Global Small Solutions to the Inviscid Hall-MHD System

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Abstract. The local existence of smooth solutions to the inviscid Hall-MHD system has been obtained since Chae, Degond and Liu (Ann. Inst. H. Poincaré Anal. Non Linéaire 31:555–565, 2014). However, as far as we know, how to construct the global small solutions to the inviscid Hall-MHD system is still an open problem. In the present paper, we give a positive answer in $\mathbb{T}^3$ when the initial magnetic field is close to a background magnetic field satisfying the Diophantine condition.

1. Introduction and Main Result

In this paper, we are concerned with the global well-posedness of the smooth solutions to the following inviscid Hall-MHD system:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p &= (\nabla \times B) \times B, \\
\partial_t B - \Delta B - \nabla \times (u \times B) &= -\nabla \times ((\nabla \times B) \times B), \\
\text{div } u = \text{div } B &= 0, \\
(u, B)|_{t=0} &= (u_0, B_0).
\end{align*}
\] (1.1)

Here, $x = (x_1, x_2, x_3) \in \mathbb{T}^3$ and $t \geq 0$ are the space and time variables, respectively. The unknown $u$ is the velocity field, $B$ is the magnetic field, $p$ is the scalar pressure, respectively.

The application of Hall-MHD system is mainly from the understanding of magnetic reconnection phenomena [11,15], where the topology structure of the magnetic field changes dramatically and the Hall effect must be included to get a correct description of this physical process. Mathematical derivations of Hall-MHD system from either two-fluids or kinetic models can be found in [1].

The study on the theory of well-posedness of solutions to the Hall-MHD system has grown enormously in recent years. For the viscous and resistive Hall-MHD system (with $-\Delta u$), there have a lot of excellent works, see instances [1–7,9–12,14,16–19]. More precisely, Chae et al. [2] showed the global existence of Leray-Hopf weak solutions. Dumas et al. [10] have been further investigated the weak solutions both for the Maxwell-Landau-Lifshitz system and for the Hall-MHD system. The temporal decay estimates for weak solutions to Hall-MHD system was established by Chae et al. [4]. They also obtained algebraic decay rates for higher order Sobolev norms of strong solutions with small initial data. It turned out that the Hall term does not affect the time asymptotic behavior, and the time decay rates behaved like those of the corresponding heat equation. In addition, a blowup criterion and the global existence of small classical solutions were also established in [2]. These results were later sharpened by [3]. In [18,19], Weng studied the long-time behaviour and obtained optimal space-time decay rates of strong solutions.

However, there has been rather few results about the inviscid Hall-MHD system (1.1). Chae et al. [2] obtained the local existence of the smooth solutions in $\mathbb{R}^3$, later Chae et al. [5] also derived the local smooth solutions of (1.1) in general dimension by considering the fractional magnetic diffusion. Jeong et al. [13] considered the regularity problem of the solutions to the axisymmetric, inviscid, and incompressible Hall-magnetohydrodynamics. To the author’s knowledge, it is still an open problem to...
construct the global solutions of the inviscid Hall-MHD system even for small initial data. The purpose of this paper is to move a step in this direction. Inspired by [8], we obtain the global small solutions of the inviscid Hall-MHD system in $\mathbb{T}^3$ when the initial magnetic field is close to a background magnetic field satisfying the Diophantine condition. Compared with the usual inviscid incompressible MHD system, (1.1) contains the extra term $\nabla \times (\nabla \times B \times B)$, which is the so called Hall term. The Hall term heightens the level of nonlinearity of the standard MHD system from a second-order semilinear to a second-order quasilinear level, significantly making its qualitative analysis more difficult.

Let $n \in \mathbb{R}^3$ satisfy the so called Diophantine condition: for any $k \in \mathbb{Z}^3 \setminus \{0\}$,

$$|n \cdot k| \geq \frac{c}{|k|^r},$$

(1.2)

for some $c > 0$ and $r > 2$.

For the simplicity, we still use the notation $B$ to denote the perturbation $B - n$. Hence, the perturbed equations can be rewritten into

$$\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p &= n \cdot \nabla B + (\nabla \times B) \times B, \\
\partial_t B - \Delta B - \nabla \times (u \times B) &= n \cdot \nabla u - n \cdot \nabla (\nabla \times B) - \nabla \times ((\nabla \times B) \times B), \\
div u &= div B = 0, \\
(u, B)|_{t=0} &= (u_0, B_0).
\end{aligned}$$

(1.3)

The main result of the paper is stated as follows.

**Theorem 1.1.** For any $N \geq 4r + 7$ with $r > 2$. Let $(u_0, B_0) \in H^N(\mathbb{T}^3)$ and

$$\int_{\mathbb{T}^3} u_0\, dx = \int_{\mathbb{T}^3} B_0\, dx = 0.$$  

(1.4)

If there exists a small constant $\varepsilon$ such that

$$\|u_0\|_{H^N} + \|B_0\|_{H^N} \leq \varepsilon.$$

Then the system (1.3) admits a global solution $(u, B) \in C([0, \infty); H^N)$. Moreover, for any $t \geq 0$ and $r + 4 \leq \beta < N$, there holds

$$\|u(t)\|_{H^\beta} + \|B(t)\|_{H^\beta} \leq C(1 + t)^{- \frac{\beta(N-r)}{2(N-r-4)}}.$$  

Remark 1.2. For any $t \geq 0$, the solutions of (1.3) will preserve this property if there holds (1.4).

Remark 1.3. The method used here is still valid for the fractional magnetic diffusion $(-\Delta)^\gamma$ with $\gamma > \frac{1}{2}$. That is to say, we can also extend the local solutions obtained in [5] to be global under the same assumptions of Theorem 1.1.

Remark 1.4. In [8, Section 2], the authors have proved that the Diophantine condition is satisfied for almost all vector fields $n$ in $\mathbb{R}^3$. However, it’s very difficult to check that would it satisfy the condition if we pick a “random” vector $n$? We are only confirmed that when the components of $n$ are rational numbers or when one component of $n$ is zero, $n$ does not satisfy the Diophantine condition.

Remark 1.5. Whether can we remove the Diophantine condition (1.2) is a challenged open problem. This is left in the future work.

2. The Proof of the Theorem

The proof of the Theorem 1.1 relies heavily on the following lemma whose proof is standard by the Plancherel formula.

**Lemma 2.1.** If $n \in \mathbb{R}^3$ satisfies the Diophantine condition (1.2), then it holds that for any $s \in \mathbb{R}$,

$$\|f\|_{H^s} \leq C\|n \cdot \nabla f\|_{H^{s+r}}.$$  

(2.2)

if $\nabla f \in H^{s+r}(\mathbb{T}^3)$ satisfies $\int_{\mathbb{T}^3} f\, dx = 0$.

Now, we begin to prove the main theorem.
2.1. L² Energy Estimate

Firstly, denote ⟨a, b⟩ the L²(Թ³) inner product of a and b. A standard energy estimate gives

\[ \frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|B\|_{L^2}^2) + \|\nabla B\|_{L^2}^2 = 0 \]  \hspace{1cm} (2.3)

where we have used the cancellations

\[ \langle u \cdot \nabla u, u \rangle = \langle u \cdot \nabla B, B \rangle = 0, \quad \langle B \cdot \nabla B, u \rangle + \langle B \cdot \nabla u, B \rangle = 0, \]

\[ \langle u \cdot \nabla B, u \rangle + \langle \nabla \cdot B, B \rangle = 0, \quad \langle \nabla \times ((\nabla \times B) \times B), B \rangle = 0. \]

The above cancellation is crucially important for the existence of global smooth solution for small data.

2.2. High Order Energy Estimate

Let \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \) be a multi-index. We operate \( D^\alpha = \partial^{(\alpha_1)} \partial x_1^{\alpha_1} \cdots \partial x_3^{\alpha_3} \) (where \( |\alpha| = \alpha_1 + \cdots + \alpha_3 \)) on the first two equations respectively and take the scalar product of them with \( D^\beta u \) and \( D^\beta B \) respectively, add them together and then sum the result over \( |\alpha| \leq m \). We obtain

\[ \frac{1}{2} \frac{d}{dt} \left( \|u\|_{H^m}^2 + \|B\|_{H^m}^2 \right) + \|\nabla B\|_{H^m}^2 = - \sum_{0 < |\alpha| \leq m} \langle D^\alpha ((\nabla \times B) \times B), D^\alpha ((\nabla \times B) \times B) \rangle \]

\[ + \sum_{0 < |\alpha| \leq m} \langle D^\alpha (u \times B), D^\alpha (\nabla \times B) \rangle \]

\[ - \sum_{0 < |\alpha| \leq m} \langle D^\alpha (u \cdot \nabla u), D^\alpha u \rangle \]

\[ + \sum_{0 < |\alpha| \leq m} \langle D^\alpha ((\nabla \times B) \times B), D^\alpha u \rangle \]

\[ \overset{\text{def}}{=} I_1 + I_2 + I_3 + I_4. \]  \hspace{1cm} (2.4)

In the following, we estimate successively each of the \( I_1-I_4 \) terms.

For \( I_1 \), in view of the cancellation

\[ \langle (D^\alpha (\nabla \times B)) \times B, D^\alpha (\nabla \times B) \rangle = 0, \]

we can get

\[ I_1 = - \sum_{0 < |\alpha| \leq m} \langle [D^\alpha ((\nabla \times B) \times B) - (D^\alpha (\nabla \times B)) \times B], D^\alpha (\nabla \times B) \rangle. \]

Using the well-known calculus inequality,

\[ \sum_{|\alpha| \leq m} \|D^\alpha (fg) - (D^\alpha f)g\|_{L^2} \leq C(\|f\|_{H^{m-1}} \|\nabla g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{H^m}), \]  \hspace{1cm} (2.5)

we have

\[ I_1 \leq C(\|B\|_{H^m} \|\nabla B\|_{L^\infty} + \|\nabla B\|_{L^\infty} \|B\|_{H^m}) \|\nabla B\|_{H^m} \]

\[ \leq \frac{1}{4} \|\nabla B\|_{H^m}^2 + C\|B\|_{H^m}^2 \|\nabla B\|_{L^\infty}^2. \]  \hspace{1cm} (2.6)
On the other hand, using Leibnitz formula and the Sobolev inequality, we obtain

\[ I_2 \leq \sum_{0 < |\alpha| \leq m} \| D^\alpha (u \times B) \|_{L^2} \| \nabla B \|_{H^m} \]
\[ \leq C (\| u \|_{L^\infty} \| B \|_{H^m} + \| u \|_{H^m} \| B \|_{L^\infty}) \| \nabla B \|_{H^m} \]
\[ \leq \frac{1}{4} \| \nabla B \|_{H^m}^2 + C \| u \|_{L^\infty}^2 \| B \|_{H^m}^2 + C \| u \|_{H^m}^2 \| B \|_{L^\infty}^2. \]

(2.7)

Then, we remark that

\[ I_3 = - \sum_{0 < |\alpha| \leq m} \langle [D^\alpha (u \cdot \nabla u) - u \cdot \nabla D^\alpha u], D^\alpha u \rangle. \]

Indeed, the second term is zero by the fact that \( u \) is divergence free. Then, similarly to the above calculation, using the calculus inequality (2.5), we obtain

\[ I_3 \leq \sum_{0 < |\alpha| \leq m} \| D^\alpha (u \cdot \nabla u) - u \cdot \nabla D^\alpha u \|_{L^2} \| u \|_{H^m} \]
\[ \leq C \| \nabla u \|_{L^\infty} \| u \|_{H^m}^2. \]

(2.8)

From (2.4) we get

\[ I_4 \leq \sum_{0 < |\alpha| \leq m} \| (\nabla \times B) \times B \|_{H^m} \| u \|_{H^m}. \]

Using Leibnitz formula, we derive

\[ I_4 \leq C (\| \nabla B \|_{L^\infty} \| B \|_{H^m} + \| \nabla B \|_{H^m} \| B \|_{L^\infty}) \| u \|_{H^m} \]
\[ \leq C \| \nabla B \|_{L^\infty} \| B \|_{H^m} \| u \|_{H^m} + \frac{1}{2} \| \nabla B \|_{H^m}^2 + C \| B \|_{L^\infty}^2 \| u \|_{H^m}^2 \]
\[ \leq \frac{1}{2} \| \nabla B \|_{H^m}^2 + C (\| \nabla B \|_{L^\infty} + \| B \|_{H^m}^2) (\| u \|_{H^m}^2 + \| B \|_{H^m}^2). \]

(2.9)

From estimates (2.6), (2.7), (2.8) and (2.9), we obtain

\[ \frac{d}{dt} (\| u \|_{H^m}^2 + \| B \|_{H^m}^2) + \| \nabla B \|_{H^m}^2 \]
\[ \leq C (\| \nabla u \|_{L^\infty} + \| \nabla B \|_{L^\infty} + \| \nabla B \|_{L^\infty}^2 + \| u \|_{L^\infty}^2 + \| B \|_{L^\infty}^2) (\| u \|_{H^m}^2 + \| B \|_{H^m}^2). \]

(2.10)

2.3. A Key Lemma

The following lemma which relies heavily on the structural characteristics of the system (1.3) is crucial to get the time decay for the velocity field.

**Lemma 2.11.** For any \( N \geq 4r + 7 \) with \( r > 2 \). Assume that

\[ \sup_{t \in [0, T]} (\| u \|_{H^N} + \| B \|_{H^N}) \leq \delta, \]

for some \( 0 < \delta < 1 \). Then there holds that

\[ \| n \cdot \nabla u \|_{H^{r+3}}^2 - \sum_{0 \leq s \leq r+3} \frac{d}{dt} \langle D^s B, D^s (n \cdot \nabla u) \rangle \]
\[ \leq C \| B \|_{H^{r+5}}^2 + C \delta^2 \| u \|_{H^3}^2. \]

(2.13)
Proof. Applying $D^s(0 \leq s \leq r + 3)$ to the second equation of \eqref{H-MHD}, and multiplying it by $D^s(n \cdot \nabla u)$ then integrating over $\mathbb{T}^3$, we obtain

$$
\|D^s(n \cdot \nabla u)\|_{L^2}^2 = \langle D^s \partial_t B, D^s(n \cdot \nabla u) \rangle - \langle D^s \Delta B, D^s(n \cdot \nabla u) \rangle \nonumber \\
+ \langle D^s(u \cdot \nabla B), D^s(n \cdot \nabla u) \rangle \nonumber \\
- \langle D^s(B \cdot \nabla u), D^s(n \cdot \nabla u) \rangle \nonumber \\
+ \langle D^s(n \cdot \nabla (\nabla \times B)), D^s(n \cdot \nabla u) \rangle \nonumber \\
+ \langle D^s(\nabla \times ((\nabla \times B) \times B)), D^s(n \cdot \nabla u) \rangle \nonumber \\
\overset{\text{def}}{=} I_5 + I_6 + I_7 + I_8 + I_9 + I_{10}.
\tag{2.14}
$$

Thanks to the Hölder inequality, Young's inequality, and the embedding relation, we have

$$
I_6 \leq C \|D^s \Delta B\|_{L^2} \|D^s(n \cdot \nabla u)\|_{L^2} \nonumber \\
\leq \frac{1}{16} \|D^s(n \cdot \nabla u)\|_{L^2}^2 + C \|\Delta B\|_{H^s}^2 \nonumber \\
\leq \frac{1}{16} \|D^s(n \cdot \nabla u)\|_{L^2}^2 + C \|B\|_{H^{r+2}}^2 \nonumber \\
\leq \frac{1}{16} \|D^s(n \cdot \nabla u)\|_{L^2}^2 + C \|B\|_{H^{r+5}}^2 . \tag{2.15}
$$

Similarly, using Leibnitz formula and the Sobolev inequality, we obtain

$$
I_7 \leq C \|D^s(u \cdot \nabla B)\|_{L^2} \|D^s(n \cdot \nabla u)\|_{L^2} \nonumber \\
\leq C(\|u\|_{L^\infty} \|\nabla B\|_{H^s} + \|\nabla B\|_{L^2} \|u\|_{H^{r+2}}) \|D^s(n \cdot \nabla u)\|_{L^2} \nonumber \\
\leq \frac{1}{16} \|D^s(n \cdot \nabla u)\|_{L^2}^2 + C(\|u\|_{H^{2}}^2 \|\nabla B\|_{H^s}^2 + \|\nabla B\|_{H^s}^2 \|u\|_{H^s}^2) \nonumber \\
\leq \frac{1}{16} \|D^s(n \cdot \nabla u)\|_{L^2}^2 + C \|u\|_{H^N}^2 \|B\|_{H^{r+1}}^2 \nonumber \\
\leq \frac{1}{16} \|D^s(n \cdot \nabla u)\|_{L^2}^2 + C\delta^2 \|B\|_{H^{r+4}}^2 , \tag{2.16}
$$

and

$$
I_8 \leq C \|D^s(B \cdot \nabla u)\|_{L^2} \|D^s(n \cdot \nabla u)\|_{L^2} \nonumber \\
\leq C(\|B\|_{L^\infty} \|\nabla u\|_{H^s} + \|\nabla u\|_{L^2} \|B\|_{H^{r+2}}) \|D^s(n \cdot \nabla u)\|_{L^2} \nonumber \\
\leq C(\|B\|_{H^2} \|u\|_{H^{r+3}} + \|\nabla u\|_{H^2} \|B\|_{H^s}) \|D^s(n \cdot \nabla u)\|_{L^2} \nonumber \\
\leq \frac{1}{16} \|D^s(n \cdot \nabla u)\|_{L^2}^2 + C \|u\|_{H^N}^2 (\|B\|_{H^s}^2 + \|B\|_{H^s}^2) \nonumber \\
\leq \frac{1}{16} \|D^s(n \cdot \nabla u)\|_{L^2}^2 + C\delta^2 \|B\|_{H^{r+3}}^2 . \tag{2.17}
$$

The estimate $I_9$ is similar to $I_6$,

$$
I_9 \leq C \|D^s \nabla^2 B\|_{L^2} \|D^s(n \cdot \nabla u)\|_{L^2} \nonumber \\
\leq \frac{1}{16} \|D^s(n \cdot \nabla u)\|_{L^2}^2 + C \|\nabla^2 B\|_{H^s}^2 \nonumber \\
\leq \frac{1}{16} \|D^s(n \cdot \nabla u)\|_{L^2}^2 + C \|B\|_{H^{r+2}}^2 \nonumber \\
\leq \frac{1}{16} \|D^s(n \cdot \nabla u)\|_{L^2}^2 + C \|B\|_{H^{r+5}}^2 . \tag{2.18}
$$
Moreover, for any $0 \leq s \leq r + 3$ and $N \geq r + 4$, there holds
\[
I_{10} \leq \langle D^s(B \cdot \nabla(\nabla \times B) - (\nabla \times B) \cdot \nabla B), D^s(n \cdot \nabla u) \rangle \\
\leq C(\|B\|_{L^\infty} \|\nabla^2 B\|_{H^r} + \|\nabla^2 B\|_{L^\infty} \|B\|_{H^r} + \|\nabla B\|_{L^\infty} \|\nabla B\|_{H^r}) \|D^s(n \cdot \nabla u)\|_{L^2} \\
\leq \frac{1}{16} \|D^s(n \cdot \nabla u)\|_{L^2}^2 + C \|B\|_{H^{r+2}}^2 \|B\|_{H^N}^2 \\
\leq \frac{1}{16} \|D^s(n \cdot \nabla u)\|_{L^2}^2 + C\delta^2 \|B\|_{H^{r+5}}^2.
\tag{2.19}
\]

Finally, we have to bound the first term on the right hand side of (2.14). In fact, exploiting the first equation in (1.3), we can rewrite this term into
\[
\langle D^s \partial_t B, D^s(n \cdot \nabla u) \rangle = \frac{d}{dt} \langle D^s B, D^s(n \cdot \nabla u) \rangle - \langle D^s B, D^s(n \cdot \nabla \partial_t u) \rangle \\
= \frac{d}{dt} \langle D^s B, D^s(n \cdot \nabla u) \rangle + \langle D^s(n \cdot \nabla B), D^s \partial_t u \rangle \\
+ \langle D^s(n \cdot \nabla B), D^s(n \cdot \nabla B) \rangle - \langle D^s(n \cdot \nabla B), D^s(u \cdot \nabla u) \rangle
\tag{2.20}
\]
where we have used the following cancellation
\[
\langle D^s(n \cdot \nabla B), D^s \nabla p \rangle = 0.
\]
The second term on the right hand side of (2.20) can be bounded as
\[
\langle D^s(n \cdot \nabla B), D^s(B \cdot \nabla B) \rangle \leq C \|D^s(n \cdot \nabla B)\|_{L^2} \left( \|B\|_{L^\infty} \|\nabla B\|_{H^r} + \|\nabla B\|_{L^\infty} \|B\|_{H^r} \right) \\
\leq C \|B\|_{H^N} \|\nabla B\|_{H^r}^2 \\
\leq C\delta \|\nabla B\|_{H^r}^2.
\tag{2.21}
\]
In the same manner, we can deal with the last term on the right hand side of (2.20)
\[
\langle D^s(n \cdot \nabla B), D^s(u \cdot \nabla u) \rangle \leq C \|D^s(n \cdot \nabla B)\|_{L^2} \left( \|u\|_{L^\infty} \|\nabla u\|_{H^r} + \|\nabla u\|_{L^\infty} \|u\|_{H^r} \right) \\
\leq C \|D^s(n \cdot \nabla B)\|_{L^2} \left( \|u\|_{H^2} \|\nabla u\|_{H^r} + \|\nabla u\|_{H^2} \|u\|_{H^r} \right) \\
\leq C \|\nabla B\|_{H^r} \|u\|_{H^N} \|u\|_{H^3} \\
\leq C \|\nabla B\|_{H^r}^2 + C\delta^2 \|u\|_{H^3}^2.
\tag{2.22}
\]
Inserting (2.21) and (2.22) into (2.20) gives
\[
\langle D^s \partial_t B, D^s(n \cdot \nabla u) \rangle \leq \frac{d}{dt} \langle D^s B, D^s(n \cdot \nabla u) \rangle + C \|\nabla B\|_{H^r}^2 + C\delta^2 \|u\|_{H^3}^2.
\tag{2.23}
\]
Plugging (2.15)–(2.18) and (2.23) into (2.14), we can arrive at (2.13). This proves the lemma. 
\]

\subsection*{2.4. Complete the Proof of the Main Theorem}

Given the initial data $(u_0, B_0) \in H^N$, the local well-posedness of the system (1.3) has been proved in [2] by using the energy method. Thus, we may assume that there exist $T > 0$ and a unique solution $(u, B) \in C([0, T]; H^N)$ of the system (1.3). Furthermore, we may assume that
\[
\sup_{t \in [0, T]} (\|u\|_{H^N} + \|B\|_{H^N}) \leq \delta,
\tag{2.24}
\]
for some $0 < \delta < 1$ to be determined later.
According to the embedding relation, we get for any $N \geq 3$ that
\[
\|\nabla u\|_{L^\infty} + \|\nabla B\|_{L^\infty} + \|\nabla B\|_{L^2}^2 + \|u\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \\
\leq C(\|\nabla u\|_{H^2} + \|\nabla B\|_{H^2} + \|\nabla B\|_{H^2}^2 + \|u\|_{H^2}^2 + \|B\|_{H^2}^2)
\leq C(\|u\|_{H^N} + \|B\|_{H^N} + \|u\|_{H^N}^2 + \|B\|_{H^N}^2)
\leq C\delta(1 + \delta)
\] (2.25)
from which and taking $m = r + 4$ in (2.10) gives
\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|_{H^{r+4}}^2 + \|B\|_{H^{r+4}}^2 \right) + \|\nabla B\|_{H^{r+4}}^2 \leq C\delta(1 + \delta)(\|u\|_{H^{r+4}}^2 + \|B\|_{H^{r+4}}^2).
\] (2.26)
Due to
\[
\int_{\mathbb{T}^3} B_0 \, dx = 0,
\]
there holds
\[
\|B\|_{H^{r+5}}^2 \leq C\|\nabla B\|_{H^{r+4}}^2.
\] (2.27)
Hence, let $A \geq 1 + 2C$ be a constant determined later, we infer from Lemma 2.11, (2.26) and (2.27) that
\[
\frac{d}{dt} \left( A(\|u\|_{H^{r+4}}^2 + \|B\|_{H^{r+4}}^2) - \sum_{0 \leq s \leq r+3} \langle D^s B, D^s (n \cdot \nabla u) \rangle \right) \\
+ A \|\nabla B\|_{H^{r+4}}^2 + \|n \cdot \nabla u\|_{H^{r+3}}^2 \\
\leq CA\delta(1 + \delta)(\|u\|_{H^{r+4}}^2 + \|B\|_{H^{r+4}}^2) + C\delta^2 \|u\|_{H^3}^2.
\] (2.28)
In view of Lemma 2.1, for any $N \geq 2r + 5$, we have
\[
\|u\|_{H^3}^2 \leq C \|n \cdot \nabla u\|_{H^{r+3}}^2 \quad \|u\|_{H^{r+4}}^2 \leq \|u\|_{H^3} \|u\|_{H^N} \leq C\delta \|n \cdot \nabla u\|_{H^{r+3}}
\] (2.29)
which gives
\[
CA\delta(1 + \delta)\|B\|_{H^{r+4}}^2 \leq CA\delta^2(1 + \delta) \|\nabla B\|_{H^{r+4}},
\] (2.30)
and
\[
CA\delta(1 + \delta)\|u\|_{H^{r+4}}^2 + C\delta^2 \|u\|_{H^3}^2 \leq CA\delta^2(1 + \delta) \|n \cdot \nabla u\|_{H^{r+3}}.
\] (2.31)
As a result, we can infer from (2.28) that
\[
\frac{d}{dt} \left( A(\|u\|_{H^{r+4}}^2 + \|B\|_{H^{r+4}}^2) - \sum_{0 \leq s \leq r+3} \langle D^s B, D^s (n \cdot \nabla u) \rangle \right) \\
+ A \|\nabla B\|_{H^{r+4}}^2 + \|n \cdot \nabla u\|_{H^{r+3}}^2 \\
\leq CA\delta^2(1 + \delta) \|n \cdot \nabla u\|_{H^{r+3}} + CA\delta^2(1 + \delta) \|\nabla B\|_{H^{r+4}}.
\] (2.32)
Define
\[
\mathcal{D}(t) = A \|\nabla B\|_{H^{r+4}}^2 + \|n \cdot \nabla u\|_{H^{r+3}}^2,
\]
\[
\mathcal{E}(t) = A(\|u\|_{H^{r+4}}^2 + \|B\|_{H^{r+4}}^2) - \sum_{0 \leq s \leq r+3} \langle D^s B, D^s (n \cdot \nabla u) \rangle.
\]
Taking $A > 1$ such that
\[
\mathcal{E}(t) \geq \|u\|_{H^{r+4}}^2 + \|B\|_{H^{r+4}}^2.
\]
Hence, by choosing $\delta > 0$ small enough, we can get that
\[
\frac{d}{dt} \mathcal{E}(t) + \frac{1}{2} \mathcal{D}(t) \leq 0.
\] (2.33)
For any $N \geq 4r+7$, by the interpolation inequality, we have
\[ \|u\|_{H^{r+4}}^2 \leq \|u\|_{H^3}^2 \|u\|_{H^N}^2 \leq C\delta^\frac{2}{7} \|n\cdot\nabla u\|_{H^{r+3}}^2 \]
which further implies that
\[
\mathcal{E}(t) \leq C(\|u\|_{H^{r+4}}^2 + \|B\|_{H^{r+4}}^2)
\leq C\|u\|_{H^3}^2 \|u\|_{H^N}^2 + C\|B\|_{H^3}^2 \|B\|_{H^N}^2
\leq C\delta^\frac{2}{7} \|n\cdot\nabla u\|_{H^{r+3}}^2 + C\delta^\frac{2}{7} \|\nabla B\|_{H^{r+4}}^2
\leq (D(t))^\frac{2}{7}.
\]
So, we get a Lyapunov-type inequality
\[ \frac{d}{dt}\mathcal{E}(t) + c(\mathcal{E}(t))^\frac{4}{7} \leq 0. \]
Solving this inequality yields
\[ \mathcal{E}(t) \leq C(1 + t)^{-\frac{3}{2}}. \] (2.34)
Taking $m = N$ in (2.10) and using the embedding relation give
\[
\frac{d}{dt}(\|u\|_{H^N}^2 + \|B\|_{H^N}^2) + \|\nabla B\|_{H^N}^2
\leq C(\|u\|_{H^3}^2 + \|u\|_{H^2}^2 + \|B\|_{H^3}^2 + \|B\|_{H^3}^2)(\|u\|_{H^N}^2 + \|B\|_{H^N}^2).
\] (2.35)
From (2.34), we have
\[ \int_0^t (\|u(\tau)\|_{H^3}^2 + \|u(\tau)\|_{H^2}^2 + \|B(\tau)\|_{H^3}^2 + \|B(\tau)\|_{H^3}^2) d\tau \leq C, \] (2.36)
thus, exploiting the Gronwall inequality implies
\[ \|u\|_{H^N}^2 + \|B\|_{H^N}^2 \leq C(\|u_0\|_{H^N}^2 + \|B_0\|_{H^N}^2) \leq C\varepsilon^2. \] (2.37)
Taking $\varepsilon$ small enough so that $C\varepsilon \leq \delta/2$, we deduce from a continuity argument that the local solution can be extended as a global one in time.

Moreover, from (2.34), we also have the following decay rate
\[ \|u(t)\|_{H^{r+4}} + \|B(t)\|_{H^{r+4}} \leq C(1 + t)^{-\frac{2}{7}}. \]
Thus, for any $\beta > r + 4$, choosing $N > \beta$ and using the following interpolation inequality
\[ \|f(t)\|_{H^\beta} \leq \|f(t)\|_{H^{r+4}}^\frac{N-\beta}{r+4} \|f(t)\|_{H^N}^\frac{2(r+4)-N}{r+4}, \]
we can get the decay rate for the higher order energy
\[ \|u(t)\|_{H^\beta} + \|B(t)\|_{H^\beta} \leq C(1 + t)^{-\frac{3(N-\beta)}{2N(r+4)}}. \]
This completes the proof of Theorem 1.1. \hfill \Box

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