MINIMUM CODEGREE THRESHOLD FOR HAMILTON ℓ-CYCLES IN
k-UNIFORM HYPERGRAPHS

JIE HAN AND YI ZHAO

Abstract. We show that for sufficiently large \( n \), every \( k \)-uniform hypergraph on \( n \) vertices with minimum codegree at least \( \frac{n}{2(k-\ell)} \) contains a Hamilton \( \ell \)-cycle. This codegree condition is best possible and improves on work of Hān and Schacht who proved an asymptotic result.

1. Introduction

A well-known result of Dirac [4] states that every graph \( G \) on \( n \geq 3 \) vertices with minimum degree \( \delta(G) \geq n/2 \) contains a Hamilton cycle. In recent years, researchers have extended this result to hypergraphs in various ways (see [17] for a survey). In order to state these results, we need to define degrees and Hamilton cycles for hypergraphs.

Given \( k \geq 2 \), a \( k \)-uniform hypergraph (in short, \( k \)-graph) consists of a vertex set \( V \) and an edge set \( E \subseteq \binom{V}{k} \), where every edge is a \( k \)-element subset of \( V \). Given a \( k \)-graph \( H \) with a set \( S \) of \( d \) vertices (where \( 1 \leq d \leq k-1 \)) we define \( \deg_H(S) \) to be the number of edges containing \( S \) (the subscript \( H \) is omitted if it is clear from the context). The minimum \( d \)-degree \( \delta_d(H) \) of \( H \) is the minimum of \( \deg_H(S) \) over all \( d \)-vertex sets \( S \) in \( H \). We refer to \( \delta_1(H) \) as the minimum vertex degree and \( \delta_{k-1}(H) \) the minimum codegree of \( H \). For \( 1 \leq \ell < k \), a \( k \)-graph is a called an \( \ell \)-cycle if its vertices can be ordered cyclically such that each of its edges consists of \( k \) consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly \( \ell \) vertices. In \( k \)-graphs, a \( (k-1) \)-cycle is often called a tight cycle while a 1-cycle is often called a loose cycle. We say that a \( k \)-graph contains a Hamilton \( \ell \)-cycle if it contains an \( \ell \)-cycle as a spanning subhypergraph. Note that a \( k \)-uniform \( \ell \)-cycle on \( n \) vertices contains exactly \( n/(k-\ell) \) edges, implying that \( k-\ell \) divides \( n \).

Confirming a conjecture of Katona and Kierstead [11], Rödl, Ruciński and Szemerédi [18, 19] showed that for any fixed \( k \), every \( k \)-graph \( H \) on \( n \) vertices with \( \delta_{k-1}(H) \geq n/2 + o(n) \) contains a tight Hamilton cycle. When \( k-\ell \) divides \( k \), a \((k-1)\)-cycle on \( V \) trivially contains an \( \ell \)-cycle on \( V \) (provided \( k-\ell \) divides \( |V| \)). Thus the result in [19] implies that for all \( 1 \leq \ell < k \) such that \( k-\ell \) divides \( k \), every \( k \)-graph \( H \) on \( n \in (k-\ell)N \) vertices with \( \delta_{k-1}(H) \geq n/2 + o(n) \) contains a Hamilton \( \ell \)-cycle. It is not hard to see that these results are best possible up to the \( o(n) \) term. With long and involved arguments, Rödl, Ruciński and Szemerédi [20] determined the minimum codegree threshold for tight Hamilton cycles in \( 3 \)-graphs.

Loose Hamilton cycles were first studied by Kühn and Osthus [14], who proved that every \( 3 \)-graph on \( n \) vertices with \( \delta_2(H) \geq n/4 + o(n) \) contains a loose Hamilton cycle. It is easy to see that this is asymptotically best possible. It was generalized to arbitrary \( k \) by Keevash, Kühn, Mycroft, and Osthus [12] and to arbitrary \( k \) and arbitrary \( \ell < k/2 \) by Hān and Schacht [7].

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Theorem 1.1. [7] Fix integers $k \geq 3$ and $1 \leq \ell < k/2$. Assume that $\gamma > 0$ and $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If $\mathcal{H} = (V, E)$ is a $k$-graph on $n$ vertices such that $\delta_{k-1}(\mathcal{H}) \geq \left(\frac{1}{(k-\ell)} + \gamma\right)n$, then $\mathcal{H}$ contains a Hamilton $\ell$-cycle.

Later Kühn, Mycroft, and Osthus [13] proved that whenever $k-\ell$ does not divide $k$, every $k$-graph on $n$ vertices with $\delta_{k-1}(\mathcal{H}) \geq \left(\frac{1}{(k-\ell)} + o(n)\right)n$ contains a Hamilton $\ell$-cycle. Since $[k/(k-\ell)] = 2$ when $\ell < k/2$, this generalizes Theorem 1.1 and is best possible up to the $o(n)$ term. Recently Buss, Hán, and Schacht [1] studied the minimum vertex degree condition and proved that every 3-graph $\mathcal{H}$ on $n$ vertices with $\delta_{1}(\mathcal{H}) \geq \left(\frac{2}{16} + o(1)\right)n$ contains a loose Hamilton cycle. Recently we [9] improved this to an exact result.

Rödl and Ruciński [17, Problem 2.9] asked for the exact minimum codegree threshold for Hamilton $\ell$-cycles in $k$-graphs. The $k = 3$ and $\ell = 1$ case was answered by Cygrinow and Molla [3] recently. In this paper we determine this threshold for all $k \geq 3$ and $\ell < k/2$.

Theorem 1.2 (Main Result). Fix integers $k \geq 3$ and $1 \leq \ell < k/2$. Assume that $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If $\mathcal{H} = (V, E)$ is a $k$-graph on $n$ vertices such that
\[ \delta_{k-1}(\mathcal{H}) \geq \frac{n}{2(k-\ell)}, \]
then $\mathcal{H}$ contains a Hamilton $\ell$-cycle.

A simple well-known construction shows that Theorem 1.2 is best possible – in fact, it works for all $\ell < k$. Let $\mathcal{H}_0 = (V, E)$ be an $n$-vertex $k$-graph in which $V$ is partitioned into sets $A$ and $B$ such that $|A| = \left\lfloor \frac{n}{(k-\ell)} \right\rfloor - 1$. The edge set $E$ consists of all $k$-sets that intersect $A$. It is easy to see (e.g. [13, Proposition 2.2]) that $\delta_{k-1}(\mathcal{H}_0) = |A|$ and $\mathcal{H}_0$ contains no Hamilton $\ell$-cycle.

Using the typical approach of obtaining exact results, our proof of Theorem 1.2 consists of an extremal case and a nonextremal case.

Definition 1.3. Let $\Delta > 0$, a $k$-graph $\mathcal{H}$ on $n$ vertices is called $\Delta$-extremal if there is a set $B \subset V(\mathcal{H})$, such that $|B| = \left\lfloor \frac{2(k-\ell)-1}{(k-\ell)} \right\rfloor n$ and $e(B) \leq \Delta n^k$.

Theorem 1.4 (Nonextremal Case). For any integer $k \geq 3$, $1 \leq \ell < k/2$ and $0 < \Delta < 1$ there exists $\gamma > 0$ such that the following holds. Suppose that $\mathcal{H}$ is a $k$-graph on $n$ vertices such that $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If $\mathcal{H}$ is not $\Delta$-extremal and satisfies $\delta_{k-1}(\mathcal{H}) \geq \left(\frac{1}{(k-\ell)} - \gamma\right)n$, then $\mathcal{H}$ contains a Hamilton $\ell$-cycle.

Theorem 1.5 (Extremal Case). For any integer $k \geq 3$, $1 \leq \ell < k/2$ there exists $\Delta > 0$ such that the following holds. Suppose $\mathcal{H}$ is a $k$-graph on $n$ vertices such that $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If $\mathcal{H}$ is $\Delta$-extremal and satisfies (1.1), then $\mathcal{H}$ contains a Hamilton $\ell$-cycle.

Theorem 1.2 follows from Theorem 1.4 and 1.5 immediately by choosing $\Delta$ from Theorem 1.5.

Let us compare our proof with those in aforementioned papers. There is no extremal case in [7, 12, 13, 14] because only asymptotic results were proved. Our Theorem 1.5 is new and more general than [3, Theorem 3.1]. Following previous work [18, 19, 20, 7, 13], we prove Theorem 1.4 by using the absorbing method initiated by Rödl, Ruciński and Szemerédi. More precisely, we find the desired Hamilton $\ell$-cycle by applying the Absorbing Lemma (Lemma 2.1), the Reservoir Lemma (Lemma 2.2), and the Path-cover Lemma (Lemma 2.3). In fact, when $\ell < k/2$, the Absorbing Lemma and the Reservoir Lemma are not very difficult and already proven in [7] (in contrast, when $\ell > k/2$, the Absorbing Lemma in [13] is more difficult to prove). Thus the main step is to prove the Path-cover Lemma. As shown in [7, 13], after the Regularity Lemma is applied, it suffices to prove that the cluster $k$-graph $\mathcal{K}$ can be tiled almost perfectly by the $k$-graph $F_{k,\ell}$, whose vertex set
consists of disjoint sets $A_1, \ldots, A_{n-1}, B$ of size $k - 1$, and whose edges are all the $k$-sets of the form $A_i \cup \{b\}$ for $i = 1, \ldots, a - 1$ and all $b \in B$, where $a = \left\lceil \frac{k}{k-1} \right\rceil (k - \ell)$. In this paper we reduce the problem to tile $\mathcal{K}$ with a much simpler $k$-graph $\mathcal{Y}_{k,2\ell}$, which consists of two edges sharing $2\ell$ vertices. Because of the simple structure of $\mathcal{Y}_{k,2\ell}$, we can easily find an almost perfect $\mathcal{Y}_{k,2\ell}$-tiling unless $\mathcal{K}$ is in the extremal case (thus the original $k$-graph $\mathcal{H}$ is in the extremal case). Interestingly $\mathcal{Y}_{3,2}$-tiling was studied in the very first paper [14] on loose Hamilton cycles but as a separate problem. Our recent paper [9] indeed used $\mathcal{Y}_{3,2}$-tiling as a tool to prove the corresponding path-cover lemma. On the other hand, the authors of [3] used a different approach (without the Regularity Lemma) to prove the Path-tiling Lemma (though they did not state such lemma explicitly).

The rest of the paper is organized as follows: we prove Theorem 1.4 in Section 2 and Theorem 1.5 in Section 3, and give concluding remarks in Section 4.

**Notation.** Given an integer $k \geq 0$, a $k$-set is a set with $k$ elements. For a set $X$, we denote by $\binom{X}{k}$ the family of all $k$-subsets of $X$. Given a $k$-graph $\mathcal{H}$ and a set $A \subseteq V(\mathcal{H})$, we denote by $e_{\mathcal{H}}(A)$ the number of the edges of $\mathcal{H}$ in $A$. In this paper we often omit the subscript that represents the underlying hypergraph if it is clear from the context. Given a $k$-graph $\mathcal{H}$ with two vertex sets $S, R$ such that $|S| < k$, we denote by $\deg_{\mathcal{H}}(S, R)$ the number of $(k - |S|)$-sets $T \subseteq R$ such that $S \cup T$ is an edge of $\mathcal{H}$ (in this case $T$ is called a *neighbor* of $S$). We define $\deg_{\mathcal{H}}(S, R) = \binom{|S|}{k-|S|} - \deg(S, R)$ as the number of *non-edges* on $S \cup R$ that contain $S$. When $R = V(\mathcal{H})$ (and $\mathcal{H}$ is obvious), we simply write $\deg(S)$ and $\overline{\deg}(S)$. When $S = \{v\}$, we use $\deg(v, R)$ instead of $\deg(\{v\}, R)$.

A $k$-graph $\mathcal{P}$ is an $\ell$-path if there is an ordering $(v_1, \ldots, v_\ell)$ of its vertices such that every edge consists of $k$ consecutive vertices and two consecutive edges intersect in exactly $\ell$ vertices. Note that this implies that $k - \ell$ divides $\ell - 1$. In this case we write $\mathcal{P} = v_1 \cdots v_\ell$ and call two $\ell$-sets $\{v_1, \ldots, v_\ell\}$ and $\{v_{\ell+1}, \ldots, v_{\ell+\ell}\}$ *ends* of $\mathcal{P}$.

## 2. Proof of Theorem 1.4

In this section we prove Theorem 1.4 by following the same approach as in [7].

2.1. Auxiliary lemmas and Proof of Theorem 1.4. We need [7, Lemma 5] and [7, Lemma 6] of Hán and Schacht, in which any linear codegree is sufficient.

**Lemma 2.1** (Absorbing lemma, [7]). For all integers $k \geq 3$ and $1 \leq \ell < k/2$ and every $\gamma_1 > 0$ there exist $\eta_0$ such that the following holds. Let $\mathcal{H}$ be a $k$-graph on $n \geq n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geq \gamma_1 n$. Then there is an $\ell$-path $\mathcal{P}$ with $|V(\mathcal{P})| \leq \gamma_1^\eta n$ such that for all subsets $U \subseteq V \setminus V(\mathcal{P})$ of size $|U| \leq \eta n$ and $|U| \in (k - \ell)\mathbb{N}$ there exists an $\ell$-path $\mathcal{Q} \subset \mathcal{H}$ with $V(\mathcal{Q}) = V(\mathcal{P}) \cup U$ such that $\mathcal{P}$ and $\mathcal{Q}$ have exactly the same ends (we say $\mathcal{P}$ *absorbs* $U$ in this case).

**Lemma 2.2** (Reservoir lemma, [7]). For all integers $k \geq 3$ and $1 \leq \ell < k/2$ and every $d, \gamma_2 > 0$ there exists an $n_0$ such that the following holds. Let $\mathcal{H}$ be a $k$-graph on $n > n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geq d n$, then there is a set $R$ of size at most $\gamma_2 n$ such that for all $(k - 1)$-sets $S \in \binom{V}{k-1}$ we have $\deg(S, R) \geq d \gamma_2 n^2/2$.

The main step in our proof of Theorem 1.4 is the following lemma, which is stronger than [7, Lemma 7].

**Lemma 2.3** (Path-cover lemma). For all integers $k \geq 3$, $1 \leq \ell < k/2$, and every $\gamma_3, \alpha > 0$ there exist integers $p$ and $n_0$ such that the following holds. Let $\mathcal{H}$ be a $k$-graph on $n > n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geq \alpha \left( \frac{k}{k-\ell} - \gamma_3 \right) n$, then there is a family of at most $p$ vertex disjoint $\ell$-paths that together cover all but at most $\alpha n$ vertices of $\mathcal{H}$, or $\mathcal{H}$ is $14\gamma_3$-extremal.

We can now prove Theorem 1.4 in a similar way as in [7].
Proof of Theorem 1.4. Given \( k \geq 3, 1 \leq \ell < k/2 \) and \( 0 < \Delta < 1 \), let \( \gamma = \min\{\frac{\Delta}{3}, \frac{1}{4k^2}\} \) and \( n \in (k - \ell)\mathbb{N} \) be sufficiently large. Suppose that \( H = (V, E) \) is a \( k \)-graph on \( n \) vertices with \( \delta_{k-1}(H) \geq (\frac{1}{2(k-\ell)} - \gamma)n \). Since \( \frac{1}{2(k-\ell)} - \gamma > \gamma \), we can apply Lemma 2.1 with \( \gamma_1 = \gamma \) and obtain \( \eta > 0 \) and an absorbing path \( \mathcal{P}_0 \) with ends \( S_0, T_0 \) such that \( \mathcal{P}_0 \) can absorb any \( u \) vertices outside \( \mathcal{P}_0 \) if \( u \leq \eta n \) and \( u \in (k - \ell)\mathbb{N} \).

Let \( V_1 = (V \setminus V(\mathcal{P}_0)) \cup S_0 \cup T_0 \) and \( H_1 = H[V_1] \). Note that \( |V(\mathcal{P}_0)| \leq \gamma^5 n \) implies that \( \delta_{k-1}(H_1) \geq (\frac{1}{2(k-\ell)} - \gamma)n - \gamma^5 n \geq \frac{1}{4k^2}n \) since \( \gamma < \frac{1}{4k^2} \) and \( \ell \geq 1 \). We next apply Lemma 2.2 with \( d = \frac{1}{2k} \) and \( \gamma_2 = \min\{\eta/2, \gamma\} \) to \( H_1 \) and get a reservoir \( R \subset V_1 \) such that for any \((k-1)\)-set \( S \subset V_1 \), we have

\[
\deg(S, R) \geq \frac{1}{2} \gamma_2 |V_1|/2 \geq \frac{1}{2} \gamma_2 n / 4.
\]

Let \( V_2 = V \setminus (V(\mathcal{P}_0) \cup R) \), \( n_2 = |V_2| \), and \( H_2 = H[V_2] \). Note that \( |V(\mathcal{P}_0) \cup R| \leq \gamma_1 n + \gamma_2 n \leq 2\gamma n \), so

\[
\delta_{k-1}(H_2) \geq \left( \frac{1}{2(k-\ell)} - \gamma \right)n - 2\gamma n \geq \left( \frac{1}{2(k-\ell)} - 3\gamma \right)n_2.
\]

Applying Lemma 2.3 to \( H_2 \) with \( \gamma_3 = 3\gamma \) and \( \alpha = \eta/2 \), we obtain at most \( p \) vertex disjoint \( \ell \)-paths that cover all but at most \( an \) vertices of \( H_2 \), unless \( H_2 \) is \( 14\gamma_3 \)-extremal. In the latter case, there exists \( B' \subseteq V_2 \) such that \( |B'| = \lfloor \frac{2k-2\ell-1}{2k}\gamma_2 n \rfloor \) and \( e(B') \leq 42\gamma_2 n_2^2 \). Then we add at most \( n - n_2 \leq 2\gamma n \) vertices from \( V \setminus B' \) to \( B' \) and obtain a vertex set \( B \subseteq V(H) \) such that \( |B| = \lfloor \frac{2k-2\ell-1}{2k}\gamma_2 n \rfloor \) and

\[
e(B) \leq 42\gamma_2 n_2^2 + 2\gamma n \cdot \left( \frac{n_1 - 1}{k - 1} \right) \leq 42\gamma n^k + \gamma n^k \leq \Delta n^k,
\]

which means that \( H \) is \( \Delta \)-extremal, a contradiction. In the former case, denote these \( \ell \)-paths by \( \{\mathcal{P}_i\}_{i \in [p']} \) for some \( p' \leq p \), and their ends by \( \{S_i, T_i\}_{i \in [p']} \). Note that both \( S_i \) and \( T_i \) are \( \ell \)-sets for \( \ell < k/2 \). We arbitrarily pick disjoint \((k - 2\ell - 1)\)-sets \( X_0, X_1, \ldots, X_{p'} \subseteq R \setminus (S_0 \cup T_0) \) (note that \( k - 2\ell - 1 \geq 0 \)). Let \( T_{p'+1} = T_0 \). By (2.1), we get for \( 0 \leq i \leq p' \),

\[
\deg \left( S_i \cup T_{i+1} \cup X_i, R \setminus \bigcup_{0 \leq i \leq p'} (S_i \cup T_i \cup X_i) \right) \geq \frac{1}{2} \gamma_2 n / 4 - (p' + 1)(k - 1) \geq p + 1,
\]

as \( n \) is large enough. So we can connect \( \mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{p'} \) by using vertices from \( R \) and get an \( \ell \)-cycle \( C \). Note that \( |V(H) \setminus V(C)| \leq |R| + an \leq \gamma n + an \leq \eta n \) and since \( n \in (k - \ell)\mathbb{N} \), \( |V(H) \setminus V(C)| \) is also a multiple of \( k - \ell \). So we can use \( \mathcal{P}_0 \) to absorb all unused vertices in \( R \) and uncovered vertices in \( V_2 \) thus obtaining a Hamilton \( \ell \)-cycle in \( H \).

The rest of this section is devoted to the proof of Lemma 2.3.

2.2. Proof of Lemma 2.3. Following the approach in [7], we use the Weak Regularity Lemma, which is a straightforward extension of Szemerédi’s regularity lemma for graphs [21].

Let \( H = (V, E) \) be a \( k \)-graph and let \( A_1, \ldots, A_k \) be mutually disjoint non-empty subsets of \( V \). We define \( e(A_1, \ldots, A_k) \) to be the number of edges with one vertex in each \( A_i, i \in [k] \), and the density of \( H \) with respect to \( (A_1, \ldots, A_k) \) as

\[
d(A_1, \ldots, A_k) = \frac{e(A_1, \ldots, A_k)}{|A_1| \cdots |A_k|}.
\]

We say a \( k \)-tuple \((V_1, \ldots, V_k)\) of mutually disjoint subsets \( V_1, \ldots, V_k \subseteq V \) is \( (\epsilon, d) \)-regular, for \( \epsilon > 0 \) and \( d \geq 0 \), if

\[
|d(A_1, \ldots, A_k) - d| \leq \epsilon
\]

for all \( k \)-tuples of subsets \( A_i \subseteq V_i, i \in [k] \), satisfying \(|A_i| \geq \epsilon |V_i|\). We say \((V_1, \ldots, V_k)\) is \( \epsilon \)-regular if it is \( (\epsilon, d) \)-regular for some \( d \geq 0 \). It is immediate from the definition that in an \( (\epsilon, d) \)-regular \( k \)-tuple
Lemma 2.7.

Theorem 2.4 (Weak Regularity Lemma). Given \( t_0 \geq 0 \) and \( \epsilon > 0 \), there exist \( T_0 = T_0(t_0, \epsilon) \) and \( n_0 = n_0(t_0, \epsilon) \) so that for every \( k \)-graph \( \mathcal{H} = (V, E) \) on \( n > n_0 \) vertices, there exists a partition \( V = V_0 \cup V_1 \cup \cdots \cup V_t \) such that

(i) \( t_0 \leq t \leq T_0 \),

(ii) \( |V_i| = |V_2| = \cdots = |V_t| \) and \( |V_0| \leq \epsilon n \),

(iii) for all but at most \( \epsilon \binom{t}{k} \) \( k \)-subsets \( \{i_1, \ldots, i_k\} \subset [t] \), the \( k \)-tuple \( (V_{i_1}, \ldots, V_{i_k}) \) is \( \epsilon \)-regular.

The partition given in Theorem 2.4 is called an \( \epsilon \)-regular partition of \( \mathcal{H} \). Given an \( \epsilon \)-regular partition of \( \mathcal{H} \) and \( d \geq 0 \), we refer to \( V_i, i \in [t] \) as clusters and define the cluster hypergraph \( \mathcal{K} = \mathcal{K}(\epsilon, d) \) with vertex set \( [t] \) and \( \{i_1, \ldots, i_k\} \subset [t] \) is an edge if and only if \( (V_{i_1}, \ldots, V_{i_k}) \) is \( \epsilon \)-regular and \( d(V_{i_1}, \ldots, V_{i_k}) \geq d \).

The following corollary shows that the cluster hypergraph inherits the minimum degree of the original hypergraph. Its proof is almost the same as in [7, Proposition 16] after we replace \( \frac{1}{2(k-t)} + \gamma \) by \( c - \gamma \) we thus omit the proof.

Corollary 2.5. [7] Given \( c, \epsilon, d > 0 \) and integers \( k \geq 3, t_0 \) such that \( 0 < \epsilon < d^2/4 \) and \( t_0 \geq 2k/d \), there exist \( T_0 \) and \( n_0 \) such that the following holds. Let \( \mathcal{H} \) be a \( k \)-graph on \( n > n_0 \) vertices such that \( \delta_{k-1}(\mathcal{H}) \geq \epsilon n \). If \( \mathcal{H} \) has an \( \epsilon \)-regular partition \( V_0' \cup V_1' \cup \cdots \cup V_t' \) with \( t_0 \leq t \leq T_0 \) and \( \mathcal{K} = \mathcal{K}(\epsilon, d) \) is the cluster hypergraph then at most \( \sqrt{\epsilon} \binom{k-1}{k} \)-subsets \( S \) of \( [t] \) violate \( \delta_k(S) \geq (c - 2d)t \).

Let \( \mathcal{H} \) be a \( k \)-partite \( k \)-graph with partition classes \( V_1, \ldots, V_k \). Then we call an \( \ell \)-path \( P \) of \( \mathcal{H} \) with edges \( \{E_1, \ldots, E_t\} \) canonical with respect to \( (V_1, \ldots, V_k) \) if

\[
E_i \cap E_{i+1} \subseteq \bigcup_{j \in [\ell]} V_j \quad \text{or} \quad E_i \cap E_{i+1} \subseteq \bigcup_{j \in [2\ell] \setminus [\ell]} V_j
\]

for \( i = 1, \ldots, t - 1 \). Note that a canonical \( \ell \)-path with an odd length \( t \) contains \( \frac{t+1}{2} \) vertices of \( V_i \) for \( i \in [2\ell] \) and \( t \) vertices of \( V_i \) for \( i > 2\ell \).

We also need the following proposition from [7].

Proposition 2.6. [7, Proposition 19] Suppose \( \mathcal{H} \) is a \( k \)-partite, \( k \)-graph with partition classes \( V_1, \ldots, V_k \), \( |V_i| = m \) for all \( i \in [k] \), and \( |E(\mathcal{H})| \geq dm^k \). Then there exists a canonical \( \ell \)-path in \( \mathcal{H} \) with \( t > \frac{dm}{\delta_k(\mathcal{H})} \) edges.

In [7] the authors used Proposition 2.6 to cover an \((\epsilon, d)\)-regular tuple \((V_1, \ldots, V_k)\) of sizes \( |V_1| = \cdots = |V_{k-1}| = (2k-2\ell-1)m \) and \( |V_k| = (k-1)m \) with vertex disjoint \( \ell \)-paths. Our next lemma shows that an \((\epsilon, d)\)-regular tuple \((V_1, \ldots, V_k)\) of sizes \( |V_1| = \cdots = |V_{2\ell}| = m \) and \( |V_i| = 2m \) for \( i > 2\ell \) can be covered with \( \ell \)-paths.

Lemma 2.7. Fix \( k \geq 3, 1 \leq \ell < k/2 \) and \( \epsilon, d > 0 \) such that \( d > 2\epsilon \). Let \( m > \frac{2\ell^2}{(d-\epsilon)^2} \). Suppose \( \mathcal{V} = (V_1, V_2, \ldots, V_k) \) is an \((\epsilon, d)\)-regular \( k \)-tuple with

\[
|V_1| = \cdots = |V_{2\ell}| = m \quad \text{and} \quad |V_{2\ell+1}| = \cdots = |V_k| = 2m.
\]

Then there are at most \( \frac{4(k-\ell)}{(d-\epsilon)^2} \) vertex disjoint \( \ell \)-paths that together cover all but at most \( 2km \) vertices of \( \mathcal{V} \).

Proof. We greedily find disjoint canonical \( \ell \)-paths of odd length by Proposition 2.6 in \( \mathcal{V} \) until less than \( \epsilon m \) vertices are uncovered in \( V_1 \). Suppose that we have obtained odd \( \ell \)-paths \( P_1, \ldots, P_p \) by Proposition 2.6 for some \( p \geq 0 \). Let \( t = \sum_{j=1}^{p} e(P_j) \). Since all \( e(P_j) \) are odd, \( \bigcup_{j=1}^{p} P_i \) contains \( \frac{t + p}{2} \).
vertices of \( V_i \) for \( i \in [2\ell] \) and \( t \) vertices of \( V_i \) for \( i > 2\ell \). For \( i \in [k] \), let \( U_i \) be the set of uncovered vertices of \( V_i \) and assume that \( |U_1| \geq em \). Using (2.2), we derive that \( |U_1| = \cdots = |U_{2\ell}| \geq em \) and
\[
|U_{2\ell+1}| = \cdots = |U_k| = 2|U_1| + p. \tag{2.3}
\]

We pick an arbitrary \( k \)-partite subhypergraph \( \mathcal{V}' \) with \( |U_1| \) vertices in each \( U_i \) for \( i \in [k] \). By regularity, \( \mathcal{V}' \) contains at least \( (d-\epsilon)|U_1|^k \) edges so that we can apply Proposition 2.6 and find an \( \ell \)-path of odd length at least \( \frac{(d-\epsilon)em}{2(k-\ell)} - 1 \) (dismiss one edge if needed). We continue this process until \( |U_1| < em \). Let \( P_1, \ldots, P_p \) be the \( \ell \)-paths obtained in \( \mathcal{V} \) after the iteration stops. Since \( |V_1 \cap V(P_j)| \geq \frac{(d-\epsilon)em}{4(k-\ell)} \) for every \( j \), we have
\[
p \leq \frac{m}{(d-\epsilon)\rho m} = \frac{4(k-\ell)}{(d-\epsilon)\epsilon}.
\]

Since \( m > \frac{2k^2}{r(\rho-\epsilon)} \), we further have
\[
p(k-2\ell) \leq \frac{4(k-\ell)(k-2\ell)}{(d-\epsilon)\epsilon} < \frac{4k^2}{(d-\epsilon)\epsilon} < 2em.
\]

By (2.3), the total number of uncovered vertices in \( \mathcal{V} \) is
\[
\sum_{i=1}^{k} |U_i| = |U_1|2\ell + (2|U_1| + p)(k-2\ell) = 2(k-\ell)|U_1| + p(k-2\ell)
< 2(k-1)em + 2em = 2kem. \quad \square
\]

Given \( k \geq 3 \) and \( 1 \leq b < k \), recall that \( \mathcal{Y}_{k,b} \) is a \( k \)-graph with two edges that share exactly \( b \) vertices. In general, given two (hyper)graphs \( G \) and \( \mathcal{H} \), a \( G \)-tiling is a sub(hyper)graph of \( \mathcal{H} \) that consists of vertex-disjoint copies of \( G \). A \( G \)-tiling is perfect if it is a spanning sub(hyper)graph of \( \mathcal{H} \). The following lemma is the main step in our proof of Lemma 2.3 and we prove it in the next subsection. Note that it generalizes [2, Lemma 3.1] of Czygrinow, DeBiasio, and Nagle.

**Lemma 2.8 (\( \mathcal{Y}_{k,b} \)-tiling Lemma).** Given integers \( k \geq 3, 1 \leq b < k \) and constants \( \gamma, \beta > 0 \), there exist \( 0 < \epsilon' < \gamma \beta \) and an integer \( n_0 \) such that the following holds. Suppose \( \mathcal{H} \) is a \( k \)-graph on \( n > n_0 \) vertices with \( \text{deg}(S) \geq \frac{1}{2k-b} - \gamma \) \( n \) for all but at most \( \epsilon' n^{k-1} \) sets \( S \in \binom{V}{k-1} \), then there is a \( \mathcal{Y}_{k,b} \)-tiling that covers all but at most \( \beta n \) vertices of \( \mathcal{H} \) unless \( \mathcal{H} \) contains a vertex set \( B \) such that \( |B| = \left| \frac{2k-b-1}{2k-b} n \right| \) and \( \epsilon(B) < 6\gamma n^{k-1} \).

Now we are ready to prove Lemma 2.3.

**Proof of Lemma 2.3.** Fix such integers \( k, \ell, 0 < \gamma_3, \alpha < 1 \). Let \( \epsilon' \) be the constant returned from Lemma 2.8 with \( b = 2\ell, \gamma = 2\gamma_3, \) and \( \beta = \alpha/2 \). So \( \epsilon' < \gamma \beta = \gamma_3 \alpha \). Furthermore, let \( p = \frac{4\rho_{0}}{(d-\epsilon)\epsilon} \), where \( \rho_{0} \) is the constant returned from Corollary 2.5 with \( c = \frac{1}{2(k-\ell)} - \gamma_3, \epsilon = (\epsilon')^2/16, \) and \( d = \gamma_3/2 \).

Let \( n \) be a sufficiently large integer and let \( \mathcal{H} \) be a \( k \)-graph on \( n \) vertices with \( \delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2(k-\ell)} - \gamma_3) n \). By applying Corollary 2.5 with the constants chosen above we obtain an \( \epsilon \)-regular partition and a cluster hypergraph \( \mathcal{K} = \mathcal{K}(\epsilon, d) \) such that for all but at most \( \sqrt{\epsilon} t^{k-1} (k-1) \)-sets \( S \in \binom{V}{k-1} \),
\[
\text{deg}_{\mathcal{K}}(S) \geq \left( \frac{1}{2(k-\ell)} - \gamma_3 - 2d \right) t = \left( \frac{1}{2(k-\ell)} - 2\gamma_3 \right) t,
\]

because \( d = \gamma_3/2 \). Let \( m \) be the size of each cluster except \( V_0 \), then \( (1-\epsilon)^2 \leq m \leq \frac{n}{2} \). Applying Lemma 2.8 with the constants chosen above, we derive that there is a \( \mathcal{Y}_{k,2\ell} \)-tiling \( \mathcal{Y} \) of \( \mathcal{K} \) which covers all but at most \( \beta t \) vertices of \( \mathcal{K} \) or there exists a set \( B \subseteq V(\mathcal{K}) \), such that \( |B| = \left| \frac{2k-2\ell-1}{2(k-\ell)} t \right| \)
and \( e_K(B) \leq 12\gamma_3 t^k \). In the latter case, let \( B' \subseteq V(H) \) be the union of the clusters in \( B \). By regularity,

\[
e_H(B') \leq e_K(B) \cdot m^k \cdot \binom{t}{k} \cdot d \cdot m^k + \epsilon \cdot \binom{t}{k} \cdot m^k + \binom{m}{2} \left( \frac{n}{k-2} \right),
\]

where the right-hand side bounds the number of edges from regular \( k \)-tuples with high density, edges from regular \( k \)-tuples with low density, edges from irregular \( k \)-tuples and edges that lie in at most \( k - 1 \) clusters. Since \( m \leq \frac{r}{\epsilon}, \epsilon < \gamma_3, d = \gamma_3/2 \), and \( t^{-2} < \gamma_0^2 < \gamma_3 \), we obtain that

\[
e_H(B') \leq 12\gamma_3 t^k \cdot m^k \cdot \binom{t}{k} \cdot d \cdot m^k + \epsilon \cdot \binom{t}{k} \cdot m^k + \frac{n}{k-2} \leq 13\gamma_3 n^k.
\]

Note that \( |B'| = \left\lfloor \frac{2k-2\ell-1}{2(k-\ell)} \right\rfloor m \leq \frac{2k-2\ell-1}{2(k-\ell)} t \cdot \frac{n}{\ell} = \frac{2k-2\ell-1}{2(k-\ell)} n \), and consequently \( |B'| \leq \left\lfloor \frac{2k-2\ell-1}{2(k-\ell)} n \right\rfloor \). On the other hand,

\[
|B'| = \frac{2k-2\ell-1}{2(k-\ell)} t \cdot \frac{n}{\ell} \geq \frac{2k-2\ell-1}{2(k-\ell)} n - en.
\]

By adding at most \( en \) vertices from \( V \setminus B' \) to \( B' \), we get a set \( B'' \subseteq V(H) \) of size exactly \( \left\lfloor \frac{2k-2\ell-1}{2(k-\ell)} n \right\rfloor \), with \( e(B'') \leq e(B') + en \cdot n^{k-1} < 14\gamma_3 n^k \). Hence \( H \) is \( 14\gamma_3 \)-extremal.

In the former case, the union of the clusters covered by \( \mathcal{Y} \) contains all but at most \( \beta tn + |V_0| \leq an/2 + en \) vertices. We apply Lemma 2.7 to each member \( \mathcal{Y}' \in \mathcal{Y} \). Suppose that \( \mathcal{Y}' \) has the vertex set \( [2k-2\ell] \) with edges \( \{1, \ldots, k\} \) and \( \{k-2\ell+1, \ldots, 2k-2\ell\} \). For \( i \in [2k-2\ell] \), let \( W_i \) denote the corresponding cluster in \( H \). We split each \( W_i \), \( i = k-2\ell+1, \ldots, k \), into two disjoint sets \( W_i' \) and \( W_i'' \) of equal size. Then the \( k \)-tuples \( (W_{k-2\ell+1}', \ldots, W_k', W_1, \ldots, W_{k-2\ell}) \) and \( (W_{k-2\ell+1}'', \ldots, W_k'', W_1, \ldots, W_{k-2\ell}) \) are \( (2\ell, d) \)-regular and of sizes \( \frac{2\ell}{2}, \ldots, \frac{2\ell}{2}, m, \ldots, m \). Applying Lemma 2.7 to these two \( k \)-tuples with \( m' = \frac{m}{2} \), we find a family of disjoint loose paths in each \( k \)-tuple covering all but at most \( 2kem' = kem \) vertices.

Since \( |\mathcal{Y}'| \leq \frac{t}{2k-2\ell} \), we obtain a path-tiling that consists of at most \( 2t \cdot \frac{4(k-\ell)}{(d-\ell)^2} \leq 4\gamma_0 \frac{4(2k-2\ell)}{(d-\ell)^2} = p \) paths and covers all but at most

\[
2kem \cdot \frac{t}{2k-2\ell} + an/2 + en < 3en + an/2 < an
\]

vertices, where we use \( 2k - 2\ell > k \) and \( \epsilon = (\epsilon')^2/16 < (\gamma_3a)^2/16 < \alpha/6 \). This completes the proof.

\[
2.3. \text{Proof of Lemma 2.8.} \text{ We first give an upper bound on the size of } k \text{-graphs containing no copy of } \mathcal{Y}_{k,b}. \text{ In its proof, we use the concept of link (hyper)graph: given a } k \text{-graph } H \text{ with a set } S \text{ of at most } k-1 \text{ vertices, the link graph of } S \text{ is the } (k - |S|) \text{-graph with vertex set } V(H) \setminus S \text{ and edge set } \{e \setminus S : e \in E(H), S \subseteq e\}. \text{ Throughout the rest of the paper, we frequently use the simple identity } \binom{m}{b} \binom{m-b}{k-b} = \binom{m}{k} \binom{k}{b}, \text{ which holds for all integers } 1 \leq b \leq k \leq m. \]

\[
\text{Fact 2.9. Let } 1 \leq b < k \leq m. \text{ If } H \text{ is a } k \text{-graph on } m \text{ vertices containing no copy of } \mathcal{Y}_{k,b}, \text{ then } e(H) < \binom{m}{k-1}.
\]
Proof. Fix any $b$-set $S \subseteq V(\mathcal{H})$ and consider its link graph $L_S$. Since $\mathcal{H}$ contains no copy of $\mathcal{Y}_{k,b}$, any two edges of $L_S$ intersect. By the Erdős–Ko–Rado Theorem [5], $|L_S| \leq \binom{m-b}{k-b-1}$. Thus,

$$e(\mathcal{H}) \leq \frac{1}{k} \binom{m}{b} \cdot \binom{m-b-1}{k-b-1} = \frac{1}{k} \binom{m}{b} \binom{m-b}{k-b} \frac{k-b}{m-b} = \binom{m}{k} \frac{k-b}{m-b} \binom{m}{k-1} \frac{k-b}{m-b} \binom{m}{k-1}.$$

Proof of Lemma 2.8. Given $\gamma, \beta > 0$, let $\epsilon' = \frac{\gamma \beta^{k-1}}{(k-1)!}$ and $n \in \mathbb{N}$ be sufficiently large. Let $\mathcal{H}$ be a $k$-graph on $n$ vertices that satisfies $\deg(S) \geq \frac{1}{2k-\gamma} n$ for all but at most $\epsilon' n^{k-1}$ $(k-1)$-sets $S$. Fix a largest $\mathcal{Y}_{k,b}$-tiling $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$ and write $V_i = V(Y_i)$ for $i \in \mathbb{N}$. Let $V' = \bigcup_{i \in \mathbb{N}} V_i$ and $U = V(\mathcal{H}) \setminus V'$.

Let $C$ be the set of vertices $v \in V'$ such that $\deg(v, U) \geq (2k-b)^2 \binom{|U|}{k-2}$. We will show that $|C| \leq \frac{n}{2k-\gamma} n^{k-2}$ and $C$ covers almost all the edges of $\mathcal{H}$, which implies that $\mathcal{H}[V \setminus C]$ is sparse and $\mathcal{H}$ is in the extremal case. We first observe that every $Y_i \in \mathcal{Y}$ contains at most one vertex in $C$. Suppose instead, two vertices $x, y \in V_i$ are both in $C$. Since $\deg(x, U) \geq (2k-b)^2 \binom{|U|}{k-2}$, by Fact 2.9, there is a copy of $\mathcal{Y}_{k-1,b}$ in the link graph of $x$ on $U$, which gives rise to $\mathcal{Y}'$, a copy of $\mathcal{Y}_{k,b}$ on $\{x \} \cup U$. Since the link graph of $y$ on $U \setminus V(\mathcal{Y}')$ has at least

$$(2k-b)^2 \binom{|U|}{k-2} \binom{|U|}{k-2} > \binom{|U|}{k-2} \binom{|U|}{k-2},$$

edges, we can find another copy of $\mathcal{Y}_{k,b}$ on $\{y \} \cup (U \setminus V(\mathcal{Y}'))$ by Fact 2.9. Replacing $\mathcal{Y}_i$ in $\mathcal{Y}$ with these two copies of $\mathcal{Y}_{k,b}$ creates a $\mathcal{Y}_{k,b}$-tiling larger than $\mathcal{Y}$, contradiction. Consequently,

$$\sum_{S \in \binom{V}{k-2}} \deg(S, V') \leq |C| \binom{|U|}{k-1} + |V' \setminus C| (2k-b)^2 \binom{|U|}{k-2},$$

$$< |C| \binom{|U|}{k-1} + (2k-b)^2 n \binom{|U|}{k-2} \text{ because } |V' \setminus C| < n$$

$$= \left( \binom{|U|}{k-1} \right) \left( |C| + \frac{(2k-b)^2 n (k-1)}{|U| - k + 2} \right). \quad (2.4)$$

Second, by Fact 2.9, $e(U) \leq \binom{|U|}{k-1}$ since $\mathcal{H}[U]$ contains no copy of $\mathcal{Y}_{k,b}$, which implies

$$\sum_{S \in \binom{U}{k-1}} \deg(S, U) \leq k \binom{|U|}{k-1}. \quad (2.5)$$

By the definition of $\epsilon'$, we have

$$\epsilon' n^{k-1} = \frac{\gamma \beta^{k-1}}{(k-1)!} n^{k-1} < \frac{\gamma |U|^{k-1}}{(k-1)!} < 2\gamma \binom{|U|}{k-1},$$

since $|U|$ is large enough. At last, by the degree condition, we have

$$\sum_{S \in \binom{U}{k-1}} \deg(S) \geq \left( \binom{|U|}{k-1} - \epsilon' n^{k-1} \right) \left( \frac{1}{2k-b} - \gamma \right) > n - \left(1 - 2\gamma \right) n - \left( \frac{1}{2k-b} - \gamma \right) n, \quad (2.6)$$

Since $\deg(S) = \deg(S, U) + \deg(S, V')$, we combine (2.4), (2.5) and (2.6) and get

$$|C| > (1 - 2\gamma) \left( \frac{1}{2k-b} - \gamma \right) n - \frac{(2k-b)^2 n (k-1)}{|U| - k + 2}.$$
Since $|U| > 16k^3/\gamma$, we get
\[
(2k - b)^2 n(k - 1) - \frac{4k^3 n}{|U|} - \frac{\gamma n}{2},
\]
As $2\gamma^2 n > k$ and $2k - b \geq 4$, it follows that $|C| > (\frac{1}{2k-b} - 2\gamma)n$.

Let $I_C$ be the set of all $i \in [m]$ such that $V_i \cap C \neq \emptyset$. Since each $V_i$, $i \in I_C$, contains one vertex of $C$, we have
\[
|I_C| = |C| \geq \left(\frac{1}{2k-b} - 2\gamma\right)n \geq m - 2\gamma n.
\]  
(2.7)

Let $A = (\bigcup_{i \in I_C} V_i \setminus C) \cup U$.

**Claim 2.10.** $\mathcal{H}[A]$ contains no copy of $\mathcal{Y}_{k,b}$, thus $e(A) \leq \binom{n}{k-1}$.

**Proof.** The first half of the claim implies the second half by Fact 2.9. Suppose instead, $\mathcal{H}[A]$ contains a copy of $\mathcal{Y}_{k,b}$, denoted by $Y_0$. Note that $V(Y_0) \subsetneq U$ because $\mathcal{H}[U]$ contains no copy of $\mathcal{Y}_{k,b}$. Without loss of generality, suppose that $V_1, \ldots, V_j$ contain the vertices of $Y_0$ for some $j \leq 2k-b$. For $i \in [j]$, let $c_i$ denote the unique vertex in $V_i \cap C$. We greedily construct vertex-disjoint copies of $\mathcal{Y}_{k,b}$ on $\{c_i\} \cup U$, $i \in [j]$ as follows. Suppose we have found $Y_1', \ldots, Y_j'$ (copies of $\mathcal{Y}_{k,b}$) for some $i < j$. Let $U_0$ denote the set of the vertices of $U$ covered by $Y_0, Y_1', \ldots, Y_i'$. Then $|U_0| \leq (i+1)(2k-b-1) \leq (2k-b)(2k-1)$.

Since $\deg(c_{i+1}, U) \geq (2k-b)^2 \binom{|U|}{k-2}$, the link graph of $c_{i+1}$ on $U \setminus U_0$ has at least
\[
(2k - b)^2 \binom{|U|}{k-2} - |U_0| \binom{|U|}{k-2} \geq \binom{|U|}{k-2}
\]
edges. By Fact 2.9, there is a copy of $\mathcal{Y}_{k,b}$ on $\{c_{i+1}\} \cup (U \setminus U_0)$. Let $Y_1'', \ldots, Y_j''$ denote the copies of $\mathcal{Y}_{k,b}$ constructed in this way. Replacing $Y_1', \ldots, Y_j'$ in $\mathcal{Y}$ with $Y_0, Y_1'', \ldots, Y_j''$ gives a $\mathcal{Y}_{k,b}$-tiling larger than $\mathcal{Y}$, contradiction.

Note that the edges not incident to $C$ are either contained in $A$ or intersect some $V_i$, $i \notin I_C$. By (2.7) and Claim 2.10,
\[
eq (2k - b) \cdot 2\gamma n \binom{n-1}{k-1} < \binom{n}{k-1} + (4k - 2b)\gamma n \binom{n}{k-1}
\]
\[
< 4k\gamma n \binom{n}{k-1} < \frac{4k}{k-1} \gamma n^k \leq 6\gamma n^k,
\]
where the last inequality follows from $k \geq 3$. Since $|C| \leq \frac{n}{2k-b}$, we can pick a set $B \subseteq V \setminus C$ of order $n/2k-b$ such that $e(B) < 6\gamma n^k$.

**3. The Extremal Theorem**

In this section we prove Theorem 1.5. Assume that $k \geq 3$, $1 \leq \ell \leq k/2$ and $0 < \Delta \ll 1$. Let $n \in (k-\ell)\mathbb{N}$ be sufficiently large. Let $\mathcal{H}$ be a $k$-graph on $V$ of $n$ vertices such that $\delta_{k-\ell}(\mathcal{H}) \geq \frac{n}{2(k-\ell)}$. Furthermore, assume that $\mathcal{H}$ is $\Delta$-extremal, namely, there is a set $B \subseteq V(\mathcal{H})$, such that $|B| = \frac{(2k-2\ell-1)n}{2(k-\ell)}$ and $e(B) \leq \Delta n^k$. Let $A = V \setminus B$. Then $|A| = \frac{n}{2(k-\ell)}$.

Let us give an outline of our proof first. We denote by $A'$ and $B'$ the sets of “typical” vertices of $A$ and $B$, respectively. Let $V_0 = V \setminus (A' \cup B')$. It is not hard to see that $A' \approx A$, $B' \approx B$, and thus $V_0 \approx \emptyset$. In the ideal case where $V_0 = \emptyset$ and $|B'| = (2k - 2\ell - 1)|A'|$, we assign a cyclic order to the vertices of $A'$, construct $|A'|$ copies of $\mathcal{Y}_{k,\ell}$ such that each copy contains one vertex of $A'$ and $2k - \ell - 1$ vertices of $B'$, and any two consecutive copies of $\mathcal{Y}_{k,\ell}$ share exactly $\ell$ vertices of $B'$. This gives rise to the desired Hamilton $\ell$-cycle of $\mathcal{H}$. In the general case, we first construct an $\ell$-path $Q$ with ends $L_0$ and $L_1$ such that $V_0 \subseteq V(Q)$ and $|B_1| = (2k - 2\ell - 1)|A_1| + 1$, where $A_1 = A' \setminus V(Q)$.
and \( B_1 = (B \setminus V(Q)) \cup L_0 \cup L_1 \). Next we complete the Hamilton \( \ell \)-cycle by constructing an \( \ell \)-path on \( A_1 \cup B_1 \) with ends \( L_0 \) and \( L_1 \).

For the convenience of later calculations, we let \( \epsilon_0 = 2k!\epsilon \Delta \ll 1 \) and claim that \( e(B) \leq \epsilon_0 \binom{|B|}{k} \).

Indeed, since \( 2(k - \ell) - 1 \geq k \), we have

\[
\frac{1}{e} \leq \left( 1 - \frac{1}{2(k - \ell)} \right)^{2(k - \ell) - 1} \leq \left( 1 - \frac{1}{2(k - \ell)} \right)^k.
\]

Thus we get

\[
e(B) \leq \frac{\epsilon_0}{2k!e} n^k \leq \epsilon_0 \left( 1 - \frac{1}{2(k - \ell)} \right)^k \frac{n^k}{2k!} \leq \epsilon_0 \binom{|B|}{k}.
\]

(3.1)

In general, given two disjoint vertex sets \( X \) and \( Y \) and two integers \( i, j \geq 0 \), a set \( S \subset X \cup Y \) is called an \( X^iY^j \)-set if \( |S \cap X| = i \) and \( |S \cap Y| = j \). When \( X, Y \) are two disjoint subsets of \( V(H) \) and \( i + j = k \), we denote by \( \mathcal{H}(X^iY^j) \) the family of all edges of \( \mathcal{H} \) that are \( X^iY^j \)-sets, and let \( e_{\mathcal{H}}(X^iY^j) = |\mathcal{H}(X^iY^j)| \) (the subscript may be omitted if it is clear from the context). We use \( \tau_{k}(X^iY^{k-i}) \) to denote the number of non-edges among \( X^iY^{k-i} \)-sets. Given a set \( L \subset X \cup Y \) with \( |L \cap X| = l_1 \leq i \) and \( |L \cap Y| = l_2 \leq k - i \), we define \( \deg(L, X^iY^{k-i}) \) as the number of edges in \( \mathcal{H}(X^iY^{k-i}) \) that contain \( L \), and \( \deg(L, X^iY^{k-i}) = \binom{|X| - l_1}{k - l_2} - \deg(L, X^iY^{k-i}) \). Our earlier notation \( \deg(S, R) \) may be viewed as \( \deg(S, S^{[S]}(R \setminus S)^{[S]}) \).

### 3.1. Classification of vertices.

Let \( \epsilon_1 = \epsilon_0^{1/3} \) and \( \epsilon_2 = 2\epsilon_1^2 \). Assume that the partition \( V(\mathcal{H}) = A \cup B \) satisfies that \( |B| = \lceil \frac{2k-2\ell-1}{2} \rceil n \) and (3.1). In addition, assume that \( e(B) \) is the smallest among all such partitions. We now define

\[ A' := \left\{ v \in V : \deg(v, B) \geq 1 - \epsilon_1 \binom{|B|}{k-1} \right\}, \]

\[ B' := \left\{ v \in V : \deg(v, B) \leq \epsilon_1 \binom{|B|}{k-1} \right\}, \]

\[ V_0 := V \setminus (A' \cup B'). \]

**Claim 3.1.** \( A \cap B' \neq \emptyset \) implies that \( B \subseteq B' \), and \( B \cap A' \neq \emptyset \) implies that \( A \subseteq A' \).

**Proof.** First, assume that \( A \cap B' \neq \emptyset \). Then there is some \( u \in A \) such that \( \deg(u, B) \leq \epsilon_1 \binom{|B|}{k-1} \). If there exists some \( v \in B \setminus B' \), namely, \( \deg(v, B) > \epsilon_1 \binom{|B|}{k-1} \), then we can switch \( u \) and \( v \) and form a new partition \( A'' \cup B'' \) such that \( |B''| = |B| \) and \( e(B'') > e(B) \), which contradicts the minimality of \( e(B) \).

Second, assume that \( B \cap A' \neq \emptyset \). Then some \( u \in B \) satisfies that \( \deg(u, B) \geq (1 - \epsilon_1) \binom{|B|}{k-1} \). Similarly, by the minimality of \( e(B) \), we get that for any vertex \( v \in A \), \( \deg(v, B) \geq (1 - \epsilon_1) \binom{|B|}{k-1} \), which implies that \( A \subseteq A' \). \( \square \)

**Claim 3.2.** \( \{|A \setminus A'|, |B \setminus B'|, |A' \setminus A|, |B' \setminus B|\} \leq \epsilon_2 |B| \) and \( |V_0| \leq 2\epsilon_2 |B| \).

**Proof.** First assume that \( |B \setminus B'| > \epsilon_2 |B| \). By the definition of \( B' \), we get that

\[
e(B) > \frac{1}{k} \epsilon_1 \binom{|B|}{k-1} \cdot \epsilon_2 |B| > 2\epsilon_0 \binom{|B|}{k},
\]

which contradicts (3.1).

Second, assume that \( |A \setminus A'| > \epsilon_2 |B| \). Then by the definition of \( A' \), for any vertex \( v \notin A' \), we have that \( \deg(v, B) > \epsilon_1 \binom{|B|}{k-1} \). So we get

\[
\tau(AB^{k-1}) > \epsilon_2 |B| \cdot \epsilon_1 \binom{|B|}{k-1} = 2\epsilon_0 |B| \binom{|B|}{k-1}.
\]
Together with (3.1), this implies that
\[ \sum_{S \in \binom{L}{k-1}} \deg(S) = k\overline{\sigma}(B) + \overline{\sigma}(AB^{k-1}) \]
\[ > k(1 - \epsilon_0) \binom{|B|}{k} + 2\epsilon_0 |B| \binom{|B|}{k-1} \]
\[ = (1 - \epsilon_0)(|B| - k + 1) + 2\epsilon_0 |B| \binom{|B|}{k-1} > |B| \binom{|B|}{k-1}. \]
where the last inequality holds because \( n \) is large enough. By the pigeonhole principle, there exists a set \( S \in \binom{B}{k-1} \), such that \( \overline{\deg}(S) > |B| = \left\lfloor \frac{2(k^2 - 2\ell - 1)n}{2(k-\ell)} \right\rfloor \), contradicting (1.1).
Consequently,
\[ |A' \setminus A| = |A' \cap B| \leq |B \setminus B'| \leq \epsilon_2 |B|, \]
\[ |B' \setminus B| = |A \cap B'| \leq |A' \setminus A| \leq \epsilon_2 |B|, \]
\[ |V_0| = |A \setminus A'| + |B \setminus B'| \leq \epsilon_2 |B| + \epsilon_2 |B| = 2\epsilon_2 |B|. \]

3.2. Classification of \( \ell \)-sets in \( B' \). In order to construct our Hamilton \( \ell \)-cycle, we need to connect two \( \ell \)-paths. To make this possible, we want the ends of our \( \ell \)-paths to be \( \ell \)-sets in \( B' \) that have high degree in \( \mathcal{H}[A'B'^{k-1}] \). Formally, we call an \( \ell \)-set \( L \subset V \) typical if \( \deg(L, B) \leq \epsilon_1 \binom{|B|}{k-\ell} \), otherwise atypical. We prove several properties related to typical \( \ell \)-sets in this subsection.

Claim 3.3. The number of atypical \( \ell \)-sets in \( B \) is at most \( \epsilon_2 \binom{|B|}{k} \).

Proof. Let \( m \) be the number of atypical \( \ell \)-sets in \( B \). By (3.1), we have
\[ m \epsilon_1 \binom{|B|}{k} \leq e(B) \leq \epsilon_0 \binom{|B|}{k}, \]
which gives that \( m \leq \frac{\epsilon_0 \binom{|B|}{k}}{\epsilon_1 \binom{|B|}{k-\ell}} = \frac{\epsilon_0 (k)}{\epsilon_1 (k-\ell)} \leq \frac{\epsilon_2 (k)}{\epsilon_1 (k-\ell)} \).
Putting these together and using Claim 3.2, we obtain that
\[
\sum_{L \subseteq D \subseteq B', |D| = k - 1} \deg(D, A') \geq \frac{|B'| - \ell}{k - \ell - 1} (|A| - |V_0|) - 2\epsilon_1 |B| \left( \frac{|B'| - \ell}{k - \ell - 1} \right)
\geq \frac{|B'| - \ell}{k - \ell - 1} (|A| - 3\epsilon_2 |B| - 2\epsilon_1 |B|).
\]
Note that \(\deg(L, A' B^{k-1}) = \sum_{L \subseteq D \subseteq B', |D| = k - 1} \deg(D, A')\). Since \(|B| \leq (2k - 2\ell - 1)|A| \leq (2k - 2\ell)|A'|\), we finally derive that
\[
\deg(L, A' B^{k-1}) \geq \left( \frac{|B'| - \ell}{k - \ell - 1} \right) (1 - (2k - 2\ell)(3\epsilon_2 + 2\epsilon_1)) |A'| \geq (1 - 4k\epsilon_1) \left( \frac{|B'| - \ell}{k - \ell - 1} \right) |A'|.
\]
as desired.

We next show that we can connect any two disjoint typical \(\ell\)-sets of \(B'\) with an \(\ell\)-path of length two while avoiding any given \(\frac{n}{4(k-\ell)}\) vertices of \(V\).

**Claim 3.5.** Given two disjoint typical \(\ell\)-sets \(L_1, L_2\) in \(B'\) and a vertex set \(U \subseteq V\) with \(|U| \leq \frac{n}{4(k-\ell)}\), there exist a vertex \(a \in A' \setminus U\) and a \((2k - 3\ell - 1)\)-set \(C \subset B' \setminus U\) such that \(L_1 \cup L_2 \cup \{a\} \cup C\) spans an \(\ell\)-path (of length two) ended at \(L_1, L_2\).

**Proof.** Fix two disjoint typical \(\ell\)-sets \(L_1, L_2\) in \(B'\). Using Claim 3.2, we obtain that \(|U| \leq \frac{n}{4(k-\ell)} \leq \frac{|A|}{2} < \frac{3}{4} |A'|\) and
\[
\frac{n}{4(k - \ell)} \leq \frac{|B| + 1}{2(2k - 2\ell - 1)} \leq \frac{1 + 2\epsilon_2 |B'|}{2k} < \frac{|B'|}{k}.
\]
Thus \(|A' \setminus U| > \frac{|A'|}{2}\) and \(|B' \setminus U| > \frac{k - 1}{k} |B'|\). Consider a \((k - \ell)\)-graph \(G\) on \((A' \cup B') \setminus U\) such that an \(A' B^{k-\ell-1}\)-set \(T\) is an edge of \(G\) if and only if \(T \cap U = \emptyset\) and \(T\) is a common neighbor of \(L_1\) and \(L_2\) in \(H\). By Claim 3.4, we have
\[
\tau(G) \leq 2 \cdot 4k\epsilon_1 \left( \frac{|B'| - \ell}{k - \ell - 1} \right) |A'| < 8k\epsilon_1 \left( \frac{k - 1}{k - \ell - 1} |B' \setminus U| \right) \cdot 3 |A' \setminus U| \leq 24k\epsilon_1 \left( \frac{k}{k - \ell - 1} \right)^{k-1} \left( \frac{|B' \setminus U|}{k - \ell - 1} \right) |A' \setminus U|.
\]
Consequently \(e(G) > \frac{1}{2} \left( \frac{|B' \setminus U|}{k - \ell - 1} \right) |A' \setminus U|\). Hence there exists a vertex \(a \in A' \setminus U\) such that \(deg_G(a) > \frac{1}{2} \left( \frac{|B' \setminus U|}{k - \ell - 1} \right)\). By Fact 2.9, the link graph of \(a\) contains a copy of \(Y_{k-\ell-1,k-\ell-1}\) (two edges of the link graph sharing \(\ell - 1\) vertices). In other words, there exists a \((2k - 3\ell - 1)\)-set \(C \subset B' \setminus U\) such that \(a \cup C\) contains two edges of \(G\) sharing \(\ell\) vertices. Together with \(L_1, L_2\), this gives rise to the desired \(\ell\)-path (in \(H\)) of length two ended at \(L_1, L_2\).

The following claim shows that we can always extend a typical \(\ell\)-set to an edge of \(H\) by adding one vertex from \(A'\) and \(k - \ell - 1\) vertices from \(B'\) such that every \(\ell\) new vertices form a typical \(\ell\)-set. This can be done even when at most \(\frac{n}{4(k-\ell)}\) vertices of \(V\) are not available.

**Claim 3.6.** Given a typical \(\ell\)-set \(L \subseteq B'\) and a set \(U \subseteq V\) with \(|U| \leq \frac{n}{4(k-\ell)}\), there exists an \(A' B^{k-\ell-1}\)-set \(C \subset V \setminus U\) such that \(L \cup C\) is an edge of \(H\) and every \(\ell\)-subset of \(C \cap B'\) is typical.

**Proof.** First, since \(L\) is typical in \(B'\), by Claim 3.4, \(\deg(L, A' B^{k-1}) \leq 4k\epsilon_1 \left( \frac{|B' \setminus U|}{k - \ell - 1} \right) |A'|\). Second, note that a vertex in \(A'\) is contained in \(\left( \frac{|B'|}{k - \ell - 1} \right) A' B^{k-\ell-1}\)-sets, while a vertex in \(B'\) is contained
in \(|A'|(\binom{|B'|}{k-\ell-2})^{-1} A'B^{k-\ell-1}\)-sets. It is easy to see that \(|A'|(\binom{|B'|}{k-\ell-2}) < (\binom{|B'|}{k-\ell-1})\) (as \(|A'| \approx \frac{n}{2^{k-2\ell}}\) and \(|B'| \approx \frac{2k-2\ell-1}{2k-2\ell} n\)). We thus derive that at most
\[
|U| \binom{|B'|}{k-\ell-1} \leq \frac{n}{4(k-\ell)} \binom{|B'|}{k-\ell-1}
\]
\(A'B^{k-\ell-1}\)-sets intersect \(U\). Finally, by Claim 3.3, the number of atypical \(\ell\)-sets in \(B\) is at most \(\epsilon_2(\binom{|B'|}{k-\ell-1})\). Using Claim 3.2, we derive that the number of atypical \(\ell\)-sets in \(B\) is at most
\[
\epsilon_2(\binom{|B'|}{\ell}) + \binom{|B'|}{\ell-1} \leq 2\epsilon_2(\binom{|B'|}{\ell}) + \epsilon_2|B|(\binom{|B'|}{\ell-1}) < 3k\epsilon_2(\binom{|B'|}{\ell}).
\]
Hence at most \(3k\epsilon_2(\binom{|B'|}{\ell})|A'(\binom{|B'|}{k-\ell-1}) A'B^{k-\ell-1}\)-sets contain an atypical \(\ell\)-set. In summary, at most
\[
4k\epsilon_2(\binom{|B'|}{k-\ell-1})|A'| + \frac{n}{4(k-\ell)} \binom{|B'|}{k-\ell-1} + 3k\epsilon_2(\binom{|B'|}{\ell}) \binom{|B'|}{k-2\ell-1}|A'|
\]
\(A'B^{k-\ell-1}\)-sets fail some of the desired properties. Since \(\epsilon_1, \epsilon_2 \ll 1\) and \(|A'| \approx \frac{n}{2^{k-2\ell}}\), the desired \(A'B^{k-\ell-1}\)-set always exists.

\[\square\]

3.3. Building a short path \(Q\). The following claim is the only place where we used the exact codegree condition (1.1).

**Claim 3.7.** Suppose that \(|A \cap B'| = q > 0\). Then there exists a family \(P_1\) of vertex-disjoint \(2q\) edges in \(B'\), each of which contains two disjoint typical \(\ell\)-sets.

**Proof.** Let \(|A \cap B'| = q > 0\). Since \(A \cap B' \neq \emptyset\), by Claim 3.1, we have \(B \subseteq B'\), and consequently \(|B'| = \frac{2k-2\ell-1}{2(k-\ell)} n + q\). By Claim 3.2, we have \(q \leq |A \setminus A'| \leq \epsilon_2|B|\).

Let \(B\) denote the family of the edges in \(B'\) that contain two disjoint typical \(\ell\)-sets. We derive a lower bound for \(|B|\) as follows. We first pick a \((k-1)\)-subset of \(B\) (recall that \(B \subseteq B'\)) that contains no atypical \(\ell\)-subset. Since \(2\ell \leq k-1\), such a \((k-1)\)-set contains two disjoint typical \(\ell\)-sets. By Claim 3.3, there are at most \(\epsilon_2(\binom{|B'|}{\ell})|A'(\binom{|B'|}{k-\ell-1})|B'\)-sets in \(B'\) and in turn, there are at most \(\epsilon_2(\binom{|B'|}{\ell})(\binom{|B'|}{k-\ell-1})|B'\)-subsets of \(B\) that contain an atypical \(\ell\)-subset. Thus there are at least
\[
\binom{|B'|}{k-1} - \epsilon_2(\binom{|B'|}{\ell}) \binom{|B'|}{k-\ell-1} = \left(1 - \binom{k-1}{\ell} \epsilon_2\right) \binom{|B'|}{k-1}
\]
\((k-1)\)-subsets of \(B\) that contain no atypical \(\ell\)-subset. After picking such a \((k-1)\)-set \(S \subseteq B\), we find a neighbor of \(S\) by the codegree condition. Since \(|B'| = \frac{2k-2\ell-1}{2(k-\ell)} n + q\), by (1.1), we have \(\deg(S, B') \geq q\). We thus derive that
\[
|B| \geq \left(1 - \binom{k-1}{\ell} \epsilon_2\right) \binom{|B'|}{k-1} \frac{q}{k},
\]
in which we divide by \(k\) because every edge of \(B\) is counted at most \(k\) times.

We claim that \(B\) contains \(2q\) disjoint edges. Suppose instead, a maximum matching in \(B\) has \(i < 2q\) edges. By the definition of \(B\), for any vertex \(b \in B'\), we have
\[
\deg(b, B') \leq \deg(b, B) + |B' \setminus B| \binom{|B'|}{k-2}
\]
\[
\leq \epsilon_1(\binom{|B'|}{k-1}) + \epsilon_2|B| \binom{|B'|}{k-2} < 2\epsilon_1(\binom{|B'|}{k-1}).
\]
(3.2)
Thus at most $2qk \cdot 2e_1\frac{|B|}{k-1}$ edges of $B'$ intersect the $i$ edges in the matching. Hence, the number of edges of $B$ that are disjoint from these $i$ edges is at least

$$\frac{q}{k} \left( 1 - \left( k - 1 \right) \epsilon_2 \right) \left( \frac{|B|}{k-1} \right) - 4k\epsilon_1q \left( \frac{|B|}{k-1} \right) \geq \left( \frac{1}{k} - (4k + 1) \epsilon_1 \right) q \left( \frac{|B|}{k-1} \right) > 0,$$

as $\epsilon_2 \ll \epsilon_1 \ll 1$. We may thus obtain a matching of size $i+1$, a contradiction. \hfill \Box

**Claim 3.8.** There exists a non-empty $\ell$-path $Q$ in $H$ with the following properties:

- $V_0 \subseteq V(Q)$,
- $|V(Q)| \leq 10k\epsilon_2|B|$,  
- two ends $L_0, L_1$ of $Q$ are typical $\ell$-sets in $B'$,
- $|B_1| = 2(k - 2\ell - 1)|A_1| + \ell$, where $A_1 = A' \setminus V(Q)$ and $B_1 = (B' \setminus V(Q)) \cup L_0 \cup L_1$.

**Proof.** We split into two cases here.

**Case 1.** $A \cap B' \neq \emptyset$.

By Claim 3.1, $A \cap B' \neq \emptyset$ implies that $B \subseteq B'$. Let $q = |A \cap B'|$. We first apply Claim 3.7 and find a family $P_1$ of vertex-disjoint $2q$ edges in $B'$. Next we associate each vertex of $V_0$ with $2k - \ell - 1$ vertices of $B$ (so in $B'$) forming an $\ell$-path of length two such that these $|V_0|$ paths are pairwise vertex-disjoint, and also vertex-disjoint from the paths in $P_1$, and all these paths have typical ends. To see it, let $V_0 = \{x_1, \ldots, x_{|V_0|}\}$. Suppose that we have found such $\ell$-paths for $x_1, \ldots, x_{i-1}$ with $i \leq |V_0|$. Since $B \subseteq B'$, it follows that $A \setminus A' = (A \cap B') \cup V_0$. Hence $|V_0| + q = |A \setminus A'| \leq \epsilon_2|B|$ by Claim 3.2. Therefore

$$(2k - \ell - 1)(i - 1) + |V(P_1)| < 2k|V_0| + 2kq \leq 2k\epsilon_2|B|$$

and consequently at most $2k\epsilon_2|B|\left(\frac{|B|}{k-2}\right) < 2k^2\epsilon_2\left(\frac{|B|}{k-1}\right)$-(k-1)-sets of $B$ intersect the existing paths (including $P_1$). By the definition of $V_0$, $\deg(x_i, B) > \epsilon_1\left(\frac{|B|}{k-1}\right)$. Let $G_{x_i}$ be the $(k-1)$-graph on $B$ such that $e \in G_{x_i}$ if

- $\{x_i\} \cup e \in E(H)$,
- $e$ does not contain any vertex from the existing paths,
- $e$ does not contain any atypical $\ell$-set.

By Claim 3.3, the number of $(k-1)$-sets in $B$ containing at least one atypical $\ell$-set is at most $\epsilon_2\left(\frac{|B|}{k-\ell}\right)\frac{|B|-\epsilon}{k-\ell-1} = \epsilon_2\left(\frac{|B|}{k-1}\right)$). Thus, we have

$$e(G_{x_i}) \geq \epsilon_1 \left( \frac{|B|}{k-1} \right) - 2k^2\epsilon_2 \left( \frac{|B|}{k-1} \right) - \epsilon_2 \left( \frac{k-1}{\ell} \right) \left( \frac{|B|}{k-1} \right) > \frac{\epsilon_1}{2} \left( \frac{|B|}{k-1} \right) > \left( \frac{|B|}{k-2} \right),$$

because $\epsilon_2 \ll \epsilon_1$ and $|B|$ is sufficiently large. By Fact 2.9, $G_{x_i}$ contains a copy of $Y_{k-1, \ell-1}$, which gives the desired $\ell$-path of length two containing $x_i$.

Denote by $P_2$ the family of $\ell$-paths we obtained so far. Now we need to connect paths of $P_2$ together to a single $\ell$-path. For this purpose, we apply Claim 3.5 repeatedly to connect the ends of two $\ell$-paths while avoiding previously used vertices. This is possible because $|V(P_2)| = (2k - \ell)|V_0| + 2kq$ and $(2k - 3\ell)(|V_0| + 2q - 1)$ vertices are needed to connect all the paths in $P_2$ - the set $U$ (when we apply Claim 3.5) thus satisfies

$$|U| \leq (4k - 4\ell)|V_0| + (6k - 6\ell)q - 2k + 3\ell \leq 6(k - \ell)\epsilon_2|B| - 2k + 3\ell.$$ 

Let $P$ denote the resulting $\ell$-path. We have $|V(P) \cap A'| = |V_0| + 2q - 1$ and

$$|V(P) \cap B'| = k \cdot 2q + (2k - \ell - 1)|V_0| + (2k - 3\ell - 1)(|V_0| + 2q - 1) = 2(2k - 2\ell - 1)|V_0| + 2(3k - 3\ell - 1)q - (2k - 3\ell - 1).$$
Let \( s = (2k - 2\ell - 1)|A'| + |V(P)| - |B' \setminus V(P)| \). We have

\[
s = (2k - 2\ell - 1)(|A'| - |V_0| - 2q + 1) - |B'| + 2(2k - 2\ell - 1)|V_0| + 2(3k - 3\ell - 1)q - (2k - 3\ell - 1)
\]

\[
= (2k - 2\ell - 1)|A'| - |B'| + (2k - 2\ell - 1)|V_0| + (2k - 2\ell)q + \ell.
\]

Since \(|A'| + |B'| + |V_0| = n\), we have

\[
s = (2k - 2\ell)(|A'| + |V_0| + q) - n + \ell. \tag{3.3}
\]

Note that \(|A'| + |V_0| + q = |A|\) and

\[
(2k - 2\ell)|A| - n = \begin{cases} 0, & \text{if } \frac{n}{2\ell} \text{ is even} \\ k - \ell, & \text{if } \frac{n}{2\ell} \text{ is odd}. \end{cases} \tag{3.4}
\]

Thus \( s = \ell \) or \( s = k \). If \( s = k \), then we extend \( P \) to an \( \ell \)-path \( Q \) by applying Claim 3.6, otherwise let \( Q = P \). Then

\[
|V(Q)| \leq |V(P)| + (k - \ell) \leq 6k\epsilon_2|B|,
\]

and \( Q \) has two typical ends \( L_0, L_1 \subset B' \). We claim that

\[
(2k - 2\ell - 1)|A' \setminus V(Q)| - |B' \setminus V(Q)| = \ell. \tag{3.5}
\]

Indeed, when \( s = \ell \), this is obvious; when \( s = k \), \( V(Q) \setminus V(P) \) contains one vertex of \( A' \) and \( k - \ell - 1 \) vertices of \( B' \) and thus

\[
(2k - 2\ell - 1)|A' \setminus V(Q)| - |B' \setminus V(Q)| = s - (2k - 2\ell - 1) + (k - \ell - 1) = \ell.
\]

Let \( A_1 = A' \setminus V(Q) \) and \( B_1 = (B' \setminus V(Q)) \cup L_0 \cup L_1 \). We derive that \(|B_1| = (2k - 2\ell - 1)|A_1| + \ell\) from (3.5).

**Case 2.** \( A \cap B' = \emptyset \).

Note that \( A \cap B' = \emptyset \) means that \( B' \subseteq B \). Then we have

\[
|A'| + |V_0| = |V \setminus B'| = |A| + |B \setminus B'|. \tag{3.6}
\]

If \( V_0 \neq \emptyset \), we handle this case similarly as in Case 1 except that we do not need to construct \( P_1 \). By Claim 3.2, \(|B \setminus B'| \leq \epsilon_2|B|\) and thus for any vertex \( x \in V_0 \),

\[
\text{deg}(x, B') \geq \text{deg}(x, B) - |B \setminus B'| \cdot \left( \frac{|B| - 1}{k - 2} \right)
\]

\[
\geq \epsilon_1 \left( \frac{|B|}{k - 1} - (k - 1)\epsilon_2 \left( \frac{|B|}{k - 1} \right) \right) > \frac{\epsilon_1}{2} \left( \frac{|B'|}{k - 1} \right). \tag{3.7}
\]

As in Case 1, we let \( V_0 = \{x_1, \ldots, x_{|V_0|}\} \) and cover them with vertex-disjoint \( \ell \)-paths of length two. Indeed, for each \( i \leq |V_0| \), we construct \( G_x \) as before and show that \( e(G_x) \geq \epsilon_1 \left( \frac{|B'|}{k - 1} \right) \). We then apply Fact 2.9 to \( G_x \), obtaining a copy of \( H_{k-1, \ell-1} \), which gives an \( \ell \)-path of length two containing \( x_i \).

As in Case 1, we connect these paths to a single \( \ell \)-path \( P \) by applying Claim 3.5 repeatedly. Then \(|V(P)| = (2k - \ell)|V_0| + (2k - 3\ell)(|V_0| - 1)\). Define \( s \) as in Case 1. Thus (3.3) holds with \( q = 0 \).

Applying (3.6) and (3.4), we derive that

\[
s = 2(k - \ell)(|A| + |B \setminus B'|) - n + \ell = \begin{cases} \ell + 2(k - \ell)|B \setminus B'|, & \text{if } \frac{n}{2\ell} \text{ is even} \\ k + 2(k - \ell)|B \setminus B'|, & \text{if } \frac{n}{2\ell} \text{ is odd}. \end{cases} \tag{3.8}
\]

which implies that \( s \equiv \ell \mod (k - \ell) \). We extend \( P \) to an \( \ell \)-path \( Q \) by applying Claim 3.6 \( \frac{s - \ell}{k - \ell} \) times. Then

\[
|V(Q)| = |V(P)| + s - \ell \leq (4k - 4\ell)|V_0| - 2k + 3\ell + k - \ell + 2(k - \ell)|B \setminus B'| \leq 10k\epsilon_2|B|
\]
by Claim 3.2. Note that \( Q \) has two typical ends \( L_0, L_1 \subset B' \). Since \( V(Q) \setminus V(P) \) contains \( \frac{s - \ell}{k - \ell} \) vertices of \( A' \) and \( \frac{s - \ell}{k - \ell} (k - \ell - 1) \) vertices of \( B' \), we have

\[
(2k - 2 \ell - 1) |A' \setminus V(Q)| - |B' \setminus V(Q)| = s - \frac{s - \ell}{k - \ell} (2k - 2 \ell - 1) + \frac{s - \ell}{k - \ell} (k - \ell - 1) = \ell.
\]

We define \( A_1 \) and \( B_1 \) in the same way and similarly we have \( |B_1| = (2k - 2 \ell - 1) |A_1| + \ell \).

When \( V_0 = \emptyset \), we pick an arbitrary vertex \( v \in A' \) and form an \( \ell \)-path \( P \) of length two with typical ends such that \( v \) is in the intersection of the two edges. This is possible by the definition of \( A' \).

Define \( s \) as in the previous case. It is easy to see that (3.8) still holds. We then extend \( P \) to \( Q \) by applying Claim 3.6 \( \frac{s - \ell}{k - \ell} \) times. Then \( |V(Q)| = 2k - \ell + s - \ell \leq 2k \epsilon_2 |B| \) because of (3.8). The rest is the same as in the previous case.

**Claim 3.9.** The \( A_1, B_1 \) and \( L_0, L_1 \) defined in Claim 3.8 satisfy the following properties:

1. \( |B_1| \geq (1 - \epsilon_1) |B| \).
2. For any vertex \( v \in A_1, \overline{\deg}(v, B_1) < 3 \epsilon_1 \left( \frac{|B_1|}{k - 1} \right) \).
3. For any vertex \( v \in B_1, \overline{\deg}(v, A_1B_1^{k-1}) < 3k \epsilon_1 \left( \frac{|B_1|}{k - 1} \right) \).
4. \( \overline{\deg}(L_0, A_1B_1^{k-1}) \leq 5k \epsilon_1 \left( \frac{|B_1|}{k - 1} \right), \overline{\deg}(L_1, A_1B_1^{k-1}) \leq 5k \epsilon_1 \left( \frac{|B_1|}{k - 1} \right) \).

**Proof.** Part (1): By Claim 3.2, we have \( |B_1 \setminus B| \leq |B' \setminus B| \leq \epsilon_2 |B| \).

Part (2): For a vertex \( v \in A_1, \overline{\deg}(v, B_1) \leq \epsilon_1 \left( \frac{|B|}{k - 1} \right) + \epsilon_2 |B| \left( \frac{|B_1| - 1}{k - 2} \right) \leq \epsilon_1 \left( \frac{|B|}{k - 1} \right) + \epsilon_2 |B| \left( \frac{|B_1| - 1}{k - 2} \right) \leq \epsilon_1 \left( \frac{|B|}{k - 1} \right) + \epsilon_1 \left( \frac{|B_1|}{k - 1} \right) < 3 \epsilon_1 \left( \frac{|B_1|}{k - 1} \right) \).

Part (3): Consider the sum \( \sum \overline{\deg}(S \cup \{v\}) \) taken over all \( S \in \binom{B' \setminus \{v\}}{k - 2} \). Since \( \delta_{k-1}(H) \geq |A| \), we have \( \sum \overline{\deg}(S \cup \{v\}) \geq \binom{|B' - 1|}{k - 2} |A| \). On the other hand,

\[
\sum \overline{\deg}(S \cup \{v\}) = \overline{\deg}(v, A'B_1^{k-1}) + \overline{\deg}(v, V_0B_1^{k-1}) + (k - 1) \overline{\deg}(v, B').
\]

By (3.2), \( \overline{\deg}(v, B') \leq 2 \epsilon_1 \left( \frac{|B_1|}{k - 1} \right) \). We thus derive that

\[
\overline{\deg}(v, A'B_1^{k-1}) \geq \left( \frac{|B_1| - 1}{k - 2} \right) |A| - \overline{\deg}(v, V_0B_1^{k-1}) - (k - 1) \overline{\deg}(v, B')
\geq \left( \frac{|B_1| - 1}{k - 2} \right) \left( |A| - 2 \epsilon_2 |B| \left( \frac{|B_1| - 1}{k - 2} \right) - 2 (k - 1) \epsilon_1 \left( \frac{|B|}{k - 1} \right) \right)
\geq \left( \frac{|B_1| - 1}{k - 2} \right) |A| - 2k \epsilon_1 \left( \frac{|B|}{k - 1} \right).
\]

Thus, by Part (1), we have

\[
\overline{\deg}(v, A_1B_1^{k-1}) \leq \overline{\deg}(v, A'B_1^{k-1}) \leq 2k \epsilon_1 \left( \frac{|B_1|}{k - 1} \right) \leq 3k \epsilon_1 \left( \frac{|B_1|}{k - 1} \right).
\]
Part (4): By Claim 3.4, for any typical \( L \subseteq B' \), we have \( \overline{\text{deg}}(L, A'B^{k-1}) \leq 4k\epsilon_1 \left( \frac{|B'| - \ell}{k - \ell - 1} \right) |A'| \).

Thus,
\[
\overline{\text{deg}}(L_0, A_1B_1^{k-1}) \leq \overline{\text{deg}}(L_0, A'B^{k-1}) \leq 4k\epsilon_1 \left( \frac{|B'| - \ell}{k - \ell - 1} \right) |A'| \leq 5k\epsilon_1 \left( \frac{|B_1|}{k - \ell} \right),
\]
where the last inequality holds because \( |B'| \leq |B_1| + |V(\mathcal{Q})| \leq (1 + \epsilon_1)|B_1| \). The same holds for \( L_1 \).

### 3.4. Completing the Hamilton cycle

We finally complete the proof of Theorem 1.5 by applying the following lemma with \( X = A_1, Y = B_1, \rho = 5k\epsilon_1 \), and \( L_0, L_1 \).

**Lemma 3.10.** Fix \( 1 \leq \ell < k/2 \). Let \( 0 < \rho \ll 1 \) and \( n \) be sufficiently large. Suppose that \( \mathcal{H} \) is a \( k \)-graph with a partition \( V(\mathcal{H}) = X \cup Y \) and the following properties:

- \(|Y| = (2k - 2\ell - 1)|X| + \ell\),
- for every vertex \( v \in X \), \( \overline{\text{deg}}(v, Y) \leq \rho |Y| \) and for every vertex \( v \in Y \), \( \overline{\text{deg}}(v, XY^{k-1}) \leq \rho \left( \frac{|Y|}{k - \ell} \right) \),
- there are two disjoint \( \ell \)-sets \( L_0, L_1 \subseteq Y \) such that
\[
\overline{\text{deg}}(L_0, XY^{k-1}), \overline{\text{deg}}(L_1, XY^{k-1}) \leq \rho \left( \frac{|Y|}{k - \ell} \right).
\]

Then \( \mathcal{H} \) contains a Hamilton \( \ell \)-path with \( L_0 \) and \( L_1 \) as ends.

In order to prove Lemma 3.10, we apply two results of Glebov, Person, and Weps [6]. Given \( 1 \leq \ell \leq k - 1 \) and \( 0 \leq \rho \leq 1 \), an ordered set \( (x_1, \ldots, x_l) \) is \( \rho \)-typical in a \( k \)-graph \( \mathcal{G} \) if for every \( i \in [l] \)
\[
\overline{\text{deg}}_{\mathcal{G}}((x_1, \ldots, x_i)) \leq \rho^{k-i} \left( \frac{|V(\mathcal{G})| - i}{k - i} \right).
\]
It was shown in [6] that every \( k \)-graph \( \mathcal{G} \) with very large minimum vertex degree contains a tight Hamilton cycle. The proof of [6, Theorem 2] actually shows that we can obtain a tight Hamilton cycle by extending any fixed tight path of constant length with two typical ends. This implies the following theorem that we will use.

**Theorem 3.11.** [6] Given \( 1 \leq \ell \leq k \) and \( 0 < \alpha \ll 1 \), there exists an \( m_0 \) such that the following holds. Suppose that \( \mathcal{G} \) is a \( k \)-graph on \( V \) with \( |V| = m \geq m_0 \) and \( \delta_1(\mathcal{G}) \geq (1 - \alpha)\left( \frac{n-1}{k-1} \right) \). Then given any two \( (22\alpha) \frac{k-\ell}{k+1} \)-typical ordered \( \ell \)-sets \( (x_1, \ldots, x_l) \) and \( (y_1, \ldots, y_l) \), there exists a tight Hamilton path \( P = x_l x_{l-1} \cdots x_1 \cdots y_{2l} \cdots y_1 \) in \( \mathcal{G} \).

We also use [6, Lemma 3], in which \( V^{2k-2} \) denotes the set of all \( (2k - 2) \)-tuples \( (v_1, \ldots, v_{2k-2}) \) such that \( v_i \in V \) (\( v_i \)'s are not necessarily distinct).

**Lemma 3.12.** [6] Let \( \mathcal{G} \) be the \( k \)-graph given in Lemma 3.11. Suppose that \( (x_1, \ldots, x_{2k-2}) \) is selected uniformly at random from \( V^{2k-2} \). Then the probability that all \( x_i \)'s are pairwise distinct and \( (x_1, \ldots, x_{k-1}), (x_k, \ldots, x_{2k-2}) \) are \( (22\alpha) \frac{k-\ell}{k+1} \)-typical is at least \( \frac{8}{\Pi} \).

**Proof of Lemma 3.10.** In this proof we often write the union \( A \cup B \cup \{x\} \) as \( ABx \), where \( A, B \) are sets and \( x \) is an element.

Let \( t = |X| \). Our goal is to write \( X \) as \( \{x_1, \ldots, x_t\} \) and partition \( Y \) as \( \{L_i, R_i, S_i, R'_i : i \in [t]\} \) with \( |L_i| = \ell, |R_i| = |R'_i| = k - 2\ell \), and \( |S_i| = \ell - 1 \) such that
\[
L_i R_i S_i x_{L_i}, S_i x_{L_i} R'_i L_{i+1} \in E(\mathcal{H})
\]
for all \( i \in [t] \), where \( L_{i+1} = L_0 \). Consequently
\[
L_1 R_1 S_1 x_1 R'_1 L_2 R_2 S_2 x_2 R'_2 \cdots L_t R_t S_t x_t R'_t L_{t+1}
\]
is the desired Hamilton $\ell$-path of $\mathcal{H}$.

Let $\mathcal{G}$ be the $(k-1)$-graph on $Y$ whose edges are all $(k-1)$-sets $S \subseteq Y$ such that $\deg_{\mathcal{H}}(S, X) > (1 - \sqrt{\rho})t$. The following is an outline of our proof. We first find a small subset $Y_0 \subseteq Y$ with a partition \{\(L_i, R_i, S_i, R_i': i \in [t_0]\)\} such that for every $x \in X$, we have $L_i R_i S_i x, S_i x R_i' L_{i+1} \in E(\mathcal{H})$ for many $i \in [t_0]$. Next we apply Theorem 3.11 to $\mathcal{G}[Y \setminus Y_0]$ and obtain a tight Hamilton path, which, in particular, partitions $Y \setminus Y_0$ into \{\(L_i, R_i, S_i, R_i': t_0 < i \leq t\)\} such that $L_i R_i S_i, S_i R_i' L_{i+1} \in E(\mathcal{G})$ for $t_0 < i \leq t$. Finally we apply the Marriage Theorem to find a perfect matching between $X$ and $[t]$ such that (3.10) holds for all matched $x_i$ and $i$.

We now give details of the proof. First we claim that

$$\delta(\mathcal{G}) \geq (1 - 2\sqrt{\rho}) \left( |Y| - \frac{1}{k - 2} \right),$$

(3.11)

and consequently,

$$\tau(\mathcal{G}) \leq 2\sqrt{\rho} \left( \frac{|Y|}{k - 1} \right).$$

(3.12)

Suppose instead, some vertex $v \in Y$ satisfies $\deg_{\mathcal{G}}(v) > 2\sqrt{\rho} \left( \frac{|Y|}{k - 2} \right)$. Since every non-neighbor $S'$ of $v$ in $\mathcal{G}$ satisfies $\deg_{\mathcal{G}}(S', X) \geq \sqrt{\rho}t$, we have $\deg_{\mathcal{H}}(v, XY^{k-1}) > 2\sqrt{\rho} \left( \frac{|Y| - \ell}{2k - 2\ell - 1} \right) \left( \frac{|Y| - 1}{k - 2} \right) > \rho \left( \frac{|Y|}{k - 1} \right) \left( \frac{|Y| - 1}{k - 2} \right) = \rho \left( \frac{|Y|}{k - 1} \right),$ contradicting our assumption (the second inequality holds because $|Y|$ is sufficiently large).

Let $Q$ be a $(2k - \ell - 1)$-subset of $Y$. We call $Q$ good (otherwise bad) if every $(k-1)$-subset of $Q$ is an edge of $\mathcal{G}$ and every $\ell$-set $L \subset Q$ satisfies

$$\deg_{\mathcal{G}}(L) \leq \rho^{1/4} \left( \frac{|Y| - \ell}{k - \ell - 1} \right).$$

(3.13)

Furthermore, we say $Q$ is suitable for a vertex $x \in X$ if $x \cup T \in E(\mathcal{H})$ for every $(k-1)$-set $T \subset Q$.

Note that if a $(2k - \ell - 1)$-set is good, by the definition of $\mathcal{G}$, it is suitable for at least $(1 - \left( \frac{2k - \ell - 1}{k - 1} \right) \sqrt{\rho})t$ vertices of $X$. Let $Y' = Y \setminus (L_0 \cup L_1)$.\n
Claim 3.13. For any $x \in X$, at least $(1 - \rho^{1/5}) \left( \frac{|Y|}{2k - \ell - 1} \right)$ $(2k - \ell - 1)$-subsets of $Y'$ are good and suitable for $x$.

Proof. Since $\rho + \rho^{1/2} + 3(\frac{2k - \ell - 1}{\ell}) \rho^{1/4} \leq \rho^{1/5}$, the claim follows from the following three assertions:

- At most $2\ell \left( \frac{|Y| - 1}{2k - \ell - 2} \right) \leq \rho \left( \frac{|Y|}{2k - \ell - 1} \right)$ $(2k - \ell - 1)$-subsets of $Y$ are not subsets of $Y'$.

- Given $x \in X$, at most $\rho^{1/2} \left( \frac{2k - |Y|}{2k - \ell - 1} \right)$ $(2k - \ell - 1)$-sets in $Y$ are not suitable for $x$.

- At most $3 \left( \frac{2k - |Y|}{\ell} \right) \rho^{1/4} \left( \frac{|Y|}{2k - \ell - 1} \right)$ $(2k - \ell - 1)$-sets in $Y$ are bad.

The first assertion holds because $|Y \setminus Y'| = 2\ell$. The second assertion follows from the degree condition of $\mathcal{H}$, namely, for any $x \in X$, the number of $(2k - \ell - 1)$-sets in $Y$ that are not suitable for $x$ is at most $\rho \left( \frac{|Y|}{k - 1} \right) \left( \frac{|Y| - k - 1}{k - \ell - 1} \right) \leq \sqrt{\rho} \left( \frac{|Y|}{2k - \ell - 1} \right)$.

To see the third one, let $m$ be the number of $\ell$-sets $L \subseteq Y$ that fail (3.13). By (3.12),

$$m \frac{\rho^{1/4} \left( \frac{|Y| - \ell}{k - \ell - 1} \right)}{\left( \frac{2k - \ell - 1}{\ell} \right)} \leq \tau(\mathcal{G}) \leq 2\sqrt{\rho} \left( \frac{|Y|}{k - 1} \right),$$

which implies that $m \leq 2\rho^{1/4} \left( \frac{|Y|}{\ell} \right)$. Thus at most

$$2\rho^{1/4} \left( \frac{|Y|}{\ell} \right) \cdot \left( \frac{|Y| - \ell}{2k - 2\ell - 1} \right)$$
(2k − ℓ − 1)-subsets of Y contain an ℓ-set L that fails (3.13). On the other hand, by (3.12), at most
\[
\tau(G)\left(\binom{|Y| - k + 1}{k - \ell}\right) \leq 2\sqrt{\rho}\left(\binom{|Y|}{k - 1}\right)\left(\binom{|Y| - k + 1}{k - \ell}\right)
\]
(2k − ℓ − 1)-subsets of Y contain a non-edge of G. Putting these together, the number of bad (2k − ℓ − 1)-sets in Y is at most
\[
2\rho^{1/4}\left(\binom{|Y|}{\ell}\right)\left(\binom{|Y| - \ell}{2k - 2\ell - 1}\right) + 2\sqrt{\rho}\left(\binom{|Y|}{k - 1}\right)\left(\binom{|Y| - k + 1}{k - \ell}\right) \leq 3\left(\frac{2k - \ell - 1}{\ell}\right)^{\rho^{1/4}}\left(\binom{|Y|}{2k - \ell - 1}\right),
\]
as \rho \ll 1.

We will pick a family of disjoint good (2k − ℓ − 1)-sets in Y′ such that for any x ∈ X, many members of this family are suitable for x. To achieve this, we pick a family F by selecting each good (2k − ℓ − 1)-sets of Y′ randomly and independently with probability \( p = 6\sqrt{|Y|}/(2k - \ell - 1) \). Since there are at most \( (2k - \ell - 1) \cdot (2k - \ell - 1) \cdot (2k - \ell - 2) \) pairs of intersecting (2k − ℓ − 1)-sets in Y, the expected number of intersecting pairs of (2k − ℓ − 1)-sets in F is at most
\[
p^2\left(\binom{|Y|}{2k - \ell - 1}\right) \cdot (2k - \ell - 1) \cdot \left(\binom{|Y| - 1}{2k - \ell - 2}\right) = 36(2k - \ell - 1)^2\rho|Y|.
\]

By applying Chernoff’s bound on the first two properties and Markov’s bound on the last one below, we can find, with positive probability, a family F of good (2k − ℓ − 1)-subsets of Y′ that satisfies

- \(|F| \leq 2p(\binom{|Y|}{2k - \ell - 1}) \leq 12\sqrt{|Y|}
- for any vertex x ∈ X, because of Claim 3.13, at least
\[
\frac{p}{2}(1 - \rho^{1/5})\left(\frac{|Y|}{2k - \ell - 1}\right) \geq 2\sqrt{\rho}|Y|
\]
  members of F are suitable for x.
- the number of intersecting pairs of (2k − ℓ − 1)-sets in F is at most 72(2k − ℓ − 1)^2ρ|Y|.
After deleting one (2k − ℓ − 1)-set from each of the intersecting pairs from F, we obtain a family \( F' \subseteq F \) consisting of at most 12\sqrt{|Y|} disjoint good (2k − ℓ − 1)-subsets of Y′ and for each x ∈ X, at least
\[
2\sqrt{\rho}|Y| - 72(2k - \ell - 1)^2\rho|Y| \geq \frac{3}{7}\sqrt{\rho}|Y|
\]
members of \( F' \) are suitable for x.

Denote \( F' \) by \{Q_2, Q_3, \ldots, Q_{2q}\} for some \( q \leq 12\sqrt{|Y|} \). We arbitrarily partition each \( Q_2i \) into \( L_{2i} \cup P_{2i} \cup L_{2i+1} \) such that \( |L_{2i}| = |L_{2i+1}| = \ell \) and \( |P_{2i}| = 2k - 3\ell - 1 \). Since \( Q_2i \) is good, both \( L_{2i} \) and \( L_{2i+1} \) satisfy (3.13). We claim that \( L_0 \) and \( L_1 \) satisfy (3.13) as well. Let us show this for \( L_0 \). By the definition of \( G \), the number of \( XY^{k-\ell-1} \)-sets \( T \) such that \( T \cup L_0 \not\in E(H) \) is at least \( \deg_G(L_0)\sqrt{\rho}t \). Using (3.9), we derive that \( \deg_G(L_0)\sqrt{\rho}t \leq \rho(\binom{|Y|}{k-\ell}) \). Since \( |Y| \leq (2k - 2\ell)t \), it follows that \( \deg_G(L_0) \leq 2\sqrt{\rho(\binom{|Y|}{k-\ell})} \leq \rho^{1/4}(\binom{|Y|}{k-\ell}) \).

Next we find disjoint \((2k - 3\ell - 1)-sets P_1, P_3, \ldots, P_{2q-1} from Y' \setminus \bigcup_{i=1}^{q} Q_{2i} such that for i \in [q], every \((k - \ell - 1)-subset of P_{2i-1} is a common neighbor of L_{2i-1} and L_{2i} in G. Since L_{1}, L_{2}, \ldots, L_{2q} all satisfy (3.13), at most
\[
2\cdot\rho^{1/4}\left(\binom{|Y| - \ell}{k - \ell - 1}\right)\left(\binom{|Y| - k + \ell + 1}{k - 2\ell}\right)
\]
(2k - 3\ell - 1)-subsets of Y contain a non-neighbor of L_{2i-1} or L_{2i}. Since \( q \leq 12\sqrt{|Y|} \) and \( \rho \ll 1 \), we can greedily find desired \( P_1, P_3, \ldots, P_{2q-1} \).
Let $Y_1 = Y' \setminus \bigcup_{i=1}^q (P_{2i-1} \cup Q_{2i})$ and $G' = G[Y_1]$. Then $|Y_1| = |Y'| - (2k - 2\ell - 1)2q$. Since $\deg_{G'}(v) \leq \deg_G(v)$ for every $v \in Y_1$, we have, by (3.11),

$$\delta_1(G') \geq \left(\frac{|Y_1| - 1}{k - 2}\right) - 2\sqrt{p}\left(\frac{|Y| - 1}{k - 2}\right) \geq (1 - 3\sqrt{p})\left(\frac{|Y_1| - 1}{k - 2}\right).$$

Let $\alpha = 3\sqrt{p}$ and $\rho_0 = (22\alpha)^{-1/4}$. We want to find two disjoint $\rho_0$-typical ordered subsets $(x_1, \ldots, x_{k-\ell-1})$ and $(y_1, \ldots, y_{k-\ell-1})$ of $Y_1$ such that

$$L_{2q+1} \cup \{x_1, \ldots, x_{k-\ell-1}\}, L_0 \cup \{y_1, \ldots, y_{k-\ell-1}\} \in E(G).$$

(3.15)

To achieve this, we choose $(x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1})$ from $Y_1^{2k-2}$ uniformly at random. By Lemma 3.12, with probability at least $\frac{n^i}{2}$, $(x_1, \ldots, x_{k-\ell-1})$ and $(y_1, \ldots, y_{k-\ell-1})$ are two disjoint ordered $\rho_0$-typical $(k - \ell - 1)$-sets. Since $L_0$ satisfies (3.13), at most $(k - \ell - 1)!p^{1/4}\left(\frac{|Y| - 1}{\ell}\right)$ ordered $(k - \ell - 1)$-subsets of $Y$ are not neighbors of $L_0$ (the same holds for $L_{2q+1}$). Thus (3.15) fails with probability at most $2(k - \ell - 1)!p^{1/4}$, provided that $x_1, \ldots, x_{k-\ell-1}, y_1, \ldots, y_{k-\ell-1}$ are all distinct. Therefore the desired $(x_1, \ldots, x_{k-\ell-1})$ and $(y_1, \ldots, y_{k-\ell-1})$ exist.

Next we apply Theorem 3.11 to $G'$ and obtain a tight Hamilton path

$$P = x_{k-\ell-1}x_{k-\ell-2} \cdots x_1 \cdots y_1y_2 \cdots y_{k-\ell-1}.$$ 

Following the order of $\mathcal{P}$, we partition $Y_1$ into

$$R_{2q+1}, S_{2q+1}, R'_{2q+1}, L_{2q+2}, \ldots, L_t, R_t, S_t, R'_t$$

such that $|L_i| = \ell$, $|R_i| = |R'_i| = k - 2\ell$, and $|S_i| = \ell - 1$. Since $\mathcal{P}$ is a tight path in $G$, we have

$$L_i, R_i, S_i, R'_i, L_{i+1} \in E(G)$$

(3.16)

for $2q + 2 \leq i \leq t - 1$. Letting $L_{t+1} = L_0$, by (3.15), we also have (3.16) for $i = 2q + 1$ and $i = t$.

We now arbitrarily partition $P_i, 1 \leq i \leq 2q$ into $R_i \cup S_i \cup R'_i$ such that $|R_i| = |R'_i| = k - 2\ell$, and $|S_i| = \ell - 1$. By the choice of $P_i$, (3.16) holds for $1 \leq i \leq 2q$.

Consider the bipartite graph $\Gamma$ between $X$ and $Z := \{z_1, z_2, \ldots, z_t\}$ such that $x \in X$ and $z_i \in Z$ are adjacent if and only if $L_i R_i S_i x, x S_i R'_i L_{i+1} \in E(\mathcal{H})$. For every $i \in [t]$, since (3.16) holds, we have $\deg_{\Gamma}(z_i) \geq (1 - 2\sqrt{p})t$ by the definition of $G$. Let $Z' = \{z_{2q+1}, \ldots, z_t\}$ and $X_0$ be the set of $x \in X$ such that $\deg_{\Gamma}(x, Z') \leq |Z'|/2$. Then

$$\frac{|X_0|}{2} \leq \sum_{x \in X} \deg_{\Gamma}(x, Z') \leq 2\sqrt{p}t \cdot |Z'|,$$

which implies that $|X_0| \leq 4\sqrt{p}t = 4\sqrt{p}\frac{|Y| - \ell}{\ell - 2\ell - 1} \leq 4\sqrt{p}|Y|$ (note that $2k - 2\ell - 1 \geq k \geq 3$).

We now find a perfect matching between $X$ and $Z$ as follows.

Step 1: Each $x \in X_0$ is matched to some $z_{2i}$, $i \in [q]$ such that the corresponding $Q_{2i} \in \mathcal{F}'$ is suitable for $x$ (thus $z_{2i}$ and $z_{2i+1}$ are adjacent in $\Gamma$) – this is possible because of (3.14) and $|X_0| \leq \frac{4}{3}\sqrt{p}|Y|$.

Step 2: Each of the unused $z_{2i}$, $i \in [2q]$ is matched to a vertex in $X \setminus X_0$ – this is possible because $\deg_{\Gamma}(z_i) \geq (1 - 2\sqrt{p})t \geq |X_0| + 2q$.

Step 3: Let $X'$ be the set of the remaining vertices in $X$. Then $|X'| = t - 2q = |Z'|$. Now consider the induced subgraph $\Gamma'$ of $\Gamma$ on $X' \cup Z'$. Since $\delta(\Gamma') \geq |X'|/2$, the Marriage Theorem provides a perfect matching in $\Gamma'$.

The perfect matching between $X$ and $Z$ gives rise to the desired Hamilton path of $\mathcal{H}$. □
4. Concluding Remarks

Let \( h^\ell_d(k, n) \) denote the minimum integer \( m \) such that every \( k \)-graph \( H \) on \( n \) vertices with minimum \( d \)-degree \( \delta_d(H) \geq m \) contains a Hamilton \( \ell \)-cycle (provided that \( k - \ell \) divides \( n \)). In this paper we determined \( h^\ell_{k-1}(k, n) \) for all \( \ell < k/2 \) and sufficiently large \( n \). Unfortunately our proof does not give \( h^\ell_{k-1}(k, n) \) for all \( k, \ell \) such that \( k - \ell \) does not divide \( k \) even though we believe that \( h^\ell_{k-1}(k, n) = \left\lceil \frac{n}{k-\ell} \right\rceil \). In fact, when \( k - \ell \) does not divide \( k \), if we can prove a path-cover lemma similar to Lemma 2.3, then we can follow the proof in [13] to solve the nonextremal case. When \( \ell \geq k/2 \), we cannot define \( Y_{k,2\ell} \) so the current proof of Lemma 2.3 fails. In addition, when \( \ell \geq k/2 \), the extremal case becomes complicated as well.

The situation is quite different when \( k - \ell \) divides \( k \). When \( k \) divides \( n \), one can easily construct a \( k \)-graph \( H \) such that \( \delta_{k-1}(H) \geq \frac{n}{2} - k \) and yet \( H \) contains no perfect matching and consequently no Hamilton \( \ell \)-cycle for any \( \ell \) such that \( k - \ell \) divides \( k \). A construction in [15] actually shows that \( h^\ell_{k-1}(k, n) \geq \frac{n}{2} - k \) whenever \( k - \ell \) divides \( k \), even when \( k \) does not divide \( n \). The exact value of \( h^\ell_d(k, n) \) is not known except for \( h^3_2(3, n) = \lfloor n/2 \rfloor \) given in [20]. In the forthcoming paper [8], the first author determines \( h^k_d/2(k, n) \) exactly for even \( k \) and any \( d \geq k/2 \).

Let \( t_2(n, F) \) denote the minimum integer \( m \) such that every \( k \)-graph \( H \) on \( n \) vertices with minimum \( d \)-degree \( \delta_d(H) \geq m \) contains a perfect \( F \)-tiling. One of the first results on hypergraph tiling was \( t_2(n, Y_{3,2}) = n/4 + o(n) \) given by Kühn and Osthus [14]. The exact value of \( t_2(n, Y_{3,2}) \) was determined recently by Czygrinow, DeBiasio, and Nagle [2]. We [10] determined \( t_1(n, Y_{3,2}) \) very recently. The key lemma in our proof, Lemma 2.8, shows that every \( k \)-graph \( H \) on \( n \) vertices with \( \delta_{k-1}(H) \geq \left( \frac{1}{2(k-b)} - o(1) \right) n \) either contains an almost perfect \( Y_{k,b} \)-tiling or is in the extremal case. Naturally this raises a question: what is \( t_{k-1}(n, Y_{k,b}) \)? Mycroft [16] recently proved a general result on tiling \( k \)-partite \( k \)-graphs, which implies that \( t_{k-1}(n, Y_{k,b}) = \frac{n}{2k-b} + o(n) \). The lower bound comes from the following construction. Let \( H_0 \) be the \( k \)-graph on \( n \in (2k-b)\mathbb{N} \) vertices such that \( V(H_0) = A \cup B \) with \( |A| = \frac{n}{2k-b} - 1 \), and \( E(H_0) \) consists of all \( k \)-sets intersecting \( A \) and some \( k \)-subsets of \( B \) such that \( H_0[B] \) contains no copy of \( Y_{k,b} \). Thus, \( \delta_{k-1}(H_0) \geq \frac{n}{2k-b} - 1 \). Since every copy of \( Y_{k,b} \) contains at least one vertex in \( A \), there is no perfect \( Y_{k,b} \)-tiling in \( H_0 \). We believe that one can find a matching upper bound by the absorbing method (similar to the proof in [2]). In fact, since we already proved Lemma 2.8, it suffices to prove an absorbing lemma and the extremal case.

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(Jie Han and Yi Zhao) DEPARTMENT OF MATHEMATICS AND STATISTICS, GEORGIA STATE UNIVERSITY, ATLANTA, GA 30303

E-mail address, Jie Han: jhan22@gsu.edu

E-mail address, Yi Zhao: yzhao6@gsu.edu