On finite energy monopole solutions in Weinberg-Salam model

D. G. Pak,1,2 P. M. Zhang,1,3 and L. P. Zou1

1Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China
2Lab. of Few Nucleon Systems, Institute for Nuclear Physics, Ulugbek, 100214, Uzbekistan
3State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

We study the problem of existence of finite energy monopole solutions in the Weinberg-Salam model starting with a most general ansatz for static axially-symmetric electroweak magnetic fields. The ansatz includes an explicit construction of field configurations with various topologies described by monopole and Hopf charges. We introduce a unique SU(2) gauge invariant definition for the electromagnetic field. It has been proved that magnetic charge of any finite energy monopole solution must be screened at far distance. This implies non-existence of finite energy monopole solutions with non-zero total magnetic charge. In a case of a special axially-symmetric Dashen-Hasslacher-Neveu ansatz we revise the structure of the sphaleron solution and show that sphaleron represents a non-trivial system of monopole and antimonopole with their centers located in one point. This is different from the known interpretation of the sphaleron as a monopole-antimonopole pair like Nambu’s “dumb-bell”. In general, the axially-symmetric magnetic field may admit a helical structure. We conjecture that such a solution exists and estimate a lower bound for its energy, $E_{\text{bound}} = 4.3$ TeV.

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I. INTRODUCTION

One of puzzling questions in modern high energy physics is whether monopoles exist in realistic theories of fundamental interactions. The well known singular monopoles [1,2] rather can not represent physical observable objects. Discovery of such solely existing monopoles of pure electromagnetic (or gravitational) nature would break the present fundamental laws of physics. In this regard, composite monopole solutions described within the framework of the standard electroweak model could be more favorable candidates for monopoles expected to be found at LHC [3]. An important issue in search of monopoles is to provide strong theoretical reasons for their existence. So far known solutions either lack regular structure or need essential extension beyond the standard model with including new fields and particles [4,5].

Yang-Mills-Higgs (YMH) theory with a complex doublet Higgs field does not admit finite energy monopole solutions. The reason is that finite energy Higgs field configurations in the asymptotic region at space infinity forms a three-dimensional sphere. Since a two-dimensional sphere is contractible on $S^3$ the relative homotopy group $\pi_2(S^3, S^2) = \pi_2(S^3)$ is trivial and can not provide topological monopole charge. It has been observed [6] that in Weinberg-Salam model the situation changes drastically due to presence of the hypermagnetic gauge symmetry $U_Y(1)$ which allows to define two-dimensional sphere as a coset $SU(2)/U_Y(1) \simeq S^2$. This leads to appearance of a non-trivial homotopy group $\pi_2(S^3)$ which provides a necessary condition for existence of topological monopole solutions. Indeed, an example of such a solution is given by singular Cho-Maison monopole [7]. The question whether there exist finite energy monopole solutions in Weinberg-Salam model has not been resolved so far and represents an important issue.

In the present paper we undertake a systematic study of possible monopole solutions in Weinberg-Salam model. Our consideration is restricted by the case of static axially-symmetric magnetic field configurations. A general ansatz for axially-symmetric solutions in the electroweak theory was suggested in [8]. We propose an alternative general axially-symmetric ansatz based on formalism of gauge invariant decomposition [9,11] in a natural basis frame $\hat{m}_i \ (i = 1, 2, 3)$ in the internal space of the group $SU(2)$. The natural basis frame is determined by $SU(2)$ triplet vector field $\hat{m} \equiv \hat{m}_3$ constructed from the Higgs field. A key point in our ansatz is an explicit parametrization of the vector field $\hat{m}$ in terms of two arbitrary functions which determine the topology of the Higgs field described by two topological invariants. Our approach allows to describe topology of the Higgs and gauge bosons and trace topological origin of magnetic like solutions. Applying our ansatz and finite energy condition one can find all possible local solutions for the gauge fields and Higgs boson in the asymptotic region at space infinity. For a wide class of static axially-symmetric field configurations we have proved that any possible finite energy monopole solution must have a totally screened magnetic charge. We show that finite energy monopole solutions with a non-zero total magnetic charge do not exist in Weinberg-Salam model. Possible finite energy magnetic solutions can be represented by monopole-antimonopole systems or pure magnetic field configurations only with vanishing total magnetic charge. We will consider simple examples of such two types of magnetic solutions.
In a special case of axially symmetric magnetic fields with vanishing azimuthal magnetic field component the general axially symmetric ansatz reduces to Dashen-Hasslacher-Neveu (DHN) ansatz \[12\] \[13\]. Within the DHN ansatz various sphaleron solutions were obtained in Yang-Mills-Higgs theory and Weinberg-Salam model \[12\] \[18\]. It was suggested to interpret the original DHN sphaleron as a monopole-antimonopole pair \[19\] \[20\]. The interpretation was conditioned by use of a special definition for electromagnetic field tensor which supposed to have ambiguity in its definition \[19\] \[21\]. One should notice, a physical concept and the respective mathematical definition for the electromagnetic field as a physical observable quantity should be unique and invariant under SU(2) gauge transformation. We show that electromagnetic vector potential can be uniquely defined in SU(2) gauge invariant manner. With this we revise the internal structure of the sphaleron and demonstrate that DHN sphaleron represents a monopole-antimonopole pair with screened monopole and antimonopole charges. Our view is different from Nambu’s "dumb-bell" monopole-antimonopole interpretation \[19\] \[20\].

In general, the magnetic field may admit non-vanishing helicity. We consider a simple possible magnetic solution in Yang-Mills-Higgs theory which possesses only azimuthal non-vanishing magnetic flux. Applying variational method we obtain an estimate for the lower energy bound, \(E_{\text{bound}} = 4.3\,\text{TeV}\). We conjecture that such a solution exists in the Weinberg-Salam model with energy comparable to the energy of the sphaleron.

The paper is organized as follows. In Section II a general ansatz for static axially-symmetric magnetic fields is proposed. In Section III the problem of existence of finite energy monopole solutions is studied. Internal structure of the sphaleron solution is revised in Section IV. We find an exact numeric magnetic solution in a simple SU(2) gauge invariant manner. With this we study the internal structure of the sphaleron and demonstrate that a possible magnetic solution in a simple SU(2) gauge invariant manner. With this we revise the internal structure of the sphaleron and demonstrate that DHN sphaleron represents a monopole-antimonopole pair with screened monopole and antimonopole charges. Our view is different from Nambu’s "dumb-bell" monopole-antimonopole interpretation \[19\] \[20\].

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## II. A GENERAL ANSATZ FOR STATIC AXIALLY SYMMETRIC MAGNETIC FIELD

The bosonic part of the Weinberg-Salam model is described by the following Lagrangian \((\mu, \nu = 0, 1, 2, 3, a, b = 1, 2, 3)\)

\[
\mathcal{L} = -\frac{1}{4} (F_{\mu\nu})^2 - \frac{1}{4} (G_{\mu\nu})^2 - |D_\mu \phi|^2 - \frac{\lambda}{2} (\phi^\dagger \phi - v^2)^2,
\]

\[
F_{\mu\nu}^a = \partial_\mu A_{\nu}^a - \partial_\nu A_{\mu}^a + i g a^{abc} A_{\mu}^b A_{\nu}^c,
\]

\[
G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,
\]

\[
D_\mu = \partial_\mu - i g_2 A_\mu - g_1 B_\mu,
\]

where \(A_\mu\) and \(B_\mu\) are the gauge fields corresponding to the electroweak gauge group \(SU(2) \times U_Y(1)\), and \(\phi\) is the Higgs complex scalar doublet.

The equations of motion have the following form

\[
D^\nu \tilde{F}_{\nu\mu}^a = \frac{ig}{2} \left( (\phi^\dagger \partial_\nu \phi) - (D_\nu \phi)^\dagger \phi \right),
\]

\[
\partial_\nu G_{\mu\nu} = \frac{ig}{2} \left( (\phi^\dagger \partial_\nu \phi) - (D_\nu \phi)^\dagger \phi \right),
\]

\[
D^\mu D_\mu \phi = \lambda (\phi^\dagger \phi - v^2)^2.
\]

To construct a most general ansatz for static axially-symmetric magnetic field configurations we apply Cho-Duan-Ge decomposition of the gauge potential in arbitrary orthonormal frame \((\hat{n}_1, \hat{n}_2, \hat{n}_3)\) in the internal space of \(SU(2)\) \[9\] \[10\]

\[
\vec{A}_\mu = \hat{A}_\mu + \vec{X}_\mu, \\
\hat{A}_\mu = A_\mu \hat{n} + \vec{C}_\mu, \\
\vec{C}_\mu = -\frac{1}{g} \hat{n} \times \partial_\mu \hat{n}, \quad \vec{X}_\mu = X_\mu^1 \hat{n}_1 + X_\mu^2 \hat{n}_2 + X_\mu^3 \hat{n}_3,
\]

where \(\vec{A}_\mu\) is a restricted gauge potential, \(\vec{X}_\mu\) contains two off-diagonal components of the gauge potential, \(\hat{n} \equiv \hat{n}_3\) is a basic \(SU(2)\) vector field which determines the basis frame \(\hat{n}_i\), \((i = 1, 2, 3)\), in the group \(SU(2)\) up to arbitrary local \(U(1)\) rotation around its direction. The vector \(\hat{n}\) satisfies the covariant constance condition

\[
\hat{D}_\mu \hat{n} = (\partial_\mu \hat{n} + g \hat{A}_\mu \times \hat{n}) = 0.
\]

For any given unit vector field \(\hat{n}\) the full \(SU(2)\) gauge field strength can be decomposed into Abelian and off-diagonal parts in a gauge invariant manner

\[
\vec{F}_{\mu\nu} = (F_{\mu\nu} + H_{\mu\nu}) \hat{n} + \vec{D}_\mu \vec{X}_\nu - \vec{D}_\nu \vec{X}_\mu + g \vec{X}_\mu \times \vec{X}_\nu,
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
\]

\[
H_{\mu\nu} = \frac{1}{g} \epsilon^{abc} \hat{n}_a \partial_\mu \hat{n}_b \partial_\nu \hat{n}_c = \partial_\mu \vec{C}_\nu - \partial_\nu \vec{C}_\mu,
\]

where the Abelian dual magnetic potential \(\vec{C}_\mu\) is defined by the vector field \(\hat{n}\) (up to dual Maxwell type \(\hat{U}(1)\) gauge transformation) \[10\].

The vector field \(\hat{n}\) can be expressed in terms of one complex function \(u(x)\) or in terms of complex projective coordinates \(\zeta_{1,2}\) on the sphere \(S^2\) by using a standard stereographic projection

\[
\hat{n} = \frac{1}{1 + u \bar{u}} \begin{pmatrix} u + \bar{u} u \bar{u} & -i (u - \bar{u}) \bar{u} \\ u \bar{u} - 1 & \end{pmatrix},
\]

\[
u = \frac{\zeta_1}{\zeta_2}, \quad \hat{n} = \zeta^* \partial \zeta,
\]

where \(\zeta_{1,2}\) form a unit complex \(SU(2)\) vector doublet.
A key point in our construction is an ansatz for the SU(2) vector field \( \hat{n} \) which can be written for the complex function \( u(x) \) in spherical coordinates as follows

\[
u(r, \theta, \varphi) = e^{-im\varphi} \left( \cot \left( \frac{n\theta}{2} \right) f(r, \theta) + i \csc \left( \frac{n\theta}{2} \right) Q(r, \theta) \right),
\]

where \((m, n)\) are integer winding numbers which specify the topological monopole charge, \( Q_m \), and the Hopf charge, \( Q_H \), of the magnetic field configuration \( H_{mn} \)

\[
Q_m = \frac{1}{A(S)} \int_{S^2} H_{ij} \cdot d\sigma^{ij},
\]

\[
Q_H = \frac{1}{32\pi^2} \int d^3x \epsilon^{ijk} \hat{C}_i H_{jk},
\]

where \( A(S) \) is the surface area of a sphere \( S^2 \).

Let us consider a parametrization for the Higgs field suitable for description of its topological properties. One can parameterize the Higgs field in terms of a scalar field \( \rho(x) \) and a unit complex SU(2) doublet \( \hat{\zeta} \) with explicit extracting \( U_Y(1) \) exponential factor containing a field variable \( \omega(x) \)

\[
\phi = \frac{1}{\sqrt{2}} \hat{\rho} \hat{e}^{i\omega(x)},
\]

\[
\hat{\zeta}^+ \hat{\zeta} = 1.
\]

In the following we will use a gauge condition \( \omega(x) = 0 \). One can define a real SU(2) triplet vector field \( \hat{m} \) constructed directly from the Higgs field

\[
\hat{m}^a = \hat{\phi}^+ \hat{\sigma}^a \hat{\phi} = \hat{\zeta}^+ \hat{\sigma}^a \hat{\zeta},
\]

\[
\hat{\phi} = \frac{\phi}{|\phi|}, \quad \hat{m}^2 = 1,
\]

where \( \hat{\sigma}^a \) are Pauli matrices.

The vector field \( \hat{n} \) in Abelian decomposition \([3]\) does not possess its own equations of motion and contains only topological degrees of freedom. Another feature of the gauge invariant Abelian decomposition is appearance of two types of SU(2) gauge symmetries, so-called “active” and “passive” ones \([22]\). Respectively, the vector \( \hat{n} \) transforms in different ways under these two types of gauge symmetries. Contrary to the case of the vector \( \hat{n} \), the vector \( \hat{m} \), \([10]\), is determined completely by a given Higgs field in a standard fundamental representation of SU(2).

So, the basis frame \( \hat{n} \) constructed with \( \hat{n}_3 \equiv \hat{m} \) represents a natural basis in the decomposition of the gauge potential. In general one can perform Abelian decomposition of the gauge potential in various basis frames defined by any vector field \( \hat{n} \). To fix the arbitrariness of choosing \( \hat{n} \) one can impose a constraint on \( \hat{n} \) and \( \hat{m} \). In practical applications it is convenient to fix the vector \( \hat{n} \) using one-to-one or double covering mapping between two spheres defined by vectors \( \hat{n} \) and \( \hat{m} \).

For the gauge potentials \( A_\mu, X^\mu_1^2 \) we adopt a most general form given by arbitrary functions depending on two spherical coordinates \((r, \theta)\). In the case of static magnetic fields one can choose a temporal gauge \( \bar{A}_0 = 0 \). It is convenient to parameterize the Higgs field in terms of three independent functions \( \beta_k(r, \theta) \) \((k = 1, 2, 3)\)

\[
\phi = \frac{1}{\sqrt{2}} (\beta_1 + i\beta_3, e^{im\varphi} \beta_2).
\]

In some cases it is suitable to change further the variables and express \( \beta_k \) in terms of a Higgs real scalar field \( \rho \) and two angle functions \( \tilde{T}(r, \theta), \tilde{S}(r, \theta) \)

\[
\beta_1(r, \theta) = \rho(r, \theta) \cos \frac{\tilde{T}(r, \theta)}{2} \sin \frac{\tilde{S}(r, \theta)}{2},
\]

\[
\beta_2(r, \theta) = \rho(r, \theta) \sin \frac{\tilde{T}(r, \theta)}{2} \cos \frac{\tilde{S}(r, \theta)}{2},
\]

\[
\beta_3(r, \theta) = \rho(r, \theta) \cos \frac{\tilde{S}(r, \theta)}{2}.
\]

Notice, parametrization \([12]\) with the functions \( \beta_{1,2,3} \) corresponds in general to topology of two-dimensional sphere \( S^2 \) for a constant valued Higgs scalar field

\[
\beta_1^2 + \beta_2^2 + \beta_3^2 = \rho^2.
\]

Our ansatz includes totally fifteen field variables which satisfy fifteen equations of motion. The functions \( \beta_{1,2,3} \) are treated as initial independent variables of the Higgs field (up to equivalent change of variables), and in the following we will always use original Weinberg-Salam equations obtained by variation of the Lagrangian \([1]\) with respect to the fields \( B_\mu, \bar{A}_\mu, \beta_{1,2,3} \).

To verify that our ansatz leads to axially symmetric configurations let us consider the structure of the Lagrangian \([1]\). It has been shown in \([10]\) that within the formalism of gauge invariant Abelian projection the Yang-Mills part in the Lagrangian \([1]\) describes a theory of a charged matter field \( X_\mu = \frac{1}{\sqrt{2}} (A_\mu^1 + iX^1_\mu) \) interacting with an Abelian gauge field \( A_\mu + \tilde{C}_\mu \). The original SU(2) gauge transformation for the gauge fields can be written in the form \([10, 22]\)

\[
\delta \hat{n} = -\alpha \times \hat{n},
\]

\[
\delta A_\mu = \frac{1}{g} \hat{n} \cdot \partial_\mu \alpha,
\]

\[
\delta \bar{X}_\mu = -\alpha \times \bar{X}_\mu,
\]

which implies that vector field \( \bar{X}_\mu \) transforms covariantly, i.e., it behaves as a matter field. By direct calculation one can check that ansatz \([7]\) implies axially symmetric configuration for the magnetic field \( H_{\mu\nu} \). So that axially symmetric form of the gauge potentials \( A_\mu, X^1_\mu \) guarantees axially-symmetric configurations for all fields and consistence with equations of motion as well.

Let us now consider definitions of gauge invariant quantities in the Weinberg-Salam model. The hypermagnetic field strength tensor \( G_{\mu\nu} \) is gauge invariant due to Abelian structure of the gauge group \( U_Y(1) \). For SU(2)
gauge potential $\vec{A}_\mu$ we apply gauge invariant decomposition in the natural basis frame ($\vec{m}_1, \vec{m}_2, \vec{m}_3 \equiv \vec{m}$). Since the vector field $\vec{m}$ belongs to adjoint representation of the group $SU(2)$, the Abelian projection onto $\vec{m}$ direction provides a field tensor which is invariant under $SU(2)$ gauge transformation

$$F_{\mu\nu}^\text{full} = \vec{F}_{\mu\nu} \vec{m} = F_{\mu\nu} + H_{\mu\nu} + gX_\mu^1X_\nu^2 - gX_\mu^2X_\nu^1.$$  \hspace{1cm} (15)

An important issue of the gauge invariant decomposition is that for a given covariant vector $\vec{m}$ one can define an Abelian gauge vector potential $A_\mu$ of Maxwell type which is gauge invariant under arbitrary $SU(2)$ transformation

$$A_\mu = A_\mu + \vec{\mathbf{C}}_\mu.$$  \hspace{1cm} (16)

This allows to define a corresponding $SU(2)$ gauge invariant Abelian field strength tensor

$$F_{\mu\nu} = F_{\mu\nu} + H_{\mu\nu}.$$  \hspace{1cm} (17)

This definition coincides with ’t Hooft-Polyakov magnetic field tensor in YMH theory \[5, 6\]. With this one can introduce a unique gauge invariant definition for the electromagnetic gauge potential and neutral gauge boson

$$A_\mu^{em} = \cos \theta_W B_\mu + \sin \theta_W A_\mu,$$

$$Z_\mu = -\sin \theta_W B_\mu + \cos \theta_W A_\mu.$$  \hspace{1cm} (18)

One should stress, that for any given Higgs field configuration the expressions for $A_\mu^{em}, Z_\mu$ are invariant under $SU(2)$ transformation and do not depend on a specific choice of a gauge condition for $\vec{m}$. In a unitary gauge, $\vec{m} = (0, 0, 1)$, or equivalently $\vec{m} = (0, 1)$, the expressions for $A_\mu^{em}, Z_\mu$ reduce to standard definitions of the electromagnetic potential and neutral gauge boson in the Weinberg-Salam model. A corresponding $SU(2)$ gauge invariant electromagnetic field tensor is given by

$$F_{\mu\nu}^{em} = \cos \theta_W G_{\mu\nu} + \sin \theta_W F_{\mu\nu}.$$  \hspace{1cm} (19)

An alternative definition for the electromagnetic field tensor with using the full Abelian field tensor $F_{\mu\nu}^\text{full}$ instead of $F_{\mu\nu}$ was suggested in \[19, 20\]. However, a corresponding definition for the electromagnetic vector potential contains $SU(2)$ gauge non-invariant term $A_\mu^\nu \delta^\nu$, and, in addition, the Bianchi identities for a respective electromagnetic field are not fulfilled anymore. So, such definition can not be accepted as a consistent concept.

The presence of the Abelian gauge potential $A_\mu$ allows to define an $SU(2)$ gauge invariant monopole charge, $Q_m$, and an analog of the Chern-Simons number, $Q_{CS}$, and an analog of the Chern-Simons number, $Q_{CS}$,

$$Q_m = \frac{1}{A(S)} \int_{S^2} \mathcal{F}_{ij} \cdot d\sigma^{ij},$$

$$Q_{CS} = \frac{1}{32\pi^2} \int d^3x \epsilon^{ijk} A_i \mathcal{F}_{jk}.$$  \hspace{1cm} (20)

where $A(S)$ is the area of a closed two-dimensional surface $S^2$.

Let us consider the reduction of the general axially-symmetric ansatz to Dashen-Hasslacher-Neveu ansatz. In Abelian decomposition of $SU(2)$ gauge potential $\vec{A}_\mu$ we use the following parametrization for the vector $\vec{m}$ in terms of one function $f(r, \theta)$, or angle function $T(r, \theta)$ equivalently

$$\zeta_1 = \cos \frac{T(r, \theta)}{2} = \cos \frac{\phi}{2} f(r, \theta) \sqrt{\cos^2 \frac{\phi}{2} f^2(r, \theta) + \sin^2 \frac{\phi}{2}},$$

$$\zeta_2 = e^{im\phi} \sin \frac{T(r, \theta)}{2}.$$  \hspace{1cm} (21)

The parametrization (21) corresponds to a special ansatz \[7\] when $Q(\rho, \theta) \equiv 0$.

For the Higgs field we apply a reduced parametrization \[12\] with the constraints

$$\tilde{S}(\rho, \phi) = \pi, \quad \tilde{T}(\rho, \theta) = pT(\rho, \theta),$$  \hspace{1cm} (22)

where the last relationship established a connection between vectors $\vec{n} (\vec{\zeta})$ and $\vec{n} (\vec{\zeta})$, $p$ is an integer number corresponding to cover mapping of the sphere $S^2$.

A simple setting

$$A_r^3 = A_\theta^3 = A_\phi^3 = 0,$$

$$A_r^2 = K_1, \quad A_\theta^2 = K_2,$$

$$A_\phi^2 = K_3, \quad A_r^1 = A_\theta^1 = A_\phi^1 = 0,$$  \hspace{1cm} (23)

leads to a modified DHN axially symmetric ansatz \[12, 13\] with four functions $K_1, ..., K_4$ and two functions $\beta_1, \beta_2$ (or $(\tilde{r}, \tilde{T})$). Notice, equations of motion are obtained by variation of the initial Lagrangian with respect to independent variables $K_1, ..., K_4, \beta_1, \beta_2$. In comparison with the original DHN ansatz one has an additional field degree of freedom represented by the function $f(r, \theta)$. A simplest choice $f(r, \theta) = 1$ leads to the original DHN ansatz. Notice, in the case of static magnetic field the DHN ansatz implies that azimuthal magnetic field component $\mathcal{F}_{r\phi}$ vanishes identically. So, within the DHN ansatz one can not describe magnetic field configurations with helical structure. In addition, the relationship \[13\] reduces to the following one

$$\beta_1^2 + \beta_2^2 = \rho^2.$$  \hspace{1cm} (24)

which constrains topology of the Higgs field configuration to a foliation $R^+ \times S^1$. $R^+$ is a half-line corresponding to positive values of the Higgs scalar $\rho$.

For further numeric purpose it is suitable to write down the energy functional corresponding to the Weinberg-Salam Lagrangian \[1\] for static magnetic field configurations in terms of dimensionless variables $r \to r m_W$, $k \equiv 2\sqrt{\lambda}/g = m_H/m_W$, $\vec{A}_\mu \to \vec{A}_\mu g/m_W$, $\vec{C}_\mu \to \vec{C}_\mu g/m_W$, $B_\mu \to B_\mu g/m_W$, $\rho \to \rho/v$. In these variables the energy functional takes the following form
expression for the gauge invariant magnetic field $F_{mn}$

\begin{align*}
F_{r\theta} &= 0, \\
F_{r\varphi} &= K_1 \sin(n\theta) + \partial_r K_2 - K_4 K_1, \\
F_{\theta\varphi} &= K_2 \sin(n\theta) + \partial_\theta K_3 - K_4 (K_2 + n). 
\end{align*}

In addition we impose a gauge condition $K_1(r, \theta) = 0$ due to presence of the residual gauge symmetry within the DHN ansatz \cite{13, 14, 25}.

Let us consider possible solutions with a set of parameters $(n = 1, \rho = 1)$. For simplicity we restrict our consideration by the limiting case $\lambda = 0$. To find proper boundary conditions for the gauge field components $B_{r}, K_2, K_3$ one can simplify the asymptotic expression for the energy functional \cite{25} in the lowest order of series expansion to the following form

\begin{equation}
E = \int d\tau d\varphi \mathcal{E}^{\text{asymp}},
\end{equation}

where the asymptotic energy density $\mathcal{E}^{\text{asymp}}$ includes the integration volume $\rho^2 \sin \varphi$, and $B_{r}^{\text{asymp}}(\theta)$, $K_2^{\text{asymp}}(\theta)$, $K_3^{\text{asymp}}(\theta)$ are asymptotic functions for the respective gauge fields at space infinity. Finite energy condition implies the following constraints

\begin{align}
K_2^{\text{asymp}}(\theta) &= 1 - \cos \varphi - B_{r}^{\text{asymp}}(\theta), \\
K_3^{\text{asymp}}(\theta) &= -1, \\
K_4^{\text{asymp}}(\theta) &= \sin \varphi. 
\end{align}

In a weak coupling limit, $g^2 = 0$, $B_m = 0$, the finite energy condition leads to vanishing the radial magnetic field component $F_{\theta r}$ \cite{25} in asymptotic region in agreement with the fact of absence of monopole solutions in pure YM theory. For the case $g^2 \neq 0$ the structure of asymptotic solutions is more rich. Substituting the asymptotic functions \cite{28} into all equations of motion one can verify that in lowest order approximation all equations are satisfied except equations for $B_{r}$ and $K_3$. The full equation of motion for the hypermagnetic field reads

\begin{align}
-2 \left[ B_{r} - 1 + \cos \theta + K_3 \right] \rho^2 + \\
\kappa \left[ \rho^2 \partial_r B_{r} + \partial_\theta B_{r} - \cot \theta \partial_\theta B_{r} \right] &= 0.
\end{align}

The equation in asymptotic region reduces to two equations enclosed in square brackets. The first one gives the same relationship between $B_{r}^{\text{asymp}}$ and $K_3^{\text{asymp}}$ as in \cite{28}, and the second one has a simple solution

\begin{equation}
B_{r}^{\text{asymp}}(\theta) = C_1 + C_2 \cos \theta, 
\end{equation}
various sets of parameter values. The asymptotic energy density has the same form for the magnetic field is given by

\[ K_3^{\text{inf}}(\theta) = 1 - C_1 - (1 + C_2) \cos \theta. \]  

(31)

With this a non-zero component of the Abelian gauge invariant magnetic field tensor takes the form

\[ \mathcal{F}_{\theta \varphi} = C_2 \sin \theta \]  

(32)

which implies a finite magnetic flux through the sphere of infinite radius, i.e., a non-zero magnetic charge. The energy density in the asymptotic region has Wu-Yang monopole like behavior

\[ \mathcal{E}^{\text{asym}} = \frac{C_2^2}{2r^2}(1 + \kappa) \sin \theta. \]  

(33)

In a similar manner one can find asymptotic solutions for the magnetic fields in the case \( p = 1, 2, n = 1, 2 \)

\[ B_\varphi^{\text{inf}}(\theta) = C_1 + C_2 \cos \theta, \]

\[ K_2^{\text{inf}}(\theta) = -pn, \]

\[ K_3^{\text{inf}}(\theta) = -\cos(n\theta) - (B_\varphi^{\text{inf}}(\theta) - 1) \cos(n(p - 1)\theta), \]

\[ K_4^{\text{inf}}(\theta) = \sin(n\theta) - (B_\varphi^{\text{inf}}(\theta) - 1) \sin(n(p - 1)\theta). \]  

(34)

A corresponding asymptotic expression for the non-vanishing component of the gauge invariant Abelian magnetic field is given by

\[ \mathcal{F}_{\theta \varphi} = -\partial_\theta B_\varphi^{\text{inf}}(\theta). \]  

(35)

The asymptotic energy density has the same form for various sets of parameter values \( (p = 1, 2, n = 1, 2) \)

\[ \mathcal{E}^{\text{asym}} = \frac{C_2^2}{2r^2}(1 + \kappa) \sin \theta. \]  

(36)

As expected, the expressions for gauge invariant quantities like the energy density and magnetic field \( \mathcal{F}_{mn} \) do not depend on choice of parameter values \( (p, n) \). Now it becomes clear that non-vanishing contribution to the magnetic flux created by \( \mathcal{F}_{\theta \varphi} \) originates only from the hypermagnetic gauge potential \( B_\varphi \). For non-zero values of the integration constant \( C_2 \) the hypermagnetic field creates non-vanishing magnetic flux through the sphere of infinite radius. This implies that one has a divergence of the hypermagnetic field at least in some point inside the sphere. Since the hypermagnetic magnetic field is gauge invariant and divergenceless by its Abelian structure, the magnetic flux through any closed two-dimensional surface must vanish, otherwise the magnetic field will possess a singularity which will lead to infinite total energy. Notice, the energy functional \[ \mathcal{E}^{\text{asym}} \] contains four terms each of them is positively defined. Due to this an infinite energy contribution of the hypermagnetic field can not be compensated by contributions of Higgs boson nor by SU(2) gauge field. We conclude, for any static axially-symmetric finite energy monopole-like solutions the magnetic charge must be screened at far distance, and the hypermagnetic field can not be a source of monopole with a non-zero magnetic charge localized inside any two-dimensional closed surface. In other words, for any possible finite energy monopole solutions the magnetic flux through the sphere of infinite radius must vanish, and magnetic flux of the hypermagnetic field through any closed surface of finite radius must be zero as well. This implies immediately non-existence of a finite energy solution representing a system of monopoles and anti-monopoles localized in different points. In particular, a finite energy Nambu type of monopole-antimonopole does not exist at least for the case of static non-rotating monopole-antimonopole pair.

**IV. SPHALERON AS A MONOPOLE-ANTIMONOPOLE PAIR WITH SCREENED MAGNETIC CHARGES**

Let us consider the case \( f = 1, Q = 0 \), when the general axially-symmetric ansatz reduces to the original DHN ansatz. In the limit of small coupling constant \( g \) the Weinberg-Salam Lagrangian for the bosonic fields turns into the Lagrangian of Yang-Mills-Higgs theory by simple setting \( B_\mu = 0 \). So that, solutions of the YMH theory represent approximate solutions of the Weinberg-Salam model in lowest order of perturbation theory. In this section we constrain our study by the case of YM theory in the limit \( \lambda = 0 \). Our goal is to study all possible monopole like solutions within the DHN ansatz, \[ \text{[21 23]}. \] With setting \( m = 1, T(r, \theta) = n\theta, \tilde{T}(r, \theta) = p\theta \) one can find the following reduction ansatz in the case of parameter values \( (p = 2, n = 1) \)

\[ K_1 = 0, \]

\[ K_2 = -1 + K(r), \]

\[ K_3 = 0, \]

\[ K_4 = (1 - K(r)) \sin \theta, \]

\[ \tilde{T} = 2\theta. \]  

(37)

One can easily verify that ansatz \[ \text{[37]} \] with a radial trial function \( K(r) \) leads to the known DHN sphaleron solution \[ \text{[12 14]}. \] In a similar manner one can find a solution applying the DHN ansatz with parameter setting \( (p = 2, n = 2) \)

\[ K_1 = 0, \]

\[ K_2 = -3 + K(r), \]

\[ K_3 = (1 + K(r)) \sin^2 \theta, \]

\[ K_4 = \frac{1}{2}(3 - K(r)) \sin(2\theta), \]

\[ \tilde{T} = 4\theta. \]  

(38)

The Higgs field \( \rho(r) \) and the function \( K(r) \) have only radial dependence, but the whole Higgs complex scalar
doublet and gauge fields have axially-symmetric configuration. Another one axially-symmetric solution can be obtained by using the ansatz with the set of parameters \((p = 1, n = 2)\)

\[
\begin{align*}
K_1 &= 0, \\
K_2 &= -3 - K(r), \\
K_3 &= (1 - K(r)) \sin^2 \theta, \\
K_4 &= \frac{1}{2}(3 + K(r)) \sin(2\theta), \\
\tilde{T} &= 2\theta. 
\end{align*}
\]

(39)

Parametrization of the functions \(K_{2,3,4}\) in terms of the trial radial function \(K(r)\) is chosen in such a way that final differential equations and boundary conditions for the functions \(K, \rho\) are the same for all three solutions. Direct substituting the ansatz into the equations of motion reduces all equations to two ordinary differential equations

\[
\begin{align*}
r^2 K'' &= K(K^2 - 1) + r^2 \rho^2(K + 1), \\
r^2 \rho'' + 2r \rho' &= \frac{1}{2} \rho(1 + K)^2. 
\end{align*}
\]

(40)

The energy density reduces to a simple expression

\[
\mathcal{E} = \frac{1}{r^2} \left(2r^2 \rho'' + K' + \frac{(1 - K^2)^2}{2r^2} + \rho^2(1 + K)^2\right). 
\]

(41)

So that one has three gauge equivalent representations for the DHN sphaleron. To solve the equations one choose boundary conditions consistent with the finite energy constraint

\[
K(r = 0) = 1, \quad K(r = \infty) = -1, \\
\rho(r = 0) = 0, \quad \rho(r = \infty) = 1. 
\]

(42)

Notice, the boundary values for the function \(K(r)\) correspond to vacuum configurations for \(SU(2)\) gauge potential near the origin and in the asymptotic region at space infinity. Numeric solution is depicted in Fig. 1.

The total energy is calculated numerically, and its value is \(7.63\) TeV in qualitative agreement with the energy estimate \(8 \text{ TeV}\) obtained in past for the sphaleron in the Weinberg-Salam model \((\lambda \neq 0)\) [14]. Notice, that expressions for the Higgs vector field \(\tilde{m}\) are different in [38-39], and the gauge field components \(K_3 = A_{\varphi}\) differ in all three solutions. However, by explicit calculating one can check that expressions for the gauge invariant Abelian potential \(A_{m, s}\) [16], are the same for all three gauge equivalent representations of the sphaleron as it should be. A respective gauge invariant magnetic field \((17)\) has two non-vanishing magnetic field components

\[
\begin{align*}
F_{\varphi \theta} &= 0, \\
F_{r \varphi} &= K' \sin^2 \theta, \\
F_{\theta r} &= -(1 + K) \sin 2\theta. 
\end{align*}
\]

(43)

The solution possesses magnetic flux through the plane \((X, Y)\) which is a multiple of \(4\pi\)

\[
\Phi_{\theta = \pm} = \int_0^\infty dr \int_0^{2\pi} d\varphi F_{\varphi r} = -4\pi. 
\]

(44)

The radial vector magnetic field corresponds to the magnetic field component \(F_{\theta r}\) shown in Fig. 2.

![Magnetic field](image)

**FIG. 2:** Magnetic field \(F_{\theta r}\).

The total magnetic flux created by \(F_{\theta r}\) through any two-dimensional sphere centered at the origin vanishes identically. However magnetic fluxes through the upper and lower hemispheres \(H^\pm\) do not vanish. One can consider magnetic flux through the closed surface \(S^\pm\) composed from the upper (lower) hemisphere \(H^\pm\) of radius \(a\) and a disc \(D^2\) : \(\{ r \leq a\}\) in the plane \((X, Y)\). This allows to define magnetic charges \(Q_m^\pm(a)\) of the monopole and antimonopole which depend on radius \(a\)

\[
\Phi_{\pm}(a) = \int_{H}^{\pm} d\theta d\varphi F_{\theta r} = \pm \int_{D^2} dr d\varphi F_{\varphi r} = 4\pi Q_m^\pm(a). 
\]

(45)

The magnetic flux through the disc \(D^2\) vanishes in the limit \(a \to 0\). So, one can compactify the hemispheres to spheres by identifying all points at the boundary of each hemisphere. Magnetic flux \(\Phi_{\pm}(a)\) through the upper (lower) hemisphere reaches a maximal value \(-4\pi\) \((+4\pi)\)
in the limit \( a \to 0 \). With this we obtain the maximal values of the magnetic charges \( Q^\text{max} = \mp 1 \) for the anti-monopole and monopole placed in one point \( r = 0 \). Magnetic fluxes through the upper and lower hemispheres at space infinity vanish, i.e., each of magnetic charges of the monopole and antimonopole is totally screened at space infinity.

It is known that total energy density of the sphaleron is spherically symmetric. However, the energy corresponding to the gauge invariant Abelian magnetic field \( F_{mn} \) has axial symmetry. Notice, the energy density of the Abelian magnetic field is not spherically symmetric, and near the origin \( r \approx 0 \) it is proportional to

\[
\mathcal{E}(F) \simeq \frac{1}{r^4} \cos^2 \theta,
\]

which shows the presence of two relative maximums located at the points \((r, \theta = 0)\) and \((r, \theta = \pi)\), i.e., the maximums merge in the limit \( r \to 0 \). The existence of a non-trivial solution for monopole and antimonopole in the limit when they collapse at one point is an essential feature of non-linear structure of the non-Abelian theory \( \text{SU}(2) \). Notice, in the case of Abelian theory like Maxwell electrodynamics the Dirac monopole and antimonopole have mutual attraction and collapse to a trivial solution.

The sphaleron solution in the Weinberg-Salam model has a non-vanishing dipole magnetic moment \( \text{SU}(2) \). To explain this it was suggested in \( \text{SU}(2) \) to interpret the sphaleron as a monopole-antimonopole pair which resembles the Nambu’s dumb-bell \( \text{SU}(2) \). Our consideration demonstrates that with a proper definition of the electromagnetic field the sphaleron solution represents a monopole and antimonopole placed in one point at the origin. One can verify, that asymptotic behavior of the magnetic field with our definition is the same as in \( \text{SU}(2) \), so that sphaleron has the same dipole magnetic moment. A principal difference from the description in \( \text{SU}(2) \) appears in the structure of the sphaleron at small distance, namely, the magnetic field in our treatment has a singularity at the origin of the type \( 1/\sqrt{r^2} \), whereas the definition for the electromagnetic tensor with \( \text{SU}(2) \) accepted in \( \text{SU}(2) \) leads to regular behavior. One should stress that singularity of the magnetic field at the origin does not imply any inconsistencies in the theory due to the following reasons. First of all, the sphaleron solution is regular everywhere and has a finite total energy. This takes place due to mutual cancelation of the contributions of the Abelian gauge field and off-diagonal components of \( \text{SU}(2) \) gauge potential in \( \text{SU}(2) \). In a fact, the magnetic field of the known ’t Hooft-Polyakov monopole in YM theory with a real triplet Higgs field has the same singularity at the origin which does not cause any inconsistencies. Secondly, the presence of such a singularity in the electromagnetic part of the classical sphaleron solution has rather a deep physical origin, and it can be treated as a reflection of the quantum structure of the Weinberg-Salam model. It is known that quantum electrodynamics itself is not a consistent quantum theory due to presence of a positive beta function which implies existence of the Landau pole in the theory. Only within unification of the electro-weak interaction one has a consistent quantum theory with asymptotically free behavior provided by a negative beta function. This effect holds due to dominant contribution of \( \text{SU}(2) \) gauge fields to the running fine coupling constant in asymptotic regime at high energy scale. That means, at short distance in quantum description of the sphaleron one should have cancellation of contributions of the electromagnetic field and off-diagonal \( \text{SU}(2) \) gauge bosons. So that, the singularity of the magnetic field of the classical sphaleron solution will disappear in quantum description.

V. ON SOLUTIONS WITH TOPOLOGICAL AZIMUTHAL MAGNETIC FLUX

To find a magnetic solution with a non-vanishing azimuthal magnetic flux having topological origin one should apply a general axially-symmetric ansatz and solve fifteen equations of motion of the Weinberg-Salam model which is a hard problem. We suppose that such a solution exists and study its properties using variational method. For simplicity we restrict our consideration by the case of pure Yang-Mills theory. In this case one can find a restricted ansatz which admits local solutions to all equations of motion near the origin, \( r \approx 0 \), and in asymptotic region, \( r \approx \infty \). Using these solutions we apply variational procedure to find field configuration minimizing the energy functional.

First we provide a qualitative argument to existence of a magnetic solution with a magnetic flux around \( Z \)-axis. Let us consider a simple \( CP^1 \) type model defined by the energy density obtained from \( \text{SU}(2) \) by keeping only terms with the Higgs field components \( \hat{m} \) and \( \rho \)

\[
E_1 = \frac{1}{4} H_{mn}^2 + \frac{1}{2} \hat{C}_m^2 r^2,
\]

where \( \hat{C}_m \) is defined by \( \text{SU}(2) \) with \( \hat{n} \equiv \hat{m} \). For simplicity we replace the Higgs scalar field \( \rho \) with its classical vacuum averaged value, \( \rho = 1 \). In this approximation the energy density \( \text{SU}(2) \) defines a modified Skyrme model which admits exact finite energy solutions \( \text{SU}(2) \). Simple scaling arguments based on the Derrick theorem \( \text{SU}(2) \) imply that such a model admits stable static solitons. We parameterize the \( CP^1 \) field \( \hat{m} \) with only one function \( Q \) whereas setting the function \( f \) to a constant value, \( f \equiv 1 \). Changing variable, \( Q = \cot(\frac{S}{2}) \), one can simplify the Euler-Lagrange equation of motion for \( S(r, \theta) \)

\[
(r^2 \cos^2 \theta + 4 \sin^2 \theta \sin^2 \frac{S}{2}) S_{rr} + \left( \frac{1}{r^2} \sin^2 \theta \sin^2 S + \cos^2 \theta \right) S_{\theta \theta} + \frac{1}{r^2} \sin^2 \theta \sin S \left( \cos SS_\theta^2 - 4 \sin^2 S \frac{S}{2} + 3 \cot \theta \sin SS_\theta \right) + \frac{1}{4 \sin \theta} \left( \cos \theta + 3 \cos(3\theta) \right) S_\theta - 2 \cos^2 \theta \sin S S_\theta = 0.
\]
Finite energy condition allows the following boundary conditions
\[ S(0, \theta) = 0, \quad S(\infty, \theta) = 2\pi \] (49)
which provide a total magnetic flux $4\pi$ for the azimuthal magnetic field through the half plane $\{y = 0; x \geq 0\}$. A regular numeric solution is obtained by using the package COMSOL for solving partial differential equations. The results for the function $S$ and energy density $E^C_{\mu\nu}$ are presented in Figs. 3, 4.

At space infinity the Higgs triplet vector field $\hat{m}$ takes asymptotic form
\[ \hat{m} = \left( \cos \varphi \sin(2\theta), \sin \varphi \sin(2\theta), \cos(2\theta) \right). \] (51)
We decompose the gauge potential in the natural basis frame
\[ \tilde{A}_m(r, \theta) = A_m(r, \theta)\hat{m} - \hat{m} \times \partial_m \tilde{A}_m W(r, \theta). \] (52)

Notice, in the case when the off-diagonal component $W(r, \theta)$ equals one, one has an ansatz with five functions $A_m, Q, \rho$ which reduces all equations of motion to five independent second order equations. This ansatz might be too restricted since by $SU(2)$ gauge transformation one can rotate the vector field $\hat{m}$ to a constant unit vector $(0, 0, 1)$ everywhere. With this the Lagrangian of YMH theory becomes formally equivalent to the Lagrangian of the Ginzburg-Landau model with a complex scalar field in the unitary gauge. The ansatz with the constraint $W = 1$ can not admit finite energy solutions with a non-zero azimuthal magnetic flux, so we keep a non-constant off-diagonal gauge field $W(r, \theta)$.

Direct substitution of the ansatz (52) into all equations of motion leads to number of independent equations more than number of unknown variables. We have solved all equations of motion near the origin and in asymptotic region in the gauge $A_r = 0$ and obtain local solutions which are consistent with finite energy condition. The solution at the origin $r \simeq 0$ can be found by perturbation theory by series expansion as follows
\[
G(r, \theta) = C_1 r - C_1 r^2 + \frac{C_1}{240\rho_0} r^3 \left( 120C_2 + \rho_0 (-37\rho_0^2 + 4C_1^2 (w_0(w_0 - 2) + 60)) - 15(24C_2 - \rho_0(5\rho_0^2 - 12C_1^2 w_0(w_0 - 2))) \cos^2 \theta \right), \\
W(r, \theta) = w_0 + \frac{w_0 - 1}{48} r^2 \left( 3(5\rho_0^2 - 12C_1^2 w_0(w_0 - 2)) + (-7\rho_0^2 + 4C_2^2 w_0(w_0 - 2)) \cos(2\theta) \right), \\
A_\varphi(r, \theta) = \frac{C_1}{3} r^3 (-\rho_0^2 + 4C_1^2 w_0(w_0 - 2)) \sin \theta, \\
A_r(r, \theta) = -\frac{1}{12} r^2 (-\rho_0^2 + 28C_1^2 w_0(w_0 - 2)) \sin^2 \theta, \\
\rho(r, \theta) = \rho_0 + \frac{1}{24} r^2 \left( 6C_2 + 4C_1^2 \rho_0 (3 + 2w_0(w_0 - 2)) + 18C_2 \cos(2\theta) \right),
\] (53)
where $w_0, \rho_0$ are free variational parameters for initial values of $W, \rho$ at the origin. We have performed a change of variables $Q(r, \theta) \rightarrow G(r, \theta) = (1 + Q(r, \theta))^{-1}$ which is suitable for numeric purpose. The solution contains only those independent integration constants which provide finite energy conditions and the symmetry under the reflection $(z \rightarrow -z)$. In a similar manner one can find a solution in asymptotic space region $r \simeq \infty$. The solution

![FIG. 3: Solution for $S(r, \theta)$](image)

![FIG. 4: Energy density plot.](image)
includes an essential singularity which can be extracted in the exponential factor
\[
G(r, \theta) = 1 + \tilde{C}_1 \left( e^{-r/(1+r)} + \text{Chi}(r) - \text{Shi}(r) \right),
\]
\[
W(r, \theta) = 1 - \tilde{C}_2 e^{-r/(1+r)},
\]
\[
A_\theta(r, \theta) = -2\tilde{C}_1 \left( e^{-r/(2+2r+r^2)} + \text{Chi}(r) - \text{Shi}(r) \right) \sin \theta,
\]
\[
A_\phi(r, \theta) = \left( 2 + \frac{\tilde{C}_3}{r} (1 + r) \right) \sin^2 \theta,
\]
\[
\rho(r, \theta) = 1 + \frac{\tilde{C}_4}{r} + \frac{\tilde{C}_5}{r^3} (1 + 3 \cos(2\theta)),
\]
where \(\text{Chi}(r), \text{Shi}(r)\) are the special cosine and sine hyperbolic integral functions, and for simplicity the Higgs coupling constant \(\lambda\) is set to zero. If we take into account a non-zero value for the coupling constant \(\lambda\) then the Higgs scalar \(\rho\) will approach its asymptotic value in exponential form as well. We will apply variational method to obtain qualitative description of the profile functions and energy density of a possible solution. To simplify variational procedure one can observe, that the minimum of the energy requires a simple condition for the azimuthal component of the gauge potential \(A_\phi(r, \theta)\), namely
\[
A_\phi(r, \theta) = -\tilde{C}_\phi(r, \theta) = \frac{2 \sin^2 \theta}{1 + Q^2}.
\]
This condition provides mutual cancelation of contributions to the energy from the fields \(H_{r\phi}, H_{\theta\phi}\) and \(F_{r\theta}, F_{\theta\phi}\) respectively as it can be seen from the Abelian structure of the field strength in (5). With this we construct trial variational functions for the fields \((G(r, \theta), A_{\theta, \phi}(r, \theta), W(r, \theta), \rho(r, \theta))\) in a consistent manner with the known local solutions taking into account first derivative terms as well. To make a qualitative estimate we consider radial dependent functions \(G(r), \rho(r)\) and factorize the angle dependence of the gauge potentials \(A_{\theta, \phi}(r, \theta)\) as follows
\[
A_\theta(r, \theta) = a_\theta(r) \sin \theta,
\]
\[
A_\phi(r, \theta) = a_\phi(r) \sin^2 \theta,
\]
where \(a_\phi(r)\) is determined by equation (55). Minimization procedure of the energy functional produces a lower bound for the total energy, \(E_{\text{bound}} \approx 4.3\text{TeV}\). Variational profile functions \(W, G, a_\theta, a_\phi, \rho\) minimizing the energy are presented in Fig. 5. The solution possesses a non-zero magnetic flux around the Z-axis
\[
\Phi_\phi = \int drd\theta F_{r\phi} = \int drd\theta H_{r\phi} = 2\pi.
\]
A gauge invariant Abelian magnetic field, \(F_{mn}\), has only one non-vanishing component, \(F_{r\theta}\), which is plotted in Fig. 6. The gauge invariant definition for the Abelian magnetic field \(F_{mn}\), allows to extract a respective energy density \(\mathcal{E}(F) = \frac{1}{4}F_{mn}^2\) shown in Fig. 7. Notice,
a total energy corresponding solely to the Abelian magnetic field $F_{mn}$ is finite, $E_{\text{magn}} = 1.58 \text{ TeV}$, contrary to the case of the sphaleron solution.

One should stress, due to additive structure of the $SU(2)$ gauge invariant Abelian field strength $F_{\mu \nu}$, one has partial mutual cancelation of contributions of the fields $F_{\mu \nu}$ and $H_{\mu \nu}$. Without such cancelation, i.e., if we neglected the field $F_{\mu \nu}$, we would have energy contribution of the Higgs field provided by the term $\frac{1}{4} H_{\mu \nu}^2$ about 4.9 TeV what is three times larger than the actual energy 1.58 TeV of the Abelian gauge field $F_{\mu \nu}$. So that interaction between the Higgs field and $SU(2)$ gauge bosons leads to significant decrease of the Abelian magnetic field energy and provides magnetic charge screening effect in the case of monopole like solutions.

VI. CONCLUSION

We propose a most general ansatz for static axially-symmetric magnetic field configurations in the Weinberg-Salam model based on gauge invariant decomposition of $SU(2)$ gauge potential. Introducing a unique $SU(2)$ gauge invariant definition for the electromagnetic vector gauge potential we have proved that for any possible finite energy monopole like solutions the magnetic charge of monopole (antimonopole) must be totally screened at space infinity. Our analysis implies that finite energy condition and equations of motion forbid existence of solutions representing a system of monopoles and antimonopoles localized in different points. We have demonstrated that known sphaleron solution represents a pair of monopole and antimonopole placed in one point. In general, an axially-symmetric magnetic field configuration can possess a helical magnetic structure. We conjecture that such a solution may exist in the standard electroweak model and describe a possible simple solution with azimuthal magnetic flux using variational method in the case of a pure Yang-Mills Higgs theory. The solution possesses a total magnetic flux $2\pi$ around the $Z$-axis. We estimate the lower bound of the solution, $E = 4.3 \text{ TeV}$. In the case of the Weinberg-Salam model the energy of such magnetic solution is expected to be of the same order as the energy of sphaleron. So the sphaleron and magnetic solution with azimuthal magnetic flux could be good candidates in search of magnetic like composite bound states in the electroweak theory.

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