Percolation on networks with weak and heterogeneous dependency

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In real networks, the dependency between nodes is ubiquitous, however, the dependency is not always complete and homogeneous. In this paper, we propose a percolation model with weak and heterogeneous dependency, i.e., different nodes could have different dependency. We find that the heterogeneous dependency strength will make the system more robust, and for various distributions of dependency strengths both the continuous and discontinuous percolation transitions can be found. For Erdős-Rényi networks, we prove that the crossing point of the continuous and discontinuous percolation transitions is dependent on the first, second and third moments of the dependency strength distribution. This indicates that the discontinuous percolation transition on networks with dependency is not only determined by the dependency strength but also its distribution. Furthermore, in the area of the continuous percolation transition, we also find that the critical point depends on the first and second moments of the dependency strength distribution.

I. INTRODUCTION

Percolation model is an important and widely used model in the study of complex networks[1]. It describes the properties of the clusters formed by some links and nodes. For instance, the bond (link) percolation has been used to study the spreading of epidemic disease[2], and the site (node) percolation has been used to study the structure and the robustness of complex networks[3]. On a tree-like network, all these models can be solved exactly by the so-called generating function technique[4]. In this way, the percolation model can also provide a theoretical approach to study the problems on/of complex networks.

In recent years, many modified percolation models have been proposed in the studying of complex networks, such as the k-core percolation, l-hop percolation and clique percolation[5–8]. In these percolation models, different dependency relationship between adjacent nodes have been proposed to represent the correlations between them in the spreading dynamics on complex networks or the organization of complex networks. Beyond the connection, the dependency relationship between nodes without a direct connection has also been taken into account as a part of the complicated interdependency in real systems[9, 10]. In general, this type of dependency is presented by the so-called dependency link. Two nodes connected by a dependency link will fail together if either of them fails (remove from the network). This mechanism makes the failures easier to spread in the network, and the network, which, with dependency links, are much more fragile than those without them[11]. More interesting, different from the continuous transition found in the classic percolation, the percolation process demonstrates a discontinuous phase transition. Therefore, this new model has attracted the attention of many physicists[11]. It is found that there is actually a hybrid phase transition in these models, meaning that the order parameter has a jump at the transition point but there are also critical phenomena related to it[12–14].

Those researches have shown that the dependency between nodes within a network or between different networks can greatly jeopardize the stability of the whole system. However, most networks in real world do not seem to be so fragile. Thus, many previous researches are concerning about networks with only a fraction of nodes having dependency links, the so-called partial dependency[15–17]. They found that a reduction in the number of dependency links would make the system more robust, and a crossover of the continuous and discontinuous percolation transitions can be found.

Here we focus on another aspect of this problem. We think that one of the key reasons for the robustness of the real dependent networks is that the strength of dependency is limited and heterogeneous. That is the failure of a node’s dependency partner sometimes can only reduce its function partially, instead of destroying it completely. For example, in a financial network, when a company loses its partner with funding requirement, it often only loses some trading links instead of going into bankruptcy. In this paper we will study this type of dependency. In our model, a failed node will not cause the node depending on it fail completely but only lose some of its connectivity links. Based on this mechanism, both homogeneous and heterogeneous dependency will be considered in this paper.

The paper is organized as follows. In Sec.II, the details of our model will be given. We will give the results of the general formalism (homogeneous dependency) in Sec.III, and then the heterogeneous dependency will be studied in Sec.IV. In the last section, we will report our conclusions.

II. MODEL

Our model displays on a network with degree distribution \(p_k\) and average degree \(\langle k \rangle\). The dependency partners are assigned randomly, and each node has only one dependency partner. For the classical percolation, we occupy a fraction
$p$ of the nodes in the network, and all the unoccupied nodes (fraction $1 - p$) are considered as failed nodes and will be removed from the network. When a node loses its dependency partner, each of its links will be removed from the network with a probability $\beta$, respectively. Note that $\beta$ could be different for different nodes. The preserved links will no longer be affected by the failed dependency partner. At the same time, all the nodes that do not belong to the giant component will also be considered as failed nodes. Repeating the two processes to ensure that there are no nodes and links can be removed from the network. To evaluate the robustness of such systems, we check the fraction of nodes in the giant component of the final network, $S$, which marks the size of the giant component.

Obviously, the parameter $\beta$ plays a key role in determining the robustness of such networks. For simplification, we call $\beta$ as dependency strength. When $\beta = 0$ for all nodes, there is no dependency between nodes in the system, and the system just takes a classical percolation on network. While $\beta = 1$ for all nodes, this model reduces to the one discussed by Parshani et al. [9], in which the dependency is very strong and the system is fragile. Furthermore, by assigning different nodes with different $\beta$, we can study a system with heterogeneous dependency. This is one of the key issues studied in this paper.

III. HOMOGENEOUS DEPENDENCY

A. General formalism

In this section, we consider the case of homogeneous dependency, i.e., all nodes have the same dependency strength $\beta$. We solve this model by considering the final state after the cascades as the method used in ref. [18, 19]. Let $R$ be the probability that a node reached by following a randomly chosen link belongs to the giant component of the final network.

Then, the size of the giant component $S$ can be written as

$$S = p^2[1 - G_0(1 - R)]^2 + p[1 - G_0(1 - R + R\beta)][1 - p + pG_0(1 - R)],$$

where $G_0(x)$ is the generating function of the degree distribution $G_0(x) = \sum k p_k x^k$. It is easy to know that $G_0(1 - R)$ gives the probability that a randomly chosen node does not connect to the giant component. Since the dependency partners are paired randomly, such a probability for a randomly chosen node’s dependency partner can also be expressed as $G_0(1 - R)$. For a node losing its dependency partner, this probability will be $G_0(1 - R + R\beta)$. In this way, the first term of eq.(1) is just for the node with a working dependency partner, and the second term is for the node with a failed dependency partner.

To solve eq.(1), we must get the equation for $R$. For this, the generating function $G_1(x) = \sum k p_k x^{k-1}/\langle k \rangle$ will be used, which describes the excess degree of the node reached by following a link. Similar with eq.(1), we can get a self-consistent equation for $R$,

$$R = p^2[1 - G_1(1 - R)][1 - G_0(1 - R)] + p(1 - \beta)[1 - G_1(1 - R + R\beta)][1 - p + pG_0(1 - R)].$$

The meanings of the two terms are same as that in eq.(1). Note that $1 - \beta$ in the second term means that the link must be preserved when the dependency partner fails.

Thus we can obtain $S$ and $R$ from eqs. (1) and (2) for a network with given $p_k$ and $\beta$. In Fig. 1, we give the simulation results for both Erdős-Rényi (ER) and scale-free (SF) networks, which agree with our theory well. We can also find that the critical point $p_c$ decreases with the increasing of dependency strength $\beta$. This indicates that the networks become more robust as $\beta$ decreases. Moreover, Fig. 1 also indicates that there exists a critical point $p_c$, above which the system will show a discontinuous percolation transition.

B. The critical point

Next, we will show how to obtain the critical point $p_c$ and $\beta_c$ of the system. First, we consider ER network, which takes the Poisson degree distribution $p_k = e^{-\langle k \rangle}\langle k \rangle^k/k!$. Then, the generating functions $G_0(x)$ and $G_1(x)$ take the same form,

$$G_0(x) = G_1(x) = e^{-\langle k \rangle(1-x)}.$$

In this case, eqs. (1) and (2) can be written as

$$S = p^2 \left(1 - e^{-R\langle k \rangle} \right)^2 + p \left[1 - e^{-1-\beta R\langle k \rangle} \right] \left(1 - p + pe^{-R\langle k \rangle} \right),$$

$$R = p^2 \left(1 - e^{-R\langle k \rangle} \right)^2 + p(1 - \beta) \left[1 - e^{-1-\beta R\langle k \rangle} \right] \left(1 - p + pe^{-R\langle k \rangle} \right).$$

These equations have a trivial solution at $R = 0$, which means that the network is completely fragmented. The nontrivial solution of $R$ can be presented by the crossing points of the curve $f(R)$ defined by eq. (5) ($f(R) = r.h.s. - R$) and $R$-axis as shown in Fig.2. We find that the system shows two different types of solutions with the increasing of $\beta$, corresponding to the two types of percolation transitions shown in Fig.1. For both cases, the critical points of the system satisfy $\partial f(R)/\partial R = 0$,
For the continuous phase transition, points 3 and discontinuous percolation transitions are met, simultaneously. According to the graphical solution shown in Fig. 3, the percolation transition is continuous with the critical point \( p_c = 0.71926 \) and a nonzero order parameter \( R_c > 0 \). The percolation transition is continuous with the critical point \( p_c = 0.44643 \) and the order parameter \( R_c = 0 \).

which gives
\[
p_c^2 \langle k \rangle e^{-R_c \langle k \rangle} [1 + \beta - 2e^{-R_c \langle k \rangle} + (1 - \beta)e^{-(1-\beta)R_c \langle k \rangle}] + p_c(1 - \beta) e^{-(1-\beta)R_c \langle k \rangle} (1 - p_c + p_c e^{-R_c \langle k \rangle}) = 1. \tag{6}
\]

For the continuous phase transition, \( R_c = 0 \), the first term of the left hand side of eq. (6) vanishes. Thus we can obtain the critical point \( p_c^H \) of the continuous phase transition,
\[
p_c^H = \frac{1}{\langle k \rangle(1 - \beta)^2}, \quad \beta < \beta_c. \tag{7}
\]

When the percolation transition is discontinuous (\( \beta > \beta_c \)), both the two terms of the left hand side of eq. (6) contribute to the critical point \( p_c^H \). In this way, we can not get a closed form of the critical point. However, together with eq. (5), we can easily obtain the numerical solution of \( p_c^H \).

At the critical point \( \beta_c \), the conditions for the continuous and discontinuous percolation transitions are met, simultaneously. According to the graphical solution shown in Fig. 2, \( \beta_c \) satisfies both eq. (7) and \( \partial^2 f(R)/\partial R^2 \big|_{R=0} = 0 \), which yields
\[
(\langle k \rangle(1 - \beta_c)^3 - 2(2 - \beta_c) / \beta_c = 0. \tag{8}
\]

This equation can also be solved numerically. A network with \( \beta < \beta_c \) undergoes a continuous percolation transition, otherwise it would be the discontinuous percolation transition.

In Fig. 3, we show the phase diagram of the system for different average degrees. As the theory predicts, the phenomenon of crossover in the percolation transition can be found, and the simulation results agree with our theory very well. In addition, we can also obtain the critical point in a similar way for SF networks, however, we can not find a simple form as eqs. (7) and (8).

\section{IV. HETEROGENEOUS DEPENDENCY}

\subsection{A. Theory}

In reality, the strength of the dependency may be different for different nodes. In ref. [20], we have also discussed this problem from the perspective of asymmetric dependency. However, the distribution of the dependency strength cannot be addressed in such a model. In this model, we can investigate this problem by allowing different nodes to have different dependency strengths. Assuming that the dependency strength \( \beta \) of nodes in the network takes a distribution \( p_\beta \), we can rewrite eqs. (1) and (2) as
\[
S_\beta = p_\beta^2 [1 - G_0(1 - R)]^2 + p_\beta [1 - G_0(1 - R + R\beta)][1 - p + pG_0(1 - R)], \tag{9}
\]
\[
R = p_\beta^2 [1 - G_1(1 - R)][1 - G_0(1 - R)] + p_\beta [1 - p + pG_0(1 - R)] \times \sum_\beta p_\beta(1 - \beta)[1 - G_1(1 - R + R\beta)]. \tag{10}
\]

Here, \( S_\beta \) is the probability that a randomly chosen node with dependency strength \( \beta \) belongs to the giant component of the final network. Thus the order parameter of the system is given by \( S = \sum_\beta p_\beta S_\beta \).

Using the same method mentioned in the last subsection, we can obtain the critical point of the continuous percolation transition \( p_\beta^H \) analytically,
\[
p_\beta^H = \frac{1}{G_1'(1) \sum_\beta p_\beta(1 - \beta)^2} = \frac{1}{G_1'(1) \langle (1 - \beta)^2 \rangle}. \tag{11}
\]

Here, \( \langle \cdot \rangle \) means the average over the distribution of the dependency strength \( p_\beta \). This equation indicates that the continuous transition point only depends on the first and second moments of the distribution \( p_\beta \) for a given network. For homogeneous dependency, \( p_\beta \) can be expressed by Dirac delta function \( p_\beta = \delta(\beta - \beta_0) \). In this way, eq. (11) reduces to eq. (7) for ER networks.

To compare with the case of homogeneous dependency,
This indicates that an ER network with heterogeneous dependency strength will always be more robust, whether the percolation transition is continuous or discontinuous.

For heterogeneous dependency strength \( \beta \), we can not get a critical point \( \beta_c \) as we did in the homogeneous case. However, in a similar way we can also obtain an equation, which meets the conditions of the discontinuous and continuous percolation, simultaneously. For ER networks, that is

\[
\langle k \rangle \left( (1 - \beta)^3 - 2 (2 - \beta) \beta \right) = 0,
\]

This indicates that the crossing point of the discontinuous and continuous percolation transitions is dependent on the first, second and third moments of the distribution of \( \beta \). In addition, it is easy to find that for homogeneous dependency this equation will reduce to eq.(8).

**B. Example**

Next, we will give a simple example for the network with heterogeneous dependency strength. We consider the case that there exists two dependency strengths \( \beta_1 = \overline{\beta} - \Delta \beta \) (fraction \( q \)) and \( \beta_2 = \overline{\beta} + \Delta \beta \) (fraction \( 1-q \)) in an ER network. As shown in Figs. 4 and 5, with the increasing of the difference \( \Delta \beta \) of \( \beta_1 \) and \( \beta_2 \), the networks become more fragile. This is consistent with our theory, i.e., the network with heterogeneous dependency strength is more robustness than the network with only one type of dependency strength. From Fig. 5, we can also find that with the increasing of \( \overline{\beta} \), the robustness of the networks will decrease, and the percolation transition changes from the continuous one to the discontinuous. For a moderate \( \overline{\beta} \), both the two types of percolation transitions can be found in the system for different \( \Delta \beta \).

From our theory, we can also obtain the fraction of each kind of nodes survived in the final state, \( S_\beta / S \). From Fig. 6, we can find that the nodes with the smaller dependency strength are more easily to survive from the cascading failures. This phenomenon is more obvious near the critical point \( p_c \), and
In our model the failed node only leads to the failures of its dependency one’s links. Specifically, when a node fails each link of its dependency partner will fail with a probability \( \beta \), respectively.

By assigning different nodes with different dependency strengths \( \beta \), we generally compare the systems with homogeneous and heterogeneous dependency strengths. We find that the heterogeneous dependency strength will make the system more robustness than the homogeneous dependency strength, and both the continuous and discontinuous percolation transitions can be found for different dependency strength distributions. This indicates that the type of the percolation transition on networks with dependency is not only determined by the dependency strength but also its distribution. Furthermore, for ER networks we prove that the crossing point of the continuous and discontinuous percolation transitions is dependent on the first, second and third moments of the distribution of dependency strengths. All these findings indicate that the distribution of dependency strengths plays an important role in the robustness of networks, and more research is needed for a deeper understanding of the heterogeneous dependency.

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Appendix A: Proof for eq.(15)

Similar with the discussion shown in Fig. 2, we define

\[
\begin{align*}
    f(R, \beta, p) &= p^2(1 - \xi)^2 + p(1 - \beta)(1 - p + p\xi)[1 - \zeta(\beta)] - R,
\end{align*}
\]

(A1)

where \( \xi = e^{-R(k)} \) and \( \zeta(\beta) = e^{-(1-\beta)R(k)} \). It is clear that both \( \xi \) and \( \zeta(\beta) \) are smaller than 1. Then, for a network with a distribution \( p_\beta \), such an function will be the linear combination of eq.(A1) with different \( \beta \), i.e.,

\[
\begin{align*}
    F(R, \beta, p) &= \sum_\beta p_\beta f(R, \beta, p) \\
    &= p(1 - p + p\xi) \sum_\beta p_\beta(1 - \beta)[1 - \zeta(\beta)] \\
    &\quad + p^2(1 - \xi)^2 - R. 
\end{align*}
\]

(A2)

Similar with that shown in Fig. 2, the tangency points of eqs.(A1) and (A2) with \( R \)-axis correspond to the critical points of the two systems, respectively.

To give a smaller critical point, \( F(R, \beta, p) \) must be larger.

V. CONCLUSIONS

In this paper we have proposed a model to study the percolation process on networks with limited dependency between nodes. As the usual setting of networks with dependency, the failure of one node will affect the function of its dependency node. However, instead of destroying the node completely,
FIG. 7. The parameter $\langle k \rangle R_{c,\beta}$ plots as a function of the dependency strength $\beta$ for different average degrees. The results are obtained by eqs. (5) and (6). The discontinuous percolation corresponds to the area in which $R_{c,\beta} > 0$.

It is easy to know that if

$$g(\beta) = (1 - \beta)\zeta(\beta) = (1 - \beta)e^{-c(1-\beta) R_{\beta}}$$

is a concave function, eq. (A4) will be satisfied for all $\beta$. Next, we will focus on this.

The first and second derivatives of $g(\beta)$ are

$$\frac{dg(\beta)}{d\beta} = \zeta(\beta)\{\langle k \rangle R(1 - \beta) - 1\},$$

$$\frac{d^2g(\beta)}{d\beta^2} = \langle k \rangle \zeta(\beta)\{\langle k \rangle R(1 - \beta) - 2\}.$$  

For a concave function, $d^2g(\beta)/d\beta^2 > 0$ for all $\beta$, that is $\langle k \rangle R > 2$. Actually, eq. (A3) only needs to be met at the critical point $p_{c,\beta}$ given by $f(R,\beta, p)$ [21]. So we simply need to prove $R_{c,\beta} < 2/\langle k \rangle$.

Let $\beta = 1$ in eq. (5), we can find that $R_{c,\beta} \approx 1.26/\langle k \rangle$, which satisfies the condition discussed above. Unfortunately, we cannot get a closed form of $R_{c,\beta}$ for $\beta < 1$. However, from the numerical solution shown in Fig. 7, we can find that $R_{c,\beta}$ is a monotonous increasing function of $\beta$ at the area of the discontinuous percolation. As the theory predicts, $\langle k \rangle R_{c,\beta}$ for different average degrees all converge to 1.26 at $\beta = 1$. Thus, we address $R_{c,\beta} < 2/\langle k \rangle$, i.e., $g(\beta)$ is a concave function at the critical point $p_{c,\beta}$.

Above all, we have proved that $F(R_{c,\beta}, p_{c,\beta}) - f(R_{c,\beta}, p_{c,\beta})$ will be always positive. Since $f(R_{c,\beta}, p_{c,\beta}) = 0$, we obtain $F(R_{c,\beta}, p_{c,\beta}) > 0$. In addition, we know that both $f(R,\beta, p)$ and $F(R,\beta, p)$ are continuous functions. Therefore, we conclude that with the increasing of $p$, before the curve $f(R,\beta, p)$ touches $R$-axis, the curve $F(R,\beta, p)$ has already had some crossing points with $R$-axis. This is just the meaning of eq. (15).

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[21] In factor, any $R$ satisfying $F(R, \beta, p, \pi) > 0$ is enough to demonstrate that $f(R, \beta, p)$ gives a smaller critical point than $f(R, \beta, p)$. The method and the choice $R = R_c, \beta$ used here is just one of the ways to find such a $R$. 