Is the Casimir effect relevant to sonoluminescence?

V. V. Nesterenko* and I. G. Pirozhenko†

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research
141980 Dubna Russia

(May 22, 2018)

The Casimir energy of a solid ball (or cavity in an infinite medium) is calculated by a direct frequency summation using the contour integration. The dispersion is taken into account, and the divergences are removed by making use of the zeta function technique. The Casimir energy of a dielectric ball (or cavity) turns out to be positive, it being increased when the radius of the ball decreases. The latter eliminates completely the possibility of explaining, via the Casimir effect, the sonoluminescence for bubbles in a liquid. Besides, the Casimir energy of the air bubbles in water proves to be immensely smaller than the amount of the energy emitted in a sonoluminescent flash. The dispersive effect is shown to be inessential for the final result.

12.20.-m, 12.20.Ds, 78.60.Mq

1. Sonoluminescence being observed during more than half century [1] has not received satisfactory explanation yet. As known this phenomenon represents the emission of visual light by spherical bubbles of air or other gas injected in water and subjected to an intense acoustic wave in such a way that the radius of bubbles changes periodically. In the last years of his life Schwinger proposed [2] that the bases of sonoluminescence is formed by the Casimir effect. While changing the size of bubbles the zero point energy of the vacuum electromagnetic field (the Casimir energy) of a cavity in a dielectric medium changes too. According to Schwinger, it is these changes of the electromagnetic energy that are emitted as a visual light in sonoluminescent flashes. In Schwinger’s calculations the Casimir energy for the configuration in hand proves to be of the same order as the energy of the photons in an individual flash (∼ 10 MeV). Other authors obtained results both consistent with Schwinger’s calculation [3] and differing from it [4, 5]. This disagreement is basically due to different methods used for removing the divergences in the problem under consideration.

In the present note the calculation of the Casimir energy of a dielectric ball placed in an endless dielectric medium (or cavity in this medium) is carried out under following conditions. In the first place a realistic description of dielectric properties of media is used which takes into account dispersion [6]. On the other hand the most simple and reliable method for removing the divergences, the zeta function technique, is applied. Till now these conditions were not combined in studies of the problem in question.

2. When calculating the Casimir energy we shall use the mode-by-mode summation of the eigenfrequencies of the vacuum electromagnetic oscillations by applying the contour integration in a complex frequency plane [7]. Consider a ball material of which is characterized by permittivity ε and permeability µ. The ball is assumed to be placed in an infinite medium with permittivity ε and permeability µ. For this configuration the frequencies of transverse-electric (TE) and transverse-magnetic (TM) modes are determined by the equations

\[
\Delta_{TE}^{s}(a\omega) \equiv \sqrt{\frac{1}{\mu_2}} \tilde{s}_1(k_1a)\tilde{e}_1(k_2a) - \sqrt{\frac{\varepsilon_2}{\mu_1}} \tilde{s}_1(k_1a)\tilde{e}_1'(k_2a) = 0, \tag{1}
\]

\[
\Delta_{TM}^{s}(a\omega) \equiv \sqrt{\frac{1}{\mu_2}} \tilde{s}_1(k_1a)\tilde{e}_1(k_2a) - \sqrt{\frac{\varepsilon_1}{\mu_1}} \tilde{s}_1(k_1a)\tilde{e}_1'(k_2a) = 0, \tag{2}
\]

where \( \tilde{s}_1(x) = \sqrt{\pi/2} J_{1+1/2}(x) \) and \( \tilde{e}_1(x) = \sqrt{\pi/2} H_{1+1/2}^{(1)}(x) \) are the Riccati-Bessel functions, \( k_i = \sqrt{\varepsilon_i \mu_i} \omega \), \( i = 1, 2 \) are the wave numbers inside and outside the ball, respectively; prime stands for the differentiation with respect to the argument \( k_1a \) or \( k_2a \) of the Riccati-Bessel functions.

As usual we define the Casimir energy by the formula

\[
E = \frac{1}{2} \sum s (\omega_s - \bar{\omega}_s), \tag{3}
\]

where \( \omega_s \) are the roots of Eqs. [1] and [2] and \( \bar{\omega}_s \) are the same roots under condition \( a \to \infty \). Here \( s \) is a collective index that stands for a complete set of indices specifying the roots of Eqs. [1] and [2]: \( s = \{l, m, n\} \quad l = 1, 2, \ldots ; m =

*Electronic address: nestr@thsun1.jinr.dubna.su
†Electronic address: pirozhen@thsun1.jinr.dubna.su
exactly. It gives their permittivities when

\[ \epsilon \]

where [9] has been considered in [9]. For definiteness we put

\[ l \ll 1, \quad \delta \ll 1, \quad \nu \ll 1, \quad \delta \ll 1 \]

except for its dependence on \( \delta \) is a static value of \( \delta \) and the parameter \( y_0 \) is determined by a "plasma" frequency \( \omega_0 \): \( y_0 = a \omega_0 \). The function describing dispersion in Eq. (6) is a standard one [one-absorption-frequency Sellmeir dispersion relation] except for its dependence on \( l \). We have introduced this dependence in order to be able to use the zeta function technique below. This complication does not contradict the main goal pursued by using this function, namely, it should simulate crudely the behaviour of \( \delta(y) \) at large \( y \). As known [10], the general theoretical principles lead to the following properties of the function \( \epsilon(\omega) \) in the upper half-plane \( \omega \). On the imaginary axis \( \omega = iy \), \( y > 0 \) the function \( \epsilon(iy) \) acquires real values, and with increasing \( y \) it steadily decreases from the static value 1 + \( \delta_0 \) to 1. Obviously formula (6) meets these requirements.

Substituting (6) into (3) and making use of the uniform asymptotic expansion for the modified Bessel functions (13) when \( l \to \infty \) one obtains

\[ E_l \sim \frac{3}{64a} \left( \frac{\delta_0}{4} \right)^2 f_1(a \omega_0) + \]

\[ \frac{9}{214\nu^2} \left( \frac{\delta_0}{4} \right)^2 \left[ 6f_2(a \omega_0) - 7f_3(a \omega_0) \left( \frac{\delta_0}{4} \right)^2 \right] + \mathcal{O}(\nu^{-4}), \]

where

\[ f_1(z) = \frac{z}{(1 + z)^4} \left( z^3 + 4z^2 + \frac{16}{3}z + \frac{4}{3} \right), \]

\[ f_2(z) = \frac{z^3}{(1 + z)^2} \left( \frac{521}{9} + \frac{1127}{27}z + \frac{593}{27}z^2 + 7z^3 + z^4 \right), \]
\[ f_3(z) = \frac{z}{(1+z)^9} \left( \frac{80}{63} + \frac{80}{7}z + \frac{928}{21}z^2 + \frac{1952}{21}z^3 + \frac{5960}{63}z^4 \
+ 80z^5 + \frac{320}{9}z^6 + 9z^7 + z^8 \right). \]  

(11)

We carry out the summation of the partial Casimir energies \[ \text{(11)} \] with the help of the zeta function technique \[ \text{(12)} \] taking into account asymptotics \[ \text{(13)} \]

\[ E = \sum_{l=1}^{\infty} E_l = \sum_{l=1}^{\infty} \left[ E_l + \frac{3}{64a} \left( \frac{\delta_0}{4} \right)^2 f_1(a\omega_0) - \frac{3}{64a} \left( \frac{\delta_0}{4} \right)^2 f_1(a\omega_0) \right] \]
\[ = \sum_{l=1}^{\infty} \bar{E}_l - \frac{3}{64a} \left( \frac{\delta_0}{4} \right)^2 f_1(a\omega_0) \sum_{l=1}^{\infty} (l+1/2)^0 \]
\[ = \sum_{l=1}^{\infty} \bar{E}_l - \frac{3}{64a} \left( \frac{\delta_0}{4} \right)^2 f_1(a\omega_0) \zeta(0,1/2) - 1]. \]

Here \[ \bar{E}_l = E_l + (3/64a) (\delta_0/4)^2 f_1(a\omega_0) \] is the renormalized partial Casimir energy, \( \zeta(s,q) \) is the Hurwitz zeta function. As \( \zeta(0,1/2) = 0 \), we get for the Casimir energy \[ \text{(13)} \]

\[ E = \sum_{l=1}^{\infty} \bar{E}_l + \frac{3}{64a} \left( \frac{\delta_0}{4} \right)^2 f_1(a\omega_0). \]

(13)

With allowance for \[ \text{(13)} \] one can obtain the estimation for the sum \( \sum_{i=1}^{\infty} \bar{E}_l \)

\[ \sum_{i=1}^{\infty} \bar{E}_l \approx 9 \left( \frac{\delta_0}{4} \right)^2 \left[ 6f_2(a\omega_0) - 7f_3(a\omega_0) \left( \frac{\delta_0}{4} \right)^2 \right] \sum_{l=1}^{\infty} \frac{1}{(l+1/2)^2} \]
\[ = \frac{9}{2^{14}} \left( \frac{\delta_0}{4} \right)^2 \left[ 6f_2(a\omega_0) - 7f_3(a\omega_0) \left( \frac{\delta_0}{4} \right)^2 \right] \left( \frac{\pi^2}{2} - 4 \right) \]
\[ = 5.135 \times 10^{-4} \left( \frac{\delta_0}{4} \right)^2 \left[ 6f_2(a\omega_0) - 7f_3(a\omega_0) \left( \frac{\delta_0}{4} \right)^2 \right]. \]

(14)

Thus the Casimir energy of a dielectric ball is

\[ E \approx \frac{3}{64a} \left( \frac{\delta_0}{4} \right)^2 \left\{ f_1(a\omega_0) + 0.066 f_2(a\omega_0) - 0.0048 \delta_0^2 f_3(a\omega_0) \right\}, \]

(15)

dispersion resulting only in the positive functions \( f_i(a\omega_0), \) \( i = 1, 2, 3. \) When \( z \) increases the functions \( f_i(z) \) approach 1 (see Fig. 1), and \[ \text{(15)} \] turns into the expression for the partial Casimir energy of a solid ball without dispersion \[ \text{(16)} \].

Considering the behaviour of the functions \( f_i(z) \) (see Fig. 1) one concludes that the main contribution, with a few percents accuracy, gives the first term in braces in Eq. \[ \text{(15)} \] with the result

\[ E \approx \frac{3}{64a} \left( \frac{\delta_0}{4} \right)^2 f_1(a\omega_0). \]

(16)

Obviously the change of the energy sign or a considerable increasing its magnitude due to the dispersion effect \[ \text{(13)} \] is out of the question.

Let us estimate the value of \( f_1(a\omega_0) \). The parameter \( \omega_0 \) can be determined by demanding that at this frequency the photons do not ‘feel’ the interface between two media. This condition will be certainly met when the wave length of photon is less than the interatomic distance in media \( d \sim 10^{-8} \) cm. Actually it is the condition of applicability of the macroscopic description of dielectric media \[ \text{(18)} \]. Sonoluminescence is observed with the air bubbles in water \[ \text{(19)} \], the radius of bubbles being \( a \sim 10^{-4} \) cm. Hence it follows that \( a\omega_0 \sim a/d = 10^4 \) and \( f_1(10^4) = 0.999 \ldots \). Thus the allowance for the dispersion in calculating the Casimir energy of a dielectric ball (or spherical cavity in a slab of a dielectric) practically has no effect on the final result.

3
Certainly the real picture of dispersion in the whole frequency range \(0 < \omega < \infty\) for any dielectric, including water, is exceedingly complicated and cannot be described by a simple equation \((7)\) with a single parameter \(\omega_0\). As known absorption of the electromagnetic waves in water and, as a consequence, their dispersion take place already in the radio frequency band. Putting in this case \(\lambda \sim 10^4\) cm, we obtain \(a \omega_0 \sim 1\) and \(f_1(1) = 0.729\ldots\). From here one can infer that the effective value of \(a \omega_0\) should be less than \(10^4\). In order for a more precise evaluation of this parameter to be done a more detailed consideration of the dispersion mechanism is needed. Obviously this may lead only to diminution of the absolute value of the Casimir energy. However this issue is beyond the scope of the present paper for the main conclusion (see below) does not depend on this point.

It is worth noting two peculiarities of the final formula \((16)\). When the radius of the bubble decreases its Casimir energy increases. This behaviour is completely opposite to one needed for explanation of sonoluminescence (as known, emission of light takes place at the end of collapsing the bubbles in liquid). Besides, this energy is immensely smaller than the amount of energy emitted in a separate sonoluminescent flash \((\sim 10\) MeV\). Actually taking \(a = 10^{-4}\) cm and \(\delta_0 = 3/4\) (water) we arrive at \(E \simeq 5 \cdot 10^{-3}\) eV.

Thus the results of this paper unambiguously testify that the Casimir effect is irrelevant to sonoluminescence.

This work was accomplished with financial support of Russian Foundation of Fundamental Research (Grant 97-01-00745).

[1] H. Frenzel and H. Schultes, Z. Phys. Chem., Abt. B 27, 421 (1934); B.P. Barber, R.A. Hiller, R. Löfstedt, S.J. Putterman, and K. Weniger, Phys. Rep. 281, 65 (1997); L.A. Crum, Physics Today 47, No. 9, 22 (1994).
[2] J. Schwinger, Proc. Nat. Acad. Sci. U.S.A. 90, 958, 2105, 4505, 7285 (1993); 91, 6473 (1994).
[3] C.E. Carlson, C. Molina-París, J. Pérez-Mercader, Matt Visser, Phys. Lett. B 395, 76 (1997); Phys. Rev. D 56, 1262 (1997); C. Molina-París and Matt Visser, Casimir effect in dielectrics: Surface area contribution, hep-th/9707073.
[4] K. A. Milton and Y. J. Ng, Phys. Rev. E 55, 4207 (1997); K. A. Milton and Y. J. Ng, Observability of the bulk Casimir effect: Can the dynamical Casimir effect be relevant to sonoluminescence?, Preprint OKHEP-97-04, hep-th/9707122.
[5] I. Brevik, V.V. Nesterenko, and I.G. Pirozhenko, Direct mode summation for the Casimir energy of a solid ball, JINR Preprint E2-97-307, Dubna (1997), hep-th/9710101.
[6] There is a point of view [14] that the dispersion effects may essentially affect the final result when calculating the Casimir energy.
[7] V.V. Nesterenko and I.G. Pirozhenko, Phys. Rev. D 57, 1284 (1998).
[8] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).
[9] Yu.S. Barash and V.L. Ginzburg, ZETP Lett. (in Russian) 15, 567 (1972), Usp. Fiz. Nauk (in Russian) 116, No. 1, 5 (1975).
[10] L.D. Landau and E.M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1984).
[11] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, D. C., 1964; reprinted by Dover, New York, 1972).
[12] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerlini, *Zeta regularization techniques with applications* (World Scientific, Singapore, 1994).
[13] I. Brevik and G. Enevoll, Phys. Rev. D 37, 2977 (1988); I. Brevik and R. Sollie, J. Math. Phys. 31, 1445 (1990); I. Brevik, H. Skurdal, and R. Sollie, J. Math. A: Math. Gen. 27, 6853 (1994), I. Brevik and V.N. Marachevsky, ‘Casimir surface force on a dilute dielectric ball’, Preprint Norwegian University of Science and Technology, Trondheim, 1998.
[14] P. Candelas, Ann. Phys. (N. Y.) 143, 241 (1982).
Fig.1. Functions $f_i(z)$, $i=1,2,3$ defining the expansion (8). When $z$ increases the functions approach their limiting value equal to 1.