DERIVATIONS OF GROUP RINGS
OREST D. ARTEMOVYCH, VICTOR A. BOVDI, MOHAMED A. SALIM

Abstract. Let \( R[G] \) be the group ring of a group \( G \) over an associative ring \( R \) with unity such that all prime divisors of orders of elements of \( G \) are invertible in \( R \). If \( R \) is finite and \( G \) is a Chernikov (torsion \( FC \)-) group, then each \( R \)-derivation of \( R[G] \) is inner. Similar results also are obtained for other classes of groups \( G \) and rings \( R \).

Dedicated to the memory of Professor V.I. Sushchansky

1. Introduction

Let \( B \) be an associative ring not necessarily with unity 1. An additive map \( \delta : B \to B \) is called a derivation of \( B \) if

\[
\delta(ab) = \delta(a)b + a\delta(b) \quad (a, b \in B).
\]

The set \( \text{Der} B \) of all derivations of \( B \) is a Lie ring with respect to a point-wise addition and a point-wise Lie multiplication of derivations. The zero map \( 0 : B \ni r \mapsto 0 \in B \) is a derivation of \( B \). A ring \( B \) having only zero derivation (i.e., \( \text{Der} B = 0 \)) is called differentially trivial (see [5]).

The rule

\[
\partial_a : B \ni x \mapsto ax - xa \in B
\]
determines a derivation \( \partial_a \) of \( B \) (so-called an inner derivation of \( B \) induced by \( a \in B \)). If \( \delta \in \text{Der} B \) is not inner, then it is called outer. The set \( \text{IDer} B := \{ \partial_a \mid a \in B \} \) of all inner derivations of \( B \) is an ideal of the derivation ring \( \text{Der} B \) (see [27]).

Throughout the paper \( p \) is a prime, \( \mathbb{Z} \) the ring of integers and \( \mathbb{N} \) the set of positive integers. If \( X, Y \subseteq B \), then a subgroup of the additive group \( B^+ \) generated by the Lie commutators \([g, h] = gh - hg\), where \( g \in X \) and \( h \in Y \), is denoted by \([X, Y]\). The commutator ideal \( C(B) \) is an ideal of \( B \) generated by the set \([B, B]\). Furthermore, \( \text{ann} K = \{ r \in B \mid rK = Kr = 0 \} \) is the annihilator of \( K \) in \( B \), \( F(B) = \{ a \in B \mid a \text{ is of finite index in } B^+ \} \) is the torsion part of \( B \), \( \mathbb{P}(B) \) is the prime radical of \( B \), i.e., the intersection of all prime ideals of \( B \), and \( Q(B) \) is the classical quotient ring of \( B \) (see e.g. [88] Chapters 2 and 5).

In the next \( R \) is an associative ring with unity 1, \( G \) a multiplicative group and \( R^\mathbb{Z} \) is a direct sum of \( \text{card} \mathbb{Z} \) copies of \( R \). The torsion part of a group \( G \) we denote by \( \tau G \) and the set of primes that divide orders of elements of \( \tau G \) by \( \pi(G) \), respectively. Let \( \Delta^+(G) \) be the set of all elements \( g \in G \) of finite orders such that \( |G : C_G(x)| < \infty \). The center of a group (respectively a ring) \( X \) we denote by \( Z(X) \).

\(^1\)Corresponding Author: V.A. Bovdi
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Let $R[G]$ be the group algebra of a group $G$ over a ring $R$. The group of units $U(R[G])$ of the group algebra $R[G]$ can be written as $U(R[G]) = U(R) \times V(R[G])$, where $U(R)$ is the group of units of the ring $R$ and

$$V(R[G]) := \left\{ \sum_{g \in G} \alpha_g g \in U(R[G]) \mid \sum_{g \in G} \alpha_g = 1 \right\}$$

is the subgroup of the normalized units of $R[G]$. It follows that

$$(2) \quad [U(R), V(R[G])] = 0.$$ 

Finally, any unexplained terminology is standard as in [14, 22, 24, 43].

If $\delta(R) = 0$, then a derivation $\delta$ of a group ring $R[G]$ is called an $R$-derivation of $R[G]$. The set $\text{Der}_R R[G]$ of all $R$-derivations of $R[G]$ is a subring of the Lie ring $\text{Der} R[G]$.

Smith (see [43]) was first who started to study the derivations in group rings. For instance, she showed that in group rings of torsion-free nilpotent groups always there exists an outer derivation. In certain papers (see [21, 44, 45]) the properties of group rings $R[G]$ such that its every derivation is inner were studied. In [21] it was proved that $R$-derivations of a group ring $R[G]$ of a center-by-finite group $G$ over a semiprime ring $R$, where char $R = 0$ or every prime char $R \notin \pi(G)$, are inner. In [16] Burkov proved that every $K$-derivation $\theta$ of a group ring $K[G]$, where $K$ is a commutative ring, $G$ is a torsion group and the order of each element of $G$ is invertible in $R$, is generalized inner (i.e., there exists $a \in (K[G])^*$ such that $\theta(x) = xa - ax$ for each $x \in K[G]$, where $(K[G])^*$ is a $(K[G], K[G])$-bimodule of all functions from $G$ in $K$). Derivations of certain rings were investigated in [2, 3, 6, 7, 8, 9, 10, 31, 32, 33, 34, 35, 36, 37, 42, 45, 48].

Our main results are the following

**Theorem 1.** Let $R$ be a finite ring. If $G$ is a Chernikov group such that $\pi(G) \cap \pi(R^+) = \emptyset$, then every $R$-derivation of $R[G]$ is inner.

**Theorem 2.** Let $R$ be a ring. If $G$ is a torsion abelian group such that all primes $p \in \pi(G)$ are invertible in $R$, then $\text{Der}_R R[G] = 0$.

**Theorem 3.** Let $R$ be a finite ring, $G$ a countable locally finite group such that every prime $p \in \pi(G)$ is invertible in $R$. If $G$ has a finite subgroup $H$ with the finite centralizer $C_{R[G]}(H)$, then every $R$-derivation of $R[G]$ is inner.

Also if $R$ is finite and $G$ is a torsion $FC$-group, then every $R$-derivation of $R[G]$ is inner (see Corollary 4).

Recall that a ring $B$ is prime if, for any ideals $X, Y$ of $B$, we have $XY = 0$ implies that $X = 0$ or $Y = 0$. A ring $B$ without nonzero nilpotent ideals is called semiprime.

In [29] Kaplansky has conjectured that the group algebra $F[G]$ of a torsion-free group $G$ over a field $F$ has no nontrivial idempotents. We prove the following.

**Theorem 4.** If $R$ is a prime ring of characteristic $\neq 2, 3$ with the non-central unit group $U(R)$ and $G$ a group such that $\Delta^+(G) = 1$, then the following conditions hold:

(i) if $G$ is non-abelian, then $R[G]$ has only trivial idempotents;

(ii) if every prime $p \in \pi(G)$ is invertible in $R$ and $R[G]$ has a non-trivial idempotent, then $G$ is abelian, $\delta(\tau G) = 0$ for each $\delta \in \text{Der}_R R[G]$ (and so $\text{IDer}_R R[G] = 0$).
A Lie ring $D$ is called nilpotent if $\gamma_{n+1}D = 0$ for some $n \in \mathbb{N}$, where

$$\gamma_1 D := D, \ldots, \gamma_{k+1} D := [\gamma_k D, D], \ldots \quad (k \in \mathbb{N}).$$

The least such $n$ is the class of nilpotency of $D$. If $n \in \mathbb{N}$ and $x_1, \ldots, x_n, x, y \in B$, then the left normed commutators are defined inductively:

- $[x, y] := [x_1 y] := xy - yx$;
- $[x, n+1 y] := [[[x, n] y], y]$;
- $[x_1, \ldots, x_{n-1}, x_n] := [[[x_1, \ldots, x_{n-1}], x_n]$.

The lower central chain of $B$ is

$$B = \gamma_1 B \supseteq \gamma_2 B \supseteq \cdots \supseteq \gamma_n B \supseteq \cdots,$$

where $\gamma_{n+1} := [\gamma_n B, B]$ for integers $n \geq 1$. If $\gamma_{n+1} = 0$ and $\gamma_n B \neq 0$ for some $n \in \mathbb{N}$, then $B$ is called Lie nilpotent of class $n$. Each associative ring $B$ can be considered as a Lie ring $B^L$ under operations “$[−, −]$” and “$+$”, which is called the associated Lie ring of $B$. It is easy to see that an associative ring $B$ is Lie nilpotent if and only if the associated Lie ring $B^L$ is nilpotent.

A Lie ring $D$ is called solvable if $D^{(n+1)} = 0$ for some $n \in \mathbb{N}$, where $D^{(1)} := D$, $D^{(n+1)} := [D^{(n)}, D^{(n)}]$. The least such $n$ is the derived length of $D$. The derived chain of $B$ is given by

$$B = B^{(0)} \supseteq B^{(1)} \supseteq \cdots \supseteq B^{(n)} \supseteq \cdots,$$

where $B^{(n)} := [B^{(n-1)}, B^{(n-1)}]$, $n \geq 1$. An associative ring $B$ is Lie solvable of length $n$ if $B^{(n)} = 0$, $n$ least. An associative ring $B$ is Lie solvable if and only if the associated Lie ring $B^L$ is solvable.

**Theorem 5.** Let $B$ be a semiprime ring. The Lie ring $\text{Der} B$ is nilpotent (respectively solvable) if and only if $\text{Der} B = 0$, i.e., $B$ is differentially trivial.

As an immediate consequence of the previous result we obtain the following.

**Theorem 6.** Let $\mathbb{F}$ be a division ring of characteristic $p \geq 0$, $G$ a group. If each prime $q \in \pi(G)$ is invertible in $\mathbb{F}$, then $\text{Der} \mathbb{F}[G]$ is a nilpotent (respectively solvable) Lie ring if and only if $\text{Der} \mathbb{F}[G] = 0$.

The set of all sequences $\{D_n \in \text{Der} R\}_{n \in \mathbb{N}}$ such that $D_n(a) = 0$ for every $a \in R$ and almost all $n \in \mathbb{N}$ we denote by $\mathcal{LF}(R)$. Evidently, $\mathcal{LF}(R)$ is a Lie ring with respect to an addition “$+$” and the Lie multiplication “$[−, −]$” defined by the rules:

$$\{D_n\}_{n \in \mathbb{N}} + \{K_n\}_{n \in \mathbb{N}} = \{D_n + K_n\}_{n \in \mathbb{N}}, \quad \{D_n\}_{n \in \mathbb{N}}, \{K_n\}_{n \in \mathbb{N}} = \{[D_n, K_n]\}_{n \in \mathbb{N}},$$

where $\{D_n\}_{n \in \mathbb{N}}, \{K_n\}_{n \in \mathbb{N}} \in \mathcal{LF}(R)$. As in \cite{L}, the family $\mathcal{LF}(R)$ is called locally finite. The next result is analogously with \cite[Theorem 2]{L}.

**Theorem 7.** Let $R$ be a ring. If $G$ is a countable torsion-free abelian group, then the following conditions hold:

(i) $\text{Der} R[G] \cong \mathcal{LF}(R) \oplus (Z(R[G]))^\mathbb{Z}$ as Lie rings;

(ii) each nonzero $R$-derivation of $R[G]$ is outer.

In section 4 we prove that the behavior of the $R$-derivations of a group ring $R[G]$ of a nilpotent group $G$ with some restriction is very similar to the nilpotency property of derivations (see Proposition 5).
2. Solders and central derivations

If \( H \leq G \), then \( \mathcal{I}_R(H) \) is a right ideal of \( R[G] \) generated by the set \( \{1 - h \mid h \in H\} \). If \( H \) is normal in \( G \), then \( \mathcal{I}_R(H) \) is an ideal of \( R[G] \) and we have a ring isomorphism

\[
R[G]/\mathcal{I}_R(H) \cong R[G/H]
\]

[14] Proposition 1, p. 17. Therefore, we have some Lie ring isomorphism

\[
\text{Der}(R[G]/\mathcal{I}_R(H)) = \text{Der} R[G/H].
\]

A ring \( R \) is called \( n \)-torsion-free (\( n \in \mathbb{N} \)) if the implication \( nx = 0 \Rightarrow x = 0 \) is true for each \( x \in R \).

**Lemma 1.** Let \( R \) be a ring. The following conditions hold:

(i) if \( U \) is a subsemigroup of the multiplicative semigroup \( (R, \cdot) \) of \( R \) and \( h : U \to R^+ \) is a semigroup homomorphism, then \( \delta_h : U \ni u \mapsto uh(u) \in R^+ \) satisfies the Leibniz rule (see Eq. [11]) if and only if \( u[R, h(u)] = 0 \) for each \( u \in U \);

(ii) if \( h : (R, \cdot) \to R^+ \) is a semigroup homomorphism, then we have:
   (a) \( h(1) = 0 = h(0) \) and \( 2h(-1) = 0 \);
   (b) if \( R \) is 2-torsion-free, then \( h(a) = h(-a) \) for each \( a \in R \);

(iii) if \( U \) is a subgroup of the unit group \( U(R) \) and \( \delta \in \text{Der} R \), then

\[
L_\delta : U \ni a \mapsto a^{-1}\delta(a) \in R^+
\]

is a group homomorphism if and only if \( [a^{-1}\delta(a), U] = 0 \) for each \( a \in U \).

**Proof.** Let \( a, b \in U \).

(i) It is easy to see that

\[
ah(a)b + abh(b) = \delta_h(a)b + a\delta_h(b) = \delta_h(ab) = abh(ab) = abh(a) + abh(b)
\]

if and only if \( a(h(a)b - bh(a)) = 0 \), so the proof is done.

(ii) Let \( h : (R, \cdot) \to R^+ \) be a semigroup homomorphism.

(a) Clearly, \( h(1) = h(1^2) = h(1) + h(1) \) and \( h(0) = h(0^2) = h(0) + h(0) \), so \( h(1) = 0 = h(0) \).

Furthermore, \( 0 = h(1) = h((-1) \cdot (-1)) = h(-1) + h(-1) \).

(b) Since \( h(-1) = 0 \), we get that \( h(-a) = h((-1)a) = h(-1) + h(a) = h(a) \) for each \( a \in R \).

(iii) It is easy to check that

\[
L_\delta(a) + L_\delta(b) = L_\delta(ab) = (ab)^{-1}\delta(ab) = b^{-1}a^{-1}\delta(b) + a^{-1}b^{-1}\delta(a)b
\]

\[
= b^{-1}\delta(b) + b^{-1}ba^{-1}\delta(a) + b^{-1}[a^{-1}\delta(a), b]
\]

\[
= L_\delta(a) + L_\delta(b) + b^{-1}[a^{-1}\delta(a), b]
\]

if and only if \( b^{-1}[a^{-1}\delta(a), b] = 0 \). Thus \( [a^{-1}\delta(a), U] = 0 \) .

A map \( h : R \to R \) is called a solder of \( R \) (see Nowicki [39], p. 46) if:

(i) \( (a + b)h(a + b) = ah(a) + bh(b) \) (\( a, b \in R \));

(ii) \( h(ab) = h(a) + h(b) \) (\( a, b \in R \setminus \{0\} \)).

**Lemma 2.** Let \( h \) be a solder of a ring \( R \). The following conditions hold:

(i) \( h(xy) = h(yx) \) for each \( x, y \in R \);
(ii) the rule \( \delta_h : R \ni x \mapsto xh(x) \in R \) determines a derivation \( \delta_h \in \text{Der} R \) if and only if \( x[h(x), R] = 0 \) for each \( x \in R \);

(iii) if \( R \) is a prime ring, then \( \delta_h \in \text{Der} R \) if and only if \( h(R) \subseteq Z(R) \) (such solder is called central);

(iv) if \( R \) is 2-torsion-free, then \( h(2) = 0 \);

(v) if \( e^2 = e \in R \), then \( h(e) = 0 \);

(vi) if \( xy = 1 \) for some \( x, y \in R \), then \( h(x) = -h(y) \); in particular, if \( x^2 = 1 \), then \( 2h(x) = 0 \).

Proof. (i) Since \( h(0) = 0 \) and \( \delta_h(xy) = xyh(x) + xyh(y) \) for each \( x, y \in R \), the result follows.

(ii) It follows from Lemma 1(i) because \( \delta_h \) is additive.

(iii) If \( x, y \in R \), then \( \delta_h \) is a derivation \( R \) if and only if

\[
xy[h(x), R] = xy[h(x), R] + xy[h(y), R] = xy[h(xy), R] = 0
\]

by the part (ii). Thus \( xR[h(x), R] = 0 \) and so \( [h(x), R] = 0 \) by the primeness of \( R \).

(iv) Obviously, \( 0 = 1 \cdot h(1) + 1 \cdot h(1) = (1 + 1)h(1 + 1) = 2h(2) \), therefore \( h(2) = 0 \).

(v) In as much as \( h(e) = h(e^2) = h(e) + h(e) \), we have that \( h(e) = 0 \).

(vi) Evidently, \( 0 = h(1) = h(xy) = h(x) + h(y) \), so \( h(x) = -h(y) \).

The subgroup of a group \( G \) generated by the set \( \{ g^n \mid g \in G \} \) we denote by \( G^n \). The exponent of a torsion group \( G \) is the following number \( \exp(G) = \min\{ n \in \mathbb{N} \mid x^n = 1, \forall x \in G \} \).

Proposition 1. Let \( R \) be a ring, \( G \) a group and \( n \in \mathbb{N} \). The following conditions hold:

(i) \( \text{ZDer} R := \{ \delta \in \text{Der} R \mid \delta(R) \subseteq Z(R) \} \) is an ideal of the Lie ring \( \text{Der} R \);

(ii) if \( \delta \in \text{ZDer} R \), then \( \delta([R, R]) = 0 \);

(iii) if \( R \) is \( n \)-torsion-free and the exponent \( \exp(G) = n \), then \( \delta(G) = 0 \) for each \( \delta \in \text{ZDer} R[G] \);

(iv) if \( nR = 0 \) and \( G \) is an abelian torsion-free group, then \( \delta(G^n) = 0 \) for each \( \delta \in \text{ZDer} R[G] \).

Proof. Let \( \delta \in \text{ZDer} R[G] \) and \( d \in \text{Der} R \).

(i) Indeed, \( [\delta, d](r) = \delta(d(r)) - d(\delta(r)) \in Z(R) \) for any \( r \in R \) and the result holds.

(ii) It follows in view of the part (i).

(iii) The map \( L_\delta : G \to R[G]^+ \) is a group homomorphism for any \( \delta \in \text{ZDer} R[G] \) and \( g \in G \) by Lemma 1(iii). Using facts that \( g^n = 1 \) for \( g \in G \) and

\[
g^{-n}\delta(g^n) = L_\delta(g^n) = nL_\delta(g),
\]

we deduce that \( L_\delta(g) = 0 \), so \( \delta(g) = 0 \).

(iv) In as much as \( \delta(G) \subseteq Z \text{Der} R[G] \), the map \( L_\delta : G \to R^+ \) is a group homomorphism by Lemma 1(iii). Hence Eqs. (3) hold for any \( g \in G \) and \( n \in \mathbb{N} \). Since \( nL_\delta(g) = 0 \), we deduce that \( \delta(g^n) = 0 \). \( \square \)

A derivation \( \delta \in \text{Der} R \) is called central if \( \delta(R) \subseteq Z(R) \) (see [49]). The set of all central \( R \)-derivations of \( R[G] \) we denote by \( \text{ZDer}_R R[G] \). It is easy to check that

\[
\text{ZDer}_R R[G] = (\text{ZDer} R[G]) \cap (\text{Der}_R R[G]).
\]

Clearly, \( Z(G) \subseteq Z(R[G]) \), \( Z(R)[Z(G)] \subseteq Z(R)[G] \) and \( d(Z(R)) \subseteq Z(R) \) for any \( d \in \text{Der} R \).
Proposition 2. Let $R$ be a ring and let $G$ be a group. The following conditions hold:

(i) if $\delta \in \text{Der}_R R[G]$, then $\delta(g) \in Z(R)[G]$ for any $g \in G$;

(ii) $Z\text{Der}_R R[G]$ is an ideal of the Lie ring $\text{Der}_R R[G]$;

(iii) if $G$ is a torsion group such that $\pi(F(R)) \cap \pi(G) = \emptyset$ (respectively $G$ is an abelian divisible torsion-free group and $nR = 0$ for some $n \in \mathbb{N}$), then $Z\text{Der}_R R[G] = 0$;

(iv) if $\pi(F(R)) \cap \pi(G) = \emptyset$, then $\delta(\pi G) = 0$ for any $\delta \in Z\text{Der}_R R[G]$;

(v) if $G$ is a torsion group and $\pi(F(R)) \cap \pi(G) = \emptyset$, then $\text{Der}_R R[G] = 0$ if and only if $\delta(G) = 0$ for any $\delta \in \text{Der}_R R[G]$.

Proof. (i) Obviously, because $\delta(g)r = \delta(gr) = \delta(rg) = r\delta(g)$ for any $g \in G$ and $r \in R$.

(ii) If $\delta \in \text{Der}_R R[G]$ and $\mu \in Z\text{Der}_R R[G]$, then

$$[\delta, \mu](x) = \delta(\mu(x)) - \mu(\delta(x)) \in Z(R[G]) \quad \text{and} \quad [\delta, \mu](r) = 0$$

for any $x \in R[G]$ and $r \in R$. Hence $[\delta, \mu] \in Z\text{Der}_R R[G]$.

(iii) Suppose that $\delta \in Z\text{Der}_R R[G]$ and $g \in G$.

a) If $G$ is torsion and $n = |g|$, then

$$0 = \delta(1) = \delta(g^n) = ng^{n-1}\delta(g),$$

this gives that $\delta(g) = 0$, so $\delta = 0$.

b) Assume that $G$ is abelian divisible torsion-free and $nR = 0$. Then $nL_\delta(g) = 0$ and Eqs. (3) imply that $\delta(g^n) = 0$ and thus $\delta = 0$.

(iv) If $g \in \pi G$ and $|g| = n$, then $\delta(g) = 0$ by Eqs. (4).

(v) ($\Rightarrow$) If $\text{Der}_R R[G] = 0$, then $\partial_g = 0$ for any $g \in G$ and so $G \subseteq Z(R[G])$. Hence $\delta(G) \subseteq Z(R[G])$ for any $\delta \in \text{Der}_R R[G]$. Since $g^n = 1$ for some $n \in \mathbb{N}$ and $L_\delta : \langle g \rangle \to R[G]^+$ is a group homomorphism by Lemma 1(iii), we have that Eqs. (3) imply that $nL_\delta(g) = 0$. Hence $L_\delta(g) = 0$, because $\pi(F(R)) \cap \pi(G) = \emptyset$. Therefore $\delta(g) = 0$, what gives $\delta(G) = 0$.

($\Leftarrow$) Obviously.

Proof of Theorem 2. Let $\delta \in \text{Der}_R R[G]$. Then $\delta(g) \in Z(R)[G] \subseteq Z(R[G])$ for any $g \in G$ by Proposition 2(i). In as much as $R$ is $|g|$-torsion-free, we conclude that $\delta(g) = 0$ be the same argument as in the proof of Proposition 1(iii). Hence $\delta(G) = 0$ and consequently $\delta = 0$.

Since $\text{Der}_R R[G] = \text{Der}_R R[G]$ in a differentially trivial ring $R$, Theorem 2 implies the following.

Corollary 1. Let $R$ be a differentially trivial ring. If $G$ is a torsion abelian group such that each prime $p \in \pi(G)$ is invertible in $R$, then $\text{Der}_R R[G] = 0$.

3. Locally inner derivations

A derivation $\delta \in \text{Der}_R R[G]$ is called locally inner (see [26, 30]) if, for every finite subset $F \subseteq R[G]$, there exists an inner derivation $\partial_x \in \text{I} \text{Der}_R R[G]$ (depending on $\delta$ and $F$) such that

$$\delta|_F = (\partial_x)|_F.$$

The set $\text{L} \text{Der}_R R[G]$ of all locally inner derivations of $R[G]$ is an ideal of the Lie ring $\text{Der}_R R[G]$ (see [25]) and

$$\text{I} \text{Der}_R R[G] \subseteq \text{L} \text{Der}_R R[G] \subseteq \text{Der}_R R[G] \quad \text{and} \quad \text{I} \text{Der}_R R[G] \subseteq \text{L} \text{Der}_R R[G] \subseteq \text{Der}_R R[G].$$
Now as a light extension of \[26\], Theorem 2.1 we have.

**Lemma 3.** Let $R$ be a ring and let $G$ be a locally finite group. If each prime $p \in \pi(G)$ is invertible in $R$, then all $R$-derivation of $R[G]$ is locally inner.

**Proof.** The same as the proof of \[26\], Theorem 2.1. We prove it here in order to have the paper more self-contained. Assume that $\delta \in \text{Der}_R R[G]$ and $F$ is a finite subset of $R[G]$. Then there exists a finite subgroup $H$ of $G$ such that $F \subseteq R[H]$. Let

\[
(5) \quad x_H = \frac{1}{|H|} \sum_{a \in H} a^{-1} \delta(a) \in R[H].
\]

If $x := x_H$, then for any $y \in F$ and $a \in H$ holds

\[
\partial_x(y) = \frac{1}{|H|} \left( \sum_{a \in H} a^{-1} \delta(a) y - \sum_{a \in H} a^{-1} a y \right) = \frac{1}{|H|} \left( \sum_{a \in H} a^{-1} \delta(ay) - \sum_{a \in H} a^{-1} a \delta(y) - \sum_{a \in H} y a^{-1} \delta(a) \right) = \frac{1}{|H|} \left( \sum_{b \in H} y a^{-1} \delta(b) - |H| \delta(y) - \sum_{a \in H} y a^{-1} \delta(y) \right) = -\delta(y).
\]

Hence $\delta|_F = (\partial_x)|_F$. \qed

**Example.** Let $S(Q) := \{f : Q \to Q \mid f \text{ is bijective} \}$ be the group of all permutations of the rational numbers field $Q$ and $\sup f := \{q \in Q \mid f(q) \neq q \}$ a support of the element $f \in S(Q)$. The set $FS(Q) := \{f \in S(Q) \mid \sup f \leq \infty \}$ is a subgroup of $S(Q)$ and $R[FS(Q)]$ is a group ring over a commutative ring $R$. Let $u = \sum_{q \in Q} f(q) \in (R[FS(Q)])^*$, where $f(q) \in FS(Q)$ and $\sup f(q) = \{q - 1, q\}$. It follows the following two statements:

(i) \[16\] Example] the rule

\[
\delta : R[FS(Q)] \ni x \mapsto [x, u] \in R[FS(Q)]
\]

determines a generalized inner derivation of $R[FS(Q)]$, which is not inner;

(ii) $\delta$ is a locally inner derivation.

In fact, if $\delta$ is an inner derivation, then $\delta = \partial_a$ for some $a = \sum_{g \in FS(Q)} \alpha_g g \in R[FS(Q)]$. Since this sum is finite, there exist $\beta, \lambda \in Q$ and $f \in R[FS(Q)]$ such that

\[
\sup f = \{\beta, \lambda\} \subseteq \bigcup_{q \in Q} \sup f(q) \quad \text{and} \quad \sup f \cup \bigcup_{g \in \text{supp } a} \sup g = \emptyset,
\]

where $\text{supp } a$ is the support of $a \in R[G]$. It follows that $\partial_a(f) = 0$ and $\delta(f) \neq 0$, a contradiction. Hence $\delta$ is not inner.

If $F$ is a finite subset of $R[FS(Q)]$, then $F \subseteq R[H]$ for a finite subgroup $H$ of $FS(Q)$. The set $\text{supp } H = \cup_{a \in H} \text{supp } a$ is finite, so there exists $b \in R[FS(Q)]$ such that

\[
\text{supp } b \subseteq \text{supp } u \cup \text{supp } H
\]

and $\delta_H = (\partial_b)|_H$. Consequently, $\delta$ is locally inner on $F$.

Notice that, if $G$ is a non-abelian group, then $gh - hg \neq 0$ for $g \in G$, so $\text{IDer}_R R[G]$ is nonzero. If $H$ is a subgroup of $G$, then $I_R(H)$ is a right ideal of $R[G]$ generated by the set $\{1 - h \mid h \in H\}$. If $H$ is normal in $G$, then $I_R(H)$ is an ideal of $R[G]$ and there exists some ring isomorphism

\[
R[G]/I_R(H) \cong R[G/H]
\]
Let $\delta \in \text{Der} R$. An ideal $I$ of $R$ is called a $\delta$-ideal of $R$ if $\delta(a) \in I$ for all $a \in I$. If $\delta(a) \in I$ for any $\delta \in \text{Der} R$, then $I$ is called a $(\text{Der} R)$-ideal.

**Lemma 4.** Let $R$ be a ring and let $G$ be a group. If $H$ is a subgroup of $G$ such that each prime $p \in \pi(H)$ is invertible in $R$, then the following conditions hold:

(i) if $\delta \in \text{Der} R[G]$ and $H$ is finite, then $\delta(H) = \partial_x(H)$ for some $x \in R[G];$

(ii) if $H$ is a normal torsion subgroup of $G$, then $\mathcal{I}_R(H)$ is a $\delta$-ideal for any $\delta \in \text{Der}_R R[G].$

**Proof.** (i) In fact, from proof of Lemma 3 it holds that $x := x_H$ is of the form \( (5) \).

(ii) If $h \in t(H)$, then $\delta(h) = \partial_x(h)$ for some $x = \sum_{t \in G} a_t t \in Z(R)[G]$ and

$$
\delta(r g(h - 1)) = r \delta(g)(h - 1) + r g \delta(h) \quad (r \in R, g \in G, h \in H).
$$

Moreover, $r g \delta(h) = \sum_{t \in G} r a_t g(th - ht) = \sum_{t \in G} r a_t g t(1 - h^{-1}t^{-1}ht) \in \mathcal{I}_R(H)$. It follows that $\delta(r g(h - 1)) \in \mathcal{I}_R(H)$ and $\mathcal{I}_R(H)$ is a $\delta$-ideal. \( \square \)

If $R$ is a differentially simple ring (i.e., $R^2 \neq 0$ and $R$ has no proper (Der $R$)-ideals) with a minimal two-sided ideal, then (see [13, Main Theorem]) the ring $R$ is simple or there exist $n \in \mathbb{N}$ and a simple ring $S$ of characteristic $p > 0$ such that $R = S[G]$, where $G$ is a direct sum of $n$ copies of a cyclic group of order $p$. Therefore, Lemma 4(ii) is not true in the modular case.

**Corollary 2.** Let $R$ be a ring. If $G$ is a finite group such that each prime $p \in \pi(G)$ is invertible in $R$, then every $R$-derivation of $R[G]$ is inner.

**Proof.** Let $H$ be a finite subgroup of $G$, $b \in H$ and $\delta \in \text{Der}_R R[G]$. It is easy to see that $\delta(b) = -\partial_x(b)$ by Lemma 3 where $x$ is the same as in Eq. (4).

This corollary earlier was proved for a finite group $G$ and the following rings $R$:

- $R$ is a semiprime ring with some restrictions (see [21 Theorem 1.1]);
- $R$ is a field of characteristic zero (see [20 Corollary 2.2]);
- $R$ is the integer numbers ring (see [15, Theorem 1]);
- $R$ is commutative with some restrictions (see [16, Corollary], [19, p. 490] and [20, p. 76]).

**Example.** Let $R$ be a commutative ring, $G$ a torsion abelian group and $\delta \in \text{Der}_R R[G]$.

(a) If $R$ is $n$-torsion-free and $\exp(G) = n \in \mathbb{N}$ (respectively $R^+$ is torsion-free), then $\text{Der}_R R[G] = 0$ in view of Proposition 1(iii).

(b) If $R$ is a domain of characteristic $0$ which fraction field $Q(R)$ is an algebraic extension of the rational numbers field $\mathbb{Q}$, then $\text{Der}_R R[G] = 0$.

Indeed, $Q(R)[G]$ is an algebraic extension over $Q(R)$ (see e.g. [41 Theorem 28.1]) and we conclude that for every element $g \in G$ there exists its minimal polynomial

$$
m_g = X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n \in Q(R)[X].
$$

Then $bm_g \in R[X]$ for some $0 \neq b \in R$. If $\delta \in \text{Der}_R R[G]$, then

$$
0 = \delta(0) = \delta(bm_g(g)) = b(ng^{n-1} + (n - 1)a_1 g^{n-2} + \cdots + a_{n-1}) \delta(g)
$$
what implies that $\delta(g) = 0$.

(c) If $R$ is a domain of characteristic $p > 0$, $R = \{x^p \mid x \in R\}$ and $p \notin \pi(G)$, then $\text{Der} R[G] = 0$.

Clearly, for every $r \in R$ there exists $x \in R$ such that $r = x^p$ and then $\delta(r) = px^{p-1}\delta(x) = 0$. Moreover, $\delta(G) = 0$ by Eqs. (3) and the assertion follows.

**Proof of Theorem**. Let $G = D \cdot F$ be a Chernikov group, where $D$ is a normal, countable, divisible abelian group and $F$ is a finite group. Obviously, $D$ has an ascending series $\{D_n\}_{n \in \mathbb{N}}$ of finite-$G$ normal subgroups, such that $D = \bigcup_{n=1}^{\infty} D_n$. Let $\delta \in \text{Der}_R R[G]$. There exist elements $x_1 := x_{D_1}$ and $x_n := x_{D_n}$ of the form (5) (see Lemma 3) such that

$$\delta(D_1) = \partial_{x_1}(D_1) \quad \text{and} \quad \delta(D_n) = \partial_{x_n}(D_n).$$

Evidently, $\partial_{x_n}(D_s) = \delta(D_s) = \partial_{x_{n-s}}(D_{n-s})$ for all $s \leq n \in \mathbb{Z}$, so $x_s - x_n \in C_{R[G]}(D_s)$, in which $C_{R[G]}(D_s)$ is the centralizer of $D_s$ in $R[G]$. Then we have the following descending chain

$$C_{R[G]}(D_1) \geq \cdots \geq C_{R[G]}(D_s) \geq \cdots.$$

Since $R[D]$ is a subgroup of finite index in the additive group of $R[G]$ and $R[D] \subseteq C_{R[G]}(D_s)$, there exists $m \in \mathbb{N}$ such that

$$C_{R[G]}(D_m) = C_{R[G]}(D_{m+1}) = \cdots.$$

It follows that $x_q - x_m \in C_{R[G]}(D_m)$ for all $q \geq m$. Hence $\delta|_D = (\partial_{x_m})|_D$ is inner. Moreover, $D_m F$ is finite and $\partial_{D_m F} = (\partial_a)|_{D_m F}$ for some $a \in R[G]$. However $x_m - a \in C_{R[G]}(D_m)$ and so $\partial_{x_m} = \partial_a$ on $D$. Consequently, $\delta = \partial_a$. □

4. **Inner R-derivations of R[G]**

If $R$ is commutative, then $\text{IDer} R[G] = \text{IDer}_R R[G]$. Additionally, if $G$ is an abelian group, then $\text{IDer}_R R[G] = 0$. Clearly, the center $Z(B)$ of $B$ is an ideal of the Lie ring $B^L$.

**Lemma 5.** Let $R$ be a ring and let $G$ be a group. There exists a Lie ring isomorphism

$$(Z(R)[G])^L/Z(Z(R)[G]) \cong \text{IDer}_R R[G].$$

**Proof.** The rule $\varphi : (Z(R)[G])^L \ni \alpha \mapsto \partial_\alpha \in \text{IDer}_R R[G]$ satisfies

$$\varphi([\alpha, \beta]) = [\partial_\alpha, \partial_\beta] = [\varphi(\alpha), \varphi(\beta)]$$

i.e., $\varphi$ is a Lie homomorphism. Finally, $\varphi(\alpha) = 0$ iff $\partial_\alpha = 0$ iff $\alpha \in Z(Z(R)[G])$. □

**Corollary 3.** Let $R$ be a ring and let $G$ be a group. The ring $\text{IDer}_R R[G]$ is an abelian Lie ring if and only if the derived subgroup $G'$ is central.

**Proof.** Since $[\partial_x, \partial_y] = \partial_{[x,y]}$ for any $x, y \in R[G]$, we conclude that $[\partial_x, \partial_y] = 0$ iff $[x, y] \in Z(R[G])$ what implies that $G' \subseteq Z(G)$. □

Since no each derivation of $R[G]$ is inner in the case of a locally finite group $G$ (see [16, Example]), we obtain the next.

**Lemma 6.** Let $R$ be a ring, let $G$ be a locally finite group such that each $p \in \pi(G)$ is invertible in $R$ and $\delta \in \text{Der}_R R[G]$. If the set $\delta(G)$ is finite, then $\delta \in \text{IDer}_R R[G]$. 

Proof. The set $Q := \{g \in G \mid \delta(g) = 0\}$ is a subgroup of $G$ because
\[0 = \delta(1) = \delta(hh^{-1}) = \delta(h)h^{-1} + h\delta(h^{-1}) = h\delta(h^{-1}),\]
so $\delta(h^{-1}) = 0$ and $\delta(gh^{-1}) = \delta(g)h^{-1} + g\delta(h^{-1}) = 0$ for any $g, h \in Q$. Clearly, $\delta(G) = \{\delta(g_i) \mid i = 1, \ldots, n\}$ and the subgroup $H = \langle g_1, \ldots, g_n \rangle \subseteq G$ is finite (where $n = |\delta(G)|$), so we conclude that $\delta|_H = (\partial_x)|_H$ for some $x \in Z(R)[G]$ by Lemma 4(ii).

\[\Box\]

**Lemma 7.** Let $R$ be a ring and let $G$ be a group such that each prime $p \in \pi(G)$ is invertible in $R$. If $\delta \in \text{Der}_R[G]$ and $g \in \tau G$, then the following conditions hold:

(i) [\text{Lemma 1}] if $t \in G$ and $\delta(g) = \sum_{h \in G} \alpha_{g,h}h \in R[G]$, then $gt = tg$ implies that $\alpha_{g,t} = 0$;

(ii) if $G$ is torsion, then $\delta(Z(G)) = 0$ for any $\delta \in \text{Der}_R[G]$.

Proof. (i) If $n = |g|$, then
\[0 = \delta(1) = \delta(g^n) = \sum_{i+j=n-1} g^i\delta(g)g^j = \delta(g^n) = \sum_{i+j=n-1} g^i(\sum_{h \in G} \alpha_{g,h}h)g^j = \sum_{h \in G, i+j=n-1} \alpha_{g,h}g^i hg^j = \sum_{i+j=n-1} \alpha_{g,h}g^i hg^j + \sum_{h \in G \setminus \{t\}, i+j=n-1} \alpha_{g,h}g^i hg^j = n\alpha_{g,t} + \sum_{h \in G \setminus \{t\}, i+j=n-1} \alpha_{g,h}g^i hg^j\]
what gives that $\alpha_{g,t} = 0$. The part (ii) it holds from (i).

We obtain the next generalization of [21] Theorem 1.1.

**Corollary 4.** Let $R$ be a ring, $G$ a torsion $FC$-group such that each prime $p \in \pi(G)$ is invertible in $R$ and $\delta \in \text{Der}_R[R[G]]$. The following conditions hold:

(i) $\delta$ is inner if and only if the set $\delta(G)$ is finite;

(ii) if $G$ is centre-by-finite, then every $R$-derivation of $R[G]$ is inner.

Proof. Let $\delta \in \text{Der}_R[R[G]]$.

(i) $(\Leftarrow)$ It follows by Lemma 6

$(\Rightarrow)$ If $\delta$ is inner, then there exists $x \in Z(R)[G]$ such that $\delta = \partial_x$. Since the index $|G : C_G(x)|$ is finite, the image $\partial_x(G)$ is finite.

(ii) In as much as $|G : Z(G)| < \infty$, we deduce that $\delta(G)$ is finite in view of Lemma 7(ii). The rest holds from the part (i).

Proof of the Theorem 3. Let $G$ be an infinite group with an ascending series of its subgroups
\[H \leq \langle H, x_1 \rangle \leq \cdots \leq \langle H, x_1, \ldots, x_n \rangle \leq \cdots\]
such that $G = \bigcup_{n=1}^{\infty} \langle H, x_1, \ldots, x_n \rangle$. There exist $x, y_n \in R[G]$ by Lemma 4(i), such that
\[\delta|_H = (\partial_x)|_H \quad \text{and} \quad \delta|_{\langle H, x_1, \ldots, x_n \rangle} = (\partial_{y_n})|_{\langle H, x_1, \ldots, x_n \rangle}.\]
Then \( y_n - x \in C_R(G)[H] \) and so there exists \( n_0 \in \mathbb{N} \) such that \( y_m = y_{n_0} \) for all \( m \geq n_0 \). This yields that \( \delta = \partial y_{n_0} \).

**Proof of the Theorem**

The group ring \( R[G] \) is prime and every its non-trivial idempotent is non-central.

(i) Let \( G \) be non-abelian. The equation \([2] \) and \([36]\) Corollary 6] imply that either \( [G, L] = 0 \) or \( [U(R), M] = 0 \), where \( L \) is some non-central Lie ideal and \( M \) is some non-central ideal of \( R \), which is impossible by \([12] \) Lemma 2].

(ii) The group \( V(R[G]) \) is central by \([36] \) Corollary 6] and \([12] \) Lemma 2] and so \( \delta(\tau G) = 0 \) for \( \delta \in \text{Der} R[G] \) by Lemma \(7\). □

### 5. Nilpotency and solvability of derivation rings

A group \( G \) is called \( p \)-abelian \((p > 0)\) if its commutator subgroup \( G' \) is a finite \( p \)-group. A ring \( R \) is called Lie hypercentral, if for each sequence \( x, x_1, \ldots, x_n, \ldots \in R \), there exists \( m \in \mathbb{N} \) such that \( [x, x_1, \ldots, x_m] = 0 \). Analogously we can defined the notion of a hypercentral Lie ring.

**Proposition 3.** Let \( R \) be a ring and let \( G \) be a group. The following conditions hold:

- (i) if \( \text{IDer}_R R[G] \) is a nilpotent (respectively solvable) Lie ring, then the unit group \( U(Z(R)[G]) \) (and consequently \( G \)) is a nilpotent (respectively solvable) group;
- (ii) if \( R \) is a division ring of characteristic \( p \geq 0 \), then:
  - (a) the Lie ring \( \text{IDer}_R R[G] \) is nilpotent if and only if \( G \) is \( p \)-abelian and nilpotent;
  - (b) for \( p \neq 2 \), \( \text{IDer}_R R[G] \) is solvable if and only if \( G \) is \( p \)-abelian;
  - (c) for \( p = 2 \), \( \text{IDer}_R R[G] \) is solvable if and only if \( G \) has a 2-abelian subgroup of index at most 2;
- (iii) the Lie ring \( \text{IDer}_R R[G] \) is hypercentral if and only if one of the following conditions holds:
  - (d) \( G \) is abelian;
  - (e) \( R \) is of characteristic \( p^m \) and \( G \) is a nilpotent \( p \)-abelian group.

**Proof.** (i) Assume that \( \text{IDer}_R R[G] \) is nilpotent (respectively solvable). The Lie ring \( (Z(R)[G])^L \) is nilpotent (respectively solvable) by Lemma \(5\). Then there exists \( n \in \mathbb{N} \) such that

\[
\gamma_n G \leq \gamma_n U(Z(R)[G]) \leq \underbrace{[Z(R)[G], \ldots, Z(R)[G]]}_{n \text{ times}} + 1 = 1
\]

by \([23] \) Theorem A] (respectively

\[
G^{(n)} - 1 \leq U(Z(R)[G])^{(n)} - 1 \leq ((Z(R)[G])^L)^{(n)} = 0
\]

in view of \([40] \) Lemma 1.2].

(ii) \( (\Rightarrow) \) If \( \text{IDer}_R R[G] \) is nilpotent (respectively solvable), then \( (Z(R)[G])^L \) is nilpotent (respectively solvable) by Lemma \(5\]. The rest follows from \([40] \) Theorem].

(iii) In as much as \( Z(R)[G] \) is Lie nilpotent (respectively Lie solvable) by \([40] \) Theorem], we deduce that \( \text{IDer}_R R[G] \) is nilpotent (respectively solvable) by Lemma \(5\]

(iii) If follows in view of \([15] \) Theorem] and Lemma \(5\]. □

**Lemma 8.** Let \( P \) be an ideal in a ring \( B \). The following conditions hold:
(i) if \( \delta(P) \subseteq P \) for some \( \delta \in \text{Der} B \), then
\[
\overline{\delta} : B/P \ni a + P \mapsto \delta(a) + P \in B/P
\]
is a derivation of the quotient ring \( B/P \);
(ii) if \( A \) is a nilpotent (respectively solvable) subring of \( \text{Der} B \) and \( \delta(P) \subseteq P \) for any \( \delta \in A \), then \( \overline{A} := \{ \overline{\delta} | \delta \in A \} \) is a nilpotent (respectively solvable) subring of \( \text{Der}(B/P) \).

**Proof.** Evidently. \( \square \)

**Lemma 9.** Let \( B \) be a ring. The following conditions hold:

(i) if \( \delta \in Z(\text{Der} B) \), then \( \delta(R) \subseteq Z(B) \);
(ii) if \( \delta \in Z(\text{Der} B) \), then \( \delta(Z(B)) \subseteq \text{ann} D := \{ r \in B \ | \ r(\text{Der} B) = 0 \} \);
(iii) if \( B \) is commutative and \( \delta \in Z(\text{Der} B) \), then:
(a) if \( \delta \) is surjective as a map, then \( B^2 = 0 \);
(b) if \( \text{ann} B = 0 \), then \( \delta = 0 \);
(iv) if \( \text{IDer} B \) is Lie nilpotent (or equivalently \( B \) is Lie nilpotent), then \( C(B) \subseteq \mathbb{P}(B) \);
(v) if \( \text{IDer} B \) is Lie solvable (or equivalently \( B \) is Lie solvable), then \( [B^{(n)}, B] \subseteq \mathbb{P}(B) \) for some integer \( n \geq 0 \);
(vi) if \( B \) is semiprime with the solvable (in particular, nilpotent) Lie ring \( \text{IDer} B \), then \( B \) is commutative.

**Proof.** Assume that \( \delta \in Z(\text{Der} B) \).

(i) In fact, \( \partial_x \delta(y) = \delta \partial_x(y) \) for any \( x, y \in B \) and so \( [x, \delta(y)] = [\delta(x), y] + [x, \delta(y)] \) what gives that \( \delta(x), y] = 0 \).

(ii) If \( a \in Z(B) \) and \( d \in \text{Der} B \), then \( ad \in \text{Der} B \). Since
\[
(6) \quad (ad)\delta = \delta(ad) = \delta(a)d + a(\delta d),
\]
we deduce that \( \delta(a)d = 0 \).

(iii) It holds from the part (ii).

(iv) Let \( P \) be a prime ideal of \( B \). Since \( \text{IDer}(B/P) \) is nilpotent, \( B_1 := B/P \) is Lie nilpotent. Clearly, \( B_1[B_1, B_1], B_1B_1 \) is a nilpotent ideal of \( B_1 \) by [28] (this is also was proved in [31, Lemma 2.2] and [47, Corollary 2.4]). It is easy to see that \( \text{ann} B_1 = 0 \), so \( B_1, B_1 \subseteq Z(B_1) \). The commutator ideal \( C(B_1) \) is nil by [11, Lemma 1.7]. Consequently, \( B_1 \) has a nilpotent ideal contained in \( C(B_1) \) and so \( C(B_1) = 0 \). This means that \( B/P \) is commutative and thus \( C(B) \subseteq \mathbb{P}(R) \).

(v) If \( \text{IDer} B \) is solvable of length \( n \), then \( B^L \) is solvable of length \( \leq n + 1 \). Let \( P \) be a prime ideal of \( B \). Obviously, \( (B/P)^{(n)} \) is a commutative Lie ideal of the prime ring \( B/P \). If \( \text{char} B/P \neq 2 \), then \( (B/P)^{(n)} \subseteq Z(B/P) \) by [12, Lemma 2].

Now let \( \text{char} B/P = 2 \). If \( Z(B/P) = 0 \), then \( (B/P)^{(n)} \) is nil of the index 2 by [37, Lemma 2] and so \( (B/P)^{(n)} = 0 \) by [37, Lemma 1]. Hence \( B^{(n)} \subseteq P \). Therefore, we assume that \( Z(B/P) \neq 0 \). Consequently, \( (B/P)Z(B/P)^{-1} \) is simple 4-dimensional over its center by [37, Theorem 4], in which \( (B/P)Z(B/P)^{-1} \) is the localization of \( B/P \) at \( Z(B/P) \), and so \( (B/P)^{(n)} \) is central by [37, Lemma 6].

In both cases \( [B^{(n)}, B] \subseteq P \), so \( [B^{(n)}, B] \subseteq \mathbb{P}(B) \).

(vi) It holds in view of the part (v). \( \square \)
Proof of the Theorem\textsuperscript{5} Let $D := \text{Der } B$.

(i) Assume that $D$ is nilpotent. The ring $B$ is commutative by Lemma\textsuperscript{5}(iv) and $\delta(B)^2 = 0$ for any $\delta \in Z(D)$ by Lemma\textsuperscript{5}(ii) and $\delta(B) = 0$ in view of the semiprimeness of $B$ and Lemma\textsuperscript{5}(ii). This means that $\delta = 0$ and thus $D = 0$.

(ii) Now assume that the Lie ring $D$ is solvable. The ring $B$ is a commutative by Lemma\textsuperscript{5}(iv) and consequently $D$ is a left $B$-algebra. Assume that the algebra $D$ is nonzero, $A$ is the last nonzero member of its derived chain, $d, \delta \in A$ and $b \in B$. In as much as $A$ is abelian and $bd \in A$, Eqs. (6) imply that $\delta(b)d = 0$. This yields that $(\delta(B)B)^2 = 0$. The semiprimeness of $B$ implies $\delta(B) = 0$, a contradiction. □

Proof of the Theorem\textsuperscript{6} If every prime $p \in \pi(G)$ is invertible in $\mathbb{F}$, then $\mathbb{F}[G]$ is a semiprime ring (see [18]). The rest holds from Theorem\textsuperscript{5} and Proposition\textsuperscript{3}.

We need the next

Lemma 10. [1] Lemma 1 Let $B$ be a ring and let $x, y \in B$. If $[[x, y], x] = 0$, then

$$[x^k, y] = kx^{k-1}[x, y] \quad (k \in \mathbb{N}).$$

A group $G$ is called n-divisible ($n \in \mathbb{N}$) if, for each $g \in G$, there exists $h \in G$ such that $g = h^n$.

Proposition 4. Let $R$ be a ring and let $G$ be a torsion group such that each $p \in \pi(G)$ is invertible in $R$ (respectively $nR = 0$ and $G$ a $n$-divisible group for some $n \in \mathbb{N}$). If $G$ is an Engel set in $R[G]$, i.e. for each $g, h \in G$ there exists $m = m(g, h) \in \mathbb{N}$ such that $[g, m, h] = 0$, then $G$ is abelian and $\text{Der}_R R[G] = 0$.

Proof. Let $g, h \in G$. Since $G$ is an Engel set, there exists $m \in \mathbb{N}$ such that $[g, m, h] := [a, h, h] = 0$, where $a := \begin{cases} [g, m-2, h] & \text{if } m \geq 3; \\ g & \text{if } m = 2. \end{cases}$

Now, if each $p \in \pi(G)$ is invertible in $R$, then

$$0 = [h^{|h|}, a] = [h|h^{|h|}-1|h, a]$$

(see Lemma\textsuperscript{10}). If $nR = 0$ and $G$ is a $n$-divisible group, then $[h^n, a] = n(h^{n-1}[h, a]) = 0$ (see Lemma\textsuperscript{10}). Thus $[g, a] = 0$ in both cases, so $[g, h] = 0$ by induction. Consequently, $G$ is abelian and $\text{Der}_R R[G] = 0$ in view of Theorem\textsuperscript{2}. □

6. Case of nilpotent groups

Each nilpotent group $G$ of class $m$ has a central series

$$1 = Z_0 < Z_1 < \cdots < Z_{m-1} < Z_m = G$$

in which $Z(G/Z_{i-1}) = Z_i/Z_{i-1}$ for all $i = 1, \ldots, m$.

Lemma 11. Let $R$ be a ring such that $nR = 0$ for some $n \in \mathbb{N}$, $G$ a divisible torsion-free nilpotent group of the nilpotent length $m$ and $\delta \in \text{Der } R[G]$. Then $\mathcal{I}_R(Z_k)$ is a $\delta$-ideal for each $k = 1, \ldots, m$.

Proof. Each subgroup $Z_k$ is divisible and normal in $G$. Moreover, $\delta(Z_k) \subseteq Z(R[G])$. Now we have

$$\delta(rg(z-1)) = \delta(r)g(h-1) + r\delta(g)(h-1) + r\delta(h) \quad \text{and} \quad \delta(h^n) = nh^{n-1}\delta(h) = 0$$
for any \( r \in R, \, g \in G \) and \( z \in Z_1 \). We deduce that \( \delta(\mathcal{I}_R(Z_1)) \subseteq \mathcal{I}_R(Z_1) \). There is a Lie ring isomorphism

\[
\text{Der } R[G/Z_1] \ni \xi \rightarrow \overline{\xi} \in \text{Der } R[G]/\mathcal{I}_R(Z_1)
\]
such that

\[
\overline{\xi} : R[G]/\mathcal{I}_R(Z_1) \ni a + \mathcal{I}_R(Z_1) \mapsto \delta(a) + \mathcal{I}_R(Z_1) \in R[G]/\mathcal{I}_R(Z_1).
\]

In as much as

\[
\overline{\xi}(\mathcal{I}_R(Z_2/Z_1)) \subseteq \mathcal{I}_R(Z_2/Z_1)
\]
and we have the following ring isomorphisms

\[
R[G/Z_1]/\mathcal{I}_R(Z_2/Z_1) \cong R[(G/Z_1)/(Z_2/Z_1)] \cong R[G/Z_2] \cong R[G]/\mathcal{I}_R(Z_2),
\]
we conclude that \( \delta(\mathcal{I}_R(Z_2)) \subseteq \mathcal{I}_R(Z_2) \). Thus the assertion follows by induction. \( \square \)

**Proposition 5.** Let \( R[G] \) be the group ring of a nilpotent group \( G \) of class \( cl(G) = m \geq 2 \) over a ring \( R \). If \( G \) is torsion and \( \pi(G) \cap \pi(F(R)) = \emptyset \) (respectively \( G \) is torsion-free and \( nR = 0 \) for some \( n \in \mathbb{N} \)), then each \( \mathcal{I}_R(Z_i) \) is a \( \delta \)-ideal (\( i \geq 1 \)) and

\[
\delta_1 \delta_2 \cdots \delta_{m-1}(R[G]) \subseteq \mathcal{I}_R(Z_1) \quad (\delta, \delta_1, \ldots, \delta_{m-1} \in \text{Der } R[G]).
\]
Moreover, if \( G \) is a torsion abelian group and \( \pi(G) \cap \pi(F(R)) = \emptyset \) (respectively \( G \) is an abelian torsion-free group and \( nR = 0 \) for some \( n \in \mathbb{N} \)), then \( \text{Der } R[G] = 0 \).

**Proof.** Assume also that \( \delta \in \text{Der } R[G], \, r \in R, \, g \in G, \, a \in Z_1 \) and \( \delta(g) = \sum_{f \in G} x_f f \in Z(R)[G] \).

Obviously, \( \delta(Z_1) \subseteq \delta(Z(R[G])) \subseteq Z(R[G]) \) and \( L_\delta : Z_1 \rightarrow \mathbb{R}^+ \) is a group homomorphism by Lemma 1(iii). Therefore, \( a^{-1}\delta(a^n) = \delta(1) = 0 \) if \( a \) is of order \( n \) (respectively \( nL_\delta(a) = 0 \) and thus \( \delta(a) = 0 \) in view of Eqs. (6)). So, we obtain that \( \delta(Z_1) = 0 \). The ideal \( \mathcal{I}_R(Z_1) \) is a \( \delta \)-ideal by Lemma 4. This implies that \( \mathcal{I}_R(Z_1) \) is a \( \delta \)-ideal and so

\[
\overline{\delta} : R[G]/\mathcal{I}_R(Z_1) \ni a + \mathcal{I}_R(Z_1) \mapsto \delta(a) + \mathcal{I}_R(Z_1) \in R[G]/\mathcal{I}_R(Z_1)
\]
is a derivation of the quotient ring \( R[G]/\mathcal{I}_R(Z_1) \). A ring isomorphism \( \varphi : R[G]/\mathcal{I}_R(Z_1) \rightarrow R[G/Z_1] \) induces a Lie ring isomorphism

\[
\text{Der}(R[G]/\mathcal{I}_R(Z_1)) \ni \xi \rightarrow \overline{\xi} \in \text{Der } R[G/Z_1]
\]
such that \( \overline{\xi}(Z_2) = \overline{0} \), where \( Z_2 := Z_2/Z_1 \), what forces that \( \delta(Z_2) \subseteq \mathcal{I}_R(Z_1) \). By induction, we deduce that \( \delta(Z_i) \subseteq \mathcal{I}_R(Z_{i-1}) \) and so

\[
[Z_i, G] \subseteq \mathcal{I}_R(Z_{i-1}) \quad (i = 2, \ldots, m).
\]
Moreover, each \( \mathcal{I}_R(Z_i) \) is a \( \delta \)-ideal by Lemma 4.

Now, since \( \text{Der } R[G/Z_{m-1}] = 0 \), we conclude that \( \delta(R[G]) \subseteq \mathcal{I}_R(Z_{m-1}) \subseteq R[Z_{m-1}] \). By induction, we obtain that \( \delta(R[Z_i]) \subseteq \mathcal{I}_R(Z_{i-1}) \) for any \( i = 2, \ldots, m \) and the result follows. \( \square \)

**Proof of Theorem 7.** Let \( R \) be a ring, \( G = \{x_n \mid n \in \mathbb{Z}\} \) a countable torsion-free abelian group. Let \( \alpha = \sum_{n \in \mathbb{Z}} a_n x_n \in R[G], \, z = \sum_{n \in \mathbb{Z}} z_n x_n \in Z(R[G]), \, \delta \in \text{Der } R[G], \, x \in G \) and \( r \in R \). Clearly, \( Z(R[G]) = Z(R)[G] \). Since \( xr = rx \),

\[
\delta(x)r + x\delta(r) = \delta(r)x + r\delta(x)
\]
and \( \delta(x)r = r\delta(x) \). Hence \( \delta(x) \in Z(R[G]) = Z(R)[G] \). In addition,
\[
0 = \delta(1) = \delta(x^{-1}) = \delta(x)x^{-1} + x\delta(x^{-1})
\]
and so \( \delta(x^{-1}) = -x^{-1}\delta(x)x^{-1} = -x^{-2}\delta(x) \). Thus \( \delta(\alpha) = \sum_{n \in \mathbb{Z}} \delta(a_n)x_n + \sum_{n \in \mathbb{Z}} a_n\delta(x_n) \).

If \( a, b \in R \subset R[G] \), then \( \delta(a), \delta(b) \in R[G] \), so we can write
\[
\delta(a) = \sum_{n \in \mathbb{Z}} D_n(a)x_n \quad \text{and} \quad \delta(b) = \sum_{n \in \mathbb{Z}} D_n(b)x_n, \quad (D_n(a), D_n(b) \in R)
\]
in which almost all coefficients \( D_n(a) \) and \( D_n(b) \) are zero. Now
\[
\sum_{n \in \mathbb{Z}} (D_n(a) + D_n(b))x_n = \delta(a) + \delta(b) = \delta(a + b) = \sum_{n \in \mathbb{Z}} D_n(a + b)x_n
\]
implies that \( D_n(a + b) = D_n(a) + D_n(b) \),
\[
\sum_{n \in \mathbb{Z}} D_n(ab)x_n = \delta(ab) = \delta(a)b + a\delta(b) = \\
= (\sum_{n \in \mathbb{Z}} D_n(a)x_n)b + a(\sum_{n \in \mathbb{Z}} D_n(b)x_n) = \\
= \sum_{n \in \mathbb{Z}} (D_n(a)b + aD_n(b))x_n
\]
implies that \( D_n(ab) = D_n(a)b + aD_n(b) \).

Thus \( D_n \in \text{Der } R \) for any \( n \in \mathbb{Z} \) and we have the following Lie ring isomorphism
\[
\text{Der } R[G] \ni \delta \mapsto (\{D_n\}_{n \in \mathbb{Z}}, \{\delta(x_n)\}_{n \in \mathbb{Z}}) \in LF(R) \oplus (Z(R[G]))^\mathbb{Z}.
\]
Consequently, \( \text{Der}_R R[G] \cong (Z(R)[G])^\mathbb{Z} \). Finally, \( \partial_\alpha(x) = 0 \) and so \( 1 \text{Der}_R R[G] = 0 \), i.e., each nonzero \( R \)-derivation of \( R[G] \) is outer. \( \square \)

For example, if \( \mathbb{Q}^+ \) is the additive group of the rational numbers field \( \mathbb{Q} \), then \( \text{Der}_R R[\mathbb{Q}^+] = 0 \).

**Corollary 5.** Let \( R \) be a ring. If \( G \) is a countable abelian group such that each prime \( p \in \pi(G) \) is invertible in \( R \), then \( \text{Der}_R R[G] = 0 \) or each nonzero \( R \)-derivation of \( R[G] \) is inner.

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Department of Applied Mathematics, Cracow University of Technology, Cracow, Poland

E-mail address: artemo@usk.pk.edu.pl

UAEU, United Arab Emirates

E-mail address: vbovdi@gmail.com; msalim@uaeu.ac.ae