Quantum search processes in the cyclic group state spaces

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Abstract

The hardness to solve an unstructured quantum search problem by a standard quantum search algorithm mainly originates from the low efficiency to amplify the amplitude of the unknown marked state in the Hilbert space of an $n$-qubit pure-state quantum system by the oracle unitary operation associated with other known unitary operations. A standard quantum search algorithm generally can achieve only a square speedup over the best known classical counterparts. In order to bypass this square speedup limitation it is necessary to develop other type of quantum search algorithms. In the paper an oracle-based quantum dynamical method has been developed to solve the quantum search problem in the cyclic group state space of the Hilbert space. The binary dynamical representation for a quantum state in the Hilbert space of the $n$-qubit quantum system is generalized to the multi-base dynamical representation for a quantum state in the cyclic group state space. Thus, any quantum state such as the marked state and its corresponding oracle unitary operation in the cyclic group state space may be described completely in terms of a set of dynamical parameters that are closely related to the symmetric property and structure of the cyclic group. The quantum search problem then may be solved through determining the set of dynamical parameters that describe completely the marked state instead by directly measuring the marked state which is a necessary step in the standard quantum search algorithm. The quantum dynamical method makes it possible to manipulate at will the unknown marked state and its oracle unitary operation. By a similar method used extensively in the hidden subgroup problems, a cyclic group state space may be formed by mapping all the group elements of a cyclic group one-to-one onto the specific states of the Hilbert space of the $n$-qubit quantum system. It carries the symmetric property and structure of a cyclic group. An unstructured quantum search process in the Hilbert space may be affected greatly by the symmetric property and structure of the cyclic group when the quantum search problem is solved in the cyclic group state space. When the cyclic group is high symmetric, that is, the cyclic...
group with order $p$ is a product group of many cyclic subgroups and each cyclic subgroup has an order $\sim O(\log p)$, the quantum search problem in the cyclic group state space could be solved better through reducing it from the cyclic group state space with dimension $p$ to the cyclic group state subspaces with dimension $\sim O(\log p)$ of these cyclic subgroups, for any quantum search problem can be efficiently solved in these subspaces due to their much small dimension. The main attempt of the paper is to make use of the symmetric properties and structures of groups to help solving the unstructured quantum search problem in the Hilbert space. It is shown how the quantum search process could be reduced efficiently from the cyclic group state space to its cyclic group state subspaces with the help of the symmetric property and structure of the cyclic group on an ideal universal quantum computer.

1. Introduction

The quantum search is tremendously valuable as it has an extensive application in computation science. In classical computation most important problems are either polynomial-time or NP-complete [1]. The conventional computers based on the classical physical principles are much suitable for solving efficiently the polynomial-time problems, but inherently they are not enough powerful to treat efficiently all the NP-hard problems [1]. On the other hand, it has been shown that all the NP-complete problems in the classical computation could be solved efficiently on a quantum computer if there existed a polynomial-time unstructured quantum search algorithm. Thus, a great progress could be achieved in quantum computation if an efficient quantum search algorithm could be found. In the past decade a great effort has been devoted to attacking this extremely important problem in quantum computation. A number of quantum search algorithms [2-13] have been proposed and developed since the standard Grover quantum search algorithm was suggested [2]. The famous include the standard Grover search algorithm [2], the quantum adiabatic search algorithm [4, 5], and the amplitude-amplification search algorithm [6]. Most of these oracle-based or block-box-based quantum search algorithms are based on the quantum-state tomographic method. These quantum search algorithms usually start with a superposition of the Hilbert space of a pure-state quantum system, then perform an iterative sequence of unitary operations which include the oracle unitary operation and other known unitary operations to amplify the amplitude of the marked state of the quantum search problem, and after the unitary operation sequence measure the generated state, in which the
marked state has a high probability ($\sim 1$), to output directly the computing result, i.e., the complete information of the marked state. Since the efficiency is low to amplify amplitude of the marked state with these unitary operation sequences these search algorithms usually can only achieve a square speedup over the best known classical counterparts. It has been proven that this square speedup for all these unstructured quantum search algorithms is optimal [3, 6, 9, 13]. More generally, it has been shown that many oracle-based quantum algorithms (not limited to the quantum search algorithms) based on the quantum-state tomography are subjected to polynomial bounds in speedup [14], that is, these quantum algorithms can only achieve a polynomial speedup over their best classical counterparts. In order to bypass this speedup obstacle inherently for the oracle-based quantum algorithms based on the quantum-state tomography it is necessary to develop other types of quantum algorithms to solve the quantum search problem and other problems [15]. Due to the fact that there is a low efficiency to amplify the amplitude of the marked state in these quantum search algorithms [2, 3, 6, 15], in developing new type of quantum search algorithms a direct quantum measurement on the marked state with a high probability ($\sim 1$) should be avoided becoming a necessary step so that amplification of the amplitude of the marked state may not be the key component in algorithm, while the quantum measurement to output the computing results could be carried out on those states which are closely related to the marked state and carry the complete information of the marked state [15]. It is particularly important to be able to manipulate at will any quantum state even the unknown marked state in the Hilbert space in developing efficient quantum algorithms. This is an important step towards the goal to realize that any quantum state in the Hilbert space of an $n-$qubit quantum system is able to be described and characterized completely in a parameterization form [15]. Such a parameterization description for a quantum state is different from the conventional quantum-state tomographic method. Since any quantum state in the Hilbert space can be described and characterized completely by a set of dynamical parameters called the quantum-state unit-number vector [15] and there is a one-to-one correspondence between the oracle unitary operation and the marked state in the quantum search problem, it becomes possible to manipulate at will the oracle unitary operation and the unknown marked state. Due to the fact that the unknown marked state can be described completely by the set of dynamical parameters it is possible to solve the quantum search problem by determining the set of dynamical parameters instead directly
through measuring the marked state [15]. This gives a possibility to avoid
a direct amplification of amplitude of the marked state which is a key com-
ponent in the conventional quantum search algorithms [2-13]. This strategy
to solve the quantum search problem opens a large space to develop new
quantum search algorithms.

Generally, the unknown marked state of the quantum search problem can
be any possible state of the Hilbert space and the quantum search space for
the problem must contain the marked state in a quantum search algorithm.
Therefore, for these conventional quantum search algorithms the quantum
search spaces usually are taken as the whole Hilbert space and hence the ini-
tial state is a superposition over the whole Hilbert space. The usual quantum
search algorithms also have showed that the low efficiency to amplify the am-
plitude of the marked state is strongly dependent on the dimensional size of
the search space, that is, the efficiency generally is inversely proportional to
the square root of the dimensional size of the search space [2, 3, 6] and it has
been shown that the efficiency is optimal [3, 6, 9, 13]. One possible scheme
to increase this efficiency therefore could be that the quantum search space
is limited to a small state subspace of the Hilbert space for a quantum search
problem [16]. Generally, this scheme will meet difficulty and is not feasible
if the marked state is not in the subspace. To make the scheme feasible one
must convert the marked state from the whole Hilbert space to the small sub-
space. Because there is the rotation symmetry in spin space in the n—qubit
quantum spin system the whole Hilbert space of the spin system can be di-
vided into (\(n+1\)) state subspaces according to the angular momentum theory
in quantum mechanics and it can be shown that any unknown quantum state
such as the marked state can be converted efficiently from a small subspace
into a larger subspace in the Hilbert space [16]. This fact directly leads to
that in a single n—qubit quantum system the quantum search problem can
be reduced efficiently from the whole Hilbert space into its largest subspace.
This search space reduction speeds up the conventional quantum search pro-
cess, although this speedup is limited and does not change essentially the
computational complexity for the quantum search problem. However, it is
very important for the fact that the symmetric properties and structures of
quantum systems may be exploited to speed up the quantum search process,
for one can go a further step to use the special group symmetric properties
and structures to help solving the quantum search problem. This idea will
play an important role to guide the construction of quantum search algo-
rithms in the cyclic group state space in the paper. Generally, the whole
Hilbert space of the $n$-qubit quantum system may not have some specific group symmetric properties and structures, but a specific state subset of the Hilbert space could carry the symmetric property and structure of a specific group such as a cyclic group. Then quantum computation may be affected greatly by the symmetric property and structure of the group if it is carried out on the state subset. In order to make use of the symmetric property and structure of a finite group in developing new quantum algorithms one may first form this specific state subspace in the Hilbert space of the $n$-qubit quantum system. By mapping all the group elements of the group one-to-one onto these specific states of the Hilbert space, which means that each group element corresponds one-to-one to a state of the specific state subspace and hence the mapping is isomorphic, then all these specific states form a state subspace of the Hilbert space and evidently this subspace is an invariant or closed state subspace under the action of the group operations. This state subspace is called the group state space of the Hilbert space. This subspace can be thought of as an artificially-formed state space of the Hilbert space which carries the information of the symmetric property and structure of the group. The dimension of the group state space is just the order of the finite group. The similar scheme to the group state space has been extensively used previously in the hidden subgroup problems [17]. If the group is high symmetric, which means that the group is a product of many its factor subgroups, then the corresponding group state space also may contain many state subspaces which one-to-one correspond to these factor subgroups of the group. Since the dimensional size of the group is a product of the dimensional sizes of all these factor subgroups, the dimension of the group state space is also a product of those of the group state subspaces of the factor subgroups. The dimensional size (denoted as $p$ here) of the group state space can be very large, $p \sim 2^n$, and may increase exponentially as the qubit number, but since it is a product of the dimensional sizes of many state subspaces of these factor subgroups the dimensional sizes of the state subspaces of the factor subgroups can be much small, $\sim O(\log p)$, and may increase only polynomially as the qubit number. Then the quantum search problem in the whole group state space would be solved efficiently if the problem could be efficiently reduced from the whole group state space to the state subspaces of these subgroups. As the main purpose, the paper intends to achieve such a search space reduction for the quantum search problem in the cyclic group state space with the help of the symmetric property and structure of a cyclic group. Here the symmetric property and structure of
a cyclic group are employed to help solving the quantum search problem as a cyclic group is one of the simplest groups and its symmetric property and structure are very simple and have been studied in detail and thoroughly [18].

2. Quantum search model in the cyclic group state space

2.1. The binary dynamical representation and the multi-base dynamical representation of quantum states

In the Hilbert state space of an $n$-qubit pure-state quantum system a quantum state can be characterized and described completely by a set of $n$ dynamical parameters $\{a_k^s = \pm 1, k = 1, 2, ..., n\}$, which has been called the quantum-state unit-number vector in the papers [15]. This is a parameterization description for quantum states in the $2^n$-dimensional Hilbert space and different from the conventional quantum-state tomographic method. By measuring the set of the dynamical parameters one can determine uniquely the corresponding quantum state. This dynamical description picture not only is able to describe completely a quantum state $|s\rangle$ in the $2^n$-dimensional Hilbert space of the pure-state quantum system but also is used to describe the corresponding quantum state $\rho_s = |s\rangle\langle s|$ in the Liouville operator space of the quantum ensemble of the quantum system. For instance, in the Hilbert state space a conventional computational basis state $|s\rangle$ can be described completely with the parameter vector $\{a_k^s\}$ by $|s\rangle = \bigotimes_{k=1}^n (\frac{1}{2}T_k + a_k^s S_k)$ with $T_k = |0_k\rangle + |1_k\rangle$ and $S_k = \frac{1}{2}(|0_k\rangle - |1_k\rangle)$, while in the corresponding Liouville operator space the quantum state is described completely by the diagonal density operator $\rho_s = |s\rangle\langle s| = \bigotimes_{k=1}^n (\frac{1}{2}E_k + a_k^s I_{kz})$ which is also determined uniquely by the vector $\{a_k^s\}$. By the parameter vector $\{a_k^s\}$ one can set up a one-to-one correspondence between a quantum state $|s\rangle$ or $\rho_s = |s\rangle\langle s|$ and the selective rotation unitary operation $C_s(\theta) = \exp(-i\theta D_s)$ with the Hermitian diagonal operator $D_s = |s\rangle\langle s| = \bigotimes_{k=1}^n (\frac{1}{2}E_k + a_k^s I_{kz})$, which selectively acts on only the quantum state $|s\rangle$ or $\rho_s$ and is described completely also by the vector $\{a_k^s\}$. The diagonal operator $D_s$ is called the quantum-state diagonal operator since it is a diagonal operator and also equals the state $\rho_s$ formally. The unitary evolution process of a quantum system or its corresponding quantum ensemble under the action of the selectively rotation unitary operation $C_s(\theta)$ is described completely by the vector parameters $\{a_k^s\}$ and in this sense the vector parameters $\{a_k^s\}$ are also called the dynamical parameters. The quantum-state diagonal operator $D_s$ makes it possible for one
manipulating at will the evolution process of an unknown quantum state in the Hilbert space of a quantum system. This representation for the quantum state via the dynamical parameter vector \( \{a_k^s\} \) is called the binary dynamical representation, for the vector parameters \( \{a_k^s\} \) can take only two values +1 and −1. In the quantum dynamics any quantum-state search problem can be reduced to determining the dynamical parameter vector \( \{a_k^s\} \) of the marked state. The direct measurement on the marked state to output the information of the marked state in the conventional quantum search algorithms [2-13] therefore could not be necessary in the quantum dynamical method, for there are a number of possible methods in quantum dynamics which work either in a pure-state quantum system or in a quantum dynamical method, to determine the dynamical parameter vectors \( \{a_k^s\} \) for quantum states including the marked state [15]. Therefore, the quantum search algorithms based on the quantum dynamics have an important difference from the conventional ones [2-13] that it is not necessary to measure directly the marked state to output the complete information of the marked state in algorithm. The quantum dynamic method opens a large space to develop new type of quantum search algorithms.

Besides the binary dynamical representation for a quantum state in the Hilbert space of an \( n \)-qubit quantum system it is possible to use other multi-base dynamical representations to describe completely a quantum state of a quantum system. The multi-base dynamical representations for a quantum state could be a better choice for the quantum search problem in the cyclic group state space. Before the multi-base dynamical representations can be described the group state space of a cyclic group is firstly defined in the Hilbert space. A cyclic group is an Abelian group in which any group elements are commutable to one another [18]. A cyclic group \( G \) can be generated by a fixed generator \( g \), that is, \( G = \langle g \rangle = \{E, g, g^2, ..., g^{n_r-1}\} \), here \( E \) is the unity element and \( n_r \) the order of the cyclic group. In an analogue way to the method used extensively in the hidden subgroup problems [17], now each group element of the cyclic group \( G \) is mapped one-to-one onto the specific state of the Hilbert state space of an \( n \)-qubit quantum system. Then these specific states of the Hilbert space that correspond to all the group elements of the cyclic group form a state subset and this state subset is the cyclic group state space of the Hilbert space. The cyclic group state space is an invariant state subspace under the action of any group operation (element) of the cyclic group. Suppose further that the unity element \( E \) of the group \( G \) is mapped onto the specific state \( |\varphi_0\rangle \) of the Hilbert space, then the cyclic
the integer states system is the conventional computational basis. This basis set consists of a convenient state basis in the Hilbert space of an $n$-qubit quantum system is the conventional computational basis. This basis set consists of the integer states $\{|Z_{2^n}\rangle\} = \{0\}, |1\rangle, |2\rangle, \ldots, |2^n - 1\rangle$. Then the state basis set of the cyclic group state space of the Hilbert space is the specific state subset of the integer state set $\{|Z_{2^n}\rangle\}$. Now consider the integer set $Z_m = \{0, 1, 2, \ldots, m - 1\}$. The integer set $Z_m$ is a Ring $(Z_m = \mathbb{Z}/m\mathbb{Z})$ under the modular arithmetic operation $(\text{mod } m)$ in number theory [19] and also an additive cyclic group under the modular $(m)$ additive operation [18, 20]. As the multiplicative operation is often used in quantum computation, the integer set $Z_{p-1} = \{0, 1, \ldots, p-2\}$ can be mapped by the modular exponentiation: $z \to g^z \text{ mod } p$ to the positive integer set $Z_p^+ = \{g^z \text{ mod } p\} = \{1, 2, \ldots, p-1\}$, where the integer $z \in Z_{p-1}$ and $p$ is a known prime and $g$ a known primitive root (mod $p$). The integer set $Z_p^+$ forms a multiplicative cyclic group under modular multiplication operation [18, 20]. Both the additive cyclic group $Z_{p-1}$ and the multiplicative cyclic group $Z_p^+$ have an order $p - 1$. Both the cyclic groups have a one-to-one correspondence. In fact, all the same order cyclic groups are isomorphic to one another [18]. Hereafter $p$ is denoted as a known prime, $g$ as a primitive root or a generator of a cyclic group, and $C_m$ the multiplicative cyclic group such as $Z_m^+$ with order $m$. If any of the two cyclic groups is mapped onto the Hilbert space, one obtains their corresponding cyclic group state spaces. For the additive cyclic group $Z_{p-1}$ the mapping between the group elements and the corresponding states in the Hilbert space may be given conveniently by $s \rightarrow |s\rangle$ for the group element $s \in Z_{p-1} = \{0, 1, \ldots, p-2\}$ (|$\varphi_0\rangle = |0\rangle$), and for the multiplicative cyclic group $C_{p-1}$ the mapping may be conveniently written as $f(s) = g^s \text{ mod } p \rightarrow |g^s \text{ mod } p\rangle$ (|$\varphi_0\rangle = |1\rangle$) for the group element $g^s \equiv g^s \text{ mod } p \in Z_p^+ = \{1, 2, \ldots, p-1\}$. Therefore, any quantum state of the cyclic group state space $S(C_{p-1}) = \{|g^s \text{ mod } p\rangle\} = \{|1\rangle, |2\rangle, \ldots, |p-1\rangle\}$ of the multiplicative cyclic group $(C_{p-1})$ can be expressed generally as $|\varphi_s\rangle = |g^s \text{ mod } p\rangle, \ s \in Z_{p-1},$ where the state $|\varphi_s\rangle$ is also a conventional computational base. Since $\varphi_s > 0$ for any $s \in Z_{p-1}$ the state $|0\rangle$ is not included in the cyclic group state space.
There is a one-to-one correspondence between the modular exponential function \( f(s) = g^s \mod p \) of the integer set \( \mathbb{Z}_p^+ \) and the index \( s \) of the integer set \( \mathbb{Z}_{p-1} \). The index \( s \) is really the discrete logarithmic function of the function \( f(s) \), that is, \( s = \log_g f(s) \) with \( g \) a logarithmic base. In other words, the index \( s \) is the inversion function of the modular exponential function \( f(s) = g^s \mod p \), i.e., \( s = f(s)^{-1} \). In classical computation the modular exponential function is not hard to be computed, but the discrete logarithmic function usually is hard and this forms the basis of the classical public secure key cryptography based on the discrete logarithms [21]. However, the Shor discrete logarithmic quantum algorithm shows that the discrete logarithmic function now is not hard yet to be computed in quantum computation [22].

Actually, the additive cyclic group state space \( S(\mathbb{Z}_{p-1}) \) is just the state subset \( \{|Z_{p-1}\} \) consisting of the first \( p-2 \) conventional computational bases of the Hilbert space \( \{|Z_{2^n}\} \). Apparently it can not see any difference between the additive cyclic group state space \( S(\mathbb{Z}_{p-1}) \) and the state subset \( \{|Z_{p-1}\} \) of the Hilbert space if one does not care about the symmetric properties and structures of the two state subsets in the quantum search problem. However, their difference could be great if their symmetric properties and structures are taken into consideration. For the multiplicative cyclic group state space \( S(C_{p-1}) \) whose state bases are the modular exponentiation states \( \{|g^s \mod p\} \), one may easily imagine that there exists difference between the two state subsets \( S(C_{p-1}) \) and \( \{|Z_{p-1}\} \). However, only from the symmetric property and structure of the cyclic group state space can one understand deeply that the difference could lead to a completely different result in quantum computation. According to the fundamental theorem of arithmetic (see the Theorem 2 in Ref. [19]) the order \( p-1 \) of the multiplicative cyclic group \( C_{p-1} \) which is also the dimension of the cyclic group state space \( S(C_{p-1}) \) can be expressed as a product of distinct primes, \( p-1 = p_1^{a_1}p_2^{a_2}...p_r^{a_r} \), where \( p_1, p_2, ..., p_r \) are distinct primes and the exponents \( a_1, a_2, ..., a_r > 0 \). Then correspondingly the cyclic group \( C_{p-1} \) can be decomposed as a product of its factor cyclic subgroups (see Chapter One and Two in Ref. [18]):

\[
C_{p-1} = C_{p_1^{a_1}} \times C_{p_2^{a_2}} \times ... \times C_{p_r^{a_r}},
\]

(1)

where the factor cyclic subgroup \( C_{p_k^{a_k}} \) has an order \( p_k^{a_k} \) for \( k = 1, 2, ..., r \). Thus, the order \( p-1 \) of the cyclic group \( C_{p-1} \) is a product of the orders \( \{p_k^{a_k}\} \) of the factor cyclic subgroups \( \{C_{p_k^{a_k}}\} \). This shows that though the order \( p-1 \) of the cyclic group \( C_{p-1} \) can be a large number (even \( p \sim 2^n \)), the orders \( \{p_k^{a_k}\} \) of
the factor cyclic subgroups \( \{C_{p_k^{a_k}}\} \) may take much small numbers \( \sim O(\log p) \).

Just like the cyclic group state space \( S(C_{p-1}) \) the cyclic group state subspace \( S(C_{p_k^{a_k}}) \) \((k = 1, 2, ..., r) \) also can be formed by mapping all the elements of the cyclic subgroup \( C_{p_k^{a_k}} \) onto the specific states of the Hilbert space, and it is a state subspace of the cyclic group state space \( S(C_{p-1}) \) and also of the Hilbert space. Since the dimensional size of a cyclic group state space is just the order of the cyclic group, the cyclic group state subspaces \( \{S(C_{p_k^{a_k}})\} \) also can be formed by mapping all the elements of the cyclic subgroup \( C_{p_k^{a_k}} \) onto the specific states of the Hilbert space, and it is a state subspace of the cyclic group state space \( S(C_{p-1}) \) and also of the Hilbert space. Since the dimensional size of a cyclic group state space is just the order of the cyclic group, the cyclic group state subspaces \( \{S(C_{p_k^{a_k}})\} \) may also have much small dimensional sizes \( \sim O(\log p) \) even if the whole cyclic group state space \( S(C_{p-1}) \) has a large dimension \( (p \sim 2^n) \). It is well known that a problem could be difficult to be solved in a large dimension, but generally it may be fast solved in a small dimension even in classical computation. Since the quantum search speed for a search problem is generally inversely proportional to the square root of the dimensional size of the problem \[2, 6\], then the quantum search problem could be efficiently solved even in the whole cyclic group state space \( S(C_{p-1}) \) if it could be efficiently reduced from the whole cyclic group state space to the state subspaces \( \{S(C_{p_k^{a_k}})\} \) of the factor cyclic subgroups \( \{C_{p_k^{a_k}}\} \). Thus, the main purpose in the paper is how to achieve the reduction for the quantum search problem from the cyclic group state space \( S(C_{p-1}) \) to the cyclic group state subspaces \( \{S(C_{p_k^{a_k}})\} \).

A quantum state \( |s\rangle \) of the additive cyclic group state space \( S(Z_{p-1}) \) may be described completely by the dynamical parameter vector \( \{a_k^s\} \). Since \( g^s \mod p \) is an integer of the positive integer set \( Z^+_p \) any quantum state \( |g^s \mod p\rangle \) of the cyclic group state space \( S(C_{p-1}) \) is a usual computational basis and also can be described completely by the dynamical parameter vector \( \{a_k^g\} \), where the parameter vector \( \{a_k^s\} \) may not be equal to the vector \( \{a_k^g\} \) and the two vectors \( \{a_k^s\} \) and \( \{a_k^g\} \) are related by the one-to-one correspondence \( s \leftrightarrow g^s \mod p \). However, a better method to describe completely a quantum state \( |g^s \mod p\rangle \) of the multiplicative cyclic group state space \( S(C_{p-1}) \) could be to use the multi-base dynamical representation in quantum computation. Notice that the cyclic group \( C_{p-1} \) is a product of the cyclic subgroups \( \{C_{p_k^{a_k}}\} \), each of which has an order \( p_k^{a_k} \). Suppose that the cyclic subgroup \( C_{p_k^{a_k}} \) is generated by a generator \( g_k \), that is, \( C_{p_k^{a_k}} = \langle g_k \rangle \). Then any element of the cyclic subgroup \( C_{p_k^{a_k}} \) can be generally written as \( g_k^{l_k} \) for the index \( l_k = 0, 1, ..., p_k^{a_k} - 1 \). Corresponding to the product decomposition (1) for the cyclic group \( C_{p-1} \) each group element \( g^s \) of the cyclic group \( C_{p-1} \) is also a product of the group elements \( \{g_k^{s_k}\} \) of the factor cyclic subgroups.
\{C_{p_k^{a_k}}\},
\quad g^s = g_1^{n_1s_1} \times g_2^{n_2s_2} \times \ldots \times g_r^{n_rs_r},

(2)

where the generator \(g_k\) of the subgroup \(C_{p_k^{a_k}}\) can written as \(g_k = g^{M_k} \mod p\) \cite{18} for \(k = 1, 2, \ldots, r\) and the positive integers \(M_k\) and \(n_k\) will be determined later. The product decomposition (2) for a group element of the cyclic group \(C_{p-1}\) is really written according to the Chinese remainder theorem (see the Theorem 1.4.3 in Ref. \cite{18}). Actually, there exists a one-to-one correspondence between the index \(s\) of the group element \(g^s\) of the cyclic group \(C_{p-1}\) and the index vector \(\{s_k\}\) of the group elements \(\{g_k^{p_k^{a_k}}\}\) of the factor cyclic subgroups \(\{C_{p_k^{a_k}}\}\). Note that the order \(p - 1\) of the cyclic group \(C_{p-1}\) is decomposed as a product of the distinct prime factors \(\{p_k^{a_k}\} : p - 1 = p_1^{a_1}p_2^{a_2}\ldots p_r^{a_r}\). For convenience, here denote that \(m_k = p_k^{a_k}\) for \(k = 1, 2, \ldots, r\), and \(p - 1 = m = m_1m_2\ldots m_r\). Evidently, the integers \(\{m_k\}\) are coprime to one another in pair, that is, the highest common divisor for any pair of the integers \(m_i\) and \(m_j\) equals one: \((m_i, m_j) = 1\) for \(1 \leq i < j \leq r\). Since the index \(s = s \mod (p - 1)\) and if the index \(s_k\) is written as \(s_k = s \mod m_k\) for \(k = 1, 2, \ldots, r\), then the index \(s\) can be uniquely expressed as a linear combination \((\mod (p - 1))\) of the indices \(\{s_k\}\) according to the Chinese remainder theorem \cite{19},

\[ s = (n_1M_1s_1 + n_2M_2s_2 + \ldots + n_rM_r s_r) \mod (p - 1), \]

(3)

where \(p - 1 = m_kM_k\) for \(k = 1, 2, \ldots, r\). Note that \((m_k, M_k) = 1\). The integer \(n_k\) is just the multiplicative inverse to the integer \(M_k \mod m_k\) that satisfies \(n_kM_k = 1 \mod m_k\). Using the known integers \(m_k\) and \(M_k\) one can efficiently calculate the integer \(n_k\) by the Euclidean algorithm \cite{20}. Because the integer \(M_k\) satisfies \(M_k = (p - 1)/m_k\), that is, \(M_k\) is a divisor of the order \(p - 1\), it follows from the Theorem 1.4.3 in Ref. \cite{18} that the generator \(g_k\) of the cyclic subgroup \(C_{p_k^{a_k}}\) is just \(g_k = g^{\frac{M_k}{m_k}} \mod p\) and the order of the subgroup \(C_{p_k^{a_k}}\) is \(p_k^{a_k}\) and hence the cyclic subgroup is written as \(C_{p_k^{a_k}} = \langle g^{\frac{M_k}{m_k}} \mod p\rangle\). Then the state subspace of the cyclic subgroup \(C_{p_k^{a_k}} = \langle g^{\frac{M_k}{m_k}} \mod p\rangle\) is given by

\[ S(C_{p_k^{a_k}}) = \{(g^{M_k})^{l_k} \mod p), l_k = 0, 1, \ldots, p_k^{a_k} - 1\}. \]

The dimension of the state subspace \(S(C_{p_k^{a_k}})\) is just the order \(p_k^{a_k}\) of the cyclic subgroup \(C_{p_k^{a_k}}\). The cyclic group state subspace \(S(C_{p_k^{a_k}})\) is an invariant subspace under the action of any group operation \(g_k^{l_k}\) of the cyclic subgroup \(C_{p_k^{a_k}}\).
Given a set of the indices \( \{s_k\} \) for the group elements \( \{(g^{M_k})^{n_k s_k} \mod p\} \) of the factor cyclic subgroups \( \{C_{p_k}^{a_k}\} \) with the generators \( \{g^{M_k} \mod p\} \), here the integers \( \{n_k\} \) are known, then according to the equations (2) one can compose a unique group element \( g^s \mod p \) for the cyclic group \( C_{p-1} \) with the index \( s \) determined by the equation (3). In turn, if one is given any element \( g^s \mod p \) of the cyclic group \( C_{p-1} \) with the index \( s \), then according to the equation (2) the element can be decomposed uniquely as a product of the elements \( \{(g^{M_k})^{n_k s_k} \mod p\} \) of the cyclic subgroups \( \{C_{p_k}^{a_k}\} \) and the indices \( \{s_k\} \) are given by \( s_k = s \mod m_k \). Therefore, there is a one-to-one correspondence between the index \( s \) of the group element \( g^s \mod p \) of the cyclic group \( C_{p-1} \) and the index set \( \{s_k\} \) of the group elements \( \{(g^{M_k})^{n_k s_k} \mod p\} \) of the factor cyclic subgroups \( \{C_{p_k}^{a_k}\} \). This one-to-one correspondence shows that one can also use the set of indices \( \{s_k\} \) to describe completely the index state \( |s\rangle \) and the cyclic group state \( |g^s \mod p\rangle \) as well in addition to the dynamical parameter vector \( \{a_k^s\} \). Furthermore, because \( s_k = s \mod p_k^{a_k} \) the index \( s_k \) can be expanded in the field \( GF(p_k^{a_k}) \) [19, 20, 21],

\[
s_k = s \mod p_k^{a_k} = \sum_{l=0}^{a_k-1} h_{kkl}^s p_k^l,
\]

where the coefficients \( \{h_{kkl}^s\} \) satisfy \( 0 \leq h_{kkl}^s < p_k \) for \( l = 0, 1, ..., a_k - 1 \) and \( k = 1, 2, ..., r \). This expansion could be thought of as the \( p_k \)-base expansion for the index \( s_k \) similar to the conventional binary expansion for a number. Clearly, given the prime \( p_k \) the index \( s_k = s \mod p_k^{a_k} \) is determined uniquely by the coefficients \( h_{kkl}^s \) for \( l = 0, 1, ..., a_k - 1 \). Therefore, it is needed \( r \) indices \( \{s_k\} \) or \( \sum_{k=1}^r a_k \) coefficients \( \{h_{kkl}^s\} \) to describe completely the index state \( |s\rangle \) or the cyclic group state \( |g^s \mod p\rangle \), while in the binary representation it need only \( n \) parameters \( \{a_k^s\} \) for the complete description for the index state \( |s\rangle \) in the Hilbert space of an \( n \)-qubit quantum system. It seems to be that the multi-base representation \( \{h_{kkl}^s = 0, 1, ..., p_k - 1\} \) or the index vector \( \{s_k\} \) for the index state \( |s\rangle \) or the cyclic group state \( |g^s \mod p\rangle \) is more complicated than the binary representation \( \{a_k^s = +1, -1\} \) in the Hilbert space. However, the importance is that the multi-base representation \( \{h_{kkl}^s\} \) or the index vector \( \{s_k\} \) is related to the symmetric property and structure of the cyclic group \( C_{p-1} \), while this symmetric property and structure could have a great impact on the quantum computation that is carried out in the cyclic group state space. Thus, it could be better in the quantum search problem in the cyclic group state space that the binary dynamical representation \( \{a_k^s\} \) is
replaced with the index vector \( \{ s_k \} \) or the multi-base dynamical representation \( \{ h_{kl}^s \} \) to represent completely the quantum states and to describe the quantum dynamics of the quantum search process. Now in the cyclic group state space the quantum search process to find the marked state is just to determine completely the index vector \( \{ s_k \} \) or the parameter vector \( \{ h_{kl}^s \} \), which is the same process as the previous one that searching for the marked state is just to determine the dynamical parameter vector \( \{ a_k^s \} \) [15].

2.2. The oracle unitary operation acting on the cyclic group state space

The quantum search process in the cyclic group state space may be carried out either in the additive cyclic group state space \( S(Z_{p-1}) \) or in the multiplicative cyclic group state space \( S(C_{p-1}) \). Correspondingly the marked state of the search problem can be represented either by the index state \( |s \rangle \) of the additive cyclic group state space \( S(Z_{p-1}) \) or the modular exponentiation state \( |g^s \mod p \rangle \) of the multiplicative cyclic group state space \( S(C_{p-1}) \). The index state \( |s \rangle \) and the modular exponentiation state \( |g^s \mod p \rangle \) can be efficiently converted into each other by a unitary transformation which is given in next section. It might be more convenient that the quantum search process is carried out in the multiplicative cyclic group state space \( S(C_{p-1}) \) as the multiplicative unitary operations usually are easily constructed and used. Suppose that the quantum search process is used to solve a specific problem such as an NP problem which has only one solution and the possible solution to the problem is within the integer set \( Z_{p}^+ = \{ g^k \mod p \} = \{ 1, 2, ..., p - 1 \} \), here assume that the possible solution can be represented with the integer variable \( x \in Z_{p}^+ \). In the quantum search problem there is a block box or an oracle to compute a function \( f(x) \) for the variable \( x = g^k \mod p \in Z_{p}^+ \). If the variable \( x = g^s \mod p \) is the solution to the problem, then the function \( f(x) = f(g^s \mod p) = 1 \); otherwise, \( f(x) = 0 \). The quantum computational process to compute the function \( f(x) \) in the block box can be represented by a unitary operation. This basic unitary operation is called the oracle unitary operation. It is the unique unitary operation that can access directly the unknown marked state \( |g^s \mod p \rangle \) in the quantum search problem, where the marked state corresponds to the unique solution \( x = g^s \mod p \) to the problem. Generally, if the marked state is defined as \( |g^s \mod p \rangle \), then the corresponding oracle unitary operation \( U_o = U_{os}(\theta) \) in the cyclic group state space
space $S(C_{p-1})$ can be defined by

$$U_{os}(\theta)|g^x \mod p\rangle|a\rangle = \exp[-i\theta f(g^x \mod p)]|g^x \mod p\rangle|a\rangle$$

$$= \begin{cases} 
\exp(-i\theta)|g^x \mod p\rangle|a\rangle, & \text{if } x = s \\
|g^x \mod p\rangle|a\rangle, & \text{if } x \neq s 
\end{cases}$$

where the auxiliary state $|a\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. According to this definition the oracle unitary operation $U_{os}(\theta)$ is really equivalent to the selective rotation operation in the cyclic group state space $S(C_{p-1})$:

$$U_{os}(\theta) = \exp[-i\theta D_s(g)].$$

Here, the quantum-state diagonal operator $D_s(g)$ [15] which is applied to the cyclic group state space $S(C_{p-1})$ can be generally expressed in terms of the cyclic group state,

$$D_s(g) = |g^s \mod p\rangle\langle g^s \mod p|.$$  

Note that the diagonal operator $D_s(g)$ is different from the conventional one $D_s = |s\rangle\langle s|$ in the Hilbert space of an $n$-qubit quantum system. Actually, the diagonal operator $D_s(g)$ can also be expressed in terms of the dynamical parameter vector $\{b^s_k\}$,

$$D_s(g) = \bigotimes_{k=1}^n \left( \frac{1}{2} E_k + b^s_k I_{kz} \right),$$

but here the vector $\{b^s_k = \pm 1\}$ corresponds to the state $|g^s \mod p\rangle$, while the conventional vector $\{a^s_k = \pm 1\}$ is assigned to the index state $|s\rangle$ and the state $D_s = |s\rangle\langle s|$. Through the quantum-state diagonal operator $D_s(g)$ one can set up one-to-one correspondence between the oracle unitary operation $U_{os}(\theta)$ and the cyclic group state $|g^s \mod p\rangle$. This correspondence makes it possible to calculate explicitly the time evolution of a quantum system under the action of the oracle unitary operation $U_{os}(\theta)$ [15], and it may also provide a convenience for manipulating at will the time evolution of a quantum system by the oracle unitary operation. If the auxiliary state is taken as $|a\rangle = |0\rangle$, then the oracle unitary operation $U_{o}$ is simply defined as

$$U_{os}|g^x \mod p\rangle|0\rangle = |g^x \mod p\rangle|f(g^x \mod p)\rangle$$

$$= \begin{cases} 
|g^x \mod p\rangle|1\rangle, & \text{if } x = s \\
|g^x \mod p\rangle|0\rangle, & \text{if } x \neq s 
\end{cases}.$$
The quantum search problem in the cyclic group state space is how to find the marked state \(|g^s \mod p\rangle\), given the oracle unitary operation \(U_{os}\). This is different from the quantum discrete logarithmic problem which states that given a positive integer \(\varphi_s\) such that \(\varphi_s = g^s \mod p\), how to compute the index \(s\), while the quantum search problem is really equivalent to that given the oracle unitary operation \(U_{os}\), how to determine the index \(s\).

The conventional quantum search process usually is carried out in the \(2^n\)-dimensional Hilbert space of a single \(n\)-qubit quantum system. If besides the given work register used for searching task there are also other auxiliary registers, then the oracle unitary operation \(U_o\) could be thought of as a non-selective oracle unitary operation with respect to any states of those auxiliary registers, since in addition to the auxiliary state \(|a\rangle\), which loads the functional values \(f(g^x \mod p)\), the oracle unitary operation \(U_o\) can only apply to the work register. If there are any other auxiliary registers the oracle unitary operation \(U_o\) will not make any effect on any states of all these auxiliary registers. If the quantum search process is carried out in such a multi-register quantum system that contains a work register and several auxiliary registers in addition to the auxiliary state \(|a\rangle\) and each register could consist of a single \(n\)-qubit quantum subsystem, then in order that the search space still has the same dimensional size as before all the states of the auxiliary registers should be set to a given state, e.g., the state \(|R_0\rangle = |00...0\rangle\). This search space is really a small subspace of the whole Hilbert space of the multi-register quantum system. Corresponding to this search subspace the subspace-selective oracle unitary operation \(\overline{U}_o\) should be defined by

\[
\overline{U}_{os}(\theta)|\Psi\rangle|g^x \mod p\rangle|a\rangle = \begin{cases} 
\exp(-i\theta)|R_0\rangle|g^s \mod p\rangle|a\rangle, \\
|R_0\rangle|g^x \mod p\rangle|a\rangle, \\
|\Psi\rangle|g^x \mod p\rangle|a\rangle, \\
\end{cases}
\]

if \(x = s\) and \(|\Psi\rangle = |R_0\rangle\);

if \(x \neq s\) and \(|\Psi\rangle = |R_0\rangle\);

if \(|\Psi\rangle \neq |R_0\rangle\);

where the states \(|\Psi\rangle\) and \(|R_0\rangle\) belong to the auxiliary registers and the oracle unitary operation works in the cyclic group state space \(S(C_{p-1})\) of the work register. Here \(|R_0\rangle\) also denotes the auxiliary register library with the specific state \(|00...0\rangle\) (see next sections). The subspace-selective oracle unitary operation \(\overline{U}_{os}(\theta)\) acts on selectively the state \(|R_0\rangle\) but does not have any effect on any other state \(|\Psi\rangle\) of the auxiliary registers. It is really equivalent to the selective rotation operation in the Hilbert space of the
multi-register quantum system where the work register is in the cyclic group state space $S(C_\rho_{-1})$,

$$U_{os}(\theta) = \exp[-i\theta \overline{D}_s(g)]$$

with the diagonal operator $\overline{D}_s(g) = |R0\rangle |g^s \text{ mod } p\rangle \langle g^s \text{ mod } p| |R0\rangle$. Why using many auxiliary registers here? This is mainly because the conventional mathematical-logic gates usually need to use a large space to perform their reversible operations, while these mathematical-logic gates have been used extensively in constructing the quantum search algorithms in the Hilbert space [16] and also in the cyclic group state space (see next sections). However, it must be careful as there could be a potential risk that the auxiliary registers could enlarge greatly the search space for the quantum search problem and as a result the quantum search process could become degraded.

The effect of the oracle unitary operation $U_o$, which acts on only the marked state, on the evolution process of an $n$-qubit quantum system is so small that it is hard to be detected quantum mechanically when the qubit number $n$ is large [2, 3, 6, 15]. This results in that the quantum search problem generally is hard to be solved in a large Hilbert space. Any superposition of the Hilbert space with dimension $N = 2^n$ could be converted partly into the marked state under the action of the oracle unitary operation associated with other known quantum operations, but each time for the action this conversion efficiency of the marked state is proportional to $1/\sqrt{N}$ [2, 3, 6]. In order to achieve an observable amplitude for the marked state a standard quantum search algorithm needs to call $\sim \sqrt{N}$ times the oracle unitary operation and thus, the quantum search time to find the marked state with a high probability ($\sim 1$) is proportional to the square root ($\sqrt{N}$) of the dimensional size $(N)$ of the search space which here is the whole Hilbert space. This search time therefore increases exponentially as the qubit number $n$. This low amplitude-amplification efficiency results in that a standard quantum search algorithm usually can have only a square speedup over the best known classical counterparts, and it has been also shown that this square speedup is optimal and hence can not be further improved essentially [3, 6, 9, 13]. A number of quantum search algorithms [2-13] have been proposed to achieve this optimal efficiency (with respect to the dimensional variable $N$) which include the standard Grover search algorithm [2], the amplitude-amplification search algorithm [6], and the quantum adiabatic search algorithm [4, 5]. All these search algorithms are based on the quantum-state tomography. A direct quantum measurement on the marked state is necessary to output the
information of the marked state in these search algorithms and hence it is required in algorithm that the amplitude of the marked state be first amplified by a suitable unitary sequence that contains $\sim \sqrt{N}$ oracle unitary operations so that the probability for the marked state is high enough ($\sim 1$) for observation. In recent years a great effort has been made to develop other type of quantum search algorithms [15, 16] in order to break through the square speedup limitation. These quantum search algorithms are based on the quantum dynamical principles. In these quantum-dynamical search algorithms a direct measurement on the marked state may not be necessary so that a direct amplification for the amplitude of the marked state could be avoided, instead the quantum measurement to output the computing results could be carried out on some other states that are closely related to the marked state and carry the complete information of the marked state and the computing results are further used to obtain the complete information of the marked state [15]. The basis behind the quantum-dynamical search algorithms is that (i) any quantum state such as the unknown marked state in the Hilbert space can be described completely in a parameterization form by a set of dynamical parameters and the quantum searching for the marked state therefore is reduced to determining the set of the dynamical parameters; (ii) by the set of the dynamical parameter the oracle unitary operation and the unknown marked state is set up a one-to-one correspondence, and it becomes possible to manipulate at will the evolution process of a quantum system under the oracle unitary operation in the quantum search process. This quantum search method may avoid a direct quantum measurement on the marked state in the quantum search problem and hence it could not be necessary to achieve an enough high probability for the marked state to be observable. Regarding the fact that there is a low efficiency to amplify the amplitude of the marked state by the oracle unitary operation and this efficiency is closely related to the dimensional size of the search space of the search problem, that is, the larger the search space, the lower the efficiency, one simple scheme to increase the efficiency is that the search space of the search problem is limited to a small subspace of the Hilbert space [16]. This scheme is feasible only if the marked state is in the subspace. Hence to make the scheme feasible one may convert the marked state from the whole Hilbert space to the subspace. It is well known that the structure of the Hilbert space of a quantum system generally is closely related to the symmetric property and structure of the quantum system. Because there is a rotation symmetry in spin space in the $n$–qubit quantum spin system according to the angular
momentum theory in quantum mechanics, the Hilbert space of the \( n \)-qubit spin system can be divided into \((n+1)\) state subspaces. Then it can be shown that the quantum search problem in the Hilbert space of the \( n \)-qubit spin system can be efficiently reduced from the whole Hilbert space to the largest subspace of the \((n + 1)\) state subspaces [16]. The conventional quantum search process therefore may be sped up, although this speedup is limited and does not change essentially the computational complexity for the search problem. However, the importance for the fact that the symmetric properties and structures of quantum systems may be employed to speed up the quantum computational process is that one may further use the symmetric property and structure of a group to help solving the quantum search problem. This is just the main purpose of the paper that the symmetric property and structure of a cyclic group are employed to help solving the quantum search problem.

2.3. The structural quantum search in the cyclic group state space

The conventional unstructured and structural search problems are referred to the problems themselves [7, 8]. The structure for a quantum search problem in a cyclic group state space has a different sense from the conventional one. It is referred to the symmetric structure of the cyclic group used to help solving the unstructured quantum search problem in the Hilbert space of the \( n \)-qubit quantum system. A cyclic group is one of the simplest groups. It is an Abelian group and any two elements of a cyclic group are commutable to one another. Its property and structure have been studied in detail and extensively [18, 20]. As shown in equation (1), a cyclic group can be decomposed as a direct product of its factor cyclic subgroups because every Abelian group can be decomposed as a direct product of cyclic groups [18]. The cyclic groups of prime order are the only Abelian simple groups. They have not any nontrivial and proper subgroup. A cyclic group of non-prime order must have a nontrivial and proper subgroup at least. Here, that a cyclic group is highly symmetric means that the cyclic group has many factor cyclic subgroups. Thus, a highly symmetric cyclic group can be expressed as a direct product of its factor cyclic subgroups, as shown in equation (1). The quantum search problem in the cyclic group state space is either unstructured or structural only dependent on the symmetric structure of the cyclic group no matter what the quantum search problem itself is unstructured or structural in the Hilbert space of the \( n \)-qubit quantum
system. If the quantum search is performed in a cyclic group state space whose cyclic group has a prime order, then it is said to be an unstructured quantum search. However, generally a quantum search is carried out in the cyclic group state space of a highly symmetric cyclic group so that the symmetric property and structure of the cyclic group can be employed to help solving the quantum search problem. Therefore, the quantum search proposed in the paper generally is structural in the cyclic group state space. Generally, the Hilbert space of the $n$-qubit quantum system does not have some specific group symmetric properties and structures, but a specific and artificial state subset of the Hilbert space which could be formed by mapping all the group elements of a specific group such as a cyclic group onto the Hilbert space may have the symmetric property and structure of the group. Then quantum computation which is carried out on the state subset may be affected greatly by the group symmetric property and structure. Consequently, though the quantum search problem in the Hilbert space of the $n$-qubit quantum system is unstructured, it is affected inevitably by the symmetric property and structure of the group if it can be reduced to and therefore is solved in the group state space. The effect of the group symmetric properties and structures could lead to a significant speedup for some quantum computation processes. How the cyclic group symmetric property and structure influence on the speedup of the unstructured quantum search process in the whole Hilbert space is important research project that comes to be investigated in detail in the paper and in the future work.

3. The efficient state transformation between the additive and multiplicative cyclic group state spaces

In the quantum factoring problem and the discrete logarithmic problem a number of reversible mathematical-logic operations such as the modular addition, modular multiplication, and modular exponentiation unitary operations have been used extensively [22, 23, 24, 25]. In the reversible computation one basic principle to construct a reversible mathematical-logic operation is that all the input states are also kept together with the output states after the logic operation [26, 27]. A mathematical-logic operation usually needs to use many auxiliary registers so that the operation process can be made reversible. Both the classical irreversible computation and the reversible computation usually are equivalent in computational complexity in time and space [27, 28]. Therefore, the classical irreversible computation generally can be efficiently simulated by the reversible one. The reversible logic operations can be per-
formed in a quantum system as well, but they usually consume much more qubits in space than the conventional unitary operators quantum mechanically in the quantum system. They could influence on the unitary evolution process of a quantum system in a different manner from the conventional unitary operators quantum mechanically. This is because a reversible logic operation usually acts on only some specific states of the quantum system, while the conventional unitary operators quantum mechanically usually act on any states of the quantum system. It must be careful to use a reversible logic operation to manipulate the quantum dynamical process of a quantum system. A reversible mathematical-logic operation could be thought of as a selective unitary operation of a quantum system because there are usually a number of quantum states of the quantum system independent of the action of the logic operation. Quantum physically there are a number of unitary evolution pathways in a multi-qubit quantum system, but under the reversible mathematical-logic operations there are only few unitary evolution pathways to be allowed in the quantum system. This just shows that the mathematical principles can make constraints on the unitary evolution process of a quantum system. On the other hand, quantum computation is a physical process or exactly a unitary evolution process quantum physically, as pointed out by Deutsch [29, 38]. Therefore, the quantum computational process for a given problem obeys not only the quantum physical laws but also is compatible with the used mathematical principles and its computational complexity not only is dependent on the quantum dynamical process but also on the used mathematical principles. One large advantage to use the reversible mathematical-logic operations in solving some mathematical problems is that one could easily trace the unitary evolution pathways for some quantum states in the Hilbert space of the quantum system under the action of the logic operations.

The discrete logarithmic problem is an important problem in classical public secure key cryptography [21]. It can be stated that given an integer \( b = a^s > 0 \), how to calculate the discrete logarithmic function \( s = (\log_a b) \mod p \), which is also called the index of the discrete logarithm, where the positive integer \( a \) is a given logarithmic base and \( p \) a known prime. In the classical computation it is hard to calculate the logarithmic function of a large integer \( b \). This is the basis for the classical public key cryptographic systems based on the discrete logarithm [21]. It has been shown [22, 25, 30] that the discrete logarithmic problem can be solved in polynomial time in quantum computation. Shor first gave an efficient quantum algorithm to calculate the index of
the discrete logarithm [22]. Later this quantum algorithm was improved in a
determination form [30a] with the help of the amplitude amplification method
[6]. Here, with the help of these quantum algorithms [6, 22, 25, 30a] an ef-
ficient unitary sequence is constructed to generate the index state $|s\rangle$ of the
discrete logarithm from the modular exponentiation state $|g^s \mod p\rangle$. By this
efficient unitary sequence any quantum state $|g^s \mod p\rangle$ of the multiplicative
cyclic group state space $S(C_{p-1})$ can be efficiently converted into the corre-
sponding index state $|s\rangle$ of the additive cyclic group state space $S(Z_{p-1})$. In
constructing this efficient unitary sequence many efficient mathematical-logic
operations have been employed extensively, such as the modular exponent-
ation operation, the modular multiplication operation, and the quantum
Fourier transform and so on, and some mathematical knowledge of number
theory are also used necessarily. Because the index $s$ and the modu-
lar exponential function $f(s) = g^s \mod p$ have a one-to-one correspondence,
there exists a unitary operator $U_{\log}(g)$ such that $U_{\log}(g)|g^s \mod p\rangle = |s\rangle$ and
$U_{\log}^+(g)|s\rangle = |g^s \mod p\rangle$, here $g$ is the logarithmic base and also a primitive
root (mod $p$) or a generator of the multiplicative cyclic group $C_{p-1}$. Note that
here there is not any extra auxiliary register to be used by the unitary oper-
or $U_{\log}(g)$. Generally, such a discrete logarithmic unitary operator $U_{\log}(g)$ is hard to be constructed. However, with the help of the reversible com-
putational techniques [22, 23, 24, 26, 27] an alternative construction to the
discrete logarithmic unitary operator $U_{\log}(g)$ could be achieved conveniently
by using many extra auxiliary registers. The construction can be divided
into two steps [23, 24, 27]. One step is to construct by using two registers
the modular exponentiation unitary operation: $V_f|s\rangle|0\rangle = |s\rangle|g^s \mod p\rangle$. It is well known that the modular exponentiation unitary operation can be ef-
ciently built up in the reversible computation [21, 22, 23, 24]. Another
is to construct the unitary operation of inversion function of the modular
exponential function: $V_{f^{-1}}|g^s \mod p\rangle|0\rangle = |g^s \mod p \rangle|s\rangle$, here also by using
two registers. Then the discrete logarithmic unitary operation $U_{\log}(g)$ may
be expressed equivalently by $U_{\log}(g) = V_f^+SV_{f^{-1}}$, where the SWAP unitary
operation $S$ is defined by $S|s\rangle|g^s \mod p\rangle = |g^s \mod p\rangle|s\rangle$. This is due to the
fact that there holds $U_{\log}(g)|g^s \mod p\rangle|0\rangle = V_f^+SV_{f^{-1}}|g^s \mod p\rangle|0\rangle = |s\rangle|0\rangle$, which further indicates that by omitting the auxiliary register with the state
$|0\rangle$ the unitary operation sequence $V_f^+SV_{f^{-1}}$ is really the discrete logarithmic
unitary operation $U_{\log}(g)$. In effect the unitary operation sequence $V_f^+SV_{f^{-1}}$
is equivalent to the unitary operator $U_{\log}(g)$ of the discrete logarithm, but it
must be careful when the unitary operation sequence is performed in a quantum system since the unitary sequence requires the auxiliary registers of the quantum system be in the specific state $|0\rangle$ before and after the operation, while the quantum system may be in any state. Though the modular exponentiation unitary operation $V_f$ can be built up efficiently [22, 23, 24], it is generally hard to build up the unitary operation $V_{f^{-1}}$ of the inversion function of the modular exponential function. It is this unitary operation $V_{f^{-1}}$ that makes it hard to construct the discrete logarithmic unitary operation $U_{\log}(g)$. A functional unitary operation may exist but its inversion-functional unitary operation could or could not, which usually is dependent on the mathematical property of the function. Obviously, some mathematical functions have their own unique inversion functions but some do not have in some given functional or variable value ranges. If the functions do not have their own unique inversion functions in some given value ranges, then the unitary operations for the inversion functions usually could not exist uniquely in these value ranges although the functions may have their own unitary operations. When both a function and its inversion function exist in a given value range, they usually have their own unitary operations, respectively, and sometime their unitary operations are the same up to the conjugate relation. But in general a functional and its inversion-functional unitary operations can be different completely. As the modular exponential function $f(s) = g^s \mod p$ and its index variable $s$ have a one-to-one correspondence, this makes the modular exponential function $f(s)$ and its inversion function, i.e., the discrete logarithmic function or the index variable $s$, have their own unitary operations. Because the unitary operation $V_f$ can be built up efficiently the discrete logarithmic unitary sequence $U_{\log}(g) = V_f^+ SV_{f^{-1}}$ is mainly dependent on the unitary operation $V_{f^{-1}}$ in computational complexity. Below it is devoted to the construction for the efficient unitary operation $V_{f^{-1}}$ of inversion function of the modular exponential function. Before building up the unitary operation $V_{f^{-1}}$ several conventional reversible mathematical-logic operations are introduced.

(i) The modular addition operation $ADD_L(\alpha, \beta)$. The modular addition operation is defined as

$$ADD_L(1, 2)|x\rangle|y\rangle = |x\rangle|x + y \mod L\rangle, \quad x, y \in \mathbb{Z}_L.$$ 

Here the integer set $\mathbb{Z}_L = \{0, 1, ..., L - 1\}$. The indices $\alpha$ and $\beta$ denote the registers that are acted on by the modular addition operation $ADD_L(\alpha, \beta)$. The modular addition operation $ADD_L(1, 2)$ is performed by adding the
integer $x$ of the first register to the second register and taking modulus $L$. It can be implemented in polynomial time $\sim O(\log L)$ [22]. The modular addition operation $\text{ADD}_L(1,2)$ is a reversible operation since the integer $y$ can be derived uniquely from $x$ and $(x+y) \mod L$ if $0 \leq x, y \leq L - 1$. As a specific modular addition unitary operation the COPY unitary operation $\text{COPY}(\alpha, \beta)$ is defined as

$$\text{COPY}(1,2)|x|0\rangle = |x|x\rangle, \quad x \in Z_L.$$  

The inverse COPY unitary operation $[\text{COPY}(\alpha, \beta)]^+$ is really the subtraction unitary operation: $[\text{COPY}(1,2)]^+|x\rangle = |x|0\rangle, \quad x \in Z_L$.

(ii) The modular multiplication unitary operation $M_L(\alpha, \beta, \gamma)$ is defined as

$$M_L(1,2,3)|x\rangle|y\rangle|0\rangle = |x\rangle|y\rangle|xy \mod L\rangle, \quad x, y \in Z_N$$

where $x, y$ are integer variables, the integer $L$ is modulus and the integer $N$ may be different from $L$. The indices $\alpha$ and $\beta$ denote the two registers whose integer variables $x$ and $y$ are multiplied to one another and the index $\gamma$ marks the third register that loads the multiplication operation result. As an example, the modular multiplication unitary operation is applied on the cyclic group state:

$$M_p(1,2,3)|g^x \mod p\rangle|g^y \mod p\rangle|0\rangle$$

$$= |g^x \mod p\rangle|g^y \mod p\rangle|g^{x+y} \mod p\rangle, \quad x, y \in Z_{p-1}.$$  

(iii) The modular exponentiation unitary operation. First consider the modular multiplication operation $U_{a,N}(\alpha)$ which is defined as

$$U_{a,N}(1)|x\rangle = |xa \mod N\rangle, \quad (a, N) = 1, \quad x \in Z_N.$$  

This operation is unitary only when the integer $a$ is coprime to the integer $N$ [17, 31], i.e., $(a, N) = 1$. This unitary operation need not any additional auxiliary register in principle. But when the unitary operation is constructed by the mathematical-logic operations it still needs many extra auxiliary registers. The index $\alpha$ denotes the register acted on by the operation $U_{a,N}(\alpha)$. Generally, the modular exponentiation operation may be taken as $[U_{a,N}(\alpha)]^l$ for any positive integer $l$. The conditional modular exponentiation operation $U^c_{a,L}(\alpha, \beta)$ may be defined with the help of the modular multiplication operation $U_{a,L}(\beta)$:

$$U^c_{a,L}(1,2)|x\rangle|y\rangle = |x\rangle[U_{a,L}(2)^x|y\rangle$$

$$= |x\rangle|ya^x \mod L\rangle, \quad x \in Z_N, \quad y \in Z_L.$$  

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This conditional modular exponentiation operation is unitary only if the integer \( a \) is coprime to the integer \( L \). However, using one more auxiliary register a general conditional modular exponentiation operation, which is unitary even if the integer \( a \) is not coprime to the integer \( L \), may be constructed by

\[
U_{a,L}^c(1, 2, 3)|x\rangle|y\rangle|0\rangle = |x\rangle|y\rangle|ya^x \mod L\rangle, \quad x, y \in \mathbb{Z}_N .
\]

In particular, the two-variable conditional modular exponentiation operation \( U_f = U_{a,b,L}^c(1, 2, 3) \) have been used extensively in the discrete logarithmic problem [22, 25, 30]: \( U_{a,b,L}^c(1, 2, 3)|x\rangle|y\rangle|0\rangle = |x\rangle|y\rangle|b^xa^y \mod L\rangle, \quad x, y \in \mathbb{Z}_N \), where \( a \) and \( b \) are constant integers and usually \( N \geq L \). These modular multiplication and modular exponentiation unitary operations may be built up efficiently by the basic reversible logic operations [21-27] and generally can be efficiently implemented in polynomial time \( \sim O(\log^2 N) \) and \( \sim O(\log^3 N) \), respectively [22]. The qubit number used to implement these modular exponentiation operations generally is \( \sim O(\log N) \) [21, 22, 23, 24].

Besides these conventional mathematical-logic unitary operations introduced above mathematically or quantum physically many important unitary operators, unitary operations, elementary propagators, or quantum gates also can be employed in construction of a unitary sequence. A large advantage for the type of unitary operations is that the unitary operations usually are non-selective unitary operators and hence need not any auxiliary qubits. But the artificial conditional unitary operations, which also can be thought of as the selective unitary operations, may need few auxiliary qubits to help achieving the specific conditional operations.

(iv) The SWAP unitary operation and other elementary quantum gates [32]. The \( SWAP(\alpha, \beta) \) unitary operation is defined as

\[
SWAP(1, 2)|x\rangle|y\rangle = |y\rangle|x\rangle, \quad x, y \in \mathbb{Z}_N .
\]

(v) The quantum Fourier transforms in the Hilbert space. The conventional quantum Fourier transform [22, 33, 34] usually is defined in the regular Hilbert space \( \{|Z_N\} \),

\[
|l\rangle \xrightarrow{Q_{NFT}} \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp[i2\pi kl/N]|k\rangle, \quad k, l \in \mathbb{Z}_N .
\] (5)

For the integer \( N = 2^n \) the quantum circuit \( Q_{NFT} \) for the quantum Fourier transform is very simple and consists of \( \sim O(n^2) \) basic quantum gates. Note
that there is not any auxiliary qubit in construction of the quantum circuit \(Q_{2^nFT}\). For the case that the integer \(N\) is not a power of two the quantum circuit \(Q_{NF^{-}}\) also can be constructed with \(\sim O(\log^2 N)\) basic quantum gates or even less [30, 31, 34, 35], but many auxiliary qubits are needed in the construction of the quantum circuit.

\((vi)\) The functional quantum Fourier transform. The functional quantum Fourier transform is really the quantum Fourier transform applying to a non-regulation state subspace of the Hilbert space. Because the functional quantum Fourier transform is related closely to the unitary operation of the inversion function of a function it could not be generally constructed efficiently for any function. Suppose that the function \(f(x)\) is a periodic function: \(f(x) = f(x + r)\), here \(r\) is the period of the function. Then the functional quantum Fourier transform \(Q_{rf}t\) for the periodic function \(f(x)\) may be defined as [36]

\[ Q_{rf}t|f(l)\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \exp[i2\pi kl/r]|f(k)\rangle, \quad k, l \in Z_r. \quad (6) \]

It can be shown that the functional quantum Fourier transform \(Q_{rf}t\) can be constructed efficiently if both the unitary operations for the periodic function \(f(x)\) and its inversion function \(f(x)^{-1}\) in the variable value range \(Z_r\) can be built up efficiently. Suppose that the functional and its inversion-functional unitary operations are defined by \(V_f|x\rangle|0\rangle = |x\rangle|f(x)\rangle\) and \(V_{f^{-1}}|f(x)\rangle|0\rangle = |f(x)\rangle|x\rangle\) for \(x \in Z_r\), respectively. Then the unitary sequence for the invertible periodic function \(f(x)\) is \(U_f = V_f^{+}SV_f\) which satisfies \(U_f|x\rangle = |f(x)\rangle\) for \(x \in Z_r\), here any auxiliary qubits are dropped and \(S\) is the SWAP operation. Using the invertible-function unitary sequence \(U_f\) the functional quantum Fourier transform \(Q_{rf}t\) is related to the conventional \(r\)-base quantum Fourier transform \(Q_{rFT}\) by

\[ Q_{rf}t = U_fQ_{rFT}U_f^{+}. \]

Thus, the quantum circuit for the functional quantum Fourier transform \(Q_{rf}t\) can be efficiently constructed if there is an efficient quantum circuit for the unitary operation \(U_f\) of the invertible function \(f(x)\).

\((vii)\) The group operations of a cyclic group. A cyclic group \(G\) can be generated by a generator \(g\), \(G = \langle g \rangle = \{E, g, g^2, ..., g^{n_r-1}\}\). If the generator \(g\) is a unitary operator which is denoted as \(U_g\) here, then all the group
elements of the cyclic group $G$ are also unitary operators. When the unitary cyclic group operation $U_g$ is applied to a cyclic group state the unitary transformation is given by

$$U_g |g^x \mod p \rangle = |g^{x+1} \mod p \rangle.$$  

The unitary operation of the cyclic group may be built up efficiently with the help of the diagonal and anti-diagonal unitary operators [16]. Actually, just like the modular multiplication unitary operation $U_{a,L}(\alpha)$ the cyclic group operation $U_g$ could also be constructed efficiently by using the basic reversible logic operations [26, 27], but this construction needs many extra auxiliary qubits. The cyclic group operation $U_g$ can also be performed in a conditional form

$$U^c_g |a \rangle |g^x \mod p \rangle = |a \rangle |g^{x+a} \mod p \rangle.$$  

With the help of these efficient unitary operations mentioned above an efficient unitary sequence will be built up below, by which the index state $|s\rangle$ of the discrete logarithm can be generated from the modular exponentiation state $|g^s \mod p \rangle$.

The oracle unitary operation in the discrete logarithmic problem is the usual conditional modular exponentiation operation $U_f = U^c_{b,g,p}(\alpha, \beta, \gamma)$:

$$U_f |x\rangle |y\rangle |b\rangle |g\rangle |0\rangle = |x\rangle |y\rangle |b\rangle |g\rangle |f(x,y)\rangle, \; x, y \in \mathbb{Z}_N.$$  

The double-variable modular exponential function $f(x,y)$ is defined by

$$f(x,y) = b^x g^y \mod p$$

where the integer $b = g^s \mod p > 0$ with the index $s \in \mathbb{Z}_{p-1}$. The Fermat little theorem (the Theorem 71 in Ref. [19]) shows that there holds $a^{p-1} \equiv 1 \mod p$ for a prime $p$ and any integer $a$ that is not divided by the prime $p$. In particular, for the integer $a = g$, $b$, or even $g^z \mod p$ with $z = sx + y$ for any integers $x$ and $y$ there also holds $a^{p-1} \equiv 1 \mod p$ since $g$ is a primitive root $\pmod{p}$. Thus, the modular exponential function $f(x,y)$ is a periodic function with the period $p-1$ by the Fermat little theorem. Since the periodic function $f(x,y)$ satisfies $f(x,y) = f_1(sx + y) = g^{sx+y} \mod p$, $f_1(z) = f_1(z + (p - 1))$ and also $f(x,y) = f(x + l, y - ls)$ for any integer $l$ [25] the Fourier transform of the functional state $|f(x,y)\rangle$ therefore takes the form

$$|\tilde{f}(l_1,l_2)\rangle = |\tilde{f}(l_2 s \mod (p-1), l_2)\rangle \delta((l_2 s - l_1) \mod (p-1))$$

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\[
\begin{align*}
\delta((l_2s - l_1) \mod (p - 1)) \\
\times \frac{1}{p - 1} \sum_{x=0}^{p-2} \sum_{y=0}^{p-2} \exp[i2\pi l_2(sx + y)/(p - 1)] |f(x, y)\rangle.
\end{align*}
\] (7)

The indices \(l_1\) and \(l_2\) in the Fourier transform state \(\tilde{f}(l_1, l_2)\) must satisfy the relation \((l_2s - l_1) = 0 \mod (p - 1)\) for \(l_1, l_2 = 0, 1, \ldots, p - 2\) due to the fact that \(f(x, y) = f(x + l, y - ls)\). In terms of the Fourier transform states (7) the functional state \(|f(x, y)\rangle\) is expressed as

\[
|f(x, y)\rangle = \frac{1}{\sqrt{p - 1}} \sum_{l=0}^{p-2} \exp[-i2\pi l(sx + y)/(p - 1)] \tilde{f}(ls, l)\).
\] (8)

If one looks the function \(f(x, y)\) as the single-variable periodic function \(f_1(z) = g^z \mod p\) with the variable \(z = sx + y = 0, 1, \ldots, p - 2\), \(f_1(z) = f_1(z + p - 1)\), then one can express the functional state \(|f(x, y)\rangle = |f_1(z)\rangle\) in terms of its Fourier transform states \(|\tilde{f}_1(l)\rangle\),

\[
|f_1(z)\rangle = \frac{1}{\sqrt{p - 1}} \sum_{l=0}^{p-2} \exp[-i2\pi lz/(p - 1)] |\tilde{f}_1(l)\rangle.
\]

By comparing it with equation (8) one can see that there holds the state identity \(|\tilde{f}(ls, l)\rangle/\sqrt{p - 1} = |\tilde{f}_1(l)\rangle\) for \(l = 0, 1, \ldots, p - 2\) and equation (7) is indeed the Fourier transform of the functional state \(|f_1(z)\rangle\) (its explanation can be seen later).

The functional Fourier transform states (7) and the functional states (8) will be used below in building up the unitary operation \(V_{f^{-1}}\) of the inversion function of the modular exponential function. There are many auxiliary registers to be used in the construction of the unitary operation \(V_{f^{-1}}\). The starting state in the construction may be taken as \(|\Psi_0\rangle = |R0\rangle \otimes |g^s \mod p\rangle\). Here \(|R0\rangle = |0\rangle|0\rangle\ldots|0\rangle\) stands for the library of auxiliary registers with the initial state \(|0\rangle\) and suppose that the register library stores sufficiently many registers to supply to the coming quantum computation. The starting state is first converted into the superposition by applying the conventional \((p - 1)\)-base quantum Fourier transforms \(Q_{(p-1)FT}\) to the first two registers, respectively, which are supplied by the register library \(|R0\rangle\). Then the oracle unitary operation \(U_f\) of the discrete logarithm is applied to the first three registers, here the oracle unitary operation \(U_f\) uses the data \(g\) and \(b =
g^s \mod p. After the oracle unitary operation $U_f$ the state of the quantum system is in the state $|\Psi_1\rangle$,

$$
|\Psi_0\rangle = |\text{R0}\rangle \bigotimes |g^s \mod p\rangle \equiv |\text{R0}\rangle \bigotimes |0\rangle |0\rangle |g^s \mod p\rangle \\
\xrightarrow{Q_{(p-1)FT}} |\text{R0}\rangle \bigotimes \frac{1}{p-1} \sum_{x=0}^{p-2} \sum_{y=0}^{p-2} |x\rangle |y\rangle |g^s \mod p\rangle \\
\xrightarrow{U_f} |\Psi_1\rangle = |\text{R0}\rangle \bigotimes \frac{1}{p-1} \sum_{x=0}^{p-2} \sum_{y=0}^{p-2} |x\rangle |y\rangle |f(x, y)\rangle |g^s \mod p\rangle.
$$

The oracle unitary operation $U_f$ of the discrete logarithm is performed in the conventional manner that the integers $g$ and $b = g^s \mod p$ are first stored in auxiliary registers, the quantum computer reads the integers $g$ and $b$ and values of the variables $x$ and $y$ in the first two registers, then performs the functional operation $f(x, y) = b^x g^y \mod p$ and puts the computing result in the third register which is provided by the register library $|\text{R0}\rangle$. Note that the data $b$ is already in the third register before the oracle unitary operation $U_f$ and in the fourth register after the oracle unitary operation, while the known data $g$ can be stored in a temporary register beforehand and after the operation $U_f$ it can be removed from the register. Using the functional Fourier transform states (7) to express the functional state $|f(x, y)\rangle$ one obtains, by inserting equation (8) into the state $|\Psi_1\rangle$,

$$
|\Psi_1\rangle = |\text{R0}\rangle \bigotimes \frac{1}{p-1} \sum_{l=0}^{p-2} \left\{ \left[ \frac{1}{\sqrt{p-1}} \sum_{x=0}^{p-2} \exp[-i2\pi lx/(p-1)] |x\rangle \right] \bigotimes \left[ \frac{1}{\sqrt{p-1}} \sum_{y=0}^{p-2} \exp[-i2\pi ly/(p-1)] |y\rangle \right] |\tilde{f}(l, s)\rangle |g^s \mod p\rangle \right\}.
$$

Now the conventional $(p-1)$–base quantum Fourier transforms $Q_{(p-1)FT}$ are applied again to the first two registers in the state $|\Psi_1\rangle$, respectively, then the quantum system is in the created state $|\Psi_2\rangle$ after the SWAP unitary operation,

$$
|\Psi_1\rangle \xrightarrow{Q_{(p-1)FT}} |\text{R0}\rangle \bigotimes \frac{1}{p-1} \sum_{l=0}^{p-2} |ls \mod (p-1)\rangle |l\rangle |\tilde{f}(l, s)\rangle |g^s \mod p\rangle \\
\xrightarrow{SWAP} |\Psi_2\rangle = |\text{R0}\rangle \bigotimes \frac{1}{p-1} \sum_{l=0}^{p-2} |l\rangle |ls \mod (p-1)\rangle |\tilde{f}(l, s)\rangle |g^s \mod p\rangle.
$$

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The state $|\Psi_2\rangle$ contains the information of the index $s$ in the last three registers. It is expected to extract the index $s$ from the second register as the quantum states in other two registers are more complicated. Therefore, the problem to be solved is how to extract the index $s$ from the state of the second register in the state $|\Psi_2\rangle$ and this is related to the construction of the unitary transformation $U_s$:

$$
|\Psi_2\rangle \xrightarrow{U_s} |R0\rangle \bigotimes_{l \geq 0} \frac{C}{p-1} |l\rangle |ls \text{ mod}(p-1)\rangle |s\rangle |f(l,s,l)\rangle |g^s \text{ mod } p\rangle,
$$

where the index $l$ runs over only some specific values in the range $0 \leq l < p-1$ and $C$ is a normalization constant (see below). In the unitary transformation $U_s$ the desired state transfer $|l\rangle |ls \text{ mod}(p-1)\rangle |0\rangle \rightarrow |l\rangle |ls \text{ mod}(p-1)\rangle |s\rangle$ usually could not be achieved by the conventional inverse multiplication operation $M_{p-1}^+ (\alpha, \beta, \gamma)$. This is because the function $f(s) = ls \text{ mod}(p-1)$ does not have a one-to-one correspondence to its variable $s$ for some integer values $l$ in the range $0 \leq l < p-1$. Actually, it is possible that the inversion function $f(s)^{-1} \neq s$ if the integer $l$ is not coprime to $p-1$. However, the inversion function $f(s)^{-1} = s$ if the integer $l$ is coprime to $p-1$, i.e., $(l, p-1) = 1$, and this is one of the two bases to achieve this unitary state transfer and obtain the real index state $|s\rangle$. It can be seen that the state $|\Psi_2\rangle$ consists of $p-1$ orthogonal states with index $l = 0, 1, ..., p-2$. Among all the $(p-1)$ orthogonal states how many orthogonal states have an index integer $l$ coprime to $(p-1)$? The question can be answered by the Euler theorem in number theory (see the Theorem 72 in reference [19]). As known in number theory [19], number for the positive integers coprime to and not greater than $p-1$ is $\phi(p-1)$, where $\phi(p-1)$ is the Euler totient function, and it is also known that the Euler totient function $\phi(p-1) > \delta(p-1)/\log \log(p-1)$ for some constant $\delta$. More exactly, if the integer $(p-1)$ has a prime factorization: $p-1 = p_1^{a_1} p_2^{a_2} ... p_r^{a_r}$, where $p_1, p_2, ..., p_r$ are distinct primes, then $\phi(p-1) = (p-1) \prod_{l=1}^{r} (1 - p_l^{-1})$. This shows that among the $(p-1)$ orthogonal states of the state $|\Psi_2\rangle$ there are $\phi(p-1)$ orthogonal states that have an index integer $l$ coprime to $p-1$. Thus, the probability for all such orthogonal states in the state $|\Psi_2\rangle$ is $\phi(p-1)/(p-1) > \delta/\log \log(p-1)$. The probability is inversely proportional to $\log \log(p-1)$ and hence is high even when the prime $p$ is very large. This is another basis to obtain the real index state $|s\rangle$. If the index integer $l$ is coprime to the integer $(p-1)$, there is a modular multiplication unitary operator $U_{l^{-1}} = U_{l,(p-1)}^+$ such that
generate the real index state $|s\rangle$ from the state $|ls\mod(p-1)\rangle$. Indeed, the unitary operation $U_{l^{-1}}$ can generate the real index state $|s\rangle$ from the state $|ls\mod(p-1)\rangle$. But the unitary operation $U_{l^{-1}}$ does depend on the integer $l$, then it is clear that for the case $l \neq l'$ the unitary operation $U_{l^{-1}}$ does not generate the index state $|s\rangle$ from the state $|l's\mod(p-1)\rangle$, that is, $U_{l^{-1}}|l's\mod(p-1)\rangle \neq |s\mod(p-1)\rangle$ if $l' \neq l$. Since all the index integers $l$ in the $(p-1)$ orthogonal states of the state $|\Psi_2\rangle$ are different it is impossible to use a single unitary operation $U_{l^{-1}}$ to generate the real index state $|s\rangle$ from these orthogonal states even if the index integer $l$ for each of these states is coprime to $(p-1)$. In order to generate the real index state $|s\rangle$ from the state $|\Psi_2\rangle$ the unitary transformation $U_s$ should be independent of any index integer $l$. The conventional Euclidean algorithm [19] could be used to construct the unitary transformation $U_s$. Suppose that the greatest common divisor for the two integers $l$ and $(p-1)$ is $d_l$, i.e., $(l, p-1) = d_l$. The Euclidean algorithm can find efficiently two integers $a_l$ and $b_l$ such that the greatest common divisor $d_l = (l, p-1) = a_l + b_l(p-1)$. Then $a_l = d_l \mod(p-1)$. If $d_l = 1$ then $a_l = 1 \mod(p-1)$ and hence $a_l$ is the inverse element of the integer $l$ ($\mod(p-1)$). Using the Euclidean algorithm the following unitary transformations can be obtained,

$$
M_{p-1}^{(2,3,4)} \cdot |l\rangle|a_l\rangle|ls\mod(p-1)\rangle|0\rangle \xrightarrow{GCD} |l\rangle|a_l\rangle|ls\mod(p-1)\rangle|0\rangle
$$

$$
= \ x |l\rangle|a_l\rangle|ls\mod(p-1)\rangle|d_l\mod(p-1)\rangle
$$

$$
(\xrightarrow{GCD}^+ ) |l\rangle|0\rangle|ls\mod(p-1)\rangle|d_l\mod(p-1)\rangle
$$

$$
= \left\{ \begin{array}{ll}
|l\rangle|0\rangle|ls\mod(p-1)\rangle|s\rangle, & \text{if } d_l = 1. \\
|l\rangle|0\rangle|ls\mod(p-1)\rangle|d_l\mod(p-1)\rangle, & \text{if } d_l > 1.
\end{array} \right.
$$

Here the Euclidean algorithm $GCD$ must be performed in a quantum parallel form. This unitary transformation could be used to build up efficiently the unitary transformation $U_s$ as the classical Euclidean algorithm can be implemented in polynomial time $\sim O(\log^3 p)$. A quantum-version extended Euclidean algorithm was given in Ref. [30b]. Another algorithm that may be used to build up the unitary transformation $U_s$ is based on the Euler theorem in number theory [19]. The Euler theorem (the Theorem 72 in Reference [19]) states that if $(a, m) = 1$, then $a^{\phi(m)} = 1 \mod m$. Thus, there holds $l^{\phi(p-1)} = 1 \mod(p-1)$ for any integer $l$ coprime to $(p-1)$, i.e., $(l, p-1) = 1$. 

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But if \((l, p-1) \neq 1\), the identity \(l^{\phi(p-1)} = 1 \text{ mod}(p-1)\) generally does not hold. Since the computation for the modular exponentiation \(l^{\phi(p-1)} \text{ mod}(p-1)\) is simpler and efficient, it could be more convenient to use the modular exponentiation operation to build up the unitary transformation \(U_s\). When the state \(|\Psi_2\rangle\) is acted on by the conditional modular exponentiation unitary operation \(U^{c}_{\phi(p-1)-1,p-1}\) it will be converted into the state \(|\Psi_3\rangle\),

\[
|\Psi_2\rangle^{U^{c}_{\phi(p-1)-1,p-1}} |\Psi_3\rangle = |R0\rangle \bigotimes \frac{1}{p-1} \sum_{l=0}^{p-2} |l\rangle |ls \text{ mod}(p-1)\rangle \bigotimes |l^{\phi(p-1)} s \text{ mod}(p-1)\rangle |f(l, s)\rangle |g^{s \text{ mod} p}\rangle
\]

where the modular exponential function \(l^{\phi(p-1)-1} \text{ mod}(p-1)\) is first computed by the conditional modular exponentiation operation \(U^{c}_{\phi(p-1)-1,p-1}\) in a quantum parallel form by using the integer \(l\) in the first register and then is put in a temporary register, then the function \(l^{\phi(p-1)-1} \text{ mod}(p-1)\) and the function \(ls \text{ mod}(p-1)\) in the second register are multiplied with one another and the result is put in the third register, and after these operations those states in the temporary registers are removed unitarily. The state \(|\Psi_3\rangle\) is written as \(|\Psi_3\rangle = |\Psi_{3s}\rangle + |\Psi_{3s'}\rangle\) and the two orthogonal states \(|\Psi_{3s}\rangle\) and \(|\Psi_{3s'}\rangle\) are given respectively by

\[
|\Psi_{3s}\rangle = |R0\rangle \bigotimes \frac{1}{p-1} \sum_{(l, p-1) = 1}^{p-2} |l\rangle |ls \text{ mod}(p-1)\rangle \bigotimes |s\rangle |f(l, s)\rangle |g^{s \text{ mod} p}\rangle,
\]

\[
|\Psi_{3s'}\rangle = |R0\rangle \bigotimes \frac{1}{p-1} \sum_{(l, p-1) > 1}^{p-2} |l\rangle |ls \text{ mod}(p-1)\rangle \bigotimes |s'\rangle |f(l, s)\rangle |g^{s \text{ mod} p}\rangle,
\]

where the sum with symbol \((l, p-1) = 1\) means that the index \(l\) takes those integers less than and coprime to the integer \((p-1)\) and the sum with \((l, p-1) > 1\) for the index \(l\) runs over those integers less than and not coprime to the integer \((p-1)\), the index \(s' = l^{\phi(p-1)} s \text{ mod}(p-1)\) for \((l, p-1) > 1\) (this also includes \(l = 0\)) and the index \(s = l^{\phi(p-1)} s \text{ mod}(p-1)\) by the Euler theorem that \(l^{\phi(p-1)} = 1 \text{ mod}(p-1)\) if \(l\) is coprime to \(p-1\). Generally, the index \(s' \neq s\). It is known that the computational complexity for the
modular exponentiation operation is $\sim O(\log^3 p)$ and hence the conditional modular exponentiation unitary operation $U^c_{\phi(p-1)-1,p-1}$ may be implemented in polynomial time $\sim O(\log^3 p)$. Now there are the desired state $|\Psi_{3s}\rangle$ which contains the real index state $|s\rangle$ and the undesired state $|\Psi_{3s'}\rangle$ which does not have the index state $|s\rangle$ in the state $|\Psi_3\rangle$. Obviously, the probability for the desired state $|\Psi_{3s}\rangle$ in the state $|\Psi_3\rangle$ is $\phi(p-1)/(p-1)$ and hence the probability for the real index state $|s\rangle$ in the state $|\Psi_3\rangle$ is $\phi(p-1)/(p-1) > \delta/\log \log(p-1)$. It is necessary to remove unitarily the undesired state $|\Psi_{3s'}\rangle$ from the state $|\Psi_3\rangle$ or to convert it into the desired state $|\Psi_{3s}\rangle$ by a unitary transformation so that the real index state $|s\rangle$ can be obtained from the desired state $|\Psi_{3s}\rangle$ in a high probability ($\sim 1$).

Here gives a simple method to convert unitarily the whole state $|\Psi_3\rangle$ into the desired state $|\Psi_{3s}\rangle$. This method is similar to the amplitude amplification method [6, 30a]. It uses simply two unitary operations, one is the inversion operation for the desired state $\Psi_{3s}$, 

$$U(|\Psi_{3s}\rangle) = \exp\{-i\pi(|\Psi_{3s}\rangle\langle\Psi_{3s}|)\}$$

and another is simply taken as 

$$U(|\Psi_3\rangle) = \exp\{-i\pi(|\Psi_3\rangle\langle\Psi_3|)\} = \exp\{-i\pi(|\Psi_{3s}\rangle + |\Psi_{3s'}\rangle)(\langle\Psi_{3s}| + \langle\Psi_{3s'}|)\}.$$ 

Firstly, the inversion for the state $|\Psi_{3s}\rangle$ can be achieved efficiently. Because $g$ is a primitive root (mod $p$), it has the inverse element $g^{-1} = g^{p-2} \mod p$ such that $g^{-1}g = 1 \mod p$. Then by making the conditional cyclic group operation $U^c_{g^{-1}}$ one obtains the following state transformation:

$$U^c_{g^{-1}}|s'\rangle|g^s \mod p\rangle|0\rangle = |s'\rangle|g^s \mod p\rangle|g^{-s+s'} \mod p\rangle, s, s' \in Z_{p-1}.$$ 

Here the operation result is put in the last register. Therefore, there holds the unitary transformation:

$$U^c_{g^{-1}}|s'\rangle|g^s \mod p\rangle|0\rangle = \begin{cases} |s\rangle|g^s \mod p\rangle|1\rangle, & \text{if } s' = s \\ |s'\rangle|g^s \mod p\rangle|g^{-s+s'} \mod p\rangle, & \text{if } s' \neq s \end{cases}$$

Because the state $|1\rangle$ is orthogonal to these states $|g^{-s+s'} \mod p\rangle$ for any indices $s' \neq s$, one can make the selective inversion operation $C_1(\pi) = \exp(-i\pi D_1)$ to invert the state $|1\rangle$, while leaving these states $|g^{-s+s'} \mod p\rangle$
with \( s' \neq s \) unchanged. If now the conditional cyclic group operation \( U_{g^{-1}}^c \) acts on the state \( |\Psi_3\rangle \), then only the desired state \( |\Psi_{3s}\rangle \) generates the state \( |1\rangle \) because it contains the index state \( |s\rangle \), while the state \( |\Psi_{3s'}\rangle \) produces the states \( |g^{-s'+s} \mod p\rangle \) with \( s' \neq s \). After the unitary operation \( U_{g^{-1}}^c \) the selective inversion operation \( C_1(\pi) \) is applied to the register whose state is either \( |1\rangle \) or \( |g^{-s'+s} \mod p\rangle \), then only the state \( U_{g^{-1}}^c |\Psi_{3s}\rangle \) is inverted, while the state \( U_{g^{-1}}^c |\Psi_{3s'}\rangle \) keeps unchanged. After the selective inversion operation \( C_1(\pi) \) the states \( U_{g^{-1}}^c |\Psi_{3s}\rangle \) and \( U_{g^{-1}}^c |\Psi_{3s'}\rangle \) are returned to the states \( |\Psi_{3s}\rangle \) and \( |\Psi_{3s'}\rangle \), respectively, by applying the inverse unitary operation \( (U_{g^{-1}}^c)^{-1} \). The inversion for the state \( |\Psi_{3s}\rangle \) therefore is achieved, while the state \( |\Psi_{3s'}\rangle \) keeps unchanged. Another unitary operation \( U(|\Psi_3\rangle) \) is generated from the oracle unitary operation: \( U_{os}(\theta) = \exp \{-i\theta(|\mathbf{R0}\rangle|g^s \mod p\rangle \langle g^s \mod p| \langle \mathbf{R0}|)\} \) with \( \theta = \pi \). It is shown above that the state \( |\mathbf{R0}\rangle|g^s \mod p\rangle \) can be efficiently converted into the state \( |\Psi_{3}\rangle = |\Psi_{3s}\rangle + |\Psi_{3s'}\rangle \) by a sequence of unitary operations which may be simply denoted as \( U_T(|\Psi_3\rangle) \). Then \( |\mathbf{R0}\rangle|g^s \mod p\rangle \rightarrow U_T(|\Psi_3\rangle) |\Psi_{3}\rangle \) and the unitary operation \( U(|\Psi_3\rangle) \) can be expressed as \( U(|\Psi_3\rangle) = U_T(|\Psi_3\rangle) U_{os}(\pi) U_T^+(|\Psi_3\rangle) \). The unitary operation sequence that converts the state \( |\Psi_3\rangle \) into the desired state \( |\Psi_{3s}\rangle \) then is given simply by

\[
R(m) = [U(|\Psi_3\rangle)C(|\Psi_{3s}\rangle)]^m,
\]

where the iterative number \( m \) takes \( \sim O(\sqrt{\log \log (p - 1)}) \) so that the state \( |\Psi_3\rangle \) is converted in a high probability (\( \sim 1 \)) into the desired state \( |\Psi_{3s}\rangle \); this is because the probability for the desired state \( |\Psi_{3s}\rangle \) in the state \( |\Psi_{3}\rangle \) is \( \phi(p - 1)/(p - 1) > \delta/\log \log (p - 1) \). Thus, under the unitary operation sequence \( R(m) \) the state \( |\Psi_3\rangle \) is converted completely into the desired state \( |\Psi_{3s}\rangle \),

\[
|\Psi_3\rangle \overset{R(m)}{\longrightarrow} |\Psi_{3s}\rangle = |\mathbf{R0}\rangle \bigotimes_{p - 1} C \sum_{l,(p - 1) = 1}^{p - 2} |l\rangle|ls \mod (p - 1)\rangle
\bigotimes |s\rangle |f(l,s,l)|g^s \mod p\rangle,
\]

where \( C \) is a normalization constant, \( C = \sqrt{(p - 1)/\phi(p - 1)} \). Now all the orthogonal states in the state \( |\Psi_{3s}\rangle \) have the index state \( |s\rangle \). The state \( |ls \mod (p - 1)\rangle \) in the second register in the state \( |\Psi_{3s}\rangle \) can be removed unitarily by making an inverse multiplication operation \( M_{p - 1}^{+}(1,3,2) \) on the state \( |\Psi_{3s}\rangle \). After the index state \( |s\rangle \) in the third register in the state \( |\Psi_{3s}\rangle \)
is moved to the last register, in which the index state $|s\rangle$ will be kept to the end, the state $|\Psi_{3s}\rangle$ is changed to the state $|\Psi_{4s}\rangle$:

$$|\Psi_{3s}\rangle \xrightarrow{M_{p-1}^{\pm}(1,3,2)SWAP} |\Psi_{4s}\rangle$$

$$= |\text{R0}\rangle \bigotimes \frac{C}{p-1} \sum_{l,(l,p-1)=1}^{p-2} |l\rangle |\widetilde{f}(l,s,l)\rangle |g^{s \bmod p}\rangle |s\rangle.$$

By inserting the inverse Fourier transform state $|\widetilde{f}(l,s,l)\rangle$ (7) the state $|\Psi_{4s}\rangle$ can be rewritten as

$$|\Psi_{4s}\rangle = |\text{R0}\rangle \bigotimes \frac{C}{p-1} \sum_{x_1=0}^{p-2} \sum_{x_2=0}^{p-2} \exp[i2\pi l(sx_1 + x_2)/(p-1)]$$

$$\times |l\rangle |f(x_1, x_2)\rangle |g^s \bmod p\rangle |s\rangle.$$

Since the functional state $|f(x_1, x_2)\rangle = |g^{sx_1 + x_2} \bmod p\rangle = |f_1(sx_1 + x_2)\rangle$, there are only $p-1$ functional states $|f(x_1, x_2)\rangle$ to be independent. However, there are $(p-1) \times (p-1)$ functional states $|f(x_1, x_2)\rangle$ in the state $|\Psi_{4s}\rangle$, then not all these $(p-1) \times (p-1)$ functional states are independent. Actually, the state $|\Psi_{4s}\rangle$ can be reduced to the simple form $|\Psi_{5s}\rangle$:

$$|\Psi_{5s}\rangle = |\text{R0}\rangle \bigotimes \frac{C}{p-1} \sum_{z=0}^{p-2} \sum_{l,(l,p-1)=1} \exp[i2\pi l z/(p-1)]$$

$$\times |l\rangle |f_1(z)\rangle |g^s \bmod p\rangle |s\rangle.$$

Why can the state $|\Psi_{4s}\rangle$ be written as the simple form $|\Psi_{5s}\rangle$? There are totally $(p-1) \times (p-1)$ different index pairs $(x_1, x_2)$ in the state $|\Psi_{4s}\rangle$ since the indices $x_1, x_2 = 0, 1, ..., p-2$. Now for each given $z = (sx_1 + x_2) \bmod (p-1)$ for $z = 0, 1, ..., p-2$ there are $(p-1)$ different index pairs $(x_1, x_2)$ to satisfy the same equation $z = (sx_1 + x_2) \bmod (p-1)$, while for all these $(p-1)$ pairs of indices $(x_1, x_2)$ the functional states $|f(x_1, x_2)\rangle$ take the same one: $|f_1(z)\rangle$ and the phase factor $\exp[i2\pi l (sx_1 + x_2)/(p-1)]$ also are the same as $\exp[i2\pi l z/(p-1)]$.

These $(p-1)$ different index pairs $(x_1, x_2)$ that fulfill the same equation: $z = (sx_1 + x_2) \bmod (p-1)$ may be taken as $(x_1, (z - sx_1) \bmod (p-1))$ for $x_1 = 0, 1, ..., p-2$. Thus, taking $x_1 = 0, 1, ..., p-2$ and $z = 0, 1, ..., p-2$ generates just all possible $(p-1) \times (p-1)$ different index pairs $(x_1, x_2)$. Then in the state $|\Psi_{4s}\rangle$ the sums over the indices $x_1$ and $x_2$ may be carried out
in such a way that the sum for the index $x_1$ is first to run over the $(p-1)$ different index pairs $(x_1, (z - sx_1) \mod (p-1))$ for $x_1 = 0, 1, \ldots, p-2$ and for any given $z = (sx_1 + x_2) \mod (p-1)$, this sum will generate a factor of $(p-1)$ as the same functional states $|f(x_1, x_2)\rangle$ and the same phase factors $\exp[i 2\pi l (sx_1 + x_2)/(p-1)]$ in the state $|\Psi_{4s}\rangle$ are taken for these $(p-1)$ index pairs, then the sum for the index $z$ is carried out for $z = 0, 1, \ldots, p-2$, and hence the state $|\Psi_{4s}\rangle$ can be written as the simple state $|\Psi_{5s}\rangle$. Now one can also understand why the Fourier transform state $|\tilde{f}_1(l)\rangle = |\tilde{f}(ls,l)\rangle/\sqrt{p-1}$ for $l = 0, 1, \ldots, p-2$ (see before).

Now observe the state $|\Psi'_{5s}\rangle$ and a series of unitary transformations:

$$|\Psi'_{5s}\rangle = |R0\rangle \bigotimes \frac{1}{p-1} \sum_{z=0}^{p-2} \sum_{l=0}^{p-2} \exp[i 2\pi l z/(p-1)]$$

$$\times |l\rangle|f_1(z)\rangle|g^{s \mod p}\rangle|s\rangle$$

$$Q_{(p-1)^{FT}}^{+} |R0\rangle \bigotimes \frac{1}{\sqrt{p-1}} \sum_{z=0}^{p-2} |z\rangle|g^z \mod p\rangle|g^{s \mod p}\rangle|s\rangle$$

$$(U_{g,p}^c)^{+} |R0\rangle \bigotimes \frac{1}{\sqrt{p-1}} \sum_{z=0}^{p-2} |z\rangle|g^s \mod p\rangle|s\rangle$$

$$Q_{(p-1)^{FT}}^{+} |R0\rangle \bigotimes |g^s \mod p\rangle|s\rangle.$$
where the sum for the index \( l \) runs over only those integers less than and coprime to the integer \((p - 1)\). Obviously, the trigonometrical sum \( h(z, z) = \sqrt{\phi(p - 1)} \) if the index \( z' = z \), for the number of the integers less than and coprime to the integer \((p - 1)\) is \( \phi(p - 1) \). Then the state \(|\Psi_{6s}\rangle\) can be rewritten as the sum of the two terms:

\[
|\Psi_{6s}\rangle = |\mathbf{R0}\rangle \bigotimes \frac{\sqrt{\phi(p - 1)}}{p - 1} \sum_{z=0}^{p-2} |z\rangle |g^z \text{ mod } p\rangle |g^s \text{ mod } p\rangle |s\rangle \\
+|\mathbf{R0}\rangle \bigotimes \frac{1}{p - 1} \sum_{z\neq z', z, z' = 0}^{p-2} h(z, z') |z'\rangle |g^{z'} \text{ mod } p\rangle |g^s \text{ mod } p\rangle |s\rangle.
\]

By making the inverse modular exponentiation operation \((U_{g,p}^c)^+\) on the first two registers the state \(|\Psi_{6s}\rangle\) is transferred to the state \(|\Psi_{7s}\rangle\):

\[
|\Psi_{7s}\rangle = |\mathbf{R0}\rangle \bigotimes \frac{\sqrt{\phi(p - 1)}}{p - 1} \sum_{z=0}^{p-2} |z\rangle |1\rangle |g^s \text{ mod } p\rangle |s\rangle \\
+|\mathbf{R0}\rangle \bigotimes \frac{1}{p - 1} \sum_{z\neq z', z, z' = 0}^{p-2} h(z, z') |z'\rangle |g^{z-z'} \text{ mod } p\rangle |g^s \text{ mod } p\rangle |s\rangle.
\]

Since the index \( z' \neq z \), the state \(|g^{z-z'} \text{ mod } p\rangle \neq |1\rangle\) and hence the two terms in the state \(|\Psi_{7s}\rangle\) are orthogonal to one another. Evidently, the first term in the state \(|\Psi_{7s}\rangle\) has a total probability \( \phi(p - 1)/(p - 1) \) which is greater than \( \delta/\log \log(p - 1) \) for some constant \( \delta \). Again using the amplitude amplification method the second term in the state \(|\Psi_{7s}\rangle\) can be converted into the first term in a high probability \((\sim 1)\) and the iterative number in the amplitude amplification process to achieve this complete state conversion needs \( \sim O(\sqrt{\log \log(p - 1)}) \). This time the selective inversion operation is applied to the state \(|1\rangle\) in the second register in the state \(|\Psi_{7s}\rangle\) and another unitary operation for the amplitude amplification process is just the unitary operator \( \exp\{-i\pi(|\Psi_{7s}\rangle\langle\Psi_{7s}|)\} \) which can be also built up efficiently because the state \(|\Psi_{7s}\rangle\) itself can be generated efficiently from the initial state \(|\Psi_0\rangle\), as shown in the state-transfer process above. After the state \(|\Psi_{7s}\rangle\) is changed to its first term completely, an inverse Fourier transform on the state \(|z\rangle\) in the first register and the state transfer \( F_1^+ : |1\rangle \rightarrow |0\rangle \) in the second register change the first term to the desired state ultimately,

\[
|\Psi_{7s}\rangle \rightarrow |\mathbf{R0}\rangle \bigotimes \frac{1}{\sqrt{p - 1}} \sum_{z=0}^{p-2} |z\rangle |1\rangle |g^s \text{ mod } p\rangle |s\rangle.
\]
\[
Q_{(p-1)FTF}^+ \rightarrow |R0\rangle \bigotimes |g^s \text{mod} p\rangle |s\rangle.
\]

Obviously, the whole unitary transformation process above really performs a unitary transformation that firstly converts the starting state \(|\Psi_0\rangle = |R0\rangle \bigotimes |g^s \text{mod} p\rangle |0\rangle\) to the state \(|\Psi_3\rangle\), then to the state \(|\Psi_{ts}\rangle\), and finally to the desired state \(|R0\rangle \bigotimes |g^s \text{mod} p\rangle |s\rangle\). Evidently, this is an efficient unitary transformation process. This unitary operation sequence is just the unitary operation \(V_{f-1}\) of the inversion function of the modular exponential function \(f(s) = g^s \text{mod} p\) if the register library \(|R0\rangle\) is dropped. Once the inversion-functional unitary operation \(V_{f-1}\) is obtained the unitary operation \(U_{log}(g)\) of the discrete logarithmic function \(s = \log_g f(s)\) can be set up by \(U_{log}(g) = V_f^{-1} SV_{f-1}\).

If the starting state is a superposition, \(|\Psi_0\rangle = \sum_s \alpha_s |R0\rangle \bigotimes |g^s \text{mod} p\rangle\), then it is required that the unitary operator \(U(|\Psi_0\rangle) = \exp\{-i\theta(|\Psi_0\rangle\langle\Psi_0|)\}\) be efficiently constructed so that the unitary operation \(U(|\Psi_3\rangle)\), etc., can be efficiently built up with the unitary operator \(U(|\Psi_0\rangle)\). In this case the superposition \(|\Psi_0\rangle\) can be efficiently converted into the superposition \(|\Psi_f\rangle = \sum_s \alpha_s |R0\rangle \bigotimes |g^s \text{mod} p\rangle |s\rangle\). For the quantum discrete logarithmic problem the integer \(b = g^s \text{mod} p\) is given beforehand and hence the oracle unitary operation \(U_{os}(\theta) = \exp[-i\theta D_s(g)]\) can be constructed efficiently in advance. Note that here the data \(b\) is used to prepare the unitary operation instead of a quantum state. Then using the above unitary operation sequence \(V_{f-1}\) the initial known state \(|\Psi_0\rangle = |R0\rangle \bigotimes |g^s \text{mod} p\rangle\) can be efficiently converted into the state \(|R0\rangle \bigotimes |g^s \text{mod} p\rangle |s\rangle\). Furthermore, by using directly the discrete logarithmic unitary operation \(U_{log}(g)\) the initial known state \(|\Psi_0\rangle\) can be efficiently transferred to the index state \(|R0\rangle \bigotimes |s\rangle\) and a quantum measurement on the index state will output directly the complete information of the index \(s\) of the integer \(b = g^s \text{mod} p\).

4. The efficient state transformations among the cyclic group state subspaces

Once it is obtained the unitary operation \(U_{log}(g)\) of the discrete logarithmic function \(x = \log_g f(x)\) with \(f(x) = g^x \text{mod} p\), one may further use it to prepare some useful auxiliary oracle unitary operations \(U_{os'}(\theta)\) where the index \(s' \neq s\) generally and the index \(s\) is of the oracle unitary operation \(U_{os}(\theta) = \exp[-i\theta D_s(g)]\). The process to generate the auxiliary oracle unitary operation \(U_{os'}(\theta)\) with index \(s' = js\) from the basic oracle unitary operation
$U_{os}(\theta)$ is related to the state unitary transformation $V_{js}$:

$$|R0\rangle \otimes |g^s \bmod p\rangle \xrightarrow{V_{js}} |R0\rangle \otimes |g^{js} \bmod p\rangle.$$ 

In the classical irreversible computation the modular exponential function $g^{js} \bmod p$ can be efficiently computed for any given integers $j$ and $b = g^s \bmod p$ [21], but it may not be so easy in the quantum search problem to generate unitarily the state $|g^{js} \bmod p\rangle$ from the state $|g^s \bmod p\rangle$ for any given integer $j$. If the integer $j$ is coprime to the integer $(p - 1)$, then there is an efficient unitary transformation such that $U_{j,p-1}(\alpha)|s\rangle = |js \bmod (p - 1)\rangle$ and the unitary transformation $V_{js}$ therefore can be achieved efficiently with the help of the unitary operation $U_{\log}(g)$ of the discrete logarithm. Hence the auxiliary oracle unitary operation $U_{ojs}(\theta)$ can be efficiently generated from the oracle unitary operation $U_{os}(\theta)$. However, in order to simplify the quantum search problem in the cyclic group state space one had better convert the marked state into a small subspace of the cyclic group state space. Then the auxiliary oracle unitary operation $U_{ojs}(\theta)$ usually is specific one and the integer $j$ takes only some specific positive integer values that are usually not coprime to the integer $(p - 1)$. How can such an auxiliary oracle unitary operation $U_{ojs}(\theta)$ be constructed from the oracle unitary operation $U_{os}(\theta)$?

Evidently, the following unitary transformations can be achieved efficiently for any integer $j$:

$$|R0\rangle \otimes |g^s \bmod p\rangle \xrightarrow{U_{\log}(g)} |R0\rangle \otimes |s\rangle \xrightarrow{F_j} |R0\rangle \otimes |j\rangle |s\rangle$$

where the unitary transformation $F_j|0\rangle = |j\rangle$ for any known integer $j$ can be built up efficiently. If the integer $j$ is coprime to $p - 1$, then a further unitary transformation sequence can be made:

$$|R0\rangle \otimes |j\rangle |s\rangle |js \bmod (p - 1)\rangle |0\rangle \xrightarrow{U_{\phi(p-1)-1,p-1}^{(1,3,4)}} |R0\rangle \otimes |j\rangle |s\rangle |js \bmod (p - 1)\rangle |s\rangle \xrightarrow{COPY(4,2)} |R0\rangle \otimes |j\rangle |js \bmod (p - 1)\rangle |s\rangle \xrightarrow{U_{\phi(p-1)-1,p-1}^{(1,2,3)+}} |R0\rangle \otimes |j\rangle |js \bmod (p - 1)\rangle$$

$$\xrightarrow{F_j^+} |R0\rangle \otimes |js \bmod (p - 1)\rangle \xrightarrow{U_{\log}(g)} |R0\rangle \otimes |g^{js} \bmod p\rangle.$$
Therefore, the state unitary transformation $V_{js}$ can be achieved too by a more complicated way. However, from these detailed state unitary transformations one may see more clearly why the state unitary transformation $V_{js}$ is not easy to be constructed if the integer $j$ is not coprime to the integer $(p-1)$.

If the integer $j$ is not coprime to the integer $(p-1)$, that is, $(j,p-1) > 1$, then situation becomes much more complicated. Firstly, the state transformation $|j⟩|js \mod (p-1)⟩⟨0| → |j⟩|js \mod (p-1)⟩⟨s|$ for any index $s \in Z_{p-1}$ usually could not be unitary. This is related to the problem whether there exists a unique inversion function of the function $f(x) = jx \mod (p-1)$ or not for any index variable $x \in Z_{p-1}$. Since the function $f(x)$ may not be a one-to-one function corresponding to its variable $x \in Z_{p-1}$ if the integer $j$ is not coprime to $(p-1)$, the inversion-functional operation $f(x)^{-1}$ therefore may not be unitary in the variable value range: $x \in Z_{p-1}$. In the same argument the state transformation $|g^{js} \mod p⟩|0| → |g^{js} \mod p⟩|s|$ for any $s \in Z_{p-1}$ usually could not be unitary if $(j,p-1) > 1$, although the state transformation $|g^{js} \mod p⟩|0| → |g^{js} \mod p⟩|js \mod (p-1)|$ is unitary. More generally, there could not be a single unitary transformation for any integer $j \in Z_{p-1}$ such that $|j⟩|js \mod (p-1)⟩⟨0| → |j⟩|js \mod (p-1)⟩⟨s|$ for any given index $s$, as shown in section 3. These may be best understood with the knowledge of number theory [19]. Suppose that one is given a set of the integers $j = a_k$ and $js \mod (p-1) = b_k$ for $k = 1, 2, ..., r$ to reproduce uniquely the index $s$. This problem is really equivalent to solving the congruence system:

$$a_kx = b_k \mod (p-1), \quad k = 1, 2, ..., r,$$

where the integers $\{a_k\}$ may not be coprime to $p-1$ and evidently $x = s$ is a solution to the congruence system. First consider a single congruence, for example, the $k-$th congruence: $a_kx \mod (p-1) = b_k$. Denote that the greatest common divisor between $a_k$ and $p-1$ is $d_k = (a_k, p-1)$. Then the single $k-$th congruence has exactly $d_k$ solutions [19, 20] as $d_k$ is a divisor of the integer $b_k = a_k s \mod (p-1)$ (i.e. $d_k | b_k$) for $k = 1, 2, ..., r$. If now the integer $a_k$ is not coprime to the integer $(p-1)$, that is, $d_k > 1$, then there are $d_k$ different index values $s$ to satisfy the same $k-$th congruence, indicating that there is not a single unitary state transformation: $|a_k⟩|a_k s \mod (p-1)⟩⟨0| → |a_k⟩|a_k s \mod (p-1)⟩⟨s|$ for any index $s \in Z_{p-1}$.

Now consider the whole congruence system (9). Obviously, the congruence system is solvable. Note that $d_k$ divides the integers $(p-1), a_k,$ and
Denote the integer \( m_k = (p - 1)/d_k, \tilde{a}_k = a_k/d_k, \) and \( \tilde{b}_k = b_k/d_k \equiv \tilde{a}_k s \mod m_k. \) Then the Theorem 57 in reference [19] shows that the congruence system is equivalent to the simpler one: \( \tilde{a}_k x \equiv \tilde{b}_k \mod m_k, k = 1, 2, ..., r. \) Since \( (\tilde{a}_k, m_k) = 1 \) there exists an inverse element \( h_k \) of \( \tilde{a}_k \) such that \( h_k \tilde{a}_k = 1 \mod m_k, \) the congruence system \( \tilde{a}_k x = \tilde{b}_k \mod m_k, k = 1, 2, ..., r, \) then can be further reduced to the standard one:

\[
x = c_k \mod m_k, \quad k = 1, 2, ..., r, \tag{10}
\]

where the coefficients \( c_k = h_k \tilde{b}_k. \) Now the Chinese remainder theorem [19, 20] shows that if \( m_1, m_2, ..., m_r \) are coprime in pair, i.e., \( (m_i, m_j) = 1 \) for \( 1 \leq i < j \leq r, \) then the standard congruence system (10) has a unique solution (mod \( m \)),

\[
x = (n_1 M_1 c_1 + n_2 M_2 c_2 + ... + n_r M_r c_r) \mod m, \tag{11}
\]

where \( m = m_1 m_2 ... m_r = m_1 M_1 = m_2 M_2 = ... = m_r M_r \) and the inverse element \( n_k \) of \( M_k \) (mod \( m_k \)) satisfies \( n_k M_k = 1 \mod m_k \) for \( k = 1, 2, ..., r \) because \( (m_k, M_k) = 1. \) Hence using the efficient Euclidean algorithm [19] the integer \( n_k \) is determined from the known integers \( M_k \) and \( m_k \) for \( k = 1, 2, ..., r. \) The solution \( x \) of equation (11) is really the index \( s \) if the index \( s \) is bounded on by \( 0 \leq s < m \) because the solution \( x \) is unique (mod \( m \)). However, the index \( s \) really belongs to the integer set \( Z_{p-1} = \{ 0, 1, ..., p-2 \}. \)

Then the solution \( x \) could not be the real index \( s \) if \( m < p - 1, \) for example, it could occur that \( s = x + m \) for \( 0 \leq s < p - 1. \) In order that the solution \( x \) of equation (11) is exactly the real index \( s \) the integer \( m \) should be equal to or greater than \( (p - 1). \) In fact, it is better to take the integer \( m \) exactly as the integer \( p - 1, \) that is, \( m = p - 1, \) as the situation is related closely to the prime factorization of the integer \( p - 1 \) and the structure of the cyclic group \( S(C_{p-1}), \) as shown in section 2. Now consider this specific case that the integer \( m = (p - 1) \) and \( (p - 1) \) has the prime factorization: \( (p - 1) = p_1^{a_1} p_2^{a_2} ... p_r^{a_r} \) (\( p_k \) are distinct primes). Take \( a_k = (p - 1)/p_k^{a_k} = M_k \) and \( b_k = a_k s \mod (p - 1) = M_k s \mod (p - 1). \) Thus, \( d_k = a_k, p - 1 = M_k \) and \( m_k = (p - 1)/d_k = p_k^{a_k}. \) Then \( \tilde{a}_k = 1 \) and \( \tilde{b}_k = s \mod m_k. \) Moreover, \( (p - 1) = m_1 m_2 ... m_r = m_i M_i, \) and \( (m_i, M_i) = 1, \) \( (m_i, M_i) = 1, \) for \( 1 \leq i < j \leq r. \) Clearly, \( h_k = 1 \) and \( c_k = s \mod m_k. \) Thus, the solution (11) is further reduced to the form

\[
x = (n_1 M_1 c_1 + n_2 M_2 c_2 + ... + n_r M_r c_r) \mod (p - 1). \tag{12}
\]
Now the solution \( x \) of equation (12) is just the real index \( s \) and the vector \( \{c_k\} \) is just the index vector \( \{s_k\} \) in the equation (3) in section 2. Actually, in comparison with the equation (3) in section 2 one now sees that the equation (12) is just the equation (3), showing once again that this solution \( x \) is just the real index \( s \). Therefore, the Chinese remainder theorem [19, 20] ensures that any index state \(|s⟩\) with \( 0 \leq s < p - 1 \) can be exactly expressed as

\[
|s⟩ \equiv |(n_1 M_1 s_1 + n_2 M_2 s_2 + \ldots + n_r M_r s_r) \mod (p - 1)⟩
\]

\[
\equiv |(n_1 M_1 + n_2 M_2 + \ldots + n_r M_r) s \mod (p - 1)⟩, \quad (13)
\]

where the identity \( M_k s_k \equiv M_k s \mod (p - 1) \) has been used for \( k = 1, 2, \ldots, r \).

The index state identity (13) could be helpful to prepare some specific auxiliary oracle unitary operations in the additive cyclic group state space \( S(Z_{p-1}) \). Now it can be shown below that the index state \(|s⟩\) can be converted unitarily into a tensor product of the \( r \) states \(|s \mod m_k⟩\) or \(|M_k s \mod (p - 1)⟩\) for \( k = 1, 2, \ldots, r \) in the \( r \) different registers. Firstly, by simply applying the reversible modular arithmetic operation \( MOD(m_k) \) on the index state \(|s⟩\) one obtains

\[
|\text{R0}⟩ \otimes |s⟩ \xrightarrow{MOD(m_k)} |Φ_0⟩ = |\text{R0}⟩ \otimes |s⟩ \otimes |s \mod m_k⟩.
\]

The reversible modular arithmetic operation can be thought of as a specific reversible modular addition operation. Evidently, the state \(|s \mod m_k⟩\in S(Z_{m_k}) \) and here \( 0 \leq s \mod m_k < m_k \) for \( k = 1, 2, \ldots, r \). Repeating the reversible modular arithmetic operation \( r \) times for \( k = 1, 2, \ldots, r \) one arrives at the state \(|Φ_1⟩\):

\[
|\text{R0}⟩ \otimes |s⟩ \rightarrow |Φ_1⟩ = |\text{R0}⟩ \otimes |s⟩ \otimes |s \mod m_1⟩
\]

\[
\otimes |s \mod m_2⟩ \otimes \ldots \otimes |s \mod m_r⟩.
\]

Here each state \(|s \mod m_k⟩ = |s_k⟩\) occupies one register for \( k = 1, 2, \ldots, r \). Now substituting the state identity (13) for the index state \(|s⟩\) the state \(|Φ_1⟩\) is expressed as

\[
|Φ_1⟩ = |\text{R0}⟩ \otimes |(n_1 M_1 s_1 + n_2 M_2 s_2 + \ldots + n_r M_r s_r) \mod (p - 1)⟩
\]

\[
\otimes |s_1⟩ \otimes |s_2⟩ \otimes \ldots \otimes |s_r⟩.
\]
In order to remove unitarily the composite state $|\sum_k n_k M_k s_k \mod(p-1)\rangle$ in the state $|\Phi_1\rangle$ one needs to perform a series of the modular multiplication operations $M_{p-1}(\alpha, \beta, \gamma)$ and inverse modular addition operations $ADD_{p-1}^+(\alpha, \beta)$ on the state $|\Phi_1\rangle$, for example,

$$|\Phi_1\rangle \xrightarrow{F_{n_1 M_1}} M_{p-1}(\alpha_1, \beta_1, \gamma_1)$$

$$|R0\rangle \otimes |n_1 M_1 \mod(p-1)\rangle |n_1 M_1 s_1 \mod(p-1)\rangle$$

$$\otimes |(n_1 M_1 s_1 + n_2 M_2 s_2 + \ldots + n_r M_r s_r) \mod(p-1)\rangle$$

$$\otimes |s_1 \rangle \otimes |s_2 \rangle \otimes \ldots \otimes |s_r \rangle$$

$$\xrightarrow{ADD_{p-1}^+(\alpha_1, \beta_1)} |R0\rangle \otimes |n_1 M_1 \mod(p-1)\rangle |n_1 M_1 s_1 \mod(p-1)\rangle$$

$$\otimes |(n_2 M_2 s_2 + n_3 M_3 s_3 + \ldots + n_r M_r s_r) \mod(p-1)\rangle$$

$$\otimes |s_1 \rangle \otimes |s_2 \rangle \otimes \ldots \otimes |s_r \rangle$$

$$\xrightarrow{M_{p-1}^+(\alpha_1, \beta_1, \gamma_1)} M_{n_1 M_1}^+ \ xrightarrow{F_{n_1 M_1}}$$

$$|R0\rangle \otimes |(n_2 M_2 s_2 + n_3 M_3 s_3 + \ldots + n_r M_r s_r) \mod(p-1)\rangle$$

$$\otimes |s_1 \rangle \otimes |s_2 \rangle \otimes \ldots \otimes |s_r \rangle.$$

The unitary transformation process in the example is stated below. The state $|n_1 M_1 \mod(p-1)\rangle$ is first created by the unitary transformation: $F_{n_1 M_1} |0\rangle = |n_1 M_1 \mod(p-1)\rangle$, then the modular multiplication operation $M_{p-1}(\alpha_1, \beta_1, \gamma_1)$ acts on both the states $|n_1 M_1 \mod(p-1)\rangle$ and $|s_1\rangle$ to generate the state $|n_1 M_1 s_1 \mod(p-1)\rangle$, and then the modular subtraction operation or the inverse modular addition operation $ADD_{p-1}^+(\alpha_1, \beta_1)$ acts on both the state $|n_1 M_1 s_1 \mod(p-1)\rangle$ and the composite state $|\sum_k n_k M_k s_k \mod(p-1)\rangle$ so that the composite state is changed to the state $|((n_2 M_2 s_2 + n_3 M_3 s_3 + \ldots + n_r M_r s_r) \mod(p-1))\rangle$. After these unitary transformations the unitary operations $M_{p-1}^+(\alpha, \beta, \gamma)$ and $F_{n_1 M_1}^+$ are used to convert the states $|n_1 M_1 \mod(p-1)\rangle$ and $|n_1 M_1 s_1 \mod(p-1)\rangle$ back to the states $|0\rangle$. Clearly, the whole unitary transformation process really cancels the term $n_1 M_1 s_1 \mod(p-1)$ in the composite state $|\sum_k n_k M_k s_k \mod(p-1)\rangle$ of the state $|\Phi_1\rangle$. If this unitary transformation process is repeated $r$ times with different unitary operations $F_{n_k M_k}$, $M_{p-1}(\alpha_k, \beta_k, \gamma_k)$, and $ADD_{p-1}^+(\alpha_k, \beta_k)$ for $k = 1, 2, \ldots, r$, then the composite state $|\sum_k n_k M_k s_k \mod(p-1)\rangle$ is ultimately converted into the state $|0\rangle$ in the state $|\Phi_1\rangle$. Therefore, it is shown that with the help of the
state identity (13) the index state $|\textbf{RO}\rangle \otimes |s\rangle$ can be efficiently converted into the state $|\Phi_2\rangle$

\[ |\textbf{RO}\rangle \otimes |s\rangle \rightarrow |\Phi_2\rangle = |\textbf{RO}\rangle \otimes |s \mod m_1\rangle \]
\[ \otimes |s \mod m_2\rangle \otimes \ldots \otimes |s \mod m_r\rangle. \]

The state $|\Phi_2\rangle$ is a tension product of the $r$ states $\{|s \mod m_k\rangle\}$ in the $r$ different registers. In an analogue way, the index state $|\textbf{RO}\rangle \otimes |s\rangle$ also can be efficiently converted into a tension product of the $r$ states $\{|M_k s \mod (p-1)\rangle\}$ in the $r$ different registers,

\[ |\textbf{RO}\rangle \otimes |s\rangle \rightarrow |\Phi_3\rangle = |\textbf{RO}\rangle \otimes |M_1 s \mod (p-1)\rangle \]
\[ \otimes |M_2 s \mod (p-1)\rangle \otimes \ldots \otimes |M_r s \mod (p-1)\rangle. \]

Since there is the identity $M_k s_k \equiv M_k s \mod (p-1)$ for $k = 1, 2, \ldots, r$, the state $|M_k s \mod (p-1)\rangle = |M_k s_k \mod (p-1)\rangle$. By using the inverse discrete logarithmic unitary operation $U_{\log}^+(g)$ the state $|M_k s_k \mod (p-1)\rangle$ can be converted into the state $|(g^{M_k})^{s_k} \mod p\rangle$ which belongs to the state subspace $S(C_{p_k}^{s_k})$ of the cyclic subgroup $C_{p_k}^{s_k}$. On the other hand, using the inverse discrete logarithmic unitary operation $U_{\log}^+(g^{M_k})$ with the logarithmic base $g^{M_k}$ the state $|s \mod m_k\rangle$, i.e., $|s_k\rangle$, can also be converted to the same state $|(g^{M_k})^{s_k} \mod p\rangle$. These results show that the auxiliary oracle unitary operation $\exp\{-i\theta|(g^{M_k})^{s_k} \mod p\rangle\langle(g^{M_k})^{s_k} \mod p|\}$ of the multiplicative cyclic group state subspace $S(C_{p_k}^{s_k})$ can be efficiently built out of the auxiliary oracle unitary operation $\exp\{-i\theta(|s \mod m_k\rangle\langle s \mod m_k|)\}$ of the additive cyclic group state subspace $S(Z_{m_k})$ or the auxiliary oracle unitary operation $\exp\{-i\theta(|M_k s \mod (p-1)\rangle\langle M_k s \mod (p-1)|)\}$ of the additive cyclic group state space $S(Z_{p-1})$.

Now consider the multiplicative cyclic group state space $S(C_{p-1})$. Suppose that the prime factors of the integer $(p-1) = p_1^{a_1}p_2^{a_2} \ldots p_r^{a_r}$ are ordered in magnitude: $p_1^{a_1} < p_2^{a_2} < \ldots < p_r^{a_r}$ and $p_r^{a_r} \sim O(\log p)$. Then $m_1 < m_2 < \ldots < m_r$ and $M_1 > M_2 > \ldots > M_r$. As shown in section 2, the cyclic group $C_{p-1}$ with order $p-1$ is the direct product of $r$ factor cyclic subgroups: $C_{p-1} = C_{p_1}^{a_1} \times C_{p_2}^{a_2} \times \ldots \times C_{p_r}^{a_r}$. Each such cyclic subgroup $C_{p_k}^{s_k}$ corresponds to a state subspace $S(C_{p_k}^{s_k})$ with dimension $p_k^{a_k}$ of the cyclic group state space $S(C_{p-1})$. For convenience, denote $S(m_k) \equiv S(C_{p_k}^{s_k})$ with
It can be proven that the state $|g^{s_{M_k}} \bmod p\rangle$ is in the state subspace $S(m_k)$ for any index integer $s$. This is because the generator and the order of the cyclic subgroup $C_{p_k}$ is $g^{M_k}$ and $m_k$, respectively, then there holds the state identity $|g^{s_{M_k}} \bmod p\rangle = |(g^{M_k})^{s \bmod m_k} \bmod p\rangle$ for any index $s$, while the latter state $|(g^{M_k})^{s_k} \bmod p\rangle$ with the index $s_k = s \bmod m_k$ is just in the state subspace $S(m_k)$. Since the dimensional size of the cyclic group state subspace $S(m_k)$ is just the order $m_k$ of the subgroup $C_{p_k}$ and $m_1 < m_2 < \ldots < m_r$, then the state $|g^{s_{M_1}} \bmod p\rangle$ is in the smallest state subspace $S(m_1)$, the state $|g^{s_{M_2}} \bmod p\rangle$ in the second smallest subspace $S(m_2)$, ..., and the state $|g^{s_{M_r}} \bmod p\rangle$ in the largest subspace $S(m_r)$ of the $r$ state subspaces $\{S(m_k)\}$. It follows from the equation (2) in section 2 that every state of the cyclic group state space $S(C_{p-1})$ can be expressed as

$$|g^{s} \bmod p\rangle \equiv |(g^{M_1})^{n_{11}} \times (g^{M_2})^{n_{22}} \times \ldots \times (g^{M_r})^{n_{rr}} \bmod p\rangle \quad (14)$$

The state identity (14) plays a similar role to the state identity (13) in decomposing any state of the cyclic group state space $S(C_{p-1})$ as a tensor product of the states of the state subspaces $\{S(m_k)\}$ of the factor cyclic subgroups $\{C_{p_k}\}$. By the modular exponentiation operation the state $|(g^{M_k})^{s} \bmod p\rangle$ of the state subspace $S(m_k)$ can be generated from the cyclic group state $|g^{s} \bmod p\rangle$,

$$|\textbf{R0}\rangle \bigotimes |g^{s} \bmod p\rangle \rightarrow |\Phi_4\rangle = |\textbf{R0}\rangle \bigotimes |g^{s} \bmod p\rangle |(g^{M_k})^{s} \bmod p\rangle$$

$$= |\textbf{R0}\rangle \bigotimes |g^{s} \bmod p\rangle |(g^{M_k})^{s_k} \bmod p\rangle.$$ 

Repeating this modular exponentiation operation $r$ times for $k = 1, 2, \ldots, r$ the state $|\textbf{R0}\rangle \bigotimes |g^{s} \bmod p\rangle$ is converted into the state $|\Phi_5\rangle$,

$$|\textbf{R0}\rangle \bigotimes |g^{s} \bmod p\rangle \rightarrow |\Phi_5\rangle = |\textbf{R0}\rangle \bigotimes |g^{s} \bmod p\rangle \bigotimes |(g^{M_1})^{s_1} \bmod p\rangle$$

$$\bigotimes |(g^{M_2})^{s_2} \bmod p\rangle \bigotimes \ldots \bigotimes |(g^{M_r})^{s_r} \bmod p\rangle.$$ 

By using the state identity (14) and the modular exponentiation, the modular multiplication, and the COPY operation the state $|g^{s} \bmod p\rangle$ in the state $|\Phi_5\rangle$ can be removed unitarily and hence the state $|\textbf{R0}\rangle \bigotimes |g^{s} \bmod p\rangle$ can be efficiently converted into a tensor product of the $r$ states $\{|(g^{M_k})^{s_k} \bmod p\rangle\}$ of the $r$ different subspaces $\{S(m_k)\}$ in the $r$ different registers,

$$|\textbf{R0}\rangle \bigotimes |g^{s} \bmod p\rangle \rightarrow |\Phi_6\rangle = |\textbf{R0}\rangle \bigotimes |(g^{M_1})^{s_1} \bmod p\rangle$$

$$\bigotimes |(g^{M_2})^{s_2} \bmod p\rangle \bigotimes \ldots \bigotimes |(g^{M_r})^{s_r} \bmod p\rangle.$$
This unitary transformation is stated below. The states \( \{|(g^{M_k})^{n_k s_k} \mod p\}\) are first generated efficiently from the states \( \{|(g^{M_k})^{s_k} \mod p\}\) by the modular exponentiation operations in temporary registers in the state \( |\Phi_5\rangle\) because the integers \( \{n_k\}\) are known, as shown before. Then by the modular multiplication operations the state \( |\prod_k(g^{M_k})^{n_k s_k} \mod p\rangle\) is created efficiently from these states \( \{|(g^{M_k})^{n_k s_k} \mod p\}\}. The state identity (14) shows that the state \( |\prod_k(g^{M_k})^{n_k s_k} \mod p\rangle\) is just the state \( |g^s \mod p\rangle\). Then using the COPY operation the state \( |g^s \mod p\rangle\) can be removed from the state \( |\Phi_5\rangle\). After these unitary operations those states \( |\prod_k(g^{M_k})^{n_k s_k} \mod p\rangle\) and \( \{|(g^{M_k})^{n_k s_k} \mod p\}\) in temporary registers are returned back to the state \( |0\rangle\) and therefore the state \( |\Phi_6\rangle\) is obtained. Note that these states \( |(g^{M_j})^s \mod p\rangle\) for different index \( j \) in the state \( |\Phi_6\rangle\) belong to different subspaces \( \{S(m_j)\}\) and also different registers. It has been shown that any unknown state can be efficiently transferred to a larger state subspace from a small subspace in the Hilbert space [16]. Then the state \( |(g^{M_j})^s \mod p\rangle\) which is in the subspace \( S(m_j) \) with the dimensional size \( m_j \) may be efficiently transferred to a larger subspace \( S(m_k) \) with dimensional size \( m_k > m_j \). Since the dimensional size \( m_k \) for any subspace \( S(m_k) \) is \( \sim O(\log p) \) and there hold \( 0 \leq s_k < m_k \) and \( m_1 < m_2 < \ldots < m_r \), the unitary operation for the state transfer \( |(g^{M_j})^s \mod p\rangle \rightarrow |(g^{M_k})^s \mod p\rangle\) for \( 1 \leq j < k \leq r \) always can be constructed efficiently [16]. Now the state transfer is carried out from a small subspace \( S(m_k) \) \( (k \neq r) \) to the largest subspace \( S(m_r) \), that is, \( |(g^{M_k})^{s_k} \mod p\rangle \rightarrow |(g^{M_r})^{s_k} \mod p\rangle\) for \( k = 1, 2, \ldots, r-1 \), then the state \( |\Phi_6\rangle\) will be directly changed to a tension product of the \( r \) states \( \{|(g^{M_r})^{s_k} \mod p\rangle, k = 1, 2, \ldots, r\} \) of the largest subspace \( S(m_r) \) in the \( r \) different registers respectively:

\[
|\Phi_6\rangle \rightarrow |\Phi_7\rangle = |\text{RO}0\rangle \bigotimes |(g^{M_r})^{s_1} \mod p\rangle \bigotimes |(g^{M_r})^{s_2} \mod p\rangle \bigotimes \ldots \bigotimes |(g^{M_r})^{s_r} \mod p\rangle,
\]

where the state transfers can be performed in a parallel manner in the first \( r-1 \) registers of the state \( |\Phi_6\rangle\). The state \( |\Phi_7\rangle\) shows that any state \( |g^s \mod p\rangle\) of the cyclic group state space \( S(C_{p-1}) \) can be efficiently converted into a tension product of the \( r \) cyclic group states of the largest subspace \( S(m_r) \). If the index state \( |s\rangle \) is unknown, then in the state \( |\Phi_7\rangle\) all these states \( \{|(g^{M_r})^{s_k} \mod p\rangle, k = 1, 2, \ldots, r\} \) are also unknown and they carry the complete information of the index state \( |s\rangle\). Evidently, if the initial index state \( |s\rangle \) or the initial cyclic group state \( |g^s \mod p\rangle\) is replaced with a superposition, then the above state transformations work as well.
For the discrete logarithmic problem it is much simple to generate unitarily the auxiliary oracle unitary operation $\mathcal{U}_{\text{oj}s}(\theta) = \exp[-i\theta \mathcal{D}_{s}(g)]$ with the diagonal operator $\mathcal{D}_{s}(g) = |\text{R0}\rangle\langle \text{R0}| \otimes |g^{js} \text{mod} p\rangle\langle g^{js} \text{mod} p|$ and $j = M_k$ or even $j = (p - 1)/p_k$ from the basic oracle unitary operation $\mathcal{U}_{\text{os}}(\theta) = \exp[-i\theta \mathcal{D}_{s}(g)]$ in polynomial time. Actually, this can be achieved directly by the state transformation: $|\text{R0}\rangle \otimes |g^s \text{mod} p\rangle \rightarrow |\Phi_4\rangle$ without using any state identity (13) or (14). This is because (i) the integer $b = g^s \text{mod} p$ is given beforehand and hence the oracle unitary operation $\mathcal{U}_{\text{os}}(\theta)$ can be efficiently constructed in advance, and (ii) the known state $|g^s \text{mod} p\rangle$ in the state $|\Phi_4\rangle$ can be efficiently converted to the state $|0\rangle$. Therefore, using the auxiliary oracle unitary operation $\mathcal{U}_{\text{oj}s}(\theta)$ and the standard quantum search algorithm one can solve efficiently the discrete logarithmic problem in polynomial time if the dimensional size $m_k$ for every cyclic group state subspace $S(m_k)$ is $\sim O(\log p)$. This quantum discrete logarithmic algorithm is similar to the classical counterpart [21]. By combining with the quantum discrete logarithmic algorithm in section 3 this algorithm will obtain much more speedup.

However, the quantum search problem is much harder than the discrete logarithmic problem. The auxiliary oracle unitary operations corresponding to the states $|\Phi_2\rangle$ and $|\Phi_7\rangle$ still may be unsuitable for the quantum search task, for these factor states $\{|s \text{mod} m_k\rangle\}$ in the state $|\Phi_2\rangle$ or $\{|(g^{M_r})^{s_k} \text{mod} p\rangle\}$ in the state $|\Phi_7\rangle$ that carry the complete information of the index state $|s\rangle$ are in the $r$ different registers and this makes the search space too large for the quantum search problem. There are two possible schemes to solve this problem. One scheme is to compress unitarily all these $r$ states in the $r$ different registers into one register only in the state $|\Phi_2\rangle$ or $|\Phi_7\rangle$, and this scheme will lead to that the quantum search space is limited to the largest cyclic group state subspace $S(Z_{m_r})$ or $S(m_r)$. Since the dimension of the state subspace $S(Z_{m_r})$ or $S(m_r)$ is $m_r \sim O(\log p)$ the quantum search process may be implemented efficiently in these state subspaces. Another is to keep only one desired state but remove unitarily the other $r - 1$ states in the state $|\Phi_2\rangle$ or $|\Phi_7\rangle$. For example, one may let all those states $|(g^{M_r})^s \text{mod} p\rangle$ for $j \neq k$ return unitarily to the known state $|0\rangle$ but only the desired state $|(g^{M_r})^{s_k} \text{mod} p\rangle$ be retained in the state $|\Phi_7\rangle$. It could be better that the two schemes are used together. In next section a possible algorithm is proposed on a universal quantum computer to further reduce the quantum search space for the state $|\Phi_7\rangle$ in the multiplicative cyclic group state $S(C_{p-1})$, while the reduction for the quantum search space on the basis of the state $|\Phi_2\rangle$ in the additive cyclic group state space $S(Z_{p-1})$ is left in the
future work.

5. An efficient reduction for the quantum search space on an ideal universal quantum computer

A universal quantum computer [29, 38, 40] should be capable of computing any recursive function in mathematics and any computational process on it obeys the unitary quantum dynamics in physics. Now a quantum computational program based on the reversible computation [26, 27] is designed to transform some states \(\{|(g^M_r)_s \mod p\}\) back to the known state \(|0\rangle\) but keep the desired state in the state \(|\Phi_7\rangle\). This quantum program \(Q_p\) may run on a universal quantum computer [29, 38, 40]. It is given by

\[
|n_h\rangle = |0\rangle \\
|b_h\rangle = |0\rangle \\
\]

For \(i = 1\) to \(m_r\)

\[
\text{If } |g_r(y)\rangle = |1\rangle \text{ then } |b_h\rangle \rightarrow |b_h + 1\rangle \text{ end if}
\]

When \(|g_r(y)\rangle = |1\rangle\), Do \(|g_r(y)\rangle|n_h\rangle = |1\rangle|0\rangle \rightarrow |0\rangle|0\rangle, \ |n_h\rangle = |0\rangle \rightarrow |1\rangle, \text{ halt}

\[
\text{If } |b_h\rangle = |0\rangle \text{ then}
\]

\[
U_{g^M_r}|f_r(x)\rangle|g_r(y)\rangle \\
U_r|f_r(x)\rangle|g_r(y)\rangle
\]

else \(U_{g^M_r}|f_r(x)\rangle|g_r(y)\rangle \) end if

end for.

The quantum program \(Q_p\) can be really written as \(Q_p = \{Q_u\}^{m_r}\) in which the basic operational unit \(Q_u\) is repeated to execute \(m_r\) times. The basic operational unit \(Q_u\) may be formally expressed as \(Q_u = \{U_r^cU_{g^M_r}P^c\}\), here the operation \(P^c\) executes the two statements: "If \(|g_r(y)\rangle = |1\rangle \) then \(|b_h\rangle \rightarrow |b_h + 1\rangle \) end if" and "When \(|g_r(y)\rangle = |1\rangle\), Do \(|g_r(y)\rangle|n_h\rangle = |1\rangle|0\rangle \rightarrow |0\rangle|0\rangle, \ |n_h\rangle = |0\rangle \rightarrow |1\rangle, \text{ halt"}", the operation \(U_r^c\) performs conditionally the unitary operation \(U_r\) if the branch-control state \(|b_h\rangle = |0\rangle\), and the operation \(U_{g^M_r}\) performs the unitary cyclic group operation of the cyclic subgroup \(C_{p^{m_r}}\).

The state \(|n_h\rangle\) is the halting state of the quantum program and belongs to an independent two-dimensional state space \(\{|0\rangle, |1\rangle\}\). The branch-control state \(|b_h\rangle\) belongs to a larger and independent state space \(\{|0\rangle, |1\rangle, |2\rangle, ...\}\)
Note that for any given indices \( i \) stands for number of the basic operational unit \( Q_u \) to have been already executed. In the quantum program the functions \( f_r(x) \) and \( g_r(x) \) are \( f_r(x) = g_r(x) = (g_{M_r})^x \mod p \) for \( 0 \leq x < m_r \). Both the functions are periodic functions, \( f_r(x) = f_r(x + m_r) \) and \( g_r(y) = g_r(y + m_r) \), and they also satisfy \( f_r(x) = g_r(x) = 1 \) for \( x = 0 \mod m_r \). In the quantum program the cyclic group operation \( U_{g_{M_r}} \) acts on only the state \( |f_r(x)\rangle\),

\[
U_{g_{M_r}} |f_r(x)\rangle |g_r(y)\rangle = |f_r(x+1)\rangle |g_r(y)\rangle,
\]

while the state transformation of the unitary operation \( U_r \) is defined by

\[
U_r |f_r(x)\rangle |g_r(y)\rangle = \begin{cases} 
|f_r(x)\rangle |g_r(y)\rangle, \text{ if } x+y \neq 0 \mod m_r, \\
|f_r(x)\rangle \ket{1}, \text{ if } x+y = 0 \mod m_r.
\end{cases} \tag{15}
\]

Note that for any given indices \( x \) and \( y (0 \leq x, y < m_r) \) there is a unique index \( i (1 \leq i \leq m_r) \) such that \( x+y+i = 0 \mod m_r \). Therefore, there is a unique index \( i (1 \leq i \leq m_r) \) such that the state \(|f_r(x+i)\rangle |g_r(y)\rangle\) can be changed to the state \(|f_r(x+i)\rangle \ket{1}\) for given indices \( x \) and \( y \) by the unitary operation \( U_r \) in the quantum program.

In order to explain clearly how the quantum program \( Q_p \) works the statement “When \(|g_r(y)\rangle = |1\rangle\), Do \(|g_r(y)\rangle |n_h\rangle = |1\rangle |0\rangle \rightarrow |0\rangle |0\rangle\), \(|n_h\rangle = |0\rangle \rightarrow |1\rangle\), halt” which involves in the halting protocol of quantum Turing machine [29] in the quantum program is not considered temporarily. The quantum program starts at the initial state \(|b_h = 0\rangle |f_r(x)\rangle |g_r(y)\rangle\) of the quantum system of a universal quantum computer. The program first checks whether the state \(|g_r(y)\rangle\) is \( |1\rangle \) or not. If yes, then the branch-control state \(|b_h = 0\rangle\) is changed to the state \(|1\rangle\), otherwise it keeps unchanged. If the branch-control state \(|b_h\rangle\) is not \( |0\rangle \), then the program performs only the cyclic group operation \( U_{g_{M_r}} \), otherwise \(|b_h\rangle = |0\rangle\) it executes another unitary operation sequence, that is, it executes first the cyclic group operation \( U_{g_{M_r}} \) and then the unitary operation \( U_r \). At the end of the step \((i = 1)\) the quantum system is either (a) in the state \(|b_h = 1\rangle |f_r(x+1)\rangle |1\rangle\) if the initial state \(|g_r(y)\rangle = |1\rangle\) or (b) in the state \(|b_h = 0\rangle |f_r(x+1)\rangle |1\rangle\) if the initial state \(|g_r(y)\rangle \neq |1\rangle\) but \( x+y+1 = 0 \mod m_r \) or (c) in the state \(|b_h = 0\rangle |f_r(x+1)\rangle |g_r(y)\rangle\) if the initial state \(|g_r(y)\rangle \neq |1\rangle\) and \( x+y+1 \neq 0 \mod m_r \). Therefore, at next step \((i = 2)\) the three situations need to be considered, respectively. For the case (a), since the state \(|g_r(y)\rangle = |1\rangle\) and the branch-control state \(|b_h = 1\rangle\) the program performs only the cyclic group operation \( U_{g_{M_r}} \) which converts the state \(|n_h = 1\rangle |f_r(x+1)\rangle |1\rangle\) into the
state $|n_h = 1\rangle |f_r(x + 2)\rangle |1\rangle$. Evidently, once the state $|g_r(y)\rangle$ is transformed to the state $|1\rangle$ and then the state $|b_h\rangle$ to the state $|1\rangle$, the two states $|g_r(y)\rangle$ and $|b_h\rangle$ are kept at the state $|1\rangle$ in following steps and even to the end of the program, and hence the program performs only the cyclic group operation $U_{g^{m_r}}$ to the end ($i = m_r$). Then at the end the quantum system is in the state $|b_h = 1\rangle |f_r(x + m_r)\rangle |1\rangle = |b_h = 1\rangle |f_r(x)\rangle |1\rangle$. For the case (b), since the state $|g_r(y)\rangle = |1\rangle$, then the branch-control state $|b_h\rangle = |0\rangle$ is changed to $|1\rangle$, that is, $|b_h = 0\rangle |f_r(x + 1)\rangle |1\rangle$ is transformed to $|b_h = 1\rangle |f_r(x + 1)\rangle |1\rangle$ which will be further changed to the state $|b_h = 1\rangle |f_r(x)\rangle |1\rangle$ at the end of the program, as explained in the case (a). For the case (c), just like at the end of the step ($i = 1$), at the end of the step ($i = 2$) there are also three situations to be considered again and these situations can be analyzed in a similar way given in the step ($i = 1$). The analysis shows that when the program is at the $k-$th step ($i = k$) such that $x + y + k = 0 \mod m_r$, the quantum system is changed from the state $|b_h = 0\rangle |f_r(x + k - 1)\rangle |g_r(y)\rangle$ with $|g_r(y)\rangle \neq |1\rangle$ at the beginning to the state $|b_h = 0\rangle |f_r(x + k)\rangle |1\rangle$ at the end of the $k-$th step by the unitary operation $U_r$. At the following step ($i = k + 1$) the branch-control state $|b_h\rangle = |0\rangle$ is transformed to the state $|1\rangle$. Then starting from the step ($i = k + 1$) the quantum system is acted on only by the cyclic group operation $U_{g^{m_r}}$ and this action continues to the end of the program. The final state ($i = m_k$) of the quantum system therefore is $|b_h = 1\rangle |f_r(x + m_r)\rangle |1\rangle = |b_h = 1\rangle |f_r(x)\rangle |1\rangle$. Thus, after execution of the whole quantum program one time the input state $|b_h = 0\rangle |f_r(x)\rangle |g_r(y)\rangle$ is changed to the output state $|b_h = 1\rangle |f_r(x)\rangle |1\rangle$.

However, there is a precondition for the quantum program to work as stated above that once the state $|g_r(y)\rangle$ is changed to the state $|1\rangle$ by the unitary operation $U_r$, the branch-control state $|b_h\rangle = |0\rangle$ is changed to the state $|1\rangle$ and since then the branch-control state $|b_h\rangle = |1\rangle$ is kept unchanged to the end of the program. This precondition may be achieved by the statement: “When $|g_r(y)\rangle = |1\rangle$, Do $|g_r(y)\rangle |n_h\rangle = |1\rangle |0\rangle \rightarrow |0\rangle |0\rangle$, $|n_h\rangle = |0\rangle \rightarrow |1\rangle$, halt” in the program. This statement is executed after the branch-control state $|b_h\rangle = |0\rangle$ is changed to the state $|1\rangle$. The statement shows that once the state $|g_r(y)\rangle$ goes to the state $|1\rangle$, the state $|g_r(y)\rangle |n_h\rangle = |1\rangle |0\rangle$ is changed to the state $|0\rangle |0\rangle$ which means that the state $|g_r(y)\rangle = |1\rangle$ is changed to the state $|0\rangle$ conditionally when the halting state $|n_h\rangle = |0\rangle$, then the halting state $|n_h\rangle = |0\rangle$ is changed to the state $|1\rangle$, and since then the halting state $|n_h\rangle = |1\rangle$ is kept unchanged to the end of the program which is executed by the instruction “halt” of the statement. There are three operations in the
statement, the first is the unitary operation $U_h : |g_r(y)\rangle|n_h\rangle = |1\rangle|0\rangle \leftrightarrow |0\rangle|0\rangle$, the second is the trigger pulse $P_c$ on the halting qubit: $|n_h\rangle = |0\rangle \leftrightarrow |1\rangle$, and the last operation $T(n) : "\text{halt}"$, which could involve in the unitary nonde-
molition measurement operation on the halting qubit [29, 40], will kept the
halting qubit at the state $|n_h\rangle = |1\rangle$ unchanged until the end of the pro-
gram. It can be shown that if the halting state $|n_h\rangle = |1\rangle$ can be kept unchanged, then the branch-control state $|b_h\rangle = |1\rangle$ can also be kept un-
changed. Suppose that at the $i-$th step of the program the state $|g_r(y)\rangle$
go to the state $|1\rangle$, then at the $(i + 1)$-th step the state $|b_h\rangle$ goes to
the state $|1\rangle$ which will stop the unitary operation $U_h$ later, and then the
state $|g_r(y)\rangle = |1\rangle$ is changed to the state $|0\rangle$ and the halting state $|n_h\rangle$
enters the state $|1\rangle$. Note that the cyclic group operation $U_{g_{m_r}}$ does not af-
fact the state $|g_r(y)\rangle$ and the unitary operation $U_h$ now is halted. Now at
the $(i + 2)$-th step the conditional unitary operation $U_h : |b_h\rangle \rightarrow |b_h + 1\rangle$
does not change the state $|b_h\rangle = |1\rangle$ because the state $|g_r(y)\rangle = |0\rangle$, and
the unitary operation $U_h : |g_r(y)\rangle|n_h\rangle = |1\rangle|0\rangle \leftrightarrow |0\rangle|0\rangle$ also has not net
effect on the quantum system because the state $|g_r(y)\rangle|n_h\rangle = |1\rangle|1\rangle$ now.
Though the unitary operation $P_c : |n_h\rangle = |0\rangle \leftrightarrow |1\rangle$ may change the halting
state $|n_h\rangle = |1\rangle$ back to the state $|0\rangle$, but the halting state $|n_h\rangle = |1\rangle$
is prevented by the halting operation $T(i + 2)$ from the action of the unitary
operation $P_c$ so that it still keeps at the same state $|1\rangle$ at the step,
and this is the key point for the whole quantum program. Thus, from the
$(i + 2)$-th step to the end of the program the halting state is kept at the state $|1\rangle$ and hence the branch-control state is also kept at the state $|1\rangle$. Ob-
viously, when the whole quantum program includes the statement: "When
$|g_r(y)\rangle = |1\rangle$, Do $|g(y)\rangle|n_h\rangle = |1\rangle|0\rangle \rightarrow |0\rangle|0\rangle$, $|n_h\rangle = |0\rangle \rightarrow |1\rangle$, halt", the output state $|n_h\rangle|b_h\rangle|f_r(x)\rangle|g_r(y)\rangle$ is $|1\rangle|1\rangle|f_r(x)\rangle|0\rangle$ if the input state is
$|0\rangle|0\rangle|f_r(x)\rangle|g_r(y)\rangle$.

One might ask one question: is the unitarity of the quantum program
destroyed?, because there are different input states $|0\rangle|0\rangle|f_r(x)\rangle|g_r(y)\rangle$ for
different states $|g_r(y)\rangle$, but the quantum program obtains the same output
state $|1\rangle|1\rangle|f_r(x)\rangle|0\rangle$. Actually, there is a different index $i$ ($1 \leq i \leq m_r$) such
that $(x + y + i) = 0 \mod m_r$ for a different input state $|0\rangle|0\rangle|f_r(x)\rangle|g_r(y)\rangle$
where the state $|f_r(x)\rangle$ may be fixed. Then there is a different time (e.g.,
the $i-$th step) for the state $|g_r(y)\rangle$ to go to the state $|1\rangle$ and for the halting
operation $T(i)$ to act on the quantum system. In effect the halting opera-
tion $T(i)$ acting on the quantum system at different time $i$ is equivalent to
that the quantum program in a different unitary operation acts on the input
According as the universal quantum computer model [29], the halting state |n_h⟩ should be periodically observed from the outside in a unitary and nondemolition form so that once the halting state is found at the state |1⟩ the halting operation T(i) starts to act on the quantum system of the quantum computer. Before the halting operation T(i) takes an action the quantum system has already been made a unitary transformation U(i) which is clearly dependent on the time i. Obviously, this unitary transformation generally is different if the halting operation T(i) takes an action at a different time, while for the current quantum program this is clearly correct as well. Therefore, the same output state |1⟩|1⟩|f_r(x)⟩|0⟩ is obtained from different input state |0⟩|0⟩|f_r(x)⟩|g_r(y)⟩ by a different unitary transformation in the current quantum program. Although different input states can not be converted to the same output state by a same unitary transformation, they are admitted to change to the same output state by different unitary transformations! Therefore, the quantum program keeps its unitarity.

The key point to make the quantum program Q_p work as stated above is that the halting protocol of quantum Turing machine is available and must be unitary. The unitarity for the halting protocol of quantum Turing machine is crucial for the quantum program when it is used to solve the quantum search problem based on the quantum unitary dynamics. Unlike the conventional measurement operation in quantum computation where the measurement operation usually could not be unitary and some information could loss during the measurement operation but these usually do not much affect the final computing results, the current halting operation must be unitary which contains the unitary nondemolition measurement operation since it could carry some information of the input state, as shown before, while the information could be necessary because in theory the inverse halting operation which contains the inverse unitary process of the nondemolition measurement operation could be necessary for solving quantum search problem based on the unitary quantum dynamics.

The quantum program Q_p is really assumed to run on an ideal universal quantum computer which has the unitary halting protocol of quantum Turing machine. Obviously, this program is trivial and could be irreversible if it runs on a conventional classical computer, but it could be simulated efficiently by the reversible computation [26, 27, 28]. The quantum program could also be efficiently performed on a quantum Turing machine (QTM) [29, 37, 40], as analyzed above. In fact, in a quantum Turing machine one may set directly the halting state |n_h⟩ in the program to be the QTM halting-control state to
control the quantum program. Once the state $|g_r(y)\rangle$ is $|1\rangle$ in the program the halting state $|n_h\rangle = |0\rangle$ is changed to the state $|1\rangle$, then the program stops performing the operational branch consisting of the two unitary operations $U_{g^M_r}$ and $U_r$ but turns to perform another operational branch of a single cyclic group operation $U_{g^M_r}$ to the end $(i = m_r)$, which ensures that the whole process of the program is unitary, as pointed out in [41c]. However, there hides a basic assumption that any input state of the quantum program is a single basis state. This basic assumption could ensure that the halting protocol of quantum Turing machine could be made available and unitary for the quantum program on an ideal universal quantum computer [29, 40, 41a-41d].

However, if the input state of the quantum program is a superposition $\sum_s \alpha_s |n_h = 0\rangle |b_h = 0\rangle |f_r(x(s))\rangle |g_r(y(s))\rangle$, there seems to be a question whether the halting protocol can be available and unitary or not on a quantum Turing machine [41a-41d] when the quantum program is run on the QTM machine. This is because in this situation there are many operational branches to be executed simultaneously, and one does not known in advance when the state $|g_r(y(s))\rangle$ is changed to the state $|1\rangle$ and actually for different index value $s$ there may be a different time (the index $i$, $1 \leq i \leq m_r$) for the state $|g_r(y(s))\rangle$ to go to the state $|1\rangle$ in the program, although for any index value $s$ the quantum program always stops at the same time when the index $i = m_r$. At present there is not a satisfactory halting protocol on a quantum Turing machine when the input state is a superposition. The detail discussion relevant to the halting problem of quantum Turing machine for this situation can be seen in Refs. [41a-41d]. However, it has been shown [29, 40, 41a-41d] that there is an acceptable halting protocol of quantum Turing machine which may be made unitary if the input state is limited to be any single basis state on a quantum Turing machine. Then there should not be any problem to run the quantum program in a unitary form on a quantum Turing machine if its input state is limited to be a single basis state. One therefore concludes that if there existed a universal quantum Turing machine (UQTM) that on it any computational process obeys the unitary quantum dynamics in physics and it is capable of computing any computable functions in mathematics such as any recursive functions which of course include the current one computed by the quantum program, then such a universal quantum Turing machine could run the current quantum program in a unitary form when the input state is limited to be any single basis state for the program.
Though the quantum program could work on a universal quantum Turing machine and it has been shown that a quantum circuit model is equivalent to a universal quantum Turing machine in computation [39], it is still a challenge to construct an efficient quantum circuit for the quantum program. From the point of view of a quantum circuit model [38] the situation may be different. A quantum circuit model usually does not use any halting protocol and its input state can be either a single basis state or a superposition. However, in order to achieve the same result as the quantum program run on a universal quantum Turing machine, the quantum circuit model should be really able to simulate faithfully and efficiently the quantum program and especially the unitary halting protocol of quantum Turing machine used in the program. According to the definition (15) of the unitary operation $U_r$ the unitary operation $U_k$ for $k = 1, 2, ..., r$ can be generally defined by

$$
|\left( g^{M_k} \right)^x \mod p \rangle |\left( g^{M_k} \right)^{-x} \mod p \rangle \leftrightarrow |\left( g^{M_k} \right)^x \mod p \rangle |1\rangle, \ 0 \leq x \leq m_k - 1,
$$

$$
|\left( g^{M_k} \right)^x \mod p \rangle |\left( g^{M_k} \right)^{y} \mod p \rangle \leftrightarrow |\left( g^{M_k} \right)^x \mod p \rangle |\left( g^{M_k} \right)^{y} \mod p \rangle,
$$

$$
x + y \neq 0 \mod m_k; \ 0 \leq x, y \leq m_k - 1,
$$

while the conditional unitary operation $U_k^c$ is defined as

$$
U_k^c |b_h \rangle |\left( g^{M_k} \right)^x \mod p \rangle |\left( g^{M_k} \right)^{y} \mod p \rangle = \begin{cases} 
|b_h \rangle U_k (|\left( g^{M_k} \right)^x \mod p \rangle |\left( g^{M_k} \right)^{y} \mod p \rangle), \text{ if } b_h = 0. \\
|b_h \rangle |\left( g^{M_k} \right)^x \mod p \rangle |\left( g^{M_k} \right)^{y} \mod p \rangle, \text{ if } b_h \neq 0.
\end{cases}
$$

The unitary operations $U_k$ and $U_k^c$ always can be built up efficiently since the dimension of the cyclic group state subspace $S(m_k) = \{|\left( g^{M_k} \right)^x \mod p \rangle\}$ is $m_k$ and $m_k \sim O(\log p)$. The conditional unitary operation $U_k^c$ is dependent on the branch-control state $|b_h \rangle$. When the branch-control state $|b_h \rangle \neq |0 \rangle$ the conditional unitary operation $U_k^c$ does not act on the state $|\left( g^{M_k} \right)^x \mod p \rangle |\left( g^{M_k} \right)^{y} \mod p \rangle$ for any indices $x$ and $y$. If the unitary operator $U_k$ can be written as $U_k = \exp[-iH_k]$ with the Hamiltonian $H_k$, then it is clear that $U_k^c = \exp[-i|b_h = 0 \rangle \langle b_h = 0|] \bigotimes H_k$. As pointed out before, the key point for the quantum circuit is to simulate faithfully the unitary halting protocol of quantum Turing machine. Since the quantum circuit does not use the halting qubit, one may use an isolated two-level state control subspace to replace it. Denote the isolated two-level state subspace as $\{|c\rangle, |0\rangle\}$. The two states in the subspace $\{|c\rangle, |0\rangle\}$ are not in the cyclic group state subspace.

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The ideal conditional state-locking pulse $P$ only when the quantum system goes to the states of the control subspace. Therefore, the state-locking pulse does not make any net effect on the quantum system because before the trigger pulse $P_t$ converted completely into the state $|\psi_i(t)\rangle$. The conditional trigger pulse connects the state $|c\rangle$ of the control subspace to the state $|1\rangle$ of the cyclic group state subspace $S(m_r)$. The time-dependent state-locking pulse $P_{SL}^c(\{\varphi_i(t)\})$, where $\{\varphi_i(t)\}$ are time-dependent control parameters, can be only applied to the control subspace and does not affect any other states in the quantum system. Then the state-locking pulse does not make any net effect on the quantum system if the quantum system is not in the control subspace. Therefore, the conditional state-locking pulse can take an action on the quantum system only when the quantum system goes to the states of the control subspace. The ideal conditional state-locking pulse $P_{SL}^c(\{\varphi_i(t)\})$ could be defined by

$$P_{SL}^c(\{\varphi_i(t)\})|b_h\rangle|f_r(x)\rangle|g_r(y)\rangle = |b_h\rangle|f_r(x)\rangle|g_r(y)\rangle, \quad t < t_0,$$

$$P_{SL}^c(\{\varphi_i(t)\})|b_h\rangle|f_r(x)\rangle|c\rangle = |b_h\rangle|f_r(x)\rangle|0\rangle, \quad t_0 \leq t < t_0 + \Delta t_0,$$

$$P_{SL}^c(\{\varphi_i(t)\})|b_h\rangle|f_r(x)\rangle|0\rangle = |b_h\rangle|f_r(x)\rangle|0\rangle, \quad t > t_0 + \Delta t_0,$$

where $t_0$ is the time at which the state $|c\rangle$ is generated completely by the trigger pulse $P_t$ and evidently there are $m_r$ different times $t_0$ at most for the quantum circuit $Q_c$ (see below). $\Delta t_0$ is the interval that the state $|c\rangle$ is converted completely into the state $|0\rangle$ and it is shorter than the interval to execute the statement: "While $|g_r(y)\rangle = |1\rangle$, Do $P_t : |g_r(y)\rangle = |1\rangle \rightarrow |c\rangle$, $P_{SL}^c : |c\rangle \rightarrow |0\rangle$" (see the quantum program $Q_c$ below). Here also assume that during the period $(t_0 = t_0 - \delta t_0 \leq t < t_0)$ of the trigger pulse $P_t$ the state-locking pulse has a negligible effect on the quantum system. The unitary transformation shows that after the state $|c\rangle$ is changed to the state $|0\rangle$, the state $|0\rangle$ is kept unchanged by the state-locking pulse and hence it will not change as the time. The conditional trigger pulse $P_t$ instructs what time the conditional state-locking pulse $P_{SL}^c(\{\varphi_i(t)\})$ starts to take an action on the quantum system because before the trigger pulse $P_t$ changes the state.
|1⟩ of the cyclic group state subspace $S(m_r)$ to the state |c⟩ of the control subspace the quantum system is not in the control subspace and hence the state-locking pulse has not a net effect on the quantum system. On the other hand, the conditional trigger pulse $P_t$ can change the state |1⟩ of the cyclic group state subspace $S(m_r)$ to the state |c⟩ of the control subspace only when the state |gr(y)⟩ goes to the state |1⟩. Therefore, the conditional state-locking pulse $P_{SL}^c(\{\varphi_i(t)\})$ can take an action on the quantum system only after the state |gr(y)⟩ goes to the state |1⟩. In effect the conditional state-locking pulse will replace the halting operation $T(i)$ of the quantum program $Q_p$ to control the quantum circuit $Q_c$, as can be seen below. This is because when the state |c⟩ is changed to the state |0⟩ of the control subspace and then the state |0⟩ is locked by the state-locking pulse, the quantum system really leaves the state |c⟩ and hence the trigger pulse $P_t$ is no longer to take an action on the quantum system. Using the conditional unitary operation $U_b : |g_r(y) = 1⟩|b_h⟩ \rightarrow |g_r(y) = 1⟩|b_h + 1⟩$, the conditional unitary operation $U_c^c$, and the cyclic group operation $U_{g^m_r}$ as well as the conditional trigger pulse $P_t$ and the conditional state-locking pulse $P_{SL}^c$ a possible unitary quantum circuit that simulates faithfully and efficiently the quantum program $Q_p$ could be constructed by

$$Q_c = \{P_{SL}^c : OFF\} \{U_c^c U_{g^m_r} P_t U_b\}^{m_r} \{P_{SL}^c : ON\}.$$ 

In fact, given any input basis state this quantum circuit in theory is exactly equivalent to the following quantum program $Q_c$:

**State-Locking Pulse : ON**

$$|b_h⟩ = |0⟩$$

For $i = 1$ to $m_r$

If $|g_r(y)⟩ = |1⟩$ then $|b_h⟩ = |b_h + 1⟩$ end if

While $|g_r(y)⟩ = |1⟩$, Do $P_t : |g_r(y)⟩ = |1⟩ \rightarrow |c⟩$, $P_{SL}^c : |c⟩ \rightarrow |0⟩$

If $|b_h⟩ = |0⟩$ then

$$U_{g^m_r} |f_r(x)⟩ |g_r(y)⟩$$

else $U_{g^m_r} |f_r(x)⟩ |g_r(y)⟩$ end if
end for

State-Locking Pulse : OFF.

The sole difference from the previous one $Q_p$ is that the halting qubit $\{n_h\}$ of the quantum program $Q_p$ is replaced with the two-level state subspace $\{|c\rangle, |0\rangle\}$ in the quantum program $Q_c$. Here the input state of the quantum circuit $Q_c$ is still limited to be a single basis state, although a quantum circuit does not limit any input state. In theory the output state of the quantum circuit is $|b_h = 1\rangle|f_r(x)\rangle|0\rangle$ if the input state is the single basis state $|b_h = 0\rangle|f_r(x)\rangle|g_r(y)\rangle$. The quantum program $Q_c$ shows that the state-locking pulse $P^c_{SL}$ is first applied to the quantum system at the beginning of the quantum circuit. Because the quantum system may not be in the control subspace at the beginning, the state-locking pulse does not make an action on the quantum system, but it keeps applying and does not start to act on the quantum system until the quantum system goes to the state $|c\rangle$, and only at the end of the quantum circuit the state-locking pulse is switched off.

The performance of the quantum circuit $Q_c$ usually may be mainly dependent on the state-locking pulse $P^c_{SL}(\{\varphi_i(t)\})$. The real unitary transformation during the period of the state-locking pulse applying to the quantum system should be generally written as

$$P^c_{SL}(\{\varphi_i(t)\})|b_h\rangle|f_r(x)\rangle|g_r(y)\rangle = |b_h\rangle|f_r(x)\rangle|g_r(y)\rangle, \quad t < t_0,$$

$$P^c_{SL}(\{\varphi_i(t)\})|b_h\rangle|f_r(x)\rangle|c\rangle = |b_h\rangle|f_r(x)\rangle|\varepsilon(t, t_0)\rangle|c\rangle + e^{-i\gamma(t, t_0)}\sqrt{1 - |\varepsilon(t, t_0)|^2}|0\rangle, \quad t \geq t_0,$$

where $\gamma(t, t_0)$ is a phase factor and the absolute amplitude value $|\varepsilon(t, t_0)|$ is zero in theory when the time $t > t_0 + \Delta t_0$ for every time $t_0$. Hereafter the absolute amplitude value $|\varepsilon(t, t_0)|$ is referred to the one with the time $t > t_0 + \Delta t_0$. The real amplitude value $|\varepsilon(t, t_0)|$ may be dependent on the real physical process of the quantum circuit. The amplitude value $|\varepsilon(t, t_0)|$ measures how close the quantum circuit $Q_c$ is to the quantum program $Q_p$, the closer the amplitude value $|\varepsilon(t, t_0)|$ to zero, the closer the quantum circuit $Q_c$ to the quantum program $Q_p$. The quantum circuit $Q_c$ is really equivalent to the quantum program $Q_p$ when the amplitude value $|\varepsilon(t, t_0)| = 0$ exactly for every time $t_0$, but this could be possibly achieved only in an ideal case. However, the amplitude value $|\varepsilon(t, t_0)|$ could not be always equal to zero for every time $t_0$ if the input state of the quantum circuit is a superposition. This
is one reason why the input state of the quantum circuit $Q_c$ is still limited to be a single basis state, although the input state is allowed to be any state such as a superposition in the quantum circuit. Therefore, the quantum circuit $Q_c$ is really an approximation to the ideal quantum program $Q_p$ in a real physical process. In practice the conditional state-locking pulses need to be designed so that the amplitude value $|\varepsilon(t, t_0)|$ is as close zero as possible for every time $t_0$. Hence this involves in quantum control in technique. The conditional state-locking pulse generally could be an amplitude- and phase-modulation time-dependent pulse. A better choice for the state-locking pulse could be an adiabatic pulse.

If there existed a universal quantum computer that in computation obeys the unitary quantum dynamics in physics and is capable of computing any computable functions in mathematics such as any recursive functions, then such an ideal universal quantum computer would be enough powerful to solve efficiently the quantum search problem in the cyclic group state space. Actually, by taking the basis state $|n_h = 0\rangle|b_h = 0\rangle|\Phi_7\rangle$ as the input state of the quantum program $Q_p$, and setting the function $f_r(x) = (g^{M_r})^x \mod p$ with $x = s_1$ and $g_r(y) = (g^{M_r})^y \mod p$ with $y = s_2$, after executing one time the quantum program the output state is given by

$$Q_p |\Phi_8\rangle = |1\rangle|1\rangle \bigotimes ((g^{M_r})^{s_1} \mod p)(g^{M_r})^{s_2} \mod p) \bigotimes |\Phi'_7\rangle,$$

where the state $|\Phi_7\rangle = |(g^{M_r})^{s_1} \mod p)(g^{M_r})^{s_2} \mod p) \bigotimes |\Phi'_7\rangle$ and the state $|\Phi'_7\rangle$ is given by

$$|\Phi'_7\rangle = |R0\rangle \bigotimes ((g^{M_r})^{s_3} \mod p) \bigotimes \ldots \bigotimes ((g^{M_r})^{s_r} \mod p).$$

This is because only the state $|(g^{M_r})^{s_1} \mod p)(g^{M_r})^{s_2} \mod p)$ in the first two registers of the state $|\Phi_7\rangle$ is made the unitary transformation by the quantum program $Q_p$, while the state $|\Phi'_7\rangle$ in other registers of the state $|\Phi_7\rangle$ keeps unchanged, and the output state of the quantum program is $|1\rangle|1\rangle|f_r(x)\rangle|g_r(y)\rangle$, as shown before. This unitary transformation removes the state $|(g^{M_r})^{s_2} \mod p)$ in the second register of the state $|\Phi_7\rangle$. Next step is to remove unitarily the state $|(g^{M_r})^{s_3} \mod p)$ in the third register of the state $|\Phi_7\rangle$. First, both the branch-control state $|b_h\rangle = |1\rangle$ and the halting state $|n_h\rangle = |1\rangle$ are changed back to the state $|0\rangle$ in the state
\[ |\Phi_8\rangle \text{ and the state } |0\rangle \text{ in the register four of the state } |\Phi_8\rangle \text{ is absorbed by the register library } |R0\rangle. \text{ After these operations the state } |\Phi_8\rangle \text{ is changed to the state } |\Phi_9\rangle : \]

\[ |\Phi_9\rangle = |n_9 = 0\rangle |b_9 = 0\rangle \bigotimes |(g^{M_r})^{s_1} \mod p\rangle |(g^{M_r})^{s_1} \mod p\rangle \bigotimes |\Phi_8\rangle \]

where the state \(|\Phi_9\rangle = |R0\rangle \bigotimes |(g^{M_r})^{s_1} \mod p\rangle \bigotimes \ldots \bigotimes |(g^{M_r})^{s_r} \mod p\rangle \). Now taking the state \(|\Phi_9\rangle \) as the input state of the quantum program \(Q_p\) and setting the function \(f_r(x) = (g^{M_r})^x \mod p\) with \(x = s_1\) and \(g_r(y) = (g^{M_r})^y \mod p\) with \(y = s_3\), the unitary transformation of the quantum program removes the state \(|(g^{M_r})^{s_3} \mod p\rangle\) of the state \(|\Phi_9\rangle\). In an analogue way, by setting the fixed function \(f_r(x) = (g^{M_r})^x \mod p\) with \(x = s_1\) and the function \(g_r(y) = (g^{M_r})^y \mod p\) with \(y = s_k\) for \(k = 2, 3, \ldots, r\), respectively, and then repeating \(r - 1\) times the application of the quantum program \(Q_p\), the states \(|(g^{M_r})^{s_k} \mod p\rangle\) with \(k = 2, 3, \ldots, r\) are one by one removed unitarily from the state \(|\Phi_{r}\rangle\) and ultimately the state \(|\Phi_{7}\rangle\) is transformed to the desired state \(|R0\rangle \bigotimes |(g^{M_r})^{s_1} \mod p\rangle\), where the branch-control state \(|0\rangle\) and the halting state \(|0\rangle\) are also absorbed by the register library. This transformation may also be carried out in a parallel manner. In an analogue way, one may obtain the desired state \(|R0\rangle \bigotimes |(g^{M_r})^{s_k} \mod p\rangle\) from the state \(|\Phi_{r}\rangle\) for \(k = 1, 2, \ldots, r\), respectively. Once the unitary state transformation \(|R0\rangle \bigotimes |g^s \mod p\rangle \rightarrow |R0\rangle \bigotimes |(g^{M_r})^{s_k} \mod p\rangle\) is efficiently achieved for \(k = 1, 2, \ldots, r\), the auxiliary oracle unitary operation \(U_{osk}(\theta) = \exp[-i\theta D_{s_k}(g^{M_r})]\) with the quantum-state diagonal operator \(D_{s_k}(g^{M_r}) = |R0\rangle \langle R0| \bigotimes |(g^{M_r})^{s_k} \mod p\rangle \langle (g^{M_r})^{s_k} \mod p|\) can be efficiently built out of the oracle unitary operation \(U_{os}(\theta)\). This auxiliary oracle unitary operation is applied only to the cyclic group state subspace \(S(m_r)\). The state \(|R0\rangle \bigotimes |(g^{M_r})^{s_k} \mod p\rangle\) may be transferred to the register of the search space by a SWAP operation so as to obtain the auxiliary oracle unitary operation \(U_{osk}(\theta)\) which is applied only to the search space with dimension \(m_r = O(\log p)\). Note that the register of the search space in which the index vector \(\{s_k\}\) is determined may be different from all those registers in the state \(|\Phi_{7}\rangle\).

6. An efficient quantum search process in the cyclic group state subspaces

When the auxiliary oracle unitary operation \(U_{osk}(\theta)\) with \(k = 1, 2, \ldots, r\) is obtained the quantum search process to find the index \(s_k\) can be efficiently constructed. As shown in the previous section 5, the initial state for the
quantum search process should be limited to be a single basis state because both the input states of the quantum program $Q_p$ and the quantum circuit $Q_c$ are limited to be a single basis state. Therefore, the standard quantum search algorithm which usually starts at a superposition will not be used here to determine the index vector $\{ s_k \}$. Because the quantum search space now is limited to the cyclic group state subspace $S(m_k)$ with dimensional size $m_k \sim O(\log p)$, one may use every basis state of the cyclic group state subspace $S(m_k)$ as the initial state of the quantum search process without changing essentially the computational complexity of the quantum search process. For convenience, now the oracle unitary operation $U_{os_k}(\theta)$ acting on a basis state of the search space $S(m_r)$ can be rewritten as

$$U_{os_k}(\theta)|((g^{M_r})^x \mod p)\rangle = \begin{cases} \exp(-i\theta)|((g^{M_r})^x \mod p)\rangle, & \text{if } x = s_k, \\ |((g^{M_r})^x \mod p)\rangle, & \text{if } x \neq s_k, \end{cases}$$

where the register library $|R0\rangle$ is dropped without confusion. On the other hand, the basis state $|((g^{M_r})^x \mod p)\rangle$ with $0 \leq x < m_r$ can also be expressed in terms of the binary dynamical parameter $\{b^x_k\}$ (see sections 2.1 and 2.2),

$$|((g^{M_r})^x \mod p)\rangle = \bigotimes_{k=1}^{n} \frac{1}{2} T_k + b^x_k S_k.$$ 

The dynamical parameters $\{b^x_k\}$ can be determined conveniently below for a given integer $(g^{M_r})^x \mod p$ and will be used later in the construction of the quantum search process. The integer $(g^{M_r})^x \mod p$ is first expressed in terms of the usual binary representation:

$$(g^{M_r})^x \mod p = a_n 2^{n-1} + a_{n-1} 2^{n-2} + ... + a_2 2^1 + a_1 2^0,$$

where the qubit number $n = \lceil \log_2 p \rceil + 1$ and $a_k = 0$ or 1. Then the dynamical parameter $b^x_k$ is given by $b^x_k = (1 - 2a_k)$ for $k = 1, 2, ..., n$. Since the oracle unitary operation $U_{os_k}(\theta)$ can generate a phase factor $\exp(-i\theta)$ only for the marked state $|((g^{M_r})^{s_k} \mod p)\rangle$ but nothing for any other states of the search space $S(m_r)$, as shown in (16), one can only use this phase factor to distinguish the marked state $|((g^{M_r})^{s_k} \mod p)\rangle$ from any other states of the search space. This search process to find the marked state can be made efficient due to the fact that the dimension of the search space $S(m_r)$ is $m_k \sim O(\log p)$. Here, an efficient quantum search process is suggested to find the marked state in the search space. It is based on the use of the multiple-quantum
unitary operators [42] in the \( n \)-qubit quantum spin system \( (n = \lfloor \log_2 p \rfloor + 1) \) whose Hilbert space contains the search space \( S(m_r) \).

A particularly important multiple-quantum transition to be used in the quantum search process is the highest-order quantum transition in the \( n \)-qubit quantum spin system. The highest-order quantum transition is defined as the transition between the ground state \( |00...0\rangle \) and the highest excited state \( |11...1\rangle \) of the \( n \)-qubit spin system. In an \( n \)-qubit spin system the highest order of quantum transition is \( \pm n \) [43] and the Hermitian highest-order quantum operators \( Q_{nx} \) and \( Q_{ny} \) may be defined by

\[
Q_{nx} = \frac{1}{2} (I_1^+ I_2^+ \ldots I_n^+ + I_1^- I_2^- \ldots I_n^-),
\]

(18)

and

\[
Q_{ny} = \frac{1}{2i} (I_1^+ I_2^+ \ldots I_n^+ - I_1^- I_2^- \ldots I_n^-),
\]

(19)

where the operators \( I_k^\pm = I_{kx} \pm iI_{ky} \) for \( k = 1, 2, ..., n \). The highest-order quantum unitary operators are defined by \( U_{n\mu}(\theta) = \exp(-i2\theta Q_{n\mu}) \) with \( \mu = x, y \). They can induce an \( n \)-order quantum transition only between the ground state \( |00...0\rangle \) and the highest excited state \( |11...1\rangle \) of the Hilbert space of the \( n \)-qubit spin system, but they do not induce any other order quantum transition between any pair of quantum states of the spin system different from the pair of the ground state and the highest excited state. This is because the transition matrix elements \( \langle k|Q_{n\mu}|r \rangle = \langle r|Q_{n\mu}|k \rangle^* = 0 \) \( (\mu = x, y) \) for any computational base \( |k\rangle \) and \( |r\rangle \) of the spin system other than the ground state \( |00...0\rangle \) or the highest excited state \( |11...1\rangle \). Since \( I_k^+ |0_l\rangle = 0, \ I_k^- |0_l\rangle = \delta_{kl} |0_k\rangle, \ I_k^+ |1_l\rangle = \delta_{kl} |1_k\rangle, \) and \( I_k^- |1_l\rangle = 0 \) [43] for \( k, l = 1, 2, ..., n \), the \( n \)-order quantum operator \( Q_{ny} \) acting on the ground state (the highest excited state) creates the highest excited state (the ground state),

\[
2Q_{ny}|00...0\rangle = i|11...1\rangle
\]

and

\[
2Q_{ny}|11...1\rangle = -i|00...0\rangle.
\]

Then it is easy to turn out that there are the unitary transformations when the \( n \)-order quantum unitary operator \( U_{ny}(\theta) = \exp(-i2\theta Q_{ny}) \) acts on the ground state and the highest excited state, respectively,

\[
\exp(-i2\theta Q_{ny})|00...0\rangle = \cos \theta |00...0\rangle + \sin \theta |11...1\rangle, \ (-\pi \leq \theta \leq \pi)
\]

(20)
\[ \exp(-i2\theta Q_{ny}|11\ldots1\rangle = \cos \theta|11\ldots1\rangle - \sin \theta|00\ldots0\rangle, \quad (-\pi \leq \theta \leq \pi). \]  

(21)

In particular, when \( \theta = \pi/4 \) the equally weighted superposition of the ground state and the highest excited state is obtained from (20),

\[ |\Psi_{0n}\rangle = \exp\left(-i\frac{\pi}{2}Q_{ny}\right)|00\ldots0\rangle = \frac{1}{\sqrt{2}}(|00\ldots0\rangle + |11\ldots1\rangle). \]

(22)

The efficient quantum circuit for the highest-order quantum unitary operator \( U_{ny}(\theta) \) is constructed below. By using the quantum-state diagonal operator \( D_0 \) the \( n \)-order quantum operator \( Q_{ny} \) may be expressed as

\[ 2iQ_{ny} = [D_0, 2^n I_{1x}I_{2x}...I_{nx}]. \]

(23)

On the other hand, the operator \( Q_{ny} \) can also be written as

\[ 2iQ_{ny} = (-i)\exp(i\varphi I_z)[D_0, 2^n I_{1x}I_{2x}...I_{nx}] + \exp(-i\varphi I_z) \]

(24)

with \( n\varphi = \pi/2 \). The relation (24) can be proved below. Since there holds the unitary transformation: \( \exp(-i\varphi I_{kz})I_k^+ \exp(i\varphi I_{kz}) = \exp(\mp i\varphi)I_k^\pm \) [43] it follows from (18) and (19) that there exists the unitary transformation when the unitary operator \( \exp(-i\varphi I_z) = \exp[-i\varphi \sum_{k=1}^n I_{kz}] \) with \( n\varphi = \pi/2 \) acts on the \( n \)-order quantum operator \( Q_{ny} \),

\[
\exp(-i\varphi I_z)Q_{ny}\exp(i\varphi I_z) = \frac{1}{2i} [\exp(-in\varphi)I_1^+I_2^+...I_n^+ - \exp(in\varphi)I_1^-I_2^-...I_n^-]
\]

\[ = -\frac{1}{2}[D_0, 2^n I_{1x}I_{2x}...I_{nx}] = -Q_{nx} \]

Obviously, the relation (24) can be obtained directly from this unitary transformation. There is a general unitary transformation identity for the selective rotation operation \( C_{t}(\theta) \) [15],

\[
C_{t}(\theta)\rho C_{t}(\theta)^{-1} = \rho - (1 - \cos \theta)[\rho, D_t]_+ + i\sin \theta[\rho, D_t]_+ + 2(1 - \cos \theta)D_t\rho D_t.
\]

(25)

Taking \( \rho = 2^n I_{1x}I_{2x}...I_{nx}, \) \( D_t = D_0, \) and \( \theta = \pi, \) and noting that there holds the operator identity \( D_t 2^n I_{1x}I_{2x}...I_{nx}D_t = 0 \) for any index \( t, \) one obtains the
following relation from the identity (25),

\[
[D_0, 2^n I_1 x I_2 x \cdots I_{nx}]^+ = \frac{1}{2} \{ 2^n I_1 x I_2 x \cdots I_{nx} - C_0(\pi) 2^n I_1 x I_2 x \cdots I_{nx} C_0(\pi)^{-1} \}. \tag{26}
\]

With the help of the relations (24) and (26) and the Trotter-Suzuki formula [44] the quantum circuit for the highest-order quantum unitary operator \( U_{ny}(\theta) \) can be constructed efficiently by

\[
U_{ny}(\theta) = \exp(-i2\theta Q_{ny})
= \exp(i\varphi I_z) \{ C_0(\pi) G C_0(\pi)^{-1} G^{-1} \}^m \exp(-i\varphi I_z) + O(m^{-1}) \tag{27}
\]

where the unitary operation \( G = \exp(-i\theta 2^n I_1 x I_2 x \cdots I_{nx} / m) \) can be decomposed efficiently into a sequence of one- and two-qubit quantum gates [15]. Note that the norms \( ||D_0|| = 1 \) and \( ||2^n I_1 x I_2 x \cdots I_{nx}|| = 1 \). For a modest integer \( m \) the decomposition (27) converges quickly.

With the help of the unitary transformations of (20) and (21) of the highest-order quantum unitary operator \( U_{ny}(\theta) \) one can set up two quantum circuits to judge whether a known quantum state is just the solution of the quantum search problem or not in polynomial time. One quantum circuit \( U_{0n}(\pi) \) is constructed with the selective inversion operation \( C_t(\pi) \) and the highest-order quantum unitary operator \( U_{ny}(\theta) \),

\[
U_{0n}(\pi) |00...0\rangle = \exp(i\frac{\pi}{2} Q_{ny}) C_t(\pi) \exp(-i\frac{\pi}{2} Q_{ny}) |00...0\rangle
= \begin{cases} 
|11...1\rangle, & \text{if } t = 0 \\
-|11...1\rangle, & \text{if } t = N - 1 \\
|00...0\rangle, & \text{if } t \neq 0, N - 1 
\end{cases}
\]

where \( N = 2^n \). The quantum circuit \( U_{0n}(\pi) \) acting on the ground state \( |00...0\rangle \) induces the highest-order quantum transition only when the selective inversion operation \( C_t(\pi) \) with \( t = 0 \) or \( N - 1 \) is applied to either the ground state \( |00...0\rangle \) or the highest excited state \( |11...1\rangle \), while for any other selective inversion operation \( C_t(\pi) \) with \( t \neq 0 \) and \( N - 1 \) which is applied to neither the ground state nor the highest excited state the quantum circuit \( U_{0n}(\pi) \) induces no transition from the ground state to the highest excited state.

Generally, the quantum circuit \( U_{0n}(\theta) \) with a general selective rotation operation \( C_t(\theta) \) \( (-\pi \leq \theta \leq \pi) \) acting on the ground state induces the \( n \)-order
quantum transition with a transition probability dependent on the rotation angle θ,

\[
\exp(i\frac{1}{2}πQ_{ny})C_t(\theta)\exp(-i\frac{1}{2}πQ_{ny})|00...0\rangle
\]

\[
= \begin{cases} 
  P_+|00...0\rangle + P_-|11...1\rangle, & \text{if } t = 0 \\
  P_+|00...0\rangle - P_-|11...1\rangle, & \text{if } t = N - 1 \\
  |00...0\rangle, & \text{if } t \neq 0, N - 1
\end{cases}
\]

with \( P_\pm = \frac{1}{2}(1 \pm \exp(-iθ)) \), but the quantum circuit does not induce any quantum transition when the selective rotation operation \( C_t(θ) \neq C_0(θ) \) and \( C_{N-1}(θ) \). When \( C_t(θ) = C_0(θ) \) or \( C_{N-1}(θ) \) the unitary operation \( U_{0n}(θ) \) does induce the highest order quantum transition with the transition probability:

\[
P_{0n}(θ) = |P_-|^2 = \frac{1}{2}(1 - \cos θ).
\]

The transition probability \( P_{0n}(θ) \geq 0.5 \) when \( \pi/2 \leq |θ| \leq π \).

Using the total quantum circuit \( U_{0n}(θ)|00...0\rangle \) (\( \pi/2 \leq |θ| \leq π \)) which includes the initial state, i.e., the ground state, one can know whether the quantum state \( |t\rangle \) is one of the two states: the ground state and the highest excited state or any other quantum state of the Hilbert space. If the quantum state \( |t\rangle \) is either the ground state or the highest excited state, then one need use further another quantum circuit \( U'_{0n} \) to determine certainly the quantum state \( |t\rangle \) to be the ground state or the highest excited state,

\[
U'_{0n}|00...0\rangle \equiv \exp(i\frac{1}{2}πQ_{ny})C_0(π/2)C_t(-π/2)\exp(-i\frac{1}{2}πQ_{ny})|00...0\rangle
\]

\[
= \begin{cases} 
  |00...0\rangle, & \text{if } t = 0 \\
  i|11...1\rangle, & \text{if } t = N - 1
\end{cases}
\]

If the quantum state \( |t\rangle \) is the highest excited state, which means that \( C_t(-π/2) = C_{N-1}(-π/2) \), then there is an \( n \)-order quantum transition from the ground state to the highest excited state under the action of the unitary operation \( U'_{0n} \) on the ground state, otherwise there is not such an \( n \)-order quantum transition and the ground state keeps unchanged. Now it is easy to judge if an unknown state \( |t\rangle \) is the state \( |00...0\rangle \) or the state \( |11...1\rangle \) or any other state of the Hilbert space by using first the quantum circuit \( U_{0n}(π)|00...0\rangle \) and then \( U'_{0n}|00...0\rangle \).

It is well known in computational complexity that an NP-hard problem is hard to be solved on a classical computer, but whether a given solution
is just the real solution to the NP problem or not can be efficiently checked computationally. This fact is also true on a quantum computer. How to confirm whether a given state is the solution to the quantum search problem on a quantum computer? Suppose that the marked state $|s\rangle$ is the real solution to the quantum search problem and the oracle unitary operation of the marked state is $C_s(\theta)$. For a given quantum state $|r\rangle$ one knows its dynamical parameter vector $\{a^r_k\}$, an example can be seen in equation (17). One first sets up an auxiliary oracle unitary operation $C_t(\theta) = U_{or}C_s(\theta)U_{or}^+$:

$$U_{or}C_s(\theta)U_{or}^+ = \begin{cases} C_0(\theta), & \text{if } |r\rangle = |s\rangle \\ C_t(\theta) (t \neq 0), & \text{if } |r\rangle \neq |s\rangle \end{cases}$$

where the known unitary operator $U_{or}$ that depends upon the dynamical parameter vector $\{a^r_k\}$ is given by [15a],

$$U_{or} = \prod_{k=1}^n \{\exp(i\pi I_{kk}/2) \exp(-i\pi a^r_k I_{kk}/2)\}.$$  

Then using the quantum circuit $U_{0n}(\pi)|00...0\rangle$ one knows whether the auxiliary oracle unitary operation $C_t(\theta)$ is just $C_0(\theta)$ or $C_{N-1}(\theta)$ or any other one. If $C_t(\theta) \neq C_0(\theta)$ and $C_{N-1}(\theta)$, then the quantum state $|r\rangle$ is not the real solution $|s\rangle$ to the quantum search problem. If $C_t(\theta) = C_0(\theta)$ or $C_{N-1}(\theta)$, then the quantum circuit $U'_{0n}|00...0\rangle$ is further used to judge whether $C_t(\theta) = C_0(\theta)$ or $C_t(\theta) = C_{N-1}(\theta)$. If $C_t(\theta) = C_{N-1}(\theta)$, then the state $|r\rangle$ is not the solution $|s\rangle$. But if $C_t(\theta) = C_0(\theta)$ one knows certainly the quantum state $|r\rangle$ is just the solution $|s\rangle$. Therefore, in polynomial time one can confirm whether a given quantum state is just the solution to the quantum search problem.

Both the ground state $|00...0\rangle$ and the highest excited state $|11...1\rangle$ of the Hilbert space of the $n$-qubit spin system with $n = [\log_2 p] + 1$ do not belong the search space $S(m_k)$. This is clear that the ground state $|00...0\rangle$ is not contained in the multiplicative cyclic group state space $S(C_{p-1})$, as shown in section 2.1. On the other hand, the prime $p$ is less than $2^n$ with $n = [\log_2 p] + 1$, that is, $p \leq 2^n - 1$, then $p - 1 \leq 2^n - 2$, which means that every cyclic group state $|g^n \mod p\rangle$ of the state space $S(C_{p-1})$ corresponds one-to-one to its own integer $g^n \mod p \in Z_p^+$ which is never greater than $2^n - 2$, while the highest excited state $|11...1\rangle$ stands for the number $2^n - 1$. Therefore, the cyclic group state space $S(C_{p-1})$ does not contain the state $|11...1\rangle$. By checking the quantum program $Q_p$ and the quantum circuit $Q_c$
in section 5 one can see that the state $|00...0\rangle$ has been used by the program $Q_p$ in the unitary transformation: $|g_r(y)\rangle |n_h\rangle = |1\rangle |0\rangle \leftrightarrow |0\rangle |0\rangle$ and by the quantum circuit $Q_c$ as the control state $|0\rangle$ of the control subspace $\{|c\rangle, |0\rangle\}$ with $|c\rangle \neq |11...1\rangle$, but that state $|00...0\rangle$ is not in the current search space $S(m_r)$, while the state $|11...1\rangle$ of the Hilbert space that contains the search space $S(m_r)$ is never used by both the program and the quantum circuit. Therefore, there hold the unitary transformations: $Q_p|00...0\rangle = |00...0\rangle$ and $Q_p|11...1\rangle = |11...1\rangle$ in the search space $S(m_r)$. This is also in agreement with the fact that the oracle unitary operation $U_{osk}(\theta)$ does not make an effect on both the states. The quantum circuit $U_{0n}(\pi)|00...0\rangle$ now can be modified so that it can be used to determine the index $s_k$ of the oracle unitary operation $U_{osk}(\theta)$. Obviously, the superposition $|\Psi_{0n}\rangle$ of (22) is not in the search space $S(m_r)$ and not affected by the quantum program $Q_p$. Now the ground state $|00...0\rangle$ in the superposition $|\Psi_{0n}\rangle$ is changed to the state $|1\rangle$ by the unitary operation $F_1$ and further changed to the cyclic group state $|(g^{M_r})^x \mod p\rangle$ by the cyclic group operation $(U_{g^{M_r}})^x$,

$$F_1 (U_{g^{M_r}})^x |\Psi_{1n}\rangle = \frac{1}{\sqrt{2}} ((g^{M_r})^x \mod p \rangle + |11...1\rangle).$$

The unitary operation $F_1$ and the cyclic group operation $(U_{g^{M_r}})^x$ do not affect the highest-level state $|11...1\rangle$. If now the superposition $|\Psi_{1n}\rangle$ is taken as the input state of the oracle unitary operation $U_{osk}(\pi)$, then in effect the input state is essentially a single basis state for the oracle unitary operation $U_{osk}(\pi)$ and also for the quantum program $Q_p$. Since the highest-level state $|11...1\rangle$ is not in the search space $S(m_r)$ and also not affected by the quantum program, there is only the single basis state $|(g^{M_r})^x \mod p\rangle$ in the state $|\Psi_{1n}\rangle$ that the quantum program can take an action, although the state $|\Psi_{1n}\rangle$ is a superposition of two states. Now it is applied the oracle unitary operation $U_{osk}(\pi)$ to the state $|\Psi_{1n}\rangle$. Note that only the basis state $|(g^{M_r})^x \mod p\rangle$ in the state $|\Psi_{1n}\rangle$ is affected by the oracle unitary operation. If the index $s_k = x$ then the state $|(g^{M_r})^x \mod p\rangle$ is inverted by the oracle unitary operation $U_{osk}(\pi)$, otherwise the state $|\Psi_{1n}\rangle$ keeps unchanged. After these unitary operations the state $|(g^{M_r})^x \mod p\rangle$ is changed back to the ground state $|0\rangle$ by the inverse operations $[(U_{g^{M_r}})^x]^+$ and $F_1^+$. At the final step the inverse $n$–order quantum unitary operation $\exp(i\frac{\pi}{2}Q_{ny})$ is applied so that it can
be judged whether the index \( x = s_k \) or not by the quantum measurement. The final result is given by

\[
Q(x, s_k)|00...0\rangle = \exp\left(i\frac{1}{2}\pi Q_{ny}\right)F_1^+[(U_{g^{m_r}})^x]\mathcal{U}_{osk}(\pi)
\]

\[\times\left[(U_{g^{m_r}})^x F_1 \exp\left(-i\frac{1}{2}\pi Q_{ny}\right)|00...0\rangle\right]
\]

\[
= \begin{cases} 
|11...1\rangle, & \text{if } x = s_k \\
|00...0\rangle, & \text{if } x \neq s_k 
\end{cases}
\]

The quantum measurement is carried out on the highest-level state \(|11...1\rangle\). Given the oracle unitary operation \(\mathcal{U}_{osk}(\pi)\) one can try \(m_r\) different index values \(x = 0, 1, ..., m_r - 1\) at most with the quantum circuit \(Q(x, s_k)|00...0\rangle\) to find the index \(s_k\) due to the fact that \(0 \leq s_k < m_r\). If the highest-level state \(|11...1\rangle\) is measured in a high probability (\(\sim 1\)), then the corresponding index value \(x\) is just the index \(s_k\). Again it is pointed out that the input state of the quantum program \(Q_p\) is essentially limited to be a single basis state during the quantum search process. When the index values \(\{s_k\}\) are obtained one may use the index identity (3) or (12) to compose the index \(s\) and hence the marked state \(|s\rangle\) is found ultimately for the quantum search problem in the cyclic group state space.

7. Discussion

In the paper an oracle-based quantum dynamical method has been set up to solve the quantum search problem in the cyclic group state space of the Hilbert space of an \(n\)-qubit pure-state quantum system. The main attempt is to make use of the symmetric properties and structures of groups to help solving a general unstructured quantum search problem in the Hilbert space.

It is known that the hardness to solve an unstructured quantum search problem by a standard quantum search algorithm mainly originates from the low efficiency to amplify the amplitude of the marked state in the Hilbert space by the oracle unitary operation associated with other known quantum operations. This low amplitude-amplification efficiency results in that a standard quantum search algorithm generally can have only a square speedup over the best known classical counterparts. In order to break through the square speedup limitation it is necessary to develop other type of quantum search algorithms. The quantum dynamical method [15] may be a better choice, for it allows a parameterization description for an unknown quantum state such as the marked state and its oracle unitary operation in the
Hilbert space of the $n-$qubit quantum system. Since the oracle unitary operation corresponds one-to-one to the unknown marked state, with the help of the parameterization description the quantum dynamical method makes it possible to manipulate at will the evolution process of the marked state in the quantum system and hence it also makes it possible to manipulate at will the oracle unitary operation. The quantum dynamical method is different from the standard quantum search algorithm in that any quantum state of the Hilbert space can be described completely by a set of dynamical parameters and hence the quantum searching for the marked state can be indirectly achieved by determining the set of dynamical parameters which describe completely the marked state instead by directly measuring the marked state. Therefore, amplification of amplitude of the marked state and the direct measurement on the marked state to obtain the complete information of the marked state, both are the key components of a standard quantum search algorithm, may not be necessary in the quantum dynamical method. In the quantum dynamical method the quantum measurement to output the computing results may be carried out on those states that carry the information of the marked state, while the complete information of the marked state can be further extracted from these computing results. In the paper the binary dynamical representation for a quantum state in the Hilbert space of an $n-$qubit quantum system is generalized to a general multi-base dynamical representation for a quantum state in a cyclic group state space and the quantum dynamical method therefore is extended to solve the quantum search problem in the cyclic group state space of the Hilbert space.

A cyclic group state space of the Hilbert space of an $n-$qubit quantum system carries the symmetric property and structure of the cyclic group. A quantum search process may be affected greatly by the symmetric property and structure of the cyclic group if the quantum search is performed in the cyclic group state space. It is known that the amplitude-amplification efficiency for the marked state by the oracle unitary operation associated with other known unitary operations generally is inversely proportional to the square root of the dimensional size of the search space of the quantum search problem and this low efficiency results in the square speedup limitation for a standard quantum search algorithm. There is naturally a possible scheme to bypass this speedup limitation that the search space of the problem is limited to a small subspace of the Hilbert space so that this speedup limitation becomes less important or even unimportant in the quantum search problem. Therefore, it is a challenge how to reduce efficiently the search
space from the whole Hilbert space to its small subspaces in the unstructured quantum search problem in the Hilbert space. It has been shown that the symmetric property and structure in spin space of an $n$-qubit spin system may be helpful for this reduction of search space. In the paper it is made a further emphasis and generalization for the idea that the symmetric property and structure of a quantum system or even a group may be employed to speed up the quantum search process through the scheme of the search-space reduction. A cyclic group is one of the simplest groups and its symmetric property and structure has been studied in detail and extensively. Therefore, it could be simplest and most convenient to exploit the symmetric property and structure of a cyclic group to help solving the quantum search problem in the cyclic group state space of the Hilbert space.

The reversible mathematical-logic operations have been used extensively in quantum computation. They may be generally thought of as selective unitary operations in a quantum system and have been employed in the construction of quantum search processes in the cyclic group state space. A large advantage for the type of unitary operations is that the time evolution process of a quantum state in a complex multi-qubit quantum system may be traced more easily under the action of the mathematical-logic operations. However, in order to be reversible and unitary a logic operation in mathematics usually needs to consume much more extra auxiliary qubits with respect to those unitary operators quantum physically. Since the dimensional size of the Hilbert space of a quantum system increases exponentially as the qubit number, it must be careful to use the mathematical-logic operations in solving a quantum search problem, otherwise these extra auxiliary qubits could lead to a large search space for the quantum search problem and make the quantum search process degraded. On the other hand, the conventional unitary operators, propagators, operations, or quantum gates in a quantum system in physics usually need not any extra auxiliary qubits except those artificial conditional unitary operations which usually need only few extra qubits to help to achieve some specific conditional operations instead of their unitarity. The time evolution process of a quantum state in a multi-qubit quantum system generally is complex and is not easy to trace under the action of the type of unitary operations. However, there is a general rule that any unknown quantum state can be efficiently transferred to a larger subspace from a small subspace in the Hilbert space of the multi-qubit quantum system. Through this general rule one could set up the connection between the Hilbert space of the $n$-qubit quantum system and its cyclic group state.
space for an unstructured quantum search problem.

It has been shown that if there existed a universal quantum computer that in computation obeys the unitary quantum dynamics in physics and is capable of computing any computable functions in mathematics such as any recursive functions, then such a universal quantum computer would be enough powerful to solve efficiently the quantum search problem in the cyclic group state space. There seems to be a question whether such an ideal universal quantum computer existed or not. This question is due to the argument that a universal quantum computer could not have a satisfactory halting protocol when its input state is a superposition. However, as far as the present quantum search process in the multiplicative cyclic group state space is concerned, there seems not to be such a question because the input state in the quantum search process can be strictly limited to be a single basis state. An ideal quantum program, which is a key component of the present quantum search process, is designed for the efficient reduction of quantum search space for the quantum search problem. It has been shown in theory that this quantum program could be run unitarily on an ideal universal quantum computer when its input state is strictly limited to be a single basis state and hence it could be used to solve efficiently the quantum search problem in the cyclic group state space. Moreover, a quantum circuit is also designed to simulate efficiently the ideal quantum program. The key point for the quantum circuit is to use the state-locking pulse and the two-level control subspace to simulate efficiently the unitary halting protocol of the quantum program. Although at present a state-locking pulse that is continuously applied to a quantum system during the whole period of the quantum circuit is not popularly used in quantum computation, a large number of similar techniques have been used extensively in the conventional NMR experiments [43]. Obviously, it is necessary to further investigate in detail the quantum circuit in some important problems such as how to design a state-locking pulse with a better performance and how the state-locking pulse affects the practical computational complexity of the quantum circuit and the whole quantum search process. Evidently, it is possible to design simpler quantum program and quantum circuit than the present ones to solve the quantum search problem in the cyclic group state space.

With the help of the symmetric property and structure of a cyclic group and the Chinese remainder theorem in number theory any quantum state in the cyclic group state space can be efficiently converted into a tension product of the states of the cyclic group state subspaces of the cyclic group.
state space. There are the relations among these states of the cyclic group state subspaces through the Chinese remainder theorem. These relations are important and may be further employed to develop efficient quantum search methods in the cyclic group state space in the future work.

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