HÖLDER CONTINUITY FOR STOCHASTIC FRACTIONAL HEAT EQUATION WITH COLORED NOISE

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ABSTRACT. In this paper, we consider semilinear stochastic fractional heat equation \( \frac{\partial}{\partial t}u_{\beta,t}(x) = \triangle^{\alpha/2}u_{\beta,t}(x) + \sigma(u_{\beta,t}(x))\eta_{\beta} \). The Gaussian noise \( \eta_{\beta} \) is assumed to be colored in space with covariance of the form \( E[\eta_{\beta}(t,x)\eta_{\beta}(s,y)] = \delta(t-s)f_{\beta}(x-y) \), where \( f_{\beta} \) is the Riesz kernel \( f_{\beta}(x) \propto |x|^{-\beta} \). We obtain the spatial and temporal Hölder continuity of the mild solution.

1. INTRODUCTION

In this paper, we consider the following stochastic fractional heat equation

\[
\begin{aligned}
\frac{\partial}{\partial t}u_{\beta,t}(x) &= \triangle^{\alpha/2}u_{\beta,t}(x) + \sigma(u_{\beta,t}(x))\eta_{\beta}, \quad t > 0, \ x \in \mathbb{R}, \\
u_{\beta,0}(x) &= \phi(x),
\end{aligned}
\]

where \( 1 < \alpha \leq 2, \triangle^{\alpha/2} := -(-\Delta)^{\alpha/2} \) denotes the fractional Laplacian defined by the Fourier transform

\[
(\mathcal{F}(-\Delta)^{\alpha/2}u)(\xi) = (2\pi|\xi|)^{\alpha}\mathcal{F}(u)(\xi),
\]

here \( \mathcal{F} \) denotes the Fourier transform,

\[
(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x}\varphi(x)dx.
\]

\( \eta_{\beta} \) is the Gaussian space time colored noise with covariance of the form

\[
E[\eta_{\beta}(t,x)\eta_{\beta}(s,y)] = \delta(t-s)f_{\beta}(x-y),
\]

where \( [9], \text{Ex.1} \)

\[
f_{\beta}(x) = c_{1-\beta}g_{\beta}(x) = (\mathcal{F}g_{1-\beta})(x), \quad g_{\beta}(x) = \frac{1}{|x|^{\beta}}, \quad \beta \in (0,1),
\]

and

\[
c_{\beta} = \frac{2\sin(\beta\pi/2)\Gamma(1-\beta)}{(2\pi)^{1-\beta}},
\]

where \( \Gamma(\cdot) \) is the Gamma function.

We assume that the following conditions hold:

(A1) \( \phi \) is bounded and \( \rho \)-Hölder continuous.

(A2) \( \sigma \) is Lipschitz continuous and there exists a constant \( K \) such that \( |\sigma(x) - \sigma(y)| \leq K|x - y| \) and \( |\sigma(x)| \leq K(1 + |x|) \).

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The mild solutions are the solutions of the integral equations

\[ u_{\beta,t}(y) = (u_{\beta,0} \ast p_t)(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_{\beta,s}(x))\eta_\beta(ds, dx). \]  

(1.6)

where the fractional heat kernel \( p_t(x) \) is the fundamental solution of

\[ v_t = \Delta^{\alpha/2} v, \]  

(1.7)

and \( \ast \) denotes the usual convolution operator, \( (f \ast g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy \). Since \( 0 < \beta < 1 \), we can get the existence and uniqueness of the mild solution of (1.1) (see, e.g., [11, 12]). It is known that \( p_t(x) \) satisfies the following inequality ([6, 5, 7])

\[ c_1 t \left( \frac{t}{t^{1/\alpha}} + |x| \right)^{1+\alpha} \leq p_t(x) \leq c_2 t \left( \frac{t}{t^{1/\alpha}} + |x| \right)^{1+\alpha}, \]  

(1.8)

where \( t > 0, x \in \mathbb{R}, c_1 \) and \( c_2 \) are positive constants depending on \( \alpha \).

In the very recent paper [1], Bezdek considered the following equations

\[
\begin{cases}
\frac{\partial}{\partial t} u_{\beta,t}(x) = \frac{\kappa}{2} \Delta u_{\beta,t}(x) + \sigma(u_{\beta,t}(x))\eta_\beta, & t > 0, \ x \in R, \\
u_{\beta,0}(x) = \phi(x),
\end{cases}
\]  

(1.9)

where \( \kappa > 0 \) and \( \eta_\beta \) is the Gaussian noise colored in space and white in time. Stochastic PDEs with colored noise has been studied in many papers (see, e.g., [3, 4, 8, 9]).

Bezdek has obtained the Hölder continuity estimates which take into account \( \beta \) as a variable, the results are novel in that sense. In this paper, based on some estimates of the fractional heat kernel, we will show the spatial and temporal Hölder continuity for the mild solution of stochastic fractional heat equations (1.1).

2. THE SPATIAL AND TEMPORAL HÖLDER CONTINUITY

2.1. SOME LEMMA. In this subsection, we will prove some lemmas, which will be used in next subsections. We use \( C \) to denote generic constant, which may change from line to line.

**Lemma 2.1.** For all \( t > 0 \) and \( x \in \mathbb{R} \),

\[
\int_{\mathbb{R}} |p_t(y-x) - p_t(y)|dy \leq C\left( \frac{|x|}{t^{1/\alpha}} \wedge 1 \right),
\]  

(2.1)

where \( C \) does not depend on \( t \) or \( x \).

**Proof.** For all \( r > 0 \), define

\[
\mu(r) = \mu(r, t) := \sup_{z \in \mathbb{R}, |z| \leq r} \int_{\mathbb{R}} |p_t(y-z) - p_t(y)|dy.
\]  

(2.2)

Then

\[
\mu(|x|) = \sup_{z \in (0, |x|)} \int_{-\infty}^{\infty} \left| \int_{y-z}^{y} \frac{\partial p_t(\xi)}{\partial \xi} d\xi \right|dy.
\]  

(2.3)

By (2.3) of [6] (or Lemma 5 in [7]), we have

\[
\left| \frac{\partial p_t(\xi)}{\partial \xi} \right| \leq C \left( \frac{t}{t^{1/\alpha} + |\xi|^{3+\alpha}} \right).
\]  

(2.4)
where $C$ only depends on $\alpha$. Taking (2.4) into (2.28) to get

$$
\mu(|x|) \leq C|x| \int_{-\infty}^{\infty} \frac{t|w|}{(t^{1/\alpha} + |x|)^{1+\alpha}} d\xi
= C|x| \int_{-\infty}^{\infty} \frac{|\nu|}{(1 + |\nu|)^{1+\alpha}} d\nu
\leq C|x| \frac{t^{1/\alpha}}{t^{1/\alpha}}.
$$

(2.5)

On the other hand, since $|p_t(y - x) - p_t(y)| \leq p_t(y - x) + p_t(y)$ and $\int_R p_t(y) dy = 1$, we have $\mu(|x|) \leq 2$. □

**Lemma 2.2.** For all $t, \varepsilon > 0$, we have

$$
\int_R |p_{t+\varepsilon}(y) - p_t(y)| dy \leq C(\log(t + \varepsilon) - \log(t)) \wedge 1).
$$

(2.6)

**Proof.** From (1.7) and Proposition 2.1 in [2], it is easy to show that

$$
\left| \frac{\partial p_t(y)}{\partial t} \right| \leq \frac{C p_t(y)}{t}.
$$

(2.7)

Then we have

$$
\int_R |p_{t+\varepsilon}(y) - p_t(y)| dy = \int_R \left| \int_t^{t+\varepsilon} \frac{\partial p_s(y)}{\partial s} ds \right| dy \leq C \int_R \int_t^{t+\varepsilon} \frac{p_s(y)}{s} ds dy
= C \int_t^{t+\varepsilon} \frac{1}{s} ds = C(\log(t + \varepsilon) - \log(t)).
$$

(2.8)

On the other hand, we have $\int_R |p_{t+\varepsilon}(y) - p_t(y)| dy \leq 2$. □

**Lemma 2.3.** Let $0 < \rho < 1$ and let $w$ be a bounded $\rho$-Hölder continuous function, then there exists $C > 0$ such that for every $t > 0$, $\delta > 0$, $x \in \mathbb{R}$, $z \in \mathbb{R}$, we have

$$
\left| \int_R (p_t(x - y) - p_t(z - y)) w(y) dy \right| \leq C |x - z|^{\rho},
$$

$$
\left| \int_R (p_{t+\delta}(x - y) - p_t(x - y)) w(y) dy \right| \leq C \delta^{\rho/\alpha}.
$$

**Proof.**

$$
\left| \int_R (p_t(x - y) - p_t(z - y)) w(y) dy \right| = \left| \int_R p_t(z - y) w(y + x - z) dy - \int_R p_t(z - y) w(y) dy \right|
= \left| \int_R p_t(z - y)(w(y + x - z) - w(y)) dy \right| \leq C |x - z|^{\rho} \int_R p_t(z - y) dy = C |x - z|^{\rho}.
$$
By the semigroup property of $p_t$ and note that $\int_R p_\delta(y)dy = \int_R p_t(x-z)dz = 1$,

$$|\int_R (p_{t+\delta}(x-y) - p_t(x-y))w(y)dy| = |\int_R (\int_R p_t(x-z)p_\delta(z-y)dz)w(y)dy - \int_R p_t(x-y)w(y)dy|$$

$$= |\int_R (\int_R p_t(x-z)p_\delta(z)dz)w(z-y)dy - \int_R p_t(x-z)w(z)dz|$$

$$= |\int_R p_\delta(y) (\int_R p_t(x-z)w(z-y)dz)dy - \int_R p_\delta(y) (\int_R p_t(x-z)w(z)dz)dy|$$

(2.9)

$$\leq \int_R p_\delta(y)||y||^\rho dy.$$

From (2.9) and (1.8), it follows that

$$\int_R (p_{t+\delta}(x-y) - p_t(x-y))w(y)dy \leq 2 \int_0^\infty \frac{\delta y^\rho}{(\delta^{1/\alpha} + y)^{1+\alpha}} dy = 2\delta^{\rho/\alpha} \int_0^\infty \frac{y^\rho}{(1+y)^{1+\alpha}} dy. \quad \square$$

2.2. Difference in the spatial variable. For a random variable $X \in L^k(P)$, define $\|X\|_{L^k(P)} = (E(|X|^k))^{1/k}$. For simplicity, we write $\|\cdot\|$ instead of $\|\cdot\|_{L^k(P)}$. We will estimate the spatial and the time difference of the following stochastic integral $I$ in this and next subsection. For $t \in [0,T]$, $x,y,z \in \mathbb{R}$, Define

$$I_{\beta,t}(x) = \int_0^t \int_R p_{t-s}(z-x)\sigma(u_{\beta,s}(z))\eta_\beta(ds,dz),$$

and denote

$$A_s(x,y) = \sigma(u_{\beta,s}(x))\sigma(u_{\beta,s}(y)),$$

$$B_s(r) = p_{t-s}(r-x) - p_{t-s}(r-y).$$

For all $k \geq 2$, the difference in the spatial variable is

$$E(|I_{\beta,t}(x) - I_{\beta,t}(y)|^k) = E\left(\left|\int_0^t \int_R (p_{t-s}(z-x) - p_{t-s}(z-y))\sigma(u_{\beta,s}(z))\eta_\beta(ds,dz)\right|^k\right).$$

**Theorem 2.4.** For all $t \in [0,T]$, $x,y \in \mathbb{R}$,

$$E(|I_{\beta,t}(x) - I_{\beta,t}(y)|^k) \leq C|x-y|^{\frac{\alpha b}{\alpha - b}},$$

where $C$ is a constant, $b \in (0,1 - \frac{1}{\alpha})$, $\alpha \in (1,2]$, $\beta \in (0,1)$.

**Proof.** We apply the Cauchy-Schwarz inequality to bound $E(\sigma(u_{\beta,s}(x))\sigma(u_{\beta,s}(y)))^{k/2}$ by $\sup_{x \in \mathbb{R}} E(\sigma(u_{\beta,s}(x)))^k$. Similar to the proof of Theorem 13 of [9], we can show that $\sup_{x \in [0,T]} \sup_{s \in \mathbb{R}} E(\sigma(u_{\beta,s}(x)))^k < \infty$. Then by (A2), we obtain $\sup_{x \in [0,T]} \sup_{s \in \mathbb{R}} E(\sigma(u_{\beta,s}(x)))^k < \infty$. It is easy to show that $p_{t-s}f$ is positive definite and continuous for all $r > 0$, then $\sup_{z \in \mathbb{R}} (p_{t-s}f_\beta)(z) = (p_{t-s}f)(0)$. Since $(B_{s*}f_\beta)(w) \leq 2 \sup_{z \in \mathbb{R}} (p_{t-s}f)(z)$, we get $\sup_{z \in \mathbb{R}} (p_{t-s}f_\beta)(z) \leq 2(p_{t-s}f)(0)$. By Burkholder inequality ([13], Theorem
5.27), Minkowski integral inequality ([14], Appendix A.1) and Lemma [2.1] we have
\[ E(|I_{\beta,t}(x) - I_{\beta,t}(y)|^k) \]
\[ \leq c_k E\left( \left| \int_0^t \int_{\mathbb{R}} f_\beta(z - w)B_s(z)B_s(w)A_s(z, w)ds dw \right|^k \right) \]
\[ \leq c_k \left| \int_0^t \sup_{x \in \mathbb{R}} |\sigma(u_{\beta,s}(x))| \right|^2 \int_0^t \int_{\mathbb{R}} |f_\beta(z - w)|B_s(z)|B_s(w)|ds dw |k/2 \]
\[ \leq C \left| \int_0^t \int_{\mathbb{R}} |f_\beta(z - w)|B_s(z)|B_s(w)|ds dw \right|^k/2 \]
\[ \leq C \left| \int_0^t (p_{t-s} * f_\beta)(0)ds \int_{\mathbb{R}} |p_{t-s}(z - x) - p_{t-s}(y - x)|dz \right|^k/2 \]
\[ \leq C \left| \int_0^t (p_{t-s} * f_\beta)(0) \left( \frac{|x - y|}{(t-s)^{1/\alpha}} + 1 \right) ds \right|^k/2. \] 

(2.12)

Since \( r \wedge 1 \leq r^{\alpha b} \) for all \( r > 0 \) and \( b \in (0, \frac{1}{\alpha}) \), by (2.12), we have
(2.13)
\[ E(|I_{\beta,t}(x) - I_{\beta,t}(y)|^k) \leq C |x - y|^\frac{\alpha k}{r} \left| \int_0^t (p_{t-s} * f_\beta)(0)(t - s)^{-b}ds \right|^k/2. \]

By (1.8), we have
\[ (p_{t-s} * f_\beta)(0) = c_{1-\beta} \int_\mathbb{R} \frac{1}{|x|^{\beta}}p_{t-s}(x)dx \leq c_{1-\beta} \int_\mathbb{R} \frac{1}{|x|^{\beta}} \cdot \frac{t - s}{((t - s)^{1/\alpha} + |x|)^{1+\alpha}} dx \]
\[ \leq c_{1-\beta} (t - s)^{-\beta/\alpha} \int_\mathbb{R} \frac{1}{r^{\beta}(1 + |r|)^{1+\alpha}} dr \leq C(t - s)^{-\beta/\alpha}. \] 

(2.14)

Put (2.14) into (2.13) to get
(2.15)
\[ E(|I_{\beta,t}(x) - I_{\beta,t}(y)|^k) \leq C |x - y|^\frac{\alpha k}{r} \left| \int_0^t (t - s)^{-\frac{\beta}{\alpha} - b}ds \right|^k/2. \]

Since \( \beta \in (0, 1) \) and \( \alpha \in (1, 2) \), we can choose \( b \in (0, 1 - \frac{1}{\alpha}) \subset (0, \frac{1}{\alpha}) \) to guarantee that \( \left| \int_0^t (t - s)^{-\frac{\beta}{\alpha} - b}ds \right| < \infty \). Therefore we obtain
(2.16)
\[ E(|I_{\beta,t}(x) - I_{\beta,t}(y)|^k) \leq C |x - y|^\frac{\alpha k}{r}, \quad t \in [0, T], \quad x, y \in \mathbb{R}. \]

\[ \square \]

2.3. Difference in the time variable. For all \( k \geq 2 \), the difference in the time variable is
\[ E(|I_{\beta,t+\delta} - I_{\beta,t}|^k) \]
\[ = E\left( \left| \int_0^{t+\delta} \int_{\mathbb{R}} p_{t+\delta-s}(z - x)\sigma(u_{\beta,s}(z))\eta_\beta(ds, dz) - \int_0^t \int_{\mathbb{R}} p_{t-s}(z - x)\sigma(u_{\beta,s}(z))\eta_\beta(ds, dz) \right|^k \right). \]

Theorem 2.5. For all \( \delta > 0 \), \( t \in [0, T] \), \( x, y \in \mathbb{R} \),
\[ E(|I_{\beta,t+\delta}(x) - I_{\beta,t}(x)|^k) \leq C \delta^{\frac{1+\alpha}{2}}, \]
where \( C \) is a constant, \( \alpha \in (1, 2) \), \( \beta \in (0, \frac{1}{2}) \).
Recall the Fourier transform of fractional heat kernel (2.20)

证明. By the elementary inequality \(|a + b|^k \leq 2^k|a|^k + 2^k|b|^k\), we have

\[
E(|I_{t+\delta}(x) - I_{t}(x)|^k)
\leq 2^k E\left(\int_0^{t+\delta} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} (p_{t+\delta-s}(z-x) - p_t(z-x)) \sigma(u_{\beta,s}(z)) \eta_\beta(ds,dz)\right)^k\right) dt\right)

+ 2^k E\left(\int_0^{t+\delta} \left(\int_{\mathbb{R}} \sigma(u_{\beta,s}(z)) \eta_\beta(ds,dz)\right)^k\right)
\]

(2.18) \quad = I_1 + I_2.

For \(I_2\), by the same technique as in the proof of Theorem 2.4, we have

\[
I_2 \leq C \left( \int_t^{t+\delta} \sup_{x \in \mathbb{R}} \|\sigma(u_{\beta,s}(x))\|^2 \int_{\mathbb{R}} \int_{\mathbb{R}} f_\beta(z-w)p_{t+\delta-s}(z-x)p_{t+\delta-s}(w-x) dsdzdw \right)^{k/2}
\]

(2.19)

\[
\leq C \left( \int_t^{t+\delta} \int_{\mathbb{R}} \int_{\mathbb{R}} f_\beta(z-w)p_{t+\delta-s}(z-x)p_{t+\delta-s}(w-x) dsdzdw \right)^{k/2}.
\]

Denote by \(S(\mathbb{R})\) the Schwartz space of rapid decreasing test-functions from \(\mathbb{R}\) to \(\mathbb{R}\), by elementary properties of convolution and Fourier transform, the following holds (see formula (10) in [10]):

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) f_\beta(x-y) \varphi(y) dx dy = \int_{\mathbb{R}} f_\beta(x) (\varphi * \tilde{\psi})(x) dx = \int_{\mathbb{R}} g_{1-\beta}(\xi) |\mathcal{F}\varphi(\xi)|^2 d\xi,
\]

for all \(\varphi, \psi \in S(\mathbb{R})\), where \(\tilde{\psi}\) is defined by \(\tilde{\psi}(x) = \psi(-x)\). Then by change of variables,

(2.20)

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} p_{t+\delta-s}(z-x) f_\beta(z-w)p_{t+\delta-s}(w-x) dz dw = \int_{\mathbb{R}} g_{1-\beta}(\xi) |\mathcal{F}p_{t+\delta-s}(\xi)|^2 d\xi.
\]

Recall the Fourier transform of fractional heat kernel \(p_{t}(x)\) (see formula (3) in [7]) and note (1.2), we have

(2.21)

\[
\mathcal{F}p_{t-s}(\xi) = e^{-(t-s)(2\pi|\xi|)^\alpha}.
\]

Thus,

\[
\int_{\mathbb{R}} g_{1-\beta}(\xi) |\mathcal{F}p_{t-s}(\xi)|^2 d\xi = \int_{\mathbb{R}} g_{1-\beta}(\xi) e^{-(t-s)2^{\alpha+1}\pi^\alpha|\xi|^\alpha} d\xi
\]

\[
= \int_{\mathbb{R}} |\xi|^{\beta-1} e^{-(t-s)2^{\alpha+1}\pi^\alpha|\xi|^\alpha} d\xi
\]

\[
= 2 \int_0^\infty \xi^{\beta-1} e^{-(t-s)2^{\alpha+1}\pi^\alpha \xi^\alpha} d\xi
\]

\[
= 2 \int_0^\infty e^{-(t-s)2^{\alpha+1}\pi^\alpha r^{\frac{\beta}{\alpha}}} r^{\frac{\beta}{\alpha}-1} dr
\]

\[
= \frac{2}{\alpha(2^{\alpha+1}\pi^\alpha (t-s))^{3/\alpha}} \int_0^\infty e^{-z^{\frac{\beta}{\alpha}-1}} dz
\]

(2.22)

\[
= \frac{2\Gamma\left(\frac{\beta}{\alpha}\right)}{\alpha(2^{\alpha+1}\pi^\alpha (t-s))^{3/\alpha}}.
\]
By (2.19), (2.20) and (2.22), we get

\begin{equation}
I_2 \leq C \left( \int_t^{t+\delta} (t + \delta - s)^{-\beta/\alpha} ds \right)^{k/2} = C \delta^{(\alpha-\beta)/2}.
\end{equation}

For $I_1$, by the similar argument and note that Lemma 2.2 and (2.14), we have

\begin{equation}
I_1 \leq C \left( \int_0^t \sup_{x \in \mathbb{R}} \|u_{\beta,s}(x)\|_k^2 (f_{\beta} \ast p_{t-s})(0) \int_{\mathbb{R}} p_{t+\delta-s}(z) - p_{t-s}(z) dz ds \right)^{k/2}.
\end{equation}

By integrating by parts,

\begin{equation}
\int_0^t s^{-\beta/\alpha} (\log(s + \delta) - \log(s)) ds
= \frac{\alpha}{\alpha - \beta} \log(1 + \delta/t)^{(\alpha-\beta)/\alpha} + \frac{\alpha}{\alpha - \beta} \int_0^t s^{(\alpha-\beta)/\alpha} \frac{\delta}{s(s + \delta)} ds
= I_3 + I_4.
\end{equation}

For $I_4$,

\begin{equation}
\frac{\alpha}{\alpha - \beta} \int_0^t s^{(\alpha-\beta)/\alpha} \frac{\delta}{s(s + \delta)} ds
= \left( \frac{\alpha}{\alpha - \beta} \right)^2 \int_0^{(\alpha-\beta)/\alpha} \frac{\delta}{\mu^{(\alpha-\beta)/\alpha} + \delta} d\mu
= \left( \frac{\alpha}{\alpha - \beta} \right)^2 \frac{\delta}{1 + \mu^{(\alpha-\beta)/\alpha}} d\mu
\leq \left( \frac{\alpha}{\alpha - \beta} \right)^2 \frac{\delta}{1 + \mu^{(\alpha-\beta)/\alpha}} d\mu
\leq C \delta^{(\alpha-\beta)/\alpha}.
\end{equation}

Next, we will prove that for any $\mu > 0$, $r \in \left[ \frac{1}{2}, 1 \right]$,

\begin{equation}
0 < \log(1 + \mu) \leq \mu^r.
\end{equation}

For $r = 1$, it is obvious that $0 < \log(1 + \mu) \leq \mu$. In the following, we only consider the case $r \in \left[ \frac{1}{2}, 1 \right]$. For $\mu > 0$, let $h(\mu) = \log(1 + \mu) - \mu^r$. Then

\begin{equation}
h'(\mu) = \frac{\mu^{1-r} - (1 + \mu)r}{(1 + \mu)^{1-r}}.
\end{equation}

Let $l(\mu) = \mu^{1-r} - (1 + \mu)r$. It is easy to get the maximum $l_{\text{max}} = l(\mu)_{\mu=(1+2r)/2} = \frac{r}{r-1} \left[ (\frac{1}{1+r})^r - \frac{1}{1+r} \right]$. Since $r \in \left[ \frac{1}{2}, 1 \right]$, we get $l_{\text{max}} \leq 0$. By (2.28), $h'(\mu) \leq 0$. This together with $h(0) = 0$ yield that $h(\mu) \leq 0$. Therefore (2.27) holds. Since $\beta \in (0, \frac{1}{2}]$, then $\frac{\alpha-\beta}{\alpha} \in \left[ \frac{1}{2}, 1 \right]$. By (2.27), we have

\begin{equation}
0 < \log(1 + \delta/t) \leq (\delta/t)^{(\alpha-\beta)/\alpha}.
\end{equation}

Thus, for $I_3$, we get

\begin{equation}
I_3 \leq C \delta^{(\alpha-\beta)/\alpha}.
\end{equation}

By (2.21), (2.23), (2.20) and (2.30), we have

\begin{equation}
E(\|I_{\beta,t+s}(x) - I_{\beta,t}(x)\|^k) \leq C \delta^{(\alpha-\beta)/2}.
\end{equation}
2.4. Spatial and temporal Hölder continuity.

**Theorem 2.6.** For all $k \geq 2$, $\alpha \in (1, 2]$, $\beta \in (0, \frac{2}{\alpha}]$, $\rho \in (0, 1)$, $b \in (0, 1 - \frac{1}{\alpha})$ and $x, y \in \mathbb{R}$,

$$
E \left( |u_{\beta,t}(x) - u_{\beta,s}(y)|^k \right) \leq C \left( |x - y|^{kc} + |t - s|^{kd} \right),
$$

where $C_1, C_2$ are positive constants, $c \in (0, \frac{\alpha}{2} \wedge \rho)$, $d \in (0, \frac{(\alpha - \beta)}{2\alpha} \wedge \frac{\rho}{\alpha})$.

**Proof.** By Lemma 2.3, Theorem 2.4 and Theorem 2.5, we can draw the conclusion. □

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