ASYMPTOTIC BEHAVIORS OF SOLUTIONS TO QUASILINEAR ELLIPTIC EQUATIONS WITH CRITICAL SOBOLEV GROWTH AND HARDY POTENTIAL

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Abstract. Optimal estimates on the asymptotic behaviors of weak solutions both at the origin and at the infinity are obtained to the following quasilinear elliptic equations

\[-\Delta_p u - \frac{\mu}{|x|^p} |u|^{p^*-2} u = Q(x)|u|^{\frac{Np}{N-p}} - 2 u, \quad x \in \mathbb{R}^N,\]

where \(1 < p < N, 0 \leq \mu < (N - p)/p\) and \(Q \in L^\infty(\mathbb{R}^N)\).

Keywords: Quasilinear elliptic equations; Hardy’s inequality; Comparison principle; Asymptotic behaviors

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1. Introduction and main results

Let \(1 < p < N, p^* = Np/(N - p)\) and \(0 \leq \mu < \bar{\mu} = ((N - p)/p)^p\). In this paper, we study the following quasilinear elliptic equations

\[-L_p u \equiv -\Delta_p u - \frac{\mu}{|x|^p} |u|^{p^*-2} u = Q(x)|u|^{\frac{Np}{N-p}} - 2 u, \quad x \in \mathbb{R}^N, \quad (1.1)\]

where

\[\Delta_p u = \sum_{i=1}^N \partial_{x_i}(|\nabla u|^{p-2} \partial_{x_i} u), \quad \nabla u = (\partial_{x_1} u, \cdots, \partial_{x_N} u)\]

is the \(p\)-Laplacian operator and \(Q \in L^\infty(\mathbb{R}^N)\). It is well known that equation (1.1) is the Euler-Lagrange equation of the energy functional \(E : \mathcal{D}^{1,p}(\mathbb{R}^N) \to \mathbb{R}\) defined by

\[E(u) = \frac{1}{p} \int_{\mathbb{R}^N} \left( |\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) - \frac{1}{p^*} \int_{\mathbb{R}^N} Q|u|^{p^*}, \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N),\]

where \(\mathcal{D}^{1,p}(\mathbb{R}^N)\) is the function space defined as

\[\mathcal{D}^{1,p}(\mathbb{R}^N) = \{v \in L^{p^*}(\mathbb{R}^N) : v \text{ is weakly differentiable and } \nabla v \in L^p(\mathbb{R}^N)\} \]
equipped with the seminorm $\|v\|_{D^{1,p}(\mathbb{R}^N)} = \|\nabla v\|_{L^p(\mathbb{R}^N)}$. All of the integrals in energy functional $E$ are well defined, due to the Sobolev inequality

$$C \left( \int_{\mathbb{R}^N} |\varphi|^p \right)^{\frac{1}{p}} \leq \int_{\mathbb{R}^N} |\nabla \varphi|^p, \quad \forall \varphi \in \mathcal{D}^{1,p}(\mathbb{R}^N),$$

where $C = C(N, p)$ is a positive constant, and due to the Hardy inequality (see [4, Lemma 1.1])

$$\left( \frac{N - p}{p} \right)^p \int_{\mathbb{R}^N} \frac{|\varphi|^p}{|x|^p} \leq \int_{\mathbb{R}^N} |\nabla \varphi|^p, \quad \forall \varphi \in \mathcal{D}^{1,p}(\mathbb{R}^N).$$

A function $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ is a weak subsolution of equation (1.1) if for any nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - \frac{\mu}{|x|^p} |u|^{p-2} u \varphi) \leq \int_{\mathbb{R}^N} Q(x)|u|^{p-2} u \varphi.$$

A function $u$ is a supersolution of equation (1.1) if $-u$ is a subsolution, and $u$ a weak solution of equation (1.1) if $u$ is both a weak subsolution and a weak supersolution.

Equation (1.1) and its variants have been extensively studied in the literature. For the existence of solutions to equation (1.1), we refer to e.g. [3, 4, 7, 8, 9, 10, 11, 16, 17, 19, 20, 26]. For the uniqueness of solutions to equation (1.1), we refer to e.g. [6, 11, 12, 18, 27]. In the present paper, we study the asymptotic behaviors of weak solutions to equation (1.1).

In the case $\mu = 0$, a prototype of equation (1.1) (when $Q \equiv 1$) is

$$-\Delta_p u = |u|^{p-2} u, \quad \text{in } \mathbb{R}^N. \quad (1.2)$$

The boundedness of weak solutions to equation (1.2) in the neighborhood of the origin is well known. As to the asymptotic behavior of solutions at the infinity, when $p = 2$, it was proved by Gidas, Ni and Nirenberg [18] that positive $C^2$ solutions (not necessarily in the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$) of equation (1.2) satisfying

$$\lim \inf_{|x| \to \infty} \left(|x|^{-2} u(x)\right) < \infty, \quad (1.3)$$

must be of the form $u(x) = \lambda^\frac{N-2}{p+2} u_0(\lambda(x - x_0))$ for some $\lambda > 0$ and $x_0 \in \mathbb{R}^N$, where

$$u_0(x) = (N(N - 2))^{\frac{N-2}{4(N+2)}} \left(1 + |x|^2\right)^{-\frac{N-2}{2}}. \quad (1.4)$$

Hypothesis (1.3) was removed by Caffarelli, Gidas and Spruck in [6]. Thus for positive $C^2$ solutions of equation (1.2), we have

$$|u(x)| \leq C|x|^{2-N} \quad \text{for } |x| > 1$$

for some positive constant $C$. This estimate has been proved to be true for all weak solutions in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ of (1.2), see Cao and Yan [10]. For $p \neq 2$, Cao, Peng and Yan [9] proved that for any weak solutions $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ of equation (1.2) we have

$$|u(x)| \leq C|x|^{\frac{N-p}{p+\theta}} \quad \text{for } |x| > 1 \quad (1.5)$$

for any $\theta > 0$ and some positive constant $C$ (depending also on $\theta$). We remark that their result can be easily extended, by the same approach in [10], to equation (1.1) ($\mu = 0$) in the presence of a bounded function $Q$. 
We will focus on the case $\mu \neq 0$. When $p = 2$, the behavior of weak solutions to equation (1.1) at origin is known. It was proved that if $u$ is a weak solution of equation (1.1), then
\[
|u(x)| \leq C|x|^{-\left(\sqrt{\bar{p}}-\sqrt{\bar{p}-\mu}\right)} \quad \text{for } |x| < 1
\] (1.6)
for some positive constant $C$, see [8, Theorem 1.1]. In addition, if the function $Q$ is nonnegative and the solution $u$ is also nonnegative, Han [21] proved that
\[
u(x) \geq C|x|^{-\left(\sqrt{\bar{p}}-\sqrt{\bar{p}-\mu}\right)} \quad \text{for } |x| < 1
\] (1.7)
for some nonnegative constant $C$. In fact, Ferrero and Gazzola [17] proposed the problem of studying the asymptotic behavior of eigenfunctions of the operator $-\Delta_2$ on bounded domain, that is, to study the asymptotic behavior of weak solutions to the linear equation as $x \to 0$
\[
-L_2u \equiv -\Delta u - \frac{\mu}{|x|^2} u = \lambda u, \quad u \in H^1_0(\Omega),
\] (1.8)
where $\lambda \in \mathbb{R}$ and $\Omega$ is a bounded domain containing the origin. Cabrè and Martel [5] obtained (1.6) for the first eigenfunctions in the case when $\Omega$ is a unit ball. Cao and Han [7] proved (1.6) for all solutions of equation (1.8). The approach of [7] is as follows: if $u$ is a weak solution to equation (1.8), then the function $\nu(x) \equiv |x|\sqrt{\bar{p}}-\sqrt{\bar{p}-\mu}u(x)$ satisfies the equation
\[
-\text{div} \left( |x|^{-2}\left( \sqrt{\bar{p}}-\sqrt{\bar{p}-\mu}\right) \nabla \nu \right) = \lambda |x|^{-2}\left( \sqrt{\bar{p}}-\sqrt{\bar{p}-\mu}\right) \nu, \quad x \in \Omega,
\]
which is a weighted elliptic equation in divergence form. By means of Moser’s iteration technique [25], $\nu$ is proved to be bounded, which is equivalent to (1.6). In [21], the author applied the same method to deal with more general type of equation than equation (1.1) when $p = 2$, and proved the estimates (1.6) and (1.7). Obviously, the approach of [7] is not applicable to general quasilinear equations when $p \neq 2$.

As to asymptotic behaviors of solutions of equation (1.1) at infinity when $p = 2$, to the best of the author’s knowledge, all known results are concerned with the particular case $Q \equiv 1$. That is, consider the equation
\[
-L_2u \equiv -\Delta u - \frac{\mu}{|x|^2} u = |u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N.
\] (1.9)
For any positive solution $u \in C^2(\mathbb{R}^N\setminus\{0\})$ of equation (1.9) satisfying
\[
\nu(x) \equiv |x|\sqrt{\bar{p}}-\sqrt{\bar{p}-\mu}u(x) \in L^\infty_{\text{loc}}(\mathbb{R}^N),
\] (1.10)
i.e., the function $\nu$ is locally bounded in $\mathbb{R}^N$, then a direct calculation verifies that $\nu$ satisfies the conditions of Theorem B of Chou and Chu [12], and thus $\nu$ is radially symmetric with respect to the origin by [12, Theorem B]. Therefore $u$ is radially symmetric with respect to the origin. Catrina and Wang [11] and Terracini [27] proved that all positive radial solutions of (1.9) are of the form $u(x) = \lambda^\frac{2}{N-2}u_0(\lambda x)$ for some $\lambda > 0$, where
\[
u(x) = (4N(\bar{p}-\mu)/(N-2))^{\frac{N-4}{4}}\left(|x|\sqrt{\bar{p}}-\sqrt{\bar{p}-\mu} + |x|\sqrt{\bar{p}}-\sqrt{\bar{p}-\mu}\right)^\frac{N-2}{2}.
\] (1.11)
Thus for any positive solution \( u \in C^2(\mathbb{R}^N \setminus \{0\}) \) to equation (1.9) satisfying (1.10), there is a \( \lambda > 0 \) such that \( u(x) = \lambda^{\frac{N-p}{2}} u_0(\lambda x) \). Consequently we have

\[
|u(x)| \leq C|x|^{-\left(\sqrt{\frac{N}{p}} + \sqrt{\frac{N}{p} - \mu}\right)} \quad \text{for } |x| > 1.
\]

for some positive constant \( C \). Cao and Yan [10, Lemma B.2] proved this estimate for all weak solutions in \( D^{1,2}(\mathbb{R}^N) \) of equation (1.9). Their method depends on the Kelvin transformation \( v(x) = |x|^{2-N} u \left( |x|^{-2} x \right) \). It seems that this approach does not work for general quasilinear equations (1.1) when \( p \neq 2 \).

Much less is known in the case \( \mu \neq 0 \) and \( p \neq 2 \). Boumediene, Veronica and Peral [4, Theorem 3.13] classified all weak positive radial solutions in \( D^{1,p}(\mathbb{R}^N) \) of equation (1.1) when \( Q \equiv 1 \). They are of the form \( u(x) = \lambda^{\frac{N-p}{2}} u_0(\lambda x) \) for some \( \lambda > 0 \), where \( u_0 \) is a particular weak positive radial solution in \( D^{1,p}(\mathbb{R}^N) \) (see [4, Theorem 3.13]) satisfying

\[
\lim_{|x| \to 0} u_0(x)|x|^\gamma_1 = C_1, \quad \lim_{|x| \to \infty} u_0(x)|x|^\gamma_2 = C_2
\]

(1.12)

for some positive constants \( C_1, C_2 \). In (1.12) and throughout the paper, \( \gamma_1, \gamma_2 \in [0, \infty), \gamma_1 < \gamma_2 \), are defined as the two roots of the equation

\[
\gamma^{p-2}[\gamma - (N - p)\gamma] + \mu = 0.
\]

(1.13)

While the exact form (1.11) of the positive radial solutions to equation (1.1) is known when \( p = 2 \) and \( Q \equiv 1 \), the exact form of the positive radial solution \( u_0 \) to equation (1.1) when \( p \neq 2 \) and \( Q \equiv 1 \) seems to be unknown.

For later use, we note that

\[
0 \leq \gamma_1 < \frac{N-p}{p} < \gamma_2 \leq \frac{N-p}{p-1}.
\]

In the case \( p = 2 \), \( \gamma_1 = \sqrt{\mu} - \sqrt{\mu - \mu} \) and \( \gamma_2 = \sqrt{\mu} + \sqrt{\mu - \mu} \), and in the case \( \mu = 0 \), \( \gamma_1 = 0 \) and \( \gamma_2 = (N-p)/(p-1) \).

In this paper, we give a complete description on the asymptotic behaviors of weak solutions to equation (1.1) at the origin and at the infinity.

**Theorem 1.1.** Let \( Q \in L^\infty(\mathbb{R}^N) \) and \( u \in D^{1,p}(\mathbb{R}^N) \) be a weak solution of equation (1.1). Then there exists a positive constant \( C \) depending on \( N, p, \mu, \|Q\|_\infty \) and \( u \), such that

\[
|u(x)| \leq C|x|^{-\gamma_1} \quad \text{for } |x| < R_0,
\]

(1.14)

and

\[
|u(x)| \leq C|x|^{-\gamma_2} \quad \text{for } |x| > R_1,
\]

(1.15)

where \( R_0 > 0 \) and \( R_1 > 0 \) depend on \( N, p, \mu, \|Q\|_\infty \) and \( u \).

In the above theorem, the positive constants \( C, R_0, R_1 \) depend on the solution \( u \). Indeed, this is the case, since equation (1.1) when \( Q \equiv 1 \) is invariant under the scaling \( v(x) = \lambda^{\frac{N-p}{2}} u(\lambda x) \), \( \lambda > 0 \). In above theorem and in the following, if we say a constant depends on the solution \( u \), it means that the constant depends on \( \|u\|_{L^p(\mathbb{R}^N)} \), the \( L^p \)–norm of \( u \), and also on the modulus of continuity of the function \( h(\rho) = \|u\|_{L^p(B_\rho(0))} + \|u\|_{L^p(\mathbb{R}^N \setminus B_\rho(0))} \) at zero. Precisely, we will choose a constant \( \epsilon_0 > 0 \) depending on \( N, p, \mu, \|Q\|_\infty \). Since \( h(\rho) \to 0 \) as \( \rho \to 0 \), there exists \( \rho_0 > 0 \) such that

\[
\|u\|_{L^p(B_{\rho_0}(0))} + \|u\|_{L^p(\mathbb{R}^N \setminus B_1(0))} < \epsilon_0.
\]
The constants \( C, R_0, R_1 \) in Theorem 1.1 depend on \( R_0 \).

We remark that our result is new even in the case \( p \neq 2, \mu = 0 \). We improve the estimate (1.5) by Cao, Peng and Yan [9].

The following theorem shows that the exponents \( \gamma_1 \) and \( \gamma_2 \) in the estimates (1.14) and (1.15) respectively in Theorem 1.1 are sharp.

**Theorem 1.2.** Let \( Q \in L^\infty(\mathbb{R}^N) \) be a nonnegative function and \( u \in D^{1,p}(\mathbb{R}^N) \) a nonnegative weak solution of equation (1.1). Then

\[
u(x) \geq m|x|^{-\gamma_1} \quad \text{for} \quad |x| < 1,
\]

and

\[
u(x) \geq m|x|^{-\gamma_2} \quad \text{for} \quad |x| > 1,
\]

where \( m = \inf_{\partial B(0)} u \).

In fact, Theorem 1.1 and Theorem 1.2 hold for solutions of more general equations. Consider the equation

\[-L_\mu u = f|u|^{p-2}u\]

in an arbitrary domain \( \Omega \subset \mathbb{R}^N \), where \( f \) is a function in the space \( L_\infty(\Omega) \). Equation (1.18) is the Euler-Lagrange equation of the energy functional \( E : D^{1,p}(\Omega) \to \mathbb{R} \) defined by

\[E(u) = \frac{1}{p} \int_\Omega \left( |\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) - \frac{1}{p} \int_\Omega f|u|^p, \quad u \in D^{1,p}(\Omega),\]

where \( D^{1,p}(\Omega) \) is the function space defined as

\[D^{1,p}(\Omega) = \left\{ v \in L^p(\Omega) : v \text{ is weakly differentiable and } \nabla v \in L^p(\Omega) \right\},\]

equipped with the seminorm \( \|v\|_{D^{1,p}(\Omega)} = \|\nabla v\|_{L^p(\Omega)} \).

A function \( u \in D^{1,p}(\Omega) \) is a weak subsolution of equation (1.18) in \( \Omega \) if

\[\int_\Omega \left( |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - \frac{\mu}{|x|^p} |u|^{p-2} u \varphi \right) \leq \int_\Omega f|u|^{p-2} u \varphi\]

for all nonnegative function \( \varphi \in C_{0}^\infty(\Omega) \). A function \( u \) is a weak supersolution of equation (1.18) in \( \Omega \) if \( -u \) is a weak subsolution. A function \( u \) is a weak solution of equation (1.18) in \( \Omega \) if \( u \) is both a weak subsolution and a weak supersolution.

The following theorem gives the asymptotic behavior of solutions to equation (1.18) at the origin.

**Theorem 1.3.** Let \( \Omega \) be a bounded domain containing the origin and \( f \) a function in \( L_\infty^N(\Omega) \) satisfying \( f(x) \leq A|x|^{-\alpha} \) in \( \Omega \) for some given constants \( A, \alpha, A > 0 \) and \( p > \alpha \). If \( u \in D^{1,p}(\Omega) \) is a weak subsolution to equation (1.18) in \( \Omega \), then there exists a positive constant \( C \) depending on \( N, p, \mu, A \) and \( \alpha \) such that

\[u(x) \leq CM|x|^{-\gamma_1} \quad \text{for} \quad x \in B_{R_0}(0),\]

where \( M = \sup_{\partial B_{R_0}(0)} u^+ \) and \( R_0 > 0 \) is a constant depending on \( N, p, \mu, A, \alpha \).

We also have the following theorem which shows that the exponent \( \gamma_1 \) in Theorem 1.3 is optimal.
Theorem 1.4. Let \( \Omega \) be a bounded domain containing the origin and \( f \in L^\infty(\Omega) \) a nonnegative function. If \( u \in D^{1,p}(\Omega) \) is a nonnegative weak supersolution of equation (1.18) in \( \Omega \), then
\[
u(x) \geq Cm|x|^{-\gamma_1} \quad \text{for } x \in \Omega,
\]
where \( m = \inf_{\partial \Omega} u \) and \( C = \inf_{\partial \Omega} |x|^{\gamma_1} \).

We also have the following corresponding results on the asymptotic behavior of weak solutions of equation (1.18) at infinity.

Theorem 1.5. Let \( \Omega \) be an exterior domain in \( \mathbb{R}^N \) such that \( \Omega^c = \mathbb{R}^N \setminus \Omega \) is bounded and function \( f \in L^\infty(\Omega) \) satisfy \( f(x) \leq A|x|^{-\alpha} \) in \( \Omega \) for some given constants \( A, \alpha, A > 0, p < \alpha \). If \( u \in D^{1,p}(\Omega) \) is a weak subsolution of equation (1.18) in \( \Omega \), then there exists a positive constant \( C \) depending only on \( N, p, \mu, A \) and \( \alpha \) such that
\[
u(x) \leq CM|x|^{-\gamma_2} \quad \text{for } x \in \mathbb{R}^N \setminus B_{R_1}(0),
\]
where \( M = \sup_{\partial B_R(0)} u^+ \) and \( R_1 > 1 \) is a constant depending on \( N, p, \mu, A, \alpha \).

Theorem 1.6. Let \( \Omega \) be an exterior domain in \( \mathbb{R}^N \) such that \( \Omega^c = \mathbb{R}^N \setminus \Omega \) is bounded and \( f \in L^\infty(\Omega) \) a nonnegative function. If \( u \in D^{1,p}(\Omega) \) is a nonnegative weak supersolution of equation (1.18) in \( \Omega \), then
\[
u(x) \geq Cm|x|^{-\gamma_2} \quad \text{for } x \in \Omega,
\]
where \( m = \inf_{\partial \Omega} u \) and \( C = \inf_{\partial \Omega} |x|^{\gamma_2} \).

Before we close this section, we outline the proof of Theorem 1.4. Other theorems are proved similarly. Suppose that \( u \in D^{1,p}(\Omega) \) is a nonnegative supersolution to equation (1.18) in a bounded domain \( \Omega \) containing the origin and \( f \) is a nonnegative function in \( L^\infty(\Omega) \). Note that the function \( v(x) = m|x|^{-\gamma_1}, \ m = \inf_{\Omega} u \), is a solution of equation
\[-L_p v = 0 \quad \text{in } \Omega.
\]
Therefore, \( v \) is also a subsolution of equation (1.18). To prove Theorem 1.4, we establish a comparison principle between subsolutions and supersolutions of equation (1.18) on \( \Omega \). The comparison principle is known in the case when \( \mu = 0 \) and \( f \equiv 0 \), see e.g. [24]. The comparison principle in the general case is established in Theorem 3.2 in Section 3. Then Theorem 1.4 follows by verifying that the supersolution \( u \) and subsolution \( v \) satisfy all the conditions required in Theorem 3.2. Our idea to prove the comparison principle for equation (1.18) is inspired by the paper [23] of Lindqvist, where he proved the simplicity of the first eigenvalue of the (minus) \( p \)-Laplacian operator. The essential point of his proof is to use the test functions of the type \( \varphi = u^{1-p}(u^p - v^p) \) and \( \psi = v^{1-p}(v^p - u^p) \).

The paper is organized as follows. In Section 2, we establish some preliminary asymptotic behaviors estimates for solutions of equation (1.11), and in Section 3 we establish comparison principles for subsolutions and supersolutions of equation (1.18) both on bounded domains and exterior domains. Section 4 is devoted to the proof of theorems listed above.

Throughout the paper, we denote domains by \( \Omega \), the complement of \( \Omega \) in \( \mathbb{R}^N \) by \( \Omega^c \). We also denote by \( B_R \) or by \( B_R(0) \) the ball centered at origin with radius \( R \). For any \( 1 \leq q \leq \infty \), \( L^q(\Omega) \) is
the Banach space of Lebesgue measurable functions \( u \) such that the norm
\[
\|u\|_{q, \Omega} = \begin{cases} \left( \int_{\Omega} |u|^q \right)^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty, \\ \text{esssup}_{\Omega} |f|, & \text{if } q = \infty \end{cases}
\]
is finite.

2. Preliminary estimates

In this section we prove the following preliminary asymptotic behavior estimates for solutions of equation (1.1). These estimates are used to prove Theorem 1.1.

**Proposition 2.1.** Let \( Q \in L^\infty(\mathbb{R}^N) \) and \( u \in D^{1,p}(\mathbb{R}^N) \) be a weak solution of equation (1.1). Then there exists a positive constant \( C \) depending on \( N, p, \mu, \|Q\|_\infty \) and \( u \), such that
\[
|u(x)| \leq C|x|^{-\left(\frac{\nu_p}{p} - \tau_0\right)} \quad \text{for } |x| < r_0,
\]
and that
\[
|u(x)| \leq C|x|^{-\left(\frac{\nu_p}{p} + \tau_1\right)} \quad \text{for } |x| > r_1,
\]
where \( \tau_0, \tau_1, r_0, r_1 > 0 \) are constants depending on \( N, p, \mu, \|Q\|_\infty \) and \( u \).

These pointwise estimates will be proved by Moser’s iteration argument [25]. We divide the proof into several lemmas.

**Lemma 2.2.** There exists a constant \( \delta_1 = \delta_1(N, p, \mu, \|Q\|_\infty) > 0 \) such that for any solution \( u \in D^{1,p}(\mathbb{R}^N) \) to equation (1.1) and for any \( \rho > 0 \) satisfying
\[
\|u\|_{L^\rho(B_{2\rho})} + \|u\|_{L^\rho(\mathbb{R}^N \setminus B_{1\rho})} \leq \delta_1,
\]
there exists a positive constant \( C \) depending on \( N, p, \mu, \|Q\|_\infty, \rho \) such that
\[
\|u\|_{L^\rho(B_R)} \leq CR^{\tau_0} \quad \text{for } R \leq \rho,
\]
and that
\[
\|u\|_{L^\rho(B_R)} \leq CR^{-\tau_1} \quad \text{for } R \geq 1/\rho,
\]
where \( \tau_0, \tau_1 > 0 \) are two constants depending on \( N, p, \mu \).

Let \( V = Q(x)|u|^{p-2} \). Equation (1.1) reads as
\[
-L_p u = V|u|^{p-2} u \quad \text{in } \mathbb{R}^N.
\]

**proof of Lemma 2.2.** We only prove the first inequality in Lemma 2.2. The second one can be proved similarly. Let \( R > 0 \) and \( \eta \in C_0^\infty(\mathbb{R}^N) \) be a cut-off function such that \( 0 \leq \eta \leq 1 \) in \( \mathbb{R}^N \), \( \eta \equiv 0 \) on \( B_R^c \), \( \eta \equiv 1 \) on \( B_{R/2} \) and \( |\nabla \eta| \leq 4/R \). Substituting test function \( \varphi = \eta^p u \) into equation (1.1), we obtain
\[
\langle -L_p u, \varphi \rangle = \int_{\mathbb{R}^N} \left( |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - \frac{\mu}{|x|^p} |u|^{p-2} u \varphi \right) = \int_{\mathbb{R}^N} V|u|^{p-2} u \varphi.
\]
Note that for any \( \delta \in (0, 1) \) there is a constant \( C_\delta > 0 \) such that
\[
\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \geq (1 - \delta) \int_{\mathbb{R}^N} |\nabla(u\eta)|^p - C_\delta \int_{\mathbb{R}^N} |\nabla \eta|^p |u|^p,
\]
and from the Hardy inequality we have

\[ \int_{\mathbb{R}^N} \frac{\mu}{|x|^p} |u|^{p-2} u \varphi \leq \frac{\mu}{\mu_r} \int_{\mathbb{R}^N} |\nabla (\varphi u)|^p. \]

Taking \( \delta = \delta(N, p, \mu) > 0 \) small enough such that \( (1 - \delta) > \mu/\mu_r \), we have

\[ \langle -L_p u, \varphi \rangle \geq (1 - \delta - \mu/\mu_r) \int_{\mathbb{R}^N} |\nabla (\varphi u)|^p - C_\delta \int_{\mathbb{R}^N} |\nabla \varphi| |u|^p \]

\[ \geq C_1 \left( \int_{\mathbb{R}^N} |\varphi u|^p \right)^{\frac{p}{p'}} - C_2 \left( \int_{B_R \setminus B_{R/2}} |u|^p \right)^{\frac{p}{p'}}, \]

where \( C_i = C_i(N, p, \mu), i = 1, 2 \). On the other hand,

\[ \left| \int_{\mathbb{R}^N} V |u|^{p-2} u \varphi \right| \leq \|V\|_{\mathcal{D}_{p, B_R}} \left( \int_{\mathbb{R}^N} |\varphi u|^p \right)^{\frac{1}{p'}} \leq \|Q\|_{\infty} \|u\|_{L^p, B_R}^{|p|-\frac{1}{p'}} \left( \int_{\mathbb{R}^N} |\varphi|^p \right)^{\frac{1}{p'}}. \]

Thus we obtain that

\[ \left( \int_{\mathbb{R}^N} |\varphi u|^p \right)^{\frac{p}{p'}} \leq C \left( \int_{B_R \setminus B_{R/2}} |u|^p \right)^{\frac{p}{p'}} + C \|Q\|_{\infty} \|u\|_{L^p, B_R}^{|p|-\frac{1}{p'}} \left( \int_{\mathbb{R}^N} |\varphi|^p \right)^{\frac{1}{p'}}. \]

Set

\[ \epsilon_1 = (4C\|Q\|_{\infty})^{-1/(p' - p)}, \]

and choose \( \rho > 0 \) small enough such that (2.1) holds. Then \( C\|Q\|_{\infty} \|u\|_{L^p, B_R} \leq 1/2 \) for all \( 0 < R \leq \rho \). We obtain

\[ \int_{B_R \setminus B_{R/2}} |u|^p \leq \int_{\mathbb{R}^N} |\varphi u|^p \leq C \int_{B_R \setminus B_{R/2}} |u|^p \quad \forall \ 0 < R \leq \rho, \]

where \( C \) depends only on \( N, p, \mu \). Denote \( \Psi(R) = \int_{B_R} |u|^p \) for \( R \leq \rho \). We get that

\[ \Psi(R/2) \leq \theta \Psi(R), \quad \forall \ 0 < R \leq \rho, \]

where \( \theta = \frac{C}{2} \in (0, 1) \) depends only on \( N, p, \mu \). Now applying Lemma 3.5 in [22, chapter 4] to \( \Psi \) on the interval \([0, \rho]\), we obtain

\[ \Psi(R) \leq \frac{1}{\theta} \Psi(\rho) \left( \frac{R}{\rho} \right)^{\tau_0} \quad \forall \ 0 < R \leq \rho, \]

where \( \tau_0 = \log(1/\theta)/\log 2 \) depends only on \( \theta \). Now the first inequality of this lemma follows by setting \( \tau_0 = \tau_0'/p^* \) and \( C = (\theta^{-1} \rho^{-\tau_0'} \Psi(\rho))^{1/p^*}. \]

Recall that we denote by \( B_R \) the ball \( B_R(0) \) centered at origin with radius \( R \). Let \( A_R = B_{8R} \setminus B_{R/8} \) and \( D_R = B_{4R} \setminus B_{R/4} \) for \( R > 0 \). We need the following uniform estimate with respect to \( R \).

**Lemma 2.3.** Let \( t \in (p^*, N/\gamma_1) \). Then there exists a constant \( \epsilon_2 = \epsilon_2(N, p, \mu, \|Q\|_{\infty}, t) > 0 \) such that for any solution \( u \in D^{1,p}(\mathbb{R}^N) \) to equation (1.1) and for any \( \rho > 0 \) satisfying

\[ \|u\|_{L^p(B_R)} + \|u\|_{L^{p^*}(B_{3R/4})} \leq \epsilon_2, \quad (2.2) \]

there holds

\[ \left( \int_{D_R} |u|^t \right)^{\frac{1}{t}} \leq C \left( \int_{A_R} |u|^p \right)^{\frac{1}{p}} \quad \forall \ R < \rho/8 \text{ or } R > 8/\rho, \quad (2.3) \]
where \( \int_R |u|^l = \frac{1}{|R|} \int_{R} |u|^l \) and \( C > 0 \) is a constant depending on \( N, p, \mu, t, \|Q\|_{\infty} \) and \( \rho \) but independent of \( R \).

**Proof.** The proof is essentially the same as that of Theorem 1.3 in [22, Chapter 4]. Here we need to show that the constant \( C \) in (2.3) is independent of \( R \). Set \( \nu(x) = u(Rx) \) for \( R > 0 \). Then \( \nu \) satisfies

\[
-L_p \nu = V_R |\nu|^{p-2} \partial_t \nu \quad \text{in} \ A_1,
\]

where \( V_R(x) = R^p V(Rx) \). Define \( \bar{\nu} = \max(\nu, 0) \) and \( \nu_m = \min(\bar{\nu}, m) \) for \( m \geq 1 \). Substituting, for any \( \eta \in C_0^\infty(A_1) \), and \( \frac{N-p}{p+y_1} > s > 1 \), the test function \( \varphi = \eta p \nu_m^{p(s-1)} \bar{\nu} \) into the equation of \( \nu \), we get

\[
\langle -L_p \nu, \varphi \rangle = \int_{A_1} (|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - \frac{\mu}{|x|^p} |u|^{p-2} u \varphi) = \int_{A_1} V_R |\nu|^{p-2} \nu \varphi.
\]

It is easy to see that for any \( \delta > 0 \) small there exist \( C_\delta > 0 \) such that

\[
\int_{A_1} |\nabla \nu|^{p-2} \nabla \nu \cdot \nabla \varphi \geq (1 - \delta) \frac{p(s-1) + 1}{s^p} \int_{A_1} |\nabla (\eta \nu_m^{p(s-1)} \bar{\nu})|^p - C_\delta \int_{A_1} |\nabla \eta|^p |\nu_m^{p(s-1)} \bar{\nu}|^p,
\]

and from the Hardy inequality,

\[
\int_{A_1} \frac{\mu}{|x|^p} |\nu|^{p-2} \nu \varphi = \int_{A_1} \frac{\mu}{|x|^p} (\eta \nu_m^{p-1} \bar{\nu})^p \leq \frac{\mu}{\mu} \int_{A_1} |\nabla (\eta \nu_m^{p-1} \bar{\nu})|^p.
\]

Since \( \frac{p(s-1)+1}{s^p} > \frac{\mu}{\mu} \) for all \( s \in (\frac{N-p}{p+y_1}, \frac{N-p}{p+y}) \), we can choose \( \delta \) small enough such that

\[
\langle -L_p \nu, \varphi \rangle \geq \left( (1 - \delta) \frac{p(s-1) + 1}{s^p} - \frac{\mu}{\mu} \right) \int_{A_1} |\nabla (\eta \nu_m^{p(s-1)} \bar{\nu})|^p - C_\delta \int_{A_1} |\nabla \eta|^p |\nu_m^{p(s-1)} \bar{\nu}|^p
\]

for some constants \( C_1, C_2 > 0 \) depending on \( N, p, \mu, s \), where \( \chi = p^s / p \). On the other hand, Hölder’s inequality gives us

\[
\int_{A_1} V_R |\nu|^{p-2} \nu \varphi \leq ||V_R||_{p^{-1}, A_1} \left( \int_{A_1} |\eta \nu_m^{p(s-1)} \bar{\nu}|^p \right)^{1/\chi} \leq ||Q||_{\infty} ||u||_{p^s, A_1} \left( \int_{A_1} |\eta \nu_m^{p(s-1)} \bar{\nu}|^p \right)^{1/\chi}.
\]

Therefore we have

\[
\left( \int_{A_1} |\eta \nu_m^{p(s-1)} \bar{\nu}|^p \right)^{1/\chi} \leq C_3 \int_{A_1} |\nabla \eta|^p |\nu_m^{p(s-1)} \bar{\nu}|^p + C_3 ||u||_{p^s, A_1} \left( \int_{A_1} |\eta \nu_m^{p(s-1)} \bar{\nu}|^p \right)^{1/\chi} \tag{2.4}
\]

for some constant \( C_3 = C_3(N, p, \mu, ||Q||_{\infty}, s) > 0 \).

Fix \( t \in (p^s, N/p_1) \) and \( k \in \mathbb{N} \) so that \( p^s \leq t < p^{s+1} \). Then there exists a positive constant \( C_3 = C_3(N, p, \mu, ||Q||_{\infty}, t) \) such that (2.4) holds for all \( 1 \leq s \leq \min \left\{ \frac{N-p}{p+y_1}, \chi k \right\} \).

Set

\[
e_2 \equiv (4C_3)^{-1/(p^s - p)}
\]

and choose \( \rho \in (0, 1) \) small enough such that (2.2) holds. Then

\[
||u||_{p^s, A_R} \leq e_2, \quad \forall 0 < R \leq \rho/8 \text{ or } R \geq 8/\rho.
\]

Therefore, for all \( 0 < R \leq \rho/8 \) or \( R \geq 8/\rho \), we have

\[
\left( \int_{A_1} |\eta \nu_m^{p(s-1)} \bar{\nu}|^p \right)^{1/\chi} \leq C \int_{A_1} |\nabla \eta|^p |\nu_m^{p(s-1)} \bar{\nu}|^p,
\]
for all $1 \leq s \leq \min\left\{ \frac{N-p}{p+1}, 1 \right\}$, where $C > 0$ depends only on $N, p, \mu, t, \|Q\|_{\infty}, \rho$. The same procedure can be applied also to $(-v)^+ = \max(-v, 0)$, the positive part of $-v$.

Now by choosing appropriate functions $\eta$ and then applying Moser’s iteration method [25], we conclude that for any $t \in (p^*, N/\gamma_1)$ fixed, after finitely many times of iteration, we can achieve the estimate

$$
\left( \int_{D_1} |v|^\mu \right)^{\frac{1}{\mu}} \leq C \left( \int_{A_1} |v|^{p^*} \right)^{\frac{1}{p^*}}.
$$

This proves (2.3).

Now we prove Proposition 2.1.

Proof of Proposition 2.1. Let $t = (p^* + N/\gamma_1)/2 \in (p^*, N/\gamma_1)$ as in Lemma 2.3 and $\epsilon_0 = \min(\epsilon_1, \epsilon_2)$, where $\epsilon_1$ and $\epsilon_2$ are as in Lemma 2.2 and Lemma 2.3 respectively. Let $\rho > 0$ such that (2.1) holds for $\epsilon_0$. Still consider the equation of $v(x) = u(Rx)$ given by

$$
-\Delta_{p^*} v + c(x)|v|^{p^*-2} v = 0 \quad \text{in } D_1,
$$

where $c(x) = -\mu|x|^{-p} - V_R(x)$. Note that $|x|^{-p}$ is a bounded function on $D_1$, and $V_R(x) = \rho^p V(Rx) \in L^q(D_1)$ with $q = \frac{p^*}{p^*-p} > \frac{N}{p}$ due to (2.5). Following the proof of Theorem 1.1 in [22, Chapter 4] one gets that

$$
\sup_{B} |v| \leq C \left( \int_{2B} |v|^p \right)^{\frac{1}{p}}
$$

(2.6)

for any ball $B = B(x, r)$ such that $2B \subset D_1$, where $C = C(N, p, \mu, \|V_R\|_{L^q(2B)})$.

We claim that the quantity $\|V_R\|_{L^q(D_1)}$ is uniformly bounded with respect to $R$. Moreover, there exists a constant $C > 0$ depending on $N, p, \mu, q, \|Q\|_{\infty}$ and $\rho$ such that

$$
\|V_R\|_{L^q(D_1)} \leq C \|u\|_{p^*, \mathbb{R}^N}^{p^*-p} \quad \forall \ 0 < R \leq \rho/8 \text{ or } R \geq 8/\rho.
$$

Indeed, by the definition of $V_R$ and (2.3),

$$
\|V_R\|_{q, D_1} = \|R^{p^* - \frac{N}{q}} V\|_{q, D_R} \leq \|Q\|_{\infty} R^{p^* - \frac{N}{q}} \|u\|_{p^*, D_R}^{p^*-p} \leq CR^{p^* - \frac{N}{q} - \frac{N}{p}} \|u\|_{p^*, \mathbb{R}^N}^{p^*-p} \leq C \|u\|_{p^*, \mathbb{R}^N}^{p^*-p},
$$

since $p - \frac{N}{q} - \frac{N}{p} - (\frac{N}{p} - \frac{N}{t})(p^* - p) = 0$. Therefore the estimate (2.6) is uniform with respect to $0 < R \leq \rho/8$ or $R \geq 8/\rho$.

Now a simple covering argument leads us to

$$
\sup_{B_2 \setminus B_1} |v| \leq C \left( \int_{D_1} |v|^p \right)^{\frac{1}{p}}.
$$

Recall that $v(x) = u(Rx)$. Equivalently we arrive at

$$
\sup_{B_{3R} \setminus B_R} |u(x)| \leq C \left( \int_{D_R} |u|^p \right)^{\frac{1}{p}} \quad \forall \ 0 < R \leq \rho/8 \text{ or } R \geq 8/\rho.
$$
Hence by Hölder’s inequality,
\[ \sup_{B_{2R} \setminus B_R} |u| \leq C \left( \int_{D_R} |u|^{p_1} \right)^{\frac{1}{p_1}} \leq C \|u\|_{L^{p_1}(D_R)} R^{\frac{p-\alpha}{\alpha}}, \quad \forall 0 < R \leq \rho/8 \text{ or } R \geq 8/\rho, \quad (2.7) \]
where \( C \) depends only on \( N, p, \mu, q, \|Q\|_{\infty}, \rho \) and \( \|u\|_{p^*} \).

Since \( A_R \subset B_\rho \) for \( 0 < R \leq \rho/8 \) and \( A_R \subset B_{1/\rho} \) for \( R \geq 8/\rho \), by Lemma 2.2, there exist \( \tau_0, \tau_1 > 0 \) depending only on \( N, p, \mu \) such that \( \|u\|_{L^{p_1}(D_R)} \leq CR^\tau_0 \) if \( 0 < R \leq \rho/8 \) and \( \|u\|_{L^{p_1}(D_R)} \leq CR^{-\tau_1} \) if \( R \geq 8/\rho \). Therefore, by letting \( r_0 = \rho/8, r_1 = 8/\rho \) and inserting the estimates of \( \|u\|_{L^{p_1}(D_R)} \) into (2.7), we complete the proof of Proposition 2.1.

Next we prove that some special functions are supersolutions.

**Proposition 2.4.**
1. Given two constants \( A, \alpha \in \mathbb{R}, A > 0 \) and \( \alpha < p \). There exist constants \( 1 > \delta, \epsilon > 0 \) depending on \( N, p, \mu, A, \alpha \) such that the function \( v(x) = |x|^{-\gamma}(1 - \delta |x|^\gamma) \in \mathcal{D}^{1,p}(B_{R_0}) \) is a positive supersolution to equation
\[ -L_p v = g(x)|v|^{\gamma-2} v, \quad x \in B_{R_0}, \quad (2.8) \]
where \( g \) is a positive function in \( L^{\frac{N}{p}}(B_{R_0}) \) satisfying
\[ g(x) \geq A|x|^{-\alpha}, \quad x \in B_{R_0}, \]
with some constant \( 1 > R_0 = R_0(N, p, \mu, A, \alpha) > 0 \).
2. Given two constants \( A, \alpha \in \mathbb{R}, A > 0 \) and \( \alpha > p \). There exist constants \( 1 > \delta, \epsilon > 0 \) depending on \( N, p, \mu, A, \alpha \) such that the function \( v(x) = |x|^{-\gamma}(1 - \delta |x|^\gamma) \in \mathcal{D}^{1,p}(B_{R_1}) \) is a supersolution to equation
\[ -L_p v = g(x)|v|^{\gamma-2} v, \quad x \in B_{R_1}, \quad (2.9) \]
where \( g \) is a positive function in \( L^{\frac{N}{p}}(B_{R_1}) \) satisfying
\[ g(x) \geq A|x|^{-\alpha}, \quad x \in B_{R_1}, \]
with some constant \( R_1 = R_1(N, p, \mu, A, \alpha) > 1 \).

**Proof.** Let \( \gamma, \delta, \epsilon \in \mathbb{R} \) and define function \( v(x) = |x|^{-\gamma}(1 - \delta |x|^\gamma) \). Direct computation shows that
\[ -L_p v = \frac{h(\delta|x|^\gamma)}{[1 - \delta |x|^\gamma]^{p-2} (1 - \delta |x|^\gamma)} |v|^{\gamma-2} v, \quad \text{for } x \neq 0, \]
where
\[ h(t) \equiv |\gamma - (\gamma + \epsilon)t|^{p-2} [k(\gamma - \epsilon)t - k(\gamma)] - \mu|1 - t|^{p-2} (1 - t), \quad t \in [0, \infty), \]
and \( k(t) \equiv (p - 1)t^2 - (N - p)t, t \geq 0 \). It is easy to prove that \( v \) is a weak solution of equation (2.8) and equation (2.9) with
\[ g(x) = \frac{h(\delta|x|^\gamma)}{[1 - \delta |x|^\gamma]^{p-2} (1 - \delta |x|^\gamma)} |x|^\gamma. \]
It remains to prove that \( g \) satisfies those properties mentioned in the proposition.

Note that \( h(0) = -|\gamma|^{p-2} k(\gamma) - \mu \), where \( k(\gamma) = (p - 1)\gamma^2 - (N - p)\gamma \). Thus by the definition of \( \gamma_1 \) and \( \gamma_2 \), as in (1.13), we have \( h(0) = 0 \) when \( \gamma = \gamma_1 \) or \( \gamma = \gamma_2 \).

Also we have
\[ h'(0) = (p - 1)\gamma^{p-2} (-p\gamma + N - p + \epsilon) \epsilon. \]
So

\[ h'(0) > 0 \quad \text{if} \quad \gamma = \gamma_1, \epsilon > 0 \quad \text{or} \quad \gamma = \gamma_2, \epsilon < 0. \]

Therefore, there exists \( 1 > \delta_h > 0 \) such that \( 2h'(0)t \geq h(t) \geq \frac{1}{2}h'(0)t > 0 \) for \( t \in (0, \delta_h) \).

To obtain (1), we choose \( \delta = \min\{\delta_h, 1/2\}, \epsilon = (p - \alpha)/2 > 0 \) and

\[ R_0 = \min \left\{ 1, \left( \frac{2A}{\delta^2}h'(0) \right)^{-1/(p-\alpha-\epsilon)} \right\}. \]

Then function \( v(x) = |x|^{-\gamma_1}(1 - \delta|x|^\alpha) \) is positive in \( D_{1,p}(B_{R_0}) \) since \( \gamma_1 < (N - p)/p, \delta > 0, \epsilon > 0 \) and function \( g \) given above satisfies

\[ A|x|^{-\alpha} \leq \frac{\delta}{2}h'(0)|x|^{\epsilon - p} \leq g(x) \leq 2^p \delta h'(0)|x|^{\epsilon - p}, \quad \forall \; x \in B_{R_0}. \]

The last inequality implies that \( g \in L^{\frac{N}{p}}(B_{R_0}) \).

(2) is obtained similarly. \( \square \)

3. Comparison principle

In this section, we prove the comparison principle for subsolutions and supersolutions to equation (1.18). We start with the following pointwise estimate.

**Lemma 3.1.** For all weakly differentiable positive functions \( u, v \) on a domain \( \Omega \), we have for \( p \geq 2 \),

\[
|\nabla u|^{p-2} \nabla u \cdot \nabla \left( u - \frac{u^p}{u^p} u \right) + |\nabla v|^{p-2} \nabla v \cdot \nabla \left( v - \frac{u^p}{v^p} v \right) \geq C_p(u^p + v^p)|\nabla \log u - \nabla \log v|^p;
\]

and also for \( 1 < p < 2 \),

\[
|\nabla u|^{p-2} \nabla u \cdot \nabla \left( u - \frac{u^p}{u^p} u \right) + |\nabla v|^{p-2} \nabla v \cdot \nabla \left( v - \frac{u^p}{v^p} v \right) \geq C_p(u^p + v^p)\left| \nabla \log u - \nabla \log v \right|^2 \left( |\nabla \log u| + |\nabla \log v| \right)^2 - p,
\]

where \( C_p \) are positive constants depending only on \( p \).

**Proof.** Let \( u, v \) be two weakly differentiable positive functions, and \( \eta_1 = u - \frac{u^p}{u^p} u, \eta_2 = v - \frac{u^p}{v^p} v \). Then

\[
|\nabla u|^{p-2} \nabla u \cdot \nabla \eta_1 = |\nabla u|^p - v^p \left( |\nabla \log u|^p + p|\nabla \log u|^{p-2} \nabla \log u \cdot \nabla \log v \right),
\]

and

\[
|\nabla v|^{p-2} \nabla v \cdot \nabla \eta_2 = |\nabla v|^p - u^p \left( |\nabla \log v|^p + p|\nabla \log v|^{p-2} \nabla \log v \cdot \nabla \log u \right).
\]

When \( p \geq 2 \), applying the following elementary inequality (see [23]):

\[
|a|^p \geq |b|^p + p|b|^{p-2}b \cdot (a - b) + \frac{|a - b|^p}{2^{p-1} - 1}, \quad \forall \; a, b \in \mathbb{R}^N,
\]

we obtain that,

\[
|\nabla u|^{p-2} \nabla u \cdot \nabla \eta_1 \geq |\nabla u|^p - v^p(|\nabla \log u|^p + C_p|\nabla \log u - \nabla \log v|^p)
= |\nabla u|^p - |\nabla v|^p + C_p v^p|\nabla \log u - \nabla \log v|^p,
\]

and that

\[
|\nabla v|^{p-2} \nabla v \cdot \nabla \eta_2 \geq |\nabla v|^p - |\nabla u|^p + C_p u^p|\nabla \log u - \nabla \log v|^p.
\]

Adding these two inequalities together, we finish the proof of the lemma in the case \( p \geq 2 \).
The case $1 < p < 2$ follows from the following elementary inequality (see [23]):
\[
|a|^p \geq |b|^p + p|b|^{p-2}b \cdot (a - b) + C_p \frac{|a - b|^2}{(|a| + |b|)^{2-p}}, \quad \forall a, b \in \mathbb{R}^N,
\]
where $C_p$ is a positive constant depending only on $p$. $\square$

In the following two theorems, Theorem 3.2 and Theorem 3.3, we prove the comparison principle. Theorem 3.2 deals with the bounded domains, and Theorem 3.3 with the exterior domains.

**Theorem 3.2.** Let $\Omega$ be a bounded domain and $f \in L^\infty(\Omega)$, Let $u \in \mathcal{D}_0^1(\Omega)$ be a weak subsolution of equation (1.18) and $v \in \mathcal{D}_0^1(\Omega)$ a weak supersolution of
\[
-L_p u = g|v|^{p-2}v \quad \text{in } \Omega
\]
such that $\inf_{\Omega} v > 0$, where $g$ is a given function in $L^\infty(\Omega)$ such that $f \leq g$ in $\Omega$. If $u \leq v$ on $\partial \Omega$, then
\[
u \leq v \quad \text{in } \Omega.
\]

**Proof.** We only prove Theorem 3.2 in the case when $p \geq 2$. We can prove similarly Theorem 3.2 in the case when $1 < p < 2$. For $m > 1$, it is easy to check that functions $\eta_1 = u^{1-p} \min((u^p - v^p)^+, m)$ and $\eta_2 = -v^{1-p} \min((u^p - v^p)^+, m)$ can be taken as test functions of equations (1.18) and equation (3.1) respectively. Therefore, substituting $\eta_1$ into equation (1.18) and $\eta_2$ into equation (3.1), and then adding together, we obtain that
\[
\langle -L_p u, \eta_1 \rangle + \langle -L_p v, \eta_2 \rangle \leq \int_{\Omega} f|u|^{p-2} u \eta_1 + \int_{\Omega} g|v|^{p-2} v \eta_2
\]
\[
= \int_{\{w^p - v^p \geq m\}} m(f - g) + \int_{\{0 \leq w^p - v^p \leq m\}} (f - g)(u^p - v^p)
\]
\[
\leq 0 \quad \text{since } f \leq g,
\]
where $\langle -L_p w, \eta \rangle$ is defined as
\[
\langle -L_p w, \eta \rangle = \int_{\Omega} \left( |\nabla w|^{p-2} \nabla w \cdot \nabla \eta - \frac{\mu}{|x|^p} |w|^{p-2} w \eta \right)
\]
for all $w, \eta \in \mathcal{D}_0^1(\Omega)$.

By the definition of $\eta_1, \eta_2$, we obtain that
\[
\langle -L_p u, \eta_1 \rangle = \int_{\{w^p - v^p \geq m\}} |\nabla u|^{p-2} \nabla u \cdot \nabla (mu^{1-p}) - \mu \frac{u^{p-1}}{|x|^p} mu^{1-p}
\]
\[
+ \int_{\{0 \leq w^p - v^p \leq m\}} |\nabla u|^{p-2} \nabla u \cdot \nabla \left( u - \frac{v^p}{u^p} u \right) - \mu \frac{u^{p-1}}{|x|^p} \left( u - \frac{v^p}{u^p} u \right),
\]
and
\[
\langle -L_p v, \eta_2 \rangle = \int_{\{w^p - v^p \geq m\}} |\nabla v|^{p-2} \nabla v \cdot \nabla (-mv^{1-p}) + \mu \frac{v^{p-1}}{|x|^p} mv^{1-p}
\]
\[
+ \int_{\{0 \leq w^p - v^p \leq m\}} |\nabla v|^{p-2} \nabla v \cdot \nabla \left( v - \frac{u^p}{v^p} v \right) - \mu \frac{v^{p-1}}{|x|^p} \left( v - \frac{u^p}{v^p} v \right).
\]
respectively. Hence we have
\[
\langle -L_p u, \eta_1 \rangle = \int_{\{u^p - v^p \geq m\}} (m(1 - p)u^{p-1}|\nabla u|^p - m\mu|x|^{p-1}) \\
+ \int_{\{0 \leq u^p - v^p \leq m\}} \left(|\nabla u|^{p-2}\nabla u \cdot \nabla \left(u - \frac{v^p}{u^p}u\right) - \mu \frac{u^p - v^p}{|x|^p}\right),
\]
and
\[
\langle -L_p v, \eta_2 \rangle \geq \int_{\{u^p - v^p \geq m\}} m\mu|x|^{p-1} + \int_{\{0 \leq u^p - v^p \leq m\}} \left(|\nabla v|^{p-2}\nabla v \cdot \nabla \left(v - \frac{u^p}{v^p}v\right) - \mu \frac{v^p - u^p}{|x|^p}\right),
\]
since $|\nabla v|^{p-2}\nabla v \cdot \nabla (-mv^{1-p}) \geq 0$. Therefore we obtain that
\[
\langle -L_p u, \eta_1 \rangle + \langle -L_p v, \eta_2 \rangle \geq \int_{\{u^p - v^p \geq m\}} m(1 - p)u^{p-1}|\nabla u|^p \\
+ \int_{\{0 \leq u^p - v^p \leq m\}} \left(|\nabla u|^{p-2}\nabla u \cdot \nabla \left(u - \frac{v^p}{u^p}u\right) + |\nabla v|^{p-2}\nabla v \cdot \nabla \left(v - \frac{u^p}{v^p}v\right)\right).
\]
Estimate the right hand side of the above equation as follows: for the first term we have
\[
(p - 1) \int_{\{u^p - v^p \geq m\}} m\mu u^{p-1}|\nabla u|^p \leq (p - 1) \int_{\{u^p \geq m\}} |\nabla u|^p \to 0 \quad \text{as } m \to \infty,
\]
since \(\lim_{m \to \infty} |\{u^p \geq m\}| = 0\), and for the second term we apply Lemma 3.1 to obtain that
\[
|\nabla u|^{p-2}\nabla u \cdot \nabla \left(u - \frac{v^p}{u^p}u\right) + |\nabla v|^{p-2}\nabla v \cdot \nabla \left(v - \frac{u^p}{v^p}v\right) \geq C_p (u^p + v^p)|\nabla \log u - \nabla \log v|^p
\]
for some positive constant \(C_p\) depending only on \(p\). Recall that \(\langle -L_p u, \eta_1 \rangle + \langle -L_p v, \eta_2 \rangle \leq 0\). Letting \(m \to \infty\), we obtain that
\[
\int_{\{0 \leq u^p - v^p\}} (u^p + v^p)|\nabla \log u - \nabla \log v|^p = 0,
\]
which implies that
\[
\log u = \log v + C \quad \text{on } \{x \in \Omega; u(x) \geq v(x)\},
\]
i.e.
\[
u = C\log v \quad \text{on } \{x \in \Omega; u(x) \geq v(x)\},
\]
for some positive constant \(C > 0\). Since we assume in the theorem that \(\inf_{\Omega} v > 0\), it follows that \(C = 1\). This implies that
\[
u \leq v, \quad \text{in } \Omega.
\]
This finishes the proof of Theorem 3.2 in the case \(p \geq 2\). \(\square\)

Corresponding comparison principle in exterior domains is given in the following theorem.

**Theorem 3.3.** Let \(\Omega\) be an exterior domain such that \(\Omega^c = \mathbb{R}^N \setminus \Omega\) is bounded and \(f \in L^{\frac{N}{N-1}}(\Omega)\). Let \(u \in D^{1,p}(\Omega)\) be a subsolution of equation (1.18) and \(v \in D^{1,p}(\Omega)\) a positive supersolution of
\[
- L_p v = g|v|^{p-2}v \quad \text{in } \Omega,
\]
such that \(\inf_{\partial \Omega} v > 0\), where functions \(g\) belongs to \(L^{\frac{N}{N-1}}(\Omega)\) and \(f \leq g\) in \(\Omega\). Moreover, assume that
\[
\lim_{R \to \infty} \frac{1}{R} \int_{B_R \setminus B_{2R}} u^p|\nabla \log v|^{p-1} = 0.
\]
If $u \leq v$ on $\partial \Omega$, then

$$u \leq v \quad \text{in } \Omega.$$  

**Proof.** We only prove Theorem 3.3 in the case when $p \geq 2$. We can prove similarly Theorem 3.3 in the case when $1 < p < 2$. Fix $R > 2\text{diam}(\Omega^*)$ and $m \geq 1$. Let $\eta \in C_0^\infty(B_{2R})$ be a cut-off function such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_R$ and $|\nabla \eta| \leq \frac{2}{R}$. Substituting test functions

$$\varphi_1 = \eta u^{1-p} \min((u^p - v^p)^+, m) \quad \text{and} \quad \varphi_2 = -\eta v^{1-p} \min((u^p - v^p)^+, m)$$

into equation (1.18) and equation (3.2) respectively, and adding together, we have

$$\langle -L_p u, \varphi_1 \rangle + \langle -L_p v, \varphi_2 \rangle \leq \int_{\Omega} f|u|^{p-2} u \varphi_1 + \int_{\Omega} g|v|^{p-2} v \varphi_2$$

$$= \int_{\{u^p - v^p \geq m\}} m(f - g)\eta + \int_{\{0 \leq u^p - v^p \leq m\}} (f - g)(u^p - v^p)\eta$$

$$\leq 0 \quad \text{since } f \leq g.$$  

On the other hand, by the definition of $\varphi_1$ and $\varphi_2$, we have

$$\langle -L_p u, \varphi_1 \rangle + \langle -L_p v, \varphi_2 \rangle$$

$$= \int_{\{0 \leq u^p - v^p \leq m\}} \left( \eta|u|^{2-p} \nabla u \cdot \nabla \left( u - \frac{v^p}{u^p} u \right) + \eta|v|^{2-p} \nabla v \cdot \nabla \left( v - \frac{u^p}{v^p} v \right) \right)$$

$$+ \int_{\{0 \leq u^p - v^p \leq m\}} \left( u - \frac{v^p}{u^p} u \right) \nabla u |^{p-2} \nabla u \cdot \nabla \eta + \left( v - \frac{u^p}{v^p} v \right) \nabla v |^{p-2} \nabla v \cdot \nabla \eta$$

$$+ \int_{\{u^p - v^p \geq m\}} \nabla u |^{p-2} \nabla u \cdot \nabla (m^2 u^{1-p}) + \int_{\{u^p - v^p \geq m\}} \nabla v |^{p-2} \nabla v \cdot \nabla (-m^2 v^{1-p})$$

$$=: I_1 + I_2 + I_3 + I_4.$$  

Lemma 3.1 implies that

$$I_1 \geq C_p \int_{\{0 \leq u^p - v^p \leq m\}} \eta(u^p + v^p)|\nabla \log u - \nabla \log v|^p$$

$$\geq C_p \int_{\{0 \leq u^p - v^p \leq m\} \cap \partial \Omega} (u^p + v^p)|\nabla \log u - \nabla \log v|^p,$$

where $C_p > 0$ is independent of $m, R$.

We estimate $I_k$ ($k = 2, 3, 4$) as follows. For $I_2$, we have

$$|I_2| \leq \int_{\{0 \leq u^p - v^p\}} (|\nabla u|^{p-1} + |\nabla v|^{p-1}) |\nabla \eta| + \int_{\{0 \leq u^p - v^p\}} u^p |\nabla \log v|^{p-1} |\nabla \eta|$$

$$\leq \frac{2}{R} \int_{B_{2R}\setminus B_R} (|\nabla u|^{p-1} + |\nabla v|^{p-1}) \eta + \frac{2}{R} \int_{B_{2R}\setminus B_R} u^p |\nabla \log v|^{p-1} =: J_1 + J_2.$$  

Hölder’s inequality implies that

$$J_1 \leq C \left( \int_{B_{2R}\setminus B_R} |\nabla u|^p \right)^{\frac{p-1}{p}} \left( \int_{B_{2R}\setminus B_R} |u|^p \right)^{\frac{1}{p}} + C \left( \int_{B_{2R}\setminus B_R} |\nabla v|^p \right)^{\frac{p-1}{p}} \left( \int_{B_{2R}\setminus B_R} |v|^p \right)^{\frac{1}{p}}$$

$$= o(1) \quad \text{as } R \to \infty,$$

and by assumption (3.3)

$$J_2 \to 0 \quad \text{as } R \to \infty.$$
Note that $J_1, J_2$ are independent of $m$, so
\[
\lim_{m,R \to \infty} I_2 = 0.
\]
For $I_3$, it is easy to see that
\[
I_3 = m \int_{\{u^p - v^p \geq m\}} \left( |\nabla u|^{p-2} \nabla u \cdot \nabla \eta u^{1-p} + (1 - p) \eta u^{-p} |\nabla u|^p \right).
\]
So
\[
|I_3| \leq \int_{\{u^p \geq m\}} \left( |\nabla u|^{p-1} |\nabla \eta| u + (p - 1) \eta |\nabla u|^p \right) \leq J_1 + (p - 1) \int_{\{u^p \geq m\}} |\nabla u|^p.
\]
Since the level set $\{u^p \geq m\}$ vanishes as $m \to \infty$, one deduces that
\[
\lim_{m,R \to \infty} I_3 = 0.
\]
For the last term $I_4$, we have
\[
I_4 = \int_{\{u^p - v^p \geq m\}} \left( m(p - 1) \eta v^{-p} |\nabla v|^p - m v^{1-p} |\nabla v|^{p-2} \nabla v \cdot \nabla \eta \right)
\]
\[
\geq -m \int_{\{u^p - v^p \geq m\}} |\nabla \log v|^{p-1} |\nabla \eta|
\]
\[
\geq - \int_{\{u^p \geq m\}} |\nabla \log v|^{p-1} |\nabla \eta| u^p \geq -J_2,
\]
which converges to zero as $m, R \to \infty$ by assumption (3.3). Hence combining together all above estimates we obtain that
\[
0 \geq \limsup_{R,m \to \infty} \left( \langle -L_p u, \varphi_1 \rangle + \langle -L_p v, \varphi_2 \rangle \right) \geq C \int_{\{u \geq v\}} (u^p + v^p) |\nabla \log u - \nabla \log v|^p.
\]
Thus
\[
\int_{\{u \geq v\}} (u^p + v^p) |\nabla \log u - \nabla \log v|^p = 0,
\]
which implies that $u \leq v$ in $\Omega$ as proved before. This finishes the proof. 

\[\square\]

4. Proofs of main results

In the following we will prove Theorem 1.3, Theorem 1.4, Theorem 1.5 and Theorem 1.6 first, and then we prove Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.3.** Let $u \in D^{1,p}(\Omega)$ be a subsolution to equation (1.18) with $f \in L^\frac{n}{p} (\Omega)$ satisfying $f(x) \leq A|x|^{-\alpha}$ in $\Omega$ for some constants $A > 0, p > \alpha$.

By Proposition 2.4, the function $v(x) = |x|^{-\gamma_1}(1 - \delta|x|^\delta)$ is a positive supersolution of equation (3.1) in $B_{R_0} \subset \Omega$ with function $g \in L^\frac{n}{p}(B_{R_0})$ satisfying $g(x) \geq A|x|^{-\alpha}$ for all $x \in B_{R_0}$, where $1 \geq \delta, \epsilon, R_0 > 0$ are constants depending on $N, p, \mu, A, \alpha$. Obviously we have $g \geq f$ in $B_{R_0}$.

Let $k > 0$ and define function $\tilde{v} = C(M + k)v$, where $C = \sup_{\partial B_{R_0}} v^{-1}, M = \sup_{\partial B_{R_0}} u^+$. Then $\tilde{v}$ is also a positive supersolution to equation (3.1) in the ball $B_{R_0}$ with the same function $g$ as above. Moreover, $\inf_{B_{R_0}} \tilde{v} = M + k > 0$ and $u \leq \tilde{v}$ on $\partial B_{R_0}$. Thus we can apply Theorem 3.2 to the subsolution $u$ of equation (1.18) and the supersolution $\tilde{v}$ of equation (3.1) on the ball $B_{R_0}$ to conclude that
\[
u(x) \leq \tilde{v}(x) \leq C \left( \sup_{\partial B_{R_0}} u^+ + k \right)|x|^{-\gamma_1},
\]
for $x \in B_{R_0}$. 


with constant $C = \sup_{\partial B_{R_0}} v^{-1}$ independent of $k$. Now Theorem 1.3 follows by taking $k \to 0$. □

**Proof of Theorem 1.4.** Suppose that $u \in D^{1,p}(\Omega)$ is a nonnegative supersolution to equation (1.18) with $f \geq 0$. Then $u$ is also a supersolution to equation (3.1) with $g = 0$. To prove the theorem, we may assume that $m \equiv \inf_{\partial \Omega} u > 0$, otherwise Theorem 1.4 is trivial since we assume that $u \geq 0$.

Define function $v(x) = Cm|x|^{-\gamma_1}$ in $\Omega$, where $C = \inf_{\partial \Omega} |x|^{-\gamma_1}$. Then $v$ is a solution of equation (1.18) with $f \equiv 0$, thus a subsolution of equation (1.18) with $f \equiv 0$. Obviously there holds $u \geq v$ on $\partial \Omega$. So applying Theorem 3.2 to the subsolution $v$ of equation (1.18) and the supersolution $u$ of equation (3.1) on the domain $\Omega$, we conclude that

$$u(x) \geq v(x) = Cm|x|^{-\gamma_1} \quad \text{for } x \in \Omega.$$  

This finishes the proof of Theorem 1.4. □

**Proof of Theorem 1.5.** Let $u \in D^{1,p}(\Omega)$ be a subsolution to equation (1.18) with $f \in L^\infty(\Omega)$ satisfying $f(x) \leq A|x|^{-\alpha}$ in $\Omega$ for some constants $A > 0$, $\alpha > p$.

By Proposition 2.4, the function $v(x) = |x|^{-\gamma_1}(1-\delta|x|^{-\epsilon})$ is a positive supersolution of equation (3.2) in $B^c_{R_1}$ with function $g \in L^\infty(\Omega)$ satisfying $g(x) \geq A|x|^{-\alpha} \geq f(x)$ for all $x \in B^c_{R_1}$, where $1 > \delta, \epsilon > 0, R_1 > 1$ are constants depending on $N, \mu, A, \alpha$.

Let $k > 0$ and define $\tilde{v} = C(M + k)v$, where $C = \sup_{\partial B_{R_0}} v^{-1}, M = \sup_{\partial B_{R_0}} u^+$. Then $\tilde{v}$ is a positive supersolution to (3.2) in $B^c_{R_1}$ with the same function $g$ given above. Moreover, $\inf_{\partial B_{R_1}} \tilde{v} = M + k > 0$ and $u \leq \tilde{v}$ on $\partial B^c_{R_1}$. We verify the condition (3.3) as follows: by Hölder’s inequality

$$\frac{1}{R} \int_{B_{2R}\setminus B_R} u^p |\nabla \log v|^p \leq \frac{C}{R^p} \int_{B_{2R}\setminus B_R} u^p \leq C \left( \int_{B_{2R}\setminus B_R} |u|^p \right)^{1/p^*},$$

where $C$ is a constant independent of $R$. The first inequality follows by noting that $|\nabla \log v(x)| \leq C|x|^{-1}$ with some constant $C$ independent of $R$. Therefore (3.3) holds since $u \in L^{p^*}(\mathbb{R}^N)$.

Thus we can apply Theorem 3.3 to the subsolution $u$ of equation (1.18) and the supersolution $\tilde{v}$ of equation (3.2) in $B^c_{R_1}$ to conclude that

$$u(x) \leq \tilde{v}(x) \leq C \left( \sup_{\partial B_{R_1}} u^+ + k \right)|x|^{-\gamma_1} \quad \text{for } x \in B^c_{R_1},$$

with constant $C = \sup_{\partial B_{R_0}} v^{-1}$ independent of $k$. Now the theorem follows by taking $k \to 0$. □

**Proof of Theorem 1.6.** Suppose that $u \in D^{1,p}(\Omega)$ is a nonnegative supersolution to (1.18) with $f \geq 0$, then $u$ is a nonnegative supersolution to equation (3.2) with $g = 0$. We may assume that $m \equiv \inf_{\partial \Omega} u > 0$, otherwise the theorem is trivial since we assume that $u \geq 0$. The positivity of $u$ in $\Omega$ is a consequence of the fact that $u$ is also a nonnegative supersolution to $p-$Laplacian equation

$$-\Delta_p u \geq 0$$

in $\Omega$ since $-\Delta_p u \geq \mu u^{p-1}/|x|^p \geq 0$. Moreover, it is well known (see [24]) that

$$\int_{B_{2R}\setminus B_R} |\nabla \log u|^p \leq CR^{N-p}, \quad (4.1)$$

for all $R$ large enough and $C > 0$ a constant independent of $R$. 

Let $C_1 = \inf_{x \in \Omega_1} |x|^{\gamma_2}$. The function $v = C_1 m |x|^{-\gamma_2}$ is a solution to (1.18) with $f = 0$, and thus a subsolution to (1.18) with $f = 0$. Condition (3.3) on $u$ and $v$ is also satisfied by (4.1) and Hölder’s inequality:

$$\begin{align*}
R^{-1} \int_{B_{2R} \setminus B_R} v^p |\nabla \log u|^{p-1} & \leq C R^{-1 - p \gamma_2} \left( \int_{B_{2R} \setminus B_R} |\nabla \log u|^p \right)^{\frac{p-1}{p}} |B_{2R}(0) \setminus B_R(0)|^{\frac{1}{p}} \\
& \leq C R^{-1 - p \gamma_2 + \frac{p-1}{p} (N - p) + \frac{N}{p}} \\
& \to 0 \quad \text{as } R \to \infty,
\end{align*}$$

since $-1 - p \gamma_2 + \frac{p-1}{p} (N - p) + \frac{N}{p} < 0$ due to the fact that $\gamma_2 > \frac{N - p}{p}$.

Thus we can apply Theorem 3.3 to the supersolution $u$ of equation (3.2) and the subsolution $v$ of equation (1.18) in $B_{R_1}^c$ to conclude that

$$u(x) \geq v(x) = C_1 m |x|^{-\gamma_2} \quad \text{for } x \in B_{R_1}^c,$$

This completes the proof of Theorem 1.6.

Now we prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Let $u$ be a weak solution of (1.1). By Proposition 2.1, there exists a positive constant $C$ depending on $N, p, \mu, ||Q||_{\infty}$ and $u$ such that

$$|u(x)| \leq C |x|^{-\left(\frac{N-p}{p} - \tau_0\right)} \quad \text{for } |x| < r_0,$$

and that

$$|u(x)| \leq C |x|^{-\left(\frac{N-p}{p} + \tau_1\right)} \quad \text{for } |x| > r_1,$$

for some constants $\tau_0, \tau_1, r_0, r_1 > 0$ depending on $N, p, \mu, ||Q||_{\infty}$ and $u$.

To prove (1.14) and (1.15), we regard both $u$ and $-u$ as subsolutions of equation (1.18) with function $f$ given by $f = Q|u|^{p^* - p}$. Then $f \in L^{\frac{N}{p^*}}(\mathbb{R}^N)$ and it holds that

$$|f(x)| \leq C |x|^{-\alpha} \quad \text{for } |x| < r_0,$$

and that

$$|f(x)| \leq C |x|^{-\beta} \quad \text{for } |x| > r_1,$$

with $\alpha = \left(\frac{N-p}{p} - \tau_0\right)(p^* - p) < p$ and $\beta = \left(\frac{N-p}{p} + \tau_1\right)(p^* - p) > p$.

Therefore, we can apply Theorem 1.3 and Theorem 1.5 to $\pm u$ in the ball $B_{r_0}$ and in the exterior ball $B_{r_1}^c$ respectively to conclude that (1.14) and (1.15) hold.

Proof of Theorem 1.2. Let $u$ be a nonnegative weak solution of (1.1) and $Q \in L^{\infty}(\mathbb{R}^N)$ be a nonnegative function. We regard $u$ as a nonnegative supersolution of equation (1.18) with function $f \equiv Q|u|^{p^* - p}$. Then (1.16) and (1.17) follow from Theorem 1.4 and Theorem 1.6 respectively. This completes the proof of Theorem 1.2.

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