Stability in MaVaN Models

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Mass-varying neutrino (MaVaN) models propose a source of dark energy in a new scalar field called the acceleron. Recent work has shown that nonrelativistic neutrino fields in these theories are unstable to inhomogeneous fluctuations, and form structures that no longer behave as dark energy. One might expect that in multiple-neutrino models, the lighter species could continue to act as a source for the acceleron, generating dark energy without the help of heavier species. This paper shows that by considering the evolution of the acceleron field for a large class of models, the result of any component becoming unstable is that all components become unstable within a short time on cosmological scales. An alternate model employing a second scalar field in a hybrid potential is shown to have stable MaVaN dark energy even in the presence of unstable heavier components.

INTRODUCTION

Recent results in precision cosmology have created a number of puzzles. Among the unexplained phenomena are several apparent coincidences in which the energy density of two components are comparable despite having different redshift properties. Another is the existence of a negative pressure cosmological fluid that accelerates the rate of expansion of the universe. The model of mass-varying neutrinos (MaVaN), introduced by Fardon et al. in [1], suggests that neutrino and dark energy densities have tracked each other throughout the lifetime of the universe through a new scalar field called the acceleron. The energy density of the scalar potential of this new field contributes to the dark energy. The authors showed that this new field is capable of explaining the present cosmological expansion.

Since the original MaVaN proposal, several authors have investigated the stability of the dark sector, composed of neutrinos and the scalar field, under perturbations to the neutrino density [2]. Both groups subject the model to a hydrodynamic analysis. In this approximate picture, the speed of sound squared in the cosmological fluid, given by \[ c_s^2 = \frac{\dot{P}}{\dot{\rho}} = w + \frac{\ddot{\rho}}{\dot{\rho}} = w - \frac{\ddot{w}}{3H(1+w)} \] (1)
is positive only if
\[ \frac{\partial w}{\partial z} \geq -\frac{3w(1+w)}{1+z} \] (2)
where \( z \) is the redshift. However, for nonrelativistic neutrinos, \( \frac{\partial w}{\partial z} \) is a negative quantity while the right hand side is positive for \( w \) close to -1. Since at least one neutrino must be nonrelativistic today, the dark sector appears unstable.

Additionally, Afshordi, et al. perform a stability analysis using kinetic theory to account for neutrino streaming [2]. This analysis also shows that perturbations in the neutrino field become unstable when the mass of the neutrino is of order the neutrino temperature. The ratio of mass to temperature when the instability occurs is a function of the acceleron potential, but is approximately 7 for the potentials considered in this paper. Afshordi, et al., also examine the result of undergoing a phase transition in a neutrino component, and suggest that the unstable neutrino field may rapidly form nonlinear structures termed neutrino nuggets, which redshift as dark matter and cannot drive the cosmic expansion.

In this paper, we will assume that the neutrinos are initially relativistic. As the universe expands and cools, the neutrinos become less relativistic until their mass and temperature are approximately equal. At this point, we assume that they decouple from the scalar field into some structure such as the one described by Afshordi et al. The resulting dark matter will not provide a large contribution to the energy density, and we will not include the contribution in the dark sector energy. Note that in this paper dark sector will always refer to the energy density contributions of the stable neutrinos and the scalar field.

In a model with multiple neutrinos, the dark energy may still be driven by relativistic species even after the heavier components have become unstable. However, the coupling between the neutrinos and acceleron create a feedback mechanism. The shift in the acceleron expectation value when a neutrino becomes unstable can be sufficient to necessarily change the mass of another species so that it goes from relativistic to nonrelativistic, causing it to become unstable as well. This cascaded instability can make all neutrinos unstable at about the same time that the heaviest neutrino becomes unstable. This paper will examine a particular class of models, and will show that see-saw MaVaN models with flat scalar potentials suffer from precisely this problem. The timing between the instability in successive neutrinos is strongly dependent on the flatness of the scalar potential, but a flat potential is also required to generate the observed dark energy. This result may point toward models that do no suffer from
a cascaded instability, and this paper concludes with a simple example.

SEE-SA W MODELS

Consider a model with $n$ active neutrinos in which each neutrino is paired with a sterile counterpart, which is coupled to a new scalar field,

$$-L \supset \sum_{i=1}^{n} (M_i^2 \nu_i N_i + \lambda A N_i N_i)$$  \hspace{1cm} (3)

where $\nu_i$ are the active neutrinos, $N_i$ are the sterile neutrinos, and $A$ is the acceleron scalar field. $M_i$ and $\lambda$ describe coupling strengths. If we assume that the normal see-saw limit holds, $\langle \lambda A \rangle \gg M_i$, the system reduces to an effective Lagrangian describing active neutrinos with a Majorana mass term,

$$-L_{\text{eff}} \supset \sum_{i=1}^{n} \frac{M_i^2}{A} \nu_i \bar{\nu}_i \equiv \sum_{i=1}^{n} m_i \nu_i \nu_i$$  \hspace{1cm} (4)

where $M_i^2 = M_i^2 / \lambda$, and $m_i$ is the effective mass.

The energy density of the dark sector is

$$\rho_d = \rho_\nu + V(A)$$  \hspace{1cm} (5)

where $V$ is the potential of the acceleron field. Assuming that the neutrino distribution function is a stretched thermal distribution, the value of the equation of state for this dark sector is (one derivation is given in [2])

$$w = \frac{\rho_d}{\rho_d} = \frac{T^4 \sum_i \left[AF \left(\frac{m_i^2}{T^2}\right) - J \left(\frac{m_i^2}{T^2}\right)\right]}{3 \left[ T^4 \sum_i F \left(\frac{m_i^2}{T^2}\right) + V(A) \right]} - 1$$  \hspace{1cm} (6)

where $i$ runs over the active neutrino species, and we have defined the distribution function and its derivative,

$$F(x) = \frac{1}{\pi^2} \int_{0}^{\infty} \frac{y^2 \sqrt{y^2 + x^2} dy}{e^y + 1}$$  \hspace{1cm} (7)

$$J(x) = \frac{x^2}{\pi^2} \int_{0}^{\infty} \frac{y^2 dy}{\sqrt{y^2 + x^2(e^y + 1)}}$$  \hspace{1cm} (8)

Fardon, et al., show in [1] that the system remains very close to the minimum of the effective potential and evolves adiabatically. For the model above, this minimization conditions becomes

$$\frac{\partial V}{\partial A} = \sum_{i=1}^{n} \int_{0}^{\infty} \frac{dy}{\pi^2} \frac{m_i T^2 y^2}{y^2 + \frac{m_i^2}{T^2}} \frac{1}{1 + e^y} (-\frac{\partial m_i}{\partial A})$$  \hspace{1cm} (9)

where $y = \frac{m_i}{T}$.

Commonly employed forms for the potential include small power law ($V = B A^k$, $k \ll 1$), logarithmic ($V = \text{Blog}(A / A_0)$) and quadratic ($V = B A^2$). After starting with the small power law case, it will be easy to generalize to all cases by assuming $\frac{\partial V}{\partial A} = B k A^{k-1}$ with $B$ and $k$ unrestricted.

APPROXIMATIONS

The expectation value of the acceleron at a particular value of $z$ is determined by the minimization equation (9). Unfortunately, this equation in general does not have a closed form solution. To examine the behavior of this equation it is useful to approximate the result in three ranges: relativistic ($m_\nu \ll T$), quasirelativistic ($m_\nu \sim T$), and nonrelativistic ($m_\nu \gg T$). In these approximation, the minimization equation becomes

$$R: \frac{\partial V}{\partial A} \simeq \frac{1}{A^3} \frac{M_i^4 T^2}{\pi^2} I_1$$  \hspace{1cm} (10)

$$NR: \frac{\partial V}{\partial A} \simeq \frac{1}{A^5} \frac{M_i^4 T^3}{\pi^2} I_2$$  \hspace{1cm} (11)

$$QR: \frac{\partial V}{\partial A} \simeq \frac{1}{A^7} \frac{M_i^4 T^4}{\pi^2} I_3$$  \hspace{1cm} (12)

with the unitless $O(1)$ integrals

$$I_1 = \int_{0}^{\infty} dy \frac{y}{1 + e^y} \simeq 0.822$$  \hspace{1cm} (13)

$$I_2 = \int_{0}^{\infty} dy \frac{y^2}{1 + e^y} \simeq 1.803$$  \hspace{1cm} (14)

$$I_3 = \int_{0}^{\infty} dy \frac{y^2}{(1 + e^y)^2} \simeq 0.670$$  \hspace{1cm} (15)

A comparison of these approximations to unapproximated numerical simulation for the two neutrino models discussed in the next section is shown in figure 1. This figure shows the contribution of the derivative of the neutrino term to the minimization equation, which are the right hand sides of equations (10-11). The relativistic and nonrelativistic approximations (dashed curves) are a good match to the numerically calculated value in their respective regions of validity at high and low redshift.

![FIG. 1: The derivative of neutrino energy density with respect to A. The solid line is a numerically determined result, and the dashed lines are approximations in the nonrelativistic (low z) and relativistic (high z) regimes.](image-url)
TWO ACTIVE NEUTRINOS, SMALL POWER LAW

As a warm-up, consider the case of two active neutrinos, with \( m_1 \gg m_2 \), and a power law potential with \( 0 < k \ll 1 \). To first order, the derivative of the potential becomes (accurate to 10% for typical small values of \( k \))

\[
\frac{\partial V}{\partial A} \approx \frac{Bk}{A} \tag{16}
\]

At large \( z \), both neutrinos will be relativistic. As the universe cools, the neutrino temperature decreases while mass increases, and at some value of \( z \) neutrino 1 becomes quasirelativistic, and will become unstable. Due to the original mass hierarchy, neutrino 2 is still relativistic at this point.

Now consider the acceleron expectation value and neutrino masses both before and after the transition of neutrino 1 to a dark matter phase. Before the transition, when neutrino 1 has mass of order the temperature, we have

\[
m_{1,\text{before}} \sim T \rightarrow A_{\text{before}} \sim \frac{M_1^2}{T} \tag{17}
\]

\[
m_{2,\text{before}} = \frac{M_2^2}{A_{\text{before}}} \sim \frac{M_2^2}{M_1^2} \tag{18}
\]

So our mass hierarchy implies \( M_2^2 \ll M_1^2 \).

The minimization equation \(19\), with the approximations from \(10\) and \(12\), yields the value of the acceleron prior to the transition,

\[
A_{\text{before}} = \frac{T}{\pi} \frac{1}{\sqrt{Bk}} \sqrt{M_1^4 I_3 + M_2^4 I_1} \tag{19}
\]

Since the acceleron will suddenly change value when neutrino 1 becomes unstable and stops sourcing it, we do not know a priori whether neutrino 2 will be R, NR or QR after the transition. By trying each in turn, we quickly find that only the QR assumption is consistent. For instance, an assumption that neutrino 2 stays relativistic yields a value for \( m_2 \) that is of order the temperature, which violates the assumption. Looking carefully at the QR case, from the \( m_1 \sim T \) condition before the transition, we have the relationship

\[
\frac{\sqrt{Bk}}{T} \frac{\pi}{\sqrt{I_3}} \sim T \tag{20}
\]

which provides the temperature at the time of transition in terms of the parameters of the system. Solving for the acceleron and neutrino 2 mass after the transition, we have

\[
A_{\text{after}} = \frac{M_2^2 T}{\pi} \sqrt{\frac{I_1}{Bk}} \tag{21}
\]

\[
m_{2,\text{after}} = \frac{\sqrt{Bk}}{T} \frac{\pi}{\sqrt{I_3}} \sim T \tag{22}
\]

The picture that emerges from this small example is that after the first neutrino becomes unstable and decouples from the acceleron, the acceleron assumes a value that pushes the second neutrino into a quasirelativistic region. Once this occurs, the second neutrino will also soon become unstable. In the end there is nothing left to drive the dark energy.

GENERALIZED MODELS

The simple two-neutrino, small-power law model generalizes easily to include a larger number of neutrino species and different potentials. Additional neutrinos do not improve the stability picture, but decreasing the flatness of the potential does.

First consider the addition of a third neutrino that is much less massive than neutrino 1. This requires \( M_2^2 \ll M_1^2 \) and \( M_3^2 \ll M_1^2 \). After the transition, the value of the acceleron becomes

\[
A_{\text{after}} = \frac{1}{\sqrt{Bk}} \frac{T}{\pi} \sqrt{M_2^4 I_1 + M_3^4 I_1} \tag{23}
\]

As a result, the mass of the second and third neutrinos become

\[
m_{2,\text{after}} = \sqrt{\frac{I_3}{I_1}} \frac{M_2^2}{\sqrt{M_2^4 I_3 + M_3^4 I_3}} T \tag{24}
\]

\[
m_{3,\text{after}} = \sqrt{\frac{I_3}{I_1}} \frac{M_3^2}{\sqrt{M_2^4 I_3 + M_3^4 I_3}} T \tag{25}
\]

Neutrino 2 is relativistic only if \( M_2 \ll M_3 \), but similarly neutrino 3 is relativistic only if \( M_3 \ll M_2 \). Since we can not satisfy both these conditions, at least one of the remaining neutrinos is quasirelativistic, and becomes unstable.

Using similar arguments, it is easy to see that for the general case of \( n \) neutrinos, all neutrinos will become unstable within a short period of each other.

Now consider a more general potential, \( V = BA^k \) (for an arbitrary \( k \)), for the two-neutrino case. The mass of neutrino 2 after neutrino 1 becomes unstable is

\[
m_{2,\text{after}} \sim \left( \frac{I_1}{I_1} \right)^{k+1} \frac{M_1^{2k}}{M_2^{2k}} m_2^2 T \tag{26}
\]

In this case, to keep neutrino 2 relativistic, we require

\[
\left( \frac{M_2}{M_1} \right)^{2k} \ll 1 \tag{27}
\]

Note that this does allow the light neutrino to continue to drive the acceleron, but only if the acceleron potential is flatter than the small-\( k \) power we examined above. Unfortunately, this potential predicts a dark energy equation of state that is in conflict with observation. The
value of \( k \) is connected to the present value of the equation of state by
\[
  k = \frac{1 - w_0}{w_0} \tag{28}
\]
Requiring a value of \( w_0 \) close to \(-0.9\) yields a value of \( k \) that lies in the region where the lighter neutrinos become unstable very quickly. Conversely, values of \( k \) that are large enough to keep the lighter neutrinos relativistic also predict a value of \( w_0 \) too large.

**NUMERICAL SIMULATION**

We verified the above relationships using an unapproximated numerical simulation. An example of the dependence of the stability of lighter neutrinos on the steepness of the acceleron potential is shown in figure 2. The plots show the evolution of the equation of state for the neutrino and dark energy components, as given in (6), calculated numerically, of a two neutrino system for two values of \( k \). Note that this \( w \) does not include contributions from either the dark matter terms that resulted from unstable neutrino components, or from components that were not included in the dark sector as defined above. As \( k \) increases, the lighter neutrino is longer-lived, and can continue to drive the dark energy closer to today (\( z = 0 \)). However, this also results in \( w \) approaching a disallowed value.

Comparing the set of approximations made in simplifying the minimization equation to the numerical results show that they are accurate to 15-50%. Noting that the order-of-magnitude arguments above already contain multiplicative \( O(1) \) corrections, the contribution of numerical inaccuracy does not affect our conclusions.

**HYBRID MODEL**

The condition for the stability of lighter neutrino species, given in equation (27), suggests that we need to find a model that does not require a flat potential to generate dark energy. There are several ways to achieve this, and here we will discuss a particularly simple extension. This new model employs a second scalar field that is not directly coupled to the neutrinos but provides large contribution to the dark energy. Since the configuration of the potential is borrowed from hybrid inflation (see [2]), the new model is called the “Hybrid MaVaN” model.

The hybrid model includes a scalar field, \( \sigma \), that acts as the waterfall field. The potential is
\[
  V = b^2 A^2 + g^2 A^2 \sigma^2 + (h^2 - \alpha \sigma^2)^2 \tag{29}
\]
where \( b, g, h \) and \( \alpha \) are new coupling constants. The precise details of the potential are unimportant, as long as it supports a false minimum as discussed below.

The hybrid potential has two distinct regions of behavior under variation of \( \sigma \). If \( 2 \alpha h^2 > g^2 A^2 \), then there are two minima at \( \pm \sqrt{\frac{2 \alpha h^2 - g^2 A^2}{2 \alpha^2}} \). Otherwise, there is a single “false minimum” at 0. Forcing the field into this false minimum by requiring
\[
  A > \sqrt{2 \alpha h/g} \tag{30}
\]
the potential becomes
\[
  V \rightarrow b^2 A^2 + h^4 \tag{31}
\]
which includes a cosmological-constant type term \( h^4 \). This term dominates the acceleron contribution to the potential if \( h \gg \sqrt{2 \alpha h/g} \), and from equations (5) and (6) may also dominate over the neutrino contribution to the energy density. If we also assume the see-saw condition, \( A \gg m_i/\lambda \), then the model described in previous sections can be used without any change other than using the form of the potential in equation (31).

The quadratic dependence on \( A \) in equation (31) means that the stability condition for the lighter neutrinos in equation (27) is easily satisfied. The allowed parameter
range is quite large, and it is straightforward to find coefficients that are produce observationally allowed values of neutrino mass and equation of state. The numerical simulation of the evolution of one such model, with a hierarchy of masses and $h = 0.06$ is shown in figure 3. The lighter two neutrinos stay relativistic until $z = 0$, despite the instability in the massive neutrino. Note this model achieves both the stability of the lighter neutrinos and has a dark sector equation of state of $w = -1$ at $z$ near 0.

There are a number of possible resolutions. One is to increase the curvature of the scalar potential. In models with a single scalar field, this makes it difficult for the scalar field potential to form dark energy. However, this is easily remedied by including a second scalar field. A simple example is illustrated above in the hybrid MaVaN model. A similar potential arises naturally in supersymmetric models, such as those in [7] and [8].

Another solution is to modify the theory so that the dark sector never reaches a state where the adiabatic condition applies. Such models do not suffer from the instability described above. One such theory is presented in [9].

Reducing the dependence of the acceleron on the heavy neutrino components also forms a class of possible solutions. If the acceleron is decoupled from each heavy component before it becomes unstable, the acceleron expectation value is not quickly driven to a new scale. The mass of the lighter neutrinos would be largely unaffected, avoiding instability.

**CONCLUSION AND DISCUSSION**

Several authors have found that a nonrelativistic MaVaN neutrino field is unstable to inhomogeneous fluctuations. By considering the evolution of the acceleron expectation value when a neutrino field becomes unstable, we have argued above that all neutrino fields in the theory are susceptible to a cascaded instability in which they all become unstable at nearly the same time. This occurs as long as the scalar potential has a nearly flat dependence on the acceleron, and the neutrino masses vary inversely with the acceleron. Including a very light neutrino is not sufficient to avoid this problem. Since there are at least three neutrino species, and the atmospheric neutrino deficit requires at least one mass scale above 1eV, the instability poses a constraint on all physical MaVaN models.

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