States of charged quantum fields and their statistical properties in the presence of critical potential steps

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Abstract

Evolution of charged quantum fields under the action of constant nonuniform electric fields is studied. To this end we construct a special generating functional for density operators of the quantum fields with different initial conditions. Then we study some reductions of the density operators. For example, reductions to electron or positron subsystems, reduction induced by measurements, and spatial reduction to the left or to the right subsystems of final particles. We calculate von Neumann entropy for the corresponding reduced density operators, estimating in such a way an information loss. Then we illustrate the obtained results by calculations in a specific background of a strong constant electric field between two infinite capacitor plates separated by a finite distance $L$.

Keywords: QED with $x$-electric potential steps, particle creation, von Neumann entropy

1 Introduction

Problems of quantum field theory with external backgrounds violating the vacuum stability are already being studied systematically for a long time. Recently, they turned out to be of special attention due to new real possible applications in astrophysics and physics of nanostructures. A nonperturbative formulation of QED with so-called $t$-electric potential steps (time-dependent potentials of special form) was developed in Refs. [1] and applied to various model and realistic physical problems, see, e.g. Refs. [2, 3, 4]. In particular, quantum entanglement in the Schwinger effect of Dirac or Klein-Gordon field due to the $t$-electric steps, between a subsystem and the rest of the system, as measured by the von Neumann entropy of the reduced density matrix, was calculated [5] (see, as well, Refs. [6, 7]).

In the graphene and similar nanostructures already laboratory electric fields can violate the vacuum stability [8]. Thus, new carriers are produced from the vacuum modifying the conductivity. Such an effect caused by an action of $t$-electric potential steps was calculated in the graphene [9]. However, there exist many interesting physical situations where external backgrounds are represented by strong time-independent nonuniform electric fields concentrated in restricted space areas. Such kind of backgrounds are called $x$-electric potential steps. In the recent work [10] a consistent nonperturbative formulation of QED with $x$-electric potential steps strong enough to violate the vacuum stability was constructed. In the latter work and in Ref. [11] some quantum effects related to a violation of the vacuum instability by $x$-electric potential steps were calculated.

In this article we study evolution of different initial states of charged quantum fields in $x$-electric critical potential steps, using the above mentioned formulation of QED [10]. To this end, we introduce a special generating functional which allows us to construct density operators for different initial states of the system of quantum fields. We consider pure initial states, the vacuum and many-particle states, and thermal initial states. Corresponding final states are studied using three type of reductions. Since $x$-electric potential steps cause natural division of created particles in...
subsystems of electrons and positrons substantially separated spatially, we first consider reductions to electron or positron subsystems. In the background under consideration, it is interesting to calculate reductions to the left and right parts of the whole system and compare the obtained states with states resulting from the previously mentioned reductions. Finally, we study reductions due to possible measurements of number of final particles. The latter kind of reductions can also occur due to some decoherence processes, such as collisions with some external sources (e.g. with impurities in the graphene). To study the loss of the information in all the reductions, we calculate von Neumann entropy for reduced density operators. In two first reduction cases this entropy can be also identified with a measure of quantum entanglement between the corresponding quantum subsystems. The article is organized as follows. In Section 2 we recall basic points of QED with $x$-electric potential steps. In Section 3 we introduce a special generating function which allow us to construct density matrices for different initial conditions by choosing appropriate sources. In Section 4 we construct density operators for different initial states of charged quantum fields. Then in Section 5 we construct a normal form of the density operator. All the above mentioned reductions are presented in Section 6. The corresponding von Neumann entropy is calculated in Section 7. Some useful relations are placed in the Appendix.

2 QED with $x$-electric potential steps

The general theory of quantization of charged fields in the presence of critical potential steps was formulated by Gitman and Gavrilov in Ref. [10]. They have constructed a special self-consistent QED with $x$-electric potential steps utilizing so-called generalized Furry picture. In the framework of this QED it is possible to take into account interaction with the external electric field exactly when the analytical solutions of the Dirac equation in the corresponding field are known. Here we repeat some crucial moments of this theory. In this article we generally adapt notations used in Ref. [10]: we utilize the system of units where $c = \hbar = 1$.

We work in $d = D + 1$ dimensional Minkowski space-time parametrized by coordinates $X$,

$$X = (X^\mu, \mu = 0, 1, \ldots, D) = (t, \mathbf{r}), \quad X^0 = t, \quad \mathbf{r} = (X^1, \ldots, X^D).$$

(1)

One of the most important points of QED with critical potential steps is that the whole space of quantum numbers $n \in \Omega$ (which are energy, momenta and spin $n = (p_0, p, \sigma)$) can be divided into five different ranges $\Omega_i, i = 1, \ldots, 5$, $n_i \in \Omega_i$, where the solutions of the corresponding Dirac equation have similar forms. The full operator of Dirac field can be presented as a sum of operators defined for each particular range $\Omega_i$,

$$\hat{\Psi}(X) = \sum_{i=1}^{5} \hat{\Psi}_i(X).$$

(2)

Operators $\hat{\Psi}_i(X)$ can be decomposed using the specific set of solutions of the Dirac equation with quantum numbers $n \in \Omega_i$. It was shown that the decompositions have the form

$$\hat{\Psi}_1(X) = \sum_{n \in \Omega_1} \mathcal{M}_n^{-1/2} \left[ a_n(\text{in}) \psi_n(X) - a_n(\text{in}) \psi_n(X) \right]$$

$$= \sum_{n \in \Omega_1} \mathcal{M}_n^{-1/2} \left[ +a_n(\text{out}) \psi_n(X) - a_n(\text{out}) \psi_n(X) \right],$$

$$\hat{\Psi}_3(X) = \sum_{n \in \Omega_3} \mathcal{M}_n^{-1/2} \left[ a_n(\text{in}) \psi_n(X) - a_n(\text{out}) \psi_n(X) \right]$$

$$= \sum_{n \in \Omega_3} \mathcal{M}_n^{-1/2} \left[ +a_n(\text{out}) \psi_n(X) + b_n^\dag(\text{in}) \psi_n(X) \right],$$

$$\hat{\Psi}_5(X) = \sum_{n \in \Omega_5} \mathcal{M}_n^{-1/2} \left[ a_n(\text{out}) \psi_n(X) - a_n(\text{out}) \psi_n(X) \right]$$

$$= \sum_{n \in \Omega_5} \mathcal{M}_n^{-1/2} \left[ +b_n^\dag(\text{in}) \psi_n(X) + b_n^\dag(\text{in}) \psi_n(X) \right]$$

(3)

in the ranges $\Omega_i, i = 1, 3, 5$, and

$$\hat{\Psi}_2(X) = \sum_{n \in \Omega_2} \mathcal{M}_n^{-1/2} \psi_n(X), \quad \hat{\Psi}_4(X) = \sum_{n \in \Omega_4} \mathcal{M}_n^{-1/2} b_n^\dag \psi_n(X),$$

(4)
in the ranges $$\Omega_i, i = 2, 4$$.

In these decompositions, $$a, a^\dagger$$ and $$b, b^\dagger$$ are interpreted as operators of creation and annihilation of particles (electrons) and antiparticles (positrons) correspondingly. Operators

$$-a_n(\text{in}) = -a_n, \quad +a_n(\text{in}) = +a_n,$$

$$-b_n^\dagger(\text{in}) = -b_n^\dagger, \quad +b_n(\text{in}) = +b_n,$$

and their conjugated ones correspond to initial electrons and positrons, while operators

$$+a_n(\text{out}) = +a_n, \quad -a_n(\text{out}) = -a_n,$$

$$+b_n(\text{out}) = +b_n, \quad -b_n(\text{out}) = -b_n,$$

correspond to final electrons and positrons.

Operators (5), (6) obey the following anticommutation relations. All operators with different sets of quantum numbers $$n$$ anticommute. This implies that all operators from different ranges $$\Omega_i$$ anticommute. Inside each range $$\Omega_i$$ exist the following non-zero anticommutation relations:

$$n \in \Omega_1: \begin{bmatrix} +a_n, +a_n^\dagger \end{bmatrix} = \delta_{nn'}, \quad n \in \Omega_3: \begin{bmatrix} -a_n, -a_n^\dagger \end{bmatrix} = \delta_{nn'}, \quad n \in \Omega_5: \begin{bmatrix} +b_n, +b_n^\dagger \end{bmatrix} = \delta_{nn'},$$

in the ranges $$\Omega_i, i = 1, 3, 5,$$ and

$$n \in \Omega_2: \begin{bmatrix} a_n, a_n^\dagger \end{bmatrix} = \delta_{nn'}, \quad n \in \Omega_4: \begin{bmatrix} b_n, b_n^\dagger \end{bmatrix} = \delta_{nn'},$$

in the ranges $$\Omega_i, i = 2, 4$$.

Initial and final vacuum vectors are defined as state vectors annihilated by operators of initial and final particles:

$$+a_n |0, \text{in}\rangle = -a_n |0, \text{in}\rangle = -b_n |0, \text{in}\rangle = +b_n |0, \text{in}\rangle = 0,$$

$$-a_n |0, \text{out}\rangle = +a_n |0, \text{out}\rangle = +b_n |0, \text{out}\rangle = -b_n |0, \text{out}\rangle = 0,$$

for quantum numbers from ranges $$\Omega_i, i = 1, 3, 5,$$ and

$$n \in \Omega_2: \quad a_n |0, \text{in}\rangle = a_n |0, \text{out}\rangle = 0, \quad n \in \Omega_4: \quad b_n |0, \text{in}\rangle = b_n |0, \text{out}\rangle = 0,$$

in ranges $$\Omega_2$$ and $$\Omega_4$$. Since all operators from different $$\Omega_i$$ anticommute, the total initial and final vacua vectors can be represented as the following tensor product,

$$|0, \text{in}\rangle = \otimes_{1,3,5} |0, \text{in}\rangle^{(i)} \otimes |0\rangle^{(2)} \otimes |0\rangle^{(4)}, \quad |0, \text{out}\rangle = \otimes_{1,3,5} |0, \text{out}\rangle^{(i)} \otimes |0\rangle^{(2)} \otimes |0\rangle^{(4)},$$

where $$|0, \text{in/out}\rangle^{(i)}$$ denote partial in- and out-vacua in ranges $$\Omega_i, i = 1, 3, 5,$$ and $$|0\rangle^{(2)}, |0\rangle^{(4)}$$ – partial vacua in ranges $$\Omega_2$$ and $$\Omega_4$$ correspondingly. It is follows from (11) and (12), that

$$|0\rangle^{(2)} = |0, \text{in}\rangle^{(2)} = |0, \text{out}\rangle^{(2)}, \quad |0\rangle^{(4)} = |0, \text{in}\rangle^{(4)} = |0, \text{out}\rangle^{(4)}.$$

In addition, inside each range $$\Omega_i$$ the partial vacua can be in turn be presented as the tensor products in quantum modes:

$$|0, \text{in}\rangle^{(i)} = \prod_{n \in \Omega_i} |0, \text{in}\rangle^{(i)}_n, \quad |0, \text{out}\rangle^{(i)} = \prod_{n \in \Omega_i} |0, \text{out}\rangle^{(i)}_n, \quad |0\rangle^{(2,4)} = \prod_{n \in \Omega_{2,4}} |0\rangle^{(2,4)}_n.$$

Each of these partial vacua vectors is destroyed only by annihilation operators with corresponding quantum numbers $$n$$.

The in- and out-sets of operators of creation and annihilation of electrons and positrons as well as in- and out-vacua are connected via the special unitary evolution operators $$V (V V^\dagger = I), |0, \text{in}\rangle = V |0, \text{out}\rangle,$$

$$\{a(\text{in}), a^\dagger(\text{in}), b(\text{in}), b^\dagger(\text{in})\} = V \{a(\text{out}), a^\dagger(\text{out}), b(\text{out}), b^\dagger(\text{out})\} V^\dagger. \quad (14)$$
This, in particular, implies that
\[ \hat{F}(\text{in}) = V \hat{F}(\text{out}) V^\dagger, \] (15)
where \( \hat{F}(\text{in}) \) is an operator-valued function written in terms of in-set of the operators of creation and annihilation operators while \( \hat{F}(\text{out}) \) is the same function written in terms of out-set. The explicit form of the operator \( V \) were found in Ref. \cite{10}. The initial partial vacua states remain vacua \cite{10} in ranges \( \Omega_{1,2,4,5} \) (i.e. the vacuum is stable in these ranges). In other words,
\[ |0, \text{in}^{(i)}\rangle = |0, \text{out}^{(i)}\rangle, \quad i = 1, 2, 4, 5. \] (16)
We can also define vacuum-to-vacuum transition amplitude as
\[ c_v = \langle 0, \text{out} | 0, \text{in} \rangle = \langle 3 \rangle^{(3)} \langle 0, \text{out} | 0, \text{in} \rangle^{(3)}. \] (17)
Taking into account relations \cite{rebrown09}, we can also introduce partial transition amplitudes for each quantum mode \( n \),
\[ c_{v,n} = \langle 3 \rangle^{(3)}_{n} \langle 0, \text{out} | 0, \text{in} \rangle^{(3)}_{n}, \quad c_v = \prod_{n \in \Omega_i} c_{v,n}. \] (18)

The connection between in- and out-operators can also be presented via linear canonical transformation (also called Bogolubov transformation), that has the following form in different ranges of quantum numbers \( \Omega_i \). In the range \( \Omega_1 \) for electrons the transformation reads as
\[ +a_n = \eta_l g (+|)^{-1} a_n + g (-|)^{-1} g (+|) - a_n, \]
\[ -a_n = g (+|)^{-1} g (-|) a_n - \eta_l g (-|)^{-1} a_n, \]
\[ +a_n = g (-|)^{-1} g (+|) a_n + \eta_l g (+|)^{-1} + a_n, \]
\[ -a_n = -\eta_R g (-|)^{-1} a_n + g (+|)^{-1} g (-|) + a_n. \] (19)
Parameters \( \eta_{l,R} = \text{sgn}(p_0 - U_{l,R}) \) denote the signs of asymptotic particle kinetic energy, where \( p_0 \) is the full energy of particle and \( U_{l,R} \) are left and right asymptotic potential energies correspondingly, \( U_L = U(x \to -\infty), \quad U_R = U(x \to +\infty). \) Canonical transformations between the initial and final pairs of creation operators of positrons in \( \Omega_5 \) can be derived from the latter expression by replacing \( \pm a_{n} \rightarrow \pm b_n^\dagger, \pm a_{n} \rightarrow \pm b_n^\dagger, \eta_{l} \leftrightarrow \eta_{R}. \) In Klein zone, \( \Omega_3, \) the canonical transformation takes the form
\[ +a_n = -g (-|)^{-1} b_n^\dagger + g (-|)^{-1} g (+|) - a_n, \]
\[ +b_n^\dagger = g (-|)^{-1} g (+|)^{-1} b_n^\dagger + g (-|)^{-1} a_n, \]
\[ -b_n^\dagger = g (+|)^{-1} g (-|) b_n^\dagger - g (+|)^{-1} a_n, \]
\[ -a_n = g (+|)^{-1} b_n^\dagger + g (+|)^{-1} g (-|) + a_n. \] (20)

Functions \( g \) are mutual decomposition coefficients of Dirac equation solutions,
\[ \eta_l \zeta \psi_n(X) = +\psi_n(X)g (+|) - \psi_n(X)g (-|), \]
\[ \eta_R \zeta \psi_n(X) = +\psi_n(X)g (+|) - \psi_n(X)g (-|), \] (21)
with respect to inner product on the \( x \)-constant hyperplane (see Ref. \cite{rebrown09} for the details), and have the following properties:
\[ \left( \zeta \psi_n, \zeta' \psi_n \right) = \delta_n, n' g (|\zeta\rangle \zeta'), \quad g (|\zeta\rangle \zeta') = g (|\zeta\rangle \zeta')^*, \]
\[ |g (-|)^2 = |g (+|)^2, \quad |g (+|)^2 = |g (-|)^2, \quad \frac{g (+|)}{g (-|)} = \frac{g (|\zeta\rangle)}{g (|\zeta\rangle^*)}. \] (22)

3 Generating functionals for density operators
We introduce special generating functionals that allow us to obtain the explicit forms of density operators (matrices) for different initial states by choosing an appropriate set of sources.
As we mentioned in the previous section, all the creation-annihilation operators (5) from different ranges $\Omega_i$ anticommute. Density operator $\hat{\rho}$ of the system in consideration is a function of quadratic combinations these creation-annihilation operators. This fact allows us to present the density operator $\hat{\rho}$ as a tensor product of partial density operators $\hat{\rho}^{(i)}$ for each range $\Omega_i$,

$$\hat{\rho} = \otimes \prod_{i=1}^{5} \hat{\rho}^{(i)}. \quad (23)$$

One can see that due to Eqs. (7) and (8) operators $\hat{\rho}^{(i)}$ anticommute and can be considered separately. Thus, it is convenient to introduce the separate partial generating functional for each range of quantum numbers $\Omega_i$. We will refer to each of these generating functionals as $R^{(i)}(J)$, and $J = \{J_n\}_{n=\Omega_i}$ is a complete set of sources in each range which fully describe (parametrize) the initial state of the system in each range $\Omega_i$. The total generating functional can be obtained as a direct tensor product of functionals $R^{(i)}(J)$,

$$R(J) = \otimes \prod_{i=1}^{5} R^{(i)}(J). \quad (24)$$

a. Generating functionals in $\Omega_1$ and $\Omega_5$.

In $\Omega_1$ the generating functional $R^{(1)}(J)$ has the form

$$R^{(1)}(J) = \prod_{n \in \Omega_1} R^{(1)}_{n}, \quad R^{(1)}_{n} = \left[ Z^{(1)}_{n} \right]^{-1} R^{(1)}_{n}, \quad \text{tr} R^{(1)}_{n} = 1,$$

$$\mathcal{R}^{(1)} = \exp \left[ +a_{n}^{\dagger} \left( J^{(1)}_{x,n} - 1 \right) +a_{n} + +a_{n}^{\dagger} \left( J_{-n}^{(1)} - 1 \right) - a_{n} \right]. \quad (25)$$

In $\Omega_5$ the generating functional $R^{(5)}(J)$ has the form

$$R^{(5)}(J) = \prod_{n \in \Omega_5} R^{(5)}_{n}, \quad R^{(5)}_{n} = \left[ Z^{(5)}_{n} \right]^{-1} R^{(5)}_{n}, \quad \text{tr} R^{(5)}_{n} = 1,$$

$$\mathcal{R}^{(5)} = \exp \left[ +b_{n}^{\dagger} \left( J^{(5)}_{x,n} - 1 \right) +b_{n} + -b_{n}^{\dagger} \left( J_{-n}^{(5)} - 1 \right) - b_{n} \right]. \quad (26)$$

Here the $Z^{(1)}_{n}$ and $Z^{(5)}_{n}$ are normalization factors (statistical sums); colons $: \ldots :$ always denote the normal form with respect to creation-annihilation operators inside them. Using the Eq. (27) one can calculate each of them as

$$Z^{(1,5)}_{n} = \left( J^{(1,5)}_{x,n} + 1 \right) \left( J^{(1,5)}_{-n} + 1 \right). \quad (27)$$

b. Generating functionals in $\Omega_2$ and $\Omega_4$.

In these ranges the corresponding generating functionals $R^{(2,4)}(J)$ have the following structure:

$$R^{(2)}(J) = \prod_{n \in \Omega_2} R^{(2)}_{n}, \quad R^{(2)}_{n} = \left[ Z^{(2)}_{n} \right]^{-1} : \exp \left[ a_{n}^{\dagger} (J^{(2)}_{n} - 1) a_{n} \right] :,$$

$$R^{(4)}(J) = \prod_{n \in \Omega_4} R^{(4)}_{n}, \quad R^{(4)}_{n} = \left[ Z^{(4)}_{n} \right]^{-1} : \exp \left[ b_{n}^{\dagger} (J^{(4)}_{n} - 1) b_{n} \right] :. \quad (28)$$

Here $J^{(2)}_{n}$ and $J^{(4)}_{n}$ are the corresponding sources in $\Omega_2$ and $\Omega_4$, and the corresponding normalization factors are

$$Z^{(2,4)}_{n} = \left( J^{(2,4)}_{n} + 1 \right). \quad (29)$$

The structure of operators $R^{(2,4)}(J)$ is trivial as there is no particle production in these ranges and all initial particles are subjected to total reflection [10]. For this reason we often omit the consideration of ranges $\Omega_2$ and $\Omega_4$ through the article.

c. Generating functional in Klein zone.
In the Klein zone, $\Omega_3$, the corresponding generating functional $R^{(3)}(J)$ has the form

\[
R^{(3)}(J) = \prod_{n \in \Omega_3} R_n^{(3)}, \quad R_n^{(3)} = \left[ Z_n^{(3)} \right]^{-1} \mathcal{E}_n^{(3)}, \quad \text{tr} R_n^{(3)} = 1,
\]

\[
R_n^{(3)} = : \exp \left[ -a_n^+ (J_{+,n}^{(3)} - 1) - a_n + b_n^+ (J_{-,n}^{(3)} - 1) - b_n \right] : ,
\]

where the normalization factor $Z_n^{(3)}$ has the form

\[
Z_n^{(3)} = \left( J_{+,n}^{(3)} + 1 \right) \left( J_{-,n}^{(3)} + 1 \right).
\]

### 4 Density operators with different initial conditions

a. Initial vacuum state

To obtain the density operator with initial vacuum state, we set $J = 0$ in $R(J)$, i.e. we set $J_{\pm,n}^{(i)} = J_n^{(i)} = 0$ in every partial generating functional $R^{(i)}(J)$. In this case, the general density operator with vacuum initial state takes the form

\[
R(J = 0) = \hat{\rho}_v = \otimes \prod_{i = 1}^5 \hat{\rho}_v^{(i)} = \prod_{n \in \Omega_3} \hat{\rho}_v^{(i)} ,
\]

where one-mode partial density operators $\hat{\rho}_v^{(i)}$ are (in terms of in-set of creation-annihilation operators)

\[
\hat{\rho}_v^{(1)} = : \exp \left[ -a_n^+ a_n + a_n^+ a_n - a_n^+ a_n \right] : ; \quad \hat{\rho}_v^{(5)} = : \exp \left[ a_n^+ b_n + b_n^+ a_n - b_n^+ b_n \right] : ;
\]

\[
\hat{\rho}_v^{(3)} = : \exp \left[ -a_n^+ a_n - b_n^+ b_n \right] : ; \quad \hat{\rho}_v^{(2)} = : \exp \left[ a_n^+ a_n \right] : ; \quad \hat{\rho}_v^{(4)} = : \exp \left[ -b_n^+ b_n \right] : .
\]

Taking into account the well-known Berezin formula\[17\]

\[
|0 \rangle \langle 0| = : \exp \left[ -a_n^+ a_n \right] : ,
\]

one can see that operators $\hat{\rho}_v^{(i)}$ are, in fact, partial vacuum projectors:

\[
\hat{\rho}_v^{(i)} = |0, \text{in}^{(i)} \rangle \langle \text{in}^{(i)}_n| , \quad i = 1, 3, 5, \quad \hat{\rho}_v^{(2,4)} = |0 \rangle^{(2,4)}_n \langle 0|^{(2,4)}_n .
\]

One can show that the differential numbers of initial electrons and positrons (see Eqs. \[5\] and \[6\] for the reference) in the state described by operator $\hat{\rho}_v$ vanish for all $n$,\[3\]

\[
\begin{align*}
\text{tr} \hat{\rho}_v - a_n^+ a_n &= \text{tr} \hat{\rho}_v + a_n^+ a_n = \text{tr} \hat{\rho}_v - b_n^+ b_n = \text{tr} \hat{\rho}_v + b_n^+ b_n = 0, \quad n \in \Omega_{1,3,5}, \n\text{tr} \hat{\rho}_v a_n^+ a_n &= \text{tr} \hat{\rho}_v b_n^+ b_n = 0, \quad n \in \Omega_{2,4}.
\end{align*}
\]

The mean differential numbers of final electrons and positrons are different from zero in range $\Omega_3$. These numbers are equal to the number of pairs created from vacuum,

\[
N_n^a = N_n^b = N_n^{cr} = \text{tr} \hat{\rho}_v + a_n^+ a_n = |g (-)^0|^{-2} , \quad n \in \Omega_3.
\]

b. Initial thermal state

Before writing the expressions for the density operator, we must remind that we consider the situation when the electric field is not zero only in the finite region $S_{\text{int}} = (x_L, x_R)$ situated between the planes $x = x_L$ and $x = x_R$. Outside of $S_{\text{int}}$ for $x \in S_L = (-\infty, x_L)$ and for $x \in S_R = [x_R, \infty)$ particles are free (i.e. their movement is unbounded at least in one direction). It should be noted that usually QFT deals with physical quantities that are presented by volume integrals on the hyperplane $t = \text{const}$. The main contribution to these integrals is coming from regions $S_L$ and $S_R$, where particles are free. This fact allows one to obtain the explicit form of kinetic energies for all particles (see details in Ref. \[10\]) and is used in what follows.
To obtain the density operator with initial thermal state, we need to set the sources $J$ as follows:

\[
J_{\pm,n} = e^{-E_{\pm,n}/\Theta}, \quad E_{\pm,n} = \beta (\varepsilon_{\pm,n} - \mu \pm), \quad \beta = \Theta^{-1}, \quad n \in \Omega_{1,3,5},
\]

\[
J_{n} = e^{-E_n/\Theta}, \quad E_n = \beta (\varepsilon_n - \mu), \quad n \in \Omega_{2,4},
\]

where $\varepsilon_{\pm,n}$ and $\varepsilon_n$ are the kinetic energies of particles and antiparticles with quantum numbers $n$; $\mu \pm$ and $\mu$ are the corresponding chemical potentials, and $\Theta$ is the absolute temperature.\footnote{Here and later in the definition of von Neumann entropy we omit Boltzmann constant $k_B$ for the sake of convenience.} For the sake of simplicity in what follows we will suppose that all chemical potentials for electrons and positrons are equal. It is easy to see that such choice of sources turns the generating functional into density operator $\hat{\rho}$ that has been in thermal state at the initial time instant. Using the relation it is easy to show that the density operator $\hat{\rho}$ can be written as

\[
\hat{\rho}_\beta = \otimes \prod_{i=1}^{5} \hat{\rho}_\beta^{(i)}, \quad \hat{\rho}_\beta^{(i)} = \prod_{n \in \Omega_i} \hat{\rho}_{\beta,n}^{(i)}.
\]

where one-mode density operators $\hat{\rho}_{\beta,n}^{(i)}$ have the form

\[
\hat{\rho}_{\beta,n}^{(1)} = \left[ Z_{n}^{(1)} \right]^{-1} \exp \left[ - a_n^+ E_n^+ + a_n - a_n^+ E_n^- - a_n \right],
\]

\[
\hat{\rho}_{\beta,n}^{(5)} = \left[ Z_{n}^{(5)} \right]^{-1} \exp \left[ - b_n^+ E_n^+ + b_n - b_n^+ E_n^- - b_n \right],
\]

\[
\hat{\rho}_{\beta,n}^{(3)} = \left[ Z_{n}^{(3)} \right]^{-1} \exp \left[ - a_n^+ E_n^+ - a_n - b_n^+ E_n^- - b_n \right],
\]

\[
\hat{\rho}_{\beta,n}^{(2)} = \left[ Z_{n}^{(2)} \right]^{-1} \exp \left[ - a_n^+ E_n a_n \right] \hat{\rho}_{\beta,n}^{(4)} = \left[ Z_{n}^{(4)} \right]^{-1} \exp \left[ - b_n^+ E_n b_n \right].
\]

The statistical sums $Z_n^{(i)}$ for this case are

\[
Z_{n}^{(1,3,5)} = \left( 1 + e^{-E_n^+} \right) \left( 1 + e^{-E_n^-} \right), \quad Z_{n}^{(2,4)} = \left( 1 + e^{-E_n} \right).
\]

Note that operators can also be presented as

\[
\hat{\rho}_\beta^{(i)} = \left[ Z_n^{(i)} \right]^{-1} \exp \left\{ - \beta \left[ \hat{H}_n^{(i)} - \mu \hat{N}_n^{(i)} \right] \right\},
\]

where for $i = 1, 3, 5$ we have

\[
\hat{H}_n^{(i)} = \begin{cases}
+ a_n^+ E_n^+ + a_n + - a_n^+ E_n^- - a_n, & n \in \Omega_1 \\
- a_n^+ E_n^+ - a_n + - b_n^+ E_n^- - b_n, & n \in \Omega_3 \\
+ b_n^+ E_n^+ + b_n + - b_n^+ E_n^- - b_n, & n \in \Omega_5
\end{cases},
\]

\[
\mu \hat{N}_n^{(i)} = \begin{cases}
\mu^+ + a_n^+ a_n + \mu^- - a_n^+ a_n, & n \in \Omega_1 \\
\mu^- - a_n^+ a_n + \mu^- - b_n^+ b_n, & n \in \Omega_3 \\
\mu^+ + a_n^+ b_n + \mu^- - b_n^+ b_n, & n \in \Omega_5
\end{cases},
\]

while for $i = 2, 4$ these operators take the form

\[
\hat{H}_n^{(i)} = \begin{cases}
a_n^+ E_n a_n, & n \in \Omega_2 \\
b_n^+ E_n b_n, & n \in \Omega_4
\end{cases}, \quad \mu \hat{N}_n^{(i)} = \begin{cases}
\mu a_n^+ a_n, & n \in \Omega_2 \\
\mu b_n^+ b_n, & n \in \Omega_4
\end{cases}.
\]

One can see that the density operators are well-known Fermi-Dirac and Bose-Einstein distributions. For example, the differential number of initial electrons in range $\Omega_3$ can be found as

\[
N_{n,\beta,-}^{(3)}(\text{in}) = \text{tr} \hat{\rho}_{\beta,n}^{(3)} \hat{N}_{n,\beta,-}^{(3)}(\text{in}) = \text{tr} \hat{\rho}_{\beta,n}^{(3)} - a_n^+ a_n = \left( e^{E_n^-} + 1 \right)^{-1}, \quad n \in \Omega_3.
\]

Other differential distributions can be calculated in the same way using corresponding operators of creation-annihilation and partial density operators.
c. Pure states with definite number of electrons/positrons as initial states.

The generating functionals $R^{(i)}$ also allow us to construct the partial density operators $\hat{\rho}^{(i)}_{m\to n}$ of the system the system which is initially found in a pure state with a definite number of particles with fixed sets of quantum numbers $\{m\}_M = \{m_1, m_2, \ldots, m_M\}$ and $\{n\}_N = \{n_1, n_2, \ldots, n_N\}$ as follows.

(I) In ranges $\Omega_1$, $\Omega_3$ and $\Omega_5$:

$$
\hat{\rho}^{(i)}_{m\to n} = \frac{\partial^{M+N} R^{(i)}(J)}{\partial \left(J^{(i)}_{m_1} \cdots J^{(i)}_{m_M}, J^{(i)}_{n_1} \cdots J^{(i)}_{n_N}\right)}_{J=0} = |\Psi^{(i)}_{m\to n}(in)\rangle^{(i)} \langle \Psi^{(i)}_{m\to n}(in)|, \ m, n \in \Omega,
$$

where the states $|\Psi^{(i)}_{m\to n}(in)\rangle^{(i)}$ are defined as

$$
|\Psi^{(i)}_{m\to n}(in)\rangle^{(i)} = a^+_m \cdots a^+_m - a^+_n - a^+_n |0, in\rangle^{(i)},
$$

$$
|\Psi^{(i)}_{m\to n}(in)\rangle^{(i)} = -a^+_m - a^+_m - b^+_n - b^+_n |0, in\rangle^{(i)},
$$

$$
|\Psi^{(i)}_{m\to n}(in)\rangle^{(i)} = b^+_m + b^+_m - b^+_n - b^+_n |0, in\rangle^{(i)}.
$$

(II) In the ranges $\Omega_2$ and $\Omega_4$:

$$
\hat{\rho}^{(i)}_{n} = \frac{\partial^{N} R^{(i)}(J)}{\partial \left(J^{(i)}_{n_1} \cdots J^{(i)}_{n_N}\right)}_{J=0} = |\Psi^{(i)}_{n}(in)\rangle^{(i)} \langle \Psi^{(i)}_{n}(in)|, \ n \in \Omega,
$$

with the states $|\Psi^{(i)}_{n}(in)\rangle^{(i)}$ having a form

$$
|\Psi^{(i)}_{n}(in)\rangle^{(i)} = a^+_n \cdots a^+_n |0\rangle^{(2)}, \ |\Psi^{(i)}_{n}(in)\rangle^{(i)} = b^+_n \cdots b^+_n |0\rangle^{(4)}.
$$

5 Normal form of generating functional

The problem of calculating the mean value $F(out)$ of an operator $F(out)$ at the final state of the system is related to the problem of calculating the quantity $\text{tr} \left[ \hat{F}(out) \hat{\rho} \right]$, which is

$$
\text{tr} \left[ \hat{F}(out) \hat{\rho} \right] = \sum_{M,N=0}^{\infty} \sum_{M!N!} \langle \Psi^{(i)}_{m\to n}(out) \rangle \hat{F}(out) \hat{\rho} \langle \Psi^{(i)}_{m\to n}(out) \rangle, 
$$

$$
|\Psi^{(i)}_{m\to n}(out)\rangle = a^+_m \cdots a^+_m b^+_n \cdots b^+_n |0, out\rangle.
$$

For this reason, it is convenient to have the expression for generating functionals $R^{(i)}(J)$ (and, subsequently, for density operators $\hat{\rho}$) in terms of out-set of creation-annihilation operators. According to (45),

$$
R(J) = VU(J)V^\dagger,
$$

where $U(J)$ are operators $R(J)$ with creation-annihilation operators from in-set replaced by corresponding operators from out-set. Taking into account Eq. (44) and the fact that evolution operator $V$ can also be factorized as $[10]

$$
V = \otimes \prod_{i=1}^{5} V^{(i)},
$$

we can write that

$$
R^{(i)}(J) = V^{(i)} U^{(i)}(J) V^{(i)^\dagger},
$$

where $U^{(i)}(J)$ are operators $R^{(i)}(J)$ with creation-annihilation operators from in-set replaced by corresponding operators from out-set for each range $\Omega_i$. Utilizing the explicit forms of operators $V^{(i)}$, $i = 1, 3, 5$ found in Ref. [10], we
can construct the expression for generating functionals $R^{(1)}(J)$ in terms of out-set. It should be noted that the unitary
evolution operators $V^{(1)}$ have the same functional form in terms of in- and out-sets of operators of particle creation
and annihilation due to the properties (14), (15).

(1) Ranges $\Omega_1$ and $\Omega_5$

In $\Omega_1$ the partial evolution operator $V^{(1)} = \prod_{n \in \Omega_1} V^{(1)}_n$ has the form:

$$
V^{(1)}_n = \exp \left[ +a^\dagger_n S_4 - a_n \right] \exp \left[ -a^\dagger_n S_3 - a_n \right] \exp \left[ +a^\dagger_n S_2 + a_n \right] \exp \left[ -a^\dagger_n S_1 + a_n \right],
$$

$$
S_4 = g(-|+)^{-1}, \quad S_3 = \ln \left[ g(-|) g(+-) \right], \quad S_2 = \ln \left[ g(+|+) g(+-)^{-1} \right], \quad S_1 = -g(-|+)^{-1}.
$$

(52)

In $\Omega_5$ the operator $V^{(5)} = \prod_{n \in \Omega_5} V^{(5)}_n$ is

$$
V^{(5)}_n = \exp \left[ +b^\dagger_n S'_4 - b_n \right] \exp \left[ -b^\dagger_n S'_3 - b_n \right] \exp \left[ +b^\dagger_n S'_2 + b_n \right] \exp \left[ -b^\dagger_n S'_1 + b_n \right],
$$

$$
S'_4 = -g(+-)^{-1}, \quad S'_3 = \ln \left[ g(+-)^{-1} g(+|+) \right], \quad S'_2 = \ln \left[ g(+|+) g(+-)^{-1} \right], \quad S'_1 = g(+|+)^{-1}.
$$

(53)

Then, one can write that

$$
R^{(1)}_n = V^{(1)}_n U^{(1)}_n V^{(1)}_n^\dagger,
$$

(54)

where $U^{(1)}_n$ is the operator that can be obtained from $R^{(1)}_n$ by simultaneous replacements $+a_n \rightarrow +a_n$ and $-a_n \rightarrow -a_n$.
Similarly we have that

$$
R^{(5)}_n = V^{(5)}_n U^{(5)}_n V^{(5)}_n^\dagger,
$$

(55)

where $U^{(5)}_n$ is the operator that can be obtained from $R^{(5)}_n$ by simultaneous replacements $-b_n \rightarrow -b_n$ and $+b_n \rightarrow +b_n$.

Let us calculate the normal form of the operator $\tilde{V}_n^{(1)}$. This can be done using the relation (163). Utilizing the anticommutation relations for the creation-annihilation operators (7), one can find that

$$
ex + a^\dagger_n S_2 + a_n \exp \left[ -a^\dagger_n S_1 + a_n \right] = \exp \left[ -a^\dagger_n S_1 e^{-S_2} + a_n \right] \exp \left[ +a^\dagger_n S_2 + a_n \right],
ex - a^\dagger_n S_3 - a_n \exp \left[ -a^\dagger_n S_1 e^{-S_2} + a_n \right] = \exp \left[ -a^\dagger_n S_1 e^{-S_2} + a_n \right] \exp \left[ -a^\dagger_n S_3 - a_n \right].
$$

(56)

Then partial operators $V^{(1)}_n$ with the help of relation (163) can be presented as

$$
V^{(1)}_n = Y_n \tilde{V}_n, \quad Y_n = \exp \left[ +a^\dagger_n S_4 - a_n \right] \exp \left[ -a^\dagger_n e^{S_3} S_1 e^{-S_2} + a_n \right],
$$

$$
\tilde{V}_n = :\exp \left[ +a^\dagger_n \left( e^{S_2} - 1 \right) + a_n + -a^\dagger_n \left( e^{S_3} - 1 \right) - a_n : \right].
$$

(57)

Using the relation (166), one can present the product $\tilde{V}_n U^{(1)}_n \tilde{V}_n^\dagger$ as follows:

$$
\tilde{V}_n U^{(1)}_n \tilde{V}_n^\dagger = :\exp \left[ +a^\dagger_n A_{++} + a_n + -a^\dagger_n A_{--} - a_n : \right], \quad A_{++} = J^{(1)}_{n, ++} \left| e^{S_2} \right|^2 - 1, \quad A_{--} = J^{(1)}_{n, --} \left| e^{S_3} \right|^2 - 1.
$$

(58)

On the other hand, with the help of relation (166), the operator $Y_n$ can be presented as

$$
Y_n = :\exp \left[ +a^\dagger_n S_4 - a_n + -a^\dagger_n \tilde{S}_1 + a_n + +a^\dagger_n S_4 \tilde{S}_1 + a_n : \right], \quad \tilde{S}_1 = e^{S_3} S_1 e^{-S_2},
$$

(59)

Then one can calculate $Y_n \tilde{V}_n U^{(1)}_n \tilde{V}_n^\dagger$ to be

$$
Y_n \tilde{V}_n U^{(1)}_n \tilde{V}_n^\dagger = :\exp \left[ +a^\dagger_n B_{++} - a_n + -a^\dagger_n B_{--} + a_n + +a^\dagger_n B_{++} + a_n + -a^\dagger_n B_{--} - a_n : \right], \quad B_{++} = A_{++} + (1 + A_{++}) S_4 \tilde{S}_1, \quad B_{--} = A_{--}, \quad B_{+-} = S_4 (1 + A_{--}), \quad B_{-+} = (1 + A_{++}) \tilde{S}_1.
$$

(60)

Finally, we can attach the last remaining operator $Y^n_{-+}$ from the right side, to obtain

$$
\tilde{F}^{(1)}_n = :\exp \left[ +a^\dagger_n C_{+-} - a_n + -a^\dagger_n C_{-+} + a_n + +a^\dagger_n C_{+-} + a_n + -a^\dagger_n C_{-+} - a_n : \right], \quad C_{++} = S_4 \tilde{S}_1 + B_{+-} S_4 + B_{++} \left( 1 + S_4 \tilde{S}_1 \right), \quad C_{--} = B_{-+} + B_{--} \tilde{S}_1, \quad C_{-+} = \tilde{S}_1 + B_{-+} + B_{++} \tilde{S}_1.
$$

(61)
Substituting $B_{\zeta\zeta'}$ and $A_{\pm\pm}$ into the latter expression, we find the explicit form of the operator $R^{(1)}$ in terms of out-operators to be

$$C_{++} = -1 + J_{n,n}^{(1)} |S_1|^2 |e_{S_1}|^2 + J_{n,n}^{(1)} \left( 1 + S_4 \tilde{S}_1 \right) \left( 1 + S_4 \tilde{S}_1 \right) |e_{S_2}|^2,$$

$$C_{--} = -1 + J_{n,n}^{(1)} |e_{S_1}|^2 + J_{n,n}^{(1)} \left| \tilde{S}_1 \right|^2 |e_{S_2}|^2,$$

$$C_{+-} = J_{n,n}^{(1)} S_4 \left( 1 + S_4 \tilde{S}_1 \right) |e_{S_2}|^2 + J_{n,n}^{(1)} S_4^* \left| e_{S_1} \right|^2,$$

$$C_{-+} = J_{n,n}^{(1)} S_4 \left( 1 + S_4 \tilde{S}_1 \right) |e_{S_2}|^2.$$

(62)

In ranges $\Omega_3$ and $\Omega_5$ the matrices $g$ are connected\textsuperscript{[10]} to the relative amplitudes of an electron/positron reflection $R_{\pm}$ and transmission $T_{\pm}$ as follows,

$$R_{+,n} = g \left( |+| \right)^{-1} G \left( |+| \right)^{-1}, \quad T_{+,n} = \eta \left( |+| \right)^{-1},$$

$$R_{-,n} = g \left( |+| \right)^{-1} G \left( |+| \right)^{-1}, \quad T_{-,n} = -\eta R \left( + | \right)^{-1}. \quad (63)$$

One can these definitions and the properties of matrices $g$ given by \textsuperscript{[22]} to present the coefficients from Eq. (62) as

$$1 + S_4 \tilde{S}_1 = 1 - |g \left( |+| \right)|^{-2} |g \left( |+| \right)|^2 = 1 - |g \left( |+| \right)|^2 = |R_{+,n}|^2,$$

$$|S_4|^2 = |g \left( |+| \right)|^2 = |T_{+,n}|^2 / |R_{+,n}|^2,$$

$$|e_{S_1}|^2 = |g \left( |+| \right)|^2 |g \left( |+| \right)|^2 = |R_{+,n}|^2,$$

$$|e_{S_2}|^2 = |g \left( |+| \right)|^2 |g \left( |+| \right)|^2 = |R_{+,n}|^2.$$

Thus coefficients $C$ in Eq. (62) take the following compact form,

$$C_{++} = -1 + J_{n,n}^{(1)} |T_n|^2 + J_{n,n}^{(1)} |R_n|^2, \quad C_{--} = -1 + J_{n,n}^{(1)} |R_n|^2 + J_{n,n}^{(1)} |T_n|^2,$$

$$C_{+-} = J_{n,n}^{(1)} g \left( |+| \right)^{-1} |R_n|^2 - J_{n,n}^{(1)} g \left( |+| \right)^{-1} |T_n|^2,$$

$$C_{-+} = J_{n,n}^{(1)} g \left( |+| \right)^{-1} |R_n|^2 - J_{n,n}^{(1)} g \left( |+| \right)^{-1} |T_n|^2,$$

(65)

where we introduced notations

$$|T_n|^2 = |T_{-,n}|^2 = |T_{+,n}|^2, \quad |R_n|^2 = |R_{-,n}|^2 = |R_{+,n}|^2,$$

(66)

where $|T_n|^2$ and $|R_n|^2$ are the absolute probability of electron transmission and the absolute probability of electron reflection correspondingly, so that $|T_n|^2 + |R_n|^2 = 1$.

The normal form of the operator $R^{(5)}$ can be constructed in the same manner, and has the form

$$R^{(5)} = \exp \left[ + b_n^1 D_{++} + b_n^1 + b_n^1 D_{+-} - b_n + b_n D_{++} + b_n + b_n D_{+-} - b_n \right]:,$$

$$D_{++} = -1 + J_{n,n}^{(1)} |T_n|^2 + J_{n,n}^{(1)} |R_n|^2,$$

$$D_{-+} = -1 + J_{n,n}^{(1)} |R_n|^2 + J_{n,n}^{(1)} |T_n|^2,$$

$$D_{++} = -1 + J_{n,n}^{(1)} |T_n|^2 + J_{n,n}^{(1)} |R_n|^2,$$

where $n \in \Omega_3$.

(II) Range $\Omega_3$

The unitary evolution operator $V^{(3)}$ has the form

$$V^{(3)} = \prod_{n \in \Omega_3} V^{(3)}_n,$$

$$v^{(3)}_4 = \exp \left\{ + b_n^1 w_n \left( |+| \right) + b_n^1 \right\}, \quad v^{(3)}_3 = \exp \left\{ + b_n^1 \ln w_n \left( |+| \right) + b_n \right\},$$

$$v^{(3)}_2 = \exp \left\{ + a_n \ln w_n \left( |+| \right) + a_n \right\}, \quad v^{(3)}_1 = \exp \left\{ - + b_n w_n \left( |+| \right) + a_n \right\}.$$

(68)
where \( w_n(\zeta|\zeta') \) are elementary amplitudes of scattering and pair creation processes, defined as
\[
\begin{align*}
  w_n(-|-) &= c_{\nu}^{-1} \langle 0, \text{out} | +b_n - b_n^\dagger | 0, \text{in} \rangle, \\
  w_n(+|+) &= c_{\nu}^{-1} \langle 0, \text{out} | +a_n - a_n^\dagger | 0, \text{in} \rangle, \\
  w_n(+|-|0) &= c_{\nu}^{-1} \langle 0, \text{out} | +a_n + b_n | 0, \text{in} \rangle, \\
  w_n(0|-|-) &= c_{\nu}^{-1} \langle 0, \text{out} | -b_n^\dagger - a_n^\dagger | 0, \text{in} \rangle.
\end{align*}
\] (69)

All these amplitudes are diagonal in quantum numbers due to Eq. (7) and can be expressed in terms of the coefficients \( g(\zeta|\zeta') \) as follows:
\[
\begin{align*}
  w_n(-|-) &= g(-|+) g(-|-)^{-1} = g(-|+) g(+|+)^{-1}, \\
  w_n(+|+) &= g(+|-) g(-|-)^{-1} = g(+|-) g(+|+)^{-1}, \\
  w_n(+|-|0) &= g(+|+)^{-1}, \\
  w_n(0|-|-) &= -g(-|-)^{-1}.
\end{align*}
\] (70)

Relative amplitude of pair creation, \( w_n (+|-|0) \), is also connected with the differential number of pairs created from vacuum,
\[
N_n^{cr} = |g(-|+)|^{-2} = \frac{|w_n(+-|0)|^2}{1 + |w_n(+-|0)|^2}, \quad |w_n(+-|0)|^2 = \frac{N_n^{cr}}{1 - N_n^{cr}}, \quad |c_{\nu,n}|^2 = 1 - N_n^{cr}.
\] (71)

We note the structure of operator \( V^{(3)} \) can be formally identified with the structure of the unitary evolution operator \( V \) for QED with time-dependent uniform electric potential steps [3] with a formal replacement \( a_n \rightarrow +a_n^\dagger \) and \( b_n \rightarrow +b_n^\dagger \). Thus, the normal form of the operator \( J^{(3)}_m \) can be obtained in the exactly same way as in Ref. [3], and has the form
\[
\begin{align*}
  R^{(3)}_n &= |w_n(-|-)|^{-2} (1 + AB) : \exp \left[ -+a_n^\dagger (1 - D_+) + a_n \right] , \\
  a_n^\dagger (1 - D_-) + b_n - +a_n^\dagger C^\dagger + b_n^\dagger - +a_n C + a_n , \\
  D_+ &= |w_n(+|+)|^2 (1 + AB)^{-1} J^{(3)}_m, \\
  D_- &= |w_n(-|-)|^2 (1 + AB)^{-1} J^{(3)}_m, \\
  A &= J^{(3)}_m B^* J^{(3)}_m, \\
  C &= w_n(-|-)^* A^* (1 + AB)^{-1} w_n(+|+) + w_n(+|-|0)^*.
\end{align*}
\] (72)

### 6 Reduced density operators

#### 6.1 Reduced density operators for the electron and the positron subsystems

In the general case, the states of the system under consideration at the final time instant contain both particles and antiparticles due to the pair creation by external fields and the structure of the initial state. However, we are often interested in physical quantities \( F_k \) that describe only electrons (+) or positrons (−) at the final state of the system. The corresponding operators \( F_k \) are functions of either only electron-creation-annihilation operators \( a_n, a_n^\dagger \) or positron ones \( b_n, b_n^\dagger \). The mean values of operators can be obtained from the so-called reduced density operators \( \hat{\rho}_\pm \), defined as reduced traces of the general density matrix \( \hat{\rho} \) over one of subsystems (positron or electron one, respectively):
\[
\hat{\rho}_\pm = \text{tr}_\mp \hat{\rho}, \quad \hat{\rho} = \sum_{\mp} \hat{\rho}_\mp, \quad \hat{\rho}_\mp = \prod_{n \in \Omega_i} \hat{\rho}_n^{(i)}.
\] (73)

In the latter expression, the reduced traces \( \text{tr}_\mp \) of the operator \( \hat{\rho} \) are defined as
\[
\begin{align*}
  \text{tr}_+ \hat{\rho} &= \sum_{M=0}^\infty \sum_{\{m\}} \frac{1}{M!} \langle \Psi^a_{\{m\},M} | \hat{\rho} | \Psi^a_{\{m\},M} \rangle, \\
  \text{tr}_- \hat{\rho} &= \sum_{M=0}^\infty \sum_{\{m\}} \frac{1}{M!} \langle \Psi^b_{\{m\},M} | \hat{\rho} | \Psi^b_{\{m\},M} \rangle,
\end{align*}
\] (74)

where \( |\Psi^a_{\{m\},M} \rangle \) are state vectors for electron/positron states,
\[
|\Psi^a_{\{m\},M} \rangle = a_{m_1}^\dagger \cdots a_{m_M}^\dagger |0, \text{out} \rangle_a, \quad |\Psi^b_{\{m\},M} \rangle = b_{m_1}^\dagger \cdots b_{m_M}^\dagger |0, \text{out} \rangle_b.
\] (75)

Here \( |0, \text{out} \rangle_a \) and \( |0, \text{out} \rangle_b \) are partial electron and positron vacua. Note that in ranges \( \Omega_1 \) and \( \Omega_2 \) (where there are only electron states exist) \( |0, \text{out} \rangle_a^{(1,2)} = |0, \text{out} \rangle^{(1,2)} \); similarly to this, in ranges \( \Omega_4 \) and \( \Omega_5 \) we have \( |0, \text{out} \rangle_b^{(4,5)} = |0, \text{out} \rangle^{(4,5)} \).
In Klein zone, $\Omega_3$, where exist both electron and positron states, the total vacuum is a product of electron and positron partial vacua, $|0, \text{out}\rangle^+(a) \otimes |0, \text{out}\rangle^+(b) = |0, \text{out}\rangle^+(3)$. Every partial electron and positron vacuum can be in turn presented as a product in quantum modes,

$$
|0, \text{out}\rangle_{a(n)}^{(i)} = \prod_{n \in \Omega_i} |0, \text{out}\rangle_{a(n)}^{(i)} , \quad |0, \text{out}\rangle_{b(n)}^{(i)} = \prod_{n \in \Omega_i} |0, \text{out}\rangle_{b(n)}^{(i)} .
$$

Due to this reason it is obvious enough that the reduced trace $\text{tr}_+$ completely traces out partial density operators $\hat{\rho}^{(1,2)}_+$ and leaves partial operators $\hat{\rho}^{(4,5)}$ unaffected. In the same manner the reduced trace $\text{tr}_-$ traces out operators $\hat{\rho}^{(4,5)}$ and leaves $\hat{\rho}^{(1,2)}$ unchanged. Therefore, the reduced density operators $\hat{\rho}_\pm$ can be presented as

$$
\hat{\rho}_+^{(3)} = \sum_{M=0}^{\infty} \sum_{\{m\}} (M!)^{-1} b_M |0, \text{out}\rangle + b_{m_1} \hat{\rho}^{(3)} + b_{m_1}^\dagger \cdots + b_{m_M}^\dagger |0, \text{out}\rangle ,
\hat{\rho}_-^{(3)} = \sum_{M=0}^{\infty} \sum_{\{m\}} (M!)^{-1} a_M |0, \text{out}\rangle + a_{m_1} \hat{\rho}^{(3)} + a_{m_1}^\dagger \cdots + a_{m_M}^\dagger |0, \text{out}\rangle ,
$$

The reduced density operators $\hat{\rho}_\pm^{(3)}$ can be obtained from the reduced generating functionals $R^{(3)}_\pm$:

$$
R^{(3)}_\pm = \text{tr}_\pm R^{(3)} ,
$$

where the partial are defined in the same way as in Eq. (77).

Using the path integral representation for traces in representation $C$ for $R^{(3)}$, we obtain that

$$
R^{(3)}_+ = \prod_{n \in \Omega_3} \left[Z_{+,n}^{(3)}\right]^{-1} : \exp \left[ - a_n^\dagger (1 - K_+(J)) + a_n \right] : ,

\quad R^{(3)}_- = \prod_{n \in \Omega_3} \left[Z_{-,n}^{(3)}\right]^{-1} : \exp \left[ - b_n^\dagger (1 - K_-(J)) + b_n \right] : ,

K_\pm(J) = D_\pm + C^\dagger (1 + D_\mp)^{-1} C , \quad \left[Z_{\pm,n}^{(3)}\right]^{-1} = |w_n(-|-)|^{-2} (1 + AB) (1 + D_\mp) .
$$

Choosing appropriate sources $J$ in the same manner as it was done in Section 4 we can obtain the corresponding partial density operators $\hat{\rho}_\pm^{(3)}$ for different initial conditions.

### 6.2 Measurement induced reduction

We can also consider a reduction of density operators, which occurs due to measurement of a physical quantity, namely, the number of final particles, by some classical tool. This kind of reduction can also occur due to some decoherence processes, such as collisions with some external sources (e.g. with impurities in the graphene). For us, there is no difference which of the mechanisms is implemented, so in what follows we talk about an intermediate measurement by a classical tool as a source of the decoherence.

We study the measurement induced deformation of density matrix for two other initial conditions, namely, when the initial state of the system is a pure state with fixed number of particles with known $n$, and when the system initially is in a thermal equilibrium. Suppose that we are measuring the number of final particles (electrons or positrons) $N$ in the state $\hat{\rho}$ of the system under consideration. According to von Neumann [18], the density operator $\hat{\rho}$ after this measurement is reduced to the operator $\hat{\rho}_N$ of a form

$$
\hat{\rho}_N = \sum_{\{s\}} W_s \hat{P}_s , \quad \hat{P}_s = |s, \text{out}\rangle \langle s, \text{out}| , \quad W_s = \langle s, \text{out}| \hat{\rho} |s, \text{out}\rangle ,
$$

where $|s, \text{out}\rangle$ are eigenstates of the operator $\hat{N}$ with the eigenvalues $s$ that represent the total number of electrons and
Positrons in the state $|s, \text{out}\rangle$,

$$\hat{N}(\text{out})|s, \text{out}\rangle = s|s, \text{out}\rangle, \ |s, \text{out}\rangle = \prod_{n \in \Omega_1} [a_n^+ + a_n]^{l_{n,1}} [a_n + a_n^+]^{k_{n,1}} \prod_{n \in \Omega_2} [a_n^+ + a_n]^{l_{n,2}}$$

$$\times \prod_{n \in \Omega_3} [b_n^+ + b_n]^{l_{n,4}} \prod_{n \in \Omega_5} [a_n + a_n^+]^{l_{n,5}} [b_n + b_n^+]^{k_{n,5}} |0, \text{out}\rangle,$$

$$s = \sum_{n \in \Omega_1} (l_{n,1} + k_{n,1}) + \sum_{n \in \Omega_2} (l_{n,2}) + \sum_{n \in \Omega_4} (l_{n,4}) + \sum_{n \in \Omega_5} (l_{n,5} + k_{n,5}) + \sum_{n \in \Omega_5} (l_{n,3} + k_{n,3}). \quad (81)$$

Note that for Dirac particles $l_{n,i}, k_{n,i} = (0, 1)$. It is convenient to introduce partial density operators for each range $\Omega_i$:

$$\hat{\rho}_N^{(i)} = \prod_{n \in \Omega_i} \hat{\rho}_{N,n}^{(i)} \hat{\rho}_{N,n}^{(i)} = \sum_{\{s_i\}} W_{s,n}^{(i)} \hat{P}_{s,n}^{(i)},$$

$$\hat{P}_{s,n}^{(i)} = |s_i, \text{out}\rangle_n^{(i)} \langle s_i, \text{out}|, \ W_{s,n}^{(i)} = \langle i | s_i, \text{out}\rangle \hat{\rho}_s^{(i)} |s_i, \text{out}\rangle^{(i)}.$$

This way the general density operator of the system can be presented as

$$\hat{\rho}_N = \otimes_{i=1}^5 \hat{\rho}_N^{(i)}. \quad (83)$$

The state vectors $|s_i, \text{out}\rangle_n^{(i)}$ introduced in Eq. (82) are different for each range $\Omega_i$:

$$|s_1, \text{out}\rangle_n^{(1)} = [a_n^+ + a_n]^{l_{n,1}} [a_n + a_n^+]^{k_{n,1}} |0, \text{out}\rangle_n^{(1)}, \ s_1 = l_{n,1} + k_{n,1},$$

$$|s_3, \text{out}\rangle_n^{(3)} = [a_n^+ + a_n]^{l_{n,3}} [b_n^+ + b_n]^{k_{n,3}} |0, \text{out}\rangle_n^{(3)}, \ s_3 = l_{n,3} + k_{n,3},$$

$$|s_5, \text{out}\rangle_n^{(5)} = [b_n^+ + b_n]^{l_{n,5}} [a_n + a_n^+]^{k_{n,5}} |0, \text{out}\rangle_n^{(5)}, \ s_5 = l_{n,5} + k_{n,5},$$

$$|s_2, \text{out}\rangle_n^{(2)} = [b_n^+ + b_n]^{l_{n,2}} |0\rangle_n^{(2)}, \ |s_4, \text{out}\rangle_n^{(4)} = [b_n + b_n^+]^{l_{n,4}} |0\rangle_n^{(4)}, \ s_2/4 = l_{n,2/4}. \quad (84)$$

The sum of all eigenvalues equals to the total number of particles in the state $|s, \text{out}\rangle$, i.e. $\sum_{i=1}^5 \sum_{n \in \Omega_i} s_i = s$.

In what follows we also use the following notation for partial vacuum projectors for Klein zone:

$$P_{v,n}^{(3)}(\text{in}) = |0, \text{in}\rangle_n^{(3)} \langle 0, \text{in}|, \ P_{v,n}^{(i)}(\text{out}) = |0, \text{out}\rangle_n^{(i)} \langle 0, \text{out}|. \quad (85)$$

(I) Initial vacuum state

Vacuum states in ranges $\Omega_{1,2,4,5}$ remain vacuum and the measurement of number of particles does not deform the partial density operators with vacuum initial conditions in these ranges. It is easy to show that in Klein zone $\Omega_3$ initial vacuum state evolves as

$$|0, \text{in}\rangle_n^{(3)} = c_{v,n} \left[1 - a_n^+ w_n (+ - |0\rangle + b_n^+) \right]|0, \text{out}\rangle_n^{(3)}. \quad (86)$$

The corresponding partial density operator $\hat{\rho}_{v,n}^{(3)}$ with initial vacuum state can be written as

$$\hat{\rho}_{v,n}^{(3)} = P_{v,n}^{(3)}(\text{in}) = |c_{v,n}|^2 \left[1 - a_n^+ w_n (+ - |0\rangle + b_n^+) \right] P_{v,n}^{(3)}(\text{out}) \left[1 - b_n w_n (+ - |0\rangle + a_n^+) \right]. \quad (87)$$

Performing reduction procedure \cite{r}, we obtain

$$\hat{\rho}_{v,n}^{(3)} = |c_{v,n}|^2 P_{v,n}^{(3)}(\text{out}) + |c_{v,n}|^2 w_n (+ - |0\rangle)^2 + a_n^+ b_n^+ P_{v,n}^{(3)}(\text{out}) + b_n^+ a_n + b_n + a_n. \quad (88)$$

The first term of this expression corresponds to the situation where we find a vacuum state after the measurement, and the second one corresponds to the situation where we find the state with electron-positron pair. Coefficients $|c_{v,n}|^2$ and $|c_{v,n}|^2 w_n (+ - |0\rangle)^2$ are classical probabilities for each of the outcomes.

(II) Pure states with definite number of initial particles

First, we consider the initial state with definite number of particles with fixed quantum numbers. It should be stressed that the fact that quantum modes evolve separately substantially simplifies the technical side of the
consideration. Suppose that the initial particles are present in only one quantum mode \( m \in \Omega_i \). In this case, the partial density operator \( \hat{\rho}^{(i)} \) for the range \( \Omega_i \) can be presented as

\[
\hat{\rho}^{(i)} = \hat{\rho}^{(i)}_m \otimes \prod_{n \neq m \in \Omega_i} \hat{\rho}^{(i)}_{v,n},
\]

where \( \hat{\rho}^{(i)}_m \) is the partial density operator for the quantum mode \( m \) corresponding to the initial state with the definite number of particles in question. Due to the structure of operator \( \hat{\rho}^{(i)} \) given by Eq. (89), the operator \( \hat{\rho}^{(i)}_N \) takes the form

\[
\hat{\rho}^{(i)}_N = \hat{\rho}^{(i)}_{N,m} \otimes \prod_{n \neq m \in \Omega_i} \hat{\rho}^{(i)}_{N,n}.
\]

One can see that the procedure of reduction for the case under consideration differs from the deformation of vacuum initial state only in the quantum mode \( m \), where initial particles are present. One can also see that it is not difficult to generalize the consideration for the case when initial particles are present in more than one quantum mode.

Let us first consider the deformations for the range \( \Omega_1 \). It is easy to verify that the only non-zero weights \( W^{(1)}_{s,n} \), \( n \neq m, \) in Eq. (90) are those where \( |s_i, \text{out}^{(1)}_m \rangle = |0, \text{out}^{(1)}_m \rangle \), i.e. partial density operators for vacuum quantum modes do not change due to measurement of number of particles; it is possible to write that

\[
\hat{\rho}^{(1)}_N = \hat{\rho}^{(1)}_{N,m} \otimes \prod_{n \neq m \in \Omega_i} \hat{\rho}^{(1)}_{v,n}.
\]

Then, all that is left is to deal with the quantum mode \( m \) where initial particles are present.

Suppose there is a single initial electron \( -a_m^\dagger \) in mode \( m \). Using the linear canonical transformation connecting in- and out-operators (14), definitions of reflection and transmission coefficients (15) and the connection between in- and out-vacua given by Eq. (14), in terms of out-set of creation-annihilation operators we find that in this case

\[
- a_m^\dagger \, |0, \text{in}^{(1)}_m \rangle = (T_{-m} - a_m^\dagger + R_{-m}^* + a_m^\dagger) \, |0, \text{out}^{(1)}_m \rangle,
\]

\[
\hat{\rho}^{(1)}_m = (T_{-m} - a_m^\dagger + R_{-m}^* + a_m^\dagger) \, \hat{\rho}^{(1)}_{v,m} (T^*_{-m} - a_m + R^*_{-m} + a_m).
\]

Similar result can be obtained for a state with a single \( +a_m^\dagger \)-electron:

\[
+ a_m^\dagger \, |0, \text{in}^{(1)}_m \rangle = (R_{+m} - a_m^\dagger + T_{+m}^* + a_m^\dagger) \, |0, \text{out}^{(1)}_m \rangle,
\]

\[
\hat{\rho}^{(1)}_m = (R_{+m} - a_m^\dagger + T_{+m}^* + a_m^\dagger) \, \hat{\rho}^{(1)}_{v,m} (R^*_{+m} - a_m + T^*_{+m} + a_m).
\]

The initial state with both left and right electrons transforms as

\[
+ a_m^\dagger - a_m^\dagger \, |0, \text{in}^{(1)}_m \rangle = (R_{+m} - a_m^\dagger + T_{+m}^* + a_m^\dagger) (T_{-m} - a_m^\dagger + R_{-m}^* + a_m^\dagger) |0, \text{out}^{(1)}_m \rangle,
\]

and the corresponding partial density operator \( \hat{\rho}^{(1)}_m \) is easy to write down, but it looks rather cumbersome, so its explicit form is omitted. Note that the same results can be obtained by applying the procedure described in Eq. (46) to the normal form of generating functional \( \mathcal{F}^{(1)} \), given by Eq. (62).

Substituting density operators \( \hat{\rho}^{(1)}_m \) into Eq. (90) and utilizing relations (63) and (66), we can show that operators \( \hat{\rho}^{(1)}_{N,m} \) are

\[
\hat{\rho}^{(1)}_{N,m} = |T_{-m}|^2 - a_{m}^\dagger \hat{\rho}^{(1)}_{v,m} - a_m + |R_{-m}|^2 + a_m^\dagger \hat{\rho}^{(1)}_{v,m} + a_m \quad \text{for (92)},
\]

\[
\hat{\rho}^{(1)}_{N,m} = |R_{+m}|^2 - a_{m}^\dagger \hat{\rho}^{(1)}_{v,m} - a_m + |T_{+m}|^2 + a_m^\dagger \hat{\rho}^{(1)}_{v,m} + a_m \quad \text{for (93)},
\]

\[
\hat{\rho}^{(1)}_{N,m} = \left[ |R_{m}|^2 + |T_{m}|^2 \right]^2 + a_{m}^\dagger \hat{\rho}^{(1)}_{v,m} - a_m^\dagger \hat{\rho}^{(1)}_{v,m} - a_m + a_m \quad \text{for (94)}
\]

From the latter expression one can see that for the case of a single initial electron we have two terms: the first term corresponds to reflection of initial electron, while the second term corresponds to transition of the electron through the barrier. Considering two electrons in the initial state, we see that they can either be simultaneously reflected or simultaneously transmitted through the barrier.
In a similar way and with the same result one can consider the range $\Omega_5$.

In ranges $\Omega_2$, $\Omega_3$ and $\Omega_4$ all initial particles are subjected to total reflection; due to this reason partial density operators for modes with one initial particle (electron or positron) are simple enough and have the form

\[
\hat{\rho}^{(2)}_n = a_n^\dagger \hat{\rho}^{(2)}_v a_n \quad \text{if } n \in \Omega_2, \quad \hat{\rho}^{(4)}_n = b_n^\dagger \hat{\rho}^{(4)}_v b_n \quad \text{if } n \in \Omega_4,
\]
\[
\hat{\rho}^{(3)}_n = -a_n^\dagger P^{(3)}_{v,n}(\text{in}) - a_n = -a_n^\dagger P^{(3)}_{v,n}(\text{out}) + a_n, \quad n \in \Omega_3,
\]
\[
\hat{\rho}^{(3)}_n = -b_n^\dagger P^{(3)}_{v,n}(\text{in}) - b_n = +b_n^\dagger P^{(3)}_{v,n}(\text{out}) + b_n, \quad n \in \Omega_3. \tag{96}
\]

It is easy to show that the measurement of number of particles does not deform them. The only exception is the case when we have an initial electron-positron pair in mode $m \in \Omega_3$. In this situation, one can show, using Eqs. (14), (20), and (58) and relations (70) and (22), that the initial state evolves as follows:

\[
-w_m^\dagger b_m^\dagger |0, \text{in}\rangle^{(3)}_m = w_m^\ast (+|+, -|0\rangle \ |0, \text{out}\rangle^{(3)}_m. \tag{97}
\]

In this expression, the first term on the right-hand side of the equation is the state vector corresponding to the situation when both initial particles are reflected from the potential step, and the second is the vacuum state vector corresponding to the situation when the initial pair annihilates. The corresponding probabilities of pair reflection and annihilation are

\[
|w_n(++)|^{-2} = |w_n(--)|^{-2} = |c_{v,n}|^2 \quad \text{for pair scattering},
\]
\[
|w_n(++)|^{-2} |w_n(+ - 0)|^2 = |c_{v,n}|^2 |w_n(+ - 0)|^2 \quad \text{for pair annihilation}. \tag{98}
\]

Therefore, the partial density operator reduced by measurement of number of particles with initial pair in mode $m \in \Omega_3$ has the form

\[
\hat{\rho}^{(3)}_{\beta,v,m} = |c_{v,m}|^2 |w_m(+ - 0)|^2 P^{(3)}_{v,m}(\text{out}) + |c_{v,m}|^2 + a_m^\dagger b_m^\dagger P^{(3)}_{v,m}(\text{out}) + b_m + a_m. \tag{99}
\]

(III) Initial thermal state

Using the normal form of the generating functionals $R^{(i)}$, we can consider the measurement-induced reduction for thermal initial state of the system. The partial density operators $\hat{\rho}^{(i)}$ are obtained from the generating functionals $R^{(i)}$ by setting the sources $J$ as in Eq. (85). The non-vanishing weights $W^{(i)}$ from Eq. (82) are the following:

Range $\Omega_1$:

\[
W^{(1)}_{1,n} = (1) \langle n | 0, \text{out} \rangle^2 \hat{\rho}^{(1)}_{\beta,n} |0, \text{out}\rangle^{(1)}_n = \left[ Z^{(1)}_n \right]^{-1},
\]
\[
W^{(1)}_{2,n} = (1) \langle n | 0, \text{out} \rangle^2 a_n \hat{\rho}^{(1)}_{\beta,n} + a_n^\dagger |0, \text{out}\rangle^{(1)}_n = \left[ Z^{(1)}_n \right]^{-1} \hat{C}_{++}, \quad \hat{C}_{++} = 1 + C_{++},
\]
\[
W^{(1)}_{3,n} = (1) \langle n | 0, \text{out} \rangle^2 a_n \hat{\rho}^{(1)}_{\beta,n} - a_n^\dagger |0, \text{out}\rangle^{(1)}_n = \left[ Z^{(1)}_n \right]^{-1} \hat{C}_{--}, \quad \hat{C}_{--} = 1 + C_{--},
\]
\[
W^{(1)}_{4,n} = (1) \langle n | 0, \text{out} \rangle^2 a_n - a_n \hat{\rho}^{(1)}_{\beta,n} - a_n^\dagger + a_n^\dagger |0, \text{out}\rangle^{(1)}_n = \left[ Z^{(1)}_n \right]^{-1} \left[ \hat{C}_{++} \hat{C}_{--} - C_{++} C_{--} \right]. \tag{100}
\]

Range $\Omega_5$:

\[
W^{(5)}_{1} = (5) \langle 0, \text{out} \rangle^2 \hat{\rho}^{(5)}_{\beta} |0, \text{out}\rangle^{(5)}_n = \left[ Z^{(5)}_n \right]^{-1},
\]
\[
W^{(5)}_{2} = (5) \langle 0, \text{out} \rangle^2 b_n \hat{\rho}^{(5)}_{\beta,n} + b_n^\dagger |0, \text{out}\rangle^{(5)}_n = \left[ Z^{(5)}_n \right]^{-1} \hat{D}_{++}, \quad \hat{D}_{++} = 1 + D_{++},
\]
\[
W^{(5)}_{3} = (5) \langle 0, \text{out} \rangle^2 b_n \hat{\rho}^{(5)}_{\beta,n} - b_n^\dagger |0, \text{out}\rangle^{(5)}_n = \left[ Z^{(5)}_n \right]^{-1} \hat{D}_{--}, \quad \hat{D}_{--} = 1 + D_{--},
\]
\[
W^{(5)}_{4} = (5) \langle 0, \text{out} \rangle^2 b_n - b_n \hat{\rho}^{(5)}_{\beta,n} - b_n^\dagger + b_n^\dagger |0, \text{out}\rangle^{(5)}_n = \left[ Z^{(5)}_n \right]^{-1} \left[ \hat{D}_{++} \hat{D}_{--} - D_{++} D_{--} \right]. \tag{101}
\]

Range $\Omega_3$:

\[
W^{(3)}_{1} = (3) \langle 0, \text{out} \rangle^2 \hat{\rho}^{(3)}_{\beta} |0, \text{out}\rangle^{(3)}_n = \bar{Z}^{(3)}_n, \quad \bar{Z}^{(3)}_n = \left[ Z^{(3)}_n \right]^{-1} |c_{v,n}|^2 (1 + AB),
\]
\[
W^{(3)}_{2} = (3) \langle 0, \text{out} \rangle^2 a_n \hat{\rho}^{(3)}_{\beta,n} + a_n^\dagger |0, \text{out}\rangle^{(3)}_n = \tilde{Z}^{(3)}_n \hat{D}_+, \quad \tilde{Z}^{(3)}_n \hat{D}_+ = \left[ Z^{(3)}_n \right]^{-1} \hat{D}_+, \quad \hat{D}_+ = 1 + D_+,
\]
\[
W^{(3)}_{3} = (3) \langle 0, \text{out} \rangle^2 b_n \hat{\rho}^{(3)}_{\beta,n} + b_n^\dagger |0, \text{out}\rangle^{(3)}_n = \tilde{Z}^{(3)}_n D_-, \quad \tilde{Z}^{(3)}_n D_- = \left[ Z^{(3)}_n \right]^{-1} \hat{D}_-, \quad \hat{D}_- = 1 + D_-,
\]
\[
W^{(3)}_{4} = (3) \langle 0, \text{out} \rangle^2 a_n + b_n \hat{\rho}^{(3)}_{\beta,n} + b_n^\dagger + a_n^\dagger |0, \text{out}\rangle^{(3)}_n = \bar{Z}^{(3)}_n \left( \hat{D}_+ + \hat{D}_- + C_+ C_- \right). \tag{102}
\]
6.3 Spatial reduction (left and right subsystems)

In this section we reduce the generating functional \( \mathcal{F}^{(2)} \) over left and right particles and construct reduced generating functionals, which allow us to obtain reduced density operators for left and right subsystems for different initial conditions.

The \( x \)-electric potential steps provide the spatial separation of the whole system in two subsystems: left subsystem and right subsystem, i.e. final particles to the left of potential step and final particles to the right of potential step. It is easy to imagine a situation when we are interested in measuring physical values only in left and right asymptotic areas. For example, we can suppose that measuring tools are situated only to the left of potential step. In this case general density operator must be averaged (reduced) over all unavailable states of final right particles.

From the general theory\(^{[10]}\) we know the following. In range \( \Omega_1 \) all the electrons (initial and final) are located on the left side of potential step, and all the positrons are on the right side.\(^{[10]}\) Therefore, one can see that reduction over left and right subsystems in range \( \Omega_1 \) coincide with reduction over electron and positron subsystems correspondingly, i.e.

\[
\mathcal{F}_{\text{left}}^{(3)} = \mathcal{F}_+^{(3)}, \quad \mathcal{F}_{\text{right}}^{(3)} = \mathcal{F}_-^{(3)}.\]  

In range \( \Omega_2 \) there are only electrons on the left side of potential step, so \( \mathcal{F}_{\text{left}}^{(2)} = \mathcal{F}_+^{(2)} \). Similarly, there are only right positrons in range \( \Omega_4 \), therefore \( \mathcal{F}_{\text{right}}^{(4)} = \mathcal{F}_-^{(4)} \).

Range \( \Omega_1 \) contain left and right electrons; range \( \Omega_5 \) contain left and right positrons. In these ranges we need to consider the reduction of partial generating functionals over the left or right final particles. Let us start with range \( \Omega_1 \). We can use the obtained expression for \( \mathcal{F}^{(1)} \) given by Eq. (104) to calculate the partial (right and left) trace over states with right electrons, i.e. states constructed with creation-annihilation operators \( +a^\dagger, +a \) or \( -a^\dagger, -a \), thus creating spatially reduced partial generating functionals as follows:

\[
\mathcal{F}_{\text{left}}^{(1)} = \text{tr}_{\text{right}}^{(1)}, \quad \mathcal{F}_{\text{right}}^{(1)} = \text{tr}_{\text{left}}^{(1)},
\]

\[
\text{tr}_{\text{right}} \hat{A} = \sum_{M=0}^{\infty} \sum_{\{m\}} (M!)^{-1} \langle \Psi^{\text{right}}_{\{m\}_M} | A | \Psi^{\text{right}}_{\{m\}_M} \rangle,
\]

\[
\text{tr}_{\text{left}} \hat{A} = \sum_{M=0}^{\infty} \sum_{\{m\}} (M!)^{-1} \langle \Psi^{\text{left}}_{\{m\}_M} | A | \Psi^{\text{left}}_{\{m\}_M} \rangle,
\]

where \( \Psi^{\text{right}}_{\{m\}_M} \) and \( \Psi^{\text{left}}_{\{m\}_M} \) are state vectors for right and left electrons, correspondingly,

\[
\Psi^{\text{right}}_{\{m\}_M} = +a_{m_1}^\dagger \cdots +a_{m_M}^\dagger |0, \text{out}\rangle^{(1)}_{\text{right}},
\]

\[
\Psi^{\text{left}}_{\{m\}_M} = -a_{m_1}^\dagger \cdots -a_{m_M}^\dagger |0, \text{out}\rangle^{(1)}_{\text{left}},
\]

\[
|0, \text{out}\rangle^{(1)}_{\text{right}} \otimes |0, \text{out}\rangle^{(1)}_{\text{left}} = |0, \text{out}\rangle^{(1)}.
\]

Note that partial left and right electron vacua can be factorized in quantum modes \( n \), and therefore generating functionals \( \mathcal{F}^{(1)} \) can be factorized as well. Taking this into account and calculating the trace, we obtain

\[
\mathcal{F}_{\text{left}, n}^{(1)} = (1 + \hat{C}_{++}) : \exp \left\{ -a_n^\dagger L_- - a_n \right\} : L_- = C_{--} - C_{+-} \left( 1 + \hat{C}_{++} \right)^{-1} C_{+-},
\]

\[
\mathcal{F}_{\text{right}, n}^{(1)} = (1 + \hat{C}_{--}) : \exp \left\{ +a_n^\dagger L_+ + a_n \right\} : L_+ = C_{++} - C_{+-} \left( 1 + \hat{C}_{--} \right)^{-1} C_{+-}.
\]

Similar results can be obtained for the range \( \Omega_5 \):

\[
\mathcal{F}_{\text{left}, n}^{(5)} = (1 + \hat{D}_{--}) : \exp \left\{ -b_n^\dagger K_- - b_n \right\} : K_- = D_{--} - D_{+-} \left( 1 + \hat{D}_{--} \right)^{-1} D_{+-},
\]

\[
\mathcal{F}_{\text{right}, n}^{(5)} = (1 + \hat{D}_{++}) : \exp \left\{ +b_n^\dagger K_+ + b_n \right\} : K_+ = D_{++} - D_{+-} \left( 1 + \hat{D}_{++} \right)^{-1} D_{+-}.
\]

Now, setting the sources \( J \) as it was discussed in Section 4 one can obtain the density operators with different initial conditions.
7 Entropy of reduced density operators

As we know from the general theory \[14\], there are two different kind of states: pure and mixed ones. Density matrices describing pure states always provide us maximum possible information about the system, while density matrices describing mixed states provide only a part of information. Reduction over one of quantum subsystems makes a part of information unavailable. Thus, reduced density operators we have introduced in the previous section always describe mixed states even when the initial state of the system was pure.

As a measure of information loss due to reduction we use von Neumann entropy, defined as

\[
S(\hat{\rho}) = - \text{tr}\hat{\rho}\ln\hat{\rho} .
\]  
(108)

7.1 Entropy corresponding to reduction over the subsystems of electrons and positrons

Especially interesting information can be obtained calculating von Neumann entropy for reduced density matrices \([77]\),

\[
S(\hat{\rho}_\pm) = - \text{tr}\hat{\rho}_\pm\ln\hat{\rho}_\pm ,
\]  
(109)

where \(\text{tr}\) denotes the full trace of the operator, \(\text{tr}\hat{A} = \text{tr}_-\text{tr}_+\hat{A}\). Entropy corresponding to these density operators can be used as a measure of information loss due to reduction, or as a measure of entanglement \[14\] between subsystems of electrons and positrons.

Using the definitions \([77]\), let us transform the operator \(\ln\hat{\rho}_\pm\) as follows

\[
\ln\hat{\rho}_+ = \ln\hat{\rho}^{(1)} + \ln\hat{\rho}^{(2)} + \ln\hat{\rho}^{(3)}, \quad \ln\hat{\rho}_- = \ln\hat{\rho}^{(5)} + \ln\hat{\rho}^{(4)} + \ln\hat{\rho}^{(3)}.
\]  
(110)

Due to the fact that partial density matrices \(\hat{\rho}^{(i)}\) and \(\hat{\rho}^{(3)}_\pm\) are normalized (\(\text{tr}\hat{\rho}^{(i)} = \text{tr}\hat{\rho}^{(3)}_\pm = 1\)), it is easy to show that Eq. \(109\) transforms into a sum of following entropies,

\[
S(\hat{\rho}_+) = S(\hat{\rho}^{(1)}) + S(\hat{\rho}^{(2)}) + S(\hat{\rho}^{(3)}_+), \quad S(\hat{\rho}_-) = S(\hat{\rho}^{(5)}) + S(\hat{\rho}^{(4)}) + S(\hat{\rho}^{(3)}_-).
\]  
(111)

We recall that in each range \(\Omega_i\) the partial density operators \(\hat{\rho}^{(i)}\), \(\hat{\rho}^{(3)}_\pm\) can be factorized in quantum modes \(n\), and each one-mode operator is also normalized. This allows us to further simplify Eq. \(111\) expressions and write

\[
S(\hat{\rho}^{(i)}) = \sum_{n \in \Omega_i} S(\hat{\rho}_n^{(i)}) = - \sum_{n \in \Omega_i} \text{tr}\hat{\rho}_n^{(i)}\ln\hat{\rho}_n^{(i)}, \quad i = 1, 2, 4, 5,
\]

\[
S(\hat{\rho}^{(3)}_\pm) = \sum_{n \in \Omega_3} S(\hat{\rho}^{(3)}_{\pm,n}) = - \sum_{n \in \Omega_3} \text{tr}\hat{\rho}^{(3)}_{\pm,n}\ln\hat{\rho}^{(3)}_{\pm,n}.
\]  
(112)

Now, setting the appropriate sources \(J\), one can obtain the entropy for the density operators with different initial conditions.

(I) Initial vacuum state, \(J^{(i)}_\pm = 0\)

In this case partial density operators \(\hat{\rho}^{(i)}_n\), \(i = 1, 2, 4, 5\) are given by Eq. \(33\). It is easy to see that the corresponding entropies vanish, i.e. \(S(\hat{\rho}^{(1,2,4,5)}_n) = 0\). The entropies of density operators \(\hat{\rho}^{(3)}_{\pm,n}\) are equal and can be calculated\[11\] to have the form

\[
S(\hat{\rho}^{(3)}_{\pm,n}) = - [(1 - N_{\text{ct}}^n)\ln(1 - N_{\text{ct}}^n) + N_{\text{ct}}^n\ln N_{\text{ct}}^n],
\]  
(113)

where \(N_{\text{ct}}^n\) is the mean differential number of pairs created from vacuum by electric field \(57\). The full entropy can be found as a sum over all quantum numbers in \(\Omega_3\),

\[
S \left( \hat{\rho}^{(3)}_\pm \right) = - \sum_{n \in \Omega_3} [(1 - N_{\text{ct}}^n)\ln(1 - N_{\text{ct}}^n) + N_{\text{ct}}^n\ln N_{\text{ct}}^n].
\]  
(114)

(II) Initial state with definite number of particles with fixed quantum numbers \(n\):

Due to the nature of reduction over electron or positron subsection the partial density operators \(\hat{\rho}^{(i)}_n\), \(i = 1, 2, 4, 5\), either are not affected by reduction or are completely traced out. The states with definite number of initial particles with fixed quantum numbers \(n\) are pure states. This means that the corresponding entropies vanish again, \(S(\hat{\rho}^{(1,2,4,5)}_n) = 0\).

In the range \(\Omega_3\) initial electrons and positrons are subjected to total reflection \[19\], i.e. states with a single initial electron or a single initial positron remain pure states. Then, applying procedure given by Eq. \(46\) to the normal
To do so, we present the right-hand side as a formal Taylor decomposition for logarithm

\[ \hat{\rho}^{(3)}_n = |\Psi\rangle_n \langle \Psi|, \quad |\Psi\rangle_n = w_n^* (+|+\rangle)^{-1} \left[ a_n^\dagger + b_n^\dagger - w_n^* (+ - |0\rangle) \right] |0, \text{out}\rangle_n, \]

and reduced partial density operators \( \hat{\rho}^{(3)}_{\pm,n} \) can be calculated as

\[
\begin{align*}
\hat{\rho}^{(3)}_{+,n} &= |w_n(+|+\rangle)^{-2} \left[ a_n^\dagger P^{(3)}_{+,n}(\text{out}) + a_n + |w_n(+ - |0\rangle)^2 P^{(3)}_{+,n}(\text{out}) \right], \\
\hat{\rho}^{(3)}_{-,n} &= |w_n(+|+\rangle)^{-2} \left[ b_n^\dagger P^{(3)}_{-,n}(\text{out}) + b_n + |w_n(+ - |0\rangle)^2 P^{(3)}_{-,n}(\text{out}) \right],
\end{align*}
\]

where \( |w_n(+|+\rangle)^{-2} = |c_{v,n}|^2 \), and projectors \( P^{(3)}_{\pm,n}(\text{out}) \) are

\[
P^{(3)}_{+,n}(\text{out}) = |0,\text{out}\rangle^{(3)}_{a,n} a_n^{(0,\text{out})}, \quad P^{(3)}_{-,n}(\text{out}) = |0,\text{out}\rangle^{(3)}_{b,n} b_n^{(0,\text{out})}.
\]

Let us calculate the corresponding von Neumann entropies, \( S(\hat{\rho}^{(3)}_{\pm,n}) \), which is

\[ S(\hat{\rho}^{(3)}_{\pm,n}) = -\text{tr}\hat{\rho}^{(3)}_{\pm,n} \ln \hat{\rho}^{(3)}_{\pm,n}. \]

To do so, we present the right-hand side as a formal Taylor decomposition for logarithm

\[
\hat{\rho}^{(3)}_{\pm,n} \ln \hat{\rho}^{(3)}_{\pm,n} = -\sum_{k=1}^{\infty} \frac{1}{k} \hat{\rho}^{(3)}_{\pm,n} \left( 1 - \hat{\rho}^{(3)}_{\pm,n} \right)^k = -\sum_{k=1}^{\infty} \frac{1}{k} \sum_{l=0}^{k} C_k^l (-1)^l \left[ \hat{\rho}^{(3)}_{\pm,n} \right]^{l+1},
\]

where \( C_k^l \) are binomial coefficients. Note that both terms in Eq. (116) are orthogonal and normalized. For this reason we have that

\[
\begin{align*}
\hat{\rho}^{(3)}_{+,n} \ln \hat{\rho}^{(3)}_{+,n} &= |c_{v,n}|^2 l + a_n^\dagger P^{(3)}_{+,n}(\text{out}) + a_n + |c_{v,n}|^2 |w_n (+ - |0\rangle)^2 l P^{(3)}_{+,n}(\text{out}), \\
\hat{\rho}^{(3)}_{-,n} \ln \hat{\rho}^{(3)}_{-,n} &= |c_{v,n}|^2 l + b_n^\dagger P^{(3)}_{-,n}(\text{out}) + b_n + |c_{v,n}|^2 |w_n (+ - |0\rangle)^2 l P^{(3)}_{-,n}(\text{out}).
\end{align*}
\]

Substituting the latter expression into Eq. (119), collecting the Taylor series back, and calculating the trace, we obtain

\[ S(\hat{\rho}^{(3)}_{\pm,n}) = -\left[ |c_{v,n}|^2 \ln |c_{v,n}|^2 + |c_{v,n}|^2 |w_n (+ - |0\rangle)^2 \ln |c_{v,n}|^2 |w_n (+ - |0\rangle)^2 \right]. \]

Now taking into account Eq. (74), we finally find

\[ S(\hat{\rho}^{(3)}_{\pm,n}) = -\left[ (1 - N_{n,\text{in}}^{(c)}) \ln (1 - N_{n,\text{in}}^{(c)}) + N_{n,\text{in}}^{(c)} \ln N_{n,\text{in}}^{(c)} \right]. \]

One can see that this result coincide with the one obtained for the case of initial vacuum state, Eq. (113).

(III) Initial thermal state

The entropies corresponding to partial density operators \( \hat{\rho}^{(i)}_{\beta,n} \) are

\[
\begin{align*}
S(\hat{\rho}^{(1,5)}_{\beta,n}) &= -\sum_{\zeta = \pm} \left( 1 - N^{(1,5)}_{n,\beta,\zeta}(\text{in}) \right) \ln \left( 1 - N^{(1,5)}_{n,\beta,\zeta}(\text{in}) \right) + N^{(1,5)}_{n,\beta,\zeta}(\text{in}) \ln N^{(1,5)}_{n,\beta,\zeta}(\text{in}), \\
S(\hat{\rho}^{(2,4)}_{\beta,n}) &= -\left[ (1 - N^{(2,4)}_{n,\beta}(\text{in}) \ln (1 - N^{(2,4)}_{n,\beta}(\text{in})) + N^{(2,4)}_{n,\beta}(\text{in}) \ln N^{(2,4)}_{n,\beta}(\text{in}) \right], \\
S(\hat{\rho}^{(3)}_{\beta,\pm,n}) &= -\left[ (1 - N^{(3)}_{n,\beta,\pm}(\text{out}) \ln (1 - N^{(3)}_{n,\beta,\pm}(\text{out}) + N^{(3)}_{n,\beta,\pm}(\text{out}) \ln N^{(3)}_{n,\beta,\pm}(\text{out}) \right],
\end{align*}
\]

where the mean differential numbers of particles from the first two lines are given by

\[
N^{(1)}_{n,\beta,\zeta}(\text{in}) = \left( e^{E_n + 1} \right)^{-1}, \quad \zeta = \pm, n \in \Omega_{1,5}, \quad N^{(2,4)}_{n,\beta}(\text{in}) = \left( e^{E_n + 1} \right)^{-1}, \quad n \in \Omega_{2,4},
\]
and $N_{n,\beta,\pm}^{(3)}(\text{out})$ are the differential mean numbers of final electrons (+) and positrons (−) in range $\Omega_3$,

\[
N_{n,\beta,+}^{(3)}(\text{out}) = \text{tr} \hat{\rho}_{\epsilon,n,\beta,+}^{(3)} + a_n^+ a_n = N_n^{cr} \left( 1 - N_{n,\beta,-}(in) \right) + (1 - N_n^{cr}) N_{n,\beta,+}(in),
\]

\[
N_{n,\beta,-}^{(3)}(\text{out}) = \text{tr} \hat{\rho}_{\epsilon,n,\beta,-}^{(3)} + b_n^+ b_n = N_n^{cr} \left( 1 - N_{n,\beta,+}(in) \right) + (1 - N_n^{cr}) N_{n,\beta,-}(in).
\]  

(125)

The differential mean numbers $N_{n,\beta,\pm}^{(3)}(\text{in})$ in Eq. (125) can be calculated similar to Eq. (45) using the corresponding creation-annihilation operators.

### 7.2 Entropy corresponding to measurement induced reduction

The measurement reduced density operators $\hat{\rho}_N^{(i)}$ with different initial conditions are given by Eq. (52). Similarly to the previous subsection, it is easy to show that von Neumann entropy can be presented as the sum over quantum modes $n$ of partial entropies,

\[
S(\hat{\rho}_N^{(i)}) = \sum_{n \in \Omega_i} S(\hat{\rho}_N^{(i),n}) = - \sum_{n \in \Omega_i} \text{tr} \hat{\rho}_N^{(i),n} \ln \hat{\rho}_N^{(i),n}.
\]  

(126)

Therefore, to obtain the total entropy it is sufficient to calculate only the entropies $S(\hat{\rho}_N^{(i),n})$ corresponding to partial density operators $\hat{\rho}_N^{(i),n}$ and then perform the summation over all quantum numbers $n \in \Omega_i$.

(I) Initial vacuum state

Let us calculate von Neumann entropy corresponding to density operator $\hat{\rho}_N^{(3),n}$. Following the procedure (118)-(122), we can show that entropy for this case takes the form

\[
S(\hat{\rho}_N^{(3),n}) = - \left[ |c_{\epsilon,n}|^2 \ln |c_{\epsilon,n}|^2 + |c_{\epsilon,n}|^2 |w_n (+ - |0\rangle)^2 \ln |c_{\epsilon,n}|^2 |w_n (+ - |0\rangle)^2 \right],
\]

\[
|c_{\epsilon,n}|^2 = 1 - N_n^{cr}, \quad |w_n (+ - |0\rangle)^2 = N_n^{cr} (1 - N_n^{cr})^{-1},
\]

(127)

which leads us to the following result

\[
S(\hat{\rho}_N^{(3),n}) = - \left[ (1 - N_n^{cr}) \ln (1 - N_n^{cr}) + N_n^{cr} \ln N_n^{cr} \right].
\]  

(128)

(II) Initial pure states with definite number of particles

For this case $\hat{\rho}_N^{(1),n}$ is given by Eq. (95). Following the procedure (118)-(122), one can show that

\[
S(\hat{\rho}_N^{(1),n}) = - \left[ |R_n|^2 \ln |R_n|^2 + |T_n|^2 \ln |T_n|^2 \right] \quad \text{for (92) and (93)},
\]

(129)

where notations (66) have been used. Entropy for (94) vanish, i.e. $S(\hat{\rho}_N^{(1),n}) = 0$, as $\hat{\rho}_N^{(1),n}$ corresponding to Eq. (94) describe pure state, despite of the fact that a measurement has been performed in the system. Taking into account that for this case the differential numbers of final particles are

\[
N_{n,-}(\text{out}) = \text{tr} \hat{\rho}_{N,n,-}^{(1)} - a_n^+ a_n = |T_n|^2, \quad |R_n|^2 = 1 - N_{n,-}(\text{out}),
\]

\[
N_{n,+}(\text{out}) = \text{tr} \hat{\rho}_{N,n,+}^{(1)} + a_n^+ a_n = |T_n|^2, \quad |R_n|^2 = 1 - N_{n,+}(\text{out}),
\]  

(130)

we can represent entropies (129) as

\[
S(\hat{\rho}_N^{(1),n}) = - \left[ (1 - N_{n,-}(\text{out})) \ln (1 - N_{n,-}(\text{out})) + N_{n,-}(\text{out}) \ln N_{n,-}(\text{out}) \right] \quad \text{for (92)},
\]

\[
S(\hat{\rho}_N^{(1),n}) = - \left[ (1 - N_{n,+}(\text{out})) \ln (1 - N_{n,+}(\text{out})) + N_{n,+}(\text{out}) \ln N_{n,+}(\text{out}) \right] \quad \text{for (93)}.
\]

(131)

The entropies for the range $\Omega_3$ have the same form,

\[
S(\hat{\rho}_N^{(5),n}) = - \left[ (1 - N_{n,-}(\text{out})) \ln (1 - N_{n,-}(\text{out})) + N_{n,-}(\text{out}) \ln N_{n,-}(\text{out}) \right],
\]

\[
S(\hat{\rho}_N^{(5),n}) = - \left[ (1 - N_{n,+}(\text{out})) \ln (1 - N_{n,+}(\text{out})) + N_{n,+}(\text{out}) \ln N_{n,+}(\text{out}) \right],
\]

(132)

for the cases of single left and single right initial positrons.

(III) Initial thermal state
Using the latter expressions, we can represent Eq. (135) as
\[ S(\hat{\rho}_{N,n}^{(i)}) = - \sum_{i=1}^{4} W_i^{(i)} \ln W_i^{(i)}. \] (133)

Sources \( J_i \), given by in Eq. (38), for the case of thermal initial state are connected to differential mean numbers of initial particles by the following relation:
\[ J_{\pm,n}^{(i)} = e^{-R_{n,\pm}^{(i)}} = N_{n,\pm}^{(i)} (\ln (1 - N_{n,\pm}^{(i)})^{-1}. \] (134)

Using then Eqs. (63), (64), (65) and (72), it is possible to present weights \( W_i^{(i)} \) via differential mean numbers of initial particles, reflection and transition probabilities \( |T_n|^2 \), \( |R_n|^2 \), and number of particles created from vacuum \( N_n^{\text{cr}} \).

### 7.3 Entropy corresponding to spatial reduction (left and right)

We can also calculate von Neumann entropy for the left and right reduced density operators, found in Eqs. (106) and (107). For the reduced generating functionals from \( \Omega_1 \) this entropy has the form
\[
S(R_{\text{left}}^{(1)}) = -\text{tr}R_{\text{left}}^{(1)} \ln R_{\text{left}}^{(1)} = \sum_{n \in \Omega_1} \left[ \ln Z_n^{(1)} - \ln \left(1 + \tilde{C}_{++} - N_{n,-}^{(1)} \ln (1 + L_-) \right) \right],
\]
\[
S(R_{\text{right}}^{(1)}) = -\text{tr}R_{\text{right}}^{(1)} \ln R_{\text{right}}^{(1)} = \sum_{n \in \Omega_1} \left[ \ln Z_n^{(1)} - \ln \left(1 + \tilde{C}_{--} - N_{n,+}^{(1)} \ln (1 + L_+) \right) \right],
\] (135)

where \( N_{n,-}^{(1)} \) and \( N_{n,+}^{(1)} \) are the differential mean numbers of left and right final electrons in \( \Omega_1 \),
\[
N_{n,-}^{(1)} = \text{tr}R_{\text{left}}^{(1)} a_n^+ a_n = [Z_n^{(1)}]^{-1} \left[ (1 + \tilde{C}_{++}) \tilde{C}_{--} - C_{++} - C_{--} \right],
\]
\[
N_{n,+}^{(1)} = \text{tr}R_{\text{right}}^{(1)} a_n^+ a_n = [Z_n^{(1)}]^{-1} \left[ (1 + \tilde{C}_{--}) \tilde{C}_{++} - C_{++} - C_{--} \right].
\] (136)

Using the fact that reduced generating functionals \( R_{\text{left}}^{(1)} \) are normalized, \( \text{tr}R_{\text{left}}^{(1)} = 1 \), one can show that the following relations hold true:
\[
(1 + L_-) = \frac{N_{n,-}^{(1)} \text{out}}{1 - N_{n,-}^{(1)} \text{out}}, \quad (1 + L_+) = \frac{N_{n,+}^{(1)} \text{out}}{1 - N_{n,+}^{(1)} \text{out}}, \quad \frac{(1 + \tilde{C}_{++})}{Z_n^{(1)}} = (2 + L_-)^{-1}.
\] (137)

Using the latter expressions, we can represent Eq. (135) as
\[
S(R_{\text{left}}^{(1)}) = -\sum_{n \in \Omega_1} \left[ (1 - N_{n,-}^{(1)} \text{out}) \ln \left(1 - N_{n,-}^{(1)} \text{out} \right) + N_{n,-}^{(1)} \text{out} \ln N_{n,-}^{(1)} \text{out} \right],
\]
\[
S(R_{\text{right}}^{(1)}) = -\sum_{n \in \Omega_1} \left[ (1 - N_{n,+}^{(1)} \text{out}) \ln \left(1 - N_{n,+}^{(1)} \text{out} \right) + N_{n,+}^{(1)} \text{out} \ln N_{n,+}^{(1)} \text{out} \right].
\] (138)

For the reduced generating functionals from \( \Omega_5 \) the result reads
\[
S(R_{\text{left}}^{(5)}) = -\text{tr}R_{\text{left}}^{(5)} \ln R_{\text{left}}^{(5)} = \sum_{n \in \Omega_5} \left[ \ln Z_n^{(5)} - \ln \left(1 + \tilde{D}_{++} - N_{n,+}^{(5)} \text{out} \ln (1 + K_+) \right) \right],
\]
\[
S(R_{\text{right}}^{(5)}) = -\text{tr}R_{\text{right}}^{(5)} \ln R_{\text{right}}^{(5)} = \sum_{n \in \Omega_5} \left[ \ln Z_n^{(5)} - \ln \left(1 + \tilde{D}_{--} - N_{n,-}^{(5)} \text{out} \ln (1 + K_-) \right) \right],
\] (139)

where \( N_{n,+}^{(5)} \) and \( N_{n,-}^{(5)} \) are the differential mean numbers of final positrons,
\[
N_{n,+}^{(5)} \text{ out} = \text{tr}R_{\text{left}}^{(5)} b_n^+ b_n = [Z_n^{(5)}]^{-1} \left[ (1 + \tilde{D}_{++}) \tilde{D}_{++} - D_{++} D_{++} \right],
\]
\[
N_{n,-}^{(5)} \text{ out} = \text{tr}R_{\text{right}}^{(5)} b_n^+ b_n = [Z_n^{(5)}]^{-1} \left[ (1 + \tilde{D}_{--}) \tilde{D}_{++} - D_{++} D_{++} \right].
\] (140)
The entropies (139) in terms of mean differential numbers (140) take the form

\[
S(R^{(5)}_{\text{left}}) = - \sum_{n \in \Omega_5} \left[ (1 - N_{n,+}^{(5)}(\text{out})) \ln (1 - N_{n,+}^{(5)}(\text{out})) + N_{n,+}^{(5)}(\text{out}) \ln N_{n,+}^{(5)}(\text{out}) \right],
\]

\[
S(R^{(5)}_{\text{right}}) = - \sum_{n \in \Omega_5} \left[ (1 - N_{n,-}^{(5)}(\text{out})) \ln (1 - N_{n,-}^{(5)}(\text{out})) + N_{n,-}^{(5)}(\text{out}) \ln N_{n,-}^{(5)}(\text{out}) \right].
\]

(141)

### 7.4 Loss of information due to electron-positron reduction in L-constant field

Here we illustrate some of the obtained formulas by considering the deformation of the quantum vacuum between two infinite capacitor plates separated by a finite distance \(L\). Some aspects of particle creation by the constant electric field between such plates (this field is also called \(L\)-constant electric field) were studied in Ref. [19]. The latter field is a particular case of \(x\)-electric potential step. Thus, we consider the \(L\)-constant electric field in \(d = D + 1\) dimensions. We chose \(E(x) = (E^i, \ i = 1, ..., D)\), \(E^1 = E_e(x)\), \(E^2, ..., D = 0\),

\[
E_e(x) = \begin{cases} 
0, & x \in (-\infty, -L/2] \n E = \text{const} > 0, & x \in (-L/2, L/2) 
0, & x \in [L/2, \infty)
\end{cases}.
\]

We consider a particular case with a sufficiently large length \(L\) between the capacitor plates,

\[
\sqrt{\varepsilon EL} \gg \max \{1, E_c/E\}.
\]

(142)

Here \(E_c = m^2/e\) is the critical Schwinger field. We conditionally call this approximation as large work approximation when \(\Delta U = EL \gg 2m\). Such kind of \(x\)-electric step represent a regularization for a constant uniform electric field and is suitable for imitating a small-gradient field.

1. Initial vacuum state

Let us calculate von Neumann entropy corresponding to subsystems of electrons and positrons created from vacuum in \(\Omega_3\).

The leading asymptotic contributions to the differential and total number of created from the vacuum particles in the large work approximation has the form [19]

\[
N_{\text{cr}} \approx \exp \left[ -\frac{\pi^2}{eE} \right], \quad \pi_1^2 = \mathbf{p}_1^2 + m^2, \quad N_{\text{cr}} \approx \frac{J_d TV(eE)^{d/2}}{(2\pi)^{d-1}} \exp \left( -\pi \frac{E_c}{E} \right),
\]

where \(V = LV_\perp\) is the volume inside of the capacitor (the volume occupied by the electric field, \(L\) is the distance between capacitor plates, and \(V_\perp\) is the transversal volume of capacitor), \(J_d = 2^{[d/2]-1}\) is a spin summation factor, and \(e > 0\) is an absolute value of electron charge.

Let us estimate the information loss of the reduced electron and positron subsystems, which can be calculated as entropies (114) of these states. Performing summation over quantum modes \(n\) (for the details of this operation see Ref. (19) and (11)), we obtain the expression

\[
S(\hat{\rho}^{(3)}_{\text{cr}}) \approx \frac{J_d TV(eE)^{d/2}}{(2\pi)^{d-1}} \exp \left( -\pi \frac{E_c}{E} \right) A(d, E_c/E) \quad \text{if} \quad d > 2,
\]

(144)

where the factor \(A(d, E_c/E)\) has the form

\[
A(d, E_c/E) = [(\pi E_c/E + d/2 - 1) + \sum_{l=1}^{\infty} \left( l^{-d/2} - l^{-1}(l + 1)^{(2-d)/2} \exp (-\pi E_c/E) \right) \exp (-\pi (l - 1) E_c/E)].
\]

(145)

Comparing Eqs. (144) and (143), one can see that the entropy is proportional to the total number of particles created, i.e.

\[
S(\hat{\rho}^{(3)}_{\text{cr}}) \approx N_{\text{cr}} A(d, E_c/E).
\]

(146)

\(^2\text{Here }[\ldots]\text{ denote the integer part of the expression.}\)
(II) Initial thermal state

Here we only consider the Klein zone $\Omega_3$ as well, as for the case of electron-positron subsystem reduction the density operators of the other quantum ranges $\Omega_i$ are either completely traced out and do not contribute to von Neumann entropy, or are undisturbed by the reduction and therefore their initial entropy does not change after the reduction.

As distributions of initial particles we take the Fermi ones. They depend on particle energy and are given by Eq. (88). In the Klein zone these distributions have the form

$$N_n^{(3)}(in) = \left[\exp \left(\beta \left(\frac{\varepsilon_n^+ - \mu^+}{k}\right)\right) + 1\right]^{-1}.$$  \hspace{1cm} (147)

At any given $p_{\perp}$ available quantum numbers $p_0$ in the Klein zone for $L$-constant field are restricted by the definition of the Klein zone [10],

$$U_L + \pi_{\perp} \leq p_0 \leq U_R - \pi_{\perp}, \quad U_R = -U_L = \Delta U/2 = eEL/2.$$  \hspace{1cm} (148)

Such that

$$\varepsilon_n^+ = \pm p_0 + \frac{\Delta U}{2} [1 - N_n^{ctr}].$$  \hspace{1cm} (149)

Here $U_L = -eA_0(x \to -\infty)$ and $U_R = -eA_0(x \to +\infty)$ are left and right asymptotic potential energies, correspondingly.

Let us analyze Eq. (147) for initial electrons. The number $N_n^{ctr}$ is even with respect of change $p_0 \to -p_0$ and has the form (143) for the large range if $|p_0|, \pi_\perp \ll \Delta U/2$. At the left (right) edge of the Klein zone asymptotic longitudinal momenta $|p_L|$ ($|p_R|$),

$$\left|p_{\perp}/R\right| = \sqrt{[\pm p_0 + \Delta U/2]^2 - \pi_{\perp}^2},$$

tends to zero and one of the following limits holds true $N_n^{ctr} \sim |p_L|/\sqrt{\varepsilon E} \to 0$ or $N_n^{ctr} \sim |p_R|/\sqrt{\varepsilon E} \to 0$, respectively. We see that kinetic energies $\varepsilon_n^+$ tend to the minimum, given by transversal energy, $\varepsilon_n^+ \to \pi_{\perp}$. Therefore, it is more likely to find a particle with a lower kinetic energy $\sim \pi_{\perp}$ just as one would expect.

For further analysis it is convenient to rewrite expressions (125) for final differential number of electrons and positrons as

$$N_{n,\beta,\pm}^{(3)}(out) = N_{n,\beta,\pm}^{(3)}(in) + N_n^{ctr} \left(1 - N_{n,\beta,\pm}^{(3)}(in) - N_n^{ctr} \right),$$

$$N_{n,\beta,\mp}^{(3)}(out) = N_{n,\beta,\mp}^{(3)}(in) + N_n^{ctr} \left(1 - N_{n,\beta,\mp}^{(3)}(in) - N_n^{ctr} \right).$$  \hspace{1cm} (150)

Note that if $\mu^+ = \mu^-$, the sum $N_{n,\beta,\pm}^{(3)}(in) + N_{n,\beta,\pm}^{(3)}(in)$ is even with respect of change $p_0 \to -p_0$. The further consideration can be easily extended to the case when, for example, $N_{n,\beta,\pm}^{(3)}(in) = 0$ or $N_{n,\beta,\mp}^{(3)}(in) = 0$, i.e. when only one type of initial particles is present. We can sum these expression over quantum numbers $n \in \Omega_3$ as follows,

$$N_{\beta,\pm}^{(3)}(out) = \sum_{n \in \Omega_3} N_{n,\beta,\pm}^{(3)}(out) = \frac{J(d)TV_{\perp}}{(2\pi)^{d-1}} \int_{p_{\perp},p_0 \in \Omega_3} d^{d-2}p_{\perp} dp_0 N_{n,\beta,\pm}^{(3)}(out).$$  \hspace{1cm} (151)

It was shown in Ref. [19] that a leading contribution to $N_{n,\beta}^{ctr}$, given by Eq. (143), comes from inner subrange $D$ defined as

$$D : \frac{\pi_{\perp}}{\sqrt{\varepsilon E}} < K_{\perp}, \quad |p_0|/\sqrt{\varepsilon E} < \sqrt{\varepsilon EL}/2 - K, \quad \sqrt{\varepsilon EL}/2 \gg K \gg K_{\perp}^2 \gg \max \{1, m^2/\varepsilon E\}.$$  \hspace{1cm} (152)

For the second terms of Eq. (150) $N_{n,\beta}^{ctr}$ acts as a cutoff factor, so we can integrate over subrange $D$ only. Note that for quantum modes $n'$ where $N_{n'}^{ctr}$ is small enough, i.e. the number of particles created is small enough, distributions $N_{n,\beta,\pm}^{(3)}(out)$ are only slightly differ from initial distributions $N_{n,\beta,\pm}^{(3)}(in)$. In this situation, the corresponding entropy will almost coincide with the initial entropy of each subsystem,

$$S(\rho_{n',\beta,\pm}^{(3)}(in)) \approx \left[\left(1 - N_{n',\beta,\pm}^{(3)}(in)\right) \ln \left(1 - N_{n',\beta,\pm}^{(3)}(in)\right) + N_{n',\beta,\pm}^{(3)}(in) \ln N_{n',\beta,\pm}^{(3)}(in)\right].$$  \hspace{1cm} (153)

To calculate impact of a pair creation we can rewrite Eq. (151) as

$$N_{\beta,\pm}^{(3)}(out) \approx \frac{J(d)TV_{\perp}}{(2\pi)^{d-1}} \int_{p_{\perp},p_0 \in D} d^{d-2}p_{\perp} dp_0 N_{n,\beta,\pm}^{(3)}(out).$$  \hspace{1cm} (154)
Let us consider, for example, the case \( N_{n}^{cr} \ll 1 \). Taking relation (149) and integrating \( N^{(3)}_{n,\beta,\pm}(in) \) over \( p_{0} \), we obtain that the leading term is

\[
\int_{D} dp_{0} N^{(3)}_{n,\beta,\pm}(in) = N^{(3)}_{\perp,\beta,\pm}(in),
\]

\[
N^{(3)}_{\perp,\beta,\pm}(in) \approx \frac{1}{\beta} \ln \frac{1 + \exp \left[ -\beta \left( \sqrt{\varepsilon E K} - \mu \right) \right]}{1 + \exp \left[ -\beta (eE L - \mu) \right]}.
\] (155)

In particular, for low temperature and not very large \( \mu, \sqrt{\varepsilon E K} \gg \mu \), we have \( \beta (eE L - \mu) \gg \beta \left( \sqrt{\varepsilon E K} - \mu \right) \gg 1 \), then

\[
N^{(3)}_{\perp,\beta,\pm}(in) \approx \frac{1}{\beta} \ln 1 + \exp \left[ -\beta \left( \sqrt{\varepsilon E K} - \mu \right) \right] \approx \frac{1}{\beta} \exp \left[ -\beta \left( \sqrt{\varepsilon E K} - \mu \right) \right].
\] (156)

For high temperature, \( 1 \gg \beta (eE L - \mu) \gg \beta \left( \sqrt{\varepsilon E K} - \mu \right) \),

\[
N^{(3)}_{\perp,\beta,\pm}(in) \approx \frac{1}{2} \left( eE L - \sqrt{\varepsilon E K} \right).
\] (157)

Integrating it over the transversal momentum, we get

\[
N^{(3)}_{\beta,\pm}(in) \approx \frac{J(d) T V_{\perp}}{(2 \pi)^{d-1}} (2K_{\perp})^{d-2} (eE)^{d/2-1} N^{(3)}_{\perp,\beta,\pm}(in) \approx \frac{J(d) T V (eE)^{d/2}}{2(2\pi)^{d-1}} (2K_{\perp})^{d-2}.
\] (158)

The second terms of (159) can be integrated in a similar way,

\[
\int_{D} d^{d-2}p_{\perp} dp_{0} N^{cr}_{n}(1 - N^{(3)}_{n,\beta,-}(in) - N^{(3)}_{n,\beta,+}(in))
\]

\[
= \int_{D} d^{d-2}p_{\perp} N^{cr}_{n} \left[ eE L - 2\sqrt{\varepsilon E K} - N^{(3)}_{\perp,\beta,+}(in) - N^{(3)}_{\perp,\beta,-}(in) \right].
\] (159)

Note that for high temperature, the second terms of (159) vanish. This means that for this particular case \( N^{(3)}_{\beta,\pm}(out) \approx N^{(3)}_{\beta,\pm}(in) \), and starting entropy of the system does not change significantly due to pair creation and subsequent reduction over one of subsystems.

Corresponding entropy for general case is not difficult to write down,

\[
S \left( \rho^{(3)}_{\pm} \right) = - \sum_{n \in D} \left[ \left( 1 - N^{(3)}_{n,\beta,\pm}(out) \right) \ln \left( 1 - N^{(3)}_{n,\beta,\pm}(out) \right) + N^{(3)}_{n,\beta,\pm}(out) \ln N^{(3)}_{n,\beta,\pm}(out) \right],
\] (160)

where the summation over the quantum numbers can be done in the same manner as in Eq. (151). However, unlike the case of vacuum initial state, the latter expression is complicated. To obtain further results from it one must utilize numerical calculations with definite parameters of a particular system configuration: temperature \( \Theta = \beta^{-1} \), field strength \( E \) and capacitor length \( L \).

### 8 Concluding remarks

In this work, we have considered the deformation of different initial states by constant nonuniform electric fields and statistical properties of the resulting states.

First, we have introduced a special generating functional that allow us to construct density operators for different initial conditions. In graphene and similar materials any electric field can be considered as supercritical due to the fact that charge carriers are massless. Because of this a significant number of carrier pairs is produced. Possible dissipative processes lead to a loss of coherence of the states arising from vacuum, and it becomes necessary to study the statistical properties of the state generated by the field. For this reason, we have considered two cases of initial states of the system other than vacuum: case when the system was initially in thermodynamic equilibrium at absolute temperature \( \Theta = \beta^{-1} \), and the case of a pure state with a certain number of particles with fixed quantum numbers. In the framework of QED with \( x \)-electric potential steps we have to introduce five partial generating functionals for...
each range of quantum numbers $\Omega$. To simplify further calculations, we construct the normal form of these generating functionals in terms of creation-annihilation operators corresponding to final particles. Setting appropriate sources in these generating functionals, we obtain density operators for different initial states of the system: vacuum state, pure states with definite number of particles with fixed quantum numbers and thermal initial state. We also note that it is formally possible to construct the generating functional for a system with different initial conditions in different areas of quantum numbers $\Omega$. For example, choosing $J_{x,n}^{(3)} = J_{n}^{(2)} = J_{n}^{(4)} = 0$ and $J_{x,n}^{(1)} = J_{x,n}^{(5)} = e^{-E_{n}^{2}}$, we can construct the following density operator:

$$\hat{\rho}_{\text{mix}} = \prod_{n \in \Omega_1} \hat{\rho}_{\beta,n}^{(1)} \otimes \prod_{n \in \Omega_2} \hat{\rho}_{v,n}^{(2)} \otimes \prod_{n \in \Omega_3} \hat{\rho}_{v,n}^{(3)} \otimes \prod_{n \in \Omega_4} \hat{\rho}_{v,n}^{(4)} \otimes \prod_{n \in \Omega_5} \hat{\rho}_{\beta,n}^{(5)}. \quad (161)$$

This density operator correspond to the case when there are particles with quantum numbers $n$ from ranges $\Omega_1$ and $\Omega_5$ in thermal equilibrium at the initial time instant, but there were no particles that belong to Klein zone $\Omega_3$ (i.e. the initial state in this range was the vacuum state) as well as in ranges $\Omega_2$ and $\Omega_4$. Moreover, the functionals $R^{(i)}(J)$ permit factorization in quantum modes $n$ and each of those modes evolve separately. This fact allows one to assemble the general density operator as a product of partial operators $R_n^{(i)}(J)$, setting their initial conditions individually for each mode $n$.

Sometimes, there are situations when only the part of the system is available for observation; in this case we need to construct reduced density operators that describe this available part only. Another possible scenario for reduction is a measurement with the classical tool, which causes decoherence and deforms the general density operator of the system. We considered three types of reduction: the reduction over electron and positron subsystems, the reduction due to measurement of number of final particles and the spatial reduction over left or right final particles. We note that the latter kind of reduction is of interest when considering the type of fields that are concentrated in restricted space areas. We can compare the situation at hands to the case of QED with $t$-electric potential steps: for time-dependent uniform fields spatial reduction always coincides with reduction over electron or positron subsystems, as formulation of the problem suggests that the field occupies the entire space. Therefore, for a field acting for sufficiently long time period all the final electrons regardless of their initial state move in the direction of the field (and all the final positrons move in the opposite direction). The same can be said about electron-positron pairs created from vacuum by external field. For this reason the electron subsystem always coincides with the left spatial subsystem, and the positron one coincides with the right spatial subsystem in uniform time dependent electric fields. However, for $x$-potential electric steps the electric field is restricted in a finite area of space. Thus, there exist initial particles in ranges $\Omega_1$ and (or) $\Omega_5$, which can go through the potential barrier and end up at the opposite side of potential barrier as free final particles. Taking that into account, we can conclude that the spatial reduction is different from electron-positron subsystem reduction in general case. However, this difference exists only when there are initial particles in ranges $\Omega_1$ and $\Omega_5$. When there are no initial particles in these ranges, e.g., for the case of initial vacuum state, spatial reduction coincide with electron-positron subsystem reduction.

We have constructed reduced density operators corresponding to each of three types of reduction. We have calculated von Neumann entropy for the reduced density operators. Using so-called $L$-constant field as an example, we have shown that for the reduced density operators of electron and positron subsystems this entropy is proportional to the total number of pairs created.

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10 Appendix

Here we provide some relations that we have used during the calculations.
I. For both Fermi and Bose cases the following relations hold\[3\]:

\[
\begin{align*}
    ae^aD &= e^aD e^a, \quad a^\dagger e^D &= e^aD e^a D,
    \\
    e^aD &= \exp \left\{ a^\dagger \left( e^D - 1 \right) a \right\},
\end{align*}
\]

where, in general case, $D$ is an arbitrary matrix. Note that in the case under consideration all products are diagonal in quantum numbers $n$, $a^\dagger Da = a^\dagger D_{nn} a$, and the matrices $D_{nn}$ are diagonal and single-rank, i.e., are just $c$-numbers. One can also easily see that the following generalization of Eq. (162) holds

\[
\begin{align*}
    e^aD &= \exp \left\{ a^\dagger \left( e^D - 1 \right) a \right\},
\end{align*}
\]

For the product of two normal-form operators the following relation is useful:

\[
\begin{align*}
    e^aD &= e^{D(a^\dagger D + D D^\dagger) a},
\end{align*}
\]

where $D$ and $D^\dagger$ are some matrices, and its simple generalization

\[
\begin{align*}
    e^bD &= e^{D(a^\dagger Dc + b^\dagger D D^\dagger c)}.
\end{align*}
\]

where for the case of Fermi-operators the decomposition of the exponent is finite and has the form

\[
\begin{align*}
    e^{b^\dagger Da + a^\dagger Dc} &= 1 + b^\dagger Da + a^\dagger Dc + b^\dagger D D^\dagger c - b^\dagger a^\dagger Dac.
\end{align*}
\]

II. The trace of a normal product of creation and annihilation operators can be calculated by using the following path integral representation. Suppose that $X(a^\dagger, a)$ be an operator expression of creation and annihilation operators. Then the trace of its normal form can be expressed as [3]:

\[
\begin{align*}
    \text{tr} \left\{ : e^{a^\dagger D a} : \right\} &= \langle 0 | \int \exp \left\{ a^\dagger \lambda^* - a^\dagger \lambda \right\} X(a^\dagger, a) : \exp \left\{ a^\dagger \lambda \right\} \prod d\lambda^* d\lambda | 0 \rangle.
\end{align*}
\]

\[
\text{References}
\]

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