Why $Z_{BH} = |Z_{\text{top}}|^2$

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Abstract

It is argued, using an M-theory lift, that the IIA partition function on a euclidean $AdS_2 \times S^2 \times CY_3$ attractor geometry computes the modified elliptic genus $Z_{BH}$ of the associated black hole in a large charge expansion. The partition function is then evaluated using the Green-Schwarz formalism. After localizing the worldsheet path integral with the addition of an exact term, contributions arise only from the center of $AdS_2$ and the north and south poles of $S^2$. These are the topological and anti-topological string partition functions $Z_{\text{top}}$ and $\bar{Z}_{\text{top}}$ respectively. We thereby directly reproduce the perturbative relation $Z_{BH} = |Z_{\text{top}}|^2$. 

August 2, 2006
1. Introduction

Several years ago it was conjectured [1] that the modified elliptic genus $Z_{BH}$ of a IIA Calabi-Yau black hole and the topological string partition function $Z_{top}$ on the associated Calabi-Yau attractor enjoy a relationship of the form

$$Z_{BH} = |Z_{top}|^2 \quad (1.1)$$

to all orders in perturbation theory. Some evidence for and refinements of this conjecture have been presented in [2-18].

The motivation given in [1] was simply a detailed comparison of the large-charge perturbation expansions of both sides of the relation (1.1). In this comparison, deriving the exponent ‘2’ appearing on the right hand side required detailed knowledge of the conventions involved. No insight was offered as to why the square should appear. Recently the origin of the square was explained [19] by lifting to M-theory. At low temperatures, the black hole partition function can be expressed as a dilute-gas sum over BPS wrapped membranes in $AdS_3 \times S^2 \times CY_3$. There are two types of such: those wrapping holomorphic cycles and localized near the north pole, and those wrapping anti-holomorphic cycles and localized near the south pole. These two types of wrapped branes were shown to give factors
$Z_{\text{top}}$ and $\bar{Z}_{\text{top}}$ respectively. This is not exactly the OSV relation (1.1) which is a statement about high temperatures, where the membranes are not dilute. A modular transformation is then used to turn this low-temperature observation into the high-temperature relation (1.1).

It is of interest to find a direct derivation of (1.1) which does not involve a modular transformation. This requires thinking about the M-theory partition function at high temperatures. At high temperatures the thermal circle is small, and M-theory reduces to IIA (because we want a $(-)^F$ insertion, the thermal circle has the correct periodic boundary conditions). This leads us back to the IIA theory on an $AdS_2 \times S^2 \times CY_3$ attractor. In this paper we will in fact argue, using worldsheet methods, that the perturbative IIA worldsheet partition function on an attractor gives precisely $|Z_{\text{top}}|^2$, with one factor coming from worldsheet instantons at the north pole and the other from worldsheet anti-instantons at the south pole.

This paper is organized as follows. In section 2 we consider the lift of the euclidean IIA attractor geometry to M-theory. The result is a quotient of $AdS_3 \times S^2 \times CY_3$ in which the asymptotic $AdS_3$ boundary is a torus. $AdS_3/CFT_2$ duality then implies (noting the fermion boundary conditions) that this path integral computes the (modified) elliptic genus for the black hole. Two notable features of this discussion is that it involves complex geometries and that, unlike in [20], no “Wilson line” insertions are required. Having argued that the path integral on the IIA attractor computes the elliptic genus, we then proceed, in section 3, to a direct evaluation in string perturbation theory using the Green-Schwarz formalism. An immediate difficulty is a factor of $\infty \times 0$ coming from bosonic and fermionic worldsheet zero modes on $AdS_2 \times S^2$. To compute this factor we localize the path integral to the north and south poles of $S^2$ and the center of $AdS_2$ by adding an exact term to the action. Relying heavily on the worldsheet computation of [21] for the internal $CY_3$ factor, we then find that the path integral gives precisely $|Z_{\text{top}}|^2$, as concluded in section 4. A number of technical points are detailed in the appendices.

2. Euclidean IIA Attractors and the Elliptic Genus

2.1. The D4-D0 Attractor

In this subsection we briefly describe the euclidean Calabi-Yau attractor geometry for the black hole with D4 fluxes $p^4$ and D0 potential $\phi^0$. The 10d string frame metric is

\[
\begin{align*}
\text{d} s_{10}^2 &= \text{d} s_{CY}^2 + \text{d} s_3^2,
\end{align*}
\]

\[\text{(2.1)}\]

\[\text{(1)}\]

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1 D2 charges are suppressed here for brevity and will be restored at the end of this section.
\[ ds_4^2 = 4\ell^2 \left( \frac{dr^2 + r^2 d\theta^2}{(1 - r^2)^2} + \frac{1}{4} d\Omega_2^2 \right), \]  
(2.2)

where the \( AdS_2 \) radius is \[ \ell = g_s \sqrt{\alpha'} \frac{\phi^0}{\pi} \]  
(2.3)

The Calabi-Yau Kähler class is \[ J = \frac{4\pi^3}{\sqrt{\alpha'}} p^A \]  
(2.4)

where \( \omega_A \) is an integral basis for \( H^2(CY_3) \). The RR field strengths are \[ F^{(4)} = \omega_{S^2} \wedge p^A \omega_A \]  
(2.5)

with the normalization \( \int_{S^2} \omega_{S^2} = 1 \) and \[ F^{(2)} = \frac{2i\phi^0}{\pi} \frac{2r d\theta \wedge dr}{(1 - r^2)^2}. \]  
(2.6)

We note that \( F^{(2)} \) is purely imaginary due to the analytic continuation to euclidean space, as is the underlying one form potential \[ A^{(1)} = -\frac{2i\phi^0}{\pi} \frac{d\theta}{1 - r^2}. \]  
(2.7)

### 2.2. IIA \( \rightarrow M \)

In this section we construct the lift of the above euclidean IIA attractor geometry to M theory. For constant dilaton the M-theory and IIA metrics are related by

\[ ds_{11}^2 = g_s^2 \alpha' (dx^{11} + A^{(1)}_\mu dx^\mu)^2 + ds_{10}^2 \]  
(2.8)

where \( x^{11} \sim x^{11} + 2\pi \). In the case at hand

\[ ds_{11}^2 = ds_{CY}^2 + 4\ell^2 \left( \frac{dr^2 + r^2 d\theta^2}{(1 - r^2)^2} + \frac{1}{4} d\Omega^2 \right) + g_s^2 \alpha' \left( dx^{11} - \frac{2i\phi^0}{\pi} \frac{d\theta}{1 - r^2} \right)^2 \]  
(2.9)

The metric is complex because the RR potential \( A^{(1)} \) is complex.

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2 At leading order \( \phi^0 = \pi \sqrt{\frac{D_{ABC} A^B B^C}{g_0}} \), where \( 6D_{ABC} \) are the Calabi-Yau intersection numbers.
Since $F^{(2)}$ is nonzero, the M-theory circle (parameterized by $x^{11}$) is fibered over $AdS_2$. The metric for this 3D fiber bundle is

$$ds_3^2 = 4\ell^2 \left[ \frac{dr^2 + r^2 d\theta^2}{(1 - r^2)^2} + \frac{\pi}{\phi^0} dx^{11} - i \frac{d\theta}{1 - r^2} \right]^2 \quad (2.10)$$

This is a complexified $AdS_3$ quotient. There is also a $G^{(4)}$ flux background coming from the lift of $F^{(4)}$.

Now let us look at the conformal boundary of this 3D metric, which is at $r = 1$. Conformally rescaling the metric by $(r^2 - 1)/\ell^2$, throwing away the normal $dr^2$ term, writing $r = 1 - \epsilon$ and looking near $\epsilon \to 0$, we obtain

$$ds_{\text{bnd}}^2 = d\theta^2 + \frac{2\pi i}{\phi^0} d\theta dx^{11} + O(\epsilon) \quad (2.11)$$

where $\theta$ and $x^{11}$ are identified modulo $2\pi$. The fermion boundary conditions are periodic around the $x^{11}$ circle but antiperiodic around the $\theta$ circle, as it is contractible in (2.10).

### 2.9. Complexified boundary tori

The metric on the torus with module parameter $\tau$ is, up to an overall conformal factor

$$ds_\tau^2 = |d\theta + \tau dx^{11}|^2 = d\theta^2 + 2\text{Re}\tau d\theta dx^{11} + (\text{Re}\tau^2 + \text{Im}\tau^2)(dx^{11})^2 \quad (2.12)$$

For periodic (antiperiodic) $x^{11}$ ($\theta$) boundary conditions, the path integral of a 2D CFT on such a torus yields the partition function

$$Z(\tau) = Tr_{NS}(-1)^F q^L \bar{q}^{\bar{L}} = Tr_{NS}(-1)^F e^{2\pi i \text{Re}\tau (L_0 - \bar{L}_0)} e^{-2\pi \text{Im}\tau (L_0 + \bar{L}_0)} \quad (2.13)$$

More generally, we may wish to compute the above partition function when $\text{Re}\tau$ and $\text{Im}\tau$ are not necessarily real, but are defined by separate analytic continuations to the complex plane. This can be accomplished by analytic continuation to a complex metric of the form (2.12), but in which the parameters $\text{Re}\tau$ and $\text{Im}\tau$ are independent complex numbers.\(^3\)

Let us now look at the boundary metric (2.11) in the light of these comments. Since there is no $(dx^{11})^2$ term at $\epsilon = 0$, this corresponds to $(\text{Re}\tau)^2 + (\text{Im}\tau)^2 = 0$ or $\text{Im}\tau = \pm i \text{Re}\tau$.

\(^3\) Complex extrema of the euclidean path integral appear in a variety of situations, see [22] for a cogent discussion.
where the sign is fixed by requiring that the real part of $Im \tau$ be positive, such that (2.13) be nondivergent. Matching also the coefficient of $d\theta dx^{11}$ one concludes that

$$Im \tau = \frac{\pi}{\phi^0}, \quad Re \tau = i \frac{\pi}{\phi^0}$$  \hspace{1cm} (2.14)$$

Hence the boundary CFT partition function on the torus (2.11) simply computes

$$Z_{IIA}(\phi^0, p^A) = Tr_{NS}(-)^F e^{-\frac{4\pi^2}{\phi^0} L_0}$$  \hspace{1cm} (2.15)$$

Invoking holographic duality of bulk M-theory to the boundary CFT on the torus, we conclude that the M-theory partition function on (2.9) computes the dual CFT partition function. Since there is no $\bar{q}_{L_0}$ dependence in (2.15), right spectral flow acts trivially and we may finally relate the IIA partition function to the elliptic genus

$$Z_{IIA}(\phi^0, \phi^A, p^A) = Tr_{R}(-)^F e^{-\frac{4\pi^2}{\phi^0} L_0 - q_A \frac{\phi^A}{\phi^0}} = Z_{BH}$$  \hspace{1cm} (2.16)$$

where we have reinstated the D2 potentials $\phi^A$ conjugate to the D2 charges $q_A$. Indeed, the holographic duality maps Wilson lines for the Narain lattice of M2-brane charges to the nontrivial $G^{(4)}$ background mentioned in the previous section.

For future reference we record the attractor equations for the black hole moduli and the OSV formula for the topological string coupling $g_{top}$

$$t^A = \frac{\pi p^A + i \phi^A}{i \phi^0}, \quad g_{top} = \frac{4\pi^2}{\phi^0}$$  \hspace{1cm} (2.17)$$

2.4. Summary

We now put together and summarize the results of this section. The string theory partition function on the euclidean D0-D2-D4 attractor geometry lifts to an M-theory partition on a complexified quotient of $AdS_3 \times S^2 \times CY_3$. Holography then relates this to a path integral of the dual CFT on the boundary torus, which in turn is identified with the partition function of the D0-D2-D4 black hole. Putting this all together and taking into account the fermion boundary conditions, we conclude that the partition function of

\[\text{In this and subsequent expression we have factored out the singleton modes which are dual to the black hole center-of-mass. Hence no } F^2 \text{ insertion is required.}\]
IIA string theory on the euclideanization of the D0-D2-D4 attractor geometry computes
the black hole elliptic genus

\[ Z_{IIA}(g_{top} = \frac{4\pi^2}{\phi_0}, p^A, \phi^A) = \text{Tr}_{R}(-)^{F} e^{-g_{top}L_0 - q_A \frac{\phi^A}{\phi_0}} \]  \ (2.18) \]

where the trace on the right hand side is over black hole microstates with center-of-mass
factored out.

While we have not done so, we expect it is possible to derive (2.18) without invoking
the lift to M-theory. This might be done along the lines of e.g. [23] by considering the
variation of the partition function with respect to a deficit angle at the horizon (the center
of \(AdS_2\)) while being careful about fermion boundary conditions and the complex field
strengths. Such a derivation would have the advantage of extending (2.18) to nonzero D6
charges, for which our M-theory discussion is not directly applicable.

3. Worldsheet instantons in attractors

In this section we derive an expression for the perturbative string loop expansion
of the IIA partition function on an euclidean attractor, in particular the contribution
from worldsheet instantons. This calculation could be set up in the RNS, Green-Schwarz
or hybrid formalism. Each of these formalisms has a different set of advantages and
disadvantages. Here we will employ the Green-Schwarz formalism as adapted to the study
of worldsheet instantons in [21] (which in the end becomes quite similar to the hybrid
formalism [24]). A great advantage for us is that the reduction of the Green-Schwarz string
to the topological string was already detailed in [21], and will not need to be repeated here.5
At the same time the AdS spacetime supersymmetries and RR background are easily dealt
with in this formalism.

5 In [21] certain subtleties concerning multiple covers in the Green-Schwarz formalism were left
unresolved. While this is an interesting issue in its own right, it is beyond the scope of the present
paper. Herein we simply assume an appropriate resolution consistent with the equivalence of the
RNS and Green-Schwarz formalisms.
3.1. Worldsheet instantons

Worldsheet instantons wrap a holomorphic cycle \( \Sigma \) in the Calabi-Yau space \( CY_3 \) and sit at a point in \( AdS_2 \times S^2 \). In the Green-Schwarz formalism we employ, the bosonic moduli space for the instanton is \( AdS_2 \times S^2 \times M^Q_g \), where \( M^Q_g \) is the appropriate moduli space of smoothly embedded, genus \( g \) holomorphic curves in the homology class \( Q \) in the Calabi-Yau threefold. There are also fermionic zero-modes. Except for the replacement of \( R^4 \) by \( AdS_2 \times S^2 \), the analysis of the Green-Schwarz string in this context is exactly as in [21]. We refer to Section 5.2 of [21] where it is shown that, for an instanton that wraps an isolated curve, the internal \( CY_3 \) degrees of freedom give rise to the correct power of \( g_{\text{top}} \) and the correct dependence on the moduli \( t^A \) to reproduce the well-known instanton contribution to the IIA free energy \( F_{\text{top}}(t^A) \). Here we take \( t^A \) to be at the attractor point. If the instanton wraps a curve which is neither isolated nor smooth, the simple analysis of [21] must be extended, but we do not address such technical issues here. Instead, we concentrate on the aspects which differ significantly from [21], namely the zero modes of the \( AdS_2 \times S^2 \) bosons and their four goldstino superpartners, corresponding to the four supersymmetries that are broken by the worldsheet instanton. This sector of the theory is the same at every genus, and gives an overall factor. In the flat space computation of [21] the corresponding goldstinos have to be absorbed by the insertion of two graviton vertex operators. We will show in the next section that these insertions are unnecessary in order to obtain a nonzero result in \( AdS_2 \times S^2 \), and the spacetime factor multiplies the free energy by a genus-independent constant.

3.2. Localization of the \( AdS_2 \times S^2 \) sector

In this subsection we compute the integral over the bosonic \( (X^\mu) \) and fermionic \( (\theta^{\alpha A}) \) zero modes which are just constants on \( \Sigma \). These are the goldstone bosons and goldstinoes associated with the breaking of the \( SU(1,1|2) \) supergroup down to the group generated by the 6 elements conventionally denoted \( L_0, \ J^3, \ \pm G^+_1, \ \pm G^-_{-\frac{1}{2}}, \ \pm G^-_{-\frac{3}{2}} \). It can be seen that

\[ 6 \quad \alpha, A \in \{1, 2\}, \text{ where } \alpha \text{ is a negative chirality spinor index and } A \text{ is an R-symmetry index, see appendix B for details.} \]

\[ 7 \quad \text{The right subscript (superscript) indicates the } J^3 (L_0) \text{ eigenvalue, while the left superscript denotes the } R\text{-charge.} \]
supersymmetry does not allow quartic fermionic terms for the zero modes. Hence the action for these zero modes vanishes, and we get a prefactor

\[ Z_4 = \int d^4X d^4\theta = 0 \times \infty. \] (3.1)

A priori this could be infinity, zero, or a finite number. In this section we will use localization to argue that it is finite, and then later fix the finite constant by comparison to supergravity.

The integral (3.1) can be regularized as follows. After gauge-fixing the local kappa symmetry (see the appendix for details) we are left with a finite-dimensional group of residual symmetries associated to the unbroken spacetime supersymmetries. Of these, four are linearly realized and four nonlinearly realized on the worldsheet. The transformation laws are

\[ \delta_i A^\mu X^\mu = 2\theta^\dagger_i \gamma^\mu P_+ \xi^i, \quad \delta_i A^B = -\sigma^B_A P_- \xi^i \] (3.2)

where \( P_\pm \) are the positive and respectively negative chirality projection operators and \( \theta^A \) satisfies \( P_+ \theta^A = 0 \). Here \( \xi^i \) are the four Killing spinors on \( AdS_2 \times S^2 \), \( \xi^i \in \{ \xi^\pm \} = \xi^j_{l_0} \).

For an instanton sitting at the north pole of the sphere, \( \delta_{A+} \) and \( \delta_{A-} \) are the linearly realized supersymmetries, while \( \delta_{A-} \) and \( \delta_{A+} \) represent the broken ones.

Now we replace (3.1) with

\[ Z_4 = \int d^4X d^4\theta \ e^{-tS_{exact}} \] (3.3)

where we choose \( S_{exact} \) to be the \( \delta_- \)-exact action

\[ S_{exact} = \delta_- \delta_{1+}^+ (\epsilon_{\alpha\beta} \theta_1^\alpha \theta_2^\beta) \] (3.4)

and \( t \) is a free parameter that we will want to take to infinity. Using the fact that \( \delta_- S_{exact} = 0 \) and that the expectation value of any \( \delta_- \) - exact operator is zero, it is trivial to show that (3.3) is independent of the parameter \( t \). Therefore (3.3) is the properly defined version of (3.1). The bosonic part of the action (3.4) is then

\[ S_{bos} = (\xi^+_{\pm})^\dagger P_- \xi^+_{\pm} = \frac{|w|^2 + |z|^2}{(1 - |w|^2)(1 + |z|^2)} \] (3.5)

\[ \text{8 The } \sigma^B_A \text{ comes from the fact that } \epsilon^1 \text{ is a linear combination of the } \xi^i, \text{ while } \epsilon^2 \text{ is a linear combination of } \gamma_5 \xi^i \text{ (see appendices A and B2).} \]
So \( e^{-tS_{bos}} \) behaves like a gaussian near the north pole of \( S^2 \) and the origin of \( AdS_2 \) and vanishes at the \( AdS_2 \) boundary \( |w| = 1 \). Therefore

\[
\int d^2w \ d^2z \exp \left[ -t(|w|^2 + |z|^2) \right] \sim \frac{\pi^2}{t^2} \quad \text{as} \quad t \to \infty
\]  

(3.6)

and localizes the action to \( w = z = 0 \). Thus we have succeeded in eliminating the infinity from integration over \( AdS_2 \). Since the bosonic term has localized the action to \( w = z = 0 \) for \( t \to \infty \), we only need to know

\[
S_f|_{w=z=0} = 2\sigma^3_{\alpha\beta} \theta^\alpha_1 \theta^\beta_2
\]  

(3.7)

It remains to evaluate the fermionic integral

\[
\int d^4\theta \exp \left[ -tS_{ferm}|_{w=z=0} \right] = \int d^4\theta \ t^2 \ S^2_{ferm}|_{w=z=0} = 8t^2 \int d^4\theta \ \theta^4 = 8t^2
\]  

(3.8)

We see that the leading factors of \( t \) in the fermionic integration cancel the factor of \( t \) in the bosonic integration and there is a finite nonzero \( t \to \infty \) limit. Since the answer is \( t \)-independent, this limiting value must be exact for all \( t \) and

\[
Z_4 = \text{finite nonzero constant.}
\]  

(3.9)

We have not kept track of numerical factors of 2, \( \pi \) etc. in this discussion.

### 3.3. Anti-instantons

As detailed in the appendix, an anti-instanton at the center of \( AdS_2 \) and the south pole of the sphere preserves the same supersymmetry as an instanton at the center of \( AdS_2 \) and at the north pole of the sphere. The goldstinoes \( \theta^A_\alpha \) now satisfy the opposite chirality condition, but the computation from the previous section goes through basically unchanged. Now we obtain

\[
S_{bos} = (\xi^+_+ \uparrow \ P_+ \xi^+_+) = \frac{1 + |w|^2|z|^2}{(1 - |w|^2)(1 + |z|^2)}
\]  

(3.10)

which localizes the action at \( w = 0, z = -\infty \) - that is, at the south pole. Performing the fermionic integral then gives a \( t \) independent finite result as for the instanton.

While such anti-instantons preserve the same supersymmetry, they couple with the opposite sign to the B-field. The action of an instanton is \( t^A q_A \), and that of an anti-instanton \( \bar{t}^A q_A \), where \( t^A \) is determined by the attractor equations. Hence the contribution to the partition function will involve the modulus \( t^A \) replaced by its complex conjugate \( \bar{t}^A \), resulting in a contribution proportional \( \bar{F}_{top} \) to the free energy.
4. Conclusion and Summary

We have seen that the contribution from holomorphic and antiholomorphic instantons to the IIA free energy in $AdS_2 \times S^2$ turns out to be

$$C \sum d_{g,Q} e^{-t^A q_A} g_{top}^{2g-2} + c.c = C (F_{top} + \bar{F}_{top}) \quad (4.1)$$

where $d_{g,Q}$ are the Gromov-Witten invariants (assuming a more precise extension of the results in [21]). We did not compute the overall constant $C$, which depends on the normalization of the worldsheet functional measure. This overall non-zero constant can be deduced using the fact that IIA string theory is described by a supergravity expansion at low energies. The easiest term to match is the one loop $R^2$ term, which contains no powers of $g_s$ and reduces in four dimensions to the Euler density. The contribution of this term to the partition function was evaluated in [25], using the fact that the Euler character of $AdS_2 \times S^2$ is 2. Agreement with this result implies $C = 1$. Hence, putting together the results of sections 2 and 3 and exponentiating the single worldsheet sum to the multi-worldsheet sum, we have finally that

$$|Z_{top}|^2 = Z^{IIA}_{AdS_2 \times S^2}(p^A, \phi^A, \phi^0) = Z_{BH}(p^A, \phi^A, \phi^0). \quad (4.2)$$

Acknowledgements

We are grateful to J. Lapan, W. Li, J. Marsano, L. Motl, A. Neitzke and J. Seo for helpful discussions. This work is supported in part by the DOE grant DE-FG02-91ER40654.

Appendix A. Killing spinors in Euclidean $AdS_2 \times S^2$

We start with the Killing spinor equation in $11d_L$

$$\nabla_\mu \epsilon + \frac{1}{288} (\Gamma^\nu_{\mu \rho \sigma \kappa} - 8 \delta^\nu_\mu [\nu \Gamma^{\rho \sigma \kappa}]) F_{\nu \rho \sigma \kappa} \epsilon = 0 \quad (A.1)$$

where the antisymmetrization procedure satisfies $F_{[\mu_1 \mu_2 ... \mu_n]} = F_{\mu_1 \mu_2 ... \mu_n}$ for an $n$-form $F$. We reduce this condition on $S^1 \times CY$, where the M-theory $S^1$ is the time circle. The metric and the four-form field strength are given by (2.8) and (2.3) respectively, and after
decomposing the Killing spinor to $4d$ as $\epsilon = \epsilon^1 \otimes \eta_+ + \epsilon^2 \otimes \eta_-$ and a few manipulations, we obtain

$$\nabla_\mu \epsilon^1 + \frac{i}{4} F \gamma_\mu \epsilon^1 = 0$$

(A.2)

where $F = \omega_{S^2}$, and $\epsilon^2$ satisfies the same equation, but with a relative minus sign. Therefore if $\epsilon^1$ is the solution we get to (A.2), the solution for $\epsilon^2$ is $\epsilon^2 = \gamma_{(4)} \epsilon^1$. Let us now work in a convenient coordinate system and find the eight Killing spinors of $AdS_2 \times S^2$.

We parametrize $AdS_2 \times S^2$ by the complex coordinates $z, w$, with metric

$$ds^2 = \frac{4 dw d\bar{w}}{(1 - w\bar{w})^2} + \frac{4 dz d\bar{z}}{(1 + z\bar{z})^2}$$

(A.3)

In these coordinates, we have

$$F = 2 i dz \wedge d\bar{z}$$

(A.4)

We can decompose the $4d_E$ gamma matrices as

$$\gamma^m = \sigma^m \otimes \sigma^3_S, \quad \gamma^i = 1 \otimes \sigma^i_S, \quad \gamma_{(4)} = \sigma^3_A \otimes \sigma^3_S$$

(A.5)

where the $m$ indices stand for $AdS_2$, and $i$ for $S^2$. Then $F = 2i 1 \otimes \sigma^3_S$. Writing the Killing spinor equation in these coordinates we obtain

$$(\nabla_m - \frac{1}{2} \sigma_m \otimes I_2) \epsilon^1 = 0, \quad (\nabla_i - \frac{1}{2} I_2 \otimes \sigma^3_S \sigma^i) \epsilon^1 = 0$$

(A.6)

from which we see that we can write the solutions as $\xi^i = \chi_{AdS_2} \otimes \chi_{S^2}$, where $\chi_{AdS_2} (\chi_{S^2})$ has to be a Killing spinor of $AdS_2 (S^2)$. Since there are two Killing spinors ($\pm$) on $AdS_2$ and two on the sphere, we get a total of four Killing spinors $\xi^i$, whose explicit forms are

$$\begin{align*}
\xi_+^+ &= N \begin{pmatrix} 1 \\
-w \\
-wz \\
-w \end{pmatrix} \\
\xi_-^+ &= N \begin{pmatrix} \bar{w} \\
-\bar{w}z \\
1 \\
z \end{pmatrix} \\
\xi_+^- &= N \begin{pmatrix} \bar{z} \\
1 \\
\bar{w}z \\
-w \end{pmatrix} \\
\xi_-^- &= N \begin{pmatrix} \bar{w}z \\
\bar{z} \end{pmatrix}
\end{align*}$$

(A.7)

where $N = [(1 + \bar{z}z)(1 - w\bar{w})]^{-\frac{1}{2}}$. Then the Killing spinor $\epsilon^1(w, \bar{w}, z, \bar{z})$ can be written as a linear combination

$$\epsilon^1(w, \bar{w}, z, \bar{z}) = c_i \xi^i = c_1 \cdot \xi_+^+ + c_2 \cdot \xi_-^+ + c_3 \cdot \xi_+^- + c_4 \cdot \xi_-^-$$

(A.8)

9 The superscript denotes the $J_3$ eigenvalue, while the subscript denotes the $L_0$ one.
We will see later that the supersymmetries preserved by an instanton wrapping a holomorphic cycle in the Calabi-Yau satisfy $P_\pm \epsilon^1 = P_\pm \epsilon^2 = 0$, while those corresponding to an instanton wrapping an antiholomorphic curve satisfy $P_\pm \epsilon^1 = P_\pm \epsilon^2 = 0$, where $P_\pm$ are the $4d_E$ chirality projectors. It is easy to check that these equations are the same (for any $w$) if we place the instanton at $(z, \bar{z})$, and the anti-instanton at $(-1/\bar{z}, -1/z)$ (diametrically opposed points on the sphere). This means that an instanton sitting at a point $(z, \bar{z})$ and an anti-instanton sitting at the diametrically opposite point of the sphere preserve the same set of supersymmetries. Nevertheless, the set of unbroken supersymmetries changes as we vary the point $z$.

Appendix B. Spinors and $\kappa$-symmetry in euclidean space

B.1. Majorana conditions in euclidean space

In usual type IIA theory, the two supersymmetry parameters are Majorana-Weyl spinors in $10d_L$, each counting with 16 real components. Since in $10d_E$ we cannot impose both these conditions simultaneously, we choose to impose a Majorana condition on $\epsilon$ - initially a Dirac spinor with 64 real components - in the case of euclidean type IIA (since we need two supersymmetries of opposite chirality) and would choose a Weyl condition for euclidean type IIB.

Generally speaking, a Majorana condition $\psi^* = B\psi$ can be imposed on a spinor $\psi$ if there exists an invertible matrix $B$ such that $B^\dagger B = 1$, and $B$ is symmetric. We would like to know what are the consequences in $4d_E$ of imposing a Majorana condition $\Psi^* = B_{(10)}\Psi$ on the $10d_E$ spinor $\Psi$, which we decompose as

$$\Psi = \psi_1 \otimes \eta_+ + \psi_2 \otimes \eta_-$$  \hfill (B.2)

To be very precise, the action of $B$ on the gamma matrices in $d$ euclidean dimensions is

$$\Gamma^* = \eta B_{(d)} \Gamma_m B_{(d)}^{-1}, \quad \Gamma^*_{d+1} = \Gamma_{(d)} = (-1)^\frac{d}{2} B_{(d)} \Gamma_{d+1} B_{(d)}^{-1} \quad B_{(d)} B_{(d)} = \epsilon, \quad \eta^2 = \epsilon^2 = 1 \hfill (B.1)$$

where the coefficients $\epsilon$ and $\eta$ are to be specified in each dimension. We can also write $B^T = \epsilon B$. In Polchinski’s conventions, $\eta = (-1)^\frac{d}{2}$ for what he calls $B_1$, and $\eta = (-1)^\frac{d}{2} + 1$ for $B_2$. We can only impose a Majorana condition on a spinor if $\epsilon = 1$; the choice of $\eta$ seems to have more to do with the ability to give a mass to that spinor. We usually have two choices for $\epsilon, \eta$ in each dimension: in $10d_E$, we can choose $\eta = \epsilon = 1$ or $\eta = \epsilon = -1$. In $6d_E$ we have $\epsilon = -\eta = \pm 1$, while in $4d_E$, $\epsilon = -1$, while $\eta = \pm 1$. We therefore see that it is impossible to impose a Majorana condition on a single $4d$ euclidean spinor.
on $\mathcal{M}_4 \times CY$, where $\eta_{\pm}$ are the covariantly constant spinors (of definite chirality) on the Calabi-Yau, satisfying $(\eta_{\pm})^* = \eta_{\mp}$. Let us decompose the $10d_E$ gamma matrices into 4 and 6-dimensional pieces as

$$\Gamma^i = \gamma^i \otimes I_6, \quad \Gamma^m = \gamma^{(4)} \otimes \gamma^m, \quad \Gamma^{(10)} = \gamma^{(4)} \otimes \gamma^{(6)} \quad (B.3)$$

where $i$ denote 4d directions and $m$ the 6d ones. If we write $B_{(10)}$ as $B_{(10)} = B_{(4)} \otimes \gamma^{(6)}$, then we get $\eta = 1$ in $4d_E$, and the spinors $\psi_1$ and $\psi_2$ get related by a *symplectic Majorana condition*

$$\psi^*_A = -B_{(4)} \epsilon_{AB} \psi^B, \quad A = 1, 2 \quad (B.4)$$

so we can always trade $\psi_1^*$ for $\psi_2$ and viceversa, which we will do in the text. Note also that we have $B_{(4)} \gamma^{(4)} B_{(4)}^{-1} = \gamma^{(4)}_*$, so the subgroups $(2^{k-1})_+$ and $(2^{k-1})_-$ are left invariant by the action of $B_{(4)}$. Therefore we can simultaneously impose a Weyl condition on $\psi_A$, which will be left unchanged by complex conjugation.\footnote{That is, complex conjugation does not change the dotted into the undotted indices that correspond to each of the $SU(2)$’s in the decomposition $SO(4) = SU(2) \times SU(2)$.}

Our choice of $B_{(4)}$ will be such that

$$\psi_{A\dot{\alpha}} = \epsilon_{\alpha\beta} \epsilon_{AB} \psi^{B\dot{\beta}}, \quad \psi_{A\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{AB} \psi^{B\dot{\beta}} \quad (B.5)$$

**B.2. The kappa-symmetry analysis**

The $\kappa$-symmetry variation of the $10d_E$ (super)coordinates $X^M$, $\Theta$ for the case of a string wrapping a holomorphic cycle in the Calabi-Yau, can be written as

$$\delta_\kappa \Theta = (1 + \Gamma)\kappa(\sigma) \equiv 2\mathcal{P}_+ \kappa(\sigma), \quad \delta_\kappa X^M = -\Theta^\dagger \Gamma^M \delta_\kappa \Theta \quad (B.6)$$

where $\Gamma$ is given by

$$\Gamma = i \frac{e^z \bar{e}^\bar{z}}{2} \partial_z X^M \partial_{\bar{z}} X^N \Gamma_{MN} \Gamma_{11} \quad (B.7)$$

On the other hand, under global supersymmetry these fields transform as

$$\delta_\epsilon \Theta = \epsilon, \quad \delta_\epsilon X^M = \Theta^\dagger \Gamma^M \epsilon \quad (B.8)$$

where $\epsilon$ satisfies the Killing spinor equation in the background under consideration, in our case $AdS_2 \times S^2 \times CY$, and can be decomposed as in (B.2). Now we would like to impose the
gauge-fixing condition $P_+ \Theta = 0$, which makes sense since for any supersymmetry variation $\delta_\epsilon \Theta_+ = \epsilon_+$, we can always find a compensating $\kappa$-symmetry transformation $2\kappa_+ = -\epsilon_+$ that would keep us in this gauge. (Note that we can always set $\kappa_- = 0$, since such parameters do not contribute to the change in $\Theta$). This gauge condition is also compatible with a Majorana condition on $\Theta$, that is

\[(P_+ \Theta)^* = B_{(10)} P_+ \Theta\]  \hspace{1cm} (B.9)

which is easily checked using the properties of $B_{(10)}$.

After gauge-fixing, the residual supersymmetries that we are left with in $10d_E$ are

\[\delta_\epsilon \Theta = P_- \epsilon, \quad \delta_\epsilon X^M = 2\Theta^\dagger \Gamma^M P_+ \epsilon\]  \hspace{1cm} (B.10)

Now, we decompose $\Theta$ as

\[\Theta = \theta^1 \otimes \eta_+ + \theta^2 \otimes \eta_+\]  \hspace{1cm} (B.11)

where $\theta^{1,2}$ are the fermionic superpartners of the $AdS_2 \times S^2$ coordinates. The $\kappa$-symmetry condition on them reduces to $P_+ \theta^1 = P_+ \theta^2 = 0$, where $P_\pm = \frac{1}{2} (1 \pm \gamma(4))$.

Decomposing (B.10), we get

\[\delta_\epsilon \theta^A = P_- \epsilon^A, \quad \delta_\epsilon X^\mu = 2\theta^\dagger_A \gamma^\mu P_+ \epsilon^A\]  \hspace{1cm} (B.12)

where $A$ is summed over $1, 2$, and $\epsilon^1 = c_i \xi^i$, $\epsilon^2 = d_i \gamma(4)\xi^i$. Note that $(\xi^{ij}_a)_A^B$ can be rewritten as $B_{(4)} \gamma(4) \xi^{ij}_a$, so our Killing spinors do satisfy the symplectic Majorana condition $\epsilon_1^* = B_{(4)} \epsilon_2$, where we chose $B_{(4)} = i \sigma^2 \otimes \sigma^1$ and the coefficients $d_i$ are a linear combination of the $c_i$. We can use this Majorana condition for $\theta^1$ and $\theta^2$ to rewrite the $X^\mu$ variation under each supersymmetry as

\[\delta_i^A \theta^{B\alpha} = -\sigma_A^B \xi^{i\alpha}, \quad \delta_i^A X^\mu = 2\theta^\dagger_A \gamma^\mu \gamma_{\alpha\dot{\alpha}} \xi^{i\dot{\alpha}} = -2\epsilon_{AB} \theta^{B\alpha} \gamma_{\alpha\dot{\alpha}} \xi^{i\dot{\alpha}}\]  \hspace{1cm} (B.13)

In the case of a string wrapping an antiholomorphic cycle, all that happens is that $\Gamma$ changes to $-\Gamma$ in (B.9). Therefore the gauge-fixing condition is now $P_- \Theta = 0$ and the analysis goes basically the same, just changing all $P_+$ into $P_-$ and viceversa.\footnote{Generally, $\Theta$ decomposes as $\Theta = \theta^1 \otimes \eta_+ + \theta^2 \otimes \eta_- + \theta^a \otimes \gamma_a \eta_+ + \theta^\dot{a} \otimes \gamma_{\dot{a}} \eta_-$. If we separate the $a$ directions into tangent and normal to the curve $\Sigma$ and impose $P_+ \Theta = 0$, then we get the same set of physical fields as in [21].}
Appendix C. The worldsheet fields

Following \[21\] for the instanton, we need only describe the physical fields that live on the worldsheet after fixing the $\kappa$ symmetry. A key point is that the resulting fields are automatically twisted if the normal bundle $N$ of $\Sigma$ in $CY_3$ is nontrivial.

The massless bosons describe fluctuations normal to $\Sigma$ in $AdS_2 \times S^2 \times CY_3$. They are valued in the rank four holomorphic bundle $O^2 \oplus N$, where $O^2$ is the trivial rank two holomorphic bundle on $\Sigma$ which describes fluctuations normal to $\Sigma$ inside $AdS_2 \times S^2$ and $N$ is the holomorphic normal bundle to $\Sigma$ inside the Calabi-Yau threefold.

To describe the physical fermions on $\Sigma$, we must first fix the kappa symmetry, which we achieve by imposing the gauge condition

$$P_+ \Theta = 0$$  \hspace{1cm} (C.1)$$

Following \[21\], we let $S_+$ be a right-moving spin bundle and $S_-$ be a left-moving spin bundle associated to the holomorphic tangent bundle of $\Sigma$. Before we impose the gauge condition in (C.1), the spinor $\Theta$ transforms as the direct sum of a positive and a negative chirality spinor in ten dimensions. In a neighborhood of the curve $\Sigma$, the ten-dimensional symmetry group $Spin(10)$ reduces to a product $Spin(2) \times Spin(8)$, where $Spin(2)$ acts on the two real tangent directions along $\Sigma$ and $Spin(8)$ acts on the eight real normal directions to $\Sigma$ in $AdS_2 \times S^2 \times CY_3$. Under this reduction from $Spin(10)$ to $Spin(2) \times Spin(8)$, the ten-dimensional spinor $\Theta$ decomposes into components transforming as

$$S_+ \otimes S_+(O^2 \oplus N), \quad S_- \otimes S_-(O^2 \oplus N),$$

$$S_- \otimes S_+(O^2 \oplus N), \quad S_+ \otimes S_-(O^2 \oplus N).$$  \hspace{1cm} (C.2)$$

Here we naturally use $S_\pm (O^2 \oplus N)$ to denote the positive and negative chirality spinors of the normal bundle $O^2 \oplus N$ to $\Sigma$ inside $AdS_2 \times S^2 \times CY_3$.

In terms of the reduction from $Spin(10)$ to $Spin(2) \times Spin(8)$, we see that the gauge-fixing condition in (C.1) imposes the constraint that $\Theta$ have definite chirality with respect to the $Spin(8)$ structure group of the normal bundle $O^2 \oplus N$. In our conventions, the gauge condition (C.1) then sets the components of $\Theta$ which live in $S_+ \otimes S_-(O^2 \oplus N)$ and $S_- \otimes S_-(O^2 \oplus N)$ in (C.2) to zero. So after we fix the kappa-symmetry, the physical fermions on $\Sigma$ that arise from $\Theta$ transform as sections of the bundles $S_+ \otimes S_+(O^2 \oplus N)$ and $S_- \otimes S_+(O^2 \oplus N)$. We can further use the manifest reduction from $Spin(8)$ to $Spin(4) \times Spin(4)$ to write these bundles as
Thus far, no twisting is evident in (C.3). As in [21], the twisting only becomes apparent when we use the Calabi-Yau condition to identify the tensor products of the various spin bundles appearing in (C.3) as vector bundles on $\Sigma$. In particular, we have the isomorphisms

\[
S_+ \otimes S_- (O^2 \oplus N) = \left[ S_+ \otimes S_+ (O^2) \otimes S_+ (N) \right] \oplus \left[ S_+ \otimes S_- (O^2) \otimes S_- (N) \right],
\]
\[
S_- \otimes S_+ (O^2 \oplus N) = \left[ S_- \otimes S_+ (O^2) \otimes S_+ (N) \right] \oplus \left[ S_- \otimes S_- (O^2) \otimes S_- (N) \right].
\]

(C.3)

Here $N^*$ denotes the conormal bundle, the dual of $N$.

By convention, the kinetic operator for a right-moving fermion on $\Sigma$ is a $\partial$ operator (coupled to the appropriate bundle), and the kinetic operator for a left-moving fermion on $\Sigma$ is a $\bar{\partial}$ operator. Hence it is very natural to describe the right-moving fermions as transforming as sections of anti-holomorphic bundles and the left-moving fermions as transforming as sections of holomorphic bundles. Although all the bundles appearing in (C.4) are holomorphic, we can easily use the holomorphic three-form and the hermitian metric on the Calabi-Yau to relate these holomorphic bundles to anti-holomorphic ones.

Hence we identify the twisted right-moving fermions on $\Sigma$ as transforming as sections of the bundles

\[
S_+ (O^2) \otimes \bar{\mathcal{N}}, \quad S_- (O^2) \otimes \mathcal{O}, \quad S_- (O^2) \otimes \Omega^1_{\Sigma}
\]

(C.5)

where $\bar{\mathcal{O}}$ is the trivial anti-holomorphic line bundle on $\Sigma$ and $\Omega^1_{\Sigma}$ is the anti-holomorphic bundle of forms of type (0,1) on $\Sigma$. We denote these right-moving fermions by $(\chi^m_\alpha, \theta^\alpha_2, \theta^\alpha_{\bar{z}})$.

Similarly, we see from (C.4) that the twisted left-moving fermions on $\Sigma$ transform as sections of the holomorphic bundles

\[
S_+ (O^2) \otimes \mathcal{N}, \quad S_- (O^2) \otimes \mathcal{O}, \quad S_- (O^2) \otimes \Omega^1_{\Sigma}.
\]

(C.6)

We denote these fermions by $(\chi^m_\alpha, \theta^\alpha_2, \theta^\alpha_{\bar{z}})$. As a small check, we observe that the world-volume theory on a Type IIA string must certainly be a non-chiral theory, and our identifications in (C.3) and (C.6) are consistent with this fact. In [21], it was shown that the self-dual part of the graviphoton field strength couples to the fields $\theta^\alpha_2$ and $\theta^\alpha_{\bar{z}}$. 
For the anti-instanton in physical gauge, one finds from a similar analysis that the twisted worldvolume fermions transform in nearly the same bundles on $\Sigma$ as in (C.3) and (C.6), but with $S_+(\mathcal{O}^2)$ exchanged for $S_-(\mathcal{O}^2)$ and vice versa. That is, on the anti-instanton we now have twisted fermions $(\bar{\chi}_m, \bar{\theta}_{\bar{\alpha}}^2, \bar{\theta}_{\bar{\alpha}}^3)$ and $(\chi_m, \theta_{\alpha}^1, \theta_{\alpha}^3)$. The anti-self-dual part of the graviphoton field strength couples to the fields $\theta_{\alpha}^3$ and $\theta_{\alpha}^3$. 
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