(Super)$^n$-Energy for Arbitrary Fields and its Interchange: Conserved Quantities

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Abstract

Inspired by classical work of Bel and Robinson, a natural purely algebraic construction of super-energy (s-e) tensors for arbitrary fields is presented, having good mathematical and physical properties.

Remarkably, there appear quantities with mathematical characteristics of energy densities satisfying the dominant property, which provides s-e estimates useful for global results and helpful in other matters.

For physical fields, higher order (super)$^n$-energy tensors involving the field and its derivatives arise. In Special Relativity, they provide infinitely many conserved quantities.

The interchange of s-e between different fields is shown. The discontinuity propagation law in Einstein-Maxwell fields is related to s-e tensors, providing quantities conserved along null hypersurfaces. Finally, conserved s-e currents are found for any minimally coupled scalar field whenever there is a Killing vector.

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The importance of the Bel-Robinson and other super-energy (s-e) tensors \[1,2\] is usually recognized, even though their physical interpretation remains somewhat obscure. Their mathematical usefulness is clear, a manifestation of which is their relevance in the proof of the global stability of Minkowski spacetime \[3\], or in the study of well-posed symmetric hyperbolic systems of differential equations including gravity (see, e.g. \[4\]). In this paper, I show how to generalize these important mathematical properties to arbitrary fields and, more importantly, try to shed some light into the physical meaning of s-e by finding non-trivial conservation laws which involve two different fields.

First, let us present the procedure to construct the s-e of any given tensor \[5\]. Consider any tensor \( t_{\mu_1...\mu_m} \) as an \( r \)-fold \((n_1, \ldots, n_r)\)-form \((n_1 + \ldots + n_r = m)\) by separating the \( m \) indices into \( r \) blocks, each containing \( n_A \) \((A = 1, \ldots, r)\) completely antisymmetric indices. This can always be done. Several examples are: \( F_{\mu\nu} = F_{[\mu\nu]} \) is a simple (2)-form, while \( \nabla_\rho F_{\mu\nu} \) is a double \((1,2)\)-form; the Riemann tensor \( R_{\alpha\beta\gamma\delta} \) is a double symmetrical \((2,2)\)-form (pairs can be interchanged); the Ricci tensor \( R_{\mu\nu} \) is a double symmetrical \((1,1)\)-form, etcetera. Schematically, \( t_{\mu_1...\mu_m} \) will be denoted by \( t_{[n_1],\ldots,[n_r]} \) where \([n_A]\) indicates the \( A \)-th block with \( n_A \) antisymmetrical indices. Duals can be defined by contracting each of the blocks with the volume element \( \ast \).

Let us define the “semi-square” \( (t_{[n_1],\ldots,[n_r]} \times t_{[n_1],\ldots,[n_r]}) \) by contracting all indices but one of each block in the product of \( t \) with itself (after reordering indices if necessary)

\[
(t \times t)_{\lambda_1\mu_1...\lambda_r\mu_r} \equiv \left( \prod_{A=1}^{r} \frac{1}{(n_A - 1)!} \right) t_{\lambda_1\rho_2...\rho_{n_1} \ldots, \lambda_r\sigma_2...\sigma_{n_r}} t_{\rho_1\mu_2...\mu_{n_1} \ldots, \sigma_1...\sigma_{n_r}}.
\]

The s-e tensor of \( t \) is \( 2r \)-covariant and defined as half of the sum of the \( 2^r \) semi-squares constructed with \( t_{[n_1],\ldots,[n_r]} \) and all its duals. Explicitly:

\[
2T_{\lambda_1\mu_1...\lambda_r\mu_r} \{t\} \equiv \left( t_{[n_1],\ldots,[n_r]} \times t_{[n_1],\ldots,[n_r]} \right)_{\lambda_1\mu_1...\lambda_r\mu_r} + \ldots + \left( t_{[4-n_1],\ldots,[4-n_r]} \times t_{[4-n_1],\ldots,[4-n_r]} \right)_{\lambda_1\mu_1...\lambda_r\mu_r}.
\]

It can be proven that \[1\] is symmetric on each \((\lambda_{A\mu A})\)-pair; if \( t_{[n_1],\ldots,[n_r]} \) is symmetric in the change of two \([n_A]-\)blocks, then \(2\) is symmetric with respect to the corresponding \((\lambda_{A\mu A})\)-pairs; and \(3\) is traceless in any \((\lambda_{A\mu A})\)-pair with \(n_A = 2\).
For any timelike unit vector $\vec{u}$, the s-e density of $t$ relative to $\vec{u}$ is defined as

$$W_t(\vec{u}) \equiv T_{\lambda_1 \mu_1 \ldots \lambda_r \mu_r} \{t\} u^\lambda_1 u^{\mu_1} \ldots u^\lambda_r u^{\mu_r}.$$ 

$W_t(\vec{u})$ is non-negative and satisfies

$$\{\exists \vec{u} \text{ such that } W_t(\vec{u}) = 0\} \iff T_{\lambda_1 \mu_1 \ldots \lambda_r \mu_r} \{t\} = 0 \iff t_{\mu_1 \ldots \mu_m} = 0. \quad (2)$$

Actually, $W_t(\vec{u})$ is the sum of the squares $|t_{\mu_1 \ldots \mu_m}|^2$ of all the components of $t$ in any orthonormal basis $\{\vec{e}_\mu\}$ with $\vec{u} = \vec{e}_0$. More importantly, the tensor (3) satisfies the dominant s-e property (DSEP) [6, 7, 8, 9], that is, for any future-pointing causal vectors $\vec{k}_1, \ldots, \vec{k}_r$ we have

$$T_{\lambda_1 \mu_1 \ldots \lambda_r \mu_r} \{t\} k^\lambda_1 k^\mu_1 \ldots k^\lambda_r k^\mu_r \geq 0.$$

This is equivalent to the “dominance” of $T_{0 \ldots 0} \{t\}$ in any orthonormal basis:

$$W_t(\vec{e}_0) = T_{0 \ldots 0} \{t\} \geq |T_{\lambda_1 \mu_1 \ldots \lambda_r \mu_r} \{t\}|, \quad \forall \lambda_1, \mu_1, \ldots, \lambda_r, \mu_r.$$

The proof of DSEP, which is one of the main properties of definition (1), has been given by Bergqvist [9] in full generality using spinors. DSEP is a helpful tool to prove causal propagation of $t_{\mu_1 \ldots \mu_m}$ (see e.g. [10, 7, 4]) and provides s-e estimates which may be useful in generalizing the theorems in [3] to the case with matter.

Applying (1) to the gravitational field, one constructs the tensors $T^{\alpha \beta \lambda \mu} \{R_{[2][2]}\}$ and $T^{\alpha \beta \lambda \mu} \{C_{[2][2]}\}$ associated to the Riemann and Weyl curvatures, getting the classical Bel [2] and Bel-Robinson [1] (BR) tensors, respectively

$$B^{\alpha \beta \lambda \mu} = R^{\alpha \rho \lambda \sigma} R^\beta \rho_\sigma + R^* R^{\alpha \rho \lambda \sigma} R^* R^\beta \rho_\sigma = R^{\alpha \rho \lambda \sigma} R^* R^\beta \rho_\sigma$$

and

$$T^{\alpha \beta \lambda \mu} = C^{\alpha \rho \lambda \sigma} C^\beta \rho_\sigma = C^* C^{\alpha \rho \lambda \sigma} C^* C^\beta \rho_\sigma.$$ 

These tensors have physical dimensions of $L^{-4}$, leading to two possible interpretations: either as energy density “square” ([1] and references therein), or as energy density per unit surface. This second interpretation seems correct, as can be deduced from several independent results [2, 13]. Notably, the relationship (2)

$$E_r = \text{const.} W_T (\vec{u}) \mid_{r=0} r^5 + O(r^6)$$

between the quasilocal energy $E_r$ of small 2-spheres of radius $r$ in vacuum and the BR s-e density $W_T (\vec{u})$ supports clearly this view ($\vec{u}$ orthogonal to the 2-sphere).
The Bel tensor satisfies
\[ \nabla_\alpha B^{\alpha \beta \lambda \mu} = R^{\beta \lambda}_{\rho \sigma} J^{\mu \rho} + R^{\beta \mu}_{\rho \sigma} J^{\lambda \rho} - \frac{1}{2} g^{\lambda \mu} R^{\beta}_{\rho \sigma \gamma} J^{\sigma \gamma} \] (4)
where \( J_{\lambda \mu} \equiv \nabla_\lambda R_{\mu \beta} - \nabla_\mu R_{\lambda \beta} \). Therefore, the Bel (as well as the BR) tensor is divergence-free in Einstein spaces (including vacuum):
\[ R_{\alpha \beta} = \Lambda g_{\alpha \beta} \implies \nabla_\alpha B^{\alpha \beta \lambda \mu} = \nabla_\alpha T^{\alpha \beta \lambda \mu} = 0 \]
implying that \( T^{\alpha \beta \lambda \mu} \) is divergence-free for any conformal Killing vectors \( \vec{\xi}, \vec{\eta}, \vec{\zeta} \). Other divergence-free tensors appear in [14], but they do not satisfy DSEP nor (2), and their lack of index symmetries prevents the existence of conserved currents.

Let us now apply our general procedure to physical fields. Take any electromagnetic field \( F_{\mu \nu} \) satisfying the source-free Maxwell equations:
\[ \nabla_\rho F^{\rho \nu} = 0, \quad \nabla_{[\rho} F^{\mu \nu]} = 0. \]
Using (1) one arrives at
\[ T_{\lambda \mu} \{ F_{[2]} \} = F_{\lambda \rho} F^{\rho \mu} - \frac{1}{4} g_{\lambda \mu} F_{\rho \sigma} F^{\rho \sigma} \]
which is the standard divergence-free energy-momentum tensor, leading to conserved currents \( T^{\mu \nu} \{ F_{[2]} \} \zeta_\nu \) for any conformal Killing vector \( \vec{\zeta} \). What about BR-like tensors? As a next step one can use the double (1,2)-form \( \nabla_\alpha F_{\mu \nu} \) as starting object to construct the corresponding tensor (1), which becomes
\[ T_{\alpha \beta \lambda \mu} \{ \nabla_{[1]} F_{[2]} \} = \nabla_\alpha F_{\lambda \rho} \nabla_\beta F^{\rho \mu} + \nabla_\alpha F_{\mu \rho} \nabla_\beta F^{\rho \lambda} - g_{\alpha \beta} \nabla_\sigma F_{\lambda \rho} \nabla^\sigma F^{\rho \mu} - \frac{1}{2} g_{\lambda \mu} \nabla_\alpha F_{\sigma \rho} \nabla_\beta F^{\sigma \rho} + \frac{1}{4} g_{\alpha \beta} g_{\lambda \mu} \nabla_\tau F_{\sigma \rho} \nabla^\tau F^{\sigma \rho}. \] (5)
Chevreton’s tensor [15] is given simply by \( T_{\alpha \beta \lambda \mu} \{ \nabla_{[1]} F_{[2]} \} + T_{\lambda \mu \alpha \beta} \{ \nabla_{[1]} F_{[2]} \} \). The procedure can be continued building the (super)\(^n\)-energy tensor \( T_{\alpha \beta \lambda \mu \tau \nu} \) associated with the triple (1,1,2)-form \( \nabla_\alpha \nabla_\beta F_{\lambda \mu} \), and so on, producing an infinite set of (super)\(^n\)-energy tensors, one for each natural number \( n \). The following fundamental result holds: the (super)\(^n\)-energy (tensor) of the electromagnetic field vanishes at a point \( p \) iff the \( n^{th} \) covariant derivative of \( F \) is zero at \( p \).

The divergences of (3) and higher order s-e tensors can be computed, and they vanish in flat spacetime providing infinitely many conserved quantities for the electromagnetic field in Special Relativity. This result is not surprising, because if \( F_{\mu \nu} \) satisfies Maxwell’s equations in flat spacetime, then the 2-forms \( \nabla_{[\alpha_1} \cdots \nabla_{[\alpha_s]} F_{\mu \nu]} \) for fixed values of \( \alpha_1, \ldots, \alpha_s \) so do, and their “energy-momentum” quantities give rise to the s-e tensors. In the full Einstein-Maxwell theory, there exists a divergence-free 4-index tensor [3] which, unfortunately, does not satisfy DSEP nor (2) and, due to its peculiar index symmetries, it does not provide conserved currents.
To explore the possibility of s-e exchange, let us consider the propagation of discontinuities \[\mathcal{F}[5]\]. Assume there is an electromagnetic field propagating in a background so that there is a (necessarily null) hypersurface of discontinuity \(\Sigma\). Let us denote by \([V]\Sigma\) the discontinuity of any object \(V\) across \(\Sigma\). From the classical Hadamard theory one proves the existence of \(c_\mu\) on \(\Sigma\) such that

\[
[F_{\mu\nu}]_\Sigma = n_\mu c_\nu - n_\nu c_\mu, \quad n_\mu c^\mu = 0
\]

where \(n_\mu\) is a null 1-form normal to \(\Sigma\) (notice that \(\vec{n}\) is in fact tangent to \(\Sigma\) because \(n_\mu n^\mu = 0\) \[16\]), as well as the following propagation equation

\[
n^\mu \partial_\mu c^2 + c^2 (K + 2\psi) = 0, \quad c^2 \equiv c_\mu c^\mu \geq 0
\]

where \(\psi\) is the factor appearing in \(n^\mu \nabla_\mu n^\nu = \psi n^\nu\) and \(K\) is the trace of the second fundamental form of \(\Sigma\) relative to \(n_\mu\) (as \(\Sigma\) is null, such a second fundamental form is orthogonal to \(\vec{n}\) and not extrinsic \[16\]). The above propagation equation ensures that if \(c_\mu = 0\) at a spacelike 2-surface cut \(S\) of \(\Sigma\), then \(c_\mu = 0\) everywhere on \(\Sigma\). For any conformal Killing vector \(\vec{\zeta}\) it is then straightforward to prove that

\[
\int_S c^2 (n_\mu \zeta^\mu)^2 \omega
\]

is a conserved quantity along \(\Sigma\), where \(\omega\) is the volume element 2-form of \(S\). This can be related to the energy-momentum of the electromagnetic field because

\[
\left[ \mathcal{T}_{\mu\nu} \mathcal{F}[2] \right]\Sigma = c^2 n_\mu n_\nu \quad \text{if the background is empty.}
\]

But what happens when \([F_{\mu\nu}]_\Sigma = 0\)? Then, \(\mathcal{T}_{\mu\nu} \mathcal{F}[2]\) is continuous and s-e quantities must come in. Now there exist \(B_{\mu\nu}\) and \(f_{\mu}\) on \(\Sigma\) such that

\[
[R_{\alpha\beta\lambda\gamma}]_\Sigma = 4n_\alpha [B_{\beta\gamma}]_\mu n_\lambda, \quad B_{\mu\nu} = B_{\nu\mu}, \quad 2n^\alpha B_{\mu\beta} + g^{\mu\nu} B_{\mu\nu} n_\beta = 0,
\]

\[
[\nabla_\alpha F_{\mu\nu}]_\Sigma = 2n_\alpha [n_\mu f_\nu], \quad n_\rho f^\rho = 0
\]

and satisfying the propagation laws \[17\]

\[
n^\mu \partial_\mu B^2 + B^2 (K + 4\psi) - 2n^\sigma F_{\sigma\rho} B^{\rho\tau} f_\tau = 0, \quad B^2 \equiv B_{\mu\nu} B^{\mu\nu} \geq 0, \quad (6)
\]

\[
n^\mu \partial_\mu f^2 + f^2 (K + 4\psi) + 2n^\sigma F_{\sigma\rho} B^{\rho\tau} f_\tau = 0, \quad f^2 \equiv f_\mu f^\mu \geq 0. \quad (7)
\]

Therefore, the discontinuities of the Riemann tensor and of \(\nabla_\alpha F_{\mu\nu}\) are a source for each other. In fact, from \[3\] and \[7\] it is immediate that \[17\]

\[
n^\mu \partial_\mu \left( B^2 + f^2 \right) + \left( B^2 + f^2 \right) (K + 4\psi) = 0
\]
from where one finds the following conserved quantity along $\Sigma$

$$
\int_S \left( B^2 + f^2 \right) (n_\mu \zeta^\mu)^4 \omega
$$

which needs both the electromagnetic and gravitational contributions. The interesting point is that (8) is related to the tensors (3) and (5): if $\nabla_\rho F_{\mu\nu}$ vanishes at one side of $\Sigma$, then $[B_{\alpha\beta\mu\nu}]_\Sigma = 2B^2 \eta_{\alpha\beta} \eta_{\mu\nu}$ and $[T_{\alpha\beta\mu\nu} \{\nabla_{[1]} F_{[2]} \}]_\Sigma = 2f^2 \eta_{\alpha\beta} \eta_{\mu\nu}$.

To check whether or not this interplay of s-e quantities is generic, consider finally a scalar field $\phi$ satisfying the Klein-Gordon equation $\nabla_\rho \nabla^\rho \phi = m^2 \phi$ for mass $m$. Expression (1) for $\phi$ would simply be $T\{\phi\} = \phi^2/2$. However, the case of physical interest arises by constructing the tensor (1) associated to $\nabla_\mu \phi$:

$$
T_{\lambda\mu} \{\nabla_{[1]} \phi\} = \nabla_\lambda \phi \nabla_\mu \phi - \frac{1}{2} g_{\lambda\mu} \nabla_\rho \phi \nabla^\rho \phi
$$

which is the standard energy-momentum tensor of a massless scalar field. When $\mu \neq 0$, $T\{m\phi\}$ has the same physical dimensions than $T_{\lambda\mu} \{\nabla_{[1]} \phi\}$, so that one must combine them (using $-g_{\mu\nu}$ which obviously satisfies DSEP) to obtain the whole energy-momentum tensor of the minimally coupled scalar field

$$
T_{\lambda\mu} \equiv T_{\lambda\mu} \{\nabla_{[1]} \phi\} + T\{m\phi\} (-g_{\lambda\mu}) = \nabla_\lambda \phi \nabla_\mu \phi - \frac{1}{2} g_{\lambda\mu} \nabla_\rho \phi \nabla^\rho \phi - \frac{1}{2} m^2 \phi^2 g_{\lambda\mu}.
$$

This procedure is systematic and the BR-like tensor for the scalar field is the sum of expression (1) for the double symmetric (1, 1)-form $\nabla_\alpha \nabla_\beta \phi$ plus the terms coming from $m$. Expanding the duals one gets

$$
T_{\alpha\beta\lambda\mu} = T_{\alpha\beta\lambda\mu} \{\nabla_{[1]} \nabla_{[1]} \phi\} - T_{\alpha\beta} \{m \nabla_{[1]} \phi\} g_{\lambda\mu} - T_{\lambda\mu} \{m \nabla_{[1]} \phi\} g_{\alpha\beta} + T\{m^2 \phi\} g_{\alpha\beta} g_{\lambda\mu} =
$$

$$
= 2\nabla_\alpha \nabla_\lambda \phi \nabla_\mu \nabla_\beta \phi - g_{\alpha\beta} \left( \nabla_\lambda \nabla^\rho \phi \nabla_\mu \nabla_\rho \phi + m^2 \nabla_\lambda \phi \nabla_\mu \phi \right) - g_{\lambda\mu} \left( \nabla_\alpha \nabla_\beta \phi \nabla_\rho \phi \right)
$$

$$
+ m^2 \nabla_\alpha \nabla_\beta \phi
$$

$$
+ \frac{1}{2} g_{\alpha\beta} g_{\lambda\mu} \left( \nabla_\sigma \nabla_\rho \phi \nabla_\sigma \nabla_\rho \phi + 2m^2 \nabla_\rho \phi \nabla_\rho \phi + m^4 \phi^2 \right)
$$

which satisfies DSEP, $T_{\alpha\beta\lambda\mu} = T_{\alpha(\beta}(\lambda\mu)} = T_{\mu\alpha\beta\lambda}$, and was previously found in Special Relativity [2]. Again one can construct (super)$n$-energy tensors associated to the higher derivatives $\nabla_{\mu_1} \cdots \nabla_{\mu_{n+1}} \phi$: the (super)$n$-energy (tensor) of the scalar field ($m \neq 0$) vanishes at a point $p$ iff the covariant derivatives of $\phi$ up to $(n+1)^{th}$ order are zero at $p$. (If $m = 0$ the result involves the $(n+1)^{th}$-derivatives exclusively).

Direct computation leads to

$$
\nabla_\alpha T_{\beta\lambda\mu}^{\alpha} = 2\nabla_\beta \nabla_\lambda \phi \nabla_\mu \phi - g_{\lambda\mu} R^{\sigma\rho}_{\beta\alpha} \nabla_\sigma \phi \nabla_\rho \phi
$$

$$
- \nabla_\sigma \phi \left( 2\nabla_\sigma \nabla_\lambda \phi R^{\sigma}_{\beta\rho\mu} + g_{\lambda\mu} R^{\sigma\rho}_{\alpha\beta} \nabla_\sigma \nabla_\rho \phi \right)
$$

and similar longer formulae for (super)$n$-energy tensors, so that they are divergence-free in flat spacetime, providing infinitely many conserved quantities there.
The situation hitherto is that the Bel tensor (3) is divergence-free in vacuum, and the s-e tensor (9) of the scalar field is divergence-free in the absence of gravitation. The natural question arises: can these tensors be combined to produce a conserved quantity in General Relativity? To answer it, assume that the spacetime satisfies the Einstein-Klein-Gordon equations, so that

\[ R_{\mu\nu} = \nabla_{\mu}\phi \nabla_{\nu}\phi + \frac{1}{2} m^2 \phi^2 g_{\mu\nu} \]  

(11)

and that \( \xi \) is a Killing vector. Then \( \xi^\mu \nabla_\mu \phi = 0 \) (if \( m \neq 0 \)) \[ \text{[19]} \] from where it also follows \( \xi^\mu \nabla_\nu \phi \nabla_\mu \nabla_\nu \phi = 0 \). Using these results with (10) and (11) one finds

\[ \nabla_\alpha \left( B^{\alpha\beta\lambda\mu} \xi_\beta \xi_\lambda \xi_\mu \right) = \nabla_\sigma \phi \left( 2 \nabla^\nu \nabla_\lambda \phi R^{\sigma}_{\mu\rho\beta} + g_{\lambda\mu} R^{\sigma}_{\rho\beta\tau} \nabla_\nu \nabla^\tau \phi \right) \xi_\beta \xi_\lambda \xi_\mu \]

and from (10) and (11)

\[ \nabla_\alpha \left( T^{\alpha\beta\lambda\mu} \xi_\beta \xi_\lambda \xi_\mu \right) = - \nabla_\sigma \phi \left( 2 \nabla^\nu \nabla_\lambda \phi R^{\sigma}_{\mu\rho\beta} + g_{\lambda\mu} R^{\sigma}_{\rho\beta\tau} \nabla_\nu \nabla^\tau \phi \right) \xi_\beta \xi_\lambda \xi_\mu \]

so that finally

\[ \nabla_\alpha j^\alpha = 0 \quad j^\alpha \equiv \left( B^{\alpha\beta\lambda\mu} + T^{\alpha\beta\lambda\mu} \right) \xi_\beta \xi_\lambda \xi_\mu . \]

This also holds when \( m = 0 \) for either case of \( \xi \) being parallel or orthogonal to \( \nabla \phi \).

The relevance of this result is that provides conserved s-e quantities (by means of the typical integration of \( j^\alpha \) and Gauss’ theorem) proving the interchange of s-e between the gravitational and scalar fields, because neither \( B^{\alpha\beta\lambda\mu} \xi_\beta \xi_\lambda \xi_\mu \) nor \( T^{\alpha\beta\lambda\mu} \xi_\beta \xi_\lambda \xi_\mu \) are divergence-free separately in general. These conservation laws make definition (4) of s-e tensors not only mathematically appealing but also physically very promising.

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