Speeding up Glauber Dynamics for Random Generation of Independent Sets

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ABSTRACT
The maximum independent set (MIS) problem is a well-studied combinatorial optimization problem that naturally arises in many applications, such as wireless communication, information theory and statistical mechanics.

MIS problem is NP-hard, thus many results in the literature focus on fast generation of maximal independent sets of high cardinality. One possibility is to combine Gibbs sampling with coupling from the past arguments to detect convergence to the stationary regime. This results in a sampling procedure with time complexity that depends on the mixing time of the Glauber dynamics Markov chain.

We propose an adaptive method for random event generation in the Glauber dynamics that considers only the events that are effective in the coupling from the past scheme, accelerating the convergence time of the Gibbs sampling algorithm.

The full paper is available on arXiv.

Categories and Subject Descriptors
G.2.2 [Discrete Mathematics]: Graph theory—graph algorithms; G.3 [Probability and Statistics]: Markov processes, Probabilistic algorithms (including Monte Carlo)

Keywords
Glauber dynamics; exact sampling; independent sets

1. COUPLING FROM THE PAST

Propp and Wilson [4] introduced in 1996 the coupling from the past (CFTP) algorithm, a perfect sampling technique for the stationary distribution of an ergodic finite state space Markov chain. CFTP generates an unbiased sample from the stationary distribution in finite expected time.

Formally, given a finite state space $S$ of cardinality $n$ and a $n \times n$ transition matrix $M$, a Markov chain over $S$ with transition matrix $M$ can be represented as

$$X_t = X_0 \cdot u_1 \cdot \ldots \cdot u_t,$$

where the $u_i$ are i.i.d. letters from a finite alphabet $A$, drawn according to a distribution $D$ defined such that the operator $\cdot : S \times A \rightarrow S$ satisfies

$$\forall x, y \in S, \quad P_{u \sim D}(x \cdot u = y) = M_{xy}.$$

Consider an i.i.d. sequence of letters $(u_t)_{t \in \mathbb{N}}$ drawn according to $D$, and define the chains $(B_{-i}^t)_{i,t \in \mathbb{N}}$ by

$$B_{-i}^t = \{ x \cdot u_{-i} \cdot \ldots \cdot u_{-t-1} : x \in S \}.$$

Theorem 1 (Propp and Wilson [4]). If there exists a $T \in \mathbb{N}$ such that $B_{-i}^T$ is a singleton, then:

1. For all $-t \leq -T$, $B_{-i}^{-t} = B_{-i}^{-T}$;
2. The unique element of $B_{-i}^{-T}$ is distributed according to the stationary distribution of $M$.

A key element in their proof is that

$$\forall t, \forall i, \quad B_{-i}^{-t-1} \subseteq B_{-i}^{-t} \quad (1)$$

CFTP algorithm consists in drawing the letters $u_i$, one by one until $B_{-i}^{-1}$ is a singleton, then returning the unique element of $B_{-i}^{-1}$. The main drawback is that at each time step $-i$, the algorithm computes $x \cdot u_{-i}$ for all $x \in B_{-i}^{-1}$, which is often prohibitive due to the cardinality of $S$. The complexity of this one step transition can be improved through the use of bounding chains and we assume here the approach in [2].

2. EXPANDING AND CONTRACTING

The complexity of the CFTP algorithm is closely linked to the rate at which the Markov chain changes state: for a given set of states $B$, call a letter $u$ passive if $B = \{ x \cdot u : x \in B \}$.

The main contribution of this paper is to propose a variant of the algorithm that avoids considering the passive letters, while still producing an unbiased sample from the stationary distribution.

Whereas such a modification is easily achievable in a simple Markov chain Monte Carlo dynamic, it is not immediate to adapt it to the CFTP approach. The main reason is that letters passive for $B_{-i}^{-1}$ are not necessarily passive for $B_{-i}^{-T}$. This is illustrated in Figure 1.

The proposed method removes passive letters as it proceeds, while reinserting active letters where they may have been removed due to being passive at some earlier stage of the algorithm. The insertion is done in a randomized manner that preserves the distribution of the output of the algorithm.

We define two operations, contraction and expansion, that remove and add passive letters in a given word. These are illustrated in Figure 2.
The proposed algorithm, called CFTP with oracle skip-
ning (OS-CFTP), is obtained by adapting the CFTP algo-

Contraction is straightforward. Expansion, on the other
hand, is more complex as it should not introduce a bias in
the distribution of the output. The key element here is to
make sure that the inclusion (1) is still valid despite new
letters being introduced. Also, when starting a new chain
from further back in time, the number of new letters must
be geometrically distributed. Due to space constraints, we
do not give all the details of this procedure here.

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Two bounding chains starting at times $u_5$ and $u_7$.
$u_4$ is passive for the former but becomes active for the later.

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Figure 3: Star graph with 1000 vertices

Figure 4: Barabasi-Albert model with 100 vertices

3. GENERATING INDEPENDENT SETS

We illustrate the efficiency of the OS-CFTP algorithm for
independent sets. An independent set in a graph is a subset
of the vertices of that graph, no two of which are linked
by an edge. The target distribution is given by

$$\pi(I) = \frac{\lambda^{\text{card}(I)}}{Z_\lambda},$$

where $\lambda$ is a parameter, called the fugacity, and $Z_\lambda$ is a
normalization constant. It is possible to express $\pi$ as the
stationary distribution of a Markov chain by using a Gibbs
sampler. This method was furthermore refined by Dyer and
Greenhill [1, 2].

Simulation results for different graph models are given in
Figures 3 and 4. Theoretical bounds for the star topology
are given in the full paper. Overall, oracle skipping does
better than previous methods, though the gain is strongly
linked to the model studied.

4. ACKNOWLEDGMENTS

The work presented in this paper has been carried out
at LINC5 (www.lincs.fr) and is supported by the French
National Research Agency grant ANR-12-MONU-0019.

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