COUNTING DESCENT PAIRS WITH PRESCRIBED COLORS IN THE COLORED PERMUTATION GROUPS

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Abstract. We define new statistics, \((c, d)\)-descents, on the colored permutation groups \(\mathbb{Z}_r \wr S_n\) and compute the distribution of these statistics on the elements in these groups. We use some combinatorial approaches, recurrences, and generating functions manipulations to obtain our results.

1. Introduction

A permutation statistic on the symmetric group \(S_n\), or more generally over a group, is a function from the group to the set of nonnegative integers. The study of permutation statistics started with Euler, who considered the number of descents of a permutation. Netto, at the beginning of the last century, considered the number of inversions, and Major Percy MacMahon [6, Vol. I, pp. 135, 186; Vol. II, p. viii], [7] considered the parameter called today major index and the excedance number.

On the symmetric group \(S_n\), a descent pair of a permutation \(\pi \in S_n\) is a pair \((i, i + 1)\) such that \(\pi_i > \pi_{i+1}\). The descent set of a permutation \(\pi = \pi_1\pi_2\cdots\pi_n \in S_n\), denoted by \(\text{Des}(\pi)\), is the set of indices \(i\) such that \((i, i + 1)\) is a descent pair. The number of descents in a permutation \(\pi\), denoted by \(\text{des}(\pi) = |\text{Des}(\pi)|\), is a classical permutation statistic. This statistic was first studied by MacMahon [6] almost a century ago, and it still plays an important role in the study of permutation statistics.

Kitaev and Remmel [5] counted descents in \(S_n\) according to the parity of the first or second element of the descent pair. In [4], they generalize it to the case where the first or the second element in the descent pair is divisible by \(k\) for some \(k \geq 2\). Some more research was done in the direction of studying the corresponding distribution for words (see [1, 2, 3]).

Let \(r, n\) be positive integers. A colored permutation on of \(n\) objects with \(r\) colors is a bijection \(\pi\) on the set

\[
\Sigma_{r,n} = \{1, \ldots, n, 1^{[1]}, \ldots, n^{[1]}, \ldots, 1^{[r-1]}, \ldots, n^{[r-1]}\}
\]
satisfying the following condition: if $\pi(i^\alpha) = j^\beta$, then $\pi(i^{\alpha+1}) = j^{\beta+1}$, where the exponents are taken modulo $r$. If $\pi_k = i^\beta$, we define $|\pi_k| = i$, where $\pi_k = \pi(k^{[0]})$ for each $k$. Moreover, the color of the position $k$, denoted by $c_k(\pi)$, is $j$. The set of all such colored permutations form a group, $G_{r,n}$, which is isomorphic to the wreath product $\mathbb{Z}_r \wr S_n = \mathbb{Z}_r^n \rtimes S_n$. The classical Weyl groups appear as special cases: the symmetric group $G_{1,n} = S_n$ and the hyperoctahedral group $G_{2,n} = B_n$.

We fix two different orders on the set $\Sigma_{r,n}$:

- The color order:
  $1^{[r-1]} < 2^{[r-1]} < \cdots < n^{[r-1]} < 1^{[1]} < \cdots < n^{[1]} < 1^{[0]} < \cdots < n^{[0]}$.

- The absolute order:
  $1^{[r-1]} < 1^{[r-2]} < \cdots < 1^{[0]} < 2^{[r-1]} < \cdots < 2^{[0]} < \cdots < n^{[r-1]} < \cdots < n^{[0]}$.

Let $\prec$ be one of the orders defined above. Let $\pi \in G_{r,n}$ and let $c, d$ be two nonnegative integers such that $c \leq d$. A $(c, d)$-descent in position $i$ is a descent (with respect to $\prec$) in position $i$ such that $c_i(\pi) = c$ and $c_{i+1}(\pi) = d$. For example, if $\pi = 6^{[1]}2^{[2]}4^{[0]}3^{[1]}5^{[2]}1^{[2]}$, then $\pi$ has two $(1, 2)$-descents with respect to the color order: $i = 1$ and $i = 4$, but only one $(1, 2)$-descent with respect to the absolute order: $i = 1$. The number of $(c, d)$-descents in $\pi$ with respect to the color order (respectively absolute order) is denoted by $\text{des}^{\text{Clr}}_{c,d}(\pi)$ (respectively $\text{des}^{\text{Abs}}_{c,d}(\pi)$).

It should be noted that actually, in the case of $(c, d)$-descents where $c < d$ with respect to the color order, one can forget about the colored permutations and deal with the underlying color words. For example, the underlying color word of $\pi = 6^{[1]}2^{[2]}4^{[0]}3^{[1]}5^{[2]}1^{[2]}$ is 120122, so our problem reduces to finding the number of words over $\{0, \ldots, r-1\}$ having a fixed number of places where the color $d$ is subsequent to $c$. Hence, the enumeration of $(c, d)$-descents in the case of the color order can be done as a special case of the work of Hall and Remmel [2] where they compute the generating function of $(X, Y)$-descents (where $X$ and $Y$ are subsets of the alphabet). Nevertheless, since our enumeration covers both orders at once, we present here a simple and direct combinatorial proof for these enumerations.

In Section 2, we study the generating function for the number of colored permutations in $G_{r,n}$ having exactly $m$ $(c, d)$-descents where $c < d$.

We get the following results.
Proposition 1.1. The number of colored permutations in $G_{r,n}$ with exactly $0$ $(c,d)$-descents ($0 \leq c < d \leq r - 1$), with respect to both orders is

$$n! \sum_{k=0}^{n} (r-1)^k + n! \sum_{k=1}^{n-1} \sum_{b=1}^{k} \binom{k}{b} \binom{n-k}{b} (r-2)^b (r-1)^{n-k-b}.$$  

The number of colored permutations in $G_{r,n}$ with exactly $m > 0$ $(c,d)$-descents ($0 \leq c < d \leq r - 1$) with respect to the color order, is given by

$$n! \sum_{k=1}^{n-1} \sum_{b=1}^{k} \binom{k}{b} \binom{n-k}{b} (r-2)^b (r-1)^{n-k-b}.$$  

The number of colored permutations in $G_{r,n}$ with exactly $m > 0$ $(c,d)$-descents ($0 \leq c < d \leq r - 1$) with respect to the absolute order, is given by

$$\frac{n!}{2^m} \sum_{k=1}^{n-1} \sum_{b=1}^{k} \binom{k}{b} \binom{n-k}{b} (r-2)^b (r-1)^{n-k-b}.$$  

Note that when $n = 2$, there will be no dependency on the value of $r$, as a simple verification shows.

The following is a special case for the hyperoctahedral group $B_n = G_{2,n}$.

Proposition 1.2. The number of signed permutations in $B_n$ with exactly $m$ $(0,1)$-descents is given by $n! \binom{n+1}{n-2m}$.

In Section 3, we study the generating function for the number of colored permutations in $G_{r,n}$ having exactly $m$ $(c,c)$-descents. Here, we obtain our results by using recurrences and manipulations with exponential generating functions.

Proposition 1.3. The number of colored permutations in $G_{r,n}$ with exactly $m$ $(c,c)$-descents, $0 \leq c \leq r - 1$, with respect to both orders is given by

$$n! \sum_{j=0}^{m} \sum_{i=0}^{m} (-1)^i (r-1)^j \binom{j+m-i}{j} \binom{n+1}{i} \frac{(m-i+j+1)^{n-j}}{(n-j)!}.$$  

2. Counting $(c,d)$-descents with $c < d$ by a direct combinatorial approach

Our approach for enumerating $(c,d)$-descents is based on a counting argument. We enumerate for both orders simultaneously.
Let \( \pi = \pi_1^{[c_1]} \pi_2^{[c_2]} \cdots \pi_n^{[c_n]} \in G_{r,n} \). Define a \( c \)-block (respectively \( \bar{c} \)-block) in \( \pi \) to be a maximal subsequence \( \pi_i^{[c_i]} \pi_{i+1}^{[c_{i+1}]} \cdots \pi_j^{[c_j]} \) of \( \pi \) such that for all \( k \in [i,j] \), \( c_k = c \) (respectively \( c_k \neq c \)). The cardinalities of the blocks form a composition of \( n \). For example, in \( G_{6,6} \), with \( c = 2 \), \( d = 3 \), if \( \pi = [2,4,2,5,3,1,6,2,3] \), then the corresponding blocks of \( \pi \) are

\[
\{2^{[2]}, 4^{[2]}\}, \{5^{[3]}, 1^{[4]}\}, \{6^{[2]}\}, \{3^{[1]}\},
\]

while the corresponding composition is \( (2, 2, 1, 1) \).

Note that \((c, d)\)-descents can appear in the transitions from a \( c \)-block to a \( \bar{c} \)-block, but not necessarily in all of these transitions.

Let \( \text{Comp}(n) \) be the set of compositions of \( n \), and let \( \text{Comp}^{\text{clr}}(n) = \text{Comp}(n) \times \{c, \bar{c}\} \). For each \( \varphi \in \text{Comp}(n) \) and \( x \in \{c, \bar{c}\} \), the element \((\varphi, x) \in \text{Comp}^{\text{clr}}(n)\) represents the composition \( \varphi \) where the first part is colored by \( x \).

Now, let \( \mu = (\varphi, x) \in \text{Comp}^{\text{clr}}(n) \). We define the following parameters:

- \( e_\varphi \) is the sum of parts in the even places of \( \varphi \). For example, if \( \varphi = (2, 3, 4, 5) \) then \( e_\varphi = 3 + 5 = 8 \).
- \( k = k_\mu = \begin{cases} n - e_\varphi & \text{if } x = c, \\ e_\varphi & \text{if } x = \bar{c}. \end{cases} \)
- \( b = b_\mu \) is the number of \( c \)-blocks and \( \bar{b}_\mu \) is the number of \( \bar{c} \)-blocks. Explicitly,

\[
b_\mu = \begin{cases} \frac{|\varphi|}{2} & \text{if } |\varphi| \text{ even}, \\ \frac{|\varphi|+1}{2} & \text{if } |\varphi| \text{ odd, } x = c, \\ \frac{|\varphi|-1}{2} & \text{if } |\varphi| \text{ odd, } x = \bar{c}, \end{cases}
\]

where \( |\varphi| \) is the number of parts of \( \varphi \), and \( \bar{b}_\mu = |\varphi| - b_\mu \).
- \( t_\mu \) is the number of transitions between a \( c \)-block to a \( \bar{c} \)-block.

Define

\[
A_{r,n}^{\text{Ord}}(q) = \sum_{\pi \in G_{r,n}} q^{\text{des}^{\text{Ord}}_{c,d}(\pi)},
\]

where \( \text{Ord} \) stands for the orders Clr or Abs. For calculating \( A_{r,n}^{\text{Ord}}(q) \), we run over the elements of \( \text{Comp}^{\text{clr}}(n) \), instead of running over the elements of \( G_{r,n} \).

**Lemma 2.1.** For all \( n \geq 0 \),

\[
A_{r,n}^{\text{Clr}}(q) = n! \sum_{\mu \in \text{Comp}^{\text{clr}}(n)} (q + r - 2)^t_\mu (r - 1)^{n - k_\mu - t_\mu}.
\]
\[ A_{r,n}^{\text{Abs}}(q) = n! \sum_{\mu \in \text{Comp}^{\text{clr}}(n)} \left( \frac{q}{2} + r - 2 \right)^{t_\mu} (r-1)^{n-k_\mu-t_\mu}. \]

**Proof.** Each \( \mu \in \text{Comp}^{\text{clr}}(n) \) gives rise to \( n!(r-1)^{n-k_\mu} \) elements of \( G_{r,n} \). \( \pi \in G_{r,n} \) contributes a \((c,d)\)-descent with respect to the color order in each transition from a \( c \)-block to a \( \bar{c} \)-block such that the first digit of the \( \bar{c} \)-block is colored by \( d \). This proves Equation (2.1).

Only half of the \((c,d)\)-descents with respect to the color order are also \((c,d)\)-descents with respect to the absolute order (see an example in the introduction), so we have Equation (2.2) too. \( \square \)

In order to get an explicit expression for the formulas appearing in Lemma 2.1, we treat separately \( \text{Comp}(n) \times \{c\} \) and \( \text{Comp}(n) \times \{\bar{c}\} \). We do this only for the color order. The other computation is similar.

Let \( \mu = (\varphi, c) \in \text{Comp}(n) \times \{c\} \).

We split the contribution into three cases according to \( |\varphi| \).

1. \( |\varphi| = 1 \): in this case we have \( k_\mu = n \) and \( t_\mu = 0 \), thus its contribution is
   \[ n! \sum_{\mu = (\varphi, c) \in \text{Comp}^{\text{clr}}(n) \atop |\varphi| = 1} (q + r - 2)^{t_\mu} (r-1)^{n-k_\mu-t_\mu} = n!. \]

2. \( |\varphi| \) is even: in this case, \( k_\mu \) can vary in \( \{1, \ldots, n-1\} \) and hence \( b_\mu \in \{1, \ldots, k_\mu\} \). Moreover, \( \bar{b}_\mu = t_\mu = b_\mu \). Then the contribution is
   \[ n! \sum_{\mu = (\varphi, c) \in \text{Comp}^{\text{clr}}(n) \atop |\varphi| \text{ even}} (q + r - 2)^{t_\mu} (r-1)^{n-k_\mu-t_\mu} = n! \sum_{k=1}^{n-1} \sum_{b=1}^{k} \binom{k-1}{k-b} \binom{n-k-1}{n-k-b} (q + r - 2)^{b} (r-1)^{n-k-b}. \]

   Note that the first binomial coefficient corresponds to the choice of dividing the \( k \) digits colored by \( c \) into \( b \) non-empty blocks, while the second binomial coefficient corresponds to the choice of dividing the remaining \( n-k \) digits (which are not colored by \( c \)) into \( b \) non-empty blocks.

3. \( |\varphi| > 1 \) odd: In this case, \( k_\mu \) can vary in \( \{2, \ldots, n-1\} \) and hence \( b_\mu \in \{2, \ldots, k_\mu\} \). Moreover, \( \bar{b}_\mu = b-1, t_\mu = b_\mu - 1 \). Then
the contribution is

$$\sum_{\mu=(\varphi,\bar{c})\in \text{Comp}^{l}(n)} (q + r - 2)^{t_{\mu}}(r - 1)^{n-k_{\mu}-t_{\mu}}$$

$$= n! \sum_{k=2}^{n-1} \sum_{b=1}^{k} \left( \frac{k-1}{k-b} \right) \left( \frac{n-k-1}{n-k-b+1} \right) (q + r - 2)^{b-1}(r - 1)^{n-k-b+1}.$$ 

Now, let us treat the set $\text{Comp}(n) \times \{\bar{c}\}$. Let $\mu = (\varphi, \bar{c}) \in \text{Comp}(n) \times \{\bar{c}\}$.

As in the case of $\text{Comp}(n) \times \{c\}$, we split the computation into three cases according to $|\varphi|$.

1. $|\varphi| = 1$:

$$n! \sum_{\mu=(\varphi,\bar{c})\in \text{Comp}^{l}(n)} (q + r - 2)^{t_{\mu}}(r - 1)^{n-k_{\mu}-t_{\mu}} = n!(r - 1)^{n}.$$ 

2. $|\varphi|$ is even: in this case, $k_{\mu} \in \{1, \ldots, n - 1\}$ and hence $b_{\mu} \in \{1, \ldots, k_{\mu}\}$. Moreover, $b_{\mu} = t_{\mu} = b_{\mu} - 1$. Then we have

$$n! \sum_{\mu=(\varphi,\bar{c})\in \text{Comp}^{l}(n)} (q + r - 2)^{t_{\mu}}(r - 1)^{n-k_{\mu}-t_{\mu}}$$

$$= n! \sum_{k=1}^{n-1} \sum_{b=1}^{k} \left( \frac{k-1}{k-b} \right) \left( \frac{n-k-1}{n-k-b} \right) (q + r - 2)^{b-1}(r - 1)^{n-k-b+1}.$$ 

3. $|\varphi| > 1$ odd: in this case, $k_{\mu}$ can vary in $\{1, \ldots, n - 2\}$ and hence $b_{\mu} \in \{2, \ldots, k_{\mu}\}$. Moreover, $b_{\mu} = b_{\mu} + 1, t_{\mu} = b_{\mu}$. Then we have

$$n! \sum_{\mu=(\varphi,\bar{c})\in \text{Comp}^{l}(n)} (q + r - 2)^{t_{\mu}}(r - 1)^{n-k_{\mu}-t_{\mu}}$$

$$= n! \sum_{k=1}^{n-2} \sum_{b=1}^{k} \left( \frac{k-1}{k-b} \right) \left( \frac{n-k-1}{n-k-b-1} \right) (q + r - 2)^{b}(r - 1)^{n-k-b}.$$
Summing up all the parts, we get
\[
A_{r,n}^{\text{Clr}}(q) = n!(1 + (r - 1)^n)
\]
\[
+ n! \sum_{k=1}^{n-1} \sum_{b=1}^{k} \binom{k}{k-b} \binom{n-k}{n-k-b} (q + r - 2)^b (r - 1)^{n-k-b}
\]
\[
+ n! \sum_{k=1}^{n-1} \sum_{b=1}^{k} \binom{k}{k-b} \binom{n-k}{n-k-b+1}
\]
\[
\cdot (q + r - 2)^{b-1} (r - 1)^{n-k-b+1}
\]
\[
= n! \sum_{k=0}^{n} (r - 1)^k
\]
\[
+ n! \sum_{k=1}^{n-1} \sum_{b=1}^{k} \binom{k}{b} \binom{n-k}{b} (q + r - 2)^b (r - 1)^{n-k-b}.
\]

Here is the analogous result for the absolute order.

**Corollary 2.2.**

\[
A_{r,n}^{\text{Abs}}(q) = n! \sum_{k=0}^{n} (r - 1)^k
\]
\[
+ n! \sum_{k=1}^{n-1} \sum_{b=1}^{k} \binom{k}{b} \binom{n-k}{b} \left( \frac{q}{2} + r - 2 \right)^b (r - 1)^{n-k-b}.
\]

Hence, we get the following result.

**Proposition 2.3.** The number of colored permutations in \(G_{r,n}\) with exactly 0 \((c,d)\)-descents, \(0 \leq c < d \leq r - 1\), is

\[
n! \sum_{k=0}^{n} (r - 1)^k + n! \sum_{k=1}^{n-1} \sum_{b=1}^{k} \binom{k}{b} \binom{n-k}{b} (r - 2)^{b-m} (r - 1)^{n-k-b}.\]

The number of colored permutations in \(G_{r,n}\) with exactly \(m > 0\) \((c,d)\)-descents \((0 \leq c < d \leq r - 1)\) with respect to the color order, is given by

\[
n! \sum_{k=1}^{n-1} \sum_{b=1}^{k} \binom{k}{b} \binom{n-k}{b} \binom{b}{m} (r - 2)^{b-m} (r - 1)^{n-k-b}.
\]

The number of colored permutations in \(G_{r,n}\) with exactly \(m > 0\) \((c,d)\)-descents \((0 \leq c < d \leq r - 1)\) with respect to the absolute order,
is given by

\[
\frac{n!}{2^m} \sum_{k=1}^{n-1} \sum_{b=1}^{k} \binom{k}{b} \binom{n-k}{b} \binom{b}{m} (r - 2)^{b-m} (r - 1)^{n-k-b}.
\]

This proves Proposition 1.1.

3. \((c, c)\)-descents

In this section, we calculate the number of permutations of \(G_{r,n}\) having a fixed number of \((c, c)\)-descents. Note that in this case, the enumeration is identical in the color order and the absolute order, since both orders satisfy

\[
1^{[c]} < 2^{[c]} < \ldots < n^{[c]}.
\]

Hence, we denote: \(\text{des}_c(\pi) = \text{des}_{c,c}^{\text{Clr}}(\pi) = \text{des}_{c,c}^{\text{Abs}}(\pi)\).

We use here a recursive approach. Let \(\pi = i_1^{[j_1]} \cdots i_n^{[j_n]} \in G_{r,n}\). Using a map \(f : G_{r,n} \to G_{r,n}\) which takes each colored digit \(i^{[j]}\) to \(i^{[(j+1) \text{ (mod } r)]}\), we are led to the following.

**Observation 3.1.** For each color \(c \in \{0, \ldots, r-1\}\), the number of colored permutations with exactly \(m\) \((c, c)\)-descents is equal to the number of colored permutations with exactly \(m\) \((0, 0)\)-descents.

By the above observation, it suffices to find the generating function for the number of colored permutations in \(G_{r,n}\) having \((0, 0)\)-descents.

Let

\[
g_{r,n}(q) = \sum_{\pi \in G_{r,n}} q^{\text{des}_0(\pi)}.
\]

In order to find a recurrence for \(g_{r,n}(q)\), we will use the following notations. Define

\[
g_{r,n}^+(q) = \sum_{\pi \in G_{r,n}, \ c_1(\pi) = 0} q^{\text{des}_0(\pi)},
\]

\[
g_{r,n}^-(q) = \sum_{\pi \in G_{r,n}, \ c_1(\pi) \neq 0} q^{\text{des}_0(\pi)}.
\]

Note that \(g_{r,n}(q) = g_{r,n}^+(q) + g_{r,n}^-(q)\).

Let \(\pi \in G_{r,n}\). If \(|\pi_n| = n\), denote \(\pi = \pi' \pi_n\) and we have

\[
\text{des}_0(\pi) = \text{des}_0(\pi').
\]

Otherwise, let \(j < n\) be such that \(|\pi_j| = n\). Denote: \(\pi = \pi' \pi_j \pi'' \in G_{r,n}\). If \(\pi_j = n^{[c]}\) for \(c \neq 0\), then \(\text{des}_0(\pi) = \text{des}_0(\pi') + \text{des}_0(\pi'')\). If
\( \pi_j = n \), we have two cases depending on the color of \( \pi_1'' \) (the first element of \( \pi'' \)):

\[
\text{des}_0(\pi) = \begin{cases} 
\text{des}_0(\pi') + \text{des}_0(\pi'') & \text{if } c_1(\pi'') \neq 0, \\
1 + \text{des}_0(\pi') + \text{des}_0(\pi'') & \text{otherwise}.
\end{cases}
\]

Since there are no restrictions on the positions of the digits \( 1, 2, \ldots, n-1 \), we have the following recurrences.

**Lemma 3.2.** For all \( n \geq 1 \), we have

\[
gr_{r,n}(q) = rgr_{r,n-1}(q) + (r - 1) \sum_{j=1}^{n-1} \binom{n-1}{j-1} gr_{r,j-1}(q) gr_{r,n-j}(q)
+ \sum_{j=1}^{n-1} \binom{n-1}{j-1} gr_{r,j-1}(q)(gr_{r,n-j}(q) + qg_{r,n-j}(q)),
\]

and for \( n \geq 2 \), we have

\[
g^+_{r,n}(q) = r g^+_{r,n-1}(q) + (r - 1) \sum_{j=2}^{n-1} \binom{n-1}{j-1} g^+_{r,j-1}(q) gr_{r,n-j}(q)
+ \sum_{j=1}^{n-1} \binom{n-1}{j-1} g^+_{r,j-1}(q)(gr_{r,n-j}(q) + qg^+_{r,n-j}(q)).
\]

**Proof.** In both expressions, the first summand corresponds to the case \( |\pi_n| = n \), the second corresponds to the case \( \pi_j = n[c] \) (\( c \neq 0 \)) for some \( j, 1 \leq j \leq n - 1 \), and the third corresponds to the case \( \pi_j = n \) for some \( j, 1 \leq j \leq n - 1 \). \( \square \)

In order to find an explicit formula for \( gr_{r,n}(q) \), we rewrite the above recurrences in terms of exponential generating functions. Define

\[
G_r(x, q) = \sum_{n \geq 0} gr_{r,n}(q) \frac{x^n}{n!}, \quad G^+_r(x, q) = \sum_{n \geq 0} g^+_{r,n}(q) \frac{x^n}{n!},
\]

and

\[
G^-_r(x, q) = \sum_{n \geq 0} g^-_{r,n}(q) \frac{x^n}{n!}.
\]

Since \( gr_{r,n}(q) = g^+_{r,n}(q) + g^-_{r,n}(q) \), we have

\[
G_r(x, q) = G^+_r(x, q) + G^-_r(x, q).
\]
Moreover, for all $n \geq 1$, Lemma 3.2 yields
\[
\frac{d}{dx} \left( \frac{g_r, n(q)}{n!} x^n \right) = \frac{r g_{r-1, n}(q)}{(n-1)!} x^{n-1}
+ (r-1) x^{n-1} \sum_{j=0}^{n-2} \frac{g_{r, j}(q)}{j!} \cdot \frac{g_{r, n-1-j}(q)}{(n-1-j)!}
+ x^{n-1} \sum_{j=0}^{n-2} \frac{g_{r, j}(q)}{j!} \cdot \frac{g_{r, n-1-j}(q) + q g_{r, n-1-j}^+(q)}{(n-1-j)!},
\]
and
\[
\frac{d}{dx} \left( \frac{g_{r, n}^+(q)}{n!} x^n \right) = \frac{r g_{r, n-1}^+(q)}{(n-1)!} x^{n-1}
+ (r-1) x^{n-1} \sum_{j=1}^{n-2} \frac{g_{r, j}(q)}{j!} \cdot \frac{g_{r, n-1-j}(q)}{(n-1-j)!}
+ x^{n-1} \sum_{j=0}^{n-2} \frac{g_{r, j}(q)}{j!} \cdot \frac{g_{r, n-1-j}(q) + q g_{r, n-1-j}^+(q)}{(n-1-j)!}.
\]
Summing over all $n \geq 1$ and using the initial conditions $g_{r, 0}(q) = g_{r, 0}^+(q) = g_{r, 1}^+(q) = 1$, $g_{r, 0}^-(q) = 0$ and $g_{r, 1}^-(q) = r - 1$, we get
\[
G_r(x, q) = G_r^+(x, q) + G_r^-(x, q),
\]
\[
\frac{d}{dx} (G_r(x, q)) = (1-q) G_r(x, q) + (r-1) (G_r(x, q))^2
+ G_r(x, q) G_r^-(x, q) + q G_r(x, q) G_r^+(x, q),
\]
\[
\frac{d}{dx} (G_r^+(x, q)) = (1-q) G_r^+(x, q) + (r-1) G_r^+(x, q) G_r(x, q)
+ G_r^+(x, q) G_r^-(x, q) + q (G_r^+(x, q))^2.
\]
Using any computer algebra package, such as Maple or Mathematica, we obtain the following result.

**Theorem 3.3.** The generating functions $G_r(x, q)$, $G_r^+(x, q)$ and $G_r^-(x, q)$ are given by
\[
G_r(x, q) = \frac{1 - q}{(r-1)x(q-1) - q + e^{(q-1)x}},
\]
\[
G_r^+(x, q) = \frac{(1-q)(1-(r-1)x)}{(r-1)x(q-1) - q + e^{(q-1)x}},
\]
\[
G_r^-(x, q) = \frac{x(r-1)(1-q)}{(r-1)x(q-1) - q + e^{(q-1)x}}.
\]
respectively.

In order to obtain an explicit formula for the number of colored permutations in $G_{r,n}$ with exactly $m$ $(0,0)$-descents, we find the coefficient of $x^n q^m$ in $G_r(x, q)$:

$$G_r(x, q) = \frac{1 - q}{e^{(q-1)x} - q} \cdot \frac{1}{e^{(q-1)x} - q}$$

$$= \sum_{j \geq 0} \frac{(r-1)^j x^j (1-q)^{j+1}}{(e^{(q-1)x} - q)^{j+1}}$$

$$= \sum_{j,i,k \geq 0} \frac{(r-1)^j x^j (1-q)^{j+1}}{(i+j+1)!} \cdot q^i (1-q)^k (i+j+1)^k$$

Thus the coefficient of $x^n q^m$ in $G_r(x, q)$ is given by

$$\sum_{j=0}^{n} \sum_{i=0}^{m} (-1)^i (r-1)^j \binom{j+m-i}{j} \binom{n+1}{i} \frac{(m-i+j+1)^{n-j}}{(n-j)!}.$$

This implies that the number of colored permutations in $G_{r,n}$ with exactly $m$ $(0,0)$-descents (or $(c,c)$-descents for any $0 \leq c \leq r-1$) is

$$n! \sum_{j=0}^{n} \sum_{i=0}^{m} (-1)^j (r-1)^j \binom{j+m-i}{j} \binom{n+1}{i} \frac{(m-i+j+1)^{n-j}}{(n-j)!},$$

as stated in Proposition 1.3.

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