Geometry of Limits of Zeros of Polynomial Sequences of Type (1,1)

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Abstract
We study the root distribution of some univariate polynomials satisfying a recurrence of order two with linear polynomial coefficients. We show that the set of non-isolated limits of zeros of the polynomials is either an arc, or a circle, or an interval, or a “lollipop.” As an application, we discover a sufficient and necessary condition for the universal real-rootedness of the polynomials, subject to certain sign condition on the coefficients of the recurrence. Moreover, we obtain the sharp bound for all the zeros when they are real.

Keywords Limit of zeros · Real-rootedness · Recurrence · Root distribution

Mathematics Subject Classification 03D20 · 26C10 · 30C15

1 Introduction
Root distribution of polynomials in a sequence discovers intensive information about the interrelations of the polynomials in the sequence, especially when the sequence satisfies a recurrence. In the study of the root distribution of sequential polynomi-
als, both the real-rootedness and the limiting distribution of zeros of the polynomials receive much attention. Some evidence for the significance of real-rootedness of polynomials can be found in Stanley [18, Sect. 4]. Bleher and Mallison [6] consider the zeros of Taylor polynomials and the asymptotics of the zeros for linear combinations of exponentials. Some studies on certain “zero attractor” of particular sequences of polynomials can be found in [7,11]. The exploration of zero attractors of Appell polynomials has been regarded as “gems in experimental mathematics” in [8]. Limiting distribution of zeros has been used to study the four-color theorem via the chromatic polynomials initiated by Birkhoff [5], which amounts to the nonexistence of a chromatic polynomial with a zero at the point 4. Beraha and Kahane [2] examined the limits of zeros for the sequence of chromatic polynomials of a special family of cubic graphs, described as to consist of an inner and outer square separated by $n$ 4-rings. It turns out that the number 4 is a limit of zeros of polynomials in this family. Brown and Tufts [10] determined the limiting curves for domination roots of complete bipartite graphs. Motivated by the LCGD conjecture from topological graph theory, Gross, Mansour, Tucker and the first author [12,13] study the root distribution of polynomials satisfying the recurrence

$$W_n(z) = A(z)W_{n-1}(z) + B(z)W_{n-2}(z), \quad (1.1)$$

where the functions $A(z)$ and $B(z)$ are polynomials such that one of them is linear and that the other is constant. They established the real-rootedness subject to some sign conditions for the coefficients of $A(z)$ and $B(z)$, which consequently confirms the LCGD conjecture for many graph families. Orthogonal polynomials and quasi-orthogonal polynomials have close relations with (1.1); see Andrews, Askey and Roy [1] and Brezinski, Driver and Redivo-Zaglia [9]. Jin and Wang [14] characterized the common zeros of polynomials $W_n(z)$ for general $A(z)$ and $B(z)$. Kuijlaars and Van Assche [16] studied asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients, where the coefficients in place of $A(z)$ and $B(z)$ depend on $n$ and are independent of $z$. Kuijlaars and McLaughlin [15] studied asymptotic zero behavior of Laguerre polynomials with negative parameter.

Following [12], a sequence $\mathcal{W} = \{W_n(z)\}_{n=0}^{\infty}$ of polynomials satisfying (1.1) is said to be of type $(\deg A(z), \deg B(z))$. It is normalized if $W_0(z) = 1$ and $W_1(z) = z$. When $A(z) = az + b$ and $B(z) = cz + d$ are linear, (1.1) reduces to

$$W_n(z) = (az + b)W_{n-1}(z) + (cz + d)W_{n-2}(z). \quad (1.2)$$

By considering the root distribution of the polynomials $(-1)^n W_n(-z)$, one may suppose without loss of generality that $c \geq 0$. We use a quadruple

$$(\text{sgn}(a), \text{sgn}(b), \text{sgn}(c), \text{sgn}(d)),$$

each coordinate of which is either $+$ or $-$ or 0, to denote the sign combination of the numbers $a, b, c, d$.

Gross et al. [12,13] establish the real-rootedness for Cases $(+, *, 0, -), (0, +, +, +)$ and $(0, +, +, -)$, where the symbol * indicates that the number $b$ might be of any
sign. In Case \((-, -, +, -)\), Wang and Zhang [20] establish the real-rootedness of all polynomials \(W_n(z)\) for when \(\Delta_g > 0\), where \(\Delta_g = (b + c)^2 + 4d(1 - a)\). In Case \((+, +, +, +)\), they [21] show that every polynomial \(W_n(z)\) is real-rooted if and only if \(ad \leq bc\).

According to Beraha, Kahane, and Weiss’ result [3,4] on limits of zeros of polynomials satisfying (1.1), polynomials satisfying (1.2) have at most two isolated limits of zeros. In this paper, we show that the set of non-isolated limits of zeros of polynomials satisfying (1.2) is either an arc, or a circle, or an interval, or a “lollipop.” Here, a lollipop is the union of a circle and a line segment on the plane such that (i) the intersection of the circle and line segment is a single point, (ii) the line containing the segment goes through the center of the circle, and (iii) the circle radius is longer than the part of the segment that lies inside the circle. As an application, we can show that in Case \((+, -, +, -)\), every polynomial is real-rooted if and only if \(ad \leq bc\). Moreover, when the isolated limits are real, the zeros approach to them in an oscillating manner in Cases \((0, +, +, +)\) and \((+, +, +, +)\), that is, the zeros converge from both the left and right sides of the isolated limits, while the convergence way is from only one side in Case \((+, -, +, -)\); see Theorem 3.8.

We should mention that the generating function of the normalized polynomials satisfying (1.1) is

\[
\sum_{n \geq 0} W_n(z)t^n = \frac{1 + (z - A(z))t}{1 - A(z)t - B(z)t^2}.
\]

In comparison, the root distribution of the polynomials generated by the function

\[
\sum_{n \geq 0} W_n(z)t^n = \frac{1}{1 - A(z)t - B(z)t^2}
\]

has been investigated in [19], in which Tran found an algebraic curve containing the zeros of all polynomials \(W_n(z)\) for large \(n\).

The main result in this paper is Theorem 2.3, which is a geometric description of the root distribution of polynomials satisfying (1.2). In Sect. 3, we provide a sufficient and necessary condition of real-rootedness in Case \((+, -, +, -)\), and the root distribution when they are real-rooted as an application of Theorem 2.3.

## 2 Geometry of the Limits of Zeros

Throughout this paper, we let \(a, b, c, d \in \mathbb{R}\) such that \(ac \neq 0\), and let

\[\mathcal{W} = \{W_n(z)\}_{n=0}^{\infty}\]

be a sequence of polynomials satisfying normalized (1.2). Then, the leading term of the polynomial \(W_n(z)\) is \(a^{n-1}z^n\).

For any complex number \(z = re^{i\theta}\) with \(\theta \in (-\pi, \pi]\), we use the square root notation \(\sqrt{z}\) to denote the number \(\sqrt{r}e^{i\theta/2}\), which lies in the right half-plane. The
general formula in Lemma 2.1 is the base of our study, which can be found from [12,13].

**Lemma 2.1** Let \( A, B \in \mathbb{C} \). Suppose that \( W_0 = 1 \) and

\[
W_n = AW_{n-1} + BW_{n-2}
\]

for \( n \geq 2 \). Then,

\[
W_n = \begin{cases} 
\alpha_+ \lambda_+^n + \alpha_- \lambda_-^n, & \text{if } \Delta \neq 0 \\
\frac{A + nh}{2} \cdot \left( \frac{A}{2} \right)^{n-1}, & \text{if } \Delta = 0
\end{cases}
\]

for \( n \geq 0 \), where \( h = 2W_1 - A \), and

\[
\lambda_\pm = \frac{A \pm \sqrt{\Delta}}{2}, \quad \alpha_\pm = \frac{\sqrt{\Delta} \pm h}{2\sqrt{\Delta}}, \quad \text{with } \Delta = A^2 + 4B.
\]

Inspired by Lemma 2.1, we employ the following notation of functions:

\[
\Delta(z) = A(z)^2 + 4B(z) = a^2 z^2 + (2ab + 4c)z + (b^2 + 4d),
\]

\[
h(z) = 2W_1(z) - A(z) = (2 - a)z - b,
\]

\[
\lambda_\pm(z) = \frac{A(z) \pm \sqrt{\Delta(z)}}{2},
\]

\[
\alpha_\pm(z) = \frac{\sqrt{\Delta(z)} \pm h(z)}{2\sqrt{\Delta(z)}},
\]

\[
g(z) = \frac{h^2(z) - \Delta(z)}{4} = (1 - a)z^2 - (b + c)z - d.
\]

The zeros of \( A(z), B(z) \), and \( \Delta(z) \) are respectively

\[
x_A = -\frac{b}{a}, \quad x_B = -\frac{d}{c}, \quad \text{and } x_\pm = x_A + \frac{-2c \pm 2\sqrt{\Delta_\Delta}}{a^2},
\]

where

\[
\Delta_\Delta = c^2 - a^2 B(x_A)
\]

is the discriminant of \( \Delta(z) \).

Consider the set of zeros of all polynomials in the sequence \( \mathcal{W} \) as a subset of \( \mathbb{R}^2 \) with the standard topology. A number \( z^* \in \mathbb{C} \) is a **limit of zeros** of the polynomials in \( \mathcal{W} \) if there is a zero \( z_n \) of the polynomial \( W_n(z) \) for each \( n \) such that

\[
\lim_{n \to \infty} z_n = z^*.
\]
We denote the set of non-isolated limits of zeros of the polynomials $W_n(z)$ by ♣ and denote the set of isolated limits of zeros by ♠. The club symbol ♣ is adopted for the leaves of a clover as it depicts which are not alone, while the spade symbol ♠ which appears as a single leaf stands for isolation. Beraha et al. [3] considered the limits of zeros of a sequence of polynomials satisfying a recurrence. Their result, when the recurrence is (1.2), can be restated as follows.

**Lemma 2.2 (Beraha et al.)** Let $W$ be the normalized polynomial sequence satisfying (1.2). Suppose that it satisfies the non-degeneracy conditions (N-i) the sequence $W$ does not satisfy a recurrence of order less than two, and (N-ii) there is no $\omega \in \mathbb{C}$ with $|\omega| = 1$ such that $\lambda_+(z) \equiv \omega \lambda_-(z)$.

Then, $♣ = \{z \in \mathbb{C} : |\lambda_+(z)| = |\lambda_-(z)|\}$ and $♠ = ♠_- \cup ♠_+$, where

$$♠_- = \{z \in \mathbb{C} : \Delta(z) \neq 0, \alpha_-(z) = 0, |\lambda_-(z)| > |\lambda_+(z)|\}, \quad \text{and}$$

$$♠_+ = \{z \in \mathbb{R} : \Delta(z) \neq 0, \alpha_+(z) = 0, |\lambda_+(z)| > |\lambda_-(z)|\}.$$

For any complex number $z$, denote by $\overline{z}$ the complex conjugate of $z$, and by $\Re(z)$ the real part of $z$.

**Theorem 2.3** Let $a, b, c, d \in \mathbb{R}$ and $ac \neq 0$. Let $W = \{W_n(z)\}_{n=0}^{\infty}$ be the polynomial sequence satisfying (1.2):

$$W_n(z) = (az + b)W_{n-1}(z) + (cz + d)W_{n-2}(z),$$

with $W_0(z) = 1$ and $W_1(z) = z$. Suppose that $W$ is not the sequence $\{z^n\}_{n=0}^{\infty}$. Then, the sets of isolated and non-isolated limits of zeros of $W$ are, respectively,

$$♠ = \{z \in \mathbb{C} : g(z) = 0, \Re\left(A(z)\overline{h(z)}\right) < 0\} \quad \text{and}$$

$$♣ = \begin{cases} x_\Delta x_A x_\Delta^+, & \text{if } \Delta_\Delta < 0 \\ C_0, & \text{if } \Delta_\Delta = 0 \\ J_\Delta, & \text{if } \Delta_\Delta > 0 \text{ and } B(x_A) \leq 0 \\ J_\Delta \cup C_0, & \text{if } \Delta_\Delta > 0 \text{ and } B(x_A) > 0, \end{cases}$$

where $x_\Delta x_A x_\Delta^+$ stands for the circular arc connecting the points $x_\Delta^-$ and $x_\Delta^+$ through the point $x_A$, the set

$$C_0 = \{z \in \mathbb{C} : |z - x_B| = |x_A - x_B|\}$$

is the circle with center $x_B$ and radius $|x_A - x_B|$, and the set

$$J_\Delta = \{x \in \mathbb{R} : \Delta(x) \leq 0\} = \{x \in \mathbb{R} : x_\Delta^- \leq x \leq x_\Delta^+\}$$

is an interval.
Proof First of all, let us verify the non-degeneracy conditions. If \((N-i)\) is not satisfied, then the sequence \(W\) satisfies a recurrence of order one. It follows that \(W_n(z) = z^n\) for each \(n\), contradicting the premise. If \((N-ii)\) is not satisfied, then \(|\lambda - (z)| = |\lambda + (z)|\) for all \(z \in C\). It follows that the cubic polynomial \(A(z) \Delta(z)\) vanishes identically, which is impossible. This verifies the non-degenerate conditions.

Second, we shall show that \(\clubsuit = S\), where

\[
S = \{z \in \mathbb{C} : g(z) = 0, \Re \left( A(z) \overline{h(z)} \right) < 0 \}.
\]

Let \(z \in \spadesuit_\ldots\). By the definitions of \(\alpha - (z)\), we find \(h(z) = \sqrt{-\Delta(z)} \neq 0\). Thus,

\[
g(z) = \frac{h^2(z) - \Delta(z)}{4} = 0.
\]

From the definitions of the functions \(\lambda_\pm (z)\), we have

\[
|\lambda_\ldots (z)| > |\lambda_+ (z)| \iff |2| A(z) - \sqrt{\Delta(z)} > |2| A(z) + \sqrt{\Delta(z)}
\]

\[
\iff |A(z) - h(z)| > |A(z) + h(z)|
\]

\[
\iff \Re \left( A(z) \overline{h(z)} \right) < 0. \tag{2.1}
\]

Now let \(z \in \spadesuit_+\). Then, \(\alpha_+ (z) = 0\) implies \(h(z) = -\sqrt{\Delta(z)}\), and thus, \(g(z) = 0\). In this case, we can deduce the equivalences

\[
|\lambda_+ (z)| > |\lambda_\ldots (z)| \iff |2| A(z) + \sqrt{\Delta(z)} > |2| A(z) - \sqrt{\Delta(z)}
\]

\[
\iff |A(z) - h(z)| > |A(z) + h(z)|
\]

\[
\iff \Re \left( A(z) \overline{h(z)} \right) < 0. \tag{2.2}
\]

Therefore, we obtain \(\spadesuit = \spadesuit_\ldots \cup \spadesuit_+ \subseteq S\).

Conversely, assume that \(z \in S\). Then,

\[0 = g(z) = \frac{h^2(z) - \Delta(z)}{4}.
\]

Thus, \(\sqrt{\Delta(z)} \in \{ \pm h(z) \}\). If \(\Delta(z) = 0\), then \(h(z) = 0\), contradicting the premise

\[
\Re \left( A(z) \overline{h(z)} \right) < 0.
\]

Therefore \(\Delta(z) \neq 0\). Now, if \(\sqrt{\Delta(z)} = h(z)\), then \(\alpha_\ldots (z) = 0\), and \(|\lambda_\ldots (z)| > |\lambda_+ (z)|\) by (2.1). Thus, \(z \in \spadesuit_\ldots\). For the other case \(\sqrt{\Delta(z)} = -h(z)\), we can deduce that \(z \in \spadesuit_+\) by (2.2). In summary, we obtain \(S \subseteq \spadesuit\).

Thirdly, let us show the formula for the set

\[
\spadesuit = \{z \in \mathbb{C} : |\lambda_\ldots (z)| = |\lambda_+ (z)|\}.
\]
Let \( z = x + iy \in \mathbb{C} \), where \( x, y \in \mathbb{R} \).

Some sporadic cases can be handled in advance. On the one hand, it is direct to verify that \( z \in \mathbb{C} \) if \( A(z)\Delta(z) = 0 \). In other words,

\[
\{x_A, x_A^-, x_A^+\} \subseteq \mathbb{C}.
\]

On the other hand, if \( y = 0 \), then \( z, A(z), \Delta(z) \in \mathbb{R} \). In this case, we can infer the equivalence that

\[
|\lambda - (z)| = |\lambda + (z)| \quad \iff \quad \Delta(z) < 0 \quad \iff \quad \Delta > 0 \text{ and } x \in (x_A^-, x_A^+).
\]

Below we can suppose that \( A(z)\Delta(z)y \neq 0 \). We will show that \( \mathbb{C} \) is the intersection of a circle and a vertical strip. We start from the following equivalences:

\[
|\lambda - (z)| = |\lambda + (z)| \quad \iff \quad 2|A(z) + \sqrt{\Delta(z)}| = 2|A(z) - \sqrt{\Delta(z)}|
\]

\[
\iff \quad \text{the vectors } A(z) \text{ and } \sqrt{\Delta(z)} \text{ are orthogonal}
\]

\[
\iff \quad \text{the vectors } A^2(z) \text{ and } \Delta(z) \text{ have opposite directions}
\]

\[
\iff \quad \text{the vectors } A^2(z) \text{ and } B(z) \text{ have opposite directions, and } |1| A^2(z) < |4B(z)|
\]

\[
\iff \quad \left\{\begin{array}{l}
\Re A^2(z) \cdot \Im B(z) = \Re B(z) \cdot \Im A^2(z) \\
|\Im A^2(z)| < 4 |\Im B(z)|
\end{array}\right.
\]

\[
\iff \quad \left\{\begin{array}{l}
(x - x_B)^2 + y^2 = (x_A - x_B)^2 \\
(x - x_A) \left(x - x_A + \frac{2c}{a^2}\right) < 0
\end{array}\right.
\]

\[
\implies z \in C_0 \cap S_0 \{x_A, x_A^-, x_A^+\},
\]

where

\[
S_0 = \left\{z \in \mathbb{C} : |\Re z - x_A| \leq \left|\frac{2c}{a^2}\right|, c \cdot (\Re z - x_A) \leq 0\right\}
\]

is the vertical strip with boundaries \( \Re z = x_A \) and \( \Re z = x_A - 2c/a^2 \). It is clear that the boundary \( \Re z = x_A \) intersects the circle \( C_0 \) at the point \( x_A \). To figure out the intersection of the other boundary with \( C_0 \), we proceed according to the sign of \( \Delta \).

Suppose that \( \Delta < 0 \). Then, \( J_\Delta = \emptyset \) from the definition, and

\[
\Re \left(x_\Delta^\pm\right) = x_A - \frac{2c}{a^2} \quad \text{and} \quad \Im \left(x_\Delta^\pm\right) = \pm \frac{2\sqrt{-\Delta}}{a^2}.
\]
It follows that

\[ |1| x_\Delta^\pm - x_B^2 = \left( x_A - \frac{2c}{a^2} - x_B \right)^2 + \left( \frac{2\sqrt{-\Delta}}{a^2} \right)^2 = (x_A - x_B)^2. \]

Thus, the points \( x_\Delta^\pm \) lie on the intersection of the boundary

\[ \Re z = x_A - \frac{2c}{a^2} \]

and the circle \( C_0 \). Since the intersection contains at most two points, the points \( x_\Delta^\pm \) constitute the intersection. Hence, the set \( \clubsuit = C_0 \cap S_0 \) is the circular arc \( x_\Delta x_A x_\Delta^+ \).

When \( \Delta = 0 \), we can compute that

\[ x_\Delta^- = x_\Delta^+ = x_A - \frac{2c}{a^2}. \]

As a consequence, \( C_0 \cap S_0 = C_0 \) and \( \clubsuit = J_\Delta \cup C_0 = C_0 \).

Below we can suppose that \( \Delta > 0 \). Note that

\[ B(x_A) = c(x_A - x_B). \quad (2.3) \]

We shall show that

\[ C_0 \cap S_0 = \begin{cases} \{x_A\}, & \text{if } B(x_A) \leq 0; \\ C_0, & \text{if } B(x_A) > 0. \end{cases} \]

Assume that \( B(x_A) \leq 0 \). Let \( z \in C_0 \cap S_0 \). If \( c > 0 \), then \( x_A \leq x_B \) by Eq. (2.3). Since \( z \in C_0 \), we have \( \Re z \geq x_A \). Since \( z \in S_0 \), we have \( c(\Re z - x_A) \leq 0 \). Therefore, we infer that \( \Re z = x_A \) and \( z = x_A \). For the other case \( c < 0 \), we have \( x_A \geq x_B \) by Eq. (2.3). In this case, \( z \in C_0 \) implies \( \Re z \leq x_A \), and \( z \in S_0 \) implies \( \Re z \geq x_A \). Hence, \( z = x_A \). Since \( \Delta(x_A) = 4B(x_A) \leq 0 \), we have \( x_A \in J_\Delta \). Hence, \( \clubsuit = J_\Delta \).

Now assume that \( B(x_A) > 0 \). Let \( z \in C_0 \). One may show \( c(\Re z - x_A) \leq 0 \) in the same fashion as when \( B(x_A) < 0 \). By geometric interpretation and the condition \( \Delta > 0 \), we deduce that

\[ |\Re z - x_A| \leq (\text{the diameter of } C_0) = 2|x_A - x_B| < |3| \frac{2c}{a^2}. \]

This proves \( \clubsuit = J_\Delta \cup C_0 \).

We remark that the interval case condition “\( \Delta > 0 \) and \( B(x_A) \leq 0 \)” can be reduced to “\( B(x_A) \leq 0 \)” since \( \Delta \leq 0 \) implies \( B(x_A) > 0 \).

The set \( \spadesuit \) of isolated limits of zeros consists of at most two points, since each isolated limit is a zero of the polynomial \( g(z) \), which is of degree at most two. From geometric point of view, we see that the set \( \spadesuit \) is either an arc, or an interval, or a circle.
or the union of an interval and a circle. Intuitively, the circle-interval union resembles a candy-stick combination such that the candy is mounted onto the stick in a certain symmetric way; see the thick part in Fig. 4 for an illustration. Below we present an example for each of these four cases; see Examples 2.4–2.7.

**Example 2.4** Let $\mathcal{W}$ be the normalized polynomial sequence satisfying

$$W_n(z) = (3z - 16)W_{n-1}(z) + (5z + 20)W_{n-2}(z).$$

Then, the polynomials in $\mathcal{W}$ have two isolated limits of zeros, approximately $2.75 \pm 1.56i$. The set of non-isolated limits of zeros of polynomials in $\mathcal{W}$ is the arc $x_\Delta \mp x_\Delta \approx 4.22 \pm 4.42i$, where

$$x_\Delta = \frac{16}{3}$$

See Fig. 1.

**Example 2.5** Let $\mathcal{W}$ be the normalized polynomial sequence satisfying

$$W_n(z) = (z - 1)W_{n-1}(z) + (-z + 2)W_{n-2}(z).$$

Then, $\mathcal{W}$ has no isolated limits of zeros, and the set of non-isolated limits of zeros is the circle $|z - 2| = 1$; see Fig. 2.

**Example 2.6** Let $\mathcal{W}$ be the normalized polynomial sequence satisfying

$$W_n(z) = (2z - 1)W_{n-1}(z) + (z - 1)W_{n-2}(z).$$

Then, the number $-1$ is the unique isolated limit of zeros of polynomials in $\mathcal{W}$, and the set of non-isolated limits of zeros is the interval $[x^-_\Delta, x^+_\Delta] \approx [-0.87, 0.87]$; see Fig. 3.
**Example 2.7** Let $\mathcal{W}$ be the normalized polynomial sequence satisfying

$$W_n(z) = (z - 2)W_{n-1}(z) + (2z - 1)W_{n-2}(z).$$

Then, $\mathcal{W}$ has no isolated limit of zeros, and the set of non-isolated limits of zeros depicts the “lollipop”

$$\{x \in \mathbb{R}: -4 \leq x \leq 0\} \cup \{z \in \mathbb{C}: |z - 1/2| = 3/2\}.$$

See Fig. 4.
In Example 2.7, we used the word “lollipop” for the intuitiveness purpose only. The definition of a lollipop will be given after Theorem 2.9. We remark that the part outside is possibly shorter than the radius. For example, when the quadruple \((a, b, c, d)\) is \((1, -2, 1.1, -1)\), the part on \(J_\Delta\) that is outside \(C_0\) is approximately 0.22, while the radius is approximately 1.09.

**Corollary 2.8** Let \(a, b, c, d \in \mathbb{R}\) and \(ac \neq 0\). Let \(W\) be the normalized polynomial sequence satisfying (1.2). If every polynomial \(W_n(z)\) for large \(n\) is real-rooted, then \(B(x_A) \leq 0\), and \(\Delta \geq 0\) as a consequence.

**Proof** Since every polynomial \(W_n(z)\) for large \(n\) is real-rooted, we have \(\heartsuit \cup \diamondsuit \subset \mathbb{R}\). By Theorem 2.3, we find either \(\heartsuit = J_\Delta\), or \(\diamondsuit = C_0\) and \(C_0\) degenerates to a single point. In the former case, we find \(B(x_A) \leq 0\). In the latter case, we have \(\Delta = 0\) and \(x_A = x_B\), which is impossible since otherwise

\[ 0 = \Delta = c^2 - a^2 B(x_A) = c^2, \]

a contradiction. This completes the proof. \(\square\)

Theorem 2.9 gives more geometric information for the “lollipops.”

**Theorem 2.9** Suppose \(\Delta > 0\) and \(B(x_A) > 0\). Then,

\[ J_\Delta \cap C_0 = \{2x_B - x_A\}, \]

and the part of \(J_\Delta\) outside the circle \(C_0\) is longer than the part of \(J_\Delta\) inside \(C_0\).

**Proof** By Theorem 2.3, we have \(\heartsuit = J_\Delta \cup C_0\). Denote by \(x_0 = 2x_B - x_A\) the real point on \(C_0\) that is not \(x_A\). First of all, we have \(x_0 \in J_\Delta\) since

\[ \Delta(x_0) = -\frac{4B(x_A)\Delta}{c^2} < 0. \]

Secondly, the center of the circle \(C_0\) is not on the interval \(J_\Delta\) since

\[ \Delta(x_B) = A^2(x_B) > 0. \]

It follows that \(J_\Delta \cap C_0 = \{x_0\}\). Thirdly, note that

\[ x_0 - \frac{x^-_\Delta + x^+_\Delta}{2} = \frac{1}{c} \cdot \frac{2\Delta}{a^2}. \quad (2.4) \]

If \(c > 0\), then \(x_B < x_A\) by Eq. (2.3). It follows that \(x_0 < x_B\). Thus, the interval \(J_\Delta\) intersects the circle \(C_0\) from the left of \(C_0\). By Eq. 2.4, we have

\[ x_0 > \frac{x^-_\Delta + x^+_\Delta}{2}. \]

Thus, the part of \(J_\Delta\) outside the circle \(C_0\) is longer than the part of \(J_\Delta\) inside. The other case \(c < 0\) can be handled in the same way. \(\square\)
For the sake of a geometric description of the circle-interval union, we define a lollipop to be the union of a circle and a line segment on the plane such that

(i) the intersection of the circle and line segment is a single point,
(ii) the line containing the segment goes through the center of the circle, and
(iii) the circle radius is longer than the part of the segment that lies inside the circle.

Under this definition, by Theorems 2.3 and 2.9, we see that when both $\Delta_\Delta$ and $B(x_A)$ are positive, the set of non-isolated limits of zeros of the polynomials in $W$ is a lollipop.

### 3 The Interlacing Zeros for Case $(+, -, +, -)$

Here is the main result of this section.

**Theorem 3.1** Let $a, c > 0$ and $b, d < 0$. Let $W$ be the normalized sequence of polynomials satisfying (1.2). Then, all polynomials $W_n(z)$ for $n \geq 2$ are real-rooted if and only if $x_A \leq x_B$.

The necessity part of Theorem 3.1 can be seen from Corollary 2.8 directly. The sufficiency part can be proved by using a celebrated result due to Wang and Yeh [22, Theorem 1], though the notion of interlacing in their paper is slightly different from that of ours which lies before Lemma 3.7; see also [17, Theorem 1.1]. For completeness, we present a proof for the sufficiency by using [13, Lemma 3.3]; see Lemma 3.7. We will handle the case $x_A < x_B$ in Theorem 3.8, and the case $x_A = x_B$ in Theorem 3.10.

Throughout this section, we suppose that $x_A \leq x_B$, which implies that $\Delta_\Delta > 0$ and $x_A^\pm \in \mathbb{R}$. The zeros of the function $g(z)$ are

$$x_g^\pm = \begin{cases} \frac{b+c}{2(1-a)} \pm \frac{\sqrt{\Delta_g}}{2(1-a)}, & \text{if } a \neq 1, \\ \frac{-d}{b+c}, & \text{if } a = 1 \text{ and } b + c \neq 0, \end{cases}$$

where $\Delta_g = (b + c)^2 + 4d(1 - a)$. We define two numbers $u$ and $v$ by

$$(u, v) = \begin{cases} (x^-_g, x^+_g), & \text{if } a < 2 \text{ and } F \leq 0; \\ (x^-_g, x^-_g), & \text{if } a > 2 \text{ and } F < 0; \\ (x^+_g, x^+_g), & \text{if } a < 1 \text{ and } F > 0; \\ (x^-_g, x^+_g), & \text{otherwise}; \end{cases} \quad (3.1)$$

where

$$F = \Delta_g - \Delta_\Delta = d(a - 2)^2 + bc(2 - a) + b^2.$$ 

Note that $(u, v) = (x^-_A, x^+_A)$ if $a = 1$ and $b + c = 0$. Furthermore, we have $u, v \in \mathbb{R}$ since $\Delta_g > \Delta_\Delta > 0$ whenever $a \geq 2$ or $F > 0$. As will be seen in Theorems 3.8 and 3.10, we have $u < v$ and the interval $(u, v)$ is the best bound for the zeros of $W_n(z)$. 

\[ \text{Springer} \]
3.1 Case $x_A < x_B$

We determine the signs of $W_n(u)$ and $W_n(v)$ in Lemma 3.2.

**Lemma 3.2** Let $a, c > 0$ and $b, d < 0$. Let $\mathcal{W}$ be the normalized polynomial sequence satisfying (1.2). Suppose that $x_A < x_B$. Then, we have

\[
\begin{align*}
    u &\leq x_A^- < x_A < x_A^+ \leq v < x_B, \\
    u &< 0 < v, \\
    W_n(u)(-1)^n > 0, \\
    W_n(v) > 0, \quad \text{and} \\
    \{u, v\} &\subseteq \clubsuit \cup \spadesuit.
\end{align*}
\]

**Proof** The premise $x_A < x_B$ implies $\Delta(x_A) = 4B(x_A) < 0$. It follows that

\[
\begin{align*}
    x_A &\in (x_A^-, x_A^+), \quad x_A^+ > 0, \quad A(x_A^+) > 0 > A(x_A^-), \quad \text{and} \\
    h(x_A^+) = (2 - a)x_A^+ - b \geq -b > 0 \quad \text{if } a \leq 2.
\end{align*}
\]

Since $\Delta(x_B) = A^2(x_B) > 0$ and $x_A^- < x_A < x_B$, we have $x_A^+ < x_B$.

To confirm Relation (3.6), by Theorem 2.3, it suffices to show that

\[
A(x)h(x) < 0, \quad \text{for any } x \in \{u, v\} \setminus \{x_A^-, x_A^+\}.
\]

Let $x_h$ be the unique zero of the function $h(z)$ when $a \neq 2$. Then,

\[
x_h = \frac{b}{2 - a}.
\]

We proceed according to the definition of the numbers $u$ and $v$.

**Case 3.3** $a < 2$, $F \leq 0$ and $[u, v] = J_\Delta$. It is routine to compute that

\[
h(x_A^-)h(x_A^+) = \frac{4F}{a^2}.
\]

Together with InEq. (3.7), we have $h(x_A^-) \leq 0$ and thus

\[
x_A^- \leq x_h = \frac{b}{2 - a} < 0,
\]

verifying InEq. (3.3). By Lemma 2.1, we have

\[
W_n(x_A^\pm) = \frac{A(x_A^\pm) + nh(x_A^\pm)}{2} \cdot \left(\frac{A(x_A^\pm)}{2}\right)^{n-1},
\]

which implies InEqs. (3.4) and (3.5).
**Case 3.4** \( a > 2, \ F < 0 \) and \([u, v] = [x_g^-, x_g^+]\). Observe that

\[
g(x_\Delta^\pm) = \frac{h^2(x_\Delta^\pm)}{4} \geq 0.
\]  

(3.11)

Since the polynomial \( g(z) \) is quadratic with leading coefficient negative, we can derive all inequalities in (3.2) except \( u < x_B \). Since \( F < 0 \), we have \( d(a - 2) - bc < 0 \) and thus

\[
g(x_B) = -\frac{d}{c^2} ((a - 1)d - bc) < -\frac{d}{c^2} ((a - 2)d - bc) < 0.
\]

Since \( x_g^- < x_A < x_B \), we infer that \( x_g^+ < x_B \).

On the other hand, by Vièta’s theorem, we have

\[
x_g^- x_g^+ = \frac{d}{a - 1},
\]  

(3.12)

whose negativity verifies InEq. (3.3). By Lemma 2.1, we have

\[
W_n\left(x_g^\pm\right) = \left(x_g^\pm\right)^n,
\]  

(3.13)

which implies InEqs. (3.4) and (3.5). It is routine to compute that

\[
h(x_g^-)h(x_g^+) = \frac{F}{a - 1} \quad \text{if} \ a \neq 1.
\]  

(3.14)

Thus, \( h(v) < 0 < h(u) \). By (3.2), we have \( A(u) < 0 < A(v) \). This proves InEq. (3.8).

**Case 3.5** \( a < 1, \ F > 0 \) and \([u, v] = [x_g^+, x_A^+]\). In view of Eqs. (3.10) and (3.7), to confirm InEqs. (3.2)–(3.5) and (3.8), we shall show that

\[x_g^+ \leq x_\Delta^-, \quad x_g^+ < 0, \quad \text{and} \quad h(x_g^+) > 0.\]

In fact, we note that the polynomial \( g(z) \) is quadratic with leading coefficient positive. On the one hand, Eq. (3.14) gives \( x_h \in (x_g^-, x_g^+) \). This confirms \( h(x_g^+) > 0 \) immediately. By Eq. (3.9), we can deduce that \( x_h < x_\Delta^- \), since otherwise one would have the absurd inequality

\[0 < x_\Delta^- < x_h = \frac{b}{2 - a} < 0.\]

Thus, InEq. (3.11) implies

\[
\left(x_g^-, x_g^+\right) \cap J_\Delta = \emptyset.
\]
Moreover, the whole interval \( (x^g_-, x^g_+) \) lies to the left of \( J_\Delta \). This proves \( x^g_+ \leq x^g_- \).

On the other hand, by InEq. (3.12) we have \( x^g_- x^g_+ > 0 \). Since \( x^g_- < x_h < 0 \), we find \( x^g_+ < 0 \).

**Case 3.6** For all remaining cases, we have \([u, v] = [x^g_-, x^g_+]\). This time, to confirm InEqs. (3.2)–(3.5) and (3.8), we shall show that

\[
x^g_- \leq x^\Delta_-, \quad x^g_- < 0, \quad h(x^g_+) \geq 0, \quad \text{and} \quad h(x^g_-) > 0.
\]

In fact, when \( a = 1 \), in view of Case (3.3), we now have \( F > 0 \) and thus \( b + c < 0 \).

Note that

\[
g(z) = -(b + c)z - d.
\]

It follows from InEq. (3.11) that \( x^g_- \leq x^\Delta_- \). Since \( g(0) = -d > 0 \), we obtain \( x^g_- < 0 \).

By InEq. (3.7), we have \( h(x^g_+) \geq 0 \). It is routine to compute that

\[
h(x^g_-) = x^g_- - b = -\frac{d}{b + c} - b = -\frac{F}{b + c} > 0.
\]

Now, in view of Cases (3.3) and (3.5), we thus may assume that \( a > 1 \). Consequently, one may derive \( J_\Delta \subseteq [x^g_-, x^g_+] \) and \( x^g_- < 0 \) as in Case (3.4). We shall handle the two inequalities involving \( h \) according to the value range of \( a \). If \( a = 2 \), then the function \( h(z) = -b \) reduces to a positive constant and we are done. Now we can suppose that \( a \neq 2 \).

(1) If \( a > 2 \), then

\[
h(x^\Delta_-) + h(x^\Delta_+) = \frac{4}{a^2}((a - 2)c - ab) > 0.
\]

In view of Case (3.4), we have \( F \geq 0 \). By Eq. (3.9), we thus may assume that

\[
h(x^\Delta_-)h(x^\Delta_+) \geq 0.
\]

Therefore, we infer that \( h(x^\Delta_+) \geq 0 \). Since the polynomial \( h(z) \) is strictly decreasing and \( x^g_- < x^\Delta_- \), we have \( h(x^g_-) > h(x^\Delta_+) > 0 \).

(2) If \( 1 < a < 2 \), by InEq. (3.7), it suffices to show that \( h(x^g_-) > 0 \). In view of Case (3.3), we thus may assume that \( F > 0 \). By InEq. (3.7) and Eq. (3.9), we have \( h(x^\Delta_-) > 0 \) and \( x_h < x^\Delta_- \). By InEq. (3.14), we have \( h(x^g_-)h(x^g_+) > 0 \). Since \( J_\Delta \subseteq [x^g_-, x^g_+] \), we deduce that \( x_h < x^g_- \), i.e., \( h(x^g_-) > 0 \).

This completes the proof. \( \square \)

Let \( X, Y \subseteq \mathbb{R} \) be finite sets such that \(|X| - |Y| \in \{0, 1\} \). We say that \( X \) *interlaces* \( Y \), if the elements \( x_i \) of \( X \) and the elements \( y_j \) of \( Y \) can be arranged so that \( x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \), and that \( X \) *strictly interlaces* \( Y \) if no equality holds in the ordering. Lemma 3.7 is Lemma 3.3 of [13], wherein used in a proof of the real-rootedness of polynomials \( W_n(z) \) defined by (1.2) with \( a > 0, b \in \mathbb{R}, c = 0 \) and \( d < 0 \) by induction.
Lemma 3.7 (Gross et al. [13]) Let \( W \) be a polynomial sequence satisfying (1.1). Denote by \( R_n \) the zero set of the polynomial \( W_n(z) \). Let \( m \geq 0 \) and \( \alpha, \beta \in \mathbb{R} \). Suppose that the polynomial \( W_{m+2}(x) \) has degree \( m + 2 \), and that \( B(x) < 0 \) for all \( x \in R_{m+1} \), \( W_m(\alpha)W_{m+2}(\alpha) > 0 \), \( W_m(\beta)W_{m+2}(\beta) > 0 \), \( |R_{m+1}| = m + 1 \), \( R_{m+1} \subset (\alpha, \beta) \), and \( R_{m+1} \) strictly interlaces \( R_m \). Then, we have \( |R_{m+2}| = m + 2 \), \( R_{m+2} \subset (\alpha, \beta) \), and \( R_{m+2} \) strictly interlaces \( R_{m+1} \).

Now we are in a position to show the real-rootedness with the interlacing property and the best bound of all zeros.

Theorem 3.8 Let \( a, c > 0 \) and \( b, d < 0 \) such that \( x_A < x_B \). Let \( W \) be the normalized polynomial sequence satisfying (1.2). Then, every polynomial \( W_n(z) \) is real-rooted. Denote by \( R_n \) the zero set of \( W_n(z) \). Then, \( R_n \subset (u, v) \), and the set \( R_{n+1} \) strictly interlaces \( R_n \). Moreover, the bound \( (u, v) \) is sharp, in the sense that both the numbers \( u \) and \( v \) are limits of zeros.

Proof We prove it by induction with the aid of Lemma 3.7 for \((\alpha, \beta) = (u, v)\). Note that \( R_1 = \{0\} \). By Lemma 3.2, we have \( u < 0 < v \). From definition, any singleton set strictly interlaces the empty set \( R_0 \). Now, we can suppose, for some \( m \geq 0 \), that \( |R_{m+1}| = m + 1 \), \( R_{m+1} \subset (u, v) \), and \( R_{m+1} \) strictly interlaces \( R_m \). From (1.2), every polynomial \( W_m(z) \) is of degree \( m \). By Lemma 3.2, we have \( B(x) < 0 \) for \( x \in R_m \), \( W_m(u)W_{m+2}(u) > 0 \) and \( W_m(v)W_{m+2}(v) > 0 \). By Lemma 3.7, we obtain the real-rootedness, the bound \((u, v)\) and the strict interlacing property. By Theorem 2.3, we have \( \{x_A^\pm\} \subseteq \diamond \). By Lemma 3.2, we have \( \{u, v\} \setminus \{x_A^\pm\} \subseteq \diamond \). Hence, both the numbers \( u \) and \( v \) are limits of zeros. This completes the proof. \( \square \)

Notice that every number in \( J_\Delta \) is a limit of zeros, and \( \bigcup_{n \geq 0} R_n \cap J_\Delta \) is dense in \( J_\Delta \). The same property holds in Theorem 3.10.

3.2 Case \( x_A = x_B \)

Suppose that \( x_A = x_B \). Then, the values of \( u \) and \( v \) in Eq. (3.1) reduce to

\[
u = \begin{cases} x_A^-, & \text{if } a < 2 \text{ and } F \leq 0 \\ x_A^+ & \text{if } a < 1 \text{ and } F > 0 \\ \text{xg}^-, & \text{otherwise} \end{cases}
\]

and
\[
u = x_A^+ = x_A = x_B.
\]

In an analogue with Lemma 3.2, we have Lemma 3.9.

Lemma 3.9 Let \( a, c > 0 \) and \( b, d < 0 \). If \( x_A = x_B \), then \( u \leq x_A^- \), \( u < 0 \), \( W_n(u)(-1)^n > 0 \), and \( u \in \diamond \) as if \( u \neq x_A^- \).

Proof Same to the proof of Lemma 3.2. \( \square \)

Now we can demonstrate the root distribution of the polynomials \( W_n(z) \).
Theorem 3.10  Let $a, c > 0$ and $b, d < 0$ such that $x_A = x_B$. Let $W$ be the normalized polynomial sequence satisfying (1.2). Then, the function

$$U_n(z) = W_n(z)A^{-[n/2]}(z)$$

is a polynomial, with all its zeros lying in the interval $(u, x_B)$. Moreover, the interval $(u, x_B)$ is sharp in the sense that both the numbers $u$ and $x_B$ are limits of zeros of the polynomials $U_n(z)$.

Proof  By (1.2), the functions $U_n(z)$ satisfy the recurrence

$$U_n(z) = \begin{cases} U_{n-1}(x) + c' \cdot U_{n-2}(x), & \text{if } n \text{ is even} \\ A(x)U_{n-1}(x) + c' \cdot U_{n-2}(x), & \text{if } n \text{ is odd,} \end{cases} \tag{3.15}$$

where $c' = c/a$, with $U_0(z) = 1$ and $U_1(z) = z$. It follows immediately that the function $U_n(z)$ is a polynomial of degree $\lceil n/2 \rceil$. Let $R_n'$ be the zero set of $U_n(z)$.

We shall show by induction that the zeros $z_j$ of $U_n(z)$ strictly interlace the zeros $x_j$ of $U_{n-1}(z)$ from the left, in the interval $(u, x_B)$, i.e.,

$$\begin{cases} u < z_1 < x_1 < z_2 < \cdots < z_{\lceil \frac{n}{2} \rceil} < x_{\lceil \frac{n-1}{2} \rceil} < x_B, & \text{if } n \text{ is even;} \\ u < z_1 < x_1 < z_2 < \cdots < z_{\lceil \frac{n-1}{2} \rceil} < x_{\lceil \frac{n-1}{2} \rceil} < x_B, & \text{if } n \text{ is odd.} \tag{3.16} \end{cases}$$

We make some preparations. First, by (3.15), it is direct to show by induction that $U_n(x_B) > 0$. Second, by Lemma 3.9, we have $u < x^-_A < x^+_A = x_B$ and $W_n(u)(-1)^n > 0$. Therefore, we have $A(u) < 0$ and thus

$$U_n(u)(-1)^{[n/2]} > 0.$$ 

In particular, we have $U_2(u) < 0$. Since $U_2(u) = z + c'$, we have $u < -c' < 0 < x_B$. This checks the truth for $n = 2$. Let $n \geq 3$. By induction hypothesis, the set $R'_{n-2}$ strictly interlaces $R'_{n-1}$ from the left. Therefore, we have

$$U_{n-2}(x_j)(-1)^{[n/2+j]} > 0 \quad \text{for } j \leq \lceil (n-1)/2 \rceil.$$ 

By (3.15), the number $U_n(x_j)$ has the same sign as the number $U_{n-2}(x_j)$, that is, $U_n(x_j)(-1)^{[n/2+j]} > 0$. By using the intermediate value theorem, we derive the desired (3.16).

Same to the proof of Theorem 3.8, one may show the minimality of the interval $(u, x_B)$ as a bound of the zeros of polynomials $W_n(z)$. Note that $x^-_A \neq x^+_A$. By Theorem 2.3, each point in the interval $J_A$ is a limit of zeros of the polynomials $W_n(z)$. Therefore, each point in $J_A$ is a limit of zeros of the polynomials $U_n(z)$, and the interval $(u, x_B) = (u, x^+_A)$ is the best bound of the union of zeros of all polynomials $U_n(z)$. This completes the proof. \qed
The polynomial $W_5(z)$ in Example 3.11 has 5 zeros: two zeros have the same value $1/2$, and the other three zeros are illustrated by circles. The interval of non-isolated limits of zeros is the ultra-thick segment with bullet ends.

**Example 3.11** Let $W$ be the normalized polynomial sequence satisfying

$$W_n(z) = (2z - 1)W_{n-1}(z) + (z - 1/2)W_{n-2}(z).$$

Then, all the zeros of $W_n(z)$ lie in the interval $(u, x_B] = (-1/2, 1/2]$. Moreover, the point $x_B = 1/2$ is the only multiple zero, and its multiplicity is $\lfloor n/2 \rfloor$; see Fig. 5.

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