Functional Integral Method in Quantum Theory of Composite Particles and Quasiparticles

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Abstract

In this series of lectures we present the universal method based on the use of the functional integrals to derive the bound state equations for different two-body systems in elementary particle physics as well as in condensed matter theory: the relativistic Nambu-Jona-Lasinio equation and the relativistic Bethe-Salpeter equation in elementary particle physics, the Bethe-Salpeter equation for the bipolaritons. The bound state equation for the three-body systems is also established.

1 Introduction

The theoretical study of the bound states of different systems of fundamental particles, their energy (mass) spectra and the corresponding many-body quantum-mechanical wave functions or quantum field-theoretical state vectors was always one among the main problems of the quantum physics. The simplest examples of the two-body quantum systems are the hydrogen atom and the positronium. In the non-relativistic approximation for the proton the energy spectrum and the wave functions of the hydrogen atom are determined by the Dirac equation with the Coulomb potential of the attractive interaction between the electron and the proton. For the formation of the bound states of two particles (including particle-antiparticle pairs) due to the boson exchange in the relativistic quantum field theory, for example the positronium in QED, we have the Bethe-Salpeter equation\(^1\). This relativistic integral equation was widely applied to study the bound states of different two-body systems: the mesons as the bound states of the quark-antiquark pairs in QCD\(^2\)−\(^4\) and the diquarks as the bound states of the quark-quark pairs in dense QCD\(^5\),\(^6\). The direct four-fermion coupling may be also the physical origin of the formation of the composite particles as the bound states of some fermion-antifermion pairs or two-fermion systems. In this case we have the Nambu-Jona-Lasinio equation\(^7\). It was widely applied to the study of the formation and the physical properties of the mesons as the bound states of the quark-antiquark pairs\(^8\)−\(^15\) . Recently this equation was used for the investigation of the instanton induced quark-quark pairing in dense QCD\(^16\)−\(^21\). In the study of the formation of the bound states of two-body systems of quasiparticles in condensed matter one can use the Schrödinger equation if the effective mass approximation is valid. However, there are quasiparticles with the complicated energy spectra so that we cannot establish the Schrödinger equation. In this case we must derive and then apply either the Bethe-Salpeter or the Nambu-Jona-Lasinio equation. For example, due to the exciton-exciton interaction there may exist the bound states of the systems of two polaritons called the bipolaritons. The Bethe-Salpeter equation for the bipolaritons was established and studied by many authors\(^22\),\(^23\).

The relativistic Bethe-Salpeter equation for the systems of two elementary particles as well as the nonrelativistic Bethe-Salpeter equation for the systems of two quasiparticles with the complicated structure of the energy spectra in condensed matters were

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derived within the framework of the perturbation theory by summing up the matrix elements of the infinite series of Feynman diagrams in the ladder approximation. The Nambu-Jona-Lasinio equation was established in the pioneering work of Nambu and Jona-Lasinio also by means of the perturbation theory\cite{7}. Subsequently the elegant derivation of this famous equation by means of the functional integral technique was given by Eguchi, Sugawara\cite{8,9} and Kikkawa\cite{10}. Within the functional integral formalism it is not necessary to sum up the infinite series of the matrix elements of the Feynman diagrams in the ladder approximation, and the derivation of the equations is simplified. In the attempt to present the functional integral technique as the universal method for the study of many different problems in the quantum theory\cite{24−29} in this series of lectures we review the derivation of both types of bound state equations in the elementary particle physics as well as in the condensed matter theory by means of the functional integral technique. The derivation of the bound state equations for the systems of two quasiparticles in condensed matters by means of the functional integral method is presented first time in these lectures. As we shall see, it is straightforward to establish the bound state equations for the three-body systems\cite{30} in the framework of the functional integral formalism.

2 Relativistic Nambu-Jona-Lasinio Equation

We start to study the equations for the bound states of many-body systems by considering the simplest examples: the formation of the mesons and the diquarks due to the direct 4-fermion coupling of the quark field with the interaction Lagrangian

\[
L_{\text{int}} (x) = \frac{1}{2} \bar{\psi}^A (x) \psi_B (x) U_{AC}^{BD} \bar{\psi}^C (x) \psi_D (x), \quad (1)
\]

\[
U_{AC}^{BD} = -U_{AB}^{CD} = U_{AC}^{DB} = U_{CA}^{DB},
\]

where \( \psi_A (x) \) denote the quark field,

\[ A = (\alpha ai), \quad B = (\beta bj), \quad C = (\gamma ck), \quad D = (\delta dl) \]

are the sets consisting of the Dirac spinor indices \( \alpha, \beta, \gamma, \delta = 1, 2, 3, 4 \), the indices of the color symmetry \( a, b, c, d = 1, 2, \ldots N_c \) and those of the flavor one \( i, j, k, l = 1, 2, \ldots N_f \).

For the convenience in the study of the diquark formation we write the interaction Lagrangian (1) also in another form

\[
L_{\text{int}} (x) = \frac{1}{2} \bar{\psi}^A (x) \bar{\psi}^C (x) V_{CA}^{BD} \psi_D (x) \psi_B (x), \quad (2)
\]

with the new coupling constants

\[ V_{CA}^{BD} = U_{AC}^{BD}. \quad (3) \]

The mathematical tool in the functional integral formalism is the functional integral of the field system

\[
Z = \int [D\psi] [D\bar{\psi}] \exp \{ iS [\psi, \bar{\psi}] \}, \quad (4)
\]

where \( S [\psi, \bar{\psi}] \) is the total action of the system

\[
S [\psi, \bar{\psi}] = S_0 [\psi, \bar{\psi}] + S_{\text{int}} [\psi, \bar{\psi}], \quad (5)
\]

\( S_0 [\psi, \bar{\psi}] \) being the action of the free quark field with some bare mass \( m \)

\[
S_0 [\psi, \bar{\psi}] = -\int d^4x \bar{\psi}^A (x) \left[ (\bar{\partial})_A^B + m\delta^B_A \right] \psi_B (x), \quad (6)
\]

\[
(\bar{\partial})_A^B = \delta^B_\alpha \delta^\alpha_i (\gamma_\mu)_i^\beta \partial^\mu
\]
\( S_{\text{int}} [\psi, \bar{\psi}] \) being the contribution of the interaction Lagrangian to the total action
\[
S_{\text{int}} [\psi, \bar{\psi}] = \int d^4x L_{\text{int}} (x) . \tag{7}
\]

For the study of the mesons as the bound states of the quark-antiquark pairs we use the interaction Lagrangian in the form (1), introduce the composite meson field \( \Phi^A_B (x) \) and set
\[
Z_0^\Phi = \int [D\Phi] \exp \left\{ -\frac{i}{2} \int d^4x \Phi^A_B (x) U^{BD}_{AC} \Phi^C_D (x) \right\}. \tag{8}
\]

Shifting the functional integration variables
\[
\Phi^A_B (x) \rightarrow \Phi^A_B (x) + \bar{\psi}^A (x) \psi_B (x) ,
\]
we establish the Hubbard-Stratonovich transformation
\[
\exp \left\{ \frac{i}{2} \int d^4x \bar{\psi}^A (x) \psi_B (x) U^{BD}_{AC} \Phi^C_D (x) \psi_D (x) \right\} = \frac{1}{Z_0^\Phi} \int [D\Phi] \exp \left\{ -\frac{i}{2} \int d^4x \Phi^A_B (x) U^{BD}_{AC} \Phi^C_D (x) \right\} \exp \left\{ -i \int d^4x \bar{\psi}^A (x) \psi_B (x) \Delta^B_A (x) \right\} \tag{9}
\]
with
\[
\Delta^B_A (x) = U^{BD}_{AC} \Phi^C_D (x) \tag{10}
\]

Using this formula, we transform the expression (4) of the functional integral of the quark field into that of the composite meson one
\[
Z = \frac{Z_0^\Phi}{Z_0} \int [D\Phi] \exp \{ iS_{\text{eff}} [\Phi] \} \tag{11}
\]
with the effective action
\[
S_{\text{eff}} [\Phi] = -\frac{1}{2} \int d^4x \Phi^A_B (x) U^{BD}_{AC} \Phi^C_D (x) + W [\Delta] , \tag{12}
\]

where
\[
Z_0 = \int [D\psi] [D\bar{\psi}] \exp \{ iS_0 [\psi, \bar{\psi}] \} \tag{13}
\]
and the functional \( W [\Delta] \) is determined by the formula

\[
\exp \{ iW [\Delta] \} = \frac{1}{Z_0^\Phi} \int [D\psi] [D\bar{\psi}] \exp \{ iS_0 [\psi, \bar{\psi}] \} \exp \left\{ -i \int d^4x \bar{\psi}^A (x) \psi_B (x) \Delta^B_A (x) \right\}. \tag{14}
\]

We can express \( W [\Delta] \) in the form of a functional power series in the field \( \Delta^B_A (x) \)
\[
W [\Delta] = \sum_{n=1}^\infty W^{(n)} [\Delta] , \tag{15}
\]

where \( W^{(n)} [\Delta] \) is a homogeneous functional of the \( n \)-th order. Calculations give, for examples,
\[
W^{(1)} [\Delta] = -i \int d^4x S^{BD}_A (0) \Delta^B_A (x) , \tag{16}
\]
we derive the field equation

\[ W^{(2)}[\Delta] = \frac{i}{2} \int d^4x \int d^4y S_B^A(x) \Delta_C^E(y) S_C^D(y - x) \Delta_D^B(x), \quad (17) \]

\[ W^{(3)}[\Delta] = \frac{i}{3} \int d^4x \int d^4y \int d^4z S_B^A(x) \Delta_C^E(y) S_C^D(y - z) \Delta_D^E(z) S_D^A(z - x), \quad (18) \]

etc, where \( S_B^A(x - y) \) is the two-point Green function of the free quark field

\[ S_B^A(x - y) = \frac{i}{Z_0} \int [D\bar{\psi}] [D\psi] \bar{\psi}_A(x) \psi^B(y) \exp \left\{ i S_0[\bar{\psi}, \psi] \right\}. \quad (19) \]

It satisfies the equation

\[ \left[ \left( \frac{\partial}{\partial x} \right)_A + m \delta_B^A \right] S_B^C(x - y) = \delta_C^2 \delta(x - y). \quad (20) \]

Denote \( \bar{S}_A^B(p) \) its Fourier transform

\[ \bar{S}_A^B(x) = \frac{1}{(2\pi)^4} \int d^4p \, e^{ipx} \bar{S}_B^A(p). \quad (21) \]

We have

\[ \bar{S}_A^B(p) = \delta_\alpha^B \delta_\beta^A \left( \frac{1}{ip + m} \right)^\beta. \quad (22) \]

From the variational principle

\[ \frac{\delta S_{\text{eff}}[\Phi]}{\delta \Phi^C_D(x)} = 0 \quad (23) \]

we derive the field equation

\[ \Delta_C^E(x) = U_{CA}^{DB} \frac{\delta W[\Delta]}{\delta \Delta_D^B(x)}. \quad (24) \]

For some definite types of composite meson fields the first order functional \( W^{(1)}[\Delta] \) either vanishes or does not give the contribution. Then in the second order with respect to \( \Delta_A^B(x) \) we can replace \( W[\Delta] \) by \( W^{(2)}[\Delta] \) and obtain the approximate field equation

\[ \Delta_C^E(x) = U_{CA}^{DB} \frac{\delta W^{(2)}[\Delta]}{\delta \Delta_D^B(x)}. \quad (25) \]

Substituting the expression (17) of \( W^{(2)}[\Delta] \) into the r.h.s. of the relation (25), we obtain immediately the Nambu-Jona-Lasinio equation

\[ \Delta_C^E(x) = iU_{CA}^{DB} \int d^4y S_B^E(x - y) \Delta_E^D(y) S_D^A(y - x). \quad (26) \]

For establishing the field equation in the general case with non-vanishing \( W^{(1)}[\Delta] \) we must take into account the contributions of the functionals \( W^{(n)}[\Delta] \) of all orders by summing up the infinite series (15). Denote

\[ S_B^A(x, y) = S_B^A(x - y) - \int d^4x_1 S_B^{A_1}(x - x_1) \Delta_{B_1}^{A_1}(x_1) S_A^{B_1}(x_1 - y) \]
\[ + \int d^4x_1 \int d^4x_2 S_B^{A_2}(x - x_1) \Delta_{B_1}^{A_2}(x_1) S_A^{B_1}(x_1 - x_2) \Delta_{B_2}^{A_2}(x_2) S_A^{B_2}(x_2 - y) \]
\[ - \int d^4x_1 \int d^4x_2 \int d^4x_3 S_B^{A_3}(x - x_1) \Delta_{B_1}^{A_3}(x_1) S_A^{B_1}(x_1 - x_2) \Delta_{B_2}^{A_3}(x_2) S_A^{B_2}(x_2 - x_3) \Delta_{B_3}^{A_3}(x_3) S_A^{B_3}(x_3 - y) + \ldots. \quad (27) \]
the two-point Green function of the quark field interacting with the composite meson field $\Delta^A_B (x)$. It is determined by the Schwinger-Dyson equation

$$S^B_A (x, y) = S^B_A (x - y) - \int d^4 z S^C_A (x - z) \Delta^D_C (z) S^B_D (z, y). \quad (28)$$

In terms of this new two-point Green function we can rewrite the field equation (24) in the form

$$\Delta^D_C (x) + i U^{DB}_{CA} S^A_B (x, x) = 0. \quad (29)$$

In general there exists some non-vanishing constant solution of the nonlinear equation (29)

$$\Phi^B_A = \text{const}, \quad \Delta^B_A = U^{BD}_{AC} \Phi^C_D = \text{const}. \quad (30)$$

With the constant meson field the expression in the r.h.s. of the formula (27) depends only on the difference $x - y$ of the coordinates $x$ and $y$

$$S^A_B (x, y) = S^A_B (x - y), \quad (31)$$

and this constant meson field is determined by the equation

$$(\Delta^0)^D_C + i U^{DB}_{CA} S^A_B (0) = 0. \quad (32)$$

Consider the quantum fluctuations of the meson field around the background constant one (30) satisfying the equation (32) and set

$$\Phi^B_A (x) = (\Phi^0)^B_A + \phi^B_A (x), \quad \Delta^B_A (x) = (\Delta^0)^B_A + \xi^B_A (x), \quad (33)$$

with

$$\xi^B_A (x) = U^{AC}_{BD} \phi^D_C (x). \quad (34)$$

Up to the second order with respect to the quantum fluctuations the effective action of the composite meson field equals

$$S_{\text{eff}} [\Phi] \approx S_{\text{eff}} [\Phi^0] + \frac{1}{2} \int d^4 x \int d^4 y \phi^A_B (x) \frac{\delta^2 S_{\text{eff}} [\Phi]}{\delta \Phi^A_B (x) \delta \Phi^C_D (y)}|_{\Phi=\Phi^0} \phi^C_D (y). \quad (35)$$

Using the expression (12) of $S_{\text{eff}} [\Phi]$, we have

$$S_{\text{eff}} [\Phi] \approx S_{\text{eff}} [\Phi^0] - \frac{1}{2} \int d^4 x \phi^A_B (x) U^{BD}_{AC} \phi^C_D (x) + \frac{1}{2} \int d^4 x \int d^4 y \xi^A_B (x) \frac{\delta^2 W [\Delta]}{\delta \Delta^A_B (x) \delta \Delta^C_D (y)}|_{\Delta=\Delta^0} \xi^C_D (y). \quad (36)$$

By summing up the infinite series

$$\sum_{n=1}^{\infty} \frac{\delta^2 W^{(n)} [\Delta]}{\delta \Delta^A_B (x) \delta \Delta^C_D (y)}|_{\Delta=\Delta^0}$$

we obtain

$$\frac{\delta^2 W [\Delta]}{\delta \Delta^A_B (x) \delta \Delta^C_D (y)}|_{\Delta=\Delta^0} = i S^C_B (x - y) S^A_B (y - x). \quad (37)$$
Then from the formula (36) for the effective action and the variational principle it follows the Nambu-Jona-Lasinio equation containing the two-point Green function of the quarks interacting with the constant background meson field

\[ \xi^A_B (x) = iU^{BD}_{AC} \int d^4y S^E_D (x - y) \xi^F_E (y) S^C_F (y - x). \]  

(38)

In the special case when the background meson field vanishes or its contribution is neglected then the equation (38) reduces to the equation (26).

Now we study the formation of the bound states of a pair of two quarks. For this purpose we use the interaction Lagrangian in the form (2), introduce the composite bosonic diquark field \( \Phi_{AB} (x) \) as well as its conjugate \( \overline{\Phi}^{BA} (x) \) and set

\[
Z_0^{\Phi, \overline{\Phi}} = \int [D\Phi] [D\overline{\Phi}] \exp \left\{ -\frac{i}{2} \int d^4x \overline{\Phi}^{AC} (x) V^{BD}_{CA} \Phi_{DB} (x) \right\}.
\]

(39)

Shifting the functional integration variables

\[
\Phi_{DB} (x) \rightarrow \Phi_{DB} (x) + \psi_D (x) \psi_B (x), \\
\overline{\Phi}^{AC} (x) \rightarrow \overline{\Phi}^{AC} (x) + \overline{\psi}^A (x) \overline{\psi}^C (x),
\]

(40)

we establish the Hubbard-Stratonovich transformation

\[
\exp \left\{ -\frac{i}{2} \int d^4x \overline{\psi}^A (x) \overline{\psi}^C (x) V^{BD}_{CA} \Phi_{DB} (x) \right\} = \frac{1}{Z_0^{\Phi, \overline{\Phi}}} \int [D\Phi] [D\overline{\Phi}] \exp \left\{ -\frac{i}{2} \int d^4x \overline{\Phi}^{AC} (x) V^{BD}_{CA} \Phi_{DB} (x) \right\} \exp \left\{ -\frac{i}{2} \int d^4x \overline{\psi}^A (x) \overline{\psi}^C (x) \Delta_{CA} (x) + \overline{\Delta}^{BD} (x) \psi_D (x) \psi_B (x) \right\},
\]

(41)

where

\[
\Delta_{CA} (x) = V^{BD}_{CA} \Phi_{DB} (x), \\
\overline{\Delta}^{BD} (x) = \overline{\Phi}^{AC} (x) V^{BD}_{CA}.
\]

(42)

Using formula (41), we can transform the expression (4) of the functional integral of the quark field into that of the diquark system

\[
Z = \frac{Z_0^{\Phi, \overline{\Phi}}}{Z_0^{\Phi, \overline{\Phi}}} \int [D\Phi] [D\overline{\Phi}] \exp \left\{ iS_{\text{eff}} [\Phi, \overline{\Phi}] \right\}
\]

(43)

with the effective action

\[
S_{\text{eff}} [\Phi, \overline{\Phi}] = -\frac{1}{2} \int d^4x \overline{\Phi}^{AC} (x) V^{BD}_{CA} \Phi_{DB} (x) + W [\Delta, \overline{\Delta}],
\]

(44)

\( W [\Delta, \overline{\Delta}] \) being determined by the formula

\[
\exp \left\{ i W [\Delta, \overline{\Delta}] \right\} = \frac{1}{Z_0^{\Phi, \overline{\Phi}}} \int [D\psi] [D\overline{\psi}] \exp \left\{ iS_0 [\psi, \overline{\psi}] \right\} \exp \left\{ -\frac{i}{2} \int d^4x \left[ \overline{\psi}^A (x) \overline{\psi}^C (x) \Delta_{CA} (x) + \overline{\Delta}^{BD} (x) \psi_D (x) \psi_B (x) \right] \right\},
\]

(45)

and having the form
If we neglect the contributions of the high order functionals $W$ we derive the field equation

$$W^{(2)}[\Delta, \Phi] = \frac{i}{2} \int d^4 x_1 \int d^4 x_2 A^{A_1 B_1} (x_1) S_{A_1}^{B_1} (x_1 - x_2) S_{B_1}^{A_2} (x_1 - x_2) \Delta_{B_2 A_2} (x_2)$$

or in another form

$$W^{(2)}[\Delta, \Phi] = \frac{i}{2} \int d^4 x_1 \int d^4 x_2 A^{A_1 B_1} (x_1) S_{B_1}^{A_2} (x_1 - x_2) \Delta_{B_2 A_2} (x_2) S_{A_1}^{A_2} (x_2 - x_1)$$

Convenient for the comparison with the higher order functionals

$$W^{(4)}[\Delta, \Phi] = \frac{i}{4} \int d^4 x_1 \int d^4 x_4 A^{A_1 B_1} (x_4) S_{A_4}^{B_4} (x_4 - x_3) \Delta_{B_4 A_4} (x_4 - x_2) \Delta_{B_2 A_2} (x_2 - x_1)$$

$$W^{(6)}[\Delta, \Phi] = \frac{i}{6} \int d^4 x_1 \int d^4 x_2 \int d^4 x_3 \int d^4 x_4 A^{A_1 B_1} (x_1) S_{B_1}^{A_2} (x_1 - x_2) \Delta_{B_2 A_2} (x_2) S_{A_4}^{A_4} (x_4 - x_3) \Delta_{B_4 A_4} (x_4 - x_2) \Delta_{B_2 A_2} (x_2 - x_1)$$

etc, where we used the notation

$$S_{A B}^{T A} (y - x) = S_{A B} (x - y)$$

From the variational principle

$$\frac{\delta S_{\text{eff}} [\Phi, \Phi]}{\delta \Phi^{A B} (x)} = 0$$

we derive the field equation

$$\frac{1}{2} \Delta_{B A} (x) = V_{C D}^{E D} \frac{\delta W [\Delta, \Phi]}{\delta \Phi^{A B} (x)}$$

If we neglect the contributions of the high order functionals $W^{(2n)}[\Delta, \Phi]$ with $n > 1$ and use the approximation

$$W [\Delta, \Phi] \approx W^{(2)}[\Delta, \Phi]$$

then the field equation (53) is the Nambu-Jona-Lasinio equation for the composite diquark field

$$\Delta_{A B} (x) = -i V_{A B}^{D C} \int d^4 y S_{E}^{C} (x - y) S_{D}^{E} (x - y) \Delta_{E F} (y)$$

In order to take into account the contributions of the high order functionals $W^{(2n)}[\Delta, \Phi]$ with $n > 1$ we search the non-vanishing constant solution of the field equation (53)

$$\Phi^0 \approx \text{const}, \quad \Phi^0 \approx \text{const},$$

$$\Delta_{A B} = V_{C A}^{B D} \Phi^0 \approx \text{const}, \quad \Delta_{A B} = V_{C A}^{B D} \Phi^0 \approx \text{const}$$
and consider the quantum fluctuations around this background constant field:

\[
\Phi_{DB}(x) = (\Phi^0)_{DB} + \varphi_{DB}(x), \quad \tilde{\Phi}^{AC}(x) = (\tilde{\Phi}^0)^{AC} + \tilde{\varphi}^{AC}(x), \\
\Delta_{CA}(x) = (\Delta^0)_{CA} + \xi_{CA}(x), \quad \tilde{\Delta}^{BD}(x) = (\tilde{\Delta}^0)^{BD} + \tilde{\xi}^{BD}(x),
\]

with

\[
\xi_{CA}(x) = V_{CA}^{BD} \varphi_{DB}(x), \quad \tilde{\xi}^{BD}(x) = \tilde{\varphi}^{AC}(x) V_{CA}^{BD}.
\]

Up to the first order with respect to each type of bosonic fluctuations \(\varphi_{DB}(x), \xi_{DB}(x)\) and \(\tilde{\varphi}^{AC}(x), \tilde{\xi}^{AC}(x)\) we have the effective action

\[
S_{\text{eff}}[\Phi, \tilde{\Phi}] \approx S_{\text{eff}}[\Phi^0, \tilde{\Phi}^0] = \frac{1}{2} \int d^4x \varphi^{AC}(x) V_{CA}^{BD} \varphi_{DB}(x) + \int d^4x \int d^4y \tilde{\xi}^{AC}(x) \frac{\delta^2 W[\Delta, \tilde{\Delta}]}{\delta \Delta^{AC}(x) \delta \Delta_{DB}(y)} |_{\Delta = \Delta^0, \tilde{\Delta} = \tilde{\Delta}^0} \xi_{DB}(x).
\]

By summing up the infinite series

\[
\sum_{n=1}^{\infty} \frac{\delta^2 W[\Delta, \tilde{\Delta}]}{\delta \Delta^{AC}(x) \delta \Delta_{DB}(y)} |_{\Delta = \Delta^0, \tilde{\Delta} = \tilde{\Delta}^0} = -\frac{i}{2} S_A^B(x-y) S_C^B(x-y),
\]

where \(S_A^B(x-y)\) is determined by the Schwinger-Dyson equation

\[
S_A^B(x-y) = S_A^B(x-y) - \int d^4z \int d^4u S_C^D(x-z) (\Delta^0)_{CD} S_E^{TD}(z-u) \left(\tilde{\Delta}^0\right)^{EF} S_F^B(u-y).
\]

Then from the expression (58) of the effective action and the variational principle it follows the Nambu-Jona-Lasinio equation

\[
\xi_{AB}(x) = -i V_{AB}^{DC} \int d^4y S_C^E(x-y) S_D^F(x-y) \xi_{EF}(y)
\]

containing the two-point Green function \(S_A^B(x-y)\) of the quark field interacting with the background bosonic constant field (55). In the special case of the vanishing background bosonic field \((\Phi^0)_{DB} = (\tilde{\Phi}^0)^{AC} = (\Delta^0)_{DB} = (\tilde{\Delta}^0)^{AC} = 0\) the equation (61) coincides with the equation (54).

In order to solve equation (60) we work in the momentum space. Denote \(\tilde{S}_A^B(p)\) and \(\tilde{S}_A^B(p)\) the Fourier transforms of the two-point Green functions \(S_A^B(x)\) and \(S_A^B(x)\),

\[
S_A^B(x) = \frac{1}{(2\pi)^4} \int d^4p e^{ipx} \tilde{S}_A^B(p), \quad S_A^B(x) = \frac{1}{(2\pi)^4} \int d^4p e^{ipx} \tilde{S}_A^B(p),
\]

\[
px = px - p_0x_0 = px - Et,
\]

and introduce the matrices \(\tilde{S}(p), \tilde{S}^T(p), \tilde{S}(p), \Delta^0\) and \(\tilde{\Delta}^0\) with the elements \(\tilde{S}_A^B(p), \tilde{S}_A^B(-p), \tilde{S}(p), (\Delta^0)_{AB}\) and \(\tilde{\Delta}^0\). Then we can rewrite the Schwinger-Dyson equation (60) in the matrix form.
\[ \frac{1}{S(p)} = \frac{1}{\overline{S(p)}} + \Delta^0 \overline{S^T} (-p) \overline{\Delta^0}. \] (63)

For the constant background composite field (55) the field equation (53) has the explicit form

\[ \left( \Delta^0 \right)_{AB} = -i V^{DC}_{AB} \left\{ \int d^4 y S_C^E (x - y) \left( \Delta^0 \right)_{EF} S_D^{TF} (y - x) - \int d^4 y \int d^4 x_1 \int d^4 y_1 S_C^E (x - y) \left( \Delta^0 \right)_{EF} S_D^{TF} (y - x_1) \cdot \left( \Delta^0 \right)_{C_i} S_{C_i}^{E_i} (x_1 - y_1) \left( \Delta^0 \right)_{E_i F_i} S_D^{TF_i} (y_1 - x) \right. \]

\[ + \int d^4 y \int d^4 x_1 \int d^4 y_2 \int d^4 x_2 \int d^4 y_2 S_C^E (x - y) \left( \Delta^0 \right)_{EF} S_D^{TF} (y - x_1) \cdot \left( \Delta^0 \right)_{C_i} S_{C_i}^{E_i} (x_1 - y_1) \left( \Delta^0 \right)_{E_i F_i} S_D^{TF_i} (y_1 - x_2) \left( \Delta^0 \right)_{D_j C_j} S_{C_j}^{E_j} (x_2 - y_2) \left( \Delta^0 \right)_{E_j F_j} S_D^{TF_j} (y_2 - x) \] (64)

\[ \left. + \int d^4 y \int d^4 x_1 \int d^4 y_1 \int d^4 x_2 \int d^4 y_2 S_C^E (x - y) \left( \Delta^0 \right)_{EF} S_D^{TF} (y - x_1) \cdot \left( \Delta^0 \right)_{C_i} S_{C_i}^{E_i} (x_1 - y_1) \left( \Delta^0 \right)_{E_i F_i} S_D^{TF_i} (y_1 - x_2) \right\}. \]

In term of the Fourier transforms of the two-point Green functions this equation becomes

\[ \left( \Delta^0 \right)_{AB} = -i V^{DC}_{AB} \frac{1}{(2\pi)^4} \int d^4 p \left\{ \tilde{S}_C^E (p) \left( \Delta^0 \right)_{EF} \tilde{S}_D^{TF} (-p) - \tilde{S}_C^E (p) \left( \Delta^0 \right)_{EF} \tilde{S}_D^{TF} (-p) \left( \Delta^0 \right)_{C_i} \tilde{S}_{C_i}^{E_i} (p) \left( \Delta^0 \right)_{E_i F_i} \tilde{S}_D^{TF_i} (-p) \right. \]

\[ + \tilde{S}_C^E (p) \left( \Delta^0 \right)_{EF} \tilde{S}_D^{TF} (-p) \left( \Delta^0 \right)_{C_i} \tilde{S}_{C_i}^{E_i} (p) \left( \Delta^0 \right)_{E_i F_i} \tilde{S}_D^{TF_i} (-p) \left( \Delta^0 \right)_{D_j C_j} \tilde{S}_{C_j}^{E_j} (p) \left( \Delta^0 \right)_{E_j F_j} \tilde{S}_D^{TF_j} (-p) \] (65)

or in the matrix form

\[ \left( \Delta^0 \right)_{AB} = -i V^{DC}_{AB} \frac{1}{(2\pi)^4} \int d^4 p \left\{ \tilde{S} (p) \Delta^0 S^T (-p) \frac{1}{1 + \Delta^0 \tilde{S} (p) \Delta^0 S^T (-p)} \right\}_{CD}. \] (66)

### 3 Relativistic Bethe-Salpeter Equation

Now we generalize the reasonings presented in the preceding Section to establish the bound state equations for the two-body systems of relativistic Dirac fermions with some effective (non-local, in general) 4-fermion interaction in the quantum field theory. For the definiteness we consider again the quark field with some effective direct 4-quark (non-local, in general) interaction induced by the dynamical mechanisms in QCD.

Introduce the Fourier transforms \( \tilde{\psi}_A (p) \) and \( \tilde{\psi}^A (p) \) of the quark field \( \psi_A (x) \) and its conjugate \( \overline{\psi}^A (x) \),

\[ \psi_A (x) = \frac{1}{(2\pi)^2} \int d^4 p e^{ipx} \tilde{\psi}_A (p), \]

\[ \overline{\psi}^A (x) = \frac{1}{(2\pi)^2} \int d^4 p e^{-ipx} \tilde{\psi}^A (p). \] (67)

Then we have
\[ S_0 \left[ \psi, \bar{\psi} \right] = - \int d^4p \bar{\psi}^A (p) \left[ i \left( \hat{p} \right)_A^B + m\delta_A^B \right] \psi_B (p) \] (68)

and

\[ \frac{i}{Z_0} \int \left[ D\bar{\psi} \right] \left[ D\psi \right] \bar{\psi}^A (p) \psi^B (q) \exp \left\{ iS_0 \left[ \psi, \bar{\psi} \right] \right\} = \delta (p - q) \bar{S}_A^B (p) \] (69)

The contribution \( S_{\text{int}} \left[ \psi, \bar{\psi} \right] \) of the 4–quark interaction to the effective action reproducing the 4–point Green function of the quark field will be represented either in the form

\[ S_{\text{int}} \left[ \psi, \bar{\psi} \right] = \frac{1}{2} \int d^4p_1 d^4q_1 \int d^4p_2 d^4q_2 \delta^{(4)} (p_1 + q_1 - p_2 - q_2) \] (70)

\[ \bar{\psi}^A_1 (p_1) \bar{\psi}^B_1 (-q_1) U_{A_1 B_2}^{B_1} (p_1, -q_1; -q_2, p_2) \bar{\psi}^B_2 (q_2) \bar{\psi}^A_2 (p_2) \]

convenient for the study of the mesons as the bound states of the quark-antiquark pairs, or in another form

\[ S_{\text{int}} \left[ \psi, \bar{\psi} \right] = \frac{1}{2} \int d^4p_1 d^4q_1 \int d^4p_2 d^4q_2 \delta^{(4)} (p_1 + q_1 - p_2 - q_2) \] (71)

\[ \bar{\psi}^A_1 (p_1) \bar{\psi}^B_1 (q_1) V_{B_2 A_1}^{A_2 B_2} (q_1, p_1; p_2, q_2) \bar{\psi}^B_2 (q_2) \bar{\psi}^A_2 (p_2) \]

convenient for the study of the bound states of two quarks - the diquarks. The covariant quantities \( U_{A_1 B_2}^{B_1} (p_1, q_1; p_2, q_2) \) and \( V_{B_2 A_1}^{A_2 B_2} (q_1, p_1; p_2, q_2) \) may be called the relativistic effective potentials. They are related each to other

\[ V_{B_2 A_1}^{A_2 B_2} (q_1, p_1; p_2, q_2) = U_{A_2 B_1}^{A_1 B_2} (p_1, q_1; p_2, q_2) \] (72)

and must be antisymmetric under the simultaneous interchanges of the upper or lower spinor indices and the corresponding 4–momenta of the quark field

\[ U_{A_1 B_2}^{B_1} (p_1, -q_1; -q_2, p_2) = - U_{B_2 A_1}^{A_1 B_2} (-q_2, -q_1; p_1, q_2) = \]

\[ = - U_{B_2 A_1}^{A_1 B_2} (p_1, p_2; -q_2, -q_1) = U_{B_2 A_1}^{A_1 B_2} (-q_2, p_2; p_1, -q_1), \]

\[ V_{B_2 A_1}^{A_2 B_2} (q_1, p_1; p_2, q_2) = - V_{A_2 B_1}^{B_1 A_2} (p_1, q_1; p_2, q_2) = \]

\[ = - V_{A_2 B_1}^{B_1 A_2} (q_1, p_1; q_2, p_2) = V_{A_2 B_1}^{B_1 A_2} (p_1, q_1; q_2, p_2). \] (73)

In order to study the mesons we start from the expression (70) of \( S_{\text{int}} \left[ \psi, \bar{\psi} \right] \), introduce the composite bi-local meson field \( \Phi^A_B (x, y) \), whose Fourier transform is denoted \( \tilde{\Phi}^A_B (p, q) \)

\[ \Phi^A_B (x, y) = \frac{1}{(2\pi)^4} \int d^4p d^4qe^{-i(px-qq)} \tilde{\Phi}^A_B (p, q) , \] (74)

and set

\[ Z_0^\Phi = \int \left[ D\Phi \right] \exp \left\{ - \frac{i}{2} \int d^4p_1 d^4q_1 \int d^4p_2 d^4q_2 \delta^{(4)} (p_1 + q_1 - p_2 - q_2) \right\} \]

\[ \tilde{\Phi}^A_{B_1} (p_1, -q_1) U_{A_1 B_2}^{B_1 A_2} (p_1, -q_1; -q_2, p_2) \tilde{\Phi}^B_{A_2} (-q_2, p_2) \] (75)

Shifting the functional integration variables
we establish the Hubbard-Stratonovich transformation

\[
\tilde{\Phi}_B^A(p, -q) \to \tilde{\Phi}_B^A(p, -q) + \tilde{\psi}^A(p) \tilde{\psi}_B(-q),
\]

\[\text{(76)}\]

Applying the transformation (77) to the expression in the r.h.s of the formula (4) it follows the field equation

\[
\begin{align*}
\exp \left\{ i \int d^4p_1 d^4q_1 \int d^4p_2 d^4q_2 \delta^{(4)}(p_1 + q_1 - p_2 - q_2) \\
\tilde{\psi}^A(p_1) \tilde{\psi}_B(-q_1) U_{B_1, B_2}^{A_1, A_2}(p_1, -q_1; -q_2, p_2) \tilde{\psi}_B^A(-q_2) \psi_{A_2}(p_2) \right\} = \\
\frac{1}{Z_0^{B}} \int [D\Phi] \exp \left\{ -\frac{i}{2} \int d^4p_1 d^4q_1 \int d^4p_2 d^4q_2 \delta^{(4)}(p_1 + q_1 - p_2 - q_2) \\
\tilde{\Phi}_B^{A_1}(p_1, -q_1) U_{B_1, B_2}^{A_1, A_2}(p_1, -q_1; -q_2, p_2) \tilde{\Phi}_B^{A_2}(-q_2, p_2) \right\} \\
\exp \left\{ -i \int d^4p d^4q \tilde{\psi}^A(p) \tilde{\psi}_B(-q) \Delta_A^B(p, -q) \right\},
\end{align*}
\]

\[\text{(77)}\]

where

\[
\Delta_A^B(p, -q) = \int d^4p' d^4q' \delta^{(4)}(p + q - p' - q') U_{A, B'}^{A', B}(p, -q; -q', p') \tilde{\Phi}_A^{B'}(-q', p')
\]

\[\text{(78)}\]

Applying the transformation (77) to the expression in the r.h.s of the formula (4) with the effective interaction action \(S_{\text{int}}[\psi, \bar{\psi}]\) of the form (70) and performing the functional integration over the fermionic integration variables \(\tilde{\psi}_A(x)\), \(\psi_A(x)\) or \(\bar{\psi}_A(p)\) and \(\tilde{\psi}^A(p)\), we rewrite the functional integral (4) of the interacting quark field in the form of the functional integral (11) of the composite bi-local meson field with the effective action

\[
S_{\text{eff}}[\Phi] = -\frac{1}{2} \int d^4p_1 d^4q_1 \int d^4p_2 d^4q_2 \delta^{(4)}(p_1 + q_1 - p_2 - q_2) \\
\tilde{\Phi}_B^{A_1}(p_1, -q_1) U_{B_1, B_2}^{A_1, A_2}(p_1, -q_1; -q_2, p_2) \tilde{\Phi}_B^{A_2}(-q_2, p_2) + W[\Delta],
\]

\[\text{(79)}\]

where \(W[\Delta]\) is a functional power series of the form (15) in the bi-local meson field. Calculations give

\[
W^{(1)}[\Delta] = -i \int d^4p S_A^B(p) \Delta_A^B(p, -p),
\]

\[\text{(80)}\]

\[
W^{(2)}[\Delta] = \frac{i}{2} \int d^4p_1 \int d^4p_2 S_A^B(p_1) \Delta_A^B(p_1, -p_2) S_A^B(p_2) \Delta_A^B(p_2, -p_1),
\]

\[\text{(81)}\]

\[
W^{(3)}[\Delta] = -\frac{i}{3} \int d^4p_1 \int d^4p_2 \int d^4p_3 S_A^B(p_1) \Delta_A^B(p_1, -p_2) S_A^B(p_2) \Delta_A^B(p_2, -p_3) S_A^B(p_3) \Delta_A^B(p_3, -p_1),
\]

\[\text{(82)}\]

etc.

From the variational principle

\[
\frac{\delta S_{\text{eff}}[\Phi]}{\delta \tilde{\Phi}_B^A(p, -q)} = 0
\]

\[\text{(83)}\]

it follows the field equation
\[ \tilde{\Delta}^B_A (p, -q) = \int d^4 p' d^4 q' \delta^{(4)} (p + q - p' - q') U^{BA'}_{AB'} (p, -q; q', p') \frac{\delta W [\Delta]}{\delta \tilde{\Delta}^A_{A'} (-q', p')} . \]

If \( W^{(1)} [\Delta] \) vanishes or does not give the contribution, then in the second order approximation the field equation (84) becomes a homogeneous linear integral equation for the bi-local meson field - the relativistic Bethe-Salpeter equation for the quark-antiquark bound states

\[ \tilde{\Delta}^B_A (p, -q) = i \int d^4 p' d^4 q' \delta^{(4)} (p + q - p' - q') U^{BA'}_{AB'} (p, -q; q', p') \tilde{\Delta}^C_{A'} (p', -q') \tilde{S}^D_{B'} (q') . \]  

In the general case, when \( W^{(1)} [\Delta] \) is non-vanishing or the contributions of the high-order functionals \( W^{(n)} [\Delta] \), \( n > 1 \), are not negligible, we start to search the solution of the field equation (84) in the class of the function of the special form

\[
\left( \tilde{\Phi}^0 (p, -q) \right)_B^A = \delta^{(4)} (p + q) \tilde{\Phi}^A_B (p), \quad \left( \tilde{\Delta}^0 (p, -q) \right)_B^A = \delta^{(4)} (p + q) \tilde{\Delta}^B_A (p),
\]

which are the Fourier transforms of the functions \( (\Phi^0 (x - y))_B^A \) and \( (\Delta^0 (x - y))_A^B \) depending only on the difference \( x - y \) of the coordinates,

\[
\left( \Phi^0 (x - y) \right)_B^A = \frac{1}{(2\pi)^4} \int d^4 p d^4 q e^{-i(p_x + q_y)} \delta^{(4)} (p + q) \tilde{\Phi}^A_B (p) = \frac{1}{(2\pi)^4} \int d^4 p e^{-i p_x} \tilde{\Phi}^A_B (p),
\]

\[
\left( \Delta^0 (x - y) \right)_A^B = \frac{1}{(2\pi)^4} \int d^4 p d^4 q e^{-i(p_x + q_y)} \delta^{(4)} (p + q) \tilde{\Delta}^B_A (p) = \frac{1}{(2\pi)^4} \int d^4 p e^{-i p_x} \tilde{\Delta}^B_A (p),
\]

and then to consider the quantum fluctuations of the bi-local meson field around the background one of the form (87)

\[
\tilde{\Phi}^A_B (p, -q) = \delta^{(4)} (p - q) \tilde{\Phi}^A_B (p) + \tilde{\varphi}^A_B (p, -q),
\]

\[
\tilde{\Delta}^B_A (p, -q) = \delta^{(4)} (p - q) \tilde{\Delta}^B_A (p) + \tilde{\xi}^B_A (p, -q),
\]

\[
\tilde{\xi}^B_A (p, -q) = \int d^4 p' d^4 q' \delta^{(4)} (p + q - p' - q') U^{BA'}_{AB'} (p, -q; -q', p') \tilde{\xi}^B_A (-q', p').
\]

For the background meson field (87) the equation (84) becomes

\[
\tilde{\Delta}^B_A (p) = -i \int d^4 q U^{B}^{BA}_A (p, -p; -q, q) \tilde{S}^C_{B} (q),
\]

where \( \tilde{S}^C_B (p) \) is the Fourier transform of the two-point Green function of the quarks interacting with the meson field (86) :
It satisfies the Schwinger-Dyson equation

\[ \overline{S}_B^B(p) = \overline{S}_A^B(p) - \overline{S}_A^{A_1}(p) \Delta_{A_1}^{B_1}(p) \overline{S}_B^{B_1}(p) + \overline{S}_A^{A_1}(p) \Delta_{A_1}^{B_1}(p) \overline{S}_B^{A_2}(p) \Delta_{A_2}^{B_2}(p) \overline{S}_B^{B_2}(p) - \ldots \]  \( (91) \)

In the second order with respect to the quantum fluctuations the effective action (79) equals

\[
S_{\text{eff}}[\Phi] = S_{\text{eff}}[\Phi^0] - \frac{1}{2} \int d^4 p_1 d^4 q_1 \int d^4 p_2 d^4 q_2 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) \\
\tilde{\varphi}_{A_1}^{A_1}(p_1, -q_1) U_{A_1, B_2}^{B_1} (p_1, -q_1; -q_2, p_2) \tilde{\varphi}_{A_2}^{B_2} (q_2, p_2) + \frac{1}{2} \int d^4 p_1 d^4 q_1 \int d^4 p_2 d^4 q_2. \quad (93)
\]

By summing up the infinite series

\[
\sum_{n=1}^{\infty} \frac{\delta^2 W^{(n)}}{\delta \Delta_{A_1}^{B_1}(p_1, -q_1) \delta \Delta_{A_2}^{B_2}(p_2, -q_2)} |\tilde{\Delta}_A^{(\tilde{\Delta})}_A = i \delta^{(4)}(p_1 + q_2) \delta^{(4)}(p_2 + q_1) \overline{S}_B^{B_1}(p_1) \overline{S}_B^{A_2}(p_2) .
\]

we obtain

\[
\frac{\delta^2 W[\Delta]}{\delta \Delta_{A_1}^{B_1}(p_1, -q_1) \delta \Delta_{A_2}^{B_2}(p_2, -q_2)} |\tilde{\Delta}_A^{(\tilde{\Delta})}_A = i \delta^{(4)}(p_1 + q_2) \delta^{(4)}(p_2 + q_1) \overline{S}_B^{B_1}(p_1) \overline{S}_B^{A_2}(p_2) .
\]

Then from the formula (93) for the effective action and the variational principle it follows the relativistic Bethe-Salpeter equation containing the two-point Green function of the quarks interacting with the background meson field

\[
\tilde{\xi}_A^B(p, -q) = \int d^4 p' d^4 q' \delta^{(4)}(p + q - p' - q') \quad U_{AB}^{BA'}(p, -q; -q', p') \overline{S}_A^{C}(p') \tilde{\xi}_C^{D}(p', -q) \overline{S}_D^{B}(q') .
\]

In the special case when the background meson field vanishes or its contribution is neglected then the equation (95) reduces to the equation (85)

In order to study the formation of the diquarks we start from the expression (71) of \( S_{\text{int}}[\bar{\psi}, \bar{\psi}] \), introduce the composite bi-local bi-spinor bosonic field \( \Phi_{AB}(x, y) \) and its conjugate \( \Phi_{AB}^\dagger(x, y) \) as well as their Fourier transforms \( \Phi_{AB}(p, q) \) and \( \Phi_{AB}^\dagger(p, q) \)

\[
\Phi_{AB}(x, y) = \frac{1}{(2\pi)^4} \int d^4 p d^4 q e^{i(px + qy)} \overline{\Phi}_{AB}(p, q) ,
\]

\[
\overline{\Phi}_{AB}^\dagger(x, y) = \frac{1}{(2\pi)^4} \int d^4 p d^4 q e^{-i(px + qy)} \Phi_{AB}^\dagger(p, q) ,
\]

and set

\[
Z_0[\Phi, \overline{\Phi}] = \int [D\Phi][D\overline{\Phi}] \exp \left\{ -\frac{i}{2} \int d^4 p_1 \int d^4 q_1 \int d^4 p_2 \int d^4 q_2 \delta^4(p_1 + q_1 - p_2 - q_2) \quad \overline{\Phi}_{A_1, B_1}^P(p_1, q_1) V_{B_1, A_1}^{A_2, B_2}(q_1, p_1; p_2, q_2) \Phi_{B_2, A_2}(q_2, p_2) \right\}
\]  \( (97) \)
Shifting the functional integration variables

\[
\Phi_{AB} (p, q) \rightarrow \Phi_{AB} (p, q) + \bar{\psi}_A (p) \bar{\psi}_B (q), \\
\Phi^A (p, q) \rightarrow \Phi^A (p, q) + \bar{\psi}^A (p) \bar{\psi}^B (q),
\]

we establish the Hubbard-Stratonovich transformation

\[
\exp \left\{ \frac{i}{2} \int d^4 p_1 \int d^4 q_1 \int d^4 p_2 \int d^4 q_2 \delta^4 (p_1 + q_1 - p_2 - q_2) \right. \\
\left. \frac{i}{2} A_{B_1}^A (p_1) \bar{\psi}^B_1 (q_1) V^{A_2 B_2}_{B_1 A_1} (q_1; p_2, q_2) \bar{\psi}^B_2 (q_2) \bar{\psi}^A_2 (p_2) \right\}
\]

\[
= \frac{1}{Z_0 [\Phi, \bar{\Phi}]} \int [D\Phi] [D\bar{\Phi}] \exp \left\{ -\frac{i}{2} \int d^4 p_1 \int d^4 q_1 \int d^4 p_2 \int d^4 q_2 \delta^4 (p_1 + q_1 - p_2 - q_2) \right. \\
\left. \frac{i}{2} A_{B_1}^A (p_1) \bar{\psi}^B_1 (q_1) V^{A_2 B_2}_{B_1 A_1} (q_1; p_2, q_2) \bar{\psi}^B_2 (q_2) \bar{\psi}^A_2 (p_2) \right\}
\]

\[
\exp \left\{ -\frac{i}{2} \int d^4 p \int d^4 q \psi^A (p) \bar{\psi}^B (q) \tilde{\Delta}^{AB} (q, p) + \tilde{\Delta}^{AB} (p, q) \bar{\psi}^B (q) \bar{\psi}^A (p) \right. \\
\left. \right\}
\]

\[
\tilde{\Delta}^{BA} (q, p) = \int d^4 p' \int d^4 q' \delta^4 (p' + q' - p - q) V^{A'B'}_{BA} (q, p; p', q') \tilde{\Phi}^{B'A'} (q', p') , \\
\tilde{\Delta}^{AB} (p, q) = \int d^4 p' \int d^4 q' \delta^4 (p' + q' - p - q) \bar{\Phi}^{A'B'} (q', p') V^{AB}_{B'A} (q', p; p, q).
\]

The bi-local bi-spinor fields \( \tilde{\Phi}_{AB} (p, q) \) and \( \bar{\Phi}^{AB} (p, q) \) are antisymmetric under the simultaneous permutations of the coordinates and the bi-spinor indices

\[
\tilde{\Phi}_{BA} (q, p) = -\tilde{\Phi}_{AB} (p, q), \quad \tilde{\Delta}_{BA} (q, p) = -\tilde{\Delta}_{AB} (p, q), \\
\bar{\Phi}^{BA} (q, p) = -\bar{\Phi}^{AB} (p, q), \quad \bar{\Delta}^{BA} (q, p) = -\bar{\Delta}^{AB} (p, q).
\]

Substituting the expression in the r.h.s. of the relation (99) into the functional integral (4) and performing the functional integration over the fermionic integration variables \( \psi_A (x) \) and \( \bar{\psi}^A (x) \) or \( \bar{\psi}_A (p) \) and \( \bar{\psi}^A (p) \), we rewrite the functional integral (4) of the interacting quark field in the form of the functional integral (43) of the composite bi-local bi-spinor bosonic diquark field with the effective action

\[
S_{\text{eff}} [\Phi, \bar{\Phi}] = -\frac{1}{2} \int d^4 p_1 \int d^4 q_1 \int d^4 p_2 \int d^4 q_2 \delta^4 (p_1 + q_1 - p_2 - q_2)
\]

\[
+ \frac{i}{2} A_{B_1}^A (p_1) \bar{\psi}^B_1 (q_1) V^{A_2 B_2}_{B_1 A_1} (q_1; p_2, q_2) \bar{\psi}^B_2 (q_2) \bar{\psi}^A_2 (p_2) + W [\Delta, \bar{\Delta}],
\]

where \( W [\Delta, \bar{\Delta}] \) is a functional power series of the form (46) in the fields \( \Delta_{AB} (x, y) \) and \( \bar{\Delta}^{AB} (x, y) \) whose Fourier transforms are \( \Delta_{AB} (p, q) \) and \( \bar{\Delta}^{AB} (p, q) \) resp.,

\[
\Delta_{AB} (x, y) = \frac{1}{(2\pi)^4} \int d^4 q d^4 p e^{i(px + qy)} \Delta_{AB} (p, q), \\
\bar{\Delta}^{AB} (x, y) = \frac{1}{(2\pi)^4} \int d^4 q d^4 p e^{-i(px + qy)} \bar{\Delta}^{AB} (p, q).
\]

Calculations give
\[ W^{(2)} [\Delta, \bar{\Delta}] = -\frac{i}{2} \int d^4 p \int d^4 q \Delta^{AB} (p, q) S_B^D (q) S_A^C (p) \bar{\Delta}_{DC} (q, p) \]  \hspace{1cm} (104)

\[ W^{(4)} [\Delta, \bar{\Delta}] = \frac{i}{4} \int d^4 p_1 \int d^4 q_1 \int d^4 p_2 \int d^4 q_2 \Delta^{A_1B_1} (p_1, q_1) S_{B_1}^D (q_1) \bar{\Delta}_{D_1C_1} (q_1, p_1) \]
\[ \cdot \Delta^{A_2B_2} (p_2, q_2) S_{B_2}^D (q_2) \bar{\Delta}_{D_2C_2} (q_2, p_2) \bar{\Delta}^{C_2} (p_1), \]  \hspace{1cm} (105)

\[ W^{(6)} [\Delta, \bar{\Delta}] = \frac{i}{6} \int d^4 p_1 \int d^4 q_1 \int \cdots \int d^4 p_3 \int d^4 q_3 \Delta^{A_1B_1} (p_1, q_1) S_{B_1}^D (q_1) \bar{\Delta}_{D_1C_1} (q_1, p_1) \]
\[ \cdot \Delta^{A_2B_2} (p_2, q_2) \cdots \Delta^{A_3B_3} (p_3, q_3) \Delta^{D_3C_3} (q_3, p_3) \bar{\Delta}^{C_3} (p_1), \]  \hspace{1cm} (106)

etc.

From the variational principle
\[ \frac{\delta S_{\text{eff}} [\Phi, \bar{\Phi}]}{\delta \bar{\Phi}^{AC} (x, y)} = 0 \]  \hspace{1cm} (107)

it follows the field equation
\[ \frac{1}{2} \bar{\Delta}_{CA} (q, p) = \int d^4 p' \int d^4 q' \delta^{(4)} (p + q - p' - q') V_{CA}^{BD} (p, q; p', q') \frac{\delta W [\Delta, \bar{\Delta}]}{\delta \bar{\Delta}^{BD} (p', q')} \]  \hspace{1cm} (108)

In the second order approximation with respect to the composite diquark field we have the linear integral equation
\[ \bar{\Delta}_{CA} (q, p) = -i \int d^4 p' \int d^4 q' \delta^{(4)} (p + q - p' - q') \]
\[ V_{CA}^{BD} (p, q; p', q') S_{B}^D (p') \bar{\Delta}^{E} (q', p'), \]  \hspace{1cm} (109)

which is the relativistic Bethe-Salpeter equation for the diquarks. In the general case, when the contributions of the high order functionals \( W^{(2n)} [\Delta, \bar{\Delta}], n > 1, \) cannot be neglected, we search the solution \( (\Phi^0 (y - x))_{DB}, (\bar{\Phi}^0 (x - y))_{BD} \) or \( (\Delta^0 (y - x))_{CA}, \)
\( (\bar{\Delta}^0 (x - y))^{AC} \) of the field equation in the class of the functions depending only on the difference \( x - y \) of the coordinates and study the quantum fluctuations around this background field:

\[ \Phi_{DB} (y, x) = (\Phi^0 (y - x))_{DB} + \varphi_{DB} (y, x), \]
\[ \bar{\Phi}_{BD} (x, y) = (\bar{\Phi}^0 (x - y))_{BD} + \bar{\varphi}_{BD} (x, y), \]
\[ \Delta_{CA} (y, x) = (\Delta^0 (y - x))_{CA} + \xi_{CA} (y, x), \]
\[ \bar{\Delta}^{AC} (x, y) = (\bar{\Delta}^0 (x - y))^{AC} + \bar{\xi}^{AC} (x, y). \]  \hspace{1cm} (110)

The Fourier transforms of the background field and its conjugate have special form
\[ (\bar{\phi}^0 (q, p))_{DB} = \delta^{(4)} (p + q) \bar{\Phi}_{DB} (p), \quad (\bar{\bar{\phi}^0} (p, q))_{BD} = \delta^{(4)} (p + q) \bar{\Phi}_{BD} (p), \]
\[ (\bar{\Delta}^0 (q, p))_{CA} = \delta^{(4)} (p + q) \bar{\Delta}_{CA} (p), \quad (\bar{\bar{\Delta}^0} (p, q))_{AC} = \delta^{(4)} (p + q) \bar{\Delta}^{AC} (p), \]  \hspace{1cm} (111)
Denote \( \tilde{\varphi}_{DB} (q, p) \), \( \tilde{\varphi}^{BD} (p, q) \) and \( \tilde{\psi}_{CA} (q, p) \), \( \tilde{\xi}^{AC} (p, q) \) the Fourier transforms of the fields \( \varphi_{DB} (y, x) \), \( \varphi^{BD} (x, y) \) and \( \psi_{CA} (y, x) \), \( \xi^{AC} (x, y) \), resp. In the second order with respect to the quantum fluctuations the effective action equals

\[
S_{\text{eff}} [\Phi, \bar{\Phi}] \approx S_{\text{eff}} [\Phi^0, \bar{\Phi}^0] - \frac{1}{2} \int d^4p_1 \int d^4q_1 \int d^4p_2 \int d^4q_2 \delta^{(4)} (p_1 + q_1 - p_2 - q_2) \nonumber
\]

\[
\tilde{\varphi}^{A_1 C_1} (p_1, q_1) V_{C_1 C_2}^{A_2 C_2} (p_1, q_1; p_2, q_2) \tilde{\varphi}^{C_2 A_2} (q_2, p_2) 
+ \int d^4p_1 \int d^4q_1 \int d^4p_2 \int d^4q_2 \nonumber
\]

\[
\tilde{\psi}^{A_1 C_1} (p, q) \frac{\delta W [\Delta^0, \bar{\Delta}]}{\delta \Delta^{A_1 C_1} (p_1, q_1) \delta \Delta^{C_2 A_2} (q_2, p_2)} |_{\Delta^{CA} = (\Delta^0)^{CA}, \bar{\Delta}^{AC} = (\bar{\Delta}^0)^{AC}} = (\bar{\Delta}^0)^{AC} \tilde{\xi}^{AC}_{C_2 A_2} (q_2, p_2). \tag{112}
\]

Summing up the infinite series

\[
\sum_{n=1}^{\infty} \frac{\delta^2 W [\Delta^0, \bar{\Delta}]}{\delta \Delta^{A_1 C_1} (p_1, q_1) \delta \Delta^{C_2 A_2} (q_2, p_2)} |_{\Delta^{CA} = (\Delta^0)^{CA}, \bar{\Delta}^{AC} = (\bar{\Delta}^0)^{AC}} = -\frac{i}{2} \delta^{(4)} (p_1 - p_2) \delta^{(4)} (q_1 - q_2) \tilde{S}^{C_1}_{C_1} (q_1) \tilde{S}^{A_2}_{A_1} (p_1), \tag{113}
\]

where

\[
\tilde{S}^c_{A} (p) = \tilde{S}^c_{A} (p) - \tilde{S}^c_{A} (p) \Delta_{C_1 A_1} (p) S^{A_1}_{A_2} (-p) \Delta^{A_2 C_2} (p) \tilde{S}^c_{C_2} (p) 
+ \tilde{S}^c_{A} (p) \Delta_{C_1 A_1} (p) S^{A_1}_{A_2} (-p) \Delta^{A_2 C_2} (p) \tilde{S}^c_{C_2} (p) \Delta_{C_2 A_4} (p) 
. S^{A_3}_{A_4} (-p) \Delta^{A_4 C_4} (p) \tilde{S}^c_{C_4} (p) - ... \tag{114}
\]

is the two-point Green function of the quark field in the presence of the background diquark one and satisfies the Schwinger-Dyson equation

\[
\tilde{S}^c_{A} (p) = \tilde{S}^c_{A} (p) - \tilde{S}^c_{A} (p) \Delta_{C_1 A_1} (p) S^{A_1}_{A_2} (-p) \Delta^{A_2 C_2} (p) \tilde{S}^c_{C_2} (p) , \tag{115}
\]

or in the matrix form

\[
\frac{1}{S(p)} = \frac{1}{\tilde{S}(p)} + \Delta(p) S^T (-p) \bar{\Delta}(p) , \tag{116}
\]

where \( \Delta(p) \) and \( \bar{\Delta}(p) \) are the matrices with the elements \( \Delta_{CA} (p) \) and \( \bar{\Delta}^{AC} (p) \).

From the expression (112) of the effective action and the relation (113) it follows the relativistic Bethe-Salpeter equation

\[
\tilde{\xi}_{CA} (q, p) = -i \int d^4p' \int d^4q' \delta^{(4)} (p + q - p' - q') V_{CA}^{BD} (p, q; p', q') \tilde{S}_B^{BE} (p') \bar{S}_D^{F} (q') \tilde{\xi}_{FE} (q', p') \tag{117}
\]
containing the two-point Green function of the quark field interacting with the background diquark one. If the later vanishes, then the equation (117) reduces to the equation (109).

The relativistic effective potential \( U^{B_1,A_2}_{A_1,B_2}(p_1,-q_1;-q_2,p_2) \) and \( V^{A_2,B_1}_{B_1,A_1}(q_1,p_1;-p_2,q_2) \) in the bound state equation (85), (95) and (109), (117) are determined by the dynamical mechanism of the effective 4-quark interaction in QCD. In the case of the instanton induced 4-quark coupling they are expressed in terms of the form-factors calculated in the work of Rapp, Schäfer, Shuryak and Velkovsky[31]. For the effective non-local 4-quark interaction generated by the one-gluon exchange mechanism we have

\[
U^{B_1,A_2}_{A_1,B_2}(p_1,-q_1;-q_2,p_2) = 4\pi \sum_I \left\{ \alpha \left[ \frac{(p_1 + q_1)^2}{(p_1 + q_1)^2} \right] \left( \gamma_\mu \otimes \lambda_I \right)^{B_2}_{A_1} \left( \gamma_\mu \otimes \lambda_I \right)^{A_2}_{B_1} - \frac{\alpha}{(p_1 - q_2)^2} \left( \gamma_\mu \otimes \lambda_I \right)^{A_2}_{B_1} \left( \gamma_\mu \otimes \lambda_I \right)^{B_2}_{A_1} \right\} \quad (118)
\]

where \( \alpha (k)^2 \) is the running quark-gluon coupling constant in QCD, \( \lambda_I \) are the Gell-Mann matrices of the color symmetry group \( SU(N_C) \); the flavor indices play no role and are omitted. In reality there do exist the contribution of both above mentioned dynamical mechanisms of the effective 4-quark interactions, and they must lie taken into account simultaneously. Note that in the special case of the direct 4-quark coupling with the interaction Lagrangian (1) or (2) the Bethe-Salpeter equation (5) and (117) reduce to the Nambu-Jona-Lasinio equations (38) and (61) resp., This means that the Nambu-Jona-Lasinio equation is a particular form of the Bethe-Salpeter equation.

4 Bethe-Salpeter Equation for Bipolaritons.

Now we apply the method presented in two preceding Sections to the study of the bound states of two quasiparticles in the condensed matters. As a typical example of the system of two quasiparticles with the complicated dispersion curves and the momentum dependent interaction potential energies we consider the bipolariton-the bound states of two excitonic polaritons in semiconductors. The excitonic polaritons are the elementary excitations whose formation is the consequence of the quantum mutual transition between the photons and the excitons-the exciton-photon mixing. Denote \( \gamma_\alpha^\dagger (p) \) and \( \gamma_\alpha (p) \) the destruction and creation operators of the photon with the momentum \( p \) and the energy \( \omega(p) \) in the polarization state labeled by the index \( \alpha = 1,2 \), where

\[
\omega(p) = \varepsilon p,
\]

\( \varepsilon \) being the background dielectric constant of the semiconductor. Due to the electromagnetic interaction of the photons with the electrons there arises the quantum transition between the photons and the excitons. Applying the second quantization formalism to describe the excitons in the local approximation, we denote \( B_\alpha (p) \) and \( B_\alpha^\dagger (p) \) the destruction and creation operators of the exciton with the momentum \( p \) and the energy \( E(p) \) in the spin state labeled also by the index \( \sigma = 1,2 \) defined such that the quantum mutual transition between the photon in the polarization state with the index \( \sigma_1 \) and the exciton in the spin state with the index \( \sigma_2 \) is allowed if \( \sigma_1 = \sigma_2 \) and forbidden if \( \sigma_1 \neq \sigma_2 \). The Hamiltonian of the photon-exciton system without the exciton-exciton interaction can be written in the form

\[
H_0 = \sum_{\sigma} \sum_p \left\{ \omega(p) \gamma_\sigma^\dagger (p) \gamma_\sigma (p) + E(p) B_\sigma^\dagger (p) B_\sigma (p) \right. \\
+ \left. \frac{g(p)}{2} \left[ \gamma_\sigma^\dagger (p) B_\sigma (p) + B_\sigma^\dagger (p) \gamma_\sigma (p) \right] \right\} \quad (119)
\]
with some function \( g(\mathbf{p}) \). The Coulomb interactions between the electrons and the holes induce some effective exciton-exciton direct coupling with the interaction Hamiltonian

\[
H_{\text{int}} = \sum_{\sigma_1, \sigma_2} \sum_{\mathbf{p}_1, \mathbf{p}_2} \sum_{\mathbf{k}} B_{\sigma_1}^+ (\mathbf{p}_1 + \mathbf{k}) B_{\sigma_2}^+ (\mathbf{p}_2 - \mathbf{k}) \tilde{U}_{\sigma_1, \sigma_2} (\mathbf{k}) B_{\sigma_1} (\mathbf{p}_1) B_{\sigma_2} (\mathbf{p}_2). \tag{120}
\]

The Fourier transform \( \tilde{U}_{\sigma_1, \sigma_2} (\mathbf{k}) \) of the effective interactions potential energies were given, for example, in Refs\[22,23\]. The functional integral of the system of interacting photons and excitons is

\[
Z = \int [D\gamma] [D\gamma^+] [DB] [DB^+] \exp \left\{ i \int dt \sum_{\sigma} \sum_{\mathbf{p}} \left( \gamma_\sigma^+ (\mathbf{p}, t) \left[ i \frac{\partial}{\partial t} - \omega (\mathbf{p}) \right] \gamma_\sigma (\mathbf{p}) + B_\sigma^+ (\mathbf{p}, t) \left[ i \frac{\partial}{\partial t} - E (\mathbf{p}) \right] B_\sigma (\mathbf{p}, t) \right.ight.
\]

\[
+ \frac{g(\mathbf{p})}{2} \left[ \gamma_\sigma^+ (\mathbf{p}, t) B_\sigma (\mathbf{p}, t) + B_\sigma^+ (\mathbf{p}, t) \gamma_\sigma (\mathbf{p}, t) \right] \right\} \exp \left\{ - \frac{i}{2} \sum_{\sigma_1, \sigma_2} \sum_{\mathbf{p}_1, \mathbf{p}_2} \sum_{\mathbf{k}} B_{\sigma_1}^+ (\mathbf{p}_1 + \mathbf{k}, t) B_{\sigma_2}^+ (\mathbf{p}_2 - \mathbf{k}, t) \tilde{U}_{\sigma_1, \sigma_2} (\mathbf{k}) B_{\sigma_1} (\mathbf{p}_1, t) B_{\sigma_2} (\mathbf{p}_2, t) \right\}. \tag{121}
\]

If we neglect the direct exciton-exciton coupling, then the functional integral of the system becomes

\[
Z_0 = \int [D\gamma] [D\gamma^+] [DB] [DB^+] \exp \left\{ i \int dt \sum_{\sigma} \sum_{\mathbf{p}} \left( \gamma_\sigma^+ (\mathbf{p}, t) \left[ i \frac{\partial}{\partial t} - \omega (\mathbf{p}) \right] \gamma_\sigma (\mathbf{p}) + B_\sigma^+ (\mathbf{p}, t) \left[ i \frac{\partial}{\partial t} - E (\mathbf{p}) \right] B_\sigma (\mathbf{p}, t) \right.ight.
\]

\[
+ \frac{g(\mathbf{p})}{2} \left[ \gamma_\sigma^+ (\mathbf{p}, t) B_\sigma (\mathbf{p}, t) + B_\sigma^+ (\mathbf{p}, t) \gamma_\sigma (\mathbf{p}, t) \right] \right\} \tag{122}
\]

In order to describe the bipolaritons we introduce the composite symmetric bi-local field

\[
\Phi_{\sigma_1, \sigma_2} (\mathbf{p}_1, \mathbf{p}_2, t) = \Phi_{\sigma_2, \sigma_1} (\mathbf{p}_2, \mathbf{p}_1, t) \tag{123}
\]

depending on two momenta \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) and being labeled by two indices \( \sigma_1, \sigma_2 \). Denote \( \Phi_{\sigma_1, \sigma_2}^+ (\mathbf{p}_1, \mathbf{p}_2, t) \) its hermitian conjugate and set

\[
Z_0^{\Phi^+} = \int [D\Phi] [D\Phi^+] \exp \left\{ \frac{i}{2} \int dt \sum_{\sigma_1, \sigma_2} \sum_{\mathbf{p}_1, \mathbf{p}_2} \sum_{\mathbf{k}} \Phi_{\sigma_1, \sigma_2}^+ (\mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2 - \mathbf{k}, t) \tilde{U}_{\sigma_1, \sigma_2} (\mathbf{k}) \Phi_{\sigma_1, \sigma_2} (\mathbf{p}_1, \mathbf{p}_2, t) \right\}. \tag{124}
\]

Shifting the functional integration variables

\[
\Phi_{\sigma_1, \sigma_2} (\mathbf{p}_1, \mathbf{p}_2, t) \rightarrow \Phi_{\sigma_1, \sigma_2} (\mathbf{p}_1, \mathbf{p}_2, t) + B_{\sigma_1} (\mathbf{p}_1, t) B_{\sigma_2} (\mathbf{p}_2, t), \tag{125}
\]

\[
\Phi_{\sigma_1, \sigma_2}^+ (\mathbf{p}_1, \mathbf{p}_2, t) \rightarrow \Phi_{\sigma_1, \sigma_2}^+ (\mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2 - \mathbf{k}, t) + B_{\sigma_1}^+ (\mathbf{p}_1 + \mathbf{k}, t) B_{\sigma_2}^+ (\mathbf{p}_2 - \mathbf{k}, t),
\]

we establish the Hubbard-Stratonovich transformation
the elementary fields, we rewrite
\[ Z = \exp \left\{ \frac{-i}{2} \int dt \sum_{\sigma_1 \sigma_2} \sum_{p_1, p_2} \sum_k B_{\sigma_1}^+ (p_1 + k, t) B_{\sigma_2}^+ (p_2 - k, t) \tilde{U}_{\sigma_1 \sigma_2} (k) B_{\sigma_1} (p_1, t) B_{\sigma_2} (p_2, t) \right\} \]

\[ = \frac{1}{Z_0^{\Phi^+}} \int [D\Phi] [D\Phi^+] \exp \left\{ \frac{i}{2} \int dt \sum_{\sigma_1 \sigma_2} \sum_{p_1, p_2} \sum_k \Phi_{\sigma_1 \sigma_2}^+ (p_1 + k, p_2 - k, t) \tilde{U}_{\sigma_1 \sigma_2} (k) \Phi_{\sigma_1 \sigma_2} (p_1, p_2, t) \right\} \]

\[ = \exp \left\{ \frac{i}{2} \int dt \sum_{\sigma_1 \sigma_2} \sum_{p_1, p_2} \sum_k \Delta_{\sigma_1 \sigma_2}^+ (p_1, p_2, t) B_{\sigma_1} (p_1, t) B_{\sigma_2} (p_2, t) \right\} \]

where

\[ \Delta_{\sigma_1 \sigma_2} (p_1, p_2, t) = \sum_k \tilde{U}_{\sigma_1 \sigma_2} (k) \Phi_{\sigma_1 \sigma_2} (p_1 - k, p_2 + k, t), \]

\[ \Delta_{\sigma_1 \sigma_2}^+ (p_1, p_2, t) = \sum_k \Phi_{\sigma_1 \sigma_2}^+ (p_1 - k, p_2 + k, t) \tilde{U}_{\sigma_1 \sigma_2} (-k). \] (127)

Using this relation to transform the last exponential in the r.h.s. of the formula (121), reversing the order of the functional integrations over the elementary fields \( \gamma_\sigma, \gamma_\sigma^+, B_\sigma, B_\sigma^+ \) and the composite fields \( \Phi_{\sigma_1 \sigma_2}, \Phi_{\sigma_1 \sigma_2}^+ \) and then integrating out over the elementary fields, we rewrite \( Z \) in the form of a functional integral over the bi-local composite field \( \Phi_{\sigma_1 \sigma_2} \) and its conjugate \( \Phi_{\sigma_1 \sigma_2}^+ \)

\[ Z = \frac{Z_0}{Z_0^{\Phi^+}} \int [D\Phi] [D\Phi^+] \exp \left\{ iS_{\text{eff}} [\Phi, \Phi^+] \right\} \] (128)

with the effective action of the bi-local composite field

\[ S_{\text{eff}} [\Phi, \Phi^+] = \frac{1}{2} \int dt \sum_{\sigma_1 \sigma_2} \sum_{p_1, p_2} \sum_k \Phi_{\sigma_1 \sigma_2}^+ (p_1 + k, p_2 - k, t) \tilde{U}_{\sigma_1 \sigma_2} (k) \Phi_{\sigma_1 \sigma_2} (p_1, p_2, t) + W [\Delta, \Delta^+] \]

where the functional \( W [\Delta, \Delta^+] \) is determined by the formula

\[ \exp \left\{ iW [\Delta, \Delta^+] \right\} = \frac{1}{Z_0} \int [D\gamma] [D\gamma^+] [DB] [DB^+] \exp \left\{ i \int dt \sum_{\sigma} \sum_k \left( \gamma_\sigma (p, t) \left[ i \frac{\partial}{\partial t} - \omega (p) \right] \gamma_\sigma (p) \right) \right. \]

\[ + B_{\sigma}^+ (p, t) \left[ i \frac{\partial}{\partial t} - E (p) \right] B_{\sigma} (p, t) \]

\[ + \frac{g(p)}{2} \left( \gamma_\sigma^+ (p, t) B_{\sigma} (p, t) + B_{\sigma}^+ (p, t) \gamma_\sigma (p, t) \right) \right\} \]

\[ \cdot \exp \left\{ \frac{i}{2} \int dt \sum_{\sigma_1 \sigma_2} \sum_{p_1, p_2} \left[ \Delta_{\sigma_1 \sigma_2}^+ (p_1, p_2, t) B_{\sigma_1} (p_1, t) B_{\sigma_2} (p_2, t) \right. \right. \]

\[ + B_{\sigma_1}^+ (p_1, t) B_{\sigma_2}^+ (p_2, t) \Delta_{\sigma_1 \sigma_2} (p_1, p_2, t) \left. \right\} \]

and can be represented in the form of a functional power series in the fields \( \Delta_{\sigma_1 \sigma_2} (p_1, p_2, t) \) and \( \Delta_{\sigma_1 \sigma_2}^+ (p_1, p_2, t) \).
\[ W \left[ \Delta, \Delta^+ \right] = \sum_{n=1}^{\infty} W^{(2n)} \left[ \Delta, \Delta^+ \right] \]

\[ W^{(2n)} \left[ \Delta, \Delta^+ \right] \] being a homogeneous functional of \( n-th \) order with respect to each type of fields \( \Delta_{\sigma_1 \sigma_2} (p_1, p_2, t) \) and \( \Delta_{\sigma_1 \sigma_2}^+ (p_1, p_2, t) \).

In order to calculate the functional integral in the formula (130) and derive the explicit expressions of \( W^{(2n)} \left[ \Delta, \Delta^+ \right] \) we diagonalize the Hamiltonian \( H_0 \) of the free polaritons by means of the Bogolubov transformation

\[
B(p) = u(p) c_1 \sigma + v(p) c_2 \sigma , \quad \gamma(p) = -v(p) c_1 \sigma + u(p) c_2 \sigma . \quad (132)
\]

It can be shown that with the transformation coefficient \( u(p) \) and \( v(p) \) are determined by the relations

\[
u(p)^2 = \frac{1}{2} \left\{ 1 + \frac{\omega(p) - E(p)}{\left[ \omega(p) - E(p) \right]^2 + g(p)^2} \right\} ,
\]

\[
v(p)^2 = \frac{1}{2} \left\{ 1 - \frac{\omega(p) - E(p)}{\left[ \omega(p) - E(p) \right]^2 + g(p)^2} \right\} , \quad (133)
\]

\[
2uv(p) = g(p) \left[ \frac{1}{\left[ \omega(p) - E(p) \right]^2 + g(p)^2} \right]^{1/2} ,
\]

the Hamiltonian \( H_0 \) becomes

\[
H_0 = \sum_{i=1,2} \sum_{\sigma} \sum_p E_i(p) c_i^{\dagger} \sigma(p) c_i \sigma(p) , \quad (134)
\]

where

\[
E_1(p) = \frac{1}{2} \left\{ \omega(p) + E(p) + \left[ \omega(p) - E(p) \right]^2 + g(p)^2 \right\}^{1/2} ,
\]

\[
E_2(p) = \frac{1}{2} \left\{ \omega(p) + E(p) - \left[ \omega(p) - E(p) \right]^2 + g(p)^2 \right\}^{1/2} . \quad (135)
\]

c_i \sigma(p) and \( c_i^{\dagger} \sigma(p) \) are the destruction and creation operators of the polariton with the momentum \( p \), energy \( E_i(p) \) and polarization \( \sigma \) in the branche \( i \). In terms of the new functional integration variables we have

\[
Z_0 = \int[Dc_i] \left[ Dc_i^{\dagger} \right] \exp \left\{ i \int dt \sum_i \sum_\sigma \sum_p c_i^{\dagger} \sigma(p,t) \left[ \frac{i}{\partial t} - E_i(p) \right] c_i \sigma(p,t) \right\} . \quad (136)
\]

Define the two-point Green functions \( G_i(p, t_1, t_2) \) of the free polaritons in the following manner

\[
\frac{1}{Z_0} \int[Dc_i] \left[ Dc_i^{\dagger} \right] c_{i_1, \sigma_1}^{\dagger}(p_1, t_1) c_{i_2, \sigma_2}^{\dagger}(p_2, t_2) \exp \left\{ i \int dt \sum_i \sum_\sigma \sum_p c_i^{\dagger} \sigma(p,t) \left[ \frac{i}{\partial t} - E_i(p) \right] c_i \sigma(p,t) \right\} \quad (137)
\]

\[= i \delta_{i_1, i_2} \delta_{\sigma_1, \sigma_2} \delta_{p_1, p_2} G_i(p_i, t_1 - t_2) \]
and denote $\tilde{G}_i(p, \omega)$ their Fourier transforms with respect to the time variable

$$G_i(p, t) = \frac{1}{2\pi} \int e^{-i\omega t} \tilde{G}_i(p, \omega) \, d\omega$$ \quad (138)

Instead of the time-dependent bi-local fields $\Phi_{\sigma_1\sigma_2}(p_1, p_2, t)$, $\Phi^+_{\sigma_1\sigma_2}(p_1, p_2, t)$ and $\Delta_{\sigma_1\sigma_2}(p_1, p_2, t)$, $\Delta^+_{\sigma_1\sigma_2}(p_1, p_2, t)$, it is also convenient to work with their Fourier transforms with respect to the time variable $\tilde{\Phi}_{\sigma_1\sigma_2}(p_1, p_2, \omega)$, $\tilde{\Phi}^+_{\sigma_1\sigma_2}(p_1, p_2, \omega)$ and $\tilde{\Delta}_{\sigma_1\sigma_2}(p_1, p_2, \omega)$, $\tilde{\Delta}^+_{\sigma_1\sigma_2}(p_1, p_2, \omega)$

$$\Phi_{\sigma_1\sigma_2}(p_1, p_2, t) = \frac{1}{2\pi} \int e^{-i\omega t} \tilde{\Phi}_{\sigma_1\sigma_2}(p_1, p_2, \omega) \, d\omega,$$

$$\Delta_{\sigma_1\sigma_2}(p_1, p_2, t) = \frac{1}{2\pi} \int e^{-i\omega t} \tilde{\Delta}_{\sigma_1\sigma_2}(p_1, p_2, \omega) \, d\omega. \quad (139)$$

In the second order with respect to the fields $\tilde{\Phi}_{\sigma_1\sigma_2}(p_1, p_2, \omega)$, $\tilde{\Phi}^+_{\sigma_1\sigma_2}(p_1, p_2, \omega)$ and $\tilde{\Delta}_{\sigma_1\sigma_2}(p_1, p_2, \omega)$, $\tilde{\Delta}^+_{\sigma_1\sigma_2}(p_1, p_2, \omega)$ the effective action (129) becomes

$$S_{\text{eff}}[\Phi, \Phi^+] \approx \frac{1}{4\pi} \int d\omega \sum_{\sigma_1\sigma_2} \sum_{p_1, p_2} \tilde{\Delta}^+_{\sigma_1\sigma_2}(p_1, p_2, \omega) \left[ \tilde{\Phi}_{\sigma_1\sigma_2}(p_1, p_2, \omega) - M(p_1, p_2, \omega) \tilde{\Delta}_{\sigma_1\sigma_2}(p_1, p_2, \omega) \right], \quad (140)$$

where

$$M(p, q, \omega) = \frac{i}{2\pi} \int \left[ u(p)^2 \tilde{G}_1(p, \omega - \omega') + v(p)^2 \tilde{G}_2(p, \omega - \omega') \right] \left[ u(q)^2 \tilde{G}_1(q, \omega') + v(q)^2 \tilde{G}_2(q, \omega') \right] d\omega'. \quad (141)$$

Using the formula

$$\tilde{G}_i(p, \omega) = \frac{1}{\omega - E_i(p) + i0} \quad (142)$$

and performing the integration over the variables $\omega$, we obtain

$$M(p, q, \omega) = \frac{u(p)^2 u(q)^2}{\omega - E_1(p) - E_2(q)} + \frac{u(p)^2 v(q)^2}{\omega - E_1(p) - E_2(q)}$$

$$+ \frac{v(p)^2 u(q)^2}{\omega - E_1(p) - E_2(q)} + \frac{v(p)^2 v(q)^2}{\omega - E_2(p) - E_2(q)} \quad (143)$$

From the expression (14) of the effective action and the variational principle

$$\frac{\delta S_{\text{eff}}[\Phi, \Phi^+]}{\delta \tilde{\Delta}_{\sigma_1\sigma_2}(p_1, p_2, \omega)} = 0 \quad (144)$$

it follows the field equation

$$\tilde{\Phi}_{\sigma_1\sigma_2}(p_1, p_2, \omega) - M(p_1, p_2, \omega) \tilde{\Delta}_{\sigma_1\sigma_2}(p_1, p_2, \omega) = 0 \quad (145)$$

or, in another form

$$L(p_1, p_2, \omega) \tilde{\Phi}_{\sigma_1\sigma_2}(p_1, p_2, \omega) - \sum_k \tilde{U}_{\sigma_1\sigma_2}(k) \tilde{\Phi}_{\sigma_1\sigma_2}(p_1 - k, p_2 + k, \omega) = 0 \quad (146)$$

with
\[ L(p_1, p_2, \omega) = \frac{1}{M(p_1, p_2, \omega)} \] (147)

it becomes the Schrödinger equation for the biexcitons

\[ [E_1(p_1) + E_2(p_2)] \tilde{\Phi}_{\sigma_1 \sigma_2}(p_1, p_2, \omega) + \]
\[ + \sum_k \tilde{U}_{\sigma_1 \sigma_2}(k) \tilde{\Phi}_{\sigma_1 \sigma_2}(p_1 - k, p_2 + k, \omega) = \omega \tilde{\Phi}_{\sigma_1 \sigma_2}(p_1, p_2, \omega) \] (148)

In terms of the functions of the space-time variables and the differential operators of the space-time coordinates we can rewrite the Bethe-Salpeter equation (146) in the form

\[ \left[ L \left( -i \frac{\partial}{\partial r_1}, -i \frac{\partial}{\partial r_2}, i \frac{\partial}{\partial t} \right) - U_{\sigma_1 \sigma_2}(r_1 - r_2) \right] \Phi_{\sigma_1 \sigma_2}(r_1, r_2, t) = 0, \] (149)

where \( U_{\sigma_1 \sigma_2}(r_1 - r_2) \) are the effective interaction potential energies between two excitons in the polarization states \( \sigma_1 \) and \( \sigma_2 \).

5 Conclusion and Discussions

In this series of lectures we have presented the universal method based on the application of functional integral technique for the study of the bound states of the systems of two relativistic particles in high energy physics or two quasiparticles with arbitrary dispersion laws in the condensed matters. The Bethe-Salpeter equation for the composite fields describing the corresponding two-body systems were derived. In the special case of the direct local 4-fermion coupling the Bethe-Salpeter equation reduces to the Nambu-Jona-Lasinio equation. In the derivation of these equations we have observed a very interesting phenomenon: there might exist some composite boson fields with non-vanishing vacuum expectation values-the dynamical Higgs fields. The existence of these fields and the spontaneous breaking of the corresponding symmetries in QCD should be studied subsequently.

On the example of the bipolaritons problem we have demonstrated how to apply the functional integral method in order to derive the Bethe-Salpeter equation for the bound states of two quasiparticles with the complicated energy spectra in the condensed matters. The solution of the established bound state equation for the bipolaritons, the derivation of the Bethe-Salpeter equations for other two-body systems in the condensed matters, for example the biphonons, the bimagnons, the electron-phonon or electron-magnon bound states etc., as well as the solution of these equations would be also, the interesting subjects for the subsequent study.

The presented functional integral method can be easily extended for the application to the study of different three-body systems in high energy physics as well as in condensed matters theory. For the definiteness and as an example let us consider the formation of the baryons as bound states of the tree-quark systems - the triquark. As the necessary tool for the application of the functional integral method with the use of the Hubbard-Stratonovich transformation we introduce the six-quark (non-local, in general) coupling induced by the interaction mechanism in QCD (instanton induced, gluon exchange etc.) with the contribution

\[ S_{\text{int}}[\psi, \bar{\psi}] = \frac{1}{6} \int d^4k_1 \int d^4p_1 \int d^4q_1 \int d^4k_2 \int d^4p_2 \int d^4q_2 \]
\[ \delta^{(4)}(k_1 + p_1 + q_1 - k_2 - p_2 - q_2) \bar{\psi}_{A_1}(k_1) \bar{\psi}_{B_1}(p_1) \bar{\psi}_{C_1}(q_1) \]
\[ V_{C_1, B_1, A_1} (q_1, p_1, k_1; k_2, p_2, q_2) \psi_{C_2}(q_2) \bar{\psi}_{B_2}(p_2) \bar{\psi}_{A_2}(k_2) \] (150)
to the effective action reproducing the 6-point Green functions of the interacting quark field. In order to describe the baryons as the triquarks we introduce the tri-local composite fermionic field and its conjugate with the Fourier transforms $\Psi_{CBA} (q, p, k)$ and $\overline{\Psi}^{ABC} (k, p, q)$. We set

$$Z_0^{\Psi, \overline{\Psi}} = \int [D\Psi] [D\overline{\Psi}] \exp \left\{ -\frac{i}{6} \int d^4k_1 \int d^4q_1 \int d^4p_1 \int d^4q_2 \delta^{(4)} \left( k_1 + p_1 + q_1 - k_2 - p_2 - q_2 \right) \overline{\Psi}^{A_1B_1C_1} (k_1, p_1, q_1) \Psi^{A_2B_2C_2} (q_1, p_1, k_1; k_2, p_2, q_2) \overline{\Psi}^{C_2B_2A_2} (q_2, p_2, k_2) \right\} \tag{151}$$

and establish the Hubbard-Stratonovich transformation

$$\exp \left\{ -\frac{i}{6} \int d^4k_1 \int d^4q_1 \int d^4p_1 \int d^4q_2 \delta^{(4)} \left( k_1 + p_1 + q_1 - k_2 - p_2 - q_2 \right) \overline{\Psi}^{A_1} (k_1) \overline{\Psi}^{B_1} (p_1) \overline{\Psi}^{C_1} (q_1) \Psi^{A_2B_2C_2} (q_1, p_1, k_1; k_2, p_2, q_2) \overline{\Psi}^{C_2} (q_2) \overline{\Psi}^{B_2} (p_2) \overline{\Psi}^{A_2} (k_2) \right\} = \frac{1}{Z_0^{\Psi, \overline{\Psi}}} \int [D\Psi] [D\overline{\Psi}] \exp \left\{ -\frac{i}{6} \int d^4k_1 \int d^4q_1 \int d^4p_1 \int d^4q_2 \delta^{(4)} \left( k_1 + p_1 + q_1 - k_2 - p_2 - q_2 \right) \overline{\Psi}^{A_1B_1C_1} (k_1, p_1, q_1) \Psi^{A_2B_2C_2} (q_1, p_1, k_1; k_2, p_2, q_2) \overline{\Psi}^{C_2B_2A_2} (q_2, p_2, k_2) \right\} \tag{152}$$

$$\exp \left\{ -\frac{i}{6} \int d^4k \int d^4p \int d^4q \left[ \overline{\Psi}^{A} (k) \overline{\Psi}^{B} (p) \overline{\Psi}^{C} (q) \frac{\Delta_{CBA}}{\Delta} (q, p, k) + \Delta^{ABC} (k, p, q) \overline{\Psi}^{C} (q) \overline{\Psi}^{B} (p) \overline{\Psi}^{A} (k) \right] \right\},$$

where

$$\Delta_{CBA} (q, p, k) = \int d^4k' \int d^4p' \int d^4q' \delta^{(4)} \left( k + p + q - k' - p' - q' \right) V^{A'B'C'}_{CBA} (q, p, k; k', p', q'),$$

$$\Delta^{ABC} (k, p, q) = \int d^4k' \int d^4p' \int d^4q' \delta^{(4)} \left( k + p + q - k' - p' - q' \right) \overline{\Psi}^{A'B'C'} (k', p', q') V^{ABC}_{C'B'A'} (q', p', k; p, q).$$

Substituting the expression (152) into the r.h.s of the formula (4), reversing the order of the functional integrations and integrating out over the quark field and its conjugate, we rewrite the functional (4) in the new form

$$Z = \frac{Z_0^{\Psi, \overline{\Psi}}}{Z_0^{\Psi, \overline{\Psi}}} \int [D\Psi] [D\overline{\Psi}] \exp \left\{ S_{\text{eff}} [\Psi, \overline{\Psi}] \right\} \tag{154}$$

with the effective action

$$S_{\text{eff}} [\Psi, \overline{\Psi}] = -\frac{1}{6} \int d^4k_1 \int d^4q_1 \int d^4q_1 \int d^4p_1 \int d^4q_2 \delta^{(4)} \left( k_1 + p_1 + q_1 - k_2 - p_2 - q_2 \right) \overline{\Psi}^{A_1B_1C_1} (k_1, p_1, q_1) V^{A_2B_2C_2} (q_1, p_1, k_1; k_2, p_2, q_2) \overline{\Psi}^{C_2B_2A_2} (q_2, p_2, k_2) + W [\Delta, \overline{\Delta}], \tag{155}$$

with $W [\Delta, \overline{\Delta}]$ being some functional
\[ W[\Delta, \bar{\Delta}] \] being a functional power series in the composite fields \( \bar{\Delta}_{CBA}(q, p, k) \) and \( \bar{\Delta}^{ABC}(k, p, q) \). In the lowest (second) order approximation we have

\[
W[\Delta, \bar{\Delta}] \approx W^{(2)}[\Delta, \bar{\Delta}] = \frac{1}{6} \int d^4k \int d^4p \int d^4q \bar{\Delta}^{A_1B_1C_1}(k, p, q) S^{C_2}_{C_1}(q) S^{B_2}_{B_1}(p) S^{A_2}_{A_1}(k) \bar{\Delta}_{C_2B_2A_2}(q, p, k).
\]

In this approximation from the variational principle

\[
\frac{\delta S_{\text{eff}}}{\delta \bar{\Delta}^{ABC}(k, p, q)} = 0
\]

it follows the Bethe-Salpeter equation for the triquarks

\[
\bar{\Delta}_{CBA}(q, p, k) = \int d^4k' \int d^4p' \int d^4q' \delta(k + p + q - k' - p' - q') V^{A'B'C'}_{CBA}(q, p, k; k', p', q') S^{C_2}_C(q') S^{B_2}_B(p') S^{A_2}_A(k') \bar{\Delta}_{C_2B_2A_2}(q', p', k').
\]

In the special case of the local direct 6-fermion coupling of the quark field the potentials \( V^{A'B'C'}_{CBA}(q, p, k; k', p', q') \) do not depend on the momenta \( k, p, q, k', p', q' \) and the Bethe-Salpeter equation (158) reduces to the Nambu-Jona-Lasinion established and discussed in Ref\cite{32}. They would be the basic equations for the theoretical study of the baryon structure in QCD.

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