LOCAL INDEX FORMULAE ON NONCOMMUTATIVE ORBIFOLDS AND EQUIVARIANT ZETA FUNCTIONS FOR THE AFFINE METAPLECTIC GROUP

ANTON SAVIN AND ELMAR SCHROHE

Abstract. We consider the algebra $\mathcal{A}$ of bounded operators on $L^2(\mathbb{R}^n)$ generated by quantizations of isometric affine canonical transformations. The algebra $\mathcal{A}$ includes as subalgebras noncommutative tori of all dimensions and toric orbifolds. We define the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{H} = L^2(\mathbb{R}^n, \Lambda(\mathbb{R}^n))$ and the Euler operator $D$, a first order differential operator of index 1. We show that this spectral triple has simple dimension spectrum: For every operator $B$ in the algebra $\Psi(\mathcal{A}, \mathcal{H}, D)$ generated by the Shubin type pseudodifferential operators and the elements of $\mathcal{A}$, the zeta function $\zeta_B(z) = \text{Tr}(B|D|^{-2z})$ has a meromorphic extension to $\mathbb{C}$ with at most simple poles. Our main result then is an explicit algebraic expression for the Connes-Moscovici cocycle. As a corollary we obtain local index formulae for noncommutative tori and toric orbifolds.

2010 Mathematics Subject Classification: 58J20, 46L87, 58B34, 13D03

Contents

1. Introduction 1
2. The Local Index Formula of Connes and Moscovici 4
3. The Metaplectic Group 6
4. Shubin Type Pseudodifferential Operators 7
5. Operators on $\mathbb{R}$ 8
6. Operators on $\mathbb{R}^n$ 11
7. Cyclic Cocycles 18
8. Applications to Noncommutative Tori and Orbifolds 21
9. Equivariant Zeta Functions for the Affine Metaplectic Group 24
References 28

In this article we present a local index formula for a spectral triple associated with the affine metaplectic group. As special cases we obtain local index formulae for noncommutative tori of arbitrary dimension and noncommutative toric orbifolds. We follow the noncommutative geometry approach laid out in the classical paper by Connes and Moscovici [11].

The key notion is that of spectral triple. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of an algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$, and an unbounded operator $D$ acting on $\mathcal{H}$. In addition, $\mathcal{A}$ acts on $\mathcal{H}$ by bounded operators, and the commutators $[D, a]$ are bounded for all $a \in \mathcal{A}$. A classical example is the spectral triple

\begin{equation}
(C^\infty(M), L^2(M, S), D),
\end{equation}

1
where \( C^\infty(M) \) is the algebra of smooth functions on a closed smooth Riemannian spin manifold \( M \), \( L^2(M, S) \) is the space of \( L^2 \)-sections of the spinor bundle, while \( D \) is the Dirac operator on spinors. Under certain conditions, spectral triples define classes in the Kasparov \( K \)-homology of \( \mathcal{A} \) and one obtains the Chern–Connes character

\[
\text{ch}(\mathcal{A}, \mathcal{H}, D) \in H^*(\mathcal{A})
\]

in periodic cyclic cohomology of \( \mathcal{A} \). The local index formula of Connes and Moscovici \[11] expresses this class in terms of periodic cyclic cocycles on \( \mathcal{A} \), which are described in terms of regularized traces on \( \mathcal{A} \). In the case of the Dirac spectral triple \[11] these regularized traces reduce to the celebrated Wodzicki residue \[17], and the local formula of Connes and Moscovici gives the classical local index formula, see \[36\]. Let us emphasize, however, that to obtain an explicit index formula in a new situation using Connes’ and Moscovici’s formula, one has to study these regularized traces and carry out their explicit computation. For applications of the Connes–Moscovici formula see \[9,13,14,32,35,38,43\].

Let us now describe the spectral triple under consideration. Denote by \( \text{Mp}^c(n) \) the complex metaplectic group (see e.g. Leray \[31\], \[17,25,41\] or Section 3). One of many equivalent definitions of this group says that this is the group of all unitary operators acting on \( L^2(\mathbb{R}^n) \) equal to quantizations of linear canonical transformations of the cotangent bundle \( T^*\mathbb{R}^n \). More generally, if we consider affine canonical transformations of \( T^*\mathbb{R}^n \), we obtain the affine complex metaplectic group. We set \( \mathcal{A} \) to be the algebra of bounded operators on \( L^2(\mathbb{R}^n) \) generated by quantizations of \textit{isometric} affine canonical transformations. It can be shown that \( \mathcal{A} \) has the following generators:

1) Heisenberg-Weyl operators: \( u(x) \mapsto e^{ikx-iax^2/2}u(x + a) \), where \( a, k, x \in \mathbb{R}^n \);
2) Shift operators: \( u(x) \mapsto u(g^{-1}x) \), where \( g \in O(n) \) is an orthogonal matrix;
3) Fractional Fourier transforms for \( \varphi \in (0, \pi) \) (see e.g. \[5\] or Section 3):

\[
u(x_1, x_2, ..., x_n) \mapsto \sqrt{\frac{1 - i \text{ctg} \varphi}{2\pi}} \int \exp \left( i \left( \frac{x_1^2 + y_1^2}{2} \text{ctg} \varphi - \frac{x_1 y_1}{\text{sin} \varphi} \right) \right) \, u(y_1, x_2, ..., x_n) \, dy_1\]

Generators of the form 1) are quantizations of shifts in \( T^*\mathbb{R}^n \), those of the form 2) are quantizations of differentials of orthogonal transformations, while those in 3) are quantizations of rotations in the \( (x_1, p_1) \) plane. This algebra includes as subalgebras noncommutative tori of all dimensions and toric orbifolds \[3,8,10,15,19,44-46\]. Moreover, we set \( \mathcal{H} = L^2(\mathbb{R}^n, \Lambda(\mathbb{R}^n)) \) and show that the elements in \( \mathcal{A} \) naturally act on the differential forms. Finally, our operator \( D \) is the well-known Euler operator, a differential operator of index one on \( \mathbb{R}^n \), see e.g. Higson, Kasparov and Trout \[27\].

Our first result asserts that this spectral triple satisfies the conditions in the local index theorem of Connes and Moscovici. Using the stationary phase approximation we show that the zeta functions

\[
\zeta_{a,b}(z) = \text{Tr}(ab|D|^{-2z}),
\]

where \( a \in \mathcal{A} \) and \( b \) is a pseudodifferential operator of Shubin type on \( \mathbb{R}^n \), admit a meromorphic continuation to \( \mathbb{C} \) with simple poles; in other words, the spectral triple has simple dimension spectrum.

Our second result is an explicit algebraic formula for the Connes–Moscovici cocycle. We express this cocycle as a sum of contributions over the fixed point sets of the canonical transformations. The computation reduces to obtaining (i) equivariant heat trace asymptotics for the quantum oscillator with respect to elements of the affine metaplectic group (bosonic part) and (ii) heat trace asymptotics for operators given by Clifford products acting on algebraic forms.
To compute these asymptotics, we use the Mehler formula and Getzler’s calculus. Furthermore, we analyze the Connes–Moscovici periodic cyclic cocycle and show that it is in fact a sum of cyclic cocycles localized at conjugacy classes in the group of isometric affine canonical transformations (this group is isomorphic to the semidirect product $\mathbb{C}^n \rtimes U(n)$). As applications, we give explicit index formulae for noncommutative tori and for noncommutative toric orbifolds. It turns out that noncommutative tori correspond to choosing lattices in $\mathbb{C}^n$, while orbifolds correspond to finite groups acting on such lattices.

As mentioned above, our algebra $\mathcal{A}$ is generated by quantized canonical transformations acting in $L^2(\mathbb{R}^n)$. Thus, this paper is part of our ongoing project to construct an index theory associated with groups of quantized canonical transformations. The articles [39] and [40] focused on operators on closed manifolds, see also Gorokhovsky, de Kleijn and Nest [23] for related work.

In the recent article [41] we treated the algebra generated by the metaplectic operators and the Shubin type pseudodifferential operators on $\mathbb{R}^n$ and obtained an index formula using K-theory. Here, in contrast, we study the algebra generated by the affine metaplectic group, define a spectral triple, and find explicit expressions for the Connes-Moscovici cocycles associated with it.

**Historical notes and relation to previous work.** The local index formula in [11] was extended to twisted spectral triples by Connes and Moscovici, [12], [34]. A conceptually different approach to the local index formula was developed by Carey, Phillips, Rennie and Sukochev in [6,7]; it allowed them to derive the local index formula without the technical condition on the rapid decay of the zeta functions along vertical lines in $\mathbb{C}$ that Connes and Moscovici had imposed; see also Higson [29] for another derivation.

As for concrete applications, Connes and Moscovici stated the formula for the case of the Dirac operator on a closed spin manifold [11] Remark II.1]. In [36], Ponge derived the formula for the Connes-Moscovici cocycle for a Dirac spectral triple from his new proof of the local index formula. Chern and Hu [9] and Azmi [2] computed the corresponding expressions for equivariant Dirac operators via heat kernel techniques, however without verifying the technical assumptions made in [11]. A complete treatment of the equivariant case was eventually given by Ponge and Wang [38]. In [39] and [41] van Suijlekom, Dabrowski, Landi, Sitarz and Varilly obtained a local index formula for the quantum $SU(2)$. For a recent survey on index theory and noncommutative geometry see Gorokhovsky and van Erp [24].

Noncommutative tori and noncommutative orbifolds are central and actively researched objects in noncommutative geometry. The local index formula for the noncommutative two torus can be found in [10]. This was extended recently by Fathizadeh, Luef and Tao [20], who established a local formula for the index of the Dirac operator of a twisted spectral triple on the two torus. However, the local index is given as a number without considering the Connes-Moscovici cocycle. Chakraborty and Luef [8], building on and partly generalizing work by Echterhoff, Lück, Phillips and Walters [15] and Walters [45,46], studied $n$-dimensional noncommutative tori with an action of a finite cyclic group. They used metaplectic representations and obtained structural and K-theoretical results, but no local index formulae. There also is a series of recent articles by Ponge and collaborators concerning pseudodifferential operators and differential geometric objects on noncommutative two tori, see e.g. [37]. In a different vein, Mathai and Rosenberg in [33] showed a Riemann–Roch theorem and a Hodge theorem for $n$-dimensional complex noncommutative tori. The proof is via deformation to the commutative case without the use of local index formulae. Another interesting development is the pseudodifferential calculus on quantum Euclidean spaces developed by Gao, Junge and McDonald [22], which they use to derive a local index formula for this situation. It is not unlikely that this calculus can also be used in the
present context. For the purposes of this article, however, the Shubin pseudodifferential calculus combined with the metaplectic operators is a much simpler and completely adequate tool.

**Structure of the article.** We start by recalling the local index formula by Connes and Moscovici in Section 2. We state their result both in terms of zeta functions and heat trace asymptotics. In two subsequent sections we recall necessary information about the metaplectic group and pseudodifferential operators of Shubin type. Then we obtain in Section 5 the local index formula in the one-dimensional case. Section 6 is central to our paper: Here we define our spectral triple in $\mathbb{R}^n$, show that it satisfies the conditions of Connes’ and Moscovici’s theorem (Theorem 3) and give explicit formulae for the Connes–Moscovici cocycle (Theorem 4). A decomposition of the Connes–Moscovici periodic cyclic cocycle into a sum of cyclic cocycles localized at conjugacy classes in $C_n \rtimes U(n)$ is obtained in Section 7, while examples are considered in Section 8. In Section 9, we prove all the necessary results about equivariant zeta functions for Shubin type pseudodifferential operators and metaplectic operators.

**Acknowledgement.** ES thanks Gerd Grubb and Jens Kaad for helpful remarks. The work of the first author was supported by RFBR, project number 21-51-12006; that of the second by DFG through project SCHR 319/8-1. We thank the referees for useful suggestions.

2. The Local Index Formula of Connes and Moscovici

**The Chern-Connes character.** Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. Here

- $\mathcal{A}$ is an algebra;
- $\mathcal{H} = H_0 \oplus H_1$ is a graded Hilbert space with a representation of $\mathcal{A}$ on it by bounded even operators;
- $D$ is an odd self-adjoint operator on $\mathcal{H}$. It is assumed that $D$ is local: $[D, a]$ is bounded for all $a \in \mathcal{A}$ and $(1 + D^2)^{-1}$ is compact.

Suppose, moreover, that the spectral triple is $p$-summable, i.e.

$$(1 + D^2)^{-1/2} \in L^p(\mathcal{H})$$

where $L^p(\mathcal{H}) = \{ T \mid T \text{ is compact and } \text{Tr} |T|^p < \infty \}$ is the Schatten von Neumann ideal. Then one defines the Chern–Connes character of the spectral triple in periodic cyclic cohomology, see [10], [29]:

$$\text{ch}(\mathcal{A}, \mathcal{H}, D) \in HP^{ev}(\mathcal{A}).$$

It contains information about the analytic indices of twisted operators. More precisely, given a projection $P \in \text{Mat}_N(\mathcal{A})$ over $\mathcal{A}$, we have

$$\text{ind}(P(D \otimes 1_N) : P\mathcal{H}_0^N \to P\mathcal{H}_1^N) = \langle \text{ch}(\mathcal{A}, \mathcal{H}, D), [P] \rangle,$$

where $[P] \in K_0(\mathcal{A})$ is the class of the projection in $K$-theory, while

$$\langle \cdot, \cdot \rangle : HP^{ev}(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{C}$$

stands for the pairing of periodic cyclic cohomology with $K$-theory. Let us recall the definition of this pairing. Given a periodic cyclic cocycle $\varphi = (\varphi_{2k})_{k=0,1,...,n}$ over $\mathcal{A}$ and a projection $p \in \text{Mat}_N(\mathcal{A})$, we have

$$\langle \varphi, p \rangle = \langle \varphi_0 \# \text{Tr}(p) \rangle(p) + \sum_{k \geq 1} (-1)^{k(k+1)/2} \langle \varphi_{2k} \# \text{Tr}((p - 1/2, p, ..., p) \rangle,$$

where

$$\langle \varphi_{2k} \# \text{Tr}(m_0 \otimes a_0, ..., m_{2k} \otimes a_{2k}) \rangle = \text{Tr}(m_0...m_{2k})\varphi_{2k}(a_0, ..., a_{2k}), \quad a_k \in \mathcal{A}, m_k \in \text{Mat}_N(\mathbb{C})$$

is a $2k + 1$-linear functional on $\text{Mat}_N(\mathcal{A})$. 
The local index formula. Connes and Moscovici \[11\] (see also Higson \[29\]) proved that the class \(\text{ch}(\mathcal{A}, \mathcal{H}, D)\) contains a special representative, which we call the Connes–Moscovici cocycle. To state their result, we introduce several notions. Let us assume for simplicity that \(D\) is invertible (for the noninvertible case see \[11\]).

The spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is supposed to be regular, see Definition 3.14 in \[29\], i.e. for every \(a \in \mathcal{A}\), a and \([D, a]\) are in the domains of all iterated commutators

\[
[D, ], [D, [D, ]], \ldots
\]

Given a regular spectral triple, one defines the algebra \(\Psi(\mathcal{A}, \mathcal{H}, D)\) as the smallest algebra of linear operators in \(\mathcal{H}^\infty = \bigcap_{j \geq 1} \text{Dom}[D]^j\) that contains \(\mathcal{A}\) and \([D, \mathcal{A}]\) and is closed under taking commutators with \(D^2\): \(B \in \Psi(\mathcal{A}, \mathcal{H}, D)\) implies that \([D^2, B] \in \Psi(\mathcal{A}, \mathcal{H}, D)\).

Given \(B \in \Psi(\mathcal{A}, \mathcal{H}, D)\), we introduce the zeta function \(\zeta_B\) by

\[
\zeta_B(z) = \text{Tr}(B|D|^{-2z}),
\]

which is defined and holomorphic for \(\text{Re} z\) sufficiently large.

We say that \((\mathcal{A}, \mathcal{H}, D)\) has simple dimension spectrum, if there is a discrete set \(F \subset \mathbb{C}\) such that \(\zeta_B(z)\) extends meromorphically to \(\mathbb{C}\) with at most simple poles in the set \(F + \text{ord} B\) for all \(B \in \Psi(\mathcal{A}, \mathcal{H}, D)\).

**Theorem 1. (Connes-Moscovici)** Suppose that the spectral triple has simple dimension spectrum. Then the Chern–Connes character \(\text{ch}(\mathcal{A}, \mathcal{H}, D) \in H^p_{ev}(\mathcal{A})\) in periodic cyclic cohomology has a representative \((\Psi_0, \Psi_2, \Psi_4, \ldots, \Psi_{2k}, \ldots)\), where

\[
\Psi_{2k}(a_0, a_1, \ldots, a_{2k}) = \sum_\alpha c_{k, \alpha} \text{Res}_{z = 0} \text{Tr}_s \left( a_0|D| a_1|D|^{\alpha_1} \cdots [D, a_{2k}]|D|^{\alpha_{2k}} |D|^{-2(\alpha_1 + k + z)} \right), \quad k \geq 1,
\]

\(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2k})\) is a multi-index, \(B^{[j]}\) stands for the \(j\)-th iterated commutator of the operator \(B\) with \(D^2\), and

\[
c_{k, \alpha} = (-1)^{|\alpha|} \frac{\Gamma(\lfloor \alpha \rfloor + k)}{\alpha! (\alpha_1 + 1)! (\alpha_1 + \ldots + \alpha_{2k} + 2k)!},
\]

while, see \[29, \text{Remark 5.7}h\],

\[
\Psi_0(a_0) = \text{Res}_{z = 0} z^{-1} \text{Tr}_s \left( a_0|D|^{-2z} \right).
\]

**Remark 1.** Connes and Moscovici additionally required the zeta functions \(\zeta_B\) to have rapid decay along vertical lines. It was shown by Carey, Phillips, Rennie and Sukcho\,ev \[6, 7\] that this assumption is not needed, see also Higson \[29\]. In the case at hand, rapid decay can be established directly with the help of the weakly parametric calculus of Grubb and Seeley, \[29\].

The local index formula and heat trace asymptotics. The following proposition will enable us to apply techniques in local index theory based on heat trace asymptotics.

**Proposition 1.** Let the assumptions in Theorem \[11\] be satisfied for the spectral triple \((\mathcal{A}, \mathcal{H}, D)\). Suppose in addition that for an operator \(B \in \Psi(\mathcal{A}, \mathcal{H}, D)\) the heat trace \(\text{Tr}(B|D|^{-2})\) exists, is a continuous function of \(t > 0\), is \(O(t^{-\infty})\) for \(t \to \infty\) and has an asymptotic expansion

\[
\text{Tr}(B|D|^{-2}) \sim \sum_{k=0}^{\infty} a_{m_k} t^{m_k} \quad \text{as } t \to 0^+
\]

for a sequence of real numbers \(m_k \not\to \infty\). Then

\[
\text{Res}_{z=0} \text{Tr}(B|D|^{-2(m+z)}) = \frac{a_m}{\Gamma(m)} \quad \text{whenever } m > 0.
\]
Moreover, for $m = 0$,
\begin{equation}
\Res_{z = 0} z^{-1} \Tr(B|D|^{-2z}) = a_0.
\end{equation}

Proof. Let $m > 0$. We use the Mellin transform $M_{t \to z}$ and obtain for $\Re z > 0$:
\begin{equation}
|D|^{-2z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-tD^2} dt, \text{ and hence } \zeta_B(z) = \frac{1}{\Gamma(z)} M_{t \to z}(\Tr(Be^{-tD^2})).
\end{equation}

The Mellin transform is well defined by our assumptions for large $\Re z$. Hence
\begin{align*}
\Tr(B|D|^{-2(m+z)}) &= \frac{1}{\Gamma(z + m)} \int_0^\infty t^{z+m-1} \Tr(Be^{-tD^2}) dt \\
&= \frac{1}{\Gamma(z + m)} \int_0^1 t^{z+m-1} \Tr(Be^{-tD^2}) dt = \frac{1}{\Gamma(z + m)} \sum_{k : m + m_k < 1} a_{m_k} \int_0^1 t^{z+m+m_k-1} dt,
\end{align*}
where $\equiv$ means equality modulo functions holomorphic for $\Re z > -1$. We then conclude that
\begin{equation}
\Res_{z = 0} \Tr(B|D|^{-2(m+z)}) = \Res_{z = 0} \frac{1}{\Gamma(z + m)} \sum_{k : m + m_k < 1} a_{m_k} \frac{t^{z+m+m_k}}{z + m + m_k} \bigg|_0^1 = \frac{a_{-m}}{\Gamma(m)}
\end{equation}

The case $m = 0$ is considered similarly.

\begin{remark}
It is possible to use properties of the heat trace and properties of the Mellin transform (see e.g. [21] Theorems 3 and 4) in order to obtain the properties of zeta functions listed in Theorem 3. For instance, if we require that $\Tr(Be^{-tD^2})$ is smooth, $O(t^{-\infty})$ for $t \to \infty$, and has an asymptotic expansion \((\ref{As})\) as $t \to 0$, then this implies that $\zeta_B(z)$ has a meromorphic continuation to $\mathbb{C}$ and rapid decay on vertical lines.

Suppose that all the operators $B = a_0[D, a_1]^{[\alpha_1]} \ldots [D, a_{2k}]^{[\alpha_{2k}]}$ in \((\ref{As})\) satisfy the assumptions in Proposition 1. Then we apply Proposition 1 and express the Connes–Moscovici cocycle $\Psi_{2k}$ in terms of heat invariants and get:
\begin{equation}
\Psi_{2k}(a_0, a_1, \ldots, a_{2k}) = \sum_{\alpha} d_{k,\alpha} \times \left\{ \text{finite part of } t^{[\alpha]+k} \Tr \left( a_0[D, a_1]^{[\alpha_1]} \ldots [D, a_{2k}]^{[\alpha_{2k}]} e^{-tD^2} \right) \right\},
\end{equation}
for all $k \geq 0$, where
\begin{equation}
d_{k,\alpha} = \frac{(-1)^{[\alpha]}}{\alpha!(\alpha_1 + 1) \ldots (\alpha_1 + \ldots + \alpha_{2k} + 2k)}.
\end{equation}

3. The Metaplectic Group

Let us recall a few facts about the symplectic and metaplectic groups from [25,31,41].

The symplectic and the metaplectic groups. The metaplectic group $\Mp(n) \subset BL^2(\mathbb{R}^n)$ is the group generated by unitary operators of the form
\begin{equation}
\exp(-i\hat{H}) \in \Mp(n),
\end{equation}
where $\hat{H}$ is the Weyl quantization of a homogeneous real quadratic Hamiltonian $H(x,p), \ (x,p) \in T^*\mathbb{R}^n$. In its turn, the complex metaplectic group $\Mp^c(n)$ is similarly generated by unitaries associated with Hamiltonians $H(x,p) + \lambda$, where $H(x,p)$ is as above, while $\lambda$ is a real constant. Elements of the metaplectic group are called metaplectic operators.

The symplectic group $\Sp(n) \subset \GL(2n, \mathbb{R})$ is the group of linear canonical transformations of $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$, i.e., linear transformations that preserve the symplectic form $dx \wedge dp$. The
symplectic group is generated by the canonical transformations equal to the evolution operator for time $t = 1$ of the Hamiltonian system
\[ \dot{x} = H_p, \quad \dot{p} = -H_x, \]
where $H(x,p)$ is a homogeneous real quadratic Hamiltonian as above.

There is a natural projection $\pi : \text{Mp}(n) \to \text{Sp}(n)$ that takes a metaplectic operator to the corresponding canonical transformation. This projection is a nontrivial double covering of $\text{Sp}(n)$. Thus, one can not represent elements of $\text{Sp}(n)$ unambiguously by metaplectic operators. However, it turns out that one can define a representation of the subgroup of isometric linear canonical transformations by operators in the complex metaplectic group. Let us describe this representation.

**Isometric linear canonical transformations and their quantization.** Consider the maximal compact subgroup $\text{Sp}(n) \cap O(2n)$ of isometric linear canonical transformations. It is well known that this intersection coincides with the group $U(n)$ of unitary transformations of $T^*\mathbb{R}^n$ if we introduce the complex structure on $T^*\mathbb{R}^n \simeq \mathbb{C}^n$ via $(x, p) \mapsto z = p + ix$, see [1].

Recall that the unitary group is generated by the matrices $\exp(B + iA)$, where $A$ is a symmetric real matrix, while $B$ is a skew-symmetric real matrix.

**Proposition 2.** The following mapping is a well-defined homomorphism of groups
\[ R : U(n) \to \text{Mp}(n) \]
\[ g = \exp(B + iA) \mapsto R_g = \exp(-i\hat{H}) \exp(i\text{Tr}A/2), \]
where $\hat{H}$ is the Weyl quantization of the Hamiltonian
\[ H(x,p) = \frac{1}{2}(x,p) \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}. \]

Since $U(n)$ is generated by the subgroups $O(n)$ and $U(1) = \{\text{diag}(z,1,\ldots,1) \mid |z| = 1\}$ (see e.g. [11 Lemma 1]), it follows that the homomorphism $R$ is characterized by the properties:

- $R_g(x) = u(g^{-1}x)$, if $g \in O(n) \subset U(n)$; in this case $R_g$ is the shift operator for an orthogonal matrix $g$.
- $R_g(x) = e^{i\varphi/2 - \hat{H}_1}u(x)$, if $g = \text{diag}(e^{\varphi}, 1, \ldots, 1)$, where $\hat{H}_1 = \frac{1}{2}(x_1^2 - \partial^2_{x_1})$. In this case, the operator $R_g$ is called the fractional Fourier transform with respect to $x_1$ and for $\varphi \in (0, \pi)$ is equal to (see [30 Corollary 4.2])
\[ R_g u(x) = \sqrt{\frac{1 - i \text{ctg} \varphi}{2\pi}} \int \exp \left( i \left( (x_1^2 + y_1^2) \frac{\text{ctg} \varphi}{2} - \frac{x_1y_1}{\sin \varphi} \right) \right) u(y_1,x_2,\ldots,x_n)dy_1. \]

4. **Shubin Type Pseudodifferential Operators**

A smooth function $a = a(x,p)$ on $T^*\mathbb{R}^n$ is a pseudodifferential symbol (of Shubin type) of order $m \in \mathbb{R}$, provided its derivatives satisfy the estimates
\[ |D_p^\alpha D_x^\beta a(x,p)| \leq c_{\alpha,\beta}(1 + |x| + |p|)^{m-|\alpha|-|\beta|} \]
for all multi-indices $\alpha, \beta$, with suitable constants $c_{\alpha,\beta}$. In this article, we only work with classical symbols where $a$ admits an asymptotic expansion $a \sim \sum_{j=0}^{\infty} a_{m-j}$. Here, each $a_{m-j}$ is a symbol of order $m- j$, which is (positively) homogeneous in $(x, p)$ for $|x, p| \geq 1$.

To a symbol $a$ as above we associate the operator $\text{op}(a)$ on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, defined by
\[ \text{op}(a)u(x) = (2\pi)^{-n/2} \int e^{ixp} a(x,p)\hat{u}(p) dp, \]
where \( \hat{u}(p) = (2\pi)^{-n/2} \int e^{-ix\cdot p} u(x) \, dx \) is the Fourier transform of \( u \). Alternatively, we have the Weyl quantization \( \text{op}^w(a) \) of a defined by

\[
\text{op}^w(a)(x) = (2\pi)^{-n} \int \int e^{i(x-y)\cdot p} a\left(\frac{x+y}{2}, p\right) u(y) \, dy \, dp.
\]

The principal symbol \( \sigma(A) \) of the operator \( A = \text{op}(a) \) is defined as the homogeneous extension of the leading term \( a_m \) to \( T^*\mathbb{R}^n \setminus \{0\} \).

A full calculus for Shubin type pseudodifferential operators, i.e. pseudodifferential operators with such symbols, has been developed in [42, Chapter IV]. We write \( \Psi^m(\mathbb{R}^n) \) for the space of all Shubin type pseudodifferential operators of order \( \leq m \) and \( \Psi(\mathbb{R}^n) \) for the algebra of all these operators.

A fact we need in several places is that the elements of \( \Psi^0(\mathbb{R}^n) \) extend to bounded operators on \( L^2(\mathbb{R}^n) \) and those of \( \Psi^m(\mathbb{R}^n) \) to trace class operators on \( L^2(\mathbb{R}^n) \) provided \( m < -2n \). This is shown in [42 Theorem 24.3 and Proposition 27.2].

Moreover, an Egorov theorem holds: Given an element \( S \in \text{Mp}(n) \) and \( A = \text{op}^w a \) a Weyl-quantized Shubin type pseudodifferential operator with symbol \( a \), then \( S^{-1}AS \) is the Weyl-quantized Shubin type pseudodifferential operator with symbol \( a \circ \pi(S) \), where \( \pi(S) \in \text{Sp}(n) \) is the canonical transformation associated with \( S \), see [25 Theorem 7.13]. As the principal symbol of the Weyl-quantized operator \( \text{op}^w(a) \) coincides with that of \( \text{op}(a) \), we find in particular that

\[
\sigma(S^{-1}AS) = \sigma(A) \circ \pi(S).
\]

5. Operators on \( \mathbb{R} \)

We start with the case \( n = 1 \), as it is simpler and the results will be useful later on.

**The Euler operator.** We introduce

\[
D = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & x - \partial_x \\ x + \partial_x & 0 \end{pmatrix} : S(\mathbb{R}, \mathbb{C}^2) \to S(\mathbb{R}, \mathbb{C}^2), \quad \text{so that} \quad D^2 = \begin{pmatrix} H - \frac{i}{2} & 0 \\ 0 & H + \frac{i}{2} \end{pmatrix},
\]

where \( H = \frac{1}{2}(x^2 - \partial_x^2) \).

**Heisenberg-Weyl operators.** For \( a, k \in \mathbb{R} \) we define the Heisenberg-Weyl operators \( T_{a,k} \) on \( L^2(\mathbb{R}) \) by

\[
T_{a,k}u(x) = e^{ikx-ik^2/2}u(x-a).
\]

We shall also write \( T_{a,k} = T_z \), where \( z = a-ik \). These operators generate the Heisenberg group, and we have the product formula

\[
T_{z_1}T_{z_2} = e^{-i\text{Im}z_1\text{Im}z_2/2}T_{z_1+z_2}.
\]

We extend the action of the Heisenberg-Weyl operator \( T_z \) to \( S(\mathbb{R}, \mathbb{C}^2) \) by \( (u,v) \mapsto \langle T_zu, T_zv \rangle \), denoting it by \( T_z \) as in the scalar case. Then the following commutation relations are true

\[
[D, T_z] = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix} T_z,
\]

\[
[H, T_z] = \frac{1}{2} ((k^2 - a^2 + 2ax)T_z - 2ikT_z \partial_x) = T_z \cdot (\text{operator of order } 1).
\]
Metaplectic operators (fractional Fourier transforms). Consider the representation

\[ R : U(1) \rightarrow Mp^c(1) \]
\[ e^{i\varphi} \mapsto R_\varphi = e^{i(1/2-H)\varphi}. \]

We obtain the commutation relation:

\[ R_\varphi T_z R_\varphi^{-1} = T_{e^{i\varphi}z}. \]

We also extend the action of the metaplectic operator \( R_\varphi \) to \( S(\mathbb{R}, \mathbb{C}^2) \) via

\[ R_\varphi \psi(u, v) = (R_\varphi u, e^{-i\varphi} R_\varphi v). \]

It turns out that \( D \) is \( U(1) \)-equivariant (see [41] for the proof), i.e.

\[ R_\varphi DR_\varphi^{-1} = D. \]

Heat asymptotics.

**Proposition 3.** We have the following asymptotics as \( t \to 0^+ \):

\[ \text{Tr}(T_z R_\varphi e^{-tH}) = \begin{cases} 
O(t^{+\infty}) & \text{if } \varphi = 0, z \neq 0; \\
\frac{1}{\sqrt{2\sqrt{\text{ch} t - 1}}} = \frac{1}{t} + O(1) & \text{if } \varphi = 0, z = 0; \\
\frac{1}{1 - e^{-i\varphi}} \exp \left( \frac{i}{4}(a^2 + k^2) \cotg(\varphi/2) \right) + O(t) & \text{if } \varphi \in (0, 2\pi).
\end{cases} \]

Here \( T_z R_\varphi e^{-tH} \) is treated as an operator on \( L^2(\mathbb{R}) \).

**Proof.** 1. In case \( \varphi = 0 \), Mehler’s formula for the heat kernel

\[ e^{-tH}(x, y) = \frac{1}{\sqrt{2\pi \text{sh} t}} \exp \left( -c\text{th} t \frac{x^2 + y^2}{2} + \frac{1}{\text{sh} t} xy \right) \]

shows that

\[ T_z e^{-tH}(x, y) = \frac{e^{i(kx-ak/2)}}{\sqrt{2\pi \text{sh} t}} \exp \left( -c\text{th} t \left( x - a \right)^2 + \frac{1}{\text{sh} t} (x - a) y \right). \]

Hence, the value of the kernel at the diagonal is equal to

\[ T_z e^{-tH}(x, x) = \frac{e^{-ik/2}}{\sqrt{2\pi \text{sh} t}} \exp \left( -x^2 \left( \frac{\text{ch} t - 1}{\text{sh} t} \right) + x \left( ik + a \frac{\text{ch} t - 1}{\text{sh} t} \right) - a^2 \frac{\text{ch} t}{2 \text{sh} t} \right) \]

and we obtain

\[ \text{Tr}(T_z e^{-tH}) = \int_{\mathbb{R}} T_z e^{-tH}(x, x) dx = \frac{1}{\sqrt{2\sqrt{\text{ch} t - 1}}} \exp \left( -a^2 \frac{\text{ch} t + 1}{4 \text{sh} t} - k^2 \frac{\text{sh} t}{4(\text{ch} t - 1)} \right). \]

This readily gives us the first two lines in (17).

2. In case \( \varphi \neq 0 \) we have

\[ T_z R_\varphi e^{-tH} = T_z e^{i(1/2-H)\varphi} e^{-tH} = e^{i\varphi/2} T_z e^{-(t+i\varphi)H}. \]

\[ \int_{\mathbb{R}} e^{-ax^2 + bx + c} dx = \sqrt{\frac{\pi}{a}} e^{b^2/4a + c}. \]
We can compute the trace of this operator by replacing \( t \) by \( t + i \varphi \) in (18). Then
\[
\lim_{t \to 0^+} \text{Tr}(T_z R_\varphi e^{-tH}) = \frac{e^{i\varphi/2}}{i\sqrt{2\sqrt{1 - \cos \varphi}}} \exp \left( \frac{i}{4} \left( a^2 \frac{1 + \cos \varphi}{\sin \varphi} + k^2 \frac{\sin \varphi}{1 - \cos \varphi} \right) \right),
\]
where \( \varphi \in (0, 2\pi) \). Here the argument of the square root is computed using the Taylor expansion at \( t = 0, \varphi = 0: \sqrt{\text{ch}(t + i\varphi)} - 1 \sim \sqrt{(t + i\varphi)^2/2} = (t + i\varphi)/\sqrt{2} \). This expression is equal to the last line in (17).

The local index formula. We denote by \( \mathcal{A} \) the algebra generated by the operators \( T_z \) and \( R_g, z \in \mathbb{C}, g \in U(1) \), acting on the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^2) \). Let us compute the Connes-Moscovici cocycle of the spectral triple \( (\mathcal{A}, \mathcal{H}, D) \) defined from the operator \( D \). It follows from the above commutation relations that
\[
a_0[D, a_1]^{[a_1]}[D, a_2k]^{[a_2k]}|D|^{-2(|a| + k + z)}
\]
is an operator of order \( \leq |\alpha| - 2(|\alpha| + k + \text{Re} \, z) = -|\alpha| - 2k - 2 \text{Re} \, z \). As operators of order \( < -2 \) are of trace class, there are only two possibilities to obtain a nontrivial contribution to the Connes-Moscovici local index formula in (21)/(23) namely a) \( k = 0 \) and b) \( k = 1, |\alpha| = 0 \). This will be important for the proof of the following theorem.

Theorem 2. The component \( \Psi_0 \) of the Connes-Moscovici cocycle is equal to
\[
\Psi_0(T_z R_\varphi) = \begin{cases} 
0, & \text{if } \varphi = 0, z \neq 0, \\
1, & \text{if } \varphi = 0, z = 0, \\
\exp \left( \frac{i}{4} (a^2 + k^2) \cot(\varphi/2) \right), & \text{if } \varphi \neq 0.
\end{cases}
\]
Here \( z = a - ik \). The component \( \Psi_2 \) of the Connes-Moscovici cocycle is equal to
\[
\Psi_2(T_{z_0} R_{\varphi_0}, T_{z_1} R_{\varphi_1}, T_{z_2} R_{\varphi_2}) = \begin{cases} 
0, & \text{if } \varphi_1 + \varphi_2 \neq 0 \text{ or } z_0 + e^{i\varphi_0} z_1 + e^{i(\varphi_0 + \varphi_1)} z_2 \neq 0, \\
\frac{e^{i\varepsilon}}{4} (z_1 \overline{z}_2 e^{-i\varphi_1} - \overline{z}_1 z_2 e^{i\varepsilon_1}), & \text{otherwise},
\end{cases}
\]
where \( \varepsilon = \text{Im}(e^{i\varphi_0} z_1 \overline{z}_0 + e^{i\varphi_1} z_2 \overline{z}_1 + e^{i(\varphi_0 + \varphi_1)} z_2 \overline{z}_0) \).

Proof. We saw that there is no contribution to the Connes-Moscovici cocycle from terms with \( \alpha \neq 0 \).

1. In order to show (20), we note that
\[
\text{Tr}_s(T_z R_\varphi e^{-tD^2}) = \text{Tr} \left( \begin{array}{cc} 1 & 0 \\
0 & -1 \end{array} \right) T_z \left( \begin{array}{cc} R_\varphi & 0 \\
0 & R_\varphi e^{-i\varphi} \end{array} \right) \left( \begin{array}{cc} e^{-t(H - \frac{1}{2})} & 0 \\
0 & e^{-t(H + \frac{1}{2})} \end{array} \right) 
\]
\[
= (e^{t/2} - e^{-i\varphi} e^{-t/2}) \text{Tr}(T_z R_\varphi e^{-tH}).
\]
According to (5) and (8) we have to compute the constant term as \( t \to 0^+ \). Substituting the heat asymptotics from Proposition 3 gives precisely (20).

2. Next let us prove (21). A direct computation using (15), (12), (13), (16) shows that
\[
\text{Tr}_s(T_{z_0} R_{\varphi_0} [D, T_{z_1} R_{\varphi_1}] [D, T_{z_2} R_{\varphi_2}] e^{-tD^2}) = \frac{1}{2} (z_1 \overline{z}_2 e^{-i\varphi_1} e^{t/2} - \overline{z}_1 z_2 e^{-i(\varphi_0 + \varphi_2)} e^{-t/2}) \text{Tr}(T_{z_0} R_{\varphi_0} T_{z_1} R_{\varphi_1} T_{z_2} R_{\varphi_2} e^{-tH})
\]
\[
= \frac{1}{2} (z_1 \overline{z}_2 e^{-i\varphi_1} e^{t/2} - \overline{z}_1 z_2 e^{-i(\varphi_0 + \varphi_2)} e^{-t/2}) \text{Tr}(e^{i\varepsilon_1} T_w R_{\varphi_0 + \varphi_1 + \varphi_2} e^{-tH}),
\]
where
\[
w = z_0 + e^{i\varphi_0} z_1 + e^{i(\varphi_0 + \varphi_1)} z_2,
\]
\[
e^{i\varepsilon_1} Id = T_{z_0} T_{e^{i\varphi_0} z_1} T_{e^{i(\varphi_0 + \varphi_1)} z_2} T_{z_2}^{-1}.
\]
By (14) and (17), $\Psi_2(T_{z_0}R_{\varphi_0}, T_{z_1}R_{\varphi_1}, T_{z_2}R_{\varphi_2})$ equals $1/2$ times the coefficient of $t^{-1}$ in the asymptotics of (22) as $t \to 0^+$. By Proposition 18 this coefficient is zero if either $\varphi_0 + \varphi_1 + \varphi_2 \neq 0$ or $w = z_0 + e^{i\varphi_0}z_1 + e^{(i\varphi_0 + \varphi_1)}z_2 \neq 0$. Otherwise, we obtain

$$
(23) \quad \text{Tr}_x(T_{z_0}R_{\varphi_0}[D, T_{z_1}R_{\varphi_1}[D, T_{z_2}R_{\varphi_2}]]e^{-iD^2}) = \frac{1}{2}(z_1\bar{z}_2e^{-i\varphi_1} - \bar{z}_1z_2e^{i\varphi_1})e^{ie't-1} + O(1),
$$

where

$$
e^{ie't}Id = T_{z_0}T_{e^{i\varphi_0}z_1}T_{e^{(i\varphi_0 + \varphi_1)}z_2} = e^{i\text{Im}(\bar{z}_0\varphi_1)}T_{z_0}T_{e^{i\varphi_0}z_1}T_{e^{(i\varphi_0 + \varphi_1)}z_2} = e^{i\text{Im}(\bar{z}_0\varphi_1)}e^{i\text{Im}(\bar{z}_0\varphi_1)}Id = e^{i\varphi_1}Id.
$$

Asymptotics (23) and Eq. (14) give the desired expression (21) for $\Psi_2$. 

\section{Operators on $\mathbb{R}^n$}

**The Euler operator.** We introduce the Euler operator

$$
(24) \quad D = \frac{1}{\sqrt{2}}(d + d^* + xdx \wedge + (xdx \wedge)^*) : S(\mathbb{R}^n, \Lambda^{even}(\mathbb{C}^n) \otimes \mathbb{C}) \to S(\mathbb{R}^n, \Lambda^{odd}(\mathbb{R}^n) \otimes \mathbb{C}).
$$

Here $d$ is the exterior differential, $xdx \wedge$ is the operator of exterior multiplication by $xdx = dr^2/2 = \sum x_jdx_j$, where $r = |x|$, while $d^*$ and $(xdx \wedge)^*$ stand for the adjoint operators. Its symbol is invertible for $|x|^2 + |p|^2 \neq 0$. We consider this operator in the Schwartz spaces of complex-valued differential forms. Below, we will use the identification $\Lambda(\mathbb{R}^n) \otimes \mathbb{C} \simeq \Lambda(\mathbb{C}^n)$. According to [27 Lemma 14]

$$
(25) \quad D^2 = H + F, \quad \text{where } H = \frac{1}{2} \sum_{j=1}^{n} \left( -\frac{\partial^2}{\partial x_j^2} + x_j^2 \right), \quad F|_{\Lambda^k} = \left( k - \frac{n}{2} \right) \text{Id}.
$$

**Heisenberg-Weyl operators.** Given $z = a - ik \in \mathbb{C}^n$, where $a, k \in \mathbb{R}^n$, we define the operators

$$
T_zu(x) = e^{ikx - iak/2}u(x - a).
$$

These operators and $e^{ie't}$ for all $\varepsilon \in \mathbb{R}$ define the so-called Schrödinger representation of the Heisenberg group. We note the following product formula

$$
(26) \quad T_{z_1}T_{z_2} = e^{-i\text{Im}(z_1, z_2)/2}T_{z_1 + z_2}, \quad \text{where } (z_1, z_2) = z_1\bar{z}_2.
$$

The Heisenberg-Weyl operators are extended to the space of forms by the trivial action on the differentials, and this extension is denoted by the same symbol.

**Metaplectic operators.** Let $g \in U(n) \mapsto R_g \in BL^2(\mathbb{R}^n)$ be the unitary representation of $U(n)$ by operators in the complex metaplectic group defined in Section 6. By Theorem 7.13 in [25]

$$
(27) \quad R_{g}T_{z}R_{g}^{-1} = T_{gz}.
$$

The metaplectic operators are also extended to the space of forms by the formula:

$$
(28) \quad R_{g} : S(\mathbb{R}^n, \Lambda^{even}(\mathbb{C}^n)) \to S(\mathbb{R}^n, \Lambda^{even}(\mathbb{C}^n)) \quad u \otimes \omega \mapsto R_{g}(u \otimes \omega) = R_{g}u \otimes g^{*^{-1}}\omega,
$$

where $g^{*^{-1}} : \Lambda^{even}(\mathbb{C}^n) \to \Lambda^{even}(\mathbb{C}^n)$ stands for the induced action on forms for $g : \mathbb{C}^n \to \mathbb{C}^n$.

One has the following properties:

$$
(29) \quad [D, T_z] = \frac{1}{\sqrt{2}}T_zc(z), \quad \text{where } c(z) = \Theta dx \wedge + zdz; \quad 2
$$

\[\text{Indeed, } \sigma(D)(x, p) = 2^{-1/2}[(ip + xdx) \wedge + ((ip + xdx)\wedge)^*]. \text{ Hence, } \sigma(D)^2(x, p) = 2^{-1}|x|^2 + |p|^2]\text{Id.} \]
\[ [D, R_g] = 0, \text{ for all } g \in U(n). \]

Equality (29) is straightforward, while (30) is [11] Lemma 4.

**Main results.** Let \( \mathcal{A} \) be the operator algebra generated by the operators \( T_z \) and \( R_g \) for \( z \in \mathbb{C}^n, g \in U(n) \) on the graded Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^n, \Lambda(\mathbb{C}^n)) \). It follows from (20) and (27) that an arbitrary element in \( \mathcal{A} \) can be written as a finite sum

\[ a = \sum_{z,g} a_{z,g} T_z R_g : L^2(\mathbb{R}^n, \Lambda(\mathbb{C}^n)) \to L^2(\mathbb{R}^n, \Lambda(\mathbb{C}^n)), \quad a_{z,g} \in \mathbb{C}. \]

By \( \Psi(\mathcal{A}, \mathcal{H}, D) \) we denote the algebra of all operators of the form

\[ B = \sum_k D_k T_{z_k} R_{g_k}, \]

where the sum is finite, \( z_k \in \mathbb{C}^n, g_k \in U(n) \) and the \( D_k \) are Shubin type pseudodifferential operators (see Section 4). This algebra \( \Psi(\mathcal{A}, \mathcal{H}, D) \) might be larger than the one defined by Connes and Moscovici (see Section 2).

**Theorem 3.** The conditions in Connes–Moscovici’s local index theorem (Theorem 2) are satisfied for the graded spectral triple \( (\mathcal{A}, \mathcal{H}, D) \). More precisely, the spectral triple is regular, finitely summable, and has simple dimension spectrum.

**Proof.** The regularity of the spectral triple follows from the invariance of the Shubin pseudodifferential calculus under the affine metaplectic group generated by all \( T_z \) and \( R_g, z \in \mathbb{C}^n, g \in U(n) \). Moreover, the explicit description of the spectrum of \( D^2 = H + F \) enables one to prove that \( |D|^{-1} \) is \( p \)-summable whenever \( p > 2n \).

The proof that zeta functions for this spectral triple extend to meromorphic functions on \( \mathbb{C} \) with simple poles is deferred to Section 9. \( \square \)

To state the main result of this paper, we recall the definition of the Berezin integral. Given a complex subspace \( L \subset \mathbb{C}^n \), we define the Berezin integral as a linear functional

\[ \int_L : \Lambda(L) \to \mathbb{C} \]

on exterior forms on \( L \) considered as a real vector space of dimension \( 2k \). To define this functional, we choose an orthonormal base \( e_1, \ldots, e_k \) in \( L \), denote the coordinates with respect to this base by \( z_j = p_j + ix_j \) and consider the volume form \( dp_1 \wedge dx_1 \wedge \ldots \wedge dp_k \wedge dx_k \in \Lambda^{2k}(L) \). Then the Berezin integral (32) is characterized by the properties: \( \int_L dp_1 \wedge dx_1 \wedge \ldots \wedge dp_k \wedge dx_k = 1 \) and \( \int_L \omega = 0 \) whenever \( \text{deg} \omega < 2k \). It is easy to show that this definition does not depend on the choice of an orthonormal base. Below we denote the coordinates in \( \mathbb{C}^n \) by \( z = p + ix \).

**Theorem 4.** The components of the Connes–Moscovici cocycle \( \Psi = (\Psi_0, \Psi_2, \ldots, \Psi_{2n}) \) of the spectral triple in Theorem 3 are equal to

\[ \Psi_{2k}(a_0, a_1, \ldots, a_{2k}) = \begin{cases} 0, & \text{if the mapping } w \mapsto gw + z \text{ has no fixed points or } k > \dim \mathbb{C}_g^n \\ \frac{i^{-k}}{(2k)!} e^{\frac{i}{2} \sum_{j=1}^m e^{\frac{i}{2}(z_j x_j)^2} \text{ctg}(x_j/2)} \int_{\mathbb{C}_g^n} \sigma(w_1) \wedge \sigma(w_2) \wedge \ldots \wedge \sigma(w_{2k}) \wedge e^{-w} \end{cases} \]

where

- \( a_j = T_{z_j} R_{g_j}, z_j \in \mathbb{C}^n, g_j \in U(n); \)
- \( \mathbb{C}_g^n \subset \mathbb{C}^n \) is the fixed point set of \( g = g_0 g_1 \ldots g_{2k}; m = n - \dim \mathbb{C}_g^n; \)
Proposition 4. of Proposition 3 and obtain the following asymptotics.

As before, \(\{e_j\}_{j=1}^{m}\) is an orthonormal system of eigenvectors of \(g\) with eigenvalues \(e^{i\varphi_j}\); \(e^{i\varphi_j} \neq 1\) for \(j = 1, \ldots, m\). Note that \((z, e_j) = (h^{-1}z)_j\) and that the condition that the fixed point set of \(w \mapsto gw + z\) is nontrivial is equivalent to \((z)_j = (z, e_j) = 0\) for all \(j > m\) (equivalently, \(z\) is orthogonal to the fixed point set of \(g\)).

Also, if \(B\) is a differential operator of order \(\text{ord} B\), then one has

\[
\text{Tr}(BT_z R_g e^{-tH}) \sim \begin{cases} O(t^{+\infty}) & \text{if the mapping } w \mapsto gw + z \text{ has no fixed points;} \\ O(t^{-(n-m+\text{ord} B)/2}) & \text{otherwise.} \end{cases}
\]
Step 2. Computation of the Connes–Moscovici cocycle for $\alpha = 0$. Given $a_j = T_{z_j} R_{g_j}$, we have

\begin{align*}
(37) \quad a_0[D, a_1]...[D, a_{2k}]e^{-tD^2} &= 2^{-k}T_{z_0} R_{g_0} c(z_1) T_{z_1} R_{g_1} ... c(z_{2k}) T_{z_{2k}} R_{g_{2k}} e^{-t(H + F)} \\
&= 2^{-k} \left( T_{z_0} R_{g_0} T_{z_1} R_{g_1} ... T_{z_{2k}} R_{g_{2k}} e^{-tH} \right) \otimes \left( g_0^{-1} c(z_1) g_1^{-1} c(z_2) g_2^{-1} ... c(z_{2k}) g_{2k}^{-1} e^{-tF} \right) \\
&= 2^{-k} \left( e^{i\epsilon} T_{z} R_{g} e^{-tH} \right) \otimes \left( c(w_1) c(w_2) ... c(w_{2k}) g^{-1} e^{-tF} \right),
\end{align*}

where 

$$z = w_0 + w_1 + w_2 + ...,$$ 

while $e^{i\epsilon} Id = T_{w_0} T_{w_1} T_{w_2} ... T_{w_{2k}} T_z^{-1}$. Hence, we obtain for the supertrace

\begin{equation}
(38) \quad \text{Tr}_s(a_0[D, a_1]...[D, a_{2k}]e^{-tD^2}) = 2^{-k} e^{i\epsilon} \text{Tr}\left( T_{z} R_{g} e^{-tH} \right) \text{Tr}_s\left( c(w_1) c(w_2) ... c(w_{2k}) g^{-1} e^{-tF} \right),
\end{equation}

The heat trace $\text{Tr}\left( T_{z} R_{g} e^{-tH} \right)$ here is computed using $(35)$. Note that $(35)$ is exponentially small if the fixed point set of the affine mapping $w \rightarrow gw + z$ is empty. It remains to compute the expansion of the supertrace in $(38)$.

We have $w_j \in \mathbb{C}^n \simeq \mathbb{R}^{2n}$ with the base $e_1, e_2, ..., e_{2n-1}, e_{2n}$:

$$e_1 = (i, 0, 0, ..., 0), \quad e_2 = (1, 0, 0, ..., 0), \quad ..., \quad e_{2n-1} = (0, 0, ..., 0, i), \quad e_{2n} = (0, 0, ..., 0, 1)$$

and let $\text{Cl}(2n)$ be the real Clifford algebra generated by these vectors with the relations

$$e_j^2 = 1 \quad \text{for all } j; \quad e_j e_k + e_k e_j = 0 \quad \text{for all } k \neq j.$$

The mapping $z \mapsto c(z) \in \text{End} \Lambda(\mathbb{C}^n)$ defined in $(29)$ enjoys the property

$$c(z_1) c(z_2) + c(z_2) c(z_1) = 2 \text{Re}(z_1 \cdot \overline{z_2}).$$

Thus, $c(e_j)^2 = 1$, and $c(e_j) c(e_k) + c(e_k) c(e_j) = 0$, $k \neq j$. Hence, this mapping uniquely extends to a homomorphism of algebras denoted by $c : \text{Cl}(2n) \rightarrow \text{End} \Lambda(\mathbb{C}^n)$.

The symbol mapping $\sigma$ is given by

$$\sigma : \text{Cl}(2n) \대로 \Lambda(\mathbb{R}^{2n})$$

\begin{align*}
& e_{j_1} e_{j_2} ... e_{j_k} \mapsto \sigma(e_{j_1}) \land \sigma(e_{j_2}) \land ... \land \sigma(e_{j_k}),
\end{align*}

where $j_1, ..., j_k$ are all different, and $\sigma(e_{2j-1}) = dx_j$, $\sigma(e_{2j}) = dp_j$.

In the sequel we shall need the Berezin lemma:

$$\text{Tr}_s(c(a)) = (-2i)^n \int_{\mathbb{C}^{2n}} \sigma(a), \quad \forall a \in \text{Cl}(2n).$$

To check that the constant here is chosen correctly, we consider the special case $n = 1$. Then $\text{Cl}(2)$ is spanned by $1, e_1, e_2, e_1 e_2$. Both sides of the formula are nontrivial only for $a = e_1 e_2$ and we have

$$c(e_1) c(e_2) = (-i dx \land + i dx \land) (dx \land + dx \land) = -i (dx \land dx \land - dx \land dx \land).$$

Hence, we get $\text{Tr}_s(c(e_1) c(e_2)) = i + i = 2i$. On the other hand, we have $\sigma(e_1 e_2) = dx \land dp$ and

$$\int_{\mathbb{C}} \sigma(e_1 e_2) dp = \frac{dz \land dp}{dxdz} = -1.$$
Proposition 5. Given \( k \leq n - m \), we have

\[
\text{Tr}_s\left(c(w_1)c(w_2)\ldots c(w_{2k})g^{-1^*}e^{-tF}\right) \\
\sim t^{n-m-k}2^k\sqrt{i}^{-k} \prod_{j=1}^{m} \left(1 - e^{-i\varphi_j}\right) \int_{\mathbb{C}^n} \sigma(w_1) \wedge \sigma(w_2) \wedge \ldots \wedge \sigma(w_{2k}) \wedge e^{-\omega},
\]

where \( \omega = dx_1 \wedge dp_1 + \ldots + dx_n \wedge dp_n \) is the symplectic form on \( \mathbb{C}^n \) with coordinate \( z = p + ix \). Also if \( k > n - m \), then the left hand side in (39) is \( O(1) \).

Proof. If \( k > n - m \), then the statement in this proposition is true, since the expression is smooth up to \( t = 0 \). Let us now obtain the asymptotics for \( k \leq n - m \). Since both sides of the formula are invariant under changes of coordinates, we choose coordinates, in which \( g \) is a diagonal matrix \( \text{diag}(e^{i\varphi_1}, \ldots, e^{i\varphi_m}, 1, \ldots, 1) \).

Lemma 1. The operators \( F, e^{-tF}, g^{-1^*} \in \text{End} \Lambda(\mathbb{C}^n) \) are expressed in terms of the Clifford multiplication as

\[
F = \frac{i}{2} \sum_{j=1}^{n} c(e_{2j-1}e_{2j}), \quad e^{-tF} = \prod_{j=1}^{n} \left( \text{ch} \frac{t}{2} - ic(e_{2j-1}e_{2j}) \text{sh} \frac{t}{2} \right)
\]

(40)

\[
g^{-1^*} = \prod_{j=1}^{m} \left( \cos \frac{\varphi_j}{2} + \sin \frac{\varphi_j}{2} c(e_{2j-1}e_{2j}) \right) e^{-i\varphi_j/2}.
\]

Proof. The proof is straightforward. \( \Box \)

We now use Lemma 1 and the Berezin lemma to obtain

\[
\text{Tr}_s\left(c(w_1)c(w_2)\ldots c(w_{2k})g^{-1^*}e^{-tF}\right) \\
= \text{Tr}_s\left(c(w_1)c(w_2)\ldots c(w_{2k}) \prod_{j=1}^{m} \left( \cos \frac{\varphi_j}{2} + \sin \frac{\varphi_j}{2} c(e_{2j-1}e_{2j}) \right) e^{-i\varphi_j/2} \prod_{j=1}^{n} \left( \text{ch} \frac{t}{2} - ic(e_{2j-1}e_{2j}) \text{sh} \frac{t}{2} \right) \right) \\
= (-2i)^n \int_{\mathbb{C}^n} \sigma \left( w_1w_2\ldots w_{2k} \prod_{j=1}^{m} \left( \cos \frac{\varphi_j}{2} + \sin \frac{\varphi_j}{2} e_{2j-1}e_{2j} \right) e^{-i\varphi_j/2} \prod_{j=1}^{n} \left( \text{ch} \frac{t}{2} - ie_{2j-1}e_{2j} \text{sh} \frac{t}{2} \right) \right).
\]
Since $2k \leq 2n - 2m$ by assumption, the main term of the expansion of (11) is of order $t^{n-m-k}$ and equal to

$$
(42) \quad (-2i)^n \int_{\mathbb{C}^n} \sigma \left( w_1 w_2 \ldots w_{2k} \prod_{j=1}^m \left( \cos \frac{\varphi_j}{2} + \sin \frac{\varphi_j}{2} e^{i \varphi_j} \right) e^{-i \varphi_j} \prod_{j=1}^n \left( \text{ch} \frac{t}{2} - i e^{i \varphi_j} \text{sh} \frac{t}{2} \right) \right)
$$

$$
\sim (-2i)^n (-it/2)^{n-m-k} \int_{\mathbb{C}^n} \sigma \left( w_1 w_2 \ldots w_{2k} \prod_{j=1}^m \left( \sin \frac{\varphi_j}{2} e^{i \varphi_j} \right) e^{-i \varphi_j} \prod_{j \in J} c_{2j-1} e^{i \varphi_j} \right) =
$$

$$
= (-2i)^n (-it/2)^{n-m-k} m \prod_{j=1}^m \sin \frac{\varphi_j}{2} e^{-i \varphi_j} \times
$$

$$
\times \int_{\mathbb{C}^n} \sigma(w_1) \wedge \sigma(w_2) \wedge \ldots \sigma(w_{2k}) \wedge dx_1 \wedge dp_1 \wedge \ldots \wedge dx_m \wedge dp_m \wedge \sum_{J \in J} \prod_{j \in J} dx_j \wedge dp_j
$$

$$
= (-2i)^n (-it/2)^{n-m-k} (-1)^m m \prod_{j=1}^m \frac{1 - e^{-i \varphi_j}}{2i} \int_{\mathbb{C}_g} \sigma(w_1) \wedge \sigma(w_2) \wedge \ldots \wedge \sigma(w_{2k}) \wedge e^\omega
$$

$$
= (-2i)^n (-it/2)^{n-m-k} e^{k \cdot \omega} \prod_{j=1}^m \left( 1 - e^{-i \varphi_j} \right) \int_{\mathbb{C}_g} \sigma(w_1) \wedge \sigma(w_2) \wedge \ldots \wedge \sigma(w_{2k}) \wedge e^{-\omega}.
$$

Here the summations $\sum_J$ are over all subsets $J \subset \{m+1, \ldots, n\}$ of $n - m - k$ different numbers.

The proof of Proposition 3 is now complete. \hfill \Box

Now we substitute the asymptotics (35) and (39) into (38) and obtain the desired expression (43) for the Connes–Moscovici cocycle.

**Step 3. Computation of the Connes–Moscovici cocycle for $\alpha \neq 0$.** We claim that for $\alpha \neq 0$ the contribution to the Connes–Moscovici cocycle is equal to zero. Indeed, similar to (37) we get

$$
(43) \quad \text{Tr}_s(a_0[D, a_1]^{[\alpha_1]} \ldots [D, a_{2k}]^{[\alpha_{2k}]} e^{-tD^2})
$$

$$
= \text{Tr}_s(T_{w_0}[D, T_{w_1}]^{[\alpha_1]} \ldots [D, T_{w_{2k}}]^{[\alpha_{2k}]} R_g e^{-t(H+F)})
$$

$$
= \text{Tr}_s(T_{w_0}[D, T_{w_1}]^{[\alpha_1]} \ldots [D, T_{w_{2k}}]^{[\alpha_{2k}]} (R_g \otimes g^s)^{-1}(e^{-tH} \otimes e^{-tF})).
$$

To study the asymptotics of this supertrace, we recall the following definition of Getzler order.

**Definition 1.** Given an operator

$$
(44) \quad B = \sum_k (B_k T_{z_k}) \otimes c(a_k) : S(\mathbb{R}^n, \Lambda(\mathbb{C}^n)) \rightarrow S(\mathbb{R}^n, \Lambda(\mathbb{C}^n)),
$$

where $B_k$ are scalar Shubin differential operators, $z_k \in \mathbb{C}^n$, $a_k \in Cl(2n)$, its **Getzler order** is equal to

$$
\text{ord} B = \max_k (\text{ord} B_k + \text{deg} a_k).
$$

Thus, in the order we count the order in $x, \partial/\partial x$ and the Clifford filtration.

\footnote{Recall that $Cl(2n)$ is a filtered algebra and we define ord $a$ for $a \in Cl(2n)$ to be equal to the Clifford filtration $d$, where $Cl(d(2n)) \subset Cl(2n)$ is the subspace of elements spanned by the products $v_1 \cdots v_d \in Cl(2n)$, where $v_j \in \mathbb{C}^n \subset Cl(n)$.}
The Getzler orders of the operators in (43) are computed in the following lemma.

**Lemma 2.** One has \( \text{ord} g^{s-1} = 2m \) and \( \text{ord}[D, T_w]^\gamma \leq 1 + \gamma \).

*Proof.* The first equality follows from (40). The second estimate is proved by induction. Indeed, if \( \gamma = 0 \), then (29) shows that \( \text{ord}[D, T_w] \leq 1 \). Let us now show that \( \text{ord}[D^2, B] \leq \text{ord} B + 1 \) for all \( B \) as in (44). We have:

\[
(45) \quad [D^2, B] = [H + F, B] = [H, B] + [F, B] = \sum_k ([H, B_k T_{z_k}] \otimes c(a_k) + B_k T_{z_k} \otimes [F, c(a_k)]).
\]

It follows from the properties of Shubin operators that \( \text{ord}[H, B_k T_{z_k}] \leq \text{ord} B_k + 1 \), and from the properties of the Clifford multiplication that \( \text{ord}[F, c(a_k)] \leq \text{ord} a_k + 1 \). These estimates and (45) imply the desired estimate

\[
\text{ord}[D^2, B] \leq \max_k (\text{ord} B_k + \text{ord} a_k + 1) = \text{ord} B + 1.
\]

The proof of Lemma 2 is now complete.

**Lemma 3.** Given an operator \( B \) as in (43), we have

\[
(46) \quad \text{Tr}_s(B R_g e^{-tD^2}) = \begin{cases} O(t^{+\infty}) & \text{if the fixed point set of } w \mapsto gw + z \text{ is trivial} \\ O(t^{-\text{ord} B/2}) & \text{otherwise}. \end{cases}
\]

*Proof.* We have

\[
(47) \quad \text{Tr}_s(B R_g e^{-tD^2}) = \sum_k \text{Tr}_s((B_k T_{z_k} \otimes c(a_k))(R_g \otimes g^{s-1})(e^{-tH} \otimes e^{-tF}))
= \sum_k \text{Tr}(B_k T_{z_k} R_g e^{-tH}) \text{Tr}_s(c(a_k)g^{s-1}e^{-tF}).
\]

On the one hand, the trace of scalar operators is estimated by Proposition 4:

\[
(48) \quad \text{Tr}(B_k T_{z_k} R_g e^{-tH}) = \begin{cases} O(t^{+\infty}) & \text{if the fixed point set of } w \mapsto gw + z \text{ is trivial} \\ O(t^{-\dim C^0_g - \text{ord} B_k/2}) & \text{otherwise}. \end{cases}
\]

On the other hand, the supertrace in (47) is computed by Proposition 5:

\[
(49) \quad \text{Tr}_s(c(a_k)g^{s-1}e^{-tF}) = \begin{cases} O(1) & \text{if ord } a_k \text{ is odd or } \dim C^0_g - \text{ord } a_k/2 < 0 \\ O(t^{\dim C^0_g - \text{ord } a_k/2}) & \text{otherwise}. \end{cases}
\]

Estimating the traces in (47) using (48) and (49), we obtain the desired estimate (46). \( \square \)

Now, we see from Lemma 2 that

\[
\text{ord}[D, a_1]^{[\alpha_1]}, ..., [D, a_{2k}]^{[\alpha_{2k}]} \leq 2k + |\alpha|.
\]

Thus, Lemma 3 implies that

\[
\text{Tr}_s(a_0[D, a_1]^{[\alpha_1]}...[D, a_{2k}]^{[\alpha_{2k}]}e^{-tD^2}) = O(t^{-2k+|\alpha|/2}).
\]

Hence

\[
t^{k+|\alpha|} \text{Tr}_s(a_0[D, a_1]^{[\alpha_1]}...[D, a_{2k}]^{[\alpha_{2k}]}e^{-tD^2}) = O(t^{|\alpha|/2})
\]

and the constant term in the asymptotic expansion is equal to zero. This implies the desired statement that the terms in the Connes–Moscovici cocycle for \( \alpha \neq 0 \) are equal to zero.

The proof of Theorem 3 is now complete.
7. Cyclic Cocycles

In this section, we show that each component of the periodic cyclic cocycle in Theorem 4 is actually a cyclic cocycle. Moreover, each of these cocycles is a sum of cyclic cocycles localized at the conjugacy classes in the semidirect product of \( C^n \) and the unitary group \( U(n) \) which we denote by \( C^n \rtimes U(n) \). Here we use the approach due to Connes to define cyclic cocycles as characters of cycles, see [10].

Noncommutative differential forms. We consider \( \mathcal{A} \) as a subalgebra of the differential graded algebra \( \Omega^* \subset BL^2(\mathbb{R}^n, \Lambda(\mathbb{R}^{2n})) \) consisting of all operators \( a \) that are finite sums

\[
a = \sum_k u_k T_{z_k} R_{g_k}, \quad z_k \in C^n, g_k \in U(n), u_k \in \Lambda(\mathbb{R}^{2n}).
\]

This algebra is graded by the degree of forms. We define the operator \( d : \Omega^* \rightarrow \Omega^{*+1} \) by

\[
d(uT_{z} R_{g}) = (-1)^{\deg u} u\sigma(z) T_{z} R_{g}, \quad \text{where} \quad \sigma(z) = \text{Im} z dx + \text{Re} zd\).
\]

**Lemma 4.** The operator \( d \) is a graded differentiation on \( \Omega^* \). More precisely, the following equalities hold:

\[
d^2 a = 0, \quad d(a_1 a_2) = (da_1) a_2 + (-1)^{\deg a_1} a_1 da_2, \quad \text{for all} \ a, a_1, a_2 \in \Omega^*.
\]

**Proof.** The first equality is seen as follows

\[
d(d(uT_{z} R_{g})) = d((-1)^{\deg u} u\sigma(z) T_{z} R_{g}) = -u\sigma(z)\sigma(z) T_{z} R_{g} = u \cdot 0 \cdot T_{z} R_{g} = 0.
\]

Before proving the second, we first show that

\[
g^{-1}(\sigma(z)) = \sigma(gz) \quad \text{for all} \ g \in U(n) \text{ and } z \in C^n.
\]

In fact, given \( g = B + iA \in U(n) \), where \( A \) and \( B \) are real matrices, we have on the one hand

\[
\sigma(gz) = \text{Re}(gzd(p + ix)) = \sum_{kl} \text{Re} ((B_{kl} + iA_{kl})(\text{Re} z_l + i \text{Im} z_l)(dp_k - idx_k))
\]

\[
= \sum_{kl} (B_{kl}(\text{Re} z_l dp_k + \text{Im} z_l dx_k) + A_{kl}(-\text{Im} z_l dp_k + \text{Re} z_l dx_k)).
\]

On the other hand, \( g^{-1} = B^t - iA^t \) with the transposed matrices \( A^t \) and \( B^t \), and

\[
g^{-1}(x) = \begin{pmatrix} B^t & -A^t \\ A^t & B^t \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}.
\]

Now \( (52) \) follows from the fact that

\[
g^{-1} g^{-1}(\sigma(z)) = \sum_l (\text{Im} z_l) g^{-1}(dx_l) + \sum_l (\text{Re} z_l) g^{-1}(dp_l)
\]

\[
= \sum_{kl} (\text{Im} z_l)(B_{kl}dx_k - A_{kl}dp_k) + \sum_{kl} (\text{Re} z_l)(A_{kl}dx_k + B_{kl}dp_k)
\]

which coincides with \( (53) \).
As for the second statement: Given $a_1 = u_1 T_{z,1} R_{g_1}$ and $a_2 = u_2 T_{z,2} R_{g_2}$, we find that
\[
d(a_1 a_2) = d((u_1 T_{z,1} R_{g_1})(u_2 T_{z,2} R_{g_2})) = d(u_1 g_1^{-1} u_2 T_{z,1} R_{g_1} T_{z,2} R_{g_2})
\]
\[
= e^{- \i \text{Im}(z_1, g_1 z_2)/2} d(u_1 g_1^{-1} u_2 T_{z,1} R_{g_1} R_{g_2} T_{z,2})
\]
\[
= (-1)^{\deg u_1 + \deg u_2} e^{- \i \text{Im}(z_1, g_1 z_2)/2} u_1 g_1^{-1} u_2 \sigma(z_1 + g_1 z_2) T_{z,1} R_{g_1} R_{g_2} T_{z,2}
\]
\[
= (-1)^{\deg u_1} u_1 \sigma(z_1) g_1^{-1} u_2 T_{z,1} R_{g_1} R_{g_2} + (-1)^{\deg u_1} u_1 g_1^{-1} u_2 \sigma(z_2) T_{z,1} T_{g_1} R_{g_2}
\]
\[
= (-1)^{\deg u_1} u_1 \sigma(z_1) T_{z,1} R_{g_1} u_2 T_{z,2} R_{g_2} + (-1)^{\deg u_1} u_1 T_{z,1} R_{g_1} (1)^{\deg u_2} u_2 \sigma(z_2) T_{z,2} R_{g_2}
\]
\[
= (da_1) a_2 + (-1)^{\deg a_1} a_1 da_2.
\]

\[\square\]

**Localized traces.** Let us fix a pair $(z_0, g_0) \in \mathbb{C}^n \times U(n)$ such that the fixed point set of the affine mapping
\[
\mathbb{C}^n \rightarrow \mathbb{C}^n, \quad w \rightarrow g_0 w + z_0
\]
is not empty. Then we define the functional

\[
\tau_{z_0,g_0} : \Omega^* \rightarrow \mathbb{C}
\]
\[
\tau_{z_0,g_0} \left( \sum_{z,g} u_{z,g} T_z R_g \right) = \sum_{(z,g) \in \{(z_0, g_0)\}} \prod_{j=1}^m e^{\varphi_j(z, e_j(g))} \cosec \varphi_j(g)/2 \int_{C^g} u_{z,g} e^{-\omega}.
\]

Here $(z_0, g_0) \subset \mathbb{C}^n \times U(n)$ stands for the conjugacy class of $(z_0, g_0)$, $m = n - \dim C^g$, $e_j(g)$ stand for the eigenvectors of $g$ with eigenvalues $e^{i\varphi_j(g)} \neq 1$.

**Lemma 5.** The functional $\tau_{z_0,g_0}$ is a closed graded trace on the differential graded algebra $\Omega^*$. More precisely, one has
\[
\tau_{z_0,g_0}(da) = 0 \quad \text{for all } a \in \Omega^*,
\]
\[
\tau_{z_0,g_0}(a_1 a_2) = (-1)^{\deg a_1} \deg a_2 \tau_{z_0,g_0}(a_2 a_1) \quad \text{for all } a_1, a_2 \in \Omega^*.
\]

**Proof.** 1. Given $a = u T_{z} R_g$, we have
\[
\tau_{z_0,g_0}(da) = (-1)^{\deg u} \tau_{z_0,g_0}(u \sigma(z) T_z R_g) \cdot \int_{C^g} u \sigma(z) e^{-\omega} = 0,
\]

where we used the assumption that the fixed point set of $w \mapsto gw + z$ is nonempty, which is equivalent to $\sigma(z)|_{C^g} = 0$. Indeed, if we choose the basis in which $g$ is diagonal, then $C^g = \{(0, ..., w_{m+1}, ..., w_n)\}$. Hence, the fixed point set of $w \mapsto gw + z$ is nonempty if and only if $z_j = 0$ whenever $j > m$. Clearly, this condition is equivalent to $\sigma(z)|_{C^g} = 0$.

2. Given $a_j = u_j T_{z,j} R_{g_j}$, $j = 1, 2$, we denote by $\gamma \subset \mathbb{C}^n \times U(n)$ the conjugacy class of $(z_2, g_2)(z_1, g_1)$ which coincides with that of $(z_1, g_1)(z_2, g_2)$. Then we have
\[
\tau_\gamma(a_1 a_2) = \tau_\gamma(u_1 T_{z,1} R_{g_1} u_2 T_{z,2} R_{g_2}) = e^{-\i \text{Im}(z_1, g_1 z_2)/2} \tau_\gamma(u_1 g_1^{-1} u_2 T_{z,1} + g_1 z_2 R_{g_1} R_{g_2})
\]
\[
= e^{-\i \text{Im}(z_1, g_1 z_2)/2} \prod_{j=1}^m e^{\varphi_j(z_1 + g_1 z_2, e_j)} \cosec \varphi_j(g)/2 \int_{C^g} u_1 g_1^{-1} u_2 e^{-\omega},
\]

where the $e_j$ are the eigenvectors of $g_1 g_2$ with eigenvalues $e^{i\varphi_j} \neq 1$. Similarly, we get
\[
\tau_\gamma(a_2 a_1) = e^{-\i \text{Im}(z_2, g_2 z_1)/2} \prod_{j=1}^m e^{\varphi_j(z_2 + g_2 z_1, f_j)} \cosec \varphi_j(g)/2 \int_{C^g} u_2 g_2^{-1} u_1 e^{-\omega},
\]

where the $f_j$ are the eigenvectors of $g_2 g_1$ with eigenvalues $e^{i\varphi_j} \neq 1$. 

\[\square\]
To prove (59), we decompose

\[ (57) \quad \int_{\mathbb{C}^n_{g_1g_2}} u_1 g_1^{-1} u_2 e^{-\omega} = (-1)^{\deg u_1 \deg u_2} \int_{\mathbb{C}^n_{g_2g_1}} u_2 g_2^{-1} u_1 e^{-\omega}. \]

Indeed, since \( g_1^{-1} \) defines an isomorphism \( \mathbb{C}^n_{g_1g_2} \to \mathbb{C}^n_{g_2g_1} \), we have

\[ (58) \quad \int_{\mathbb{C}^n_{g_2g_1}} u_2 g_2^{-1} u_1 e^{-\omega} = \int_{\mathbb{C}^n_{g_1g_2}} (g_1^{-1} u_2) g_1^{-1} g_2^{-1} u_1 e^{-\omega} = \int_{\mathbb{C}^n_{g_1g_2}} (g_1^{-1} u_2) u_1 e^{-\omega} = (-1)^{\deg u_1 \deg u_2} \int_{\mathbb{C}^n_{g_1g_2}} u_1 g_1^{-1} u_2 e^{-\omega}, \]

where we used the fact that \( g_1^{-1} g_2^{-1} = (g_1 g_2)^{-1} = 1 \) on \( \mathbb{C}^n_{g_1g_2} \).

To compare the exponential functions in (55) and (56), we set \( f_j = g_1^{-1} e_j \) and claim that

\[ (59) \quad \frac{\text{Im}(z_2, g_2 z_1) - \text{Im}(z_1, g_1 z_2)}{2} + \frac{1}{4} \sum_{j=1}^{m} (|(z_1 + g_1 z_2, e_j)|^2 - |(z_2 + g_2 z_1, f_j)|^2) \cot \varphi_j/2 = 0. \]

To prove (59), we decompose \( z_1 + g_1 z_2 \) and \( z_1 \) as

\[ z_1 = \sum_j d_j e_j + z_{10}, \quad d_j = (z_1, e_j), \quad z_{10} \in \mathbb{C}^n_{g_1g_2}, \]

\[ z_1 + g_1 z_2 = \sum_j c_j e_j, \quad c_j = (z_1 + g_1 z_2, e_j). \]

Note that \( z_1 + g_1 z_2 \) has no component in \( \mathbb{C}^n_{g_1g_2} \) since by our assumption the affine mapping \( w \mapsto g_1 g_2 w + z_1 + g_1 z_2 \) has nontrivial fixed point set. Let us now compute the left hand side in (59). We have

\[ (z_2 + g_2 z_1, f_j)|^2 = |(z_2 + g_2 z_1, g_1^{-1} e_j)|^2 = |(g_1 z_2 + g_1 g_2 z_1, e_j)|^2 = |(g_1 z_2 + z_1 + (g_1 g_2 - 1) z_1, e_j)|^2 = |c_j + (e^{i\varphi_j} - 1) d_j|^2 = |c_j|^2 + |d_j|^2 2(1 - \cos \varphi_j) + 2 \text{Re}(c_j d_j (e^{-i\varphi_j} - 1)). \]

Hence,

\[ (61) \quad \text{Im}(z_2, g_2 z_1) - \text{Im}(z_1, g_1 z_2) = \text{Im}(g_1 z_2, g_1 g_2 z_1) - \text{Im}(z_1, g_1 z_2) \]

\[ = \text{Im} \left( \sum_j (c_j - d_j) e_j - z_{10}, z_{10} + \sum_j e^{i\varphi_j} d_j e_j \right) - \text{Im} \left( \sum_j d_j e_j + z_{10}, \sum_j (c_j - d_j) e_j - z_{10} \right) \]

\[ = \text{Im} \sum_j (c_j - d_j) \overline{d_j} e^{-i\varphi_j} - \text{Im} \sum_j d_j (c_j - d_j) = \sum_j \left( \text{Im}(c_j d_j (e^{-i\varphi_j} + 1)) + |d_j|^2 \sin \varphi_j \right), \]
Thus, the left hand side in (63) is equal to

\[
\frac{1}{2} \sum_j \left( \text{Im}(c_j \overline{d_j}(e^{-i\varphi_j} + 1)) + |d_j|^2 \sin \varphi_j \right) \\
+ \frac{1}{4} \sum_j \left( |c_j|^2 - |c_j|^2 - |d_j|^2(1 - \cos \varphi_j) - 2 \Re(c_j \overline{d_j}(e^{-i\varphi_j} - 1)) \right) \cotg(\varphi_j/2) \\
= \frac{1}{2} \sum_j |d_j|^2(\sin \varphi_j - (1 - \cos \varphi_j) \cotg(\varphi_j/2)) + \frac{1}{2} \sum_j \left[ \text{Im}(c_j \overline{d_j}(e^{-i\varphi_j} + 1)) - \Re(c_j \overline{d_j}(e^{-i\varphi_j} + 1))(-i) \right] \\
= 0 + \frac{1}{2} \sum_j \left[ \text{Im}(c_j \overline{d_j}(e^{-i\varphi_j} + 1)) - \Re(c_j \overline{d_j}(e^{-i\varphi_j} + 1)) \right] = 0.
\]

Here we used the identities

\[
\sin \varphi_j - (1 - \cos \varphi_j) \cotg(\varphi_j/2) = 0, \quad (e^{-i\varphi_j} - 1) \cotg(\varphi_j/2) = -i(e^{-i\varphi_j} + 1).
\]

Equations (57) and (59) show that the left hand sides in (55) and (56) are related as follows:

\[
\tau_\gamma(a_1a_2) = (-1)^{\deg a_1 \deg a_2} \tau_\gamma(a_2a_1).
\]

This equality is precisely the graded trace property. The proof of Lemma 5 is now complete. \(\square\)

**Cyclic cocycles.** Lemmas 4 and 5 imply that \((\Omega^*, \tau_{v_0, g_0})\) is a cycle in the sense of Connes [10]. In a standard way, we define the character of this cycle as the following cyclic cocycle:

\[
\Phi_{k, v_0, g_0}(a_0, a_1, ..., a_k) = \tau_{v_0, g_0}(a_0 da_1 ... da_k), \quad a_j \in \mathcal{A}.
\]

Theorem 4 together with Lemmas 4 and 5 implies the following corollary.

**Corollary 1.** Each component \(\Psi_{2k}\) of the Connes–Moscovici periodic cyclic cocycle (63) is a cyclic cocycle and has the following decomposition

\[
\Psi_{2k} = \frac{i^{-k}}{(2k)!} \sum_{(z, g)} \Phi_{2k; z, g}
\]

into the sum of localized cyclic cocycles (63), where the summation is over all conjugacy classes in \(\mathbb{C}^n \ltimes U(n)\) with nontrivial fixed point set.

8. **Applications to Noncommutative Tori and Orbifolds**

Here we specialize our spectral triple \((\mathcal{A}, \mathcal{H}, D)\) to subalgebras in \(\mathcal{A}\) and obtain as corollaries of Theorem 4 local index formulas on noncommutative tori of arbitrary dimension and noncommutative orbifolds.

**The local index formula for noncommutative tori.** Let \(v_j \in \mathbb{C}^n, 1 \leq j \leq N\) be a collection of vectors linearly independent over \(\mathbb{Q}\). They generate the lattice

\[
\left\{ \sum_j \ell_j v_j \mid \ell_j \in \mathbb{Z} \right\} \subset \mathbb{C}^n
\]

isomorphic to \(\mathbb{Z}^N\). We define the algebra \(\mathcal{A}_v \subset \mathcal{A}\) of ‘functions on an \(N\)-dimensional noncommutative torus’ by

\[
\mathcal{A}_v = \left\{ \sum_{\ell} c_{\ell} T_{v_1}^{\ell_1} \cdots T_{v_N}^{\ell_N} \mid c_{\ell} \in \mathbb{C}, \ell = (\ell_1, \ell_2, ..., \ell_N), \ell_j \in \mathbb{Z} \right\}.
\]
This is the algebra generated by the $N$ unitaries
\[ T_{v_j}u(x) = e^{i(k_j x - a_j k_j/2)}u(x - a_j), \quad \text{where } a_j = \text{Re } v_j, \ k_j = -\text{Im } v_j, \]
acting on \( \mathcal{H} = L^2(\mathbb{R}^n, \Lambda(\mathbb{C}^n)) \) with the commutation relations (cf. [16])
\[ T_{v_k}T_{v_j} = e^{-i\text{Im}(v_k,v_j)}T_{v_j}T_{v_k}. \]

Then we consider the spectral triple \( (\mathcal{A}, \mathcal{H}, D) \), where \( D \) was defined in [24]. Corollary 1 implies that the Connes–Moscovici periodic cyclic cocycle of this spectral triple decomposes into cyclic cocycles
\[ (66) \quad \Psi_{2k}(a_0, \ldots, a_{2k}) = \frac{i^{-k}}{(2k)!} \int_{\mathbb{C}^n} (a_0 da_1 \ldots da_{2k})(0) \wedge e^{-\omega}, \quad k \leq n, \ a_j \in \mathcal{A}. \]

Here \( a_j \) are treated as elements in the differential graded algebra \( \Omega^* \) (see [50] and [51]), \( (a_0 da_1 \ldots da_{2k})(0) \in \Lambda(\mathbb{C}^n) \) stands for the component corresponding to \( \ell = 0 \) in \( (65) \), which is the only one with nontrivial fixed point set, \( \omega = dx \wedge dp \) is the symplectic form, while \( \int_{\mathbb{C}^n} \) is the Berezin integral.

For \( n = 1 \) Eq. (66) coincides with the Connes cyclic cocycles in [10], while for \( n \geq 1 \) this result is a refinement of the Riemann–Roch theorem on noncommutative tori of arbitrary dimension (see [33]).

**The local index formula for noncommutative \( \mathbb{Z}_4 \)-orbifolds.** Choose complex numbers \( z_1 = k, z_2 = ik, k > 0 \) and \( g = i \in U(1) \). We define the square lattice \( L = \{ n_1 z_1 + n_2 z_2 \in \mathbb{C} \mid n_1, n_2 \in \mathbb{Z} \} \) on which the group \( \mathbb{Z}_4 = \{ i^\beta \mid \beta \in \mathbb{Z} \} \) acts by rotations.

To these elements, we associate the unitary operators \( U = T_{z_1}, V = T_{z_2}, R = R_g \):
\[ U f(x) = f(x - k), \quad V f(x) = e^{-ikx} f(x), \quad R u(x) = (2\pi)^{-1/2} \int f(y) e^{-ixy} dy. \]

We obtain the commutation relations:
\[ VU = e^{i\theta} UV, \quad VR^{-1} = V, \quad RV^{-1} = U^{-1}, \quad \text{where } \theta = -k^2. \]

Hence, the algebra generated by \( U \) and \( V \) is just the noncommutative torus \( \mathcal{A}_\theta \), while the algebra generated by \( U, V, R \) is the crossed product \( \mathcal{A}_\theta \rtimes \mathbb{Z}_4 \) with respect to the action of the generator of \( \mathbb{Z}_4 \) on the generators \( U, V \in \mathcal{A}_\theta \) as:
\[ U \mapsto R^{-1} U^{-1} = V, \quad V \mapsto RV^{-1} = U^{-1}. \]

This crossed product is known as a noncommutative orbifold for the group \( \mathbb{Z}_4 \) and was studied earlier in operator algebras and noncommutative geometry (see [19],[19],[44]).

It follows from the commutation relations that elements \( a \in \mathcal{A}_\theta \rtimes \mathbb{Z}_4 \) can be uniquely written as
\[ a = \sum_{(z, \alpha) \in L \times \mathbb{Z}_4} a(z, \alpha) T_z R_\alpha. \]

Consider the spectral triple \( (\mathcal{A}_\theta \rtimes \mathbb{Z}_4, \mathcal{H}, D) \), which is the restriction of the spectral triple \( (\mathcal{A}, \mathcal{H}, D) \) in Section 6 to the subalgebra \( \mathcal{A}_\theta \rtimes \mathbb{Z}_4 \subset \mathcal{A} \). Then Corollary 1 shows that the Connes–Moscovici periodic cyclic cocycle decomposes as a sum of cyclic cocycles
\[ \Phi_{2k,z,\alpha} \in H^2_{CL}(\mathcal{A}_\theta \rtimes \mathbb{Z}_4), \quad l = 0, 1 \]
for each \( (z, \alpha) \in L \times \mathbb{Z}_4 \). Let us describe these cocycles explicitly.
First, if $\alpha = 0$, then the cocycles are nontrivial only if $z = 0$. In this case the fixed point set of the affine mapping $w \mapsto i^\alpha w + z = w$ is equal to $\mathbb{C}$ and we have

$$\Phi_{0,0,0}(a) = a(0,0), \quad \Phi_{2,0,0}(a_0, a_1, a_2) = \int_{\mathbb{C}} (a_0 da_1 da_2)(0,0),$$

where the exterior differential is that described above, while $\int_{\mathbb{C}}$ stands for the Berezin integral.

Second, if $\alpha \neq 0$, then the fixed point set of the affine mapping $w \mapsto i^\alpha w + z$ is always zero dimensional; hence the cocycles $\Phi_{2,0,0}$ are trivial by (33). Let us describe the trace $\Phi_{0,0,0}$. A direct computation shows that the conjugacy class $\langle (z, i^\alpha) \rangle \subset \mathbb{Z}^2 \times \mathbb{Z}_4$ is equal to

$$\langle (z, i^\alpha) \rangle = (i^2 z + L(1 - i^\alpha)) \times \{i^\alpha\}.$$ 

Hence, the cyclic cocycle is equal to

$$\Phi_{0,0,0}(f) = \sum_{z' \in i^2 z + L(1 - i^\alpha)} \exp \left( i \frac{|z'|^2}{4} \frac{\pi \alpha}{4} \right) f(z', \alpha).$$

A computation shows that there are actually eight different conjugacy classes in (67), see the following table:

| $z$ | $i^\alpha$ | conjugacy class $\langle (z, i^\alpha) \rangle$ |
|-----|------------|-----------------------------------------------|
| 0   | 1          | $\{0\} \times \{1\}$                         |
| 0   | $i$        | $k[(1-i)\mathbb{Z} + (1+i)\mathbb{Z}] \times \{i\}$ |
| $k$ | $i$        | $k[1 + (1-i)\mathbb{Z} + (1+i)\mathbb{Z}] \times \{i\}$ |
| 0   | $i^2$      | $k[2\mathbb{Z} + 2i\mathbb{Z}] \times \{i^2\}$ |
| $k$ | $i^2$      | $k[1 + (1-i)\mathbb{Z} + (1+i)\mathbb{Z}] \times \{i^2\}$ |
| $k(1+i)$ | $i^2$ | $k[1 + i + 2\mathbb{Z} + 2i\mathbb{Z}] \times \{i^2\}$ |
| 0   | $i^3$      | $k[(1-i)\mathbb{Z} + (1+i)\mathbb{Z}] \times \{i^3\}$ |
| $k$ | $i^3$      | $k[1 + (1-i)\mathbb{Z} + (1+i)\mathbb{Z}] \times \{i^3\}$ |

Thus, the eight different traces $\Phi_{0,0,0}$ coming from the decomposition of the Connes–Moscovici local index formula form a basis of the eight-dimensional space of traces on $\mathcal{A}_0 \times \mathbb{Z}_4$ (see [44]).

The local index formula for noncommutative $\mathbb{Z}_6$-orbifolds. Choose complex numbers $z_1 = k, z_2 = z \xi, g = \varepsilon \in U(1)$, where $k > 0$ and $\varepsilon = e^{\pi i/3}$. We consider the triangular lattice $L = \{n_1 z_1 + n_2 z_2 \in \mathbb{C} \mid n_1, n_2 \in \mathbb{Z}\}$ with the group $\mathbb{Z}_6 = \{e^\beta \mid \beta \in \mathbb{Z}\}$ acting on $L$ by rotations.

Consider the unitary operators $U = T_{z_1}, V = T_{z_2}, R = R_g$:

$$uf(x) = f(x - k), \quad Vf(x) = e^{i(-xk\sqrt{3}/2 + k\sqrt{3}/8)}f(x - k/2),$$

$$Ru(x) = \sqrt{1 - \frac{\sqrt{3}}{2\pi}} \int \exp \left( i \left( (x^2 + y^2) \frac{1}{2\sqrt{3}} - \frac{2xy}{\sqrt{3}} \right) \right) u(y)dy.$$ 

We have the commutation relations:

$$VU = e^{i\theta}UV, \quad RU R^{-1} = V, \quad RVR^{-1} = e^{-i\theta/2}U^{-1}V,$$

where $\theta = -\frac{\sqrt{3}}{2} k^2$.

Hence, the algebra generated by $U$ and $V$ is just the noncommutative torus $\mathcal{A}_0$, while the algebra generated by $U, V, R$ is the crossed product $\mathcal{A}_0 \times \mathbb{Z}_6$ with respect to the action of the generator of $\mathbb{Z}_6$ on the generators $U, V \in \mathcal{A}_0$ as:

$$U \mapsto RU R^{-1} = V, \quad V \mapsto RVR^{-1} = e^{-i\theta/2}U^{-1}V.$$ 

This crossed product is known as a noncommutative orbifold for the group $\mathbb{Z}_6$ and was studied earlier in operator algebras and noncommutative geometry (see [14, 15, 45, 46]).
It follows from the commutation relations that elements $f \in \mathcal{A}_0 \ltimes \mathbb{Z}_6$ can be uniquely written as

$$f = \sum_{(z, \alpha) \in L \times \mathbb{Z}_6} f(z, \alpha)T_zR^\alpha.$$

Consider the spectral triple $(\mathcal{A}_0 \ltimes \mathbb{Z}_6, \mathcal{H}, D)$, which is the restriction of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ in Section 3 to the subalgebra $\mathcal{A}_0 \ltimes \mathbb{Z}_6 \subset \mathcal{A}$. Then Corollary 4 shows that the Connes–Moscovici periodic cyclic cocycle decomposes as a sum of cyclic cocycles

$$\Phi_{2l; z, \alpha} \in HC^2(\mathcal{A}_0 \ltimes \mathbb{Z}_6), \quad l = 0, 1,$$

for each $(z, \alpha) \in L \times \mathbb{Z}_6$. Let us describe these cocycles explicitly.

First, if $\alpha = 0$, then the cocycles are nontrivial only if $z = 0$. In this case the fixed point set of the rotation $w \mapsto \varepsilon^\alpha w$ is equal to $\mathbb{C}$ and we have

$$\Phi_{0,0,0}(a) = a(0,0), \quad \Phi_{2;0,0}(a_0, a_1, a_2) = \int_{\mathbb{C}} (a_0 da_1 da_2)(0,0),$$

where the exterior differential is that described above, and $\int_{\mathbb{C}}$ stands for the Berezin integral.

Second, if $\alpha \neq 0$, then the fixed point set of the affine mapping $w \mapsto \varepsilon^\alpha w + z$ is of dimension zero and the cocycle $\Phi_{2z, \alpha}$ is trivial by (68). Let us describe the trace $\Phi_{0; z, \alpha}$. A direct computation shows that the conjugacy class $\{(z, \varepsilon^\alpha)\} \subset L \times \mathbb{Z}_6$ is equal to

$$\{(z, \varepsilon^\alpha)\} = [\varepsilon^z z + L(1 - \varepsilon^\alpha)] \times \{\varepsilon^\alpha\}.\tag{69}$$

Hence, the trace is equal to

$$\Phi_{0; z, \alpha}(f) = \sum_{z' \in \varepsilon^z z + L(1 - \varepsilon^\alpha)} \exp\left(\frac{i}{4} z'|^2 \varepsilon^\alpha \frac{\pi \alpha}{6}\right) f(z', \alpha).$$

A computation shows that there are actually nine different conjugacy classes in (68), see the following table:

| $z$  | $\varepsilon^\alpha$ | conjugacy class $\{(z, \varepsilon^\alpha)\}$ |
|------|----------------------|---------------------------------------------|
| 0    | 1                    | $\{0\} \times \{1\}$                      |
| 0    | $\varepsilon$        | $L \times \{\varepsilon\}$                |
| 0    | $\varepsilon^2$      | $k[(\varepsilon^2 - 1)\mathbb{Z} + (\varepsilon + 1)\mathbb{Z}] \times \{\varepsilon^2\}$ |
| $k$  | $\varepsilon^2$      | $(L \setminus k[(\varepsilon^2 - 1)\mathbb{Z} + (\varepsilon + 1)\mathbb{Z}]) \times \{\varepsilon^2\}$ |
| 0    | $\varepsilon^3$      | $2L \times \{\varepsilon^3\}$             |
| $k$  | $\varepsilon^3$      | $(L \setminus 2L) \times \{\varepsilon^3\}$ |
| 0    | $\varepsilon^4$      | $k[(\varepsilon + 1)\mathbb{Z} + (\varepsilon^2 + \varepsilon)\mathbb{Z}] \times \{\varepsilon^4\}$ |
| $k$  | $\varepsilon^4$      | $(L \setminus k[(\varepsilon + 1)\mathbb{Z} + (\varepsilon^2 + \varepsilon)\mathbb{Z}]) \times \{\varepsilon^4\}$ |
| 0    | $\varepsilon^5$      | $L \times \{\varepsilon^5\}$              |

Thus, the nine different traces $\Phi_{0; z, \alpha}$ coming from the decomposition of the Connes–Moscovici local index formula form a basis of the nine-dimensional space of traces on $\mathcal{A}_0 \ltimes \mathbb{Z}_6$ (see [4]).

9. Equivariant Zeta Functions for the Affine Metaplectic Group

Let $A \in \Psi(\mathbb{R}^n)$ be a Shubin type pseudodifferential operator of order ord $A$, $g \in U(n), w \in \mathbb{C}^n$. For $z \in \mathbb{C}$ with Re $z$ sufficiently large consider the zeta function

$$\zeta_{A,g,w}(z) = \text{Tr}(R_g T_w AH^{-z}).\tag{69}$$

**Theorem 5.** The zeta function $\zeta_{A,g,w}(z)$ has the following properties:

1. It is well defined and holomorphic in the half-plane ord $A - 2 \text{Re } z < -2n$;
(2) It has a meromorphic continuation to \( \mathbb{C} \) with possibly simple poles at the points

\[
z = \dim \mathbb{C}^n_j + (\text{ord} \, A - j)/2, \quad j \in \mathbb{Z}_+,
\]

where \( \mathbb{C}^n_j \) is the fixed point set of \( g : \mathbb{C}^n \to \mathbb{C}^n \). Moreover, if the fixed point set of the affine mapping \( \mathbb{C}^n \to \mathbb{C}^n, v \mapsto gv + w \) is empty, then the zeta function has no poles.

**Proof.**

1. The operator \( AH^{-Re z} \) is a Shubin type pseudodifferential operator of order \( \leq \text{ord} \, A - 2 \text{Re} \, z \) by the assumption. Hence, it is of trace class. Thus, the zeta function is well defined and holomorphic in \( z \), since \( R_g, T_w, H^{-1} \text{Im} \, z \) are bounded operators.

2. Let us now show that the Schwartz function has a meromorphic continuation to \( \mathbb{C} \). Without loss of generality we can assume that \( g \) is a diagonal matrix. Indeed, if this is not the case, then we have \( g = u_0 u^{-1} \), where \( u \) is unitary, while \( g_0 \) is diagonal and unitary. Hence:

\[
(70) \quad \zeta_{A,g,w}(z) = \text{Tr}(R_g T_w A H^{-z}) = \text{Tr}(R_u R_g R_u^{-1} T_w A H^{-z})
\]

\[
= \text{Tr}(R_g R_u^{-1} T_w (R_u R_u^{-1}) A (R_u R_u^{-1}) H^{-z} R_u)
\]

\[
= \text{Tr}(R_g (R_u^{-1} T_w R_u) (R_u^{-1} A R_u) (R_u^{-1} H^{-z} R_u))
\]

\[
= \text{Tr}(R_g T_w A' H^{-z}) = \zeta_{A',g',w'}(z)
\]

Here \( A' = R_u^{-1} A R_u \) is a Shubin type operator by Egorov's theorem, \( R_u^{-1} T_w R_u = T_w' \), where \( w' = u^{-1} w \), and we used the fact that \( H \) commutes with \( R_u \).

Let now \( w = (w_1, \ldots, w_m) = a - ik \), where \( a, k \in \mathbb{R}^n \), and consider diagonal element

\[
(71) \quad g = \text{diag}(e^{i \varphi_1}, \ldots, e^{i \varphi_m}, i, \ldots, i, -i, \ldots, -i, -1, \ldots, -1, 1, \ldots, 1),
\]

where \( \varphi_j \neq \pi Z/2 \) and \( m_5 = \dim(\mathbb{C}^n)^g \). For later purposes we also let \( \varphi_j = \pi/2 \) for \( j = m_1 + 1, \ldots, m_1 + m_2 \) and \( \varphi_j = 3\pi/2 \) for \( j = m_1 + m_2 + 1, \ldots, m_1 + m_2 + m_3 \).

The Schwartz kernel of \( AH^{-z} \) is written as

\[
(72) \quad K_{AH^{-z}}(x, x') = \int e^{i(x-x')p} q(x', p; z) dp,
\]

where \( q(x', p; z) \) is a classical symbol of order \( \text{ord} \, A - 2z \). Then the Schwartz kernel of \( T_w A H^{-z} \) is equal to

\[
(73) \quad K_{T_w A H^{-z}}(x, x') = \text{Const} \int e^{i(x-a-x')p+kx} q(x', p; z) dp.
\]

Since \( g \) is a diagonal matrix, the operator \( R_g \) is a product of fractional Fourier transforms in the variables \( x_1, \ldots, x_n \) and its Schwartz kernel is given by the Mehler formula and, hence, the Schwartz kernel of \( R_g T_w A H^{-z} \) is equal to

\[
(74) \quad K_{R_g T_w A H^{-z}}(x, x') = \text{Const} \int e^{i \phi_1} q(x', p; z) dp dx'' \quad x'' = (x''_1, \ldots, x''_{m_1+m_2+m_3}),
\]

where

\[
\phi_1 = \sum_{j=1}^{m_1+m_2+m_3} \left( \frac{-x_j x''}{\sin \varphi_j} + \frac{ctg \varphi_j}{2} (x''_j^2 + x''_{j'}) + (x''_j - a_j - x'_j) p_j + k_j x''_j \right)
\]

\[
+ \sum_{j=m_1+m_2+m_3+1}^{n-m_5} ((-x_j - a_j - x'_j) p_j - k_j x_j) + \sum_{j=n-m_5+1}^{n} ((x_j - a_j - x'_j) p_j + k_j x_j).
\]
Hence, the zeta function is equal to

\[ \zeta_{A,g,w}(z) = \int K_{R_gT_w}A\zeta(x,x)dx = \text{Const} \int e^{i\phi_2}q(x,p;z)dpdx''dx, \]

where

\[ \phi_2 = \sum_{j=1}^{m_1+m_2+m_3} \left( -\frac{x_jx_j'}{\sin\varphi_j} + \frac{\ctg\varphi_j}{2}(x_j'^2 + x_j''^2) + (x_j'' - a_j - x_j)p_j + k_jx_j'' \right) \]

\[ - \sum_{j=m_1+m_2+m_3+1}^{n-m_5} ((2x_j + a_j)p_j + k_jx_j) + \sum_{j=n-m_5+1}^{n} (-a_jp_j + k_jx_j). \]

Note that if some \( a_j \neq 0 \) (or \( k_j \neq 0 \)) for \( j > n - m_5 \), then integrations by parts in (75) with respect to \( p_j \) (respectively \( x_j \)) show that \( \zeta_{A,g,w}(z) \) can also be represented by an integral, where the pseudodifferential symbol has very negative order. This proves that in this case the zeta function is in fact an entire function in \( C \).

Thus, below we suppose that \( a_j = k_j = 0 \) for all \( j > n - m_5 \). Let us now compute the Gaussian integral over \( x'' \) in (73):

\[ \int \exp \left( i \left( \frac{\ctg\varphi_j}{2} x_j'^2 + x_j'' \left( p_j + k_j - \frac{x_j}{\sin\varphi_j} \right) \right) \right) dx_j'' \]

\[ = \begin{cases} 
\text{Const} \exp \left( -\frac{i}{2} \text{tg}\varphi_j \left( p_j + k_j - \frac{x_j}{\sin\varphi_j} \right)^2 \right), & \text{if } \varphi_j \notin \pi\mathbb{Z}/2, \\
\text{Const} \ \delta \left( p_j + k_j + x_j \right), & \text{if } \varphi_j = \pm\pi/2,
\end{cases} \]

and obtain

\[ \zeta_{A,g,w}(z) = \text{Const} \int e^{i\phi_3(x,p')}q(x,p',p'';z)dxdp'. \]

Here we decomposed \( p \) as follows: \( p = (p', p'') \), where \( p' = (p_1, \ldots, p_{m_1}, p_{m_1+m_2+m_3+1}, \ldots, p_n) \) and \( p'' = (p_{m_1+1}, \ldots, p_{m_1+m_2+m_3}) \). Note that the integration of the \( \delta \)-functions gives us \( p_j = \pm x_j - k_j \) for all \( j = m_1 + 1, \ldots, m_1 + m_2 + m_3 \). The phase function in (76) is equal to

\[ \phi_3(x,p') = \sum_{j=1}^{m_1} \left( x_j'^2 \left( \frac{\ctg\varphi_j}{2} - \frac{1}{\sin2\varphi_j} \right) + x_jp_j \left( \frac{1}{\cos\varphi_j} - 1 \right) - \frac{\ctg\varphi_j}{2} x_j + \frac{k_j}{\cos\varphi_j} p_j(1 + k_j \text{tg}\varphi_j) - \frac{\ctg\varphi_j k_j^2}{2} \right) \]

\[ - \sum_{j=m_1+1}^{m_1+m_2+m_3} (x_j + a_j) \left( \frac{x_j}{\sin\varphi_j} - k_j \right) + \sum_{j=m_1+m_2+m_3+1}^{n-m_5} (2x_jp_j + a_jp_j + k_jx_j). \]

A change of variables \( (x, p') = Bv + b \), where \( B \in O(\nu) \), \( \nu = 2n - m_2 - m_3 \), is an orthogonal matrix and \( v, b \in \mathbb{R}^\nu \), makes the phase function \( \phi_3 \) quadratic in \( v \) plus a constant:

\[ \phi_3(x,p') = \sum_{j=1}^{\nu} \lambda_j v_j^2 + \text{Const}. \]

\[ \text{This condition is equivalent to the condition that the affine mapping } z \mapsto g z + w \text{ has no fixed points.} \]
Note that \( B \) and \( b \) depend only on \( g \) and \( w \). We introduce spherical coordinates \( v = r \theta \), where \( r \geq 0 \) and \( \theta \in S^\nu-1 \) in (76), and obtain

\[
\int e^{i \phi_\lambda(x,p')} q(x,p',p'';z) dx dp' = \text{Const} \int_0^\infty \left( \int_{S^{\nu-1}} \exp \left( i \sum_{j=1}^\nu \lambda_j v_j^2 \right) q(Br\theta + b, p''; z) d\theta \right) r^{\nu-1} dr
\]

\[
\equiv \text{Const} \int_0^\infty c(r; z) r^{\nu-1} dr
\]

where

\[
c(r; z) = \int_{S^{\nu-1}} \exp \left( i \sum_{j=1}^\nu \lambda_j v_j^2 \right) q(Br\theta + b, p''; z) d\theta.
\]

The asymptotics of \( c(r; z) \) as \( r \to \infty \) can be computed by the stationary phase formula. To state it, we denote by \( \{ \mu_l \} \) all the different numbers \( \lambda_1, \ldots, \lambda_\nu \) in (78), and let \( \{ \kappa_l \} \) be their multiplicities.

**Lemma 6.** We have an asymptotic expansion as \( r \to \infty \):

\[
c(r; z) \sim r^{\text{ord} A - 2\zeta^2 - \nu} \sum_l r^{\kappa_l \nu - l} e^{ir^2 \mu_l} \sum_{j \geq 0} c_{l,j}(z) r^{-j},
\]

where the coefficients \( c_{l,j}(z) \) are entire functions of \( z \).

**Proof.** One shows that the stationary points of the phase function in (80) are just the unit length eigenvectors of the diagonal matrix \( \text{diag}(\lambda_1, \ldots, \lambda_\nu) \). Hence, the set of stationary points is just the disjoint union of spheres \( S^{\kappa_l - 1} \subseteq S^{\nu-1} \) over all distinct eigenvalues. Moreover, these critical submanifolds are nondegenerate. Thus, application of the stationary phase formula (with large parameter equal to \( r^2 \)) together with the fact that \( a(x,p;z) \) is a classical symbol of order \( \text{ord} A - 2\zeta^2 \), gives us the desired asymptotic expansion:

\[
c(r; z) \sim r^{\text{ord} A - 2\zeta^2} \sum_l r^{\kappa_l \nu - l} e^{ir^2 \mu_l} \sum_{j \geq 0} c_{l,j}(z) r^{-j}.
\]

\[\square\]

We now substitute (81) in (79) and obtain that modulo entire functions \( \zeta_{A,g,w}(z) \) in (76) is equal to the series:

\[
\zeta_{A,g,w}(z) \equiv \text{Const} \sum_l \sum_{j \geq 0} c_{l,j}(z) \int_1^\infty r^{\text{ord} A - 2\zeta^2 + \mu_l - j} e^{ir^2 \mu_l} dr.
\]

Integration by parts shows that the integral

\[
\int_1^\infty r^{\text{ord} A - 2\zeta^2 + \mu_l - j} e^{ir^2 \mu_l} dr
\]

is an entire function of \( z \) unless \( \mu_l = 0 \). Suppose for definiteness that \( \mu_1 = 0 \). Thus, we obtain the following equality modulo entire functions:

\[
\zeta_{A,g,w}(z) \equiv \text{Const} \sum_{j \geq 0} c_{1,j}(z) \int_1^\infty r^{\text{ord} A - 2\zeta^2 + \mu_1 - j} dr \equiv \text{Const} \sum_{j \geq 0} \frac{-c_{1,j}(z)/2}{z - (\text{ord} A + \kappa_1 - j)/2}.
\]

It remains to note that \( \kappa_1 = 2 \dim C_0^n \) is the real dimension of the fixed point set of \( g \in U(n) \) (this follows from (77) and (71)).

This completes the proof of Theorem 5. \[\square\]
References

[1] V. I. Arnold. *Mathematical Methods of Classical Mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, second edition, 1989.

[2] F. Azmi. The equivariant Dirac cyclic cocycle, *Rocky Mountain J. Math.* 30(4):1171–1206, 2000.

[3] Ch. Bönicke, S. Chakraborty, Z. He, H.-C. Liao. Isomorphism and Morita equivalence classes for crossed products of irrational rotation algebras by cyclic subgroups of $SL_2(\mathbb{Z})$. *J. Funct. Anal.*, 275(11):3208–3243 (2018).

[4] J. Buck and S. Walters. Connes-Chern characters of hexic and cubic modules. *J. Operator Theory*, 57(1):35–65, 2007.

[5] A. Bultheel and H. Martínez-Sulbaran. Recent developments in the theory of the fractional Fourier and linear canonical transforms. *Bull. Belg. Math. Soc. Simon Stevin*, 13(5):971–1005, 2006.

[6] A. L. Carey, J. Phillips, A. Rennie, and F. A. Sukochev. The local index formula in semifinite von Neumann algebras. II. The even case. *Adv. Math.*, 202(2):517–554, 2006.

[7] A. L. Carey, J. Phillips, A. Rennie, and F. A. Sukochev. The local index formula in noncommutative geometry revisited. Noncommutative geometry and physics. 3, 3–36, Keio COE Lect. Ser. Math. Sci., 1, World Sci. Publ., Hackensack, NJ, 2013.

[8] S. Chakraborty, F. Luef. Metaplectic transformations and finite group actions on noncommutative tori. *J. Operator Theory*, 82(1):147–172 (2019).

[9] Sh. Chern and X. Hu. Equivariant Chern character for the invariant Dirac operator. *Michigan Math. J.*, 44(3):451–473, 1997.

[10] A. Connes. *Noncommutative geometry*. Academic Press Inc., San Diego, CA, 1994.

[11] A. Connes and H. Moscovici. The local index formula in noncommutative geometry. *Geom. Funct. Anal.*, 5(2):174–243, 1995.

[12] A. Connes and H. Moscovici. Type III and spectral triples. In *Traces in number theory, geometry and quantum fields*, Aspects Math., E38, pages 97–110. Friedr. Vieweg, Wiesbaden, 2008.

[13] A. Connes and H. Moscovici. Hopf algebras, cyclic cohomology and the transverse index theorem. *Comm. Math. Phys.*, 198(1):199–246, 1998.

[14] L. Dąbrowski. The local index formula for quantum SU(2). In *Traces in number theory, geometry and quantum fields*, Aspects Math., E38, pages 99–110. Friedr. Vieweg, Wiesbaden, 2008.

[15] S. Echterhoff, W. Lück, N. C. Phillips, and S. Walters. The structure of crossed products of irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$. *J. Reine Angew. Math.*, 639:173–221, 2010.

[16] A. Gorokhovsky, N. de Kleijn and R. Nest. Equivariant algebraic index theorem. *J. Inst. Math. Jussieu* 20(3):929–955, 2021.

[17] G. B. Folland. *Harmonic analysis in phase space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[18] E. Getzler. Pseudodifferential operators on supermanifolds and the Atiyah–Singer index theorem. *Comm. Math. Phys.*, 92:163–183, 1983.

[19] C. Farsi and N. Watling. Quartic algebras. *Canad. J. Math.*, 46(4):1167–1191, 1992.

[20] F. Fathizadeh, F. Luef, and J. Tao. A twisted local index formula for curved noncommutative two tori. Preprint arXiv:1904.03819, 2019.

[21] Ph. Flajolet, X. Gourdon, and Ph. Dumas. Mellin transforms and asymptotics: harmonic sums. volume 144, pages 3–58, 1995. Special volume on mathematical analysis of algorithms.

[22] Li Gao, M. Junge, and E. McDonald. Quantum Euclidean spaces with noncommutative derivatives. *J. Noncommut. Geom.* 16(1):153–213, 2022.

[23] A. Gorokhovsky, N. de Kleijn and R. Nest. Equivariant algebraic index theorem. *J. Inst. Math. Jussieu* 20(3):929–955, 2021.

[24] A. Gorokhovsky and E. van Erp. Index theory and noncommutative geometry: a survey. *Advances in noncommutative geometry—on the occasion of Alain Connes’ 70th birthday*, 421–462, Springer, Cham, 2019.

[25] M. de Gosson. *Symplectic geometry and quantum mechanics*, volume 166 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2006.

[26] G. Grubb and R. T. Seeley. Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems. *Invent. Math.*, 121(3):481–529, 1995.

[27] N. Higson, G. Kasparov, and J. Trout. A Bott periodicity theorem for infinite dimensional Euclidean space. *Adv. Math.*, 135(1):1–40, 1998.
[28] N. Higson. The local index formula in noncommutative geometry. In *Contemporary developments in algebraic K-theory*, ICTP Lect. Notes, XV, pages 443–536. Abdus Salam Int. Cent. Theor. Phys., Trieste, 2004.

[29] N. Higson. The residue index theorem of Connes and Moscovici. In *Surveys in noncommutative geometry*, volume 6 of Clay Math. Proc., pages 71–126. Amer. Math. Soc., Providence, RI, 2006.

[30] L. Hörmander. Symplectic classification of quadratic forms, and general Mehler formulas. *Math. Z.* 219:413–449, 1995.

[31] J. Leray. *Analyse Lagrangienne et Mécánique Quantique*. IRMA, Strasbourg, 1978.

[32] J.-M. Lescure. Triplets spectraux pour les variétés à singularité conique isolée. *Bull. Soc. Math. France*, 129(4):593–623, 2001.

[33] V. Mathai and J. Rosenberg. The Riemann-Roch theorem on higher dimensional complex noncommutative tori, *J. Geom. Phys.* 147, 103534, 9 (2020).

[34] H. Moscovici. Local index formula and twisted spectral triples. In *Quanta of maths*, volume 11 of Clay Math. Proc., pages 465–500. Amer. Math. Soc., Providence, RI, 2010.

[35] S. Neshveyev and L. Tuset. A local index formula for the quantum sphere. *Comm. Math. Phys.*, 254(2):323–341, 2005.

[36] R. Ponge. A new short proof of the local index formula and some of its applications. *Comm. Math. Phys.*, 241(2-3):215–234, 2003.

[37] R. Ponge. Noncommutative residue and canonical trace on noncommutative tori. Uniqueness results. *SIGMA* Symmetry Integrability Geom. Methods Appl. 16, Paper No. 061, 31 pp., 2020.

[38] R. Ponge and H. Wang. Noncommutative geometry and conformal geometry. I. Local index formula and conformal invariants. *J. Noncommut. Geom.*, 12(4):1573–1639, 2018.

[39] A. Savin, E. Schrohe, and B. Sternin. Elliptic operators associated with groups of quantized canonical transformations. *Bull. Sci. Math.*, 155:141–167, 2019.

[40] A. Savin and E. Schrohe. Analytic and algebraic indices of elliptic operators associated with discrete groups of quantized canonical transformations. *J. Funct. Anal.*, 278(5):108400, 45, 2020.

[41] A. Savin and E. Schrohe. An index formula for groups of isometric linear canonical transformations. *Doc. Math.*, 27:983-1013, 2022.

[42] M. A. Shubin. *Pseudodifferential Operators and Spectral Theory*. Springer–Verlag, Berlin–Heidelberg, 1985.

[43] W. van Suijlekom, L. Dąbrowski, G. Landi, A. Sitarz, and J. C. Várilly. The local index formula for SU_q(2). *K-Theory*, 35(3-4):375–394, 2005.

[44] S. G. Walters. Chern characters of Fourier modules. *Canad. J. Math.*, 52(3):633–672, 2000.

[45] S. Walters. Periodic integral transforms and C*-algebras. *C. R. Math. Acad. Sci. Soc. R. Can.*, 26(2):55–61, 2004.

[46] S. Walters. Toroidal orbifolds of Z_3 and Z_6 symmetries of noncommutative tori. *Nuclear Phys. B*, 894:496–526, 2015.

[47] M. Wodzicki. Noncommutative residue. I. Fundamentals. *Lecture Notes in Math.*, 1289. Berlin, New York: Springer-Verlag, pp. 320–399, 1987.

A. Savin. Peoples’ Friendship University of Russia (RUDN University), 6 Miklukho-Maklaya St, Moscow, 117198, Russia

Email address: a.yu.savin@gmail.com

E. Schrohe. Leibniz University Hannover, Institute of Analysis, Welfengarten 1, 30167 Hannover, Germany

Email address: schrohe@math.uni-hannover.de