POWERS OF $t$-SPREAD PRINCIPAL BOREL IDEALS

CLAUDIA ANDREI, VIVIANA ENE, BAHAREH LAJMIRI

ABSTRACT. We prove that $t$-spread principal Borel ideals are sequentially Cohen-Macaulay and study their powers. We show that these ideals possess the strong persistence property and compute their limit depth.

INTRODUCTION

In this paper, we study $t$-spread principal Borel ideals. They have been recently introduced in [2]. Let $K$ be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring in $n$ variables over $K$. Let $t \geq 0$ be an integer. A monomial $x_{i_1} \cdots x_{i_d} \in S$ with $i_1 \leq \cdots \leq i_d$ is called $t$-spread if $i_j - i_{j-1} \geq t$ for $2 \leq j \leq d$. For example, every monomial in $S$ is 0-spread and every squarefree monomial is 1-spread.

We recall from [2] that a monomial ideal $I \subset S$ with the minimal system of monomial generators $G(I)$ is called $t$-spread strongly stable if it satisfies the following condition: for all $u \in G(I)$ and $j \in \text{supp}(u)$, if $i < j$ and $x_i(u/x_j)$ is $t$-spread, then $x_i(u/x_j) \in I$. A monomial ideal $I \subset S$ is called $t$-spread principal Borel if there exists a monomial $u \in G(I)$ such that $I = B_t(u)$ where $B_t(u)$ denotes the smallest $t$-spread strongly stable ideal which contains $u$. For example, for an integer $d \geq 2$, if $u = x_{n-(d-1)t} \cdots x_{n-t}x_n$, then $B_t(u)$ is minimally generated by all the $t$-spread monomials of degree $d$ in $S$. In this case, we call $B_t(u)$ the $t$-spread Veronese ideal generated in degree $d$ and denote it $I_{n,d,t}$.

Throughout this paper we consider $t$-spread monomial ideals with $t \geq 1$. In particular, they are squarefree monomial ideals. As it was observed in [2], if $u = x_{i_1} \cdots x_{i_d}$ is a $t$-spread monomial in $S$ then a monomial $x_{j_1} \cdots x_{j_d} \in G(B_t(u))$ if and only if the following two conditions hold:

(i) $j_k \leq i_k$ for $1 \leq k \leq d$,
(ii) $j_k - j_{k-1} \geq t$ for $2 \leq k \leq t$.

In [2] Theorem 1.4] it was shown that every $t$-spread strongly stable ideal has linear quotients with respect to the pure lexicographic order in $S$. In addition, in [2 Corollary 2.5] it was shown that a $t$-spread strongly stable ideal generated in a single degree is Cohen-Macaulay if and only if it is a $t$-spread Veronese ideal. In

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particular, it follows that every $t$-spread principal Borel ideal has linear quotients and such an ideal is Cohen-Macaulay if and only if it is $t$-spread Veronese.

Since $B_t(u)$ is a squarefree monomial ideal, we may interpret it as the Stanley-Reisner ideal of a simplicial complex $\Delta$. In the proof of Theorem 1.1 we show that if the generator $R$ ring and such an ideal is Cohen-Macaulay if and only if it is particular, it follows that every $t$-spread principal Borel ideal has linear quotients.

In Section 3.1 we study the limit behavior of the depth for the powers of $B_t(u)$, that is, the Krull dimension of the fiber ring $\mathcal{R}(B_t(u))/m\mathcal{R}(B_t(u))$ where $m = (x_1, \ldots, x_n)$.

1. $t$-spread principal Borel ideals are sequentially Cohen-Macaulay

Let $u = x_{i_1} x_{i_2} \cdots x_{i_d} \in S = K[x_1, \ldots, x_n]$ be a $t$-spread monomial. Suppose that $i_d = n$ and $I = B_t(u)$. For a monomial $u \in I$, we denote \( \text{supp}(u) = \{j \in [n] : x_j \mid u\} \).

Let $\Delta$ be the simplicial complex such that $I_\Delta = I$. We denote $\Delta^\vee$ the Stanley-Reisner ideal of $\Delta$.

**Theorem 1.1.** Let $t \geq 1$ be an integer and $I = B_t(u)$ where $u = x_{i_1} x_{i_2} \cdots x_{i_d}$ is a $t$-spread monomial. We assume that $\bigcup_{v \in \mathcal{G}(I)} \text{supp}(v) = [n]$. Then $I^\vee$ is generated by the monomials of the following forms

\[
(1) \quad \prod_{k=1}^n \frac{x_k}{(v_{j_1} \cdots v_{j_{d-1}})}
\]

with $j_l \leq i_l$ for $1 \leq l \leq d - 1$ and $j_l - j_{l-1} \geq t$ for $2 \leq l \leq d - 1$, where $v_{j_k} = x_{j_k} \cdots x_{j_{k+(t-1)}}$ for $1 \leq k \leq d - 1$.

\[
(2) \quad \prod_{k=1}^{i_1} x_k.
\]

\[
(3) \quad \prod_{k=1}^{i_s} x_k/(v_{j_1} \cdots v_{j_{s-1}})
\]
with \(2 \leq s \leq d - 1\), \(j_l \leq i_l\) for \(1 \leq l \leq s - 1\), \(j_l - j_{l-1} \geq t\) for \(2 \leq l \leq s - 1\), where \(v_{j_k} = x_{j_k} \cdots x_{j_k + (t-1)}\) for \(1 \leq k \leq s - 1\).

Proof. Let \(\Delta\) be the simplicial complex whose Stanley-Reisner ideal is \(I\) and let \(\mathcal{F}(\Delta)\) be the set of the facets of \(\Delta\). We prove that every facet of \(\Delta\) is of one of the following forms:

(i) \(F_1 = \{j_1, j_1 + 1, \ldots, j_1 + (t-1), j_2, j_2 + 1, \ldots, j_2 + (t-1), \ldots, j_{d-1}, j_{d-1} + 1, \ldots, j_{d-1} + (t-1)\}\), for some \(j_1, j_2, \ldots, j_{d-1}\) such that \(j_i \leq i_l\) for \(1 \leq l \leq d - 1\) and \(j_i - j_{i-1} \geq t\) for \(2 \leq l \leq d - 1\).

(ii) \(F_2 = \{i_1 + 1, i_1 + 2, \ldots, n\}\).

(iii) \(F_3 = \{j_1, j_1 + 1, \ldots, j_1 + (t-1), \ldots, j_s, j_s + 1, \ldots, j_s + (t-1), j_s, j_s + 1, \ldots, n\}\), for some \(j_1, j_2, \ldots, j_s\) such that \(2 \leq s \leq d - 1\), \(j_i \leq i_l\) for \(1 \leq l \leq s - 1\), \(j_s = i_s + 1\) and \(j_i - j_{i-1} \geq t\) for \(2 \leq l \leq s\).

Since \(I = B_t(u)\) has the primary decomposition

\[
I = \bigcap_{F \in \mathcal{F}(\Delta)} P_{[n] \setminus F},
\]

where \(P_{[n] \setminus F}\) is the prime ideal generated by all variables \(x_j\) with \(j \in [n] \setminus F\), by [4, Corollary 1.5.5], the statement holds.

Since \((x_1, \ldots, x_i) \in \text{Min}(I)\) by [2, Theorem 2.4], we obtain \(F_2 \in \mathcal{F}(\Delta)\).

We have \(F_1, F_3 \in \Delta\), since \(x_{F_1} = \prod_{i \in F_1} x_i\) and \(x_{F_3} = \prod_{i \in F_3} x_i\) do not belong to \(I\). Indeed, if \(x_{F_1} \in I\), then there exists \(v = x_{k_1} \cdots x_{k_d} \in G(I)\) such that \(v \mid x_{F_1}\).

Since \(v\) is the product of \(d\) distinct variables and \(F_1\) consists of \(d - 1\) intervals of the form \([j_r, j_r + (t-1)]\), \(1 \leq r \leq d - 1\), there exist \(l \in \{2, \ldots, d\}\) and \(1 \leq r \leq d - 1\) such that \(k_{l-1}, k_l \in [j_r, j_r + (t-1)]\). It follows that \(k_l - k_{l-1} < t\), which is false.

If \(x_{F_3} \in I\), then there exists \(v = x_{k_1} \cdots x_{k_d} \in G(I)\) such that \(v \mid x_{F_3}\). Since \(k_l - k_{l-1} \geq t\) for \(2 \leq l \leq s\) and \(F_3\) consists of \(s - 1\) intervals of the form \([j_{s}, j_s + (t-1)]\), \(1 \leq r \leq s - 1\), and only one interval of the form \([j_s, n]\), we have \(k_s \in [j_s, n]\). It follows that \(j_s = i_s + 1 \leq k_s\), which is false because \(v \in G(I)\) and \(k_s \leq i_s\).

We claim that \(F_i \cup \{j\} \notin \Delta\) for every \(j \in [n] \setminus F_i\) and \(i \in \{1, 3\}\). This will prove that every set of the form (i) or (iii) is a facet of \(\Delta\).

Let \(j \in [n] \setminus F_1\). We consider the following cases:

(a) \(j < j_1\). Then \(x_j x_{j_1 + (t-1)} \cdots x_{j_{d-1} + (t-1)} \in I\) because \(j < j_1 \leq i_1\) and \(j_l + (t-1) \leq i_l + (t-1)\) for \(1 \leq l \leq d - 1\).

(b) \(j \geq j_{d-1} + t\). Then \(x_{j_1} x_{j_2} \cdots x_{j_{d-1}} x_j \in I\).

(c) There exists \(1 \leq l \leq d - 2\) such that \(j_l + (t-1) < j < j_{l+1}\). Then

\[
x_{j_1} x_{j_2} \cdots x_{j_l} x_j x_{j_{l+1} + (t-1)} \cdots x_{j_{d-1} + (t-1)} \in I,
\]

since \(j_k \leq i_k\) for \(1 \leq k \leq l\), \(j < j_{l+1} \leq i_{l+1}\) and \(j_k + (t-1) \leq i_k + (t-1) < i_{k+1}\) for \(l + 1 \leq k \leq d - 1\).

Therefore, we have proved that for \(j \in [n] \setminus F_1\), \(F_1 \cup \{j\} \notin \Delta\) which implies that \(F_1\) is a facet in \(\Delta\).

Let \(j \in [n] \setminus F_3\). We show that \(x_j x_{F_3} \in I\), thus \(F_3 \cup \{j\} \notin \Delta\).
(a) If $j < j_1$, then $x_j x_{j_1} + (t-1) \cdots x_{j_s - 1} + (t-1) x_{j_s} + (t-1) x_{j_r} + (t-1) \cdots x_j + (d-s) t - 1 \in I$ because $j < j_1 < i_1$, $j_k + (t-1) \leq i_k + (t-1) < i_{k+1}$ for $2 \leq k \leq s-1$ and $j_s + (kt-1) = i_s + 1 + (kt-1) = i_s + kt \leq i_{s+k}$ for $1 \leq k \leq d-s$.

(b) If there exists $2 \leq l \leq s$ such that $j_{l-1} + (t-1) < j < j_l$, then

$$x_{j_l} x_{j_{l+1}} \cdots x_{j_s} x_{j_l} + (t-1) \cdots x_{j_s - 1} + (t-1) x_{j_s} + (t-1) \cdots x_j + (d-s) t - 1 \in I.$$ 

Indeed, $j_k \leq i_k$ for $1 \leq k \leq l$. If $l = s$, then $j < j_s = i_s + 1$ and $j_s + (kt-1) = i_s + 1 + (kt-1) = i_s + kt \leq i_{s+k}$ for $1 \leq k \leq d-s$. In the case that $l < s$, we have $j < j_l \leq i_l$, $j_k + (t-1) \leq i_k + (t-1) < i_{k+1}$ for $1 \leq k \leq s-1$ and $j_s + (kt-1) = i_s + 1 + (kt-1) \leq i_{s+k}$ for $1 \leq k \leq d-s$.

It remains to show that the sets of the forms (i)–(iii) are the only facets of $\Delta$. This is equivalent to showing that for every face $G \in \Delta$, there exists $F \in \mathcal{F}(\Delta)$ of one of the forms (i)–(iii) which contains $G$.

Let $G \in \Delta$ and $j_1 = \min \{ j : j \in G \}$. Inductively, for $l \geq 2$, we set

$$j_l = \min \{ j : j \in G \text{ and } j \geq j_{l-1} + 1 \}.$$ 

In the case that $j_l \leq i_l$ for all $l$, then the sequence $j_1 < j_2 < \ldots < j_l < \ldots$ has at most $d-1$ elements. Otherwise, $x_{j_1} \cdots x_{j_d} \in I$, which implies that $G \notin \Delta$, a contradiction. Let $j_1 < j_2 < \ldots < j_r$ with $r \leq d-1$. Then $G \subset F_1 = \{ j_1, j_1 + (t-1), \ldots, j_r, j_r + (t-1), i_{r+1}, i_{r+1} + (t-1), \ldots, i_d, i_d + (t-1) \} \in \mathcal{F}(\Delta)$.

If there exists $l \leq d$ such that $j_l > i_l$, then we denote by $s$ the smallest index with this property. In the case that $s = 1$, $j_1 = \min \{ j : j \in G \}$ and $G \subset G_2 = \{ i_1 + 1, i_2, i_3, \ldots, n \} \in \mathcal{F}(\Delta)$. If $s > 1$, then $G \subset F_3 = \{ j_1, j_1 + 1, \ldots, j_s, j_s + 1, \ldots, j_{s-1}, j_{s-1} + (t-1), i_{s-1} + 1, i_s, i_s + 2, \ldots, n \} \in \mathcal{F}(\Delta)$.

**Theorem 1.2.** Let $t \geq 1$ be an integer, $u = x_{i_1} x_{i_2} \cdots x_{i_d} \in S$ a $t$-spread monomial, and suppose that $i_d = n$. Then the $t$-spread principal Borel ideal $I = B_t(u)$ is sequentially Cohen-Macaulay.

**Proof.** By [4] Theorem 8.2.20, it is enough to show that $I^\vee$, that is, the Stanley-Reisner ideal of $\Delta^\vee$, is componentwise linear. By [4] Corollary 8.2.21, it suffices to prove that $I^\vee$ has linear quotients.

Let $w_1 = \prod_{k=1}^1 x_k$. Let $w_2, w_3, \ldots, w_q$ be the minimal monomial generators of $I^\vee$ of the form (3), ordered decreasingly with respect to the pure lexicographic order. Let $w_{q+1}, w_{q+2}, \ldots, w_r$ be the remaining minimal monomial generators of $I^\vee$, namely those of form (1), ordered decreasingly with respect to the pure lexicographic order.

We show that $I^\vee$ has linear quotients with respect to the above order of its minimal generators. This is equivalent to proving the following claim:

\[ \text{for every } 1 < j \leq r \text{ and } g < j, \text{ there exists } w > \text{lex } w_j \text{ and} \]
\[ 1 \leq l \leq n \text{ such that } x_l = \frac{w}{\gcd(w, w_j)} \text{ and } x_l \mid \frac{w_g}{\gcd(w_g, w_j)}. \]

Since $w_2$ is the largest generator of the form (3) with respect to the pure lexicographic order, we have $x_1 x_2 \cdots x_{i_{j-1}} \mid w_2$. Thus, we obtain $x_{i_1} = w_1 / \gcd(w_1, w_2)$ and claim (4) is proved for $g = 1 < j = 2$. 


Let \( 2 < j \leq q, g = 1 \) and \( w_j = \prod_{k=1}^{s} x_k/v_{j_1}v_{j_2}\cdots v_{j_{s-1}} \) with \( s \leq d - 1 \). If \( j_1 = i_1 \), then we have \( x_i = w_1/gcd(w_1, w_j) \). Let \( j_1 < i_1 \). Then \( x_{j_1} | w_1/gcd(w_1, w_j) \). We have to find \( w >_{\text{lex}} w_j \) such that \( x_{j_1} = w/gcd(w, w_j) \).

(a) If there exists a least integer \( 1 \leq l \leq s - 2 \) such that \( j_{l+1} > j_l + t \), then we take \( w = \prod_{k=1}^{s} x_k/v_{j_1}v_{j_2}\cdots v_{j_{l+1}}v_{j_{l+1}+1}\cdots v_{j_{s-1}} \).

(b) If \( j_{l+1} = j_l + t \) for \( 1 \leq l \leq s - 2 \), then
\[
    w = \prod_{k=1}^{s} x_k/v_{j_1}v_{j_2}\cdots v_{j_{l+1}}v_{j_{l+1}+1}\cdots v_{j_{s-1}} + 1.
\]

Let \( 1 < g < j \). Then \( w_g >_{\text{lex}} w_j \). Let
\[
    w_g = \frac{\prod_{k=1}^{s} x_k}{v_{g_1}v_{g_2}\cdots v_{g_{s-1}}}.
\]
with \( s' \leq d - 1 \). Thus, we obtain \( v_{g_1}v_{g_2}\cdots v_{g_{s-1}} <_{\text{lex}} v_{j_1}v_{j_2}\cdots v_{j_{s-1}} \). It implies that there exists \( h \geq 1 \) such that \( j_1 = g_1, j_2 = g_2, \ldots, j_{h-1} = g_{h-1} \) and \( j_h < g_h \leq i_1 \). Since \( x_{j_h} | v_{j_1}v_{j_2}\cdots v_{j_{h-1}} \) and \( x_{j_h} v_{g_1}v_{g_2}\cdots v_{g_{s-1}} \), we get \( x_{j_h} | w_g/gcd(w_g, w_j) \). Then we must find a generator \( w \) of \( I' \) with \( w >_{\text{lex}} w_j \) such that \( x_{j_h} = w/gcd(w, w_j) \).

(a) If there exists a least integer \( h \leq l \leq s - 2 \) such \( j_{l+1} > j_l + t \), then
\[
    w = \frac{\prod_{k=1}^{s} x_k}{v_{j_1}v_{j_2}\cdots v_{j_{l+1}}v_{j_{l+1}+1}\cdots v_{j_{s-1}}}.
\]

(b) If \( j_{l+1} = j_l + t \) for \( h \leq l \leq s - 2 \), then \( w = \prod_{k=1}^{s} x_k/v_{j_1}v_{j_2}\cdots v_{j_{h-1}}v_{j_{h+1}}v_{j_{h+1}+1}\cdots v_{j_{s-1}} \).

With similar arguments, one proves claim (4) for \( q + 1 \leq j \leq r \).

**Example 1.3.** Let \( B = B_2(x_2x_4x_9) \subset K[x_1, \ldots, x_9] \). Then
\[
    I^\vee = (x_1x_2, x_1x_4, x_3x_4, x_1x_6x_7x_8x_9, x_3x_6x_7x_8x_9, x_5x_6x_7x_8x_9),
\]
where we ordered the generators as indicated in the proof of the above theorem.

**2. The Rees algebra of t-spread Borel principal ideals.**

In this section we consider the Rees algebra of t-spread Borel principal ideals.

For two monomials \( v, w \in S \), we write \( vw = x_{i_1}x_{i_2}\cdots x_{i_{2d}} \) with \( i_1 \leq i_2 \leq \cdots \leq i_{2d} \). Then the sorting of the pair \((v, w)\) is the pair of monomials \((v', w')\) where \( v' = x_{i_1}x_{i_3}\cdots x_{i_{2d-1}} \) and \( w' = x_{i_2}x_{i_4}\cdots x_{i_{2d}} \). The map sort : \( S_d \times S_d \to S_d \times S_d \) with sort \((v, w) = (v', w')\) is called the sorting operator. A subset \( B \subset S_d \) is called sortable if sort \((B \times B) \subset B \times B \). An \( r \)-tuple \((u_1, \ldots, u_r)\) of monomials of degree \( d \) is sorted if for all \( 1 \leq i < j \leq r \), the pair \((u_i, u_j)\) is sorted, that is, sort \((u_i, u_j) = (u_i, u_j)\). This means that if we write \( u_1 = x_{i_1}\cdots x_{i_a}, u_2 = x_{j_1}\cdots x_{j_d}, \ldots, u_r = x_{k_1}\cdots x_{k_d} \), then \((u_1, \ldots, u_r)\) is sorted if and only if \( 1 \leq i_1 \leq j_1 \leq \cdots \leq k_1 \leq i_2 \leq j_2 \leq \cdots \leq k_2 \leq \cdots \leq i_d \leq j_d \leq \cdots \leq k_d \). By [I, Theorem 6.12], for every \( r \)-tuple \((u_1, \ldots, u_r)\) of monomials of degree \( d \), there exists a unique sorted \( r \)-tuple \((u_1', \ldots, u_r')\) such that \( u_1 \cdots u_r = u_1' \cdots u_r \). When \((u_1', \ldots, u_r') = (u_1, \ldots, u_r)\), we say that the product \( u_1 \cdots u_r \) is sorted.
Let $t \geq 1$ be an integer and $I = B_t(u)$ where $u = x_{i_1} \cdots x_{i_d}$ is a $t$-spread monomial. It was shown in [2] Proposition 3.1] that the minimal set of monomial generators of a $t$-spread principal Borel ideal is sortable. Therefore, if $v, w \in G(I)$, and sort$(v, w) = (v', w')$, then $v', w' \in G(I)$.

Let $R(I) = \bigoplus_{j \geq 0} I^j$ be the Rees algebra of the ideal $I = B_t(u)$. Since the minimal generators of $B_t(u)$ have the same degree, the fiber $R(I)/\mathfrak{m}R(I)$ of the Rees ring $R(I)$ is isomorphic to $K[G(I)]$. Let $\psi : T = K[[t_v : v \in G(I)]] \to K[G(I)]$ be the homomorphism given by $t_v \mapsto v$ for $v \in G(I)$. As it was proved in [2] Theorem 3.2], the set of binomials

$$G = \{t_v t_w - t_{v'} t_{w'} : (v, w) \text{ unsorted, } (v', w') = \text{sort}(v, w)\}$$

is a Gröbner basis of the toric ideal $J_u = \ker \psi$ with respect to the sorting order on $T$. For more details about sorting order we refer to [3, Chapter 14] or [1, Section 6.2]. The initial monomial of the binomial $t_v t_w - t_{v'} t_{w'} \in G$ with respect to the sorting order is $t_v t_w$.

We now recall the definition of $\ell$-exchange property from [5]; see also [1, Section 6.4] and [3, Section 3].

Let $I \subset S$ be a monomial ideal generated in a single degree and $K[[u : u \in G(I)]]$ the polynomial ring in $|G(I)|$ variables endowed with a monomial order $<$. Let $P$ be the kernel of the $K$-algebra homomorphism

$$K[[u : u \in G(I)]] \to K[G(I)], u \mapsto u, u \in G(I).$$

A monomial $t_{u_1} \cdots t_{u_N}$ is called standard with respect to $<$ if it does not belong to $\in_{\ell}(P)$.

**Definition 2.1.** [5] The monomial ideal $I \subset S$ satisfies the $\ell$-exchange property with respect to $<$ if the following condition holds: let $t_{u_1} \cdots t_{u_N}, t_{v_1} \cdots t_{v_N}$ be two standard monomials with respect to $<$ of the same degree $N$ satisfying

(i) $\deg_{x_i} u_1 \cdots u_N = \deg_{x_i} v_1 \cdots v_N$ for $1 \leq i \leq q - 1$ with $q \leq n - 1$,

(ii) $\deg_{x_q} u_1 \cdots u_N < \deg_{x_q} v_1 \cdots v_N$.

Then there exists integers $\delta, j$ with $q < j \leq n$ and $j \in \text{supp}(u_\delta)$ such that $x_q u_\delta / x_j \in I$.

**Proposition 2.2.** Let $u = x_{i_1} \cdots x_{i_d}$ be a $t$-spread monomial in $S$. Then the $t$-spread principal Borel ideal $B_t(u)$ satisfies the $\ell$-exchange property with respect to the sorting order $<_{\text{sort}}$.

**Proof.** Let $T = K[[t_v : v \in G(B_t(u))]]$ and $t_{u_1} \cdots t_{u_N}, t_{v_1} \cdots t_{v_N} \in T$ be two standard monomials with respect to $<_{\text{sort}}$ of the same degree $N$ such that the two conditions of Definition 2.1 are fulfilled. As the chosen monomials are standard with respect to $<_{\text{sort}}$, it follows that the products $u_1 \cdots u_N, v_1 \cdots v_N$ are sorted. Since $\deg_{x_i} u_1 \cdots u_N = \deg_{x_i} v_1 \cdots v_N$ for $1 \leq i \leq q - 1$, we have $\deg_{x_i} u_\nu = \deg_{x_i} (v_\nu)$ for all $1 \leq \nu \leq N$ and $1 \leq i \leq q - 1$. The condition $\deg_{x_q} u_1 \cdots u_N < \deg_{x_q} v_1 \cdots v_N$ implies that there exists $1 \leq \ell \leq N$ such that $\deg_{x_q}(u_\delta) < \deg_{x_q}(v_\ell)$.

Let $u_\delta = x_{j_1} \cdots x_{j_d}, v_\delta = x_{l_1} \cdots x_{l_d}$, and assume that $q = \ell_{\mu}$ for some $1 \leq \mu < d$. It follows that $j_1 = l_1, \ldots, j_{\mu - 1} = l_{\mu - 1}$ and $j_{\mu} > l_{\mu} = q$. If $j_{\mu} \not\in \text{supp}(v_\delta)$, then we
take \( j = j_\mu \). Then the monomial \( x_q u_\delta / x_j \) is \( t \)-spread, thus it belongs to \( B_t(u) \), since 
\[
q = \ell_\mu \geq \ell_{\mu-1} + t = j_{\mu-1} + t \quad \text{and} \quad j_{\mu+1} \geq j_\mu + t > q + t.
\]
If \( j_\mu \in \text{supp}(u_\delta) \), we may choose any \( j \in \text{supp}(u_\delta) \setminus \text{supp}(v_\delta) \) with \( j > q \). We should note that such an integer \( j \) exists since \( \deg(u_\delta) = \deg(v_\delta) \) and \( \deg_{x_q}(u_\delta) < \deg_{x_q}(v_\delta) \). Then \( x_q u_\delta / x_j \) is \( t \)-spread since \( q, j_\mu \in \text{supp}(v_\delta) \).

The Rees ring \( \mathcal{R}(I) \) where \( I = B_t(u) \) has the presentation
\[
\varphi : R = S[\{t_v : v \in G(I)\}] \to \mathcal{R}(I),
\]
given by
\[
x_i \mapsto x_i, 1 \leq i \leq n, t_v \mapsto vt, v \in G(I).
\]
Let \( J = \ker \varphi \) be the toric ideal of \( \mathcal{R}(I) \). We consider the sorting order \( <_{\text{sort}} \) on the ring \( T = K[\{t_v : v \in G(I)\}] \) and the lexicographic order \( <_{\text{lex}} \) on the ring \( S \). Let \( < \) be the monomial order on \( R = S[\{t_v : v \in G(I)\}] \) defined as follows: if \( m_1, m_2 \) are monomials in \( S \) and \( v_1, v_2 \) are monomials in \( T \), then
\[
m_1 v_1 > m_2 v_2 \text{ if } m_1 >_{\text{lex}} m_2 \text{ or } m_1 = m_2 \text{ and } v_1 >_{\text{sort}} v_2.
\]

**Theorem 2.3.** Let \( I = B_t(u) \) be a \( t \)-spread principal Borel ideal. The reduced Gröbner basis of the toric ideal \( J \) with respect to \( < \) consists of the set of binomials \( t_v t_w - t_v' t_w' \) where \( (v, w) \) is unsorted and \( (v', w') = \text{sort}(v, w) \), together with the binomials of the form \( x_i t_v - x_j t_w \) where \( i < j, x_i v = x_j w \), and \( j \) is the largest integer for which \( x_q u_\delta / x_j \in G(I) \).

**Proof.** By Proposition 2.2, we know that the principal Borel ideal \( B_t(u) \) satisfies the \( t \)-exchange property with respect to the sorting order \( <_{\text{sort}} \). Thus, the statement follows by applying [5] Theorem 5.1. \( \square \)

**Proposition 2.4.** All the powers of \( B_t(u) \) have linear quotients. In particular, all the powers of \( B_t(u) \) have a linear resolution.

**Proof.** By Theorem 2.3 it follows that every binomial in the reduced Gröbner basis of the toric ideal \( J \) has the degree in variables \( x_i \) at most 1 which means that \( B_t(u) \) satisfies the \( x \)-condition. Then, by [4] Theorem 10.1.9], all the powers of \( B_t(u) \) have linear quotients. \( \square \)

Theorem 2.3 shows that the Rees algebra \( \mathcal{R}(B_t(u)) \) has a quadratic Gröbner basis and the initial ideal of the toric ideal \( J \) is squarefree. Therefore, we get the following consequences.

**Corollary 2.5.** The Rees algebra \( \mathcal{R}(B_t(u)) \) is Koszul.

**Corollary 2.6.** The Rees algebra \( \mathcal{R}(B_t(u)) \) is a normal Cohen-Macaulay domain. In particular, \( B_t(u) \) satisfies the strong persistence property. Therefore, \( B_t(u) \) satisfies the persistence property.

**Proof.** Since the initial ideal of the toric ideal \( J \) of \( \mathcal{R}(B_t(u)) \) is squarefree, by a theorem due to Sturmfels [3], it follows that \( K[B_t(u)] \) is a normal domain. Next, by a theorem of Hochster [7], it follows that \( K[B_t(u)] \) is Cohen-Macaulay. The second part follows by using [6] Corollary 1.6. \( \square \)
3. Depth of powers of \( t \)-spread principal Borel ideals

In this section, we consider the asymptotic behavior of the depth for \( t \)-spread principal Borel ideals.

Let \( u = x_{i_1} \cdots x_{i_d} \) be the generator of the \( t \)-spread principal Borel ideal \( I = B_t(u) \subset S = K[x_1, \ldots, x_n] \). We would like to compute the limit of \( \text{depth}(S/I^k) \) when \( k \to \infty \). Obviously, we may consider \( i_d = n \) since otherwise we may reduce to the study of depth \( S'/I^k \) where \( S' = K[x_1, \ldots, x_{i_d}] \).

**Theorem 3.1.** Let \( t \geq 1 \) be an integer and \( I = B_t(u) \subset S \) the \( t \)-spread principal Borel ideal generated by \( u = x_{i_1} \cdots x_{i_d} \) where \( t + 1 \leq i_1 < i_2 < \cdots < i_{d-1} < i_d = n \). Then

\[
\text{depth} \frac{S}{I^k} = 0, \quad \text{for } k \geq d.
\]

In particular, the analytic spread of \( I \) is \( \ell(I) = n \).

**Proof.** We first observe that the second claim of the theorem follows since it is known that, for any non-zero graded ideal \( I \), \( \lim_{k \to \infty} \text{depth}(S/I^k) = n - \ell(I) \) if the Rees algebra \( \mathcal{R}(I) \) is Cohen-Macaulay; see [4, Proposition 10.3.2]. But the Rees algebra of \( B_t(u) \) is indeed Cohen-Macaulay by Corollary 2.6.

In order to prove the first claim, by Auslander-Buchsbaum theorem, we have to show that

\[
\text{proj dim} \frac{S}{I^k} = n \quad \text{for } k \geq d.
\]

Since \( I \) satisfies the \( \ell \)-exchange property (Proposition 2.2), by using [3, Theorem 3.6], it follows that \( I^k \) has linear quotients with respect to \( >_{\text{lex}} \) for all \( k \geq 1 \). This implies that we may compute the projective dimension of \( S/I^k \) by using [4, Corollary 8.2.2]. More precisely, if \( G(I^k) = \{w_1, \ldots, w_m\} \) with \( w_1 >_{\text{lex}} \cdots >_{\text{lex}} w_m \) and for \( 1 \leq j \leq m \), \( r_j \) is the number of variables which generate the ideal quotient \( (w_1, \ldots, w_{j-1}) : w_j \), then

\[
\text{proj dim} \frac{S}{I^k} = \max\{r_1, \ldots, r_m\} + 1.
\]

Thus, in order to prove (5), we only need to find a monomial \( w \in G(I^k) \) such that the ideal quotient \( (w' \in G(I^k) : w' >_{\text{lex}} w) : w \) is generated by \( n - 1 \) variables. Let \( w_0 = v_1v_2 \cdots v_{d-1}v_d \) where

\[
\begin{align*}
v_1 &= x_1x_{t+1} \cdots x_{(d-3)t+1}x_{(d-2)t+1}x_n, \\
v_2 &= x_1x_{t+1} \cdots x_{(d-3)t+1}x_{id-1}x_n, \\
& \quad \vdots \\
v_{d-1} &= x_1x_{i_2} \cdots x_{id-2}x_{id-1}x_n, \\
v_d &= u = x_1x_{i_2} \cdots x_{id-2}x_{id-1}x_n.
\end{align*}
\]

Obviously, all the monomials \( v_1, \ldots, v_d \) belong to \( G(I) \). Let \( k \geq d \) and \( w = w_0u^{k-d} \in G(I^k) \). If we show that \( (w' \in G(I^k) : w' >_{\text{lex}} w) : w \supseteq (x_1, \ldots, x_{n-1}) \) then the proof of equality (5) is completed.
Let $J = (w' \in G(I^k) : w' >_{\text{lex}} w)$. We show that $x_j w \in J$ for all $1 \leq j \leq n - 1$. Let $1 \leq s \leq d$ such that $i_{s-1} \leq j < i_s$, where we set $i_0 = 1$. We consider the monomial

$$v'_{d-s+1} = \frac{x_j v'_{d-s+1}}{x_{i_s}} = x_1 x_{t+1} \cdots x_{(s-2)t+1} x_j x_{i_s+1} \cdots x_{i_d-1} x_n.$$  

Clearly, $v'_{d-s+1} \in G(I)$ since $j < i_s$ and $v'_{d-s+1}$ is a $t$-spread monomial. Indeed, we have $i_{s+1} - j > i_{s+1} - i_s \geq t$ and

$$j - (s-2)t - 1 \geq i_{s-1} - (s-2)t - 1 \geq (i_1 + (s-2)t) - (s-2)t - 1 = i_1 - 1 \geq t.$$  

Let $w'_0$ be monomial obtained from $w_0$ by replacing the monomial $v'_{d-s+1}$ with $v'_{d-s+1}$ and let $w' = w'_0 u^{k-d}$. Then $w' >_{\text{lex}} w$, thus $w' \in J$ and since $x_j w = x_i w'$, we have $x_j w \in J$, which implies that $x_j \in J : w$. 

The analytic spread $\ell(I)$ of a graded ideal $I \subset S$ is the Krull dimension of the fiber ring $R(I)/mR(I)$ where $m = (x_1, \ldots, x_n)$. In our case, $I = B_t(u)$ is generated in the same degree $d$. Thus the fiber ring is actually $K[G(B_t(u))]$. Therefore, we get the following consequence. 

**Corollary 3.2.** Let $t \geq 1$ be an integer and $B_t(u) \subset S$ the $t$-spread principal Borel ideal generated by $u = x_{i_1} \cdots x_{i_d}$ where $t + 1 \leq i_1 < i_2 < \cdots < i_{d-1} < i_d = n$. Then $\dim K[G(B_t(u))] = n$. 

**Remark 3.3.** Note that the order with respect to which we derived that all the powers of $B_t(u)$ have linear quotients in Proposition 2.4 does not coincide with the decreasing lexicographic order that we used in the proof of Theorem 3.1. In fact, the toric ideal of $K[B_t(u)]$ does not have a quadratic Gröbner basis with respect to the lexicographic order as we can see in the following example. 

Let $u = x_6 x_8 x_{10} \in K[x_1, \ldots, x_{10}]$ and $B_2(u)$ the 2-spread principal Borel ideal defined by $u$. Let $f = t_{u_1} t_{u_2} u_3 - t_{u_1} t_{u_2} t_{v_0}$ with $u_1 = x_1 x_3 x_8$, $u_2 = x_1 x_7 x_9$, $u_3 = x_2 x_4 x_6$ and $v_1 = x_1 x_3 x_9$, $v_2 = x_1 x_6 x_8$, $v_3 = x_2 x_4 x_7$. Then $f$ is a binomial in the toric ideal of $K[B_2(u)]$ whose initial monomial with respect to the lexicographic order is $t_{u_1} t_{u_2} t_{u_3}$. One can easily see that there is no quadratic monomial in the initial ideal of the toric ideal which divides $t_{u_1} t_{u_2} t_{u_3}$. This shows that, with respect to the lexicographic order, the reduced Gröbner basis of the toric ideal of $K[B_2(u)]$ is not quadratic. 

**Remark 3.4.** Condition $i_1 \geq t + 1$ in the hypothesis of Theorem 3.1 is essential. Indeed, let us consider $I = B_t(u)$ where $u = x_{i_1} x_{2t} \cdots x_{dt}$, the $t$-spread Veronese ideal in $n = dt$ variables. We claim that

$$\text{depth} \frac{S}{I^k} = d - 1, \text{ for } k \geq d,$$

which implies that

$$\lim_{k \to \infty} \text{depth} \frac{S}{I^k} = d - 1.$$  

In particular, this implies that the analytic spread of $I$ is $\ell(I) = n - d + 1 = \dim K[B_t(u)]$. Indeed, we show that

$$\text{proj dim} \frac{S}{I^k} = n - d + 1, \text{ for } k \geq d.$$
As in the proof of Theorem 3.1, one considers the monomials
\[ v_1 = x_1 x_{t+1} \cdots x_{(d-2)t+1} x_{dt}, \ldots, v_{d-1} = x_1 x_{2t} \cdots x_{dt}, v_d = u = x_t x_{2t} \cdots x_{dt}, \]
and shows that, for \( k \geq d \),
\[ (w' \in G(I^k) : w' >_{\text{lex}} w) : w \supseteq (x_j : j \in [n] \setminus \text{supp}(u)), \]
where \( w = v_1 \cdots v_d u^{k-d} \).

On the other hand, we show that for every \( j \in \text{supp}(u) \), \( x_j \) does not appear among the variables which generate the colon ideals of \( I^k \). Let us assume that there exists \( k \geq 1 \) and some monomial \( \mu \in I^k \), \( \mu = \mu_1 \cdots \mu_k \) with \( \mu_1, \ldots, \mu_k \in I \), such that \( x_j \in (\mu' \in I^k : \mu' >_{\text{lex}} \mu) \) for some \( j \in \text{supp}(u) \). This implies that there exists \( \mu' = \mu'_1 \cdots \mu'_k \in I^k \) and some integer \( s > j \) such that \( x_j \mu = \mu' x_s \). Let \( j = qt \) for some \( 1 \leq q \leq d \). Then
\[
\sum_{\ell=(q-1)t+1}^{qt} \deg_{x_\ell}(x_j \mu) = \sum_{\ell=(q-1)t+1}^{qt} \deg_{x_\ell}(x_j \mu_1 \cdots \mu_k) = k + 1 > k =
= \sum_{\ell=(q-1)t+1}^{qt} \deg_{x_\ell}(\mu'_1 \cdots \mu'_k x_s)
\]
contradiction. Therefore, the maximal number of variables which generate the colon ideals of \( I^k \) for \( k \geq d \) is \( n - d \), which implies that \( \text{proj dim} \frac{S}{J} = n - d + 1 \), for \( k \geq d \).

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Claudia Andrei, Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, 010014 Bucharest, Romania
E-mail address: claudiaandrei1992@gmail.com

Viviana Ene, Faculty of Mathematics and Computer Science, Ovidius University, Bd. Mamaia 124, 900527 Constanta, Romania
E-mail address: vivian@univ-ovidius.ro

Bahareh Lajmiri, Department of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Ave, Tehran, Iran
E-mail address: bahareh.lajmiri@aut.ac.ir