Investigating and addressing student difficulties with the corrections to the energies of the hydrogen atom for the strong and weak field Zeeman effect

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Abstract
Understanding when and how to make limiting case approximations and why they are valid in a particular situation is a hallmark of expertise in physics. Using limiting cases can simplify the problem-solving process significantly and they often provide a means to check that the results obtained are reasonable. We discuss an investigation of student difficulties with the corrections to the energy spectrum of the hydrogen atom for the limiting cases of the strong and weak field Zeeman effects using degenerate perturbation theory. This investigation was carried out in advanced quantum mechanics courses by administering written free-response and multiple-choice questions and conducting individual interviews with students. Here we first discuss the common student difficulties related to these concepts. We then describe how the research on student difficulties was used as a guide to develop and evaluate a quantum interactive learning tutorial (QuILT) which strives to help students develop a functional understanding of the concepts necessary for finding the corrections to the energy spectrum of the hydrogen atom for the strong field and weak field Zeeman effects. The development of the QuILT and its evaluation in the undergraduate and PhD level courses are presented.

Keywords: perturbation theory, degenerate perturbation theory, physics education, tutorial, quantum mechanics
1. Introduction

A major goal of physics courses, especially those for the physics majors, is to help students learn to think like a physicist \([1]\). In many physics courses, in addition to helping students learn physics content, there is emphasis on helping them develop problem-solving, reasoning, and metacognitive skills \([2–8]\). In physics, problem-solving, reasoning, and metacognitive skills can involve, for example, planning a solution to a problem, monitoring one’s problem solving, considering limiting cases appropriately and evaluating the final answer. In particular, physicists often utilize limiting cases when appropriate to simplify the problem-solving process and to check whether the results in those limits make sense. Students often learn about limiting cases throughout the physics curriculum, from introductory physics to advanced undergraduate and graduate level courses. We have been developing several quantum interactive learning tutorials (QuILTs) that strive to help students develop problem-solving, reasoning, and metacognitive skills including learning to use limiting cases and understanding their utility and applicability.

Here we discuss student understanding of limiting cases in the context of degenerate perturbation theory (DPT) for finding the corrections to the energy spectrum of the hydrogen atom for the Zeeman effect and the development and validation of a research-based QuILT to improve student understanding. The origin of the Zeeman term in the hydrogen atom involves the potential energy of the magnetic moments due to the orbital and spin angular momentum in an external magnetic field. The Zeeman effect is the shift in the energy spectrum of the hydrogen atom due to the presence of an external magnetic field, and it is proportional to the strength of the external magnetic field. In addition, the fine structure term in the hydrogen atom includes corrections due to the spin–orbit coupling and a relativistic correction for the kinetic energy. We focus on two limiting cases: the strong and weak field Zeeman effects. The strong field Zeeman effect occurs when the corrections to the energies due to the Zeeman term are much greater than the corrections to the energies due to the fine structure term. The weak field Zeeman effect occurs when the corrections to the energies due to the fine structure term are much greater than the corrections to the energies due to the Zeeman term.

The time-independent Schrödinger equation (TISE) for the Hamiltonian with the fine structure and Zeeman corrections cannot be solved exactly. Nevertheless, since the fine structure term and, in general, the Zeeman term are significantly smaller than the unperturbed Hamiltonian, perturbation theory (PT) is an excellent method for determining the approximate solutions to the TISE and the corrections to the energy spectrum of the hydrogen atom. Due to the degeneracy in the hydrogen atom energy spectrum, DPT must be used to find the corrections for the strong and weak field Zeeman effect.

It is important to help students develop a functional understanding of DPT in order to find the corrections to the energies for the strong and weak field Zeeman effects. However, quantum mechanics (QM) is challenging for upper-level undergraduate and PhD level students (e.g. see \([9–29]\)). Since advanced students often struggle with the DPT for the limiting cases of the strong and weak field Zeeman effect, we investigated student difficulties with finding the first-order corrections to the energies of the hydrogen atom for the strong and weak field Zeeman effects using DPT.

There have been a number of prior research studies aimed at investigating student reasoning in QM \([30–40]\) and using the findings as resources for improving student understanding \([41–48]\). However, there have been relatively few studies investigating student understanding of DPT \([49]\). We have been developing a set of research-based learning tools.
that are inspired by research studies conducted to identify student difficulties with QM and findings of cognitive research. One such research-based tool is the QuILTs which strive to help students develop a solid grasp of QM \[50–60\]. In this paper we describe the development and validation of the research-based QuILT focusing on DPT that uses student difficulties as resources. The QuILT strives to help students learn to find the corrections to the energy spectrum of the hydrogen atom for the limiting cases of the strong and weak field Zeeman effect\(^1\).

2. Background

We first discuss the requisite knowledge students must have to use DPT in general and in the limiting contexts of the strong and weak field Zeeman effects in particular.

2.1. Basics for DPT

PT is a useful approximation method for finding the energies and the energy eigenstates for a system for which the TISE is not exactly solvable. The Hamiltonian \(\hat{H}\) for the system can be expressed as the sum of two terms, the unperturbed Hamiltonian \(\hat{H}^0\) and the perturbation \(\hat{H}'\), i.e. \(\hat{H} = \hat{H}^0 + \hat{H}'\). The TISE for the unperturbed Hamiltonian, \(\hat{H}^0\psi^0_n = E^0_n \psi^0_n\), where \(\psi^0_n\) is the \(n\)th unperturbed energy eigenstate and \(E^0_n\) is the \(n\)th unperturbed energy, is exactly solvable. The energies can be approximated as \(E_n = E^0_n + E^n_1 + E^n_2 + \ldots\) where \(E_i\) for \(i = 1, 2, 3..\) are the \(i\)th order corrections to the \(n\)th energy of the system.

In nondegenerate PT, the first-order correction to the \(n\)th energy is

\[
E_n^1 = \langle \psi^0_n | \hat{H}' | \psi^0_n \rangle
\]

and the first-order correction to the \(n\)th unperturbed energy eigenstate is

\[
|\psi_n^1\rangle = \sum_{m \neq n} \frac{\langle \psi^0_m | \hat{H}' | \psi^0_n \rangle}{(E^0_n - E^0_m)} |\psi^0_m\rangle,
\]

in which \(|\psi^0_m\rangle\) is a complete set of eigenstates of the unperturbed Hamiltonian \(\hat{H}^0\). When the eigenvalue spectrum of \(\hat{H}^0\) has degeneracy (i.e. two or more eigenstates of \(\hat{H}^0\) have the same energy so that two or more diagonal elements of \(\hat{H}^0\) are equal), equations (1) and (2) from nondegenerate PT are still valid provided one uses a good basis. For a given \(\hat{H}^0\) and \(\hat{H}'\), we define a good basis as consisting of a complete set of eigenstates of \(\hat{H}^0\) that diagonalizes \(\hat{H}'\) in each degenerate subspace of \(\hat{H}^0\). In a good basis, \(\hat{H}'\) is diagonal in each degenerate subspace of \(\hat{H}^0\). Therefore, the terms \(\langle \psi^0_m | \hat{H}' | \psi^0_n \rangle\) in equation (2) for the wavefunction are zero when \(m \neq n\) so that the expression for the corrections to the wavefunction in equation (2) does not have terms that diverge. In a good basis, equation (1) is also valid for finding the first-order corrections to the energies (which are the diagonal elements of the \(\hat{H}'\) matrix as given by equation (1)).

\(^1\) Parts of sections 1, 2.1, 5.1 and 5.2 reproduce work originally published in the conference proceedings paper discussing student difficulties with DPT in three dimensions \[49\].
2.2. Background for DPT involving the limiting cases of the Zeeman effect

For a hydrogen atom in an external magnetic field, one can use DPT to find the corrections to the energy spectrum. Using standard notations, the unperturbed Hamiltonian $\hat{H}^0$ of a hydrogen atom is $\hat{H}^0 = \frac{\mu^2}{2m} - \frac{e^2}{4\pi\epsilon_0 r^2} \left( \frac{1}{\ell} \right)$, where $m$ is the reduced mass and $r$ is the radial distance between the proton and electron which accounts for the interaction of the electron with the nucleus via Coulomb attraction (other symbols also have their usual meaning).

The solution for the TISE for the hydrogen atom with Coulomb potential energy gives the unperturbed energies $E_n^0 = -\frac{13.6 \text{ eV}}{n^2}$, where $n$ is the principal quantum number. The perturbation is $\hat{H}' = \hat{H}'_Z + \hat{H}'_0$, in which $\hat{H}'_Z$ is the Zeeman term and $\hat{H}'_0$ is the fine structure term. The Zeeman term accounts for the potential energy of the magnetic moments due to the orbital and spin angular momenta in the external magnetic field. The Zeeman term is given by $\hat{H}'_Z = \frac{\mu B}{\hbar} (L_z + 2S_z)$, in which $\hat{B}_{\text{ext}} = B_{\text{ext}} \hat{z}$ is a uniform, time-independent external magnetic field along the $\hat{z}$-direction, $\mu_B$ is the Bohr magneton and $\hat{L}_z$ and $\hat{S}_z$ are the operators corresponding to the $z$ component of the orbital and spin angular momenta, respectively.

The fine structure term includes the spin–orbit coupling and a relativistic correction for the kinetic energy and is expressed as $\hat{H}'_0 = \hat{H}'_1 + \hat{H}'_{\text{SO}}$. Here, $\hat{H}'_1 = -\frac{\hat{p}^2}{8mc^2}$ is the relativistic correction term and $\hat{H}'_{\text{SO}} = \left( \frac{e}{8\pi\epsilon_0} \right) \frac{1}{m'c^2} (\hat{S} \cdot \hat{L})$ is the spin–orbit interaction term (all notations are standard).

We note that $\hat{H}^0$ for the hydrogen atom is diagonal when ANY complete set of orthogonal states with the same $n$ is chosen for the angular part of the basis (consisting of the product states of orbital and spin angular momenta). Thus, so long as the radial part of the basis is always chosen to be stationary state wavefunctions $R_{nl}$ for the hydrogen atom (for a given principle quantum number $n$ and azimuthal quantum number $\ell$), the choice of a good basis amounts to choosing the angular part of the basis (the part of the basis that involves the product states of the orbital and spin angular momenta) appropriately. Therefore, we focus on the angular part of the basis for the $n = 2$ degenerate subspace of $\hat{H}^0$ to find a good basis and the corrections to the energies for the perturbation $\hat{H}'$ corresponding to the limiting cases of the Zeeman corrections to the hydrogen atom. The total angular momentum is defined as $\hat{J} = \hat{L} + \hat{S}$. For the angular part of the basis, states in the coupled representation $|l, j, m_j\rangle$ are labelled by the quantum numbers $l$, $j$, and $m_j$ where $l$ is the total angular momentum number, $j$ is the spin quantum number, $j$ is the total angular momentum number, and $m_j$ is the $z$ component of the total angular momentum quantum number (all notations are standard and $s = 1/2$ has been suppressed from the states $|l, j, m_j\rangle$ since $s = 1/2$ is a fixed value for a neutral hydrogen atom). On the other hand, states in the uncoupled representation $|l, m_l, m_m\rangle$ are labelled by the quantum numbers $l$, $m_l$, and $m_m$ (the quantum numbers $m_l$ and $m_m$ correspond to the $z$ component of the orbital and spin angular momenta, respectively).

In the limiting cases of the strong and weak field Zeeman effect, the perturbation $\hat{H}'$ can be separated into two terms $\hat{H}' = \hat{H}'_{\text{strong}} + \hat{H}'_{\text{weak}}$, in which $\hat{H}'_{\text{strong}}$ is the stronger perturbation and $\hat{H}'_{\text{weak}}$ is the weaker perturbation. The corrections to the energies due to the stronger perturbation $\hat{H}'_{\text{strong}}$ are larger than the corrections due to the weaker perturbation $\hat{H}'_{\text{weak}}$. In these limiting cases, in order to find the corrections to the energies, one useful approach is to use DPT via a two-step approximation. In the first step, the stronger perturbation $\hat{H}'_{\text{strong}}$ is treated as the only perturbation. A good basis for step 1 is one that diagonalizes the unperturbed Hamiltonian $\hat{H}^0$ and also diagonalizes the stronger perturbation $\hat{H}'_{\text{strong}}$ in each degenerate subspace of the unperturbed Hamiltonian $\hat{H}^0$. After a good basis has been
identified for step 1, the first-order corrections for the stronger perturbation $\hat{H}_\text{strong}$ are determined. In the second step of the two-step approximation, $\hat{H}_\text{strong}^0 = \hat{H}^0 + \hat{H}_\text{strong}^\prime$ is the new unperturbed Hamiltonian and the weaker perturbation $\hat{H}_\text{weak}^\prime$ is treated as the perturbation. For step 2, a good basis is one that diagonalizes the unperturbed Hamiltonian $\hat{H}_\text{strong}^0$ and also diagonalizes $\hat{H}_\text{weak}^\prime$ in each degenerate subspace of $\hat{H}_\text{strong}^0$. Once a good basis for step 2 has been identified, the first-order corrections to the energies due to the weaker perturbation can be determined. The total first-order corrections to the energies are the sum of the corrections from steps 1 and 2.

As an example of one limiting case, the following steps describe how to determine a good basis and the first-order corrections to the energies for the strong field Zeeman effect. (1) Treat the stronger perturbation $\hat{H}_Z$ as the only perturbation on the unperturbed Hamiltonian $\hat{H}^0$. Identify that a basis consisting of states in the uncoupled representation forms a good basis for the unperturbed Hamiltonian $\hat{H}^0$ and the stronger perturbation $\hat{H}_Z$ (since both $\hat{H}^0$ and $\hat{H}_Z$ are diagonal in uncoupled representation, $\hat{H}_Z^\prime$ is diagonal in each degenerate subspace of $\hat{H}_\text{strong}^0$ in the uncoupled representation). Determine the first-order corrections to the energies due to the stronger perturbation $\hat{H}_Z$. After step 1, the first-order corrections to the energies break some of the degeneracy that is present from only considering $\hat{H}^0$. (2) Treat the weaker perturbation $\hat{H}_l$ as the perturbation on $\hat{H}_Z^0 = \hat{H}^0 + \hat{H}_Z$. Identify that a basis consisting of states in the uncoupled representation forms a good basis for the unperturbed Hamiltonian $\hat{H}_Z^0$ and the weaker perturbation $\hat{H}_l$ (since $\hat{H}_Z$ is diagonal in the uncoupled representation and $\hat{H}_l$ is diagonal in the degenerate subspaces of $\hat{H}_Z^0$ in the uncoupled representation). Determine the first-order corrections to the energies due to the weaker perturbation $\hat{H}_l$. (3) The sum of the first-order corrections obtained in steps 1 and 2 is the first-order corrections to the energy spectrum of the hydrogen atom.

In step 1, when the stronger perturbation $\hat{H}_Z^\prime$ is treated as the only perturbation on the unperturbed Hamiltonian $\hat{H}^0$, basis states in the uncoupled representation $(l, m_l, m_s)$ diagonalize both $\hat{H}^0$ and $\hat{H}_Z$. For the $n = 2$ subspace, if $l = 1$ then $m_l = -1, 0, 1$ or if $l = 0$ then $m_l = 0$ and for $s = \frac{1}{2}$, we have $m_s = -\frac{1}{2}$ or $\frac{1}{2}$. By taking all the combinations of $l, m_l, s,$ and $m_s$ one finds that the $n = 2$ subspace is eight-dimensional with basis states in the uncoupled representation $(l, m_l, m_s)$ given by $\begin{bmatrix} 0, 0, \frac{1}{2} \\ 0, 0, -\frac{1}{2} \\ 1, 1, \frac{1}{2} \\ 1, 1, -\frac{1}{2} \\ 1, 0, \frac{1}{2} \\ 1, 0, -\frac{1}{2} \\ 1, -1, \frac{1}{2} \\ 1, -1, -\frac{1}{2} \end{bmatrix}$. The matrices for the operators $\hat{H}^0$ and $\hat{H}_Z^\prime$ are given below which correspond to the $n = 2$ subspace in which the basis states are chosen in the uncoupled representation in the order $\begin{bmatrix} 0, 0, \frac{1}{2} \\ 1, 0, \frac{1}{2} \\ 0, 0, -\frac{1}{2} \\ 1, 0, -\frac{1}{2} \\ 1, 1, \frac{1}{2} \\ 1, 1, -\frac{1}{2} \\ 1, -1, \frac{1}{2} \\ 1, -1, -\frac{1}{2} \end{bmatrix}$: \[
\begin{align*}
\hat{H}^0 &= -\frac{13.6 \text{ eV}}{4}
\end{align*}
\]
\[ \hat{H}_Z' = \mu_B B_{\text{ext}}. \]  

(4)

Since \( \hat{H}_Z' \) is diagonal in the uncoupled representation, it is diagonal in each degenerate subspace of \( \hat{H}^0 \) and basis states chosen in the uncoupled representation form a good basis for step 1. In fact, since the basis consisting of states in the uncoupled representation simultaneously diagonalizes \( \hat{H}^0 \) and \( \hat{H}_Z' \), the energies including the first-order corrections to the energies obtained after step 1 are the exact result for the energies. In step 2, the new unperturbed Hamiltonian is \( \hat{H}_Z^0 = \hat{H}^0 + \hat{H}_Z' \) and the weaker perturbation is \( \hat{H}_\text{weak} = \hat{H}_\text{lc} \). Basis states chosen in the uncoupled representation also form a good basis for step 2 for the strong field Zeeman effect because \( \hat{H}_Z^0 = \hat{H}_Z' + \hat{H}_\text{lc} \) has lower degeneracy than \( \hat{H}_Z' \) and \( \hat{H}_\text{weak} = \hat{H}_\text{lc} \) is diagonal in each degenerate subspace of \( \hat{H}_Z^0 \).

Below are the matrices for the \( n = 2 \) subspace in which basis states are chosen in the same order as the earlier matrices above (\( \beta = \mu_B B_{\text{ext}} \) and \( \alpha \) is the fine structure constant):

\[
\hat{\hat{H}}_Z^2 = \begin{bmatrix}
E_2 + \beta & 0 & 0 & 0 & 0 & 0 \\
0 & E_2 + \beta & 0 & 0 & 0 & 0 \\
0 & 0 & E_2 - \beta & 0 & 0 & 0 \\
0 & 0 & 0 & E_2 - \beta & 0 & 0 \\
0 & 0 & 0 & 0 & E_2 & 0 \\
0 & 0 & 0 & 0 & 0 & E_2 + 2\beta \\
0 & 0 & 0 & 0 & 0 & E_2 - 2\beta
\end{bmatrix}
\]

(5)

\[
\hat{H}_3' = \frac{(-13.6 \text{ eV})\alpha^2}{192}
\begin{bmatrix}
15 & 0 & 0 & 0 & 0 & 0 \\
0 & 7 & 0 & 4\sqrt{2} & 0 & 0 \\
0 & 0 & 15 & 0 & 0 & 0 \\
0 & 0 & 0 & 7 & 0 & 4\sqrt{2} \\
0 & 4\sqrt{2} & 0 & 0 & 11 & 0 \\
0 & 0 & 4\sqrt{2} & 0 & 11 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 3
\end{bmatrix}
\]

(6)

All of the unperturbed energy eigenvalues in the \( n = 2 \) subspace have energy \( E_2 = \frac{-13.6 \text{ eV}}{4} \), as given by the diagonal matrix elements in equation (3). States in the uncoupled representation are eigenstates of \( \hat{H}_Z' \) with energy \( \mu_B B_{\text{ext}}(m_l + 2m_s) \). For example, the state with \( l = 1, m_l = -1, \) and \( m_s = \frac{1}{2} \) has energy \( \mu_B B_{\text{ext}}\left(-1 + 2\left(\frac{3}{2}\right)^2\right) = 0 \) (see equation (4)). Thus, in step 1, the corrections to the unperturbed energies due to the Zeeman term in the strong field Zeeman limit are the diagonal matrix elements of \( \hat{H}_Z' \) in equation (4).
Moreover, even though the weaker perturbation $\hat{H}_{fs}^t$ is not diagonal if basis states are chosen in the uncoupled representation, $\hat{H}_{fs}^t$ is diagonal in each degenerate subspace of $\hat{H}_Z^t = \hat{H}_0^t + \hat{H}_Z^t$ as noted by the boxed matrix elements for $n = 2$ (see equation (6)). Therefore, a basis consisting of states in the uncoupled representation is a good basis for both steps 1 and 2 for the strong field Zeeman effect. In step 2, the diagonal matrix elements of $\hat{H}_Z^t$ in the uncoupled representation, which are the fine structure corrections in the strong field Zeeman limit, are given by 

$$\frac{(13.6\text{ eV})^2}{h^2} \left\{ \frac{3}{4n^2} - \frac{l(l+1)}{(l+1/2)(l+1)} \right\}$$

for $l = 1$ [61].

On the other hand, for the weak field Zeeman effect, the dominant fine structure term is the only perturbation on $\hat{H}_0^t$ in step 1 and the weaker perturbation $\hat{H}_Z^t$ is the perturbation on the Hamiltonian $\hat{H}_{fs}^t = \hat{H}_0^t + \hat{H}_Z^t$ in step 2. In the weak field Zeeman effect, the coupled representation forms a good basis for both steps 1 and 2.

### 3. Methodology

Student difficulties with the corrections to the energies of the hydrogen atom for the strong and weak field Zeeman effects using DPT were investigated using four years of data involving responses from 52 upper-level undergraduate students and 42 first-year PhD level students to open-ended and multiple-choice questions administered after traditional, lecture-based instruction in relevant concepts. The undergraduates were enrolled in an upper-level, undergraduate QM course. Most PhD students enrolled in the PhD level course had already learned this material in an undergraduate QM course but this was the first exposure beyond the undergraduate level. The PhD level course was a required core course for the enrolled PhD student who were in their first year of physics PhD program (the completion of six core courses including two courses in QM is mandatory for PhD students to obtain a masters’ degree while pursuing PhD and advance to candidacy for the PhD research—note that most research universities in the US do not admit students only for a masters in physics so students enrol in PhD program and take PhD core courses such as the QM course discussed here in their first year after finishing their undergraduate degree). Traditional instruction was used in both the undergraduate and PhD level courses and consisted of lecture style instruction along with traditional textbook homework problems. The textbook used in the undergraduate course was authored by Townsend [62], and the textbook used in the PhD course was authored by Sakurai [63]. For the majority of the classes, students listened to lectures and took notes (except in the undergraduate course when students engaged with the QuILT). The instructor for the undergraduate course was the same in both years, and the instructor for the PhD level course was the same in both years. Additional insight about the difficulties was gained from 13 individual think-aloud interviews (a total of 45 hours). Interviewed students were provided with all relevant information discussed in the introduction and background sections and had traditional, lecture-based instruction in relevant concepts. Similar percentages of undergraduate and PhD level students displayed difficulties with DPT.

After analysing responses of 32 undergraduates on similar questions administered in two previous years, we posed the following question to 20 undergraduate and 42 PhD level students in the following two years as part of an in-class quiz after traditional lecture-based instruction to examine student difficulties. Question Q1 was posed to identify whether students were able to determine a good basis for the limiting case of the strong field Zeeman effect (in which the limiting cases of the strong field and weak field Zeeman effects were
listed individually in two separate questions). Students selected all the representations that form a good basis for strong field Zeeman effect in the multiple-choice question and were then asked to provide explanation for the options they chose in Q1:

Q1. A perturbation \( \hat{H}' = \hat{H}'_\text{fs} + \hat{H}'_\text{Z} \) acts on a hydrogen atom with the unperturbed Hamiltonian \( \hat{H}^0 = -\frac{\hbar^2}{2m} \nabla^2 - \frac{e^2}{4\pi\epsilon_0} \frac{1}{r} \). For the perturbation \( \hat{H}' = \hat{H}'_\text{fs} + \hat{H}'_\text{Z} \), circle ALL of the representations that form a good basis for the strong field Zeeman effect and explain your reasoning. Assume that for all cases the principal quantum number \( n = 2 \).

i. Coupled representation,
ii. Uncoupled representation,
iii. ANY arbitrary orthonormal basis constructed with a linear combination of states in the coupled representation,
iv. ANY arbitrary orthonormal basis constructed with a linear combination of states in the uncoupled representation,
v. Neither coupled nor uncoupled representation.

Students were also asked to determine a good basis for the limiting case of the weak field Zeeman effect (in a question identical to Q1 except that the phrase ‘strong field Zeeman effect’ was replaced by ‘weak field Zeeman effect’). The correct answer for the strong field Zeeman effect is option ii and the correct answer for the weak field Zeeman effect is option i. Below, we discuss difficulties with corrections to the energies due to the strong and weak field Zeeman effects.

4. Student difficulties

Students had several difficulties with DPT in general (i.e. not restricted to the limiting context of the strong and weak field Zeeman effects only). For example, when students were asked to determine a good basis for finding the corrections to the energies of the hydrogen atom due to fine structure, many students did not even realize that DPT should be used. Other students knew that they had to use DPT to find corrections to the wavefunction, but they did not use DPT to find the first-order corrections to the energies. These students incorrectly claimed that they did not need to use DPT since no terms in \( E^0_{n\ell} = \langle \psi^0_n | \hat{H} | \psi^0_n \rangle \) ‘blow up’. Some students only focused on the Zeeman term \( \hat{H}'_\text{Z} \) when asked to determine a good basis for finding the corrections to the energies of the hydrogen atom in the limiting cases of the strong and weak field Zeeman effect. In particular, they ignored the fine structure term \( \hat{H}'_\text{fs} \) altogether and focused on the Zeeman term as the only term in the perturbation. Moreover, even if students realized that DPT should be used for the limiting cases of the strong and weak field Zeeman effects, many of them admitted that they had memorized which representation was a good basis in a given situation. Memorization of which basis to use often masked the fact that students did not have a deep understanding of DPT. Table 1 shows that many students

| Limiting case | U (N = 20) | P (N = 42) |
|---------------|------------|------------|
| Strong field  | 40%        | 29%        |
| Weak field    | 25%        | 31%        |
struggled to identify a good basis for finding the corrections to the energy spectrum due to the limiting cases of the strong and weak field Zeeman effects. Below, we discuss some specific student difficulties:

A. Not focusing on both $\hat{H}^0$ and $\hat{H}'$ when determining a good basis: Students with this type of difficulty typically focused on the basis that make $\hat{H}^0$ diagonal but did not give consideration to $\hat{H}'$ when finding a good basis. For example, in the first step of the two-step approximation for the weak field Zeeman effect, some students incorrectly claimed that the uncoupled representation forms a good basis because it diagonalizes the operator $\hat{H}^0$. Interviews suggest that these students often did not realize that $\hat{H}'$ is not diagonal in the degenerate subspace of $\hat{H}^0$ if the uncoupled representation is chosen as the basis and the corrections to the energies using this representation will yield incorrect values inconsistent with experiments.

B. Focusing on the degeneracy in the perturbation $\hat{H}'$ instead of the degeneracy in the unperturbed Hamiltonian when determining a good basis in step 1 or 2: When determining whether DPT should be used and whether a basis is a good basis, some students incorrectly focused on the degenerate subspaces of $\hat{H}'$ instead of $\hat{H}^0$. For example, when students were asked to find the energy corrections in the first step of the two-step approximation, some students incorrectly focused on the degeneracy in the stronger perturbation $\hat{H}'_{\text{strong}}$ to determine whether DPT should be used and whether the basis provided was good. In particular, they focused on whether the degenerate subspaces in $\hat{H}'_{\text{strong}}$ were diagonal to determine if the basis was good (instead of whether $\hat{H}'_{\text{strong}}$ was diagonal in the degenerate subspaces of $\hat{H}^0$). An analogous student difficulty was also prevalent in step 2 of the two-step approximation. In particular, in order to determine whether a basis is a good basis for the strong or weak field Zeeman effect in step 2, students must identify the degenerate subspaces of $\hat{H}'_{\text{strong}} = \hat{H}' + \hat{H}'_{\text{strong}}$ and determine whether or not the weaker perturbation $\hat{H}'_{\text{weak}}$ is diagonal in each degenerate subspace of $\hat{H}'_{\text{strong}}$. However, many students incorrectly focused on the degeneracy and degenerate subspaces of $\hat{H}'_{\text{weak}}$ instead of the degenerate subspaces of $\hat{H}'_{\text{strong}}$ to determine whether DPT should be used and if the basis provided was good.

For example, during the portion of the interview regarding the strong field Zeeman effect, in step 2, students were given the strong field Zeeman Hamiltonian $\hat{H}'_{\text{strong}} = \hat{H}' + \hat{H}'_{\text{strong}}$ from step 1 and the weaker perturbation $\hat{H}'_{\text{weak}}$ in matrix form in the uncoupled representation for $n = 2$ (since the uncoupled representation is a good basis for step 1 of the two-step approximation method). The students were then asked to identify the $\hat{H}'_{\text{strong}}$ matrix in each degenerate subspace of $\hat{H}'_{\text{strong}} = \hat{H}' + \hat{H}'_{\text{strong}}$ and explain whether or not the uncoupled representation forms a good basis in step 2 of the 2-step approximation method. In the $n = 2$ subspace with $s = \frac{1}{2}$, the $\hat{H}'_{\text{strong}} = \hat{H}' + \hat{H}'_{\text{strong}}$ matrix provided to students to probe their understanding is the following in which the basis states are chosen in the uncoupled representation $(l, m_l, m_m)$ in the order $\left|0, 0, 0\right\rangle, \left|0, 0, -1\right\rangle, \left|1, 1, 1\right\rangle, \left|1, 1, -1\right\rangle, \left|1, 0, 1\right\rangle, \left|1, 0, -1\right\rangle, \left|1, -1, 1\right\rangle, \text{ and } \left|1, -1, -1\right\rangle$: 
The basis states were chosen in a different order than in equations (3)–(6) to probe whether students were able to identify the degenerate subspace of \( \hat{H}_Z^0 \) when basis states with the same energies are not in the adjacent rows/columns.

The \( \hat{H}_Z^0 \) matrix has three separate two-fold degeneracies for the energies \( E_2 + \beta, E_2 - \beta, \) and \( E_2 \) as indicated by the boxed, underlined, and circled matrix elements of \( \hat{H}_Z^0 \) above. In order to determine whether a basis consisting of states in the uncoupled representation forms a good basis, \( \hat{H}_s' \) must be diagonal in each of these three degenerate subspaces of \( \hat{H}_Z^0 \). The \( \hat{H}_s' \) matrix in the \( n = 2 \) subspace in which the basis states are chosen in the same order as they were for the \( \hat{H}_Z^0 \) matrix above is as follows:

\[
\hat{H}_s' = \begin{bmatrix}
  E_2 + \beta & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & E_2 - \beta & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & E_2 + 2\beta & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & E_2 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & E_2 + \beta & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & E_2 - \beta & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & E_2 - 2\beta \\
\end{bmatrix}.
\] (7)

From the boxed matrix elements, \( \hat{H}_s' \) in the degenerate subspace of \( \hat{H}_Z^0 \) for the degenerate energy \( E_2 + \beta \) is \( \begin{bmatrix} 15 & 0 \\ 0 & 7 \end{bmatrix} \). Similarly, as given by the underlined matrix elements, \( \hat{H}_s' \) in the degenerate subspace of \( \hat{H}_Z^0 \) for the degenerate energy \( E_2 - \beta \) is \( \begin{bmatrix} 15 & 0 \\ 0 & 7 \end{bmatrix} \) and from the circled matrix elements, \( \hat{H}_s' \) in the degenerate subspace of \( \hat{H}_Z^0 \) for the degenerate energy \( E_2 \) is \( \begin{bmatrix} 15 & 0 \\ 0 & 7 \end{bmatrix} \).

However, students often did not realize that they should focus on the degeneracy of the Hamiltonian \( \hat{H}_Z = \hat{H}_s^0 + \hat{H}_s' \) and instead they focused on the degeneracy of the weak perturbation \( \hat{H}_s' \) by examining the diagonal matrix elements of \( \hat{H}_s' \) that were equal. In particular, they incorrectly focused on whether the degenerate subspaces of \( \hat{H}_s' \) were diagonal in order to determine whether a given basis is a good basis. For example, they focused on the degenerate subspace \( \begin{bmatrix} 15 & 0 \\ 0 & 7 \end{bmatrix} \) in \( \hat{H}_s' \). However, the degeneracy of the weaker perturbation \( \hat{H}_s' \) is not relevant to determining a good basis. Instead, one should identify the degenerate subspaces of \( \hat{H}_Z = \hat{H}_s^0 + \hat{H}_s' \) and determine if the weaker perturbation \( \hat{H}_s' \) is diagonal in the degenerate subspaces of \( \hat{H}_Z = \hat{H}_s^0 + \hat{H}_s' \) to decide whether a given basis is a good basis in step 2 of the two-step process.
C. Incorrectly claiming that $\hat{H}_{\text{weak}}$ must be diagonal in each degenerate subspace of $\hat{H}^0$ (instead of $\hat{H}^0_{\text{strong}}$) in a good basis when using the two-step approximation: Many students incorrectly claimed that, in a good basis for step 2 of the two-step approximation, $\hat{H}_{\text{weak}}$ must be diagonal in the degenerate subspace of $\hat{H}^0$ as opposed to the degenerate subspaces of $\hat{H}^0_{\text{strong}}$. They did not realize that when using the two-step approximation in the limiting cases in step 2, the weaker perturbation $\hat{H}_{\text{weak}}$ need only be diagonal in each degenerate subspace of the stronger Hamiltonian $\hat{H}^0_{\text{strong}}$ obtained after step 1 (as opposed to each degenerate subspace of $\hat{H}^0$). In the strong field Zeeman effect, a basis consisting of states in the uncoupled representation forms a good basis. The matrices to be considered in step 2 for the new unperturbed Hamiltonian $\hat{H}^0_{\text{strong}} = \hat{H}^0_Z$ and the weaker perturbation $\hat{H}_{\text{fs}}$ are given below for the $n = 2$ subspace, in which basis states are chosen in the uncoupled representation ($l, m_l, m_s$) in the order $\left| 0, 0, \frac{1}{2} \right>, \left| 1, 0, \frac{1}{2} \right>, \left| 0, 0, -\frac{1}{2} \right>, \left| 1, 0, -\frac{1}{2} \right>, \left| 1, 1, -\frac{1}{2} \right>, \left| 1, 1, \frac{1}{2} \right>, \left| 1, -1, \frac{1}{2} \right>, \left| 1, 1, \frac{1}{2} \right>, \text{and} \left| 1, -1, -\frac{1}{2} \right>$:

$$\hat{H}^0_Z = \begin{pmatrix}
E_2 + \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E_2 + \beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & E_2 - \beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & E_2 - \beta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & E_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & E_2 + 2\beta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & E_2 - 2\beta
\end{pmatrix}$$

(9)

$$\hat{H}_{\text{fs}} = \frac{(-13.6 \text{ eV})\alpha^2}{192} \begin{pmatrix}
15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 7 & 0 & 0 & 4\sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 15 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 7 & 0 & 4\sqrt{2} & 0 & 0 \\
0 & 4\sqrt{2} & 0 & 0 & 11 & 0 & 0 & 0 \\
0 & 0 & 0 & 4\sqrt{2} & 0 & 11 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix}$$

(10)

In step 2, despite the fact that the weaker perturbation $\hat{H}_{\text{weak}} = \hat{H}_{\text{fs}}$ is not diagonal in the degenerate subspace of $\hat{H}_{\text{strong}}$ (as seen by the matrix elements $4\sqrt{2}$ above for the $n = 2$ subspace) when the uncoupled representation is chosen as the basis, the weaker perturbation $\hat{H}_{\text{fs}}$ is diagonal in each degenerate subspace of $\hat{H}_{\text{strong}} = \hat{H}^0_Z = \hat{H}^0 + \hat{H}_{\text{fs}}$ after accounting for the splitting of the energy levels due to the stronger perturbation $\hat{H}^0_Z$ (as seen by the boxed subspaces in the weaker perturbation $\hat{H}_{\text{fs}}$). Many students struggled with the fact that the weaker perturbation $\hat{H}_{\text{fs}}$ must only be diagonal in each degenerate subspace of $\hat{H}_{\text{fs}}$ in step 2. For example, one interviewed student claimed ‘the uncoupled is not a good basis (for strong field Zeeman effect) since $\hat{H}_{\text{fs}}$ is not diagonal in the uncoupled representation. So we will have
off-diagonal (matrix) elements (in $\hat{H}_{\text{Z}}^i$). Thus, many students had difficulty with the two-step approximation involving the limiting cases (strong or weak field Zeeman effect). They struggled to identify when it was valid to use the two-step approximation and in connecting these limiting cases with the intermediate field Zeeman effect in the appropriate limit.

D. Not realizing that some of the degeneracy is broken after taking into account the stronger perturbation, allowing $\hat{H}_{\text{weak}}^i$ to be diagonal in each degenerate subspace of $\hat{H}_{\text{strong}}^0$. Many students struggled with the fact that the utility of the two-step approximation for the limiting cases of the strong and weak field Zeeman effects lies in the fact that some of the degeneracy is broken in step 1 of the two-step approximation when the stronger perturbation $\hat{H}_{\text{strong}}^0$ is treated as the only perturbation on the unperturbed Hamiltonian $\hat{H}^0$. They did not realize that, in general, after taking into account the stronger perturbation in step 1, the dimension of some of the degenerate subspaces is reduced. Therefore, in step 2, when $\hat{H}_{\text{strong}}^0 = \hat{H}^0 + \hat{H}_{\text{strong}}^0$ is treated as the new unperturbed Hamiltonian, the degeneracy of energy spectrum $\hat{H}_{\text{strong}}^0 = \hat{H}^0 + \hat{H}_{\text{strong}}^0$ is less than the degeneracy of $\hat{H}^0$, making it possible for the weaker perturbation $\hat{H}_{\text{weak}}^i$ to be diagonal in the degenerate subspaces of $\hat{H}_{\text{strong}}^0 = \hat{H}^0 + \hat{H}_{\text{strong}}^0$. For example, in the strong field Zeeman effect, a basis consisting of states in the uncoupled representation forms a good basis for $\hat{H}^0$ and $\hat{H}_{\text{Z}}^i$ in step 1 and also in step 2 for $\hat{H}_{\text{strong}}^0$ and $\hat{H}_{\text{weak}}^i$. However, students often did not realize that for $n = 2$, the degeneracy in the new unperturbed Hamiltonian $\hat{H}_{\text{Z}}^0 = \hat{H}^0 + \hat{H}_{\text{Z}}^i$ is reduced to three separate two-fold degeneracies and two energies with degeneracy of one (instead of an 8-fold degeneracy in the unperturbed Hamiltonian $\hat{H}^0$). In particular, they did not realize that in step 2, in the uncoupled representation, the weaker perturbation $\hat{H}_{\text{Z}}^i$ is diagonal in each of these $2 \times 2$ subspaces of the Hamiltonian $\hat{H}_{\text{Z}}^0 = \hat{H}^0 + \hat{H}_{\text{Z}}^i$ so that the uncoupled representation is a good basis for finding the corrections.

In interviews, students often argued that neither a basis consisting of states in the coupled representation nor a basis consisting of states in the uncoupled representation is a good basis even in the limiting cases since neither is a good basis for both the Zeeman term $\hat{H}_{\text{Z}}^i$ and the fine structure term $\hat{H}_{\text{fs}}^i$. They claimed that even in the limiting cases, one must find a basis that diagonalizes both the Zeeman term $\hat{H}_{\text{Z}}^i$ and the fine structure term $\hat{H}_{\text{fs}}^i$. Further probing suggests that they often did not realize that in the limiting cases, some of the degeneracy is lifted after step 1 in the two-step process so that the basis chosen in step 1 remains a good basis in step 2.

E. Difficulty connecting the first-order corrections to the energy spectrum in the intermediate field Zeeman effect with the two-step approximation in the appropriate limit: Prior to considering the limiting cases of the strong and weak field Zeeman effect during the interview, students worked through examples involving the intermediate field Zeeman effect (in which $\hat{H}_{\text{Z}}^i \approx \hat{H}_{\text{fs}}^i$). For the intermediate field Zeeman effect, the Zeeman term and the fine structure term are on equal footing and must be treated simultaneously as the perturbation on the unperturbed Hamiltonian $\hat{H}^0$. Since neither a basis consisting of states in the coupled representation nor a basis consisting of states in the uncoupled representation form a good basis for the intermediate field Zeeman effect, students must choose an initial basis (either the coupled or uncoupled representation) and then diagonalize $\hat{H}^i = \hat{H}_{\text{Z}}^i + \hat{H}_{\text{fs}}^i$ in each degenerate subspace of $\hat{H}^0$ in order to determine a good basis and find the first-order corrections to the energy spectrum.

After working through examples involving the intermediate field Zeeman effect in the interview, students considered the two limiting cases of the strong and weak field Zeeman
effects. For the intermediate field Zeeman effect, one obtains first-order corrections to the energies in which both the Zeeman term, $\hat{H}_Z$, and the fine structure term, $\hat{H}_{fs}$, are treated as perturbations simultaneously on the unperturbed Hamiltonian, $\hat{H}^0$. However, in the limit that the energy corrections due to one of these two perturbations are much larger than the other, one can use the expressions for the energy corrections for the intermediate field Zeeman effect and perform a Taylor series expansion about the small parameter that accounts for the smaller correction. The results obtained by the two-step approximation method in the limiting cases (strong and weak field Zeeman effects) yield the same corrections to the energy spectrum as those obtained by the Taylor series expansion of the corrections to the energy spectrum in the intermediate field Zeeman effect when retaining terms of the same order. Many students struggled to connect these limiting cases to the intermediate field Zeeman effect. They did not realize that under the appropriate limits, the first-order corrections to the intermediate field Zeeman effect for the hydrogen atom are consistent with the first-order corrections in the strong and weak field Zeeman effects when using the two-step approximation method. In fact, interviews suggest that some students viewed the limiting cases of the strong and weak field Zeeman effect as entirely separate problems and did not think of these limiting cases as related at all to their previous work on the intermediate field Zeeman effect.

Other students had difficulty correctly expressing a Taylor series expansion of the intermediate field Zeeman effect in order to determine the first-order corrections to the energy spectrum for the limiting cases in the strong and weak field Zeeman effects. For example, one interviewed student claimed that in order to obtain the results for the weak field Zeeman effect, ‘for the Taylor expansion (of the intermediate field Zeeman corrections to the first-order energies), we can set $B_{ext} = 0$ and all the terms for the magnetic field drop out.’ Another interviewed student claimed that in order to obtain the results for the strong field Zeeman effect from the general case, we should simply ‘let $B_{ext}$ go to infinity’ in the expression for the energy corrections for the intermediate field Zeeman effect. When asked what the first-order corrections to the energies would be if we let $B_{ext}$ go to infinity, the student responded, ‘Well, they would go to infinity. Wait that’s not right, is it? I think I need to go back and review how to do a Taylor series expansion.’

It is important that students realize that the first-order corrections to the energy spectrum in the limiting cases of the strong and weak field Zeeman effects match those obtained in the intermediate field Zeeman effect when taking the appropriate limits. This realization would help students make sense of the use of the two-step approximation method for the limiting cases and recognize that the errors in the two-step approximation are comparable to those obtained by retaining terms of a certain order in a Taylor series expansion.

5. Methodology for development of the QuILT

5.1. Development and validation of the QuILT

The difficulties described show that many students struggle in determining a good basis for finding the corrections to the energies in the limiting cases of the strong and weak field Zeeman effects. Therefore, we developed a QuILT that takes into account these difficulties and strives to help students build a robust knowledge structure of these concepts. The development of the DPT QuILT started with an investigation of student difficulties via open-ended and multiple-choice questions administered after traditional instruction to advanced undergraduate and PhD level students and conducting a cognitive task analysis from an expert perspective [64]. The cognitive task analysis was conducted by three physics education
researchers together and discussed with members of the physics faculty who teach (or had taught) QM. It described not only the requisite knowledge and skills one would need in order to have a functional understanding of DPT, but also the order in which the material should be presented to help ensure that the material in each section built upon that in previous sections. The investigation of student difficulties also informed the cognitive task analysis in that we were able to fill in any ‘gaps’ (due to expert blind spots) in the expert cognitive task analysis based upon students’ perspectives. The QuILT strives to help students build on their prior knowledge and addresses common difficulties found via research, some of which were discussed in the previous section.

The QuILT is inspired by Piaget’s ‘optimal mismatch’ framework as well as the preparation for future learning framework of Bransford and Schwartz. In Piaget’s ‘optimal mismatch’ framework, students are intentionally placed in a situation in which their current knowledge structures are inadequate and the students are required to reorganize existing structures or develop new structures to reconcile this conflict [65]. Bransford and Schwartz’s preparation for future learning framework emphasizes that learning occurs when elements of innovation and efficiency are both present [66]. Innovation and efficiency describe two orthogonal components of instruction. Innovation describes aspects that are new to students, such as new concepts or new problem-solving skills. Efficiency is a measure of the structure and organization of the material, as well as how proficient the student is with the material. Instruction that incorporates only one of these elements leads to students becoming disengaged. If instruction is too innovative, students cannot connect the material with their prior knowledge and become frustrated. When the instruction is too efficient, students interact with repetitious material that does not provide intellectual stimulation and may become routine experts. However, they will not be able to transfer their learning to new situations.

In the QuILT, students are presented with innovative tasks that strive to create a cognitive conflict. The QuILT then provides scaffolding aimed at resolving their cognitive conflict. For example, students are asked to consider conversations between hypothetical students in which one student makes an incorrect statement involving a common difficulty while other students make statements that illuminate inconsistencies in the incorrect statement. They must decide which hypothetical students are correct and explain their reasoning. The goal is to create a cognitive conflict and have students realize that there is some inconsistency between their thoughts and the correct reasoning. After this when students want to resolve the conflict, further scaffolding is provided in order to resolve the inconsistencies and to help students reconcile their initial reasoning with the correct reasoning. Whether it be examples, hypothetical conversations, or calculations, the QuILT strives to help students develop a deeper understanding by actively working through the inquiry-based learning sequences. Student difficulties are incorporated in these examples and conversations to create a cognitive conflict in which they are then guided through future tasks designed to resolve these issues. Efficiency is addressed in the QuILT in several ways. First, the QuILT follows the sequence laid out in the cognitive task analysis. It is organized in a manner which attempts to build on the students’ prior knowledge, and each section in the QuILT builds upon the previous section. Second, students are provided scaffolding designed to help address common difficulties, thus reducing the cognitive conflict. Third, the QuILT progressively reduces the scaffolding to help students solve problems without any assistance. Finally, as the students work through the different tasks, they develop more proficiency at identifying the concepts and answering the questions.

The development of the QuILT went through a cyclic, iterative process. The preliminary version was developed based upon the cognitive task analysis and knowledge of common student difficulties. Next, the QuILT underwent many iterations among the three researchers
and then was iterated several times with three physics faculty members to ensure that they agreed with the content and wording. It was also administered to graduate and advanced undergraduate students in individual think-aloud interviews to ensure that the guided approach was effective, the questions were unambiguously interpreted, and to better understand the rationale for student responses. During these semi-structured interviews, students were asked to think aloud while answering the questions [67]. Students first read the questions on their own and answered them without interruptions except that they were prompted to think aloud if they were quiet for a long time (in order to not disrupt their thought processes). After students had finished answering a particular question to the best of their ability, they were asked to further clarify and elaborate on issues that they had not clearly addressed earlier. The next step involved evaluating the impact of the QuILT on student learning and determining if the difficulties remained. Finally, modifications and improvements were made based upon the student and faculty feedback before it was administered to students in various QM courses.

5.2. Structure of the QuILT

The QuILT uses a guided inquiry-based approach to learning and actively engages students in the learning process. In an inquiry-based approach, students take an active role by answering questions throughout the task which requires that they work through problems and reflect upon the underlying concepts. A guided inquiry-based approach has these same features, but it also incorporates scaffolding based upon the expert and novice cognitive task analyses that provides students with support to help ensure they develop robust understanding while working on the task. It includes a pretest to be administered in class after traditional instruction in DPT. Then, students engage with the tutorial in small groups in class (or alone when using it as a self-paced learning tool in homework), and then a posttest is administered in class. As students work through the tutorial, they are asked to predict what should happen in a given situation. Then, the tutorial strives to provide scaffolding and feedback as needed to bridge the gap between their initial knowledge and the level of understanding that is desired. Students are also provided checkpoints to reflect upon what they have learned and make explicit connections between what they are learning and their prior knowledge. They are given an opportunity to reconcile differences between their predictions and the guidance provided in the checkpoints of the QuILT before proceeding further.

The DPT QuILT uses a blend of guided learning sequences involving both qualitative and quantitative reasoning to improve students’ understanding. For example, the QuILT requires qualitative reasoning while students reason about hypothetical student conversations and quantitative reasoning to determine the matrix elements of the operators $\hat{H}_{SO}$ and $\hat{H}_{Z}$ in the coupled and uncoupled representations.

5.3. Addressing student difficulties via guided learning sequences in the QuILT

In the guided inquiry-based learning sequences in the QuILT, students actively engage with examples involving DPT in which they consider the terms $\hat{H}_r^I$, $\hat{H}_{SO}^I$, $\hat{H}_c^I$, $\hat{H}_Z^I$, and $\hat{H}_{c}^I + \hat{H}_Z^I$ as perturbations on $\hat{H}^1$ ($\hat{H}_r^I$ is the relativistic correction term and $\hat{H}_{SO}^I$ is the spin–orbit interaction term). In this manner, students focus on the fundamental concepts for determining a good basis for the fine structure and Zeeman corrections to the energy spectrum of the hydrogen atom. In particular, for the unperturbed Hamiltonian $\hat{H}^0$ and the perturbation $\hat{H}' = \hat{H}_c + \hat{H}_Z$, students learn about (1) why DPT must be used, (2) why care must be taken to choose a good basis, and (3) how to determine a good basis using the two-step approximation method for the limiting cases of
the strong and weak field Zeeman effects. Below, we discuss how the QuILT strives to address student difficulties and help them learn about the perturbative corrections to the energy spectrum of the hydrogen atom due to the strong and weak field Zeeman effects using the DPT.

Students first work through a warm-up for the tutorial that strives to help them identify the bases that diagonalize each of the operators $\hat{H}^0$, $\hat{H}^I$, $\hat{H}_{SO}^I$, $\hat{H}_{I},$ and $\hat{H}^Z$ in the $n = 2$ degenerate subspace of $\hat{H}^0$. In addition, students also work through examples in which they must determine matrix elements of each of the operators $\hat{H}^I_{SO}$ or $\hat{H}^I_{Z}$. For example, they calculate several diagonal and off-diagonal matrix elements of $\hat{H}^I_{SO}$ and $\hat{H}^Z$ in both a basis consisting of states in the coupled representation and a basis consisting of states in the uncoupled representation. The warm-up strives to help students learn the prerequisites for finding a good basis for the hydrogen atom in the limiting cases of the strong and weak field Zeeman effect in the context of DPT.

Helping students identify $\hat{H}_{weak}$ in the degenerate subspace of $\hat{H}_{strong}$: In the QuILT, students are provided scaffolding that strives to help them develop systematic reasoning and build upon their prior knowledge for each step in the two-step approximation method of the DPT for the limiting cases of the strong and weak field Zeeman effects. In particular, students work through several inquiry-based sequences in which they must determine a good basis for step 1 and the corresponding first-order corrections to the energies. Next, they are provided guidance and asked to identify a good basis for step 2. Students are then provided the matrices for the operators $\hat{H}^I_{strong}$ and $\hat{H}^I_{weak}$ which allow them to identify degeneracy in the new unperturbed Hamiltonian $\hat{H}^I_{strong}$ and also identify $\hat{H}^I_{weak}$ in each degenerate subspace of $\hat{H}^I_{strong}$. They can reconcile whether their initial choice of a good basis is correct and also determine the first-order corrections to the energy spectrum for step 2 if the given basis is a good basis.

For example, students work through an inquiry-based sequence for the strong field Zeeman effect in which they start by determining a good basis for the first step when the stronger perturbation $\hat{H}^I_Z$ is treated as the only perturbation on the unperturbed Hamiltonian $\hat{H}^0$ (in this case the answer is exact for the first-order correction since $\hat{H}^0$ and $\hat{H}^Z$ commute and if states in the uncoupled representation is chosen as the basis, they are simultaneous eigenstates of both). Once the students identify a good basis for step 1, they engage with an inquiry-based sequence to identify $\hat{H}^I_{weak} = \hat{H}^I_{c}$ in the degenerate subspace of $\hat{H}^I_{strong} = \hat{H}^0 + \hat{H}^I_Z$. The following is an example of an inquiry-based sequence students work through for the two-step approximation method in the context of the strong field Zeeman effect.

Q4. STEP 1:

For the case $\hat{H}^I_Z \gg \hat{H}^I_{c}$, we treat only $\hat{H}^I_Z$ as the perturbation on $\hat{H}^0$.

Q4(a) For the case $\hat{H}^I_Z \gg \hat{H}^I_{c}$, what is a good basis for step 1 when we ignore $\hat{H}^I_{c}$? Explain.

Q4(b) Write an expression for the first-order corrections to the energies due to only the stronger perturbation $\hat{H}^I_Z$ acting on the unperturbed Hamiltonian $\hat{H}^0$ (once you have found a good basis). Here the first-order corrections are the exact results for the energies after STEP 1.

Q5. STEP 2:

In the strong field Zeeman effect when $\hat{H}^I_Z \gg \hat{H}^I_{c}$ in step 2, the unperturbed Hamiltonian includes the Zeeman term and becomes

$$\hat{H}^0 = \hat{H}^0 + \hat{H}^Z = -\frac{\hbar^2}{2m} \nabla^2 - \frac{e^2}{4\pi \epsilon_0 r} + \frac{e}{2m} B_{ext}(\hat{L}_z + 2\hat{S}_z).$$

(11)
Is the $\hat{H}_Z^0$ matrix a diagonal matrix if the coupled representation or the uncoupled representation is chosen as the basis? Explain your reasoning.

**Q6.** Now for the $n=2$ subspace, take a look at the $\hat{H}_Z^0 = \hat{H}_Z^0 + \hat{H}_f^\prime$ and $\hat{H}_f^\prime$ matrices given below in which $E_2 = -\frac{13.6 \text{ eV}}{4}$ and the basis vectors are chosen in the **uncoupled** representation ($\langle l, m_l, m_s \rangle$ in the order $|\psi_1\rangle = |0, 0, \frac{1}{2}\rangle$, $|\psi_2\rangle = |0, 0, -\frac{1}{2}\rangle$, $|\psi_3\rangle = |1, 1, \frac{1}{2}\rangle$, $|\psi_4\rangle = |1, 1, -\frac{1}{2}\rangle$, $|\psi_5\rangle = |1, 0, \frac{1}{2}\rangle$, $|\psi_6\rangle = |1, 0, -\frac{1}{2}\rangle$, $|\psi_7\rangle = |1, -1, \frac{1}{2}\rangle$, and $|\psi_8\rangle = |1, -1, -\frac{1}{2}\rangle$. Then answer questions Q6(a)–(c) for the strong field Zeeman effect.

\[
\hat{H}_Z^0 = \begin{bmatrix}
E_2 + \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E_2 - \beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & E_2 + 2\beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & E_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & E_2 + \beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E_2 - \beta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & E_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & E_2 - 2\beta
\end{bmatrix}
\]

(12)

\[
\hat{H}_f^\prime = \frac{(-13.6 \text{ eV})\alpha^2}{192}
\begin{bmatrix}
15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 15 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 11 & 4\sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 4\sqrt{2} & 7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 7 & 4\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 4\sqrt{2} & 11 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0
\end{bmatrix}
\]

(13)

**Q6(a)** Determine the degeneracy of the energy eigenvalues of the new unperturbed Hamiltonian $\hat{H}_Z^0 = \hat{H}_Z^0 + \hat{H}_f^\prime$ after accounting for the stronger perturbation and circle the corresponding degenerate subspaces of $\hat{H}_Z^0 = \hat{H}_Z^0 + \hat{H}_f^\prime$ (for $n=2$ subspace) in the preceding matrix representation.

**Q6(b)** Circle $\hat{H}_f^\prime$ in each degenerate subspace of $\hat{H}_Z^0 = \hat{H}_Z^0 + \hat{H}_f^\prime$ and determine if $\hat{H}_f^\prime$ in any of these subspaces of $\hat{H}_Z^0$ is diagonal.

**Q6(c)** Determine whether the uncoupled representation chosen as the basis in question Q5 is a good basis for the unperturbed Hamiltonian $\hat{H}_Z^0 = \hat{H}_Z^0 + \hat{H}_f^\prime$ and the perturbation $\hat{H}_f^\prime$. Explain how you made the determination.

This inquiry-based sequence is designed to help students focus on the necessary requirements for a **good** basis in steps 1 and 2. For example, Q6(a) strives to help students realize that some of the degeneracy has been lifted after step 1 and that one needs to focus on the degeneracy of $\hat{H}_Z^0$. To determine whether a basis consisting of states in the uncoupled representation is a **good** basis for step 2, the students work to identify whether $\hat{H}_f^\prime$ is diagonal in each degenerate subspace of $\hat{H}_Z^0$ in Q6(b). In Q6(c), the students are asked to describe whether a basis consisting of states in the uncoupled representation forms a **good** basis. Based upon their answers to Q6(a) and (b), the QuILT strives to help students identify $\hat{H}_f^\prime$ in the degenerate subspace of $\hat{H}_Z^0$ and find that a basis consisting of states in the uncoupled
representation is a good basis for the strong field Zeeman effect. Students work through a similar inquiry-based sequence for the weak field Zeeman effect.

Helping students realize that some of the degeneracy is broken after step 1 and that $\hat{H}_{\text{weak}}$ need only be diagonal in each degenerate subspace of $\hat{H}_{\text{strong}}^0$: It is also important that students realize that the utility of the two-step approximation lies in the fact that some of the degeneracy is broken in the first step, which allows for the weaker perturbation to be diagonal in the degenerate subspace of $\hat{H}_{\text{strong}}^0 = \hat{H}^0 + \hat{H}_{\text{strong}}^1$ in step 2. In the QuILT, students consider the following hypothetical students’ statements in the context of the strong field Zeeman effect that are intended to help students realize that some of the degeneracy is broken after the first step and that in step 2, the weaker perturbation is diagonal in each degenerate subspace of $\hat{H}_{\text{strong}}^0$.

**Student 1:** In step 1, when we only consider $\hat{H}_{\text{Z}}^0$ as the perturbation on $\hat{H}^0$, we choose the uncoupled representation as the good basis. Once the uncoupled representation is chosen as the good basis, we are guaranteed to have off-diagonal matrix elements in the weaker fine structure perturbation matrix $\hat{H}_{\text{fs}}^0$.

**Student 2:** Actually, once we treat the stronger Zeeman perturbation $\hat{H}_{\text{Z}}^0$ in the first step, we lift some of the degeneracy in the energy spectrum of $\hat{H}^0$. There is still degeneracy in the energy spectrum $E_n^0 = E_n + \mu_B B_{\text{ext}} (m_l + 2m_s)$ after the first step, but now the degeneracy is present in smaller subspaces of $\hat{H}^0$. For example, for the $n=2$ subspace in step 2, $\hat{H}_{\text{Z}}^0 = \hat{H}^0 + \hat{H}_{\text{Z}}^1$ is

$$\hat{H}_{\text{Z}}^0 = \begin{bmatrix}
E_2 + \beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E_2 - \beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & E_2 + 2\beta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (E_2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & E_2 + \beta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E_2 - \beta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & E_2 - 2\beta
\end{bmatrix}.$$ (14)

In the uncoupled representation, $\hat{H}_{\text{Z}}^0$ is not diagonal in the entire $n=2$ subspace, but it is diagonal in each degenerate subspace of $\hat{H}_{\text{Z}}^0 = \hat{H}^0 + \hat{H}_{\text{Z}}^1$. In the $\hat{H}_{\text{fs}}^0$ matrix below, the elements in the degenerate subspace of $\hat{H}_{\text{Z}}^0$ corresponding to the degenerate energy $E_2 + \beta$ are boxed. We see $\hat{H}_{\text{fs}}^0$ is diagonal in the 2 × 2 subspace corresponding to the degenerate energy $E_2 + \beta$.

$$\hat{H}_{\text{fs}}^0 = \frac{(-13.6 \text{ eV})\alpha^2}{192} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 15 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 11 & 4\sqrt{2} & 0 \\
0 & 0 & 4\sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 7 & 4\sqrt{2} \\
0 & 0 & 0 & 0 & 4\sqrt{2} & 11 \\
0 & 0 & 0 & 0 & 0 & 3
\end{bmatrix}.$$ (15)

In the uncoupled representation, $\hat{H}_{\text{Z}}^1$ is not diagonal in the entire $n=2$ subspace, but it is diagonal in each degenerate subspace of $\hat{H}_{\text{Z}}^1 = \hat{H}^0 + \hat{H}_{\text{Z}}^1$. In the $\hat{H}_{\text{fs}}^1$ matrix below, the elements in the degenerate subspace of $\hat{H}_{\text{Z}}^1$ corresponding to the degenerate energy $E_2 + \beta$ are boxed. We see $\hat{H}_{\text{fs}}^1$ is diagonal in the 2 × 2 subspace corresponding to the degenerate energy $E_2 + \beta$. 
Similarly, $\hat{H}_f$ is diagonal in the degenerate subspace of $\hat{H}_Z^0$ for the degenerate energies $E_2$ and $E_2 - \beta$. Therefore, the uncoupled representation does form a good basis in this two-step process.

In the preceding hypothetical conversation, Student 1 is correct in that the weaker perturbation $\hat{H}_s$ will have off-diagonal matrix elements if a basis consisting of states in the uncoupled representation is chosen. However, Student 1 does not realize that when using the two-step approximation method for the strong field Zeeman effect, $\hat{H}_s$ need only be diagonal in each degenerate subspace of $\hat{H}_Z^0$. Student 2’s statement is designed to have students reflect upon the fact that some of the degeneracy is lifted after step 1 and as a result, the off-diagonal matrix elements of $\hat{H}_s$ are not in any of the degenerate subspaces of $\hat{H}_Z^0$. Thus, a basis consisting of states in the uncoupled representation is a good basis for the strong field Zeeman effect.

Helping to connect the first-order corrections to the energies in the intermediate field Zeeman effect to those found in the limiting cases using the two-step approximation method: The QuILT strives to help students identify a good basis and determine the first-order corrections to the energy spectrum in the limiting cases of the strong and weak field Zeeman effects. Additionally, the QuILT strives to help students learn that the resulting first-order corrections to the energies in the two limiting cases are consistent with the first-order corrections to the energies in the intermediate field Zeeman effect when one takes the appropriate limit. In an effort to help students make these connections, the QuILT asks the following question:

**Q7.** The splitting of the energy levels for the $n = 2$ states of the hydrogen atom in the intermediate field Zeeman effect are given in table 2 below. Use the appropriate Taylor series expansion to check that the corrections to the energies in the intermediate field Zeeman effect are consistent with the corrections found in the limiting cases of the strong and weak field Zeeman effects earlier. $(E_2 = \frac{13.6 \text{eV}}{4}, \gamma = \frac{\alpha}{\hbar} = 13.6 \text{eV}, \alpha = \frac{e^2}{4\pi\varepsilon_0\hbar}, \beta = \mu_B B_{\text{ext}}$ were defined previously in the QuILT.)

| Table 2. Energy levels in the intermediate field Zeeman effect ($n = 2$). |
|---------------------------------------------------------------|
| $\epsilon_1 = E_2 - 5\gamma + \beta$                         |
| $\epsilon_2 = E_2 - 5\gamma - \beta$                        |
| $\epsilon_3 = E_2 - \gamma + 2\beta$                       |
| $\epsilon_4 = E_2 - \gamma - 2\beta$                       |
| $\epsilon_5 = E_2 - 3\gamma + \beta/2 + \sqrt{4\gamma^2 + (2/3)\gamma\beta + \beta^2/4}$ |
| $\epsilon_6 = E_2 - 3\gamma + \beta/2 - \sqrt{4\gamma^2 + (2/3)\gamma\beta + \beta^2/4}$ |
| $\epsilon_7 = E_2 - 3\gamma - \beta/2 + \sqrt{4\gamma^2 - (2/3)\gamma\beta + \beta^2/4}$ |
| $\epsilon_8 = E_2 - 3\gamma - \beta/2 - \sqrt{4\gamma^2 - (2/3)\gamma\beta + \beta^2/4}$ |

For example, in the strong field limit ($\gamma \ll \beta$), we have $\sqrt{4\gamma^2 + \frac{2}{3}\gamma\beta + \frac{1}{2}\beta^2} \approx \frac{1}{2}\beta \pm \frac{2}{3}\gamma$. Using the Taylor series expansion of the energy levels in the intermediate field Zeeman effect (see table 2) and taking the appropriate limit for the strong field Zeeman effect, one can show that the energy levels match those found using the two-step approximation method.

Students are also given an opportunity to make sense of a graph depicting the relationship between the splitting of the energy levels and strength of the external magnetic field. They are asked to compare the results they obtained for the intermediate field Zeeman effect and the limiting cases of the strong and weak field Zeeman effect and discuss whether their
results are consistent with the graph and whether the intermediate field expression yields the limiting values in the appropriate limits.

6. Evaluation of the QuILT

Once the researchers determined that the QuILT was successful in one-on-one implementation using a think-aloud protocol, it was administered in PhD level and upper-level undergraduate QM courses. Students in both PhD level and upper-level undergraduate courses were given a pretest after traditional instruction in relevant concepts in DPT but before working through the tutorial. The pretests were not returned to the students after grading. The undergraduates worked through the tutorial in class for two days and were asked to work on the remainder of the tutorial as homework. The PhD level students were given the tutorial as their only homework assignment for the week. After working through and submitting the completed tutorial, both groups were given the posttest in class. Students were given enough time in class to work through the pretest and posttest.

The pre/posttest results for Q1 (as shown in section III) are summarized in table 3 and suggest that the QuILT was helpful in reducing student difficulties with these concepts. In particular, over 80% of PhD level students and nearly 80% of the undergraduate students were able to correctly identify that a basis consisting of states in the uncoupled representation is a good basis for the strong field Zeeman effect and a basis consisting of states in the coupled representation is a good basis for the weak field Zeeman effect.

Table 4 shows the performance of undergraduate and PhD level students on the pretest and posttest. The average score includes both the answer for the multiple-choice question and the students’ explanation for Q1. Table 4 also includes the average gain, G, and normalized gain \[
\text{gain} = \frac{\text{posttest percent} - \text{pretest percent}}{(100 - \text{pretest percent})}.
\] Both undergraduate and PhD level students struggled with this topic as can be seen by the scores on the pretest. However, both groups showed significant improvement after working through the QuILT.

Q1 was graded using a rubric which was developed by the researchers together. Each question was worth six points. A maximum of four points were awarded for the multiple-choice portion of Q1 and two points were awarded for their explanation. For example, when grading the multiple-choice portion of Q1 for the weak field Zeeman effect, students were given four points for correctly choosing only the coupled representation (option i). If they chose the coupled representation (option i) and ANY arbitrary orthonormal linear combination of states in the coupled representation (option iii), they were given two out of four points for the multiple-choice portion of Q1. We found that some interviewed students correctly reasoned that the degeneracy in the energy spectrum of \( H_{\text{fs}} \) allowed for linear combinations of

| Limiting case | Undergraduate students (%) | PhD level students (%) |
|---------------|-----------------------------|------------------------|
|               | Pretest | Posttest | Pretest | Posttest |
| Strong field  | 40     | 79       | 39     | 83       |
| Weak field    | 25     | 79       | 33     | 81       |
states in the coupled representation with the same $n$, $l$ and $j$ to diagonalize $\hat{H}_\text{fs}$. However, it is not the case that ANY linear combination of states in the coupled representation diagonalizes $\hat{H}_\text{fs}$ in the degenerate subspace of $\hat{H}^0$. While these students did not show entirely correct reasoning, they were correctly thinking about issues caused by the degeneracy in the energy spectrum but incorrectly overgeneralized these concepts to reason that ANY linear combination of states in the coupled representation diagonalizes $\hat{H}_\text{fs}$ in the degenerate subspace of $\hat{H}^0$. Students were given one out of four points if they chose both the coupled and uncoupled representation (options i and ii) as a basis that diagonalizes $\hat{H}_\text{fs}$ in the degenerate subspace of $\hat{H}^0$. Some students incorrectly claimed that the operator $\hat{H}_\text{fs} = \hat{H}_\text{fs} + \hat{H}_\text{SO}$ is diagonal in both the coupled and uncoupled representations because $\hat{H}_\text{SO} = \frac{1}{2}(\hat{J}^2 - \hat{S}_z^2 - \hat{L}_z^2) = \frac{1}{2}(\hat{L}_z\hat{S}_z + \hat{L}_z\hat{S}_z) + \hat{L}_z\hat{S}_z$. Although states in the coupled representation are eigenstates of the operators $\hat{J}^2$, $\hat{L}^2$, and $\hat{S}_z^2$, states in the uncoupled representation are eigenstates of $\hat{L}_z$ and $\hat{S}_z$ but they are not eigenstates of the operators $\hat{L}_z$ and $\hat{S}_z$. Therefore, $\hat{H}_\text{SO}$ is diagonal in the coupled representation in the degenerate subspace of $\hat{H}^0$ but not diagonal in the uncoupled representation. This type of response shows some correct reasoning ($\hat{H}_\text{fs}$ is diagonal in the degenerate subspace of $\hat{H}^0$ in the coupled representation) and was awarded partial credit. No points were awarded for any other combination of answers for the weak field Zeeman effect in Q1. The explanations that provided correct reasoning were awarded two points, responses with mostly correct reasoning were awarded one point, and responses with little or no correct reasoning were given zero points.

A subset of questions was graded separately by the researchers with a final inter-rater reliability of nearly 100%.

The following are written responses to Q1 for the weak field Zeeman effect ($\hat{H}_\text{fs} \ll \hat{H}_\text{fs}^0$) taken from two students’ posttests who correctly chose a basis consisting of states in the coupled representation as a good basis: ‘Neither really form a good basis, but we can use a 2-step process. 1st considering only the $\hat{H}_\text{fs}^0$ perturbation then some of the degeneracy is lifted. Then solve a simplified problem with $\hat{H}_\text{fs}^0$ as a perturbation on $\hat{H} + \hat{H}_\text{fs}$.’ 2-step process: $\hat{H}' = \hat{H}_\text{fs}^0$ is diagonal (in the $n = 2$ degenerate subspace of $\hat{H}^0$) in coupled representation. $\hat{H}'$ diagonal in the degenerate subspace of $\hat{H}_\text{fs}^0$ (lifted degeneracy).’ Student responses on the posttests were analogous in the context of the strong field Zeeman effect. These types of responses by students on the posttest demonstrate that they had learned to reason about how to find a good basis in the limiting cases of the strong and weak field Zeeman effect.
7. Summary

Using the common difficulties of advanced students with finding the corrections to the energies of the hydrogen atom in the limiting cases of the strong and weak field Zeeman effects, we developed and evaluated a research-based QuILT which strives to help students learn to reason about and find a good basis for these limiting cases. In particular, the QuILT strives to help students learn that when the corrections to the energies due to either the Zeeman term $\hat{H}_Z$ or the fine structure term $\hat{H}_f$ are much larger than the other, one can perform DPT using a two-step approximation. The QuILT strives to help students learn about these limiting cases to determine when it is appropriate to use this two-step approximation and be able to reason that these results in the limiting cases are consistent with the corrections to the energies in the intermediate field Zeeman effect in the appropriate regimes. Students learn to reason that not only are the results in various limits consistent with the general case, but there is a benefit to using the two-step approximation method in that a good basis can be determined without explicitly diagonalizing the perturbation $\hat{H}'$ in each degenerate subspace of $\hat{H}^0$ by ‘brute force’. The QuILT uses these limiting case examples to help students develop expert-like reasoning skills and help them learn to think like a physicist. In particular, learning to use limiting cases and understanding why the limits can be obtained from the general case (i.e. intermediate field Zeeman effect) can be useful for developing the problem-solving and metacognitive skills of an expert physicist. The QuILT strives to provide appropriate scaffolding and feedback using a guided inquiry-based approach to help students develop a functional understanding of relevant concepts. The evaluation shows that the QuILT is effective in improving students’ understanding of the perturbative corrections to the energy spectrum of the hydrogen atom placed in an external magnetic field in the limiting cases of the strong and weak field Zeeman effect in the context of DPT.

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