Cosmological applications of Padé approximant

Hao Wei, Xiao-Peng Yan and Ya-Nan Zhou

School of Physics, Beijing Institute of Technology, Beijing 100081, China
E-mail: haowei@bit.edu.cn, 764644314@qq.com, 675346557@qq.com

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Abstract. As is well known, in mathematics, any function could be approximated by the Padé approximant. The Padé approximant is the best approximation of a function by a rational function of given order. In fact, the Padé approximant often gives better approximation of the function than truncating its Taylor series, and it may still work where the Taylor series does not converge. In the present work, we consider the Padé approximant in two issues. First, we obtain the analytical approximation of the luminosity distance for the flat XCDM model, and find that the relative error is fairly small. Second, we propose several parameterizations for the equation-of-state parameter (EoS) of dark energy based on the Padé approximant. They are well motivated from the mathematical and physical points of view. We confront these EoS parameterizations with the latest observational data, and find that they can work well. In these practices, we show that the Padé approximant could be an useful tool in cosmology, and it deserves further investigation.

Keywords: cosmology of theories beyond the SM, dark energy theory, dark energy experiments

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1 Introduction

Since the great discovery of the current accelerated expansion of our universe, dark energy has become one of the most active fields in physics and astronomy [1–7]. As is well known, Type Ia supernovae (SNIa) have been considered as a powerful probe to investigate this mysterious phenomenon [1–9]. We can constrain cosmological models by comparing their theoretical luminosity distances (or distance modulus equivalently) with the observational ones of SNIa. Therefore, computing the luminosity distance is fairly important in dark energy cosmology. On the other hand, it is also well known that the equation-of-state parameter (EoS) plays an important role in cosmology. The evolution of energy density of dark energy mainly depends on its EoS. Determining EoS of dark energy is one of the key tasks in cosmology [1–7].

As is well known, any function $f(x)$ can be approximated by the Taylor series expansion, namely, $f(x) = f(x_0) + f_1(x-x_0) + f_2(x-x_0)^2 + \cdots + f_n(x-x_0)^n$. Taylor expansion has been extensively used in many fields of physics. The so-called Padé approximant can be regarded as a generalization of Taylor polynomial [10–17]. In mathematics, a Padé approximant is the best approximation of a function by a rational function of given order [13]. In fact, the Padé approximant often gives better approximation of the function than truncating its Taylor series, and it may still work where the Taylor series does not converge [13]. For any function $f(x)$, its corresponding Padé approximant of order $(m, n)$ is given by the rational function $[10–17]$

$$f(x) = \frac{\alpha_0 + \alpha_1 x + \cdots + \alpha_m x^m}{1 + \beta_1 x + \cdots + \beta_n x^n},$$

where $m$ and $n$ are both non-negative integers; $\alpha_i$ and $\beta_i$ are all constants. Obviously, it reduces to the Taylor polynomial when all $\beta_i = 0$.

In the present work, we are interested to study the luminosity distance and EoS of dark energy by using the Padé approximant. In fact, computing the luminosity distance involves repeated numerical integral, and hence consumes a large amount of time and computing power, especially in the massive computation. So, to have a rapid computation, it is desirable to find an analytical approximation of the luminosity distance, in place of integral. In section 2, we will discuss this issue by the help of Padé approximant. On the other hand, although there are many parameterizations for the EoS of dark energy in the literature, most
of them are \textit{ad hoc} and purely written by hand. Motivated by the Padé approximant, in
section 3 we propose two types of Padé parameterizations for the EoS of dark energy, and
constrain them by using the latest observational data. Finally, some discussions are given in
section 4.

2 Padé analytical approximation of the luminosity distance

2.1 Status of the art

As mentioned above, computing the luminosity distance involves repeated numerical integral,
or elliptic functions [18]. In order to accelerate a massive computation (e.g. Monte Carlo
simulation), an analytical approximation of the luminosity distance is desirable. To our
knowledge, in 1999, Pen [19] obtained the first analytical approximation of the luminosity
distance for the flat ΛCDM model, namely
\[ d_L = \frac{c}{H_0} (1 + z) \left[ \eta(1, \Omega_{m0}) - \eta\left(\frac{1}{1+z}, \Omega_{m0}\right) \right], \tag{2.1} \]
where
\[ \eta(a, \Omega_{m0}) = 2 \sqrt{s^3 + 1} \left[ \frac{1}{a^4} - 0.1540 \frac{s}{a^3} + 0.4304 \frac{s^2}{a^2} + 0.19097 \frac{s^3}{a} + 0.066941 s^4 \right]^{-1/8}, \tag{2.2} \]
\[ s^3 = \frac{1 - \Omega_{m0}}{\Omega_{m0}}, \tag{2.3} \]
and \( \Omega_{m0} \) is the present fractional energy density of the pressureless matter; \( c \) is the speed
of light; \( H_0 \) is the Hubble constant; \( z \) is the redshift; \( a = (1 + z)^{-1} \) is the scale factor (we
have set \( a_0 = 1 \); the subscript “0” indicates the present value of corresponding quantity).
It is claimed that this formula has a relative error of less than 0.4\% for \( 0.2 < \Omega_{m0} < 1 \) for
any redshift, and a global relative error of less than 4\% for any choice of parameters [19].

More than ten years passed, and dark energy cosmology has been developed significantly. In
the recent years, the repeated computation of the luminosity distance has become more and
more massive, while the observational data accumulated significantly. Wickramasinghe and
Ukwatta [20] found a new analytical approximation of the luminosity distance for the flat
ΛCDM model, namely
\[ d_L = \frac{c}{H_0} \frac{1 + z}{(1 - \Omega_{m0})^{1/6} \Omega_{m0}^{1/3}} \left[ \Psi(x(0, \Omega_{m0})) - \Psi(x(z, \Omega_{m0})) \right], \tag{2.4} \]
where
\[ \Psi(x) = 3x^{1/3}2^{2/3} \left( 1 - \frac{x^2}{252} + \frac{x^4}{21060} \right), \tag{2.5} \]
\[ x(z, \Omega_{m0}) = \ln \left( \alpha + \sqrt{\alpha^2 - 1} \right), \quad \alpha(z, \Omega_{m0}) = 1 + \frac{1 - \Omega_{m0}}{\Omega_{m0}} \frac{2}{(1 + z)^3}. \tag{2.6} \]
They claimed that this formula has a relative error smaller than the one of Pen [19]. Then,
Adachi and Kasai [16] found another analytical approximation of the luminosity distance for
the flat ΛCDM model, namely
\[ d_L = \frac{2c}{H_0 \sqrt{\Omega_{m0}}} \left[ \Phi(x(0, \Omega_{m0})) - \frac{1}{\sqrt{1 + z}} \Phi(x(z, \Omega_{m0})) \right], \tag{2.7} \]
where
\[ \Phi(x) = \frac{1 + 1.320x + 0.4415x^2 + 0.02656x^3}{1 + 1.392x + 0.5121x^2 + 0.03944x^3}, \]
\[ x(z, \Omega_{m0}) = 1 - \frac{\Omega_{m0}}{(1 + z)^3}. \]  
(2.8)

They claimed that for a wide range of \( \Omega_{m0} \) and redshift \( z \), this formula has a relative error even smaller than the one of Wickramasinghe and Ukwatta [20]. It is worth noting that Zhang et al. [21, 22] also discussed the computation of the luminosity distance. However, they did not obtain the analytical approximation of the luminosity distance. Instead, they considered the numerical algorithms to compute the elliptic integrals of the luminosity distance. So, the works of Zhang et al. [21, 22] are not tightly relevant.

2.2 Analytical approximation of the luminosity distance for the flat XCDM model

To our knowledge, the relevant works in the literature concentrated on the flat ΛCDM model, namely the role of dark energy is played by a cosmological constant (its EoS \( w_{de} = -1 \) exactly). However, dynamical dark energy with \( w_{de} \neq -1 \) is extensively considered in cosmology [1–7]. So, it is of interest to find also an analytical approximation of the luminosity distance for dynamical dark energy models. In the present work, we would like to consider a flat XCDM model and find an analytical approximation of the luminosity distance in this case.

The flat XCDM model describes a flat Friedmann-Robertson-Walker (FRW) universe containing only pressureless matter and dark energy with \( w_{de} = w_X = \text{const.} \). To accelerate the cosmic expansion, \( w_X < 0 \) is required. By definition, the luminosity distance reads (see e.g. [16, 19, 20, 23–38])
\[ d_L \equiv c \frac{(1 + z)}{H(z)} = \frac{c}{H_0} (1 + z) \int_a^1 \frac{d\tilde{a}}{\tilde{a}^2 E(\tilde{a})}, \]  
(2.9)
where \( E \equiv H/H_0 \) and \( H \equiv \dot{a}/a \) is the Hubble parameter (a dot denotes the derivative with respect to cosmic time \( t \)). For convenience, we introduce a new function
\[ \Psi \equiv \frac{1}{2} \int_0^a \frac{\Omega_{m0}^{1/2} d\tilde{a}}{\tilde{a}^2 E(\tilde{a})}. \]  
(2.10)

Using this function, the luminosity distance can be recast as
\[ d_L = \frac{2c}{H_0 \Omega_{m0}^{1/2}} \left[ \Psi(a = 1) - \Psi(a) \right]. \]  
(2.11)
For the flat XCDM model, the corresponding \( E = H/H_0 \) reads (see e.g. [23–38])
\[ E = \left[ \Omega_{m0} a^{-3} + (1 - \Omega_{m0}) a^{-3(1 + w_X)} \right]^{1/2} = \left[ \Omega_{m0}(1 + z)^3 + (1 - \Omega_{m0}) (1 + z)^3(1 + w_X) \right]^{1/2}. \]  
(2.12)
Substituting it into eq. (2.10), we obtain
\[ \Psi = \frac{1}{2} \int_0^a \frac{1}{\sqrt{a} \sqrt[3]{1 + s a^{-3 w_X}}} \frac{d\tilde{a}}{\tilde{a}} = \int_0^x \left( -\frac{1}{6w_X} \right) \left( \frac{\tilde{x}}{s} \right)^{-1/(6w_X)} \frac{d\tilde{x}}{\tilde{x} \sqrt[3]{1 + \tilde{x}}}, \]  
(2.13)
where

\[ x = s a^{-3w_X} = s(1 + z)^{3w_X}, \quad s \equiv \frac{1 - \Omega_{m0}}{\Omega_{m0}}. \]  

(2.14)

Considering the Padé approximant of \( \Psi(x) \) up to order (3, 3), and noting \( a = (x/s)^{-1/(3w_X)} \), we have

\[ \Psi = \sqrt{a} \cdot \frac{1 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3}{1 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3}, \]  

(2.15)

where the constant coefficients are given by

\[ \alpha_1 = 80 \xi^{-1}(-1 + 12w_X)(-1 + 18w_X)(-1 + 24w_X)(-1 + 30w_X)(1 + 36w_X(-4 + 3w_X(107 + 4w_X(-1216 + 9w_X(3805 + 12w_X(-5888 + 3w_X(25945 + 48w_X(-4469 + 16947w_X))))))))), \]

\[ \alpha_2 = 24 \xi^{-1}(-1 + 12w_X)(-1 + 24w_X)(1 + 36w_X(-5 + w_X(491 + 36w_X(-800 + 9w_X(3383 + 4w_X(-22237 + w_X(427301 + 72w_X(-82459 + 8w_X(97171 + 18w_X(-31315 + 99918w_X)))))))))), \]

\[ \alpha_3 = \xi^{-1}(1 + 72w_X(-3 + 2w_X(175 + 6w_X(-2098 + 3w_X(33719 + 24w_X(-48254 + 3w_X(413179 + 3w_X(-2674436 + 3w_X(12787417 + 72w_X(-1828153 + 16w_X(837203 + 9w_X(-495569 + 2012094w_X))))))))))), \]

\[ \beta_1 = 112 \xi^{-1}(1 - 6w_X)(-1 + 12w_X)(-1 + 24w_X)(-1 + 30w_X) \times (1 + 24w_X(-5 + 6w_X(58 + 3w_X(-688 + 3w_X(4687 + 48w_X(-1243 + 12105w_X)))))), \]

\[ \beta_2 = 56 \xi^{-1}(-1 + 6w_X)(-1 + 24w_X)(1 + 6w_X(-5 + 36w_X))^2 \times (1 + 12w_X(-11 + 18w_X(47 + 2w_X(-925 + 3w_X(6989 + 648w_X(-151 + 1586w_X)))))), \]

\[ \beta_3 = 7 \xi^{-1}(1 - 18w_X)^2(1 - 6w_X)^2 \times (1 + 72w_X(-2 + 3w_X(57 + 2w_X(-1252 + 9w_X(3647 + 144w_X(-421 + 5323w_X)))))), \]  

(2.16)

in which

\[ \xi \equiv 64(-1 + 6w_X)(-1 + 12w_X)(-1 + 18w_X)(-1 + 24w_X)(-1 + 30w_X)(-1 + 36w_X) \times (1 + 108w_X(-1 + 4w_X(16 + w_X(-521 + 9w_X(1105 + 24w_X(555 + 1586w_X))))))). \]  

(2.17)

Although the constant coefficients in eqs. (2.16) and (2.17) look awesome, it is not a problem when one calculate the luminosity distance using computer (see the discussions in section 4). It is easy to check that if \( w_X = -1 \), our results can reduce to the one of Adachi and Kasai [16], namely eqs. (2.7) and (2.8).

Since our analytical approximation of the luminosity distance for the flat XCDM model is obtained for the first time (to our knowledge), we can only compare it with the exact integral in eq. (2.9). The relative error reads

\[ \Delta \equiv \frac{d_L^{\text{Padé}} - d_L^{\text{int}}}{d_L^{\text{int}}}, \]  

(2.18)

where \( d_L^{\text{Padé}} \) is calculated using eq. (2.11), while \( \Psi \) is given in eq. (2.15); \( d_L^{\text{int}} \) is calculated using eq. (2.9), while \( E \) is given in eq. (2.12). We present the 3D plots of the relative error \( \Delta \) as a function of \( w_X \), \( \Omega_{m0} \) and redshift \( z \) in figures 1 and 2. In figure 1, we fixed \( \Omega_{m0} = 0.3 \).
Figure 1. The 3D plot of the relative error $\Delta$ as a function of $w_X$ and redshift $z$ while $\Omega_{m0} = 0.3$.

In the whole history $0 \leq z < \infty$, for any EoS of dark energy in the range $-1.5 \leq w_X \leq -0.5$, the relative error $\Delta$ is always smaller than 0.34%. The larger $w_X$, the smaller relative error $\Delta$ is. It is easy to see that the relative error $\Delta$ is not sensitive to the EoS of dark energy $w_X$ in fact. The relative error $\Delta$ increases only when $z \leq 1$. To see clearly, in figure 3, we present the 2D plots of $\Delta$ for $z \geq 1$. From the left panel of figure 3, for a fixed $\Omega_{m0} = 0.3$ and any EoS of dark energy in the range $-1.5 \leq w_X \leq -0.5$, the relative error $\Delta$ is always smaller than 0.06% for redshift $z \geq 1$. For $w_X$ around $-1$, the relative error $\Delta$ is always smaller than 0.03%. For high redshift, the relative error $\Delta$ is smaller than 0.008%. On the other hand, in figure 2, we fixed $w_X = -0.95$ instead. The relative error $\Delta$ is sensitive to $\Omega_{m0}$ when it is smaller than 0.25. In the whole history $0 \leq z < \infty$, for any $\Omega_{m0}$ in the range $0.2 \leq \Omega_{m0} \leq 0.4$, the relative error $\Delta$ is always smaller than 2%. The smaller $\Omega_{m0}$, the larger relative error $\Delta$ is. Fortunately, the latest Planck 2013 data [39] favors a large $\Omega_{m0} \sim 0.315$. For this large $\Omega_{m0}$, the relative error $\Delta$ is always smaller than 0.3% in the whole history $0 \leq z < \infty$. From the right panel of figure 3, for a fixed $w_X = -0.95$ and any $\Omega_{m0}$ in the range $0.2 \leq \Omega_{m0} \leq 0.4$, the relative error $\Delta$ is always smaller than 0.27% for redshift $z \geq 1$. For $\Omega_{m0}$ around 0.3, the relative error $\Delta$ is always smaller than 0.03%. For high redshift, the relative error $\Delta$ is smaller than 0.02%.

3 Padé parameterizations for the EoS of dark energy

3.1 Various EoS parameterizations in the literature

Now, we turn to another issue, namely the parameterizations for the EoS of dark energy. Today, there are many EoS parameterizations in the literature. In the early researches, the
Figure 2. The 3D plot of the relative error $\Delta$ as a function of $\Omega_{m0}$ and redshift $z$ while $w_X = -0.95$.

Figure 3. The 2D plot of the relative error $\Delta$ for redshift $z \geq 1$. In the left panel, $\Omega_{m0} = 0.3$ is fixed, while the lines from top to bottom correspond to $w_X = -1.5$, $-1$, $-0.5$, respectively. In the right panel, $w_X = -0.95$ is fixed, while the lines from top to bottom correspond to $\Omega_{m0} = 0.2$, $0.25$, $0.3$, $0.4$, respectively.
popular parameterization is given by [40, 41]

\[ w_{\text{de}} = w_0 + w_1 z. \]  

(3.1)

Its generalized parameterization is \( w_{\text{de}} = w_0 + w_1 z + w_2 z^2 \) [42]. They can be regarded as the Taylor series expansion of \( w_{\text{de}} \) with respect to redshift \( z \) up to first or second order. However, these two parameterizations cannot work well when redshift \( z \) is high. So, they have been soon replaced by the well-known Chevallier-Polarski-Linder (CPL) parameterization [43, 44]

\[ w_{\text{de}} = w_0 + w_a (1 - a) = w_0 + w_a \frac{z}{1 + z}. \]  

(3.2)

Its generalized parameterization is \( w_{\text{de}} = w_0 + w_a (1 - a) + w_b (1 - a)^2 \) [45]. They can be regarded as the Taylor series expansion of \( w_{\text{de}} \) with respect to \((1 - a)\) (or scale factor \( a \)) up to first or second order. In the passed ten years, the CPL parameterization is the most popular one and has been extensively used in the literature. Although the CPL parameterization dominated most works, there are still many exotic parameterizations in the literature. For instance, in e.g. [46, 47], the following parameterization has been proposed, namely

\[ w_{\text{de}} = w_0 + w_1 \frac{z}{(1 + z)^\alpha}, \]  

(3.3)

where \( \alpha \) usually was taken to be 2. An interesting parameterization was considered in e.g. [48, 49], i.e.,

\[ w_{\text{de}} = w_0 + w_1 \frac{1 - a^\beta}{\beta}. \]  

(3.4)

The logarithm parameterization [50, 51],

\[ w_{\text{de}} = w_0 + w_1 \ln a, \]  

(3.5)

can be seen in many works in the literature. Another logarithm parameterization reads [52]

\[ w_{\text{de}} = w_0 \left[ 1 + b \ln(1 + z) \right]^\alpha, \]  

(3.6)

where \( \alpha \) usually was taken to be 1 or 2.

Although the CPL parameterization works well, it will diverge when \( a \to \infty \) (or equivalently \( z \to -1 \)). This is also a common feature of many existing parameterizations. To overcome this divergence, several ad hoc parameterizations have been proposed in the literature. For example, almost six years ago the following parameterization [53–55] has been proposed to avoid the divergence at \( z \to -1 \), namely

\[ w_{\text{de}} = w_0 + w_1 \frac{z (1 + z)}{1 + z^2}. \]  

(3.7)

Only one month later, a similar parameterization [56] has been proposed, i.e.,

\[ w_{\text{de}} = \frac{w_0}{1 + (w_1 z)^2}. \]  

(3.8)

Recently, this issue regained attention. In [57, 58], the following parameterization was proposed to avoid the divergence at \( z \to -1 \), namely

\[ w_{\text{de}} = w_0 + w_1 \left[ \frac{\ln(2 + z)}{1 + z} - \ln 2 \right]. \]  

(3.9)
Two parameterizations similar to the one in eq. (3.7) were considered in [59], i.e.,

\[ w_{de} = w_0 + w_1 \frac{z}{1 + z^2}, \quad w_{de} = w_0 + w_1 \frac{z^2}{1 + z^2}. \]  

(3.10)

Actually, in the literature, there are other \textit{ad hoc} parameterizations to this end. We finally mention two of them [60], namely

\[ w_{de} = w_0 + w_1 \frac{z}{1 + z^2}, \quad w_{de} = w_0 + w_1 \frac{1 - a}{1 + a}. \]  

(3.11)

We refer to e.g. [1–7, 61] and the references therein for other exotic parameterizations.

3.2 Type (I) \textit{Pad\`e} parameterization

As mentioned above, most of the existing EoS parameterizations are \textit{ad hoc} and purely written by hand. In particular, all the parameterizations without the divergence at \( z \to -1 \) (\( a \to \infty \)) are not well motivated from mathematics or fundamental physics. So, the well-motivated parameterizations are still welcome. As mentioned in section 1, in mathematics, any function could be approximated by the \textit{Pad\`e} approximant. The \textit{Pad\`e} approximant is the best approximation of a function by a rational function of given order [13]. In fact, the \textit{Pad\`e} approximant often gives better approximation of the function than truncating its Taylor series, and it may still work where the Taylor series does not converge [13]. This fact motivates us to propose several novel EoS parameterizations here.

Firstly, we consider the type (I) \textit{Pad\`e} parameterization,

\[ w_{de} = \frac{w_0 + wa(1 - a)}{1 + wb(1 - a)}, \]  

(3.12)

where \( w_0, w_a \) and \( w_b \) are all constants. In fact, it is the \textit{Pad\`e} approximant of \( w_{de} \) with respect to \( (1 - a) \) (or scale factor \( a \)) up to order \( (1, 1) \). Noting that it can be recast as

\[ w_{de} = \frac{w_0 + (w_0 + w_a)z}{1 + (1 + w_b)z}, \]  

(3.13)

it is also the \textit{Pad\`e} approximant of \( w_{de} \) with respect to redshift \( z \) up to order \( (1, 1) \). It is worth noting that if \( w_b = 0 \), our type (I) \textit{Pad\`e} parameterization (3.12) or (3.13) can reduce to the well-known CPL parameterization (3.2). If \( w_b \neq 0 \), our type (I) \textit{Pad\`e} parameterization can avoid the divergence at \( a \to \infty \) (or \( z \to -1 \) equivalently), unlike the CPL parameterization. In fact, it is easy to see that

\[ w_{de} = \begin{cases} \frac{w_0 + wa}{1 + w_b}, & \text{for } a \to 0 \ (z \to \infty, \ \text{the early time}), \\ w_0, & \text{for } a = 1 \ (z = 0, \ \text{now}), \\ \frac{wa}{w_b}, & \text{for } a \to \infty \ (z \to -1, \ \text{the far future}), \end{cases} \]  

(3.14)

where \( w_b \neq 0 \) and \( w_b \neq -1 \) are required. In addition, to avoid the denominator in eq. (3.12) being zero for any physical scale factor \( a \geq 0 \), we require that

\[ -1 < w_b < 0. \]  

(3.15)
So, the denominator in eq. (3.12) is always positive. Under the condition in eq. (3.15), our type (I) Padé parameterization is always regular for the whole $0 \leq a < \infty$ (or $-1 \leq z < \infty$ equivalently). Note that $w_{de}$ can cross the so-called phantom divide $w_{de} = -1$ at

$$a_* = 1 + \frac{1 + w_0}{w_a + w_b},$$

or equivalently,

$$z_* = \frac{-1 - w_0}{1 + w_0 + w_a + w_b}.$$  \hspace{1cm} (3.16)

Naturally, it is important to confront our Padé parameterization with the latest observational data. The Union2.1 compilation [8, 9] is the largest published and spectroscopically confirmed Type Ia supernovae (SNIa) sample to date. The 580 data points of Union2.1 SNIa compilation are given in terms of the distance modulus $\mu_{\text{obs}}(z_i)$. On the other hand, the theoretical distance modulus is defined by

$$\mu_{\text{th}}(z_i) \equiv 5 \log_{10} D_L(z_i) + \mu_0,$$  \hspace{1cm} (3.17)

where $\mu_0 \equiv 42.38 - 5 \log_{10} h$ and $h$ is the Hubble constant $H_0$ in units of 100 km/s/Mpc, while

$$D_L(z) = (1 + z) \int_0^z \frac{dz}{E(z; \textbf{p})},$$  \hspace{1cm} (3.18)

in which $\textbf{p}$ denotes the model parameters, and $E \equiv H/H_0$. Correspondingly, the $\chi^2$ from 580 Union2.1 SNIa is given by

$$\chi^2_{\mu}(\textbf{p}) = \sum_i \frac{[\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i; \mu_0 = 0, \textbf{p})]^2}{\sigma^2_{\mu_{\text{obs}}}(z_i)},$$  \hspace{1cm} (3.19)

where $\sigma$ is the corresponding $1\sigma$ error. The parameter $\mu_0$ is a nuisance parameter but it is independent of the data points. One can perform a uniform marginalization over $\mu_0$. However, there is an alternative way. Following [62–64], the minimization with respect to $\mu_0$ can be made by expanding the $\chi^2_{\mu}$ of eq. (3.19) with respect to $\mu_0$ as

$$\chi^2_{\mu}(\textbf{p}) = \tilde{A} - 2 \mu_0 \tilde{B} + \mu_0^2 \tilde{C},$$  \hspace{1cm} (3.20)

where

$$\tilde{A}(\textbf{p}) = \sum_i \frac{[\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i; \mu_0 = 0, \textbf{p})]^2}{\sigma^2_{\mu_{\text{obs}}}(z_i)},$$

$$\tilde{B}(\textbf{p}) = \sum_i \frac{\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i; \mu_0 = 0, \textbf{p})}{\sigma^2_{\mu_{\text{obs}}}(z_i)},$$

$$\tilde{C} = \sum_i \frac{1}{\sigma^2_{\mu_{\text{obs}}}(z_i)}.$$  

Eq. (3.20) has a minimum for $\mu_0 = \tilde{B}/\tilde{C}$ at

$$\tilde{\chi}^2_{\mu}(\textbf{p}) = \tilde{A}(\textbf{p}) - \frac{\tilde{B}(\textbf{p})^2}{\tilde{C}}.$$  \hspace{1cm} (3.21)

Since $\tilde{\chi}^2_{\mu,\text{min}} = \chi^2_{\mu,\text{min}}$ (up to a constant) obviously, we can instead minimize $\tilde{\chi}^2$ which is independent of $\mu_0$. In addition to SNIa, the other useful observations include the cosmic microwave background (CMB) anisotropy [39] and the large-scale structure (LSS) [45, 65–67]. However, using the full data of CMB and LSS to perform a global fitting consumes a large amount of computation time and power. As an alternative, one can instead use the shift.
parameter $R$ from CMB, and the distance parameter $A$ of the measurement of the baryon acoustic oscillation (BAO) peak in the distribution of SDSS luminous red galaxies. In the literature, the shift parameter $R$ and the distance parameter $A$ have been used extensively. It is argued in e.g. [68] that they are model-independent and contain the main information of the observations of CMB and BAO, respectively. As is well known, the shift parameter $R$ of CMB is defined by [68, 69]

$$R \equiv \Omega_{m0}^{1/2} \int_{0}^{z_{\text{rec}}} \frac{d\bar{z}}{E(\bar{z})},$$  \hspace{1cm} (3.22)$$

where the redshift of recombination $z_{\text{rec}} = 1090.48$ which was determined by the latest Planck 2013 data [39], and $\Omega_{m0} \equiv 8\pi G \rho_{m0}/(3H_{0}^{2})$ is the present fractional density of pressureless matter. The value of $R$ has been determined to be $1.7407 \pm 0.0094$ from the Planck 2013 data [70] (note that in the original Planck paper [39] the value of $R$ has not been given directly, and it was obtained in [70] later by other authors using the Planck 2013 data [39]). On the other hand, the distance parameter $A$ of the measurement of the BAO peak in the distribution of SDSS luminous red galaxies [45, 65–67] is given by

$$A \equiv \Omega_{m0}^{1/2} E(z_{b})^{-1/3} \left[ \frac{1}{z_{b}} \int_{0}^{z_{b}} \frac{d\bar{z}}{E(\bar{z})} \right]^{2/3},$$  \hspace{1cm} (3.23)$$

where $z_{b} = 0.35$. In [71], the value of $A$ has been determined to be $0.469 (n_{s}/0.98)^{-0.35} \pm 0.017$. Here the scalar spectral index $n_{s}$ is taken to be $0.9662$, which comes from the Planck 2013 data [70]. So, the total $\chi^{2}$ is given by

$$\chi^{2} = \chi_{\mu}^{2} + \chi_{\text{CMB}}^{2} + \chi_{\text{BAO}}^{2},$$  \hspace{1cm} (3.24)$$

where $\chi_{\mu}^{2}$ is given in eq. (3.21), $\chi_{\text{CMB}}^{2} = (R - R_{\text{obs}})^{2}/\sigma_{R}^{2}$ and $\chi_{\text{BAO}}^{2} = (A - A_{\text{obs}})^{2}/\sigma_{A}^{2}$. The best-fit model parameters are determined by minimizing the total $\chi^{2}$. As in [23–38, 62, 63], the 68.3% confidence level is determined by $\Delta \chi^{2} \equiv \chi^{2} - \chi^{2}_{\text{min}} \leq 1.0, 2.3, 3.53, 4.72$ for $n_{p} = 1, 2, 3, 4$ respectively, where $n_{p}$ is the number of free model parameters. Similarly, the 95.4% confidence level is determined by $\Delta \chi^{2} \equiv \chi^{2} - \chi^{2}_{\text{min}} \leq 4.0, 6.18, 8.02, 9.72$ for $n_{p} = 1, 2, 3, 4$ respectively.

Now, let us come back to our type (I) Padé parameterization, namely eq. (3.12). Substituting this $w_{de}$ into the energy conservation equation $\dot{\rho}_{de} + 3H \rho_{de} (1 + w_{de}) = 0$, we find that

$$\rho_{de} = \rho_{de,0} a^{-3(1+w_{0}+w_{a}+w_{b})/(1+w_{b})} \left[ 1 + w_{b}(1-a) \right]^{-3(w_{a}-w_{0}w_{b})/[w_{a}(1+w_{b})]}.$$  \hspace{1cm} (3.25)$$

Note that $w_{b} \neq 0$ and $w_{b} \neq -1$ as mentioned above. In this work, we consider a flat FRW universe containing only pressureless matter and dark energy. Substituting eq. (3.25) into Friedmann equation, we finally obtain

$$E^{2} = \Omega_{m0}(1+z)^{3} + (1 - \Omega_{m0}) (1+z)^{3}(1+w_{0}+w_{a}+w_{b})/(1+w_{b}) \left( 1 + \frac{w_{b} z}{1+z} \right)^{-3(w_{a}-w_{0}w_{b})/[w_{a}(1+w_{b})]}.$$  \hspace{1cm} (3.26)$$

There are four free parameters in this model, namely $\Omega_{m0}$, $w_{0}$, $w_{a}$ and $w_{b}$. Note that $-1 < w_{b} < 0$ is required in eq. (3.15). By minimizing the corresponding total $\chi^{2}$ in eq. (3.24), we find the best-fit parameters $\Omega_{m0} = 0.280$, $w_{0} = -0.995$, $w_{a} = -0.020$, and $w_{b} = -0.052$, while $\chi^{2}_{\text{min}} = 562.256$. In figure 4, we present the 68.3% and 95.4% confidence level contours in the $w_{0} - w_{a}$, $w_{0} - w_{b}$, $w_{a} - w_{b}$, $\Omega_{m0} - w_{0}$, $\Omega_{m0} - w_{a}$, and $\Omega_{m0} - w_{b}$ planes for the type (I)
Figure 4. The 68.3% and 95.4% confidence level contours in the $w_0 - w_a$, $w_0 - w_b$, $w_a - w_b$, $\Omega_{m0} - w_0$, $\Omega_{m0} - w_a$, and $\Omega_{m0} - w_b$ planes for the type (I) Padé parameterization in eq. (3.12). The best-fit parameters are also indicated by the black solid points.
Padé parameterization in eq. (3.12). Noting eqs. (3.14) and (3.16), for the best-fit parameters, \( w_{de} = -1.071 \) in the very beginning \((a \to 0, z \to \infty)\), and then crossed the phantom divide at \( a_s = 0.933 \) \((z_s = 0.071)\); today \((a = 1, z = 0)\), we have \( w_{de} = -0.995\); in the far future \((a \to \infty, z \to -1)\), \( w_{de} = 0.393\). We plot this \( w_{de} \) as a function of redshift \( z \) in the left panel of figure 5, and we can easily see that \( w_{de} \) crossed the phantom divide \( w_{de} = -1 \). Obviously, the type (I) Padé parameterization works very well in fact.

Note that the Padé parameterization (I) in eq. (3.12) has three free parameters, namely \( w_0, w_a \) and \( w_b \). We can simplify it by setting \( w_a = 0 \). The simplified parameterization (I) reads

\[
 w_{de} = \frac{w_0}{1 + w_b (1 - a)},
\]

which has only two free parameters \( w_0 \) and \( w_b \). It is in fact the Padé approximant of \( w_{de} \) with respect to \((1 - a)\) (or scale factor \( a \)) up to order \((0, 1)\). Of course, the condition \(-1 < w_b < 0 \) in eq. (3.15) is still required to avoid any singularity. In this case, eqs. (3.14), (3.16) and (3.25), (3.26) are still valid but one should set \( w_a = 0 \) in them. By minimizing the corresponding total \( \chi^2 \) in eq. (3.24), we find the best-fit parameters \( \Omega_m0 = 0.280, w_0 = -0.995 \), and \( w_b = -0.072 \), while \( \chi^2_{\text{min}} = 562.256 \). In figure 6, we present the 68.3% and 95.4% confidence level contours in the \( w_0 - w_b, \Omega_m0 - w_0 \), and \( \Omega_m0 - w_b \) planes for the simplified parameterization (I) in eq. (3.27). For the best-fit parameters, \( w_{de} = -1.073 \) in the very beginning \((a \to 0, z \to \infty)\), and then crossed the phantom divide at \( a_s = 0.932 \) \((z_s = 0.073)\); today \((a = 1, z = 0)\), we have \( w_{de} = -0.995\); in the far future \((a \to \infty, z \to -1)\), \( w_{de} \to 0 \). We plot this \( w_{de} \) as a function of redshift \( z \) in the right panel of figure 5, and we can easily see that \( w_{de} \) crossed the phantom divide \( w_{de} = -1 \).

### 3.3 Type (II) Padé parameterization

In the literature, the most familiar time variables are cosmic time \( t \), scale factor \( a \), redshift \( z \), and the so-called \( e \)-folding time \( N = \ln a \). As is well known, in many cases it is more convenient to express cosmological quantities in terms of the \( e \)-folding time \( N = \ln a \). Therefore, it is reasonable to consider the Padé approximant of \( w_{de} \) with respect to the \( e \)-folding time \( N = \ln a \) up to order \((1, 1)\), and propose the type (II) Padé parameterization

\[
 w_{de} = \frac{w_0 + w_1 \ln a}{1 + w_2 \ln a},
\]

where \( w_0, w_1 \) and \( w_2 \) are all constants. Obviously, it reduces to the parameterization in eq. (3.5) if \( w_2 = 0 \), and reduces to the parameterization in eq. (3.6) with \( \alpha = 1 \) if \( w_1 = 0 \). If \( w_2 \neq 0 \), our type (II) Padé parameterization can avoid the divergences at \( a \to \infty \) \((or \ z \to -1 \ equivalently)\), and \( a \to 0 \) \((or \ z \to \infty \ equivalently)\), unlike the logarithm parameterization in eq. (3.5). It is easy to see that

\[
 w_{de} = \begin{cases} 
 \frac{w_1}{w_2}, & \text{for } a \to 0 \ (z \to \infty, \ \text{the early time}), \\
 w_0, & \text{for } a = 1 \ (z = 0, \ \text{now}), \\
 \frac{w_1}{w_2}, & \text{for } a \to \infty \ (z \to -1, \ \text{the far future}),
\end{cases}
\]

where \( w_2 \neq 0 \) is required. However, this parameterization unfortunately has an unavoidable singularity at \( a = \exp(-1/w_2) \) \((or \ z = -1 + \exp(1/w_2) \ equivalently)\). It is a so-called \( w \)-singularity \([72, 73]\) (see below) in fact. This type of singularity also exists in the well-known
Figure 5. $w_{\text{de}}$ as a function of redshift $z$ for the models (left panel: parameterization (I) in eq. (3.12); right panel: simplified parameterization (I) in eq. (3.27)) with their best-fit parameters, respectively. See the text for details.

Figure 6. The 68.3% and 95.4% confidence level contours in the $w_0 - w_b$, $\Omega_m0 - w_0$, and $\Omega_m0 - w_b$ planes for the simplified parameterization (I) in eq. (3.27). The best-fit parameters are also indicated by the black solid points.
logarithm parameterization in eq. (3.6). Nevertheless, if we require
\[ w_2 < 0, \quad (3.30) \]
the singularity will occur in the future, namely \( a > 1 \) (\( z < 0 \)). Under the condition (3.30), the type (II) Padé parameterization works well at least in the whole past history \( 0 \leq a \leq 1 \) (\( 0 \leq z < \infty \)), and hence we can still employ it as a workhorse. The smaller \( |w_2| \), the longer term of service is. Note that \( w_{de} \) can cross the phantom divide \( w_{de} = -1 \) at
\[ a_* = \exp \left( -\frac{1 + w_0}{w_1 + w_2} \right), \quad \text{or equivalently,} \quad z_* = -1 + \exp \left( \frac{1 + w_0}{w_1 + w_2} \right). \quad (3.31) \]
Substituting this \( w_{de} \) into the energy conservation equation \( \dot{\rho}_{de} + 3H\rho_{de} (1 + w_{de}) = 0 \), we find that
\[ \rho_{de} = \rho_{de,0} a^{-3(w_1+w_2)/w_2} (1 + w_2 \ln a)^{3(w_1-w_0w_2)/w_2^2}. \quad (3.32) \]
Note that \( w_2 \neq 0 \) as mentioned above. Again, in the present work, we consider a flat FRW universe containing only pressureless matter and dark energy. Substituting eq. (3.32) into Friedmann equation, we finally obtain
\[ E^2 = \Omega_{m0} (1 + z)^3 + (1 - \Omega_{m0}) (1 + z)^3(w_1+w_2)/w_2 [1 - w_2 \ln(1 + z)]^3(w_1-w_0w_2)/w_2^2. \quad (3.33) \]
There are four free parameters in this model, namely \( \Omega_{m0}, w_0, w_1 \) and \( w_2 \). Note that \( w_2 < 0 \) is required in eq. (3.30). By minimizing the corresponding total \( \chi^2 \) in eq. (3.24), we find the best-fit parameters \( \Omega_{m0} = 0.280, w_0 = -0.996, w_1 = 0.200, \) and \( w_2 = -0.139, \) while \( \chi^2_{\text{min}} = 562.254 \). In figure 7, we present the 68.3% and 95.4% confidence level contours in the \( w_0 - w_1, w_0 - w_2, w_1 - w_2, \Omega_{m0} - w_0, \Omega_{m0} - w_1, \) and \( \Omega_{m0} - w_2 \) planes for the type (II) Padé parameterization in eq. (3.28). Noting eqs. (3.29) and (3.31), for the best-fit parameters, \( w_{de} = -1.438 \) in the very beginning \( a = 0, z \to \infty \), and crossed the phantom divide at \( a_* = 0.926 \) \( (z_* = 0.076) \); today \( (a = 1, z = 0) \), we have \( w_{de} = -0.996; \) and then \( w_{de} \) will monotonously increase to \( +\infty \) at the singularity when \( a = 1328.1 \) \( (z = -0.999) \). This is in fact a so-called \( w \)-singularity [72, 73]. According to [72, 73], if this \( w \)-singularity is weak, the spacetime can be extended continuously beyond the singularity. From the physical point of view, a finite object is not necessarily crushed on crossing a weak singularity [72, 73]. If our type (II) Padé parameterization can cross this \( w \)-singularity, \( w_{de} \) will suddenly drop to \( -\infty \) when it just crosses the singularity, and then it will rapidly increase. Finally, \( w_{de} \to -1.438 \) again in the far future \( (a \to \infty, z \to -1) \). We plot this \( w_{de} \) as a function of redshift \( z \) in figure 8, and we can easily see that \( w_{de} \) crossed the phantom divide \( w_{de} = -1 \). The type (II) Padé parameterization works well.

4 Conclusion and discussions

As is well known, in mathematics, any function could be approximated by the Padé approximant. The Padé approximant is the best approximation of a function by a rational function of given order [13]. In fact, the Padé approximant often gives better approximation of the function than truncating its Taylor series, and it may still work where the Taylor series does not converge [13]. In the present work, we considered the Padé approximant in two issues. First, we obtained the analytical approximation of the luminosity distance for the flat XCDM model, and found that the relative error is fairly small. Second, we proposed several
Figure 7. The 68.3% and 95.4% confidence level contours in the $w_0 - w_1$, $w_0 - w_2$, $w_1 - w_2$, $\Omega_{m0} - w_0$, $\Omega_{m0} - w_1$, and $\Omega_{m0} - w_2$ planes for the type (II) Padé parameterization in eq. (3.28). The best-fit parameters are also indicated by the black solid points.
parameterizations for the EoS of dark energy based on the Padé approximant. They are well motivated from the mathematical and physical points of view. We confronted these EoS parameterizations with the latest observational data, and found that they can work well. In these practices, we showed that the Padé approximant could be an useful tool in cosmology, and it deserves further investigation.

Here, let us further clarify the physical motivations to use the Padé approximant in this work one by one. In the case of EoS parameterization, the physical motivation is to avoid divergence. As is well known, the singularity is not welcome in physics. However, as we mentioned in section 3.1, the EoS parameterizations extensively considered in the literature diverge in some special cases. For example, the familiar parameterization (3.1), $w_{de} = w_0 + w_1 z$, diverges when redshift $z \to \infty$, and hence this parameterization is unsuitable to describe the early universe at high redshift $z$. On the other hand, the most popular CPL parameterization (3.2), $w_{de} = w_0 + w_a (1 - a)$, diverges when $a \to \infty$ (or $z \to -1$ equivalently). If dark energy is phantom-like (its $w_{de} < -1$), as is well known, the scale factor $a$ of the universe will diverge ($a \to \infty$) in a finite future time (this singularity is the well-known big rip). Therefore, the CPL parameterization, $w_{de} = w_0 + w_a (1 - a)$, is unsuitable to describe the phantom-like dark energy when the scale factor $a$ is large in the future before the big rip. Even if dark energy is quintessence-like (its $w_{de} > -1$), the CPL parameterization is also unsuitable to describe the late time universe when the scale factor $a$ is very large. On the contrary, as is shown in section 3, the Padé parameterizations proposed in this work can easily avoid the divergences when $a \to 0$ ($z \to \infty$) and $a \to \infty$ ($z \to -1$), unlike the familiar parameterization (3.1) and the most popular CPL parameterization (3.2). In particular, as mentioned above, the type (I) Padé parameterization proposed in eq. (3.12) is completely free of singularity in the whole range $0 \leq a < \infty$ $(-1 \leq z < \infty)$. Therefore, using the Padé approximant in EoS parameterization is physically motivated, not only mathematically motivated.

Then, let us further clarify the motivation to use the Padé approximant in the case of the analytical approximation of the luminosity distance. The motivation is to improve the calculating efficiency and hence we can save the computing time and power. As is well
known, the numerical integral in the luminosity distance usually employs the algorithms of Romberg Integration or Gaussian Quadratures [79]. These algorithms repeat the calling of the subroutine which implements the extended trapezoidal rule and Simpson’s rule [79], and hence it usually consumes longer time than calculating an analytical expression straightforwardly. As is shown in e.g. [16, 19, 20], for a given goal accuracy, calculating an analytical approximation of the luminosity distance is significantly faster than the numerical integral employing the algorithms of Romberg Integration or Gaussian Quadratures. In particular, as is shown in [16], the Padé analytical approximation of the luminosity distance is the most efficient one in the three existing analytical approximations of the luminosity distance for the flat ΛCDM model. This is important. Let us explain in more details. When we confront dark energy model with the observational data, say, the latest Union2.1 compilation [8, 9] which consists of 580 SNIa, we need to calculate the theoretical luminosity distance for 580 times to compare it with 580 SNIa at different redshift $z_{\text{obs}}$. When we scan the parameter space or run a Monte Carlo simulation, we need to compare dark energy model with the observational data for typically $10^6$ times or even more. So, we have to repeat the calculation of the theoretical luminosity distance for $5.8 \times 10^8$ times or even more. Therefore, even a little improvement in calculating efficiency can make a big difference in the computing time and power. As mentioned above, an analytical approximation of the luminosity distance is significantly efficient than the numerical integral employing the algorithms of Romberg Integration or Gaussian Quadratures [16, 19, 20], while the Padé analytical approximation of the luminosity distance is the most efficient one [16]. Therefore, using the Padé approximant in this case is well motivated in fact. Of course, we understand the worry about the complicated expressions of $\alpha_i$ and $\beta_i$ in eqs. (2.16) and (2.17). However, it is not a problem in fact. Noting that the constants $\alpha_i$ and $\beta_i$ in eqs. (2.16) and (2.17) depend only on $w_X$, they are simple numerical values for a given $w_X$. For instance, if $w_X = -0.95$, it is easy to find that $\alpha_1 = 1.31874$, $\alpha_2 = 0.43988$, $\alpha_3 = 0.0262761$, $\beta_1 = 1.39336$, $\beta_2 = 0.513621$, $\beta_3 = 0.0397332$. When we write a computer code, these constants are very simple for a given $w_X$, and can be written directly or by using simply a few lines of code. So, the apparently complicated expressions of the constants $\alpha_i$ and $\beta_i$ in eqs. (2.16) and (2.17) are not a problem in fact.

It is of interest to compare our Padé EoS parameterizations with the well-known CPL parameterization. Since these models have different free parameters and the correlations between model parameters are fairly different, it is not suitable to directly compare their confidence level contours. Instead, as in the literature, it is more appropriate to compare them from the viewpoint of goodness-of-fit. A conventional criterion for model comparison in the literature is $\chi^2_{\text{min}}/\text{dof}$, in which the degree of freedom $\text{dof} = N - k$, while $N$ and $k$ are the number of data points and the number of free model parameters, respectively. On the other hand, there are other criterions for model comparison in the literature. The most sophisticated criterion is the Bayesian evidence (see e.g. [80, 81] and references therein). However, the computation of Bayesian evidence usually consumes a large amount of time and power. As an alternative, one can consider some approximations of Bayesian evidence, such as the so-called Bayesian Information Criterion (BIC) and Akaike Information Criterion (AIC). The BIC is defined by [82]

$$BIC = -2 \ln L_{\text{max}} + k \ln N,$$

(4.1)

where $L_{\text{max}}$ is the maximum likelihood. In the Gaussian cases, $\chi^2_{\text{min}} = -2 \ln L_{\text{max}}$. So, the difference in BIC between two models is given by $\Delta BIC = \Delta \chi^2_{\text{min}} + \Delta k \ln N$. The AIC is
defined by \[83\]
\[
\text{AIC} = -2 \ln L_{\text{max}} + 2k.
\] (4.2)

The difference in AIC between two models is given by \[ \Delta \text{AIC} = \Delta \chi^2_{\text{min}} + 2 \Delta k. \] As is well known, the corresponding \( E \equiv H/H_0 \) for CPL parameterization (3.2) is given by (see e.g. [23–38])
\[
E(z) = \left[ \Omega_{m0}(1 + z)^3 + (1 - \Omega_{m0})(1 + z)^{3(1+w_0+w_a)} \exp\left( \frac{-3w_a z}{1+z} \right) \right]^{1/2}.
\] (4.3)

There are three independent parameters. By minimizing the corresponding total \( \chi^2 \) in eq. (3.24), we find the best-fit parameters \( \Omega_{m0} = 0.280, w_0 = -0.995 \) and \( w_a = -0.074 \), while \( \chi^2_{\text{min}} = 562.256 \). In table 1, we present \( \chi^2_{\text{min}}/\text{dof} \), \( \Delta \text{BIC} \) and \( \Delta \text{AIC} \) for CPL parameterization, type (I) Padé parameterization (PI), simplified type (I) Padé parameterization (SPI), and type (II) Padé parameterization (PII). Note that CPL parameterization has been chosen to be the fiducial model when we calculate \( \Delta \text{BIC} \) and \( \Delta \text{AIC} \). From table 1, we see that the rank of models is coincident in all the three criterions, namely, \( \chi^2_{\text{min}}/\text{dof} \), BIC and AIC. The CPL parameterization is slightly better than type (I) and (II) Padé parameterizations given in eqs. (3.12) and (3.28). However, the simplified type (I) Padé parameterization given in eq. (3.27) is as good as CPL parameterization.

Some remarks are in order. First, it is important to obtain an analytical approximation of the luminosity distance for dark energy models with variable EoS \( w_{\text{de}} \). We have tried the dark energy model with CPL EoS \( w_{\text{de}} = w_0 + w_a(1 - a) \) but failed because the relative error is unacceptably large. This issue deserves further attempts, and we leave it as an open question. Second, it is also of interest to find an analytical approximation of the luminosity distance for a non-flat FRW universe. Even, one can further consider an inhomogeneous or anisotropic universe, say, a Lemaitre-Tolman-Bondi (LTB) universe. Third, since the angular diameter distance \( d_A = d_L(1 + z)^{-2} \), the corresponding analytical approximation of the angular diameter distance is also ready. Fourth, in both issues of the luminosity distance and EoS parameterization, one can of course use a Padé approximant of higher order to improve the accuracy, rather than just order (3, 3) in \( d_L \), or order (1, 1) in EoS parameterization as we done. However, this will bring the drawback of heavier computation, or too many free model parameters. Fifth, in section 3.2, motivated by eq. (3.14) and the best-fit parameters,

| Model | CPL  | PI   | SPI  | PII  |
|-------|------|------|------|------|
| \( \chi^2_{\text{min}} \) | 562.256 | 562.256 | 562.256 | 562.254 |
| \( k \) | 3    | 4    | 3    | 4    |
| \( \chi^2_{\text{min}}/\text{dof} \) | 0.971081 | 0.972761 | 0.971081 | 0.972758 |
| \( \Delta \text{BIC} \) | 0    | 6.36647 | 0    | 6.36447 |
| \( \Delta \text{AIC} \) | 0    | 2    | 0    | 1.998 |
| Rank | 1    | 3    | 1    | 2    |

Table 1. Comparing CPL parameterization with type (I) Padé parameterization (PI), simplified type (I) Padé parameterization (SPI), and type (II) Padé parameterization (PII). Note that CPL parameterization has been chosen to be the fiducial model when we calculate \( \Delta \text{BIC} \) and \( \Delta \text{AIC} \). See the text for details.
we can consider other two simplified versions of the type (I) Padé parameterization, namely

\[ w_{de} = \frac{-1 + w_a(1 - a)}{1 + w_b(1 - a)}, \tag{4.4} \]

by setting \( w_0 = -1 \), or

\[ w_{de} = \frac{w_0 - w_b(1 - a)}{1 + w_b(1 - a)}, \tag{4.5} \]

by setting \( w_a = -w_b \), while \(-1 < w_b < 0\) still holds. They have only two free parameters, and different interesting behaviors. Sixth, although there exists a so-called \( w \)-singularity [72, 73] in the type (II) Padé parameterization, it might be not so serious. According to [72, 73], if this \( w \)-singularity is weak, the spacetime can be extended continuously beyond the singularity. From the physical point of view, a finite object is not necessarily crushed on crossing a weak singularity [72, 73]. So, it is of interest to study whether this type of \( w \)-singularity is weak, and we leave it to the future works. Seventh, in fact, the present work is not the first one using the Padé approximant in cosmology. We refer to e.g. [74–78, 84] (see also [16, 17]) for the previous relevant works. In these works, the Padé approximant has been used in the slow-roll inflation, the reconstruction of the scalar field potential from SNIa, the data fitting of luminosity distance, a special EoS parameterization with respect to redshift \( z \), and the cosmological perturbation in LSS. Anyway, the issues discussed in the present work are different from the previous works in the literature. Eighth, since our type (I) Padé parameterization in eq. (3.12) is well motivated from mathematics, and completely free of singularity in the whole range \( 0 \leq a < \infty \) \((-1 \leq z < \infty\) unlike the CPL parameterization, we recommend the community to use it or its variants in the relevant works. Finally, as shown in this work, the Padé approximant is useful. It is of interest to consider its other applications in cosmology.

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