STOCHASTIC FLOWS OF SDES WITH IRREGULAR COEFFICIENTS AND
STOCHASTIC TRANSPORT EQUATIONS

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ABSTRACT. In this article we study (possibly degenerate) stochastic differential equations (SDE)
with irregular (or discontinuous) coefficients, and prove that under certain conditions on the co-
efficients, there exists a unique almost everywhere stochastic (invertible) flow associated with
the SDE in the sense of Lebesgue measure. In the case of constant diffusions and BV drifts,
we obtain such a result by studying the related stochastic transport equation. In the case of non-
constant diffusions and Sobolev drifts, we use a direct method. In particular, we extend the recent
results on ODEs with non-smooth vector fields to SDEs. Moreover, we also give a criterion for
the existence of invariant measures for the associated transition semigroup.

1. Introduction

Consider the following Itô’s stochastic differential equation (SDE):
\[ dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \]  
(1.1)
where \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m \) are two Borel measurable functions, and \((W_t)_{t \geq 0}\) is the \( m \)-dimensional standard Brownian motion on the classical Wiener space \((\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0})\), i.e., \( \Omega \) is the space of all continuous functions from \( \mathbb{R}^+ \) to \( \mathbb{R}^m \) with locally uniform convergence topology, \( \mathcal{F} \) is the Borel \( \sigma \)-field, \( P \) is the Wiener measure, \((\mathcal{F}_t)_{t \geq 0}\) is the natural filtration generated by the coordinate process \( W_t(\omega) = \omega(t) \).

It is by now a classical result that if \( b \) and \( \sigma \) are globally Lipschitz continuous, then there exists a unique bi-continuous solution \((t, x) \mapsto X_t(x)\) to SDE (1.1) such that for almost all \( \omega \) and any \( t \geq 0, x \mapsto X_t(\omega, x) \) is a homeomorphism. Thus, \((X_t(x), x \in \mathbb{R}^d_{t \geq 0})\) forms a stochastic homeomorphism flow (cf. [16]). Recently, there are increasing interests for studying the stochastic homeomorphism flow property associated with SDE (1.1) under various non-Lipschitz assumptions on \( b \) and \( \sigma \) (cf. [21, 1, 20, 25, 8, 9, 10, 11, 28], etc.). Here, the non-Lipschitz conditions may be less smooth or not global Lipschitz.

On the other hand, when \( \sigma \) is non-degenerate and \( b \) is not continuous and even singular, SDE (1.1) may have a unique strong solution for each starting point \( x \in \mathbb{R}^d \) (cf. [14, 17, 26], etc.). But it is not known whether it still defines a stochastic homeomorphism flow. In the completely degenerate case \((\sigma = 0)\), a celebrated theory established by DiPerna and Lions [6] says that ordinary differential equation (ODE)
\[ dX_t = b(X_t)dt, \quad X_0 = x \]
(1.2)
defines a regular Lagrangian flow in the sense of Lebesgue measure when \( b \) is a Sobolev vector field with bounded divergence. This theory was later extended to the case of BV vector fields by

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Ambrosio [2]. The central of DiPerna and Lions’ theory are based on the connection between ODE and the Cauchy problem for the transport equation:

$$\partial_t u + b^i \partial_i u = 0, \quad u|_{t=0} = u_0. \quad (1.3)$$

Here and below, we use the usual convention: the repeated indices will be summed. By introducing a new notion of renormalized solutions, DiPerna and Lions showed the uniqueness and stability of $L^\infty$-distributional solutions for (1.3) when $b$ is Sobolev regular so that they can go back to ODE and show the well posedness of (1.2) with Sobolev vector field $b$ in the distributional sense.

We now back to SDE (1.1). It is also well known that SDE (1.1) is connected with the following stochastic transport equation (cf. [16, 24]):

$$du = \frac{1}{2} \sigma^{il} \sigma^{jl} \partial_{x_i}^2 u - (b^j - \sigma^{ij} \partial_j \sigma^{il}) \partial_l u dt - \sigma^{il} \partial_i u dW_t^j, \quad u|_{t=0} = u_0. \quad (1.4)$$

Thus, it is natural to ask whether we can extend the DiPerna and Lions theory to the case of SDEs. Notice that (1.4) is always a degenerate second order stochastic parabolic equation whatever $\sigma$ is or not degenerate. More general second order linear stochastic partial differential equation has been recently studied in [28]. In general, it is hard to solve equation (1.4) if $b$ and $\sigma$ are not smooth (cf. [24]). The source of difficulty clearly comes from the degeneracy. Nevertheless, we can extend the well known theory about the transport equation to the case of constant $\sigma$ and BV vector field $b$. In this case, it will be shown that we can also go back to SDE (1.1) from stochastic transport equation (1.4) and obtain the well posedness of SDE (1.1) with BV drift. We remark that in another direction, Flandoli, Gubinelli and Priola [13] studied the well posedness of (1.4) when $b$ is Hölder continuous and $\sigma$ is the unit matrix, where their proofs benefit from the stochastic flow associated with SDE (1.1). We emphasize that when $\sigma$ is constant, SDE (1.1) can be directly solved by transferring it to a time dependent ODE. But, this will lose some “stochastic flavor”.

Recently, Crippa and De Lellis [5] derived some new estimates for ODEs with Sobolev coefficients. These estimates allowed them to give a direct and simple treatment for DiPerna-Lions flows. The key ingredient of their method is to give some control for the following quantity in terms of $\|\nabla b\|_{L^p}$ ($p > 1$):

$$\int_{B_r} \sup_{x \in \mathbb{R}^d} \sup_{\tau \in [0, T] \cap \mathbb{Z}} \left[ \int_{B_{\delta r}} \log \left( \frac{|X_t(x) - X_t(x + y)|}{\delta} + 1 \right) dy \right]^\theta dx,$$

where $B_r := \{ x \in \mathbb{R}^d : |x| \leq r \}$ denotes the ball with radius $r$ and center 0. For estimating this quantity, the Hardy-Littlewood maximal function was used to control the difference $|b(X_t(x)) - b(X_t(x + y))|$. Moreover, the stability was also derived in [5] by using a similar quantity. We remark that the above quantity was first introduced in [3] in order to prove the approximative differentiability of regular Lagrangian flows. The second part of this paper is to extend Crippa and De Lellis’ result to the stochastic case so that $\sigma$ can be non-constant.

We also mention that Figalli [12] has already developed a stochastic counterpart for DiPerna-Lions theory. Therein, the martingale solution (or weak solution) in the sense of Stroock-Varadhan was considered corresponding to the Fokker-Planck equation. Moreover, the non-degenerate condition on $\sigma$ is required when $\sigma$ is non-constant. Compared with [12], we can directly construct the “strong” solution of SDE (1.1) with Sobolev drift and possibly degenerate diffusion coefficients in the sense of Lebesgue measure. Moreover, as an easy consequence, we can uniquely solve the SDE in the classical sense when the initial value is an absolutely continuous $\mathcal{F}_0$-measurable random variable (see Corollary 6.4 and Corollary 6.5 below). It should be noted that for the simplicity, we only consider the time independent coefficients in the present paper. Clearly, our results can be extended to the time dependent case by requiring some integrability in the time variable.
In the study of stochastic dynamical systems, an important problem is to prove the existence of equilibrium point (invariant measure). Since we are dealing with non-smooth stochastic differential equations, it is not expected to have the Feller property for the associated transition semigroup. Thus, it seems that the classical coercivity condition is not enough to guarantee the existence of an invariant probability measure for SDE (1.1) (cf. [4, 16]). In the present paper, we shall give a criterion for the existence of an invariant probability measure in terms of the classical coercivity condition as well as some divergence condition (see Theorem 2.8 below). We want to emphasize that in our result, such an invariant measure is indeed absolutely continuous with respect to the Lebesgue measure.

This paper is organized as follows: in Section 2, after introducing the notion of almost everywhere stochastic (invertible) flow, we give two direct consequences of this notion and then state our main results. In Section 3, we give some necessary preliminaries for later use. In Section 4, we study stochastic transport equation (1.4) in case that $b \in BV_{loc}$ has bounded divergence and $\sigma$ is constant. In Section 5, we apply the results of Section 4 to the study of stochastic flows of SDE with BV drift and constant diffusion coefficients. In Section 6, we extend the result of [5] to the stochastic case. Here, an SDE with discontinuous coefficients is provided to show our result. This section can be read independently of Sections 4 and 5. In Section 7, we prove our main results. In the appendix, we give a detailed proof about the flow property as well as the Markov property when SDE (1.1) admits a unique almost everywhere stochastic flow in the sense of Definition 2.1 below.

2. Main Results

We first introduce some necessary notations. Let $(E, \mathcal{E}, \mu)$ be a measure space and $\mathcal{T} : E \to E$ a measurable transformation. We shall use $\mu \circ \mathcal{T}$ to denote the image measure of $\mu$ under $\mathcal{T}$, i.e., for any nonnegative measurable function $\varphi$,

$$\int_E \varphi(x) \mu(T(dx)) := \int_E \varphi(T(x)) \mu(dx).$$

By $\mu \circ \mathcal{T} \ll \mu$ we mean that $\mu \circ \mathcal{T}$ is absolutely continuous with respect to $\mu$. Let $C_0^\infty(\mathbb{R}^d)$ be the set of all smooth functions on $\mathbb{R}^d$ with compact supports, $C_b(\mathbb{R}^d)$ the set of all bounded continuous functions, and $\mathcal{L}^*(\mathbb{R}^d)$ the set of all nonnegative Borel measurable functions. Below, we shall denote the Lebesgue measure by $\mathcal{L}(dx)$ or $dx$.

Convention: The repeated indices will be summed. The letter $C$ with or without subscripts will denote a positive constant whose value is not important and may change in different occasions. Moreover, all the derivatives, gradients and divergences are taken in the distributional sense.

We introduce the following notion of almost everywhere stochastic (invertible) flows, which is inspired by LeBris and Lions [18] and Ambrosio [2].

**Definition 2.1.** Let $X_t(\omega, x)$ be a $\mathbb{R}^d$-valued measurable stochastic field on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d$. We say $X$ an almost everywhere stochastic flow of (1.1) corresponding to $(b, \sigma)$ if

(A) For $\mathcal{L}$-almost all $x \in \mathbb{R}^d$, $t \mapsto X_t(x)$ is a continuous ($\mathcal{F}_t$)-adapted stochastic process satisfying that for any $T > 0$

$$\int_0^T |b(X_s(x))| ds + \int_0^T |\sigma(X_s(x))|^2 ds < +\infty, \quad P-a.s.,$$

and solves

$$X_t(x) = x + \int_0^t b(X_s(x)) ds + \int_0^t \sigma(X_s(x)) dW_s, \quad \forall t \geq 0.$$
(B) For any $t \geq 0$ and $P$-almost all $\omega \in \Omega$, $L \circ X_t(\omega, \cdot) \ll L$. Moreover, for any $T > 0$, there exists a constant $K_{T,b,\sigma} > 0$ such that for all $\varphi \in L^+_{\mathbb{R}^d}$

$$\sup_{t \in [0,T]} \mathbb{E} \int_{\mathbb{R}^d} \varphi(X_t(x))dx \leq K_{T,b,\sigma} \int_{\mathbb{R}^d} \varphi(x)dx. \quad (2.1)$$

We say $X$ an almost everywhere stochastic invertible flow of (1.1) corresponding to $(b, \sigma)$ if in addition to the above (A) and (B).

(C) For any $t \geq 0$ and $P$-almost all $\omega \in \Omega$, there exists a measurable inverse $X^{-1}_t(\omega, \cdot)$ of $X_t(\omega, \cdot)$ so that $L \circ X^{-1}_t(\omega, \cdot) = \rho_t(\omega, \cdot) L$, where the density $\rho_t(\omega)$ is given by

$$\rho_t(x) := \exp \left\{ \int_0^t \left[ \text{div} b - \frac{1}{2} \partial_i \sigma^j \partial_j \sigma^i \right](X_s(x))ds + \int_0^t \text{div} \sigma(X_s(x))dW_s \right\}. \quad (2.2)$$

Here, $\text{div} \sigma := \partial_i \sigma^i$ and we require that for any $T > 0$ and $L$-almost all $x \in \mathbb{R}^d$,

$$\int_0^T \left[ |\text{div} b| + |\partial_i \sigma^j \partial_j \sigma^i| + |\text{div} \sigma|^2 \right](X_s(x))ds < +\infty, \quad P - a.s.$$

Remark 2.2. If $\sigma = \text{constant}$ and $\text{div} b \in L^\infty(\mathbb{R}^d)$, then (C) clearly implies (B). In fact, in this case we have

$$L \circ X_t(\omega, \cdot) = \rho_t^{-1}(\omega, X^{-1}_t(\omega, \cdot))L$$

and by (2.2)

$$|\rho_t^{-1}(\omega, X^{-1}_t(\omega, x))| \leq e^{f|\text{div} b|}_\omega.$$ 

In what follows, for the simplicity of notations, we shall drop the time variable $t$ and the spatial variable $x$ if there are no confusions. For examples, for a function $f_t(x)$, we simply write

$$\int_0^T \int_{\mathbb{R}^d} f := \int_0^T \int_{\mathbb{R}^d} f_t(x)dxd\sigma$$

and

$$\int_0^T \int_{\mathbb{R}^d} f dW_s := \int_0^T \int_{\mathbb{R}^d} f_t(x)dxdW_s.$$

The following result is an easy consequence of Definition 2.1

Proposition 2.3. Assume that $b \in L^1_{\text{loc}}(\mathbb{R}^d)$ with $\text{div} b \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\sigma \in C^2(\mathbb{R}^d)$. Let $X$ be an almost everywhere stochastic invertible flow of (1.1) in the sense of Definition 2.1. Let $u_0 \in L^\infty(\mathbb{R}^d)$ and set $u_t(x) := u_t(X^{-1}_t(x))$. Then $u_t(x)$ solves the following stochastic transport equation in the distributional sense:

$$du = \frac{1}{2} \sigma^j \sigma^i \partial^2_{ij} u - b^j \partial_j u dt - \sigma^j \partial_j u dW^j_t,$$

where $b^j_{ij} := b^j - \sigma^j \partial^i \sigma^i$. In particular, $\bar{u}_t(x) := \mathbb{E}u_0(X^{-1}_t(x))$ is a distributional solution of the following second order parabolic differential equation:

$$\partial_t \bar{u} = \frac{1}{2} \sigma^j \sigma^i \partial^2_{ij} \bar{u} - b^j_{ij} \partial_i \bar{u}.$$

Proof. Let $\varphi \in C^\infty_c(\mathbb{R}^d)$. By (C) of Definition 2.1, we have

$$- \int_0^T \int \left( b^j_{ij} \partial_i u \right) \varphi = \int_0^T \int u_0(X^{-1}) \cdot \text{div}(b^j \varphi) = \int_0^T \int u_0 \cdot \text{div}(b^j \sigma \varphi)(X) \cdot \rho =$$

$$= \int_0^T \int u_0 \cdot (b^j_{ij} \partial_i \varphi)(X) \cdot \rho + \int_0^T \int u_0 \cdot (\varphi \text{div} b^j \sigma)(X) \cdot \rho.$$
Moreover, by stochastic Fubini’s theorem, we have
\[
- \int_0^t \int (\sigma^i_t \partial_i u)_t \phi dW^i_s = \int_0^t \int u_0 \cdot (\sigma^i_t \partial_i \phi)(X_t) \cdot \rho dW^i_s + \int_0^t \int u_0 \cdot (\partial_i \sigma^i_t \phi)(X_t) \cdot \rho dW^i_s
\]
and
\[
\frac{1}{2} \int_0^t \int \sigma^i_t \partial^i_j u \phi dW^i_s = \frac{1}{2} \int_0^t \int u_0 \cdot [\partial^2_{ij} \sigma^i_t \phi](X_t) \cdot \rho.
\]
Moreover, by stochastic Fubini’s theorem, we have
\[
\int_0^t \int u_0 \cdot (b^i_t \partial_i \phi)(X_t) \cdot \rho + \int_0^t \int u_0 \cdot (\sigma^i_t \partial_i \phi)(X_t) \cdot \rho dW^i_s
\]
\[
= \int u_0 \left( \int (b^i_t \partial_i \phi)(X_t) \cdot \rho ds + \int (\sigma^i_t \partial_i \phi)(X_t) \cdot \rho dW^i_s \right)
\]
and
\[
\int_0^t \int u_0 \cdot (\text{div} b^i_t \cdot \phi)(X_t) \cdot \rho + \int_0^t \int u_0 \cdot (\text{div} \sigma^i_t \cdot \phi)(X_t) \cdot \rho dW^i_s
\]
\[
= \int u_0 \left( \int (\text{div} b^i_t \cdot \phi)(X_t) \cdot \rho ds + \int (\text{div} \sigma^i_t \cdot \phi)(X_t) \cdot \rho dW^i_s \right).
\]

On the other hand, by (2.2) and Itô’s formula, we have
\[
\rho_t = 1 + \int_0^t \rho_s \left[ \text{div} b^i_t + \frac{1}{2} \partial^2_{ij} (\sigma^i_t \sigma^j_t) \right](X_s) ds + \int_0^t \rho_s \partial_i \sigma^i_t(X_s) dW^i_s,
\]
and
\[
d[\phi(X_t) \rho_t] = \left[ b^i_t \partial_i \phi + \frac{1}{2} \sigma^i_t \sigma^j_t \partial^2_{ij} \phi \right](X_t) \rho_t dt + (\sigma^i_t \partial_i \phi)(X_t) \rho_t dW^i_t
\]
\[
+ \phi \text{div} b^i_t \rho_t dt + \frac{1}{2} \phi \partial^2_{ij} (\sigma^i_t \sigma^j_t) \rho_t dt + [\phi \partial_i \sigma^i_t](X_t) \rho_t dW^i_t
\]
\[
= b^i_t \partial_i \phi(X_t) \rho_t dt + (\sigma^i_t \partial_i \phi)(X_t) \rho_t dW^i_t
\]
\[
+ \phi \text{div} b^i_t \rho_t dt + [\phi \partial_i \sigma^i_t](X_t) \rho_t dW^i_t
\]
\[
+ \frac{1}{2} \partial^2_{ij} (\sigma^i_t \sigma^j_t)(X_t) \rho_t dt.
\]

Combining the above calculations, we get
\[
\frac{1}{2} \int_0^t \int \sigma^i_t \partial^i_j u \phi - \int_0^t \int (b^i_t \partial_i u)_t \phi - \int_0^t \int (\sigma^i_t \partial_i u)_t \phi dW^i_s
\]
\[
= \int u_0 \left( \int_0^t d[\phi(X_t) \rho_t] \right) = \int u_0 [\phi(X_t) \rho_t - \phi]
\]
\[
= \int u_0 (X_t^{-1}) \phi - \int_0^t u_0 \phi = \int u_t \phi - \int_0^t u_0 \phi.
\]

The proof is complete. \(\square\)

The following proposition is much technical. We shall prove it in the appendix.

**Proposition 2.4.** Assume that SDE (1.1) admits a unique almost everywhere stochastic (or invertible) flow. Then the following flow property holds: for any \(s \geq 0\) and \((P \times \mathcal{L})\)-almost all \((\omega, x) \in \Omega \times \mathbb{R}^d\),
\[
X_{t+s}(\omega, x) = X_t(\theta_s \omega, X_s(\omega, x)), \quad \forall t \geq 0,
\]
(2.3)
where \( \theta_t \omega := \omega(s + t) - \omega(s) \). Moreover, for any bounded measurable function \( \varphi \) on \( \mathbb{R}^d \), define
\[
T_t \varphi(x) := \mathbb{E} \varphi(X_t(x)),
\]
then for any \( t, s \geq 0 \)
\[
\mathbb{E}(\varphi(X_{t+s}(x))|\mathcal{F}_s) = T_t \varphi(X_s(x)), \quad (P \times \mathcal{L}) - \text{a.e.} \quad (2.4)
\]
In particular, \( \mathbb{T}_t \) forms a bounded linear operator semigroup on \( L^p(\mathbb{R}^d) \) for any \( p \geq 1 \).

**Remark 2.5.** Here, an open question is that whether the following stronger flow property holds: For \( (P \times \mathcal{L}) \)-almost all \( (\omega, x) \in \Omega \times \mathbb{R}^d \)
\[
X_{t+s}(\omega, x) = X_t(\theta_t \omega, X_s(\omega, x)), \quad \forall t, s \geq 0.
\]
In the language of random dynamical systems (cf. [4], Definition 1.1.1]), property (2.3) is called “crude”, and property (2.5) is called “perfect”. A deep result of Arnold and Scheutzow (cf. [4], p.17, Theorem 1.3.2]) asserted that a crude cocycle admits an indistinguishable and perfect version. But, it seems that we can not use their result to deduce (2.5) since it is not clear how to endow a structure on the set of all measurable transformations so that it becomes a Hausdorff topological group with countable topological base.

Our main result of this paper is:

**Theorem 2.6.** Assume that
\[
\frac{|b(x)|}{1 + |x|^\ell} \text{div} b(x) \in L^\infty(\mathbb{R}^d) \quad (2.6)
\]
and one of the following conditions holds:
\[
b(x) \in BV_{\text{loc}} \text{ and } \sigma \text{ is independent of } x; \quad (2.7)
\]
\[
\left\{ \begin{aligned}
|\nabla b(x)| &\in (L \log L)_{\text{loc}}(\mathbb{R}^d), \\
|\nabla \sigma(x)|, \sup_{|z| < 1} |\sigma(x-z) \cdot |\nabla \sigma|(x) &\in L^\infty(\mathbb{R}^d). 
\end{aligned} \right. \quad (2.8)
\]

Then there exists a unique almost everywhere stochastic invertible flow of (1.1) corresponding to \( (b, \sigma) \) in the sense of Definition 2.1.

**Remark 2.7.** By definitions, \( b \in BV_{\text{loc}} \) means that \( \nabla b \) is a locally finite vector valued Radon measure on \( \mathbb{R}^d \); and \( |\nabla b| \in (L \log L)_{\text{loc}}(\mathbb{R}^d) \) means that \( |\nabla b| \log(|\nabla b| + 1) \in L^1_{\text{loc}}(\mathbb{R}^d) \). In particular, for any \( p > 1 \),
\[
L^p_{\text{loc}}(\mathbb{R}^d) \subset (L \log L)_{\text{loc}}(\mathbb{R}^d) \subset L^1_{\text{loc}}(\mathbb{R}^d).
\]
In (2.8), the second condition on \( \sigma \) is certain growth restriction of \( \sigma \) and \( \nabla \text{div} \sigma \).

About the existence of invariant measure of \( \mathbb{T}_t \), we have the following criterion.

**Theorem 2.8.** Assume that SDE (1.1) admits a unique almost everywhere stochastic flow with \( K_{T,b,\sigma} = K_{b,\sigma} \) in (2.1) independent of \( T \), and \( (b, \sigma) \) satisfies
\[
\langle x, b(x) \rangle_{\mathbb{R}^d} + ||\sigma(x)||_{H^2_{\text{HS}}}^2 < 0 \text{ (or } -C_1|x|^2 + C_2), \quad (2.9)
\]
where \( C_1, C_2 > 0 \), and \( ||\sigma(x)||_{H^2_{\text{HS}}} \) denotes the Hilbert-Schmidt norm of matrix \( \sigma(x) \). Then \( \mathbb{T}_t \) admits an invariant probability measure \( \mu(dx) = \gamma(x)dx \) with \( \gamma \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \) so that for all \( \varphi \in L^1(\mathbb{R}^d) \) and \( t \geq 0 \)
\[
\int_{\mathbb{R}^d} T_t \varphi(x) \gamma(x)dx = \int_{\mathbb{R}^d} \varphi(x) \gamma(x)dx. \quad (2.10)
\]

**Remark 2.9.** It is well known that if \( \mathbb{T}_t \) is a Feller semigroup, then under (2.9), there exists an invariant probability measure for \( \mathbb{T}_t \). In our case, \( \mathbb{T}_t \) may be not a Feller semigroup. In Theorem 6.3 below, we shall give a condition such that \( K_{T,b,\sigma} = K_{b,\sigma} \) in (2.7) is independent of \( T \).

These two theorems will be proved in Section 7.
3. Preliminaries

In this section, we prepare some lemmas for later use. Below, we consider SDE (1.1) and assume that \( b, \sigma \in C^\infty_b(\mathbb{R}^d) \) are \( C^\infty \)-smooth, which together with their derivatives of all orders are bounded. It is well known that the family of solutions \( \{X_t(x), t \geq 0\}_{x \in \mathbb{R}^d} \) to SDE (1.1) forms a \( C^\infty \)-diffeomorphism flow (cf. [15, 16]). We have the following simple result about the Jacobian determinant of stochastic flow.

**Lemma 3.1.** Let \( \rho_t(x) \) be defined by (2.2). Then

\[
\det(\nabla X_t(x)) = \rho_t(x) \tag{3.1}
\]

and for any \( T > 0 \) and \( p \geq 1 \),

\[
\mathbb{E}|\det(\nabla X_T^{-1}(x))|^p \leq \exp \left( p T \left( \| -\text{div} b + \frac{1}{2} \partial_i \sigma^j \partial_j \sigma^i + \sigma^j \partial_j \sigma^i + \frac{p}{2} |\text{div} \sigma|^2 \right\|_\infty \right) \tag{3.2}
\]

where for a real number \( a \), \( a^+ := a \vee 0 := \max(a, 0) \).

**Proof.** Let \( \tilde{b} := b - \frac{1}{2} \sigma \partial_j \sigma^j \). We write equation (1.1) as Stratonovich form:

\[
dX = \tilde{b}(X)dt + \sigma(X) \circ dW_t, \quad X_0 = x.
\]

Let \( W^n_t \) be the linearized approximation of \( W_t \). Consider the following ODE:

\[
dX_n(x) = \tilde{b}(X_n)dt + \sigma(X_n)W^n_t dt.
\]

Then,

\[
\det(\nabla X_n(t,x)) = \exp \left\{ \int_0^t \text{div} \tilde{b}(X_n(s,x)) ds + \int_0^t \text{div} \sigma(X_n(s,x)) W^n_s ds \right\}.
\]

By the limit theorem (cf. [15, 16]), we get

\[
\det(\nabla X_t(x)) = \exp \left\{ \int_0^t \text{div} \tilde{b}(X(s,x)) ds + \int_0^t \text{div} \sigma(X(s,x)) \circ dW_s \right\}.
\]

Lemma 3.1 then follows by rewriting the Stratonovich integral as Itô’s integral.

On the other hand, fix \( T > 0 \) and let \( Y_t \) solve the following SDE:

\[
dY_t = -\tilde{b}(Y_t)dt + \sigma(Y_t) \circ dW^T_t, \quad Y_0 = x,
\]

where \( W^T_t := W_{T-t} - W_T \). It is well known that (cf. [15, 16])

\[
X_T^{-1}(x) = Y_T(x).
\]

As above, we have

\[
\det(\nabla Y_T) = \exp \left\{ -\int_0^T \text{div} \tilde{b}(Y_s) ds + \int_0^T \text{div} \sigma(Y_s) \circ dW^T_s \right\}
\]

\[
= \exp \left\{ \int_0^T \left[ -\text{div} \tilde{b} + \frac{1}{2} \sigma^j \partial_j \sigma^i \right](Y_s) ds + \int_0^T \text{div} \sigma(Y_s) dW^T_s \right\}.
\]

Note that for any \( p \geq 1 \)

\[
t \mapsto \exp \left\{ p \int_0^t \text{div} \sigma(Y_s) dW^T_s - \frac{p^2}{2} \int_0^t |\text{div} \sigma(Y_s)|^2 ds \right\}
\]

is a continuous exponential martingale. Estimate (3.2) then follows by Hölder’s inequality. □

Let \( C^\infty_p(\mathbb{R}^d) \) be the set of all smooth functions with polynomial growth. The following proposition is an easy consequence of Proposition 2.3 (see also [24, p.180, Theorem 1]).
Proposition 3.2. For any \( u_0 \in C^\infty_c(\mathbb{R}^d) \), let \( u_t(x) := u_0(X_t^{-1}(x)) \). Then \( u_t(x) \) solves the following stochastic transport equation in the classical sense:

\[
\frac{du}{dt} = \frac{1}{2} \sigma^{ij} \sigma^{kl} \partial_{ij} \partial_{kl} u - b^i \partial_i u dt - \sigma^{ij} \partial_t u dW^i_t, \quad u|_{t=0} = u_0,
\]

where \( b^i := b^i - \sigma^{ij} \partial_j \sigma^{ij} \).

The following result can be found in [16] and [24, p. 180, Theorem 1].

Proposition 3.3. Let \( X_{s,t}(x) \) solve

\[
X_{s,t}(x) = x + \int_s^t b(X_{s,r}) dr + \int_s^t \sigma(X_{s,r}) dW_r, \quad t \geq s \geq 0.
\]

Fix \( t > 0 \). For any \( v_0 \in C^\infty_c(\mathbb{R}^d) \), let \( v_{s,t}(x) := v_0(X_{s,t}(x)) \), where \( s \in [0,t] \). Then \( v_{s,t}(x) \) solves the following backward stochastic Kolmogorov equation in the classical sense:

\[
dv + \frac{1}{2} \sigma^{ij} \sigma^{kl} \partial_{ij} \partial_{kl} v + b^i \partial_i v ds + \sigma^{ij} \partial_t v * dW^i_s = 0, \quad v|_{t=1} = v_0,
\]

where the asterisk denotes the backward Itô's integral.

Let \( C^+_c(\mathbb{R}^d) \) be the set of all non-negative continuous functions on \( \mathbb{R}^d \) with compact support and \( C \) a countable and dense subset of \( C^+_c(\mathbb{R}^d) \) with respect to the uniform norm \( \| \varphi \|_{\infty} := \sup_{x \in \mathbb{R}^d} |\varphi(x)| \). We need the following simple lemma.

Lemma 3.4. Let \( X, Y : \mathbb{R}^d \to \mathbb{R}^d \) be two measurable transformations.

(i) Let \( \varphi \in L^1(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d) \). Assume that for any \( \varphi \in \mathcal{C} \),

\[
\int \varphi(X) \leq \int \varphi \cdot \gamma.
\]

Then this inequality still holds for all \( \varphi \in L^1(\mathbb{R}^d) \). In particular, \( \mathcal{L} \circ X \ll \mathcal{L} \).

(ii) Let \( \rho : \mathbb{R}^d \to \mathbb{R}^+ \) be a positive measurable function with \( \rho \in L^1_{loc}(\mathbb{R}^d) \). Assume that for any \( \varphi, \psi \in \mathcal{C} \),

\[
\int \varphi(Y) \cdot \psi = \int \varphi \cdot \psi(X) \cdot \rho.
\]

Then \( X \) admits a measurable invertible \( Y \), i.e., \( X^{-1}(x) = Y(x) \) a.e.. Moreover,

\[
\mathcal{L} \circ X^{-1} = \rho \mathcal{L}, \quad \mathcal{L} \circ X = \rho^{-1}(X^{-1}) \mathcal{L}.
\]

Proof. (i) Thanks to the density of \( \mathcal{C} \) in \( C^+_c(\mathbb{R}^d) \), by Fatou’s lemma and the dominated convergence theorem, one sees that \([3.3]\) holds for all \( \varphi \in C^+_c(\mathbb{R}^d) \). Now, let \( O \subset \mathbb{R}^d \) be a bounded open set. Define

\[
\varphi_n(x) := 1 - \left( \frac{1}{1 + \text{distance}(x, O^c)} \right)^n.
\]

Then \( \varphi_n \in C^+_c(\mathbb{R}^d) \) and for every \( x \in \mathbb{R}^d \),

\[
\varphi_n(x) \uparrow 1_O(x) \text{ as } n \to \infty.
\]

By the monotone convergence theorem, we find that \([3.3]\) holds for \( \varphi = 1_O \). Thus, the desired conclusion follows by the monotone class theorem.

(ii) As above, one sees that \([3.4]\) holds for all \( \varphi, \psi \in L^1(\mathbb{R}^d) \). Thus, we have for all \( \varphi, \psi \in L^1(\mathbb{R}^d) \),

\[
\int \varphi(X \circ Y) \cdot \psi = \int \varphi(X) \cdot \psi(X) \cdot \rho = \int \varphi \cdot \psi
\]

and

\[
\int \varphi \cdot \psi(Y \circ X) \cdot \rho = \int \varphi(Y) \cdot \psi(Y) = \int \varphi \cdot \psi \cdot \rho.
\]
By the monotone class theorem, we obtain that for any Borel measurable set \( A \subset \mathbb{R}^d \times \mathbb{R}^d \),
\[
\int_{\mathbb{R}^d} 1_A(X \circ Y(x), x) \cdot e^{-|x|} \, dx = \int_{\mathbb{R}^d} 1_A(x, x) \cdot e^{-|x|} \, dx
\]
and
\[
\int_{\mathbb{R}^d} 1_A(x, Y \circ X(x)) \cdot e^{-|x|} \, dx = \int_{\mathbb{R}^d} 1_A(x, x) \cdot e^{-|x|} \, dx.
\]
Hence, letting \( A = \{(x, y) : x \neq y\} \) yields that \( X \circ Y(x) = x \) and \( Y \circ X(x) = x \) for \( \mathcal{L}^d \)-almost all \( x \in \mathbb{R}^d \). The result follows. \( \square \)

The following lemma will play a crucial role for taking limits below.

**Lemma 3.5.** Let \( X_n(\omega, x) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d \), \( n \in \mathbb{N} \) be a family of measurable mappings, which are uniformly bounded in \( L_{\text{loc}}^p(\mathbb{R}^d; L^p(\Omega)) \) for any \( p \geq 1 \). Suppose that for \( P \)-almost all \( \omega \in \Omega \), \( \mathcal{L} \circ X_n(\omega, \cdot) \ll \mathcal{L} \) and the density \( \gamma_n(\omega, x) \) satisfies
\[
\sup_n \text{ess. sup}_{x \in \mathbb{R}^d} E|\gamma_n(x)|^2 \leq C_1. \tag{3.5}
\]
If for \((P \times \mathcal{L})\)-almost all \( (\omega, x) \in \Omega \times \mathbb{R}^d \), \( X_n(\omega, x) \to X(\omega, x) \) as \( n \to \infty \), then for \( P \)-almost all \( \omega \in \Omega \), \( \mathcal{L} \circ X(\omega, \cdot) \ll \mathcal{L} \) and the density \( \gamma \) also satisfies
\[
\text{ess. sup}_{x \in \mathbb{R}^d} E|\gamma(x)|^2 \leq C_1. \tag{3.6}
\]
Moreover, let \( (\psi_n)_{n \in \mathbb{N}} \) be a family of measurable functions on \( \mathbb{R}^d \) and satisfy that for some \( C_2 > 0 \) and \( \alpha \geq 1 \)
\[
\sup_{n \in \mathbb{N}} \text{ess. sup}_{x \in \mathbb{R}^d} \frac{|\psi_n(x)|}{1 + |x|^\alpha} \leq C_2. \tag{3.7}
\]
If \( \psi_n \) converges to some \( \psi \) in \( L_{\text{loc}}^1(\mathbb{R}^d) \), then for any \( N > 0 \),
\[
\lim_{n \to \infty} E \int_{B_N} |\psi_n(X_n) - \psi(X)| = 0. \tag{3.8}
\]

**Proof.** Fix \( \varphi \in C_c^0(\mathbb{R}^d) \) with support contained in \( B_N \) for some \( N > 0 \). Then by Fubini’s theorem and Fatou’s lemma, we have for \( P \)-almost all \( \omega \in \Omega \),
\[
\int \varphi(X(\omega)) \leq \lim_{n \to \infty} \int \varphi(X_n(\omega)) = \lim_{n \to \infty} \int \varphi \cdot \gamma_n(\omega) =: \lim_{n \to \infty} J^\varphi_n(\omega). \tag{3.9}
\]
By (3.5), there exists a subsequence still denoted by \( n \) and a \( \gamma_0 \in L_{\text{loc}}^2(\mathbb{R}^d; L^2(\Omega)) \) satisfying (3.6) such that
\( \gamma_n \) weakly * converges to \( \gamma_0 \) in \( L_{\text{loc}}^2(\mathbb{R}^d; L^2(\Omega)) \).

Since \( \gamma_n \) also weakly converges to \( \gamma_0 \) in \( L^2(B_N \times \Omega) \), by Banach-Saks’ theorem, there is another subsequence still denoted by \( n \) such that its Cesàro mean \( \tilde{\gamma}_n := \frac{1}{n} \sum_{k=1}^n \gamma_k \) strongly converges to \( \gamma_0 \) in \( L^2(B_N \times \Omega) \). Thus, there is another subsequence still denoted by \( n \) such that for \( P \)-almost all \( \omega \in \Omega \),
\[
\tilde{\gamma}_n(\omega) \overset{n \to \infty}{\longrightarrow} \gamma_0(\omega) \text{ in } L^2(B_N).
\]
Hence,
\[
\tilde{J}^\varphi_n(\omega) := \frac{1}{n} \sum_{k=1}^n J^\varphi_k(\omega) = \int \varphi \cdot \tilde{\gamma}_n(\omega) \overset{n \to \infty}{\longrightarrow} \int \varphi \cdot \gamma_0(\omega),
\]
which together with (3.9) yields that for \( P \)-almost all \( \omega \),
\[
\int \varphi(X(\omega)) \leq \lim_{n \to \infty} J^\varphi_n(\omega) \leq \lim_{n \to \infty} \tilde{J}^\varphi_n(\omega) = \int \varphi \cdot \gamma_0(\omega).
\]
Since \( C \) is countable, we may find a common null set \( \Omega' \subset \Omega \) such that the above inequality holds for all \( \omega \notin \Omega' \) and \( \varphi \in C \). The first conclusion then follows by (i) of Lemma 3.4.

We now prove (3.8). We make the following decomposition:

\[
\int_{B_N} |\psi_n(X_n) - \psi(X)| \leq \int_{B_N} |\psi_n(X_n) - \psi(X)| + \int_{B_N} |\psi(X_n) - \psi(X)| =: I_n + J_n.
\]

By (3.7) and \( \psi_n \to \psi \) in \( L^1_{\text{loc}}(\mathbb{R}^d) \), we also have

\[
\text{ess. sup}_{x \in \mathbb{R}^d} \frac{\|\psi(x)\|}{1 + \|x\|^\alpha} \leq C_2.
\]

Let \( (\phi_m)_{m \in \mathbb{N}} \) be a family of bounded continuous functions such that \( \phi_m \to \psi \) in \( L^1_{\text{loc}}(\mathbb{R}^d) \) as \( m \to \infty \) and

\[
\text{sup}_{m \in \mathbb{N}} \text{ess. sup}_{x \in \mathbb{R}^d} \frac{|\phi_m(x)|}{1 + |x|^\alpha} \leq C_2. \tag{3.10}
\]

We have

\[
J_n \leq \int_{B_N} |\phi_m(X_n) - \psi(X_n)| + \int_{B_N} |\phi_m(X) - \psi(X)| + \int_{B_N} |\phi_m(X_n) - \phi_m(X)| =: J_{1nm} + J_{2m} + J_{3nm}.
\]

For any \( R > 0 \), we may write

\[
J_{1nm} = \int_{B_N \cap \{|X_n| < R\}} |\phi_m(X_n) - \psi(X_n)| + \int_{B_N \cap \{|X_n| > R\}} |\phi_m(X_n) - \psi(X_n)| =: J_{1nm}^1 + J_{1nm}^2.
\]

By the change of variable and (3.5), we have

\[
\mathbb{E}J_{1nm}^1 \leq \int_{B_R} |\phi_m - \psi| \cdot \gamma_n \leq C_1 \int_{B_R} |\phi_m - \psi|.
\]

By Chebyshev’s inequality and (3.10), we have

\[
\mathbb{E}J_{2nm}^2 \leq \frac{C_{N,\alpha}}{R} \sup_{x \in B_N} \mathbb{E}(1 + |X_n(x)|^\alpha) \leq \frac{C_{N,\alpha}}{R}.
\]

Combining the above two estimates, we obtain

\[
\lim_{m \to \infty} \sup_{n \in \mathbb{N}} \mathbb{E}J_{1nm} = 0. \tag{3.11}
\]

Similarly, we also have

\[
\lim_{m \to \infty} \mathbb{E}J_{2m} = 0
\]

and for fixed \( m \in \mathbb{N} \), by the dominated convergence theorem,

\[
\lim_{n \to \infty} \mathbb{E}J_{3nm} = 0.
\]

Hence,

\[
\lim_{n \to \infty} \mathbb{E}J_n = 0.
\]

As proving (3.11), we also have

\[
\lim_{n \to \infty} \mathbb{E}I_n = 0.
\]

The proof is then complete. \( \square \)
The following lemma will be used to prove the strong convergence in Theorem 4.7 below.

**Lemma 3.6.** Let $\mathcal{B}$ be a separable and uniformly convex Banach space. Let $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence in $L^1(\Omega; C([0, T]; \mathcal{B}))$. Assume that for some $u \in L^1(\Omega; C([0, T]; \mathcal{B}))$,

$$
\lim_{n \to \infty} \mathbb{E} \left( \sup_{t \in [0, T]} |\mathbb{E} \cdot \langle \phi, u_n(t) - u(t) \rangle_{\mathcal{B}}| \right) = 0, \ \forall \phi \in \mathcal{B}^*, \quad (3.12)
$$

where $\mathcal{B}^*$ is the dual space of $\mathcal{B}$, and

$$
\lim_{n \to \infty} \mathbb{E} \left( \sup_{t \in [0, T]} \|\|u_n(t)\|_{\mathcal{B}} - \|u(t)\|_{\mathcal{B}}\| \right) = 0. \quad (3.13)
$$

Then $\sup_{t \in [0, T]} \|u_n(t) - u(t)\|_{\mathcal{B}}$ converges to zero in probability as $n \to \infty$.

**Proof.** It is enough to prove that for any subsequence $n_k$, there exists a subsequence $n'_k$ such that $\sup_{t \in [0, T]} \|u_{n'_k}(t) - u(t)\|_{\mathcal{B}}$ converges to zero $P$-almost surely as $k \to \infty$. We now fix a subsequence $n_k$ below. Since $\mathcal{B}^*$ is separable, by (3.12) and (3.13), we may find a subsequence $n'_k$ and a measurable set $\Omega' \subset \Omega$ with $P(\Omega') = 1$ such that for all $\omega \in \Omega'$, $u(\omega, \cdot) \in C([0, T]; \mathcal{B})$ and

$$
\lim_{k \to \infty} \sup_{t \in [0, T]} |\mathbb{E} \cdot \langle \phi, u_{n'_k}(\omega, t) - u(\omega, t) \rangle_{\mathcal{B}}| = 0, \ \forall \phi \in \mathcal{B}^*. \quad (3.14)
$$

and

$$
\lim_{k \to \infty} \sup_{t \in [0, T]} \|\|u_{n'_k}(\omega, t) - u(\omega, t)\|_{\mathcal{B}}\| = 0. \quad (3.15)
$$

We want to show that for such $\omega \in \Omega'$,

$$
\lim_{k \to \infty} \sup_{t \in [0, T]} \|u_{n'_k}(\omega, t) - u(\omega, t)\|_{\mathcal{B}} = 0.
$$

Suppose that this is not true. Then, there exist a $\delta > 0$ and a sequence $(t_k)_{k \in \mathbb{N}} \subset [0, T]$ such that

$$
\|u_{n'_k}(\omega, t_k) - u(\omega, t_k)\|_{\mathcal{B}} \geq \delta, \ \forall k \in \mathbb{N}. \quad (3.16)
$$

Without loss of generality, we assume that $t_k$ converges to $t_0$. By (3.14), (3.15) and $u(\omega, \cdot) \in C([0, T]; \mathcal{B})$, we have

$$
\lim_{k \to \infty} \|u_{n'_k}(\omega, t_k) - u(\omega, t_0)\|_{\mathcal{B}} = 0,
$$

which together with $u(\omega, \cdot) \in C([0, T]; \mathcal{B})$ yields

$$
\lim_{k \to \infty} \|u_{n'_k}(\omega, t_k) - u(\omega, t_k)\|_{\mathcal{B}} = 0.
$$

This is a contradiction with (3.16). The proof is complete. \qed

We also recall some facts about local maximal functions. Let $f$ be a locally integrable function on $\mathbb{R}^d$. For every $R > 0$, the local maximal function is defined by

$$
M_R f(x) := \sup_{0 < r < R} \frac{1}{|B_r|} \int_{B_r} f(x + y) dy =: \sup_{0 < r < R} \frac{1}{|B_r|} \int_{B_r} f(x + y) dy.
$$

The following result can be found in [7] p.143, Theorem 3] and [5] Appendix A.

**Lemma 3.7.** (i) (Morrey’s inequality) Let $f \in L^1_{loc}(\mathbb{R}^d)$ be such that $\nabla f \in L^q_{loc}(\mathbb{R}^d)$ for some $q > d$. Then there exist $C_{q,d} > 0$ and a negligible set $A$ such that for all $x, y \in A'$ with $|x - y| \leq R$,

$$
|f(x) - f(y)| \leq C_{q,d} \cdot |x - y| \cdot \left( \int_{|z| > R} |\nabla f|^q(x + z) dz \right)^{1/q}
\leq C_{q,d} \cdot |x - y| \cdot (M_R |\nabla f|^q(x))^{1/q}. \quad (3.17)
$$
(ii) Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ be such that $\nabla f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then there exist $C_d > 0$ and a negligible set $A$ such that for all $x,y \in A^c$ with $|x - y| \leq R$,

$$|f(x) - f(y)| \leq C_d \cdot |x - y| \cdot (M_R|\nabla f|(x) + M_R|\nabla f|(y)). \tag{3.18}$$

(iii) Let $f \in (L \log L)_{\text{loc}}(\mathbb{R}^d)$. Then for any $N,R > 0$ and some $C_{d,N}, C_d > 0$,

$$\int_{B_N} M_R|f| \leq C_{d,N} + C_d \int_{B_{N+R}} |f| \log(|f| + 1). \tag{3.19}$$

(iv) Let $f \in L^p_{\text{loc}}(\mathbb{R}^d)$ for some $p > 1$. Then for some $C_{d,p} > 0$ and any $N,R > 0$,

$$\left(\int_{B_N} (M_R|f|)^p\right)^{1/p} \leq C_{d,p} \left(\int_{B_{N+R}} |f|^p\right)^{1/p}. \tag{3.20}$$

4. Stochastic Transport Equations

In this section we work on $[0,T]$ and mainly study the following stochastic transport equation:

$$\text{d}u = \left[\frac{1}{2}\sigma^{ij}\sigma^{jl}\partial_{ij}^2 u + b^i \partial_i u\right]\text{d}t + \sigma^{il} \partial_i u \text{d}W_t^l, \quad u|_{t=0} = u_0, \tag{4.1}$$

where $\sigma \in \mathbb{R}^d \times \mathbb{R}^m$ does not depend on $x$, and $b$ is a BV vector field and satisfies

$$\frac{b(x)}{1 + |x|}, \text{ div } b(x) \in L^\infty(\mathbb{R}^d), \quad b \in \text{BV}_{\text{loc}}. \tag{4.2}$$

We first introduce the following notion of renormalized solutions for equation (4.1).

**Definition 4.1.** A measurable and $(\mathcal{F}_t)$-adapted stochastic field $u : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ is called a renormalized solution of (4.1) if for any $\beta \in C^2(\mathbb{R})$,

$$v_t(\omega, x) := \beta(\text{arctan } u_t(\omega, x))$$

solves (4.1) in the distributional sense, i.e., for any $\phi \in C^\infty_c(\mathbb{R}^d)$

$$\int v_t \phi = \int v_0 \phi + \frac{1}{2} \int_0^t \int_0^1 v \sigma^{ij} \sigma^{jl} \partial_{ij}^2 \phi - \int_0^t \int v \text{ div } b \phi + b \partial_t \phi - \int_0^t \int v \sigma^{il} \partial_i \phi \text{d}W_t^l. \tag{4.3}$$

**Remark 4.2.** Since $v$ is bounded, it is clear that both sides of (4.3) are well defined.

Our main result in this section is that

**Theorem 4.3.** Assume that condition (4.2) holds.

(Existence and Uniqueness) For any measurable function $u_0$, there exists a unique renormalized solution $u$ to stochastic transport equation (4.1) with $u|_{t=0} = u_0$ in the sense of Definition 4.1. Moreover, for any $p > 1$ and $N > 0$,

$$\text{arctan } u \in L^p(\Omega; C([0,T]; L^p(B_N))).$$

(Stability) Let $b_n \in L^1_{\text{loc}}(\mathbb{R}^d)$ be such that $\text{div } b_n \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $b_n, \text{div } b_n$ converge to $b, \text{div } b$ respectively in $L^1_{\text{loc}}(\mathbb{R}^d)$. Let $u_0^n, \mathcal{L}$-almost everywhere converge to $u_0$. Let $u^n$ and $u$ be the renormalized solutions corresponding to $(b^n, u_0^n)$ and $(b, u_0)$ in the sense of Definition 4.1. Then for any $p > 1$ and $N > 0$,

$$\text{arctan } u^n \to \text{arctan } u \text{ strongly in } L^p(\Omega; C([0,T]; L^p(B_N))).$$

For proving this theorem, we first study the following more general stochastic partial differential equation:

$$\text{d}u = \left[\frac{1}{2}\sigma^{ij}\sigma^{jl}\partial_{ij}^2 u + b^i \partial_i u + cu\right]\text{d}t + (\sigma^{il} \partial_i u + h^i u) \text{d}W_t^l, \quad u|_{t=0} = u_0, \tag{4.4}$$
where $\sigma$ and $b$ are as above and
\[ c, h, \nabla h \in L^\infty(\mathbb{R}^d). \] \hspace{1cm} (4.5)

As Definition [4.1] we also introduce the following notion about the renormalized solutions for equation (4.4).

**Definition 4.4.** We say $u \in L^\infty([0, T] \times \Omega \times \mathbb{R}^d)$ a renormalized solution of (4.4) if for any $\beta \in C^2(\mathbb{R})$, it holds that in the distributional sense
\[
d\beta(u) = \left[ \frac{1}{2} \sigma^{il} \sigma^{jl} \partial_t \beta(u) + b^i \partial_t \beta(u) + cu \beta'(u) \right] dt
+ \frac{1}{2} h^2 \beta''(u) u^2 + h^i \sigma^{il} \partial_t \beta'(u) u dt
+ (\sigma^{il} \partial_t \beta(u) + h^i u \beta'(u)) dW_t^l.
\]

We remark that for equation (4.4), the renormalized solution is a nonlinear notion, whereas the distributional solution is a linear notion. However, under (4.2) and (4.5), we can show that these two notions are equivalent. For this aim, we need the following class of regularized functions:
\[ \mathcal{N} := \left\{ \varphi \in C^\infty_c(B_1), \varphi \geq 0, \int \varphi = 1 \right\}. \hspace{1cm} (4.6)\]

We now establish the following equivalence between the distributional solution and renormalized solution.

**Proposition 4.5.** Let $u \in L^\infty([0, T] \times \Omega \times \mathbb{R}^d)$ be a distributional solution of (4.4). Then under (4.2) and (4.5), $u$ is also a renormalized solution of (4.4) in the sense of Definition 4.4.

**Proof.** Let $\varphi \in \mathcal{N}$ and set $\varphi_\varepsilon(x) := \varepsilon^{-d} \varphi(x/\varepsilon)$. Define
\[ u_\varepsilon := u_{t,\varepsilon}(x) := u_t \ast \varphi_\varepsilon(x) = \int u_t(y) \varphi_\varepsilon(x - y) dy. \]

Taking convolutions for both sides of (4.4), we obtain
\[ du_\varepsilon = \left[ \frac{1}{2} \sigma^{il} \sigma^{jl} \partial_t^2 u_\varepsilon + (b^i \partial_t u_\varepsilon) \varphi_\varepsilon + (cu) \ast \varphi_\varepsilon \right] dt + [\sigma^{il} \partial_t u_\varepsilon + (h^i u_\varepsilon) \ast \varphi_\varepsilon] dW_t^l.
\]

Let $\beta \in C^2(\mathbb{R})$. By Itô’s formula, we have
\[
d\beta(u_\varepsilon) = \left[ \frac{1}{2} \sigma^{il} \sigma^{jl} \partial_t^2 \beta(u_\varepsilon) + ((b^i \partial_t u_\varepsilon) \ast \varphi_\varepsilon + (cu) \ast \varphi_\varepsilon) \cdot \beta'(u_\varepsilon) \right] dt
+ \frac{1}{2} \beta''(u_\varepsilon)((h^i u_\varepsilon) \ast \varphi_\varepsilon)^2 + \beta''(u_\varepsilon) \sigma^{il} \partial_t u_\varepsilon \cdot (h^i u_\varepsilon) \ast \varphi_\varepsilon dt
+ [\sigma^{il} \partial_t \beta(u_\varepsilon) + (h^i u_\varepsilon) \ast \varphi_\varepsilon \cdot \beta'(u_\varepsilon)] dW_t^l.
\]

Write
\[ r_\varepsilon := ((b^i \partial_t u_\varepsilon) \ast \varphi_\varepsilon - b^i \partial_t (u \ast \varphi_\varepsilon)) \cdot \beta'(u_\varepsilon) \]
and
\[ r_\varepsilon \varepsilon = (h^i u \ast \varphi_\varepsilon - h^i (u \ast \varphi_\varepsilon)). \]

Let $\phi \in C^\infty_c(\mathbb{R}^d)$. Multiplying both sides by $\phi$ and integrating over $\mathbb{R}^d$, by the integration by parts formula, we get
\[ \int \beta(u_{t,\varepsilon}) \phi = \int \beta(u_{0,\varepsilon}) \phi + \int_0^t \int \beta(u_\varepsilon) \left[ \frac{1}{2} \sigma^{il} \sigma^{jl} \partial_t^2 \phi - \nabla \cdot b \phi - b^i \partial_t \phi \right] dt
+ \int_0^t \int r_\varepsilon \phi + \int_0^t (cu) \ast \varphi_\varepsilon \cdot \beta'(u_\varepsilon) \phi dt. \]
Proof. Since the left hand side of the above inequality does not depend on $\phi$, it suffices to show that

$$\inf_{\varepsilon \in \mathbb{N}} \limsup_{\varepsilon \to 0} \left| \int_0^t r^\varepsilon \phi \right| = 0.$$ 

This has been proved in the proof of [2] Theorem 3.5].

Using Proposition 4.5, we can prove the uniqueness of distributional solutions.

Proposition 4.6. Let $u \in L^\infty([0, T] \times \Omega; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ be a distributional solution of (4.4). If $u|_{t=0} = 0$, then

$$u_t(\omega, x) = 0, \ a.e.$$ 

Proof. Let $\chi \in C^\infty_c(\mathbb{R}^d)$ be a nonnegative cutoff function with

$$\|\chi\|_\infty \leq 1, \ \chi(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 2. \end{cases} \quad (4.7)$$

Set $\chi_n(x) := \chi(x/n)$. By Proposition 4.5 and Definition 4.4, we have

$$\mathbb{E} \int u_n^2 \chi_n = \mathbb{E} \int_0^t \int \left[ -u^2 \sigma^\varepsilon \sigma^\beta \partial^\varepsilon \chi_n + u^2 b^\varepsilon \partial^\varepsilon \chi_n \\
+ \mathbb{E} \int_0^t \left[ -u^2 \text{div} \chi_n + 2 cu^2 \chi_n \\
+ \mathbb{E} \int_0^t \left[ |h|^2 u^2 \chi_n - u^2 \partial^\varepsilon (h^\varepsilon \chi_n) \sigma^\beta \right]. \right]$$

Observe that by (4.2)

$$|b^\beta \partial^\varepsilon \chi_n| \leq \frac{|b| \cdot 1_{\{n \leq |x| \leq 2n\} \cdot \|\nabla \chi\|_\infty}{n} \leq C_1 \cdot 1_{\{x \geq n\}}, \quad (4.8)$$

where $C_1 = 3|b/(1 + |x|)|_\infty \cdot \|\nabla \chi\|_\infty$, and

$$|h^\beta \partial^\varepsilon \chi_n| \leq \|h\|_\infty \cdot \|\nabla \chi\|_\infty.$$ 

Since $u^2 \in L^\infty([0, T] \times \Omega; L^1(\mathbb{R}^d))$, by letting $n \to \infty$, we obtain

$$\mathbb{E} \int u_t^2 = \mathbb{E} \int_0^t \left[ (-\text{div} b + 2c + |h|^2 - \sigma^\beta \partial^\varepsilon h) u_t^2 \right] \leq \|2c + |h|^2 - \text{div} b - \sigma^\beta \partial^\varepsilon h\|_\infty \int_0^t \left( \mathbb{E} \int u_s^2 \right) ds,$$
which gives by Gronwall’s inequality that
\[ \mathbb{E} \int u_t^2 = 0. \]
The uniqueness follows. \(\square\)

In general, it is not expected to have a bounded solution for SPDE \((4.4)\) because of the presence of stochastic integral \(\int h^i u dW_t^i\) (cf. [24]). We now turn back to stochastic transport equation \((4.1)\), and prove the existence-uniqueness and stability of \(L^\infty\)-distributional solutions when the initial value belongs to \(L^\infty(\mathbb{R}^d)\).

**Theorem 4.7.** Assume that condition \((4.2)\) holds.

(Existence and Uniqueness) For any \(u_0 \in L^\infty(\mathbb{R}^d)\), there exists a unique distributional solution \(u \in L^\infty([0, T] \times \Omega \times \mathbb{R}^d)\) (also a renormalized solution in the sense of Definition \((4.4)\)) to stochastic transport equation \((4.7)\) satisfying
\[ \|u_t(\omega)\|_{\infty} \leq \|u_0\|_{\infty}. \] (4.9)
Moreover, there is a version still denoted by \(u\) such that for any \(p > 1\) and \(N > 0\)
\[ u \in L^p(\Omega; C([0, T]; L^p(B_N))). \] (4.10)

(Stability) Let \(b_n \in L^1_{loc}(\mathbb{R}^d)\) and \(u_{n0} \in L^\infty(\mathbb{R}^d)\) be such that \(\text{div} b_n \in L^1_{loc}(\mathbb{R}^d)\) and \(b_n, \text{div} b_n, u_{n0}\) converge to \(b, \text{div} b, u_0\) respectively in \(L^1_{loc}(\mathbb{R}^d)\). Let \(u_n, u \in L^\infty([0, T] \times \Omega \times \mathbb{R}^d)\) be the distributional solutions of \((4.7)\) corresponding to \((b_n, u_{n0})\) and \((b, u_0)\) and satisfy \((4.10)\). Assume that
\[ \sup_n \|u_n\|_{L^\infty([0, T] \times \Omega \times \mathbb{R}^d)} < +\infty. \] (4.11)
Then for any \(p > 1\) and \(N > 0\),
\[ u_n \to u \text{ strongly in } L^p(\Omega; C([0, T]; L^p(B_N))). \] (4.12)

**Proof.** (Existence) Fix a \(\varrho \in \mathcal{N}\) and a cutoff function \(\chi\) satisfying \((4.7)\). Let
\[ \varrho_n(x) := n^d \varrho(nx), \quad \chi_n(x) = \chi(x/n) \]
and define
\[ b_n = b * \varrho_n \cdot \chi_n. \] (4.13)
Let \(X_n\) solve the following SDE:
\[ dX_n = -b_n(X_n)dt - \sigma dW_t, \quad X_n|_{t=0} = x. \]
By Proposition \(3.2\), \(u_{n, t} := u_0(X^{-1}_{n,t})\) solves the following SPDE:
\[ du_n = \left[ \frac{1}{2} \sigma^{ij} \sigma^{kl} \frac{\partial^2}{\partial x^i \partial x^j} u_n + b_n^i \frac{\partial}{\partial x^i} u_n \right] dt + \sigma^{ij} \frac{\partial}{\partial x^i} u_n dW^j_t, \quad u_n|_{t=0} = u_0. \]
Clearly, \(u_n \in L^\infty([0, T] \times \Omega \times \mathbb{R}^d)\) and
\[ \|u_{n,t}(\omega, \cdot)\|_{\infty} \leq \|u_0\|_{\infty}. \]
Therefore, for some \(u \in L^\infty([0, T] \times \Omega \times \mathbb{R}^d)\) and some subsequence \(n_k\),
\[ u_{n_k} \to u \text{ weakly* in } L^\infty([0, T] \times \Omega \times \mathbb{R}^d). \]
Taking weakly* limits, it is easy to see that \(u\) is a distributional solution of \((4.1)\). Moreover, \((4.9)\) holds. As for \((4.10)\), it can be seen from the proof of the following stability.

(Uniqueness) Let \(u\) and \(\hat{u}\) be two distributional solutions of \((4.1)\) with the same initial value. Then \(v := u - \hat{u} \in L^\infty([0, T] \times \Omega \times \mathbb{R}^d)\) is still a distributional solution of \((4.1)\) with zero initial
value. Since $v$ does not belong to $L^\infty([0, T] \times \Omega; L^1(\mathbb{R}^d))$, we cannot directly use Proposition 4.6 to obtain $v = 0$. Below, we use a simple trick. Let

$$\tilde{\lambda}(x) := \frac{1}{(1 + |x|^2)^{\alpha}}, \quad \tilde{v}_t := v_t \cdot \tilde{\lambda}.$$

It is easy to see that

$$\tilde{v}_t \in L^\infty([0, T] \times \Omega; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)).$$

Moreover, noting that

$$\partial_t \tilde{\lambda}(x) = -\frac{2dx_i}{1 + |x|^2} \tilde{\lambda}(x), \quad \partial_i \partial_j \tilde{\lambda}(x) = \left(\frac{4d(d + 1)x_ix_j}{(1 + |x|^2)^2} - \frac{2d\delta_{ij}}{1 + |x|^2}\right) \tilde{\lambda}(x),$$

we can check that $\tilde{v}_t$ is a distributional solution of

$$d\tilde{v}_t = \left[\frac{1}{2} \sigma^{jl} \sigma^{il} \partial^2_{tj} \tilde{v} + \tilde{b}^j \partial_j \tilde{v} + c \tilde{v}\right]dt + (\sigma^{jl} \partial_j \tilde{v} + h^l \tilde{v})dW^l_t, \quad \tilde{v}_{t=0} = 0,$$

where

$$\tilde{b}^j(x) = b^j(x) + \frac{2dx_i \sigma^{il} \sigma^{jl}}{1 + |x|^2}, \quad h^l(x) = \frac{2dx_i \sigma^{il}}{1 + |x|^2}.$$ 

and

$$c(x) = \frac{2dx_i b^l(x)}{1 + |x|^2} + \left(\frac{2d(d - 1)x_ix_j}{(1 + |x|^2)^2} + \frac{d\delta_{ij}}{1 + |x|^2}\right) \sigma^{jl} \sigma^{il}.$$ 

By (4.2), one sees that $\tilde{b}$ still satisfies (4.2) and $c, h$ satisfy (4.3). Thus, we can use Proposition 4.6 to get $\tilde{v} = 0$. The uniqueness follows.

(Stability) We follow DiPerna-Lions’ argument [6, p.523]. Fix an even number $2n + 1$ and let $v_n := u_n^p$ for any $p > 1$. Then, by Definition 4.4, $v_n$ is a distributional solution of

$$dv_n = \left[\frac{1}{2} \sigma^{jl} \sigma^{il} \partial^2_{tj} v_n + b^j \partial_j v_n\right]dt + \sigma^{jl} \partial_j v_n dW^l_t, \quad v_n|_{t=0} = (u_0^p)^p.$$

By (4.11), we have

$$\sup_n \|u_n\|_{L^\infty([0, T] \times \Omega \times \mathbb{R}^d)} + \sup_n \|v_n\|_{L^\infty([0, T] \times \Omega \times \mathbb{R}^d)} < +\infty.$$

Without loss of generality, we may assume that $u_n$ and $v_n$ converges weakly* in $L^\infty([0, T] \times \Omega \times \mathbb{R}^d)$ to $u$ and $v$, which are distributional solutions of (4.1) corresponding to $u|_{t=0} = u_0$ and $v|_{t=0} = u_0^p$ by the assumptions. By Proposition 4.5 and the uniqueness proved above, we have

$$u^p = v.$$

Thus,

$$u_n^p \rightharpoonup u^p \text{ weakly* in } L^\infty([0, T] \times \Omega \times \mathbb{R}^d).$$

Hence, for any $N > 0$,

$$\mathbb{E} \int_0^T \int_{B_N} u_n^p \to \mathbb{E} \int_0^T \int_{B_N} u^p.$$

By virtue of

$$u_n \rightharpoonup u \text{ weakly in } L^p([0, T] \times \Omega \times B_N),$$

we thus obtain that for any $N > 0$,

$$u_n \to u \text{ strongly in } L^p([0, T] \times \Omega \times B_N). \quad (4.14)$$

We now strengthen this convergence to (4.12). Let $w_n = u_n - u$. Then we have for any $N > 0$ and $\phi \in C^\infty_c(B_N)$,

$$\left|\int w_n \phi \right| = \left|\int w_{n,0} \phi + \int_0^N \int w_n \frac{1}{2} \sigma^{jl} \sigma^{il} \partial^2_{tj} \phi - w_n (\text{div} \phi + b^j \partial_j \phi)\right|$$
then there exists a unique almost everywhere stochastic invertible flow to SDE (5.1) in the sense of Definition 4.1. By another approximation, we further have for any $\phi \in L^p(\mathbb{R}^d) 
abla w_n \to 0$, $0 < p < \infty$.

In this section, we use Theorem 4.3 to prove the following result.

**Proposition 4.6.** We are now in a position to give:

Proof of Theorem 4.3: (Uniqueness) Let $u$ and $\hat{u}$ be two renormalized solutions of SPDE (4.1) corresponding to the initial value $u_0$ in the sense of Definition 4.1. Then $\arctan u$ and $\arctan \hat{u}$ are two distributional solutions of SPDE (4.1) corresponding to the initial value $\arctan u_0$. By Proposition 4.6, we have

$$\arctan u = \arctan \hat{u}.$$ 

Hence,

$$u = \hat{u}.$$ 

(Existence) Let $v$ be the unique renormalized solution of SPDE (4.1) given in Proposition 4.7 corresponding to the initial value $\arctan u_0 \in [-\pi/2, \pi/2]$. Since

$$|v(t,\omega)|_{L^\infty} \leq \|\arctan u_0\|_{L^\infty} \leq \pi/2,$$

we may define

$$u(t,\omega, x) = \tan v(t, \omega, x)$$

so that $u$ is a renormalized solution of (4.1) in the sense of Definition 4.1. (Stability) It follows from the stability in Theorem 4.7.

5. **Stochastic Flows with BV Drifts and Constant Diffusions**

Consider the following SDE:

$$dX_t(x) = b(X_t(x))dt + \sigma dw_t, \quad X_0 = x.$$ (5.1)

In this section, we use Theorem 4.3 to prove the following result.

**Theorem 5.1.** Assume that $b$ is a BV vector field and satisfies

$$\frac{b(x)}{1 + |x|} \in L^p(\mathbb{R}^d), \quad b \in BV_{loc}.$$ 

Then there exists a unique almost everywhere stochastic invertible flow to SDE (5.1) in the sense of Definition 2.7.
Proof. (Existence): Define $b_n$ as in (4.13). Let $X_{n,s,t}(x)$ solve the following SDE:

$$X_{n,s,t}(x) = x + \int_s^t b_n(X_{n,s,r}(x))dr + \sigma(W_r - W_s), \quad \forall t \geq s \geq 0.$$  \hfill (5.2)

We divide the proof into two steps.

(Step 1): Fix $t > 0$. By Proposition 3.3, $v_{n,s,t}^k(x) := X_{n,s,t}^k(x)$ solves the following backward stochastic Kolmogorov equation:

$$dv_n^k + \frac{1}{2} \sigma^{il} \sigma^{jl} \partial_{ij} v_n^k ds + b_l^j \partial_l v_n^k ds + \sigma^{il} \partial_l v_n^k \ast dW_s = 0, \quad v_n^k|_{s=t} = x^k,$$

and by Proposition 3.2, $u_{n,s,t}^k(x) := [X_n^{-1}(x)]^k$ solves the following equation:

$$du_n^k = \frac{1}{2} \sigma^{il} \sigma^{jl} \partial_{ij} u_n^k - b_l^j \partial_l u_n^k dt - \sigma^{il} \partial_l u_n^k dt \ast dW_t, \quad u_n^k|_{t=0} = x^k,$$

where $x^k$ is the $k$-th coordinate of spatial variable $x$.

By Theorem 4.3, let $v_n^k$ and $u_n^k$ be the unique renormalized solutions of the following SPDEs in the sense of Definition 2.1:

$$dv_n^k + \frac{1}{2} \sigma^{il} \sigma^{jl} \partial_{ij} v_n^k ds + b_l^j \partial_l v_n^k ds + \sigma^{il} \partial_l v_n^k \ast dW_s = 0, \quad v_n^k|_{s=t} = x^k,$$

$$du_n^k = \frac{1}{2} \sigma^{il} \sigma^{jl} \partial_{ij} u_n^k - b_l^j \partial_l u_n^k ds - \sigma^{il} \partial_l u_n^k \ast dW_t, \quad u_n^k|_{s=0} = x^k.$$

Then by the stability result in Theorem 4.3, we have for any $p > 1$ and $N > 0$,

$$\lim_{n \to \infty} \mathbb{E} \left( \sup_{x \in [0,t]} \int_{B_N} |\arctan v_{n,s,t}^k - \arctan v_{s,t}^k|^p \right) = 0$$  \hfill (5.3)

and

$$\lim_{n \to \infty} \mathbb{E} \left( \sup_{x \in [0,t]} \int_{B_N} |\arctan u_{n,s,t}^k - \arctan u_{s,t}^k|^p \right) = 0.$$  \hfill (5.4)

Define

$$X_t(\omega, x) := v_{0,t}(\omega, x), \quad Y_t(\omega, x) := u_t(\omega, x)$$

Below, we want to show that $X_t(x)$ satisfies (A), (B) and (C) of Definition 2.1 and $X_t^{-1}(\omega, x) = Y_t(\omega, x)$.

(Step 2): By (5.3), we have for any $p > 1$ and $N > 0$

$$\lim_{n \to \infty} \mathbb{E} \left( \int_{0 \in B_N} |\arctan v_{n,0,s}^k - \arctan v_{0,s}^k|^p \right) = 0.$$  \hfill (5.3)

Hence, there exists a subsequence still denoted by $n$ such that for almost all $(s, \omega, x) \in [0,t] \times \Omega \times \mathbb{R}^d$ and any $k = 1, \cdots, d$

$$\lim_{n \to \infty} \arctan v_{n,0,s}^k(\omega, x) = \arctan v_{0,s}^k(\omega, x),$$

i.e.,

$$\lim_{n \to \infty} X_{n,0,s}(\omega, x) = X_s(\omega, x),$$

as well as for $(P \times \mathcal{F})$-almost all $(\omega, x) \in \Omega \times \mathbb{R}^d$,

$$\lim_{n \to \infty} X_{n,0,t}(\omega, x) = X_t(\omega, x).$$  \hfill (5.5)

Note that by (3.2) and (4.8), for any $p \geq 1$,

$$\mathbb{E} |\det(\nabla X_n^{-1})|^p \leq c \rho_1^{1/d} \langle \nabla b \rangle_{\infty} \leq c \rho_1^{1/d} (\langle |b| + 1 + |x| \rangle_{\infty}).$$
By Lemma 3.5, it is easy to see that (A) and (B) of Definition 2.1 hold, and for any \( N > 0 \),
\[
\lim_{n \to \infty} \mathbb{E} \int_0^t \int_{B_N} |\text{div} b_n(X_{n,0,t}) - \text{div} b(X_t)| = 0.
\]
Thus, for \((P \times \mathcal{L})\)-almost all \((\omega, x) \in \Omega \times \mathbb{R}^d \),
\[
\det(\nabla X_{n,0,t}(\omega, x)) = \exp \left\{ \int_0^t \text{div} b_n(X_{n,0,t}(\omega, x)) \, ds \right\} \to \exp \left\{ \int_0^t \text{div} b(X_t(\omega)) \, ds \right\} =: \rho_t(\omega, x).
\]
On the other hand, for fixed \( t \geq 0 \) and \( P \)-almost all \( \omega \in \Omega \), it holds that for all \( \varphi, \psi \in C^+_c(\mathbb{R}^d) \),
\[
\int \varphi(u_{n,t}(\omega)) \cdot \psi = \int \varphi(\omega) \cdot \psi = \int \varphi(\omega) \cdot \psi(X_{n,0,t}(\omega)) \cdot \det(\nabla X_{n,0,t}(\omega)).
\]
If necessary, by extracting a subsequence and then taking limits \( n \to \infty \) for both sides of (5.7), by (5.4), (5.5) and (5.6), we obtain that for \( P \)-almost all \( \omega \in \Omega \) and all \( \varphi, \psi \in C^+_c(\mathbb{R}^d) \),
\[
\int \varphi(Y_t(\omega)) \cdot \psi = \int \varphi(\omega) \cdot \psi(X_t(\omega)) \cdot \rho_t(\omega).
\]
Thus, by (ii) of Lemma 5.4 one sees that (C) of Definition 2.1 holds.

(Uniqueness): It follows from Propositions 2.3 and 4.7.

6. Stochastic Flows with Sobolev Drifts and Non-Constant Diffusions

We first prove the following key estimate.

**Lemma 6.1.** Let \( X_t(x) \) and \( \hat{X}_t(x) \) be two almost everywhere stochastic flows of (1.1) corresponding to \((b, \sigma)\) and \((\hat{b}, \hat{\sigma})\) in the sense of Definition 2.7 where
\[
b, \hat{b} \in L^1_{\text{loc}}(\mathbb{R}^d), \quad |\nabla \hat{b}| \in (L \log L)_{\text{loc}}(\mathbb{R}^d)
\]
and
\[
\sigma, \hat{\sigma} \in L^2_{\text{loc}}(\mathbb{R}^d), \quad |\nabla \hat{\sigma}| \in L^2_{\text{loc}}(\mathbb{R}^d).
\]
Then, for any \( T, N, R > 0 \), there exist constants \( C_1, C_2 \) given below such that for all \( \delta > 0 \),
\[
\mathbb{E} \int_{B_N \cap G^R_T} \log \left( \frac{\sup_{t \in [0,T]} |X_t - \hat{X}_t|^2}{\delta^2} + 1 \right) \leq C_1 + \frac{C_2}{\delta} \left( \int_{B_R} |b - \hat{b}| + \left[ \int_{B_R} |\sigma - \hat{\sigma}|^2 \right]^{1/2} \right),
\]
where
\[
G^R_T(\omega) := \left\{ x \in \mathbb{R}^d : \sup_{t \in [0,T]} |X_t(\omega, x)| \vee |\hat{X}_t(\omega, x)| \leq R \right\},
\]
\[
C_1 := C_{d,R,N} \cdot T \cdot (K_{T,b,\sigma} + K_{T,\hat{b},\hat{\sigma}} \left( 1 + \int_{B_{2R}} |\nabla \hat{b}| \log(|\nabla \hat{b}| + 1) + \left[ \int_{B_{2R}} |\nabla \hat{\sigma}|^2 \right]^{1/2} \right)),
\]
and \( C_2 := C_N \cdot T \cdot K_{T,b,\sigma} \). Here, \( K_{T,b,\sigma} \) is from (2.1), \( C_{d,R,N} \) only depends on \( d, R, N \), and \( C_N \) only depends on \( N \).
Proof. Set 

\[ Z_t(x) := X_t(x) - \hat{X}_t(x). \]

By Itô’s formula, we have

\[
\log \left( \frac{|Z_t|^2}{\delta^2} + 1 \right) = 2 \int_0^t \langle Z, b(X) - \hat{b}(\hat{X}) \rangle \, ds + 2 \int_0^t \frac{\langle Z, (\sigma(X) - \hat{\sigma}(\hat{X}))dW_s \rangle}{|Z|^2 + \delta^2} \\
+ \int_0^t \frac{|\sigma(X) - \hat{\sigma}(\hat{X})|^2}{|Z|^2 + \delta^2} \, ds - 2 \int_0^t \frac{|(\sigma(X) - \hat{\sigma}(\hat{X}))(X - \hat{X})|^2}{(|Z|^2 + \delta^2)^2} \, ds \\
= : I_1(t) + I_2(t) + I_3(t) + I_4(t).
\]

For \( I_1(t) \), we have

\[ I_1(t) \leq \frac{1}{\delta} \int_0^t |b(X) - \hat{b}(X)| \, ds + 2 \int_0^t \frac{\hat{b}(X) - \hat{b}(\hat{X})}{\sqrt{|Z|^2 + \delta^2}} \, ds =: I_{11}(t) + I_{12}(t). \]

Below, we write for a continuous function \( f : \mathbb{R}_+ \to \mathbb{R}, \)

\[ f^\ast(T) := \sup_{t \in [0, T]} |f(t)|. \]

Noting that

\[ G^R_T(\omega) \subset \{ x : |X_t(\omega, x)| \leq R \} \cap \{ x : |\hat{X}_t(\omega, x)| \leq R \}, \quad \forall t \in [0, T], \]

by (2.1), we have

\[
\mathbb{E} \int_{G^R_T} |I_{12}(T)| \leq \frac{1}{\delta} \mathbb{E} \int_0^T \int_{|X| \leq R} |b(X) - \hat{b}(X)| \leq \frac{\tilde{K}_{T, b, \sigma}}{\delta} \int_{B_R} |b - \hat{b}|, \]

where \( \tilde{K}_{T, b, \sigma} := T \cdot K_{T, b, \sigma} \), and by \( \mathcal{L} \circ X \ll \mathcal{L} \) and \( \mathcal{L} \circ \hat{X} \ll \mathcal{L} \),

\[
\mathbb{E} \int_{G^R_T} |I_{12}(T)| \leq C_d \mathbb{E} \int_0^T \int_{G^R_T} \left( [M_R|\nabla \hat{b}|](X) + [M_R|\nabla \hat{b}|](\hat{X}) \right) \\
\leq C_d \mathbb{E} \int_0^T \left( \int_{|X| \leq R} [M_R|\nabla \hat{b}|](X) + \int_{|\hat{X}| \leq R} [M_R|\nabla \hat{b}|](\hat{X}) \right) \\
\leq C_d \cdot (\tilde{K}_{T, b, \sigma} + \tilde{K}_{T, b, \hat{\sigma}}) \int_{B_R} M_R|\nabla \hat{b}| + \frac{\tilde{K}_{T, b, \sigma}}{\delta} \int_0^T \int_{B_R} |b - \hat{b}|. \]

Hence,

\[
\mathbb{E} \int_{G^R_T} |I_1(T)| \leq C_d \cdot (\tilde{K}_{T, b, \sigma} + \tilde{K}_{T, b, \hat{\sigma}}) \left( 1 + \int_{B_R} |\nabla \hat{b}| \log(|\nabla \hat{b}| + 1) \right) \\
+ \frac{\tilde{K}_{T, b, \sigma}}{\delta} \int_0^T \int_{B_R} |b - \hat{b}|.
\]

For \( I_2(t) \), set

\[ \tau_R(\omega, x) := \inf \{ t \geq 0 : |X_t(\omega, x)\| \vee |\hat{X}_t(\omega, x)| > R \}, \]

then

\[ G^R_T(\omega) = \{ x : \tau_R(\omega, x) > T \}. \]

By BDG’s inequality, we have

\[
\mathbb{E} \int_{B^N \cap G^R_T} |I_2^2(T)| \leq \int_{B_N} \mathbb{E} \left( \sup_{t \in [0, T \wedge \tau_R]} \left| \int_0^t \frac{\langle Z, (\sigma(X) - \hat{\sigma}(\hat{X}))dW_s \rangle}{|Z|^2 + \delta^2} \right| \right).
\]
\[ \leq C \int_{B_N} \mathbb{E} \left[ \int_0^{T \wedge \tau_R} \frac{|\sigma(X) - \hat{\sigma}(\hat{X})|^2}{|Z|^2 + \delta^2} \, ds \right]^{\frac{1}{2}} \]

\[ \leq C_N \left[ \mathbb{E} \int_0^T \int_{|x \cdot T(y) > 0)} \frac{|\sigma(X) - \hat{\sigma}(\hat{X})|^2}{|Z|^2 + \delta^2} \right]^{\frac{1}{2}} . \]

As the treatment of \( I_1(t) \), we can prove that

\[ \mathbb{E} \int_{B_N \cap G^R_T} |I_2(T)| \leq C_d \cdot (\bar{K}_{T,b,n} + \bar{K}_{T,b,\sigma}) \left[ \int_{B_R} |\nabla \hat{\sigma}|^2 \right]^{\frac{1}{2}} + \frac{C_N \cdot \bar{K}_{T,b,\sigma}}{\delta} \left[ \int_{B_R} |\sigma - \hat{\sigma}|^2 \right]^{\frac{1}{2}} . \]

\( I_3(t) \) is dealt with similarly and \( I_4(t) \) is negative and abandoned. The proof is thus complete. \( \Box \)

**Lemma 6.2.** Let \( \Phi(\omega, x) := \sup_{t \in [0,T]} |X_t(\omega, x) - \hat{X}_t(\omega, x)|^2 \). Assume that for some \( M > 0 \),

\[ \int_{B_N \cap G^R_T} \log \left( \frac{|\Phi(\omega)|}{\delta^2} + 1 \right) \leq M, \]

where \( G^R_T(\omega) \) is as in Lemma 6.1. Then,

\[ \int_{B_N \cap G^R_T} |\Phi(\omega)| \leq \frac{4R^2}{M} + \delta^2 (e^{M^2} - 1) |B_N|, \]

where \( |B_N| \) denotes the volume of the ball \( B_N \).

**Proof.** It follows from

\[ \log \left( \frac{|\Phi(\omega, x)|}{\delta^2} + 1 \right) \leq M^2 \implies |\Phi(\omega, x)| \leq \delta^2 (e^{M^2} - 1) \]

and Chebyshev’s inequality.

\( \Box \)

We introduce the following assumptions on \( b \) and \( \sigma \):

**\( \text{(H1)} \)** \( b \in L^1_\text{loc}(\mathbb{R}^d), |\nabla b| \in (L \log L)_{\text{loc}}(\mathbb{R}^d) \) and \( \sigma \in L^2_\text{loc}(\mathbb{R}^d), |\nabla \sigma| \in L^2_\text{loc}(\mathbb{R}^d) \).

**\( \text{(H2)} \)** There exist \( b_n, \sigma_n \in C^\infty_b(\mathbb{R}^d) \) such that

(i) For any \( R > 0 \)

\[ \lim_{n \to \infty} \int_{B_R} |b_n - b| = 0, \quad \lim_{n \to \infty} \int_{B_R} |\sigma_n - \sigma|^2 = 0 \]  \( (6.1) \)

and

\[ \sup \left( \int_{B_R} |\nabla b_n| \left( \log(|\nabla b_n| + 1) \right) + \int_{B_R} |\nabla \sigma_n|^2 \right) < + \infty . \]  \( (6.2) \)

(ii) For some \( C_1, C_2 > 0 \) independent of \( n \),

\[ \|[-\text{div}b_n + \frac{1}{2} \partial_i \sigma_n^i \sigma_n^j + \sigma_n^i \partial_j \sigma_n^j + |\text{div} \sigma_n|^2] \|_{\infty} \leq C_1 \]  \( (6.3) \)

and

\[ \langle x, b_n(x) \rangle_{\mathbb{R}^d} + 2 |\sigma_n(x)|_{H^1}^2 \leq C_2 (|x|^2 + 1), \quad \forall x \in \mathbb{R}^d . \]  \( (6.4) \)

We are now in a position to prove our main result of this section.

**Theorem 6.3.** Assume that \( \text{(H1)} \) and \( \text{(H2)} \) hold. Then there exists a unique almost everywhere stochastic flow of \( \{X_t\} \) in the sense of Definition 2.1. Moreover, the constant \( K_{T,b,\sigma} \) in \( (2.1) \) is less than \( e^{C_1 T} \), where \( C_1 \) is from \( (6.3) \). In particular, if \( C_1 = 0 \), then \( K_{T,b,\sigma} \leq 1 . \)
Proof. (Existence): Let $b_n$ and $\sigma_n$ be as in (H2). Let $X_n$ solve the following SDE:

$$
\mathrm{d}X_n = b_n(X_n)\mathrm{d}t + \sigma_n(X_n)\mathrm{d}W_t, \; X_n|_{t=0} = x.
$$

We want to prove that for any $T, N > 0$ and $q \in [1, 2]$,

$$
\lim_{n,m \to \infty} \mathbb{E} \int_{B_N} \sup_{t \in [0,T]} \left| X_{n,t}(x) - X_{m,t}(x) \right|^q \mathrm{d}x = 0. \tag{6.5}
$$

First of all, by (6.4), it is standard to prove that

$$
\sup_n \sup_{x \in B_n} \mathbb{E} \left( \sup_{t \in [0,T]} \left| X_{n,t}(x) \right|^2 \right) < +\infty. \tag{6.6}
$$

Thus, for proving (6.5), it suffices to prove that for any $\eta > 0$,

$$
\lim_{n,m \to \infty} P \left\{ \omega : \int_{B_N} \sup_{t \in [0,T]} \left| X_{n,t}(\omega, x) - X_{m,t}(\omega, x) \right|^2 \mathrm{d}x \geq 2\eta \right\} = 0. \tag{6.7}
$$

Fix $\varepsilon, \eta, T > 0$ below and set

$$
\Phi_{n,m}(\omega, x) := \sup_{t \in [0,T]} \left| X_{n,t}(\omega, x) - X_{m,t}(\omega, x) \right|^2
$$

and

$$
G_{n,m}^R(\omega) := \left\{ x \in \mathbb{R}^d : \sup_{t \in [0,T]} \left| X_{n,t}(\omega, x) \right| \vee \left| X_{m,t}(\omega, x) \right| \leq R \right\}.
$$

Then,

$$
P \left\{ \omega : \int_{B_N} \Phi_{n,m}(\omega) \geq 2\eta \right\} \leq P \left\{ \omega : \int_{B_N \cap G_{n,m}^R(\omega)} \Phi_{n,m}(\omega) \geq \eta \right\} + P \left\{ \omega : \int_{B_N \cap G_{n,m}^R(\omega)} \Phi_{n,m}(\omega) \geq \eta \right\} = I_{n,m}^R + I_{n,m}^R. \tag{6.8}
$$

For $I_{n,m}^R$, by Chebyshev’s inequality and (6.6), we may choose $R > 0$ large enough such that for all $n, m \in \mathbb{N}$,

$$
I_{n,m}^R \leq \frac{1}{\eta} \mathbb{E} \int_{B_N \cap G_{n,m}^R(\omega)} \Phi_{n,m} \leq \frac{1}{\eta} \mathbb{E} \int_{B_N} \left( \mathbb{E} \Phi_{n,m}^2 \cdot P(\omega : x \notin G_{n,m}^R(\omega)) \right)^{\frac{1}{2}} \leq \varepsilon. \tag{6.9}
$$

Fixing such a $R$, we look at $I_{n,m}^R$. Set

$$
\xi_{n,m}^\delta := \int_{B_N \cap G_{n,m}^R} \log \left( \frac{\Phi_{n,m}}{\delta^2} + 1 \right).
$$

By (3.2) and (6.3), we have

$$
\sup_n \sup_{t \in [0,T], x \in \mathbb{R}^d} \mathbb{E} |\det(\nabla X_{n,t}(x))|^q \leq e^{2TC_1}, \tag{6.10}
$$

which yields that the constant $K_{T,b_n,\sigma_n}$ in (2.1) is bounded by $e^{C_1T}$. Hence, in Lemma 6.1, if we choose

$$
\delta = \delta_{n,m} = \int_{B_n} |\overline{b}_n - b_m| + \left[ \int_{B_n} |\overline{\sigma}_n - \sigma_m|^2 \right]^\frac{1}{2},
$$

then by (6.2), we have for some $C_{T,R,N}$ independent of $n, m$,

$$
\mathbb{E} \xi_{n,m}^\delta \leq C_{T,R,N}.
$$

Thus, there exists an $M_1 > 0$ such that for all $M \geq M_1$ and all $n, m$,

$$
P(\xi_{n,m}^\delta > M) \leq \varepsilon.
$$
Now, by Lemma 6.2 and (6.1), we may choose $M > M_1 \vee 8R^2/\eta$ and $n, m$ large enough such that
\[
\delta_{n,m} < \sqrt{\frac{\eta}{4(eM^2 - 1)|B_N|}},
\]
which leads to
\[
\Omega_{n,m}^M := \left\{ \omega : \int_{B_N \cap G_{n,m}^\delta(\omega)} \Phi_{n,m}(\omega) \geq \eta; \xi_{n,m}^{\delta_{n,m}}(\omega) \leq M \right\} = \emptyset.
\]
Hence, first letting $M$ large enough and then $n, m$ large enough, we obtain
\[
\mathcal{J}_{n,m}^R \leq P(\Omega_{n,m}^M) + P(\xi_{n,m}^{\delta_{n,m}} > M) \leq \varepsilon. \tag{6.11}
\]
Combining (6.8), (6.9) and (6.11), by the arbitrariness of $\varepsilon$, we get (6.7) as well as (6.5). So, for $q \in (1, 2)$, there exists a stochastic field $X \in L^q(\mathbb{R}^d; L^q(\Omega; C([0, T])))$ such that for any $N > 0$
\[
\lim \mathbb{E} \int_{B_N} \sup_{t \in [0, T]} |X_t(x) - X_t(\omega)|^q dx = 0.
\]
In particular, there is a subsequence still denoted by $n$ such that for $(P \times \mathcal{L})$-almost all $(\omega, x) \in \Omega \times \mathbb{R}^d$
\[
\lim \sup_{n \to \infty} \sup_{t \in [0, T]} |X_t(\omega, x) - X_t(\omega, x)| = 0. \tag{6.12}
\]
In view of (6.6), (6.10) and (6.12), by Lemma 3.5 and (6.1), it is easy to check that $X_t(\omega, x)$ satisfies (A) and (B) of Definition 2.1

(Uniqueness): Let $X_t(x)$ and $\tilde{X}_t(x)$ be two almost everywhere stochastic flows of (1.1). Then, by Lemma 6.1 we have for any $T, N, R > 0$ and $\delta > 0$,
\[
\mathbb{E} \int_{B_N \cap G_{T}^\delta} \log \left( \frac{\sup_{t \in [0, T]} |X_t - \tilde{X}_t|^2 \delta^2}{\gamma} + 1 \right) \leq C_{T,N,R},
\]
where $C_{T,N,R}$ is independent of $\delta$. Letting $\delta$ go to zero, we obtain
\[
1_{G_{T}^{\delta}(\omega)}(x) \cdot \sup_{t \in [0, T]} |X_t(\omega, x) - \tilde{X}_t(\omega, x)| = 0 \text{ a.e. on } \Omega \times B_N
\]
The uniqueness then follows by letting $R \to \infty$. \hfill \square

The following example is inspired by [22, 19].

**Example:** Let $d \geq 3$. Consider the following SDE in $\mathbb{R}^d$ with discontinuous and degenerate coefficients:
\[
dX_t = \frac{\beta X_t}{|X_t|^2} dt + \frac{X_t \otimes X_t}{|X_t|^2} dW_t, \quad X_0 = x,
\]
where $\beta \geq (4d^2 + 5d)/(d - 2)$. Define
\[
b(x) := \frac{\beta x}{|x|^2}, \quad \sigma(x) := \frac{x \otimes x}{|x|^2}
\]
and
\[
b_n(x) := \frac{\beta x}{|x|^2 + 1/n}, \quad \sigma_n(x) := \frac{x \otimes x}{|x|^2 + 1/n}.
\]
By virtue of $d \geq 3$, one sees that for any $q \in (1, 3/2)$
\[
|\nabla b| \in L^q_{loc}(\mathbb{R}^d) \subset (L \log L)_{loc}(\mathbb{R}^d), \quad |\nabla \sigma| \in L^2_{loc}(\mathbb{R}^d).
\]
Thus, (H1) is true for $b$ and $\sigma$.

Let us verify (H2). First of all, (6.1), (6.2) and (6.4) are easily checked. We look at (6.3). Noting that
\[
\tilde{\partial}_i \sigma_{ij}^n(x) = \frac{\partial_n(x^i x^j)(|x|^2 + 1/n) - 2 x^i x^j x^j}{(|x|^2 + 1/n)^2},
\]
Moreover, we have
\[
\text{div}\sigma^j = \partial_i\sigma^j(x) = \frac{(d-1)|x|^2 + (d+1)/n}{(|x|^2 + 1/n)^2}.
\]
Hence,
\[
\sum_i |\text{div}\sigma^j_i(x)|^2 = \frac{(d-1)|x|^2 + (d+1)/n}{(|x|^2 + 1/n)^4} \leq \frac{(d-1)|x|^2 + (d+1)/n}{(|x|^2 + 1/n)^3} \leq \frac{4d^2}{|x|^2 + 1/n}
\]
and
\[
\partial_i\sigma^j_i(x)\partial_j\sigma^j_i(x) = \frac{(d+3)|x|^2 + (d+1)/n}{(|x|^2 + 1/n)^4} \leq \frac{d+3}{|x|^2 + 1/n}.
\]
Furthermore,
\[
\text{div}b_n(x) = \frac{\beta(d-2)}{|x|^2 + 1/n} + \frac{2\beta}{n(|x|^2 + 1/n)^2}.
\]
Thus, combining the above calculations and by \(\beta \geq (4d^2 + 5d)/(d-2)\), we have
\[-\text{div}b_n + \frac{1}{2}\partial_i\sigma^j_i\partial_j\sigma^j_n + \sigma^j_n\partial_j\sigma^j_n + |\text{div}\sigma_n|^2 < 0,
\]
and so, (6.3) holds. Thus, (H2) is also true.

We now give two corollaries of Theorem 6.3.

**Corollary 6.4.** Assume that (H1) and (H2) hold. Let \(Y_0 \in L^2(\Omega, F_0)\) be such that \(P \circ Y_0 \ll \mathcal{L}\) and the density \(\gamma_0 \in L^\infty(\mathbb{R}^d)\). Then there exists a unique continuous \((\mathcal{F}_t)\)-adapted process \(Y_t(\omega)\) such that
\[
P \circ Y_t \ll \mathcal{L} \text{ with the density } \gamma_t \in L^\infty_\text{loc}(\mathbb{R}^d; L^\infty(\mathbb{R}^d)) \quad (6.13)
\]
and \(Y_t\) solves
\[
Y_t = Y_0 + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s)dW_s, \quad \forall t \geq 0. \quad (6.14)
\]
Moreover,
\[
Y_t(\omega) = X_t(\omega, Y_0(\omega)),
\]
where \(X_t(x)\) is the unique almost everywhere stochastic flow given in Theorem 6.3.

**Proof.** As in the proof in the appendix, we can check that \(Y_t(\omega) := X_t(\omega, Y_0(\omega))\) solves equation (6.14). Moreover, since \(X_t(x)\) is independent of \(Y_0\), by (2.21), we have for any \(\varphi \in L^\infty(\mathbb{R}^d)\) and \(t \in [0, T]\),
\[
\mathbb{E}\varphi(Y_t) = \mathbb{E}(\varphi(X_t(x)))|_{x=Y_0} = \int_{\mathbb{R}^d} \mathbb{E}\varphi(X_t(x))\gamma_0(x)dx
\]
\[
\leq ||\gamma_0||_\infty \int_{\mathbb{R}^d} \mathbb{E}\varphi(X_t(x))dx \leq ||\gamma_0||_\infty \cdot K_{T,b,\sigma} \int_{\mathbb{R}^d} \varphi(x)dx,
\]
which implies that \(P \circ Y_t \ll \mathcal{L}\) and the density \(\gamma_t\) satisfies
\[
\sup_{t \in [0,T]} ||\gamma_t||_\infty \leq ||\gamma_0||_\infty \cdot K_{T,b,\sigma}.
\]
Let us now look at the uniqueness. Let $\hat{Y}_t$ be another solution of (6.14) with $\hat{Y}_0 = Y_0$ and satisfy that

$$P \circ \hat{Y}_t \ll \mathcal{L}$$

with the density $\hat{\gamma}_t \in L^\infty_{loc}(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$.\hspace{1cm} (6.15)

It is now standard to prove that for any $T > 0$,

$$\mathbb{E}\left(\sup_{t \in [0, T]} |Y_t|^2\right) + \mathbb{E}\left(\sup_{t \in [0, T]} |\hat{Y}_t|^2\right) < +\infty. \hspace{1cm} (6.16)$$

Set

$$Z_t := Y_t - \hat{Y}_t$$

and for $R > 0$

$$\tau_R := \inf\{t \geq 0 : |Y_t| \vee |\hat{Y}_t| \geq R\}.$$ 

Then by (6.16), we have

$$P\left\{\omega : \lim_{R \to \infty} \tau_R(\omega) = +\infty\right\} = 1.$$

As in the proof of Lemma 6.1 we have

$$\mathbb{E}\log\left(\frac{|Z_{\tau\wedge \tau_R}|^2}{\delta^2} + 1\right) \leq 2\mathbb{E} \int_0^{\tau\wedge \tau_R} \frac{(Z_t b(Y_t) - b(\hat{Y}_t))^2}{|Z_t|^2 + \delta^2} ds + \mathbb{E} \int_0^{\tau\wedge \tau_R} \frac{||\sigma(Y_t) - \sigma(\hat{Y}_t)||^2}{|Z_t|^2 + \delta^2} ds \hspace{1cm} (6.17)$$

which yields the uniqueness by first letting $\delta \to 0$ and then $R \to \infty$. \hspace{1cm} $\Box$

**Corollary 6.5.** In addition to (H1) and (H2), we also assume that for some $q > d$,

$$|\nabla b| \in L^q_{loc}(\mathbb{R}^d).$$

Let $Y_0 \in L^2(\Omega, \mathcal{F}_0)$ be such that $P \circ Y_0 \ll \mathcal{L}$ and the density $\gamma_0 \in L^\infty(\mathbb{R}^d)$. Then $Y_t(\omega) := X_t(\omega, Y_0(\omega))$ uniquely solves SDE (6.14), where $X_t(x)$ is the unique almost everywhere stochastic flow given in Theorem 6.3.

**Proof.** Following the proof of Corollary 6.4 we only need to prove the uniqueness. Let $\hat{Y}$ be another solution of SDE (6.14) with the same initial value $\hat{Y}_0 = Y_0$. Choosing $q' \in (d, q)$, and using (3.17) in (6.17), we have

$$\mathbb{E}\log\left(\frac{|Z_{\tau\wedge \tau_R}|^2}{\delta^2} + 1\right) \leq C_{q'} \mathbb{E} \int_0^{\tau\wedge \tau_R} |\nabla b|^{q'} (y) dy + C_T \hspace{1cm} (6.18)$$

$$\leq C_{q', T} \int_{|y| < R} |\nabla b|^{q'} (y) dy + C_T \hspace{1cm} (6.19)$$

and

$$\mathbb{E}\log\left(\frac{|Z_{\tau\wedge \tau_R}|^2}{\delta^2} + 1\right) \leq C_{q', q} \mathbb{E} \int_0^{\tau\wedge \tau_R} |\nabla b|^{q'} (y) dy + C_T \hspace{1cm} (6.20)$$

which in turn implies the uniqueness as Corollary 6.4. \hspace{1cm} $\Box$
7. Proofs of Main Results

We first give:

**Proof of Theorem 2.6.** Under (2.6) and (2.7), it has been proven in Theorem 5.1. We now consider the case of (2.6) and (2.8). Let us define $b_n := b * \mathcal{Q}_n \cdot \chi_n$ and $\sigma_n := \sigma * \mathcal{Q}_n \cdot \chi_n$ as in (4.13). Note that as in estimating (4.8),

\[
|\nabla b_n| \leq |\nabla b| * \mathcal{Q}_n \cdot \chi_n + |b| * \mathcal{Q}_n \cdot |\nabla \chi_n| \\
\leq |\nabla b| * \mathcal{Q}_n + 2||\nabla \chi||_{\infty} \cdot ||b/(1 + |x|)||_{\infty}
\]

Hence,

\[
\rho_n \leq |\nabla b| * \mathcal{Q}_n + C_1.
\]

If we define

\[
\Psi(r) := (r + C_1) \log(r + C_1 + 1),
\]

then $r \rightarrow \Psi(r)$ is a convex function on $\mathbb{R}_+$. Thus, by Jensen’s inequality, we have for any $R > 0$,

\[
\int_{B_R} |\nabla b_n| \log(|\nabla b_n| + 1) \leq \int_{B_R} \Psi(|\nabla b| * \mathcal{Q}_n) \leq \int_{B_R} \Psi(|\nabla b|) * \mathcal{Q}_n \leq \int_{B_R} \Psi(|\nabla b|).
\]

(7.1)

Moreover, by (2.6) and (2.8), it is easy to check that

\[
\sup_n \left( \frac{1}{1 + |x|} \right) \leq 1 \text{ and } \leq 1 + ||\nabla \sigma_n||_{\infty} + ||\sigma_n \cdot |\nabla \div \sigma_n||_{\infty} < +\infty.
\]

(7.2)

Hence, (H1) and (H2) hold.

By Theorem 6.3 there exists a unique almost everywhere stochastic flow. Following the proof of Theorem 6.3, we only need to check (C) of Definition 2.1.

Fix a $T > 0$ and let

\[
\rho_n := \exp \left\{ \int_0^T \left( \div b_n - \frac{1}{2} \partial_i \sigma_n^{ij} \partial_j \sigma_n^{kl} \right) (X_n) ds + \int_0^T \div \sigma_n (X_n) dW_s \right\}.
\]

As in Lemma 3.1 and by (7.2), we have for any $p \geq 1$,

\[
\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}^d} \mathbb{E}[\rho_n(x)]^p < +\infty.
\]

(7.3)

In view of (6.6), (6.10) and (6.12), by Lemma 3.3 we have for any $N > 0$,

\[
\lim_{n \to \infty} \mathbb{E} \int_0^T \int_{B_N} |\div b_n - \div b| (X_n) = 0,
\]

\[
\lim_{n \to \infty} \mathbb{E} \int_0^T \int_{B_N} |\partial_i \sigma_n^{ij} \partial_j \sigma_n^{kl} - \partial_i \sigma_n^{il} \partial_j \sigma_n^{kl}| (X_n) = 0,
\]

\[
\lim_{n \to \infty} \mathbb{E} \int_{B_N} \left| \div \sigma_n (X_n) - \div \sigma (X) dW_s \right| = 0.
\]

So, there is a subsequence still denoted by $n$ such that for almost all $(\omega, x)$,

\[
\lim_{n \to \infty} \rho_n (\omega, x) = \rho_T (\omega, x),
\]

(7.4)

where $\rho_T (x)$ is defined by (2.2). By (7.3) and (7.4), we further have for any $p \geq 1$ and $N > 0$,

\[
\lim_{n \to \infty} \mathbb{E} \int_{B_N} |\rho_n - \rho_T|^p = 0.
\]

(7.5)

Now, let $Y_n$ solve the following SDE

\[
dY_n = -\ddot{b}_n (Y_n) dt + \sigma_n (Y_n) dW^T_t, \quad Y_n |_{t=0} = x,
\]

where $\ddot{b}_n = b^j_n - \sigma_n^{il} \partial_j \sigma_n^{kl}$ and $W^T_t := W_{T-t} - W_T$. As in the proof of Theorem 6.3 there exists

\[
Y \in L_{loc}^2 (\mathbb{R}^d; L^2 (\Omega; C([0, T])))
\]
such that for any $N > 0$,

$$
\lim_{n \to \infty} \mathbb{E} \int_{B_N \setminus \{0,T\}} |Y_{n,t}(x) - Y_t(x)|^2 \, dx = 0. \quad (7.6)
$$

Note that for any $\varphi, \psi \in C_c^+(\mathbb{R}^d)$ (see the proof of Lemma 3.1),

$$
\int \varphi(Y_{n,T}(\omega)) \cdot \psi = \int \varphi \cdot \psi(X_{n,T}(\omega)) \cdot \rho_n(\omega), \quad P \text{-a.s.} \quad (7.7)
$$

By (6.12), (7.5) and (7.6), if necessary, extracting a subsequence and then taking limits $n \to \infty$ in $L^1(\Omega)$ for both sides of (7.7), we get that for all $\varphi, \psi \in \mathcal{C} \subset C_c^+(\mathbb{R}^d)$ and $P$-almost all $\omega \in \Omega$,

$$
\int \varphi(Y_{T}(\omega)) \cdot \psi = \int \varphi \cdot \psi(X_{T}(\omega)) \cdot \rho_T(\omega). \quad (7.8)
$$

Since $\mathcal{C}$ is countable, one may find a common null set $\Omega' \subset \Omega$ such that (7.8) holds for all $\omega \notin \Omega'$ and $\varphi, \psi \in \mathcal{C}$. Thus, by (ii) of Lemma 3.4 one sees that (C) of Definition 2.1 holds.

We next give:

**Proof of Theorem 2.8**: We follow the classical Krylov-Bogoliubov’s method. Let $Y_0$ be an $\mathcal{F}_0$-measurable $\mathbb{R}^d$-valued random variable. Suppose that the probability law of $Y_0$ is absolutely continuous with respect to $\mathcal{L}$ with the density $\gamma_0 \in L^\infty(\mathbb{R}^d)$. Define $Y_t(\omega) := X_t(\omega, Y_0(\omega))$ and

$$
\mu_n(\varphi) := \frac{1}{n} \int_0^n \mathbb{E}\varphi(Y_s)ds = \frac{1}{n} \int_0^n \mathbb{E}\left(\int \mathcal{T}_s \varphi(X_0)\right)ds,
$$

where $\{X_s(x), x \in \mathbb{R}^d\}_{s \geq 0}$ is the unique almost everywhere stochastic flow of (1.1).

Noting that $Y_t(\omega)$ solves the following SDE (see Corollary 6.4)

$$
Y_t = Y_0 + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s)dW_s,
$$

by (2.9) and Itô’s formula, it is standard to prove that

$$
\mathbb{E}|Y_t|^2 \leq \mathbb{E}|Y_0|^2 \quad \text{or} \quad \frac{1}{t} \int_0^t \mathbb{E}|Y_s|^2ds \leq \frac{\mathbb{E}|Y_0|^2}{C_1t} + \frac{C_2}{C_1}.
$$

From this, we derive that the family of probability measures $\mu_n$ is tight.

On the other hand, for any $\varphi \in \mathcal{L}^+(\mathbb{R}^d)$, we have

$$
\mu_n(\varphi) \leq \int_0^n \int_{\mathbb{R}^d} \mathcal{T}_s \varphi(x) \cdot \gamma_0(x)dxds \leq \|\gamma_0\|_{\infty} \int_0^n \int_{\mathbb{R}^d} \mathcal{T}_s \varphi(x)dxds \leq \|\gamma_0\|_{\infty} \cdot K_{b,\sigma} \cdot \int_{\mathbb{R}^d} \varphi(x)dx,
$$

which means that

$$
\mu_n \ll \mathcal{L}
$$

and the density $\gamma_n$ satisfies

$$
\|\gamma_n\|_{\infty} \leq \|\gamma_0\|_{\infty} \cdot K_{b,\sigma}.
$$

Hence, there exists a subsequence $n_k$, $\gamma \in L^\infty(\mathbb{R}^d)$ and a probability measure $\mu$ such that

$$
\gamma_{n_k} \text{ weakly * converges to } \gamma \text{ in } L^\infty(\mathbb{R}^d)
$$

and $\mu_{n_k}$ weakly converges to $\mu$ in the sense that for any $\varphi \in C_b(\mathbb{R}^d)$

$$
\lim_{k \to \infty} \int_{\mathbb{R}^d} \varphi(x)\mu_{n_k}(dx) = \int_{\mathbb{R}^d} \varphi(x)\mu(dx).
$$
Since for all $\varphi \in C_c(\mathbb{R}^d)$,
\[ \int_{\mathbb{R}^d} \varphi(x)\mu(dx) = \int_{\mathbb{R}^d} \varphi(x)\gamma(x)dx, \]
we have $\mu(dx) = \gamma(x)dx$.

Let us verify (2.10). For $\varphi \in L^1(\mathbb{R}^d)$ and $t \geq 0$, since $T_t\varphi \in L^1(\mathbb{R}^d)$, we have
\[ \int_{\mathbb{R}^d} T_t\varphi(x)\gamma(x)dx = \lim_{k \to +\infty} \int_{\mathbb{R}^d} T_{t_n}\varphi(x)\gamma_{n_k}(x)dx = \lim_{k \to +\infty} \frac{1}{n_k} \int_{\mathbb{R}^d} T_{t+n_k}T_t\varphi(x)\gamma_0(x)dxds \]
\[ = \lim_{k \to +\infty} \frac{1}{n_k} \int_{0}^{\infty} \int_{\mathbb{R}^d} T_{t+s}\varphi(x)\gamma_0(x)dxds \]
\[ = \lim_{k \to +\infty} \frac{1}{n_k} \left( \int_{0}^{n_k} \int_{\mathbb{R}^d} T_{t+s}\varphi(x)\gamma_0(x)dxds + \int_{n_k}^{n_k+t} \int_{\mathbb{R}^d} T_{t+s}\varphi(x)\gamma_0(x)dxds + \int_{n_k+t}^{\infty} \int_{\mathbb{R}^d} T_{t+s}\varphi(x)\gamma_0(x)dxds \right) \]
\[ = \lim_{k \to +\infty} \frac{1}{n_k} \int_{\mathbb{R}^d} \varphi(x)\gamma_{n_k}(x)dx = \int_{\mathbb{R}^d} \varphi(x)\gamma(x)dx. \]
The proof is thus complete.

8. Appendix

Before proving Proposition 2.4, we need the following simple lemma.

**Lemma 8.1.** Let $\mathcal{G}$ and $\mathcal{A}$ be two independent $\sigma$-subalgebras of $\mathcal{F}$. Let $G : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a bounded $\mathcal{G} \times \mathcal{B}(\mathbb{R}^d)$-measurable function and $X : \Omega \times \mathbb{R}^d \to \mathbb{R}$ a $\mathcal{A} \times \mathcal{B}(\mathbb{R}^d)$-measurable mapping. Suppose that for $P$-almost all $\omega$, $\mathcal{L} \circ X(\omega, \cdot) \ll \mathcal{L}$. Then for $\mathcal{L}$-almost all $x \in \mathbb{R}^d$,
\[ (\mathbb{E}(G(\cdot, X(\cdot, x)))|\mathcal{A}) = (\mathbb{E}G(\cdot, y))|_{y=X(\cdot, x)}. \]  
(8.1)

**Proof.** Define $G_\varepsilon(\omega, y) := G(\omega, \cdot) * g_\varepsilon(y)$, where $g_\varepsilon$ is a family of regularized kernel functions as in Section 4. It is easy to see that
\[ (\mathbb{E}(G_\varepsilon(\cdot, X(\cdot, x)))|\mathcal{A}) = (\mathbb{E}G_\varepsilon(\cdot, y))|_{y=X(\cdot, x)}. \]  
(8.2)

Since for $(P \times \mathcal{L})$-almost all $(\omega, y) \in \Omega \times \mathbb{R}^d$,
\[ \lim_{\varepsilon \to 0} G_\varepsilon(\omega, y) = G(\omega, y) \]
and
\[ (P \times \mathcal{L}) \circ (\cdot, X(\cdot, \cdot)) \ll P \times \mathcal{L}. \]
By taking limits $\varepsilon \to 0$ for both sides of (8.2), we get (8.1). \qed

**Proof of Proposition 2.4.** Consider the case of almost everywhere stochastic invertible flow. Fix an $s > 0$ below. By (B) of Definition 2.1, one sees that
\[ (P \times \mathcal{L}) \circ (\theta_s(\cdot), X_s(\cdot, \cdot)) \ll P \times \mathcal{L}. \]  
(8.3)

Therefore, there exists a null set $A_s \subset \Omega \times \mathbb{R}^d$ such that for all $(\omega, x) \not\in A_s$,
\[ \tilde{X}_t(\omega, x) := \begin{cases} X_t(\omega, x), & t \in [0, s], \\ X_{t-s}(\theta_s \omega, X_s(\omega, x)), & t \in [s, \infty) \end{cases} \]
is well defined. We now check that $\tilde{X}$ still satisfies (A), (B) and (C) of Definition 2.1.

**Verification of (A) for $\tilde{X}$:** It is clear that for $\mathcal{L}$-almost all $x \in \mathbb{R}^d$, $t \mapsto \tilde{X}_t(x)$ is a continuous and $(\mathcal{F}_t)$-adapted process. We just need to show that for any $t > s$,
\[ \int_s^t \|b(\tilde{X}_t(x))\|dr + \int_s^t \|\sigma(\tilde{X}_t(x))\|^2dr < +\infty, \quad (P \times \mathcal{L}) - a.e., \]  
(8.4)
and for $\mathcal{L}$-almost all $x \in \mathbb{R}^d$,
\[
\tilde{X}_r(x) = X_s(x) + \int_s^r b(\tilde{X}_r(x))dr + \int_s^r \sigma(\tilde{X}_r(x))dW_r, \quad P - a.s. \tag{8.5}
\]

First of all, by (8.3) it is easy to see that (8.4) is true. We look at (8.5). Write
\[
Y_{s,t}(\omega, x) := X_{t-s}(\theta_s \omega, x), \quad t \geq s
\]
and for $M > 0$, set
\[
\tau_M(\omega, x) := \inf \left\{ t \geq 0 : \int_0^t |\sigma(X_{r}(\omega, x))|^2 dr > M \right\}.
\]
Then for $\mathcal{L}$-almost all $x, y \in \mathbb{R}^d$,
\[
\tau_M(\theta_s(\cdot, y)) \text{ and } Y_{s,t}(\omega, y) \text{ are independent of } X_s(x). \tag{8.6}
\]

By (A) for $X$ and (8.3), we have for $(P \times \mathcal{L})$-almost all $(\omega, x) \in \Omega \times \mathbb{R}^d$,
\[
\lim_{M \to \infty} \tau_M(\theta_s(\omega), X_s(\omega, x)) = +\infty. \tag{8.7}
\]
Observe that $Y_{s,t}(x)$ solves
\[
Y_{s,t}(x) = x + \int_s^t b(Y_{s,r}(x))dr + \int_s^t \sigma(Y_{s,r}(x))dW_r, \quad t \geq s.
\]
For verifying (8.5), by (8.7) it suffices to show that for $\mathcal{L}$-almost all $x \in \mathbb{R}^d$,
\[
\int_s^{\tau_M(\theta_s(\cdot, y))} \sigma(Y_{s,r}(y))dW_r \bigg|_{y=X_s(x)} = \int_s^{\tau_M(\theta_s(\cdot, X_s(x)))} \sigma(Y_{s,r}(X_s(x)))dW_r, \quad P - a.s. \tag{8.8}
\]
We extend $\sigma(Y_{s,r}(y)) = 0$ for $r < s$ and define for $h > 0$
\[
f_r^h(y) := \frac{1}{h} \int_{r-h}^r \sigma(Y_{s,r}(y))dr'.
\]
Then $r \to f_r^h(y)$ is a continuous and $(\mathcal{F}_t)$-adapted process and
\[
\int_s^t |f_r^h(y)|^2 dr \leq \int_s^t |\sigma(Y_{s,r}(y))|^2 dr, \lim_{h \to 0} \int_s^t |f_r^h(y) - \sigma(Y_{s,r}(y))|^2 dr = 0. \tag{8.9}
\]
Hence, for any $R > 0$, by (8.6) and Lemma 8.1 we have
\[
\begin{align*}
\mathbb{E} \int_{|X_s(x)| \leq R} \left( \int_s^{\tau_M(\theta_s(\cdot, X_s(x)))} (f_r^h(X_s(x)) - \sigma(Y_{s,r}(X_s(x))))dW_r \right)^2 dx \\
= \int \mathbb{E} \left( \int_s^{\tau_M(\theta_s(\cdot, X_s(x)))} (f_r^h(X_s(x)) - \sigma(Y_{s,r}(X_s(x)))) \cdot 1_{|X_s(x)| \leq R} dW_r \right)^2 dx \\
= \int \mathbb{E} \left( \int_s^{\tau_M(\theta_s(\cdot, X_s(x)))} |f_r^h(X_s(x)) - \sigma(Y_{s,r}(X_s(x)))|^2 \cdot 1_{|X_s(x)| \leq R} dr \right) dx \\
= \int \mathbb{E} \left( \int_s^{\tau_M(\theta_s(\cdot, y))} |f_r^h(y) - \sigma(Y_{s,r}(y))|^2 \cdot 1_{|y| \leq R} dr \right) y \\
\leq 2.1 \int B_R \mathbb{E} \left( \int_s^{\tau_M(\theta_s(\cdot, y))} |f_r^h(y) - \sigma(Y_{s,r}(y))|^2 dr \right) dy \\
\to 0, \quad \text{as } h \to 0, \tag{8.10}
\end{align*}
\]
Hence, by (2.1), we have for any function $\phi$ and $\sigma$ thus, for proving (8.8), we only need to prove that for fixed $h > 0$,

$$\lim_{h \to 0} \mathbb{E} \int_{X_t(x) \in R} \left( \int_{s \in \omega(x, y)} (f^h_r(y) - \sigma(Y_{s,t}(y)))dW_r \right) dy = 0,$$

Thus, for proving (8.8), we only need to prove that for fixed $h > 0$,

$$\int_{s \in \omega(x, y)} f^h_r(y)dW_r \bigg|_{y=X_t(x)} = \int_{s \in \omega(x, y)} f^h_r(X_t(x))dW_r, \quad P-a.s. \quad (8.11)$$

Let $\Delta_n = \{s = r_0 < r_1 < \cdots < r_n = t\}$ be a division of $[s, t]$. Write

$$F^h_n(y) := \sum_{r_k \in \Delta_n \setminus \{r_1\}} f^h_{r_k}(y) (W_{r_{k+1}} - W_{r_k}) \cdot 1_{r_k \in \omega(x, y)}$$

and

$$F^h(y) := \int_{s \in \omega(x, y)} f^h_r(y)dW_r.$$

Then $F^h_n(y)$ and $F^h(y)$ are independent of $X_t(x)$ and for $\mathcal{L}$-almost all $y \in \mathbb{R}^d$,

$$\mathbb{E}|F^h_n(y)|^2 \leq C_{h,M}, \quad \lim_{|\Delta_n| \to 0} \mathbb{E}|F^h_n(y) - F^h(y)|^2 = 0, \quad (8.12)$$

where $|\Delta_n| := \min_{r_k \in \Delta_n \setminus \{r_1\}} |r_{k+1} - r_k|$. Thus, as in estimating (8.10), by (2.1) and (8.12), we have

$$\lim_{|\Delta_n| \to 0} \mathbb{E} \int_{X_t(x) \in R} \left( F^h_n(X_t(x)) - \int_{s \in \omega(x, y)} f^h_r(X_t(x))dW_r \right)^2 dx = 0$$

and

$$\lim_{|\Delta_n| \to 0} \mathbb{E} \int_{X_t(x) \in R} \left( F^h_n(X_t(x)) - \int_{s \in \omega(x, y)} f^h_r(y)dW_r \bigg|_{y=X_t(x)} \right)^2 dx = 0,$$

which in turn yields (8.11).

**Verification of (B) for $\tilde{X}$**: By (8.6) and Lemma [8.1], we have for any bounded measurable function $\varphi$,

$$\mathbb{E}\varphi(Y_{s,t}(X_t(x))) = (\mathbb{E}\varphi(y)|_{y=X_t(x)}).$$

Hence, by (2.1), we have for any $s \leq t \leq T$

$$\int_{\mathbb{R}^d} \mathbb{E}\varphi(\tilde{X}_t(x))dx = \int_{\mathbb{R}^d} \mathbb{E}\varphi(Y_{s,t}(X_t(x)))dx \leq K_{T,b,\sigma} \int_{\mathbb{R}^d} \mathbb{E}\varphi(Y_{s,t}(y))dy \leq K^2_{T,b,\sigma} \int_{\mathbb{R}^d} \varphi(x)dx.$$

**Verification of (C) for $\tilde{X}$**: Fixing $t \geq s$, we have for $\varphi \in \mathcal{L}_{c}^{+}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \varphi(\tilde{X}_s^{-1}(\omega, x))dx = \int_{\mathbb{R}^d} \varphi(X_s^{-1}(\omega, X_t^{-1}(\theta_s, \omega, x)))dx$$

$$= \int_{\mathbb{R}^d} \varphi(X_s^{-1}(\omega, x)) \rho_{t-s}(\theta_s, \omega, x)dx$$

$$= \int_{\mathbb{R}^d} \varphi(x) \rho_{t-s}(\theta_s, X_s(\omega, x)) \rho_s(x) dx.$$

Noticing that

$$\rho_{t-s}(\theta_s, x) = \exp \left\{ \int_{s}^{t} \left[ \text{div} b - \frac{1}{2} \frac{\partial \sigma}{\partial x} \frac{\partial \sigma}{\partial x} \right](Y_{s,t}(x))dr + \int_{s}^{t} \text{div} \sigma(Y_{s,t}(x))dW_r \right\},$$

as in verifying (8.5), we have

$$\rho_{t-s}(\theta_s, X_s(x)) = \exp \left\{ \int_{s}^{t} \left[ \text{div} b - \frac{1}{2} \frac{\partial \sigma}{\partial x} \frac{\partial \sigma}{\partial x} \right](\tilde{X}_s(x))dr + \int_{s}^{t} \text{div} \sigma(\tilde{X}_s(x))dW_r \right\}.$$
Therefore, 
\[
\tilde{\rho}_t(x) = \rho_{t-s} (\theta_s, \tilde{X}_s(x)) \rho_s(x) = \\
= \exp \left\{ \int_0^t \left[ \text{div} b - \frac{1}{2} \frac{\partial \sigma^j}{\partial x^i} \frac{\partial \sigma^i}{\partial x^j} \right] (\tilde{X}_s(x)) ds + \int_0^t \text{div} \sigma (\tilde{X}_s(x)) dW_s \right\}.
\]

Finally, by the uniqueness, we have for \((P \times \mathcal{L})\)-almost all \((\omega, x) \in \Omega \times \mathbb{R}^d\),
\[
\tilde{X}_t(\omega, x) = X_t(\omega, x), \forall t \geq 0,
\]
that is, (2.3) holds.

**Markov Property (2.4):** It follows from (2.3) and Lemma 8.1 as well as the independence of \(X_t(\theta_s, x)\) and \(\mathcal{F}_s\).

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**References**

[1] Airault, H., Ren, J.: Modulus of continuity of the canonic Brownian motion “on the group of diffeomorphisms of the circle”. J. of Func. Anal., 196 (2002), No.2, pp. 395-426.
[2] Ambrosio, L.: Transport equation and Cauchy problem for BV vector fields. Invent. Math., 158 (2004), no. 2, 227-260.
[3] Ambrosio, L., Lecumberry, M. and Maniglia, S.: Lipschitz regularity and approximate differentiability of the DiPerna-Lions flow. Rend. Sem. Univ. Padova, 114 (2005), 29-50.
[4] Arnold, L.: Random dynamical systems. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.
[5] Crippa G. and De Lellis C.: Estimates and regularity results for the DiPerna-Lions flow. J. reine angew. Math. 616 (2008), 15-46.
[6] DiPerna R.J. and Lions P.L.: Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math., 98, 511-547 (1989).
[7] Evans, L.C. and Gariepy, R.F.: Measure theory and fine properties of functions. Studies in Advanced Mathematics, CRC Press, London, 1992.
[8] Fang, S. and Zhang T.: A study of a class of stochastic differential equations with non-Lipschitzian coefficients. Prob. Theory Relat. Fields, 132, 356-390 (2005).
[9] Fang, S. and Zhang, T.: Isotropic stochastic flow of homeomorphisms on \(S^d\) for the critical Sobolev exponent. J. Math. Pures Appl. (9) 85 (2006), no. 4, 580–597.
[10] Fang, S. and Luo, D.: Flow of homeomorphisms and stochastic transport equations. Stoch. Anal. and Appl., Vol. 25:1079-1108 (2007).
[11] Fang, S., Imkeller, P. and Zhang, T.S.: Global flows for stochastic differential equations without global Lipschitz conditions. Ann. of Prob., Vol. 35, No.1, 180-205 (2007).
[12] Figalli, A.: Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients. J. Funct. Anal. 254 (2008), no. 1, 109–155.
[13] Flandoli, F., Gubinelli, M. and Priola, E.: Well-posedness of the transport equation by stochastic perturbation. http://arxiv.org/abs/0809.1310.
[14] Gyöngy I., Martinez, T.: On stochastic differential equations with locally unbounded drift. Czechoslovak Math. J. 51(126) (2001), no. 4, 763-783.
[15] Ikeda, N. and Watanabe S.: Stochastic differential equations and diffusion processes. North-Holland/Kodanska, Amsterdam/Tokyo, 1981.
[16] Kunita, H.: Stochastic flows and stochastic differential equations. Cambridge, Cambridge University Press, 1990.
[17] Krylov, N.V. and Röckner, M.: Strong solutions of stochastic equations with singular time dependent drift. Probab. Theory Related Fields 131 (2005), no. 2, 154–196.
[18] Le Bris, C. and Lions, P.L.: Renormalized solutions of some transport equations with partially \(W^{1,1}\) velocities and applications. Annali di Matematica, 183, 97-130 (2004).
[19] Le Bris, C. and Lions, P.L. : Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients. Comm. in Partial Diff. Equ., 33:1272-1317,2008.
[20] Le Jan Y. and Raimond O.: Integration of Brownian vector fields. Ann. of Prob., 30(2002)826-873.
[21] Malliavin, P.: The canonic diffusion above the diffeomorphism group of the circle. C.R. Acad. Sci. Paris, Série I, 329 (1999), pp. 325-329.
[22] Ottinger, H.C.: Stochastic processes in polymeric fluids. Springer-Verlag, Berlin, 1996.
[23] Revuz, D. and Yor, M.(1994): Continuous martingales and Brownian motion. Second Edition, Springer-Verlag.
[24] Rozovskii, B.L. : Stochastic evolution systems. Linear theory and applications to nonlinear filtering. Mathematics and its Applications (Soviet Series), 35, Kluwer Academic Publishers, 1990.
[25] Zhang, X.: Homeomorphic flows for multi-dimensional SDEs with non-Lipschitz coefficients. Stoch. Proc. and Appl., 115, 435-448(2005) and Erratum: 116, 873-875(2006).
[26] Zhang, X.: Strong solutions of SDEs with singular drift and Sobolev diffusion coefficients. Stoch. Proc. and Appl., 115/11 pp. 1805-1818(2005).
[27] Zhang, X.: Stochastic partial differential equations with unbounded and degenerate coefficients. Preprint.
[28] Zhang, X.: Stochastic flows and Bismut formulas for non-Lipschitz stochastic Hamiltonian systems. Preprint.