Classification of Irreducible Weight Modules with a Finite-dimensional Weight Space over the Twisted Schrödinger-Virasoro Lie algebra

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Abstract. It is shown that the support of an irreducible weight module over the Schrödinger-Virasoro Lie algebra with an infinite-dimensional weight space, coincides with the weight lattice and that all nontrivial weight spaces of such a module are infinite-dimensional. As a side-product, it is obtained that every simple weight module over the Schrödinger-Virasoro Lie algebra with a nontrivial finite-dimensional weight space, is a Harish-Chandra module.

Key Words: The Schrödinger-Virasoro Lie algebra, weight modules, irreducible modules
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1. Introduction

The twisted Schrödinger-Virasoro Lie algebra is the infinite-dimensional Lie algebra $\mathcal{L}$ with $\mathbb{C}$-basis $\{L_n, Y_m, M_p | m, n, p \in \mathbb{Z}\}$ and the following relations

\[
\begin{align*}
[L_n, L_p] &= (p - n)L_{n+p}, \\
[L_n, Y_m] &= (m - \frac{n}{2})Y_{n+m}, \\
[L_n, M_p] &= pM_{n+p}, \\
[Y_m, Y_{m'}] &= (m' - m)M_{m+m'}, \\
[Y_m, M_p] &= [M_n, M_p] = 0.
\end{align*}
\]

where $m, m', n, p \in \mathbb{Z}$. This infinite-dimensional Lie algebra is the twisted deformation of the Schrödinger-Virasoro Lie algebra, which was originally introduced in [3] in the context of non-equilibrium statistical physics, as a by-product of the computation of $n$-point functions that are covariant under the action of the Schrödinger group, containing as subalgebras both the Lie algebra of invariance of the free Schrödinger equation and the central charge-free Virasoro algebra. Both original and twisted Schrödinger-Virasoro Lie algebras are closely

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related to the Schrödinger Lie algebras and the Virasoro Lie algebra (cf. [2,5-7,10,11,14]).

They should consequently play a role akin to that of the Virasoro Lie algebra in two-dimensional equilibrium statistical physics.

One attempt of introducing two-dimensional conformal field theory is to understand the universal behaviour of two-dimensional statistical systems at equilibrium and at the critical temperature, where they undergo a second-order phase transition. A systematic investigation of the theory of representations of the Virasoro algebra in the 80es led to introduce a class of physical models (called unitary minimal models), corresponding to the unitary highest weight representations of the Virasoro algebra with central charge less than one. Miraculously, covariance alone is enough to allow the computation of the statistic correlators (or so-called ‘n-point functions’) for these highly constrained models.

Motivated by the search for deformations and central extensions of both original and twisted Schrödinger-Virasoro Lie algebras, Roger and Unterberger presented the sets of generators provided by the cohomology classes of the cocycles in the fifth part of [9]. In [12], Unterberger constructed vertex algebra representations of the Schrödinger-Virasoro Lie algebras out of a charged symplectic boson and a free boson and its associated vertex operators.

A weight module $M$ over $L$ is an $\mathfrak{sh}$-diagonalizable module, where

$$\mathfrak{sh} := \text{Span}_\mathbb{C}\{L_0, M_0\}$$

is the Cartan subalgebra of $L$. If, in addition, $M$ is irreducible and all weight spaces $M_\lambda$ (cf. (1.1)) are finite-dimensional, the module is called a Harish-Chandra module.

If $M$ is an irreducible weight $L$-module, then $M_0$ must act as some complex number $h_M$ on $M$. Furthermore, $M$ has the weight space decomposition

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda, \quad \text{where } M_\lambda = \{v \in M \mid L_0v = \lambda v\},$$

where $M_\lambda$ is called a weight space with weight $\lambda$. Denote the set of weights $\lambda$ of $M$ by

$$\text{Supp}(M) := \{\lambda \in \mathbb{C} \mid M_\lambda \neq 0\},$$

called the support of $M$. Obviously, if $M$ is an irreducible weight $L$-module, then there exists $\lambda \in \mathbb{C}$ such that $\text{Supp}(M) \subset \lambda + \mathbb{Z}$.

An irreducible weight module $V$ is called a pointed module if there exists a weight $\lambda \in \mathbb{C}$ such that $\dim V_\lambda = 1$. Xu posted the following natural problem in [13]:

**Problem 1.1** *Is any irreducible pointed module over the Virasoro algebra a Harish-chandra module?*

An irreducible weight module $V$ is called a mixed module if there exist $\lambda \in \mathbb{C}$ and $i \in \mathbb{Z}$ such that $\dim V_\lambda = \infty$ and $\dim V_{\lambda+i} < \infty$. The following conjecture was posted in [7]:

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Conjecture 1.2 There are no irreducible mixed modules over the Virasoro algebra.

Mazorchuk and Zhao [8] gave the positive answers to the above question and conjecture. In [10], the authors gave positive answers to the above question and conjecture for the twisted Heisenberg-Virasoro algebra. In this note, we also give the positive answers to the above question and conjecture for the Schrödinger-Virasoro Lie algebra. Our main result is the following:

Theorem 1.3 Let $M$ be an irreducible weight $L$-module. Assume that there exists $\lambda \in \mathbb{C}$ such that $\dim M_\lambda = \infty$. Then $\text{Supp}(M) = \lambda + \mathbb{Z}$ and for every $k \in \mathbb{Z}$, we have $\dim M_{\lambda+k} = \infty$.

Theorem 1.3 also implies the following classification of all irreducible weight $L$-modules which admit a nontrivial finite dimensional weight space:

Corollary 1.4 Let $M$ be an irreducible weight $L$-module. Assume that there exists $\lambda \in \mathbb{C}$ such that $0 < \dim M_\lambda < \infty$. Then $M$ is a Harish-Chandra module.

2. Proof of Theorem 1.3

We first recall a main result about the weight Virasoro-modules in [8].

Theorem 2.1 Let $V$ be an irreducible weight Virasoro-module. Assume that there exists $\lambda \in \mathbb{C}$ such that $\dim V_\lambda = \infty$. Then $\text{Supp}(V) = \lambda + \mathbb{Z}$ and for every $k \in \mathbb{Z}$, we have $\dim V_{\lambda+k} = \infty$.

Lemma 2.2 Assume that there exists $\mu \in \mathbb{C}$ and a non-zero element $v \in M_\mu$, such that

$$L_1 v = L_2 v = Y_1 v = M_1 v = 0 \quad \text{or} \quad L_{-1} v = L_{-2} v = Y_{-1} v = M_{-1} v = 0.$$  \tag{2.1}

Then $M$ is a Harish-Chandra module.

Proof. Indeed, under these conditions, $v$ is either a highest or a lowest weight vector and hence $M$ is a Harish-Chandra module (see, e.g. [6]).

Assume now that $M$ is an irreducible weight $L$-module such that there exists $\lambda \in \mathbb{C}$ satisfying $\dim M_\lambda = \infty$.

Lemma 2.3 There exists at most one $i \in \mathbb{Z}$ such that $\dim M_{\lambda+i} < \infty$.

Proof. Assume that $\dim M_{\lambda+i} < \infty$ and $\dim M_{\lambda+j} < \infty$ for different $i, j \in \mathbb{Z}$. Without loss of generality we may assume $i = 1$ and $j > 1$. Let $V$ be the following $L$-module:

$$\text{Ker}(L_1 : M_\lambda \to M_{\lambda+1}) \cap \text{Ker}(Y_1 : M_\lambda \to M_{\lambda+1}) \cap \text{Ker}(M_1 : M_\lambda \to M_{\lambda+1})$$

$$\cap \text{Ker}(L_j : M_\lambda \to M_{\lambda+j}) \cap \text{Ker}(Y_j : M_\lambda \to M_{\lambda+j}) \cap \text{Ker}(M_j : M_\lambda \to M_{\lambda+j}). \tag{2.1}$$
Since
\[ \dim M_\lambda = \infty, \quad \dim M_{\lambda+1} < \infty \quad \text{and} \quad \dim M_{\lambda+j} < \infty, \]
all cokernels of the maps appearing in (2.1) are finite-dimensional. Thus
\[ \dim V = \infty. \]  
(2.2)
Since
\[ [L_1, L_p] = (p - 1)L_{p+1} \neq 0, \quad [L_1, M_r] = rM_{r+1} \neq 0 \quad \text{and} \quad [L_1, Y_q] = (q - \frac{1}{2})Y_{q+1} \neq 0 \]
for \( 1 \neq p \in \mathbb{Z}, q \in \mathbb{Z}, 1 \leq r \in \mathbb{Z}, \) we get
\[ L_p V = 0, \quad p = 1, j, j + 1, j + 2, \ldots, \quad \text{and} \]
\[ Y_q V = M_r V = 0, \quad q, r = 1, 2, \ldots. \]  
(2.3)
If there would exist \( 0 \neq v \in V \) such that \( L_2v = 0, \) then \( Y_1v = M_1v = L_1v = L_2v = 0 \) and \( \mathcal{M} \) would be a Harish-Chandra module by Lemma 2.2, a contradiction. Hence \( L_2v \neq 0 \) for all \( v \in V. \) In particular,
\[ \dim L_2 V = \infty. \]
Since \( \dim M_{\lambda+1} < \infty, \) there exists \( 0 \neq w \in L_2 V \) such that \( Y_{-1}w = M_{-1}w = L_{-1}w = 0 \) (as in the proof of (2.2)). Let \( w = L_2u \) for some \( u \in V. \) For all \( k \geq j, \) using (2.3), we have
\[ L_k w = L_k L_2 u = L_2 L_k u + (2 - k) L_{k+2} u = 0. \]
Hence \( L_k w = 0 \) for all \( k = 1, j, j + 1, j + 2, \ldots. \) Since
\[ [L_{-1}, L_l] = (l + 1)L_{l-1} \neq 0, \quad [L_l, M_{-1}] = M_{l-1} \neq 0 \quad \text{and} \]
\[ [Y_{-1}, L_l] = (1 + \frac{l}{2}) Y_{l-1} \neq 0 \]
for all \( l > 1, \) we get inductively \( L_k w = Y_k w = M_k w = 0 \) for all \( k = 1, 2, \ldots. \) Hence \( \mathcal{M} \) is a Harish-Chandra module by Lemma 2.2, again a contradiction. The lemma follows. \( \square \)

Because of Lemma 2.3, we can now fix the following notation: \( \mathcal{M} \) is an irreducible weight \( \mathcal{L} \)-module, \( \mu \in \mathbb{C} \) is such that \( \dim M_\mu < \infty \) and \( \dim M_{\mu+i} = \infty \) for all \( i \in \mathbb{Z} \setminus \{0\}. \)

**Lemma 2.4** Let \( 0 \neq v \in M_{\mu-1} \) be such that \( Y_1v = M_1v = L_1v = 0. \) Then

1. \( (L_1^3 - 6L_2L_1 + 6L_3)L_2v = 0, \)
2. \( Y_kv = M_kv = 0, \)
3. \( Y_kL_2v = M_kL_2v = 0, \)

where \( k = 1, 2, \ldots. \)
Proof. (1) is the conclusion of Lemma 4 in [8].

(2) Since \( Y_1v = M_1v = L_1v = 0 \) and \([L_1, Y_l] = (l - \frac{1}{2})Y_{l+1} \neq 0, [L_1, M_l] = lM_{l+1} \neq 0 \) for \( l \geq 1 \), we inductively get (2).

(3) follows from \( Y_kL_2 = L_2Y_k - (k - 1)Y_{k+2} \), \( M_kL_2 = L_2M_k - kM_{k+2} \) and (2). \( \Box \\

Proof of Theorem 1.3. Denote

\[ V := \ker\{ L_1 : M_{\mu-1} \to M_{\mu} \} \cap \ker\{ Y_1 : M_{\mu-1} \to M_{\mu} \} \cap \ker\{ M_1 : M_{\mu-1} \to M_{\mu} \}. \]

Since \( \dim M_{\mu-1} = \infty \) and \( \dim M_{\mu} < \infty \), similar with the discussion of (2.2), we have \( \dim V = \infty \). For any \( 0 \neq v \in V \), consider the element \( L_2v \). By Lemma 2.2, \( L_2v = 0 \) would imply that \( M \) is a Harish-Chandra module, a contradiction. Hence \( L_2v \neq 0 \), in particular, \( \dim L_2V = \infty \). This implies that there exists \( w \in L_2V \) such that \( w \neq 0 \) and \( Y_{-1}w = M_{-1}w = L_{-1}w = 0 \). Using

\[ [L_{-1}, M_{-k+1}] = (1 - k)M_{-k} \quad \text{and} \]
\[ [L_{-1}, Y_{-k+1}] = (\frac{3}{2} - k)Y_{-k}, \]

we obtain

\[ Y_{-k}w = M_{-k}w = 0, \quad k = 1, 2, \ldots. \]

From Lemma 2.4, we have \( (L_1^3 - 6L_2L_1 + 6L_3)w = 0, \) and \( Y_kw = M_kw = 0, \quad k = 1, 2, \ldots. \)

Therefore,

\[ Y_0w = \frac{2}{3}[L_{-1}, Y_1]w = 0 \quad \text{and} \]
\[ M_0w = [L_{-1}, M_1]w = 0. \]

The above relations mean that \( Y_k, M_k \) act trivially on \( M \) for all \( k \in \mathbb{Z} \). Therefore, \( M \) is simply a module over the Virasoro algebra \( \mathcal{V}ir \). Thus, Theorem 1.3 follows from Theorem 2.1. \( \Box \\

Proof of Corollary 1.4. Assume that \( M \) is not a Harish-Chandra module. Then there should exist \( i \in \mathbb{Z} \) such that \( \dim M_{\lambda+i} = \infty \). In this case, Theorem 1.3 implies \( \dim M_{\lambda} = \infty \), a contradiction. Hence \( M \) is a Harish-Chandra module, and the rest of the statement follows from [6]. \( \Box \\

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