Category decomposition of $\text{Rep}_k(\text{SL}_n(F))$

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Abstract

Let $F$ be a non-archimedean local field with residual characteristic $p$, and $k$ an algebraically closed field of characteristic $\ell \neq p$. We establish a category decomposition of $\text{Rep}_k(\text{SL}_n(F))$ according to the $\text{GL}_n(F)$-inertially equivalent supercuspidal classes of $\text{SL}_n(F)$, and we give a block decomposition of the supercuspidal sub-category of $\text{Rep}_k(\text{SL}_n(F))$. Finally we give an example to show that in general a block of $\text{SL}_n(F)$ is not defined according to a unique inertially equivalent supercuspidal classes of $\text{SL}_n(F)$, which is different from the case when $\ell = 0$.

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1 Introduction

Let $F$ be a non-archimedean local field with residual characteristic $p$, and $k$ an algebraically closed field with characteristic $\ell$ different from $p$. We say $G$ is a $p$-adic group if it is the group of $F$-rational points of a connected reductive group $G$ defined over $F$. Let $\text{Rep}_k(G)$ be the category of smooth $k$-representations of $G$. In this article, we always denote by $M'$ a Levi subgroup of $\text{SL}_n(F)$, and we study the category $\text{Rep}_k(M')$.

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For arbitrary $p$-adic group $G$, we say that $\text{Rep}_k(G)$ has a **category decomposition** according to an index set $\mathcal{A}$, if there exists an equivalence:

$$\text{Rep}_k(G) \cong \prod_{\alpha \in \mathcal{A}} \text{Rep}(G)_\alpha,$$

where $\text{Rep}_k(G)_\alpha$ are full sub-categories of $\text{Rep}_k(G)$. The equivalence implies that:

- Each object $\Pi \in \text{Rep}_k(G)$ can be decomposed as a direct sum $\Pi \cong \oplus_{\alpha \in \mathcal{A}} \Pi_\alpha$, where $\Pi_\alpha \in \text{Rep}_k(G)_\alpha$.

- For $i = 1, 2$ and $\alpha_i \in \mathcal{A}$, if $\alpha_1 \neq \alpha_2$, then $\text{Hom}_G(\Pi_1, \Pi_2) = 0$ for $\Pi_i \in \text{Rep}_k(G)_{\alpha_i}$.

Furthermore, if

- for $\alpha \in \mathcal{A}$, there is no such decomposition for $\text{Rep}_k(G)_\alpha$, we say that $\text{Rep}_k(G)_\alpha$ is **non-split**.

If $\text{Rep}_k(G)_\alpha$ is non-split for each $\alpha \in \mathcal{A}$, we call this category decomposition a **block decomposition** of $\text{Rep}_k(G)$, which means the finest category decomposition of $\text{Rep}_k(G)$, and we call each $\text{Rep}_k(G)_\alpha$ a **block** of $\text{Rep}_k(G)$.

When $\ell = 0$, a block decomposition of $\text{Rep}_k(G)$ has been established according to $\mathcal{A} = \mathcal{SC}_G$, where $\mathcal{SC}_G$ is the set of $G$-inertially equivalent supercuspidal classes of $G$ (see Section \ref{section:2.1} for the definition). Let $[M, \pi]_G \in \mathcal{SC}_G$, where $(M, \pi)$ is a supercuspidal pair of $G$ (see Section \ref{section:2.1}). The sub-category $\text{Rep}_k(G)_{[M, \pi]_G}$ consists of the objects whose irreducible sub-quotients have supercuspidal supports (see Section \ref{section:2.1}) in $[M, \pi]_G$.

When $\ell$ is positive, a block decomposition has been established when $G = \text{GL}_n$ (He) and its inner forms (SeSt). For $G = \text{GL}_n$, the block decomposition is according to $\mathcal{SC}_G$ as well, which is the same as the case when $\ell = 0$. It is worth noting that when we restrict the block decomposition in Equation \ref{equation:1} to the set of irreducible $k$-representations of $G$, the block decomposition according to $\mathcal{SC}_G$ requires that the supercuspidal support of an irreducible $k$-representation of $G$ belongs to a unique $G$-inertially equivalent supercuspidal class, which can be deduced from the uniqueness of supercuspidal support proved in \cite{V2} §V.4 for $\text{GL}_n(F)$. However the uniqueness of supercuspidal support is not true in general, in \cite{Da} an irreducible $k$-representation of $\text{Sp}_8(F)$ that the supercuspidal support is not unique up to $\text{Sp}_8(F)$-conjugation has been found. As for $\text{SL}_n(F)$, the uniqueness of supercuspidal support holds true and has been proved in \cite{C2}, hence the block decomposition according to $\text{SL}_n(F)$-inertially equivalent supercuspidal classes was expected. However in this article, we show that this is **not** always true by providing a counter-example in Section \ref{section:4}.

### 1.1 Main results

Now we describe the work in this article with more details. Let $M'$ be a Levi subgroup of $\text{SL}_n(F)$, and $M$ be a Levi subgroup of $\text{GL}_n(F)$ such that $M \cap \text{SL}_n(F) = M'$, we establish a category decomposition of $\text{Rep}_k(M')$ according to $M'$-inertially equivalent supercuspidal classes $\mathcal{SC}^{M'}_G$ (see Section \ref{section:2.1} for the definitions), which is different from $\mathcal{SC}_M$ the set of $M'$-inertially equivalent supercuspidal classes . In fact, let $L$ be a Levi subgroup of $M$ and $L' = L \cap M'$ a Levi subgroup of $M'$, and let $\tau$ be an irreducible supercuspidal $k$-representation of $L$. Denote by $I(\tau)$ the set of isomorphic classes of irreducible components of $\tau|_{L'}$. Let $\tau' \in I(\tau)$, denote by $[L', \tau']_{M'}$ the $M'$-inertially equivalent supercuspidal class defined by $(M', \tau')$. The $M'$-inertially equivalent supercuspidal class of $(L', \tau')$ is $\bigcup_{\tau' \in I(\tau)} [L', \tau']_{M'}$, and we denote it by $[L', \tau']_{M'}$. 

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**Theorem 1.1** (Theorem 3.13). Let $\mathcal{SC}^M_{M'}$ be the set of $M$-inertially equivalent supercuspidal classes of $M'$. There is a category decomposition of $\text{Rep}_k(M')$ according to $\mathcal{SC}^M_{M'}$.

In particular, let $[L', \tau'_M] \in \mathcal{SC}^M_{M'}$, a $k$-representation $\Pi$ of $M'$ belongs to the full subcategory $\text{Rep}_k(M'',[L', \tau'_M])$, if and only if the supercuspidal support of each of its irreducible sub-quotients belongs to $[L', \tau'_M]$.

The above theorem gives a category decomposition

$$\text{Rep}_k(M') \cong \text{Rep}_k(M')_{\text{SC}} \times \text{Rep}_k(M')_{\text{non-SC}},$$

where a $k$-representation $\Pi$ of $M'$ belongs to $\text{Rep}_k(M')_{\text{SC}}$ (resp. $\text{Rep}_k(M')_{\text{non-SC}}$) if each (resp. non) of its irreducible sub-quotients is supercuspidal. We call $\text{Rep}_k(M')_{\text{SC}}$ the supercuspidal sub-category of $\text{Rep}_k(M')$. In Section 4 we establish a block decomposition of $\text{Rep}_k(M')_{\text{SC}}$.

Let $\pi$ be an irreducible supercuspidal $k$-representation of $M$, and let $\mathcal{I}(\pi)$ be the set of isomorphic classes of irreducible components of $\pi|_{M'}$. In Section 4 we introduce an equivalence relation $\sim$ on $\mathcal{I}(\pi)$. For $\pi' \in \mathcal{I}(\pi)$, an irreducible supercuspidal $k$-representation of $M'$, let $(\pi', \sim)$ be the connected component of $\mathcal{I}(\pi)$ containing $\pi'$ under this equivalence relation, and let $[\pi', \sim]$ be the union of $M'$-inertially equivalent supercuspidal classes of $\pi'_j \in (\pi', \sim)$. Denote by $\mathcal{SC}_{M', \sim}$ the set of pairs in the form of $[\pi', \sim]$. We establish a block decomposition of $\text{Rep}_k(M')_{\text{SC}}$:

**Theorem 1.2** (Theorem 4.12). There is a block decomposition of $\text{Rep}_k(M')_{\text{SC}}$ according to $\mathcal{SC}_{M', \sim}$. In particular, let $[\pi', \sim] \in \mathcal{SC}_{M', \sim}$, a $k$-representation $\Pi$ belongs to $\text{Rep}_k(M')_{[\pi', \sim]}$ if and only if each of its irreducible sub-quotients belongs to $[\pi', \sim]$.

This article ends with Example 4.13 of a $k$-representation in the supercuspidal sub-category of $\text{Rep}_k(SL_2(F))$ when $\ell = 3$. In this example, we construct a finite length projective $k$-representation of $SL_2(F)$ which is induced from a projective cover of a maximal simple supercuspidal $k$-type of depth zero. By using the theory of $k$-representations of finite $SL_n$ group, we compute the irreducible sub-quotients of this projective cover, and we show that there exists two different supercuspidal $k$-representations of $SL_2(F)$, which are not inertially equivalent, such that they belong to a same block. Or equivalently, this example shows that there exists an irreducible supercuspidal $k$-representation $\pi'$ of $SL_2(F)$, such that $[\pi', \sim]$ is not a unique $SL_2(F)$-inertially equivalent supercuspidal class, hence the equivalence relation defined on $\mathcal{I}(\pi)$ is non-trivial in general. This example shows that a block decomposition of $\text{Rep}_k(G')$ (resp. $\text{Rep}_k(M')$) according to $G'$-inertially equivalent supercuspidal classes $\mathcal{SC}_{G'}$ (resp. $\text{Rep}_k(M')$)-inertially equivalent supercuspidal classes $\mathcal{SC}_{M'}$ is not always possible in general.

### 1.2 Structure of this paper

The author is inspired by the method in [He]. We use the theory of maximal simple $k$-types, which has been firstly established for $\mathbb{C}$-representations of $GL_n(F)$ in [BuKu] and generalised by the author to the cuspidal $k$-representations of $M$ a Levi subgroup of $SL_n(F)$ in [C1]. In this article, we construct a family of projective objects defined from the projective cover of maximal simple $k$-types. In Section 3.3. we show that the projective cover of a maximal simple $k$-type of $M'$ is an indecomposable direct summand of the restriction of the projective cover of a maximal simple $k$-type of $M$. We apply the compact induction functor $\text{ind}^M_{M'}$ to these projective objects and show their decomposition under the block
decomposition of \( \text{Rep}_k(M) \). The above two parts leads to a family of injective objects verifying the conditions stated in Proposition \( \text{[2.1]} \) which gives the category decomposition in Theorem \( \text{[1.1]} \).

Section 4 concentrates on the supercuspidal sub-category of \( \text{Rep}_k(M') \), where \( M' \) is a Levi subgroup of \( \text{SL}_n(F) \). We introduce an equivalence relation generated by putting all the irreducible sub-quotients of the projective cover of a maximal simple supercuspidal \( k \)-type of \( M' \) into a same equivalent class. Let \( \pi \) be an irreducible supercuspidal \( k \)-representation of \( M \), the above equivalence relation on maximal simple supercuspidal \( k \)-types induces an equivalence relation on \( \mathbb{I}(\pi) \), which is the equivalence relation \( \sim \) needed in Theorem \( \text{[1.2]} \).

There is a natural conjecture that a block decomposition of \( \text{Rep}_k(G') \) can be given according to the set of \( G' \)-conjugacy classes of elements in \( SC_{M',\sim} \) for all Levi subgroup \( M' \), which involves the study of projective cover of maximal simple \( k \)-types (non-supercuspidal) of \( M' \) and leads to a study of semisimple \( k \)-types of \( G' \).

2 Preliminary

2.1 Notations

Let \( F \) be a non-archimedean local field with residual characteristic equal to \( p \).

- \( \mathfrak{o}_F \): the ring of integers of \( F \), and \( p_F \): the unique maximal ideal of \( \mathfrak{o}_F \).
- \( k \): an algebraically closed field with characteristic \( \ell \neq p \).
- Let \( K \) be a closed subgroup of a \( p \)-adic group \( G \), then \( \text{ind}_K^G \): compact induction, \( \text{Ind}_K^G \): induction, \( \text{res}_K^G \): restriction.
- Fix a split maximal torus of \( G \), and \( M \) be a Levi subgroup, then \( r_M^G, r_M^G \): normalised Parabolic induction and normalised Parabolic restriction.
- Denote by \( \delta_G \) the module character of \( G \).

In this article, without specified we always denote by \( G \) the group of \( F \)-rational points of \( \text{GL}_n \) and by \( G' \) the group of \( F \)-rational points of \( \text{SL}_n \). Let \( \iota \) be the canonical embedding from \( G' \) to \( G \), which induces an isomorphism between the Weyl group of \( G' \) and \( G \), hence gives a bijection from the set of Levi subgroups of \( G' \) to those of \( G \). In particular, suppose \( M \) is a Levi subgroup of \( G \), we always denote by \( M' \) the Levi subgroup \( M \cap G' \) of \( G' \). We say an irreducible \( k \)-representation \( \pi \) of a \( p \)-adic group \( G \) is **cuspidal**, if \( r_M^G \pi \) is zero for every proper Levi subgroup \( M \); we say \( \pi \) is **supercuspidal** if it does not appear as a sub-quotient of \( r_M^G \tau \) for each proper Levi subgroup \( M \) and its irreducible representation \( \tau \).

Let \( \pi \) be an irreducible \( k \)-representation of \( G \), its restriction \( \pi|_{G'} \) is semisimple with finite length, and each irreducible \( k \)-representation \( \pi' \) of \( G' \) appears as a direct component of \( \pi|_{G'} \). A pair \((M, \tau)\) is called a **cuspidal** (resp. **supercuspidal**) **pair** if \( M \) is a Levi subgroup and \( \tau \) is an irreducible cuspidal (resp. supercuspidal) of \( M \). Let \((M_1', \tau_1'), (M_2', \tau_2')\) be two cuspidal pairs of \( G' \) and \( K \) be a subgroup of \( G \), we say they are \( K \)-**inertially equivalent**, if there exists an element \( g \in K \) such that \( g(M_1') = M_2' \) and there exists an unramified \( k \)-quasicharacter \( \theta \) of \( F^\times \) such that \( g(\tau_1') \cong \tau_2' \otimes \theta \). Denote by \([M', \tau']_K \) the \( K \)-inertially equivalent class defined by \((M', \tau')\), and we call it a \( K \)-inertially equivalent supercuspidal (resp. cuspidal) class if \((M', \tau') \) is a supercuspidal (resp. cuspidal) pair.
same definition of \([M, \tau]_G\) is applied for cuspidal pairs of \(G\). We always abbreviate \([M', \tau']_{G'}\) as \([M', \tau']\), and abbreviate \([M, \tau]_G\) as \([M, \tau]\).

We say that a cuspidal (resp. supercuspidal) pair \((M, \tau)\) belongs to the **cuspidal** (resp. **supercuspidal**) support of \(\pi\), if \(\pi\) appears as a sub-representation or a quotient-representation (resp. sub-quotient representation) of \(i^G_M \tau\). When \(\pi\) is an irreducible \(k\)-representation of \(G\) (resp. \(G'\)), its supercuspidal support as well as its cuspidal support is unique up to \(G\)-(resp. \(G'\))-conjugation.

To decompose \(\text{Rep}_k(G')\) as a direct product of a family of full-subcategories, we constructs a family of injective objects and follow the method as below, which is the same strategy as in [He, Proposition 2.4]. We state it here for convenient reason.

**Proposition 2.1.** Let \(I_1, I_2\) be two injective objects in \(\text{Rep}_k(G')\), and denote by \(S_1, S_2\) the sets of irreducible \(k\)-representations of \(G'\) which appears as a sub-quotient of \(I_1\) and \(I_2\) respectively. Suppose the following conditions are verified:

- an object in \(S_1\) can be embedded into \(I_1\);
- an object in \(S_1\) does not belong to \(S_2\) up to isomorphism;
- an irreducible \(k\)-representation of \(G'\), which does not belong to \(S_1\) up to isomorphism, can be embedded into \(I_2\).

Then \(\text{Rep}_k(G')\) can be decomposed as a direct product of two full subcategories \(R_1\) and \(R_2\), such that

- every object \(\Pi \in \text{Rep}_k(G')\) is isomorphic to a direct sum \(\pi_1 \oplus \pi_2\), where each irreducible sub-quotient of \(\pi_1\) belongs to \(S_1\) and each irreducible sub-quotient of \(\pi_2\) belongs to \(S_2\);
- every object in \(R_1\) has an injective resolution by direct sums of copies of \(I_1\), and every object in \(R_2\) has an injective resolution by direct sums of copies of \(I_2\) (copies means direct product by itself).

**Remark 2.2** (Projective version). Let \(P_1, P_2\) be two projective objects in \(\text{Rep}_k(G')\), and denote by \(S_1, S_2\) the sets of irreducible \(k\)-representations of \(G'\) which appears as a sub-quotient of \(P_1\) and \(P_2\) respectively. Suppose the following conditions are verified:

- an object in \(S_1\) is a quotient of \(P_1\);
- an object in \(S_1\) does not belong to \(S_2\) up to isomorphism;
- an irreducible \(k\)-representation of \(G'\), which does not belong to \(S_1\) up to isomorphism, can be realised as a quotient of \(P_2\).

Then \(\text{Rep}_k(G')\) can be decomposed as a direct product of two full subcategories \(R_1\) and \(R_2\), such that every object \(\Pi \in \text{Rep}_k(G')\) is isomorphic to a direct sum \(\pi_1 \oplus \pi_2\), where each irreducible sub-quotient of \(\pi_1\) belongs to \(S\) and each irreducible sub-quotient of \(\pi_2\) belongs to \(S_2\).

The proof of Remark 2.2 is in the same manner as in Proposition 2.4 of [He] by changing injective objects to projective objects as suggested in Remark 2.5 of [He].
2.2 Maximal simple $k$-types of $M'$

In this section, we recall notations and definitions in the theory of maximal simple $k$-types of Levi subgroups $M'$ of $G'$ which has been studied in [CI]. It requires the theory of maximal simple $k$-types of $G$ which has been established in [BuKuII] for complex case. The later is related to modulo $\ell$ maximal simple types in [VI] §III by considering the reduction modulo $\ell$, while [MS] gives a more intrinsic description. We state some useful properties which will be needed for the further use.

A maximal simple $k$-type of $G$ is a pair $(J, \lambda)$, where $J$ is an open compact subgroup of $G$ and $\lambda$ is an irreducible $k$-representation of $J$. We have a groups inclusion:

$$H^1 \subset J^1 \subset J,$$

where $J^1$ is a normal pro-$p$ open subgroup $J^1$ of $J$, such that the quotient $J/J^1$ is isomorphic to $GL_m(F_q)$, where $F_q$ is a field extension of the residue field of $F$, and $H^1$ is open. The $k$-representation $\lambda$ is a tensor product $\kappa \otimes \sigma$, where $\kappa$ is irreducible whose restriction to $H^1$ is a multiple of a $k$-character, and $\sigma$ is inflated from a cuspidal $k$-representation of $J/J^1$. By [VI] §III 4.25 or [MS] Proposition 3.1, for an irreducible $k$-cuspidal representation $\pi$ of $G$, there exists a maximal simple $k$-type $(J, \lambda)$, and compact modulo centre subgroup $K$ and an irreducible representation $\Lambda$ of $K$, where $\pi$ is the unique largest compact open subgroup of $K$ and $\Lambda$ is an extension of $\lambda$, such that $\pi \cong \text{ind}^K_A \Lambda$. Since a Levi subgroup of $G$ is a tensor product of $GL$-groups in lower rank, so we can define maximal simple $k$-types $(J_M, \lambda_M)$ and obtain the same property for cuspidal $k$-representations of $M$ as above.

For the reason that a Levi subgroups $M'$ of $G'$ is not a product of $SL$-groups of lower rank, so it is not sufficient to consider only the maximal simple $k$-types of $G'$. Let $(J_M, \lambda_M)$ be a maximal simple $k$-type of $M$, the group of projective normaliser $\tilde{J}_M$ contains $J_M$ as a normal subgroup, which is defined in [CI] 2.15 and [BuKuII] 2.2. In particular, for any $g \in \tilde{J}_M$, the conjugate $g(\lambda_M) \cong \lambda_M \otimes \chi$, where $\chi$ is a $k$-quasicharacter of $F'^\times$. As in [CI] 2.48, a maximal simple $k$-type of $M'$ is a pair in the form of $(\tilde{J}'_M, \tilde{\lambda}'_M)$, where $\tilde{\lambda}'_M$ is an irreducible direct component of $(\text{ind}_{\tilde{J}_M}\lambda_M)|_{\tilde{J}'_M}$, and we denote by $\tilde{\lambda}'_M$ as $\text{ind}^{\tilde{J}'_M}_{\tilde{J}_M}\lambda_M$, which is irreducible as proved in [CI] Theorem 2.47. For any irreducible cuspidal $k$-representation $\pi'$ of $M'$, there exists an irreducible cuspidal $k$-representation $\pi$ of $M$ such that $\pi'$ is a direct component of $\pi|_M$. Let $(J_M, \lambda_M)$ be a maximal simple $k$-types contained in $\pi$, then there exists a maximal simple $k$-type $(\tilde{J}_M, \tilde{\lambda}_M)$ as well as an open compact modulo centre subgroup $N_M(\tilde{\lambda}_M)$, the normaliser group of $\tilde{\lambda}_M$ in $M'$, containing $\tilde{J}_M$ as its largest open compact subgroup, as well as an extension $\Lambda_M$ of $\tilde{\lambda}_M$ to $N_M(\tilde{\lambda}_M)$, such that $\pi' \cong \text{ind}_{N_M(\tilde{\lambda}_M)}^{M'} \Lambda_M$. We call $(N_M(\tilde{\lambda}_M), \Lambda'_M)$ an extended maximal simple $k$-type.

**Proposition 2.3** (C.). Let $\pi'$ be an irreducible cuspidal $k$-representation of $M'$, there exists a cuspidal $k$-representation $\pi$ of $M$, such that $\pi'$ is a direct component of $\pi|_M$. Then $\pi'$ is supercuspidal if and only if $\pi$ is supercuspidal. When $\pi$ is supercuspidal, we call a $k$-type $(J_M, \lambda_M)$ (resp. $(\tilde{J}'_M, \tilde{\lambda}'_M)$) contained in $\pi$ (resp. $\pi'$) a maximal simple supercuspidal $k$-type.

Let $K_1, K_2$ be two open subgroups of $M'$ and $\rho_1, \rho_2$ two irreducible $k$-representations of $K_1, K_2$ respectively. We say that $\rho_1$ is weakly intertwined with $\rho_2$ in $M'$, if there exists an element $m \in M'$ such that $\rho_1$ is isomorphic to a sub-quotient of $\text{ind}^{K_1}_{K_1 \cap m(K_2)} \text{res}^{m(K_2)}_{K_1 \cap m(K_2)} m(\rho_2)$.

**Recall that:**

**Proposition 2.4** (C.). 1. $\tilde{J}_M = \tilde{J}'_M J_M$. 6
2. Let \((\tilde{J}_{M,1}', \tilde{\lambda}_{M,1}')\) and \((\tilde{J}_{M,2}', \tilde{\lambda}_{M,2}')\) be two maximal simple \(k\)-types of \(M'\), they are weakly intertwined in \(M'\) if and only if they are \(M'\)-conjugate.

3  

Category decomposition

In this section, to simplify the notations, we denote by \(G\) a Levi subgroup of \(\text{GL}_n(F)\) and \(G' = G \cap \text{SL}_n(F)\), which is a Levi subgroup of \(\text{SL}_n(F)\). Let \(M\) be a Levi subgroup of \(G\), we denote by \(M' = M \cap G'\) a Levi subgroup of \(G'\), and let \(K\) be an open subgroup of \(G\); we always denote by \(K' = K \cap G'\). If \(\pi\) is an irreducible \(k\)-representation of \(K\), then \(\pi'\) is one of its irreducible summands of \(\pi|_{K'}\).

3.1 Projective objects

In this section, we will follow the strategy as in \([He]\) to construct some projective objects of \(\text{Rep}_k(G')\). We study first the projective cover of maximal simple \(k\)-types of Levi subgroups \(M'\), then we consider their induced representations. Proposition 3.6 and Corollary 3.7 give the relation between these projective objects and irreducible \(k\)-representations whose cuspidal support is given by the corresponding maximal simple \(k\)-type. The later properties will be used in Section 3.2.

Let \((J_M, \lambda_M)\) be a maximal simple \(k\)-type of \(M\), and \(\tilde{J}_M\) be the group of projective normaliser of \((J_M, \lambda_M)\) (see Section 2.2). Write \(\lambda_M\) as \(\kappa_M \otimes \sigma_M\). Let \(P_{\lambda_M}\) be the projective cover of \(\lambda_M\), from \([He]\) Lemma 4.8 we know that \(P_{\lambda_M}\) is isomorphic to \(P_{\sigma_M} \otimes \kappa_M\), where \(P_{\sigma_M}\) is the projective cover of \(\sigma_M\). Denote by \(\tilde{\lambda}_M\) the irreducible \(k\)-representation \(\text{ind}_{\tilde{J}_M}^{J_M} \lambda_M\). Let \((\tilde{J}_M, \tilde{\lambda}_M')\) be a maximal simple \(k\)-type of \(M'\) defined from \((J_M, \lambda_M)\) as in Section 2.2. Since \(P_{\lambda_M}\) has finite length, we have \(P_{\lambda_M}|_{\tilde{J}_M} = \oplus_{i=1}^r P_i\), where \(P_i\) is an indecomposable projective \(k\)-representation of \(J'_M\) for each \(i\).

**Remark 3.1.** The projective cover \(P_{\sigma_M}\) is given by the theory of \(k\)-representations of finite general linear groups. When \(\sigma_M\) is inflated from a supercuspidal \(k\)-representation of \(M\), which means \((J_M, \lambda_M)\) is a maximal simple supercuspidal \(k\)-type of \(M\), according to the construction of \(P_{\sigma_M}\) (in Lemma 5.11 of \([Geck]\) or see Corollary 3.5 of \([C2]\) ) as well as Deligne-Lusztig theory, we conclude that the irreducible subquotients of \(P_{\sigma_M}\) are isomorphic to \(\sigma_M\).

Let \(\pi\) be an irreducible cuspidal \(k\)-representation of \(M\) which contains \((J_M, \lambda_M)\), and \(\pi'\) be an irreducible cuspidal \(k\)-representation of \(M'\) which contains \((\tilde{J}_M, \tilde{\lambda}_M')\) such that \(\pi' \hookrightarrow \pi_{|M'}\). We denote by

\[P_{[M, \pi]} = i_M^{G'} \text{ind}^M_{J_M} P_{\lambda_M},\]

and by

\[P_{[M', \pi']} = i_M'^{G'} \text{ind}^{M'}_{\tilde{J}'_M} P_{\tilde{\lambda}_M'}.\]

**Lemma 3.2.** \(P_{[M', \pi']}\) is a direct summand of \(P_{[M, \pi]|_{M'}}\).

**Proof.** We have

\[i_M^{G'} \text{ind}^M_{J_M} P_{\lambda_M}|_{G'} \cong i_M'^{G'} \text{ind}^{M'}_{\tilde{J}'_M} P_{\tilde{\lambda}_M'}.\]

Since \(\text{ind}^M_{J_M} P_{\lambda_M}|_{J'_M}\) is projective and has a surjection to \(\tilde{\lambda}_M'\), it is deduced that \(P_{\tilde{\lambda}_M'}\) is a direct summand of \(\text{ind}^M_{J_M} P_{\lambda_M}|_{J'_M}\). Hence \(P_{\tilde{\lambda}_M'}\) is a direct summand of \(P_{\lambda_M}|_{M'}\) where \(P_{\lambda_M} \cong \text{ind}^M_{\tilde{J}_M} P_{\lambda_M}\), and \(P_{[M', \pi']}\) is a direct summand of \(P_{[M, \pi]|_{G'}}\). \(\square\)
Let $(J_M, \lambda_M)$ be a maximal simple supercuspidal $k$-type of $M$, and $(\tilde{J}_M, \tilde{\lambda}_M)$ be a maximal simple supercuspidal $k$-type of $M'$ defined from $(J_M, \lambda_M)$ as in Section 2.2.

**Lemma 3.3.** Let $\pi$ be an irreducible supercuspidal $k$-representation of $M$ which contains $(J_M, \lambda_M)$, and $\tau'$ be an irreducible subquotient of the projective cover $P_{\tilde{\lambda}_M}$ of $\tilde{\lambda}_M$. Then $(\tilde{J}_M, \tau')$ is also a maximal simple supercuspidal $k$-type defined by $(J_M, \lambda_M)$, and there exists an irreducible direct component $\pi_0'$ of $\pi|_{M'}$ which contains $(\tilde{J}_M, \tau')$. In particular, when $M' = G' = SL_n(F)$, if $\tau'$ is different from $\tilde{\lambda}_M'$, and suppose $\pi'$ is an irreducible direct component of $\pi|_{M'}$ containing $\tilde{\lambda}'_M$, then $\pi_0'$ is different from $\pi'$.

**Proof.** Recall that $P_{\lambda_M}$ is the projective $k$-cover of $\lambda_M$, as explained in Remark 3.1 its irreducible subquotients are isomorphic to $\lambda_M$. As in the proof of Lemma 2.2 we know that the projective representation $P_{\tilde{\lambda}_M}$ is an indecomposable direct component of $\text{ind}_{\tilde{J}_M}^G P_{\lambda_M}|_{\tilde{J}_M}$. As in Section 2.2 the induced representation $\tilde{\lambda}_M := \text{ind}_{\tilde{J}_M}^G \lambda_M$ is irreducible. By the exactness of induction functor, we know that the irreducible subquotients of $\text{ind}_{\tilde{J}_M}^G P_{\lambda_M}$ are isomorphic to $\tilde{\lambda}_M$, which implies that an irreducible subquotient of $P_{\tilde{\lambda}_M}$ is isomorphic to an irreducible direct component of $\tilde{\lambda}_M|_{\tilde{J}_M}$. Since $\pi$ contains $\tilde{\lambda}_M$ after restricting to $\tilde{J}_M$, by the Mackey’s theory, any irreducible direct component of $\tilde{\lambda}_M|_{\tilde{J}_M}$ must be contained in an irreducible direct component of $\pi|_{M'}$.

When $M' = G' = SL_n(F)$, by Mackey’s theory we have the induction $\text{ind}_{\tilde{J}_M}^{G'} \text{res}_{\tilde{J}_M}^{G} \tilde{\lambda}_G$ is a sub-representation of $\pi|_{G'}$, and each irreducible component of $\tilde{\lambda}_G|_{\tilde{J}_M}$ is irreducibly induced to $G'$. The second statement is directly from the fact that $\pi|_{M'}$ is multiplicity-free as proved in Proposition 2.35 of [11].

**Remark 3.4.** When $M'$ is a proper Levi subgroup of $G'$, it is possible that two different maximal simple supercuspidal $k$-types $(\tilde{J}_M', \tilde{\lambda}_M')$ and $(\tilde{J}_M', \tilde{\lambda}_M')$, which are defined from a same maximal simple supercuspidal $k$-type, are $M'$-conjugate to each other, which implies that they may be contained in a same irreducible supercuspidal $k$-representation of $M'$.

**Lemma 3.5.** 1. Let $\alpha \in \tilde{J}_M$, then $\alpha(P_{\lambda_M}) \cong P_{\alpha(\lambda_M)} \cong P_{\lambda_M} \otimes \theta$, where $\theta \in \det(J_M)^\wedge$ and $\alpha(\lambda_M) \cong \lambda_M \otimes \theta$.

2. Let $(\tilde{J}_M, \tilde{\lambda}_M)$ and $(\tilde{J}_M', \tilde{\lambda}_M')$ be two different maximal simple $k$-types defined from $(J_M, \lambda_M)$.

Let $\alpha \in \tilde{J}_M$ such that $\alpha(\tilde{\lambda}_M) \cong \tilde{\lambda}_M'$, then for the projective covers we have $\alpha(P_{\tilde{\lambda}_M}) \cong P_{\tilde{\lambda}_M'}$.

**Proof.** For the first part, there is a surjective morphism from $P_{\lambda_M} \otimes \theta$ to $\lambda_M \otimes \theta$ and is indecomposable. Moreover, the projectivity can be easily deduced directly from the definition. Since $\alpha(P_{\lambda_M})$ is the projective cover of $\alpha(\lambda_M)$, we obtain the expected equality. The second part can be deduced in a similar way.

**Proposition 3.6.** Recall that $G'$ is a Levi subgroup of $SL_n(F)$. Let $\rho'$ be an irreducible $k$-representation of $G'$ and $(M', \pi')$ be a cuspidal pair of $G'$ inside the cuspidal support of $\rho'$, then there is a surjective morphism

$$P_{[M', \pi']} \rightarrow \rho'.$$

**Proof.** Let $(\tilde{J}_M, \tilde{\lambda}_M)$ be a maximal simple $k$-type contained in $\pi'$, hence there is an injection $\tilde{\lambda}_M' \rightarrow \text{res}_{\tilde{J}_M}^{G'} \pi'$. By Frobenius reciprocity, it gives a surjection $\text{ind}_{\tilde{J}_M}^{G'} P_{\tilde{\lambda}_M} \rightarrow \pi'$, which induces a surjection $P_{[M', \pi']} \rightarrow \mathbb{I}_M' \pi'$, hence a surjection from $P_{[M', \pi']}$ to $\rho'$ by [11] II, 2.20.
Corollary 3.7. Let $I_{[M', \pi']}$ be the contragredient of $P_{[M', \pi']}$, where $\pi' \tau$ is the contragredient of $\pi'$. Suppose that the cuspidal support of $\tau'$ is $[M', \pi']$, then $\tau'$ is embedding to $I_{[M', \pi']}$. 

3.2 Category decomposition

Recall that in this section $G'$ is a Levi subgroup of $SL_n(F)$ and $G$ is a Levi subgroup of $GL_n(F)$ such that $G' = G \cap SL_n(F)$. A decomposition of $Rep_k(G')$ by its full sub-categories will be given in Theorem 3.15 according to the G-twist equivalent supercuspidal classes of $G'$ (see the paragraph below Proposition 3.12 for G-twist equivalence). This will not be a block decomposition in general, which means it does not always verify the last condition of Equation (1), however we will see in Section 4 that it is not always possible to decompose $Rep_k(G')$ according to the $G'$-inertially equivalent supercuspidal classes as for $Rep_k(G)$ in Equation (1).

Let $A$ be a family of $G$-inertially equivalent supercuspidal classes of $G$, and denote by $Rep_k(G)_A$ the union of blocks $\bigcup_{[M, \pi] \in A} Rep_k(G)_{[M, \pi]}$. Let $A'$ be a family of $G'$-inertially equivalent supercuspidal classes of $G'$, verifying that $[M', \pi'] \in A'$ if and only if there exists $[M, \pi] \in A$ such that $M' = M \cap G'$ and $\pi' \rightarrow \pi|_{M'}$. Let $L$ be a Levi subgroup of $G$ which contains $M$, denote by $A_L$ the family of $L$-inertially equivalent supercuspidal classes in the form of $[w(M), w(\pi)]_\mathbb{A}$, where $[M, \pi] \in A$, and recall that $[\cdot, \cdot]_L$ is the $L$-inertially equivalent class, and $w \in G$ such that $w(M) \in L$.

Lemma 3.8. Let $P \in Rep_k(G)_{[M, \pi]}$, and $L$ be a Levi subgroup of $G$. Then $r_L^{G}P \in \bigcup_{w \in G, w(M) \in L} Rep_k(G)_{[w(M), w(\pi)]_L}$.

Proof. Suppose $P$ is an irreducible sub-quotient of $r_L^{G}P$, whose cuspidal support is $(N, \tau)$, where $N$ is a Levi subgroup of $L$ and $\tau$ is a cuspidal representation of $N$. Let $P_{[N, \tau]}$ be the projective object defined from the maximal simple $k$-type of $\tau$, then there is a non-trivial morphism $\mathcal{P}_{[N, \tau]} \rightarrow r_L^{G}P$. By the second adjunction of Bernstein, we have a non-trivial morphism from $r_L^{G}P_{[N, \tau]}$ to $P$, where $r_L^{G}$ is the opposite normalised parabolic induction from $L$ to $G$. Since the module character $\delta_L$ is an unramified character on $L$, the $k$-representation $r_L^{G}P_{[N, \tau]}$ belongs to the same block as $r_L^{G}P_{[M, \pi]}$, which implies that the supercuspidal support of $\tau$ belongs to the union $\bigcup_{w \in G, w(M) \in L} (w(M), w(\pi))$. We finish the proof.

Lemma 3.9. Let $P \in Rep_k(G)_A$, and $\tau'$ be an irreducible subquotient of $P|_{G'}$, then the supercuspidal support of $\tau'$ belongs to $A'$.

Proof. Suppose firstly that $\tau'$ is cuspidal, then there exists a maximal simple $k$-type $(J, \lambda)$ of $G$, such that an irreducible component $(J', \lambda')$ of $\lambda|_{G'}$ is contained in $\tau'$ as a subrepresentation. By [C1] Lemma 2.14, up to twist a $k$-character of $F^\times$, we can assume that $\lambda$ is a subquotient of $P|_J$. Hence there is a non-trivial morphism from the projective cover $\mathcal{P}_\lambda$ of $\lambda$ to $P|_J$, which implies that for any irreducible cuspidal $k$-representation $\tau$ of $G$ which contains $(J, \lambda)$, its supercuspidal support must belong to $A$. In particular, we can choose $\tau$ such that $\tau' \rightarrow \tau|_{G'}$, hence by [C2] Proposition 4.4 we know the supercuspidal support of $\tau'$ must belong to $A'$. Now suppose $\tau'$ is not cuspidal. Let $(L', \rho')$ belong to its cuspidal support. The $\rho'$ appears as a subquotient of $r_L^{G}P|_{G'} \cong (r_L^{G}P)|_{L'}$. By Lemma 3.8 and the previous paragraph, we know that the supercuspidal support of $\rho'$ must belong to $A_L'$, from which we deduces the desired property of supercuspidal support of $\tau$. 

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Lemma 3.10. Let \( \pi \) and \( \pi' \) be cuspidal \( k \)-representation of \( M \) and \( M' \) respectively and \( \pi' \hookrightarrow \pi_M \). Let \((J_M, \lambda_M)\) be a maximal simple \( k \)-type of \( \pi \) and \((\tilde{J}_M, \tilde{\lambda}_M)\) be a maximal simple \( k \)-type of \( \pi' \) defined from \((J_M, \lambda_M)\). Suppose \([L, \tau]\) is the supercuspidal support of \([M, \pi]\), then we have

\[
\text{ind}_{G'}^G \mathcal{P}_{[M', \pi']} \in \prod_{\chi \in (G')^\vee} \text{Rep}_k(G)_{[L, \tau \otimes \chi]}.
\]

Proof. We denote by \( \mathcal{P}' = \mathcal{P}_{[M', \pi']} \), \( \mathcal{P}'_M = \text{ind}_{J_M'} \mathcal{P}_{\lambda_M} \), and \( \mathcal{P}_\lambda = \text{ind}_{J_M} \mathcal{P}_{\lambda_M} \), this proof. Recall that \( \mathcal{P}' = \text{ind}_{J_M'} \mathcal{P}'_M \). Since the module character \( \delta_{M'} = \delta_{M|M'} \), we have

\[
\text{ind}_{G'}^G \mathcal{P}' \cong \text{ind}_{G}^G \mathcal{P}'_M \hookrightarrow \text{ind}_{G}^G \mathcal{P}_\lambda \oplus \text{ind}_{J_M} \mathcal{P}_\lambda.
\]

where

\[
\text{res}_{J_M} \mathcal{P}_\lambda \cong \text{res}_{J_M} \mathcal{P}_\lambda \oplus \text{res}_{J_M} \mathcal{P}_\lambda = \bigoplus_{\alpha \in \tilde{J}_M \setminus J_M} \text{res}_{J_M} \mathcal{P}_\lambda \otimes \oplus_{j_M \setminus J_M} \text{ind}_{J_M} \mathcal{P}_\lambda.
\]

Since \( J_M \) is a \( p \)-group, and by the definition of \( J_M \) the above representation is semisimple whose direct components are in the form of \( \eta \otimes \theta \), where \( \eta \) is the Heisenberg representation of the simple character of \( \lambda_M \), and \( \theta \in (\det(J_M))^\vee \) which can be extend to an character of \((F^\times)^\vee \) and we fix one of such extension by denoting it as \( \theta \) as well. Hence we have the decomposition

\[
\text{res}_{J_M} \mathcal{P}_\lambda \cong \bigoplus_{\alpha \in \tilde{J}_M \setminus J_M} \text{res}_{J_M} \mathcal{P}_\lambda \otimes \oplus_{j_M \setminus J_M} \text{ind}_{J_M} \mathcal{P}_\lambda.
\]

Lemma 3.11. Let \( A \) be as above, and \( P \in \text{Rep}_k(G)_A \). Then \( P^{\vee} \in \text{Rep}_k(G)_{A^\vee} \), where \( A^\vee \) consists of the \( G \)-inertially equivalent supercuspidal classes \([M, \pi]\) such that \([M, \pi] \in A\).

Proof. Suppose there exists an irreducible sub-quotient of \( P^{\vee} \). Denote by \([\tilde{M}_0, \tau_0]\) its supercuspidal support and by \([L_0, \pi_0]\) is cuspidal support. There is non-trivial morphism
Proposition 3.12. We take the same notations as in Lemma 3.10. Let $\rho'$ be an irreducible subquotient of the contragredient $P^\vee$, then the supercuspidal support of $\rho'$ is contained in union of $G$-conjugation classes of $[L', \tau'^\vee]$. In other words, let $\tau|_{L'} = \oplus_{i \in I} \tau_i^\vee$, then the supercuspidal support of $\rho'$ is contained in $\bigcup_{i \in I} [L', \tau_i^\vee]$.

Proof. Let $P'$ be $P_{[M', \pi']}$ in this proof. Since there is no non-trivial character on $G'$, we have $(\text{ind}_{G'}^G P')^\vee \cong \text{Ind}_{G'}^G P'^\vee$. By Lemma 3.10 and Lemma 3.11, we have $(\text{Ind}_{G'}^G P')^\vee \in \prod_{\chi \in (\mathbb{O} \times \mathbb{F}) \vee \text{Rep}_k(G)[L', \tau^\vee \otimes \chi]}$. By the surjective morphism $\text{res}_{G'}^G \text{Ind}_{G'}^G P'^\vee \to P'^\vee$ and Lemma 3.9, we conclude that the supercuspidal support of an arbitrary irreducible sub-quotient of $P'^\vee$ belongs to the $G$-conjugation of $[L', \tau']$.

Definition 3.13. Let $(L_1, \tau_1)$ and $(L_2, \tau_2)$ be cuspidal pairs of $G$, we say they are $G$-twist equivalent, if there exists $g \in G$ such that $g(L_1) = L_2$ and $g(\tau_1)$ is isomorphic to $\tau_2$ up to a $k$-quasicharacter of $\mathbb{F} \times$, which is an equivalence relation and denote by $[L_1, \tau_1]^\text{tw}$ the $G$-twist equivalent class defined by $(L_1, \tau_1)$.

Noting that in the above definition, we do not require the $k$-quasicharacter of $\mathbb{F} \times$ is unramified, which is different comparing to the relation of $G$-inertially equivalence. We define the depth of a $G$-twist equivalent class as the minimal depth among all the pairs inside this class. Denote by $C_{[L, \tau]^\text{tw}}$ the set of $G$-twist equivalent cuspidal classes whose supercuspidal support belong to $[L, \tau]^\text{tw}$ up to an isomorphism, and denote by $\overline{C_{[L, \tau]^\text{tw}}}$ the set of $G$-twist equivalent cuspidal classes whose supercuspidal support does not belong to $[L, \tau]^\text{tw}$ up to isomorphism and a twist of a $k$-quasicharacter of $\mathbb{F}^\times$. It is worth noticing that

1. $C_{[L, \tau]^\text{tw}}$ is a finite set;
2. fix a positive number $n \in \mathbb{N}$, there are only finitely many object in $C_{[L, \tau]^\text{tw}}$ whose depth is smaller than $n$.

Define

$$I_{(L, \tau)} = \bigoplus_{[M', \pi'] \in C'_{[L, \tau]^\text{tw}}} P^\vee_{[M', \pi'^\vee]},$$

$$I_{\overline{(L, \tau)}} = \bigoplus_{[M', \pi'] \in \overline{C'_{[L, \tau]^\text{tw}}}} P^\vee_{[M', \pi'^\vee]},$$

where the relation between $C_{[L, \tau]^\text{tw}}$ and $C'_{[L, \tau]^\text{tw}}$ is as explained in the beginning of Section 3.2.

Lemma 3.14. $I_{\overline{(L, \tau)}}$ is injective.
Proof. In fact \( I_{(L, \tau)} \) is the smooth part of the contragredient \( \prod_{[M', \pi'] \in C_{[L, \tau]}^w} P^*_{[M', \pi']} \) of \( \bigoplus_{[M', \pi'] \in C_{[L, \tau]}^w} P_{[M', \pi']} \), where \( P^*_{[M', \pi']} \) is the contragredient (not necessarily smooth) of \( P_{[M', \pi']} \). Fix an open compact subgroup \( K' \) of \( G' \), there exist finitely many \([M', \pi'] \in C_{[L, \tau]}^w\) such that the \( K \)-invariant part of \( P^*_{[M', \pi']} \) is non-trivial, which implies the same property for the contragredient \( P^*_V_{[M', \pi']} \) by [Y1 §4.15]. Hence an \( K \)-invariant non-trivial linear form \( f \) in the smooth part of \( \prod_{[M', \pi'] \in C_{[L, \tau]}^w} P^*_{[M', \pi']} \) must belong to \( \bigoplus_{[M', \pi'] \in C_{[L, \tau]}^w} P^*_{[M', \pi']} \), which finishes the proof. 

\[ \text{Theorem 3.15.} \] Let \( G \) be a Levi subgroup of \( GL_n(F) \) and \( G' \) be a Levi subgroup of \( SL_n(F) \), such that \( G' = G \cap SL_n(F) \). We have a category decomposition

\[
\text{Rep}_k(G') \cong \prod_{[L, \tau]^w \in SC^w_G} \text{Rep}_k(G')_{[L, \tau]^w},
\]

where

1. \( SC^w_G \) is the set of \( G \)-twist equivalent supercuspidal classes in \( G \);

2. \( \text{Rep}_k(G')_{[L, \tau]^w} \) is the full sub-category consisting with the objects whose irreducible sub-quotients have supercuspidal support belonging to \([L', \tau']_G\) and \( \tau' \) is an irreducible direct component of \( \tau|_{L'} \).

In particular, for each object in \( \text{Rep}_k(G')_{[L, \tau]^w} \), it has an injective resolution with direct sums of copies of \( I_{(L, \tau)} \).

Proof. By the definition of \( I_{(L, \tau)} \), Corollary 3.14 and Proposition 3.12 each irreducible sub-quotient of \( I_{(L, \tau)} \) is a sub-representation of \( I_{(L, \tau)} \), and non of the irreducible sub-quotient of \( I_{(L, \tau)} \) appears as a sub-quotient of \( I_{(L, \tau)} \). Furthermore, each irreducible \( k \)-representation is either a sub-representation of \( I_{(L, \tau)} \) or a sub-representation of \( I_{(L, \tau)} \) by the unicity of cuspidal support as well as the unicity of supercuspidal support. Hence by Proposition 2.4.1 for any object \( \Pi \in \text{Rep}_k(G') \) and any \( G \)-twist equivalent supercuspidal class \([L, \tau]^w \) of \( G \), define \( \Pi_{[L, \tau]^w} \) be the largest sub-representation of \( \Pi \) belonging to \( \text{Rep}_k(G')_{[L, \tau]^w} \), we have \( \Pi \cong \bigoplus_{[L, \tau]^w \in SC^w_G} \Pi_{[L, \tau]^w} \), and by applying Proposition 2.4.1 we know that there is no morphism between objects of sub-categories defined from different \( G \)-twist equivalent supercuspidal classes, hence we finish the proof. 

\[ \text{Remark 3.16.} \] Let \((L, \tau)\) be a supercuspidal pair of \( G \), and \( \tau|_{L'} \cong \bigoplus_{j=1}^s r'_j \), where \((L', r'_j)\) are supercuspidal pairs of \( G' \). Denote by \( \text{Rep}_k(G')_{[L', \tau'_j]} \) the full sub-category of \( \text{Rep}_k(G') \), consisting of objects of which any irreducible subquotient has supercuspidal support belonging to the \( G' \)-inertially equivalent class \([L', \tau'_j] \). The sub-category \( \text{Rep}_k(G')_{[L', \tau'_j]} \) is generated by sub-categories \( \text{Rep}_k(G')_{[L', \tau'_j]} \), for all \( 1 \leq j \leq s \). In other words, let \( SC^w_G \) be the set of \( G' \)-inertially equivalent supercuspidal classes of \( G' \), Theorem 3.15 gives a category decomposition of \( \text{Rep}_k(G') \) according to \( SC^w_G \).

\[ \text{Corollary 3.17.} \] Let \([L, \tau] \) be a \( G' \)-inertially equivalent class of \( G \), where \( G \) is a Levi subgroup of \( GL_n(F) \). The functor \( \text{res}_G^G \) gives functors from blocks \( \text{Rep}_k(G/[L, \tau] \otimes \chi) \) for any \( k \)-quasicharacter \( \chi \) of \( F^* \) to the sub-category \( \text{Rep}_k(G')_{[L, \tau]^w} \).

Proof. It is directly from Theorem 3.15 and Lemma 3.9.
Corollary 3.18. Let $G'$ be a Levi subgroup of $\text{SL}_n(F)$. There is a category decomposition
\[ \text{Rep}_k(G') \cong \text{Rep}_k(G')_{\text{SC}} \times \text{Rep}_k(G')_{\text{non-SC}}, \]
where

1. an object belongs to $\text{Rep}_k(G')_{\text{SC}}$, if and only if its irreducible sub-quotients are supercuspidal;

2. an object belongs to $\text{Rep}_k(G')_{\text{non-SC}}$, if and only if none of its irreducible sub-quotients is supercuspidal.

Proof. Directly from Theorem 3.15 \HBox{ule{0.5em}{0.5em}}

Definition 3.19. We call $\text{Rep}_k(M')_{\text{SC}}$ the supercuspidal sub-category of $\text{Rep}_k(M')$, and the blocks of $\text{Rep}_k(M')_{\text{SC}}$ are called supercuspidal blocks of $\text{Rep}_k(M')$.

4 Supercuspidal sub-category of $\text{Rep}_k(M')$

In this section, let $G$ be $\text{GL}_n(F)$ and $G'$ be $\text{SL}_n(F)$. In the previous section, Theorem 3.15 gives a category decomposition of $\text{Rep}_k(G')$, according to which we define the supercuspidal sub-category $\text{Rep}_k(G')_{\text{SC}}$. In this section, Theorem 4.12 gives a description of the blocks of the supercuspidal sub-category of $\text{Rep}_k(G')$ and $\text{Rep}_k(M')$, where $M'$ is a Levi subgroup of $G'$.

4.1 $M'$-inertially equivalent supercuspidal classes

In this section, we give a bijection between $M'$-conjugacy classes of maximal simple $k$-types of $M'$, and $M'$-inertially equivalent cuspidal classes of $M'$. The most complexity of this section comes from the fact that the Levi subgroup of $G'$ is not a special linear group in lower rank.

Let $M$ be the Levi subgroup of $G$ such that $M' = M \cap G'$. Let $(\tilde{J}_M', \tilde{\lambda}_M')$ be a maximal simple $k$-type of $M'$ defined from a maximal simple $k$-type $(J_M, \lambda_M)$ of $M$. As explained in Section 2.2, let $\pi$ be an irreducible cuspidal $k$-representation of $M$ containing $(J_M, \lambda_M)$, then there exists a direct component $\pi'$ of $\pi|_{M'}$, such that $\pi'$ contains $(\tilde{J}_M', \tilde{\lambda}_M')$.

Lemma 4.1. Let $E$ be a field extension of $F$, such that there is an embedding $E^\times \hookrightarrow \text{GL}_n(F)$. Let $\omega_E$ be a uniformiser of $E$, and $Z_{\omega_E}$ be a subgroup of $\text{GL}_n(F)$ generated by the image of $\omega_E$ under the embedding. Then a $k$-character of $Z_{\omega_E}$ can be extended to a character of $\text{GL}_n(F)$.

Proof. A $k$-character of $Z_{\omega_E}$ factors through determinant of $\text{GL}_n(F)$ \HBox{ule{0.5em}{0.5em}}

Under the assumption on $E$ as in Lemma 4.1, denote by $Z_{\mathcal{O}_{E}}$ the group generated by the image of $\mathcal{O}_E^\times$ under the embedding. For general Levi subgroup $M$ of $G = \text{GL}_n(F)$. Suppose $M$ is a direct product of $m$ general linear groups, and there exist field extensions $E_i, 1 \leq i \leq m$ of $F$, such that $\prod_{i=1}^m E_i^\times \hookrightarrow M$. Then after fixing a uniformiser $\omega_i$ for each $E_i$, we denote by $Z_{\omega_{E_i}}$ the group generated by the image of $\{1 \times \ldots \times \omega_i \times \ldots \times 1, 1 \leq i \leq m\}$ under the embedding, and by $Z_{\mathcal{O}_{E_i}}$ the group generated by the image of $\prod_{i=1}^m \mathcal{O}_i^\times$, where $\mathcal{O}_i$ is the ring of integers of $E_i$. It is obvious that the image of $\prod_{i=1}^m E_i^\times$ can be decomposed as a direct product $Z_{\omega_{E_i}} \times Z_{\mathcal{O}_{E_i}}$. In particular, when $E_i = F$ for $1 \leq i \leq m$, we consider
the canonical embedding, which is the equivalence between \((F^\times)^m\) and the centre of \(M\). Then the centre of \(M\) decomposes as \(Z_{\omega_{FM}} \times Z_{\nu_{FM}}\). We denote by \(Z'_{\omega_{FM}}\) as \(Z_{\omega_{FM}} \cap M'\) and \(Z'_{\nu_{FM}}\) as \(Z_{\nu_{FM}} \cap M'\).

**Remark 4.2.** Lemma 4.1 implies that a \(k\)-character of \(Z_{\omega_{FM}}\) can be extended to a \(k\)-character of \(M\). In particular, for two irreducible \(k\)-representations of \(M\), if their central characters coincide to each other on \(Z_{\nu_{FM}}\), then up to modifying by an unramified \(k\)-character, they share the same central character.

**Proposition 4.3.** Let \(\pi_1, \pi_2\) be two irreducible cuspidal \(k\)-representations of \(M'\) which contain \((\tilde{J}_M, \tilde{\lambda}_M)\). Then there exists an unramified \(k\)-character \(\chi\) of \(F^\times\), such that \(\pi_1 \cong \pi_2 \otimes \chi\).

**Proof.** Let \(N_M(\tilde{\lambda}_M)\) be the normaliser of \(\tilde{\lambda}_M\) in \(M'\), which contains the centre \(Z_{\nu_{FM}}\) of \(M'\) as mentioned in Section 2.2 by Theorem 4.4 of [C1]. There exists an \(\Lambda_{M',1}, \Lambda_{M',2}\) of \(\tilde{\lambda}_M\) to \(N_M(\tilde{\lambda}_M)\), such that \(\pi_1 \cong \text{ind}_{M'} N_M(\tilde{\lambda}_M) \Lambda_{M',1}\) and \(\pi_2 \cong \text{ind}_{M'} N_M(\tilde{\lambda}_M) \Lambda_{M',2}\).

After modifying an unramified \(k\)-character of \(M'\), we can assume that \(\Lambda_{M',1}\) and \(\Lambda_{M',2}\) have the same central character on \(Z_{\nu_{FM}}\). In fact, we have \(Z_{\nu_{FM}} \subset J_M \subset \tilde{J}_M\), hence the central characters of \(\Lambda_{M',1}\) and \(\Lambda_{M',2}\) coincide on \(Z_{\nu_{FM}}\). On the other hand, since \(Z_{\nu_{FM}} \cong \mathbb{Z}^m\) for an integer \(m\) decided by \(M\), a character of a sub-\(\mathbb{Z}\)-module of \(Z_{\nu_{FM}}\) can be extended to \(Z_{\nu_{FM}}\). In particular, we can extend a character of \(Z_{\nu_{FM}}\) to \(Z_{\nu_{FM}}\), then to \(\text{M}\) by Lemma 4.1 finally restricting to \(M'\). Hence we prove that a character of \(Z_{\nu_{FM}}\) can be extended to \(M'\). Combining with the above discussion, we conclude that there is an unramified \(k\)-character \(\chi_1\) of \(M'\), such that \(\Lambda_{M',1} \otimes \chi_1|_{Z_{\nu_{FM}} \tilde{J}_M} \cong \Lambda_{M',2}|_{Z_{\nu_{FM}} \tilde{J}_M}\). By the Frobenius reciprocity, there is an injection

\[
\Lambda_{M',1} \otimes \chi_1 : Z_{\nu_{FM}} \tilde{J}_M 
\] to \(\text{M}\).

As in Remark 2.42 of [C1], the group \(N_M(\tilde{\lambda}_M)\) (see Section 2.2 for definition) is a subgroup with finite index of \(E_M^\times \tilde{J}_M \cap M'\), where \(E_M^\times \cong \prod_{i=1}^m E_i^\times\) and \(E_i\) is a field extension of \(F\) for each \(1 \leq i \leq m\). Since the quotient group \(N_M(\tilde{\lambda}_M)/Z_{\nu_{FM}} \tilde{J}_M\) is isomorphic to a subquotient group of \(Z_{\omega_{FM}}\), hence a character of \(\text{ind}_{N_M(\tilde{\lambda}_M)} Z_{\nu_{FM}} \tilde{J}_M\) can be extended to a character of \(M\) by Lemma 4.1.

Now we look back to Equation 4. The \(k\)-representation \(\text{ind}_{N_M(\tilde{\lambda}_M)} Z_{\nu_{FM}} \tilde{J}_M\) has finite length and each of its irreducible subquotients is a character of \(\text{ind}_{N_M(\tilde{\lambda}_M)} Z_{\nu_{FM}} \tilde{J}_M\), hence can be viewed as a character of \(M'\). By the unicity of Jordan-Hölder factors, there exists a character \(\chi_2\) of \(M'\), such that \(\Lambda_{M',1} \otimes \chi_1 \cong \Lambda_{M',2} \otimes \chi_2\), since \(\chi_1, \chi_2\) are \(k\)-characters of \(M'\), applying the induction functor \(\text{ind}_{N_M(\tilde{\lambda}_M)}^M\) on both sides gives an equivalence that \(\pi_1 \otimes \chi_1 \cong \pi_2 \otimes \chi_2\). Define \(\chi\) to be \(\chi_2 \chi_1^{-1}\), which is the required unramified \(k\)-character of \(M'\).

**Proposition 4.4.** Let \((\tilde{J}_M, \tilde{\lambda}_M)\) be a maximal simple \(k\)-type of \(M'\), and \(\pi'\) an irreducible \(k\)-representation of \(M'\) containing \((\tilde{J}_M, \tilde{\lambda}_M)\). Then any irreducible subquotient of \(\text{ind}_{\tilde{J}_M} \tilde{\lambda}_M\) must belong to \([M', \pi']_{M'}\), or equivalently saying, be \(M'\)-inertially equivalent to \(\pi'\).

**Proof.** By Proposition IV.1.6 of [V2], we know that \(\text{ind}_{\tilde{J}_M} \tilde{\lambda}_M\) is cuspidal, hence its subrepresentation \(\text{ind}_{\tilde{J}_M} \tilde{\lambda}_M\) is cuspidal as well. Then an irreducible subquotient \(\pi_0\) is cuspidal, which contains a maximal simple \(k\)-type \((J'_0, \lambda'_0)\), which is weakly intertwining with \((\tilde{J}_M, \tilde{\lambda}_M)\) in \(M'\) by Mackey’s theory. By the property of weakly intertwining implying conjugacy of maximal simple \(k\)-types of \(M'\) in Theorem 3.25 of [C1], we conclude that a maximal simple \(k\)-type contained in \(\pi_0\) must \(M'\)-conjugate to \((\tilde{J}_M, \tilde{\lambda}_M)\), and hence \(\pi_0\) contains \((\tilde{J}_M, \tilde{\lambda}_M)\). By Proposition 4.3 we conclude that \(\pi_0\) is \(M'\)-inertially equivalent to \(\pi'\).
Remark 4.5. 1. Lemma 4.3 and Proposition 4.4 give a bijection between the set of $M'$-conjugacy classes of maximal simple $k$-types and the set of $M'$-inertially equivalent cuspidal classes:

$$\nu : [J_{M'}^i, \lambda_M^i]_{M'} \mapsto [M', \pi']_{M'},$$

where $[J_{M'}^i, \lambda_M^i]_{M'}$ is the $M'$-conjugacy class of $(J_{M'}^i, \lambda_M^i)$, and $\pi'$ is an irreducible cuspidal $k$-representation contains $(J_{M'}^i, \lambda_M^i)$.

2. Let $(J_{M'}^i, \lambda_M^i)$ and $(J_{M'}^j, \lambda_M^j)$ be two different maximal simple $k$-types defined by $(J_M, \lambda_M)$.

When $M' = G' = \text{SL}_n(F)$ by Lemma 3.3, the associated $G'$-inertially equivalent cuspidal classes defined of $(J_{M'}^i, \lambda_i)$, $i = 1, 2$ are different. When $M'$ is a proper Levi of $\text{SL}_n(F)$ by Remark 3.4, the associated $G'$-inertially equivalent cuspidal classes may be the same.

4.2 Supercuspidal blocks of $\text{Rep}_k(M')$

In this Section, we give a block decomposition of the supercuspidal sub-category $\text{Rep}_k(M')_{\text{SC}}$ of $\text{Rep}_k(M')$, of which the blocks are called supercuspidal blocks of $\text{Rep}_k(M')$ as defined in the end of Section 3.2. Let $\{M'^i_{\lambda M}, \pi_{\lambda M}^i\}$ be a sub-category $\text{Rep}_k(M')$ as defined in this proof. By Theorem 3.15, the irreducible subquotients of $\text{Rep}_k(M')_{\text{SC}}$ is irreducible and (3.4) - types defined by $(J_M, \lambda_M)$.

Let $(J_M, \lambda_M)$ be a maximal simple supercuspidal $k$-type of $M$, and $(J_M^i, \lambda^i_M)$ be a maximal simple supercuspidal $k$-type defined from $(J_M, \lambda_M)$ as explained in Section 4.2. Recall that $\mathcal{P}_{\lambda_M^i}$ is the projective cover of $\lambda_M^i$. By Lemma 3.3, its irreducible subquotients are maximal simple supercuspidal $k$-types of $M'$ as well, and we denote by $\mathcal{I} (\lambda_M^i)$ the set of isomorphic classes of irreducible subquotients of $\mathcal{P}_{\lambda_M^i}$. We define a set of $M'$-inertially equivalent supercuspidal classes $\mathcal{SC} (\lambda_M^i)$, such that there is a bijection

$$\nu : \mathcal{I} (\lambda_M^i) \to \mathcal{SC} (\lambda_M^i),$$

which is given as in Remark 3.5.

Proposition 4.6. Suppose the image $\mathcal{SC} (\lambda_M^i)$ is not a singleton. For any non-trivial disjoint union $\mathcal{SC} (\lambda_M^i) = \mathcal{SC}_1 \cup \mathcal{SC}_2$, and let $\mathcal{I} (\lambda_M^i) = \mathcal{I}_1 \cup \mathcal{I}_2$ such that $\mathcal{SC}_1 = \nu (\mathcal{I}_1)$ and $\mathcal{SC}_2 = \nu (\mathcal{I}_2)$, it is not possible to decompose $\text{ind}_{\mathcal{P}_{\lambda_M^i}} M' \mathcal{P}_{\lambda_M^i}$ as $P_1 \oplus P_2$, where any irreducible subquotients of $P_1$ belongs to $\mathcal{SC}_1$ and any irreducible subquotients of $P_2$ belongs to $\mathcal{SC}_2$.

Proof. We abbreviate $\text{ind}_{\mathcal{P}_{\lambda_M^i}} M' \mathcal{P}_{\lambda_M^i}$ by $\mathcal{P}_{M'}$ in this proof. By Theorem 3.15, the irreducible subquotients of $\mathcal{P}_{M'}$ are supercuspidal. Suppose the contrary that, there exists a non-trivial disjoint union $\mathcal{SC} (\lambda_M^i) = \mathcal{SC}_1 \cup \mathcal{SC}_2$, such that $\mathcal{P}_{M'} = P_1 \oplus P_2$ verifying the conditions in the statement of the proposition. Without loss of generality, we suppose $\lambda_M^i \in \mathcal{I}_1$. Let $\psi_M^i$ be a maximal simple supercuspidal $k$-type in $\mathcal{I}_2$, and $\psi'$ be a supercuspidal $k$-representation of $M'$ containing $\psi_M^i$. Hence $\psi'$ is a sub-representation of $\text{ind}_{\mathcal{P}_{\lambda_M^i}} M' \mathcal{P}_{\lambda_M^i}$, and the later is a subquotient of $\mathcal{P}_{M'}$, hence $P_2$ is non-trivial ($P_1$ is also non-trivial since $\lambda_M^i \in \mathcal{I}_1$).

By Lemma 3.3, there exists a filtration of $\{0\} = W_0 \subset W_1 \subset \cdots \subset W_s = \mathcal{P}_{\lambda_M^i}$ for an $s \in \mathbb{N}$, such that each quotient $\hat{\lambda}_i := W_i/W_{i-1}, 1 \leq i \leq s$ is irreducible and $(\hat{J}_M^i, \hat{\lambda}_i)$ is a maximal simple supercuspidal $k$-type of $M'$ defined also from $(J_M, \lambda_M)$. In particular, $\lambda_M$ as well as...
\(\tilde{\lambda}'_M\) are isomorphic to  \(\tilde{\lambda}'_i\) for some \(0 \leq i \leq s\) respectively. Now define \(\tilde{\lambda}'_M\) to be null, and denote by \(V_i = \text{ind}^{M'}_{J^i_M} W_i\), then \(\{V_i\}_{0 \leq i \leq s}\) is a filtration of \(\mathcal{P}_M\) and \(V_i/V_{i-1} \cong \text{ind}^{M'}_{J^i_M} \tilde{\lambda}'_i, 1 \leq i \leq s\). Denote by \(V_{i,1}\) the image of \(V_i\) in \(P_1\) under the canonical projection, and \(V_{i,2}\) the image of \(V_i\) in \(P_2\) under the canonical projection. Hence \(\{V_{i,1}\}_{0 \leq i \leq s}\) (resp. \(\{V_{i,2}\}_{0 \leq i \leq s}\)) forms a filtration of \(P_1\) (resp. \(P_2\)). By Proposition 4.4, the quotient \(V_{i,1}/V_{i-1,1}\) (resp. \(V_{i,2}/V_{i-1,2}\)) is non-trivial if and only if \(\tilde{\lambda}'_i \in \mathcal{I}_1\) (resp. \(\tilde{\lambda}'_i \notin \mathcal{I}_2\)).

Now we consider the canonical injective morphism

\[\alpha : \mathcal{P}_{\tilde{\lambda}'_M} \hookrightarrow \text{res}^{M'}_{J^i_M} \mathcal{P}_M\).

Under the above assumption, we have \(\text{res}^{M'}_{J^i_M} \mathcal{P}_M \cong \text{res}^{M'}_{J^i_M} P_1 \oplus \text{res}^{M'}_{J^i_M} P_2\). Since we consider a representation of infinite length, the unicity of Jordan-Hölder factors is not sufficient, and we need a simple but practical lemma as below to continue the proof:

**Lemma 4.7.** Let \(G\) be a locally pro-finite group, and \(\pi\) a \(k\)-representation of \(G\). Let \(\pi_1\) be a sub-representation of \(\pi\). Suppose \(\tau\) is an irreducible subquotient of \(\pi\), then \(\tau\) is either isomorphic to an irreducible subquotient of \(\pi_1\) or to an irreducible subquotient of \(\pi/\pi_1\).

**Proof.** Easy to check. \(\square\)

**Continue the proof of Proposition 4.6.** Suppose \(\alpha(\mathcal{P}_{\tilde{\lambda}'_M}) \subset \text{res}^{M'}_{J^i_M} P_1\). Let \(\tilde{\lambda}'_M \in \mathcal{I}_2\) be an irreducible subquotient of \(\mathcal{P}_{\tilde{\lambda}'_M}\) by Lemma 4.7 there exists \(1 \leq i \leq s\), such that \(\tilde{\lambda}'_M\) is an irreducible subquotient of \(V_{i,1}/V_{i-1,1}\), and the later is a subquotient of \(\text{ind}^{M'}_{J^i_M} \tilde{\lambda}'_i\). In other words, \(\tilde{\lambda}'_M\) is an irreducible subquotient of \(\text{ind}^{M'}_{J^i_M} \tilde{\lambda}'_i\). Applying Mackey’s theorem, it is equivalent to say that \(\tilde{\lambda}'_M\) is weakly intertwined with \(\tilde{\lambda}'_i\) in \(M'\) (see Section 2.2 for weakly intertwining), hence by Theorem 3.25 of [C1] they are \(M'\)-conjugate to each other, hence they define the same \(M'\)-inertially equivalent class as in Remark 4.5. Meanwhile, by the above analysis, we know that \(\nu(\tilde{\lambda}'_i) \in SC_1\) and \(\nu(\tilde{\lambda}'_M) \in SC_2\), which is a contradiction. Hence \(\alpha(\mathcal{P}_{\tilde{\lambda}'_M}) \cap \text{res}^{M'}_{J^i_M} P_1 \neq \alpha(\mathcal{P}_{\tilde{\lambda}'_M})\).

Now we consider \(\alpha(\mathcal{P}_{\tilde{\lambda}'_M})/(\alpha(\mathcal{P}_{\tilde{\lambda}'_M}) \cap \text{res}^{M'}_{J^i_M} P_1)\), which is non-null as above, and is a subrepresentation of \(\text{res}^{M'}_{J^i_M} \mathcal{P}_M/\text{res}^{M'}_{J^i_M} P_1 \cong \text{res}^{M'}_{J^i_M} P_2\). By the same manner as above, we conclude that each irreducible subquotient of \(\alpha(\mathcal{P}_{\tilde{\lambda}'_M})/(\alpha(\mathcal{P}_{\tilde{\lambda}'_M}) \cap \text{res}^{M'}_{J^i_M} P_1)\) belongs to \(\mathcal{I}_2\), which implies that there exists \(\tilde{\lambda}'_M\) such that \(\mathcal{P}_{\tilde{\lambda}'_M} \to \tilde{\lambda}'_M\). Since \(\tilde{\lambda}'_M\) is different from \(\tilde{\lambda}'_M\), the maximal semisimple quotient of \(\mathcal{P}_{\tilde{\lambda}'_M}\) contains \(\tilde{\lambda}'_0 \oplus \tilde{\lambda}'_M\), which contradicts to the fact that \(\mathcal{P}_{\tilde{\lambda}'_M}\) is the projective cover of \(\tilde{\lambda}'_M\) by Proposition 41 c) [Ser]. Hence we finish the proof. \(\square\)

**Lemma 4.8.** Let \((\tilde{\mathcal{J}}_M, \tilde{\lambda}'_1)\) and \((\tilde{\mathcal{J}}_M, \tilde{\lambda}'_2)\) be two maximal simple supercuspidal \(k\)-types. Suppose \(\tilde{\lambda}'_2 \in \mathcal{I}(\tilde{\lambda}'_1)\), then \(\tilde{\lambda}'_2 \in \mathcal{I}(\tilde{\lambda}'_2)\) (see the begining of this section for the definition of \(\mathcal{I}(\cdot)\)).

**Proof.** Let \(W(k)\) be the ring of Witt vectors of \(k\), and \(K\) be the fractional field of \(W(k)\). Let \(\tilde{K}\) be a finite field extension of \(K\), such that \(\tilde{K}\) contains the \(|\mathcal{J}_M/N|\)-th roots, where \(N\) is the kernel of \(\mathcal{P}_{\lambda_1}\), and let \(\mathcal{O}\) be its ring of integers. Consider the \(\ell\)-modular system \((\tilde{K}, \mathcal{O}, k)\), we have that \(\mathcal{P}_{\lambda_1} \otimes_{\mathcal{O}} \tilde{K}\) is semisimple, whose direct components are absolutely irreducible.

By Proposition 42 of [Ser], the projective cover \(\mathcal{P}_{\lambda_1}\) can be lifted over \(\mathcal{O}\), and we denote the lifting to \(\tilde{\mathcal{O}}\) by \(\mathcal{P}_{\lambda_1}\) as well. Now we consider \(\mathcal{P}_{\lambda_1} \otimes_{\tilde{\mathcal{O}}} \tilde{K}\), which is semisimple with finite length. Suppose \(P\) is an irreducible component of \(\mathcal{P}_{\lambda_1} \otimes_{\tilde{\mathcal{O}}} \tilde{K}\), then the semisimplification of
its reduction modulo $\ell$ must contain $\tilde{\lambda}_1$, otherwise it will induce a surjection from $P_{\tilde{\lambda}_1}$ to an irreducible $k$-representation different from $\tilde{\lambda}_1$, which contradicts with the fact the $P_{\tilde{\lambda}_1}$ is the projective cover of $\tilde{\lambda}_1$ by Proposition 41 of [SG]. Since $\tilde{\lambda}_1$ is a subquotient of $P_{\tilde{\lambda}_1}$, their exists an irreducible direct component $P_2'$ of $P_{\tilde{\lambda}_1} \otimes_{\tilde{O}} \tilde{K}$, of which the semisimplification of reduction modulo-$\ell$ contains $\tilde{\lambda}_1$ as well as $\tilde{\lambda}_2'$. Let $\alpha \in \tilde{J}_M$, such that $\alpha(\tilde{\lambda}_1') \cong \tilde{\lambda}_2'$. By the second part of Lemma 3.5 we have $\alpha(P_{\tilde{\lambda}_1'}) \cong P_{\tilde{\lambda}_2'}$, which implies that $\alpha(P_2')$ is a direct component of $P_{\tilde{\lambda}_1'}$. We state that $\alpha$ stabilises $P_2'$. In fact, by the proof of Lemma 5.2 we have $P_{\tilde{\lambda}_1'}$ is an indecomposable direct factor of $P_{\tilde{\lambda}_M}$. In particular, the reduction modulo-$\ell$ of each irreducible components of $P_{\tilde{\lambda}_M} \otimes_{\tilde{O}} \tilde{K}$ is isomorphic to $\tilde{\lambda}_M$. By the unicity of Jordan-Holdar factors, there exists an irreducible component $P_2'$ of $P_{\tilde{\lambda}_M} \otimes_{\tilde{O}} \tilde{K}$, such that $P_2'$ is an irreducible component of $P_2|_{J_M}$. Since $\alpha(\tilde{\lambda}_1') \cong \tilde{\lambda}_2'$, the semisimplification of the reduction modulo-$\ell$ of $\alpha(P_2')$ contains $\tilde{\lambda}_2'$. Since $\alpha(P_2')$ is isomorphic to an irreducible component of $P_2|_{J_M}$, and the reduction modulo-$\ell$ of $P_2$ is isomorphic to $\tilde{\lambda}_M$, combining with the fact that $\tilde{\lambda}_M|_{J_M} = 1$ is multiplicity-free, we conclude that $\alpha(P_2') \cong P_2'$. Hence $\tilde{\lambda}_1' \in \mathcal{I}(\tilde{\lambda}_2')$.

**Definition 4.9.** Let $(J_M, \tilde{\lambda}_M)$ be a maximal simple supercuspidal $k$-type of $M$, and denote by $\mathcal{I}(\tilde{\lambda}_M)$ the set of isomorphic classes of maximal simple supercuspidal $k$-types of $M'$ defined by $(J_M, \tilde{\lambda}_M)$. Suppose $(J_{M}, \gamma')$ and $(J_{M}', \tau')$ be two elements in $\mathcal{I}(\tilde{\lambda}_M)$, we say

1. $\gamma'$ is related to $\tau'$, if $\gamma' \in \mathcal{I}(\tau')$ (or equivalently $\tau' \in \mathcal{I}(\gamma')$) by Lemma 3.8 and we denote by $\gamma' \leftrightarrow \tau'$;

2. $\gamma' \sim \tau'$ if there exists a serie $(\tilde{J}_M, \tilde{\lambda}_i')$, $1 \leq i \leq t$ for an integer $t$, such that

$$\gamma' \leftrightarrow \tilde{\lambda}_1' \leftrightarrow \ldots \leftrightarrow \tilde{\lambda}_t' \leftrightarrow \tau',$$

and we call the series $\{\tilde{\lambda}_i', 1 \leq i \leq t\}$ a connected relation of $\gamma'$ and $\tau'$. The relation $\sim$ defines an equivalence relation on $\mathcal{I}(\tilde{\lambda}_M)$ (By Proposition 2.6 of [C], the relation $\leftrightarrow$ and $\sim$ on $\mathcal{I}(\tilde{\lambda}_M)$ does not depend on the choice of $\tilde{\lambda}_M$).

3. Denote by $[\tilde{\lambda}_M', \sim]$ the subset of $\mathcal{I}(\tilde{\lambda}_M)$ consisting of all $\tau'$ such that $\tau' \sim \tilde{\lambda}_M'$, or equivalently the connected component containing $\tilde{\lambda}_M'$ defined by $\sim$.

Let $\pi$ be an irreducible supercuspidal $k$-representation of $M$, denote by $\mathcal{I}(\pi)$ the isomorphic classes of the irreducible direct components of $\pi|_{M'}$. Let $(J_M, \lambda_M)$ be a maximal simple supercuspidal $k$-type contained in $\pi$. The above equivalence relation $\sim$ on $\mathcal{I}(\lambda_M)$ induces an equivalence relation on $\mathcal{I}(\pi)$.

**Definition 4.10.** Let $\pi_1', \pi_2' \in \mathcal{I}(\pi)$, we say $\pi_1' \sim \pi_2'$ if there exists a maximal simple supercuspidal $k$-type $(J_{M}, \lambda_M)$ contained in $\pi$, and two maximal simple supercuspidal $k$-types $(J_{M,1}, \lambda_{M,1})$ and $(J_{M,2}, \lambda_{M,2})$ defined from $(J_M, \lambda_M)$, such that $\pi_i'$ contains $\lambda_{M,i}$ for $i = 1, 2$, and $\lambda_{M,1} \sim \lambda_{M,2}$. By the unicity property of intertwining implying conjugacy in Theorem 3.25 of [C], $\sim$ defines an equivalence relation on $\mathcal{I}(\pi)$.

**Remark 4.11.** Let $\pi' \in \mathcal{I}(\pi)$, define $[\pi', \sim]$ to be a subset of $\mathcal{I}(\pi)$, consisting of the elements that are equivalent to $\pi'$. In other words, $(\pi', \sim)$ is the connected component containing $\pi'$ under the equivalence relation $\sim$ on $\mathcal{I}(\pi)$. In particular, there exists a subset $\{\pi_j', 1 \leq j \leq s\}$ of $\mathcal{I}(\pi)$ for an integer $s$, such that $(\pi_j', \sim)$ are two-two disjoint, and $\bigcup_{j=1}^s (\pi_j', \sim) = \mathcal{I}(\pi)$. Denote by $[\pi_j', \sim]$ the family of $M'$-inertiually equivalent classes of $\pi' \in (\pi_j', \sim)$, and we call $[\pi_j', \sim]$ a connected $M'$-inertiually equivalent class of $\pi_j'$. 

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By Theorem 3.15 to give a block decomposition of $\text{Rep}_k(M)_{SC}$ is equivalent to give a block decomposition of $\text{Rep}_k(M'_{[M,\pi]^{tw}})$ for each irreducible supercuspidal $k$-representation $\pi$ of $M$.

**Theorem 4.12** (Block decomposition of $\text{Rep}_k(M'_{SC})$). Let $\pi$ be an irreducible supercuspidal $k$-representation of $M$, and we occupy the notations in Remark 4.11. For each $1 \leq j \leq s$, define the full sub-category $\text{Rep}_k(M'_{[\pi_j,\sim]})$, consisting of the objects, of which each irreducible subquotient belongs to $[\pi_j,\sim]$. Then $\text{Rep}_k(M'_{[\pi_j,\sim]})$ is a block, and the subcategory $\text{Rep}_k(M'_{[M,\pi]^{tw}}) \cong \prod_{j=1}^s \text{Rep}_k(M'_{[\pi_j,\sim]})$.

**Proof.** First we prove that $\text{Rep}_k(M'_{[\pi_j,\sim]})$ is non-split. By Proposition of [V2][§III], we only need to prove that for any non-trivial disjoint union $[\pi_j,\sim] = I_1 \sqcup I_2$, where $I_1, I_2$ are two non-trivial families of $M'$-inertially equivalence classes, then there exists an object $P \in \text{Rep}_k(M'_{[\pi_j,\sim]})$, such that $P$ cannot be decomposed as $P_1 \oplus P_2$, where $P_1 \in \text{Rep}_k(M'_{I_1})$ and $P_2 \in \text{Rep}_k(M'_{I_2})$. Without loss of generality, we assume that $\pi_j \vert I_1$ and $\pi_j \vert I_2$ such that $\pi_j \vert I_1 \in I_2$. Since $\pi_j \sim \pi_j'$, there exists a maximal simple supercuspidal $k$-type $(J_{\lambda_M}, \lambda_M)$ of $\pi$ and two maximal simple supercuspidal $k$-types $(J_{\lambda_M}', \lambda_M')$ of $\pi_j \vert I_1$, such that $\lambda_M \sim \lambda_M'$ in $T(\lambda_M)$. By the second part of Definition 4.9 let $\{\lambda_{i,1}^t, 1 \leq i \leq t\}$ be a series of a connected relation of $\lambda_M$ and $\lambda_M'$. Define a new series $\{\lambda_i^t, 0 \leq i \leq t+1\}$, by putting $\lambda_{i,1}^t = \lambda_i^t$ and $\lambda_{i+1}^t = \lambda_{i+1}^t \lambda_{i+1}^t$. There exists $0 \leq i \leq t$, such that $\nu(\lambda_{i+1}^t) \in I_1$ but $\nu(\lambda_{t+1}^t) \in I_2$, where $\nu$ is defined as in Remark 4.5. Now we consider $\mathcal{P}_{M'} := \prod_{t \in J_0} \mathcal{P}_{\lambda_i^t} \in \text{Rep}_k(M'_{SC(\lambda_i^t)})$ (see the beginning of Section 4.2 for the definition of $\mathcal{P}_{\lambda_i^t}$), hence $\mathcal{P}_{M'} \in \text{Rep}_k(M'_{[\pi_j,\sim]})$. Assume contrarily that $\mathcal{P}_{M'} \cong P_1 \oplus P_2$, where $P_1 \in \text{Rep}_k(M'_{I_1})$ and $P_2 \in \text{Rep}_k(M'_{I_2})$. Then $P_1 \in \text{Rep}_k(M'_{I_1 \cap SC(\lambda_i^t)})$ and $P_2 \in \text{Rep}_k(M'_{I_2 \cap SC(\lambda_i^t)})$. Since the union of $I_1 \cap SC(\lambda_i^t)$ and $I_2 \cap SC(\lambda_i^t)$ is a non-trivial disjoint union of $SC(\lambda_i^t)$, the decomposition $\mathcal{P}_{M'} \cong P_1 \oplus P_2$ is contradicted with Proposition 4.6.

Secondly, we prove that $\text{Rep}_k(M'_{[M,\pi]^{tw}}) \cong \prod_{j=1}^s \text{Rep}_k(M'_{[\pi_j,\sim]})$. We use the projective version in Remark 2.2. Now fix $j_0$, let $(J_{\lambda_M}', \lambda_{j_0}^t)$ be a maximal simple supercuspidal $k$-type contained in $\pi_{j_0}$, defined from a maximal simple supercuspidal $k$-type $(J_{\lambda_M}, \lambda_M)$ of $M$. By Definition 4.11 and Remark 4.11, we fix a maximal simple supercuspidal $k$-type for each $M'$-inertially equivalent supercuspidal classes contained in $[\pi_{j_0},\sim]$, and denote by $I_{j_0}$ the finite set of these maximal simple supercuspidal $k$-types. Define $\mathcal{P}_{(\pi_{j_0},\sim)^t} := \prod_{\tau \in I_{j_0}} \text{ind}_{J_{\lambda_M}^t}^{J_{\lambda_M}^t} \mathcal{P}_{\tau}$, where $\mathcal{P}_{(\pi_{j_0},\sim)^t}$ is the projective cover of $\tau$. For each $1 \leq j \leq s$ different from $j_0$, let $[\pi_j,\sim] = \cup_{i=1}^t [M', \pi_j, \sim]$ where $\pi_j \vert \equiv i$ are irreducible supercuspidal and $t \in \mathbb{N}$. Fix a maximal simple supercuspidal $k$-type $(J_{\lambda_M}^t, \lambda_{j_0}^t)$ contained in $\pi_{j_0}$, define $[\pi_{j_0},\sim] = \cup_{j \neq j_0}[\pi_{j_0},\sim]$ and $\mathcal{P}_j[\pi_{j_0},\sim] := \cup_{j \neq j_0} \prod_{J_{\lambda_M}^{t_j}}^{\mu_{J_{\lambda_M}^{t_j}}} \mathcal{P}_{J_{\lambda_M}^{t_j}}$, such that we show that $\mathcal{P}_j[\pi_{j_0},\sim]$ and $\mathcal{P}_j[\pi_{j_0},\sim]^{-1}$ verify the conditions in Remark 2.2. By Proposition 4.4 and Lemma 4.7, we know that an irreducible sub-quotient of $\mathcal{P}_j[\pi_{j_0},\sim]^{-1}$ belong to $[\pi_{j_0},\sim]$. Meanwhile an irreducible sub-quotient of $\mathcal{P}_j[\pi_{j_0},\sim]^{-1}$ belong to $[\pi_{j_0},\sim]^{-1} := \cup_{j \neq j_0}[\pi_{j_0},\sim]$. Condition 1 and 3 of Remark 2.2 can be deduced from Proposition 3.6. Condition 2 of remark 2.2 is verified from Remark 4.11 that “$\sim$” defines an equivalent relation, and $[\pi_{j_0},\sim]$ is disjoint with $[\pi_{j_0},\sim]^{-1}$. Hence by repeating the same operation on $\text{Rep}_k(M'_{[\pi_{j_0},\sim]})$, and after finite times we obtain the desired decomposition.

**Example 4.13.** For $G' = \text{SL}_n(F)$, when $\ell$ is positive,

- it is not always true that the reduction modulo $\ell$ of an irreducible $\ell$-adic supercuspidal representation of $G'$ is irreducible;
\begin{itemize}
    \item it is not always true that \( \text{Rep}_p(G') \) can be decomposed according to the \( G' \)-inertially equivalent supercuspidal classes as in Equation \( 7 \) in the case where \( \ell = 0 \).
\end{itemize}

**Proof.** Let \( p = 5, n = 2, \ell = 3 \), and denote by \( G = \text{GL}_2(F) \) and by \( G' = \text{SL}_2(F) \). From \cite{Bonn} §11.3.2] we know that there exists two irreducible supercuspidal \( \bar{G}_\ell \)-representations \( \pi_1, \pi_2 \) of \( G \) (\( \pi_1 \) corresponding to \(-j^\wedge\) and \( \pi_2 \) corresponding to \( \theta_0 \) as in \cite{Bonn} §11.3.2]), such that the reduction modulo \( \ell \) of \( \pi_1 \) and \( \pi_2 \) are irreducible and coincide to each other. Meanwhile, the restriction \( \pi_{1|\ell} \) is irreducible but \( \pi_{2|\ell} \) is semisimple with length \( 2 \). We denote by \( \pi_2 \) the reduction modulo \( \ell \) of \( \pi_2 \). By \cite{Bonn} §11.3.2] the length of \( \pi_{1|\ell} \) is two, and denote by \( \pi_{2,1}, \pi_{2,2} \) the two irreducible direct components of \( \pi_{2|\ell} \) (in the notation of \cite{Bonn} §11.3.2], \( \pi_{2,1} \) and \( \pi_{2,2} \) corresponds to the reduction modulo \( \ell \) of \( R_\ell'(\theta_0) \) and \( R_\ell'(\theta_0) \) respectively). In other words, the reduction modulo \( \ell \) of the irreducible supercuspidal \( \bar{G}_\ell \)-representation \( \pi_{1|\ell} \) is reducible, and its Jordan-Hölder components consist of \( \pi_{2,1} \) and \( \pi_{2,2} \). Both of \( \pi_{2,1} \) and \( \pi_{2,2} \) are supercuspidal by \cite{C2} §3.2], from the fact that their projective covers are cuspidal.

We consider the \( \bar{G}_\ell \)-projective cover \( \mathcal{Y}_{\pi_{2,1}} \) of \( \pi_{2,1} \). The strategy is to prove that the irreducible \( \bar{G}_\ell \)-representation \( \pi_{1|\ell} \) is a subquotient of \( \mathcal{Y}_{\pi_{2,1}} \otimes \bar{G}_\ell \), from which we deduce that \( \pi_{2,2} \) is a subquotient of \( \mathcal{Y}_{\pi_{2,1}} \otimes k \), then we apply Proposition 4.4.

Let \( U \) be the subgroup of upper triangular matrices of \( G \), the reduction modulo \( \ell \) gives a bijection between non-degenerate \( \bar{G}_\ell \)-characters of \( U \) and non-degenerate \( k \)-characters of \( U \). Let \( \theta_{\pi_{1}} \) be a non-degenerate \( \bar{G}_\ell \)-character of \( U \), and \( \theta_\ell \) be the reduction modulo \( \ell \) of \( \theta_{\pi_{1}} \), which is a non-degenerate \( k \)-character of \( U \), such that \( \pi_{2,1} \) is generic according to \( \theta_\ell \). By the unicity of Whittaker model, it follows that \( \pi_{2,2} \) is not generic according to \( \theta_\ell \). By \cite{C2}, \( \mathcal{Y}_{\pi_{2,1}} \otimes \bar{G}_\ell \) is semisimple, and can be written as \( \oplus_{s \in S_{\mu}} \pi_s \otimes \mu \). Here \( s_0 \) is the \( \ell' \)-semisimple conjugacy class in \( G \) corresponding to \( \pi_2 \) by the theory of Deligne-Lusztig (or equivalently \( s_0 \) corresponds to \( \theta_0 \) under the notations of \cite{Bonn} §11.3.2]), and \( S_{\mu} \) is the set of semisimple conjugacy classes in \( G \) whose \( \ell' \)-part is equal to \( s_0 \). Denote by \( \pi_s \) the irreducible supercuspidal \( \bar{G}_\ell \)-representation corresponding to \( s \), and \( \pi_{s,\theta} \) the unique irreducible component of \( \pi_s \) which is generic according to \( \theta_\ell \). Hence \( \pi_1 \) is a subrepresentation of \( \mathcal{Y}_{\pi_{2,1}} \otimes \bar{G}_\ell \), which implies that \( \pi_{2,2} \) is a sub-quotient of \( \mathcal{Y}_{\pi_{2,1}} \otimes k \), which is the \( k \)-projective cover of \( \pi_{2,1} \).

To go further to the \( p \)-adic groups, we conclude that the semisimplification of \( \mathcal{Y}_{\pi_{2,1}} \otimes k \) consists with a non-trivial multiple of \( \pi_{2,1} \) and a non-trivial multiple of \( \pi_{2,2} \).

Now we consider the \( p \)-adic groups \( G = \text{GL}_2(F) \) and \( G' = \text{SL}_2(F) \), suppose that \( F = \mathbb{Q}_5 \), and \( k = \mathbb{F}_5 \). Let \( J = \text{GL}_2(\mathbb{Z}_5) \), \( J^1 = 1 + M_2(5\mathbb{Z}_5) \) and \( J' = J \cap G', J'^1 = J^1 \cap G' \). We have \( J/ J^1 \cong \bar{G}_5 \), and \( J'/ J'^1 \cong \bar{G}_5 \). We still denote by \( \pi_{1}, \pi_{2}, \pi_{2,1}, i = 1, 2 \) the corresponding inflation to \( J' \) respectively. Hence \( (J, \pi_{i}) \), \( i = 1, 2 \) are maximal simple supercuspidal \( k \)-types of \( G \). According to \cite{C1} 3.18 and the fact that there are 4 \( G' \)-conjugacy classes of non-degenerate characters on \( U \), we deduce from the unicity of Whittaker models that for an irreducible cuspidal \( \ell \)-representation \( \pi \) of \( G \), the length of \( \pi|_{C'} \) is a divisor of 4, hence is prime to 5. By Theorem 3.18 of \cite{C1}, the index \( [J : J] \) is a \( p \)-power and a divisor of the length \( \pi|_{C'} \), which implies that \( J = J' \). We deduce firstly that \( (J', \pi_{2,1}) \) is a maximal simple supercuspidal \( \bar{G}_5 \)-type of \( G' \), and \( (J', \pi_{2,2}) \), \( i = 1, 2 \) are a maximal simple supercuspidal \( k \)-types of \( G' \). Hence \( \text{ind}_J^{G'} \pi_{i,\ell} \) is irreducible, but its reduction modulo \( \ell \) had length two, with two factors \( \text{ind}^J_{J^1} \pi_{2,i}, i = 1, 2 \), which is the first of this example. Secondly, we have that \( (J', \pi_{2,1}) \) and \( (J', \pi_{2,2}) \) are non \( G' \)-conjugate by the second part of Remark 3.35 By \cite{C1} Proposition 2.35, Theorem 3.30, \( \Pi_1 := \text{ind}_J^{G'} \pi_{2,1} \) and \( \Pi_2 := \text{ind}_J^{G'} \pi_{2,2} \) are different irreducible supercuspidal \( k \)-representations, and they defines different \( G' \)-inertially equivalent classes since there is no non-trivial \( k \)-characters on \( G' \). The inflation of \( \mathcal{Y}_{\pi_{2,1}} \) to \( J' \) is the \( \bar{G}_5 \)-projective cover of \( \pi_{2,1} \).
By the previous paragraphs, $\pi_{2,2}$ appears as a sub-quotient of $Y_{2,1}$. Apply Theorem 4.12, we conclude that two full sub-categories $\text{Rep}_k(G'[\Pi_1])$ and $\text{Rep}_k(G'[\Pi_1])$ belong to the same block.

References

[Bonn] C. Bonnafé Representations of $\text{SL}_2(\mathbb{F}_q)$. Algebra and Applications, 13. Springer-Verlag London, Ltd., London, 2011. xxii+186 pp. ISBN: 978-0-85729-156-1.

[BuKuI] C.J. Bushnell, P.C. Kutzko, The admissible dual of $\text{GL}(N)$ via compact open subgroups. Annals of Mathematics Studies, 129. Princeton University Press, Princeton, NJ, 1993.

[BuKuII] C.J. Bushnell; P.C. Kutzko, The admissible dual of $\text{SL}(N)$ II. Proc. London Math. Soc. (3) 68 (1994), no. 2, 317-379.

[C1] P. Cui; Modulo $\ell$-representations of $p$-adic groups $\text{SL}_n(F)$: maximal simple $k$-types. arXiv:2012.07392

[C2] P. Cui; Supercuspidal support of irreducible modulo $\ell$-representations of $\text{SL}_n(F)$. arXiv:2109.10248

[Da] J.-F. Dat, Simple subquotients of big parabolically induced representations of $p$-adic groups, J. Algebra 510 (2018), 499-507, with an Appendix: Non-uniqueness of supercuspidal support for finite reductive groups by O. Dudas

[Geck] M. Geck, Modular Harish-Chandra series, Hecke algebras and (generalized) $q$-Schur algebras. Modular representation theory of finite groups (Charlottesville, VA, 1998), 1-66, de Gruyter, Berlin, 2001.

[He] D. Helm, The Bernstein center of the category of smooth $W(k)[\text{GL}_n(F)]$-modules. Forum Math. Sigma 4, 2016.

[MS] A. Mínguez; V. Sécherre, Types modulo $\ell$ pour les formes intérieures de $\text{GL}_n$ sur un corps local non archimédien. With an appendix by Vincent Sécherre and Shaun Stevens. Proc. Lond. Math. Soc. (3) 109 (2014), no. 4, 823-891.

[Ser] J.-P. Serre, Linear representations of finite groups. Translated from the second French edition by Leonard L. Scott. Graduate Texts in Mathematics, Vol. 42. Springer-Verlag, New York-Heidelberg, 1977.

[SeSt] V. Sécherre; S. Stevens, Block decomposition of the category of $\ell$-modular smooth representations of $\text{GL}(n,F)$ and its inner forms Ann. Sci. Éc. Norm. Supér. (4) 49 (2016), no. 3, 669-709.

[V1] M.-F. Vignéras, Représentations $\ell$-modulaires d’un groupe réductif $p$-adique avec $\ell \neq p$. Progress in Mathematics, 137. Birkhäuser Boston, Inc., Boston, MA, 1996.

[V2] M.-F. Vignéras, Induced $R$-representations of $p$-adic reductive groups. Selecta Math. (N.S.) 4 (1998), 549-623, with an appendix by A. Arabia.