Practical and Robust Privacy Amplification with Multi-Party Differential Privacy

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Abstract
When collecting information, local differential privacy (LDP) alleviates privacy concerns of users, as users' private information is randomized before being sent to the central aggregator. However, LDP results in loss of utility due to the amount of noise that is added. To address this issue, recent work introduced an intermediate server and with the assumption that this intermediate server did not collude with the aggregator. Using this trust model, one can add less noise to achieve the same privacy guarantee; thus improving the utility.

In this paper, we investigate this multiple-party setting of LDP. We first analyze the threat model and identify potential adversaries. We then make observations about existing approaches and propose new techniques that achieve a better privacy-utility tradeoff than existing ones. Finally, we perform experiments to compare different methods and demonstrate the benefits of using our proposed method.

1 Introduction
To protect data privacy in the context of data publishing, the concept of differential privacy (DP) has been proposed, and has been widely adopted [21]. DP mechanisms add noise to the aggregated result such that the difference of whether or not an individual is included in the data is bounded. Recently, local differential privacy (LDP) has been deployed by industry. LDP differs from DP in that random noise is added by each user before sending the data to the central server. Thus, users do not need to rely on the trustworthiness of the company hosting the server. This desirable feature of LDP has led to wider deployment by industry (e.g., by Google [24], Apple [1], and Microsoft [19]). Meanwhile, DP is still deployed in settings where the centralized server can be trusted (e.g., the US Census Bureau plans to deploy DP technologies for the 2020 census [3]).

However, removing the trusted central party comes at the cost of utility. Since every user adds some independently generated noise, the effect of noise adds up when aggregating the result. While noise of scale (standard deviation) $\Theta(1)$ suffices for DP, LDP has noise of scale $\Theta(\sqrt{n})$ on the aggregated result ($n$ is the number of users). This gap is essential for eliminating the trust in the centralized server, and cannot be removed [13] by algorithmic improvement.

Recently, researchers introduced settings where one can achieve a middle ground between DP and LDP, in terms of both privacy and utility. This is typically achieved by introducing additional parties [7, 15, 16, 23]. One such setting is called the shuffler model, which introduces another party called the shuffler. Users perturb their information minimally, and then send encrypted version of the perturbed information to the shuffler, who shuffles the users' information, and then sends them to the server. The server then decrypts the reports and aggregates the information. If the shuffler and the server collude, there is no privacy amplification, and the user obtains privacy only from perturbation. If the shuffler and the server can be trusted not to collude, then the shuffler learns nothing about the reported data (because of semantic-security of the encryption scheme), and the server learns less about each individual's report because it cannot link a user to a report. In short, the role of the shuffler is to break the linkage between the users and the reports, thus providing some privacy boost. Due to the privacy boost, users can add less noise, while achieving the same level of privacy for the server. This boost, however, requires trusting that the shuffler will not collude with the server. This new model of LDP, which we call Multi-party DP (MDP), offers a different trade-off between trust and utility than DP and LDP.

Besides the shuffle-based approach, in the MDP model, there is also another interesting direction that uses homomorphic encryption [16]. In particular, each user homomorphically encrypts his/her value using one-hot encoding. The auxiliary server then multiplies the ciphertexts in each location to get the aggregated result (i.e., a histogram), and adds noise to the histogram to provide DP privacy guarantee. Finally, the results are sent to the server. As one-hot encoding is used, the communication cost is large for big domains.

Since the MDP model involves many parties, there could
be different patterns of interaction and collusion among the parties. The possibilities of these colluding parties and the consequences have not been systematically analyzed. For example, existing work proves the privacy boost obtained by shuffling under the assumption that the adversary observes the shuffled reports and knows the input values of the users (except the victim). However, if the other users collude with the adversary, they could also provide their locally perturbed reports, invalidating any privacy boost due to shuffling. For another example, while the homomorphic encryption-based approach provides privacy guaranteed when the adversary consists of the users (except the victim) colluding with the server, there is no privacy when the adversary consists of the server colluding with the auxiliary server. In this paper, we analyze the interaction and potential threats of the MDP models in more detail. We present a unified view of privacy that generalizes DP and LDP. Different parties and possible collusions are then presented and analyzed. Existing protocols are also cast in this framework and analyzed.

Based on our observations, we propose a protocol called MURS (stands for Multi Uniform Random Shufflers). MURS is built on top of the shuffler-based approach. But compared to existing methods, which uses random response, MURS supports Optimized Local Hash (OLH), which provides better utility when the domain size is large. In OLH, each user reports a randomly selected hash function, together with a perturbation of the hashed result of their sensitive value. To enable the usage of OLH, we show that the essence of the privacy amplification [7] is a distribution from the LDP mechanism that is independent of the input value. Then we configure OLH to identify such an independent distribution.

As a result of our analysis of the MDP model, we propose to have the auxiliary server also introduce noise. In MURS, the auxiliary server, besides shuffling the received reports, adds some uniformly random reports so that when all other users collude with the server, there is still privacy guarantee. Moreover, we suggest having more auxiliary servers, which mitigates the threat that a single auxiliary server colludes with the server. As long as not all of the auxiliary servers collude with the server, the privacy amplification guarantee still holds.

To summarize, the main contributions of this paper are:

- We present a unified view of DP, LDP, and recent enhancements, and propose a general framework of MDP.
- We design a protocol MURS that achieves better trust and better utility-privacy trade-off. The design of MURS relies on (1) a theoretical improvement; (2) a thorough analysis of MDP; and (3) existing ideas from related fields.
- We provide implementation details and measure utility and performance of MURS on real datasets. Moreover, we will open source our implementation so that other researchers can build on our results.

Roadmap. In Section 2 we present the requisite background. Existing work whose goal is amplifying privacy guarantees is presented in Section 3. We then analyze our multi-party DP model in Section 4. Based on our analysis, we present our proposal in Section 5. Our evaluation is presented in Section 6. Related work is discussed in Section 7. Finally, we end with some concluding remarks in Section 8.

2 Background

We first review the privacy definitions and the corresponding algorithms used to satisfy these definitions. We then describe some cryptographic primitives that will be used throughout this paper.

2.1 Differential Privacy

Differential privacy is a rigorous notion about individual’s privacy in the setting where there is a trusted data curator, who gathers data from individual users, processes the data in a way that satisfies DP, and then publishes the results. Intuitively, the DP notion requires that any single element in a dataset has only a limited impact on the output.

Definition 1 (Differential Privacy). An algorithm \( A \) satisfies \((\varepsilon, \delta)\)-DP, where \( \varepsilon, \delta > 0 \), if and only if for any neighboring datasets \( D \) and \( D' \), and any set \( R \) of possible outputs of \( A \), we have

\[
\Pr \left[ A(D) \in R \right] \leq e^{\varepsilon} \Pr \left[ A(D') \in R \right] + \delta
\]

Denote a dataset as \( D = \langle v_1, v_2, \ldots, v_n \rangle \). Two datasets \( D = \langle v_1, v_2, \ldots, v_n \rangle \) and \( D' = \langle v'_1, v'_2, \ldots, v'_n \rangle \) are said to be neighbors, or \( D \approx D' \), if there exists at most one \( i \in [n] = \{1, \ldots, n\} \) such that \( v_i \neq v'_i \), and for other \( j \neq i, v_j = v'_j \). When \( \delta = 0 \), we simplify the notation and call \((\varepsilon,0)\)-DP as \( \varepsilon \)-DP. To satisfy \( \varepsilon \)-DP, one can use the Laplace Mechanism, described below:

Laplace Mechanism. The Laplace mechanism computes a function \( f \) on the dataset \( D \) in a differentially private manner, by adding to \( f(D) \) a random noise. The magnitude of the noise depends on \( GS_f \), the global sensitivity or the \( \ell_1 \) sensitivity of \( f \). When \( f \) outputs a single element, such a mechanism \( \hat{A} \) is given below:

\[
\hat{A}(D) = f(D) + \text{Lap} \left( \frac{GS_f}{\varepsilon} \right)
\]

where

\[
GS_f = \max_{(D, D') : D \approx D'} \| f(D) - f(D') \|_1,
\]

and

\[
\Pr[\text{Lap}(B) = x] = \frac{1}{2\beta} e^{-|x|/\beta}
\]

In the above, \( \text{Lap}(B) \) denotes a random variable sampled from the Laplace distribution with scale parameter \( \beta \). When \( f \) outputs a vector, \( \hat{A} \) adds a fresh sample of \( \text{Lap} \left( \frac{GS_f}{\varepsilon} \right) \) to each element of the vector. In the histogram counting queries, \( f(D) = \{ \frac{1}{n} \sum_{v \in D} 1\}_{v \in \mathcal{V}} \), and \( GS_f = 2/n \), where \( 1\}_{v \in \mathcal{V}} \) is
the indicator function that the $i$-th user’s value $v_i$ equals $v$ (more generally, $\mathbb{1}_{\{\text{True}\}} = 1$, $\mathbb{1}_{\{\text{False}\}} = 0$).

2.2 Local Differential Privacy

Compared to the centralized setting, the local version of DP offers a stronger level of protection, because each user only reports the perturbed data. Each user’s privacy is still protected even if the aggregator is malicious.

In the local setting, each user perturbs the input value $v$ using an algorithm $A_i$ and reports $A_i(v)$ to the aggregator.

**Definition 2** (Local Differential Privacy). An algorithm $A(\cdot)$ satisfies $(\epsilon, \delta)$-local differential privacy ($(\epsilon, \delta)$-LDP), where $\epsilon, \delta \geq 0$, if and only if for any input $v, v' \in \mathcal{D}$, and any set $R$ of possible outputs of $A$, we have

$$\Pr[A(v) \in R] \leq e^\epsilon \Pr[A(v') \in R] + \delta$$

Typically, the $\delta$ value used is 0 (thus $\epsilon$-LDP). In LDP, most problems can be reduced to frequency estimation. We present two state-of-the-art protocols for this problem.

**Random Response.** The basic mechanism in LDP is called random response [40]. It was introduced for the binary case, but can be easily generalized to the categorical setting. Here we present the generalized version of random response (GRR), which enables the estimation of the frequency of any given value $v \in \mathcal{D}$.

Here each user with private value $v \in \mathcal{D}$ sends the true value $v$ with probability $p$, and with probability $1-p$ sends a randomly chosen $v' \in \mathcal{D}$ s.t. $v' \neq v$. More formally, the perturbation function is defined as

$$\forall_{y \in \mathcal{D}} \Pr[\text{GRR}(v) = y] = \begin{cases} p = \frac{e^\epsilon}{\epsilon + d - 1}, & \text{if } y = v \\ q = \frac{1}{\epsilon + d - 1}, & \text{if } y \neq v \end{cases}$$

This satisfies $\epsilon$-LDP since $\frac{q}{p} = e^\epsilon$. To estimate the frequency of $\tilde{f}_v$ for $v \in \mathcal{D}$, one counts how many times $v$ is reported, denoted by $\sum_{i \in [n]} \mathbb{1}_{\{y_i = v\}}$, and then computes

$$\tilde{f}_v = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{\{y_i = v\}} - \frac{1}{p - q}$$

where $\mathbb{1}_{\{y_i = v\}}$ is the indicator function that the report of the $i$-th user $y_i$ equals $v$, and $n$ is the total number of users.

**Optimized Local Hashing.** This protocol deals with a large domain size $d = |\mathcal{D}|$ by first using a hash function to compress the input domain into a smaller domain of size $d' < d$, and then applying randomized response to the hashed value. In this protocol, both the hashing step and the randomization step result in information loss. The choice of the parameter $d'$ is a tradeoff between loss of information during the hashing step and loss of information during the randomization step. It is shown in [37] that the estimation variance as a function of $d'$ is minimized when $d' = e^\epsilon + 1$.

In OLH, one reports $\langle H, \text{GRR}(H(v)) \rangle$ where $H$ is randomly chosen from a family of hash functions that hash each value in $\mathcal{D}$ to $\{1 \ldots d'\}$, and GRR(·) is the perturbation function for random response, while operating on the domain $\{1 \ldots d'\}$ (thus $p = \frac{e^\epsilon}{\epsilon + d' - 1}$ in Equation (1)). Let $\langle H_i, y_i \rangle$ be the report from the $i$-th user. For each value $v \in \mathcal{D}$, to compute its frequency, one first computes $|\{i \mid H_i(v) = y_i\}| = \sum_{i \in [n]} \mathbb{1}_{\{H_i(v) = y_i\}}$, and then transforms it to its unbiased estimation

$$\tilde{f}_v = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{\{H_i(v) = y_i\}} - \frac{1}{p - 1/d'}$$

In [37], the estimation variances of GRR and OLH are analyzed and compared. Unlike GRR, OLH has a variance that does not depend on $|\mathcal{D}|$. As a result, for smaller $|\mathcal{D}|$ (such that $|\mathcal{D}| - 2 < 3e^\epsilon$), GRR is better; but for large $|\mathcal{D}|$, OLH is preferable.

2.3Cryptographic Primitives

In addition to the mechanisms used to ensure privacy, in the setting where DP and LDP are commonly used, cryptographic primitives are also necessary to protect data from breaches. For example, when LDP is used, the reports are transmitted over the Internet, and the security of the communication channel relies on modern cryptography (more specifically, TLS). Cryptographic protocols can also be used in place of LDP and DP for privacy.

**Additive Homomorphic Encryption.** In Additive Homomorphic encryption [32], one can apply an algebraic operation (denoted by $\times$) to two ciphertexts $c_1$ and $c_2$, so that the result $c_1 \times c_2$ is a ciphertext of the sum of the corresponding plaintexts.

**Mix-net (Shuffler).** A mix-net (introduced in [14]) is a server that receives users’ inputs and sends them to another server in a way that breaks any linkage from the identifier of a user to the input of the user. In particular, the $i$-th user’s data is encrypted with two layers, i.e., $c_i = \text{Enc}_{k_2}(\text{Enc}_{k_1}(v_i))$. The shuffler possesses $k_2$ and can decrypts the outer layer to obtain $\langle \text{Dec}_{k_2}(c_i) \rangle_i$. The shuffler then generates a random permutation $\pi$ from $[n]$ to $[n]$ and send $\langle \text{Dec}_{k_2}(\text{Enc}_{k_1}(v_i)) \rangle_{i \in [n]}$ to the server ($n$ is the number of users and $[n]$ is omitted when clear from context). The server decrypts the inner layer with $k_1$ to get the plaintext $\langle \text{Dec}_{k_2}(\text{Dec}_{k_1}(v_i)) \rangle_i = \langle v_{\pi(i)} \rangle_i$. During this process, the shuffler learns nothing about $b_i$ as it is encrypted. On the other hand, even of the server eavesdrops the network communication and learns $c_i$ was sent from user $i$, the server cannot link the reported data to the individual user as it is shuffled.
3 Existing Privacy Amplification Techniques

Due to the large amount of noise and poor utility of LDP (especially compared to centralized DP), people investigated relaxations of the LDP trust model to achieve better utility. The intuition is to introduce an auxiliary party to hide fine-grained information. Users only need to trust that the auxiliary server does not collude with the original server. This trust model is better (which means safer for users) than DP’s where the centralized server is trusted, but worse than that of LDP where no party needs to be trusted.

3.1 Privacy Amplification via Shuffling

The shuffling idea was originally proposed in Prochlo [12], where a shuffler is inserted between the users and the server to break the linkage between the report and the user identification. The authors of Prochlo showed the intuition of the privacy benefit of shuffling; but the formal proof was given later [23]. Intuitively, if the users send data with LDP using privacy budget $ε_l$; after shuffling, it is proved the users’ data is protected by $ε_c$-DP where $ε_c < ε_l$. This phenomenon was also investigated by two other groups [7, 15], Table 1 gives a summary of these results. Among them, [7] provides the strongest result in the sense that the $ε_c$ is the smallest, and the use case is the most general.

| Method | Condition | $ε_c$ |
|--------|-----------|-------|
| [23]   | $ε_l < 1/2$ | $\sqrt{144\ln(1/δ) \cdot \frac{ε_l}{n}}$ |
| [15]   | $\sqrt{\frac{32}{n} \ln(4/δ)} < ε_c < 1$, binary | $\sqrt{32\ln(4/δ) \cdot \frac{ε^2}{l + 1}}$ |
| [7]    | $\sqrt{\frac{144\ln(2/δ)}{n-1}} < ε_c < 1$ | $\sqrt{14\ln(2/δ) \cdot \frac{l + 1}{n-1}}$ |

Table 1: Guarantee comparison. Each row corresponds to a method. The amplified $ε_c$ only differs in constants. But the circumstances under which the method can be used are different. In the condition column, “binary” means only binary random response can be used.

In [7], the authors introduce a technique called blanket decomposition. The idea is to decompose the distribution of an LDP report $y$ into a linear composition of two distributions, one equivalent to the true value and the other independent (uniformly random). In particular, GRR given in Equation (1) is decomposed into

$$\forall v \in \mathcal{D}, \Pr[\text{GRR}(v) = y] = (1 - γ)\mathbb{I}_{v = y} + γ\Pr[\text{Uni}(d) = y]$$

where $γ = \frac{d}{\sqrt{d} + 1}$ and $\Pr[\text{Uni}(d) = y] = 1/d$. After shuffling, the histogram of $n - 1$ (except the victim’s) such random variables can be regarded as some unbiased inputs (determined by the sensitive input) plus freshly sampled noise added to each element of the histogram, where the noise comes from the blanket (i.e., the $γ\Pr[\text{Uni}(d) = y]$ part) and follows $\text{Bin}(n - 1, γ/d)$. Intuitively, the binomial distribution is close to Gaussian, which is a standard mechanism for satisfying $(ε, δ)$-DP. Theorem 1 gives this effect.

**Theorem 1** (Binomial Mechanism, derived from Theorem 3.1 of [7]). The binomial mechanism which adds independent noise $\text{Bin}(n, p)$ to each component of the histogram satisfies $(ε_c, δ)$-DP where

$$ε_c = \sqrt{\frac{14\ln(2/δ)}{np}}$$

Thus, given GRR, the authors of [7] can derive the guarantee of $ε_c$. Note that the derived $ε_c$ is not tight. It is indicated in [7] that with other tighter bounds, or even numerical calculation, smaller $ε_c$ can be derived. One limitation of [7] is that $ε_c$ cannot be arbitrarily small, as $p = γ/d = 1/(ε^l + d - 1)$ is determined by $d$. In the extreme case, even if $ε_l$ is 0, $ε_c = \sqrt{\frac{14\ln(2/δ)}{n-1}} > 0$.

3.2 DP from Homomorphic Encryption

When an additional party is introduced, we can leverage cryptographic tools to separate the information so that collectively they can achieve DP with higher utility but they cannot glean sensitive information from their own share of information. [16] proposes one instantiation of such an idea via homomorphic encryption, and we call it HE for short. In this approach, users first encode their data via one-hot encoding, i.e., encode $v \in \mathcal{D}$ using a bit-vector $B = (b^j)_{j \in [d]}$ where only the $ν$-th bit $b^ν$ is 1 and others are all 0’s. Then each bit is encrypted with homomorphic encryption; and the ciphertext is sent to the auxiliary party via a secure channel. Denote $c^j_i$ as the ciphertext of user $i$’s report on his/her $j$’s bit, the auxiliary server computes $C^j = \prod_{i \in [n]} c^j_i$ for each location $j$. Then for each location $j$, the auxiliary server generates random noise in a way that satisfies centralized DP, encrypts the noise, and multiply it with $C^j$ and sends the result to the server. The server decrypts the aggregated ciphertext. Since the auxiliary server only knows the amount of noise he added, the server must add another DP noise before publishing the information, in order to prevent the auxiliary server from learning the true results. Researches have abstracted the operations and provided general recipes of writing DP programs in this framework [16].

In this design, if the auxiliary server and the server collude, the auxiliary server can send the ciphertext of the individual data to the server; and the privacy guarantee is completely broken. To counter this threat, each user can add local noise to trade-off utility for the worst-case privacy guarantee. The key issues are that, due to one-hot encoding, (1) the communication/computation overhead is large, and (2) one cannot use OLH to obtain better utility.
4 The Multi-party Differential Privacy Model

In this section, we formalize the model of multi-party differential privacy (which we call MDP). The goal is to examine different aspects of the model, and identify the important factors one should consider when designing a protocol in this model. We start by analyzing different aspects of the model. We then cast existing privacy amplification work reviewed in Section 3 in our model. Finally, we present new observations that lay the basis for our proposed method described in Section 5.

4.1 System Model

The Parties. There are four types of parties: users, the server, the auxiliary server, and the analyst. Figure 1 shows the interactions among them. Users send information to the auxiliary server. The auxiliary server processes the users’ information and then forwards the assembled information to the server. From the aggregated information, the server finally derives results for queries issued by the analyst. The auxiliary server does not exist in the traditional models of DP and LDP. In MN (in Section 3.1), the auxiliary server is the shuffler; and in HE (in Section 3.2), the auxiliary server is a cryptographic server.

The Adversaries. From the point of view of a single user, other parties, including the auxiliary server, the server, the analyst, and other users, could all be malicious. In DP and LDP, there is no auxiliary server, and identifying the adversary model is straightforward. In LDP, the adversary is assumed to control all parties other than the user, including the server and other users. In DP, the adversary is the data analyst, who is assumed to control all other users, but the server is trusted. The goal of introducing multi-party models is to both avoid placing trust in any single entity, and achieve better utility than LDP. As a result, one has to consider multiple adversaries, each controlling a different combination of parties involved in the protocol. A user’s levels of privacy against different adversaries are different.

We assume all parties have the same level of (very powerful) background knowledge, i.e., all other users’ information except the victim’s. This assumption essentially enables us to argue DP-like guarantee for each party, which will be introduced later.

The prominent adversary is the server. Other parties can also be adversaries but are not the focus because they have less information. In particular, the analyst’s knowledge is strictly less than the server’s, because the analyst’s queries are answered by the server. The auxiliary server knows some additional knowledge (e.g., the linkage as in MN), but such information is meaningless unless the server colludes with it.

Additional Threat of Collusion. Existing work of MDP only considers the privacy guarantee against the server, but we note that in the multi-party setting, one needs to consider the consequence when different parties collude. In general, there are many combinations of colluding parties. And understanding the consequences of these scenarios enables us to better analyze and compare different approaches.

In particular, the server can collude with the auxiliary server. In this case, the system model is reduced to that for LDP. If the protocol does not consider the possibility of this collusion (e.g., HE), the guarantee will then be worse than the LDP guarantee. On the other hand, if the server colludes with the users (except the victim), the privacy guarantee could downgrade to LDP as well. The server can also collude with both the auxiliary server and other users.

Other combinations are possible but less severe. Specifically, we ignore the analyst as its information is strictly less than the analyst’s. And there is no benefit if the auxiliary server colludes with the users.

To summarize, we consider all potential collusions and identify three important (sets of) adversaries: (1) the server itself, (2) the server colluding with other users, and (3) the server with the auxiliary server.

4.2 Multi-party Differential Privacy

Given the four possible (sets of) adversaries, directly analyzing the privacy property of a method becomes challenging. Existing methods only prove the DP guarantee for the server only. To quantitatively compare different methods, we propose a unified DP notion, called MDP (Multi-party DP), that models different parties by specifying its views.

In particular, we assume the parties follow the protocol. Each party is associated with an algorithm; and the party observes the output of this algorithm. Although the party’s view is interactive, i.e., there is timing information about observations and they may depend on what the party sends out, we argue that the distribution of the observation is time and input independent. To argue for the privacy guarantee against each party, one proves DP bound of the algorithm whose output is the party’s observation.

Definition 3 (Multi-party Differential Privacy). We say that
a protocol satisfies \((\epsilon, \delta)\)-DP against a set \(P\) of parties, where \(\epsilon, \delta \geq 0\), when the following is true. Let \(A\) denote the (randomized) function that takes the input dataset(s) and outputs the views of all parties in \(P\). Then for any neighboring datasets \(D\) and \(D'\), and any set \(R\) of possible outputs of \(A\), we have

\[
\Pr[A(D) \in R] \leq \epsilon \Pr[A(D') \in R] + \delta
\]

Note that the cryptographic primitives are assumed to be safe (i.e., the adversaries are computationally bounded and cannot learn any information from the ciphertext).

This abstraction from (sets of) parties’ views to algorithms relies on the protocol being data-independent, i.e., the behavior of each party does not depend on its input. If the parties can deviate from the prescribed procedure, one should consider the deviation procedure and prove the guarantee under the deviated procedure. We discuss the possible deviations in Section 4.4.

Atomic Algorithms. To simplify the analysis of the potential adversaries in a protocol, we first introduce a set of atomic operations. Given that, the views of different potential adversaries can be modeled as combinations of these operations. Table 2 gives the list of atomic operations.

| Algorithm | Procedure                        |
|-----------|----------------------------------|
| \(A_s\)   | Shuffle input                    |
| \(A_l\)   | Perturb a single record (LDP)    |
| \(A_a\)   | Aggregate input to a histogram   |
| \(A_c\)   | Perturb the aggregated result (DP)|

Table 2: List of atomic operations.

Among the possible local randomizer \(A_l\), two of them are given in Section 2.2. For DP algorithm \(A_c\), one needs to aggregate the results first (i.e., run the aggregation function \(A_a\)), and then add independent noise to each count of the histogram (we consider the basic task of histogram estimation or counting query, which serves as the building block for other tasks).

Note that \(A_l\) is described for each value and \(A_l(v)\) is \((\epsilon, \delta)\)-LDP. The server actually receives \(A_l(D) = \langle A_l(v_1), \ldots, A_l(v_n) \rangle\). We can argue that \(A_l\) is DP to the server.

**Theorem 2.** If \(A_l(t)\) is \((\epsilon, \delta)\)-LDP, then \(A_l(D) = \langle A_l(v_1), \ldots, A_l(v_n) \rangle\) is \((\epsilon, \delta)\)-DP.

The intuition is that as each singleton \(\{v\}\) is protected by DP, the whole dataset \(D\) is also protected by DP. The proof is deferred to Appendix A.

4.3 Analyzing Existing Methods

Now we identify interactions and possible threats for MN and HE introduced in Section 3, and model them with different algorithms. Without loss of generality, we assume that the \(n\)-th user is the victim. And we denote \(D_{-n} = \langle v_1, \ldots, v_{n-1}, \bot \rangle\) as the vector of \(D\) without the \(n\)-th item. We use the concatenation operator \(\circ\) to denote the composition of algorithms, e.g., \(A_c \circ A_l(D) = A_l(A_c(D))\). The analysis of MN and HE are given as follows:

**Analyzing MN.** The server can be modeled by \(A_l(D) = A_s \circ A_l(D)\), as the shuffling operation is applied to the LDP reports. If the users collude with the server, the server’s view can be modeled as \(\langle A_s \circ A_l(D), A_l(D)_{-n} \rangle\). Note that the two instances of \(A_l(D)\) are the same. The adversary can subtract \(A_l(D)_{-n}\) from the aggregated \(A_s \circ A_l(D)\) to obtain \(A_l(t_n)\), which is equivalent to saying \(A_2(D) = A_l(D)\). Finally, if the auxiliary server colludes with the server, the model fall back to the DP setting and server’s view can be modeled as \(A_3(D) = A_l(D)\).

**Analyzing HE.** In this method, the server has access to \(A_l(D) = A_s \circ A_l(D)\), which satisfies DP. If the users collude with the server, it can be modeled as \(A_2(D) = \langle A_s \circ A_l(D), D_{-n} \rangle\). The server can then subtract \(A_s(D_{-n})\) from \(A_s \circ A_l(D)\) and obtain \(A_s \circ A_l(t_n)\). As \(A_s\) satisfies DP, \(A_s\) also satisfies DP. If the auxiliary server colludes with the server, the server’s view can be modeled as \(A_3(D) = D\), because the auxiliary server can send the ciphertexts to the server, which are encrypted version of the users’ values; and the server can decrypt them to obtain \(D\), thus there is no privacy guarantee for \(A_3\) (and we use \(\epsilon = \infty\) to present its privacy guarantee). Table 3 gives the summary of algorithms and guarantees of the four types of adversaries.

|        | \(A_1\) Server | \(A_2\) Server + Users | \(A_3\) Server + Auxi. |
|--------|-----------------|------------------------|------------------------|
| MN Guarantee | \(A_s \circ A_l(D)\) | \(A_l(v_n)\) | \(A_l(v_n)\) |
| HE Guarantee | \(A_s \circ A_l(D)\) | \(A_s \circ A_l(t_n)\) | \(D\) |

Table 3: List of adversaries and their privacy guarantees in MN and HE. The value of \(\delta\) are the same and thus ignored. We use \(\epsilon = \infty\) when there is no privacy guarantee.

4.4 Robustness to Perturbation

So far, we mostly assume all the parties to be honest but curious. It is also interesting to examine the consequences when parties deviate from the protocol. There could be multiple reasons for each party to be malicious to (1) interrupt the process, (2) infer sensitive information (break privacy), and (3) downgrade the utility. We analyze the three concerns one by one.

The first concern is to mitigate. If a user blocks the protocol, his report can be ignored. If the auxiliary server denies the service, the server can find another auxiliary server and redo the protocol. Note that in this case, users need to remember their report to avoid averaging attacks. Finally,
the server has no incentive to disrupt the process started by himself.

Second, it is possible that the auxiliary server deviates from the protocol by not shuffling things (in the MN method) or not adding noise (in the HE method), thus the server has access to $A_i(D)$ (in the MN method) or $A_a(D)$ (in the HE method). In these cases, the server can learn more information, but the auxiliary server does not have benefits except saving some computational power. And if the auxiliary server colludes with the server, they can learn more information without this deviation. Thus we assume the auxiliary server will not deviate to infer sensitive information, and do not bother with this concern. Note that however, trusted hardware or verifiable computation techniques can help mitigate this concern.

Third, we note that any party can downgrade the utility. The most straight-forward way is to generate many fake users, and let them join the protocol. This kind of sybil attack is hard to defend against without some offline authentication, which is orthogonal to the focus of this paper. As there are many users, we want to limit each user’s contribution such that the most the user can do to downgrade the utility is to change his original value or register fake accounts. MN satisfies this property. For HE, as one-hot encoding is used, the user has to prove by zero knowledge that his report is well-formed. On the other hand, the power of the auxiliary server is much greater as it collects all the data. For this paper, we assume the auxiliary server follows the protocol. Whether it is possible and how to verify this fact are interesting open questions.

To summarize, we assume the server and the auxiliary server follow the protocol, as there is no benefit deviating from the procedure. We are mainly concerned about the users adding too much noise disrupting the utility.

### 4.5 Discussion and Key Observations

In this section, we first systematically analyze the system model of the MDP model. From the privacy perspective, we then propose Definition 3 to quantitatively analyze different methods. Finally, we discuss the potential concern of malicious parties. Several observations and lessons are worth noting:

**Consider Different Threats.** Throughout our analysis, we have identified three potential adversaries, modeled as $A_1$ through $A_3$. We note that in the MDP model, one needs to consider all possible threats.

**When Auxiliary Server Colludes: No Amplification.**
When the aggregator colludes with the auxiliary server, the MDP falls back to the original DP model. In MN, there is still LDP guarantee as each user adds local noise. In HE, there is no privacy protection at all, as each user sends their true values.

**When Users Collude: Possibility Missed in MN.** When we prove for DP/LDP guarantees, we assume the adversary has access to all users’ true sensitive values except the victim, i.e., $D_{\neg n}$. We note that in MN, such an adversary may also have access to $A_i(D)_{\neg n}$. Such cases include the users (except the victim) collude with the server; or the server is controlling the users (except the victim). Of course there could also be cases that the adversary knows $D_{\neg n}$ but not $A_i(D)_{\neg n}$, e.g., the users report true values on another website, which colludes with the server. But we note that in this case, the victim’s true value may also be leaked.

Thus, it is possible but less meaningful to assume the adversary knows $D_{\neg n}$ but not $A_i(D)_{\neg n}$, as in MN, which makes the shuffle-based amplification less intuitive in real-world scenarios. On the other hand, in HE, there is still strong privacy guarantee when users collude.

### 5 MURS: Multi Uniform Random Shufflers

Based on the observations described in Section 4.5, we advocate adding noise from both the user side and the server side. In this section, we describe our proposed protocol MURS, which stands for Multi Uniform Random Shufflers. MURS adopts two existing techniques, i.e., (1) the MN model and (2) introduction of multiple auxiliary servers. Based on that, MURS has two novelties: (1) we theoretically improve the privacy amplification technique in MN; and (2) we propose to have both the auxiliary servers and the users add noise, and develop corresponding privacy guarantees.

The MURS protocol improves over existing techniques in both utility and privacy. Moreover, MURS is flexible and allows configuration of different levels of resistance to the three adversaries in MDP.

In what follows, we first present the reasons behind the design choices. Then we provide the details of our techniques.

#### 5.1 Design Choices

**Choosing the Model.** We choose the MN model instead of the HE model mainly because it is more computation- and communication-efficient, and it can scale well with the domain $\mathcal{D}$.

In HE, as one-hot encoding is used, the communication and computation overheads is large. In particular, for $n$ users with domain $\mathcal{D}$, the communication bandwidth is $n \cdot |\mathcal{D}|$ times the size of the HE ciphertext. In addition, each estimation requires $n$ expensive HE operations.

**Multiple Auxiliary Servers.** As there is no privacy advantage once the two servers collude, we borrow the idea of involving more auxiliary servers, which is a standard approach in other settings (e.g., voting, private messaging), to mitigate this threat. However, as long as the server cannot collude with all the auxiliary servers, there is still some privacy amplification effect, but this introduces more communication cost.
5.2 Extend MN with OLH

As pointed in Section 2.2, GRR has poor utility when the domain size \(|\mathcal{D}|\) is large. Moreover, if the central privacy budget \(\varepsilon\) is small, there is no privacy amplification. To overcome these two shortcomings, we propose to use OLH in MN.

Each user runs the perturbation function of OLH locally. That is, each user first selects a random hash function \(H\), and then hashes the value \(v\) into a smaller domain \(H(v) \in [d']\). Different from OLH, in this setting, we require \(d' = e^{\varepsilon_1/2} + 1\) rather than \(e^{\varepsilon_1} + 1\). For simplicity, we assume \(d'\) is an integer. The user then perturbs \(H(v)\) to \(y\), and reports \(H\) and \(y\) together.

Specifically, given \(r\) auxiliary servers, the user obtains their public keys \(pk_1, \ldots, pk_r\) and the server’s public key \(pk_s\). The user then encrypts \((H, y)\) with reverse layered public-key encryption \(c = \text{Enc}_{pk_s}((\ldots \text{Enc}_{pk_r}(\text{Enc}_{pk_i}((H, y))))))\) and sends \(c\) to the first auxiliary server. Each auxiliary server decrypts one layer of the reports using the corresponding secret key \(sk\), shuffles the result, and sends them to the next auxiliary server. Finally, the server decrypts the inner-most layer to obtain the results and evaluates them as described in Equation (3).

Now we analyze the privacy guarantee of using OLH by examine the three adversaries (1) to (3) as listed in Table 3. Given that each user applies an \(\varepsilon_1\)-DP OLH locally, the adversary (3) \(A_3\) is \(\varepsilon_1\)-DP. This also holds for adversary (2) \(A_2\), as the server can obtain the user report by colluding with the users. But for adversary (1), the privacy guarantee is different from MN. In particular, we have:

**Theorem 3.** When using \(\varepsilon_1\)-DP OLH in MN, \(A_1\) is \(\varepsilon_c\)-DP, where

\[
\varepsilon_c = 2 \sqrt{14 \ln(2/\delta)} \cdot \frac{e^{\varepsilon_1/2} + 1}{n-1} \tag{4}
\]

**Proof.** We utilize Theorem 1 in the proof. The key is to derive the privacy blanket which follows the Binomial distribution.

Given the OLH report \((H', y')\) for each user \(i\), we can re-construct a bit-vector \(B\) of size \(d\) where \(B[j] = 1\) if \(H'(j) = y'\). As the \(H'\) is a random hash function, the bits can be regarded as independent, i.e., \(\Pr[B[j] = 1] = \frac{1}{2} \approx \frac{1}{e^{\varepsilon_1/2} + 1}\), if \(j \neq v'\), and \(\Pr[B[j] = 1] = \frac{e^{\varepsilon_1}}{e^{\varepsilon_1/2} + 1}\) if \(j = v'\). Now for any two neighboring datasets \(D \approx D'\), w.l.o.g., we assume they differ in the \(n\)-th value, and \(v'' = 1\) in \(D\), \(v'' = 2\) in \(D'\). By the independence of the bits, other locations are equivalent and can be canceled out. Thus we only need to examine the summation of bits for location 1 and 2. For each location, there are \(n - 1\) users, each reporting the bit with probability

\[
\forall y \in \{0, 1\} \quad \Pr[B[j] = y] = (1 - \gamma) \cdot (1 - \gamma) + \gamma \Pr[\text{Uni}(2) = y]
\]

where \(\gamma = \frac{2}{e^{\varepsilon_1/2} + 1}\), \(\Pr[\text{Uni}(2) = y] = 1/2\), and \(\oplus\) denotes the xor operation. After shuffling, the histogram of \(n - 1\) (except the victim’s) such random variables follows Bin\((n - 1, \gamma/d)\).

Algorithm 1 Auxiliary Server \(k \in [r]\)

| Input: \(sk, n, n_u, \varepsilon_1, D\) |
|---|
| \(d \leftarrow |D|, d' \leftarrow e^{\varepsilon_1/2} + 1\) |
| \(p \leftarrow e^{\varepsilon_1} + 1, q \leftarrow 1\) |
| Init \(C^k \leftarrow \langle c_{k}^1 \rangle_{i \in [n+k-n_u]}\) |
| for \(i \leftarrow 1\) to \(n + k - n_u\) do |
| Receive \(c_i^{k-1}\) |
| \(c_i^k \leftarrow \text{Dec}_{sk_i}(c_i^{k-1})\) |
| for \(j \leftarrow 1\) to \(n_u\) do |
| \(u \leftarrow \xi \in \mathcal{D} \text{, } H \leftarrow \|\) |
| \(x \leftarrow [0, 1] \text{, } \|\) |
| if \(x < p - q\) then |
| \(y \leftarrow H(u) \text{ mod } d'\) |
| else |
| \(y \leftarrow \|\) |
| \(c_{n+k+n_u+j}^k \leftarrow \text{Enc}_{pk_{k+1}}((\ldots \text{Enc}_{pk_r}(\text{Enc}_{pk_i}((H, y)))))\) |
| Shuffle \(C^k\) |
| if \(k = r\) then |
| Send \(C^k\) to server |
| else |
| Send \(C^k\) to auxiliary server \(k + 1\) |

Algorithm 2 Server

| Input: \(sk, n, n_r, \varepsilon_1, D\) |
|---|
| \(d \leftarrow |D|, d' \leftarrow e^{\varepsilon_1/2} + 1\) |
| \(p \leftarrow e^{\varepsilon_1} + 1, q \leftarrow 1\) |
| Init \(Y \leftarrow \langle y_i \rangle_{i \in [n+n_u]}\) |
| Receive \(C'\) from auxiliary server \(r\) |
| for \(i \leftarrow 1\) to \(n + n_u\) do |
| \(\langle H_i, y_i \rangle \leftarrow \text{Dec}_{sk_i}(c_i^{r-1})\) |
| for \(v \in \mathcal{D}\) do |
| \(\tilde{f}_v \leftarrow \frac{1}{n} \sum_{i \in [n+n_u]} \frac{1 - (y_i = v)}{p-q} - \frac{q}{p}\) |

As there are two locations, by Theorem 1, we have \(\varepsilon_c/2 = \sqrt{14 \ln(2/\delta)} \cdot \frac{e^{\varepsilon_1/2} + 1}{n-1}\). \(\Box\)

5.3 Fake Response from Auxiliary Server

We propose to equip the auxiliary servers with an additional function of adding some random reports. These random reports serve the purpose of DP noise. If the server colludes with some auxiliary servers, he recovers part of the added values. Specifically, each auxiliary server adds \(n_u\) values drawn from the input domain uniformly at random. Given \(r\) auxiliary servers, there are \(n_s = n_u \cdot r\) uniform reports. The added values are then perturbed by the same algorithm used by the users, and shuffled with the received reports. The
server, after aggregation, subtracts \( n_r/d \) from the estimation, where \( d \) is the size of the domain. Algorithms 1 and 2 give the procedure for the auxiliary server and server in MURS, respectively. The algorithms by default use OHL assuming \( d \) is large. When \( d \) is smaller, MURS will switch to GRR to have better utility. We will give more guideline on when to switch later.

To argue about privacy guarantee in this method, we need to identify \( A_1 \) through \( A_3 \). First, the server’s view is the shuffled reports; and the reports are from users and some randomly drawn input. We have \( A_1(D) = A_s(A_1(D)||A_1 \circ A_u(n_r)) \), where \( || \) here concatenates two tuples. If the users collude, the server can subtract \( A_1(D)_{\sim u} \) from \( A_1 \) and obtain \( A_2(D) = A_s(A_1(v_n)||A_1 \circ A_u(n_r)) \). If all the auxiliary server colludes with the server, the permutation and the added reports are revealed, and one have \( A_3(D) = A_1(D) \). The following theorem gives the precise privacy guarantee for each of the adversaries:

**Theorem 5.** In MURS, given that \( A_3 \) is \( \varepsilon_l\text{-DP} \), \( A_2 \) is \( \varepsilon_y\text{-DP} \), and \( A_1 \) is \( \varepsilon_c\text{-DP} \), where

\[
\varepsilon_s = 2 \sqrt{14 \ln(2/\delta) \cdot \frac{e^{\varepsilon_l/2} + 1}{n_r}} \\
\varepsilon_c = 2 \sqrt{14 \ln(2/\delta) \cdot \frac{e^{\varepsilon_l/2} + 1}{n_1 + n_r}}
\]

Note that one can also use GRR in MURS, and we have a similar theorem:

**Theorem 4.** In MURS with GRR, given that \( A_3 \) is \( \varepsilon_l\text{-DP} \), \( A_2 \) is \( \varepsilon_y\text{-DP} \), and \( A_1 \) is \( \varepsilon_c\text{-DP} \), where

\[
\varepsilon_s = \sqrt{14 \ln(2/\delta) \cdot \frac{d}{n_r}} \\
\varepsilon_c = \sqrt{14 \ln(2/\delta) \cdot \frac{n_1 + n_r}{d}}
\]

The proofs are deferred to Appendix A. The intuition is that, \( A_1 \) and \( A_2 \), essentially add binomial noise to the histogram; and one can follow Theorem 3 to argue they satisfy DP.

### 5.4 Utility Analysis

We analyze the utility of MURS from the perspective of the server. In particular, we measure the expected squared error

\[
\frac{1}{d} \sum_{v \in \mathcal{D}} \mathbb{E}[(\tilde{f}_v - f_v)^2]
\]

where \( f_v \) is the true frequency of value \( v \in \mathcal{D} \), and \( \tilde{f}_v \) from Algorithm 2 is the server’s estimation of \( f_v \). We first show \( \tilde{f}_v \) is unbiased.

**Lemma 6.** The server’s estimation \( \tilde{f}_v \), from Algorithm 2 is an unbiased estimation of \( f_v \), i.e.,

\[
\mathbb{E}[(\tilde{f}_v)] = f_v
\]

The proof is deferred to Appendix A. Given that, the expected squared error equals variance, i.e.,

\[
\frac{1}{d} \sum_{v \in \mathcal{D}} \mathbb{E}[(\tilde{f}_v - f_v)^2] = \frac{1}{d} \sum_{v \in \mathcal{D}} \text{Var}[(\tilde{f}_v)]
\]

Now we derive the expected squared error of \( \tilde{f}_v \). For OHL alone, \( \tilde{f}_v \) was shown in Equation (3). Its variance has been shown in [37], but note that our setting is slight different, as we have a total population of \( n + n_r \) users, and we need to subtract \( n_r/d \) from the result.

**Theorem 7.** If \( A_1 \) is \( \varepsilon_c\text{-DP} \), and fix \( n \) and \( n_r \), the expected squared error is bounded by

\[
\frac{e^{\varepsilon_l/2}}{(e^{\varepsilon_l/2} - 1)^2} \frac{n + n_r}{n^2} + \frac{d - 1}{d} \frac{n_r}{n^2}
\]

where \( \varepsilon_l \) satisfies the Equation (6).

**Proof.** With \( d' = e^{\varepsilon_l/2} + 1 \), \( p = \frac{e^{\varepsilon_l/2}}{e^{\varepsilon_l/2} - 1} = \frac{e^{\varepsilon_l/2}}{e^{\varepsilon_l/2} + 1} \), and \( q = \frac{1}{e^{\varepsilon_l/2} + 1} \), we adapt the proof of [37]:

\[
\text{Var}([\tilde{f}_v]) = \text{Var} \left[ \frac{1}{n} \sum_{i \in [n + n_r]} \frac{1_{[y_i = H(v_i)]} - q}{p - q} - \frac{n_r}{d} \right]
\]

\[
= \frac{1}{n^2} \text{Var} \left[ \sum_{i \in [n + n_r]} \frac{1_{[y_i = H(v_i)]}}{p - q} \right]
\]

\[
= \sum_{i \in [n + n_r]} \text{Var} \left[ 1_{[y_i = H(v_i)]} \right] \frac{(p - q)^2}{n^2}
\]

Here for each of the \( n \) users, if his true value is \( v \), we have \( \text{Var}[1_{[y_i = H(v_i)]}] = p(1-p) \), and there are \( n_f \) of them; otherwise, we have \( \text{Var}[1_{[y_i = H(v_i)]}] = q(1-q) \) for the rest \( n(1 - f_v) \) users. For the uniform responses, as their values are randomly sampled, we have \( \text{Var}[1_{[y_i = H(v_i)]}] = p'(1-p') \),
where $p' = \frac{1}{d} p + \frac{d-1}{d} q = \frac{\epsilon^2/2 + d-1}{d(\epsilon^2/2 + 1)}$. Together, we have

$$
\text{Var} \left[ \hat{f}_v \right] = \frac{n_f \rho (1 - p) + n(1 - f_v) q (1 - q) + n_v p' (1 - p')}{n^2 (p - q)^2} = \frac{n q (1 - q) + n_f (p (1 - p) - q (1 - q)) + n_v p' (1 - p')}{n^2 (p - q)^2} \tag{7}
$$

Finally, as $\text{Var} \left[ \hat{f}_v \right]$ is independent of $v$,

$$
\mathbb{E} \left[ (\hat{f}_v - f_v)^2 \right] = \frac{1}{d} \sum_{v \in \mathcal{G}} \text{Var} \left[ \hat{f}_v \right] = \frac{\epsilon^2/2 n + n_r}{(\epsilon^2/2 - 1)^2} + \frac{d - 1 - n_r}{d^2 n^2}.
$$

If GRR is used in MURS, we have similar analyses, presented below. The proofs are deferred to Appendix A.

**Lemma 8.** The server’s estimation $\hat{f}_v$ with GRR is an unbiased estimation of $f_v$, i.e.,

$$
\mathbb{E} \left[ \hat{f}_v \right] = f_v
$$

**Theorem 9.** If $A_1$ is $\epsilon_1$-DP, and fix $n$ and $n_r$, using GRR, the expected squared error is bounded by

$$
\frac{\epsilon^2/2}{n(\epsilon^2/2 - 1)^2} + \frac{d - 2}{dn(\epsilon^2/2 - 1)} + \frac{n_r (d - 1)(\epsilon^2/2 + d - 1)^2}{d^2 n^2 (\epsilon^2/2 - 1)^2}
$$

where $\epsilon_1$ satisfies the Equation (6).

### 5.5 Discussion

The proposed MURS strengthens MN from three perspectives: First, it improves utility of MN by applying a configured OLH primitive. Second, it provides better privacy guarantee when users collude with the server, which is an assumption made in DP. Third, it makes the threat of the server colluding with the auxiliary server more difficult. Given the main techniques, there are issues and extensions we want to discuss.

**Choosing OLH or GRR.** In [37], there is clear guideline for choosing GRR or OLH, based on domain size $d$. Here, as given in Theorems 7 and 9, the choice depends on more parameters, and thus is more complicated. We can numerically compare the utility of GRR and OLH.

**Choosing Parameters.** Given the desired privacy level $\epsilon_1, \epsilon_2, \epsilon_3$ against the three adversaries $A_1, A_2, A_3$, respectively. Also given the domain size $d$, number of users $n$, and $\delta$, we want to find what is the best configuration in terms of server’s utility?

Local perturbation is necessary to satisfy $\epsilon_3$-DP against adversary (3), i.e., $\epsilon_3 \geq \epsilon_3$. To achieve $\epsilon_2$ when other users collude, noise from auxiliary servers are also necessary, i.e., $\epsilon_3 \geq \epsilon_2$. Given that, to satisfy $\epsilon_3 \geq \epsilon_1$, if we have to add more noise, we have the option to configure the two noise kinds of noise. Note that this holds even when satisfying $\epsilon_3 \geq \epsilon_2$. That is, the natural way is to add noisy reports from the auxiliary server, but we can also lower $\epsilon_2$ at the same time. As we have the privacy and utility expressions, we can numerically search the optimal configuration of $n_r$ and $\epsilon_2$. Note that we also needs to consider the choice of GRR or OLH as utility is depended on it.

**Extensions.** When $n_r = 0$ (no noise from auxiliary parties is needed), there is one vulnerability when the shufflers process data sequentially [27]. That is, the first shuffler can keep the victim’s data while replacing all other users’ report with dummy reports. The server can find the victim’s report and privacy amplification effect disappear. In this case, we can have users choose the path randomly to mitigate this threat. Note that this also load-balances the overhead to all the auxiliary servers across the hops. It can be proved that if the user selects the path randomly, there is no additional privacy leakage compared to the sequenced setting.

Another extension is to allow different configurations of collusion probabilities for the auxiliary servers. We assume all the auxiliary servers are equal and let them generate equal amount of noise (i.e., $n_u = n_r/r$, where $r$ is the number of auxiliary servers).

### 6 Evaluation

The purpose of the evaluation is two-fold. First, we want to measure the utility of MURS, i.e., how much it improves over MN; and how practical it is. Second, we want to measure the communication and computation overhead of MURS, to see whether the technique is applicable in practice.

As a highlight, our MURS can make estimations that has absolute errors of $< 0.01\%$ in reasonable settings, improving orders of magnitude over MN. The overhead is small and practical. We have implemented a prototype system of MURS and will open source the code.

#### 6.1 Experimental Setup

**Datasets.** We run experiments on two real datasets.
• IPUMS [35]: The US Census data for the year 1940. We sample 1% of users, and use the city attribute (N/A are discarded). This results in $n = 602325$ users and $d = 915$ cities.

• Kosarak [2]: A dataset of 1 million click streams on a Hungarian website that contains around one million users with 42178 possible values. For each stream, one item is randomly chosen.

**Competitors.** We compare the following methods:

• MN: The shuffler-based approach using mix-net from [7].

• MN-OLH: We use OLH instead of GRR in MN.

• MURS: Our proposed method. Compared to MN-OLH, (1) multiple shufflers are introduced; (2) GRR and OLH are adaptively chosen based on the utility; and (3) $n_r$, the total number of random values added by the shufflers, is optimized.

**Implementation.** The prototype was implemented using Python3.6 with fastecdsa 1.7.4, pycryptodome 2.6.1, python-xxhash 1.3.0 and numpy 1.15.3 libraries.

We used the elliptic curve secp256r1 for nested public key encryption and AES-256-CBC. By default, there were three auxiliary servers. For the layered encryption through them, we used the ElGamal encryption scheme initiated with Elliptic Curve (EC). In particular, given the message, a random AES key was generated to encrypt it. The asymmetric key was then utilized to encrypt the AES key. Both AES and EC satisfied 256-bit security.

**Metrics.** We use mean squared error (MSE) of the estimates as metrics. For each value $v$, we compute its estimated frequency $\hat{f}_v$ and the ground truth $f_v$, and calculate their squared difference. Specifically,

$$\text{MSE} = \frac{1}{d} \sum_{v \in D} (f_v - \hat{f}_v)^2$$

**Methodology.** The experiments were conducted on servers running Linux kernel version 5.0 with Intel Xeon Silver 4108 CPU @ 1.80GHz and 128GB memory. All servers used were connected via 1 Gbps of network bandwidth and 50ms delay.

For each dataset and each method, we repeat the experiment 100 times, with result mean and standard deviation reported. The standard deviation is typically very small, and barely noticeable in the figures.

### 6.2 Improvement of using OLH in MN

We first show that in terms of utility, how much better one can get by adopting OLH in MN. Besides MN and MN-OLH, we also evaluate a baseline method (Base) that always outputs a uniform distribution, and the Laplace mechanism (Lap) that represents the lower bound.

Figure 2 shows the utility comparison of the four methods. We vary the overall privacy guarantee $\varepsilon_c$ against the server only ($A_1$) from 0.1 to 1, and plot MSE. First of all, there is no privacy amplification for MN when $\varepsilon_c$ is below a threshold. The threshold is linear in $\sqrt{d}$. In particular, when $\varepsilon_c < \sqrt{\frac{14\ln(2/\delta)(d-1)}{n-1}}$, we only show results on the IPUMS dataset because for the Kosarak dataset, $d$ is too large so that MN cannot benefit from amplification. When there is no amplification, the utility of MN is poor, even worse than the random guess baseline method. Compared to MN, our improved MN-OLH method can always enjoy the privacy amplification advantage, and gets better utility result, especially when $\varepsilon_c$ is small.

There is a gap of around 2 orders of magnitude between MN-OLH and the lower bound Lap, achieved by the central DP Laplace mechanism (Section 2.1). Moreover, Lap has $\delta = 0$ while MN requires $\delta > 0$. Note that the current analysis of the amplification effect (i.e., Theorem 3) is not tight. And MN-OLH can benefit from tighter analysis. It is unknown what is the lower bound of MN-OLH and how close it is to Lap.

### 6.3 Utility Evaluation of MURS

We show the MSE of MURS in Figures 3, varying the number of added uniform samples $n_r$ in the evaluation. In this case, MN-OLH is a special case where $n_r = 0$. In Figures 3, each line corresponds to a value of $\varepsilon_c$. Overall, MSE of MURS gets smaller with larger $\varepsilon_c$. For each specific $\varepsilon_c$, when $n_r$ increases, utility always gets worse. Note that here we do not limit the range of $n_r$, which corresponds to the case where there is no constraints on $\varepsilon_c$ (for the second adversary composed of the server and other users) and $\varepsilon_l$ (the LDP parameter). Fixing $\varepsilon_c$, a larger $n_r$ provides better privacy guarantee against $A_2$ (smaller $\varepsilon_c$), while each individual user can add less noise locally (larger $\varepsilon_l$).

Fixing $\varepsilon_c$, the estimation variance is composed of two parts, one from the LDP perturbation (e.g., OLH), and the other from the $n_r$ uniformly drawn samples. By increasing $n_r$, the second variance increases; but the first variance is reduced. The empirical results from Figure 3 show that in those settings, one should always set $n_r = 0$. Note that in practice, one can always numerically compute the optimal $n_r$ from the theorems in Section 5.

### 6.4 Complexity Evaluation

We evaluate the computational and communication costs of MURS, focusing on the overhead introduced by the shuffling and layered encryption schemes.

**Computation.** First, the user-side computation is fast as it
only involves sampling and several encryption operations. We measure the client-side computational time for encryption with different number of layers ranging from 1 to 10. Second, for the auxiliary servers, we measure the processing time for decrypting $n$ user’s ciphertexts (for one layer), shuffling, and then add $n_k$ random reports (with necessary encryption). Note that both the decryption and adding new reports operations can be done in parallel and thus easily scalable. In the implementation, we use 32 threads for demonstration, but with more resource, the processing time can be faster. We vary the number of users and record the time used for decrypting the outer-most layer for the number of layers ranging from 1 to 10. We fix $n_k$ to be 1k. Finally, there is no computation overhead introduced at the server side.

The result is shown in Figure 4. The running time is stable. For the client, its computation is linear in the number of layers. For the auxiliary servers, on the other hand, as only the outer-most layer is decrypted, the time is more close to a constant.

Communication. As the computational overhead is small in MURS, we measure the communication overhead, which is more focused in the MDP setting. Since we take 256-bit security for both AES and EC. The EC ciphertext is 96 bytes for each layer. That is, in EC with ElGamal, there are two part for ciphertext $⟨P, C⟩$. $P$ is a point in the secp256r1 curve and thus can be represented by $256 \times 2$ bits; and $C$ is a number in $\{0, 1\}^{256}$. For OLH, we let each user randomly select a 8-byte seed as the random hash function. After padding, each message is $32 + 96k$ bytes, where $k$ is the number of layers used for auxiliary servers.

Given $n = 1$ million users and 3 auxiliary servers, there will be $10^6 \times 3 \times (32 + 96k) = 672$ MB data sent to the three auxiliary servers. Given the network configuration of 1 Gbps bandwidth, we expect to have the communication time of 5.376 seconds. In the actual evaluation, we typically see 100 to 112 MB throughput bandwidth, which gives an overall of 6 to 7 seconds communication time.

7 Related Work

We review related topics in LDP and the other related models.

Frequency Oracle. One basic mechanism in LDP is to estimate frequencies of values. There have been several mechanisms [4, 8, 9, 24, 37, 41] proposed for this task. Among them, [37] introduces OLH, which achieves low estimation errors and low communication costs. The application of OLH is crucial for the utility of other application such as heavy hitter identification [8] and frequent itemset mining [33, 39]. And one more contribution of this paper is to enable OLH to enjoy the privacy amplification effect.

Relaxed Definitions. Rather than introducing the shuffler, another direction to boost the utility of LDP is to relax its semantic meaning. In particular, Wang et al. propose to relax the definition by taking into account the distance between the true value and the perturbed value [36]. More formally, given the true value, with high probability, it will be perturbed to a nearby value (with some pre-defined distance function); and with low probability, it will be changed to a value that is far apart. A similar definition is proposed in [25]. Both usages are similar to the geo-indistinguishability notion in the centralized setting [6]. A similar definition is also proposed in [31], which considers some answers to be sensitive while some not (there is also a DP counterpart called One-sided DP [20]). Our work applied to the standard LDP definition, and we conjecture that these definitions can also benefit from introducing a shuffler without much effort.

There also exist relaxed models that seem incompatible with the shuffler model, i.e., [10] considers the inferring probability as the adversary’s power; and [38] utilizes the linkage between each user’s sensitive and public attributes.

Distributed DP. In the distributed setting of DP, each data owner (or proxy) has access to a disjoint subset of users. For example, each patient’s information is possessed by a hospital. The DP noise is added at the level of the intermediate data owners (e.g., [29]). A special case (two-party computation) is also considered [26, 34]. [28] studies the limitation of two-
8 Conclusions

In this paper, we study the Multi-party DP model. We first analyze the different threats and propose a unified view of the privacy definition that models the adversary as an algorithm. We then make observations about existing work. We propose MURS, which can resist more severe threat models, while achieving better privacy-utility tradeoff than existing ones. Finally, we perform experiments to compare different methods and demonstrate the advantage of our proposed method.
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A Proofs

Proof of Theorem 2.

Proof. For any neighboring datasets $D$ and $D'$, w.l.o.g., assume they differ in the $n$-th value. For and any set of possible output $R$ of $A_l$, we have:

$$Pr\left[A_l(D) \in R\right] = \sum_{R \in R} Pr\left[A_l(D) = R\right] = \sum_{R \in R} \prod_{i \in [n]} Pr\left[A_l(t_i) = R_i\right]$$

$$\leq \sum_{R \in R} \prod_{i \in [n]} Pr\left[A_l(t'_i) = R_i\right] (e^\varepsilon) \cdot \sum_{R \in R} Pr\left[A_l(t'_i) = R_i\right] \cdot \delta$$

$$= e^\varepsilon \sum_{R \in R} \prod_{i \in [n]} Pr\left[A_l(t'_i) = R_i\right] + \sum_{R \in R} \prod_{i \in [n-1]} Pr\left[A_l(t'_i) = R_i\right] \cdot \delta$$

$$\leq e^\varepsilon \cdot Pr\left[A_l(D') \in R\right] + \sum_{R \in R} \prod_{i \in [n-1]} Pr\left[A_l(t'_i) = R_i\right] \cdot \delta$$

$$\leq e^\varepsilon \cdot Pr\left[A_l(D') \in R\right] + \delta$$

\[\square\]
Proof of Theorem 4.

Proof. As in the analysis, \( A_1(D) = A_1(A_i(D)||A_i \circ A_B(n_2)) \) and \( A_2(D) = A_2(A_i(v_r)||A_i \circ A_B(n_2)) \). That is, \( A_2 \) outputs the victim’s report shuffled with \( n_r \) random ones added by the auxiliary server. For \( A_1 \), the \( n-1 \) reports from other users are also shuffled together.

We assume that for the \( n_r \) reports, the server knows the originally sampled values. That is, we prove for the case where the server has strictly more knowledge. This can be done by modifying MURS so that the auxiliary servers notify the server about the original distribution (but not the OLH reports) of the \( n_r \) reports. Now this is reduced to the case of MN with OLH, and we can apply Theorem 3 to obtain

\[
\varepsilon_v = 2\sqrt{14\ln(2/\delta) / \left( \frac{n_r}{2\delta} \right)^{1/2}}.
\]

The argument for \( A_1 \) is similar. One can regard the \( n_r \) reports as from common users (thus there are \( n_r + n - 1 \) users except the victim), and use Theorem 3 to obtain

\[
\varepsilon_c = 2\sqrt{14\ln(2/\delta) / \left( \frac{n_r+n_2}{2\delta} \right)^{1/2}}.
\]

\( \Box \)

Note that in the proof, we assume the adversary has strictly more information, so the derived guarantee may not be tight.

Proof of Theorem 5. For Theorem 5, one can also follow the proof of Theorem 4 to obtain similar guarantees. Here we utilize the fact of GRR to obtain better guarantees:

Proof. Similar to the proof of Theorem 4, we have \( A_2 \) that outputs the the victim’s report shuffled with \( n_r \) random ones added by the auxiliary server, and \( A_1 \) that also mixes the \( n-1 \) reports from other users.

Specifically, in the output of \( A_2 \), there are \( n_r \) random reports that follow uniform distribution over \( [d] = \{1, \ldots, d\} \) added by the auxiliary server. This can be viewed as the Binomial mechanism with \( \text{Bin} \left( n_r, \frac{1}{d} \right) \). By Theorem 1, we have \( A_2 \) is \( \sqrt{14\ln(2/\delta) \cdot \frac{n_r}{2\delta}} \)-DP. For \( A_1 \), there are \( n-1 \) random reports from users, and \( n_r \) reports from the auxiliary server. This can be viewed as the Binomial mechanism with \( \text{Bin} \left( n - 1 + n_r, \frac{n-1+1}{n-1+n_r} \right) = \text{Bin} \left( n - 1 + n_r, \frac{1}{2} \right) \). And by Theorem 1, \( A_1 \) is \( \varepsilon_c \)-DP.

\( \Box \)

Proof of Lemma 6.

Proof.

\[
\mathbb{E} [\hat{f}_v] = \mathbb{E} \left[ \frac{1}{n} \left( \sum_{i \in [n+n_r]} \mathbb{I}_{\{H_i(v) = v_r\}} - q \right) \right]
\]

\[
= \frac{1}{n} \left( \mathbb{E} \left[ \sum_{i \in [n+n_r]} \mathbb{I}_{\{H_i(v) = v_r\}} \right] - (n+n_r)q \right) - \frac{n_r}{d}
\]

Thus, the \( n+n_r \) reports can be decomposed into two parts, \( n \) of them are from the users (because of the shuffler, we cannot identify them, but this does not affect our analysis), and \( n_r \) are from the randomly sampled values. For the \( n \) reports from users, \( n_f \) of them have original value \( v \); and for the \( n_r \) reports, in expectation, \( n_r/d \) of them have original value \( v \). After perturbation, these \( n_f + n_r/d \) reports will have \( H(v) = y \) with probability \( p \). The rest \( n + n_r - (n_f + n_r/d) \) reports will have \( H(v) = y \) with probability \( q \). By the linearity of expectations, we have

\[
\mathbb{E} \left[ \sum_{i \in [n+n_r]} \mathbb{I}_{\{H_i(v) = v_r\}} \right] = (n_f + n_r/d)(p - q) + (n+n_r)q
\]

\[
\Rightarrow \mathbb{E} [\hat{f}_v] = \frac{1}{n} (n_f + n_r/d - \frac{n_r}{d}) = f_v
\]

\( \square \)

Proof of Lemma 8.

Proof. We can reuse the proof of Lemma 6 with different parameters of \( p = \frac{\varepsilon^2_t}{1} \) and \( q = \frac{\varepsilon^2_t}{1} \).