On the application of optimal control strategies to a generalized SVIR model

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Abstract. In this paper, a deterministic optimal control problem involving a Susceptible - Vaccinated - Infected - Recovered (SVIR) epidemic model is considered. The optimal control problem is characterized using the Pontryagin's maximum principle involving three control strategies namely, social mobilization, screening and sanitation. The derived optimality system is numerically solved using the forward - backward Runge - Kutta fourth order method via the computational software matlab. The numerical simulations depict that each of the control strategy has its significance in minimizing the spread of diseases, but the optimal combination of these controls are more effective in stemming the emergence and spread of an epidemic.

1. Introduction

Optimal control involves deriving a control law for a biological or physical dynamical system with respect to time such that a certain optimality condition is established. It includes a cost functional representing states and control variables, and it is achieved using the Pontryagins maximum (minimum) principle. Optimal control is an important area in infectious disease modelling in analyzing the spread and control of infectious diseases for which either treatment and vaccines or other forms of disease controls are readily available to combat diseases like Influenza, Measles, Cholera, Malaria, Ebola etc. Optimal control applied to epidemiological processes has been an important area where several authors have worked. The work of Lenhart and Workman [5] and Joshi et al. [8] gave a deep insight on the application of optimal control to biological and epidemiological processes. Their work also gave the description of how diseases evolve and transmit as well as the application of medical or healthy measures to curtail and eliminate disease in human and environmental host population. In their work, a forward - backward sweep numerical scheme was developed to solve co-state problems. Optimal control of SIR and SVIR models can also be seen in [1, 2, 3, 4, 6, 9]. In turn, this work considered and extended the work of [7], by imposing three control strategies of social mobilization, screening and sanitation on the model to explore the effect of these controls in minimizing infections together with the cost of implementation. Section 2 presents the optimal control model derivation, characterization and the optimality system, while Section 3 discusses the simulations and graphical view presentation of the optimality system and finally, Section 4 present the conclusion of the work.
2. Optimal Control Model Derivation

The model considered is seen in [7], while the application of controls of social mobilisation $u_1$, screening $u_2$ and sanitation $u_3$ to the deterministic model gives.

\[ \begin{align*}
\dot{S} &= A - \mu S - \tau SI - \rho S + \delta_1 R + \delta_2 V - u_1 S, \\
\dot{V} &= \rho S - \mu V - \delta_2 V - u_3 V, \\
\dot{I} &= \tau SI - (u + \alpha + \gamma) I - \rho I - u_2 I, \\
\dot{R} &= \gamma I - uR - \delta_1 R + u_1 S + u_2 I + u_3 V. \\
\end{align*} \]  

(1)

Together with the initial conditions $S(t) > 0, V(t) > 0, I(t) > 0, R(t) > 0$.

The aim of the application of these controls to the model is to keep at minimum, the number of infected humans and the cost of implementation related to the use of social mobilization, medical screening and sanitation on $[0, T]$. Thus, the objective functional is defined as

\[ J_{\text{minimum}}(u_1,u_2,u_3) = \int_0^T \left( Z_1 I + Z_2 \frac{u_1^2}{2} + Z_3 \frac{u_2^2}{2} + Z_4 \frac{u_3^2}{2} \right) dt \]  

(2)

Where $Z_1 I$ represent the total number of individuals who are infected, taken as a measure of death associated with disease outbreak, while $Z_2, Z_3, Z_4$ are the weight parameters which describe the comparative significance of the three terms in the objective functional, where quantity $Z_2 \frac{u_1^2}{2}$ denote the cost of social mobilization, $Z_3 \frac{u_2^2}{2}$ denote the cost of medical screening and $Z_4 \frac{u_3^2}{2}$ denote the cost of sanitation. In this work, the quadratic cost control is considered which follows a nonlinear representation. Also $u_1, u_2$ and $u_3$ are measurable functions and piece wise continuous, such that the control constraints are given by

\[ U_0 = \{(u_1(t), u_2(t), u_3(t)) | 0 < u_1 < 1, 0 < u_2 < 1, 0 < u_3 < 1, t \in [0, T]\}. \]  

(3)

2.1. Characterization of the Optimal Control Model

An optimal control pair $U_0^* = (u_1^*, u_2^*, u_3^*)$ and the associated model system (1) is given. We seek the minimum value of a Lagragian $L_g$ defined by
The Hamiltonian \((H)\) of the control model is given by

\[
H(S, I, R, V, u_1, u_2, u_3, \lambda_1, \lambda_2, \lambda_3, \lambda_4, t) = L_g(I, u_1, u_2, u_3 + \lambda_3 \dot{S}, \lambda_V \dot{V}, \lambda_I \dot{I}, \lambda_R \dot{R})
\]

where \(\lambda_2, \lambda_V, \lambda_I, \lambda_R\) are adjoint functions to be obtained. The optimal control problem (1) is characterized by employing the Pontryagin's minimum principle to determine the Hamiltonian given by

\[
H(S, I, R, V) = (Z_1 I + Z_2 \frac{u_1^2}{2} + Z_3 \frac{u_2^2}{2} + Z_4 \frac{u_3^2}{2}) + \lambda S(A - \mu S - \tau SI - \rho S + \delta_1 R + \delta_2 V - \mu_1 S) + 
\lambda V(\rho S - \mu V - \delta_2 V - \mu_3 V) + \lambda I(\tau SI - (\mu + \alpha + \gamma) I - r I - \mu_2 I) + \lambda R(\gamma I - \mu R - \delta_1 R + \mu_1 S + \mu_2 I + \mu_3 V).
\]

Differentiating (6) with respect to each of the classes of Hamiltonian, we obtain the following adjoint variables given by

\[
\begin{align*}
\dot{\lambda}_S &= \lambda_s(\mu + \tau I + \rho + \mu_1) - \lambda_I \tau I - \lambda_R u_1 - \lambda_V \rho, \\
\dot{\lambda}_V &= -\lambda_S \delta_2 - \lambda_R \mu + \lambda_V (\mu + \delta_2 + u_3), \\
\dot{\lambda}_I &= -Z_1 - \lambda_S \lambda_S - \lambda_I (\lambda_S - (\mu + \alpha + \gamma + \mu_2)) - (\gamma + u_2), \\
\dot{\lambda}_R &= -\lambda_S \delta_1 - \lambda_R (\mu + \delta_1).
\end{align*}
\]

Subject to the transversality conditions \(\lambda_S(T) = \lambda_I (T) = \lambda_R (T) = \lambda_V (T) = 0\) where

\[
\begin{align*}
u^*_1 &= \min \left\{ \max \left(0, \frac{\lambda_S - \lambda_R}{\lambda_V}, 1\right) \right\}, \\
u^*_2 &= \min \left\{ \max \left(0, \frac{\lambda_I - \lambda_R}{\lambda_V}, 1\right) \right\}, \quad (8) \]
\]

2.2. Derivation of the Optimality System

The resulting optimality system given by the state and adjoint equations yields

\[
\begin{align*}
\dot{S} &= A - \mu S - \tau SI - \rho S + \delta_1 R + \delta_2 V - u_1 S, \\
\dot{V} &= \rho S - \mu V - \delta_2 V - u_3 V, \\
\dot{I} &= \tau SI - (u + \alpha + \gamma) I - r I - u_2 I, \\
\dot{R} &= \gamma I - uR - \delta_1 R + u_4 S + u_2 I + u_3 V, \\
\lambda_S &= \lambda_S (\mu + \tau I + \rho + \mu_1) - \lambda_I \tau I - \lambda_R u_1 - \lambda_V \rho, \\
\lambda_V &= -\lambda_S \delta_2 - \lambda_R \mu + \lambda_V (\mu + \delta_2 + u_3), \\
\lambda_I &= -Z_1 - \lambda_S \lambda_S - \lambda_I (\lambda_S - (\mu + \alpha + \gamma + \mu_2)) - (\gamma + u_2), \\
\lambda_R &= -\lambda_S \delta_1 - \lambda_R (\mu + \delta_1)
\end{align*}
\]

Together with the initial conditions \(S(t) \geq 0, I(t) \geq 0, V(t) \geq 0, R(t) \geq 0\) and transversality conditions \(\lambda_S (T) = \lambda_I (T) = \lambda_R (T) = \lambda_V (T) = 0\).

3. Numerical Simulations and Explanations
The optimality model system (9) is solved using the forward – backward Runge – Kutta fourth order through the computational software matlab, see [5, 8]. Tables 1 and 2 displays the values of variables and parameters involved in the simulations of the control model. Figures 2 - 5 describes the behaviour of the imposed controls on the model system equations. The impact of the absence and presence of control $u_t$ on the sub - population of susceptible individuals $S(t)$ in Figure 1 describes the behavioural change in susceptible individuals by being more aware of the presence of epidemic menace through social mobilization (media coverage and campaign), leading to combating and eliminating diseases in human host population. Figure 2 depict the impact of absence and presence of screening control ($u_2 = 0, u_2 \neq 0$). As time increases in the absence of control $u_2$, infected humans will be on the rise, but as soon as screening control is imposed, infected individuals with level of infection will be ascertained to

| Table 1. Variable Descriptions |
|--------------------------------|
| **Descriptions** | **Variables** | **Values** |
| Susceptible humans | $S_h$ | 100 |
| Vaccinated humans | $I_{hl}$ | 50 |
| Infected humans | $I_{hh}$ | 70 |
| Recovered humans | $T_h$ | 55 |

| Table 2. Parameter Descriptions |
|--------------------------------|
| **Descriptions** | **Parameters** | **Values** | **Sources** |
| Human recruitment rate | $A$ | 100 | [7] |
| Natural Death rate | $\mu$ | 0.112 | [3] |
| Vaccinated rate of Susceptibles | $\rho$ | 0.21 | [7] |
| Rate of loss of immunity | $\delta_1$ | 0.21 | [7] |
| Rate of waning in vaccinated individual | $\delta_2$ | 0.34 | [7] |
| Rate of infection induced mortality | $\alpha$ | 0.0125 | [3] |
| Rate of recovery from infection | $\gamma$ | 0.11 | [3] |
| Treatment function | $r$ | 0.016 | [7] |
Figure 2. Effect of the absence and presence of control $u_1 = 0, u_1 \neq 0$ on $S(t)$

Figure 3. Effect of the absence and presence of control $u_2 = 0, u_2 \neq 0$ on $I(t)$

Figure 4. Impact of the absence and presence of the combined controls $u_1, u_2, u_3$.

Figure 5. Impact of the absence and presence of control $u_3 = 0, u_3 \neq 0$ on $V(t)$

Know the type of treatment to be applied. Figure 5 depict the impact of absence and presence of sanitation control ($u_3 = 0, u_3 \neq 0$). As time increases, vaccination wanes in individuals, but effective sanitation of the environment drastically reduces the presence of disease leading to a full-blown epidemic in the human host community. Figure 4 displays the combined effect of the three controls ($u_1, u_2, u_3$). This shows that these controls are effective optimum strategies needed to combat the menace of disease epidemic spread, which remain a public and global health challenge.

4. Conclusion

An optimal control model is derived to explain the impact of controls of social mobilization, screening and sanitation on a generalized SVIR model depicting the spread and the reduction of infection in human host population. The results show that each of the controls have an impact in reducing the spread of disease, but the combination of these three controls are effective control measures needed to be implemented by public health administrators in stemming the emergence and re-emergence of infectious diseases in the human and environmental host population.

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