BOUNDARY LAYER PROBLEM AND QUASINEUTRAL LIMIT OF COMPRESSIBLE EULER-POISSON SYSTEM

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ABSTRACT. We study the boundary layer problem and the quasineutral limit of the compressible Euler-Poisson system arising from plasma physics in a domain with boundary. The quasineutral regime is the incompressible Euler equations. Compared to the quasineutral limit of compressible Euler-Poisson equations in whole space or periodic domain, the key difficulty here is to deal with the singularity caused by the boundary layer. The proof of the result is based on a $\lambda$-weighted energy method and the matched asymptotic expansion method.

1. Introduction. This paper studies the boundary layer problem and the quasineutral limit of the isentropic Euler-Poisson system in plasma for $t > 0$ and $x = (x_1, x_2, x_3) = (y, x_3) \in \mathbb{R}_+^3 := \mathbb{R}^2 \times \mathbb{R}_+$,

\begin{align*}
\partial_t n^\lambda + \text{div}(n^\lambda u^\lambda) &= 0, \\
\partial_t u^\lambda + u^\lambda \cdot \nabla u^\lambda + \nabla h(n^\lambda) &= -\nabla \phi^\lambda, \\
-\lambda^2 \Delta \phi^\lambda &= n^\lambda - 1,
\end{align*}

where $\lambda$ is the (scaled) Debye length, $n^\lambda, u^\lambda = (u^\lambda_1, u^\lambda_2, u^\lambda_3) = (u^\lambda_y, u^\lambda_3)$, $-\phi^\lambda$ denote the electron density, the electron velocity and the electric potential, respectively. The function $h = h(n)$ is the enthalpy for the system and satisfies

$h'(n) = \frac{p'(n)}{n}$, $n > 0$,

where $p = p(n)$ is the pressure-density function, and we assume

$p(n) = a^2 n^\gamma$, $n > 0$, $a \neq 0$, $\gamma \geq 1$.

The boundary conditions of the system read as:

$u^\lambda_3|_{x_3=0} = 0$, $\phi^\lambda|_{x_3=0} = \phi_b$,
where $\phi_b = \phi_b(y)$ is some prescribed potential and smooth and compactly supported.

The Euler-Poisson system plays a vital role in the mathematical modeling of charge transport in semiconductors, see Markowich [18]. In recent years, the Euler-Poisson system and its asymptotic limits have been widely studied. System (1)–(3) describes the movement of electron with the act of its electric field against a constant charged ion background (see Guo [6]). Majda [17] proved that, for any fixed $\lambda$, the Euler-Poisson system has a unique smooth solution locally in time when the initial data $n(0, x) \geq 0$. Guo [6] studied the global existence and uniqueness of the smooth solution to the Euler-Poisson system for irrotational flow in $\mathbb{R}^{3+1}$. The global existence of smooth solutions with their long time stability for one-dimensional and multi-dimensional Euler-Poisson models were studied in Ali et al. [1] and [2] respectively for solutions near constant equilibrium states. Hsiao et al. [9] studied the asymptotic behavior of smooth solutions close to constant equilibrium for multi-dimensional Euler-Poisson model. The existence of global smooth solutions near a given steady state has been studied in Guo et al. [8]. Global smooth electron dynamics were constructed in Guo [6] due to dispersive effect of the electric field. Guo [7] constructed global smooth irrotational solutions with small amplitude for ion dynamics in the Euler-Poisson system.

Compared to the characteristic observation length $L$ of physical interest, the Debye length $\lambda_D$ is very small, with the scaled Debye length $\lambda = \frac{\lambda_D}{L} \ll 1$, in such case the density of electrons almost equals to the density of ions, and the zero Debye length limit $\lambda \to 0$ is also called quasineutral limit. Many results have been obtained about the quasineutral limits of corresponding Euler-Poisson systems. These results may be divided into two classes according to their boundary conditions: the case of without the boundary layer and the case of with the boundary layer. In the first case, there are many mathematical studies on quasineutral limits of the corresponding Euler-Poisson system in a domain without boundary. In Peng [20], Wang [29], Loeper [16], Peng et al. [23] and Peng et al. [24], the authors considered the systems describing electrons with fixed ions. Cordier et al. [3] and Pu [25] studied the models of ions with electrons following a Maxwell-Boltzmann law. Jiang et al. ([10] and [11]), Peng et al. [21] and Li et al. [14] considered the quasineutral limits in two-fluid isentropic Euler-Poisson systems, where the quasineutral regime is governed by compressible Euler equations.

In the second case, some authors studied the quasineutral limits of Euler-Poisson systems in domains with boundaries. Slenrod et al. [26] considered the quasineutral limit for the steady-state Euler-Poisson system on a bounded interval in $\mathbb{R}^1$, proved that the quasineutral limit is obtained until the ion velocity reaches the ion-sound speed. Peng [19] studied the zero-electron mass limit, the quasineutral limit and the zero-relaxation-time limit for one dimensional steady-state Euler-Poisson system, proved the strong convergence of a sequence of solutions and gave the corresponding convergence rate. Peng and Wang [22] and Violet [28] studied the quasineutral limit in the steady-state Euler-Poisson system for a potential flow, justified the asymptotic expansions up to first order in one-dimensional case and to any order in multidimensional case respectively. Suzuki [27] was concerned with boundary layers of a multicomponent plasma which consists of electrons and several positive ion species. Gérard-Varet et al. ([4] and [5]) studied the quasineutral limit of the isothermal Euler-Poisson system for the ions in a domain with boundaries, and proved the quasineutral regime is given by the compressible Euler equations. The
corresponding result on the quasineutral limit is extended to the case of the two-
fluid isothermal Euler-Poisson system in a bounded domain of $\mathbb{R}^3$ by Ju et al. [12].
Recently, the quasineutral limit for a model of a three dimensional Euler-Poisson
system with linear enthalpy function has been studied in Liu et al. [15].
In this paper, we study the boundary layer problem and the quasineutral limit
of nonstationary isentropic Euler-Poisson system (1)–(3) for a rotational flow of the
electron (the density of ions being prescribed as a constant background) in a domain
of $\mathbb{R}^3$ with the boundary condition (4).
The rest of our paper is organized as follows. In section 2 we state the main
result. In section 3 we construct the boundary layer approximations. Section 4 is
devoted to the convergence to the incompressible Euler equations.
In this paper, the operator $\partial_i$ stands for $\frac{\partial}{\partial x_i}$, $i = 1, 2, 3$. For convenience,
we shall omit the spatial domain $\mathbb{R}^3$ in integrals.

2. Main result. For Poisson equation (3), letting $\lambda = 0$, we first get the neutrality:
\[ n = 1. \]
From (1)–(2) in the limit $\lambda \to 0$ and the fact that $n = 1$, we have the following
incompressible Euler system:
\[ \text{div} u = 0, \]
\[ \partial_t u + u \cdot \nabla u = -\nabla \phi. \]
We expect that the Euler–Poisson system (1)–(3) converges to this system. System
(5)–(6) and initial data is well-posed with the boundary condition
\[ u_3|_{x_3=0} = 0, \]
\[ u|_{t=0} = u_0. \]
However, the solution $(u^0, \phi^0)$ of (5)–(6) cannot in general satisfy the boundary
condition $\phi^0|_{x_3=0} = \phi_b$. In order to correct this boundary condition, we expect the
formation of a boundary layer. This will be given in the following section 3. Let us
state our main result of this paper as follows:

**Theorem 2.1.** Let $(u^0, \phi^0)$ be a solution to (5)–(6) such that
\[ u^0 \in C^0([0, T], H^s(\mathbb{R}_x^3)) \text{ with } s \text{ large enough.} \]
There is $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0]$, there exists $(n^\lambda, u^\lambda, \phi^\lambda)$, a unique solution to (1)–(3) also defined on
$[0, T]$ such that
\[ \limsup_{\lambda \to 0} \sup_{[0, T]} (\|n^\lambda - 1\|_{L^2(\mathbb{R}_x^3)} + \|u^\lambda - u^0\|_{L^2(\mathbb{R}_x^3)} + \|\lambda \nabla \phi^\lambda - \lambda \nabla \phi^0\|_{L^2(\mathbb{R}_x^3)}) \to 0. \]

The proof of Theorem 2.1 is based on the $\lambda$- weighted energy method. This
is a boundary layer problem. The main difficulties are to deal with oscillation
terms like $u \cdot \nabla n_a$, caused by the boundary layer function $n_a = n^0(x_1, x_2, x_3, t) +\n N^0(x_1, x_2, x_3, t) + \lambda(n^1 + N^1) + \cdots$, given in section 3, and the general $\gamma$- law
pressure term $\nabla h(n_a + n) = h'(n_a + n)(\nabla n_a + \nabla n)$. Here our key point is to control
the oscillating behavior of the electric field due to the loss of the damping term
in the Poisson equations. These are required to overcome by some new techniques
different from the papers in Gérard-Varet et al. [4], [5] and Ju et al. [12]. Also, the
quesineutral regime here is incompressible Euler flow, which is different from the
previous results on quasineutral limits in a domain with boundary.
3. Construction of boundary layer approximations. We establish the approximate solution of the Euler-Poisson system (1)–(3) under the form of an asymptotic expansion of a power series in $\lambda$

\[
(n_\alpha, u_\alpha, \phi_\alpha) = \sum_{i=0}^{K} \lambda_i (n_i(t, x), u_i(t, x), \phi_i(t, x)) + \sum_{i=0}^{K} \lambda_i (N_i(t, y, \frac{x_3}{\lambda}), U_i(t, y, \frac{x_3}{\lambda}), \Phi_i(t, y, \frac{x_3}{\lambda})) ,
\]

where $K$ is an arbitrarily large integer. The coefficients $(n^i, u^i, \phi^i)$ of the first sum and $(N^i, U^i, \Phi^i)$ of the second sum should describe the macroscopic behaviour and a boundary layer of size $\lambda$ near the boundary of the solutions respectively. Nevertheless, $(n^i, u^i, \phi^i)(0 \leq i \leq K)$ depend on the independent variables $(t, x)$, $(N^i, U^i, \Phi^i)(0 \leq i \leq K)$ not only on the independent variables $(t, y)$, but also on a stretched variable $z = \frac{x_3}{\lambda}$, at the same time, we shall assume that

\[
\partial_j^2 (N_i(t, y, z), U_i(t, y, z), \Phi_i(t, y, z)) \to 0, \quad \forall j \geq 0 \text{ as } z \to +\infty.
\]

According to the boundary conditions (4) of the system (1)–(3), for the whole approximation, the coefficients shall further satisfy the following boundary conditions:

\[
u^i_j(t, y, 0) + U^i_j(t, y, 0) = 0, \quad \text{for all } i \geq 0.
\]

\[
\phi^0_j(t, y, 0) + \Phi^0_j(t, y, 0) = \phi_b, \quad \phi^j_3(t, y, 0) + \Phi^j_3(t, y, 0) = 0, \quad \text{for all } i \geq 1.
\]

Thus the leading terms $(n^0, u^0, \phi^0)$ will determine the dynamics of these approximate solutions. From the previous section, we need to prove that $(u^0, \phi^0)$ is the solution of the incompressible Euler system

\[
\text{div} u^0 = 0, \tag{12}
\]

\[
\partial_t u^0 + u^0 \cdot \nabla u^0 = -\nabla \phi^0, \tag{13}
\]

with $n^0 = 1$ and the non-penetration condition

\[
u_3^0(t, y, 0) = 0. \tag{14}
\]

We have the following result:

**Theorem 3.1.** Assume the initial velocity satisfies $u^0_\alpha$ satisfy $u^0_\alpha \in H^{m+2K+3}, K \in N_+, m \in N_+, m \geq 3$, div$u^0_0 = 0$, and some compatibility conditions on boundary. Then there exists $T > 0$ and an approximate solution formed (8) of the Euler-Poisson system (1)–(3) such that

1) $(u^i, \phi^i)$ is a solution to the incompressible Euler system (5)–(6) with initial datum $u^0_\alpha$,

2) $\forall 1 \leq i \leq K, (n^i, u^i, \phi^i) \in C^0([0, T], H^{m+3+2K-2i}(R_3))$.

3) $\forall 1 \leq i \leq K, (N^i, U^i, \Phi^i)$ and their derivatives are uniformly exponentially decreasing functions with respect to the last variable $z$.

In addition, let us assume $(n^\lambda, u^\lambda, \phi^\lambda)$ be a solution to Euler-Poisson system (1)–(3) and define

\[
n = n^\lambda - n_\alpha, \quad u = u^\lambda - u_\alpha, \quad \phi = \phi^\lambda - \phi_\alpha. \tag{15}
\]
Then \((n,u,\phi)\) satisfies the system of the error equations:

\[
\begin{align*}
\partial_t n + \text{div}[n_a u + n(u_a + u)] &= \lambda^K R_n, \\
\partial_t u + u_a \cdot \nabla u + u \cdot \nabla (u_a + u) + \nabla h(n_a + n) - \nabla h(n_a) &= -\nabla \phi + \lambda^K R_u, \\
-\lambda^2 \Delta \phi &= n + \lambda^{K+1} R_\phi,
\end{align*}
\]

(16) where \(R_n, R_u, R_\phi\) are remainders satisfying:

\[
\sup_{[0,T]} \|\nabla^\alpha R_{n,u,\phi}\|_{L^2} \leq C_\alpha \lambda^{-\alpha_3}, \ \forall \alpha = (\alpha_1, \alpha_2, \alpha_3) \in N^3, \ |\alpha| \leq m
\]

with \(C_\alpha > 0\) independent of \(\lambda\).

The rest of the section is the proof of Theorem 3.1.

To obtain the approximate solution \((8)\) of \((1)-(4)\), we solve the equations which the leading terms of \((8)\) satisfy. By substituting the expansion \((8)\) into the Euler-Poisson system \((1)-(4)\) and considering the same amplitude terms, we get series of equations on the coefficients \((n^i, u^i, \phi^i)\) and \((N^i, U^i, \Phi^i)\).

We divide the domain of variabilities \(R^3\) into two zones: one is the inner zone and the other is the outer zone (the boundary layer zone). In the inner zone, as \(\lambda \to 0, z = \frac{z}{\lambda^2} \to +\infty\), the boundary layer correctors \((N^i, U^i, \Phi^i)\) \(\to 0\). In the outer zone, by using Taylor expansion

\[
n^i(t, y, x_3) = n^i(t, y, \lambda z) = n^i(t, y, 0) + \lambda z \frac{\partial n^i}{\partial x_3} |_{x_3=0} + \frac{(\lambda z)^2}{2} \frac{\partial^2 n^i}{\partial x_3^2} |_{x_3=0} + \ldots
\]

(the same for \(u^i\) and \(\phi^i\)), we receive the boundary layer equations in variables \((t, y, z)\). For brevity, we denote \(f|_{x_3=0} = \Gamma f\).

In the inner zone, by collecting the \(O(\lambda^0)\) amplitude terms, we get the relations

\[
\begin{align*}
\text{div} u^0 &= 0, \\
\partial_t u^0 + u^0 \cdot \nabla u^0 &= -\nabla \phi^0,
\end{align*}
\]

(19) (20) with \(n^0 = 1\) and the boundary condition \(u_3|_{x_3=0} = 0\).

For the incompressible Euler system \((19)-(20)\), we refer to Kato [13] for well-posedness.

In the outer zone, by collecting the \(O(\lambda^{-1})\) amplitude terms, we get the relations

\[
\begin{align*}
\partial_z [h(\Gamma n^0 + N^0)(\Gamma u^0_3 + U^0_3)] &= 0, \\
(\Gamma u^0_3 + U^0_3) \partial_z U^0_3 + h'(\Gamma n^0 + N^0) \partial_z N^0 &= -\partial_z \Phi^0.
\end{align*}
\]

(21) (22) Equation \((21)\) implies that \((\Gamma n^0 + N^0)(\Gamma u^0_3 + U^0_3)\) does not depend on \(z\). From boundary condition \((10)\), we have

\[
(\Gamma n^0 + N^0)(\Gamma u^0_3 + U^0_3) = 0.
\]

Since \((\Gamma n^0 + N^0) \neq 0\), we obtain that

\[
\Gamma u^0_3 + U^0_3 = 0.
\]

By \((9)\), we deduce \(U^0_3 = \Gamma u^0_3 = 0\). Then we obtain that \((n^0, u^0, \phi^0)\) satisfies the incompressible equations \((12)-(13)\) together with \(n^0 = 1\) and the boundary condition \((14)\). Now, equation \((22)\) can be rewritten as follows

\[
\partial_z [h(\Gamma n^0 + N^0) - h(\Gamma n^0)] = -\partial_z \Phi^0.
\]
By using (9), we get
\[ h(\Gamma n_0 + N_0) = -\Phi^0 + h(\Gamma n_0). \] (23)

Still in the outer zone, collecting the \(O(\lambda^0)\) terms in the Poisson equation (3), we get
\[ -\frac{\partial^2 \Phi^0}{\partial z^2} = N_0. \] (24)

Combining (23) with (24), we have
\[ h(\Gamma n_0 - \partial^2 \Phi^0) = -\Phi^0 + h(\Gamma n_0) \] (25)
with the boundary conditions
\[ \Phi^0|_{z=0} = \phi_b - \phi_0|_{x_3=0}, \quad \Phi^0|_{z=\infty} = 0. \] (26)

This should determine completely \(\Phi^0\). Once \(\Phi^0\) is determined, then \(N_0\) can be solved by (24). For the nonlinear boundary layer system (25)–(26), in view of the definition of enthalpy \(h(n)\), we get an ordinary differential equation in \(z\).
\[ \frac{a^2}{\gamma - 1}(1 - \Phi^0'')^{\gamma - 1} = -\Phi^0 + \frac{a^2}{\gamma - 1} \] (27)

We rewrite (27) as the corresponding Hamiltonian system,
\[ \frac{d}{dz} \left( \begin{array}{c} p \\ \Phi^0 \end{array} \right) = \nabla H(p, \Phi^0), \quad p := \Phi^0', \quad H(p, \Phi^0) := \frac{p^2}{2} + T(\Phi^0) \] (28)
where
\[ T(\Phi^0) = -a^2\left(-\frac{\gamma - 1}{a^2\gamma}\Phi^0 + 1\right)^{\gamma - 1} - \Phi^0 \]

By linearizing the Hamiltonian system at point \((p, \Phi^0) = (0, 0)\), we have
\[ \frac{d}{dz} \left( \begin{array}{c} \tilde{p} \\ \Phi^0 \end{array} \right) = \left( \begin{array}{cc} 0 & \frac{1}{a^2\gamma} \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} \tilde{p} \\ \Phi^0 \end{array} \right) \]

Since the determinant of the Jacobi matrix is a negative number, \((p, \Phi^0) = (0, 0)\) is a saddle fixed point and the stable manifold is locally a curve which is tangent to \(\left(\frac{1}{a} - \frac{1}{\sqrt{\gamma - 1}}\right)\). As \(H(p, \Phi^0) = 0\), we obtain the stable manifold,
\[ p = -\sqrt{2} \sqrt{a^2\left(1 - \frac{\gamma - 1}{a^2\gamma}\Phi^0\right)^{\gamma - 1} + \Phi^0}, \quad \text{for} \quad \Phi^0 > 0 \]
\[ p = \sqrt{2} \sqrt{a^2\left(1 - \frac{\gamma - 1}{a^2\gamma}\Phi^0\right)^{\gamma - 1} + \Phi^0}, \quad \text{for} \quad \Phi^0 < 0 \]

Obviously, combining equation (27), on this branch, the solutions and all their derivatives decay exponentially to 0.

From the above statement, \((n_0, u_0, \phi_0)\), and \((U_0^0, \Phi^0, N_0^0)\) have been derived. Next, we obtain the equations for \((n_1, u_1, \phi_1)\) and \((U_1^0, U_1^y, \Phi_1, N_1^1)\).

In the inner zone, collecting the amplitude \(O(\lambda^1)\) terms in (1)–(3), we have
\[ \text{div} u^1 = 0, \] (29)
\[ \partial_t u^1 + u^0 \cdot \nabla u^1 + u^1 \cdot \nabla u^0 = -\nabla \phi^1, \] (30)
with \(n_1 = 0\).

To solve the system, we need a boundary condition.
In the outer zone, collecting the $O(\lambda^0)$ terms in equation (2), we have
\[
\begin{align*}
\frac{\partial_t (\Gamma u^0_1 + U^0_1)}{\partial x_1} + (\Gamma u^0_1 + U^0_1) \frac{\partial (\Gamma u^0_1 + U^0_1)}{\partial x_1} + (\Gamma u^0_2 + U^0_2) \frac{\partial (\Gamma u^0_2 + U^0_2)}{\partial x_2} \\
+ (z\Gamma \partial_3 u^0_3 + \Gamma u^0_3 + U^0_3) \frac{\partial (\Gamma u^0_3 + U^0_3)}{\partial z} + \frac{\partial h(\Gamma n^0 + N^0)}{\partial x_1} = - \frac{\partial (\Gamma \phi^0 + \Phi^0)}{\partial x_1}, \\
\frac{\partial_t (\Gamma u^0_2 + U^0_2)}{\partial x_1} + (\Gamma u^0_2 + U^0_2) \frac{\partial (\Gamma u^0_2 + U^0_2)}{\partial x_1} + (\Gamma u^0_2 + U^0_2) \frac{\partial (\Gamma u^0_2 + U^0_2)}{\partial x_2} \\
+ (z\Gamma \partial_3 u^0_3 + \Gamma u^0_3 + U^0_3) \frac{\partial (\Gamma u^0_3 + U^0_3)}{\partial z} + \frac{\partial h(\Gamma n^0 + N^0)}{\partial x_2} = - \frac{\partial (\Gamma \phi^0 + \Phi^0)}{\partial x_2}, \\
\frac{\partial}{\partial z} \left[ (\Gamma n^0 + N^0)(z\partial_3 n^0 + \Gamma n^1 + N^1) \right] = - \frac{\partial \Phi^1}{\partial z} + F^0_3,
\end{align*}
\]}

where
\[
F^0_3 = -\Gamma \partial_3 \phi^0.
\]

We notice that $U^0_1 = 0$ and $U^0_2 = 0$ are the trivial solutions to the above system by combining (13), (23) and the fact that $h(\Gamma n^0) \equiv const$. Collecting the $O(\lambda^0)$ terms in equation (1), we obtain
\[
\begin{align*}
\partial_1[(\Gamma n^0 + N^0)\Gamma u^1_1] + \partial_2[(\Gamma n^0 + N^0)\Gamma u^1_2] + \partial_3[(\Gamma n^0 + N^0)(\Gamma u^1_3 + U^1_3)] = F^0_N,
\end{align*}
\]}

where
\[
F^0_N = - (\partial_1 N^0 + \partial_2[(\Gamma n^0 + N^0)z\Gamma \partial_3 u^0_3]).
\]

Note that (9) requires the following compatibility conditions:
\[
(\partial_1(n^0 u^0_1) + \partial_2(n^0 u^0_2))_{x_3 = 0} = F^0_N|_{z = \pm \infty}.
\]

Rewrite equation (34),
\[
\begin{align*}
\partial_2[(\Gamma n^0 + N^0)(\Gamma u^1_3 + U^1_3)] = F^0_N - \partial_1[(\Gamma n^0 + N^0)\Gamma u^1_1] - \partial_2[(\Gamma n^0 + N^0)\Gamma u^1_2].
\end{align*}
\]

Integrate above equation from 0 to $z$, by using (10), we get
\[
\begin{align*}
\Gamma u^1_3 + U^1_3 = \frac{1}{\Gamma n^0 + N^0} \int_0^z (F^0_N - \partial_1[(\Gamma n^0 + N^0)\Gamma u^1_1] - \partial_2[(\Gamma n^0 + N^0)\Gamma u^1_2]).
\end{align*}
\]}

By using the decay condition (9), we can get
\[
\begin{align*}
u^1_3|_{x_3 = 0} = \frac{1}{\Gamma n^0} \int_0^{\pm \infty} (F^0_N - \partial_1[(\Gamma n^0 + N^0)\Gamma u^0_1] - \partial_2[(\Gamma n^0 + N^0)\Gamma u^0_2]).
\end{align*}
\]}

So we get the boundary condition of system (29)–(30). From system (29)–(30), we can determine $(n^1, u^1, \phi^1)$ together with the initial value and the boundary condition (36). Using (35), we get $U^1_3$.

Collecting the $O(\lambda^1)$ terms in equation (3), we obtain
\[
\begin{align*}
- \frac{\partial^2 \Phi^1}{\partial x^2} = N^1.
\end{align*}
\]}

Combining (33) and (37), together with the decay condition (9) and the Dirichlet condition
\[
\begin{align*}
\Phi^1|_{x_3 = 0} = -\phi^1|_{x_3 = 0},
\end{align*}
\]}

we can get $\Phi^1$ and $N^1$ eventually.

More generally, we derive for all $i \geq 2$: ...
In the inner zone, collecting amplitude $O(\lambda^i)$ terms in (1)–(3), we obtain systems of type:

$$\partial_t n^i + \text{div}(u^i + n^i u^0) = f_n^i,$$

(39)

$$\partial_t u^i + u^0 \cdot \nabla u^i + u^i \cdot \nabla u^0 + h'(n^0)\nabla n^i = -\nabla \phi^i + f_u^i,$$

(40)

$$n^i = f_n^i,$$

(41)

where

$$f_n^i = -\text{div}(n^1 u^{i-1} + n^2 u^{i-2} + \cdots + n^{i-1} u^1),$$

$$f_u^i = (u^1 \cdot \nabla u^{i-1} + u^2 \cdot \nabla u^{i-2} + \cdots + u^{i-1} \cdot \nabla u^1) + \nabla (\frac{1}{n!} \frac{\partial^k h(n)}{\partial \lambda^k} |_{\lambda=0} - h'(n^0)\nabla n^i),$$

$$f_{\phi}^i = -\Delta \phi^{i-2}.$$

In the outer zone, collecting $O(\lambda^{-1})$ terms in equations (1)–(2), we get:

$$\partial_t [(\Gamma n^0 + N^0)(\Gamma u_1^{i-1} + U_1^{i-1})] + \partial_2 [(\Gamma n^0 + N^0)(\Gamma u_2^{i-1} + U_2^{i-1})] + \partial_3 [(\Gamma n^0 + N^0)(\Gamma u_3^{i-1} + U_3^{i-1})] = F_{\phi}^{i-1},$$

(42)

$$\partial_t (\Gamma u_1^{i-1} + U_1^{i-1}) + (\Gamma u_0^0 + U_0^0)\partial_1 (\Gamma u_1^{i-1} + U_1^{i-1}) + (\Gamma u_0^0 + U_0^0)\partial_1 (\Gamma u_0^0 + U_0^0) + (\Gamma u_2^{i-1} + U_2^{i-1})\partial_2 (\Gamma u_2^{i-1} + U_2^{i-1}) + (\Gamma u_3^{i-1} + U_3^{i-1})\partial_3 (\Gamma u_3^{i-1} + U_3^{i-1}) = F_1^{i-1},$$

(43)

$$\partial_t (\Gamma u_2^{i-1} + U_2^{i-1}) + (\Gamma u_0^0 + U_0^0)\partial_1 (\Gamma u_2^{i-1} + U_2^{i-1}) + (\Gamma u_0^0 + U_0^0)\partial_1 (\Gamma u_0^0 + U_0^0) + (\Gamma u_2^{i-1} + U_2^{i-1})\partial_2 (\Gamma u_2^{i-1} + U_2^{i-1}) + (\Gamma u_3^{i-1} + U_3^{i-1})\partial_3 (\Gamma u_3^{i-1} + U_3^{i-1}) = F_2^{i-1},$$

(44)

$$\partial_t (\Gamma u_3^{i-1} + U_3^{i-1}) + (\Gamma u_0^0 + U_0^0)\partial_1 (\Gamma u_3^{i-1} + U_3^{i-1}) + (\Gamma u_0^0 + U_0^0)\partial_1 (\Gamma u_0^0 + U_0^0) + (\Gamma u_2^{i-1} + U_2^{i-1})\partial_2 (\Gamma u_2^{i-1} + U_2^{i-1}) + (\Gamma u_3^{i-1} + U_3^{i-1})\partial_3 (\Gamma u_3^{i-1} + U_3^{i-1}) = F_3^{i-1},$$

(45)

The source terms $F_N^{i-1}, F_1^{i-1}, F_2^{i-1}$ and $F_3^{i-1}$ depend on $(n^k, u^k, \phi^k)$, and on $(N^k, U_3^k, U_y^k, \phi^k)$, for indices $k \leq i - 1$.

We notice that $U_1^{i-1} = 0$ and $U_2^{i-1} = 0$ are the trivial solutions to the system (42)–(45) by combining (13) and (23).

Collecting terms with amplitude $O(\lambda^i)$ in (3), we have:

$$-\frac{\partial^2 \phi^i}{\partial z^2} = \Gamma n^i + N^i + F_{\phi}^i,$$

(46)

where $F_{\phi}^i$ depends on $(n^k, \phi^k)$ and on $(N^k, \phi^k)$, $k \leq i - 1$.

Note that (9) requires the following compatibility conditions:

$$\partial_1 (n^0 u_1^{i-1}) + \partial_2 (n^0 u_2^{i-1}) |_{z=0} = F_{N}^{i-1} |_{z=+\infty},$$

$$\phi^i |_{z=0} = F_3^{i-1} |_{z=+\infty},$$

$$-n^i |_{z=0} = F_{\phi}^i |_{z=+\infty}.$$

Rewrite equation (42),

$$\partial_1 [(\Gamma n^0 + N^0)(\Gamma u_1^0 + U_1^0)] = F_{N}^{i-1} - \partial_1 [(\Gamma n^0 + N^0)\Gamma u_1^{i-1}] - \partial_2 [(\Gamma n^0 + N^0)\Gamma u_2^{i-1}].$$
Integrate above equation from 0 to \( z \), by using (10), we get
\[
\Gamma u_i^3 + U_i^3 = \frac{1}{\Gamma n^0 + N^0} \int_0^z (F_i^{r-1} - \partial_1[(\Gamma n^0 + N^0)\Gamma u_i^{r-1}] - \partial_2[(\Gamma n^0 + N^0)\Gamma u_2^{r-1}]).
\]
(47)

From the decay condition (9), we deduce that
\[
u_i^3\big|_{x_3=0} = 1 \quad \Gamma n_0 + N_0 \int_0^\infty (F_i^{r-1} - \partial_1[(\Gamma n^0 + N^0)\Gamma u_i^{r-1}] - \partial_2[(\Gamma n^0 + N^0)\Gamma u_2^{r-1}]).
\]
(48)

So we get the boundary condition of system (39)–(41). From system (39)–(41), we can determine \((n^i, u^i, \phi^i)\) together with a good initial value and the boundary condition (48). Using (47), we get \(U_i^3\).

Combining (45) and (46), together with the decay condition (9) and the Dirichlet condition
\[
\Phi^i\big|_{x_3=0} = -\phi^i\big|_{x_3=0},
\]
(49)
we can get \(\Phi^i\) and \(N^i\) eventually.

As regards the linear hyperbolic systems (29)–(36) and (39)–(48), it is easy to get the well-posedness. For brevity, we won’t cover them in this article.

4. **Stability estimates.** In this section, we study the stability of the boundary layer approximations built in the previous section.

Let us write the solution \((n^\lambda, u^\lambda, \phi^\lambda)\) of (1)–(3) in the form
\[
n^\lambda = n_a + n, \quad u^\lambda = u_a + u, \quad \phi^\lambda = \phi_a + \phi.
\]
Then, we get the error equations:
\[
\partial_t n + \text{div}[n_a u + n(u_a + u)] = \lambda K R_n, \quad (50)
\]
\[
\partial_t u + u_a \cdot \nabla u + u \cdot \nabla (u_a + u) + \nabla h(n_a + n) - \nabla h(n_a) = -\nabla \phi + \lambda K R_u, \quad (51)
\]
\[
-\lambda^2 \Delta \phi = n + \lambda K^{+1} R_\phi, \quad (52)
\]
with the boundary conditions:
\[
u_3\big|_{x_3=0} = 0, \quad \phi\big|_{x_3=0} = 0,
\]
(53)
and the initial conditions:
\[
u|_{t=0} = \lambda K^{+1} u_0, \quad n|_{t=0} = \lambda K^{+1} n_0
\]
(54)

Our main result of this section is stated as follows:

**Theorem 4.1.** Let \( m \geq 3, \theta \geq 3, \) and \((n_0, u_0) \in H^m(R^3_+)\) be the initial data for (54), satisfying some suitable compatibility conditions. Let \( K \in \mathbb{N}^*, \ K \geq \theta m + 1 \) and \( n_a, u_a, \phi_a \) an approximate solution at order \( K \) given by Theorem 3.1 which is defined on \([0, T_0]\). There exists \( \lambda_0 \) such that for every \( \lambda \in (0, \lambda_0] \), the solution of (50)–(52) is defined on \([0, T_0]\) and satisfies the estimates
\[
\lambda^{\theta |\alpha|} \|\partial^\alpha (n, u, \lambda \nabla \phi)\|_{L^2(R^3_+)} \leq C \lambda^{K-1}, \quad \forall \alpha \in \mathbb{N}^3, \ |\alpha| \leq m.
\]

As a simple rephrase of Theorem 4.1, we obtain:
**Proposition 1.** Let \( (n_a, u_a, \phi_a) \) be the approximate solution constructed in Theorem 3.1 and some smooth \((n, u, \phi)\) such that \(u_3|_{x_3=0} = 0\) and
\[
n_a + n \geq \frac{1}{M}, \quad |n| + |u| \leq M,
\]
Then there exist $C(M)$ and $C(C_a, M)$ independent of $\lambda$ such that we have on $[0, T]$ the estimate
\[
\frac{d}{dt} \int ((n_a + n) \frac{\dot{u}^2}{2} + \frac{1}{2} \mathcal{H} h'(n_a + n) + \frac{1}{2} \lambda \nabla \phi^2)
\leq C(C_a, M)(1 + \|\nabla_t x(n, u)\|_{L^\infty} + \frac{1}{\lambda} \|n\|_{L^\infty}) (\|\dot{u}\|_{L^2} + \|\dot{u}\|_{L^2})
+ \|\lambda \nabla \phi\|_{L^2}^2 + \|r_u\|_{L^2}^2 + \|r_n\|_{L^2}^2 + \frac{1}{\lambda} \|\phi\|_{L^2}^2 + \int r_n \dot{\phi} + \int \partial_t \partial_x \phi. \tag{62}
\]

Proof. In the proof, we shall denote by $C_a$ a number which may change from line to line but which is uniformly bounded for $\lambda \in (0, T_0]$ where $T_0$ is the interval of time on which the approximate solution is defined. Since the leading boundary layer term of $u_a$ vanishes, we have
\[
\sup_{(0, T) \times \mathbb{R}^3} |u_a| + |\nabla_t x u_a| \leq C_a, \quad C_a > 0. \tag{63}
\]

For $n_a, \phi_a$, we have
\[
\sup_{(0, T) \times \mathbb{R}^3} |n_a| + |\nabla_{t, x_1, x_2} n_a| \leq C_a, \quad \sup_{(0, T) \times \mathbb{R}^3} |\partial_3 n_a| \leq \frac{1}{\lambda} C_a; \tag{64}
\]
\[
\sup_{(0, T) \times \mathbb{R}^3} |\phi_a| + |\nabla_{t, x_1, x_2} \phi_a| \leq C_a, \quad \sup_{(0, T) \times \mathbb{R}^3} |\partial_3 \phi_a| \leq \frac{1}{\lambda} C_a. \tag{65}
\]

Let us now prove the energy estimate. Multiplying the velocity equation by $(n_a + n) \dot{u}$ and performing standard manipulations, we obtain:
\[
\frac{d}{dt} \int (n_a + n) \frac{\dot{u}^2}{2} = \int (n_a + n) \dot{u} \partial_t \dot{u} + \int \partial_t (n_a + n) \frac{\dot{u}^2}{2}
= \int \partial_t (n_a + n) \frac{\dot{u}^2}{2} + \int r_u \cdot (n_a + n) \dot{u} - \int \dot{u} \cdot \nabla u_a \cdot [(n_a + n) \dot{u}]
- \int O(n) \dot{n} \nabla n_a \cdot (n_a + n) \dot{u} + I_1 + I_2 + I_3, \tag{66}
\]
where
\[
I_1 = -\int (u_a + u) \cdot \nabla \dot{u} \cdot [(n_a + n) \dot{u}],
I_2 = -\int \nabla \phi \cdot [(n_a + n) \dot{u}],
I_3 = -\int \lambda h'(n_a + n) \dot{n} \nabla n_a \cdot (n_a + n) \dot{u}.
\]

The first four terms at the r.h.s. of (66) can be easily estimated by using (63), (64) and (65):
\[
\int \partial_t (n_a + n) \frac{\dot{u}^2}{2} \leq C(C_a, M)(1 + \|\partial_t n\|_{L^\infty}) \|\dot{u}\|_{L^2}^2, \tag{67}
\]
\[
\int r_u \cdot (n_a + n) \dot{u} \leq C(C_a, M) \|r_u\|_{L^2} \|\dot{u}\|_{L^2}, \tag{68}
\]
\[
\int \dot{u} \cdot \nabla u_a \cdot [(n_a + n) \dot{u}] \leq C(C_a, M) \|\dot{u}\|_{L^2}^2, \tag{69}
\]
\[
- \int O(n) \dot{n} \nabla n_a \cdot (n_a + n) \dot{u} \leq C(C_a, M) \frac{1}{\lambda} \|n\|_{L^\infty} \|\dot{n}\|_{L^2} \|\dot{u}\|_{L^2}. \tag{70}
\]
Let us turn to the treatment of $I_1$.

$$I_1 = -\int (u_a + u) \cdot \nabla \dot{u} \cdot [(n_a + n)\ddot{u}] = -\int (n_a + n)(u_a + u) \cdot \nabla \frac{\|\dot{u}\|^2}{2}$$

$$= \int \text{div}[(n_a + n)(u_a + u)] \frac{\|\dot{u}\|^2}{2}.$$  

Relying on (63), (64) and $\|u \cdot \nabla u_a\|_{L^2} \leq C_a \|\nabla u\|_{L^\infty}$, we infer that:

$$I_1 \leq C(C_a, M)(1 + \|\nabla (n, u)\|_{L^\infty}) \|\dot{u}\|^2_{L^2}. \quad (71)$$  

Next, we estimate $I_2$. Integrating by parts, we first have

$$I_2 = -\int (n_a + n)\ddot{u} \cdot \nabla \dot{\phi} = \int \text{div}[(n_a + n)\ddot{u}] \dot{\phi}.$$  

By using equation (57) to express $\text{div}[(n_a + n)\ddot{u}]$, we have

$$I_2 = \int r_n \dot{\phi} - \int \partial_t \ddot{u} \dot{\phi} - \int \text{div}[\dot{u}(u_a + u)] \dot{\phi}.$$  

By applying $\partial_t$ to the Poisson equation (59), we get that

$$-\lambda^2 \partial_t \Delta \dot{\phi} = \partial_t \ddot{u} + \partial_t r_{\phi}.$$  

So $I_2$ can be expressed as follows:

$$I_2 = \int r_n \dot{\phi} + \int \partial_t r_{\phi} \dot{\phi} + \lambda^2 \int \partial_t \Delta \dot{\phi} - \int \text{div}[\dot{u}(u_a + u)] \dot{\phi}.$$  

Integrating by parts and multiplying the Poisson equation by $(u_a + u)$, we get

$$I_2 = \int r_n \dot{\phi} + \int \partial_t r_{\phi} \dot{\phi} - \frac{1}{2} \frac{d}{dt} \int |\lambda \nabla \dot{\phi}|^2$$

$$- \int \lambda^2 \Delta \dot{\phi}(u_a + u) \cdot \nabla \dot{\phi} - \int r_{\phi}(u_a + u) \cdot \nabla \dot{\phi}$$

$$= \int r_n \dot{\phi} + \int \partial_t r_{\phi} \dot{\phi} - \frac{1}{2} \frac{d}{dt} \int |\lambda \nabla \dot{\phi}|^2 + I_2^1 + I_2^2.$$  

For $I_2^1$, integrating by parts, we have

$$I_2^1 = \lambda^2 \int \nabla \dot{\phi} \cdot \nabla \dot{\phi} (u_a + u)$$

$$= \lambda^2 \int \nabla \dot{\phi} \cdot [\nabla \dot{\phi} (u_a + u)] + \lambda^2 \int (u_a + u) \cdot \nabla \frac{|\nabla \dot{\phi}|^2}{2}$$

$$= \lambda^2 \int \nabla \dot{\phi} \cdot [\nabla \dot{\phi} (u_a + u)] - \lambda^2 \int \text{div}(u_a + u) \frac{|\nabla \dot{\phi}|^2}{2}$$

$$\leq \lambda^2 \int |\nabla \dot{\phi}|^2 |\nabla (u_a + u)| - \lambda^2 \int \text{div}(u_a + u) \frac{|\nabla \dot{\phi}|^2}{2}.$$  

Using (61) and (63), we get

$$I_2^1 \leq C(C_a, M)(1 + \|\nabla u\|_{L^\infty}) \int |\lambda \nabla \dot{\phi}|^2.$$  

For $I_2^2$, thanks to (63), we obtain

$$I_2^2 \leq (C_a + \|u\|_{L^\infty}) \frac{1}{\lambda} |r_{\phi}|_{L^2} \|\lambda \nabla \dot{\phi}\|_{L^2}.$$
Combining the previous inequalities, we have
\[
I_2 + \frac{1}{2} \frac{d}{dt} \int |\lambda \nabla \phi|^2 \leq C(C_a, M) \left( 1 + \|\nabla u\|_{L^\infty} + \|u\|_{L^\infty} \right) \|\lambda \nabla \phi\|_{L^2}^2 + \int r_n \dot{\phi} + \int \partial_t r_n \dot{\phi}.
\]
Finally, we need to estimate \( I_3 \). Since
\[
div((n_a + n) \dot{u} h'(n_a + n) \dot{n})
= (n_a + n) \dot{u} h'(n_a + n) \cdot \nabla \dot{n} + (n_a + n) \dot{u} \cdot \nabla h'(n_a + n) \dot{n}
+ \text{div}((n_a + n) \dot{u}) h'(n_a + n) \dot{n}
= (n_a + n) \dot{u} h'(n_a + n) \cdot \nabla \dot{n} + \text{div}((n_a + n) \dot{u}) h'(n_a + n) \dot{n}
+ [h''(n_a) + h^{(3)}(n_a) n + O(n^2)] \cdot (\nabla n + \nabla n_a) \cdot \dot{n}(n_a + n) \dot{u},
\]
we have
\[
I_3 = \int h'(n_a + n) \dot{n} \text{div}[(n_a + n) \dot{u}] + \int h''(n_a) \nabla \cdot \dot{n}(n_a + n) \dot{u}
+ \int h^{(3)}(n_a) n(\nabla n + \nabla n_a) \cdot \dot{n}(n_a + n) \dot{u}
+ \int O(n^2)(\nabla n + \nabla n_a) \cdot \dot{n}(n_a + n) \dot{u}
= I_3^1 + I_3^2 + I_3^3 + I_3^4.
\]
For \( I_3^1 \), we rewrite (57) in the form
\[
div[(n_a + n) \dot{u}] = r_n - \partial_t \dot{n} - \text{div}[\dot{n}(u_a + u)].
\]
So,
\[
I_3^1 = \int r_n h'(n_a + n) \dot{n} - \int \partial_t \dot{n} h'(n_a + n) \dot{n} - \int \text{div}[\dot{n}(u_a + u)] h'(n_a + n) \dot{n}.
\]
Integrating by parts, we obtain:
\[
I_3^1 = \int r_n h'(n_a + n) \dot{n} - \frac{1}{2} \frac{d}{dt} \int \dot{n}^2 h'(n_a + n) + \frac{1}{2} \int \dot{n}^2 h''(n_a + n) \partial_t (n_a + n)
- \int \dot{n} \text{div}(u_a + u) h'(n_a + n) \dot{n} - \int \nabla \dot{n} \cdot (u_a + u) h'(n_a + n) \dot{n}
= \int r_n h'(n_a + n) \dot{n} - \frac{1}{2} \frac{d}{dt} \int \dot{n}^2 h'(n_a + n) + \frac{1}{2} \int \dot{n}^2 h''(n_a + n) \partial_t (n_a + n)
- \frac{1}{2} \int \dot{n}^2 \text{div}(u_a + u) h'(n_a + n) + \frac{1}{2} \int \dot{n}^2 h''(n_a + n) \nabla (n_a + n) \cdot (u_a + u)
= \int r_n h'(n_a + n) \dot{n} - \frac{1}{2} \frac{d}{dt} \int \dot{n}^2 h'(n_a + n) - \frac{1}{2} \int \dot{n}^2 \text{div}(u_a + u) h'(n_a + n)
+ \frac{1}{2} \int \dot{n}^2 h''(n_a + n) \partial_t (n_a + n) + \nabla (n_a + n) \cdot (u_a + u).
\]
For \( \int \dot{n}^2 h''(n_a + n) \partial_t (n_a + n) + \nabla (n_a + n) \cdot (u_a + u) \), since we have the equation
\[
\partial_t n_a + \nabla n_a \cdot u_a = -n_a \text{div} u_a - \lambda R_n,
\]
with together with (61), we have
\[
|\partial_t n_a + \nabla n_a \cdot u_a| \leq C_a.
\]
Moreover, since
\[ u_3(t, y, x_3) = \int_0^{x_3} \frac{\partial u_3}{\partial x_3} dx_3 \leq x_3 |\frac{\partial u_3}{\partial x_3}|_{L^\infty}, \]
here we used \( u_3|_{x_3=0} = 0 \), and by using (63), we have
\[ ||u \cdot \nabla n_a||_{L^2} \leq C_a ||\nabla u||_{L^\infty}. \]

Hence relying on (61), (63) and (64), we have
\[ I_3^3 + \frac{1}{2} d\int \dot{n}^2 h'(n_a + n) \leq C(C_a, M)((1 + ||\nabla n||_{L^\infty} + ||\nabla u||_{L^\infty})(||\dot{n}||^2_{L^2} + ||r_n||_{L^2}||\dot{u}||_{L^2}) + \frac{1}{\lambda}||n||_{L^\infty} + ||\nabla n||_{L^\infty})||\dot{n}||_{L^2}||\dot{u}||_{L^2}. \] (73)

For \( I_3^2, I_3^3, I_3^4 \), using (61),(63) and (64) again, we have
\[ I_3^2 \leq C(C_a, M)||\nabla n||_{L^\infty} ||\dot{n}||_{L^2}||\dot{u}||_{L^2}, \] (74)
\[ I_3^3 \leq C(C_a, M)\frac{1}{\lambda}||n||_{L^\infty} + ||\nabla n||_{L^\infty})||\dot{n}||_{L^2}||\dot{u}||_{L^2}, \] (75)
\[ I_3^4 \leq C(C_a, M)(\frac{1}{\lambda}||n||_{L^\infty} + ||\nabla n||_{L^\infty})||\dot{n}||_{L^2}||\dot{u}||_{L^2}. \] (76)

Combining the previous inequalities, we have
\[ I_3 + \frac{d}{dt}(n_a + n) \leq C(C_a, M)(1 + \frac{1}{\lambda}||n||_{L^\infty} + ||\nabla n||_{L^\infty} + ||\nabla u||_{L^\infty}) \times (||\dot{n}||^2_{L^2} + ||\dot{u}||^2_{L^2}) + ||r_n||^2_{L^2}. \] (77)

Combining (77) with (71)–(72), we obtain
\[ \frac{d}{dt}(n_a + n) \frac{\dot{u}^2}{2} + \frac{1}{2} \frac{d}{dt} \int \dot{n}^2 h'(n_a + n) + \frac{1}{2} \frac{d}{dt} \int |\lambda \nabla \phi|^2 \leq C(C_a, M)(1 + ||\nabla n||_{L^\infty} + \frac{1}{\lambda}||n||_{L^\infty})(||\dot{n}||^2_{L^2} + ||\dot{u}||^2_{L^2} + \lambda ||\nabla \phi||^2_{L^2}) \]
\[ + ||r_n||^2_{L^2} + ||r_n||^2_{L^2} + \frac{1}{\lambda^2}||r_n||^2_{L^2} + \int r_n \dot{\phi} + \int \partial_t r_n \dot{\phi}. \]

The proof of Proposition 1 is complete. 

**Proposition 2.**
\[ ||(n, u, \lambda \nabla \phi(t))||^2_{L^2} \leq C(C_a, M)(||n_0, u_0||^2_{L^2} + \lambda^{2(K-1)}t) \]
\[ + C(C_a, M) \int_0^t \left(1 + ||\nabla n(u, u)||_{L^\infty} + \frac{1}{\lambda}||(n, u)||_{L^\infty} ||(n, u, \lambda \nabla \phi(t))||^2_{L^2} \right) \] (78)

**Proof.** We rewrite the density equation (50) into the following form:
\[ \partial_t n + \text{div}[(n_a + n)u] + \text{div}[n(u_a + u)] = \text{div}(nu) + \lambda^K R_n. \]

Using directly the estimate of (62), we get that
\[
\frac{d}{dt} \int (n_a + n) \frac{|u|^2}{2} + \frac{1}{2} \frac{d}{dt} \int n^2 h'(n_a + n) + \frac{1}{2} \frac{d}{dt} \int |\lambda \nabla \phi|^2 \\
\leq C[\xi, M_t](1 + \|\nabla t, u\|_{L^\infty} + \frac{1}{\lambda} \|\phi\|_{L^\infty}) (\|n\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\lambda \nabla \phi\|_{L^2}^2) \\
+ \|r_u\|_{L^2}^2 + \|r_\phi\|_{L^2}^2 + \frac{1}{\lambda^2} \|r_\phi\|_{L^2}^2 + \int r_n \phi + \int \partial_t r_\phi \phi, 
\]
where we have set
\[
r_n = \text{div}(nu) + \lambda^K R_n, \quad r_u = \lambda^K R_u, \quad r_\phi = \lambda^{K+1} R_\phi.
\]
We deal with the term \(\|r_\phi\|_{L^2}^2\) and the last two terms in the right hand side of (79) with integrating by parts and using the Poincaré inequality as follows:
\[
\int r_n \phi = \int \phi(\text{div}(nu) + \lambda^K R_n) = - \int \nabla \phi \cdot (nu) + \int \phi \lambda^K R_n \\
\leq - \int \nabla \phi \cdot (nu) + \|\lambda \nabla \phi\|_{L^2}^2 + \lambda^{2(K-1)} \|R_n\|_{L^2}^2 \\
\leq \frac{1}{\lambda} \|\nabla \phi\|_{L^2}^2 (\|\lambda \nabla \phi\|_{L^2} + \|u\|_{L^2}^2) + \|\lambda \nabla \phi\|_{L^2}^2 + \lambda^{2(K-1)} \|R_n\|_{L^2}^2, 
\]
\[
\int \partial_t r_\phi \phi \leq \|\lambda \nabla \phi\|_{L^2}^2 + \frac{1}{\lambda^2} \|\partial_t r_\phi\|_{L^2}^2 \leq \|\lambda \nabla \phi\|_{L^2}^2 + \lambda^{2K} \|\partial_t R_\phi\|_{L^2}^2. 
\]
Therefore, (79) is proved by using the estimate in Proposition 1. The proof of Proposition 2 is complete. \(\square\)

4.2. Nonlinear stability. Thanks to the well-posedness in \(H^m\) for \(m \geq 3\) of the system (50)-(52), we can define
\[
T^\alpha = \sup\{T \in [0, T_0], \forall t \in [0, T], \|(n, u, \lambda \nabla \phi)\|_{H^m_t(R^2_+)} \leq \lambda^\tau, \text{and (56) is verified}\}
\]
where \(\tau\) is chosen such that
\[
\frac{5}{2} < \tau < K 
\]
and the \(H^m_\lambda\) norm is defined by
\[
\|f\|_{H^m_\lambda(R^2_+)} = \sum_{|\alpha| \leq m} \lambda^{|\alpha|} \|\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} f\|_{L^2(R^2_+)}.
\]
We shall also use the norms:
\[
\|f\|_{H^m_{\text{co}, \lambda}(R^2_+)} = \sum_{|\alpha| \leq m} \|Z_0^\alpha Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3} f(t)\|_{L^2(R^2_+)}
\]
where the vector fields \(Z_i\) are defined by
\[
Z_0 = \lambda \partial_{x_1}, \quad Z_i = \lambda \partial_{x_i}, \quad i = 1, 2, \quad Z_3 = \lambda \varphi(x_3) \partial_{x_3}
\]
and
\[
\|f\|_{H^m_t(R^2_+)} = \sum_{|\alpha| \leq m} \lambda^{|\alpha|} \|\partial_t^{\alpha_0} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} f(t)\|_{L^2(R^2_+)}, 
\]
\[
\varphi(x_3) = \frac{x_3}{1 + x_3}.
\]
For the sake of brevity, we will also use the following notation:
\[
Z_\alpha = Z_0^\alpha Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}, \text{ for } \alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3).
\]
Finally, we set
\[ Q_m(t) = \| (n, u, \lambda \nabla \phi) \|_{H^m_{\lambda}(R_1^2)} + \| w(t) \|_{H^{m-1}_{\lambda}(R_1^2)} \]
where we have set \( w = \lambda \text{curl} u \) i.e.
\[ w = \lambda \left( \begin{array}{c} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{array} \right) \quad (86) \]

4.2.1. Estimate of the \( H^m_{\lambda} \) norm. Let us recall a classical estimate for products in dimension 3:
\[ \|uv\|_{L^2(R_1^3)} \leq \|u\|_{H^{s_1}(R_1^3)} \|v\|_{H^{s_2}(R_1^3)}, \quad (87) \]
with \( s_1 + s_2 = \frac{3}{2} \) and \( s_1 \neq 0, s_2 \neq 0 \).

We shall first prove that by using the equation, we can estimate the \( H^m_{\lambda} \) norm of the solution of (50)–(52) on \([0, T^\lambda] \):

**Proposition 3.** For \( m \geq 3 \), we have for every \( \lambda \in (0, 1] \), for \( t \in [0, T^\lambda] \), \( \forall \delta > 0 \),
\[ \|(n, u, \lambda \nabla \phi)(t)\|_{L^\infty(R_1^3)} \leq C\lambda^{-(\frac{3}{2} + \delta)}, \quad \|\nabla(n, u, \lambda \nabla \phi)(t)\|_{L^\infty(R_1^3)} \leq C\lambda^{-(\frac{3}{2} + \delta)} \quad (88) \]
for some \( C > 0 \) independent of \( \lambda \) and
\[ \|(n, u, \lambda \nabla \phi)(t)\|_{H^m_{\lambda}(R_1^3)} \leq C[C_a, M](\lambda^{K+1} + \lambda^\gamma) \quad (89) \]
where \( C \) stands for a continuous non-decreasing function with respect to all its arguments which does not depend on \( \lambda \).

**Proof.** In 3- dimensional space, for \( \forall \delta > 0 \), we have \( H^{\frac{3}{2} + \delta} \hookrightarrow L^\infty \), then
\[ \|(n, u, \lambda \nabla \phi)(t)\|_{L^\infty} \leq C\|(n, u, \lambda \nabla \phi)(t)\|_{H^{\frac{3}{2} + \delta}} \]
By the definitions of \( H^m_{\lambda} \)- norm and \( T^\lambda \), for \( m \geq 3 \), we obtain
\[ C\|(n, u, \lambda \nabla \phi)(t)\|_{H^\frac{3}{2}} \leq C\lambda^{-(\frac{3}{2} + \delta)}\|(n, u, \lambda \nabla \phi)(t)\|_{H^{\frac{3}{2} + \delta}} \]
\[ \leq C\lambda^{-(\frac{3}{2} + \delta)}\|(n, u, \lambda \nabla \phi)(t)\|_{H^m_{\lambda}} \]
\[ \leq C\lambda^{-(\frac{3}{2} + \delta)} \]
In the same way, we can get
\[ \|\nabla(n, u, \lambda \nabla \phi)(t)\|_{L^\infty(R_1^3)} \leq C\lambda^{-(\frac{3}{2} + \delta)} \]
By the definitions of \( H^m_{\lambda} \) norm and \( H^m_{\lambda} \) norm, we have
\[ \|(n, u, \lambda \nabla \phi)\|_{H^m_{\lambda}} = \|(n, u, \lambda \nabla \phi)\|_{H^m_{\lambda}} + \|\lambda \partial_t (n, u, \lambda \nabla \phi)(t)\|_{H^{m-1}_{\lambda}} \]
\[ + \|\lambda \partial_t^2 (n, u, \lambda \nabla \phi)(t)\|_{H^{m-2}_{\lambda}} + \cdots \]
\[ + \|\lambda \partial_t^{m-1} (n, u, \lambda \nabla \phi)(t)\|_{H^1_{\lambda}} + \|\lambda \partial_t^m (n, u, \lambda \nabla \phi)(t)\|_{L^2} \]
\[ = \|(n, u, \lambda \nabla \phi)\|_{H^m_{\lambda}} + \sum_{k=1}^{m} \|\lambda \partial_t^k (n, u, \lambda \nabla \phi)(t)\|_{H^{m-k}_{\lambda}} \]
For \( k = 1 \), we proceed as follows.
Multiplying (50) and (51) by \(\lambda\), we have
\[
\lambda \partial_t n = -\lambda (n_a \text{div} u + \nabla n_a \cdot u + n \text{div} u + \nabla n \cdot u_a + \nabla n \cdot u + \nabla n \cdot u) + \lambda^{K+1} R_n,
\]
\[
\lambda \partial_t u = -\lambda (u_a \cdot \nabla u + u \cdot \nabla (u_a + u) + \nabla h(n_a + n) - \nabla h(n_a)) - \lambda \nabla \phi + \lambda^{K+1} R_u.
\]
Then we compute \(\|\lambda \partial_t (n, u)\|_{H^{m-1}_x}\). By using the method referenced in [4], together with the induction assumption, we get that, for \(k\)
\[
\lambda^{\frac{k}{2}} \|n, u\|_{H^{m-1}_x} + \|\lambda \nabla \phi\|_{H^{m-1}_x} + \|\lambda^{K+1} R_n\|_{H^{m-1}_x} + \|\lambda^{K+1} R_u\|_{H^{m-1}_x} + \lambda^{K+1}
\]
(90)
By applying \(\lambda \partial_t\) to the Poisson equation (52), we have
\[-\lambda^2 \triangle (\lambda \partial_t \phi) = \lambda \partial_t n + \lambda^{K+1} \lambda \partial_t R_n, \quad \lambda \partial_t \phi|_{x_3=0} = 0\]
and by applying next (\(\lambda \partial_x\))^{\alpha-1},
\[-(\lambda \partial_x)^\alpha \lambda \text{div} \lambda \partial_t \lambda \nabla \phi = (\lambda \partial_x)^\alpha - \lambda \partial_t n + \lambda^{K+1} (\lambda \partial_x)^{\alpha-1} \lambda \partial_t R_n\]
So we compute \(\|\lambda \partial_t \lambda \nabla \phi\|_{H^{m-1}_x}\),
\[
\|\lambda \partial_t \lambda \nabla \phi\|_{H^{m-1}_x} \leq C(\|\lambda \partial_t n\|_{H^{m-2}_x} + \lambda^{K+1} \|\lambda \partial_t R_n\|_{H^{m-2}_x})
\]
(91)
Therefore, by combining the last estimate and (90), we get that
\[
\|\lambda \partial_t (n, u, \lambda \nabla \phi)\|_{H^{m-1}_x(R^+_1)} \leq C[C_a, M, \|n, u\|_{L^\infty(R^+_1)}, \|\lambda \nabla (n, u)\|_{L^\infty(R^+_1)},
\lambda^{-\frac{3}{2}} \|n, u\|_{H^{m}_x(R^+_1)} + \|\lambda \nabla \phi\|_{H^{m-1}_x(R^+_1)} + \lambda^{K+1} + \lambda^7
\]
(92)
Since \(\|(n, u, \lambda \nabla \phi)\|_{H^{m}_x(R^+_1)} \leq \lambda^7\), and (88), the above inequality can rewrite as follow:
\[
\|\lambda \partial_t (n, u, \lambda \nabla \phi)\|_{H^{m-1}_x(R^+_1)} \leq C[C_a, M, \lambda^{-\frac{3}{2}} (\lambda^{K+1} + \lambda^7)
\]
(93)
So we complete the proof of \(k = 1\). By induction, we assume the following inequality has been proven:
\[
\|(\lambda \partial_t)^k (n, u, \lambda \nabla \phi)\|_{H^{m-k}_x(R^+_1)} \leq C[C_a, M, \lambda^{-\frac{3}{2}} (\lambda^{K+1} + \lambda^7)
\]
(94)
By applying \(\lambda (\lambda \partial_t)^k\) to the equation (50) and (52), we have
\[
(\lambda \partial_t)^{k+1} n = -(\lambda \partial_t)^k \lambda (n_a \text{div} u + \nabla n_a \cdot u + n \text{div} u + \nabla n \cdot u_a + n \text{div} u
\]
\[
+ \nabla n \cdot u) + \lambda^{K+1} (\lambda \partial_t)^k R_n,
\]
\[
(\lambda \partial_t)^{k+1} u = -(\lambda \partial_t)^k \lambda (u_a \cdot \nabla u + u \cdot \nabla (u_a + u) + \nabla h(n_a + n) - \nabla h(n_a))
\]
\[-(\lambda \partial_t)^k \lambda \nabla \phi + \lambda^{K+1} (\lambda \partial_t)^k R_u.
\]
Thanks to Lemma 1 in [4], together with the induction assumption, we get that, for \(k + 1 \leq m, k \geq 1\)
\[
\|(\lambda \partial_t)^{k+1} (n, u)\|_{H^{m-k-1}_x(R^+_1)} \leq C[C_a, M, \lambda^{-\frac{3}{2}} (\lambda^{K+1} + \lambda^7)
\]
(95)
Applying \(\lambda (\lambda \partial_t)^{k+1}\) to the equation, we have
\[-(\lambda \partial_t)^{k+1} \lambda \text{div} (\lambda \nabla \phi) = (\lambda \partial_t)^{k+1} n + \lambda^{K+1} (\lambda \partial_t)^{k+1} R_n, \quad (\lambda \partial_t)^{k+1} \phi|_{x_3=0} = 0\]
and by applying next \((\lambda \partial_z)^{\alpha-1}\),
\[
- (\lambda \partial_z)^{\alpha-1} \lambda \text{div}(\lambda \partial_z) k^{+1}(\lambda \nabla \phi) = (\lambda \partial_z)^{\alpha-1}(\lambda \partial_z)^{k^{+1}n + \lambda K^{+1}(\lambda \partial_z)^{\alpha-1}(\lambda \partial_z)^{k^{+1}R_\phi})
\]
Together with (95), we obtain the following estimate
\[
\|(\lambda \partial_t)^{k^{+1}} \lambda \nabla \phi \|_{H^{m-k-1}(R^2_+)} \leq C[C_a, M, \lambda^{\gamma - \frac{3}{2}}](\lambda K^{+1} + \lambda^\gamma) \tag{96}
\]
So we complete the proof of proposition 3 by induction. \(\square\)

By using similar arguments as above, we shall also get:

**Lemma 4.2.** For \(\lambda \in (0, 1]\), we have the estimate
\[
\|(n, u, \lambda \nabla \phi)(0)\|_{H^{\alpha}(R^2_+)} \leq C[C_a, M]\lambda^\gamma.
\]

### 4.2.2. Normal derivatives estimate.

**Proposition 4.** There exists \(\lambda_0\) such that for every \(\lambda \in (0, \lambda_0], \ t \in [0, T^\lambda)\), \(0 \leq k \leq m\),
\[
\|(\lambda \partial_t)^k(n, u, \lambda \nabla \phi)\|_{H^{m-k}(R^2_+)} \leq C[C_a, M](\lambda K^{+1} + Q_m(t))
\]

**Note that we can reformulate the above inequality into**
\[
\|(n, u, \lambda \nabla \phi)\|_{H^{m-k}(R^2_+)} \leq C[C_a, M](\lambda K^{+1} + Q_m(t)), \ \forall t \in [0, T^\lambda) \tag{97}
\]

**Proof.** Since \(w = \lambda \nabla \times u\)
\[
\|(\lambda \partial_t)^k(\nu_1, u_2)\|_{H^{m-1}(R^2_+)} \leq \|w\|_{H^{m-1}(R^2_+)} + \|u\|_{H^m(R^2_+)} \leq Q_m(t) \tag{98}
\]

Notice that
\[
\|(\lambda \partial_t)^2 \lambda \nabla \phi\|_{H^{m-1}(R^2_+)} \leq \|\lambda \nabla \phi\|_{H^{m}(R^2_+)} + \|(\lambda \partial_t)^2 \phi\|_{H^{m-1}(R^2_+)}
\]
\[
\leq \|\lambda \nabla \phi\|_{H^{m}(R^2_+)} + \|\lambda^2 \Delta \phi\|_{H^{m-1}(R^2_+)}
\]
By using the Poisson equation (52), the last term of above inequality can be controlled by \(\|n\|_{H^{m-1}(R^2_+)}\) and \(\lambda K^{+1}\), so we get
\[
\|(\lambda \partial_t)^2 \lambda \nabla \phi\|_{H^{m-1}(R^2_+)} \leq \|\lambda \nabla \phi\|_{H^{m}(R^2_+)} + \|n\|_{H^{m-1}(R^2_+)} + C_a \lambda K^{+1}
\]
\[
\leq C_a(\lambda K^{+1} + Q_m). \tag{99}
\]

We can rewrite the equation (50) and the equation on \(u_3\) in (51) into:
\[
A_n \begin{pmatrix} \partial_3 u_3 \\ \partial_2 n \end{pmatrix} = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}
\]
where
\[
A_n = - \begin{pmatrix} n_a + n & \chi(u_a + u_3) \\ \chi(u_a + u_3) & h'(n_a + n) \end{pmatrix}.
\]
\[
D_1 = \partial_t h + \nabla n_a \cdot u + (n_a + n) \text{div}(u_1, u_2) + \nabla_{1,2} \cdot (u_a + u)_{1,2}
\]
\[
+ (1 - \chi)(u_a + u_3) \partial_3 n + n \text{div}u_a - \lambda^k R_n
\]
\[
D_2 = \partial_t u_3 + (u_a + u)_{1,2} \cdot \nabla_{1,2} u_3 + u \cdot \nabla u_3
\]
\[
+ (1 - \chi)(u_a + u_3) \partial_3 u_3 + h'(n_a + n) \partial_3 n_a - \partial_3 h(n_a) + \partial_3 \phi - \lambda^k R_u.
\]
Combining (56), we can get that
\[ A_n^{-1} = C_{A_n} \begin{pmatrix} h'(n_a + n) & -\chi(u_a + u)_3 \\ -\chi(u_a + u)_3 & (n_a + n) \end{pmatrix} \]
where \( C_{A_n} = -\frac{1}{h'(n_a + n)(n_a + n) - |\chi(u_a + u)_3|^2} \). Thanks to (56), \( C_{A_n} \) is bounded by \( C_a \) and \( M \). Therefore, we get the equation on \( \lambda \partial_3 n \) and \( \lambda \partial_3 u_3 \) as follows
\[ \begin{pmatrix} \lambda \partial_3 u_3 \\ \lambda \partial_3 n \end{pmatrix} = A_n^{-1} \begin{pmatrix} \lambda D_1 \\ \lambda D_2 \end{pmatrix} \]  
(100)
Combining (56) with (100) yields that
\[ \| \lambda \partial_3 (n, u_3) \|_{H^{m-1}_0(R^3_+)} \leq C[C_a, M](\| \lambda D_1 \|_{H^{m-1}_0(R^3_+)} + \| \lambda D_2 \|_{H^{m-1}_0(R^3_+)} + \lambda) \]

We decompose the terms into
1. linear terms about \( \lambda \partial_{1, 2, 3} (n, u_3) \), \( \lambda \partial_1 u_1 \) and \( \lambda \partial_2 u_2 \). Thanks to (63)–(65), these terms are bounded by \( C[C_a, u_1] \|(n, u)\|_{H^{m-1}_0(R^3_+)} \).
2. linear term about \( n, u \), whose coefficients depend on \( \lambda \nabla (n_a, u_1) \) and \( u_a \). These terms are bounded by \( C[C_a, M, \lambda] \|(n, u)\|_{H^{m-1}_0(R^3_+)} \).
3. quadratic terms about \( n \nabla_{1, 2} \cdot u_1, \lambda \nabla_{1, 2, 3} n \cdot u \) and \( u \nabla_{1, 2, 3} \cdot u_3 \). Using lemma 3 in [4], these terms can be bounded by
\[ \| (n, u) \|_{L^\infty} + \lambda^{-\frac{1}{2}} \| (n, u) \|_{H^{m-1}_0(R^3_+)} + \lambda \| \nabla (n, u) \|_{L^\infty} \| (n, u) \|_{H^{m-1}_0(R^3_+)} \]
4. high-order terms. They are bounded by
\[ C[\|n\|_{L^\infty}] \|n\|_{H^{m-1}_0(R^3_+)} + C[\|n\|_{L^\infty}, \lambda^{-\frac{1}{2}} \|n\|_{H^{m-1}_0(R^3_+)}] \| \lambda \partial_3 n \|_{H^{m-1}_0(R^3_+)} \]
with lemma 3 in [4] again (\( k = 0 \), \( |\alpha| = m - 1 \)).
5. terms of \( \lambda^{K+1} R_{n, u, 3} \) and \( \lambda \partial_3 \phi \). They are bounded by
\[ \| \lambda \nabla \phi \|_{H^{m-1}_0(R^3_+)} + \lambda^{K+1} \]
By using the definition of \( T^k \) and proposition 3, we have
\[ \| \lambda \partial_3 (n, u_3) \|_{H^{m-1}_0(R^3_+)} \leq \| (n, u) \|_{H^{m-1}_0(R^3_+)} + \| \lambda \nabla \phi \|_{H^{m-1}_0(R^3_+)} + \lambda^{K+1} \leq \lambda^{K+1} + Q_m(t) \]
(101)
We combine the estimates (98), (99) with (101), and obtain the estimate for \( k = 1 \). By induction on \( k \), we assume
\[ \| ((\lambda \partial_3)^k (n, u, \lambda \nabla \phi) \|_{H^{m-k}_0(R^3_+)} \leq C[C_a, M](\lambda^{K+1} + Q_m(t)) \]

has been proven.
Applying an argument similar to the case \( k = 1 \), we get the estimate
\[ \| ((\lambda \partial_3)^{k+1} (n, u, \lambda \nabla \phi) \|_{H^{m-k-1}_0(R^3_+)} \) with lemma 3 in [4] again and the induction assumption.

Thus, Proposition 4 is proved. \( \square \)
4.2.3. Main energy estimate.

**Proposition 5.** There exists $\lambda_0$ such that for every $\lambda \in (0, \lambda_0]$, we have the estimate

$$Q_{\text{sm}}(t)^2 \leq C(C_a, M)(\lambda^{2K} + \lambda^{2(K-1)}t) + \int_0^t Q_{\text{sm}}(s)^2 ds$$

where $\theta$ is a constant, and $\theta \geq 3$

*Proof.* First we estimate $\|n, u, \lambda \nabla \phi\|_{H_{\text{sm}}(R^3)}$. Applying the differential operator $Z^\alpha$ to the error equation (50)–(52), we have

$$\begin{align*}
\partial_t Z^\alpha n + \text{div}[(n + Z^\alpha u)] + Z^\alpha n(u_a + u) &= C_n + \lambda^K Z^\alpha R_n, \\
\partial_t Z^\alpha u + (u_a + u) \cdot \nabla Z^\alpha u + Z^\alpha u \cdot \nabla u_a + Z^\alpha n(h''(n_a) + O(n))\nabla n_a + h'(n_a + n)\nabla Z^\alpha n &= -\nabla Z^\alpha \phi + C_u + \lambda^K Z^\alpha R_u,
\end{align*}$$

where $C_n$, $C_u$ and $C_\phi$ are remainders due to commutators. The expressions list as follows:

$$\begin{align*}
C_n &= -[Z^\alpha, n_a \text{div}]u - [Z^\alpha, \nabla n_a]u - C_\alpha^3 Z^\alpha - nZ^\alpha \text{div}(u_a + u) \\
&\quad - [Z^\alpha, \nabla]n \cdot (u_a + u) - C_\alpha^3 Z^\alpha - \nabla Z^\alpha (u_a + u) - n[Z^\alpha, \text{div}], \\
C_u &= -[Z^\alpha, \nabla] \phi - [Z^\alpha, (u_a + u) \cdot \nabla]u - C_\alpha^3 Z^\alpha - \nabla Z^\alpha u_a \\
&\quad - [Z^\alpha, h'(n_a + n)\nabla]n - C_\alpha^3 Z^\alpha - nZ^\alpha (h''(n_a) + O(n))\nabla n_a, \\
C_\phi &= \lambda^2[Z^\alpha, \Delta] \phi.
\end{align*}$$

Comparing system (102)–(104) to system (50)–(52), we observe that they are similar to each other. So we have, for system (102)–(104),

$$\begin{align*}
\frac{d}{dt} \int (n + n_a)^2 \left| \frac{Z^\alpha u}{2} \right|^2 + \frac{1}{2} \int (Z^\alpha n)^2 h'(n_a + n) + \frac{1}{2} \int \left| \lambda \nabla Z^\alpha \phi \right|^2 \\
\leq C(C_a, M)(1 + \|\nabla_{t,x}(n, u)\|_{L^\infty} + \frac{1}{\lambda} \|n\|_{L^\infty})(\|Z^\alpha n\|_{L^2}^2 + \|Z^\alpha u\|_{L^2}^2) \\
&\quad + \|\lambda \nabla Z^\alpha \phi\|_{L^2}^2 + \|r_n \|_{L^2}^2 + \|r_n \|_{L^2}^2 + \frac{1}{\lambda^2} \|r_\phi\|_{L^2}^2 + \int r_n Z^\alpha \phi + \int \partial_t r_\phi Z^\alpha \phi,
\end{align*}$$

where $r_n = C_n + \lambda^K Z^\alpha R_n$, $r_u = C_u + \lambda^K Z^\alpha R_u$, and $r_\phi = C_\phi + \lambda^K + 1 Z^\alpha R_\phi$.

According to Reference [4], we have

$$\|C_n, C_u\|_{L^2} \leq C(C_a, M)(\lambda^{K+1} + Q_m(t)).$$

(105)

Next, we estimate the commutator $C_\phi$. Since

$$[Z_3, \lambda^2 \Delta] \phi = -2\lambda^3 \varphi' \partial_3 \partial_3 \phi - \lambda^3 \varphi'' \partial_3 \phi,$$

using the property repeatedly, we obtain that

$$\begin{align*}
\|C_\phi\|_{L^2} &\leq \lambda \|\lambda \partial_3 \phi\|_{H_{c,\lambda}^{m-1}} + \lambda^2 \|\lambda \partial_3 \phi\|_{H_{c,\lambda}^{m-1}} \\
&\leq \lambda \|\lambda \nabla \phi\|_{H_{c,\lambda}^{m-1}} + \lambda^2 \|\lambda \nabla \phi\|_{H_{c,\lambda}^{m-1}}.
\end{align*}$$

(106)

In the same way, we have

$$\|\partial_t C_\phi\|_{L^2} \leq \|\lambda \nabla \phi\|_{H_{c,\lambda}^{m-1}} + \lambda \|\lambda \nabla \phi\|_{H_{c,\lambda}^{m-1}}.$$
For $\int r_n Z^n \phi$, using the Poincaré inequality, we have
$$\int r_n Z^n \phi = \int (C_n + \lambda^2 R_n) Z^n \phi \leq \|C_n\|_{L^2}^2 + \lambda^2 + \|\nabla Z^n \phi\|_{L^2}^2.$$ With the help of the regularity theory of the Poisson equation (104), we easily get that
$$\|\nabla Z^n \phi\|_{L^2}^2 \leq \lambda^{-4} \|n\|_{L^2}^2 + \lambda^{-2} \|\lambda \nabla \phi\|_{L^2}^2 + \|\lambda \nabla \phi\|_{L^2}^2 + \lambda^2 (K-1) \|Z^n R\|_{L^2}^2.$$ Combine above inequality,
$$\int r_n Z^n \phi \leq \|C_n\|_{L^2}^2 + \lambda^{-4} \|n\|_{L^2}^2 + \lambda^{-2} \|\lambda \nabla \phi\|_{L^2}^2 + \|\lambda \nabla \phi\|_{L^2}^2 + \lambda^2 (K-1).$$

With the similar argument, we have that
$$\int \partial_t r_n Z^n \phi \leq \|\partial_t C_n\|_{L^2}^2 + \lambda^{-4} \|n(t)\|_{L^2}^2 + \lambda^{-2} \|\lambda \nabla \phi(t)\|_{L^2}^2 + \|\lambda \nabla \phi(t)\|_{L^2}^2 + \lambda^2 (K-1).$$ By using Proposition 3 and inequalities (105)–(109), we obtain that
$$\frac{d}{dt} \|(n, u, \lambda \nabla \phi)\|_{H^m}^2 \leq C[C_n, M] (Q_m(t) + \lambda^2 (K-1)) + \lambda^{-4} \|n(t)\|_{L^2}^2 + \lambda^{-2} \|\lambda \nabla \phi(t)\|_{L^2}^2 + \lambda^2 (K-1).$$

Applying Gronwall inequality and Lemma 4.2, we have
$$\|(n, u, \lambda \nabla \phi)\|_{H^m}^2 \leq C[C_n, M] (\lambda^2 K + \lambda^2 (K-1) t + \int_0^t Q_m(t)) + \frac{d}{dt} \|(n, u, \lambda \nabla \phi)\|_{H^m}^2 \leq C[C_n, M] (\lambda^2 K + \lambda^2 (K-1) t + \int_0^t Q_m(t)) + \lambda^{-4} \|n(t)\|_{L^2}^2 + \lambda^{-2} \|\lambda \nabla \phi(t)\|_{L^2}^2 + \lambda^2 (K-1).$$

We need to estimate the second term of $Q_m(t)$. Applying $\lambda \nabla \times$ to equation (51), we have that
$$\partial_t w + (u_a + u) \cdot \nabla w = (w_a + w) \cdot \nabla u - u \cdot \nabla w_a + w \cdot \nabla u_a + \lambda K R_w,$$ where $R_w = \lambda \nabla \times R_a$. Then, applying $(\lambda \partial)^\alpha$ to above equation, using the similar argument as Reference [4], we have that
$$\|w\|_{H^m}^2 \leq C[C_n, M] (\lambda^2 K + \lambda^2 (K-1) t + \int_0^t Q_m(t))$$

In order to eliminate the singular terms in (110), we define the following norms:
$$\|f\|_{H^m} = \sum_{|\alpha| \leq m} \lambda^{\theta(\alpha)} \|\partial_\alpha f\|_{L^2} + \int_0^t Q_m(t))$$
$$\|f\|_{H^m_{x_1, x_2}} = \sum_{|\alpha| \leq m} \lambda^{\theta(\alpha)} \|Z^n f(t)\|_{L^2} + \int_0^t Q_m(t))$$

and
$$Q_{\theta m}(t) = \|(n, u, \lambda \nabla \phi)\|_{H^m_{x_1, x_2}} + \|w(t)\|_{H^{m-1}_{x_2}} + \lambda \|\nabla \phi(t)\|_{L^2}^2 + \lambda^2 (K-1).$$
With the new norms, combining Propositions 3 and 4, we can easily prove that
\[ ||(n, u, \lambda \nabla \phi)||_{H^{2/(\alpha-1)}(\Omega^2)} \leq C[C_{a}, M](\lambda^{K+\theta} + Q_{\theta m}(t)), \quad \forall t \in [0, T^\lambda) \] (112)

Corresponding inequality (110), we can obtain that
\[ ||(n, u, \lambda \nabla \phi)(t)||_{H^{2/(\alpha-1)}(\Omega^2)} + \int_0^t \lambda^{2(\theta-1)-4}||Q_{\theta m}(t)||_{H^{2/(\alpha-1)}(\Omega^2)} \leq C[C_{a}, M](\lambda^{2K} + \lambda^{2(K-1)t} + t) \] (113)

In order to ensure that the singular terms vanish away, we set \( \theta \geq 3 \). Thus by combining (110) and (111), we have
\[ Q_{\theta m}^2(t) \leq C[C_{a}, M](\lambda^{2K} + \lambda^{2(K-1)t} + \int_0^t Q_{\theta m}^2(t)) \]

Here we complete the proof of Proposition 5. \( \square \)

4.2.4. **Proof of Theorem 4.1.** Since
\[ Q_{\theta m}^2(t) \leq C[C_{a}, M](\lambda^{2K} + \lambda^{2(K-1)t} + \int_0^t Q_{\theta m}^2(t)), \]
we use Gronwall inequality to get
\[ Q_{\theta m}(t) \leq C[C_{a}, M]e^{C[C_{a}, M]t} \quad \forall t \in [0, T^\lambda] \]

So by the definition of \( T^\lambda \), for \( \lambda \) small enough, such that \( T^\lambda \geq T_0 \), we have
\[ \lambda^{\theta} ||f(t, n, u, \lambda \nabla \phi)||_{L^2} \leq \lambda^{K-1} \quad \forall t \in [0, T_0]. \]
Thus we complete the proof of Theorem 4.1.

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