A twisted integrable hierarchy with $D_2$ symmetry

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May 10, 2014

Abstract

A loop algebra approach to the Gerdjikov-Mikhailov-Valchev (GMV) equation is provided to exploit the associated twisted integrable structure and a new twisted integrable hierarchy is discovered. Using the twisted loop algebra structure, we obtain a transparent treatment of the associated scattering and inverse scattering theory and solve the initial value problem for the GMV equation.

1 Introduction

Symmetry is a novel phenomenon in Nature. It is also an important tool for scientists to unravel complicated dynamics. In the theory of integrable systems, studying symmetries has been one of the central problems and yields rewarding results even beyond the field itself. Roughly speaking, two main approaches are adopted: (1) try to understand specific problems by identifying simple, symmetric structures that lie within them [6], [21], etc; (2) try to classify integrable systems to provide a framework for ordering or understanding more general situation [17], [5], [19], etc.

One successful attempt in classification theory is the study of reduction groups, formulated in [16], [17], and developed in [18], [20], [8], [7]. Recent results have characterized Lax pairs with finite reduction groups of fractional-linear transformations, i.e., $\mathbb{Z}_N$, $D_N$, $T$, $O$ and $I$ and aroused interest in the classification theory of automorphic Lie algebras [12], [13], [14].

Despite progress made in the classification theory of algebraic structures, the analytic properties such as the construction of solutions, the investigation of the inverse
scattering theory, of the above integrable systems remain mostly open. In particular, one of the simplest systems with $D_2$-symmetry is the anisotropic deformation of a multicomponent generalization of the classical Heisenberg ferromagnetic equation:

$$
in \vec{u}_t = (\vec{u}_x - \vec{u}(\vec{u}^* \cdot \vec{u}_x))_x + 4\epsilon \vec{u}(\vec{u}^* \cdot \vec{J}\vec{u}) + A\vec{u},$$

$$\vec{u}^* \vec{u} = 1, \quad \vec{u} \in \mathbb{C}^{N-1}, \quad J^2 = 1, \quad [A, J] = 0, \quad \epsilon > 0. \quad (1.1)$$

Many interesting algebraic and analytic properties of (1.1) are provided in [10] but a complete resolution of the inverse problem and Cauchy problem is still demanded.

On the other hand, in studying symmetries of the generalized sine-Gordon equations (GSGE), famous for being connected to submanifold geometry in Euclidean spaces, Terng introduced twisted $U/K$-hierarchies via a loop group approach [22]. The inverse scattering problem of one prototypical class of twisted $U/K$-hierarchies is then solved by encoding the loop algebra structures into the inverse scattering theory of GSGE [1] and associated submanifold geometry in Minkowski spaces is derived [15]. Twisted $U/K$-hierarchies are integrable hierarchies with $D_2$ symmetry.

Compared to the study of reduction groups, the loop group approach puts more emphasis on an organic assembling of ingredients of symmetries in integrable systems [2], [24], [23], [22], [15]. To illustrate, given three involutions $\tau$, $\sigma_0$ and $\sigma_1$ on a simple Lie group, let $U$ be the real form of $\tau$, $U/K$ be the symmetric space defined by $\sigma_0$, $U = K + P$, and $(L_+, L_-)$ be a splitting of the loop algebra $L(U/K)$ such that

$$\sigma_0(\xi(-\lambda)) = \xi(\lambda), \quad \sigma_1(\xi(1/\lambda)) = \xi(\lambda)$$

for $\xi \in L_+$. Denote the corresponding projection map to $L_\pm$ as $\hat{\pi}_\pm$. A twisted $U/K$-hierarchy is then defined by the Lax pair

$$[\partial_x + \hat{\pi}_+ (mJ_1m^{-1}) , \partial_t + \hat{\pi}_+ (mJ_2m^{-1}) ] = 0, \quad (1.2)$$

for some $m = m(x, t, \lambda) \in L_-$, the loop group corresponding to the loop algebra $L_-$, and constant loops $J_j \in L_+$ with coefficients in a Cartan subalgebra in $P$. The loop group approach is rooted in, enhanced and enriched by the inverse scattering theory.

The purpose of this paper is to provide a loop algebra approach to the anisotropic deformation of the multicomponent generalization of the Heisenberg ferromagnetic equation (1.1), $N=3$, called the GMV equation for simplicity from now on, and to solve the inverse scattering theory. Distinct features discovered are:

- The loop algebra factorization $(L_+, L_-)$ is not of a splitting type. Thus the twisted hierarchy associated with the GMV equation, i.e., the twisted $U^{U(3)}_{U(1)\times U(2)}$-hierarchy, generalizes the twisted $U/K$-hierarchies defined in [22].

- One needs to introduce an extended spectral problem and extended scattering data to solve the inverse problem. The extended spectral problem chosen is that for the twisted $U^{U(4)}_{U(2)\times U(2)}$-hierarchy which shares the same reduction group and can be ”“projected”” to a twisted $U^{U(3)}_{U(1)\times U(2)}$-spectral problem when the scattering data is an extended one.
The paper is organized as follows: in Section 2, we define the twisted $\frac{U(3)}{U(1) \times U(2)}$ hierarchy via a non-split factorization of the loop algebra and compute the explicit formula of a decisive coefficient, for the GMV equation, in the Lax pair. Section 3 is the discussion of the GMV equation and its relation with the twisted $\frac{U(3)}{U(1) \times U(2)}$ hierarchy. Section 4 and 5 are devoted to the scattering and inverse scattering theory of the twisted $\frac{U(3)}{U(1) \times U(2)}$-hierarchy. The Cauchy problems of twisted $\frac{U(3)}{U(1) \times U(2)}$-flows and the GMV equation are solved in Section 6.

We make two special remarks at last. Though the discussion in Section 2 and 3 should be extended for general $N$ by analogy, the spectral problem for (1.1) is no longer defined by an oblique direction [1], [15] and the associated direct problem cannot be solved when $N > 3$. The other remark is a B"acklund transformation theory for the twisted $\frac{U(4)}{U(2) \times U(2)}$-flows should be obtained by adapting the amazing computation and theory for the GSGE [4]. However, the extended scattering data is not preserved under these transformations. So the approach yields no GMV solitons.

Acknowledgements

The author would like to thank Professor Zixiang Zhou for initiating the work at Fudan University in November, 2011 and for many stimulating discussions afterwards which made this work possible. The author was partially supported by NSC 100-2115-M-001-001.

2 The twisted $\frac{U(3)}{U(1) \times U(2)}$-hierarchy

Let $\sigma_i, i = 1, 2$, be involutions on $U(3)$ defined by

$$\sigma_i(x) = J_i x J_i^{-1}, \quad x \in U(3),$$

$$J_1 = \text{diag}(1, -1, -1), \quad J_2 = \text{diag}(1, -1, 1)$$

(2.1)

and $u(3) = K_i \oplus P_i, i = 1, 2$, the Cartan decompositions for $\sigma_i$. Let $K_i$ be the Lie algebras of $K_i$, i.e.,

$$K_1 = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \mid |a_{11}| = 1, \left( \begin{array}{cc} a_{22} & a_{23} \\ a_{32} & a_{33} \end{array} \right) \in U(2) \right\},$$

$$K_2 = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix} \mid |a_{22}| = 1, \left( \begin{array}{cc} a_{11} & a_{13} \\ a_{31} & a_{33} \end{array} \right) \in U(2) \right\},$$

$$P_1 = \left\{ i \begin{pmatrix} 0 & u & v \\ u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix} \in u(3) \right\}, \quad P_2 = \left\{ i \begin{pmatrix} 0 & u & 0 \\ u^* & 0 & v \\ 0 & v^* & 0 \end{pmatrix} \in u(3) \right\}.$$

Hence

$$S = K_1 \cap K_2 = \left\{ \text{diag} \left( e^{i \alpha_1}, e^{i \alpha_2}, e^{i \alpha_3} \right) \mid \alpha_i \in \mathbb{R} \right\},$$

$$S = \left\{ i \text{ diag} \left( \alpha_1, \alpha_2, \alpha_3 \right) \mid \alpha_i \in \mathbb{R} \right\},$$

(2.2)
and

\[ K_1 = S \times_{S_1} K_1', \quad K_1' = 1 \otimes SU(2), \]
\[ K_1 = S +_{S_1} K_1', \quad K_1' = 0 \oplus su(2), \]
\[ S_1 = S \cap K_1' = \{ \text{diag } (1, e^{i\alpha}, e^{-i\alpha}) \mid \alpha \in \mathbb{R} \}, \]
\[ S_1 = S \cap K_1 = \{ i \text{ diag } (0, \alpha, -\alpha) \mid \alpha \in \mathbb{R} \}. \]

Here \( K_1 = S \times_{S_1} K_1' \) means for \( \forall x \in K_1, x = \xi \eta \) with \( \xi \in S, \eta \in K_1' \) and if

\[ x = \xi \eta = \tilde{\xi} \tilde{\eta}, \quad \xi, \tilde{\xi} \in S, \quad \eta, \tilde{\eta} \in K_1', \]

then \( \xi^{-1} \tilde{\xi} = \eta \tilde{\eta}^{-1} \in S_1 \). By analogy \( K_1 = S +_{S_1} K_1' \) is defined by factoizations of elements in \( K_1 \) up to factors in \( S_1 \).

Furthermore, for a fixed \( \epsilon > 0 \), define the loop groups

\[ L^\epsilon = \{ f : \mathfrak{A}_{\sqrt{\epsilon}, \sqrt{\delta}} \to GL_3(\mathbb{C}) \mid (f(\tilde{\lambda}))^* f(\lambda) = I, \sigma_1(f(-\lambda)) = f(\lambda) \}, \]
\[ L^\epsilon_+ = \{ f \in L \mid \sigma_2(f(\epsilon/\lambda)) = f(\lambda) \}, \]
\[ L^\epsilon_- = \{ f \in L \mid f : \mathbb{C}/\mathfrak{D}_{\sqrt{\epsilon}} \to GL_3(\mathbb{C}), f(\infty) \in K_1' \}. \]

Here \( 0 < \delta < 1 \), \( \mathfrak{D}^r \) is the circle of radius \( r \) centered at 0, \( \mathfrak{D}_r \) is the disk of radius \( r \), and \( \mathfrak{A}_{r_1, r_2} \) is the annulus with boundaries \( \mathfrak{D}^{r_1} \) and \( \mathfrak{D}^{r_2} \). Then the Lie algebras of \( L^\epsilon, L^\epsilon_+, L^\epsilon_- \) are

\[ \mathcal{L}^\epsilon = \{ \xi(\lambda) = \sum_{j \leq n_0} \xi_j \lambda^j \mid \xi_j \in \mathcal{K}_1 \text{ if } j \text{ is even, } \xi_j \in \mathcal{P}_1 \text{ if } j \text{ is odd} \}, \]
\[ \mathcal{L}^\epsilon_+ = \{ \xi(\lambda) = \sum_{|j| \leq n_0} \xi_j \lambda^j \in \mathcal{L}^\epsilon \mid \xi_{-j} = \sigma_2(\xi_j) e^j, \xi_0 \in \mathcal{S} \}, \]
\[ \mathcal{L}^\epsilon_- = \{ \xi(\lambda) = \sum_{j \leq 0} \xi_j \lambda^j \in \mathcal{L}^\epsilon \mid \xi_0 \in \mathcal{K}_1' \}. \]

Similarly, we have a non-splitting decomposition \( \mathcal{L}^\epsilon = \mathcal{L}^\epsilon_+ +_{S_1} \mathcal{L}^\epsilon_- \) and can define projections \( \tilde{\pi}_\pm \) of \( \xi \in \mathcal{L}^\epsilon \) onto \( \mathcal{L}^\epsilon_+, \mathcal{L}^\epsilon_- \), up to factors in \( S_1 \), by the following relations:

\[ \tilde{\pi}_+(\xi) = \pi_{S}(\xi_0) + \sum_{0<j\leq n_0} \left( \xi_j \lambda^j + \sigma_2(\xi_j) \left( \frac{\epsilon}{\lambda} \right)^j \right), \]
\[ \tilde{\pi}_-(\xi) = \pi_{\mathcal{K}_1}(\xi_0) + \sum_{0<j\leq n_0} (\xi_{-j} - \sigma_2(\xi_j) e^j) \lambda^{-j}, \]
\[ \xi = \tilde{\pi}_+(\xi) +_{S_1} \tilde{\pi}_-(\xi), \quad \xi_0 = \pi_{S}(\xi_0) +_{S_1} \pi_{\mathcal{K}_1}(\xi_0). \]

Let \( \xi(\lambda) \sim_{S_1} \tilde{\xi}(\lambda) \) mean that \( \xi(\lambda) - \tilde{\xi}(\lambda) \) is a constant loop in \( S_1 \). Finally, let

\[ \mathcal{A} = \{ i \begin{pmatrix} d_1 & r & 0 \\ r & d_1 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \in u(3) : r, d_1, d_3 \in \mathbb{R} \}. \]

(2.8)
be a maximal abelian subalgebra in $u(3)$,

$$a = i \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{P}_1 \cap \mathcal{A},$$

(2.9)

and

$$\hat{J}_{1,0} = a\lambda + \sigma_2(a)\left(\frac{\epsilon}{\lambda}\right) \in \mathcal{P}_1 \cap \mathcal{A} \cap \mathcal{L}_+^\epsilon,$$

$$\hat{J}_k = i^{k-1}a^k\lambda^k - i \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_3 \end{pmatrix} + i^{k-1}\sigma_2(a^k)\left(\frac{\epsilon}{\lambda}\right)^k \in \mathcal{A} \cap \mathcal{L}_+^\epsilon,$$

(2.10)

(2.11)

for $k \in \{1, 2, \ldots\}$. Thus we have the commutativity condition

$$[\hat{J}_{1,0}, \hat{J}_k] = 0.$$

(2.12)

**Definition 1.** The $k$-th twisted $U(3)$-flow, parametrized by $(d_1, d_3)$, in the twisted $U(1) \times U(2)$-hierarchy is the compatibility condition

$$[L, M] = 0,$$

(2.13)

where

$$L = \partial_x - \frac{\partial \Psi}{\partial x} \Psi^{-1} = \partial_x - (\lambda bab^{-1} + \frac{\epsilon}{\lambda} \sigma_2(bab^{-1})),$$

$$M = \partial_t - \frac{\partial \Psi}{\partial t} \Psi^{-1},$$

$$\Psi(x, t, \lambda) = m(x, t, \lambda) e^{x \hat{J}_{1,0} + t \hat{J}_k},$$

(2.14)

(2.15)

(2.16)

for some $m = m(x, t, \cdot) \in L_\epsilon^\times$ and $b(x, t) = m(x, t, \infty) \in \mathfrak{P}_1$ or $\mathfrak{P}_2$. Here

$$\mathfrak{P}_1 = \{ f : R^2 \to K_1'| f(\cdot, t) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{S}, \ \forall t \},$$

$$\mathfrak{P}_2 = \{ f : R^2 \to K_1'| f(\cdot, t) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathcal{S}, \ \forall t \},$$

(2.17)

(2.18)

and $\mathcal{S}$ is the Schwartz space.

We remark that the inverse scattering theory derived in this report shows $m(x, t, \lambda)$ is determined by $\partial_1 b(x, t)$. Hence the $k$-th twisted $U(3)$-flow is a system in $b(x, t)$
and \( b(x, t) \) is also called the \( k \)-th twisted \( U(3) \times U(1) \times U(2) \)-flow if no ambiguity occurs. Theorem 8 will provide the existence theorem by solving the initial value problem of the \( k \)-th twisted \( U(3) \times U(1) \times U(2) \)-flows.

By the definition of twisted \( U(3) \times U(1) \times U(2) \)-flows, one has

\[
\frac{\partial \Psi}{\partial x}^{-1} = \left\{ \left[ \frac{\partial m}{\partial x} + m \left( \lambda a + \frac{\epsilon}{\lambda} \sigma_2(a) \right) \right] e^{xj_{1,0} + tj_k - tj_k m^{-1}} \right\} e^{-xj_{1,0} - tj_k m^{-1}}
\]

\[
= \frac{\partial m}{\partial x} m^{-1} + m \left( \lambda a + \frac{\epsilon}{\lambda} \sigma_2(a) \right) m^{-1}
\]

\[
\sim_{S_1} \hat{\pi}_+ \left( m\hat{j}_{1,0} m^{-1} \right).
\]

Similarly, \( \frac{\partial \Psi}{\partial t}^{-1} \in L_+^t \) as well and

\[
\frac{\partial \Psi}{\partial t}^{-1} = \sum_{1}^{k} \left( P_j \lambda^j + \sigma_2(P_j) \left( \frac{\epsilon}{\lambda} \right)^j \right) + P_0
\]

\[
= \frac{\partial m}{\partial t} m^{-1} + m\hat{j}_k m^{-1},
\]

\[
\sim_{S_1} \hat{\pi}_+ \left( m\hat{j}_k m^{-1} \right)
\]

Thus the \( k \)-th twisted \( U(3) \times U(1) \times U(2) \)-flows satisfy

\[
L \sim_{S_1} \partial_x - \hat{\pi}_+(m\hat{j}_{1,0} m^{-1}),
\]

\[
M \sim_{S_1} \partial_t - \hat{\pi}_+(m\hat{j}_k m^{-1})
\]

which are similar to (1.2).

**Lemma 2.1.** The first twisted \( U(3) \times U(1) \times U(2) \)-flow is the linear system

\[
\frac{\partial}{\partial x} \left( bab^{-1} \right) - \frac{\partial}{\partial t} \left( bab^{-1} \right) = \begin{bmatrix} bab^{-1}, i \left( \begin{array}{ccc} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{array} \right) \end{bmatrix}
\]

where \( c_i \) are real constants.

**Proof.** Following (2.13), (2.14), and (2.19), we obtain

\[
\frac{\partial}{\partial x} \left( bab^{-1} \right) - \frac{\partial}{\partial t} \left( bab^{-1} \right) - \left[ bab^{-1}, P_0 \right] = 0,
\]

\[
\frac{\partial}{\partial x} P_0 - \left[ bab^{-1}, \epsilon \sigma_2(bab^{-1}) \right] - \left[ \epsilon \sigma_2(bab^{-1}), bab^{-1} \right] = 0.
\]

Thus \( P_0 \) is independent of \( x \) (and \( \lambda \)). Therefore \( P_0 \) is constant by taking the limit of (2.20) when \( x \to -\infty \), \( \lambda \to \infty \), and \( m(x = -\infty, t, \lambda = \infty) = b(x = -\infty, t) = 1 \) or

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.
\]
We proceed to characterizing the $k$-th twisted $\mathcal{U}^{(3)}_{\mathcal{U}(1) \times \mathcal{U}(2)}$-flow, $k \geq 2$, by showing that the coefficients $P_j$, $j \neq 0$, defined by (2.19), of $M$ can be computed explicitly in terms of $x$-derivatives of entries of $b(x,t)$. As for $P_0$, owing to the non-splitting factorization $L_\epsilon = L_\epsilon^+ + S_1 L_\epsilon^-$ (up to factors in $S_1$), only the first diagonal entry of $P_0$ can be computed explicitly (in terms of $x$-derivatives of entries of $b(x,t)$). The last two diagonal entries of $P_0$ are (inexplicit) functions in $\partial_j x b(x,t)$ as we have remarked earlier. This phenomenon is distinct from that of twisted flows defined in [15], [22].

**Lemma 2.2.** For the $k$-th twisted $\mathcal{U}^{(3)}_{\mathcal{U}(1) \times \mathcal{U}(2)}$-flow,

$$m(\partial_x - \hat{J}_{1,0})m^{-1} = \partial_x - (bab^{-1}\lambda + \sigma_2(bab^{-1})\frac{\epsilon}{\lambda}), \quad (2.24)$$

$$m(\partial_t - \hat{J}_k)m^{-1} = \partial_t - \sum_{i=1}^{k} \left( P_j \lambda^i + \sigma_2(P_j)\left(\frac{\epsilon}{\lambda}\right)^i\right) - P_0. \quad (2.25)$$

Here $P_i$ are defined by (2.19).

**Proof.** By (2.14), we have

$$(\partial_x \Psi) \Psi^{-1} = bab^{-1}\lambda + \sigma_2(bab^{-1})\frac{\epsilon}{\lambda}.$$

So (2.16) implies

$$(\partial_x m + m \hat{J}_{1,0})m^{-1} = bab^{-1}\lambda + \sigma_2(bab^{-1})\frac{\epsilon}{\lambda},$$

which is equivalent to (2.24). The identity (2.25) can be proved similarly. \hfill \Box

**Lemma 2.3.** For the $k$-th twisted $\mathcal{U}^{(3)}_{\mathcal{U}(1) \times \mathcal{U}(2)}$-flow,

$$[\partial_x - bab^{-1}\lambda - \sigma_2(bab^{-1})\frac{\epsilon}{\lambda}, m\hat{J}_km^{-1}] = 0. \quad (2.26)$$

**Proof.** Because $\hat{J}_k$ are loops with constant coefficients, by (2.12), we have

$$[\partial_x - \hat{J}_{1,0}, \hat{J}_k] = 0,$$

which implies

$$[m(\partial_x - \hat{J}_{1,0})m^{-1}, m(\hat{J}_km^{-1})] = 0.$$

By (2.24), we derive (2.26). \hfill \Box

**Lemma 2.4.** The coefficients $P_j$, $j \neq 0$, and the first diagonal element of $P_0$ of $M$ are fixed functions of components of $\partial_x b$, $0 \leq s \leq k - j$.  

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By the decomposition property \((2.7)\), Lemma \(2.3\) and defining

\[
m\hat{J}_k m^{-1} = \sum_{j=0}^{k} \left( P_j \lambda^j + \sigma_2(P_j)(\frac{\epsilon}{\lambda})^j \right) + P_0 + \sum_{j=0}^{\infty} R_j \lambda^{-j},
\]

we obtain the recursive formula on \(P_j\), \(0 < j \leq k\):

\[
\begin{align*}
P_k &= i^{k-1}ba^kb^{-1} \\
\partial_x P_k - [bab^{-1}, P_{k-1}] &= 0, \\
\partial_x P_{k-1} - [bab^{-1}, P_{k-2}] - [\epsilon \sigma_2(bab^{-1}), P_k] &= 0, \\
\vdots \\
\partial_x P_2 - [bab^{-1}, P_1] - [\epsilon \sigma_2(bab^{-1}), P_3] &= 0, \\
\partial_x P_1 - [bab^{-1}, P_0 + R_0] - [\epsilon \sigma_2(bab^{-1}), P_2] &= 0.
\end{align*}
\]

(2.28)

Therefore, we can adapt the argument of the proof of Lemma \(2.3\) in [15] to prove Lemma \(2.4\). We skip the proof. Instead, we compute the case \(k = 2\) for the purpose of solving the Cauchy problem of the GMV equation in this report.

**Lemma 2.5.** Write

\[
b(x,t) = m(x,t,\infty) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & -\bar{v} \\ 0 & v & \bar{u} \end{pmatrix} \in \mathfrak{P}_1 \cup \mathfrak{P}_2, \quad \tilde{u} = \begin{pmatrix} u \\ v \end{pmatrix},
\]

then

\[
P_1 = \begin{pmatrix} 0 \\ (1 - \tilde{u}\tilde{u}^*)\tilde{u}_x \\ (1 - \tilde{u}\tilde{u}^*)\bar{u}_x \end{pmatrix} (2.30)
\]

for the second twisted \(U(3)\) flow.

**Proof.** We first define \(T = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}\), so

\[
T^{-1}aT = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix},
\]

(2.31)

\[
bT = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (bT)^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -v & u \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},
\]

(2.32)

\[
P_2 = -i \begin{pmatrix} 1 & 0 & 0 \\ 0 & |u|^2 & u\bar{v} \\ 0 & \bar{u}v & |v|^2 \end{pmatrix}, \quad \partial_x P_2 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\bar{u}v)_x & |v|^2_x \\ 0 & (\bar{u}v)_x & |v|^2_x \end{pmatrix}.
\]

(2.33)
By (2.28), we have
\[ [bab^{-1}, P_1] = \partial_x P_2. \] (2.34)

Taking the conjunction \((b^T)^{-1} \cdot (b^T)\) on both sides of (2.34) and using (2.29), (2.31)-(2.33), we obtain
\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & i
\end{pmatrix}
, \begin{pmatrix}
0 & \bar{u}ar{v}_x - \bar{u}_x \bar{v} & 0 \\
0 & 0 & \bar{v}_x - \bar{u}_x \bar{v} & 0
\end{pmatrix}

\]

Thus the off diagonal part of \((b^T)^{-1} P_1 (b^T)\), denoted as \([((b^T)^{-1} P_1 (b^T))^o\), is
\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & i
\end{pmatrix}
, \begin{pmatrix}
0 & \bar{u}ar{v}_x - \bar{u}_x \bar{v} & 0 \\
0 & 0 & \bar{v}_x - \bar{u}_x \bar{v} & 0
\end{pmatrix}

\]

On the other hand, using the minimal polynomial of \((b^T)^{-1} m \hat{J}^2 m^{-1} (b^T)\) is
\[
(X + i(\lambda^2 + d_1 + \frac{\epsilon^2}{\lambda^2})I)(X + id_3 I),
\]
we obtain
\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}
\lambda^2 + (b^T)^{-1} P_1 (b^T) \lambda + (b^T)^{-1} (P_0 + R_0) m^{-1} (b^T) + \cdots

+ i(\lambda^2 + d_1 + \frac{\epsilon^2}{\lambda^2})I \times \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}
\lambda^2 + (b^T)^{-1} P_1 (b^T) \lambda

+ (b^T)^{-1} (P_0 + R_0) m^{-1} (b^T) + \cdots + id_3 I = 0.

Equating the \(\lambda^3\)-coefficient of (2.36) yields the diagonal part of \((b^T)^{-1} P_1 (b^T)\), denoted as \([((b^T)^{-1} P_1 (b^T))^d\), which is 0. Therefore, \((b^T)^{-1} P_1 (b^T) = [((b^T)^{-1} P_1 (b^T))^o\). Together with (2.32) and (2.35), we obtain
\[
P_1 = \begin{pmatrix}
0 & \bar{v}(\bar{u}ar{v}_x - \bar{u}_x \bar{v}) - u(\bar{u}ar{v}_x - \bar{u}_x \bar{v}) \\
-\bar{v}(uvx - u_x v) & 0 & 0 \\
\bar{u}(uvx - u_x v) & 0 & 0
\end{pmatrix}

= \begin{pmatrix}
0 & (1 - \bar{u}ar{u}^*) \bar{u}_x \\
(1 - \bar{u}ar{u}^*) \bar{u}_x & 0_{2 \times 2}
\end{pmatrix}.
\]

\[\square\]
In studying integrable systems with reductions, one of the simplest nontrivial systems introduced by Gerdjikov, Mikhailov, Valchev [10], [11], [9], is the anisotropic multicomponent generalization of the classical Heisenberg ferromagnetic equation:

\[ i\ddot{\mathbf{u}}_t = (\mathbf{u}_x - \ddot{\mathbf{u}}(\mathbf{u}^* \cdot \mathbf{u}_x))_x + 4\epsilon \dot{\mathbf{u}}(\mathbf{u}^* \cdot J \dot{\mathbf{u}}) + A \ddot{\mathbf{u}}, \]  

(3.1)

where

\[ \mathbf{u}^* \mathbf{u} = 1, \quad \mathbf{u}(x, t) \in \mathbb{C}^2, \]  

(3.2)

\[ J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}. \]

The equation (3.1), called the GMV equation for simplicity, has a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) reduced Lax representation

\[ [\mathbf{L}, \mathbf{M}] = 0, \]  

(3.3)

\[ \mathbf{L} = \partial_x - ba^{-1}\lambda - \sigma_2(ba^{-1}) \frac{\epsilon}{\lambda}, \]  

(3.4)

\[ \mathbf{M} = \partial_t - iba^{-1} \lambda^2 - p_1 \lambda - p_0 - \sigma_2(p_1) \frac{\epsilon}{\lambda} - i\sigma_2(ba^{-1}) \epsilon^2, \]  

(3.5)

with

\[ a = i \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{A}, \quad b(x, t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & -\bar{v} \\ 0 & v & \bar{u} \end{pmatrix} \in \mathcal{K}, \]  

(3.6)

\[ p_1(x, t) = \begin{pmatrix} 0 \\ -\bar{a} \\ 0 \end{pmatrix} \in \mathcal{P}, \quad \bar{a} = (1 - \bar{u}\bar{u}^*) \mathbf{u}_x, \quad \dddot{\mathbf{u}} = \begin{pmatrix} u \\ v \end{pmatrix}, \]  

(3.7)

\[ p_0 = -i \begin{pmatrix} -2\epsilon \bar{u}^* J\mathbf{u} & 0 \\ 0 & \epsilon (J\bar{u}\bar{u}^* + \bar{u}\bar{u}^* J) \end{pmatrix} - i \text{diag}(0, \alpha, \beta) \in \mathcal{S}. \]  

(3.8)

It is readily to see that \( \mathbf{L} - \partial_x \in \mathcal{L}^+_+, \mathbf{M} - \partial_t \in \mathcal{L}^+_+. \)

**Lemma 3.1.** Suppose

\[ [\mathbf{L}, \mathbf{M}'] = 0, \]  

(3.9)

where \( \mathbf{L} \) is defined by (3.4), (3.6), and \( \mathbf{M}' - \partial_t \in \mathcal{L}^+_+ \) with \(-iba^{-1} \lambda^2 \) as its leading term. Then there exist real functions \( \gamma(x, t), \alpha_1(t), \alpha_2(t) \) and \( \alpha_3(t) \), such that

\[ \mathbf{M}' = \partial_t - iba^{-1} \lambda^2 - p'_1 \lambda - p'_0 - \sigma_2(p'_1) \frac{\epsilon}{\lambda} - i\sigma_2(ba^{-1}) \epsilon^2, \]  

(3.10)

with

\[ p'_1 - p_1 = \gamma bab^{-1} \in \mathcal{P}, \]  

(3.11)

\[ p'_0 + i \begin{pmatrix} -2\epsilon \bar{u}^* J\mathbf{u} & 0 \\ 0 & \epsilon (J\bar{u}\bar{u}^* + \bar{u}\bar{u}^* J) \end{pmatrix} = -i \text{diag}(\alpha_1, \alpha_2, \alpha_3), \]  

(3.12)
and $p_1$ being the coefficient of $M$ defined by (3.7). Moreover,

$$i\ddot{u}_t = (\dddot{u}_x - \ddot{u}(uu^* \dddot{u}_x) + i\gamma \ddot{u})x + 4\epsilon \ddot{u}(uu^* \cdot J \ddot{u}) + A' \ddot{u},$$

$$\begin{array}{c}
A' = \text{diag} (\alpha, \beta), \\
\alpha = \alpha_2(t) - \alpha_1(t), \\
\beta = \alpha_3(t) - \alpha_1(t).
\end{array}$$

(3.13)

**Proof.** By assumption, one can set

$$L = \partial_x - q_1 \lambda - \sigma_2(q_1) \frac{\epsilon}{\lambda},$$

(3.14)

$$M' = \partial_t - p_2 \lambda^2 - p_1' \lambda - p_0 - \sigma_2(p_1') \frac{\epsilon}{\lambda} - \sigma_2(p_2')(\frac{\epsilon}{\lambda})^2,$$

(3.15)

with

$$q_1 = bab^{-1}, \quad p_2' = iba^2b^{-1},$$

(3.16)

$$p_1' = \begin{pmatrix} 0 & -\bar{\theta}^* \\ \bar{\theta} & 0 \end{pmatrix} \in \mathcal{P}_1, \quad \bar{\theta}(x,t) = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},$$

(3.17)

$$p_0' = -i \begin{pmatrix} -2\epsilon \ddot{u}^* J \ddot{u} & 0 \\ 0 & \epsilon (J \ddot{u}u^* + \ddot{u}u^* J) \end{pmatrix} - i \text{diag} (\alpha_1, \alpha_2, \alpha_3),$$

(3.18)

and $\alpha_i = \alpha_i(x,t) \in \mathbb{R}$. The compatibility condition (3.9) then yields

$$\begin{align*}
\partial_x p_2' - [q_1, p_1'] &= 0, \\
\partial_x p_1' - \partial_t q_1 - [q_1, p_0'] - [\epsilon \sigma_2(q_1), p_2'] &= 0, \\
\partial_x p_0' - [q_1, \epsilon \sigma_2(p_1')] - [\epsilon \sigma_2(q_1), p_1'] &= 0.
\end{align*}$$

(3.19)-(3.21)

Let $\ddot{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Then (3.19) implies

$$\begin{align*}
\ddot{u}_1 \theta_1 + \ddot{u}_2 \theta_2 + u_1 \ddot{\theta}_1 + u_2 \ddot{\theta}_2 &= 0, \\
\partial_x (|u_1|^2) - u_1 \ddot{\theta}_1 - \dddot{u}_1 \theta_1 &= 0, \\
\partial_x (u_1 \ddot{u}_2) - u_1 \ddot{\theta}_2 - \dddot{u}_2 \theta_1 &= 0.
\end{align*}$$

(3.22)-(3.24)

It is an under-determined linear system. One then obtains

$$\bar{\theta} = (1 - \dddot{u}u^*) \dddot{u}_x + i\gamma(x,t) \ddot{u}, \quad \gamma(x,t) \in \mathbb{R}.$$  

(3.25)

On the other hand, (3.20), (3.22)-(3.24) imply

$$- \partial_x \bar{\theta} + i \partial_t \bar{\theta} - A' \bar{\theta} - 2\epsilon \dddot{u}u^* J \ddot{u} - 2\epsilon \dddot{u}u^* J \ddot{u} = 0,$$

(3.26)

with $A' = \text{diag} (\alpha(x,t), \beta(x,t))$, $\alpha = \alpha_2 - \alpha_1$, and $\beta = \alpha_3 - \alpha_1$. Hence we obtain

$$i\ddot{u}_t = (\dddot{u}_x - \ddot{u}(uu^* \dddot{u}_x) + i\gamma \ddot{u})x + 4\epsilon \dddot{u}(uu^* \cdot J \ddot{u}) + A' \ddot{u},$$

(3.27)
by (3.25). Moreover, (3.21) is equivalent to
\[
\epsilon^{-1}\partial_x\alpha_1 - 2\partial_x(-|u_1|^2 + |u_2|^2) + 2u^*J\bar{\theta} + 2\bar{\theta}^*Ju = 0, \tag{3.28}
\]
\[
\partial_x A'' + J\left(\partial_x(\bar{u}u^*) - (\bar{\theta}u^* + u\bar{\theta}^*)\right) + \left(\partial_x(\bar{u}u^*) - (\bar{\theta}u^* + u\bar{\theta}^*)\right)J = 0, \tag{3.29}
\]
with \( A'' = \epsilon^{-1}\text{diag}(\alpha_2(x,t), \alpha_3(x,t)) \). Note (3.22) and (3.23) imply
\[
-2\partial_x(-|u_1|^2 + |u_2|^2) + 2u^*J\bar{\theta} + 2\bar{\theta}^*Ju = 0.
\]
Together with (3.28) yields \( \alpha_1 = \alpha(t) \). Besides,
\[
\begin{align*}
\partial_x(\bar{u}u^*) - (\bar{\theta}u^* + u\bar{\theta}^*) &= \bar{u}_xu^* - \bar{\theta}u^* + u\bar{u}_x^* - \bar{\theta}\bar{u}^* \\
&= \bar{u}_xu^* - [(1 - \bar{u}u^*) \bar{u}_x + i\gamma\bar{u}] u^* + \bar{u}\bar{u}_x^* - \bar{u}[u^*(1 - \bar{u}u^*) - i\gamma\bar{u}^*] \\
&= \bar{u}_xu^* - (1 - \bar{u}u^*) \bar{u}_x^* + u\bar{u}_x^* - \bar{u}_x^*(1 - \bar{u}u^*) \\
&= \bar{u}\bar{u}_x^* - \bar{u}_x^* \bar{u}^* + u\bar{u}_x^* - \bar{u}u^* \\
&= \bar{u} (|u|^2)_x u^* \\
&= 0.
\end{align*}
\]
Here we have used (3.25) and \( b \in 1 \times SU(2) \). Combining (3.29) and (3.30), we obtained \( \alpha_i = \alpha_i(t), i = 2, 3. \)

Given \( \epsilon > 0 \), for the GMV equation parametrized by \( (4\epsilon, \beta) \) or \( (\alpha, -4\epsilon) \) with arbitrary real constants \( \alpha, \beta \), we have

Theorem 1. Write the second twisted \( U(3) \overline{U(1) \times U(2)} \)-flow \( b(x,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & -\bar{v} \\ 0 & v & \bar{u} \end{pmatrix} \), then
\[
\bar{u} = \begin{pmatrix} u \\ v \end{pmatrix}
\]
satisfies the GMV equation
\[
i\bar{u}t = (\bar{u}_x - \bar{u}(\bar{u}^* \cdot \bar{u}_x)) + 4\epsilon\bar{u}(\bar{u}^* \cdot J\bar{u}) + A'\bar{u},
\]
\[
A' = \begin{pmatrix} 4\epsilon & 0 \\ 0 & 2\epsilon + d_3 - d_1 \end{pmatrix}
\]
or
\[
\begin{pmatrix} -2\epsilon + d_3 - d_1 & 0 \\ 0 & -4\epsilon \end{pmatrix}.
\]

Proof. By Definition 11 and Lemma 11, the second twisted \( U(3) \overline{U(1) \times U(2)} \)-flow satisfies
\[
i\bar{u}t = (\bar{u}_x - \bar{u}(\bar{u}^* \cdot \bar{u}_x)) + i\gamma\bar{u} + 4\epsilon\bar{u}(\bar{u}^* \cdot J\bar{u}) + A'\bar{u},
\]
\[
A' = \text{diag}(\alpha, \beta), \quad \alpha = \alpha_2(t) - \alpha_1(t), \quad \beta = \alpha_3(t) - \alpha_1(t),
\]
where \( \gamma(x,t), \alpha_i(t) \) are defined by (3.15), (3.17), (3.18), and (3.25). By Lemma 2.5 we conclude \( \gamma(x,t) \equiv 0 \). Hence the theorem reduces to showing \( \alpha_i, i \in \{1, 2, 3\}, \)
are certain constants determined by $d_1$ and $d_3$. This can be seen by taking the limit of (2.20) when $x \to -\infty$ and $\lambda \to \infty$ since $\alpha_i$ are independent of $x$. Moreover, $m(x,t,\lambda) \to b(x,t)$ as $\lambda \to \infty$ will be shown in Theorem 6. Thus we have

$$
\begin{pmatrix}
    d_1 & 0 & 0 \\
    0 & d_1 & 0 \\
    0 & 0 & d_3
\end{pmatrix}
= \lim_{x \to -\infty} b^{-1} \left[ \begin{pmatrix} -2\epsilon \bar{u}^* J \bar{u} & 0 \\
    0 & \epsilon (J \bar{u}^* + \bar{u}^* J) \end{pmatrix} + \text{diag} (\alpha_1, \alpha_2, \alpha_3) \right] b
$$

$$
= \begin{cases} 
    \begin{pmatrix} 2\epsilon + \alpha_1 & 0 & 0 \\
    0 & -2\epsilon + \alpha_2 & 0 \\
    0 & 0 & \alpha_3 \end{pmatrix}, & \text{if } b \in \mathfrak{P}_1, \\
    \begin{pmatrix} -2\epsilon + \alpha_1 & 0 & 0 \\
    0 & 2\epsilon + \alpha_3 & 0 \\
    0 & 0 & \alpha_2 \end{pmatrix}, & \text{if } b \in \mathfrak{P}_2.
\end{cases}
$$

So $\alpha_i$ are constant and $(\alpha, \beta) = (4\epsilon, 2\epsilon + d_3 - d_1)$ or $(-2\epsilon + d_3 - d_1, -4\epsilon)$.

**Remark 3.1.** Theorem 7 implies that each second twisted $U(3)_{U(1) \times U(2)}$-flow, for a fixed $(d_1, d_3)$, gives a GMV solution. We will prove that different $(d_1, d_3)$’s give different second solutions to the same GMV solution (3.31) once $d_3 - d_1$’s are equal (Theorem 6). Thus the GMV equation is only part of the constraints in the second twisted $U(3)_{U(1) \times U(2)}$-flows. However, the twisted $U(3)_{U(1) \times U(2)}$-hierarchy is still called the (generalized) associated hierarchy for the GMV equation in this paper.

**Remark 3.2.** The obstruction to constructing the GMV equation parametrized by an arbitrary pair $(\alpha, \beta)$ by the second twisted $U(3)_{U(1) \times U(2)}$-flow is the commutativity condition (2.12).

### 4 The direct problem

#### 4.1 The spectral problem

Let $\sigma_2, a$ be defined by (3.6),

$$
b = b(x) = \begin{pmatrix} 1 & 0 & 0 \\
    0 & u & -\bar{v} \\
    0 & v & \bar{u} \end{pmatrix} \in K_1',
$$

and $b - 1 \in \mathbb{S}$. Consider the spectral problem

$$
\partial_x \Psi = \lambda bab^{-1} \Psi + \frac{\epsilon}{\lambda} \sigma_2 (bab^{-1}) \Psi,
$$

$$
\Psi(x, \lambda) e^{-x(\lambda a + \frac{\epsilon}{\lambda} \sigma_2 (a))} \to 1 \text{ as } x \to -\infty.
$$
By introducing the normalizations

\[ \Psi(x, \lambda) = m(x, \lambda)e^{x(\lambda a + \frac{1}{\lambda} \sigma_2(a))} \]

(4.3)

\[ = b(x)m'(x, \lambda)e^{x(\lambda a + \frac{1}{\lambda} \sigma_2(a))}, \]

(4.4)

the partial differential equation in (4.2) turns into

\[ \frac{\partial m}{\partial x} = \lambda (ba^{-1}m - ma) + \frac{\epsilon}{\lambda} \left( \sigma_2(ba^{-1})m - m_2(a) \right), \]

(4.5)

\[ \frac{\partial m'}{\partial x} = [\lambda a + \frac{\epsilon}{\lambda} \sigma_2(a), m'(x, \lambda)] + Q(x, \lambda)m'(x, \lambda), \]

(4.6)

with

\[ Q(x, \lambda) = \frac{\epsilon}{\lambda} \left( ba^{-1} \sigma_2(ba^{-1})b - \sigma_2(a) \right) - ba^{-1} \frac{\partial b}{\partial x}. \]

(4.7)

**Definition 2.** We define the operator \( J_\lambda = J_{a, \lambda} \) on \( gl(n, \mathbb{C}) \) by

\[ J_\lambda f = \left[ \lambda a + \frac{\epsilon}{\lambda} \sigma_2(a), f \right], \]

and \( \pi_0^\lambda, \pi_\pm^\lambda \) to be the orthogonal projections of \( gl(3, \mathbb{C}) \) to the \( 0^- \), \( \pm \)-eigenspaces of \( Re J_\lambda = \frac{1}{2} (J_\lambda + (J_\lambda)^*) \). Besides, the characteristic curve of (3.6) is defined by

\[ \Sigma_a = \{ \lambda \in \mathbb{C} | \text{the projections } \pi_0^\lambda, \pi_\pm^\lambda \text{ are not continuous at } \lambda \}. \]

(4.8)

A direct direct computation yields the characteristic curve \( \Sigma_a \) of (3.6) is \( \mathbb{R} \). Therefore we can follow the argument as that in [15] to derive

**Theorem 2.** Let \( b(x) \in K_1' \) and \( b - 1 \in S \). Then there exists a bounded set \( Z \subset C \), such that \( Z \cap (C \setminus \mathbb{R}) \) is discrete in \( C \setminus \mathbb{R} \) and for all \( \lambda \in C \setminus (\mathbb{R} \cup Z) \), there exists uniquely a solution \( m(x, \lambda) \) of (4.7) satisfying

\[ m(\cdot, \lambda) \text{ is bounded for each } \lambda \in C \setminus (\mathbb{R} \cup Z), \]

(4.9)

\[ m(x, \lambda) \to 1 \text{ as } x \to -\infty \text{ for each } \lambda \in C \setminus (\mathbb{R} \cup Z), \]

(4.10)

\[ m(x, \cdot) \text{ is meromorphic in } C \setminus \mathbb{R} \text{ with poles at } \lambda \in Z, \]

(4.11)

\[ m(x, \lambda) \to b(x) \text{ uniformly as } |\lambda| \to \infty. \]

(4.12)

Furthermore we have

**Theorem 3.** For generic \( b(x) \) satisfying the assumption of Theorem 2 the set \( Z \) is a finite set contained in \( C \setminus \mathbb{R} \), and \( m(x, \lambda) \) has a continuous extension, denoted as \( m_\pm(x, \lambda) \), to \( \mathbb{R} \) from \( C^\pm \). In addition, there exists \( V(\lambda), \lambda \in \mathbb{R} \cup Z \), such that

\[ m_+(x, \lambda) = m_-(x, \lambda)e^{x(\lambda a + \frac{1}{\lambda} \sigma_2(a))}V(\lambda)e^{-x(\lambda a + \frac{1}{\lambda} \sigma_2(a))}, \lambda \in \mathbb{R}, \]

(4.13)

\[ m(x, \lambda) \left( 1 - e^{x(\lambda a + \frac{1}{\lambda} \sigma_2(a))}V(\lambda_0)e^{-x(\lambda a + \frac{1}{\lambda} \sigma_2(a))} \right) \text{ is regular at } \lambda_0 \in Z \].

(4.14)
and for \(\lambda \in \mathbb{R}\)

\[
\partial_\alpha^\lambda (V - 1) \text{ is } \mathcal{O}(\lambda^N) \text{ as } \lambda \to 0 \text{ and } \mathcal{O}(\lambda^{-N}) \text{ as } |\lambda| \to \infty
\]  

(4.15)

for all positive integer \(N\) and nonnegative integer \(\alpha\),

\[
det V \equiv 1,
\]

(4.16)

\[
V(\bar{\lambda})^* V(\lambda)^{-1} = 1,
\]

(4.17)

\[
\sigma_1(V(-\lambda))V(\lambda) = 1,
\]

(4.18)

\[
\sigma_2(V(\epsilon/\lambda))V(\lambda) = 1,
\]

(4.19)

and for \(\lambda \in \mathbb{Z}\),

\[
V(\lambda)^2 = 0,
\]

(4.20)

\[
V(\lambda) = -V(\bar{\lambda})^*,
\]

(4.21)

\[
V(\lambda) = -\sigma_1(V(-\lambda)),
\]

(4.22)

\[
V(\lambda) = -\frac{\lambda^2}{\epsilon} \sigma_2(V(\epsilon/\lambda)).
\]

(4.23)

Proof. The generic property for simple pole with residue satisfying (4.20) has be shown in [3]. The reality conditions (4.17)-(4.19), (4.21)-(4.23) can be proved by showing

\[
m(x, \bar{\lambda})^* = m(x, \lambda)^{-1},
\]

(4.24)

\[
\sigma_1(m(x, -\lambda)) = m(x, \lambda),
\]

(4.25)

\[
\sigma_2(m(x, \epsilon/\lambda)) = m(x, \lambda),
\]

(4.26)

and using the properties (4.13), (4.14), and (4.20). Finally, by the same argument as that in the proof of Proposition 2.1 in [3], one can prove

\[
det \Psi = det m \equiv 1.
\]

(4.27)

The statement (4.16) follows from (4.27) and (4.13).

\[
\sqrt{\omega} \text{ is } \mathcal{O}(\lambda^N) \text{ as } \lambda \to 0 \text{ and } \mathcal{O}(\lambda^{-N}) \text{ as } |\lambda| \to \infty
\]  

(4.15)

\[
det V \equiv 1,
\]

(4.16)

\[
V(\bar{\lambda})^* V(\lambda)^{-1} = 1,
\]

(4.17)

\[
\sigma_1(V(-\lambda))V(\lambda) = 1,
\]

(4.18)

\[
\sigma_2(V(\epsilon/\lambda))V(\lambda) = 1,
\]

(4.19)

and for \(\lambda \in \mathbb{Z}\),

\[
V(\lambda)^2 = 0,
\]

(4.20)

\[
V(\lambda) = -V(\bar{\lambda})^*,
\]

(4.21)

\[
V(\lambda) = -\sigma_1(V(-\lambda)),
\]

(4.22)

\[
V(\lambda) = -\frac{\lambda^2}{\epsilon} \sigma_2(V(\epsilon/\lambda)).
\]

(4.23)

Definition 3. The associated scattering data of the generic potential \(b(x)\) is defined by the matrix function \(V(\lambda), \lambda \in \mathbb{R} \cup \mathbb{Z}\), provided \(b\) satisfies the assumption of Theorem 3. Moreover, \(V(\lambda)\) is called a scattering data, if \(V(\lambda), \lambda \in \mathbb{R} \cup \mathbb{Z}\), satisfies (4.15)-(4.23).

4.2 An extended direct problem

We need to consider an extended spectral problem of (4.2) for solving the inverse problem. The first criteria for an extended spectral problem is preserving the reality conditions with respect to involutions \(\sigma_1, \sigma_2\) and the self-adjointness. So there could
be multiple choices for extended spectral problems. We choose a (splitting type) twisted $\frac{U(4)}{U(2) \times U(2)}$ spectral problem to be our extended system. Since the inverse scattering problem of twisted $\frac{U(4)}{U(2) \times U(2)}$-flows is almost the same as that of the twisted $\frac{O(n,n)}{O(n)}$ which has been tackled in [1], [15].

Let $\tilde{\sigma}_i, i = 1, 2,$ be involutions on $U(4)$ defined by

$$\tilde{\sigma}_i(x) = J_i x J_i^{-1}, \quad x \in U(4),$$

$$J_1 = \text{diag}(1, 1, -1, -1), \quad J_2 = \text{diag}(1, 1, -1, 1)$$

and $u(4) = \tilde{K}_i \oplus \tilde{P}_i, i = 1, 2,$ the Cartan decompositions for $\tilde{\sigma}_i$. Let $\tilde{K}_i$ be the Lie algebras of $\tilde{K}_i$, i.e.,

$$\tilde{K}_1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix} \in U(4) : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in U(2) \right\},$$

$$\tilde{K}_2 = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & 0 \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix} \in U(4) : |a_{33}| = 1, \begin{pmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} \in U(3) \right\},$$

$$\tilde{P}_1 = \left\{ \begin{pmatrix} 0 & 0 & u_1 & v_1 \\ 0 & 0 & u_2 & v_2 \\ u_1^* & u_2^* & 0 & 0 \\ v_1^* & v_2^* & 0 & 0 \end{pmatrix} \in u(4) \right\}, \quad \tilde{P}_2 = \left\{ \begin{pmatrix} 0 & 0 & u_1 & 0 \\ 0 & 0 & u_2 & 0 \\ u_1^* & u_2^* & 0 & u_3 \\ 0 & 0 & u_3^* & 0 \end{pmatrix} \in u(4) \right\}.$$

Let

$$\tilde{S} = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in U(4) : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in U(2) \right\} \subset \tilde{K}_1 \cap \tilde{K}_2,$$

$$\check{S} = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in U(4) : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in u(2) \right\},$$

and

$$\tilde{K}_1 = \tilde{S} \otimes \tilde{K}_1', \quad \tilde{K}_1' = 1_{2 \times 2} \otimes U(2),$$

$$\check{K}_1 = \check{S} \oplus \check{K}_1', \quad \check{K}_1' = 0_{2 \times 2} \oplus U(2).$$

The extended spectral problem of (4.2) is

$$\partial_x \check{\Psi} = \lambda \check{b} \check{a}_1 \check{b}^{-1} \check{\Psi} + \frac{\epsilon}{\lambda} \check{\sigma}_2 (\check{b} \check{a}_1 \check{b}^{-1}) \check{\Psi},$$

$$\check{\Psi}(x, \lambda) e^{-x (\lambda \check{a}_1 + \frac{\epsilon}{\lambda} \check{b}_2 (\check{a}_1))} \to 1 \text{ as } x \to -\infty,$$
with
\[ \tilde{a}_1 = i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{b}(x, t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u & -\bar{v} \\ 0 & 0 & v & \bar{u} \end{pmatrix} \in \tilde{K}_1. \quad (4.32) \]

We note that \( \tilde{a}_1 \) is not an oblique direction for solving the twisted \( U(4) \times U(2) \)-spectral problem (cf [11], [15]). However, for the extended spectral problem \( (4.31), (4.32) \), we have

**Lemma 4.1.** The spectral equation \((4.2)\) is satisfied by \( \Psi(x, \lambda) = (\Psi_{ij})_{1 \leq i, j \leq 3} \) if and only if \((4.31)\) is satisfied by

\[ \tilde{\Psi}(x, \lambda) = \begin{pmatrix} \Psi_{11} & 0 & \Psi_{12} & \Psi_{13} \\ 0 & 1 & 0 & 0 \\ \Psi_{21} & 0 & \Psi_{22} & \Psi_{23} \\ \Psi_{31} & 0 & \Psi_{32} & \Psi_{33} \end{pmatrix}. \quad (4.33) \]

Moreover, let \( m = (m_{ij})_{1 \leq i, j \leq 3} \), \( m' = (m'_{ij})_{1 \leq i, j \leq 3} \) be the normalized eigenfunctions defined by \((4.3)\) and \((4.4)\) and

\[ \tilde{\Psi}(x, \lambda) = \tilde{m}(x, \lambda)e^{x(\lambda \tilde{a}_1 + \frac{i}{2}\tilde{\sigma}_2(\tilde{a}_1))} = \tilde{b}(x)\tilde{m}'(x, \lambda)e^{x(\lambda \tilde{a}_1 + \frac{i}{2}\tilde{\sigma}_2(\tilde{a}_1))}. \quad (4.34) \]

Then

\[ \tilde{m}(x, \lambda) = \begin{pmatrix} m_{11} & 0 & m_{12} & m_{13} \\ 0 & 1 & 0 & 0 \\ m_{21} & 0 & m_{22} & m_{23} \\ m_{31} & 0 & m_{32} & m_{33} \end{pmatrix}, \quad \tilde{m}'(x, \lambda) = \begin{pmatrix} m'_{11} & 0 & m'_{12} & m'_{13} \\ 0 & 1 & 0 & 0 \\ m'_{21} & 0 & m'_{22} & m'_{23} \\ m'_{31} & 0 & m'_{32} & m'_{33} \end{pmatrix}. \quad (4.35) \]

Finally, for generic \( b \), there exists a finite set \( Z \subset \mathbb{C} \setminus \mathbb{R} \) and

\[ \tilde{m}_+(x, \lambda) = \tilde{m}_-(x, \lambda)e^{x(\lambda \tilde{a}_1 + \frac{i}{2}\tilde{\sigma}_2(\tilde{a}_1))}\tilde{V}(\lambda)e^{-x(\lambda \tilde{a}_1 + \frac{i}{2}\tilde{\sigma}_2(\tilde{a}_1))}, \quad \lambda \in \mathbb{R}, \quad (4.36) \]

\[ \tilde{m}(x, \lambda) \left( 1 - \frac{e^{x(\lambda \tilde{a}_1 + \frac{i}{2}\tilde{\sigma}_2(\tilde{a}_1))}\tilde{V}(\lambda)e^{-x(\lambda \tilde{a}_1 + \frac{i}{2}\tilde{\sigma}_2(\tilde{a}_1))}}{\lambda - \lambda_0} \right) \text{ is regular at } \lambda_0 \in Z. \quad (4.37) \]

with

\[ \tilde{V}(\lambda) = \begin{pmatrix} V_{11} & 0 & V_{12} & V_{13} \\ 0 & 1 & 0 & 0 \\ V_{21} & 0 & V_{22} & V_{23} \\ V_{31} & 0 & V_{32} & V_{33} \end{pmatrix}. \quad (4.38) \]

and for \( \lambda \in \mathbb{R} \),

\[ \partial^\alpha_x (\tilde{V} - 1) \text{ is } O(\lambda^N) \text{ as } \lambda \to 0 \text{ and } O(\lambda^{-N}) \text{ as } |\lambda| \to \infty. \quad (4.39) \]
\[
det \tilde{V} \equiv 1, \quad \tilde{V}(\lambda)^* \tilde{V}(\lambda)^{-1} = 1, \quad \tilde{\sigma}_1(\tilde{V}(-\lambda)) \tilde{V}(\lambda) = 1, \quad \tilde{\sigma}_2(\tilde{V}(\epsilon/\lambda)) \tilde{V}(\lambda) = 1, \quad (4.41)
\]

for \( \lambda \in \mathbb{Z} \),

\[
\tilde{V}(\lambda)^2 = 0, \quad \tilde{V}(\lambda) = -\tilde{V}(\bar{\lambda})^*, \quad \tilde{V}(\lambda) = -\sigma_1(\tilde{V}(-\lambda)), \quad (4.42)
\]

\[
\tilde{\sigma}_1(\tilde{V}(-\lambda)) \tilde{V}(\lambda) = 1, \quad (4.43)
\]

\[
\tilde{\sigma}_2(\tilde{V}(\epsilon/\lambda)) \tilde{V}(\lambda) = 1, \quad (4.44)
\]

Proof. The statements can be proved by a direct computation. \( \square \)

**Definition 4.** The associated extended scattering data of \( b \) is defined by the matrix function \( \tilde{V}(\lambda) \), \( \lambda \in \mathbb{R} \cup \mathbb{Z} \), provided \( b \) satisfies the assumption of Theorem 3. Moreover, \( \tilde{V}(\lambda) \) is called an extended scattering data, if \( \tilde{V}(\lambda), \lambda \in \mathbb{R} \cup \mathbb{Z} \), satisfies (4.39)-(4.48).

**Remark 4.1.** If \( b(x) \in K'_1 \), \( b(x) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in S \), then the spectral problem needed to be considered is

\[
\partial_x \Psi = \lambda bab^{-1} \Psi + \frac{\epsilon}{\lambda} \sigma_2(bab^{-1}) \Psi, \quad (4.49)
\]

\[
\Psi(x, \lambda)e^{-x(\lambda a + \frac{\epsilon}{\lambda} \sigma_2(a))} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{as } x \rightarrow -\infty.
\]

It is more convenient to use a change of variables to turn (4.49) into

\[
\partial_x \Psi = \lambda' bab'^{-1} \Psi + \frac{\epsilon}{\lambda} \sigma_2(bab'^{-1}) \Psi, \quad (4.50)
\]

\[
\Psi(x, \lambda)e^{-x(\lambda a + \frac{\epsilon}{\lambda} \sigma_2(a))} \rightarrow 1 \quad \text{as } x \rightarrow -\infty,
\]

with

\[
a' = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{P}_1, \quad b'(x, t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{v} & u \\ 0 & -\bar{u} & v \end{pmatrix} \in K'_1. \quad (4.51)
\]

By analogy, one can derive the existence theorem of the eigenfunction \( \Psi(x, \lambda) \), extract continuous and discrete scattering data, and solve the associated extended direct problem

\[
\partial_x \tilde{\Psi} = \lambda bab^{-1} \tilde{\Psi} + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(bab^{-1}) \tilde{\Psi},
\]

\[
\tilde{\Psi}(x, \lambda)e^{-x(\lambda a + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(a))} \rightarrow 1 \quad \text{as } x \rightarrow -\infty.
\]
with

$$\tilde{a}_2 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \tilde{b}(x, t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \tilde{v} & u \\ 0 & 0 & -\bar{u} & v \end{pmatrix} \in \tilde{K}_1', \quad (4.53)$$

and define the extended scattering data $\tilde{V}(\lambda)$, $\lambda \in \mathbb{R} \cup \mathbb{Z}$.

These two different boundary conditions, (4.2) and (4.49), are the only cases which can be tackled by our approach. Since the associated spectral operators are perturbation of diagonalizable operators. That is, $a$ and $\sigma(a)$ ($a'$ and $\sigma(a')$ respectively) can be simultaneously diagonalized.

5 The inverse problem

In this section, the normalized eigenfunction $\tilde{m}'(x, \lambda)$ will be constructed from the scattering data by solving a Riemann-Hilbert problem. To find the gauge $\tilde{b}(x)$ to reconstruct $\tilde{m}(x, \lambda)$, one needs to understand the symmetries between coefficients of $(\partial_x \tilde{m}'(\tilde{m}'))^{-1}$ which is equivalent to solving an over-determined differential systems. Inspired by the result of [1], [15], we reconstruct the gauge via solving an exterior differential system derived from one-dimensional systems associated with Cartan sub-algebras with higher ranks (cf. Definition 3.1 in [15]). This is the motivation for us to study the extended twisted $\mathcal{U}(4)_{U(2) \times U(2)}$-spectral problem in §4.2.

The major differences between loop algebra structures associated with twisted $O(2,2)_{O(2) \times O(2)}$ and with twisted $\mathcal{U}(4)_{U(2) \times U(2)}$-hierarchies are the symmetric and antisymmetric properties of $P_0$ and $\tilde{P}_1$. However, the proof of the inverse problem in Section 6 of [15] mainly involves with the involution properties of $\sigma_i$, the commutativity property $[a_i, a_j] = 0$ and the self-adjointness of $K_0$, and has nothing to do with the symmetric property of $P_0$. As a result, the inverse scattering problem of twisted $\mathcal{U}(4)_{U(2) \times U(2)}$-hierarchy can be solved by the same argument. We will state the results, leave analogous details to [1], [15], and only give the proof for projecting the extended inverse results to that of a twisted $\mathcal{U}(3)_{U(1) \times U(2)}$-spectral problem in this section.

Write $\tilde{a} = \begin{pmatrix} 0 & D \\ -D^* & 0 \end{pmatrix}$, and $D = \text{diag}(w_1, w_2)$. Define

$$\tilde{x} = (x_1, x_2) = x(w_1, w_2), \quad \tilde{V}(\lambda) \in \mathbb{R} \cup \mathbb{Z}.$$

$$X = x_1 \tilde{a}_1 + x_2 \tilde{a}_2,$$

$$\tilde{a}_1 = i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{a}_2 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$
Theorem 4. Let \( \tilde{V}(\lambda) \), \( \lambda \in \mathbb{R} \) satisfy the analytical constraints (4.40) and the algebraic constraints (4.41)-(4.48). Then there exists uniquely \( M(\vec{x}, \lambda) \) such that
\[
M_+(\vec{x}, \lambda) = M_-^{*}(\vec{x}, \lambda) e^{\lambda X + \frac{\hat{\sigma}_2(\lambda)}{\lambda}} \tilde{V}(\lambda) e^{-\lambda X - \frac{\hat{\sigma}_2(\lambda)}{\lambda}}, \quad \lambda \in \mathbb{R},
\]
\[
M(\vec{x}, \lambda) \left(1 - \frac{e^{\lambda X + \frac{\hat{\sigma}_2(\lambda)}{\lambda}} \tilde{V}(\lambda_0) e^{-(\lambda X + \frac{\hat{\sigma}_2(\lambda)}{\lambda})}}{\lambda - \lambda_0}\right) \text{ is regular at } \lambda_0 \in \mathbb{Z},
\]
\[
M(\vec{x}, \lambda) \text{ is holomorphic for } \lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \mathbb{Z}) , \quad M(\vec{x}, \lambda) \to 1 \text{ as } |\lambda| \to \infty, \quad (5.6)
\]
and \( M(\vec{x}, \lambda) \) satisfies the analytical and algebraic conditions
\[
det M \equiv 1, \quad (5.7)
\]
\[
M(\vec{x}, \lambda)^* = M(\vec{x}, \lambda)^{-1}, \quad (5.8)
\]
\[
\bar{\sigma}_1(M(\vec{x}, -\lambda)) = M(\vec{x}, \lambda), \quad \bar{\sigma}_2(M^{-1}(\vec{x}, 0) M(\vec{x}, \frac{\epsilon}{\lambda})) = M(\vec{x}, \lambda), \quad (5.9)
\]
\[
x^k \partial^k_x \lambda^k (M(\vec{x}, \lambda) - 1) \in L^2(\mathbb{R}) \text{ for } \forall k, k' \text{ and tends to 0 uniformly } \quad (5.10)
\]
as \( x \to -\infty \); \( \exists \delta(\lambda) \) diagonal, s.t. \( x^k \partial^k_x \lambda^k (M(\vec{x}, \lambda) - \delta(\lambda)) \in L^2(\mathbb{R}) \) for \( \forall k, k' \) and tends to 0 uniformly as \( x \to \infty \).

Moreover, if \( \tilde{V}(\lambda) \) is an extended scattering data, i.e. is of the form (4.39), then
\[
M((x, 0), \lambda) = \begin{pmatrix} M_{11} & 0 & M_{12} & M_{13} \\ 0 & 1 & 0 & 0 \\ M_{21} & 0 & M_{22} & M_{23} \\ M_{31} & 0 & M_{32} & M_{33} \end{pmatrix}, \quad (5.11)
\]

Proof. The existence of \( M(\vec{x}, \lambda) \) satisfying (5.4)-(5.6), and (5.10) can be proved by the same argument as that in the proof of Theorem 5.1 in [15]. Properties (4.41), (4.45) imply \( \det M \) is continuous for \( \lambda \in \mathbb{C} \). Thus Condition (5.7) is shown by noting \( \partial^k_x \det M = 0 \) for \( \lambda \in \mathbb{C}^\pm \) and applying Liouville’s theorem. The statements (5.8), (5.9), and (5.11) can be proved by the uniqueness property of \( M(\vec{x}, \lambda) \).

Defining the asymptotic expansions
\[
M(\vec{x}, \lambda) \to 1 + \sum_{k=1}^{\infty} M_k^i(\vec{x}) \lambda^{-k} \text{ as } |\lambda| \to \infty, \quad (5.12)
\]
\[
M(\vec{x}, \lambda) \to \sum_{k=0}^{\infty} M_k^i(\vec{x}) \lambda^k \text{ as } |\lambda| \to 0. \quad (5.13)
\]
and applying the same argument as that in the proof of Lemma 6.1 - 6.3, and Theorem 6.1 in [15], one can derive the following four lemmas.

Lemma 5.1. Suppose \( M(\vec{x}, \lambda) \) is derived by Theorem 4. Then \[
\frac{\partial M}{\partial x_j} = [\lambda \tilde{a}_j + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{a}_j), M] + \frac{\epsilon}{\lambda} (B_{j}(\vec{x}) - \tilde{\sigma}_2(\tilde{a}_j)) M - C_{j}(\vec{x}) M, \quad (5.14)
\]
with \[
B_{j}(\vec{x}) \in \bar{\mathcal{P}}_1 \cap C^\infty, \quad C_{j}(\vec{x}) \in \bar{\mathcal{K}}_1 \cap C^\infty. \quad (5.15)
\]
Lemma 5.2. The compatibility conditions of (3.14) are
\[ \partial_x C_i - \partial_x C_j - [C_i, C_j] = \epsilon \left[ \tilde{a}_i, B_j \right] - \epsilon \left[ \tilde{a}_j, B_i \right]. \tag{5.16} \]

Lemma 5.3. For any constant \( \mu \in \mathbb{R} \), there exists uniquely
\[ \tilde{b}(\vec{x}) \in \tilde{K}'_1 \cap C^\infty, \tag{5.17} \]
such that
\[ \tilde{b}(x(w_1, w_2)) \to \begin{pmatrix} 1_{2 \times 2} & 0 \\ 0 & e^{-i\mu/2}1_{2 \times 2} \end{pmatrix} \in \tilde{K}'_1 \quad \text{as} \quad x \to -\infty, \tag{5.18} \]
\[ -\tilde{b} C_j \tilde{b}^{-1} + (\partial_j \tilde{b}) \tilde{b}^{-1} \in \tilde{S} \quad \text{for} \quad \forall j. \tag{5.19} \]

Proof. We remark the boundary condition (5.18) can be chosen for arbitrary element in \( \tilde{K}'_1 \).

Lemma 5.4. Suppose the assumption of Theorem 4 holds. For any constant \( \mu \in \mathbb{R} \), let
\[ \tilde{\Psi}(x, \lambda) = \tilde{b}(\vec{x}) M(\vec{x}, \lambda) e^{\lambda X + \tilde{b}^2(\vec{x})} \tag{5.20} \]
Here \( x, \vec{x}, X, M \) satisfy (5.1)-(5.3). Then
\[ \frac{\partial \tilde{\Psi}}{\partial x} = \lambda \tilde{b} \tilde{a} \tilde{b}^{-1} \tilde{\Psi} + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{b} \tilde{b}^{-1}) \tilde{\Psi} + \tilde{v} \tilde{\Psi}, \tag{5.21} \]
with
\[ \tilde{v}(\vec{x}) = \sum_{j=1}^{2} w_j (-\tilde{b} C_j \tilde{b}^{-1} + (\partial_x \tilde{b}) \tilde{b}^{-1}) \in \tilde{S} \cap S, \tag{5.22} \]
where \( C_j, \tilde{b}(\vec{x}) \) are defined by Lemma 5.1, 5.3, respectively. Moreover,
\[ \tilde{b}(\vec{x}) = \begin{pmatrix} 1_{2 \times 2} & 0 \\ 0 & e^{-i\mu/2}1_{2 \times 2} \end{pmatrix} \in \tilde{K}'_1 \cap S, \tag{5.23} \]
\[ \tilde{v} \text{ is independent of } \mu \text{ defined by (5.18).} \tag{5.24} \]

Proof. Once the boundary condition (5.18) is a diagonal element in \( \tilde{K}'_1 \), the argument in proving Theorem 6.1 in [15] works well in proving all statements in Lemma 5.4 except (5.24). The property (5.24) follows from the fact that changing \( \mu \) could only alter the \( \tilde{K}'_1 \) part of the right hand side of (5.22) and \( \tilde{v} \in \tilde{S} \).

Note (5.21) and (5.20) imply
\[ \frac{\partial M}{\partial x} = [\lambda \tilde{a} + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{a}), \ M(x, \lambda)] + Q(x, \lambda) M(x, \lambda) \tag{5.25} \]
\[ Q(x, \lambda) = \frac{\epsilon}{\lambda} \left( \tilde{b}^{-1} \tilde{\sigma}_2(\tilde{b} \tilde{b}^{-1}) \tilde{b} - \tilde{\sigma}_2(\tilde{a}) \right) - \tilde{b}^{-1} \frac{\partial \tilde{b}}{\partial x} + \tilde{b}^{-1} \tilde{v} \tilde{b}. \tag{5.26} \]
Thus the Schwartz properties (5.22) and (5.23) follow from (3.10), (5.25), and (5.26). \( \square \)
Lemma 5.4 solves the inverse problem of a general twisted $U(4)/U(2)\times U(2)$-spectral problem for scattering data $\tilde{V}(\lambda)$ satisfying (4.40)-(4.48). The following theorem says that when $\tilde{V}(\lambda)$ is an extended scattering data, the above result can be projected to be a solvability of the inverse problem for a twisted $U(3)/U(1)\times U(2)$-spectral problem.

**Theorem 5.** Suppose the assumption of Theorem 4 holds for either

$$
\tilde{V}(\lambda) = \begin{pmatrix} V_{11} & 0 & V_{12} & V_{13} \\ 0 & 1 & 0 & 0 \\ V_{21} & 0 & V_{22} & V_{23} \\ V_{31} & 0 & V_{32} & V_{33} \end{pmatrix},
$$

or

$$
\tilde{V}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & V_{11} & V_{12} & V_{13} \\ 0 & V_{21} & V_{22} & V_{23} \\ 0 & V_{31} & V_{32} & V_{33} \end{pmatrix}.
$$

Then there exist a unique $\Psi(x, \lambda) \in L^\epsilon_+$ and a unique $b(x) \in K'_1$ satisfying $b(x) - 1 \in S$ (or $b(x) - \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \in S$) such that

$$
\frac{\partial \Psi}{\partial x} = \lambda bab^{-1}\Psi + \epsilon \sigma_2(bab^{-1})\Psi
$$

(5.29)

with $a$ defined by (3.6), and the associated extended scattering data of $b(x)$ is $\tilde{V}(\lambda)$.

**Proof.** We only prove the case (5.27). Case (5.28) can be argued by analogy. Let $M(\bar{x}, \lambda), \tilde{b}(\bar{x}), \tilde{\Psi}(x, \lambda)$ be derived from Theorem 4, Lemma 5.3 and 5.4 by specially choosing $(w_1, w_2) = (1, 0)$ in (5.1), i.e. $\bar{x} = (x, 0), X = x\tilde{a}_1, \tilde{a} = \tilde{a}_1 \in \tilde{A}$. Define

$$
M(\bar{x}, \lambda) = \tilde{m}'(x, \lambda)
$$

(5.30)

Applying Theorem 4 and (5.27), we have

$$
\tilde{m}'(x, \lambda) = \begin{pmatrix} m'_{11} & 0 & m'_{12} & m'_{13} \\ 0 & 1 & 0 & 0 \\ m'_{21} & 0 & m'_{22} & m'_{23} \\ m'_{31} & 0 & m'_{32} & m'_{33} \end{pmatrix}.
$$

(5.31)

Together with $\tilde{b} \in \tilde{K}'_1$, we find $\tilde{\psi}$ in (5.21) is of the form diag $(i\nu, 0, 0, 0)$. Hence (5.21) can be gauged to

$$
\frac{\partial \tilde{\psi}_1}{\partial x} = \lambda \tilde{b}_1\tilde{a}_1^{-1}\tilde{\psi}_1 + \epsilon \sigma_2(\tilde{b}_1\tilde{a}_1^{-1})\tilde{\psi}_1,
$$

(5.32)

$$
\tilde{b}_1 = \tilde{b} \cdot \operatorname{diag}(1, 1, e^{i\nu}, 1) \in \tilde{K}'_1 = 1 \otimes U(2),
$$

(5.33)

$$
\tilde{\psi}_1 = \operatorname{diag}(e^{i\nu}, 1, 1, 1)\tilde{\Psi}.
$$

(5.34)
Applying the same argument as that in Proposition 2.1 in [15] to (5.31), we obtain that \( \det(\tilde{\Psi}_1) \) is constant. Together with (5.7), (5.20), and (5.33), we conclude \( e^{i\nu} \det \tilde{b} \) is constant which equals to \( e^{i(\nu(x=-\infty)-\mu)} \) by (5.18) and (5.33). Equation (5.32) then yields

\[
\det \tilde{b}_1 = e^{i\nu} \det \tilde{b} \equiv e^{i(\nu(x=-\infty)-\mu)}.
\]

Besides, noting \( \nu(x=-\infty) \) exists (by (5.22)) and is determined by \( \tilde{v} \) (independent of \( \mu \)). Consequently

\[
\tilde{b}_1 = \begin{pmatrix} 1_{2\times2} & 0 \\ 0 & \omega \end{pmatrix}, \quad \omega \in SU(2), \quad \tilde{b}_1 - 1 \in S
\]

by choosing \( \mu = \nu(x=-\infty) \) and using (5.24), (5.32).

However, by solving the direct problem of (5.31) with \( \tilde{b}_1 \) satisfying (5.34) and applying (4.12), (5.6), (5.32), and (5.33), as matter of fact, \( \nu = 0 \).

Therefore the theorem is proved by defining

\[
\Psi(x, \lambda) = b(x)m'(x, \lambda)e^{x(\lambda a + \frac{\epsilon}{\lambda} \sigma_2(a))}
\]

with \( a \) defined by (3.6), \( m'(x, \lambda) = (m'_{ij}) \), and \( b = \begin{pmatrix} 1_{1\times1} & 0 \\ 0 & \omega \end{pmatrix} \in K'_1 \).

\[
6 \quad \text{The Cauchy problem}
\]

We first apply the inverse scattering theory established in Section 4 and 5 to solve the initial value problem of the \( k \)-th twisted \( U(3) \times U(2) \)-flow.

**Theorem 6.** Given \( d_1, d_3 \in \mathbb{R} \), and \( b_0(x) \in K'_1 \) such that either \( b_0 - 1 \in S \) or \( b_0 - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in S \). If the scattering data for \( b_0(x) \) is generic, then the initial value problem of the \( k \)-th twisted \( U(3) \times U(2) \)-flow admits a unique solution \( b(x, t) \in \mathcal{P}_1 \) or \( \mathcal{P}_2 \). More precisely, there uniquely exist \( m(x, t, \lambda) \in L^\epsilon_\pm \) and \( \Psi(x, t, \lambda) = m(x, t, \lambda)e^{xj_1,0+tj_2} \in L^\epsilon_\pm \) such that

\[
[L, M] = 0 \quad (6.1)
\]

with

\[
L = \partial_x - \frac{\partial \Psi}{\partial x}, \quad M = \frac{\partial \Psi}{\partial t}
\]

\[
b(x, 0) = b_0(x), \quad b(x, t) = m(x, t, \infty) \in \mathcal{P}_1 \quad (or \ b(x, t) \in \mathcal{P}_2 ).
\]
Proof. We first apply Theorem 2 to solve the eigenfunction of (4.2) for \( a, b(x, 0) = b_0(x) \) defined by (5.6) and (1.11). Applying Definition 3 and Theorem 3, we obtain the scattering data \( V(\lambda, 0), \lambda \in \mathbb{R} \cup \mathbb{Z} \) for the potential \( b(x, 0) \). Define
\[
V(\lambda, t) = e^{\hat{J}_k} V(\lambda, 0) e^{-t \hat{J}_k}, \quad \text{for } \lambda \in \mathbb{R} \cup \mathbb{Z}
\] (6.3)
So \( V(\lambda, t) \) satisfies the assumption of Theorem 5 and there exist uniquely smooth \( m'(x, t, \lambda) \in L^\epsilon_x, b(x, t) \in \mathfrak{P}_1 \) (or \( b(x, t) \in \mathfrak{P}_2 \)) satisfying
\[
m(x, t, \lambda) = b(x, t)m'(x, t, \lambda),
\]
\[
\Psi(x, t, \lambda) = b(x, t)m'(x, t, \lambda) e^{x \hat{J}_1, 0 + t \hat{J}_k},
\]
(6.4) and
\[
\frac{\partial \Psi}{\partial x}(x, t, \lambda) = \lambda bab^{-1}\Psi(x, t, \lambda) + \frac{\epsilon}{\lambda} \sigma_2(bab^{-1})\Psi(x, t, \lambda).
\] (6.5)
Consequently, we can solve the initial value problem for the GMV equation for \((\alpha, \beta) = (\alpha, -4\epsilon)\) or \((\alpha, \beta) = (4\epsilon, \beta)\).

Corollary 6.1. Given \( \epsilon > 0, \beta \in \mathbb{R} \), and a (generic) function \( \bar{u}_0(x) - \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \in \mathcal{S} \), the initial value problem of the GMV equation
\[
i \bar{u}_t = (\bar{u}_x - \bar{u}(\bar{u}^* \cdot \bar{u}_x))_x + 4\epsilon \bar{u}(\bar{u}^* \cdot J \bar{u}) + A \bar{u},
\]
\[
\bar{u}^* \bar{u} = 1, \quad \bar{u} \in \mathcal{C}^2, \quad \bar{u}(x, 0) = \bar{u}_0(x),
\]
\[
J = \text{diag}(-1, 1), \quad A = \text{diag}(4\epsilon, \beta),
\]
(6.7)
admits one family of global solutions.

Proof. The solvability follows from setting \( \hat{J}_k \) to be
\[
\hat{J}_2 = i(a^2 \lambda^2 - \left( \begin{array}{ccc}
2\epsilon + \alpha_1 & 0 & 0 \\
0 & 2\epsilon + \alpha_1 & 0 \\
0 & 0 & \beta + \alpha_1
\end{array} \right)) + \sigma_2(a^2)(\frac{\epsilon}{\lambda})^2), \quad \alpha_1 \in \mathbb{R},
\]
and applying Theorem 11 and 3. Different \( \alpha_1 \)'s correspond to different \( b(x, t) \)'s since the scattering data differ when \( t > 0 \) by (6.3). \qed

Corollary 6.2. Given \( \epsilon > 0, \alpha \in \mathbb{R} \), and a (generic) function \( \bar{u}_0(x) - \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \in \mathcal{S} \), the initial value problem of the GMV equation
\[
i \bar{u}_t = (\bar{u}_x - \bar{u}(\bar{u}^* \cdot \bar{u}_x))_x + 4\epsilon \bar{u}(\bar{u}^* \cdot J \bar{u}) + A \bar{u},
\]
\[
\bar{u}^* \bar{u} = 1, \quad \bar{u} \in \mathcal{C}^2, \quad \bar{u}(x, 0) = \bar{u}_0(x),
\]
\[
J = \text{diag}(-1, 1), \quad A = \text{diag}(\alpha, -4\epsilon),
\]
(6.8)
admits one family of global solutions.
Proof. Set $\hat{J}_k$ to be

$$
\hat{J}_2 = i(a^2\lambda)^2 - \begin{pmatrix} -2\epsilon + \alpha_1 & 0 & 0 \\ 0 & -2\epsilon + \alpha_1 & 0 \\ 0 & 0 & \alpha + \alpha_1 \end{pmatrix} + \sigma_2(a^2)(\frac{\epsilon}{\lambda})^2), \quad \alpha_1 \in \mathbb{R},
$$

and apply Theorem 1 and 6. Different $\alpha_1$’s correspond to different $b(x,t)$’s since the scattering data differ when $t > 0$ by $(6.3)$. \qed

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