High order mixed finite elements with mass lumping for
elasticity on triangular grids *

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Abstract

A family of conforming mixed finite elements with mass lumping on triangular
grids are presented for linear elasticity. The stress field is approximated by symmetric
$H(\text{div}) - P_k$ ($k \geq 3$) polynomial tensors enriched with higher order bubbles so as to allow
mass lumping, which can be viewed as the Hu-Zhang elements enriched with higher
order interior bubble functions. The displacement field is approximated by $C^{-1} - P_{k-1}$
polynomial vectors enriched with higher order terms to ensure the stability condition.
For both the proposed mixed elements and their mass lumping schemes, optimal error
estimates are derived for the stress with $H(\text{div})$ norm and the displacement with $L^2$
norm. Numerical results confirm the theoretical analysis.

Keywords linear elasticity, mixed finite element, mass lumping, error estimate

AMS subject classifications. 65N15, 65N30, 74H15, 74S05

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a polygonal region with boundary $\partial \Omega$. We consider the following
mixed variational system of linear elasticity based on the Hellinger-Reissner principle: Find
$(\sigma, u) \in \Sigma \times V := H(\text{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^2)$, such that

\[
\begin{aligned}
(A\sigma, \tau) + (\text{div}\tau, u) &= 0 \quad \forall \tau \in \Sigma, \\
-(\text{div}\sigma, v) &= (f, v) \quad \forall v \in V.
\end{aligned}
\]  

(1.1)

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Here \( \sigma: \Omega \rightarrow \mathbb{S} := \mathbb{R}^{2 \times 2}_{\text{sym}} \) denotes the symmetric \( 2 \times 2 \) stress tensor field, \( u : \Omega \rightarrow \mathbb{R}^2 \) the displacement field, and \( \mathcal{A}\sigma \in \mathbb{S} \) the compliance tensor with

\[
\mathcal{A}\sigma := \frac{1}{2\mu} \left( \sigma - \frac{\lambda}{2\mu + 2\lambda} \text{tr}(\sigma)I \right),
\]

where \( \lambda > 0, \mu > 0 \) are the Lamé coefficients, \( \text{tr}(\sigma) \) the trace of \( \sigma \), \( I \) the \( 2 \times 2 \) identity matrix, and \( f \) the body force. \( H(\text{div}, \Omega; \mathbb{S}) \) denotes the space of square-integrable symmetric matrix fields with square-integrable divergence, and \( L^2(\Omega; \mathbb{R}^2) \) the space of square-integrable vector fields. The \( L^2 \) inner products on vector and matrix fields are given by

\[
(v,w) := \int_{\Omega} v \cdot w \, dx = \int_{\Omega} \sum_{i=1}^{2} v_i w_i \, dx, \quad v = (v_1, v_2), \; w = (w_1, w_2) \in V,
\]

\[
(\sigma,\tau) := \int_{\Omega} \sigma : \tau \, dx = \int_{\Omega} \sum_{1 \leq i,j \leq 2} \sigma_{ij} \tau_{ij} \, dx, \quad \sigma = (\sigma_{ij}), \; \tau = (\tau_{ij}) \in \Sigma,
\]

respectively.

According to the standard theory of mixed methods [11], a mixed finite element discretization of the weak problem (1.1) requires the pair of stress and displacement approximations to satisfy two stability conditions, i.e. a coercivity condition and an inf-sup condition. These stability constraints make it challenging to construct stable finite element pairs with symmetric stresses. In this field, we refer to [1–5,7,12,21] for some conforming mixed methods and to [6,19,22,25,30] for some nonconforming methods. In [23,24] Hu and Zhang designed a family of conforming symmetric mixed finite elements with optimal convergence orders for linear elasticity on triangular and tetrahedral grids. Later Hu [20] extended the elements to simplicial grids in \( \mathbb{R}^n \) for any positive integer \( n \). In these elements, the stress is approximated by symmetric \( H(\text{div}, \Omega; \mathbb{S}) - P_k \) polynomial tensors and the displacement is approximated by \( L^2(\Omega; \mathbb{R}^n) - P_{k-1} \) polynomial vectors for \( k \geq n + 1 \).

However, for a mixed finite element discretization based on (1.1), a computational drawback is the need to solve an algebraic system of saddle point type like

\[
\begin{pmatrix}
A & B^T \\
-B & 0
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2
\end{pmatrix}
= \begin{pmatrix}
O \\
F
\end{pmatrix},
\]

where \( A \) is a symmetric and positive definite (SPD) matrix corresponding to the term \( (\mathcal{A}\sigma,\tau) \) in (1.1), and \( X_1 \) and \( X_2 \) are the vectors of unknowns for the discrete stress and displacement approximations, respectively. One possible approach to resolve this difficulty is to apply ‘mass lumping’ on \( (\mathcal{A}\sigma,\tau) \) so as to get a diagonal or block-diagonal matrix approximation, \( \tilde{A} \), of the ‘mass matrix’ \( A \). Replacing \( A \) with \( \tilde{A} \) in the discrete system (1.3), we obtain

\[
X_1 = -\tilde{A}^{-1}B^T X_2
\]
and then
\[ B\tilde{A}^{-1}B^T X_2 = F. \] (1.4)

Notice that \( \tilde{A} \) is diagonal or block-diagonal, so is \( \tilde{A}^{-1} \). This means that the Schur complement \(-B\tilde{A}^{-1}B^T \) is SPD. As a result, by mass lumping the saddle point system (1.3) is reduced to the SPD system (1.4), which can be solved efficiently by many fast algorithms.

The key to achieve mass lumping is to select appropriate numerical quadrature rule, in which the quadrature nodes are required to match the finite element basis functions as well as maintain sufficient numerical integration accuracy. It has been shown that mass lumping schemes can be constructed for some finite elements [8,9,13–18,26–28,31]. In [8,14,17] the standard linear triangular/tetrahedral elements with mass lumping were analyzed, where the quadrature nodes are the vertices of the elements. Traditional higher order elements are not suitable for mass lumping due to the requirements of numerical accuracy and stability, and one has to use finite element spaces enriched with some bubble functions to adapt mass lumping [13,16,18,26–28]. We note that a family of mixed rectangular and cubic finite elements with mass lumping were constructed in [9] for linear elastodynamic problems, where the stress and displacement are approximated by symmetric \( H(\text{div}) - Q_k \) polynomial tensors and \( L^2 - Q_{k-1} \) polynomial vectors, respectively, and the locations of the degrees of freedom for the finite element spaces correspond to tensor products of one-dimensional quadrature nodes associated with Gauss-Lobatto (for stress) or Gauss-Legendre (for velocity) quadrature formulas.

In this paper, we first modify Hu-Zhang’s mixed conforming finite elements [23] to obtain a family of new elements which allow mass lumping. The stress field is approximated by symmetric \( H(\text{div}) - P_k \) \((k \geq 3)\) polynomial tensors enriched with higher order bubbles, and the displacement field by \( C^{-1} - P_{k-1} \) polynomial vectors enriched with higher order terms. Error analysis is carried out for the new elements as well as their mass lumping schemes.

The remainder of this paper is organized as follows. Section 2 introduces some preliminary results of mixed finite elements, including Hu-Zhang’s elements. Sections 3 and 4 are devoted to the construction and analysis of the new mixed elements and their mass lumping schemes, respectively. Finally, Section 5 gives some numerical experiments to verify the theoretical results.

2 Preliminaries

2.1 Notations

For integer \( m \geq 0 \), let \( H^m(\Omega; X) \) be the Sobolev spaces consisting of functions with domain \( \Omega \), taking values in \( X = \mathbb{S} \) or \( \mathbb{R}^2 \), and with all derivatives of order at most \( m \) square-integrable. The norm and semi-norm on \( H^m(\Omega; X) \) are denoted respectively by \( \| \cdot \|_m \) and \( | \cdot |_m \). In particular, \( H^0(\Omega; X) = L^2(\Omega; X) \).
Suppose $\mathcal{T}_h = \bigcup \{K\}$ to be a conforming and shape-regular triangulation of the domain $\Omega$ consisting of triangles. For any $K \in \mathcal{T}_h$, let $h_K$ denote its diameter, and set $h := \max_{K \in \mathcal{T}_h} h_K$. We use $P_m(K; X)$ to denote the set of all polynomials on $K$ with degree at most $m$ and taking values in $X$.

Throughout the paper, we use $a \lesssim b$ ($a \gtrsim b$) to denote $a \leq Cb$ ($a \geq Cb$), where $C$ is a generic positive constant independent of mesh parameters $h$.

### 2.2 Mixed finite element discretization

Let $\Sigma_h \subset \Sigma, V_h \subset V$ be two finite-dimensional spaces for the stress and displacement approximations, respectively. Then the mixed finite element discretization of (1.1) reads:

Find $(\sigma_h, u_h) \in \Sigma_h \times V_h$ such that

$$
\begin{align*}
(A\sigma_h, \tau_h) + (\text{div}\tau_h, u_h) &= 0 \quad \forall \tau_h \in \Sigma_h, \\
-(\text{div}\sigma_h, v_h) &= (f, v_h) \quad \forall v_h \in V_h.
\end{align*}
$$

According to the standard theory of mixed finite element methods [10, 11], the pair of finite element spaces $\Sigma_h$ and $V_h$ needs to satisfy the following stability conditions:

- **K-ellipticity condition**

  $$(A\tau_h, \tau_h) \gtrsim \|\tau_h\|^2_{H(\text{div})} \quad \forall \tau_h \in Z_h := \{\tau_h \in \Sigma_h | (\text{div}\tau_h, v) = 0, \forall v \in V_h\},$$

  where $\|\cdot\|_{H(\text{div})}$ is the norm on the space $\Sigma$ defined by

  $$
  \|\tau\|^2_{H(\text{div})} := \|\tau\|^2_0 + \|\text{div}\tau\|^2_0 \quad \forall \tau \in \Sigma.
  $$

- **Discrete BB (inf-sup) condition**

  $$
  \sup_{\tau_h \in \Sigma_h} \frac{(\text{div}\tau_h, v_h)}{\|\tau_h\|_{H(\text{div})}} \gtrsim \|v_h\|_0 \quad \forall v_h \in V_h.
  $$

### 2.3 Hu-Zhang’s mixed conforming elements

For each $K \in \mathcal{T}_h$, define an $H(\text{div})$ bubble function space, $B_{k,K}$, of polynomials of degree $k$ by

$$
B_{k,K} := \{\tau \in P_k(K; \mathbb{S}) : \tau\nu|_{\partial K} = 0\},
$$

where $\nu$ is the normal vector along $\partial K$. Introduce the local rigid motion space

$$
R(K) := \{v \in H^1(K; \mathbb{R}^2) : \nabla v + (\nabla v)^T = 0\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}\right\}
$$

and its orthogonal complement space with respect to $P_{k-1}(K; \mathbb{R}^2)(k \geq 2)$,

$$
R^\perp_{k}(K) = \{v \in P_{k-1}(K; \mathbb{R}^2) : (v, w)_K = 0, \forall w \in R(K)\}. 
$$

The following result holds.
Lemma 2.1. [20] For any \( K \in \mathcal{T}_h \) and \( k \geq 2 \), it holds that
\[
R^\perp_k(K) = \text{div} B_{k,K}.
\] (2.6)

For \( k \geq 3 \), introduce the following global finite element spaces [23]:
\[
\Sigma_{k,h} := \bar{\Sigma}_{k,h} + B_{k,h},
\]
\[
V_{k,h} := \{ v \in L^2(\Omega; \mathbb{R}^2) : v|_K \in P_{k-1}(K; \mathbb{R}^2), \ \forall K \in \mathcal{T}_h \},
\]
where
\[
B_{k,h} := \{ \tau \in H(\text{div}, \Omega; S) : \tau|_K \in B_{k,K}, \ \forall K \in \mathcal{T}_h \},\]
(2.9)
\[
\bar{\Sigma}_{k,h} := \{ \tau \in H^1(\Omega; S) : \tau|_K \in P_k(K; S), \ \forall K \in \mathcal{T}_h \}.
\]
(2.10)

It is easy to see that \( S \) has a canonical basis:
\[
T_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_3 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]
(2.11)

Let \( T_{E,j}^\perp \in S \) \( (j = 1, 2) \) be two orthogonal complement matrices of \( T_E \) with
\[
T_{E,j}^\perp : T_E = 0, \quad T_{E,j}^\perp : T_{E,j}^\perp = 1 \quad \text{and} \quad T_{E,1}^\perp : T_{E,2}^\perp = 0.
\]
(2.12)

Here \( A : B = \sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ij} \) for \( A = (a_{ij})_{i,j=1}^n \) and \( B = (b_{ij})_{i,j=1}^n \). It has been shown in [23] that
\[
S = \text{span}\{T_1, T_2, T_3\} = \text{span}\{T_E, T_{E,1}^\perp, T_{E,2}^\perp\}.
\]

Let \( \{\chi_i\}_{i=1}^l \) be the set of nodes for the Lagrange element of order \( k \) and \( \{\zeta_i\}_{i=1}^l \) be their associated Lagrange node basis functions such that
\[
\zeta_i(\chi_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots, l.
\]
(2.13)

Then the basis functions of \( \Sigma_{k,h} \) on \( K \) fall into the following four classes [23]:

(1) Vertex-based basis functions. If \( \chi_i \) is a vertex, the three associated basis functions of \( \Sigma_{k,h} \) are \( \zeta_i T_{j}, j = 1, 2, 3 \).

(2) Volumed-based functions. If \( \chi_i \) is a node inside \( K \), the three associated basis functions of \( \Sigma_{k,h} \) are \( \zeta_i T_{j}, j = 1, 2, 3 \).
(3) Edge-based basis functions with nonzero fluxes. If \( \chi_i \) is a node on edge \( E \) (not the vertex), the two associated basis functions of \( \Sigma_{k,h} \) are \( \zeta_i T_{E,j}^\perp, j = 1, 2 \).

(4) Edge-based bubble functions. If \( \chi_i \in K \) is a node on edge \( E \) (not the vertex) shared by elements \( K_1 \) and \( K_2 \), then \( \zeta_i T_{E,j}^\nu \mid_E \equiv 0 \) due to (2.11), and then the \( H(\text{div}) \) bubble functions in \( \Sigma_{k,h} \) are \( \zeta_i \mid_{K_j} T_{E,j}, j = 1, 2 \).

**Theorem 2.1.** \([20, 23]\)** Let \( (\sigma, u) \in \Sigma \times V \) and \( (\sigma_h, u_h) \in \Sigma_h \times V_h \), with \( \Sigma_h = \Sigma_{k,h} \) and \( V_h = V_{k,h} \), solve (1.1) and (2.1), respectively. If \( \sigma \in H^{k+1}(\Omega; \mathbb{S}) \) and \( v \in H^k(\Omega; \mathbb{R}^2) \), then

\[
\|\sigma - \sigma_h\|_{H(\text{div})} + \|u - u_h\|_0 \lesssim h^k (\|\sigma\|_{k+1} + \|u\|_k) \tag{2.14}
\]

and

\[
\|\sigma - \sigma_h\|_0 \lesssim h^{k+1} \|\sigma\|_{k+1}. \tag{2.15}
\]

### 2.4 Mass lumping for Hu-Zhang elements?

To solve the discrete system (2.1), we need to compute the inverse of the mass matrix corresponding to the term \( (A \sigma_h, \tau_h) \).

Let us first consider the local mass matrix on element \( K \in T_h \). Recall that \( \chi_i \) (\( i = 1, 2, \ldots, l \)) are the nodes of the Lagrange element of order \( k \) and \( \zeta_i \) (\( i = 1, 2, \ldots, l \)) are the associated Lagrange node basis functions. Thus, the basis functions of \( \Sigma_{k,h} \) on \( K \) can be denoted by

\[
\varphi_{3(m-1)+s} := \zeta_m T_s, \quad m = 1, 2, \ldots, l; s = 1, 2, 3,
\]

where \( T_s \in \{T_1, T_2, T_3\} \) if \( \chi_m \) is a vertex or a node inside \( K \), and \( T_s \in \{T_E, T_{E,1}, T_{E,2}\} \) if \( \chi_m \) is a node on edge \( E \) (not the vertex). Then the local mass matrix \( A_K \) on \( K \) is given by

\[
(A_K)_{ij} := (A \varphi_i, \varphi_j) \mid_K, \quad i, j = 1, 2, \ldots, 3l.
\]

During the finite element method, we commonly evaluate the integrals approximately by using a numerical integration formula in each element \( K \). To achieve mass lumping, the usual way is to choose the quadrature points to be the nodes \( \{\chi_i\}_{i=1}^l \) on \( K \), and the quadrature rule is of the form

\[
\int_K f dx \approx I_{k,K}(f) := \sum_{i=1}^l w_i f(\chi_i), \tag{2.16}
\]

where \( \{w_i\}_{i=1}^l \) are the weights. Then we have

\[
I_{k,K}(\zeta_i, \zeta_j) = w_i \delta_{ij}, \quad i, j = 1, 2, \ldots, l. \tag{2.17}
\]
For $m, n = 1, 2, \cdots, l$ and $s, q = 1, 2, 3$, set
\[ i = 3(m - 1) + s, \quad j = 3(n - 1) + q, \]
then from (2.17) it follows
\[
(A_K)_{ij} \approx (\tilde{A}_K)_{ij} := I_{k,K} (A \varphi_i, \varphi_j) = I_{k,K} (A \zeta_m T_s, \zeta_n T_q)
\]
\[
= \begin{cases} 
0, & m \neq n, \\
 w_m (AT_s : T_q) & m = n.
\end{cases} \tag{2.18}
\]
This means that the approximate local mass matrix $\tilde{A}_k$ is block-diagonal and of the form
\[
\tilde{A}_K = \text{diag}(w_1 B_1, w_2 B_2, \cdots, w_l B_l), \tag{2.19}
\]
where $B_m (m = 1, 2, \cdots, l)$ are $3 \times 3$ SPD matrices. For example, if $\chi_m$ is a vertex or a node inside $K$, then
\[
B_m = \frac{1}{4\mu(\mu + \lambda)} \begin{pmatrix} 2\mu + \lambda & 0 & -\lambda \\ 0 & 4(\mu + \lambda) & 0 \\ -\lambda & 0 & 2\mu + \lambda \end{pmatrix}.
\]
However, the accuracy of numerical integration has to be taken into account. From the standard theory [8, 14, 29], the following condition is required to satisfy so as to maintain the accuracy of the scheme (2.1):

(A1) The quadrature rule (2.16) must be exact for $P_{2k-2}$.

Unfortunately, the standard $P_k$ Lagrange elements fail to satisfy this condition for $k \geq 3$ (cf. [16]). In other words, Hu-Zhang’s elements do not allow mass lumping without loss of numerical accuracy.

3 Modified mixed conforming finite elements for elasticity

3.1 $P_{k,k'}$-Lagrange finite elements for mass lumping

As mentioned before, the standard $P_k$ Lagrange elements with $k \geq 3$ fail to satisfy the accuracy condition, (A1), of the quadrature rule (2.16) for mass lumping. For wave problems, as shown in [13, 16, 18, 26–28], an efficient way to address this difficulty is to construct a slightly larger finite element space

\[
P_{k,k'}(K; \mathbb{R}) := P_k(K; \mathbb{R}) + bP_{k-3}(K; \mathbb{R}) = P_k(K; \mathbb{R}) \oplus b \sum_{i=k-2}^{k'-3} P_i^{\text{hom}}(K; \mathbb{R}). \tag{3.1}
\]
Here \( k' > k \), and \( b = \lambda_1 \lambda_2 \lambda_3 \) is the bubble function on the element \( K \) with \( \lambda_i \) \( (i = 1, 2, 3) \) being the barycentric coordinates. \( P_{i}^{\text{hom}}(K; \mathbb{R}) \) denotes the set of homogeneous polynomials on \( K \) of degree \( i \). The symbol “\( \oplus \)” means that \( P_{k}(K; \mathbb{R}) \bigcap bP_{i}^{\text{hom}}(K; \mathbb{R}) = \{0\} \) for \( i = k-2, k-1, \cdots, k'-3 \).

Let \( \{\chi_i\}_{i=1}^{r} \) be the set of nodes for the \( P_{k,k'} \)-Lagrange element. Then the corresponding quadrature rule is of the form

\[
\int_{K} f \, dx \approx I_{k,k',K}(f) := \sum_{i=1}^{r} w_i f(\chi_i),
\]

where \( \{w_i\}_{i=1}^{r} \) are the weights, and \( \sum_{i=1}^{r} w_i = \text{meas}(K) \).

To maintain the accuracy and stability of finite element scheme, the following two conditions are required (cf. [13,16]):

(B1) The weights \( w_i (i = 1, \cdots, r) \) in (3.2) should be strictly positive;

(B2) The quadrature rule (3.2) must be exact for \( P_{k+k'-2} \).

Table 1 lists several \( P_{k,k'} \)-finite elements which satisfy (B1) and (B2) with \( 3 \leq k \leq 5 \). In the table, a given node \( (\alpha_1, \alpha_2, \alpha_3) \) represents an equivalence class which includes all the nodes obtained by taking all the permutations of the barycentric coordinates \( \alpha_i \). For instance, the class \((0, 0, 1)\) includes three points, \((0, 0, 1)\), \((0, 1, 0)\), and \((1, 0, 0)\); the class \((\alpha, 0, 1 - \alpha)\) includes

\[
(\alpha, 0, 1 - \alpha), (0, \alpha, 1 - \alpha), (\alpha, 1 - \alpha, 0), (1 - \alpha, \alpha, 0), (1 - \alpha, 0, \alpha), (0, 1 - \alpha, \alpha).
\]

### 3.2 Modified mixed element spaces for elasticity

Inspired by the \( P_{k,k'} \)-Lagrange elements which allow mass lumping, in this subsection we shall construct a family of new mixed conforming element spaces based on the modification of Hu-Zhang’s elements.

For \( k' > k \geq 3 \), set

\[
\Lambda_{k,k'} := \left\{ \tau \in H(\text{div}; \Omega) : \tau|_K = \sum_{i=k-2}^{k'-3} bP_{i}^{\text{hom}}(K; \mathbb{S}), \ \forall K \in \mathcal{T}_h \right\}.
\]

Then the modified global finite element spaces for the stress and displacement are given by

\[
\Sigma_{k,k',h} := \Sigma_{k,h} \oplus \Lambda_{k,k'},
\]

\[
V_{k,k',h} := V_{k,h} + \text{div} \Lambda_{k,k'}.
\]

Obviously we have \( \text{div} \Sigma_{k,k',h} \subset V_{k,k',h} \).
Table 1: $P_{k,k'}$-Lagrange triangular elements.

| space  | $k$ | $k'$ | class | weight | position parameters |
|--------|-----|------|-------|--------|---------------------|
| $P_{3,4}$ [13,16] | 3   | 4    | (0,0,1) | $(8 - \sqrt{7})/720$ | \(1/2 - \sqrt{1/(3\sqrt{7}) - 1/12}\) |
|        |     |      | $\alpha,0,1-\alpha$ | $(7 + 4\sqrt{7})/720$ | \((7 - \sqrt{7})/21\) |
|        |     |      | $\alpha,\alpha,1-2\alpha$ | $7(14 - \sqrt{7})/720$ | \((7 - \sqrt{7})/21\) |
| $P_{3,5}$ [13] | 3   | 5    | (0,0,1) | 0.00356517965360224101681201 | 0.307745941625991646104616 |
|        |     |      | $\alpha,0,1-\alpha$ | 0.017847080884026469663777 | 0.11861366396592868190663 |
|        |     |      | $\alpha,\alpha,1-2\alpha$ | 0.050942326513475907057019 | 0.425340125989747152025431 |
| $P_{4,5}$ [13,28] | 4   | 5    | (0,0,1) | 1/315 | \((1/2 - 1/\sqrt{3})\) |
|        |     |      | $\alpha,0,1-\alpha$ | 4/315 | \((5 - \sqrt{7})/18\) |
|        |     |      | $\alpha,\alpha,1-2\alpha$ | 163/2520 - 47\sqrt{7}/8820 | \((5 + \sqrt{7})/18\) |
| $P_{4,6}$ [28] | 4   | 6    | (0,0,1) | 0.00150915593385883937469324 | 0.199632107119457219140683 |
|        |     |      | $\alpha,0,1-\alpha$ | 0.010187148126178846308014 | 0.0804959191700374444460458 |
|        |     |      | $\alpha,\alpha,1-2\alpha$ | 0.0069954014635896201 | 0.10759182178486752062175, |
|        |     |      | $\alpha,\alpha,1-2\alpha$ | 0.006095591593093891810431 | 0.3029127830386363411733216 |
| $P_{5,7}$ [13,28] | 5   | 7    | (0,0,1) | 0.000709423970679245979296007 | 0.13226458163271398535888 |
|        |     |      | $\alpha,0,1-\alpha$ | 0.0034805786404821065844268 | 0.36329807415368045705506 |
|        |     |      | $\alpha,0,1-\alpha$ | 0.0069554014635896201 | 0.05752768441410105660175 |
|        |     |      | $\alpha,\alpha,1-2\alpha$ | 0.011626135459617571394984 | 0.2568517076195076063891 |
|        |     |      | $\alpha,\alpha,1-2\alpha$ | 0.04590123762857377019 | \(0.45783683807916110193503\) |
|        |     |      | $\alpha,\alpha,1-2\alpha$ | 0.0345304303772827935283885 | \(0.0781925846255170219988860\) |
|        |     |      | $\alpha,\alpha,1-2\alpha$ | 0.027278575969962595486715 | \(0.221001218759890007978128\) |
Remark 3.1. If we define
\[ B_{k,k',h} := \{ \tau \in P_{k,k'}(K; S) : \tau_{\nu}|_{\partial K} = 0, \forall K \in \mathcal{T}_h \} = B_{k,h} + \Lambda_{k,k'}, \tag{3.7} \]
\[ \Sigma_{k,k',h} := \{ \tau \in H^1(\Omega; S) : \tau|_K \in P_{k,k'}(K; S), \forall K \in \mathcal{T}_h \} = \Sigma_{k,h} + \Lambda_{k,k'}, \tag{3.8} \]
then we can also write
\[ \Sigma_{k,k',h} = \Sigma_{k,k',h} + B_{k,k',h}. \tag{3.9} \]

Let \( \{\chi_i\} \) be set of the nodes for the \( P_{k,k'} \)-Lagrange element, and \( \{\zeta_i\} \) be the corresponding nodal basis functions satisfying
\[ \zeta_i(\chi_j) = \delta_{ij}. \tag{3.10} \]
Similarly to Hu-Zhang’s elements described in Section 2.3, for each node \( \chi_i \) the associated basis functions of \( \Sigma_{k,k',h} \) on \( K \) are given as follows:
(1) \( \zeta_i T_j \) \( (j = 1, 2, 3) \), if \( \chi_i \) is a vertex or a node inside \( K \);
(2) \( \zeta_i T_E j \) \( (j = 1, 2) \) and \( \zeta_i|_K T_E j \) \( (j = 1, 2) \), if \( \chi_i \) is a node on edge \( E \) (not the vertex) shared by elements \( K_1 \) and \( K_2 \).

3.3 Stability results

This subsection is devoted to the stability analysis and error estimation of the mixed finite element scheme (2.1) with
\[ \Sigma_h = \Sigma_{k,k',h} \quad \text{and} \quad V_h = V_{k,k',h}. \]

Let \( \hat{K} \) be the reference element with vertexes \((0,0), (0,1), (1,0)\). For each \( K \in \mathcal{T}_h \), let \( F_K \) denote the affine map from \( \hat{K} \) onto \( K \) so that \( F_K(\hat{K}) = K \). Let \( \chi_0, \chi_1, \chi_2 \) be the vertices of triangle \( K \in \mathcal{T}_h \). The referencing mapping is then of the form
\[ x = F_K(\hat{x}) = \chi_0 + (\chi_1 - \chi_0 \chi_2 - \chi_0) \hat{x} := \chi_0 + B_K \hat{x}, \quad \forall \hat{x} \in \hat{K}. \]

By the shape regularity of \( \mathcal{T}_h \), it holds that
\[ \|B_K\|_0 \lesssim h, \quad \|B_K^{-1}\|_0 \lesssim h^{-1}. \tag{3.11} \]

We need to introduce the Piola transform as follows. Given \( \hat{\tau} : \hat{K} \mapsto S, \tau : K \mapsto S \) is defined by
\[ \tau(x) := B_K \hat{\tau}(\hat{x}) B_K^T. \tag{3.12} \]
Clearly this sets up a one-to-one correspondence between \( L^2(\hat{K}; S) \) and \( L^2(K; S) \) with
\[ \text{div} \tau(x) = B_K \text{div} \hat{\tau}(\hat{x}). \tag{3.13} \]
Standard scaling arguments yield the following lemma.
Lemma 3.1. For any $K \in T_h$ and $\hat{\tau} \in \hat{P}_{k,k'}(\hat{K}; S)$, let $\tau$ be given by (3.12). Then for $1 \leq q \leq k$,

$$
|\tau|_{q,K} \lesssim h^{2-q}|\det B_K|^\frac{1}{2} |\hat{\tau}|_{q,\hat{K}},
$$

(3.14)

$$
|\hat{\tau}|_{q,\hat{K}} \lesssim h^{q-2}|\det B_K|^{-\frac{1}{2}} |\tau|_{q,K}.
$$

(3.15)

Assumption 3.1. For any $\tau \in \Sigma_{k,k',h} \subset H(\div, \Omega; S)$, if

$$
\div \tau |_K = 0 \quad \forall K \in T_h,
$$

then $\tau = 0$.

Define the piecewise $m$-order semi-norm $|\cdot|_{m,h}$ ($1 \leq m \leq k'$) on $\Sigma_{k,k',h}$ as follows:

$$
|\tau_h|_{m,h} := \left( \sum_{K \in T_h} |\tau_h|_{m,K}^2 \right)^{\frac{1}{2}}, \quad \tau_h \in \Sigma_{k,k',h}.
$$

(3.16)

Lemma 3.2. Suppose that $\tau \in \Sigma_{k,k',h}$ satisfies Assumption 3.1, then

$$
\|\tau\|_0 \lesssim h |\tau|_{1,h} \lesssim h \|\div \tau\|_0.
$$

(3.17)

Proof. For any $K \in T_h$, let $\hat{K}$ be the reference element. By (3.12), we have

$$
\hat{\tau}(\hat{x}) = B_{\hat{K}}^{-1} \tau(x)(B_{\hat{K}}^{-1})^T,
$$

and $\hat{\tau}(\hat{x})$ satisfies Assumption 3.1. Thus, both $\|\hat{\div} \hat{\tau}\|_{0,\hat{K}}$ and $|\hat{\tau}|_{1,\hat{K}}$ are norms on $\hat{K}$. Then it holds

$$
\|\hat{\tau}\|_{0,\hat{K}} \lesssim |\hat{\tau}|_{1,\hat{K}} \lesssim \left\|\hat{\div} \hat{\tau}\right\|_{0,\hat{K}}
$$

which, together with (3.14) and (3.15), implies

$$
\|\tau\|_{0,K} = \left\| B_{\hat{K}} \hat{\tau} B_K^T \right\|_{0,K}
\leq \|B_K\| |\det B_K|^{\frac{1}{2}} \|\hat{\tau}\|_{0,\hat{K}} \left\| B_K^T \right\|
\lesssim h^2 |\det B_K|^{\frac{1}{2}} \|\hat{\tau}\|_{1,\hat{K}}
\lesssim h |\tau|_{1,K}
$$

and

$$
|\tau|_{1,K} \lesssim h |\det B_K|^{\frac{1}{2}} |\hat{\tau}|_{1,\hat{K}}
\lesssim h |\det B_K|^{\frac{1}{2}} \left\|\hat{\div} \hat{\tau}\right\|_{0,\hat{K}} = h |\det B_K|^{\frac{1}{2}} \|B_K^{-1}\div \tau\|_{0,K}
\lesssim \left\|\div \tau\right\|_{0,K}.
$$

This completes the proof.  \qed
In view of the definitions in (3.7) and (2.4), integration by part yields
\[
\int_K \text{div}\tau_h \cdot w_h dx = 0 \quad \forall \tau_h \in B_{k,k',h}, \ w_h \in R(K), \ K \in \mathcal{T}_h.  
\] (3.18)

Analogous to (2.5), we define
\[
R^\perp_{k,k'}(K) := \{ v \in V_{k,k',h} : (v,w)_K = 0, \ \forall w \in R(K)\}.  
\] (3.19)

By following the same routines as in [20], we can easily derive the following two lemmas.

**Lemma 3.3.** For any \( K \in \mathcal{T}_h \) and \( k \geq 2 \), it holds that
\[
R^\perp_{k,k'}(K) = \text{div} B_{k,k',h}\big|_K.  
\] (3.20)

**Lemma 3.4.** For any \( v_h \in V_{k,k',h} \), there exists a \( \tau_h \in \Sigma_{k,k',h} \) such that
\[
\int_K (\text{div}\tau_h - v_h) \cdot p dx = 0 \quad \forall p \in R(K), \ K \in \mathcal{T}_h
\]
and
\[
\|\tau_h\|_{H(\text{div})} \lesssim \|v_h\|_0. 
\]

We are now in a position to show the existence and uniqueness result.

**Theorem 3.1.** The mixed finite element scheme (2.1) with \( \Sigma_h = \Sigma_{k,k',h} \) and \( V_h = V_{k,k',h} \) admits a unique solution \( (\sigma_h, u_h) \in \Sigma_h \times V_h \).

**Proof.** It suffices to prove the K-ellipticity (2.2) and the discrete BB inequality (2.3). Note that (2.2) follows from the fact \( \text{div}\Sigma_{k,k',h} \subset V_{k,k',h} \).

We follow a similar way in [20] to show (2.3). For any given \( v_h \in V_{k,k',h} \), by Lemma 3.4, there exists \( \tau_1 \in \Sigma_{k,k',h} \) with
\[
\int_K (\text{div}\tau_1 - v_h) \cdot p dx = 0 \quad \forall p \in R(K), \ K \in \mathcal{T}_h
\]
and
\[
\|\tau_1\|_{H(\text{div})} \lesssim \|v_h\|_0. 
\]

Then, by Lemma 3.3 there exists \( \tau_2 \in B_{k,k',h} \) with
\[
\text{div}\tau_2 = v_h - \text{div}\tau_1, \quad \|\tau_2\|_0 = \min\{\|\tau\|_0 : \text{div}\tau = v_h - \text{div}\tau_1, \ \tau \in B_{k,k',h}\}.
\]

Thus, if \( \text{div}\tau_2 = 0 \), then \( \tau_2 = 0 \), i.e. \( \tau_2 \) satisfies Assumption 3.1. Hence, by Lemma 3.2 we have
\[
\|\tau_2\|_{H(\text{div})} \lesssim \|v_h - \text{div}\tau_1\|_0 \lesssim \|v_h\|_0.  
\] (3.21)

Finally, set \( \tau_h := \tau_1 + \tau_2 \), which implies that
\[
\text{div}\tau_h = v_h \quad \text{and} \quad \|\tau_h\|_{H(\text{div})} \lesssim \|v_h\|_0.  
\] (3.22)

This means that the discrete BB inequality (2.3) holds.
Remark 3.2. According to Lemma 3.3 and 3.4, we can derive that there exists an interpolation \( \Pi_h : H^1(\Omega; \mathbb{S}) \mapsto \Sigma_{k,k',h} \) such that for any \( \tau \in H^1(\Omega; \mathbb{S}) \),

\[
(\operatorname{div}(\tau - \Pi_h \tau), v_h)_K = 0, \quad \forall K \in T_h, \forall v_h \in V_{k,k',h}.
\]

Furthermore, if \( \tau \in H^{k+1}(\Omega; \mathbb{S}) \), then

\[
\|\tau - \Pi_h \tau\|_0 \leq C h^{k+1} \|\tau\|_{k+1}.
\] (3.23)

This shows that the operator \( \Pi_h : H^1(\Omega; \mathbb{S}) \mapsto \Sigma_{k,k',h} \) has the following commutative property:

\[
P_h \operatorname{div} \tau = \operatorname{div} \Pi_h \tau \quad \forall \tau \in H^1(\Omega; \mathbb{S}).
\] (3.24)

Here \( P_h : L^2(\Omega; \mathbb{R}^2) \mapsto V_{k,k',h} \) is the \( L^2 \) projection operator.

By the stability conditions (2.2)-(2.3) and Remark 3.2, we easily obtain the following error estimates.

Theorem 3.2. Let \( (\sigma, u) \in (\Sigma \cap H^{k+1}(\Omega; \mathbb{S})) \times (V \cap H^k(\Omega; \mathbb{R}^2)) \) and \( (\sigma_h, u_h) \in \Sigma_h \times V_h = \Sigma_{k,k',h} \times V_{k,k',h} \) solve (1.1) and (2.1), respectively. Then it holds that

\[
\|\sigma - \sigma_h\|_{H(\operatorname{div})} + \|u - u_h\|_0 \lesssim h^k (\|\sigma\|_{k+1} + \|u\|_k)
\] (3.25)

and

\[
\|\sigma - \sigma_h\|_0 \lesssim h^{k+1} \|\sigma\|_{k+1}.
\] (3.26)

4 Mass lumping mixed finite element method

4.1 Mass lumping scheme

As mentioned before, the mixed scheme (2.1) leads to an algebraic system of saddle point type. One approach to address this issue is applying mass lumping.

The mass lumping scheme for (2.1) is described as follows: Find \( (\sigma_h, u_h) \in \Sigma_{k,k',h} \times V_{k,k',h} \), such that

\[
\begin{aligned}
(A\sigma_h, \tau_h)_h + (\operatorname{div} \tau, u_h)_h &= 0 \quad \forall \tau_h \in \Sigma_{k,k',h}, \\
-(\operatorname{div} \sigma_h, v_h) &= (f, v_h) \quad \forall v_h \in V_{k,k',h}.
\end{aligned}
\] (4.1)

Here \( (A\sigma_h, \tau_h)_h := \sum_{K \in T_h} (A\sigma_h, \tau_h)_{h,K} \) with

\[
(A\sigma_h, \tau_h)_{h,K} := I_{k,k',K}(A\sigma_h : \tau_h),
\]

and \( I_{k,k',K} \) is the quadrature operator in (3.2) satisfying the conditions (B1) and (B2).

The following lemma shows that the quadrature rule (3.2) produces a coercive bilinear form \( (\cdot, \cdot)_h \).
Lemma 4.1. It holds that
\[
(A\tau, \tau)_{h} \gtrsim \|\tau\|_{0,K}^2 \quad \forall \tau \in \Sigma_{k,k',h}.
\] (4.2)

Proof. Recall that \(\{\chi_i\}_{i=1}^r\) are the nodes for the \(P_{k,k'}\)-Lagrange element, and \(\{\zeta_i\}_{i=1}^r\) are the corresponding nodal basis functions satisfying (3.10). Then, for any \(\tau \in \Sigma_{k,k',h}\) we can denote
\[
\tau|_K = \sum_{i=1}^r \sum_{j=1}^3 c_{ij} \zeta_i T_j,
\]
where \(T_j \in \{T_1, T_2, T_3\}\) if \(\chi_m\) is a vertex or a node inside \(K\), and \(T_j \in \{T_E, T_{E,1}, T_{E,2}\}\) if \(\chi_m\) is a node on edge \(E\) (not the vertex). Thus,
\[
(A\tau, \tau)_{h,K} = \sum_{i=1}^r \left( \sum_{j=1}^3 c_{ij} A T_j \right) \zeta_i : \sum_{s=1}^r \left( \sum_{t=1}^3 c_{st} T_t \right) \zeta_s
\]
\[
= \sum_{i=1}^r \sum_{s=1}^r I_{k,k',K} \left( (\zeta_i \sum_{j=1}^3 c_{ij} A T_j) : (\zeta_s \sum_{t=1}^3 c_{st} T_t) \right)
\]
\[
= \sum_{i=1}^r w_i \left( \sum_{j=1}^3 c_{ij} A T_j : \sum_{t=1}^3 c_{it} T_t \right)
\]
\[
\gtrsim \sum_{i=1}^r w_i \sum_{j=1}^3 c_{ij}^2 \gtrsim h^2 \|\tau\|_{0,K}^2,
\]
where \(w_i\) are the weights in (3.2). As a result,
\[
(A\tau, \tau)_h \gtrsim \sum_{K \in T_h} h^2 \|\tau\|_{0,K}^2 \gtrsim \|\tau\|_{0}^2,
\]
which completes the proof. \(\square\)

This coercivity lemma, together with the discrete BB condition (2.3), yields the following conclusion.

Lemma 4.2. The mass lumping scheme (4.1) admits a unique solution.

4.2 Error estimation

In light of the stability conditions (4.2) and (2.3) and standard techniques, we easily derive the following result.
Lemma 4.3. Let \((\sigma, u) \in \Sigma \times V\) and \((\sigma_h, u_h) \in \Sigma_{k,k',h} \times V_{k,k',h}\) be the solutions of (1.1) and (4.1), respectively. Then

\[
\|\sigma - \sigma_h\|_{H(\text{div})} + \|u - u_h\|_0 \lesssim \|u - P_h u\|_0 + \inf_{\tilde{\tau}_h \in \Sigma_{k,k',h}} \left( \|\sigma - \tilde{\tau}_h\|_{H(\text{div})} + \sup_{\tau_h \in \Sigma_{k,k',h}} \frac{E_h(\tilde{\tau}_h, \tau_h)}{\|\tau_h\|_{H(\text{div})}} \right),
\]

where \(P_h : L^2(\Omega; \mathbb{R}^2) \mapsto V_{k,k',h}\) is the \(L^2\) projection operator, and

\[
E_h(\tilde{\tau}_h, \tau_h) := (A\tilde{\tau}_h, \tau_h) - (A\tilde{\tau}_h, \tau_h)_h = \sum_{K \in T_h} \left( \int_K A\tilde{\tau}_h : \tau_h dx - I_{k,k',K}(A\tilde{\tau}_h : \tau_h) \right).
\]

Let \(W_h\) be a space satisfying

\[
\Sigma_{k,h} \subseteq W_h \subseteq \Sigma_{k,k',h}
\]

and consisting of piecewise polynomial tensors of degree at most \(\tilde{k}\), \(k \leq \tilde{k} \leq k'\). Then we have the following estimate for \(E_h(\tilde{\tau}_h, \tau_h)\), which can be viewed as an extended version of [16, Lemma 5.2].

Lemma 4.4. If

\[
1 \leq p \leq k - 1 + (k' - \tilde{k}), \quad 0 \leq q \leq k - 1,
\]

then for all \((\tilde{\tau}_h, \tau_h) \in \Sigma_{k,k',h} \times W_h\), it holds

\[
|E_h(\tilde{\tau}_h, \tau_h)| \lesssim h^p q|\tilde{\tau}_h|_{p,h} \cdot |\tau_h|_{q,h}.
\]

Proof. For any \(K = F_K(\hat{x}) \in T_h\) with \(x = F_K(\hat{x})\), we set

\[
\hat{\tilde{\tau}}_h(\hat{x}) := \tilde{\tau}_h(x)|_K, \quad \hat{\tau}_h(\hat{x}) = \tau_h(x)|_K.
\]

By scaling arguments we have

\[
|\hat{\tilde{\tau}}_h|_{p,\hat{K}} \lesssim h^p |\det B_K|^{-\frac{1}{2}} |\tilde{\tau}_h|_{p,K}, \quad |\hat{\tau}_h|_{p,\hat{K}} \lesssim h^p |\det B_K|^{-\frac{1}{2}} |\tau_h|_{p,K}.
\]

Then

\[
|E_h(\tilde{\tau}_h, \tau_h)| = \sum_{K \in T_h} |E_h,K(\tilde{\tau}_h, \tau_h)| = \sum_{K \in T_h} |\det B_K| \hat{E}_{h,K} (\hat{\tilde{\tau}}_h, \hat{\tau}_h).
\]

From (4.5) it follows

\[
0 \leq p - 1 + \tilde{k} \leq k + k' - 2, \quad 0 \leq q - 1 + k' \leq k + k' - 2, \quad 0 \leq p - 1 + q - 1 \leq k + k' - 2.
\]
Let $\hat{\Pi}_j$ denote the $L^2$ projection from $\hat{L}^2(\hat{K}; \mathbb{S})$ onto $\hat{P}_j(\hat{K}; \mathbb{S})$. By (B2), the quadrature rule (3.2) is exact for $P_{k+k'-2}$. Thus,

$$
\left| \hat{E}_{h,K}(\hat{\tau}_h, \tau_h) \right| = \left| \hat{E}_{h,K}(\hat{\tau}_h - \hat{\Pi}_{p-1}\hat{\tau}_h, \tau_h - \hat{\Pi}_{q-1}\hat{\tau}_h) \right|
\leq \left\| \hat{\tau}_h - \hat{\Pi}_{p-1}\hat{\tau}_h \right\|_{0,K} \cdot \left\| \tau_h - \hat{\Pi}_{q-1}\tau_h \right\|_{0,K}
\leq h^p |\det B_K|^{-\frac{1}{2}} |\tau_h|_{p,K} \cdot h^q |\det B_K|^{-\frac{1}{2}} |\tau_h|_{q,K},
$$

which, together with (4.8), yields the desired result. 

**Remark 4.1.** If taking $W_h = \Sigma_{k,k',h}$ and $p = k - 1, q = 0$ in Lemma 4.4, then we obtain

$$
|E_h(\hat{\tau}_h, \tau_h)| \leq h^{k-1} |\tau_h|_{k-1,h} \cdot |\tau_h|_{0,h}, \quad \forall \hat{\tau}_h, \tau_h \in \Sigma_{k,k',h},
$$

which yields

$$
\sup_{\tau_h \in \Sigma_{k,k',h}} \frac{E_h(\hat{\tau}_h, \tau_h)}{\|\tau_h\|_{H(\text{div})}} \leq h^{k-1} |\tau_h|_{k-1,h}.
$$

This inequality, together with Lemma 4.3, leads to an error estimate like

$$
\|\sigma - \sigma_h\|_{H(\text{div})} + \|u - u_h\|_0 \leq h^{k-1} (\|\sigma\|_k + \|u\|_{k-1}),
$$

provided that $\sigma \in H^k(\Omega, \mathbb{S})$ and $u \in H^{k-1}(\Omega, \mathbb{R}^2)$. Note that such an estimate is not optimal.

In what follows we will apply a more elaborate analysis to get a better estimate for the consistency error than (4.9). To this end, we set, for any $K \in \mathcal{T}_h$,

$$
\Xi_j := bP_j^{\text{hom}}(K; \mathbb{S}), \quad k - 2 \leq j \leq k' - 3.
$$

Here we recall that $P_j^{\text{hom}}(K; \mathbb{S})$ denotes the set of homogeneous polynomial tensors of degree $j$. On the reference element $\hat{K}$ with vertexes $(0,0)$, $(1,0)$ and $(0,1)$, the bubble function reads $\hat{b} = \hat{x}_1\hat{x}_2(1 - \hat{x}_1 - \hat{x}_2)$. Let $\left\{ \hat{\psi}_i \right\}_{i=0}^j$ be the basis of the space $b\hat{P}_j^{\text{hom}}(\hat{K}; \mathbb{R})$, then

$$
\hat{\psi}_i = \hat{x}_1^{j-i}\hat{x}_2^{i} \hat{b} = \hat{x}_1^{j+1}\hat{x}_2^{i+1}(1 - \hat{x}_1 - \hat{x}_2), \quad i = 0, 1, \cdots, j
$$

and

$$
\Xi_j = \text{span}\{\hat{\psi}_i \mathcal{T}_s : i = 0, 1, \cdots, j; \ s = 1, 2, 3\}.
$$

**Lemma 4.5.** For any $j \geq 1$, $\tau \in \Xi_j$ satisfies Assumption 3.1.
Proof. We first show that the functions \( \left\{ \frac{\partial \hat{\psi}_i}{\partial x_1}, \frac{\partial \hat{\psi}_i}{\partial x_2} \right\}_{i=0}^j \) are linear independent. It is easy to obtain
\[
\frac{\partial \hat{\psi}_1}{\partial x_1} = (i + 1)x_1^{j+1} - (j + 2)x_2^{j+1} - (i + 1)x_1^{j+2},
\frac{\partial \hat{\psi}_2}{\partial x_2} = (j - i + 1)x_1^{j-i} - (j - i + 1)x_2^{j-i} - (j - i + 2)x_1^{j-i}x_2^{j-i}.
\]

Suppose that there are constants \( \{c_i\}_{i=0}^j \) such that
\[
\sum_{i=0}^j c_i \frac{\partial \hat{\psi}_i}{\partial x_1} + \sum_{i=0}^j d_i \frac{\partial \hat{\psi}_i}{\partial x_2} = 0,
\]
which indicates, for \( 0 \leq i \leq j + 1, \)
\[
(i + 1)c_i + (j - i + 2)d_{i-1} = 0,
(i + 1)c_i + (i + 1)c_{i-1} + (j - i + 3)d_{i-2} + (j - i + 3)d_{i-1} = 0.
\]

Here we set \( c_{-1} = d_{-1} = d_{-2} = c_{j+1} = 0. \) Simple calculations show that
\[
c_i = d_i = 0, \quad i = 0, 1, \ldots, j,
\]
i.e. \( \left\{ \frac{\partial \hat{\psi}_i}{\partial x_1}, \frac{\partial \hat{\psi}_i}{\partial x_2} \right\}_{i=0}^j \) are linear independent.

Second, for any \( \hat{\tau} \in \hat{\mathbb{Z}}_j \), there exist constants \( c_{i} (i = 0, 1, \ldots, j; s = 1, 2, 3) \), such that
\[
\hat{\tau} = \sum_{i=0}^j \sum_{s=1}^3 c_{is} \hat{\psi}_i \hat{T}_s,
\]
which means
\[
\hat{\text{div}} \hat{\tau} = \left( \sum_{i=0}^j \left( c_1 \frac{\partial \hat{\psi}_i}{\partial x_1} + c_3 \frac{\partial \hat{\psi}_i}{\partial x_2} \right) \right) + \left( \sum_{i=0}^j \left( c_2 \frac{\partial \hat{\psi}_i}{\partial x_2} + c_3 \frac{\partial \hat{\psi}_i}{\partial x_1} \right) \right).
\]

If \( \hat{\text{div}} \hat{\tau} = 0 \), then we get
\[
\sum_{i=0}^j \left( c_1 \frac{\partial \hat{\psi}_i}{\partial x_1} + c_3 \frac{\partial \hat{\psi}_i}{\partial x_2} \right) = 0, \quad \sum_{i=0}^j \left( c_2 \frac{\partial \hat{\psi}_i}{\partial x_2} + c_3 \frac{\partial \hat{\psi}_i}{\partial x_1} \right) = 0.
\]

Thus, from the linear independence of \( \left\{ \frac{\partial \hat{\psi}_i}{\partial x_1}, \frac{\partial \hat{\psi}_i}{\partial x_2} \right\}_{i=0}^j \) it follows
\[
c_{is} = 0, \quad i = 0, 1, \ldots, j, \quad s = 1, 2, 3,
\]
i.e. \( \hat{\tau} = 0. \) This completes the proof. \( \square \)
Thanks to Lemma 4.5, we can obtain the following estimate for the consistency error.

Lemma 4.6. For any $\tilde{\tau}_h \in \Sigma_{k,k',h}$, it holds

$$\sup_{\tau_h \in \Sigma_{k,k',h}} \frac{|E_h(\tilde{\tau}_h, \tau_h)|}{\|\tau_h\|_{H(\text{div})}} \lesssim h^k |\tilde{\tau}_h|_{k,k,h}. \quad (4.11)$$

Proof. In view of the definition, (3.5), of $\Sigma_{k,k',h}$, we have for any $\tilde{\tau}_h \in \Sigma_{k,k',h}$,

$$\sup_{\tau_h \in \Sigma_{k,k',h}} \frac{E_h(\tilde{\tau}_h, \tau_h)}{\|\tau_h\|_{H(\text{div})}} \leq \sup_{\tau_h \in \Sigma_{k,k',h} \setminus \Xi_{k' - 3}} \frac{E_h(\tilde{\tau}_h, \tau_h)}{\|\tau_h\|_{H(\text{div})}} + \sup_{\tau_h \in \Xi_{k' - 3}} \frac{E_h(\tilde{\tau}_h, \tau_h)}{\|\tau_h\|_{H(\text{div})}} =: M_1 + M_2.$$

We first estimate $M_1$. Since the degree of polynomials contained in $\Sigma_{k,k',h} \setminus \Xi_{k' - 3}$ is at most $k' - 1$, we can take $k = k' - 1, p = k, q = 0$ in Lemma 4.4 to get

$$M_1 = \sup_{\tau_h \in \Sigma_{k,k',h} \setminus \Xi_{k' - 3}} \frac{E_h(\tilde{\tau}_h, \tau_h)}{\|\tau_h\|_{H(\text{div})}} \lesssim h^k |\tilde{\tau}_h|_{k,k,h}. \quad (4.12)$$

For $M_2$, take $k = k', p = k - 1, q = 1$ in Lemma 4.4, then by Lemma 3.2 and Lemma 4.5 we obtain

$$M_2 = \sup_{\tau_h \in \Xi_{k' - 3}} \frac{E_h(\tilde{\tau}_h, \tau_h)}{\|\tau_h\|_{H(\text{div})}} \lesssim \sup_{\tau_h \in \Xi_{k' - 3}} \frac{h^k |\tilde{\tau}_h|_{k-1,h} |\tau_h|_{1,h}}{\|\tau_h\|_0} \lesssim \sup_{\tau_h \in \Xi_{k' - 3}} \frac{h^k |\tilde{\tau}_h|_{k-1,h} |\tau_h|_{1,h}}{|\tau_h|_{1,h}}$$

$$\lesssim h^k |\tilde{\tau}_h|_{k-1,h},$$

which, together with (4.12), yields the desired conclusion.

Finally, combining Lemma 4.6 and Lemma 4.3 immediately yields the following optimal error estimate for the mass lumping mixed finite element scheme.

Theorem 4.1. Let $(\sigma, u) \in (\Sigma \cap H^{k+1}(\Omega; \mathbb{S})) \times (V \cap H^k(\Omega; \mathbb{R}^2))$ and $(\sigma_h, u_h) \in \Sigma_{k,k',h} \times V_{k,k',h}$ be the solutions of (1.1) and (4.1), respectively. Then

$$\|\sigma - \sigma_h\|_{H(\text{div})} + \|u - u_h\|_0 \lesssim h^k \left(\|\sigma\|_{k+1} + \|u\|_k\right). \quad (4.13)$$

The following theorem shows that the optimal error estimate for the stress in $L^2$ norm can be achieved for some special cases.

Theorem 4.2. Let $(\sigma_h, v_h) \in \Sigma_{k,k',h} \times V_{k,k',h}$ be the solution of (4.1) with $k' \geq k + 2$. If the tensor functions contained in the space $\Xi_{k' - 3} \cup \Xi_{k' - 4}$ satisfy Assumption 3.1, then

$$\|\sigma - \sigma_h\|_0 \lesssim h^{k+1} \|\sigma\|_{k+1}. \quad (4.14)$$
Proof. Let $\Pi_h$ be the same operator as in (3.24), then it suffices to show
\[ \|\Pi_h \sigma - \sigma_h\|_0 \lesssim h^{k+1} \|\sigma\|_{k+1}. \]
In fact, we can write
\[ \Pi_h \sigma - \sigma_h = \sigma_1 + \sigma_2 \]
with $\sigma_1 \in \Sigma_{k,k',h} \setminus (\Xi_{k'-3} \cup \Xi_{k'-4})$ and $\sigma_2 \in \Xi_{k'-3} \cup \Xi_{k'-4}$. By the community property (3.24), we get
\[ \text{div}(\sigma_1 + \sigma_2) = \text{div}(\Pi_h \sigma - \sigma_h) = 0. \]
Since the degree of the polynomial vector $\text{div}\sigma_1$ is $k' - 3$, then the degree of $\text{div}\sigma_2$ is also no more than $k' - 3$. According to Assumption 3.1, we can derive that $\sigma_2 \in \Xi_{k'-4} = bP^k_{k'-4}(K;S)$. This implies that $\text{div}\sigma_2$ is of degree $k' - 2$ if $\sigma_2 \neq 0$, which conflicts the conclusion that $\text{div}\sigma_2$ is no more than $k' - 3$. Hence, $\sigma_2 = 0$. As a result, $\Pi_h \sigma - \sigma_h = \sigma_1$ is of degree at most $k' - 2$.

Now we set $\tilde{k} = k' - 2, p = k + 1, q = 0$ in Lemma 4.4, then
\[ E_h(\Pi_h \sigma, \Pi_h \sigma - \sigma_h) \lesssim h^{k+1} \|\Pi_h \sigma\|_{k+1,0} \|\Pi_h \sigma - \sigma_h\|_0 \lesssim h \|\sigma\|_{k+1} \|\Pi_h \sigma - \sigma_h\|_0. \]
From (1.1), (4.1) and Lemma 4.1, it follows
\[ \|\Pi_h \sigma - \sigma_h\|_0^2 \lesssim (\mathcal{A}(\Pi_h \sigma - \sigma_h), \Pi_h \sigma - \sigma_h)_h = -(\mathcal{A}(\sigma - \Pi_h \sigma), \Pi_h \sigma - \sigma_h) - E_h(\Pi_h \sigma, \Pi_h \sigma - \sigma_h). \]
Combining the two estimates above indicates
\[ \|\Pi_h \sigma - \sigma_h\|_0 \lesssim \|\sigma - \Pi_h \sigma\|_0 + \frac{E_h(\Pi_h \sigma, \Pi_h \sigma - \sigma_h)}{\|\Pi_h \sigma - \sigma_h\|_0} \lesssim h^{k+1} \|\sigma\|_{k+1}. \]
This finishes the proof.

Remark 4.2. We can verify that the space $\Xi_1 \cup \Xi_2$ satisfies Assumption 3.1. Thus, for the solution $(\sigma_h, v_h) \in \Sigma_{3,5,h} \times V_{3,5,h}$ of (4.1), it holds
\[ \|\sigma - \sigma_h\|_0 \lesssim h^4 \|\sigma\|_4. \] (4.15)

5 Numerical results

In this section, we shall give a numerical example to verify our theoretical analysis for the scheme (2.1), of the modified mixed element $\Sigma_{k,k',h} - V_{k,k',h}$, and the mass lumping scheme (4.1) in three cases: $k = 3, k' = 4; k = 4, k' = 5; k = 3, k' = 5.
Take $\Omega = [0, 1] \times [0, 1]$ and Lamé constants $\lambda = 1, \mu = \frac{1}{2}$ in the model problem (1.1). Let the exact solution $(\sigma, u)$ be of the following form:

\begin{align*}
  u_1 &= -x_1^2 x_2 (2x_2 - 1)(x_1 - 1)^2(x_2 - 1), \\
  u_2 &= x_1 x_2^2 (2x_1 - 1)(x_2 - 1)^2(x_1 - 1), \\
  \sigma_{11} &= -\sigma_{22} = -2x_1 x_2 (2x_1^2 - 3x_1 + 1)(2x_2^2 - 3x_2 + 1), \\
  \sigma_{12} &= \sigma_{21} = x_1 x_2^2 (x_2 - 1)^2(2x_1^2 - \frac{3}{2}) - x_1^2 x_2 (x_1 - 1)^2(2x_2 - \frac{3}{2}) \\
  &\quad - \frac{x_1^2}{2}(2x_2 - 1)(x_1 - 1)^2(x_2 - 1) + \frac{x_2^2}{2}(2x_1 - 1)(x_1 - 1)(x_2 - 1)^2.
\end{align*}

We use $N \times N$ uniform triangular meshes for the computation (cf. Figure 1), and list the error results of the stress and displacement approximations in Tables 2-4.

Table 2 gives the results of Hu-Zhang’s element $\Sigma_{k,h} - V_{k,h}$ [23], the modified element $\Sigma_{k,k',h} - V_{k,k',h}$ and the mass lumping scheme for $k = 3, k' = 4$. Table 3 gives the results of the three methods for $k = 4, k' = 5$. And Table 4 gives the results of the modified element $\Sigma_{k,k',h} - V_{k,k',h}$ and the mass lumping scheme for $k = 3, k' = 5$. From the numerical results we have the following observations:

• As same as Hu-Zhang’s element, the modified element $\Sigma_{k,k',h} - V_{k,k',h}$ for $k = 3, 4$ yields the $k$-th order of convergence for $||\text{div}(\sigma - \sigma_h)||_0$ and $||u - u_h||_0$, and $k+1$-th order of convergence for $||\sigma - \sigma_h||_0$. This is conformable to the theoretical results in Theorem 3.2.

• The mass lumping scheme of the modified element $\Sigma_{k,k',h} - V_{k,k',h}$ yields the $k$-th order of convergence for $||\text{div}(\sigma - \sigma_h)||_0$ and $||u - u_h||_0$, as is conformable to the theoretical result in Theorem 4.1.

• The mass lumping scheme of $\Sigma_{k,k',h} - V_{k,k',h}$, with $k = 3, k' = 4$ and $k = 4, k' = 5$, yields the $k$-th order of convergence for $||\sigma - \sigma_h||_0$, one order lower than the original scheme, while the mass lumping scheme with $k = 3, k' = 5$ yields the $k+1$-th order of convergence, which is consistent with Remark 4.2.

• Though the proposed modified element $\Sigma_{k,k',h} - V_{k,k',h}$ is of more degrees of freedom than Hu-Zhang’s element $\Sigma_{k,h} - V_{k,h}$, its mass lumping scheme leads to a SPD system that is much easier to solve.

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Figure 1: The domain with uniform triangular meshes

Table 2: History of convergence: $k = 3, k' = 4$

|                | $N$ | $\|\sigma - \sigma_h\|_\Omega$ | $\|\sigma\|_0$ | Order | $\|\text{div}(\sigma - \sigma_h)\|_0$ | $\|\text{div}\sigma\|_0$ | Order | $\|u - u_h\|_0$ | Order |
|----------------|-----|---------------------------------|----------------|-------|-----------------------------------|-----------------|-------|----------------|-------|
| **Hu-Zhang’s element** $\Sigma_{3,h} - V_{3,h}$ | 2   | 9.361e-2                         | -              | 9.256e-2 | 2.64                              | 1.409e-1        | -     | 1.948e-2       | 2.85  |
|                | 4   | 9.035e-3                         | 3.37           | 1.480e-2 | 2.64                              | 2.590e-3        | 2.91  | 3.296e-4       | 2.97  |
|                | 8   | 6.498e-4                         | 3.79           | 1.953e-3 | 2.92                              | 3.102e-5        | 2.99  | 4.139e-5       | 2.99  |
|                | 16  | 4.289e-5                         | 3.92           | 2.473e-4 | 2.98                              | 4.139e-5        | 2.99  | 4.139e-5       | 2.99  |
|                | 32  | 2.742e-6                         | 3.96           | 3.102e-5 | 2.99                              | 4.139e-5        | 2.99  | 4.139e-5       | 2.99  |
| **Modified element** $\Sigma_{3,4,h} - V_{3,4,h}$ | 2   | 1.065e-1                         | -              | 5.414e-2 | -                                 | 7.038e-2        | -     | -              | -     |
|                | 4   | 1.120e-2                         | 3.25           | 7.438e-3 | 2.86                              | 9.685e-3        | 2.86  | 1.240e-3       | 2.96  |
|                | 8   | 8.296e-4                         | 3.75           | 9.496e-4 | 2.96                              | 1.240e-3        | 2.96  | 1.565e-4       | 2.98  |
|                | 16  | 5.551e-5                         | 3.90           | 1.193e-4 | 2.99                              | 1.565e-4        | 2.98  | 1.962e-5       | 2.99  |
|                | 32  | 3.573e-6                         | 3.95           | 1.493e-5 | 3.00                              | 1.962e-5        | 2.99  | 1.962e-5       | 2.99  |
| **Mass lumping** $\Sigma_{3,4,h} - V_{3,4,h}$ | 2   | 1.219e-1                         | -              | 6.417e-2 | -                                 | 8.983e-2        | -     | -              | -     |
|                | 4   | 1.731e-2                         | 2.81           | 7.880e-3 | 3.02                              | 1.327e-2        | 2.75  | 2.232e-4       | 2.97  |
|                | 8   | 2.0759e-3                        | 3.06           | 9.741e-4 | 3.01                              | 1.758e-3        | 2.91  | 2.232e-4       | 2.97  |
|                | 16  | 2.466e-4                         | 3.07           | 1.213e-4 | 3.00                              | 2.232e-4        | 2.97  | 2.232e-4       | 2.97  |
|                | 32  | 2.981e-5                         | 3.04           | 1.515e-5 | 3.00                              | 2.801e-5        | 2.99  | 2.801e-5       | 2.99  |
Table 3: History of convergence: \( k = 4, k' = 5 \)

| \( N \) | \( \frac{\| \sigma - \sigma_h \|_0}{\| \sigma \|_0} \) | \( \frac{\| \text{div}(\sigma - \sigma_h) \|_0}{\| \text{div}\sigma \|_0} \) | \( \frac{\| u - u_h \|_0}{\| u \|_0} \) |
|------|----------------|----------------|----------------|
|      | Error Order    | Error Order    | Error Order    |
| Hu-Zhang’s element |                |                |                |
| \( \Sigma_{4,h} - V_{4,h} \) |                |                |                |
| 2    | 1.919e-2 – 2.505e-2 – 2.583e-2 – |                |                |                |
| 4    | 7.329e-4 4.71 | 1.724e-3 3.86 | 2.655e-3 3.28 |                |
| 8    | 2.481e-5 4.88 | 1.101e-4 3.96 | 1.860e-4 3.83 |                |
| 16   | 8.043e-7 4.94 | 6.919e-6 3.99 | 1.194e-5 3.96 |                |
| 32   | 2.557e-8 4.97 | 4.330e-7 4.00 | 7.519e-7 3.99 |                |
| Modified element |                |                |                |
| \( \Sigma_{4,5,h} - V_{4,5,h} \) |                |                |                |
| 2    | 2.602e-2 – 4.862e-2 – 1.403e-2 – |                |                |                |
| 4    | 9.792e-4 4.73 | 2.393e-4 4.44 | 6.087e-4 4.52 |                |
| 8    | 3.302e-5 4.88 | 1.243e-5 4.17 | 3.298e-5 4.20 |                |
| 16   | 1.069e-6 4.94 | 7.508e-6 4.04 | 1.980e-6 4.05 |                |
| 32   | 3.401e-8 4.97 | 4.650e-8 4.01 | 1.225e-7 4.01 |                |
| Mass lumping |                |                |                |
| \( \Sigma_{4,5,h} - V_{4,5,h} \) |                |                |                |
| 2    | 3.679e-2 – 6.097e-3 – 1.751e-2 – |                |                |                |
| 4    | 2.377e-3 3.95 | 2.532e-4 4.58 | 1.753e-3 3.32 |                |
| 8    | 1.499e-4 3.98 | 1.308e-5 4.27 | 1.223e-4 3.84 |                |
| 16   | 9.369e-6 4.00 | 7.690e-7 4.08 | 7.853e-6 3.96 |                |
| 32   | 5.843e-7 4.00 | 4.727e-8 4.02 | 4.942e-7 3.99 |                |

Table 4: History of convergence: \( k = 3, k' = 5 \)

| \( N \) | \( \frac{\| \sigma - \sigma_h \|_0}{\| \sigma \|_0} \) | \( \frac{\| \text{div}(\sigma - \sigma_h) \|_0}{\| \text{div}\sigma \|_0} \) | \( \frac{\| u - u_h \|_0}{\| u \|_0} \) |
|------|----------------|----------------|----------------|
|      | Error Order    | Error Order    | Error Order    |
| Modified element |                |                |                |
| \( \Sigma_{3,5,h} - V_{3,5,h} \) |                |                |                |
| 2    | 1.140e-1 – 3.226e-2 – 5.201e-2 – |                |                |                |
| 4    | 1.185e-2 3.26 | 4.176e-3 2.95 | 5.757e-3 3.17 |                |
| 8    | 8.745e-4 3.76 | 5.248e-4 2.99 | 6.694e-4 3.10 |                |
| 16   | 5.841e-5 3.90 | 6.567e-5 3.00 | 8.354e-5 3.00 |                |
| 32   | 3.757e-6 3.95 | 8.211e-6 3.00 | 1.045e-5 3.00 |                |
| Mass lumping |                |                |                |
| \( \Sigma_{3,5,h} - V_{3,5,h} \) |                |                |                |
| 2    | 1.131e-1 – 3.575e-2 – 6.825e-2 – |                |                |                |
| 4    | 1.184e-2 3.25 | 4.621e-3 2.95 | 6.751e-3 3.33 |                |
| 8    | 8.751e-4 3.75 | 5.809e-4 2.99 | 7.929e-4 3.08 |                |
| 16   | 5.850e-5 3.90 | 7.271e-5 3.00 | 9.862e-5 3.00 |                |
| 32   | 3.764e-6 3.95 | 9.091e-6 3.00 | 1.233e-5 3.00 |                |
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