \int_{\mathbb{R}^n} \frac{\left[d_k^2 + 2d_k \nu - (d_k^2 + 2d_k - 2(1 + d_k)y_n + |y|^2)_+\right]^{p-1}}{|y|^{n+ps}} dy \right)
\]

\[
= \frac{C_{n,s,p}}{2} \int_{\mathbb{R}^n} \frac{\left[(d_k + 2)_+^{\nu} - (d_k + 2 + (1 + d_k)z_n + d_k|z|^2)_+^{\nu}\right]^{p-1}}{|z|^{n+ps}} dz \quad (y = d_k z)
\]

\[
= \frac{2^{(p-1)\nu - 1} C_{n,s,p}}{2} \int_{\mathbb{R}^n} \frac{\left[1 - (1 + z_n)_+^{\nu}\right]^{p-1} + \left[1 - (1 - z_n)_+^{\nu}\right]^{p-1}}{|z|^{n+ps}} dz
\]

\[
= 2^{(p-1)\nu} (-\Delta)^{s}(x_n)_+^{\nu}|_{x_n = 1}
\]

\[
= 2^{(p-1)\nu} C_{\nu,n} > 0.
\]

This is a contradiction with (39).

In addition to the barrier function, we also need a comparison principle for the fractional p-Laplacian (see [23 Lemma 9]).

**Lemma 3.3** Let \( \Omega \) be bounded in \( \mathbb{R}^n \), \( p > 2 \) and \( s \in (0,1) \). Assume that \( u, v \in L_{ps} \). If

\[
\begin{cases}
(-\Delta)^{s}_p u \leq (-\Delta)^{s}_p v, & x \in \Omega, \\
u \leq v, & x \in \Omega^C,
\end{cases}
\]

then \( u \leq v \) in \( \Omega \).
Let’s prove Theorem 2.

**Proof.** Briefly speaking, the proof consists of two parts. In part one, using the comparison principle we show that

\[ \|u\|_{L^\infty(\Omega)} < \infty. \]

In part two, we construct an auxiliary function that is Lipschitz continuous near the boundary so as to cover \( u(x) \) from above.

Let \( g(x) = \min\{2 - x_n, 5\} \). Then

\[ g(x) = (2 - x_n)^+ - (2 - x_n)^+- (2 - y_n)^+ \]

By [23, Lemma 3.1], we know

\[ (-\Delta)^s_p(2 - x_n)^+ = 0, \quad x \in B_1. \]

Hence for \( x \in B_1 \), we have

\[
(-\Delta)^s_p g(x) \geq c > 0 \text{ in } B_1(0).
\]

Let \( \tilde{g}(x) = g(\frac{x}{R})C \) with \( R > 0, C > 0 \) sufficiently large so that \( \Omega \subset B_R(0) \) and

\[ (-\Delta)^s_p \tilde{g}(x) = \frac{C^{p-1}}{R^{ps}} \left[ (-\Delta)^s_p g\left(\frac{x}{R}\right) \right] \geq \frac{cC^{p-1}}{R^{ps}} \geq \|f\|_{L^\infty(\Omega)}. \]

Then it is obvious that

\[
\begin{cases}
(-\Delta)^s_p \tilde{g}(x) \geq (-\Delta)^s_p u(x), & x \in B_R, \\
\tilde{g}(x) \geq u(x), & x \in \mathbb{R}^n \setminus B_R.
\end{cases}
\]

From Lemma 3.3 it follows

\[ u(x) \leq \tilde{g}(x) \leq c \text{ in } \Omega. \]
Similarly we can show that

\[ -u(x) \leq \tilde{g}(x) \leq c \text{ in } \Omega. \]

This proves that

\[ \|u\|_{L^\infty(\Omega)} \leq c. \]

Next we show that \( u(x) \) is \( C^\nu(\bar{\Omega}) \) for \( \nu \in (0, s) \). Here \( C^\nu \) denotes the Lipschitz space. Given \( x^o \in \Omega \) and close to \( \partial \Omega \), let \( \bar{x}^o \in \partial \Omega \) be such that \( \text{dist}(x^o, \partial \Omega) = |x^o \bar{x}^o| \). We show that there exists a constant \( c > 0 \) such that

\[ |u(x^o) - u(\bar{x}^o)| \leq c |x^o - \bar{x}^o|^\nu. \]

(43)

Without loss of generality, we relocate the origin \( O \) so that it is on the line \( x^o \bar{x}^o \) and is outside of \( \Omega \) with \( |ox^o| = 1 \). Let \( \phi(x) = (|x|^2 - 1)^\nu_+ \). Choose \( \xi(x) \) to be a smooth cut-off function so that \( \xi(x) = 0 \) in \( B_1(0) \), \( \xi(x) = 1 \) in \( R^n \setminus B_{1+\epsilon}(0) \) with the same \( \epsilon \) appeared in Lemma 3.2 and \( \xi(x) \in [0, 1] \) in \( R^n \). Let

\[ A(x) = C\phi(x) + \xi(x). \]

Then it is easy to see that \( A(x) \) is \( C^\nu(B_1(0)) \). Without loss of generality, Let

\[ D = B_{1+\epsilon}(0) \setminus B_1(0) \cap \Omega. \]

Given that \( \bar{x}^o \) is near \( \partial \Omega \), it is reasonable to say that \( \bar{x}^o \in D \).

Our goal is to show that

\[
\begin{align*}
\{ & (-\Delta)^s_+ A(x) \geq (-\Delta)^s_+ u(x), \quad x \in D, \\
& A(x) \geq u(x), \quad x \in R^n \setminus D. \quad (44)
\end{align*}
\]

We postpone the proof of (44) for the moment. Together with Lemma 3.3, it yields

\[ A(x) \geq u(x), \quad x \in D. \]

Since

\[ u(x)|_{\partial \Omega} = \xi(x)|_{\partial B_1(0)} = 0, \]

and \( \xi \) is smooth everywhere, we have

\[
|u(x^o) - u(\bar{x}^o)| = |u(x^o)| \leq |A(x^o)| \leq |A(x^o) - \xi(\bar{x}^o)| \leq |C(|x^o|^2 - 1)^\nu_+ + \xi(x^o) - \xi(\bar{x}^o)| \leq C|x^o - \bar{x}^o|^\nu.
\]
This implies $u \in C^\nu(\bar{\Omega})$.

What remains is to show (44). On one hand, it’s easy to see that the boundary condition is satisfied because the $A(x)$ controls $u(x)$ on $\mathbb{R}^n \setminus D$ for $C$ sufficiently large. On the other hand, the fractional inequality on $D$ is valid for $\epsilon$ small because of $\nu < s$ and

\[ (-\Delta)^s_p A(x) \geq C_0(|x| - 1)^{\nu(p-1)-ps}, \quad x \in B_{1+\epsilon}(0) \setminus B_1(0). \tag{45} \]

To verify this, we use an argument similar to that in the proof of Lemma 3.2. Suppose otherwise, then there exists a sequence $\{x^k\} \in D$ so that $|x^k| \to 1$ and

\[ (-\Delta)^s_p A(x^k)(|x^k| - 1)^{\nu(p-1)-ps} \to 0, \quad k \to \infty. \tag{46} \]

Without loss of generality, let $x^k = (0, 1 + d_k)$. Then

\[ d_k = |x^k| - 1 \to 0, \quad k \to \infty. \]

By Lemma 3.1 we have

\[
\begin{align*}
&\frac{(|x^k| - 1)^{ps-\nu(p-1)}(-\Delta)^s_p A(x^k)}{2} \\
&= \frac{d_k^{ps-\nu(p-1)}}{2} C_{n,s,p} \int_{\mathbb{R}^n} \frac{[A(x^k) - A(x^k + y)]^{p-1} + [A(x^k) - A(x^k - y)]^{p-1}}{|y|^{n+ps}} dy \\
&= C_{n,s,p} d_k^{ps-\nu(p-1)} \left( \int_{\mathbb{R}^n} \frac{C(|x^k|^2 - 1)^\nu + C(|x^k + y|^2 - 1)^\nu + \xi(x^k) - \xi(x^k + y)}{|y|^{n+ps}} dy \right. \right. \\
&\quad \left. \left. + \int_{\mathbb{R}^n} \frac{C(|x^k|^2 - 1)^\nu + C(|x^k - y|^2 - 1)^\nu + \xi(x^k) - \xi(x^k - y)}{|y|^{n+ps}} dy \right) \right) \right) \\
&= C_{n,s,p} \frac{d_k^{ps-\nu(p-1)}}{2} \int_{\mathbb{R}^n} \left( \frac{[C(d_k + 2)^\nu - C(d_k + 2 + 2(1 + d_k)z_n + d_k|z|^2)^\nu + \xi(x^k) - \xi(x^k + d_k z)]^{p-1}}{|z|^{n+ps}} \\
&\quad + [C(d_k + 2)^\nu - C(d_k + 2 - 2(1 + d_k)z_n + d_k|z|^2)^\nu + \xi(x^k) - \xi(x^k - d_k z)]^{p-1}} \frac{d_k^{ps-\nu(p-1)}}{dz} \right) dz
\end{align*}
\]

\[ \text{21} \]
\[
C_{n,s,p}^{p-1} \int_{\mathbb{R}^n} \left( \frac{C(d_k + 2)^\nu_+ - C(d_k + 2 + 2(1 + d_k)z_n + d_k|z|^2)^\nu_+ + \nabla \xi(\tilde{z}) \cdot zd_k^{-\nu}p^{-1}}{|z|^{n+ps}} + \frac{C(d_k + 2)^\nu_+ - C(d_k + 2 - 2(1 + d_k)z_n + d_k|z|^2)^\nu_+ + \nabla \xi(\hat{z}) \cdot zd_k^{-\nu}p^{-1}}{|z|^{n+ps}} \right) dz
\]

\[
\rightarrow \frac{C^{p-1}C_{n,s,p}}{2} \int_{\mathbb{R}^n} \frac{(2^\nu - (2 + 2z_n)^\nu)_+p^{-1}}{|z|^{n+ps}} dz
\]

\[
= (2^\nu C)^{p-1} \frac{C_{n,s,p}}{2} \int_{\mathbb{R}^n} \frac{(1 - (1 + z_n)^\nu)_+p^{-1} + (1 - (1 - z_n)^\nu)_+p^{-1}}{|z|^{n+ps}} dz
\]

\[
= (2^\nu C)^{p-1}C_{\nu,n} > 0,
\]

where \(\tilde{z}\) is between \(x^k\) and \(x^k - d_kz\), \(\hat{z}\) is between \(x^k\) and \(x^k + d_kz\). This proves (46) and thus completes the proof of the theorem.

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Authors’ Addresses and E-mails:

Lingyu Jin
College of Science
South China Agricultural University
Guangdong, Guangzhou, 510640, P.R. China
13822276656@126.com

Yan Li
Department of Mathematics
Baylor University
Waco, Texas, 76706, U.S.
Yan_Li1@baylor.edu