Spectral Functionals, Nonholonomic Dirac Operators, and Noncommutative Ricci Flows

Sergiu I. Vacaru

"Al. I. Cuza" Iași University, Science Department, 54 Lascar Catargi street, 700107, Iași, Romania

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Abstract

We formulate a noncommutative generalization of the Ricci flow theory in the framework of spectral action approach to noncommutative geometry. Grisha Perelman’s functionals are generated as commutative versions of certain spectral functionals defined by nonholonomic Dirac operators and corresponding spectral triples. We derive the formulas for spectral averaged energy and entropy functionals and state the conditions when such values describe (non)holonomic Riemannian configurations.

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1 Introduction

The Ricci flow equations [1] and Perelman functionals [2] can be re-defined with respect to moving frames subjected to nonholonomic constraints [3].

Considering models of evolution of geometric objects in a form adapted to certain classes of nonholonomic constraints, we proved that metrics and connections defining (pseudo) Riemannian spaces may flow into similar nonholonomically deformed values modelling generalized Finsler and Lagrange configurations [4], with symmetric and nonsymmetric metrics, or possessing noncommutative symmetries [5].

The original Hamilton–Perelman constructions were for unconstrained flows of metrics evolving only on (pseudo) Riemannian manifolds. There were proved a set of fundamental results in mathematics and physics (for instance, Thurston and Poincaré conjectures, related to spacetime topological properties, Ricci flow running of physical constants and fields etc), see Refs. [6 7 8 9], for reviews of mathematical results, and [10 11 3], for some applications in physics. Nevertheless, a number of important problems in geometry and physics are considered in the framework of classical

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1there are used also some other equivalent terms like anholonomic, or non–integrable, restrictions/ constraints; we emphasize that in classical and quantum physics the field and evolution equations play a fundamental role but together with certain types of constraints and broken symmetries; a rigorous mathematical approach to modern physical theories can be elaborated only following geometric methods from ‘nonholonomic field theory and mechanics’
and quantum field theories with constraints (for instance, the Lagrange and Hamilton mechanics, Dirac quantization of constrained systems, gauge theories with broken symmetries etc). With respect to the Ricci flow theory, to impose constraints on evolution equations is to extend the research programs on manifolds enabled with nonholonomic distributions, i.e. to study flows of fundamental geometric structures on nonholonomic manifolds.\footnote{on applications of the geometry of nonholonomic manifolds (which are manifolds enabled with nonholonomic distributions/structures) to standard theories of physics, see Refs. \cite{22,3}; historical reviews of results and applications to Finsler, Lagrange, Hamilton geometry and generalizations are considered in Refs. \cite{13,14,15}}

Imposing certain noncommutative conditions on physical variables and coordinates in an evolution theory, we transfer the constructions and methods into the field of noncommutative geometric analysis on nonholonomic manifolds. This also leads naturally to various problems related to noncommutative generalizations of the Ricci flow theory and possible applications in modern physics.

In this work, we follow the approach to noncommutative geometry when the spectral action paradigm \cite{16}, with spectral triples and Dirac operators, gives us a very elegant formulation of the standard model in physics. We cite here some series of works on noncommutative Connes–Lott approach to the standard model and further developments and alternative approaches \cite{17,18,19,20,21,22,23,24,25,26,27,28,29}, see also a recent review of results in Ref. \cite{30} and monographs \cite{31,32,33}.

Following the spectral action paradigm, all details of the standard models of particle interactions and gravity can be ”extracted” from a noncommutative geometry generated by a spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) by postulating the action

\[ \text{Tr} \ f(D^2/\Lambda^2)+<\Psi|\mathcal{D}|\Psi>, \tag{1} \]

where ”spectral” is in the sense that the action depends only on the spectrum of the Dirac operator \(\mathcal{D}\) on a certain noncommutative space defined by a noncommutative associative algebra \(\mathcal{A} = C^\infty(V) \otimes P\mathcal{A}\). In formula (1), \(\text{Tr}\) is the trace in operator algebra and \(\Psi\) is a spinor, all defined for a Hilbert space \(\mathcal{H}\), \(\Lambda\) is a cutoff scale and \(f\) is a positive function. For a number of physical applications, \(P\mathcal{A}\) is a finite dimensional algebra and \(C^\infty(V)\) is the algebra of complex valued and smooth functions over a ”space” \(V\), a topological manifold, which for different purposes can be enabled with various necessary geometric structures. The spectral geometry of \(\mathcal{A}\) is given

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\[ \text{Tr} \ f(D^2/\Lambda^2)+<\Psi|\mathcal{D}|\Psi>, \tag{1} \]
by the product rule $\mathcal{H} = L^2(V, S) \otimes P\mathcal{H}$, where $L^2(V, S)$ is the Hilbert space of $L^2$ spinors and $P\mathcal{H}$ is the Hilbert space of quarks and leptons fixing the choice of the Dirac operator $PD$ and the action $PA$ for fundamental particles. Usually, the Dirac operator from (1) is parametrized $D = VD \otimes 1 + \gamma_5 \otimes PD$, where $VD$ is the Dirac operator of the Levi–Civita spin connection on $V$.

In order to construct exact solutions with noncommutative symmetries and noncommutative gauge models of gravity [5, 34] and include dilaton fields [35], one has to use instead of $VD$ certain generalized types of Dirac operators defined by nonholonomic and/or conformal deformations of the ‘primary’ Levi–Civita spin connection. In a more general context, the problem of constructing well defined geometrically and physically motivated nonholonomic Dirac operators is related to the problem of definition of spinors and Dirac operators on Finsler–Lagrange spaces and generalizations [36, 37, 38]; for a review of results see [39, 40] and Part III in the collection of works [12], containing a series of papers and references on noncommutative generalizations of Riemann–Finsler and Lagrange–Hamilton geometries, nonholonomic Clifford structures and Dirac operators and applications to standard models of physics and string theory.

The aims and results of this article are outlined as follow: Section 2 is devoted to an introduction to the geometry of nonholonomic (commutative) Riemannian manifolds and definition of spinors on such manifolds. Nonholonomic Dirac operators and related spectral triples are considered in Section 3. We show how to compute distances in such nonholonomic spinor spaces.

The main purpose of this paper (see Section 4) is to prove that the Perel-man’s functionals [2] and their generalizations for nonholonomic Ricci flows in [3] can be extracted from corresponding spectral functionals defining flows of a generalized Dirac operator and their scalings. Finally, in Section 5 we discuss and conclude the results of the paper. Certain important component formulas are outlined in Appendix.

## 2 Nonholonomic Manifolds and Spinor Structures

The concept of nonholonomic manifold was introduced independently by G. Vranceanu [11] and Z. Horak [12] for geometric interpretations of nonholonomic mechanical systems (see modern approaches and historical remarks

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4In this work, we shall use left ”up” and ”low” abstract labels which should not be considered as tensor or spinor indices written in the right side of symbols for geometrical objects.
in Refs. [13, 12, 43]). They called a pair \((V, \mathcal{N})\), where \(V\) is a manifold and \(\mathcal{N}\) is a nonintegrable distribution on \(V\), to be a nonholonomic manifold and considered new classes of linear connections (which were different from the Levi–Civita connection). Three well known classes of nonholonomic manifolds, when the nonholonomic distribution defines a nonlinear connection (\(N\)-connection) structure, are defined by Finsler spaces [44, 45, 46] and their generalizations as Lagrange and Hamilton spaces and higher order models [47, 14, 15, 48] (usually such geometries are modelled on a tangent bundle). More recent examples, related to exact off–diagonal solutions and nonholonomic frames in Einstein/ string/ gauge/ quantum/ noncommutative gravity and nonholonomic Fedosov manifolds [5, 49, 12, 50] also emphasize nonholonomic geometric structures but on generic spacetime manifolds and generalizations.

The aim of this section is to formulate the geometry of nonholonomic Clifford structures in a form adapted to generalizations for noncommutative spaces.

2.1 Nonholonomic distributions and nonlinear connections

We consider a \((n + m)\)-dimensional manifold \(V\), with \(n \geq 2\) and \(m \geq 1\) (for simplicity, in this work, \(V\) is a real smooth Riemannian space). The local coordinates on \(V\) are denoted \(u = (x, y)\), or \(u^\alpha = (x^i, y^a)\), where the ”horizontal” (h) indices run the values \(i, j, k, \ldots = 1, 2, \ldots, n\) and the ”vertical” (v) indices run the values \(a, b, c, \ldots = n + 1, n + 2, \ldots, n + m\). We parameterize a metric structure on \(V\) in the form

\[
g = g_{\alpha\beta}(u) du^\alpha \otimes du^\beta \tag{2}
\]

defined with respect to a local coordinate basis \(du^\alpha = (dx^i, dy^a)\) by coefficients

\[
g_{\alpha\beta} = \left[ g_{ij}(u) + \frac{N^a_i(u) N^b_j(u) h_{ab}(u)}{N^e_i(u) h_{be}(u)} \frac{N^e_c(u) h_{ae}(u)}{h_{ab}(u)} \right]. \tag{3}
\]

We denote by \(\pi^\top : TV \to TV\) the differential of a map \(V \to V\) defined by fiber preserving morphisms of the tangent bundles \(TV\) and \(TV\), where \(V\) is a \(n\)-dimensional manifold of necessary smooth class\(^5\). The kernel of \(\pi^\top\) is

\(^5\)For simplicity, we restrict our considerations for a subclass of nonholonomic distributions \(\mathcal{N}\) modelling certain fibered structures \(V \to V\) with constant rank \(\pi\). In such a case, the map \(\pi^\top\) is similar to that for a vector bundle with total space \(V\) and base \(V\). In general, we can use any map \(\pi^\top\) for which the kernel defines a corresponding vertical subspace as a nonholonomic distribution.
just the vertical subspace $v\mathbb{V}$ with a related inclusion mapping $i : v\mathbb{V} \rightarrow T\mathbb{V}$.

**Definition 2.1** A nonlinear connection ($N$–connection) $N$ on a manifold $\mathbb{V}$ is defined by the splitting on the left of an exact sequence

$$0 \rightarrow v\mathbb{V} \xrightarrow{i} T\mathbb{V} \rightarrow T\mathbb{V}/v\mathbb{V} \rightarrow 0,$$

i.e. by a morphism of submanifolds $N : T\mathbb{V} \rightarrow v\mathbb{V}$ such that $N \circ i$ is the unity in $v\mathbb{V}$.

Locally, a $N$–connection is defined by its coefficients $N_i^a(u)$,

$$N = N_i^a(u)dx^i \otimes \frac{\partial}{\partial y^a}. \quad (4)$$

Globalizing the local splitting, one prove that any $N$–connection is defined by a Whitney sum of conventional horizontal (h) subspace, $(h\mathbb{V})$, and vertical (v) subspace, $(v\mathbb{V})$,

$$T\mathbb{V} = h\mathbb{V} \oplus v\mathbb{V}. \quad (5)$$

The sum (5) states on $T\mathbb{V}$ a nonholonomic distribution of horizontal and vertical subspaces. The well known class of linear connections consists on a particular subclass with the coefficients being linear on $y^a$, i.e. $N_i^a(u) = \Gamma_{bj}^a(x)y^b$.

For simplicity, we shall work with a particular class of nonholonomic manifolds:

**Definition 2.2** A manifold $\mathbb{V}$ is $N$–anholonomic if its tangent space $T\mathbb{V}$ is enabled with a $N$–connection structure (3).

There are also two important examples of $N$–anholonomic manifolds modelled on bundle spaces:

**Example 2.1** a) A vector bundle $E = (E, \pi, M, N)$ defined by a surjective projection $\pi : E \rightarrow M$, with $M$ being the base manifold, $\dim M = n$, and $E$ being the total space, $\dim E = n + m$, and provided with a $N$–connection splitting (5) is a $N$–anholonomic vector bundle.

b) A particular case is that of $N$–anholonomic tangent bundle $T\mathbb{M} = (TM, \pi, M, N)$, with dimensions $n = m$.

Following our unified geometric formalism, we can write that in the above mentioned examples $\mathbb{V} = E$, or $\mathbb{V} = T\mathbb{M}$.

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Footnote 6: For the tangent bundle $T\mathbb{M}$, we can consider that both type of indices run the same values.
A N–anholonomic manifold is characterized by its curvature:

**Definition 2.3** The N–connection curvature is defined as the Neijenhuis tensor, $\Omega(X, Y) \doteq [vX, vX] + v[X, Y] - v[vX, Y] - v[X, vX]$. 

In local form, we have for $\Omega = \frac{1}{2} \Omega^a_{ij} d^i \wedge d^j \otimes \partial_a$ the coefficients

$$\Omega^a_{ij} = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_b^j \frac{\partial N_i^a}{\partial y^b} - N_b^i \frac{\partial N_j^a}{\partial y^b}. \tag{6}$$

Performing a frame (vielbein) transform $e^\alpha = e^\alpha_\alpha \partial/\partial u^\alpha$ and $e^\beta = e^\beta_\beta du^\beta$, where we underline the local coordinate indices, when $\partial/\partial u^\alpha = (\partial_i = \partial/\partial x^i, \partial_a = \partial/\partial y^a)$, with coefficients

$$e^\alpha_\nu(u) = \begin{bmatrix} e^i_\nu(u) & N^b_i(u)e^a_b(u) \\ 0 & e^a_\nu(u) \end{bmatrix}, \quad e^\beta_\mu(u) = \begin{bmatrix} e^i_\mu(u) & -N^k_i(u)e^k_\nu(u) \\ 0 & e^a_\mu(u) \end{bmatrix},$$

we transform the metric (5) into a distinguished metric (d–metric)

$$g = h g + \,^vh = g_{ij}(x, y) \; e^i \otimes e^j + h_{ab}(x, y) \; e^a \otimes e^b, \tag{7}$$

for an associated, to a N–connection, frame (vielbein) structure $e_\nu = (e_i, e_a)$, where

$$e_i = \frac{\partial}{\partial x^i} - N^a_i(u) \frac{\partial}{\partial y^a} \quad \text{and} \quad e_a = \frac{\partial}{\partial y^a}. \tag{8}$$

and the dual frame (coframe) structure $e^\nu = (e^i, e^a)$, where

$$e^i = dx^i \quad \text{and} \quad e^a = dy^a + N^a_i(u)dx^i. \tag{9}$$

A vector field $X \in TV$ can be expressed

$$X = (hX, vX), \quad \text{or} \quad X = X^a e_a = X^i e_i + X^a e_a,$$

where $hX = X^i e_i$ and $vX = X^a e_a$ state, respectively, the adapted to the N–connection structure horizontal (h) and vertical (v) components of the vector. In brief, $X$ is called a distinguished vector (in brief, d–vector). 7

7The vielbeins (8) and (9) are called respectively N–adapted frames and coframes. In order to preserve a relation with some previous our notations [7, 12], we emphasize that $e_\nu = (e_i, e_a)$ and $e^\nu = (e^i, e^a)$ are correspondingly the former “N–elongated” partial derivatives $\delta_\nu = \partial/\partial u^\nu = (\delta_i, \delta_a)$ and “N–elongated” differentials $\delta^\nu = \partial u^\nu = (d^i, d^a)$. They define certain “N–elongated” differential operators which are more convenient for tensor and integral calculations on such nonholonomic manifolds.
**Convention 2.1** The geometric objects on $V$ like tensors, spinors, connections etc are called respectively $d$–tensors, $d$–spinors, $d$–connections etc if they are adapted to the $N$–connection splitting (5).

The vielbeins (9) satisfy the nonholonomy relations

$$[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha\beta} e_\gamma$$

(10)

with (antisymmetric) nontrivial anholonomy coefficients $W^b_{ia} = \partial_a N^b_i$ and $W^a_{ji} = \Omega^a_{ij}$.

On any commutative nonholonomic manifold $V$, we can work equivalently with an infinite number of $d$–connections $^N\mathbf{D} = ^N\mathbf{\Gamma}(\mathbf{g})$, which are $d$–metric compatible, $^N\mathbf{D} \mathbf{g} = 0$, and uniquely defined by a given metric $\mathbf{g}$.

Writing the deformation relation

$$^N\mathbf{\Gamma}(\mathbf{g}) = \mathbf{\Gamma}(\mathbf{g}) + ^N\mathbf{Z}(\mathbf{g}),$$

(11)

where the deformation tensor $^N\mathbf{Z}(\mathbf{g})$ is also uniquely defined by $\mathbf{g}$, we can transform any geometric construction for the Levi–Civita connection $\mathbf{\Gamma}(\mathbf{g})$ equivalently into corresponding constructions for the $d$–connection $^N\mathbf{\Gamma}(\mathbf{g})$, and inversely, see details in Refs. [3, 4, 5, 12, 51]. From a formal point of view, there is a nontrivial torsion $^N\mathbf{T}(\mathbf{g})$, computed following formula (A.2), with coefficients (A.3), all considered for $^N\mathbf{\Gamma}(\mathbf{g})$. This torsion is induced nonholonomically as an effective one (by anholonomy coefficients, see (10) and (6)) and constructed only from the coefficients of metric $\mathbf{g}$. Such a torsion is completely deferent from that in string, or Einstein–Cartan, theory when the torsion tensor is an additional (to metric) field defined by an antisymmetric $H$–field, or spinning matter, see Ref. [52].

The main conclusion of this section is that working with nonholonomic distributions on formal Riemannian manifolds we can model, by anholonomic frames and adapted geometric objects, various types of geometric structures and physical theories with generic off–diagonal gravitational interactions, constrained Lagrange–Hamilton dynamics, Finsler and Lagrange spaces etc.

### 2.2 $N$–anholonomic spin structures

The spinor bundle on a manifold $M$, $\text{dim}M = n$, is constructed on the tangent bundle $TM$ by substituting the group $SO(n)$ by its universal covering $Spin(n)$. If a horizontal quadratic form $^h\mathbf{g}_{ij}(x, y)$ is defined on $T_x hV$ we can consider $h$–spinor spaces in every point $x \in hV$ with fixed $\mathbf{g}^a$. The constructions can be completed on $TV$ by using the $d$–metric $\mathbf{g}$ (7). In this case, the
group $SO(n + m)$ is not only substituted by $Spin(n + m)$ but with respect to $N$–adapted frames there are emphasized decompositions to $Spin(n) \oplus Spin(m)$.

### 2.2.1 Clifford $N$–adapted modules (d–modules)

A Clifford d–algebra is a $\wedge V^{n+m}$ algebra endowed with a product

$$uv + vu = 2g(u, v) \mathbb{I}$$

distinguished into $h$–, $v$–products

$$h_u h_v + h_v h_u = 2 h g(u, v) \mathbb{I},$$

$$v_u v_v + v_v v_u = 2 v h(u, v) v v I,$$

for any $u = (h_u, v_u), v = (h_v, v_v) \in V^{n+m}$, where $I, h I$ and $v I$ are unity matrices of corresponding dimensions $(n + m) \times (n + m)$, or $n \times n$ and $m \times m$.

A metric $h g$ on $hV$ is defined by sections of the tangent space $T hV$ provided with a bilinear symmetric form on continuous sections $\Gamma(T hV)$. This allows us to define Clifford $h$–algebras $h Cl(T_x hV)$, in any point $x \in T hV$,

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2 g_{ij} h I.$$

For any point $x \in hV$ and fixed $y = y_0$, there exists a standard complexification, $T_x hV^C \cong T_x hV + i T_x hV$, which can be used for definition of the 'involuntion' operator on sections of $T_x hV^C$,

$$h \sigma_1 h \sigma_2(x) \cong h \sigma_2(x) h \sigma_1(x), \quad h \sigma^*(x) \cong h \sigma(x)^*, \forall x \in hV,$$

where "*" denotes the involuntion on every $h Cl(T_x hV)$.

**Definition 2.4** A Clifford d–space on a nonholonomic manifold $V$ enabled with a $d$–metric $g(x, y)$ and a $N$–connection $N$ is defined as a Clifford bundle $Cl(V) = h Cl(hV) \oplus v Cl(vV)$, for the Clifford $h$–space $h Cl(hV) \cong h Cl(T^* hV)$ and Clifford $v$–space $v Cl(vV) \cong v Cl(T^* vV)$.

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8It should be noted here that spin bundles may not exist for general holonomic or nonholonomic manifolds. For simplicity, we do not provide such topological considerations in this paper, see Ref. [53] on nontrivial topological configurations with nonholonomic manifolds. We state that we shall work only with $N$–holonomic manifolds for which certain spinor structures can be defined both for the h- and v–splitting; the existence of a well defined decomposition $Spin(n) \oplus Spin(m)$ follows from $N$–connection splitting [5].

9For simplicity, we shall consider only "horizontal" geometric constructions if they are similar to "vertical" ones.
For a fixed N–connection structure, a Clifford N–anholonomic bundle on $V$ is defined $N\text{Cl}(V) \doteq N\text{Cl}(T^*V)$. Let us consider a complex vector bundle $E: E \rightarrow V$ on an N–anholonomic space $V$ when the N–connection structure is given for the base manifold. The Clifford d–module of a vector bundle $E$ is defined by the $C(V)$–module $\Gamma(E)$ of continuous sections in $E$, $c : \Gamma(N\text{Cl}(V)) \rightarrow \text{End}(\Gamma(E))$.

In general, a vector bundle on a N–anholonomic manifold may be not adapted to the N–connection structure on base space.

2.2.2 h–spinors, v–spinors and d–spinors

Let us consider a vector space $V^n$ provided with Clifford structure. We denote such a space $hV^n$ in order to emphasize that its tangent space is provided with a quadratic form $h g$. We also write $h\text{Cl}(V^n) \equiv \text{Cl}(hV^n)$ and use subgroup $SO(hV^n) \subset O(hV^n)$.

**Definition 2.5** The space of complex h–spins is defined by the subgroup

$$h\text{Spin}^c(n) \equiv \text{Spin}^c(hV^n) \equiv h\text{Spin}^c(V^n) \subset \text{Cl}(hV^n),$$

determined by the products of pairs of vectors $w \in hV^c$ when $w = \lambda u$ where $\lambda$ is a complex number of module 1 and $u$ is of unity length in $hV^n$.

Similar constructions can be performed for the v–subspace $vV^m$, which allows us to define similarly the group of real v–spins.

A usual spinor is a section of a vector bundle $S$ on a manifold $M$ when an irreducible representation of the group $\text{Spin}(M) \doteq \text{Spin}(T^*_x M)$ is defined on the typical fiber. The set of sections $\Gamma(S)$ is an irreducible Clifford module. If the base manifold is of type $hV$, or is a general N–anholonomic manifold $V$, we have to define spinors on such spaces in a form adapted to the respective N–connection structure.

**Definition 2.6** A h–spinor bundle $hS$ on a h–space $hV$ is a complex vector bundle with both defined action of the h–spin group $h\text{Spin}(V^n)$ on the typical fiber and an irreducible representation of the group $h\text{Spin}(V) \equiv \text{Spin}(hV) \doteq \text{Spin}(T^*_x hV)$. The set of sections $\Gamma(hS)$ defines an irreducible Clifford h–module.

The concept of "d–spinors" has been introduced for the spaces provided with N–connection structure \[36, 37\].
**Definition 2.7** A distinguished spinor (d–spinor) bundle \( S \equiv ( { h S, \ v S} ) \) on a \( N\)–anholonomic manifold \( V \), \( \dim V = n+m \), is a complex vector bundle with a defined action of the spin d–group \( \text{Spin}(V) \equiv \text{Spin}(V^n) \oplus \text{Spin}(V^m) \) with the splitting adapted to the \( N\)–connection structure which results in an irreducible representation \( \text{Spin}(V) \equiv \text{Spin}(\mathbb{T}^*V) \). The set of sections \( \Gamma(S) = \Gamma({ h S}) \oplus \Gamma({ v S}) \) is an irreducible Clifford d–module.

If we study algebras through theirs representations, we also have to consider various algebras related by the Morita equivalence.\(^{10}\)

The possibility to distinguish the \( \text{Spin}(n) \) (or, correspondingly \( \text{Spin}(hV), \text{Spin}(V^n) \oplus \text{Spin}(V^m) \)) allows us to define an antilinear bijection \( { h J : h S \to h S} \) (or \( v J : \ v S \to v S \) and \( J : \ S \to \ S \)) with properties of type:

\[
{ h J( h a \psi) = h \chi( h a) \ h J h \psi, \text{ for } h a \in \Gamma^\infty(\text{Cl}(hV));}
\]

(12)

\[
( h J h \phi | h J h \psi) = ( h \psi | h \phi) \text{ for } h \phi, h \psi \in h S.
\]

The considerations presented in this Section consists the proof of:

**Theorem 2.1** Any d–metric and N–connection structure defines naturally the fundamental geometric objects and structures (such as the Clifford h–module, v–module and Clifford d–modules, or the h–spin, v–spin structures and d–spinors) for the corresponding nonholonomic spin manifold and/or \( N\)–anholonomic spinor (d–spinor) manifold.

We note that similar results were obtained in Refs. \([36, 37]\) for the standard Finsler and Lagrange geometries and theirs higher order generalizations. In a more restricted form, the idea of Theorem 2.1 can be found in Ref. \([5]\), where the first models of noncommutative Finsler geometry and related gravity were analyzed.

### 3 Nonholonomic Dirac Operators and Spectral Triples

The Dirac operator for a class of (non) commutative nonholonomic spaces provided with d–metric structure was introduced in Ref. \([5]\) following previous constructions for the Dirac equations on locally anisotropic spaces

\(^{10}\)The Morita equivalence can be analyzed by applying in \( N\)–adapted form, both on the base and fiber spaces, the consequences of the Plymen’s theorem (see Theorem 9.3 in Ref. \([33]\); in this work, we omit details of such considerations).
(various variants of Finsler–Lagrange and Cartan–Hamilton spaces and generalizations), see [36, 37, 39, 40] and Part III in [12]. In this Section, we define nonholonomic Dirac operators for general N–anholonomic manifolds.

3.1 N–anholonomic Dirac operators

The geometric constructions depend on the type of linear connections considered for definition of such Dirac operators. They are metric compatible and N–adapted if the canonical d–connection is used (similar constructions can be performed for any deformation which results in a metric compatible d–connection).

3.1.1 Noholonomic vielbeins and spin d–connections

Let us consider a Hilbert space of finite dimension. For a local dual coordinate basis \( e^i = dx^i \) on \( hV \), we may respectively introduce certain classes of orthonormalized vielbeins and the N–adapted vielbeins, \( e^i \approx e^i(x, y) e^x \) and \( e^i \approx e^i(x, y) e^y \), when \( g_{ij} e^i e^j = \delta^i_0 \) and \( g_{ij} e^i e^j = g^{ij} \).

We define the algebra of Dirac’s gamma horizontal matrices (in brief, gamma h–matrices defined by self–adjoint matrices \( M_k(\mathbb{C}) \) where \( k = 2^n/2 \) is the dimension of the irreducible representation of \( Cl(hV) \) from relation \( \gamma^i \gamma^j + \gamma^j \gamma^i = 2\delta^{ij} h\mathbb{I} \). The action of \( dx^i \in Cl(hV) \) on a spinor \( h\psi \in hS \) is given by formulas

\[
h_c(dx^i) \gamma^i \text{ and } h_c(dx^i) h\psi \approx \gamma^i h\psi \equiv e^i e^i h\psi. \quad (13)
\]

Similarly, we can define the algebra of Dirac’s gamma vertical matrices related to a typical fiber \( F \) (in brief, gamma v–matrices defined by self–adjoint matrices \( M'_k(\mathbb{C}) \), where \( k' = 2^m/2 \) is the dimension of the irreducible representation of \( Cl(F) \) from relation \( \gamma^a \gamma^b + \gamma^b \gamma^a = 2\delta^{ab} v\mathbb{I} \). The action of \( dy^a \in Cl(F) \) on a spinor \( v\psi \in vS \) is

\[
v_c(dy^a) \gamma^a \text{ and } v_c(dy^a) v\psi \approx \gamma^a v\psi \equiv e^a e^a \gamma^a v\psi.
\]

A more general gamma matrix calculus with distinguished gamma matrices (in brief, gamma d–matrices\(^{11}\)) can be elaborated for N–anholonomic manifolds \( V \) provided with d–metric structure \( g = h g \oplus v h \) and for d–spinors \( \dot{\psi} = (h\psi, v\psi) \in S \oplus (hS, vS) \). In this case, we consider d–gamma

\(^{11}\) in some our previous works [36, 37] we wrote \( \sigma \) instead of \( \gamma \).
matrix relations $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\delta^{\alpha\beta}I$, with the action of $du^\alpha \in \text{Cl}(V)$ on a d–spinor $\tilde{\psi} \in S$ resulting in distinguished irreducible representations

$$c(du^\hat{\alpha}) \doteq \gamma^\hat{\alpha} \quad \text{and} \quad c = (du^\alpha) \tilde{\psi} \doteq \gamma^\alpha \tilde{\psi} \equiv e^\alpha_\beta \gamma^\hat{\alpha} \tilde{\psi}$$

which allows us to write $\gamma^\alpha(u)\gamma^\beta(u) + \gamma^\beta(u)\gamma^\alpha(u) = 2g^{\alpha\beta}(u)I$.

In the canonical representation, we have the irreducible form $\tilde{\gamma} \doteq h_\gamma \oplus v_\gamma$ and $\tilde{\psi} \doteq h_\psi \oplus v_\psi$, for instance, by using block type of h– and v–matrices. We can also write such formulas as couples of gamma and/or h– and v–spinor objects written in N–adapted form, $\gamma^\alpha \doteq (h_\gamma, v_\gamma)$ and $\tilde{\psi} \doteq (h_\psi, v_\psi)$.

The spin connection $s\nabla$ for Riemannian manifolds is induced by the Levi–Civita connection $\Gamma$, $s\nabla \doteq d - \frac{1}{4} \Gamma^\gamma_{jk} \gamma^j \, dx^k$. On N–anholonomic manifolds, spin d–connection operators $s\nabla$ can be similarly constructed from any metric compatible d–connection $\Gamma^\alpha_{\beta\mu}$ using the N–adapted absolute differential $\delta$ acting, for instance, on a scalar function $f(x,y)$ in the form

$$\delta f = (e_\nu f) \delta u^\nu = (e_i f) \, dx^i + (e_a f) \delta y^a,$$

for $\delta u^\nu = e^\nu$, see N–elongated operators (8) and (9).

**Definition 3.1** The canonical spin d–connection is defined by the canonical d–connection,

$$s\hat{\nabla} \doteq \delta - \frac{1}{4} \hat{\Gamma}^\alpha_{\beta\mu} \gamma^\alpha \gamma^\beta \delta u^\mu,$$

(15)

where the N–adapted coefficients $\hat{\Gamma}^\alpha_{\beta\mu}$ are given by formulas (A.7).

We note that the canonical spin d–connection $s\hat{\nabla}$ is metric compatible and contains nontrivial d–torsion coefficients induced by the N–anholonomy relations (see formulas (A.3) proved for arbitrary d–connection).

### 3.1.2 Dirac d–operators

We consider a vector bundle $E$ on a N–anholonomic manifold $V$ (with two compatible N–connections defined as h– and v–splittings of $TE$ and $TV$)). A d–connection $D : \Gamma^\infty(E) \to \Gamma^\infty(E) \otimes \Omega^1(V)$ preserves by parallelism splitting of the tangent total and base spaces and satisfy the Leibniz condition $D(f\sigma) = f(D\sigma) + \delta f \otimes \sigma$, for any $f \in C^\infty(V)$, and $\sigma \in \Gamma^\infty(E)$ and $\delta$ defining an N–adapted exterior calculus by using N–elongated operators (8) and (9) which emphasize d–forms instead of usual forms on $V$, with the coefficients taking values in $E$.
The metricity and Leibniz conditions for $\mathcal{D}$ are written respectively
\[ g(\mathcal{D}X, Y) + g(X, \mathcal{D}Y) = \delta[g(X, Y)], \tag{16} \]
for any $X, Y \in \chi(V)$, and
\[ \mathcal{D}(\sigma \beta) \doteq \mathcal{D}(\sigma) \beta + \sigma \mathcal{D}(\beta), \tag{17} \]
for any $\sigma, \beta \in \Gamma(\mathcal{E})$. For local computations, we may define the corresponding coefficients of the geometric d–objects and write
\[ \mathcal{D}\sigma \beta \doteq \Gamma^\alpha_{\beta \mu} \sigma_\alpha \otimes \delta u^\mu = \Gamma^\alpha_{\beta i} \sigma_\alpha \otimes dx^i + \Gamma^\alpha_{\beta a} \sigma_\alpha \otimes dy^a, \]
where fiber "acute" indices are considered as spinor ones.

The respective actions of the Clifford d–algebra and Clifford h–algebra can be transformed into maps $\Gamma(\mathcal{S}) \otimes \Gamma(Cl(V))$ and $\Gamma(\mathcal{S}) \otimes \Gamma(Cl(hV))$ to $\Gamma(\mathcal{S})$ and, respectively, $\Gamma(\mathcal{S})$ by considering maps of type (13) and (14)
\[ \hat{c}(\bar{\psi} \otimes a) = c(a) \bar{\psi} \text{ and } h \hat{c}(h\psi \otimes h_a) = h c(h_a) h \psi. \]

**Definition 3.2** The Dirac d–operator (Dirac h–operator, or v–operant) on a spin N–anholonomic manifold $(V, S, J)$ (on a h–spin manifold $(hV, hS, hJ)$, or on a v–spin manifold $(vV, vS, vJ)$) is defined
\[ \mathcal{D} \doteq -i(\hat{c} \circ s \nabla) \tag{18} \]
\[ = \left( h \mathcal{D} = -i(\hat{h}c \circ h s \nabla), \quad v \mathcal{D} = -i(\hat{v}c \circ v s \nabla) \right) \]
Such N–adapted Dirac d–operators are called canonical and denoted $\hat{\mathcal{D}} = (\hat{h} \mathcal{D}, \hat{v} \mathcal{D})$ if they are defined for the canonical d–connection (13) and respective spin d–connection (14).

We formulate:

**Theorem 3.1** Let $(V, S, J)$ ( $(hV, hS, hJ)$) be a spin N–anholonomic manifold (h–spin space). There is the canonical Dirac d–operator (Dirac h–operator) defined by the almost Hermitian spin d–operator
\[ s \hat{\nabla} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}) \otimes \Omega(V) \]
(spin h–operator $\hat{h} s \hat{\nabla} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}) \otimes \Omega(hV)$) commuting with $J$ (h–J), see (12), and satisfying the conditions
\[ (s \hat{\nabla} \psi | \phi) + (\bar{\psi} | s \hat{\nabla} \phi) = \delta(\bar{\psi} | \phi) \]
and
\[
\mathbf{s}\hat{\nabla}(\mathbf{c}(a)\psi) = \mathbf{c}(\hat{\mathbb{D}}a)\psi + \mathbf{c}(a)\ s\hat{\nabla}\psi
\]
for \( a \in \text{Cl}(\mathbf{V}) \) and \( \psi \in \Gamma^\infty(\mathbf{S}) \)

\[
(\ ( h\hat{\nabla} h\phi | h\phi ) + ( h\psi | \ k\hat{\nabla} h\phi ) = h\delta( h\psi | h\phi )
\]
and
\[
\mathbf{h}\hat{\nabla}(h\mathbf{c}(h\mathbf{a})h\psi) = h\mathbf{c}(h\hat{\mathbb{D}}h\mathbf{a})h\psi + h\mathbf{c}(h\mathbf{a})k\hat{\nabla}h\psi \text{ for } h\mathbf{a} \in \text{Cl}(h\mathbf{V})
\]
and \( \psi \in \Gamma^\infty( h\mathbf{S} ) \) determined by the metricity (16) and Leibnitz (17) conditions.

Proof. We sketch the main ideas of such Proofs. There are two possibilities:

The first one is similar to that given in Ref. [33], Theorem 9.8, for the Levi–Civita connection. We have to generalize the constructions for d–metrics and canonical d–connections by applying N–elongated operators for differentials and partial derivatives. The formulas have to be distinguished into h– and v–irreducible components. Such an approach can be elaborated for d–connections with arbitrary torsions (it is not a purpose of this work to consider such general constructions).

In a more particular case, we work with nonholonomic deformations of linear connections of type (11). In such a case, there is a a second possibility to provide a very simple proof on existence of the canonical Dirac d–operator. The Levi–Civita connection, a metric compatible d–connection and a corresponding distorsion tensor from (11) are completely defined by a d–metric structure. Using the standard spin and Dirac operator (defined by the Levi–Civita connection), we can construct a unique nonholonomic deformation into the canonical Dirac d–operator \( \hat{\mathbb{D}} \) using three steps:

1) For a given d–metric \( g \), we can compute \( \Gamma_\gamma^\alpha_\beta \), then \( \hat{\Gamma}_\gamma^\alpha_\beta \), see formulas (A.7), and \( Z_\gamma^\alpha_\beta \), see formulas (A.9), determining a unique deformation of linear connections, \( \Gamma_\gamma^\alpha_\beta = \hat{\Gamma}_\gamma^\alpha_\beta + Z_\gamma^\alpha_\beta \) (A.8).

2) Introducing splitting (A.8) into formulas for \( \mathbf{s}\nabla \) and \( \mathbf{s}\hat{\nabla} \), see formula (15) and related explanations, we define a unique splitting \( \mathbf{s}\nabla = \mathbf{s}\hat{\nabla} + Z\hat{\nabla} \), where \( Z\hat{\nabla} \) is completely determined by \( Z_\gamma^\alpha_\beta \) (and, as a consequence, by \( g \)).

For simplicity, we omit explicit formulas for operators \( \mathbf{s}\nabla \) and \( Z\hat{\nabla} \).

3) Following Definition 3.2 the prvious splitting sum for spin d–operators results in a corresponding splitting formula for the Dirac operator, see explicitly formula (18), when \( \mathbf{s}\mathbb{D} = \mathbb{D} + Z\mathbb{D} \), for \( \mathbb{D} \), defined by the Levi–Civita connection, and \( Z\mathbb{D} \), induced by \( Z_\gamma^\alpha_\beta \) (A.9). For simplicity, we omit explicit

\[\text{inverse constructions are similar}^{12}\]
formulas for operators $\mathcal{D}$ and $\mathcal{Z}$. Such a splitting has an associated splitting for the corresponding almost Hermitian spin $d$–operator, mentioned in the formulation of this Theorem.

We conclude that on a spin $N$–anholonomic manifold we can work equivalently with two canonical Dirac operators, $\mathcal{S}\mathcal{D}$ and $\mathcal{D}$. From a formal point of view, the canonical Dirac $d$–operator $\mathcal{D}$ encode a nonholonomically induced torsion $d$–tensor, but such a distorsion $d$–tensor is completely defined by a $d$–metric $g$, which is quite similar to constructions with the Levi–Civita connection. □

The geometric information of a spin manifold (in particular, the metric) is contained in the Dirac operator. For nonholonomic manifolds, the canonical Dirac $d$–operator has $h$– and $v$–irreducible parts related to off–diagonal metric terms and nonholonomic frames with associated structure. In a more special case, the canonical Dirac $d$–operator is defined by the canonical $d$–connection. Nonholonomic Dirac $d$–operators contain more information than the usual, holonomic, ones.

**Proposition 3.1** If $\tilde{\mathcal{D}} = (\tilde{h}\mathcal{D}, \tilde{v}\mathcal{D})$ is the canonical Dirac $d$–operator then

$$\left[ \tilde{\mathcal{D}}, f \right] = i\mathcal{C}(\delta f),$$

equivalently,

$$\left[ \tilde{h}\mathcal{D}, f \right] + \left[ \tilde{v}\mathcal{D}, f \right] = i \left( h\mathcal{C}(dx^i \frac{\delta f}{\partial x^i}) + v\mathcal{C}(\delta y^a \frac{\partial f}{\partial y^a}) \right),$$

for all $f \in C^\infty(V)$.

**Proof.** It is a straightforward computation following from Definition 3.2. □

The canonical Dirac $d$–operator and its $h$– and $v$–components have all the properties of the usual Dirac operators (for instance, they are self–adjoint but unbounded). It is possible to define a scalar product on $\Gamma^\infty(S)$,

$$\langle \psi, \phi \rangle = \int_V \langle \psi | \phi \rangle |\nu_g|$$

where $\nu_g = \sqrt{\text{det}|g|} \sqrt{\text{det}|h|} \ dx^1 ... dx^n \ dy^{n+1} ... dy^{n+m}$ is the volume $d$–form on the $N$–anholonomic manifold $V$.

### 3.2 $N$–adapted spectral triples and distance in $d$–spinor spaces

We denote $N\mathcal{H} \simeq L^2(V, S) = [h\mathcal{H} = L^2(hV, \ hS), \ v\mathcal{H} = L^2(vV, \ vS)]$ the Hilbert $d$–space obtained by completing $\Gamma^\infty(S)$ with the norm defined by the
scalar product (19). Similarly to the holonomic spaces, by using formulas (18) and (15), one may prove that there is a self–adjoint unitary endomorphism \([\text{cr}]\Gamma\) of \(N\mathcal{H}\), called "chirality", being a \(\mathbb{Z}_2\) graduation of \(N\mathcal{H}\), which satisfies the condition \(\hat{D}\ [\text{cr}]\Gamma = - [\text{cr}]\Gamma \hat{D}\). Such conditions can be written also for the irreducible components \(h\hat{D}\) and \(v\hat{D}\).

**Definition 3.3** A distinguished canonical spectral triple (canonical spectral d–triple) \((N\mathcal{A}, N\mathcal{H}, \hat{D})\) for a d–algebra \(N\mathcal{A}\) is defined by a Hilbert d–space \(N\mathcal{H}\), a representation of \(N\mathcal{A}\) in the algebra \(N\mathcal{B}(N\mathcal{H})\) of d–operators bounded on \(N\mathcal{H}\), and by a self–adjoint d–operator \(N\mathcal{H}\), of compact resolution, such that \([N\mathcal{H}, a] \in N\mathcal{B}(N\mathcal{H})\) for any \(a \in N\mathcal{A}\).

Every canonical spectral d–triple is defined by two usual spectral triples which in our case corresponds to certain h– and v–components induced by the corresponding h– and v–components of the Dirac d–operator. For such spectral h(v)–triples we, can define the notion of \(KR_n\)–cycle and \(KR_m\)–cycle and consider respective Hochschild complexes. To define a noncommutative geometry the h– and v– components of a canonical spectral d–triples must satisfy certain well defined seven conditions (see Refs. [16, 33] for details, stated there for holonomic configurations): the spectral dimensions are of order \(1/n\) and \(1/m\), respectively, for h– and v–components of the canonical Dirac d–operator; there are satisfied the criteria of regularity, finiteness and reality; representations are of 1st order; there is orientability and Poincaré duality holds true. Such conditions can be satisfied by any Dirac operators and canonical Dirac d–operators (in the last case we have to work with d–objects).

**Definition 3.4** A spectral d–triple is a real one satisfying the above mentioned seven conditions for the h– and v–irreversible components and defining a (d–spinor) N–anholonomic noncommutative geometry stated by the data \((N\mathcal{A}, N\mathcal{H}, \hat{D}, J, [\text{cr}]\Gamma)\) and derived for the Dirac d–operator (18).

For N–adapted constructions, we can consider d–algebras \(N\mathcal{A} = h\mathcal{A} \oplus v\mathcal{A}\). We generate N–anholonomic commutative geometries if we take \(N\mathcal{A} = C^\infty(V)\), or \(h\mathcal{A} = C^\infty(hV)\).

Let us show how it is possible to compute distance in a d–spinor space:

---

13 We use the label \([\text{cr}]\) in order to avoid misunderstanding with the symbol \(\Gamma\) used for linear connections.

14 An operator \(D\) is of compact resolution if for any \(\lambda \in \text{sp}(D)\) the operator \((D - \lambda I)^{-1}\) is compact, see details in [33].

15 We omit in this paper the details on axiomatics and related proofs for such considerations.
Theorem 3.2 Let \( (N, A, \mathcal{N}, \mathcal{H}, \hat{D}, J, [\epsilon r] \Gamma) \) defines a noncommutative geometry being irreducible for \( N, A = C^\infty(V) \), where \( V \) is a compact, connected and oriented manifold without boundaries, of spectral dimension \( \dim V = n + m \). In this case, there are satisfied the conditions:

1. There is a unique \( d \)-metric \( g(\hat{D}) = (h^g, v^g) \) of type \( ]7[ \), with the "nonlinear" geodesic distance on \( V \) defined by
   \[
   d(u_1, u_2) = \sup_{f \in C(V)} \left\{ f(u_1, u_2)/ || \hat{D}, f || \leq 1 \right\},
   \]
   for any smooth function \( f \in C(V) \).

2. A \( N \)-anholonomic manifold \( V \) is a spin \( N \)-anholonomic space, for which the operators \( \hat{D}' \) satisfying the condition \( g(\hat{D}') = g(\hat{D}) \) define an union of affine spaces identified by the \( d \)-spinor structures on \( V \).

3. The functional \( S(\hat{D}) \equiv \int |\hat{D}|^{-n-m+2} \) defines a quadratic \( d \)-form with \( (n + m) \)-splitting for every affine space which is minimal for \( \hat{D} = \hat{D} \)
   as the canonical Dirac \( d \)-operator corresponding to the \( d \)-spin structure with the minimum proportional to the Einstein–Hilbert action constructed for the canonical \( d \)-connection with the \( d \)-scalar curvature \( ^sR \).

Proof. This Theorem is a generalization for \( N \)-anholonomic spaces of a similar one formulated in Ref. [16], with a detailed proof presented in [33], for the noncommutative geometry defined for a triple \( (A, \mathcal{H}, sD, J, [\epsilon r] \Gamma) \)\(^{16}\). That (holonomic) Dirac operator \( sD \) is associated to a Levi–Civita connection and any integral with \( sD \rightarrow D \) and computed following formula
   \[
   \int |D|^{-n+2} = \frac{1}{2^n/2\Omega_n} Wres|D|^{-n+2},
   \]
   where \( \Omega_n \) is the integral of the volume on the sphere \( S^n \) and \( Wres \) is the Wodzicki residue, see details in Theorem 7.5 [33]. As we sketched in the proof for Theorem 3.1, we get equivalent nonholonomical configurations by distorting canonically the constructions for the Levi–Civita connection, and related spin and Dirac operators, into those with associated canonical \( d \)-connections, using splitting \( \Gamma_{\alpha\beta} = \hat{\Gamma}_{\alpha\beta} + Z_{\alpha\beta} \)\(^{18}\). To such holonomic/ nonholonomic configurations, we can associate

\(^{16}\) In the mentioned monographs, there are provided formulations/proofs of Theorem 3.2 re-defined in terms of usual holonomic Levi–Civita structures for Riemann/ spin manifolds.
a unique distance/metric determined in unique forms by two equivalent, holonomic and/or nonholonomic Dirac operators. For trivial nonholonomic distortions, the conditions and proof of this theorem for the canonical d–connection transform into those for the Levi–Civita connection [16]. □

The existence of a canonical d–connection structure which is metric compatible and constructed from the coefficients of the d–metric and N–connection structure is of crucial importance allowing the formulation and proofs of the main results of this work. As a matter of principle, we can consider any splitting of connections of type (11) and compute a unique distance like we stated in the above Theorem 3.2, but for a ”non–canonical” Dirac d–operator. This holds true for any noncommutative geometry induced by a metric compatible d–connection supposed to be uniquely induced by a metric tensor.

In more general cases, we can consider any metric compatible d–connection with arbitrary d–torsion. Such constructions can be also elaborated in N–adapted form by preserving the respective h– and v–irreducible decompositions. For the Dirac d–operators, we have to start with the Proposition 3.1 and then to repeat all constructions from [16, 33], both on h– and v–subspaces. In this article, we do not analyze (non) commutative geometries enabled with general torsions but consider only nonholonomic deformations when distortions are induced by a metric structure.

Finally, we note that Theorem 3.2 allows us to extract from a canonical nonholonomic model of noncommutative geometry various types of commutative geometries (holonomic and N–anholonomic Riemannian spaces, Finsler–Lagrange spaces and generalizations) for corresponding nonholonomic Dirac operators.

4 Spectral Functionals and Ricci Flows

The goal of this section is to prove that the Perelman’s functionals [2] and their generalizations for nonholonomic Ricci flows in [8] [in the second reference, see formulas (29), for commutative holonomic configurations, and (30) and (31), for commutative nonholonomic configurations, and Theorems 4.1 and 4.2 in this work] can be extracted from flows of a generalized Dirac operator \( ^N\mathcal{D}(\chi) = \mathcal{D}(\chi) \otimes 1 \) included in spectral functionals of type

\[
\text{Tr} \ b f\left( ^N\mathcal{D}^2(\chi)/\Lambda^2\right),
\]

where \( b f(\chi) \) are testing functions labelled by \( b = 1, 2, 3 \) and depending on a real flow parameter \( \chi \), which in the commutative variant of the Ricci flow
theory corresponds to that for R. Hamilton’s equations \[1\]. For simplicity, we shall use one cutoff parameter \( \Lambda \) and suppose that operators under flows act on the same algebra \( \mathcal{A} \) and Hilbert space \( \mathcal{H} \), i.e. we consider families of spectral triples of type \((\mathcal{A}, \mathcal{H}, \mathcal{N} \mathcal{D}(\chi))\)\[17\].

**Definition 4.1** The normalized Ricci flow equations (R. Hamilton’s equations) generalized on nonholonomic manifolds are defined in the form

\[
\frac{\partial g_{\alpha\beta}(\chi)}{\partial \chi} = -2 N R_{\alpha\beta}(\chi) + \frac{2}{5} g_{\alpha\beta}(\chi),
\]

where \( g_{\alpha\beta}(\chi) \) defines a family of \( d \)-metrics parametrized in the form \[7\] on a \( N \)-anholonomic manifold \( V \) enabled with a family of \( N \)-connections \( N \Gamma \).

The effective ”cosmological” constant \( 2r/5 \) in \[21\] with normalizing factor \( r = \int_V s N R dv/v \) is introduced with the aim to preserve a volume \( v \) on \( V \), where \( s N R \) is the scalar curvature of type \( A.6 \), see basic definitions and component formulas in Appendix\[18\].

The corresponding family of Ricci tensors \( N R_{\alpha\beta}(\chi) \), in \[21\], and nonholonomic Dirac operators \( N \mathcal{D}(\chi) \), in \[11\], are defined for any value of \( \chi \) by a general metric compatible linear connection \( N \Gamma \) adapted to a \( N \)-connection structure. In a particular case, we can consider the Levi–Civita connection \( \Gamma \), which is used in standard geometric approaches to physical theories. Nevertheless, for various purposes in modelling evolution of off–diagonal Einstein metrics, constrained physical systems, effective Finsler and Lagrange geometries, Fedosov quantization of field theories and gravity etc\[19\], it is convenient to work with a ”\( N \)-adapted” linear connection \( N \Gamma(g) \). If such a connection is also uniquely defined by a metric structure \( g \), we are able to re–define the constructions in an equivalent form for the corresponding Levi–Civita connection.

In noncommutative geometry, all physical information on generalized Ricci flows can be encoded into a corresponding family of nonholonomic Dirac operators \( N \mathcal{D}(\chi) \). For simplicity, in this work, we shall consider that \( P^* D = 0 \), i.e. we shall not involve into the (non)commutative Ricci flow

\[17\] we shall omit in this section the left label ”\( N \)” for algebras and Hilbert spaces if that will not result in ambiguities.

\[18\] We note that in Ref. [3] we use two mutually related flow parameters \( \chi \) and \( \tau \); for simplicity, in this work we write only \( \chi \) even, in general, such parameters should be rescaled for different geometric analysis constructions.

\[19\] the coefficients of corresponding \( N \)-connection structures being defined respectively by the generic off–diagonal metric terms, anholonomy frame coefficients, Finsler and Lagrange fundamental functions etc.
theory the particle physics. We cite here the work on Ricci–Yang–Mills flow \cite{54} with evolution equations which can be extracted from generalized spectral functionals \cite{20} with corresponding Yang–Mills nontrivial component in $P^D$. Perhaps a "comprehensive" noncommutative Ricci flow theory should include as a stationary case the "complete" spectral action \cite{1} parametrized for the standard models of gravity and particle physics. Following such an approach, the (non)commutative/ (non)holonomic/ quantum/ classical evolution scenarios are related to topological properties of a quantum/ classical spacetime and flows and actions of fundamental matter fields.

4.1 Spectral flows and Perelman functionals

Let us consider a family of generalized d–operators

$$D^2(\chi) = -\left\{ \frac{1}{2} g^{\alpha\beta}(\chi) [e_{\alpha}(\chi)e_{\beta}(\chi) + e_{\beta}(\chi)e_{\alpha}(\chi)] + A^{\nu}(\chi)e_{\nu}(\chi) + B(\chi) \right\},$$

where the real flow parameter $\chi \in [0, \chi_0)$ and, for any fixed values of this parameter, the matrices $A^{\nu}(\chi)$ and $B(\chi)$ are determined by a N–anholonomic Dirac operator $D$ induced by a metric compatible d–connection $D$, see (A.1) and Definition 3.2; for the canonical d–connection, we have to put "hats" on symbols and write $\hat{D}^2, \hat{A}^{\nu}$ and $\hat{B}$. We introduce two functionals $\mathcal{F}$ and $\mathcal{W}$ depending on $\chi$,

$$\mathcal{F} = Tr \left[ 1 f(\chi) (\phi D^2(\chi)/\Lambda^2) \right] \simeq \sum_{k \geq 0} 1 f_{(k)}(\chi) 1 a_{(k)}(\phi D^2(\chi)/\Lambda^2)$$

and

$$\mathcal{W} = 2\mathcal{W} + 3\mathcal{W},$$

for $\mathcal{W} = Tr \left[ e f(\chi)(\phi D^2(\chi)/\Lambda^2) \right]$

$$= \sum_{k \geq 0} e f_{(k)}(\chi) e a_{(k)}(\phi D^2(\chi)/\Lambda^2),$$

where we consider a cutting parameter $\Lambda^2$ for both cases $e = 2, 3$. Functions $b f$, with label $b$ taking values 1, 2, 3, have to be chosen in a form which insure that for a fixed $\chi$ we get certain compatibility with gravity and particle physics and result in positive average energy and entropy for Ricci flows of
geometrical objects. For such testing functions, ones hold true the formulas

\[ b_f(0)(\chi) = \int_{0}^{\infty} b_f(\chi, u) \, du, \quad b_f(2)(\chi) = \int_{0}^{\infty} b_f(\chi, u) \, du, \]

\[ b_f(2k+4)(\chi) = (-1)^k \cdot b_f(k)(0), \quad k \geq 0. \]  

(25)

We will comment the end of this subsection on dependence on \( \chi \) of such functions.

The coefficients \( b(a(k)) \) can be computed as the Seeley – de Witt coefficients [55] (we chose such notations when in the holonomic case the scalar curvature is negative for spheres and the space is locally Euclidean). In functionals (23) and (24), we consider dynamical scaling factors of type \( b(2) = \Lambda \exp(\phi) \), when, for instance,

\[ 1^\phi D^2 = e^{-\phi} D^2 e^{\phi} \]

\[ = -\left\{ \frac{1}{2} \phi g^{\alpha\beta} \left[ 1^\phi e_\alpha \, 1^\phi e_\beta + 1^\phi e_\beta \, 1^\phi e_\alpha \right] + 1^\phi A^{\alpha\nu} \, 1^\phi e_\nu + 1^\phi B \right\}, \]

for \( 1^\phi A^{\nu} = e^{-2^\phi} A^{\nu} - 2^\phi g^{\nu\mu} \times 1^\phi e_{\beta} \times 1^\phi (1^\phi) \),

\[ 1^\phi B = e^{-2^\phi} \left( B - A^{\nu} \, 1^\phi e_{\beta} \times (1^\phi) \right) + 1^\phi A^{\nu} \times 1^\phi W^{\gamma}_{\nu\mu} \times 1^\phi e_{\gamma}, \]

for re-scaled d–metric \( 1^\phi g^{\alpha\beta} = e^{2^\phi} g^{\alpha\beta} \) and N–adapted frames \( 1^\phi e_{\alpha} = e^{^\phi} e_{\alpha} \) satisfying anholonomy relations of type (10), with re-scaled non-holonomy coefficients \( 1^\phi W^{\gamma}_{\mu\nu} \). We emphasize that similar formulas can be written by substituting respectively the labels and scaling factors containing \( 2^\phi \) and \( 3^\phi \). For simplicity, we shall omit left labels 1, 2, 3 for \( \phi \) and \( f, a \) if that will not result in ambiguities.

Let us denote by \( ^dR(g_{\mu\nu}) \) and \( C_{\mu\nu\lambda\gamma}(g_{\mu\nu}) \), correspondingly, the scalar curvature [16] and conformal Weyl d–tensor [21].

\[ C_{\mu\nu\lambda\gamma} = R_{\mu\nu\lambda\gamma} + \frac{1}{2} \left( R_{\mu\lambda} g_{\nu\gamma} - R_{\nu\lambda} g_{\mu\gamma} - R_{\mu\gamma} g_{\nu\lambda} + R_{\nu\gamma} g_{\mu\lambda} \right) \]

\[ -\frac{1}{6} \left( g_{\mu\lambda} g_{\nu\gamma} - g_{\nu\lambda} g_{\mu\gamma} \right) \cdot s R, \]

\[ ^{20} \text{similar constructions with dilaton fields are considered in Refs. [18] and [35], but in our case we work with N–anholonomic manifolds, d–metrics and d–connections when instead "dilatons" there are used scaling factors for a corresponding N–adapted Ricci flow model.} \]

\[ ^{21} \text{for any metric compatible d–connection D, the Weyl d–tensor can be computed by formulas similar to those for the Levi–Civita connection V; here we note that if a Weyl d–tensor is zero, in general, the Weyl tensor for V does not vanish (and inversely).} \]
defined by a d–metric $g_{\mu\nu}$ and a metric compatible d–connection $D$ (in our approach, $D$ can be any d–connection constructed in a unique form from $g_{\mu\nu}$ and $N_i^a$ following a well defined geometric principle). For simplicity, we shall work on a four dimensional space and use values of type

$$\int d^4 u \sqrt{|g_{\mu\nu}|} R(e^2 \phi \mathbf{g}_{\mu\nu}) =$$

$$\int d^4 u \sqrt{|g_{\mu\nu}|} R^*(g_{\mu\nu}) =$$

$$\frac{1}{4} \int d^4 u \left( \frac{1}{\sqrt{|g_{\mu\nu}|}} \right)^{-1} \epsilon_{\mu\nu\alpha\beta} \epsilon_{\rho\sigma\gamma\delta} R^{\rho\sigma}_{\mu\nu} R^{\gamma\delta}_{\alpha\beta},$$

for the curvature d–tensor $R^{\rho\sigma}_{\mu\nu}$ (A.4), where sub–integral values are defined by Chern-Gauss–Bonnet terms of type

$$\int d^4 u e^{2\phi} \sqrt{|g_{\mu\nu}|} \left( \frac{1}{4} \sqrt{|g_{\mu\nu}|} \epsilon_{\mu\nu\alpha\beta} \epsilon_{\rho\sigma\gamma\delta} R^{\rho\sigma}_{\mu\nu} R^{\gamma\delta}_{\alpha\beta}.\right.$$
de Witt coefficient is
\[
 a_{(2)}(D^2(\chi)/\Lambda^2) = \frac{\Lambda^2}{16\pi^2} \int_V \delta V \, Tr \left( -\frac{R}{6} \right).
\]
This coefficient can be used for evaluating
\[
 a_{(2)}(\phi D^2(\chi)/\Lambda^2)
\]
following the method of conformal transforms for operators and functionals contained in spectral actions which was developed in sections II and III of Ref. [35]. We have only to perform a similar calculus on N–anholonomic manifolds using d–connections and N–adapted frames/operators.

For a conformally transformed inverse d–metric \( \phi g \), when \( \phi g^{\mu\nu} = e^{-2\phi} g^{\mu\nu} \), \( g^{\mu\nu} \) being inverse to coefficients of \( g_{\mu\nu} \), we have the formula for conformal transform of scalar of curvature \( (A.6) \),
\[
 sR(\phi g) = e^{-2\phi} [ sR(g) + 3 g^{\mu\nu} ( g_{\mu\nu} g_{\rho\sigma} g^{\rho\sigma} g_{\mu\nu} + e_\mu e_\nu + e_\nu e_\mu ) \phi ],
\]
where \( g_{\mu\nu} \) is a metric compatible d–connection completely determined by \( g \). Using the identity
\[
 a_{(2)} \left( u, e^{-\phi} D^2 e^{-\phi} \right) = a_{(2)} \left( u, D^2 e^{-2\phi} \right) = a_{(2)} \left( u, e^{-2\phi} D^2 \right),
\]
which can be verified by straightforward computations with operator \( D^2 \) containing N–adapted derivatives \( e_\mu \), and putting together all terms we get that
\[
 a_{(2)} (\phi D^2(\chi)/\Lambda^2) = \frac{15}{16\pi^2} \int_V \delta V e^{2\phi} \times
\]
\[
 \left[ sR(\phi g) + 3 \phi g^{\alpha\beta} (e_\alpha \phi e_\beta \phi + e_\beta \phi e_\alpha \phi) \right],
\]
i.e. the coefficient before \( f_{(2)} \) in (27).

Finally, we note that generalizing the calculus from [35] for d–connections and N–adapted frames, we can similarly compute the coefficients
\[
 a_{(0)} (\phi D^2(\chi)/\Lambda^2) \quad \text{and} \quad a_{(2)} (\phi D^2(\chi)/\Lambda^2),
\]
for any chosen conformal transform \( \phi \) (in general, with labels \( \beta \) and parameter \( \chi \). Summarizing all necessary terms, we get the approximation (27) . □

Let us state some additional hypotheses which will be used for proofs of the theorems in this section: Hereafter we shall consider a four dimensional compact N–anholonomic manifold \( V \), with volume forms \( \delta V = \sqrt{\det |g_{\mu\nu}|} \delta^4 u \) and normalization \( \int_V \delta V \, \mu = 1 \) for \( \mu = e^{-f(4\pi \chi)^{-\mu}} \) with \( f \) being a scalar function \( f(\chi, u) \) and \( \chi > 0 \).

Now, we are able to formulate the main results of this Paper:
Theorem 4.1 For the scaling factor $\phi = -f/2$, the spectral functional (23) can be approximated $F = P_F(g, D, f)$, where the first Perelman functional (in our case for $N$-anholonomic Ricci flows) is

$$P_F = \int_V \delta V \ e^{-f} \left[ sR(e^{-f} g_{\mu\nu}) + \frac{3}{2} e^f g^{\alpha\beta}(e_{\alpha} f \ e_{\beta} f + e_{\beta} f \ e_{\alpha} f) \right].$$

Proof. We introduce $\phi = -f/2$ into formula (27) from Lemma 4.1. We can rescale the flow parameter $\chi \rightarrow \tilde{\chi}$ such way that $\frac{3}{2} \exp[f(\chi)] = \exp[\tilde{f}(\tilde{\chi})]$. We get that up to a scaling factor $\tilde{f}$ and additional fixing of a new test function to have the coefficients $\tilde{f}(2) = \frac{16\pi^2}{15}$ and $\tilde{f}(0) = \tilde{f}(4) = 0$, computed for "inverse hat" values by choosing necessary values of $f$ and $\tilde{\chi}$ in formulas (25), the value

$$P_F \sim \int_V \delta V \ e^{-\tilde{f}} (sR + |Df|^2)$$

is just the $N$-anholonomic version of the first Perelman functional (formula (30) in Ref. [3]). Taking $D = \nabla$, we get the well-known formula for Ricci flows of Riemannian metrics [2].

One should be noted here that the coefficient $\frac{3}{2}$ was re-scaled by imposing a corresponding nonholonomic constraint on functionals under consideration, which is possible for Ricci flows (such re-definitions of flow parameters were considered in Perelman’s work [2]; additional nonholonomic constraints and evolutions being introduced in [3]). Such "re-scaled" approximations are not possible if we extract certain commutative physical models from noncommutative spectral actions (i.e. not from evolution functionals) like in Refs. [29, 30, 35, 56, 57]. If we fix from the very beginning a Ricci flow parameter $\chi$ (not allowing re-scalings), we have to correct the resulting (non) holonomic Perelman like functionals by introducing certain additional coefficients like $3/2$ etc, which can be interpreted as some contributions from noncommutative geometry for certain evolution models. □

Sketching the proof of the above theorem and further theorems in this section, we can use the techniques elaborated in Refs. [2] but generalized for functionals depending on a flow parameter and performing necessary approximations on $N$-anholonomic manifolds. There are some important remarks.

Remark 4.1 For nonholonomic Ricci flows of (non)commutative geometries, we have to adapt the evolution to certain $N$-connection structures (i.e. nonholonomic constraints). This results in additional possibilities to
re-scale coefficients and parameters in spectral functionals and their commutative limits:

1. The evolution parameter $\chi$, scaling factors $b^f$ and nonholonomic constraints and coordinates can be re-scaled/ redefined (for instance, $\chi \rightarrow \tilde{\chi}$ and $b^f \rightarrow \tilde{b}^f$) such a way that the spectral functionals have limits to some 'standard' nonholonomic versions of Perelman functionals (with prescribed types of coefficients) considered in Ref. [3].

2. Using additional dependencies on $\chi$ and freedom in choosing scaling factors $b^f(\chi)$, we can prescribe such nonholonomic constraints/ configurations on evolution equations (for instance, with $\tilde{f}_2(2) = 16\pi^2/15$ and $\tilde{f}_0(0) = \tilde{f}_4(4) = 0$) when the spectral functionals result exactly in necessary types of effective Perelman functionals (with are commutative, but, in general, nonholonomic).

3. For simplicity, we shall write in brief only $\chi$ and $f$ considering that we have chosen such scales, parametrizations of coordinates and $N$-adapted frames and flow parameters when coefficients in spectral functionals and resulting evolution equations maximally correspond to certain generally accepted commutative physical actions/ functionals.

4. For nonholonomic Ricci flow models (commutative or noncommutative ones) with a fixed evolution parameter $\chi$, we can construct certain effective nonholonomic evolution models with induced noncommutative corrections for coefficients.

5. Deriving effective nonholonomic evolution models from spectral functionals, we can use the technique of "extracting" physical models from spectral actions, elaborated in [29, 30, 35, 56, 57], see also references therein. For commutative and/or noncommutative geometric/ physical models of nonholonomic Ricci flows, we have to generalize the approach to include spectral functionals and $N$-adapted evolution equations depending on the type of nonholonomic constraints, normalizations and re-scalings of constants and effective conformal factors.

We "extract" from the second spectral functional [24] another very important physical value:
**Theorem 4.2** The functional \( \mathcal{W} \) is approximated

\[
\mathcal{W} = \int_V \delta V \mu \times \left[ \chi \left( s \mathbf{R} (e^{-f} \left[ g_{\mu\nu} \right]) + \frac{3}{2} e^f g^{\alpha\beta} (e_\alpha f e_\beta f + e_\beta f e_\alpha f) \right) + f - (n + m) \right],
\]

for scaling \( 2\phi = -f/2 \) in \( 2\mathcal{W} \) and \( 3\phi = (\ln|f - (n + m)| - f)/2 \) in \( 3\mathcal{W} \), from (27).

**Proof.** Let us compute \( \mathcal{W} = 2\mathcal{W} + 3\mathcal{W} \), using formula (27), for \( 2\mathcal{W} \) defined by \( 2\phi = -f/2 \) with \( 2f_{(0)}(\chi) = 2f_{(4)}(\chi) = 0 \) and \( 2f_{(2)}(\chi) = 16\pi^2/[15(4\pi\chi)^{(n+m)/2}] \) and \( 3\mathcal{W} \) defined by \( 3\phi = (\ln|f - (n + m)| - f)/2 \) with \( 3f_{(2)}(\chi) = 3f_{(4)}(\chi) = 0 \) and \( 3f_{(0)}(\chi) = 4\pi^2/[45(4\pi\chi)^{(n+m)/2}] \). The possibility to use parametrizations of scaling factors and imposed types of nonholonomic constraints on evolution functionals follows from Remark 4.1 and, in this case, the approximations are similar to those performed in the proof of Theorem 4.1. After a corresponding redefinition of coordinates, we get

\[
\mathcal{W} \sim \int_V \delta V \mu \left[ \chi (s \mathbf{R} + |Df|^2) + f - (n + m) \right]
\]

which is just the N–anholonomic version of the second Perelman functional (formula (31) in Ref. [3]). Taking \( D = \nabla \), we obtain a formula for Ricci flows of Riemannian metrics [2]. □

The nonholonomic version of Hamilton equations (21) can be derived from commutative Perelman functionals \( \mathcal{F} \) and \( \mathcal{W} \), see Theorems 3.1 and 4.1 in Ref. [3]. The original Hamilton–Perelman Ricci flows constructions can be generated for \( D = \nabla \). The surprising result is that even we start with a Levi–Civita linear connection, the nonholonomic evolution will result almost sure in generalized geometric configurations with various \( \mathbf{N} \) and \( \mathbf{D} \) structures.

### 4.2 Spectral functionals for thermodynamical values

Certain important thermodynamical values such as the average energy and entropy can be derived directly from noncommutative spectral functionals as respective commutative configurations of spectral functionals of type (23) and (24) but with different testing functions than in Theorems 4.1 and 4.2.
Theorem 4.3 Using a scaling factor of type \( \frac{1}{2} \phi = -f/2 \), we extract from the spectral functional (23) a nonholonomic version of average energy, \( \mathcal{F} \rightarrow \mathcal{E} \), where

\[
\langle \mathcal{E} \rangle = -\chi^2 \int \delta V \, \mu \left[ sR(e^{-f}g_{\mu
u}) + \frac{3}{2} e^\alpha g^{\alpha\beta}(e_\alpha f \, e_\beta f + e_\beta f \, e_\alpha f) \right]
\]

if the testing function is chosen to satisfy the conditions \( f(0)(\chi) = 4\pi^2(n + m)\chi/45(4\pi\chi)^{(n+m)/2} \), \( f(2)(\chi) = 16\pi^2\chi^2/15(4\pi\chi)^{(n+m)/2} \) and \( f(4)(\chi) = 0 \).

**Proof.** It is similar to that for Theorem 4.1, but for different coefficients of the testing function. Here, we note that, in general, the statement of this theorem if for a different parametrization of \( \chi \) and \( f \), see point 3 in Remark 4.1. Re-defining coordinates and nonholonomic constraints, we can write (28) in the form

\[
\langle \hat{\mathcal{E}} \rangle \sim -\chi^2 \int \delta V \, \mu \left[ sR + |Df|^2 - \frac{n + m}{2\chi} \right]
\]

which is the N–anholonomic version of average energy from Theorem 4.2 in Ref. [3]). We get the average energy for Ricci flows of Riemannian metrics [2] if \( D = \nabla \). □

Similarly to Theorem 4.2 (inverting the sign of nontrivial coefficients of the testing function) we prove:

**Theorem 4.4** We extract a nonholonomic version of entropy of nonholonomic Ricci flows from the functional (24), \( \mathcal{W} \rightarrow S \), where

\[
S = -\int \delta V \, \mu \times \left[ \chi \left( sR(e^{-f}g_{\mu\nu}) - \frac{3}{2} e^f g^{\alpha\beta}(e_\alpha f \, e_\beta f + e_\beta f \, e_\alpha f) \right) + f - (n + m) \right],
\]

if we introduce \( \delta V = \delta^4 u \) and \( \mu = e^{-f}(4\pi\chi)^{-(n+m)/2} \) into formula (27), for \( \chi > 0 \) and \( \int V \, dV \, \mu = 1 \) in (27), for scaling \( 2\phi = -f/2 \) in \( \mathcal{W} \) and \( 3\phi = (\ln |f - (n + m)| - f)/2 \) in \( \mathcal{W} \), from (24).

**Proof.** This Theorem is a "thermodynamic" analog of Theorem 4.2, in general, with different parameterizations of the evolution parameter and scaling factor (as we noted in points 3–5 of Remark 4.1). We compute \( S = \mathcal{W} + 3\mathcal{W} \), using formula (27), for \( \mathcal{W} \) defined by \( 2\phi = -f/2 \) with
\(2f(0)(\chi) = 2f(4)(\chi) = 0\) and \(2f(2)(\chi) = -16\pi^2/[15(4\pi\chi)^{(n+m)/2}]\) and \(3\mathcal{W}\) defined by \(3\phi = (\ln |f - (n + m)| - f)/2\) with \(3f(2)(\chi) = 3f(4)(\chi) = 0\) and \(3f(0)(\chi) = -4\pi^2/[45 (4\pi\chi)^{(n+m)/2}]\). After corresponding re-parametrization and re-definition of scaling factor and redefinition of N–adapted frames/ nonholonomic constraints, we transform \(\mathcal{S}\) into

\[
\tilde{\mathcal{S}} \sim \int_V \delta V \mu \left[ \chi \left( sR + |Df|^2 \right) + f - (n + m) \right],
\]

i.e. we obtain the N–anholonomic version of Perelman’s entropy, see Theorem 4.2 in Ref. [3]). For \(D = \nabla\), we get the corresponding formula for the entropy Ricci flows of Riemannian metrics [2]. □

We can formulate and prove a Theorem alternative to Theorem 4.3 and get the formula (28) from the spectral functional \(3\mathcal{W} + 3\mathcal{W}\). Such a proof is similar to that for Theorem 4.2 but with corresponding nontrivial coefficients for two testing functions \(2f(\chi)\) and \(3f(\chi)\). The main difference is that for Theorem 4.3 it is enough to use only one testing function. We do not present such computations in this work.

It is not surprising that certain 'commutative' thermodynamical physical values can be derived alternatively from different spectral functionals because such type 'commutative' thermodynamical values can be generated by a partition function

\[
\tilde{\mathcal{Z}} = \exp \left\{ \int_V \delta V \mu \left[ -f + \frac{n + m}{2} \right] \right\}, \tag{29}
\]

associated to any \(Z = \int \exp(-\beta E) d\omega(E)\) being the partition function for a canonical ensemble at temperature \(\beta^{-1}\), which in it turn is defined by the measure taken to be the density of states \(\omega(E)\). In this case, we can compute the average energy, \(<E> = -\partial \log Z/\partial \beta\), the entropy \(S = \beta <E> + \log Z\) and the fluctuation \(\sigma = <(E - <E>)^2 > = \partial^2 \log Z/\partial \beta^2\).

**Remark 4.2** Following a straightforward computation for (29) (similarly to constructions from [2], but following a N–adapted calculus, see Theorem 4.2 in Ref. [3]) we prove that

\[
\tilde{\sigma} = 2\chi^2 \int_V \delta V \mu \left[ R_{ij} + D_i D_j f - \frac{1}{2\chi} g_{ij} \right]^2 + \left[ R_{ab} + D_a D_b f - \frac{1}{2\chi} g_{ab} \right]^2. \tag{30}
\]

\[\text{we emphasize that in this section we follow a different system of denotations for the Ricci flow parameter and normalizing functions}\]
Using formula $R^2_{\mu\nu} = \frac{1}{2} C_{\mu\nu\rho\sigma} - \frac{1}{2} R^{*} R^{\ast} + \frac{1}{3} s R^2$ (it holds true for any metric compatible d–connections, similarly to the formula for the Levi–Civita connection; see, for instance, Ref. [35]), we expect that the formula for fluctuations (30) can be generated directly, by corresponding re-scalings, from a spectral action with nontrivial coefficients for testing functions when $f_{(4)} \neq 0$, see formula (27). Here we note that in the original Perelman’s functionals there were not introduced terms being quadratic on curvature/ Weyl / Ricci tensors. For nonzero $f_{(4)}$, such terms (see Lemma 4.1) may be treated as certain noncommutative / quantum contributions to the classical commutative Ricci flow theory. For simplicity, we omit such considerations in this work.

The framework of Perelman’s functionals and generalizations to corresponding spectral functionals can be positively applied for developing statistical analogies of (non) commutative Ricci flows. For instance, the functional $W$ is the ”opposite sign” entropy, see formulas from Theorems 4.2 and 4.4. Such constructions may be considered for a study of optimal ”topological” configurations and evolution of both commutative and noncommutative geometries and relevant theories of physical interactions.

Here, one should be emphasized that the formalism of Perelman functionals and associated thermodynamical values can not be related directly to similar concepts in black hole physics (as it is discussed in [2,3]) or to quantum mechanical systems as generalized Bost–Connes systems [56, 57]. The approach is not related directly to alternative constructions in geometric and nonequilibrium thermodynamics, locally anisotropic kinetics and stochastic processes [58, 59, 60, 61] for which the nonholonomic geometric methods play an important role. Nevertheless, spectral functional constructions seem to be important for certain noncommutative versions of stochastic processes and kinetics of particles in constrained phase spaces and for noncommutative mechanics models.

5 Discussion and Conclusions

To summarize, we have shown that an extension of the spectral action formalism to spectral functionals with nonholonomic Dirac operators includes naturally the Ricci flow theory and gravitational field equations and various types of generalized geometric configurations modelled by nonholonomic frames and deformations of linear connections. This unification of the spectral triple approach to noncommutative geometry [16, 30, 56] with the Hamilton–Perelman Ricci flow theory [11,2], with certain new applications in
physics, emphasizes new advantages obtained previously following the nonlinear connection formalism and anholonomic frame method, elaborated for standard models of physics in Refs. [4, 5, 12, 40, 50, 62, 51].

We conclude that the paradigm of spectral action and spectral functionals with nonholonomic Dirac operators is a very general one containing various types of locally anisotropic, noncommutative, nonsymmetric space-time geometries and that all the correct features of the standard physical interactions and evolution models are obtained. Such results support the idea that all geometric and physical information about spacetime, physical fields and evolution scenarios can be extracted from a corresponding generalized Dirac operator, and its flows and/or stationary configurations, on appropriated noncommutative spaces.

Let us outline the some important motivations for a systematic approach to noncommutative Ricci flow theory provided by certain directions in modern particle and mathematical physics.

1. The theory of Ricci flows with nonholonomic constraints:

In a series of papers on Ricci flows and exact solutions in gravity, we proved that if the evolution (Hamilton’s) equations are subjected to nonholonomic constraints the Riemannian metrics and connections positively transform into geometric objects defining generalized Lagrange–Finsler, nonsymmetric, noncommutative and various other spaces.

2. The theory of spinors on Riemann–Finsler spaces:

Finsler geometry is not only a straightforward generalization of the concept of Riemannian space to nonlinear metric elements on tangent bundle. There were developed a set of new geometric constructions with nonlinear connection structures and by introducing the concept of nonholonomic manifold. It is well known that the first example of (later called) Finsler metric is contained in the famous B. Riemann thesis from 1856, where, for simplicity, the considerations were restricted only to quadratic forms, see historical remarks and reviews in Refs. [14, 15, 45, 46, 48] and [10, 12], on application of Lagrange–Finsler methods to standard models of physics. But real physical nonlinear phenomena can not be restricted only to quadratic metric elements and linear connections. It was a very difficult task to define spinors and write the Dirac equation on Finsler–Lagrange spaces (and generalizations) working, for instance, with the Cartan–Finsler canonical nonlinear and linear connections, see results outlined in Refs. [36, 37, 39, 40].
Having defined the Dirac–Finsler/Lagrange operators, induced by the canonical distinguished connection, it was not a problem to construct noncommutative versions of spaces with generic local anisotropy (for instance, different models of noncommutative Riemann–Finsler geometry, noncommutative geometric mechanics, the constructions are summarized in the Part III of monograph [12]).

3. String theory and gauge gravity models:

Effective locally anisotropic (super) gravity models were derived in low energy limits of string/M–theory, see Ref. [64, 65]. The so–called absolute anti–symmetric torsion is a ”source” for noncommutative coordinate relations in such theories; on such nonholonomic configurations, see Chapters 13 and 14 in monograph [12]. Here we note that noncommutative gauge gravity models can be generated by applying the Seiberg–Witten transform [20] to gauge theories [34, 5, 25, 22, 21, 23, 26]. Beta functions and renormalization problems in such theories result, in general, in nonholonomic and noncommutative Ricci flow evolution equations.

4. Exact solutions with generic off–diagonal metrics and nonholonomic variables in gravity:

There were constructed and analyzed a number of exact solutions in modern gravity theories, see reviews of results and references in [51, 4, 12], following the idea that considering nonholonomic distributions on a commutative Einstein manifold, defined by nonholonomic moving frames, it is possible to model Finsler like structures and generalizations in Einstein/ string/ gauge ... gravity theory. Such constructions are not for vector/tangent bundles, but for the (pseudo) Riemannian/Einstein and Riemann–Cartan manifolds with local fibered structure. Geometrically, a nonholonomic structure induces a formal torsion even on (pseudo) Riemannian manifolds. In such cases, it is possible to work equivalently both with the Levi–Civita and the Cartan connection, or other metric linear connection structures completely defined by a metric. For the Levi–Civita case, the torsion is zero, but in other cases the effective torsions are induced by certain off-diagonal coefficients of the metric, via nonholonomic deformations. Constructing noncommutative analogs of exact off-diagonal solutions in different models of gravity, one obtains noncommutative models of Finsler geometries and generalizations. We emphasize that the approach can be elaborated for standard commutative and noncommutative mod-
5. Fedosov quantization of Einstein gravity and quantum Lagrange–Finsler spaces:

In a series of recent works, see [50, 49, 62] and discussed there reference, it was proved that the Einstein gravity can be alternatively described in the so-called Finsler-Lagrange and almost Kähler variables (similarly, there are equivalent formulations of the general relativity theory in spinor, tetradic, differential forms, tensorial form etc) and quantized following the methods of deformation quantization. Applying to nonholonomic (pseudo) Riemannian manifolds the geometric technique developed by Fedosov for deformation quantization, we proved that the Einstein, Lagrange–Finsler, Hamilton–Cartan and generalized spaces can be quantized following such methods. Using the corresponding nonholonomic Dirac operators and spinor structures, it is possible to define generalized Finsler like spectral triples and to define noncommutative Fedosov–Einstein, Fedosov–Finsler etc spaces which for corresponding special cases result in already quantized (in the meaning of deformation quantization) geometries and their Ricci flows.

As future directions, it might be worthwhile to pursue the results of this paper for computing noncommutative Ricci flow corrections to physically valuable exact solutions in gravity and elaborating noncommutative versions of quantum gravity models in almost Kähler variables quantized following Fedosov methods.

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A N–adapted Linear Connections

The class of linear connection on a N–anholonomic manifolds splits into two subclasses of those which are adapted or not to a given N–connection structure.

A distinguished connection (d–connection, or N–adapted linear connection) $\mathbf{D}$ on a N–anholonomic manifold $\mathbf{V}$ is a linear connection conserving under parallelism the Whitney sum $\mathbf{5}$. For any d–vector $\mathbf{X}$, there is a
decomposition of $D$ into $h$– and $v$–covariant derivatives,

$$D_X \doteq X[D = hX][D + vX][D = DhX + DvX = hD_X + vD_X,$$

where the symbol "$|$" denotes the interior product. We shall write conventionally that $D = (hD, vD)$, or $D_{\alpha} = (D_i, D_a)$. With respect to $N$–adapted bases $\mathcal{S}$ and $\mathcal{S}$, the local formulas for $d$–connections a parametrized in the form: $D = \{\Gamma^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})\}$, with $hD = (L^i_{jk}, L^a_{bk})$ and $vD = (C^i_{jc}, C^a_{bc})$.

The $N$–adapted components $\Gamma^\alpha_{\beta\gamma}$ of a $d$–connection $D_{\alpha} = (e_{\alpha}|D)$, where the symbol "$|$" denotes the interior product, are computed following equations

$$D_{\alpha}e_\beta = \Gamma^\gamma_{\alpha\beta}e_\gamma,$$

where, by definition, $L^i_{jk} = (D_k e_j)|e_i$, $L^a_{bk} = (D_k e_b)|e^a$, $C^i_{jc} = (D_c e_j)|e^i$, $C^a_{bc} = (D_c e_b)|e^a$ are computed for $N$–adapted frames $\mathcal{S}$ and $\mathcal{S}$.

In the subclass of $d$–connections $D$ on $V$, for standard physical applications, it is convenient to work with $d$–metric compatible $d$–connections (metrical $d$–connections) satisfying the condition $Dg = 0$ including all $h$– and $v$–projections $D_j g_{kl} = 0, D_a g_{kl} = 0, D_j h_{ab} = 0, D_a h_{bc} = 0$.

The torsion of a $d$–connection $D = (hD, vD)$, for any $d$–vectors $X, Y$ is defined by the $d$–tensor field

$$T(X, Y) \doteq D_X Y - D_Y X - [X, Y].$$

One has a $N$–adapted decomposition

$$T(X, Y) = T(hX, hY) + T(hX, vY) + T(vX, hY) + T(vX, vY).$$

The $d$–torsions $hT(hX, hY), vT(vX, vY), ...$ are called respectively the $h (hh)$–torsion, $v (vv)$–torsion and so on.

We can also consider a $N$–adapted differential 1–form $\Gamma^\alpha_{\beta \gamma} = \Gamma^\alpha_{\beta \gamma} e_\gamma$, from which we can compute the torsion $T^\alpha \doteq D e^\alpha = de^\alpha + \Gamma^\alpha_{\beta \gamma} e_\gamma$. Locally, we get the $N$–adapted $d$–torsion coefficients

$$T^i_{jk} = L^i_{jk} - L^i_{kj}, T^i_{ja} = -T^i_{aj} = C^i_{ja}, T^a_{ji} = \Omega^a_{ji},$$

$$T^a_{bi} = -T^a_{ib} = \frac{\partial N^a_{ib}}{\partial y^b} - L^a_{bi}, T^a_{bc} = C^a_{bc} - C^a_{cb}.$$  

(A.3)

The curvature of a $d$–connection $D$ is defined

$$R(X, Y) \doteq D_X D_Y - D_Y D_X - D_{[X, Y]}$$

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for any d–vectors \( X, Y \). By a straightforward d–form calculus, we can find the N–adapted components of the curvature

\[
R^\alpha_{\beta\gamma\delta} = D\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \wedge \Gamma^\gamma_{\delta\alpha} = R^\alpha_{\beta\gamma\delta} e^\gamma \wedge e^\delta, \tag{A.4}
\]
of a d–connection \( D \), i.e. the d–curvatures:

\[
\begin{align*}
R^i_{hjk} &= e_k L^i_{hj} - e_j L^i_{hk} + L^m_{hk} L^i_{mj} - L^m_{hk} L^i_{mj} - C^i_{ha} \Omega^a_{kj}, \\
R^a_{bjk} &= e_k L^a_{bj} - e_j L^a_{bk} + L^c_{bk} L^a_{cj} - L^c_{bk} L^a_{cj} - C^a_{bc} \Omega^c_{kj}, \\
R^i_{jka} &= e_a L^i_{jk} - D_k C^i_{ja} + C^i_{jb} T^b_{ka}, \\
R^c_{bka} &= e_a L^c_{bk} - D_k C^c_{ba} + C^c_{bd} T^d_{ka}, \\
R^i_{jbc} &= e_c C^i_{jb} - e_b C^i_{jc} + C^i_{jh} C^j_{hc} - C^i_{jh} C^j_{hc}, \\
R^a_{bcd} &= e_d C^a_{bc} - e_c C^a_{bd} + C^a_{be} C^b_{ed} - C^a_{be} C^b_{ed}.
\end{align*}
\]

The Ricci tensor \( R_{\alpha\beta} \) is characterized by h– v–components, i.e. d–tensors,

\[
R_{ij} \doteq R^k_{ijk}, \quad R_{ia} \doteq -R^k_{ika}, \quad R_{ai} \doteq R^b_{aib}, \quad R_{ab} \doteq R^c_{abc}. \tag{A.5}
\]
The scalar curvature of a d–connection is

\[
sR \doteq g^{\alpha\beta} R_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} R_{ab}, \tag{A.6}
\]
defined by a sum the h– and v–components of (A.5) and d–metric (7).

For any metric structure \( g \) on a manifold \( V \), there is the unique metric compatible and torsionless Levi–Civita connection \( \nabla \) for which \( \nabla \Gamma^\alpha = 0 \) and \( \nabla g = 0 \). This is not a d–connection because it does not preserve under parallelism the N–connection splitting (5) (it is not adapted to the N–connection structure).

**Theorem A.1** For any d–metric \( g = [hg, vg] \) on a N–anholonomic manifold \( V \), there is a unique metric canonical d–connection \( \hat{\mathbf{D}} \) satisfying the conditions \( \hat{\mathbf{D}} g = 0 \) and with vanishing \( h(hh) \)–torsion, \( v(vv) \)–torsion, i.e. \( h\hat{T}(hX, hY) = 0 \) and \( v\hat{T}(vX, vY) = 0 \).

**Proof.** By straightforward calculations, we can verify that the d–connection with coefficients \( \hat{\Gamma}^\alpha_{\beta\gamma} = (\hat{L}^i_{jk}, \hat{L}^a_{bk}, \hat{C}^i_{jc}, \hat{C}^a_{bc}) \), for

\[
\begin{align*}
\hat{L}^i_{jk} &= \frac{1}{2} g^{ij} (e_k g_{jr} + e_j g_{kr} - e_r g_{jk}), \\
\hat{L}^a_{bk} &= e_b (N^a_k) + \frac{1}{2} h^{ac} \left( e_k h_{bc} - h_{dc} e_b N^d_k - h_{db} e_c N^d_k \right), \\
\hat{C}^i_{jc} &= \frac{1}{2} g^{ik} (e_c g_{jk}, \hat{C}^a_{bc} = \frac{1}{2} h^{ad} (e_c h_{bd} + e_c h_{cd} - e_d h_{bc}).
\end{align*}
\]
satisfies the condition of Theorem. □

In modern classical and quantum gravity theories defined by a (pseudo) Riemannian metric structure $g$ (2), it is preferred to work only with the Levi–Civita connection $\nabla(g) = \{ \Gamma(g) \}$, which is uniquely defined by this metric structure. Nevertheless, for a given $\tilde{N}$–connection splitting $\tilde{N}$ on a nonholonomic manifold $V$, with redefinition of the metric structure in the form $g$ (3), there is an infinite number of metric compatible $\tilde{d}$–connections uniquely defined by $g$ (2), equivalently by $g$ (3) and $\tilde{N}$.

The Levi–Civita linear connection $\nabla = \{ \Gamma^{\alpha}_{\beta\gamma} \}$, uniquely defined by the conditions $\tilde{T} = 0$ and $\nabla g = 0$, is not adapted to the distribution (5). There is an extension of the Levi–Civita connection $\nabla$ to a canonical $\tilde{d}$–connection $\tilde{D} = \{ \tilde{\Gamma}^{\gamma}_{\alpha\beta} \}$ (A.7), which is metric compatible and defined only by a metric $g$ when $\tilde{T}_{jk} = 0$ and $\tilde{T}_{bc} = 0$ but $\tilde{T}_{ja}, \tilde{T}_{ji}$ and $\tilde{T}_{bi}$ are not zero, see (A.3).

A straightforward calculus shows that the coefficients of the Levi–Civita connection can be expressed in the form

$$\Gamma^{\gamma}_{\alpha\beta} = \tilde{\Gamma}^{\gamma}_{\alpha\beta} + Z^{\gamma}_{\alpha\beta}, \quad (A.8)$$

where

$$Z^{i}_{jk} = 0, \quad Z^{a}_{jk} = -C^{a}_{gb} g^{\alpha}_{h} h_{\alpha b}^{i} - \frac{1}{2} \Omega^{a}_{jk}; \quad Z^{i}_{bk} = \frac{1}{2} \delta^{i}_{jk} h_{\alpha b}^{i} + \Xi^{i}_{jk} C^{i}_{h b},$$

$$Z^{a}_{bk} = \Xi^{ab}_{cd} \left[ L^{c}_{bk} - e_{b} (N^{c}_{k}) \right], \quad Z^{i}_{kb} = \frac{1}{2} \delta^{i}_{jk} h_{\alpha b}^{i} + \Xi^{i}_{jk} C^{i}_{h b}, \quad (A.9)$$

$$Z^{a}_{jb} = - \Xi^{ab}_{cb} \left[ L^{c}_{db} - e_{b} (N^{c}_{b}) \right], \quad Z^{i}_{ab} = - \frac{1}{2} \delta^{ij}_{ab} \left[ L^{c}_{ab} + L^{c}_{ba} \right],$$

$$\Xi^{i}_{jk} = \frac{1}{2} \left( \delta^{i}_{jk} h^{i}_{jk} - g_{jk} g^{ih} \right), \quad \Xi^{ab}_{cd} = \frac{1}{2} \left( \delta^{a}_{cd} h^{ab} + h_{cd} h^{ab} \right),$$

for $\Omega^{a}_{jk}$ computed as in formula (6), $\left[ L^{c}_{ab} + L^{c}_{ba} \right]$ and $\left[ \delta^{a}_{cd} h^{ab} + h_{cd} h^{ab} \right]$. It should be emphasized that all components of $\Gamma^{\gamma}_{\alpha\beta}, \tilde{\Gamma}^{\gamma}_{\alpha\beta}$ and $Z^{\gamma}_{\alpha\beta}$ are defined by the coefficients of $\tilde{d}$–metric $g$ (7) and $\tilde{N}$–connection $\tilde{N}$ (3), or equivalently by the coefficients of the corresponding generic off–diagonal metric (3).

For instance, such a principle can be defined by any condition to construct from the given metric coefficients and a $(n + n)$–splitting a unique
d–connection compatible to the canonical almost complex structure (this is the so–called Cartan connection), or admitting a straightforward application of Fedosov quantization in Einstein gravity, or of Finsler–Lagrange geometry. Such constructions were recently developed in order to construct more general classes of exact solutions in gravity [51, 5] (see also Part II in [12]), physical applications of Ricci flow theory [10, 11, 3] and Fedosov quantization of Einstein gravity in almost Kähler and/or Finsler–Lagrange variables [50, 62].

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