The method of $\Gamma$-operators and the tertiary quantization of quantum electrodynamics equations

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Abstract. In cases of solving the problem described by the equations of quantum electrodynamics within the second quantization formalism, it is proposed to quantize these equations one more time, and then solve this problem within the tertiary quantization formalism. In optics of the dispersive media, this procedure makes it possible to avoid the problem of the forced breaking of the photon quantum correlators and to discover many correlation effects previously unknown.

1. Introduction

When solving the equations of quantum electrodynamics using the methods of perturbation theory, no problems occur. But if we consider the dynamics of electromagnetic fields in the dispersive mediums, the presence of many particles in the system and numerous interactions between them require using the forced breaking procedure for the quantum correlators in calculations. Currently, it is not possible to avoid this procedure. It seems to be natural to use this procedure both in the Bogolyubov chains method and in the Feynman diagram technique, when the Wick algebraic theorem [1] is substituted for its thermodynamic variant [2]. It is impossible to estimate the errors arising in this case, just because they can be hundreds of percent. In classical optics, the breaking of correlators can be used because the rarefied mediums are considered. In quantum theory, using such a procedure cannot be justified, because the wave function of two identical photons $\hat{\alpha}^+ \hat{\alpha}^+$ can never be represented as a product of the single-photon wave functions $\hat{\alpha}^+ \hat{\alpha}^+$, because of their mutual orthogonality. The impossibility of breaking quantum correlators also explains the thirty-year history of creating the theory of superconductivity. It would seem that, in this field of research, the quantum electrodynamics is applicable, but, in this case, it is not enough to use only the standard methods for problem solution.

We propose a method for solving the equations of quantum electrodynamics, which makes it possible to avoid using the forced breaking procedure for quantum correlators and predicts the existence of new optical correlation effects.

The idea of the method is as follows. Let us consider an example. The nonrelativistic theory of hydrogen atom is based on the Schrödinger equation. But it is possible to use the secondary quantization procedure for the equations of quantum electrodynamics. In the nonrelativistic approximation, we will obtain the same result. Similarly, the solutions of the secondary quantized equations can be found both by directly solving these equations, and by solving them after they are preliminarily quantized one more time. This is a cumbersome way of solving the equations. Attempts to perform the "third" quantization have been made many times [3], but, as far as we know,
no considerable success was achieved. And, in the course of those studies, no fundamentally new natural phenomena were discovered. We use the "tertiary" quantization in cases, when its effects coincide with those described by the secondary quantized equations. But it will be shown that, in such calculations, it is possible to avoid the forced breaking procedure for quantum correlators. Moreover, the use the "tertiary" quantization makes it possible to predict a large number of new correlation quantum effects. One of such effects is the possibility of existence of the bound states of photon pairs in the thermally excited mediums [5].

2. Investigating model
As a model, we consider the transverse quantum electromagnetic field interacting with homogeneous non-relativistic atomic gas, which is nondegenerate with respect to the temperature and occupies the entire space. For the sake of simplicity, we assume that each atom has one valence electron. The relativistic and spin effects will be neglected. The interaction is assumed to be quasi-resonant.

In the secondary quantization representation, the Schrödinger equation for the wave function of the entire system \( \Psi(t) \) takes the following form

\[
i\hbar \frac{\partial \Psi(t)}{\partial t} = \left[ \hat{H}_a + \hat{H}_{ph} - \frac{e}{mc} \int \psi^* (r, R) \tilde{p}^* \hat{A}^r (r) \psi (r, R) d\mathcal{R} \right] \Psi(t). \tag{1}
\]

Here, the electromagnetic field (in the rationalized Gaussian system of units) corresponds to the field operator of the vector potential.

\[
\hat{A}^r (r) = \sum_{k,l} \hat{A}^v_{kl} (r), \quad \hat{A}^v_{kl} (r) = \frac{\hbar c}{2kV} e_{kl}^v \left( \hat{\alpha}_{kl} e^{ikr} + \hat{\alpha}^*_{kl} e^{-ikr} \right),
\]

where \( e_{kl}^v \) are the unit vectors of linear polarization, \( m \) is the electron mass, \( V \) is the volume of quantization. The \( \hat{\alpha}_{kl} \) and \( \hat{\alpha}^*_{kl} \) operators are the photon annihilation and creation operators in the photon states described by the wave vector \( k \) and the linear polarization index \( \lambda \), and, as usual, they are subject to the commutation relations for the Bose–Einstein fields

\[
\left[ \hat{\alpha}_{kl}; \hat{\alpha}^*_{kl'} \right] = \delta_{\lambda\lambda'} \delta (k; k').
\]

For the transverse electromagnetic field, we assume \( \lambda = 1,2 \). In the Schrödinger representation, gas atoms are associated with the field operator

\[
\tilde{\psi} (r, R) = \sum_j \psi_j (r - R) \exp (i p R / \hbar) \hat{b}_{jp} / \sqrt{V},
\]

where \( R \) is the coordinate of the center of gravity of individual atom, \( r \) is the coordinate of the valence electron in this atom, and \( p \) is the atom momentum. The wave function \( \psi_j (r - R) \) describes the atomic state of electron with energy \( \epsilon_j \). We assume that in the Maxwellian gas, the annihilation operators \( \hat{b}_{jp} \) and creations operators \( \hat{b}^*_{jp} \) of atom in the state \( (j, p) \) are subject to the commutation relations \( \left[ \hat{b}_{jp}; \hat{b}^*_{jp'} \right] = \delta_{jj'} \delta (p; p') \). For the gas, which is nondegenerate with respect to temperature, the concrete form of the commutation relations determining its statistical properties is insignificant. Next, we obtain

\[
\hat{H}_a = \sum_{jp} \epsilon_j (p) \hat{b}^*_{jp} \hat{b}_{jp}, \quad \epsilon_j (p) = \epsilon_j + \frac{p^2}{2M}, \quad \hat{H}_{ph} = \sum_{kl} \hbar c k \hat{\alpha}_{kl} \hat{\alpha}^*_{kl}.
\]

Here, we imply the summation over the index \( v \). \( M \) is the mass of individual atom of the gas medium.
3. Investigation method

Let us look at the problem from the other side. Our goal is to avoid the forced breaking procedure for quantum correlators. This procedure appears only when solving the many-particle problems. If there is only one particle, it is not necessary to break the correlators. For example, in the theory of the hydrogen atom, it is not necessary to break correlators. Thus, if it is possible to substitute a many-particle problem for a one-particle problem, then the main problem will be solved. Let us make some auxiliary theoretical constructs. We consider the free photon gas. We assume that the \((\mathbf{k}, \mathbf{\lambda})\) mode consists of \(N_{k\mathbf{\lambda}}\) photons. In the secondary quantization representation, the wave function of such a one-mode state coincides with the wave function of the quantum oscillator \(\varphi_{N_{k\mathbf{\lambda}}}(\xi_{k\mathbf{\lambda}})\). We assume the following forms of operators .

\[
\hat{a}_{k\mathbf{\lambda}} = \frac{1}{\sqrt{2}} \left( \xi_{k\mathbf{\lambda}} + \frac{\partial}{\partial \xi_{k\mathbf{\lambda}}} \right), \quad \hat{a}_{k\mathbf{\lambda}}^\dagger = \frac{1}{\sqrt{2}} \left( \xi_{k\mathbf{\lambda}} - \frac{\partial}{\partial \xi_{k\mathbf{\lambda}}} \right). \tag{3}
\]

The set of occupied numbers \(N_{k\mathbf{\lambda}}\) in all modes determines the electromagnetic field conglomerate as a whole, which is described by the wave function \(\Phi_N(\mathbf{\xi}) = \prod_{k\mathbf{\lambda}} \varphi_{N_{k\mathbf{\lambda}}}(\xi_{k\mathbf{\lambda}})\). Obviously, the \(\Phi_0(\mathbf{\xi}) = \prod_{k\mathbf{\lambda}} \varphi_0(\xi_{k\mathbf{\lambda}})\) wave function describes the state of electromagnetic vacuum (physical vacuum in the secondary quantization representation). We introduce the many-dimensional vectors \(\mathbf{N} = \ldots, N_{k\mathbf{\lambda}}, \ldots\), and \(\mathbf{\xi} = \ldots, \xi_{k\mathbf{\lambda}}, \ldots\). Using the \((\mathbf{k}, \mathbf{\lambda})\) axes, we construct an auxiliary many-dimensional space. Let us call it the \(\Gamma\)-space. In the \(\Gamma\)-space, the \(\mathbf{N}\) vector, determining the electromagnetic field conglomerate, determines only one point. In this space, the evolution of all photons is described by the only one curved line. Using the \(\Gamma\)-space, we construct the \(\Gamma\)-representation of the wave functions.

We assume that, in this representation, the \(\varphi_0(\mathbf{\xi}) = \prod_{N} \varphi_0(\xi_N)\) function is the wave function of electromagnetic vacuum (mathematical vacuum). We assume that, in the \(\Gamma\)-representation, the operator \(\hat{A}^+(\mathbf{N}) = (\xi_N - \partial / \partial \xi_N) / \sqrt{2}\) defines the creation operator of the photon conglomerate \(\mathbf{N}\), so that the construction \(\hat{A}^+(\mathbf{N})\varphi_0(\mathbf{\xi}) = \varphi_1(\mathbf{\xi}) \prod_{N} \varphi_0(\xi_N)\) is a wave function of this conglomerate in the \(\Gamma\)-representation. We note that the wave functions \(\Phi_N(\mathbf{\xi})\) and \(\hat{A}^+(\mathbf{N})\varphi_0(\mathbf{\xi})\) describe one and the same physical state of free photons, though they are defined in different representations.

By the way, if all components of the \(\mathbf{N}\) vector are equal to zero, then we deal with the physical vacuum state in the \(\Gamma\)-representation.

In terms of mathematics, at \(n > 1\), the wave functions \((\hat{A}^+(\mathbf{N}))^n\varphi_0(\mathbf{\xi})\) can exist, but they are absent among the solutions of the Schrödinger equation. For this reason, in the \(\Gamma\)-representation, the set of wave functions \((\hat{A}^+(\mathbf{N}))^n\varphi_0(\mathbf{\xi})\) forms the complete basis of wave functions for the realizable physical states of the electromagnetic field, described by different \(\mathbf{N}\) vectors. For arbitrary photon field state, the solution \(\varphi(\mathbf{N}, t)\) of the Schrödinger equation can be decomposed into the basis functions.

We introduce such annihilation operator of the photon conglomerate \(\hat{A}(\mathbf{N}) = (\xi_N + \partial / \partial \xi_N) / \sqrt{2}\) that the following relation is true: \(\hat{A}(\mathbf{N})\hat{A}^+(\mathbf{N})\varphi_0(\mathbf{\xi}) = \varphi_0(\mathbf{\xi})\delta_{NN}\). The algebraic properties of the \(\hat{A}^+(\mathbf{N})\) and \(\hat{A}(\mathbf{N})\) operators are similar to those of the single photon operators \(\hat{a}_{k\mathbf{\lambda}}\) and \(\hat{a}_{k\mathbf{\lambda}}^\dagger\). In particular,

\[
[\hat{A}(\mathbf{N}); \hat{A}^+(\mathbf{N})^\prime] = \delta_{NN}. \tag{4}
\]
The remarkable property of the wave functions $\Psi_f(t) = \Psi_f^{(t)}$, which are the solutions of the Schrödinger equation in the $\Gamma$-representation, is the fact that, at any degree of annihilation operator $\hat{A}^a(N)$, except for the first degree, these functions vanish. Thus, when using the Wick theorem [1], all members of the normal product of operators, except for those containing annihilation operators in the first degree, vanish. The coefficient at the first-degree-operator $\hat{A}^a(N)$ can be calculated explicitly. In dispersive mediums, this makes it possible to avoid using the thermodynamic variant of the Wick theorem [2], as well as to avoid the procedure of breaking the photon-photon correlators [4]. In essence, when using this technique, we deal with the single conglomerates of photons.

There is one-to-one correspondence between the basis functions in the standard secondary quantization representation $\Phi_N(\xi) = \prod_{k_i} \phi_{\alpha_k}(\xi_{k_i})$ and the basis functions $\hat{A}^*\!\!(N)\!\!\!\!(0)_{\Gamma_f}$ in the $\Gamma$-representation, which is determined by the following unitary operator:

$$\hat{O}(\xi) = \hat{\Phi}(\xi)^{0}_{\Gamma_f}$$

where $\hat{\Phi}(\xi) = \sum_{\Phi_N(\xi)} \hat{A}^*\!\!(N)\Phi_N(\xi)$.

It is easy to see that

$$\hat{O}(\xi)\Phi_N(\xi) = \sum_{N}^{0} \hat{A}^*\!\!(N')\Phi_N(\xi)^{0}_{\Gamma_f} \Phi_N(\xi) d\xi = \hat{A}^*\!\!(N)\!\!\!\!(0)_{\Gamma_f}, \quad d\xi = \Pi_{k_i} d\xi_{k_i},$$

$$\hat{O}(\xi)\hat{A}^\dagger (N)\!\!\!\!(0)_{\Gamma_f} = \sum_{N}^{0} \langle \hat{A}(N')\Phi_N(\xi)\hat{A}^\dagger (N)\!\!\!\!(0)_{\Gamma_f} = \Phi_N(\xi).$$

Now all calculations can be performed in the $\Gamma$-representation. In particular, in the $\Gamma$-representation, the Schrödinger equation takes the following form:

$$i\hbar \frac{\partial \Psi_f(t)}{\partial t} = \hat{H}_f \Psi_f(t), \quad (5)$$

where

$$\hat{H}_f = \hat{H}_a + \int \hat{\Phi}^\dagger (\xi)\hat{H}_p \hat{\Phi}(\xi) d\xi, \quad \hat{H}_f = \hat{H}_a + \int \hat{\Phi}^\dagger (\xi)\hat{H}_p \hat{\Phi}(\xi) d\xi - \frac{e}{mc} \int \hat{\Phi}^\dagger (\xi)\hat{\psi}^\dagger (r, R)\hat{P}_r \hat{A}^\dagger (r, \xi)\hat{\psi}(r, R)\hat{\Phi}(\xi) d\xi d\xi' dr dR.$$  

The $\hat{A}^\dagger (r, \xi)$ operator is given by expression (2), in which the $\hat{a}_{k_i}$ and $\hat{a}^a_{k_i}$ operators are substituted for the differential operators (3). In the secondary quantization representation, the general photon operators can be written as $\hat{K}(\xi, \xi')$. In the $\Gamma$-representation, the quantum average value of any operator $\hat{K}(\xi, \xi')$ can be found using the following formula:

$$\langle \hat{K} \rangle = \int \langle \hat{\Phi}^\dagger (\xi')\hat{K}(\xi', \xi)\hat{\Phi}(\xi) \rangle d\xi d\xi'.$$

Here, we used the following equality:

$$\hat{\Phi}^\dagger (\xi)\hat{K}(\xi, \xi')\hat{\Phi}(\xi') = \hat{O}(\xi)\hat{K}(\xi, \xi')\hat{D}^\dagger (\xi'),$$

which turns into identity when acting on any basis vector $\hat{A}^\dagger (N)\!\!\!\!(0)_{\Gamma_f}$. Thus, the construct

$$\rho(\xi, \xi') = \langle \hat{\Phi}^\dagger (\xi')\hat{\Phi}(\xi) \rangle_{\Gamma_f}$$

(6)

can be considered as the matrix of the electromagnetic field density in the medium.

In [5], using the $\Gamma$-operator method, the concrete examples were considered and the solutions of the equations of quantum electrodynamics were found, which were previously unknown.
In the $\Gamma$-representation, it is of interest to go from the Schrödinger representation to the Heisenberg representation, which can be performed using the unitary operator $\hat{U}(t) = \exp\left(-\hat{H}_f t / i\hbar\right)$. In this representation, the $\hat{\Phi}(\xi)$ and $\hat{\psi}(r, \mathbf{R})$ operators have the following form:

$$\hat{\Phi}(\xi, t) = \exp\left(-\frac{1}{i\hbar}\hat{H}_f t\right)\Phi(\xi)\exp\left(\frac{1}{i\hbar}\hat{H}_f t\right), \quad \hat{H}_f = \hat{H}_f,$$

$$\hat{\psi}(r, \mathbf{R}, t) = \exp\left(-\frac{1}{i\hbar}\hat{H}_f t\right)\psi(\xi)\exp\left(\frac{1}{i\hbar}\hat{H}_f t\right).$$

Thus, we obtain

$$i\hbar \frac{\partial}{\partial t} \hat{\Phi}(\xi, t) = \left[\hat{H}_f - \frac{e}{mc}\int \hat{\psi}^\dagger(\mathbf{r}, \mathbf{R}, t)\hat{A}^\dagger(\mathbf{r}, \xi)\hat{\psi}(\mathbf{r}, \mathbf{R}, t)d\mathbf{r}d\mathbf{R}\right]\hat{\Phi}(\xi, t),$$

$$i\hbar \frac{\partial}{\partial t} \hat{\psi}(\mathbf{r}, \mathbf{R}, t) = \left[\hat{\psi}; \hat{H}_f\right]_\varepsilon = \left[\hat{H}_f - \frac{e}{m}\int \hat{\Phi}^\dagger(\xi, t)\hat{A}^\dagger(\mathbf{r}, \xi)\hat{\Phi}(\xi, t)d\xi\right]\hat{\psi}(\mathbf{r}, \mathbf{R}, t),$$

$$\left[\hat{\Phi}(\xi, t); \hat{\Phi}^\dagger(\xi', t)\right] = \delta(\xi, \xi') = \prod_{k \neq k'} \delta(\xi_{kk'} - \xi'_{kk'}), \quad \left[\hat{\psi}(\mathbf{r}, \mathbf{R}, t); \hat{\psi}^\dagger(\mathbf{r}', \mathbf{R}', t)\right] = \delta(\mathbf{r} - \mathbf{r}')\delta(\mathbf{R} - \mathbf{R}').$$

Let us make a comment. We consider the evolution of a free particle with mass $m$. In the quantum theory, its state is determined by the wave function $\psi(\mathbf{r}, t)$. In the Schrödinger representation, this function satisfies the Schrödinger equation $i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = -\hbar^2\nabla^2 \psi / 2m$. When performing the secondary quantization, the $\psi$ wave function determining the particle state is substituted for the field operator $\hat{\psi}$ satisfying the same equation $i\hbar \frac{\partial}{\partial t} \hat{\psi}/ \partial t = -\hbar^2\nabla^2 \hat{\psi} / 2m$. In the case of the Bose-Einstein particles, the resulting field operators should satisfy the following commutation relation:

$$\left[\hat{\psi}(\mathbf{r}, t); \hat{\psi}^\dagger(\mathbf{r}', t)\right] = \delta(\mathbf{r} - \mathbf{r}').$$

Now, in the Schrödinger representation, the $\Psi$ wave function determining the state of the system satisfies the following Schrödinger equation: $i\hbar \frac{\partial \Psi}{\partial t} = \left\{\mathbf{p}^2 / 2m\right\}\hat{\beta}^\dagger_{\mathbf{p}}\hat{\beta}_{\mathbf{p}}\Psi\},$ where $\hat{\beta}^\dagger_{\mathbf{p}}$ and $\hat{\beta}_{\mathbf{p}}$ are the operators of creation and annihilation of particles with the momentum $\mathbf{p}$, respectively. If we again substitute the $\Psi$ function determining the state of the system for the operator that satisfies the same equation, then we will perform the tertiary quantization.

The same scheme works, when, in the description of the secondary quantized free electromagnetic field, we go from the Schrödinger representation (1) to the Heisenberg representation in the $\Gamma$-space (7). According to Eq. (1), the free electromagnetic field state is described by the wave function $\Psi$ satisfying the Schrödinger equation $i\hbar \frac{\partial \Psi(t)}{\partial t} = \hat{H}_{\phi\Psi}(t)\Psi(t)$. In the $\Gamma$-representation, according to (7), the operator function $\hat{\Phi}$, subject to the commutation relation (8), also satisfies this equation. Now, the wave function satisfies the equation (5), which takes the following form:

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = \sum_{\mathbf{N}} \varepsilon_{\mathbf{N}} \hat{\mathbf{A}}_{\mathbf{N}}^{\dagger} \hat{\mathbf{A}}_{\mathbf{N}} \hat{H}_f \Psi(t), \quad \varepsilon_{\mathbf{N}} = \sum_{k \neq k'} \hbar\epsilon_k N_{k \neq k'},$$

where the $\hat{\mathbf{A}}(\mathbf{N})$ and $\hat{\mathbf{A}}^{\dagger}(\mathbf{N})$ operators satisfy the commutation relation (4). So, we conclude that the procedure of going from the standard secondary quantization representation to the $\Gamma$-representation is equivalent to the procedure of the third quantization of the secondary quantized field.
4. Summary
Thus, before starting the procedure of solving the quantum electrodynamics equations, it is reasonable
to preliminary quantize these equations one more time and search for solutions of the secondary
quantized set of equations in the framework of the tertiary quantization formalism. In this case,
the forced breaking procedure for the photon correlators can be avoided.

References
[1] Wick G C 1950 Phys. Rev. 80 268
[2] Matzubara T A 1955 Progr. Theor. Phys. 14 351
[3] Lomsadze Yu M, Krivsky I Yu and Khimich I V 1981 Izv. Vuzov: Fizika 4 113
[4] Veklenko B A 1989 JETP 96 457
[5] Veklenko B A 2018 Engineering Physics 1 30