Algorithms for linear time reconstruction by discrete tomography in three dimensions

Matthew Ceko\textsuperscript{a,1}, Silvia M.C. Pagani\textsuperscript{b,2,∗}, Rob Tijdeman\textsuperscript{c}

\textsuperscript{a}School of Physics and Astronomy, Monash University, Melbourne, Australia
\textsuperscript{b}Dipartimento di Matematica e Fisica “N. Tartaglia”, Università Cattolica del Sacro Cuore, via Musei 41, 25121 Brescia, Italy
\textsuperscript{c}Mathematical Institute, Leiden University, 2300 RA Leiden, P.O. Box 9512, The Netherlands

Abstract

The goal of discrete tomography is to reconstruct an unknown function $f$ via a given set of line sums. In addition to requiring accurate reconstructions, it is favourable to be able to perform the task in a timely manner. This is complicated by the presence of switching functions, or ghosts, which allow many solutions to exist in general. Previous work has shown that it is possible to determine all solutions in linear time (with respect to the number of directions and grid size) regardless of whether the solution is unique. In this work, we show that a similar linear algorithm exists in three dimensions. This is achieved by viewing the three-dimensional grid along each 2D coordinate plane, effectively solving the problem with a series of 2D linear algorithms. By that, it is possible to solve the problem of 3D discrete tomography in linear time.

Keywords: discrete tomography; lattice direction; linear time algorithm; reconstruction algorithm; switching function; three-dimensional reconstruction

∗Corresponding author

Email addresses: matthew.ceko@monash.edu (Matthew Ceko), silvia.pagani@unicatt.it (Silvia M.C. Pagani), tijdeman@math.leidenuniv.nl (Rob Tijdeman)

\textsuperscript{1}Research supported from the Monash University Postgraduate Publications Award.
\textsuperscript{2}Research supported by D1 Research line of Università Cattolica del Sacro Cuore.
1. Introduction

Tomography is the process of reconstructing an object from a set of its projected views. In 1917, Radon showed that a differentiable function can be represented by the set of its integral projections [12]. Then in 1937, Kaczmarz gave the convergence of an algorithm to approximate solutions for systems of linear equations. This allowed the Radon transform to be used for approximating a solution from a finite number of projections [9]. This process requires many projections for accurate solutions.

Imaging sensitive objects such as fine nanostructures or biological matter often limit the number of projections that can be taken, as to not perturb or destroy the object. The methods of discrete tomography permit useful reconstructions or approximations to be obtained with relatively little projection information. Discrete tomography considers a function $f$ on a finite grid $A$ of $\mathbb{Z}^2$ representing the object. Instead of continuous integral projections, discrete line sums are used. These sum the $f$-values at grid points along lines in a finite number of directions $d$. In this work, we assume that these line sums are free of noise or errors.

Discrete tomography began from analysing the problem of recovering binary matrices from their row and column sums. In 1957, Ryser gave a necessary and sufficient condition, and an algorithm for reconstruction of this problem [13]. By 1978, Katz showed that reconstruction is possible for any number of directions in the absence of a nontrivial function with vanishing line sums over the set of directions, known as a switching function or ghost [10]. Hajdu and Tijdeman offered an algebraic interpretation of discrete tomography in 2001, which viewed switching elements as polynomials [7]. Switching functions of minimal size are termed primitive switching functions. They showed that all switching functions are thereby a linear combination of primitive switching functions. Hence, points of the grid $A$ have uniquely determinable values if they are not in the domain of the union of the primitive switching domains. Furthermore, they showed that arbitrary values can be assigned to a subset of the switching domain, which allows unique values to be applied to all other points.

This fact was exploited by Dulio and Pagani in [6], wherein a rounding theorem was proven which allowed exact and unique binary tomographic reconstructions from the minimum Euclidean norm solution. It also motivated the construction of boundary ghosts by Ceko, Petersen, Svalbe and Tijdeman [2]. Boundary ghosts are switching components that consist of a thin boundary of switching elements, and a largely empty interior of uniquely determinable values.
An important aspect of discrete tomography is the time in which reconstructions can be obtained. It was shown by Gardner, Gritzmann and Prangenberg in 1999 that a function \( f : A \to \mathbb{N} \) can be reconstructed in linear time for two directions, but that the problem is NP-complete for \( d \geq 3 \). They also showed that the problem is NP-complete for \( d \geq 2 \) when the function contains six or more different values. In 2015 Dulio, Frosini and Pagani proved that the corners of \( A \) can be uniquely determined in linear time for \( d = 2 \) [3], and gave conditional results for \( d = 3 \) [4, 5]. This result was generalised by Pagani and Tijdeman for any number of directions [11]. They showed that the values of \( A \) outside of the convex hull of the union of all the switching domains can be computed in linear time, and in the absence of a switching function all values may be obtained in linear time.

The coalescence of these results yielded an algorithm provided by the authors which allows for reconstruction of all solutions from the line sums in linear time in terms of the product of the number of directions and the size of the product [1]. It implies the reconstruction of any unknown function \( f \) from its line sums, provided \( f \) takes values in a unique factorization domain and the line sums are noise free. In the present work, we extend the algorithm to three dimensions, and show that a 3D discrete reconstruction can be computed in linear time with respect to the product of the grid size and number of directions. This is achieved through repeated applications of the aforementioned 2D algorithm along coordinate planes in 3D.

We begin by recalling the relevant results of the two-dimensional case in Section 2. In Section 3 we extend these results to permit non-primitive directions which is necessary for the 3D case, as we will show. In Section 4 we formalise the extension of the problem to three dimensions. In Section 5 we analyse the convex hull of the three-dimensional switching domain, which is central to the operation of the algorithm. In Section 6 we give the 3D algorithm to determine the set of solutions, with examples provided. Section 7 offers an analysis of the computational complexity of the algorithm. Finally we state some conclusions.

2. Results in the two-dimensional case

We give a survey of the relevant results in the two-dimensional case. In our figures, if not differently specified, the \( y \)-axis is oriented downwards. Let \( A \) be an \( m \) by \( n \) grid of points

\[
A = \{(p, q) \in \mathbb{Z}^2 : 0 \leq p < m, 0 \leq q < n \}
\]
and $D$ a finite set of directions $(a_h, b_h)$ for $h = 1, 2, \ldots, d := |D|$. By definition directions are pairs $(a, b)$ of coprime integers with $a \geq 0$ and $b = 1$ if $a = 0$. We call $(A, D)$ valid if $\sum_{h=1}^d a_h < m$ and $\sum_{h=1}^d |b_h| < n$ and nonvalid otherwise. If $B \subset \mathbb{Z}^2$ and $(a, b) \in \mathbb{Z}^2$, then $B + (a, b) = \{(p, q) + (a, b) : (p, q) \in B\}$.

A lattice line $L$ is a line containing at least two points in $\mathbb{Z}^2$. Let $f : A \to \mathbb{R}$. The line sum of $f$ along the lattice line $L(a, b, p_0, q_0) = \{(p_0, q_0) + t(a, b) : t \in \mathbb{Z}\}$ in the direction $(a, b)$ is defined as

$$\ell(a, b, p_0, q_0, f) = \sum_{(p,q) \in L(a,b,p_0,q_0) \cap A} f(p, q).$$

A nontrivial function $g : A \to \mathbb{R}$ is called a switching function of $(A, D)$ if all the line sums of $g$ in all the directions of $D$ are 0. Observe that $g$ is a switching function if and only if $f$ and $f + g$ have the same line sums in the directions of $D$. Therefore the existence of a switching function implies that $f$ is not uniquely determined by its line sums in the directions of $D$ and conversely. The support of a switching function is called a switching domain. In [7] the structure of switching functions has been described. In that paper an elementary switching function is constructed, $g_0 : A_{(0,0)} \to \mathbb{R}$ with $A_{(0,0)} \subseteq [0, \sum_h a_h] \times [0, \sum_h |b_h|] \cap \mathbb{Z}^2$. Obviously, the corresponding function with shifted domain $g_{(p,q)} : A_{(0,0)} + (p, q) \to \mathbb{R}$ is also an elementary switching function. The authors of [7] proved that every switching function of $(A, D)$ is of the form $\sum_{(p,q) \in U} c(p,q)g_{(p,q)}$ with $c(p,q) \in \mathbb{R}$ and $U$ is the subset of $(p, q) \in A$ for which the domain of $g_{(p,q)}$ fits into $A$. It follows that there are

$$\left(m - \sum_{h=1}^d a_h\right) \times \left(n - \sum_{h=1}^d |b_h|\right)$$

free choices of $f^*$-values, where $f^*$ is a function with the same line sums as $f$. A function $f$ is uniquely determined by its line sums in the directions of $D$ outside the union $T$ of its switching domains and not elsewhere. In particular, the function $f$ is uniquely determined by its line sums in the directions of $D$ if and only if $(A, D)$ is invalid, a result already proved in 1978 by Katz [10].

Let $D = \{(a_1, b_1), \ldots, (a_d, b_d)\}$ be ordered such that

$$\frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_d}{a_d}$$

where $1/0$ is considered as $\infty$. Then it follows from the construction in [7]
as worked out in [11] that the convex hull $C$ of $T$ is a polygon with vertices

$$(p_0, q_0), (p_1, q_1), \ldots, (p_{2d-1}, q_{2d-1}), (p_{2d}, q_{2d}) = (p_0, q_0)$$

such that $(p_0, q_0) = (0, \sum_{b_h < 0} |b_h|)$, $(p_h, q_h) - (p_{h-1}, q_{h-1}) = (a_h, b_h)$ for $1 \leq h \leq d$ and $(p_h, q_h) - (p_{h-1}, q_{h-1}) = -(a_{h-d}, b_{h-d})$ for $d + 1 \leq h \leq 2d$. In Corollary 19 of [11] the authors give an algorithm to compute the $f$-values of the points of $A$ outside $C$ starting from the line sums in the directions of $D$.

The principle of the algorithm is to find an ordering of the integer points outside $C$ and for each such point to choose a direction $d$ from $D$ so that if it is the turn of a point $P \in A$ in this ordering, then $P$ is the only point on the lattice line in the direction $d$ of which the $f$-value is unknown. Then the $f$-value of $P$ can be computed by subtracting all the known $f$-values of other points on that lattice line from the line sum of that lattice line. The ordering of the points $P$ is determined by their weights. The weight of $P$ is the ratio of the distance of $P$ to some corner point $Q$ of $A$ and the distance of $Q$ along the line through $Q$ and $P$ to $C$.

The algorithm is extended in the paper [1] of the present authors. Theorems 5 and 6 of that paper state that given a function $f : A \to \mathbb{R}$ of which the line sums in the directions of $D$ are known, a function $f^* : A \to \mathbb{R}$ can be computed in linear time which has the same line sums as $f$ in the directions of $D$. Hence, by using the theory in [7], the set of all functions $F : A \to \mathbb{R}$ which have the same line sums as $f$ in the directions of $D$ are known. Linear time means here that there is an algorithm to compute $f^*$ in $O(dnm)$ operations such as addition, multiplication and division. A special feature of the algorithm is that the $f^*$-values are obtained by only subtractions so that the original $f$-values are found for the pixels outside $T$, if the computer performs these operations exactly.

The extension in [1] means that not only the $f$-values of the integer points outside $C$ can be computed, but also the $f$-values of the points inside $C$ which are not in $T$. Ceko, Petersen, Svalbe and Tijdeman [2] studied so-called boundary ghosts which are switching functions defined on $c^n$ points with $1 < c < 2$ enclosing a set of almost $2^n$ points, for any $n$. The new algorithm enables one to compute the $f$-values of those interior points in linear time. In the literature alternative names for switching function are ghost and phantom.
3. Extension to non-primitive directions

In our treatment of the three-dimensional case we shall use previous results in the two-dimensional case. However, there is a complication. The results in the two-dimensional case are derived for primitive directions, i.e., directions \((a, b)\) with \(a\) and \(b\) coprime. In the three-dimensional case we shall use the two-dimensional results for non-primitive directions too. In this section we show that this is not a restriction.

Let again \(A = \{(p, q) \in \mathbb{Z}^2 : 0 \leq p < m, 0 \leq q < n\}\). Let \(D\) be a set of directions \((a_1, b_1), \ldots, (a_k, b_k)\), not necessarily primitive, with \(k \geq 2\), where \(a_1, \ldots, a_k\) are positive integers and \(b_1, \ldots, b_k\) negative integers ordered such that

\[
\frac{b_1}{a_1} < \frac{b_2}{a_2} < \ldots < \frac{b_k}{a_k}. \tag{1}
\]

We call the points

\[
\left(\sum_{h=1}^{k} a_h, 0\right), \left(\sum_{h=2}^{k} a_h, |b_1|\right), \ldots, \left(0, \sum_{h=1}^{k} |b_h|\right)
\]

the border points \((P_0, Q_0), (P_1, Q_1)\), \ldots, \((P_k, Q_k)\), respectively. We define a weight function.

**Definition 1.** For a point \((p, q) \in A\) we define its weight \(w(p, q)\) by

\[
w(p, q) = \frac{a H q + |b H| p}{a H (\sum_{h=1}^{H} |b_h|) + |b H| (\sum_{h=H+1}^{k} a_h)},
\]

where \(H\) is determined by

\[
\frac{\sum_{h=1}^{H-1} |b_h|}{\sum_{h=H}^{k} a_h} \leq \frac{q}{p} \leq \frac{\sum_{h=1}^{H} |b_h|}{\sum_{h=H+1}^{k} a_h}.
\]

Obviously \(w(p, q) = 1\) if \((p, q)\) is a border point. Observe that the lines with equation

\[
y \sum_{h=H+1}^{k} a_h = x \sum_{h=1}^{H} |b_h| \quad \text{for } H = 0, 1, \ldots, k
\]

are the lines connecting the origin with the border points. The lines generate \(k\) full triangles \(V_1, V_2, \ldots, V_k\) with as vertices of \(V_H\) the origin and two consecutive border points \((P_{H-1}, Q_{H-1})\) and \((P_H, Q_H)\) for \(H = 1, 2, \ldots, k\) (see Figure 1). The weight function is linear on each \(V_H\) and is on \(V_H\) only
equal to 1 on the line segment between \((P_{H-1}, Q_{H-1})\) and \((P_H, Q_H)\). Thus for a point \((p, q) \in V_H\) it represents the quotient of the distance between the origin and the point \((p, q)\) and the distance between the origin and the intersection of the line through the origin and \((p, q)\) and the line through \((P_{H-1}, Q_{H-1})\) and \((P_H, Q_H)\).

![Figure 1: The triangles \(V_1, V_2, V_3\) for the set \(D = \{(3, -2), (4, -3), (1, -2)\}\). The border points are \((P_0, Q_0) = (8, 0), (P_1, Q_1) = (5, 2), (P_2, Q_2) = (1, 5), (P_3, Q_3) = (0, 7)\). For every \(H\) the line through \((P_{H-1}, Q_{H-1})\) and \((P_H, Q_H)\) is an edge of triangle \(V_H\), and intersects each other triangle \(V_h\), since the slopes increase with increasing \(h\) by the ordering in (1).](image)

We order the points of \(V_1 \cup V_2 \cup \cdots \cup V_k\) according to nondecreasing weights; see Figure 2. If weights are equal, we order the points according to increasing \(y\)-value. (In fact, points with equal weights can be handled simultaneously.)

The next result states that if we follow the assigned order numbers and reach the point \((p, q) \in V_H\), then we can compute the \(f\)-value of \((p, q)\) by subtracting all the (known) \(f\)-values of other points on the lattice line through \((p, q)\) in the direction \((a_H, b_H)\) from its line sum. The idea of the proof is that if \((p, q) \in V_H\), then \((p, q) + t(a_H, b_H)\) has a lower weight than \((p, q)\) for all \(t \in \mathbb{Z} \neq 0\) and therefore a smaller order number.

**Theorem 2.** Let \(A\) be the \(m \times n\) grid of points \(A = \{(p, q) \in \mathbb{Z}^2 : 0 \leq p < m, 0 \leq q < n\}\). Let \(D\) be a set of not necessarily primitive directions \((a_1, b_1), \ldots, (a_k, b_k)\) where \(a_1, \ldots, a_k\) are positive integers and \(b_1, \ldots, b_k\) negative integers ordered as in (1). Then \((p, q) \in V_H\), not a border point, has a larger weight than all the lattice points \((p, q) + t(a_H, b_H) \in A\) with \(t \neq 0\).
The new feature of Theorem 2 is that $a_j$ and $b_j$ are not necessarily coprime for $j = 1, 2, \ldots, k$. It need not longer be true that there are no lattice points in between two consecutive border points. We give a geometric proof which is much simpler than the proof in [11]. It follows from the theorem that if it is the turn of the point $(p, q)$ in the ordering by weights, all points on the line $(p, q) + t(a_H, b_H)$ outside $V_H$ have lower weights and hence their $f$-values are already known.

**Proof.** Consider a point $(p, q) \in V_H$ and a point $(p', q') = (p, q) + t(a_H, b_H) \in A$ with $t \in \mathbb{Z}, t \neq 0$. The triangle $V_H$ has an edge of length $(P_H, Q_H) - (P_{H-1}, Q_{H-1}) = (a_H, b_H)$ so that the shift of $(p, q)$ by a nonzero multiple of $(a_H, b_H)$ cannot be in $V_H$. This implies that $(p', q') \in V_h$ for some $h \neq H$. The weight of $(p, q)$ is by definition equal to the quotient of the distance of the origin to $(p, q)$ and the length of the line segment between the origin and the intersection of the line through the origin and $(p, q)$ and the line through $(P_{H-1}, Q_{H-1})$ and $(P_H, Q_H)$. Hence the weight of $(p', q')$ is equal to the quotient of the distance of the origin to $(p', q')$ and the length of the line segment between the origin and the intersection of the line through the origin and $(p', q')$ and the line through $(P_{h-1}, Q_{h-1})$ and $(P_h, Q_h)$. By the ordering (1) the line segment between the border points $(P_{h-1}, Q_{h-1})$ and
\((P_h, Q_h)\) is farther away from the origin than the intersection of \(V_h\) with the line through \((P_{H-1}, Q_{H-1})\) and \((P_H, Q_H)\) (cf. Figure 3). It follows that the weight of \((p, q)\) is larger than the weight of \((p', q')\). 

\[
\begin{array}{c}
\text{Figure 3: The triangles } V_1, V_2, V_3 \text{ for the set } D = \{(3, -2), (2, -2), (3, -6)\}. \text{ The border points are } (P_0, Q_0) = (8, 0), (P_1, Q_1) = (5, 2), (P_2, Q_2) = (3, 4), (P_3, Q_3) = (0, 10). \text{ The translates } V_2 \pm (2, -2) \text{ have only border points in common with } V_2. \\
\text{The points } (4, 2) + (2, -2), (4, 2) - (2, -2), (4, 2) - 2(2, -2) \text{ have lower weights than } (4, 2), \text{ viz. } .750, .750, .600, .857, \text{ respectively. Note that the other lattice points on the line } x + y = 6 \text{ are not included in the theorem. E.g. } (3, 3) \text{ has also weight } .857.
\end{array}
\]
4. Set up in the three-dimensional case

We consider an $m$ by $n$ by $o$ grid of points

$$A = \{(p, q, r) \in \mathbb{Z}^3 : 0 \leq p < m, 0 \leq q < n, 0 \leq r < o\}$$

and a finite set $D$ of directions $(a_h, b_h, c_h)$ for $h = 1, 2, \ldots, d := |D|$. Directions are triples $(a, b, c)$ of coprime integers with $a \geq 0, b \geq 0$ if $a = 0$ and $c = 1$ if $a = b = 0$. We call $(A, D)$ valid if $\sum_{h=1}^d a_h < m, \sum_{h=1}^d b_h < n$ and $\sum_{h=1}^d c_h < o$ and nonvalid otherwise. If $B \subset \mathbb{Z}^3$ and $(a, b, c) \in \mathbb{Z}^3$, then $B + (a, b, c) = \{(p, q, r) + (a, b, c) : (p, q, r) \in B\}$.

A lattice line $L$ is a line containing at least two points in $\mathbb{Z}^3$. Let $f : A \to \mathbb{R}$. The line sum of $f$ along the lattice line $L(a, b, c, p_0, q_0, r_0) = \{(p_0, q_0, r_0) + t(a, b, c) : t \in \mathbb{Z}\}$ in the direction $(a, b, c)$ is defined as

$$\ell(a, b, c, p_0, q_0, r_0, f) = \sum_{(p, q, r) \in L(a, b, c, p_0, q_0, r_0) \cap A} f(p, q, r).$$

A nontrivial function $g : A \to \mathbb{R}$ is called a switching function of $(A, D)$ if all the line sums of $g$ in all the directions of $D$ are 0. Observe that $g$ is a switching function if and only if $f$ and $f + g$ have the same line sums in the directions of $D$, for all $f$. The support of a switching function is called a switching domain. In [8] the structure of switching functions has been described. In that paper an elementary switching function $g_0 : A_0 \to \mathbb{Z}$ with $A_0 = [0, \sum h a_h] \times [0, \sum h |b_h|] \times [0, \sum h |c_h|] \cap \mathbb{Z}^3$ is defined as follows. For $(p, q, r) \in A$ the number $g_0(p, q, r)$ counts how often $(p, q - \sum_{h < 0} |b_h|, r - \sum_{c < 0} |c_h|)$ can be written as sum of distinct direction vectors where the counting is with factor 1 if the number of terms is even and with factor $-1$ if the number of terms is odd. By definition $g_0(0, \sum_{h < 0} |b_h|, \sum_{c < 0} |c_h|) = 1$. Obviously, the corresponding function $g_{(p, q, r)} : A_0 + (p, q, r) \to \mathbb{R}$ is a switching function with a shifted switching domain of the same size. The authors of [8] prove that every switching function is of the form $\sum_{(p, q, r) \in U} c_{(p, q, r)} g_{(p, q, r)}$ with $c_{(p, q, r)} \in \mathbb{R}$ and $U$ is the subset of $(p, q, r) \in A$ for which the domain of $g_{(p, q, r)}$ fits into $A$. It follows that the dimension of the linear space of functions satisfying the line sums in the directions of $D$ equals

$$\left( m - \sum_{h=1}^d a_h \right) \times \left( n - \sum_{h=1}^d |b_h| \right) \times \left( o - \sum_{h=1}^d |c_h| \right). \quad (2)$$
Denote by $T$ the union of all switching domains. It follows that a function $f$ is uniquely determined by its line sums in the directions of $D$ outside $T$ and not elsewhere. In particular, the function $f$ is uniquely determined by its line sums in the directions of $D$ if and only if $(A, D)$ is nonvalid.

5. The convex hull $C$

In the sequel the convex hull $C$ of $T$ plays an important role. It is obvious that the vertices of $C$ are points of $T$. Suppose first that

$$
\sum_{h=1}^{d} a_h = m, \sum_{h=1}^{d} |b_h| = n, \sum_{h=1}^{d} |c_h| = o.
$$

(3)

Then, by the convexity of $C$, the only switching domain is elementary and every point of $T$ is of the form

$$(0, \sum_{b_h < 0} |b_h|, \sum_{c_h < 0} |c_h|) + \varepsilon_1 d_1 + \varepsilon_2 d_2 + \ldots + \varepsilon_d d_d : \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d \in \{0, 1\}.
$$

(4)

We call this representations of the elements of $T_0$. We write $T_0$ for this special $T$ and call its convex hull $C_0$. Let $P$ and $Q$ be vertices of $C_0$ connected by an edge. Then, by the convexity of $C_0$, there are infinitely many hyperplanes $\tilde{H}$ through this edge of which the intersection with $C_0$ is just this edge, hence $C_0$ is entirely on one side of $\tilde{H}$. Let $d'_1, \ldots, d'_p \in \pm D$ such that $Q - P$ is of the form $\sum_{j=1}^{p} d'_j$, where $\pm D = \{\pm d_1, \pm d_2, \ldots, \pm d_d\}$. Observe that if $d'_j \in D$, then $d'_j$ is in the (chosen) representation of $Q$, but not in the (chosen) representation of $P$, whereas if $d'_j \notin D$, then $d'_j$ is in that representation of $P$, but not in that representation of $Q$. Consider the points $P, P + d'_1, P + d'_1 + d'_2, \ldots, Q$ and the first point $P + \sum_{j=1}^{p} d'_j$ in this sequence which is not in $\tilde{H}$. Then $P + d'_j$ and $Q - d'_j$ are not in $\tilde{H}$ and on different sides of $\tilde{H}$, but both points are elements of $T_0$, a contradiction. Thus all the points of the sequence have to be on $\tilde{H}$. Since $\tilde{H}$ is not unique, all the points of the sequence are on the edge between $P$ and $Q$. Since the directions are not collinear, it implies $Q - P \in \pm D$.

Actually each face of $C_0$ is a parallelogram. This can be seen as follows. Let the line segment between the vertices $P$ and $P + d_1$ of $T_0$ be an edge of $C_0$ such that $d_1$ is not in the representation of $P$ as sum of a subset of $D$. Then each other edge with $P$ as endpoint is of the form $P + d_2$ with $d_2$ not in the representation of $P$ or of the form $P - d_2$ with $d_2$ in the representation of $P$. In the former case $P + d_1 + d_2$ is also a point of $T$ and therefore
the parallelogram with vertices $P, P + d_1, P + d_2, P + d_1 + d_2$ belongs to $C_0$. In the latter case $P + d_1 - d_2$ is also a point of $T_0$ and therefore the parallelogram with vertices $P, P + d_1, P - d_2, P + d_1 - d_2$ belongs to $C_0$, for each pair of directions $d_1, d_2$.

Now we drop the restriction $\{3\}$. If $(A, D)$ is nonvalid, then $T$ and $C$ are empty. Otherwise $\sum_{h=1}^d a_h \leq m, \sum_{h=1}^d |b_h| \leq n, \sum_{h=1}^d |c_h| \leq o$. The elementary switching domains are shifts of $T_0$ by a vector $(p, q, r)$ with $p \leq m - \sum_{h=1}^d a_h, q \leq n - \sum_{h=1}^d |b_h|, r \leq o - \sum_{h=1}^d |c_h|$. Thus the faces of the polytope $C$ are rectangles in the $x$-$y$-plane, in the $x$-$z$-plane and the $y$-$z$-plane and further shifts of the sides of $C_0$.

We have proved

**Theorem 3.** The faces of the convex hull $C$ of $T$ are parallelograms of which the edges are in $\pm D$ or are parallel to the $x$-axis, $y$-axis or $z$-axis.

Obviously, for every pair of directions there are two parallelograms in $C_0$ with these direction vectors as edges such that $C_0$ is bounded by the hyperplanes through the parallelograms. Thus $C_0$ consists of $d(d + 1)$ parallelograms.

### 6. Determination of the set of solutions

Let be given an $m$ by $n$ by $o$ grid of points $A = \{(p, q, r) \in \mathbb{Z}^3 : 0 \leq p < m, 0 \leq q < n, 0 \leq r < o\}$ and a finite set $D$ of primitive directions $(a_h, b_h, c_h)$ for $h = 1, 2, \ldots, d := |D|$. Assume that all the line sums of an unknown function $f : A \rightarrow \mathbb{R}$ in the directions of $D$ are known. In this section we show how the algorithms for the two-dimensional case can be used to find a function $f^* : A \rightarrow \mathbb{R}$ which has the same line sums in the directions of $D$ as $f$ has. By the theory in $[3]$ this means that all such functions are known. In particular, if $(A, D)$ is nonvalid, then $f$ can be computed. We extend the definition of $f$ to $f : \mathbb{Z}^3 \rightarrow \mathbb{R}$ by setting $f(p, q, r) = 0$ for $(p, q, r) \notin A$.

The proof uses induction. We show how to compute all the $f^*$-values of points $(p, q, o - 1) \in A$. After that we apply the same procedure with $o$ replaced with $o - 1$, and so on. Let $T$ denote the union of all the switching domains of $(A, D)$ and $C$ its convex hull. Of course, in the following procedure only $f^*$-values have to be computed which are not yet known.

The first step is to consider the projection of $C$ in the $x$-direction, on the $y$-$z$-plane, say. Its boundary is a convex polygon $X$ with vertices $(q, r)$ so that $(p, q, r) \in T$ for some $p$ and edges in directions $\pm (b_h, c_h)$ such that $(a_h, b_h, c_h) \in D$ for some $a_h$. The theory in $[13]$, combined with Theorem 2, provides an ordering of the points $\{(q, r) \in \mathbb{Z}^2 : 0 \leq q < n, 0 \leq r < o\}$
outside $X$ by considering their weights such that if it is the turn of point $P_0 = (q_0, r_0)$ in this ordering, then $P_0$ is the only point with unknown $f^*$-value on the line through it in the coupled direction $(b_h, c_h)$. By the construction in [11] the lines $(q_0, r_0) + t(b_h, c_h)$ ($t \in \mathbb{R}$) have no point in common with $X$. Therefore the lines $(p, q_0, r_0) + t(a_h, b_h, c_h)$ ($t \in \mathbb{R}$) have no point in common with $C$. Thus the $f^*$-values of the points $(p, q_0, r_0)$ for all $p$ with $0 \leq p < m$ can be computed if it is the turn of $(q_0, r_0)$ in the ordering of [11]. Applying this procedure the $f^*$-values of all the points $(p, q, r)$ for which $(q, r)$ is outside $X$ can be computed by subtractions.

The second step is to follow the same procedure for the projection of $C$ in the $y$-direction on the $x$-$z$-plane. We call its boundary $Y$. The result will be that we know all the $f^*$-values of the points $(p, q, r) \in A$ for which $(p, r)$ is outside $Y$.

We still have to consider the remaining points of $A$. These are the points $(p_0, q_0, r_0)$ such that there exist $p, q$ for which both $(p, q, r_0)$ and $(p_0, q, r_0)$ belong to $C$. We start the third step with the points of the form $(p_0, q_0, o - 1)$ for which there exist $p, q$ such that both $(p, q, o - 1)$ and $(p_0, q, o - 1)$ belong to $T$. These points form the maximal lattice rectangle such that every row and every column contains a point of $T$. Because of (4) the elements $(p, q, o - 1)$ of $T$ have a representation which involves every direction with positive $c_h$ and no direction with negative $c_h$. The freedom is in the choice of directions with $c_h = 0$. Therefore the intersection of $C$ and the plane $z = o - 1$ is a shift of the union of all the switching domains of the directions $(a_h, b_h, 0) \in D$. Since the directions in $D$ are primitive, the corresponding directions $(a_h, b_h)$ are primitive. Hence we can apply the algorithm in [1] to compute the (remaining) $f^*$-values of all the points of the form $(p, q, o - 1)$.

In this process there is free choice of

$$
\left( m - \sum_{h=1}^{d} a_h \right) \times \left( n - \sum_{h=1}^{d} |b_h| \right)
$$

(5) $f^*$-values. The next step is to apply the above procedure to

$$
A - (0, 0, 1) := \{(p, q, r) \in \mathbb{Z}^3 : 0 \leq p < m, 0 \leq q, -1 \leq r < o - 1\},
$$

then to $A - (0, 0, 2)$, to $A - (0, 0, 3), \ldots$, to $A - (o - \sum_{h=1}^{d} |c_h| + 1)$. Each time there is free choice of (5) $f^*$-values. After the Step 3 $f^*$-values have been chosen for [2] points. Different choices generate different solutions and the ultimately found solution will only depend on these choices. But we are not ready yet.
The fourth and last step is to consider the remaining points \((p_0, q_0, r_0)\) with \(0 \leq r_0 < \sum_{h=1}^{d} |c_h|\). This is a nonvalid situation, because of the range of \(r_0\). Thus the union of the switching domains is empty. We follow the same procedure as in the first step, but this time for the projection of the rectangle of remaining points \((p_0, q_0, r_0)\) in the \(z\)-direction on the \(x-y\)-plane. The final result will be that we know the \(f^*\)-values of all the points of \(A\).

**Example 4.** Let be given \(A = \{(0, 4] \times [0, 4] \times [0, 5] \cap \mathbb{Z}^3\}\) and \(D = \{(1, 1, 2), (1, -2, 1), (1, 1, -2), (1, 0, 0)\}\). Because of the negative entries we have a shift by \((0, 2, 2)\). The switching function is elementary. By (4) we have

\[
T = \{(0, 2, 2), (1, 0, 3), (1, 2, 2), (1, 3, 0), (1, 3, 4), (2, 0, 3), (2, 1, 1), (2, 1, 5), (2, 3, 0), (2, 3, 4), (2, 4, 2), (3, 1, 1), (3, 1, 5), (3, 2, 3), (3, 4, 2), (4, 2, 3)\}.
\]

Note that \(T\) is point symmetric in \((2, 2, 5/2)\). The vertices of \(C\) are the points of \(T\) except for the internal points \((1, 2, 2)\) and \((3, 2, 3)\). (See Figure 4).

![Figure 4: The polytope C. The straight line segments are visible, the broken line segments are on the backside. Observe the point symmetry in (2, 2, 5/2).](image)

In Step 1 we consider projection along the \(x\)-axis. We have directions \((1, 2), (-2, 1), (1, -2), (0, 0)\) and a shift by \((2, 2)\). The boundary \(X\) of the projection of \(C\) is given in Figure 5(a). The \(f^*\)-values of the points \((p, q, r)\) can
be computed for all \( p \) with \((q, r) \in \{(0,0), (0,1), (0,2), (0,4), (0,5), (1,0), (2,0), (2,5), (3,5), (4,0), (4,1), (4,3), (4,4), (4,5)\}\).

In Step 2 we consider projection along the \( y \)-axis. We have directions \((1,2), (1,1), (1, -2), (1,0)\) and a shift by \((0,2)\). The boundary \( Y \) of the projection of \( C \) is given in Figure 5(b). The \( f^* \)-values of the points \((p,q,r)\) can be computed for all \( q \) with \((p,r) \in \{(0,0), (0,1), (0,3), (0,4), (0,5), (1,5), (3,0), (4,0), (4,1), (4,2), (4,4), (4,5)\}\).

In Step 3 we have a two-dimensional problem with directions \((a,b,c)\) from \( D \) with \( c = 0 \), that is with \((a,b) = (1,0)\). By Steps 1 and 2 the only points \((p,q,5)\) for which the \( f^* \)-values are still unknown are \((2,1,5)\) and \((3,1,5)\). We have one free choice because of the elementary switching component and we can compute the \( f^* \)-value of the other point by using the direction \((1,0,0)\).

What is left is a nonvalid situation. In Step 4 we consider projection along the \( z \)-axis. We have directions \((1,1), (1, -2), (1,1), (1,0)\) and a shift by \((0,2)\). The \( f^* \)-values of the points \((p,q,r)\) can be computed for all \( r \) by applying the algorithm in \([1]\) to order the pairs \((p,q)\). If it is the turn of \((p,q)\) the \( f^* \)-values of all the remaining points \((p,q,r)\) can be computed by subtractions.
7. Complexity

In [1] the two-dimensional algorithm to find a solution for an \(m\times n\) rectangle \(A\) for which the line sums in \(d\) directions are known turned out to require \(O(dmn)\) elementary operations, where an elementary operation is an addition, subtraction, multiplication, division, determination of the largest of two given quantities or assignment. A closer analysis gives that the number of multiplications and divisions is \(O(d + m + n)\) and that the \(f^*\)-values themselves are computed only by subtractions. Thus if the computer performs additions and subtractions exactly, the final solution is exact.

Step 1 of the three-dimensional algorithm requires \(O(dmno)\) elementary operations of which \(O(d + n + o)\) multiplications and divisions, Step 2 also \(O(dmno)\) elementary operations of which \(O(d + m + o)\) multiplications and divisions, Steps 3 and 4 require \(O(dmno)\) elementary operations of which \(O(d + m + n)\) multiplications and divisions. Thus the three-dimensional algorithm requires \(O(dmno)\) operations in the above defined sense of which \(O(d + m + n + o)\) multiplications and divisions. Hence it is linear in the problem size \(dmno\). Again to compute the \(f^*\)-values themselves only subtractions are needed.

8. Conclusion

In this work we have presented a linear time algorithm for the reconstruction of a three-dimensional function from its line sums, with values in a unique factorization domain. This was achieved through the extension of permitting non-primitive directions in the two-dimensional algorithm presented in our previous work. The characterisation of switching domain convex hull in 3D allows repeated applications of the 2D algorithm to solve the 3D reconstruction problem in linear time with respect to the size of the grid, and number of directions. After the procedure errors can easily be detected by checking the line sums of \(f^*\) along \(D\) and comparing them with those of \(f\).

While this method works for exact line sums, in many practical applications there is noise. Our method works badly for noisy situations, since errors increase exponentially. Therefore, future work will look towards finding a well fitting consistent set of line sums for an inconsistent set of line sums in linear time. After that, the method described in this paper can be applied.

Here, we distinguish between two kinds of noise. The first case is mistakes in measurement, for example if beams are taken of a function which
is defined on a real rectangle without the restriction of an integer lattice. Secondly, errors which are made in a sequence of integers (e.g. binary) or other discrete set of values that are transmitted.

References

References

[1] M. Ceko, S.M.C. Pagani, R. Tijdeman, Algorithms for linear time reconstruction by discrete tomography II, to appear in Discr. Appl. Math.

[2] M. Ceko, T. Petersen, I. Svalbe, R. Tijdeman, Boundary ghosts for discrete tomography, to appear in J. Math. Imaging Vision.

[3] P. Dulio, A. Frosini, S.M.C. Pagani, A geometrical characterization of regions of uniqueness and applications to discrete tomography, Inverse problems 31 (12) (2015), 125011.

[4] P. Dulio, A. Frosini, S.M.C. Pagani, Geometrical characterization of the uniqueness regions under special sets of three directions in discrete tomography, Discrete Geometry for Computer Imaginary, LNCS 9647 (2016), Springer-Verlag, pp. 105-116.

[5] P. Dulio, A. Frosini, S.M.C. Pagani, Regions of uniqueness quickly reconstructed by three directions in discrete tomography, Fundamenta Informaticae 155(4) (2017), 407-423.

[6] P. Dulio, S.M.C. Pagani, A rounding theorem for unique binary tomographic reconstruction, Discrete Appl. Math. 268 (2019), 54-69.

[7] L. Hajdu, R. Tijdeman, Algebraic aspects of discrete tomography, J. reine angew. Math. 534 (2001), 119-128.

[8] L. Hajdu, R. Tijdeman, Algebraic discrete tomography. In Advances in Discrete Tomography and its Applications, ed. by G.T. Herman and A. Kuba, Applied Numerical Harmonic Analysis, Birkhäuser, 2007, pp. 55-81.

[9] S. Kaczmarz, Approximate solution of systems of linear equations, Intern. J. Control. 57 (1993), 1269-1271.

[10] M.B. Katz, Questions of uniqueness and resolution in reconstruction from projections, Lecture Notes in Biomathematics 26 (1978), Springer-Verlag.
[11] S.M.C. Pagani, R. Tijdeman, Algorithms for linear time reconstruction by discrete tomography, Discrete Appl. Math. 271 (1999), 152-170.

[12] J. Radon, Über die Bestimmung von Funktionen durch ihre Integralwerte langs gewisser Mannigfaltigkeiten, Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Phys. Kl., 69 (1917), 262-277.

[13] H.J. Ryser, Combinatorial properties of matrices of zeros and ones, Canad. J. Math. 9 (1957), 1073-1082.