ON FINITENESS AND RIGIDITY OF $J$-HOLOMORPHIC CURVES IN SYMPLECTIC THREE-FOLDS

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Abstract. Given a symplectic three-fold $(M, \omega)$ we show that for a generic almost complex structure $J$ which is compatible with $\omega$, there are finitely many $J$-holomorphic curves in $M$ of any genus $g \geq 0$ representing a homology class $\beta$ in $H_2(M, \mathbb{Z})$ with $c_1(M) \cdot \beta = 0$, provided that the divisibility of $\beta$ is at most 4 (i.e. if $\beta = n\alpha$ with $\alpha \in H_2(M, \mathbb{Z})$ and $n \in \mathbb{Z}$ then $n \leq 4$). Moreover, each such curve is embedded and 4-rigid.

1. Introduction

Let $(M, \omega)$ be a symplectic three-fold, and $J$ be an element in the space $\mathcal{J}^\infty(M, \omega)$ of smooth almost complex structure on $M$ which are compatible with $\omega$. For any given homology class $\beta \in H_2(M, \mathbb{Z})$ and any given genus $g \geq 0$, the virtual dimension of the moduli space of $J$-holomorphic curves representing $\beta$ which have genus $g$ is equal to $c_1(M) \cdot \beta$. In particular, when $c_1(M) \cdot \beta = 0$ (e.g. if $c_1(M) = 0$) this moduli space is expected to be zero-dimensional. In fact, a conjecture of Ionel and Parker predicts that for a generic almost complex structure $J$ all such moduli spaces are compact zero dimensional manifolds (see section 7.4 of [2]). In this paper we will present a proof when the homology class $\beta$ is not divisible by large integers.

Definition 1.1. For a homology class $\beta \in H_2(M, \mathbb{Z})$ define the divisibility $|\beta|$ to be the largest integer $n$ such that $\beta = n\alpha$ for some $0 \neq \alpha \in H_2(M, \mathbb{Z})$.

The following is the main result of this paper:

Theorem 1.2. For any $J$ in a subset $\mathcal{J}^\infty_{\text{rigid}}(M, \omega) \subset \mathcal{J}^\infty(M, \omega)$ of second category, any given genus $g \geq 0$, and any homology class $\beta \in H_2(M, \mathbb{Z})$ which satisfies $c_1(M) \cdot \beta = 0$ and $|\beta| \leq 4$, the moduli space $\mathcal{M}_g(M, \beta; J)$ of somewhere injective $J$-holomorphic curves of genus $g$ representing the homology class $\beta$ consists of finitely many elements. Moreover, every curve in this moduli space is embedded and 4-rigid in the symplectic category.

An embedded $J$-holomorphic curve $C \subset M$ with holomorphic normal bundle $N_C$ is called $n$-rigid (in symplectic category) if for any holomorphic branched covering map $\pi : \Sigma \rightarrow C$ from a smooth Riemann surface $\Sigma$ to $C$, of degree less than or equal to $n$, there is no no-zero section $X \in \Gamma(\Sigma, \pi^* N_C)$ satisfying the linearized Cauchy-Riemann equation

$$\nabla X + J\nabla j_{\Sigma} X + (\nabla X J) d\pi j_{\Sigma} = 0,$$

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where $\nabla$ is the covariant derivative associated with a metric connection, $j_\Sigma$ is the complex structure on $\Sigma$, and the equation takes place in the vector space $\Gamma(\Sigma, \Omega^{0,1}_\Sigma \otimes J^{-1}(\pi^* N_C))$. In particular, such a curve is called rigid if it is 1-rigid, and is called super-rigid if it is $n$-rigid for all $n \in \mathbb{Z}^+$. If the Hermitian connection corresponding to $\nabla$ is so that the parallel translation commutes with $J$, equation 1 may be re-written in terms of the Nijenhuis tensor $N_J$ for $J$ as

\begin{equation}
\nabla X + J \nabla_{j_\Sigma} X + \frac{1}{4} N_J (\partial_J (\iota_C), X) = 0,
\end{equation}

where $\iota_C$ denotes the embedding of $C$ in $M$. Thus for integrable $J$, the notions of super-rigidity from [1] and symplectic super-rigidity agree.

The $n$-rigidity of a $J$-holomorphic curve $C \subset M$ implies that for any genus $g \geq 0$ and any homology class $\beta \in H_2(M, \mathbb{Z})$ which is of the form $d[C]$ for some integer $0 < d \leq n$, the compactified moduli space $\overline{M}_g(M, \beta; J)$ (of $J$-holomorphic curves $f: \Sigma \to M$ of genus $g$ representing the homology class $\beta$) has an open component which may be identified with the moduli space $\overline{M}_g(C, d[C])$ of genus $g$, degree $d$ branched covers of $C$. Thus the contribution of $C$ to the Gromov-Witten invariants $N_g(M, \beta)$ is well-defined. This contribution will be denoted by $C_g(C, d; J)$. For instance, $C_0(C, d; J)$ is equal to $1/d^3$ when $J$ is an integrable complex structure and $C = \mathbb{CP}^1$ is a rational curve with normal bundle $O(-1) \oplus O(-1)$. When $J$-holomorphic curves are super-rigid, global Gromov-Witten theory of $M$ is thus reduced to the local Gromov-Witten theory of such $J$-holomorphic curves.

The significance of super-rigid curves in the holomorphic category was observed by Bryan and Pandharipande [1] in the study of a local version of Gopakumar-Vafa conjecture which describes the Gromov-Witten invariants of a Calabi-Yau three-fold $M$ in terms of (not mathematically defined) integer valued invariants, called the Gopakumar-Vafa invariants [2]. It was observed in [11] by Pandharipande that the contribution of a super-rigid holomorphic curve to the Gromov-Witten invariants of higher genus (corresponding to the multiples of the homology class the curve represents) is independent of its normal bundle. Thus, for an ideal Calabi-Yau three-fold (where for an integrable almost complex structure $J$ and associated with each homology class and genus there are only finitely many $J$-holomorphic curves and all of them are embedded and super-rigid) the Gopakumar-Vafa conjecture is equivalent to its local version.

Using the spectral flow from the Dolbeault $\bar{\mathcal{J}}$ operator to the linearized Cauchy-Riemann operator defined by the left hand side of equation 1 one may assign a sign in $\{-1, 1\}$ to each element of $\mathcal{M}_g(\beta; J)$ for $\beta \in H_2(M, \mathbb{Z})$ with $c_1(M) \cdot \beta = 0$, provided that $J \in \mathcal{J}^{\text{rigid}}(M, \omega)$ and $|\beta| \leq 4$. Counting such curves with the assigned sign we obtain the integer numbers $e_g(\beta; J)$. 


It is an interesting question to investigate the dependence of $e_g(\beta; J)$ on the almost complex structure $J$, and possible evaluation of the local contributions $C_g(C, d; J)$, at least when $d < 5$. Note that with integrability assumption on $J$, the local contributions $C_g(C, d; J)$ are independent of the normal bundle, and may be denoted by $C_g(C, d)$ (see [1], also section [7]). If a similar conclusion could be extended to arbitrary $J \in J_{\text{rigid}}(M, \omega)$, the independence of $e_g(\beta; J)$ from $J$ would have been an immediate corollary, using a Möbius inversion formula as in [1]. However, the work of Taubes in dimension two [14] suggests that a wall-crossing phenomena may appear in this case, forcing non-trivial dependence of $e_g(\beta; J)$ on $J \in J_{\text{rigid}}(M, \omega)$.

Another obvious open direction for further investigation is the case of curves with higher divisibility of the associated homology class $\beta \in H_2(M, \mathbb{Z})$, and the question of finiteness and (super)-rigidity for such objects. The main technical issue in extending the results of the current paper is the following. If a smooth $J$-holomorphic curve $C \subset M$ is not super-rigid, there is a Riemann surface $\Sigma$ admitting an action of a finite group $\mathcal{G}$ which gives a branched covering map $\pi : \Sigma \rightarrow \Sigma/\mathcal{G} = C$ with the property that the pull-back of the normal bundle $N_C$ of $C$ to $\Sigma$ via $\pi$ admits a non-trivial section satisfying equation [1]. In general, $\mathcal{G}$ may admit large irreducible real representations that damage the transversality argument of section [11] which is the heart of our proof. Although this obstacle seems hard to overcome, the author hopes that the techniques and the setup used in this paper may be applied in other moduli problems.

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2. Moduli space of embedded Riemann surfaces

Fix the integers $k, p > 1$ and $\ell \geq k$. We will sometimes drop these integers from our notation for simplicity, and will mention them only in the final statements.

Let $(M, \omega)$ be a compact symplectic three-fold and fix a second homology class $\beta \in H_2(M, \mathbb{Z})$. Let $\overline{M}_g$ denote the Deleign-Mumford compactification of the moduli space $M_g$ of Riemann surfaces of genus $g$. Let $\overline{\mathcal{X}}$ be the fiber bundle over $\overline{M}_g$ whose fiber over $[\Sigma]$ is given by

$$\overline{\mathcal{X}}_\Sigma := \{ f : \Sigma \rightarrow M \mid f \in W^{k,p}(\Sigma, M) \& f_*[\Sigma] = \beta \},$$

and let $\mathcal{X}$ be the union of fibers over the moduli space $M_g$ of smooth curves. Here $W^{k,p}(\Sigma, M)$ denotes the Sobolev manifold of maps of type $W^{k,p}$ from the surface $\Sigma$ to the manifold $M$. Denote the subspace of $\mathcal{X}$ consisting of the somewhere injective maps by $\mathcal{X}^\circ$. Let $\mathcal{J} = \mathcal{J}^\ell(M, \omega)$ be the $C^\ell$ completion of the space of smooth almost complex structures on $M$ which
are compatible with $\omega$. Let $E$ be the vector bundle over $\mathcal{Y} = \mathcal{X} \times \mathcal{J}$ whose fiber over the pair $(f : \Sigma \to M, J)$ is the vector space

$$E_f := \Gamma^{k-1,p}(\Sigma, \Omega^{0,1}_{j_\Sigma} \otimes J f^*TM),$$

where $j_\Sigma$ is the complex structure on $\Sigma$. Here $\Omega^{0,1}_{j_\Sigma}$ denotes the vector space of $(0,1)$-forms on $\Sigma$ determined by the complex structure $j_\Sigma$, and $\Gamma^{k-1,p}$ denotes the space of sections of type $W^{k-1,p}$. The almost complex structure $J$ defines a section $\overline{\mathcal{D}}_J$ of the vector bundle $E|_{\mathcal{X} \times \{J\}}$ over $\mathcal{X} \times \{J\}$, given by $\overline{\mathcal{D}}_J(f) = df + J \circ df \circ (J^-1)$. The sections $\overline{\mathcal{D}}_J$ glue together and define a smooth section of $E \to \mathcal{Y}$ given by $\overline{\mathcal{D}}(f, J) = \overline{\mathcal{D}}_J(f)$. The following lemma is proved in [10] (see the proof of theorem 1.4):

**Lemma 2.1.** With the above notation fixed, the intersection of $\overline{\mathcal{D}}$ with the zero section of $E \to \mathcal{Y}$ is transverse.

We define:

$$\mathcal{M}_g(\beta, J) := \left\{ f : \Sigma \to M \mid \overline{\mathcal{D}}_J(f) = 0, \ f \in \mathcal{X} \right\} \subset \mathcal{X} \times \{J\}, \quad \&$$

$$\mathcal{M}_g(\beta) := \left\{ (f, J) \in \mathcal{Y} \mid \overline{\mathcal{D}}_J(f) = 0 \right\} = \bigcup_{J \in \mathcal{J}} \mathcal{M}_g(\beta, J) \subset \mathcal{Y}.$$

Thus the moduli space $\mathcal{M}_g(\beta)$ is a smooth, Banach manifold. The projection map $\mathcal{Y} = \mathcal{X} \times \mathcal{J} \to \mathcal{J}$ induces a map $\pi_{g,\beta} : \mathcal{M}_g(\beta) \to \mathcal{J}$, which will be Fredholm of index $2c_1(M,\beta)$. By Sard-Smale theorem, the set of regular values for the projection map $\pi_{g,\beta} : \mathcal{M}_g(\beta) \to \mathcal{J}$ is a subset $\mathcal{J}_{\text{reg}}(g, \beta) \subset \mathcal{J}$, which is of second category. For every $J \in \mathcal{J}_{\text{reg}}(g, \beta)$, $\mathcal{M}_g(\beta, J) = \pi^{-1}_{g,\beta}(J)$ is a smooth manifold whose dimension is equal to the index of the projection map $\pi_{g,\beta}$. By the regularity of $J$, $\overline{\mathcal{D}}_J : \mathcal{X} \times \{J\} \to E$ intersects the zero section transversely over $\mathcal{M}_g(\beta, J)$. Set

$$\mathcal{J}_{\text{reg},0} := \bigcap_{g \geq 0} \bigcap_{\beta \in H^2(M,\mathbb{Z})} \bigcap_{c_1(M,\beta) = 0} \mathcal{J}_{\text{reg}}(g, \beta).$$

Let $T_J$ be the tangent space to $\mathcal{J}$ at $J$, consisting of the linear homomorphisms $u : TM \to TM$ which satisfy

$$u \circ J + J \circ u = 0 \quad \& \quad \omega_{x}(u(X), Y) + \omega_{x}(Y, u(Y)) = 0, \ \forall X, Y \in T_xM.$$

The derivative of $\overline{\mathcal{D}}_J$ at a zero $(f, J)$ of this section is a linear map

$$d\overline{\mathcal{D}} : T_J \mathcal{X} \oplus T_J \to T_{(f,J)} \mathcal{Y} = T_f \mathcal{X} \oplus T_J \oplus E_f.$$

Projection over the last factor in this decomposition, composed with the differential $d\overline{\mathcal{D}}$, gives a linear operator (after multiplication by 2)

$$L : T_J \mathcal{X} \oplus T_J = \Gamma^{k,p}(f^*TM) \oplus H^1(T\Sigma) \oplus T_J \to E_f = \Gamma^{k-1,p}(\Omega^{0,1}_{\Sigma} \otimes J f^*TM).$$

The intersection of $\overline{\mathcal{D}}$ and the zero section at $(f, J)$ is transverse if and only if this linear map $L$ is surjective. If $\nabla$ denotes the Levi-Civita connection of
the metric on \(M\) (defined by \(\langle X, Y \rangle = w(X, JY)\)), we may write down an explicit formula for this linear map

\[
\mathbf{L}(X, \eta, u) = \nabla X + J\nabla_j X + (\nabla_X J) df_j + J df \eta + u df_j
\]

\[=: \mathcal{L}(X, \eta) + u df_j =: \mathcal{L}(X, \eta) + \mathcal{L}(u).
\]

Note that \(\mathcal{L} = \mathcal{L}_{(f, J)}\) is in fact a linear operator

\[
\mathcal{L} : \Gamma^{k,p}(f^*TM) \oplus H^1(T_\Sigma) \longrightarrow \Gamma^{k-1,p}(\Omega^{0,1}_\Sigma \otimes J f^*T'M),
\]

which is Fredholm of index \(2c_1(M, \beta)\). By integrating the point-wise Hermitian inner product of \(\Omega^{0,1}_\Sigma \otimes f^*T'M\) we obtain a Hermitian inner product on \(\Gamma^{k-1,p}(\Omega^{0,1}_\Sigma \otimes f^*T'M)\). Similarly, we have an inner product on \(\Gamma^{k,p}(f^*TM) \oplus H^1(T_\Sigma)\). Using these inner products we may define an adjoint operator for \(\mathcal{L}\)

\[
\mathcal{L}^*: \Gamma^{k,q}(\Omega^{0,1}_\Sigma \otimes J f^*T'M) \longrightarrow \Gamma^{k-1,q}(f^*T'M) \oplus H^1(T_\Sigma),
\]

where \(q\) satisfies \(\frac{1}{p} + \frac{1}{q} = 1\). It follows from elliptic regularity that every section \(\delta \in \Gamma^{k-1,p}(\Sigma, \Omega^{0,1}_\Sigma \otimes J f^*T'M)\) which is annihilated by the image of the operator \(\mathcal{L}\) in the sense that

\[
\langle \mathcal{L}(X, \eta), \delta \rangle = 0, \quad \forall X \in \Gamma^{k-1,p}(\Sigma, f^*T'M), \quad \eta \in H^1(\Sigma, T_\Sigma),
\]

is automatically in the kernel of \(\mathcal{L}^*\), and is of class \(W^{k,q'}\) for all values of \(q'\). Thus, the cokernel of \(\mathcal{L}\) may be identified with the kernel of \(\mathcal{L}^*\). Also, note that if \(f : \Sigma \to M\) is in \(\mathcal{M}_g(\beta, J)\), then it is at least of class \(C^{\ell + 1}\) by elliptic regularity, and the moduli space \(\mathcal{M}_g(\beta, J)\) is thus independent of \(k\).

The following is one of the main results of [10] (theorem 1.4):

**Proposition 2.2.** For any \(J\) in a subset \(\mathcal{J}_{reg}^l(M, \omega) \subset \mathcal{J}^l(M, \omega)\) of second category the following is true for any \(\beta \in H_2(M, \mathbb{Z})\). If \(c_1(M, \beta) < 0\) the moduli space \(\mathcal{M}_g(J, \beta)\) is empty, while for \(c_1(M, \beta) = 0\) \(\mathcal{M}_g(J, \beta)\) consists of embeddings. Moreover, if \(c_1(M, \beta) = c_1(M, \beta') = 0\) and if \(f \in \mathcal{M}_g(\beta, J)\) and \(f' \in \mathcal{M}_g(\beta', J)\) have distinct images, then their images are disjoint.

Fix \(J \in \mathcal{J}_{reg}^l(M, \omega) \cap \mathcal{J}^\infty(M, \omega)\). For the moment, we are not worried about the existence of such almost complex structures, and will return to this issue in section 6. Let \(\beta \in H_2(M, \mathbb{Z})\) be a homology class with \(c_1(M, \beta) = 0\). The existence of a sequence of embedded \(J\)-holomorphic curves \(f_i : \Sigma_i \to M\) in \(\mathcal{M}_g(\beta, J)\) implies that the domains converge to a possibly singular curve \(\Sigma \in \overline{\mathcal{M}}_g\), and that a subsequence of \(f_i\) will converge to a \(J\)-holomorphic map \(f : \Sigma \to M\), by Gromov compactness theorem. The homology class \(\alpha \in H_2(M, \mathbb{Z})\) associated with the map \(f\) may be different from \(\beta\). The class \(\beta - \alpha\) will be represented by \(J\)-holomorphic bubbles (i.e. \(J\)-holomorphic maps from \(\mathbb{C} \mathbb{P}^1\) to \(M\)). Thinking of these bubbles as the components of \(\Sigma\) and dropping the stability condition on the domain, we may assume that \(f_i(\Sigma) = \beta\). The normalization of the domain \(\Sigma\) will be a union \(\Sigma^1 \cup \Sigma^2 \cup \ldots \cup \Sigma^k\) of smooth components. The map \(f\) induces a \(J\)-holomorphic
map \( f^i : \Sigma^i \to M \) on each component \( \Sigma^i \) which decomposes as \( f^i = i^i \circ \pi^i \), where \( i^i \) is a somewhere injective map from a domain \( C^i \) of some genus \( h_i \) to \( M \) and \( \pi^i : \Sigma^i \to C^i \) is a branched covering (which may be \( Id_{\Sigma^i} \)). When \( \Sigma^i \) is a collapsed component, we will assume that \( C^i = \Sigma^i \), \( i^i : C^i \to M \) is the collapsing (constant) map, and \( \pi^i \) is the identity map of \( \Sigma^i = C^i \).

If the homology class represented by \( i^i \) is \( \alpha_i \), \( i^i \) would be in \( M_{h_i}(\alpha_i, J) \). Since this moduli space is non-empty, by proposition 2.2 we would have \( c_1(M) \cdot \alpha_i \geq 0 \). Since \( \beta = m_1 \alpha_1 + \ldots + m_k \alpha_k \) for positive integers \( m_i \in \mathbb{Z}^+ \), \( i = 1, \ldots, k \) and \( c_1(M) \cdot \beta = 0 \), we conclude that \( c_1(M) \cdot \alpha_i = 0 \) for \( i = 1, \ldots, k \). Proposition 2.2 implies that the images of \( i^i \) are disjoint or identical. Since \( \Sigma \) is connected, there is a \( J \)-holomorphic curve \( C \) in \( M \) such that each map \( i^i \) is in fact the embedding of \( C^i \simeq C \) in \( M \). Thus the map \( f : \Sigma \to M \) may be regarded as the composition of a branched covering map \( \pi : \Sigma \to C \) (which is equal to \( \pi^i \) on the component \( \Sigma^i \)) with the \( J \)-holomorphic embedding \( \iota_C : C \to M \).

A neighbourhood of \( C \) in \( M \) may be (symplectically) identified with a neighbourhood of the zero section in its normal bundle \( N_C \). For \( m \) sufficiently large we may assume that the image of \( f_m \) is included in this neighbourhood. We may thus work in \( N_C \) and assume that the images of all \( f_m \) are in the unit disk bundle around the zero section in \( N_C \) (implicitly, we are fixing a metric on the normal bundle \( N_C \)).

For any constant \( \epsilon > 0 \), let \( \chi_\epsilon : \mathbb{R} \to [0, 1] \) be a smooth non-decreasing function with \( \chi_\epsilon(x) = \epsilon \) for \( x \leq 1 \) and \( \chi_\epsilon(x) = 1 \) for \( x \geq 2 \). Furthermore, assume that \( \chi_\epsilon \) varies smoothly with \( \epsilon \) for \( 0 \leq \epsilon \leq 1 \). Let \( F_\epsilon : N_C \to N_C \) be the diffeomorphism defined by

\[
F_\epsilon(z; X) := (z; \chi_\epsilon(\|X\|^2).X), \quad \forall \ z \in C \ & \ X \in (N_C)_z.
\]

Let \( J_\epsilon \) denote the almost complex structure \( F_\epsilon^*J \) on \( N_C \). For any embedding \( f_m \) as above, let \( \epsilon(m) \) denotes the supremum norm of \( f_m \) as a multi-section of \( N_C \). The embedding \( f_m \) may then be composed with \( F_{\epsilon(m)}^{-1} \) to obtain a \( J_{\epsilon(m)} \)-holomorphic embedding of \( \Sigma_m \) in \( N_C \), which will be denoted by \( h_m : \Sigma_m \to N_C \). By the above construction, the multi-sections \( h_m \) will have supremum norm equal to 1.

As the sequence \( \epsilon(m) \) converges to zero with \( m \) going to infinity, the almost complex structures \( J_{\epsilon(m)} \) converge to a limit denoted by \( J' \). Gromov compactness theorem then tells us that the sequence \( h_m : \Sigma_m \to N_C \) converges to a limit \( h : \Sigma \to N_C \), which is \( J' \)-holomorphic. The composition of \( h \) with the projection of \( N_C \) over \( C \) is the branched covering map \( \pi : \Sigma \to C \), and the map \( h \) is determined by a section \( X \in \Gamma(\Sigma, \pi^*N_C) \). The section \( X \)
is in fact a union of sections
\[ X^j \in \Gamma^\infty (\Sigma^j, (\pi^j)^*N_C) \subset \Gamma^{k,p} (\Sigma^j, (\pi^j)^*N_C), \quad j = 1, \ldots, k. \]

The map \( h \) is \( J^i \)-holomorphic if and only if for \( j = 1, \ldots, k \), the section \( X^j \) satisfies the following equation
\[ K_{\pi^j}(X^j) := \nabla X^j + J \nabla_{j_{\Sigma^j}} X^j + (\nabla_{X^j} J) \, d\pi^j_{j_{\Sigma^j}} = 0 \]
in \( \Gamma^{k-1,p}(\Sigma^j, \Omega^{0,1}_{\Sigma^j} \otimes J (\pi^j)^*N_C) \). This means that \( X^j \) gives an element in the kernel of the operator
\[ K_{\pi^j} : \Gamma^{k,p}(\Sigma^j, (\pi^j)^*N_C) \rightarrow \Gamma^{k-1,p}(\Sigma^j, \Omega^{0,1}_{\Sigma^j} \otimes (\pi^j)^*N_C). \]

Thus the existence of a sequence as above which converges to an embedded \( J \)-holomorphic curve \( C \) of genus \( h \leq g \) implies that there is a branched covering map \( \pi_S : S \rightarrow C \) (with \( S = \Sigma^j \) and \( \pi_S = \pi^j \) for some \( j \)), and a non-trivial section \( X \) in the kernel of the elliptic operator
\[ K_{\pi_S} : \Gamma^{k,p}(S, \pi_S^*N_C) \rightarrow \Gamma^{k-1,p}(S, \Omega^{0,1}_S \otimes J (\pi_S)^*N_C). \]

Based on this observation, we make the following definition, which should be compared with the concepts of rigidity and super-rigidity in the integrable case by Bryan and Pandharipande \[1\].

**Definition 2.3.** Let \( J \in \mathcal{J}^\infty(M, \omega) \), and \( C \) be a smooth embedded \( J \)-holomorphic curves of genus \( h \geq 0 \) in \( M \) with normal bundle \( N_C \). The curve \( C \) is called \((m, n)\)-rigid for the integers \( m \geq 0 \) and \( n > 0 \), if for every branched covering map \( \pi_S : S \rightarrow C \) from a smooth Riemann surface \( S \) whose genus \( g \) satisfies \( g \leq m + h \), and with \( \deg(\pi_S) \leq n \), the kernel of the linear operator
\[ K_{\pi_S} : \Gamma^{k,p}(S, \pi_S^*N_C) \rightarrow \Gamma^{k-1,p}(S, \Omega^{0,1}_S \otimes J (\pi_S)^*N_C) \]

is trivial. For a positive integer \( n > 0 \), the curve \( C \) is called \( n \)-rigid if it is \((m, n)\)-rigid for all \( m \in \mathbb{Z}^{\geq 0} \). The curve \( C \) is called super-rigid if it is \( n \)-rigid for every integer \( n > 0 \). This definition does not depend on the particular choice of the integers \( k, p \geq 2 \) by elliptic regularity. In a similar way, we may define the notions of \( n \)-rigidity and super-rigidity if \( J \) is an almost complex structure of class \( C^\ell \) and the embedding of \( C \) in \( M \) is of class \( W^{k,p} \) for some integers \( k, p \geq 2 \) such that \( k \leq \ell \).

Our discussion in this section implies that for an almost complex structure \( J \) in \( \mathcal{J}^\ell_{\text{reg}}(M, \omega) \cap \mathcal{J}^\infty(M, \omega) \), if an embedded \( J \)-holomorphic curve \( C \in \mathcal{J}^\ell_{\text{reg}}(M, \omega) \cap \mathcal{J}^\infty(M, \omega) \), then the curve \( C \) is \((m, n)\)-rigid for all \( m \in \mathbb{Z}^{\geq 0} \) and \( n > 0 \).
\( \mathcal{M}_{h}(M, \alpha, J) \) is the limit of a sequence of embedded \( J \)-holomorphic curves \( f_{i} : \Sigma_{i} \to M \) representing the homology class \( \beta \) (which is forced to be of the form \( n\alpha \)) in the sense that these curves converge to a multiple cover \( \Sigma \) of \( C \) of some genus \( g = h + m \), then \( C \) is not \( (m, n) \)-rigid.

3. Rigidity; a few reductions

In this section, we will study a few possible reductions of the concept of rigidity, which would be useful in our later considerations.

**Reduction 1.** Suppose that \( \pi : \Sigma \to C \) is an arbitrary branched covering map between smooth Riemann surfaces. Suppose that \( B \in \text{Div}(\Sigma) \) is the branching divisor of \( \pi \), and that \( \pi(B) = \{q_{1}, ..., q_{l}\} \subset C \) is the set of critical values of \( \pi \). Fix \( q \in C - \pi(B) \) and let \( p_{1}, ..., p_{n} \) denote all the points with \( \pi(p_{i}) = q \). The group \( \pi_{1}(C - \pi(B), q) \) acts on \( \pi^{-1}(q) = \{p_{1}, ..., p_{n}\} \) as follows. Suppose that \( \gamma : [0, 1] \to C - \pi(B), \ \gamma(0) = \gamma(1) = q \) is a loop representing an element \([\gamma] \in \pi_{1}(C - \pi(B), q)\). For \( i = 1, ..., n \), we may find the lift \( \gamma_{i} \) of \( \gamma \) under the covering map \( \pi \) such that

\[
\gamma_{i} : [0, 1] \to \Sigma - B, \ \gamma_{i}(0) = p_{i}, \ \pi \circ \gamma_{i} = \gamma.
\]

We may then define the action of \([\gamma] \) on \( \{p_{1}, ..., p_{n}\} \) by

\[
[\gamma] \star p_{i} := \gamma_{i}(1) \in \pi^{-1}(q) = \{p_{1}, ..., p_{n}\}, \ \forall \ i \in \{1, ..., n\}.
\]

This gives a homomorphism

\[
\rho : \pi_{1}(C - \pi(B), q) \to S_{n}
\]

from the fundamental group of \( C - \pi(B) \) to the group \( S_{n} \) of permutations in \( n \) letters. The kernel \( \text{Ker}(\rho) \) of this homomorphism, which is a normal subgroup of the fundamental group, determines a finite regular covering map \( \pi^{\circ} : \Sigma^{\circ} \to C - \pi(B) \), which may be extended to a branched covering map \( \tilde{\pi} : \tilde{\Sigma} \to C \) from a compact Riemann surface \( \tilde{\Sigma} \) to \( C \). Moreover, this map decomposes as \( \tilde{\pi} = \pi \circ \tau \) for some branched covering map \( \tau : \tilde{\Sigma} \to \Sigma \). Since \( \text{Ker}(\rho) \) is normal in the fundamental group, the group of deck transformations for \( \pi^{\circ} \) may be computed as

\[
\text{Dec}(\pi^{\circ}) = \frac{\pi_{1}(C - \pi(B), q)}{\pi_{1}(\Sigma^{\circ}, \pi^{\circ})} \simeq \frac{\pi_{1}(C - \pi(B), q)}{\text{Ker}(\rho)} \simeq \text{Im}(\rho) < S_{n}.
\]

Denote this later subgroup of \( S_{n} \) by \( \mathfrak{G} \). Note that the degree of \( \tilde{\pi} \) is equal to the order of the finite group \( \mathfrak{G} \). Any deck transformation of \( \Sigma^{\circ} \) may be extended to \( \tilde{\Sigma} \) as an automorphism with possible fixed points, and \( C \) may be realized as the quotient of \( \tilde{\Sigma} \) under the action of \( \mathfrak{G} < \text{Aut}(\tilde{\Sigma}) \).

Now suppose that the curve \( C \) is an embedded \( J \)-holomorphic curve in \( M \) for some almost complex structure \( J \in \mathcal{J}^{\infty}(M, \omega) \), and let \( N_{C} \) denote
the normal bundle of $C$. Fix the branched covering map $\pi : \Sigma \to C$ and consider the operator

$$\mathcal{K}_\pi : \Gamma^{k,p}(\Sigma, \pi^* N_C) \to \Gamma^{k-1,p}(\Sigma, \Omega^{0,1}_\Sigma \otimes J \pi^* N_C)$$

introduced earlier. If $X \in \text{Ker}(\mathcal{K}_\pi)$ is non-trivial, then $\tau^* X \in \text{Ker}(\mathcal{K}_\pi)$ is non-trivial as well. Thus, if an embedded $J$-holomorphic curve $C$ is not rigid, there is a Riemann surface $S = \Sigma$ admitting an action of a subgroup $\mathcal{G}$ of $S_n$, with $\pi_S : S \to C = S/\mathcal{G}$ the corresponding branched covering map, so that the kernel of $\mathcal{K}_{\pi_S}$ is non-trivial.

**Reduction 2.** Let us now assume that $\pi = \pi_S : S \to S/\mathcal{G} = C$ is a branched covering map coming from the action of a finite group, and assume that $X \in \text{Ker}(\mathcal{K}_\pi)$ is a non-trivial section. For any $\sigma \in \mathcal{G}$, it is then clear that $\sigma^* X$ is also in $\text{Ker}(\mathcal{K}_\pi)$. Thus the action of the group ring $\mathbb{R}\mathcal{G}$ on $\Gamma^{k,p}(S, \pi^* N_C)$ defined by

$$a.Y := \sum_{\sigma \in \mathcal{G}} a_\sigma \sigma^* Y, \quad \forall a = \sum a_\sigma \sigma^{-1} \in \mathbb{R}\mathcal{G}, \quad \&\ Y \in \Gamma^{k,p}(S, \pi^* N_C)$$

induces an action of $\mathbb{R}\mathcal{G}$ on $\text{Ker}(\mathcal{K}_\pi)$. In other words, if $X \in \text{Ker}(\mathcal{K}_\pi)$ and $a \in \mathbb{R}\mathcal{G}$, then $a.X \in \text{Ker}(\mathcal{K}_\pi)$. Let us denote by $m_X$ the left ideal of $\mathbb{R}\mathcal{G}$ consisting of the elements $a \in \mathbb{R}\mathcal{G}$ such that $a.X = 0$. Let $I(\mathcal{G})$ be the set of irreducible real representations $\iota$ of $\mathcal{G}$. The group ring $\mathbb{R}\mathcal{G}$ may then be decomposed (by Artin-Wedderburn and Maschke theorems) as

$$(5) \quad \mathbb{R}\mathcal{G} \simeq \bigoplus_{\iota \in I(\mathcal{G})} M_{\ell(\iota)}(R_{\iota}), \quad R_{\iota} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\},$$

where $M_{\ell}(R)$ denotes the space of $\ell \times \ell$ matrices with entries in the ring $R$, $\ell(\iota)$ is the dimension of the representation $\iota$ as an algebra over $R_{\iota}$, and $\mathbb{H}$ denotes the ring of quaternions. Let us denote by $t_\iota \in \mathbb{R}\mathcal{G}$ the identity matrix in the matrix algebra associated with the representation $\iota$. We thus have

$$1_{\mathbb{R}\mathcal{G}} = \sum_{\iota \in I(\mathcal{G})} t_\iota \implies X = \sum_{\iota \in I(\mathcal{G})} t_\iota X =: \sum_{\iota \in I(\mathcal{G})} X_\iota.$$

If $X$ is in $\text{Ker}(\mathcal{K}_\pi)$, so are $X_\iota$. Moreover, at least one of the sections $X_\iota$ is non-zero. Fix such a representation $\iota \in I(\mathcal{G})$ and let $R = R_{\iota}$ and $\ell = \ell(\iota)$. Let $\epsilon_i$, $i = 1, \ldots, \ell$ denotes the matrix in $M_{\ell}(R)$ with 1 as its $(i,i)$ entry and zeros elsewhere. Note that we may think of $\epsilon_i$ as an element of $\mathbb{R}\mathcal{G}$. Since

$$0 \neq X_\iota = t_\iota X = (\epsilon_1 + \ldots + \epsilon_\ell).X,$$

at least one of the sections $\epsilon_i.X$ is a non-zero element in $\text{Ker}(\mathcal{K}_\pi)$. Denote one such section by $Y$. The left ideal $m_Y$ is then a maximal left ideal $m_{t_\iota}$ which consists of all elements of $\mathbb{R}\mathcal{G}$ which are characterized using the presentation of equation $5$ as those matrices which have zeros in the $i$-th
column in the matrix presentation corresponding to the representation $i$. The maximal left ideal $m$ determines a restriction of the operator $\mathcal{K}_\pi$:

$$
\mathcal{K}_\pi^m : \Gamma_{m}^{k,p}(S, \pi^*N_C) \longrightarrow \Gamma_{m}^{k-1,p} \left( S, \Omega_S^{0,1} \otimes \pi^*N_C \right).
$$

Here $\Gamma_{m}^{k}(S, \star)$ denotes the vector space of sections $Z$ of the bundle $\star$ such that the left ideal $mZ$ contains $m$. The section $Y$ is then in the kernel of the operator $\mathcal{K}_\pi$. Note that associated with any covering map $\pi : S \rightarrow C = S/\mathcal{G}$ as above, there are finitely many maximal left ideals of the form $m = m_i$.

We define $\ell(m) = \ell(i)$ and $R_m = R_i$. We denote the finite set of such maximal left ideals by $I(\pi)$.

The kernel $\text{Ker}(\rho)$ of the representation $\rho : \mathcal{G} \rightarrow M_\ell(R)$ is a normal subgroup of $\mathcal{G}$ and determines a degeneration of the covering map $\pi : S \rightarrow C = S/\mathcal{G}$ as a composition

$$
S \xrightarrow{\pi_1} S' = S/\text{Ker}(\rho) \xrightarrow{\pi_2} C = S/\mathcal{G} = S'/\text{Im}(\rho).
$$

Moreover, since the elements of $\text{Ker}(\rho)$ preserve the section $Y$ of the bundle $\pi^*N_C$, $Y$ should be of the form $\pi_1^*Z$ for some $Z \in \Gamma_{m}^{k,p}(S', \pi_2^*N_C)$. Clearly, $Z \in \text{Ker}(\mathcal{K}_{\pi_2})$ is non-trivial, since $Y$ is non-trivial. We may thus replace $S$ by $S'$, $\mathcal{G}$ by $\mathcal{G}' = \text{Im}(\rho)$, and the section $Y$ by $Z$. The above considerations imply that if an embedded $J$-holomorphic curve $C$ is not $n$-rigid, then there is a Riemann surface $S$ admitting an action of a finite group $\mathcal{G} < S_n$ (corresponding to a branched covering map $\pi : S \rightarrow C = S/\mathcal{G}$), and a maximal left ideal $m \in I(\pi)$ of the group ring $\mathbb{R}_\mathcal{G}$, so that the kernel of the operator

$$
\mathcal{K}_\pi^m : \Gamma_{m}^{k,p}(S, \pi^*N_C) \longrightarrow \Gamma_{m}^{k-1,p} \left( S, \Omega_S^{0,1} \otimes \pi^*N_C \right)
$$

is non-trivial. Moreover, the representation

$$
\rho : \mathcal{G} \rightarrow M_\ell(R), \quad \ell = \ell(m) \& R = R_m \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \}
$$

is faithful.

In the above situation, for any $Y \in \Gamma_{m}^{k,p}(S, \pi^*N_C)$, we introduce the formal sum

$$
\alpha(Y) := \sum_{\sigma \in \mathcal{G}} (\sigma^*Y) \sigma.
$$

Any element $a = \sum_{\sigma \in \mathcal{G}} a_\sigma \sigma^{-1}$ of the group ring $\mathbb{R}_\mathcal{G}$ acts on any such expression from left and right by

$$
a \odot \alpha(Y) := \sum_{\sigma, \tau \in \mathcal{G}} a_\sigma (\tau^* Y) \sigma^{-1} \tau, \quad \alpha(Y) \odot a := \sum_{\sigma, \tau \in \mathcal{G}} a_\sigma (\tau^* Y) \tau \sigma^{-1}.
$$
From the above two definitions we may compute
\[ a \odot a(Y) = \sum_{\sigma, \tau \in \mathfrak{G}} a_{\sigma} ((\sigma \tau)^* Y) . \tau = \sum_{\tau \in \mathfrak{G}} \tau^* (a.Y) . Y = a(a.Y), \quad \& \]
\[ \tau^* \alpha(Y) = \tau^* \left( \sum_{\sigma \in \mathfrak{G}} \sigma^* Y. \sigma \right) = \sum_{\sigma \in \mathfrak{G}} \tau^* \sigma^*(Y).\sigma = \alpha(Y) \odot \tau^{-1} \quad \forall \tau \in \mathfrak{G}. \]

**Reduction 3.** In order to study 4-rigidity, as observed in the first reduction, we only need to consider the case where the group \( \mathfrak{G} \) is isomorphic to a subgroup of \( S_4 \). The subgroups of \( S_4 \) are isomorphic to one of the following groups:

\[ \mathbb{Z}/k\mathbb{Z}, \quad k = 1, 2, 3, 4, \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \quad S_3, \quad D_8, \quad A_4 \quad \& \quad S_4. \]

Except for the last 4 in this list, the other groups are abelian and the matrix algebras corresponding to their irreducible real representations are either \( M_1(\mathbb{R}) \) or \( M_1(\mathbb{C}) \). The group \( A_4 \) has 3 real irreducible representations with the corresponding matrix algebras equal to \( M_1(\mathbb{R}), M_1(\mathbb{C}) \) and \( M_3(\mathbb{R}) \). For \( D_8 \), the dihedral group, we may distinguish 4 one dimensional irreducible representations with the associated matrix algebra equal to \( M_1(\mathbb{R}) \), and one two dimensional irreducible representation with the corresponding matrix algebra equal to \( M_2(\mathbb{R}) \). Finally for the groups \( S_3 \) and \( S_4 \) the associated matrix algebra of any irreducible real representation is one of \( M_1(\mathbb{R}), M_2(\mathbb{R}) \) and \( M_3(\mathbb{R}) \). These observations imply that if \( \mathfrak{G} \) is a subgroup of \( S_4 \) acting on a Riemann surface \( S \), and if \( \pi \) denotes the branched covering map \( S \to C = S/\mathfrak{G} \), then for any \( m \in I(\pi) \) the associated matrix algebra is one of \( M_i(\mathbb{R}), \quad i = 1, 2, 3 \) or \( M_1(\mathbb{C}) = \mathbb{C} \).

4. Moduli space of multiply covered curves

Once again, fix the integers \( k, p > 1 \) and \( \ell \geq k \) throughout this section. For \( J \in \mathcal{J}_{reg}^\ell (M, \omega) \) let \( C \in \mathcal{M}_h(M, \beta; J) \) be a \( J \)-holomorphic curve of genus \( h \) and class \( W^{k,p} \) (and hence automatically of class \( W^{\ell+1,p} \)) in \( M \) which represents a homology class \( \alpha \in H_2(M, \mathbb{Z}) \) with \( c_1(M).\alpha = 0 \). Suppose that \( \pi : \Sigma \to C \) is a branched covering map, and \( B \in \text{Div}(\Sigma) \) is the branching divisor. Let \( \{q_1, \ldots, q_l\} \) be the set of points in the image of \( B \) (i.e. the set of critical values for \( \pi \)). When we move these \( l \) punctures on \( C \), the complex structure on the complement of \( q_1, \ldots, q_l \) on the Riemann surface \( C \) determines a corresponding complex structure on \( \Sigma \). This gives a family of branched covering maps from Riemann surfaces of genus \( g = q \Sigma \) to \( C \) with the same monodromy data as the original branched covering map \( \pi \). Note that some branched covering maps may have multiple representatives in this description due to automorphisms. This determines an open family of covers of \( C \), which has dimension \( l \) (note that the embedded curve \( C \) is parametrized). Moreover, as \( (C, J) \) moves in \( \mathcal{M}_h(\alpha) \) the above open families glue together and form a moduli space which will be denoted by
\(M_\pi(\beta)\), with \(\beta = \deg(\pi).\alpha\). Any point in this moduli space corresponds to a branched covering map with the same branching behaviour and monodromy information as the initial map \(\pi\). We will say in short that any branched covering map in this family has the same topological type as \(\pi\). Having fixed the target genus \(h\), the domain genus \(g \geq h\) and the degree \(n\), there are only finitely many topological types of branched covering maps of degree \(n\) from a smooth domain of genus \(g\) to a smooth target of genus \(h\). If the initial branched covering map \(\pi\) comes from the action of a group \(\mathcal{G}\) on \(\Sigma\) (so that \(\pi : \Sigma \to C = \Sigma/\mathcal{G}\) is the quotient map), it is clear that all the maps of the same topological type as \(\pi\) come from an action of \(\mathcal{G}\), since the action may be regarded as the action of deck transformations of the honest covering map obtained by removing the critical values from \(C\) and their pre-images from \(\Sigma\). In this situation we say that the topological type of the group action for other branched covering maps in the family is the same as the topological type of the group action for \(\mathcal{G}\) and the branched covering map \(\pi : \Sigma \to C = \Sigma/\mathcal{G}\).

Fix the topological type of the group action and the corresponding branched covering map \(\pi : \Sigma \to C = \Sigma/\mathcal{G}\) as above. Let \(n = \deg(\pi) = |\mathcal{G}|, \beta = n\alpha \in H_2(M,\mathbb{Z})\), and assume that \(g\) denotes the genus of \(\Sigma\). The moduli space \(M_\pi(\beta)\) is then fibered over \(M_h(\alpha)\), where the fiber over \((\iota_C : C \to M, J)\) consists of the \(l\) dimensional family of branched covering maps with target \(C\) which are of the same topological type as \(\pi\). The moduli space \(M_\pi(\beta)\) is a smooth Banach manifold and the fibration \(q_\pi : M_\pi(\beta) \to M_h(\alpha)\) is a Fredholm map of index \(l\). We will denote the composition of \(q_\pi\) with the projection map from \(M_h(\alpha)\) to \(\mathcal{F}(M,\omega)\) by \(\Pi_\pi\), which is again a Fredholm map of index \(l\). Consider the subset \(M_\pi^0(\beta)\) of \(M_\pi(\beta)\) consisting of the points of the form \(\phi = (\pi : \Sigma \to C, \iota_C : C \to M, J)\) such that \(\iota_C\) is an embedding. This open subset of \(M_\pi(\beta)\) fibers over the subset \(M_h^0(\alpha)\). Abusing the notation, we re-define the bundles \(\mathcal{E}\) and \(\mathcal{F}\) over \(M_\pi^0(\beta)\) by defining the fibers at a point \(\phi = (\pi : \Sigma \to C, \iota_C : C \to M, J) \in M_\pi^0(\beta)\) by

\[
\mathcal{F}_\phi = \Gamma^{k,p}(\Sigma, \pi^*N_C), \quad \mathcal{E}_\phi = \Gamma^{k-1,p}(\Sigma, \Omega^{0,1}_\Sigma, \otimes_J N_C).
\]

The bundle \(\mathcal{E}\) may be pulled back over \(\mathcal{F}\) using the projection map \(\mathcal{F} \to M_\pi^0(\beta)\). We abuse the notation and will denote this pull back by \(\mathcal{E}\) as well. Then the operator \(\mathcal{K}\) defines a section of \(\mathcal{E} \to \mathcal{F}\).

For a fixed maximal left ideal \(m \in I(\pi)\) of \(\mathbb{R}\) let \(\mathcal{F}^m\) denote the subbundle of \(\mathcal{F}\) consisting of the points \((\phi; X)\) with \(\phi \in M_\pi^0(\beta)\) and \(X \in \mathcal{F}_\phi\) such that \(m \subset m_X\). Let \(\mathcal{F}^m \subset \mathcal{F}^m\) denote the subspace which consists of those \((\phi; X)\) with \(m = m_X\). Since \(m\) is maximal, \(\mathcal{F}^m\) is the complement of \((\phi; 0)\) in \(\mathcal{F}_\phi\). Note that the later vector space may be identified with \(\Gamma^k_m(\Sigma, \pi^*N_C)\). Let \(\mathcal{E}^m\) be the sub-bundle of \(\mathcal{E}|_{\mathcal{F}^m} \to \mathcal{F}^m\) consisting of the tuples \((\phi; X; \delta)\) with \((\phi; X) \in \mathcal{F}^m\) and \(\delta \in \mathcal{E}_\phi\) so that \(m \subset m_\delta\). This gives
a vector bundle $\mathcal{E}^m \to \mathcal{F}^m$ and a section $\mathcal{K}^m : \mathcal{F}^m \to \mathcal{E}^m$. For a point $(\phi; X) \in \mathcal{F}^m$ with

$$\phi = (\pi : \Sigma \to C, \iota_C : C \hookrightarrow M, J) \in \mathcal{M}^0_\pi(\beta), \quad X \in \Gamma_m^{k,p}(\Sigma, \pi^*N_C)$$

the section $\mathcal{K}^m$ is defined by

$$\mathcal{K}^m(\phi; X) = \mathcal{K}^m_\pi(X) \in \Gamma_m^{k-1,p}(\Sigma, \Omega^{0,1}_\Sigma \otimes J \pi^*N_C) = \mathcal{E}^m_{\phi(X)}.$$

**Proposition 4.1.** With the above notation fixed, for any maximal left ideal $m$ of $\mathbb{R}_\phi$ such that the associated irreducible representation is faithful and the corresponding matrix algebra is either $M_i(\mathbb{R})$ for $i = 1, 2, 3$ or $M_1(\mathbb{C}) = \mathbb{C}$, the intersection of $\mathcal{K}^m : \mathcal{F}^m \to \mathcal{E}^m$ with the zero section of the vector bundle $\mathcal{E}^m \to \mathcal{F}^m$ is transverse.

**Proof.** Suppose that $\phi = (\pi, \iota_C, J)$ is a point of $\mathcal{M}^0_\pi(\beta)$ and that $(\phi; X)$ is a point of $\mathcal{F}^m$ such that $\mathcal{K}^m(\phi; X) = 0$. We should show that the differential of $\mathcal{K}^m$, projected over the fiber, defines a surjective operator

$$d\mathcal{K}^m : T_{(\phi; X)}\mathcal{F}^m \simeq T_{\phi}\mathcal{M}_\pi(\beta) \oplus \overline{\mathcal{F}}^m_\phi \to \mathcal{E}^m_{\phi}.$$ 

The tangent space $T_{\phi}\mathcal{M}_\pi(\beta)$ has a subspace which consists of the elements $u$ of $T_J = T_J\mathcal{J}^\ell(\Sigma, \omega)$ such that $u|_{\text{Im}(\iota_C)} = 0$ (and the tangent vector is trivial in the direction of $\mathcal{X}$). Denote this subspace by $\mathcal{H}_J(\iota_C)$. The restriction of $d\mathcal{K}^m$ to $\mathcal{H}_J(\iota_C) \oplus \overline{\mathcal{F}}^m_\phi$ may be easily computed:

$$d\mathcal{K}^m : \mathcal{H}_J(\iota_C) \oplus \Gamma_m^{k,p}(\Sigma, \pi^*N_C) \to \Gamma_m^{k-1,p}(\Sigma, \Omega^{0,1}_\Sigma \otimes J \pi^*N_C)$$

$$d\mathcal{K}^m(u, Y) = \mathcal{K}^m(Y) + (\nabla_X u).d\pi.j.$$ 

The following lemma then completes the proof of this proposition. 

**Lemma 4.2.** With the above notation fixed, suppose that $0 \neq X \in \text{Ker}(\mathcal{K}^m)$ and that $m = m_X$ is a maximal left ideal of $\mathbb{R}_\phi$ such that the associated irreducible representation is faithful and the corresponding matrix algebra is either $M_i(\mathbb{R})$ for $i = 1, 2, 3$ or $M_1(\mathbb{C}) = \mathbb{C}$. Then operator $\nabla_X : \mathcal{H}_J(\iota_C) \to \text{Coker}(\mathcal{K}^m)$ defined by $\nabla_X(u) = (\nabla_X u)d\pi.j$ is surjective.

**Proof.** We will present the proof in the cases where the associated matrix algebra of $m$ is $M_3(\mathbb{R})$ or $M_1(\mathbb{C})$. The other two cases are in fact easier (and similar). Let us denote the linear operator $\mathcal{K}^m_\pi$ by $L$, and the normal bundle $N_C$ by $N$ for simplicity. Suppose that $\nabla_X$ is not surjective. Identify the cokernel of $L$ with the kernel of its adjoint $L^*$. Choose the non-zero section $\delta$ in $\text{Coker}(L)$ so that it is orthogonal to the image of $\nabla_X$. Since $L^*(\delta) = 0$, $\delta$ is not identically zero on any open subset of $\Sigma$. Choose an open ball $U \subset C^\circ = C - \pi(B)$. Let $p \in U$ be a point in $U$ and let $\{p_\sigma\}_{\sigma \in \mathcal{G}}$ be the set of pre-images, which are indexed so that $\tau(p_\sigma) = p_{\tau\sigma}$ for all $\sigma, \tau \in \mathcal{G}$. We may assume that $\pi^{-1}(U) = \cup_{\sigma \in \mathcal{G}} U_\sigma$, with $U_\sigma$ a ball around $p_\sigma$ such that the balls $U_\sigma$ are disjoint and $\pi : U_\sigma \to U$ is a diffeomorphism. Let $z$
denote the local coordinate over $U$ and $z_\sigma$ denote the corresponding local coordinate over $U_\sigma$. Assume that $\delta$ is non-zero in $U_e$ (where $e$ denotes the identity element of $G$). We will denote $U_e$ by $V$ and $z_e$ by $w$. Fix a bump function $\lambda$ with support inside $U$ and lift it to $\Sigma$, keeping the same name $\lambda$ for it. By choosing $U$ small enough, we may assume that there is a section $u \in \mathcal{H}_{\Sigma}(C)$, such that $\nabla_X(u)$ is any given section over $V$. The section $\lambda u$ is then supported on $\pi^{-1}(U)$, and we have

$$0 = \langle \nabla_X(\lambda u), \delta \rangle = \frac{1}{2\sqrt{-1}} \sum_{\sigma \in \mathfrak{G}} \int_{U_\sigma} \lambda(z_\sigma) \langle \nabla_X(u), \delta \rangle_{z_\sigma} dz_\sigma \wedge d\bar{z}_\sigma$$

$$= \frac{1}{2\sqrt{-1}} \int_V \lambda(w) \left( \sum_{\sigma \in \mathfrak{G}} \langle \nabla_{\sigma^* X}(u), \sigma^* \delta \rangle_w \right) dw \wedge d\bar{w}.$$  

For the second equality in the above equation we use the fact that the restriction of $u$ to the image of $C$ is zero, and the function $\lambda$ may thus be brought out of the differentiation.

The complex anti-linear 1-form $\delta$ can locally be written as $\delta = Z ds - JZ dt$ where $Z \in \Gamma^{k,q}(\pi^{-1}(U), \pi^* N)$ is a given vector field, and $z = s + t\sqrt{-1}$ is the local coordinate over $U$. Thus, for any $p \in C^0$ and any linear transformation $B : N_p \to N_p$, by letting $\lambda$ converge to the delta function supported above $p$ and choosing $u$ so that $\nabla_X(u)(p)$ corresponds to $B(X(p))ds - JB(X(p))dt$ we will have

$$(8) \quad F_B(p) := \sum_{\pi(q)=p} \langle BX, Z \rangle_q = 0.$$  

If $U$ is small, $N$ may be trivialized over $U$ as $C \oplus C$. The vector fields $X$ and $Z$ will then have a corresponding decomposition $X = (X_1, X_2)$ and $Z = (Z_1, Z_2)$. Varying the matrix $B$ in the equation $F_b(p) = 0$ we conclude that for all $i, j \in \{1, 2\}$ and all $p \in C^0$

$$(9) \quad F_{ij}(p) := \sum_{\pi(q)=p} (X_i, Z_j)(q) = 0 \; \& \; \overline{F}_{ij}(p) := \sum_{\pi(q)=p} (X_i, \overline{Z_j})(q) = 0.$$  

Let us first assume that the matrix algebra associated with $\mathfrak{m}$ is $M_1(C) = C$.

Since the corresponding representation is faithful, we have

$$G = \mathbb{Z}/n\mathbb{Z} = \langle \sigma \rangle, \quad \text{with } \rho(\sigma) = \zeta \in S^1 \setminus \{ \pm 1 \} \subset \mathbb{C}^*.$$  

Here $\zeta$ denotes a primitive $n$-th root of unity. We may thus find a second section $Y \in \Gamma^{k,p}(\Sigma, \pi^* N)$ such that

$$(10) \quad \begin{pmatrix} (\sigma^m)^* X \\ (\sigma^m)^* Y \end{pmatrix} = \begin{pmatrix} \text{Re}(\zeta^m) & \text{Im}(\zeta^m) \\ -\text{Im}(\zeta^m) & \text{Re}(\zeta^m) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$  

Similarly, there is a section $\epsilon \in \Gamma^{k,q}(\pi^{-1}(U), \Omega^{0,1}_\Sigma \otimes_J \pi^* N)$ such that

$$(11) \quad \begin{pmatrix} (\sigma^m)^* Z \\ (\sigma^m)^* W \end{pmatrix} = \begin{pmatrix} \text{Re}(\zeta^m) & \text{Im}(\zeta^m) \\ -\text{Im}(\zeta^m) & \text{Re}(\zeta^m) \end{pmatrix} \begin{pmatrix} Z \\ W \end{pmatrix}.$$
Rewriting equation 9 using equations 10 and 11 we obtain
\[
0 = \sum_{m=0}^{n-1} \left( \text{Re}(\zeta^m)^2 X_i Z_j + \text{Im}(\zeta^m)^2 Y_i W_j \right) + \text{Im}(\zeta^m)\text{Re}(\zeta^m)(X_iW_j + Y_iZ_j)q
\]
\[
= \left( \sum_{m=0}^{n-1} \text{Re}(\zeta^m)^2 \right) (X_iZ_j)q + \left( \sum_{m=0}^{n-1} \text{Im}(\zeta^m)^2 \right) (Y_iW_j)q
+ \left( \sum_{m=0}^{n-1} \text{Im}(\zeta^m)\text{Re}(\zeta^m) \right) (X_iW_j + Y_iZ_j)q
\]
\[
= \left( \sum_{m=0}^{n-1} \text{Re}(\zeta^m)^2 \right) (X_iZ_j + Y_iW_j)q, \quad \forall \ q \in V.
\]

Here the last equality follows since from \( n \neq 2 \) we have \( \sum_{m=0}^{n-1} \zeta^{2m} = 0 \) and
\[
\zeta^{2m} = (\text{Re}(\zeta^m)^2 - \text{Im}(\zeta^m)^2) - 2\sqrt{-1} (\text{Re}(\zeta^m)\text{Im}(\zeta^m)).
\]
Since \( \sum_{m=0}^{n-1} \text{Re}(\zeta^m)^2 \) is a positive real number, we conclude that \( X_iZ_j + Y_iW_j \) is identically zero on \( V \). Similarly, from the second equality in equation 9 we may conclude that
\[
(X_iZ_j + Y_iW_j) |_V \cong 0.
\]

These two relations imply that
\[
(\text{Re}(Z_j)X + \text{Re}(W_j)Y) (w) = 0, \quad \forall \ w \in V.
\]

Since \( X \) and \( Y \) are both in the kernel of \( L \), and the zeros of both of them are isolated, equation 13 implies that \( \tilde{T}_w \left( \frac{\text{Re}(W_j)}{\text{Re}(Z_j)} \right) = 0 \) on \( V \), and hence \( \text{Re}(W_j)/\text{Re}(Z_j) \) is constant on \( V \) (as a real valued holomorphic function). This means that \( Y \) is a constant multiple of \( X \) over \( V \), and hence on all of \( \Sigma \), unless \( \text{Re}(Z_j) = \text{Re}(W_j) = 0 \). Since the former can not happen, we should have \( \delta = 0 \) on \( V \), which is also a contradiction. This completes the proof when the matrix algebra is \( \mathbb{C} \).

Let us now assume that the matrix algebra associated with \( \mathfrak{m} \) is \( M_3(\mathbb{R}) \). Then associated with any \( \sigma \in \mathfrak{G} \) is an orthogonal matrix \( A_\sigma \in O_3(\mathbb{R}) \subseteq M_3(\mathbb{R}) \). Furthermore, assuming \( \mathfrak{m} \) corresponds to the matrices with zero in the first column, we may write
\[
\alpha(X) = X^1\alpha_1 + X^2\alpha_2 + X^3\alpha_3, \quad \& \quad \alpha(Z) = Z^1\alpha_1 + Z^2\alpha_2 + Z^3\alpha_3,
\]
where we have

\[
\begin{align*}
\alpha_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\alpha_2 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\alpha_3 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{align*}
\]

\[X^i \in \Gamma^k(\Sigma, \pi^* N) \quad \& \quad Z^j \in \Gamma^k(\pi^{-1}(U), \pi^* N), \quad \forall i, j \in \{1, 2, 3\}.
\]

From the presentations of equation 14 setting \(\beta(X) = (X^1, X^2, X^3)^t\) and \(\beta(Z) = (Z^1, Z^2, Z^3)^t\) we have

\[
\sigma^*(\beta(X)) = A_\sigma \beta(X), \quad \& \quad \sigma^*(\beta(Z)) = A_\sigma \beta(Z).
\]

There is a fixed vector \(v = (v_1, v_2, v_3) \in \mathbb{R}^3\) such that \(X = v \beta(X)\) and \(Z = v \beta(Z)\). If \(X_i\) and \(Z_i\) denote the \(i\)-th components of \(X\) and \(Z\) in a trivialization of \(N\) over \(U\) as \(\mathbb{R}^4 \times U\), for a fixed point \(q \in V\) which satisfies \(\pi(q) = p\), equation 15 reads as

\[
0 = \sum_{\pi(z) = p} Z_i(z)X(z) = \sum_{\sigma \in \mathfrak{m}} Z_i(\sigma(q))X(\sigma(q))
\]

\[
= \beta(X)^t(q) \left( \sum_{\sigma \in \mathfrak{m}} A_\sigma^t.v.A_\sigma \right).\beta(Z_i)(q) =: \beta(X)^t(q).B.\beta(Z_i)(q).
\]

For the \(3 \times 3\) matrix \(B\) and any choice of \(\tau \in \mathfrak{m}\) we have

\[
A_\tau^{-1}BA_\tau = A_\tau^t.BA_\tau = \sum_{\sigma \in \mathfrak{m}} A^t_{\sigma\tau}.v.A_\sigma = B \Rightarrow BA_\tau = A_\tau B, \quad \forall \tau \in \mathfrak{m}.
\]

Since the matrices \(\{A_\tau\}_{\tau \in \mathfrak{m}}\) generate the algebra \(M_3(\mathbb{R})\), \(B\) should be a multiple of identity. On the other hand, the trace of \(B\) may be computed via

\[
\text{tr}(B) = \sum_{\sigma \in \mathfrak{m}} \text{tr}(A^t_{\sigma\tau}.v.A_\sigma) = \sum_{\sigma \in \mathfrak{m}} \text{tr}(v.A_\sigma.A^t_{\sigma\tau}.v) = 3|\mathfrak{m}|\|v\|^2 \neq 0
\]

\[\Rightarrow B = \|v\|^2|\mathfrak{m}|.I_{3\times 3} \neq 0.
\]

Combining equations 15 and 16 it follows that \(\beta(X)\beta(Z_i)^t\) is identically zero over \(\pi^{-1}(U)\) for \(i = 1, 2, 3, 4\).

The three vectors \(X^1(q), X^2(q)\) and \(X^3(q)\) will thus linearly depend on each other (over the point \(q\)). If \(r(q)\) denotes the rank of the vector space spanned by these three vectors we will have \(r(q) \in \{1, 2\}\) for a generic choice of \(q\). If \(r(q) = 2\), then for points in an open neighbourhood of \(q\) the same will be true. For any point \(z\) in this open neighbourhood, the four vectors

\[
\beta(Z_i)(z) = (Z^1_i(z), Z^2_i(z), Z^3_i(z)) \in \mathbb{R}^3, \quad i = 1, 2, 3, 4
\]

are thus multiples of one-another, and \(Z^1\) is thus a real multiple of \(Z^2\) over this open neighbourhood, i.e. \(Z^1(z) = \lambda(z).Z^2(z)\) for some real valued function \(\lambda\). Since \(Z^1\) and \(Z^2\) satisfy perturbed Cauchy-Riemann equations, this means that \(\partial_t(\lambda) = 0\), implying that \(\lambda\) is constant (since it is real valued). Repeating this argument for the other pairs, it is implied that
there is a constant vector $0 \neq w = (w_1, w_2, w_3) \in \mathbb{R}^3$ such that $w, \beta(X) = 0$ over a small open neighbourhood on $\Sigma$, and hence on all of $\Sigma$. We thus have

$$0 = \sigma^* (w, \beta(X)) = (w, A_\sigma) \beta(X) \quad \forall \sigma \in \mathfrak{G}$$

$$\Rightarrow \beta(X) = 0, \text{ which implies } X = 0.$$ From this contradiction, we should have $r(q) = 1$ for a generic point $q$ on $\Sigma$. If $r \leq 1$ in a neighbourhood of $q$, it is implied that $X^2(z) = \lambda(z)X^1(z)$ for $z$ near $q$ and for a real valued function $\lambda$. Again, since $X^1$ and $X^2$ satisfy perturbed Cauchy-Riemann equations, this means that $\mathcal{D}_z(\lambda) = 0$, implying that $\lambda$ is constant near $q$ (since it is real valued). The equation $X^2(z) = \lambda X^1(z)$ (with $\lambda$ a real constant) thus extends to all of $\Sigma$, and the same argument as above may be repeated for $w = (-\lambda, 1, 0)$. The resulting contradiction then completes the proof. \hfill \square

**Theorem 4.3.** For any $J$ in a subset $\mathcal{J}_{s,\omega}^\ell(M, \omega) \subset \mathcal{J}_{reg}^\ell(M, \omega)$ which is of second category as a subset of $\mathcal{J}^\ell(M, \omega)$, any genus $h \geq 0$, and any homology class $\alpha \in H_2(M, \mathbb{Z})$ which satisfies $c_1(M) \cdot \alpha = 0$, all the curves in the moduli space $\mathcal{M}_h(\alpha, J)$ are smooth 4-rigid embeddings.

**Proof.** Let $\alpha \in H_2(M, \mathbb{Z})$ be a class which satisfies $c_1(M) \cdot \alpha = 0$, and $g \geq h \geq 0$ be fixed integers. Let $\pi : \Sigma \to \Sigma / \mathfrak{G} = C$ denote the topological type of a branched covering map coming from a group action on a surface $\Sigma$ of genus $g$ and with quotient a surface $C$ of genus $h$. Furthermore, assume that $\mathfrak{G}$ is a subgroup of $S_4$. Let $\beta = |\mathfrak{G}| \cdot \alpha \in H_2(M, \mathbb{Z})$. Fix $\mathbf{m} \in I(\pi)$ and assume that the corresponding representation is faithful. Let $\mathcal{N}_m^\ell(\beta)$ be the zero locus of the section $\mathcal{K}_m^\ell$, which is fibered over $\mathcal{J}^\ell(M, \omega)$. By proposition 4.1, each one of the moduli spaces $\mathcal{N}_m^\ell(\beta)$ is a smooth infinite dimensional manifold. The set of regular values of the Fredholm projection map

$$\Pi_m^\ell(\beta) : \mathcal{N}_m^\ell(\beta) \to \mathcal{J}^\ell(M, \omega)$$

is of second category. Denote the set of regular values for the projection map $\Pi_m^\ell(\beta)$ by $\mathcal{J}_{reg}^\ell(\mathbf{m}, \alpha)$ and set

$$\mathcal{J}_{\mathbf{n}}^\ell(M, \omega) = \mathcal{J}_{reg}^\ell(M, \omega) \bigcap \left( \bigcap_{\mathbf{m}, \alpha \in H_2(M, \mathbb{Z})} \bigcap_{c_1(M) \cdot \alpha = 0} \mathcal{J}_{reg}^\ell(\mathbf{m}, \alpha) \right).$$

Here, the first intersection inside the parentheses is over all the topological types of the branched covering maps $\pi$ coming from the action of a subgroup of $S_n$ and all $\mathbf{m} \in I(\pi)$ which correspond to faithful representations. When $n = 4$, the corresponding matrix algebra of $\mathbf{m}$ is thus one of $M_i(\mathbb{R})$, $i = 1, 2, 3$ or $M_1(\mathbb{C}) = \mathbb{C}$.  

\hfill \Box
For an almost complex structure $J \in \mathcal{J}_\ell^t(M, \omega)$ and any homology class $\alpha \in H_2(M, \mathbb{Z})$ satisfying $c_1(M) . \alpha = 0$, $\mathcal{M}_h(\alpha, J)$ is a zero dimensional manifold since $\mathcal{J}_t(M, \omega) \subset \mathcal{J}_{reg}^t(M, \omega)$. If an embedded curve $C \in \mathcal{M}_h(\alpha, J)$ is not 4-rigid, we may construct a branched covering map $\pi : \Sigma \to \Sigma \mathcal{G} = C$ for a subgroup $\mathcal{G} \subset S_4$, so that the kernel of $\mathcal{K}_m^\pi$ is non-trivial for some $m \in I(\pi)$. Furthermore, we may assume that the representation associated with $m$ is faithful (all these reductions were discussed in section 3). This non-trivial kernel gives a non-empty subset of $\mathcal{N}_m^\pi(\beta, J)$. The moduli space $\mathcal{N}_m^\pi(\beta, J)$ is a smooth manifold of dimension $\text{Index}(\Pi_m^\pi(\beta))$, by regularity of the almost complex structure $J$. This later index is equal to the sum of the indices of the projection map from $\mathcal{M}_\pi(\beta)$ to $\mathcal{J}_\ell^t(M, \omega)$ and the projection map from $\mathcal{N}_m^\pi(\beta)$ to $\mathcal{M}_\pi(\beta)$. This later projection map has the same index as the operator $\mathcal{K}_m$. In the following section, we will estimate $\text{Index}(\mathcal{K}_m)$ and prove proposition 4.4.

**Proposition 4.4.** If $m \in I(\pi)$ corresponds to a faithful representation, the index $\text{Index}(\mathcal{K}_m)$ is at most $-l$, where $l$ is the number of points in the image $\pi(B)$ of the branched locus of $\pi$.

Assuming proposition 4.4, the index of $\Pi_m^\pi(\beta)$ is equal to $l + \text{Index}(\mathcal{K}_m)$, which is at most 0. If this manifold is non-empty, and $(\phi, Y) \in \mathcal{N}_m^\pi(\beta, J)$, then for any $0 \neq \lambda \in \mathbb{R}$, $(\phi, \lambda Y)$ is also in $\mathcal{N}_m^\pi(\beta, J)$. This contradicts the fact that this moduli space is at most zero dimensional. The contradiction proves that every curve $C \in \mathcal{M}_h(\alpha, J)$ is 4-rigid.

5. Index computation

Fix a holomorphic map $\pi : \Sigma \to C$ of degree $n > 1$ which is determined by the action of a group $\mathcal{G}$ on the Riemann surface $\Sigma$ (so that $C = \Sigma/\mathcal{G}$). Let $L \to C$ be a holomorphic vector bundle over $C$ and $\pi^* L \to \Sigma$ be its pullback over $\Sigma$. The group ring $\mathbb{R}[\mathcal{G}]$ acts on the cohomology groups $H^i(\Sigma, \pi^* L)$. For $\eta \in H^i(\Sigma, \pi^* L)$ we may set $m_\eta$ to be the left ideal of $\mathbb{R}[\mathcal{G}]$ consisting of the elements $a \in \mathbb{R}[\mathcal{G}]$ such that $a \eta = 0$. For any $m \in I(\pi)$ and any vector bundle of the form $\pi^* L$ let

$$H^i_m(\pi^* L) := \{ \eta \in H^i(\Sigma, \pi^* L) \mid m \subset m_\eta \}, \quad i = 0, 1.$$ 

Define $h^i_m(\pi^* L) = \dim_{\mathbb{C}}(H^i_m(\pi^* L))$, $i = 0, 1$, and let $\chi_m(\pi^* L) = h^0_m(\pi^* L) - h^1_m(\pi^* L)$. For any $m \in I(\pi)$ the Fredholm map $\mathcal{K}_m = \mathcal{K}_m^\pi$ is a zero order perturbation of the operator

$$\mathcal{D} : \Gamma^k_0(\Sigma, \pi^* N_C) \to \Gamma^{k-1,p}_0(\Sigma, \Omega^{0,1}_\Sigma \otimes J \pi^* N_C),$$

i.e. the operator obtained when $J$ is the integrable complex structure on a neighbourhood of the zero section in $N_C$. The index of $\mathcal{K}_m$ may thus be computed by a computation for $\mathcal{D}$. In this case (i.e. for $\mathcal{K}_m = \mathcal{D}$) we have

$$\text{Ker}(\mathcal{K}_m) = H^0_m(\Sigma, \pi^* N_C), \quad \text{Coker}(\mathcal{K}_m) = H^1_m(\Sigma, \pi^* N_C).$$
Since $N_C$ may be deformed to $\mathcal{O}_C \oplus K_C$, where $K_C$ denotes the canonical bundle of $C$ (see \[1\] and \[11\]), we have

$$\text{Index}(\mathcal{K}^m) = \chi_m(\pi^*K_C) + \chi_m(\pi^*\mathcal{O}_C).$$

Let $B$ denote the branching divisor of the map $\pi$ and $\mathcal{O}_B$ denote the corresponding sheaf over $\Sigma$ with support in $B$. The group ring $\mathbb{R}_G$ will also act on $H^0(\Sigma, \mathcal{O}_B)$ and it thus makes sense to talk about $H^0_m(\mathcal{O}_B)$ and $h^0_m(\mathcal{O}_B)$.

**Lemma 5.1.** For any $m \in I(\pi)$ we have $\text{Index}(\mathcal{K}^m) = -2h^0_m(\mathcal{O}_B)$.

**Proof.** Since $K_\Sigma$ is invariant under the action of $\mathfrak{G}$, we may apply Serre duality to obtain $\chi_m(\pi^*K_C) = -\chi_m(K_\Sigma - \pi^*K_C) = -\chi_m(B)$. The line bundle associated with $B$ sits in a short exact sequence

$$0 \rightarrow \mathcal{O}_\Sigma \rightarrow B \rightarrow \mathcal{O}_B \rightarrow 0.$$

Considering the cohomology long exact sequence associated with this short sequence, and restricting attention to the sections trivialized by the maximal left ideal $m$, we obtain $\chi_m(B) = \chi_m(\mathcal{O}_\Sigma) + h^0_m(\mathcal{O}_B)$. Since $\pi^*\mathcal{O}_C = \mathcal{O}_\Sigma$, the proof is complete. \(\square\)

**Proof.** (of proposition \[14\]) By the above lemma, the computation of the index of the operator $\mathcal{K}^m$ is reduced to the computation of $h^0_m(\mathcal{O}_B)$. Let $x \in C$ be a point in the image of the branched locus of $\pi$. The monodromy around $x$ gives a partition of $n$ as $n = d_1 + d_2 + \ldots + d_\ell$ with $d_1 \geq d_2 \geq \ldots \geq d_\ell$. Associated with each $d_i$ is a point $y_i$ in $\pi^{-1}(x)$ which has multiplicity $d_i$ and is a branched point if $d_i > 1$. Let $B_x$ be the divisor $(d_1 - 1)y_1 + \ldots + (d_\ell - 1)y_\ell$. Since $\pi$ comes from the action of a group, we should have $d_1 = \ldots = d_m = d > 1$, and that the degree of $B_x$ is $m(d - 1)$. Set $B_x = \mathcal{O}_B_x$ and note that $B = \sum_x B_x$ and $\mathcal{O}_B = \sum_x \mathcal{O}_B_x$. From $H^0_m(\mathcal{O}_B) = \oplus_x H^0_m(B_x)$, it suffices to show $h^0_m(B_x) \geq 1$.

The elements of $H^0(B_x)$ are the germs $\phi = \sum_{i=1}^\ell \sum_{j=1}^{d-1} A_{i,j}/z_i^j$ where $z_i$ is the pull-back of the local coordinate $z$ around $x$ to a neighborhood of $y_i$. If $\zeta = \exp(2\pi \sqrt{-1}/d)$ is the primitive $d$-th root of unity, there is a unique $\tau \in \mathfrak{G}$ such that $\tau(z_1) = \zeta z_1$ is satisfied near $y_1$. We may also assume, after re-parametrization, that there are elements $\sigma_1, \ldots, \sigma_m \in \mathfrak{G}$ such that $\sigma_j(z_1) = z_j$ is satisfied for $j = 1, \ldots, m$. Note that $\sigma_1$ would be the identity element of $\mathfrak{G}$. The group $\mathfrak{G}$ is then identified as

$$\mathfrak{G} = \{ \sigma(i,j) := \sigma_i \tau^j \mid i = 1, \ldots, m, j = 1, \ldots, d \}.$$

For $(i,j) \in \{1, \ldots, m\} \times \{1, \ldots, d\}$ we define $1_{i,j} := \sum_{p=1}^{d-1} \zeta^{pj}z_i^{-p} \in H^0(B_x)$. For $\sigma \in \mathfrak{G}$, $\sigma(z_i) = \zeta^a z_{\sigma(i)}$, where $a \in \{1, \ldots, d\}$ and $\sigma(i) \in \{1, \ldots, m\}$ are integers which depend on $\sigma$ and $i$. Thus

$$\sigma(1_{i,j}) = \sum_{p=1}^{d-1} \zeta^{pj} (\zeta^a \zeta_{\sigma(i)})^{-p} = \sum_{p=1}^{d-1} \zeta^{pj-a} z_{\sigma(i)}^{-p} = 1_{\sigma(i),j-a}.$$
Thus $\mathfrak{G}$ permutes the elements $\mathfrak{r}_{i,j}$ among themselves. If we denote $\sigma(\mathfrak{r}_{1,d})$ by $\mathfrak{r}_\sigma$ we will have
\[
\sigma_1(\mathfrak{r}_{\sigma_2}) = \mathfrak{r}_{\sigma_1\sigma_2}, \quad \forall \sigma_1, \sigma_2 \in \mathfrak{G}.
\]
There are precisely $m$ relations among the sections $\mathfrak{r}_\sigma$, indexed by $i = 1, \ldots, m$:
\[
\sum_{j=1}^{d} \mathfrak{r}_{\sigma(i,j)} = \sigma_i \left( \sum_{j=1}^{d} \sum_{p=1}^{d-1} \tau^j (z_1)^{-p} \right) = \sigma_i \left( \sum_{p=1}^{d-1} \sum_{j=1}^{d} \zeta^{-p} \right) = 0.
\]
The sections $\{\mathfrak{r}_\sigma\}_{\sigma \in \mathfrak{G}}$ generate $H^0(B_x)$. An element $\phi \in H^0(B_x)$ of the form $\phi = \sum_{\sigma \in \mathfrak{G}} a_\sigma \mathfrak{r}_\sigma$ is in $H^0_m(B_x)$ if and only if
\[
\sum_{\sigma \in \mathfrak{G}} b_\sigma a_\sigma \mathfrak{r}_\sigma = 0, \quad \forall b = \sum_{\sigma \in \mathfrak{G}} b_\sigma \sigma^{-1} \in m.
\]
This means that $H^0_m(B_x)$ is non-trivial if and only if the sum of the subspace $m$ of the vector space $\mathbb{R}_\mathfrak{G}$ with the subspace
\[
\left\langle \alpha(i) := \sum_{j=1}^{d} \sigma(i,j)^{-1} \big| i = 1, \ldots, m \right\rangle_{\mathbb{R}}
\]
is not all of $\mathbb{R}_\mathfrak{G}$. Note that this sum should be considered as the sum of sub-vector-spaces, and not ideals.
Let us assume that the vector space $m \in \mathbb{R}_\mathfrak{G}$ is of codimension $\ell$. Thus the subspace $\mathbb{R}_\mathfrak{G}/m$ generated by $\sigma_1^{-1}, \ldots, \sigma_m^{-1}$ is at most of dimension $\ell$, and we may assume that it is generated by $\sigma_1^{-1}, \ldots, \sigma_\ell^{-1}$. Thus there are real numbers $c_{i,k} \in \mathbb{R}, i = 1, \ldots, m$ and $k = 1, \ldots, \ell$ such that $\sigma_i^{-1} = \sum_{k=1}^{\ell} c_{i,k} \sigma_k^{-1}$ for $i = 1, \ldots, m$. We thus have
\[
\alpha(i) = \sum_{k=1}^{\ell} c_{i,k} \alpha(k), \quad \text{(modulo $m$),} \quad \forall \ i = 1, \ldots, m,
\]
which implies that $H^0_m(B_x) \neq 0$ if and only if (again as vector spaces)
\[
m + \langle \alpha(i) \big| i = 1, \ldots, \ell \rangle_{\mathbb{R}} \neq \mathbb{R}_\mathfrak{G}.
\]
It thus suffices to show that there is some non-zero $a = (a_1, \ldots, a_\ell) \in \mathbb{R}^\ell$ such that
\[
\sum_{i=1}^{\ell} a_i \alpha(i) = \left( \sum_{j=0}^{d-1} \tau^j \right) \left( \sum_{i=1}^{\ell} a_i \sigma_i^{-1} \right) \in m.
\]
This criteria should be investigated over the distinguished component of the maximal left ideal $m$ in the decomposition of equation $\mathbb{R}$. Over this component, we have an isomorphism with a matrix algebra of the form $M_k(R)$ for an integer $k = \ell, \ell/2$ or $\ell/4$ depending on whether $R = \mathbb{R}, \mathbb{C}$ or
The ideal $\mathfrak{m}$ determines a left ideal of $M_k(R)$, which may be specified by a vector $v_\mathfrak{m} \in R^k$ in the sense that
$$\mathfrak{m} \cap M_k(R) = \{ A \in M_k(R) \mid A.v_\mathfrak{m} = 0 \}.$$ 
Let $S_1, ..., S_\ell$ denote the matrices in $M_k(R)$ which correspond to $\sigma_1, ..., \sigma_\ell$. Suppose that the criteria is not satisfied. Thus $e_i = S_i v_\mathfrak{m}$, for $i = 1, ..., \ell$ generate $R^k$ over $\mathbb{R}$. Let $A$ be the matrix in $M_k(R)$ which corresponds to $(1 + \tau + ... + \tau^{d-1})/d$. The above criteria would be satisfied if $A(a_1 e_1 + ... + a_\ell e_\ell) = 0$. In other words, we only need to show that the kernel of $A$ is non-trivial. Suppose otherwise that $A$ is non-singular. Since $A^2 = A$ we should have $A = I_{M_k(R)}$ is the identity matrix in $M_k(R)$. Let $T$ denote the matrix which corresponds to $\tau$. Then we have
$$\frac{1 + \tau + ... + \tau^{d-1}}{d} = \frac{1 + \tau + ... + \tau^{d-1}}{d} \Rightarrow TA = A \Rightarrow T = I_{M_k(R)}.$$ 
In other words, $\tau$ is in the kernel of the representation corresponding to $\mathfrak{m}$, which is a contradiction (with the assumption that this representation is faithful). The contradiction completes the proof.

Remark 5.2. Note that the index computation does not make any use of the assumptions on the size of the irreducible real representation, and is thus completely general.

6. Elliptic regularity and smooth complex structures

So far, we have only considered $C^\ell$ almost complex structures and $n$-rigidity for embedded curves, and sections over them which are of class $W^{k,p}$, where $k \leq \ell$. Note that if $J$ is an almost complex structure in $\mathcal{J}^\ell(M, \omega)$ and $f : \Sigma \rightarrow M$ is a $J$-holomorphic curve of class $W^{1,p}$ for some $p > 2$, then it will be of class $W^{\ell+1,p}$ and there is a constant $c(J, \ell)$ such that $\|f\|_{W^{\ell+1,p}} \leq c(J, \ell)\|f\|_{W^{1,p}}$.

Fix $\ell \in \mathbb{Z}^{\geq 0} \cup \{ \infty \}$, the integers $n, K > 0$, and an almost complex structure $J \in \mathcal{J}^\ell(M, \omega)$. Fix an identification of $H_2(M, \mathbb{Q})$ with $Q^b(M)$ once for all, and for a homology class $\beta \in H_2(M, \mathbb{Z})$ let $\|\beta\|$ denote its Euclidean norm under this identification. Let $|\beta|$ denote the divisibility of $\beta$.

Definition 6.1. A $J$-holomorphic map $f : C \rightarrow M$ of class $W^{k,p}$ with (possibly disconnected and nodal) domain $C$ of total genus $h \leq K$ which is somewhere injective on any irreducible component of $C$ is called a critical $(J,n,K)$ curve if $f(C)$ is connected, $\|df\|_{\infty} < K$, $\|f_*[C]\| < K$, $|f_*[C]| \leq n$ and either of the following happens:
1. The domain $C$ is nodal (i.e. not smooth)
2. $C$ is smooth, but there are $x, y \in C$ with $x \neq y$ and $f(x) = f(y)$.
3. $C$ is smooth and $f$ is one-to-one, but there is some $x \in C$ with $df_x = 0$.
4. $C$ is smooth and $f$ is an embedding, but the image of $f$ is not $(n,K)$-rigid.
The $J$-holomorphic map $f : C \to M$ is called a critical $(J, n)$ curve if it is a critical $(J, n, K)$ curve for some $K > 0$.

Thus $\mathcal{J}_n^\ell(M, \omega)$ is included in the subset of $\mathcal{J}^\ell(M, \omega)$ consisting of the almost complex structures $J$ so that there are no critical $(J, n)$ curves in $M$. Abusing the notation, we will use $\mathcal{J}_n(\omega)$ to denote this larger (bigger) subset of $\mathcal{J}^\ell(M, \omega)$, which is of second category by theorem 4.3 in [4,5] in $\mathcal{J}^\ell(M, \omega)$, provided that $\ell \neq \infty$ and $n \leq 4$. Let $\mathcal{J}_n(\omega) \subset \mathcal{J}^\ell(M, \omega)$ be the subset of almost complex structures $J$ of class $C^\ell$ such that there are no critical $(J, n, K)$ curves $f : C \to M$ of class $W^{\ell+1,2}$. We will simply denote $\mathcal{J}_n^\infty(K)$ by $\mathcal{J}_n(K)$. It is then clear that $\mathcal{J}_n^\infty(M, \omega) = \cap_{K \in \mathbb{Z} > 0} \mathcal{J}_n(K)$.

**Lemma 6.2.** For any positive real number $K > 0$, $\mathcal{J}_n(K)$ is open in $\mathcal{J}^\infty(M, \omega)$ with respect to $C^\infty$ topology.

**Proof.** Suppose otherwise that there is a sequence $J_i$ of smooth almost complex structures outside $\mathcal{J}_n(K)$ which converges to a point $J \in \mathcal{J}_n(K)$. We may thus find a sequence $f_i : \Sigma_i \to M$ of critical $(J_i, n, K)$ curves satisfying

$$\|df_i\|_\infty \leq K, \quad \|(f_i)_*|\Sigma_i|\| \leq K, \quad \& \quad \|(f_i)_*|\Sigma_i|\| \leq n.$$ 

Furthermore, if $f_i$ is an embedding, it is not $(n, K)$-rigid. After passing to a subsequence, we may assume that all $f_i$ represent the same homology class $\beta$ with $|\beta| \leq n$, and that their domains $\Sigma_i$ have the same genus $g \leq K$. The sequence $\{f_i\}_i$ will thus converge to a $J$-holomorphic curve $f : \Sigma \to M$ which satisfies

$$\|df\|_\infty \leq K, \quad \|f_*|\Sigma|\| = \|\beta\| \leq K, \quad \& \quad |f_*|\Sigma|\| \leq |\beta| \leq n.$$ 

Furthermore, the genus of $\Sigma$ is at most $K$. The map $f$ decomposes as $f = \iota_C \circ \pi$ where $\pi : \Sigma \to C$ is a branched covering map and $\iota_C : C \to M$ is $J$-holomorphic and somewhere injective on all its components. Since $J \in \mathcal{J}_n(K)$, $C$ is smooth, $\iota_C$ is an embedding and the image of $C$ under $\iota_C$ is $(n, K)$-rigid. Note also that $\deg(\pi) \leq n$.

We may identify a neighbourhood of $C$ with a neighbourhood of the zero section in its normal bundle $N_C$ in $M$. For $i$ sufficiently large, the image of $f_i$ will be in this neighbourhood and we may thus regard the images of the critical $(J_i, n, K)$ curves $f_i : \Sigma_i \to M$ as multi-sections of this normal bundle. As in section 2 we may re-scale these images so that their supremum norm is equal to 1. Similar to our discussion in section 2 this convergence gives a section

$$(X, j) \in \Gamma^\infty(\Sigma, \pi^*N_C) \oplus \text{Hom}^{0,1}_J(T_C, N_C),$$

which satisfies the following equation in $\Gamma^\infty(\Sigma, \Omega^{0,1}_\Sigma \otimes_J \pi^*N_C)$:

$$K_\pi(X) + j \circ d\pi \circ j_\Sigma = 0.$$
Here $\text{Hom}^{0,1}_J(T_C, N_C)$ is the space of bundle homomorphisms $j$ from the tangent space $T_C$ of $C$ to the normal bundle $N_C$ which anti-commute with the complex structure, i.e. $J \circ j + j \circ j_C = 0$.

Since $C$ is $(n, K)$-rigid, for any branched covering map $\zeta : S \rightarrow C$ such that $\text{deg}(\zeta) \leq n$ and the genus $g_S$ of $S$ satisfies $g_S \leq K + g_C$ the operator $K_{\zeta}$ is injective. In particular, for $\zeta = \text{Id}_C : C \rightarrow C$, since the index of $K_{\zeta}$ is zero, this operator is an isomorphism. Thus there is some $Y \in \Gamma(C, N_C)$ such that $K_{\zeta}(Y) + j \circ \text{Id}_{T_C} \circ j_C = 0$. This means that $X - \pi^*Y \in \text{Ker}(K_{\pi^*})$. By $(n, K)$-rigidity of $C$ we have $X = \pi^*Y$.

Since $\gamma_C : C \rightarrow M$ is $(n, K)$-rigid (and hence 1-rigid), there are $J_i$-holomorphic curves $C_i$ close to $C$, which converge to $C$ as $i$ goes to infinity. We may then regard both $C_i$ and $\Sigma_i$ as sections and multi-sections of $N_C$. For large values of $i$, the projection map $\pi_i$ from $C_i$ to $C$ is a diffeomorphism, and the projection map from $\Sigma_i$ to $C$ may thus be composed with $\pi_i^{-1}$ to give the projection maps $\pi_i : \Sigma_i \rightarrow C_i$. The map $\pi_i$ may be used to define a new complex structure $j_i$ on $\Sigma_i$ which makes $\pi_i$ holomorphic. Next, the bundle $N = N_C$ may be pulled back over $C_i$ using $\pi_i$. Finally, the map $f_i : \Sigma_i \rightarrow M$ may be regarded as a section $X_i \in \Gamma^\infty(\Sigma_i, \pi_i^*N)$. The section $X_i$ describes the distance of $\Sigma_i$ from $C_i$. Since $X$ is of the form $\pi^*Y$, we have

$$\lim_{i \rightarrow \infty} \frac{\|X_i\|_{C^0}}{\|J_i - J\|_{C^0}} = 0.$$ 

Since $\Sigma_i$ is $J_i$-holomorphic we have

$$\nabla X_i + J\nabla_{j_i} X_i + (\nabla X_i, J) \circ d\pi_i \circ j_i = O(\|J_i - J\|_{C^0}, \|X_i\|_{W^{k,p}}) + O(\|X_i\|^2_{W^{k,p}}),$$

for some fixed values of $k, p$, say $k = p = 2$. If $X_i \neq 0$, re-scaling $X_i$ we obtain the section $\overline{X_i}$ with $\|\overline{X_i}\|_{W^{k,p}} = 1$ in the domain of $K_{\pi_i, J_i}$ such that $\|K_{\pi_i, J_i}(\overline{X_i})\|$ becomes arbitrarily small as $i$ goes to infinity. However, this implies that the kernel of $K_{\pi, J}$ is non-trivial, unless $X_i$ are zero for $i$ large enough. Since $C$ is $(n, K)$-rigid, the later should be the case, i.e. the image of $\Sigma_i$ is included in $C_i$ for $i$ large enough. The somewhere injectivity assumption on $f_i : \Sigma_i \rightarrow M$ then implies that $\Sigma_i = C_i$.

Thus the limit curve $f : \Sigma \rightarrow C \subset M$ is of degree 1 and is an embedding. Furthermore, except for finitely many values of $i$, $f_i : \Sigma_i \rightarrow M$ is an embedding as well. Since $f_i : \Sigma_i \rightarrow M$ are critical $(J_i, n, K)$ curves, there are branched covering maps $p_i : \Sigma'_i \rightarrow \Sigma_i$ of bounded degree (bounded by $n$) such that $\text{Ker}(K_{p_i}) \neq 0$. Furthermore, the genus of $\Sigma'_i$ is at most $h + K$, where $h$ denotes the genus of $C$. By passing to a further sub-sequence we may assume that $\text{deg}(p_i) = d \leq n$, and that the genus of all of $\Sigma_i$ is $g \leq K + h$. We may also assume that the maps $p_i$ converge to a degree $d$ covering map $p : \Sigma' \rightarrow \Sigma$. Since the kernel of each $K_{p_i}$ is non-trivial, the kernel of $K_p$ is non-trivial as well, violating the assumption that $J \in J_n(K)$. This contradiction completes the proof. \qed
As long as proposition 4.1 is true for a positive integer \( n \), the open subsets \( J_n(K) \) are also dense in \( J^\infty(M, \omega) \).

**Lemma 6.3.** The subspace \( J_n(K) \subset J^\infty(M, \omega) \) is dense with respect to \( C^\ell \) topology for any \( \ell > 1 \) and \( n \leq 4 \).

**Proof.** Suppose that \( \ell > 1 \) and \( n \leq 4 \). By elliptic regularity
\[
J_n(K) = J^\infty(M, \omega) \cap J^\ell_n(K).
\]
The argument used in the proof of lemma 6.2 in the smooth case may be used to show that \( J^\ell_n(K) \) is an open subset of \( J^\ell(M, \omega) \) with respect to the \( C^\ell \) topology. Moreover, \( J^\ell_n(K) \) contains \( J^\ell_n(M, \omega) \) which is dense in \( J^\ell(M, \omega) \) by theorem 4.3, and is thus an open dense subset. Since \( J_n(K) \) is the intersection of \( J^\infty(M, \omega) \) with an open dense subset of \( J^\ell(M, \omega) \), the claim of the lemma is implied.

The above two lemmas imply the following stronger version of theorem 1.2.

**Theorem 6.4.** The subset \( J^\infty_n(M, \omega) \subset J^\infty(M, \omega) \) is of second category for \( n = 1, 2, 3, 4 \).

**Proof.** Since lemma 6.3 is true for all \( \ell \), \( J_n(K) \) is dense in \( J^\infty(M, \omega) \) in \( C^\infty \) topology and
\[
J_n(M, \omega) = \bigcap_{K \in \mathbb{Z}^>0} J_n(K)
\]
is the intersection of a countable collection of open dense subsets of \( J^\infty(M, \omega) \).

7. **Conclusion and final remarks**

Let us fix a generic almost complex structure
\[
J \in J^\infty_{\text{rigid}}(M, \omega) := J^\infty_4(M, \omega),
\]
and let \( C \subset M \) be any (smooth) \( J \)-holomorphic curve of genus \( h \geq 0 \) which is 4-rigid. In this case, \( C \) determines an open component of the moduli space \( \overline{M}_g(\beta, J) \) for any homology class \( \beta = d[C] \in H_2(M, \mathbb{Z}) \) with \( 0 < d < 5 \). This open component may be identified with the moduli space \( \overline{M}_g(C, d[C]) \) of branched covering maps of \( C \) of total genus \( g \) and total degree \( d \). Theorem 6.4 tells us that for a generic \( J \) as above, for any homology class \( \alpha \in H_2(M, \mathbb{Z}) \) in \( M \) with \( c_1(M) \cdot \alpha = 0 \) and \( |\alpha| < 5 \), all \( J \)-holomorphic curves representing \( \alpha \) are embedded, 4-rigid, and they are all disjoint from each-other. Thus for any embedded \( J \)-holomorphic curve \( C \) in \( M \) of genus \( h \) and representing the homology class \( \alpha \) as above, and for any \( g \geq h \) and \( 0 < d < 5 \) the compactified moduli space \( \overline{M}_g(C, d[C]) \) is a component of the Delign-Mumford compactification \( \overline{M}_g(M, d\alpha; J) \) of \( M_g(M, d\alpha; J) \).
The fiber of the obstruction bundle $\Upsilon = \Upsilon_g(C, d[C])$ is the cokernel of the operator $K_\pi$. After a suitable stabilization of this bundle, we may extend it as a virtual obstruction bundle over $\overline{\mathcal{M}}_g(C, d[C])$ (see [7], or [3]). Denote the Euler class of this obstruction bundle by $\chi_g(C, d; J)$. Since the rank of the obstruction bundle and the dimension of the virtual moduli cycle are both $4((g - 1) - d(h - 1))$, $\chi_g(C, d; J)$ may be integrated against the virtual class $[\overline{\mathcal{M}}_g(C, d[C])]^{\text{vir}}$ to give the rational numbers

$$
C_g(C, d; J) := \int_{[\overline{\mathcal{M}}_g(C, d[C])]^{\text{vir}}} \chi_g(C, d; J) \in \mathbb{Q}.
$$

The number $C_g(C, d; J)$ is the local contribution of the curve $C$ to the Gromov-Witten invariant $N_g(M, d\alpha)$, and depends not only on the normal bundle of the $J$-holomorphic curve $C$, but also on $\nabla J$ in normal directions, or equivalently, on the Nijenhuis tensor $N_J$ associated with $J$ along $C$. In the particular case of degree one curves, $C_h(C, 1; J)$ is always equal to $\pm 1$, depending on the sign of the spectral flow from the Dolbeault $\overline{\partial}$-operator to $K_{Id}$. Moreover, the argument of [11] may be used to compute all $C_g(C, 1; J)$ for $C \in \mathcal{M}_h(\beta; J)$ and $g \geq 0$ via the following generating function formula

$$
\sum_{g \geq h} C_g(C, 1; J) \lambda^{2g-2} = (2 \sin(\frac{\lambda}{2}))^{2h-2} C_h(C, 1; J).
$$

Let us now assume that $C = \mathbb{C}P^1$ is a rational curve. Note that the moduli space $\overline{\mathcal{M}}_0(\mathbb{C}P^1, d[\mathbb{C}P^1])$, after moding out by the automorphism group of the domain, may be identified with $\mathbb{C}P^{2d-2}$. In fact, for every map $\pi : \mathbb{C}P^1 \to \mathbb{C}P^1$ of degree $d$, the pre-image of three generic points (which may be labeled $0, 1$ and $\infty$) consist of $3d$ points, $d$ in the pre-image of each one of them. There are thus $d^3$ triples $(x_0, x_1, x_\infty)$ on $\mathbb{C}P^1$ which are mapped to $(0, 1, \infty)$ by $\pi$. Each triple as above determines a re-parametrization of the domain of $\pi$ which takes $(0, 1, \infty)$ to itself. Thus, associated with any degree $d$ map in $\mathbb{C}P^{2d-2}$, there are $d^3$ elements in $\overline{\mathcal{M}}_0(C, d[C])$ which differ only by automorphisms of the domain. The fibers of the obstruction bundle over these points are naturally mapped to each other under the automorphism. Using the frame-work developed by Li and Tian [7], we may thus write

$$
\int_{[\overline{\mathcal{M}}_0(C, d[C])]^{\text{vir}}} \chi_0(C, d[C]; J) = \frac{1}{d^3} \int_{\mathbb{C}P^{2d-2}} \chi_0(C, d; J) := \frac{c(C, d; J)}{d^3},
$$

where $\chi_0(C, d; J) \in H^{4d-4}(\mathbb{C}P^{2d-2}, \mathbb{Z})$, and $c(C, d; J)$ is thus an integer (see [L5]). For $0 \neq \alpha \in H_2(M, \mathbb{Z})$ satisfying $c_1(M).\alpha = 0$ and $|\alpha| \leq 4$ let

$$
e_0(\alpha, d; J) := \sum_{C \in \mathcal{M}_0(\alpha; J)} c(C, d; J).$$
Let $\mathcal{I}_5$ denote the subspace generated by the formal powers $q^\beta$ over $\mathbb{Q}$, where $\beta \in H_2(M, \mathbb{Z})$ satisfies $c_1(M).\beta = 0$ and $|\beta| \geq 5$. Replacing the computation of equation [15] in the generating function for genus zero Gromov-Witten invariants we obtain

$$\sum_{0 \neq \beta \in H_2(M, \mathbb{Z})}^0 N_0(\beta)q^\beta = \sum_{0 \neq \alpha \in H_2(M, \mathbb{Z})}^0 \sum_{d > 0} \frac{e_0(\alpha, d; J)}{d^3} q^{d\alpha} \quad \text{(modulo } \mathcal{I}_5)\text{).}$$

This presentation of genus zero Gromov-Witten invariants of $(M, \omega)$ is a re-statement of conjecture 7.4.5 from [2] for homology classes with divisibility less than 5.

In the integrable case, if $M$ is equipped with a complex structure $J$ and $C \subset M$ is an isolated $d$-rigid smooth $J$-holomorphic curve of genus $h$, the local contributions $C_g(C, d, J)$ are proved by Bryan and Pandharipande [1] to be independent of the normal bundle, and the complex structure on $C$. They show that these contributions only depend on the genus $h$ of $C$ and the integers $g \geq h$ and $d > 0$. We may thus denote them by $C_g(h, d)$.

However, when the almost complex structure $J \in J^{\text{rigid}}(M, \omega)$ is not integrable, the independence of the local contributions $C_g(C, d; J)$ from $J$ is not clear. If $\{J_t\}_{t \in [0, 1]}$ is a generic path of almost complex structures with $J_0, J_1 \in J^{\text{rigid}}(M, \omega)$, we can not guarantee that $\{J_t\}_t$ determines a compact one-dimensional oriented cobordism from $M_h(\beta; J_0)$ to $M_h(\beta; J_1)$. The smooth moduli space corresponding to such path may have ends different from $M_h(\beta; J_0) \cup M_h(\beta; J_1)$. One may show, however, that every other limit point is in correspondence with an embedded $J_t$-holomorphic curve $C$, for $t$ in a finite subset of $[0, 1]$, and that for any such curve $C$, there is a degree 2 branched covering map $\pi : \Sigma \to C$ so that the kernel of $K_\pi$ is one dimensional. Moreover, if $X$ is the generator of this kernel and $\sigma$ is the natural involution of $\Sigma$ then $\sigma^*X = -X$. As we vary $J$ through the generic path $\{J_t\}_t$ and pass through one of the above finitely many values of $t$, the signed count of points in the moduli space $M_h(\beta; J)$ may change by 1. Formulating a precise wall-crossing formula is not straight-forward, although it is possible in principle to do so, following the method used by Taubes [13] in complex dimension 2.

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