I. INTRODUCTION

Symmetries and their breaking phenomena are essential concepts in modern physics. Especially after the theoretical prediction [1] and experimental discovery [2] of the Higgs particle, research on symmetry breaking has become one of the main study areas of particle physics. The idea of symmetry breaking has been vigorously studied also in cosmology [3–9]. However, there have not been enough attempts to apply the concept to the geometrical symmetry of spacetime itself in the cosmological context. For example, [10] studies how the diffeomorphism may be broken explicitly or spontaneously in some modified gravity models like Chern-Simons gravity and bimetric gravity and investigate their post-Newtonian limits, but did not consider the cosmological situation. The diffeomorphism invariance in $f(R)$ gravity was studied in [11], but the idea of symmetry breaking was not applied. Hence, we are motivated to study how symmetry breaking may affect modified cosmological models, especially in the primordial era.

Meanwhile, a new class of theories satisfying a new geometrical symmetry known as Weyl geometry has attracted the interest of many cosmologists. It originated from the philosophy that the physical theory should allow and include the gauge symmetry as large as possible. [12] [13]. In this context, Weyl geometry can be viewed as an extension of Einstein’s general relativity (GR) with just a bigger gauge group [14]. On the other hand, it can be derived from the Brans-Dicke theory when one applies the Palatini formalism, which claims that the metric and the connection are independent of each other [15]. Since the theory has many connections with other research themes of cosmology, many studies have applied it to many problems of cosmology, such as inflationary cosmology [16] [20].

Recently, We attempted to apply the idea of symmetry breaking in Weyl geometry. In [21], we have shown the possibility of distinguishing two types of primordial symmetry breaking, model A and B, with an observation of cosmic microwave background (CMB) anisotropy. Model A originates from Zee’s broken-symmetric theory of gravity [22], and model B is an application of Palatini formalism to model A. Significantly, the symmetry breaking of Weyl geometry, appeared in model B, has a particular interest because it has a new salient geometrical feature that the standard explanation for symmetry breaking cannot have.

In this paper, we expand this previous study on geometrical symmetry breaking in Weyl geometry to the case where the Einstein-Hilbert action is replaced with an arbitrary scalar curvature function, i.e., $f(R)$ gravity. Currently, many $f(R)$ gravity models are vigorously studied in cosmology and quantum gravity, including attempts to find an appropriate inflation model and search for a possible resolution for the dark energy problem [23]. In this context, it would be of some worth to study how the geometrical symmetry breaking has a distinctive effect on these subjects.

Furthermore, we explore the possibility of resolving or alleviating Hubble tension, one of the most puzzling problems in current cosmology by studying Integrated-Sachs-Wolfe (ISW) effect of CMB anisotropy TT spectra. As many observations have been conducted, many cosmologists believed the value of Hubble factor from different observations would coincide with one value. However, the discrepancy between results from CMB anisotropy data and results from others, such as observation of supernovae, has been increasing. Hence to this day, there has been much research to find a remedy to this problem. They include the modified gravity models we have mentioned [24]. In this article, we tackle this issue with the idea of primordial symmetry breaking in $f(R)$ gravity.

This paper consists as follows: In Section II, we construct a class of $f(R)$ theories satisfying the geometrical gauge symmetry of Weyl geometry. Section
III shows that the constructed model is equivalent to the Einstein gravity with one additional scalar field, which is non-minimally coupled with the new scalar field and the Weyl field only at the perturbational scale. In Section IV, as an important example, we study how the evolution in the first-order perturbation theory in Starobinsky gravity differs from the original without primordial Weyl symmetry breaking. As one interesting feature of the model, we found that the existence of primordial gravitational waves (GW) may also affect observational outcomes of the scalar perturbation, not only of the tensor perturbation. In addition, we compare the results with our previous study of the Weyl symmetry breaking in [21]. Section V investigates how the results from section IV are reflected in the CMB anisotropy impacts is studied in detail. In section VI, we summarize our study and discuss its advantages and drawbacks. We also suggest a few possible extensions of our research.

II. \( f(R) \) GRAVITY FROM WEYL GEOMETRY

Here we show how to construct \( f(R) \) gravity theory satisfying Weyl gauge symmetry. In the standard formalism of GR, a covariant derivative \( \nabla \) is taken to be dependent on the metric tensor \( g_{\mu\nu} \). This dependency is called the metric compatibility and is given by the following relationship:

\[
\nabla_\alpha g_{\mu\nu} = 0. \tag{1}
\]

From (1), we can obtain an expression of the connection \( \Gamma \) only using the metric tensor as follows:

\[
\Gamma^\alpha_{\mu\nu} = \frac{1}{2} \theta^{\alpha\gamma}(g_{\gamma\mu,\nu} + g_{\gamma\nu,\mu} - g_{\mu\nu,\gamma}), \tag{2}
\]

which is known as the Levi-Civita connection. However, one may allow more degrees of freedom to obtain a broader class of theories. One of them we adopt here is the Weyl integrable geometry, which introduces a new scalar field \( \varphi \) to define a new covariant derivative \( \nabla \) (or a new connection \( \bar{\Gamma} \)):

\[
\nabla_\alpha g_{\mu\nu} = g_{\mu\nu} \frac{\partial \varphi}{\partial x^\alpha} \bigg|_{M_p}, \tag{3}
\]

where \( M_p \) is a Planck mass. One intriguing property of (3) is that it manifests a new kind of symmetry. That is, (3) is invariant under the following transformation:

\[
g_{\mu\nu}, \varphi \to (\Omega^2 g_{\mu\nu}, \frac{\varphi}{M_p} + 2 \ln \Omega), \tag{4}
\]

where \( \Omega \) is a gauge fixing parameter. Therefore, we can understand (3) as a scale symmetry associated with the scalar field \( \varphi \). This symmetry is often called the Weyl gauge symmetry. We call the scalar field \( \varphi \) giving Weyl gauge symmetry to spacetime a Weyl scalar field.

As the theory is invariant, one may choose a specific frame for one’s purpose. The frame we have used until now \((g_{\mu\nu}, \varphi)\) is called a Weyl frame. However, it is helpful to choose another frame when we derive equations of motion. This new frame \((\gamma_{\mu\nu}, 0)\), where \( \gamma_{\mu\nu} \equiv e^{-\varphi/M_p} g_{\mu\nu}, \) called a Riemannian frame. One crucial property of the Riemannian frame is that the connection \( \bar{\Gamma} \) behaves like \( \Gamma \), i.e., the standard Levi-Civita connection. To be explicit, we rewrite so that the metric compatibility holds with the following modified metric:

\[
\nabla_\alpha \gamma_{\mu\nu} = 0, \tag{5}
\]

or in the language of the connection,

\[
\bar{\Gamma}^\alpha_{\mu\nu} = \frac{1}{2} \gamma^{\alpha\gamma} (\gamma_{\mu\nu,\gamma} + \gamma_{\nu\gamma,\mu} - \gamma_{\mu\nu,\gamma}), \tag{6}
\]

This property provides some convenience when we conduct a computation under a theory satisfying (4) since we can treat the derivative with the rescaled metric \( \gamma_{\mu\nu} \) just as we do in GR. Therefore, with an analogy with the standard \( f(R) \) gravity, We write an action for the gravity side in the Riemannian frame:

\[
S_\gamma = \frac{M_p^2}{2} \int dx^4 \sqrt{-\gamma} f(\bar{\Gamma}), \tag{7}
\]

where we have expressed \( f \) with \( \mathcal{R} \), the scalar curvature for \( \bar{\Gamma} \). Including an action for the Weyl scalar field \( \varphi \), we now write the entire action as follows:

\[
S = \frac{M_p^2}{2} \int dx^4 \sqrt{-\gamma} \left[ f(\bar{\Gamma}) - \frac{1}{2} \gamma^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - e^{2\varphi} V(\varphi) \right], \tag{8}
\]

where \( V(\varphi) \) is a potential for \( \varphi \). We call this class of theories Weyl \( f(R) \) gravity.

Taking a variation (8) with the rescaled metric and the Weyl scalar field, we obtain equations of motion for the metric and the Weyl scalar field:

\[
F(\bar{\Gamma}) \mathcal{R}_{\mu\nu} - \frac{1}{2} f(\bar{\Gamma}) \gamma_{\mu\nu} - \nabla_\mu \nabla_\nu F(\bar{\Gamma}) + \gamma_{\mu\nu} \Box F(\bar{\Gamma}) = \kappa T_{\mu\nu} + T^{(\varphi)}_{\mu\nu}, \tag{9}
\]

\[
\nabla_\mu (\gamma^{\mu\nu} \nabla_\nu \varphi) = 2 e^{2\varphi} (V + \frac{1}{2} \frac{dV}{d\varphi}), \tag{10}
\]

where \( F = \partial f/\partial \mathcal{R} \), \( T_{\mu\nu} \) is an energy-momentum (EM) tensor for ordinary matter including the cosmological constant and in Riemannian frame and \( T^{(\varphi)}_{\mu\nu} \) is defined by

\[
T^{(\varphi)}_{\mu\nu} = \frac{1}{2} \left[ \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{2} \gamma_{\mu\nu} \gamma^{\alpha\beta} (\nabla_\alpha \varphi \nabla_\beta \varphi + e^{2\varphi} V(\varphi)) \right]. \tag{11}
\]
III. THE EQUIVALENCE WITH THE NON-MINIMALLY COUPLED MODEL AFTER SYMMETRY BREAKING

We adopt the following Higgs-type potential to invoke primordial symmetry breaking at the Planck scale:

\[ V(\varphi) = V_0(\varphi^2 - M_p^2)^2. \]  

(12)

After the symmetry breaking, we expand \( \varphi = M_p(1 + \delta \varphi) \) where \( \delta \varphi \) is a perturbation of \( \varphi \). In the same manner, \( \mathcal{R} \) can be expanded up to the first order of \( \delta \varphi \) as follows:

\[ \mathcal{R} = R - 3 \Box \delta \varphi. \]  

(13)

Note that here we regard \( \gamma_{\mu \nu} \) as fixed and not perturbed by the expansion of \( \varphi \) so as to exploit the metric compatibility (5) in the Riemannian frame. Moreover, after the symmetry breaking the equation of motion for \( \varphi \) becomes

\[ \Box \delta \varphi \approx m^2_\varphi \delta \varphi, \]  

(14)

where \( m^2_\varphi \equiv 4V_0M_p^2 \). Here one should note that this is not an exact form of the equation of motion for \( \delta \varphi \), as we did not consider the effect of the symmetry breaking in the gravity action yet. However, we will use this equation as an approximation when we transform the action as a standard gravity with a non-minimally coupled field at the non-perturbative scale. This assumption is, of course, not the completely exact way to compute the action. However, we will see that this has some benefits as we may reduce the high-order derivative part of the action.

Our goal in this section is to find an equivalent form of (8) whose action is Einstein-Hilbert with a non-minimally coupled scalar field at the non-perturbative scale. To find such a theory, let us first review how one can find an equivalent theory with scalar field in general \( f(R) \) theories without symmetry breaking, following the methodology of [23]. We claim that the following action is equivalent to the action of \( f(R) \) theory, that is, \( S_g = \int d^4x \sqrt{-g} (M_p^2/2)f(R) \):

\[ S_g = \int d^4x \sqrt{-g} \frac{M_p^2}{2} [f(\chi) + f'(\chi)(R - \chi)], \]  

(15)

where we have adopted an auxiliary field \( \chi \). Varying (15) with respect to the new field \( \chi \), one finds that \( f''(\chi)(R - \chi) = 0 \) so that \( \chi = R \) and our claim is proved provided that \( f''(\chi) \neq 0 \). If we define a new field \( \zeta \equiv f'(\chi) \), one may rewrite (15) as

\[ S_g = \int d^4x \sqrt{-g} \frac{M_p^2}{2} \zeta R - U(\zeta), \]  

(16)

where the potential \( U(\zeta) \) is given by

\[ U(\zeta) \equiv \frac{M_p^2}{2} [\chi(\zeta) - f(\chi(\zeta))]. \]  

(17)

Now the action (16) can be expressed as a non-minimally coupled one when adopting an appropriate metric rescaling. However, when one tries to find an equivalent action in the presence of Weyl symmetry breaking, one needs to pay more attention because the newly defined field \( \zeta \) is clearly associated with the perturbative terms with respect to \( \delta \varphi \). Here we want to restrict that the non-minimally coupled scalar field only comes from the non-perturbative part of \( f(R) \) so that we can see new effects from the symmetry breaking more clearly. So it would be of some comfort to abstract the perturbative part from the original action. To this end, we expand the gravity part of the action (7) up to the first order of \( \delta \varphi \) to obtain

\[ S_g = \int d^4x \sqrt{-\gamma} \frac{M_p^2}{2} f(\gamma_{\mu \nu} R_{\mu \nu} - 3 \Box \delta \varphi) \]

\[ = \int d^4x \sqrt{-\gamma} \frac{M_p^2}{2} f(\gamma_{\mu \nu} R_{\mu \nu}) \]

\[ - \int d^4x \sqrt{-\gamma} \frac{M_p^2}{2} [3f'(\gamma_{\mu \nu} R_{\mu \nu}) m^2_\varphi \delta \varphi], \]  

(18)

Here we used the relation (14) to reduce the high derivative order of the field \( \delta \varphi \). The readers should be aware that we have expanded the action only up to the first order of the perturbed Weyl field. As a matter of fact, to be consistent with the procedure when we compute a perturbation of the action, we would have to consider up to the second order of \( \delta \varphi \). However, since our current goal is merely to consider the effect of the primordial symmetry breaking in the \( f(R) \) theory, we expect that only considering up to the first order in the function \( f \) would be enough for our current purpose. Note that we suppress the Weyl field part of the action and only consider \( S_g \) in this section for consistency and convenience.

From this expression, we introduce a new scalar field that only comes from the non-perturbative part of \( f(R) \):

\[ \sigma \equiv f'(\gamma_{\mu \nu} R_{\mu \nu}). \]  

(19)

Proceeding similar steps as we saw in (15)-(17), we express (18) as

\[ S_g = \int d^4x \sqrt{-\gamma} \left[ \frac{M_p^2}{2} (\sigma + \psi) R - U(\sigma, \psi) \right], \]  

(20)

where we have defined

\[ \psi \equiv -3m^2_\varphi f''(\gamma_{\mu \nu} R_{\mu \nu}) \delta \varphi, \]  

(21)

and the potential is given by

\[ U(\sigma, \psi) = \frac{M_p^2}{2} [ (\sigma + \psi) R - f(R) ]. \]  

(22)

To clarify the effect of the Weyl symmetry breaking on the potential, we split the potential as \( U(\sigma, \psi) = U_\sigma(\sigma) + \delta U(\sigma, \psi) \), where \( \delta U \) is valid only at the perturbative scale.
Expanding (22) with $\delta \varphi$ to the first order at most, we obtain the following:

$$U_\sigma(\sigma) = \frac{M_P^2}{2} [\sigma \gamma^{\mu
u} R_{\mu\nu} - f(\gamma^{\mu
u} R_{\mu\nu})]$$

$$= \frac{M_P^2}{2} [\sigma (f')^{-1}(\sigma) - f((f')^{-1}(\sigma))]$$

(23)

$$\delta U(\sigma, \psi) = \frac{M_P^2}{2} \psi \gamma^{\mu\nu} R_{\mu\nu}$$

$$= \frac{M_P^2}{2} \psi (f')^{-1}(\sigma).$$

(24)

Now we may perform a conformal transformation to find an action that is non-minimally coupled with the scalar field $\sigma$ at the non-perturbative scale. First, let us define the rescaled metric

$$\hat{\gamma}_{\mu\nu} = \sigma \gamma_{\mu\nu}. \quad (25)$$

Note that this transformation does not correspond to the gauge transformation (4). It is merely a technic to obtain the non-minimally coupled action. Rewriting our action (20) with respect to the newly defined metric (25), we finally arrive at the following result:

$$S_\gamma = \int d^4 x \sqrt{-\gamma} \left[ \frac{M_P^2}{2} (1 + \psi e^{-\sqrt{2/3} \phi/M_P}) \bar{R} - \frac{1}{2} \hat{\gamma}^{\mu\nu} \nabla_{\mu} \phi \nabla_{\nu} \phi - V(\phi) - \delta V(\phi, \psi) \right],$$

(26)

where the terms with hat denote that it is with respect to the rescaled metric $\hat{\gamma}_{\mu\nu}, \phi/M_P \equiv \sqrt{3/2} \ln \sigma$, and

$$V(\phi) \equiv e^{-2\sqrt{2/3} \phi/M_P} U_\sigma(e^{\sqrt{2/3} \phi/M_P})$$

$$\delta V(\phi, \psi) \equiv e^{-2\sqrt{2/3} \phi/M_P} \delta U(e^{\sqrt{2/3} \phi/M_P}, \psi).$$

(27)

(28)

Hence, the effect of Weyl symmetry breaking is manifested as an additional non-minimal coupling only valid at the perturbative scale, with an additional correction to the potential.

**IV. STAROBSKINSY INFLATION WITH A BROKEN WEYL SYMMETRY**

To clarify how our modification differs from the original $f(R)$ gravity, we study an example of $f(R)$ gravity known as Starobinsky gravity [25], whose function $f$ is given by

$$f(R) = R + \frac{R^2}{6M^2},$$

(29)

where $M$ is a mass scale constant. The Weyl gauge symmetry becomes broken at the Planck energy scale due to the Higgs-type potential. After the symmetry breaking, the transformation formula (4) is only valid at the perturbative scale, and the Weyl scalar field becomes a perturbative variable. At non-perturbative scales, the potential $V(\phi)$ is known as Starobinsky potential:

$$V(\phi) = \frac{3}{4} M_P^2 (1 - e^{-\sqrt{2/3} \phi/M_P})^2.$$ \hspace{1cm} (30)

For the Starobinsky model, we find that

$$\psi = -\left( \frac{m_\sigma}{M} \right)^2 \delta \varphi,$$

(31)

$$\delta V(\phi, \delta \varphi) \equiv \frac{3}{2} M_P^2 e^{-2\sqrt{2/3} \phi/M_P} (1 - e^{\sqrt{2/3} \phi/M_P}) \delta \varphi.$$ \hspace{1cm} (32)

From now on, let us restrict our interest to the first-order perturbation theory. Varying the action (26) with $\hat{\gamma}_{\mu\nu}$ and $\phi$, we obtain equations of motion:

$$\{1 + \psi e^{-\sqrt{2/3} \phi/M_P}\} \hat{G}_{\mu\nu} = \kappa T_{\mu\nu} + T^{\text{int}}_{\mu\nu}, \quad (33)$$

$$\bar{\Box} \phi = \partial_\mu [V(\phi) + \delta V(\phi, \psi)], \quad (34)$$

where $\kappa \equiv 2/M_P^2$ and

$$T_{\mu\nu}^{(\phi)} \equiv \nabla_\mu \phi \nabla_\nu \phi - \hat{\gamma}_{\mu\nu} \hat{\gamma}^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi + V(\phi),$$

(35)

$$T^{\text{int}}_{\mu\nu} \equiv -\left( \frac{m_\sigma}{M} \right)^2 (\nabla_\mu \nabla_\nu - \hat{\gamma}_{\mu\nu} \bar{\Box}) (e^{-\sqrt{2/3} \phi/M_P} \delta \varphi) - \hat{\gamma}_{\mu\nu} \delta V(\phi, \psi).$$ \hspace{1cm} (36)

where $T^{\text{int}}_{\mu\nu}$ denotes the interaction between $\delta \varphi$ and $\phi$. One might think that the additional terms may affect the perturbation of the inflaton field $\phi$. However, we can fix $\delta \varphi = \delta \phi = 0$ if we adopt the following gauge:

$$g_{\mu\nu} = -dt^2 + a(t)^2 dx^2 [(1 - 2A) \delta_{ij} + h_{ij}],$$

(37)

where $\partial_i h_{ij} = h_i^i = 0$, and hence one can prove that there is no effect from Weyl gauge symmetry breaking on quantum fluctuations during inflation.

After the inflation, the gravity sector of the action becomes

$$S_\gamma = \int d^4 x \sqrt{-\gamma} \left[ \frac{M_P^2}{2} \{1 - (\frac{m_\sigma}{M})^2 \delta \varphi \} \bar{R} \right].$$ \hspace{1cm} (38)

Here one must note that we may use the original metric before the conformal transformation at the first order scale since $\sigma = e^{\sqrt{2/3} \phi/M_P} \approx 1$, so $\hat{\gamma}_{\mu\nu} \approx \gamma_{\mu\nu}$. From action (38), we can see that the non-minimal coupling of the Weyl field $\delta \varphi$ and the scalar curvature still survives. Furthermore, this coupling gives us one of the crucial imprints of the inflation theory when one recalls that the mass $M$ inside the coupling constant is related to the tensor power spectrum $P_T$ by the result of the Starobinsky inflation [23]:

$$P_T \approx \frac{4}{\pi} \left( \frac{M_P}{M} \right)^2.$$ \hspace{1cm} (39)
Hence, it is evident that the existence of primordial gravitational waves affects the results, including scalar perturbation. This feature clearly distinguishes the effect of Weyl symmetry breaking in Starobinsky gravity from our previous result \[21\], wherein the results cannot be associated with the inflation theory.

Now let us bring the Weyl field action again and consider it together with (38) to observe how this affects the cosmological observables.

\[
S = \int d^4x \sqrt{-\gamma} \left\{ \frac{M^2}{2} \left[ 1 - \left( \frac{m_{\varphi}}{M} \right)^2 \right] \delta \varphi \right\} \mathcal{R} + \frac{1}{2} \left( \frac{m_{\varphi}}{M} \delta \varphi \nabla_{\mu} \delta \varphi \nabla_{\nu} \delta \varphi + \frac{1}{2} m_{\varphi}^2 \delta \varphi^2 \right) + S_M, \quad (40)
\]

where \( S_M \) is the action for ordinary matters, including matters coming from the fluctuation during inflation. Up to the first order of \( \delta \varphi \), we obtain the equation of motion for the metric:

\[
[1 - \left( \frac{m_{\varphi}}{M} \right)^2 \delta \varphi] G_{\mu \nu} = \kappa T_{\mu \nu} - \left( \frac{m_{\varphi}}{M} \right)^2 \left( \nabla_{\mu} \nabla_{\nu} - \mathcal{R}_{\mu \nu} \right) \delta \varphi, \quad (41)
\]

where \( G_{\mu \nu} \equiv \mathcal{R}_{\mu \nu} - \mathcal{R} \gamma_{\mu \nu}/2 \) is the Einstein tensor in the Riemannian frame and \( T_{\mu \nu}^{(M)} \) is the energy-momentum tensor of ordinary matters. Meanwhile, one must be careful when one tries to derive an equation of motion for \( \delta \varphi \), since it is only valid at the perturbative level. That is, the gravity action coupled with \( \delta \varphi \) is not the whole part of the scalar curvature but only its first-order perturbative part, namely \( \delta R \). Hence, the new equation of motion for \( \delta \varphi \) is given as follows:

\[
\Box \delta \varphi = m_{\varphi}^2 \left[ \delta \varphi + M^{-2} \delta \tau \right]. \quad (42)
\]

Now, let us find the complete form of the Einstein equation at the perturbative scale. We expand the Einstein tensor up to the first order of \( \delta \varphi \):

\[
G_{\mu \nu} = G_{\mu \nu} - \left( \nabla_{\mu} \nabla_{\nu} \delta \varphi - \mathcal{R}_{\mu \nu} \Box \delta \varphi \right), \quad (43)
\]

where \( G_{\mu \nu} \) is the standard Einstein tensor with respect to the standard Levi-Civita connection. Furthermore, since we only consider the first-order theory, we find \( \nabla_{\alpha} \delta \varphi = \nabla_{\alpha} \delta \varphi \) and \( \Box \delta \varphi = \Box \delta \varphi \). Finally, we obtain the following results:

\[
\delta G_{\mu \nu} = \kappa \delta T_{\mu \nu}^{(M)} + \kappa \left( \frac{m_{\varphi}}{M} \right)^2 \left\langle T_{\mu \nu}^{(M)} \right\rangle \delta \varphi + \left[ 1 - \left( \frac{m_{\varphi}}{M} \right)^2 \right] \left( \nabla_{\mu} \nabla_{\nu} - \mathcal{R}_{\mu \nu} \right) \delta \varphi, \quad (44)
\]

where \( \left\langle T_{\mu \nu}^{(M)} \right\rangle \) is an average value of the energy-momentum tensor. Taking a Trace of (44) to find \( \delta R \) and substituting it into (42), we have

\[
\Box \delta \varphi = \left[ 1 + 3 \left( \frac{m_{\varphi}}{M} \right)^2 \right] \left[ 1 - \left( \frac{m_{\varphi}}{M} \right)^2 \right]^{-1} \times \left[ m_{\varphi}^2 - \left( \frac{m_{\varphi}}{M} \right)^2 \kappa T \right] \delta \varphi - \kappa \left( \frac{m_{\varphi}}{M} \right)^2 \delta T, \quad (45)
\]

where \( \delta T \) is a trace of \( T_{\mu \nu}^{(M)} \) and \( T \) is a trace of \( \left\langle T_{\mu \nu}^{(M)} \right\rangle \).

For comparison with the Weyl symmetry breaking in the case \( f(\mathcal{R}) = \mathcal{R} \), we recall the result of [21]:

\[
\delta G_{\mu \nu} = \kappa \delta T_{\mu \nu}^{(M)} + \kappa \left( \frac{\gamma_{\mu \nu}}{M} \right) \delta \varphi, \quad (46)
\]

\[
\Box \delta \varphi = m_{\varphi}^2 \delta \varphi. \quad (47)
\]

These equations correspond to model B in our previous paper, wherein we adopted the initial value of \( \delta \varphi \) as \( \delta \varphi_{\text{ini}} = \sqrt{2} A \), and \( \langle \delta \varphi \rangle_{\text{ini}} = 0 \), where \( A \) is the initial amplitude of the scalar fluctuation. With the limit \( m_{\varphi} \rightarrow \infty \), (46) and (47) approach to the results of GR. However, we expect that with such initial values, we cannot find any reasonable limit by which we can obtain the results of GR because of the term \( \left( m_{\varphi}/M \right)^2 \). Hence, for the initial values of the current theory (44) and (45), we adopt the following:

\[
\delta \varphi_{\text{ini}} = 0, \quad (48)
\]

\[
\langle \delta \varphi \rangle_{\text{ini}} = 0. \quad (49)
\]

Note that, in our previous study of model B, all the observational results would be equivalent to GR with this choice of initial values, as there can be no evolution of the Weyl field. However, since we now have an additional source term \( \delta R \) in the equation of motion for the Weyl field, we expect that there must be some evolution of the Weyl field even with this simplest initial value choice. In addition, one can clearly see that one can obtain the results of GR in the limit \( m_{\varphi} \rightarrow 0 \), contrary to our previous study.

V. CMB OBSERVABLES AND THEIR SIGNIFICANCE

In this section, we establish how one can observationally verify the model discussed with CMB anisotropy observables. We mainly focus on how Weyl symmetry breaking in Starobinsky gravity can affect the ISW effect on CMB anisotropy. For numerical computations, we have used the code CAMB, which computes perturbational quantities in cosmological background and CMB multipoles at high accuracy [26]. For numerical values in the code, we have used Planck 2018 best fit values [27]: \( H_0 = 67.32 \), \( \Omega_b h^2 = 0.022383 \), \( \Omega_c h^2 = 0.12011 \), \( \tau = 0.0543 \), \( \ln(10^{10} A_s) = 3.0448 \), \( n_s = 0.96605 \), and assumed the standard ΛCDM model with a cosmological constant and a flat spatial curvature, i.e., \( K = 0 \).

First, let us study how the perturbed Weyl field \( \delta \varphi \) evolves through time. Figure 1 shows how it varies with the values of \( m_{\varphi} \). It is clear from the evolution equation (45) that the field have more oscillation when the mass of the field gets heavier. In addition, as \( m_{\varphi} \)}
FIG. 1. Evolution of the perturbed Weyl field $\delta \varphi$ through time for $k = 0.3$ with varying $m_{\varphi}$. Time is in unit of Megaparsec and we fixed $r = 0.1$. We also plot the difference of the field values $\Delta \delta \varphi$. As the mass becomes heavier, there are more odds that the value of the field become larger.

FIG. 2. Evolution of the perturbed Weyl field $\delta \varphi$ through time for $k = 0.3$. Time is in unit of Megaparsec and we fixed $m_{\varphi} = 10^{-2} M_{\text{P}}$. We see that there is no meaningful numerical change with varying $r$ in a reasonable range. As the Weyl field mass becomes heavier, $\delta \varphi$ also tends to be more oscillative, or more importantly, gets more positive values. We show this tendency at the bottom of Figure 1 with the field difference $\Delta \delta \varphi$.

FIG. 3. CMB temperature power spectra $\ell (\ell + 1) C_{\ell}/2\pi$ in unit of $\mu K^2$ and its lensing effect, with varying value of the Weyl field mass $m_{\varphi}$. For some small values of $m_{\varphi}$, the power spectra decrease overall. However, at some large enough value for the mass, the power spectra at low-$\ell$ region increase dramatically. The spectra at high-$\ell$ region ($\ell \gtrsim 10^2$) become smaller as the mass increases. The effect on the lensing shows a similar behavior as the total power spectra. In general, the lensing effect diminishes as the field mass has large enough value.

Next, we plot the CMB TT spectrum by varying the Weyl field mass to understand how the change manifests in CMB observables. From now on, we will fix $r = 0.01$. As Figure 3 shows, the effect of the symmetry breaking is most dominant in the low-$\ell$ region of the multipoles. As the Weyl field mass $m_{\varphi}$ increases, the values at the low-$\ell$ region increase dramatically, whereas the values at the high-$\ell$ region and the lensing effect, defined by

$$\Delta C_{\ell}^{\text{TT}} \equiv \frac{C_{\ell}^{\text{TT+lensing}} - C_{\ell}^{\text{TT+original}}}{C_{\ell}^{\text{TT+lensing}}}$$

overall decrease, although, for a small enough value for $m_{\varphi}$, the effects differ, showing smaller values also at the low-$\ell$ region. For the detailed analysis, we investigate how the perturbational values and multipoles evolved in the early era of the universe.

To this end, we adopt the conformal Newtonian gauge:

$$\gamma_{\mu\nu} = -a^2(\eta)[{1+2\Psi(\eta, \mathbf{x})}] d\eta^2 + [1 - 2\Phi(\eta, \mathbf{x})] \delta_{ij} dx^i dx^j,$$

where $a$ is the scale factor of the universe, and $d\eta \equiv dt/a$ is conformal time. Also, we make a normal mode
decomposition. First, let us denote $Q(k, x)$ as an eigenmode of the Helmholtz equation
\[ \nabla^2 Q(k, x) = k^2 Q(k, x), \]
where $\nabla^2 \equiv \delta_{ij} \nabla^i \nabla^j$ is a Laplacian and $k^2 \equiv |k|^2$. With this function, we may decompose an arbitrary quantity $X$ as follows:
\[ X(\eta, x) = \int \frac{d^3k}{(2\pi)^3} X_k(\eta, k)Q(k, x), \]
where the subscript $k$ denotes that it is a decomposed form. However, from now on, we assume all the quantities are decomposed and suppress the subscript unless we need it.

From the Einstein equation (44), the anisotropic stress part is given as follows:
\[ k^2(\Phi - \Psi) = 12\pi G(\bar{\rho} + \bar{p})\pi_k + [1 - \left(\frac{m^2}{M}\right)^2]k^2\delta\phi, \]
where $\bar{\rho}$ and $\bar{p}$ are the average energy density and pressure and $(\rho + p)\pi_k \equiv -(k^i k^j - \delta^{ij}/3)\Pi_{ij}$ where $k_i$ is the unit vector of $k$ and $\Pi_{ij}$ is the anisotropic part of the energy-momentum tensor. To connect metric fluctuations with the CMB temperature anisotropy, let us expand the photon energy density perturbation $\delta\phi \equiv -\bar{\rho}/\rho_\gamma$ with Legendre polynomials, where $\rho_\gamma$ and $\bar{\rho}_\gamma$ are the photon energy density and its average.

\[ \Theta_t = (-i)^{-l} \int_{-1}^{1} \frac{d\mu}{2} P_l(\mu)\delta\phi(\mu), \]
where $\mu = k \cdot \hat{z}$ is a projection of the vector $k$ to the $z$-axis. The CMB TT power spectra $C_l$ is given by
\[ C_l = 4\pi \int \frac{dk}{k} |\Theta_l(\eta_0, k)|^2 \Delta_R(k), \]
where $\eta_0$ is a current conformal time, and $\Delta_R(k)$ denotes a dimensionless matter power spectra. Meanwhile, it is well known that $\Theta_t$ can be separated into a few parts [28]. One of them is called ISW part, written as
\[ \Theta_t^{(ISW)} = \int_{\eta_*}^{\eta_0} d\eta [\Phi'(k, \eta) + \Psi'(k, \eta)]j_l(k(\eta_0 - \eta)), \]
where $\eta_*$ is a conformal time at decoupling, and $j_l$ is a spherical Bessel function. The critical point of the ISW term is that it contributes to CMB power spectra when the gravitational potential, i.e., $\Phi$ or $\Psi$, varies through time. In GR, it is known that this phenomenon happens only in the radiation-dominated era and the dark-energy-dominated era. From (54), we may write
\[ \Theta_t^{(ISW, Weyl)} = \Theta_t^{(ISW, GR)} - \left[\left(\frac{m^2}{M}\right)^2 - 1\right] \int_{\eta_*}^{\eta_0} d\eta \delta\phi' j_l(k(\eta_0 - \eta)), \]

where $\Theta_t^{(ISW, Weyl)}$ denotes the ISW effect in our current model and $\Theta_t^{(ISW, GR)}$ denotes one in GR. We expect that effects from the field oscillation are likely to cancel themselves largely because they are almost symmetric to the $\eta$-axis. Hence, the changes with $m_\phi$ would come from the difference in the field value $\Delta\phi$. So it is not so inaccurate to adopt an approximation scheme of our previous paper [21] for our goal to describe the ISW effect, to assume that $\Delta\phi \approx 0$. Now, from the approximation scheme, one obtains the following:
\[ \delta\phi \approx \frac{1}{a} \exp[-\int d\eta'' \int d\eta' (k^2 + a^2M - \frac{a''}{a})], \]
where we have defined
\[ M \equiv [1 + 3(\frac{m^2}{M})^2(1 - (\frac{m^2}{M})^2)]^{-1} \times [m_\phi^2 + 2(\frac{m^2}{M})^2\kappa\Lambda], \]
where $\Lambda$ is the cosmological constant. From (59) and (60), we expect that $\delta\phi'$ in the approximated form is negative, and the latter term in (58) becomes more significant if the field mass become heavier. One can see that one obtains actually a stronger ISW effect when $m_\phi$ gets bigger in Figure 4. We also demonstrate that the perturbed photon energy density amplitude becomes more noticeable with the heavier Weyl field mass in Figure 5. It is also notable that the perturbed Weyl field affects both amplitude and frequency of $\delta\phi$ in the early era, whereas, in the late era, the frequencies synchronize.
FIG. 5. Oscillation of the perturbed photon density perturbation $\delta_\gamma$ by varying value of the Weyl field mass $m_\phi$. One can observe that the heavier mass gives larger amplitudes of $\delta_\gamma$.

with the original value even though the Weyl field mass gets heavier.

Since these changes in the ISW effect may give us a different value of the current Hubble parameter when comparing our model with data, we propose that this might be a novel solution to Hubble tension, one of the most notorious problems in cosmology today. Especially we would like to emphasize that our model suggests neither a new model for dark energy nor an unnecessary modification of the background equations. Of course, we think there must be additional changes when adopting dark energy models differing from a cosmological constant. However, when the equation of state for the dark energy model converges to $w \approx -1$ enough, we expect that one could obtain almost the same result independent of the models. Also, it is presumably natural to consider high-curvature correction to the Einstein-Hilbert action from the motivation of quantum gravity. Particularly the Starobinsky-type of inflation model is one of the most successive inflationary theories consistent with observational data. Moreover, symmetry-breaking phenomena are universal in physics, and there is no reason not to consider them also in gravity. Although we have assumed some hand-wavy assumptions by expanding the theory, we believe that our model could imply some hints for Hubble tension at least.

In Figure 6, we plot the CMB polarization EE spectrum and its cross-spectrum with the temperature TE spectrum. As shown in the figure, changes with varying $m_\phi$ in EE and TE spectra are relatively negligible compared with the TT spectra. We though find some interesting phenomena in these plots. For EE spectra, a higher value of $m_\phi$ gives smaller spectra values. For TE spectra, as $m_\phi$ gets larger, amplitudes for odd number peaks and troughs decrease, whereas for even numbers they increase. However, these changes are relatively negligible compared with the TT spectra.

FIG. 6. CMB EE and TE power spectra in unit of $\mu K^2$, with varying value of the Weyl field mass $m_\phi$. For EE spectra, higher value of $m_\phi$ gives a smaller value. For TE spectra, as $m_\phi$ gets larger, amplitudes for odd number peaks and troughs decrease, whereas for even numbers they increase. However, these changes are relatively negligible compared with the TT spectra.

FIG. 7. CMB BB power spectra in unit of $\mu K^2$, with varying value of the Weyl field mass $m_\phi$. The effects are significant at the all region of $\ell$, lowering the amplitude of the spectra.

For comparison with other gravity models, we mainly discuss three examples, our previous study [21], Brans-Dicke theory, and standard $f(R)$ models. First, we briefly review our previous research for comparison. We have studied how primordial symmetry could be broken in two different situations. The first one, namely model A, originates from Zee’s broken-symmetric theory of gravity [22], a Brans-Dicke-type theory with a Higgs-type spectra, as $m_\phi$ gets larger, amplitudes for odd number peaks and troughs decrease, whereas, for even numbers, they increase. In Figure 7, the BB power spectra decrease when $m_\phi$ becomes larger.
potential, and the second one, model B, is a modification of the first one by adopting Palatini formalism to the action of model A. With this method, we could establish new gravity models with an additional geometrical gauge symmetry called Weyl geometry. For now, let us discuss model B first, and model A will be discussed after with the Brans-Dicke theory.

Contrary to model B, in our current research, we have not assumed Palatini formalism, adopted the Weyl gauge symmetry first and found \( f(R) \) models satisfying this symmetry. One could apply Palatini formalism also to the action in the Weyl frame, that is, an action with respect to \( g_{\mu\nu} \), wherein the Weyl field appears as if it has a non-minimal coupling with the scalar curvature. However, it would bring a different result that would not coincide with Weyl geometry due to higher-order curvature. For instance, there is no coupling with the Weyl field for the quadratic term, i.e., the \( R^2 \) term in our action (7), even in the Weyl frame. In this case, we would have to study Palatini formalism in \( f(\varphi, R) \) gravity, where there would be no such simple geometrical symmetry as (4). Also, it is worth mentioning that there are changes only at the high-\( \ell \) region of the CMB TT spectra in our previous models, both A and B.

Moreover, one must note that we have adopted different initial values of the perturbed Weyl field and its time derivative of the current model, i.e., \( \delta \varphi_{\text{ini}} = 0 \) and \( (\delta \varphi)_{\text{ini}} = 0 \), whereas, in our previous study, we adopted \( \delta \varphi_{\text{ini}} = \sqrt{2} A_s \) and \( (\delta \varphi)_{\text{ini}} = 0 \) to give initial evolution for the field. In the model in this article, we did not need to adopt such values since there exists an external source from the ordinary matters due to the non-minimal coupling from higher curvature terms. Furthermore, these values would not give us meaningful results for the CMB observable because of the term \((m_\varphi/M)^2\) in the perturbed Einstein equation (44). For the perturbed Weyl field to have a small enough effect with such nonzero initial conditions, we must impose significant enough values, apparently large above Planck mass. Nevertheless, (44) would then prevent the results from converging at the limit \( m_\varphi \to \infty \). Moreover, the initial conditions in our model were chosen to avoid this issue; hence one might think that this is too intentional because in [21], we could find these non-minimal initial conditions by computing the action under the assumption that \( \|k\tau\| \ll 1 \) We suspect this kind of problem could be alleviated by adopting a coupling constant for the Weyl scalar field like \( \omega \) in Brans-Dicke theory and leave this as a future study topic.

Now we compare our results with studies on CMB anisotropy in Brans-Dicke theory, including model A from our study. CMB anisotropy in the original version of Brans-Dicke theory, which does not include the potential for the Brans-Dicke field, was studied in detail in [29, 30]. In this study, there is one additional free parameter, the coupling constant \( \omega \). It appears that both the amplitude and the location of peaks in the CMB TT spectra depend on the value of \( \omega \). However, in our model, there is no apparent change in the peak locations. Also, the Brans-Dicke theory shows no significant ISW effect in the CMB TT spectra. In addition to that, as the free parameter in our theory is the Weyl field mass and not the coupling constant, it might be inappropriate to compare our study with the original Brans-Dicke theory directly.

There are many studies regarding Brans-Dicke-type theories with potential, for example, see: [31] [32]. This research includes many interesting new phenomena, such as an additional ISW effect like our paper. Nevertheless, this model also requires background modification, and they clearly differ from our model. Especially many of them are motivated to resolve the dark energy problem. Therefore it is natural to assume that there will be a difference in the ISW effect in the late era. In fact, some of the dark energy models in Brans-Dicke-type theories are suggested to cure Hubble tension in the first place. However, in our model, the non-minimal coupling appears only at the perturbative level; there is no reason to suspect the change in the ISW part of the CMB spectra at face value. The modified temporal evolution of the gravitational potential comes purely from the perturbation theory. Model A in our previous study has the same feature. That is, the non-minimal coupling only valid at the perturbative scale. However, the effect of symmetry was relevant only in the early era of the universe, whereas the current model affects all times of the universe, although the effects are mainly from the late era.

Finally, let us close our discussion by briefly comparing our model with the standard \( f(R) \) theories. As many models in \( f(R) \) gravity also require modification of the background equations like Brans-Dicke theory, we find that comparing our model with all types of \( f(R) \) gravity models would not be so meaningful. In addition, we focused only on the Starobinsky model in our paper. So it would be enough only to compare with the original Starobinsky inflation model. One interesting feature is that even though we have adopted a new scalar field, there is no change in the matter power spectra originating from the inflation. However, we might be able to observe the presence of the primordial gravitational waves, the smoking gun for inflation, although it is nearly impossible in practice because the mass scale of the perturbed Weyl field is much larger than the inflation energy scale. Also, while there are many studies on Palatini formalism in \( f(R) \) gravity and higher-derivative theories of gravity [33] [34], there were not so many studies on Palatini formalism in \( f(\varphi, R) \) gravity, primarily focused on the geometrical symmetry. Hence, it would be interesting to study an application of Palatini formalism in \( f(\varphi, R) \) gravity to find their new possible symmetry and its breaking phenomena.
VI. CONCLUSION

In this article, we have studied how Weyl gauge symmetry breaking can arise in $f(R)$ gravity and how this affects the CMB observable. We assumed the connection satisfies a new kind of geometrical gauge symmetry, and from that, we have derived the corresponding version of $f(R)$ gravity satisfying that symmetry. In addition, we invoked the symmetry breaking at the Planck scale of the universe with the Higgs-type potential for the Weyl scalar field. We considered the perturbed field $\delta \varphi$ after the symmetry breaking up to first order in the gravity side of the action and found an additional non-minimal coupling with the perturbed Weyl field only valid at the perturbative level.

We next studied how Starobinsky inflation works in our model as a case study. We found no change in the quantum fluctuation of the metric, and hence the matter power spectra stay the same. However, it appears that the perturbed Weyl field can affect the CMB observables, especially the ISW effect in the late era of the universe. We could see that the CMB TT power spectra increase overall in the low-$\ell$ region; hence, our model might be a new proposal to resolve or alleviate Hubble tension. Of course, to verify that our idea works, we need to conduct a Monte-Carlo simulation to compare with data such as from the Planck satellite. So we would like to note that our current purpose of this paper is to suggest a tentative solution to Hubble tension using the concept of primordial symmetry breaking. The detailed analysis with such data will be followed and reported soon after.

Moreover, our study exploits an additional assumption that could be inappropriate. We have used an approximated equation of motion (14) when we find an equivalent action with a scalar field. This strategy was to reduce the higher-order derivatives in our theory. However, in principle, one must abandon such an assumption to construct a completely exact form of the theory. Also, we had to adopt Weyl geometry in the first place and did not use Palatini formalism, which gave us the original motivation to extend the symmetry of GR [14]. Of course, there is no fundamental difference or advantage to using Palatini formalism. However, at least it would be worthy of applying it to the action (8) to investigate whether there would be some new class of geometrical gauge symmetry. Furthermore, last but not least, the expected value of the perturbed Weyl field is quite close to the Planck mass even though with the factor $\sim 10^{-2}$ and much bigger than the inflation scale, although it is much less value when compared with our previous study, where we supposed $m_\varphi \gtrsim 10^3 M_p$.

The physics at the energy scale we have to deal with is still totally unknown to us, and although the higher-order correction of the action we considered could arise from the high-energy quantum correction, this surely could turn out to be not the exact model. Furthermore, it is even needless to say that the inflationary scenario itself is still not fully confirmed, although many cosmologists today believe it would be true. Nevertheless, we might be able to think of our model as one of the practical descriptions of gravity under the assumption that there existed inflation. Only a more profound study could tell us how this theory would be valid.

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[1] Peter W. Higgs. Broken symmetries and the masses of gauge bosons. Phys. Rev. Lett., 13:508–509, Oct 1964.
[2] ATLAS collaboration. Observation of a new particle in the search for the standard model higgs boson with the atlas detector at the lhc. Physics Letters B, 716(1):1–29, 2012.
[3] Andreas Albrecht and Paul J. Steinhardt. Cosmology for grand unified theories with radiatively induced symmetry breaking. Phys. Rev. Lett., 48:1220–1223, Apr 1982.
[4] Daniel Boyanovsky. Spontaneous symmetry breaking in inflationary cosmology: On the fate of goldstone bosons. Phys. Rev. D, 86:023509, Jul 2012.
[5] Eric Greenwood, Evan Halstead, Robert Poltis, and Dejan Stojkovic. Electroweak vacua, collider phenomenology, and possible connection with dark energy. Phys. Rev. D, 79:103003, May 2009.
[6] H. Mohseni Sadjadi, M. Honardoost, and H.R. Sepangi. Symmetry breaking and onset of cosmic acceleration in scalar field models. Physics of the Dark Universe, 14:40–47, 2016.
[7] D Kazanas. Dynamics of the universe and spontaneous symmetry breaking. Astrophys. J., Lett. Ed.; (United States), 241(2), 10 1980.
[8] PANKAJ JAIN and SUBHADIP MITRA. Cosmological symmetry breaking, pseudo-scale invariance, dark energy and the standard model. Modern Physics Letters A, 22(22):1651–1661, 2007.
[9] M. Sami and Radouane Gannouji. Spontaneous symmetry breaking in the late universe and glimpses of the early universe phase transitions à la baryogenesis. International Journal of Modern Physics D, 30(13):2130005, 2021.
[10] Robert Bluhm. Explicit versus spontaneous diffeomorphism breaking in gravity. Phys. Rev. D,
91:065034, Mar 2015.
[11] Amir Ghalee. Notes on diffeomorphisms symmetry of \( f(r) \) gravity in the cosmological context. \emph{The European Physical Journal C} volume 76, 156, 2016.
[12] Hermann Weyl. Eine neue erweiterung der relativitätstheorie. \textit{Annalen der Physik}, 364, 101-133, 1919.
[13] Hermann Weyl. \emph{Space, Time, Matter}. Dover, New York, 1952.
[14] J. B. Fonseca-Neto C. Romero and M. L. Pucheu. General relativity and weyl geometry. \textit{Class. Quantum Grav.} 29 155015, 2012.
[15] T. S. Almeida, M. L. Pucheu, C. Romero, and J. B. Formiga. From brans-dicke gravity to a geometrical scalar-tensor theory. \textit{Phys. Rev. D}, 89:064047, Mar 2014.
[16] D.M. Ghilencea. Spontaneous breaking of weyl quadratic gravity to einstein action and higgs potential. \textit{J. High Energ. Phys.} 2019, 49, 2019.
[17] Steffano Lucat Alexander Barnaveli and Tomislav Prokopec. Inflation as a spontaneous symmetry breaking of weyl symmetry. \textit{JCAP} 01, 022, 2019.
[18] D. M. Ghilencea and Hyun Min Lee. Weyl gauge symmetry and its spontaneous breaking in the standard model and inflation. \textit{Phys. Rev. D}, 99:115007, Jun 2019.
[19] M. L. Pucheu, C. Romero, M. Bellini, and José Edgar Madriz Aguilar. Gauge invariant fluctuations of the metric during inflation from a new scalar-tensor weyl-integrable gravity model. \textit{Phys. Rev. D}, 94:064075, Sep 2016.
[20] M. L. Pucheu, F. A. P. Alves Junior, A. B. Barreto, and C. Romero. Cosmological models in weyl geometrical scalar-tensor theory. \textit{Phys. Rev. D}, 94:064010, Sep 2016.
[21] Jiwon Park and Tae Hoon Lee. Probing primordial symmetry breaking with the cosmic microwave background anisotropy. \textit{Phys. Rev. D}, 101:123528, Jun 2020.
[22] A. Zee. Broken-symmetric theory of gravity. \textit{Phys. Rev. Lett.}, 42:417–421, Feb 1979.
[23] De Felice A. and S. Tsujikawa. \( f(r) \) theories. \textit{Living Rev. Relativ.} 13, 3, 2010.
[24] Eleonora Di Valentino, Olga Mena, Supriya Pan, Luca Visinelli, Weiqiang Yang, Alessandro Melchiorri, David F. Mota, Adam G. Riess, and Joseph Silk. In the realm of the hubble tension—a review of solutions. \textit{Class. Quantum Grav.} 38 153001, 2021.
[25] A. A. Starobinsky. Spectrum of relict gravitational radiation and the early state of the universe. \textit{Journal of Experimental and Theoretical Physics Letters}, 30, 682, 1979.
[26] Antony Lewis, Anthony Challinor, and Anthony Lasenby. Efficient computation of cosmic microwave background anisotropies in closed friedmann-robertson-walker models. \textit{ApJ} 538 473, 2000.
[27] Planck Collaboration. Planck 2018 results - vi. cosmological parameters. \textit{A&A}, 641:A6, 2020.
[28] Oliver Piattella. \textit{Lecture Notes in Cosmology}. Springer, 2018.
[29] Wu Feng-Quan, Qiang Li-e, Wang Xin, and Chen Xuelei. Cosmic microwave background with brans-dicke gravity. i. covariant formulation. \textit{Phys. Rev. D}, 82:083002, Oct 2010.
[30] Feng-Quan Wu and Xuelei Chen. Cosmic microwave background with brans-dicke gravity. ii. constraints with the wmap and sdss data. \textit{Phys. Rev. D}, 82:083003, Oct 2010.
[31] Alex Zucca, Levon Pogosian, Alessandra Silvestri, Yuting Wang, and Gong-Bo Zhao. Generalized brans-dicke theories in light of evolving dark energy. \textit{Phys. Rev. D}, 101:043518, Feb 2020.
[32] M. Ballardini, D. Sapone, C. Umità, F. Finelli, and D. Paoletti. Testing extended jordan-brans-dicke theories with future cosmological observations. \textit{Journal of Cosmology and Astroparticle Physics}, 2019(05):049–049, may 2019.
[33] Mónica Borunda, Bert Janssen, and Mar Bastero-Gil. Palatini versus metric formulation in higher-curvature gravity. \textit{Journal of Cosmology and Astroparticle Physics}, 2008(11):008, nov 2008.
[34] Thomas P Sotiriou. Constraining \( f(r) \) gravity in the palatini formalism. \textit{Classical and Quantum Gravity}, 23(4):1253–1267, feb 2006.
[35] Xin-Juan Yang and Da-Ming Chen. \( f(R) \) gravity theories in the Palatini formalism constrained from strong lensing. \textit{Monthly Notices of the Royal Astronomical Society}, 394(3):1449–1458, 04 2009.
[36] Kucukakca Y. and Camci U. Noether gauge symmetry for \( f(r) \) gravity in palatini formalism. \textit{Astrophys Space Sci} 338, 211–216, 2012.