Generic Gaussian ideals

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Abstract

The content of a polynomial $f(t)$ is the ideal generated by its coefficients. Our aim here is to consider a beautiful formula of Dedekind–Mertens on the content of the product of two polynomials, to explain some of its features from the point of view of Cohen–Macaulay algebras and to apply it to obtain some Noether normalizations of certain toric rings. Furthermore, the structure of the primary decomposition of generic products is given and some extensions to joins of toric rings are considered.

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1 Introduction

If $R$ is a commutative ring and $f = f(t) \in R[t]$ is a polynomial, say $f = a_0 + \cdots + a_m t^m$, the content of $f$ is the $R$-ideal $(a_0, \ldots, a_m)$. It is denoted by $c(f)$. Given another polynomial $g$, the Gaussian ideal of $f$ and $g$ is the $R$-ideal

$$G(f, g) = c(fg).$$

(1)

This ideal bears a close relationship to the ideal $c(f)c(g)$, one aspect of which is expressed in the classical lemma of Gauss: If $R$ is a PID then

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\[ c(fg) = c(f)c(g). \] (2)

In fact, if \( R \) is a domain, then this equality holds for arbitrary pairs of polynomials if and only if \( R \) is a Prüfer domain. In general, these two ideals are very different but one aspect of their relationship is given by (see [10])

\[ c(fg)c(g)^m = c(f)c(g)^{m+1}. \] (3)

One of our purposes in this note is to ‘explain’ this formula, originally due to Dedekind–Mertens, in terms of the theory of Cohen–Macaulay rings, and to consider some extensions of it. More precisely, we study the ideal \( G(f, g) \) in the case when \( f \) and \( g \) are generic polynomials. It turns out that several aspects of the theory of Cohen–Macaulay rings—e.g., \( a \)-invariants and linkage theory—show up very naturally when we closely examine \( G(f, g) \).

One path to our analysis and its applications to Noether normalizations of some semigroup rings starts by multiplying both sides of (3) by \( c(f)^m \); we obtain

\[ c(fg)[c(f)c(g)]^m = c(f)c(g)[c(f)c(g)]^m. \] (4)

It is this ‘decayed’ content formula that will be the focus of our observations. One result (namely, Theorem 1) will show that (4) is sharp in terms of the exponent \( m = \deg f \) (and, therefore, (3) as well). It will be the outgrow of looking for Noether normalizations of certain rings generated by monomials and basic facts of the theory of Cohen–Macaulay rings. In particular (4) is shown to be a direct consequence of the lemma of Gauss.

To make this connection, we recall the notion of a reduction of an ideal (see [11]). Let \( R \) be a commutative Noetherian ring and let \( I \) be an ideal. A reduction of \( I \) is an ideal \( J \subset I \) such that, for some non-negative integer \( r \), the equality \( I^{r+1} = JI^r \) holds. The smallest such integer is the reduction number \( r_J(I) \) of \( I \) relative to \( J \). Thus (4) says that \( J = c(fg) \) is a reduction for \( I = c(f)c(g) \), and that the reduction number is at most \( \min \{ \deg f, \deg g \} \).

One of the advantages of reductions is that they contain much of the information carried by \( I \) but often with great deal fewer generators. We indicate how this may come about, with the notion of minimal reduction. Let \( (R, \mathfrak{m}) \) be a Noetherian local ring and let \( I \) be an ideal (or a homogeneous ideal of a graded ring). The special fiber of the Rees algebra \( R[It] \) is the ring

\[ \mathcal{F}(I) = R[It] \otimes_R R/\mathfrak{m}. \]

\[ ^4 \text{See also [12, Section 3].} \]
Its Krull dimension is called the analytic spread of \( I \), and is denoted \( \ell(I) \).

If \( R/\mathfrak{m} \) is an infinite field, minimal reductions of \( I \) arise from the standard Noether normalizations of the graded algebra \( \mathcal{F}(I) \). The number of minimal generators of such reductions is \( \ell(I) \). Let

\[
A = k[z_1, \ldots, z_{\ell}] \hookrightarrow \mathcal{F}(I),
\]

where \( \ell = \ell(I) \), be a Noether normalization with the \( z_j \)'s chosen in degree 1. Let further \( b_1, \ldots, b_s \) be a minimal set of homogeneous module generators of \( \mathcal{F}(I) \) over the algebra \( A \)

\[
\mathcal{F}(I) = \sum_{1 \leq q \leq s} A b_q.
\]

If \( J = (y_1, \ldots, y_{\ell}) \), where \( y_i \) is a lift in \( R \) of \( z_i \), it is easy to see that \( J \) is a reduction of \( I \) and \( r_J(I) = \sup \{ \deg b_q \} \). In the case that the algebra \( \mathcal{F}(I) \) is Cohen–Macaulay, \( \mathcal{F}(I) \) is a free module over \( A \) so that \( r_J(I) \) can be read off its Hilbert–Poincaré series.

We shall now outline our results. In Section 2, we relate the exponent in the Dedekind–Mertens' formula directly to the \( a \)-invariant of the Segre product of two rings of polynomials (Theorem 1). The application of the formula to Noether normalization is also pointed out in [5]. After remarks on Gaussian ideals defined through algebras which are not polynomial rings in Section 3, we give in Section 4 the primary decomposition of the Gaussian ideal defined by two generic polynomials. The components have the pleasing property that they all are Gorenstein ideals (Theorem 6). In the final Section, we study the normality of algebras associated to graphs; that includes the toric algebras connected directly to (3). There are some natural Noether normalizations for some of these extensions but not the most general ones.

2 Graphs and determinantal ideals

If \( G \) is a graph with vertices labelled by \( x_0, \ldots, x_m \), its monomial subring \( k[G] \) is the subring of \( k[x_0, \ldots, x_m] \) generated by all monomials \( x_i x_j \) where \((x_i, x_j) \) is an edge of \( G \). In parallel, there exists another algebra attached to \( G \), defined by the ideal of \( k[x_0, \ldots, x_m] \) generated by those monomials (see [15]). In general, it is difficult to find Noether normalizations of any of these two families of algebras.

The following ‘explains’ (4) at the same time that solves the question of
Noether normalizations\(^5\) for maximal bipartite graphs. It would be nice to find explicit normalizations for other classes of graphs.

**Theorem 1** Let \(X = \{x_0, \ldots, x_m\}\) and \(Y = \{y_0, \ldots, y_n\}\) be distinct sets of indeterminates and let

\[
f = \sum_{i=0}^{m} x_i t^i \quad \text{and} \quad g = \sum_{j=0}^{n} y_j t^j
\]

be the corresponding generic polynomials over a field \(k\). Set \(R = k[X, Y]\), \(I = c(f)c(g)\), and \(J = c(fg)\) and suppose \(m \leq n\). Then

(a) \(J\) is a minimal reduction of \(I\), \(\ell(I) = m + n + 1\), and \(r_J(I) = m\).

(b) The polynomials

\[
h_q = \sum_{i+j=q} x_i y_j
\]

are algebraically independent and \(k[h_q's]\) is a Noether normalization of \(k[x_i y_j's]\).

In particular, the factor \(c(f)^m\) in the content formula (3) is sharp.

**PROOF.** We note that the ideal \(I = (x_i y_j's)\) is the edge ideal associated to the graph \(G\) which is the join of two discrete graphs, one with \(m + 1\) vertices and another with \(n + 1\) vertices; \(G\) is, therefore, bipartite.

Since \(J\) is already a reduction of \(I\) by (4), we may assume that \(k\) is an infinite field. On the other hand, as \(I\) is generated by homogeneous polynomials of the same degree, \(\mathcal{F}(I) \simeq k[x_i y_j's] = k[G]\) (see [14]). Let \(Q_{ij}, 0 \leq i \leq m, 0 \leq j \leq n\) be distinct indeterminates and map

\[
\psi : k[Q_{ij}'s] \rightarrow k[x_i y_j's], \quad \psi(Q_{ij}) = x_i y_j.
\]

We claim that the kernel of \(\psi\) is generated by the \(2 \times 2\) minors of a generic \((m + 1) \times (n + 1)\) matrix. Indeed let \(Q = (Q_{ij})\). It is clear that the ideal \(I_2(Q)\), generated by the \(2 \times 2\) minors of \(Q\), is contained in \(Q = \ker(\psi)\). On the other hand, since the graph is bipartite, \(\dim(k[G]) = m + n + 1\) (see [14]) and, therefore,

\[
\text{height}(Q) = (m + 1)(n + 1) - (m + n + 1) = mn = \text{height}(I_2(Q)),
\]

\(^5\) After a first draft of this note, we have found that [5, Part 0] already points out this Noether normalization. In addition, it has a delightful historical account of (3). Our contribution on this point is to explain the meaning of the exponent.
the latter by the classical formula for determinantal ideals (see [3, Theorem 2.5]). Since they are both prime ideals, we have $I_2(Q) = Q$.

To complete the proof we note that the $a$-invariant of $k[Q_{ij}]/I_2(Q)$ is $-n - 1$ according to [1], and therefore the reduction number of $\mathcal{F}(I)$ is $(m + n + 1) - n - 1 = m$. □

**Remark 2** Another approach to the computation of the $a$-invariant is through the theory of Segre products, and then appealing directly to [6]. The Cohen–Macaulayness and Gorensteiness of algebras that include $k[x,y_j]$’s, has been dealt with in great detail already in [2].

### 3 Generalized contents

Let $R$ be a commutative ring and let $A$ be an $R$-algebra which is free as an $R$-module. Let $\{e_i\}$ be an $R$-basis with attached structure constants $c_{ijk}$. Given an element $f \in A$, define now $c(f)$ as the $R$-ideal generated by the coefficients of the expression of $f$ as a linear combination of the $e_i$. This ideal is independent of the choice of basis being the usual order ideal of an element of a free module.

We would like to know which condition on the $c_{ijk}$’s implies that $c(fg)$ is a reduction of $c(f)c(g)$. Here is one instance

**Proposition 3** Let $A$ be an algebra which is a free module over the integral domain $R$. Suppose $A$ has an $R$-basis indexed by a well-ordered monoid. If for each pair of indices $i, j \sum_k R c_{ijk} = R$, then for any two elements $f, g \in A$, $c(fg)$ is a reduction of $c(f)c(g)$.

**PROOF.** We may replace $R$ by one of its valuation overrings $V$ (see [16, p. 350]). It will then suffice to show that $c(f)c(g)V = c(fg)V$.

The assertion will follow from

**Lemma 4 (Gauss Lemma)** Let $A$ be an algebra as above and let $f$ and $g$ be two unimodular elements (i.e., $c(f) = c(g) = R$). Then $fg$ is unimodular.

**Remark 5** The condition on the well-ordering of the index set of the basis is too restrictive, although it can be used for bases change (for instance, even in the case of $R[t]$ one could use other bases than $\{t^n, n \geq 0\}$, with a compatible ordering). More precisely, once Gauss Lemma holds for a basis it will hold
for any other bases: all that requires is that for each prime \( p \) of \( R \) the fiber \( A \otimes_R k(p) \) is an integral domain.

4 Primary decomposition

The generic form of the ideal \( c(fg) \) has an interesting primary decomposition.

**Theorem 6** Let \( R \) be a Noetherian integral domain and let

\[
\begin{align*}
  f &= x_0 + x_1 t + \cdots + x_m t^m & \text{and} & \quad g &= y_0 + y_1 t + \cdots + y_n t^n,
\end{align*}
\]

be generic polynomials of degrees \( m \) and \( n \) over \( R \). The Gaussian ideal \( G(f, g) = c(fg) \) has a primary decomposition

\[
  c(fg) = c(f) \cap c(g) \cap [c(fg) + c(f)^{n+1} + c(g)^{m+1}]. \tag{5}
\]

Furthermore, if \( R \) is a Gorenstein ring then

\[
  L(f, g) = c(fg) + c(f)^{n+1} + c(g)^{m+1} \tag{6}
\]

is a Gorenstein ideal.

**PROOF.** The primary decomposition is easy to verify

\[
\begin{align*}
  c(f) \cap c(g) \cap [c(fg) + c(f)^{n+1} + c(g)^{m+1}] &= \\
  c(f) \cap [c(g) \cap [c(fg) + c(f)^{n+1} + c(g)^{m+1}]] &= \\
  c(f) \cap [c(fg) + c(g) \cap c(f)^{n+1} + c(g)^{m+1}] &= \\
  c(f) \cap [c(fg) + c(g)^{m+1}] &= \\
  c(fg) + c(f) \cap c(g)^{m+1} = c(fg),
\end{align*}
\]

where in the third and fourth equalities we used (3).

To prove that \( L(f, g) = c(fg) + c(f)^{n+1} + c(g)^{m+1} \) is Gorenstein, we show that it is a proper specialization of the Gorenstein ideal described in [8, Example 3.4].

The building blocks of this ideal are a sequence \( X = (X_1, \ldots, X_r) \) and a \( r \times s \) matrix \( \varphi \), \( s \geq r \). For the generic sequence and matrix define

\[
  J = (X \cdot \varphi) + (X)^{s-r+1} + I_s(\varphi),
\]
where \((X \cdot \varphi)\) denotes the ideal generated by the entries of the product of the sequence by the matrix, and \(I_r(\varphi)\) is the ideal generated by the minors of order \(r\) of the matrix \(\varphi\). In [8] it is shown that \(J\) is a Gorenstein ideal of codimension \(s + 1\).

In our case,

\[
X = (x_0, x_1, \ldots, x_m)
\]

and \(\varphi\) is the \((m + 1) \times (m + n + 1)\) matrix

\[
\varphi = \begin{bmatrix}
y_0 & y_1 & y_2 & \cdots & y_n & 0 & \cdots & 0 \\
0 & y_0 & y_1 & \cdots & y_{n-1} & y_n & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & y_0 & y_1 & y_2 & \cdots & y_n
\end{bmatrix}.
\]

Now we note that

\[
(X \cdot \varphi) = c(fg),
\]

\[
(X)^{s-r+1} = c(f)^{n+1},
\]

\[
I_r(\varphi) = c(g)^{m+1},
\]

and the codimension of \(L(f, g)\) is \(s + 1 = m + n + 2\). This means that \(L(f, g)\) is a proper specialization of \(J\) and is therefore Gorenstein as well. \(\square\)

It is natural to define the Gaussian ideal associated to any finite set of polynomials. In the generic case, these ideals share similar properties to \(G(f, g)\). Let us consider the case of 3 polynomials, where an open question arises.

**Theorem 7** Let \(X = \{x_0, \ldots, x_m\}\), \(Y = \{y_0, \ldots, y_n\}\), and \(Z = \{z_0, \ldots, z_p\}\) be 3 sets of indeterminates. Defining the polynomials

\[
f = \sum_{i=0}^{m} x_i t^i, \quad g = \sum_{j=0}^{n} y_j t^j, \quad \text{and} \quad h = \sum_{k=0}^{p} z_k t^k,
\]

one has that the primary decomposition of \(c(fgh)\) is given by

\[
c(fgh) = c(f) \cap c(g) \cap c(h) \cap L(f, g) \cap L(f, h) \cap L(g, h) \cap L(f, g, h), \quad (7)
\]

where
\[ L(f, g) = c(fg) + c(f)^{n+1} + c(g)^{m+1}, \]
\[ L(f, h) = c(fh) + c(f)^{p+1} + c(h)^{m+1}, \]
\[ L(g, h) = c(gh) + c(g)^{p+1} + c(h)^{n+1}, \]
\[ L(f, g, h) = c(fgh) + c(fg)^{p+1} + c(fh)^{n+1} + c(gh)^{m+1} + c(f)^{n+p+1} + c(g)^{m+p+1} + c(h)^{m+n+1}. \]

**PROOF.** The proof follows from a repeated use of Theorem 6 and the Dedekind–Mertens formula. Indeed, one easily verifies that

\[
c(f) \cap c(g) \cap c(h) \cap L(f, g) \cap L(f, h) \cap L(g, h) \cap L(f, g, h) =
\]
\[
= c(fg) \cap c(fh) \cap c(gh) \cap L(f, g, h)
= c(fg) \cap c(fh) \cap [c(fgh) + c(gh) \cap c(fg)^{p+1} + c(gh) \cap c(fh)^{n+1} + c(gh)^{m+1} + c(gh) \cap c(f)^{n+p+1} + c(gh) \cap c(g)^{m+p+1} + c(gh) \cap c(h)^{m+n+1}]
= c(fg) \cap c(fh) \cap [c(fgh) + c(fh) \cap c(gh) \cap c(fg)^{p+1} + c(gh) \cap c(fh)^{n+1} + c(gh)^{m+1} + c(fgh) \cap c(f)^{n+p} + c(gh) \cap c(g)^{m+p+1} + c(gh) \cap c(h)^{m+n+1}]
= c(fg) \cap [c(fgh) + c(fh) \cap c(gh) \cap c(fg)^{p+1} + c(gh) \cap c(fh)^{n+1} + c(fh) \cap c(gh)^{m+1} + c(gh) \cap c(fgh)c(g)^{m+p} + c(fh) \cap c(gh) \cap c(h)^{m+n+1}]
= c(fgh) + c(fh) \cap c(gh) \cap c(fg)^{p+1} + c(fg) \cap c(gh) \cap c(fh)^{n+1} + c(fg) \cap c(fh) \cap c(gh)^{m+1} + c(fh) \cap c(gh) \cap c(fgh)c(h)^{m+n}
= c(fgh) + c(h) \cap c(fg)^{p+1} + c(g) \cap c(fh)^{n+1} + c(f) \cap c(gh)^{m+1}
= c(fgh) + c(fgh) \cap c(fg)^{p} + c(fgh) \cap c(fh)^{n} + c(fgh) \cap c(gh)^{m}
= c(fgh),
\]

as claimed. \(\square\)

**Remark 8** Experiments show that the ideals \(L(f, g, h)\) are Gorenstein. Perhaps they can be obtained by specialization of sums of Huneke–Ulrich ideals.
5 Multiproducts and joins

In order to see a different explanation of (4), we extend it to the product of 3 (or more) polynomials, but use the theory of Segre products as a tool.

Let \( X = \{x_0, \ldots, x_m\} \), \( Y = \{y_0, \ldots, y_n\} \), and \( Z = \{z_0, \ldots, z_p\} \) be 3 sets of indeterminates. Defining the polynomials

\[
\begin{align*}
 f &= \sum_{i=0}^{m} x_i t^i, \quad g = \sum_{j=0}^{n} y_j t^j, \quad \text{and} \quad h = \sum_{k=0}^{p} z_k t^k,
\end{align*}
\]

one has that \( J = c(fgh) \) is a reduction of \( I = c(f)c(g)c(h) \) by Gauss Lemma. If \( m \leq n \leq p \), a simple calculation will show that \( \ell(I) = m + n + p + 1 \) and \( r_J(I) \leq m + n \). We now resolve this inequality.

**Proposition 9** Let \( X = \{x_0, \ldots, x_m\} \), \( Y = \{y_0, \ldots, y_n\} \), and \( Z = \{z_0, \ldots, z_p\} \) be sets of distinct indeterminates, let \( R = k[X, Y, Z] \) be a polynomial ring over a field \( k \), and let \( I = (x_iy_jz_k | x_i \in X, y_j \in Y, z_k \in Z) \). Then \( I \) is a normal ideal of \( R \).

**PROOF.** We will show that \( I^q \) is complete for all \( q \geq 1 \). Let \( I^q \) be the integral closure of \( I^q \) and let \( f \in I^q \) be a monomial. We write

\[
 f = x_1^{a_1} \cdots x_r^{a_r} y_1^{b_1} \cdots y_s^{b_s} z_1^{c_1} \cdots z_t^{c_t}.
\]

Since \( f^w \in I^{qw} \) for some \( w > 0 \) we can write

\[
 f^w = x_1^{d_1} \cdots x_r^{d_r} M,
\]

where \( M \) is a monomial whose support is contained in \( Y \cup Z \). We obtain \( w \sum_{i=1}^{r} a_i = \sum_{i=1}^{\lambda} d_i \geq wq \), which implies \( \sum_{i=1}^{r} a_i \geq q \), and a similar argument shows \( \sum_{i=1}^{s} b_i \geq q \) and \( \sum_{i=1}^{t} c_i \geq q \). Therefore \( f \in I^q \). □

Note that by Hochster’s theorem (see [7]), the algebra \( R[I] \) is Cohen–Macaulay. Furthermore, since \( \mathcal{F}(I) = k[x_iy_jz_k | x_i \in X, y_j \in Y, z_k \in Z] \) is a direct summand of \( R[I] \), it is also normal and therefore Cohen–Macaulay by [7]. We may thus more easily compute the reduction number of \( \mathcal{F}(I) \).

**Theorem 10** The reduction number of the ideal \( I \) above is \( m + n \).
PROOF. Since $\mathcal{F}(I)$ is Cohen–Macaulay, its reduction number can also be obtained from the degrees of the generators of its canonical module. But $\mathcal{F}(I)$ is a Segre product of standard Cohen–Macaulay algebras and the canonical module is given by an explicit formula from the canonical modules of the factors (see [6, Theorem 4.3.1]). Entering the data we get $r_J(I) = m + n$. □

Remark 11 Semigroup rings attached to more general bipartite graphs are obtained by deleting some of the generators in $k[x_iy_j's]$. These rings are still normal but we do not know what their reductions are like.

The join of two normal ideals

Definition 12 Let $I$ and $J$ be two monomial ideals of the polynomial rings $k[x_0, \ldots, x_m]$ and $k[y_0, \ldots, y_n]$ respectively. The join of $I$ and $J$ is:

$$I * J = I + J + K; \text{ where } K = (x_iy_j| 0 \leq i \leq m \text{ and } 0 \leq j \leq n).$$

Theorem 13 Let $R = k[x_0, \ldots, x_m]$ and $S = k[y_0, \ldots, y_n]$ be polynomial rings over a field $k$, and let $I$, $J$ be two ideals of $R$ and $S$ respectively. If $I$ and $J$ are normal ideals generated by square-free monomials of the same degree $t \geq 2$ then their join $I * J$ is normal.

PROOF. Set $X = \{x_0, \ldots, x_m\}$, $Y = \{y_0, \ldots, y_n\}$ and $L = I + J + K$, where $K = (X)(Y)$. By induction on $p$ we will show that $L_a^p = L^p$ for all $p \geq 1$, where $L_a^p$ denotes the integral closure of $L^p$. If $p = 1$ then $L$ is a radical ideal (see [4, Prop. 1]), hence $L$ is integrally closed. Assume $L_a^i = L^i$ for $i < p$ and $p \geq 2$. Using the results of [9] we have

$$L_a^p = (\{z | z \text{ is a monomial in } k[X,Y] \text{ and } z^q \in L_q \text{ for some } q \geq 1\}).$$

Let $z$ be a monomial in $L_a^p$, then $z^q \in L_q$, $q > 0$. Let us show $z \in L^p$. Since $L_a^p \subseteq L_a^{p-1} = L^{p-1}$ we can write

$$z = Mh_1 \cdots h_s g_1 \cdots g_r f_1 \cdots f_{p-r-s-1},$$

where $M$ is a monomial, the $h_i$’s are monomials of degree two in $K$, the $g_i$’s and $f_i$’s are degree $t$ monomials in $J$ and $I$ respectively. Likewise we can write

$$z^q = Nh'_1 \cdots h'_{s_1} g'_1 \cdots g'_{r_1} f'_1 \cdots f'_{p-r-1-s_1},$$

where $N$ is a monomial, $\deg(h'_i) = 2$ and $h'_i$ is a monomial in $K$ for all $i$, $g'_i$
and $f'_j$ are degree $t$ monomials in $J$ and $I$ respectively for all $i, j$. From the last two equalities we have

$$z^q = M^q(h_1 \cdots h_s)^q(g_1 \cdots g_r)^q(f_1 \cdots f_{p-r-s-1})^q$$
$$= Nh'_1 \cdots h'_{s_1} g'_1 \cdots g'_{s_2} f'_1 \cdots f'_{q_3 - r_1 - s_1}.$$  

From (8) one readily derives the inequality

$$(qs - s_1)(t - 2) + qt \leq q \deg(M). \tag{9}$$

We may assume $M = x^\alpha$ or $M = y^a$, otherwise $z \in L^p$. By symmetry we may assume $M = x^\alpha$, where $\alpha \geq 0$.

(a) If $t \geq 3$, then $r = 0$ or $p - r - s - 1 = 0$, otherwise $g_1 f_1 \in L^3$ and hence $z \in L^p$. First we treat the case $r = 0$. By taking degrees in (8) w.r.t. the variables $y_0, \ldots, y_n$ one has $s_1 + tr_1 \leq qs$, which together with (9) yields $\deg M \geq t$. Let $z_1$ be the result of evaluating $z$ at $y_i = 1$. From (8) we derive $z_1^q \in I^{qp - s_1 - r_1} \subseteq I^{q(p-3)}$, and $z_1 \in I_a^{p-s} = I^{p-s}$. We may write $z = y^\beta z_1$ and $z_1 = x^\beta w$, where $\deg(y^\beta) = s$ and $w$ is a monomial in $I^{p-s}$ of degree $t(p-s)$. Since $\deg(z_1) = \deg(M) + s + t(p-s - 1)$ we obtain $\deg(z_1) \geq s + t(p-s)$, hence $\deg(x^\beta) \geq s$. Altogether we derive $z = y^\beta x^\beta w \in L^p$. Next we consider the case two $r = p - s - 1 \geq 1$, observe that $\deg(M) \leq 1$, otherwise $z \in L^p$. Therefore either $z = h_1 \cdots h_s g_1 \cdots g_r$, or we may rewrite $z = y^\beta h_1 \cdots h_s h_{s+1} g_1 \cdots g_{r-1}$, where $\deg(h_{s+1}) = 2$ and $h_{s+1} \in K$, interchanging the $x_i$ and $y_i$ variables we may apply the arguments above to conclude $z \in L^p$.

(b) Assume $t = 2$. Using $z^q \in L^{qp}$ one rapidly obtains $\deg(M) \geq 2$, hence we may assume $r = 0$ (otherwise $z \in L^p$) and the arguments of case (a) can be applied to conclude $z \in L^p$. □

The following Corollary generalizes the normality assertion of [13, Theorem 4.8(v)].

**Corollary 14** Let $X = \{x_0, \ldots, x_m\}$ and $\{y_0, \ldots, y_n\}$ be two disjoint sets of indeterminates over a field $k$. Let $I$ be a normal ideal of $k[X]$ generated by square free monomials of degree $t$ and let $L = I + K$, where $K = (X)(Y)$. Then $L$ is a normal ideal.

**PROOF.** Proceed as in the proof of Theorem 13 and notice that in this case $r = p - s - 1$. □
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References

[1] W. Bruns and J. Herzog, On the computation of $a$-invariants, Manuscripta Math. 77 (1992), 201–213.
[2] W. Bruns, A. Simis, and N. V. Trung, Blow-up of straightening-closed ideals in ordinal Hodge algebras, Trans. Amer. Math. Soc. 326 (1991), 507–528.
[3] W. Bruns and U. Vetter, Determinantal Rings, Lecture Notes in Mathematics 1327, Springer-Verlag, Berlin–Heidelberg–New York, 1988.
[4] J. A. Eagon and M. Hochster, $R$-sequences and indeterminates, Quart. J. Math. Oxford 25 (1974), 61–71.
[5] H. Edwards, Divisor Theory, Birkhäuser, Boston, 1990.
[6] S. Goto and K. Watanabe, On graded rings, I, J. Math. Soc. Japan 30 (1978), 179–213.
[7] M. Hochster, Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes, Ann. of Math. 96 (1972), 318–337.
[8] C. Huneke and B. Ulrich, Residual intersections, J. reine angew. Math. 390 (1988) 1–20.
[9] G. Kempf, F. Knudsen, D. Mumford, and B. Saint–Donat, Toroidal Embeddings I, Lecture Notes in Mathematics 339, Springer-Verlag, Berlin–Heidelberg–New York, 1973.
[10] D. G. Northcott, A generalization of a theorem on the contents of polynomials, Proc. Camb. Phil. Soc. 55 (1959), 282–288.
[11] D. G. Northcott and D. Rees, Reductions of ideals in local rings, Proc. Camb. Phil. Soc. 50 (1954), 145–158.
[12] L. J. Ratliff and D. E. Rush, Two notes on reduction of ideals, Indiana Univ. Math. J. 27 (1978), 929–934.
[13] A. Simis, Topics in Rees algebras of special ideals, in Commutative Algebra, Proceedings, Salvador 1988 (W. Bruns and A. Simis, Eds.), Lecture Notes in Mathematics 1430, Springer-Verlag, Berlin–Heidelberg–New York, 1990, 98–114.
[14] A. Simis, W. V. Vasconcelos and R. Villarreal, On the ideal theory of graphs, J. Algebra 167 (1994), 389–416.

[15] R. Villarreal, Cohen–Macaulay graphs, Manuscripta Math. 66 (1990), 277–293.

[16] O. Zariski and P. Samuel, *Commutative Algebra*, Vol. II, Van Nostrand, Princeton, 1960.