Boundary charges in gauge theories: using Stokes theorem in the bulk

Glenn Barnich

Physique Théorique et Mathématique, Université Libre de Bruxelles, Campus Plaine CP 231, B-1050 Bruxelles, Belgium

Received 17 April 2003
Published 28 July 2003
Online at stacks.iop.org/CQG/20/3685

Abstract
Boundary charges in gauge theories (such as the ADM mass in general relativity) can be understood as integrals of linear conserved \( n - 2 \) forms of the free theory obtained by linearization around the background. These forms are associated one-to-one with reducibility parameters of this background (such as the time-like Killing vector of Minkowski spacetime). In this paper, closed \( n - 2 \) forms in the full interacting theory are constructed in terms of a one-parameter family of solutions to the full equations of motion that admits a reducibility parameter. These forms thus allow one to apply Stokes theorem without bulk contributions and, provided appropriate fall-off conditions are satisfied, they reduce asymptotically near the boundary to the conserved \( n - 2 \) forms of the linearized theory. As an application, the first law of black-hole mechanics in asymptotically anti-de Sitter spacetimes is derived.

PACS numbers: 11.15.-q, 11.30.-j, 04.20.Ha, 04.70.Bw

1. Introduction
Both Lagrangian [1–3] and Hamiltonian [4, 5] approaches involve in some way the idea that it is the linearized theory around the background that determines the asymptotically conserved \( n - 2 \) forms used for the construction of boundary charges in gauge theories in \( n \) spacetime dimensions. Recent results from variational calculus [6] (see [7] for a review and also [8, 9]) corroborate this point of view:

• when restricted to solutions of the equations of motion, equivalence classes of closed, local, \( n - 2 \) forms up to exact, local, \( n - 2 \) forms correspond one-to-one to non-trivial reducibility parameters; in this context, reducibility parameters are possibly field-dependent gauge parameters such that the associated gauge transformations vanish on solutions of the equations of motion; such parameters are trivial if they vanish themselves on solutions of the equations of motion;

1 Research Associate of the National Fund for Scientific Research (Belgium).
• in standard interacting gauge theories such as general relativity or semi-simple Yang–
Mills theories in spacetime dimensions strictly higher than 2, there are no non-trivial
reducibility parameters and thus no non-trivial conserved \( n - 2 \) forms; in other words,
every local \( n - 2 \) form that is closed on solutions of the equations of motion is given by
the exterior derivative of a local \( n - 3 \) form on solutions of the equations of motion;
• in linear gauge theories however, reducibility parameters may very well exist; for instance
in general relativity linearized around some background, particular reducibility parameters
are given by the Killing vectors of the background; furthermore, for the flat background in
spacetime dimensions strictly higher than 2, they can be shown to be the only non-trivial
ones [10].

In [11], the one-to-one correspondence has been extended to (suitable equivalence classes
of ) asymptotically conserved \( n - 2 \) forms on one hand and asymptotic reducibility parameters
on the other. Furthermore, for given reducibility parameters, the asymptotically conserved
\( n - 2 \) forms have been explicitly constructed out of the linearized equations of motion and
of the gauge transformations evaluated at the background. That the associated charges have
all the standard properties such as time independence or independence of the form or position
of the closed \( n - 2 \) dimensional hypersurface used in their definition is a direct consequence
of on-shell closure and of Stokes theorem. By construction however, the \( n - 2 \) forms are only
closed near the boundary, when evaluated for asymptotic solutions, i.e., deviations from the
background that satisfy the boundary conditions and the linearized field equations to leading
order. Hence, the application of Stokes theorem to relate the boundary charges to integrals
over surfaces deep in the bulk will in general involve bulk contributions.

From the point of view of the full interacting theory, different expressions for
asymptotically conserved \( n - 2 \) forms are considered as equivalent to the linear \( n - 2 \) forms
discussed above, if asymptotically near the boundary, all nonlinear terms in the field deviations
from the background vanish and if the linear terms belong to the same equivalence class (and
thus define the same boundary charges) as the \( n - 2 \) forms of the linearized theory. This
leaves of course a lot of freedom in the definition of these forms, and allows one to show,
for instance, that the expressions derived in [1, 2, 8, 12] for energy–momentum and angular
momentum in asymptotically flat general relativity are all equivalent (see also, e.g., [13] for a
recent discussion).

Motivated by the work of Wald and Iyer [14–16] (see also, e.g., [17]) on the formulation
of the first law of black-hole mechanics in terms of Noether charge, we will construct in this
work \( n - 2 \) forms of the full interacting theory that are closed in a region of the bulk, provided
that

• the \( n - 2 \) form is constructed using a one-parameter family of solutions to the full equations
  of motion valid in the region of the bulk where one wants to use Stokes theorem;
• the exact reducibility parameters of the background are simultaneously exact reducibility
  parameters of the one-parameter family of solutions to the full equations of motion.

As a result, for these one-parameter family of solutions, the boundary charges are related
to the integrals of the \( n - 2 \) forms over surfaces deep in the bulk. Furthermore, if the Taylor
expansions in the parameter of the \( n - 2 \) forms satisfy suitable fall-off conditions near the
boundary, these forms reduce to the \( n - 2 \) forms of the linearized theory discussed previously
and thus correctly describe the boundary charges.

In the following section, we briefly review, in the context of the linearized theory that is
supposed to describe the full theory asymptotically near the boundary, the expression for the
linear conserved \( n - 2 \) forms associated with the reducibility parameters of the background.
Section 3 contains the main result on how the conserved \( n - 2 \) forms of the linearized
theory should be modified so that Stokes theorem can be used without bulk contributions. In section 4, the theorem of section 3 is applied to the well-known cases of Yang–Mills theory and Einstein gravity with cosmological constant. It is shown explicitly how the improved $n-2$ forms can be used to express conservation of total energy. As an application, a derivation of the first law of black-hole mechanics for asymptotically anti-de Sitter spacetimes is presented. In the conclusion, the results that have been obtained are discussed from the point of view of the original derivation of the first law and comments on the relation to other approaches are made.

2. Construction of the linear conserved $n-2$ forms of the free theory

Let $R^a_\alpha[\phi]$ denote a generating set of gauge transformations \[18\], with associated gauge symmetries $\delta_f \phi^i = R^a_\alpha(f^a)$, where the parameters $f^a$ are local functions, i.e., they may depend on $x^\mu$, the fields $\phi^i$ and a finite number of their derivatives.

For example, in Einstein–Maxwell theory with cosmological constant $\Lambda$, described by the action,

$$S = \int d^n x \sqrt{|g|} \left[ R - 2\Lambda - F_{\mu\nu}F^{\mu\nu} \right],$$

the fields $\phi^i$ correspond to $g_{\mu\nu}, A_\mu$, the metric and the electromagnetic gauge potentials, respectively; the gauge parameters $f^a$ correspond to $\xi^\mu, \lambda$, the parameters of an infinitesimal diffeomorphism and an infinitesimal $U(1)$ gauge transformation, respectively, while $\delta_f \phi^i = R^a_\alpha(f^a)$ corresponds to

$$\delta_{\xi,\lambda} g_{\mu\nu} = L_\xi g_{\mu\nu},$$

$$\delta_{\xi,\lambda} A_\mu = L_\xi A_\mu + \partial_\mu \lambda,$$

with $L_\xi$ denoting the Lie derivative.

For all $Q_i$ $d^n x$ with $Q_i$ local functions, we define the current $n-1$ form $S^\mu_\alpha[\phi](Q_i, f^a) = S^\mu_\alpha(Q_i, f^a)(d^{n-1} x)_\mu$ through

$$Q_i R^a_\alpha(f^a) = f^a R^a_i(Q_i) d^n x + d_H S^\mu_\alpha(Q_i, f^a).$$

In this equation, $R^a_\alpha$ denote the associated generating set of Noether operators, obtained from the gauge symmetries by the integrations by parts that move the derivatives from the gauge parameters to the $Q_i$. The operator $d_H = dx^\mu \partial_\mu$, with $\partial_\mu$ the total derivative with respect to $x^\mu$, is the horizontal differential and $d^{n-1} x = \epsilon_{\mu_1 \cdots \mu_n} dx^{\mu_1} \cdots dx^{\mu_n}$ with $\epsilon_{0 \cdots n-1} = 1$. For example, if

$$Q_i R^a_\alpha(f^a) = Q_i f^a R^a_\alpha + Q_i R^a_\mu \partial_\mu f^a,$$

then

$$f^a R^a_i(Q_i) = f^a R^a_i Q_i - f^a \partial_\mu \left[R^a_\mu(Q_i) \right],$$

$$S^\mu_\alpha(Q_i, f^a) = Q_i R^a_\mu f^a.$$

The defining property of the Noether operators are the Noether identities

$$R^a_i \left( \frac{\delta L}{\delta \phi^i} \right) = 0.$$

In the Einstein–Maxwell example considered above, these identities correspond to the contracted Bianchi identities $D_\mu(G^{\mu\nu} + \Lambda g^{\mu\nu}) = 0$ and $D_\mu D_\nu F^{\mu\nu} = 0$. 

When the $Q_i$ are replaced with the left-hand side of the Euler–Lagrange equations of motion in (4), one gets, because of the Noether identities (8),

$$
\frac{\delta L}{\delta \phi^i} R^i_a(f^a) = d_H S_f, \quad S_f \equiv S_a \left( \frac{\delta L}{\delta \phi^i}, f^a \right).
$$

(9)

In other words, $S_f$ is a weakly vanishing representative for the Noether current $n-1$ form associated with the gauge symmetry $\delta_f \phi^i = R^i_a(f^a)$.

In the Einstein–Maxwell example, $S_f$ is explicitly given by

$$
S_{\xi, \lambda} = \sqrt{\frac{16}{16\pi}} \left[ \left( -G^{\mu\nu} + \Lambda g^{\mu\nu} + 8\pi T_{em}^{\mu\nu} 2\xi_{\nu} + 4D_{\nu} F^{\mu\nu}(\xi^\rho A_\rho + \lambda) \right) \left( d x^{\mu-1} x \right)_\mu, \right]
$$

(10)

where the electromagnetic energy–momentum tensor is

$$
T_{em}^{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta L_{em}}{\delta g^{\mu\nu}} = \frac{1}{4\pi} \left[ F^{\mu\alpha} F_{\alpha}^{\nu} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right].
$$

(11)

Let $\bar{\phi}^i(x)$ be a background solution to the Euler–Lagrange equations of motion and $\phi^i = \bar{\phi}^i(x) + \phi^i$. We denote by $S_{\delta L/\delta \phi^i} f^a$ the equations of motion linearized around $\bar{\phi}^i(x)$, obtained from the quadratic piece of the Lagrangian in an expansion according to $\phi^i$. If $f^{0\alpha} = f^a [x, \bar{\phi}]$, the linearization around $\bar{\phi}^i(x)$ of (9) gives

$$
\frac{\delta L}{\delta \phi^i} R^i_a(f^{0\alpha}) = d_H S_f, \quad S_f = S_a \left( \frac{\delta L_{free}}{\delta \phi^i}, f^{0\alpha} \right),
$$

(12)

where $S^{0i}_a = S_a \left[ \bar{\phi}(x) \right]$. For field independent parameters $f^a$ that satisfy

$$
R^i_a(f^{0\alpha}) = 0,
$$

(13)
equation (12) implies that

$$
d_H S_f = 0.
$$

(14)

In the Einstein–Maxwell example, solutions to (13) are given by the Killing vectors of the background metric, $L_{\xi} \bar{\phi}^{\mu\nu} = 0$, which satisfy in addition $L_{\xi} \bar{A}_{\mu} + \partial_\mu \lambda = 0$ for some gauge parameter $\lambda$.

In the case of trivial topology, equation (14) implies that $s_f = d_H s_f$, for some $n-2$ form $\bar{k}_f[\psi; \bar{\phi}]$. Provided the equations of motion $\frac{\delta L}{\delta \phi^i}$ and the gauge transformations $R^i_a(f^a)$ are local, in the sense that they depend only on a finite number of derivatives of the fields, the $n-2$ form $\bar{k}_f[\psi; \bar{\phi}]$ can be constructed to be local as well, in the sense that it depends (linearly) on the $\phi^i$ and a finite number of their derivatives, and on a finite number of derivatives of the background $\bar{\phi}^i$. How to explicitly construct $\bar{k}_f$ out of $s_f$ has been explained in many references, see, e.g., [19–28]. In the general relativity literature for instance, an algorithm has been given in [29]. In fact, the whole theory of ‘black-hole entropy from Noether charge’ [14–17] relies crucially on this algorithm. Whereas in these references, the algorithm is used in terms of arbitrary gauge parameters, we will use the explicit formula for the ‘contracting homotopy’ involving higher-order Euler operators due to Anderson [30, 31], in terms of the fields $\phi^i$ of the linearized theory below, and in terms of the fields $\phi^i$ of the full theory in the following section. Indeed, in the case of trivial topology, for local forms $\omega^p$ of degree $p$.

2 The general case of field-dependent gauge parameters $\tilde{f}^a$ that satisfy $R^i_a(f^{0\alpha}) \approx_{free} 0$, where $\approx_{free}$ means an equality that holds on solutions of the equations of motion of the linearized theory, is treated in section 3 of [11]. It can be shown that, when evaluated on solutions of the linearized theory, the expression of the $n-2$ forms constructed below is still valid in the more general case.
strictly lower than \( n \) that vanish when the \( \phi^i \) and their derivatives are set to zero, there exists an operator \( \rho_{H,\phi} \) such that
\[
d_H \left( \rho_{H,\phi}^n \omega^p \right) + \rho_{H,\phi}^{n+1} (d_H \omega^p) = \omega^p.
\]
(15)

It follows that the \( n - 2 \) form of the linearized theory defined by
\[
\tilde{k}_j = \rho_{H,\phi}^{n-1} s^j,
\]
(16)

is closed when the linearized equations of motion hold:
\[
d_H \tilde{k}_j = s^j \approx \text{free 0}.
\]
(17)

The explicit expression for \( \tilde{k}_j[\phi; \bar{\phi}] \) is
\[
\tilde{k}_j[\phi; \bar{\phi}] = \sum_{k=0}^{\infty} \frac{k+1}{k+2} \partial_{\mu_1} \cdots \partial_{\mu_k} \left[ \phi^j \frac{\delta}{\delta \phi^j_{\mu_1} \cdots \mu_k} \right],
\]
(18)

with the understanding that the first term in the sum does not contain an index \( \mu \) and no total derivative. Furthermore, the notation \( \frac{\delta}{\delta \phi^j_{\mu_1} \cdots \mu_k} \) stands for the contraction of a form with the vector \( \frac{\partial}{\partial \phi^j} \), while the higher-order Euler operators \( \frac{\delta}{\delta \phi^j_{\mu_1} \cdots \mu_k} \) are constructed out of the symmetrized partial derivatives \( \frac{\partial}{\partial \phi^j_{\mu_1} \cdots \mu_k} \). The detailed combinatorial factors involved in the definitions of these operators can be found in [31] or in appendix A of [11]. Note, however, that the definition of \( \tilde{k}_j[\phi; \bar{\phi}] \) used here differs by an overall minus sign from the one used in [11].

The \( n - 2 \) forms \( \tilde{k}_j[\phi; \bar{\phi}] \) have been explicitly computed in [11] for Yang–Mills theory and Einstein gravity with cosmological constant and the results of [2, 3] and [8] have been recovered. From the point of view of [2, 3] the contracting homotopy \( \rho_{H,\phi}^n \) provides a systematic way of converting ‘volume integrals to surface integrals’.

In the following section, we construct an \( n - 2 \) form in the full theory that, under suitable assumptions, is closed in the bulk and reduces asymptotically to the \( n - 2 \) form (18) of the linearized theory if appropriate fall-off conditions are satisfied.

3. Construction of the closed \( n - 2 \) forms of the full theory

The most general form of the homotopy formula for \( d_H \) (given in [31], chapter 4, pp 119–22) allows one to interpolate between a form evaluated at two different field configurations by using a (not necessarily straight) path that connects these configurations: if \( \phi^i(x), s \in [0, 1] \) is a one-parameter family of field configurations and \( \phi^i_j(x) = \frac{d \phi^i_j(x)}{d s} \), one can show that
\[
S_f \left[ \phi_1(x) \right] - S_f \left[ \phi_0(x) \right] = \int_0^1 d s \frac{d}{d s} (S_f[\phi_s])
\]
\[
= \int_0^1 d s \left( \sum_{k=0}^{n} \partial_{\mu_1} \cdots \partial_{\mu_k} \phi^j_s \left[ \frac{\delta}{\delta \phi^j_{\mu_1} \cdots \mu_k} S_f \right] \right)
\]
\[
= d_H \rho_{H,s}^{n-1} S_f + \rho_{H,s}^n d_H S_f,
\]
(19)

where for a \( p \) form \( \omega^p[\phi] \),
\[
\rho_{H,s}^p \omega^p = \int_0^1 d s I^p_{\phi_s(x)} (\omega^p)[\phi_s(x)],
\]
(20)

with
\[
I^p_{\phi_s}(\omega^p) = \sum_{k=0}^{n} \frac{k+1}{n-p+k+1} \partial_{\mu_1} \cdots \partial_{\mu_k} \phi^j_s(x) \frac{\delta}{\delta \phi^j_{\mu_1} \cdots \mu_k} \omega^p \left( \frac{\partial}{\partial d x^\rho} \right).
\]
(21)
If we define
\[ K_f = \rho_H^{n+1} S_f, \tag{22} \]
it follows using (9) that
\[ d_H K_f = -\rho_H^n \left( \frac{\delta L}{\delta \phi^i} R^a_a(f^a) \, \delta \phi^i \right) \, d^n x + S_f[\phi_1(x)] - S_f[\phi_0(x)], \tag{23} \]
for an arbitrary configuration \( \phi_i(x) \). Using the explicit expression of \( \rho_H^n \), one can then show:

**Theorem 1.** For a given solution \( \phi^i(x) \) of the full equations of motion, the \( n-2 \) form \( K_f \), defined in (22), is closed in the bulk,
\[ d_H K_f = 0, \tag{24} \]
provided

- a one-parameter family \( \phi^i_s(x) \) of solutions to the full equations of motion interpolating between \( \phi^i(x) \) and the background solution \( \bar{\phi}^i(x) \) is used,
\[ \frac{\delta L}{\delta \phi^i} [\phi_s(x)] = 0, \quad \forall \, s \in [0, 1], \tag{25} \]
- the gauge parameters \( f^a = \hat{f}^a \) are reducibility parameters for this interpolating solution,
\[ R^a_a[\phi_s(x)](\hat{f}^a) = 0, \quad \forall \, s \in [0, 1]. \tag{26} \]

The proof of the theorem is given in the appendix.

The \( n-2 \) form \( K_f \) depends in general on the path \( \gamma \) in the space of solutions chosen to interpolate between \( \bar{\phi}(x) \) and \( \phi(x) \), but not on the parametrization chosen for this path\(^3\). This justifies the following notation:
\[ K_f = \int_{\gamma} I_{\phi(x)}^{n-1}(S_f)[\phi], \tag{27} \]
where \( d_V \phi^i \) stands for an infinitesimal variation of the fields. More precisely, \( K_f \) is the integral along \( \gamma \) of the vertical 1 form and horizontal \( n-2 \) form in the variational bicomplex [30]–[32].

Given appropriate fall-off conditions and an analytic expansion in \( s \), the \( n-2 \) form \( K_f \) coincides asymptotically near the boundary with the linear \( n-2 \) form \( \hat{K}_f \) discussed in the previous section: indeed, if \( \phi_s(x) = \bar{\phi}(x) + s \phi^i(x) + s^2 \phi^i_2(x) + \cdots \) and the fall-off conditions are such that, in an expansion according to \( s \), only the term independent of \( s \) in \( I_{\phi(x)}^{n-1}(S_f)[\phi_s(x)] \) contributes because all the other terms fall off too fast near the boundary, we have
\[ K_f \longrightarrow I_{\phi(x)}^{n-1}(S_f)[\hat{\phi}(x)]. \tag{28} \]
In the appendix, it is shown that this expression agrees with expression (18) of the linearized theory.

This reasoning can also be turned around to see how \( K_f \) can be constructed from \( \hat{K}_f[\psi; \hat{\phi}] \) of the linearized theory: because \( s_f = (d_V S_f)[\psi; \hat{\phi}] \), where \( d_V \) denotes a variation of the fields \( \phi^i \) and their derivatives and the argument \([\psi; \hat{\phi}]\) means that the variations of the fields are substituted by \( \psi^i \) and the fields by the background solution \( \phi^i(x), I_{\phi(x)}^{n-1}(S_f)[\phi_s(x)] \) is given by the right-hand side of (18), where \( \psi^i(x), \phi^i(x) \) are replaced by \( \psi^i_0(x), \phi^i_0(x) \). We have thus shown the following corollary to theorem 1:

\(^3\) The author wants to thank J Zanelli for drawing his attention to this point.
Corollary 1. The closed forms $K_f$ associated with a one-parameter family of solutions $\phi^i(x)$ with reducibility parameters $f^a$ can be obtained from the conserved $n-2$ forms $k_f[\psi; \hat{f}]$ of the linearized theory defined in (18) by substituting the background for the one-parameter family of solutions $\phi^i(x)$, by substituting the field deviations $\phi$ by $\phi^i(x) = \frac{d\phi^i(x)}{ds}$ and by integrating over the parameter:

$$K_f = \int_0^1 ds \, k_f[\psi^s; \hat{f}].$$

(29)

Again, if $\gamma$ is the path in the space of solutions interpolating between $\phi_i(x)$ and $\phi_i(x)$, reparametrization invariance allows one to write

$$K_f = \int_{\gamma} k_f[dV \phi; \hat{f}].$$

(30)

Consider now the path $\gamma + \delta\gamma$ where $\delta\gamma$ is the ‘straight path’ between the solution $\phi_i(x)$ and the infinitesimally close solution $\phi_i(x) + \delta\phi^i(x)$. If we define $\delta K_f$ to be the variation of $K_f$ when evaluated on $\gamma + \delta\gamma$ and on $\gamma$, it follows that

$$\delta K_f = \int_{\gamma + \delta\gamma} k_f[dV \phi; \hat{f}] - \int_{\gamma} k_f[dV \phi; \hat{f}] = k_f[\delta\phi(x); \phi(x)].$$

(31)

4. Standard applications

The $n-2$ forms of the linearized theory given by (18) have been explicitly computed, up to a conventional overall sign difference, in [11] section 6 for Yang–Mills theory and general relativity. The corresponding $n-2$ forms in the full theory constructed using (29) are briefly discussed in the following two subsections.

4.1. Yang–Mills theory

In the Yang–Mills case, the $n-2$ forms of the linearized theory agree with the ones found in [3] if $D_\mu f = 0$ is taken into account. Here $D_\mu$ is the background covariant derivative and $\tilde{f} = \tilde{f}^a T_a$ are non-Abelian gauge parameters. In the full nonlinear theory, application of (29) then gives

$$K_f = (d^{n-2}x)_{\mu\nu} \int_0^1 ds \, Tr \left( \tilde{f} f^{\mu\nu}_{stu} \right)$$

$$= (d^{n-2}x)_{\mu\nu} Tr(\tilde{f}(F^{\mu\nu}(x) - \tilde{F}^{\mu\nu}(x))),$$

(32)

where

$$D_\mu[A_s(x)] f = 0,$$

(33)

has been taken into account and $f^{\mu\nu}_{stu}(x) = D^\mu[A_s(x)] a^u_s(x) - D^\nu[A_s(x)] a^u_s(x)$ with $a^u_s = \frac{dA_s^u}{dx}$.

Equivalently,

$$K_f = (d^{n-2}x)_{\mu\nu} \int_0^1 ds \left( \partial^\mu Tr(\tilde{f} a^\nu_s) - \partial^\nu Tr(\tilde{f} a^\mu_s) \right)$$

$$= 2(d^{n-2}x)_{\mu\nu} \partial^\mu Tr(\tilde{f}(A^{\nu}(x) - \tilde{A}^{\nu}(x))).$$

(34)

In this case, the result does not depend on the path $A_{\mu}(x)$ in the space of solutions but only on the end points $A_{\mu}(x)$ and $A_{\mu}(x)$.
4.2. General relativity with cosmological constant

In the case of gravity with Lagrangian
\[ L = \frac{1}{16\pi} \sqrt{-g} (R - 2\Lambda), \]
direct application of (18) gives \( n - 2 \) forms in the linearized theory that agree with those of [2]. According to (29), the \( n - 2 \) forms in the nonlinear theory are obtained by replacing the background \( \bar{g}_{\mu\nu} \) by a one-parameter family of solutions \( g^s_{\mu\nu} \) and the metric deviations \( h_{\mu\nu} \) by \( h^s_{\mu\nu} = \frac{dg^s_{\mu\nu}}{ds} \) and integrating the resulting expression over \( s \). Dropping for notational simplicity the \( s \) dependence, one finds

\[ K_{\xi} = \frac{1}{16\pi} (d^{n-2}x)_{\mu\nu} \int_0^1 ds \sqrt{-g} \left( \tilde{\xi}^\nu D^\mu h + \tilde{\xi}^\mu D_\nu h^{\sigma\mu} + \tilde{\xi}_\sigma D^\nu h^{\sigma\mu} \right. \]
\[ + \left. \frac{1}{2} h D^\nu \tilde{\xi}^\mu + \frac{1}{2} h^{\mu\sigma} D_\sigma \tilde{\xi}^\nu + \frac{1}{2} h^{\nu\sigma} D_\sigma \tilde{\xi}^\mu - (\mu \leftrightarrow \nu) \right) \]
\[ = -\frac{1}{16\pi} (d^{n-2}x)_{\mu\nu} \int_0^1 ds \sqrt{-g} \left( \tilde{\xi}_\mu D_\nu H^{\rho\sigma\mu\nu} + \frac{1}{2} H^{\rho\sigma\mu\nu} \partial_{\rho} \tilde{\xi}_\sigma \right), \]
\[ (35) \]

where \( H^{\rho\sigma\mu\nu}[h; g] \) has the symmetries of the Riemann tensor:
\[ H^{\rho\sigma\mu\nu}[h; g] = -\hat{h}^{\rho\beta} g^{\nu\sigma} - \hat{h}^{\mu\nu} g^{\rho\beta} + \hat{h}^{\mu\nu} g^{\rho\beta} + \hat{h}^{\rho\beta} g^{\nu\sigma}, \]
\[ (37) \]
\[ \hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h. \]
\[ (38) \]

In these expressions, it is understood that \( g_{\alpha\beta} = g^s_{\alpha\beta}(x) \) is the metric used to define the covariant derivative and, together with its inverse, to lower and raise the indices, with \( h = h^s_{\mu\nu} \). In [11], it has been pointed out that using the Killing equation \( D_\mu \tilde{\xi}_\nu + D_\nu \tilde{\xi}_\mu = 0 \), the corresponding expression in the linearized theory agrees with the simplified expression given in equation (11) of [8]. Accordingly, in the full theory, using the Killing equation
\[ D_\mu \tilde{\xi}_\nu + D_\nu \tilde{\xi}_\mu = 0, \]
\[ (39) \]
an equivalent expression is
\[ K_{\xi} = \frac{1}{16\pi} (d^{n-2}x)_{\mu\nu} \int_0^1 ds \sqrt{-g} \left( h^{\mu\sigma} D_\nu h^{\alpha\sigma} - D^{\nu} h^{\mu\sigma} \right. \]
\[ + \left. \tilde{\xi}^\mu (D_\nu h^{\alpha\sigma} - D^{\nu} h) - (\mu \leftrightarrow \nu) \right) \]
\[ = -\frac{1}{16\pi} (d^{n-2}x)_{\mu\nu} \int_0^1 ds \sqrt{-g} \left[ K^g_{\xi} [g_\nu(x)] + \frac{1}{16\pi} K^g_{\xi} [\bar{g}(x)] - \int_0^1 ds \tilde{\xi}^\nu \frac{\partial}{\partial x^\nu} \Theta[g_\nu(x), h_\sigma(x)] \right], \]
\[ (41) \]
where
\[ K^g_{\xi} [g] = (d^{n-2}x)_{\mu\nu} \sqrt{-g} (D^\mu \tilde{\xi}^\nu - (\mu \leftrightarrow \nu)), \]
\[ (42) \]

As pointed out in [8], their expression (11) in the linearized theory agrees with \( d_\nu Q - \tilde{\xi} \cdot \Theta \), where \( d_\nu \) denotes a field variation and \( Q \) and \( \Theta \) are defined in equations (61) respectively (63) of [15]. Using this observation in the context of the full theory, one finds that \( K_{\xi} \) can be decomposed into a piece involving the Komar integrands depending only on the end points of the path and a path-dependent piece that is proportional to the undifferentiated Killing vector:

\[ K_{\xi} = \int_0^1 ds \left( -\frac{1}{16\pi} \frac{d}{dx} K^g_{\xi} [g_\nu(x)] - \tilde{\xi}^\nu \frac{d}{dx^\nu} \Theta[g_\nu(x), h_\sigma(x)] \right) \]
\[ = -\frac{1}{16\pi} K^g_{\xi} [g_\nu(x)] + \frac{1}{16\pi} K^g_{\xi} [\bar{g}(x)] - \int_0^1 ds \tilde{\xi}^\nu \frac{\partial}{\partial x^\nu} \Theta[g_\nu(x), h_\sigma(x)], \]
\[ (41) \]
is the Komar integrand and
\[ \Theta[g, h] = \frac{1}{16\pi} (d^{n-1} x)_{\mu} \sqrt{-g} (D_{\mu} h^{\mu \nu} - D^{\mu} h). \] (43)

4.3. The first law of black-hole mechanics

As an application, we consider four-dimensional asymptotically anti-de Sitter, stationary and axisymmetric black-hole spacetimes, such as the (uncharged) Kerr anti-de Sitter black holes [33] (see also, e.g., [34, 35] for recent discussions of the first law in this context). In this particular case, the path that interpolates between anti-de Sitter space \( \tilde{g}_{\mu \nu} \) and a given Kerr anti-de Sitter black hole \( g_{\mu \nu}(x) = g_{\mu \nu}^1(x) \) with fixed \( M \) and \( a \) can for instance be chosen to be \( g_{\mu \nu}(x) = g_{\mu \nu}^1(s M, s a) \). (Note that this interpolation only needs to be valid in the region of spacetime where one wants to apply Stokes theorem.) The total energy difference \( E \) between the background and the given solution is
\[ E = \oint_{S}\int_{\Sigma} K \xi, \] (44)
where \( S\infty \) is the 2 sphere at infinity given in Boyer–Lindquist type coordinates by \( t = t_0, r = R \to \infty \), with \( t_0, R \) constant, and the stationary Killing vector field \( k \) in these coordinates can (for instance) be chosen to be \( k = \frac{\partial}{\partial t} \).

As a consequence of Stokes theorem and the fact that \( d K \xi = 0 \), conservation of total energy reduces to
\[ E = \oint_{S} K \xi, \] (45)
where \( S \) is another closed two-dimensional surface such that \( S\infty \) and \( S \) are the boundaries of some three-dimensional volume \( \Sigma \). In particular, time independence of \( E \) follows by choosing \( S \) to be the 2 sphere at infinity for some other time \( t = t_1 \).

Let \( S = H \) be the intersection of \( \Sigma \) with the event horizon \( \mathcal{H}^+ \), \( k = \xi - \Omega_H m \), where \( m \) is the axial Killing vector field given in Boyer–Lindquist type coordinates by \( \frac{\partial}{\partial \phi} \), \( \xi \) is the null generator of the horizon associated with the solution \( g_{\mu \nu}(x) = g_{\mu \nu}^1(x) \) and \( \Omega_H \) its constant angular velocity. Using expression (41) and the fact that \( m \) is tangent to \( S\infty \), a Smarr-type formula can be obtained as follows:
\[ E = \oint_{S}\int_{\Sigma} [K \xi - \Omega_H K_m] \]
\[ = \oint_{H} K \xi + \Omega_H J \]
\[ = \frac{\kappa}{8\pi} A_H + \frac{1}{16\pi} \oint_{H} K_{\xi} \left[ \tilde{g}(x) \right] - \oint_{H} \int_{0}^{1} dx \frac{\partial}{\partial x} \Theta + \Omega_H J, \] (46)
with \( A_H \) the area of the horizon, \( \kappa \) its surface gravity and
\[ J = -\oint_{S}\int_{\Sigma} K_m = \frac{1}{16\pi} \oint_{S}\int_{\Sigma} \left( K_m \left[ \tilde{g}(x) \right] - K_m^\theta \left[ \tilde{g}(x) \right] \right). \] (47)
the total angular momentum.

In order to derive the first law of black-hole mechanics in this context, we will follow closely the reasoning of [14, 15] in the asymptotically flat case. The same steps as in the previous paragraph will now be applied to \( \delta K \xi \) given in (31), with \( H \) chosen to be the bifurcation
surface $B$ of the Killing horizon of the solution $g_{\mu\nu}(x) = g^1_{\mu\nu}(x)$. If

$$\delta J = - \oint_{S^\infty} \tilde{k}_m [\delta g(x); g(x)]$$

$$= \frac{1}{16\pi} \oint_{S^\infty} \delta K^K_m$$

$$= \frac{1}{16\pi} \oint_{S^\infty} \left( \delta g \right)_{\mu\nu} \sqrt{-g} \left( \delta g^{\mu\sigma} D_\sigma \xi^\nu - \frac{1}{2} \delta g D^\mu \tilde{\xi}^\nu - (\mu \leftrightarrow \nu) \right), \quad (48)$$

with $\delta g = \delta g^\mu_\mu$, we get

$$\delta E = \oint_{S^\infty} \tilde{k}_\xi [\delta g(x); g(x)]$$

$$= \oint_{S^\infty} (\tilde{k}_\xi [\delta g(x); g(x)]) - \Omega H \tilde{k}_m [\delta g(x); g(x)]$$

$$= \oint_{B} \tilde{k}_\xi [\delta g(x); g(x)] + \Omega H \delta J$$

$$= \frac{\kappa}{8\pi} \delta A_H + \Omega H \delta J. \quad (49)$$

The last line follows if one can show that $\int_B \tilde{k}_\xi [\delta g(x); g(x)] = \frac{\kappa}{8\pi} \delta A_H$. Because $\xi$ vanishes on the bifurcation surface $B$, one sees by comparing expressions (40) and (41) (without integral over $s$ and $h_{\mu\nu}$ replaced by $\delta g_{\mu\nu}$) that only the variation of the Komar integrand can contribute,

$$\int_B \tilde{k}_\xi [\delta g(x); g(x)] = - \frac{1}{16\pi} \int_B \delta K^K_\xi. \quad \text{That this last expression reduces to} \quad \frac{\kappa}{8\pi} \delta A_H \text{can be shown by following, for instance, the proof of theorem 6.1 of [15].}$$

5. Conclusion

In standard interacting gauge theories such as semi-simple Yang–Mills theory or general relativity in spacetime dimensions strictly higher than 2, all local $n - 2$ forms that are closed on-shell are trivial in the sense that they are given, on-shell, by the exterior derivative of local $n - 3$ forms. In the linearized theory around the background, however, non-trivial on-shell closed $n - 2$ forms do exist and they are in one-to-one correspondence with non-trivial reducibility parameters of the background.

In the case where there exists a path in the space of solutions that connects the background to the solution of interest and that admits reducibility parameters, we have constructed in this paper closed $n - 2$ forms of the full interacting theory, that under suitable assumptions, reduce asymptotically to the conserved $n - 2$ forms of the linearized theory used in the definition of the boundary charges. As a consequence, Stokes theorem can be used to easily relate the boundary charges to the integral of these $n - 2$ forms on surfaces deep in the bulk, as needs to be done for instance in a derivation of the first law of black-hole mechanics.

The standard derivation of the first law is based on Komar integrals [36, 37] (see, e.g., [38] for a review). Komar integrals are very useful in this context because they allow one to directly relate integrals defined over the 2 sphere at infinity to integrals defined over the horizon, the bulk contribution that arises in a direct application of Stokes theorem being easily expressible in terms of the energy–momentum tensor. However, the Komar integrals do not provide a complete theory for boundary charges: their normalization has to be fixed by comparing to ADM-type expressions and their validity is restricted to the asymptotically flat case.
Systematic Lagrangian approaches to constructing the charges of interacting gauge theories such as general relativity are based on the linearized theory around the background \cite{2, 8, 11}. When applying Stokes theorem directly to the corresponding $n-2$ forms in order to reach surfaces deep in the bulk such as the black-hole horizon, one has to take into account the complicated bulk contribution of the nonlinear part of the field equations that have been shuffled into the right-hand side of the equations of motion in the form of an effective energy–momentum tensor.

In this paper, we have shown how complicated bulk contributions can be avoided rather easily by a straightforward improvement of the $n-2$ forms constructed in the linearized theory. Furthermore, because the $n-2$ forms constructed here are directly related to the boundary charges for non-flat backgrounds, they allow one to generalize the approach of \cite{14–17} to the first law of black-hole mechanics to more general backgrounds.

In future work, we plan to study general conditions under which the $n-2$ forms are path independent and apply the formalism to more complicated theories, for instance, the higher-curvature gravity theories considered in \cite{8, 39–41}, or more exotic black-hole solutions involving scalar fields such as those discussed in \cite{42, 43}.

Acknowledgments

The author wants to thank R Aros, M Bañados, X Bekaert, F Brandt, C Chryssomalakos, V Frolov, M Henneaux, C Martínez, R Olea, R Troncoso and J Zanelli for useful discussions. This work is supported in part by the ‘Actions de Recherche Concertées’ of the ‘Direction de la Recherche Scientifique-Communauté Française de Belgique’, by a ‘Pôle d’Attraction Interuniversitaire’ (Belgium), by IISN-Belgium (convention 4.4505.86), by Proyectos FONDECYT 1970151 and 7960001 (Chile) and by the European Commission RTN programme HPRN-CT00131, in which the author is associated with K U Leuven.

Appendix

In the first part of this appendix, we prove theorem 1 and derive sufficient conditions that guarantee that $K_f$ defined by (22) is closed in the bulk. If $\phi_i^0(x) \equiv \tilde{\phi}^0(x)$ and $\phi_i^1(x) \equiv \phi^1(x)$ are solutions to the full equations of motion, the last two terms on the right-hand side of (23) vanish. Taking into account the explicit expression for the higher-order Euler operators, the first term on the right-hand side of (23) is given by

$$\rho_{\alpha}^1 \left( \frac{\delta L}{\delta \phi^t} R_{\alpha}^i (f^\alpha) \right) d^n x = \int_0^1 ds \partial_\mu \left[ \phi_i^1 (x) \frac{\delta}{\delta \phi^t} \left( \frac{\delta L}{\delta \phi^t} \frac{\delta L}{\delta \phi^t} \right) R_{\alpha}^i (f^\alpha) \right] \left[ \phi_i^1 (x) \right] (d^{n-1} x)_\rho.$$  

(A.1)

In this equation, a multi-index notation has been used: $(\mu)$ stands for $\mu_1 \cdots \mu_k$, $|\mu|$ is the length of the multi-index, i.e., if $(\mu) = \mu_1 \cdots \mu_k$, $|\mu| = k$ and $\partial_\mu = \partial_{\mu_1} \cdots \partial_{\mu_k}$. Furthermore, repeated multi-indices involve sums over both individual indices and the length. The binomial factors in the above expression come from the definition of the higher-order Euler operators. The terms in the last line vanish if $\phi_i^1 (x)$ is a one-parameter family of solutions to the full equations of motion, i.e., if equation (25) holds, whereas the terms in the second line
vanish if the (possibly field-dependent) gauge parameters are reducibility parameters of this one-parameter family of solutions, i.e., \( f^\nu = \tilde{f}^\nu \) and equation (26) holds. This finishes the proof of theorem 1.

In the second part of the appendix, we will show that

\[
I_{\phi(x)}(S_{\tilde{f}})[\tilde{\phi}(x)] = \left[ \frac{|\mu| + 1}{|\mu| + 2} \theta(\mu) \frac{\delta S_{\tilde{f}}}{\delta \phi(\mu, x)} \right][\tilde{\phi}(x)]
\]

(A.2)

agrees with (18) when the latter is evaluated on solutions \( \phi^i(x) \) of the linearized theory. The only terms that will contribute to \( K_{\tilde{f}} \) are those for which the derivatives with respect to the fields of the higher-order Euler operators act on \( \partial(\lambda) \frac{\delta L}{\delta \phi^i} \) contained in \( S_{\tilde{f}} \) because all other terms will vanish when evaluated on solutions \( \phi^i(x) \) of the equations of motion. Taking into account that

\[
L_{\text{free}} = \frac{1}{2} \frac{\partial^2 L}{\partial \phi^i(\sigma) \partial \phi^j(\tau)}[\bar{\phi}(x)] \delta(\sigma) \partial(\tau) \partial(\sigma) \partial(\tau) \phi^i \phi^j
\]

(A.3)

(A.2) agrees with (18) evaluated at \( \tilde{\phi}(x) \) because

\[
\left( \frac{\partial^S}{\partial \phi^i(\sigma)} \frac{\partial^S L}{\partial \phi^j(\tau)} \right)[\tilde{\phi}(x)] = \frac{\partial^S}{\partial \phi^i(\sigma)} \left( \frac{\partial^S L}{\partial \phi^j(\tau)} \right)[\tilde{\phi}(x)] \delta(\sigma) \partial(\tau) \phi^j
\]

(A.4)

Furthermore, if \( \phi^i(x) \) is a one-parameter family of solutions to the full equations of motion, \( \phi^i(x) \) is a solution to the linear equations of motion defined by \( L_{\text{free}} \). This can be verified by differentiating (25) with respect to \( s \) and putting \( s \) to zero.

References

[1] Misner C, Thorne K and Wheeler J 1973 Gravitation (New York: W H Freeman)
[2] Abbott L F and Deser S 1982 Stability of gravity with a cosmological constant Nucl. Phys. B 195 76
[3] Abbott L F and Deser S 1982 Charge definition in nonabelian gauge theories Phys. Lett. B 116 259
[4] Arnoffit R, Deser S and Misner C W 1961 Coordinate invariance and energy expressions in general relativity Phys. Rev. 122 997–1006
[5] Regge T and Teitelboim C 1974 Role of surface integrals in the Hamiltonian formulation of general relativity Ann. Phys., NY 88 286
[6] Barnich G, Brandt F and Henneaux M 1995 Local BRST cohomology in the antifield formalism. I. General theorems Commun. Math. Phys. 174 57–92 (Preprint hep-th/9405109)
[7] Barnich G, Brandt F and Henneaux M 1999 Local BRST cohomology in gauge theories Phys. Rep. 338 439–569 (Preprint hep-th/0002245)
[8] Anderson I M and Torre C G 1996 Asymptotic conservation laws in field theory Phys. Rev. Lett. 77 4109–13 (Preprint hep-th/9706008)
[9] Torre C G 1997 Local cohomology in field theory with applications to the Einstein equations Preprint hep-th/9706092. Lectures given at 2nd Mexican School on Gravitation and Mathematical Physics, Taxcala, Mexico, 1–7 Dec 1996
[10] Boulanger N, Damour T, Gualtieri L and Henneaux M 2001 Inconsistency of interacting, multigraviton theories Nucl. Phys. B 597 127–71 (Preprint hep-th/0007220)
[11] Barnich G and Brandt F 2002 Covariant theory of asymptotic symmetries, conservation laws and central charges Nucl. Phys. B 633 3–82 (Preprint hep-th/0111246)
[12] Landau L D and Lifshitz E M 1962 The Classical Theory of Fields (London: Pergamon)
[13] Bak D, Cangemi D and Jackiw R 1994 Energy–momentum conservation in gravity theories Phys. Rev. D 49 5173–81
[14] Wald R M 1993 Black hole entropy is Noether charge Phys. Rev. D 48 3427–31 (Preprint gr-qc/9307038)
[15] Iyer V and Wald R M 1994 Some properties of Noether charge and a proposal for dynamical black hole entropy Phys. Rev. D 50 846–64 (Preprint gr-qc/9403028)
[16] Iyer V and Wald R M 1995 A comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes Phys. Rev. D 52 4430–9 (Preprint gr-qc/9503052)
[17] Jacobson T, Kang G and Myers R C 1994 On black hole entropy Phys. Rev. D 49 6587–98 (Preprint gr-qc/9312023)
[18] Henneaux M and Teitelboim C 1992 Quantization of Gauge Systems (Princeton, NJ: Princeton University Press)
[19] Vinogradov A 1977 On the algebra-geometric foundations of Lagrangian field theory Sov. Math. Dokl. 18 1200
[20] Takens F 1979 A global version of the inverse problem to the calculus of variations J. Diff. Geom. 14 543
[21] Tulczyjew W 1980 The Euler-Lagrange resolution Lecture Notes in Mathematics 836 22
[22] Anderson I and Duchamp T 1980 On the existence of global variational principles Am. J. Math. 102 781
[23] Wilde M D 1981 On the local Chevalley cohomology of the dynamical Lie algebra of a symplectic manifold Lett. Math. Phys. 5 351
[24] Tsujishita T 1982 On variational bicomplexes associated to differential equations Osaka J. Math. 19 311
[25] Brandt F, Dragon N and Kreuzer M 1990 Completeness and nontriviality of the solutions of the consistency conditions Nucl. Phys. B 332 224–49
[26] Dubois-Violette M, Henneaux M, Talon M and Viallet C-M 1991 Some results on local cohomologies in field theory Phys. Lett. B 267 81–7
[27] Dickey L 1992 On exactness of the variational bicomplex Cont. Math. 132 307
[28] Dragon N 1996 BRS symmetry and cohomology Preprint hep-th/9602163
[29] Wald R 1990 On identically closed forms locally constructed from a field J. Math. Phys. 31 2378
[30] Olver P 1993 Applications of Lie Groups to Differential Equations 2nd edn (New York: Springer)
Olver P 1986 Applications of Lie Groups to Differential Equations 1st edn (New York: Springer)
[31] Anderson I 1989 The variational bicomplex Tech. Rep. Formal Geometry and Mathematical Physics, Department of Mathematics, Utah State University, webpage http://www.math.usu.edu/~fgmp/Pages/Publications/Publications.html.
[32] Anderson I 1992 Introduction to the variational bicomplex Mathematical Aspects of Classical Field Theory (Contemporary Mathematics vol 132) ed M Gotay, J Marsden and V Moncrief (Providence, RI: American Mathematical Society)
[33] Carter B 1968 Hamilton-Jacobi and Schrodinger separable solutions of Einstein’s equations Commun. Math. Phys. 10 280
[34] Caldarelli M M, Cognola G and Klemm D 2000 Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories Class. Quantum Grav. 17 399–420 (Preprint hep-th/9908022)
[35] Silva S 2002 Black hole entropy and thermodynamics from symmetries Class. Quantum Grav. 19 3947–62 (Preprint hep-th/0204179)
[36] Bardeen J M, Carter B and Hawking S W 1973 The four laws of black hole mechanics Commun. Math. Phys. 31 161–70
[37] Carter B 1973 Black hole equilibrium states Black Holes ed C De Witt and B De Witt (New York: Gordon and Breach) pp 58–214 (1972 Les Houches Lectures)
[38] Townsend P K 1997 Black holes Preprint gr-qc/9707012
[39] Jacobson T and Myers R C 1993 Black hole entropy and higher curvature interactions Phys. Rev. Lett. 70 3684–7 (Preprint hep-th/9305016)
[40] Deser S and Tekin B 2002 Gravitational energy in quadratic curvature gravities Phys. Rev. Lett. 89 101101 (Preprint hep-th/0205318)
[41] Deser S and Tekin B 2002 Energy in generic higher curvature gravity theories Preprint hep-th/0212292
[42] Henneaux M, Martínez C, Troncoso R and Zanelli J 2002 Black holes and asymptotics of 2+1 gravity coupled to a scalar field Phys. Rev. D 65 104007 (Preprint hep-th/0201170)
[43] Martínez C, Troncoso R and Zanelli J 2003 de Sitter black hole with a conformally coupled scalar field in four dimensions Phys. Rev. D 67 024008 (Preprint hep-th/0205319)