Anderson’s localization in a random metric: applications to cosmology

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Abstract: It is considered an equation for the Lyapunov exponent $\gamma$ in a random metric for a scalar propagating wave field. At first order in frequency this equation is solved explicitly. The localization length $L_c$ (reciprocal of Re($\gamma$)) is obtained as function of the metric-fluctuation-distance $\Delta R$ (function of disorder) and the frequency $\omega$ of the wave. Explicitly, low-frequencies propagate longer than high, that is $L_c \omega^2 = \text{Cte}$. Direct applications with cosmological quantities like background radiation microwave ($\lambda \sim 1/2 \times 10^{-3} \text{[m]}$) and the Universe-length ('localization length' $L_c \sim 1.6 \times 10^{25} \text{[m]}$) permits to evaluate the metric-fluctuations-distance as $\Delta R \sim 10^{-35} \text{[m]}$, a number at order of the Planck’s length.

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I Introduction: Anderson’s localization.

Anderson’s localization [1-3] is a general non dissipative phenomena related to wave localization due to interference in random (disordered) media. Depending on the degree of disorder ($\sigma$), frequency ($\omega$) and dimension of the space ($D$) waves could remain exponentially localized in some region characterized by the so-called localization length ($L_c$). Certainly, the calculation of the localization length and the spectrum for these systems is not an easy task, nevertheless, some general properties are today well understood [2,3]. It is known that Anderson’s localization is very sensitive to correlation between different random parts of the media [4-9] and decoherence [10,11]; it has been studied in systems like seismic waves (coda) [12,14], electromagnetic [15] and sound waves [16], electronic systems [17], chaos [18] and others. Anderson’s localization is a general phenomena related to waves and it seems natural to investigate its application on waves propagating in a given random metric, explicitly in a general relativity context as, for instance, waves in the whole space (Universe). Indeed being the curvature of the space-time described by the metric tensor, that is directly related to the matter distribution, it is reasonable to assume that, when the matter is distributed in a random way, the metric also can be considered random. In this case wave localization properties in these random systems could be expected.

In this paper we will found an expression for the localization length $L_c$ for waves propagating in a random metric (ground $\oplus$ disorder). The metric will be characterized and the basic parameter related to it, the metric-fluctuations-distance $\Delta R$ could be calculated from the ground metric and disorder (both assumed translation invariant). In the limit of slow frequency, the localization length $L_c$ as function of the metric-fluctuations will be obtained and given by (section III. $c$ light speed)

$$L_c = \frac{c^2}{\omega^2 \Delta R}.$$  \hspace{1cm} (1)

An expression quite reasonable since $L_c = \infty$ when $\Delta R = 0$ (no disorder) or when $\omega = 0$. In fact, is well known that for classical waves with long wavelength are less localized since it does not ‘feel’ inhomogeneities (disorder).

It seems quite intriguing that when we consider the background microwave radiation ($\lambda \sim 1/2 \times 10^{-3}[\text{m}]$) and identifying the universe radii with
the localization length \((L_c \sim 1.6 \times 10^{25} \text{[m]})\) the above expression (1) gives a metric-fluctuation-distance similar to the Planck’s length \((\Delta R \sim 10^{-35} \text{[m]}, \text{section IV})\).

In the next section (II), the basic equation (Riccati) defining the Lyapunov exponent (vector) in a random metric will be presented. A formal frequency-expansion will be proposed for the real and imaginary part of the Lyapunov exponent. The real part is identified with the reciprocal of the localization length \(L_c\). In section III the equation is solved formally for uncorrelated disorder in the small frequency limit. The assumption of independent random variables for the temporal and spatial part of the metric is always assumed. The localization length is expressed in term of Green functions. Nevertheless, defining the metric-fluctuation-distance, the localization length could be expressed in a more intuitive way (equation (1) or (21)). As pointed out in this introduction, in section IV we will study a direct application using cosmological parameters. We find that disorder fluctuations are of order of the Planck’s length, a quite intriguing result.

II Lyapunov exponent.

Let us consider a scalar field \(\phi(x_\mu)\) defined by a wave equation in a four dimensional time independent symmetric metric \(g^{\alpha\beta}\), where we assume \(g^{01} = g^{02} = g^{03} = 0\). Explicitly, consider the wave-equation [19,20] for the scalar field:

\[
-\omega^2 g^{oo} \phi + g^{ik} \phi_{;i;k} = 0, \quad i, k = 1, 2, 3. \tag{2}
\]

where \(g^{oo} > 0\) and the symbol \(;i\) define the covariant derivative respect to the spatial coordinate \(x_i\). Latin indices like \(i, j\), etc., run over three spatial coordinate. As usual, the controvariant tensors are related to covariant ones by the metric tensor. We will assume that the field \(\phi\) does not affect the metric, namely, \(g \neq g(\phi)\) as expected for weak propagating fields. Also note that we use units where the light velocity \(c = 1\).

From a general point of view, we consider a random tensor metric like to \(g = g_{(o)} + \Delta g\) where \(g_{(o)}\) denotes the translation-invariant background deterministic metric and \(\Delta g\) a random tensor which will be specified later.
Since we are interested in Anderson’s localization phenomena it is useful to consider the (local) Lyapunov exponent $\gamma_i$, which is a vector, defined by

$$\gamma_i = \frac{\phi_i}{\phi} R,$$

where $R$ is a parameter with dimension of spatial distance and assumed an invariant of the system, it will be specified later. In fact, it does not play an important role and we use it only for dimensional reasons. The reciprocal of the norm of real part of $\gamma_i$ will be related with the localization length. From (2), the equation for the components of $\gamma$ becomes

$$R\gamma^i = R^2 \omega^2 g^{00} - \gamma^i \gamma_i.$$

We stress that the covariant derivative in (4) is respect to the spatial metric (i.e. $g^{ij}$). Eq.(4) is the start-point to our calculations and corresponds to a Riccati type equation.

The real and the imaginary part of the Lyapunov exponent are defined as $a_i = Re \gamma_i$ and $b_i = Im \gamma_i$. Namely,

$$\gamma_i = a_i + \sqrt{-1} b_i,$$

and the norm of the real part of $\gamma_i$ will be identified with the inverse of the localization length and the imaginary part with the wave-vector. From the equation (4) for the Lyapunov exponent, we have the pair of coupled equations for these components

$$R a^i = R^2 \omega^2 g^{00} - a^i a_i + b^i b_i,$$

and

$$R b^i = -2 a^i b_i,$$

which are exact. Nevertheless, taking in account that the metric is random, for any realization we must use the corresponding covariant derivative. For the euclidean metric the wave-plane solution of (6,7) is $a_i = 0$ and $b_i=constant$, with the restriction $R^2 \omega^2 + b^i b_i = 0$ on the norm of $b$.

It is worthy to stress that for the above pair of equations, the solution $a = b = 0$ corresponds to the frequency $\omega = 0$ and is formally an extended state ($\phi = constant$) so that we could expect for low frequency a large localization length. This fact is well known for classical waves but not necessarily true.
for electronics waves where disorder does not multiplies the frequency. In fact this limit (slow frequency) can be treated analytically and it will be the object of the next section. We consider the formal expansion on frequency:

\[ a_i = \sum_{n=1}^{\infty} (R\omega)^{2n} A_i^{(n)}, \quad (8) \]

and

\[ b_i = \sum_{n=1}^{\infty} (R\omega)^{2n-1} B_i^{(n)}, \quad (9) \]

where the news amplitudes \( A \) and \( B \) must be determined by solving (6,7). The choice \( \omega \)-even for the function \( a \) and \( \omega \)-odd for \( b \) becomes directly from the form of these equation. We do not write explicitly the evolution equation for all order in the amplitude \( A \) or \( B \) because is not relevant for our purposes. In the next section we will consider the first order only.

III Small frequency case: calculation of the localization length.

As said before, in this section we will consider the small frequency expansion and the localization length will be evaluate. Consider the first order expansion of equations (8,9) for the Lyapunov exponent given by (section II)

\[ a_i = R^2 \omega^2 A_i + O(\omega^4), \quad \text{and} \quad b_i = R\omega B_i + O(\omega^3). \quad (10) \]

Where \( A \) and \( B \) are auxiliary variables. From (10) we have the pair of equations:

\[ RA_{i_i} = (g^{00} - 1) + (1 + B^i B_i), \quad \text{and} \quad B^i_{i} = 0. \quad (11) \]

The amplitude \( B \) will be the choose as a constant with \( B^i B_i + 1 = 0 \). In fact, this amplitude is related to the wave vector for homogenous systems. On the other hand, the equation for \( A \) becomes

\[ RA_{i_i} = \varepsilon(x), \quad \text{where} \quad g^{00} = 1 + \varepsilon(x) \quad \text{and} \quad x \equiv (x_1, x_2, x_3). \quad (12) \]
The random independent variables \( \varepsilon(x) \) (the disorder) are assumed well characterized and depending only of spatial coordinates. Explicitly, we assume:

\[
\langle \varepsilon(x) \rangle = 0 \quad \text{and} \quad \langle \varepsilon(x) \varepsilon(x') \rangle = R^D \sigma^2 \delta(x - x'),
\]

where the symbol \( \langle \cdot \rangle \) denotes disorder average, \( \sigma \) is the dimensionless disorder parameter (dispersion) and \( \delta \) is the usual Dirac distribution in the corresponding spatial dimension \( D \), actually \( D = 3 \). The above equations define our disordered model for the temporal part of the metric.

The formal solution of (12) is given by

\[
A_i(x) = \frac{1}{R} \int dV' G_i(x, x') \varepsilon(x'),
\]

where \( dV \) is an invariant element of volume and the spatial Green function \( G \) is solution of \( G_{ii}(x, x') = \delta(x - x') \). Note that it depends on the random spatial metric which will not be explicitly considered. In fact, Gauss theorem ensures that \( G_{ii} G_i = 1/S^2 \) where \( S \) is the surface of the ‘sphere’ of radius \( r \) in the corresponding metric. Taking the norm of this vector:

\[
\| A_i(x) \|^2 \equiv -A_i(x) A_i(x) = -\frac{1}{R} \int dV' dV'' G_i(x, x') G_i(x, x'') \varepsilon(x') \varepsilon(x''),
\]

at this stage this scalar quantity is depending on the spatial variables. Since we assume that the random components of the metric tensor (spatial and temporal) are independents, using the Eq. (13), the ensemble average of the norm of \( A \) becomes

\[
\langle A_i A_i \rangle = \sigma^2 R^{D-2} \int \langle dV G_{ii} \rangle,
\]

which is a space invariant quantity since we assume homogenous disorder and translation invariant background metric, so it is not depending on a particular positions. Note that when \( \langle g^{00} \rangle \neq 1 \) the above result is valid for the dispersion \( \langle (A_i^* A_i - \langle A_i^* \rangle \langle A_i \rangle) \rangle \).

The norm of the real part for the Lyapunov exponent satisfies

\[
\langle a_i a_i \rangle = \omega^4 \sigma^2 R^{D+2} \int \langle dV G_{ii} \rangle,
\]

and defining formally the localization length \( L_c \) as
the following expression is obtained:

\[ L_c^{-2} = -\frac{1}{R^2} \left\langle a^i a_i \right\rangle, \]  

(18)

It is relevant to note that:

(i) The integral is only on spatial coordinates and we need more specification to do it. Moreover, it depends on the dimension \( D \).

(ii) Low frequencies have large localization length as expected for classical waves in random media. In fact, \( L_c^{-2} \omega^4 \) is constant, for a given random system.

(iii) The localization length is a global property of the systems like to the averaged Lyapunov exponent (no position depending).

(iv) The invariant distance \( R \) is assumed, for instance, as related to the inverse mean density \( \rho \) of matter (i.e. \( R \sim \rho^{-1/D} \)). Nevertheless, eventually, it could be also considered other relevant distance parameter of the systems.

Defining the disorder parameter \( \Delta R \) (the metric-fluctuations-distance) as

\[ (\Delta R)^2 \equiv -\sigma^2 R^D \int \left\langle dV G^i G_i \right\rangle, \]  

(20)

the localization length becomes in term of this parameter given by

\[ L_c^{-2} = \omega^4 \Delta R^2 / c^4, \]  

(21)

where we have restored the velocity of light \( c \). Note that the metric-fluctuations-distance \( \Delta R \) is a parameter which must be calculated using the distribution of spatial disorder. Expression (1) in the section I corresponds essentially to (21).
IV The microwave radiation background, localization length, and metric fluctuation estimations: Planck’s length.

As application of the above results, consider scalar waves propagating in the whole Universe assumed as a media with disordered-metric. We will consider explicitly microwave background radiation and the radii of the Universe identified as its associated localization length. It must be noted here that the background radiation is a electromagnetic vector field, not a scalar-field. Nevertheless, every component of this field can be formally reduced to a scalar evolution equation. We assume that the metric is slowly changing in time compared to others time scale.

The expression for the localization length (21) could be written in term of the wavelength \( \lambda \) as \( L_c = (L_c^{-2})^{-1/2} \)

\[
L_c = \frac{\lambda^2}{(2\pi)^2} \Delta R.
\] (22)

At this point a remark becomes in order, for disordered systems the wave vector is not a parameter characterising the wave since the whole system is not translation invariat. So, the definition of the wavelength is in the sense of \( \lambda = 2\pi c/\omega \). For instance, assuming a wavelength \( \lambda \sim \frac{1}{2} \times 10^{-3} \text{[m]} \) for the microwave radiation background at the universe, and \( L_c \sim 1.6 \times 10^{25} \text{[m]} \) [20] for the universe size then, we obtain for the metric fluctuations \( \Delta R \sim 3.9 \times 10^{-35} \text{[m]} \). Namely, fluctuations in the metric in order of the Planck’s length. It is a surprising result since no direct relationship a prior exist between Anderson’s localization and Planck’s length. Note that the condition of small frequencies hold since the metric fluctuations are smaller than the wave length.

**Conclusion:** Anderson’s localization was directly considered from a Riccati type equation (4) with a random metric with separated spatial and temporal disorder (uncorrelated). The localization length in the low-frequency limit was evaluated (eq.(1) or (21)). It is a quit reasonable expression, that is, for low-frequency the localization length goes to infinite and also for small
disorder. A direct application to the background microwave radiation at the Universe, identifying the universe length with the localization length, gives us an estimation of the metric-length-fluctuations. It corresponds to the Planck’s length. A surprising result since Planck’s length is not put in the calculation as input.

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References

[1] P. W. Anderson, Phys. Rev. 109, 1492 (1958).

[2] C. P. Enz, A Course on Many-Body Theory Applied to Solid-State Physics (Worl-Scientific, Singapure, 1992).

[3] Y. Imry, Introduction to Mesoscopic Physics (Oxford University Press, 1997). And references therein.

[4] J. C. Flores, J. Phys.: Condens. Matter 1, 8471 (1989).

[5] D. H. Dunlap, H.-L. Wu and P. Phillips, Phys. Rev. Lett. 65, 88 (1990).

[6] F. A. B. F. de Moura and M. L. Lyra, Phys. Rev. Lett. 81, 3735 (1998).

[7] E. Lazo and M. E. Onell, Phys. Lett. A 283, 376 (2001).

[8] V. Bellani et al, Phys. Rev. Lett. 82, 2159 (1999).

[9] F. M. Izrailev and A. A. Krokhin, Phys. Rev. Lett. 82, 4062 (1999).

[10] J. C. Flores, Phys. Rev. B 60, 30 (1999). Phys. Rev. B 69, 012201 (2004).

[11] S. A. Gurvitz, Phys. Rev. Lett. 85, 812 (2000).
[12] R. Hennino et al, Phys. Rev. Lett. 86, 3447 (2001).
[13] E. Larose et al, Phys. Rev. Lett. 93, 048501 (2004).
[14] M. van der Bran, Geophys. J. Int. 145, 631 (2001).
[15] D. S. Wiersma, P. Bartolini, Ad Lagendijk and R. Righini, Nature 390, 671 (1997).
[16] P. Sheng, Scattering and Localization of Classical Waves in Random Media (World-Scientific, Singapure (1990)).
[17] T. Brandes and Kettermann (Eds.), Anderson’s localization and its Ramifications (Springer, Berlin (2003)).
[18] F. Haake, Quantum Signatures of Chaos (Springer, Berlin (2000)).
[19] S. Weinberg, Gravitation and Cosmology (John Wiley and Sons, N. Y. 1972).
[20] L. D. Landau and E. M. Lifshitz, The Classical Theory of fields (Pergamon Press, Oxford 1962).