Abstract

We discuss the quantum state structure using the standard model for three colored quarks in the fundamental representations of $SU(3)_c$ making up the singlet ground state of the hadrons. This allows us to calculate a finite von Neumann entropy from the quantum reduced density matrix, which we explicitly evaluate for the quarks in a model for the meson and baryon states. Finally we look into the general effects and implications of entanglement in the $SU(3)_c$ color space.
1 Introduction

The well known theorem of Nernst, which is often times referred to as the Third Law of Thermodynamics, has the generally accepted interpretation in the theory of gases that the entropy vanishes in the zero temperature limit. Schrödinger [1] had pointed out long ago that when two states contribute to the ground state of a many particle system that a finite constant term could appear in the entropy. In particular, for a system with $2^N$ states making up the ground state of a system of $N$ particles one should expect a ground state entropy of $N \ln 2$. We can now understand his result in terms of the $SU(2)$ symmetry for the $N$ particles. In this sense we should expect an internal symmetry provided by the quantum structure to yield an entropy following the prescription of von Neumann [2].

The standard model has the color charge carried by the quarks as the fundamental property of the strong nuclear interaction [3]. In clear contrast to the other known charges the color charge cannot be easily isolated and separately measured. In nature it always appears as part of selective states of the $SU(3)_c$, wherein the quarks and antiquarks are placed in the fundamental $3$ and antifundamental $3^*$ representations of this group. These two representations together with the adjoint representation of $SU(3)_c$ make up the symmetry structure of Quantum Chromodynamics (QCD) [4]. The two main categories of strong interacting particles (hadrons) are the mesons, which may be written as a product of the fundamental and the antifundamental representations $3 \otimes 3^*$ and the baryons, which are a product of three fundamental representations $3 \otimes 3 \otimes 3$. Although the different quarks have other properties like spin, electrical charge, mass as well as a very special property called flavor, we shall not presently go into these aspects here [3].

In this work we shall write the quark and antiquark color states as follows: $|0\rangle, |1\rangle, |2\rangle$ and $|0^*\rangle, |1^*\rangle, |2^*\rangle$. We shall use this notation to describe the orthonormal bases of the fundamental and the antifundamental representations of $SU(3)_c$ instead of the more common color names. From these color states we can construct a representation for the color hadronic wavefunctions— in particular for the singlet meson $\Psi_{M,s}$ and baryon $\Psi_{B,s}$ groundstates. We also mention the construction for the eight density matrices for the color octet states of the mesons and baryons. From these color wavefunctions we are able to construct the corresponding density matrices $\rho_{M,s}$ and $\rho_{B,s}$ for the color singlet states [5]. In the following work we shall arrive at the single quark reduced density matrix $\rho_q$, which is of particular interest in all further calculations. From $\rho_q$ we can directly calculate the quantum entropy in the sense of von Neumann [5,2]. The results of this calculation show a significant contribution of order one to the entropy of the quarks in the hadronic singlet and octet states. This value is given as a pure number without physical dimensions when we use the usual high energy units with $\hbar$, $c$ and Boltzmann’s constant $k$ all set to the value one.

The further implications of these results can be brought together with some of our earlier work. After a discussion of the entropy density we look at the thermodynamics of
the quark states at very low temperatures, which means at temperatures much lower than the lightest quark masses. We rewrite the energy density equation in a form involving the trace of the energy momentum tensor. This allows us to include the vacuum condensates in the low temperature limit. Next we relate this work to some earlier work involving a finite number of baryons at finite temperatures. We discuss some recent work involving the hadron resonance mass spectrum and its relationship to lattice QCD thermodynamics. At high temperatures the entropy density $s(T)$ from gluons is well known from SU(3) lattice simulations. Finally we mention some recent work relating to the quantum mechanics of entangled states as it is often applied to quantum information theory.

## 2 Singlet Quark Structure

The starting point for the ground state is the evaluation of the density matrix \[\rho\] for the singlet quark structure. Here we only consider the color part of the wavefunctions $\Psi_{M,s}$ and $\Psi_{M,s}^*$ coming from the representation $3 \otimes 3^*$ for the color singlet wavefunctions of the mesons,

$$\Psi_{M,s} = \frac{1}{\sqrt{3}} (|0^*0\rangle + |1^*1\rangle + |2^*2\rangle). \quad (2.1)$$

and keeping the left to right order of the quark and antiquark for its conjugate wavefunction

$$\Psi_{M,s}^* = \frac{1}{\sqrt{3}} (\langle 0^*0| + \langle 1^*1| + \langle 2^*2|). \quad (2.2)$$

Similarly we may write a wavefunction for the baryons $\Psi_{B,s}$ coming from the representation $3 \otimes 3 \otimes 3$ for the color singlet state of the baryons,

$$\Psi_{B,s} = \frac{1}{\sqrt{6}} (|0^*1^*2\rangle + |1^*2^*0\rangle + |2^*0^*1\rangle - |0^*2^*1\rangle - |1^*0^*2\rangle - |2^*1^*0\rangle). \quad (2.3)$$

The conjugate state wavefunction in the order of the tensor product for the baryons is given by

$$\Psi_{B,s}^* = \frac{1}{\sqrt{6}} (\langle 0^*1^*2| + \langle 1^*2^*0| + \langle 2^*0^*1| - \langle 0^*2^*1| - \langle 1^*0^*2| - \langle 2^*1^*0|). \quad (2.4)$$

We can now write down the density matrices $\rho$ for the hadrons using the direct product of $\Psi$ and $\Psi^*$. This gives for the color singlet mesons and baryons the density matrices in the following forms:

$$\rho_{M,s} = \Psi_{M,s}\Psi_{M,s}^* \quad (2.5)$$

and

$$\rho_{B,s} = \Psi_{B,s}\Psi_{B,s}^*. \quad (2.6)$$

Up until now we have only considered the hadronic states as being made out of the
quark states. The resulting density matrices are for the hadrons pure states \[5\]. However, for the quarks we look at the one quark reduced density matrices, which give the statistical state of the individual quark within the hadron. In order to get the reduced density matrices for the mesons, we project out all the antiquark states \( \langle i^* | \) and \( | j^* \rangle \) by using the orthonormality and the completeness properties. Similarly for the baryons we project onto the other two quark states resulting in two contributions for each color. Thus both the meson and the baryon reduced density matrices for the quark states take on the same form:

\[
\rho_q = \frac{1}{3} (|0\rangle\langle0| + |1\rangle\langle1| + |2\rangle\langle2|).
\]

(2.7)

This is the reduced density matrix for the quarks in the color singlet state. It yields a completely mixed state where each color contribution has the same eigenvalue \( \lambda_i \) equal to the value \( 1/3 \). The reduced density matrices can also be calculated for each quark state in the octet representations. A more detailed discussion will appear in a later work.

### 3 Entropy for Quark States

We can calculate the entropy \( S \) of the quantum states using the prescription of von Neumann \[5,2\], which makes direct use of the density matrix \( \rho \). It is simply written as

\[
S = -\text{Tr}( \rho \ln \rho ),
\]

(3.1)

where the trace "\( \text{Tr} \)" is taken over the quantum states. When, as is presently the case, the eigenvectors are known for \( \rho \), we may write this form of the entropy in terms of the eigenvalues \( \lambda_i \) as follows:

\[
S = -\lambda_i \ln \lambda_i.
\]

(3.2)

It is obviously important to have positive eigenvalues. For a zero eigenvalue we use the fact that \( x\ln x \) vanishes in the small \( x \) limit. Then for the density matrix \( \rho \) we may interpret \( \lambda_i \) as the probablitiy of the state \( i \) or \( p_i \). This meaning demands that \( 0 < p_i \leq 1 \). Thus the orthonormality condition for the given states results in the trace condition

\[
\text{Tr} \rho = \sum_i p_i = 1.
\]

(3.3)

This is a very important condition for the entropy.

We now apply these definitions to the entropy for the quark states. It is clear that the original hadron states are pure colorless states which posses zero entropy. For the meson it is immediately obvious since each colored quark state has the opposing colored antiquark state for the resulting colorless singlet state. The sum of all the cycles determine the colorlessness of the baryon singlet state thereby giving no entropy. However, the reduced density matrix for the individual quarks (antiquarks) \( \rho_q \) or \( \rho_{\bar{q}} \) has a finite entropy. For \( SU(3)_c \) all the eigenvalues \( \lambda_i \) in Equation (2.4) have the same value \( 1/3 \). Thus we find for all the quarks (antiquarks) in singlet states

\[
S_q = \ln 3.
\]

(3.4)
As a further point we may draw a qualitative comparison of this result for the singlet state with the entropies of the quark octet states. The octet density matrices \( \rho_{o,i} \) may be constructed from the eight Gell-Mann matrices \( (\lambda)_i \) with \( i = 1, 2, \ldots, 8 \). The density matrix for each state is constructed by using the properties of \( \Psi(\lambda)_i \Psi^\ast \). From the reduced density matrix, the first seven of these all give the same value for the entropy \( S_{o,i} \), since all of these states are constructed only from the Pauli matrices. However, the eighth diagonal Gell-Mann matrix yields a larger entropy \( S_{o,8} \) from the fact that it involves all three colors although not with equal weights as it was the case for the color singlet state.

Hereupon we may discuss the entropy in some more detail for the main examples of the colorless hadronic ground states– the mesons and the baryons. As we have discussed above for the density matrix, all the mesons consist of a quark-antiquark pair bound together as a sum of all the three colors. Since each single quark or antiquark state is equally weighted in the reduced density matrix, therefore each state possesses equal probability of \( 1/3 \). Thus we easily get the entropy of \( \ln 3 \). The baryon has the doubly reduced density matrix for each single quark state appearing twice so that with the normalization factor of \( 1/6 \) the probability of each colored quark state is again \( 1/3 \), which yields the same result for the entropy, \( \ln 3 \). This value gives the maximal entropy for a completely mixed state. We know that the singlet and the octet states make up a major contribution to all the hadronic states.

It is the colored quark entropy density which physically distinguishes the thermodynamics of the baryons from the mesons. If we use the generally accepted values for the root mean squared charge radius of the hadrons \( 8 \), we take for the mesons (pions) \( \sqrt{\langle r^2_M \rangle} = 0.66 \pm 0.02 \text{ fm} \), while for the baryons (protons) \( 6 \) we use \( \sqrt{\langle r^2_B \rangle} \) the value \( 0.870 \pm 0.008 \text{ fm} \). These values of the charge radii are small when put on the nuclear size scale, where we would generally expect sizes well over 1 fm. These mean charge radii give spherical charge volumes for the mesons ranging from \( 1.098 \text{ fm}^3 \) to \( 1.317 \text{ fm}^3 \) or about \( 1.20 \text{ fm}^3 \) as the average mean volume. Similarly for the baryons we find an average mean volume of \( 2.76 \text{ fm}^3 \). The mean entropy density of the quarks in the singlet ground state of the mesons (pions) is given by

\[
s_M = \frac{\ln 3}{1.20 \text{ fm}^3} = 0.912 \frac{1}{\text{fm}^3}. \tag{3.5}
\]

Similarly for the baryons in the singlet ground state we arrive at a mean value for the entropy density

\[
s_B = \frac{\ln 3}{2.76 \text{ fm}^3} = 0.398 \frac{1}{\text{fm}^3}. \tag{3.6}
\]

These entropy densities represent the most probable distributions of quarks in the given volume for the charged portions of the hadrons. In the next section we shall discuss the relationship of these results to the thermodynamics of massive quark systems.
As a last remark in this section on the meaning of this type of entropy for the quantum ground state we should note that the effects of this type appear in other systems with internal symmetries. In quantum spin chains [7] the effects of the ground state entanglement show strong correlations in a block of L spins giving entropies proportional to the logarithm of the size L in the various special cases of the quantum Heisenberg model. These results are then related to the entropy in a 1 + 1 dimensional conformal field theory. Furthermore, one could, perhaps, extend these results to a three state model like the $Z(3)$ symmetric spin models or the extended Potts models [8] to find analogous properties for the entropy in the low temperature limit.

4 Thermodynamics for confined Quarks

We start with the form of the First Law of Thermodynamics for a quark gluon system at very low temperatures $T$. In terms of the densities we write

$$s(T)T = \epsilon(T) + p(T) - \mu_q n_q(T), \quad (4.1)$$

where $\mu_q$ is the quark chemical potential and $n_q(T)$ is quark density distribution function. We may rewrite this equation using the fact that the thermal average of the trace of the energy-momentum tensor is written as the equation of state [5]

$$\langle \Theta_{\mu}^{\mu} \rangle_T = \epsilon(T) - 3p(T), \quad (4.2)$$

where the sum is taken on the Lorentz indices $\mu$. We now rewrite equation (4.1) as

$$\langle \Theta_{\mu}^{\mu} \rangle_T = s(T)T - 4p(T) + \mu_q n_q(T). \quad (4.3)$$

In the limit that the temperature goes to zero we use $p = -B$ where $B$ is the bag constant [3]. Clearly $s(T)T$ vanishes. Thus our equation of state in the low temperature limit the Fermi distribution function $f(\mu_q,T)$ simply becomes a step function so that

$$\langle \Theta_{\mu}^{\mu} \rangle_0 = 4B + \mu_q n_0 f(\mu_q). \quad (4.4)$$

The trace of the energy-momentum tensor can be related to the gluon and quark vacuum expectation values arising from the operator product expansion [3,9,10] taking the following form:

$$\langle \Theta_{\mu}^{\mu} \rangle_0 = \langle G^2 \rangle_0 + m_q \langle \bar{\psi}_q \psi_q \rangle_0 \quad (4.5)$$

Thus we have the important vacuum contributions of operator dimension four to the equation of state, which are independent of the temperature.

We choose as a simple special case that of the meson with the same light quarks surrounded by nuclear matter as an example for this investigation. Then we have from the Fermi statistics using the quark-antiquark symmetry $\mu_{\bar{q}} = -\mu_q$ and for the antiquark quantum density distribution function $n_{\bar{q}}(T) = n_0(1 - f(T))$, where $n_0$ is the
quark density at zero temperature. Thus in the limit of zero temperature we have simply

\[ \langle \Theta^\mu_\mu \rangle_0 = 4B. \] (4.6)

We know from Leutwyler’s estimate [10] that the gluon condensate is about \( 2\text{GeV}/f m^3 \) as well as the quark condensate for light quarks is almost negligible. Thus we find that for this case the bag constant \( B \) is rather big. However, if we take \( T \) finite and small, the gluon condensate does not change much [10][11], but the entropy does play a very important role. The equation of state becomes with \( 2s_M \) for both quarks and antiquarks

\[ \langle \Theta^\mu_\mu \rangle_T = (2s_M + s(T))T - 4(p(T) - B) + \mu_q n_0(2f(T) - 1). \] (4.7)

At very low temperatures we would expect that \( s(T) \), \( p(T) \) and \( n_q(T) \) all to remain insignificant so that the main change in the equation of state is the value of \( s_M \). This effect could relate to a lowered bag constant or a raised chemical potential. However, we presently do not know how much the finite temperature changes the actual value of \( s_M \).

5 Discussions and Conclusions

We have calculated the entropy for a single quark in the color singlet ground state of the hadrons. We saw that the singlet state is a completely mixed state with the maximum value of the entropy given by \( \ln 3 \). In the theory of information it is known that the completely mixed state is that of minimal information, which is consistent with the idea of confinement. If we were to consider a gas of \( N \) hadrons in the sense that Schrödinger [1] considered a gas of two level atoms, we would generally expect an entropy of the form \( N\ln 3 \) for the color singlet ground state.

In an earlier collaborative work [12] we considered the thermodynamical properties of a class of models with non-abelian internal symmetries at finite temperature and baryon number. In this model we calculated the entropy density for \( N \) quarks \( s_N \) for fairly high temperatures and small quark numbers \( N \). We noted that the quantum effects became increasingly important when we took larger quark number with lower temperatures and smaller volumes. However, we did not then further investigate this issue.

Further work on the transition from the hadronic phase to the quark-gluon plasma has been recently carried out [13]. Although the temperatures are generally in the range above 100MeV for comparison with the lattice data, the hadron resonance gas model can be used for much lower temperatures. It is also useful to take into account the values [14] for \( s(T) \) from the pure \( SU(3)_C \) lattice simulations [15] as a comparison even though the critical temperature \( T_c \) is much higher at about 264MeV. However, the MILC two light quark flavors data [16] shows quite different behavior at much lower temperatures beginning around 125MeV as does the four flavor data [17]. Furthermore, some very interesting three quark simulations have recently appeared [18].
Finally we mention quantum information theory as another field of science where similar techniques involving the density matrices and the corresponding entropies are used. Although the difficulties arising from the quantum mechanical structure present in the statistical physics have been recognized for many years, it is only quite recently that it has received much attention outside these fields—particularly in quantum information theory. Here elaborate systems of codes are treated like quantum spin states. Also further studies in mathematical physics relating to the entanglement problem itself have been recently carried out. Nevertheless, our objective here has been naturally provided by the entanglement of the quark color structure as the extension of this approach to the three basis states of the fundamental representations of $SU(3)$.c.

Acknowledgements

The author would like to thank Philippe Blanchard, Frithjof Karsch, Krzysztof Redlich, Dieter Schildknecht and especially Abdelnasser Tawfik for many very helpful discussions. He is also very grateful to the Pennsylvania State University Hazleton for the sabbatical leave of absence and to the Fakultät für Physik der Universität Bielefeld.

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