Exact solutions for the (3+1)-dimensional Kudryashov-Sinelshchikov equation

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Abstract. In this work, (3+1)-dimensional Kudryashov-Sinelshchikov equation is investigated by using the sine-cosine method and modification of the truncated expansion method. A variety of exact solutions are obtained.

1. Introduction
In this paper, we study the (3+1)-dimensional Kudryashov-Sinelshchikov equation \cite{1}
\begin{equation}
\frac{u_t + \alpha uu_x + \gamma u_{xxx}}{x} + du_{yy} + eu_{zz} = 0,
\end{equation}
where $\alpha$ represents the nonlinearity, $\gamma$ is dispersion term, while $d$ and $e$ stand for transverse variation of wave in $y$ and $z$ directions. The equation (1) describes the physical characteristics of nonlinear waves in a bubbly liquid. In the case $e = 0$, equation (1) reduces to two-dimensional Korteweg-de Vries equation and for case $d = e = 0$ we obtain the one-dimensional Korteweg-de Vries equation. The equation (1) was studied by the modified tanh-coth method \cite{2}, Backlund transformation \cite{3}, bifurcation analysis was presented in \cite{4}, and density fluctuation symbolic computation in \cite{5}. In one-dimensional and two-dimensional cases the Kudryashov-Sinelshchikov equations were studied by the $G'/G$ expansion method in \cite{6}, the first integral method was applied in \cite{7}, modification of truncated expansion method \cite{8}, the modified exp-function method \cite{9}.

The purpose of this work is to find exact solutions for the (3+1)-dimensional Kudryashov-Sinelshchikov equation. The methods for finding exact solution of nonlinear partial differential equations are known. Some of them are the Darboux transformation \cite{10–13}, Hirota bilinear method \cite{14–17}, Kudryashov method \cite{18}, extended tanh method \cite{19, 20}, sine-cosine method \cite{20}. To obtain the exact solution for the (3+1)-dimensional Kudryashov-Sinelshchikov equation we use the two methods such as the sine-cosine method \cite{20} and the modification of the truncated method \cite{8}.

The organization of the paper is as follows: In Section 2, the description of the sine-cosine method and exact solution are given. In section 3, we study the Kudryashov-Sinelshchikov equation by the modification of the truncated method. Finally, the conclusion is given in Section 4.
2. The Sine-cosine method

2.1. Review of the sine-cosine method

In this section we describe sine-cosine method that is presented in [20]. According to method the partial differential equation

\[ E_1(u_t, u_x, u_{xx}, u_{xxx}, \ldots) = 0, \quad (2) \]

can be converted to ODE

\[ E_2(u, u', u'', u''', \ldots) = 0, \quad (3) \]

by using a wave variable

\[ u(x, t) = u(\xi), \quad \xi = x - ct. \quad (4) \]

Then equation (3) is integrated as long as all terms contain derivatives where integration constants are considered zeros. The solutions of ODE can be expressed in the form

\[ u(x, t) = \lambda \cos^\beta(\mu \xi), \quad |\xi| \leq \frac{\pi}{2\mu}, \quad (5) \]

or

\[ u(x, t) = \lambda \sin^\beta(\mu \xi), \quad |\xi| \leq \frac{\pi}{\mu}, \quad (6) \]

where the parameters \( \lambda, \mu \) and \( \beta \) will be determined, and \( \mu \) is wave number and \( c \) is wave speed respectively. The derivatives of (5) become

\[ (u^n)_\xi = -n\beta\mu\lambda^n \cos^{n\beta-1}(\mu \xi) \sin(\mu \xi), \quad (7) \]

\[ (u^n)_{\xi\xi} = -n^2\mu^2\beta^2\lambda^n \cos^{n\beta}(\mu \xi) + n\mu^2\lambda^n\beta(n\beta - 1) \cos^{n\beta-2}(\mu \xi), \quad (8) \]

and the derivatives of (6) have next forms

\[ (u^n)_\xi = -n\beta\mu\lambda^n \sin^{n\beta-1}(\mu \xi) \cos(\mu \xi), \quad (9) \]

\[ (u^n)_{\xi\xi} = -n^2\mu^2\beta^2\lambda^n \sin^{n\beta}(\mu \xi) + n\mu^2\lambda^n\beta(n\beta - 1) \sin^{n\beta-2}(\mu \xi), \quad (10) \]

and so on for the other derivatives. Using (5)-(10) into the reduced ODE gives a trigonometric equation of \( \cos^R(\mu \xi) \) or \( \sin^R(\mu \xi) \) terms. Then, we determine the parameters by first balancing the exponents of each pair of cosine or sine to determine \( R \). Next, we collect all coefficients of the same power in \( \cos^k(\mu \xi) \) or \( \sin^k(\mu \xi) \), where these coefficients have to vanish. The system of algebraic equations among the unknown \( \beta, \lambda \), and \( \mu \) will be given and from that, we can determine coefficients.

2.2. Application the sine-cosine method

In this section, we apply sine-cosine method to the \((3+1)\)-dimensional equation (1). By wave variable

\[ u(x, y, z, t) = u(\xi), \quad \xi = x + y + z - ct, \quad (11) \]

the equation (1) can be converted to

\[ (-c + d + e)u + \frac{\alpha}{2}u^2 + \gamma u'' = 0. \quad (12) \]
Seeking the solution in (5) and (8)
\[
\cos^\beta(\mu \xi) \left[ (-c + d + e)\lambda - \gamma \mu^2 \beta^2 \lambda \right] + \\
\frac{\alpha}{2} \lambda^2 \cos^{2\beta}(\mu \xi) + \mu^2 \lambda^2 \beta(\beta - 1) \cos^{2\beta - 2}(\mu \xi) = 0.
\] (13)

Equating the exponents and the coefficients of each pair of the \(\cos(\mu \xi)\) functions we find the following algebraic system:
\[
2\beta = \beta - 2, \rightarrow \beta = -2.
\] (14)

Substituting equation (14) into equation (13) to get
\[
\cos^{-2}(\mu \xi) \left[ (-c + d + e)\lambda - \gamma \mu^2 \beta^2 \lambda \right] + \\
\frac{\alpha}{2} \lambda^2 \cos^{-4}(\mu \xi) + \mu^2 \lambda^2 \beta(\beta - 1) \cos^{-4}(\mu \xi) = 0.
\] (15)

Equating the exponents and the coefficients of each pair of the \(\cos(\mu \xi)\) functions, we obtain a system of algebraic equations
\[
\cos^{-2}(\mu \xi) : (-c + d + e)\lambda - \gamma \mu^2 \beta^2 \lambda = 0,
\] (16)
\[
\cos^{-2}(\mu \xi) : \frac{\alpha}{2} \lambda^2 + \mu^2 \lambda^2 \beta(\beta - 1) = 0.
\] (17)

Solving the algebraic system (16)-(17), we get:
\[
\lambda = \frac{-12\mu^2}{\alpha}, \quad \mu = -\frac{1}{2} \sqrt{\frac{d + e - c}{\gamma}}.
\] (18)

The result (18) can be easily obtained if we also use the sine method (6) and then we obtain the following exact solutions
\[
u_1(x, y, z, t) = \frac{-12\mu^2}{\alpha} \cos^{-2}\left(\frac{1}{2} \sqrt{\frac{d + e - c}{\gamma}} (x + y + z - ct)\right),
\] (19)
\[
u_2(x, y, z, t) = \frac{-12\mu^2}{\alpha} \sin^{-2}\left(\frac{1}{2} \sqrt{\frac{d + e - c}{\gamma}} (x + y + z - ct)\right),
\] (20)

where \(c \neq d + e\).

3. Modification of the truncated expansion method

3.1. Review modification of the truncated expansion method

The partial differential equation
\[
E_1(u_t, u_x, u_{xx}, u_{xxx}, ...) = 0,
\] (21)

can be converted to ODE
\[
E_2(u, u', u'', u''', ...) = 0,
\] (22)

by using a wave variable
\[
u(x, t) = u(\xi), \quad \xi = x - ct.
\] (23)
To find dominant terms we substitute

\[ u = \xi^{-p}, \quad (24) \]

into all terms of equation (22). Then we ought to compare degrees of all terms of equations and choose two or more with the highest degree. The maximum value of \( p \) is called the pole of the equation (22) and we denote it as \( N \). The method can be applied when \( N \) is integer. The exact solution of equation (22) is looked in the form

\[ u = a_0 + a_1 Q(\xi) + a_2 Q(\xi)^2 + \ldots + a_N Q(\xi)^N, \quad (25) \]

where \( Q(\xi) \) is the following function

\[ Q(\xi) = \frac{1}{1 + e^\xi}. \quad (26) \]

We can calculate number of derivatives by

\[ u_\xi = \sum_{n=0}^{N} a_n n Q^n (Q - 1), \quad (27) \]

\[ u_{\xi\xi} = \sum_{n=0}^{N} a_n n Q^n (Q - 1) [(n + 1)Q - n], \quad (28) \]

\[ u_{\xi\xi\xi} = \sum_{n=0}^{N} a_n n Q^n (Q - 1) [(n^2 + 3n + 2)Q^2 - (2n^2 + 3n + 1)Q + n^2]. \quad (29) \]

**3.2. Application to Kudryashov-Sinelshchikov Equation**

In this section, we apply modification of truncated expansion method to the \((3+1)\)-dimensional Kudryashov-Sinelshchikov equation (1). By wave variable

\[ u(x, y, z, t) = u(\xi), \quad \xi = (x + y + z - ct), \quad (30) \]

the equation (1) can be converted to ODE

\[ (-c + d + e)u + \frac{\alpha}{2} u^2 + \gamma u'' = 0. \quad (31) \]

From equation (31) we find \( N = 2 \) then we look for the solution of equation (31) in the form

\[ u = a_0 + a_1 Q(\xi) + a_2 Q(\xi)^2. \quad (32) \]

The second derivative of equation (32) is

\[ u_{\xi\xi} = a_1 Q + (4a_2 - 3a_1)Q^2 + (2a_1 - 10a_2)Q^3 + 6a_2 Q^4. \quad (33) \]

Substituting (32)-(33) into (31) we obtain the system of algebraic equations in the form

\[ Q^4 : \quad \frac{1}{2} a_2^2 \alpha + 6a_2 \gamma, \quad (34) \]

\[ Q^3 : \quad a_1 a_2 \alpha + 2a_1 \gamma - 10a_2 \gamma, \quad (35) \]

\[ Q^2 : \quad -a_2 c + a_2 d + a_2 e + a_0 a_2 \alpha + \frac{1}{2} a_1^2 \alpha - 3a_1 \gamma + 4a_2 \gamma, \quad (36) \]

\[ Q^1 : \quad a_0 a_1 \alpha - a_1 c + a_1 d + a_1 e + a_1 \gamma, \quad (37) \]

\[ Q^0 : \quad -a_0 c + a_0 d + a_0 e + \frac{1}{2} a_0^2 \alpha. \quad (38) \]
From the system (34)-(38) we can find coefficients with two cases as

\[ \begin{align*}
1) \quad a_0 &= 0, \quad a_1 = \frac{12\gamma}{\alpha}, \quad a_2 = -\frac{12\gamma}{\alpha}, \quad c = d + e + \gamma, \\
2) \quad a_0 &= -\frac{2\gamma}{\alpha}, \quad a_1 = \frac{12\gamma}{\alpha}, \quad a_2 = -\frac{12\gamma}{\alpha}, \quad c = -\gamma + d + e.
\end{align*} \]  

(39) \hspace{1cm} (40)

Substituting (39)-(40) in (32) we obtain exact solutions of equations (1) in the form

\[ \begin{align*}
u_3(x, y, z, t) &= \frac{12\gamma}{\alpha} \frac{1}{1 + e^\xi} - \frac{12\gamma}{\alpha} \left( \frac{1}{1 + e^\xi} \right)^2, \\
u_4(x, y, z, t) &= -\frac{2\gamma}{\alpha} + \frac{12\gamma}{\alpha} \frac{1}{1 + e^\xi} - \frac{12\gamma}{\alpha} \left( \frac{1}{1 + e^\xi} \right)^2,
\end{align*} \]  

(41) \hspace{1cm} (42)

where \( \xi = (x + y + z - ct) \). The graphical representation of \( u_3 \) and \( u_4 \) is shown in Fig. 1

![Graphs](image1.png)

**Figure 1**: Solutions corresponding to \( u_3 \) and \( u_4 \) for \( \alpha = 1, \gamma = 1, d = 0.5, e = 0.5, x = 0, t = 0. \)

4. Conclusion

In this paper, the (3+1)-dimensional Kudryashov-Sinelshchikov equation was studied using the two methods such as the sine-cosine method and the modification of the truncated method. The schemes of the two methods were presented. New exact solutions for the Kudryashov-Sinelshchikov equation were obtained. These methods can be applied to other kinds of nonlinear problems.

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