A GEOMETRIC COUNTERPART OF THE BAUM-CONNES MAP FOR $GL(n)$.

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Abstract. We describe a geometric counterpart of the Baum-Connes map for the $p$-adic group $GL(n)$.

INTRODUCTION

Let $F$ be a nonarchimedean local field and let $G = GL(n) = GL(n, F)$. The goal of this paper is to construct the following commutative diagram:

$$
\begin{array}{ccc}
K^\text{top}_*(G) & \xrightarrow{\mu} & K_*(C^*_r(G)) \\
\downarrow\text{ch} & & \downarrow\text{ch} \\
\text{HP}_*(\mathcal{H}(G)) & \xrightarrow{\iota_*} & \text{HP}_*(S(G)) \\
\downarrow & & \downarrow \\
\text{H}^*_c(\Pi(G); \mathbb{C}) & \longrightarrow & \text{H}^*_c(\Pi'(G); \mathbb{C})
\end{array}
$$

The topological $K$-theory $K^\text{top}_*(G)$ is defined to be the equivariant $K$-homology of the universal example $EG$. As a model for $EG$ we will take the affine building $\beta G$ of $G$. Then $\beta G$ is the product of a simplicial complex by an affine line. In addition $\beta G$ is a contractible space on which $G$ acts properly and the quotient $\beta G/G$ is compact. Explicitly,

$$K^\text{top}_*(G) = KK_*^G(C_0(\beta G), \mathbb{C})$$

The map $\mu$ from the topological $K$-theory of $G$ to the $K$-theory of the reduced $C^*$-algebra of $G$ has been shown by V. Lafforgue to be an isomorphism for all reductive $p$-adic groups [19]. The Hecke algebra $\mathcal{H}(G)$ is the algebra of all complex-valued, compactly supported, uniformly locally constant functions on $G$ with the convolution product. The Schwartz algebra $S(G)$ is the algebra of all rapid decay functions on $G$, again with the convolution product. There is a natural inclusion $\mathcal{H}(G) \to S(G)$. The periodic cyclic homology $\text{HP}_*(\mathcal{H}(G))$ is to be understood in a purely algebraic sense. The periodic cyclic homology

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of $\text{HP}_*(\mathcal{S}(G))$ is to be understood in terms of topological algebras and the inductive tensor products. It is proved in [1] that the map $\iota$ is an isomorphism, and a more detailed proof is given below. The Chern character on the right hand side of the diagram was studied in [10] and shown to be an isomorphism after tensoring over $\mathbb{Z}$ with $\mathbb{C}$.

An outline of the construction of the Chern character on the left hand side of the diagram is given in [1]. In this construction the Chern character factors through the chamber homology of $\beta G$:

$$ch : K^{\text{top}}_*(G) \longrightarrow H_*(G; \beta G) \longrightarrow \text{HP}_*(\mathcal{H}(G)).$$

We could also define the Chern character indirectly as the composite of three maps so as to make the upper part of the diagram commutative. If denote the Chern character on the left hand side of the diagram by $\text{ch}_L$ and the Chern character on the right hand side of the diagram by $\text{ch}_R$ then the definition of $\text{ch}_L$ is

$$\text{ch}_L = \iota_*^{-1} \circ \text{ch}_R \circ \mu.$$

This creates the following commutative diagram

$$
\begin{array}{ccc}
K^{\text{top}}_*(G) \otimes_\mathbb{Z} \mathbb{C} & \longrightarrow & K_*(C^*_r(G)) \otimes_\mathbb{Z} \mathbb{C} \\
\text{ch}_L & & \text{ch}_R \\
\text{HP}_*(\mathcal{H}(G)) & \longrightarrow & \text{HP}_*(\mathcal{S}(G))
\end{array}
$$

in which each map is an isomorphism of complex vector spaces.

From the point of view of noncommutative geometry it is natural to seek the spaces which underlie the noncommutative algebras $\mathcal{H}(G)$ and $\mathcal{S}(G)$. We prove that, at the level of periodic cyclic homology, these spaces reveal themselves in terms of representation theory. The smooth dual $\Pi(G)$ has, with the the aid of Langlands parameters, a natural structure of complex manifold. The tempered dual $\Pi^t(G)$ is, with the aid of Harish-Chandra parameters, a disjoint union of compact orbifolds. These are the two spaces which underlie $\mathcal{H}(G)$ and $\mathcal{S}(G)$. Not only that but there is a deformation retraction of the smooth dual onto the tempered dual which, in the context of our main commutative diagram, induces the Baum-Connes map.

We give a detailed description of the $q$-projection introduced in [9]. In the last section of this paper we track the fate of supercuspidal representations of $G$ through the commutative diagram. In particular, the index map $\mu$ manifests itself as an example of Ahn reciprocity.

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1. The complex structure on the smooth dual of $GL(n)$

The field $F$ is a nonarchimedean local field, so that $F$ is a finite extension of $\mathbb{Q}_p$, for some prime $p$ or $F$ is a finite extension of the function field $\mathbb{F}_p((x))$. The residue field $k_F$ of $F$ is the quotient $o_F/m_F$ of the ring of integers $o_F$ by its unique maximal ideal $m_F$. Let $q$ be the cardinality of $k_F$.

The essence of local class field theory, see [22, p.300], is a pair of maps

$$(d : G \to \hat{\mathbb{Z}}, v : F^\times \to \mathbb{Z})$$

where $G$ is a profinite group, $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$, and $v$ is the valuation.

Let $\bar{F}$ be a separable algebraic closure of $F$. Then the absolute Galois group $G(\bar{F}|F)$ is the projective limit of the finite Galois groups $G(E|F)$ taken over the finite extensions $E$ of $F$ in $\bar{F}$. Let $\bar{F}$ be the maximal unramified extension of $F$. The map $d$ is in this case the projection map

$$d : G(\bar{F}|F) \to G(\bar{F}|F) \cong \hat{\mathbb{Z}}$$

The group $G(\bar{F}|F)$ is procyclic. It has a single topological generator: the Frobenius automorphism $\phi_F$ of $\bar{F}|F$. The Weil group $W_F$ is by definition the pre-image of $<\phi_F>$ in $G(\bar{F}|F)$. We thus have the surjective map

$$d : W_F \to \mathbb{Z}$$

The pre-image of 0 is the inertia group $I_F$. In other words we have the following short exact sequence

$$1 \longrightarrow I_F \longrightarrow W_F \longrightarrow \mathbb{Z} \longrightarrow 0$$

The group $I_F$ is given the profinite topology induced by $G(\bar{F}|F)$. The topology on the Weil group $W_F$ is dictated by the above short exact sequence. The Weil group $W_F$ is a locally compact group with maximal compact subgroup $I_F$. The map

$$W_F \to G(\bar{F}|F)$$

is a continuous homomorphism with dense image.

A detailed account of the Weil group for local fields may be found in [28]. For a topological group $G$ we denote by $G^{ab}$ the quotient $G^{ab} = G/G^c$ of $G$ by the closure $G^c$ of the commutator subgroup of $G$. Thus $G^{ab}$ is the maximal abelian Hausdorff quotient of $G$. The local reciprocity laws [22, p.320]

$$r_{E|F} : G(E|F)^{ab} \cong F^\times /N_{E|F}E^\times$$
now create an isomorphism \[23, \text{p.69}]:
\[ r_F : W_F^{ab} \cong F^\times \]

We have \( W_F = \sqcup \Phi^n I_F, n \in \mathbb{Z} \). The Weil group is a locally compact, totally disconnected group, whose maximal compact subgroup is \( I_F \). This subgroup is also open. There are three models for the Weil-Deligne group.

One model is the crossed product \( W_F \rtimes \mathbb{C} \), where the Weil group acts on \( \mathbb{C} \) by \( w \cdot x = \|w\| x \), for all \( w \in W_F \) and \( x \in \mathbb{C} \).

The action of \( W_F \) on \( \mathbb{C} \) extends to an action of \( W_F \) on \( SL(2, \mathbb{C}) \). The semidirect product \( W_F \rtimes SL(2, \mathbb{C}) \) is then isomorphic to the direct product \( W_F \times SL(2, \mathbb{C}) \), see [17, p.278]. Then a complex representation of \( W_F \times SL(2, \mathbb{C}) \) is determined by its restriction to \( W_F \times SU(2) \), where \( SU(2) \) is the standard compact Lie group.

From now on, we shall use this model for the Weil-Deligne group:
\[ \mathcal{L}_F = W_F \times SU(2). \]

In the next definition, the complex general linear group \( GL(n, \mathbb{C}) \) is equipped with the discrete topology.

1.1. Definition. An \( L \)-parameter is a continuous homomorphism
\[ \phi : \mathcal{L}_F \rightarrow GL(n, \mathbb{C}) \]
such that \( \phi(w) \) is semisimple for all \( w \in W_F \). Two \( L \)-parameters are equivalent if they are conjugate under \( GL(n, \mathbb{C}) \). The set of equivalence classes of \( L \)-parameters is denoted \( \Phi(G) \).

1.2. Definition. A representation of \( G \) on a complex vector space \( V \) is smooth if the stabilizer of each vector in \( V \) is an open subgroup of \( G \). The set of equivalence classes of irreducible smooth representations of \( G \) is the smooth dual \( \Pi(G) \) of \( G \).

1.3. Theorem. Local Langlands Correspondence for \( GL(n) \). There is a natural bijection between \( \Phi(GL(n)) \) and \( \Pi(GL(n)) \).

The naturality of the bijection involves compatibility of the \( L \)-factors and \( \epsilon \)-factors attached to the two types of objects.

The local Langlands conjecture for \( GL(n) \) was proved by Stuhler [20] when \( F \) has positive characteristic and by Harris-Taylor [15] and Henniart [16] when \( F \) has characteristic zero.

We recall that a matrix coefficient of a representation \( \rho \) of a group \( G \) on a vector space \( V \) is a function on \( G \) of the form \( f(g) = \langle \rho(g)v, w \rangle \), where \( v \in V, w \in V^* \), and \( V^* \) denotes the dual space of \( V \). The inner product is given by the duality between \( V \) and \( V^* \). A representation \( \rho \)
of $G$ is called supercuspidal if and only if the support of every matrix coefficient is compact modulo the centre of $G$.

Let $\tau_j = \text{spin}(j)$ denote the $(2j + 1)$-dimensional complex irreducible representation of the compact Lie group $SU(2)$, $j = 0, 1/2, 1, 3/2, 2, \ldots$.

For $GL(n)$ the local Langlands correspondence works in the following way.

- Let $\rho$ be an irreducible representation of the Weil group $W_F$. Then $\pi_F(\rho \otimes 1)$ is an irreducible supercuspidal representation of $GL(n)$, and every irreducible supercuspidal representation of $GL(n)$ arises in this way. If $\det(\rho)$ is a unitary character, then $\pi_F(\rho \otimes 1)$ has unitary central character, and so is pre-unitary.
- We have $\pi_F(\rho \otimes \text{spin}(j)) = Q(\Delta)$, the Langlands quotient associated to the segment $\{ | \cdot |^{-(j-1)/2} \pi_F(\rho), \ldots, | \cdot |^{(j-1)/2} \pi_F(\rho) \}$. If $\det(\rho)$ is unitary, then $Q(\Delta)$ is in the discrete series. In particular, if $\rho = 1$ then $\pi_F(1 \otimes \text{spin}(j))$ is the Steinberg representation $St(2j + 1)$ of $GL(2j + 1)$.
- If $\phi$ is an $L$-parameter for $GL(n)$ then $\phi = \phi_1 \oplus \cdots \oplus \phi_m$ where $\phi_j = \rho_j \otimes \text{spin}(j)$. Then $\pi_F(\phi)$ is the Langlands quotient $Q(\Delta_1, \ldots, \Delta_m)$. If $\det(\rho_j)$ is a unitary character for each $j$, then $\pi_F(\phi)$ is a tempered representation of $GL(n)$.

This correspondence creates, as in [18, p. 381], a natural bijection $\pi_F : \Phi(GL(n)) \rightarrow \Pi(GL(n))$.

A quasi-character $\psi : W_F \rightarrow \mathbb{C}^\times$ is unramified if $\psi$ is trivial on the inertia group $I_F$. Recall the short exact sequence

$$0 \rightarrow I_F \rightarrow W_F \rightarrow \mathbb{Z} \rightarrow 0$$

Then $\psi(w) = zd(w)$ for some $z \in \mathbb{C}^\times$. Note that $\psi$ is not a Galois representation unless $z$ has finite order in the complex torus $\mathbb{C}^\times$, see [28]. Let $\Psi(W_F)$ denote the group of all unramified quasi-characters of $W_F$. Then $\Psi(W_F) \simeq \mathbb{C}^\times$,

$$\psi \mapsto z$$

Each $L$-parameter $\phi : \mathcal{L}_F \rightarrow GL(n, \mathbb{C})$ is of the form $\phi_1 \oplus \cdots \oplus \phi_m$ with each $\phi_j$ irreducible. Each irreducible $L$-parameter is of the form $\rho \otimes \text{spin}(j)$ with $\rho$ an irreducible representation of the Weil group $W_F$.

1.4. Definition. The orbit $\mathcal{O}(\phi) \subset \Phi_F(G)$ is defined as follows

$$\mathcal{O}(\phi) = \left\{ \bigoplus_{r=1}^m \psi_r \phi_r \mid \psi_r \in \Psi(W_F), 1 \leq r \leq m \right\}$$

where each $\psi_r$ is an unramified quasi-character of $W_F$. 
An equivalent definition of the orbit $O(\phi)$: Let $^L M$ be the minimal Levi subgroup of the $^L G$ which contains the image $\phi(L_F)$, see [6, Prop. 8.6, p.41]. Now twist the image $\phi(L_F)$ by all unramified quasi-characters of $^L M$. This creates the orbit $O(\phi)$.

1.5. Definition. Let $\det \phi_r$ be a unitary character, $1 \leq r \leq m$ and let $\phi = \phi_1 \oplus \ldots \oplus \phi_m$. The compact orbit $O^c(\phi) \subset \Phi^c(G)$ is defined as follows:

$$O^c(\phi) = \{ \bigoplus_{r=1}^{m} \psi_r \phi_r \mid \psi_r \in \Psi(W_F), 1 \leq r \leq m \}$$

where each $\psi_r$ is an unramified unitary character of $W_F$.

We note that $I_F \times SU(2) \subset W_F \times SU(2)$ and in fact $I_F \times SU(2)$ is the maximal compact subgroup of $L_F$. Now let $\phi$ be an $L$-parameter. Moving (if necessary) to another point in the orbit $O(\phi)$ we can write $\phi$ in the canonical form

$$\phi = \phi_1 \oplus \ldots \oplus \phi_1 \oplus \ldots \oplus \phi_k \oplus \ldots \oplus \phi_k$$

where $\phi_1$ is repeated $l_1$ times, ..., $\phi_k$ is repeated $l_k$ times, and the representations

$$\phi_j |_{I_F \times SU(2)}$$

are irreducible and pairwise inequivalent, $1 \leq j \leq k$. We will now write $k = k(\phi)$. This natural number is an invariant of the orbit $O(\phi)$. We have

$$O(\phi) = \text{Sym}^{l_1} \mathbb{C}^x \times \ldots \times \text{Sym}^{l_k} \mathbb{C}^x$$

the product of symmetric products of $\mathbb{C}^x$.

1.6. Theorem. The set $\Phi(GL(n))$ has the structure of complex algebraic variety. Each irreducible component $O(\phi)$ is isomorphic to the product of a complex affine space and a complex torus

$$O(\phi) = \mathbb{A}^l \times (\mathbb{C}^x)^k$$

where $k = k(\phi)$.

Proof. Let $Y = \mathbb{V}(x_1y_1 - 1, \ldots, x_ny_n - 1) \subset \mathbb{C}^{2n}$. Then $Y$ is a Zariski-closed set in $\mathbb{C}^{2n}$, and so is an affine complex algebraic variety. Let $X = (\mathbb{C}^x)^n$. Set $\alpha : Y \to X, \alpha(x_1, y_1, \ldots, x_n, y_n) = (x_1, \ldots, x_n)$ and $\beta : X \to Y, \beta(x_1, \ldots, x_n) = (x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$. So $X$ can be embedded in affine space $\mathbb{C}^{2n}$ as a Zariski-closed subset. Therefore $X$ is an affine algebraic variety, as in [27, p.50].

Let $A = \mathbb{C}[X]$ be the coordinate ring of $X$. This is the restriction to $X$ of polynomials on $\mathbb{C}^{2n}$, and so $A = \mathbb{C}[X] = \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, the ring of Laurent polynomials in $n$ variables $x_1, \ldots, x_n$. Let $S_n$ be
the symmetric group, and let \( Z \) denote the quotient variety \( X/S_n \). The variety \( Z \) is an affine complex algebraic variety.

The coordinate ring of \( Z \) is

\[ \mathbb{C}[Z] \simeq \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]^{S_n}. \]

Let \( \sigma_i, i = 1, \ldots, n \) be the elementary symmetric polynomials in \( n \) variables. Then from the last isomorphism we have

\[ \mathbb{C}[Z] \simeq \mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}[\sigma_n^{-1}] \]

\[ \simeq \mathbb{C}[\sigma_1, \ldots, \sigma_n] \otimes \mathbb{C}[\sigma_n^{-1}] \]

\[ \simeq \mathbb{C}[\sigma_1, \ldots, \sigma_n-1] \otimes \mathbb{C}[\sigma_n, \sigma_n^{-1}] \]

\[ \simeq \mathbb{C}[A^{n-1}] \otimes \mathbb{C}[\mathbb{A} - \{0\}] \]

\[ \simeq \mathbb{C}[A^{n-1} \times (\mathbb{A} - \{0\})] \]

where \( \mathbb{A}^n \) denotes complex affine \( n \)-space. The coordinate ring of the quotient variety \( \mathbb{C}^\times/n/S_n \) is isomorphic to the coordinate ring of \( \mathbb{A}^{n-1} \times (\mathbb{A} - \{0\}) \). Now the categories of affine algebraic varieties and of finitely generated reduced \( \mathbb{C} \)-algebras are equivalent, see \( [27, \text{p.26}] \). Therefore the variety \( \mathbb{C}^\times/n/S_n \) is isomorphic to the variety \( \mathbb{A}^{n-1} \times (\mathbb{A} - \{0\}) \).

Consider \( \mathbb{A} - \{0\} = \mathbb{V}(f) \) where \( f(x) = x_1x_2 - 1 \). Then \( \partial f/\partial x_1 = x_2 \neq 0 \) and \( \partial f/\partial x_2 = x_1 \neq 0 \) on the variety \( \mathbb{V}(f) \). So \( \mathbb{A} - \{0\} \) is smooth. Then \( \mathbb{A}^{n-1} \times (\mathbb{A} - \{0\}) \) is smooth. Therefore the quotient variety \( \mathbb{C}^\times/n/S_n \) is a smooth complex affine algebraic variety of dimension \( n \). Now each orbit \( \mathcal{O}(\phi) \) is a product of symmetric products of \( \mathbb{C}^\times \). Therefore each orbit \( \mathcal{O}(\phi) \) is a smooth complex affine algebraic variety. We have

\[ \mathcal{O}(\phi) = \text{Sym}^{l_1} \mathbb{C}^\times \times \ldots \times \text{Sym}^{l_k} \mathbb{C}^\times = \mathbb{A}^l \times (\mathbb{C}^\times)^k \]

where \( l = l_1 + \ldots + l_k - k \) and \( k = k(\phi) \).

We now transport the complex structure from \( \Phi(GL(n)) \) to \( \Pi(GL(n)) \) via the local Langlands correspondence. This leads to the next result.

1.7. Theorem. The smooth dual \( \Pi(GL(n)) \) has a natural complex structure. Each irreducible component is a smooth complex affine algebraic variety.

The smooth dual \( \Pi(GL(n)) \) has countably many irreducible components of each dimension \( d \) with \( 1 \leq d \leq n \). The irreducible supercuspidal representations of \( GL(n) \) arrange themselves into the \( 1 \)-dimensional tori.

Each irreducible component is a smooth affine scheme, i.e. of the form \( \text{Spec}(R) \) where \( R \) is a commutative unital ring. In fact each \( R \) is a reduced finitely generated \( \mathbb{C} \)-algebra. From the point of view of
noncommutative geometry, the smooth dual $\Pi(GL(n))$ is a noncommutative affine scheme underlying the noncommutative non-unital Hecke algebra $\mathcal{H}(GL(n))$.

It follows from Theorems 1.6 and 1.7 that the smooth dual $\Pi(GL(n))$ is a complex manifold. Then $\mathbb{C} \times \Pi(GL(n))$ is a complex manifold. So the local $L$-factor $L(s, \pi_v)$ and the local $\epsilon$-factor $\epsilon(s, \pi_v)$ are functions of several complex variables:

$$L : \mathbb{C} \times \Pi(GL(n)) \rightarrow \mathbb{C}$$

$$\epsilon : \mathbb{C} \times \Pi(GL(n)) \rightarrow \mathbb{C}.$$
Let
\[ \Pi(\Omega) = (\text{inf.ch.})^{-1}\Omega. \]

Then \( \Pi(\Omega) \) is a smooth complex algebraic variety with finitely many irreducible components. We have the following Bernstein decomposition of \( \Pi(G) \):
\[ \Pi(G) = \bigsqcup \Pi(\Omega). \]

Let \( M \) be a compact \( C^\infty \) manifold. Then \( C^\infty(M) \) is a Fréchet algebra, and we have Connes' fundamental theorem [14, Theorem 2, p. 208]:
\[ HP_* (C^\infty(M)) \cong H^*(M; \mathbb{C}). \]

Now the ideal \( H(\Omega) \) is a purely algebraic object, and, in computing its periodic cyclic homology, we would hope to find an algebraic variety to play the role of the manifold \( M \). This algebraic variety is \( \Pi(\Omega) \).

2.1. Theorem. Let \( \Omega \) be a component in the Bernstein variety \( \Omega(G) \). Then the periodic cyclic homology of \( H(G) \) is isomorphic to the periodised de Rham cohomology of \( \Pi(\Omega) \):
\[ HP_* H(\Omega) \cong H^*(\Pi(\Omega); \mathbb{C}). \]

Proof. We can think of \( \Omega \) as a vector \((\tau_1, \ldots, \tau_r)\) of irreducible supercuspidal representations of smaller general linear groups, the entries of this vector being only determined up to tensoring with unramified quasicharacters and permutation. If the vector is equivalent to \((\sigma_1, \ldots, \sigma_1, \ldots, \sigma_r, \ldots, \sigma_r)\) with \( \sigma_j \) repeated \( e_j \) times, \( 1 \leq j \leq r \), and \( \sigma_1, \ldots, \sigma_r \) are pairwise distinct, then we say that \( \Omega \) has exponents \( e_1, \ldots, e_r \).

Then there is a Morita equivalence
\[ \mathcal{H}(\Omega) \sim \mathcal{H}(e_1, q_1) \otimes \cdots \otimes \mathcal{H}(e_r, q_r) \]
where \( q_1, \ldots, q_r \) are natural number invariants attached to \( \Omega \).

This result is due to Bushnell-Kutzko [11, 12, 13]. We describe the steps in the proof. Let \((\rho, W)\) be an irreducible smooth representation of the compact open subgroup \( K \) of \( G \). As in [12, 4.2], the pair \((K, \rho)\) is an \( \Omega \)-type in \( G \) if and only if, for \((\pi, V) \in \Pi(G)\), we have \( \text{inf.ch.}(\pi) \in \Omega \) if and only if \( \pi \) contains \( \rho \). The existence of an \( \Omega \)-type in \( GL(n) \), for each component \( \Omega \) in \( \Omega(GL(n)) \), is established in [13, 1.1]. So let \((K, \rho)\) be an \( \Omega \)-type in \( GL(n) \). As in [12, 2.9], let
\[ e_\rho(x) = (\text{vol}K)^{-1} (\dim \rho) \text{Trace}_W(\rho(x^{-1})) \]
for \( x \in K \) and 0 otherwise.

Then \( e_\rho \) is an idempotent in the Hecke algebra \( \mathcal{H}(G) \). Then we have

\[
\mathcal{H}(\Omega) \cong \mathcal{H}(G) * e_\rho * \mathcal{H}(G)
\]
as in \([12, 4.3]\) and the two-sided ideal \( \mathcal{H}(G) * e_\rho * \mathcal{H}(G) \) is Morita equivalent to \( e_\rho * \mathcal{H}(G) * e_\rho \). Now let \( \mathcal{H}(K, \rho) \) be the endomorphism-valued Hecke algebra attached to the semisimple type \((K, \rho)\). By \([12, 2.12]\) we have a canonical isomorphism of unital \( \mathbb{C} \)-algebras:

\[
\mathcal{H}(G, \rho) \otimes_\mathbb{C} \text{End}_\mathbb{C} W \cong e_\rho * \mathcal{H}(G) * e_\rho
\]

so that \( e_\rho * \mathcal{H}(G) * e_\rho \) is Morita equivalent to \( \mathcal{H}(G, \rho) \). Now we quote the main theorem for semisimple types in \( GL(n) \) \([13, 1.5]\): there is an isomorphism of unital \( \mathbb{C} \)-algebras

\[
\mathcal{H}(G, \rho) \cong \mathcal{H}(G_1, \rho_1) \otimes \cdots \otimes \mathcal{H}(G_r, \rho_r)
\]
The factors \( \mathcal{H}(G_i, \rho_i) \) are (extended) affine Hecke algebras whose structure is given explicitly in \([11, 5.6.6]\). This structure is in terms of generators and relations \([11, 5.4.6]\). So let \( \mathcal{H}(e, q) \) denote the affine Hecke algebra associated to the affine Weyl group \( \mathbb{Z}^e \rtimes S_e \). Putting all this together we obtain a Morita equivalence

\[
\mathcal{H}(\Omega) \sim \mathcal{H}(e_1, q_1) \otimes \cdots \otimes \mathcal{H}(e_r, q_r)
\]
The natural numbers \( q_1, \ldots, q_r \) are specified in \([11, 5.6.6]\). They are the cardinalities of the residue fields of certain extension fields \( E_1/F, \ldots, E_r/F \).

Using the K"unneth formula the calculation of \( H_P^*(\mathcal{H}(\Omega)) \) is reduced to that of the affine Hecke algebra \( \mathcal{H}(e, q) \). When \( q = 1 \) the Hecke algebra \( \mathcal{H}(e, q) \) is isomorphic to the complex group algebra of the affine Weyl group \( \mathbb{Z}^e \rtimes S_e \). We now quote the main result in \([4, 5]\):

\[
H_P^* \mathcal{H}(e, q) \cong H_P^* \mathcal{H}(e, 1) \cong H_P^* \mathbb{C}[\mathbb{Z}^e \rtimes S_e].
\]
We have also \([3, 4]\)

\[
H_P^* \mathbb{C}[\mathbb{Z}^e \rtimes S_e] \sim H^*(\mathbb{T}^e/S_e; \mathbb{C})
\]
where \( \mathbb{T}^e/S_e \) is the extended quotient of the torus \( \mathbb{T}^e \) by the symmetric group \( S_e \). Therefore we have

\[
H_P^* (\mathcal{H}(e, q)) = H^*(\mathbb{T}^e/S_e; \mathbb{C}).
\]

If \( \Omega \) has exponents \( e_1, \ldots, e_r \) then \( e_1 + \cdots + e_r = d(\Omega) = \dim_\mathbb{C} \Omega \), and \( W(\Omega) \) is a product of symmetric groups:

\[
W(\Omega) = S_{e_1} \times \cdots \times S_{e_r}
\]
Form the semidirect product $\mathbb{Z}^{d(\Omega)} \rtimes W(\Omega)$. Then we have

$$HP_*(\mathcal{H}(\Omega)) \cong HP_*(C[\mathbb{Z}^{d(\Omega)} \rtimes W(\Omega)]).$$

We therefore have

$$HP_j(\mathcal{H}(\Omega)) \cong \oplus_l H^{j+2l}(\mathbb{T}^{d(\Omega)}/W(\Omega); \mathbb{C})$$

with $j = 0, 1$. By [9, p. 217] we have $\Pi(\Omega) \cong \Omega^+$. It now follows that

$$HP_*(\mathcal{H}(\Omega)) = H^*(\Pi(\Omega); \mathbb{C}).$$

2.2. Lemma. Let $\Omega$ be a component in the variety $\Omega(G)$. Then we have the dimension formula

$$\dim_{\mathbb{C}} HP_*(\mathcal{H}(\Omega)) = 2^{k(\phi_1)-1} + \ldots + 2^{k(\phi_r)-1}.$$

Proof. Let

$$\Phi(\Omega) = (inf.ch. \circ \pi_F)^{-1} \Omega = (\pi_\gamma \circ \alpha)^{-1} \Omega.$$

Then $\Phi(\Omega)$ is a disjoint union of orbits [9, p.217] and we have

$$\Phi(\Omega) = \bigcup (\phi_1) \sqcup \ldots \sqcup \bigcup (\phi_r) \cong \Omega^+.$$

By Theorem 2.1 we have

$$HP_*(\mathcal{H}(\Omega)) \cong H^*(\Omega^+; \mathbb{C})$$

and the lemma easily follows.

2.3. Theorem. The inclusion $\mathcal{H}(G) \longrightarrow S(G)$ induces an isomorphism at the level of periodic cyclic homology:

$$HP_*(\mathcal{H}(G)) \cong HP_*(S(G)).$$

3. The $q$-projection

Let $\Omega$ be a component in the Bernstein variety. This component is an ordinary quotient $D/\Gamma$. We now consider the extended quotient $\tilde{D}/\Gamma = \bigsqcup D^\gamma/Z_\gamma$, where $D$ is the complex torus $\mathbb{C}^m$. Let $\gamma$ be a permutation of $n$ letters with cycle type

$$\gamma = (1 \ldots \alpha_1) \cdots (1 \ldots \alpha_r)$$
where \( \alpha_1 + \cdots + \alpha_r = m \). On the fixed set \( D^\gamma \) the map \( \pi_q \), by definition, sends the element \((z_1, \ldots, z_1, \ldots, z_r, \ldots, z_r)\) where \( z_j \) is repeated \( \alpha_j \) times, \( 1 \leq j \leq r \), to the element
\[
(q^{(\alpha_1-1)/2}z_1, \ldots, q^{(1-\alpha_1)/2}z_1, \ldots, q^{(\alpha_r-1)/2}z_r, \ldots, q^{(1-\alpha_r)/2}z_r)
\]
The map \( \pi_q \) induces a map from \( D^\gamma/Z_\gamma \) to \( D/\Gamma \), and so a map, still denoted \( \pi_q \), from the extended quotient \( \tilde{D}/\Gamma \) to the ordinary quotient \( D/\Gamma \). This creates a map \( \pi_q \) from the extended Bernstein variety to the Bernstein variety:
\[
\pi_q : \Omega^+(G) \longrightarrow \Omega(G).
\]

3.1. Definition. The map \( \pi_q \) is called the \textit{q-projection}.

The \textit{q-projection} \( \pi_q \) occurs in the following commutative diagram \([\Phi]\):
\[
\begin{array}{ccc}
\Phi(G) & \longrightarrow & \Pi(G) \\
\alpha \downarrow & & \downarrow \text{inf. ch.} \\
\Omega^+(G) & \xrightarrow{\pi_q} & \Omega(G)
\end{array}
\]

Let \( A, B \) be commutative rings with \( A \subset B, 1 \in A \). Then the element \( x \in B \) is \textit{integral} over \( A \) if there exist \( a_1, \ldots, a_n \in A \) such that
\[
x^n + a_1x^{n-1} + \cdots + a_n = 0.
\]

Then \( B \) is \textit{integral} over \( A \) if each \( x \in B \) is integral over \( A \). Let \( X, Y \) be affine varieties, \( f : X \longrightarrow Y \) a regular map such that \( f(X) \) is dense in \( Y \). Then the pull-back \( f^\# \) defines an isomorphic inclusion \( \mathbb{C}[Y] \longrightarrow \mathbb{C}[X] \). We view \( \mathbb{C}[Y] \) as a subring of \( \mathbb{C}[X] \) by means of \( f^\# \). Then \( f \) is a \textit{finite} map if \( \mathbb{C}[X] \) is integral over \( \mathbb{C}[Y] \), see \([20]\). This implies that the pre-image \( F^{-1}(y) \) of each point \( y \in Y \) is a finite set, and that, as \( y \) moves in \( Y \), the points in \( F^{-1}(y) \) may merge together but not disappear. The map \( \mathbb{A}^1 - \{0\} \longrightarrow \mathbb{A}^1 \) is the classic example of a map which is \textit{not} finite.

3.2. Lemma. Let \( X \) be a component in the extended variety \( \Omega^+(G) \). Then the \textit{q-projection} \( \pi_q \) is a finite map from \( X \) onto its image \( \pi_q(X) \).

Proof. Note that the fixed-point set \( D^\gamma \) is a complex torus of dimension \( r \), that \( \pi_q(D^\gamma) \) is a torus of dimension \( r \) and that we have an isomorphism of affine varieties \( D^\gamma \cong \pi_q(D^\gamma) \). Let \( X = D^\gamma/Z_\gamma, Y = \pi_q(D^\gamma)/\Gamma \) where \( Z_\gamma \) is the \( \Gamma \)-centralizer of \( \gamma \). Now each of \( X \) and \( Y \) is a quotient of the variety \( D^\gamma \) by a finite group, hence \( X, Y \) are affine varieties \([20]\, p.31\). We have \( D^\gamma \longrightarrow X \longrightarrow Y \) and \( \mathbb{C}[Y] \longrightarrow \mathbb{C}[X] \longrightarrow \mathbb{C}[D^\gamma] \).

According to \([26]\, p.61\), \( \mathbb{C}[D^\gamma] \) is integral over \( \mathbb{C}[Y] \) since \( Y = D^\gamma/\Gamma \).
Therefore the subring $\mathbb{C}[X]$ is integral over $\mathbb{C}[Y]$. So the map $\pi_q : X \rightarrow Y$ is finite.

**Example.** $GL(2)$. Let $T$ be the diagonal subgroup of $G = GL(2)$ and let $\Omega$ be the component in $\Omega G$ containing the cuspidal pair $(T, 1)$. Then $\sigma \in \Pi(GL(2))$ is *arithmetically unramified* if $\inf.ch.\sigma \in \Omega$. If $\pi_F(\phi) = \sigma$ then $\phi$ is a 2-dimensional representation of $L_F$ and there are two possibilities:

- $\phi$ is reducible, $\phi = \psi_1 \oplus \psi_2$ with $\psi_1, \psi_2$ unramified quasicharacters of $W_F$. So $\psi_j(w) = z_j^{d(w)}, z_j \in \mathbb{C}^\times, j = 1, 2$. We have $\pi_F(\phi) = Q(\psi_1, \psi_2)$ where $\psi_1$ does not precede $\psi_2$. In particular we obtain the 1-dimensional representations of $G$ as follows:

$$\pi_F(\phi) = Q(\psi_1, \psi_2) = \psi \otimes spin(1/2).$$

**Example.** $GL(3)$. In the above example, the $q$-projection is stratified-injective, i.e. injective on each orbit type. This is not so in general, as shown by next example (due to J-F. Dat). Let $T$ be the diagonal subgroup of $GL(3)$ and let $\Omega$ be the component containing the cuspidal pair $(T, 1)$. Then $\Omega = Sym^3 \mathbb{C}^\times$ and $\Omega^+ = Sym^3 \mathbb{C}^\times \sqcup (\mathbb{C}^\times)^2 \sqcup \mathbb{C}^\times$. The $q-$projection works as follows:

$$\pi_q : \{z_1, z_2\} \mapsto \{z_1, z_2\}$$

$$\pi_q : z \mapsto \{q^{1/2}z, q^{-1/2}z\}$$

where $q$ is the cardinality of the residue field of $F$. 

The map $\pi_q$ works as follows:

$$\{z_1, z_2, z_3\} \mapsto \{z_1, z_2, z_3\}$$

$$(z, w, w) \mapsto \{z, q^{1/2}w, q^{-1/2}w\}$$

$$(z, z, z) \mapsto \{qz, z, q^{-1}z\}.$$ 

Consider the $L$-parameter

$$\phi = \psi_1 \otimes 1 \oplus \psi_2 \otimes spin(1/2) \in \Phi(GL(3)).$$
If \( \psi(w) = z^{d(w)} \) then we will write \( \psi = z \). With this understood, let

\[
\phi_1 = q \otimes 1 \oplus q^{-1/2} \otimes \text{spin}(1/2)
\]

\[
\phi_2 = q^{-1} \otimes 1 \oplus q^{1/2} \otimes \text{spin}(1/2).
\]

Then \( \alpha(\phi_1), \alpha(\phi_2) \) are distinct points in the same stratum of the extended quotient, but their image under the \( q \)-projection \( \pi_q \) is the single point \( \{ q^{-1}, 1, q \} \in Sym^3 \mathbb{C}^\times \).

Let

\[
\phi_3 = 1 \otimes \text{spin}(3/2)
\]

\[
\phi_4 = q^{-1} \otimes 1 \oplus 1 \otimes 1 \oplus q \otimes 1.
\]

Then the distinct \( L \)-parameters \( \phi_1, \phi_2, \phi_3, \phi_4 \) all have the same image under the \( q \)-projection \( \pi_q \).

4. The Geometric Counterpart

Given an \( L \)-parameter \( \phi : \mathcal{L}_F \rightarrow GL(n, \mathbb{C}) \) we have

\[
\phi = \phi_1 \oplus \ldots \oplus \phi_m
\]

with each \( \phi_j \) an irreducible representation. We have \( \phi_j = \rho_j \otimes \text{spin}(j) \) where each \( \rho_j \) is an irreducible representation of the Weil group \( W_F \). We shall assume that \( \det \rho_j \) is a unitary character. Let \( \mathcal{O}(\phi) \) be the orbit of \( \phi \) as in Definition 1.4. The map \( \mathcal{O}(\phi) \rightarrow \mathcal{O}^t(\phi) \) is now defined as follows

\[
\psi_1 \phi_1 \oplus \ldots \oplus \psi_m \phi_m \mapsto |\psi_1|^{-1} \cdot \psi_1 \phi_1 \oplus \ldots \oplus |\psi_m|^{-1} \cdot \psi_m \phi_m.
\]

This map is a deformation retraction of the complex orbit \( \mathcal{O}(\phi) \) onto the compact orbit \( \mathcal{O}^t(\phi) \). Since \( \Pi(G) \) is a disjoint union of such complex orbits this formula determines, via the local Langlands correspondence for \( GL(n) \), a deformation retraction of \( \Pi(G) \) onto the tempered dual \( \Pi^t(GL(n)) \).

The results of the paper may be summarized as follows.

4.1. Theorem. The smooth dual \( \Pi(GL(n)) \) has a natural complex structure: it is a complex manifold, with infinitely many components.
In the context of the commutative diagram

\[
\begin{align*}
K_\text{top}^*(G) & \xrightarrow{\mu} K_\text{c}^*(G) \\
\downarrow \text{ch} & \quad \downarrow \text{ch} \\
HP_\text{c}(\mathcal{H}(G)) & \xrightarrow{\iota} HP_\text{c}(S(G)) \\
\downarrow & \quad \downarrow \\
H_\text{c}^c(\Pi(G); \mathbb{C}) & \longrightarrow H_\text{c}^c(\Pi^c(G); \mathbb{C})
\end{align*}
\]

the Baum-Connes map has a geometric counterpart: it is induced by the deformation retraction of \( \Pi(GL(n)) \) onto the tempered dual \( \Pi^c(GL(n)) \).

5. \textbf{Supercuspidal representations of } GL(n) \textbf{ }

In this section we track the fate of supercuspidal representations of \( GL(n) \) through the commutative diagram. Let \( \rho \) be an irreducible \( n \)-dimensional complex representation of the Weil group \( W_F \) such that \( \det \rho \) is a unitary character and let \( \phi = \rho \otimes \chi \). Then \( \phi \) is the \( L \)-parameter for a pre-unitary supercuspidal representation \( \omega \) of \( GL(n) \). Let \( \mathcal{O}(\phi) \) be the orbit of \( \phi \) and \( \mathcal{O}^c(\phi) \) be the compact orbit of \( \phi \). Then \( \mathcal{O}(\phi) \) is a component in the Bernstein variety isomorphic to \( \mathbb{C}^* \) and \( \mathcal{O}^c(\phi) \) is a component in the tempered dual, isomorphic to \( T \). The \( L \)-parameter \( \phi \) now determines the following data.

Let \( (J, \lambda) \) be a maximal simple type for \( \omega \) in the sense of Bushnell and Kutzko \([11, \text{chapter 6}]\). Then \( J \) is a compact open subgroup of \( G \) and \( \lambda \) is a smooth irreducible complex representation of \( J \).

We will write

\[ T = \{ \psi \otimes \omega : \psi \in \Psi^c(G) \} \]

where \( \Psi^c(G) \) denotes the group of unramified unitary characters of \( G \).

\textbf{5.1. Theorem.} Let \( K \) be a maximal compact subgroup of \( G \) containing \( J \) and form the induced representation \( W = \text{Ind}_J^K(\lambda) \). We then have

\[ \ell^2(G \times_K W) \simeq \text{Ind}_K^G(W) \simeq \text{Ind}_J^G(\lambda) \simeq \int_T \pi d\pi. \]

\textit{Proof.} The supercuspidal representation \( \omega \) contains \( \lambda \) and, modulo unramified unitary twist, is the only irreducible unitary representation with this property \([11, 6.2.3]\). Now the Ahn reciprocity theorem expresses \( \text{Ind}_J^G(\lambda) \) as a direct integral \([21, \text{p.58}]\):

\[ \text{Ind}_J^G(\lambda) = \int n(\pi, \lambda) \pi d\pi \]
where $d\pi$ is Plancherel measure and $n(\pi, \lambda)$ is the multiplicity of $\lambda$ in $\pi|_J$. But the Hecke algebra of a maximal simple type is commutative (a Laurent polynomial ring). Therefore $\omega|_J$ contains $\lambda$ with multiplicity 1 (thanks to C. Bushnell for this remark). We then have $n(\psi \otimes \omega, \lambda) = 1$ for all $\psi \in \Psi^t(G)$. We note that Plancherel measure induces Haar measure on $T$, see [24].

The affine building of $G$ is defined as follows [29, p. 49]:

$$\beta G = \mathbb{R} \times \beta SL(n)$$

where $g \in G$ acts on the affine line $\mathbb{R}$ via $t \mapsto t + \text{val}(\det(g))$. Let $G^o = \{g \in G : \text{val}(\det(g)) = 0\}$. We use the standard model for $\beta SL(n)$ in terms of equivalence classes of $\mathfrak{o}_F$-lattices in the $n$-dimensional $F$-vector space $V$. Then the vertices of $\beta SL(n)$ are in bijection with the maximal compact subgroups of $G^o$, see [24, 9.3]. Let $P \in \beta G$ be the vertex for which the isotropy subgroup is $K = GL(n, \mathfrak{o}_F)$. Then the $G$-orbit of $P$ is the set of all vertices in $\beta G$ and the discrete space $G/K$ can be identified with the set of vertices in the affine building $\beta G$. Now the base space of the associated vector bundle $G \times_K W$ is the discrete coset space $G/K$, and the Hilbert space of $\ell^2$-sections of this homogeneous vector bundle is a realization of the induced representation $\text{Ind}^G_K(W)$.

The $C_0(\beta G)$-module structure is defined as follows. Let $f \in C_0(\beta G)$, $s \in \ell^2(G \times_K W)$ and define

$$(fs)(v) = f(v)s(v)$$

for each vertex $v \in \beta G$. We proceed to construct a $K$-cycle in degree 0. This $K$-cycle is

$$(C_0(\beta G), \ell^2(G \times_K W) \oplus 0, 0)$$

interpreted as a $\mathbb{Z}/2\mathbb{Z}$-graded module. This triple satisfies the properties of a (pre)-Fredholm module [14, IV] and so creates an element in $K_0^{\text{top}}(G)$. By Theorem 5.1 this generator creates a free $C(\mathbb{T})$-module of rank 1, and so provides a generator in $K_0(C^*_r(G))$.

5.2 The Hecke algebra of the maximal simple type $(J, \lambda)$ is commutative (the Laurent polynomials in one complex variable). The periodic cyclic homology of this algebra is generated by 1 in degree zero and $dz/z$ in degree 1.

The corresponding summand of the Schwartz algebra $S(G)$ is Morita equivalent to the Fréchet algebra $C^\infty(\mathbb{T})$. By an elementary application of Connes’ theorem [14, Theorem 2, p. 208], the periodic cyclic homology of this Fréchet algebra is generated by 1 in degree 0 and $d\theta$ in degree 1.
5.3 The corresponding component in the Bernstein variety is a copy of $\mathbb{C}^\times$. The cohomology of $\mathbb{C}^\times$ is generated by 1 in degree 0 and $d\theta$ in degree 1.

The corresponding component in the tempered dual is the circle $\mathbb{T}$. The cohomology of $\mathbb{T}$ is generated by 1 in degree 0 and $d\theta$ in degree 1.

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