Tunable chiral bound states in a dimer chain of coupled resonators

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Abstract

We study an excitation hopping on a one-dimensional (1D) dimer chain of coupled resonators with the alternate on-site photon energies, which interacts with a two-level emitter (TLE) by a coupling point or two adjacent coupling points. In the single-excitation subspace, this system not only possesses two energy bands with propagating states, but also possesses photonic bound states. The number of bound states depends on the coupling forms between the TLE and the dimer chain. It is found that when the TLE is locally coupled to one resonator of the dimer chain, the bound-state that has mirror reflection symmetry. When the TLE is nonlocally coupled to two adjacent resonators, three bound states with preferred direction arise due to the mirror symmetry breaking. By using chirality to measure the asymmetry, it is found that the chirality of these bound states can be tuned by changing the energy differences of single photon in the adjacent resonators, the coupling strengths and the transition energy of the TLE.

1. Introduction

The quantum electrodynamics of light-matter interactions in waveguide systems [1–3] has attracted considerable interest. In a waveguide, the electromagnetic field is confined spatially in two dimensions and propagates along the remaining one, which is called guided modes, the interference of spontaneously emitted waves from a quantum emitter (QE) and the incident wave leads to total reflection of single photons in the one-dimensional (1D) waveguide with linear [4] or nonlinear [5] dispersion relation. One of the most intriguing is the existence of bound states for photons: The single-photon bound states in continuum [6–10], where the photon is trapped between the QEs or the mirror and a QE; The single-photon bound states with energies slightly outside the continuum [11–17], where a photon is localized and symmetrical around the resonator coupling to the quantum emitter; Multiple-photon bound states [18–23], which give rise to strong correlations between photons.

Nowadays, widespread attention has been paid on giant atoms. Different from a local interaction between QEs and light field [24–31], giant atoms nonlocally couples to light field [32–39] at multiple points. Such unconventional light–matter interaction occurs when the atomic size is comparable or even larger than the wavelength of the light. The interference effects between these multiple coupling points leads to unconventional phenomenon such as frequency dependent relaxation and Lamb shift [32], non-Markovian atomic dissipation [40–42] and the decoherence free interatomic interaction [43, 44]. The symmetry bound states outside the band are found numerically for a giant atom coupling to a 1D coupled-resonator waveguide [45]. And the asymmetric bound state close to an atomic transition frequency is found in a giant atom interacting with the photonic mode of an energy band [46]. In this paper, we study a waveguide quantum electrodynamics system composed of a two-level emitter (TLE) and 1D coupled resonators arranged in a dimer chain due to their alternate on-site photon energies. Different from the 1D
coupled-resonator waveguide with uniform-hopping rates, this dimer chain possesses two energy bands, and the point-like TLE coupled non-locally to two adjacent resonators of the dimer chain. To establish the relation of bound states with the mirror symmetry, we present the exact analytical solution of the bound states in real space for arbitrary atomic transition frequency. The asymmetry of bound states stems from the nonlocal coupling of the quantum emitter to two resonators with different on-site energies. Although we show how to judge the symmetry or asymmetry of bound states from the mirror symmetry, the properties of single-photon bound states include more details, for example, how many bound states in this system; whether they all have the same preferred direction or not? Is it possible for a bound state to localized at the one-side of the symmetry axe? How to tune the asymmetry of the bound state?

The paper is organized as follows. In section 2, we propose the model describing the interaction between a TLE and a chain of coupled resonators, and the equations of the probability amplitudes are presented in single-excitation subspace. In section 3, we derive the condition for the single-photon bound state and discuss the asymmetry of all bound states by introducing the chirality. Finally, a summary has been made in section 4.

2. Model for a quantum emitter nonlocal coupled to a dimer chain

We consider a system consisting of a one-dimensional (1D) waveguide and a TLE. The 1D waveguide consists of a series of coupled resonators in which light propagates due to the coupling between the adjacent resonators (see figure 1(a)). The system can be implemented by nano-electromechanical resonator arrays where two nearest resonators with ferromagnetic particles in the tips are coupled to a localized spin [47], a side-defected cavity with double couplings to a waveguide of coupled defected cavity arrays [25, 48, 49], a superconducting atoms coupled to a Josephson photonic-crystal waveguide [46, 50], or a quantum emitter coupled to a surface plasmon polaritiation waveguide [51, 52]. For the last system the surface plasmon polaritiation supported by a metallic waveguide and can be strongly coupled with quantum emitter [53, 54] and the dual-band surface polariton waveguide can be realized by composite structures, which facilitates the emergence of the bound states. In contrast to the previous 1D waveguide with identical resonators [5], the two adjacent resonators (shown as the green and blue cavities in figure 1(a)) have on-site photon energies \(\omega_g - \delta\) and \(\omega_c + \delta\), respectively. Here, 26 denotes the the energy differences of single photon in the adjacent resonators. We cast this 1D waveguide as a dimer chain of discrete bosonic sites with equally spaced sites but two kinds of on-site photon energies, and the dimer chain is assumed to be infinitely long in both direction. Then the continuum of modes in the waveguide are the Bloch modes. We use \(\hat{a}_{j \sigma}\) and \(\hat{b}_{j \sigma}\) as the bosonic annihilation (creation) operators of a single photon for the green and blue resonators at the \(j\)th cell, respectively. Its corresponding real-space Hamiltonian reads

\[
\hat{H}_c = (\omega_c - \delta) \sum_{j=-\infty}^{+\infty} \hat{a}_{j \sigma}^\dagger \hat{a}_{j \sigma} + (\omega_g + \delta) \sum_{j} \hat{b}_{j \sigma} \hat{b}_{j+1 \sigma} - \sum_{j} \lambda \left( \hat{a}_{j \sigma} \hat{b}_{j+1 \sigma} + \hat{a}_{j+1 \sigma}^\dagger \hat{b}_{j \sigma}^\dagger \right) + h.c. \tag{1}
\]

where \(\lambda\) is the coupling constant between adjacent resonators in the dimer chain. Two bands of propagating photons have the following dispersion relation as (see appendix A)

\[
\omega_{\pm}^k = \omega_c \pm \sqrt{\delta^2 + 4\lambda^2 \cos^2(k/2)}. \tag{2}
\]

with wave number \(k \in [-\pi, \pi]\). The bands of propagating photons with different \(\delta\) in the first Brillouin zone is depicted in figure 1(b). The time-reversal symmetry is satisfied for the propagating modes in the dimer chain since \(\omega_{\pm}^k = \omega_{\mp}^k\).

The TLE’s ground and excited states \(|g\rangle\) and \(|e\rangle\), respectively, are separated in energy by \(\Omega\). The transition \(|g\rangle \leftrightarrow |e\rangle\) of the TLE is dipole coupled with coupling strength \(g_{d} \langle g | |e\rangle\) to the resonator at the 0th cell. Defining the rising operators \(\hat{\sigma}_+ = |e\rangle\langle g|\), and its adjoint \(\hat{\sigma}_-\), the Hamiltonian for the free TLE part and the interaction between the TLE and the field within the rotating-wave approximation reads

\[
\hat{H}_1 = \Omega \langle e | g \rangle \langle e | + \hat{\sigma}_+ \left( g_{d} \hat{a}_0 + g_{b} \hat{b}_0 \right) + h.c. \tag{3}
\]

Here, we have introduced the nonlocal interaction between the TLE and the light field.

The number operator \(\hat{N} = \sum_j (\hat{a}_{j \sigma}^\dagger \hat{a}_{j \sigma} + \hat{b}_{j \sigma}^\dagger \hat{b}_{j \sigma} + 1)\) commutes with the total Hamiltonian \(\hat{H} = \hat{H}_c + \hat{H}_1\). We restrict the analysis to the subspaces with single excitation hereafter. In the single-excitation subspace, there are two mutual exclusive possibilities: the particle either is propagating...
inside the cavity or is absorbed by the TLE. Letting $|\emptyset\rangle = |0g\rangle$ be the state without photon while the TLE stays on its ground state, the eigenstate of the Hamiltonian reads

$$|\epsilon\rangle = \left(\sum_j \alpha_{2j} \hat{a}_{2j}^\dagger + \sum_j \beta_{2j+1} \hat{b}_{2j+1}^\dagger + u_c \hat{\sigma}_+\right) |\emptyset\rangle,$$

(4)

where $\alpha_{2j}$, $\beta_{2j+1}$ are the probability amplitudes to find a photon in $a$ and $b$ resonators of the $j$th cell, respectively, and $u_c$ is the probability amplitude of the TLE in the excited state while no photon in the dimer chain. From the stationary Schrödinger equation, one can obtain the equations for the amplitudes. By removing $u_c$, the equations for the photonic amplitudes reduce to

$$\begin{align}
(\epsilon - \omega_c + \delta) \alpha_{2j} &= -\lambda \left(\beta_{2j+1} + \beta_{2j-1}\right) + \delta \beta \left(G \alpha_0 + V_b \beta_1\right), \\
(\epsilon - \omega_c - \delta) \beta_{2j+1} &= -\lambda \left(\alpha_{2j} + \alpha_{2j+2}\right) + \delta \alpha \left(G^* \alpha_0 + V_b \beta_1\right),
\end{align}

(5a, b)

which lead to a nonlocal energy-dependent delta-like potentials $V_n$, $n = a, b$ and the effective dispersive coupling strength

$$V_n = \frac{\xi_n^a \xi_n^b}{\epsilon - \Omega}G = \frac{\xi_n^a \xi_n^b}{\epsilon - \Omega},$$

(6)

Obviously, the effective dispersive coupling strength $G$ vanishes when the TLE only interacts with one resonator of the unit cell, which plays an important role in the emergence of the chiral bound states.

3. Single-photon bound states

The presence of the TLE breaks down the translational symmetry of the dimer chain, which leads to the highly-localized states. We plot the energy spectrum versus coupling strengths in the single excitation subspace in figures 2(a)–(d) by numerical diagonalization of the Hamiltonian in the real space and plot the energy versus wave number $k$ in figures 2(e)–(f). Two energy bands of scattering states are symmetrically formed above and below $\epsilon = \omega_c$ with $2\delta$ as the band gap. Obviously the two bands merge to one band when there is no energy difference between resonators ($\delta = 0$). The three curves, one above all bands, one below all bands and the other inside the band gap, are the bound states. As coupling strengths increase, the energy differences between the bound states and the band edges also increase. However, the increment of the energy differences between the bound states and the band edges is dependent on transition frequency $\Omega$, especially for the emerging bound state inside the gap. When $\Omega < \omega_c$, the energy of the emerging bound state increases as the coupling strengths increase. When $\Omega > \omega_c$, the energy of the emerging bound state decreases as the coupling strengths increase. In figures 2(c) and (d) the TLE only interacts with one resonator of the unit cell. It is noted that figure 2(d) can be obtained by the mirror reflection of the figure 2(c) with mirror surface locating at $\epsilon = \omega_c$, which actually indicates the same mirror reflection symmetry of the dimer resonator energies and the TLE’s energy when $a \rightarrow b$ and $\Omega - \omega_c \rightarrow - (\Omega - \omega_c)$. We plot the energy versus wave number $k$ in figures 2(e) and (f). Hereafter, for brevity, the bound state above, below the bands in the dimer chain are denoted as bound state I, II, and the bound state in the band gap is denoted as bound state III.

The highly-localized states in previous studies [11–17, 45] decay exponentially and symmetrically in both directions, actually, the symmetry of the bound state is guaranteed by the mirror reflection symmetry around
the quantum emitter. However, the mirror reflection symmetry is broken in our model with all $g_0 \neq 0$. Since only the energy bands are modified, there should still be localized modes of photons around the cell where the TLE is embedded. So we assume the following damped wave

$$\alpha_{2j} = \begin{cases} A_F^e e^{ik_0+\kappa j}, & j < 0 \\ A_R^e e^{ik_0-\kappa j}, & j > 0 \end{cases}, \quad \beta_{2j+1} = \begin{cases} B_F^e e^{ik_0+(j+1)/2}, & j < 0 \\ B_R^e e^{ik_0-(j+1)/2}, & j > 0 \end{cases}$$

(7)

with $k_0 = 0, \pi$, which decreases exponentially with the distance from the 0th cell. As shown in appendix B, the imaginary wave vector $\kappa > 0$ labels the energy

$$\epsilon = \omega_c \pm \sqrt{\delta^2 + \lambda^2 (2 + e^{ik_0+\kappa} + e^{-ik_0-\kappa})}$$

(8)

of a localized photon outside of the bands. At the region far away from the TLE, we obtain

$$\frac{A_F^e}{B_F^e} = -\lambda \frac{e^{\delta/2} + e^{-\kappa/2}}{\epsilon - \omega_c + \delta}, \quad \frac{A_R^e}{B_R^e} = -\lambda \frac{e^{-\kappa/2} + e^{-\kappa/2}}{\epsilon - \omega_c + \delta}.$$  

(9)

By applying equations (7) and (9) to the discrete scattering Equation at $j = \pm 1$, the amplitudes for the $j = 0$ cell can be obtained as

$$\alpha_0 = A_F^e, \quad \beta_1 = B_R^e e^{-\kappa/2}.$$  

(10)

Substituting equation (10) and the spatial exponential-decay solution (7) to equation (5) at $j = 0$ yields the condition for the energy of the bound state

$$|G - \lambda|^2 = \left( \epsilon - \omega_c + \delta - V_a + \frac{\lambda B_F^e}{A_F^e} e^{-ik_0-\kappa/2} \right) \left( \epsilon - \omega_c - \delta - V_b + \frac{\lambda A_R^e}{B_R^e} e^{ik_0-\kappa/2} \right).$$

(11)

Once $\kappa$ is obtained from equation (11), so does the ratio in equation (9). To give an intuitive knowledge on the bound state, we rewrite the wave function of the bound states as

$$\frac{\alpha_{2j}}{A_R^e} = \begin{cases} A_F^e e^{ik_0+\kappa j}, & j < 0 \\ A_R^e j, & j = 0 \\ A_F^e e^{ik_0-\kappa j}, & j > 0 \end{cases}, \quad \frac{\beta_{2j+1}}{A_R^e} = \begin{cases} B_F^e e^{ik_0+(j+1)/2}, & j < 0 \\ B_R^e e^{-\kappa/2}, & j = 0 \\ B_F^e e^{ik_0-(j+1)/2}, & j > 0 \end{cases}.$$  

(12)
The bound state distribution above all the bands (a), (d), (g), inside the band gap (b), (e), (h) and below all the bands (c), (f), (i). In panels (a)–(c), \( g_a = 0.5, g_b = 0, \delta = 0.2, \Omega = 99.8 \), i.e. the TLE only interacts with the resonator at the 0th site. In panels (d)–(f), \( g_a = 0, g_b = 0.5, \delta = 0.2, \Omega = 100.2 \), the TLE only interacts with the resonator at the 1st site. In panels (g)–(i), the TLE interacts with the adjacent resonators at the 0th cell, \( g_a = g_b = 0.7, \Omega = 102, \delta = 1 \) (h) \( \Omega = 99.5, \delta = 0.3 \), (i) \( \Omega = 98.5, \delta = 0.5 \). All parameters are in units of the hop strength \( \lambda \) and \( \omega_c = 100 \).

Figure 3. The photonic probability distribution of the bound state I (a,d,g), III (b,e,h) and II (c,f,i). It can be found from figures 3(a)–(f) that all the bound states are symmetry around the resonator. When the TLE only interacts with single resonator, the same mirror reflection symmetry of the dimer resonator energies and the TLE’s energy under \( a \rightarrow b \) and \( \Omega \rightarrow -\omega_c \rightarrow -(\Omega - \omega_c) \) still can be found in panels (a–c) and (d–f) (e.g. figures 3(a) and (j)). However, when the TLE interacts with two adjacent resonators at one unit cell, the bound states no longer has the mirror reflection symmetry as shown in figures 3(g)–(i).

If the TLE is located at the middle of the unit cell at \( j = 0 \), e.g. position \( x_0 = l/4 \), the line at the TLE’s location divides the space into left- and right-hand side of the TLE. The photonic component of the bound state in figure 3(g) are strongly localized at the right-hand side of the TLE, and that in figure 3(h) mostly distributes to the left-hand side of the TLE, the asymmetry of the photonic probability on both side of the \( x_0 \) axis can also be found in figure 3(i). According to the reflection symmetry in geometry, we introduce the chirality [46]

\[
S = \frac{s_L - s_R}{s_L + s_R}
\]
Figure 4. Tuning the chirality of the bound state above all the bands (a), (d), below all the bands (b), (e) and inside the band gap (c), (f). (a)–(c) Chirality versus $\delta$ for fixed coupling strength $g_a = g_b = 0.7, \Omega = 102$ (red), $\Omega = 101.5$ (blue), $\Omega = 100.5$ (orange), $\Omega = 99.5$ (green) and $\Omega = 98.5$ (purple). (d)–(f) Chirality versus $g_b$ for fixed $\delta = 0.2, g_a = 0.7, \Omega = 101.5$ (blue), $\Omega = 100.5$ (orange), $\Omega = 99.5$ (green) and $\Omega = 98.5$ (purple). All parameters are in units of the hop strength $\lambda$ and $\omega_c = 100$.

to depict the asymmetry of the bound state in left- and right-hand side of the TLE

\[
s_L = \sum_{j=-\infty}^{\infty} \left( |\alpha_{2j}|^2 + |\beta_{2j+1}|^2 \right) + |\alpha_0|^2, \quad (16a)
\]

\[
s_R = \sum_{j=1}^{\infty} \left( |\alpha_{2j-1}|^2 + |\beta_{2j}|^2 \right) + |\beta_1|^2, \quad (16b)
\]

where $S > 0$ ($S < 0$) indicates that the left-handed (right-handed) chirality, and $S \to 1$ ($S \to -1$) corresponds to the perfect left-handed (right-handed) chirality. By applying the wave function in equation (12), we can obtain the expression of the chirality in terms of the ratios

\[
S = \frac{|A_L^x/A_L^y|^2 + \left( |A_L^x/A_L^y|^2 - 1 \right) \frac{e^{-\kappa} + |A_L^x/A_L^y|^2}{1 - e^{-2\kappa}} e^{-\kappa}}{|A_R^x/A_R^y|^2 + \left( |A_R^x/A_R^y|^2 + 1 \right) \frac{e^{-\kappa} + |A_R^x/A_R^y|^2}{1 - e^{-2\kappa}} e^{-\kappa}}. \quad (17)
\]

To show the dependence of the chirality on the parameters, we have plotted the chirality $S$ of the bound state versus $\delta$ in figures 4(a)–(c) and the coupling strength in figures 4(d)–(f). Since the TLE-resonator coupling strengths are equal in panels (a–c), the break of the reflection symmetry depends on whether $\delta$ vanishes or not. It is shown that all chirality vanishes at $\delta = 0$ in panels (a,b) for ordinary bound states I and II. While since the merging bound state III disappear for $\delta = 0$, the critical chirality of the emerging bound state III in panel (c) is quite different for different transition energies of the TLE, while the chirality of ordinary bound states tends to coincidence. One can tune the chirality by adjusting the transition energy $\Omega$ for a given $\delta$: the left-handed chirality decreases as $\Omega$ increases for bound state II, the right-handed chirality can increase as $\Omega$ increases under certain parameters for bound state I. It should be aware that the chirality of bound state I prefers the right-hand side of dimer chain and bound state II prefers the left hand side of dimer chain. Although the chirality of all bound states can be continuously tuned by adjusting $\delta$, the right (left) hand side chirality remains unchanged for bound state I (II), only bound state III can change its sign of chirality for an appropriate $\Omega$ (see the green line in panel c). The perfect right (left) chirality can be achieved only for the bound state I (III). For a given $\delta$, one can change the sign of chirality of the bound states I and II by adjusting one of the coupling strengths, see figures 4(d) and (e). These changes can be easily explained by following reasons: When $g_b \ll g_a$, bound states are mainly localized around the 0th site, however the
symmetry line is positioned at \( x_0 = l/4 \) not \( x_0 = 0 \), they present left handed chirality; When \( g_b \gg g_a \), bound states are mainly localized around the 1th site, thence the symmetry line positioned at \( x_0 = l/4 \) gives rise to a right handed chirality of bound state I and II. It can be also found that the left handed chirality increases (decreases) as \( \Omega \) increases when \( g_b \) is smaller than \( g_a \) and the right handed chirality increases (decreases) as \( \Omega \) increases when \( g_b \) is larger than \( g_a \) for bound state I (II), the perfect right (left) charity can be still achieved only for the bound state I (III).

4. Conclusion

In summary, we consider a 1D dimer chain of coupled resonators, where two resonators in each dimer have different on-site photon energies. In the single-excitation subspace, this system has propagating states forming two energy bands and bound-states. We mainly focus on the chiral feature of the bound states. After obtaining the analytical solutions of the bound states in real space, we found that 1) the bound-state distributes symmetrically around the coupling point when the TLE is locally coupled to one resonator of the dimer chain; 2) the mirror-reflection symmetry breaking leads to the formation of three chiral bound states when the TLE is nonlocal coupled to two adjacent resonators. The chirality of the bound states inherits from the geometry aspects, which are characterized by either the difference of on-site energies in each unit cell or the coupling strengths between the TLE and the resonators, but it can also be tuned by the transition energy of the TLE. When the coupling strengths are identical, the nonvanishing difference of on-site energies lead to right handed chirality of bound state I and left handed chirality of bound state II, and bound state III can change its preferred chirality by adjusting on-site difference together with appropriate transition energy of the TLE. For given on-site energies, one can change the preferred chirality the bound states I and II by adjusting one of the coupling strengths.

The storage, the manipulation, and the transmission of quantum information are the three principal components of quantum information processing. For storage, the bound state trap the excitation around the TLS. For manipulation, when multiple TLSs are coupled to the same waveguide, the directional dipole–dipole interaction can be induced by the chiral bound states since their interaction strength is determined by the overlap between the decaying evanescent fields of the bound states, which means that the atoms only interact with the atoms in their chiral preferred directions, but cannot interact with those in the opposite direction. More exotic quantum phenomena, such as topological phase transition, might be realized.

This model can be realized in the circuit QED system: a superconducting qubit coupled to an array of transmission line resonators. The eigenfrequency of a transmission line resonator can be tuned from \( 2\pi \times 4 \text{ GHz} \) to \( 2\pi \times 4.8 \text{ GHz} \) [55], the coupling strength among two adjacent resonator is about \( 2\pi \times 44 \text{ MHz} \) [56], and it can be further increased by using larger capacitors to connect two transmission line resonators or by a qubit ultrastrong coupling to two transmission line resonators [57].

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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Appendix A

We begin with Hamiltonian in equation (1) which reads

\[
\hat{H}_c = (\omega_c - \delta) \sum_{j=-\infty}^{+\infty} \hat{a}_{2j}^\dagger \hat{a}_{2j} + (\omega_c + \delta) \sum_{j} \hat{b}_{2j+1}^\dagger \hat{b}_{2j+1} - \sum_{j} \lambda \left( \hat{a}_{2j}^\dagger \hat{b}_{2j+1} + \hat{a}_{2j} \hat{b}_{2j-1}^\dagger \right) + h.c. \tag{18}
\]
By applying the Fourier transform under the periodic boundary condition

$$\hat{a}_{2j} = \frac{1}{\sqrt{N}} \sum_k \hat{a}_k e^{jkx_{2j}}, \hat{b}_{2j+1} = \frac{1}{\sqrt{N}} \sum_k \hat{b}_k e^{jkx_{2j+1}}$$  \hspace{1cm} (19)$$

where $x_{2j} = j$ and $x_{2j+1} = j + \frac{1}{2}$. Substituting equation (18) into equation (19), we have

$$\hat{H}_c = \sum_k \left( \hat{a}_k \hat{a}_k^\dagger \hat{b}_k^\dagger \right) \left( \begin{array}{cc} \omega_c - \delta & -2\lambda \cos \frac{k}{2} \\ -2\lambda \cos \frac{k}{2} & \omega_c + \delta \end{array} \right) \left( \begin{array}{c} \hat{a}_k \\ \hat{b}_k \end{array} \right).$$  \hspace{1cm} (20)$$

Now we can introduce the new operator

$$\hat{c}_{k+} = \cos \theta \hat{a}_k - \sin \theta \hat{b}_k, \hat{c}_{k-} = \sin \theta \hat{a}_k + \cos \theta \hat{b}_k$$  \hspace{1cm} (21)$$

with the parameters

$$\cos \theta = \frac{\cos \frac{k}{2}}{\sqrt{\left( \delta + \sqrt{\delta^2 + 4\lambda^2 \cos^2 \frac{k}{2}} \right)^2 + 4\lambda^2 \cos^2 \frac{k}{2}}}$$  \hspace{1cm} (22)$$

$$\sin \theta = \frac{\delta + \sqrt{\delta^2 + 4\lambda^2 \cos^2 \frac{k}{2}}}{\sqrt{\left( \delta + \sqrt{\delta^2 + 4\lambda^2 \cos^2 \frac{k}{2}} \right)^2 + 4\lambda^2 \cos^2 \frac{k}{2}}}$$  \hspace{1cm} (23)$$

to write Hamiltonian $\hat{H}_c$ in the diagonal form

$$\hat{H}_c = \sum_k \omega_k^+ \hat{c}_k^+ \hat{c}_k^+ + \sum_k \omega_k^- \hat{c}_k^- \hat{c}_k^-.$$  \hspace{1cm} (24)$$

Then, the energy spectrum is obtained as

$$\omega_k^\pm = \omega_c \pm \sqrt{\delta^2 + 4\lambda^2 \cos^2 \left( k/2 \right)}.$$  \hspace{1cm} (25)$$

**Appendix B**

In section 2 of the main text, we have given the equations for the amplitudes by the stationary Schrödinger equation. At the region far away from the TLE, equation (5) can be written as

$$(\epsilon - \omega_c + \delta) \alpha_{2j} = -\lambda \left( \beta_{2j+1} + \beta_{2j-1} \right),$$  \hspace{1cm} (26a)$$

$$(\epsilon - \omega_c - \delta) \beta_{2j+1} = -\lambda \left( \alpha_{2j} + \alpha_{2j+2} \right).$$  \hspace{1cm} (26b)$$

Substituting the following test wavefunction

$$\alpha_{2j} = \left\{ \begin{array}{ll} A_R^c e^{ikj} & j < 0 \\ A_R^c e^{-ikj} & j > 0 \end{array} \right., \beta_{2j+1} = \left\{ \begin{array}{ll} B_R^c e^{ik(j+1)/2} & j < 0 \\ B_R^c e^{-ik(j+1)/2} & j > 0 \end{array} \right.$$  \hspace{1cm} (27)$$

with $k_0 = 0, \pi$ into equation (26) far away from the resonators $j = \pm 1$, the relation between the left-side amplitudes of the TLS as well as that of its right-side amplitudes reads

$$0 = (\epsilon - \omega_c + \delta) A_R^c + \lambda B_R^c \left( e^{\kappa j/2} + e^{-i\kappa j} e^{-\kappa j/2} \right),$$  \hspace{1cm} (28a)$$

$$0 = (\epsilon - \omega_c - \delta) B_R^c + \lambda A_R^c \left( e^{-\kappa j/2} + e^{i\kappa j} e^{\kappa j/2} \right),$$  \hspace{1cm} (28b)$$

$$(\epsilon - \omega_c + \delta) A_R^c = -\lambda B_R^c \left( e^{-\kappa j/2} + e^{-i\kappa j} e^{\kappa j/2} \right),$$  \hspace{1cm} (29a)$$

$$(\epsilon - \omega_c - \delta) B_R^c = -\lambda A_R^c \left( e^{\kappa j/2} + e^{i\kappa j} e^{-\kappa j/2} \right).$$  \hspace{1cm} (29b)$$
According to equations (28) and (29), we can obtain equation (9). The non-zero amplitudes of equations (28) and (29) requires the determinant of coefficients matrix to be zero. Therefore, the energy of the bound state read

$$\epsilon = \omega_c \pm \sqrt{\delta^2 + \lambda^2 \left( 2 + e^{i \theta} + e^{-i \theta - \kappa} \right)}.$$ (30)

The upper one of equation (26) at \( j = 1 \) and the lower one of equation (26) at \( j = -1 \) read

$$\lambda \beta_1 = -\lambda \beta_3 - (\epsilon - \omega_c + \delta) \alpha_2,$$ (31a)

$$\lambda \alpha_0 = -\lambda \alpha_2 - (\epsilon - \omega_c - \delta) \beta_1.$$ (31b)

By applying equations (27) and (9) to the equation (31), the amplitudes for \( j = 0 \) cell can be obtained as

$$\alpha_0 = A^L_0, \beta_1 = B^R_0 e^{-\kappa/2}.$$ (32)

Substituting equations (32) and (27) to equation (5) at \( j = 0 \) leads to the following equations

$$0 = (\epsilon - \omega_c + \delta - V_a) A^L_0 + (\lambda - G) B^R_0 e^{-\kappa/2} + \lambda B^L_0 e^{-i \theta_0} e^{-\kappa/2},$$ (33a)

$$0 = (\epsilon - \omega_c - \delta - V_b) B^R_0 + (\lambda - G^*) A^L_0 e^{-\kappa/2} + \lambda A^R_0 e^{i \theta_0} e^{-\kappa/2}.$$ (33b)

After some algebra, we then obtain the condition for the energy of the bound state

$$|G - \lambda|^2 = \left( \epsilon - \omega_c + \delta - V_a + \frac{\lambda B^R_0}{A^L_0} e^{-i \theta_0 - \kappa/2} \right) \left( \epsilon - \omega_c - \delta - V_b + \frac{\lambda A^R_0}{B^R_0} e^{i \theta_0 - \kappa/2} \right).$$ (34)

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