Inviscid limit for the derivative Ginzburg–Landau equation with small data in higher spatial dimensions

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Abstract. We study the Cauchy problem for derivative Ginzburg–Landau equation

\[ u_t = (\nu + i)\Delta u + \lambda_1 \cdot \nabla (|u|^2 u) + (\lambda_2 \cdot \nabla u)|u|^2 + \alpha |u|^{2\delta} u, \]

where \( u \) is a complex valued function of \((t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+ = [0, +\infty]; \nu > 0, \alpha \in \mathbb{C}, \delta \in \mathbb{N}, \lambda_1, \lambda_2 \) are complex vectors.

The DCGL equation (1.1) arises as the envelope equation for a weakly subcritical bifurcation to counter-propagating waves, and it is also important for a number of physical systems including the onset of oscillatory convection in binary fluid mixture; cf. [3]. In the case of one or two dimensions, the global existence of solutions, finite dimensional global attractors, Gevery regularity of solutions have been studied extensively for equation (1.1); cf. [8, 14, 13, 30, 39]. Taking \( \nu = 0 \), (1.1) can be written as

\[ u_t - i\Delta u = \lambda_1 \cdot \nabla (|u|^2 u) + (\lambda_2 \cdot \nabla u)|u|^2 + \alpha |u|^{2\delta} u, \quad u(0, x) = u_0(x), \]
which is the well-known derivative nonlinear Schrödinger equation (DNLS). There are some recent works which have been devoted to equation (1.2); cf. [25, 26, 28, 29, 32, 37]. N. Hayashi and Ozawa in [18] proposed the method of gauge transformation which is useful to avoid the loss of derivatives for equation (1.2) in one spatial dimension.

A natural question between Eqs. (1.1) and (1.2) is the inviscid limit. Let \( u \) and \( v \) be the solutions of the Cauchy problems of Eqs. (1.1) and (1.2), respectively. Does \( u \) converge to \( v \) as the parameter \( \nu \) tends to 0?

When \( \lambda_1 = \lambda_2 = 0 \), Eq. (1.1) can be rewritten as

\[
\begin{align*}
  u_t = (\nu + i)\Delta u + \alpha |u|^{2s} u, \\
  u(0, x) = u_0(x),
\end{align*}
\]

which is the well-known complex Ginzburg–Landau equation. Eq. (1.3) is an important model equation in the description of spatial pattern formation and of the onset of instabilities in nonequilibrium fluid dynamical systems; cf. [6]. For Eq. (1.3), there are some recent results devoted to the global well-posedness and limit behavior, see Ginibre and Velo [11], Wu [42], Bechouche and Jungel [2], Wang [38], Machihara and Nakamura [31], Wang and Huang [17].

For the derivative complex Ginzburg–Landau equation (1.1), using Bourgain’s \( X^{s,b} \) method, Huo and Jia [16] obtained the inviscid limit for the solutions in \( C([0, T]; H^s) \) \((s > 1/2)\) in one spatial dimension, where the bilinear estimate condition \( 2\lambda_1 + \lambda_2 = 0 \) and some energy estimate conditions on coefficients and \( \|u_0\|_{L^2} \ll 1 \) are required. B. Wang and Y. Wang in [41] also considered the inviscid limit for the solutions, when initial data belong to \( \dot{H}^3 \cup \dot{H}^{-\frac{1}{2}} \), in one spatial dimension. As far as the authors can see, there are no result on the inviscid limit of Eq. (1.1) in high dimension case \( n \geq 2 \).

It was well known that \( H^{s+\epsilon + n/2} \subset M^s_{2,1} \subset H^s \), for \( \forall \epsilon > 0 \). In this paper, we will show that Eq. (1.1) is uniformly globally well posed on the parameter \( \nu \geq 0 \) in modulation space \( M^s_{2,1}(\mathbb{R}^n) \), \( n \geq 3 \), \( s > 3 \) with the sufficiently small Cauchy data in \( L^2 \). As \( \nu \to 0 \), we prove that the solutions of Eq. (1.1) will converge to that of the derivative Schrödinger equation. When \( n = 2 \), we also show local well-posedness results and inviscid limit in modulation space \( M^s_{1,1} \), \( s > 5/2 \). The techniques used in this paper are the anisotropic global smooth effect estimates and maximal inequality estimates which are independent of parameter \( \nu \geq 0 \), those estimates in the case \( \nu = 0 \) were obtained in our earlier work [37], where global well-posedness for equation (1.2) is showed in \( M^s_{2,1}(\mathbb{R}^n) \), \( s \geq 5/2 \), for small Cauchy data.

Finally, we consider the quadratic derivative Ginzburg–Landau equation:

\[
\begin{align*}
  u_t - (\nu + i)\Delta u - \lambda \cdot \nabla (u^2) = 0, \\
  u(0, x) = u_0(x).
\end{align*}
\]

Its limit equation is

\[
\begin{align*}
  u_t - i\Delta u - \lambda \cdot \nabla (u^2) = 0, \\
  u(0, x) = u_0(x).
\end{align*}
\]
When \( n = 1 \), Christ in [1] showed that for Eq. (1.5), the flow map \( u_0 \to u \) is not continuous in any Sobolev space \( H^s(\mathbb{R}) \) (\( s \in \mathbb{R} \)) for any short time lifespan (\( \| u_0 \|_{H^s} \ll 1 \) but \( \| u(t) \|_{H^s} \gg 1 \) for some \( t \ll 1 \)). In [30], Stefanov showed the existence for the weak solutions in \( H^1 \) space with small total disturbance \( u_0 \in H^1(\mathbb{R}^1) \cap L^1(\mathbb{R}^1) \cap \{ f : \sup_x |f(y)dy| \leq \epsilon \} \).

In this paper, we will show that Eqs. (1.4) and (1.5) are locally well posed in modulation space \( M^3_{1,1}(\mathbb{R}^n) \) and the inviscid limit between Eqs. (1.4) and (1.5) also holds in the space \( M^3_{1,1}(\mathbb{R}^n) \) for the solutions. From this point of view, \( M^3_{1,1} \) seems to be a proper space to deal with the solutions of quadratic derivative nonlinear Schrödinger equation.

### 1.1 Main results

**Theorem 1.1** Let \( n \geq 3 \). Assume initial data \( u_0 \in M^2_{2,1}, s > 3 \) and \( \| u_0 \|_{L^2} \leq \delta \) for some small \( \delta > 0 \). Then Eq. (1.1) has a unique global solution \( u_\nu \in C(\mathbb{R}^+, M^2_{2,1}) \cap X_s \) satisfying

\[
\| u_\nu \|_{X_s} \leq C \| u_0 \|_{M^2_{2,1}}.
\]

where \( C \) is independent of \( \nu \), \( X_s \) is defined in (1.1).

**Theorem 1.2** Let \( n \geq 3 \). Assume initial data \( u_0 \in M^2_{2,1} \) and \( \| u_0 \|_{L^2} \leq \delta \) for some small \( \delta > 0 \). \( u_\nu \) is the solution of (1.1), and let \( v \) is the solution of (1.2) with the same initial data, then for any \( T > 0 \) we have

\[
\| u_\nu - v \|_{C(0,T;L^2)} \lesssim \| u_\nu - v \|_{C(0,T;M^2_{2,1})} \lesssim \nu T, \quad \nu \ll 1.
\]

**Theorem 1.3** Let \( n = 1, 2 \). Assume initial data \( u_0 \in M^s_{1,1}, s > 5/2 \) and \( \| u_0 \|_{L^1} \leq \delta \) for some small \( \delta > 0 \). Then Eq. (1.1) has a unique local solution

\[
\begin{align*}
\nu \in C([0,1], M^s_{2,1}) & \cap C([0,1], M^{s-1/2}_{1,1}) \cap X^1_s \\
\| u_\nu \|_{X^1_s} & \leq C \| u_0 \|_{M^s_{1,1}}, \quad \text{where } C \text{ is independent of } \nu, \ X^T \text{ is defined in (6.1)}.
\end{align*}
\]

Moreover, if \( u_0 \in M^3_{1,1} \), then we have

\[
\| u_\nu - v \|_{C(0,T;L^1)} \lesssim \| u_\nu - v \|_{C(0,T;M^1_{1,1})} \lesssim \nu T, \quad \nu \ll 1.
\]

where \( v \) is the solution of the DNLS (1.2) with the same initial data.

**Theorem 1.4** Let \( n \in \mathbb{N} \). Assume initial data \( u_0 \in M^s_{1,1}, s > 3 \) and \( \| u_0 \|_{L^1} \leq \delta \) for some small \( \delta > 0 \). Then Eq. (1.4) has a unique solution

\[
\begin{align*}
u \in C([0,1], M^2_{2,1}) & \cap C([0,1], M^{s-1/2}_{1,1}) \cap X^1_s \\
\end{align*}
\]

\(^1\) \( u_0 \in M^2_{2,1} \) implies that \( u_0 \in L^2, \delta > 0 \) may depends on \( \| u_0 \|_{M^2_{2,1}} \).

\(^2\) \( u_0 \in M^1_{1,1} \) implies that \( u_0 \in L^1, \delta > 0 \) may depends on \( \| u_0 \|_{M^1_{1,1}} \).
satisfying $\|u_\nu\|_{L^1} \leq C\|u_0\|_{M^+_{1,1}}$, where $C$ is independent of $\nu$, $\hat{X}^T$ is defined in (7.3). Meanwhile, we have
\[ \|u_\nu - v\|_{C(0,T;L^1)} \lesssim \|u_\nu - v\|_{C(0,T;M^+_{1,1})} \lesssim \nu T, \quad \nu \ll 1. \]
where $u_\nu$ and $v$ is the solution of the DCGL (1.4) and DNLS (1.5) with the same initial data.

Now we give a brief explanation to the proof of our main results. We rewrite (1.1) into an integral equation:
\[
\begin{align*}
    u &= G_\nu(t)u_0 - \mathcal{A}_\nu[\vec{\lambda}_1 \cdot \nabla(|u|^2u) + (\vec{\lambda}_2 \cdot \nabla u)|u|^2 + \alpha |u|^{2\nu}u], \\
    G_\nu(t) &= \mathcal{F}^{-1} e^{-\nu|\xi|^2 - \nu|\xi|^2} \mathcal{F}, \quad \mathcal{A}_\nu f(t, x) = \int_0^t G_\nu(t - \tau) f(\tau, x) d\tau.
\end{align*}
\]
We will use the smooth effect techniques to prove our result. Comparing with the Schrödinger equation, the semigroup of Ginzburg-Landau equation $G_\nu(t)$ doesn’t have conjugate symmetry property, this means
\[ G_\nu(t) \neq G_\nu(-t), \]
we can not use standard $TT^*$ argument to get the smooth effect estimates, maximal function estimates and their relations with the Strichartz estimates for $G_\nu(t)$ and $\mathcal{A}_\nu$. It is known that $TT^*$ method is a basic tool for those estimates in the case $\nu = 0$.

The crucial estimates are the uniform anisotropic global smooth effect estimates for semigroup $G_\nu(t)$ and integral operator $\mathcal{A}_\nu$:
\[
\begin{align}
    &\|D_{x_i}^{1/2} G_\nu(t) \Box k u_0\|_{L^2_{t\infty} L^2_{x\in R^n}} \leq C\|\Box k u_0\|_2, \quad |k_i| \geq 4, \quad (1.6) \\
    &\|\mathcal{A}_\nu \partial_{x_i} f\|_{L^2_{t\infty} L^2_{x\in R^n}} \leq C\|f\|_{L^1_{t\infty} L^2_{x\in R^n}} \leq C\|f\|_{L^1_{t\infty} L^2_{x\in R^n}} L^2_{x\in R^n}, \quad (1.7)
\end{align}
\]
where those estimates in the case $\nu = 0$ were established in [25, 33, 37]. The main difficulty arises in the fact that the constant $C$ in (1.6) and (1.7) should be independent of parameter $\nu \geq 0$. We also need to show the uniform maximal function estimates for $G_\nu(t)$:
\[
    \Box k G_\nu(t) u_0 \|_{L^2_{t\infty} L^2_{x\in R^n}} \|_{L^2_{t\infty} L^2_{x\in R^n}} \lesssim C(k_i)^{1/2} \|\Box k u_0\|_2, \quad n \geq 3. \quad (1.8)
\]
In order to show (1.8), we will use the maximal operator estimates in anisotropic Lebesgue spaces as in [34]. After establishing those uniform estimates, we can use the idea in [37] to carry out the uniform global well posedness of Eq. (1.1). The limit behavior can be shown by using the techniques as in [13].

This paper is organized as follows. In Section 2 and Section 3 we prove the uniform anisotropic global smooth effect estimates, maximal inequality estimates, Strichartz type estimates for semigroup $G_\nu(t)$ and integral operator $\mathcal{A}_\nu$. In Section 4 we show the proof of Theorem 1. In Section 5 we show the proof of inviscid limit results. In Sections 6 and 7, we show the proof of Theorems 1.3 and 1.4.
1.2 Notation

In the sequel \( C \) will denote a universal positive constant which can be different at each appearance. \( x \lesssim y \) (for \( x, y > 0 \)) means that \( x \leq Cy \), and \( x \sim y \) stands for \( x \lesssim y \) and \( y \lesssim x \). For any \( p \in [1, \infty] \), \( p' \) denotes the conjugate number of \( p \), i.e., \( 1/p + 1/p' = 1 \).

Now we introduce the spaces used in our paper. Let \( \mathcal{S}(\mathbb{R}^n) \) be Schwartz space. We will use the Lebesgue spaces \( L^p := L^p(\mathbb{R}^n) \) with the norm \( \| f \|_p := \| f \|_{L^p(\mathbb{R}^n)} \) and the function spaces \( L^q_{I} \) and \( L^q_{I} \) for which the norms are defined by:

\[
\| f \|_{L^q_{I}} = \left( \int_{I} |f|^q \, dx \right)^{1/q}, \quad \| f \|_{L^q_{I}'} = \left( \int_{I} |f|^q \, dx \right)^{1/q}.
\]

\( I \) will be omitted if \( I = \mathbb{R} \), i.e., we simply write \( L^q_{\mathbb{R}} := L^q_{\mathbb{R}^n} \), \( L^q_{\mathbb{R}^n} := L^q_{\mathbb{R}^n} \) and \( L^q_{\mathbb{R}^n} \) are defined by \( L^q_{\mathbb{R}^n} := L^q_{\mathbb{R}^n} \). In high dimension case, we denote by \( L^p_{\mathbb{R}^n} \) the anisotropic Lebesgue space for which the norm is

\[
\| f \|_{L^p_{\mathbb{R}^n} (I \times \mathbb{R}^n)} = \left( \int_{I} \| f \|_{L^p_{\mathbb{R}^n}}^p \, dx \right)^{1/p}.
\]

The homogeneous Sobolev space \( \dot{H}^s(\mathbb{R}^n) \) is defined by \((-\Delta)^{-s/2} L^2(\mathbb{R}^n)\).

Modulation spaces were first introduced by Feichtinger [9]. We will use an equivalent norm on the modulation space \( M^2_{I,1} \):

\[
\| f \|_{M^2_{I,1}} = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \| \mathcal{F} f \|_{L^2(Q_k)},
\]

where \( \langle k \rangle = 1 + |k| \), \( Q_k = \{ \xi : -1/2 \leq \xi_i - k_i \leq 1/2, i = 1, \ldots, n \} \). Let \( \{ \sigma_k \}_{k \in \mathbb{Z}^n} \) satisfies:

\[
\begin{align*}
\sigma_k(\xi) &\geq c, & \forall \xi \in Q_k; \\
\text{supp} \sigma_k &\subset \{ \xi : |\xi - k| \leq \sqrt{n} \}, \\
\sum_{k \in \mathbb{Z}^n} \sigma_k(\xi) &\equiv 1, & \forall \xi \in \mathbb{R}^n; \\
|D^a \sigma_k(\xi)| &\leq C_m, & \forall \xi \in \mathbb{R}^n, |a| \leq m \in \mathbb{N}.
\end{align*}
\]

Denote

\[
\mathcal{Y} = \{ \{ \sigma_k \}_{k \in \mathbb{Z}^n} : \{ \sigma_k \}_{k \in \mathbb{Z}^n} \text{ satisfies (1.11) } \}.
\]

Let \( \{ \sigma_k \}_{k \in \mathbb{Z}^n} \in \mathcal{Y} \) be a function sequence. Then we can define the frequency-uniform decomposition operators \( \square_k \) as:

\[
\square_k := \mathcal{F}^{-1} \sigma_k \mathcal{F}, \quad k \in \mathbb{Z}^n,
\]

and we have

\[
\| f \|_{M^2_{I,1}} \sim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \| \square_k f \|_{L^2(\mathbb{R}^n)}.
\]
Using the operators $\Box_k$, we can equivalently define the modulation space $M_{1,1}^s$ in the following way:

$$\|f\|_{M_{1,1}^s} = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \|\Box_k f\|_{L^1(\mathbb{R}^n)}. \quad (1.15)$$

For simplicity, we use function space $l_1^s(\mathcal{L}^\nu(\mathbb{R}^+; \mathcal{L}^r(\mathbb{R}^n)))$ which contains all of the functions $f(t, x)$ so that the following norm is finite:

$$\sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \|\Box_k f\|_{L^1(\mathbb{R}^n)} \quad (1.16)$$

2 Anisotropic Global smooth effect with $\Box_k$-decomposition

In this section, we always denote

$$G_\nu(t) = \mathcal{F}^{-1}e^{-it|\xi|^2}e^{-\nu|\xi|^2}e^{|t|}, \quad \mathcal{A}f(t, x) = \int_0^t G_\nu(t-\tau)f(\tau, x)d\tau. \quad (2.1)$$

For convenience, we will use the following function sequence $\{\sigma_k\}_{k \in \mathbb{Z}^n}$:

**Lemma 2.1** Let $\eta_k : \mathbb{R} \to [0, 1](k \in \mathbb{Z})$ be a smooth-function sequence satisfying $[1.11]$. Denote

$$\sigma_k(\xi) := \eta_{k_1}(\xi_1)\ldots\eta_{k_n}(\xi_n), \quad k = (k_1, \ldots, k_n). \quad (2.2)$$

Then we have $\{\sigma_k\}_{k \in \mathbb{Z}^n} \in \Upsilon$.

**Lemma 2.2** $[37]$ We have for any $\delta \in \mathbb{R}$ and $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $|k_i| \geq 4$,

$$\|\Box_k D_{\delta x_i}^\nu u\|_{L^p_x L^q_y(\mathbb{R}_+ \times \mathbb{R}^n)} \lesssim \langle k \rangle^s \|\Box_k u\|_{L^p_x L^q_y(\mathbb{R}_+ \times \mathbb{R}^n)}, \quad (2.3)$$

Replacing $D_{\delta x_i}^\nu$ by $\partial_{\delta x_i}^\nu (\delta \in \mathbb{N})$, the above inequality holds for all $k \in \mathbb{Z}^n$.

In order to obtain global smooth-effect estimates, we need the following Lemma in the case $n = 1$:

**Lemma 2.3** Let $n = 1$. Then there exists $C > 0$, which is independent of $\nu > 0$ such that

$$\|\mathcal{F}_{\xi}^{-1}e^{-it|\xi|^2}e^{-\nu|\xi|^2}\mathcal{F}_{\xi}^x \Box_k \phi\|_{L^p_x L^q_y(\mathbb{R} \times \mathbb{R})} \leq C\langle k \rangle^{1/2} \|\Box_k \phi\|_{L^2(\mathbb{R})}. \quad (2.4)$$

**Proof.** We may assume that $k \geq 4$. By changing the variable, we have:

$$\mathcal{F}_{\xi}^{-1}e^{-it|\xi|^2}e^{-\nu|\xi|^2}\mathcal{F}_{\xi}^x \Box_k \phi$$
For convenience, we denote by \( \eta_k \) was defined in Lemma 2.1.

From Plancherel’s identity, Fubini theorem and Young’s inequality we have

\[
\| \mathcal{F}_\xi^{-1} e^{-it\xi^2} e^{-iu|\xi|^2} \mathcal{F}_x \Box_k \phi \|_{L_\infty^1 L_2^1(\mathbb{R} \times \mathbb{R})} \\
\leq \left\| \int_\mathbb{R} \mathcal{F}_t(e^{-u|\eta|}(\tau + \eta)) \eta_k(\sqrt{\eta}) \hat{\phi}(\sqrt{\eta}) e^{ix\sqrt{\eta}} \frac{1}{2\sqrt{\eta}} d\eta \right\|_{L_\infty^1 L_2^1} \\
= \left\| \int_\mathbb{R} \left( \frac{1}{\nu \eta} \frac{1}{1 + \frac{1}{\nu \eta}} \right) \eta_k(\sqrt{\eta}) \hat{\phi}(\sqrt{\eta}) \frac{1}{2\sqrt{\eta}} d\eta \right\|_{L_2^1} \\
\leq \left\| \int_\mathbb{R} \left( \frac{1}{\nu((k) - 1/2)^2} \frac{1}{1 + \frac{1}{\nu ((k) + 1/2)^2}} \right) \eta_k(\sqrt{\eta}) \hat{\phi}(\sqrt{\eta}) \frac{1}{2\sqrt{\eta}} d\eta \right\|_{L_2^1} \\
\leq C(k)^{-1/2} \| \Box_k \phi \|_{L_2^1},
\]

where we have used

\[
\left\| \mathcal{F}_t(e^{-u|\eta|})(\tau) \right\|_{L_1^1} = \int \frac{1}{1 + \tau^2} d\tau \leq C,
\]

where \( C \) is independent of \( \nu, \eta \). In equation (2.6), \( |k| \geq 4 \) is necessary. \( \square \)

**Lemma 2.4**

\[
\sup_{s \geq 0} \left\| \mathcal{F}_\xi^{-1} \frac{\xi}{\tau + \xi^2 - i\nu(\xi^2 + s)} \mathcal{F}_{t,x} f \left\|_{L_\infty^1 L_2^1(\mathbb{R}^{1+1})} \leq C \| f \|_{L_2^1 L_2^1(\mathbb{R}^{1+1})}.
\]

The constant \( C \) in (2.8) is independent of \( \nu > 0 \).

**Proof.** For convenience, we denote by \( \mathcal{F}_{t,x}, \mathcal{F}_t, \mathcal{F} \) the Fourier transforms on \( (t, x) \), \( t \), \( x \), respectively. From Plancherel’s identity, we have:

\[
\left\| \mathcal{F}_\xi^{-1} \frac{\xi}{\tau + \xi^2 - i\nu(\xi^2 + s)} \mathcal{F}_{t,x} f \right\|_{L_2^1} \\
= \left\| \int_\mathbb{R} \int_\mathbb{R} e^{i(x-y)\xi} \frac{\xi}{\tau + \xi^2 - i\nu(\xi^2 + s)} (\mathcal{F}_t f)(\tau, y) d\xi dy \right\|_{L_2^1} \\
:= \left\| \int_\mathbb{R} K(\tau, x-y) (\mathcal{F}_t f)(\tau, y) dy \right\|_{L_2^1},
\]

where the integral

\[
K(\tau, z) = \int_\mathbb{R} e^{iz\xi} \frac{\xi}{\tau + \xi^2 - i\nu(\xi^2 + s)} d\xi
\]
is taken in the P.V. meaning. Now we only need to show that
\[
\sup_{s \geq 0} \|K(\tau, z)\|_{L^\infty_{\tau,z}} \lesssim 1, \tag{2.11}
\]
We only consider case $\tau < 0$, (the case $\tau \geq 0$ do not contain singular point, so it is easy to handle). For $\tau < 0$, we have
\[
K(\tau, z) = \int_\mathbb{R} e^{iz\sqrt{-\tau} \eta} \frac{\eta}{1 - \eta^2 + i\nu(\eta^2 + s_1)} d\eta
= \int_\mathbb{R} e^{iz\sqrt{-\tau} \eta} \eta[1 - \eta^2 - i\nu(\eta^2 + s_1)]\nu^2(\eta^2 + s_1)^2 + (1 - \eta^2)^2 d\eta, \tag{2.12}
\]
where $s_1 = -s/\tau > 0$.

Since when $s_1 \to 0, \nu \to 0, \eta \to 1$, $K(\tau, z)$ is difficult to handle, we will divide $\eta$ into different cases: Let $\psi_1, \psi_2, \psi_3 \in C^\infty_0(\mathbb{R})$ satisfy $\text{supp} \psi_1 \subseteq \{ \eta : |\eta| \geq 3/2 \}$, $\psi_3(-\cdot) = \psi_1(\cdot)$, $\text{supp} \psi_2 \subseteq (-2, 1/2]$, $\text{supp} \psi_3 \subseteq (0, 2)$, $\sum_{i=1}^3 \psi_i = 1$. Define
\[
K_i(\tau, z) = \int_\mathbb{R} e^{iz\sqrt{-\tau} \eta} \eta[1 - \eta^2 - i\nu(\eta^2 + s_1)]\eta \psi_i(\eta)\nu^2(\eta^2 + s_1)^2 + (1 - \eta^2)^2 d\eta, \quad i = 1, 2, 3. \tag{2.13}
\]
\[
K_1(\tau, z) = \int_{|\eta| \geq 3/2} e^{iz\sqrt{-\tau} \eta} \frac{\eta}{\eta^2 + s_1} + (1 - \eta^2)^2 d\eta
- (1 + i\nu) \int_{|\eta| \geq 3/2} e^{iz\sqrt{-\tau} \eta} \frac{\eta^3}{\eta^2 + s_1} + (1 - \eta^2)^2 d\eta
- i \int_{|\eta| \geq 3/2} e^{iz\sqrt{-\tau} \eta} \frac{\nu \eta s_1}{\eta^2 + s_1} + (1 - \eta^2)^2 d\eta
:= K^1_1(\tau, z) + K^2_1(\tau, z) + K^3_1(\tau, z). \tag{2.14}
\]
It is easy to see
\[
|K^1_1(\tau, z)| \lesssim \int_{|\eta| \geq 3/2} \frac{1}{\eta^2} d\eta \leq C. \tag{2.15}
\]
From variable changing, we have:
\[
|K^1_1(\tau, z)| \leq 2 \int_{|\eta| \geq 3/2} \frac{\nu s_1 \eta}{(\nu s_1)^2 + \eta^4} d\eta \leq \arctan \eta_{|\eta|^\infty} \lesssim C, \tag{2.16}
\]
\[
|K^2_1(\tau, z)| \lesssim \int_{|\eta| \geq 3/2} e^{iz\sqrt{-\tau} \eta} \frac{\eta^3}{\eta^2 + s_1} + (1 - \eta^2)^2 d\eta. \tag{2.17}
\]
We derive $K^2_1(\tau, z)$ into two parts $I, II$, from (2.16) we have:
\[
I = \int_{3/2 \leq |\eta| \leq 10 \sqrt{\nu s_1}} e^{iz\sqrt{-\tau} \eta} \frac{\eta^3}{\eta^2 + s_1} + (1 - \eta^2)^2 d\eta
\]
\[
\left| \int_{3/2 \leq |\eta| \leq 10\sqrt{s_1}} \frac{\nu s_1 \eta}{(\nu s_1)^2 + \eta^4} \right| \leq \arctan \eta \bigg|_{\eta/\nu s_1}^\infty \leq C. \quad (2.18)
\]

From variable changing, we have:
\[
|II| = \left| \int_{\eta \geq 10\sqrt{s_1}} \frac{\sin(z \sqrt{-\tau} \eta)}{(1 + \nu^2)\eta + \frac{\nu^2 s_1^2 + 1}{\eta^3} + \frac{2(\nu^2 s_1 - 2)(z \sqrt{-\tau})^2}{\eta}} \, d\eta \right|
= \left| \int_{\eta \geq 10\sqrt{s_1}} \frac{\sin(\eta)}{(1 + \nu^2)\eta + \frac{(z \sqrt{-\tau})^4(\nu^2 s_1^2 + 1)}{\eta^3} + \frac{2(\nu^2 s_1 - 2)(z \sqrt{-\tau})^2}{\eta}} \, d\eta \right|. \quad (2.19)
\]

Now we prove (2.19) is bounded. Write
\[
F(\eta) := \frac{1}{(1 + \nu^2)\eta + \frac{(z \sqrt{-\tau})^4(\nu^2 s_1^2 + 1)}{\eta^3} + \frac{2(\nu^2 s_1 - 2)(z \sqrt{-\tau})^2}{\eta}}.
\]

For any \( \epsilon > 0 \), when \( A' > A > 1/(1 + \nu^2)\epsilon \), we have:
\[
F(A) \leq \epsilon, \quad F(A')' \leq \epsilon.
\]

Notice that \( F(\eta) \) is monotonous decreasing when \( \eta \geq 10\sqrt{s_1} \) and for any \( \eta \in [A, A'] \), \( \int_A^{A'} \sin(\eta) \, d\eta \leq C \). So from the second integral-mean-value theorem, we have:
\[
\left| \int_A^{A'} \frac{1}{(1 + \nu^2)\eta + \frac{(z \sqrt{-\tau})^4(\nu^2 s_1^2 + 1)}{\eta^3} + \frac{2(\nu^2 s_1 - 2)(z \sqrt{-\tau})^2}{\eta}} \, d\eta \right| \leq C\epsilon. \quad (2.20)
\]

Then from the Cauchy convergence theorem, we can get (2.19) is bounded. So
\[
||K_1(\tau, z)||_{L^\infty_r} \lesssim 1.
\]

Notice that \( \nu^2(\eta^2 + s_1^2) + (1 - |\eta|)^2 \geq 3/4 \), when \( \eta \in (-3/2, 1/2] \), so it is easy to estimate \( K_2(\tau, z) \):
\[
||K_2(\tau, z)||_{L^\infty_r} \lesssim \int_R e^{i z \sqrt{-\tau} \eta} \frac{\eta \psi_2(\eta)}{\nu^2(\eta^2 + s_1^2) + (1 - \eta^2)^2} \, d\eta
\lesssim \left| \eta \frac{\eta \psi_2(\eta)}{\nu^2(\eta^2 + s_1^2) + (1 - \eta^2)^2} \right| \leq C. \quad (2.21)
\]

\[
K_3(\tau, z) = \int_R e^{i z \sqrt{-\tau} \eta} \frac{\eta \psi_3(\eta)}{1 - \eta^2 + i\nu(\eta^2 + s_1)} \, d\eta
= \int_R e^{i z \sqrt{-\tau} \eta} \frac{\eta \psi_3(\eta)}{1 - \eta^2} \, d\eta - \int_R e^{i z \sqrt{-\tau} \eta} \frac{i\nu(\eta^2 + s_1) \psi_3(\eta)}{1 - \eta^2 + i\nu(\eta^2 + s_1)(1 - \eta^2)} \, d\eta
:= K_3^1(\tau, z) + K_3^2(\tau, z), \quad (2.22)
\]

\[
|K_3^1(\tau, z)| = \left| \int_R e^{i z \sqrt{-\tau} \eta} \frac{1}{1 - \eta} \cdot \frac{\eta \psi_3(\eta)}{1 + \eta} \, d\eta \right| \sim \left| \left( \text{sgn} * \mathcal{F}^{-1} \left[ \frac{\eta \psi_3(\eta)}{1 + \eta} \right] \right)(\sqrt{-\tau}z) \right|, \quad (2.23)
\]
We have

\[
\left| K_3^2(\tau, z) \right| = \left| \int_{\mathbb{R}} e^{i\sqrt{-1} \eta \cdot \tau} \frac{1}{1 - \eta} \cdot \frac{iv\eta(\eta^2 + s_1)[1 - \eta^2 - iv(\eta^2 + s_1)]\psi_3(\eta)}{[(1 - \eta^2)^2 + \nu^2(\eta^2 + s_1)^2](1 + \eta)} \, d\eta \right|
\]

\sim \left| \left( \text{sgn} \ast \mathcal{F}^{-1} \left[ \frac{iv\eta(\eta^2 + s_1)[1 - \eta^2 - iv(\eta^2 + s_1)]\psi_3(\eta)}{[(1 - \eta^2)^2 + \nu^2(\eta^2 + s_1)^2](1 + \eta)} \right] \right) (\sqrt{-1} \tau) \right|.
\]

(2.24)

Noticing that

\[
\left| \frac{iv\eta(\eta^2 + s_1)[1 - \eta^2 - iv(\eta^2 + s_1)]\psi_3(\eta)}{[(1 - \eta^2)^2 + \nu^2(\eta^2 + s_1)^2](1 + \eta)} \right| \leq C,
\]

from (2.22) (2.23), we obtain that

\[
\|K_3(\tau, z)\|_{L^\infty_z} \leq \left\| \mathcal{F}^{-1} \left[ \frac{\psi_3(\eta)}{1 + |\eta|} \right] \right\|_{L^1} + \left\| \mathcal{F}^{-1} \left[ \frac{iv\eta(\eta^2 + s_1)[1 - \eta^2 - iv(\eta^2 + s_1)]\psi_3(\eta)}{[(1 - \eta^2)^2 + \nu^2(\eta^2 + s_1)^2](1 + \eta)} \right] \right\|_{L^1} \lesssim 1.
\]

(2.25)

It follows that (2.11) holds.

**Proposition 2.5** For any \( i = 1, \ldots, n, |k_i| \geq 4, \) we have

\[
\|D_{x_i}^{1/2} \square_k G_\nu(t) u_0\|_{L^\infty_t L^2_{(x_j)_{j \neq i}} L^2_t(\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \| \square_k u_0 \|_2.
\]

(2.26)

**Proof.** As in the proof of Lemma 2.3, we only need to prove that

\[
\|D_{x_i}^{1/2} \square_k G_\nu'(t) u_0\|_{L^\infty_t L^2_{(x_j)_{j \neq i}} L^2_t(\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \| \square_k u_0 \|_2,
\]

(2.27)

where \( G_\nu(t)' = \mathcal{F}^{-1} e^{-it|\xi|^2 - \nu|\xi|^2} \mathcal{F} \). It suffices to show the case \( i = 1 \). By Plancherel’s identity and Minkowski’s inequality,

\[
\|D_{x_i}^{1/2} G_\nu(t)' \square_k u_0\|_{L^\infty_t L^2_{x_1} L^2_{x_2, \ldots, x_n} L^2_t(\mathbb{R}^{1+n})} \\
\leq \|D_{x_1}^{1/2} \mathcal{F}^{-1} e^{-it|\xi|^2} e^{-\nu t|\xi|^2} \mathcal{F}_{x_1} F_{x_2, \ldots, x_n} \square_k u_0\|_{L^\infty_t L^2_{x_2, \ldots, x_n} L^2_t(\mathbb{R}^{1+n})} \\
\leq \|D_{x_1}^{1/2} \mathcal{F}^{-1} e^{-it|\xi|^2} e^{-\nu t|\xi|^2} \mathcal{F}_{x_1} F_{x_2, \ldots, x_n} \square_k u_0\|_{L^\infty_t L^2_{x_2, \ldots, x_n} L^2_t(\mathbb{R}^{1+n})}.
\]

In view of Lemma 2.3 in one spatial dimension, using Plancherel’s identity, we immediately obtain (2.24). \( \square \)

**Proposition 2.6** For any \( k = (k_1, \ldots, k_n) \in \mathbb{Z}^n, |k_i| \geq 4, \) we have

\[
\|\partial_{x_i} \square_k \mathcal{F} f\|_{L^\infty_t L^2_{x_i} L^2_{x_1} L^2_t(\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \left( k_i \right)^{1/2} \| \square_k f\|_{L^2_t L^2_{(x_j)_{j \neq i}} L^2_t(\mathbb{R}^+ \times \mathbb{R}^n)}.
\]

(2.28)

**Proof.** For \( |k_i| \geq 4, \) from Proposition 2.5 (2.26) has the following dual estimate:

\[
\left\| \square_k D_{x_i}^{1/2} \int_0^t G_\nu(t - \tau) f(\tau) \, d\tau \right\|_{L^\infty_t L^2_{x_i} L^2_t(\mathbb{R}^n)} \lesssim \| \square_k f\|_{L^2_t L^2_{(x_j)_{j \neq i}} L^2_t(\mathbb{R}^+ \times \mathbb{R}^n)}.
\]

(2.29)

Then from Lemma 2.5 which implies (2.28) holds, as desired. \( \square \)
Proposition 2.7 For any $i = 1, \ldots, n$ and $k = (k_1, \ldots, k_n)$, there exist $C > 0$, which are independent of $\nu > 0$ such that

$$
\| \Box_k \mathcal{A}_t \partial_x f \|_{L^\infty_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)} \leq C \| \Box_k f \|_{L^1_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)}. \tag{2.30}
$$

Proof. In order to prove (2.30), assume $f(t, x) = 0$ for $t < 0$, so we only need to prove

$$
\| \Box_k \mathcal{A}_t \partial_x f \|_{L^\infty_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)} \leq C \| \Box_k f \|_{L^1_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)}. \tag{2.31}
$$

We have

$$
\partial_x \mathcal{A}_t f = C \mathcal{F}^{-1} \left\{ \frac{\xi_1}{\xi_1^2 - \nu (\xi_1^2 + |\xi|^2) + \tau + |\xi|^2} \right\} \mathcal{F}_t \mathcal{F}_x f, \tag{2.32}
$$

where we assume that the right hand side of (2.32) is zero as $t = 0$. It follows that

$$
\| \partial_x \mathcal{A}_t f \|_{L^\infty_t L^2_x L^2_\nu L^1_t (\mathbb{R}^{1+n})} \leq \left\| \mathcal{F}^{-1} \left\{ \frac{\xi_1}{\xi_1^2 - \nu (\xi_1^2 + |\xi|^2) + \tau + |\xi|^2} \right\} \mathcal{F}_t \mathcal{F}_x f \right\|_{L^\infty_t L^2_x L^1_\nu L^2_t (\mathbb{R}^{1+n})}. \tag{2.33}
$$

Now changing the variable $\tau + |\xi|^2 \to \mu$, we have

$$
\left\| \mathcal{F}^{-1} \left\{ \frac{\xi_1}{\xi_1^2 - \nu (\xi_1^2 + |\xi|^2) + \tau + |\xi|^2} \right\} \mathcal{F}_t \mathcal{F}_x f \right\|_{L^\infty_t L^2_x L^1_\nu L^2_t (\mathbb{R}^{1+n})} \leq \sup_{s > 0} \left\| \mathcal{F}^{-1} \left\{ \frac{\xi_1}{\xi_1^2 - \nu (\xi_1^2 + |\xi|^2) + \tau + |\xi|^2} \right\} \mathcal{F}_t \mathcal{F}_x f \right\|_{L^\infty_t L^2_x L^1_\nu L^2_t (\mathbb{R}^{1+n})}. \tag{2.34}
$$

From the uniform smooth effect estimate as in Lemma 2.4,

$$
\sup_{s > 0} \left\| \mathcal{F}^{-1} \left\{ \frac{\xi}{\xi^2 - \nu (\xi^2 + s) + \tau} \right\} \mathcal{F}_t \mathcal{F}_x f \right\|_{L^\infty_t L^2_x L^2_\nu L^1_t (\mathbb{R}^{1+n})} \leq C \| f \|_{L^1_t L^2_x (\mathbb{R}^{1+n})}. \tag{2.35}
$$

From (2.33), (2.34) and (2.35), we have that

$$
\| \partial_x \mathcal{A}_t f \|_{L^\infty_t L^2_x L^2_\nu L^1_t (\mathbb{R}^{1+n})} \lesssim \| e^{\nu |\xi|^2} \mathcal{F}_{x_2, \ldots, x_n} f \|_{L^2_t L^2_{x_2, \ldots, x_n} L^1_t (\mathbb{R}^{1+n})}. \tag{2.36}
$$

Using Minkowski’s inequality and Plancherel’s identity, we immediately have

$$
\| \partial_x \mathcal{A}_t f \|_{L^\infty_t L^2_x L^2_\nu L^1_t (\mathbb{R}^{1+n})} \lesssim \| f \|_{L^1_t L^2_x L^2_\nu L^1_t (\mathbb{R}^{1+n})}. \tag{2.37}
$$

Other cases can be shown in a similar way.

Generally, the right hand side in (2.32) is not equal to zero for $t = 0$:

$$
\left. \frac{1}{\tau + |\xi|^2 - \nu |\xi|^2} \mathcal{F}_t \mathcal{F}_x f \right|_{t=0} = -\left. \frac{\nu |\xi|^2}{(\tau + |\xi|^2)^2 + (\nu |\xi|^2)^2} \mathcal{F}_t \mathcal{F}_x f \right|_{t=0} + \left. \frac{|\xi|^2}{(\tau + |\xi|^2)^2 + (\nu |\xi|^2)^2} \mathcal{F}_t \mathcal{F}_x f \right|_{t=0}
$$
Noticing that $\mathcal{F}(e^{-i|\tau|\xi})(\tau) = \frac{1}{\nu|\xi|^2} \mathcal{F}(\partial_x e^{-i|\tau|\xi})(\tau)\frac{1}{\nu|\xi|^2}$ and changing the variable, we have

$$u_2(0, x) = C \int_{\mathbb{R}^n} e^{ix\xi} \int_{\mathbb{R}} \frac{\nu|\xi|^2}{(\tau + |\xi|^2)^2 + (\nu|\xi|^2)^2} e^{-irs} \hat{f}(s, \xi) ds d\tau d\xi$$

Then from (2.26), (2.39) and (2.29) we have

$$\|\Box_k \partial_x G_\nu(t) u_2(0, x)\|_{L^2_{x, t}(\mathbb{R}^n \times \mathbb{R}^n)} \leq \|D^{1/2}_{x, t} \Box_k u_2(0, x)\|_{L^2}$$

Noticing that

$$\mathcal{F}_1(\partial_t e^{-i|\tau|\xi})(\tau) = \frac{1}{\nu|\xi|^2} \mathcal{F}_1(\partial_x e^{-i|\tau|\xi})(\tau) = \frac{1}{\nu|\xi|^2} \frac{1}{\nu|\xi|^2 + 1} \frac{\tau}{\tau^2 + (\nu|\xi|^2)^2} = \frac{\tau}{\tau^2 + (\nu|\xi|^2)^2},$$

similar to (2.39), we have

$$u_1(0, x) = C \int_{\mathbb{R}^n} e^{ix\xi} \int_{\mathbb{R}} \frac{\tau + |\xi|^2}{(\tau + |\xi|^2)^2 + (\nu|\xi|^2)^2} e^{-irs} \hat{f}(s, \xi) ds d\tau d\xi$$

Then from (2.26), (2.41) and (2.29) we have

$$\|\Box_k \partial_x G_\nu(t) u_1(0, x)\|_{L^2_{x, t}(\mathbb{R}^n \times \mathbb{R}^n)} \leq \|D^{1/2}_{x, t} \Box_k u_1(0, x)\|_{L^2}$$

Collecting (2.42), (2.40), we can obtain the result, as desired.
3 Other estimates with $\square_k$-decomposition

In this section, we consider the Strichartz estimates, the maximal function estimates and derivative interaction estimates for the solutions of Ginzburg-Laundau equation by using the frequency-uniform decomposition operators.

Using Lemma 3.3 and the property of frequency-uniform decomposition operators (cf. [39]), we can establish the following Strichartz estimates in a class of function spaces by using the frequency-uniform decomposition operators.

**Proposition 3.1** Let $2 \leq r < \infty$, $q > \nu \geq 2 \lor \nu(r)$, then we have

$$
\|G_\nu(t)f\|_{L^q(\mathbb{R}^r;L^r(\mathbb{R}^n))} \lesssim C\|f\|_{M_2(\mathbb{R}^n)},
$$

$$
\|\partial_\nu f\|_{L^q(\mathbb{R}^r;L^r(\mathbb{R}^n))} \lesssim C\|f\|_{L^q(\mathbb{R}^r;L^r(\mathbb{R}^n))}.
$$

**Proposition 3.2** Let $2 \leq r \leq \infty$, $2/\nu(r) = n(1/2 - 1/r)$, $q > \nu > \nu(r) \lor 2$, we have

$$
\|\square_k G_\nu u_0\|_L^1(\mathbb{R}^r;L^r(\mathbb{R}^n)) \lesssim \|\square_k u_0\|_L^2(\mathbb{R}^n),
$$

$$
\|\square_k \partial_\nu f\|_{L^q(\mathbb{R}^r;L^r(\mathbb{R}^n))} \lesssim C\|\square_k f\|_{L^q(\mathbb{R}^r;L^r(\mathbb{R}^n))}.
$$

**Proof.** Proposition 3.1 implies (3.3) and (3.4) directly. \qed

Define the semigroup of Schrödinger equation

$$
S(t) = \mathcal{F}^{-1} e^{-it\sum_{j=1}^n \xi_j^2},
$$

**Proposition 3.3** $\square_k G_\nu(t) : L^p \to L^p$ is uniformly bounded. More precisely,

$$
\|\square_k G_\nu(t)u_0\|_{L^p(\mathbb{R}^n)} \lesssim (1 + |t|^{n/2})\|\square_k u_0\|_{L^p(\mathbb{R}^n)}
$$

uniformly holds for all $k \in \mathbb{Z}^n$, $\nu \geq 0$, $p \geq 1$.

**Proof.** It is well known that $e^{-|t|^2}$ is a multiplier on $L^p$, i.e., $e^{-|t|^2} \in M_p$ ($M_p$ denotes Hörmander’s multiplier space, see [1]). Since $M_p$ is isometrically invariant under affine transformations of $\mathbb{R}^n$, we have $\|e^{-|t|^2}\|_{M_p} = \|e^{-\nu|t|^2}\|_{M_p}$, $1 \leq p \leq \infty$. We have

$$
\|\square_k G_\nu(t)f\|_p \lesssim \|\square_k S(t)f\|_p \leq \sum_{|\ell| \leq 1} \|\mathcal{F}^{-1}(\sigma_{\ell} e^{it|\xi|^2}) f\|_p
$$

$$
\leq \sum_{|\ell| \leq 1} \|\mathcal{F}^{-1}(\sigma_{\ell} e^{it|\xi|^2})\|_1 \|\square_k f\|_p.
$$

So, it suffices to show that $\|\mathcal{F}^{-1}(\sigma_{k} e^{it|\xi|^2})\|_1$ is uniformly bounded.

$$
\|\mathcal{F}^{-1}(\sigma_{k} e^{it|\xi|^2})\|_1 = \|\mathcal{F}^{-1}(\sigma_{0} e^{it|\xi|^2})\|_1
$$

13
\[ \| \sigma_0 \|_2^{1-n/2L} \sum_{|\alpha|=L} \| D^\alpha (\sigma_0 e^{it|\xi|^2}) \|_2^{n/2L} \lesssim (1 + |t|^{n/2}). \tag{3.8} \]

So we have the result, as desired. \[ \square \]

37 shows that \( S(t) \) has the following maximal function estimate:

**Lemma 3.4** (37) Let \( 4/n < p \leq \infty, \ p \geq 2, \) \( S(t) \) is defined as (3.24), then we have

\[ \| \square_k S(t)u_0 \|_{L^p_t L^\infty_x (\mathbb{R} \times \mathbb{R}^n)} \leq C(k_1)^{1/p} \| \square_k u_0 \|_{L^2(\mathbb{R}^n)}. \tag{3.9} \]

**Lemma 3.5** Define maximal operator \( M \) as following:

\[ (Mf)(x) = \sup_{r>0} c_n r^{-n} \int_{|y|<r} |f(x-y)|dy. \]

Let \( \phi \) satisfies \( \int_{\mathbb{R}^n} \phi \, dx = 1, \) then for any \( f, \) \( f \in L^p, \) \( 1 < p \leq \infty, \) we have

\[ \sup_{t>0} |f * \phi_t(x)| \leq Mf(x) \int_{\mathbb{R}^n} \phi \, dx \tag{3.10} \]

\[ \| Mf \|_{L^p} \leq C \| f \|_{L^p}. \tag{3.11} \]

Where \( \phi_t(x) = t^{-1} \phi(x/t) \).

The proof can be found in [34], Page 51, [35], Page 3.

**Proposition 3.6** Let \( 4/n < p \leq \infty, \ p \geq 2, \) we have

\[ \| \square_k G_\nu(t)u_0 \|_{L^p_t L^\infty_x (\mathbb{R} \times \mathbb{R}^n)} \leq C(k_1)^{1/p} \| \square_k u_0 \|_{L^2(\mathbb{R}^n)}. \tag{3.12} \]

**Proof.** Take \( i = 1 \) for example. When \( t = 0, \) (3.12) holds obviously. For \( t > 0, \)

\[ \square_k G_\nu(t)u_0 = \mathcal{F}^{-1} (e^{-|\xi|^2} \mathcal{F} \mathcal{F}^{-1} (e^{-|\xi|^2} \square_k u_0)) = \mathcal{F}^{-1} (e^{-|\xi|^2}/2) \sqrt{2\pi} \mathcal{F}^{-1} (e^{-|\xi|^2} \square_k u_0). \tag{3.13} \]

Notice that \( \mathcal{F}^{-1} (e^{-|\xi|^2}/2) = e^{-|x|^2/2}, \int_{\mathbb{R}^n} e^{-|x|^2/2} \, dx = C, \) then from (3.10), (3.13) we have

\[ \| \square_k G_\nu(t)u_0 \|_{L^p_t L^\infty_x (\mathbb{R} \times \mathbb{R}^n)} \leq \| \mathcal{F}^{-1} (e^{-|\xi|^2}/2) \sqrt{2\pi} \mathcal{F}^{-1} (e^{-|\xi|^2} \square_k u_0) \|_{L^p_t L^\infty_x (\mathbb{R} \times \mathbb{R}^n)} \leq C(k_1)^{1/p} \| \square_k u_0 \|_{L^2(\mathbb{R}^n)}. \tag{3.14} \]

Define \( M_{x_1}, M_{x_2} \) were the maximal operators for variable \( x_1 \) and the other variables:

\[ (M_{x_1}f)(x_1, \bar{x}) = \sup_{r>0} c_1 r^{-1} \int_{|y_1|<r} |f(x_1 - y_1, \bar{x})| \, dy_1. \]
\[(M_{\xi}f)(x_1,\bar{x}) = \sup_{r>0} c_{n-1} r^{-(n-1)} \int_{|y|<r} |f(x_1, x_2 - y_2, \ldots, x_n - y_n)|dy.\]

From the definition of maximal operators and Lemma 3.5 we have
\[\|Mf(x_1, \bar{x})\|_{L^p_{x_1}L^\infty_x} \leq \|M_{x_1}\|_{L^p_{x_1}L^\infty_x} \|f(x_1, \bar{x})\|_{L^p_{x_2}} \leq \|M_{x_1}\|_{L^p_{x_2}} \|f(x_1, \bar{x})\|_{L^p_{x_1}} \leq \|f\|_{L^p_{x_1}L^\infty_x}, \quad (3.15)\]

where \(p \geq 2\). From Lemma 3.4 and (3.15) we obtain
\[\|M[\mathcal{F}^{-1}(e^{-it\xi^2} \mathcal{F}(\square_k u_0))]\|_{L^p_{x_1}L^\infty_x(L^\infty_{x_1} \cap \mathbb{R}^n \times \mathbb{R}^n)} \leq C\|\mathcal{F}^{-1}(e^{-it\xi^2} \mathcal{F}(\square_k u_0))\|_{L^p_{x_1}L^\infty_x(L^\infty_{x_1} \cap \mathbb{R}^n \times \mathbb{R}^n)} \leq C(k_1)^{1/p}\|\square_k u_0\|_{L^2_\xi},\]

which implies the result, as desired. \(\square\)

**Proposition 3.7** For \(n = 1, 2\), we have
\[\|\square_k G_\nu(t)u_0\|_{L^2_\xi L^\infty_x} \leq C\langle k\rangle^{1/2} \ln 4\langle T\rangle\|\square_k u_0\|_{L^1_\xi}, \quad n = 1;\]
\[\|\square_k G_\nu(t)u_0\|_{L^2_\xi L^\infty_{x_1}} L^\infty_x(L^\infty_{x_1} \cap \mathbb{R}^n) \leq \langle k\rangle^{1/2}\|\square_k u_0\|_{L^1_\xi}, \quad n = 2.\]

**Proof.** We take \(i = 1\) for example.
\[\|\square_k G_\nu(t)u_0\|_{L^2_\xi L^\infty_{x_1}} L^\infty_x(L^\infty_{x_1} \cap \mathbb{R}^n) = \|\mathcal{F}^{-1}_\xi e^{-it|\xi|^2} e^{-\nu t|\xi|^2} \tilde{\sigma}_k(\xi) + \mathcal{F}^{-1}_\xi \sigma_k(\xi)\|_{L^2_\xi L^\infty_{x_1}} L^\infty_x(L^\infty_{x_1} \cap \mathbb{R}^n)}, \quad (3.16)\]

Where \(\tilde{\sigma}_k(\xi) = \sum_{|l-k|<C(n)} \sigma_l(\xi)\). For brevity, we still write \(\tilde{\sigma}_k\) as \(\sigma_k\). Now we estimate \(\|\mathcal{F}^{-1}_\xi e^{-it|\xi|^2} e^{-\nu t|\xi|^2} \tilde{\sigma}_k(\xi)\|_{L^2_\xi L^\infty_{x_1}} L^\infty_x(L^\infty_{x_1} \cap \mathbb{R}^n)\). First, consider the basic \(L^p - L^{p'}\) estimates for the semigroup of DGL equation \(G_\nu(t) = \mathcal{F}^{-1}e^{-it|\xi|^2} e^{-\nu t|\xi|^2} \mathcal{F}\). We have
\[\|G_\nu(t)\varphi\|_{L^\infty} \lesssim \|S(t)\varphi\|_{L^\infty} \lesssim CT^{-n/2}\|\varphi\|_{L^1}, \quad (3.17)\]

and so,
\[\|\mathcal{F}^{-1}_\xi e^{-it|\xi|^2} e^{-\nu t|\xi|^2} \eta_{k_1}(\xi_1)\tilde{\sigma}_k(\xi)\|_{L^\infty_x(L^\infty_x \cap \mathbb{R}^n)} \leq C(1 + |t|)^{-n/2}. \quad (3.18)\]

On the other hand, using oscillatory integral techniques, we have
\[\mathcal{F}^{-1}_{\xi_1} e^{-it\xi_1^2} e^{-\nu t\xi_1^2} \eta_{k_1}(\xi_1) = \int_{\mathbb{R}} e^{ix_1(\xi_1 + \frac{\xi_1^2}{2\pi} \nu)} e^{-\nu t\eta_{k_1}(\xi_1)} d\xi_1.\]
The result follows.

Remark 3.8 For $A$ integral operators $\nu, t$ where $\phi$ is independent of $c$. Meanwhile, it is easy to see

\[
\int_{\mathbb{R}} |\psi(|\xi|)|d\xi \leq C, \quad \int_{\mathbb{R}} |\psi'(|\xi|)|d\xi \leq C, \quad \int_{\mathbb{R}} |\psi''(|\xi|)|d\xi \leq C.
\]

$C$ is independent of $\nu, t$ and $k_1$. So integrating by part, we obtain

\[
|\mathcal{F}_{\xi_1}^{-1}e^{-it\xi_1^2}e^{-it\xi_1^2}\eta_k(|\xi_1|)| \lesssim (1 + |x_1|)^{-2}.
\]

from (3.18), (3.19), we have

\[
\|\mathcal{F}^{-1}e^{-it|\xi|^2}\eta_k(|\xi|)\|_{L^2_x L^\infty_t L^\infty_T(\mathbb{R}^n)} \leq \left[ \int (1 + |x|)^{-4}dx \right]^{1/2} + \left[ \int_{|x| \leq 4(k_1)T} \langle k_1 \rangle^n ((k_1) + |x|)^{-n}dx \right]^{1/2}.
\]

The result follows. □

Using similar method as in Proposition 3.8 we have

Remark 3.8 For $n \in \mathbb{N}$, we have

\[
\|\Box_k G(t)u_0\|_{L^1_x L^\infty_{t,j\neq i} L^\infty_T(\mathbb{R}^n)} \leq C \begin{cases} 
\langle k_1 \rangle \langle 4T \rangle^{1/2} \|\Box_k u_0\|_{L^1(\mathbb{R}^n)}, & n = 1; \\
\langle k_1 \rangle \ln \langle 4T \rangle \|\Box_k u_0\|_{L^1(\mathbb{R}^n)}, & n = 2; \\
\langle k_1 \rangle \|\Box_k u_0\|_{L^1(\mathbb{R}^n)}, & n \geq 3.
\end{cases}
\]

Next, we consider the estimates between time-space norm and space-time norm for integral operators $A$. Since the semigroup of Ginzburg-Landau equation does not have conjugate symmetry property as Schrödinger equation, we can not apply $TT^*$ argument to obtain some good estimates as those of the Schrödinger equation, see [37].

Proposition 3.9 Let $2 \leq q < \infty$, $q > 4/n$, $\lambda = 0, 1$, we have

\[
\|\Box_k A_r \partial_x f\|_{L^q_x L^\infty_{t,j\neq i} L^\infty_T(\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \langle k_1 \rangle^{\lambda + 1/2} \|\Box_k f\|_{L^q_x L^2_t(\mathbb{R}^+ \times \mathbb{R}^n)},
\]

(3.21)

\[
\|\Box_k A_r \partial_x f\|_{L^q_x L^\infty_{t,j\neq i} L^\infty_T(\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \langle k_1 \rangle^{1/2} \|\Box_k f\|_{L^q_x L^2_t(\mathbb{R}^+ \times \mathbb{R}^n)};
\]

(3.22)

where in (3.21), condition $|k_1| > 4$ is required.

Proof. From (3.12), (2.26), Lemma 2.2 and Minkowski’s inequality we have (3.21), (3.22) hold, as desired. □

Similar to Proposition 3.9 from Proposition 3.8 we have
Proposition 3.10 For $n = 1, 2, T \leq 1$, we have
\[ \|kA_x \partial_x f\|_{L^2_{x_1} L^\infty_{x_2} L^\infty_T(R^n)} \lesssim \langle k \rangle^{3/2} \|k f\|_{L^1_T L^1_R(R^n)}. \] (3.23)

Remark 3.11 For $n \in \mathbb{N}$, we have
\[ \|kA_x \partial_x f\|_{L^1_{x_1} L^\infty_{x_2} L^\infty_T(R^n)} \leq C \left\{ \begin{array}{ll}
\langle k \rangle^2 (4T)^{1/2} \|k f\|_{L^1_T L^2_R(R^n)}, & n = 1; \\
\langle k \rangle^2 \ln(4T) \|k f\|_{L^1_T L^2_R(R^n)}, & n = 2; \\
\langle k \rangle^2 \|k f\|_{L^1_T L^2_R(R^n)}, & n = 3.
\end{array} \right. \]

From Propositions 3.9, 2.6 and 2.7 we can obtain the following derivative interaction estimates:

Lemma 3.12 Let $i = 2, \ldots, n$, we have
\[ \|kA_x \partial_x f\|_{L^\infty_{x_1} L^2_{x_2} L^\infty_{x_2} L^\infty_T(R^n+\mathbb{R}^n)} \leq C \|\partial_x, \partial^{-1}_x \|_{L^1_{x_1} L^2_{x_2} L^\infty_{x_2} L^\infty_T(R^n+\mathbb{R}^n)}; \] (3.24)
\[ \|kA_x \partial_x f\|_{L^\infty_{x_1} L^2_{x_2} L^\infty_{x_2} L^\infty_T(R^n+\mathbb{R}^n)} \leq C \|\partial_x, \partial^{-1}_x \|_{L^1_{x_1} L^2_{x_2} L^\infty_{x_2} L^\infty_T(R^n+\mathbb{R}^n)}; \] (3.25)
\[ \|kA_x \partial_x f\|_{L^\infty_{x_1} L^2_{x_2} L^\infty_{x_2} L^\infty_T(R^n+\mathbb{R}^n)} \leq C \langle k \rangle \langle k \rangle^{1/2} \|k f\|_{L^1_{x_1} L^2_{x_2} L^\infty_{x_2} L^\infty_T(R^n+\mathbb{R}^n)}. \] (3.26)

Since the smooth-effect estimates for Ginzburg-Landau equation (2.30) is almost the same with the Schrödinger equation (see [37]). Follow the same method as [37], we have

Lemma 3.13 Let $\psi : [0, \infty) \rightarrow [0, 1]$ be a smooth bump function satisfying $\psi(x) = 1$ as $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \leq 2$. Denote $\psi_1(\xi) = \psi(\xi_2/2\xi_1), \psi_2(\xi) = 1 - \psi(\xi_2/2\xi_1), \xi \in \mathbb{R}^n$. Then we have for $\sigma \geq 0$,
\[ \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle^\sigma \|\mathcal{F}^{-1}_{\xi_1, \xi_2} \psi_1 \mathcal{F}_{x_1, x_2} k \partial_x A_x f\|_{L^\infty_{x_1} L^2_{x_2} L^\infty_T(R^n+\mathbb{R}^n)} \lesssim \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle^\sigma \|k f\|_{L^1_{x_1} L^2_{x_2} L^\infty_T(R^n+\mathbb{R}^n)}, \] (3.27)

and for $\sigma \geq 1$,
\[ \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle^\sigma \|\mathcal{F}^{-1}_{\xi_1, \xi_2} \psi_2 \mathcal{F}_{x_1, x_2} k \partial_x A_x f\|_{L^\infty_{x_1} L^2_{x_2} L^\infty_T(R^n+\mathbb{R}^n)} \lesssim \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle^\sigma \|k f\|_{L^1_{x_1} L^2_{x_2} L^\infty_T(R^n+\mathbb{R}^n)}. \] (3.28)

4 Global well-posedness results for $n \geq 3$

In this section, we will give the details of the proof of Theorem 1.1. Define
\[ \rho_1(u) = \sum_{i=1}^n \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle^{s-1/2} \|k u\|_{L^\infty_{x_1} L^2_{x_2} L^2_T(R^n+\mathbb{R}^n)} := \sum_{i=1}^n \rho_1^i(u) \]
\[
\rho_2(u) = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n} \|\Box_k u\|_{L^2_{t} L^{\infty}_{x,(x,j) \neq k)}(\mathbb{R}^+ \times \mathbb{R}^n)}
\]
\[
\rho_3(u) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{s-3/2} \|\Box_k u\|_{L^{\infty}_{t} L^{2}_{x} \cap L^{1}_{t} L^{3}_{x}(\mathbb{R}^+ \times \mathbb{R}^n)}
\]

Define resolution space as following:

\[
X_s := \{u \in \mathcal{S}'(\mathbb{R}^+ \times \mathbb{R}^n) : \|u\|_{X_s} := \sum_{l=1}^{s} \sum_{\lambda, \xi} \sum_{j=1}^{n} \rho_l(\partial_{x_j}^\lambda u) \leq \delta_0\}
\]

**Proof of Theorem 1.1:** Using Lemma 4.1, we have for any \(s > 3\), there exist \(\theta, \theta' > 0\) such that

\[
\|u_0\|_{M^s_{2,1}} \leq C \|u_0\|_{1/2}^{\theta'} \|u_0\|_{L^2}^{\theta}.
\]

With the conditions that \(u_0 \in M^s_{2,1}\), \(\|u_0\|_{L^2}\) small enough, we can obtain \(u_0 \in M^s_{2,1}\) and \(\|u_0\|_{M^{3/2}_{2,1}}\) sufficiently small.

We only prove the result for the case \(s = 3\), we write \(X_3 = X\) for short. Considering the following mapping:

\[
\mathcal{F} : u(t) \rightarrow G_\nu(t) u_0 - i \mathcal{A}[\lambda_1 \cdot \nabla (|u|^2 u) + (\lambda_2 \cdot \nabla u)|u|^2 + \alpha |u|^{26} u],
\]

from (2.26), (3.12), (3.3) and Lemma 2.2 we have

\[
\begin{align*}
\rho_1(\partial_{x_j}^\lambda G_\nu(t) u_0) & \leq \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n, |k|| > 4} \langle k_i \rangle^2 \langle k_j \rangle^\lambda \|\Box_k u_0\|_{L^2(\mathbb{R}^n)} \lesssim \|u_0\|_{M^3_{2,1}}, \\
\rho_2(\partial_{x_j}^\lambda G_\nu(t) u_0) & \leq \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n} \langle k_i \rangle^{1/2} \langle k_j \rangle^\lambda \|\Box_k u_0\|_{L^2(\mathbb{R}^n)} \lesssim \|u_0\|_{M^{3/2}_{2,1}}, \\
\rho_3(\partial_{x_j}^\lambda G_\nu(t) u_0) & \leq \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \langle k_j \rangle^\lambda \|\Box_k u_0\|_{L^2(\mathbb{R}^n)} \lesssim \|u_0\|_{M^{5/2}_{2,1}}.
\end{align*}
\]

So, we obtain that

\[
\|G_\nu(t) u_0\|_X \lesssim \|u_0\|_{M^3_{2,1}}.
\]

For the estimate of the nonlinear terms, noticing that

\[
\lambda_1 \cdot \nabla (|u|^2 u) + (\lambda_2 \cdot \nabla u)|u|^2 = \sum_{i=1}^{n} (\lambda_1^i (\partial_{x_i} \tilde{u}) u^2 + (2 \lambda_1^i + \lambda_2^i) (\partial_{x_i} u) \tilde{u} u),
\]

and \(\|u\|_X = \|\tilde{u}\|_X\), we only need to estimate \(\|\mathcal{A}_\nu(\lambda_1^i (\partial_{x_i} \tilde{u}) u^2)\|_X\) and \(\|\mathcal{A}_\nu(\alpha |u|^{25} u)\|_X\).

**Lemma 4.1** ([Hö], Lemma 7.2) Let \(s \geq 0, p \geq 1, \ p_i, \gamma_i \leq \infty\) satisfy

\[
\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_N}, \quad \frac{1}{\gamma} = \frac{1}{\gamma_1} + \ldots + \frac{1}{\gamma_N},
\]

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then we have
\[
\sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \| \Box_k (u_1 \ldots u_N) \|_{L_x^2 L_t^{6}([R^1+n])} \lesssim \prod_{i=1}^{N} \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \| \Box_k u_i \|_{L_x^2 L_t^{6}([R^1+n])} \right). \tag{4.4}
\]

From (3.21), Lemma 4.1 and Hölder’s inequality we have
\[
\rho_2 (\partial^\lambda x_j (\mathcal{A}_\nu (\lambda_1^i (\partial_x \bar{u}) u^2))) \lesssim \sum_{l=1}^{n} \sum_{k \in \mathbb{Z}^n} \| \Box_k \partial^\lambda x_j (\mathcal{A}_\nu (\partial_x \bar{u}) u^2) \|_{L_x^2 L_t^{6}([R^1+n\times R^n])} \lesssim n \| u \|_{L_x^2 L_t^{6}([R^1+n\times R^n])}
\]
\[
\lesssim n \left[ \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k \partial_x \bar{u} \|_{L_x^2 L_t^{6}([R^1+n\times R^n])} \right] \times \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k u \|_{L_x^2 L_t^{6}([R^1+n\times R^n])} \right)^2 \lesssim n \rho_3 (\partial_x \bar{u}) \rho_3 (u)^2 \tag{4.5}
\]

Noticing that \( \| \Box_k u \|_{L_x^2 L_t^{6}([R^1+n\times R^n])} \lesssim \| \Box_k u \|_{L_x^2 L_t^{3} \cap L_x^3 L_t^{6}([R^1+n\times R^n])} \), follow the same process as (4.5), we have
\[
\rho_2 (\partial^\lambda x_j (\mathcal{A}_\nu (\alpha u^{\delta+1} \bar{u}^\delta))) \leq \rho_3 (u)^{2\delta+1} \tag{4.6}
\]

From (4.6), (4.6), we obtain
\[
\rho_2 \left[ \partial^\lambda x_j (\mathcal{A}_\nu (\lambda_1^i (\partial_x \bar{u}) u^2 + \alpha |u|^{2\delta} u)) \right] \lesssim \| u \|_X^3 + \| u \|_X^{2\delta+1}. \tag{4.7}
\]

Next, denote
\[
S_{\ell,1}^{(i)} := \{ (k^{(1)}, \ldots, k^{(\ell+1)}) \in (\mathbb{Z}^n)^{\ell+1} : |k^{(1)}| \vee \ldots \vee |k^{(\ell+1)}| > 4 \},
\]
\[
S_{\ell,2}^{(i)} := \{ (k^{(1)}, \ldots, k^{(\ell+1)}) \in (\mathbb{Z}^n)^{\ell+1} : |k^{(1)}| \vee \ldots \vee |k^{(\ell+1)}| \leq 4 \}.
\]

Using the frequency-uniform decomposition, we have
\[
\prod_{r=1}^{\ell+1} u_r = \sum_{S_{\ell,1}^{(i)}} \Box_k u_1 \ldots \Box_k u_{\ell+1} + \sum_{S_{\ell,2}^{(i)}} \Box_k u_1 \ldots \Box_k u_{\ell+1}. \tag{4.8}
\]

Where we divide \((\mathbb{Z}^n)^{\ell+1}\) into two parts \(S_{\ell,1}^{(i)}\) and \(S_{\ell,2}^{(i)}\) by considering variable \(x_i\). \(S_{\ell,1}^{(i)}\) denotes the high frequency part, so we will apply smooth effect estimates; while \(S_{\ell,2}^{(i)}\) denotes the low frequency part, we will apply Strichartz-type estimates.

Now we estimate \(\rho_1 (\partial^\lambda x_j (\mathcal{A}_\nu (\lambda_1^i (\partial_x \bar{u}) u^2)))\). It suffices to show that
\[
\rho_1 (\partial^\lambda x_j (\mathcal{A}_\nu (\lambda_1^i (\partial_x \bar{u}) u^2))) \lesssim \| u \|_X^3, \quad j = 1, 2.
\]
Then applying the decomposition (4.8), where we consider the variable $x_i$, we leave the details of the proof into Appendix A.

Collecting (A.1)–(A.6), from symmetry, we can obtain that

\[
\sum_{\lambda=0,1} \sum_{j=1}^n \rho_1(\partial_{x_j}^\lambda \mathcal{A}_\nu(\lambda_1^j (\partial_{x_j} \bar{u}) u^2)) \leq \|u\|_X^3. \tag{4.9}
\]

The estimate of $\rho_1(\partial_{x_j}^\lambda \mathcal{A}_\nu(\lambda_1^j (\partial_{x_j} \bar{u}) u^2))$ is similar to the proof in [37], we leave the details of the proof into Appendix A. In (A.3), we will apply

\[
\|\Box_k^{(1)} u \ldots \Box_k^{(2r+1)} u\|_{L^2_x L^2_{t,x}} \leq \|\Box_k^{(1)} u\|_{L^\infty_t L^2_x} \prod_{s=2}^{2r+1} \|\Box_k^{(s)} u\|_{L^2_t L^2_{t,x}} \leq \|k^{(1)} u\|_{L^\infty_t L^2_x} \prod_{s=2}^{2r+1} \|k^{(s)} u\|_{L^2_t L^2_{t,x}}. \tag{4.10}
\]

In (A.3), we will apply

\[
\|\Box_k u\|_{L^p_t L^p_{t,x}} \lesssim \|k u\|_{L^\infty_t L^p_x L^p_t L^p_{t,x}}, \quad p \geq 3. \tag{4.11}
\]

Finally, we consider the estimate of $\rho_3(\partial_{x_j}^\lambda \mathcal{A}_\nu(\lambda_1^j (\partial_{x_j} \bar{u}) u^2))$. From Lemma [2.2] and (3.4) (where we let $\nu = 3, r = 6, q = 2\sigma + 2, \sigma \geq 1$), we can obtain

\[
\|\Box_k \partial_{x_j}^\lambda \mathcal{A}_\nu(\lambda_1^j (\partial_{x_j} \bar{u}) u^2)\|_{L^\infty_t L^2_{t,x} \cap L^4_t L^4_{t,x}} \lesssim \|k^{(1)} \partial_{x_j}^\lambda (\lambda_1^j (\partial_{x_j} \bar{u}) u^2)\|_{L^{2\sigma+2}_{t,x} L^2_{t,x}} \lesssim \langle k_j \rangle^\lambda \|k^{(1)} (\partial_{x_j} \bar{u}) u\|_{L^{2\sigma+2}_{t,x} L^2_{t,x}} \tag{4.12}
\]

Let $\sigma = 1$ in (4.12), applying Young’s inequality we have:

\[
\rho_3(\partial_{x_j}^\lambda \mathcal{A}_\nu(\lambda_1^j (\partial_{x_j} \bar{u}) u^2)) \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \langle k_j \rangle^\lambda \sum_{k^{(1)}, k^{(2)}, k^{(3)}} \|\Box_k (\Box_{k^{(1)}} (\partial_{x_j} \bar{u}) \Box_{k^{(2)}} u \Box_{k^{(3)}} u)\|_{L^{4/3}_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \sum_{k \in \mathbb{Z}^n} \sum_{k^{(1)}, k^{(2)}, k^{(3)}} \|\Box_k (\Box_{k^{(1)}} (\partial_{x_j} \bar{u}) \Box_{k^{(2)}} u \Box_{k^{(3)}} u)\|_{L^{4/3}_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \sum_{j=1}^n \sum_{k \in \mathbb{Z}^n} \langle k_j \rangle^{5/2} \sum_{k^{(1)}, k^{(2)}, k^{(3)}} \|\Box_k (\Box_{k^{(1)}} (\partial_{x_j} \bar{u}) \Box_{k^{(2)}} u \Box_{k^{(3)}} u)\|_{L^{4/3}_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)}
\]

Then applying the decomposition (4.8) (where we consider the variable $x_j$), and obtain

\[
\sum_{k \in \mathbb{Z}^n} \langle k_j \rangle^{5/2} \sum_{k^{(1)}, k^{(2)}, k^{(3)}} \|\Box_k (\Box_{k^{(1)}} (\partial_{x_j} \bar{u}) \Box_{k^{(2)}} u \Box_{k^{(3)}} u)\|_{L^{4/3}_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \sum_{k \in \mathbb{Z}^n, |k_j| > 4} \langle k_j \rangle^{5/2} \sum_{k^{(1)}, k^{(2)}, k^{(3)}} \|\Box_k (\Box_{k^{(1)}} (\partial_{x_j} \bar{u}) \Box_{k^{(2)}} u \Box_{k^{(3)}} u)\|_{L^{4/3}_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)}
\]
\[
+ \sum_{k \in \mathbb{Z}^n, |k_j| > 4} \langle k_j \rangle^{5/2} \sum_{\mathcal{S}_{2,2}^{(j)}} \| \Box_k (\Box_k^{(1)} (\partial_{x_j} \bar{u}) \Box_k^{(2)} u \Box_k^{(3)} u) \|_{L^{4/3}_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)} \\
+ \sum_{k \in \mathbb{Z}^n, |k_j| \leq 4} \langle k_j \rangle^{5/2} \sum_{k^{(1)}, k^{(2)}, k^{(3)} \in \mathbb{Z}^3} \| \Box_k (\Box_k^{(1)} (\partial_{x_j} \bar{u}) \Box_k^{(2)} u \Box_k^{(3)} u) \|_{L^{4/3}_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)} = V + VI + VII.
\]

(4.13)

For the term \( VI, VII \), from definition of \( \mathcal{S}_{2,2}^{(j)} \), we can see \( \langle k_j \rangle \leq C \). Follow the same process of \( (A,3) \), we have

\[
VI + VII \lesssim \sum_{k^{(1)}, k^{(2)}, k^{(3)} \in \mathbb{Z}^n} \| \Box_k^{(1)} (\partial_{x_j} \bar{u}) \Box_k^{(2)} u \Box_k^{(3)} u \|_{L^{4/3}_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)}
\]

\[
\lesssim \sum_{k^{(1)}, k^{(2)}, k^{(3)} \in \mathbb{Z}^n} \| \Box_k^{(1)} (\partial_{x_j} \bar{u}) \|_{L^8_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)} \| \Box_k^{(2)} u \|_{L^4_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)} \| \Box_k^{(3)} u \|_{L^4_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)}
\]

\[
\lesssim \sum_{k^{(1)} \in \mathbb{Z}^n} \langle k^{(1)} \rangle^{3/2} \| \Box_k^{(1)} (\partial_{x_j} \bar{u}) \|_{L^8_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)} \left( \sum_{k^{(2)} \in \mathbb{Z}^n} \langle k^{(2)} \rangle^{3/2} \| \Box_k^{(2)} u \|_{L^4_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)} \right)^2
\]

\[
\lesssim \rho_3 (\| \partial_{x_j} u \|) \rho_3 (u)^2. \tag{4.14}
\]

At the last step of (4.14), we apply

\[
\| \Box_k u \|_{L^p_t L^p_x (\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \| \Box_k u \|_{L^\infty_t L^2_x \cap L^2_t L^4_x (\mathbb{R}^+ \times \mathbb{R}^n)}, \quad p \geq 4. \tag{4.15}
\]

Now, we consider the estimate of \( V \), from Hölder’s inequality and (1.11), we can obtain

\[
\| \Box_k^{(1)} (\partial_{x_j} \bar{u}) \Box_k^{(2)} u \Box_k^{(3)} u \|_{L^{4/3}_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)}
\]

\[
\leq \| \Box_k^{(1)} (\partial_{x_j} \bar{u}) \Box_k^{(2)} u \Box_k^{(3)} u \|^{1/2}_{L^8_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)} \| \Box_k^{(2)} u \|^{1/2}_{L^4_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)}
\]

\[
\leq \| \Box_k^{(1)} (\partial_{x_j} \bar{u}) \|_{L^\infty_{t,x} L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)} \| \Box_k^{(2)} u \|_{L^4_{t,x} (\mathbb{R}^+ \times \mathbb{R}^n)} \| \Box_k^{(3)} u \|_{L^\infty_{t,x} L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)}
\]

\[
\times \| \Box_k^{(2)} u \|_{L^\infty_{t,x} L^2_x \cap L^2_t L^4_x (\mathbb{R}^+ \times \mathbb{R}^n)}. \tag{4.16}
\]

In this way the estimate of \( V \) reduces to the estimate of \( I \) as in (A.2). Since \( |k_j - k_j^{(1)} - k_j^{(2)} - k_j^{(3)}| \leq C \), without loss of generality, we can assume \( |k_j^{(1)}| = \max_{r=1,2,3} |k_r^{(r)}| \), from (4.10) and Hölder’s inequality we have

\[
V \lesssim \sum_{k^{(1)} \in \mathbb{Z}^n, |k_j^{(1)}| > 4} \langle k_j^{(1)} \rangle^{5/2} \| \Box_k^{(1)} (\partial_{x_j} \bar{u}) \|_{L^\infty_{t,x} L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)}
\]

\[
\times \sum_{k^{(2)} \in \mathbb{Z}^n} \left( \| \Box_k^{(2)} u \|_{L^\infty_{t,x} L^4_x (\mathbb{R}^+ \times \mathbb{R}^n)} + \| \Box_k^{(2)} u \|_{L^\infty_{t} L^2_x \cap L^4_x (\mathbb{R}^+ \times \mathbb{R}^n)} \right)
\]

\[
\times \sum_{k^{(3)} \in \mathbb{Z}^n} \left( \| \Box_k^{(3)} u \|_{L^\infty_{t,x} L^4_x (\mathbb{R}^+ \times \mathbb{R}^n)} + \| \Box_k^{(3)} u \|_{L^\infty_{t} L^2_x \cap L^4_x (\mathbb{R}^+ \times \mathbb{R}^n)} \right)
\]

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\[ \lesssim \rho_1^2(\partial_{x_t} u)(\rho_2(u) + \rho_3(u))^2. \] (4.17)

The estimate for \( \rho_3(\partial^j_x \mathcal{A}(\alpha |u|^{2j} u)) \) is similar to \( \rho_3(\partial^j_x \mathcal{A}((\partial_{x_t} u)u^2)) \), the difference is that in (4.10), we will use

\[ \| \mathcal{A}(u) \|_{L^\infty_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)} \leq \| \mathcal{A}(u) \|_{L^\infty_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)} \]

In addition to (4.15) and (4.10), we can obtain the estimate of \( \rho_3(\partial^j_x \mathcal{A}(\alpha |u|^{2j} u)) \). Until now, we have obtain

\[ \rho_3(\partial^j_x \mathcal{A}(\alpha |u|^{2j} u) + \rho_3(\partial^j_x \mathcal{A}(\lambda \partial_{x_t} u^2))) \lesssim \| u \|_{L^\infty_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)} \]

Collecting (4.7), (4.9), (4.18)

\[ \mathcal{A}(u) \|_{L^\infty_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \| u \|_{M_{2,1}} + n(\| u \|_{L^3_t L^\infty_x (\mathbb{R}^+ \times \mathbb{R}^n)} + \| u \|_{L^{2j+1}_t L^\infty_x (\mathbb{R}^+ \times \mathbb{R}^n)}). \] (4.19)

Using standard contraction mapping argument, we can obtain that Eq. (1.1) has a unique solution \( u \in X \) with \( \| u \|_{X} \leq C\| u_0 \|_{M_{2,1}}^3 \).

Finally, for the general case \( s > 3 \), using similar way as in the above, we have

\[ \| u \|_{X_s} \lesssim \| u_0 \|_{M_{2,1}^3} + \| u \|_{X_s} \lesssim \| u \|_{L^3_t L^\infty_x (\mathbb{R}^+ \times \mathbb{R}^n)} \]

Since in the right hand side of (4.20), Using the fact that \( \| u \|_{X_3} \) is sufficiently small, we can get that

\[ \| u \|_{X_s} \lesssim \| u_0 \|_{M_{2,1}^3}. \]

Finally, we show that \( u \in L^\infty_t (\mathbb{R}^+; M_{2,1}^s (\mathbb{R}^n)). \) From Proposition 2.6 and Proposition 3.9, we have

\[ \sum_{k \in \mathbb{Z}^n} (k)^s \| \mathcal{A}(\lambda_1 \cdot \nabla |u|^2 u) + (\lambda_2 \cdot \nabla u)|u|^2 \|_{L^\infty_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)} \]

\[ \lesssim \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}^n, |k_j| > 4} \langle k_j \rangle^{s-1/2} \sum_{S_{2,1}^{(1)}} \| \mathcal{A}(\mathcal{A}(\lambda_1 \cdot \nabla |u|^2 u)|u|^2 \|_{L^\infty_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)} \]

\[ + \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}^n, |k_j| > 4} \sum_{S_{2,2}^{(1)}} \| \mathcal{A}(\mathcal{A}(\lambda_1 \cdot \nabla |u|^2 u)|u|^2 \|_{L^\infty_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)} \]

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\[ + \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}^n, |k| \leq 4} \langle k_j \rangle \| \Box_k (\Box_{k(1)} \partial_x \bar{u} \Box_{k(2)} u \Box_{k(3)} u) \|_{L^1_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)}. \] (4.22)

From (4.22) and the estimate of \( \rho_1 \) and part IIIV in \( \rho_3 \) above, we obtain
\[ \sum_{k \in \mathbb{Z}^n} \langle k \rangle \| \Box_k u \|_{L^\infty_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \| u_0 \|_{M^2 \lambda} + \| u \|_{X^{2,\lambda}}^3 + \| u \|_{X^2}^{2\lambda+1}, \] (4.23)
which implies that \( u \in L^\infty_t (\mathbb{R}^+; M^2 \lambda (\mathbb{R}^n)). \)

5 Limit behavior as \( \nu \to 0 \)

In this section, we will prove Theorem 1.2. Letting initial data \( u_0 \) belong to \( M^2 \lambda \) and \( \| u_0 \|_{L^2} \) small enough, we prove that the solution of derivative Ginzburg-Landau equation (1.1) will converge to that of derivative Schrödinger equation (1.2) as \( \nu \to 0 \).

Let \( S(t) = \mathcal{F}^{-1} e^{-it|\xi|^2} \mathcal{F} \) denote the semi-group of derivative Schrödinger equation and \( \mathcal{L} f(t, x) = \int_0^t S(t - \tau) f(\tau, x) d\tau \). Rewrite DCGL equation (1.1) as
\[ u = S(t)u_0 + \mathcal{L}[\lambda_1 \cdot \nabla (|u|^2 u) + (\lambda_2 \cdot \nabla u)|u|^2 + \alpha |u|^{2\delta} u] + \nu \mathcal{L}(\Delta u). \] (5.1)

Then define:
\[ \rho^1_T(u) = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n, |k_i| > 4} \langle k_i \rangle \| \Box_k u \|_{L^\infty_t L^2_x (\mathbb{R}^+ \times \mathbb{R}^n)} \],
\[ \rho^2_T(u) = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n} \| \Box_k u \|_{L^2_t L^\infty_x (\mathbb{R}^+ \times \mathbb{R}^n)} \],
\[ \rho^3_T(u) = \sum_{k \in \mathbb{Z}^n} \| \Box_k u \|_{L^\infty_t L^2_x \cap L^2_t L^\infty_x (\mathbb{R}^n)} \].

\[ \| u \|_{Y_T} := \sum_{\lambda=0,1} \sum_{j=1}^{n} (\rho^1_\lambda (\partial_{\lambda j}^1 u) + \rho^2_\lambda (\partial_{\lambda j}^2 u) + \rho^3_\lambda (\partial_{\lambda j}^3 u)) \] (5.2)

\[ \lambda_1 \cdot \nabla (|u|^2 u) + (\lambda_2 \cdot \nabla u)|u|^2 = \sum_{i=1}^{n} [\lambda^1_i (\partial_{x_i} \bar{u} u) u^2 + (2\lambda^1_i + \lambda^2_i) (\partial_{x_i} u) u \bar{u}]. \]

Denote \( v \) is the solution of derivative nonlinear Schrödinger equation (1.2) with the same initial data. Combining the method in [15], we only need to estimate the following
\[ \| u_{\nu} - v \|_{Y_T} \]
\[ \lesssim \sum_{i=1}^{n} \| \mathcal{L} \lambda^1_i ((\partial_{x_i} \bar{u}_\nu) u^2 - (\partial_{x_i} \bar{v}) v^2) \|_{Y_T} + \sum_{i=1}^{n} \| \mathcal{L} (2\lambda^1_i + \lambda^2_i) ((\partial_{x_i} u_{\nu}) u_{\nu}^2 - (\partial_{x_i} v) v^2) \|_{Y_T} \]
\[ + \| \mathcal{L} (\bar{u}^\sigma_{\nu} u^{\sigma+1} - \bar{v}^{\sigma+1}) \|_{Y_T} + \nu \| \mathcal{L} \Delta u_{\nu} \|_{Y_T}. \] (5.3)
Similar to the method in [15], take the first and third term in (5.3) for example. The second term can be treated in similar way.

\[
(\partial_x, \bar{u}_\nu) u^2_\nu - (\partial_x, v)v^2 = \partial_x(\bar{u}_\nu - \bar{v}) u^2_\nu + \partial_x v(u_\nu - v)(u_\nu + v),
\]

(5.4)

\[
\bar{u}^2_\nu u^\sigma - \bar{v}^\sigma v^\sigma + 1 = \bar{u}^\sigma_\nu (u_\nu - v) \sum_{q=0}^{\sigma} u^q_\nu v^{\sigma-q} + v^{\sigma+1}(\bar{u}_\nu - \bar{v}) \sum_{q=0}^{\sigma-1} \bar{u}^q_\nu v^{\sigma-1-q}.
\]

(5.5)

Using the decomposition in (1.8) and combine the proof in Section 4, we only need to substitute \(\partial_x, \bar{u}\) with \(\partial_x, (\bar{u}_\nu - \bar{v})\) in the proof of \(\|\mathcal{L}^1_1(\partial_x, \bar{u}) u^2\|_X\), then we have

\[
\|\mathcal{L}^1_1(\partial_x, \bar{u}_\nu - \bar{v}) u^2\|_Y_T \leq C\|u_\nu - v\|_Y_T \|u_\nu\|_X^2.
\]

Then substitute \(u^2\) with \((u_\nu - v)(u_\nu + v)\), we have

\[
\|\mathcal{L}^1_1(\partial_x, \bar{v}(u_\nu - v))(u_\nu + v)\|_Y_T \leq C\|u_\nu - v\|_Y_T \|u_\nu\|_X \|v\|_X + \|u_\nu\|_X.
\]

Repeat the argument in (4.2), we have \(\|u_\nu\|_X, \|v\|_X\) are sufficiently small. Then we have

\[
\|\mathcal{L}^1_1(\partial_x, \bar{u}) u^2_\nu - (\partial_x, \bar{v}) v^2\|_Y_T \lesssim \frac{1}{10} \|u_\nu - v\|_Y_T.
\]

(5.6)

Similarly, in the estimate of \(\|\alpha u^{\delta+1} v\|_Y\), we will substitute \(u^{\delta+1} v\) with \(\bar{u}_\nu^{\delta}(u_\nu - v)v^{\delta-q}\) and \(v^{\delta+1}(\bar{u}_\nu - \bar{v})\bar{v}^{\delta-1-q}\), we have

\[
\|\mathcal{L}(\bar{u}_\nu^{\delta}(u_\nu - v)v^{\delta-q} - \bar{v}^{\delta+1}(\bar{u}_\nu - \bar{v})\bar{v}^{\delta-1-q})\|_Y_T \leq C\|u_\nu - v\|_Y_T \||u_\nu\|^2_1 \|v\|_X^{\delta-1-q} + \|u_\nu\|_X \|v\|_X^{2\delta-q}
\]

\[
\lesssim \frac{1}{10} \|u_\nu - u\|_Y_T.
\]

Moving the first three term in the right of (5.3) to the left, then from the definition of \(Y_T\) we obtain

\[
\|u_\nu - v\|_Y_T \leq C\nu\|\mathcal{L}_\Delta u\|_Y_T
\]

\[
\lesssim \sum_{\lambda=0,1} \sum_{j=1}^n \sum_{i=1}^n \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k_i \rangle \|\partial^\lambda_{x_j} \square_k (\mathcal{L} \Delta u)\|_{L^\infty_t L^2_x} \|L^2_{(0,T) \times \mathbb{R}^n})
\]

\[
+ \sum_{\lambda=0,1} \sum_{j=1}^n \sum_{i=1}^n \sum_{k \in \mathbb{Z}^n} \|\partial^\lambda_{x_j} \square_k (\mathcal{L} \Delta u)\|_{L^\infty_t L^2_x} \|L^\infty_{(0,T) \times \mathbb{R}^n})
\]

\[
+ \sum_{\lambda=0,1} \sum_{j=1}^n \sum_{i=1}^n \|\partial^\lambda_{x_j} \square_k (\mathcal{L} \Delta u)\|_{L^\infty_t L^2_x \cap L^1_{(0,T) \times \mathbb{R}^n})
\]

\[
:= A_1 + A_2 + A_3.
\]

(5.7)

Similar to [15], applying Minkowski’s inequality and (5.3), (5.12) and (2.20), we can obtain

\[
A_1 \lesssim \nu T \sum_{\lambda=0,1} \sum_{j=1}^n \sum_{i=1}^n \sum_{k \in \mathbb{Z}^n} \langle k_i \rangle^{1/2} (k)^2 \|\square_k (\partial^\lambda_{x_j} u)\|_{L^\infty_t L^2_{\mathbb{R}^n})},
\]

(5.8)
$$A_2 \lesssim \nu T \sum_{\lambda=0,1} \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}^n} \langle k \rangle^2 \langle k \rangle^{1/2} \| \Box_k (\partial^\lambda_{x_j} u) \|_{L_T^\infty L_2^\infty (\mathbb{R}^n)}.$$  
(5.9)

$$A_3 \lesssim \nu T \sum_{\lambda=0,1} \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\lambda} \| \Box_k (\partial^\lambda_{x_j} u) \|_{L_T^\infty L_2^\infty (\mathbb{R}^n)}.$$  
(5.10)

From the argument in (4.20), and initial data belong to $M_{2,1}^4$, we have

$$\sum_{\lambda=0,1} \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k (\partial^\lambda_{x_j} u) \|_{L_T^\infty L_2^\infty (\mathbb{R}^n)} \lesssim C.$$  

Collection (5.7)–(5.10), we finally obtain

$$\| u_\nu - v \|_{Y_T} \lesssim \nu T \sum_{\lambda=0,1} \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k (\partial^\lambda_{x_j} u) \|_{L_T^\infty L_2^\infty (\mathbb{R}^n)}.$$  
(5.11)

In this way, we obtain the results of limit behavior

$$\| u_\nu - v \|_{Y_T} \to 0, \quad \nu \to 0.$$  

6 Local well-posedness results for $n = 1, 2$

When $n = 1, 2, T \leq 1$, define

$$\rho^T_1(u) = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n, \langle k \rangle > 4} \langle k \rangle^{s-1/2} \| \Box_k u \|_{L_T^\infty L_2^2 (\mathbb{R}^n)} : \sum_{i=1}^{n} \rho_i^1(u)$$

$$\rho^T_2(u) = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n} \| \Box_k u \|_{L_T^\infty L_2^2 (\mathbb{R}^n)}$$

$$\rho^T_3(u) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{s-1} \| \Box_k u \|_{L_T^\infty L_2^2 (\mathbb{R}^n)}.$$  

Define resolution space as following:

$$X^T_s := \{ u \in \mathcal{S}'([0,T] \times \mathbb{R}^n) : \| u \|_{X^T_s} := \sum_{i=1}^{3} \sum_{\lambda=0,1} \sum_{j=1}^{n} \rho^T_i (\partial^\lambda_{x_j} u) \leq \delta_0 \}.$$  
(6.1)

We write $X^T_{5/2}$ as $X^T$ for short.

Similar to the proof of global well-posedness results, we only need to consider the case $u_0 \in M^s_{1,1}, s = 5/2$ is small enough and we have:

$$\rho^T_2 (\partial^\lambda_{x_j} G_\nu (t) u_0) \lesssim \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \| \Box_k u_0 \|_{L_1(\mathbb{R}^n)} \lesssim \| u_0 \|_{M^s_{1,1}}$$
\[\rho_2^T (\partial^\lambda \phi, G_v(t)u_0) \text{ and } \rho_3^T (\partial^\lambda \phi, G_v(t)u_0) \text{ are similar to section } 4. \text{ So we have} \]
\[\|G_v(t)u_0\|_{X^T} \lesssim \|u_0\|_{M_{1,1}^{s/2}} \quad (6.2)\]

Notice for any \(p \geq 1, q \geq 2\), we have
\[\|\Box_k u\|_{L^p_t L^q_k(\mathbb{R}^n)} \lesssim T^{1/p}\|\Box_k u\|_{L^p_t L^q_k(\mathbb{R}^n)} \quad (6.3)\]

From \((3.23)\), Lemma \(4.1\) and Hölder’s inequality we have
\[\rho_2^T (\partial^\lambda \phi, (\mathcal{A}(\lambda_1(\partial_x, u))^2)) \]
\[\lesssim n \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\lambda + 3/2 - 1/2} \sum_{k^{(1)}, k^{(2)}, k^{(3)}} \|\Box_k (\Box_{k^{(1)}} \partial_x, \Box_{k^{(2)}} u \Box_{k^{(3)}} u)\|_{L^2_t L^2_x(\mathbb{R}^n)} \]
\[+ n \sum_{k \in \mathbb{Z}^n, |k| \leq 4} \langle k \rangle^{5/2} \sum_{k^{(1)}, k^{(2)}, k^{(3)}} \|\Box_k (\Box_{k^{(1)}} \partial_x, \Box_{k^{(2)}} u \Box_{k^{(3)}} u)\|_{L^4_t L^4_x(\mathbb{R}^n)} \]
\[\lesssim \rho_1^T (\partial_x u) \rho_3^T (u)^2 + T \rho_3^T (\partial_x u) \rho_3^T (u)^2. \quad (6.5)\]

\[\rho_1^T (\partial^\lambda \phi, (\mathcal{A}(\lambda_1(\partial_x, u))^2)) \text{ is similar to section } 4, \text{ we omit the detail. The estimate for} \]
\[\|\mathcal{A}(\alpha |u|^{2\delta}) u\|_{X^T} \text{ is similar, we do not repeat here.} \]

From the above, we obtain for any \(T \leq 1\),
\[\|u\|_{X^T} \lesssim (1 + T)\|u_0\|_{M_{1,1}^{s/2}} + (1 + T)(\|u\|_{X^T}^3 + \|u\|_{X^T}^{2\delta + 1}). \quad (6.6)\]

Using the small initial data which is independent of \(T\), we have \(u \in X_T\) and \(\|u\|_{X_T}\) small enough. Similar to \((4.20)\), we obtain that
\[\|u\|_{X_T} \lesssim \|u_0\|_{M_{1,1}^{s}} + (1 + T)\|u\|_{X^T}^2 (\|u\|_{X_T}^2 + \|u\|_{X_T}^{2\delta + 1}), \quad s > 5/2. \]

Now, we show that \(u \in L^\infty_t (M_{1,1}^{s-1/2}(\mathbb{R}^n))\). Similar to the estimate of \(\rho_3\) in Section 4, we also have
\[\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{s - 1/2} \|\Box_k \lambda_1 \cdot \nabla (|u|^2 u) + (\lambda_2 \cdot \nabla u)|u|^2\|_{L^2_t L^2_x(\mathbb{R}^n)} \]

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In this section, we will prove local well-posedness results for equation

\[ u_t = (\nu + i) \Delta u + \vec{A} \cdot \nabla (u^2), \quad u(0, x) = u_0(x), \]  

(7.1)

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as in Section 5.

We obtain the local well-posedness results. The limit behavior results are almost the same as in Section 5.

\[ \| k \| L^2 \leq k \| L^2 \]

(6.7)

where we use

\[ \| k \| L^2 \leq k \| L^2 \]

(6.8)

From Proposition 5.3 and Minkowski’s inequality, we have

\[ \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{s-1/2} \| \mathcal{D}_k u \| L^2 \leq \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{s-1/2} \| \mathcal{D}_k G_{\nu}(t) u_0 \| L^2 \]

\[ + \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{s-1/2} \| \mathcal{D}_k \vec{A} \cdot \nabla \| L^2 \leq \| u_0 \| L^{2,1,1 \rightarrow 2}_{\mathbb{Z}_2^1} + \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{s-1/2} \| \mathcal{D}_k \vec{A} \cdot \nabla \| L^2 \]

(6.9)

We obtain the local well-posedness results. The limit behavior results are almost the same as in Section 5.

\[ \text{7 Local well-posedness for the quadratic DNLS} \]
and equation
\[ v_t = i \nabla \nu + \nabla (v^2), \quad v(0, x) = v_0(x). \] (7.2)

When \( n \in \mathbb{N}, T \leq 1 \), define
\[
\rho^T_1(u) = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle^{s-1} ||\Box_k u||_{L^2_x L^\infty_t \mathbb{R}^n}.
\]
\[
\rho^T_2(u) = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n} ||\Box_k u||_{L^1_x L^\infty_t \mathbb{R}^n},
\]
\[
\rho^T_3(u) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{s-1} ||\Box_k u||_{L^2_x \mathbb{R}^n}.
\]

Define resolution space as following:
\[
\tilde{X}_T^s := \{ u \in \mathcal{S}'([0, T] \times \mathbb{R}^n) : ||u||_{\tilde{X}_T} := \sum_{l=1}^{n} \sum_{j=1}^{n} \rho^T_l(u) \leq \delta_0 \}. \] (7.3)

We write \( \tilde{X}_T^s \) as \( \tilde{X}_T \) for short.

We solve equation (7.1) first. Similar to the proof of global well-posedness results, we only need to consider the case \( u_0 \in M_{1,1}^3, s = 3 \) is small enough. Similar to section 4, We have
\[
||G_\nu(t)u_0||_{\tilde{X}_T} \lesssim ||u_0||_{M_{1,1}^3}.
\] (7.4)

From (3.23), Lemma 4.1 and Hölder’s inequality we have
\[
\rho^T_2(\mathcal{A}(\partial_j u^2)) \lesssim \sum_{l=1}^{n} \sum_{k \in \mathbb{Z}^n} \langle k \rangle \langle k_1 \rangle \langle k_2 \rangle ||\Box_k u^2||_{L^2_t L^1_x \mathbb{R}^n}
\]
\[
\lesssim n \left[ \sum_{k \in \mathbb{Z}^n} \langle k \rangle^2 ||\Box_k u||_{L^2_x L^2_t \mathbb{R}^n} \right]^2
\]
\[
\lesssim nT \rho^T_3(u)^2.
\] (7.5)

Similar to (A.1),
\[
\rho^T_1(\mathcal{A}(\partial_j u^2))
\]
\[
\lesssim \sum_{k \in \mathbb{Z}^n, |k| > 4} \langle k \rangle^{5/2} \sum_{S^{(1)}_{2,1}} ||\Box_k (\Box_{k(1)} u \Box_{k(2)} u)||_{L^1_{1,1} L^2_{2,2} \cdots L^2_{n} \mathbb{R}^n}
\]
\[
+ \sum_{k \in \mathbb{Z}^n, |k| \leq 4} \langle k \rangle^{7/2} \sum_{S^{(1)}_{2,2}} ||\Box_k (\Box_{k(1)} u \Box_{k(2)} u)||_{L^1_{1,1} L^2_{2,2} \mathbb{R}^n}
\]
\[
:= I + II.
\] (7.6)
Then as the estimate of (6.7) and (6.9) in Section 6, we have

\[ \|\Box_k u\|_{L^2_t L^\infty_{x_1,\ldots,x_n} L^\infty_x (\mathbb{R}^n)} \leq \|\Box_k u\|_{L^1_t L^\infty_{x_1,\ldots,x_n} L^\infty_x (\mathbb{R}^n)}. \]  

(7.7)

Similar to (A.2), (A.3), we have

\[ I \leq \rho^T(u)\rho^T_2(u), \quad II \leq \rho^T_2(u)^2. \]  

(7.8)

So, \( \rho^T(u) (\mathcal{A}(\partial_x u^2)) \leq \rho^T_1(u)\rho^T_2(u) + \rho^T_2(u)^2 \). We estimate \( \rho^T_3(\mathcal{A}(\partial_x u^2)) \) via a similar way as in (6.5):

\[ \rho^T_3(\mathcal{A}(\partial_x u^2)) \]

\[ \lesssim n \sum_{k \in \mathbb{Z}^n \mid |k| > 4} (k_j)^3 \sum_{k(1), k(2), k(3)} \|\Box_k (\Box_{k(1)} u \Box_{k(2)} u)\|_{L^1_t L^2_{x_j} L^2_x (\mathbb{R}^n)} + n \sum_{k \in \mathbb{Z}^n \mid |k| \leq 4} (k_j)^{7/2} \sum_{k(1), k(2), k(3)} \|\Box_k (\Box_{k(1)} u \Box_{k(2)} u)\|_{L^4_{t} L^{4/3}_x (\mathbb{R}^n)} \]

\[ \lesssim \rho^T_1(u)\rho^T_2(u) + T \rho^T_3(u)^2. \]  

(7.9)

So we obtain

\[ \|u\|_{\mathcal{X}_T^3} \leq \|u_0\|_{M^{3}_{1,1}} + (1 + T) \|u\|_{\mathcal{X}_T^3}^2, \]

\[ \|u\|_{\mathcal{X}_T^3} \] is sufficiently small, and also

\[ \|u\|_{\mathcal{X}_T^3} \leq \|u_0\|_{M^{3}_{1,1}} + (1 + T) \|u\|_{\mathcal{X}_T^3} \|u\|_{\mathcal{X}_T^3}. \]

In this way, we can also obtain local well-posedness results for the solution \( v \) of Schrödinger equation (7.2). The inviscid limit for (7.1) is almost the same as section 5. We can obtain

\[ \|u_\nu - v\|_{\mathcal{X}_T^3} \to 0, \quad \nu \to 0. \]  

(7.10)

We omit the detail here.

Via a similar way as in (6.7) and (6.9), we show that \( u \in L^\infty_t (M^{s-1/2}_{1,1}(\mathbb{R}^n)) \), \( s > 3 \). where we use

\[ \|\Box_k (\Box_{k(1)} u \Box_{k(2)} u)\|_{L^1_t L^1_x (\mathbb{R}^n)} \leq \|\Box_{k(1)} u \Box_{k(2)} u\|_{L^{1/2}_t L^2_x (\mathbb{R}^n)} \|\Box_{k(2)} u\|_{L^2_t L^2_x (\mathbb{R}^n)} \]

\[ \lesssim T^{1/2} \|\Box_{k(1)} u\|_{L^\infty_t L^{2/3}_{x_j} L^2_x (\mathbb{R}^n)} \|\Box_{k(2)} u\|_{L^{2/3}_{t} L^{2/3}_x (\mathbb{R}^n)} \|\Box_{k(2)} u\|_{L^2_t L^2_x (\mathbb{R}^n)}. \]

Then as the estimate of (6.7) and (6.9) in Section 6, we have

\[ \|u\|_{K} \lesssim \|u_0\|_{M^{s-1/2}_{1,1}} + \|u\|_{\mathcal{X}_T^3} \|u\|_{\mathcal{X}_T^3}^{1/2} \|u\|_{K}^{1/2}, \]  

(7.11)
where

\[ \| u \|_K := \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{s-1/2} \| \Box_k u \|_{L_1^\infty L_2^1(\mathbb{R}^+ \times \mathbb{R}^n)}. \] (7.12)

Using \( \| u \|_{X_3^T} \) is sufficiently small, we obtain \( u \in K \) and so \( u \in L_1^\infty(M_{1,1}^{s-1/2}(\mathbb{R}^n)) \).
A Appendix

A.1 A nonlinear estimate

In this section, we will show the detail proof of (4.3). We follow the idea in our earlier work [37]. From (2.30), (3.22), Lemma 2.2 and (4.8), (where we put $\ell = 2$), obtained:

$$
\rho_1((\mathcal{A} \lambda_1^\gamma (\partial_u \bar{u} u^2)) \\
\lesssim \sum_{k \in \mathbb{Z}^n, |k| > 4} (k_1)^{\gamma + \frac{3}{2}} \left\| \Box_k (\Box_k (\partial_u \bar{u} \Box_k u) u^2) \right\|_{L^1_t L^2_x \ldots L^2_x((\mathbb{R}^+ \times \mathbb{R}^n)} \\
+ \sum_{k \in \mathbb{Z}^n, |k| > 4} (k_1)^{\gamma + \frac{1}{2}} \left\| \Box_k (\Box_k \partial_u \bar{u} \Box_k u) u^2 \right\|_{L^1_t L_2^x((\mathbb{R}^+ \times \mathbb{R}^n)}
:= I + II
$$

(A.1)

In view of the support property of $\Box_k u$, we can see

$$
\Box_k (\Box_k (\partial_u \bar{u} \Box_k u) u^2) = 0, \quad \text{if} \quad |k - k^{(1)} - k^{(2)} - k^{(3)}| \geq C.
$$

For $I$, since $|k - k^{(1)} - k^{(2)} - k^{(3)}| \leq C$, it is easy to see $|k_1| \leq C \max_{r=1,2,3} |k_1^r|$, we can assume $|k_1^{(1)}| = \max_{r=1,2,3} |k_1^r|$. From Hölder’s inequality:

$$
I = \sum_{k \in \mathbb{Z}^n, |k| > 4} (k_1)^{\gamma + \frac{3}{2}} \left\| \Box_k (\partial_u \bar{u} \Box_k u) u^2 \right\|_{L^1_t L^2_x \ldots L^2_x((\mathbb{R}^+ \times \mathbb{R}^n)} \\
\lesssim \sum_{k^{(1)} \in \mathbb{Z}^n, |k_1^{(1)}| > 4} (k_1^{(1)})^{\gamma + \frac{3}{2}} \left\| \Box_k (\partial_u \bar{u}) u^2 \right\|_{L^2_t L^\infty_x \ldots L^\infty_x((\mathbb{R}^+ \times \mathbb{R}^n)} \\
\times \left( \sum_{k^{(2)} \in \mathbb{Z}^n} \left\| \Box_k u \right\|_{L^2_t L^\infty_x \ldots L^\infty_x((\mathbb{R}^+ \times \mathbb{R}^n)} \right)^2
\lesssim \rho_1(\partial_u \bar{u}) \rho_2(\bar{u})^2.
$$

(A.2)

For $II$, from the definition of $S_{k_2}^{(1)}$, and $|k - k^{(1)} - k^{(2)} - k^{(3)}| \leq C$, we have $|k_1| \leq C$. If we fix $k^{(1)}, k^{(2)}, k^{(3)}$, then $k$ in the sum is finite. So from Holder inequality we have:

$$
II \lesssim \sum_{k^{(1)} \in \mathbb{Z}^n} \left\| \Box_k (\partial_u \bar{u} \Box_k u) u^2 \right\|_{L^1_t L^2_x((\mathbb{R}^+ \times \mathbb{R}^n)} \\
\lesssim \sum_{k^{(1)} \in \mathbb{Z}^n} \left\| \Box_k (\partial_u \bar{u}) \right\|_{L^2_t L^\infty_x((\mathbb{R}^+ \times \mathbb{R}^n)} \left( \sum_{k^{(2)} \in \mathbb{Z}^n} \left\| \Box_k u \right\|_{L^2_t L^\infty_x((\mathbb{R}^+ \times \mathbb{R}^n)} \right)^2
\lesssim \rho_3(\partial_u \bar{u}) \rho_3(\bar{u})^2.
$$

(A.3)

Now we estimate $\rho_1^1((\mathcal{A} \lambda_1^\gamma (\partial_u \bar{u} u^2)))$, $\lambda = 1$ (when $\lambda = 0$ then it is the same to the case $j = 1$). Let $P_i := \mathcal{F}_i^{-1} \psi_1 \mathcal{F}_j x_1, x_2$, where $\psi_1(i = 1, 2)$ be as in Lemma 3.13. From
Lemma 3.13, we have
\[
\rho_1^2 (\partial_{x_2} \mathcal{A}(\lambda_1^2 (\partial_{x_1} \tilde{u}) u^2) \\
\lesssim \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k \rangle^{5/2} \left\| P_{k} \partial_{x_2} \square_k (\mathcal{A}(\lambda_1^2 (\partial_{x_1} \tilde{u}) u^2) \right\|_{L^\infty_1 L^2_{x_2, \ldots, x_n} L^4_2(\mathbb{R}^+ \times \mathbb{R}^n)} \\
+ \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k \rangle^{5/2} \left\| P_{k} \partial_{x_2} \square_k (\mathcal{A}(\lambda_1^2 (\partial_{x_1} \tilde{u}) u^2) \right\|_{L^\infty_1 L^2_{x_2, \ldots, x_n} L^4_2(\mathbb{R}^+ \times \mathbb{R}^n)} \\
: = III + IV. \quad (A.4)
\]

For III, applying decomposition (4.8) (where we consider variable \(x_1\), namely, divide into \(S_{2,1}^{(1)}\) and \(S_{2,2}^{(1)}\) in (4.8)), then from (3.27), and (3.25), we have
\[
III \lesssim \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k \rangle^{5/2} \left\| P_{k} \partial_{x_2} \square_k \mathcal{A} \left( \sum_{k_2} (\square_{k(1)} \lambda_1^2 (\partial_{x_1} \tilde{u}) u \square_{k_2} (\mathcal{A}(\lambda_1^2 (\partial_{x_1} \tilde{u}) u^2)) \right) \right\|_{L^\infty_1 L^2_{x_2, \ldots, x_n} L^4_2(\mathbb{R}^+ \times \mathbb{R}^n)} \\
+ \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k \rangle^{5/2} \left\| P_{k} \partial_{x_2} \square_k \mathcal{A} \left( \sum_{k_2} (\square_{k(1)} \lambda_1^2 (\partial_{x_1} \tilde{u}) u \square_{k_2} (\mathcal{A}(\lambda_1^2 (\partial_{x_1} \tilde{u}) u^2)) \right) \right\|_{L^\infty_1 L^2_{x_2, \ldots, x_n} L^4_2(\mathbb{R}^+ \times \mathbb{R}^n)} \\
\lesssim \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k \rangle^{5/2} \left\| \square_k (\square_{k(1)} \lambda_1^2 (\partial_{x_1} \tilde{u}) u \square_{k_2} (\mathcal{A}(\lambda_1^2 (\partial_{x_1} \tilde{u}) u^2)) \right\|_{L^1_1 L^2_{x_2, \ldots, x_n} L^4_2(\mathbb{R}^+ \times \mathbb{R}^n)} \\
+ \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k \rangle^{5/2} \left\| \square_k (\square_{k(1)} \lambda_1^2 (\partial_{x_1} \tilde{u}) u \square_{k_2} (\mathcal{A}(\lambda_1^2 (\partial_{x_1} \tilde{u}) u^2)) \right\|_{L^1_1 L^2_{x_2, \ldots, x_n} L^4_2(\mathbb{R}^+ \times \mathbb{R}^n)} \\
\lesssim \rho_1(\partial_{x_2} u) \rho_2(u)^2 + \rho_3(\partial_{x_2} u) \rho_3(u)^2. \quad (A.5)
\]

At the last step of (A.5), notice the definition of \(S_{2,2}^{(1)}\), it is easy to see \(|k_2| \lesssim |k_1| \lesssim C\), then it comes back to (A.2), (A.3), so repeat the proof of (A.2), (A.3), we can obtain (A.5), as desired.

For the estimates of IV, applying the decomposition (4.8) (where we consider variable \(x_2\), namely, divide into \(S_{2,1}^{(2)}\) and \(S_{2,2}^{(2)}\) in (4.3)), in addition to (3.28) and (3.29), we have
\[
IV \lesssim \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k \rangle^{5/2} \left\| P_{k} \partial_{x_2} \square_k \mathcal{A} \left( \sum_{k_2} (\square_{k(1)} \lambda_1^2 (\partial_{x_1} \tilde{u}) u \square_{k_2} (\mathcal{A}(\lambda_1^2 (\partial_{x_1} \tilde{u}) u^2)) \right) \right\|_{L^\infty_1 L^2_{x_2, \ldots, x_n} L^4_2(\mathbb{R}^+ \times \mathbb{R}^n)} \\
+ \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k \rangle^{5/2} \left\| P_{k} \partial_{x_2} \square_k \mathcal{A} \left( \sum_{k_2} (\square_{k(1)} \lambda_1^2 (\partial_{x_1} \tilde{u}) u \square_{k_2} (\mathcal{A}(\lambda_1^2 (\partial_{x_1} \tilde{u}) u^2)) \right) \right\|_{L^\infty_1 L^2_{x_2, \ldots, x_n} L^4_2(\mathbb{R}^+ \times \mathbb{R}^n)} \\
\lesssim \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k \rangle^{5/2} \left\| \square_k (\square_{k(1)} \lambda_1^2 (\partial_{x_1} \tilde{u}) u \square_{k_2} (\mathcal{A}(\lambda_1^2 (\partial_{x_1} \tilde{u}) u^2)) \right\|_{L^1_1 L^2_{x_2, \ldots, x_n} L^4_2(\mathbb{R}^+ \times \mathbb{R}^n)} \\
+ \sum_{k \in \mathbb{Z}^n, |k_1| > 4} \langle k \rangle^{5/2} \left\| \square_k (\square_{k(1)} \lambda_1^2 (\partial_{x_1} \tilde{u}) u \square_{k_2} (\mathcal{A}(\lambda_1^2 (\partial_{x_1} \tilde{u}) u^2)) \right\|_{L^1_1 L^2_{x_2, \ldots, x_n} L^4_2(\mathbb{R}^+ \times \mathbb{R}^n)} \\
\lesssim \rho_1^2 (\partial_{x_2} u) \rho_2(u)^2 + \rho_3 (\partial_{x_2} u) \rho_3(u)^2. \quad (A.6)
\]
At the last step of (A.6), notice the definition of $S_{2,2}^{(2)}$, it is easy to see $|k_1| \lesssim |k_2| \leq C$. This way, it comes back to (A.2), (A.3), follow the same process there, we can obtain (A.6), as desired.

A.2 Appendix

Lemma A.1 For any $s \in \mathbb{R}$ and any $s^+ > s$, there exist $\theta > 0$ such that

$$\|f\| M_{2,1}^s \leq C^\theta \|f\|^{1-2\theta} M_{2,1}^s \|f\|^\theta \|L^2, \tag{A.7}$$

where $s^+ = \frac{s+2\theta}{1-2\theta}$.

Proof:

$$\|f\| M_{2,1}^s = \sum_{k \in \mathbb{Z}^n} \|\Box_k f\| H^s \leq \sum_{k \in \mathbb{Z}^n} \|\Box_k f\|^{1-\theta} \|\Box_k f\|^\theta \|L^2 \leq \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{(s+\epsilon)(1-\theta)} \|\Box_k f\|^{1-2\theta} \|\Box_k f\|^{2\theta} \right)^\frac{1}{2} \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle \|\Box_k f\|^{1-\theta} \|\Box_k f\|^{2\theta} \right)^\frac{1}{2} \leq \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{(s+\epsilon + \frac{2\theta}{1-2\theta})} \|\Box_k f\|^{1-2\theta} \|\Box_k f\|^{2\theta} \right)^\frac{1}{2} \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle \|\Box_k f\|^{1-\theta} \|\Box_k f\|^{2\theta} \right)^\frac{1}{2} \leq C^\theta \|f\|^{1-2\theta} M_{2,1}^s \|f\|^\theta \|L^2 \leq C^\theta \frac{\epsilon}{s+\epsilon}, s^+ = \frac{1-\theta}{1-2\theta}(s + \epsilon + \frac{2\theta}{1-2\theta}). \quad \square$$

Lemma A.2 For any $s \in \mathbb{R}$ and any $s^+ > s$, there exist $\theta > 0$ such that

$$\|f\| M_{1,1}^s \leq C^\theta \|f\|^{1-\theta} M_{1,1}^s \|f\|^\theta \|L^2, \tag{A.8}$$

where $s^+ = \frac{s+2\theta}{1-\theta}$.

Proof:

$$\|f\| M_{1,1}^s = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \|\Box_k f\| L^1 \leq \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \|\Box_k f\|^{1-\theta} \|\Box_k f\|^\theta \|L^1 \leq \sup_k \|\Box_k f\|^\theta \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \|\Box_k f\|^{1-\theta} \leq \|f\|^\theta \|L^1 \| \langle k \rangle^{(s+2\theta)} \|\Box_k f\|^{1-\theta} \|L^1 \| \langle k \rangle^{-2\theta} \|L^\frac{1}{\theta} \leq \|f\|^\theta \|L^1 \| \langle k \rangle^{(s+2\theta)} \|\Box_k f\|^{1-\theta} \|L^1 \| \langle k \rangle^{-2\theta} \|L^\frac{1}{\theta}.$$
\[ \leq C^\theta \|f\|^{\theta}_{L^1} \left\| \left\langle k \right\rangle^{\frac{s+2\theta}{1-\theta}} \|\Box_k f\|_{L^1} \right\|^{1-\theta}_{L^1} \]
\[ \leq C^\theta \|f\|^{1-\theta}_{M^{s+2\theta}_{1,1}} \|f\|^\theta_{L^1}, \]
where \( s^+ = \frac{s+2\theta}{1-\theta} \).

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