Abstract: The problem of providing complete presentations of reduction algebras associated to a pair of Lie algebras \((\mathfrak{g}, \mathfrak{g})\) has previously been considered by Khoroshkin and Ogievetsky in the case of the diagonal reduction algebra for \(\mathfrak{gl}(n)\). In this paper, we consider the diagonal reduction algebra of the pair of Lie superalgebras \((\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2), \mathfrak{osp}(1|2))\) as a double coset space having an associative diamond product and give a complete presentation in terms of generators and relations. We also provide a PBW basis for this reduction algebra along with Casimir-like elements and a subgroup of automorphisms.

1. Introduction

Zhelobenko [Zhe94] developed a localized version of Mickelsson’s step algebra [Mic73] in considering the representation theory of reductive Lie algebras. An overview of the application of Mickelsson-Zhelobenko algebras in determining branching rules for classical Lie algebras and bases of Gelfand-Tsetlin type is found in [Mol06]. An important generalization made by Zhelobenko is a representation-free determination of lowering and raising operators associated to a Lie algebra \(\mathfrak{g}\) (reductive in a Lie algebra \(\mathfrak{g}\)) and an associative algebra \(U\) containing the universal enveloping algebra \(U(\mathfrak{g})\), with the operators satisfying certain dynamical relations. This algebra of lowering and raising operators provides a foundation for describing dynamical Weyl groups, as mentioned in [EV02], and yields an alternative route to the developments of [TV00]. Recent directions of applications can be seen in harmonic analysis, such as in [DER17], where the algebra is called the transvector algebra. These class of algebras are also known as symmetry algebras due to their importance in understanding symmetries of extremal systems [Zhe97] key to mathematical physics, including solutions to the usual Laplace operator, Dirac equation, and Maxwell equations.

Moreover, the algebras themselves have been an object of study with much progress made by Khoroshkin, Ogievetsky, et al. [KO08; KO14]. We follow Khoroshkin and Ogievetsky’s naming convention and use the term reduction algebra over the aforementioned choices. The general construction of reduction algebras extends to super, quantum, and affine cases [AM15; MM14; van75; Zhe89] and the embedding of the reductive algebra \(\mathfrak{g}\) into the larger associative algebra \(U\) gives rise to types of reduction algebras. In [KO17], a main result is the complete presentation of generators and relations for the diagonal reduction algebra for \(\mathfrak{gl}(n)\). Determining complete presentations of diagonal reduction superalgebras connected to Lie superalgebras is an open problem in the study of associative superalgebras and super representation theory. Particularly, we address the diagonal reduction algebra associated with the first Lie superalgebra in the \(B(m, n)\) series, [Kac77] that being \(\mathfrak{osp}(1|2)\). Our main result is a complete presentation of the diagonal reduction algebra \(Z(\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2), \mathfrak{osp}(1|2))\).
In Section 2, we establish the constructions of [Zhe89] and [AST73] in defining the reduction algebra $Z(\mathfrak{g}, \mathfrak{g})$ and extremal projector associated with a Lie superalgebra $\mathfrak{g}$, where $\mathfrak{g}$ is reductive in $\mathfrak{g}$.

Section 3 recalls descriptions of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ before we present the diagonal reduction algebra for $\mathfrak{osp}(1|2)$, a PBW basis theorem with respect to the diamond product of [Ko08], and an explicit realization of $Z(\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2), \mathfrak{osp}(1|2))$ in terms of generators and relations.

Section 4 concludes with applications, including a determination of Casimir-like elements and an infinite subgroup of the group of automorphisms of $Z(\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2), \mathfrak{osp}(1|2))$.

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2. Reduction algebras for Lie superalgebras

In this section, we model the constructions of [Zhe89] and offer a categorical perspective of the extremal projector of [AST73] and related works.

2.1. Reduction algebra. Let $\mathfrak{g}$ (Fraktur G) be a Lie superalgebra and $\mathfrak{g} \subset \mathfrak{g}$ a Lie subsuperalgebra reductive in $\mathfrak{g}$ with a triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$. Let $D$ be a multiplicative subset of $U(\mathfrak{h}) \setminus \{0\}$, and let $U = D^{-1}U(\mathfrak{g}) = D^{-1}U(\mathfrak{h}) \otimes_{U(\mathfrak{h})} U(\mathfrak{g})$ be the localization of $U(\mathfrak{g})$ at the set $D$. Let $I = U\mathfrak{g}_+$ be the left ideal in $U$ generated by $\mathfrak{g}_+$, employing natural inclusions, $\mathfrak{g}_+ \hookrightarrow U$, for example. Let $N = N_U(I) = \{x \in U \mid Ix \subset I\} = \{x \in U \mid \mathfrak{g}_+x \subset I\}$ be the normalizer of $I$ in $U$. Note that $D^{-1}U(\mathfrak{h}) \subset N$ since $xh = hx - [h, x] \in I$ for all $x \in \mathfrak{g}_+$ and $h \in \mathfrak{h}$. The (D-localized) reduction algebra is defined to be $Z = Z(\mathfrak{g}, \mathfrak{g}; D) = Z(U, \mathfrak{g}) = N/I$. We often write $Z(\mathfrak{g}, \mathfrak{g})$ for $Z(\mathfrak{g}, \mathfrak{g}; D)$ when the multiplicative set $D$ is understood.

2.2. Category $\mathcal{C}(U, \mathfrak{g}_+).$ A left $U$-module $V$ is locally $\mathfrak{g}_+$-finite if $\dim_{\mathbb{C}} U_+(\mathfrak{g}_+)v < \infty$ for all $v \in V$. Let $\mathcal{C} = \mathcal{C}(U, \mathfrak{g}_+)$ be the category whose objects are locally $\mathfrak{g}_+$-finite $U$-modules and morphisms are $U$-module homomorphisms.

2.3. Invariants and coinvariants. We consider two covariant functors from $\mathcal{C}$ to the category of super vector spaces $\text{SVect}_{\mathbb{C}}$:

\begin{align*}
(-)^+: & \mathcal{C} \rightarrow \text{SVect}_{\mathbb{C}}, & V^+ = \{v \in V \mid \mathfrak{g}_+v = 0\}, \\
(-)_-: & \mathcal{C} \rightarrow \text{SVect}_{\mathbb{C}}, & V_- = V/\mathfrak{g}_-V.
\end{align*}

The set $V^+$ is the space of $\mathfrak{g}_+$-invariants of $V$, and $V_-$ is the space of $\mathfrak{g}_-$-coinvariants of $V$. The elements of $V^+$ are also called primitive (or singular) vectors. Associated to these functors are natural transformations to and from the forgetful functor $F: \mathcal{C} \rightarrow \text{SVect}_{\mathbb{C}}$, given by inclusion, respectively, canonical projection:

\begin{align*}
\iota: & (-)^+ \Rightarrow F, & \iota_V: V^+ \hookrightarrow V, \\
\pi: & F \Rightarrow (-)_-, & \pi_V: V \twoheadrightarrow V_-. \quad (2.3)
\end{align*}

Define $Q$ the be the vertical composition of the natural transformations above:

\begin{align*}
Q: & (-)^+ \Rightarrow (-)_-, & Q_V = \pi_V \circ \iota_V. \quad (2.5)
\end{align*}
There exists a natural endomorphism $\iota$ defined as:

\[ \iota: V \rightarrow V \] 

We say that the category $\text{C}$ has an extremal projector if the natural transformation $Q$ is invertible. In this case we denote the inverse by $Q^{-1}$. We denote the inverse by $P: (-)_- \Rightarrow (-)_+$. 

2.4. Extremal projectors.

Definition 2.1. We say that the category $\text{C}$ has an extremal projector if the natural transformation $Q$ is invertible. In this case we denote the inverse by $P: (-)_- \Rightarrow (-)_+$. 

Proposition 2.2. The following statements are equivalent:

(a) $\text{C}$ has an extremal projector.

(b) There exists a natural endomorphism $P$ of the forgetful functor $\mathbf{F}: \text{C} \rightarrow \text{SVect}_\mathbb{C}$ such that for all objects $V$ in $\text{C}$:

(i) $P_V \circ \iota_V = \iota_V$, 
(ii) $\pi_V \circ P_V = \pi_V$, 
(iii) $\text{im} P_V \subset \text{im} \iota_V$, 
(iv) $\ker \pi_V \subset \ker P_V$.

(c) For every $U$-module $V$ in $\text{C}$ there exists a map of super vector spaces $P_V: V \rightarrow V$ such that,

(i) if $g_+ v = 0$, then $P_V(v) = v$, 
(ii) $P_V(v) \in v + g_- V$, 
(iii) $g_+ P_V(v) = 0$, 
(iv) $P_V(g_- v) = 0$.

Furthermore, if these statements hold, then for all objects $V$ in $\text{C}$:

\[ P_V^2 = P_V. \] (2.7)

Proof. Statements (b) and (c) are equivalent by definition of the involved notions. (a)$\Rightarrow$(b): Suppose $\text{C}$ has an extremal projector and let $P: (-)_- \Rightarrow (-)_+$ be the inverse of $Q$. For every object $V$ of $\text{C}$, define $P_V = \iota_V \circ P_V \circ \pi_V$. Then $P$ is the vertical composition of the natural transformations $\iota$, $P$, and $\pi$, hence is a natural endomorphism of $\text{F}$. We check the properties:

(i) $P_V \circ \iota_V = \iota_V \circ P_V \circ \pi_V \circ \iota_V = \iota_V \circ P_V \circ Q_V = \iota_V$. (ii) $\pi_V \circ P_V = \pi_V \circ \iota_V \circ P_V \circ \pi_V = Q_V \circ P_V \circ \pi_V = \pi_V$. (iii) and (iv) are immediate by definition of $P_V$.

(b)$\Rightarrow$(a): Suppose $P$ is a natural endomorphism of $\text{F}$ satisfying (i)-(iv). Let $V$ be an object in $\text{C}$. By (iii) and (iv), $P_V$ induces a unique map $P_V: V \rightarrow V^+$ such that $\iota_V \circ P_V \circ \pi_V = P_V$. By uniqueness of $P_V$ and the naturality of $P_V$, $\iota_V$, and $\pi_V$, the maps $P_V$ define a natural transformation $P: (-)_- \Rightarrow (-)_+$. It remains to show that $P$ is the inverse of $Q$.

We have $\iota_V \circ P_V \circ Q_V = \iota_V \circ P_V \circ \pi_V \circ \iota_V = P_V \circ \iota_V = \iota_V$ by (i). Since $\iota_V$ is monic, this implies $P_V \circ Q_V = \text{Id}_{V^+}$. Similarly, $Q_V \circ P_V \circ \pi_V = \pi_V \circ P_V \circ \pi_V = \pi_V \circ \iota_V = \iota_V$ by (ii). Since $\pi_V$ is epic, this implies $Q_V \circ P_V = \text{Id}_V$. Finally, $P_V^2 = (\iota_V \circ P_V \circ \pi_V)^2 = \iota_V \circ P_V \circ Q_V \circ P_V \circ \pi_V = \iota_V \circ P_V \circ \pi_V = P_V$. 

Definition 2.3. If $V$ is an object of $\text{C}$, we call $P_V$ the extremal projector at $V$. 

\[ \begin{array}{c}
\text{C} \xrightarrow{\iota} \text{SVect} \\
\downarrow P \\
\text{C} \end{array} \] 

Explicitly, for each object $V$ of $\text{C}$, we have a map of super vector spaces $Q_V: V^+ \rightarrow V_-$ given by $Q_V(v) = v + g_- V$ for all $v \in V^+$. 

2.5. Universal highest weight module. The universal $\mathfrak{g}_+\text{-highest weight } U\text{-module}$ is

$$M = U/I.$$  

(2.8)

This is a left $U\text{-module}$ generated by the vector $1 = 1_U + I \in M$ which satisfies $\mathfrak{g}_+1 = 0$. Since $IN \subset I$, the space $M$ is a right $N\text{-module}$. Furthermore $MI = 0$. Thus $M$ is a right $Z\text{-module}$. Together, this makes $M$ a $(U,Z)\text{-bimodule}$.

Let $V$ be any left $U\text{-module}$. Restricting the action to $N$ and using that $IV^+ = 0$ we may regard $V^+$ as a $Z\text{-module}$. On the other hand, since $M$ is a $(U,Z)\text{-bimodule}$, $\text{Hom}_U(M,V)$ is a left $Z\text{-module}.$

**Lemma 2.4.** For any left $U\text{-module } V$, there is a natural isomorphism of left $Z\text{-modules}$$$
\psi: V^+ \cong \text{Hom}_U(M,V).$$  

(2.9)

**Proof.** Send $v \in V^+$ to the unique left $U\text{-module}$ map $\psi_v: M \to V$ determined by requiring $\psi_v(1) = v$. The inverse sends $\psi: M \to V$ to $\psi(1)$. \hfill \Box

Of particular importance is the case $V = M$, which gives another realization of the reduction algebra $Z$.

**Lemma 2.5.** We have the following two descriptions of $Z$:  

(i) $Z = M^+$.  

(ii) There is a natural isomorphism of associative superalgebras

$$Z \cong \text{End}_U(M)^{\text{op}}.$$  

(2.10)

**Proof.** (i) A coset $u + I \in M$ is in $M^+$ if and only if $Iu \subset I$ which by definition means $u \in N$.

(ii) Let $\psi: M^+ \to \text{End}_U(V)$ be the super vector space isomorphism (2.9) for the special case of $V = M$. Let $X,Y \in Z$, say $X = x + I$, $Y = y + I$. Then $\psi_{XY}(1) = XY$ while $(\psi_Y \circ \psi_X)(1) = \psi_Y(X) = \psi_Y(x+I)$. Since $\psi_Y$ is a left $U\text{-module}$ endomorphism, we have $\psi_Y(x+I) = x\psi_Y(1_U+I) = xY = xy + I = XY$. \hfill \Box

**Remark 2.6.** By tensor-hom adjunction we have $(M \otimes Z -) \dashv \text{Hom}_U(M,-)$. More precisely,

$$\text{Hom}_U(M \otimes Z X,V) \cong \text{Hom}_Z(X,\text{Hom}_U(M,V)) \cong \text{Hom}_Z(X,V^+)$$  

(2.11)

for any left $U\text{-module } V$ and left $Z\text{-module } X$.

**Lemma 2.7.** If $\mathcal{C}$ has an extremal projector, then $M$ is a projective object of $\mathcal{C}$.

**Proof.** Consider a diagram in $\mathcal{C}$

$$
\begin{array}{ccc}
M & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{\phi} & 0
\end{array}
$$

where the row is exact. We will show that $f$ can be lifted to a map $g: M \to X$ such that the diagram commutes. Let $y_0$ be the image of $1$. Since $\phi$ is surjective there is an $x \in X$ such that $\phi(x) = y_0$. Define $x_0 = P_X(x)$ where $P_X$ is the extremal projector at $X$. Then

$$\phi(x_0) = \phi \circ P_X(x) = P_Y \circ \phi(x) = P_Y(y_0) = P_Y(f(1)) = f(P_M(1)) = f(1) = y_0.$$  

(Here we used that $P_M(1) = 1$ which follows from property (i’) of Proposition 2.2.) Moreover $g_+x_0 = g_+P_Xx = 0$. So there is a unique $U\text{-module}$ map $g: M \to X$ satisfying $g(1) = x_0$. Clearly $\psi \circ g = f$ since both sides coincide on the generator $1$ of $M$. \hfill \Box
2.6. **Diamond product on the double coset space.** The space of coinvariants $M_- = M/\mathfrak{g}_-M$ is naturally identified with the double coset space $\mathfrak{g}_- U \backslash U/\mathfrak{g}_+ = U/\Pi$ where $\Pi = U\mathfrak{g}_+ + \mathfrak{g}_- U$. Explicitly, $(u + I) + \mathfrak{g}_- M \mapsto u + \Pi$ for $u \in U$.

Suppose $\mathfrak{c}$ has an extremal projector. Then we may define a product $\Diamond$ on $M_-$ by requiring that the super vector space isomorphism $P_M : M_- \rightarrow M^+ = Z$ is an algebra isomorphism. That is, define for all $x, y \in M_-$:

$$x \Diamond y = Q_M(P_M(x)P_M(y))$$

Since $M$ is a right $Z$-module and $\mathfrak{g}_- M$ is a right $Z$-submodule, the quotient space $M_-$ is a right $Z$-module. The following describes the diamond product in terms of the extremal projector and this right action of $Z$ on $M_-$, remembering that $M^+ = Z$.

**Lemma 2.8** ($[KO08]$). For $x, y \in M_-$, we have

$$x \Diamond y = xP_M(y) \quad (2.12)$$

**Proof.** Let $z \in Z$. Since $M$ is a $(U, Z)$-bimodule, the right action of $z$ on $M$ is a $U$-module endomorphism. By the functoriality of $(\cdot)^+$ and $(\cdot)_-$, this induces a super vector space endomorphisms $z^+ : M^+ \rightarrow M^+$ and $z_- : M_- \rightarrow M_-$. Explicitly, $z^+(x + I) = xz + I$ for $x + I \in M^+$ and $z_-(x + \Pi) = xz' + \Pi$ for $x + \Pi \in M_-$, where $z' \in N$ is any representative of $z = z' + I \in Z$. By the naturality of $Q$, we have a commutative diagram

$$
\begin{array}{ccc}
M^+ & \xrightarrow{Q_M} & M_- \\
\downarrow{z^+} & & \downarrow{z_-} \\
M^+ & \xrightarrow{Q_M} & M_-
\end{array}
$$

In other words, $Q_M : M^+ \rightarrow M_-$ is a map of right $Z$-modules. With this in mind, and that $Z = M^+$, we have for all $x, y \in M_-$:

$$x \Diamond y = Q_M(P_M(x)P_M(y)) = Q_M(P_M(x))P_M(y) = xP_M(y). \quad \Box$$

2.7. **Generators for the algebra $Z$.** Since $\mathfrak{g}$ is reductive in $\mathfrak{g}$, there exists a $\mathfrak{g}$-module complement $p$ of $\mathfrak{g}$ in $\mathfrak{g}$, i.e. $\mathfrak{g} = \mathfrak{g} \oplus p$ as $\mathfrak{g}$-modules. Consider the composition of maps

$$p \hookrightarrow \mathfrak{g} \hookrightarrow U \rightarrow U/\Pi = M_- \xrightarrow{P_M} M^+ \cong Z \quad (2.13)$$

Recall that $D^{-1}U(\mathfrak{h}) \subset N$, hence $Z$ is a $D^{-1}U(\mathfrak{h})$-ring (meaning there is an algebra map $D^{-1}U(\mathfrak{h}) \rightarrow Z$ whose image is not necessarily contained in the center of $Z$).

**Lemma 2.9.** The image of $p$ under the map (2.13) generates $Z$ as a $D^{-1}U(\mathfrak{h})$-ring. Hence the image of $p$ in $U/\Pi$ generates $U/\Pi$ as a $D^{-1}U(\mathfrak{h})$-ring with respect to the diamond product.

**Proof.** By the PBW theorem applied to the decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{p} \oplus \mathfrak{g}_+$, we have

$$U \cong U(\mathfrak{g}_- \otimes D^{-1}U(\mathfrak{h}) \otimes U(\mathfrak{p}) \otimes U(\mathfrak{g}_+)).$$

Writing $U(\mathfrak{g}_+) = \mathbb{C} \oplus U(\mathfrak{g}_+)\mathfrak{g}_+$ and using that $U(\mathfrak{g}_+)\mathfrak{g}_+ \subset I$ we see that $U(\mathfrak{g}_-) \otimes D^{-1}U(\mathfrak{h}) \otimes U(\mathfrak{p})$ still maps onto $U/I$. On the other hand, since $P_M\mathfrak{g}_- = 0$, the subspace $D^{-1}U(\mathfrak{h}) \otimes U(\mathfrak{p})$ of $U$ still maps onto $M^+$. This proves the claim. \quad \Box
2.8. Irreducibility of the $Z$-modules $V^+$.

**Proposition 2.10.** Let $V$ be an object of the category $\mathcal{C}$. If $v \in V^+$ generates $V$ as a $U$-module then $v$ generates $V^+$ as a $Z$-module. In particular, if $V$ is a simple $U$-module then $V^+$ is a simple $Z$-module.

**Proof.** Since $V$ is generated by $v$, the map $\psi_v$ is surjective. Let $K$ be the kernel of $\psi_v$. The short exact sequence of $U$-modules

$$0 \to K \to M \to V \to 0$$

gives rise to a long exact sequence

$$0 \to \text{Hom}_U(M,K) \to \text{Hom}_U(M,M) \to \text{Hom}_U(M,V) \to \text{Ext}_U^1(M,K) \to \cdots$$

Since $M$ is projective by Lemma 2.7, we have $\text{Ext}_U^1(M,K) = 0$. Therefore the map $\text{Hom}_U(M,M) \to \text{Hom}_U(M,V)$ given by $z \mapsto \psi_v \circ z$ is surjective. Note that $\text{Hom}_U(M,M) = \text{End}_U(M) \cong Z^{\op}$ by Lemma 2.3 and $\text{Hom}_U(M,V) \cong V^+$ by Lemma 2.4. Under these identifications the map in question is simply the map $Z \to V^+$ given by $z \mapsto zv$. To say that this is surjective is equivalent to saying that $v$ generates $V^+$ as a $Z$-module. \hfill $\blacksquare$

3. DIAGONAL REDUCTION ALGEBRA OF $osp(1|2)$

Here we initiate a concrete exploration of the algebra $Z = Z(\mathfrak{g}, g; D)$, where $\mathfrak{g}$ is the Lie superalgebra $osp(1|2) \times osp(1|2)$ and $g$ is the image in $\mathfrak{g}$ of $osp(1|2)$ under the diagonal embedding. The denominator set $D$ of Section 2 is the multiplicative set generated by $\{H - n \mid n \in \mathbb{Z}\}$, with $H = (h, h)$ in $osp(1|2) \times osp(1|2)$, chosen to allow for the existence of the extremal projector for $g$ (see [Tot85], [BT81]) described in Section 3.2. We investigate the structure of the superalgebra $Z$ by providing generators and relations and determining a PBW basis, with respect to the diamond ($\diamondsuit$) product, to justify a complete presentation. All (super) vector spaces and (super)algebras will be considered as objects over the field of complex numbers, unless otherwise stated. Our basic references for Lie superalgebras are [FSS00], [Mus12], [CW12].

3.1. The Lie superalgebra $osp(1|2)$.

3.1.1. **Definition.** The Type II basic classical Lie superalgebra $osp(1|2)$ is the super vector space spanned by $\{x_{-2\alpha}, h, x_{2\alpha}; x_{-\alpha}, x_{\alpha}\}$ of dimension $(3|2)$ (super dimension 1) preserving an even, nondegenerate, supersymmetric bilinear form on a super vector space of dimension $(1|2)$ (superdimension -1). Equivalently, let $\mathfrak{gl}(1|2)$ be the set of all linear transformations on $\mathbb{C}^{1|2}$ expressed as matrices with respect to a standard basis $\{v_0; v_1, v_2\}$ of $\mathbb{C}^{1|2}$, with even vector

$$v_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and odd vectors

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$  

Elements of $\mathfrak{gl}(1|2)$ are block matrices $\begin{bmatrix} a & r \\ c & A \end{bmatrix}$; here, $a$ is a scalar, $r$ is a row vector, $c$ is a column vector, and $A$ is a $2 \times 2$ square matrix. The orthosymplectic Lie superalgebra $osp(1|2)$ is the
relations on which are where, \( c \) Lie subsuperalgebra of \( gl \).

The even part \( sl \) osp 3.1.2. is an element of \( sl(2) \). For the sake of completeness, we give the supercommutator relations on

\[
\text{osp}(1|2) = \text{osp}(1|2)_0 \oplus \text{osp}(1|2)_1 = (\mathbb{C}x_{-2a} \oplus \mathbb{C}h \oplus \mathbb{C}x_{2a}) \oplus (\mathbb{C}x_{-\alpha} \oplus \mathbb{C}x_{\alpha}),
\]

where,

\[
x_{-2a} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad x_{-\alpha} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad x_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad x_{2a} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

which are

\[
[h,x_{k\alpha}] = -kx_{k\alpha}, \quad k \in \{\pm 1, \pm 2\} \tag{3.2}
\]

\[
[x_{\alpha},x_{\alpha}] = -2x_{2a} \tag{3.3}
\]

\[
[x_{-\alpha},x_{-\alpha}] = 2x_{-2a} \tag{3.4}
\]

\[
[x_{\alpha},x_{-2a}] = x_{-\alpha} \tag{3.5}
\]

\[
[x_{-\alpha},x_{2a}] = x_{\alpha} \tag{3.6}
\]

\[
[x_{\alpha},x_{-\alpha}] = h \tag{3.7}
\]

\[
[x_{-2a},x_{2a}] = h \tag{3.8}
\]

\[
[x_{\alpha},x_{2a}] = 0 \tag{3.9}
\]

\[
[x_{-\alpha},x_{-2a}] = 0. \tag{3.10}
\]

The even part \( \text{osp}(1|2)_0 \) of \( \text{osp}(1|2) \) is a Lie algebra isomorphic to \( sl(2) \), and the odd part \( \text{osp}(1|2)_1 \) is isomorphic to the natural \( sl(2) \)-module \( \mathbb{C}^2 \). The Cartan subalgebra \( h \) of \( \text{osp}(1|2) \) is equal to the Cartan subalgebra of \( \text{osp}(1|2)_0 \), which is \( \mathbb{C}h \). In Figure 1 we summarize some useful iterated brackets in \( \text{osp}(1|2) \) that will be used in connection with the extremal projector.

3.1.2. Odd roots of \( \text{osp}(1|2) \). The \( BC_1 \) root system \( \Phi \) of \( \text{osp}(1|2) \) is given by the union of even (bosonic) roots \( \Phi_0 = \{\pm 2\delta_1\} \) with odd (fermionic) roots \( \Phi_1 = \{\pm \delta_1\} \). Moreover, we choose a non-standard set of positive roots \( \Phi^+ = \{-\delta_1, -2\delta_1\} \) associated to the base \( \Pi = \{-\delta_1\} \) for \( \Phi \). Positive even (respectively, odd) roots, as well as their negative counterparts, are defined with the appropriate intersection with \( \Phi_0 \) (respectively, \( \Phi_1 \)). In particular, \( \text{osp}(1|2) \) has a lone positive odd root vector \( x_\alpha \), for \( \alpha = -\delta_1 \) identified with -1.
It follows that \( \mathfrak{osp}(1|2) \) has a triangular decomposition:
\[
\mathfrak{osp}(1|2) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,
\]
with \( \mathfrak{n}_\pm = \mathbb{C}x_{\pm 2\alpha} \oplus \mathbb{C}x_{\pm \alpha} \).

The Killing form on the Lie superalgebra \( \mathfrak{osp}(1|2) \) is non-degenerate, and it induces a bilinear form on \( \mathfrak{h}^* \) such that \((\alpha, \alpha) = 1\).

3.1.3. **Diagonal embedding.** Throughout the rest of this section, \( \mathfrak{g} \) is a reductive embedding of \( \mathfrak{osp}(1|2) \) into \( \mathfrak{g} = \mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2) \) as the image of \( \delta: \mathfrak{osp}(1|2) \to \mathfrak{g}, \ x \mapsto (x, x) \), for all \( x \) in \( \mathfrak{osp}(1|2) \).

To heed the remarks of the previous section, we provide a linear complement of \( \mathfrak{g} \): Namely,
\[
\mathfrak{p} = \{(x, -x) \mid x \in \mathfrak{osp}(1|2)\},
\]
which is the image of \( \delta_-: \mathfrak{osp}(1|2) \to \mathfrak{g}, \ x \mapsto (x, -x) \), for all \( x \) in \( \mathfrak{osp}(1|2) \). Then \( \mathfrak{G} = \mathfrak{g} \oplus \mathfrak{p} \) as \( \mathfrak{g} \)-modules. Even more, the map \( \mathfrak{g} \to \mathfrak{p} \) sending \((x, x)\) to \((x, -x)\), for all \( x \) in \( \mathfrak{osp}(1|2) \), is an isomorphism of \( \mathfrak{g} \)-modules.

A root vector \( x_\beta \) of \( \mathfrak{osp}(1|2) \) will be identified as \( X_\beta \), respectively, \( \bar{x}_\beta \), for its image in \( \mathfrak{g} \subset \mathfrak{G} \), respectively, in \( \mathfrak{p} \subset \mathfrak{G} \). That is, \( X_\beta = (x_\beta, x_\beta) \) and \( \bar{x}_\beta = (x_\beta, -x_\beta) \). We also put \( H \) for \((h, h)\) and \( \bar{h} = (h, -h) \). These identifications remain in \( \mathfrak{G} \to \mathfrak{H} \) after we fix the multiplicative set \( D \) generated by products of factors \((H - m)^n\), where \( m \) is an integer and \( n \) is a natural number.

Again for the sake of lucidity: The left ideal \( I \) in \( \mathfrak{U} = \mathfrak{U}(\mathfrak{CX}_\alpha + \mathfrak{CX}_{-2\alpha}) \) and we write \( \mathbb{I} \) for the sum \((\mathfrak{CX}_\alpha + \mathfrak{CX}_{-2\alpha})\mathfrak{U} + \mathfrak{U}(\mathfrak{CX}_\alpha + \mathfrak{CX}_{2\alpha}) \) of subspaces.

The algebra \( \mathfrak{N}_U = \mathfrak{U}(\mathfrak{CX}_\alpha + \mathfrak{CX}_{2\alpha})/\mathfrak{U}(\mathfrak{CX}_\alpha + \mathfrak{CX}_{2\alpha}) \), denoted here as \( Z(\mathfrak{G}, \mathfrak{g}) \), is the diagonal reduction algebra of \( \mathfrak{osp}(1|2) \) associated to the embedding \( \delta: \mathfrak{osp}(1|2) \subset \mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2) \).

3.1.4. **Universal enveloping algebra and PBW basis.** An ordered basis for \( \mathfrak{G} \) is the set
\[
\{X_{-2\alpha}, X_{-\alpha}, H, \bar{x}_{-2\alpha}, \bar{x}_{-\alpha}, \bar{h}, \bar{\alpha}, \bar{x}_{2\alpha}, X_{\alpha}, X_{2\alpha}\}.
\]  
(3.11)

The choice of ordered basis for \( \mathfrak{G} \) is compatible with \( \mathbb{I} \).

A direct calculation shows that if \((x, y) = z\), then
\[
[X, \bar{y}] = \bar{z}, \quad [\bar{x}, \bar{y}] = Z, \quad [X, Y] = Z,
\]
(3.12)

for all root vectors \( x \) and \( y \) in \( \mathfrak{osp}(1|2) \).

Considering the PBW basis theorem for Lie superalgebras with regards to (3.11) gives a basis for the universal enveloping algebra \( \mathfrak{U}(\mathfrak{G}) \). That is, \( \mathfrak{U}(\mathfrak{G}) \) is spanned by a linearly independent set of monomials where each monomial is of the form
\[
X_{-2\alpha}^a X_{-\alpha}^b H^{c_{-2\alpha}} \bar{x}_{-2\alpha}^{p_{-\alpha}} \bar{x}_{2\alpha}^{q_{\alpha}} \bar{h}^{r_{-\alpha}} \bar{x}_{2\alpha}^{t_{\alpha}} X_{\alpha}^d X_{2\alpha}^e
\]
(3.13)

with the exponents \( b, d, q, s \) no greater than 1 and \( a, c, e, p, r, t \) any natural number.

Thus, \( \mathfrak{U} \) is a free left (right) \( D^{-1}U(\mathfrak{h}) \)-ring.

3.1.5. **Anti-automorphism.** In the aid of computation, we define a Lie superalgebra anti-automorphism of \( \mathfrak{osp}(1|2) \) given by:
\[
\theta(x_{\pm \alpha}) = \sqrt{-1} x_{\mp \alpha}, \quad \theta(x_{\pm 2\alpha}) = -x_{\mp 2\alpha}, \quad \theta(h) = h.
\]
(3.14)

We have \( \theta([x, y]) = (-1)^{|x||y|}\theta(y)\theta(x) \), and \( \theta \) is a map of super vector spaces. Thus, it extends to a superalgebra anti-homomorphism
\[
\Theta: \mathfrak{U} \to \mathfrak{U}, \ \Theta(xy) = (-1)^{|x||y|} \Theta(y)\Theta(x).
\]
Furthermore, $\Theta(\mathfrak{H}) \subseteq \mathfrak{H}$ and hence $\Theta$ induces a linear endomorphism on $U/\mathfrak{H}$. In fact, $\Theta$ is an anti-automorphism of $U/\mathfrak{H}$ with respect to the diamond product.

3.2. **Extremal projector for $\mathfrak{osp}(1|2)$**. Refer to [Tol11] for a general background on extremal projectors associated with various algebraic objects.

For the convenience of the reader, we deduce the expression for the extremal projector of $\mathfrak{osp}(1|2)$ using the fixed notation of the current section.

The Taylor extension $TU$ of $U$ can be defined as the projective limit

$$ TU = \lim_{\leftarrow} \frac{U}{g^n U + U g^n}. \quad (3.15) $$

Define $P \in TU$ by

$$ P = \sum_{n=0}^{\infty} \varphi_n(h) x_{-\alpha}^n x_{\alpha}^n \quad (3.16) $$

where $\varphi_0(h) = 1$ and for $n > 0$, $\varphi_n(h) \in \mathbb{C}(h)$ are rational functions to be determined. Introduce $\kappa_n(h) \in \mathbb{C}[h]$ by requiring in $U(\mathfrak{osp}(1|2))$:

$$ x_{\alpha} x_{-\alpha} - (-1)^n x_{-\alpha} x_{\alpha} = [x_{\alpha}, x_{-\alpha}] = \kappa_n(h) x_{-\alpha}^{n-1}, \quad n \geq 1; \quad \kappa_0(h) = 0. \quad (3.17) $$

Explicitly, by induction and super Leibniz rule,

$$ \kappa_n(h) = \sum_{k=0}^{n-1} (-1)^k (h-k) = \begin{cases} n/2, & n \text{ even}, \\ h-(n-1)/2, & n \text{ odd}. \end{cases} \quad (3.18) $$

We have

$$ x_{\alpha} P = \sum_{n=0}^{\infty} \varphi_n(h+1) x_{\alpha} x_{-\alpha}^n x_{\alpha}^n $$

$$ = \sum_{n=0}^{\infty} \left( \varphi_n(h+1)(-1)^n x_{-\alpha}^n x_{\alpha}^{n+1} + \varphi_n(h+1) \kappa_n(h) x_{-\alpha}^{n-1} x_{\alpha}^n \right) $$

$$ = \sum_{n=0}^{\infty} \left( (-1)^n \varphi_n(h+1) + \varphi_{n+1}(h+1) \kappa_{n+1}(h) \right) x_{-\alpha}^n x_{\alpha}^{n+1}. $$

Thus we see that $x_{\alpha} P = 0$ if and only if

$$ \varphi_n(h) = \frac{(-1)^n}{\kappa_n(h-1)} \varphi_{n-1}(h), \quad n \geq 1, \quad (3.19) $$

or equivalently,

$$ \varphi_{2n-1}(h) = \frac{-1}{h-n} \varphi_{2n-2}(h), \quad \varphi_{2n}(h) = \frac{1}{n} \varphi_{2n-1}(h), \quad n \geq 1. \quad (3.20) $$

Together with the initial condition $\varphi_0(h) = 1$ this determines the rational functions $\varphi_n(h)$ uniquely. The first few values of $\varphi_n(h)$ are

$$ \varphi_0(h) = 1, \quad \varphi_1(h) = \varphi_2(h) = \frac{-1}{h-1}, \quad \varphi_3(h) = \frac{1}{(h-2)(h-1)}, \quad \varphi_4(h) = \frac{1}{2} \varphi_3(h). \quad (3.21) $$

We see that the minimal multiplicative set $D$ satisfying the Ore conditions and such that $\varphi_n(h) \in D^{-1} U(h)$ is the multiplicative submonoid of $U(h) \setminus \{0\}$ generated by $\{(h-n) \mid n \in \mathbb{Z}\}$. 

3.3. Generators. The diamond product on $U/\mathfrak{P}$ is defined as follows. For $u \in U$, let $\bar{u} = u + \mathfrak{P} \in U/\mathfrak{P}$. For all $u, v \in U$, define
\[
\bar{u} \diamond \bar{v} = uPv + \mathfrak{P}.
\] (3.22)
This is well-defined because $\mathfrak{P}P = 0 = P\mathfrak{P}$. This may also be expressed as follows:

\[
\bar{u} \diamond \bar{v} = uv + [u, X_{-\alpha}] \cdot \varphi_1(H + 1) \cdot [X_{\alpha}, v] + [[u, X_{-\alpha}], X_{-\alpha}] \cdot \varphi_2(H + 2) \cdot [X_{\alpha}, [X_{\alpha}, v]] + \cdots + \mathfrak{P}
\] (3.23)

Proposition 3.1. The algebra $Z(\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2), \mathfrak{osp}(1|2))$ is generated as a $D^{-1}U(\mathfrak{h})$-ring by the following elements:

\[
\begin{align*}
P\bar{x}_{2\alpha} + I &= \bar{x}_{2\alpha} + I \\
P\bar{x}_\alpha + I &= \bar{x}_\alpha - 2\varphi_1(H)X_{-\alpha}\bar{x}_{2\alpha} + I \\
P\bar{h} + I &= \bar{h} + \varphi_1(H)X_{-\alpha}\bar{x}_\alpha - 2\varphi_2(H)X_{2\alpha}\bar{x}_{2\alpha} + I \\
P\bar{x}_{-\alpha} + I &= \bar{x}_{-\alpha} + \varphi_1(H)X_{-\alpha}\bar{h} + \varphi_2(H)X_{2\alpha}\bar{x}_\alpha - 2\varphi_3(H)X_{3\alpha}\bar{x}_{2\alpha} + I \\
P\bar{x}_{-2\alpha} + I &= \bar{x}_{-2\alpha} + \varphi_1(H)X_{-\alpha}\bar{x}_{-\alpha} + \varphi_2(H)X_{2\alpha}\bar{h} + \varphi_3(H)X_{3\alpha}\bar{x}_\alpha - 2\varphi_4(H)X_{4\alpha}\bar{x}_{2\alpha} + I
\end{align*}
\]

Proof. Applying Lemma 2.9 \{ $P(\bar{u} + I) \mid \bar{u} \in \{\bar{x}_{-2\alpha}, \bar{x}_{-\alpha}, \bar{h}, \bar{x}_\alpha, \bar{x}_{2\alpha}\}$ \} yields a set of generators for $Z$. Furthermore,

\[
P(\bar{u} + I) = \bar{u} + \varphi_1(H)X_{-\alpha}[X_{\alpha}, \bar{u}] + \varphi_2(H)X_{2\alpha}[X_{\alpha}, [X_{\alpha}, \bar{u}]] + \cdots + I.
\]

Now we may use the bracket \[\{3.12\}\] along with the right half of the table in Figure [1] \hfill \square

We revisit the discussion of Section 2.8 through a concrete example of an irreducible $Z$-module related to oscillator representations. For each non-negative integer $\lambda$, there exists a finite-dimensional irreducible representation $V(\lambda)$ of $\mathfrak{osp}(1|2)$. The dimension of $V(\lambda)$ equals $2\lambda + 1$ and this list exhausts all finite-dimensional irreducible representations up to equivalence. The spectrum of $h$ on $V(\lambda)$ equals $\{\lambda, \lambda - 1, \ldots, -\lambda\}$. The associative superalgebra $\mathbb{C}\{x\}$, where $x$ is declared odd, carries an action of $\mathfrak{osp}(1|2)$ determined by $x_{\alpha} \mapsto \frac{1}{\sqrt{2\alpha}}\bar{h}$ and $x_{-\alpha} \mapsto \frac{1}{\sqrt{2\alpha}}x$. The spectrum of $h$ on $\mathbb{C}\{x\}$ is $\lambda + \mathbb{Z}_{\geq 0}$. Therefore the spectrum of $h$ on $\tilde{V}(\lambda) = \mathbb{C}\{x\} \otimes V(\lambda)$ is a subset of $\frac{1}{2} + \mathbb{Z}$ and hence the action of $U(\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2))$ on $\tilde{V}(\lambda)$ extends to the localization $U$ (at all $(h - n)$ for $n \in \mathbb{Z}$). Furthermore, $\tilde{V}(\lambda)$ is locally finite with respect to the action of $x_{\alpha}$, hence is an object of the category $\mathcal{C}$. This means that the space of primitive vectors (g$_{\cdot}$-invariants) $\tilde{V}(\lambda)^+$ is an irreducible representation of the diagonal reduction algebra $Z$.

Example 3.2. The superspace $V = \mathbb{C}\{x\} \otimes \mathbb{C}^{1|2}$ is an object of $\mathcal{C}$. Define the elements $w_1 = 1 \otimes v_2$ and $w_2 = 1 \otimes v_0 + \sqrt{2}x \otimes v_2$ and $w_3 = -x^2 \otimes v_2 + \sqrt{2}x \otimes v_0 - 1 \otimes v_1$. One can show that $V^+ = \text{Span}\{w_1, w_2, w_3\}$ (see [Fer13] for the case of $\mathfrak{osp}(1|2n)$ with $n > 1$). The irreducible submodules corresponding to higher-dimensional analogues of $w_1$ and $w_2$ were studied in [W120]. Note that $w_2$ is even while $w_1$ and $w_3$ are odd. In the ordered basis $(w_2, w_1, w_3)$, the irreducible representation
Lemma 3.3.

The proofs of (3.28), (3.29), and (3.30) are similar. Application of \( \Theta \) to (3.28) and (3.27) yields (3.31) and (3.32), respectively. Complete details of the proof are found in Appendix A.

It will be useful to invert these equations.

\[ \rho: Z \to \text{End}_\mathbb{C}(V^+) \text{ is given by} \]

\[
\rho(x_\alpha) = \begin{bmatrix} 0 & 0 & -6 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \rho(x_{2\alpha}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \rho(\bar{h}) = \begin{bmatrix} 2 \bar{\pi} & 0 & 0 \\ \bar{\pi} & 0 & 0 \\ 0 & 0 & -2 \bar{\pi} \end{bmatrix}, \quad \rho(\bar{x}_{-\alpha}) = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix}, \quad \rho(\bar{x}_{-2\alpha}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad \rho(f(H)) = \begin{bmatrix} f(\frac{1}{2}) & 0 & 0 \\ 0 & f(-\frac{1}{2}) & 0 \\ 0 & 0 & f(\frac{3}{2}) \end{bmatrix}.
\]

3.4. Ordered monomials and PBW bases.

3.4.1. Ordered diamond products. We compute the ordered products we need in order to deduce the commutation relations in the next section. We order the generators of the diagonal reduction algebra as follows.

\[ x_{-2\alpha} < x_{-\alpha} < \bar{h} < x_\alpha < x_{2\alpha}. \] (3.24)

We compute the lexicographically ordered diamond product of any two generators. Put \( \bar{y} = \delta_-(y) \) for \( y \in \{x_{\pm\alpha}, x_{\pm2\alpha}, h\} \).

Lemma 3.3.

\[ \bar{y} \diamond x_{2\alpha} = \bar{y}x_{2\alpha} + \Pi, \quad \forall y \in \{x_{\pm\alpha}, x_{\pm2\alpha}, h\}, \] (3.25)

\[ x_{-2\alpha} \diamond \bar{y} = \bar{x}_{-2\alpha}\bar{y} + \Pi, \quad \forall y \in \{x_{\pm\alpha}, x_{\pm2\alpha}, h\}, \] (3.26)

\[ x_\alpha \diamond \bar{x}_\alpha = (\bar{x}_\alpha)^2 - 2\varphi_1(H + 1)\bar{h}\bar{x}_{2\alpha} + \Pi, \] (3.27)

\[ \bar{h} \diamond x_\alpha = \bar{h}x_\alpha - 2\varphi_1(H)\bar{x}_{-\alpha}\bar{x}_{2\alpha} + \Pi, \] (3.28)

\[ x_{-\alpha} \diamond x_\alpha = x_{-\alpha}x_\alpha - 4\varphi_1(H - 1)\bar{x}_{-2\alpha}\bar{x}_{2\alpha} + \Pi, \] (3.29)

\[ \bar{h} \diamond \bar{h} = \bar{h}^2 + \varphi_1(H)\bar{x}_{-\alpha}\bar{x}_\alpha - 4\varphi_2(H)\bar{x}_{-2\alpha}\bar{x}_{2\alpha} + \Pi. \] (3.30)

\[ x_{-\alpha} \diamond \bar{h} = x_{-\alpha}\bar{h} + 2\varphi_1(H - 1)\bar{x}_{-2\alpha}\bar{x}_\alpha + \Pi, \] (3.31)

\[ x_{-\alpha} \diamond x_{-\alpha} = (x_{-\alpha})^2 + 2\varphi_1(H - 1)\bar{x}_{-2\alpha}\bar{h} + \Pi. \] (3.32)

Proof. (3.25) and (3.26) follow directly from the formula (3.23) since \( [x_\alpha, \bar{x}_{2\alpha}] = 0 \) and \( [\bar{x}_{-2\alpha}, X_{-\alpha}] = 0 \). For (3.27) we have

\[ x_\alpha \diamond \bar{x}_\alpha = (\bar{x}_\alpha)^2 + [\bar{x}_\alpha, X_{-\alpha}]\varphi_1(H + 1)[X_\alpha, \bar{x}_\alpha] + \Pi = (\bar{x}_\alpha)^2 - 2\bar{h}\varphi_1(H + 1)\bar{x}_{2\alpha} + \Pi. \]

The proofs of (3.28), (3.29), and (3.30) are similar. Application of \( \Theta \) to (3.28) and (3.27) yields (3.31) and (3.32), respectively. Complete details of the proof are found in Appendix A.

It will be useful to invert these equations.
Lemma 3.4.

\[ \tilde{y}\tilde{x}_{2\alpha} + \Pi = \tilde{y} \diamond \tilde{x}_{2\alpha}, \quad \forall y \in \{x_{\pm\alpha}, x_{\pm2\alpha}, h\}, \quad (3.33) \]
\[ \tilde{x}_{-2\alpha}\tilde{y} + \Pi = \tilde{x}_{-2\alpha} \diamond \tilde{y}, \quad \forall y \in \{x_{\pm\alpha}, x_{\pm2\alpha}, h\}, \quad (3.34) \]
\[ \tilde{x}_\alpha\tilde{x}_\alpha + \Pi = \tilde{x}_\alpha \diamond \tilde{x}_\alpha + 2\varphi_1(H + 1)\tilde{h} \diamond \tilde{x}_{2\alpha}, \quad (3.35) \]
\[ \tilde{h}\tilde{x}_\alpha + \Pi = \tilde{h} \diamond \tilde{x}_\alpha + 2\varphi_1(H)\tilde{x}_{-\alpha} \diamond \tilde{x}_{2\alpha}, \quad (3.36) \]
\[ \tilde{x}_{-\alpha}\tilde{x}_\alpha + \Pi = \tilde{x}_{-\alpha} \diamond \tilde{x}_\alpha + 4\varphi_1(H - 1)\tilde{x}_{-2\alpha} \diamond \tilde{x}_{2\alpha}, \quad (3.37) \]
\[ \tilde{h}\tilde{h} + \Pi = \tilde{h} \diamond \tilde{h} - \varphi_1(H)\tilde{x}_{-\alpha} \diamond \tilde{x}_\alpha + 4(\varphi_2(H) - \varphi_1(H)\varphi_1(H - 1))\tilde{x}_{-2\alpha} \diamond \tilde{x}_{2\alpha}, \quad (3.38) \]
\[ \tilde{x}_{-\alpha}\tilde{h} + \Pi = \tilde{x}_{-\alpha} \diamond \tilde{h} - 2\varphi_1(H - 1)\tilde{x}_{-2\alpha} \diamond \tilde{x}_\alpha, \quad (3.39) \]
\[ \tilde{x}_{-\alpha}\tilde{x}_{-\alpha} + \Pi = \tilde{x}_{-\alpha} \diamond \tilde{x}_{-\alpha} - 2\varphi_1(H - 1)\tilde{x}_{-2\alpha} \diamond \tilde{h}. \quad (3.40) \]

Proof. By (3.25) and (3.27),

\[ \tilde{x}_\alpha\tilde{x}_\alpha + \Pi = \tilde{x}_\alpha \diamond \tilde{x}_\alpha + (2\varphi_1(H + 1)\tilde{h}\tilde{x}_{2\alpha} + \Pi) \]
\[ = \tilde{x}_\alpha \diamond \tilde{x}_\alpha + 2\varphi_1(H + 1)\tilde{h} \diamond \tilde{x}_{2\alpha}. \]

Continued manipulation of the relations in Lemma (3.3) gives the rest of the inversions. Complete details of the proof are found in Appendix A. \[\square\]

3.5. Presentation. We are now able to deduce the relations.
Theorem 3.5. The following relations hold in $U/\mathfrak{U}$:

$$
\begin{align*}
    f(H) \diamond g(H) &= g(H) \diamond f(H), \quad \forall f(H), g(H) \in D^{-1}U(h), \\
    \bar{x}_{k\alpha} \diamond f(H) &= f(H + k) \diamond \bar{x}_{k\alpha}, \quad \forall k \in \{\pm 1, \pm 2\}, \forall f(H) \in D^{-1}U(h), \\
    \bar{h} \diamond f(H) &= f(H) \diamond \bar{h}, \quad \forall f(H) \in D^{-1}U(h), \\
    \bar{x}_{2\alpha} \diamond \bar{x}_\alpha &= (1 - \frac{2}{H + 1})\bar{x}_\alpha \diamond \bar{x}_{2\alpha} \\
    \bar{x}_\alpha \diamond \bar{x}_\alpha &= \frac{2}{H} \bar{h} \diamond \bar{x}_{2\alpha} \\
    \bar{x}_{-\alpha} \diamond \bar{x}_\alpha &= -\frac{2}{H - 2} \bar{x}_{-2\alpha} \diamond \bar{h} \\
    \bar{x}_{2\alpha} \diamond \bar{h} &= \left(1 - \frac{2}{H + 1}\right)\bar{h} \diamond \bar{x}_{2\alpha} \\
    \bar{x}_{2\alpha} \diamond \bar{x}_{-\alpha} &= \left(1 - \frac{2}{H(H - 1)}\right)\bar{x}_{-\alpha} \diamond \bar{x}_{2\alpha} + \frac{2}{H + 1}\bar{h} \diamond \bar{x}_\alpha \\
    \bar{x}_{2\alpha} \diamond \bar{x}_{-2\alpha} &= \left(1 + \frac{H^3 + H^2 - 6H + 4}{(H - 2)(H - 1)(H + 1)(H + 2)}\right)\bar{x}_{-2\alpha} \diamond \bar{x}_{2\alpha} \\
    \bar{x}_{-\alpha} \diamond \bar{x}_{-\alpha} &= (1 - \frac{2}{H - 2})\bar{x}_{-2\alpha} \diamond \bar{x}_{-\alpha} \\

\end{align*}
$$

Proof. Put $\bar{y} = \delta_\gamma(y)$ for $y \in \{x_{\pm 1}, x_{\pm 2}, 0\}$. Putting $h = x_0$ for a moment, note that for any $\beta, \gamma \in \{\pm 1, \pm 2\}$ such that $\beta + \gamma \neq 0$ we have

$$
\bar{x}_\beta \bar{x}_\gamma + \mathbb{I} = (-1)^{\beta |\beta|} \bar{x}_\alpha \bar{x}_\beta + \mathbb{I} 
$$

(3.42)

This is due to the fact that, in $U$, we have $[\bar{x}_\beta, \bar{x}_\gamma] \in \mathbb{C}X_{\beta + \gamma} \subseteq \mathbb{I}$ when $\beta + \gamma \neq 0$. Thus, we have

$$
\bar{x}_{2\alpha} \diamond \bar{x}_\alpha = \bar{x}_{2\alpha} \bar{x}_\alpha + [\bar{x}_{2\alpha}, X_{-\alpha}]\varphi_1(H + 1)[X_\alpha, \bar{x}_\alpha] + \mathbb{I} \\
= \bar{x}_\alpha \bar{x}_{2\alpha} + 2\varphi_1(H + 2)\bar{x}_\alpha \bar{x}_{2\alpha} + \mathbb{I} \\
= (1 + 2\varphi_1(H + 2))\bar{x}_\alpha \diamond \bar{x}_{2\alpha},
$$

where we used (3.25).

For a complete proof of the remaining relations, see Appendix A. \qed
3.6. PBW basis for $Z$. By the PBW theorem for $U(\mathfrak{g})$, it is immediate that $U/\mathfrak{I}$ is a free left $D^{-1}U(\mathfrak{h})$-module on
\[
\{ \tilde{x}_{-2a}^p \tilde{x}_{-a}^q \tilde{h}^r \tilde{x}_a^s \tilde{x}_{2a}^t + \mathfrak{I} \mid p, q, r, s, t \in \mathbb{Z}_{\geq 0}, q, s \leq 1 \}.
\] (3.43)

Proposition 3.6. $U/\mathfrak{I}$ is a free left $D^{-1}U(\mathfrak{h})$-module on the following set of monomials with respect to the diamond product:
\[
\{ \tilde{x}_{-2a}^p \tilde{x}_{-a}^q \tilde{h}^r \tilde{x}_a^s \tilde{x}_{2a}^t \mid p, q, r, s, t \in \mathbb{Z}_{\geq 0}, q, s \leq 1 \}.
\] (3.44)

Proof. By Theorem 3.5, the set spans $U/\mathfrak{I}$. By induction, each diamond monomial can be written
\[
\tilde{x}_{-2a}^p \tilde{x}_{-a}^q \tilde{h}^r \tilde{x}_a^s \tilde{x}_{2a}^t + \text{ (lower terms) + } \mathfrak{I}
\] (3.45)
where we order the monomials lexicographically with respect to $x_{-2a} < x_{-a} < h < x_a < x_{2a}$. □

3.7. Main theorem.

Theorem 3.7. Let $D$ be the multiplicative subset of $U(\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2))$ generated by \{ $H - n \mid n \in \mathbb{Z}$ \} where $H = h \otimes 1 + 1 \otimes h$ and let $Z$ be the diagonal reduction algebra $Z(\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2), \mathfrak{osp}(1|2))$. Then $Z$ is generated as a $\mathbb{C}$-algebra by $D^{-1}U(\mathfrak{h}) \cup \{ \tilde{x}_{-2a}, \tilde{x}_{-a}, \tilde{h}, \tilde{x}_a, \tilde{x}_{2a} \}$ subject to relations (3.41).

Proof. Let $A$ be the $\mathbb{C}$-algebra generated by $D^{-1}U(\mathfrak{h}) \cup \{ \tilde{x}_{-2a}, \tilde{x}_{-a}, \tilde{h}, \tilde{x}_a, \tilde{x}_{2a} \}$ modulo relations (3.41). By Theorem 3.5, there exists a surjective $\mathbb{C}$-algebra homomorphism $\varphi : A \to Z$. Suppose $a \in A$ belongs to the kernel of this map. Using the relations (3.41) we map write $a$ as a linear combination of ordered monomials with coefficients on the left from $D^{-1}U(\mathfrak{h})$. Applying $\varphi$ we get a corresponding linear combination of ordered monomials in $Z$. But by Proposition 3.6 these are linearly independent over $D^{-1}U(\mathfrak{h})$, so all the coefficients are zero. Thus $a = 0$. This proves that $\varphi$ is an isomorphism. □

4. Applications

4.1. Dilation automorphisms of the reduction algebra.

Proposition 4.1. Any automorphism $\tau$ of the diagonal reduction algebra of $\mathfrak{osp}(1|2)$ of the form $\tau(\tilde{x}_{\beta}) = k_3 \tilde{x}_{\beta}$, $\tau(\tilde{h}) = k_h \tilde{h}$, and $\tau(H) = H$ is given by $k_h = \epsilon \in \{ \pm 1 \}$, $k_3 = \epsilon$, for some nonzero $\xi \in \mathbb{C}$, $k_{-a} = \epsilon^{-1}$, $k_{2a} = \epsilon \xi^2$, and $k_{-2a} = \epsilon \xi^{-2}$. Conversely, for any $\epsilon \in \{ \pm 1 \}$ and any nonzero $\xi \in \mathbb{C}$, there is an automorphism $\tau$ given by $\tau(\tilde{x}_{\pm \alpha}) = \xi^{\alpha} \tilde{x}_{\pm \alpha}$, $\tau(\tilde{h}) = \epsilon \tilde{h}$, $\tau(\tilde{x}_{\pm 2a}) = \epsilon \xi^{\pm 2} \tilde{x}_{\pm 2a}$, and $\tau(H) = H$. The group of these dilation automorphisms is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{C}^*$.

Proof. Using (3.41b) to write $\tau(\tilde{x}_{\alpha} \otimes \tilde{x}_{-\alpha}) - k_3 k_{-a} \tilde{x}_{\alpha} \otimes \tilde{x}_{-\alpha}$ in the PBW basis, the PBW Theorem (Proposition 3.6) implies that $k_3 k_{-a} = k_2 k_{-2a} = k_h^2 = 1$. So $k_h = \epsilon$ for some $\epsilon \in \{ \pm 1 \}$. By (3.41b), we get $k_3^2 = k_h k_{2a}$. Since $k_2^2 = 1$, the previous equation implies $k_{2a} = k_h k_{-2a}$. Conversely, we encourage the reader to verify that any such $\tau$ preserves the relations in Theorem 3.5 as we omit the calculations here. These automorphisms commute (like complex numbers under multiplication) and are uniquely determined by a choice of $\epsilon \in \{ \pm 1 \}$ and a complex unit $\xi \in \mathbb{C}^*$, proving the final statement. □

Example 4.2. The square $\Theta^2$ of the anti-automorphism $\Theta$ (Section 3.1.5) is an automorphism of the form above corresponding to $\epsilon = 1$ and $\xi = -1$.
4.2. Linear Casimir. Put
\[ C^{(1)} = \hat{h} = (H - 1)\hat{h}. \]  
(4.1)
Use of relations (3.41c) and (3.41j) show \( C^{(1)} \) commutes with \( \hat{h} \), \( H \), and \( \bar{x}_\alpha \). Similarly, \( C^{(1)} \) commutes with \( \bar{x}_{2\alpha} \). By applying the anti-automorphism \( \Theta \), we see \( C^{(1)} \) is a central element as it also commutes with \( \bar{x}_- \quad \bar{x}_{-2\alpha} \).

**Example 4.3.** Continuing Example 3.2 we can check \( \bar{\alpha} \) also commutes with \( \Theta \).

\[ \rho(C^{(1)}) = \rho ((H - 1)\hat{h}) = \begin{bmatrix} -\frac{9}{4} & 0 & 0 & 0 \\ 0 & -\frac{9}{4} & 0 & 0 \\ 0 & 0 & -\frac{9}{4} & 0 \\ 0 & 0 & 0 & -\frac{9}{4} \end{bmatrix} . \]

Precisely,

**4.3. Quadratic anti-Casimir.** Recall that an even element is called anti-central if and only if the following equations hold:

\[ \omega(1) = \rho(1) = 0. \]
\[ \omega(2) = \rho(2) = 0. \]
\[ \omega(4) = \rho(4) = 0. \]
\[ \omega(9) = \rho(9) = 0. \]
\[ \omega(15) = \rho(15) = 0. \]

**Lemma 4.4.** Consider the following ansatz of a quadratic anti-central element in \( U/\Pi \):

\[ C^{(2)} = f_2(H) \bar{x}_{-2\alpha} \diamond \bar{x}_{2\alpha} + f_1(H) \bar{x}_- \diamond \bar{x}_\alpha + f_0(H) \hat{h} \diamond \hat{h} + g(H). \]

(4.2)
Then \( C^{(2)} \) is anti-central if and only if the following equations hold:

\[ f_1(H - 1) = \frac{H - 1}{H - 2} f_1(H) + \frac{H - 1}{H - 1} f_2(H) \]  
(4.3)
\[ f_2(H - 1) = -\frac{4(H - 1)}{(H - 2)(H - 3)} f_1(H) - \frac{H - 1}{H - 3} f_2(H) \]  
(4.4)
\[ f_0(H - 1) = -f_0(H) + \frac{1}{(H - 1)(H - 2)} f_1(H) \]  
(4.5)
\[ g(H - 1) = -g(H) - f_1(H)(H - 1) \]  
(4.6)

**Proof.** The details of the proof are found in Appendix B.

Next we will show existence and uniqueness (up to complex scalar multiple) of the above system of equations.

Consider the following element of \( U(\mathfrak{osp}(1|2)) \) from [Les95]:

\[ L = -\frac{1}{2}(x_\alpha x_- - x_- x_\alpha - \frac{1}{2}) \]
\[ = x_- x_\alpha - \frac{1}{2} h + \frac{1}{4} \]  
(4.7)
One can check that

\[ L x_\alpha + x_\alpha L = 0. \]  
(4.8)
Like so, using (3.3), (3.4):

\[ L x_\alpha + x_\alpha L = x_- x_\alpha x_\alpha - \frac{1}{2} h x_\alpha + \frac{1}{4} x_\alpha + x_\alpha x_- x_\alpha - \frac{1}{2} x_\alpha h + \frac{1}{4} x_\alpha \]
\[ = x_- x_\alpha x_\alpha - x_- x_\alpha x_\alpha + h x_\alpha - h x_\alpha - \frac{1}{2} x_\alpha + \frac{1}{2} x_\alpha \]
\[ = 0. \]

\[ ^1 \text{The following converts the basis of } \mathfrak{osp}(1|2) \text{ of Lesniewski to ours: } L_3 = \frac{1}{2} h, \quad G_{\pm} = \mp \frac{1}{2} x_\pm, \quad L_{\pm} = -x_{\mp 2\alpha}. \]
Since $L$ is fixed by the anti-automorphism $\theta$, $L$ also anti-commutes with $x_{-\alpha}$, hence the commuting of $L$ with $x_{-2\alpha}$ and $h$. Recall $U(\mathfrak{osp}(1|2)) = U(\mathfrak{osp}(1|2))_{\mathfrak{p}} \oplus U(\mathfrak{osp}(1|2))_{\mathfrak{r}}$ is a superalgebra, so the statements above imply $L$ belongs to the center of the subalgebra $U(\mathfrak{osp}(1|2))_{\mathfrak{p}}$ and anti-commutes with elements of $U(\mathfrak{osp}(1|2))_{\mathfrak{r}}$. Thus $L$ is an anti-central element of $U(\mathfrak{osp}(1|2))$.

Consider now

$$
\mathcal{L} = L \otimes L \in U.
$$

Then $\mathcal{L}$ anti-commutes with $X_\alpha$, hence commutes with $X_{2\alpha}$, hence $\mathcal{L}$ belongs to the normalizer $N$ of $I = U\mathfrak{g}_+$ in $U$. Thus

$$
P_M(\mathcal{L} + \mathfrak{H}) = \mathcal{L} + I,
$$

or equivalently,

$$
P_M(\mathcal{L} + I) = \mathcal{L} + I.
$$

We will now simplify $\bar{\mathcal{L}} = \mathcal{L} + \mathfrak{H}$ and write it in the PBW basis for $U/\mathfrak{H}$. Below let $\equiv$ stand for congruence modulo $\mathfrak{H}$

$$
\mathcal{L} \equiv (L \otimes 1)(1 \otimes L)
\equiv (x_{-\alpha}x_\alpha - \frac{1}{2}h + \frac{1}{4}) \otimes 1 \cdot 1 \otimes (x_{-\alpha}x_\alpha - \frac{1}{2}h + \frac{1}{4})
\equiv \left(\frac{1}{4}(X_{-\alpha} + \bar{x}_{-\alpha})(X_\alpha + \bar{x}_\alpha) - \frac{1}{4}(H + \bar{h}) + \frac{1}{2}(\frac{1}{4}(X_{-\alpha} - \bar{x}_{-\alpha})(X_\alpha - \bar{x}_\alpha) - \frac{1}{4}(H - \bar{h}) + \frac{1}{4})
\equiv \left(\frac{1}{4}(\bar{x}_{-\alpha}(X_\alpha + \bar{x}_\alpha) - \frac{1}{4}(H + \bar{h}) + \frac{1}{4}(\frac{1}{4}(X_{-\alpha} + \bar{x}_{-\alpha})\bar{x}_\alpha - \frac{1}{4}(H - \bar{h}) + \frac{1}{4}).
$$

Thus

$$
16\mathcal{L} \equiv \bar{x}_{-\alpha}(X_\alpha + \bar{x}_\alpha)(-X_{-\alpha} + \bar{x}_{-\alpha})\bar{x}_\alpha
+ \bar{x}_{-\alpha}(X_\alpha + \bar{x}_\alpha)(1 - H + \bar{h})
+ (1 - H - \bar{h})(-X_{-\alpha} + \bar{x}_{-\alpha})\bar{x}_\alpha
+ (1 - H - \bar{h})(1 - H + \bar{h})
\equiv \bar{x}_{-\alpha}(X_\alpha + \bar{x}_\alpha)(-X_{-\alpha} + \bar{x}_{-\alpha})\bar{x}_\alpha
+ \bar{x}_{-\alpha}\bar{x}_\alpha(1 - H + \bar{h}) + \bar{x}_{-\alpha}[X_\alpha, \bar{h}]
+ (1 - H - \bar{h})\bar{x}_{-\alpha}\bar{x}_\alpha + [\bar{h}, X_{-\alpha}]\bar{x}_\alpha
+ (1 - H - \bar{h})(1 - H + \bar{h})
\equiv \bar{x}_{-\alpha}(X_\alpha + \bar{x}_\alpha)(-X_{-\alpha} + \bar{x}_{-\alpha})\bar{x}_\alpha
+ \bar{x}_{-\alpha}\bar{x}_\alpha(-H + \bar{h}) + 4\bar{x}_{-\alpha}\bar{x}_\alpha
+ (1 - H - \bar{h})(1 - H + \bar{h}).
$$

Using that $X_\alpha + \bar{x}_\alpha = 2x_\alpha \otimes 1$ and $X_{-\alpha} - \bar{x}_{-\alpha} = 2 \cdot 1 \otimes x_{-\alpha}$ anti-commute, we get

$$
16\mathcal{L} \equiv \bar{x}_{-\alpha}(-\bar{x}_{-\alpha}X_\alpha + X_{-\alpha}X_\alpha - \bar{x}_{-\alpha}\bar{x}_\alpha + X_{-\alpha}\bar{x}_\alpha)\bar{x}_\alpha
+ [\bar{x}_{-\alpha}\bar{x}_\alpha, \bar{h}] - 2(H - 2)\bar{x}_{-\alpha}\bar{x}_\alpha
- \bar{h}h + (H - 1)^2.
$$
Since $\overline{x}_a \overline{x}_a = \frac{1}{2}[\overline{x}_a, \overline{x}_a] = X_{-2a} \in g_-$ and similarly $\overline{x}_a \overline{x}_a \in g_+$, we get

$$16\mathcal{C} \equiv \overline{x}_a X_{-a} X_a \overline{x}_a$$

$$+ [\overline{x}_a, \hat{h}] \overline{x}_a + \overline{x}_a [\overline{x}_a, \hat{h}] - 2(H - 2) \overline{x}_a \overline{x}_a - \hat{h} \hat{h} + (H - 1)^2$$

$$\equiv (-X_{-a} \overline{x}_a + 2 \overline{x}_{-2a})(-\overline{x}_a X_a - 2 \overline{x}_{2a}) - 2(H - 2) \overline{x}_a \overline{x}_a - \hat{h} \hat{h} + (H - 1)^2$$

$$\equiv -4 \overline{x}_{-2a} \overline{x}_{2a} - 2(H - 2) \overline{x}_a \overline{x}_a - \hat{h} \hat{h} + (H - 1)^2.$$

The following theorem gives the explicit expression of $\mathcal{C}$ in the PBW basis.

**Theorem 4.5.** The unique (up to complex scalar multiple) anti-central element $C^{(2)}$ of the form in Lemma 4.4 is:

$$16\mathcal{C} = \frac{H - 2}{H - 1} \overline{x}_a \overline{x}_a - \left(2(H - 2) + \frac{1}{H - 1}\right) \overline{x}_a \overline{x}_a - \frac{1}{(H - 1)^2} \hat{h} \hat{h} + (H - 1)^2. \quad (4.12)$$

**Proof.** As already observed, $\mathcal{C}$ is an anti-central element. Using Lemma 3.3 one can write $\mathcal{C}$ in the PBW basis from Proposition 2.6 obtaining the identity (4.12). Therefore, by Lemma 4.4, the coefficients of $\mathcal{C}$ in the right hand side of (4.12) satisfy equations (4.3)-(4.6). The uniqueness follows from the fact that, given initial values for $f_i(\frac{1}{2})$, $i = 0, 1, 2$, and $g(\frac{1}{2})$, the system (4.3)-(4.6) uniquely determines $f_i(a)$ and $g(a)$ for every half-integer $a \in \frac{1}{2} + Z$. Indeed, the $2 \times 2$ coefficient matrix has determinant $-\frac{H^3 - 3}{H - 2}$. The rational functions $f_i$ and $g$ are then uniquely determined by those values $f_i(a)$ and $g(a)$. 

**Example 4.6.** Continuing Example 3.2 one can check that $\rho(\mathcal{C}) = \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$: 

$$\rho(16\mathcal{C}) = \rho \left( \frac{H - 2}{H - 1} \overline{x}_a \overline{x}_a - \left(2(H - 2) + \frac{1}{H - 1}\right) \overline{x}_a \overline{x}_a - \frac{1}{(H - 1)^2} \hat{h} \hat{h} + (H - 1)^2 \right)$$

$$= \left[ \begin{array}{ccc} 12 & 0 & 0 \\ 0 & 20 & 3 \\ 0 & 0 & -4 \end{array} \right] \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] = \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right].$$
Appendix A

Details for the proof of Lemma 3.3. For (3.28) we have
\[
\dot{h} \odot \bar{x}_{\alpha} = \tilde{h}\bar{x}_{\alpha} + [\tilde{h}, X_{-\alpha}] \varphi_1(H + 1)[X_{\alpha}, \bar{x}_{\alpha}] + \Pi
\]
\[
= \tilde{h}\bar{x}_{\alpha} - 2\bar{x}_{-\alpha}\varphi_1(H + 1)\bar{x}_{2\alpha} + \Pi
\]
\[
= \tilde{h}\bar{x}_{\alpha} - 2\varphi_1(H)\bar{x}_{-\alpha}\bar{x}_{2\alpha} + \Pi
\]
For (3.29) we have
\[
\bar{x}_{-\alpha} \odot \bar{x}_{\alpha} = \bar{x}_{-\alpha}\bar{x}_{\alpha} + [\bar{x}_{-\alpha}, X_{-\alpha}] \varphi_1(H + 1)[X_{\alpha}, \bar{x}_{\alpha}] + \Pi
\]
\[
= \bar{x}_{-\alpha}\bar{x}_{\alpha} - 4\bar{x}_{-2\alpha}\varphi_1(H + 1)\bar{x}_{2\alpha} + \Pi
\]
\[
= \bar{x}_{-\alpha}\bar{x}_{\alpha} - 4\varphi_1(H - 1)\bar{x}_{-2\alpha}\bar{x}_{2\alpha} + \Pi
\]
To prove (3.30) we use three terms from the expansion (3.23):
\[
\tilde{h} \odot \tilde{h} = (\tilde{h})^2 + [\tilde{h}, X_{-\alpha}] \varphi_1(H + 1)[X_{\alpha}, \tilde{h}] +
\]
\[
+ [[\tilde{h}, X_{-\alpha}], X_{-\alpha}] \varphi_2(H + 2)[X_{\alpha}, [X_{\alpha}, \tilde{h}]] + \Pi
\]
\[
= (\tilde{h})^2 + \bar{x}_{-\alpha}\varphi_1(H + 1)\bar{x}_{\alpha} - 4\bar{x}_{-2\alpha}\varphi_2(H + 2)\bar{x}_{2\alpha} + \Pi
\]
\[
= (\tilde{h})^2 + \varphi_1(H)\bar{x}_{-\alpha}\bar{x}_{\alpha} - 4\varphi_2(H)\bar{x}_{-2\alpha}\bar{x}_{2\alpha} + \Pi
\]
Using (3.28) we have, recalling that \(\theta(\bar{x}_{2\alpha}) = -\bar{x}_{-2\alpha}\) and \(\theta(x_{\alpha}) = \sqrt{-1} x_{-\alpha}\),
\[
\bar{x}_{-\alpha} \odot \tilde{h} = -\sqrt{-1}\Theta(\tilde{h} \odot \bar{x}_{\alpha})
\]
\[
= \bar{x}_{-\alpha}\tilde{h} + 2\bar{x}_{-2\alpha}\varphi_1(H + 1)\bar{x}_{2\alpha} + \Pi
\]
\[
= \bar{x}_{-\alpha}\tilde{h} + 2\varphi_1(H - 1)\bar{x}_{-2\alpha}\bar{x}_{2\alpha} + \Pi
\]
Similarly, using (3.27) we have
\[
\bar{x}_{-\alpha} \odot \bar{x}_{-\alpha} = \Theta(\bar{x}_{\alpha} \odot \bar{x}_{\alpha})
\]
\[
= (\bar{x}_{-\alpha})^2 + 2\bar{x}_{-2\alpha}\varphi_1(H + 1)\tilde{h} + \Pi
\]
\[
= (\bar{x}_{-\alpha})^2 + 2\varphi_1(H - 1)\bar{x}_{-2\alpha}\tilde{h} + \Pi
\]
\[
\square
\]

Details of the proof for Lemma 3.4. By (3.28),
\[
\tilde{h}\bar{x}_{\alpha} + \Pi = \tilde{h} \odot \bar{x}_{\alpha} + (2\varphi_1(H)\bar{x}_{-\alpha}\bar{x}_{2\alpha} + \Pi)
\]
\[
= \tilde{h} \odot \bar{x}_{\alpha} + 2\varphi(H)\bar{x}_{-\alpha} \odot \bar{x}_{2\alpha}.
\]
By (3.29),
\[
\bar{x}_{-\alpha}\bar{x}_{\alpha} + \Pi = \bar{x}_{-\alpha} \odot \bar{x}_{\alpha} + (4\varphi_1(H - 1)\bar{x}_{-2\alpha}\bar{x}_{2\alpha} + \Pi)
\]
\[
= \bar{x}_{-\alpha} \odot \bar{x}_{\alpha} + 4\varphi(H - 1)\bar{x}_{-2\alpha} \odot x_{2\alpha}.
\]
By (3.30) and (3.29),
\[
\bar{h} \bar{h} + \Pi = \bar{h} \bar{h} - (\varphi_1(H)\bar{x}_\alpha\bar{x}_\alpha - 4\varphi_2(H)\bar{x}_{-2\alpha} + \Pi)
\]
\[
= \bar{h} \bar{h} - \varphi_1(H)(\bar{x}_\alpha \bar{x}_\alpha + 4\varphi_1(H - 1)\bar{x}_{-2\alpha} \bar{x}_{-2\alpha} + 4\varphi_2(H)\bar{x}_{-2\alpha} \bar{x}_{-2\alpha})
\]
\[
= \bar{h} \bar{h} - \varphi_1(H)\bar{x}_\alpha \bar{x}_\alpha + 4(\varphi_2(H) - \varphi_1(H)\varphi_1(H - 1))\bar{x}_{-2\alpha} \bar{x}_{-2\alpha}.
\]

By (3.36),
\[
\bar{x}_\alpha \bar{h} + \Pi = \Theta(\bar{h}\bar{x}_\alpha + \Pi)
\]
\[
= \Theta(\bar{h} \bar{x}_\alpha + 2\varphi_1(H)\bar{x}_\alpha \bar{x}_\alpha)
\]
\[
= \bar{x}_\alpha \bar{h} - \bar{x}_{-2\alpha} \bar{x}_\alpha \cdot 2\varphi_1(H)
\]
\[
= \bar{x}_\alpha \bar{h} - 2\varphi_1(H - 1)\bar{x}_{-2\alpha} \bar{x}_\alpha.
\]

By (3.35),
\[
\bar{x}_\alpha \bar{x}_\alpha + \Pi = \Theta(\bar{x}_\alpha \bar{x}_\alpha + \Pi)
\]
\[
= \Theta(\bar{x}_\alpha \bar{x}_\alpha + 2\varphi_1(H + 1)\bar{h} \bar{x}_{-2\alpha})
\]
\[
= \bar{x}_\alpha \bar{x}_\alpha - \bar{x}_{-2\alpha} \bar{x}_\alpha \cdot 2\varphi_1(H + 1)
\]
\[
= \bar{x}_\alpha \bar{x}_\alpha - 2\varphi_1(H - 1)\bar{x}_{-2\alpha} \bar{h}.
\]

\[\square\]

**Details of the proof of Theorem 3.7** We have
\[
\bar{x}_\alpha \bar{x}_\alpha = \bar{x}_\alpha P\bar{x}_\alpha + \Pi
\]
\[
= \bar{x}_\alpha^2 + [\bar{x}_\alpha, X_{-\alpha}]\varphi_1(H + 1)[X_{\alpha}, \bar{x}_\alpha] + \Pi
\]
\[
= \bar{x}_\alpha^2 - 2\bar{h}\varphi_1(H + 1)\bar{x}_{-2\alpha} + \Pi
\]
\[
= -2\varphi_1(H + 1)\bar{h} \bar{x}_{-2\alpha}
\]

where we used that \(\bar{x}_\alpha^2 = \frac{1}{2}[\bar{x}_\alpha, \bar{x}_\alpha] = -X_{2\alpha} \in g_+ \subset \Pi\), and (3.2a). Applying \(\Theta\) to (3.41c) gives (3.41b).

Next,
\[
\bar{x}_{2\alpha} \bar{h} = \bar{x}_{2\alpha} \bar{h} + [\bar{x}_{2\alpha}, X_{-\alpha}]\varphi_1(H + 1)[X_{\alpha}, \bar{h}] + [[\bar{x}_{2\alpha}, X_{-\alpha}], X_{-\alpha}]\varphi_2(H + 2)[X_{\alpha}, [X_{\alpha}, \bar{h}]] + \Pi
\]
\[
= \bar{x}_{2\alpha} \bar{h} + (-\bar{x}_{2\alpha})\varphi_1(H + 1)\bar{x}_{-2\alpha} + (-\bar{h})\varphi_2(H + 2)(-2\bar{x}_{2\alpha}) + \Pi
\]
\[
= \bar{h}\bar{x}_{2\alpha} - \varphi_1(H + 2)\bar{x}_{2\alpha} \bar{x}_\alpha \bar{x}_\alpha + 2\varphi_2(H + 2)\bar{h} \bar{x}_{2\alpha} + \Pi
\]
\[
= (1 + 2\varphi_2(H + 2))\bar{h} \bar{x}_{2\alpha}
\]

where we used (3.2a) and that \(\bar{x}_\alpha^2 \in \Pi\) as before.
Next,

\[
\bar{x}_{2a} \triangleleft \bar{x}_{-a} = \bar{x}_{2a} \bar{x}_{-a} + [\bar{x}_{2a}, X_{-a}]\varphi_1(H + 1)[X_a, \bar{x}_{-a}]
\]

\[
+ [[\bar{x}_{2a}, X_{-a}], X_{-a}]\varphi_2(H + 2)[X_a, [X_a, \bar{x}_{-a}]]
\]

\[
+ [\bar{x}_{2a}, X_{-a}]^3(x_{2a}) \cdot \varphi_3(H + 3) \cdot [X_a, \bar{x}_{-a}]^3(x_{-a}) + \Pi
\]

\[
= \bar{x}_{2a} \bar{x}_{-a}
\]

\[
+ (-\bar{x}_{a})\varphi_1(H + 1)\bar{h}
\]

\[
+ (-\bar{h})\varphi_2(H + 2)\bar{x}_{a}
\]

\[
+ (-\bar{x}_{-a})\varphi_3(H + 3)(-2\bar{x}_{2a}) + \Pi
\]

\[
= \left(1 + 2\varphi_3(H + 2)\right)\bar{x}_{-a}x_{2a} + \left(-\varphi_1(H + 2) - \varphi_2(H + 2)\right)\bar{h}x_{a} + \Pi
\]

\[
= \left(1 + 2\varphi_3(H + 2)\right)\bar{x}_{-a} \triangleleft x_{2a}
\]

\[
+ \left(-\varphi_1(H + 2) - \varphi_2(H + 2)\right)\left(\bar{h} \triangleleft x_{a} + 2\varphi_1(H)\bar{x}_{-a} \triangleleft x_{2a}\right)
\]

\[
= \left(1 + 2\varphi_3(H + 2) - 2\varphi_1(H + 2)\varphi_1(H) - 2\varphi_2(H + 2)\varphi_1(H)\right)\bar{x}_{-a} \triangleleft x_{2a}
\]

\[
+ \left(-\varphi_1(H + 2) - \varphi_2(H + 2)\right)\bar{h} \triangleleft x_{2a}
\]

using (3.25) and (3.28).

Next,

\[
\bar{x}_{2a} \triangleleft \bar{x}_{-2a} = \sum_{n=0}^{4} [\bar{x}_{2a}, X_{-a}]^n(x_{2a}) \cdot \varphi_2(H + n) \cdot [X_a, \bar{x}_{-2a}]^n(x_{-2a}) + \Pi
\]

\[
= \bar{x}_{2a} \bar{x}_{-2a} - \bar{x}_{a}\varphi_1(H + 1)\bar{x}_{-a} - \bar{h}\varphi_2(H + 2)\bar{h} - \varphi_3(H + 3)\bar{x}_{a}
\]

\[
+ 4\bar{x}_{-2a}\varphi_4(H + 4)\bar{x}_{2a} + \Pi
\]

\[
= \bar{x}_{-2a} \bar{x}_{2a} - H - \varphi_1(H + 2)\left(-\bar{x}_{-a}\bar{x}_{a} + H\right) - \varphi_2(H + 2)\bar{h}^2 - \varphi_3(H + 3)\bar{x}_{-a}\bar{x}_{a}
\]

\[
+ 4\varphi_4(H + 2)\bar{x}_{-2a} \bar{x}_{2a} + \Pi
\]

\[
= -H\left(1 + \varphi_1(H + 2)\right) + \left(1 + 4\varphi_4(H + 2)\right)\bar{x}_{-2a} \bar{x}_{2a}
\]

\[
+ \left(\varphi_4(H + 2) - \varphi_3(H + 2)\right)\left(\bar{x}_{-a} \triangleleft \bar{x}_{a} + 4\varphi_1(H - 1)\bar{x}_{2a} \triangleleft \bar{x}_{2a}\right)
\]

\[
- \varphi_2(H + 2)\left(\bar{h} \triangleleft \bar{h} - \varphi_1(H)\bar{x}_{-a} \triangleleft \bar{x}_{a} + (4\varphi_2(H) - 4\varphi_1(H)\varphi_1(H - 1))\bar{x}_{-2a} \triangleleft \bar{x}_{2a}\right)
\]

\[
= -H\left(1 + \varphi_1(H + 2)\right) + \left(1 + 4\varphi_4(H + 2) + 4\varphi_1(H + 2)\varphi_1(H - 1)\right)\bar{x}_{-2a} \triangleleft \bar{x}_{2a}
\]

\[
- 4\varphi_3(H + 3)\varphi_1(H - 1) - 4\varphi_2(H + 2)\varphi_2(H) + 4\varphi_2(H + 2)\varphi_1(H)\varphi_1(H - 1)\bar{x}_{-2a} \triangleleft \bar{x}_{2a}
\]

\[
+ \left(\varphi_1(H + 2) - \varphi_3(H + 2) + \varphi_2(H + 2)\varphi_1(H)\right)\bar{x}_{-a} \triangleleft \bar{x}_{a} - \varphi_2(H + 2)\bar{h} \triangleleft \bar{h}
\]
where we used (3.25), (3.30), and (3.29) in the penultimate equality to express products in the tilde variables in terms of the diamond products.

Next,

\[
\bar{x}_\alpha \circ \tilde{h} = \bar{x}_\alpha \tilde{h} + [\bar{x}_\alpha, X_{-\alpha}] \varphi_1(H + 1)[X_{\alpha}, \tilde{h}] + [[\bar{x}_\alpha, X_{-\alpha}], X_{-\alpha}] \varphi_2(H + 2)[X_{\alpha}, [X_{\alpha}, \tilde{h}]] + \mathbb{I} \\
= (1 + \varphi_1(H + 1))\tilde{h}\bar{x}_\alpha - 2\varphi_2(H + 1)\bar{x}_{-\alpha} \tilde{x}_{2\alpha} \\
= (1 + \varphi_1(H + 1))(\tilde{h} \circ \bar{x}_\alpha + 2\varphi_1(H)\bar{x}_{-\alpha} \circ \tilde{x}_{2\alpha}) - 2\varphi_2(H + 1)\bar{x}_{-\alpha} \circ \tilde{x}_{2\alpha} \\
= (1 + \varphi_1(H + 1))\tilde{h} \circ \bar{x}_\alpha + (2\varphi_1(H) + 2\varphi_1(H + 1)\varphi_1(H) - 2\varphi_1(H + 1))\bar{x}_{-\alpha} \circ \tilde{x}_{2\alpha}
\]

where we used (3.28) and (3.29). Substituting \( \varphi_1(H) = \varphi_2(H) = \frac{1}{H} \), one checks that \( 2\varphi_1(H) + 2\varphi_1(H + 1)\varphi_1(H) - 2\varphi_2(H + 1) = 0 \), proving (3.41).

Next,

\[
\bar{x}_\alpha \circ \bar{x}_{-\alpha} = \sum_{n=0}^{3} [\cdot, X_{-\alpha}]^{n}(\bar{x}_\alpha) \cdot \varphi_n(H + n) \cdot [X_{\alpha}, \cdot]^{n}(\bar{x}_{-\alpha}) + \mathbb{I} \\
= -\bar{x}_{-\alpha} \bar{x}_\alpha + H + \varphi_1(H + 1)\tilde{h}^2 + \varphi_2(H + 1)\bar{x}_{-\alpha} \bar{x}_\alpha - 4\varphi_3(H + 1)\bar{x}_{-2\alpha} \tilde{x}_{2\alpha} + \mathbb{I} \\
= H - 4\varphi_3(H + 1)\bar{x}_{-2\alpha} \circ \tilde{x}_{2\alpha} + (-1 + \varphi_2(H + 1))(\bar{x}_{-\alpha} \circ \bar{x}_\alpha + 4\varphi_1(H - 1)\bar{x}_{-2\alpha} \circ \tilde{x}_{2\alpha}) \\
+ \varphi_1(H + 1)(\tilde{h} \circ \bar{x}_\alpha - \varphi_1(H)\bar{x}_{-\alpha} \circ \tilde{x}_{2\alpha} + (4\varphi_2(H) - 4\varphi_1(H)\varphi_1(H - 1))\bar{x}_{-2\alpha} \circ \tilde{x}_{2\alpha}) \\
= H + (-1 + \varphi_2(H + 1) - \varphi_1(H + 1)\varphi_1(H))\bar{x}_{-\alpha} \circ \bar{x}_\alpha \\
+ (-4\varphi_3(H + 1) - 4\varphi_1(H - 1) + 4\varphi_2(H + 1)\varphi_1(H - 1) + 4\varphi_1(H + 1)\varphi_2(H) \\
- 4\varphi_1(H + 1)\varphi_1(H)\varphi_1(H - 1))\bar{x}_{-2\alpha} \circ \tilde{x}_{2\alpha} + \varphi_1(H + 1)\tilde{h} \circ \tilde{h}
\]

using (3.26), (3.29), and (3.30). Substituting expressions for \( \varphi_1(H) \), and simplifying, gives (3.41).

Relation (3.41) follows directly from (3.41), by applying the anti-automorphism \( \Theta \) to both sides. Similarly, (3.41) follows from (3.41), and (3.41) follows from (3.41).

\[ \square \]

**APPENDIX B**

*Details of the proof for Lemma 4.4.* The following expressions hold in the double coset algebra \((U/\mathbb{I} \circ)\), where we suppress any adorning bars of basis elements and \( \circ \) in products. We also write \( \tilde{h} = (H - 1)h \) for simplicity. Thus, we write \( C^{(2)} \) as

\[
C^{(2)} = f_2(H)\bar{x}_{-2\alpha} \bar{x}_{2\alpha} + f_1(H)\bar{x}_{-\alpha} \bar{x}_\alpha + f_0(H)\tilde{h}\tilde{h} + g(H).
\] (4.13)
We have

\[
C^{(2)}_{-\alpha} = f_2(H)x_{-2\alpha}(x_{2\alpha}x_{-\alpha}) + f_1(H)x_{-\alpha}(x_{\alpha}x_{-\alpha}) + f_0(H)x_{-\alpha}\hat{h}\hat{h} + g(H)x_{-\alpha}
\]

\[
= f_2(H)x_{-2\alpha}\left(1 - \frac{2}{H(H-1)}\right)x_{-\alpha}x_{2\alpha} + \frac{2}{H+1}hx_{\alpha}
\]

\[
+ f_1(H)x_{-\alpha}\left(-1 + \frac{1}{H-1}\right)x_{-\alpha}x_{\alpha} + \frac{4H}{(H-1)(H-2)}x_{-2\alpha}x_{2\alpha} - \frac{1}{H(H-1)^2}\hat{h}\hat{h} + H
\]

\[
+ f_0(H)x_{-\alpha}\hat{h}\hat{h} + g(H)x_{-\alpha}
\]

\[
= f_2(H)\left(1 - \frac{2}{(H-2)(H-3)}\right)x_{-2\alpha}x_{-\alpha}x_{2\alpha}
\]

\[
+ f_2(H)\frac{2}{H-1}x_{-2\alpha}hx_{\alpha}
\]

\[- f_1(H)(1 + \frac{1}{H-2})x_{-\alpha}x_{-\alpha}x_{\alpha}
\]

\[+ f_1(H)\frac{4(H-1)}{(H-2)(H-3)}\left(1 - \frac{2}{H-2}\right)x_{-2\alpha}x_{-\alpha}x_{2\alpha}
\]

\[+ f_0(H)x_{-\alpha}\hat{h}\hat{h} - f_1(H)\frac{1}{(H-1)(H-2)^2}x_{-\alpha}\hat{h}\hat{h}
\]

\[+ g(H)x_{-\alpha} + f_1(H)(H-1)x_{-\alpha}
\]

\[
\overset{3.410}{=} \left(f_2(H)\left(1 - \frac{2}{(H-2)(H-3)}\right) + f_1(H)\frac{4(H-1)}{(H-2)(H-3)}\left(1 - \frac{2}{H-2}\right)\right)x_{-2\alpha}x_{-\alpha}x_{2\alpha}
\]

\[
+ \left(f_2(H)\frac{2}{H-1} + f_1(H)(1 + \frac{1}{H-2})\frac{2}{H-2}\right)x_{-2\alpha}hx_{\alpha}
\]

\[+ \left(f_0(H) - f_1(H)\frac{1}{(H-1)(H-2)^2}\right)x_{-\alpha}\hat{h}\hat{h}
\]

\[+ \left(g(H) + f_1(H)(H-1)\right)x_{-\alpha}.
\]

On the other hand,

\[
x_{-\alpha}C^{(2)} = f_2(H-1)(x_{-\alpha}x_{-2\alpha})x_{2\alpha} + f_1(H-1)(x_{-\alpha}x_{-\alpha})x_{\alpha} + f_0(H-1)x_{-\alpha}\hat{h}\hat{h} + g_0(H-1)x_{-\alpha}
\]

\[
\overset{3.410}{=} f_2(H-1)\left(1 - \frac{2}{H-2}\right)x_{-2\alpha}x_{-\alpha}x_{2\alpha}
\]

\[- f_1(H-1)\frac{2}{H-2}x_{-2\alpha}hx_{\alpha}
\]

\[+ f_0(H-1)x_{-\alpha}\hat{h}\hat{h}
\]

\[+ g(H-1)x_{-\alpha}.
\]
By the PBW theorem, Proposition 3.6, $C^{(2)}$ anti-commutes with $x_{-\alpha}$ if and only if
\[
 f_1(H - 1)\frac{2}{H - 2} = f_2(H)\frac{2}{H - 1} + f_1(H)\left(1 + \frac{1}{H - 2}\right)\frac{2}{H - 2}
\]
\[
 -f_2(H - 1)\left(1 - \frac{2}{H - 2}\right) = f_2(H)\left(1 - \frac{2}{(H - 2)(H - 3)}\right) + f_1(H)\frac{4(H - 1)}{(H - 2)(H - 3)}\left(1 - \frac{2}{H - 2}\right)
\]
\[
 -f_0(H - 1) = f_0(H) - f_1(H)\frac{1}{(H - 1)(H - 2)^2}
\]
\[
 -g(H - 1) = g(H) + f_1(H)(H - 1)
\]
Simplifying these equations gives (4.29), (4.30), (4.31), (4.32). 

\[\square\]

References

[AM15] Thomas Ashton and Andrey Mudrov. “R-Matrix and Mickelsson Algebras for Orthosymmetric Quantum Groups”. Journal of Mathematical Physics 56.8 (Aug. 2015), p. 081701. issn: 0022-2488, 1089-7658. doi: 10.1063/1.4927582.

[AST73] R. M. Asherova, Yu. F. Smirnov, and V. N. Tolstoy. “Projection Operators for Simple Lie Groups: II. General Scheme for Constructing Lowering Operators. The Groups SU(n)”. Theoretical and Mathematical Physics 15.1 (Apr. 1973), pp. 392–401. issn: 0040-5779, 1573-9333. doi: 10.1007/BF01028268.

[BT81] F. A. Berezin and V. N. Tolstoy. “The Group with Grassmann Structure UOSP(1,2)”. Communications in Mathematical Physics 78.3 (Jan. 1981), pp. 409–428. issn: 0010-3616, 1432-0916. doi: 10.1007/BF01942332.

[CW12] Shun-Jen Cheng and Weiqiang Wang, Dualities and Representations of Lie Superalgebras. Vol. 144. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012. isbn: 978-0-8218-9118-6. doi: 10.1090/gsm/144.

[DER17] Hendrik De Bie, David Eelbode, and Matthias Roels. “The Harmonic Transvector Algebra in Two Vector Variables”. Journal of Algebra 473 (Mar. 2017), pp. 247–282. issn: 00218693. doi: 10.1016/j.jalgebra.2016.10.039.

[EV02] P. Etingof and A. Varchenko. “Dynamical Weyl Groups and Applications”. Advances in Mathematics 161.1 (Apr. 2002), pp. 74–127. issn: 00018708. doi: 10.1006/aima.2001.2034.

[Fer15] Thomas Ferguson. “Weight Modules of Orthosymplectic Lie Superalgebras”. PhD Thesis. The University of Texas at Arlington, 2015.

[FSS00] L. Frappat, A. Sciarrino, and P. Sorba. Dictionary on Lie Algebras and Superalgebras. San Diego: Academic Press, 2000. isbn: 978-0-12-265340-7.

[Kac77] V. G. Kac. “Lie Superalgebras”. Advances in Mathematics 26.1 (Oct. 1977), pp. 8–96. issn: 00018708. doi: 10.1016/0001-8708(77)90017-2.

[KO08] S. Khoroshkin and O. Ogievetsky. “Mickelsson Algebras and Zhelobenko Operators”. Journal of Algebra 319.5 (Mar. 2008), pp. 2113–2165. issn: 00218693. doi: 10.1016/j.jalgebra.2007.04.020.

[KO14] S. Khoroshkin and O. Ogievetsky. “Rings of Fractions of Reduction Algebras”. Algebras and Representation Theory 17.1 (Feb. 2014), pp. 265–274. issn: 1386-923X, 1572-9079. doi: 10.1007/s10468-012-9397-4.

[KO17] S. Khoroshkin and O. Ogievetsky. “Diagonal Reduction Algebra and the Reflection Equation”. Israel Journal of Mathematics 221.2 (Sept. 2017), pp. 705–729. issn: 0021-2172, 1565-8511. doi: 10.1007/s11856-017-1571-2.
[Leś95] Andrzej Leśniewski. “A Remark on the Casimir Elements of Lie Superalgebras and Quantized Lie Superalgebras”. *Journal of Mathematical Physics* 36.3 (Mar. 1995), pp. 1457–1461. ISSN: 0022-2488, 1089-7658. DOI: [10.1063/1.531133](https://doi.org/10.1063/1.531133).

[Mic73] Jouko Mickelsson. “Step Algebras of Semi-Simple Subalgebras of Lie Algebras”. *Reports on Mathematical Physics* 4.4 (June 1973), pp. 307–318. ISSN: 0034-4877. DOI: [10.1016/0034-4877(73)90006-2](https://doi.org/10.1016/0034-4877(73)90006-2).

[MM14] Takuya Matsumoto and Alexander Molev. “Representations of centrally extended Lie superalgebra $\mathfrak{psl}(2|2)$”. *Journal of Mathematical Physics* 55.9 (Sept. 2014), p. 091704. ISSN: 0022-2488, 1089-7658. DOI: [10.1063/1.4896396](https://doi.org/10.1063/1.4896396).

[Mol06] A.I. Molev. “Gelfand–Tsetlin Bases for Classical Lie Algebras”. *Handbook of Algebra*. Vol. 4. Elsevier, 2006, pp. 109–170. ISBN: 978-0-444-52213-9. DOI: [10.1016/S1570-7954(06)80006-9](https://doi.org/10.1016/S1570-7954(06)80006-9).

[Mus12] Ian M. Musson. *Lie Superalgebras and Enveloping Algebras*. Graduate Studies in Mathematics v. 131. Providence, R.I: American Mathematical Society, 2012. ISBN: 978-0-8218-6867-6.

[Tol11] V. N. Tolstoy. “Extremal Projectors for Contragredient Lie (Super)Symmetries (Short Review)”. *Physics of Atomic Nuclei* 74.12 (Dec. 2011), pp. 1747–1757. ISSN: 1063-7788, 1562-692X. DOI: [10.1134/S1063778811070155](https://doi.org/10.1134/S1063778811070155).

[Tol85] V. N. Tolstoy. “Extremal Projections for Reductive Classical Lie Superalgebras with a Non-Degenerate Generalized Killing Form”. *Russian Mathematical Surveys* 40.4 (Aug. 1985), pp. 241–242. ISSN: 0036-0279, 1468-4829. DOI: [10.1070/RM1985v040n04ABEH003668](https://doi.org/10.1070/RM1985v040n04ABEH003668).

[TV00] V. Tarasov and A. Varchenko. “Difference Equations Compatible with Trigonometric KZ Differential Equations”. *International Mathematics Research Notices* 2000.15 (2000), p. 801. ISSN: 10737928. DOI: [10.1155/S1073792800000441](https://doi.org/10.1155/S1073792800000441).

[van75] A van den Hombergh. “A Note on Mickelsson’s Step Algebra”. *Indagationes Mathematicae (Proceedings)* 78.1 (1975), pp. 42–47. ISSN: 13857258. DOI: [10.1016/1385-7258(75)90013-X](https://doi.org/10.1016/1385-7258(75)90013-X).

[Wil20] Dwight Anderson Williams II. “Bases of Infinite-Dimensional Representations of Orthosymplectic Lie Superalgebras”. PhD thesis. Arlington, TX: The University of Texas at Arlington, 2020.

[Zhe89] D. P. Zhelobenko. “Extremal Projectors and Generalized Mickelsson Algebras over Reductive Lie Algebras”. *Mathematics of the USSR-Izvestiya* 33.1 (Feb. 1989), pp. 85–100. ISSN: 0025-5726. DOI: [10.1070/IM1989v033n01ABEH000815](https://doi.org/10.1070/IM1989v033n01ABEH000815).

[Zhe94] D. P. Zhelobenko. *Representations of reductive Lie algebras*. Moscow: Nauka, 1994. ISBN: 978-5-02-014249-7.

[Zhe97] D. P. Zhelobenko. “Hypersymmetries of Extremal Equations”. *Nova Journal of Theoretical Physics* 5.4 (1997), pp. 243–258.