ON THE DEFORMED BOTT-CHERN COHOMOLOGY

WEI XIA

Abstract. Given a compact complex manifold $X$ and an integrable Beltrami differential $\phi \in A^{0,1}(X, T^1_X)$, we introduce a double complex structure on $A^{p,q}(X)$ naturally determined by $\phi$ and study its Bott-Chern cohomology. In particular, we establish a deformation theory for Bott-Chern cohomology and use it to compute the deformed Bott-Chern cohomology for the Iwasawa manifold and the holomorphically parallelizable Nakamura manifold. The $\partial\bar{\partial}$-lemma is studied and we show a compact complex manifold satisfying $\partial\bar{\partial}$-lemma is formal.

Key words: deformation of complex structures, Bott-Chern cohomology, $\partial\bar{\partial}$-lemma.

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1. Introduction

The Bott-Chern cohomology are important invariants of complex manifolds [BC65]. It has been studied by many authors in recent years [Ang13, AT15b, AT15a, AT17, ADT16, AK17a]. For example, Schweitzer studied the Hodge theory for Bott-Chern cohomology and gave a hypercohomology interpretation to it [Sch07]. Angella-Tomassini proved Fröhlicher type inequalities for Bott-Chern cohomology and gave a beautiful characterization of the $\partial\bar{\partial}$-lemma [AT13]. Recently, S. Yang and X. Yang proved a blow-up formula for the Bott-Chern cohomology and they showed that satisfying the $\partial\bar{\partial}$-Lemma is a bimeromorphic invariant for threefolds [YY20], see [RYY19, ASTT20, Ste18a, Ste18b, Men19] for related works.

Let $X$ be a complex manifold and $X_t$ a small deformation (of $X$) whose complex structure is represented by a Beltrami differential $\phi \in A^{0,1}(X, T^1_X)$. In this paper, we will study the Bott-Chern cohomology of the double complex $(A^{p,q}(X), \partial, \bar{\partial})$:

\begin{equation}
H^{p,q}_{BC,\phi}(X) := \frac{\ker d_\phi \cap A^{p,q}(X)}{\text{Im } \partial\bar{\partial}_\phi \cap A^{p,q}(X)},
\end{equation}

which we called the deformed Bott-Chern cohomology, where $d_\phi = \partial + \bar{\partial}_\phi$ and $\bar{\partial}_\phi = \bar{\partial} - L^{1,0}_\phi$. In Section 3, we will show that there are similar hypercohomology interpretations to the deformed Bott-Chern cohomology as to the usual Bott-Chern cohomology.

Let $\pi : (X, X) \to (B, 0)$ be a deformation of a compact complex manifold $X$ such that for each $t \in B$ the complex structure on $X_t$ is represented by Beltrami differential
\( \phi(t) \). Given a Bott-Chern class \([y] \in H^{p,q}_{BC}(X)\), as motivated by our previous work on deformation of Dolbeault cohomology classes [Xia19a], we try to construct a family of \((p, q)\)-forms \(\sigma(t)\) (on an analytic subset \(T\) of \(B\)) such that

1. \(\sigma(t)\) is holomorphic in \(t\);
2. \(\partial\sigma(t) = \overline{\partial}\phi(t)\sigma(t) = 0, \ \forall t \in T;\)
3. \([\sigma(0)] = [y] \in H^{p,q}_{BC}(X)\).

We will develop a deformation theory for Bott-Chern cohomology in this respect, see Section 4. Among other things, we show the following

**Theorem 1.1** (=Theorem 4.11). Let \(\pi: (\mathcal{X}, X) \to (B, 0)\) be a deformation of a compact complex manifold \(X\) such that for each \(t \in B\) the complex structure on \(X_t\) is represented by Beltrami differential \(\phi(t)\). Then the set \(\{t \in B \mid \dim H^{p,q}_{BC\phi(t)}(X) \geq k\}\)

is an analytic subset of \(B\) for any nonnegative integer \(k\).

In [AT15a, Thm. 1 and 2], Angella-Tomassini generalized their previous result [AT13] to arbitrary double complex [AT15b]. This result, when applied to our situation, will give rise to the following

**Theorem 1.2** (=Theorem 5.3). Let \(X\) be a compact complex manifold and \(X_t\) a small deformation (of \(X\)) whose complex structure is represented by a Beltrami differential \(\phi \in A^{0,1}(X, T^1_X)\). Then for every \((p, q) \in \mathbb{N} \times \mathbb{N}\), we have

\[
\dim H^{p,q}_{BC\phi}(X) + \dim H^{p,q}_{A\phi}(X) \geq \dim H^{p,q}_{\partial\phi}(X_t) + \dim H^{p,q}_{\overline{\partial}\phi}(X).
\]

In particular, for every \(k \in \mathbb{N}\), we have

\[
\sum_{p+q=k} \dim H^{p,q}_{BC\phi}(X) + \sum_{p+q=k} \dim H^{p,q}_{A\phi}(X) \geq 2 \dim H^{k}_{dR}(X),
\]

and equality holds if and only if \(X\) satisfies the \(\partial\overline{\partial}\)-lemma.

Note that when \(X_t\) is a trivial deformation, i.e. \(\phi = 0\), Theorem 1.2 is reduced to the result in [AT13]. Combine Theorem 1.2 with Theorem 1.1, we get

**Corollary 1.3.** Let \(\pi: (\mathcal{X}, X) \to (B, 0)\) be a small deformation of the compact complex manifold \(X\) such that for each \(t \in B\) the complex structure on \(X_t\) is represented by Beltrami differential \(\phi(t)\). Then the set

\[ T := \{t \in B \mid X \text{ satisfies the } \partial\overline{\partial}_{\phi(t)}\text{-lemma}\}\]

is an analytic open subset (i.e. complement of analytic subset) of \(B\). In particular, if \(B \subset \mathbb{C}\) is a small open disc with \(0 \in B\) and \(T\) is not empty, then \(T = B\) or \(T = B \setminus \{0\}\).

It is known that satisfying the \(\partial\overline{\partial}\)-lemma is a deformation open property and not a deformation closed property in the sense of Popovici [Pop14], see [Wu06, AT13, AK17b] and the references therein. But it is still not clear whether satisfying the \(\partial\overline{\partial}\)-lemma is an analytically open property, i.e. does the corresponding statement in Corollary 1.3 holds for the \(\partial\overline{\partial}\)-lemma? On the other hand, we see from Corollary 1.3 that if \(X\) satisfies the \(\partial\overline{\partial}\)-lemma then \(X\) also satisfies the \(\partial\overline{\partial}_{\phi(t)}\)-lemma for small \(t\). But conversely, if \(X\) satisfies the \(\partial\overline{\partial}_{\phi(t)}\)-lemma for all small \(t \neq 0\) it is possible
that $X$ does not satisfy the $\partial\bar{\partial}$-lemma\(^1\). Hence the following Theorem generalize the corresponding well-known result of Deligne-Griffiths-Morgan-Sullivan [DGMS75]:

**Theorem 1.4.** Let $X$ be a compact complex manifold and $X_t$ a small deformation (of $X$) whose complex structure is represented by a Beltrami differential $\phi \in A^{0,1}(X, T_X^{1,0})$. If $X$ satisfies the $\partial\bar{\partial}$-lemma, then $X_t$ is formal.

The dimensions of the deformed Bott-Chern cohomology is computed for the Iwasawa manifold and the holomorphically parallelizable Nakamura manifold, see Section 6. Comparing this with the computations of Angella-Kasuya [AK17b], we see that there exists compact complex manifold $X$ and its small deformation $X_t$ such that $X_t$ satisfy the $\partial\bar{\partial}$-lemma but $X$ does not satisfy the $\partial\bar{\partial}(\phi(t))$-lemma.

There are many questions regarding the $\partial\bar{\partial}$-lemma may be asked:

**Question 1.5.** Let $\pi : (X, X) \to (B, 0)$ be a small deformation of the compact complex manifold $X$ such that for each $t \in B$ the complex structure on $X_t$ is represented by Beltrami differential $\phi(t)$.

1. Is it true that
   \[
   \dim H_{BC\phi(t)}^{p,q}(X) \geq \dim H_{BC}^{p,q}(X_t)
   \]
   for any $t \in B$ and $(p, q) \in \mathbb{N} \times \mathbb{N}$? If this holds, then $X$ satisfies the $\partial\bar{\partial}(\phi(t))$-lemma will imply $X_t$ satisfy the $\partial\bar{\partial}$-lemma. Note that (1.4) is true for the examples considered in Section 6;

2. If $B \subset \mathbb{C}$ is a small open disc with $0 \in B$, can we find an example such that $T = B \setminus \{0\}$ (in the notation of Corollary 1.3)? According to Corollary 1.3, there should be many such examples. In this case, the Fröhlicher spectral sequence on the central fiber $X$ must degenerates at $E_1$, see Remark 5.4;

3. If $X_t$ is Kähler, is it true that $X$ must satisfy the $\partial\bar{\partial}(\phi(t))$-lemma?

2. **The deformed double complex $(A^{\bullet\bullet}(X), \partial, \bar{\partial})$ and its Bott-Chern cohomology**

Let $X$ be a complex manifold and $X_t$ a small deformation (of $X$) whose complex structure is represented by a Beltrami differential $\phi \in A^{0,1}(X, T_X^{1,0})$. Recall the following useful facts [LRY15, Xia19b]:

\[
e^{-i\phi} de^{i\phi} = d - \mathcal{L}^{1,0}_\phi - \mathcal{L}^{0,1}_\phi - i\frac{1}{2} [\phi, \phi] \text{ and } \mathcal{L}^{0,1}_\phi = -i\bar{\partial}_\phi .
\]

Since $\phi$ satisfy the Maurer-Cartan equation $\bar{\partial}_\phi - \frac{1}{2} [\phi, \phi] = 0$, we have

\[
d_\phi := e^{-i\phi} de^{i\phi} = \partial + \bar{\partial}_\phi, \text{ with } \bar{\partial}_\phi = \bar{\partial} - \mathcal{L}^{1,0}_\phi ,
\]
and

\[
d_\phi := e^{-i\phi} de^{i\phi} = \partial + \bar{\partial}, \text{ with } \bar{\partial} = \partial - \mathcal{L}^{0,1}_\phi .
\]

\(^1\)Though it is still not known whether such examples exist, we think they should be large in number.
Since $[\bar{\partial}, \tilde{\partial}_\phi] = [\partial_\phi, \bar{\partial}] = 0$, the deformed Bott-Chern cohomology can be defined as follows:

\[
(2.3) \quad H_{BC\phi}^{p,q}(X) := \frac{\ker d_\phi \cap A^{p,q}(X)}{\mathrm{Im} \partial_\phi \cap A^{p,q}(X)}, \quad H_{BC\bar{\phi}}^{p,q}(X) := \frac{\ker d_{\bar{\phi}} \cap A^{p,q}(X)}{\mathrm{Im} \partial_{\bar{\phi}} \cap A^{p,q}(X)}, \quad \forall p, q \geq 0,
\]

and $h_{BC\phi}^{p,q} := \dim H_{BC\phi}^{p,q}(X), h_{BC\bar{\phi}}^{p,q} := \dim H_{BC\bar{\phi}}^{p,q}(X)$. The conjugation gives a natural isomorphism between $H_{BC\phi}^{p,q}(X)$ and $H_{BC\bar{\phi}}^{p,q}(X)$, we thus have $h_{BC\phi}^{p,q} = h_{BC\bar{\phi}}^{p,q}$.

3. Hypercohomology interpretations to the deformed Bott-Chern cohomology

It is clear that the Poincaré lemma holds for $d_\phi$ and $\tilde{\partial}_\phi$ (for the latter, see [Xia19a, Thm.3.4]). The sheaf of germs of $\bar{\partial}_\phi$-closed $p$-forms will be denoted by $\Omega_\phi^p$. The following Lemma is essentially proved in [Sch07]:

Lemma 3.1. Let $U \subset \mathbb{C}^n$ be an open ball.

1. Let $\theta \in A^k(U)$ with $k \geq 1$ such that $\theta^{p,q} = 0$ except $p_1 \leq p \leq p_2 (p_1 < p_2)$.
   If $\theta$ is $d_{\bar{\phi}}$-closed, then $\theta = \alpha^p \alpha$ for some $\alpha \in A^{k-1}(U)$ with $\alpha^{p,q} = 0$ except $p_1 \leq p \leq p_2 - 1$.

2. Assume $\theta \in A^{p,q}(U)$ is $d_{\bar{\phi}}$-closed.
   i) If $p \geq 1$ and $q \geq 1$, then $\theta \in \bar{\partial}_{\phi} A^{p-1,q-1}(U)$.
   ii) If $p \geq 1$ and $q = 0$, then $\theta \in \bar{\partial}_{\phi} A^{p-1,0}(U)$.
   iii) If $p = 0$ and $q \geq 1$, then $\theta \in \bar{\partial}_{\phi} A^{0,q-1}(U)$.

3. Assume $\theta \in A^{p,q}(U)$ is $\bar{\partial}_{\phi}$-closed.
   i) If $p \geq 1$ and $q \geq 1$, then $\theta \in \bar{\partial}_{\phi} A^{p-1,q-1}(U) + \partial A^{p-1,q}(U)$.
   ii) If $p \geq 1$ and $q = 0$, then $\theta \in \Omega_{\phi}^{p,q}(U) + \bar{\partial} A^{p-1,0}(U)$.
   iii) If $p = 0$ and $q \geq 1$, then $\theta \in \bar{\partial}_{\phi} A^{0,q-1}(U) + \bar{\partial} A^{0,q}(U)$.

4. Let $\theta \in A^k(U)$ with $k \geq 1$ and $p_1, q_1, p_2, q_2$ be two positive integers with $p_1 + q_1 = p_2 + q_2 = k$. If $(d_\phi \theta)^{p,q} = 0$ for $p + q = k + 1$, $p_1 + 1 \leq p \leq p_2$ and $q_1 \geq q \geq q_2 + 1$, then there exists $\gamma^{p_1,q_1}, \alpha^{p_1,q_1-1}, \alpha^{p_1+1,q_1-2}, \ldots, \alpha^{p_2-1,q_2}, \gamma^{p_2,q_2}$ such that $\gamma^{p_1,q_1}$ is $\partial$-closed, $\gamma^{p_2,q_2}$ is $\bar{\partial}_{\phi}$-closed and

\[
\begin{align*}
\theta^{p_1,q_1} &= \gamma^{p_1,q_1} + \bar{\partial}_{\phi} \alpha^{p_1,q_1-1}, \\
\theta^{p_1+1,q_1-1} &= \partial \alpha^{p_1,q_1-1} + \bar{\partial}_{\phi} \alpha^{p_1+1,q_1-2}, \\
&\vdots \\
\theta^{p_2-1,q_2+1} &= \partial \alpha^{p_2-2,q_2+1} + \bar{\partial}_{\phi} \alpha^{p_2-1,q_2}, \\
\theta^{p_2,q_2} &= \partial \alpha^{p_2-1,q_2} + \gamma^{p_2,q_2},
\end{align*}
\]

in particular, we have

\[
\theta^{p_1,q_1} + \theta^{p_1+1,q_1-1} + \ldots + \theta^{p_2,q_2} = \gamma^{p_1,q_1} + \partial \alpha + \gamma^{p_2,q_2},
\]

where $\alpha = \alpha^{p_1,q_1-1} + \alpha^{p_1+1,q_1-2} + \ldots + \alpha^{p_2-1,q_2}$. 


Proof. 1. First, by the $d_\phi$-Poincaré lemma, we can write $\theta = d_\phi \beta$ for some $\beta \in A^{k-1}(U)$. If $p_1 = 0$ and $p_2 = k$ there is nothing to prove, so we assume $p_1 > 0$ or $p_2 < k$. We first consider the case $p_1 > 0$. We deduce from $\theta = d_\phi \beta$ that $\partial_\phi \beta^{0,k-1} = \theta^{0,k-1} = 0$, and by applying the $\partial_\phi$-Poincaré lemma, one can write $\beta^{0,k-1} = \partial_\phi \gamma^{0,k-2}$. Set $\tilde{\beta} := \beta - d_\phi \gamma^{0,k-2}$, we have $d_\phi \tilde{\beta} = \theta$ but $\tilde{\beta}^{0,k-1} = 0$. We can therefore assume that $\beta$ does not have components of type $(0,k-1)$. Now if $p_1 > 1$, then since $\beta^{0,k-1} = 0$ we have $0 = \theta^{1,k-1} = \partial_\phi \beta^{1,k-2} + \partial \beta^{0,k-1} = \partial_\phi \beta^{1,k-2}$. By the $\partial_\phi$-Poincaré lemma, one can write $\beta^{1,k-2} = \partial_\phi \gamma^{0,k-3}$. Set $\tilde{\beta} := \beta - d_\phi \gamma^{0,k-3}$, we have $d_\phi \tilde{\beta} = \theta$ but $\tilde{\beta}^{1,k-2} = 0$. We can therefore assume that $\beta$ does not have components of type $(1,k-2)$. By repeating this reasoning, we can assume that $\beta$ does not have components of type $(p,q)$ for $p < p_1$. The case $p_2 < k$ can be proved in the same way by applying the $\partial_\phi$-Poincaré lemma.

2.iii) is obvious. We first assume $p \geq 1$. We apply 1. to the form $\theta$ for $p_1 = p - 1$, $p_2 = p$: there exists $\alpha \in A^{p-1,q}(U)$ s.t. $\theta = d_\phi \alpha$ and so $\theta = \partial_\phi \alpha$ with $\partial_\phi \alpha = 0$. This is ii). If furthermore $q \geq 1$, by the $\partial_\phi$-Poincaré lemma, we can write $\alpha = \tilde{\alpha}$ $\bar{\beta}$ and so $\theta = \partial \tilde{\alpha}$. This is i). For $\tilde{\alpha}$), we apply 1. to $\theta$ for $p_1 = 0, p_2 = 1$: there exists $\alpha \in A^{0,q}(U)$ s.t. $\theta = d_\phi \alpha$ and so $\theta = \partial_\phi \alpha$ with $\partial_\phi \alpha = 0$.

3. Set $\theta^{p+1,q} := \partial_\phi \alpha^{p,q}$ then $\theta^{p+1,q}$ is $d_\phi$-closed. By 2.i) and ii), there exists $\alpha \in A^{p,q}(U)$ s.t. $\theta^{p+1,q} = \partial_\phi \alpha$ with $\partial_\phi \alpha = 0$. Note that $\partial (\theta - \alpha) = 0$ and $\theta = (\theta - \alpha) + \alpha$.

Then 3. follows from the $\partial_\phi$-Poincaré lemma and the $\partial_\phi$-Poincaré lemma.

4. First from the assumption we see that $(d_\phi \theta)^{p_1+1,q_1} = \partial \theta^{p_1,q_1} + \partial_\phi \theta^{p_1+1,q_1-1} = 0$. In particular, $\theta^{p_1,q_1}$ is $\partial \partial_\phi$-closed. By 3.i) and ii) there exists $\gamma^{p_1,q_1}$ s.t. $\gamma^{p_1,q_1} = \partial_\phi \theta^{p_1,q_1}$.

Then from the assumption we see that $(d_\phi \theta)^{p_1+2,q_1-1} = \partial \theta^{p_1,q_1-1} + \partial_\phi \theta^{p_1+2,q_1-2} = 0$, and note that $\partial_\phi (\theta^{p_1+1,q_1-1} + \partial_\phi \theta^{p_1+1,q_1-2}) = \partial (\theta^{p_1+1,q_1-1} + \partial_\phi \alpha^{p_1+1,q_1-2}) = -\partial^2 \alpha^{p_1+1,q_1-1} = 0$, we have $\theta^{p_1+2,q_1-2} = \partial \alpha^{p_1+1,q_1-2} + \partial_\phi \alpha^{p_1+2,q_1-3}$.

Continuing in this way, we get the desired results. In the last two steps, from $(d_\phi \theta)^{p_2-1,q_2+2} = 0$ we get $\theta^{p_2-1,q_2+1} = \partial_\phi \alpha^{p_2-2,q_2+1} + \partial \alpha^{p_2-1,q_2}$ and from $(d_\phi \theta)^{p_2,q_2+1} = 0$ we get $\theta^{p_2,q_2+1} = \partial_\phi \alpha^{p_2+1,q_2} + \gamma^{p_2,q_2}$. 

Let $X$ be a complex manifold. For fixed $p \geq 1$ and $q \geq 1$, we define a sheaf complex $\mathcal{L}_\phi^\bullet$ (which depend on $(p,q)$) as follows:

$$
\begin{align*}
\mathcal{L}_\phi^k &= \bigoplus_{r+s=k, r < p, s < q} A^{r,s}, & \text{for } k \leq p + q - 2, \\
\mathcal{L}_\phi^{k-1} &= \bigoplus_{r+s=k, r \geq p, s \geq q} A^{r,s}, & \text{for } k \geq p + q.
\end{align*}
$$

\footnote{We may further assume that $k \geq 2$ because the case $k = 1$ is trivial.}

\footnote{$\partial$-$p_1,q_1$ is $\partial$-exact if $p_1 \geq 1$.}
The differential is given by

\[
0 \longrightarrow L_\phi^0 \xrightarrow{\Pi_\phi \cdot d_\phi} L_\phi^1 \xrightarrow{\Pi_\phi \cdot 2d_\phi} L_\phi^2 \longrightarrow \cdots
\]

\[
0 \longrightarrow L_\phi^0 \xrightarrow{\Pi_\phi \cdot p+q-2d_\phi} L_\phi^{p+q-2} \xrightarrow{\partial \bar{\phi}_\phi} L_\phi^{p+q-1} \xrightarrow{d_\phi} L_\phi^{p+q} \xrightarrow{d_\phi} L_\phi^{p+q} \longrightarrow \cdots
\]

where \(\Pi_{L^k} : \bigoplus_{r+s=k} A^r s \longrightarrow L^k_\phi\) is the projection. In particular, we find that

\[
L_\phi^{p+q-2} = A^{p-1} q-1 \xrightarrow{\partial \bar{\phi}_\phi} L_\phi^{p+q-1} = A^{p} q \xrightarrow{d_\phi} L_\phi^{p+q} = A^{p+1} q,
\]

and so \(\mathbb{H}^{p+q-1}(X, L_\phi^\bullet) \cong H^{p+q-1}(L_\phi^\bullet(X)) = H^p_{BC\phi}(X)\). The sheaf complex \(L_\phi^\bullet\) has the following subcomplexes

\[
(\mathcal{H}_\phi^\bullet, \partial) : \mathcal{O}_\phi \xrightarrow{\partial} \Omega^1_\phi \xrightarrow{\partial} \Omega^2_\phi \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^p_\phi \longrightarrow 0,
\]

\[
(\mathcal{H}_\phi^n, \bar{\partial}_\phi) : \bar{\mathcal{O}} \xrightarrow{\bar{\partial}_\phi} \bar{\Omega}^1 \xrightarrow{\bar{\partial}_\phi} \bar{\Omega}^2 \xrightarrow{\bar{\partial}_\phi} \cdots \xrightarrow{\bar{\partial}_\phi} \bar{\Omega}^q-1 \longrightarrow 0,
\]

and

\[
\mathcal{H}_\phi^k := (\mathcal{H}_\phi^\bullet, \partial) + (\mathcal{H}_\phi^n, \bar{\partial}_\phi)^4.
\]

Note that by Lemma 3.1, the complex \((\mathcal{H}_\phi^\bullet, \partial)\) is exact for \(0 < k < p - 1\) where \(\mathcal{H}_\phi^k = \Omega^k_\phi\).

**Proposition 3.2.** The inclusion \(\mathcal{H}_\phi^\bullet \hookrightarrow L_\phi^\bullet\) induces an isomorphism \(\mathcal{H}^k(\mathcal{H}_\phi^\bullet) \cong \mathcal{H}^k(L_\phi^\bullet), \forall k \geq 0\), and we have\(^5\)

\[
\mathcal{H}^k(\mathcal{H}_\phi^\bullet) \cong \mathcal{H}^k(L_\phi^\bullet) = \begin{cases} \mathbb{C}, \text{ for } k = 0, p > 1, q > 1, \\ \mathcal{O}_\phi, \text{ for } k = 0, p = 1, q > 1, \\ \mathcal{O}, \text{ for } k = 0, p > 1, q = 1, \\ \mathcal{O}_\phi \oplus \mathcal{O}, \text{ for } k = 0, p = 1, q = 1, \\ \Omega^{p-1}/\partial \Omega^{p-2}, \text{ for } 0 < k = p - 1 \text{ and } p \neq q, \\ \Omega^{p-1}/\bar{\partial}_\phi \bar{\Omega}^{p-2}, \text{ for } 0 < k = q - 1 \text{ and } p \neq q, \\ \Omega^{p-1}/\partial \Omega^{p-2} \oplus \bar{\Omega}^{p-1}/\bar{\partial}_\phi \bar{\Omega}^{p-2}, \text{ for } 0 < k = p - 1 = q - 1, \\ 0, \text{ otherwise.} \end{cases}
\]

**Proof.** First, we show that \(\mathcal{H}^k(\mathcal{H}_\phi^\bullet) = 0\) for \(k \geq \max\{p, q\}\). In fact, for \(k \geq p + q\), this follows from Lemma 3.1 1.; for \(k = p + q - 1\), this follows from Lemma 3.1 2.; for \(k = p + q - 2\), this follows from Lemma 3.1 3.; for \(k < p + q - 2\), this follows\(^6\) from Lemma 3.1 4. .

\(^4\)The sum is direct except \(k = 0\) and \(\mathcal{O}_\phi + \bar{\mathcal{O}} \longrightarrow \Omega^1_\phi \oplus \bar{\Omega}^1 : f + g \rightarrow (\partial f, \bar{\partial} g)\).

\(^5\)See also [Koo11, pp. 31].

\(^6\)We apply Lemma 3.1 4. for \(p_1 = k + q + 1\), \(q_1 = q - 1\), \(p_2 = p - 1\), \(q_2 = k + p + 1\). Note that we have \(\partial^{k-q-1} = \partial^{k-q-1} + \bar{\partial}_\phi \alpha^{k-q-1, q-2}\) and \(d_{\mathcal{R}^{k-1} \gamma^{k-q-1}} = \bar{\partial}_\phi \gamma^{k-q-1}\), where \(d_{\mathcal{R}^{k-1}} = \Pi_{\mathcal{R}^{k-1}} d_\phi\). Similarly, \(\partial^{p-1, k-p+1} = \partial^{p-1, k-p+1} + \bar{\partial}_\phi \gamma^{p-1, k-p}\) and \(d_{\mathcal{R}^{k-1} \gamma^{p-1, k-p}} = \bar{\partial}_\phi \gamma^{p-1, k-p}\).
Now we discuss the cases when \( k < p \) or \( k < q \).

For \( k = p - 1 \geq q \), if \( \theta = \theta^{p-q,1} + \cdots \theta^{p-1,0} \in L^{p-1}(U) \) is \( d_{\partial \phi} \)-closed where \( U \subset X \) is an open ball. By Lemma 3.1 4., we can write
\[
\theta^{p-q,1} = \gamma^{p,q,1} + \partial_{\theta} \alpha^{p-q,2}, \quad \ldots, \theta^{p-1,0} = \partial \alpha^{p-2,0} + \gamma^{p-1,0},
\]
where \( \gamma^{p,q,1} \) is \( \partial \)-closed and \( \gamma^{p-1,0} \) is \( \partial_{\theta} \)-closed. Since \( p-q \geq 1 \), we have \( \gamma^{p,q,1} = \partial \gamma^{p,q,1} = d_{\partial \phi} \gamma^{p,q,1} \) and so
\[
\theta = d_{\partial \phi} (\gamma^{p-1,q-1} + \alpha) + \gamma^{p-1,0}, \text{ with } \alpha = \alpha^{p,q-2} + \cdots + \alpha^{p-2,0}.
\]
On the other hand, if \( \theta \) is \( d_{\partial \phi} \)-exact, then there exists \( u = u^{p-1,q-1} + \cdots + u^{p-2,0} \in L^{p-2}(U) \) s.t.
\[
d_{\partial \phi} u = (d_{\partial \phi} u)^{p,q,q-1} + \cdots + (d_{\partial \phi} u)^{p-1,0} = \theta = \theta^{p,q,q-1} + \cdots + \theta^{p-1,0}.
\]
Therefore \( \partial d^{p-2,0} = \theta^{p-1,0} = \partial \alpha^{p-2,0} + \gamma^{p-1,0} \Rightarrow \gamma^{p-1,0} = \partial(u^{p-2,0} - \alpha^{p-2,0}) \) and \( u^{p-2,0} - \alpha^{p-2,0} \) is \( \partial \partial_{\theta} \)-closed. By Lemma 3.1 3.ii), we see that \( \gamma^{p-1,0} \in \partial \Omega^{p-2}(U) \).
We thus have
\[
\mathcal{H}^{p-1}(\mathcal{L}^{\bullet}_{\phi}) = \frac{\text{Im } d_{\partial \phi} - \Omega^{p-1}}{\text{Im } d_{\partial \phi} + \partial \Omega^{p-2}} = \frac{\Omega^{p-1}}{\partial \Omega^{p-2}} = \mathcal{H}^{p-1}(\mathcal{L}_{\phi}^{\bullet}).
\]
For \( k = p - 1 < q - 1 \), if \( \theta = \theta^{p-1,p-1} + \cdots \theta^{p-1,0} \in L^{p-1}(U) \) is \( d_{\partial \phi} \)-closed, by Lemma 3.1 4., we can write
\[
\theta^{p-1,p-1} = \gamma^{0,1-p} + \partial_{\phi} \alpha^{0,2-p}, \quad \ldots, \theta^{p-1,0} = \partial \alpha^{p-2,0} + \gamma^{p-1,0},
\]
where \( \gamma^{0,1-p} \) is \( \partial \)-closed and \( \gamma^{p-1,0} \) is \( \partial_{\theta} \)-closed. Note that since \( k = p - 1 < q - 1 \), we have \( d_{\partial \phi} \theta = 0 \Rightarrow \partial_{\theta} \theta^{p-1,p} = (d_{\partial \phi} \theta)^{p-1,0} = 0 \Rightarrow \gamma^{0,1-p} \in \partial_{\phi} \Omega^{p-2} \) by Lemma 3.1 2.iii). Hence \( \gamma^{0,1-p} \in \text{Im } d_{\partial \phi}^{p-2} \). On the other hand, if \( \theta \) is \( d_{\partial \phi} \)-exact, then one can show as above that \( \gamma^{p-1,0} \in \partial \Omega^{p-2}(U) \). We thus have
\[
\mathcal{H}^{p-1}(\mathcal{L}^{\bullet}_{\phi}) = \frac{\text{Im } d_{\partial \phi} - \Omega^{p-1}}{\text{Im } d_{\partial \phi} + \partial \Omega^{p-2}} = \frac{\Omega^{p-1}}{\partial \Omega^{p-2} + \partial_{\phi} \Omega^{p-2}} = \mathcal{H}^{p-1}(\mathcal{L}_{\phi}^{\bullet}).
\]
For \( k = p - 1 = q - 1 \), we have
\[
\mathcal{H}^{p-1}(\mathcal{L}^{\bullet}_{\phi}) = \frac{\text{Im } d_{\partial \phi} - \Omega^{p-1} + \Omega^{p-1}}{\text{Im } d_{\partial \phi} + \partial \Omega^{p-2} + \partial_{\phi} \Omega^{p-2}} = \frac{\Omega^{p-1} + \Omega^{p-1}}{\partial \Omega^{p-2} + \partial_{\phi} \Omega^{p-2}} = \mathcal{H}^{p-1}(\mathcal{L}^{\bullet}_{\phi}).
\]
Consider the complex $R_\phi^\bullet$ which is a modification of $\mathcal{S}_\phi^\bullet$ given by

$$
R_\phi^\bullet : \mathbb{C} \longrightarrow \mathcal{O}_\phi \oplus \tilde{\mathcal{O}} \xrightarrow{\partial \oplus \partial} \Omega^1_\phi \oplus \Omega^1_\phi \xrightarrow{\partial \oplus \partial} \Omega^2_\phi \oplus \Omega^2_\phi \longrightarrow \cdots
$$

$$
\longrightarrow \Omega^{q-1}_\phi \oplus \Omega^{q-1}_\phi \xrightarrow{\partial \oplus \partial} \Omega^q_\phi \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^{p-1}_\phi \longrightarrow 0,
$$

where the first morphism is defined by

$$
\mathbb{C} \longrightarrow \mathcal{O}_\phi \oplus \tilde{\mathcal{O}} : a \mapsto (a, -a).
$$

**Proposition 3.3.** The natural map from $R_\phi^\bullet$ to $\mathcal{S}_\phi^\bullet[1]$, where $R_\phi^0 = \mathcal{O}_\phi \oplus \tilde{\mathcal{O}} \longrightarrow \mathcal{S}_\phi^1[1] = \mathcal{O}_\phi + \tilde{\mathcal{O}} : (a, b) \mapsto a - b$,

induces an isomorphism $H^k(\mathcal{S}_\phi^\bullet[1]) \cong H^k(R_\phi^\bullet), \forall k \geq 0$.

*Proof.* Note that $H^1(R_\phi^\bullet) = \mathbb{C} \oplus \mathbb{C}/\mathbb{C}(1, -1) \longrightarrow \mathbb{C} = H^1(\mathcal{S}_\phi^\bullet[1]) : (a, b) \mapsto a - b$ is an isomorphism. □

It follows that

$$
H_{BC_\phi}^{p,q}(X) \cong \mathbb{H}_p^q(M, \mathcal{S}_\phi^\bullet[1]) \cong \mathbb{H}_p^q(M, \mathcal{S}_\phi^\bullet[1]) \cong \mathbb{H}^{p,q}(M, \mathcal{S}_\phi^\bullet).
$$

Note that (3.2) and Proposition 3.3 is just a slight generalization of the result obtained by Schweitzer. In fact, Proposition 3.3 reduce to [Sch07, Prop. 4.3] when $\phi = 0$.

Similarly, for fixed $p \geq 1$ and $q \geq 1$, we define a sheaf complex $L_\phi^\bullet$ as follows:

$$
\left\{ \begin{array}{ll}
L_\phi^k = \bigoplus_{r+s=k, r<p, s<q} A^{r,s}, & \text{for } k \leq p+q-2, \\
L_\phi^{k-1} = \bigoplus_{r+s=k, r\geq p, s\geq q} A^{r,s}, & \text{for } k \geq p+q.
\end{array} \right.
$$

The differential is given by

$$
\begin{array}{cccccccc}
0 & \longrightarrow & L_\phi^0 & \xrightarrow{\Pi} & L_\phi^1 & \xrightarrow{\Pi} & L_\phi^2 & \longrightarrow & \cdots \\
& & L_\phi^{p+q-3} & \xrightarrow{\Pi} & L_\phi^{p+q-2} & \xrightarrow{\partial} & L_\phi^{p+q-1} & \xrightarrow{d} & L_\phi^{p+q} & \longrightarrow & \cdots.
\end{array}
$$

We have $\mathbb{H}^{p,q-1}(X, L_\phi^\bullet) \cong H^{p+q-1}(L_\phi^\bullet(X)) = H^{p+q}_{BC_\phi}(X)$. The sheaf complex $L_\phi^\bullet$ has the following subcomplex

$$
\mathcal{S}_\phi : \mathcal{O} \oplus \tilde{\mathcal{O}} \xrightarrow{\partial_\phi \oplus \tilde{\partial}} \Omega^1_\phi \oplus \tilde{\Omega}^1_\phi \xrightarrow{\partial_\phi \oplus \tilde{\partial}} \Omega^2_\phi \oplus \tilde{\Omega}^2_\phi \longrightarrow \cdots
$$

$$
\longrightarrow \Omega^{p-1}_\phi \oplus \tilde{\Omega}^{p-1}_\phi \xrightarrow{0 \oplus \tilde{\partial}} \tilde{\Omega}^p_\phi \xrightarrow{\partial} \cdots \xrightarrow{\partial} \tilde{\Omega}^{q-1}_\phi \longrightarrow 0.
$$

This is the case when $p \geq q$, the case $p < q$ is similar. To make our notations clear and simple, we will only write explicitly one of the cases in what follows.
Proposition 3.4. The inclusion \( \mathcal{I}_\phi^* \hookrightarrow \mathcal{L}_\phi^* \) induces an isomorphism \( \mathcal{H}^k(\mathcal{I}_\phi^*) \cong \mathcal{H}^k(\mathcal{L}_\phi^*), \forall k \geq 0 \), and we have

\[
\mathcal{H}^k(\mathcal{I}_\phi^*) \cong \mathcal{H}^k(\mathcal{L}_\phi^*) = \begin{cases} 
\mathbb{C}, & \text{for } k = 0, p > 1, q > 1, \\
\mathcal{O}, & \text{for } k = 0, p = 1, q > 1, \\
\mathcal{O}_\phi, & \text{for } k = 0, p > 1, q = 1, \\
\mathcal{O} \oplus \mathcal{O}_\phi, & \text{for } k = 0, p = 1, q = 1, \\
\Omega^p_\phi / \partial_\phi \Omega^{p-2}, & \text{for } 0 < k = p - 1 \neq q - 1, \\
\Omega^p_\phi / \partial_\phi \Omega^{p-2}, & \text{for } 0 < k = q - 1 \neq p - 1, \\
\Omega^p_\phi / \partial_\phi \Omega^{p-2} \oplus \Omega^p_\phi / \partial_\phi \Omega^{p-2}, & \text{for } 0 < k = p - 1 = q - 1, \\
0, & \text{otherwise}.
\end{cases}
\]

Consider the complex \( B_\phi^* \) which is a modification of \( \mathcal{I}_\phi^* \) given by

\[
B_\phi^*: \mathbb{C} \longrightarrow \mathcal{O} \oplus \mathcal{O}_\phi \rightarrow \Omega^1_\phi \oplus \Omega^2_\phi \longrightarrow \cdots
\]

It follows that

\[
\mathcal{H}^k(\mathcal{I}_\phi^*[1]) \cong \mathcal{H}^k(\mathcal{B}_\phi^*), \forall k \geq 0.
\]

Proposition 3.5. The natural map from \( B_\phi^* \) to \( \mathcal{O}_\phi^*[1] \), where

\[
B_\phi^1 = \mathcal{O}_\phi \oplus \mathcal{O} \longrightarrow \mathcal{O}_\phi^*[1] = \mathcal{O}_\phi + \mathcal{O} : (a, b) \mapsto a - b,
\]

induces an isomorphism \( \mathcal{H}^k(\mathcal{I}_\phi^*[1]) \cong \mathcal{H}^k(\mathcal{B}_\phi^*), \forall k \geq 0. \)

It follows that

\[
H^p_{BC\phi}(X) \cong H^p_{BC}(M, \mathcal{L}_\phi^*[1]) \cong H^p_{BC}(M, \mathcal{I}_\phi^*[1]) \cong H^p_{BC}(M, \mathcal{B}_\phi^*).
\]

Remark 3.6. There are natural isomorphisms

\[
H^0_{BC\phi}(X) \cong H^0_{BC}(X_t), \quad H^0_{BC\phi}(X) \cong H^0_{BC}(X_t) : \sigma \mapsto e^{i\phi} \sigma, \quad H^0_{BC\phi}(X) \cong H^0_{BC}(X_t) : \sigma \mapsto e^{i\phi} \sigma,
\]

and note also that

\[
H^0_{BC\phi}(X) = H^0_{BC}(X), \quad H^0_{BC\phi}(X) = H^0_{BC}(X).
\]

3.1. The Bott-Chern cohomology on \( X_t \). Let \( X \) be a complex manifold and \( X_t \) a small deformation (of \( X \)) whose complex structure is represented by a Beltrami differential \( \phi \in A^{0,1}(X, T^{1,0}_X) \), then by [Xia19a, Th. 4.3] or [RZ18, Prop. 2.13] we know that there are isomorphism of sheaves

\[
e^{i\phi} : \Omega^p_{X_t} \longrightarrow \Omega^p_{X_t}, \quad p = 0, 1, 2, \ldots, n,
\]

which give rise to the following commutative diagram

\[
\begin{array}{ccc}
C^* : \mathbb{C} \longrightarrow (\mathcal{O}_\phi \oplus \mathcal{O}_\phi) & \longrightarrow (\Omega^1_\phi \oplus \Omega^1_\phi) & \longrightarrow \cdots \\
\downarrow \text{id} & \downarrow e^{i\phi} \oplus e^{i\phi} & \\
B^*_X : \mathbb{C} \longrightarrow (\mathcal{O}_X \oplus \mathcal{O}_X) & \longrightarrow (\Omega^1_X \oplus \Omega^1_X) & \longrightarrow \cdots
\end{array}
\]

The differential is given by
\[
\begin{array}{ccccccccc}
\cdots & \Omega^2_{\phi} & \oplus & \Omega^1_{\phi} & \partial_{\phi=0} & \Omega^0_{\phi} & \cdots & \partial & \Omega^p_{\phi} & \cdots \rightarrow 0 \\
\downarrow & e^{\phi} & & e^{\phi} & & e^{\phi} & & e^{\phi} & & e^{\phi} \\
\cdots & \Omega^2_{X_t} & \oplus & \Omega^1_{X_t} & \partial_{t=0} & \Omega^0_{X_t} & \cdots & \partial & \Omega^p_{X_t} & \cdots \rightarrow 0.
\end{array}
\]

We see that
\[H^{p,q}_{BC}(X_t) = H^{p+q-1}(L_{X_t}^\bullet (M)) \cong \mathbb{H}^{p+q-1}(M, \mathcal{L}_{X_t}^\bullet) \cong \mathbb{H}^{p+q}(M, \mathcal{O}_{X_t}^\bullet) \cong \mathbb{H}^{p+q}(M, \mathcal{O}^\bullet),\]
where \(M\) is the underlying smooth manifold of \(X\) and \(X_t\).

3.2. The case of Aeppli cohomology. The deformed Aeppli cohomology can be defined as follows:

\[(3.5) \quad H^{p,q}_{A\phi}(X) := \frac{\ker \partial_{\phi} \cap A^{p,q}(X)}{\text{Im} \, d_{\phi} \cap A^{p,q}(X)}, \quad H^{p,q}_{A\phi}(X) := \frac{\ker \partial_{\phi} \cap A^{p,q}(X)}{\text{Im} \, d_{\phi} \cap A^{p,q}(X)}, \quad \forall p, q \geq 0,
\]

and \(h^{p,q}_{A\phi} := \dim H^{p,q}_{A\phi}(X)\). The conjugation gives a natural isomorphism between \(H^{p,q}_{A\phi}(X)\) and \(H^{q,p}_{A\phi}(X)\), we thus have \(h^{p,q}_{A\phi} = h^{q,p}_{A\phi}\).

For fixed \(p \geq 0\) and \(q \geq 0\), similar to the constructions for the Bott-Chern cohomology we define a sheaf complex which still denoted by \(\mathcal{L}^\bullet_{\phi}\) as follows:

\[(3.6) \quad \begin{cases} \mathcal{L}^k_{\phi} = \bigoplus_{r+s=k, r < p+1, s < q+1} A^r,s, & \text{for } k \leq p+q, \\ \mathcal{L}^{k-1}_{\phi} = \bigoplus_{r+s=k, r \geq p+1, s \geq q+1} A^r,s, & \text{for } k \geq p+q+2. \end{cases}
\]

The differential is given by
\[
0 \rightarrow \mathcal{L}^0_{\phi} \xrightarrow{\Pi_{\mathcal{L}^1_{\phi}} d_{\phi}} \mathcal{L}^1_{\phi} \xrightarrow{\Pi_{\mathcal{L}^2_{\phi}} d_{\phi}} \mathcal{L}^2_{\phi} \cdots
\]

\[
\xrightarrow{\Pi_{\mathcal{L}^{p+q-1}_{\phi}} d_{\phi}} \mathcal{L}^{p+q}_{\phi} \xrightarrow{\partial_{\phi}} \mathcal{L}^{p+q+1}_{\phi} \xrightarrow{d_{\phi}} \mathcal{L}^{p+q+2}_{\phi} \cdots,
\]

In particular, we find that
\[
\mathcal{L}^{p+q-1}_{\phi} = A^{p,q-1} \oplus A^{p-1,q} \xrightarrow{\Pi_{\mathcal{L}^{p+q}_{\phi}} d_{\phi}} A^{p,q} \xrightarrow{\partial_{\phi}} \mathcal{L}^{p+q+1}_{\phi} = A^{p+1,q+1},
\]

and so \(H^{p+q}(X, \mathcal{L}^\bullet_{\phi}) \cong H^{p+q}(\mathcal{L}^\bullet_{\phi}(X)) = H^{p,q}_{A\phi}(X)\). The other hypercohomology interpretations of the deformed Bott-Chern cohomology holds similarly for the deformed Aeppli cohomology. The Hodge star operator induces the following duality between the deformed Bott-Chern cohomology and the deformed Aeppli cohomology [Sch07, pp. 10]:

\[(3.7) \quad H^{p,q}_{BC}(X) \cong H^{n-q,n-p}_{A\phi}(X), \quad \text{and} \quad H^{p,q}_{BC}(X) \cong H^{n-q,n-p}_{A\phi}(X).
\]
4. Deformations of Bott-Chern classes

Let $\pi : (X, X) \to (B, 0)$ be a small deformation of a compact complex manifold $X$ such that for each $t \in B$ the complex structure on $X_t$ is represented by Beltrami differential $\phi(t)$. In this section, power series will always be written in homogenous form, e.g. we write $\phi(t) = \sum_k \phi_k$ where each $\phi_k$ is a homogeneous polynomial of degree $k$ with coefficients in $A^{0,1}(X, T^{1,0})$. The Bott-Chern Laplacian operator is defined as

$$\Box_{BC} := (\partial \bar{\partial})(\partial \bar{\partial})^* + (\partial \bar{\partial})(\partial \bar{\partial}) + (\bar{\partial}^* \partial)(\bar{\partial}^* \partial)^* + (\bar{\partial}^* \partial)(\bar{\partial}^* \partial)^* + \bar{\partial}^* \bar{\partial} + \partial^* \partial,$$

and the deformed Bott-Chern Laplacian operator is defined as

$$\Box_{BC,\phi} := (\partial \bar{\partial}_{\phi})(\partial \bar{\partial}_{\phi})^* + (\partial \bar{\partial}_{\phi})(\partial \bar{\partial}_{\phi}) + (\bar{\partial}^* \partial_{\phi})(\bar{\partial}^* \partial_{\phi})^* + (\bar{\partial}^* \partial_{\phi})(\bar{\partial}^* \partial_{\phi})^* + \bar{\partial}^* \bar{\partial}_{\phi} + \partial^* \partial_{\phi},$$

where $\phi = \phi(t)$. Both $\Box_{BC}$ and $\Box_{BC,\phi}$ are 4-th order self-adjoint elliptic differential operator [Sch07, MK06]. We have

$$\mathcal{H}_{BC} := \ker \Box_{BC} = \ker \partial \cap \ker \bar{\partial} \cap \ker (\partial \bar{\partial})^*$$

and the following orthogonal direct sum decomposition holds:

$$A^{\bullet, \bullet}(X) = \ker \Box_{BC} \oplus \operatorname{Im} \partial \bar{\partial} \oplus (\operatorname{Im} \partial^* + \operatorname{Im} \bar{\partial}^*),$$

which is equivalent to the existence of the Green operator $G_{BC}$ such that

$$1 = \mathcal{H}_{BC} + \Box_{BC} G_{BC}.$$

The same is true for the deformed Bott-Chern Laplacian operator $\Box_{BC,\phi}$. It follows from (4.4) that

$$\ker (\partial \bar{\partial})^* = \mathcal{H}_{BC} \oplus (\operatorname{Im} \partial^* + \operatorname{Im} \bar{\partial}^*) \quad \text{and} \quad \ker d = \mathcal{H}_{BC} \oplus \operatorname{Im} \partial \bar{\partial}.$$

The Aeppli Laplacian operator is defined as

$$\Box_{A} := (\partial \bar{\partial})(\partial \bar{\partial}) + (\partial \bar{\partial})(\partial \bar{\partial})^* + (\bar{\partial} \partial^*)(\bar{\partial} \partial^*)^* + (\bar{\partial} \partial^*)(\bar{\partial} \partial^*) + \bar{\partial}^* \bar{\partial} + \partial^* \partial,$$

and we have correspondingly

$$A^{\bullet, \bullet}(X) = \ker \Box_{A} \oplus \operatorname{Im} (\partial \bar{\partial})^* \oplus (\operatorname{Im} \partial + \operatorname{Im} \bar{\partial}),$$

or $1 = \mathcal{H}_{A} + \Box_{A} G_{A}$ where $G_{A}$ is the Green operator for $\Box_{A}$. Since for any $x \in A^{\bullet, \bullet}(X)$, we have $\Box_{BC} G_{BC} \partial \bar{\partial} x = \partial \bar{\partial} x$ and $\Box_{BC} \partial \bar{\partial} G_{A} x = \partial \bar{\partial} x$ which implies

$$G_{BC} \partial \bar{\partial} = \partial \bar{\partial} G_{A}.$$

Similarly, we have

$$(\partial \bar{\partial})^* G_{BC} = G_{A}(\partial \bar{\partial})^*.$$

Let $\varphi \in A^{p,q}(X)$ and $G_{BC} : A^{p,q}(X) \to A^{p,q}(X)$ be the Green operator, then for $k \geq 2$ we have

$$\|G_{BC} \varphi\|_{k+\alpha} \leq C \|\varphi\|_{k-4+\alpha},$$

where $C > 0$ is independent of $\varphi$ and $\| \cdot \|_{k+\alpha}$ is the Hölder norm.

We have the following observation:
Proposition 4.1. 1. \( \forall \sigma \in A^{p,q}(X) \), if \( d_{\phi(t)}\sigma = d\sigma - L_{\phi(t)}^{1,0} \sigma = 0 \) and \( (\partial\bar{\partial})^* \sigma = 0 \), then we must have

\[
\sigma = H_{BC}\sigma - G_{BC}A\bar{\partial}_{\phi(t)}\sigma,
\]

where \( H_{BC} : A^{p,q}(X) \to H_{BC}^{p,q}(X) \) is the projection operator to harmonic space and \( A := \bar{\partial}^*\partial^* + \bar{\partial}^* \).

2. For any fixed \( \sigma_0 \in H_{BC}^{p,q}(X) \), the equation

\[
(4.11) \quad \sigma = \sigma_0 - G_{BC}A\bar{\partial}_{\phi(t)}\sigma,
\]

has an unique solution given by \( \sigma = \sum_k \sigma_k \in A^{p,q}(X) \) and \( \sigma_k = -G_{BC}A\sum_{i+j=k} \partial_i\phi_j\sigma_i \) for \( |t| \) small where each \( \sigma_k \) is a homogeneous polynomial of degree \( k \) with coefficients in \( A^{p,q}(X) \).

Proof. The first assertion follows from the Hodge decomposition:

\[
\sigma = H_{BC}\sigma + G_{BC}\Box_{BC}\sigma = H_{BC}\sigma + G_{BC}A\sum_{i+j=0} \partial_i\phi_j\sigma_i = H_{BC}\sigma - G_{BC}A\bar{\partial}_{\phi(t)}\sigma,
\]

where we have used the fact that \( d_{\phi(t)}\sigma = 0 \) if and only if \( \partial\sigma = \partial\bar{\partial}\sigma \).

For the second assertion, substitute \( \sigma = \sigma(t) = \sum_k \) in (4.11), we have

\[
(4.12) \quad \left\{
\begin{array}{l}
\sigma_1 = -G_{BC}A\bar{\partial}_{\phi_1}\sigma_0, \\
\sigma_2 = -G_{BC}A(\partial_{\phi_2}\sigma_0 + \partial_{\phi_1}\sigma_1), \\
\vdots \\
\sigma_k = -G_{BC}A\sum_{i+j=k} \partial_i\phi_j\sigma_i, \quad \forall k > 0.
\end{array}
\right.
\]

For the convergence of \( \sigma(t) \), we note that

\[
(4.13) \quad \|\sigma_j\|_{k+\alpha} = \|G_{BC}A\sum_{a+b=j} \partial_{i\phi_k}\sigma_a\|_{k+\alpha} \leq C \sum_{a+b=j} \|\phi_a\|_{k+\alpha} \|\sigma_b\|_{k+\alpha},
\]

for some constant \( C \) depends only on \( k \) and \( \alpha \). Now it is left to show the uniqueness.

Let \( \sigma \) and \( \sigma' \) be two solutions to \( \sigma = \sigma_0 - G_{BC}A\bar{\partial}_{\phi}\sigma \) and set \( \tau = \sigma - \sigma' \). Then

\[
(4.14) \quad \|\tau\|_{k+\alpha} \leq C\|\phi(t)\|_{k+\alpha}\|\tau\|_{k+\alpha},
\]

for some constant \( c > 0 \). When \( |t| \) is sufficiently small, \( \|\phi(t)\|_{k+\alpha} \) is also small. Hence we must have \( \tau = 0 \). For smoothness of the solution, note that we have

\[
(4.15) \quad \Box_{BC}\sigma = -(\bar{\partial}^*\partial^* + \bar{\partial}^*)\bar{\partial}_{\phi(t)}\sigma = 0,
\]

which implies

\[
\Box_{BC}\sigma + (\bar{\partial}^*\partial^* + \bar{\partial}^*)\bar{\partial}_{\phi(t)}\sigma = 0,
\]

which is a standard elliptic equation for small \( t \).

Note that the solution \( \sigma \) of (4.11) automatically satisfies \( (\partial\bar{\partial})^* \sigma = 0 \) in view of (4.9).

Lemma 4.2. The natural map

\[
(4.16) \quad \frac{\ker(\partial\bar{\partial})^* \cap \ker d_{\phi(t)} \cap A^{p,q}(X)}{\ker(\partial\bar{\partial})^* \cap \ker d_{\phi(t)} \cap A^{p,q}(X)} \to H_{BC}^{p,q}(X)
\]

is an isomorphism.
Moreover, we have the following orthogonal direct sum decomposition

\[ A^{p,q}(X) = (\ker d_{\phi(t)} \cap \ker (\partial \bar{\partial})^*) \oplus (\text{Im } \partial^* + \text{Im } \bar{\partial}^* + \text{Im } \partial \bar{\partial}) \]

which implies

\[ \ker d_{\phi(t)} \cap A^{p,q}(X) = (\ker d_{\phi(t)} \cap \ker (\partial \bar{\partial})^*) \oplus \left( \ker d_{\phi(t)} \cap (\text{Im } \partial^* + \text{Im } \bar{\partial}^* + \text{Im } \partial \bar{\partial}) \right), \]

and

\[ \text{Im } \partial \bar{\partial}_{\phi(t)} \cap A^{p,q}(X) = (\text{Im } \partial \bar{\partial}_{\phi(t)} \cap \ker (\partial \bar{\partial})^*) \oplus (\text{Im } \partial \bar{\partial}_{\phi(t)} \cap (\ker (\partial \bar{\partial}_{\phi(t)})^* + \text{Im } \partial \bar{\partial})). \]

Moreover, for any \( x \in \text{Im } \partial \bar{\partial} \), there exists unique \( y \in \ker d_{\phi(t)} \) and unique \( z \in (\text{Im } \partial^* + \text{Im } \bar{\partial}^*) \) such that \( x = y + z \). This defines a surjective homomorphism

\[ \text{Im } \partial \bar{\partial} \longrightarrow \ker d_{\phi(t)} \cap (\text{Im } \partial^* + \text{Im } \bar{\partial}^*), \]

with kernel equal to \( \text{Im } \partial \bar{\partial} \cap (\text{Im } \partial^* + \text{Im } \bar{\partial}^*) \). It follows that

\[ \ker d_{\phi(t)} \cap (\text{Im } \partial^* + \text{Im } \bar{\partial}^* + \text{Im } \partial \bar{\partial}) \cong \frac{\text{Im } \partial \bar{\partial}}{\text{Im } \partial \bar{\partial} \cap (\text{Im } \partial^* + \text{Im } \bar{\partial}^*)}. \]

Similarly, we have

\[ \text{Im } \partial \bar{\partial}_{\phi(t)} \cap (\ker (\partial \bar{\partial}_{\phi(t)})^* + \text{Im } \partial \bar{\partial}) \cong \frac{\text{Im } \partial \bar{\partial}}{\text{Im } \partial \bar{\partial} \cap \ker (\partial \bar{\partial}_{\phi(t)})^*}. \]

Hence,

\[ H^{p,q}_{BC\phi(t)}(X) \cong \frac{\ker (\partial \bar{\partial})^* \cap \ker d_{\phi(t)}}{\ker (\partial \bar{\partial})^* \cap \text{Im } \partial \bar{\partial}_{\phi(t)}} \oplus \frac{\text{Im } \partial \bar{\partial} \cap (\text{Im } \partial^* + \text{Im } \bar{\partial}^*)}{\text{Im } \partial \bar{\partial} \cap (\text{Im } \partial^* + \text{Im } \bar{\partial}^*)}. \]

We claim \( \text{Im } \partial \bar{\partial} \cap \ker (\partial \bar{\partial}_{\phi(t)})^* = 0 \). Indeed, let \( \sigma \in \ker \partial \bar{\partial} \cap \ker (\partial \bar{\partial}_{\phi(t)})^* \), then it follows from the same proof of Proposition 4.1 that \( \sigma \) is the solution of the equation

\[ \sigma = \sigma_0 + G_{BC} \partial \bar{\partial}(\partial L_{\phi(t)}^{1,0})^* \sigma, \quad \sigma_0 := \mathcal{H}_{BC} \sigma \]

which is uniquely determined by \( \sigma_0 \). If \( \sigma \in \text{Im } \partial \bar{\partial} \cap \ker (\partial \bar{\partial}_{\phi(t)})^* \), then \( \sigma_0 = \mathcal{H}_{BC} \sigma = 0 \Rightarrow \sigma = 0. \)

\[ \square \]

**Proposition 4.3.** 1. For any fixed \( t \in B \), the following homomorphism

\[ g_t : \ker \partial \bar{\partial} \cap A^{p,q}(X) \longrightarrow \ker (\partial \bar{\partial})^* \cap \text{Im } \partial \bar{\partial}_{\phi(t)} \cap A^{p+1,q+1}(X) : x_0 \longmapsto \partial \bar{\partial}_{\phi(t)} x(t), \]

is surjective with \( \ker g_t = \ker \partial \bar{\partial} \cap (\ker \partial \bar{\partial}_{\phi(t)} + \text{Im } (\partial \bar{\partial})^*) \cap A^{p,q}(X) \), where \( x(t) \) is the unique solution of \( x(t) = x_0 + (\partial \bar{\partial})^* G_{BC} \partial i_{\phi(t)} \partial x(t) \).

2. Let \( \hat{g}_t : H^{p,q}_{BC}(X) \longrightarrow \ker (\partial \bar{\partial})^* \cap \text{Im } \partial \bar{\partial}_{\phi(t)} \cap A^{p+1,q+1}(X) \) be the restriction of \( g_t \) on \( H^{p,q}_{BC}(X) \), then \( \hat{g}_t \) is surjective with \( \ker \hat{g}_t = H^{p,q}_{BC}(X) \cap (\ker \partial \bar{\partial}_{\phi(t)} + \text{Im } (\partial \bar{\partial})^*). \)

Moreover, we have

\[ \dim H^{p,q}_{BC}(X) = \dim \ker \partial \bar{\partial}_{\phi(t)} \cap (H^{p,q}_{BC}(X) + \text{Im } (\partial \bar{\partial})^*) \cap A^{p,q}(X) \]

\[ + \dim \ker (\partial \bar{\partial})^* \cap \text{Im } \partial \bar{\partial}_{\phi(t)} \cap A^{p+1,q+1}(X). \]
Proof. 1. Let \( x \in A^{p,q}(X) \), then by Hodge decomposition we have
\[
\partial \bar{\partial}_{\phi(t)} x = \partial \bar{\partial} x - \partial i_{\phi(t)} \partial x = \partial \bar{\partial} x - \mathcal{H}_{BC} \partial i_{\phi(t)} \partial x - \Box_{BC} G_{BC} \partial i_{\phi(t)} \partial x,
\]
thus
\[
\partial \bar{\partial}_{\phi(t)} x \in \ker(\partial \bar{\partial})^* \iff \partial \bar{\partial} x - \partial \bar{\partial}(\partial \bar{\partial})^* G_{BC} \partial i_{\phi(t)} \partial x = 0.
\]
Set \( x_0 = x - (\partial \bar{\partial})^* G_{BC} \partial i_{\phi(t)} \partial x \), then \( x \) is a solution to the equation \( x = x_0 + (\partial \bar{\partial})^* G_{BC} \partial i_{\phi(t)} \partial x \) which is uniquely determined by \( x_0 \) in view of the proof of Proposition 4.1.

It is left to show \( \ker g_t = \ker \partial \bar{\partial} \cap (\ker \partial \bar{\partial}_{\phi(t)} + \text{Im } (\partial \bar{\partial})^*) \cap A^{p,q}(X) \). In fact, obviously we have \( \ker g_t \subseteq \ker \partial \bar{\partial} \cap (\ker \partial \bar{\partial}_{\phi(t)} + \text{Im } (\partial \bar{\partial})^*) \cap A^{p,q}(X) \). Conversely, let us consider the following surjective homomorphism
\[
\ker \partial \bar{\partial}_{\phi(t)} \longrightarrow \ker \partial \bar{\partial} \cap (\ker \partial \bar{\partial}_{\phi(t)} + \text{Im } (\partial \bar{\partial})^*)
\]
\[
x \mapsto x - (\partial \bar{\partial})^* \partial \bar{\partial} G_{BC} x = x - (\partial \bar{\partial})^* G_{BC} \partial \bar{\partial} x = x - (\partial \bar{\partial})^* G_{BC} \partial i_{\phi(t)} \partial x,
\]
whose kernel is \( \ker \partial \bar{\partial}_{\phi(t)} \cap \ker (\partial \bar{\partial})^* = 0 \) by Proposition 4.1. Its inverse is given by
\[
\ker \partial \bar{\partial} \cap (\ker \partial \bar{\partial}_{\phi(t)} + \text{Im } (\partial \bar{\partial})^*) \longrightarrow \ker \partial \bar{\partial}_{\phi(t)} : x_0 \mapsto x(t),
\]
where \( x(t) \) is the unique solution of \( x(t) = x_0 + (\partial \bar{\partial})^* G_{BC} \partial i_{\phi(t)} \partial x(t) \). So let \( x_0 \in \ker \partial \bar{\partial} \cap (\ker \partial \bar{\partial}_{\phi(t)} + \text{Im } (\partial \bar{\partial})^*) \) and \( x(t) \) be the unique solution of \( x(t) = x_0 + (\partial \bar{\partial})^* G_{BC} \partial i_{\phi(t)} \partial x(t) \), we must have \( x(t) \in \ker \partial \bar{\partial}_{\phi(t)} \Rightarrow x_0 \in \ker g_t \).

2. It can be proved in the same way that \( \ker g_t = \mathcal{H}_{BC}^{p,q}(X) \cap (\ker \partial \bar{\partial}_{\phi(t)} + \text{Im } (\partial \bar{\partial})^*) \). To show \( g_t \) is surjective it is enough to show
\[
\frac{\mathcal{H}_{BC}^{p,q}(X)}{\mathcal{H}_{BC}(X) \cap (\ker \partial \bar{\partial}_{\phi(t)} + \text{Im } (\partial \bar{\partial})^*)} \cong \frac{\ker \partial \bar{\partial} \cap A^{p,q}(X)}{\ker \partial \bar{\partial} \cap (\ker \partial \bar{\partial}_{\phi(t)} + \text{Im } (\partial \bar{\partial})^*) \cap A^{p,q}(X)}.
\]
Indeed, we have
\[
\ker \partial \bar{\partial} = \mathcal{H}_{BC}^{p,q}(X) \oplus \{ \ker \partial \bar{\partial} \cap \text{Im } (\partial \bar{\partial}) + \text{Im } \partial^* + \text{Im } \bar{\partial}^* \},
\]
and
\[
\ker \partial \bar{\partial} \cap (\ker \partial \bar{\partial}_{\phi(t)} + \text{Im } (\partial \bar{\partial})^*) \cong \ker \partial \bar{\partial}_{\phi(t)}
\]
\[
= \{ \ker \partial \bar{\partial}_{\phi(t)} \cap (\mathcal{H}_{BC}^{p,q}(X) + \text{Im } (\partial \bar{\partial})^*) \} \oplus \{ \ker \partial \bar{\partial}_{\phi(t)} \cap \text{Im } (\partial \bar{\partial}_{\phi(t)})^* + \ker \partial \bar{\partial} \cap (\text{Im } \partial \bar{\partial} + \text{Im } \partial^* + \text{Im } \bar{\partial}^*) \}
\]
\[
\cong \{ \mathcal{H}_{BC}^{p,q}(X) \cap (\ker \partial \bar{\partial}_{\phi(t)} + \text{Im } (\partial \bar{\partial})^*) \} \oplus \{ \ker \partial \bar{\partial} \cap (\text{Im } \partial \bar{\partial}) + \text{Im } \partial^* + \text{Im } \bar{\partial}^* \}.
\]

\[\square\]

Remark 4.4. It can be proved in a similar way that \( g_t \) when restricted on \( \mathcal{H}_{BC}^{p,q}(X) \) is also surjective with kernel equal to \( \mathcal{H}_{BC}^{p,q}(X) \cap (\ker \partial \bar{\partial}_{\phi(t)} + \text{Im } (\partial \bar{\partial})^*) \) and
\[
\dim \mathcal{H}_{BC}^{p,q}(X) = \dim \ker \partial \bar{\partial}_{\phi(t)} \cap (\mathcal{H}_{BC}^{p,q}(X) + \text{Im } (\partial \bar{\partial})^*) \cap A^{p,q}(X)
\]
\[+\dim \ker(\partial \bar{\partial})^* \cap \text{Im } \partial \bar{\partial}_{\phi(t)} \cap A^{p+1,q+1}(X).
\]
Definition 4.5. For any $t \in B$ and a vector subspace $V = \mathbb{C}\{\sigma_0^1, \cdots, \sigma_0^N\} \subseteq \mathcal{H}_{BC}^{p,q}(X)$, we set

$$V_t := \{ \sum_{l=1}^N a_l \sigma_l^0 \in V \mid (a_1, \cdots, a_N) \in \mathbb{C}^N \text{ s.t. } \sigma(t) \in \ker d_{\phi(t)} \},$$

where $\sigma(t) = \sum_k \sigma_k$ with $\sigma_0 = \sum_l a_l \sigma_l^0$ and $\sigma_k = -G_{BC} A \sum_{i+j=k} \partial i \phi_j \sigma_i$, $\forall k \neq 0$.

Note that $V_t$ consists of those vectors of the form $\sum_l a_l \sigma_l^0$ such that the coefficients $a_l$ satisfy the following linear equation:

$$\sum_{l=1}^N a_l d_{\phi(t)} \sigma_l^0(t) = 0,$$

where $\sigma_l(t) = \sum_k \sigma_k^l$ with $\sigma_k^l = -G_{BC} A \sum_{i+j=k} \partial i \phi_j \sigma_i$, $\forall k \neq 0$.

Definition 4.6. We set

$$f_t : V_t \longrightarrow \frac{\ker(\overline{\partial\partial})^* \cap \ker d_{\phi(t)} \cap A^{p,q}(X)}{\ker(\overline{\partial\partial})^* \cap \text{Im } \partial i \phi(t) \cap A^{p,q}(X)} \cong H_{BC}^{p,q}(X),$$

$$\sigma_0 \mapsto \sigma(t) = \sum_k \sigma_k, \text{ where } \sigma_k = -G_{BC} A \sum_{i+j=k} \partial i \phi_j \sigma_i, \forall k \neq 0.$$

Proposition 4.7. If $V = \mathcal{H}_{BC}^{p,q}(X)$, then $f_t$ is surjective.

Proof. By Proposition 4.1, the map

$$\tilde{f}_t : V_t \longrightarrow \ker(\overline{\partial\partial})^* \cap \ker d_{\phi(t)} \cap A^{p,q}(X),$$

$$\sigma_0 \mapsto \sigma(t) = \sum_k \sigma_k, \text{ where } \sigma_k = -G_{BC} A \sum_{i+j=k} \partial i \phi_j \sigma_i, \forall k \neq 0,$$

is an isomorphism.

Theorem 4.8. Let $X$ be a compact complex manifold and $\pi : (X, X) \rightarrow (B, 0)$ a small deformation of $X$ such that for each $t \in B$ the complex structure on $X_t$ is represented by Beltrami differential $\phi(t)$. For any $p, q \geq 0$, let $V = \mathbb{C}\{\sigma_0^1, \cdots, \sigma_0^N\}$ be a linear subspace of $\mathcal{H}_{BC}^{p,q}(X)$ and $\sigma_l(t) = \tilde{f}_t \sigma_0^l, l = 1, \cdots, N$. Define a subset $B(V)$ of $B$ by

$$B(V) := \{ t \in B \mid d_{\phi(t)} \sigma_l(t) = 0, l = 1, \cdots, N \},$$

Then $B(V)$ are analytic subsets of $B$ and we have

$$B(V) = \{ t \in B \mid \dim V = \dim \text{Im } f_t + \dim \ker f_t \}.$$

In particular, we have

$$B' = B(\mathcal{H}_{BC}^{p,q}(X)) = \{ t \in B \mid \dim H_{BC}^{p,q}(X) = \dim H_{BC}^{p,q}(X) + \dim \ker f_t \}.$$

Proof. First, let $\{ U_\alpha \}$ be a finite open cover of $X$ and $u_1^\alpha, u_2^\alpha, \cdots$, a local unitary frames of $p + q + 1$-forms on the $U_\alpha$, then $\forall \alpha = 1, \cdots, N$, we have

$$d_{\phi(t)} \sigma_l^\alpha(t) = 0 \iff d_{\phi(t)} \sigma_l^\alpha(t) |_{U_\alpha}, u_j^\alpha > = 0, \forall j, \alpha,$$
where \(<\cdot,\cdot>\) is the \(L^2\)-inner product on the space \(A^{p+q+1}(U_α)\). We see that each \(a_j^α(t)\) is holomorphic in \(t\) and so

\[B(V) = \{t \in B \mid a_j^α(t) = 0, \forall j, t, α\}\]

is an analytic subset of \(B\).

Furthermore, note that

\[t \in B(V) \iff V_t = V.\]

So (4.17) follows from the fact that \(\dim V_t = \dim \text{Im } f_t + \dim \ker f_t\). If \(V = H^{0,q}(X,E)\), then \(f_t : V_t \to H^{0,q}_{\partial φ(t)}(X,E)\) is surjective by Proposition 4.7 and (4.18) follows. □

**Remark 4.9.** From the above proof, we can see that \(V_t \subseteq V\) varies holomorphically with \(t\). In fact, in the notations of Definition 4.5, \(V_t\) consists of those vectors of the form \(\sum a_t^α \sigma_0^α\) such that the coefficients \(a_t^α\) satisfy

\[\sum_{l=1}^{N} a_l \cdot a_j^α(t) = 0, \quad j = 1, \ldots, m; \forall α ,\]

where \(a_j^α(t)\) are holomorphic functions in \(t\). In particular, \(\{t \in B \mid \dim V_t \geq k\}\) is an analytic subset of \(B\) for any nonnegative integer \(k\).

For the same reason, \(\{t \in B \mid \dim \ker (φ) \cap (H^{p,q}_{BC}(X) + \text{Im } (\partial \bar{∂})^*) \cap A^{p,q}(X) \geq k\}\) is also an analytic subset of \(B\) for any nonnegative integer \(k\). It follows from this and Proposition 4.3 that \(\{t \in B \mid \dim \ker (\partial \bar{∂})^* \cap \text{Im } (\partial \bar{∂}) \cap A^{p,q}(X) \leq k\}\) is an analytic subset of \(B\) for any nonnegative integer \(k\).

**Definition 4.10.** Let \(π : (X, X) \to (B, 0)\) be a deformation of a compact complex manifold \(X\) such that for each \(t \in B\) the complex structure on \(X_t\) is represented by Beltrami differential \(φ(t)\). Given \(y \in \ker d \cap A^{p,q}(X)\) and \(T \subseteq B\), which is a complex subspace of \(B\) containing 0, a (Bott-Chern) deformation of \(y\) (w.r.t. \(π : (X, X) \to (B, 0)\)) on \(T\) is a family of \((p,q)\)-forms \(σ(t)\) such that

1. \(σ(t)\) is holomorphic in \(t \in T\) and \(σ(0) = y\);
2. \(d_{φ(t)}σ(t) = 0, \forall t \in T.\)

A deformation of \([y] \in H^{p,q}_{BC}(X)\) (w.r.t. \(π\)) on \(T\) is a triple \((y, σ(t), T)\) which consisting of a representative \(y \in [y]\) and a deformation \(σ(t)\) of \(y\) (w.r.t. \(π\)) on \(T\). Two deformations \((y, σ(t), T)\) and \((y', σ'(t), T)\) of \([y]\) on \(T\) are equivalent if

\[|σ(t) - σ'(t)| = 0 \in H^{p,q}_{BC}(X), \forall t \in T.\]

A deformation \(σ(t)\) of \(y\) on \(T\) is called canonical if

\[σ(t) = σ_0 - G_{BC}(\bar{∂}^*∂^* + ∂^*)d_{φ(t)}σ(t), \quad ∀t \in T.\]

By Proposition 4.1, canonical deformation is unique on its existence domain.

For a given small deformation \(π : (X, X) \to (B, 0)\) with smooth \(B\), we say \(y \in \ker d \cap A^{p,q}(X)\) is (canonically) unobstructed w.r.t. \(π\) if a (canonical) deformation of \(y\) (w.r.t. \(π\)) exists on \(B\) and a class \(α \in H^{p,q}_{BC}(X)\) is (canonically) unobstructed w.r.t. \(π\) if there is a \(y \in α\) such that \(y\) is (canonically) unobstructed w.r.t. \(π\). If every Bott-Chern classes in \(H^{p,q}_{BC}(X)\) have canonically unobstructed deformation w.r.t. \(π\), then we say the deformations of classes in \(H^{p,q}_{BC}(X)\) is canonically unobstructed w.r.t. \(π\).
\(\pi\). If these holds for any small deformation of \(X\), we will drop the term “w.r.t. \(\pi\)”.

For example, we say \(y \in \ker d \cap A^{p,q}(X)\) is (canonically) unobstructed if for any small deformation \(\pi : (X, X) \to (B, 0)\) with smooth \(B\) there is a (canonical) deformation of \(y\) on \(B\).

Although a Bott-Chern deformation \(\sigma(t)\) of \(y \in \ker d \cap A^{p,q}(X)\) can also be viewed as a Dolbeault deformation with the additional requirement \(\partial\sigma(t) = 0\), the ways we identify deformations in these two cases is very different. We want to point out another difference between the deformation theory of Dolbeault cohomology \cite{Xia19} and that of Bott-Chern cohomology. Let \(\sigma^{BC}(t)\) and \(\sigma^D(t)\) be the canonical Bott-Chern/Dolbeault deformation of \(y \in \ker d \cap A^{p,q}(X)\) respectively, it is known that \(L^{1,0}_{\phi(t)} a^D(t) \in \ker \tilde{\partial}\) for any \(t \in B\), see \cite[Prop. 5.2]{Xia19}. This seems does not hold for the Bott-Chern deformation in general. More precisely, it is not guaranteed that \(L^{1,0}_{\phi(t)} a^{BC}(t) \in \ker d_{\phi(t)}\).

In the remainder of this section, We confine ourselves to sketching the essential points of the deformation theory of Bott-Chern cohomology. Since this part of the theory is very similar to the case of Dolbeault cohomology, the proofs will be omitted.

A notable consequence of the deformation theory for Bott-Chern classes is the following

**Theorem 4.11.** Let \(\pi : (X, X) \to (B, 0)\) be a deformation of a compact complex manifold \(X\) such that for each \(t \in B\) the complex structure on \(X_t\) is represented by Beltrami differential \(\phi(t)\). Then the set \(\{t \in B \mid \dim H^{p,q}_{BC,\phi(t)}(X) \geq k\}\) is an analytic subset of \(B\) for any nonnegative integer \(k\).

**Proof.** It follows from Proposition 4.7 that

\[
\{t \in B \mid \dim H^{p,q}_{BC,\phi(t)}(X) \geq k\} = \{t \in B \mid \dim V_t/\ker f_t \geq k\} = \{t \in B \mid \dim V_t - \dim (\ker \partial \tilde{\partial}^* \cap \Im \partial \tilde{\partial}_{\phi(t)}) \geq k\}.
\]

The conclusion then follows from Remark 4.9. \(\square\)

The canonical deformations has the following properties:

**Theorem 4.12.** Let \(\pi : (X, X) \to (B, 0)\) be a deformation of a compact complex manifold \(X\) such that for each \(t \in B\) the complex structure on \(X_t\) is represented by Beltrami differential \(\phi(t)\).

\((i)\) Assume \(S\) is an analytic subset of \(B\) with \(0 \in S\) and \(y \in \ker d \cap A^{p,q}(X)\). If the canonical deformation of \(y\) exists on \(S\) then we must have \(S \subseteq B(\mathcal{C}H_{BC}\bar{y})\);

\((ii)\) For any deformed Bott-Chern cohomology class \([u] \in H^{p,q}_{BC,\phi(t)}(X)\), there exists \(\sigma_0 \in \mathcal{H}^{p,q}_{BC}(X)\) such that \([u] = [\sigma(t)]\) where \(\sigma(t)\) is the canonical deformation of \(\sigma_0\).

**Proof.** \((i)\) follows from Theorem 4.8 and \((ii)\) follows from Lemma 4.2. \(\square\)

We end this section with the following result which is of particular interests.
Theorem 4.13. Let \( \pi : (X, X) \to (B, 0) \) be a small deformation of the compact complex manifold \( X \) such that for each \( t \in B \) the complex structure on \( X_t \) is represented by Beltrami differential \( \phi(t) \). For each \( (p, q) \in \mathbb{N} \times \mathbb{N} \), set

\[
\nu_{t}^{p,q} := \dim H^{p,q}_{BC}(X) - \dim \ker d_{\phi(t)} \cap \ker (\bar{\partial} \partial) \cap A^{p,q}(X) \geq 0,
\]

and

\[
u_{t}^{p,q} := \dim H^{p,q}_{BC}(X) - \dim \ker \bar{\partial} \partial_{\phi(t)} \cap (H^{p,q}_{BC}(X) + \text{Im} (\partial \bar{\partial})) \cap A^{p,q}(X) \geq 0,
\]

then we have

\[
dim H^{p,q}_{BC}(X) = \dim H^{p,q}_{BC\phi}(X) + \nu_{t}^{p,q} + \nu_{t}^{p-1,q-1}.
\]

Proof. This follows immediately from Lemma 4.2 and Proposition 4.3. \( \square \)

5. The deformed Fröhlicher spectral sequences and the \( \partial \bar{\partial} \phi \)-lemma

Let \( X \) be a complex manifold and \( X_t \) a small deformation (of \( X \)) whose complex structure is represented by a Beltrami differential \( \phi \in A^{0,1}(X, T_{X}^{1,0}) \). Set the deformed de Rahm cohomology as

\[
H^{\bullet}_{d_{\phi}}(X) := \ker d_{\phi}/\text{Im} d_{\phi},
\]

then it is clear that \( e^{\phi} : H^{\bullet}_{d_{\phi}}(X) \to H^{\bullet}_{d_{\bar{\phi}}}(X) \) is an isomorphism and the identity map induces the following commutative diagram:

\[
\begin{array}{ccc}
H^{\bullet}_{d_{\bar{\phi}}}(X) & \xrightarrow{e^{\phi}} & H^{\bullet}_{d_{\phi}}(X) \\
\downarrow & & \downarrow \\
H^{\bullet}_{\partial}(X) & \xrightarrow{\bar{\partial}} & H^{\bullet}_{d_{\phi}}(X)
\end{array}
\]

Definition 5.1. The spectral sequence associated to the double complex \( (A^{\bullet,\bullet}(X), \partial, \bar{\partial}_{\phi}) \) will be called the deformed Fröhlicher spectral sequence and we say \( X \) satisfies the \( \partial \bar{\partial} \phi \)-lemma if the homomorphism \( H^{\bullet,\bullet}_{BC\phi}(X) \to H^{\bullet,\bullet}_{d_{\phi}}(X) \) in (5.1) is injective, i.e.

\[
\ker \partial \cap \ker \bar{\partial}_{\phi} \cap \text{Im} d_{\phi} = \text{Im} \partial \bar{\partial}_{\phi}.
\]

Set \( d_{\phi}^{c} := J^{-1}d_{\phi}J = \sqrt{-1}(\bar{\partial}_{\phi} - \partial) \), where \( J \) is the almost complex structure on \( X \).

It is easy to see that \( \ker \partial \cap \ker \bar{\partial}_{\phi} = \ker d_{\phi} \cap \ker d_{\phi}^{c} \) and \( \text{Im} \partial \bar{\partial}_{\phi} = \text{Im} d_{\phi}d_{\phi}^{c} \). Hence, \( X \) satisfies the \( \partial \bar{\partial} \phi \)-lemma if and only if

\[
\ker d_{\phi} \cap \ker d_{\phi}^{c} \cap \text{Im} d_{\phi} = \text{Im} d_{\phi}d_{\phi}^{c} \text{ ,}
\]

or

\[
\ker d_{\phi} \cap \ker d_{\phi}^{c} \cap \text{Im} d_{\phi}^{c} = \text{Im} d_{\phi}d_{\phi}^{c} \text{ .}
\]
There are two natural filtrations on $A^{••}(X)$:

$$F^p A^k(X) = \bigoplus_{p \leq r \leq k} A^{r,k-r}(X), \quad \bar{F}^p A^k(X) = \bigoplus_{p \leq s \leq k} A^{k-s,s}(X),$$

which induces two filtrations on the deformed de Rahm cohomology $H^k_{d\phi}(X)$ for each $k \geq 0$:

$$F^p H^k_{d\phi}(X) = \{ \alpha \in H^k_{d\phi}(X) | \exists u \in F^p A^k(X) s.t. \alpha = [u] \},$$

and

$$\bar{F}^p H^k_{d\phi}(X) = \{ \alpha \in H^k_{d\phi}(X) | \exists u \in \bar{F}^p A^k(X) s.t. \alpha = [u] \}.$$

As usual, there are many ways to characterize the $\partial \bar{\partial}_{\phi}$-lemma:

**Proposition 5.2.** The following statements are equivalent:

1. $X$ satisfies the $\partial \bar{\partial}_{\phi}$-lemma;
2. The maps in (5.1) induced by the identity map are all isomorphisms;
3. The deformed Frölicher spectral sequence degenerates at $E_1$ and there is a Hodge decomposition

$$H^k_{d\phi}(X; \mathbb{C}) = \bigoplus_{p+q=k} F^p H^k_{d\phi}(X) \cap \bar{F}^q H^k_{d\phi}(X), \forall k.$$

**Proof.** This follows directly from [DGMS75, pp. 268]. \hfill \Box

**Theorem 5.3.** Let $X$ be a compact complex manifold and $X_t$ a small deformation (of $X$) whose complex structure is represented by a Beltrami differential $\phi \in A^{0,1}(X, T^1_X)$. Then for every $(p, q) \in \mathbb{N} \times \mathbb{N}$, we have

$$\dim H^{p,q}_{BC\phi}(X) + \dim H^{p,q}_{A\phi}(X) \geq \dim H^{p,q}_{\partial\bar{\partial}\phi}(X_t) + \dim H^{p,q}_{\partial\phi}(X).$$

In particular, for every $k \in \mathbb{N}$, we have

$$\sum_{p+q=k} \dim H^{p,q}_{BC\phi}(X) + \sum_{p+q=k} \dim H^{p,q}_{A\phi}(X) \geq 2 \dim H^k_{dR}(X),$$

and equality holds if and only if $X$ satisfies the $\partial \bar{\partial}_{\phi}$-lemma.

**Proof.** This follows from similar arguments as in [AT13]. In fact, this theorem is a direct consequence of [AT15a, Thm. 1 and 2] by noting that $\dim H^{p,q}_{\partial\phi}(X) = \dim H^{p,q}_{\partial\phi}(X_t)$ [Xia19a, Thm. 4.4]. \hfill \Box

**Remark 5.4.**

1. From the work of Angella-Tardini [AT17, Thm. 3.1] we know that $X$ satisfies the $\partial \bar{\partial}_{\phi}$-lemma if and only if

$$\sum_{p+q=k} \dim H^{p,q}_{BC\phi}(X) = \sum_{p+q=k} \dim H^{p,q}_{A\phi}(X);$$

2. From Proposition 5.2 we see that if $X$ satisfies the $\partial \bar{\partial}_{\phi}$-lemma, then for every $(p, q) \in \mathbb{N} \times \mathbb{N}$, we have

$$\dim H^{p,q}_{BC\phi}(X) = \dim H^{p,q}_{A\phi}(X) = \dim H^{p,q}_{\partial\phi}(X) = \dim H^{p,q}_{\partial\phi}(X_t) = \dim H^{p,q}_{\partial\phi}(X).$$
In particular, by Theorem 5.3 we have $h^k_{BC\phi} = h^k_{A\phi} = h^k_{\partial t}(X_t) = h^k_\partial = b_k$ \footnote{We follow the notations as given in [AT13], e.g. $h^k_{BC\phi} := \sum_{p+q=k} \dim H^{p,q}_{BC\phi}(X)$ and $b_k$ is the $k$-th Betti number.}, namely, the Fröhlicher spectral sequence of $(A^{\bullet,\bullet}(X), \partial, \overline{\partial})$ degenerates at $E_1$.

**Corollary 5.5.** Let $\pi : (X, X) \to (B, 0)$ be a small deformation of the compact complex manifold $X$ such that for each $t \in B$ the complex structure on $X_t$ is represented by Beltrami differential $\phi(t)$. Then the set

$$T := \{ t \in B \mid X \text{ satisfies the } \partial\overline{\partial}_{\phi(t)} \text{-lemma} \}$$

is an analytic open subset (i.e. complement of analytic subset) of $B$. In particular, if $B \subset \mathbb{C}$ is a small open disc with $0 \in B$ and $T$ is not empty, then $T = B$ or $T = B \setminus \{0\}$.

**Proof.** First, by Theorem 5.3, $X$ satisfies the $\partial\overline{\partial}_{\phi(t)}$-lemma if and only if

$$h^k_{BC\phi(t)} + h^k_{A\phi(t)} = 2b_k. \tag{5.6}$$

We note that by Theorem 4.11 the set $\{ t \in B \mid (5.6) \text{ holds} \}$ is an analytic open subset of $B$ since

$$\{ t \in B \mid h^k_{BC\phi(t)} + h^k_{A\phi(t)} = 2b_k, \ \forall \ k \} = B \setminus \{ t \in B \mid h^k_{BC\phi(t)} + h^k_{A\phi(t)} \geq 2b_k + 1, \ \forall \ k \}. \tag{5.7}$$

In particular, if $X$ satisfies the $\partial\overline{\partial}$-lemma, then by the above corollary $X$ also satisfies the $\partial\overline{\partial}_{\phi(t)}$-lemma for any small $t \in B$. Combining this with Remark 5.4, we get that the Hodge numbers $\dim H^{p,q}_{\partial t}(X_t)$ and $\dim H^{p,q}_{BC\phi(t)}(X)$ are independent of $t$.

Recall that a smooth manifold $X$ is called formal if its de Rham complex $(A^\bullet(X), d)$ is formal as a differential graded algebra (DGA for short). The later means that there is a sequence of quasi-isomorphisms from $(A^\bullet(X), d)$ to its cohomology algebra $(H^\bullet_{dR}(X), 0)$ \footnote{Here, $(H^\bullet(X), 0)$ is considered as a differential graded algebra with trivial differential.}, see [DGMS75, FHT01].

**Theorem 5.6.** Let $X$ be a compact complex manifold and $X_t$ a small deformation (of $X$) whose complex structure is represented by a Beltrami differential $\phi \in A^{0,1}(X, T^1_X)$. If $X$ satisfies the $\partial\overline{\partial}_{\phi}$-lemma, then $X$ is formal.

**Proof.** Consider the following homomorphisms of DGA

$$\xymatrix{(A^\bullet(X), d_{\phi}) \ar[r]^-i & (A^\bullet(X) \cap \ker d_{\phi}, d_{\phi}) \ar[r]^-p & (H^\bullet_{d_{\phi}}(X), d_{\phi} = 0),}$$

where $i$ is the inclusion and $p$ is the projection. We claim that the induced map $i^*$ is an isomorphism on cohomology. Indeed, $\forall x \in \ker d_{\phi} \cap \ker d_{\phi}^c$ if $x \in \Im d_{\phi}$ then by (5.2), $x \in \Im d_{\phi}d_{\phi}^c = \ker d_{\phi}$; on the other hand, by (5.3) $\forall x \in \ker d_{\phi}$ there exist $y \in A^\bullet(X)$ such that $x - d_{\phi}y \in \ker d_{\phi} \cap \ker d_{\phi}^c$, this shows that $i^*$ is surjective. Similarly, one shows that $p^*$ is an isomorphism on cohomology and $d_{\phi} = 0$ on $H^\bullet_{d_{\phi}}(X)$. The conclusion then follows since $(A^\bullet(X), d_{\phi})$ is isomorphic to $(A^\bullet(X), d)$ and $H^\bullet_{d_{\phi}}(X) \cong H^\bullet_{d_{\phi}}(X) \cong H^\bullet_{dR}(X)$. \hfill $\square$
6. The deformed Bott-Chern cohomology of the Iwasawa manifold and the holomorphically parallelizable Nakamura manifold

**Example 6.1.** Case III-(2). Let \( G \) be the matrix Lie group defined by

\[
G := \left\{ \begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}(3; \mathbb{C}) \mid z^1, z^2, z^3 \in \mathbb{C} \right\} \cong \mathbb{C}^3.
\]

Consider the discrete subgroup \( \Gamma \) defined by

\[
\Gamma := \left\{ \begin{pmatrix} 1 & \omega^1 & \omega^3 \\ 0 & 1 & \omega^2 \\ 0 & 0 & 1 \end{pmatrix} \in G \mid \omega^1, \omega^2, \omega^3 \in \mathbb{Z}[[\sqrt{-1}]] \right\}.
\]

The quotient \( X = G/\Gamma \) is called the Iwasawa manifold. A basis of \( H^0(X, \Omega^1) \) is given by

\[
\phi_1 = dz^1, \quad \phi_2 = dz^2, \quad \phi_3 = dz^3 - z^1 dz^2,
\]

and a dual basis \( \theta^1, \theta^2, \theta^3 \in H^0(X, T^1_X) \) is given by

\[
\theta^1 = \frac{\partial}{\partial z^1}, \quad \theta^2 = \frac{\partial}{\partial z^2} + z^1 \frac{\partial}{\partial z^3}, \quad \theta^3 = \frac{\partial}{\partial z^3}.
\]

\( X \) is equipped with the Hermitian metric \( \sum_{i=1}^3 \phi_i \otimes \overline{\phi_i} \). The Beltrami differential of the Kuranishi family of \( X \) is

\[
\phi(t) = t_{i\lambda} \theta^i \overline{\varphi^\lambda} - D(t) \theta^3 \overline{\varphi^3}, \quad \text{with} \quad D(t) = t_{11} t_{22} - t_{21} t_{12},
\]

and the Kuranishi space of \( X \) is

\[
\mathcal{B} = \{ t = (t_{11}, t_{12}, t_{21}, t_{22}, t_{31}, t_{32}) \in \mathbb{C}^6 \mid |t_{i\lambda}| < \epsilon, i = 1, 2, 3, \lambda = 1, 2 \},
\]

where \( \epsilon > 0 \) is sufficiently small. Set

\[
\phi_1 = \sum_{i=1}^3 \sum_{\lambda=1}^2 t_{i\lambda} \theta^i \overline{\varphi^\lambda}, \quad \phi_2 = D(t) \theta^3 \overline{\varphi^3},
\]

and write the canonical deformation of \( \sigma_0 \in H^{p,q}_{BC}(X) \) by \( \sigma(t) = \sum_k \sigma_k \) with each

\[
\sigma_k = -G_{BC} A \sum_{i+j=k} \partial_i \phi_j \sigma_i,
\]

being the homogeneous term of degree \( k > 0 \) in \( t \in \mathcal{B} \). We will use the isomorphism \( H_{BC,\phi(t)}^{p,q}(X) \cong \text{dim} \, V_t / \ker f_t \) proved in Proposition 4.7 to compute \( \text{dim} \, H_{BC,\phi(t)}^{p,q}(X) \).

Since \( \mathcal{B} \) is a polydisc, it is sufficient to check the coefficients of \( d_{\phi(t)} \sigma(t) = 0 \), that is,

\[
(6.1) \quad \partial \sigma_k = \overline{\partial} \sigma_k + \sum_{j=1}^k \partial_i \phi_j \sigma_{k-j} = 0, \quad k > 0.
\]

Let us now consider Bott-Chern deformations of forms in the harmonic space:

\[
H_{BC}^{2,2}(X) = \mathbb{C}\{\varphi^2, \varphi^1, \varphi^3, \overline{\varphi^2}, \overline{\varphi^1}, \overline{\varphi^3}, \varphi^{12}, \varphi^{13}, \varphi^{23}, \varphi^{21}, \varphi^{32}, \varphi^{31}, \varphi^{23}, \varphi^{21}, \varphi^{32}, \varphi^{31} \}.
\]
Set $\sigma_0 = \sum a_{ijkl} \varphi^{ijkl} \in \mathcal{H}_{BC}^{2,2}(X)$, then

$$\partial i_{\phi_1} \sigma_0 = (-t_{12}a_{1313} + t_{11}a_{1323} - t_{22}a_{2313} + t_{21}a_{2323}) \varphi^{1213}$$

is $\bar{\partial}$-exact if and only if

$$(6.2) \quad t_{12}a_{1313} - t_{11}a_{1323} + t_{22}a_{2313} - t_{21}a_{2323} = 0,$$

and in this case

$$\sigma_1 = -G_{BC} A \partial i_{\phi_1} \sigma_0 = 0.$$

But

$$\partial i_{\phi_2} \sigma_0 = 0 \implies \sigma_2 = -G_{BC} A \partial (i_{\phi_2} \sigma_0 + i_{\phi_1} \sigma_1) = 0,$$

and $\phi_k = 0$, $k > 2$ we thus have $\sigma_k = 0$, $k > 2$.

Therefore, for $V = \mathcal{H}_{BC}^{2,2}(X)$ we have (see Definition 4.5)

$$V_t = \{ \sum a_{ijkl} \varphi^{ijkl} \in \mathcal{H}_{BC}^{2,2}(X) \mid (a_{1213}, a_{1223}, a_{1312}, a_{1323}, a_{2312}, a_{2313}, a_{2323}) \in \mathbb{C}^8$$

s.t. $\sigma(t) \in \ker d_{\phi(t)}$, where $\sigma(t) = \sum_k \sigma_k$ with $\sigma_0 = \sum a_{ijkl} \varphi^{ijkl}$

and $\sigma_k = -G_{BC} A \sum_{i+j=k} \partial i_{\phi_j} \sigma_i$, $\forall k \neq 0$}

$$= \{ \sum a_{ijkl} \varphi^{ijkl} \mid (a_{1213}, a_{1223}, a_{1312}, a_{1323}, a_{2312}, a_{2313}, a_{2323}) \in \mathbb{C}^8 \text{ satisfy } (6.2) \}. $$

On the other hand, $\text{Im} \bar{\partial} \phi(t) = \mathbb{C} \{ \partial \bar{\partial} \phi(t) \varphi^{33} \} = \mathbb{C} \{ \varphi^{1213} \}$ (Since $X$ is parallelizable, we only need to consider left invariant forms. See the discussions in the last paragraph of this section) and

$$(\bar{\partial} \bar{\partial})^* \varphi^{1213} = (\ast \bar{\partial} \ast)(\ast \bar{\partial} \ast) \varphi^{1213} = -\ast \bar{\partial} \bar{\partial} \varphi^{33} = \varphi^{33} \neq 0,$$

implies

$$\ker f_t \cong \ker (\bar{\partial} \bar{\partial})^* \cap \text{Im} \bar{\partial} \phi(t) \cap A^{2,2}(X) = 0.$$

By Proposition 4.7 we have

$$(6.3) \quad \dim H_{BC\phi(t)}^{2,2}(X) = \dim V_t - \dim \ker f_t = \begin{cases} 8, & (t_{11}, t_{12}, t_{21}, t_{22}) = 0 \\ 7, & (t_{11}, t_{12}, t_{21}, t_{22}) \neq 0. \end{cases}$$

The other deformed Bott-Chern cohomology can be computed in the same way. Write $(i), (ii), (iii)$ for the three cases when $(t_{11}, t_{12}, t_{21}, t_{22}) = 0$, $(t_{11}, t_{12}, t_{21}, t_{22}) \neq 0$ and $D(t) = 0$, $D(t) \neq 0$, respectively. Then we have the following (where $h^{p,q} := \dim H_{BC\phi(t)}^{p,q}(X)$ and $t \in (i), (ii), (iii)$, respectively)

| $h^{1,0}$ | $h^{0,1}$ | $h^{2,0}$ | $h^{1,1}$ | $h^{0,2}$ | $h^{3,0}$ | $h^{2,1}$ | $h^{1,2}$ | $h^{0,3}$ | $h^{3,1}$ | $h^{2,2}$ | $h^{1,3}$ | $h^{3,2}$ | $h^{2,3}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 3 | 4 | 3 | 1 | 6 | 6 | 1 | 2 | 8 | 2 | 3 | 3 |
| 2 | 2 | 2 | 4 | 3 | 1 | 6 | 6 | 1 | 2 | 7 | 2 | 3 | 3 |
| 2 | 2 | 1 | 4 | 3 | 1 | 6 | 6 | 1 | 2 | 7 | 2 | 3 | 3 |
Comparing this with the computations made by Angella [Ang13] we see that $\dim H_{BC\phi(t)}^{p,q}(X) = \dim H_{BC}^{p,q}(X_t)$ is not true in general for $p = q$.

**Example 6.2.** Case III-(3b). Let $X = \mathbb{C}^3/\Gamma$ be the solvable manifold constructed by Nakamura in Example III-(3b) of [Nak75]. We have

$$H^0(X, \Omega^1_X) = \mathbb{C}\{\varphi^1 = dz^1, \varphi^2 = e^{z_1}dz^2, \varphi^3 = e^{-z_1}dz^3\},$$

$$H^0(X, T^{1,0}_X) = \mathbb{C}\{\theta^1 = \frac{\partial}{\partial z^1}, \theta^2 = e^{-z_1} \frac{\partial}{\partial z^2}, \theta^3 = e^{z_1} \frac{\partial}{\partial z^3}\},$$

$$\mathcal{H}^{0,1}(X) = \mathbb{C}\{\psi^1 = dz^1, \psi^2 = e^{z_1}dz^2, \psi^3 = e^{-z_1}dz^3\},$$

$$\mathcal{H}^{0,1}(X, T^{1,0}_X) = \mathbb{C}\{\theta^i \psi^\lambda, i = 1, 2, 3, \lambda = 1, 2, 3\},$$

where $X$ is equipped with the Hermitian metric $\sum_{i=1}^3 \varphi^i \otimes \bar{\varphi}^i$. The Beltrami differential of the Kuranishi family of $X$ is

$$\phi(t) = \phi_1 = t_{i\lambda} \theta^i \bar{\psi}^\lambda$$

and the Kuranishi space of $X$ is

$$\mathcal{B} = \{t = (t_{11}, t_{12}, t_{13}, t_{21}, t_{22}, t_{23}, t_{31}, t_{32}, t_{33}) \in \mathbb{C}^9 \mid |t_{i\lambda}| < \epsilon, i = 1, 2, 3, \lambda = 1, 2, 3\},$$

where $\epsilon > 0$ is sufficiently small. We will restrict to the one parameter family defined by $t_{12} = t_{13} = t_{21} = t_{22} = t_{23} = t_{31} = t_{32} = t_{33} = 0$ and in this case the Beltrami differential is $\phi = \phi(t) = t \frac{\partial}{\partial z^1} dz^1$ where $t = t_{11}$.

Let us consider the Bott-Chern deformations of forms in

$$\mathcal{H}_{BC}^{2,1}(X) = \mathbb{C}\{e^{z_1}dz^{12T}, e^{2z_1}dz^{12\bar{T}}, e^{-z_1}dz^{13T}, e^{-z_1}dz^{13\bar{T}}, e^{-2z_1}dz^{13\bar{T}}, e^{z_1}dz^{13\bar{T}}, e^{z_1}dz^{12\bar{T}}\}.$$

Set

$$\sigma_0 = a_{121} e^{z_1} dz^{12\bar{T}} + a_{122} e^{2z_1} dz^{12\bar{T}} + a_{123} e^{-z_1} dz^{13T} + a_{131} e^{-z_1} dz^{13\bar{T}} + a_{132} e^{-2z_1} dz^{13\bar{T}}$$

$$+ a_{133} e^{z_1} dz^{13\bar{T}} + a_{231} e^{z_1} dz^{13T} + b_{131} e^{z_1} dz^{13\bar{T}} + b_{121} e^{-z_1} dz^{12\bar{T}},$$

then

$$\partial_i \sigma_0 = -2a_{122} i e^{2z_1} dz^{12\bar{T}} + 2a_{133} i e^{-z_1} dz^{13\bar{T}}$$

is $\bar{\partial}$-exact if and only if $t = 0$. Therefore, for $V = \mathcal{H}_{BC}^{2,1}(X)$ and $t \neq 0$ we have

$$V_t = \{\sigma_0 \in \mathcal{H}_{BC}^{2,1}(X) \mid (a_{121}, a_{122}, a_{123}, a_{131}, a_{132}, a_{133}, a_{231}, b_{131}, b_{121}) \in \mathbb{C}^9 \text{ s.t.}$$

$$\sigma(t) \in \ker d_{\phi(t)}, \text{where } \sigma(t) = \sum_k \sigma_k \text{ with } \sigma_k = -G_{BC}A \sum_{i+j+k} \partial_i \sigma_j \sigma_k, \forall k \neq 0\}$$

$$= \mathbb{C}\{e^{z_1}dz^{12T}, e^{-z_1}dz^{13T}, e^{-z_1}dz^{13\bar{T}}, e^{z_1}dz^{13\bar{T}}, e^{-z_1}dz^{12\bar{T}}\}.$$
and
$$\dim \ker f_t = \dim \ker (\partial \bar{\partial})^* \cap \text{Im } \partial \bar{\partial}_{\phi(t)} \cap A^{2,2}(X) = 2.$$ 

By Proposition 4.7 we have
$$\dim H^{2,1}_{BC\phi(t)}(X) = \dim V_t - \dim \ker f_t = \begin{cases} 9, & t = 0 \\ 5, & t \neq 0 \end{cases}.$$ 

We summarise the computations of the deformed Bott-Chern cohomology in this case as follows (where $h^{p,q} := \dim H^{p,q}_{BC\phi(t)}(X)$ and $t = 0, \neq 0$, respectively):

| $h^{1,0}$ | $h^{0,1}$ | $h^{2,0}$ | $h^{1,1}$ | $h^{0,2}$ | $h^{3,0}$ | $h^{2,1}$ | $h^{1,2}$ | $h^{0,3}$ | $h^{3,1}$ | $h^{2,2}$ | $h^{1,3}$ | $h^{3,2}$ | $h^{2,3}$ |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 1         | 1         | 3         | 7         | 3         | 1         | 9         | 9         | 1         | 3         | 11        | 3         | 5         | 5         |
| 1         | 1         | 1         | 5         | 3         | 1         | 5         | 7         | 1         | 1         | 7         | 3         | 3         | 3         |

From this table and [AK17b], we notice that $X_t$ satisfy the $\partial \bar{\partial}$-lemma but $X$ does not satisfy the $\partial \bar{\partial}_{\phi(t)}$-lemma for any $t \neq 0$.

We need to point out that in the above computations (especially those concerning $\ker (\partial \bar{\partial})^* \cap \text{Im } \partial \bar{\partial}_{\phi(t)}$), only invariant forms is considered. This is valid because the Bott-Chern cohomology of complex parallelizable manifold may be computed by left invariant forms [Ang13] and given a family of deformations $\{X_t\}_{t \in B}$ of such manifolds the set of $t$ for which the deformed Bott-Chern cohomology may be computed by left invariant forms is an open subset of $B$ (this will be proved in [Xia20]).

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Wei Xia, Mathematical Science Research Center, Chongqing University of Technology, Chongqing, P.R. China, 400054.

Email address: xiaweiwei3@126.com