TOPOLOGICAL DIMENSION OF A SPACE IS DETERMINED BY THE POINTWISE TOPOLOGY OF ITS FUNCTION SPACE

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Abstract. We show that the spaces of continuous real-valued functions on the unit interval and on the unit square, equipped with the pointwise topology, are not homeomorphic. In fact, we prove a much more general result concerning function spaces $C_p(X)$ endowed with the pointwise topology: If, for Tychonoff spaces $X$ and $Y$, the spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic then $\dim X = \dim Y$.

1. Introduction

One of the main results of this paper is the following theorem which solves a basic problem concerning function spaces with the pointwise topology that had remained outstanding for a long period of time, cf. [5, page 10], [6, Question 3.15], [7, page 132], [22, Question 2.14].

Theorem 1.1. The spaces of continuous real-valued functions on the unit interval and on the unit square, equipped with the pointwise topology, are not homeomorphic.

This is in a sharp contrast to the weak topology in function spaces: the celebrated Miljutin’s theorem [11, 4.4.8] asserts that all function spaces on uncountable compact metric spaces, equipped with the weak topology, are linearly homeomorphic. We shall show that among such function spaces with the pointwise topology there are continuum many different topological types.

In fact, we establish a much more general result. Let $C_p(X)$ denote the spaces of all continuous real-valued functions on a Tychonoff space $X$, equipped with the pointwise topology, and let $\dim X$ be the covering dimension of $X$. We prove the following.

Theorem 1.2. Let $X, Y$ be Tychonoff spaces. If $C_p(X)$ is homeomorphic to $C_p(Y)$, then $\dim X = \dim Y$.

There is a vast literature investigating connections between topological properties of $C_p(X)$ and the underlying space $X$, and in particular, of great importance to us was the work of Cauty [10], Marciszewski [21] and Okunev [26] on this subject.

It seems in place to present here briefly the history of the problem and some connections to other topics in functional analysis or topology.

Classical results, going back to the fundamental Banach treatise [8], demonstrated that the topology of a compact Hausdorff space $X$ is completely determined by the metric structure of the Banach space $C(X)$ of real-valued continuous functions on $X$ (Banach–Stone theorem, cf. [11]).
Similar results about the ring structure or the lattice structure of \( C(X) \) were proved later by Gelfand and Kolmogoroff \[14\], and Kaplansky \[17\]. Further, a theorem of Katětov \[15\] provided a characterization of dimension of \( X \) in terms of analytic dimension of \( C(X) \), cf. \[15\].

After the results of Bessaga–Pelczyński \[9\] and Miljutin \[24\] establishing the linear–topological classification of spaces \( C(X) \), for compact metrizable \( X \), cf. \[1\], the interest shifted towards relations between \( X \) and \( C_p(X) \)–the space \( C(X) \) equipped with the pointwise topology. This subject was very strongly influenced by the work of A.V. Arhangel’skii and his students, cf. \[2\], \[3\], and the problem considered in the paper was quite early recognized as one of the most interesting in this area, cf. \[2\] Problem 35, \[1\] Problem 20, \[22\] Problem 2.9, \[25\] Remark 6.9.5.

Important results in this direction, dealing with the linear or uniform structure of \( C_p(X) \) were established by Pestov \[27\] and Gul’ko \[16\], and subsequent deep research of Cauty \[10\], Marciszewski \[21\] and Okunev \[26\] laid the groundwork for the definitive answer provided in our paper. An essential intermediate step towards this result was a refinement of an approach by Okunev \[26\], presented by the author in \[19\].

This paper is organized as follows. Section 2 introduces basic notation used throughout the article. Section 3 describes results of Marciszewski \[21\] in a more general setting; we deliberately omitted some proofs: the interested reader should consult the book \[25\] Chapter 6.11. The latter section is a groundwork for Section 4, where we prove the central result of the paper – Theorem 4.1. Finally, Section 5 presents some corollaries to Theorem 4.1, in particular Theorems 1.1 and 1.2.

2. Preliminaries

All spaces under considerations are Tychonoff. We will consider the real line \( \mathbb{R} \) as a subspace of its two-point compactification \( I = \mathbb{R} \cup \{-\infty, +\infty\} \). The symbol \( \mathbb{N} \) stands for all positive integers and \( \mathbb{Q} \) for all rationals. By \( C^*_p(X) \) we denote the subspace of \( C_p(X) \) consisting of all bounded functions.

To prove Theorem 1.2 for all Tychonoff spaces, we shall first establish a version of \[16\] Proposition 1.4] for homeomorphisms instead of uniform homeomorphisms (cf. Theorem 4.1 below). This is the central theorem of the paper. Then applying directly a method developed by Gul’ko in \[16\] Section 2] we will reduce Theorem 1.2 to Theorem 4.1. To make the reduction work we need to use the following concept of \( \mathbb{Q}S \)-algebra.

**Definition 2.1.** A \( \mathbb{Q}S \)-algebra is a subspace \( \mathcal{A}(X) \subseteq C_p(X) \) satisfying the following properties:

1. For \( f, g \in \mathcal{A}(X) \) and \( \lambda \in \mathbb{Q} \), the functions \( f + g \), \( f \cdot g \) and \( \lambda f \) belong to \( \mathcal{A}(X) \),
2. For every point \( x \in X \) and its neighborhood \( U \), there is a function \( f \in \mathcal{A}(X) \) such that \( f \upharpoonright (X \setminus U) = 0 \) and \( f(x) = 1 \).

Of course \( C_p(X) \) and \( C^*_p(X) \) are \( \mathbb{Q}S \)-algebras. It is not difficult to check that any \( \mathbb{Q}S \)-algebra \( \mathcal{A}(X) \) has the following property.

**Proposition 2.2.** For any finite set \( \{x_1, \ldots, x_n\} \subseteq X \) and its neighborhood \( U \), and for any rational numbers \( \lambda_1, \ldots, \lambda_n \in \mathbb{Q} \), there is \( f \in \mathcal{A}(X) \) such that \( f \upharpoonright (X \setminus U) = 0 \) and \( f(x_i) = \lambda_i \), for \( i = 1, \ldots, n \).

**Proposition 2.3.** Suppose that \( f \in \mathcal{A}(X) \) and \( A \subseteq X \) is finite. Let \( U \supseteq A \) be an open neighborhood of \( A \). There is \( g \in \mathcal{A}(X) \) such that \( g \upharpoonright A = f \upharpoonright A \) and \( g \upharpoonright (X \setminus U) \equiv 0 \).
Proof. By Proposition 2.2 there is \( h \in \mathcal{A}(X) \) such that \( h \upharpoonright A \equiv 1 \) and \( h \upharpoonright (X \setminus U) \equiv 0 \). It is enough to put \( g = h \cdot f \). \( \square \)

For a continuous function \( f : X \to \mathbb{R} \), the function \( \tilde{f} : \beta X \to I \) is the continuous extension of \( f \). For a space \( X \) we will denote by \([X]^{<\omega}\) the hyperspace of all finite subsets of \( X \) equipped with the Vietoris topology. Similarly, \([X]^{\leq n}\) (resp. \([X]^n\)), where \( n \in \mathbb{N} \) is a subspace of \([X]^{<\omega}\) consisting of all at most \( n \)-element subsets of \( X \) (resp. exactly \( n \)-element subsets of \( X \)).

For a \( QS \)-algebra \( \mathcal{A}(X) \), a finite set \( F \in [\beta X]^{<\omega} \) and \( \varepsilon > 0 \), we put

\[
O_X(F; \varepsilon) = \{ f \in \mathcal{A}(X) : \forall x \in F \ [\tilde{f}(x)] < \varepsilon \}.
\]

Observe that if \( F \in [X]^{<\omega} \) then the above set is simply a basic open neighborhood of the zero function in \( \mathcal{A}(X) \). For a point \( x \in X \), we put

\[
\overline{O}_X(x; \varepsilon) = \{ f \in \mathcal{A}(X) : |f(x)| \leq \varepsilon \}.
\]

The above set is obviously closed in \( \mathcal{A}(X) \).

Lemma 2.4. Let \( E \in [X]^{<\omega} \), \( F \in [\beta X \setminus X]^{<\omega} \), and let \( \varepsilon > 0 \). The set \( O_X(E \cup F; \varepsilon) \) is dense in \( O_X(E; \varepsilon) \).

Proof. Take \( f \in O_X(E; \varepsilon) \). Let \( A \in [X]^{<\omega} \) and \( \delta > 0 \). It is enough to show that there is \( g \in O_X(E \cup F; \varepsilon) \) satisfying \( |g(x) - f(x)| < \delta \), for any \( x \in A \). To this end, let \( U \subseteq \beta X \) be an open neighborhood of \( A \cup E \) with \( \overline{U} \cap F = \emptyset \), where \( \overline{U} \) is the closure of \( U \) in \( \beta X \). By Proposition 2.3 there is \( g \in \mathcal{A}(X) \) such that \( g \upharpoonright A \cup E = f \upharpoonright A \cup E \) and \( g \upharpoonright (X \setminus U) \equiv 0 \). From \( \overline{U} \cap F = \emptyset \) it follows that \( g(x) = 0 \) for any \( x \in F \). So \( g \) is as required. \( \square \)

Let \( \Phi : \mathcal{A}(X) \to \mathcal{A}(Y) \) be a homeomorphism between \( QS \)-algebras \( \mathcal{A}(X) \subseteq C_p(X) \) and \( \mathcal{A}(Y) \subseteq C_p(Y) \). Since \( QS \)-algebras are homogeneous, we may without loss of generality assume that \( \Phi \) takes the zero function on \( X \) to the zero function on \( Y \).

For \( m, n, k \in \mathbb{N} \) we put

\[
Z_{k,m,n} = \{(E,y) \in [X]^{\leq k} \times Y : \Phi(O_X(E; \frac{1}{m})) \subseteq \overline{O}_Y(y; \frac{1}{n})\}.
\]

By \( \pi_Y : [X]^{\leq k} \times Y \to Y \) we denote the projection onto \( Y \), and by \( \pi_X : [X]^{\leq k} \times Y \to [X]^{\leq k} \) the projection onto \( [X]^{\leq k} \). We put

\[
C(k,m,n) = \pi_Y(Z_{k,m,n}).
\]

Remark 2.5. For reader’s convenience, we adopted some of the notation from [21] (cf. [23] Chapter 6.11). However, there is a small difference between the definitions of the sets \( C(k,m,n) \), given above, and their definitions in [21] (cf. [23] Chapter 6.11). Namely, we used the concept of \( QS \)-algebra, which was not done in [21]. To get the same sets as in [21] we would have to redefine sets \( O_X(F; \varepsilon) \) and \( \overline{O}_X(x; \varepsilon) \) replacing in their definitions \( \mathcal{A}(X) \) by \( C_p(X) \).

3. Auxiliary results

In this section we assume that \( X \) and \( Y \) are metric. For the reason given in Remark 2.5, the proposition below is not the same statement as [21] Lemma 3.2. To prove it we need to modify slightly the proof of [21] Lemma 3.2 (cf. [23] 6.11.7).

Proposition 3.1. For any \( k, m, n \in \mathbb{N} \), the set \( C(k,m,n) \) is closed in \( Y \).
Proof. We begin exactly as in [25, 6.11.7]. Let \( (y_i)_i \) be a sequence in \( C(k, m, n) \) converging to \( y \in Y \). For each \( i \), let \( A_i \subseteq X \) be a set of cardinality at most \( k \) such that
\[
\Phi(O_X(A_i; \frac{1}{m})) \subseteq \overline{O}_Y(y; \frac{1}{n}).
\]

Claim 1. There are an infinite set \( I \subseteq \mathbb{N} \) and a (possibly empty) subset \( A^0 \) of \( X \) and for every \( i \in I \) a partition \( A^0_i, A^i_1 \) of \( A_i \) such that one of the following statements is true:

(A) Every \( x \in X \) has a neighborhood meeting finitely many terms of the sequence \( (A_i)_i \) only.
(B) \( A^0 \neq \emptyset \) and

1. the sequence \( (A^0_i)_i \) converges to \( A^0 \) in the Vietoris topology.
2. Every \( x \in X \) has a neighborhood meeting finitely many terms of the sequence \( (A^i_1)_i \) only.

For the proof of Claim 1, see the solution of Exercise 1.11.20 in [25, page 537].

If (A) holds then we redefine \( A^0 \) to be the empty set. We also put \( A^0_i = \emptyset \) and \( A^i_1 = A_i \) for every \( i \in I \). So, every \( x \in X \) has a neighborhood \( U \) meeting finitely many terms of the sequence \( (A^i_1)_i \) only. We may without loss of generality assume that \( I = \mathbb{N} \).

We claim that
\[
\Phi(O_X(A^0; \frac{1}{m})) \subseteq \overline{O}_Y(y; \frac{1}{n}),
\]
which shows that \( y \in C(k, m, n) \).

Observe that this also shows that
\[(*) \quad A^0 \neq \emptyset, \text{ and hence (A) does not hold.}\]

Striving for a contradiction, suppose that there is a function \( f \in O_X(A^0; \frac{1}{m}) \) such that \( |\Phi(f)(y)| > \frac{1}{n} \). By continuity of \( \Phi \), there are \( p \in \mathbb{N} \) and a finite set \( B \subseteq X \) such that, for every \( g \in A(X) \) with \( f \equiv g \in O_X(B; \frac{1}{p}) \), we have \( |\Phi(g)(y)| > \frac{1}{n} \). We may assume that \( A^0 \subseteq B \). Since every point in \( X \) has a neighborhood meeting finitely many \( A^i_1 \)'s only, there is a neighborhood \( V \) of the finite set \( B \) and an integer \( i_0 \) such that
\[
V \cap A^i_1 = \emptyset,
\]
for every \( i > i_0 \). By Proposition 2.3 there is a function \( g \in A(X) \) such that \( g(x) = f(x) \) for every \( x \in B \), and \( g \equiv (X \setminus V) \equiv 0 \). Since \( g \) is continuous, there is an open set \( U \subseteq X \) containing \( A^0 \) such that
\[
g(U) \subseteq (-\frac{1}{m}, \frac{1}{m}).
\]

Put \( U' = U \cap V \). If (A) holds then \( A^i_1 = \emptyset \) for every \( i \), and so \( A^0_i \subseteq U' \) for every \( i \).
If (B) holds then \( A^i_1 \to A^0 \) which implies that, for all but finitely many \( i \), \( A^0_i \subseteq U' \).
So there is \( i_1 > i_0 \) such that
\[
A^i_1 \cap V = \emptyset \text{ and } A^0_i \subseteq U',
\]
for every \( i > i_1 \). It follows that \( g \equiv A^i_1 \equiv 0 \) and \( g(A^i_1) \subseteq (-\frac{1}{m}, \frac{1}{m}) \), for \( i > i_1 \). Hence \( g \in O_X(A_i; \frac{1}{m}) \). But then \( |\Phi(g)(y_i)| \leq \frac{1}{n} \) while, on the other hand, \( f - g \in O_X(B; \frac{1}{p}) \) and so \( |\Phi(g)(y)| > \frac{1}{n} \). This contradicts the continuity of \( \Phi(g) \). \( \square \)

For natural numbers \( m, n, k \in \mathbb{N} \), we define
\[
E(1, m, n) = C(1, m, n) \quad \text{and} \quad E(k, m, n) = C(k, m, n) \setminus C(k - 1, m, n), \text{ for } k > 1.
\]
It follows from Proposition 5.1 that the sets $E(k, m, n)$ are $F_\sigma$ in $Y$. Moreover, the continuity of $\Phi$ implies that for any $n \in \mathbb{N}$ we have

$$\bigcup_{m,k} E(k, m, n) = Y.$$  

For $y \in E(k, m, n)$, let us put

$$\mathcal{E}(y, m, n) = \pi_X(\pi_Y^{-1}(y) \cap Z_{k,m,n}),$$

i.e. $\mathcal{E}(y, m, n)$ is the family of all exactly $k$-element subsets $E \subseteq X$ satisfying $\Phi(O_X(E; \frac{1}{m})) \subseteq \overline{O}_Y(y; \frac{1}{n})$ (this follows from the assumption $y \in E(k, m, n)$).

Remark 2.4 also applies to the sets $E(k, m, n)$ and $\mathcal{E}(y, m, n)$, so lemmas given below are not the same statements as [21, Lemma 3.3] and [21, Lemma 3.4] (cf. [25, 6.11.8], [25, 6.11.9]). However, their proofs are almost the same.

**Lemma 3.2.** Suppose that a sequence $(y_i)_i$ in $E(k, m, n)$ converges to $y \in E(k, m, n)$, for some $k, m, n \in \mathbb{N}$. If $A_i \in \mathcal{E}(y_i, m, n)$, then $(A_i)_i$ contains a subsequence $(A_j)_j$ convergent to some element of $\mathcal{E}(y, m, n)$.

**Proof.** We just repeat the proof given in [25, 6.11.8]: Using Claim 1 and Claim 4 from the previous proof, we get a nonempty finite sets $A_0 \subseteq X$ and $A_1 \subseteq A_0$ such that $A_0 \to A_0$ in the Vietoris topology, where $A_0$ witnesses the fact that $y \in C(k, m, n)$. Since $y \in E(k, m, n)$, we have $|A_0| \geq k$ and hence $|A_0| \geq k$ for all but finitely many $i$. But then, for those infinitely many $i$, we have $|A_0| = k$ since $|A_i| = k$ for every $i$. This implies that $|A_0| \leq k$ so $|A_0| = k$ and therefore $A_0 \in \mathcal{E}(y, m, n)$.

**Proposition 3.3.** For all $k, m, n \in \mathbb{N}$ and $y \in E(k, m, n)$, the family $\mathcal{E}(y, m, n)$ is finite.

**Proof.** Repeat the proof of [21, Lemma 3.4] (cf. [25, 6.11.9]) applying Proposition 2.3 to get a function $g \in \mathcal{A}(X)$ used in that proof.

For $y \in E(k, m, n)$ we put

$$\alpha_{m,n}(y) = \bigcup \mathcal{E}(y, m, n).$$

Now, for natural numbers $p \geq k$ and $q \geq 1$, we define $G(k, m, n, p, q)$ to be the set of all $y \in E(k, m, n)$ such that $|\alpha_{m,n}(y)| = p$ and for all $x, x' \in \alpha_{m,n}(y)$ we have $d(x, x') \geq \frac{1}{q}$, where $d(\cdot, \cdot)$ is a metric on $X$.

From Proposition 3.3 we infer that, for any $n \in \mathbb{N},$

$$\bigcup_{k,m,p,q} G(k, m, n, p, q) = Y.$$  

Interchanging the roles of $X$ and $Y$, we can symmetrically define sets $H(k, m, n, p, q)$ covering $X$.

Once again, Remark 2.4 also applies to the sets $G(k, m, n, p, q)$ and $H(k, m, n, p, q)$. Therefore the lemma below, which summarizes what was proved about the sets $G(k, m, n, p, q)$ and $H(k, m, n, p, q)$ in [21] (cf. [25, Chapter 6.11]), requires a proof. However, as above, one can repeat the reasoning from [21] making some obvious changes.

**Lemma 3.4.** (Marciszewski) Let $k, m, p, q \in \mathbb{N}$. We have $G(k, m, 1, p, q) = \bigcup_{r \in \mathbb{N}} G_r$, where for each $r \in \mathbb{N}$ the set $G_r$ is open in $G(k, m, 1, p, q)$ and, for each $r \in \mathbb{N}$, there are continuous mappings $f_i : G_r \to X$, $i = 1, \ldots, p$, such that $\alpha_{m,1}(y) = \{f_1(y), \ldots, f_p(y)\}$ for $y \in G_r$. Also $X = \bigcup_{s \in \mathbb{N}} H_s$, where each of the sets $H_s$ is open in $H(k', m', m + 1, p', q')$, for some $k', m', p', q' \in \mathbb{N}$ and such that, for every $s \in \mathbb{N}$, there are continuous mappings $g^j_s : H_s \to Y$, $j = 1, \ldots, p'_s$, satisfying the
following property: for every \( y \in G_r \) and \( s_1, \ldots, s_p \in \mathbb{N} \) such that \( f_i(y) \in H_{s_i} \), there exist \( i \leq p \) and \( j \leq p'_{s_i} \), such that \( g_i^{s_i}(f_i(y)) = y \).

4. The main result

In this section we also assume that spaces \( X \) and \( Y \) are metric. We are going to prove the following theorem.

**Theorem 4.1.** Suppose that for metric spaces \( X \) and \( Y \), the spaces \( A(X) \) and \( A(Y) \) are homeomorphic. Then \( Y \) (respectively \( X \)) is a countable union of closed subsets which are homeomorphic to subsets of \( X \) (respectively \( Y \)).

First, we will show that the sets \( G(k, m, n, p, q) \) and \( H(k, m, n, p, q) \) (covering \( Y \) and \( X \), respectively, for a fixed \( n \)) are \( F_r \) (in [21] it was proved that they are \( G_{\delta} \)).

To this end, we need to fix some notation first.

For \( k, m, n, r \in \mathbb{N} \) let us define

\[ W_{k, m, n, r} = \{(F, y) \in [[X]^\leq k]^{\leq r} \times Y : \forall E \in F \Phi(O_X(E; \frac{1}{m})) \subseteq \overline{O_Y(y; \frac{1}{n})}\} \]

**Lemma 4.2.** The set \( W_{k, m, n, r} \) is closed in \([[X]^\leq k]^{\leq r} \times Y \).

**Proof.** Let \((F_0, y) \in \left([[X]^\leq k]^{\leq r} \times Y \right) \setminus W_{k, m, n, r} \). Then, for some \( E_0 \in F \), we have \( \Phi(O_X(E_0; \frac{1}{m})) \not\subseteq \overline{O_Y(y; \frac{1}{n})} \); hence there is \( f \in O_X(E_0; \frac{1}{m}) \) such that \( |\Phi(f)(y)| > \frac{1}{n} \).

Let

\[ U_0 = \{E \in [X]^\leq k : E \subseteq f^{-1}(-\frac{1}{m}, \frac{1}{m})\} \]

Clearly, \( U_0 \) is an open neighborhood of \( E_0 \) in the space \([X]^\leq k \). The set

\[ U = \{F \in [[X]^\leq k]^{\leq r} : F \cap U_0 \neq \emptyset\} \]

is an open (in the Vietoris topology of \([[X]^\leq k]^{\leq r} \times Y \)) neighborhood of \( F_0 \).

Now, it is easy to see that \( U \times (Y \setminus \Phi(f)^{-1}(-\frac{1}{m}, \frac{1}{m})) \) is an open neighborhood of \((F_0, y)\) disjoint from \( W_{k, m, n, r} \).

Let us denote by \( S_{k, m, n, r} \) the closure of \( W_{k, m, n, r} \) in \([[\beta X]^\leq k]^{\leq r} \times \beta Y \). The following two lemmas are analogous to [26, Lemma 1.4] and [26, Lemma 1.5] respectively.

**Lemma 4.3.** If \((F, y) \in S_{k, m, n, r} \) and \( y \in Y \) then, for each \( E \in F \), we have \( \Phi(O_X(E; \frac{1}{m})) \subseteq \overline{O_Y(y; \frac{1}{n})} \).

**Proof.** Otherwise, there is \( E \in F \) and \( f \in A(X) \) such that \( \tilde{f}(E) \subseteq (\frac{1}{m}, \frac{1}{n}) \) and \( |\Phi(f)(y)| > \frac{1}{n} \).

Put

\[ U_0 = \{D \in [\beta X]^\leq k : D \subseteq \tilde{f}^{-1}(-\frac{1}{m}, \frac{1}{m})\} \]

Clearly \( U_0 \) is an open neighborhood of \( E \) in \([\beta X]^\leq k \). The set

\[ U = \{H \in [[\beta X]^\leq k]^{\leq r} : H \cap U_0 \neq \emptyset\} \]

is an open neighborhood of \( F \) in \([[\beta X]^\leq k]^{\leq r} \) and hence \( U \times (\beta Y \setminus \overline{\Phi(f)^{-1}(-\frac{1}{m}, \frac{1}{m})}) \) has a nonempty intersection with \( W_{k, m, n, r} \) which contradicts the definition of \( S_{k, m, n, r} \).

**Lemma 4.4.** If \((F, y) \in S_{k, m, n, r} \) and \( y \in E(k, m, n) \), then \( F \in [[X]^\leq k]^{\leq r} \).

**Proof.** Otherwise, there exists \( E \in F \) such that \( E \cap (\beta X \setminus X) \neq \emptyset \). Let \( E_1 = E \setminus (\beta X \setminus X) \). By **Lemma 2.4** the set \( O_X(E; \frac{1}{m}) \) is dense in \( O_X(E_1; \frac{1}{m}) \) so, by the continuity of \( \Phi \) and **Lemma 4.3**, we have

\[ \Phi(O_X(E_1; \frac{1}{m})) \subseteq \Phi(O_X(E_1; \frac{1}{m})) = \Phi(O_X(E; \frac{1}{m})) \subseteq \overline{O_Y(y; \frac{1}{n})} \]

Since \( |E_1| \leq k - 1 \) and \( E(k, m, n) = C(k, m, n) \setminus \overline{C(k - 1, m, n)} \), we get a contradiction.

\[ \square \]
Let $\rho_{\beta Y} : [[\beta X]^{\leq r}]^{\leq r} \times \beta Y \to \beta Y$ be the projection. For $k, m, n, r \in \mathbb{N}$ let
$$E(k, m, n, r) = E(k, m, n) \cap \rho_{\beta Y}(S_{k, m, n, r}).$$
Since $S_{k, m, n, r}$ is compact, the set $E(k, m, n, r)$ is $F_\sigma$ in $Y$ (recall that $E(k, m, n)$ is $F_\sigma$ in $Y$). Moreover, by Proposition 3.3 and the definition of $W_{k, m, n, r}$, we have

$$(1) \quad \bigcup_r E(k, m, n, r) = E(k, m, n).$$

Let us put
$$V_{k, m, n, r} = \rho_{\beta Y}^{-1}(E(k, m, n, r)) \cap S_{k, m, n, r}.$$ Compactness of $S_{k, m, n, r}$ implies that the mapping $p_{k, m, n, r} = \rho_{\beta Y} \mid V_{k, m, n, r}$ is perfect (the restriction of a perfect mapping to the full preimage is perfect). Moreover, from Lemmas 4.4 and 4.2 it follows that $V_{k, m, n, r} \subseteq W_{k, m, n, r}$. In particular, the set $E(k, m, n, r)$ is a perfect image of a subset of $[[X]^{\leq k}]^{\leq r} \times Y$.

Let $\bigcup [[X]^{\leq k}]^{\leq r} \to [X]^{\leq k}$ be the “union operator” i.e. $\bigcup F = \bigcup F$. It is well known that this function is continuous (see [25, 1.11.7]).

Note that for a natural number $p \leq k \cdot r$, the set $[X]^p$ consisting of exactly $p$-element subsets of $X$ is $F_\sigma$ in $[X]^{\leq k \cdot r}$. Hence, the set $\bigcup^{-1}([X]^p)$ is $F_\sigma$ in $[[X]^{\leq k}]^{\leq r}$ (and, of course, the space $[[X]^{\leq k}]^{\leq r}$ is a finite union of preimages $\bigcup^{-1}([X]^p)$). Therefore the set
$$(\pi_X')^{-1}(\bigcup^{-1}([X]^p)) \cap V_{k, m, n, r},$$
where $\pi_X' : [X]^{\leq k}]^{\leq r} \times Y \to [[X]^{\leq k}]^{\leq r}$ is the projection, is $F_\sigma$ in $V_{k, m, n, r}$.

Put
$$E(k, m, n, r, p) = p_{k, m, n, r}((\pi_X')^{-1}(\bigcup^{-1}([X]^p))) \cap V_{k, m, n, r}.$$ Since the mapping $p_{k, m, n, r}$ is perfect, the sets defined above are $F_\sigma$ in $E(k, m, n, r)$ and hence in $Y$. In addition, they have the following important property:

$$(2) \quad \text{For } y \in E(k, m, n, r, p), \text{ the set } \alpha_{m, n}(y), \text{ defined in Section 3, has exactly } p \text{ elements.}$$

We need one more refinement. For $y \in E(k, m, n, r, p)$ and $q \in \mathbb{N}$, let
$$E(k, m, n, r, p, q) = \{ y \in E(k, m, n, r, p) : [x, x' \in \alpha_{m, n}(y), x \neq x'] \Rightarrow d(x, x') \geq \frac{1}{q} \},$$
where $d(\cdot, \cdot)$ is a metric on $X$.

We need to show that this set is closed in $E(k, m, n, r, p)$. To this end, we follow the proof of Claim 1 in [25, 6.11.10]. Since our notation is a little bit different, we enclose a proof for the convenience of the reader.

**Proposition 4.5.** For any $k, m, n, r, p, q \in \mathbb{N}$, the set $E(k, m, n, r, p, q)$ is closed in $E(k, m, n, r, p)$ and hence it is $F_\sigma$ in $Y$.

**Proof.** Let $(y_i)$, be a sequence in $E(k, m, n, r, p, q)$ convergent to $y \in E(k, m, n, r, p)$. We need to prove that $y \in E(k, m, n, r, p, q)$. So, for every $i$, let
$$\alpha_{m, n}(y_i) = \{ x_1^i, \ldots, x_p^i \}.$$ Consider the sequence $(x_1^i)_i$. For every $i$, there is $E_i \in \mathcal{E}(y_i, m, n)$ such that $x_1^i \in E_i$. By Lemma 4.2 there is a subsequence $(E_i)_{i \in J}$ converging to $E \in \mathcal{E}(y, m, n)$. It follows from the definition of the Vietoris topology that there is an infinite $I_1 \subseteq J$ and a point $x^1 \in \alpha_{m, n}(y)$, such that the sequence $(x_1^i)_{i \in I_1}$ converges to $x^1$.

Let us consider now the sequence $(x_2^i)_{i \in I_1}$. By the same argument, it contains a subsequence converging to some $x^2 \in \alpha_{m, n}(y)$. Proceeding up to $p$ we get that
$$(x_i^j)_{i \in I_p} \to x^j \in \alpha_{m, n}(y).$$
Put $C = \{x^1, \ldots, x^p\}$. By the definition of $E(k,m,n,r,p,q)$, the elements of $\alpha_{m,n}(y_i)$ are $\frac{1}{q}$ apart, for each $i$. This implies that $|C| = p$ and elements of $C$ are also $\frac{1}{q}$ apart. Since $C \subseteq \alpha_{m,n}(y)$ and $y \in E(k,m,n,r,p)$, we conclude from (2) that $C = \alpha_{m,n}(y)$ and hence $y \in E(k,m,n,r,p,q)$.

Now, using (1), (2) and the definition of $E(k,m,n,r,p,q)$, it is easy to see that

$$\bigcup_r E(k,m,n,r,p,q) = G(k,m,n,p,q),$$

and hence, by Proposition 4.5, we conclude that $\dim Y$ (by [12, 7.3.2] and [12, 7.3.4]). Applying the countable sum theorem (see [12, 7.2.1]),

Proof. From Theorem 4.1 it follows that $G(k,m,n,p,q)$ and $H(k,m,n,p,q)$ are also $G_\delta$ (see [21] or [25] Chapter 6.11).

Now it is easy to prove Theorem 4.1.

Proof of Theorem 4.1. It suffices to prove that $Y$ is a countable union of $F_\sigma$ sets which are homeomorphic to subsets of $X$.

Since sets $G(k,m,1,p,q)$ and $H(k',m',m+1,p',q')$ are $F_\sigma$, the sets $G_r$ and $H_s$ defined in Lemma 3.4 are also $F_\sigma$, for any $r, s \in \mathbb{N}$. For $r, s \in \mathbb{N}$, $i \leq p$, put $G^i_r = f^{-1}_i(H_s)$, where $f_i : G_r \to X$ is the function defined in Lemma 3.4. We have

$$(3) \quad G_r = \bigcup_{i \leq p, s \in \mathbb{N}} G^i_r,$$

and the sets $G^i_r$ are $F_\sigma$, being continuous preimages of $F_\sigma$ sets $H_s$.

For $j \leq p$, consider the mapping $(g^i_r \circ f_j)$ on $G^i_r$ into $Y$, where $g^i_r$ is a function from Lemma 3.4. The set of all fixed points of this function is closed in $G^i_r$, hence $F_\sigma$ in $Y$, and is mapped by $f_j$ homeomorphically onto a subset of $H_s$.

Since there are countably many functions of the form $g^i_r \circ f_j$, we get countably many $F_\sigma$ sets in $Y$ homeomorphic to subsets of $X$. By (3) and Lemma 3.4 it follows that those sets cover $Y$.

Remark 4.7. From Remark 4.6 it follows easily that the set of fixed points of the function $(g^i_r \circ f_j)$ on $G^i_r$ is not only $F_\sigma$, but also $G_\delta$. Hence [21] Proposition 2.2 implies that the closed sets, on which the space $Y$ (respectively $X$) is partitioned, are homeomorphic to $G_\delta$ subsets of $X$ ($G_\delta$ subsets of $Y$, respectively).

5. Corollaries

From Theorem 4.1 we immediately obtain the following.

Corollary 5.1. Suppose that $X$ and $Y$ are metrizable spaces with homeomorphic $\mathbb{Q}$-algebras $A(X)$ and $A(Y)$ (in particular, we can assume that $C_p(X)$ and $C_p(Y)$ or $C^*_p(X)$ and $C^*_p(Y)$ are homeomorphic). Then $\dim X = \dim Y$.

Proof. From Theorem 4.1 it follows that $Y = \bigcup_{n \in \mathbb{N}} F_n$, for each $n \in \mathbb{N}$ the set $F_n$ is closed in $Y$ and homeomorphic to a subset of $X$. So for each $n \in \mathbb{N}$, $\dim F_n \leq \dim X$ (by [12] 7.3.2 and [12] 7.3.4). Applying the countable sum theorem (see [12] 7.2.1), we get $\dim Y \leq \dim X$. Interchanging the roles of $X$ and $Y$, we obtain the opposite inequality.

As a corollary we obtain Theorem 4.1.

Recall that Gul’ko proved that, for any Tychonoff space $X$, the covering dimension is determined by the uniform structure of $C_p(X)$. He did it by proving a
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result analogous to Corollary 5.1 above (see [10, Proposition 1.4]), where instead of a homeomorphism between QS-algebras a stronger notion of a uniform homeomorphism was considered. Then, assuming that $C_p(X)$ and $C_p(Y)$ are uniformly homeomorphic, where $X$ and $Y$ are arbitrary Tychonoff spaces, he was able to deduce that $\dim X = \dim Y$ by considering uniformly homeomorphic QS-algebras $A(\varphi(X))$ and $A(\psi(Y))$, where $\varphi(X)$, $\psi(Y)$ are metrizable spaces depending on $X$ and $Y$, respectively (see [10, Section 2]).

Using exactly the same technique and applying Corollary 5.1 instead of [10, Proposition 1.4] we can prove Theorem 1.2.

Applying Theorem 4.1 exactly as it was done in the proof of Corollary 5.1 and using Remark 4.7, the following general result can be concluded (cf. [22, 2.7]).

Corollary 5.2. Let $\mathcal{P}$ be the class of metrizable spaces with the following properties:

(i) if $X \in \mathcal{P}$ and $Z$ is a $G_\delta$-subset of $X$ then $Z \in \mathcal{P}$,
(ii) if $X$ is a metrizable space which is a countable union of closed subspaces $X_n \in \mathcal{P}$ then $X \in \mathcal{P}$.

Then, for metrizable spaces $X$ and $Y$ which are $t$-equivalent, $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$.

It was proved by Marciszewski in [21] that if, for metrizable $X$ and $Y$, spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic, then $X$ is countable-dimensional if and only if $Y$ is countable-dimensional. Recall that a normal space is countable-dimensional if it can be represented as a countable union of normal finite-dimensional subspaces; it is strongly countable-dimensional if it can be represented as a countable union of closed finite-dimensional subspaces. The theorem of Marciszewski can be easily derived from Theorem 4.1 just as it was done in [21]. Moreover, since the sets in Theorem 4.1 into which we are partitioning space $Y$ (and $X$, respectively), are closed, we can conclude also the following.

Corollary 5.3. Let $X$ and $Y$ be metrizable spaces such that the spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic. Then $X$ is strongly countable-dimensional if and only if $Y$ is strongly countable-dimensional.

Proof. Apply Corollary 5.2 with $\mathcal{P}$ being the class of strongly countable-dimensional spaces: property (i) follows from [13, 5.2.4] whereas property (ii) from [13, 5.2.14].

We say that a metric space is strongly $\sigma$-complete if it is a countable union of closed completely metrizable subspaces. From Corollary 5.2 we can easily deduce similar result concerning strong $\sigma$-completeness.

Corollary 5.4. If $X$ and $Y$ are metric spaces and $C_p(X)$ is homeomorphic to $C_p(Y)$, then $X$ is strongly $\sigma$-complete if and only if $Y$ is strongly $\sigma$-complete.

The latter corollary generalizes a theorem of Marciszewski and Pelant, who proved analogous result assuming that function spaces are uniformly homeomorphic (see [23, Section 4]).

Using Theorem 4.1 we can also produce continuum many metrizable continua, i.e. compact connected spaces, with pairwise nonhomeomorphic function spaces (cf. [22, 2.8]).

Example 5.5. There exists a family $\{K_\alpha : \alpha < 2^{\aleph_0}\}$ of 1-dimensional metrizable continua such that $C_p(K_\alpha)$ is not homeomorphic to $C_p(K_\beta)$ for $\alpha \neq \beta$.

To construct the required family we use the method described in [22, page 349]: Let $M$ be a Cook continuum, i.e. a hereditarily indecomposable continuum such that, for every subcontinuum $K \subseteq M$, every continuous mapping $f : K \to M$ is
either the identity or is constant (see [11]). Since $M$ is hereditarily indecomposable, it has continuum many pairwise disjoint nontrivial subcontinua $K_{\alpha} \subseteq M$ (see [20, §48.VI]). A straightforward application of Theorem 4.1 combined with the Baire category theorem and Janiszewski’s theorem (see [20, §47.III.1]) shows that the family $\{K_{\alpha} : \alpha < 2^{\aleph_0}\}$ is as required.

**Remark 5.6.** It is well known that, for a separable metrizable space $X$, $\dim X = \operatorname{Ind} X = \operatorname{Ind} X$ and there is a metrizable space $M$ with $\dim M \neq \operatorname{Ind} M$ (cf. [12, page 409]). It is claimed in [22] that the small inductive dimension of a metrizable space $X$ is entirely determined by a uniform structure of $C_p(X)$ and that this can be proved by a direct application of [22, 2.7] (which is a version of Corollary 5.2 for uniform homeomorphism between function spaces). However, this argument does not work. Namely, the countable sum theorem is not valid for the dimension $\dim$ in the class of all metrizable spaces (cf. [12, page 400]). Even the question whether, for metrizable spaces $X$ and $Y$, $\dim X = \dim Y$ provided $C_p(X)$ is linearly homeomorphic to $C_p(Y)$, seems to be open.

Of course, for metrizable spaces $X$ and $Y$, if $C_p(X)$ is homeomorphic to $C_p(Y)$ then $\operatorname{Ind} X = \operatorname{Ind} Y$. This follows from Corollary 5.1 and the well-known fact that $\operatorname{Ind} X = \dim X$, for any metrizable space $X$.

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