Bounds for the positive or negative inertia index of a graph

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Abstract: Let $G$ be a graph and let $A(G)$ be adjacency matrix of $G$. The positive inertia index (respectively, the negative inertia index) of $G$, denoted by $p(G)$ (respectively, $n(G)$), is defined to be the number of positive eigenvalues (respectively, negative eigenvalues) of $A(G)$. In this paper, we present the bounds for $p(G)$ and $n(G)$ as follows:

$$m(G) - c(G) \leq p(G) \leq m(G) + c(G), \quad m(G) - c(G) \leq n(G) \leq m(G) + c(G),$$

where $m(G)$ and $c(G)$ are respectively the matching number and the cyclomatic number of $G$. Furthermore, we characterize the graphs which attain the upper bounds or the lower bounds respectively.

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1 Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix $A(G)$ of $G$ is defined to be an $n \times n$ symmetric matrix $[a_{ij}]$ such that $a_{ij} = 1$ if $v_i v_j \in E(G)$, and $a_{ij} = 0$ otherwise. The eigenvalues of $G$ will be referred to the eigenvalues of $A(G)$. The positive inertia index (respectively, the negative inertia index) of $G$, denoted by $p(G)$ (respectively, $n(G)$), is defined to be the number of positive eigenvalues (respectively, negative eigenvalues) of $A(G)$. The rank of $G$, denoted by $r(G)$, is exactly the sum of $p(G)$ and $n(G)$.

According to Hückel theory, the eigenvalues of a chemical graph (i.e. the connected graph with maximum degree at most three) specify the allowed energies of the $\pi$ molecular orbitals available for occupation by electrons. Such a graph or corresponding molecule is said to be (properly) closed-shell if exactly half of its eigenvalues are positive (requiring an even number of vertices), which indicates a stable $\pi$-system (see [4]). Chemists are interested in whether the molecular graph of an unsaturated hydrocarbon is (properly) closed-shell, having exactly half of its eigenvalues greater than zero, because this designates a stable electron configuration.

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In the mathematics itself, one would like to know or bound \( p(G) \) or \( n(G) \) for a graph \( G \). The problem is closed related to the nullity \( \eta(G) \) of \( G \), which is defined to be the number of zero eigenvalues of \( A(G) \), since \( p(G) + n(G) = |V(G)| - \eta(G) \). Smith [8] proved that a connected graph has exactly one positive eigenvalue if and only if it is complete multipartite. Later Torgašv [9] characterized the graphs with fixed number of negative eigenvalues. Recently, Yu et al. [11] investigated the minimum positive inertia index among all bicyclic graphs of fixed order with pendant vertices, and characterized the bicyclic graphs with positive index 1 or 2. Ma et al. [6] discussed the positive or the negative inertia index for a graph with at most three cycles, and proved that \(|p(G) - n(G)| \leq c_1(G)\) for any graph \( G \), where \( c_1(G) \) denotes the number of odd cycles contained in \( G \). They conjectured that

\[ -c_3(G) \leq p(G) - n(G) \leq c_5(G), \tag{1.1} \]

where \( c_3(G) \) and \( c_5(G) \) denote the number of cycles having length 3 modulo 4 and length 1 modulo 4 respectively. In [10] we proved that the conjecture (1.1) holds for line graphs and power trees. In addition, Ma et al. [7] proved that the positive inertia index of the line graph of a tree \( T \) lies between the interval \([\varepsilon(T)+1, \varepsilon(T) + 1]\), where \( \varepsilon(T) \) denotes the number of non-pendant edges of \( T \).

We specify that Daugherty [3] characterized the positive or negative inertia of unicyclic graphs in terms of the matching number; see Theorem 2.6 below. This motivates us to give a characterization for general graphs in terms of the matching number. Denote by \( m(G) \) the matching number of a graph \( G \), and \( c(G) \) the cyclomatic number of \( G \) defined by \( c(G) = |E(G)| - |V(G)| + \theta(G) \), where \( \theta(G) \) is the number of connected components of \( G \). In Section 3 we will give the main result of this paper (see Theorem 3.2), that is,

\[ m(G) - c(G) \leq p(G) \leq m(G) + c(G), \quad m(G) - c(G) \leq n(G) \leq m(G) + c(G). \]

In Section 4 we will characterize the extremal graphs which attain the four bounds respectively. The main result is proved by a few words, but the characterization of extremal graphs costs a lot of work. However, through the discussion of extremal graphs, we get a more clear understanding of the graph structure.

### 2 Preliminaries

Let \( G \) be a graph. The degree of a vertex \( v \in V(G) \) is denoted by \( d(v) \). A vertex of \( G \) is said pendant if it has degree 1, and is said quasi-pendant if it is adjacent to a pendant vertex unless it itself is pendant. Denote by \( G - W \), for \( W \subseteq V(G) \), the induced subgraph obtained from \( G \) by deleting all vertices in \( W \) together with edges incident to them. For an induced subgraph \( G_1 \) and a vertex \( x \) outside \( G_1 \), denote by \( G_1 + x \) the subgraph of \( G \) induced by the vertices of \( V(G_1) \cup \{x\} \). Similarly, the subgraph of \( G \) induced by the vertices of \( V(G) \setminus \{x\} \) is simply written as \( G - x \). The cycle on \( n \) vertices is denoted by \( C_n \).
Lemma 2.1. Let \( G \) be a graph containing a pendant vertex, and let \( H \) be the induced subgraph of \( G \) by deleting the pendant vertex and the vertex adjacent to it. Then \( p(G) = p(H) + 1, n(G) = n(H) + 1 and \eta(G) = \eta(H) \).

Lemma 2.2. Let \( G \) be an acyclic graph. Then \( p(G) = n(G) = m(G) \), and \( r(G) = 2m(G) \).

Lemma 2.3. Let \( G = G_1 \cup G_2 \cup \ldots \cup G_q \) (disjoint union). Then \( p(G) = \sum_{i=1}^{q} p(G_i) \) and \( n(G) = \sum_{i=1}^{q} n(G_i) \).

Lemma 2.4. Let \( G \) be a graph containing a pendant vertex, and let \( H \) be the induced subgraph of \( G \) by deleting the pendant vertex and the vertex adjacent to it. Then \( p(G) = p(H) + 1, n(G) = n(H) + 1 and \eta(G) = \eta(H) \).

Lemma 2.5. Let \( G \) be a graph with a quasi-pendant vertex \( v \). Then \( m(G - v) = m(G) - 1 \).

Theorem 2.6. Let \( G \) be a unicyclic graph containing the cycle \( C_q \). Then

\[
(n(G), p(G)) = \begin{cases} 
(m(G) - 1, m(G) - 1), & \text{if } q = 4k \text{ and } M \cap F(G) = \emptyset \text{ for any maximum matching } M \text{ of } G; \\
(m(G), m(G) + 1), & \text{if } q = 4k + 1 \text{ and } m(G) = m(G - C_q) + \frac{q-1}{2}; \\
(m(G) + 1, m(G)), & \text{if } q = 4k + 3 \text{ and } m(G) = m(G - C_q) + \frac{q-1}{2}; \\
(m(G), m(G)), & \text{otherwise.}
\end{cases}
\]

3 Bounds for the positive or negative inertia index

By the Cauchy interlacing theorem (or see [2]), we easily get the following result.

Lemma 3.1. Let \( G \) be a graph with a vertex \( v \). Then

\[
p(G) - 1 \leq p(G - v) \leq p(G), \quad n(G) - 1 \leq n(G - v) \leq n(G).
\]

Theorem 3.2. Let \( G \) be a graph. Then

\[
m(G) - c(G) \leq p(G) \leq m(G) + c(G), \quad m(G) - c(G) \leq n(G) \leq m(G) + c(G).
\]

Proof. We proceed by induction on \( c(G) \). If \( c(G) = 0 \), then \( G \) is an acyclic graph, and hence \( p(G) = n(G) = m(G) \) by Lemma 2.2 which confirms the theorem. Now suppose that \( c(G) \geq 1 \). Then \( G \) contains at least one cycle. Let \( v \) be a vertex lying on a cycle of \( G \) and denote \( H := G - v \). Thus \( c(H) \leq c(G) - 1 \). Applying the induction to \( H \), we have

\[
m(H) - c(H) \leq p(H) \leq m(H) + c(H).
\]

By Lemma 3.1

\[
p(G) \leq p(H) + 1 \leq m(H) + c(H) + 1 \leq m(G) + (c(G) - 1) + 1 = m(G) + c(G),
\]

\[
p(G) \geq p(H) \geq m(H) - c(H) \geq (m(G) - 1) - (c(G) - 1) = m(G) - c(G),
\]

which completes the proof for \( p(G) \). The discussion for \( n(G) \) is similar and is omitted. \( \square \)
Corollary 3.3. Let $G$ be a graph which contains at least one cycle. If $p(G) = m(G) + c(G)$, then for any vertex $v$ lying on a cycle of $G$,

(i) $p(G - v) = p(G) - 1$;
(ii) $p(G - v) = m(G - v) + c(G - v)$;
(iii) $m(G - v) = m(G)$;
(iv) $c(G - v) = c(G) - 1$;
(v) $v$ is not a quasi-pendant vertex.

Proof. The assertions (i)-(iv) hold by considering the equality cases of the inequalities (2.1). If $v$ is a quasi-pendant vertex, then $m(G - v) = m(G) - 1$ by Lemma 2.2, contradicting to (iii). Hence, the assertion (v) holds.

Corollary 3.4. Let $G$ be a graph that contains at least one cycle. If $p(G) = m(G) - c(G)$, then for any vertex $v$ lying on a cycle of $G$,

(i) $p(G - v) = p(G)$;
(ii) $p(G - v) = m(G - v) - c(G - v)$;
(iii) $m(G - v) = m(G) - 1$;
(iv) $c(G - v) = c(G) - 1$;
(v) $v$ is not a quasi-pendant vertex.

Proof. The assertions (i)-(iv) hold by considering the equality cases of the inequalities (2.2). If $v$ is a quasi-pendant vertex that is adjacent to a pendant vertex $u$, then $p(G - v) = p(G - v - u) = p(G) - 1$ by Lemma 2.4, contradicting to (i). So the assertion (v) holds.

4 The extremal graphs

In this section, we will characterize the graphs $G$ with $p(G) = m(G) + c(G)$ or $p(G) = m(G) - c(G)$. The results for $n(G)$ can be obtained similarly and the proof is omitted. If $G$ is a union of disjoint trees, surely the equalities holds by Lemma 2.2. If $G$ is a union of disjoint cycle, then by Lemma 2.1, $p(G) = m(G) + c(G)$ if and only if the length $l$ of each cycle holds $l \equiv 1 \mod 4$; and $p(G) = m(G) - c(G)$ if and only if the length $l$ of each cycle holds $l \equiv 0 \mod 4$.

By Corollaries 3.3 and 3.4, we assert that any two cycles of $G$ share no common vertices. Assume to the contrary that two cycles of $G$ have a common vertex, say $v$. Then $c(G - v) \leq c(G) - 2$, which yields a contradiction by the inequality (2.1) or (2.2).

Lemma 4.1. If $G$ is a graph satisfying $p(G) = m(G) + c(G)$ or $p(G) = m(G) - c(G)$, then any two cycles of $G$ share no common vertices.

Based on the above discussion, it suffices to consider the graphs $G$ in the class $\mathcal{G}$ which holds the following properties: (1) $G$ contains at least one cycle but is not the disjoint union the disjoint cycles and/or trees, (2) any two cycles of $G$ share no common vertices if $G$ contains more than one cycle. Contracting each cycle of a graph $G \in \mathcal{G}$ into a vertex (called cyclic vertex), we obtain a forest denoted by $T_G$. Denote by $[T_G]$ the subgraph of $T_G$ induced by all non-cyclic vertices. We begin with a lemma about the nullity of trees.
**Lemma 4.2.** Let $T$ be a tree with at least two vertices. Then $\eta(T) \leq s(T) - 1$, where $s(T)$ denotes the number of pendant vertices in $T$.

**Proof.** We use induction on the order of $T$. Suppose that $P = u_1u_2 \cdots u_l$ is a longest path in $T$, where $l \geq 2$, and $u_iu_{i+1}$ is the edge of $P$ for each $i = 1, 2, \ldots, l - 1$. If $l = 2$ or $l = 3$, then $T$ is a star and the lemma clearly holds, as $\eta(T) = |V(T)| - 2$ and $s(T) = |V(T)| - 1$.

Suppose that $l \geq 4$ and denote $T_1 := T - u_1$. If $d(u_2) \geq 3$, then all neighbors (including $u_1$) of $u_2$ except $u_3$ have the same neighborhood, i.e. $\{u_2\}$. So $r(T) = r(T_1)$ (or see [1, Proposition 1]). Thus $\eta(T_1) = \eta(T) - 1$. The induction hypothesis implies that $\eta(T_1) \leq s(T_1) - 1$, which leads to the desired inequality $\eta(T) \leq s(T) - 1$.

Now suppose that $d(u_2) = 2$. Let $T_2 = T - \{u_1, u_2\}$. By induction we have $\eta(T_2) \leq s(T_2) - 1$, and by Lemma 4.2 we get $\eta(T_2) = \eta(T)$. If $d(u_3) \geq 3$, then $s(T_2) = s(T) - 1$, from which it follows that $\eta(T) = \eta(T_2) \leq s(T_2) - 1 = s(T) - 2$. If $d(u_3) = 2$, then $s(T_2) = s(T)$, from which it follows that $\eta(T) = \eta(T_2) \leq s(T_2) - 1 = s(T) - 1$, as desired. \hfill $\Box$

By Lemma 4.2, we obtain the following result on the matching number of trees. Note that the result can also be obtained by a pure graph discussion based on augmenting paths.

**Corollary 4.3.** Let $T$ be a tree with at least two vertices, and let $\tilde{T}$ be obtained from $T$ by deleting all its pendant vertices. Then $m(\tilde{T}) < m(T)$.

**Proof.** Assume to the contrary that $m(\tilde{T}) \geq m(T)$. By Lemma 2.2 we have

$$r(\tilde{T}) = 2m(\tilde{T}) \geq 2m(T) = r(T),$$

from which it follows that $|V(\tilde{T})| - \eta(\tilde{T}) \geq |V(T)| - \eta(T)$. Consequently,

$$\eta(T) \geq \eta(T) - \eta(\tilde{T}) \geq |V(\tilde{T})| - |V(T)| = s(T),$$

contradicting to Lemma 4.2. \hfill $\Box$

Applying Corollary 4.3, we obtain an easy property of $T_G$, which is fundamental for characterization of extremal graphs.

**Lemma 4.4.** Let $G \in \mathcal{G}$. If $m(T_G) = m([T_G])$, then $T_G$ contains a non-cyclic pendant vertex. If $v$ is the vertex in $T_G$ adjacent to such pendant vertex, then $v$ is also non-cyclic. In other words, $G$ contains at least one pendant vertex, and any quasi-pendant vertex of $G$ lies outside of cycles.

**Proof.** Observe that $T_G$ contains at least one connected component of order at least 2. If all pendant vertices of $T_G$ are cyclic vertices, then by Corollary 4.3 we have

$$m([T_G]) \leq m(\tilde{T}_G) < m(T_G),$$

a contradiction, where $\tilde{T}_G$ is obtained from $T_G$ by deleting all its pendant vertices.
Now suppose that \( u \) is a non-cyclic pendant vertex of \( T_G \). Let \( v \) be vertex in \( T_G \) that is adjacent to \( u \). Surely \( u \) is a pendant vertex of \( G \). If \( v \) is a cyclic vertex of \( T_G \), then by Lemma 2.5

\[
m([T_G]) \leq m(T_G - v) = m(T_G) - 1 < m(T_G),
\]

which yields a contradiction. So \( v \) is also also non-cyclic.

\[\square\]

**Lemma 4.5.** Let \( G \in \mathcal{G} \). If there exists a maximum matching \( M(G) \) of \( G \) such that \( M(G) \cap \mathcal{F}(G) = \emptyset \), then \( m(G) = m([T_G]) + \sum_{C \subseteq G} m(C) \), where \( C \) goes through all cycles of \( G \). If in addition, each cycle of \( G \) has an odd length, then \( m(T_G) = m([T_G]) \).

**Proof.** Suppose that \( C_1, C_2, \ldots, C_i \) are all the cycles of \( G \). The condition on \( M(G) \) shows that

\[
M(G) = (M(G) \cap E([T_G])) \cup (M(G) \cap E(C_1)) \cup \ldots \cup (M(G) \cap E(C_i)),
\]

which implies the first assertion.

Note that \( \bar{M} := M(G) \cap E([T_G]) \) is a maximum matching of \([T_G]\), and also a matching of \( T_G \). If \( \bar{M} \) is not a maximum matching of \( T_G \), there exists an augmenting path \( P \) in \( T_G \) with respect to \( \bar{M} \). Returning to the graph \( G \), the path \( P \) starts from a vertex \( u \) of a cycle \( C_i \) and ends at a vertex \( v \) of another cycle \( C_j \), and contains no other vertices of cycles by the definition of \( \bar{M} \). As \( C_i \) and \( C_j \) are both odd, we can adjust the matching \( M(G) \cap E(C_i) \) (respectively, \( M(G) \cap E(C_j) \)) such that \( u \) (respectively, \( v \)) is not covered by the resulting matching. Now \( P \) is an augmenting path in \( G \) with respect to \( M \), so \( M \) is not a maximum matching of \( G \), a contradiction. Hence, \( \bar{M} \) is a maximum matching of \( T_G \), and then \( m(T_G) = m([T_G]) \).

\[\square\]

**Corollary 4.6.** Let \( G \in \mathcal{G} \) be a graph which contains only odd cycles. Then \( m(T_G) = m([T_G]) \) if and only if there exists a maximum matching \( M(G) \) of \( G \) such that \( M(G) \cap \mathcal{F}(G) = \emptyset \).

**Proof.** The sufficiency is follows from Lemma 4.5. So we only consider the necessity. We apply induction on the order of \( G \). By Lemma 4.4, \( G \) contains a pendant vertex, say \( u \), and a quasi-pendant vertex adjacent to \( u \), say \( v \) that is not lying on any cycle. Let \( H := G - \{u, v\} \).

By Lemma 2.5, \( m(T_G) = m(T_H) + 1 \) and \( m([T_G]) = m([T_H]) + 1 \). So, \( m(T_H) = m([T_H]) \). By induction there exists a maximum matching \( M(H) \) of \( H \) such that \( M(H) \cap \mathcal{F}(H) = \emptyset \). Let \( M(G) := \{uv\} \cup M(H) \). Note that \( m(G) = m(H) + 1 \) also by Lemma 2.5, \( M(G) \) is a maximum matching of \( G \) such that \( M(G) \cap \mathcal{F}(G) = \emptyset \).

We now characterize the extremal graphs which attain the upper bound for \( p(G) \).

**Theorem 4.7.** Let \( G \) be a graph. Then \( p(G) = m(G) + c(G) \) if and only if the following three conditions all hold.

(i) Any two cycles of \( G \) share no common vertices;

(ii) Each cycle of \( G \) has length 1 modulo 4;

(iii) \( m(T_G) = m([T_G]) \).

**Proof.** (Sufficiency.) We will use induction on the order of \( G \). If \( G \) is a disjoint union of trees and/or cycles of length 1 modulo 4, clearly the result holds by Lemma 2.1 and 2.2. So
we assume $G \in \mathcal{G}$. As $m(T_G) = m([T_G])$, by Lemma 4.4 $G$ contains a pendant vertex $u$ and a quasi-pendant vertex $v$ adjacent to $u$, and $v$ lies outside any cycle of $G$. Let $H := G - \{u, v\}$. By Lemma 2.5, $m(T_H) = m([T_H])$, and $H$ satisfies the three conditions (i-iii) of this theorem. By induction we have $p(H) = m(H) + c(H)$. Now by Lemmas 2.4 and 2.5

$$p(G) = p(H) + 1 = m(H) + c(H) + 1 = m(G) + c(G).$$

(Necessity.) Let $G$ be a graph such that $p(G) = m(G) + c(G)$. If $G$ is a forest then $G$ clearly satisfies (i)-(iii) of this theorem. Assume that $G$ has at least one cycle. The assertion (i) follows from Lemma 4.1.

We assert that each cycle of $G$ has length $1$ modulo $4$. If $c(G) = 1$, the result holds by Theorem 2.6. Now assume $c(G) = l$, where $l \geq 2$. If there exists a cycle, say $C_1$, whose length is not $1$ modulo $4$, by deleting an arbitrary vertex of each cycle of $G$ except $C_1$, we get a graph $H$ with $c(H) = 1$ and $p(H) \leq m(H)$ by Theorem 2.6. By Lemma 3.1

$$p(G) \leq p(H) + l - 1 \leq m(H) + l - 1 < m(G) + c(G),$$

a contradiction.

We prove the assertion (iii) by the induction on the order of $G$. If $G$ is a disjoint union of cycles, the result follows. So we assume that $G \in \mathcal{G}$. First suppose that $G$ contains a pendant vertex, say $x$, and a quasi-pendant vertex $y$ that is adjacent to $x$. By Corollary 3.3(v), $y$ is not lying on any cycle. Let $H := G - \{x, y\}$. Then by Lemma 2.4 and 2.5

$$p(H) = p(G) - 1 = m(G) + c(G) - 1 = m(H) + c(H).$$

By induction we have $m(T_H) = m([T_H])$, and hence by Lemma 2.5

$$m(T_G) = m(T_H) + 1 = m([T_H]) + 1 = m([T_G]).$$

Now suppose that $G$ contains no pendant vertices. Then $G$ contains a pendant cycle, say $C$ such that $C$ has exactly one vertex say $u$ that is adjacent to a vertex $v$ outside $C$. Let $K := G - C$. By Corollary 3.3, we have $p(G - u) = m(G - u) + c(G - u)$ and $m(G - u) = m(G)$, which implies that $p(K) = m(K) + c(K)$ from the former equality, and $m(G) = m(C) + m(K)$ from the latter equality. By the induction we have $m(T_K) = m([T_K])$. So $K$ has a maximum matching $M(K)$ such that $M(K) \cap \mathcal{F}(K) = \emptyset$ by Corollary 4.6. Let $M(C)$ be a maximum matching of $C$. Then $M(G) := M(K) \cup M(C)$ is a maximum matching of $G$, which satisfies $M \cap \mathcal{F}(G) = \emptyset$. Again by Corollary 4.6 we get $m(T_G) = m([T_G])$. $\Box$

By Corollary 4.6 we have an alternative version of Theorem 4.7

**Theorem 4.8.** Let $G$ be a graph. Then $p(G) = m(G) + c(G)$ if and only if the following three conditions all hold.

(i) Any two cycles of $G$ share no common vertices;

(ii) Each cycle of $G$ has length $1$ modulo $4$;

(iv) There exists a maximum matching $M(G)$ of $G$ such that $M(G) \cap \mathcal{F}(G) = \emptyset$. 


For the negative inertia index of a graph, we have a similar result by using a parallel discussion to the proof of Theorem 4.7.

**Theorem 4.9.** Let $G$ be a graph. Then $n(G) = m(G) + c(G)$ if and only if $G$ satisfies both of the first two conditions and either one of the last two conditions:

(i) Any two cycles of $G$ share no common vertices;
(ii) Each cycle of $G$ has length 3 modulo 4;
(iii) $m(T_G) = m([T_G])$;
(iv) There exists a maximum matching $M$ of $G$ such that $M \cap F(G) = \emptyset$.

Next, we will characterize the extremal graph $G$ satisfying $p(G) = m(G) - c(G)$ (resp., $n(G) = m(G) - c(G)$). Before announcing the main result, we investigate the property of a special class of graphs $G$ with $p(G) = m(G) - c(G)$.

**Lemma 4.10.** Let $K$ be a graph such that any two cycles share no common vertices. Let $G$ be a graph obtained from $K$ and a cycle $C_s$ (disjoint to $K$) by adding an edge between a vertex $x$ of $C_s$ and a vertex $y$ of $K$. If $p(G) = m(G) - c(G)$, then

(i) $s$ is a multiple of 4;
(ii) the edge $xy$ does not belong to any maximum matching of $G$;
(iii) each maximum matching of $K$ covers $y$;
(iv) $m(K + x) = m(K)$;
(v) $m(G) = m(C_s) + m(K) = m(C_s) + m(K + x)$.

**Proof.** We use induction on the order of $G$ to prove (i). By Corollary 3.4(v), $y$ is not an isolated vertex of $K$. So $K$ contains at least 2 vertices. If $K$ contains exactly 2 vertices, $y$ and another vertex say $z$, then $yz$ is a pendant edge of $G$. If $C_s = G - \{y, z\}$, The result follows by Lemma 2.4, Lemma 2.5 and Lemma 2.1. If $K$ is a forest, let $u$ be a pendant vertex of $K$ other than $y$, that is adjacent to a vertex $v$. Let $H := K - \{u, v\}$. By Lemma 2.4 and 2.5 $p(H) = m(H) - c(H)$. The induction hypothesis shows $4|s$. Otherwise, $K$ contains a cycle. Pick a vertex $w$ lying on a cycle of $K$, and denote $I := G - w$. By Corollary 3.4 $p(I) = m(I) - c(I)$. The induction hypothesis shows again $4|s$.

For the assertion (ii), assume to the contrary that $xy$ belongs to a maximum matching $M$ of $G$. As $4|s$, a vertex $u$ in $C_s$ is not covered by $M$. Thus we have $m(G - u) = m(G)$, a contradiction to Corollary 3.4(iii).

The assertion (iii) follows from (ii), and (iv) follows from (iii), and (v) follows from (ii) and (iv) immediately. $\square$

**Lemma 4.11.** Let $G \in \mathcal{G}$. If $p(G) = m(G) - c(G)$, then for any maximum matching $M(G)$ of $G$, $M(G) \cap F(G) = \emptyset$.

**Proof.** We will use induction on the order of $G$ to prove the result. If $G$ contains a pendant vertex $x$, and $y$ is the unique neighbor of $x$. Then $y$ is not on the cycle by Corollary 3.4(v). Let $H := G - \{x, y\}$. Then by Lemmas 2.4 and 2.5 we have $p(H) = m(H) - c(H)$. Now let
Let $M(G)$ be a maximum matching of $G$. If $xy \in M(G)$, then $M(G) \setminus \{xy\}$ is a maximum matching of $H$. Applying the induction on $H$, $(M(G) \setminus \{xy\}) \cap \mathcal{F}(H) = \emptyset$, and hence $M(G) \cap \mathcal{F}(G) = \emptyset$. Otherwise, $yz \in M(G)$, where $z \in V(H)$ is a neighbor of $y$ other than $x$, as a quasi-pendant vertex is always covered by any maximum matching. So, $M(G) \setminus \{yz\}$ is a maximum matching of $H$, which also implies that $(M(G) \setminus \{yz\}) \cap \mathcal{F}(H) = \emptyset$. Furthermore, observing that $m(H - z) = m(H)$, so $z$ is not lying on any cycle of $H$ (and $G$) by Corollary 3.4(iii) and the fact $p(H) = m(H) - c(H)$. Combining the above discussion, we also get $M(G) \cap \mathcal{F}(G) = \emptyset$.

If $G$ contains no pendant vertices, then $G$ contains a pendant cycle of $C$ which contains exactly one vertex says $u$ adjacent to a vertex $v$ outside $C$. Let $K := G - C$. By Corollary 3.4, $p(G - u) = m(G - u) - c(G - u)$, from which it follows that $p(K) = m(K) - c(K)$. Let $M(G)$ be a maximum matching of $G$. By Lemma 4.10, we have $M(K) = m(K)$. We now prove $M(G) \cap \mathcal{F}(G) = \emptyset$, which implies that $M(G) \cap \mathcal{F}(G) = \emptyset$.

Now we are ready to present another main result.

**Theorem 4.12.** Let $G$ be a graph. Then $p(G) = m(G) - c(G)$ if and only if the following three conditions all hold.

(i) Any two cycles of $G$ share no common vertices;

(ii) Each cycle of $G$ has length 0 modulo 4;

(iii) $m(T_G) = m([T_G])$.

**Proof.** (Sufficiency.) We will use induction on the order of $G$. If $G$ is a disjoint union of trees and/or cycles of length 0 modulo 4, clearly the result holds by Lemma 2.1 and 2.2. So we assume $G \in \mathcal{G}$. By Lemma 4.4, the condition $m(T_G) = m([T_G])$ implies that $G$ has a pendant vertex, say $x$, which is adjacent to a vertex $y$ lying outside any cycle. Let $H := G - \{x, y\}$. By Lemma 2.6, we have $m(T_H) = m([T_H])$. By the induction, $p(H) = m(H) - c(H)$, and hence by Lemmas 2.4 and 2.5, $p(G) = p(H) + 1 = m(H) - c(H) + 1 = m(G) - c(G)$.

(Necessity.) Let $G$ be a graph such that $p(G) = m(G) - c(G)$. The proof for (i) and (ii) goes parallel as in Theorem 4.7, thus omitted. We now proved $m(T_G) = m([T_G])$ by the induction on the order of $G$. First assume that $G$ contains a pendant vertex $x$ that is adjacent to a vertex $y$. Then $y$ is lying outside any cycle of $G$ by Corollary 3.4(v). Let $H := G - \{x, y\}$. By Lemmas 2.4 and 2.5, $p(H) = m(H) - c(H)$. So, by induction $m(T_H) = m([T_H])$, and hence $m(T_G) = m(T_H) + 1 = m([T_H]) + 1 = m([T_G])$ by Lemma 2.5.

If $G$ contains no pendant vertices, then there exists a pendant cycle $C$ of $G$, which contains exactly one vertex, say $u$ that is adjacent to a vertex $v$ outside $C$. Let $K := G - C$, and let $H := K + u$. Let $w$ be a vertex of $C$ adjacent to $u$. By Corollary 3.4(ii), $p(G - w) = m(G - w) - c(G - w)$. Repeatedly deleting the pendant and the quasi-pendant vertices of $C - \{w\}$ until we arrive at the graph $H$, we get $p(H) = m(H) - c(H)$ by Lemmas 2.4 and 2.5. By the induction, $m(T_H) = m([T_H])$. Suppose that $C = C_1, C_2, \ldots, C_l$ are all cycles contained in $G$. By Lemma 4.11 and Lemma 4.5

$$m(G) = m([T_G]) + \sum_{i=1}^{l} \frac{|V(C_i)|}{2}.$$
By a similar discussion, we also have

\[ m(H) = m([T_H]) + \frac{\sum_{i=2}^{l} |V(C_i)|}{2}. \]

Obviously, \( T_G \) is isomorphic to \( T_H \). Thus \( m(T_G) = m(T_H) \). Noting that \( m(H) = m(K) \) and \( m(G) = m(C_1) + m(K) \) by Lemma 4.10 we finally have

\[
m(T_G) = m(T_H) = m([T_H]) = m(H) - \frac{\sum_{i=2}^{l} |V(C_i)|}{2} = m(K) - \frac{\sum_{i=2}^{l} |V(C_i)|}{2} = (m(G) - m(C_1)) - \frac{\sum_{i=2}^{l} |V(C_i)|}{2} = m(G) - \frac{\sum_{i=1}^{l} |V(C_i)|}{2} = m([T_G]).
\]

Similar result holds for the negative inertia index of a graph and the proof is omitted.

**Theorem 4.13.** Let \( G \) be a graph. Then \( n(G) = m(G) - c(G) \) if and only if the three conditions in Theorem 4.12 all hold for \( G \).

**Remark 1.** One may wish to find an equivalent condition for (iii) in Theorem 4.12 or 4.13 just like the condition (iv) in Theorem 4.8 or 4.9. According to Lemma 4.11, we have a stronger one:

(iv) \( M \cap F(G) = \emptyset \) for any maximum matching \( M \) of \( G \).

However, if a connected graph \( G \) satisfies (i), (ii) of Theorem 4.12 and the above (iv), it is possible that \( p(G) \neq m(G) - c(G) \) or \( n(G) \neq m(G) - c(G) \). For example, let \( G \) be the graph obtained from two vertex-disjoint cycles of length 4 by joining a vertex of a cycle to a vertex of another cycle with an edge. But, \( p(G) = n(G) = 3, m(G) = 4, c(G) = 2 \).

**Remark 2.** Let \( G \) be a graph such that \( p(G) = m(G) + c(G) \). By Theorem 4.7 or Theorem 4.12 \( m(T_G) = m([T_G]) \), and by Lemma 4.4 \( G \) contains a pendant vertex. So the case of \( G \) containing no pendant vertices does not exist in the proof of Theorem 4.7.

In addition, as any quasi-pendant vertex of \( G \) lies outside the cycles. As shown in the proof of Theorem 4.7, if letting \( u \) be a pendant of \( G \) and \( v \) be the vertex adjacent to \( u \). Let \( H := G - \{u, v\} \). Then \( p(H) = m(H) + c(H) \). Repeating the same procession on \( H \), we finally arrive at a graph which are union of isolated vertices or disjoint cycles of length 1 modulo 4.

By this observation, any graphs \( G \) with \( p(G) = m(G) + c(G) \) can be constructed from isolated vertices and/or disjoint cycles of length 1 modulo 4 by adding a pendant vertex and a quasi-pendant vertex at each step such that no new cycles appear; see Fig. 4.1 for an illustration, where the ‘square’ vertices are added in the order written in the square boxes.

We have a similar result for the graphs \( G \) with \( p(G) = m(G) - c(G) \) or \( n(G) = m(G) + c(G) \) or \( n(G) = m(G) - c(G) \). If replacing each cycle of Fig. 4.1 by a cycle of length 3 modulo 4, the resulting graph \( \tilde{G} \) holds that \( n(\tilde{G}) = m(\tilde{G}) + c(\tilde{G}) \). If replacing each cycle of Fig. 4.1 by a cycle of length 0 modulo 4, the resulting graph \( \tilde{G} \) holds that \( p(\tilde{G}) = n(\tilde{G}) = m(\tilde{G}) - c(\tilde{G}) \).
Fig. 4.1 An illustration of construction of graphs $G$ with $p(G) = m(G) + c(G)$

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