MONOTONICITY AND SYMMETRY OF POSITIVE SOLUTIONS
TO DEGENERATE QUASILINEAR ELLIPTIC SYSTEMS IN
HALF-SPACES AND STRIPS

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Abstract. By means of the method of moving planes, we study the monotonicity of positive solutions to degenerate quasilinear elliptic systems in half-spaces. We also prove the symmetry of positive solutions to the systems in strips by using similar arguments. Our work extends the main results obtained in [16, 20] to the system, in which substantial differences with the single cases are presented.

1. Introduction. This paper is concerned with the following system of quasilinear elliptic equations

\begin{equation}
\begin{aligned}
-\Delta_p u &= f(u, v) \quad \text{in } \mathbb{R}^N_+,
-\Delta_q v &= g(u, v) \quad \text{in } \mathbb{R}^N_+,
\end{aligned}
\end{equation}

\[ u, v > 0 \quad \text{in } \mathbb{R}^N_+,
\]
\[ u = v = 0 \quad \text{on } \partial \mathbb{R}^N_+,
\]

where $\mathbb{R}^N_+ = \{ x = (x', y) \in \mathbb{R}^N \mid x' \in \mathbb{R}^{N-1}, y > 0 \}$, $p, q > 2$ and $f$ and $g$ satisfy the following assumptions:

(H1) $f$ and $g$ are locally Lipschitz continuous functions in $[0, \infty) \times [0, \infty)$,
(H2) $f$ and $g$ are positive, i.e., $f(s, t) > 0$ and $g(s, t) > 0$ for all $s, t > 0$,
(H3) $(s, t) \mapsto f(s, t)$ is non-decreasing in $t$, $(s, t) \mapsto g(s, t)$ is non-decreasing in $s$,
(H4) $\lim_{s \rightarrow 0^+} \frac{f(s, t)}{s^p} = 0$ locally uniformly in $t$, $\lim_{t \rightarrow 0^+} \frac{g(s, t)}{t^q} = 0$ locally uniformly in $s$.

We recall that assumption (H3) is usually referred to as cooperativity condition in the literature and is a natural assumption in the study of symmetry and monotonicity properties of the solutions, see [25, 10]. We also remark that condition (H4) implies $f(0, t) = g(s, 0) = 0$ for all $s, t \geq 0$.

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Throughout the paper, we always denote a generic point $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N_+$ by $x = (x', y)$, where $x' = (x_1, x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $y = x_N > 0$. Furthermore, by the regularity results in [12, 24] and a reflection argument, we may assume that solutions $u, v \in C^{1,\alpha}_{loc} (\mathbb{R}^N_+)$ and fulfill system (1.1) in the weak distributional meaning. Our aim is to study the monotonicity properties of solutions to (1.1) with respect to the $y$-direction. This is an important task that naturally occurs in many applications: blow-up analysis, a priori estimates and also in the proofs of Liouville type theorems.

It is well known that the moving plane method is a very effective tool to prove the monotonicity and symmetry properties of the solutions to general PDEs. The moving plane method was introduced by Alexandrov [1] in the context of differential geometry and by Serrin [23] in the PDEs framework, for an overdetermined problem. The variant used in this paper is due to Berestycki, Nirenberg [3] and to Gidas, Ni, Nirenberg [18]. The symmetry of solutions to semilinear elliptic equations in bounded domains was obtained via this method in [3, 18]. For the case of semilinear elliptic equations $-\Delta u = f(u)$ in $\mathbb{R}^N_+$, where monotonicity of the solutions is expected, we refer to [4, 2, 17]. For monotonicity results in the case of problems involving system (1.1) with $p = q = 2$, we refer to [11, 10, 6] and the references therein.

The case of quasilinear elliptic equations and systems is more complicated to study since comparison principles are not equivalent to maximum principles for the $p$-Laplacian. Furthermore, the singular or degenerate nature of the operator (corresponding to $1 < p < 2$ and $p > 2$, respectively) also causes the lack of regularity of the solutions. First results in bounded domains were obtained by Damascelli and Pacella [7] for the case $1 < p < 2$. The case $p > 2$ requires the use of weighted Sobolev spaces and was solved by Damascelli and Sciunzi [8] for positive nonlinearities.

For the unbounded domains, some assumptions of global boundedness of the gradients of solutions are usually imposed to easily control the term $|\nabla u|^{p-2}$ in the $p$-Laplacian. In the case of the whole space, we refer the readers to the recent results in [22, 26], where positive $D^{1,p}(\mathbb{R}^N)$ solutions to the critical $p$-Laplace equation were proved to be radially symmetric and hence completely classified. The monotonicity of positive solutions to the $p$-Laplace equation $-\Delta_p u = f(u)$ in $\mathbb{R}^N_+$ was obtained in the singular case $1 < p < 2$ in [13] where positive locally Lipschitz continuous nonlinearities are considered. The tools used in [13] include a weak comparison principle in narrow domains and a careful analysis of the local symmetry regions of the solutions. Similar result was obtained in the degenerate case $p > 2$ in [16, 15] where $f$ is a positive locally Lipschitz continuous function such that $\lim_{t \to 0^+} \frac{f(t)}{t^p} = f_0 \in [0, \infty)$. The proofs in [16] make use of weighted Sobolev spaces and a weighted Poincaré type inequality with weight $|\nabla u|^{p-2}$. This inequality involves constants that may blow up when the solution approaches zero. Hence the lack of compactness plays an important role in this case.

Regarding the quasilinear systems, Montoro et al. [20] proved the monotonicity of solutions to system (1.1) in the case $1 < p, q < 2$ by exploiting the method of moving planes and the technique introduced in [13]. To the best of our knowledge, the work [20] is currently the only article involving monotonicity results for quasilinear systems in $\mathbb{R}^N_+$ in the literature. In our paper, we will prove the following analogous result for the case $p, q > 2$. 


Theorem 1. Let $p, q > 2$ and $u, v \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N_+)$ be solutions to (1.1) with $|\nabla u|, |\nabla v| \in L^\infty(\mathbb{R}^N_+)$. Assume that (H1)--(H4) are satisfied. Then $u$ and $v$ are monotone increasing with respect to the $y$-direction. Furthermore, 
$$
\frac{\partial u}{\partial y} > 0 \quad \text{and} \quad \frac{\partial v}{\partial y} > 0 \quad \text{in} \ \mathbb{R}^N_+.
$$
As a consequence, $u, v \in C^{2,\alpha'}_{\text{loc}}(\mathbb{R}^N_+)$ for some $0 < \alpha' < 1$.

In particular, our monotonicity result holds for the power-like system
$$
\begin{cases}
-\Delta_p u = u^av^b & \text{in} \ \mathbb{R}^N_+, \\
-\Delta_q v = u^cv^d & \text{in} \ \mathbb{R}^N_+, \\
u, v > 0 & \text{in} \ \mathbb{R}^N_+,
\end{cases}
$$
where $a > p - 1$, $d > q - 1$ and $b, c \in \{0\} \cup [1, \infty)$. Notice that Theorem 1 is proved without a priori assumptions on the boundedness of the solutions, or the behavior of the solutions near infinity. Furthermore, only local regularity on the solution is required in our result. It is also worth emphasizing that we prove the first step of the moving plane procedure in a very general setting. More precisely, we prove in Theorem 9 that any solutions of system (1.1) are monotone increasing near the boundary of $\mathbb{R}^N_+$ for any $p, q > 1$ and assuming only that the nonlinearities $f,g$ are merely continuous in $[0,1) \times [0,1)$ such that $|f(s,t)| \leq Cs^p-1$ and $|g(s,t)| \leq Ct^q-1$ for all $s, t \in [0, T]$, where $C, T > 0$.

In this paper, we are also interested in the quasilinear system in strips
$$
\begin{cases}
-\Delta_p u = f(u,v) & \text{in} \ \Sigma_h, \\
-\Delta_q v = g(u,v) & \text{in} \ \Sigma_h, \\
u, v > 0 & \text{in} \ \Sigma_h, \\
u = v = 0 & \text{on} \ \partial\Sigma_h,
\end{cases}
$$
where $\Sigma_h = \mathbb{R}^{N-1} \times (0,h)$. Notice that if $|\nabla u|, |\nabla v| \in L^\infty(\Sigma_h)$, then $u, v \in W^{1,\infty}(\Sigma_h)$. The symmetry of positive solutions to $p$-Laplace equations in strips with $\frac{2N+2}{N+2} < p \leq 2$ was obtained in [14, Theorem 1.5]. Arguing as in the proof of Theorem 1 we can deduce the following symmetry result, which is new even in the case of a single equation.

Theorem 2. Let $p, q > 2$ and $u, v \in C^{1,\alpha}_{\text{loc}}(\Sigma_h) \cap W^{1,\infty}(\Sigma_h)$ be solutions to (1.2). Assume that (H1)--(H4) are satisfied. Then $u$ and $v$ are symmetric with respect to the hyperplane $\{y = h/2\}$. Furthermore,
$$
\frac{\partial u}{\partial y} > 0 \quad \text{and} \quad \frac{\partial v}{\partial y} > 0 \quad \text{in} \ \Sigma_{h/2}.
$$

The paper is organized as follows. In Section 2 we recall some basic notions and known results which will be used later. In Section 3 we prove the monotonicity of solutions near the boundary of $\mathbb{R}^N_+$. In Section 4 we prove a weak comparison principle in narrow domains. The proof of Theorem 1 is given in Section 5. The last section is devoted to the proof of Theorem 2.
2. Preliminaries. Our proof relies on the method of moving planes and some techniques introduced in [16, 15]. We recall here some basic notions and known results in the literature. Generic positive constants will be denoted by $C$ (with subscripts in some cases) and they will be allowed to vary within a single line or formula.

For $\alpha < \beta$, we define the strip
$$\Sigma_{\alpha, \beta} = \mathbb{R}^{N-1} \times (\alpha, \beta).$$
For brevity, we will write $\Sigma_\beta = \Sigma_{0, \beta}$ for $\beta > 0$, i.e.,
$$\Sigma_\beta = \mathbb{R}^{N-1} \times (0, \beta).$$
We also define the cylinder
$$\Sigma_{\alpha, \beta}^R = B_R \times (\alpha, \beta),$$
where $B'_R$ is the ball in $\mathbb{R}^{N-1}$ of radius $R$ and center at zero. Given $\lambda \in \mathbb{R}$, we define $u_\lambda$ and $v_\lambda$ by
$$u_\lambda(x) = u_\lambda(x', y) := u(x', 2\lambda - y) \quad \text{and} \quad v_\lambda(x) = v_\lambda(x', y) := v(x', 2\lambda - y)$$
in $\Sigma_{2\lambda}$.

The strong maximum principle and Hopf’s lemma were first proved by Vázquez and play an important role in our paper. We recall their statement here.

**Theorem 3** (Strong maximum principle and Hopf’s lemma [21]). Let $\Omega$ be a domain in $\mathbb{R}^N$ and suppose that $u \in C^1(\Omega)$, $u \geq 0$ in $\Omega$, weakly solves
$$-\Delta_p u + cu^q = g \geq 0 \quad \text{in} \quad \Omega,$$
where $1 < p < \infty$, $q \geq p - 1$, $c \geq 0$ and $g \in \mathcal{L}^\infty_{\text{loc}}(\Omega)$. If $u \neq 0$ then $u > 0$ in $\Omega$. Moreover for any point $x_0 \in \partial \Omega$ where the interior sphere condition is satisfied, and such that $u \in C^1(\Omega \cup \{x_0\})$ and $u(x_0) = 0$ we have that $\frac{\partial u}{\partial \nu} > 0$ for any inward directional derivative (this means that if $y$ approaches $x_0$ in a ball $B \subset \Omega$ that has $x_0$ on its boundary, then $\lim_{y \to x_0} \frac{u(y) - u(x_0)}{|y - x_0|} > 0$).

Another important ingredient of our later proof is the strong comparison principle for $p$-Laplace equations with $p > \frac{2N+2}{N+2}$ obtained by Damascelli and Sciunzi.

**Theorem 4** (Strong comparison principle [9]). Let $u, v \in C^1(\Omega)$, where $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$ with $\frac{2N+2}{N+2} < p < \infty$. Suppose that either $u$ or $v$ is a weak solution of $-\Delta_p w = f(w)$ with $f$ positive and locally Lipschitz continuous. Assume
$$-\Delta_p u - f(u) \leq -\Delta_p v - f(v) \quad \text{and} \quad u \leq v \quad \text{in} \quad \Omega.$$ 
Then, $u = v$ in $\Omega$ or $u < v$ in $\Omega$.

Next, we recall that the linearized operator $L_u(v, \varphi)$ at a fixed solution $u$ of $-\Delta_p u = f(u)$ in $\Omega$ is well defined, for every $v, \varphi \in H^{1,2}_p(\Omega)$ with $\rho = |\nabla u|^{p-2}$, by
$$L_u(v, \varphi) := \int_\Omega \left[ |\nabla u|^{p-2} (\nabla v, \nabla \varphi) + (p - 2)|\nabla u|^{p-4} (\nabla u, \nabla \varphi)(\nabla u, \nabla \varphi) - f'(u)v \varphi \right].$$
Moreover, $v \in H^{1,2}_p(\Omega)$ is a weak solution of the linearized equation if $L_u(v, \varphi) = 0$ for all $\varphi \in H^{1,2}_0(\Omega)$. Here the weighted Sobolev space $H^{1,2}_p(\Omega)$ is defined as the space of functions $v$ such that
$$\|v\|_{H^{1,2}_p(\Omega)} := \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega, \rho)} < \infty.$$
It can be also defined as the completion of smooth functions under the norm above. The space $H_{0,\rho}^{1,2}(\Omega)$ is obtained by taking the closure of $C_c^\infty(\Omega)$ under the same norm and $\|\nabla v\|_{L^2(\Omega,\rho)}$ is an equivalent norm in $H_{0,\rho}^{1,2}(\Omega)$.

By [8] we have that $u_{x_i} := \frac{\partial u}{\partial x_i} \in H_{1,\rho}^{1,2}(\Omega)$ for $i = 1, \ldots, N$ and

$$L_u(u_{x_i}, \varphi) = 0$$

for all $\varphi \in H_{0,\rho}^{1,2}(\Omega)$. In other words, the derivatives of $u$ are weak solutions to the linearized equation. Furthermore, we have the following strong maximum principle for the linearized operator.

**Theorem 5** (Strong maximum principle for the linearized operator [9]). Let $u \in C^1(\Omega)$ be a weak solution of $-\Delta_p u = f(u)$ in a bounded smooth domain $\Omega$ of $\mathbb{R}^N$ with $\frac{2N+2}{N+2} < p < \infty$, and $f$ positive ($f(s) > 0$ for $s > 0$) and locally Lipschitz continuous. Then, for any $i = 1, \ldots, N$ and any domain $\Omega' \subset \Omega$ with $u_{x_i} \geq 0$ in $\Omega'$, we have that either $u_{x_i} \equiv 0$ in $\Omega'$ or $u_{x_i} > 0$ in $\Omega'$.

We state now the weighted Poincaré type inequality that will be useful in the sequel.

**Theorem 6** (Weighted Poincaré type inequality [8]). Let $\rho$ be a positive function such that

$$\int_{\Omega} \frac{1}{\rho^\tau}|x-y|^\tau dy \leq C_1 \quad \text{for all } x \in \Omega, \quad (2.1)$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $\max\{p-2,0\} \leq \tau < p-1$ and $\gamma < N-2$ ($\gamma = 0$ if $N = 2$). Let $w \in H_{\rho}^{1,2}(\Omega)$ be such that

$$|w(x)| \leq C_2 \int_{\Omega} \frac{|\nabla w(y)|}{|x-y|^{N-\gamma}} dy \quad \text{for all } x \in \Omega. \quad (2.2)$$

Then we have

$$\int_{\Omega} w^2 \leq C_P \int_{\Omega} \rho|\nabla w|^2,$$

where $C_P = C_P(diam(\Omega), C_1)$. Furthermore, $C_P \to 0$ as $diam(\Omega) \to 0$.

We will exploit the weighted Poincaré type inequality with the weight $\rho = |\nabla u|^{p-2}$. Next proposition gives some sufficient conditions in order to make (2.1) hold for such $\rho$.

**Proposition 1.** Let $p > 1$ and $u \in C^{1,\alpha}(\Omega)$ be a weak solution to

$$-\Delta_p u = h(x) \quad \text{in } \Omega,$$

where $h \in W^{1,\infty}(\Omega)$. Let $\Omega' \subset \subset \Omega$ and $0 < \delta < dist(\Omega', \partial \Omega)$ and assume that $h > 0$ in $\Omega'_d$, where $\Omega'_d = \{ x \in \Omega \mid \delta(x, \Omega') < \delta \}$ is the $\delta$-neighborhood of $\Omega'$. Let us fix $\beta_1, \beta_2 > 0$ such that

$$\inf_{\Omega'_d} h \geq \beta_1 \quad \text{and} \quad \delta \geq \beta_2.$$

Then there exists a positive constant $C^* = C^*(\beta_1, \beta_2)$ such that

$$\int_{\Omega'} \frac{1}{|\nabla u|^\tau |x-y|^\gamma} dy \leq C^*, \quad$$

with $\max\{p-2,0\} \leq \tau < p-1$.\]
For the proof of Proposition 1, we refer to Proposition 1 in Section 4 of [15] (see also [9]). Note that we prefer to omit the dependence of the constant $C^*$ on other parameters that are fixed and therefore not relevant in our application.

Later, we will frequently exploit the classical Harnack inequality and a boundary type Harnack inequality for $p$-Laplace equations. Therefore, we recall here an adapted version of the more general and deep result of Bidaut-Véron, Borghol and Véron.

**Theorem 7** (Boundary Harnack inequality [5]). Let $R_0 > 0$ and let $u$ be such that

$$-\Delta_p u = c(x)u^{p-1} \quad \text{in } \Sigma^{2R_0},$$

with $u$ vanishing on $\partial \Sigma^{2R_0} \cap \{y = 0\}$ and with $\|c\|_{L^\infty(\Sigma^{2R_0})} \leq C_0$. Then

$$\frac{1}{C} \frac{u(z_2)}{\rho(z_2)} \leq \frac{u(z_1)}{\rho(z_1)} \leq C \frac{u(z_2)}{\rho(z_2)}$$

for all $z_1, z_2 \in B_{R_0} \cap \Sigma^{2R_0} \setminus \{0\}$ such that $\frac{1}{2} \leq \frac{|z_1|}{|z_2|} \leq 2$, where $C = C(p, N, C_0)$ and $\rho(\cdot)$ is the distance function to $\partial \mathbb{R}^n$.

Finally, we state an elementary lemma that will be useful in the proof of the weak comparison principle in narrow domains (see Proposition 2 in Section 4).

**Lemma 8** (see Lemma 2.1 in [14]). Let $\theta > 0$ and $\nu > 0$ such that $\theta < 2^{-\nu}$. Let $L : (1, \infty) \to \mathbb{R}$ be a non-negative and non-decreasing function such that

$$\begin{cases} 
L(R) \leq \theta L(2R) & \text{for all } R > 1, \\
L(R) \leq CR^\nu & \text{for all } R > 1.
\end{cases}$$

Then $L(R) \equiv 0$.

3. A monotonicity result near $\partial \mathbb{R}^n_+$. In this section, we prove a weak version of Theorem 1. More precisely, we show that solutions of (1.1) are monotone increasing with respect to the $y$-direction near the boundary of $\mathbb{R}^n_+$. We prove such a result for systems involving a more general class of nonlinearities and for any $p, q > 1$. This is the first step to start the moving plane procedure which will be used in the proof of Theorem 1 later.

**Theorem 9.** Let $p, q > 1$ and $u, v \in C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$ be solutions to (1.1) with $|\nabla u|, |\nabla v| \in L^\infty(\mathbb{R}^N)$. Assume that $f, g \in C([0, \infty) \times [0, \infty))$ and, for some $T > 0$, it holds that

$$|f(s, t)| \leq C s^{p-1} \quad \text{and} \quad |g(s, t)| \leq C t^{q-1}$$

for $s, t \in [0, T]$, where $C = C(T) > 0$. Then it follows that there exists $\lambda > 0$ such that

$$\frac{\partial u}{\partial y} > 0 \quad \text{and} \quad \frac{\partial v}{\partial y} > 0 \quad \text{in } \Sigma_\lambda.$$

Consequently, for all $0 < \theta \leq \frac{1}{2}$, it holds that

$$u \leq u_\theta \quad \text{and} \quad v \leq v_\theta \quad \text{in } \Sigma_\theta.$$

In particular the result holds true under assumptions (H1)–(H4).

**Proof.** By contradiction, we assume that there exists a sequence of points $(x'_n, y_n)$ such that $y_n \to 0$ as $n \to \infty$ and

$$\frac{\partial u}{\partial y}(x'_n, y_n) \leq 0 \quad \text{or} \quad \frac{\partial v}{\partial y}(x'_n, y_n) \leq 0 \quad \text{for all } n \in \mathbb{N}^*.$$
Passing to a subsequence, we may assume that
\[
\frac{\partial u}{\partial y}(x_n', y_n) \leq 0 \quad \text{for all } n \in \mathbb{N}^*.
\] (3.1)

Now we fix some \( R > 0 \) and set
\[
w_n(x', y) = \frac{u(x' + x_n', y)}{u(x_n', R)}. \tag{3.2}
\]
Then \( w_n(0, R) = 1 \) and \( w_n \) verifies the equation
\[- \Delta_n w_n = c_n(x) w_n^{p-1} \quad \text{in } \mathbb{R}^N \tag{3.3}
\]
in the weak sense, where
\[c_n(x) = \frac{f(u(x' + x_n', y), v(x' + x_n', y))}{u(x_n', R)}.\]

By the Dirichlet condition and boundedness of \(|\nabla u| \text{ and } |\nabla v| \) in \( \mathbb{R}^N \), we have that \( u, v \in L^\infty(\Sigma_L) \) for any \( L > 0 \). Therefore, by the assumption on the nonlinearity \( f \), we obtain that
\[\|c_n\|_{L^\infty(\Sigma_L)} \leq C_0(L). \tag{3.4}\]

Since \( w_n(0, R) = 1 \), by the classical Harnack inequality, see [21, Theorem 7.2.1], we have that
\[\|w_n\|_{L^\infty(\Sigma^R_{0,R/2})} \leq C_H(R). \tag{3.5}\]

Next, let any \( x = (x', y) \in \Sigma^R_{0,R/2} \). Notice that we can find a point \( z = (z', 0) \) such that \( z' \in B_R' \) and \( |x - z| = R \). By the boundary Harnack inequality (see Theorem 7), we have
\[
\frac{w_n(x)}{y} \leq C \frac{w_n(z', R)}{R}.
\]
Combining this with (3.5), we obtain
\[\|w_n\|_{L^\infty(\Sigma^R_{0,R/2})} \leq \frac{C}{2} \cdot C_H(R). \tag{3.6}\]

From (3.5) and (3.6), it follows that
\[\|w_n\|_{L^\infty(\Sigma^R_{0,2R})} \leq C(R). \tag{3.7}\]

Now consider \( u \) and \( v \) defined on the entire space \( \mathbb{R}^N \) by odd reflection. That is
\[u(x', y) = -u(x', -y) \quad \text{and} \quad v(x', y) = -v(x', -y) \quad \text{in } \{y < 0\}.
\]
We also define
\[f(s, t) = -f(-s, -t) \quad \text{and} \quad g(s, t) = -g(-s, -t) \quad \text{in } \{s < 0, t < 0\}.
\]
Then \( u, v \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N) \) solve the first two equations of (1.1) in \( \mathbb{R}^N \). Hence \( w_n \) defined on the entire space \( \mathbb{R}^N \) by odd reflection solves
\[-\Delta_n w_n = c_n(x)|w_n|^{p-2}w_n \quad \text{in } \mathbb{R}^N,
\]
where \( c_n \) is defined on the entire space \( \mathbb{R}^N \) by even reflection, that is,
\[c_n(x) = \frac{f(u(x' + x_n', y), v(x' + x_n', y))}{|u(x' + x_n', y)|^{p-2}u(x' + x_n', y)}.
\]
From (3.7), we deduce \( \|w_n\|_{L^\infty(\Sigma^R_{2R,2R})} \leq C(R) \). Therefore, by the standard regularity theory, see [24, Theorem 1], we have that
\[\|w_n\|_{C^{1,\alpha}_{\text{loc}}(\Sigma^R_{2R,2R})} \leq C(R).
\]
for some $0 < \alpha < 1$. Hence we can use Ascoli-Arzelà’s theorem to get that

$$w_n \to w \quad \text{in } C^{1,\alpha'}_{\text{loc}}(\Sigma_{-2R,2R})$$

up to a subsequence for $\alpha' < \alpha$. On the other hand, by (3.4) and the fact that $c_n$ is even, we have $\|c_n\|_{L^\infty(\Sigma_{-2R,2R})} \leq C_1(2R)$. Hence

$$c_n(\cdot) \rightharpoonup c(\cdot) \quad \text{in } L^\infty(\Sigma_{-2R,2R})$$

up to a subsequence. Therefore, passing to the limit in (3.3), we obtain that $w$ weakly solves

$$\begin{cases}
-\Delta_p w = c(x)w^{p-1} & \text{in } \Sigma_{0,2R}, \\
w \geq 0 & \text{in } \Sigma_{0,2R}, \\
w = 0 & \text{on } \partial \Sigma_{0,2R} \cap \mathbb{R}_+^N.
\end{cases}$$

Notice that $w(0,R) = 1$ since $w_n(0,R) = 1$. By the strong maximum principle and Hopf’s Lemma (Theorem 3), we infer that $w > 0$ in $\Sigma_{0,2R}$ and

$$\frac{\partial w}{\partial y}(0,0) > 0.$$

However, from (3.1) and (3.2), we have

$$\frac{\partial w_n}{\partial y}(0, y_n) = \frac{1}{u(x'_{n}, R)} \cdot \frac{\partial u}{\partial y}(x'_{n}, y_n) \leq 0,$$

which implies $\frac{\partial w}{\partial y}(0,0) < 0$ after passing to the limit. This contradiction concludes the proof.

4. Weak comparison principle in narrow domains. We define by $\mathcal{P}$ the orthogonal projection from $\mathbb{R}^N$ onto $\mathbb{R}^{N-1}$, i.e., $\mathcal{P}(x) = x'$ for all $x = (x', y) \in \mathbb{R}^{N-1} \times \mathbb{R}$. We prove the following weak comparison principle for system (1.1) in narrow domains.

**Proposition 2.** Let $p, q > 2$ and $u, v \in C^{1,\alpha}_{\text{loc}}(\Sigma_{+2})$ be solutions to (1.1) with $|\nabla u|, |\nabla v| \in L^\infty(\Sigma_{+})$ and $f, g$ satisfy (H1), (H2). Fix $\lambda > 0$ and let $\lambda \in [\lambda, \lambda + 1]$. Assume that

$$u \leq u_\lambda \quad \text{and} \quad v \leq v_\lambda \quad \text{on } \partial \Sigma_{\alpha,\beta},$$

where $0 \leq \alpha \leq \beta \leq \lambda$. Assume furthermore that

$$u \geq \gamma \text{ on } I_{\lambda,\lambda}^1 \quad \text{and} \quad v \geq \gamma \text{ on } I_{\lambda,\lambda}^2,$$

(4.1)

where $\gamma > 0$, $I_{\lambda,\lambda}^1 = \left\{(x', \lambda) \mid x' \in \mathcal{P}(\text{supp}(u - u_\lambda)^+)\right\}$, $I_{\lambda,\lambda}^2 = \left\{(x', \lambda) \mid x' \in \mathcal{P}(\text{supp}(v - v_\lambda)^+)\right\}$.

Then, there exists $h_0 = h_0(p, q, f, g, \gamma, N, \lambda, \lambda) \|\nabla u\|_{L^\infty(\Sigma_{2\lambda}^+)} \|\nabla v\|_{L^\infty(\Sigma_{2\lambda}^+)}$ such that

$$u \leq u_\lambda \quad \text{and} \quad v \leq v_\lambda \quad \text{in } \Sigma_{\alpha,\beta},$$

whenever $\beta - \alpha \leq h_0$. 

Proof. Since $|∇u|$ is bounded and $u = 0$ on $∂R^N_+$, we have $(u - u_\lambda)^+ \in L^\infty(Σ_{α,β})$. Now we define

$$Ψ(x', y) := (u - u_\lambda)^+ ϕ^2_R(x'),$$

where $R > 0$ and $ϕ_R ∈ C^\infty_c(ℝ^N)$ is a nonnegative function such that

$$\begin{cases}
ϕ_R = 1 & \text{in } B'_R, \\
ϕ_R = 0 & \text{in } ℝ^{N-1} \setminus B'_{2R}, \\
|∇ϕ_R| ≤ \frac{C}{R} & \text{in } B'_{2R} \setminus B'_R.
\end{cases}$$

From $u ≤ u_\lambda$ on $∂Σ_{α,β}$, it follows that $Ψ ∈ C^1_c(Σ^2_{α,β})$.

Since $u, v$ are solutions to problem (1.1), we deduce that $u, v, u_\lambda, v_\lambda$ satisfy

$$\begin{align*}
-Δ_p u &= f(u, v) \quad \text{in } Σ_{α,β}, \\
-Δ_q v &= f(u, v) \quad \text{in } Σ_{α,β}, \\
-Δ_p u_λ &= f(u_λ, v_λ) \quad \text{in } Σ_{α,β}, \\
-Δ_q v_λ &= f(u_λ, v_λ) \quad \text{in } Σ_{α,β}, \\
u &≤ u_λ \quad \text{on } ∂Σ_{α,β}, \\
v &≤ v_λ \quad \text{on } ∂Σ_{α,β},
\end{align*}$$

(4.2)

Using $Ψ$ as test functions in the first and the third equations of problem (4.2) and substracting, we get

$$\begin{align*}
\int_{Σ^2_{α,β}} (|∇u|^{p−2}∇u - |∇u_λ|^{p−2}∇u_λ, ∇(u - u_λ)^+)ϕ^2_R \\
+ \int_{Σ^2_{α,β}} (|∇u|^{p−2}∇u - |∇u_λ|^{p−2}∇u_λ, ∇ϕ^2_R)(u - u_λ)^+
\end{align*}$$

$$= \int_{Σ^2_{α,β}} (f(u, v) - f(u_λ, v_λ))(u - u_λ)^+ ϕ^2_R.$$ 

(4.3)

We recall that for all $η, η' ∈ ℝ^N$ with $|η| + |η'| > 0$, there exist positive constants $C_1, C_2$ depending on $p$ such that

$$\begin{align*}
(|η|^{p−2}η - |η'|^{p−2}η', η - η') &≥ C_1(|η| + |η'|)^{p−2}−2|η - η'|^2, \\
|η|^{p−2}η - |η'|^{p−2}η' &≤ C_2(|η| + |η'|)^{p−2}−2|η - η'|.
\end{align*}$$

(4.4)

It follows from (4.3) and (4.4) that

$$\begin{align*}
C_1 \int_{Σ^2_{α,β}} (|∇u| + |∇u_λ|)^{p−2}|∇(u - u_λ)^+|^2 ϕ^2_R \\
≤ \int_{Σ^2_{α,β}} (|∇u|^{p−2}∇u - |∇u_λ|^{p−2}∇u_λ, ∇(u - u_λ)^+)ϕ^2_R \\
= - \int_{Σ^2_{α,β}} (|∇u|^{p−2}∇u - |∇u_λ|^{p−2}∇u_λ, ∇ϕ^2_R)(u - u_λ)^+ \\
+ \int_{Σ^2_{α,β}} (f(u, v) - f(u_λ, v_λ))(u - u_λ)^+ ϕ^2_R \\
≤ C_2 \int_{Σ^2_{α,β}} (|∇u| + |∇u_λ|)^{p−2}|∇(u - u_λ)^+| |∇ϕ^2_R|(u - u_λ)^+
\end{align*}$$
By Young’s inequality, we have
\[ C_2 \int_{\Sigma_{a,b}^R} (\|\nabla u\| + |\nabla u_\lambda|)^{p-2} |\nabla (u - u_\lambda)^+| |\nabla \varphi_R^2|(u - u_\lambda)^+ \]
\[ = 2C_2 \int_{\Sigma_{a,b}^R} (\|\nabla u\| + |\nabla u_\lambda|)^{p-2} |\nabla (u - u_\lambda)^+| |\nabla \varphi_R| \varphi_R(u - u_\lambda)^+ \]
\[ \leq \frac{C_1}{2} \int_{\Sigma_{a,b}^R} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla (u - u_\lambda)^+|^2 \varphi_R^2 \]
\[ + \frac{2C_2^2}{C_1} \int_{\Sigma_{a,b}^R} (\|\nabla u\| + |\nabla u_\lambda|)^{p-2} |\nabla \varphi_R|^2 [(u - u_\lambda)^+]^2. \]
(4.6)

On the other hand, using the local Lipschitz continuity of \( f \), we can estimate the last term in (4.5) as follows
\[ \int_{\Sigma_{a,b}^R} (f(u,v) - f(u_\lambda,v_\lambda))(u - u_\lambda)^+ \varphi_R^2 \]
\[ = \int_{\Sigma_{a,b}^R} (f(u,v) - f(u_\lambda,v))(u - u_\lambda)^+ \varphi_R^2 \]
\[ + \int_{\Sigma_{a,b}^R} (f(u_\lambda,v) - f(u_\lambda,v_\lambda))(u - u_\lambda)^+ \varphi_R^2 \]
\[ \leq C \int_{\Sigma_{a,b}^R} [(u - u_\lambda)^+]^2 \varphi_R^2 + C \int_{\Sigma_{a,b}^R} (u - u_\lambda)^+(v - v_\lambda)^+ \varphi_R^2 \]
\[ \leq C \int_{\Sigma_{a,b}^R} [(u - u_\lambda)^+]^2 \varphi_R^2 + C \int_{\Sigma_{a,b}^R} [(v - v_\lambda)^+]^2 \varphi_R^2, \] (4.7)
where \( C = C(f, \lambda, \|\nabla u\|_{L^\infty(\Sigma_{a,b}^R)}, \|\nabla v\|_{L^\infty(\Sigma_{a,b}^R)}) \). In fact, \( C \) depends on the Lipschitz constant of \( f \) in the compact set
\[ [0, \max\{\|u\|_{L^\infty(\Sigma_{a,b}^R)}, \|u_\lambda\|_{L^\infty(\Sigma_{a,b}^R)}\}] \times [0, \max\{\|v\|_{L^\infty(\Sigma_{a,b}^R)}, \|v_\lambda\|_{L^\infty(\Sigma_{a,b}^R)}\}]. \]

Substituting (4.6) and (4.7) into (4.5), we obtain
\[ \int_{\Sigma_{a,b}^R} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla (u - u_\lambda)^+|^2 \varphi_R^2 \]
\[ \leq C \int_{\Sigma_{a,b}^R} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla \varphi_R|^2 [(u - u_\lambda)^+]^2 \]
\[ + C \int_{\Sigma_{a,b}^R} [(u - u_\lambda)^+]^2 \varphi_R^2 + C \int_{\Sigma_{a,b}^R} [(v - v_\lambda)^+]^2 \varphi_R^2. \] (4.8)

Since \( |\nabla u|, |\nabla u_\lambda| \) are bounded and \( |\nabla \varphi_R| \leq \frac{C}{R} < C \) for \( R > 1 \), we have
\[ \int_{\Sigma_{a,b}^R} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla \varphi_R|^2 [(u - u_\lambda)^+]^2 \leq C \int_{\Sigma_{a,b}^R} [(u - u_\lambda)^+]^2. \]
Hence (4.8) yields
\[ \int_{\Sigma_{a,b}^R} (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla (u - u_\lambda)^+|^2 \varphi_R^2 \]
\[ \leq C \int_{\Sigma^{2R}_{\alpha,\beta}} [(u - u\lambda)^+]^2 + C \int_{\Sigma^{2R}_{\alpha,\beta}} [(v - v\lambda)^+]^2 \]  

(4.9)

for all \( R > 1 \). Following the ideas in [16], we exploit the weighted Poincaré type inequality (Theorem 6) and a covering argument to estimate the last two terms in (4.9). Let us consider the open hypercubes \( Q_i \) of \( \mathbb{R}^N \) defined by

\[ Q_i = Q'_i \times (\alpha, \beta), \]

where \( Q'_i \) are open hypercubes of \( \mathbb{R}^{N-1} \) with edge \( \beta - \alpha \) such that \( \bigcup_i Q'_i = \mathbb{R}^{N-1} \). Moreover we assume that \( Q_i \cap Q_j = \emptyset \) for \( i \neq j \) and

\[ \Sigma^{2R}_{\alpha,\beta} \subset \bigcup_{i=1}^m Q_i. \]  

(4.10)

It follows that each set \( Q_i \) has diameter \( \text{diam}(Q_i) = \sqrt{N}(\beta - \alpha) \), \( i = 1, \ldots, m \). As in [16], we will show that

\[ \int_{Q_i} [(u - u\lambda)^+]^2 \leq C_P(Q_i) \int_{Q_i} (|\nabla u| + |\nabla u\lambda|)^{p-2} |\nabla (u - u\lambda)^+|^2, \]  

(4.11)

where \( C_P(Q_i) \) is given by Theorem 6 and has the property that it goes to zero if the diameter of \( Q_i \) goes to zero.

To this end, we define

\[ w(x) := \begin{cases} 
(u - u\lambda)^+(x', y) & \text{if } (x', y) \in Q_i; \\
-(u - u\lambda)^+(x', 2\beta - y) & \text{if } (x', y) \in Q'_i, 
\end{cases} \]

where \( Q'_i = \{(x', y) \in \mathbb{R}^N \mid (x', 2\beta - y) \in Q_i \} \). Note that, if \( w \) vanishes identically in \( Q_i \), then (4.11) automatically holds. Therefore, in what follows, we may assume \( w \neq 0 \) in \( Q_i \). Since \( \int_{Q_i \cup Q'_i} w = 0 \), we have

\[ w(x) = \hat{C} \int_{Q_i \cup Q'_i} \frac{(x_i - z_i) \frac{\partial w}{\partial z_i}(z)}{|x - z|^N} dz \quad \text{for a.e. } x \in Q_i \cup Q'_i \]

where \( \hat{C} = \hat{C}(\beta - \alpha, N) \) is a positive constant. Hence for a.e. \( x \in Q_i \), we have

\[ |w(x)| \leq \hat{C} \int_{Q_i} \frac{|\nabla w(z)|}{|x - z|^{N-1}} dz + \hat{C} \int_{Q'_i} \frac{|\nabla w(z)|}{|x - z|^{N-1}} dz \]

\[ = \hat{C} \int_{Q_i} \frac{|\nabla w(z)|}{|x - z|^{N-1}} dz + \hat{C} \int_{Q_i} \frac{|\nabla w(z)|}{|x - (z', 2\beta - z_N)|^{N-1}} dz \]

\[ \leq 2\hat{C} \int_{Q_i} \frac{|\nabla w(z)|}{|x - z|^{N-1}} dz. \]

Therefore, (2.2) holds.

Next, we show that (2.1) holds with \( \rho = |\nabla u\lambda|^{p-2} \). By assumption (4.1) and the classical Harnack inequality, it follows that there exists \( \tau > 0 \) such that

\[ u \geq \tau \quad \text{and} \quad v \geq \tau \quad \text{in } \tilde{Q}'_i \times [\tilde{X}/2, 5\tilde{X}/2 + 2], \]  

(4.12)

where \( \tilde{Q}'_i = \{ x \in \mathbb{R}^{N-1} \mid \text{dist}(x, Q'_i) < \tilde{X}/2 \} \).

Let \( Q^\lambda_i \) be the reflection of \( Q_i \) with respect to the hyperplane \( \{(x', y) \in \mathbb{R}^N \mid y = \lambda \} \). Since \( \lambda \in [\tilde{X}, \tilde{X} + 1] \), one may check that the \( \tilde{X}/2 \)-neighborhood of \( Q^\lambda_i \) is
contained in \( Q_i \times [\lambda/2, 5\lambda/2 + 2] \). Therefore, Proposition 1 applies with

\[
\beta_1 = \min_{\gamma \leq s \leq \|u\|_{L^\infty(B_{2R})}} f(s, t) \quad \text{and} \quad \beta_2 = \lambda/2.
\]

Then we obtain that

\[
\int_{Q_i} \frac{1}{|\nabla u|^{p-2}|x-y|} \, dy \leq C^*(\beta_1, \beta_2) \quad \text{for} \quad x \in Q_i^\lambda.
\]

By change of variables, we have

\[
\int_{Q_i} \frac{1}{|\nabla u|^{p-2}|x-y|} \, dy \leq C^*(\beta_1, \beta_2) \quad \text{for} \quad x \in Q_i.
\]

That is, (2.1) holds with \( \rho = |\nabla u\|_{p-2} \). Now we can exploit Theorem 6 to obtain

\[
\int_{Q_i} w^2 \leq C_P(Q_i) \int_{Q_i} |\nabla u|^{p-2} |\nabla w|^2,
\]

and then (4.11) follows.

Using (4.10) and (4.11), we deduce that

\[
\int_{\Sigma^R_{a, \beta}} [(u - u\lambda)^+]^2 \leq \max_{1 \leq i \leq m} C_P(Q_i) \sum_{i=1}^m \int_{\Sigma^R_{a, \beta} \cap Q_i} (|\nabla u| + |\nabla u\|)^{p-2} |\nabla (u - u\lambda)^+]^2
\]

\[= C_P^* \int_{\Sigma^R_{a, \beta}} (|\nabla u| + |\nabla u\|)^{p-2} |\nabla (u - u\lambda)^+]^2, \tag{4.13}\]

where

\[
C_P^* := \max_{1 \leq i \leq m} C_P(Q_i) \to 0 \quad \text{as} \quad \beta - \alpha \to 0 \tag{4.14}\]

thanks to Theorem 6.

In the same way, we have

\[
\int_{\Sigma^R_{a, \beta}} [(v - v\lambda)^+]^2 \leq C_P^* \int_{\Sigma^R_{a, \beta}} (|\nabla v| + |\nabla v\|)^{q-2} |\nabla (v - v\lambda)^+]^2. \tag{4.15}\]

Collecting (4.9), (4.13), (4.15) and using the fact that \( \varphi_R = 1 \) in \( B_{2R}^c \), we derive

\[
\int_{\Sigma^R_{a, \beta}} (|\nabla u| + |\nabla u\|)^{p-2} |\nabla (u - u\lambda)^+]^2 \leq C_P^* \int_{\Sigma^R_{a, \beta}} (|\nabla v| + |\nabla v\|)^{q-2} |\nabla (v - v\lambda)^+]^2, \tag{4.16}\]

Repeating the above arguments with the second and the fourth equations of system (4.2), we have

\[
\int_{\Sigma^R_{a, \beta}} (|\nabla v| + |\nabla v\|)^{q-2} |\nabla (v - v\lambda)^+]^2 \leq C_P^* \int_{\Sigma^R_{a, \beta}} (|\nabla u| + |\nabla u\|)^{p-2} |\nabla (u - u\lambda)^+]^2 + C_P^* \int_{\Sigma^R_{a, \beta}} (|\nabla v| + |\nabla v\|)^{q-2} |\nabla (v - v\lambda)^+]^2.
\]

Adding (4.16) and (4.17), we deduce that

\[
L(R) \leq \theta L(2R) \quad \text{for all} \quad R \geq 1,
\]
where

\[ L(R) := \int_{\Sigma_{\alpha,\beta}} (|\nabla u| + |\nabla u_{\lambda}|)^{p-2} |\nabla(u - u_{\lambda})|^2 + \int_{\Sigma_{\alpha,\beta}} (|\nabla v| + |\nabla v_{\lambda}|)^{q-2} |\nabla(v - v_{\lambda})|^2 \]

and

\[ \theta := 2C^*_p. \]

Due to (4.14), we can choose

\[ h_0 = h_0(p, q, f, g, \lambda, \Sigma, \|\nabla u\|_{L^{\infty}(\Sigma_{\alpha,\beta})}, \|\nabla v\|_{L^{\infty}(\Sigma_{\alpha,\beta})}) \]

such that if \( \beta - \alpha \leq h_0 \), then \( \theta < 2^{-N} \). Moreover, from the boundedness of \( |\nabla u| \) and \( |\nabla u_{\lambda}| \) we clearly have \( L(R) \leq CR^N \). Therefore, Lemma 8 can be applied to yield

\[ L(R) \equiv 0. \]

This means that \( (u - u_{\lambda})^+ \equiv (v - v_{\lambda})^+ \equiv 0 \) and hence \( u \leq u_{\lambda} \) and \( v \leq v_{\lambda} \) in \( \Sigma_{\alpha,\beta} \). \( \square \)

Proposition 2 is a key ingredient in the moving plane procedure. As it will be clear later, we will use Proposition 2 to show that the plane can move a little bit from the limiting position and hence a contradiction will occur. To make that argument work, we have to show that the constant \( \gamma \) in Proposition 2 can be chosen independently of \( \lambda \in [\bar{\lambda}, \bar{\lambda} + 1] \). This implies that \( h_0 \) is also independent of \( \lambda \).

**Lemma 10.** Let \( p, q > 1 \) and \( u, v \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N_+) \) be solutions to (1.1) with \( |\nabla u|, |\nabla v| \in L^\infty(\mathbb{R}^N_+) \). Assume that \( f, g \in C([0, \infty) \times [0, \infty)) \) satisfy (H4). Then for each \( \lambda > 0 \), there exist \( \varepsilon_0 > 0 \) and \( \gamma > 0 \) such that

\[ u \geq \gamma \text{ on } I_{\lambda,\lambda+\varepsilon}^1 \quad \text{and} \quad v \geq \gamma \text{ on } I_{\lambda,\lambda+\varepsilon}^2 \]

for all \( 0 \leq \varepsilon \leq \varepsilon_0 \), where \( I_{\lambda,\lambda+\varepsilon}^1 \) and \( I_{\lambda,\lambda+\varepsilon}^2 \) are defined in Proposition 2.

**Proof.** By contradiction, assume that there exist a number \( \lambda > 0 \), a sequence \( \varepsilon_n \to 0 \) and a sequence of points \( (x'_n, y_n) \in \Sigma_{\lambda+\varepsilon_n} \) such that for each \( n \in \mathbb{N}^+ \) we have \( \varepsilon_n < \frac{1}{n} \) and one of the following conditions holds

(i) \( u(x'_n, y_n) \geq u_{\lambda+\varepsilon_n}(x'_n, y_n) \) and \( u(x'_n, \lambda) < \frac{1}{n} \),

(ii) \( v(x'_n, y_n) \geq v_{\lambda+\varepsilon_n}(x'_n, y_n) \) and \( v(x'_n, \lambda) < \frac{1}{n} \).

Passing to a subsequence if necessary, we may assume that (i) holds and \( y_n \) is convergent. That is,

\[ u(x'_n, y_n) \geq u_{\lambda+\varepsilon_n}(x'_n, y_n), \quad \lim_{n \to \infty} u(x'_n, \lambda) = 0, \quad \lim_{n \to \infty} y_n = y_0 \in [0, \lambda]. \quad (4.18) \]

Let us set

\[ w_n(x', y) = \frac{u(x', x'_n, y)}{u(x'_n, \lambda)}. \quad (4.19) \]

Then

\[ w_n(0, \lambda) = 1 \quad (4.20) \]

and \( w_n \) verifies the equation

\[ -\Delta_p w_n = c_n(x)w_n^{p-1} \quad \text{in } \mathbb{R}^N_+ \quad (4.21) \]

in the weak sense, where

\[ c_n(x) = \frac{f(u(x' + x'_n, y), v(x' + x'_n, y))}{w^{p-1}(x' + x'_n, y)}. \quad (4.22) \]
By the Dirichlet condition and boundedness of $|\nabla u|$ and $|\nabla v|$ in $\mathbb{R}^N$, we have that $u, v \in L^\infty(\Sigma_L)$ for any $L > 0$. Therefore, by the assumption on the nonlinearity $f$, we obtain that $||c_n||_{L^\infty(\Sigma_L)} \leq C_0(L)$.

Arguing as in the proof of Theorem 9, we can consider $u$ defined on the entire space $\mathbb{R}^N$ by odd reflection and use the standard regularity theory and the Ascoli-Arzelà theorem to get that $w_n \rightarrow w_{L,R}$ in $C^{1,\alpha'}_{\text{loc}}(\Sigma_{R-L,L})$

up to a subsequence for $\alpha' < \alpha$. By performing a standard diagonal process, we can define $w$ in the entire space $\mathbb{R}^N$ in such a way that $w$ is locally the limit of subsequences of $w_n$.

Since $w_n$ is uniformly bounded on compact sets, by (4.18) and (4.19) it follows that $u(x' + x'_n, y) \rightarrow 0$ as $n \rightarrow \infty$. By (4.22) and recalling that $\lim_{s \rightarrow 0^+} \frac{f(s, t)}{s^{p-1}} = 0$ locally uniformly in $t$, it follows that $c_n(x) \rightarrow 0$ on compact sets. Therefore, letting $n \rightarrow \infty$ in (4.21), we obtain

$$\begin{cases}
-\Delta_p w = 0 & \text{in } \mathbb{R}^N_+,
 w \geq 0 & \text{in } \mathbb{R}^N_+,
 w = 0 & \text{on } \partial\mathbb{R}^N_+.
\end{cases}$$

From (4.20) and the uniform convergence of $w_n$, we have $w(0, \lambda) = 1$. Hence $w > 0$ by the strong maximum principle. Now we can apply [19, Theorem 3.1] to deduce that $w$ must be affine linear, i.e., $w(x', y) = ky$ for some $k > 0$ by the Dirichlet condition.

If $y_0 \in [0, \lambda)$, by (4.18) and the uniform convergence of $w_n$, we would have $w(0, y_0) \leq w_{\lambda}(0, y_0)$.

This is a contradiction since $w(x', y) = ky$ for some $k > 0$.

Therefore, let us assume that $y_n \rightarrow \lambda$ and note that, by (4.18) and the mean value theorem, at some point $\xi_n$ lying on the segment from $(0, y_n)$ to $(0, 2(\lambda + \varepsilon_n) - y_n)$, it would hold that

$$\frac{\partial w_n}{\partial y}(0, \xi_n) \leq 0.$$ 

Since $w_n \rightarrow w$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$, we have that

$$\frac{\partial w}{\partial y}(0, 0) \leq 0.$$ 

Again this is a contradiction since $w(x', y) = ky$ for some $k > 0$ and the lemma is proved.

5. **Monotonicity of solutions in half-spaces.** The results proved in previous sections allow us to conclude the proof of Theorem 1 in this section.

**Proof of Theorem 1.** Let us consider

$$\Lambda := \{ t > 0 \mid u \leq u_\theta \text{ and } v \leq v_\theta \text{ in } \Sigma_\theta \text{ for all } 0 < \theta \leq t\}.$$ 

Thanks to Theorem 9, we have $\Lambda \neq \emptyset$. Hence we can define $\bar{\lambda} = \sup \Lambda$. 


To conclude the proof, we have to show that 
\[
\bar{\lambda} = +\infty.
\]

By contradiction, in what follows, we assume that \(\bar{\lambda} < +\infty\). Then we have \(u \leq u_{\bar{\lambda}}\) and \(v \leq v_{\bar{\lambda}}\) in \(\Sigma_{\bar{\lambda}}\).

We claim that: for all \(0 < \delta < \frac{\bar{\lambda}}{2}\), there exists \(\varepsilon_0\) such that for all \(0 < \varepsilon \leq \varepsilon_0\) it follows that
\[
u \leq u_{\bar{\lambda} + \varepsilon} \quad \text{and} \quad v \leq v_{\bar{\lambda} + \varepsilon} \quad \text{in} \quad \Sigma_{\delta, \bar{\lambda} - \delta}.
\]

The proof of this claim follows by an analysis of the limiting profile at infinity. Indeed, if \((\varepsilon)\) is a sequence of positive numbers \(\varepsilon_0\) converging to 0 and a sequence of points \(x_n = (x_n', y_n) \in \Sigma_{\delta, \bar{\lambda} - \delta}\) such that
\[
u(x_n', y_n) < u(x_n', y_n) \quad \text{or} \quad \nu(x_n', y_n) > v(x_n', y_n).
\]

Passing to a subsequence if necessary, we may and do assume that \(\lim_{n \to \infty} y_n = y_0 \in [\delta, \bar{\lambda} - \delta]\) and
\[
u(x_n', y_n) < u(x_n', y_n)
\]
for all \(n \in \mathbb{N}\). Let us define
\[
u_n(x', y) = u(x' + x_n', y), \quad \nu_n(x', y) = v(x' + x_n', y).
\]

We can consider \(u, v\) (and consequently \(\nu_n, \nu_n\)) defined on the entire space \(\mathbb{R}^N\) by odd reflection. That is
\[
u(x', y) = -u(x', -y) \quad \text{and} \quad v(x', y) = -v(x', -y) \quad \text{in} \quad \{y < 0\}
\]
with
\[
f(s, t) = -f(-s, -t) \quad \text{and} \quad g(s, t) = -g(-s, -t) \quad \text{in} \quad \{s < 0, t < 0\}.
\]

Notice that for every \(\eta > 0\), we have \(\|\nu_n\|_{L^\infty(\Sigma_n)} = \|\tilde{\nu}\|_{L^\infty(\Sigma_n)} \leq C\) and \(\|\nu_n\|_{L^\infty(\Sigma_n)} = \|\tilde{\nu}\|_{L^\infty(\Sigma_n)} \leq C\). Moreover, \(\tilde{\nu}_n\) and \(\tilde{\nu}_n\) satisfy
\[
\begin{cases}
-\Delta_p \tilde{\nu}_n = f(\tilde{\nu}_n, \tilde{\nu}_n) & \text{in} \ \mathbb{R}^N, \\
-\Delta_q \tilde{\nu}_n = g(\tilde{\nu}_n, \tilde{\nu}_n) & \text{in} \ \mathbb{R}^N.
\end{cases}
\]

Therefore, by the standard regularity theory \cite{24}, we have that
\[
\|\tilde{\nu}_n\|_{C^{1, \alpha}_{\text{loc}}(\mathbb{R}^N)} \leq C, \quad \|\tilde{\nu}_n\|_{C^{1, \alpha}_{\text{loc}}(\mathbb{R}^N)} \leq C
\]
for some \(0 < \alpha < 1\). By Ascoli-Arzelà’s theorem and a diagonal process, we have
\[
\tilde{\nu}_n \to \tilde{\nu} \ \text{in} \ C^{1, \alpha'}_{\text{loc}}(\mathbb{R}^N), \quad \tilde{\nu}_n \to \tilde{\nu} \ \text{in} \ C^{1, \alpha'}_{\text{loc}}(\mathbb{R}^N)
\]
up to a subsequence for \(\alpha' < \alpha\). Moreover, we have
- \(\tilde{\nu}, \tilde{\nu} \geq 0 \text{ in } \mathbb{R}^N_+\) and \(\tilde{\nu} = \tilde{\nu} = 0 \text{ on } \partial \mathbb{R}^N_+\),
- \(\tilde{\nu} \leq \tilde{\nu}_{\tilde{\lambda}}\) and \(\tilde{\nu} \leq \tilde{\nu}_{\tilde{\lambda}}\) in \(\Sigma_{\tilde{\lambda}}\),
- \(\tilde{\nu}(0, y_0) \geq \tilde{\nu}_{\tilde{\lambda}}(0, y_0)\).

By standard arguments, we can pass to the limit of the first equation of (5.2) and we obtain
\[
-\Delta_p \tilde{\nu} = f(\tilde{\nu}, \tilde{\nu}) \quad \text{in} \ \mathbb{R}^N_+
\]
in the weak form. By the strong maximum principle (see Theorem 3), it follows that \(\tilde{\nu} > 0\) or \(\tilde{\nu} = 0\).
Case 1: \( \tilde{u} > 0 \). Since \( \tilde{v} \leq \tilde{w} \) in \( \Sigma_\omega \) and \((s, t) \mapsto f(s, t)\) is non-decreasing in \( t \), it follows that
\[
-\Delta_p \tilde{u} - f(\tilde{u}, \tilde{v}) = -\Delta_p \tilde{w} - f(\tilde{w}, \tilde{v}) \leq -\Delta_p \tilde{u} - f(\tilde{u}, \tilde{v}) \quad \text{in } \Sigma_\omega.
\]

By the strong comparison principle (see Theorem 4), we get that \( \tilde{u} \leq \tilde{w} \) implies \( \tilde{u} < \tilde{w} \), since the case \( \tilde{u} = \tilde{w} \) is clearly impossible being \( \tilde{u}(x', 0) = 0 \). This is a contradiction since \( \tilde{u}(0, y_0) \geq \tilde{w}(0, y_0) \).

Case 2: \( \tilde{u} = 0 \).

Let us set
\[
\tilde{w}_n(x', y) = \frac{w(x' + x'_n, y)}{u(x'_n, y_n)}.
\]
Then
\[
w_n(0, y_n) = 1 \quad (5.3)
\]
and \( w_n \) verifies the equation
\[
-\Delta_p w_n = c_n(x) u_{n}^{p-1} \quad \text{in } \mathbb{R}_+^N
\]
(5.4)
in the weak sense, where
\[
c_n(x) = \frac{f(u(x' + x'_n, y), v(x' + x'_n, y))}{u^{p-1}(x' + x'_n, y)}.
\]

As in the proof of Theorem 9, we have \( \|c_n\|_{L^\infty(\Sigma_\omega)} = C \) for \( L > 0 \).

We can therefore exploit the Harnack inequality to get for any compact set \( K \),
\[
\sup_{K \cap \{y \geq \delta\}} w_n \leq C_H \inf_{K \cap \{y \geq \delta\}} w_n \leq C_H.
\]

Also, by the monotonicity of \( u \) in \( \Sigma_\omega \) we have
\[
\sup_{K \cap \{y \geq 0\}} w_n \leq \sup_{K \cap \{y \geq \delta\}} w_n \leq C_H.
\]

We can therefore use \( C^{1, \alpha} \) estimates, Ascoli’s theorem, and a standard diagonal process to show that
\[
w_n \to w \text{ in } C^{1, \alpha'}(\mathbb{R}^N)
\]
up to a subsequence for \( \alpha' < \alpha \). Arguing exactly as above, we see that

- \( w \geq 0 \) in \( \mathbb{R}_+^N \) and \( w = 0 \) on \( \partial \mathbb{R}_+^N \),
- \( w \leq w_\omega \) in \( \Sigma_\omega \),
- \( w(0, y_0) \geq w_\omega(0, y_0) \).

Since
\[
\tilde{u}_n(x', y) = u(x' + x'_n, y) \to 0, \quad \tilde{v}_n(x', y) = v(x' + x'_n, y) \to \tilde{v} \quad \text{in } C^{1, \alpha'}_{loc}(\mathbb{R}^N)
\]
and
\[
\lim_{s \to 0^+} \frac{f(s, t)}{s^{p-1}} = 0 \text{ locally uniformly in } t,
\]
we can apply [19, Theorem 3.1] to deduce that \( w \) must have the form \( w(x', y) = ky \) for some \( k > 0 \). This is a contradiction since by construction we would also get \( w(0, y_0) \geq w_\omega(0, y_0) \).
Therefore, (5.1) has been proved.

Next, we can choose \( \delta \) and \( \varepsilon_0 < 1 \) sufficiently small such that Proposition 2 applies in \( \Sigma_\delta \) and \( \Sigma_{\lambda - \delta, \lambda + \varepsilon} \) with \( \lambda = \bar{\lambda} + \varepsilon \). Thanks to Lemma 10, the parameter \( h_0 \) in the statement of Proposition 2 can be chosen independently of \( \varepsilon \) since there \( \gamma \) does not depend on \( \varepsilon \). Then we conclude that

\[
u \leq u_{\lambda + \varepsilon} \quad \text{and} \quad v \leq v_{\lambda + \varepsilon} \quad \text{in} \quad \Sigma_\delta \cup \Sigma_{\lambda - \delta, \lambda + \varepsilon}.
\]

Combining this with (5.1), we have that \( u \leq u_{\lambda + \varepsilon} \) and \( v \leq v_{\lambda + \varepsilon} \) in \( \Sigma_{\lambda + \varepsilon} \) for all \( 0 < \varepsilon \leq \varepsilon_0 \). This is a contradiction with the definition of \( \bar{\lambda} \). Therefore, it must hold that \( \bar{\lambda} = +\infty \).

This implies the monotonicity of \( u \) and \( v \) in the half-space, that is \( \partial u / \partial y \geq 0 \) and \( \partial v / \partial y \geq 0 \) in \( \mathbb{R}^N_+ \). By Theorem 5, since \( u \) and \( v \) are not trivial, we must have

\[
\frac{\partial u}{\partial y} > 0 \quad \text{and} \quad \frac{\partial v}{\partial y} > 0 \quad \text{in} \quad \mathbb{R}^N_+.
\]

Therefore, the set of critical points of \( u \) and \( v \) are empty (i.e., \( \{ \nabla u = 0 \} = \{ \nabla v = 0 \} = \emptyset \) and consequently the equations are no more degenerate. Then the fact that \( u, v \in C^{2,\alpha'}(\mathbb{R}^N_+) \) follows from the standard regularity results for non-degenerate elliptic equations. \( \square \)

6. Symmetry of solutions in strips. In this section, we use the moving plane method to study the symmetry of positive solutions to system (1.2). First of all, we have

**Theorem 11.** Let \( p, q > 2 \) and \( u, v \in C^{1,\alpha}_{\text{loc}}(\Sigma_h) \cap W^{1,\infty}(\Sigma_h) \) be solutions to (1.2). Assume that \( f, g \in C([0, \infty) \times [0, \infty)) \) and, for some \( T > 0 \), it holds that

\[
|f(s, t)| \leq Cs^{p-1} \quad \text{and} \quad |g(s, t)| \leq Ct^{q-1} \quad \text{for} \quad s, t \in [0, T],
\]

where \( C = C(T) > 0 \). Then it follows that there exits \( \lambda \in (0, h) \) such that

\[
\frac{\partial u}{\partial y} > 0 \quad \text{and} \quad \frac{\partial v}{\partial y} > 0 \quad \text{in} \quad \Sigma_\lambda.
\]

Consequently, for all \( 0 < \theta \leq \frac{1}{2} \), it holds that

\[
u \leq u_\theta \quad \text{and} \quad v \leq v_\theta \quad \text{in} \quad \Sigma_\theta.
\]

In particular the result holds true under assumptions (H1)–(H4).

For the proof, we may argue as in the proof of Theorem 9 with \( R = h/4 \). Therefore, the proof of Theorem 11 will be omitted. Next, we need the following weak comparison principle for system (1.2) in narrow domains.

**Proposition 3.** Let \( p, q > 2 \) and \( u, v \in C^{1,\alpha}_{\text{loc}}(\Sigma_h) \cap W^{1,\infty}(\Sigma_h) \) be solutions to (1.2) and \( f, g \) satisfy (H1), (H2). Fix \( \bar{\lambda} \in (0, h/2) \) and let \( \lambda \in [\bar{\lambda}, \bar{\lambda}/2 + h/4] \). Assume that

\[
u \leq u_\lambda \quad \text{and} \quad v \leq v_\lambda \quad \text{on} \quad \partial \Sigma_{\alpha, \beta},
\]

where \( 0 \leq \alpha \leq \beta \leq \lambda \). Assume furthermore that

\[
u \geq \gamma \quad \text{on} \quad I^1_{\bar{\lambda}, \lambda} \quad \text{and} \quad v \geq \gamma \quad \text{on} \quad I^2_{\bar{\lambda}, \lambda}, \quad (6.1)
\]

where \( \gamma > 0 \),

\[
I^1_{\bar{\lambda}, \lambda} = \{(x', \bar{\lambda}) \mid x' \in \mathcal{P}(\text{supp}(u - u_\lambda)^+)\},
\]

\[
I^2_{\bar{\lambda}, \lambda} = \{(x', \bar{\lambda}) \mid x' \in \mathcal{P}(\text{supp}(v - v_\lambda)^+)\}.
\]
Then, there exists \( h_0 = h_0(p, q, f, g, \gamma, N, \lambda, \|\nabla u\|_{L^\infty(\Sigma_h)}, \|\nabla v\|_{L^\infty(\Sigma_h)}) \) such that
\[
  u \leq u_\lambda \quad \text{and} \quad v \leq v_\lambda \quad \text{in} \ \Sigma_{\alpha, \beta}
\]
whenever \( \beta - \alpha \leq h_0 \).

**Proof.** We may reuse almost all arguments in the proof of Theorem 2 noticing that \( \Sigma_{2\lambda + 2} \) would be replaced with \( \Sigma_h \). Instead of (4.12), now we apply the classical Harnack inequality to deduce
\[
  u \geq \gamma \quad \text{and} \quad v \geq \gamma \quad \text{in} \ \tilde{Q}_\lambda \times [\lambda - \beta_2, \lambda + h/2 + \beta_2],
\]
where \( \beta_2 = \min \{\lambda/2, h/4 - \lambda/2\} \) and \( \tilde{Q}_\lambda = \{x \in \mathbb{R}^{N-1} \mid \text{dist}(x, Q_\lambda) < \beta_2\} \). Since \( \lambda \in [\lambda, \lambda/2 + h/4] \), one may check that the \( \beta_2 \)-neighborhood of \( Q_\lambda \) is contained in \( \tilde{Q}_\lambda \times [\lambda - \beta_2, \lambda + h/2 + \beta_2] \). Therefore, Proposition 1 applies with the above \( \beta_2 \) and
\[
  \beta_1 = \min_{\tau \leq s \leq \|u\|_{L^\infty(\Sigma_h)}}, f(s, t).
\]
The rest of the proof is unchanged. \(\square\)

Finally, we point out that the constant \( \gamma \) in (6.1) can be chosen independently of \( \lambda \).

**Lemma 12.** Let \( p, q > 2 \) and \( u, v \in C_{\text{loc}}^{1,\gamma}(\Sigma_h) \cap W^{1,\infty}(\Sigma_h) \) be solutions to (1.2). Assume that \( f, g \in C([0, \infty) \times [0, \infty)) \) satisfy (H4). Then for each \( \lambda \in (0, h/2) \), there exist \( 0 < \varepsilon_0 < h/2 - \lambda \) and \( \gamma > 0 \) such that
\[
  u \geq \gamma \quad \text{on} \quad \Gamma_{\lambda, \lambda + \varepsilon} \quad \text{and} \quad v \geq \gamma \quad \text{on} \quad \Gamma_{\lambda, \lambda + \varepsilon}
\]
for all \( 0 \leq \varepsilon \leq \varepsilon_0 \), where \( \Gamma_{\lambda, \lambda + \varepsilon} \) and \( \Gamma_{\lambda, \lambda + \varepsilon} \) are defined in Proposition 3.

**Proof.** First of all, we define \( u, v \) on the entire space \( \mathbb{R}^N \) by consecutive odd reflection. That is,
\[
  u(x', y) = \begin{cases} 
    u(x', y - 2kh) & \text{if } y \in [2kh, (2k + 1)h], \quad k \in \mathbb{Z}, \\
    -u(x', (2k + 2)h - y) & \text{if } y \in [(2k + 1)h, (2k + 2)h], \quad k \in \mathbb{Z},
  \end{cases}
\]
with
\[
  f(s, t) = -f(-s, -t) \quad \text{and} \quad g(s, t) = -g(-s, -t) \quad \text{in} \ \{s < 0, t < 0\}.
\]
Then \( u, v \) solve a similar system
\[
\begin{cases}
  -\Delta_p u = f(u, v) \quad \text{in} \ \mathbb{R}^N, \\
  -\Delta_q v = g(u, v) \quad \text{in} \ \mathbb{R}^N.
\end{cases}
\]
Arguing by contradiction as in the proof of Theorem 10, we can define \( w_n \) and pass to the limit \( w \) to obtain \( w(0, \lambda) = 1 \) and
\[
  -\Delta_p w = 0 \quad \text{in} \ \mathbb{R}^N. \tag{6.2}
\]
However, this is a contradiction since the only solution \( w \) of (6.2) with \( w = 0 \) on \( \partial \Sigma_h \) is zero thanks to [19, Theorem 2.2]. \(\square\)

With the above preparation, we are ready to prove Theorem 1.2.
Proof of Theorem 2. By Theorem 11, we have
\[ \Lambda := \{ 0 < t \leq h/2 \mid u \leq u_0 \text{ and } v \leq v_0 \text{ in } \Sigma_0 \text{ for all } 0 < \theta \leq t \} \neq \emptyset. \]
Hence we can define
\[ \bar{\Lambda} = \sup \Lambda. \]

We will show that
\[ \overline{\Lambda} = \frac{h}{2}. \]

By contradiction, in what follows, we assume that \( \overline{\Lambda} < \frac{h}{2} \). By continuity, we have \( u \leq u_\bar{\Lambda} \) and \( v \leq v_\bar{\Lambda} \) in \( \Sigma_\bar{\Lambda} \).

We claim that: for all \( 0 < \delta < \overline{\Lambda} \), there exists \( 0 < \varepsilon_0 < \frac{h}{2} - \overline{\Lambda} \) such that for all \( 0 < \varepsilon \leq \varepsilon_0 \) it follows that
\[ u \leq u_{\overline{\Lambda} + \varepsilon} \quad \text{and} \quad v \leq v_{\overline{\Lambda} + \varepsilon} \quad \text{in } \Sigma_{\delta, \overline{\Lambda} - \delta}. \quad (6.3) \]

Indeed, if (6.3) does not hold, then for some \( 0 < \delta < \overline{\Lambda} \), we can find a sequence of positive numbers \( \varepsilon_n \) converging to 0 and a sequence of points \( x_n = (x'_n, y_n) \in \Sigma_{\delta, \overline{\Lambda} - \delta} \) such that
\[ u_{\overline{\Lambda} + \varepsilon_n}(x'_n, y_n) < u(x'_n, y_n) \quad \text{or} \quad v_{\overline{\Lambda} + \varepsilon_n}(x'_n, y_n) < v(x'_n, y_n). \]

Passing to a subsequence if necessary, we may and do assume that \( \lim_{n \to \infty} y_n = y_0 \in [\delta, \overline{\Lambda} - \delta] \) and
\[ u_{\overline{\Lambda} + \varepsilon_n}(x'_n, y_n) < u(x'_n, y_n) \]
for all \( n \in \mathbb{N}^* \). Let us define
\[ \tilde{u}_n(x', y) = u(x' + x'_n, y), \quad \tilde{v}_n(x', y) = v(x' + x'_n, y). \]

We can consider \( u, v \) (and consequently \( \tilde{u}_n, \tilde{v}_n \)) defined on the entire space \( \mathbb{R}^N \) by consecutive odd reflection as in the proof of Lemma 12. Then \( \tilde{u}_n \) and \( \tilde{v}_n \) satisfy
\[ \begin{cases} -\Delta p \tilde{u}_n = f(\tilde{u}_n, \tilde{v}_n) & \text{in } \mathbb{R}^N, \\ -\Delta q \tilde{v}_n = g(\tilde{u}_n, \tilde{v}_n) & \text{in } \mathbb{R}^N. \end{cases} \quad (6.4) \]

Therefore, by the standard regularity theory [24], we have that
\[ \| \tilde{u}_n \|_{C^{1, \alpha}_{\text{loc}}(\mathbb{R}^N)} \leq C, \quad \| \tilde{v}_n \|_{C^{1, \alpha}_{\text{loc}}(\mathbb{R}^N)} \leq C \]
for some \( 0 < \alpha < 1 \). By Ascoli-Arzelà’s theorem and a standard diagonal process, we have
\[ \tilde{u}_n \to \tilde{u} \text{ in } C^{1, \alpha'}_{\text{loc}}(\mathbb{R}^N), \quad \tilde{v}_n \to \tilde{v} \text{ in } C^{1, \alpha'}_{\text{loc}}(\mathbb{R}^N) \]
up to a subsequence for \( \alpha' < \alpha \). Moreover, we have
- \( \tilde{u}, \tilde{v} \geq 0 \) in \( \Sigma_h \) and \( \tilde{u} = \tilde{v} = 0 \) on \( \partial \Sigma_h \),
- \( \tilde{u} \leq \tilde{u}_\overline{\Lambda} \) and \( \tilde{v} \leq \tilde{v}_\overline{\Lambda} \) in \( \Sigma_\overline{\Lambda} \),
- \( \tilde{u}(0, y_0) \geq \tilde{u}_\overline{\Lambda}(0, y_0) \).

By standard arguments, we can pass to the limit of the first equation of (6.4) to obtain
\[-\Delta p \tilde{u} = f(\tilde{u}, \tilde{v}) \quad \text{in } \Sigma_h \]
in the weak form. By the strong maximum principle (see Theorem 3), it follows that \( \tilde{u} > 0 \) or \( \tilde{u} = 0 \) in \( \Sigma_h \).

Case 1: \( \tilde{u} > 0 \) in \( \Sigma_h \). Since \( \tilde{v} \leq \tilde{v}_\overline{\Lambda} \) in \( \Sigma_\overline{\Lambda} \) and \( (s, t) \mapsto f(s, t) \) is non-decreasing in \( t \), it follows that
\[ -\Delta p \tilde{u} - f(\tilde{u}, \tilde{v}) = -\Delta p \tilde{u}_\overline{\Lambda} - f(\tilde{u}_\overline{\Lambda}, \tilde{v}_\overline{\Lambda}) \leq -\Delta p \tilde{u}_\overline{\Lambda} - f(\tilde{u}_\overline{\Lambda}, \tilde{v}) \quad \text{in } \Sigma_\overline{\Lambda}. \]
By the strong comparison principle (see Theorem 4), we get that \( \hat{u} \leq \hat{u}_\bar{X} \) implies \( \hat{u} < \hat{u}_\bar{X} \), since the case \( \hat{u} = \hat{u}_\bar{X} \) is clearly impossible being \( \hat{u}(x', 0) = 0 \). This is a contradiction since \( \hat{u}(0, y_0) \geq \hat{u}_\bar{X}(0, y_0) \).

**Case 2:** \( \hat{u} = 0 \) in \( \Sigma_h \).

Let us set\[ w_n(x', y) = \frac{u(x' + x_n', y)}{u(x_n', y_n)}. \]

Then \[ w_n(0, y_n) = 1 \quad (6.5) \]

and \( w_n \) verifies the equation\[ -\Delta_p w_n = c_n(x)u^{p-1}_n \quad \text{in} \quad \Sigma_h \quad (6.6) \]
in the weak sense, where\[ c_n(x) = \frac{f(u(x' + x_n', y), v(x' + x_n', y))}{u^{p-1}(x' + x_n', y)}. \]

As in the proof of Theorem 9, we have \( \|c_n\|_{L^\infty(\Sigma_h)} \leq C \) for \( L > 0 \).

We can therefore exploit the Harnack inequality to get for any compact set \( K \),\[ \sup_{K \cap \{0 \leq y \leq h\}} w_n \leq \sup_{K \cap \{0 \leq y \leq h-\delta\}} w_n \leq C_H \quad \inf_{K \cap \{y \leq h-\delta\}} w_n \leq C_H, \]

where, to deduce \( \sup_{K \cap \{0 \leq y \leq h\}} w_n \leq \sup_{K \cap \{y \leq h-\delta\}} w_n \) we used the fact that \( u \) is monotone increasing in the \( y \)-direction in \( \Sigma_h \) and monotone decreasing in the \( y \)-direction in \( \Sigma_{h-\theta, h} \) for sufficiently small \( \theta < h/2 \) (see Theorem 11).

We can therefore use \( C^{1,\alpha} \) estimates, Ascoli’s theorem, and a standard diagonal process to show that \[ w_n \to w \quad \text{in} \quad C^{1,\alpha'}_{loc}(\mathbb{R}^N) \]

up to a subsequence for \( \alpha' < \alpha \). Since\[ \bar{u}_n(x', y) = u(x' + x_n', y) \to 0, \quad \bar{v}_n(x', y) = v(x' + x_n', y) \to \bar{v} \quad \text{in} \quad C^{1,\alpha}_{loc}(\mathbb{R}^N) \]

and\[ \lim_{s \to 0^+, \frac{f(s, t)}{s^{p-1}} = 0 \quad \text{locally uniformly in} \quad t, \]

passing to the limit in (6.6), we obtain\[ -\Delta_p w = 0 \quad \text{in} \quad \mathbb{R}^N. \]

Now we can apply [19, Theorem 2.2] to deduce that \( w \equiv 0 \) taking into account \( w = 0 \) in \( \partial \Sigma_h \). This contradicts the fact that \( w(0, y_0) = 1 \) thanks to (6.5).

Therefore, (6.3) has been proved.

Next, we can choose \( \delta \) and \( \varepsilon_0 < h/4 - \bar{X}/2 \) sufficiently small such that Proposition 3 applies in \( \Sigma_\delta \) and \( \Sigma_{\bar{X}-\delta, \bar{X}+\varepsilon'} \) with \( \lambda = \bar{X} + \varepsilon \). Thanks to Lemma 12, the parameter \( h_0 \) in the statement of Proposition 3 can be chosen independently of \( \varepsilon \) since there \( \gamma \) does not depend on \( \varepsilon \). Then we conclude that\[ u \leq u_{\bar{X}+\varepsilon} \quad \text{and} \quad v \leq v_{\bar{X}+\varepsilon} \quad \text{in} \quad \Sigma_\delta \cup \Sigma_{\bar{X}-\delta, \bar{X}+\varepsilon}. \]

Combining this with (6.3), we have that \( u \leq u_{\bar{X}+\varepsilon} \) and \( v \leq v_{\bar{X}+\varepsilon} \) in \( \Sigma_{\bar{X}+\varepsilon} \) for all \( 0 < \varepsilon \leq \varepsilon_0 \). This is a contradiction with the definition of \( \bar{X} \). Therefore, it must hold that \( \bar{X} = h/2 \).
This implies the monotonicity of $u$ and $v$ in $\Sigma_{h/2}$, that is $\frac{\partial u}{\partial y} \geq 0$ and $\frac{\partial v}{\partial y} \geq 0$ in $\Sigma_{h/2}$. By Theorem 5, since $u$ and $v$ are not trivial, we must have

$$\frac{\partial u}{\partial y} > 0 \quad \text{and} \quad \frac{\partial v}{\partial y} > 0 \quad \text{in} \quad \Sigma_{h/2}.$$ 

Moreover, we have $u \leq u_{h/2}$ and $v \leq v_{h/2}$ in $\Sigma_{h/2}$.

Similarly, we can move the plane from $\{y = h\}$ in the $-x_N$ direction to its limiting position $\{y = h/2\}$ to obtain $u \geq u_{h/2}$ and $v \geq v_{h/2}$ in $\Sigma_{h/2}$.

Hence $u$ and $v$ are symmetric with respect to the hyperplane $\{y = h/2\}$. \qed

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