Parity oscillations and photon correlation functions in the $Z_2/U(1)$ Dicke model at a finite number of atoms or qubits

Yu Yi-Xiang$^{1,2,3}$, Jinwu Ye$^{1,2,4}$ and CunLin Zhang$^2$

$^1$ Department of Physics and Astronomy, Mississippi State University, P. O. Box 5167, Mississippi State, MS, 39762
$^2$ Key Laboratory of Terahertz Optoelectronics, Ministry of Education, Department of Physics, Capital Normal University, Beijing 100048, China
$^3$ School of Instrument Science and Opto-electronics Engineering, Institute of Optics and Electronics, Beijing 100191, China
$^4$ Kavli Institute of Theoretical Physics, University of California, Santa Barbara, Santa Barbara, CA 93106

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In this work, by using the strong coupling expansion and exact diagonalization (ED), we study the $Z_2/U(1)$ Dicke model with independent rotating wave (RW) coupling $g$ and counter-rotating wave (CRW) coupling $g'$ at a finite $N$. This model includes the four standard quantum optics model: Rabi, Dicke, Jaynes-Cummings (JC) and Tavis-Cummings (TC) model as its various special limits. We show that in the super-radiant phase, the system’s energy levels are grouped into doublets with even and odd parity. Any anisotropy $\beta = g/g' \neq 1$ leads to the oscillation of parities in both the ground and excited doublets as the atom-photon coupling strength increases. The oscillations will be pushed to the infinite coupling strength in the isotropic $Z_2$ limit $\beta = 1$. We find nearly perfect agreements between the strong coupling expansion and the ED in the super-radiant regime when $\beta$ is not too small. We also compute the photon correlation functions, squeezing spectrum, number correlation functions which can be measured by various standard optical techniques.

Introduction: There are several well known quantum optics models to study atom-photon interactions$^{1,2}$. In the Rabi model$^{3}$, a single mode photon interacts with a two level atom with equal rotating wave (RW) and counter rotating wave (CRW) strength. When the coupling strength is well below the transition frequency, the CRW term is effectively much smaller than that of RW, so it was dropped in the Jaynes-Cummings (JC) model$^{4}$. Then the Rabi and JC model were extended to an assembly of $N$ two level atoms to the Dicke model$^{5}$ and the Tavis-Cummings (TC) model$^{6}$ respectively. Despite their relative simple forms and many previous theoretical works$^{7–9}$, their solutions at a finite $N$, especially inside the superradiant regime, remain unknown. Here, we address this outstanding problem. It is convenient to classify the four well known quantum optics models from a simple symmetry point of view: the TC and Dicke model as the $U(1)$ and $Z_2$ Dicke model$^{8}$ respectively, while JC and Rabi model are just as the $Z_2$ symmetry. Any anisotropy $\beta = g/g' \neq 1$ leads to the oscillation of parities in both the ground and excited doublet states in superradiant phase as the $g$ increases. In the $Z_2$ limit $\beta = 1$, all the oscillations are pushed to $g = \infty$. We find nearly perfect agreements between the strong coupling expansion and the ED in the super-radiant regime when $\beta$ is not too small. We compute the photon correlation functions, squeezing spectrum and number correlation functions which can be detected by fluorescence spectrum, phase sensitive homodyne detection and Hanbury-Brown-Twiss (HBT) type of experiments respectively$^{1,2,30}$. Experimental realizations are discussed. New perspectives are outlined.

Strong coupling expansion — In the strong coupling limit, it is more convenient to rewrite the $U(1)/Z_2$ Dicke model$^{8}$ in its dual $Z_2/U(1)$ presentation:

$$H_{Z_2/U(1)} = \omega_a a^\dagger a + \omega_b J_z + \frac{g(1+\beta)}{\sqrt{N}} (a^\dagger + a) J_x - \frac{g(1-\beta)}{\sqrt{N}} (a^\dagger - a)iJ_y$$

(1)

where $\omega_a, \omega_b$ are the cavity photon frequency and the energy difference of the two atomic levels respectively, the $g$ and $g' = \beta g, 0 \leq \beta \leq 1$ are the atom-photon rotating wave (RW) and the counter-rotating wave (CRW) coupling respectively. If $\beta = 0$, Eq. 1 reduces to the $U(1)$ Dicke model$^7$ with the $U(1)$ symmetry $a \rightarrow ae^{i\theta}, \sigma^- \rightarrow \sigma^- e^{i\theta}$ leading to the conserved quantity $P = a^\dagger a + J_z$. The CRW $g'$ term breaks the $U(1)$ to the $Z_2$ symmetry $a \rightarrow -a, \sigma^- \rightarrow -\sigma^-$ with the conserved parity operator $\Pi = e^{i\pi(a^\dagger J_x + J_y)}$. If $\beta = 1$, it becomes the $Z_2$ Dicke model$^{12,13,51}$. 
After performing a rotation around the \( J_y \) axis by \( \pi/2 \), one can write \( H = H_0 + V \) where \( H_0 = \omega_a [a^\dagger a + G(a^\dagger + a)J_z] \), \( G = g_\lambda (1 + \lambda) / \omega_\lambda \sqrt{N} \) and the perturbation \( V = -\frac{\omega_b}{2}[J_x + \lambda (J_y - J_x) + J_z (1 - \lambda - \lambda J_x J_y)] \) where \( \lambda = \frac{g_\lambda}{\omega_\lambda} \) is a dimensionless parameter of order 1 when \( 1 - \beta \) is small in the large \( g \) limit. In principle, the strong coupling expansion is performed in the large \( g \) limit \( G \gg 1 \), but with a small \( 1 - \beta \) such that \( \lambda \) is of order 1. In practice, as compared to ED, the method works well also when \( g \) is not too close to \( \omega_a \) or \( \omega_b \).

Define \( A = a + G J_z \), then \( H_0 = \omega_a [A^\dagger A - (G J_z)^2] \). Because \( [A, J_z] = 0 \), we denote the simultaneous eigenstates of \( A \) and \( J_z \) as \( |l \rangle_m |j \rangle \), \( m = -j, \cdots, j, l = 0, 1, \cdots \). The first excited state is \( E_1 g \), the matrix element \( G_j |m \rangle \rangle \) where \( A_m = a + G m \), \( |l \rangle_m \rangle = D_m^l (g_m) |l \rangle \rangle \) where \( D_m^l \) is the largest photon Fock state. The zeroth order eigen-energies are \( H_0 |l \rangle_m |j \rangle \rangle = E_0 |l \rangle_m |j \rangle \rangle \), \( E_0 = \omega_a (l - g_m)^2 \). All the eigenstates can be grouped into even or odd under the parity operator \( \Pi = e^{i \sigma_z (a - a^\dagger)} \): \( |e \rangle \rangle = \frac{1}{\sqrt{2}} \) \(|l \rangle_m |j \rangle, m \rangle \rangle (1 - 1)^j |l - m \rangle |j - m \rangle \rangle \)

The ground state is a doublet at \( |l \rangle = 0 \). In large \( g \) limit, the excited states can be grouped into two sectors: (1) The atomic sector with the eigenstates \( |l \rangle = 0 \), \( \omega_a \). The first excited state \( l = 1 \) with the energy \( \omega_a \) is the remnant of the pseudo-Goldstone mode in the \( U(1) \) regime. (2) The optical sector with the eigenstates \( |l \rangle_m |j \rangle, m \rangle \rangle = j \). The first excited state has the energy \( \omega_0 = E_{0, l = j} - E_{0, l = j - 1} = \omega_a G^2 (2j - 1) = \frac{\omega_a^2 (1 + \beta)^2}{\omega_a^2} (2j - 1) \)

So in the strong coupling limit, there is a wide separation between the atomic sector and the optical sector. This makes the strong coupling expansion very effective to exploit the physical phenomena in the superradiant regime.

**Ground state (\( l = 0 \) splitting)**: The two degenerate ground state are \( |1 \rangle = \{ l = 0 \} |j \rangle, j = 0 \}, \{ 2 \rangle = \{ l = 0 \} |j \rangle, j = 1 \} \) with the zeroth order energy \( E_0 = -\omega_a (G_j)^2 \). By second order perturbation, one finds a new diagonal matrix element \( V_{11} = V_{22} = V_0 (\lambda) = -\frac{\omega_a}{\omega_0} G^2 (1 + \lambda) \) \( < 0 \). However, one needs to perform a \( N = 2j \) order perturbation to find the first- non-zero contribution to the off-diagonal matrix element \( V_{12} = V_{21} = \Delta_2 (\lambda) \):

\[ \Delta_2 (\lambda) = \frac{-N^2 \omega_a}{2} (\frac{\omega_b}{2 \omega_0 G^2})^{N^2} \sum_{l = 0}^{N - 1} (-1)^l (N^2)^{l} (1/2) \]

where \( [l/2] \) is the closest integer to \( l/2 \) and \( \frac{\lambda}{\omega_0} = \frac{\omega_b}{2 \omega_0 G^2} \).

Setting \( \lambda = 0 \) in Eq.3 leads to the splitting in the \( \tilde{Z}_2 \) Dicke model at \( \beta = 1 \) (Fig. S1d):

\[ \Delta_0 (\lambda) = \frac{- \omega_b}{2} (\frac{\omega_b}{2 \omega_0 G^2})^{N^2} \sum_{l = 0}^{N - 1} (-1)^l (N^2)^{l} (1/2) \]

which is always a negative quantity, so leads to the even and odd parity as the ground state and the excited state in the \( l = 0, m = j \) doublet in Eq.2 having the energies \( E_{0/e} = E_0 + V_0 \pm \Delta_0 \) (Fig. S1).

Now we study the dramatic effects of the anisotropy \( \lambda > 0 \) encoded in Eq.3. If removing the exponential factor \( e^{-\lambda N^2} \) where \( G' = NG, \lambda \) is a \( 2N \) th polynomial of \( g \). We find that it always has \( N \) positive zeros in \( g \) beyond the \( g_c \) ( namely, fall into the super-radiant regime ). Higher than the \( N \) th order perturbations will lead to other zeros at larger \( g \) shown in Fig.1). Any changing of \( \Delta_0 (\lambda) \) leads to the exchange of the parity in the ground state \( l = 0, m = j \) in Eq.2 ( namely, Eq.S1 ) with the energies \( E_{0/e} = E_0 + V_0 (\lambda) \) \( \pm \Delta_0 (\lambda) \) in Fig.1). So any \( \lambda > 0 \) will lead to infinite number of level crossings with alternative parities in the ground state, which is indeed observed in the ED results Fig.S1 for the energy levels at \( N = 2 \) and \( \beta = 0.1, 0.5, 0.9 \). It is the anisotropy which leads to the parity oscillations in the superradiant regime. However, at \( \beta = 1 \), the infinite level crossings are pushed to infinity, so no parity oscillations in Fig.S1d anymore.

**Doublet splitting at \( l > 0 \)**: Now, we look at the energy splitting at \( L > 0 \). The diagonal matrix element at \( l = 0 \) can be easily generalized to \( l > 0 \) case: \( V_{11} = V_{22} = V_l (\lambda) = -\frac{\omega}{\omega_0} G^2 (1 + \lambda) \) \( < 0 \). By performing a \( N = 2j \) order perturbation, we also find a general ( but a little bit complicated ) expression for the off-diagonal matrix element \( V_{12} = V_{21} = \Delta_2 (\lambda) \). However, in the \( \lambda \gg 1 \) limit, it can be simplified to:

\[ \Delta_2 (\lambda) \sim \left( \frac{-1}{\lambda} \right)^j G (\lambda) \]

where \( \Delta_2 (\lambda) \) is given in Eq.3. It is enhanced due to the large prefactor \( G^2 \). Note that it this oscillating sign \(-1 \) which leads to the event/odd parity state with a extra \(-1 \) in Eq.2 with \( m = j \) ( namely Eq.S2 ).

The \( \lambda \) th level has the energies \( E_{0/e} = E_0 + V_0 (\lambda) \) \( \pm \Delta_0 (\lambda) \), \( E^0 = \omega_0 l - \lambda G^2 \) \( < 0 \). By performing a \( N = 2j \) order perturbation, we also find a general ( but a little bit complicated ) expression for the off-diagonal matrix element \( V_{12} = V_{21} = \Delta_2 (\lambda) \). However, in the \( \lambda \gg 1 \) limit, it can be simplified to:

\[ \Delta_2 (\lambda) \sim \left( \frac{-1}{\lambda} \right)^j G (\lambda) \]
Comparison with Exact Diagonization (ED) results:—
In Fig. 1 (b)-(d), we compare Eq. 3 and 4 with the ED results on the energy level splitting between the doublets (the Schrödinger Cats states with even and odd parity) for $N = 2$ at $\beta = 0.1, 0.5, 0.9, l = 0, 1, 2$. We find the first $N$ zeros (or parity oscillations) from the strong coupling expansion match those from the ED nearly perfectly well at $\beta = 0.5, 0.9$ in the super-radiant regime. Of course, the ED may not be precise anymore when $g$ gets too close to the upper cutoff introduced in the ED calculation as shown in Fig. 1. In fact, the first $N = 2$ zeros of Eq. 3 can be found exactly as the two positive roots $G_\pm = (\sqrt{1 + \frac{8 + 2\sqrt{2}}{1-\beta^2} + 1})/4$ falling in the superradiant regime. The spacing between the two roots $\Delta (\frac{2\beta}{1-\beta}) = \frac{1}{\sqrt{2}}$ is independent of $\beta$ as shown in Fig. 1(d). As $\beta \to 1^-$, both roots $\sim (1-\beta)^{-1/2}$ are pushed into the infinity.

Eq. 3 is also confirmed by the ED shown in Fig. 1 for $N = 2, \beta = 0.9, l = 0, 1, 2$ where the positions of the first $N = 2$ zeros only depend on $l$ very weakly. So between the two zeros, at $l = 0, 1, 2, \cdots$, the energy levels are in the pattern $(e, o), (e, o), \cdots$ when $\Delta_0 (\lambda) < 0$ shown in Fig. 1(b) (or $(o,e), (o,e), \cdots$ when $\Delta_0 (\lambda) > 0$).

Photon, squeezing and number correlation functions — In order to calculate the photon correlation functions in the strong coupling limit, one not only needs to find the energy levels as done in the previous sections and in Fig. 1, but also the wavefunctions listed in the SM. Using the Lehmann representations, we find there is no first order correction to the normal photon correlation function, but there is one $\sim 1/G^2$ to the anomalous photon correlation function:

$$\langle a(t)a^\dagger (0) \rangle = Ae^{-\Delta_0 |t|} + e^{-\Delta_0 (l) |t|}$$

$$\langle a(t)a(0) \rangle = Ae^{-\Delta_0 |t|} - Be^{-\Delta_0 (l) |t|}$$

where $A = (Gj)^2 = N \frac{(1+\beta)^2}{2-\beta^2} \sim G^2$ is the photon number in the ground state $|37\rangle$ and $B = \frac{x^2}{G^2} \frac{(2\Delta + 1)(\Delta + 1)}{2\Delta - 1} \sim 1/G^2$ and $\Delta_0 = (V_1 - V_0) + \frac{1}{2}(|\Delta_1| + |\Delta_0|)$ shown in

![Figure 1](image-url)
Fig. 1. One can see the anomalous spectral weight $-B \sim (\lambda/G)^2$ is negative and completely due to $\lambda$ (away from the $Z_2$ limit). So the $B$ term in the anomalous photon correlation function can reflect precisely the anisotropy $\beta$ and can be easily detected in phase sensitive Homodyne measurements.

Similarly, we also find the first order correction $\sim 1/G^2$ to the photon number correlation function:

$$\langle n(t)n(0) \rangle - \langle n \rangle^2 = A[1 + B]^2 e^{-\Gamma(\omega_0 + \Delta_0)t}$$

where $\Delta_n = \langle V_1 - V_0 \rangle - \frac{1}{2}(|\Delta| - |\Delta_0|)$ shown in Fig. 1b, and $\langle n \rangle = A$ is the photon number in the ground state which does not receive first-order correction. From Eq. 6 and 7, one can see that the $\Delta_0$ can be directly extracted from the very first frequency in Eq. 6 while $|\Delta| = \Delta_a - \Delta_n$ and $V_1 - V_0 = (\Delta_a + \Delta_n)/2 - |\Delta_0|$. So all the parameters of the cavity systems such as the doublet splittings $\Delta_0(\lambda), \Delta_1(\lambda)$ and energy level shifts $V_1 - V_0$ in Fig. 1c can be extracted from the photon normal and anomalous Green functions Eq. 6 and number photon correlation functions Eq. 7. They can be measured by photoluminescence, phase sensitive homodyne and Hanbury-Brown-Twiss (HBT) type of experiments respectively.

Experimental realizations: There have been extensive efforts to realize the Schrodinger Cat state in trapped ions and superconducting qubit systems. Here, the Schrodinger Cats in Eq. 2 with $l = 0, 1, 2$... and $m = j$ can be prepared in the superradiant regime, its size can be continuously tuned from $N \sim 3 - 9$, it involves all the $N$ number of atoms (qubits) and photons strongly coupled inside the cavity and could have important applications in quantum information processes.

In order to observe the parity oscillation effects, one has to move away from the $Z_2$ limit realized in the experiments, namely, $0 < \beta < 1$. This has been realized in the recent experiment with cold atoms inside an optical cavity which can tune $\beta$ from 0 to 1. It should be straightforward to reduce the number of atoms to a few. In circuit QED systems, there are various experimental set-ups such as charge, flux, phase qubits or qutrits, the couplings could be capacitive or inductive through $\lambda, V, \Xi$ or the $\Delta$ shape. Especially, continuously changing $0 < \beta < 1$ has been achieved in the recent experiment. An shown in Fig. 1, the repulsive qubit-qubit interaction also reduces the critical coupling $g_c$.

From Fig. 1-1, at $N = 2, \beta = 0.1, l = 0$, one can estimate the maximum splitting between the first two zeros $\Delta_0 \sim 0.1\omega_0$ which is easily experimentally measurable. $\Delta_0$ increases as $l = 1, 2$ as shown in Fig. 1-2,3. At $\beta = 0.5$ in Fig. 1c1, $\Delta_0$ decreases to $\sim 0.01\omega_0$ which is still easily measurable. At $\beta = 0.9$ in Fig. 1c1, $\Delta_0$ decreases to $\sim 10^{-11}\omega_0$ which may become difficult to measure. However, in view of recent advances in the precision measurements in the detection of the elusive gravitational waves, it is also possible to measure such a tiny splitting by the phase sensitive homodyne detection. So the parity oscillations can be easily experimentally measured when $\beta$ is not too close to the $Z_2$ limit.

Conclusions and discussions: The four standard quantum optics models at a finite $N$ were proposed by the old generation of great physicists many decades ago. Their importance in quantum and non-linear optics ranks the same as the bosonic or fermionic Hubbard models and Heisenberg models in strongly correlated electron systems and the Ising models in Statistical mechanics. Despite their relative simple forms and many previous theoretical works, their solutions at a finite $N$, especially inside the superradiant regime, remain unknown. In this work, we addressed this outstanding historical problem by using the strong coupling expansion and ED. We are able to analytically calculate various photon correlation functions in the superradiant regime remarkably accurate except when $\beta$ is too small where the (non)-degenerate perturbations near the $U(1)$ limit ($\beta = 0$) works well. The present work may inspire several new directions. From the wavefunctions listed in SM, it would be interesting to evaluate the effects of the parity oscillations on the atom-photon entanglements in the Schrodinger cats at a given $l = 0, 1, 2$... It is important to incorporate the effects of the external pumping and cavity photon decays to study the de-coherences of the Schrodinger cats in the non-equilibrium $U(1)/Z_2$ Dicke model. It would be tempting to study the arrays of cavities leading to the $Z_2/U(1)$ Dicke lattice models with general $0 < \beta < 1$.

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There is a "formal" exact solution on the dynamic photon correlation functions Eq.6,7. There is a formal" exact solution even at the simplest case in the present paper in the super-radiant regime from the QCP [13] and also any new phenomenon achieved in the first line in Eq.6, setting $a_0=0$ and using $[a,a^\dagger]=1$, one can also see the photon number in the ground state $\langle n \rangle = A$.

It is tempting to try to sort out if there is an extra symmetry at the $Z_2$ limit $\beta = 1$. Taking the Eq.1 one can see the last two terms exchange under the transformation $a \rightarrow e^{i\pi/2}a, \sigma_- \rightarrow e^{i\pi/2}\sigma_-$. It is also easy to see the Hamiltonian has the symmetry $T \times R(y,\pi)$ or $\mathcal{T} \times R(x,\pi) \times Z_2$ where $\mathcal{T}$ is the Time reversal symmetry, $R(y,\pi),R(x,\pi)$ are the spin rotations by $\pi$ around $y$ or $x$ axis respectively and $Z_2$ is $a \rightarrow -a$. So there is no extra symmetry at the $Z_2$ limit.

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