Computer simulation of solutions of polyharmonic equations in plane domain

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Abstract. A systematic study of plane problems of the theory of polyharmonic functions is presented. A method of reducing boundary problems for polyharmonic functions to the system of integral equations on the boundary of the domain is given and a numerical algorithm for simulation of solutions of this system is suggested. Particular attention is paid to the numerical solution of the main tasks when the values of the function and its derivatives are given. Test examples are considered that confirm the effectiveness and accuracy of the suggested algorithm.

1. Introduction
The polyharmonic functions were the most completely studied using the theory of the analytic functions in [1]. Some statements of the boundary value problems of the theory of polyharmonic functions are given in papers [2, 3]. In [4] the representations for polyharmonic functions are received and the theorems of the existence and uniqueness of the solution of the some boundary value problems are proved. Some additional information on polyharmonic functions can be found in [5].

It is effective and useful in studying the properties of harmonic functions and particularly the development of various numerical methods to use an integral Green's formula. It enabled the development of efficient numerical algorithms for the harmonic equations’ solution and calculation of a wide range of planar and axisymmetric problems of continuum mechanics [6]. The numerical solution of biharmonic equations in the plane with respect to media with very high viscosity is given in [7].

This paper presents a systematic study of plane problems of the theory of polyharmonic functions. It is shown that the polyharmonic equation is reduced to a system of integral that the boundary element method can be represented as a system of linear algebraic equations. A classification of boundary value problems is suggested. It is shown that the main boundary value problem is equivalent to a mixed boundary value problem, and therefore the same computer simulation can be applied. The numerical method is illustrated by solving polyharmonic equations of up to the fourth order.

2. The integral relations for polyharmonic functions
Let us introduce the following notations:

\[ G_k = \frac{1}{2\pi k!} r^{2k} \left( \ln \frac{1}{r} + \sum_{m=0}^{k-1} \frac{1}{m!} \right), \quad H_k = \frac{\partial G_k}{\partial n} = \Delta^k u, \quad u_k = \Delta^k u, \quad v_k = \frac{\partial u_k}{\partial n}, \quad k = 0, n-1, \]

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where $\Delta' G_k = G(r)$, $\Delta$ is the Laplace operator, $r(P, P_0)$ is the distance between points $P$ and $P_0$. $G(r) = G_0 = -\ln r / 2\pi$ is Green’s function in plane domain, $\partial / \partial n$ is the normal derivative.

If $u = u_0$ satisfies $n$-harmonic equation $\Delta^n u = 0$, then function $u_k$ is $(n - k)$-harmonic function. According to Green’s identity, polyharmonic function $u$ is expressed by boundary values of functions $u_k$ and their normal derivatives $v_k$:

$$\mathcal{E}u(P_0) = \sum_{k=0}^{n-1} \int (v_k(P) G_k(P, P_0) - u_k(P) H_k(P, P_0)) d\sigma(P)$$  \hspace{1cm} (1)

where $P_0$ is a fixed point, $P$ is a variable integration point, $\varepsilon = 0.5$ if $P_0 \notin \partial T$, $\varepsilon = 1$ if $P_0 \in T$.

All functions $u_k$ also satisfy the relations similar to (1). Finally, let us obtain a system of integral equations of the form:

$$\mathcal{E}u_j = \sum_{k=0}^{n-1} \int (v_{j+k} G_k - u_{j+k} H_k) d\sigma, \hspace{1cm} (j = 0, n-1).$$  \hspace{1cm} (2)

Thus, polyharmonic function $u = u_0$ is defined by the values of functions $u_k$ and their normal derivatives $v_k$ on the domain’s boundary. In addition, $n$ integral relations (2) must be satisfied.

It is easy to notice that each function $u_k$ is a solution of the Poisson equation, $\Delta u_k = u_{k+1}$; the first equation in (2) is an integral relation for the $n$-harmonic function, and the last one – for the Poisson equation with a harmonic function as a right part of it.

3. **Classification of the boundary value problems for the polyharmonic function**

Integral relations (2) make it possible to classify the boundary value problems for the polyharmonic function: it is necessary to find function $u(x, y)$, which satisfies polyharmonic equation $\Delta^n u = 0$ in domain $T$ and which is continuous together with its derivatives up to the $n-1$ order inclusive in closed domain $\overline{T} = T \cup \partial T$ and satisfying the boundary $\partial T$ conditions:

1. **the Dirichlet problem**: functions $u_k$ are given;
2. **the Neumann problem**: normal derivatives $v_k$ are given;
3. **the mixed problem**: a part of functions $u_k$ is given and a part of functions $v_k$ is given;
4. **the main boundary value problem**:

$$u_{|_{br}} = g_0(s), \hspace{1cm} \frac{\partial^k u}{\partial n^k} = g_k(s), \hspace{1cm} s \in \partial T, \hspace{1cm} (k = 1, n-1).$$  \hspace{1cm} (3)

The main boundary value problem is very important in terms of applications in mechanics [8, 9].

4. **Transition from the main problem to the mixed boundary value problem**

Let boundary $\partial T$ has a parametric representation with functions of arc abscissa $(s \in \partial T)$ $x = x(s), y = y(s)$. Functions $u(s), u_k(s)$ and their derivatives are expressed in terms of partial derivatives of function $u(x, y)$ linearly with coefficients, which are defined by functions $x(s), y(s)$, which are continuous and differentiable together with their derivatives up to $(n-1)$ order inclusive. Functions $u_k(s)$ and $v_k(s)$ are also linearly expressed in terms of partial derivatives of function $u(x, y)$. Combining these relationships, let us obtain a closed system of linear equations in functions $u_k(s), v_k(s)$ and the partial derivatives of function $u(x, y)$ on the boundary. By given boundary
conditions (3), the first $n$ values of functions $u_i(s)$ and $v_i(s)$ can be found. Thus, the main problem is reduced to solving a system of $n$ integral equations (2) with mixed boundary conditions.

The unit tangent vector and normal to boundary $\partial T$ have the following components:

$$\tau = (\tau_x, \tau_y) = (x'y', y'x')$$

$$\mathbf{n} = (n_x, n_y) = (y'x' - x'y'), \quad l = (x'^2 + y'^2)^{-1/2}.$$  

1. Harmonic equation $\Delta u = 0$. Condition (3) is the known Dirichlet condition:

$$u|_{\partial T} = u_0(s), \quad s \in \partial T.$$  

2. Biharmonic equation $\Delta^2 u = 0$. The boundary conditions of a mixed boundary value problem follow from (3):

$$u|_{\partial T} = f_0(s), \quad v|_{\partial T} = f_1(s), \quad s \in \partial T.$$  

3. The polyharmonic equation of the third order, $\Delta^3 u = 0$. Boundary conditions (3) of the main problem are of the form:

$$u|_{\partial T} = f_0(s), \quad u_s|_{\partial T} = f_1(s), \quad u_{ss}|_{\partial T} = f_2(s), \quad s \in \partial T. \quad (4)$$

From the first two boundary conditions (4), the first two boundary conditions for the mixed boundary problem directly follow. To record the third condition, let us find the relations, which give the system of linear equations for derivatives of function $u$. It is possible to write this system in a matrix form:

$$M_3 X_3 = Y_3,$$  

$$M_3 = \begin{pmatrix}
  x' & y' & 0 & 0 & 0 \\
  y' & x' & 0 & 0 & 0 \\
  x'' & y'' & x'^2 & 2x'y' & y'^2 \\
  y'' & -x'' & x'y' & y'^2 - x'^2 & -x'y' \\
  0 & 0 & y'^2 & -2x'y' & x'^2 
\end{pmatrix}, \quad X_3 = \begin{pmatrix}
  u_x \\
  u_y \\
  u_{xx} \\
  u_{xy} \\
  u_{yy} 
\end{pmatrix}, \quad Y_3 = \begin{pmatrix}
  u_{11} \\
  u_{12} \\
  u_{ss} \\
  u_{ss} \\
  u_{ss} 
\end{pmatrix}.$$

Hence let us calculate vector-matrix $X_3 = M_3^{-1} Y_3$, and then function $u_1$ on boundary $\partial T$ is defined by formula $u_1 = \Delta u = u_{xx} + u_{yy}$.

4. The polyharmonic equation of the fourth order, $\Delta^4 u = 0$. In addition to boundary conditions (4) a given value of derivative $u_{ssss} = f_3(s)$ on boundary $\partial T$ is added.

Values of derivatives $u_{sss}, u_{ssss}, u_{ssss}$ are computed as derivatives with respect to curvilinear coordinate $s$. From system (5), derivatives of the first and second orders can be found. Then, to determine the third derivatives, let us obtain matrix equation $M_4 X_4 = Y_4$, where:

$$M_4 = \begin{pmatrix}
  x'^3 & 3x'^2y' & 3x'y'^2 & y'^3 \\
  x'^3y' & 2x'^2y'^2 - x'^3 & y'^3 - 2x'^2y' & -x'y'^2 \\
  x'y'^2 & y'^3 - 2x'^2y' & x'^3 - 2x'y'^2 & x'^2y' \\
  y'^3 & -3x'y'^2 & 3x'^2y' & -x'^3 
\end{pmatrix}, \quad X_4 = \begin{pmatrix}
  u_{xxx} \\
  u_{xys} \\
  u_{yys} \\
  u_{yyy} 
\end{pmatrix}, \quad Y_4 = \begin{pmatrix}
  u_{ssss} \\
  u_{ssss} \\
  u_{ssss} \\
  u_{ssss} 
\end{pmatrix}.$$

From the first two boundary conditions (4), the first two boundary conditions for the mixed boundary problem directly follow. To record the third condition, let us find the relations, which give the system of linear equations for derivatives of function $u$. It is possible to write this system in a matrix form:

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$$M_3 = \begin{pmatrix}
  x' & y' & 0 & 0 & 0 \\
  y' & x' & 0 & 0 & 0 \\
  x'' & y'' & x'^2 & 2x'y' & y'^2 \\
  y'' & -x'' & x'y' & y'^2 - x'^2 & -x'y' \\
  0 & 0 & y'^2 & -2x'y' & x'^2 
\end{pmatrix}, \quad X_3 = \begin{pmatrix}
  u_x \\
  u_y \\
  u_{xx} \\
  u_{xy} \\
  u_{yy} 
\end{pmatrix}, \quad Y_3 = \begin{pmatrix}
  u_{11} \\
  u_{12} \\
  u_{ss} \\
  u_{ss} \\
  u_{ss} 
\end{pmatrix}.$$
\[ g_1 = 3x^2u_{xx} + 3\left(x'y' + y'x'\right)u_{xy} + 3y'x''u_{yy} + x''u_x + y'u_y, \]
\[ g_2 = 2x'y' + x'y''u_{xx} + 3\left(x'y' - x'x''\right)u_{xy} - \left(x'y' + 2x'y''\right)u_{yy} + y''u_x - x''u_y, \]
\[ g_3 = 2y'y''u_{xx} - 2\left(x''y' + x'y''\right)u_{xy} + 2x'x''u_{yy}. \]

Solution \( X = M^T_iY_k \) contains the partial derivatives of the third order, and function \( v_i \) on boundary \( \partial T \) can be found by formula \( v_i = \frac{\partial u_i}{\partial n} - \left[y'(u_{xx} + u_{yy}) - x'(u_{xy} + u_{yx})\right]I. \)

The process can be continued for polyharmonic functions of higher orders, but the number of mathematical operations increases. So conditions (3) are equivalent to the mixed boundary conditions.

5. Numerical algorithm

The integral relations obtained are rather useful for constructing the numerical algorithm for solving the polyharmonic equation with a boundary elements method [10], whose essence is to approximate the boundary with a system of the finite number of sufficiently small elements and to approximate the functions on each element. Then system (2) can be reduced to a system of linear equations for the values of the unknown functions at the control points and be presented in a matrix form:

\[
\begin{align*}
(eE + A_0)U_{n-1} - B_0V_{n-1} &= 0, \\
(eE + A_0)U_{n-2} - B_0V_{n-2} + A_1U_{n-1} - B_1V_{n-1} &= 0, \\
&\vdots \\
(eE + A_0)U_0 - B_0V_0 + A_1U_1 - B_1V_1 + \cdots + A_{n-1}U_{n-1} - B_{n-1}V_{n-1} &= 0,
\end{align*}
\]

where \( E \) is a unity matrix, \( U_i, V_i \) are vectors, whose components are values at the control points:

\[ U_i^j = u_k(p), \quad V_i^j = v_k(p), \quad i = 1, N, \quad k = 0, n - 1, \]

\( A_k, B_k \) are matrixes, whose elements are calculated by integration of functions on boundary elements:

\[ A_k^{ij} = \int_{\partial T_1} H_k(p, p_i)ds, \quad B_k^{ij} = \int_{\partial T_1} G_k(p, p_i)ds, \quad i, j = 1, N, \quad k = 0, n - 1. \]

If the contour is divided into \( N \) elements, system (6) is a system of \( Nn \) linear algebraic equations for \( 2Nn \) components. To solve this system, it is necessary to set \( Nn \) values of the above-mentioned components or their linear combinations. By solving system (6) and determining the unknown values of functions \( u_k, v_k \) on the boundary of the domain, according to formula (1), it is possible to calculate the value of desired function \( u \) at any interior point \( p_0 \in T \).

6. Computer simulation of test examples

Example 1. Let us consider a polyharmonic function of the fourth order, \( u = xy(x^6 + y^6) \), in ellipse with semiaxes \( a = 1, b = 0.75 \). Figure 1a shows the dependence of boundary values \( u, u_x, u_{xx}, u_{xxx} \) on normalized curvilinear coordinate \( s / p \) of the main problem, and figure 1b shows the mixed boundary conditions for system (6): \( p = 5.526 \) is the perimeter of the ellipse. It is easy to see that the boundary values have period \( T = 0.5p \) and they are antisymmetric with respect to \( s_0 = T/2 = 0.25p \).

Example 2. Let the unknown function be composed of harmonic, biharmonic functions and the polyharmonic function of the third order, \( u = \text{Im}(x + iy)^4 + 28.5x(x^2 + y^2) \). Figure 2 shows the results of calculations for the basic boundary value problem of the polyharmonic function of the third order in ellipse with semiaxes \( a = 1, b = 0.75 \). Boundary values are shown in figure 2 (a,
b, c), and calculation results – in figure 2 (d, e, f). The number of elements on the ellipse is $N = 40$.

The numerical results in the control points are plotted with the points.

**Example 3.** Let us consider Dirichlet problem for the biharmonic equation in a doubly-connected domain between two ellipses with semiaxes $a_1 = 2a = 2$, $b_1 = 2h_1 = 1.5$ and distance between centers $d = 0.5$. The boundary conditions are compiled by biharmonic function

$$u = x^3\left(x^2 - 5y^2\right) + 25xy\left(x^2 + y^2\right)/12.$$ All the derivatives can be found analytically and can be compared with numerical results. This comparison is shown in figure 3; the number of elements on each contour is $N = 50$.

![Figure 1](image1)

Figure 1. The boundary values of the unknown function and its derivatives: a – for the main boundary value problem; b – for the mixed boundary value problem.

![Figure 2](image2)

Figure 2. The results of solving the main problem for the polyharmonic equation of the
third order: a, b, c - boundary values, d, e, f - the results of calculations (solid lines correspond to the analytical formulas, the points – to the numerical solution).

7. Conclusion
The considered examples show that there is a good agreement between numerical and analytical results, which indicates the effectiveness of the suggested numerical algorithm. It can be used to solve various problems for polyharmonic equations in the plane domains. Moreover, the integral relations obtained in this work can be used in the theory of polyharmonic functions.

Figure 3. The results of solving the Dirichlet problem for the biharmonic equation in the doubly-connected domain: a – boundary values of function $v = \partial u / \partial n$, b – boundary values of function $v_1 = \partial u_1 / \partial n$ (solid lines correspond to the analytical formulas on the external contour, dot-dash lines – to the analytical formulas on the internal contour, the points – to the numerical solution).

References
[1] Vekua I N 1967 New Methods for Solving Elliptic Equations (Amsterdam: North–Holland) 358
[2] Dimitrov D K 1996 Mathematics of Computation 65 1269–1281
[3] Dall'Acqua A and Sweers G 2004 Differential Equations 205(2) 466–487
[4] Hayman W and Korenblum B 1993 Journal Analyse Math. 60 113–133
[5] Balk M B 1991 Itogi nauki i tekhniki VINITI 85 187–246
[6] Terentiev A G, Kirschner I N and Uhlman J S 2011 The Hydrodynamics of Cavitating Flows
(Paramus: Backbone) 598

[7] Elliott L A, Ingham D B and El Bashir T B A 1994 Proc. Boundary Element Methods in Fluid Dynamics II (Southampton) pp 16–24

[8] Kazakova A O and Terent’ev A G 2014 J. of Appl. Math. and Mech. 78(5) 518–523

[9] Kazakova A O and Petrov A G 2016 Fluid Dynamics 51(3) 311–320

[10] Brebbia C A, Telles J C F and Wrobel L 1984 Boundary Element Techniques. Theory and Applications in Engineering (Heidelberg: Springer-Verlag Berlin) 464