Exact and Approximation Algorithms for the Domination Problem

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Abstract

In a simple connected graph $G = (V, E)$, a subset of vertices $S \subseteq V$ is a dominating set if any vertex $v \in V \setminus S$ is adjacent to some vertex $x$ from this subset. A number of real-life problems including facility location problems can be modeled using this problem, known to be among the difficult NP-hard problems in its class. We propose exact enumeration and approximation algorithms for the domination problem. The exact algorithm has solved optimally problem instances with over 1000 vertices within 6 minutes. This is a drastic breakthrough compared to the earlier known exact state-of-the-art algorithm which was capable to solve the instances up to 300 vertices within the range of 8 hours. Among the instances that were solved by both algorithms, in average, our exact algorithm was about 170 times faster than the former state-of-the-art algorithm. Our approximation algorithm, in 98.62\% of the analyzed instances, improved the earlier known state-of-the-art solutions. It was able to solve problem instances with more than 2000 vertices in less than 1 minute, whereas it found an optimal solution for 61.54\% of the instances. For the instances where the optimum was not found, the approximation error was 1.18.

Keywords: graph theory, dominating set, enumeration algorithm,
1. Introduction

Finding a minimum dominating set in a graph is a traditional discrete optimization problem. In a simple connected graph $G = (V, E)$ with $|V| = n$ vertices and $|E| = m$ edges, a subset of vertices $S \subseteq V$ is a dominating set in graph $G$ if any vertex $v \in V$ is adjacent to some vertex $x$ from this subset (i.e., there is an edge $(v, x) \in E$) unless vertex $v$ itself belongs to set $S$. Any subset $S$ with this property will be referred to as a feasible solution, whereas any subset of vertices from set $V$ will be referred to as a solution. The number of vertices in a solution will be referred to as its size (or order). The objective is to find an optimal solution, a feasible solution with the minimum possible, size $\gamma(G)$.

Below we mention a few real-life problems that can be modeled as the above domination problem or its variations: In the $k$-domination set, every uncovered vertex must possess at least $k$ neighbors from this set, the so-called $k$-dominating set. In the total dominating set, every vertex must have at least one neighbor from that set. In both variations, the objective is to minimize the number of vertices. Alvarez Miranda & Sinnl (2021) study the weighted version of the total domination problem, in which edges and vertices are assigned positive weights and the objective is to find a total dominating set with the minimum total weight. The algorithm, proposed by the authors solves optimally problem instances with up to 125 vertices within a time-limit of 30 minutes. Corcoran & Gagarin (2021) adopted heuristic from Parekh (1991) for the domination problem to the $k$-domination problem, and use the obtained solution for facility location problems in street networks. Haynes (2017) and Joshi et al. (1994) apply dominating sets and $k$-dominating sets, respectively, in the solution of the Facility Location Problem. Haynes (2017) also indicates how other related graph-theoretic problems including set cover, maximum inde-
pendent set and chromatic number can be reduced to the dominance problem. Liao & Lee (2005) apply the dominating sets to monitor electric power system to minimize the phase measurement units. Haynes et al. (2002) also study domination in graphs applied to electric power networks. Different forms of graph domination have been used to model clustering in wireless ad hoc networks Balasundaram & Butenko (2006), Wan et al. (2004), Wu et al. (2003) and Wu (2002). Another area of application for dominating sets is the robustness analysis of food webs Parra Inza et al. (2021b). Davidson et al. (2018) proposed integer linear programming models and two greedy heuristics for a weighted version of independent domination problem. The solutions delivered by the heuristics were compared to exact solutions obtained by the integer linear programs for instances with up to 100 nodes.

The domination problem is known to be NP-hard Garey & Johnson (1979), and is among the hardest problems in the family. A complete enumeration of all feasible solutions would imply an exponential cost of $O(2^n)$ (for a graph with only 20 nodes, such an enumeration would take centuries on modern computers). The decision version of the problem asks if there exists a dominating set with at most $k$ vertices, for some integer $k$. For a given $k$, there are about $n^{k+1}$ subsets to verify. This is basically the best currently known way to tackle the problem: not only it is not fixed parameter tractable but it does not admit fixed-parameter tractable approximation algorithms Chalermsook et al. (2020); Feldmann et al. (2020); Lin (2018). To the best of our knowledge, the only exact algorithm which does a bit better than a complete enumeration was suggested in Van Rooij & Bodlaender (2011) (we shall return to this algorithm and compare it to our results just a bit later).

As to the known non-exact solution methods, Campan et al. (2015) and Eubank et al. (2004) present heuristics with some experimental study. The authors do not present the approximation factor and the time complexity of the proposed algorithms. For a closely related weighted set cover problem,
Chvatal (1979) proposed another approximation algorithm for which they also gave approximation ratio \( \ln \left( \frac{n\text{OPT}}{\text{OPT}} \right) + 1 + \frac{1}{\text{OPT}} \). Parekh (1991) adopted the latter algorithm for the domination problem, and showed that the cardinality of the dominating set created by that algorithm is upper bounded by \( n + 1 - \sqrt{2m + 1} \).

Recently, an improved two-stage heuristic algorithm was proposed in Mira et al. (2022). It also includes an overview of the state-of-the-art solutions methods and results on domination problems.

**Our Contributions.** Here we propose an exact implicit enumeration algorithm that is capable to solve problem instances with more than 1000 vertices within a few minutes, which is a drastic improvement over the earlier state-of-the-art results (see below). We also present a new approximation algorithm that solves instances with more than 2000 vertices in less than 1 minute, and improves the state-of-the-art results in solution quality. The practical performance of our algorithms was verified for more than 600 existing benchmark instances Parra Inza (2021). In addition to that, 500 new problem instances, publicly available at Parra Inza (2022), were generated.

Our algorithms rely on lower and upper bounds on the number of vertices in an optimal solution. The initial lower bound \( L \) is obtained based on earlier results from Haynes (2017); Cabrera Martínez et al. (2020). We let the initial upper bound \( U \) be the size of the feasible solution constructed by the heuristic from Mira et al. (2022). We enumerate the solutions of size from the range \([L, U]\) applying binary search within this interval. Solutions of the same size are enumerated in a specially determined priority order.

Our approximation algorithm combines depth-first search with a breadth-first search. Special type of partial solutions of size \( \beta = \lfloor \alpha(U - s) \rfloor + s \), for \( 0 < \alpha < 1 \), are enumerated using breadth-first search, where \( s \) is the total number of support vertices in graph \( G \). The extensions of these solutions are generated in the depth-first fashion.
We give a more detailed statement of our computational results, starting with the exact enumeration algorithm. Remarkably, it was able to solve problem instances with almost 1100 vertices within the time range of 6 minutes. This is a drastic breakthrough compared to the earlier mentioned state-of-the-art exact algorithm \cite{Van_Rooij_Bodlaender_2011}, which does not report an experimental study, but derives an exact expression $O(1,4969^n)$ for the running time of the algorithm. This bound is obviously better than $2^n$, though it remains impractical (for instance, for graphs with 100 vertices, modern computers will require almost 7 years to enumerate $1,4969^{100}$ feasible solutions). However, the algorithm in \cite{Van_Rooij_Bodlaender_2011} not necessarily attains this upper bound. We coded the algorithm to estimate its practical performance (this code, together with the codes of both of our algorithms and the obtained solutions, are publicly available at \cite{Parra_Inza_2021}). Within the range of 8 hours, the algorithm was able to solve problem instances with up to 300 vertices. Compared to our exact algorithm, in average, among the instances that were solved by the both algorithms, our algorithm was about 170 times faster.

As to the proposed approximation algorithm, it was able to solve problem instances with more than 2000 vertices delivering optimal or close to optimal solutions in less than 1 minute, improving the earlier known state-of-the-art approximation solutions from \cite{Mira_et_al_2022} in 98.62\% of the tested instances. Remarkably, it found an optimal solution in 61.54\% of the instances, whereas for the instances where the optimum was not found, the average approximation error was 1.18.

The rest of the paper is organized as follows. In the next section we give the necessary preliminaries. In Sections 3 and 4 we describe our exact and approximation algorithms, respectively. Section 5 presents the conducted experimental study, and Section 6 contains a few final remarks. Preliminary version of this work was presented at the First On-Line Conference on Algorithms IOCA 2021.
2. Basic properties

In this section we define some basic properties on graphs that we use in our implicit enumeration algorithm. We accompany them with the necessary definitions.

The diameter $d(G)$ of graph $G$ is the maximum number of edges on the shortest path between any pair of vertices in that graph, and the radius $r(G)$ in graph $G$ is the minimum number of edges on the shortest path (one with the minimum number of edges) between any pair of vertices in that graph. A leaf vertex is a degree one vertex, and a support vertex is a vertex adjacent to a leaf. The sets of leaf and support vertices in graph $G$ will be denoted by $\text{Supp}(G)$ and $\text{Leaf}(G)$, respectively, and the degree of a vertex with the maximum number of neighbors in graph $G$ by $\Delta(G)$.

The initial lower bound $L$ on the size of an optimal solution that our algorithms employs is obtained based on the following known results.

**Theorem 1.** [Haynes (2017)] If $G(V, E)$ is a connected graph of order $n$ and maximum degree $\Delta(G)$, then $\gamma(G) \geq \frac{n}{\Delta(G)+1}$.

**Theorem 2.** [Haynes (2017)] If $G(V, E)$ is a connected graph of ratio $r$ and diameter $d$, then $\gamma(G) \geq \frac{2r}{3}$ and $\gamma(G) \geq \frac{d+1}{3}$.

**Theorem 3.** [Haynes (2017)] If $G(V, E)$ is a connected graph of order $n$ and $\text{Leaf}(G)$ and $\text{Supp}(G)$ are the sets of leaf and support vertices of $G$, respectively, then $|\text{Supp}(G)| \leq \gamma(G) \leq n - |\text{Leaf}(G)|$.

The next corollary is an immediate consequence of the Theorems 1, 2, and 3.

**Corollary 1.** If $G(V, E)$ is a connected graph of order $n$, ratio $r$, diameter $d$, and $s = |\text{Supp}(G)|$. Then, $L = \max\{\frac{n}{\Delta(G)+1}, \frac{2r}{3}, \frac{d+1}{3}, s\}$ is a lower bound on the size of an optimal solution.
3. The implicit enumeration algorithm

A feasible solution generated by the approximation algorithm from Mira et al. (2022) defines our initial upper bound $U$ on the size of a feasible solution. Given the lower bound $L$ from Corollary 1 and this upper bound, we restrict our search for feasible solutions with the size in the range $[L, U]$.

The solutions of the size $\nu \in [L, U]$ are generated and tested for the feasibility based on the specially formed priority list of solutions. The sizes of feasible solutions are derived by the standard binary division search accomplished in the interval $[L, U]$. For each created solution $\sigma$ of size $\nu$, feasibility condition is verified, i.e., it is verified if the solution forms a dominating set. Below we describe the general framework of the algorithm. In the next subsection, we specify how the priority lists are created.

Let $\sigma$ be the the current solution of size $\nu$ (initially, $\sigma$ is the feasible solution delivered by the algorithm from Mira et al. (2022)):

- If solution $\sigma$ is feasible, then the current upper bound $U$ is updated to $\nu$. The algorithm proceeds with the reduced in this way time range continuing with the next to $\nu$ (smaller) trial value from the interval $[L, \nu)$ which is derived by the binary division. If all trial $\nu$s were already tested, then $\sigma$ is an optimal solution. The algorithm returns solution $\sigma$ and halts.

- If the current solution $\sigma$ of size $\nu$ is not feasible, then $Procedure\_Next(\nu)$ is called and the next to $\sigma$ solution of size $\nu$ from the corresponding priority list is tested.

- If $Procedure\_Next(\nu)$ returns NIL, i.e., all the solutions of size $\nu$ were already tested for the feasibility (none of them being feasible), the current lower bound $L$ is updated to $\nu$. The algorithm proceeds with the reduced in this way time range continuing with the next to $\nu$ (larger) trial value from the interval $[\nu, U)$ derived by the binary division. If all trial $\nu$s
were already tested, then \( \sigma \) is an optimal solution. The algorithm returns solution \( \sigma \) and halts.

3.1. Procedure\_Next(\( \nu \)) and Procedure\_Priority\_LIST()

Next, we describe in which order the solutions of a given size \( \nu \) are considered during the enumeration process. For each trial value \( \nu \in [L, U] \), the solutions of size at most \( \nu \) are generated in a special priority order that is intended to help in a faster convergence to a feasible solution. Two procedure are employed. Subroutine Procedure\_Priority\_LIST() generates a priority list of vertices. The earlier included vertices in this list “potentially cover” a major number of yet uncovered vertices. Based on the order determined by Procedure\_Priority\_LIST(), Procedure\_Next(\( \nu \)) determines the solution \( \sigma_h(\nu) \) of the current trial size \( \nu \) of iteration \( h \). First we describe Procedure\_Priority\_LIST().

Neither vertices from set \( \text{Supp}(G) \) nor vertices from set \( \text{Leaf}(G) \) are considered while forming the priority list of vertices. The vertices from the first set are to be included into any dominating set, hence they form part of any solution that is considered during the enumeration process; hence, none of the vertices from set \( \text{Leaf}(G) \) need to be included into any formed solution (see Theorem 3).

The priority list of vertices is formed based on their active degrees. The notion of an active degree was introduced in Mira et al. (2022). The active degree of a vertex is determined dynamically at every iteration \( r \). Let \( \text{LIST}_r \) be the priority list of iteration \( r \) in Procedure\_Priority\_LIST() (initially, \( \text{LIST}_0 := \emptyset \)). The active degree of vertex \( v \) at iteration \( r \) is the number of neighbors of vertex \( v \) in set \( V(G) \setminus \{\text{LIST}_{r-1} \cup \text{Supp}(G)\} \) not counting the neighbors of vertex \( v \) which are adjacent to a vertex in \( \{\text{LIST}_{r-1} \cup \text{Supp}(G)\} \).

Suppose \( S \) is a feasible solution (a dominating set). Priority list of vertices is iteratively formed by the vertices from the set \( S \setminus \text{Supp}(G) \) sorted in non-
increasing order of their active degrees. The remaining vertices, i.e., ones from the set \( V \setminus \{ S \cup \text{Supp}(G) \cup \text{Leaf}(G) \} \), are iteratively inserted in non-increasing order of their degrees in graph \( G \). A formal description of the procedure follows.

**Algorithm 1 Procedure_Priority_LIST**

Input: A graph \( G(V,E) \) and a feasible solution (dominating set) \( S \).
Output: Priority list \( \text{LIST} := \text{LIST}_r \).

\[ r := 0; \]
\[ \text{LIST}_r := \emptyset; \]
\[ S_r := S \setminus \text{Supp}(G); \]
\{ iterative step \}

\[ \text{while } S_r \neq \emptyset \text{ do} \]
\[ \quad r := r + 1; \]
\[ \quad v_r := \text{any vertex of set } S_{r-1} \text{ with the maximum active degree}; \]
\[ \quad S_r := S_{r-1} \setminus \{v_r\}; \]
\[ \quad \text{LIST}_r := \text{LIST}_{r-1} \cup \{v_r\}; \]
\[ \text{end while} \]
\[ D_r := V \setminus (S \cup \text{Leaf}(G)); \]
\[ \text{while } D_r \neq \emptyset \text{ do} \]
\[ \quad r := r + 1; \]
\[ \quad v_r := \text{any vertex of set } D_{r-1} \text{ with the maximum degree}; \]
\[ \quad D_r := D_{r-1} \setminus \{v_r\}; \]
\[ \quad \text{LIST}_r := \text{LIST}_{r-1} \cup \{v_r\}; \]
\[ \text{end while} \]

Now we describe \( \text{Procedure}_\text{Next}(\nu) \) that, iteratively, verifies if the currently formed solution \( \sigma_h(\nu) \) is feasible. (Note that the feasibility of every generated solution of a given size can be verified in time \( O(n) \).) If it is feasible, the it returns that solution and continues with the next trial \( \nu \); the procedure halts if all trial sizes have been already considered. For every new trial value \( \nu \), the first solution of that size contains all the vertices from set \( \text{Supp}(G) \) complemented by the first \( \nu - |\text{Supp}(G)| \) vertices from the list \( \text{LIST}_r \) delivered by \( \text{Procedure}_\text{Priority списка}() \).

If solution \( \sigma_h(\nu) \) is not feasible, the next solution \( \sigma_{h+1}(\nu) \) of size \( \nu \) is obtained from solution \( \sigma_h(\nu) \) by a vertex interchange as follows. Let \( v \notin \sigma_h(\nu) \) be the
next vertex from the priority list, and let $v' \in \sigma_h(\nu)$ be the vertex with the smallest active degree. Vertex $v'$ is substituted by vertex $v$ in solution $\sigma_{h+1}(\nu)$ (note that the active degree of $v$ must be less than that of vertex $v'$):

$$\sigma_{h+1}(\nu) := (\sigma_h(\nu) \setminus \{v'\}) \cup \{v\}.$$

**Algorithm 2** Procedure\_Next\_Aux($\nu'$)

Input: $\nu'$, $\sigma_h(\nu')$, \textit{LIST}, \textit{index\_LIST}, $h$.

Output: $\sigma^*(\nu)$ or \textit{NIL}.

if $\nu' = 0$ then
  if $\sigma_h(\nu') \cup \text{Supp}(G)$ is feasible solution then
    $\sigma^*(\nu) := \text{Supp}(G) \cup \sigma_h(\nu')$;
    return $\sigma^*(\nu)$ and stop;
  end if
  $h := h + 1$;
  return $\sigma_{h-1}(\nu')$;
end if

$i := \text{index\_LIST}$;

while $i \leq |\text{LIST}|$ and $\text{Supp}(G) \cup \sigma_h(\nu')$ is not a feasible solution do
  Add \text{LIST}[i] at the end of $\sigma_h(\nu')$;
  Procedure\_Next\_Aux($\nu' - 1$, $i + 1$, $\sigma_h(\nu')$, $h$);
  Removes the last element in $\sigma_h(\nu')$;
  $i := i + 1$;
end while

**Algorithm 3** Procedure\_Next($\nu$)

Input: $\nu$, \textit{LIST}, $\text{Supp}(G)$.

Output: $\sigma^*(\nu)$ or \textit{NIL}.

$h := 0$;

\textit{index\_LIST} := 0;

$\nu' := \nu - s$;

$\sigma_h(\nu) := \emptyset$;

Procedure\_Next\_Aux($\nu'$, $\sigma_h(\nu)$, \textit{LIST}, \textit{index\_LIST}, $\text{Supp}(G)$, $h$);

Now we can give a formal description of our enumeration algorithm.
Algorithm 4 Algorithm_BDS

Input: A graph $G$.
Output: A $\gamma(G)$-set $S$.

$\text{Supp}(G) :=$ Set of support vertex of graph $G$;
$\text{Leaf}(G) :=$ Set of leaf vertex of graph $G$;
$L := \max\{\frac{n}{\Delta(G) + 1}, 2r, \frac{d+1}{3}, |\text{Supp}|\}$;
$S := \sigma$; \{Feasible solution proposed in [Mira et al. 2022]\}
$U := |S|$;
$\nu := \lfloor (L + U)/2 \rfloor$;

Procedure_Priority_LIST();  \{ iterative step \}
while $U - L > 1$ do
    if Procedure_Next($\nu$) returns NIL then
        $L := \nu$;
        $\nu := \lfloor (L + U)/2 \rfloor$;
    else \{ A feasible solution was found ($\sigma_h(\nu)$)\}
        $U := \nu$;
        $\nu := \lfloor (L + U)/2 \rfloor$;
        Procedure_Priority_LIST();
    end if
end while

Let $s = |\text{Supp}(G)|$ and $l = |\text{Leaf}(G)|$. Below we give an exact time complexity expression for our implicit enumeration algorithm.

Theorem 4. The time complexity of Algorithm 4 is

$$O \left( n \log \left( \frac{n}{2} - 1 \right) \binom{n}{n/4} \right).$$

Proof: Since binary search in the interval $[L, U]$ is carried out, the total number of external iterations in Algorithm 4 (i.e., the number of different sizes $\nu$) is at most $\log(U - L)$. For a given size $\nu$, the number of the generated solutions of that size is bounded by $\left( \binom{n-s-1}{\nu-s} \right)$. Indeed, in the worst case all solutions of cardinality $\nu$ are considered. By Theorem 3, the set $\text{Supp}(G)$ forms part of all generated solutions, whereas no leaf vertex belongs to any created solution and hence $\left( \binom{n-s-1}{\nu-s} \right)$ is an upper bound on the number of solutions of size $\nu$ that the algorithm creates. To establish the feasibility of solution $\sigma_h(\nu)$,
Procedure \( \text{Next}(\nu) \) verifies if every vertex \( x \in V(G) \) is in \( \sigma_h(\nu) \) or if it is adjacent to a vertex in \( \sigma_h(\nu) \), which clearly takes time \( O(n) \). For the purpose of this estimation, let us assume that \( \nu = \lfloor (U + L)/2 \rfloor \) (as the maximum number of combinations is reached for this particular \( \nu \)). We may also express \( \nu \) in terms of \( n \) as \( \nu = \lfloor (n^2 + 1)/2 \rfloor = n^4 \) using \( U \leq n/2, L \geq 1, s \geq 0 \) and \( l \geq 0 \). Summing up the above, we have an overall bound \( O \left( n \log \left( \frac{n}{2} - 1 \right) \binom{n}{n/4} \right) \) on the cost of the algorithm.

4. The approximation algorithm

From here on, let \( \sigma \) the feasible solution obtained by the greedy algorithm from Mira et al. (2022). We define an auxiliary parameter

\[
\beta = \lfloor \alpha(U - s) \rfloor + s,
\]

for \( 0 < \alpha < 1 \) \((s = |\text{Supp}(G)|)\). \( \beta \) is the size of a base solution \( \sigma^h(\beta) \), a (partial) solution that serves us as a basis for following larger sized solutions. These solutions, which will be referred to as extensions of solution \( \sigma^h(\beta) \), share the \( \beta \) vertices with solution \( \sigma^h(\beta) \). In case none of these extensions of solution \( \sigma^h(\beta) \) turn out to be feasible, the current base solution is replaced by another base solution of size \( \beta \) and the search similarly continues.

The set of vertices in a base solution is determined according to one of the following alternative rules:

1. All support vertices from set \( \text{Supp}(G) \) and the first \( \beta - s \) non-support vertices from solution \( \sigma \).
2. All support vertices from set \( \text{Supp}(G) \) and \( \beta - s \) randomly generated non-support vertices from solution \( \sigma \).
3. All support vertices from set \( \text{Supp}(G) \) and \( \beta - s \) randomly generated vertices from set \( V \setminus \{ \text{Supp}(G) \cup \text{Leaf}(G) \} \).
For each of these options, the vertices in a newly determined base solution are selected in such a way that it does not coincide with any of the earlier formed base solutions. Typically, the procedure creates the first base solution by rule (1) or rule (2). Next base solutions are obtained by rule (3) unless the last such generated base solution coincides with an earlier created one. If this happens, then the remaining base solutions are created just in the lexicographic order (according to their binary representations).

Every created base solution is iteratively extended by one vertex per iteration and each of these extensions is checked for feasibility until either (i) one of them turns out to be feasible or (ii) an extension of size $U - 1$ is created. In the latter case (ii), if the corresponding extension of size $U - 1$ is not feasible, the next base solution with size $\beta$ is constructed and the procedure is repeated for the newly created base solution (note that for each base solution, at most one extension of size $U - 1$ is generated). In the former case (i), the current upper bound is updated; correspondingly, the parameter $\beta$ is also updated, the first base solution of the new size $\beta$ is created and the procedure is again repeated for this newly created base solution. The algorithm halts when all possible base solutions of the current size $\beta$ are considered, a feasible solution of size $U - 1$ for each of them is created and none of them turned out to be feasible. In this case, $\gamma(G) = U$ and the procedure returns the corresponding feasible solution of size $U$. 
Figure 1: The complete solution tree with base solutions of size $\beta$

Figure 1 represents the complete solution tree for a given $\beta$. The root represents a feasible solution $\sigma$ of size $U$. In every branch of this tree, the size of every solution (except the base solution) is the size of its predecessor solution plus 1. Note that all the represented solutions must have been infeasible except solution $\sigma$ and possibly solution $\sigma_{\beta+1}^k$.

It remains to describe how the extensions of each base solution are generated. A base solution $\sigma_h(\beta)$ is iteratively extended in at most $U - 1 - \beta$ iterations, by one vertex per iteration. Let $\sigma_i^h(\beta)$ be the extension of solution $\sigma^h(\beta)$ by iteration $i$, $\beta < i < U$, where we let $\sigma_0^h(\beta) = \sigma^h(\beta)$. Then $\sigma_i^h(\beta) = \sigma_{i-1}^h(\beta) \cup x_i$, where $x_i \in V \setminus (Leaf(G) \cup \sigma_{i-1}^h(\beta))$ is determined by one of the following selection rules. In case solution $\sigma^h(\beta)$ was formed by rule (1), vertex $x_i$ is selected randomly; if solution $\sigma^h(\beta)$ was formed by either of the rules (2) and (3), $x_i$ is set to be a vertex with the maximum active degree from set $V \setminus (Leaf(G) \cup \sigma_{i-1}^h(\beta))$. Below we give a formal description of this sub-procedure that generates the extensions of every base solution in depth-first fashion:
Algorithm 5 \textbf{Procedure Extensions}(\sigma^h(\beta))

\begin{tabular}{l}
\textbf{Input:} \sigma^h(\beta), U, \text{Leaf}(G) . \\
\textbf{Output:} A feasible solution \sigma^h_i(\beta) or NIL. \\
i := |\sigma^h(\beta)|; \\
\sigma^h_0(\beta) := \sigma^h(\beta); \\
\textbf{while } i < U - 1 \textbf{ do } \{ \text{ Generate new extension of } \sigma^h(\beta) \} \\
i := i + 1; \\
\textbf{if } h == 0 \textbf{ then} \\
x_i := \text{ randomly selected vertex from } V \setminus (\text{Leaf}(G) \cup \sigma^h_{i-1}(\beta)); \\
\textbf{else} \\
x_i := \text{ any vertex with the maximum active degree from set } V \setminus (\text{Leaf}(G) \cup \sigma^h_{i-1}(\beta)); \\
\textbf{end if} \\
\sigma^h_i(\beta) = \sigma^h_{i-1}(\beta) \cup x_i; \\
\textbf{if } \sigma^h_i(\beta) \text{ is a feasible solution } \textbf{ then} \\
\text{ return } \sigma^h_i(\beta) \text{ and stop; } \\
\textbf{end if} \\
\textbf{end while} \\
\text{ return NIL; }
\end{tabular}

Below we give a description of our approximation algorithm that combines depth-first search with a breadth-first search: while the base solutions are generated by the breadth-first rule, the extensions of every base solution are generated in depth-first fashion (see Algorithm 6 and Figure 1).
Algorithm 6 Algorithm\_DBS

Input: A graph $G$.
Output: A feasible solution $S^\ast$.

Generate feasible solution $\sigma$;

$U := |\sigma|$;

$\beta = [\alpha(U - s)] + s; \quad \{\text{for } 0 < \alpha < 1\}$

\{ Iterative step \}

$h := 0$;

$h_{\text{max}} := (|V(G) \setminus \text{Supp}(G) \setminus \text{Leaf}(G)|)$;

while $h < h_{\text{max}}$ do

\{ Generate new base solution $\sigma^h(\beta)$ \}

if $h == 0$ then

$\sigma^h(\beta) := \text{Supp}(G) \cup \{\text{the first } \beta - s \text{ vertices from solution } \sigma \setminus \text{Supp}(G)\}$;

else

if $h < (|\sigma \setminus \text{Supp}(G)|)$ then

$\sigma^h(\beta) := \text{Supp}(G) \cup \{\text{randomly selected } \beta - s \text{ vertices from solution } \sigma \setminus \text{Supp}(G)\}$;

else

$\sigma^h(\beta) := \text{Supp}(G) \cup \{\text{randomly selected } \beta - s \text{ vertices from } V(G) \setminus (\text{Supp}(G) \cup \text{Leaf}(G))\}$;

end if

end if

if $\sigma^h(\beta)$ is a feasible solution then

$\sigma := \sigma^h(\beta)$;

$h := 0$;

Updated $U$, $\beta$ and $h_{\text{max}}$;

else

\{ Generate extensions of $\sigma^h(\beta)$ \}

if Procedure\_Extensions($\sigma^h(\beta)$) returns NIL then

$h := h + 1$;

else

\{ A feasible solution was found ($\sigma^h(\beta)$) \}

$\sigma := \sigma^h(\beta)$;

$h := 0$;

Updated $U$, $\beta$ and $h_{\text{max}}$;

end if

end if

end while

$S^\ast := \sigma$;
We complete this section with an observation that immediately follows from the construction of Algorithm 6.

**Observation 1.** If no extension of any base solution of current size $\beta$ is feasible and $\beta > L$, then $\beta < \gamma(G) \leq U$.

5. Experimental results

In this section, we describe our computational experiments. We implemented the three algorithms in C++ using Visual Studio for Windows 10 (64 bits) on a personal computer with Intel Core i7-9750H (2.6 GHz) and 12 GB of RAM DDR4. The order and the size of an instance were generated randomly utilizing function `random()`. To complete the set $E(G)$, each new edge was added in between two yet non-adjacent vertices randomly until the corresponding size was reached.

The exact algorithm from Van Rooij & Bodlaender (2011) and our exact algorithm are referred to as MSC and BDS, respectively, and our approximation algorithm will be referred to as DBS. In total, more than 1100 instances were analyzed. A complete summary of the results for these instances can be found at Parra Inza (2022). In this section we illustrate results of about 60 randomly selected instances.

A comparative study for the two exact enumeration algorithms MSC and BDS is presented in Table 1. This table represents instances where both algorithms succeeded to halt, i.e., they created an optimal solution.

| No. | $|V(G)|$ | $|E(G)|$ | Time BDS (s) | Time MSC (s) | Lower Bounds | Upper Bounds |
|-----|--------|--------|--------------|--------------|--------------|--------------|
|     |        |        |              |              | $\frac{\beta}{\Delta(G)+1}$ | $\frac{d+1}{3}$ | $\frac{2d^2}{3}$ | $|\text{Supp}(G)|$ | $\gamma(G)$ | $|S| - N(G)$ |
| 1   | 185    | 6849   | 24.667       | 666.839      | 1            | 1            | 1            | 1            | 5         | 6            | 93           |
| 2   | 189    | 7147   | 23.8574      | 770.36       | 1            | 1            | 1            | 1            | 5         | 7            | 91           |
| 3   | 194    | 7529   | 28.294       | 915.361      | 1            | 1            | 1            | 1            | 5         | 6            | 96           |
Table 1: Graphs with density $\approx 0.5$.

Table 2 presents results for larger instances where algorithm MSC failed to halt within first 8 hours of the execution, whereas algorithm BDS completed within 6 minutes for the largest instance from the table.

Table 2

| No. | $|V(G)|$ | $|E(G)|$ | Time BDS (s) | Time MSC (s) | Lower Bounds | Upper Bounds |
|-----|---------|---------|-------------|--------------|--------------|--------------|
| 1   | 1012    | 452294  | 227.346     | 1041.98      | 2 1 1 1 1    | 6 101        |
| 2   | 1014    | 454116  | 275.1       | 1036.67      | 2 1 1 1 2    | 5 103        |
| 3   | 1018    | 406867  | 135.863     | 1361.77      | 2 1 1 1 1    | 5 105        |
| 4   | 1022    | 461340  | 291.408     | 1361.09      | 1 1 1 1 1    | 5 103        |
| 5   | 1026    | 413355  | 375.123     | 1981.35      | 1 1 1 1 3    | 5 107        |
| 6   | 1030    | 416585  | 633.985     | 2011.15      | 2 1 1 1 6    | 6 115        |
| 7   | 1036    | 474219  | 198.763     | 1454.44      | 1 2 1 1 1    | 5 104        |
| 8   | 1040    | 477909  | 248.368     | 1810.09      | 1 1 1 1 5    | 5 103        |
| 9   | 1042    | 426451  | 127.957     | 1798.15      | 1 1 1 3 7    | 107          |
| 10  | 1046    | 483492  | 231.502     | 1750.39      | 1 1 1 1 5    | 6 152        |

$\gamma(G)$ and $\gamma(G)$ are the lower and upper bounds, respectively. $\Delta(G)$ is the maximum degree of the graph.
No. | | | Time BDS (s) | Lower Bounds | γ(G) | Upper Bounds |
|---|---|---|---|---|---|---|
| | | | | \( \frac{n}{\Delta + 1} \) | \( \frac{d + 1}{3} \) | \( \frac{e}{3} \) | | | | | | | |
| 11 | 1058 | 439752 | 128.976 | 1 | 2 | 2 | 2 | 5 | 7 | 162 |
| 12 | 1064 | 500420 | 310.403 | 1 | 2 | 2 | 2 | 5 | 6 | 83 |
| 13 | 1066 | 446505 | 155.74 | 1 | 2 | 2 | 2 | 5 | 6 | 173 |
| 14 | 1068 | 504205 | 92.061 | 1 | 2 | 2 | 2 | 4 | 6 | 81 |
| 15 | 1074 | 453307 | 610.302 | 1 | 2 | 2 | 2 | 5 | 7 | 174 |
| 16 | 1080 | 515722 | 333.098 | 1 | 2 | 2 | 2 | 5 | 6 | 79 |
| 17 | 1082 | 460117 | 723.124 | 1 | 2 | 2 | 2 | 5 | 7 | 183 |
| 18 | 1086 | 463560 | 480.903 | 1 | 2 | 2 | 2 | 5 | 7 | 182 |
| 19 | 1096 | 531243 | 347.718 | 1 | 2 | 2 | 2 | 5 | 6 | 87 |
| 20 | 1098 | 533220 | 349.208 | 1 | 2 | 2 | 2 | 5 | 6 | 81 |
| 21 | 1108 | 543039 | 1462.33 | 1 | 2 | 2 | 2 | 5 | 6 | 85 |

Table 2: Graphs with density > 0.7

A comparative study for the two approximation algorithms, the one from Mira et al. (2022) and our approximation algorithm DBS for the instances with up to 2098 vertices is presented in Table 3. In 98.62% of the instances, the solution delivered by algorithm from Mira et al. (2022) was improved. Algorithm DBS found an optimal solution for 61.54% of the analyzed instances, and for the remaining instances, the average approximation error was 1.18.

No. | \(|V(G)|\) | \(|E(G)|\) | \(|S|\) | \(\beta\) | \(\sigma_i(\beta)\) generates | Time(s) | \(|DS|\) |
|---|---|---|---|---|---|---|---|
| 1 | 600 | 84557 | 12 | 4 | 43 | 37.815 | 11 |
| 2 | 610 | 87490 | 12 | 4 | 3225 | 2876.67 | 11 |
| 3 | 620 | 90472 | 13 | 3 | 21 | 20.953 | 12 | 3 | 6 | 25.745 | 11 |
| 4 | 630 | 93505 | 13 | 4 | 107 | 90.532 | 12 | 3 | 35266 | 29750 | 11 |
| 5 | 640 | 96587 | 13 | 4 | 22 | 23.207 | 11 |
| 6 | 650 | 102571 | 9 | 2 | No solution found |
| 7 | 660 | 105798 | 10 | 3 | 4080 | 4635.46 | 8 |
| 8 | 670 | 109076 | 9 | 2 | 18 | 24.966 | 8 |
| 9 | 680 | 109417 | 12 | 4 | 65 | 86.754 | 11 |
| 10 | 690 | 117488 | 10 | 3 | 812 | 1055.51 | 9 |
| 11 | 700 | 116132 | 13 | 4 | 39 | 55.176 | 12 |
6. Conclusions

We described exact enumeration and approximation algorithms for the domination problem in general graphs which outperform the state-of-the-art exact and approximation algorithms from Van Rooij & Bodlaender (2011) and Mira et al. (2022), respectively. We believe that, based on our approach, efficient exact and approximation algorithms for more general graph domination problems can be developed. In particular, global dominant, \( k \)-dominant, total dominant, global total and global total \( k \)-dominant settings can be considered.

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