Conformally reducible 1+3 spacetimes

Jaume Carot\textsuperscript{1}, Aidan J Keane\textsuperscript{2} and Brian O J Tupper\textsuperscript{3}

\textsuperscript{1} Departament de Física, Universitat de les Illes Balears, Cra. Valldemossa km 7.5, E-07122 Palma de Mallorca, Spain
\textsuperscript{2} 87 Carlton Place, Glasgow G5 9TD, Scotland, UK
\textsuperscript{3} Department of Mathematics and Statistics, University of New Brunswick, Fredericton, NB E3B 5A3, Canada

E-mail: jcarot@uib.es, aidan@countingthoughts.com and bt32@rogers.com

Received 25 August 2007, in final form 4 January 2008
Published 12 February 2008
Online at stacks.iop.org/CQG/25/055002

Abstract

Spacetimes which are conformally related to reducible 1+3 spacetimes are considered. We classify these spacetimes according to the conformal algebra of the underlying reducible spacetime, giving in each case canonical expressions for the metric and conformal Killing vectors, and providing physically meaningful examples.

PACS numbers: 02.40.Ky, 04.20.Jb

1. Introduction

In a previous paper [1], conformally reducible 2+2 spacetimes, i.e., spacetimes conformal to reducible (decomposable) 2+2 spacetimes were classified according to their conformal symmetries. In this paper, we extend our study to conformally reducible 1+3 spacetimes. Reducible 1+3 spacetimes have been the subject of previous studies. Coley and Tupper [2] found the general form for reducible 1+3 spacetimes which admit proper conformal Killing vectors. Tsamparlis \textit{et al} [3] related the conformal Killing vectors of reducible 1+3 spacetimes to the conformal Killing vectors of the underlying 3-space and showed that the latter can be obtained from a combination of a gradient conformal Killing vector and a Killing vector or homothetic Killing vector. Capocci and Hall [4] studied the conformal symmetries in terms of the holonomy group classification.

A spacetime \((\mathcal{M}, g)\) is said to be a \textit{conformally reducible 1+3 spacetime} if, for every point \(p \in \mathcal{M}\), there exists a coordinate chart \(\{x^a\}\) such that the line element takes the form

\[ ds^2 = \exp(2\mu(x^a)) \left( d\sigma_0^2 + d\sigma^2 \right), \]

where

\[ d\sigma_0^2 = \epsilon_0 \, d\eta^2, \quad \epsilon_0 = \pm 1 \]

\[ 0264-9381/08/055002+43$30.00 © 2008 IOP Publishing Ltd Printed in the UK \]
with \(d\sigma_0^2\) of signature plus or minus one and \(d\sigma^2\) of signature +1 or +3, respectively. That is, \((M, g)\) is conformally related to a reducible 1+3 spacetime, say \((M, \hat{g})\) whose associated line element shall be written, from now on, as

\[
d\Sigma^2 = d\sigma_0^2 + d\sigma^2
\]

or equivalently, and in the above coordinate system

\[
d\Sigma^2 = \epsilon_0 d\eta^2 + h_{AB}(x^C) dx^A dx^B, \quad \epsilon_0 = \pm 1, \quad A, B, \ldots = 1, 2, 3.
\]

Henceforth, the 3-space with metric

\[
d\sigma^2 = h_{AB}(x^C) dx^A dx^B
\]

will be referred to as \((V, h)\).

Now, the spacetime \((M, \hat{g})\) is (locally) reducible 1+3 if and only if it admits a global, non-null, nowhere vanishing covariantly constant vector field \(\vec{\eta}\), and one can then distinguish between 1+3-spacelike (whenever \(\vec{\eta}\) is timelike, hence \((V, h)\) is spacelike and \(\epsilon_0 = -1\)) or 1+3-timelike (\(\vec{\eta}\) spacelike, \((V, h)\) Lorentz and \(\epsilon_0 = +1\)). A characterization by means of the spacetime holonomy group is also possible; thus in the first case (1+3-spacelike) the holonomy type is \(R_{13}\), whereas in the second case it is \(R_{10}\). If other non-null covariantly constant vector fields exist, the spacetime reduces still further, and the holonomy types are different, but the spacetimes can be described in a similar fashion (see [5] for details).

Before proceeding we recall that (see for example [6]) a vector field \(\vec{X}\) is said to be a conformal Killing vector (CKV) iff

\[
\mathcal{L}_{\vec{X}}g = 2\phi g
\]

where \(\phi\) is some function of the coordinates (conformal scalar), \(g\) is the metric tensor, and \(\mathcal{L}_{\vec{X}}\) stands for the Lie derivative operator with respect to the vector field \(\vec{X}\). The above equation can also be written in an arbitrary coordinate chart as

\[
X_{a;b} = \phi g_{ab} + F_{ab},
\]

where \(X_a\) and \(g_{ab}\) are the covariant components of \(\vec{X}\) and the metric in the chosen chart and \(F_{ab} = -F_{ba}\) is the conformal bivector. When \(\phi \neq 0\) the CKV is said to be proper, and if \(\phi_{ab} = 0\) the CKV is a special CKV (SCKV). When \(\phi\) is a constant, \(\vec{X}\) is a homothetic vector (HV) and \(F_{ab}\) is called the homothetic bivector. When \(\phi = 0\), \(\vec{X}\) is a Killing vector (KV) and \(F_{ab}\) is the Killing bivector. If \(F_{ab} = 0\), i.e., \(X_a = X_a\), the CKV is a gradient CKV (GCKV). Similarly, gradient HV and gradient KV are referred to as GHV and GKV, respectively. The set of CKVs admitted by a spacetime \((M, g)\) form, under the usual Lie bracket operation, a Lie algebra of vector fields which we shall designate as \(\mathfrak{C}_r(M, g)\), \(r\) being its dimension. Also the SCKV, HV and KV admitted by \((M, g)\) form Lie algebras designated as \(\mathfrak{S}_r(M, g)\), \(\mathfrak{H}_r(M, g)\) and \(\mathfrak{G}_r(M, g)\), respectively. Similar notations are used to designate the corresponding Lie algebras for the spacetime \((M, \hat{g})\) and the 3-space \((V, h)\). It follows that in any given spacetime (or 3-space) \(\mathfrak{C}_r \supseteq \mathfrak{S}_m \supseteq \mathfrak{H}_s \supseteq \mathfrak{G}_n\), with \(r \geq m \geq s \geq n\). We refer the reader to [7] for further details on CKV and their Lie algebra.

The invariant characterization of \((M, \hat{g})\) in terms of the existence of a non-null covariantly constant vector field provides an invariant characterization of \((M, g)\) which was given in [8] by the following theorem:

**Theorem 1.** The necessary and sufficient condition for \((M, g)\) to be conformally related to a reducible 1+3 spacetime \((M, \hat{g})\) is that it admits a non-null, nowhere vanishing global conformal Killing vector (CKV) \(\vec{X}\) which is hypersurface orthogonal.
We note that warped spacetimes of class A, as defined in [10] and [8], are particular instances of conformally reducible 1+3 spacetimes.

If \((M, g)\) is a conformally reducible 1+3 spacetime its conformal algebra will be the same as that of the underlying reducible 1+3 spacetime \((M, \hat{g})\) thus providing a classification for conformally reducible 1+3 spacetimes. An investigation of the conformal algebra of \((M, \hat{g})\) was carried out in [8]. We will not repeat the details of that investigation here, but we will present its conformal Lie algebra:

\[
\text{Theorem 2. Let } (M, \hat{g}) \text{ be a reducible 1+3 spacetime; the following results hold regarding its conformal Lie algebra:}
\]

1. If \((V, h)\) admits no GCKV then the only CKV that \((M, \hat{g})\) admits are HV and KV.
2. \((M, \hat{g})\) can admit a proper CKV \(\vec{Y}\) if and only if \((V, h)\) admits a GCKV, which can be either a GKV \(\vec{\xi}\), or a GHV \(\vec{\gamma}\), or a (proper) GCKV \(\vec{\zeta}\). However, a non-special proper CKV exists only if \((M, \hat{g})\) is a conformally flat or conformally reducible 2+2 spacetime (see case 4 below).
3. Two or more GCKV are admitted by \((V, h)\) if and only if \((V, h)\) is of constant curvature (hence conformally flat) and it is flat if one of the GCKV admitted is a GHV. The resulting reducible 1+3 spacetime \((M, \hat{g})\) is conformally flat also. The converse also holds: given a conformally flat reducible 1+3 spacetime, the three-dimensional subspace \((V, h)\) is necessarily of constant curvature.
4. If only one GCKV is admitted by \((V, h)\), then \((M, \hat{g})\) may admit a proper CKV or SCKV which is unique up to the addition of HV and the following situations may arise:
   a. If it is a GKV \(\vec{\xi}\) then, it must be null else \((M, \hat{g})\) degenerates into a 1+1+2 reducible spacetime [8]. Thus \(\vec{\xi}\) is a covariantly constant null KV and \((M, \hat{g})\) is a pp-wave spacetime specialized to a 1+3 spacetime with metric (see section 35.1 of [6])
      \[
      \frac{ds^2}{\eta^2} = 2 du dv - 2H(u, x) du^2 + dx^2
      \]
      and the null KV \(\vec{\xi}\) = \(\vec{\partial}_v\). Note that this metric is of isometry class I(i) in [11] where it is shown that no member of this class can admit a non-special proper CKV. However, such a spacetime can admit a proper SCKV \(\vec{\bar{Y}}\), iff the metric function \(H(u, x)\) and two other functions \(f(u), g(u)\), can be found satisfying the differential equation
      \[
      (\rho u^2 + \alpha u + \beta) H(u, x)_u + \left[\frac{1}{2}(2\rho u + \alpha + k) + f(u)\right] H(u, x)_{\bar{x}} + x f(u)_{uu} + g(u)_{uu} + (2\rho u + \alpha - k) H(u, x) = 0
      \]
      where \(\rho, \alpha, \beta, k\) are constants and \(\rho \neq 0\) for a proper SCKV to exist (see [11] for details). If \(\rho = 0\), a translation along the \(u\)-axis and a rescaling of the \(u, v\) coordinates changes (8) into the form
      \[
      (u^2 + \sigma) H(u, x)_u + \left[x \left(u + \frac{1}{2} k\right) + f(u)\right] H(u, x)_{\bar{x}} + x f(u)_{uu} + g(u)_{uu} + (2 u - k) H(u, x) = 0
      \]
      where \(\sigma\) is an arbitrary constant. The SCKV \(\vec{\bar{Y}}\) is given by
      \[
      \vec{\bar{Y}} = \eta(u + \frac{1}{2} k) \partial_u + (u^2 + \sigma) \partial_u + \left[kv + \frac{1}{2} x^2 + \frac{1}{2} u^2 + x f(u)_{uu} + g(u)\right] \partial_v + \left[(u + \frac{1}{2} k) x + f(u)\right] \partial_x
      \]
      with \(\phi = u + \frac{1}{2} k\). The corresponding metric and SCKV \(\bar{X}\) in \((V, h)\) are
      \[
      d\Sigma^2 = -2 dv dx - 2H(u, x) du^2 + dx^2.
      \]
\begin{equation}
\vec{X} = (u^2 + \sigma)\partial_u + \left[(u + \frac{1}{2}k)x + f(u)\right]\partial_x + \left[kv + \frac{1}{2}x^2 + xf(u),u + g(u)\right]\partial_v.
\end{equation}

The SCKV \( \vec{Y} \) can be written as
\begin{equation}
\vec{Y} = \eta(u + \frac{1}{2}k)\partial_\eta + \frac{1}{2}\eta^2 \vec{\xi} + \vec{X}.
\end{equation}

If in (8), \( \rho = 0 \), then no proper SCKV exist, only HV and, possibly, KV.

(b) If it is a GHV then \((M, \hat{g})\) admits a proper SCKV \( \vec{Y} \) and if it is a GCKV then \((M, \hat{g})\) admits a proper CKV \( \vec{Y} \). In each case \( \vec{Y} \) is unique up to the addition of HV. The metric in each of these cases is given by
\begin{equation}
ds^2 = \epsilon_0 d\eta^2 + \epsilon_1 du^2 + M^2(u)\Omega^2(v, x)[\epsilon_2 dv^2 + dx^2]
\end{equation}
where \( M(u) = u \) in the GHV case while in the GCKV case, depending on the sign of \( \epsilon_\alpha \), \( M(u) \) is given by one of the four functions
\begin{equation}
sin ku, \sinh ku, \cosh ku, e^{ku}(k \neq 0)
\end{equation}
which coincide with the expressions found in [2], apart from the expression \( M(u) = e^{ku} \) which was omitted in [2]. Note that the metric (13) is a conformally reducible 2+2 spacetime and so has been dealt with in [1].

In case (3) in the above theorem, namely; whenever \((M, \hat{g})\) is conformally flat, \( C(M, \hat{g}) \) is 15-dimensional and is the same as that of Minkowski spacetime. This case is also contained in the study carried out in [1] and the reader is referred there, where expressions for the metric and generators of the conformal algebra are given in different coordinate gauges.

Thus the cases of interest in theorem 2 are case (1) in which \((M, \hat{g})\) admits only HV and KV, as is always so for a reducible 2+2 spacetime, and case (4a), the only case in which \((M, \hat{g})\) admits a proper SCKV without degenerating into a conformally flat or conformally reducible 2+2 spacetime. This latter case, i.e., (4a), occurs iff \((V, h)\) admits a null GKV and a SCKV, in which case \((M, \hat{g})\) is a pp-wave spacetime.

In section 2, we briefly deal with those reducible 1+3 spacetimes admitting KVs and HVs with fixed points. In the subsequent sections we will tacitly restrict to the cases in which the KVs or HVs in the reducible 1+3 spacetime have no fixed points (or at least one of the KVs or HVs has no fixed points, so that one could choose coordinates adapted to it). We investigate the possible solutions that can arise in case (1) of theorem 2, i.e., when \((V, h)\) admits KVs or HVs and we also give examples of the pp-wave case (4a). In section 3, we obtain the canonical metrics admitting maximal \( G_r(V, h) \) and the corresponding reducible 1+3 spacetimes. In section 4, we obtain the canonical metrics admitting maximal \( H_r(V, h) \) and the corresponding reducible 1+3 spacetimes. Finally, in section 5 some examples of reducible and conformally reducible 1+3 spacetimes are presented.

2. Fixed points of KV and HV in 1+3 spacetimes

Here we will turn our attention briefly to the cases in which the spacetime admits KVs and/or HVs which do have fixed points. We shall not attempt to give detailed coordinate expressions for the metric and the relevant vector field admitting a fixed point, as this would be tedious and lengthy, in any case this can be done following the same methods as in [1]. Instead we shall focus on the general results that can be mostly gathered from [12].

Given any vector field \( \vec{X} \) on a manifold \( M \), a point \( p \in M \) is said to be a fixed point of \( \vec{X} \) iff \( \vec{X}(p) = 0 \). This is equivalent to saying that the 1-parameter group of (local) diffeomorphisms \( \{\psi_t\} \) that \( \vec{X} \) generates is such that \( \psi_t(p) = p \) for all values of \( t \) where this makes sense.
We shall be interested in the case in which $\vec{X}$ is an affine vector field (AVF), KVs and HVs being then special instances of that, and therefore it satisfies

$$X_{;bc}^{a} = R_{bcd}^{a} X^{d}$$

or equivalently, decomposing $X_{;a}^{b}$ into its symmetric and skew-symmetric parts $h_{ab}$ and $F_{ab}$, respectively,

$$X_{;a}^{b} = h_{ab} + F_{ab}$$

If $h_{ab} = k g_{ab}$ with $k$ constant then $\vec{X}$ is a HV (KV if $k = 0$).

Next we shall consider the case of a 1+3 decomposable spacetime $(M, g)$ admitting an AVF $\vec{X}$ which has a fixed point. We shall distinguish between the cases 1+3 spacelike and 1+3 timelike decomposable, see [12] for details.

2.1. 1+3 spacelike decomposable

Suppose the line element of $(M, g)$ is given in coordinates $x^{a} = \eta, x^{B}, B = 1, 2, 3$ by

$$ds^{2} = -d\eta^{2} + h_{AB}(x^{C}) dx^{A} dx^{B}. \quad (15)$$

Clearly, $\vec{\eta} = \partial/\partial \eta$ is a nowhere vanishing covariantly constant KV such that $\eta_{a} = \eta_{a}$. Suppose that an AVF $\vec{X}$ exists which has fixed points and decompose it as $\vec{X} = \kappa \vec{\eta} + \vec{k}$, where $k_{a} = (g_{ab} + \eta_{a} \eta_{b})X^{b}$ is a vector field tangent everywhere to $(V, h)$. It is then immediate to see that

$$k_{a;b} = \alpha(g_{ab} + \eta_{a} \eta_{b}) + F_{ab}, \quad (16)$$

where $\alpha \in \mathbb{R}$ and $F_{ab}$ is the affine bivector (i.e., $F_{ab} = X_{[a;b]}$), hence $\vec{k}$ is a HV (KV if $\alpha = 0$) of $(V, h)$. Now, $\vec{X}(p) = 0$ implies $\kappa(p)\vec{\eta}(p) = \vec{k}(p) = 0$, and it then follows that the only non-trivial possibility compatible with $\vec{X}$ being either a KV or a HV is that $\vec{X} = \vec{k}$ is a KV (this corresponds to $\alpha = \beta = \gamma = 0$ and $F^{a} = 0$), the set of its fixed points is then a two-dimensional submanifold (since $F^{a} = 0$ and is necessarily simple as $\vec{k}$ is tangent to $(V, h)$) through $p$ containing the integral curve of $\vec{\eta}$ that passes through that point. Other non-trivial possibilities exist but they all correspond to $\vec{X}$ being a proper AVF.

2.2. 1+3 timelike decomposable

Choose coordinates $x^{a} = \eta, x^{B}$ such that

$$ds^{2} = d\eta^{2} + h_{AB}(x^{C}) dx^{A} dx^{B}, \quad (17)$$

that is, $(V, h)$ is now Lorentz. Equation (16) and all of the comments in the previous subsection still hold writing now $\vec{X} = \kappa \vec{\eta} + \vec{k}$. In this case there exist some more possibilities for an
AVF $\vec{X}$ with a fixed point, but only two of them give rise to KV or HV, the rest of them corresponding to $\vec{X}$ being a proper AVF. The ones we are interested in are as follows:

1. $\alpha = \beta = \gamma = 0$ and $\vec{X} = \vec{k}$ is then a KV everywhere tangent to $(V, h)$ whose set of fixed points is a two-dimensional submanifold through $p$ containing the integral curve of $\vec{n}$ that passes through that point.

2. $\alpha \neq 0, \beta = 0$ and $F^a_k(p) \neq 0$ (necessarily timelike). Then $\vec{k} \neq 0$ is a proper HV on $(V, h)$ vanishing at $p$ (which is then necessarily isolated in $(V, h)$). $\vec{X}$ is a HV and has an isolated zero at $p$ in the hypersurface $\eta = -\gamma/\alpha$.

3. Isometry groups on $(V, h)$

We wish to enumerate all relevant isometry groups on three-dimensional manifolds $(V, h)$ of either signature. Isometry groups will be denoted as $G_{r, r}$ being as usual the dimension of the group (and that of the associated algebra). Note that a $G_{r, 1}$ acting on $(V, h)$ will lead to a $G_{r+1, 1}$ on $(M, \tilde{g})$ on account of the existence of the KV $\partial_\eta$. However, in the case of a pp-wave with $(V, h)$ conformally flat, but not of constant curvature, a $G_{r, 1}$ on $(V, h)$ leads to a $G_{r+2, 1}$ on $(M, \tilde{g})$.

We denote by a slash ("/"") the covariant derivative with respect to the 3-metric $h$.

The isometry groups $G_r$ admitted by three-dimensional spaces $(V, h)$ of either signature are as follows. We first consider the case with a $G_1$ acting on null orbits and then a $G_1$ acting on non-null orbits. Developing this, we consider $G_2$ acting on non-null orbits, and $G_2$ acting on null orbits (there are three distinct cases, see section 3.3). When we come to consider the $G_3$ algebras we must distinguish between those which admit a $G_2$ subalgebra and those which do not. Those $G_3$ that do admit a $G_2$ subalgebra can be derived from those metrics admitting an Abelian $G_2$ or non-Abelian $G_2$ structure, with either null or non-null orbits. Finally, every $G_4$ admits a $G_3$ subalgebra and so every 3-space admitting a $G_3$ structure can be derived from an appropriate 3-space metric admitting a $G_3$.

The isometry structures on $(V, h)$ can be deduced from Petrov’s work (chapter 5 of [13]) on four-dimensional manifolds. However, this needs to be refined for the case of reducible 1+3 spacetimes. $G_1$ on two-dimensional null orbits are also examined in detail in Barnes [14]. The $G_1$ acting transitively on $(V, h)$ correspond to the Bianchi types, see [6]. The group actions on three-dimensional null orbits are irrelevant for a reducible 1+3 spacetime. There are no $G_3$ groups with three-dimensional null orbits (Fubini’s theorem, theorem 57.1 of [15], Hall [16]). Of course, if a $(V, h)$ admits a $G_6$ then it is necessarily a space of constant curvature and the corresponding 1+3 spacetime will be conformally flat and need not be considered here. The Lie algebras are shown in table 1.

Any conformally reducible 2+2 spacetimes obtained in this analysis can be discarded since they are treated in [1].

We will treat separately the cases of null orbits and non-null orbits. The results are summarized in table 2

3.1. $(V, h)$ admits a $G_1$ on null orbits

Let $\vec{l}$ be a null KV; i.e., $l^A l_A = 0$ and put $l_{AB} = F_{AB}$ its associated Killing bivector. Choose two other vector fields to complete a null triad, say $\{\vec{l}, \vec{n}, \vec{X}\}$ such that: $l^A l_A = n^A n_A = 0, l^A n_A = -1, n^A X_A = 1$ and the remaining products are zero.

From $(l^A l_A)_{AB} = 0$ it follows $l^A F_{AB} = 0$ and therefore

$$F_{AB} = \alpha (l_A X_B - x_A l_B) \Rightarrow l_{[A;B;C]} = 0,$$

that is, $\vec{l}$ is a null KV which is hypersurface orthogonal and geodesic (the latter follows from the above form for the Killing bivector).
However, if \( \vec{d} \) change, the line element reads (dropping primes for convenience)
\[
P( u, x), w( u, x)
\]
where the metric functions \( P, w, m, H \) and it follows that
\[
f( u, x, m)
\]
and it follows that one may choose coordinates \( v, u, x \) so that
\[
\vec{l} = \partial_v, \quad l_A \, dx^A = -w( u, x) \, dv\]
\[
d\sigma^2 = P^{-2} \, dx^2 + 2 \, dv \, dw + H \, du
\]
where the metric functions \( P, w, m, H \) depend on \( u, x \). (See [6], p 380).

Note that one can still perform the following coordinate changes that preserve the form of \( \vec{l} \) and \( l_A \, dx^A \), namely: \( v = v', u = u' \) and \( x = f( u', x') \); by performing one such coordinate change, the line element reads (dropping primes for convenience)
\[
d\sigma^2 = P^{-2} f_{,u}^2 \, dx^2 + 2 \, f_{,u} (m + f_{,u} P^{-2}) \, dv
\]
\[
+ \left( P^{-2} f_{,u}^2 + 2 m f_{,u} - 2 H \right) \, du^2 - 2 w \, du \, dv
\]
and it follows that \( f \) can be chosen so that \( m + f_{,u} P^{-2} = 0 \). Rewriting \( 2H + m^2 P^2 \) as \( 2H \) the metric becomes
\[
d\sigma^2 = P^{-2} (u, x) \, dx^2 - 2 \, du \left[ w( u, x) \, dv + H( u, x) \, du \right]
\]
and the null KV is given by equation (19).

The 3-space \( (V, h) \) with metric (22) admits only the KV \( \vec{l} = \partial_v \) for general functions \( P( u, x), w( u, x) \) and \( H( u, x) \) and the corresponding spacetime \( (M, g) \) admits a \( G_2 \) only. However, if \( \vec{l} \) is covariantly constant then \( w( u, x) \) is a constant that can be rescaled to unity and \( (M, g) \) is a 1+3 pp-wave spacetime with metric given by equation (7). In this case,
although \((V, h)\) admits only a \(\mathcal{G}_1\), the corresponding \((M, \tilde{g})\) admits a \(\mathcal{G}_3\), the additional KVs being
\[
\tilde{X}_2 = \partial_y, \quad \tilde{X}_3 = \eta \partial_y + u \partial_v.
\]

If, in addition, functions \(f(u), g(u)\) can be found such that \(H(u, x)\) satisfies equation (8) with \(\rho \neq 0\), i.e., equation (9), then \((V, h)\) admits a \(S_1\) with the SCKV given by equation (12) and \((M, \tilde{g})\) admits a \(\mathcal{H}_2\) and \((M, \tilde{g})\) admits a \(\mathcal{H}_4 \supset \mathcal{G}_3\). However, if \(\rho = 0\) then \((V, h)\) admits a \(\mathcal{H}_2\) and \((M, \tilde{g})\) admits a \(\mathcal{H}_4 \supset \mathcal{G}_3\). Thus the condition that \(l\) is covariantly constant implies that the dimension of the conformal algebra of \((M, \tilde{g})\) is two more than that of \((V, h)\), as mentioned earlier.

3.2. \((V, h)\) admits a \(\mathcal{G}_1\) on non-null orbits

Let \(\tilde{X}\) be a non-null KV, i.e.,
\[
X_{A/B} = F_{AB},
\]  
(23)
where \(F_{AB} = -F_{BA}\) is the Killing bivector. There are two possibilities:

(1) \(\tilde{X}\) is not hypersurface orthogonal (h.o.). Assuming one is not at a fixed point, one can choose an adapted coordinate system, say \(u, v, w\) such that \(\tilde{X} = \partial_v\), and the metric reads then of the form
\[
d\sigma^2 = h_{AB}(u, w) \, dx^A \, dx^B.
\]  
(24)
The coordinate change \(v' \mapsto v + f(u, w)\) allows one to set \(h'_{vw} = 0\) without altering the form of the KV \(\tilde{X}\) and a transformation of the form \(u = g(u', w')\), \(w = h(u', w')\) can be used to render the 2-metric in the \(u, w\) plane in an explicitly conformally flat form. Thus, dropping the primes, the metric of \((V, h)\) can be written as
\[
d\sigma^2 = H(u, w)(\epsilon_1 \, du^2 + \epsilon_2 \, dw^2) + 2K(u, w) \, du \, dw + L(u, w) \, dv^2.
\]  
(25)

(2) \(\tilde{X}\) is h.o., i.e. \(\mathcal{X}_A X_B/C = 0\). Then, using the metric (25), we find that
\[
6X_{A/[X_B/C]} = (LK_u - KL_u)\epsilon_{ABC}
\]
with \(\epsilon_{uvw} = +1\). Thus \((LK_u - KL_u) = 0\) which implies that \(K(u, w) = f(w)L(u, w)\) for some function \(f(w)\). Substituting this into metric (25), making the transformation \(v = v' - \int f(w) \, dw\), and dropping the primes, the metric of \((V, h)\) takes the diagonal form
\[
d\sigma^2 = A(u, w) \, du^2 + B(u, w) \, dv^2 + C(u, w) \, dw^2.
\]  
(26)
3.3. \((V, h)\) admits a group \(G_2\) of isometries

A group \(G_2\) can act on two-dimensional spacelike (\(S_2\)), timelike (\(T_2\)) or null (\(N_2\)) orbits, and in each case there exist two inequivalent structures; namely

\[ G_2 I : \quad [\vec{X}, \vec{Y}] = 0 \quad G_2 II : \quad [\vec{X}, \vec{Y}] = k\vec{X}, \]  

(27)

where \(\vec{X}, \vec{Y}\) are two independent KVs generating the group and \(k\) is a constant that can be set equal to unity without loss of generality.

There are four sub-cases (a)–(d) according to the nature of the \(G_2\) orbits. Type (a) has non-null orbits \(S_2\) or \(T_2\), types (b)–(d) have null orbits. Type (b) contains a subgroup \(G_1\) generated by a null KV, type (c) has no null subgroup \(G_1\), with a null vector orthogonal to the orbits of the \(G_2\), and type (d) has no null subgroup \(G_1\), with a null vector in the orbits of the \(G_2\).

Some of the \((V, h)\) metrics found in this section correspond to pp-wave \((M, \tilde{g})\) spacetimes and, in general, such \((V, h)\) admit a \(G_r\) with \(r > 2\). However, they are included here because they arise naturally in our search for \((V, h)\) admitting a \(G_2\).

This section is organized in the following way: the Abelian and non-Abelian cases are dealt with separately, considering each sub-case (a)–(d) as follows:

3.3.1. The Abelian case, \(G_2 I\). In the following three sub-cases (a), (b) and (c) we may choose coordinates such that \(\vec{X} = \partial_v, \vec{Y} = \partial_w\), so that the metric of \((V, h)\) is of the form

\[
\underset{(V, h)}{d\sigma^2} = h_{uv}(u) \, du^2 + 2 h_{uw}(u) \, du \, dv + 2 h_{vw}(u) \, du \, dw + h_{vv}(u) \, dv^2 + 2 h_{vw}(u) \, dv \, dw + h_{ww}(u) \, dw^2. \]  

(28)

(a) \((V, h)\) admits a group \(G_2 I\) acting on non-null orbits \(S_2\) or \(T_2\). We can use a coordinate transformation of the form

\[ v \mapsto v + f(u), \quad w \mapsto w + g(u) \]  

(29)

to eliminate the \(h_{uw}\) and \(h_{vw}\) terms from the metric (28) provided that \(h_{vv} h_{ww} - (h_{vw})^2 \neq 0\). Rescaling the \(u\)-coordinate the metric takes the form

\[
\underset{(V, h)}{d\sigma^2} = \epsilon_1 \, du^2 + A(u) \, dv^2 + 2B(u) \, dv \, dw + C(u) \, dw^2. \]  

(30)
If one of the KVs is h.o. then, by an argument similar to that in section 3.2, the metric of \((V, h)\) can be diagonalized, i.e.,

\[ ds^2 = \epsilon_1 \, du^2 + A(u) \, dv^2 + C(u) \, dw^2. \]  

(31)

Note that, in equation (30), if \(A(u) = 0\) (equivalently \(C(u) = 0\)), then \(\bar{X}_1 = \partial_v\) (equivalently \(\bar{X}_2 = \partial_u\)) is a non-covariantly constant null vector and \(\bar{X}_1, \bar{X}_2\) are KVs (see section 5). If, in addition, \(B(u)\) in equation (30) is a constant then \(\bar{X}_1 = \partial_v\) is covariantly constant and \((V, h)\) is a three-dimensional pp-wave spacetime admitting a \(G_2I\) acting on non-null orbits, and so does not contradict the statement in (b) below. The coordinate change \(u \mapsto x, \, w \mapsto -u\) transforms the metric of \((V, h)\) into

\[ ds^2 = -2H(x) \, du^2 - 2 \, dv + dx^2. \]  

(32)

The corresponding spacetime \((M, \hat{g})\) is an isometry class 8 \((\epsilon = 0)\) pp-wave solution [11] admitting a \(G_4\) with basis

\[ \bar{X}_1 = \partial_v, \quad \bar{X}_2 = \partial_u, \quad \bar{X}_3 = \partial_{\eta}, \quad \bar{X}_4 = \eta \partial_u + u \partial_v. \]  

(33)

In the special case when \(H(x) = x^2\), \(n = 0\) is constant, there is an additional HV given by

\[ \tilde{H} = \frac{1}{2} (2 - n) u \partial_u + \frac{1}{2} (2 + n) \eta \partial_{\eta} + x \partial_x + \eta \partial_{\eta} \]  

(34)

with \(\psi = 1\) and \((M, \hat{g})\) admits a \(\mathcal{H}_5 \supset G_4\). If \(n = -2\) there is a further symmetry, namely a SCKV of the form

\[ \tilde{S} = u^2 \partial_u + \frac{1}{2} (x^2 + \eta^2) \partial_{\eta} + ux \partial_x + u \eta \partial_{\eta} \]  

(35)

so that \((M, \hat{g})\) admits a \(S_5 \supset \mathcal{H}_5 \supset G_4\). If \(n = 2\), \((V, h)\) is the special case (iii) of the following subsection (b) and admits a \(\mathcal{H}_7 \supset G_6\). In this case there exists a group \(G_2I\) on null orbits, but \(\bar{X}_1 = \partial_v\) does not lie on a null orbit.

(b) \((V, h)\) admits a group \(G_2I\) acting on null orbits \(N_2\) containing a subgroup \(G_1\) generated by a null KV. This is clearly a special case of the one discussed in section 3.1, but it is nevertheless an interesting case to be analysed. Let the null KV be \(\bar{I}\) and suppose that \(\bar{X}\) is another KV which generates the group \(G_2I\) along with \(\bar{I}\). It is then easy to show that \(\bar{X}\) must be spacelike and orthogonal to \(\bar{I}\) for if any of these two conditions failed, a second null vector, independent of \(\bar{I}\), would exist at every point on the orbits thus contradicting the hypothesis that they are null.

Now, from \(l^A X_A = 0\) and Killing’s equation for both \(X_A\) and \(l_A\) it follows

\[ [\bar{l}, \bar{X}]^A = -2l^B A_{AB} X^B. \]

Recalling now section 3.1, we may write \(l_{A:B} = \alpha (l_A x_B - x_A l_B)\) where \(x^A x_A = 1\) and \(x^A l_A = 0\). It is now easy to show\(^4\) that \(\bar{x}\) can be chosen parallel to the KV \(\bar{X}\), say \(\bar{X} = \lambda \bar{x}\). The above commutator gives then:

\[ [\bar{l}, \bar{X}]^A = -2 \alpha \lambda l^A \]

and the two inequivalent Lie algebra structures arise then depending on the value of \(-2 \alpha \lambda\). If it is non-zero, then it must be constant and we have a \(G_2II\) structure which we discuss later. If it is zero, which implies \(\alpha = 0\), then \(\bar{l}\) is covariantly constant, in which case \((M, \hat{g})\) is a pp-wave spacetime with metric of the form (7). However, rather than use this metric, it is more convenient to note that the group is Abelian \((G_2I)\) and we can set up coordinates such

\(^4\) e.g., choose a null triad \(l_A, n_A, x_A\) such that \(-l_A n^A = x^A x_A = 1\) and the rest of the products zero. Now \(X_A = \alpha l_A + c x_A\), and a null rotation around \(l_A\) can be used to set \(X_A \propto x_A\).
that \( \vec{l} = \partial_v \) and \( \vec{X} = \partial_x \), then \( L_I l^A = l_B X^B = 0 \) together with the fact that they are KVs, leads immediately to the metric of \((V, h)\) in the form

\[
d\sigma^2 = P^{-2}(u) dx^2 - 2\omega(u) du dv + 2m(u) du dx - 2H(u) du^2. \tag{36}
\]

Rescaling \( u \) so that \( \omega(u) = 1 \) and changing \( x \mapsto x - \int m(u) P^2(u) du \), replacing \( 2H + m(u) P^2(u) \) by \( 2H(u) \) the metric becomes

\[
d\sigma^2 = P^{-2}(u) dx^2 - 2 du dv - 2H(u) du^2, \tag{37}
\]

a form that will be useful later. The KVs \( \vec{l} = \partial_v \) and \( \vec{X} = \partial_x \) are unchanged and the metric admits a third KV, namely \( \vec{Y} = x \partial_v + u \partial_x \) and a HV, \( \vec{Z} = (2v + 2 \int H(u) du) \partial_v + x \partial_x \), and thus admits a \( H_4 \supset G_3 \) on orbits \( N_2 \). In fact, defining new coordinates by

\[
u \mapsto \nu, \quad x \mapsto P(u) x, \quad v \mapsto v - \int H(u) du + \frac{1}{2} P^{-1}(u) P(u) x^2
\]

the metric becomes

\[
d\sigma^2 = -A(u) x^2 du^2 - 2 du dv + dx^2, \tag{38}
\]

where \( A(u) = P(u)^{-1} P(u)_{uu} - 2 P^{-2}(u)(P(u)_{u})^2 \). The corresponding reducible 1+3 spacetime \((M, \hat{g})\), i.e.,

\[
d\sigma^2 = -A(u) x^2 du^2 - 2 du dv + dx^2 + dy^2 \tag{39}
\]

is a special case of an isometry class 10 pp-wave spacetime \([11]\). For arbitrary \( A(u) \), \((M, \hat{g})\) admits a \( H_6 \supset G_5 \). The five KVs are

\[
\vec{X}_1 = \partial_v, \quad \vec{X}_2 = f(u)_{,x} \partial_v + f(u) \partial_x, \quad \vec{X}_3 = g(u)_{,x} \partial_v + g(u) \partial_x, \quad \vec{X}_4 = y \partial_v + u \partial_y, \quad \vec{X}_5 = \partial_y,
\]

where \( f(u), g(u) \) are independent solutions of the differential equation

\[
C(u)_{,uu} + A(u) C(u) = 0,
\]

and the HV is

\[
\vec{H} = 2v \partial_v + x \partial_x + y \partial_y.
\]

For certain special forms of \( A(u) \) there exists an additional symmetry. From table 4 of reference \([11]\) we find that for a 1+3 pp-wave spacetime the only possible choices for \( A(u) \) leading to an additional symmetry are:

(i) \( A(u) = (u^2 + \delta)^{-2} \) which results in a SCKV given in \((V, h)\) by

\[
\vec{S} = (u^2 + \delta) \partial_u + \frac{1}{2} x^2 \partial_x + u x \partial_x
\]

and in \((M, \hat{g})\) by

\[
\vec{S} = (u^2 + \delta) \partial_u + \frac{1}{2} (x^2 + \eta^2) \partial_x + u x \partial_x + u \eta \partial_\eta
\]

so that \((M, \hat{g})\) admits a \( H_7 \supset G_6 \). This is an isometry class 10(ii) spacetime in the classification of \([11]\).

(ii) \( A(u) = \alpha u^{-2}, \alpha = \text{constant} \), which results in a KV given by \( \vec{Y} = u \partial_v - v \partial_x \), so that \((V, h)\) admits a \( H_5 \supset G_4 \) and \((M, \hat{g})\), which is an isometry class 11 spacetime \([11]\), admits a \( H_7 \supset G_6 \).

(iii) \( A(u) = \alpha, \) constant, which results in the KV \( \vec{X} = \partial_u \) so that \((V, h)\) admits a \( H_5 \supset G_4 \). \((M, \hat{g})\) admits a \( H_7 \supset G_6 \) and is an isometry class 13 spacetime \([11]\).
(c) \((V, h)\) admits a group \(G_2\) acting on null orbits \(N_2\) with no null subgroup \(G_1\), with a null vector orthogonal to the orbits of the \(G_2\). In this case the metric (28) satisfies

\[
h_{vw} h_{uw} - (h_{vw})^2 = 0 \tag{41}\]

so that the \((v, w)\) space is degenerate. The coordinate transformations (29) cannot be used to eliminate \(h_{vw}\) and \(h_{uw}\), but can be used to eliminate \(h_{vw}\) and \(h_{uw}\) to obtain

\[
d\sigma^2 = 2 \, du \, dw + (A(u) \, dv + B(u) \, dw)^2, \tag{42}\]

where \(A(u) \neq 0\) for non-degeneracy. We note that in this case there exists a null vector orthogonal to the orbits of the \(G_2\). The null vector, \(\hat{u}\), is orthogonal to the null 2-space spanned by \(v\) and \(w\).

Note that if \(B(u) = 0\) or if \(B(u)\) and \(A(u)\) are proportional to each other, a simple coordinate change transforms the metric into precisely the form (38). Thus \((M, \hat{g})\) is the special isometry class 10 pp-wave spacetime considered in detail in case (b) above. In further consideration of the 3-space with metric (42) we will assume that \(B(u)\) is neither zero nor proportional to \(A(u)\).

(d) \((V, h)\) admits a group \(G_2\) acting on null orbits \(N_2\) with no null subgroup \(G_1\), with a null vector in the orbits of the \(G_2\). If there exists a null vector in the orbits of the \(G_2\) then we must choose coordinates such that \(\hat{X} = \hat{u}\) is one KV and require a second non-null KV \(\hat{Z}\) satisfying \([\hat{X}, \hat{Z}] = 0\). We choose \(\hat{Z} = u \hat{h} + \hat{w}\). Starting with the general metric for \((V, h)\), applying the Killing equations for \(X\) and \(Z\), and making a coordinate transformation of the form \(v \mapsto v + f(u)\) leads to a metric for \((V, h)\) of the form

\[
d\sigma^2 = (A(u)w^2 - 2B(u)w + C(u)) \, dv^2 - 2(A(u)w - B(u)) \, du \, dv + A(u) \, dv^2 + 2D(u) \, du \, dw. \tag{43}\]

The corresponding reducible 1+3 spacetime \((M, \hat{g})\) is equivalent to metric (29.4) in [13].

The results of this section are summarized in table 3.

### Table 3. The reducible 1+3 spacetimes \( (M, \hat{g}) \) with \((V, h)\) metrics admitting an Abelian \(G_2\).

\[ \epsilon_0 = \pm 1 \text{ and } \epsilon_1 = \pm 1 \text{ are not both negative. The type } G_2I(b) \text{ cannot occur, since imposition of this condition leads to a } G_3 \text{ on } (V, h). \]

| Algebra | \((V, h)\) Spacetime metric/\(G_1\) basis | Condition |
|---------|------------------------------------------|-----------|
| \(G_2I(a)\) | \(d\sigma^2 = \epsilon_0 \, du^2 + \epsilon_1 \, dv^2 + A(u) \, du^2 + 2B(u) \, dv \, dw + C(u) \, dw^2\) | \(\epsilon_0 \epsilon_1 (A(u)C(u) - B^2(u))\) |
| \(\hat{X}_0 = \hat{u}, \hat{X} = \hat{h}, \hat{Y} = \hat{w}\) | \(< 0\) |
| \(G_2I(c)\) | \(d\sigma^2 = \epsilon_0 \, du^2 + 2 \, dv \, dw + (A(u) \, dv + B(u) \, dw)^2\) | \(\epsilon_0 A(u) > 0\) |
| \(\hat{X}_0 = \hat{u}, \hat{X} = \hat{u}, \hat{Y} = \hat{v}\) | |
| \(G_2I(d)\) | \(d\sigma^2 = \epsilon_0 \, du^2 + (A(u)w^2 - 2B(u)w + C(u)) \, dv^2 + A(u) \, dv^2 - 2(A(u)w - B(u)) \, du \, dv + 2D(u) \, du \, dw\) | \(\epsilon_0 A(u)D^2(u) > 0\) |
| \(\hat{X}_0 = \hat{u}, \hat{X} = \hat{u}, \hat{Y} = u \hat{h} + \hat{w}\) | |

### 3.3.2. The non-Abelian case, \(G_2II\). We set \(k = 1\) in (27), i.e., the group structure is \([\hat{X}, \hat{Y}] = \hat{X}\). In the following three sub-cases (a), (b) and (c) we choose coordinates \(v, w\) adapted to the orbits such that \(\hat{X} = \hat{v}, \hat{Y} = v \hat{h} + \hat{w}\). It is then easy to see that the metric
takes the form
\[ \mathrm{d}s^2 = h_{uu}(u) \, \mathrm{d}u^2 + 2 h_{uw}(u) \, e^{-w} \, \mathrm{d}u \, \mathrm{d}v + 2 h_{vw}(u) \, \mathrm{d}v \, \mathrm{d}w + h_{ww}(u) \, \mathrm{d}w^2. \] 
(44)

(a) \( G_JI \) acting on non-null orbits \( S_2 \) or \( T_2 \). We can use a coordinate transformation of the form
\[ v \mapsto v + f(u) \, e^w, \quad w \mapsto w + g(u) \] 
(45)
to eliminate the \( h_{uv} \) and \( h_{uw} \) terms from the metric (44) provided that \( h_{uv} h_{ww} - (h_{vw})^2 \neq 0 \). Rescaling the \( u \)-coordinate the metric takes the form
\[ \mathrm{d}s^2 = \epsilon_1 \, \mathrm{d}u^2 + A(u) \, e^{-2w} \, \mathrm{d}v^2 + 2 B(u) \, e^{-w} \, \mathrm{d}v \, \mathrm{d}w + C(u) \, \mathrm{d}w^2. \] 
(46)
If \( \tilde{X} \) is a hypersurface orthogonal KV, then again \( 6X_{[A}X_{B;C]} = (AB_\mu - BA_\mu) \epsilon_{ABC} = 0 \) implies that \( B(u) = aA(u) \) and the coordinate change \( v = v' - a \, e^w \) and \( w = w' \) allows setting \( B(u) = 0 \). If \( \tilde{Y} \) is hypersurface orthogonal, then
\[ 6Y_{[A}Y_{B;C]} = e^{-w}[v^2 \, e^{-2w}(AB_\mu - BA_\mu) + v \, e^{-w}(AC_\mu - CA_\mu)] + (BC_\mu - CB_\mu) \epsilon_{ABC} = 0 \] 
(47)
implies \( B(u) = aA(u) \) and \( C(u) = bA(u) \) for some constants \( a, b \). The above change of coordinates (along with a constant re-scaling of the coordinate \( w \)) produces then the following line element:
\[ \mathrm{d}s^2 = \epsilon_1 \, \mathrm{d}u^2 + A(u) [e^{-2w} \, \mathrm{d}v^2 + \mathrm{d}w^2]. \] 
(48)
Note that this implies that \( (M, \hat{g}) \) is conformally related to a reducible 2+2 spacetime. Thus we consider only the case in which \( \tilde{X} \) is not h.o., i.e., metric (46).

Note that if \( A(u) = 0 \) in equation (46), the KV \( \tilde{X} = \partial_v \) is a non-covariantly constant null vector. If, in addition, \( B(u) \) is a constant, which we can scale to unity, then \( \tilde{X} \) is covariantly constant and \( (V, h) \) is a three-dimensional pp-wave space with metric
\[ \mathrm{d}s^2 = \mathrm{d}u^2 + 2 \, e^{-w} \, \mathrm{d}v \, \mathrm{d}w + C(u) \, \mathrm{d}w^2. \] 
The transformations
\[ u \mapsto x, \quad w \mapsto -\ln|u|, \quad v \mapsto v \] 
change this metric into the form
\[ \mathrm{d}s^2 = \mathrm{d}x^2 - 2 \, \mathrm{d}u \, \mathrm{d}v - 2K(x) u^{-2} \, \mathrm{d}u^2 \] 
(49)
and the corresponding \( (V, h) \) is an isometry class 5 pp-wave spacetime [11]. For an arbitrary \( K(x) \), \( (V, h) \) admits only the two KVs
\[ \tilde{X}_1 = \partial_v, \quad \tilde{X}_2 = u \partial_u - v \partial_v. \]
There are two special cases of \( (V, h) \) that admit an additional symmetry:
(1) If \( K(x) = \alpha \, e^{\epsilon x} + 2k^{-1}x \), where \( \alpha, k \) are constants, there is a third KV, \( \tilde{X}_3 \), given by
\[ \tilde{X}_3 = \partial_v + 2k^{-1}(1 - kx) u^{-2} \partial_u + 2k^{-1} u^{-1} \partial_x \]
and \( (V, h) \) admits a \( G_3 \) \( V \).
(2) If \( K(x) = \alpha \ln|x| \), where \( \alpha \) is a constant, the additional symmetry is a SCKV given by
\[ \tilde{S} = u^2 \partial_u + (\frac{1}{2} x^2 - \alpha \ln|u|) \partial_x + u x \partial_v \]
with \( \psi = u \), so that \( (V, h) \) admits a \( S_3 \supset G_2 \). The corresponding \( (M, \hat{g}) \) admits the SCKV
\[ \tilde{S} = \eta u \partial_u + u^2 \partial_v + \left[ \frac{1}{2} (x^2 + \eta^2) - \alpha \ln|u| \right] \partial_x + u x \partial_v \]
which corresponds to the expression (10) with \( k = 0 \), \( g(u) = -\alpha \ln|u| \), \( f(u) = 0 \).
(b) $G_2 I$ acting on null orbits $N_2$ containing a null subgroup $G_1$. We set up coordinates so that $J = \partial_v$ and $\vec{X} = v \partial_u + \partial_x$, then $l_\lambda l_\lambda = l_\lambda X^\lambda = 0$ together with the fact that they are KV implies
\[\text{d} s^2 = P^{-2}(u) \text{d} x^2 - 2 e^{-x} \text{d} u \text{d} v - 2 H(u) \text{d} u^2. \tag{50}\]

Unlike the corresponding $G_2 I$ case, $(V, h)$ does not admit, in general, any further KV. In fact the 3-space (50) is conformally related to the pp-wave 3-space with metric (37) as can be seen by making the coordinate transformation
\[u \mapsto u, \quad v \mapsto v + \frac{1}{2} P^{-1}(u) P(u)_u x^2, \quad x \mapsto \frac{1}{2} P(u)! x\]
which changes (50) into
\[\text{d} s^2 = 4 P^{-2}(u) x^{-2} [\text{d} x^2 - 2 \text{d} u \text{d} v - A(u) x^2 \text{d} u^2], \tag{51}\]
where $A(u) = -2 P^{-2}(u) (P(u)_u)^2 + P^{-1}(u) P(u)_u + \frac{1}{4} H(u) P^2(u)$. The conformal factor $4 P^{-2}(u) x^{-2}$ leaves $\vec{X}_1 = \partial_u$ as a KV and changes the HV, $\vec{H} = 2 v \partial_u + x \partial_x$ into a KV, but all other KVs of (37) are transformed into proper CKVs of (51), which are not symmetries of the corresponding $(M, \hat{g})$. Note that if $P$ is a constant we can rescale the coordinate $x$ so that the metric takes the form
\[\text{d} s^2 = -2 e^{-2 a x} \text{d} u \text{d} v - 2 H(u) \text{d} u^2 + \text{d} x^2, \tag{52}\]
where $a$ is a constant. In this case $(V, h)$ is a 3-space of constant curvature. Thus the corresponding $(M, \hat{g})$ is conformally flat. The transformation
\[e^{a x} \mapsto a((x')^2 + (y')^2)^{1/2}, \quad y' \mapsto \frac{1}{a} \tan^{-1} \frac{y'}{x'} \quad u \mapsto u' \quad v \mapsto a v'\]
changes the metric of $(M, \hat{g})$ into (dropping the primes)
\[\text{d} s^2 = (x^2 + y^2)^{-1} [-2 \text{d} u \text{d} v - 2 H(u) (x^2 + y^2) \text{d} u^2 + \text{d} x^2 + \text{d} y^2], \tag{53}\]
where the quantity in the square brackets is the standard metric for the conformally flat pp-wave spacetime. However, the spacetime with metric (53) is not a pp-wave spacetime as it does not admit a covariantly constant vector.

(c) $G_2 I$ acting on null orbits $N_2$ with no null subgroup $G_1$, with a null vector orthogonal to the orbits of the $G_2$. In this case the metric (44) satisfies equation (41) so that, again, the $(v, w)$ space is degenerate. We use the coordinate transformations (45) to eliminate $h_{uw}$ and $h_{vu}$, and, rescaling the $u$-coordinate, we obtain
\[\text{d} s^2 = 2 \text{d} u \text{d} w + [A(u) e^{-w} \text{d} v + B(u) \text{d} w]^2. \tag{54}\]

We note that in this case there exists a null vector orthogonal to the orbits of the $G_2$. The null vector, $\partial_u$, is orthogonal to the null 2-space spanned by $v$ and $w$.

(d) $G_2 I$ acting on null orbits $N_2$ with no null subgroup $G_1$, with a null vector in the orbits of the $G_2$. In this case we could choose coordinates so that $\vec{Y}$ is one KV and find another non-null KV, $\vec{Z}$, such that $[\vec{Y}, \vec{Z}] = \vec{Y}$. Instead we choose coordinates such that $\vec{X} = \partial_u$ is the null vector and $\vec{Y} = \partial_u, \vec{Z} = \partial_u + w \partial_w$ are the two non-null KV satisfying $[\vec{Y}, \vec{Z}] = \vec{Y}$. The basic metric is the analog of metric (44) with $v$ and $w$ interchanged. From equation (41)
We consider separately $G$ can be obtained from the appropriate 3-space metric admitting a $G$ down the forms of the line element, (see for instance [6], p 227) group. Furthermore, since the orbits are of constant curvature, we can immediately write or in the transformed form (38).

$\vec{X}$ and the fact that $\vec{X}$ is null, it follows that $h_{vv} = h_{wv} = 0$, and a coordinate transformation $w \mapsto w$, $v \mapsto v + f(u)$, together with a rescaling of the $u$-coordinate leads to the metric

$$d\sigma^2 = 2 \, du \, dv + 2A(u) \, e^{-\nu} \, du \, dw + B(u) \, e^{-2\nu} \, dw^2. \quad (55)$$

The results of this section are summarized in table 4.

### Table 4

| Algebra | $(V, h)$ | Spacetime metric/$G_3$ basis | Condition |
|---------|---------|-----------------------------|-----------|
| $G_{31}(a)$ | (46) | $ds^2 = \epsilon_0 d\eta^2 + \epsilon_1 du^2 + A(u) e^{-2\nu} \, dv^2 + 2B(u) e^{-\nu} \, dw \, dv + C(u) \, dw^2$ | $\epsilon_0 \epsilon_1 (A(u)C(u) - B^2(u)) < 0$ |
| $G_{31}(b)$ | (50) | $ds^2 = \epsilon_0 d\eta^2 + \epsilon_1 du^2 + B(u) e^{-\nu} \, dv \, dw - 2H(u) \, du^2$ | $\epsilon_0 > 0$ |
| $G_{31}(c)$ | (54) | $ds^2 = \epsilon_0 d\eta^2 + 2 \, du \, dv + (A(u) e^{-\nu} \, dv + B(u) \, dw)^2$ | $\epsilon_0 A^2(u) > 0$ |
| $G_{31}(d)$ | (55) | $ds^2 = \epsilon_0 d\eta^2 + 2 \, du \, dv + 2A(u) e^{-\nu} \, du \, dw + B(u) e^{-2\nu} \, dw^2$ | $\epsilon_0 B(u) > 0$ |

and the fact that $\vec{X}$ is null, it follows that $h_{vv} = h_{wv} = 0$, and a coordinate transformation $w \mapsto w$, $v \mapsto v + f(u)$, together with a rescaling of the $u$-coordinate leads to the metric

$$d\sigma^2 = 2 \, du \, dv + 2A(u) \, e^{-\nu} \, du \, dw + B(u) \, e^{-2\nu} \, dw^2. \quad (55)$$

The results of this section are summarized in table 4.

#### 3.4. $(V, h)$ admits a group $G_3$ of isometries acting on two-dimensional orbits

In this case the orbits are of constant curvature and they can be either null or non-null. If they are null, then it can be shown that a null KV exists necessarily, and therefore they are special cases of that discussed in section 3.1, see also [13] and [14]. The line element can be written either in the form (37) with KVs

$$\vec{X} = \partial_u, \quad \vec{Y} = \partial_\epsilon, \quad \vec{Y} = x \partial_u + u \partial_\epsilon$$

or in the transformed form (38).

If the orbits are non-null, then their normal is geodesic and invariant under the isometry group. Furthermore, since the orbits are of constant curvature, we can immediately write down the forms of the line element, (see for instance [6], p 227)

$$d\sigma^2 = \epsilon_1 \, du \, dv + Y(u)[d\chi^2 + \epsilon_2 \Sigma^2(x, k) \, dy^2], \quad (56)$$

where $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1$ and $\Sigma(x, k) = \sin x$, $x$ or sinh $x$ for $k = 1, 0, -1$, respectively. However, the spacetimes $(M, \tilde{g})$ corresponding to the 3-space metric (56) are conformally reducible 2+2 spacetimes and can be discarded.

#### 3.5. $G_3$ of isometries acting transitively on $(V, h)$

We consider separately $G_3 \supset G_2$ and $G_3 \not\supset G_2$. Each 3-space metric admitting a $G_3 \supset G_2$ can be obtained from the appropriate 3-space metric admitting a $G_3$, in which case, using the coordinate system $[u, v, w]$ from section 3.3, we shall take the general form for the third KV to be

$$\vec{X}_3 = X^u(u, v, w) \partial_u + X^v(u, v, w) \partial_v + X^w(u, v, w) \partial_w.$$
There is only one $G_3$ type which does not admit a $G_2$ subalgebra, that being type $G_3 IX$. The following diagram illustrates the analysis of the $G_3$ structures.

The case $G_3 \supset G_2$ (b) need not be considered here because, as was shown in section 3.3.1, imposing the $G_2$ (b) condition automatically admits a $G_3$ on $N_2$. The only $G_3 \supset G_2$ (b) structure which needs to be considered here is the $G_3$ (VIII) $\supset G_2$ (II).

3.5.1. $G_3$ (from $G_2$ case (a)). Within the $G_3 \supset G_2$ only the type $G_3$ VIII has a non-Abelian $G_2$. The 3-space admitting an Abelian $G_2$ is given by equation (30) with basis

\[ \vec{X}_1 = \partial_v, \quad \vec{X}_2 = \partial_w. \]

$G_3 I$. This is flat 3-space.

$G_3 II$. The 3-space is

\[ d\sigma^2 = \epsilon_1 a^2 du^2 + (dv - u dw)^2 + k dw^2, \quad (57) \]

where $k \neq 0$ is a constant. The $G_3$ basis is

\[ \vec{X}_1 = \partial_v, \quad \vec{X}_2 = \partial_w, \quad \vec{X}_3 = \partial_u + w \partial_v. \]

$G_3 III$. The 3-space is

\[ d\sigma^2 = \epsilon_1 a^2 du^2 + \alpha e^{-2u} dv^2 + 2\beta e^{-u} dv dw + \gamma dw^2, \quad (58) \]

where $a \neq 0, \alpha$, $\beta$ and $\gamma$ are constants such that $\alpha \gamma - \beta^2 \neq 0$. If $\alpha = \gamma = 0$ then $(M, \hat{g})$ is conformally flat. The $G_3$ basis is

\[ \vec{X}_1 = \partial_v, \quad \vec{X}_2 = \partial_w, \quad \vec{X}_3 = \partial_u + w \partial_v. \]

$G_3 IV$. The 3-space is

\[ d\sigma^2 = \epsilon_1 a^2 du^2 + e^{-2u} [\alpha dv^2 - 2(\alpha u - \beta) dv dw + (\alpha u^2 - 2\beta u + \gamma) dw^2], \quad (59) \]
where $a \neq 0$, $\alpha$, $\beta$, and $\gamma$ are constants such that $\alpha \gamma - \beta^2 \neq 0$. The $G_3$ basis is
\[
\vec{X}_1 = \partial_{v}, \quad \vec{X}_2 = \partial_{w}, \quad \vec{X}_3 = \partial_{u} + (v + w) \partial_{v} + w \partial_{w}.
\]

$G_3 V$. The 3-space is
\[
d\sigma^2 = \epsilon_1 a^2 \, du^2 + e^{-2u} [\alpha \, dv^2 + 2\beta \, dv \, dw + \gamma \, dw^2].
\] (60)
where $a \neq 0$, $\alpha$, $\beta$, and $\gamma$ are constants such that $\alpha \gamma - \beta^2 \neq 0$. If $\alpha = \gamma = 0$ then $(M, \hat{g})$ is conformally flat. The $G_3$ basis is
\[
\vec{X}_1 = \partial_{v}, \quad \vec{X}_2 = \partial_{w}, \quad \vec{X}_3 = \partial_{u} + v \partial_{v} + w \partial_{w}.
\]

$G_3 VI q$. The 3-space is
\[
d\sigma^2 = \epsilon_1 a^2 \, du^2 + \alpha \, e^{-2u} \, dv^2 + 2\beta \, e^{-(1+q)u} \, dv \, dw + \gamma \, e^{-2qu} \, dw^2,
\] (61)
where $a$ is a non-zero constant and $\alpha$, $\beta$, and $\gamma$ are constants such that $\alpha \gamma - \beta^2 \neq 0$. The $G_3$ basis is
\[
\vec{X}_1 = \partial_{v}, \quad \vec{X}_2 = \partial_{w}, \quad \vec{X}_3 = \partial_{u} + v \partial_{v} + qw \partial_{w}.
\]

$G_3 VII p$. The 3-space is given by
\[
d\sigma^2 = \epsilon_1 a^2 \, du^2 + e^{-pu(D(u) \, dv^2 + 2E(u) \, dv \, dw + F(u) \, dw^2)}
\] (62)
\[
D(u) = \frac{1}{2} \left[ 2 + ((p^2 - 2)c_1 + spc_2) \cos(su) + ((p^2 - 2)c_2 - spc_1) \sin(su) \right]
\] (63)
\[
E(u) = \frac{1}{2} \left[ p + (sc_2 + pc_1) \cos(su) - (sc_1 - pc_2) \sin(su) \right]
\] (64)
\[
F(u) = 1 + c_1 \cos(su) + c_2 \sin(su),
\] (65)
where $a$ is a non-zero constant, $c_1$ and $c_2$ are arbitrary constants and $p$ and $s$ are constants such that $p^2 < 4$ and $s = \sqrt{4 - p^2}$, and the metric is non-degenerate. The $G_3$ basis is
\[
\vec{X}_1 = \partial_{v}, \quad \vec{X}_2 = \partial_{w}, \quad \vec{X}_3 = \partial_{u} - w \partial_{v} + (pw + v) \partial_{w}.
\]

$G_3 VIII$. In this case we use the metric (46) of the 3-space admitting a non-Abelian $G_2$. We find that
\[
c(A(u)C(u),u - 2B(u)) = 0
\] (66)
\[
A(u) f(u) + B(u) g(u) = 0
\] (67)
\[
B(u) f(u) + C(u) g(u) = -\epsilon_1 c,
\] (68)
where $c$ is a constant which is necessarily non-zero else $(M, \hat{g})$ is conformal to a 2+2 reducible spacetime, and the functions $f(u)$ and $g(u)$ are given by
\[
f(u) = -\frac{1}{2} [c B(u) + 2C(u)] A^{-1}(u)
\] (69)
\[
g(u) = \frac{1}{2} [c A(u) + 4B(u)] A^{-1}(u).
\] (70)
Note that \( A(u)C(u) - B^2(u) \neq 0 \), and the non-degeneracy condition requires \( A(u) \neq 0 \) (since \( A(u) = 0 \) implies \( B(u) = 0 \)). Condition (66) integrates to \( A(u)C(u) - B^2(u) = k = \text{constant} \), \( k \neq 0 \) and equations (67) and (68) give

\[
\begin{align*}
A(u)_{,u} &= \epsilon_1 c k^{-1} B(u), \\
g(u)_{,u} &= \epsilon_1 c k^{-1} A(u).
\end{align*}
\]

Differentiating (69) and (70) and equating the expressions for \( f(u)_{,u} \) and \( g(u)_{,u} \) to those above gives

\[
\begin{align*}
A(u)_{,u} &= \pm A(u) \left[ \beta - 4 \epsilon_1 k - 16 \epsilon_1 c^2 A^{-2}(u) - 8 \alpha c^{-2} A^{-1}(u) \right]^{1/2} \\
B(u) &= -c^{-1} A(u) \left[ \int (4k A^{-2}(u) + \alpha A^{-1}(u)) \, du + \beta \right]
\end{align*}
\]

(71) (72)

\[
C(u) = (k + B^2(u)) A^{-1}(u), \quad (73)
\]

where \( \alpha \) and \( \beta \) are arbitrary constants.

The \( G_3 \) basis is

\[
\begin{align*}
\vec{X}_1 &= \partial_v, \\
\vec{X}_2 &= \partial_w, \\
\vec{X}_3 &= c e^u [v^2 + e^u f(u)] \partial_v + [2v + e^u g(u)] \partial_w.
\end{align*}
\]

Petrov presents this metric in an alternative coordinate system where the KV \( \vec{X}_3 \) takes on a simpler form.

The canonical metric forms admitting transitive \( G_3 \)(a) algebras on \((V, h)\) are listed in table 5.

3.5.2. \( G_3 \) (from \( G_2 \) case (b)). There is only one possibility to consider here, \( G_3 VIII \), that is, the only algebra type admitting a \( G_2 II \) subalgebra. As noted earlier the case (b) \( G_2 I \) automatically admits a \( G_3 \) on \( N_2 \), which is considered in section 3.4. However, it is straightforward to show that type \( G_3 VIII \)(b) cannot occur since the imposition of the algebra type on the \( G_2 II \)(b) metric leads to a contradiction.

3.5.3. \( G_3 \) (from \( G_2 \) case (c)). The 3-space with Abelian \( G_2 \) is given by equation (42) with basis

\[
\begin{align*}
\vec{X}_1 &= \partial_v, \\
\vec{X}_2 &= \partial_w.
\end{align*}
\]

Note that the metric function \( A(u) \neq 0 \) for non-degeneracy.

\( G_3 I \). This is flat 3-space and so \((M, \hat{g})\) is Minkowski spacetime.

\( G_3 II \). Imposing the third KV condition leads to \( G_3 I \) automatically (see metric (109)), that is, there is no maximal \( G_3 II \).

\( G_3 III \). The 3-space is

\[
\begin{align*}
d\sigma^2 &= 2 \, du \, dw + (\alpha e^{-w} \, dv + \beta \, dw)^2,
\end{align*}
\]

where \( \alpha \neq 0 \) and \( \beta \) are constants

\[
\begin{align*}
\vec{X}_1 &= \partial_v, \\
\vec{X}_2 &= \partial_w, \\
\vec{X}_3 &= \partial_u + v \partial_v.
\end{align*}
\]

\( G_3 IV \). The 3-space is

\[
\begin{align*}
d\sigma^2 &= 2 \, du \, dw + k^2 u^2 [dv + \ln|u| \, dw]^2,
\end{align*}
\]

(76)
Table 5. Reducible 1+3 spacetimes admitting transitive $G_3$ algebras on $(V, h)$, with orbits of type (a). $e_0 = \pm 1$ and $e_1 = \pm 1$ are not both negative. Those which are conformally reducible 2+2 spacetimes are omitted from the table.

| Algebra | $(V, h)$ | Spacetime metric/$G_2$ basis | Condition |
|---------|----------|-------------------------------|-----------|
| $G_3II(a)$ | $(57)$ $d\sigma^2 = e_0 \,dy^2 + \epsilon_1 \,du^2 + (\alpha - u \,dv)^2 + k \,dw^2$ | $e_0 \epsilon_1 k < 0$ |
| $G_3III(a)$ | $(58)$ $d\sigma^2 = e_0 \,dy^2 + \epsilon_1 \,du^2 + \alpha e^{-2\alpha} \,dv^2 + 2\beta e^{-\beta} \,dv + \gamma \,dw^2$ | $e_0 \epsilon_1 \alpha^2 (\alpha \gamma - \beta^2) < 0$ |
| $G_3IV(a)$ | $(59)$ $d\sigma^2 = e_0 \,dy^2 + \epsilon_1 \,du^2 + e^{-2\beta}(\alpha - u \,dv + (\alpha^2 - 2\beta u + \gamma) \,dv) \,dw$ | $e_0 \epsilon_1 \alpha^2 (\alpha \gamma - \beta^2) < 0$ |
| $G_3V(a)$ | $(60)$ $d\sigma^2 = e_0 \,dy^2 + \epsilon_1 \,du^2 + e^{-2\alpha}(\alpha^2 - 2\beta \,dv + \gamma \,dw)$ | $e_0 \epsilon_1 \alpha^2 (\alpha \gamma - \beta^2) < 0$ |
| $G_3VI(a)$ | $(61)$ $d\sigma^2 = e_0 \,dy^2 + \epsilon_1 \,du^2 + e^{-2\alpha}(\alpha^2 - 2\beta \,dv + \gamma \,dw)$ | $e_0 \epsilon_1 \alpha^2 (\alpha \gamma - \beta^2) < 0$ |
| $G_3VII(a)$ | $(62)$ $d\sigma^2 = e_0 \,dy^2 + \epsilon_1 \,du^2$ | $e_0 \epsilon_1 \alpha^2$ |

where $k \neq 0$ is a constant, and

$$\tilde{X}_1 = \partial_v, \quad \tilde{X}_2 = \partial_u, \quad \tilde{X}_3 = -u \partial_u + (v + w) \partial_v + w \partial_w.$$

$G_3V$. The 3-space is

$$d\sigma^2 = 2 \,du \,dw + u^2 (\beta \,dv + \gamma \,dw)^2,$$

where $\beta \neq 0$ and $\gamma$ are constants, and

$$\tilde{X}_1 = \partial_v, \quad \tilde{X}_2 = \partial_u, \quad \tilde{X}_3 = -u \partial_u + v \partial_v + w \partial_w.$$

This is a flat 3-space and so $(M, \hat{g})$ is Minkowski spacetime.

$G_3VIq$. The 3-space is

$$d\sigma^2 = 2 \,du \,dw + [k|u|^{1/q} \,dv + l|u| \,dw]^2,$$

where $k \neq 0$ and $l$ are constants. The $G_3$ basis is

$$\tilde{X}_1 = \partial_v, \quad \tilde{X}_2 = \partial_u, \quad \tilde{X}_3 = -qu \partial_u + v \partial_v + qw \partial_w.$$
Once a solution for \( a(u) \) is determined then \( A(u), B(u) \) and \( f(u) \) can be found using

\[
[a(u)A(u)]_u = -pA(u), \quad B(u) = -a(u)A(u)_u, \quad f(u)_u = -A^{-2}(u).
\]  

For three-dimensional orbits we require \( a(u) \neq 0 \). Note that \( a(u) = ku \), where \( k \) is a constant, is a solution of equation (81) but no real metric exists because equation (81) and the condition \( p^2 < 4 \) lead to non-real values for \( k \). The general solution of (81) is unknown.

\( \mathcal{G}_3 \text{III.} \) Using the metric (54) of the 3-space with non-Abelian \( \mathcal{G}_3 \) we find that the imposition of this type leads to a contradiction in the Killing equations, and so is not possible.

The canonical metric forms admitting transitive \( \mathcal{G}_3 \) algebras on \((V, h)\) are listed in Table 6.
3.5.4. $G_3$ (from $G_2$ case (d)). The 3-space with Abelian $G_2$ is given by the metric (43) with basis $X_1 = \partial_v, X_2 = u \partial_u + \partial_w$. Note that the determinant of this metric is $-A(u)D^2(u)$ so that $A(u) \neq 0$ and $D(u) \neq 0$ for non-degeneracy.

$G_3I$. This case cannot occur since the third KV is forced to be a linear combination (with constant coefficients) of $\widetilde{X}_1$ and $\widetilde{X}_2$.

$G_3II$. Imposing the $G_3II$ conditions leads to a $G_3III(s = 0)$ solution, that is, no maximal $G_3III$ solution is possible. Details are given in section 3.6.4.

$G_3III$. The conditions on the functions are $A(u) = \alpha u^{-2}, D(u) = \delta u^{-1}$ where $\alpha, \delta$ are non-zero constants, with $B(u), C(u)$ arbitrary, i.e., the metric of $(V, h)$ is

$$
\begin{align*}
\sigma^2 &= [\alpha u^{-2}w^2 - 2B(u)w + C(u)]du^2 - 2[\alpha u^{-2}w - B(u)]du\,dv \\
&\quad + \alpha u^{-2}dv^2 + 2\delta u^{-1}du\,dw.
\end{align*}
$$

The basis for the $G_3$ is

$$
\begin{align*}
\widetilde{X}_1 &= \partial_v, \\
\widetilde{X}_2 &= u \partial_u + \partial_w, \\
\widetilde{X}_3 &= u \partial_u + [v + b(u)]\partial_w + c(u)\partial_v,
\end{align*}
$$

where the functions $b(u), c(u)$ are given by

$$
\begin{align*}
b(u), u &= \frac{1}{2\alpha\delta}[\alpha u^{-2}B(u), u - \delta u^2B(u) + u^4B^2(u) - \alpha u^2C(u)] \\
c(u) &= \frac{1}{2\alpha\delta}[\alpha u^{-2}B(u) - \alpha u^2C(u)].
\end{align*}
$$

Note that $u \neq 0$ for three-dimensional orbits. Note the special case in which $B(u) = C(u) = 0$. Then $c(u) = 0$ and $b(u)$ is constant. The KV $\widetilde{X}_3 = u \partial_u + v\partial_v$ and the metric is

$$
\begin{align*}
\sigma^2 &= \alpha u^{-2}(w\,du - dv)^2 + 2\delta u^{-1}du\,dw.
\end{align*}
$$

$G_3IV$. The metric functions are $A(u) = \alpha e^{2u}, D(u) = \delta e^{u}$ where $\alpha$ and $\delta$ are non-zero constants, with $B(u), C(u)$ arbitrary, i.e., the metric of $(V, h)$ is

$$
\begin{align*}
\sigma^2 &= [\alpha e^{2u}w^2 - 2B(u)w + C(u)]du^2 - 2[\alpha e^{2u}w - B(u)]du\,dv + \alpha e^{2u}dv^2 + 2\delta e^u\,du\,dw.
\end{align*}
$$

The basis for the $G_3$ is

$$
\begin{align*}
\widetilde{X}_1 &= \partial_v, \\
\widetilde{X}_2 &= u \partial_u + \partial_w, \\
\widetilde{X}_3 &= -\partial_u + (v + b(u))\partial_v + (w + c(u))\partial_w,
\end{align*}
$$

where the functions $b(u), c(u)$ are given by

$$
\begin{align*}
b(u), u &= c(u) + \alpha^{-1}e^{-u}[B(u) e^{-u}], u \\
c(u), u &= \frac{1}{2\alpha\delta}e^{-u}[\alpha C(u) - B^2(u) e^{-2u}], u,
\end{align*}
$$

In the special case $B(u) = C(u) = 0$ we have $c(u) = 0$ and $b(u) = cu + \beta$, where $\beta$ is a constant. We can take $\widetilde{X}_3 = -\partial_u + v\partial_v + w\partial_w$ and the metric of $(V, h)$ is

$$
\begin{align*}
\sigma^2 &= \alpha e^{2u}(w\,du - dv)^2 + 2\delta e^u\,du\,dw.
\end{align*}
$$

$G_3V$. This case cannot occur since $D(u) = 0$ which contradicts the non-degeneracy condition.

$G_3V I_q$. The functions are $A(u) = \alpha u^{-2p}, D(u) = \delta u^p$ where $\alpha$ and $\beta$ are non-zero constants, $p = 1/(q - 1)$ with $B(u), C(u)$ arbitrary, i.e., the metric of $(V, h)$ is

$$
\begin{align*}
\sigma^2 &= [\alpha u^{-2p}w^2 - 2B(u)w + C(u)]du^2 - 2[\alpha u^{-2p}w - B(u)]du\,dv \\
&\quad + \alpha u^{-2p}dv^2 + 2\delta u^p\,du\,dw.
\end{align*}
$$
The basis for the $G_3$ is
\[ \vec{X}_1 = \partial_v, \quad \vec{X}_2 = u \partial_v + \partial_w, \]
\[ \vec{X}_3 = (1 - q)u \partial_u + (v + b(u)) \partial_v + (qw + c(u)) \partial_w, \]
where the functions $b(u), c(u)$ are given by
\[ b(u), u = \frac{(q - 1)u B(u) + (q - 2)B(u)}{u}, \]
\[ c(u), u = \frac{(q - 1)^2 \alpha - 1}{2u^2} B(u) - B^2(u)u^2 p(q - 2). \]

In the special case $B(u) = C(u) = 0$ we have $c(u) = c$ is constant and $b(u) = cu + \beta$, where $\beta$ is a constant. Eliminating multiples of $\vec{X}_1$ and $\vec{X}_2$ we have
\[ \vec{X}_3 = (1 - q)u \partial_u + v \partial_v + qw \partial_w, \]
and the metric of $(V, h)$ is
\[ d\sigma^2 = au^{2p}(w du - dv)^2 + 2\delta u^p du dw. \]

$G_3$ VII. The metric functions are
\[ A(u) = \alpha(u^2 - pu + 1)^{-1} \exp[-2pq^{-1} \tan^{-1}((2u - p)/q)], \]
\[ D(u) = \delta(u^2 - pu + 1)^{-1/2} \exp[-pq^{-1} \tan^{-1}((2u - p)/q)], \]
where $\alpha$ and $\delta$ are non-zero constants with $B(u), C(u)$ arbitrary, i.e., the metric of $(V, h)$ is given by (43) with $A(u), D(u)$ as above. The $G_3$ basis is
\[ \vec{X}_1 = \partial_v, \quad \vec{X}_2 = u \partial_v + \partial_w, \]
\[ \vec{X}_3 = a(u) \partial_u + (uv + b(u)) \partial_v + (v + (p - u)w + c(u)) \partial_w, \]
where $X^u = a(u) = (u^2 - pu + 1)$ and the functions $b(u), c(u)$ are given by
\[ a(u)B(u) + [a(u), u]B(u) + (b(u), u - c(u))A(u) + D(u) = 0, \]
\[ a(u)C(u) + 2a(u)C(u) + 2(b(u), u - c(u))B(u) + 2c(u)D(u) = 0. \]

The special case with $B(u) = C(u) = 0$ has $c(u) = c$ is constant and
\[ b(u) = cu - \alpha^{-1} \int (u^2 - pu + 1)^{1/2} \exp[pq^{-1} \tan^{-1}((2u - p)/q)] du. \]

Eliminating multiples of $\vec{X}_1$ and $\vec{X}_2$ we have
\[ \vec{X}_3 = (u^2 - pu + 1) \partial_u + (v + (p - u)w) \partial_w \]
\[ + \left( uv - \alpha^{-1} \int (u^2 - pu + 1)^{1/2} \exp[pq^{-1} \tan^{-1}((2u - p)/q)] du \right) \partial_v. \]

$G_3$ VIII. In this case we use the metric (55) of the 3-space admitting a non-Abelian $G_2$. However, this leads to $B(u) = 0$ which contradicts the non-degeneracy condition, so this case cannot occur.

The canonical metric forms admitting transitive $G_3(d)$ algebras on $(V, h)$ are listed in table 7.
Table 7. Reducible 1+3 spacetimes admitting transitive $G_2$ algebras on $(V, h)$, with orbits of type (d). $e_0 = \pm 1$. Those which are conformally reducible 2+2 spacetimes are omitted from the table.

| Algebra | $(V, h)$ | Spacetime metric/$G_2$ basis | Condition |
|---------|----------|-----------------------------|-----------|
| $G_3 I I I (d)$ | (83) | $dx^2 = e_0 dy^2 + [a u^2 w^2 - 2 B(u) w + C(u)] du^2$ | $e_0 \delta^2 > 0$ |
| | | $- 2 [a u^2 w - B(u)] dv + 2 u a^2 dv^2 + 2 u a b u du$ | $u \neq 0$ |
| | | $\vec{X}_0 = \partial_y$, $\vec{X}_1 = \partial_x$, $\vec{X}_2 = u \partial_x + \partial_u$, $\vec{X}_3 = a u \partial_y + c(u) \partial_u$, $b(u), c(u)$ given by (84) |
| $G_3 I V (d)$ | (86) | $dx^2 = e_0 dy^2 + [a b^2 w^2 - 2 B(u) w + C(u)] du^2$ | $e_0 \delta^2 < 0$ |
| | | $- 2 [a b^2 w - B(u)] dv + 2 u a^2 dv^2 + 2 b^2 u du$ | $u \neq 0$ |
| | | $\vec{X}_0 = \partial_y$, $\vec{X}_1 = \partial_x$, $\vec{X}_2 = u \partial_x + \partial_u$, $\vec{X}_3 = - a u + (v + b(u)) \partial_y + (u + c(u)) \partial_u$, $b(u), c(u)$ given by (87) |
| $G_3 I V I (d)$ | (89) | $dx^2 = e_0 dy^2 + [a u^2 w^2 - 2 B(u) w + C(u)] du^2$ | $e_0 \delta^2 u^2 b^2 > 0$, $\delta \neq 0$ |
| | | $- 2 [a u^2 w - B(u)] dv + 2 u a^2 dv^2 + 2 b^2 u du$ | $q \neq 0, q \neq 1$ |
| | | $\vec{X}_0 = \partial_y$, $\vec{X}_1 = \partial_x$, $\vec{X}_2 = u \partial_x + \partial_u$, $\vec{X}_3 = (1 - q) \partial_y + (v + b(u)) \partial_y + (u + c(u)) \partial_u$, $b(u), c(u)$ given by (90) |
| $G_3 I V I I (d)$ | (43) | $dx^2 = e_0 dy^2 + [A(u) w^2 - 2 B(u) w + C(u)] du^2$ | $e_0 \delta^2 (u^2 - p u + 1)^{-2} > 0$ |
| | | $- 2 [A(u) w - B(u)] dv + 2 A(u) dv^2 + 2 D(u) du$ | $p^2 < 4$ |
| | | $\vec{X}_0 = \partial_y$, $\vec{X}_1 = \partial_x$, $\vec{X}_2 = u \partial_x + \partial_u$, $\vec{X}_3 = (u^2 - p u + 1) \partial_y$, $\partial_y + (u + b(u)) \partial_x + (v + (p - u) w + c(u)) \partial_u$, $A(u), D(u)$ given by (92), $b(u), c(u)$ given by (93) |

3.5.5. $G_3 I X$. The 3-space and KVs are (from Petrov [13])

$$d \sigma^2 = \frac{1}{2} [\alpha \{dx^2 + \sin^2 x \ dy^2\} + \beta \{- \cos(2z) \ dx^2 + \cos(2z) \ sin^2 x \ dx \ dy\}$$
$$+ \gamma \{dz^2 + \cos^2 x \ dy^2 + 2 \ cos x \ dx \ dy\} + \delta \{- \sin(2z) \ dx^2$$
$$+ \sin(2z) \ sin^2 x \ dy^2 + \cos(2z) \ sin x \ dx \ dy\} + \epsilon [2 \ cos z \ sin x \ cos x \ dy^2$$
$$- \sin x \ cos \ dx \ dy - 2 \ sin z \ dx \ dz + 2 \ cos z \ sin x \ dz \ dy\}$$
$$+ \lambda [2 \ sin z \ sin x \ cos x \ dy^2 + 2 \ cos z \ cos x \ dx \ dy + 2 \ cos z \ dx \ dz$$
$$+ 2 \ sin z \ sin x \ dz \ dy\}]. \quad (94)$$

A basis for the $G_3$ is

$$\vec{X}_1 = \partial_x$$
$$\vec{X}_2 = \ cos x \partial_x - \ cot x \ sin x \partial_y + \ sin y \ csc x \partial_z$$
$$\vec{X}_3 = - \ sin y \partial_x - \ cot x \ cos y \partial_y + \ cos y \ csc x \partial_z$$

3.5.6. $G_3$ (constant curvature 2-space admitting $G_2$). The metric given by (26) represents a $(V, h)$ which admits at least one hypersurface orthogonal Killing vector. This metric can be written

$$d \sigma^2 = \epsilon_1 e^{2 \Omega(x^1)} du^2 + \Omega^2(x^1)[\epsilon_2 (dx^1)^2 + \epsilon_3 (dx^2)^2]. \quad (95)$$
where \( k = 1, 2 \) and \( \epsilon_1, \epsilon_2 \) and \( \epsilon_3 \) are either all \(+1\) or at most one of them is \(-1\). Due to this metric’s properties, it is amenable to further investigation. We consider the possible situations that arise when the 2-space is a space of constant curvature. In this case the isometry groups of \((V, h)\) can be determined in a straightforward manner. The three-dimensional metric is

\[
d\sigma^2 = \epsilon \psi_{20(y, z)} \, du^2 + \Omega_2(y, z) \left[ \epsilon_2 \, dy^2 + \epsilon_3 \, dz^2 \right].
\]

Consider first the case \( \epsilon_2 \epsilon_3 = -1 \), i.e., the 2-space is of signature zero. If the curvature of the 2-space is zero the metric of the 2-space can be written \([1]\)

\[
d\tau^2 = 2 \epsilon \psi dv \, dw,
\]

where \( \epsilon = \pm 1 \), and if the 2-space is of non-zero curvature \( \epsilon \lambda^2 \) its metric can be written \([1]\)

\[
d\tau^2 = \epsilon \psi_4 \psi dv \, dw.
\]

When \( \epsilon_2 \epsilon_3 = +1 \), i.e., the 2-space is of signature \(+2\), the zero-curvature 2-space has the form

\[
d\tau^2 = 2 \lambda^2 \psi \, dz \, d\bar{z},
\]

while, if the 2-space is of non-zero curvature \( \epsilon \lambda^2 \) its metric can be written

\[
d\tau^2 = 2 \epsilon \lambda^2 \psi \left( \epsilon_1 \, du^2 + 2 e^{-4F(v, w)} \, dz \, d\bar{z} \right).
\]

The 3-space metrics corresponding to (96)–(99), respectively, can be written in the forms

\[
d\sigma^2 = e^{2F(v, w)} \, du^2 + 2 \epsilon \psi dv \, dw,
\]

\[
d\sigma^2 = \frac{2 e^{2F(v, w)}}{\lambda^2 (v + w)^2} (du^2 + 2 \epsilon e^{-2F(v, w)} \, dv \, dw),
\]

\[
d\sigma^2 = \epsilon \psi_1 e^{2F(z, \bar{z})} \, du^2 + 2 \epsilon e^{-2F(z, \bar{z})} \, dz \, d\bar{z},
\]

\[
d\sigma^2 = \frac{2 e^{2F(z, \bar{z})}}{\lambda^2 (1 + \epsilon z \, \bar{z})^2} \left( \epsilon_1 \, du^2 + 2 e^{-4F(z, \bar{z})} \, dz \, d\bar{z} \right).
\]

where \( z, \bar{z} \) are conjugate complex coordinates and \( \epsilon_1 = \pm 1 \). If the three KVs of the 2-spaces (96)–(99) are labelled \( Z_1, Z_2 \) and \( Z_3 \) then, from Killing’s equations the KV of the 3-spaces (100)–(103) are of the form

\[
\vec{Y} = Y^a \partial_a + a(u)Z_1 + b(u)Z_2 + c(u)Z_3,
\]

where \( Y^a = Y^a(u, v, w) \) or \( Y^a = Y^a(u, z, \bar{z}) \), as appropriate, \( Z_i = Z_i(v, w) \) or \( Z_i = Z_i(z, \bar{z}) \).

Substituting the expression (104) into Killing’s equations, we find that the function \( F \) satisfies the appropriate Liouville equation and this leads to 3-space solutions admitting a number of KV. Some of these solutions admit only two KV and are special cases of the metrics discussed in section 3.3. Some admit four KV, but all of these are conformally related to reducible 2+2 spacetimes which will not be considered further. However, several solutions admitting three KV arise and we will now present one of these solutions for each Bianchi type that arises.

The first example has a metric of the form

\[
d\sigma^2 = H(v, w) \, du^2 - \frac{4}{\lambda^2 (v + w)^2} \, dv \, dw.
\]

\( G_{3VIIq=0} \)

\[
H = \frac{k^2 (v^2 + 1)(w^2 + 1)}{\lambda^2 (v + w)^2} G^2(v, w), \quad G = \tan^{-1} \left( \frac{w - v}{1 + vw} \right).
\]
The basis vectors for the Lie algebra are
\[ \vec{X}_1 = \partial u, \]
\[ \vec{X}_2 = 2k^{-1}G^{-1} \sin ku \partial_u + (v^2 + 1) \cos ku \partial_v - (w^2 + 1) \cos ku \partial_w, \]
\[ \vec{X}_3 = -2k^{-1}G^{-1} \cos ku \partial_u + (v^2 + 1) \sin ku \partial_v - (w^2 + 1) \sin ku \partial_w. \]

The following two examples have a metric of the form:
\[ ds^2 = H(v, w) \, du^2 + \frac{4}{\lambda^2(v+w)^2} \, dv \, dw. \]

3G3VIII
\[ H = \frac{k^2(v^2 + 1)(w^2 + 1)}{\mu^2 \lambda^2(v+w)^2} \cosh^2 G(v, w), \quad G = \mu \tan^{-1} \left( \frac{w-v}{1+vw} \right). \]

The basis vectors for the Lie algebra are
\[ \vec{X}_1 = \partial u, \]
\[ \vec{X}_2 = 2k^{-1} \mu \tanh G \sin ku \partial_u - (v^2 + 1) \cos ku \partial_v + (w^2 + 1) \cos ku \partial_w, \]
\[ \vec{X}_3 = -2k^{-1} \mu \tanh G \cos ku \partial_u - (v^2 + 1) \sin ku \partial_v + (w^2 + 1) \sin ku \partial_w. \]

3G3VII
\[ ds^2 = P^{-1}(v) \, du^2 + 2\epsilon \, dv \, dw. \]

The basis vectors for the Lie algebra are
\[ \vec{X}_1 = \partial_u, \quad \vec{X}_2 = -\epsilon P(v) \partial_u + u \partial_w, \quad \vec{X}_3 = \partial_w. \]

3.6. \( G_4 \) of isometries acting multiply transitively on \((V, h)\)

Each \( G_4 \) algebra contains a \( G_3 \) subalgebra. Thus each of the 3-spaces admitting a \( G_4 \) are specializations of 3-spaces admitting \( G_3 \).

However, two cases require special mention.

As discussed in sections 3.3.1 and 3.5.2, the \( G_3 \supset G_2 \) (b) algebras have two-dimensional null orbits and there are no metrics admitting \( G_3 \supset G_2 \) (b) algebras. Therefore, it is only necessary to consider specializations possessing \( G_2 \) (b) subalgebras. These are pp-wave spacetimes.

Metrics admitting \( G_4 \) algebras with orbit types (a), (c) and (d) are derived from the appropriate \( G_3 \) metric for types I–VII. However, Petrov’s type IX metric (94) does not possess a \( G_2 \) subalgebra and so cannot be used to classify the \( G_4 \) metrics according to the
$G_2$ orbit types. Therefore, since the $G_2VIII$ algebra contains two-dimensional Abelian subalgebras, we treat the metrics as specializations of the appropriate $G_2$ metrics.

The following diagram illustrates the analysis of the $G_4$ structures.

![Diagram illustrating the analysis of the $G_4$ structures.](image)

3.6.1. $G_4$ (from $G_3 \supset G_2(a)$). In this section, we derive all $(V, h)$ admitting a $G_4$ with subalgebra $G_3$ possessing a type (a) $G_2$ subalgebra. We find that the cases $G_3I$, $G_3II$, $G_3IV$, $G_3V$ and $G_3VI$ lead either to conformally reducible 2+2 spacetime, which may be flat, or have no solution because the Killing equations for the fourth KV lead to a contradiction. The only cases of interest are $G_3III(p)$ and $G_3VII$.

$G_3III(p)$. This case is a specialization of the 3-space of type $G_3II$ with metric (57). The condition for the existence of the fourth KV is $p = 0$ and $\epsilon_1 = k = +1$. Thus only type $G_3III(p = 0)$ is possible. The metric of the 3-space is

$$ds^2 = du^2 + dw^2 + (dv - u \, du)^2$$

and the metric of $(M, \tilde{g})$ is

$$ds^2 = -d\eta^2 + du^2 + dw^2 + (dv - u \, du)^2.$$ 

The $G_3III(p = 0)$ basis is

$$\tilde{X}_1 = \partial_v, \quad \tilde{X}_2 = \partial_u, \quad \tilde{X}_3 = \partial_u + w \partial_v, \quad \tilde{X}_4 = w \partial_u + \frac{1}{2}(w^2 - u^2) \partial_v - u \partial_w.$$ 

$G_3VII$. This case is a specialization of the 3-space of type $G_3VIII$ which has the metric (46), i.e.,

$$ds^2 = \epsilon_1 \, du^2 + e^{-2w}A(u) \, dv^2 + 2B(u) \, e^{-w} \, dv \, dw + C(u) \, dw^2$$

governed by equations (66)–(70).

Imposing the conditions $[X_1, X_4] = [X_2, X_4] = 0$ and the KV equations give a fourth KV of the form

$$\tilde{X}_4 = h \partial_u + k \, e^w \partial_v + l \partial_w$$

where $h$, $k$ and $l$ are constants. If $h = 0$ we find that either $k = l = 0$, i.e., $\tilde{X}_4 = 0$, or $AC - B^2 = 0$ which implies that the metric is degenerate. Hence $h \neq 0$ and we shall put $h = 1$. The final condition $[X_1, X_4] = 0$ gives

$$lc = 0, \quad f(u)_{,u} + 2lf(u) - kg(u) = 0, \quad g(u)_{,u} + lg(u) + 2k = 0.$$
Class. Quantum Grav. 25 (2008) 055002

| Algebra | (V, h) | Spacetime metric/\(\mathcal{G}_d\) basis | Condition |
|---------|--------|----------------------------------------|-----------|
| \(\mathcal{G}_d\) | | | |
| | (105) | \(dx^2 = -d\sigma^2 + du^2 + dw^2 + (dv - u dw)^2\) | \(2\epsilon_0 \epsilon_1 < 0\) |
| \(\mathcal{G}_d\) | | | |
| | (107) | \(dx^2 = d\sigma^2 + (\epsilon_1 + A_0 k^2)u^2 dw^2 + A_0 k^2 a^2 e^{w} dw^2 + A_0 k^2 a^2 e^{w} du dv + \frac{4}{3} A_0 k^2 a^2 e^{w} du^2 + m dw^2\) | \(k^2 > 0\) |
| \(\mathcal{G}_d\) | | | |
| | (110) | \(dx^2 = d\sigma^2 + \{k w^2 - 2(B(u)w + C(u)) du^2 - 2(k w - B(u)) du dv + k du^2 + 2k dw^2\) | \(\epsilon_1 = 0\) |

From the first condition either \(l = 0\) or \(c = 0\). If \(c = 0\) then the corresponding spacetime \((M, \bar{g})\) is a conformally reducible 2+2 spacetime, which we discard. Hence we have \(l = 0\) and the KV equations give

\[
A(u) = A_0, \quad B(u) = -k A_0 u + B_0, \quad C(u) = k^2 A_0 u^2 - 2k B_0 u + C_0
\]

such that

\[
2k(A_0 C_0 - B_0^2) = \epsilon_1 c A_0. \quad (106)
\]

We note that \(AC - B^2 = A_0 C_0 - B_0^2\).

Thus the 3-space is

\[
d\sigma^2 = \epsilon_1 du^2 + A_0 e^{-2w} du^2 + 2(-k A_0 u + B_0) e^{-w} dw^2 + (k^2 A_0 u^2 - 2k B_0 u + C_0) dv^2
\]

such that (106) holds. The coordinate transformations \(u = au' + k^{-1} A_0^{-1} B_0, v = ka(u' e^{-w} + v'/2), w = -w'\), where \(a\) is a constant, transform \(d\sigma^2\) into the form (dropping primes)

\[
d\sigma^2 = (\epsilon_1 + A_0 k^2) a^2 du^2 + A_0 k^2 a^2 e^{w} du dv + \frac{4}{3} A_0 k^2 a^2 e^{w} dw^2 + m dw^2, \quad (107)
\]

where \(m = C_0 - B_0^2 A_0^{-1}\) is a non-zero constant. The \(\mathcal{G}_d\) basis is

\[
\vec{X}_1 = \bar{u} \delta_u, \quad \vec{X}_2 = v \delta_v - \delta_w, \quad \vec{X}_4 = \delta_w.
\]

\[
\vec{X}_3 = 2a^{-1} \epsilon_1 k m e^{-w} \delta_u + \left(\frac{4}{3} k a u^2 - 2k^{-1} A_0^{-1} m e^{-2w}\right) \delta_v - k a v \delta_w. \quad (108)
\]

The canonical metric forms admitting transitive \(\mathcal{G}_d\) algebras on \((V, h)\) are listed in table 8.
3.6.2. $G_4$ (from $G_3 \supset G_2(b)$) As shown in section 3.3.1, $(V, h)$ of the form (37) will admit a $G_4$ provided that either $A(u) = au^{-2}$ or $A(u) = \alpha$, where $\alpha$ is a constant. In the first case $(V, h)$ admits a $G_4I$ if $\alpha < 0$ or if $\alpha > 4$, a $G_4II$ if $\alpha = 4$, or a $G_4III$ if $0 < \alpha < 4$. In the second case $(V, h)$ admits a $G_4I$ or a $G_4III$ depending on whether $\alpha < 0$ or $\alpha > 0$, respectively.

3.6.3. $G_4$ (from $G_3 \supset G_2(c)$). In this section, we derive all $(V, h)$ admitting a $G_4$ with subalgebra $G_3$ possessing a type (c) $G_2$ subalgebra.

$G_4I_s$. The four-dimensional type I algebra is a specialization of the three-dimensional type II algebra (see table 1). As mentioned in section 3.5.3 there is no maximal $G_3II$. In fact, imposition of the $G_3II$ algebra leads immediately to a $G_4I_0$. The 3-space is

$$\text{d}\sigma^2 = 2\text{d}u \text{d}w + k^2(\text{d}v + u \text{d}w)^2,$$

(109)

where $k$ is a non-zero constant and

$$\vec{X}_1 = \partial_v, \quad \vec{X}_2 = \partial_u, \quad \vec{X}_3 = -\partial_u + w\partial_v, \quad \vec{X}_4 = -u\partial_u + w\partial_w.$$

No further $G_4$ 3-spaces of this class exist because they lead directly to the $G_4I_0$ above, or because the corresponding $G_3$ space does not exist or is flat, or the Killing equations lead to a contradiction.

3.6.4. $G_4$ (from $G_3 \supset G_2(d)$). In this section, we derive all $(V, h)$ admitting a $G_4$ with subalgebra $G_3$ possessing a type (d) $G_2$ subalgebra. As in section 3.6.3 we find that only a $G_4I_0$ 3-space exists; all other $G_4$ are excluded for the same reasons as above.

$G_4I_s$. The four-dimensional algebra type I is a specialization of the three-dimensional type II algebra. However, imposing the $G_3III$ conditions leads to a $G_4I(s = 0)$ solution, that is, no maximal $G_3III$ solution is possible. The three-dimensional type II algebra is a specialization of the two-dimensional type I algebra. The type $G_2I(d)$ 3-space metric is given by (43). The $G_3III$ conditions lead to the metric functions $A(u) = k, D(u) = h$, where $h, k$ are non-zero constants, with $B(u), C(u)$ arbitrary, i.e., the metric of $(V, h)$ is

$$\text{d}\sigma^2 = [kw^2 - 2B(u)w + C(u)]\text{d}u^2 - 2[kw - B(u)]\text{d}u \text{d}v + k \text{d}v^2 + 2h \text{d}u \text{d}w,$$

(110)

and the basis for the $G_3III$ is

$$\vec{X}_1 = \partial_v, \quad \vec{X}_2 = u\partial_v + \partial_u, \quad \vec{X}_3 = -\partial_u + f(u)\partial_v + g(u)\partial_w,$$

(111)

with $f(u), g(u)$ given by

$$f(u) = k^{-1}B(u) + \int g(u) \text{d}u,$$

(112)

$$g(u) = \frac{1}{2hk}(kC(u) - B^2(u)).$$

(113)

where we have eliminated multiples of $\vec{X}_1$ and $\vec{X}_2$. With no further restrictions on $B(u)$ and $C(u), (V, h)$ also admits a fourth KV given by

$$\vec{X}_4 = -u\partial_u + uf(u)\partial_v + [w + ug(u) + \int g(u) \text{d}u]\partial_w.$$

(114)

No $G_4I(s \neq 0)$ solution is possible. The special case with $B(u) = C(u) = 0$ has $f(u) = g(u) = 0$. The metric of $(V, h)$ is

$$\text{d}\sigma^2 = k(w \text{d}u - \text{d}v)^2 + 2h \text{d}u \text{d}w$$

and the basis is

$$\vec{X}_1 = \partial_v, \quad \vec{X}_2 = u\partial_v + \partial_u, \quad \vec{X}_3 = -\partial_u, \quad \vec{X}_4 = -u\partial_u + w\partial_w.$$
3.6.5. $G_4VIII$. As discussed at the beginning of this section, metrics with $G_4VIII$ algebras with orbit types (a), (c) and (d) will be considered as specializations of the $G_2I$ metrics. It can be shown that only type (a) exists, with metric

$$d\sigma^2 = du^2 + (k^{-1}\rho\sin(ku)\,dv + dw)^2 + k^{-2}\cos^2(ku)\,dv^2.$$  \hspace{1cm} (115)

The $G_4$ basis is

$$\tilde{X}_1 = \partial_v, \hspace{1cm} \tilde{X}_4 = \partial_w,$$

$$\tilde{X}_2 = k^{-1}\sin v\partial_u - \tan(ku)\cos v\partial_v + \rho k^{-1}\sec(ku)\cos v\partial_w,$$

$$\tilde{X}_3 = k^{-1}\cos v\partial_u + \tan(ku)\sin v\partial_v - \rho k^{-1}\sec(ku)\sin v\partial_w.$$  

The corresponding $(M, \hat{g})$ is

$$ds^2 = -d\eta^2 + du^2 + k^{-2}(\rho^2\sin^2(ku) + \cos^2(ku))\,dv^2 + 2\rho k^{-1}\sin(ku)\,dv\,dw + dw^2.$$  

4. Homothety groups on $(V, h)$

We wish to enumerate all relevant homothety algebras on three-dimensional manifolds $(V, h)$. Homothetic algebras will be denoted $H_r$, where $r$ is the dimension the algebra. Note that an $H_r$ on $(V, h)$ will lead to a $H_{r+1}$ on $(M, \hat{g})$ on account of the existence of $KV\partial_u$.

As it is well known, if a proper HV exists it is unique in the sense that any other proper HV say $\tilde{X}$ will be a linear combination of $X$ and KVs. Further, the Lie bracket of a HV and a KV is always a KV (it can also be zero, as zero is a KV). From now on, we will always refer to a proper HV simply as a HV, unless confusion may arise.

The above facts imply that an $r$-dimensional Lie algebra of homotheties, say $H_r$, always contains an $(r-1)$-dimensional Lie algebra of isometries $G_{r-1}$.

In $(V, h)$, the maximal Killing algebra is six-dimensional and the metric $h$ is then of constant curvature, i.e., the Ricci scalar, $R$, is a constant. If a HV $X$ exists and if $\psi \neq 0$ is the homothetic constant, then $L_X R = -2\psi R = 0$ which implies that $R = 0$ and $(V, h)$ is locally flat.

Assuming there are no proper homothetic fixed points, the case in which $H_5$ acts on $(V, h)$ is forbidden, see for example [17].

We first consider separately the cases with an $H_1$ with null orbits and with non-null orbits. The $H_1$ metrics are obtained in a similar way to the $G_2$ metrics. An $H_1$ acting multiply transitively on a 2-space is impossible [17]. To determine the relevant homothety algebras $H_1$ acting on $(V, h)$ we take the $G_2$ metrics $(V, h)$ and demand that in addition each case admits an HV. Note that the $H_1$ metrics were not taken as the starting point since an $H_1$ does not necessarily admit a $H_2$ subalgebra. There is only one $H_4$ algebra relevant to this work, as shall be shown presently. Hall and Steele [17] have shown that an $H_4$ acting transitively on the $(V, h)$ must possess a $G_3$ subalgebra with two-dimensional orbits. However, such a $G_3$ acting on non-null two-dimensional orbits leads to a $(M, \hat{g})$ which is a conformally reducible 2+2 spacetime, and need not be considered any further. The metric (37) represents a $(V, h)$ with $G_3$ acting on null two-dimensional orbits and, in fact, the corresponding $(M, \hat{g})$ is given by metric (39) and admits an $H_6 \supset G_5$.

Any conformally reducible 2+2 spacetimes obtained in this analysis can be discarded since they are treated in [1].

4.1. $(V, h)$ admits a group $H_1$ of homotheties

We will treat separately the cases of null orbits and non-null orbits.
4.1.1. \( H_1 \) on null orbits. Let \( \vec{l} \) be a null HV and choose a null triad, say \( \{\vec{l}, \vec{n}, \vec{x}\} \) such that:

\[
l^A l_A = n^A n_A = 0, l^A n_A = -1, x^A x_A = 1 \quad \text{and the remaining products are zero.}
\]

From

\[
l_{A/B} = \psi g_{A/B} + F_{A/B}
\]

it follows upon contraction with \( l^A \) that \( F_{A/B} l^B = \psi l_A \), hence

\[
l_{A/B} = -2\psi l_A n_B + \psi x_A x_B
\]  

(116) and one then has for \( \vec{n} \) and \( \vec{x} \) in the triad

\[
n_{A/B} = 2\psi n_A n_B + \mu x_A n_B + \nu x_A x_B
\]  

(117) \( x_{A/B} = \mu l_A l_B + \nu l_A n_B + \rho l_A x_B + \psi n_A x_B \)

(118) and then the various Lie brackets can be evaluated to get

\[
[\vec{l}, \vec{n}] = -2\psi \vec{n}, \quad [\vec{l}, \vec{x}] = -\psi \vec{x} - \vec{v}, \quad [\vec{n}, \vec{x}] = -2\mu \vec{l} - \rho \vec{x}.
\]  

(119)

Now, a null rotation can be used in order to simplify the above expressions, thus putting

\[
\vec{l}' = \vec{l}, \quad \vec{n}' = \frac{1}{2} \vec{p}^2 \vec{l} + \vec{n} + \vec{p} \vec{x}, \quad \vec{x}' = \vec{p} \vec{l} + \vec{x}
\]  

(120) and dropping primes for convenience, one gets after some straightforward calculations

\[
[\vec{l}, \vec{n}] = -2\psi \vec{n}, \quad [\vec{l}, \vec{x}] = -\psi \vec{x} - \vec{v}, \quad [\vec{n}, \vec{x}] = \mu \vec{l} + \beta \vec{n} + \gamma \vec{x}.
\]  

(121)

Note that \( \vec{l} \) and \( \vec{n} \) are surface forming, and so are \( \vec{l} \) and \( \vec{x} \), thus, we can choose coordinates, say \( x^A = v, u, x \) such that

\[
\vec{l} = \partial_v, \quad \vec{n} = A(u, v, x) \partial_v + B(u, v, x) \partial_u, \quad \vec{x} = C(u, v, x) \partial_v + D(u, v, x) \partial_x
\]  

(122) and from (121) we get

\[
\vec{n} = \exp(-2\psi v) (A_0(u, x) \partial_v + B_0(u, x) \partial_u), \quad \vec{x} = \exp(-\psi v) (C_0(u, x) \partial_v + D_0(u, x) \partial_x).
\]

Further, from the orthogonality relations \( l^A l_A = n^A n_A = 0 \), etc. it follows

\[
h_{A/B} = \exp(2\psi v) \left[ \begin{array}{cc} 0 & -B_0^{-1} \\ -B_0^{-1} & 2A_0B_0^{-2} & C_0B_0^{-1}D_0^{-1} \\ 0 & C_0B_0^{-1}D_0^{-1} & D_0^{-2} \end{array} \right]
\]  

(123) and a coordinate transformation \( v \rightarrow v, u \rightarrow u \) and \( x \rightarrow x(u, x) \) exists such that \( h_{ux} = 0 \) in the new coordinates, that is,

\[
h_{A/B} = \exp(2\psi v) K(u, x) \left[ \begin{array}{ccc} 0 & -1 & 0 \\ -1 & M(u, x) & 0 \\ 0 & 0 & N(u, x) \end{array} \right] \quad \vec{l} = \partial_v.
\]  

(124)

4.1.2. \( H_1 \) on non-null orbits. Let now \( \vec{X} \) be a non-null proper HV, that is,

\[
X_{A/B} = \psi h_{A/B} + F_{A/B}, \quad F_{A/B} = -F_{B/A} \quad \text{homothetic bivector.}
\]  

(125) There are now two possibilities:

1. \( \vec{X} \) is h.o.; i.e., \( X_{[A} X_{B/C]} = 0 \) or equivalently \( F_{A/B} = X_{[A} V_{B]} \) for some covector \( V_B \), that is, \( \vec{X} \) is contained in the blade of its own bivector (note this includes the case of \( \vec{X} \) being a gradient, i.e., \( F_{A/B} = 0 \). Assuming it has no fixed points, we can set up a coordinate
Table 9. Canonical metric forms admitting $H_1$ on $(V, h)$. The term hypersurface orthogonal is abbreviated to h.o. $\epsilon_0 = \pm 1$ and $\epsilon_1 = \pm 1$ are not both negative.

| Algebra | $(V, h)$ | Spacetime metric/4-h basis | Condition |
|---------|----------|-----------------------------|-----------|
| $H_1$ on $N_1$ | (124) | $\text{d}x^2 = \epsilon_0 \text{d}y^2 + \exp(2\psi(x))M(u, x) \text{d}v + 2N(u, x) \text{d}x^2$ | $\epsilon_0 K(u, x)N(u, x) > 0$ |
| $\tilde{X}$ | $\tilde{X}_0 = \partial_y, \tilde{l} = \partial_v$ |
| $H_1$ on $S_1$ or $T_1$ | (126) | $\text{d}x^2 = \epsilon_0 \text{d}y^2 + \exp(2\psi(u)\epsilon_1 \text{exp} V(x)) \text{d}u^2$ | $\epsilon_0 \epsilon_1 \det \tilde{h} < 0$ |
| $\tilde{X}$ is h.o. | $\tilde{X}_0 = \partial_u, \tilde{X} = \partial_x$ |
| $H_1$ on $S_1$ or $T_1$ | (128) | $\text{d}x^2 = \epsilon_0 \text{d}y^2 + \exp(2\psi(u)\epsilon_1 A^2(v, w) \text{d}u^2 + 2B(v, w) \text{d}v \text{d}w + C^2(v, w)(\epsilon_2 \text{d}v^2 + \text{d}w^2))$ | $\epsilon_0 C^2(v, w) < 0$ |
| $\tilde{X}$ is not h.o. | $\tilde{X}_0 = \partial_u, \tilde{X} = \partial_x$ |

The system $x^A = u, x^a$ adapted to $\tilde{X}$ so that $\tilde{X} = \partial_u$ and the metric in these coordinates is such that $h_{u\alpha} = 0$, imposing next that $\tilde{X}$ is a HV we easily get

$$\text{d}\sigma^2 = \exp(2k\psi)\epsilon \exp V(x) \text{d}u^2 + \tilde{h}_{\alpha\beta}(x) \text{d}x^\alpha \text{d}x^\beta,$$

where $\tilde{h}_{\alpha\beta}(x)$ is a 2-metric and can therefore be diagonalized.

(2) $\tilde{X}$ is not h.o. Assuming one is not at a fixed point, one can choose an adapted coordinate system, say $u, v, w$ such that $\tilde{X} = \partial_u$, and the metric reads then

$$\text{d}\sigma^2 = \exp(2\psi\epsilon)\tilde{h}_{AB}(v, w) \text{d}x^A \text{d}x^B.$$ (127)

Now, $\tilde{h}_{AB}$ depends just on $v$ and $w$ and coordinate changes can be performed so as to bring it to a simpler form, thus for instance, a change

$$u' = u + \alpha(v, w)$$

allows one to set $\tilde{h}_{u\alpha} = 0$ without altering the form of the HV (i.e., $\tilde{X} = \partial_u$). Next, a change of the form

$$v' = \beta(v, w), \quad \text{and} \quad w' = \gamma(v, w)$$

can be used to render the 2-metric in the $v, w$ plane in an explicitly conformally flat form, thus the whole line element can be written as

$$\text{d}\sigma^2 = \exp(2\psi\epsilon)\epsilon_1 A^2(v, w) \text{d}u^2 + 2B(v, w) \text{d}v \text{d}w + C^2(v, w)(\epsilon_2 \text{d}v^2 + \text{d}w^2).$$ (128)

Table 9 lists the corresponding 1+3 reducible spacetimes $(M, \tilde{g})$.

4.2. $(V, h)$ admits a group $H_2$ of homotheties

The Lie brackets will have the form

$$[H, X] = \lambda X$$

where $\lambda = 0, 1$. 

31
The metrics admitting $\mathcal{H}_2$ algebras are derived in a similar way to the metrics admitting $\mathcal{G}_2$ algebras: the Abelian and non-Abelian cases are dealt with separately, considering each sub-case (a)–(d) as follows:

4.2.1. The Abelian case, $\mathcal{H}_2 I$. Starting with the general $(V, h)$ metric, in cases (a), (b) and (c) we choose coordinates such that the KV $\vec{X} = \partial_v$ and HV $\vec{H} = \partial_w$. Similarly, starting with the $(V, h)$ metric, in case (d) we choose coordinates such that the KV $\vec{X} = \partial_v$ and HV $\vec{H} = u\partial_v + \partial_w$. The corresponding metrics are all of the form

$$d\Sigma^2 = e^{2\psi w} \sigma^2,$$

where $d\sigma^2$ is the 3-space metric of the corresponding $\mathcal{G}_2$ spacetime and $\psi$ is the homothetic scalar. The $(V, h)$ metrics admitting an Abelian $\mathcal{H}_2$ algebra are as follows:

(a) $d\Sigma^2 = e^{2\psi w} (\epsilon_1 d\epsilon^2 + A(u) d\epsilon^2 + 2B(u) d\epsilon d\omega + C(u) d\omega^2)$

(b) $d\Sigma^2 = e^{2\psi w} (P^{-2}(u) d\epsilon^2 - 2d\epsilon d\omega - 2H(u) d\omega^2)$

(c) $d\Sigma^2 = e^{2\psi w} (2d\epsilon d\omega + (A(u) d\epsilon + B(u) d\omega)^2)$

(d) $d\Sigma^2 = e^{2\psi w} [(A(u) w^2 - 2B(u) w + C(u)) d\epsilon^2 + A(u) d\omega^2 - 2(A(u) w - B(u)) d\epsilon d\omega + 2D(u) d\epsilon d\omega],$

corresponding to metrics (30), (37), (42) and (43), respectively. We note that there is some freedom in the choice of the KV and the HV. However, it can be shown that the above choice, and corresponding metrics, encapsulate all possibilities. Note that since (37) admits a $\mathcal{G}_4$, the metric (130) admits a $\mathcal{G}_4 \supset \mathcal{H}_2$. The reducible 1+3 spacetimes $(\mathcal{M}, \tilde{g})$ with $(V, h)$ metrics admitting an Abelian $\mathcal{H}_2$ algebra are given in table 10.

4.2.2. The non-Abelian case, $\mathcal{H}_2 II$. Since the Lie bracket of a KV and an HV is a KV then in cases (a), (b) and (c) we set $\vec{X} = \partial_v$ and $\vec{H} = v\partial_v + \partial_w$, and in case (d) we set $\vec{X} = \partial_w$ and $\vec{H} = \partial_v + w\partial_w$. The $(V, h)$ metrics admitting a non-Abelian $\mathcal{H}_2$ algebra are all conformally related to the corresponding non-Abelian $\mathcal{G}_2$ metrics, and are as follows:

(a) $d\Sigma^2 = e^{2\psi w} (\epsilon_1 d\epsilon^2 + A(u) e^{-2w} d\epsilon^2 + 2B(u) e^{-w} d\epsilon d\omega + C(u) d\omega^2)$

(b) $d\Sigma^2 = e^{2\psi w} (P^{-2}(u) d\epsilon^2 - 2e^{-w} d\epsilon d\omega - 2H(u) d\omega^2)$

corresponding to metrics (30), (37), (42) and (43), respectively. We note that there is some freedom in the choice of the KV and the HV. However, it can be shown that the above choice, and corresponding metrics, encapsulate all possibilities. Note that since (37) admits a $\mathcal{G}_4$, the metric (130) admits a $\mathcal{G}_4 \supset \mathcal{H}_2$. The reducible 1+3 spacetimes $(\mathcal{M}, \tilde{g})$ with $(V, h)$ metrics admitting an Abelian $\mathcal{H}_2$ algebra are given in table 10.
Table 10. The reducible 1+3 spacetimes \((M, \hat{g})\) with \((V, h)\) metrics admitting an Abelian \(H_2\). \(\epsilon_0 = \pm 1\) and \(\epsilon_1 = \pm 1\) are not both negative.

| Algebra | \((V, h)\) | Spacetime metric/\(H_3\) basis | Condition |
|---------|-------------|---------------------------------|------------|
| \(H_2I(a)\) | \((129)\) | \(dx^2 = \epsilon_0 dx^2 + e^{\psi w}(\epsilon_1 dw^2 + A(u) e^{-2w} dw + C(u) dw^2) + B(u) dw du + B(u) du dw + C(u) du^2\) | \(\epsilon_0 \epsilon_1 (A(u) C(u) - B^2(u)) < 0\) |
| \(H_2I(b)\) | \((130)\) | \(dx^2 = \epsilon_0 dx^2 + e^{\psi w}(p^{-2}(u) dw^2 - 2 du dw - 2H(u) du^2)\) | \(\epsilon_0 p^{-2}(u) > 0\) |
| \(H_2I(c)\) | \((131)\) | \(dx^2 = \epsilon_0 dx^2 + e^{\psi w}(2 du dw + (A(u) dv + B(u) du) dw^2)\) | \(\epsilon_0 A^2(u) > 0\) |
| \(H_2I(d)\) | \((132)\) | \(dx^2 = \epsilon_0 dx^2 + e^{\psi w}[(A(u) w^2 - 2B(u) w + C(u)) du^2 + A(u) dw^2 - 2(A(u)w - B(u)) du dw + 2D(u) du dw]\) | \(\epsilon_0 A(u) D^2(u) > 0\) |

Table 11. The reducible 1+3 spacetimes \((M, \hat{g})\) with \((V, h)\) metrics admitting a non-Abelian \(H_2\). \(\epsilon_0 = \pm 1\) and \(\epsilon_1 = \pm 1\) are not both negative.

| Algebra | \((V, h)\) | Spacetime metric/\(H_3\) basis | Condition |
|---------|-------------|---------------------------------|------------|
| \(H_2II(a)\) | \((133)\) | \(dx^2 = \epsilon_0 dx^2 + e^{\psi w}(\epsilon_1 dw^2 + A(u) e^{-2w} dw + C(u) dw^2) + B(u) dw du + C(u) du^2\) | \(\epsilon_0 \epsilon_1 (A(u) C(u) - B^2(u)) < 0\) |
| \(H_2II(b)\) | \((134)\) | \(dx^2 = \epsilon_0 dx^2 + e^{\psi w}(p^{-2}(u) dw^2 - 2 e^{-w} dw du - 2H(u) du^2)\) | \(\epsilon_0 > 0\) |
| \(H_2II(c)\) | \((135)\) | \(dx^2 = \epsilon_0 dx^2 + e^{\psi w}(2 du dw + (A(u) e^{-w} dw + B(u) du) dw^2)\) | \(\epsilon_0 A^2(u) > 0\) |
| \(H_2II(d)\) | \((136)\) | \(dx^2 = \epsilon_0 dx^2 + e^{\psi w}[(2 du dw + 2A(u) e^{-w} dw + B(u) e^{-2w} dw^2)\] | \(\epsilon_0 B(u) > 0\) |

\[
(c) \quad d\Sigma^2 = e^{2\psi w} [2 du dw + (A(u) e^{-w} dw + B(u) du)^2]
\]

\[
(d) \quad d\Sigma^2 = e^{2\psi w} [2 du dw + 2A(u) e^{-w} dw + B(u) e^{-2w} dw^2]
\]

Corresponding to metrics \((46), (50), (54)\) and \((55)\), respectively. The reducible 1+3 spacetimes \((M, \hat{g})\) with \((V, h)\) metrics admitting a non-Abelian \(H_2\) algebra are given in table 11.

4.3. \((V, h)\) admits a group \(H_3\) of homotheties

A homothety algebra \(H_r, r > 1\) will admit an isometry subalgebra \(G_{r-1}\). Therefore, \(H_3\) metrics will be obtained from the corresponding \(G_2\), rather than the \(H_2\) metrics. Type \(H_3 I X\) cannot occur since Lie algebra type \(IX\) admits no two-dimensional subalgebra. Lie algebra type \(VIII\) is not permitted as a homothety algebra because it does not satisfy the requirement that the Lie bracket of a KV and an HV must be a KV. Within the three-dimensional Lie algebras admitting a two-dimensional subalgebra, only the type \(VII\) has a non-Abelian two-dimensional subalgebra. Therefore, only those homothety algebras \(H_3 \supset G_2 I\) need to be considered, that is, types \(I-VII\). Further, orbit type (b) need not be considered since it was established in section 3.3 that \(G_2 I(b)\) cannot occur since it leads directly to an \(H_4 \supset G_3\), and
$G_2 II (b)$ has been ruled out from the discussion above. Therefore, only orbit types (a), (c) and (d) need to be considered as follows:

\[ \mathcal{H}_3 \text{ on } V^3 \]

\[ \mathcal{H}_3 \supset G_2 I \]

(a) (c) (d)

\[ \text{types } I - VII_p \]

4.3.1. $\mathcal{H}_3$ (from $G_2 (a)$). The 3-space with Abelian $G_2$ is given by equation (30) with basis $\vec{X}_1 = \partial_v, \vec{X}_2 = \partial_w$.

$\mathcal{H}_3 I$. There is no $(V, h)$ with a maximal $\mathcal{H}_3 I$. The 3-space metric can be written

\[ d\sigma^2 = \epsilon_1 du^2 + u^2 (dv^2 + dw^2) \]

and admits at least an $\mathcal{H}_4$. The corresponding spacetime $(M, \tilde{g})$ is conformally flat and can be rejected.

$\mathcal{H}_3 III$. The 3-space is

\[ d\sigma^2 = \epsilon_1 du^2 + u^2 [\alpha \, dv^2 + 2(-\alpha \ln |u| + \beta) \, dv \, dw + (\alpha (\ln |u|)^2 - 2\beta \ln |u| + \gamma) \, dw^2] \]

where $\alpha, \beta$ and $\gamma$ are constants such that $\beta^2 - \alpha\gamma \neq 0$. The $\mathcal{H}_3$ basis is

$\vec{X}_1 = \partial_v, \quad \vec{X}_2 = \partial_w, \quad \vec{H} = u \partial_u + w \partial_w$.

$\mathcal{H}_3 IV$. The 3-space is

\[ d\sigma^2 = \epsilon_1 du^2 + e^{2(\psi - 1)/\psi} [\alpha \, dv^2 + 2\psi^{-1} (-\alpha \ln |u| + \beta) \, dv \, dw + \psi^{-2} (\alpha (\ln |u|)^2 - 2\beta \ln |u| + \gamma) \, dw^2], \]

where $\alpha, \beta$ and $\gamma$ are constants such that $\beta^2 - \alpha\gamma \neq 0$. The $\mathcal{H}_3$ basis is

$\vec{X}_1 = \partial_v, \quad \vec{X}_2 = \partial_w, \quad \vec{H} = \psi u \partial_u + (v + w) \partial_v + w \partial_w$. 
\( \mathcal{H}_3 V \). The 3-space is
\[
d\sigma^2 = \epsilon_1 \, du^2 + u^2(\alpha^2 - \beta^2) \, dv^2 + 2\beta u \, dv \, dw + \gamma u^2 \, dw^2 ,
\] (141)
where \( \alpha, \beta \) and \( \gamma \) are constants such that \( \beta^2 - \alpha^2 \neq 0 \). The corresponding spacetime \( (M, \tilde{g}) \) is a conformally reducible 2+2 spacetime and can be rejected.

\( \mathcal{H}_3 VI_p \). The 3-space is
\[
d\sigma^2 = \epsilon_1 \, du^2 + u^2(\alpha^2 - \beta^2) \, dv^2 + 2\beta u \, dv \, dw + \gamma u^2 \, dw^2 ,
\] (142)
where \( q \neq 0 \), \( q \neq 1 \) and \( \alpha, \beta \) and \( \gamma \) are constants such that \( \beta^2 - \alpha^2 \gamma \neq 0 \). The \( \mathcal{H}_3 \) basis is
\[
\vec{X}_1 = \partial_v , \quad \vec{X}_2 = \partial_w , \quad \vec{H} = \psi u \partial_u + v \partial_v + q u \partial_w .
\]

\( \mathcal{H}_3 VII_p \). The 3-space metric is
\[
d\sigma^2 = \epsilon_1 \, du^2 + A(u) \, dv^2 + 2B(u) \, dv \, dw + C(u) \, dw^2
\] (143)
with functions \( A(u) \), \( B(u) \) and \( C(u) \) restricted by
\[
\psi u A(\mu) + 2B(u) - 2\psi A(u) = 0 , \quad \psi u C(\mu) - 2B(u) + 2(p - \psi) C(u) = 0.,
\]
\[
\psi u A(\mu) + 2B(u) - 2(p - \psi) B(u) + C(u) = 0 .
\]
The general solution for this system of equations is
\[
A(u) = u^{2-\rho/\psi} \left[ \alpha_1 + \alpha_2 \left\{ \left( 1 - \frac{1}{2} s^2 \right) \cos \left( \frac{s}{\psi} \ln |u| \right) - \frac{1}{2} r s \sin \left( \frac{s}{\psi} \ln |u| \right) \right\} \right. \left. + \alpha_3 \left\{ \left( 1 - \frac{1}{2} s^2 \right) \sin \left( \frac{s}{\psi} \ln |u| \right) + \frac{1}{2} r s \cos \left( \frac{s}{\psi} \ln |u| \right) \right\} \right] ,
\] (144)
\[
B(u) = \frac{1}{2} u^{2-\rho/\psi} \left[ \alpha_1 + \alpha_2 \left\{ p \cos \left( \frac{s}{\psi} \ln |u| \right) - s \sin \left( \frac{s}{\psi} \ln |u| \right) \right\} \right. \left. + \alpha_3 \left\{ p \sin \left( \frac{s}{\psi} \ln |u| \right) + s \cos \left( \frac{s}{\psi} \ln |u| \right) \right\} \right] ,
\] (145)
\[
C(u) = u^{2-\rho/\psi} \left[ \alpha_1 + \alpha_2 \cos \left( \frac{s}{\psi} \ln |u| \right) + \alpha_3 \sin \left( \frac{s}{\psi} \ln |u| \right) \right] ,
\] (146)
where \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are arbitrary constants and \( s = \sqrt{4 - \rho^2} \). The \( \mathcal{H}_3 \) basis is
\[
\vec{X}_1 = \partial_v , \quad \vec{X}_2 = \partial_w , \quad \vec{H} = \psi u \partial_u + v \partial_v + q u \partial_w .
\]

The canonical metric forms admitting transitive \( \mathcal{H}_3 \) type (a) algebras on \( (V, h) \) are listed in table 12.

4.3.2. \( \mathcal{H}_3 \) (from \( \mathcal{G}_2 \) (c)). The 3-space with Abelian \( \mathcal{G}_2 \) is given by equation (42) with basis
\[
\vec{X}_1 = \partial_v , \quad \vec{X}_2 = \partial_w .
\] As noted in sections 3.3.1(c) and 3.6.2, if \( B(u) = 0 \), the corresponding \( (M, \tilde{g}) \) is an isometry class 10 (or class 11 or 13) pp-wave spacetime. Where such cases arise in the following analysis they will not be considered in further detail.

\( \mathcal{H}_3 I \). There is no \( (V, h) \) with a maximal \( \mathcal{H}_3 I \). The 3-space can be written
\[
d\sigma^2 = 2 \, du \, dw + u(\alpha \, dv + \beta \, dw)^2,
\]
Table 12. Reducible 1+3 spacetimes admitting transitive $\mathcal{H}_3$ type (a) algebras on $(V, h)$. Those which are conformally reducible 2+2 spacetimes are omitted from the table.

| $\mathcal{H}_3$ algebra | $(V, h)$ | Spacetime metric and $\mathcal{H}_4$ basis | Condition |
|--------------------------|----------|-------------------------------------------|------------|
| $\mathcal{H}_3 I I (a)$  | (138)    | $d\sigma^2 = e_0 \, dx^2 + e_1 \, dy^2 + u^2 \{\alpha \, dx^2 + 2(\alpha \ln|u| + \beta) \, dw \}$ | $\epsilon_0 \epsilon_1 (\alpha \gamma - \beta^2) u^2 < 0$ |
|                          |          | $\tilde{X}_0 = \partial_u$, $\tilde{X}_1 = \partial_y$, $\tilde{X}_2 = \partial_w$, $\tilde{H} = u \partial_u + w \partial_w$ |            |

$\mathcal{H}_3 I I I (a)$

$\mathcal{H}_3 I V (a)$

$\mathcal{H}_3 V I I (a)$

$\mathcal{H}_3 V I I I (a)$

where $\alpha \neq 0$ and $\beta$ are constants. Since $A(u)$ and $B(u)$ are proportional, the corresponding $(M, \tilde{g})$ is a class 11 pp-wave spacetime admitting a $\mathcal{H}_5 \supset \mathcal{G}_4$ as considered in section 3.6.2.

$\mathcal{H}_3 I I$. The 3-space is

$$d\sigma^2 = 2 \, du \, dw + u \{\alpha \, dv + (\alpha \ln|u| + \beta) \, dw \}^2,$$

where $\alpha \neq 0$ and $\beta$ are constants. The $\mathcal{H}_3$ basis is

$\tilde{X}_1 = \partial_u$, $\tilde{X}_2 = \partial_w$, $\tilde{H} = u \partial_u + w \partial_w$.

$\mathcal{H}_3 I I I$. The 3-space is

$$d\sigma^2 = 2 \, du \, dw + u \{\alpha u^{-1/2} \, dv + \beta \, dw \}^2,$$

where $\alpha \neq 0$ and $\beta$ are constants. The $\mathcal{H}_3$ basis is

$\tilde{X}_1 = \partial_u$, $\tilde{X}_2 = \partial_w$, $\tilde{H} = 2 \psi u \partial_u + v \partial_v$.

$\mathcal{H}_3 I V$. There are two cases. For $\psi \neq 1/2$ the 3-space is

$$d\sigma^2 = 2 \, du \, dw + u \{\alpha u^{-1/2} \, dv + (\alpha (2 \psi - 1) - 1) \ln|u| + \beta) \, dw \}^2,$$

where $\alpha \neq 0$ and $\beta$ are constants. The $\mathcal{H}_3$ basis is

$\tilde{X}_1 = \partial_u$, $\tilde{X}_2 = \partial_w$, $\tilde{H} = (2 \psi - 1)u \partial_u + (v + w) \partial_v + w \partial_w$. 36
For $\psi = 1/2$ the 3-space is
\[ ds^2 = 2 du dw + e^{-u}[\alpha du + (-\alpha u + \beta) dw]^2, \]
where $\alpha \neq 0$ and $\beta$ are constants. The $H_3$ basis is
\[ \vec{X}_1 = \partial_u, \quad \vec{X}_2 = \partial_w, \quad \vec{H} = \partial_u + (v + w)\partial_v + w\partial_w. \]

$H_3 V$. There are two cases. For $\psi \neq 1/2$ the 3-space is
\[ ds^2 = 2 du dw + u^{2(\psi-1)/(2\psi-1)}[\alpha du + \beta dw]^2, \]
where $\alpha \neq 0$ and $\beta$ are constants. The corresponding $(M, g)$ is a type 11 pp-wave spacetime admitting a $H_3 \supset G_2$. When $\psi = 1$ the 3-space is flat. When $\psi = 1/2$ the 3-space metric is
\[ ds^2 = 2 du dw + e^{-u} dv^2. \]
The corresponding $(M, g)$ is a type 13 pp-wave spacetime admitting a $H_3 \supset G_2$.

$\tau_3 V I_i$. We note that $q \neq 0, 1$. For $\psi \neq q/2$ the 3-space metric is
\[ ds^2 = 2 du dw + u^{2(\psi-(-q+1)/(2\psi-1))}[\alpha du + \beta u^{(1-q)/(2\psi-1)} dw]^2, \]
where $\alpha \neq 0$ and $\beta$ are constants. The $H_3$ basis is
\[ \vec{X}_1 = \partial_u, \quad \vec{X}_2 = \partial_w, \quad \vec{H} = (2\psi - q)u\partial_u + v\partial_v + qw\partial_w. \]

For $\psi = q/2$ the 3-space is
\[ ds^2 = 2 du dw + [\alpha e^{(q-2u)/(2c)} du + \beta e^{-qu/(2c)} dw]^2, \]
where $\alpha \neq 0$, $\beta$ and $c \neq 0$ are constants. The $H_3$ basis is
\[ \vec{X}_1 = \partial_u, \quad \vec{X}_2 = \partial_w, \quad \vec{H} = c\partial_u + v\partial_v + qw\partial_w. \]

$\tau_3 V I I_p$. The 3-space is
\[ ds^2 = 2 du dw + [A(u) du + B(u) dw]^2, \]
where the functions $A(u)$ and $B(u)$ are restricted by
\[ A(u)u a(u) + B(u) - \psi A(u) = 0, \]
\[ B(u) [B(u)u a(u) - A(u) + (p - \psi)B(u)] = 0, \]
\[ f(u)u + A^{-2}(u) = 0, \]
\[ a(u)u + A(u)B(u) f(u)u + p - 2\psi = 0. \]

The $H_3$ basis is
\[ \vec{X}_1 = \partial_u, \quad \vec{X}_2 = \partial_w, \quad \vec{H} = a(u)\partial_u + (-w + f(u))\partial_v + (pw + v)\partial_w. \]

From equations (154)–(157) we find that $a(u)$ satisfies the equation
\[ a(u) a(u),u - (a(u),u)^2 = (p - 2\psi) a(u),u + 1 - 2p\psi + 4\psi^2 \]
corresponding to equation (81) in the $G_3 V I I_p$ case. One solution of equation (158) is $a(u) = ku$, where $k$ is a constant given in terms of $p$ and $\psi$, but this leads to $B(u) A^{-1}(u) = \text{constant}$, i.e., to a pp-wave solution. No general solution of equation (158) has been found. Note that $a(u) \neq 0$ for three-dimensional orbits.

The canonical metric forms admitting transitive $H_3$ type (c) algebras on $(V, h)$ are listed in table 13.
4.3.3. \( \mathcal{H}_3 \) \((\mathbb{G}_2 \ (d))\). The 3-space with Abelian \( \mathbb{G}_2 \) is given by metric (43) with basis \( X_1 = \partial_u, X_2 = u \partial_d + \partial_w \). Note that both metric functions \( A(u) \) and \( D(u) \) are non-zero for non-degeneracy.

\( \mathcal{H}_3 I \). This type is not possible.

\( \mathcal{H}_3 II \). The 3-space metric is

\[ d\Sigma^2 = e^{-2\phi_u} \, da^2, \]

where \( da^2 \) is the metric (110), i.e., the 3-space metric of the corresponding \( \mathbb{G}_3 I(d) \). (Recall that the corresponding \( \mathbb{G}_3 I(d) \) does not exist, see section 3.6.4.) The KV \( \vec{X}_3 \) of \( da^2 \), given by equation (114), becomes a proper CKV of \( d\Sigma^2 \) but does not give rise to a symmetry of \((M, \tilde{g})\). The KVs \( \vec{X}_1, \vec{X}_2 \) of \( da^2 \) remain as KVs of \( d\Sigma^2 \) and thus of \((M, \tilde{g})\) while the KV \( \vec{X}_3 \), given by equation (111) becomes a HV of \( d\Sigma^2 \) and leads to the HV of \((M, \tilde{g})\) given by

\[ \vec{H} = \psi \eta \partial_u - \partial_u + f(u) \partial_d + g(u) \partial_w, \]

where \( f(u) \) and \( g(u) \) are defined by (112), (113).

\( \mathcal{H}_3 III \). The 3-space metric is

\[ d\Sigma^2 = a^{2\psi} \, \sigma^2, \]
where $d\sigma^2$ is metric (83), i.e., the 3-space metric of the corresponding $G_{III}(d)$. The $H_3$ basis is given by the corresponding $G_{III}(d)$ basis, with homothety $X_3$.

$H_3 IV$. The 3-space metric is

$$d\Sigma^2 = u^{-2\phi} d\sigma^2,$$

(161)

where $d\sigma^2$ is metric (86), i.e., the 3-space metric of the corresponding $G_{IV}(d)$. The $H_3$ basis is given by the corresponding $G_{IV}(d)$ basis, with homothety $X_3$.

$H_3 V$. This type is not possible.

$H_3 VI q$. The 3-space metric is

$$d\Sigma^2 = u^{2\phi/(1-q)} d\sigma^2,$$

(162)

where $d\sigma^2$ is metric (89), i.e., the 3-space metric of the corresponding $G_{VI}(d)$. The $H_3$ basis is given by the corresponding $G_{VI}(d)$ basis, with homothety $X_3$.

$H_3 VII p$. The 3-space metric is

$$d\Sigma^2 = e^{4\phi(u)/q} d\sigma^2, \quad f(u) = \tan^{-1}[(2u - p)/q],$$

(163)

where $q = \sqrt{4 - p^2}$ and $d\sigma^2$ is metric (43) with functions $A(u), D(u)$ given by (92), i.e., the 3-space metric of the corresponding $G_{VII}(d)$. The $H_3$ basis is given by the corresponding $G_{VII}(d)$ basis, with homothety $X_3$.

The canonical metric forms admitting transitive $H_3$ type (d) algebras on $(V, h)$ are listed in table 14.

5. Discussion and examples

We have shown that the following possibilities exist for 1+3 reducible spacetimes.

1. If $(V, h)$ is of constant curvature, then it admits two or more GCKV and $(M, \hat{g})$ is conformally flat and thus is a conformally reducible 2+2 spacetime.

2. If $(V, h)$ admits only one GCKV, which is not a null GKV, then $(M, \hat{g})$ is not conformally flat, but is a conformally reducible 2+2 spacetime.

3. If $(V, h)$ admits only one GCKV, which is a null GKV, then $(M, \hat{g})$ is a pp-wave spacetime and the dimension, $r$, of the symmetry group $(S_r, H_r$ or $G_r)$ of $(V, h)$ is increased to $r + 2$ in $(M, \hat{g})$.

4. If $(V, h)$ admits no GCKV, then $(M, \hat{g})$ admits only HV and KV and the dimension, $r$, of the symmetry group $(H_r$ or $G_r)$ of $(V, h)$ is increased to $r + 1$ in $(M, \hat{g})$.

The first two possibilities were dealt with in [1]. The second two possibilities are discussed in sections 3 and 4, where we enumerated all possible isometry and homothety Lie algebras on the three-dimensional spacelike and timelike hypersurfaces of the 1+3 spacetimes.

Throughout this work we have pointed out instances where the spacetimes in question are conformally reducible 2+2 spacetimes. There may be conformally reducible 1+3 spacetimes present which are also conformally reducible 2+2 spacetimes although this may not be apparent from their particular coordinate representation. The condition for a spacetime to be reducible 2+2 is that it admits two independent null recurrent vector fields. Further, only Petrov type $D$ or $O$ are possible. oktem [18] shows that, for a non-zero Weyl tensor, the two independent recurrent vector fields are the principal null directions of the Weyl tensor. It follows that the recurrent vector fields will also be the principal null directions of the Weyl tensor for any conformally related spacetime, i.e., a conformally reducible 2+2 spacetime. If the spacetime is type $O$ then it is automatically a conformally reducible 2+2 spacetime. If the spacetime is
Theorem 3. Let \((M, g)\) be a spacetime. If there exists a function \(\mu : M \rightarrow \mathbb{R}\) and null vectors \(l^a\) and \(k^a\) \((l^a k_a = -1)\) satisfying
\[
I_{a;b} = \alpha e^{-\mu l^a l_b} - \mu_{,a} l_b + (\mu_{,c} l^c) g_{ab}, \quad k_{a;b} = -\alpha e^{-\mu} k_a l_b - \mu_{,a} k_b + (\mu_{,c} k^c) g_{ab},
\]
then \((M, g)\) is conformally related to a reducible 2+2 spacetime with conformal factor \(e^{2\mu}\). Here \(\alpha\) is some real function of the coordinates associated with the integral distribution spanned by \(l^a = e^\mu l^a\) and \(k^a = e^\mu k^a\).

We now present some examples of reducible and conformally reducible 1+3 spacetimes which, in general, are well known and which illustrate some of the results presented here.

### Table 14. Reducible 1+3 spacetimes admitting transitive \(H_3\) type (d) algebras on \((V, h)\). Those which are conformally reducible 2+2 spacetimes are omitted from the table.

| \(H_3\) algebra | \((V, h)\) | Spacetime metric and \(H_3\) basis | Condition |
|-----------------|---------|--------------------------------|-----------|
| \(H_{3,II}(d)\) | (159)   | \(\delta^2 = e_0 \delta_{a}^a + e^{\Sigma a b c d} [(e^2 a^2 - 2B(u)w + C(u))] d\delta^2\) | \(e_0 h^2 \delta^2 > 0\) |
| \(H_{3,III}(d)\) | (160)   | \(\delta^2 = e_0 \delta_{a}^a + e^{\Sigma a b c d} [(e^2 a^2 - 2B(u)w + C(u))] d\delta^2\) | \(e_0 h^2 \delta^2 > 0\) |
| \(H_{3,IV}(d)\) | (161)   | \(\delta^2 = e_0 \delta_{a}^a + e^{\Sigma a b c d} [(e^2 a^2 - 2B(u)w + C(u))] d\delta^2\) | \(e_0 h^2 \delta^2 > 0\) |
| \(H_{3,V}(d)\) | (162)   | \(\delta^2 = e_0 \delta_{a}^a + e^{\Sigma a b c d} [(e^2 a^2 - 2B(u)w + C(u))] d\delta^2\) | \(e_0 h^2 \delta^2 > 0\) |
| \(H_{3,V/l}(d)\) | (163)   | \(\delta^2 = e_0 \delta_{a}^a + e^{\Sigma a b c d} [(e^2 a^2 - 2B(u)w + C(u))] d\delta^2\) | \(e_0 h^2 \delta^2 > 0\) |

a type D conformally reducible 2+2 spacetime, then the principal null directions will satisfy the conditions of theorem 1 in [1], which we shall quote here.
Example 1. Consider the \( g_{2I} \) metric of \((V, h)\) given by (107) with KVs given by (108). The special case given by the choice \( \epsilon_1 = 1, A_0 k^2 = -2, m = a^2 \) results in the spacetime \((M, \hat{g})\) with metric
\[
d s^2 = a^2 \left[ d\eta^2 - (d\nu + e^w dv)^2 + \frac{1}{2} e^{2w} dv^2 + dw^2 \right]
\] (164)
which is the Gödel metric admitting the five KVs
\[
\vec{X}_1 = \partial_\nu, \quad \vec{X}_2 = v \partial_\nu - \partial_w, \quad \vec{X}_4 = \partial_\nu, \quad \vec{X}_5 = \partial_\eta,
\]
\[
\vec{X}_3 = -2e^{-w} \partial_\nu - (e^{-2w} + v^2/2) \partial_v + v \partial_w.
\] (165)
The spacetime with metric
\[
d \Sigma^2 = \eta^{-2} ds^2,
\]
where \( ds^2 \) is the Gödel metric (164), also satisfies the field equations for a comoving perfect fluid with a non-zero cosmological constant \( \Lambda \). The density and pressure are given by
\[
\mu = \frac{1}{2} a^{-2} \eta^{-2} - 3a^{-2} - \Lambda, \quad p = \frac{1}{2} a^{-2} \eta^2 + 3a^{-2} + \Lambda,
\]
and the appropriate energy conditions are satisfied if \( \Lambda \leq -3a^{-2} \). The KV remain as KV except for \( X_3 \) which becomes a proper CKV with \( \psi = -\eta^{-1} \).

Example 2. The perfect fluid spacetime (Senovilla [19]) with metric
\[
d s^2 = G^2(\gamma) u \Sigma^2, \tag{166}
\]
where \( G(y) = a^{-1} k \sin(\alpha y) \) and the metric of \((M, \hat{g})\) is given by
\[
d \Sigma^2 = G^{-2}(\gamma) u \Sigma^2 + F^{-1}(x) dx^2 + F(x) d\phi^2 - x^2 (d\nu + bx^{-2} d\phi)^2 \tag{167}
\]
is a conformally reducible 1+3 spacetime. Here \( F(x) = m \ln(x/c) + b^2 x^{-2} - k^2 x^2 \) and \( a, b, c, k \) and \( m \) are constants. The 3-space \((V, h)\) admits only the two KV \( \vec{X}_1 = \partial_\nu, \vec{X}_2 = \partial_\phi \), which is an example of the \( g_{2I} \) structure discussed in section 3.3.1. The spacetime \((M, \hat{g})\) admits the additional KV \( \vec{X}_3 = G^{-1}(\gamma) \partial_\nu \) and the conformally related spacetime \((M, g)\) retains the KVs \( \vec{X}_1, \vec{X}_2 \) but \( \vec{X}_3 \) becomes a proper CKV with \( \psi = -G^{-2} G_{\gamma} \).

Example 3. Consider the conformally reducible 1+3 perfect fluid spacetime of Allnutt [20], namely
\[
d s^2 = e^{-2ax} u \Sigma^2, \tag{168}
\]
where the metric of \((M, \hat{g})\) is given by
\[
d \Sigma^2 = e^{2ax} dx^2 + f^{1/n} dy^2 + t^{1-n} dz^2 - dt^2, \tag{169}
\]
\((M, g)\) is a Bianchi VIh solution and so admits three KVs.

As in the previous example, the 3-space \((V, h)\) admits two KVs, \( \vec{X}_1 = \partial_\nu, \vec{X}_2 = \partial_\nu \), but also admits a HV, \( \vec{X}_3 = i \partial_t + \frac{1}{2} (1 - n) y \partial_y + \frac{1}{2} (1 + n) z \partial_z \), and thus is an example of a \( \mathcal{H}_3 V\) space given by equation (151) after a suitable coordinate transformation. We find that \((M, \hat{g})\) admits the KV \( \vec{X}_4 = e^{-at} \partial_t \) in addition to \( \vec{X}_1 \) and \( \vec{X}_2 \) [8]. Restoring the conformal factor \( e^{-2ax} \) we find that for \((M, g)\) \( \vec{X}_1 \) and \( \vec{X}_2 \) remain as KV, \( \vec{X}_3 \) becomes the third KV while \( \vec{X}_4 \) becomes a proper CKV with \( \psi = a e^{-at} \).

Example 4. The null electromagnetic field solution found by Datta and Raychaudhuri [21] has metric
\[
d s^2 = \rho^{-1/2} [d\rho^2 + dx^2 + f(\rho)^{-1} \rho^{5/2} d\phi^2 - f(\rho) \rho^{1/2} (dt^2 - \rho f(\rho)^{-1} d\phi^2)],
\]
where \( f(\rho) = 4b^2\rho^2 + c\rho \ln|\rho| \) and \( b, c \) are constants. The corresponding \((V, h)\), i.e.,
\[
d\sigma^2 = d\rho^2 - f(\rho)\rho^{1/2} \, dt^2 + 2\rho^{3/2} \, d\phi
\]
amits a \( G_2 \) with basis, \( \vec{X}_1 = \partial_\rho, \vec{X}_2 = \partial_t \) where \( \vec{X}_1 \) is null and \( \vec{X}_1, \vec{X}_2 \) are not orthogonal.

The spacetime \((M, g)\) admits three KVs and no other symmetries.

If the constant \( c = 0 \) the metric of \((V, h)\) becomes
\[
dx^2 = d\rho^2 - 4b^2\rho^{5/2} \, dt^2 + 2\rho^{3/2} \, d\phi
\]
and \((V, h)\) admits an HV, \( \vec{H} = \rho \partial_\rho + \frac{b}{2} \phi \partial_\phi - \frac{1}{2} t \partial_t \), and so is of type \( H_3V \). The corresponding spacetime \((M, g)\), i.e.,
\[
\rho^{-1/2}(d\rho^2 + dz^2) - 4b^2\rho^2 \, dt^2 + 2\rho \, d\phi
\]
amits the three KVs \( \vec{X}_1, \vec{X}_2, \vec{X}_3 = \partial_z \) and the HV
\[
\vec{H} = 4z\partial_z + 4\rho \partial_\rho + 3\phi \partial_\phi - t \partial_t, \quad \psi = 3.
\]

**Example 5.** We now illustrate theorem 2, part 4(a), by considering a special type of plane wave spacetime \((M, g)\) with metric (the conformal symmetries of which are given in [11])
\[
dx^2 = dy^2 + dx^2 - 2du \, dv - 2u^{-4}x^2 \, du^2. \tag{170}
\]
The energy–momentum tensor for this spacetime is that of a null fluid. The 3-space \((V, h)\) given by
\[
d\sigma^2 = dx^2 - 2du \, dv - 2u^{-4}x^2 \, du^2 \tag{171}
\]
amits a null GKV, \( \vec{\xi} = \partial_z \), and a SCKV \( \vec{X}_1 = u^2 \partial_u + \frac{1}{x^2 + y^2} \partial_x + ux \partial_y \). Thus the requirements of the theorem are satisfied and \((M, g)\) admits a proper SCKV given by
\[
\vec{Y} = u^2 \partial_u + \frac{1}{2} (x^2 + y^2) \partial_x + ux \partial_y + uy \partial_y.
\]
The 3-space \((V, h)\) also admits a HV which leads to a HV of \((M, g)\) given by \( \vec{H} = 2v \partial_v + x \partial_x + y \partial_y \), and two KVs, which are also KVs of \((M, g)\) given by \( \vec{X}_2 = p(u) \partial_u + p(u) \partial_v \) and \( \vec{X}_3 = q(u) \partial_u + q(u) \partial_v \) where \( p(u) = u \cos(\sqrt{2}/u) \) and \( q(u) = u \sin(\sqrt{2}/u) \). Apart from the obvious KV \( \vec{X}_4 = \partial_z \), \((M, g)\) also admits a fifth KV \( \vec{X}_5 = y \partial_y + u \partial_y \). Thus \((M, g)\) admits one proper SCKV, one HV and five KV, including a null GKV.

**Example 6.** Siklos [22] found a family of pure radiation solutions with non-zero cosmological constant \( \Lambda \); the metric of \((M, g)\) is
\[
dx^2 = \frac{3}{|\Lambda|x^2} \, d\Sigma^2,
\]
where \( d\Sigma^2 \) is the metric of \((M, \hat{g})\) given by
\[
d\Sigma^2 = dx^2 + dy^2 - 2dv \, du - 2x^{2k} \, du^2,
\]
where \( k(2k - 3) > 0 \) for positive energy. \((M, \hat{g})\) is a pp-wave spacetime of isometry class I(i) [11] and is an example of the case \( \rho = 0 \) in theorem 2, case (4a), so no proper SCKV exists. The corresponding \((V, h)\) admits two KVs, \( \vec{X}_1 = \partial_u, \vec{X}_2 = \partial_v \) and a HV, \( \vec{X}_3 = (k - 1)u \partial_u - (k + 1)v \partial_v - x \partial_x \), while \((M, \hat{g})\) admits the additional KVs, \( \vec{X}_4 = \partial_y, \vec{X}_5 = y \partial_y + x \partial_x \) and the HV \( \vec{X}_3 = (k - 1)u \partial_u - (k + 1)v \partial_v - x \partial_x - y \partial_y \). Restoring the conformal factor \( 3/|\Lambda|x^2 \), we find that \( \vec{X}_1, \vec{X}_2, \vec{X}_4, \vec{X}_5 \) remain as KVs while \( \vec{X}_3 \) becomes a fifth KV so that \((M, \hat{g})\) admits a \( G_5 \).
Acknowledgments

JC acknowledges financial support from the Spanish Ministry of Education through grant no. FPA 2004-0366 and also the ‘Govern de les Illes Balears’ through its programme ‘Suport als Grups d’Investigació Competitius’.

References

[1] Carot J and Tupper B O J 2002 Class. Quantum Grav. 19 4141
[2] Coley A A and Tupper B O J 1992 J. Math. Phys. 33 1754
[3] Tsampanis M, Nikolopoulos D and Apostolopoulos P S 1998 Class. Quantum Grav. 15 2909
[4] Capocci M S and Hall G S 1997 Gravit. Cosmol. 3 1
[5] Hall G S and Kay W 1988 J. Math. Phys. 29 420
[6] Stephani H, Kramer D, MacCallum M A H, Hoenselaers C and Herlt E 2004 Exact Solutions to Einstein’s Field Equations 2nd edn (Cambridge: Cambridge University Press)
[7] Hall G S and Steele J D 1991 J. Math. Phys. 32 1847
[8] Ramos M P M, Vaz E G L R and Carot J 2003 J. Math. Phys. 44 4839
[9] Stephani H 1982 General Relativity: An Introduction to the Theory of the Gravitational Field (Cambridge: Cambridge University Press)
[10] Carot J and da Costa J 1993 Class. Quantum Grav. 10 461
[11] Keane A J and Tupper B O J 2004 Class. Quantum Grav. 21 2037
[12] Hall G S, Low D J and Pulham J R 1994 J. Math. Phys. 35 5930
[13] Petrov A Z 1969 Einstein Spaces (Oxford: Pergamon)
[14] Barnes A 1979 J. Phys. A: Math. Gen. 12 1493
[15] Eisenhart L. P 1933 Continuous Groups of Transformations (Princeton, NJ: Princeton University Press)
[16] Hall G S 2003 Class. Quantum Grav. 20 3745
[17] Hall G S and Steele J D 1990 Gen. Rel. Grav. 22 457
[18] Oktien F 1976 Nuovo Cimento 34B 169
[19] Senovilla J M 1987 Class. Quantum Grav. 4 L115
[20] Allnutt J A 1980 PhD Thesis Queen Elizabeth College, London
[21] Datta B K and Raychaudhuri A K 1968 J. Math. Phys. 9 1715
[22] Siklos S T C 1985 Galaxies, axisymmetric systems and relativity ed M A H MacCallum (Cambridge: Cambridge University Press)