SYMMETRIC GROTHENDIECK POLYNOMIALS, SKEWH CAUCHY IDENTITIES, AND DUAL FILTERED YOUNG GRAPHS

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Abstract. Symmetric Grothendieck polynomials are analogues of Schur polynomials in the K-
theory of Grassmannians. We build dual families of symmetric Grothendieck polynomials using
Schur operators. With this approach we prove skew Cauchy identity and then derive various
applications: skew Pieri rules, dual filtrations of Young’s lattice, generating series and enumerative
identities. We also give a new explanation of the finite expansion property for products of
Grothendieck polynomials.

1. Introduction

Symmetric Grothendieck polynomials, also known as stable Grothendieck polynomials, are cer-
tain K-theoretic deformations of Schur functions. These functions were first studied by Fomin and
Kirillov [8] as a stable limit of more general Grothendieck polynomials that generalize Schubert
polynomials in another direction.

The symmetric Grothendieck polynomial $G_{\lambda}$ can be defined by the following combinatorial for-

\[ G_{\lambda}(x_1, x_2, \ldots) = \sum_T (-1)^{|T|-|\lambda|} \prod_{i \geq 1} x_i^{\text{#}i’s \text{ in } T}, \]

where the sum runs over shape $\lambda$ set-valued tableaux $T$, a generalization of semistandard Young
tableaux so that boxes contain sets of integers.

Being a generalization of the Schur basis, symmetric Grothendieck polynomials share with it
many similarities. However $\{G_{\lambda}\}$ is inhomogeneous and of unbounded degree when defined for
infinitely many variables $(x_1, x_2, \ldots)$, i.e., it is an element of the completion $\hat{\Lambda}$ of the ring $\Lambda$ of
symmetric functions. For example, $G_{(1)} = e_1 - e_2 + e_3 - \cdots$, where $e_k$ is the $k$th elementary
symmetric function. It is thus surprising that $\{G_{\lambda}\}$ is a basis of a certain ring: each product $G_{\mu}G_{\nu}$
is a finite linear combination of $\{G_{\lambda}\}$. When $\mu$ is a single row or column partition, finite expansion
was a consequence of Pieri rules proved by Lenart [13]. In a general case, this property follows from
a Littlewood-Richardson rule given by Buch [4]. The ring spanned by Grothendieck polynomials is
related to the K-theory of Grassmannians [4]. There is also an important basis $\{g_{\lambda}\}$ of $\Lambda$, dual to
$\{G_{\lambda}\}$, that was described via plane partitions and studied by Lam and Pylyavskyy [11].

In this paper we study symmetric skew Grothendieck polynomials via noncommutative Schur
operators. We used these operators in [22] to prove dualities for certain two-parameter deformations
of Grothendieck polynomials. Employing classical Schur operators turns out to be beneficial for
obtaining a number of new properties.

Our main results are the following.
1.1. **Skew Cauchy identity.** We prove the following identity that becomes our central object.

**Theorem 1.1.** Let $\mu, \nu$ be any fixed partitions, then

$$
\sum_{\lambda} G_{\lambda/\mu}(x_1, x_2, \ldots) g_{\lambda/\nu}(y_1, y_2, \ldots) = \prod_{i,j} \frac{1}{1 - x_i y_j} \sum_{\kappa} G_{\nu/\kappa}(x_1, x_2, \ldots) g_{\mu/\kappa}(y_1, y_2, \ldots).
$$

We give a number of applications of this identity using it in both operator and generating function forms. Our approach is based on Schur operators as in Fomin [6]. For Schur functions such an identity was given by Zelevinsky in the Russian translation of Macdonald’s book [14]. Macdonald [14, Ch.1 notes] mentioned that this result has apparently been discovered independently by Lascoux, Towber, Stanley, Zelevinsky. It is also known for analogues, e.g., shifted Schur functions [6, 17]. Borodin’s symmetric functions [2] generalizing Hall-Littlewood polynomials, also satisfy Cauchy identities which is important in certain stochastic models [3]; special cases of these symmetric functions have similarity with Grothendieck polynomials [2].

1.2. **Skew Pieri rules.** One can immediately obtain some Pieri-type formulas from skew Cauchy identities (e.g., for $\mu = \emptyset$ or $\nu = \emptyset$, which is quite useful). But there is more: it is a general principle that skew Cauchy identities imply skew Pieri formulas. For Hall-Littlewood polynomials this was shown by Warnaar [21] using the $q$-binomial theorem for Macdonald polynomials [12]. In the same way we formulate a general skew Pieri-type formula for dual families of symmetric functions (Lemma 7.8) and obtain various Pieri formulas for Grothendieck polynomials. We then prove skew Pieri rules (Theorem 7.1) for multiplying the skew Grothendieck polynomials $G_{\lambda/\mu}$ and $g_{\mu/\nu}$ on elements indexed by single row or column partitions. A skew Pieri rule for Schur functions was first proved by Assaf and McNamara [1]. Generalizations were studied in [9, 10, 20].

1.3. **Basis phenomenon.** It might seem miraculous that products $G_{\mu}G_{\nu}$ expand finitely in the basis $\{G_{\lambda}\}$, as known proofs of this property rely on a Littlewood-Richardson (LR) rule [4, 16], i.e., explicit combinatorial interpretations of multiplicative structure coefficients. But is there is a more direct way to see this on the level of symmetric functions? We give a quite transparent explanation of this property without appealing to any LR rule. It is what we call a damping condition of a dual basis is crucial here. Combined with a duality automorphism, the following property of a dual basis prevents infinite expansions: if for a symmetric polynomial we have $f_{\lambda/\mu}(x_1, \ldots, x_n) \neq 0$ then $\lambda_1$ or $\ell(\lambda)$ is bounded from above by a constant depending on $\mu$ and $n$. Symmetric polynomials whose formulas are defined via tableaux with strict row or column conditions have this property. However the dual Grothendieck polynomial $g_{\lambda/\mu}$ does not satisfy it. This issue can be resolved using the polynomials $\omega(g_{\lambda/\mu})$ instead. See Section 8 for details and general conditions.

1.4. **Dual filtered Young graphs.** While the homogeneous Schur case corresponds to a graded ring and a combinatorial object behind this is self-dual graded Young’s lattice, Grothendieck polynomials correspond to a filtered ring and dual filtered Young graphs. Dual filtered graphs introduced by Patrias and Pylyavskyy [15] is a K-theoretic analogue of Stanley’s differential posets [18] and Fomin’s dual graded graphs [5]. Our approach provides new types of dual filtered Young graphs. As it was mentioned in [15], apparently the most important filtration of dual graded graphs is the so-called Möbius deformation as it is related to K-theoretic insertion and LR rules. Even though

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$^1$The shape $\lambda/\mu$ is not the usual skew shape $\lambda/\mu$. 
this deformation (sometimes) produces a dual filtered graph, it is unclear why. We show that the Möbius deformation of Young’s lattice can be obtained from our Cauchy deformation by a natural transform related to Möbius inversion. This reveals the presence of a Möbius deformation on Young’s lattice. In addition, the constructions of dual filtered graphs give enumerative identities as applications to the normal ordering of differential operators.

Acknowledgements. I am grateful to Alexei Borodin, Askar Dzhumadil’daev, Thomas Lam, Igor Pak, Leonid Petrov, and Pavlo Pylyavskyy for stimulating and helpful conversations.

2. Partitions and Young diagrams

A partition is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ of nonnegative integers with only finitely many nonzero terms. The weight of a partition $\lambda$ is the sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots$. Any partition $\lambda$ is represented as a Young diagram which contains $\lambda_i$ boxes in its $i$th row ($i = 1, 2, \ldots$); equivalently, it is the set $\{(i, j) : 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\}$, where $\ell = \ell(\lambda)$ is the number of nonzero parts of $\lambda$. The partition $\lambda'$ denotes the conjugate of $\lambda$ obtained by transposing its diagram. We use English notation for Young diagrams, index columns from left to right and rows from top to bottom.

The following notation is used throughout the paper.

- Let $I(\mu)$ be the set of removable boxes of $\mu$, it corresponds to inner corners. Denote $i(\mu) := \#I(\mu)$ the number of removable boxes of $\mu$. For partitions $\lambda \supset \mu$, define the following extension of a skew shape that will be important:

$\frac{\lambda}{\mu} := \lambda \cup I(\mu)$.

E.g. $(5331)/(432)$ consists of the skew shape $(5331)/(432) = \{\square\}$ and $I(432) = \{\square\}$.

- Denote by $a(\lambda/\mu)$ the number of open boxes (o) of $I(\mu)$ that do not lie in the same column with any box of $\lambda/\mu$. Equivalently, it is just the number of columns of $\lambda/\mu$ that are not columns of $\lambda/\mu$. E.g. $a((5331)/(432)) = 2$ (see the diagram above).

- Denote by $c(\lambda/\mu)$ and $r(\lambda/\mu)$ the number of columns and rows of $\lambda/\mu$. E.g. $c((5331)/(432)) = r((5331)/(432)) = 2$ and $r((5331)/(433)) = 3$.

We also use the following standard terminology: $\lambda/\mu$ is called a horizontal (resp. vertical) strip if no two boxes of $\lambda/\mu$ lie in the same column (resp. row), equivalently, $|\lambda/\mu| = c(\lambda/\mu)$ (resp. $|\lambda/\mu| = r(\lambda/\mu)$); $\lambda/\mu$ is a rook strip if no two boxes lie in the same row or column, equivalently, $\lambda/\mu \subset I(\lambda)$.

Let $\mathbb{Y}$ be the Young lattice, i.e., an infinite graph whose vertices are indexed by partitions and edges are given by $(\lambda, \lambda + \square)$. The Möbius function of Young’s lattice $\mathbb{Y}$ is given by (e.g., [19])

$$\mu(\lambda, \mu) = \begin{cases} (-1)^{|\lambda/\mu|}, & \text{if } \lambda/\mu \text{ is a rook strip;} \\ 0, & \text{otherwise.} \end{cases}$$

Hence for functions $f, g$ defined on $\mathbb{Y}$, the Möbius inversion takes the form

$$f(\lambda) = \sum_{\mu \subset \lambda} g(\mu), \quad g(\lambda) = \sum_{\lambda/\mu \text{ rook strip}} (-1)^{|\lambda/\mu|} f(\mu).$$
3. Schur operators

Consider the free \( \mathbb{Z} \)-module \( \mathbb{Z}P = \bigoplus_{\lambda} \mathbb{Z} \cdot \lambda \) with a basis of all partitions.

**Definition 3.1.** Let \( u = (u_1, u_2, \ldots) \) and \( d = (d_1, d_2, \ldots) \) be sets of linear operators on \( \mathbb{Z}P \), called *Schur operators*, that act on bases for each \( i \geq 1 \) as follows:

\[
\begin{align*}
    u_i \cdot \lambda &= \begin{cases} 
        \lambda + \square \text{ in column } i, & \text{if possible}, \\
        0, & \text{otherwise}
    \end{cases}, \\
    d_i \cdot \lambda &= \begin{cases} 
        \lambda - \square \text{ in column } i, & \text{if possible}, \\
        0, & \text{otherwise}.
    \end{cases}
\end{align*}
\]

The operators \( u, d \) build Young diagrams by adding or removing boxes. These operators are noncommutative but they satisfy the following commutation relations that can easily be checked on bases. Let \( [a, b] = ab - ba \) denote the commutator.

**Lemma 3.2** ([6]). The following commutation relations hold for the operators \( u, d \):

- **non-local:** \( [u_j, u_i] = [d_j, d_i] = 0, \ |i - j| \geq 2 \)
- **local Knuth:** \( [u_{i+1}, u_i] = [u_{i+1}u_i, u_{i+1}] = [d_{i+1}, d_i] = [d_{i+1}d_i, d_{i+1}] = 0 (i \geq 1) \)
- **conjugate:** \( [d_j, u_i] = 0 (i \neq j), \ d_{i+1}u_{i+1} = u_id_i (i \geq 1), \ d_1u_1 = 1. \)

Schur operators build Schur polynomials and provide a unified approach for studying their various properties such as Cauchy identities and RSK [6]. We generalize this approach for a K-theoretic setting of Grothendieck polynomials.

### 3.1. Grothendieck-Schur operators

Let \( \beta \) be a (scalar) parameter and consider the free \( \mathbb{Z}[\beta] \)-module \( \mathbb{Z}[\beta]P = \bigoplus_{\lambda} \mathbb{Z}[\beta] \cdot \lambda \).

**Definition 3.3.** Let \( \tilde{u} = (\tilde{u}_1, \tilde{u}_2, \ldots) \) and \( \tilde{d} = (\tilde{d}_1, \tilde{d}_2, \ldots) \) be linear operators acting on \( \mathbb{Z}[\beta]P \) and defined via the Schur operators as follows:

\[
\begin{align*}
    \tilde{u}_i := u_i - \beta u_id_i = u_i(1 - \beta d_i), \\
    \tilde{d}_i := d_i + \beta d_i^2 + \beta^2 d_i^3 + \cdots = (1 - \beta d_i)^{-1}d_i.
\end{align*}
\]

For example,

\[
\begin{align*}
    \tilde{u}_2 \cdot \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline \end{array} &= \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline \end{array}, & \tilde{u}_2 \cdot \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline \end{array} &= \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline \end{array}, \\
    \tilde{d}_2 \cdot \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline \end{array} &= \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline \end{array}, & \tilde{d}_2 \cdot \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline \end{array} &= \begin{array}{|c|c|c|} \hline 1 & 0 & \beta \\ \hline \end{array} + \beta^2 \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline \end{array}.
\end{align*}
\]

For a diagram, the operator \( \tilde{u}_i \) adds a box in the \( i \)th column if possible or applies the following *loop* condition: if the lowest box in the \( i \)th column is removable it multiplies the result by \(-\beta\), since the operator \( u_id_i \) results 1 (an identity) if the box in the \( i \)th column is removable, and 0 otherwise. The operator \( d_i \) removes boxes from the \( i \)th column graded by \( \beta \) in all possible ways. We defined these operators in [22] when we studied duality properties of stable Grothendieck polynomials.

**Lemma 3.4.** The following commutation relations hold for the operators \( \tilde{u}, \tilde{d} \):

- **non-local:** \( [\tilde{u}_i, \tilde{u}_j] = [\tilde{d}_i, \tilde{d}_j] = 0, \ |i - j| \geq 2 \)
- **local Knuth:** \( [\tilde{u}_{i+1}\tilde{u}_i, \tilde{u}_i + \tilde{u}_i] = [\tilde{d}_i\tilde{d}_i, \tilde{d}_i + \tilde{d}_i] = 0 (i \geq 1) \)
- **conjugate:** \( [\tilde{u}_i, \tilde{d}_j] = 0 |i - j| \geq 2, \ [\tilde{u}_{i+1}, \tilde{d}_i] = 0 (i \geq 1), \ \tilde{d}_1\tilde{u}_1 = 1. \)

**Proof.** In the appendix. \( \square \)
Remark 3.5. For $\beta = 0$ everything turns into Schur operators. For general $\beta$, the identities are different than in Lemma 3.2. Local Knuth relations do not hold for $\tilde{u}, \tilde{d}$, but we have the given local relations instead. Conjugate relations are also different. In general, relations for $\tilde{u}, \tilde{d}$ are more complicated than for $u, d$. E.g., the proof of local relations uses the identity $[u_i d_i, u_{i+1} u_i] = 0$ for Schur operators that is not from the list of Lemma 3.2.

4. Symmetric skew Grothendieck polynomials

Let $x$ be an indeterminate (central variable, commuting with the $u, d$) and define the series

$$A(x) = \cdots + x \tilde{u}_2)(1 + x \tilde{u}_1), \quad B(x) = (1 + x \tilde{d}_1)(1 + x \tilde{d}_2)$$

From non-local and local relations given in Lemma 3.4 it is standard (e.g., [6, 7]) to deduce that

$$[A(x), A(y)] = 0, \quad [B(x), B(y)] = 0.$$

Let $\langle \cdot, \cdot \rangle$ be a bilinear pairing on $\mathbb{Z}[\beta]P$ given by $\langle \lambda, \mu \rangle = \delta_{\lambda \mu}$.

Definition 4.1. Define the skew Grothendieck polynomials $\{G_{\lambda/\mu}^\beta\}, \{g_{\lambda/\mu}^\beta\}$ as follows

$$G_{\lambda/\mu}^\beta(x_1, \ldots, x_n) := \langle A(x_n) \cdots A(x_1) \cdot \mu, \lambda \rangle, \quad g_{\lambda/\mu}(x_1, \ldots, x_n) := \langle B(x_n) \cdots B(x_1) \cdot \lambda, \mu \rangle.$$

Since the series $[A(x_i), A(x_j)] = [B(x_i), B(x_j)] = 0$ commute, the functions $G^\beta, g^\beta$ (indexed by pairs of partitions) are well-defined polynomials symmetric in $x_1, \ldots, x_n$. We can then extend these symmetric functions for infinitely many variables $(x_1, x_2, \ldots)$ by letting $n \to \infty$. Notice that $G_{\lambda/\mu}^\beta = g_{\lambda/\mu}^\beta = 0$ if $\mu \not\subset \lambda$.

Remark 4.2. The reason why we use $G_{\lambda/\mu}$ and not $G_{\lambda/\mu}$ is in boundary conditions. The function $G_{\lambda/\mu}$ depends on the shape $\lambda/\mu = \lambda/\mu \cup I(\mu)$, where $I(\mu)$ is the set of removable boxes of $\mu$. For example, we can compute that

$$G_{\lambda/\lambda}^\beta(x_1, x_2, \ldots) = \prod_{k \geq 1} (1 - \beta x_k)^{i(\lambda)} \neq G_{\varnothing} = 1.$$

This is also consistent with the notation in [4] (though the functions are defined in a different way).

Proposition 4.3. The following branching formulas hold:

$$G_{\lambda/\mu}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \sum_\nu G_{\lambda/\nu}(x_1, \ldots, x_n) G_{\nu/\mu}(y_1, \ldots, y_m),$$

$$g_{\lambda/\mu}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \sum_\nu g_{\lambda/\nu}(x_1, \ldots, x_n) g_{\nu/\mu}(y_1, \ldots, y_m).$$

For a single variable $x$ we have

$$G_{\lambda/\mu}^\beta(x) = \begin{cases} (1 - \beta x)^{a(\lambda/\mu)} x^{\lambda/\mu}, & \text{if } \lambda/\mu \text{ is a horizontal strip;} \\ 0, & \text{otherwise} \end{cases}$$

$$g_{\lambda/\mu}^\beta(x) = \begin{cases} \beta^{\lambda/\mu - c(\lambda/\mu)} x^{c(\lambda/\mu)}, & \text{if } \mu \subset \lambda; \\ 0, & \text{otherwise.} \end{cases}$$
Proof. The branching formulas are immediate from the definition. For a single variable $x$ we have the following. For any $i$, the operator $1 + xu_i$ applied to $\mu$ can either do nothing, or add a box weighted $x$ in the $i$th column (if possible), or put $-\beta x$ to the removable box of $I(\mu)$ in the $i$th column without growing $\mu$. The operator $A(x)$ applied to $\mu$ repeats this procedure for $i = 1, 2, \ldots$ subsequently. Observe that if some $u_\ell$ did not add a box but put $-\beta x$ to the removable box in the column $\ell$, then this box will be open, i.e., there will be no box strictly below it as the operator proceeds with the indices $k > \ell$. On the other hand, if $u_\ell$ added a new box in the column $\ell$, then the last box of $\mu$ in this column will not be open. Furthermore, it is easy to see that the operator $A(x)$ can grow $\mu$ by only horizontal strips. If $\lambda$ is obtained from $\mu$ after applying $A(x)$, then $|\lambda/\mu|$ boxes were added (with the weight $x^{(\lambda/\mu)}$) and some open boxes from $I(\mu)$ received the weight $-\beta x$. Recall that there are $a(\lambda//\mu)$ open boxes. Hence we have

$$A(x) \cdot \mu = \sum_{\lambda//\mu \text{ hor. strip}} (1 - \beta x)^{a(\lambda//\mu)}x^{(\lambda//\mu)} \cdot \lambda = \sum_{\lambda//\mu \text{ hor. strip}} G^\beta_{\lambda//\mu}(x) \cdot \lambda.$$  

The operator $\tilde{d}_{i_1} \cdots \tilde{d}_{i_k}$ applied to $\lambda$, removes boxes from the $k$ columns $i_1 < \cdots < i_k$ of $\lambda$ in the order $i_k, \ldots, i_1$ in all possible ways, and thus gives the sum through all $\mu \subset \lambda$ such that $c(\lambda//\mu) = k$ with the corresponding weight $\beta^{(\lambda//\mu) - k}$. Therefore,

$$B(x) \cdot \lambda = \prod_{i \geq 1} (1 + x(d_i + \beta d_i^2 + \cdots)) \cdot \lambda = \sum_{\mu \subset \lambda} \beta^{(\lambda//\mu) - k}x^{(\lambda//\mu)} \cdot \mu = \sum_{\mu \subset \lambda} g^\beta_{\lambda//\mu}(x) \cdot \mu.$$  

In particular, we now obtain the following branching formulas:

$$G^\beta_{\lambda//\nu}(x_1, \ldots, x_n, x) = \sum_{\frac{\nu}{\mu} \text{ hor. strip}} G^\beta_{\lambda//\nu}(x_1, \ldots, x_n)(1 - \beta x)^{a(\frac{\nu}{\mu})}x^{(\frac{\nu}{\mu})} = \sum_{\frac{\nu}{\mu} \text{ hor. strip}} (1 - \beta x)^{a(\frac{\nu}{\mu})}x^{(\frac{\lambda}{\mu})}G^\beta_{\frac{\nu}{\mu}}(x_1, \ldots, x_n),$$  

$$g^\beta_{\lambda//\mu}(x_1, \ldots, x_n, x) = \sum_{\nu} g^\beta_{\lambda//\nu}(x_1, \ldots, x_n)x^{(\lambda//\nu)} = \sum_{\nu} \beta^{(\lambda//\nu) - c(\lambda//\mu)}x^{(\lambda//\nu)}g^\beta_{\nu//\mu}(x_1, \ldots, x_n).$$

Definition 4.4. A set-valued tableaux (SVT) of shape $\lambda//\mu$ is a filling of boxes of $\lambda//\mu = \lambda//\mu \cup I(\mu)$ (skew shape $\lambda//\mu$ and removable boxes of $\mu$) by sets of positive integers such that if one replaces each set by any of its elements the resulting tableau is semistandard (i.e., has weakly increasing rows from left to right and strictly increasing columns from top to bottom). When filling $\lambda//\mu$, sets in boxes should be nonempty, however when filling the boxes of $I(\mu)$ it is allowed to have empty sets. For a set-valued tableau $T$, the corresponding monomial is defined as $x^T = \prod_{i \geq 1} x_i^{a_i}$, where $a_i$ is the number of $i$’s in $T$ and let $|T| = \sum_i a_i$. See Fig. 1.

Definition 4.5. A reverse plane partition (RPP) of shape $\lambda//\mu$ is a filling of a Young diagram of $\lambda//\mu$ by positive integers weakly increasing in rows from left to right and columns from top to bottom. For a reverse plane partition $T$, the corresponding monomial is defined as $x^T = \prod_{i \geq 1} x_i^{c_i}$ where $c_i$ is the number of columns of $T$ containing $i$ and let $|T| = \sum_i c_i$.  

Theorem 4.6. The following combinatorial formulas hold:
\[
G_{\lambda/\mu}^\beta = \sum_{T \in SVT(\lambda/\mu)} (-\beta)^{|T|-|\lambda/\mu|} x^T, \quad g_{\lambda/\mu}^\beta = \sum_{T \in RPP(\lambda/\mu)} \beta^{|\lambda/\mu|-|T|} x^T.
\]

Proof. We construct any set-valued tableau recursively by applying the operators \(A(x_1), A(x_2), \ldots\) and recording the entries 1, 2, \ldots in diagrams. Applying the factor \((1 + x_k u_i - x_k \beta u_i d_i)\) from the operator \(A(x_k)\) to the current tableau, we either do nothing or add a new box in the \(i\)th column and record the entry \(k\) in this box (notice that this box is removable and will be treated so further), or if the box in the \(i\)th column is removable we add \(k\) to the existing entries of this box. It is clear that semistandard inequalities are preserved during these operations. The formula for \(g_{\lambda/\mu}^\beta\) can be explained similarly or it is easily seen by combining the branching and single variable formulas. \(\Box\)

Remark 4.7. Using the the operators \(\tilde{u}, \tilde{d}\) we obtained combinatorial formulas due to Buch [4] and Lam-Pylyavskyy [11]. For \(\mu = \emptyset\) and \(\beta = 1\) they coincide with dual families of stable Grothendieck polynomials \(\{G_\lambda\}, \{g_\lambda\}\). They are Hopf-dual or dual via the Hall inner product \(\langle G_\lambda, g_\mu \rangle = \delta_{\lambda\mu}\) for which Schur functions form an orthonormal basis.

Remark 4.8. Let \(\beta = 1\) and \(G_{\lambda/\mu}\) be the symmetric function defined via SVT formula of skew shape \(\lambda/\mu\) (without the removable boxes \(I(\mu)\) of \(\mu\)). Then the two functions are related [4] via the M"obius inversion
\[
G_{\lambda/\mu} = \sum_{\nu \subseteq \mu} G_{\lambda/\nu}, \quad g_{\lambda/\nu} = \sum_{\nu/\mu \text{ rook strip}} (-1)^{|\nu/\mu|} G_{\lambda/\mu}.
\]

5. Skew Cauchy Identity

We use the notation \(x = (x_1, x_2, \ldots)\) and \(y = (y_1, y_2, \ldots)\).

Theorem 5.1. Let \(\mu, \nu\) be any fixed partitions, then
\[
\sum_{\lambda} G_{\lambda/\mu}^\beta(x) g_{\lambda/\nu}^\beta(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} \sum_{\kappa} G_{\nu/\kappa}^\beta(x) g_{\mu/\kappa}^\beta(y).
\]

The identity is equivalent to the following commutation relation for the series \(A, B\) defined in Sec. 4.
**Theorem 5.2.** The following commutation relation holds

\[ B(y)A(x) = \frac{1}{1 - xy} A(x)B(y). \]

The proof uses the following Yang-Baxter-type local identity for the operators \( \tilde{u}, \tilde{d} \).

**Lemma 5.3.** For all \( i \geq 1 \) we have

\[ (1 - xy\tilde{u}_i\tilde{d}_i)^{-1}(1 + x\tilde{u}_i)(1 + y\tilde{d}_{i+1}) = (1 - xy\tilde{d}_{i+1}\tilde{u}_{i+1})^{-1}(1 + y\tilde{d}_{i+1})(1 + x\tilde{u}_i). \]

(*)

**Proof.** See the Appendix.

**Proof of Theorem 5.2.** Using Lemmas 3.4, 5.3 and the identity

\[ (1 + a)(1 - ba)^{-1}(1 + b) = (1 + b)(1 - ab)^{-1}(1 + a) \]

(\( ** \)) holding for any non-commuting \( a \) and \( b \), we have

\[ A(x)(1 - xy)^{-1}B(y) = \cdots (1 + x\tilde{u}_2)(1 + x\tilde{u}_1)(1 - xy\tilde{d}_1\tilde{u}_1)^{-1}(1 + y\tilde{d}_1)(1 + y\tilde{d}_2) \cdots \]

(\( *** \))

\[ = \cdots (1 + x\tilde{u}_2)(1 + y\tilde{d}_1)(1 - xy\tilde{u}_1\tilde{d}_1)^{-1}(1 + x\tilde{u}_1)(1 + y\tilde{d}_2) \cdots \]

(*)

\[ = (1 + y\tilde{d}_1) \cdots (1 + x\tilde{u}_3)(1 + x\tilde{u}_2)(1 - xy\tilde{d}_2\tilde{u}_2)^{-1}(1 + y\tilde{d}_2)(1 + x\tilde{u}_1)(1 + y\tilde{d}_3) \cdots \]

\[ = (1 + y\tilde{d}_1) \cdots (1 + x\tilde{u}_3)(1 + x\tilde{u}_2)(1 - xy\tilde{d}_2\tilde{u}_2)^{-1}(1 + y\tilde{d}_2)(1 + y\tilde{d}_3) \cdots (1 + x\tilde{u}_1) \]

\[ \cdots \]

\[ = B(y)A(x). \]

\( \square \)

**Proof of Theorem 5.1.** Using Theorem 5.2 and the commutativity \( [A(x_i), A(x_j)] = [B(y_i), B(y_j)] = 0 \) we have

\[ \sum_\lambda G_{\lambda\beta\mu}^\beta(x)g_{\lambda/\nu}^\beta(y) = \prod_j B(y_j) \prod_i A(x_i) \cdot \mu, \nu \]

\[ = \prod_{i,j} \frac{1}{1 - x_i y_j} \prod_i A(x_i) \prod_j B(y_j) \cdot \mu, \nu \]

\[ = \prod_{i,j} \frac{1}{1 - x_i y_j} \sum_\kappa G_{\nu\beta\kappa}^\beta(x)g_{\mu/\kappa}^\beta(y). \]

\( \square \)

**Remark 5.4.** In fact, Theorem 5.2 is equivalent to the single variable Cauchy identity

\[ \sum_\lambda G_{\lambda\beta\mu}^\beta(x)g_{\lambda/\nu}^\beta(y) = \frac{1}{1 - xy} \sum_\kappa G_{\nu\beta\kappa}^\beta(x)g_{\mu/\kappa}^\beta(y). \]

So to prove Theorem 5.1 it is enough to prove it just for single variables \( x, y \).
5.1. Corollaries. First note that by setting $\beta = 0$, the results generalize corresponding properties of skew Schur polynomials, since $G^0_{\lambda/\mu} = g^0_{\lambda/\mu} = s_{\lambda/\mu}$.

**Corollary 5.5** (Cauchy identity). For $\mu = \nu = \emptyset$ we obtain the usual Cauchy identity

$$\sum_{\lambda} G^\beta_{\lambda}(x) g^\beta_{\lambda}(y) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$  

This identity is equivalent to the duality of families $\{G_{\lambda}\}, \{g_{\mu}\}$ via the standard Hall inner product for which Schur functions are orthonormal, i.e., $\langle G_{\lambda}, g_{\mu} \rangle = \delta_{\lambda\mu}$.

**Corollary 5.6** (Pieri-type formulas). For $\mu = \emptyset$ or $\nu = \emptyset$ we have

$$\sum_{\lambda} G^\beta_{\lambda}(x) g^\beta_{\lambda/\nu}(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} G^\beta_{\nu}(x),$$

$$\sum_{\lambda} G^\beta_{\lambda/\mu}(x) g^\beta_{\lambda}(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} g^\beta_{\mu}(y).$$

In particular, the last identity gives the following formulas for $y = (1, 0, 0, \ldots)$.

$$\sum_{\lambda} \beta^{\lambda|-c(\lambda)} G^\beta_{\lambda/\mu} = \beta^{\mu|-c(\mu)} \prod_{i} \frac{1}{1 - x_i}, \quad \sum_{\lambda} G^\beta_{\lambda/\mu} = \prod_{i} \frac{1}{1 - x_i}.$$

Recall in contrast a similar identity for Schur functions (e.g., [14]):

$$\sum_{\lambda} s_{\lambda/\mu} = \prod_{i} \frac{1}{1 - x_i} \prod_{i < j} \frac{1}{1 - x_i x_j} \sum_{\kappa} s_{\mu/\kappa}.$$  

The Pieri-type formulas specialize to the following identities for $y = (q, 0, 0, \ldots)$

$$\sum_{\lambda} q^{c(\lambda/\nu)} \beta^{\lambda/\nu|-c(\lambda/\nu)} G^\beta_{\lambda} = \prod_{i} \frac{1}{1 - q x_i} G^\beta_{\nu},$$

$$\sum_{\lambda/\mu \text{ hor. strip}} (1 - \beta q)^{a(\lambda/\mu)} q^{\lambda/\mu} g^\beta_{\lambda} = \prod_{i} \frac{1}{1 - q x_i} g^\beta_{\mu}.$$  

Consider another specialization:

$$\sum_{\lambda} G^\beta_{\lambda}(x) g^\beta_{\lambda}(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} \prod_{i} \frac{1}{1 - q x_i},$$

$$g^\beta_{\lambda} := g^\beta_{\lambda}(q, y) = \sum_{\nu \subset \lambda} q^{c(\lambda/\nu)} \beta^{\lambda/\nu|-c(\lambda/\nu)} g^\beta_{\lambda/\nu}(y).$$

Let $d(\lambda) := \#\{\mu : \mu \subset \lambda\}$ be the number of subdiagrams of $\lambda$. Set $y = (1, 0, \ldots)$ and $\beta = q = 1$:

$$\sum_{\lambda} d(\lambda) G^\beta_{\lambda} = \prod_{i} \frac{1}{(1 - x_i)^2}.$$  

For a ‘pure’ skew shape $\lambda/\mu$ we obtain the following generating series.

**Proposition 5.7**. We have

$$\sum_{\lambda} G^\beta_{\lambda/\mu}(x) g^\beta_{\lambda}(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} g^\beta_{\mu}(1, y).$$
Proof. Recall that $G_{\lambda/\mu} = \sum_{\nu \subset \mu} G_{\lambda/\nu}$. From (3) we have

$$\prod_{i,j} \frac{1}{1 - x_i y_j} g_{\mu}(1, y) = \prod_{i,j} \frac{1}{1 - x_i y_j} \sum_{\nu \subset \mu} g_{\nu}(y)$$

$$= \sum_{\lambda} \sum_{\nu \subset \mu} G_{\lambda/\nu}(x) g_{\lambda}(y)$$

$$= \sum_{\lambda} G_{\lambda/\mu}(x) g_{\lambda}(y).$$

This gives the next formulas including a curious identity involving the Catalan numbers $\text{Cat}_n = \binom{2n}{n} / (n + 1)$.

**Corollary 5.8.**

$$\sum_{\lambda} G_{\lambda/\mu} = d(\mu) \prod_{i} \frac{1}{1 - x_i}, \quad \sum_{\lambda} G_{\lambda/\delta_n} = \text{Cat}_n \prod_{i} \frac{1}{1 - x_i}, \quad \delta_n = (n, n - 1, \ldots, 1).$$

6. Dual functions via the standard involution

We also need to describe the following dual families of symmetric functions. Let

$$\overline{A}(x) = (1 - x\overline{u}_1)^{-1}(1 - x\overline{u}_2)^{-1} \cdots, \quad \overline{B}(x) = (1 - x\overline{d}_2)^{-1}(1 - x\overline{d}_1)^{-1}$$

so that $A(x)\overline{A}(-x) = B(x)\overline{B}(-x) = 1$. Again, using non-local and local commutativity given in Lemma 3.4 we deduce that

$$[\overline{A}(x), \overline{A}(y)] = 0, \quad [\overline{B}(x), \overline{B}(y)] = 0.$$

Hence we can define the symmetric functions $\{J^\beta_{\lambda/\mu}\}, \{j^\beta_{\lambda/\mu}\}$ via the series

$$J^\beta_{\lambda/\mu}(x_1, \ldots, x_n) := \langle \overline{A}(x_n) \cdots \overline{A}(x_1) \cdot \mu, \lambda \rangle, \quad j^\beta_{\lambda/\mu}(x_1, \ldots, x_n) := \langle \overline{B}(x_n) \cdots \overline{B}(x_1) \cdot \lambda, \mu \rangle.$$

As for $G, g$, it is not hard to obtain the following formulas for a single variable $x$

$$J^\beta_{\lambda/\mu}(x) = \begin{cases} \frac{1}{(1 + \beta x)^{a(\lambda/\mu)}} \left( \frac{x}{1 + \beta x} \right)^{|\lambda/\mu|} & \text{if } \lambda/\mu \text{ is a vert. strip}, \\ 0 & \text{otherwise,} \end{cases}$$

(4)

$$j^\beta_{\lambda/\mu}(x) = \begin{cases} x_{\alpha(\lambda/\mu)} (\beta + x)^{|\lambda/\mu| - c(\lambda/\mu)} & \text{if } \lambda/\mu \text{ is a vert. strip}, \\ 0 & \text{otherwise.} \end{cases}$$

(5)

Define multiset-valued tableaux (MSVT) of shape $\lambda/\mu$ to be the filling as for set-valued tableaux but allowing multisets (i.e., sets with repeated elements) in boxes. The corresponding monomial is defined in the same way: $x^T = \sum_i x_i^{a_i}$, where $a_i$ is the number of $i$’s in $T$ and $|T| = \sum_i a_i$. For a semistandard Young tableaux (SSYT) $T$, let $r_i$ be the number of rows containing $i$ and $a_i$ be the total number of $i$’s.

**Proposition 6.1.** The following combinatorial formulas hold

$$J^\beta_{\lambda/\mu} = \sum_{T \in \text{MSVT}(\lambda/\mu')} (-\beta)^{|T| - |\lambda/\mu|} x^T, \quad j^\beta_{\lambda/\mu} = \sum_{T \in \text{SSYT}(\lambda/\mu')} \prod_{i} x_i^{r_i} (x_i + \beta)^{a_i - r_i}.$$
Let $\omega$ be the standard ring automorphism satisfying $\omega : s_\lambda \mapsto s_{\lambda'}$, where $\lambda'$ is the conjugate partition. To establish the next result we use the method of Fomin and Greene [7] of noncommutative Schur functions. Let $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \ldots)$ and define the noncommutative Schur functions as follows

$$s_\lambda(\tilde{u}) := \det[e_{\lambda'} - i + j(\tilde{u})] = \sum_{\sigma \in S_{\lambda_1}} \text{sgn}(\sigma)e_{\lambda'_1 + \sigma(1) - 1}(\tilde{u}) \cdots e_{\lambda'_1 + \sigma(\lambda_1) - \lambda_1}(\tilde{u}),$$

$$e_k(\tilde{u}) := \sum_{i_1 > \cdots > i_k \geq 1} \tilde{u}_{i_1} \cdots \tilde{u}_{i_k}.$$

Since the operators $\tilde{u}$ satisfy non-local and local commutativity relations (Lemma 3.4)

$$[\tilde{u}_i, \tilde{u}_j] = 0, \ |i - j| \geq 2, \quad [\tilde{u}_{i+1}, \tilde{u}_i + \tilde{u}_{i+1}] = 0,$$

we conclude that these functions commute:

$$[e_i(\tilde{u}), e_j(\tilde{u})] = [s_\lambda(\tilde{u}), s_\mu(\tilde{u})] = 0, \ \forall i, j, \lambda, \mu$$

and the following noncommutative versions of Cauchy identities hold:

$$\cdots A(x_2)A(x_1) = \prod_i \prod_j (1 + x_i \tilde{u}_j) = \sum_\lambda s_\lambda(x)s_\lambda(\tilde{u}),$$

$$\cdots \overline{A}(x_2)\overline{A}(x_1) = \prod_i \prod_j (1 - x_i \tilde{u}_j)^{-1} = \sum_\lambda s_\lambda(x)s_\lambda(\tilde{u})$$

The same holds for the operators $\tilde{d}$ and the corresponding series $B(x), \overline{B}(x)$.

**Theorem 6.2.** We have $\omega(G_{\lambda/\mu}^\beta) = J_{\lambda/\mu}^\beta$ and $\omega(g_{\lambda/\mu}^\beta) = J_{\lambda/\mu}^\beta$.

**Proof.** Using noncommutative Schur functions and Cauchy identities we have

$$\omega(G_{\lambda/\mu}^\beta) = \omega(\prod_i A(x_i) \cdot \mu, \lambda)$$

$$= \omega(\sum_\nu s_\nu(x)s_\nu(\tilde{u}) \cdot \mu, \lambda)$$

$$= \sum_\nu \omega(s_\nu(x))\langle s_\nu(\tilde{u}) \cdot \mu, \lambda \rangle$$

$$= \sum_\nu s_\nu(x)\langle s_\nu(\tilde{u}) \cdot \mu, \lambda \rangle$$

$$= \langle \prod_\nu \overline{A}(x_i) \cdot \mu, \lambda \rangle$$

$$= J_{\lambda/\mu}^\beta.$$

The property $\omega(g_{\lambda/\mu}^\beta) = J_{\lambda/\mu}^\beta$ follows in the same way. \(\square\)
Corollary 6.3 (Dual skew Cauchy identities). We have
\[ \sum_{\lambda} J_{\lambda}^{\beta} (x) g_{\lambda}^{\beta} (y) = \prod_{i,j} \left( 1 + x_i y_j \right) \sum_{\kappa} J_{\kappa}^{\beta} (x) g_{\mu_\kappa}^{\beta} (y), \]
\[ \sum_{\lambda} G_{\lambda}^{\beta} (x) j_{\lambda}^{\beta} (y) = \prod_{i,j} \left( 1 + x_i y_j \right) \sum_{\kappa} G_{\nu_\kappa}^{\beta} (x) j_{\mu_\kappa}^{\beta} (y), \]
\[ \sum_{\lambda} J_{\lambda}^{\beta} (x) j_{\lambda}^{\beta} (y) = \prod_{i,j} \left( \frac{1}{1 - x_i y_j} \right) \sum_{\kappa} J_{\nu_\kappa}^{\beta} (x) j_{\mu_\kappa}^{\beta} (y). \]

Remark 6.4. The functions \( J, j \) were first introduced and studied in [11] using noncommutative operators that are different to ours.

Remark 6.5. It can be proved that canonical deformations of symmetric Grothendieck polynomials studied in [22], the functions \( G_{\lambda}^{(0, \beta)} \), satisfying \( \omega(G_{\lambda}^{(0, \beta)}) = G_{\lambda}^{(\beta, \alpha)} \), \( \omega(G_{\lambda}^{(\alpha, \beta)}) = G_{\lambda}^{(\beta, \alpha)} \) and \( G_{\lambda}^{(1, \beta)} = G_{\lambda}^{(0, \beta)} \), also satisfy skew Cauchy identities.

7. Skew Pieri formulas

Theorem 7.1 (Skew Pieri rules). The following formulas hold
\[ G_{(1^k)}^{\beta} G_{\mu/\nu}^{\beta} = \sum_{\lambda/\mu \text{ vert strip } \eta/\nu} W_{\lambda/\mu}^{\lambda/\mu} G_{\lambda/\nu}^{\beta} \quad W_{\nu/\kappa}^{\lambda/\mu} = (-1)^{1/\nu} \frac{\mu k - k}{\nu k} \frac{(\lambda/\mu + \nu/\eta - k)}{\lambda/\mu + \nu/\eta - k}, \]
\[ g_{(k)}^{\beta} g_{\mu/\nu}^{\beta} = \sum_{\lambda/\mu \text{ hor strip } \eta/\nu} w_{\lambda/\mu}^{\lambda/\mu} g_{\lambda/\nu}^{\beta} \quad w_{\nu/\kappa}^{\lambda/\mu} = (-1)^{1/\nu} \frac{\mu k - k}{\nu k} \frac{(\lambda/\mu + \nu/\eta - k)}{\lambda/\mu + \nu/\eta - k}. \]

Remark 7.2. Note that these expansions are finite. For \( \nu = \emptyset \) and \( G_{\mu} \) we recover Pieri formulas proved in [13]. One could interpret the coefficients \( w, W \) in various ways: as a number of certain tableaux or as a number of walks in dual graphs defined later.

Consider the automorphisms \( \tilde{\tau} : G_{\lambda}^{\beta} \mapsto G_{\lambda}^{\beta} \) and \( \tau : g_{\lambda}^{\beta} \mapsto g_{\lambda}^{\beta} \). (Such automorphisms exist [4, 22].)

Proposition 7.3. We have \( \tilde{\tau}(G_{\lambda/\mu}^{\beta}) = G_{\lambda/\mu}^{\beta} \) and \( \tau(g_{\lambda/\mu}^{\beta}) = g_{\lambda/\mu}^{\beta} \).

Proof. From the Cauchy identity we have
\[ \tau_y \tilde{\tau}_x \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} G_{\lambda}^{\beta} (x) g_{\lambda}^{\beta} (y) = \prod_{i,j} \frac{1}{1 - x_i y_j}. \]
Applying \( \tau_y \tilde{\tau}_x \) on the Pieri-type formula (2) and then by using it again we obtain
\[ \sum_{\lambda} G_{\lambda}^{\beta} (x) \tau_y (g_{\lambda/\mu}^{\beta} (y)) = \tau_y \tilde{\tau}_x \left( \prod_{i,j} \frac{1}{1 - x_i y_j} \right) G_{\nu/\kappa}^{\beta} (x) \]
\[ = \prod_{i,j} \frac{1}{1 - x_i y_j} G_{\nu/\kappa}^{\beta} (x) \]
\[ = \sum_{\lambda} G_{\lambda}^{\beta} (x) g_{\lambda/\mu}^{\beta} (y) \]
from which we conclude that \( \tau_y (g_{\lambda/\mu}^{\beta} (y)) = g_{\lambda/\mu}^{\beta} (y) \). The proof of the first formula is the same. \( \square \)
Corollary 7.4 (Dual skew Pieri rules). Applying $\hat{\tau}, \tau$ we obtain
\[
G_\beta^{(k)} G_\mu^{(k)} = \sum_{\lambda/\mu \text{ hor strip} \atop \eta/\nu \text{ vert strip}} V_{\lambda/\eta}^\mu G_{\lambda/\eta}^\beta V_{\nu/\eta}^\lambda = (-1)^{|\lambda/\mu|} \beta_{\lambda/\mu} G_{\lambda/\eta}^{k(1)} \left( r(\lambda/\mu) + r(\nu/\eta) - k \right)
\]
\[
g_\beta^{(k)} g_\mu^{(k)} = \sum_{\lambda/\mu \text{ rook strip} \atop \eta/\nu \text{ hor strip}} v_{\lambda/\eta}^\mu g_{\lambda/\eta}^\beta v_{\nu/\eta}^\lambda = (-1)^{k-|\lambda/\mu|} \beta_{\lambda/\mu} G_{\lambda/\eta}^{k(1)} \left( a(\lambda/\mu) - a(\nu/\eta) - |\nu/\eta| \right)
\]

Corollary 7.5 (Simple skew Pieri rules). Let $\beta = 1$. We have
\[
g_{(1)} g_{\mu/\nu} = (-i(\mu) + i(\nu)) g_{\mu/\nu} + \sum_{\lambda=\mu+\Box} g_{\lambda/\nu} - \sum_{\eta=\nu-\Box} g_{\lambda/\eta} \]
\[
G_{(1)} G_{\mu/\nu} = \sum_{\lambda/\mu \text{ rook strip} \atop \eta/\nu \text{ hor strip}} (-1)^{|\lambda/\mu|} G_{\lambda/\eta}.
\]

Corollary 7.6. $\beta = 0$ gives the skew Pieri rule for Schur functions $s_{\lambda/\mu}$ [1]
\[
s_{(k)} s_{\mu/\nu} = \sum_{\lambda/\mu \text{ hor strip} \atop \nu/\eta \text{ vert strip}} (-1)^{|\nu/\eta|} s_{\lambda/\eta}, \quad |\lambda/\mu| + |\nu/\eta| = k.
\]

7.1. Dual skew families.

Definition 7.7. The families $\{F_{\lambda/\mu}\}, \{f_{\lambda/\mu}\}$ of symmetric functions (generally lying in the completion $\hat{\Lambda}$ of the ring of symmetric functions) indexed by pairs of partitions (with boundary conditions $f_{\emptyset/\mu} = \delta_{\emptyset,\mu}$) are called **dual** if they satisfy the following properties:

(i) skew Cauchy identity
\[
\sum_{\lambda} F_{\lambda/\mu}(x) f_{\lambda/\nu}(y) = \Omega(x,y) \sum_{\kappa} F_{\nu/\kappa}(x) f_{\mu/\kappa}(y),
\]
for some **Cauchy kernel** $\Omega$ such that there is an automorphism $\tilde{\omega} : \hat{\Lambda} \to \hat{\Lambda}$ satisfying
\[
\tilde{\omega}_x : \Omega(x,y) \mapsto \Omega(x,y)^{-1}
\]

(ii) the elements $\{f_\lambda\}$ are linearly independent, where $f_\lambda := f_{\lambda/\emptyset}$.

Lemma 7.8 (Skew Pieri-type formula). Let $\{F_{\lambda/\mu}\}, \{f_{\lambda/\mu}\}$ be dual families of symmetric functions. Then the following formula holds:
\[
\sum_{\lambda,\eta} F_{\nu/\eta}(x) F_{\lambda/\mu}(x) f_{\lambda/\nu}(y) = \Omega(x,y) f_{\mu/\nu}(y), \quad \tilde{F}_{\nu/\eta}(x) := \tilde{\omega}(F_{\nu/\eta}(x)).
\]

Proof. Specializing $\nu = \emptyset$ in the skew Cauchy identity we obtain the dual Pieri-type formulas
\[
\sum_{\lambda} F_{\lambda/\mu}(x) f_{\lambda}(y) = \Omega(x,y) f_{\mu}(y), \quad \sum_{\lambda} \tilde{F}_{\lambda/\mu}(x) f_{\lambda}(y) = \Omega(x,y)^{-1} f_{\mu}(y).
\]

Using these formulas we have
\[
\sum_{\lambda,\rho} F_{\lambda/\rho}(x) \tilde{F}_{\rho/\mu}(x) f_{\lambda}(y) = \Omega(x,y) \sum_{\rho} \tilde{F}_{\rho/\mu}(x) f_{\rho}(y) = \Omega(x,y) \Omega(x,y)^{-1} f_{\mu}(y) = f_{\mu}(y).
\]

Since the family $\{f_\lambda\}$ is linearly independent we conclude the following orthogonality relation
\[
\sum_{\rho} F_{\lambda/\rho}(x) \tilde{F}_{\rho/\mu}(x) = \delta_{\lambda\mu}.
\]
Now we have the following
\[
\sum_{\lambda,\eta} \overline{F}_{\nu/\eta}(x) F_{\lambda/\mu}(x) f_{\lambda/\eta}(y) = \sum_{\eta} \overline{F}_{\nu/\eta}(x) \sum_{\lambda} F_{\lambda/\mu}(x) f_{\lambda/\eta}(y)
= \Omega(x, y) \sum_{\eta} \overline{F}_{\nu/\eta}(x) \sum_{\kappa} F_{\eta/\kappa}(x) f_{\mu/\kappa}(y)
= \Omega(x, y) \sum_{\kappa} f_{\mu/\kappa}(y) \sum_{\eta} \overline{F}_{\nu/\eta}(x) F_{\eta/\kappa}(x)
= \Omega(x, y) f_{\mu/\nu}(y),
\]
\[= \delta_{\mu, \nu} \text{ by (a)} \]
\[\square\]

Remark 7.9. This is a general formulation of the method in [21] for Hall-Littlewood polynomials.

7.2. Back to Grothendieck. Recall that \(J^\beta_{\lambda/\mu} = \omega(G^\beta_{\lambda/\mu})\) and \(j^\beta_{\lambda/\mu} = \omega(g^\beta_{\lambda/\mu})\), where \(\omega\) is the standard involution defined on the Schur basis by \(\omega : s_\lambda \mapsto s_{\lambda'}\) (in case of \(G \in \Lambda\) it is extended for infinite linear combinations).

Theorem 7.10 (Skew Pieri-type formulas). We have
\[
\sum_{\lambda,\eta} g^\beta_{\nu/\eta}(y) j^\beta_{\nu/\eta}(\lambda) G^\beta_{\lambda/\eta}(x) = \prod_{i,j} \frac{1}{1 - x_i y_j} G^\beta_{\mu/\nu}(x),
\]
\[
\sum_{\lambda,\eta} G^\beta_{\lambda/\mu}(x) j^\beta_{\nu/\eta}(\lambda) g^\beta_{\lambda/\eta}(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} g^\beta_{\mu/\nu}(y)
\]
and the following dual formulas
\[
\sum_{\lambda,\eta} j^\beta_{\lambda/\mu}(y) g^\beta_{\nu/\eta}(\lambda) G^\beta_{\lambda/\eta}(x) = \prod_{i,j} (1 + x_i y_j) G^\beta_{\mu/\nu}(x),
\]
\[
\sum_{\lambda,\eta} G^\beta_{\lambda/\mu}(x) j^\beta_{\nu/\eta}(\lambda) g^\beta_{\lambda/\eta}(y) = \prod_{i,j} (1 + x_i y_j) g^\beta_{\mu/\nu}(y).
\]

Proof. The formulas imply from Lemma 7.8 for corresponding dual skew families \(G, g, J, j\) and the automorphism \(\tilde{\omega} : f(x) \mapsto \omega(f)(-x)\) satisfying \(\tilde{\omega} \prod_{i,j} (1 - x_i y_j)^{-1} = \prod_{i,j} (1 - x_i y_j).\)

\[\square\]

Corollary 7.11. Specializing for single variables we have
\[
\prod_i \frac{1}{1 - x_i y_j} G^\beta_{\mu/\nu}(x) = \sum_{\lambda,\eta \text{ vert. strip}} g^\beta_{\lambda/\mu}(y) j^\beta_{\nu/\eta}(\lambda) G^\beta_{\lambda/\eta}(x)
\]
\[\prod_j \frac{1}{1 - x_i y_j} g^\beta_{\mu/\nu}(y) = \sum_{\lambda,\eta \text{ vert. strip}} G^\beta_{\lambda/\mu}(x) j^\beta_{\nu/\eta}(\lambda) g^\beta_{\lambda/\eta}(y),
\]
\[\prod_i \frac{1}{1 - x_i y_j} G^\beta_{\mu/\nu}(x) = \sum_{\lambda,\eta \text{ hor. strip}} (-1)^{\nu/\eta} (1 - \beta x)^{a(\lambda/\mu) - a(\lambda/\eta)} (\nu/\eta) x^{\lambda/\mu} + (\nu/\eta) F^\beta_{\lambda/\eta}(y),
\]
\[\prod_j \frac{1}{1 - x_i y_j} g^\beta_{\mu/\nu}(y) = \sum_{\lambda,\eta \text{ hor. strip}} (-1)^{\nu/\eta} (1 - \beta x)^{a(\lambda/\mu) - a(\lambda/\eta)} (\nu/\eta) x^{\lambda/\mu} + (\nu/\eta) g^\beta_{\lambda/\eta}(y),
\]
as well as the dual formulas

\[
\prod_i (1 + x_i y) G_{\mu/\nu}^\beta(x) = \sum_{\lambda/\mu \text{ vert strip}} J_{\lambda/\mu}^\beta(y) g_{\nu/\eta}^\beta(-y) G_{\lambda/\eta}^\beta(x),
\]

(9) 

\[
\prod_j (1 + x_j y) g_{\mu/\nu}^\beta(y) = \sum_{\lambda/\mu \text{ hor strip}} J_{\lambda/\mu}^\beta(x) G_{\nu/\eta}^\beta(-x) g_{\lambda/\eta}^\beta(y),
\]

(10) 

Hence we also obtain the following skew rules

\[
h_k G_{\mu/\nu}^\beta = \sum_{\lambda/\mu \text{ vert strip}} (-1)^{k - c(\lambda/\mu) - c(\nu/\eta) + k - c(\lambda/\mu) - c(\nu/\eta)} G_{\lambda/\eta}^\beta,
\]

(11) 

\[
h_k g_{\mu/\nu} = \sum_{\lambda/\mu \text{ hor strip}} (-1)^{k - c(\lambda/\mu) - c(\nu/\eta) + k - c(\lambda/\mu) - c(\nu/\eta)} G_{\lambda/\eta}^\beta,
\]

(12) 

\[
e_k G_{\mu/\nu}^\beta = \sum_{\lambda/\mu \text{ vert strip}} (-1)^{c(\nu/\eta) + k - c(\lambda/\mu) - c(\nu/\eta) + k - c(\lambda/\mu) - c(\nu/\eta)} G_{\lambda/\eta}^\beta,
\]

(13) 

\[
e_k g_{\mu/\nu}^\beta = \sum_{\lambda/\mu \text{ hor strip}} (-1)^{c(\nu/\eta) + k - c(\lambda/\mu) - c(\nu/\eta) + k - c(\lambda/\mu) - c(\nu/\eta)} G_{\lambda/\eta}^\beta.
\]

(14) 

**Proof of Theorem 7.1.** Using the Cauchy identity and the fact that \( j_\rho^\beta(y) = 0 \) unless \( \rho \) is a single column and \( j_{(1^k)}(y) = y(\beta + y)^{k-1} \), we obtain

\[
\sum_{\rho} G_\rho^\beta(x) j_\rho^\beta(y) = 1 + \sum_{k \geq 1} G_{(1^k)}^\beta(x) y(\beta + y)^{k-1} = \prod_i (1 + x_i y).
\]

Therefore using (7) we have

\[
\prod_i (1 + x_i y) G_{\mu/\nu}^\beta(x) = \left(1 + \sum_{k \geq 1} G_{(1^k)}^\beta(x) y(\beta + y)^{k-1}\right) G_{\mu/\nu}^\beta(x)
\]

\[
= \sum_{\lambda/\mu \text{ vert strip}} (-1)^{c(\nu/\eta) + c(\nu/\eta) + c(\lambda/\mu) - c(\lambda/\mu) + c(\nu/\eta) - c(\nu/\eta)} G_{\lambda/\eta}^\beta(x)
\]

The expansion for \( G_{(1^k)}^\beta G_{\mu/\nu}^\beta \) is obtained by comparing the coefficients at \( [y(\beta + y)^{k-1}] \) from both sides. Notice that \( g_{(k)}^\beta = h_k \) and hence (12) gives the needed skew Pieri rule for \( g_{(k)}^\beta g_{\mu/\nu}^\beta \).
8. Basis phenomenon

A natural question is when a family of symmetric functions of unbounded degree, i.e., belonging to the completion of the ring of symmetric functions forms a basis of a certain ring, like \( \{G_\lambda\} \) does. In this section we prove that products \( G_\mu G_\nu \) expand finitely in \( \{G_\lambda\} \) without appealing to a Littlewood-Richardson rule as in \([4, 16]\). We provide a general sufficient condition for this situation.

Say that a family \( \{f_\lambda\} \) presents a basis phenomenon if its elements are linearly independent and for all \( \mu, \nu \), the product expansion \( f_\mu f_\nu = \sum_\lambda c^\lambda_{\mu\nu} f_\lambda \) exists and is finite.

**Theorem 8.1.** The family \( \{G_\lambda\} \) presents a basis phenomenon.

**Definition 8.2.** Say that a family \( \{F_\lambda\} \) of symmetric functions \( F_\lambda \in \hat{\Lambda} \) (assume \( F_\emptyset = 1 \) for simplicity) is damping if

(i) it is linearly independent in \( \hat{\Lambda} \) and each element of \( \hat{\Lambda} \) can uniquely be expressed as (possibly) an infinite linear combination of \( \{F_\lambda\} \)

(ii) there is an involutive automorphism \( \hat{\omega} : F_\lambda \mapsto F_{\lambda'} \)

(iii) there is a Pieri-type formula

\[
\sum_\lambda F_\lambda(x) F_{\mu\nu}(y) = \Omega(x, y) F_\mu(x)
\]

(15) for some (non-degenerate) kernel \( \Omega \) and a dual family \( \{f_{\lambda\mu}\} \) that satisfies

(a) the damping condition: For each \( \mu \) and \( n \in \mathbb{N} \), there exists a constant \( k = k(\mu, n) \) such that if \( f_{\lambda\mu}(x_1, \ldots, x_n) \neq 0 \) then \( \lambda_1 < k \)

(b) \( \{f_\lambda\} \) are linearly independent (where \( f_\lambda := f_{\lambda\emptyset} \)).

**Remark 8.3.** The damping condition is important here. It is a natural property of symmetric functions whose combinatorial presentations have strict row conditions. For example, Schur polynomials satisfy it: \( s_\lambda/\mu'(x_1, \ldots, x_n) \neq 0 \) implies that \( \lambda_1 \leq k(\mu, n) = \mu_1 + n \). On the other hand, the dual Grothendieck polynomials \( g_{\lambda/\mu} \) do not have the damping condition, i.e., if \( g_{\lambda/\mu}(x_1, \ldots, x_n) \neq 0 \) then \( \lambda_1 \) (and \( \ell(\lambda) \)) can be arbitrarily large regardless of \( \mu \) and \( n \), as their formula is based on plane partitions that have weak inequalities in rows and columns.

**Theorem 8.4.** Every damping family of symmetric functions presents a basis phenomenon.

**Proof.** Let \( \{F_\lambda\} \) be a damping family. By (15), combining it for \( \mu = \emptyset \), we have

\[
\sum_\lambda F_\lambda(x) F_{\mu\nu}(y) = \Omega(x, y) F_\mu(x) = \sum_\nu F_\nu(x) f_{\nu}(y) F_\mu(x).
\]

Therefore,

\[
F_\mu F_\nu = \sum_\lambda d^\lambda_{\mu\nu} F_\lambda \implies F_{\lambda\mu} = \sum_\nu d^\lambda_{\mu\nu} f_{\nu}.
\]

For any fixed \( \mu, \nu \) consider \( \lambda \) so that \( d^\lambda_{\mu\nu} \neq 0 \). Let \( n = n(\nu) \) be a minimal number such that \( f_{\nu}(x_1, \ldots, x_n) \neq 0 \). By the damping condition we have

\[
f_{\lambda\mu}(x_1, \ldots, x_n) = \sum_\nu d^\lambda_{\mu\nu} f_{\nu}(x_1, \ldots, x_n) \neq 0 \implies \lambda_1 < k = k(\mu, n(\nu)),
\]

\[2\]Which is a special case of a skew Cauchy for \( \mu = \emptyset \).
which means that \( \lambda_1 \) is bounded from above by a constant depending on \( \mu, \nu \). Since there is an automorphism \( \tilde{\omega} \) mapping \( F_\lambda \) to \( F_\lambda' \), we also have \( d^\lambda_{\mu\nu} = d^{\lambda'}_{\mu'\nu'} \neq 0 \) and hence by the same argument we obtain that \( \lambda'_1 = \ell(\lambda) < k(\mu', n(\nu')) \) is bounded from above as well. Therefore, for every pair \( \mu, \nu \) there are only finitely many \( \lambda \) so that \( d^\lambda_{\mu\nu} \neq 0 \). \( \square \)

Consider a damping family \( \{ F_\lambda \} \) with a dual \( \{ f_\lambda \} \), so that there is an automorphism \( f_\lambda \mapsto f_\lambda' \). Define the ring \( \Phi := \bigoplus_\lambda K \cdot F_\lambda \) with the basis \( \{ F \} \). Suppose \( \{ F \} \) has a skew extension \( \{ F_{\lambda\mu} \} \) so that the following dual formula holds as well:

\[
\sum_\lambda F_{\lambda\mu}(x)f_\lambda(y) = \Omega(x, y)f_\mu(y).
\]

**Proposition 8.5.** \( F_{\lambda\mu} \in \Phi \).

**Proof.** The proof is similar to the previous result. The dual formula gives

\[
F_{\lambda\mu} = \sum_\nu d^{\mu\nu}_{\lambda} F_\nu \implies f_\mu f_\nu = \sum_\lambda d^{\mu\nu}_{\lambda} f_\lambda.
\]

For fixed \( \lambda, \mu \) consider \( \nu \) so that \( d^{\mu\nu}_{\lambda} \neq 0 \) and let \( n = n(\lambda) \) be a minimal number so that \( f_\lambda(x_1, \ldots, x_n) \neq 0 \). Then \( f_\mu(x_1, \ldots, x_n)f_\nu(x_1, \ldots, x_n) \neq 0 \) and hence from the damping condition we have \( \nu_1 < k = k(n(\lambda)) \) is bounded. Since \( d^{\mu\nu}_{\lambda} = d^{\mu'\nu'}_{\lambda'} \) by the same argument we have \( \nu'_1 = \ell(\nu) < k(n(\nu')) \) is bounded from above as well. Therefore, for every \( \lambda, \mu \) there exists only finitely many \( \nu \) for which \( d^{\mu\nu}_{\lambda} \neq 0 \). \( \square \)

**Proof of Theorem 8.1.** Let us show that \( \{ G_\lambda \} \) is a damping family. The conditions (i), (ii) are satisfied (see e.g. [4, 22]). For (iii) we take the following Pieri-type formula that we obtained earlier:\(^3\)

\[
\sum_\lambda G_\lambda(x) j_{\lambda/\mu}(y) = \prod_{i,j}(1 + x_i y_j) G_\mu(x).
\]

The family \( \{ j_{\lambda/\mu} \} \) satisfies the damping condition: by definition (see Sec. 6) \( j_{\lambda/\mu} \) has a combinatorial formula over certain tableaux that are row strict (the operator \( \overline{B}(x) \lambda \) removes vertical strips from \( \lambda \)), which means that if \( j_{\lambda/\mu}(x_1, \ldots, x_n) \neq 0 \) then \( \lambda_1 \leq \mu_1 + n \) is bounded from above.

**Corollary 8.6.** Let \( \Gamma := \bigoplus_\lambda \mathbb{Z} \cdot G_\lambda \). We have \( G_{\lambda/\mu} \in \Gamma \) and \( G_{\lambda\mu} \in \Gamma \).

**Proof.** For \( G_{\lambda/\mu} \), the result follows by Proposition 8.5 since \( \{ G_\lambda \} \) is damping and for \( G_{\lambda\mu} \) since \( G_{\lambda/\mu} = \sum_{\nu \subset \lambda} G_{\lambda/\nu} \) is a finite sum. \( \square \)

9. **Dual filtered Young graphs**

Following [15], a *weighted filtered graph* is a digraph \( G = (V, r, E, w) \) where \( V \) is a set of countably many vertices together with a *rank function* \( r : V \to \mathbb{Z} \) satisfying \( r(a) \leq r(b) \) for every (directed) edge \( (a, b) \in E \), and \( w : E \to \mathbb{R} \) is some weight function.

For a pair \( G_1 = (V, r, E_1, w_1), G_2 = (V, r, E_2, w_2) \) of filtered graphs on the same (ranked) vertex set \( V \) construct a digraph \( G = (V, E) \) so that \( E = E_1 \cup \overline{E_2} \) is a union of edges \( E_1 \) and edges

\(^3\)We did not take another dual formula containing \( g_\lambda \), since it does not satisfy the damping condition.
Figure 2. The dual filtered Young graph $\beta Y$. The graph on the left corresponds to up edges and on the right to down edges. Here each loop has the weight $-\beta$.

of $E_2$ but taken in opposite direction. Let $\mathbb{R}V$ be the free abelian group on $V$ (formal $\mathbb{R}$-linear combinations of vertices $V$). Define the up and down operators $U, D \in \text{End}(\mathbb{R}V)$ on $G$ as follows:

$$Uv = \sum_{e=(v \to u) \in E} w_1(e)u, \quad Dv = \sum_{e=(u \to v) \in E} w_2(e)u.$$

Say that $G$ is a dual filtered graph if there exist scalars $\alpha, \beta \in \mathbb{R}$ such that for all $v \in V$ we have

$$[D, U]v = (DU - UD)v = (\alpha + \beta D)v.$$

Remark 9.1. Up to normalizations, there are only three distinct types of $(\alpha, \beta) \in \{(1, 1), (0, 1), (1, 0)\}$. In addition, the relation $[D, U] = D$ can be shifted with $D' = D - 1$ which gives $[D', U] = 1 + D'$.

Remark 9.2. If the rank function $r$ satisfies $r(a) + 1 = r(b)$ for every edge $(a, b)$ and $(\alpha, \beta) = (1, 0)$, then the corresponding graphs $G$ are called dual graded graphs studied by Fomin [5] and by Stanley [18] as differential posets.

Remark 9.3. An associative algebra generated by $U, D$ subject to $[D, U] = 1$ is called the first Weyl algebra. One may consider $D = \frac{d}{dx}$ as a differential operator and $U = x$ acting on a polynomial ring $K[x]$. The relation $[D, U] = 1 + D$ corresponds to the difference operator $Df(x) = f(x + 1) - f(x)$.

9.1. New constructions of filtered Young graphs. Recall that $Y$ is the Young lattice, i.e., an infinite graph whose vertices are indexed by partitions and edges are given by $(\lambda, \lambda + \Box)$. We think of $Y$ as a self-dual graph with up and down directed edges $(\lambda \to \lambda \pm \Box)$.

I. First define the following $\beta$-filtration $\beta Y$ of Young’s lattice $Y$ (see Fig. 2):

(i) vertices $V$ are integer partitions ranked by the number of boxes $r(\lambda) = |\lambda|$,

(ii) up edges (of $E_1$) are as in Young’s lattice $(\lambda \to \lambda + \Box)$ with the weight $w = 1$ but there are also $i(\lambda)$ many loops $(\lambda \to \lambda)$ each with the weight $w = -\beta$ (recall that $i(\lambda)$ is the number of inner corners of $\lambda$),

(iii) down edges (of $E_2$) are given by $(\lambda \to \mu)$ iff all boxes $\lambda/\mu$ are on a single column, and the corresponding weight is $w = \beta |\lambda/\mu|^{-1}$.

II. Next, let $x$ be a scalar parameter and define the Cauchy filtration of $Y$ denoted by $xY$ that satisfies exactly the same conditions (i) and (ii) as $\beta Y$ but its down edges are given by $(\lambda \to \mu)$ iff $\lambda \supset \mu$ with the weight $w = x^{c(\lambda/\mu)} \beta |\lambda/\mu|^{-1} - c(\lambda/\mu)$.
Theorem 9.4. We have

(i) $\beta Y$ is a dual filtered graph satisfying $[D, U] = 1$.
(ii) $\kappa Y$ is a dual filtered graph satisfying $[D, U] = \kappa (1 + D)$.

Proof. These constructions are natural consequences of the Cauchy identity. Suppose the operator
series
$$A(x) = 1 + \sum_{i \geq 1} U_i x^i, \quad B(y) = 1 + \sum_{i \geq 1} D_i y^i,$$
satisfy the Cauchy identity
$$B(y)A(x) = (1 - xy)^{-1}A(x)B(y).$$
By comparing coefficients at $xy$ and $x$ after plugging $y = \kappa$ we obtain that
$$[D_1, U_1] = 1 \quad \text{and} \quad [D(\kappa), U_1] = \kappa (1 + D(\kappa)), \quad D(\kappa) = D_1 \kappa + D_2 \kappa^2 + \cdots.$$
Let $\beta \in \mathbb{R}$ and define the operators
$$\tilde{U} = \tilde{u}_1 + \tilde{u}_2 + \cdots, \quad \tilde{D} = \tilde{d}_1 + \tilde{d}_2 + \cdots, \quad \tilde{D} = -1 + (1 + \kappa \tilde{d}_1)(1 + \kappa \tilde{d}_2) \cdots$$
Then the Cauchy identity gives $[\tilde{D}, \tilde{U}] = 1$ and $[\tilde{D}, \tilde{U}] = \kappa (1 + \tilde{D})$. Observe that $\tilde{U}$ defines the up edges of $\beta Y$ and $\kappa Y$, the operator $\tilde{D}$ corresponds to the down operator of $\beta Y$ as it defines its down edges. The operator $\tilde{D}$ defines the down edges of $\kappa Y$ as $x^k \tilde{d}_{i_1} \cdots \tilde{d}_{i_k}$ removes boxes from the $k$ columns $i_1 < \cdots < i_k$ in all possible ways giving the corresponding weight $\kappa^k \beta |\lambda/\mu|-k$. □

Corollary 9.5. For $\beta = 0$ we have the following special cases.

(i) $\beta Y = Y$ is the self-dual graded Young graph.
(ii) $\kappa Y$ gives the Pieri deformation of Young’s graph: up edges are as in the usual Young’s graph $Y$ and down edges are given by $(\lambda \to \mu)$ iff $\lambda/\mu$ is a horizontal strip.

As it was mentioned in [15], apparently the most interesting and mysterious type of dual filtered graphs is the so-called Möbius deformation that is related to K-theoretic insertion and LR rules. For $Y$ it is defined as follows. The defining conditions (i) and (ii) of the Möbius deformation $\mu Y$ are the same as for the $\beta$-filtration $\beta Y$ but loops have positive weight 1, and down edges are given by $(\lambda \to \mu)$ iff $\lambda/\mu$ is a rook strip (i.e., no two boxes lie on the same row or column) with the corresponding weight $w = 1$.

Besides new examples of dual filtered Young’s graphs, another consequence of our approach is the following result: Möbius deformation of Young’s lattice is related to the Cauchy deformation and can be obtained from it via a natural transformation. In particular, this result reveals the presence of a Möbius deformation for Young’s lattice and the transform is in fact related to the Möbius inversion.

Lemma 9.6. Suppose $[D, U] = -(1 + D)$. Then $[\tilde{D}, \tilde{U}] = 1 + \tilde{D}$ for $\tilde{D} = -D(1 + D)^{-1}$.

Proof. See the Appendix. □

Theorem 9.7. The Möbius deformation $\mu Y$ is a dual filtered graph satisfying $[D, U] = 1 + D$ and it can be obtained from the Cauchy deformation $\kappa Y$ for $\kappa = \beta = -1$ via the map
$$D \mapsto -D(1 + D)^{-1}.$$
Proof. Recall that for $\beta = \kappa = -1$, the down operator of $\kappa Y$ (eq. (16)) is given by

$$D = -1 + (1 - \tilde{d}_1)(1 - \tilde{d}_2) \cdots$$

$$= -1 + \left(1 - \frac{d_1}{1 + d_1}\right) \left(1 - \frac{d_1}{1 + d_1}\right) \cdots$$

$$= -1 + \frac{1}{(1 + d_1)(1 + d_2)} \cdots$$

Hence,

$$\hat{D} = \frac{-D}{1 + D} = \frac{1}{1 + D} - 1 = \cdots (1 + d_2)(1 + d_1) - 1.$$ 

Notice that $\hat{D} \lambda = (\cdots (1 + d_2)(1 + d_1) - 1) \cdot \lambda$ removes rook strips from $\lambda$ in all possible ways. Therefore, $\hat{D}$ is a down operator of $\mu Y$. For $\kappa Y$ we have $[\mathcal{D}, \tilde{U}] = -(1 + \mathcal{D})$ and by Lemma 9.6 we have $[\hat{D}, \tilde{U}] = 1 + \hat{D}$. 

10. Enumerative identities

Define increasing set-valued tableaux (ISVT) as an SVT that if after replacing each set by any of its element, the resulting tableau is increasing both in rows and columns. Let $F_{\lambda/\mu}(n)$ be the number of ISVT of shape $\lambda/\mu$ that contain all numbers from $[n] := \{1, \ldots, n\}$.

A strict tableaux (ST) of skew shape $\lambda/\mu$ is a filling of a Young diagram of $\lambda/\mu$ by positive integers so that entries strictly increase in rows from left to right, weakly increase from top to bottom, and each element can appear only on a single column. Let $f_{\lambda/\mu}(n)$ be the number of ST of shape $\lambda/\mu$ that contain all numbers from $[n]$.

An increasing tableaux (IT) is a filling of a skew diagram by positive integers so that they strictly increase in both rows and columns. Let now $g_{\lambda/\mu}(n)$ be the number of IT of skew shape $\lambda/\mu$ that contain all numbers $[n]$ (some numbers may appear several times).

Theorem 10.1. We have

$$\sum_{\lambda} (-1)^{m-|\lambda/\mu|} F_{\lambda/\mu}(m)f_{\lambda/\mu}(n) = \sum_{i} \epsilon! \binom{m}{i} \binom{n}{i} \sum_{\kappa} (-1)^{n-i-|\nu/\kappa|} F_{\nu/\kappa}(m-i)f_{\mu/\kappa}(n-i),$$

$$\sum_{\lambda} F_{\lambda/\mu}(m)g_{\lambda/\mu}(n) = \sum_{i,j} q_{n}(i,j) \binom{m}{i} \binom{n}{j} \sum_{\kappa} F_{\nu/\kappa}(m-j)g_{\mu/\kappa}(n-i),$$

where

$$q_{n}(i,j) := \sum_{\ell} \binom{i-j+\ell}{\ell} A_{i,n-\ell}$$

and $A_{i,s}$ is the Eulerian number, i.e., the number of permutations of $(1, \ldots, i)$ with $s$ descents.

Proof. Note that $(-1)^{m-|\lambda/\mu|} F_{\lambda/\mu}(m)$ is equivalently the number of signed up walks from $\mu$ to $\lambda$ of length $m$ in the 1-filtration $\beta Y$, i.e., $\beta = 1$, loops have weight $-1$, and a sign of a walk is negative if it uses an odd number of loops, otherwise it is positive. Similarly, $f_{\lambda/\mu}(n)$ is the number of down walks from $\lambda$ to $\mu$ of length $n$ in $\beta Y$. For example, $F_{(21),(2)}(2) = -3, f_{(21),(1)}(2) = 2$. Enumerator of the down graph of $\mu Y$ is $g_{\lambda/\mu}$. Then the formulas are applications of the normal ordering of
differential operators $U, D$ given in the lemma below. Then the graph $\beta Y$ with $\beta = 1$ satisfies $[D, U] = 1$ and from Lemma 10.2 we obtain:

$$\sum_\lambda (-1)^{m-|\lambda/\mu|} F_{\lambda/\mu}(m) f_{\lambda/\nu}(n) = \langle D^n U^m \mu, \nu \rangle$$

$$= \sum_i i! \binom{m}{i} \binom{n}{i} (U^{m-i} D^{n-i} \mu, \nu)$$

$$= \sum_i i! \binom{m}{i} \binom{n}{i} (U^{m-i} D^{n-i} \mu, \nu) \sum_i (-1)^{n-i-|\nu/\kappa|} F_{\nu/\kappa}(m - i) f_{\mu/\kappa}(n - i).$$

The second formula can be obtained in the same way using the normal ordering for $[D, U] = 1 + D$.

These formulas are analogous to the formula (see [17, 18, 19])

$$\sum_{\lambda \vdash n} f_{\lambda/\mu} f_{\lambda/\nu} = \sum_{i \geq 0} i! \binom{m}{i} \binom{n}{i} \sum_{|\nu/\kappa| = n-i} f_{\nu/\kappa} f_{\mu/\kappa},$$

where $f_{\lambda/\mu}$ is the number of standard Young tableaux (SYT) of shape $\lambda/\mu$. It generalizes the classical Frobenius identity

$$\sum_{\lambda \vdash n} f_{\lambda/\mu}^2 = n!$$

From the second identity one can also obtain the formula given in [15]

$$\sum_\lambda F_\lambda(n) g_\lambda(n) = \# \text{ ordered set partitions of } [n].$$

**Lemma 10.2** (Normal ordering). The following ordering formulas hold:

$$[D, U] = 1 \implies D^n U^m = \sum_i i! \binom{m}{i} \binom{n}{i} U^{m-i} D^{n-i}.$$

$$[D, U] = 1 + D \implies D^n U^m = \sum_{i,j} q_{\alpha}(i, j) \binom{m}{i} \binom{n}{j} U^{m-i} D^{n-j}.$$


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Proof of Lemma 3.4. Non-local identities (i) follow directly from non-local identities in Lemma 3.2. Let us prove the local identities. First, we have

\[ u_{i+1} \tilde{u}_i (u_{i+1} + \tilde{u}_i) = (\tilde{u}_{i+1} + u_i) u_{i+1} \tilde{u}_i \]

\[ \iff (u_{i+1} - \beta u_{i+1} d_{i+1})(u_i - \beta u_i d_i)(u_{i+1} - \beta u_{i+1} d_{i+1} + u_i - \beta u_i d_i) \]

\[ = (u_{i+1} - \beta u_{i+1} d_{i+1} + u_i - \beta u_i d_i)(u_{i+1} - \beta u_{i+1} d_{i+1})(u_i - \beta u_i d_i) \]

The free coefficients (at \( \beta^0 \)) from both sides are \( u_{i+1} u_i (u_{i+1} + u_i) \) and \( (u_{i+1} + u_i) u_{i+1} u_i \) that are equal. The coefficients at \( -\beta \) are equal iff

\[ u_{i+1} u_i (u_{i+1} d_{i+1} + u_i d_i + d_i u_{i+1} + d_{i+1} u_i + d_{i+1} u_{i+1}) \]

\[ = (u_{i+1} d_{i+1} + u_i d_i) u_{i+1} u_i + (u_{i+1} + u_i) u_{i+1} u_i d_i \]

\[ \iff u_{i+1} u_i (u_{i+1} + u_i) d_{i+1} + u_{i+1} u_i (u_i d_i + d_i u_{i+1} + d_{i+1} u_i + d_{i+1} u_{i+1}) \]

\[ = (u_{i+1} d_{i+1} + u_i d_i) u_{i+1} u_i + (u_{i+1} + u_i) u_{i+1} u_i d_i \]

\[ \iff u_{i+1} u_i (u_i + u_{i+1}) d_i = u_{i+1} u_i (d_i u_i + d_i u_{i+1} + d_{i+1} u_i) \]

\[ = (u_{i+1} d_i + u_i d_i) u_{i+1} u_i + (u_{i+1} + u_i) u_{i+1} u_i d_i \]

\[ \iff u_{i+1} u_i (d_i u_i + d_{i+1} + u_i u_{i+1}) = (u_{i+1} d_{i+1} + u_i d_i) u_{i+1} u_i \]

\[ \iff u_{i+1} u_i d_i = u_{i+1} u_i u_{i+1} u_i = u_i d_i u_{i+1} u_i + u_{i+1} u_i u_i d_i \]

It is easy to check that the last identity \([u_{i+1} u_i, u_i d_i] = 0\) is always true on the basis elements. The coefficients at \( \beta^2 \) are equal iff

\[ u_{i+1} u_i ((d_i + d_{i+1})(u_{i+1} d_{i+1} + u_i d_i) + d_i u_{i+1} (u_i + u_{i+1})) \]

\[ = (u_{i+1} + u_i) u_{i+1} u_i d_i + (u_{i+1} d_{i+1} + u_i d_i) u_{i+1} u_i (d_i + d_{i+1}) \]

Note that \( u_{i+1} d_i = d_i + 2u_i + 2 \) commutes with \( u_i \) and \( d_i \). Let us match the monomials on the l.h.s with the monomials on the r.h.s so that they are equal. We have

\[ u_{i+1} u_i d_i = u_{i+1} d_i + u_i u_i d_i = u_i u_i + u_i d_i = u_i u_i + d_i = u_i u_i + d_i = u_i u_i + d_i = u_i u_i + d_i \]

\[ u_{i+1} u_i d_i = u_{i+1} d_i + u_i u_i d_i = u_i u_i + u_i d_i = u_i u_i + u_i d_i = u_i u_i + u_i d_i = u_i u_i + u_i d_i \]

\[ u_{i+1} u_i d_i = u_{i+1} d_i + u_i u_i d_i = u_i u_i + u_i d_i = u_i u_i + u_i d_i = u_i u_i + u_i d_i = u_i u_i + u_i d_i \]

\[ u_{i+1} u_i d_i = u_{i+1} d_i + u_i u_i d_i = u_i u_i + u_i d_i = u_i u_i + u_i d_i = u_i u_i + u_i d_i = u_i u_i + u_i d_i \]

The coefficients at \( -\beta^3 \) are equal iff

\[ u_{i+1} d_i (u_{i+1} d_i + u_i d_i) = (u_{i+1} d_i + u_i d_i) u_{i+1} d_i \]

and since the elements \( \{u_i d_i\} \) commute, \([u_i d_i, u_{i+1} d_i] = [u_i d_i, d_i + 2u_i + 2] = 0\), the identity is true.
Let us prove now the local identity for $\tilde{d}$. The identity
\[ \tilde{d}_i \tilde{d}_{i+1} (\tilde{d}_i + \tilde{d}_{i+1}) = (\tilde{d}_i + \tilde{d}_{i+1}) \tilde{d}_i \tilde{d}_{i+1} \]
is equivalent to
\[ \sum_{k,\ell,m \geq 1} \beta^{k+\ell+m-3} d_i^k d_{i+1}^\ell (d_i^m + d_{i+1}^m) = \sum_{k',\ell',m' \geq 1} \beta^{k'+\ell'+m'-3} (d_i^{m'} + d_{i+1}^{m'}) d_i^{k'} d_{i+1}^{\ell'}. \]
Since $[d_i d_{i+1}, d_i] = [d_i d_{i+1}, d_{i+1}] = 0$ and for all $k, \ell, m \geq 1$ we have
\[ d_i^k d_{i+1}^\ell d_i^m = d_i^{k-1} d_{i+1}^{\ell-1} d_i^{m+1} d_{i+1} = \cdots = d_i^{k-t} d_{i+1}^{\ell+1} d_i^{m+t} d_{i+1}, \]
where $t = \min(k, \ell)$. If $t = \ell \leq k$ then $d_i^k d_{i+1}^\ell d_i^m = d_i^m d_i^k d_{i+1}^\ell$ and if $t = k < \ell$ then $d_i^k d_{i+1}^\ell d_i^m = d_i^{k-1} d_{i+1}^{\ell-k} d_i^{m+k}$.

Observe that $(d_i d_{i+1})^s = d_i^s d_{i+1}^s$ and then
\[ d_i^k d_{i+1}^\ell d_i^m = \delta_i^{k-1} (d_i^{d_{i+1}}) d_i^{\ell-1} d_i^{m+1} = d_i^{k-t} d_{i+1}^{\ell+1} d_i^{m+t} d_{i+1}. \]
Again, if $t = k \leq \ell$ then $d_i^k d_{i+1}^\ell d_i^m = d_i^m d_i^k d_{i+1}^\ell$ and if $t = \ell < k$ then $d_i^k d_{i+1}^\ell d_i^m = d_i^{k-1} d_{i+1}^{\ell-1} d_i^{m+1} d_{i+1} = d_i^{k-\ell} d_{i+1}^{m+\ell}$. Moreover, every element of the l.h.s. of (18) can be matched with the elements of the r.h.s. as follows
\[ d_i^k d_{i+1}^\ell d_i^m = d_i^m d_i^k d_{i+1}^\ell, \quad \text{if } \ell \leq k \]
\[ d_i^k d_{i+1}^\ell d_i^m = d_i^{m+k} d_i^{k-1} d_{i+1}^\ell, \quad \text{if } \ell > k \]
\[ d_i^k d_{i+1}^\ell d_i^m = d_i^{m+1} d_i^{k-1} d_{i+1}^\ell, \quad \text{if } k \leq \ell \]
\[ d_i^k d_{i+1}^\ell d_i^m = d_i^{k-\ell} d_{i+1}^{m+k} d_{i+1}^\ell, \quad \text{if } k > \ell \]
Note that the degree $k + \ell + m - 3$ is preserved and it is easy to check that we have defined a $\beta$ degree preserving bijection between the elements of l.h.s. and r.h.s of (18).

Let us finally verify the conjugate relations. The relation $[\tilde{u}_i, \tilde{d}_j] = 0$ for $|i - j| \geq 2$ is an easy consequence of a non-local commutativity. Now $\tilde{d}_i \tilde{u}_1 = (1 - \beta d_i)^{-1} d_i u_1 (1 - \beta d_{i+1}) = 1$ and $[\tilde{u}_{i+1}, \tilde{d}_i] = [u_{i+1} - \beta u_{i+1} d_i + d_i (1 - \beta d_i)^{-1}] = [u_{i+1} - \beta d_i + 2 u_{i+2}, d_i (1 - \beta d_i)^{-1}] = 0$.

\[ \Box \]

Proof of Lemma 5.3. Expanding the identity
\[ (1 - xy \tilde{u}_i \tilde{d}_i)^{-1} (1 + x \tilde{u}_i) (1 + y \tilde{d}_{i+1}) = (1 - xy \tilde{d}_{i+1} \tilde{u}_i)^{-1} (1 + y \tilde{d}_{i+1}) (1 + x \tilde{u}_i). \]
we need to show that
\[ \sum_k (xy)^k (\tilde{u}_i \tilde{d}_i)^k (1 + x \tilde{u}_i + y \tilde{d}_{i+1} + xy \tilde{u}_i \tilde{d}_{i+1}) = \sum_k (xy)^k (\tilde{d}_{i+1} \tilde{u}_i)^k (1 + x \tilde{u}_i + y \tilde{d}_{i+1} + xy \tilde{d}_{i+1} \tilde{u}_i) \]
or, equivalently, that the following identities hold (the first one for $k \geq 1$ and others for $k \geq 0$)
\[ (\tilde{u}_i \tilde{d}_i)^k + (\tilde{u}_i \tilde{d}_i)^{k-1} \tilde{u}_i \tilde{d}_{i+1} = (\tilde{d}_{i+1} \tilde{u}_i)^k + (\tilde{d}_{i+1} \tilde{u}_i)^{k-1} \tilde{d}_{i+1} \tilde{u}_i \]
\[ (\tilde{u}_i \tilde{d}_i)^k \tilde{d}_{i+1} = (\tilde{d}_{i+1} \tilde{u}_i)^k \tilde{d}_{i+1} \]
\[ (\tilde{u}_i \tilde{d}_i)^k \tilde{u}_i = (\tilde{d}_{i+1} \tilde{u}_i)^k \tilde{u}_i \]
Recall that $\tilde{u}_i = u_i (1 - \beta d_i) = u_i - \beta u_i d_i$, $\tilde{d}_i = d_i (1 - \beta d_i)^{-1} = \sum_{t \geq 1} \beta^{t-1} d_i^t$, and hence $\tilde{u}_i \tilde{d}_i = u_i d_i$.
Recall also that $u_i d_i = d_{i+1} u_{i+1}$ and $[u_i, d_j] = 0$ for $i \neq j$. 


Let us show (19). For $k = 1$ we have

$$
\tilde{u}_i d_i + \tilde{u}_i \tilde{d}_{i+1} = \tilde{d}_{i+1} \tilde{u}_{i+1} + \tilde{d}_{i+1} \tilde{u}_i
$$

$$
\iff u_i d_i + (u_i - \beta u_i d_i) \sum_{\ell \geq 1} \beta^{\ell-1} d_{i+1}^{\ell} = \sum_{\ell \geq 1} \beta^{\ell-1} d_{i+1}^{\ell} (u_i - \beta u_{i+1} d_i + u_i - \beta u_i d_i)
$$

$$
\iff u_i d_i + \sum_{\ell \geq 1} \beta^{\ell-1} u_i d_{i+1}^{\ell} - \beta^\ell u_i d_i d_{i+1}^{\ell} = \sum_{\ell \geq 1} \beta^{\ell-1} d_{i+1}^{\ell} u_{i+1} - \beta^\ell d_i d_{i+1}^{\ell} + \beta^\ell d_{i+1}^{\ell} u_{i+1} - \beta^\ell d_{i+1}^{\ell} u_i d_i
$$

$$
\iff u_i d_i - \sum_{\ell \geq 1} \beta^\ell u_i d_i d_{i+1}^{\ell} = d_{i+1} u_{i+1} - \sum_{\ell \geq 1} \beta^\ell d_{i+1}^{\ell} u_{i+1} + d_{i+1} u_{i+1} d_{i+1} + u_{i+1} d_{i+1} + u_i - \beta u_i d_i
$$

$$
\iff \sum_{\ell \geq 1} \beta^\ell u_i d_i d_{i+1}^{\ell} = \sum_{\ell \geq 1} \beta^\ell d_{i+1}^{\ell} u_{i+1} d_{i+1} + u_{i+1} d_{i+1} + u_i - \beta u_i d_i
$$

Notice that from $[d_i d_{i+1}, d_{i+1}] = 0$ we have

$$
u_i d_1 d_{i+1}^{\ell} = u_i d_{i+1}^{\ell-1} u_{i+1} + d_{i+1}^{\ell-1} u_1 d_{i+1}^{\ell} = d_{i+1}^{\ell-1} u_i d_{i+1}^{\ell} = d_{i+1}^{\ell-1} u_i d_{i+1}^{\ell}
$$

meaning that the preceding identities are indeed true. Next, note that $(\tilde{u}_i \tilde{d}_i)^k = (u_i d_i)^k = u_i d_i = \tilde{u}_i \tilde{d}_i$. For $k = 2$ we need to show that

$$(\tilde{u}_i \tilde{d}_i)^2 + \tilde{u}_i \tilde{d}_i \tilde{u}_i \tilde{d}_i = (\tilde{d}_{i+1} \tilde{u}_{i+1})^2 + \tilde{d}_{i+1} \tilde{u}_{i+1} \tilde{d}_{i+1} \tilde{u}_i
$$

$$
\iff \tilde{u}_i \tilde{d}_i + \tilde{u}_i \tilde{d}_i \tilde{u}_i \tilde{d}_i = (\tilde{d}_{i+1} \tilde{u}_{i+1})^2 + \tilde{d}_{i+1} \tilde{u}_{i+1} \tilde{d}_{i+1} \tilde{u}_i
$$

$$
\iff u_i d_i + u_i d_i \tilde{u}_i \tilde{d}_i = \tilde{d}_{i+1} \tilde{u}_{i+1} \tilde{d}_{i+1} \tilde{u}_i + \tilde{d}_{i+1} \tilde{u}_{i+1} \tilde{d}_{i+1} \tilde{u}_i
$$

$$
\iff u_i d_i + u_i d_i (u_i - \beta u_i d_i) \sum_{\ell \geq 1} \beta^{\ell-1} d_{i+1}^{\ell} = \sum_{\ell \geq 1} \beta^{\ell-1} d_{i+1}^{\ell} u_{i+1} + \tilde{d}_{i+1} \tilde{u}_{i+1} \tilde{d}_{i+1} \tilde{u}_i
$$

$$
\iff u_i d_i + \sum_{\ell \geq 1} \beta^{\ell-1} u_i d_i u_i d_i d_{i+1}^{\ell} - \beta^\ell u_i d_i d_{i+1}^{\ell} = \sum_{\ell \geq 1} \beta^{\ell-1} d_{i+1}^{\ell} u_{i+1} + \beta^\ell d_i d_{i+1}^{\ell} u_{i+1} - \beta^\ell d_{i+1}^{\ell} u_i d_i
$$

Notice that $u_i d_i d_{i+1} u_i d_i = d_{i+1} u_i d_i$ (removal of a box in the column $i + 1$ does not change the property if the box in the column $i$ was removable). Hence for $\ell \geq 2$

$$
d_{i+1}^{\ell-1} u_i d_i u_i d_i = d_{i+1}^{\ell-1} u_i d_i = d_{i+1}^{\ell-2} d_{i+1} u_i d_i = d_{i+1}^{\ell-2} d_{i+1} u_i d_i = d_{i+1}^{\ell-2} d_{i+1} u_i d_i = d_{i+1}^{\ell-2} d_{i+1} u_i d_i
$$

and then the last sum is equivalent to the following simplified form

$$
\iff u_i d_i + \sum_{\ell \geq 1} \beta^{\ell-1} u_i d_i u_i d_i d_{i+1}^{\ell} - \beta^\ell u_i d_i d_{i+1}^{\ell} = \tilde{d}_{i+1} \tilde{u}_{i+1} u_i d_i
$$

$$
+ \sum_{\ell \geq 1} \beta^\ell d_{i+1}^{\ell} u_i d_i d_{i+1}^{\ell} + \sum_{\ell \geq 1} \beta^{\ell-1} d_{i+1}^{\ell} u_i d_i u_i d_i d_{i+1}^{\ell} = \tilde{d}_{i+1} \tilde{u}_{i+1} u_i d_i
$$

which is true. Now for $k > 2$ first recall that $(\tilde{u}_i \tilde{d}_i)^k = \tilde{u}_i \tilde{d}_i$ and hence

$$(\tilde{d}_{i+1} \tilde{u}_{i+1})^k = \tilde{d}_{i+1} (\tilde{u}_{i+1} \tilde{d}_{i+1})^{k-1} u_{i+1} = (\tilde{d}_{i+1} \tilde{u}_{i+1})^2$$
and so we obtain
\[
(k, d_i) + (\bar{u}_i d_i)^{k-1} = \bar{u}_i d_i + \bar{u}_i \bar{d}_i \bar{u}_i \bar{d}_i
\]
\[
= (d_i \bar{u}_i + \bar{d}_i \bar{u}_i) + (d_i \bar{u}_i + \bar{d}_i \bar{u}_i)^{k-1}
\]
\[
= (d_i \bar{u}_i + \bar{d}_i \bar{u}_i)^k + (d_i \bar{u}_i + \bar{d}_i \bar{u}_i)^{k-1}
\]
which gives (19).

Let us now prove (20). Again, for \( k = 1 \) we need to prove that
\[
\bar{u}_i \bar{d}_i \bar{u}_i = \bar{d}_i + \bar{u}_i \bar{d}_i \bar{u}_i
\]
\[
\iff u_i d_i u_i (1 - \beta d_i) = (1 - \beta d_i) - d_i + u_i d_i u_i
\]
\[
\iff u_i d_i u_i (1 - \beta d_i) = (1 - \beta d_i) - d_i + u_i d_i u_i
\]
which is true since \([d_i, d_i, d_i] = 0\). For \( k \geq 2 \) we have
\[
(\bar{u}_i d_i)^k \bar{d}_i = \bar{u}_i \bar{d}_i \bar{u}_i = \bar{d}_i + \bar{u}_i \bar{d}_i \bar{u}_i
\]
which gives (20).

Finally, let us show (21). For \( k = 1 \) we need to show that
\[
\bar{u}_i \bar{d}_i \bar{u}_i = \bar{d}_i + \bar{u}_i \bar{d}_i \bar{u}_i
\]
\[
\iff u_i d_i u_i (1 - \beta d_i) = (1 - \beta d_i) - d_i + u_i d_i u_i
\]
\[
\iff u_i d_i u_i (1 - \beta d_i) = (1 - \beta d_i) - d_i + u_i d_i u_i
\]
For \( k = 2 \) we need to show that
\[
(\bar{u}_i d_i)^2 \bar{u}_i = (d_i + \bar{u}_i + d_i) \bar{u}_i
\]
\[
\iff u_i d_i u_i (1 - \beta d_i) = (1 - \beta d_i) - d_i + u_i d_i u_i
\]
and then it follows the same way as the previous chain for \( k = 1 \). Now for \( k > 2 \) we have
\[
(\bar{u}_i d_i)^k \bar{u}_i = \bar{u}_i \bar{d}_i \bar{u}_i = \bar{d}_i + \bar{u}_i \bar{d}_i \bar{u}_i
\]
which gives (20).
Proof of Lemma 9.6. Note that $\hat{D} = (1 + D)^{-1} - 1$. We have
\[
DU - UD = -(1 + D)
\]
\[
\implies (1 + D)U - U(1 + D) = -(1 + D)
\]
\[
\implies (1 + D)U(1 + D)^{-1} - U = -1
\]
\[
\implies U((1 + D)^{-1} - 1) - ((1 + D)^{-1} - 1)U = -(1 + D)^{-1} + 1 - 1
\]
\[
\implies U\hat{D} - \hat{D}U = -\hat{D} - 1
\]
\[
\implies [\hat{D}, U] = 1 + \hat{D}.
\]
\[
\square
\]

Proof of Lemma 10.2. The part (i) is standard and well-known. To prove (ii) first observe that
\[
(1 + D)U - U(1 + D) = 1 + D
\]
and then by induction $(1 + D)^nU = (U + n)(1 + D)^n$. Now we have
\[
D^a U^m = ((1 + D)^{-1} - 1)^a U^m
\]
\[
= \sum_k (-1)^{n-k} \binom{n}{k} (1 + D)^k u^m
\]
\[
= \sum_k (-1)^{n-k} \binom{n}{k} (U + k)^m (1 + D)^k
\]
\[
= \sum_k (-1)^{n-k} \binom{n}{k} \sum_i \binom{m}{i} k_{m-i} U^i \sum_j \binom{k}{j} D^j
\]
\[
= \sum_{i,j} U^i D^j \sum_k (-1)^{n-k} \binom{n}{k} \binom{m}{i} \binom{k}{j} k_{m-i}
\]
\[
= \sum_{i,j} \binom{m}{i} \binom{n}{j} U^i D^j \sum_k (-1)^{n-k} \binom{n-j}{k-j} k_{n-k}
\]
\[
= \sum_{i,j} \binom{m}{i} \binom{n}{j} U^{m-i} D^{n-j} \sum_k (-1)^{n-k} \binom{j}{n-k} k^i.
\]
Note that $q_n(i,j) = \sum_k (-1)^{n-k} \binom{j}{n-k} k^i$ is the coefficient at $[z^n]$ of the series
\[
\sum_{\ell} (-1)^{\ell} \binom{j}{\ell} z^\ell \sum_m z^m m^i = (1 - z)^j \frac{A_i(z)}{(1 - z)^{i+1}} = (1 - z)^{i-j+1} A_i(z),
\]
where $A_i(z) = \sum_{s=0}^{i-1} A_{i,s} z^s$ is the Eulerian polynomial. And thus $q_n(i,j) = \sum_{\ell} \binom{i-j+\ell}{\ell} A_{i,n-\ell}$ as needed.

\[
\square
\]