Error-Trellis Construction for Tailbiting Convolutional Codes

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Abstract—In this paper, we present an error-trellis construction for tailbiting convolutional codes. A tailbiting error-trellis is characterized by the condition that the syndrome former starts and ends in the same state. We clarify the correspondence between code subtrellises in the tailbiting code-trellis and error subtrellises in the tailbiting error-trellis. Also, we present a construction of tailbiting backward error-trellises. Moreover, we obtain the scalar parity-check matrix for a tailbiting convolutional code. The proposed construction is based on the adjoint-obvious realization of a syndrome former and its behavior is fully used in the discussion.

I. INTRODUCTION

In this paper, we always assume that the underlying field is \( F = \mathbb{GF}(2) \). Let \( G(D) \) be a generator matrix of an \((n, k)\) convolutional code \( C \). Let \( H(D) \) be a corresponding \( r \times n \) parity-check matrix of \( C \), where \( r = n - k \). Both \( G(D) \) and \( H(D) \) are assumed to be canonical \([1, 5]\). Denote by \( L \) the memory length of \( G(D) \) (i.e., the maximum degree among the polynomials of \( G(D) \)) and by \( M \) the memory length of \( H(D) \). Then \( H(D) \) is expressed as

\[
H(D) = H_0 + H_1 D + \cdots + H_M D^M. \tag{1}
\]

Consider a terminated version of \( C \) with \( N \) trellis sections. That is, each codeword is a path starting from the all-zero state at time \( t = 0 \) and ending in the all-zero state at time \( t = N \). In this case, \( C \) is specified by the following scalar parity-check matrix \([1, 6]\):

\[
H_{\text{scalar}} = \begin{pmatrix}
H_0 \\
H_1 & H_0 \\
\vdots & \vdots & \ddots \\
H_M & \cdots & \cdots & H_0 \\
H_M & \cdots & \cdots & H_1 \\
\vdots & \ddots & \cdots & \ddots \\
H_M & \cdots & \cdots & \cdots & \cdots & H_M
\end{pmatrix} \tag{2}
\]

with size \((N + M)r \times Nn \) (blanks indicate zeros).

Tailbiting is a technique by which a convolutional code can be used to construct a block code without any loss of rate \([4, 7, 10]\). Let \( C_{tb} \) be a tailbiting convolutional code with an \( N \)-section code-trellis \( T_{tb}^{(c)} \). The fundamental idea behind tailbiting is that the encoder starts and ends in the same state, i.e., \( \beta_0 = \beta_N \) (\( \beta_k \) is the encoder state at time \( k \)). Suppose that \( T_{tb}^{(c)} \) has \( \Sigma_0 \) initial (or final) states, then it is composed of \( \Sigma_0 \) subtrellises, each having the same initial and final states. We call these subtrellises tailbiting code subtrellises. For example, a tailbiting code-trellis of length \( N = 5 \) based on the generator matrix

\[
G_1(D) = (1, 1 + D^2, 1 + D + D^2) \tag{3}
\]

is shown in Fig. 1. Since \( \Sigma_0 = 4 \), this tailbiting code-trellis is composed of 4 code subtrellises. In Fig. 1, bold lines correspond to the code subtrellis with \( \beta_0 = \beta_5 = (1, 0) \).

On the other hand, it is reasonable to think that an error-trellis \( T_{tb}^{(e)} \) for the tailbiting convolutional code \( C_{tb} \) can be constructed. In this case, each error subtrellis should have the same initial and final states like a code subtrellis. In this paper, taking this property into consideration, we present an error-trellis construction for tailbiting convolutional codes. We also clarify the correspondence between code subtrellises in \( T_{tb}^{(e)} \) and error subtrellises in \( T_{tb}^{(c)} \). In this relationship, we see that dual states (i.e., syndrome-former states corresponding to encoder states) play an important role. Also, a kind of superposition rule associated with a syndrome former is used.

Next, we present a construction of tailbiting backward error-trellises. Using the backward error-trellis, each tailbiting error path is represented in time-reversed order. Moreover, we derive the general structure of the scalar parity-check matrix for a
tailbiting convolutional code. Similar to a scalar generator matrix, it is shown that the obtained scalar parity-check matrix has a cyclic structure. In general, unlike code-trellises, error-trellises enable decoding with remarkably low average complexity [1]. Hence, we think an error-trellis construction presented in this paper is very important.

II. SYNDROME FORMER $H^T(D)$

A. Adjoint-Obvious Realization of a Syndrome Former

Consider the adjoint-obvious realization (observer canonical form [2], [3]) of the syndrome former $H^T(D)$ ($T$ means transpose). Let $e_k = (e_k^{(1)}, e_k^{(2)}, \ldots, e_k^{(n)})$ and $\zeta_k = (\zeta_k^{(1)}, \zeta_k^{(2)}, \ldots, \zeta_k^{(r)})$ be the input error at time $k$ and the corresponding output syndrome at time $k$, respectively. Denote by $\sigma_k^{(q)}$ the contents of the memory elements in the above realization. Here, the contents of the memory array corresponding to the syndrome bit $\zeta_k^{(q)}$ are labeled with $q$. For any fixed $q$, $\sigma_k^{(q)}$ corresponds to the memory element which is closest to the $q$th output of the syndrome former (i.e., $\zeta_k^{(q)}$). If a memory element is missing, the corresponding $\sigma_k^{(q)}$ is set to zero. Using $\sigma_k^{(q)}$, the syndrome-former state at time $k$ is defined as

$$\sigma_k \triangleq (\sigma_k^{(1)}, \ldots, \sigma_k^{(r)}, \sigma_k^{(1)}_{k_M}, \ldots, \sigma_k^{(r)}_{k_M}).$$ (4)

(Remark: The effective size of $\sigma_k$ is equal to the overall constraint length of $H(D)$.)

Let $\xi_k = (\zeta_k, \sigma_k)^T$ be the extended state augmented with the syndrome $\zeta_k$. Then $\xi_k$ has an expression [8], [9]:

$$\xi_k = \begin{pmatrix} H_M & H_{M-1} & \cdots & H_1 & H_0 \\ 0 & H_M & \cdots & H_2 & H_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & H_M & H_{M-1} \\ 0 & 0 & \cdots & 0 & H_M \end{pmatrix} \times (e_{k-M}, e_{k-M+1}, \ldots, e_k)^T \\ \triangleq H^* \times (e_{k-M}, e_{k-M+1}, \ldots, e_k)^T. \tag{5}$$

From this expression, we have

$$\sigma_k \triangleq (\sigma_k^{(1)}, \sigma_k^{(2)}, \ldots, \sigma_k^{(M)}) \\ = (e_{k-M+1}, \ldots, e_{k-1}, e_k) \\ \times \begin{pmatrix} H_T^{(M)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & H_T^{(1)} \end{pmatrix} \\ \triangleq (e_{k-M+1}, \ldots, e_{k-1}, e_k) \times H^{*T}. \tag{6}$$

Note that $\sigma_k$ has an alternative expression:

$$\sigma_k = (\sigma_k^{(2)}, \ldots, \sigma_k^{(M)}, 0) + e_k(H_T^{(1)}H_T^{(2)}, \ldots, H_T^{(M)}). \tag{7}$$

Similarly, $\zeta_k$ is expressed as

$$\zeta_k = e_{k-M}H_T^{(M)} + \cdots + e_{k-1}H_T^{(1)} + e_kH_T^{(0)} \tag{8}$$

$$\sigma_k + \sigma_k' \triangleq (\sigma_k^{(2)}, \ldots, \sigma_k^{(M)}, \sigma_k^{(0)}) \\ + (e_k + e_k')H_T^{(1)}H_T^{(2)}, \ldots, H_T^{(M)}). \tag{18}$$

B. Dual States

The encoder states can be labeled by the syndrome-former states (i.e., dual states [2]). The dual state $\beta_k'$ corresponding to the encoder state $\beta_k$ is obtained by replacing $e_k$ in $\sigma_k$ by $y_k = u_kG(D)$ ($u_k$ is the information at time $k$). We have

$$\beta_k' = (y_{k-M+1}, \ldots, y_{k-1}, y_k) \times \begin{pmatrix} H_T^{(M)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & H_T^{(1)} \end{pmatrix} \tag{10}$$

Example 1: Consider the parity-check matrix

$$H_1(D) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{11}$$

corresponding to $G_1(D)$. $H_1(D)$ is expressed as

$$H_1(D) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} D \tag{12}$$

Hence ($M = 1$), the dual state corresponding to the encoder state $\beta_k = (u_k-1, u_k)$ is obtained as follows.

$$\beta_k' = y_kH_1^T \\ = (y_k^{(1)}, y_k^{(2)}, y_k^{(3)}) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{13}$$

C. Behavior of a Syndrome Former

Lemma 1: Let $\sigma_{k-1}$ be the syndrome-former state at time $k-1$. Here, assume that an error $e_k$ is inputted to the syndrome former and it moves to the state $\sigma_k$ at time $k$. Also, assume that the syndrome $\zeta_k$ is outputted according to this transition. (This relation is denoted as

$$\sigma_{k-1} + e_k \xrightarrow{\zeta_k} \sigma_k.$$

Similarly, assume the relation

$$\sigma_{k-1} + e_k \xrightarrow{\sigma'_k} \sigma'_k.$$

Then we have

$$\sigma_{k-1} + \sigma'_k \xrightarrow{\zeta_k + \sigma'_k} \sigma_k + \sigma'_k.$$

Proof: From the assumption, the relations

$$\sigma_k = (\sigma_k^{(2)}, \ldots, \sigma_k^{(M)}, 0) + e_k(H_T^{(1)}H_T^{(2)}, \ldots, H_T^{(M)}). \tag{16}$$

$$\sigma'_k = (\sigma'_k^{(2)}, \ldots, \sigma'_k^{(M)}, 0) + e'_k(H_T^{(1)}H_T^{(2)}, \ldots, H_T^{(M)}). \tag{17}$$

hold. Hence, we have

$$\sigma_k + \sigma'_k \triangleq (\sigma_k^{(2)}, \ldots, \sigma_k^{(M)}, \sigma_k^{(0)}) \\ + (e_k + e_k')H_T^{(1)}H_T^{(2)}, \ldots, H_T^{(M)}). \tag{18}$$
On the other hand, using the relations
\begin{align}
\zeta_k &= \sigma_k^{(1)} + e_k H_0^T \\
\zeta_k' &= \sigma_k'^{(1)} + e'_k H_0^T,
\end{align}
we have
\[\zeta_k + \zeta_k' = (\sigma_k^{(1)} + \sigma_k'^{(1)}) + (e_k + e'_k) H_0^T.\]  
These expressions imply that
\[\sigma_{k-1} + \sigma_{k-1}' + e_k + e'_k \sigma_k + \sigma_k' \]
holds.

**Lemma 2:** Let \( \beta_0 \) and \( \beta_N \) be the initial and final states of the code-trellis, respectively. Denote by \( y \) a code path connecting these states. (This is denoted as \( \beta_0 \overset{y}{\rightarrow} \beta_N \).)

Then we have
\[\beta_0 \overset{y}{\rightarrow} \beta_N.\]  
That is, assume that the syndrome former is in the dual state \( \beta_0 \) of \( \beta_0 \). In this case, if \( y \) is inputted to the syndrome former, then it moves to the dual state \( \beta_N \) of \( \beta_N \) and the syndrome \( \zeta = 0 \) is outputted.

**Proof:** By extending the code-trellis in both directions by \( L \) sections, if necessary, we can assume the condition
\[\beta_0 = 0 \overset{y}{\rightarrow} \beta_L \overset{y''}{\rightarrow} \beta_{N+L+2L} = 0,\]  
where \( y' \) and \( y'' \) are augmented code paths (initial and final states are both 0). Hence, we can apply the standard scalar parity-check matrix \( H_{\text{scalar}} \) (cf. (2)). Then we have
\[\beta_0^* = 0 \overset{y}{\rightarrow} \beta_L^* \overset{y''}{\rightarrow} \beta_{N+L+2L}^* = 0.\]  
That is, the output of the syndrome former is zero for all time. In the above relation, we can note the following subsection:
\[\beta_L^* \overset{y}{\rightarrow} \beta_{N+L}^*.\]

**Proof:** From the assumption, we have
\[\sigma_{\text{fin}} \overset{z+y+e}{\rightarrow} \zeta^* \overset{\zeta}{\rightarrow} \sigma_{\text{fin}}.\]

Also, from Lemma 2,
\[\beta^* \overset{y}{\rightarrow} \beta^*.\]
is obtained. Hence, by applying Lemma 1, we have
\[\sigma_{\text{fin}} + \beta^* \overset{z+y+e}{\rightarrow} \zeta \overset{\zeta+0=\zeta}{\rightarrow} \sigma_{\text{fin}} + \beta^*.\]

### III. ERROR-TRELLISES FOR TAILBITING CONVOLUTIONAL CODES

#### A. Error-Trellis Construction

Suppose that the tailbiting code-trellis based on \( G(D) \) is defined in \([0, N]\), where \( N \geq M \). In this case, the corresponding tailbiting error-trellis based on \( {H^T}(D) \) is constructed as follows.

**Step 1:** Let \( z = \{z_k\}_{k=1}^N \) be a received data. Denote by \( \sigma_0 \) the initial state of the syndrome former \( {H^T}(D) \). Let \( \sigma_{\text{fin}}(= \sigma_N) \) be the final syndrome-former state corresponding to the input \( z \). Note that \( \sigma_{\text{fin}} \) is independent of \( \sigma_0 \) and is uniquely determined only by \( z \).

**Step 2:** Set \( \sigma_0 \) to \( \sigma_{\text{fin}} \) and input \( z \) to the syndrome former. Here, assume that the syndrome sequence \( \zeta = \{\zeta_k\}_{k=1}^N \) is obtained. (Remark: \( \zeta_k \) \((k \geq M + 1)\) has been obtained in Step 1.)

**Step 3:** Concatenate the error-trellis modules corresponding to the syndromes \( \zeta_k \). Then we have the tailbiting error-trellis.

**Example 2:** Again, consider the parity-check matrix \( H_1(D) \).

Let \( z = z_1, z_2, z_3, z_4, z_5 = 111110111000 \) be the received data. According to Step 1, let us input \( z \) to the syndrome former \( {H^T}(D) \). Then we have \( \sigma_{\text{fin}}(= (0, 0)) \). Next, we set \( \sigma_0 \) to \( \sigma_{\text{fin}} = (0, 0) \) and input \( z \) to the syndrome former. In this case, the syndrome sequence
\[\zeta = \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5 = 0000100111 \]
is obtained. The tailbiting error-trellis is constructed by concatenating the error-trellis modules corresponding to \( \zeta_k \). The obtained tailbiting error-trellis is shown in Fig.2.

##### B. Correspondence Between Code Subtrellises and Error Subtrellises

With respect to the correspondence between tailbiting code subtrellises and tailbiting error subtrellises, we have the following.

**Proposition 2:** Let \( \beta_0(= \beta_N) = \beta \) be the initial (final) state of a tailbiting code subtrellis. Then the initial (final) state of the corresponding tailbiting error subtrellis is given by \( \sigma_{\text{fin}} + \beta^* \).

**Proof:** Direct consequence of Proposition 1.
Example 2 (Continued): Consider the tailbiting error-trellis in Fig. 2. In this example, we have $\sigma_{\text{fin}} = (0, 0)$. The corresponding tailbiting code-trellis based on $G_1(D)$ is shown in Fig. 1. In Fig. 1, take notice of the code subtrellis with initial (final) state $\alpha = (1, 0)$ (bold lines). The dual state of $\alpha = (1, 0)$ is calculated as $\beta^* = u_{-1} + u_0, u_0 = (1 + 0, 0) = (1, 0)$. Hence, the initial (final) state of the corresponding error subtrellis is given by $\sigma_{\text{fin}} + \beta^* = (0, 0) + (1, 0) = (1, 0)$ (bold lines in Fig. 2).

C. Backward Error-Trellis Construction

Let $\tilde{G}(D)$ and $\tilde{H}(D)$ be the reciprocal encoder and the reciprocal dual encoder [6] associated with $G(D)$, respectively. Then the tailbiting backward error-trellis corresponding to the original tailbiting error-trellis is constructed as follows.

Step 1: Let $\tilde{z} = \{\tilde{z}_k\}_{k=1}^N = \{z_{N-k+1}\}_{k=1}^N$ be the time-reversed received data. Denote by $\tilde{\sigma}_0$ the initial state of the syndrome former $\tilde{H}(D)$. Let $\tilde{\sigma}_{\text{fin}} (= \tilde{\sigma}_N)$ be the final syndrome former state corresponding to the input $\tilde{z}$. Note that $\tilde{\sigma}_{\text{fin}}$ is independent of $\tilde{\sigma}_0$ and is uniquely determined only by $\tilde{z}$.

Step 2: Set $\tilde{\sigma}_0$ to $\tilde{\sigma}_{\text{fin}}$ and input $\tilde{z}$ to the syndrome former. Here, assume that the syndrome sequence $\eta = \{\eta_k\}_{k=1}^N$ is obtained.

Remark: It is shown that $\zeta = \{\zeta_k\}_{k=1}^N$ and $\eta = \{\eta_k\}_{k=1}^N$ have the following correspondence:

$$\eta = \eta_1 \eta_2 \cdots \eta_M \eta_{M+1} \cdots \eta_N = \zeta_M \zeta_{M-1} \cdots \zeta_{M+1} \cdots \zeta_1 \cdots \zeta_N$$

(32)

Step 3: Concatenate the error-trellis modules corresponding to the syndromes $\eta_k$. Then we have the tailbiting backward error-trellis.

Example 3: Take notice of Example 2. The reciprocal dual encoder $\tilde{H}_1(D)$ associated with $G_1(D)$ is given by $\tilde{H}_1(D) = \begin{pmatrix} 1 + D & 1 & 1 + D \\ 1 & D & D \end{pmatrix}$.

(33)

Let $\tilde{z} = \tilde{z}_1 \tilde{z}_2 \tilde{z}_3 \tilde{z}_4 \tilde{z}_5 = 000 111 110 110 111$ be the time-reversed received data. According to Step 1, let us input $\tilde{z}$ to the syndrome former $\tilde{H}(D)$. Then we have $\tilde{\sigma}_{\text{fin}} = (0, 0)$. Next, we set $\tilde{\sigma}_0$ to $\tilde{\sigma}_{\text{fin}} = (0, 0)$ and input $\tilde{z}$ to the syndrome former. In this case, the syndrome sequence $\eta = \eta_1 \eta_2 \eta_3 \eta_4 \eta_5 = 00 11 01 10 00$ (35) is obtained. Since $M = 1$, we see that the correspondence

$$\eta = \eta_1 \eta_2 \eta_3 \eta_4 \eta_5 = \zeta_1 \zeta_5 \zeta_4 \zeta_3 \zeta_2$$

(36)

holds. The tailbiting backward error-trellis is constructed by concatenating the error-trellis modules corresponding to $\eta_k$. The obtained tailbiting backward error-trellis is shown in Fig. 3.

Next, consider the correspondence between forward error subtrellises and backward error subtrellises. First, note the following.

Proposition 3: Let $\tilde{\beta}_0(= \tilde{\beta}_N) = \tilde{\bar{\beta}}$ be the initial (final) state of a tailbiting backward code subtrellis. Then the initial (final) state of the corresponding backward error subtrellis is given by $\tilde{\sigma}_{\text{fin}} + \tilde{\beta}^*$.

Proof: Direct consequence of Proposition 1.

Example 3 (Continued): Consider the reciprocal encoder $\tilde{G}_1(D) = (D^2, 1 + D^2, 1 + D + D^2)$ (37) and the reciprocal dual encoder $\tilde{H}_1(D)$ associated with $G_1(D)$. $\tilde{H}_1(D)$ is expressed as $\tilde{H}_1(D) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} D$ (38)
Hence, the syndrome \( \mathbf{H}(D) \) becomes

\[
\mathbf{H}(D) = \begin{pmatrix}
H_0 & H_M & \ldots & H_2 & H_1 \\
H_1 & H_0 & \ldots & \ldots & H_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
H_{M-1} & \ldots & H_0 & H_M \\
H_M & H_{M-1} & \ldots & H_1 & \ldots \\
\vdots & \vdots & \ldots & \ldots & \ldots \\
H_H & H_{M-1} & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]

with size \( N_R \times N_n \).

Proof: Consider the tailbiting convolutional code \( C_{tb} \) with \( N \) trellis sections specified by a parity-check matrix \( H(D) \). \( C_{tb} \) can be regarded as an \( (Nn, Nk) \) block code [4]. In this case, we have the following.

**Proposition 5:** Assume that \( H(D) \) has the form (1). Then the scalar parity-check matrix \( H_{scalar} \) for \( C_{tb} \) is given by

\[
H_{scalar} = \begin{pmatrix}
H_0 & H_M & \ldots & H_2 & H_1 \\
H_1 & H_0 & \ldots & \ldots & H_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
H_{M-1} & \ldots & H_0 & H_M \\
H_M & H_{M-1} & \ldots & H_1 & \ldots \\
\vdots & \vdots & \ldots & \ldots & \ldots \\
H_H & H_{M-1} & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]

Similarly, we have

\[
\zeta_2 = (e_{-M+1}H^T_{M} + \ldots + e_1H^T_2 + e_0H^T_1) + e_1H^T_0 = (e_{N-M+2}H^T_M + \ldots + e_NH^T_2) + e_1H^T_0
\]

The same argument can be applied to \( \zeta_k \) \((3 \leq k \leq N)\). Then we see that \( H_{scalar}^{T} \) is written as

\[
\begin{pmatrix}
H^T_0 \\
H^T_1 \\
\vdots \\
H^T_M \\
H^T_{M-1} \\
\vdots \\
H^T_H \\
\end{pmatrix}
\]

By transposing this matrix, \( H_{ scalar } \) is obtained.

**V. Conclusion**

In this paper, we have presented an error-trellis construction for tailbiting convolutional codes. A tailbiting error-trellis is characterized by the condition that the syndrome former starts and ends in the same state. We have clarified the correspondence between code subtrellises in the tailbiting code-trellis and error subtrellises in the tailbiting error-trellis. Also, we have presented a construction of tailbiting backward error-trellises. Moreover, we have obtained the general structure of the scalar parity-check matrix for a tailbiting convolutional code. We see that the obtained results correspond to those for tailbiting code-trellises in the natural manner.

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