Identification and Control of Symmetric Systems

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Abstract

In a recent paper [Phys. Rev. E 57, 1550 (1998)] we demonstrated that the symmetries of the evolution equation and the target state have a profound effect on the selection of the admissible control parameters. In the present paper we extend these results to the case of time-periodic target trajectories and inexact symmetries. We also argue that the problem of phase space reconstruction is affected by the presence of symmetries similarly to the control problem.

I. INTRODUCTION

The desire to improve performance of many practically important systems and devices often calls for shifting their operating range into a highly nonlinear area, which after a series of bifurcations usually leads to irregular chaotic behavior. This kind of behavior, however, is rarely desired, while substantial benefits could be obtained by making the dynamics regular. This goal can typically be achieved by applying small preprogrammed perturbations to steer the system towards a periodic orbit with desired properties, which is broadly referred to as chaos control.

The goal of the present paper is to highlight some of the problems that arise in controlling systems which possess some kind of symmetry. Even though the importance of symmetries in chaotic dynamics has been recognized by a number of authors [1,2], symmetric systems did not receive adequate treatment in the general framework of chaos control primarily because the question of symmetry is largely ignored by the theory of deterministic chaos as well as data analysis and control theory. All three disciplines regard symmetric systems as nongeneric and, therefore, not very interesting and important.

However, many practically important dynamical systems, such as spatially extended chaotic ones, are symmetric, and thus cannot be successfully treated using the formalism developed for generic systems. Indeed, such phenomena as fluid flows, convection or chemical reactions often take place inside symmetric containers — cylinders, spheres, pipes and annuli. As a result, the dynamical equations also show rotational and translational symmetries. Even the dynamics of unbounded systems is often significantly influenced by the symmetries of the physical space. Although the presence of symmetries usually simplifies the analysis of the dynamics, it also makes system identification and control more complicated due to the inherent degeneracies of the evolution operators. In fact, the presence of symmetries,
explicit or implicit, makes a number of single-control-parameter methods fail [3–5], calling
for multi-parameter control [6,9,7,8].

In order to see how the control problem is affected by symmetries, we consider (following
the analysis conducted in [10]) a general discrete-time system (the arguments for continuous-
time systems are very similar), whose evolution is described by the map
\[ F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}, \]
such that
\[ x_{t+1} = F(x^t, u), \tag{1} \]
where \( x^t \) is the \( n_x \)-dimensional state of the system and \( u \) is the \( n_u \)-dimensional vector of
system parameters. The objective of control is to make the system follow the (possibly
unstable) periodic target trajectory \( \bar{x}^t \). Let us linearize equation \((1)\) about this target
trajectory to obtain
\[ \Delta x^{t+1} = A^t \Delta x^t + B^t \Delta u^t, \tag{2} \]
where
\[ A^t = D_x F(\bar{x}^t, \bar{u}) \tag{3} \]
is the Jacobian matrix, which determines the stability properties of the target trajectory,
and
\[ B^t = D_u F(\bar{x}^t, \bar{u}) \tag{4} \]
is the control matrix, which defines the linear response of the system to perturbation of
system parameters. According to linear system theory [11], if the target trajectory \( \bar{x}^t \) is
unstable, it can be stabilized by an appropriate feedback through the time-dependent control
perturbation \( \Delta u^t \), provided the matrices \( A^t \) and \( B^t \) satisfy certain conditions. In the present
study we concentrate on selecting from the complete set of available system parameters a
minimal set of control parameters, whose perturbation allows the stabilization of the target
state, i.e., on making an appropriate choice of the control matrix \( B^t \), given the Jacobian
\( A^t \). We will see below that the constraints affecting the choice of control parameters can
be easily obtained from the symmetry properties of the system and the controlled state.
What is more interesting, symmetry allows one to determine the minimal number of control
parameters even when the Jacobian \( A^t \) describing the local dynamics is unknown.

Discrete-time evolution equations of type \((1)\) are often obtained as a result of phase space
reconstruction of a continuous-time system when the dynamical equations
\[ \dot{s}(t) = \Phi(s(t), u) \tag{5} \]
are unknown. Here \( s(t) \in \mathcal{Q} \) denotes the \( n_s \)-dimensional internal state of the system, and \( \Phi \) is an unknown vector field on the phase space manifold \( \mathcal{Q} \). Generically, such reconstruction is possible when the measurement of a single scalar
time-dependent signal \( y(t) \), which is a function of the system state \( s(t) \), is available. Many
practically interesting systems, symmetric ones in particular, are, however, extremely non-
generic and require a number of independent scalar signals for the complete reconstruction.
Eckmann and Ruelle [12] acknowledged that the choice of signals has to be made carefully
by trial and error. Certain general rules concerning this choice, however, can be established
on purely theoretical grounds, since this problem too can be effectively treated based on the
knowledge of underlying symmetries [13].

The outline of the paper is as follows. In section II we discuss the implications of sym-
metry for control of discrete-time linear systems in the vicinity of steady target states. The
results are generalized for time-periodic target trajectories in section III and for continuous-
time systems in section IV. Section V is devoted to the problem of weak symmetry violation.
The issue of phase space reconstruction is considered in section VI. The theoretic results are
illustrated using several simple examples in section VII. Finally, we pre-
sent our conclusions in section VIII.

II. TIME-INvariant STATES

A. Stabilizability and Controllability

Although our analysis is applicable to time-varying systems, we start for simplicity by
assuming that the target state is time-invariant, \( \bar{x}^t = \bar{x} \). Then the matrices \( A^t \) and \( B^t \)
become constant, and we can drop the time index in (2) to obtain

\[
\Delta x^{t+1} = A \Delta x^t + B \Delta u^t.
\]

(6)

It is useful to introduce and compare two characterizations of the linearized evolution equa-
tion (6), which extremely simplify the analysis of feedback control algorithms: stabilizability
and controllability.

The dynamical system (6) or the pair \((A, B)\) is said to be stabilizable, if there exists a
state feedback

\[
\Delta u^t = -K \Delta x^t,
\]

(7)

making the system (6) stable, i.e., it is possible to find a feedback gain matrix \( K \), such
that all eigenvalues \( \lambda_k \) of the matrix \( A' = A - BK \) lie within a unit circle of the complex
plane, \( |\lambda_k| < 1, \forall k \). Otherwise the system or the pair \((A, B)\) is called unstabilizable. Indeed,
substituting the feedback (7) into (6) one obtains the linearized evolution equation for the
closed-loop system

\[
\Delta x^{t+1} = (A - BK) \Delta x^t,
\]

(8)

with \( \Delta x = 0 \) becoming the stable fixed point of the map (8), if and only if \( A - BK \) is stable.

Since the magnitude of the control perturbation \( \Delta u^t \) is proportional to the deviation
\( \Delta x^t \) of the system from the target state, feedback of the form (7) is often called proportional
in the physics literature, although there are a number of other terms used to denote this
type of feedback. Control theory uses the term state feedback to refer to the fact that the
state of the system is used to determine the control perturbation. At first sight equation (7)
seems to impose strict limitations on the allowed form of the feedback law. However, this is
precisely the form demanded by a number of widely used control algorithms [3,4,14].

Stabilizability is a property, which usually sensitively depends on the values of system
parameters. In the majority of practical applications, however, it is preferable to have
an adaptive control that would stabilize a given steady state $\mathbf{x}(\mathbf{u})$ for arbitrary values of system parameters. This is especially important, if one is to track the trajectory $\mathbf{x}$ as parameters slowly vary, which might be advantageous in many applications, e.g., for moving the operating point of a nonlinear device across a bifurcation, from the stable region to the chaotic region. Such a control scheme can be obtained, if the more restrictive condition of controllability, which is essentially parameter-independent, is imposed on the matrices $A$ and $B$. On the other hand, it can be demonstrated [11] that the controllability condition guarantees that the eigenvalues of the matrix $A - BK$ can be freely assigned (with complex ones in conjugate pairs) by an appropriate choice of the matrix $K$. Therefore, if the system is controllable, it is stabilizable as well, and by requiring controllability we satisfy both conditions at once.

The $n_x$-dimensional linear system (6) or the pair $(A, B)$ is said to be controllable if, for any initial state $\Delta \mathbf{x}_{t_i} = \Delta \mathbf{x}_i$, times $t_f - t_i \geq n_x$, and final state $\Delta \mathbf{x}_f$, there exists a sequence of control perturbations $\Delta \mathbf{u}^t, \cdots, \Delta \mathbf{u}^{t-1}$ such that the solution of equation (6) satisfies $\Delta \mathbf{x}_{t_f} = \Delta \mathbf{x}_f$. Otherwise, the system or the pair $(A, B)$ is called uncontrollable.

The controllability condition can be represented in a number of different equivalent forms. To obtain one particularly convenient form, we make the trivial observation that, if it is possible to drive the linear system from an arbitrary initial state $\Delta \mathbf{x}_i$ to an arbitrary final state $\Delta \mathbf{x}_f$ in $n_x$ steps, it is possible to do the same in any number of steps $n$ exceeding $n_x$. Suppose we let the system evolve under control for $n_x$ steps from the initial state $\Delta \mathbf{x}_{t_i}$. The final state will be given by

$$\Delta \mathbf{x}_{t_f} = (A)^{n_x} \Delta \mathbf{x}_i + \sum_{k=1}^{n_x} (A)^{n_x-k} B \Delta \mathbf{u}^{t+k-1}. \quad (9)$$

Denote $\mathbf{b}_m$ the $m$th column of the matrix $B$:

$$B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_{n_u}]. \quad (10)$$

Regarding the terms $(A)^{n_x-k} \mathbf{b}_m$ as vectors in the $n_x$-dimensional tangent space $\mathcal{T}$,

$$\mathbf{h}_m^k = (A)^{n_x-k} \mathbf{b}_m, \quad k = 1, \cdots, n_x, \quad m = 1, \cdots, n_u, \quad (11)$$

and the control perturbations $\Delta \mathbf{u}_m^{t+k-1}$ as coordinates, we immediately conclude that equation (9) rewritten as

$$\Delta \mathbf{x}_f - (A)^{n_x} \Delta \mathbf{x}_i = \sum_{k=1}^{n_x} \sum_{m=1}^{n_u} \Delta \mathbf{u}_m^{t+k-1} \mathbf{h}_m^k \quad (12)$$

can only be satisfied, if and only if there are $n_x$ linearly independent vectors in the set $\{\mathbf{h}_m^k\}$, i.e., the set $\{\mathbf{h}_m^k\}$ spans the tangent space $\mathcal{T}$. This is equivalent to requiring that

$$\text{rank}(\mathcal{C}) = n_x, \quad (13)$$

1Here and below in the text we use the notation $(A)^n$ to indicate that $A$ is taken to the power of $n$ to differentiate it from the notation $A^t$, where index $t$ defines the time dependence.
where the matrix
\[
C \equiv \begin{bmatrix} B & AB & (A)^2B & \cdots & (A)^{n_x-1}B \end{bmatrix} \tag{14}
\]
is called the controllability matrix. Condition (13) was introduced into the physics literature from linear systems theory by Romeiras et al. \[\text{[4]}\] as a simple, but practical test of the controllability.

In contrast, the stabilizability condition requires that the set (11) spans only the unstable subspace \(L^u \subseteq T\) of the Jacobian \(A\), instead of the whole tangent space \(T\). Stabilizability can be formally expressed in the form identical to (13). Let us define the number of stable and unstable eigenvalues of the Jacobian \(n_s^x\) and \(n_u^x\), respectively (one obviously has \(n_s^x + n_u^x = n_x\)). For instance, if \(A\) is a diagonalizable matrix, it has \(n_s^x\) linearly independent stable eigenvectors which we denote \(e_{is}^s, i = 1, \cdots, n_s^x\). It can, therefore, be shown using an appropriate coordinate transformation that the pair \((A, B)\) is stabilizable if and only if
\[
\text{rank}(S) = n_x, \tag{15}
\]
where the matrix
\[
S \equiv \begin{bmatrix} e_{i1}^s & \cdots & e_{n_s^x}^s & B & AB & \cdots & (A)^{n_u^x-1}B \end{bmatrix} \tag{16}
\]
can be called the stabilizability matrix by analogy with the controllability matrix.

In order to better understand the restrictions imposed on the control scheme by symmetries, it is beneficial to look at the controllability condition from the geometrical point of view, assuming \(n_u = 1\) and, consequently, \(B = b\). The controllability in this context is equivalent to the vectors \(h^1, h^2, \cdots, h^{n_x}\) spanning the tangent space \(T\). Generically, the matrix \(A\) is nondegenerate (has a nondegenerate spectrum), so one can always find a vector \(b\), such that the resulting set (11) forms a basis. However, if \(A\) is degenerate, which is a usual consequence of symmetry, there will exist an eigenspace of the Jacobian, \(L^r \subset T\), such that \(x^\dagger A = \lambda x^\dagger, \forall x \in L^r\) with the dimension \(d_r = \text{dim}(L^r) > 1\), where \(\dagger\) denotes (complex conjugate) transpose of a matrix or vector. The dynamics of the system in such an eigenspace cannot be controlled with just one control parameter (see \[\text{[4]}\] for an example of such a situation), because the vectors \(h^k\) only span a one-dimensional subspace of \(L^r\). Indeed, since \(d_r > 1\) there will exist \(d_r - 1\) adjoint eigenvectors \(f_j \in L^r\) orthogonal to \(b\) and each other. Then
\[
(f_j \cdot h^k) = f_j^\dagger(A)^{n_u-1}b = \lambda_r^{n_u-1} f_j^\dagger b = \lambda_r^{n_u-1} (f_j \cdot b) = 0, \tag{17}
\]
so every basis vector \(h^k\) is orthogonal to every eigenvector \(f_j, j = 1, \cdots, d_r - 1\).

It is often convenient to define the notion of controllability for individual eigenvectors. We will say that the adjoint eigenvector \(f\) of the Jacobian \(A\) is controllable, if there exists \(m, 1 \leq m \leq n_u\), such that \((f \cdot b_m) \neq 0\). Respectively, the eigenvector that is orthogonal to every column of the control matrix \(B\) is called uncontrollable. Using these definitions we

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\(\text{2For the purpose of control we regard the central directions, defined by the eigenvalues } \lambda \text{ such that } |\lambda| = 1 \text{ (Re}(\lambda) = 0 \text{ in the continuous-time case), as unstable.} \]
can, therefore, conclude that the controllability of the linearized system is equivalent to the controllability of each and every adjoint eigenvector of the Jacobian matrix (also see [13]). Similarly, the stabilizability is equivalent to the controllability of each and every unstable adjoint eigenvector.

If the system dynamics in $L^r$ happens to be stable (e.g., when the system is stabilizable, but uncontrollable), the system can still be stabilized similarly to the nondegenerate case, but we have to ensure the controllability in case the dynamics in this eigenspace is unstable. This can be achieved by increasing the number of control parameters $n_u$, which extends the set (11), until it spans every eigenspace of $\mathcal{T}$. This would lead one to assume that the minimal value of $n_u$ should be defined by the highest degeneracy of the Jacobian matrix $A$.

We will see, however, that various kinds of degeneracy have a somewhat different effect on the controllability of the system.

### B. Symmetries of the System

Symmetries usually significantly simplify the analysis of system dynamics, and the control problem is no exception. In particular, even when the exact form of the Jacobian matrix is unknown, the structure of the symmetry group describing the symmetries of the system allows one to reduce the controllability condition (13) to a set of much simpler conditions, which provide a number of system-independent results. The discussion below is based on bifurcation theory [2] and closely parallels the treatment of degeneracy in quantum mechanics and spontaneous symmetry breaking in quantum field theory and phase transitions.

In general we call the system symmetric, if the nonlinear evolution equation preserves its form under a set of linear transformations $g: \mathbf{x} \to \mathbf{x}' = g(\mathbf{x})$ of the phase space. More formally, we say that the evolution equation (1) possesses a structural symmetry described by a symmetry group $\mathcal{G}$, if the map $F$ commutes with all group actions:

$$
F(g(\mathbf{x}), \mathbf{u}) = g(F(\mathbf{x}, \mathbf{u})), \quad \forall g \in \mathcal{G}, \forall \mathbf{x} \in \mathcal{T}
$$

or, in other words, if the function $F(\mathbf{x}, \mathbf{u})$ is $\mathcal{G}$-equivariant with respect to its first argument. The group $\mathcal{G}$ is usually a byproduct of symmetries of the underlying physical space, such as rotational and translational symmetry (domain symmetry), and symmetries of the phase space, such as phase symmetry $\phi \to \phi + 2\pi$ (range symmetry). Since all interesting physical symmetries are unitary (such rare exceptions as the Lorentz group are hardly relevant in the context of control problem), we will assume that $\mathcal{G}$ is a unitary group.

Usually, the symmetry demonstrates itself in more than just one way: often steady (as well as time-periodic) states $\bar{\mathbf{x}}$ of symmetric systems too will be symmetric with respect to transformations $g \in \mathcal{H}_{\bar{\mathbf{x}}}$, where $\mathcal{H}_{\bar{\mathbf{x}}} \subseteq \mathcal{G}$ is an isotropy subgroup of $\bar{\mathbf{x}}$. In general, the target state $\bar{\mathbf{x}}$ might also be symmetric with respect to transformations which do not belong to $\mathcal{G}$. However, considering those does not provide any additional information, so we assume that

$$
g(\bar{\mathbf{x}}) = \bar{\mathbf{x}}, \quad \forall g \in \mathcal{H}_{\bar{\mathbf{x}}}.\quad (19)
$$

For the purpose of control it is important to observe that upon linearization about the target state $\bar{\mathbf{x}}$ the structural symmetry of the evolution equation (1) does not disappear, but is replaced with a related dynamical symmetry. Indeed, using the definitions (18), (19) and
the fact that symmetry transformations are linear, one obtains in the linear approximation for an arbitrary \( g \in \mathcal{H}_\bar{x} \):

\[
\bar{x} + g(A\Delta x) = g(\bar{x}) + g(A\Delta x) = g(\bar{x} + A\Delta x) \\
= g(F(\bar{x}, \bar{u}) + A\Delta x) = g(F(\bar{x} + \Delta x, \bar{u})) \\
= F(g(\bar{x} + \Delta x), \bar{u}) = F(\bar{x} + g(\Delta x), \bar{u}) \\
= F(\bar{x} + g(\Delta x), \bar{u}) = F(F(\bar{x} + \Delta x, \bar{u})), \quad \forall g \in \mathcal{H}_\bar{x}.
\]

(20)

Defining \( \mathcal{L} \) the full symmetry group of the linearized equation (3) in the absence of control \((\Delta u^t = 0)\):

\[
g(A\Delta x) = Ag(\Delta x), \quad \forall g \in \mathcal{L},
\]

one concludes that the group \( \mathcal{L} \) describing the dynamical symmetry of the system in the vicinity of the target state \( \bar{x} \) includes all transformations \( g \in \mathcal{H}_\bar{x} \), and therefore:

\[
\mathcal{H}_\bar{x} \subseteq \mathcal{L}.
\]

(22)

One can speculate that typically \( \mathcal{L} \) will coincide with \( \mathcal{H}_\bar{x} \). As a consequence, if the target state \( \bar{x} \) has low symmetry, the symmetry of the evolution equation will be reduced upon linearization to a subgroup of \( \mathcal{G} \). However, as we will see in section [VII], \( \mathcal{L} \) might be equal to \( \mathcal{G} \), or even include \( \mathcal{G} \) as a subgroup for highly symmetric target states, with the apparent symmetry increased by linearization.

It turns out that with the help of group representation theory one can substantially simplify the controllability condition (13) and, as a result, obtain a number of useful restrictions on the set of control parameters. Consider the matrix representation \( T \) generated in the tangent space \( \mathcal{T} \) by the action of transformations \( g \) from an arbitrary subgroup \( \mathcal{L}' \) of the full dynamical symmetry group \( \mathcal{L} \):

\[
(g(x))_i = (T(g)x)_i = \sum_{j=1}^{n_x} T_{ij}(g)x_j, \quad \forall x \in \mathcal{T},
\]

(23)

where, according to (21), all matrices \( T(g) \) commute with the Jacobian

\[
T(g)A = AT(g), \quad \forall g \in \mathcal{L}' \subseteq \mathcal{L}.
\]

(24)

The knowledge of the representation \( T \) is enough to derive a very simple criterion for the admissibility of the control matrix. Observe that, if \( T(g)B = B \) for an arbitrary transformation \( g \in \mathcal{L}' \), then

\[
\mathcal{C} = [T(g)g, \quad AT(g)B, \quad \cdots, \quad (A)^{n_x-1}T(g)B] = [T(g)B, \quad T(g)AB, \quad \cdots, \quad T(g)(A)^{n_x-1}B] = T(g)\mathcal{C}.
\]

(25)

As a result, since \( T_{ij}(g) \neq \delta_{i,j} \) for any \( g \neq e \) (where we defined \( e \) as the identity transformation: \( e(x) = x \)), the rows \( \tilde{c}_j \) of the controllability matrix become linearly dependent,

\[
\sum_{j=1}^{n_x} (T_{ij}(g) - \delta_{i,j})\tilde{c}_j = 0,
\]

(26)
and the controllability condition (13) is violated. Therefore, we obtain a necessary condition on the control matrix:

\[ T(g)B \neq B, \quad \forall g \in \mathcal{L}' \setminus \{e\}. \quad (27) \]

In other words, the control arrangement should be chosen such that the symmetry of the linearized evolution equation (6) is completely broken for (almost all) nonzero control perturbations \( \Delta u \neq 0 \).

C. Group Coordinates

Though simple and general, criterion (27) is not very helpful for finding the minimal set of control parameters satisfying the controllability condition. In order to derive a more practically useful criterion one has to make a few more steps. We begin with the reduction of the controllability condition to a set of simpler conditions which can be performed [11] by constructing the Jordan block decomposition of the Jacobian matrix. This task can be greatly simplified by transforming to the “group coordinates,” defined with respect to the basis set composed of vectors which transform according to different irreducible representations contained in \( T \), in which the Jacobian is block-diagonal. In practice, it is usually impossible to determine whether the isotropy group \( \mathcal{H}_{x} \) exhausts the dynamical symmetries of the system or the group \( \mathcal{L} \) contains some hidden symmetries as well. It is, therefore, important to show that a number of restrictions on the set of control parameters can be obtained using an arbitrary unitary subgroup \( \mathcal{L}' \) of \( \mathcal{L} \).

Decomposing the representation \( T \) into a sum of irreducible representations \( T^{r} \) of the group \( \mathcal{L}' \) with respective dimensionalities \( d_{r} \), we obtain:

\[ T = p_{1}T^{1} \oplus p_{2}T^{2} \oplus \cdots \oplus p_{q}T^{q} \quad (28) \]

with

\[ n_{x} = p_{1}d_{1} + p_{2}d_{2} + \cdots + p_{q}d_{q}, \quad (29) \]

where \( p_{r} \) denotes the number of equivalent representations \( T^{r} \) present in the decomposition (28), and \( q \) is the total number of nonequivalent irreducible representations. Since \( \mathcal{L}' \) is unitary, all irreducible representations \( T^{r} \) in (28) can be chosen as unitary [16].

The tangent space \( \mathcal{T} \) is similarly decomposed into a sum of invariant subspaces \( L_{\mathcal{L}}^{r_{\alpha}} \) such that \( T(g)x \in L_{\mathcal{L}}^{r_{\alpha}}, \forall x \in L_{\mathcal{L}}^{r_{\alpha}} \) and \( \forall g \in \mathcal{L}' \):

\[ \mathcal{T} = L_{\mathcal{L}}^{r_{1}} \oplus L_{\mathcal{L}}^{r_{2}} \oplus \cdots \oplus L_{\mathcal{L}}^{r_{q}}, \quad (30) \]

where

\[ L_{\mathcal{L}}^{r_{\alpha}} = L_{\mathcal{L}'}^{r_{1}} \oplus L_{\mathcal{L}'}^{r_{2}} \oplus \cdots \oplus L_{\mathcal{L}'}^{r_{p}}, \quad (31) \]

and \( \alpha = 1, \cdots, p_{r} \) indexes different invariant subspaces, which correspond to the same group of equivalent irreducible representations \( T^{r} \). It should be noted that even though the decomposition (30) is unique, the decomposition (31) is not, unless \( p_{r} = 1 \). Let us introduce
a basis in each invariant subspace \( L_{r}^{\alpha} \) and denote the basis vectors \( e_{i}^{\alpha} \), \( i = 1, \cdots, d_r \). We choose the basis vectors such that they transform according to the irreducible representation \( T^{r} \), i.e.,

\[
T(g) e_{i}^{\alpha} = \sum_{j=1}^{d_r} T_{ij}^{r}(g) e_{j}^{\alpha}, \quad \forall g \in L'.
\] (32)

For unitary \( T^{r} \) a generalized orthogonality condition between basis vectors \( e_{i}^{\alpha} \) can be established [10] as a consequence of (32):

\[
(e_{i}^{\beta} \cdot e_{j}^{\alpha}) = \delta_{r,s} \delta_{i,j} (e_{i}^{\beta} \cdot e_{i}^{\alpha}).
\] (33)

In addition, for \( p_r > 1 \) the decomposition (31) can always be performed in such a way that \( (e_{i}^{\beta} \cdot e_{j}^{\alpha}) = \delta_{\alpha,\beta} \) (this, however, still leaves some freedom in choosing the invariant subspaces \( L_{r}^{\alpha} \)), so that the complete set of basis vectors \( \{e_{i}^{\alpha}\} \), where \( r = 1, \cdots, q \), \( \alpha = 1, \cdots, p_r \), and \( i = 1, \cdots, d_r \) is made orthonormal. We, therefore, conclude that the matrix \( P \) defined by

\[
P = \begin{pmatrix}
P^{1} \\
\vdots \\
P^{q}\end{pmatrix}, \quad P^{r} = \begin{pmatrix}
P_{1}^{r} \\
\vdots \\
P_{d_r}^{r}\end{pmatrix}, \quad P_{i}^{r} = \begin{pmatrix}
(e_{i}^{r1})^{\dagger} \\
\vdots \\
(e_{i}^{rp})^{\dagger}\end{pmatrix},
\] (34)

is orthogonal, \( (P)^{-1} = P^{\dagger} \) (or, more generally, unitary).

Furthermore, according to the Wigner-Eckart theorem [10], the matrix elements of an arbitrary matrix (and the Jacobian \( A \), in particular) invariant with respect to any group transformation

\[
T(g)AT^{-1}(g) = A, \quad \forall g \in L',
\] (35)

satisfy the following general formula:

\[
(e_{i}^{\beta} \cdot Ae_{j}^{\alpha}) = \delta_{r,s} \delta_{i,j} (e_{i}^{\beta} \cdot e_{i}^{\alpha}),
\] (36)

and the scalar product

\[
(\bar{\Lambda}^{r})_{\alpha\beta} \equiv (e_{i}^{\alpha} \cdot Ae_{i}^{\beta})
\] (37)

is independent of the index \( i = 1, \cdots, d_r \) (but depends on the decomposition (31)). As a result, on transformation to the group coordinates the Jacobian matrix becomes block diagonal:

\[
\bar{A} = PAP^{-1} = \begin{bmatrix}
\bar{A}^{1} & \cdots \\
\vdots & \ddots \\
\bar{A}^{q}
\end{bmatrix},
\] (38)

where each block \( \bar{A}^{r} \) is itself block-diagonal

\[
\bar{A}^{r} = \begin{bmatrix}
\bar{\Lambda}^{r} & \cdots \\
\vdots & \ddots \\
\bar{\Lambda}^{r}
\end{bmatrix}
\] (39)
and consists of $d_r$ identical $p_r \times p_r$ blocks $\Lambda^r$ with the matrix elements defined by the scalar product (37).

If no irreducible representation $T^r$ of $L'$ enters the decomposition (28) more than once, i.e., $p_1 = \cdots = p_q = 1$, the structure of the Jacobian matrix is completely resolved: the transformed Jacobian is diagonal and its spectrum consists of eigenvalues $\lambda_r = \Lambda^r$, $r = 1, \cdots, q$ with multiplicity $d_r$, while the basis vectors $e_{i^r\alpha}$ become the corresponding eigenvectors (and, consequently, define the normal modes of the linearized system). In this case the invariant subspaces of the group $L'$ define the eigenspaces of the Jacobian, $L^r = L^r_{L'}$. Clearly, the spectrum becomes degenerate, if the symmetry is sufficiently high (such that $T$ contains at least one irreducible representation $T^r$ with dimensionality larger than one).

Degeneracy should not necessarily be associated with symmetry and might be accidental (with respect to the group $L'$). For instance, it can happen that $\Lambda^r = \Lambda^r'$ for some $r \neq r'$, so that the multiplicity of the eigenvalue $\lambda_r$ is increased respectively to $d_r + d_{r'}$. Accidental degeneracies can be alternatively thought of as a consequence of hidden symmetries contained in the full symmetry group $L$ of which $L'$ is a subgroup. However, the degeneracies not associated with some physical symmetry are likely to disappear under a typical perturbation, such as a change of system parameters and, therefore, are most conveniently regarded as accidental. Since the full symmetry group $L$, in general, depends on system parameters and cannot be directly deduced from the structural symmetry group $G$, it is usually more convenient to use its parameter-independent subgroup $L' = H_{\xi}$ instead.

D. Jordan Decomposition

If the symmetry described by $L'$ is low, a number of equivalent irreducible representations will typically be found in the decomposition (28), i.e., we will have $p_r > 1$ for certain $r$. In this case the knowledge of the dynamical symmetries alone is not sufficient to completely determine the structure of the Jacobian matrix, which is, in general, system-dependent. As a result, one has to solve a secular equation

$$(\Lambda^r - \lambda_{ra} I)e_{a}^r = 0,$$  \hspace{1cm} (40)

for each block $\Lambda^r$ with $p_r > 1$ in order to find the eigenvectors in the invariant subspace $L^r_{L'}$ and the respective eigenvalues. Here, unlike the case of quantum mechanics, the Jacobian matrix does not have to be Hermitian and, therefore, might not be diagonalizable. However, $\Lambda^r$ can always be reduced to the Jordan normal form by finding the coordinate transformation $\bar{Q}^r$ such that

$$\Lambda^r = \bar{Q}^r \Lambda^r (\bar{Q}^r)^{-1} = \begin{bmatrix} \Lambda^{r1} & \cdots \\ \vdots & \ddots \\ \Lambda^{rp'} & \cdots \end{bmatrix},$$  \hspace{1cm} (41)

where $p_r' \leq p_r$ is the number of distinct eigenvalues and the Jordan superblock

$$\Lambda^{r\alpha} = \begin{bmatrix} \Lambda^{r\alpha}_{1} & \cdots \\ \vdots & \ddots \\ \Lambda^{r\alpha}_{p_{r'}} & \cdots \end{bmatrix}$$  \hspace{1cm} (42)
corresponding to the eigenvalue $\lambda_{ra}$ consists of $j_{ra}$ Jordan blocks

$$\Lambda_i^{r\alpha} = \begin{bmatrix} \lambda_{ra} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$  (43)

In the absence of accidental degeneracy all eigenvalues of $\bar{\Lambda}^r$ are different, so that $p'_r = p_r$ and $j_{ra} = 1$ for all $\alpha$, i.e., $\Lambda^r$ is diagonal.

Since each block $\bar{A}^r$ of the transformed Jacobian $\bar{\Lambda}^r$ consists of $d_r$ identical blocks $\bar{\Lambda}^r$, applying the coordinate transformation defined by the block-diagonal matrix assembled from $d_r$ blocks $\bar{Q}^r$,

$$Q^r = \begin{bmatrix} \bar{Q}^r & & \\ & \ddots & \\ & & \bar{Q}^r \end{bmatrix},$$  (44)

reduces $\bar{A}^r$ to the Jordan normal form:

$$\bar{A}^r = Q^r \bar{A}^r (Q^r)^{-1} = \begin{bmatrix} \Lambda^r & \cdots \\ & \ddots \\ & & \Lambda^r \end{bmatrix}.$$  (45)

The Jordan blocks on the diagonal of $\bar{A}^r$ will not, in general, be arranged in superblocks with the same eigenvalue. This, however, can be trivially corrected by permuting the rows and columns of $\bar{A}^r$ to obtain the matrix

$$\hat{A}^r = R^r \bar{A}^r (R^r)^{-1} = \begin{bmatrix} \hat{A}^{r1} & \cdots \\ & \ddots \\ & & \hat{A}^{rp'} \end{bmatrix},$$  (46)

where $R^r$ is the permutation matrix arranging the identical Jordan blocks next to each other, and the Jordan superblock corresponding to the eigenvalue $\lambda_{ra}$ has the form

$$\hat{A}^{r\alpha} = \begin{bmatrix} \hat{A}^{r\alpha}_1 & \cdots \\ & \ddots \\ & & \hat{A}^{r\alpha}_{j_{ra}} \end{bmatrix}.$$  (47)

Each block $\hat{A}^{r\alpha}_i$ is, in turn, composed of $d_r$ identical Jordan blocks $\Lambda_i^{r\alpha}$, defined by (43):

$$\hat{A}^{r\alpha}_i = \begin{bmatrix} \Lambda_i^{r\alpha} & \cdots \\ & \ddots \\ & & \Lambda_i^{r\alpha} \end{bmatrix}.$$  (48)

Defining the block-diagonal coordinate transformation matrices $Q$ and $R$

$$Q = \begin{bmatrix} Q^1 & \cdots \\ & \ddots \\ & & Q^q \end{bmatrix}, \quad R = \begin{bmatrix} R^1 & \cdots \\ & \ddots \\ & & R^q \end{bmatrix},$$  (49)
we eventually obtain the sequence of coordinate transformations reducing the Jacobian matrix \( A \) to the Jordan normal form:

\[
\hat{A} = (RQP)A(RQP)^{-1} = \begin{bmatrix}
\hat{A}_1 & & \\
& \ddots & \\
& & \hat{A}_q \\
\end{bmatrix},
\]

where each block \( \hat{A}_r, r = 1, \ldots, q \) is defined by (46).

### E. Conditions for Controllability

Once the Jacobian is reduced to the Jordan normal form, we can turn to the problem of reducing the controllability condition to a set of simpler conditions that will give us the restrictions on the admissible set of control parameters. Since the controllability is a property of the system which does not depend on the choice of the coordinate system, condition (13) is invariant with respect to any (nonsingular) coordinate transformation \([11]\), and hence is satisfied for the pair \((A, B)\), if and only if it is satisfied for the pair \((\hat{A}, \hat{B})\), where \( \hat{B} = (RQP)B \) is the transformed control matrix. Let us partition the transformed control matrix \( \hat{B} \) according to the block structure of \( \hat{A} \):

\[
\hat{B} = \begin{bmatrix}
\hat{B}_1 \\
\vdots \\
\hat{B}_q
\end{bmatrix}, \quad \hat{B}^r = \begin{bmatrix}
\hat{B}^{r1} \\
\vdots \\
\hat{B}^{r_\alpha}
\end{bmatrix}, \quad \hat{B}^{\alpha} = \begin{bmatrix}
\hat{B}^{\alpha_1} \\
\vdots \\
\hat{B}^{\alpha_\alpha}
\end{bmatrix}, \quad \hat{B}^{\alpha}_i = \begin{bmatrix}
\hat{b}^{\alpha_1}_i \\
\vdots \\
\hat{b}^{\alpha_\alpha}_i
\end{bmatrix}
\]

and denote \( \hat{b}^{\alpha_\beta}_{ij} \) the first row of the matrix \( \hat{B}^{\alpha}_i \). Next, define the matrix \( \hat{B}^{\alpha} \) using the relations

\[
\hat{B}^{\alpha} = \begin{bmatrix}
\hat{B}^{\alpha_1} \\
\vdots \\
\hat{B}^{\alpha_\alpha}
\end{bmatrix}, \quad \hat{B}^{\alpha}_i = \begin{bmatrix}
\hat{b}^{\alpha_1}_i \\
\vdots \\
\hat{b}^{\alpha_\alpha}_i
\end{bmatrix}.
\]

In the absence of accidental degeneracy between the eigenvalues that correspond to different invariant subspaces \( L^r \), equation (47) ensures that there are exactly \( d_{r, j_{\alpha}} \) Jordan blocks \( \Lambda^r_{\alpha} \) with the same eigenvalue \( \lambda_{r, \alpha} \). If, however, there is such an accidental degeneracy involving \( s \) different invariant subspaces \( L^1_{L^r}, \ldots, L^s_{L^r} \), such that for certain \( \alpha_1, \ldots, \alpha_s \)

\[
\lambda_{r_1, \alpha_1} = \cdots = \lambda_{r_s, \alpha_s},
\]

the number of Jordan blocks corresponding to the eigenvalue \( \lambda_{r, \alpha} \) increases to

\[
\hat{j}_r^\alpha = \sum_{r', \alpha' : \lambda_{r', \alpha'} = \lambda_{r, \alpha}} d_{r', j_{r', \alpha'}}.
\]

The knowledge of the number of Jordan blocks is very useful, since, according to the standard result of linear system theory \([11]\), it ultimately determines the minimal number of control parameters. Specifically, it can be shown that the controllability condition for the pair of matrices \((\hat{A}, \hat{B})\) is satisfied, if and only if for every \( r \) and \( \alpha \) (taken equal to \( r_1 \) and \( \alpha_1 \) below)
\[
\text{rank } \begin{bmatrix} \hat{B}^{r_1\alpha_1} \\ \vdots \\ \hat{B}^{r_s\alpha_s} \end{bmatrix} = j'_r = d_r j_{r_1\alpha_1} + \cdots + d_r j_{r_s\alpha_s},
\]

where the indices \( r_i \) and \( \alpha_i \) are chosen according to (53). This, in turn, can be achieved, if and only if \( n_u \geq j'_r \) for every \( r \) and \( \alpha \). Hence, in the most general case the minimal number \( \bar{n}_u \) of independent control parameters should equal the maximal number of Jordan blocks with the same eigenvalue \( \lambda_{r\alpha} \):

\[
\bar{n}_u = \max_{r=1,\ldots,q} \max_{\alpha=1,\ldots,p_r} j'_r.
\]

Note that, since the block \( \hat{B}^{r\alpha} \) has \( d_r j_{r\alpha} \) rows, \( \text{rank}(\hat{B}^{r\alpha}) \leq d_r j_{r\alpha} \) for every \( r \) and \( \alpha \).

Using this fact, the trivial matrix inequality

\[
\text{rank } \begin{bmatrix} \hat{B}^{r_1\alpha_1} \\ \vdots \\ \hat{B}^{r_s\alpha_s} \end{bmatrix} \leq \text{rank}(\hat{B}^{r_1\alpha_1}) + \cdots + \text{rank}(\hat{B}^{r_s\alpha_s}),
\]

and equation (55) one obtains

\[
\text{rank}(\hat{B}^{r\alpha}) = d_r j_{r\alpha}, \quad r = 1, \ldots, q, \quad \alpha = 1, \ldots, p_r.
\]

Furthermore, according to the definition (52) of the matrix \( \hat{B}^{r\alpha} \), for every \( r \) and \( \alpha \) \( \text{rank}(\hat{B}^{r\alpha}) \geq \text{rank}(\hat{B}^{r\alpha}) \), so one can write

\[
\text{rank}(\hat{B}^{r\alpha}) = d_r j_{r\alpha}, \quad r = 1, \ldots, q, \quad \alpha = 1, \ldots, p_r.
\]

In addition, since \( Q^r \) and \( R^r \) are nonsingular coordinate transformations which do not change the rank of a matrix,

\[
\text{rank}(\hat{B}^{r\alpha}) = \text{rank}(R^r Q^r P^r B) = \text{rank}(P^r B),
\]

where we got rid of all system-specific information, which was contained in the matrices \( Q^r \) and \( R^r \).

The symmetry information alone is insufficient to determine the values of either \( j_{r\alpha} \) or \( j'_r \). However, by definition one has \( p_r \geq j_{r\alpha} \geq 1 \) so that \( j'_r \geq d_r \). As a consequence, we obtain two necessary conditions for controllability. First of all, equation (66) yields the lower bound on the minimal number of control parameters

\[
\bar{n}_u \geq \max_{r=1,\ldots,q} d_r.
\]

Second, inequality (59) combined with equality (60) imposes a number of restriction on the control matrix \( B \),

\[
\text{rank}(P^r B) \geq d_r, \quad r = 1, \ldots, q.
\]
which can be interpreted as the requirement of the mutual independence of control parameters. We can therefore, conclude that an arbitrary (unitary) subgroup $L'$ of the full dynamical symmetry group $L$ does not completely define the minimal set of control parameters. It does, however, define a set of necessary conditions required for controllability. In general, the knowledge of all dynamical symmetries, both unitary and nonunitary, described by the group $L$ is required in order to completely resolve the structure of the Jacobian matrix and obtain the necessary and sufficient condition for controllability.

Nevertheless, even without knowing the full symmetry group $L$ one can obtain the necessary and sufficient conditions by making a number of assumptions. First, assume that there are no accidental degeneracies (it is usually safe to do so if, e.g., $L'$ is taken to coincide with $H_K$: we ensure that all physical symmetries are taken into account, and accidental degeneracies should only appear for certain special values of system parameters). Then $j\alpha = 1$, $j'_\alpha = d_r$, and $\bar{B}^{\alpha} = \bar{B}'^{\alpha}$ for all $r$ and $\alpha$, so condition (56) is equivalent to

$$\bar{n}_u = \max_{r=1,\ldots,q} d_r.$$  \hfill (63)

If, in addition, no irreducible representation $T^r$ of $L'$ enters the decomposition (28) more than once, such that $p_r = 1$ for all $r$, instead of inequality (62) one obtains the equality:

$$\text{rank}(P^r B) = d_r, \quad r = 1, \ldots, q.$$  \hfill (64)

Conditions (62) and (64) can be simplified even further by defining the projection operator $\hat{P}^r \equiv (P^r)^\dagger P^r$ onto the invariant subspace $L'_r \subset T$. This operator can be obtained directly from the matrix representation $T$ for most symmetry groups of interest. For finite discrete groups it is given by

$$\hat{P}^r = \frac{d_r}{n_g} \sum_{g \in L'} \chi^r(g) T(g),$$  \hfill (65)

where $n_g$ is the number of elements of the group $L'$ and $\chi^r(g)$ is the character of the group element $g$ in the representation $T^r$. Similarly, for compact continuous groups we have

$$\hat{P}^r = d_r \int_{L'} \chi^r(g) T(g) \, d\mu(g),$$  \hfill (66)

where $d\mu(g)$ is the group measure \cite{16}. Observing that $\text{rank}((P^r)^\dagger P^r B) = \text{rank}(P^r B)$, we can use the projection operators to rewrite the condition (64) in an equivalent form

$$\text{rank}(\hat{P}^r B) = d_r, \quad r = 1, \ldots, q.$$  \hfill (67)

Summing up, we conclude that with the two assumptions made above the system is controllable, if and only if the two conditions are met. The first one requires the number $n_u$ of control parameters to be greater or equal to the dimensionality $d_r$ of the largest irreducible representation $T^r$ present in the decomposition of the matrix representation $T$ of the subgroup $L' \subseteq L$ in the tangent space $T$. The second one requires the control parameters to be independent: the columns $b_m$ of the control matrix $B$ have to be chosen such that $d_r$ of the projections $\hat{P}^r b_m$, $m = 1, \ldots, n_u$ are linearly independent (and, therefore, span
the eigenspace $L^r = L^r_{r'}$ for every $r = 1, \ldots, q$. The last requirement imposes a number of restrictions on the admissible form of the linear response of the system to perturbations of control parameters.

A number of comments are in order. First of all, as we have just seen, the number of control parameters is determined by the number of Jordan blocks with the same eigenvalue, not the multiplicity of that eigenvalue. It becomes intuitively clear why this is so, if one compares the action of different Jacobians already reduced to the Jordan form. For instance, the Jacobian

$$A_1 = \begin{bmatrix} \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda \\ \end{bmatrix}$$ (68)

generates the set of three linearly dependent vectors $h^0 = b$, $h^1 = \lambda b$, $h^2 = \lambda^2 b$ (compare to (11)), that span a one-dimensional subspace of $\mathbb{R}^3$ for an arbitrary choice of $b$. As a result, three control parameters and a control matrix with three linearly independent columns, $B = [b_1 \ b_2 \ b_3]$, are necessary to control the system. On the contrary, the Jacobian

$$A_2 = \begin{bmatrix} \lambda & \lambda & \lambda \\ 1 & \lambda & \lambda \\ \end{bmatrix}$$ (69)

generates a linearly independent set of basis vectors that spans $\mathbb{R}^3$, requiring just one control parameter and a control matrix with a single column $B = b$.

Second, symmetry does not always make the Jacobian degenerate, and the nondegenerate case can be handled in the same way as the one with no symmetries. Neither does the degeneracy by itself imply that multi-parameter control is required: even if the eigenvalue $\lambda_{r'\alpha'}$ is degenerate, but $\tilde{j}_{r\alpha} = d_r = 1$ for every $r$ and $\alpha$ (the degeneracy is accidental and limited to a single invariant subspace $L^r_{r'}$), one control parameter is sufficient to ensure the controllability. In both cases, however, the dynamical symmetry should be rather low. Specifically, the decomposition (28) of the matrix representation $T$ should not contain any multi-dimensional irreducible representations.

Finally, the conditions on the set of control parameters that were obtained above are imposed by the controllability condition and guarantee that control can be achieved. However, in general, only the weaker stabilizability condition has to be satisfied which, according to section II A, requires that every unstable normal mode of the system is controllable, so that, only $r$ and $\alpha$ such that $|\lambda_{r\alpha}| \leq 1$ have to be considered in the conditions (55) and (56). As a consequence, it might be possible to stabilize highly symmetric states of compact extended systems with strong spatial correlations using a single control parameter — if only a small number of modes is excited, there is a chance that all unstable modes will correspond to one-dimensional irreducible representations $T^r$. In strongly chaotic systems a large number of modes will be unstable and many of them will inevitably be degenerate, calling for multi-parameter control. Similar considerations apply to weakly chaotic systems with large spatial extent.
III. TIME-PERIODIC STATES

The results obtained above for the time-invariant case can be generalized for the time-varying and, in particular, time-periodic case, but first we have to define the notions of controllability and dynamical symmetry in the context of time-varying trajectories. Indeed, in the time-varying case the Jacobian $A^t$ and the control matrix $B^t$ in the linearized evolution equation (2) are time-dependent and, as a consequence, neither the definition of controllability given in section II A nor the condition (13) holds. Besides, it is not at all clear that the symmetry of the target trajectory, and hence the dynamical symmetry group $L$ can be uniquely and consistently defined.

We will see that all these notions generalize in a rather straightforward way, so that the same formalism as we used in the previous sections applies here as well. To begin with, we define the controllability of a general time-varying linear system. Expanding the definition given for time-invariant target states, we call the $n_x \times n_x$-dimensional linear system (2) or the sequences of matrices $\{A^t, B^t\}$ controllable if, for any initial state $\Delta x_{ti} = \Delta x_i$, times $t_f - t_i \geq n_x$, and final state $\Delta x_{tf}$, there exists a sequence of control perturbations $\Delta u_{ti}, \ldots, \Delta u_{tf-1}$ such that the solution of equation (2) satisfies $\Delta x_{tf} = \Delta x_f$.

The controllability condition can be restated in terms of the matrices $A^t$ and $B^t$ conducting the analysis similar to that of section II A. Applying the map (2) $n_x$ times yields

$$\Delta x^{t+n_x} = J^{t+n_x}_{n_x} \Delta x^t + \sum_{k=0}^{n_x-1} J^{t+n_x-1-k}_{n_x-1-k} B^{t+k} \Delta u^{t+k},$$

(70)

where we have introduced a shorthand notation

$$J_k^t = A^t A^{t-1} \cdots A^{t-k+1}$$

(71)

for the product of $k$ consecutive Jacobians. Arguments identical to those used to derive the controllability condition (13) from equation (9) allow us to conclude that for time-varying states the controllability condition can again be written in the matrix form:

$$\text{rank}(C_t) = n_x, \ \forall t,$$

(72)

where the controllability matrix (14) is now replaced with the sequence of matrices

$$C_t \equiv [B^t \ J_1^t B^{t-1} \ J_2^t B^{t-2} \ \cdots \ J_{n_x-1}^t B^{t-n_x+1}].$$

(73)

Next we have to define the dynamic symmetry group $L$. Suppose the target trajectory $\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^\tau$ has period $\tau$, and the symmetry of the point $\tilde{x}^t$ on the target trajectory is described by the group $H_{\tilde{x}^t} \subseteq G$. We can then write

$$g(\tilde{x}^{t+1}) = g(F(\tilde{x}^t, \tilde{u})) = F(g(\tilde{x}^t), \tilde{u}) = F(\tilde{x}^t, \tilde{u}) = \tilde{x}^{t+1}$$

(74)

for every $g \in H_{\tilde{x}^t}$. Consequently,

$$H_{\tilde{x}^1} \subseteq H_{\tilde{x}^2} \subseteq \cdots \subseteq H_{\tilde{x}^\tau} \subseteq H_{\tilde{x}^1},$$

(75)
which means that the symmetry properties of all the points on the target trajectory are the
same and the isotropy symmetry group of the trajectory $\mathcal{H}_x$ can be uniquely defined using
an arbitrary point $\bar{x}^t, \mathcal{H}_x = \mathcal{H}_{\bar{x}^t}$.

Using the arguments that lead to equation (20) we obtain for an arbitrary $g \in \mathcal{H}_{\bar{x}^t}$:

\[
\bar{x}^{t+1} + g(A^t \Delta x) = g(\bar{x}^{t+1}) + g(A^t \Delta x) = g(\bar{x}^{t+1} + A^t \Delta x) \\
= g(F(\bar{x}^t, \bar{u}) + A^t \Delta x) = g(F(\bar{x}^t + \Delta x, \bar{u})) \\
= F(g(\bar{x}^t + \Delta x), \bar{u}) = F(g(\bar{x}^t) + g(\Delta x), \bar{u}) \\
= F(\bar{x}^t + g(\Delta x), \bar{u}) = F(\bar{x}^t, \bar{u}) + A^t g(\Delta x) \\
= \bar{x}^{t+1} + A^t g(\Delta x). \tag{76}
\]

This, in turn, means that the symmetry group $\mathcal{L}_t$ of the Jacobian $A^t$ satisfies

\[
\mathcal{H}_x \subseteq \mathcal{L}_t, \quad t = 1, \ldots, \tau. \tag{77}
\]

Again, typically, we expect $\mathcal{L}_t = \mathcal{H}_x$, so that $\mathcal{L}$ too would be unique for any given periodic trajectory as would the matrix representation $T$, such that

\[
T(g) A^t = A^t T(g), \quad \forall g \in \mathcal{L}. \tag{78}
\]

It is, therefore, enough to know the symmetry properties of an arbitrary point of the periodic trajectory in order to establish the requirements on the control scheme similarly to the time-invariant case. If $\mathcal{L}_t$ is not unique, we can still use the commutation relation (78) for the subgroup $\mathcal{L}' = \mathcal{H}_x$ to obtain a lower bound on the minimal number of control parameters.

Finally, we note that although it is possible to obtain certain results for time-varying control matrices $B^t$, we assume, as is often the case in real systems, that $B^t$ is constant and drop the time index. As we will discover below, in the time-periodic case the restrictions imposed by symmetry on the structure of the matrix $B$ can typically be determined without the detailed knowledge of the Jacobian matrices, but based on the symmetry properties alone, similarly to the time-invariant case. Indeed, let us construct the representation $T$ of the group $\mathcal{L}'$ in the tangent space $\mathcal{T}$ and decompose it into the sum of irreducible representations. This again defines a set of invariant subspaces $L_{r \alpha}^\mathcal{L}'$ and a set of basis vectors $\{e_{r \alpha}^i\}$, which we use to construct the coordinate transformation matrix $P$ according to the definition (34).

Since the rank of the matrix (73) does not change under a coordinate transformation, the controllability condition (72) is equivalent to the condition

\[
\text{rank}(\bar{C}_t) = n_x, \quad t = 1, \ldots, \tau \tag{79}
\]

where

\[
\bar{C}_t = \begin{bmatrix} \bar{B} & J^t_1 \bar{B} & \cdots & J^t_{n_x-1} \bar{B} \end{bmatrix}, \tag{80}
\]

$\bar{B} = PB$ and $J^t_k = PJ^t_k(P)^{-1}$. The products $J^t_k$ have the same symmetry properties as the Jacobian matrices $A^t$ for arbitrary $k$ and $t$, and, therefore, both the matrices $A^t$ and the products $J^t_k$ block-diagonalize in exactly the same way:

\[
\bar{A}^t = PA^t(P)^{-1} = \begin{bmatrix} A^{t,1} & \cdots & A^{t,q} \end{bmatrix}, \tag{81}
\]
and

\[
\tilde{J}_k^t = PJ_k^t(P)^{-1} = \begin{bmatrix}
J_k^t\alpha & \cdots & J_k^t\beta
\end{bmatrix}.
\]

Similarly to the time-invariant case, the blocks \(\tilde{A}^{t,r}\) and \(\tilde{J}^{t,r}\) are themselves block-diagonal

\[
\tilde{A}^{t,r} = \begin{bmatrix}
\tilde{\Lambda}_1^{t,r} & \cdots & \tilde{\Lambda}_r^{t,r}
\end{bmatrix}, \quad \tilde{J}^{t,r} = \begin{bmatrix}
\tilde{\Gamma}_1^{t,r} & \cdots & \tilde{\Gamma}_r^{t,r}
\end{bmatrix}
\]

and consist of \(d_r\) identical \(p_r \times p_r\) blocks \(\tilde{\Lambda}_r^{t,r}\) and \(\tilde{\Gamma}_r^{t,r}\), respectively, whose matrix elements are defined by the scalar products

\[
(\tilde{\Lambda}_r^{t,r})_{\alpha\beta} \equiv (e_i^{r\alpha} \cdot A^t e_i^{r\beta}),
(\tilde{\Gamma}_r^{t,r})_{\alpha\beta} \equiv (e_i^{r\alpha} \cdot J^t_k e_i^{r\beta}).
\]

Using the definition (71) one can check that for any \(t, k\) and \(r\) the matrix \(\tilde{\Gamma}_r^{t,r}\) can be represented as the product

\[
\tilde{\Gamma}_r^{t,r} = \tilde{\Lambda}_1^{t,r} \tilde{\Lambda}_2^{t-1,r} \cdots \tilde{\Lambda}_r^{t-k+1,r}.
\]

Let us partition the transformed control matrix \(\tilde{B}\) into blocks \(\tilde{B}^r = P^r B\) and define the reduced controllability matrices

\[
\tilde{C}_i^r = [\tilde{B}^r \quad \tilde{J}_1^{t,r} \tilde{B}^r \quad \cdots \quad \tilde{J}_{n_x-1}^{t,r} \tilde{B}^r].
\]

Using relation (29) and the fact that the matrix \(\tilde{C}_i^r\) has \(d_r p_1\) rows one can write

\[
\text{rank}(\tilde{C}_i) = \text{rank} \begin{bmatrix}
\tilde{C}_1^r \\
\vdots \\
\tilde{C}_q^r
\end{bmatrix} \leq \text{rank}(\tilde{C}_1^r) + \cdots + \text{rank}(\tilde{C}_q^r) \leq d_1 p_1 \cdots d_q p_q = n_x
\]

(87)

to obtain as a consequence of (79) the set of reduced controllability conditions

\[
\text{rank}(\tilde{C}_i^r) = d_r p_r, \quad r = 1, \cdots, q.
\]

(88)

The blocks \(\tilde{J}_k^t \tilde{B}^r\) of the matrix (80) can become linearly dependent for certain \(\tau, d_r\) and \(p_r\). Indeed, it is trivial to see that for a sequence of \(n\) arbitrary \(p \times p\) matrices \(R_i\), it is always possible to find a set of coefficients \(\mu_0, \mu_1, \cdots, \mu_n\) such that

\[
\mu_0 I + \mu_1 R_1 + \cdots + \mu_n R_n = 0,
\]

as long as \(n \geq p^2\). Equally easy to establish is the fact that, if the matrices \(R_i\) are not arbitrary, but satisfy the condition

\[
R_i = W_1 W_2 \cdots W_i,
\]

(90)
where $W_i$ is a sequence of arbitrary $p \times p$ matrices, such that $W_{i+\tau} = W_i$, equation (89) can always be satisfied for $n \geq \min(p^2, p\tau)$. The $p_r \times p_r$ matrices $\Gamma_{t,r}^{1}, \ldots, \Gamma_{t,r}^{n-1}$ form precisely the sequence satisfying the condition (90). Besides, if the condition (89) is satisfied for $R_i = \Gamma_{t,r}^{i}$, it is satisfied for the sequence $R_i = J_{t,r}^{i}$ as well. As a result,

$$\text{rank}(\bar{C}_r) = \text{rank} [ \bar{B}^r \ J_{t,r}^{1} \bar{B}^r \ \cdots \ J_{t,r}^{n-1} \bar{B}^r ]$$

(91)

for $n = \min(p_r^2, p_r\tau)$ and arbitrary $\bar{B}^r$. Therefore, in order for the conditions (88), and hence (72), to be satisfied, one should have

$$\text{rank}(P^r B) \geq \text{ceil} \left( \max_{r=1,\ldots,q} \left( \frac{d_r}{p_r}, \frac{d_r}{\tau} \right) \right), \quad r = 1, \ldots, q,$$

(92)

where ceil$(x)$ denotes the smallest integer number $n$ such that $n \geq x$. The necessary conditions on the control matrix $B$, defined by (92) are the generalization of the time-invariant result (62). Instead of (61) one respectively obtains the restriction on the minimal number of independent control parameters required to satisfy the controllability condition (72) for a periodic target trajectory:

$$\bar{n}_u \geq \text{ceil} \left( \max_{r=1,\ldots,q} \max \left( \frac{d_r}{p_r}, \frac{d_r}{\tau} \right) \right).$$

(93)

It is interesting to note that a periodic trajectory can be made controllable using the number of control parameters $n_u$ that could be smaller than the number required for a steady state with the same symmetry.

Three special cases deserve separate consideration. First of all, suppose that the Jacobian matrices $A^t$ commute with each other, so they can be simultaneously diagonalized. In this case the condition (89) can be satisfied by an appropriate choice of coefficients $\mu_1, \ldots, \mu_n$ for $n \geq p_r$, so the necessary conditions (93) and (92) will reduce to (61) and (62), respectively, and $\bar{n}_u$ will no longer depend on the period $\tau$ of the target trajectory.

Next, suppose there are no accidental degeneracies between the eigenvalues of the Jacobians $A^t$ and their products $J_{k}^{t}$, and no irreducible representation of $L'$ appears in the decomposition (28) more than once (so that Jacobian matrices can again be simultaneously diagonalized). Now, however, identically to the time-invariant case one obtains the necessary and sufficient conditions (63) and (64) instead of the necessary conditions (61) and (62).

Finally, although we used the fact that the trajectory is periodic to derive the above results, this requirement could be lifted, provided the symmetry of all points on the target trajectory is the same, and, therefore, the condition (78) is satisfied. A nonperiodic trajectory could then be treated as a periodic one, with period $\tau = \infty$, and the condition (89) will be satisfied by an appropriate choice of coefficients $\mu_1, \ldots, \mu_n$ for $n \geq p_r^2$. As a result, instead of the restriction (92) one will obtain

$$\bar{n}_u \geq \text{ceil} \left( \max_{r=1,\ldots,q} \frac{d_r}{p_r} \right).$$

(94)
IV. CONTINUOUS-TIME SYSTEMS

Most of the results obtained in the previous sections can be directly and naturally generalized to continuous-time systems. This is a rather valuable asset of the developed theory, since continuous-time control is, in general, a much more flexible and powerful technique than discrete-time control. In the presence of a decent continuous-time mathematical model (5), continuous-time control can often achieve far superior results. It is, however, a much more complicated technique as well. For simplicity we only discuss the control of time-invariant target states. Linearizing the evolution equation (5) around the steady target state $\bar{s}$, one obtains

$$\Delta \dot{s}(t) = A\Delta s(t) + B\Delta u(t),$$

where similarly to the discrete-time case we define the Jacobian

$$A = D_s \Phi(\bar{s}, \bar{u})$$

and the control matrix

$$B = D_u \Phi(\bar{s}, \bar{u}).$$

The symmetries of the nonlinear evolution equation (5), the target state $\bar{s}$, and the linearization (55) are determined identically to the discrete-time case using the relations (18), (19), and (21), yielding the symmetry groups $G$, $H_{\bar{s}}$, and $L$, respectively. The definitions of the notions of stabilizability and controllability in the continuous-time case are completely analogous to the ones given in section II A for the discrete-time case.

The dynamical system described by equation (55) or the pair $(A, B)$ is said to be controllable if, for any initial state $\Delta s(t_i) = \Delta s_i$, times $t_f - t_i > 0$ and final state $\Delta s_f$, there exists a (piecewise continuous) control perturbation $\Delta u(t)$ such that the solution of equation (55) satisfies $\Delta s(t_f) = \Delta s_f$. Otherwise the system or the pair $(A, B)$ is called uncontrollable.

Similarly, the dynamical system or the pair $(A, B)$ is said to be stabilizable, if there exists a state feedback $\Delta u(t) = -K \Delta s(t)$ making the system stable, such that all eigenvalues of the matrix $A' = A - BK$ have a negative real part, $\text{Re} (\lambda_k') < 0$, $\forall k$. Otherwise the system or the pair $(A, B)$ is called unstabilizable.

The controllability of the pair $(A, B)$ again ensures that all eigenvalues of $A'$ can be chosen appropriately, so that any controllable continuous-time system is stabilizable as well. The controllability of a continuous-time system is also established using the same criterion (13) used to test for the controllability in the discrete-time case. As a result, the conditions imposed on the control matrix $B$ by the controllability condition in the presence of symmetry are exactly the same as those obtained for discrete-time systems.

V. SYMMETRY VIOLATION

In reality symmetries of physical systems displaying dynamical instabilities are almost never exact. Indeed, the cylinders in a Taylor-Couette experiment are never perfectly circular, the temperature inside a chemical reactor is never absolutely uniform, neither are
the rotor blades of a turbocompressor exactly identical. The above analysis, on the other hand, has been conducted in the assumption of exact symmetry. Therefore, it is essential to understand how the obtained results change, if the symmetry is not exact or, in other words, what the effect of a weak symmetry violation is. Such an analysis is also crucial in the vicinity of points in the parameter space where symmetry increasing bifurcations or accidental degeneracies occur.

For simplicity let us again consider the time-invariant case. The Jacobian $A$ of a weakly perturbed symmetric system takes the form

$$A = A_0 + \epsilon A_1,$$  \hspace{1cm} (98)

where $\epsilon$ denotes the magnitude of the perturbation and the unperturbed Jacobian $A_0$ is exactly symmetric with respect to all transformations $g$ of the group $\mathcal{L}$. For the group representation $T$ we thus have

$$T(g) A_0 - A_0 T(g) = 0, \quad \forall g \in \mathcal{L}. \quad (99)$$

In general, the perturbation $\epsilon A_1$ will not be symmetric with respect to any element of the group $\mathcal{L}$, except the identity transformation $e$:

$$T(g) A_1 - A_1 T(g) \neq 0, \quad \forall g \in \mathcal{L} \setminus \{e\}. \quad (100)$$

Therefore, since

$$T(g) A - A T(g) = \epsilon(T(g) A_1 - A_1 T(g)),$$  \hspace{1cm} (101)

the perturbation (98) completely destroys the symmetry of the linearized evolution equation (6) for any $\epsilon \neq 0$. As a result, the perturbed system can be made controllable using a single control parameter, irrespectively of the properties of the original symmetry group $\mathcal{L}$. For instance, calculating the controllability matrix of the perturbed system with $n_u = 1$ and $B = b$ one obtains

$$C = C_0 + \epsilon C_1 + o(\epsilon^2),$$  \hspace{1cm} (102)

where we defined

$$C_0 = \begin{bmatrix} b & A_0 b & \cdots & (A_0)^{n_x-1} b \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & A_1 b & \cdots & ((A_0)^{n_x-2} A_1 + \cdots + A_1 (A_0)^{n_x-2}) b \end{bmatrix}. \quad (103)$$

$C_0$ is clearly the controllability matrix of the unperturbed system with full symmetry, which does not have a full rank, if the decomposition (28) contains at least one irreducible representation $T^r$ with the dimensionality $d_r > 1$. Indeed, in the absence of accidental degeneracies that would mean

$$n_0 \equiv \text{rank}(C_0) \leq \sum_{r=1}^{q} p_r < n_x. \quad (104)$$

The controllability matrix $C$ of the perturbed system, on the other hand, has full rank for any $\epsilon \neq 0$ because the symmetry is completely destroyed by the perturbation. Therefore,
the perturbed linear system becomes controllable even though the unperturbed system is not, for arbitrarily small perturbations.

The controllability ensures that for any initial and final states of the linear system \((6)\) the control can be found mapping the initial state to the final state in \(n_x\) iterations. Using \((4)\) one obtains explicitly

\[
\Delta U^t \equiv \begin{bmatrix} \Delta u^{t+n_x-1} \\ \vdots \\ \Delta u^t \end{bmatrix} = (C)^{-1}(\Delta x^{t+n_x} - (A)^{n_x} \Delta x^t).
\] (105)

Formally, if the system is controllable, the controllability matrix is invertible, and the solution \((105)\) is well defined for any \(\Delta x^t\) and \(\Delta x^{t+n_x}\). However, when the matrix \(C\) is close to being singular its inverse is not well defined. It is convenient to use the singular value decomposition of the controllability matrix

\[
C = Q\Sigma R^t, \tag{106}
\]

where \(Q = [q_1 \quad q_2 \quad \cdots \quad q_{n_x}]\) and \(R = [r_1 \quad r_2 \quad \cdots \quad r_{n_x}]\) are some orthogonal \(n_x \times n_x\) matrices, and

\[
\Sigma = \begin{bmatrix} \sigma_1(\epsilon) \\ \vdots \\ \sigma_{n_x}(\epsilon) \end{bmatrix}. \tag{107}
\]

The singular values are ordered such that \(\sigma_1(\epsilon) \geq \sigma_2(\epsilon) \geq \cdots \geq \sigma_{n_x}(\epsilon)\) for \(\forall \epsilon\). Additionally, equation \((104)\) requires

\[
\lim_{\epsilon \to 0} \sigma_i(\epsilon) = 0, \quad i = n_0 + 1, \cdots, n_x. \tag{108}
\]

In terms of the matrices \(Q, \Sigma,\) and \(R\) we can write the inverse of \(C\) as

\[
(C)^{-1} = R(\Sigma)^{-1}Q^t = \sum_{i=1}^{n_x} \sigma_i^{-1}(\epsilon)r_i q_i^t \tag{109}
\]

and, therefore, for small \(\epsilon\) equation \((109)\) gives

\[
\Delta U^t \approx \sum_{i=n_0+1}^{n_x} \frac{(q_i \cdot \Delta x^{t+n_x}) - (q_i \cdot (A)^{n_x} \Delta x^t)}{\sigma_i(\epsilon)} r_i. \tag{110}
\]

As a consequence, we obtain

\[
\lim_{\epsilon \to 0} |\Delta U^t| = \infty. \tag{111}
\]

This relation means that at least one control perturbation of the feedback sequence \(\Delta u^t, \cdots, \Delta u^{t+n_x-1}\) diverges as the symmetry breaking perturbation \(\epsilon A_1\) of the Jacobian vanishes. Since no specific relation between the initial and the final state of the system was implied, the obtained result is general, and does not depend on the control method used to calculate the feedback.
In fact, a more general statement holds. Suppose the symmetry is violated only partially, such that the perturbed Jacobian \((98)\) remains exactly symmetric with respect to a subgroup \(L'\) of the full symmetry group \(L\). Denote \(\tilde{n}_u\) and \(\tilde{n}'_u\) the minimal number of control parameters required (assuming exact symmetry) by the groups \(L\) and \(L'\), respectively. Then it can be shown that, similarly to the single-parameter case, at least one control perturbation of the feedback sequence \(\Delta u^t, \ldots, \Delta u^{t+n_u-1}\) diverges as the symmetry breaking perturbation \(\epsilon A_1\) of the Jacobian vanishes whenever \(\tilde{n}'_u \leq n_u < \tilde{n}_u\). The same result is obtained if the independent with respect to the group \(L'\) control parameters become dependent with respect to the group \(L\), as indicated by the violation of the general independence condition \((55)\). The time-periodic generalization is also straightforward. We will call this situation parametric deficiency.

In other words, although it might be possible to control a linear system with approximate symmetry using a number of control parameters which is smaller than that required in the assumption of exact symmetry, the stabilization requires feedback of very large magnitude. Such systems are called weakly controllable in the language of control theory. However, the linear system is only an abstraction. The linear approximation \((2)\) of the evolution equation \((1)\) is only valid for small perturbations \(\Delta u^t\) of the control parameters and small deviations \(\Delta x^t\) from the target trajectory. Besides, additional restrictions on the magnitude of the feedback are usually imposed by practical limitations, size and energy constraints, etc., at the implementation stage. One can, therefore, conclude that, since the feedback scales linearly with the deviation from the target trajectory, a nonlinear system with parametric deficiency can be stabilized using linear control only in an asymptotically contracting neighborhood of the target trajectory.

Finally, consider the vicinity of the point \(\bar{u}_0\) in the parameter space \(\mathbb{R}^{n_u}\) at which an accidental degeneracy occurs, such that the dynamical symmetry is described by the group \(L'\) for \(\bar{u} \neq \bar{u}_0\) and is increased to \(L\) (of which \(L'\) is a subgroup) for \(\bar{u} = \bar{u}_0\). In this case \(L\) can be considered approximate symmetry in the vicinity of \(\bar{u}_0\), and the distance to that point determines how strongly (or weakly) the symmetry \(L\) is violated. Suppose the control scheme is such that there is a parametric deficiency. Then the system will remain controllable for \(\bar{u} \neq \bar{u}_0\). However, the strength of feedback required to control the system will diverge as \(\bar{u}\) approaches \(\bar{u}_0\), at which point the system will become uncontrollable.

\section*{VI. SYSTEM IDENTIFICATION}

Finally, let us consider the problem of phase space reconstruction (sometimes also called system identification) in the context of symmetric systems. The need to use phase space reconstruction stems from the fact that few experimental systems exist whose dynamical equations are known (let alone with precision required by the control algorithms). Even the complete information about the state of these systems is rarely available, so one typically has to contend with having a measurement of a single scalar output (sometimes called an observable) for the description of the dynamics. This output, which we call \(y(t)\) is, in general, a function of the (unknown) internal state \(s(t)\) of the system described by the (also unknown) continuous-time evolution equation \((5)\). In other words,

\begin{equation}
\label{eq:112}
y(t) = G(s(t)).
\end{equation}
It turns out that it is possible to reconstruct both the internal state of the system and its dynamics based on the time series measurement of the output $y(t)$ using the procedure originally proposed by Packard et al. [17]. The easiest way to recreate the state of the system from the scalar output signal is to use time delays. Let us choose different delay times $T_1, T_2, \ldots, T_{n_x}$ and construct an $n_x$-dimensional delay coordinate vector

$$x(t) = \left[ y(t + T_1) \\ y(t + T_2) \\ \vdots \\ y(t + T_{n_x}) \right].$$

(113)

Takens showed [18] that for a scalar output (112) and conveniently chosen delay times $T_i$, if the dimension $n_x$ of the embedding space is such that $n_x \geq 2n_h^h + 1$, where $n_h^h$ is the Hausdorff dimension of the attractor $A$, the map $P : s(t) \rightarrow x(t)$ generically provides a global one-to-one representation of the attractor and, hence, the system state. Finally, one recovers the map (1) by defining $x = x(t_k)$, where $t_0, t_1, t_2, \ldots$ denote the times of consecutive crossings of a conveniently chosen Poincaré surface of section in the embedding space $\mathbb{R}^{n_x}$ by the reconstructed trajectory $x(t)$.

However, although the Takens’ embedding theorem holds for a typical system without symmetries, it is not satisfied for most of the symmetric systems due to the symmetry-related degeneracy of the evolution operators. In other words, most symmetric systems are nongeneric in the sense of Takens. As a result of this nongenericity the state of the system becomes impossible to reconstruct using a single scalar output, no matter how large the dimensionality $n_x$ of the embedding space is, locally in the vicinity of highly symmetric periodic trajectories. The attractor of the system remains folded at certain points and along certain curves in the phase space, which prevents the global reconstruction as well. Using the language of control theory we will say that such systems are unobservable locally as well as globally.

The question of symmetry-caused nongenericity in the framework of phase space reconstruction of a general symmetric system was first considered by King and Stewart [13], who determined that the reason for the failure of the embedding theorem is the violation of one of Takens’ assumptions that the flow defined by equation (2) has simple eigenvalues for low-period periodic trajectories. As we have seen above, symmetric systems typically (but not always) have degenerate eigenvalues (due to the fact that most nontrivial irreducible representations are multi-dimensional) and, as a consequence, are nongeneric. King and Stewart went on to formulate and prove a generalization of the Takens’ embedding theorem, which required the output to be a vector, not a scalar, function of the actual state of the system $s(t)$:

$$y(t) = G(s(t)), \quad (114)$$

mapping the phase space $Q$ of the original system onto an $n_y$-dimensional Euclidean space. The state of the system can then be similarly represented by a delay coordinate vector

$$x(t) = \left[ y(t + T_1) \\ y(t + T_2) \\ \vdots \\ y(t + T_{n_x}) \right].$$

(115)
where now the dimensionality of the embedding space is \( n = n_x n_y \). The question we have to answer is what conditions should the function \( G \) satisfy in order to allow a local (or global) one-to-one embedding. Since the exact form of the evolution equations is assumed unknown, in order to find the answer one can only exploit the symmetry properties of the system, which are often easy to establish based on the underlying symmetries of the physical space. Fortunately, the symmetry provides most of the necessary information.

Since we are interested in the issue of phase space reconstruction only as far as it applies to the problem of linear control, we will assume a local character for the observability property, unless explicitly stated otherwise. According to the analysis conducted in \([13]\), local embedding in the vicinity of the periodic trajectory \( \bar{s}(t) \) requires \( \mathbb{R}^{n_y} \) to contain at least one copy of every invariant subspace \( L_r^\alpha \) generated by the (unitary) irreducible representation \( T_r \) of the respective isotropy symmetry group \( L' = H_s \). This would lead one to assume that the minimal dimension \( n_y \) of the output signal should be determined by the dimension of the largest irreducible representation \( T_r \).

This assumption can be trivially verified using the formalism developed above for the control problem in the presence of symmetry. Indeed, let us again consider a time-invariant target state \( \bar{s} \). Linearizing the output (114) in the vicinity of this target state and denoting the displacement \( \Delta y(t) = G(s(t)) - G(\bar{s}) \) one obtains:

\[
\Delta y(t) = C \Delta s(t),
\]  

where the constant matrix \( C \) is defined thus:

\[
C = D_s G(\bar{s}).
\]

The dynamical system defined by equations (95) and (116) or the pair \((A, C)\) is said to be observable if, for any times \( t_f - t_i > 0 \), the initial state \( \Delta s(t_i) = \Delta s_i \) can be determined from the measurement of control perturbation \( \Delta u(t) \) and output \( \Delta y(t) \) in the interval \( t \in [t_i, t_f] \). Otherwise, the system or the pair \((A, C)\) is said to be unobservable.

It can be shown that the notion of observability is dual to the notion of controllability. The crucial benefit of this duality is the fact that the observability condition for the pair \((A, C)\) is equivalent \([11]\) to the controllability condition for the pair \((A^\dagger, C^\dagger)\). Since the commutation relation (24) directly implies that

\[
T(g)^\dagger A^\dagger = A^\dagger T(g)^\dagger, \quad \forall g \in L',
\]

the symmetry properties of the matrices \( A \) and \( A^\dagger \) are essentially identical (as are the structures of their spectra, Jordan normal forms, etc.). As a result, all restriction imposed on the control matrix \( B \) by the controllability condition in the presence of symmetry should be satisfied for the matrix \( C^\dagger \) as well.

For instance, in case there are no accidental degeneracies and the representation \( T \) contains at most one copy of each irreducible representation of the group \( L' \), one can claim that in order to reconstruct the dynamics in the vicinity of the time-invariant symmetric target state \( \bar{s} \) the number \( n_y \) of measured scalar output signals \( y_i(t) \) should be the same as the minimal number \( n_u \) of independent control parameters, i.e., \( n_y = n_u \). Furthermore, the outputs have to be independent, so that \( d_r \) of the projections \( \hat{P}^r c_i, i = 1, \cdots, n_y \) are linearly independent for every \( r \), where \( \hat{P}^r \) is the projection operator defined by (65) and (66), and
\( \mathbf{c}_i \) is the \( i \)th row of the matrix \( \mathbf{C} \), which imposes a number of restrictions on the allowed form of the function \( \mathbf{G} \). Local observability can be defined in the same way for discrete-time systems. Careful consideration shows that symmetry produces similar constraints independent of the particular description.

In addition to the notion of controllability of individual eigenvectors it is often also convenient to define the notion of their observability. We will say that the eigenvector \( \mathbf{e} \) of the Jacobian \( \mathbf{A} \) is observable, if there exists \( i, 1 \leq i \leq n_y \), such that \( (\mathbf{e} \cdot \mathbf{c}_i) \neq 0 \). Respectively, an eigenvector that is orthogonal to every row of the matrix \( \mathbf{C} \) is called unobservable. Clearly, the observability of the linearized system is equivalent to the observability of each and every eigenvector of the Jacobian matrix.

In the conclusion of this section we make a few comments regarding the problem of global phase space reconstruction. Often it is important to know how the symmetry of the continuous-time experimental system transpires in the structure of the discrete-time map (114), obtained as a result of the time delay embedding produced by a general output signal (114). King and Stewart [13] recognized that it is as important to preserve the symmetry of the attractor as it is to preserve its topology during the reconstruction. According to (114), using an arbitrary vector output \( \mathbf{y}(t) \) to generate the delay coordinate representation of the system state corresponds to picking a function \( \mathbf{G} \) which, in general, distorts the symmetry. In order to preserve the symmetry of the original attractor the function \( \mathbf{G} \) has to be \( \mathcal{G} \)-equivariant:

\[
\mathbf{G}(g(s)) = g(\mathbf{G}(s)), \quad \forall g \in \mathcal{G}, \quad \forall s \in \mathcal{Q},
\]

where \( \mathcal{G} \) is the structural symmetry group of the system (5) which will, in general, act differently in the phase space \( \mathcal{Q} \) and Euclidean space \( \mathbb{R}^{n_y} \). In addition, the dimensionality \( n_y \) of the Euclidean space has to be chosen high enough to avoid local folding (obviously, \( n_y \) should be no smaller than the number \( \bar{n}_y \) evaluated for the steady or periodic trajectory with the highest isotropy symmetry). Finally, a global one-to-one embedding can be achieved by choosing \( n_e \geq 2n_y^2 + 1 \) to preserve the topology of the attractor. The map (11) constructed using this embedding will preserve all dynamical symmetries of the original system. However, the structural symmetry of the differential equation (5) and the map (11) will, in general, be different.

VII. EXAMPLES

A. Coupled Map Lattice

Now that the formal theory is constructed, we can illustrate it by applying to a few simple symmetric systems. First, consider the deterministic coupled map lattice (CML) with nearest neighbor diffusive coupling [19], which is described by the following evolution equation

\[
x_{i+1}^{t+1} = \epsilon f(x_{i-1}^{t}, a) + (1 - 2\epsilon) f(x_{i}^{t}, a) + \epsilon f(x_{i+1}^{t}, a).
\]

Coupled map lattices of this type are often used as models for such spatiotemporally chaotic phenomena as surface growth, population dynamics, and turbulence. Here the index \( i = \)
1, 2, · · · , \(n_x\) labels the lattice sites, and the periodic boundary condition is imposed. The choice of the map function \(f(x, a)\) is usually motivated by the local dynamics of the physical system under consideration. In principle \(f(x, a)\) can be chosen as an arbitrary (nonlinear) function with parameter \(a\), which typically represents the process of generation and growth of local fluctuations, while diffusive coupling typically plays the opposite role of dissipating these fluctuations. Therefore, the parameters \(a\) and \(\epsilon\) specify the degree of instability and the strength of dissipation in the system, respectively. For the purpose of control, however, details of the local map are not important. The only aspect of the control problem affected by any particular choice is the set of existing unstable periodic trajectories.

The analysis of the controllability condition conducted in sections II and III shows that, if the system is symmetric, certain symmetry-imposed restrictions on the choice of control parameters should be satisfied in order to achieve control. In fact, the coupled map lattice is by construction highly symmetric. The symmetry is that of the spatial lattice: the evolution equation (120) is invariant with respect to translations by an integer number of lattice sites (periodic boundary condition makes the group finite) and reflections about any site (or midplane between any adjacent sites), which map the lattice back onto itself without destroying the adjacency relationship. The corresponding point group \(C_{n_x}\) (we assume \(n_x\) – even) has a total of \(n_x/2 + 3\) nonequivalent irreducible representations. The first four are one-dimensional, \(d_1 = d_2 = d_3 = d_4 = 1\), while the rest \(n_x/2 - 1\) are two-dimensional, \(d_r = 2\), \(r \geq 5\). In comparison, breaking the reflection symmetry reduces the group to \(C_{n_x/2}\), which only has one-dimensional irreducible representations.

The dynamical symmetry group can be trivially obtained using the observation that the Jacobian matrix in the linearization (2) constructed for the CML (120) can always be represented as a product of two matrices, \(A^t = MN^t\), where

\[
M_{ij} = (1 - 2\epsilon)\delta_{i,j} + \epsilon(\delta_{i,j-1} + \delta_{i,j+1})
\]  

(121)

describes diffusive coupling, and

\[
N_{ij}^t = \partial_x f(\bar{x}_i^t, a)\delta_{ij}
\]  

(122)

defines the strength of local instability, with \(\delta_{i,j\pm1}\) extended to comply with periodic boundary condition. This partition of the Jacobian explicitly shows how the symmetry group \(L\) depends on the symmetry properties of the nonlinear evolution equation (120) and those of the controlled state \(\bar{x}^t\). The matrix \(M\) has all the symmetries imposed by the chosen inter-site couplings of the nonlinear model:

\[
T(g)M = MT(g), \quad \forall g \in \mathcal{G},
\]  

(123)

while the matrix \(N^t\) has all the symmetries of the target state \(\bar{x}^t\):

\[
T(g)N^t = N^tT(g), \quad \forall g \in \mathcal{H}_\bar{x},
\]  

(124)

where, \(T\) denotes the matrix representation of \(L\), and \(\mathcal{H}_\bar{x}\) is defined as a subgroup of \(\mathcal{G}\). Since the Jacobian \(A^t\) commutes with all matrices that commute with both \(M\) and \(N^t\), we conclude that generically \(L = \mathcal{H}_\bar{x} \subseteq \mathcal{G}\), in agreement with the general result (77).

Since the symmetry analysis conducted above is valid for every subgroup \(\mathcal{L}'\) of the dynamical symmetry group, we take \(\mathcal{L}' = \mathcal{L}\). Constructing the \(n_x\)-dimensional representation...
of $\mathcal{L}$ and decomposing it into the sum of the irreducible representations of $C_{n,s,v}$ we easily determine the restrictions imposed by the symmetry on the minimal number of control parameters $n_u$ and the structure of the control matrix $B$. For instance, a zigzag state gives $\mathcal{L} = C_{n,v}$ with $n = n_x/2$ and, according to (13), $n_u = 2$; a non-reflection-invariant state with spatial period $s$ corresponds to $\mathcal{L} = C_n$ with $n = n_x/s$ and $n_u = 1$, etc.

Let us consider the uniform target state, which has the highest symmetry possible, $\mathcal{L} = C_{n,s,v}$, in more detail. The decomposition (28) gives

$$T = T^1 \oplus T^4 \oplus T^5 \oplus \cdots \oplus T^{n_x/2+3},$$

(125)

and the corresponding basis of normal modes which transform according to these irreducible representations is given by the eigenvectors of the operators of translation and reflection, i.e., Fourier modes $g^i$:

$$(g^i)_j = \cos(jk_i + \phi_i).$$

(126)

Here $\phi_i$ are arbitrary phase shifts, and $k_i$ are the wavevectors defined thus: $k_1 = 0$, $k_i = k_{i+1} = \pi i/n_x$ for $i = 2, 4, 6, \cdots$, and, for $n_x$ - even, $k_{n_x} = \pi$. Fourier modes with the same wavevectors $k$ define invariant subspaces $L^k \subset \mathcal{T} = \mathbb{R}^{n_x}$. The subspaces $L^k$ with $0 < k < \pi$ correspond to the representations $T^r$ with $r \geq 5$, $L^0$ corresponds to $T^1$, and $L^\pi$ to $T^4$.

Since the two-dimensional irreducible representations are present in the decomposition (125), $n_u = 2$. Therefore, in order to control an unstable uniform steady state of the coupled map lattice we need at least two control parameters. This is the reflection of symmetric coupling in the model (120). Note that, since every two-dimensional irreducible representation occurs in the decomposition (125) once, $p_5 = \cdots = p_{n_x/2+3} = 1$, according to the results of section (11) the minimal number of control parameters remains the same for a spatially uniform target trajectory of arbitrary time period $\tau$.

On the other hand, since for any length $n_x$ of the lattice the group $\mathcal{G} = C_{n,s,v}$ only has one- and two-dimensional irreducible representations and $\mathcal{L}$ is a subgroup of $\mathcal{G}$, it is sufficient to have just two control parameters to make the dynamics of the coupled map lattice controllable in the vicinity of a target state with arbitrary symmetry properties and temporal period. Choosing the minimal number of control parameters, $n_u = 2$, we can determine the conditions making them independent with respect to a particular target state: the linear response of the CML to perturbation of the two parameters, given by the columns of the control matrix $B = [b_1 \ b_2]$, has to satisfy conditions (27) and (77).

Failure to satisfy the necessary condition (24) rules out the possibility of using global parameters, such as the coupling $\epsilon$ or parameter $a$ of the local map $f(x,a)$ for control of symmetric steady states. Taking $u = (a, \epsilon)$, so that

$$b_1 = \partial_a F(\bar{x},0,\bar{u}) = M \begin{bmatrix} \partial_a f(\bar{x}_1,\bar{a}) \\ \vdots \\ \partial_a f(\bar{x}_{n_x},\bar{a}) \end{bmatrix},$$

(127)

and

$$b_2 = \partial_\epsilon F(\bar{x},0,\bar{u}) = (\epsilon)^{-1}(M - I) \begin{bmatrix} f(\bar{x}_1,\bar{a}) \\ \vdots \\ f(\bar{x}_{n_x},\bar{a}) \end{bmatrix},$$

(128)

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we observe that condition (27) is only satisfied, if the group \( \mathcal{L} \) is trivial, \( \mathcal{L} = \{ e \} \). This result holds for time-periodic symmetric target states as well.

Alternatively, one can make the system controllable by directly perturbing the system at the sites \( i_1 \) and \( i_2 \). The positions of the “control” sites cannot be chosen arbitrary, again due to symmetry. However, if the target state is spatially uniform, it is trivial to show that choosing, e.g., \( i_1 = l \) and \( i_2 = l + 1 \) satisfies the controllability condition for an arbitrary length of the lattice \( n_x \). The control matrix corresponding to this choice of control parameters can be written in the form \( B_{ij} = \delta_{j,1} \delta_{i,l} + \delta_{j,2} \delta_{i,l+1} \).

Such localized control also has its downside. In the weak coupling limit, \( \epsilon \to 0 \), the coupled map lattice with local feedback becomes a weakly controllable system. The symmetry of the lattice of uncoupled maps is described by the permutation group \( \mathcal{G} = S_{n_x} \), while the linearization about a uniform target state increases the symmetry to \( \mathcal{L} = \text{GL}(n_x) \): since the respective Jacobian is a multiple of the unit matrix, \( A_{ij} = \partial_{x} f(\bar{x}, \bar{a}) \delta_{i,j} \), the linearized system is symmetric with respect to all (complex) nonsingular coordinate transformations. When coupling is restored, \( \epsilon > 0 \), the symmetry of both the nonlinear evolution equation (120) and its linearization (2) reduces to \( \mathcal{G}' = \mathcal{L}' = C_{n_x} \).

The matrix representation \( T \) of the group \( \text{GL}(n_x) \) in \( T \) is already irreducible. Consequently, \( n_u = n_x \) independent control parameters are required to control the steady uniform state of the uncoupled lattice. This result is rather intuitive. Obviously, one can no longer control the system applying control perturbations at just two lattice sites, \( i_1 \) and \( i_2 \). Since the control perturbation does not propagate to adjacent sites of the lattice, feedback has to be applied directly at each site.

If the coupling is nonzero, but very small, the controllability property is restored for \( n_u = 2 \), but, according to section V, feedback of very large magnitude is required to control the system due to parametric deficiency. Indeed, in order to affect the dynamics at site \( i \) away from \( i_1 \) and \( i_2 \) the control has to propagate a certain distance decaying by roughly a factor of \( \epsilon \) per iteration. As a result, the magnitude of the perturbation required to control an arbitrary site of the lattice diverges approximately as \( \epsilon^{-n_x/2} \) for \( \epsilon \to 0 \), resulting in the loss of control [9].

B. Particle in a Symmetric Potential

The motion of a particle in a symmetric potential, such as a point charge in electric field, serves as another example of the relation between the structural symmetry group \( \mathcal{G} \) and the dynamical symmetry group \( \mathcal{L} \). This and many other interesting physical systems, e.g., inverted pendulum, or a satellite in orbit, are described by the second order ordinary differential equation

\[
\dot{\mathbf{r}} = -\nabla V(\mathbf{r}),
\]

which can be trivially reduced to a system of first order differential equations of the form [10] introducing additional coordinate \( \mathbf{v} = \dot{\mathbf{r}} \). Suppose the potential \( V(\mathbf{r}) \) possesses the cubic symmetry (described by the group \( O \) which is a subgroup of \( \text{SO}(3) \)), but is not spherically symmetric, for instance:

\[
V(\mathbf{r}) = V_0 \cosh(kx) \cosh(ky) \cosh(kz).
\]
The group \( G = O \) defines the structural symmetry of the evolution equation (123). Linearizing the latter about the equilibrium position \( \bar{r} = 0 \) we obtain
\[
\partial_t \begin{bmatrix} \bar{r} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \omega^2 I & 0 \end{bmatrix} \begin{bmatrix} \bar{r} \\ \bar{v} \end{bmatrix},
\]
where \( \omega^2 = -V_0 k^2 / m \), while \( 0 \) and \( I \) are \( 3 \times 3 \) zero and unit blocks, respectively. If \( V_0 < 0 \) the equilibrium is unstable, and control should be applied to keep the system close to the equilibrium state.

Equation (131) is spherically symmetric, with \( \mathcal{L}' = SO(3) \) and, therefore, \( G \subset \mathcal{L} \), i.e., the symmetry of the linearized equation is higher than the symmetry of the original nonlinear evolution equation. (In fact, the full symmetry group of equation (131) is \( \mathcal{L} = GL(3) \), but we choose to use its subgroup \( \mathcal{L}' = SO(3) \), since it is physically more relevant, completely resolves the structure of the Jacobian matrix and, as such, correctly represents the effect of symmetry on the control setup.)

Next we notice that the representation \( T \) of the group \( \mathcal{L}' \) in the six-dimensional tangent space \( \{ \bar{r}, \bar{v} \} \) can be decomposed into a sum of two equivalent three-dimensional irreducible representations of \( SO(3) \) (vector representations, which coincide with the respective irreducible representation of \( GL(3) \)):
\[
T = 2T^1, \quad d^1 = 3.
\]
This indicates that in order to control the unstable steady state \( \bar{r} = \bar{v} = 0 \) one needs at least three independent control parameters, \( \bar{n}_u = 3 \).

Arguably the simplest way to control such a system is to re-adjust the potential (applying external fields, shifting support point, etc.) based on the instantaneous values of the position \( \bar{r} \) and velocity \( \bar{v} \) of the particle. This corresponds to picking the control matrix in the following form:
\[
B = \begin{bmatrix} 0 & 0 & 0 \\ b_1 & b_2 & b_3 \end{bmatrix},
\]
where \( b_1, b_2, b_3 \) could be chosen as any three linearly independent vectors in \( \mathbb{R}^3 \).

**VIII. CONCLUSIONS**

Summarizing, we have determined that if the system under consideration is symmetric, it cannot be considered generic with respect to conventional chaos control techniques, and its symmetry properties should be understood prior to constructing a control scheme, even if the symmetry is only approximate. The failure to observe the restrictions imposed by the symmetry on the structure of the measured output signal will usually prevent the experimental reconstruction of the system dynamics. Similarly, an inappropriate choice of control parameters will result in weak controllability and, as a result, extreme sensitivity to noise, or even worse, complete loss of control.

From the practical point of view, the main result of the symmetry analysis is that the minimal number of independent control parameters required for control, as well as the
minimal number of independent scalar observables required for the reconstruction of local dynamics, can typically be determined without any knowledge of the evolution equations governing the dynamics of the system. One only needs to know the symmetry properties, such as spatial and temporal periodicity, of the target state, and the structural symmetry of the dynamical equations, which in the case of extended chaotic systems is often uniquely defined by the geometry of the underlying physical space. One should, however, realize that this typical pattern does not apply to all symmetric systems without exception. The dynamical equations might, in principle, be symmetric with respect to transformations unrelated to “geometrical” symmetries, such as rotational, reflectional, or translational invariance. Additional “nonphysical” symmetries can also be introduced as a result of the linearization procedure.

A number of comments have to be made regarding accidental degeneracies. We found that when accidental degeneracies are present, restrictions obtained using symmetry considerations alone provide only the necessary conditions for controllability. In particular, one obtains a lower bound on the minimal number of control parameters. Exact determination of that number in this case requires additional information about the structure of the Jacobian matrix, which can be gathered using experimental reconstruction. On the other hand, experimental reconstruction itself is only possible, if there is an adequate number of independent scalar observables. This number, however, is similarly undetermined. In practice, though, one rarely has to worry about such complications, since accidental degeneracies are not common and unlikely to be a problem for most actual experimental systems. Besides, an estimate for the minimal number of observables and control parameters can always be easily obtained using combinatorial arguments. Also, one should be careful in equating the minimal number of observables or control parameters with the highest degeneracy of the Jacobian matrix, especially if this degeneracy is at least partially accidental. It can be argued that accidental degeneracies between eigenvalues from the same irreducible invariant subspace typically will not increase the dimensionality of the respective eigenspace and, therefore will not lead to additional degeneracy in the local dynamics.

We also established that it is not enough to find an adequate number of control parameters (or observables). These control parameters (observables) have to satisfy certain conditions. In particular, perturbation of the control parameters should completely break the dynamical symmetry. The more strict independence condition is specific to each target trajectory and, on the one hand, requires the knowledge of the system’s response to variation of different control parameters (which can be obtained experimentally, if necessary), but, on the other hand, allows one to choose the minimal set of control parameters systematically, avoiding trial and error search. For example, in case of extended dynamical systems with local feedback the independence condition usually imposes restrictions on the mutual arrangement of “control” sites, while the number of “control” sites coincides with the number of control parameters.

Finally, we discovered that the conventional approach to system identification also has to be modified in the presence of symmetries. In particular, in order to preserve not only the topology of the original attractor, but also the symmetry of the original dynamical equations, one has to use a number of simultaneously measured observables, which have to be the components of an equivariant vector function of the actual state of the system. The restrictions on the output can be relaxed somewhat in the case of local reconstruction in the
vicinity of some target trajectory. However, even then a number of independent observables should be used instead of just a single one, as long as the symmetry of the target state is nontrivial, leading to the increase in the dimension of the embedding space. Otherwise, the conventional approach carries over with minor modifications.
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