Representation theorems for generators of quadratic BSDEs

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Abstract: In this paper, we establish representation theorems for generators of backward stochastic differential equations (BSDE in short) with bounded terminal time and finite terminal time, respectively, whose generator is linear growth in $y$ and quadratic growth in $z$, one of which generalizes a representation theorem in Ma and Yao (2010). As some applications, we obtain a general converse comparison theorem of such quadratic BSDEs and uniqueness theorem, translation invariance for quadratic $g$-expectation.

Keywords: Backward stochastic differential equation; representation theorem of generator; converse comparison theorem; $g$-expectation

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1 Introduction

It is well-known that Pardoux and Peng (1990) firstly establishes the existence and uniqueness of solution of the nonlinear backward stochastic differential equation (BSDE in short) under the Lipschitz condition on generator $g$. Since then, the theory of BSDEs has gone through rapid development in many different areas such as probability, PDE, stochastic control, and mathematical finance, etc. An important study in BSDEs theory is to interpret the relation between the generators and solutions of BSDEs. In this topic, representation theorem of generator of BSDE is established, which shows a relation between generators and solutions of BSDEs in limit form. Representation theorem for generator plays an important role in the study in BSDEs theory and nonlinear expectation theory.

Representation theorem of generator is firstly established by Briand et al. (2000) for BSDEs with Lipschitz generators under two additional assumptions that $E[\sup_{0 \leq t \leq T} |g(t, 0, 0)|^2] < \infty$ and $(g(t, y, z))_{t \in [0,T]}$ is continuous in $t$, in order to study the converse comparison theorem of BSDEs. After a series of studies by Jiang (2005a, b), two assumptions mentioned above in Briand et al. (2000) are eliminated by Jiang (2005c, 2006, 2008), which establish representation theorems of generators under the condition that $g$ only satisfies Lipschitz condition. Since then, representation theorem for generators of BSDEs is studied further under some more general conditions.

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conditions on generator. One can see Jia (2008), Fan and Jiang (2010), Fan et al. (2011), Ma and Yao (2010) and Zhang and Fan (2013), etc.

BSDEs with quadratic growth have been firstly studied by Kobylanski (2000), which shows that BSDE has at least a solution when the generator \( g \) is linear growth in \( y \) and quadratic growth in \( z \), then by many papers (see Briand and Hu (2006, 2008), etc). BSDEs with quadratic growth in the variable \( z \) have many applications in PDE, control and finance. Ma and Yao (2010) establish a representation theorem for generators of BSDEs when generator satisfies conditions similar as in Kobylanski (2000) and two additional assumptions (see conditions \((g1)\) and \((g2)\) in Ma and Yao (2010, Theorem 4.1)). The main aim of this paper is to establish representation theorems for generators of BSDEs considered in Kobylanski (2000). We firstly establish a general representation theorem for generator of BSDE with bounded terminal time in almost surely sense, whose generator only satisfies the conditions in Kobylanski (2000), which generalizes the representation theorem in Ma and Yao (2010). Then the representation theorems are obtained in \( L^p \) and \( H^p \) sense for \( p > 0 \), respectively. The representation theorems for generators of such quadratic BSDEs with finite terminal time considered in Kobylanski (2000) are also established in this paper.

The famous comparison theorems of BSDEs shows that we can compare the solutions of BSDEs through comparing generators. While, converse comparison theorems for BSDEs shows we can compare the generators through comparing the solutions of BSDEs, which is firstly studied by Chen (1997), then by Briand et al. (2001), Coquet et al. (2001), Jiang (2005b, c, 2006) and Jia (2008), Fan et al. (2011), etc. Using representation theorems obtained in this paper, we establish general converse comparison theorems for quadratic BSDEs.

The notion of \( g \)-expectation is introduced by Peng (1997), which is a nonlinear expectation induced by BSDE with Lipschitz generator. Ma and Yao (2010) consider \( g \)-expectation induced by quadratic BSDE, called quadratic \( g \)-expectation, and study its properties. Quadratic \( g \)-expectation is also discussed in Hu et al. (2008). Recently, Kazi-Tani et al. (2014) consider quadratic \( g \)-expectation induced by quadratic BSDE with jumps. Using representation theorems obtained in this paper, we study uniqueness theorem, translation invariance, subadditivity, positive homogeneity for quadratic \( g \)-expectation induced by quadratic BSDEs considered in Kobylanski (2000).

This paper is organized as follows. In section 2, we will present some basic assumptions. In section 3, we will prove some important lemmas and establish representation theorems for quadratic BSDE with bounded terminal time. In section 4, we will establish representation theorems for quadratic BSDE with finite terminal time. In section 5 and section 6, we will give some applications of representation theorem for quadratic BSDEs and quadratic \( g \)-expectation, respectively.

## 2 Preliminaries

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space carrying a \(d\)-dimensional standard Brownian motion \((B_t)_{t \geq 0}\), starting from \(B_0 = 0\), let \((\mathcal{F}_t)_{t \geq 0}\) denote the natural filtration generated by \((B_t)_{t \geq 0}\) augmented by the \(P\)-null sets of \(\mathcal{F}\), let \(|z|\) denote its Euclidean norm, for \(z \in \mathbb{R}^n\), let \(T > 0\) be a given real number. We define the following usual spaces:

\[
L^p(\mathcal{F}_T) = \{\xi : \mathcal{F}_T\text{-measurable random variable}; \|\xi\|_L^p = (E[|\xi|^p])^{\frac{1}{p}} < \infty\}, \quad p > 0;
\]

\[
L^\infty(\mathcal{F}_T) = \{\xi : \mathcal{F}_T\text{-measurable random variable}; \|\xi\|_\infty = \text{esssup}_{\omega \in \Omega} |\xi| < \infty\};
\]
Let us consider a function $g$

$$g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

such that $(g(t, y, z))_{t \in [0,T]}$ is progressively measurable for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ and $g$ always takes the following form:

$$g(t, y, z) = g_1(t, y, z)y + g_2(t, y, z).$$

we give the following assumptions for $g$:

(A1). For each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, both $g_1$ and $g_2$ are progressively measurable and for each $(t, w) \in [0, T] \times \Omega$, both $g_1$ and $g_2$ are continuous in $(y, z)$;

(A2). There exist constants $\alpha, \beta, b$ and a continuous increasing function $l : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that for $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$P-a.s., \quad \beta \leq g_1(t, y, z) \leq \alpha \quad \text{and} \quad |g_2(t, y, z)| \leq b + l(\|y\|)|z|^2;$$

**Remark 2.1** In fact, if $g$ satisfies assumption (A2), then it also satisfies the following assumption (A2)*.

(A2)*. For any constant $M > 0$, there exists a constant $\lambda_M = \max\{\|\alpha\|, \|\beta\|, \|b\|, l(M)\}$ such that $\forall(t, y, z) \in [0, T] \times [-M, M] \times \mathbb{R}^d$,

$$P-a.s., \quad |g(t, y, z)| \leq \lambda_M(1 + |y| + |z|^2).$$

In this paper, we consider the following BSDE:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad t \in [0, T] \tag{1}$$

where $g$ is called generator, $\xi$ and $T$ are called terminal variable and terminal time, respectively. The BSDE (1) is introduced by Purdoux and Peng (1990), which is usually called BSDE with parameter $(g, T, \xi)$, for convention.

Now, we introduce a stochastic differential equation (SDE). Suppose $b(\cdot, \cdot, \cdot) : \Omega \times [0, T] \times \mathbb{R}^m \mapsto \mathbb{R}^m$ and $\sigma(\cdot, \cdot, \cdot) : \Omega \times [0, T] \times \mathbb{R}^m \mapsto \mathbb{R}^{m \times d}$ satisfy the following two conditions:

(H1) there exists a constant $\mu \geq 0$ such that $P-a.s., \forall t \in [0, T], \forall x, y \in \mathbb{R}^n$,

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq \mu|x - y|;$$

(H2) there exists a constant $\nu \geq 0$ such that $P-a.s., \forall t \in [0, T], \forall x \in \mathbb{R}^n$,

$$|b(t, x)| + |\sigma(t, x)| \leq \nu(1 + |x|).$$

Given $(t, x) \in [0, T] \times \mathbb{R}^n$, by SDE theory, the following SDE:

$$\begin{cases}
X^t_{s,x} = x + \int_t^s b(u, X^t_{u,x})du + \int_t^s \sigma(u, X^t_{u,x})dB_u, & s \in [t, T], \\
X^t_{t,x} = x & s \in [0, t],
\end{cases}$$

for $s \in [0, T]$.
have a unique continuous adapted solution $X^{t,x}_s$.

In the end of this section, we introduce the Lebesgue Lemma which play an important role in this paper.

**Lemma 2.1** (Lebesgue Lemma, see Hewitt and Stromberg (1978, Lemma 18.4)) Let $f$ be a Lebesgue integrable function on the interval $[0,T]$. Then for almost every $t \in [0,T]$, we have
\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} |f(u) - f(t)| ds = 0.
\]

### 3 Representation theorems in bounded terminal time case

In this section, we will establish representation theorems for generators of quadratic BSDEs with bounded terminal time. Without loss of generality, we only consider quadratic BSDEs with deterministic terminal time $T < \infty$ for convention. In fact, when $T$ is replaced by any bounded stopping time $\tau$, we will have the same arguments and results.

Firstly, we recall some known results in Kobylanski (2000). By Kobylanski (2000, Theorem 2.3), if $g$ assumptions (A1) and (A2), for each $\xi \in L^\infty(F_T)$, BSDE with parameter $(g,T,\xi)$ has a maximal solution $(Y_t, Z_t) \in S^\infty_T(R) \times H^2_T(R^d)$ and a minimal solution $(Y_t, Z_t) \in S^\infty_T(R) \times H^2_T(R^d)$ in the sense that for any solution $(Y_t, Z_t)$ of BSDE with parameter $(g,T,\xi)$, we have $Y_t \leq Y_t \leq Y_t$. Moreover, we have the following two facts:
\[
\|Y\|_\infty \leq (\|\xi\|_\infty + |b|T) e^{\alpha T}, \quad (2)
\]
and for each $\varepsilon \in [0,T-t]$, if $(Y^{t+\varepsilon}_s, Z^{t+\varepsilon}_s)$ is a solution of BSDE with parameter $(g,t+\varepsilon,0)$, then
\[
\sup_{t \leq s \leq t+\varepsilon} |Y^{t+\varepsilon}_s| \leq |b|\varepsilon e^{\alpha \varepsilon}. \quad (3)
\]
In fact, since $\|Y_t\|_\infty < \infty$, we define a continuous function $\phi(y) : R \to [0,1]$, such that
\[
\phi(y) := \begin{cases} 1, & |y| \leq \|Y_t\|_\infty, \\ 0, & |y| > \|Y_t\|_\infty + 1. \end{cases}
\]
Set
\[
\tilde{g}(r,y,z) := g_1(r,y,z)y + \phi(y)g_2(r,y,z),
\]
then by (A2), we can check that $\tilde{g}(r,y,z)$ satisfies condition (H0) in Kobylanski (2000, Proposition 2.1 and Corollary 2.2). Clearly, $(Y_t, Z_t)$ is also a solution of BSDE with parameter $(\tilde{g}, T, \xi)$. Then we can get (2) and (3) from Kobylanski (2000, Proposition 2.1 and Corollary 2.2), immediately.

Now, we prove the following three lemmas.

**Lemma 3.1** Let $g$ satisfy the assumptions (A1) and (A2), for each stopping time $\tau \in [0, T-t]$ and each $t \in [0, T]$, there exists a constant $\delta > 0$ such that for each $\varepsilon \leq \delta$,
\[
E \left[ \int_t^{t+\varepsilon \wedge \tau} |Z^{t+\varepsilon \wedge \tau}_r|^2 dr | F_t \right] \leq c \varepsilon^2.
\]
where \((Y^{t+\varepsilon\wedge \tau}_s, Z^{t+\varepsilon\wedge \tau}_s)\) is an arbitrary solution of BSDE with parameter \((g, t + \varepsilon \wedge \tau, 0)\) and \(c\) is a constant only depending on \(\alpha, \beta, b, T, l(\cdot)\).

**Proof.** For each \(t \in [0, T], \varepsilon \in [0, T-t]\) and any stopping time \(\tau \in [0, T-t]\), we consider the following BSDEs with parameter \((g, t + \varepsilon \wedge \tau, 0)\)

\[
Y^{t+\varepsilon\wedge \tau}_s = \int_s^{t+\varepsilon\wedge \tau} g(r, Y^{t+\varepsilon\wedge \tau}_r, Z^{t+\varepsilon\wedge \tau}_r)dr - \int_s^{t+\varepsilon\wedge \tau} Z^{t+\varepsilon\wedge \tau}_r dB_r.
\]

By (2), we have

\[
\sup_{0 \leq s \leq t+\varepsilon\wedge \tau} |Y^{t+\varepsilon\wedge \tau}_s| \leq |b| T e^{\alpha T}.
\]

We set \(M^0_g := |b| T e^{\alpha T}\). Applying Itô formula to \(|Y^{t+\varepsilon\wedge \tau}_s|^2\) for \(s \in [t, t + \varepsilon \wedge \tau]\), and in view of (A2)*, we can deduce there exists a constant \(\lambda_{M^0_g} = \max\{|\alpha|, |\beta|, |b|, l(M^0_g)\}\) such that

\[
|Y^{t+\varepsilon\wedge \tau}_t|^2 + \int_t^{t+\varepsilon\wedge \tau} |Z^{t+\varepsilon\wedge \tau}_r|^2 dr \\
\leq 2 \int_t^{t+\varepsilon\wedge \tau} |Y^{t+\varepsilon\wedge \tau}_r|^2 dr + 2 \lambda_{M^0_g} \int_t^{t+\varepsilon\wedge \tau} |Y^{t+\varepsilon\wedge \tau}_r|^2 dr + 2 \lambda_{M^0_g} \int_t^{t+\varepsilon\wedge \tau} |Z^{t+\varepsilon\wedge \tau}_r|^2 dr \\
- 2 \int_t^{t+\varepsilon\wedge \tau} Y^{t+\varepsilon\wedge \tau}_r Z^{t+\varepsilon\wedge \tau}_r dB_r.
\]

Then by (3), we can select \(\delta\) small enough such that for each \(\varepsilon \leq \delta\), we have

\[
\sup_{t \leq s \leq t+\varepsilon\wedge \tau} |Y^{t+\varepsilon\wedge \tau}_s| \leq \frac{1}{4 \lambda_{M^0_g}}.
\]

Then by (3)-(5), for each \(\varepsilon \leq \delta\), we have

\[
E \left[ \int_t^{t+\varepsilon\wedge \tau} |Z^{t+\varepsilon\wedge \tau}_r|^2 dr \middle| \mathcal{F}_t \right] \\
\leq 2 \lambda_{M^0_g} E \left[ \int_t^{t+\varepsilon\wedge \tau} |Y^{t+\varepsilon\wedge \tau}_r|^2 dr + \int_t^{t+\varepsilon\wedge \tau} |Y^{t+\varepsilon\wedge \tau}_r||Z^{t+\varepsilon\wedge \tau}_r|^2 dr \middle| \mathcal{F}_t \right] \\
\leq 2 \varepsilon^2 \lambda_{M^0_g}^2 e^{\varepsilon \lambda_{M^0_g}} + 2 \varepsilon^3 \lambda_{M^0_g}^3 e^{2\varepsilon \lambda_{M^0_g}} + \frac{1}{2} E \left[ \int_t^{t+\varepsilon\wedge \tau} |Z^{t+\varepsilon\wedge \tau}_r|^2 dr \middle| \mathcal{F}_t \right].
\]

From above inequality, the proof is complete. \(\Box\)

Inspired by Fan and Jiang (2010, Proposition 3) and Fan et al. (2011, Lemma 3 and Proposition 2), we have the following Lemma 3.2.

**Lemma 3.2** Let \(g\) satisfy the assumptions (A1) and (A2), for any constant \(M > 0\) and \((y, x, q) \in [-M, M] \times \mathbb{R}^m \times \mathbb{R}^m\), then there exists a non-negative process sequence \(\{(\psi^n_t)_{t \in [0, T]}\}_{n=1}^\infty\) depending on \((y, x, q)\), which is bounded uniformly in \(n\) and \(\lim_{n \to \infty} \psi^n_t = 0\) for each \(t \in [0, T]\), such that for each \(n \in \mathbb{N}\) and \((t, \tilde{y}, \tilde{z}, \tilde{x}) \in [0, T] \times [-M, M] \times \mathbb{R}^{d+m}\), we have

\[
|g(t, \tilde{y}, \tilde{z} + \sigma^*(t, \tilde{x})q) - g(t, y, \sigma^*(t, x)q)| \leq 2n \lambda_M (|\tilde{y} - y| + |\tilde{z}|^2 + |\tilde{x} - x|^2) + \psi^n_t.
\]
where \( \tilde{\lambda}_M = 4\lambda_M(1 + |q|^2|\nu|^2) \), \( \lambda_M \) is the constant in (A2)* and \( \nu \) is the constant in (H1).

**Proof.** The method of proof derives from Fan and Jiang (2010, Proposition 3) and Fan et al. (2011, Lemma 3 and Proposition 2). For any constant \( M > 0 \), and \((y, x, q) \in [-M, M] \times \mathbb{R}^m \times \mathbb{R}^m\), we set

\[
f(t, y, z, x) := g(t, y, z + \sigma(t, x)q).
\]

Then by (A2)* and (H2), there exists a constant \( \lambda_M = \max\{|\alpha|, |\beta|, |b|, l(M)\} \) such that

\[
|f(t, y, z, x)| \leq \lambda_M(1 + |y| + |z + \sigma(t, x)q|^2) \\
\leq \lambda_M(1 + |y| + 2|z|^2 + 4|q|^2|\nu|^2(1 + |x|^2)) \\
\leq 4\lambda_M(1 + |q|^2|\nu|^2)(1 + |y| + |z|^2 + |x|^2).
\]

we set \( \tilde{\lambda}_M = 4\lambda_M(1 + |q|^2|\nu|^2) \) and

\[
f^1_n(t, y, z, x) := \sup_{(u, v, w) \in \{Q_n[-M, M]\} \times Q^{d+n}} \{f(t, u, v, w) - 2n\tilde{\lambda}_M(|u - y| + |v - z|^2 + |w - x|^2)\}.
\]

\[
f^2_n(t, y, z, x) := \inf_{(u, v, w) \in \{Q_n[-M, M]\} \times Q^{d+n}} \{f(t, u, v, w) + 2n\tilde{\lambda}_M(|u - y| + |v - z|^2 + |w - x|^2)\}.
\]

Then by argument of Fan et al. (2011, Lemma 3), we can deduce the following fact:

(i) \( |f^1_n(t, y, z, x)| \leq 2\tilde{\lambda}_M(1 + |y| + |z|^2 + |x|^2) \), \( i = 1, 2 \);

(ii) \( f^1_n(t, y, z, x) \searrow \) and \( f^2_n(t, y, z, x) \nearrow \), as \( n \to \infty \);

(iii) \( f^1_n(t, y, z, x) \to f(t, y, z, x) \) as \( n \to \infty \), \( i = 1, 2 \).

Setting

\[
\psi^1_n(t) := f^1_n(t, y, 0, x); \\
\psi^2_n(t) := f^2_n(t, y, 0, x).
\]

By (i), we have

\[
|\psi^1_n(t)| \leq 2\tilde{\lambda}_M(1 + |y| + |x|^2);
|\psi^2_n(t)| \leq 2\tilde{\lambda}_M(1 + |y| + |x|^2).
\]

Then by (iii), we have

\[
\lim_{n \to \infty} \psi^1_n(t) = \lim_{n \to \infty} \psi^2_n(t) = f(t, y, 0, x).
\]

By the definition of \( f^1_n \) and \( f^2_n \), we also have

\[
f(t, \bar{y}, \bar{z}, \bar{x}) - f(t, y, 0, x) \leq 2n\tilde{\lambda}_M(|\bar{y} - y| + |\bar{z}|^2 + |\bar{x} - x|^2) + \psi^1_n(t) - f(t, y, 0, x);
\]

\[
f(t, \bar{y}, \bar{z}, \bar{x}) - f(t, y, 0, x) \geq -2n\tilde{\lambda}_M(|\bar{y} - y| + |\bar{z}|^2 + |\bar{x} - x|^2) + \psi^2_n(t) - f(t, y, 0, x);
\]

By setting

\[
\psi^n_n := |\psi^1_n(t) - f(t, y, 0, x)| + |\psi^2_n(t) - f(t, y, 0, x)|,
\]

we have

\[
|f(t, \bar{y}, \bar{z}, \bar{x}) - f(t, y, 0, x)| \leq 2n\tilde{\lambda}_M(|\bar{y} - y| + |\bar{z}|^2 + |\bar{x} - x|^2) + \psi^n_n.
\]

The proof is complete. \( \square \)
Lemma 3.3 Let \((\varphi_t)_{t \in [0,T]}\) be a \(\mathbb{R}\)-valued progressively measurable process such that \(\|\varphi_t\|_\infty < \infty\), then for any stopping time \(\tau \in [0,T-t]\) and almost every \(t \in [0,T]\), we have

\[
P - \text{a.s., } \varphi_t = \lim_{\varepsilon \to 0^+} E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon\wedge \tau} \varphi_r \, dr | F_t \right].
\]

Proof. By Jensen inequality, Fubini Theorem and Lemma 2.1, we have

\[
\int_0^T E \left[ \left| \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon\wedge \tau} \varphi_r \, dr | F_t \right| - \varphi_t \right] \, dt
\]

\[
= \int_0^T E \left[ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon\wedge \tau} \varphi_r \, dr - \varphi_t | F_t \right] \, dt
\]

\[
\leq \int_0^T \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon\wedge \tau} \varphi_r \, dr - \varphi_t \, dt
\]

\[
= E \left[ \int_0^T \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon\wedge \tau} \varphi_r \, dr - \varphi_t \right] dt
\]

\[
= E \left[ \int_0^T \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \frac{1}{\varepsilon} \int_t^{t+\varepsilon\wedge \tau} \varphi_r \, dr - \varphi_t \right] dt
\]

\[
= 0.
\]

By above inequality and Lebesgue dominated convergence theorem, we have, for almost every \(t \in [0,T]\),

\[
P - \text{a.s., } \lim_{\varepsilon \to 0^+} E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon\wedge \tau} \varphi_r \, dr | F_t \right] = E \left[ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon\wedge \tau} \varphi_r \, dr | F_t \right] = \varphi_t.
\]

The proof is complete. \(\square\)

The following Theorem 3.1 is a general representation theorem for generators of quadratic BSDEs, which is the main result of this paper.

Theorem 3.1 Let \(g\) satisfy the assumptions (A1) and (A2), then for each \((y, x, q) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m\) and almost every \(t \in [0,T]\), we have

\[
P - \text{a.s., } g(t, y, \sigma^*(t, x)q) + q \cdot b(t, x) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y^{t+\varepsilon\wedge \tau}_t - y \right),
\]

where \(\tau = \inf \{ s \geq 0 : |X^{t,x}_{t+s}| > C_0 \} \wedge (T-t)\) for constant \(C_0 > |x|\) and \((Y^{t+\varepsilon\wedge \tau}_s, Z^{t+\varepsilon\wedge \tau}_s)\) is an arbitrary solution of BSDE with parameter \((g, t+\varepsilon \wedge \tau, y+q \cdot (X^{t,x}_{t+\varepsilon\wedge \tau} - x))\).

Proof. For each \((t, x) \in [0,T] \times \mathbb{R}^m\) and a constant \(C_0 > |x|\), we define the following stopping time:

\[
\tau := \inf \{ s \geq 0 : |X^{t,x}_{t+s}| > C_0 \} \wedge (T-t)
\]

By the continuity of \(X^{t,x}_{t+s}\), we have \(0 < \tau \leq T-t\) and

\[
|X^{t,x}_{t+s \wedge \tau}| \leq C_0, \quad \forall s \in [0,T]. \tag{6}
\]
For $\varepsilon \in [0, T-t]$, let $(Y^{t+\varepsilon,\tau}, Z^{t+\varepsilon,\tau})$ be a solution of BSDE with parameter $(g, t + \varepsilon \wedge \tau, y + q \cdot (X^{t,x}_{t+\varepsilon,\tau} - x))$ and set for $s \in [t, t + \varepsilon \wedge \tau]$,

$$
\tilde{Y}^{t+\varepsilon,\tau}_s := Y^{t+\varepsilon,\tau}_s - (y + q \cdot (X^{t,x}_s - x)), \quad \tilde{Z}^{t+\varepsilon,\tau}_s := Z^{t+\varepsilon,\tau}_s - \sigma^*(t, X^{t,x}_s)q.
$$

By (2) and (6), there exists a constant $\tilde{M}$ such that for $\varepsilon \in [0, T-t]$, we have

$$
\|\tilde{Y}^{t+\varepsilon,\tau}_s\|_\infty \leq \|Y^{t+\varepsilon,\tau}_s\|_\infty + \|y + q \cdot (X^{t,x}_{t+\varepsilon,\tau} - x)\|_\infty \leq (\|y + q \cdot (X^{t,x}_{t+\varepsilon,\tau} - x)\|_\infty + |b(T)e^{\alpha T} + \|y + q \cdot (X^{t,x}_{t+\varepsilon,\tau} - x)\|_\infty \leq \tilde{M}.
$$

Applying Itô formula to $\tilde{Y}^{t+\varepsilon,\tau}_s$ for $s \in [t, t + \varepsilon \wedge \tau]$, we have

$$
\tilde{Y}^{t+\varepsilon,\tau}_s = \int_s^{t+\varepsilon,\tau} \left( g(r, \tilde{Y}^{t+\varepsilon,\tau}_r + y + q \cdot (X^{t,x}_r - x), \tilde{Z}^{t+\varepsilon,\tau}_r + \sigma^*(r, X^{t,x}_r)q) + q \cdot b(r, X^{t,x}_r) \right) dr - \int_s^{t+\varepsilon,\tau} \tilde{Z}^{t+\varepsilon,\tau}_r dB_r.
$$

Now, we define a continuous function $\tilde{\phi}(\tilde{y}) : \mathbb{R} \to [0, 1]$, such that

$$
\tilde{\phi}(\tilde{y}) := \begin{cases} 1, & |\tilde{y}| \leq \tilde{M}, \\ 0, & |\tilde{y}| > \tilde{M} + 1. 
\end{cases}
$$

Set

$$
\tilde{g}(r, \tilde{y}, \tilde{z}) := \tilde{g}_1(r, \tilde{y}, \tilde{z})y + \tilde{g}_2(r, \tilde{y}, \tilde{z}),
$$

where

$$
\tilde{g}_1(r, \tilde{y}, \tilde{z}) := g_1(r, \tilde{y}, \tilde{y} + y + q \cdot (X^{t,x}_r - x), \tilde{z} + \sigma^*(r, X^{t,x}_r)q),
$$

$$
\tilde{g}_2(r, \tilde{y}, \tilde{z}) := \tilde{\phi}(\tilde{y})g_2(r, \tilde{y}, \tilde{y} + y + q \cdot (X^{t,x}_r - x), \tilde{z} + \sigma^*(r, X^{t,x}_r)q) + q \cdot b(r, X^{t,x}_r).
$$

Clearly, $\tilde{g}$ satisfy the assumptions (A1). Moreover, by (A2), (H2), (6) and (8), we have for $r \in [0, t + \varepsilon \wedge \tau]$,

$$
\beta \leq |\tilde{g}_1(r, \tilde{y}, \tilde{z})| \leq \alpha,
$$

and

$$
|\tilde{g}_2(r, \tilde{y}, \tilde{z})| = |\tilde{\phi}(\tilde{y})g_2(r, \tilde{y}, \tilde{y} + y + q \cdot (X^{t,x}_r - x), \tilde{z} + \sigma^*(r, X^{t,x}_r)q) + q \cdot b(r, X^{t,x}_r)|
\leq |b| + l(|\tilde{\phi}(\tilde{y})g_2(r, \tilde{y}, \tilde{y} + y + q \cdot (X^{t,x}_r - x), \tilde{z} + \sigma^*(r, X^{t,x}_r)q) + q \cdot b(r, X^{t,x}_r)|
\leq \tilde{b} + 2l(|\tilde{\phi}(\tilde{y})| + M)|\tilde{z}|^2,
$$

where $\tilde{b} = |b| + 2q^2 \nu^2(1 + C_0)^2(2\tilde{M} + 1) + |q|\nu(1 + C_0)$. Then, we get that $\tilde{g}$ also satisfy the assumption (A2). Then by (A1), (A2) and (9), $(\tilde{Y}^{t+\varepsilon,\tau}_s, \tilde{Z}^{t+\varepsilon,\tau}_s)$ is a solution of BSDE with parameter $(\tilde{g}, t + \varepsilon \wedge \tau, 0)$ in $[t, t + \varepsilon \wedge \tau]$. By (3) and Lemma 3.1, there exists a constant $\delta > 0$ such that for each $\varepsilon \leq \delta$, we have

$$
\sup_{t \leq s \leq t+\varepsilon,\tau} |\tilde{Y}^{t+\varepsilon,\tau}_s| \leq \tilde{b}e^{\alpha^+\varepsilon} \quad \text{and} \quad E \left[ \int_t^{t+\varepsilon,\tau} |\tilde{Z}^{t+\varepsilon,\tau}_s|^2 d\mathcal{F}_t \right] \leq \tilde{c}e^{2\delta},
$$
where $\tilde{c}$ is a constant only depending on $\alpha, \beta, b, T, l(\cdot), y, x, q, \nu, C_0$.

Set

$$M_{t}^{\varepsilon, \tau} := \frac{1}{\varepsilon} E \left[ \int_{t}^{t+\varepsilon \wedge \tau} g(r, \tilde{Y}_{r}^{t+\varepsilon \wedge \tau} + y + q \cdot (X_{r}^{t,x} - x), \tilde{Z}_{r}^{t+\varepsilon \wedge \tau} + \sigma^*(r, X_{r}^{t,x})q) dr \right]_{F_t}$$

$$P_{t}^{\varepsilon, \tau} := \frac{1}{\varepsilon} E \left[ \int_{t}^{t+\varepsilon \wedge \tau} g(r, y, \sigma^*(r, x)q) dr \right]_{F_t},$$

$$U_{t}^{\varepsilon, \tau} := \frac{1}{\varepsilon} E \left[ \int_{t}^{t+\varepsilon \wedge \tau} q \cdot b(r, X_{r}^{t,x}) dr \right]_{F_t},$$

Then by (7) and (9), we have

$$\frac{1}{\varepsilon} \left( Y_{t}^{t+\varepsilon \wedge \tau} - y \right) - g(t, y, \sigma^*(t, x)q) - q \cdot b(t, x)$$

$$= \frac{1}{\varepsilon} \tilde{Y}_{t}^{t+\varepsilon \wedge \tau} - g(t, y, \sigma^*(t, x)q) - q \cdot b(t, x)$$

$$= (M_{t}^{\varepsilon, \tau} - P_{t}^{\varepsilon, \tau}) + (P_{t}^{\varepsilon, \tau} - g(t, y, \sigma^*(t, x)q)) + (U_{t}^{\varepsilon, \tau} - q \cdot b(t, x)).$$

(11)

By Jensen inequality, we have

$$|M_{t}^{\varepsilon, \tau} - P_{t}^{\varepsilon, \tau}|$$

$$\leq \frac{1}{\varepsilon} E \left[ \int_{t}^{t+\varepsilon \wedge \tau} \left( g(r, \tilde{Y}_{r}^{t+\varepsilon \wedge \tau} + y + q \cdot (X_{r}^{t,x} - x), \tilde{Z}_{r}^{t+\varepsilon \wedge \tau} + \sigma^*(r, X_{r}^{t,x})q) - g(r, y, \sigma^*(r, x)q) \right) dr \right]_{F_t}$$

Then by (7), (8) and Lemma 3.2, we get there exists a non-negative process sequence $\{(\psi^n_t)_{t \in [0,T]}\}_{n=1}^\infty$ depending on $(y, x, q)$, which is bounded uniformly in $n$ and $\lim_{n \to \infty} \psi^n_t = 0$ for each $t \in [0, T]$, such that for each $n \in \mathbb{N}$

$$|M_{t}^{\varepsilon, \tau} - P_{t}^{\varepsilon, \tau}|$$

$$\leq \frac{1}{\varepsilon} E \left[ \int_{t}^{t+\varepsilon \wedge \tau} \left( 2n\tilde{\lambda}_{2M}(\tilde{Y}_{r}^{t+\varepsilon \wedge \tau} + |q||X_{r}^{t,x} - x| + |\tilde{Z}_{r}^{t+\varepsilon \wedge \tau}|^2 + |X_{r}^{t,x} - x|^2 + \psi^n_t \right) dr \right]_{F_t}$$

$$\leq 2 \frac{n\tilde{\lambda}_{2M}}{\varepsilon} E \left[ \int_{t}^{t+\varepsilon \wedge \tau} \tilde{Y}_{r}^{t+\varepsilon \wedge \tau} dr \right]_{F_t} + \frac{2}{\varepsilon} n\tilde{\lambda}_{2M} E \left[ \int_{t}^{t+\varepsilon \wedge \tau} |\tilde{Z}_{r}^{t+\varepsilon \wedge \tau}|^2 dr \right]_{F_t} + \frac{2}{\varepsilon} n\tilde{\lambda}_{2M} E \left[ \int_{t}^{t+\varepsilon \wedge \tau} (|q||X_{r}^{t,x} - x| + |X_{r}^{t,x} - x|^2) dr \right]_{F_t} + \frac{1}{\varepsilon} E \left[ \int_{t}^{t+\varepsilon} |\psi^n_t| dr \right]_{F_t}$$

(12)

where $\tilde{\lambda}_{2M} = 4\lambda_{2M}(1 + |q|^2|\nu|^2)$, $\lambda_{2M}$ is the constant in (A2)* and $\nu$ is the constant in (H2). By (10), we can deduce

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} E \left[ \int_{t}^{t+\varepsilon \wedge \tau} |\tilde{Y}_{r}^{t+\varepsilon \wedge \tau}| dr \right]_{F_t} = E \left[ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{t}^{t+\varepsilon \wedge \tau} |\tilde{Y}_{r}^{t+\varepsilon \wedge \tau}| dr \right]_{F_t} = 0,$$

(13)

and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} E \left[ \int_{t}^{t+\varepsilon \wedge \tau} |\tilde{Z}_{r}^{t+\varepsilon \wedge \tau}|^2 dr \right]_{F_t} = 0.$$

(14)
By (6), Lebesgue dominated convergence theorem and the continuity of $X^{t,x}_r$ in $r$, we have,

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} E \left[ \int_t^{t+\varepsilon} (|q||X^{t,x}_r - x| + |X^{t,x}_r - x|^2)dr | \mathcal{F}_t \right]$$

$$= E \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} (|q||X^{t,x}_r - x| + |X^{t,x}_r - x|^2)dr | \mathcal{F}_t$$

$$= 0. \quad (15)$$

Since $\{(\psi^n_t)_{t \in [0,T]}\}_{n=1}^{\infty}$ is uniformly bounded in $n$ and $\lim_{n \to \infty} \psi^n_t = 0$, for each $t \in [0,T]$, then by (12)-(15) and Lemma 3.3, we get that for almost every $t \in [0,T]$,

$$P - a.s., \quad \lim_{\varepsilon \to 0^+} |M^{e,T}_t - P^{e,T}_t| \leq \lim_{n \to \infty} \inf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} E \left[ \int_t^{t+\varepsilon} |\psi^n_t|dr | \mathcal{F}_t \right] = \lim_{n \to \infty} |\psi^n_t| = 0. \quad (16)$$

by (A2), (H2) and Lemma 3.3, we have for almost every $t \in [0,T]$,

$$P - a.s., \quad \lim_{\varepsilon \to 0^+} |P^{e,T}_t - g(t,y,\sigma^*(t,x)q)| = 0. \quad (17)$$

By Jensen inequality, we have,

$$\lim_{\varepsilon \to 0^+} |U^{e,T}_t - q \cdot b(t,x)|$$

$$= \lim_{\varepsilon \to 0^+} \left| E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon} (q \cdot b(r, X^{t,x}_r) - q \cdot b(r, x) + q \cdot b(r, x))dr | \mathcal{F}_t \right] - q \cdot b(t, x) \right|$$

$$\leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} E \left[ \int_t^{t+\varepsilon} (q \cdot b(r, X^{t,x}_r) - q \cdot b(r, x))dr | \mathcal{F}_t \right]$$

$$+ \lim_{\varepsilon \to 0^+} \left| E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon} q \cdot b(r, x)dr | \mathcal{F}_t \right] - q \cdot b(t, x) \right|$$

$$\leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} E \left[ \int_t^{t+\varepsilon} q \cdot b(r, X^{t,x}_r) - q \cdot b(r, x)dr | \mathcal{F}_t \right]$$

$$+ \lim_{\varepsilon \to 0^+} \left| E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon} q \cdot b(r, x)dr | \mathcal{F}_t \right] - q \cdot b(t, x) \right|. \quad (18)$$

By (H2) and Lemma 3.3, we have, for almost every $t \in [0,T]$,

$$P - a.s., \quad \lim_{\varepsilon \to 0^+} E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon} q \cdot b(r, x)dr | \mathcal{F}_t \right] - q \cdot b(t, x) = 0. \quad (19)$$

Then by (18), (19), (6), Lebesgue dominated convergence theorem, (H1) and the continuity of $X^{t,x}_r$ in $r$, we have, for almost every $t \in [0,T]$, $P - a.s.,$

$$\lim_{\varepsilon \to 0^+} |U^{e,T}_t - q \cdot b(t,x)| \leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} E \left[ \int_t^{t+\varepsilon} |q \cdot b(r, X^{t,x}_r) - q \cdot b(r, x)|dr | \mathcal{F}_t \right]$$

$$\leq E \left[ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} |q \cdot b(r, X^{t,x}_r) - q \cdot b(r, x)|dr | \mathcal{F}_t \right]$$

$$\leq E \left[ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} |q|\mu|X^{t,x}_r - x|dr | \mathcal{F}_t \right]$$

$$= 0. \quad (20)$$
By (11), (16), (17) and (20), The proof is complete. □

Now, by Theorem 3.1, we will give two representation theorems in $L^p$ space and $\mathcal{H}^p$ space for $p > 0$, respectively.

**Theorem 3.2** Let $p > 0$ and $g$ satisfy the assumptions (A1) and (A2), then for each $(y, x, q) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ and almost every $t \in [0, T]$, we have

$$g(t, y, \sigma^*(t, x) q) + q \cdot b(t, x) = L^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left(Y^{t+\varepsilon \wedge \tau}_t - y\right),$$

where $\tau = \inf\{s \geq 0 : |X^{t,x}_{t+s}| > C_0\} \wedge (T-t)$ for constant $C_0 > |x|$ and $(Y_{s}^{t+\varepsilon \wedge \tau}, Z_{s}^{t+\varepsilon \wedge \tau})$ is an arbitrary solution of BSDE with parameter $(g, t + \varepsilon \wedge \tau, y + q \cdot (X^{t,x}_{t+\varepsilon \wedge \tau} - x))$.

**Proof.** By (7), (10), for $\varepsilon \in [0, T-t]$, we have

$$\frac{1}{\varepsilon} |Y^{t+\varepsilon \wedge \tau}_t - y| = \frac{1}{\varepsilon} |Y^{t+\varepsilon \wedge \tau}_t| \leq \tilde{b} e^{\alpha \varepsilon} \leq \tilde{b} e^{\alpha T}. \quad (21)$$

For each $p > 0$, by (21), (A2), (H2) and Lebesgue dominated convergence theorem, we have

$$\lim_{\varepsilon \to 0^+} E \left[ \frac{1}{\varepsilon} \left(Y^{t+\varepsilon \wedge \tau}_t - y\right) - g(t, y, \sigma^*(t, x) q) + q \cdot b(t, x) \right]^p = E \left[ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left(Y^{t+\varepsilon \wedge \tau}_t - y\right) - g(t, y, \sigma^*(t, x) q) + q \cdot b(t, x) \right]^p .$$

Then by Theorem 3.1, we complete the proof. □

**Theorem 3.3** Let $p > 0$ and $g$ satisfy the assumptions (A1) and (A2), then for each $(y, x, q) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$, we have

$$g(t, y, \sigma^*(t, x) q) + q \cdot b(t, x) = \mathcal{H}^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left(Y^{t+\varepsilon \wedge \tau}_t - y\right),$$

where for each $t \in [0, T]$, $\tau_t = \inf\{s \geq 0 : |X^{t,x}_{t+s}| > C_t\} \wedge (T-t)$ for constant $C_t > |x|$ and $(Y_{s}^{t+\varepsilon \wedge \tau_t}, Z_{s}^{t+\varepsilon \wedge \tau_t})$ is an arbitrary solution of BSDE with parameter $(g, t + \varepsilon \wedge \tau_t, y + q \cdot (X^{t,x}_{t+\varepsilon \wedge \tau_t} - x))$.

**Proof.** For each $p > 0$, by (21), (A2), (H2), Lebesgue dominated convergence theorem and Fubini Theorem, we have

$$\lim_{\varepsilon \to 0^+} E \int_0^T \left| \frac{1}{\varepsilon} \left(Y^{t+\varepsilon \wedge \tau_t}_t - y\right) - g(t, y, \sigma^*(t, x) q) + q \cdot b(t, x) \right|^p dt = E \int_0^T \left| \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left(Y^{t+\varepsilon \wedge \tau_t}_t - y\right) - g(t, y, \sigma^*(t, x) q) + q \cdot b(t, x) \right|^p dt = \int_0^T E \left| \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left(Y^{t+\varepsilon \wedge \tau_t}_t - y\right) - g(t, y, \sigma^*(t, x) q) + q \cdot b(t, x) \right|^p dt .$$

Then by Theorem 3.1, we complete the proof. □
Let \( q = z, b(t, x) = 0, \sigma(t, x) = 1, x = 0 \) in Theorem 3.1-Theorem 3.3, then we have the following corollaries, immediately.

**Corollary 3.1** Let \( g \) satisfy the assumptions (A1) and (A2), then for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\) and almost every \( t \in [0, T]\), we have

\[
P - \text{a.s.,} \quad g(t, y, z) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y^{t+\varepsilon \wedge \tau} - y \right),
\]

where \( \tau = \inf\left\{ s \geq 0 : |B_{t+s} - B_t| > C_0 \right\} \wedge (T - t) \) for constant \( C_0 > 0 \) and \((Y^{s+\varepsilon \wedge \tau})_s, (Z^{s+\varepsilon \wedge \tau})_s \) is an arbitrary solution of BSDE with parameter \((g, t + \varepsilon \wedge \tau, y + z \cdot (B_{t+\varepsilon \wedge \tau} - B_t))\).

**Corollary 3.2** Let \( p > 0 \) and \( g \) satisfy the assumptions (A1) and (A2), then for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\) and almost every \( t \in [0, T]\), we have

\[
g(t, y, z) = L^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y^{t+\varepsilon \wedge \tau} - y \right),
\]

where \( \tau = \inf\left\{ s \geq 0 : |B_{t+s} - B_t| > C_0 \right\} \wedge (T - t) \) for constant \( C_0 > 0 \) and \((Y^{s+\varepsilon \wedge \tau})_s, (Z^{s+\varepsilon \wedge \tau})_s \) is an arbitrary solution of BSDE with parameter \((g, t + \varepsilon \wedge \tau, y + z \cdot (B_{t+\varepsilon \wedge \tau} - B_t))\).

**Corollary 3.3** Let \( p > 0 \) and \( g \) satisfy the assumptions (A1) and (A2), then for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\), we have

\[
g(t, y, z) = \mathcal{H}^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y^{t+\varepsilon \wedge \tau_t} - y \right),
\]

where for each \( t \in [0, T], \tau_t = \inf\left\{ s \geq 0 : |B_{t+s} - B_t| > C_t \right\} \wedge (T - t) \) for constant \( C_t > 0 \) and \((Y^{s+\varepsilon \wedge \tau_t})_s, (Z^{s+\varepsilon \wedge \tau_t})_s \) is an arbitrary solution of BSDE with parameter \((g, t + \varepsilon \wedge \tau, y + z \cdot (B_{t+\varepsilon \wedge \tau} - B_t))\).

**Remark 3.1** Representation theorems of BSDEs with Lipschitz or linear growth generators are established in \( L^p \) or \( \mathcal{H}^p \) space for \( 1 \leq p < 2 \) (or \( p = 2 \)) (see Briand et al. (2000), Jiang (2008), Jia (2008), Fan and Hu (2008) and Fan and Jiang (2010), etc), while, in this section, representation theorems of quadratic BSDEs are established in \( L^p \) or \( \mathcal{H}^p \) sense for \( p > 0 \) (see Theorem 3.2 and Theorem 3.3), due to the boundedness of solutions of such quadratic BSDEs.

**Remark 3.2** To my knowledge, representation theorem for generators of BSDEs in \( L^p \) space in literatures are all established under the additional condition that \( b(t, x) \) and \( \sigma(t, x) \) in SDEs are both right continuous in \( t \) or both independent on \( t \) (see Briand et al. (2000), Jiang (2005), Jia (2008), etc). One can see this condition is eliminated in our results.

Now, we introduce the following assumptions for \( g \).

(A3). For any \( M > 0 \), there exist a function \( k(t) \in L^2[0, T] \) and a constant \( C \) such that \( \forall (t, y, z) \in [0, T] \times [-M, M] \times \mathbb{R}^d \),

\[
P - \text{a.s.,} \quad \left| \frac{\partial g}{\partial z}(t, y, z) \right| \leq k(t) + C|z|;
\]
(A4). For any \( \varepsilon > 0 \), there exists a function \( h_\varepsilon(t) \in L^1[0,T] \), such that for \((t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d\),
\[
P - \text{a.s.}, \quad \frac{\partial g}{\partial y}(t,y,z) \leq h_\varepsilon(t) + \varepsilon|z|^2.
\]

**Remark 3.3** Ma and Yao (2010, Theorem 4.1) establish a representation theorem for generators of quadratic BSDEs using a different method under some more stronger assumptions than (A1)-(A4). In fact, by Kobylanski (2000, Theorem 2.3 and Theorem 2.6), if \( g \) assumptions (A1)-(A4), for each \( \xi \in L^\infty(F_T) \), then BSDE with parameter \((g,T,\xi)\) has a unique solution. Consequently, one can see Corollary 3.1 generalizes Ma and Yao (2010, Theorem 4.1).

### 4 Representation theorems in finite terminal time case

Let \( \tau \) be a finite stopping time \((\tau < \infty, \text{a.s.})\). In this section, we will establish representation theorems for BSDEs with finite terminal time \( \tau \). In fact, the method of proof of Theorem 3.1 also works for such quadratic BSDEs. Firstly, we give the following assumption (A2)‘:

(A2)‘. (A2) in which \( \alpha < 0 \).

By Kobylanski (2000, Theorem 2.3), if \( \tau \) is a finite stopping time \((\tau < \infty, \text{a.s.})\), \( g \) satisfies assumptions (A1) and (A2)‘, then for each \( \xi \in L^\infty(F_\tau) \), BSDE with parameter \((g,\tau,\xi)\) has a maximal solution \((Y_{t\wedge \tau},Z_{t\wedge \tau}) \in S^\infty_t(R) \times \mathcal{H}^2_t(R^d)\) and a minimal solution \((Y'_{t\wedge \tau},Z'_{t\wedge \tau}) \in S^\infty_t(R) \times \mathcal{H}^2_t(R^d)\), in the sense that for any solution \((Y_{t\wedge \tau},Z_{t\wedge \tau})\) of such BSDE, we have \(Y_{t\wedge \tau} \leq Y'_{t\wedge \tau} \leq Z_{t\wedge \tau} \leq Z'_{t\wedge \tau}\). By Kobylanski (2000, Proposition 2.1 and Corollary 2.2) and the same argument as (2) and (3), we can deduce
\[
\|Y\|_\infty \leq \|\xi\|_\infty + \frac{|b|}{|\alpha|}.
\]

and for any stopping time \( \tau_t \in [t \wedge \tau, \tau - t \wedge \tau] \), if \((Y^{t\wedge \tau + \varepsilon \wedge \tau_t\wedge \tau},Z^{t\wedge \tau + \varepsilon \wedge \tau_t\wedge \tau})\) is a solution of BSDE with parameter \((g,t \wedge \tau + \varepsilon \wedge \tau_t,0)\), then we have
\[
\sup_{t \leq s \leq t \wedge \tau + \varepsilon \wedge \tau_t} |Y^{t\wedge \tau + \varepsilon \wedge \tau_t\wedge \tau}_s| \leq |b|\varepsilon.
\]

By (22), (23) and the same argument of Lemma 3.1, we also can get the following similar result as Lemma 3.1. We omit it proof.

**Lemma 4.1** Let \( g \) satisfy the assumptions (A1) and (A2)‘, \( \tau \) is a finite stopping time \((\tau < \infty, \text{a.s.})\). Then for each \( t \in [0,\infty) \) and each stopping time \( \tau_t \in [t \wedge \tau, \tau - t \wedge \tau] \), there exists a constant \( \delta > 0 \) such that for each \( \varepsilon \leq \delta \),
\[
E \left[ \int_{t \wedge \tau}^{t \wedge \tau + \varepsilon \wedge \tau_t} |Z^{t \wedge \tau + \varepsilon \wedge \tau_t}_r|^2 dr \bigg| \mathcal{F}_t \right] \leq c_1 \varepsilon^2,
\]
where \((Y^{t \wedge \tau + \varepsilon \wedge \tau_t},Z^{t \wedge \tau + \varepsilon \wedge \tau_t})\) is an arbitrary solution of BSDE with parameter \((g,t \wedge \tau + \varepsilon \wedge \tau_t,0)\) and \( c_1 \) is a constant only depending on \( \alpha, \beta, b, l(\cdot)\).

Let \( g \) satisfy the assumptions (A1) and (A2)‘, \( \tau \) be a finite stopping time \((\tau < \infty, \text{a.s.})\). Similarly, combining (22), (23), Lemma 3.2, Lemma 3.3, Lemma 4.1 and the same argument
of Theorem 3.1-3.3, we can establish the following representation theorems for generators of BSDEs under assumptions (A1) and (A2). We omit its proof.

**Theorem 4.1** Let $g$ satisfy the assumptions (A1) and (A2)', $\tau$ be a finite stopping time ($\tau < \infty$, a.s.). Then for each $(y, x, q) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ and almost every $t \in [0, \infty)$, we have

$$P - a.s., \quad g\left(t \wedge \tau, y, \sigma^\ast(t \wedge \tau, x)q + q \cdot b(t \wedge \tau, x)\right) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y_{t \wedge \tau}^{t \wedge \tau + \varepsilon \wedge \tau_l} - y \right),$$

where

$$\tau_l = \inf\{s \geq 0 : |X_{t \wedge \tau + s}^{t \wedge \tau, x}| > C_t \} \land (\tau - t \wedge \tau),$$

for constant $C_t > |x|$ and $(Y_{t \wedge \tau + \varepsilon \wedge \tau_l}^{t \wedge \tau + \varepsilon \wedge \tau_l}, Z_{t \wedge \tau + \varepsilon \wedge \tau_l})$ is an arbitrary solution of BSDE with parameter $(g, t \wedge \tau + \varepsilon \wedge \tau_l, y + q \cdot (X_{t \wedge \tau + \varepsilon \wedge \tau_l}^{t \wedge \tau, x} - x)).$

**Theorem 4.2** Let $p > 0$ and $g$ satisfy the assumptions (A1) and (A2)', $\tau$ be a finite stopping time ($\tau < \infty$, a.s.). Then for each $(y, x, q) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ and almost every $t \in [0, \infty)$, we have

$$g\left(t \wedge \tau, y, \sigma^\ast(t \wedge \tau, x)q + q \cdot b(t \wedge \tau, x)\right) = \mathcal{H}^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y_{t \wedge \tau}^{t \wedge \tau + \varepsilon \wedge \tau_l} - y \right),$$

where

$$\tau_l = \inf\{s \geq 0 : |X_{t \wedge \tau + s}^{t \wedge \tau, x}| > C_t \} \land (\tau - t \wedge \tau),$$

for constant $C_t > |x|$ and $(Y_{t \wedge \tau + \varepsilon \wedge \tau_l}^{t \wedge \tau + \varepsilon \wedge \tau_l}, Z_{t \wedge \tau + \varepsilon \wedge \tau_l})$ is an arbitrary solution of BSDE with parameter $(g, t \wedge \tau + \varepsilon \wedge \tau_l, y + q \cdot (X_{t \wedge \tau + \varepsilon \wedge \tau_l}^{t \wedge \tau, x} - x)).$

**Theorem 4.3** Let $p > 0$ and $g$ satisfy the assumptions (A1) and (A2)', $\tau$ be a finite stopping time ($\tau < \infty$, a.s.). Then for each $(y, x, q) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$, we have

$$g\left(t \wedge \tau, y, \sigma^\ast(t \wedge \tau, x)q + q \cdot b(t \wedge \tau, x)\right) = \mathcal{H}^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y_{t \wedge \tau}^{t \wedge \tau + \varepsilon \wedge \tau_l} - y \right),$$

where for each $t \in [0, \infty),

$$\tau_l = \inf\{s \geq 0 : |X_{t \wedge \tau + s}^{t \wedge \tau, x}| > C_t \} \land (\tau - t \wedge \tau),$$

for constant $C_t > |x|$ and $(Y_{t \wedge \tau + \varepsilon \wedge \tau_l}^{t \wedge \tau + \varepsilon \wedge \tau_l}, Z_{t \wedge \tau + \varepsilon \wedge \tau_l})$ is an arbitrary solution of BSDE with parameter $(g, t \wedge \tau + \varepsilon \wedge \tau_l, y + q \cdot (X_{t \wedge \tau + \varepsilon \wedge \tau_l}^{t \wedge \tau, x} - x)).$

5 Some applications for quadratic BSDEs

In this section, we will give some applications of representation theorem for quadratic BSDEs.

**Theorem 5.1** (Converse comparison theorem) Let generators $g_1$ and $g_2$ satisfy assumptions (A1) and (A2), and for any stopping time $\tau \in [0, T]$, $\xi \in L^\infty(\mathcal{F}_\tau)$, BSDEs with parameter $(g_1, \tau, \xi)$ and $(g_2, \tau, \xi)$ exist solutions $(Y_{t \wedge \tau}^{\tau_{1, \tau}}, Z_{t \wedge \tau}^{\tau_{1, \tau}})$ and $(Y_{t \wedge \tau}^{\tau_{2, \tau}}, Z_{t \wedge \tau}^{\tau_{2, \tau}})$, respectively, such that $\forall t \in [0, T],

$$P - a.s., \quad Y_{t \wedge \tau}^{\tau_{1, \tau}} \geq Y_{t \wedge \tau}^{\tau_{2, \tau}},$$

(24)
then for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\), and almost every \(t \in [0, T]\), we have

\[ P - a.s., \quad g_1(t, y, z) \geq g_2(t, y, z). \]

**Proof.** By Corollary 3.1, for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\) and almost every \(t \in [0, T]\), there exists a stopping time \(\tau > 0\), such that

\[ P - a.s., \quad g_1(t, y, z) = \lim_{n \to +\infty} n(Y_t^{t+\frac{1}{n}\wedge \tau,i} - y), \quad i = 1, 2. \tag{25} \]

where \((Y_t^{t+\frac{1}{n}\wedge \tau,i}, Z_t^{t+\frac{1}{n}\wedge \tau,i})\) is an arbitrary solution of BSDE with parameter \((g_i, t + \frac{1}{n} \wedge \tau, y + z \cdot (B_t + \frac{1}{n} \wedge \tau - B_t))\), \(i = 1, 2\), respectively. By (24) and (25), we can deduce,

\[ P - a.s., \quad g_1(t, y, z) \geq g_2(t, y, z). \]

The proof is completed. \(\square\)

**Theorem 5.2** (Converse comparison theorem II) Let generators \(g_1\) and \(g_2\) satisfy assumptions (A1) and (A2)', \(\tau\) be a finite stopping time \((\tau < \infty, \ a.s.)\) and for any stopping time \(\varrho \in [0, \tau]\), \(\xi \in L^\infty(\mathcal{F}_\varrho)\), BSDEs with parameter \((g_1, \varrho, \xi)\) and \((g_2, \varrho, \xi)\) exist solutions \((Y_{t\wedge \varrho}^{\varrho,1}, Z_{t\wedge \varrho}^{\varrho,1})\) and \((Y_{t\wedge \varrho}^{\varrho,2}, Z_{t\wedge \varrho}^{\varrho,2})\), respectively, such that \(\forall t \in [0, \infty]\)

\[ P - a.s., \quad Y_{t\wedge \varrho}^{\varrho,1} \geq Y_{t\wedge \varrho}^{\varrho,2}. \tag{26} \]

then for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\), and almost every \(t \in [0, \infty]\), we have

\[ P - a.s., \quad g_1(t \wedge \tau, y, z) \geq g_2(t \wedge \tau, y, z). \]

**Proof.** By Theorem 4.1 with \(q = z, b(t, x) = 0, \sigma(t, x) = 1, x = 0\), for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\) and almost every \(t \in [0, \infty]\), there exists a stopping time \(\tau_t > 0\), such that

\[ P - a.s., \quad g_1(t \wedge \tau_t, y, z) = \lim_{n \to +\infty} n\left(Y_{t \wedge \tau_t + \frac{1}{n} \wedge \tau_i} - y\right), \quad i = 1, 2. \tag{27} \]

where \((Y_{t \wedge \tau_t + \frac{1}{n} \wedge \tau_i,i}, Z_{t \wedge \tau_t + \frac{1}{n} \wedge \tau_i,i})\) is an arbitrary solution of BSDE with parameter \((g_i, t \wedge \tau + \frac{1}{n} \wedge \tau_t, y + z \cdot (B_t + \frac{1}{n} \wedge \tau_t - B_t))\), \(i = 1, 2\), respectively. By (26) and (27), we can deduce,

\[ P - a.s., \quad g_1(t, y, z) \geq g_2(t, y, z). \]

The proof is completed. \(\square\)

Self-financing condition and Zero-interest condition are considered in Jia (2008), Fan and Jiang (2010), and Fan et al. (2011) for BSDEs with linear growth generators. By Corollary 3.1, we get the following similar results for quadratic BSDEs with bounded terminal time.

**Theorem 5.3** (Self-financing condition) Let \(g\) satisfy assumptions (A1) and (A2), then the following two statements are equivalent:

(i) For almost every \(t \in [0, T]\),

\[ P - a.s., \quad g(t, 0, 0) = 0; \]
(ii) There exist a solution \((Y_t, Z_t)\) of BSDE with parameter \((g, T, 0)\) such that
\[ P - a.s., \quad Y_t = 0, \quad \forall t \in [0, T]. \]

**Theorem 5.4 (Zero-interest condition)** Let \(g\) satisfy assumptions \((A1)\) and \((A2)\), then the following two statements are equivalent:
(i) For almost every \(t \in [0, T]\),
\[ P - a.s., \quad g(t, y, 0) = 0, \quad \forall y \in \mathbb{R}; \]
(ii) There exist a solution \((Y_t, Z_t)\) of BSDE with parameter \((g, T, y)\) such that
\[ P - a.s., \quad Y_t = y, \quad \forall t \in [0, T]. \]

In fact, by the same argument of Theorem 5.3 and Theorem 5.4, we can get similar results for quadratic BSDEs with finite terminal time. We leave it for interested readers.

### 6 Some applications for quadratic \(g\)-expectation

The notion of \(g\)-expectation is firstly introduced in Peng (1997), which is a nonlinear expectation induced by BSDEs with Lipschitz generators. It is considered in quadratic BSDEs case by Ma and Yao (2010), called quadratic \(g\)-expectation. In this section, using representation theorem obtained in this paper, we will give some properties of quadratic \(g\)-expectation in general case. Then method of this section mainly derives from Peng (1997), Jiang (2005c, 2006, 2008) and Ma & Yao (2010). Firstly we will give the following condition.

\[ \text{(A5). } P\text{-a.s., } \forall (t, y) \in [0, T] \times \mathbb{R}^d, \quad g(t, y, 0) = 0. \]

By Kobylanski (2000, Theorem 2.3 and Theorem 2.6), if \(g\) satisfies \((A1)-(A4)\), for each \(\xi \in L^\infty(\mathcal{F}_T)\), BSDE with parameter \((g, T, \xi)\) has a unique solution \((Y^T_0, Z^T_0)\). Moreover, if \(g\) also satisfies \((A5)\), we set \(\mathcal{E}^T_g(\xi) := Y^T_0\), called quadratic \(g\)-expectation of \(\xi\), and \(\mathcal{E}^T_g[\xi|\mathcal{F}_t] = Y^T_t\) for \(t \in [0, T]\), called conditional quadratic \(g\)-expectation of \(\xi\).

**Theorem 6.1 (Uniqueness theorem)** Let \(g_1\) and \(g_2\) satisfy \((A1)-(A5)\), then the following two statements are equivalent:
(i) For each \(\xi \in L^\infty(\mathcal{F}_T)\), we have
\[ \mathcal{E}^T_{g_1}(\xi) = \mathcal{E}^T_{g_2}(\xi); \]
(ii) For each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\), and almost every \(t \in [0, T]\), we have
\[ P - a.s., \quad g_1(t, y, z) = g_2(t, y, z). \]

**Proof.** The proof is the same to Jiang (2006) for BSDEs under Lipschitz condition. We sketch it. By a comparison theorem for quadratic BSDEs (see Kobylanski (2000, Theorem 2.6) or Ma and Yao (2010, Theorem 3.2)), we get (ii) \(\Rightarrow\) (i). By Corollary 3.1, Ma and Yao (2009, Proposition 3.4) and the fact: if \((A1)-(A5)\) holds for \(g\), then \(\forall t \in [0, T], \ \xi \in L^\infty(\mathcal{F}_t)\), we have
\[ \mathcal{E}^T_g[\xi|\mathcal{F}_s] = \mathcal{E}^T_g[\xi|\mathcal{F}_s], \quad s \in [0, t]. \]
we can prove (i) \implies (ii). □

The following Theorem 6.2 is a general converse comparison theorem for quadratic \(g\)-expectation, in which (i) \iff (iii) generalizes Ma and Yao (2010, Theorem 4.2) and (i) \iff (ii) is new.

**Theorem 6.2** (Converse comparison theorem) Let \(g_1\) and \(g_2\) satisfy (A1)-(A5), then the following three statements are equivalent:

(i) For each \(\xi \in \mathbb{L}_\infty(\mathcal{F}_T)\), and each \(t \in [0, T]\), we have

\[
P - a.s., \quad \mathcal{E}_{g_1}^T[\xi | \mathcal{F}_t] \leq \mathcal{E}_{g_2}^T[\xi | \mathcal{F}_t];
\]

(ii) For each \(\xi \in \mathbb{L}_\infty(\mathcal{F}_T)\), and each \(t \in [0, T]\), we have

\[
E \left[ \mathcal{E}_{g_1}^T[\xi | \mathcal{F}_t] \right] \leq E \left[ \mathcal{E}_{g_2}^T[\xi | \mathcal{F}_t] \right];
\]

(iii) For each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\), and almost every \(t \in [0, T]\), we have

\[
P - a.s., \quad g_1(t, y, z) \leq g_2(t, y, z).
\]

**Proof.** By comparison theorem for quadratic BSDEs (see Kobylanski (2000, Theorem 2.6) or Ma and Yao (2010, Theorem 3.2)), Theorem 5.1 and (28), we can get (i) \iff (iii). (i) \implies (ii) is trivial. So, we only need show (ii) \implies (i). In fact, If (i) does not hold, then there exist \(\xi \in \mathbb{L}_\infty(\mathcal{F}_T)\) and \(s \in [0, T]\) such that for some constant \(\delta > 0,
\]

\[
P \left( \mathcal{E}_{g_1}^T[\xi | \mathcal{F}_s] \geq \mathcal{E}_{g_2}^T[\xi | \mathcal{F}_s] + \delta \right) > 0.
\]

We set \(A := \left\{ \mathcal{E}_{g_1}^T[\xi | \mathcal{F}_s] \geq \mathcal{E}_{g_2}^T[\xi | \mathcal{F}_s] + \delta \right\}\). Clearly \(A \in \mathcal{F}_s\), then by the properties of quadratic \(g\)-expectation (see Ma and Yao (2010)), we have

\[
\mathcal{E}_{g_1}^T[1_A \xi | \mathcal{F}_s] = 1_A \mathcal{E}_{g_1}^T[\xi | \mathcal{F}_s] \geq 1_A \mathcal{E}_{g_2}^T[\xi | \mathcal{F}_s] + 1_A \delta = \mathcal{E}_{g_2}^T[1_A \xi | \mathcal{F}_s] + 1_A \delta.
\]

Taking expectation on both sides, we have

\[
E \left[ \mathcal{E}_{g_1}^T[1_A \xi | \mathcal{F}_s] \right] > E \left[ \mathcal{E}_{g_2}^T[1_A \xi | \mathcal{F}_s] \right],
\]

which contradicts (ii). The proof is complete. □

By Corollary 3.1 and the argument of Jiang (2008, Theorem 3.1) or Ma and Yao (2010, Proposition 4.3), we can get the following Theorem 6.3, translation invariance of \(g\)-expectation, which is a generalization of Ma and Yao (2010, Proposition 4.3). Ma and Yao (2010, Proposition 4.3) is proved under more stronger conditions.

**Theorem 6.3** (Translation invariance) Let \(g\) satisfy (A1)-(A5), then the following four statements are equivalent:

(i) \(g\) is independent on \(y\);

(ii) For each \(\xi \in \mathbb{L}_\infty(\mathcal{F}_T)\) and constant \(C\), we have

\[
\mathcal{E}_g^T(\xi + C) = \mathcal{E}_g^T(\xi) + C;
\]

(iii) For each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\), and almost every \(t \in [0, T]\), we have

\[
P - a.s., \quad g_1(t, y, z) \leq g_2(t, y, z).
\]

By Corollary 3.1 and the argument of Jiang (2008, Theorem 3.1) or Ma and Yao (2010, Proposition 4.3), we can get the following Theorem 6.3, translation invariance of \(g\)-expectation, which is a generalization of Ma and Yao (2010, Proposition 4.3). Ma and Yao (2010, Proposition 4.3) is proved under more stronger conditions.

By Corollary 3.1 and the argument of Jiang (2008, Theorem 3.1) or Ma and Yao (2010, Proposition 4.3), we can get the following Theorem 6.3, translation invariance of \(g\)-expectation, which is a generalization of Ma and Yao (2010, Proposition 4.3). Ma and Yao (2010, Proposition 4.3) is proved under more stronger conditions.
(iii) For each $\xi \in L^\infty(\mathcal{F}_T)$, $t \in [0,T]$ and constant $C$, we have
$$P-a.s., \quad E^T_g[\xi + C|\mathcal{F}_t] = E^T_g[\xi|\mathcal{F}_t] + C;$$
(iv) For each $\xi \in L^\infty(\mathcal{F}_T)$, $t \in [0,T]$ and $\eta \in \mathcal{F}_t$, we have
$$P-a.s., \quad E^T_g[\xi + \eta|\mathcal{F}_t] = E^T_g[\xi|\mathcal{F}_t] + \eta.$$

**Remark 6.1** Let $\tau$ be a finite stopping time ($\tau < \infty, a.s.$). As Theorem 6.1-6.3 in this section, we can get similar results for quadratic $g$-expectation induced by quadratic BSDEs with finite terminal time $\tau$.

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