Research on Synchronization of Fractal Behaviors in 2-D Logistic Map

Pei Wang

Department of Electrical Engineering and Automation, Qilu University of Technology (Shandong Academy of Sciences), Jinan 250353, China

Corresponding author: Pei Wang (e-mail: wppink@126.com).

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ABSTRACT In this paper, the fractal behaviors in 2-D Logistic map are discussed. First, the definition and some properties of fractal set in 2-D Logistic map are given. Moreover, several synchronous definitions between different 2-D Logistic maps are introduced in this paper, such as complete synchronization, coupled synchronization, and adaptive synchronization. The synchronization of the fractal set between different Logistic maps is accomplished by synchronizing their iterative tracks. The simulations illustrate the effectiveness and correctness of these methods.

INDEX TERMS 2-D Logistic Map, Fractal, Synchronization

I. INTRODUCTION

Logistic map is a classic model for studying the behaviors of complex systems such as dynamic systems, chaos, and fractals. It is mainly used in epidemiology, such as it can be used to explore the risk factors of a disease. And it can also be used to simulate the growth behavior of biological populations. Therefore, Logistic map is also called "insect population model", its expression is \(x_{n+1} = \mu x_n (1-x_n)\), where \(x_n \in (0,1)\) and \(\mu \in (0,4)\) is a tunable parameter. There are lots of achievements about the chaotic behaviors of Logistic map, which relate to chaotic control and applications [1], chaotic cryptography [2]-[7], the image encryption [8] and many other research areas.

Zhang [9] achieved the control of the Julia set of the complex Henon system by using the gradient control and the auxiliary reference feedback control, respectively. Liu [10] achieved the control and synchronization of the Julia set in coupled map lattice by using the gradient control and the optimal control. Zhang [11] achieved the control and synchronization of the Mandelbrot set using the feedback control. Wang [12,13] discussed the bifurcation and fractal behaviors of the coupled Logistic maps.

The fractal behaviors in 2-D Logistic map, with abundant nonlinear characteristics such as typical self-similarity and initial value sensitivity, can be widely used in the research of secure communication. Therefore, the synchronous research of fractal behaviors in 2-D Logistic map is particularly important. However, there are fewer results on the synchronization of fractal behaviors in 2-D Logistic map. In this paper, Julia set in 2-D Logistic map is taken as an example, to discuss the synchronous control under different definitions in different application scenarios.

The outline of the paper is as follows. In the section of Julia set in 2-D Logistic map, the definitions of Julia set in Logistic map and the fixed points are given. In the section of Synchronization, several definitions of synchronization of the Julia set in 2-D Logistic map are given. In the section of Simulations, the synchronizations of two different Julia set under these definitions are achieved. The example shows the feasibility of these methods. Finally, the conclusions are given in the section of Conclusions.

II. JULIA SET IN 2-D LOGISTIC MAP

In this section, some necessary knowledge about fractional theory and Julia set is reviewed briefly. More relevant contents can be found in monographs [14]-[18] and references contained therein.

In this paper, in order to facilitate the discussion of the fractal set in 2-D Logistic map, its discrete form is taken as follows,

\[
\begin{align*}
    x_{n+1} &= \mu x_n (1-x_n), \\
    y_{n+1} &= x_n,
\end{align*}
\]

where \(\mu \in (0,4)\), and \((x, y)\) are plural. Let \(L: \mathbb{C} \rightarrow \mathbb{C}\) be the complex mapping of System (1), that is \(L(x, y) = (\mu x(1-y), y)\). Let \(L^n(x, y)\) be the \(n\)th iteration \(L(L(...(L(x, y))))\)
Let \( K = \{(x, y) \in \mathbb{C} | L^n(x, y)\text{is bounded when } n \to \infty\}\). Some definitions related to fractal set of System (1) are given below.

**Definition 1:** From the definition [14] of Julia set, the filled Julia set of System (1) is the set of the points \((x, y) \in \mathbb{C}\) whose trajectories are limited, that is the Julia set of System (1) is the boundary of the filled-in Julia set \(K\), which is denoted by \(J\), that is \(J = \partial K\).

The Julia set \(J\) of System (1) has the following properties [15].
1. \(J\) is nonempty and bounded;
2. \(J\) is fully invariant, \(J = L(J) = L^{-1}(J)\);
3. \(J = J_p\) for any positive integer \(p\);
4. If \(x^*\) is an attractive fixed point of System (1), then \(J = \partial A(x^*)\), where \(\partial A(x^*)\) is the attractive domain of the attractive fixed point \(x^*\). It is also the same as \(x^* = \infty\).

**Remark:** \(J_p\) denotes the Julia set of a family of iterative functions \(L^p\) with the period \(p\).

In other words, \(J\) is the Julia set of System (1). According to the definition of the Julia set in Logaristic map, the structure of its Julia set is closely related to the iterative trajectory of System (1). Therefore, the synchronous control of the Julia set in Logistic map can be achieved by controlling its iterative trajectory.

### III. Definitions of Different Types of Synchronization

In order to elicit the definition of synchronization, two different 2-D Logistic maps with controller are taken,

\[
x_{n+1} = \mu x_n (1 - x_{n-1}) + u_n, \quad (2)
\]
\[
w_{n+1} = \lambda w_n (1 - w_{n-1}) + v_n, \quad (3)
\]
where \(\mu \neq \lambda\). To achieve the synchronization of Julia set in Systems (2) and (3), \(u_n\) and \(v_n\) are controllers to be designed, where \(u_n = f(x_n, w_n, \mu, \lambda; k)\) and \(v_n = g(x_n, w_n, \mu, \lambda; k)\). The Julia sets of Systems (2) and (3) are denoted as \(J_2\) and \(J_3\) respectively.

Some definitions [19] of synchronization of Julia set between two different 2-D Logistic maps will be reviewed.

**Definition 2:** If \(u_n \neq 0\) and \(v_n \neq 0\) in Systems (2) and (3), when \(k\) tends to a certain constant \(k_c\), Julia sets \(J_2\) and \(J_3\) will become the same, that is,

\[
\lim_{k \to k_c} \left[(J_2 \cap J_3) \left(\{J_2 \cup J_3\}\right)\right] = \emptyset, \quad (4)
\]
then the Julia sets of Systems (2) and (3) achieve coupled synchronization.

If \(u_n = 0\) is taken in System (2), it can be referred to as a drive system, whose Julia set is denoted as \(J_{2}^o\), and System (3) with \(v_n \neq 0\) can be denoted as a response system.

**Definition 3:** If \(u_n \neq 0\) and \(v_n = 0\) in Systems (2) and (3), when \(k\) tends to a certain constant \(k_c\), Julia sets \(J_2\) and \(J_3\) will become the same, that is,

\[
\lim_{k \to k_c} \left[(J_{2} \cap J_3) \left(\{J_2 \cup J_3\}\right)\right] = \emptyset,
\]
then the Julia sets of Systems (2) and (3) achieve complete synchronization.

The synchronous error between Systems (2) and (3) is

\[
e_n = w_n - x_n. \quad (5)
\]

**Definition 4:** If \(u_n = 0, v_n \neq 0\), and suppose parameter \(\mu\) in System (2) is unknown. For some given \(k, k_0 \in \mathbb{R}\) satisfying \(|k + k_0| < 1\), the adaptive controller \(v_n\) is designed as

\[
v_n = \hat{\mu}_n x_n (1 - x_n) - \lambda w_n (1 - w_n) - k e_n, \quad (6)
\]

the update law of \(\hat{\mu}\) is

\[
\hat{\mu}_{n+1} = \hat{\mu}_n - k_0 \left[\frac{e_{n+1}}{x_{n+1} (1 - x_{n+1})} - \frac{e_n}{x_n (1 - x_{n-1})}\right], \quad (7)
\]
where \(\hat{\mu}\) is the estimated value \(\mu\). For Systems (2) and (3) with any initial conditions \((x_0, y_0) \in \mathbb{C}\), there exists

\[
w_n \to x_n \quad \text{when} \quad n \to \infty,
\]
then the adaptive synchronization of Julia set between Systems (2) and (3) is realized.

**Explanation:** Substitute Equations (6) and (7) into Equation (5), there is

\[
e_{n+1} = w_{n+1} - x_{n+1} = \lambda w_n (1 - w_{n-1}) + \hat{\mu}_n x_n (1 - x_{n-1}) - \lambda w_n (1 - w_n) - ke_n. \quad (8)
\]

From Equation (7), we have

\[
\hat{\mu}_{n+1} - \mu = \frac{-k_0 e_{n+1}}{x_{n+1} (1 - x_{n+1})} + \frac{k_0 e_n}{x_n (1 - x_{n-1})}. \quad (9)
\]

Obviously, the equivalent expression of Equation (9) is

\[
\hat{\mu}_{n+1} - \mu = \frac{-k_0 e_{n+1}}{x_{n+1} (1 - x_{n+1})} - \frac{k_0 e_n}{x_n (1 - x_{n-1})}. \quad (10)
\]
There exists \(\hat{\mu}_n - \mu = \frac{k_0 e_n}{x_n (1 - x_{n-1})}\) such that Equation (10) holds. Therefore, we have
\[ e_{n+1} = (\hat{\mu} - \mu) x_n (1-x_{n-1}) - k e_n \]
\[ = -k e_n x_n (1-x_{n-1}) - k e_n = -(k_0 + k)e_n. \]

Then we have \( e_{n+1} = -(k_0 + k)^{n+1} e_n \). If \( |k_0 + k| < 1 \), there exist \( \lim_{n \to \infty} |e_n| = 0 \) as \( n \to \infty \). According to Definition 4, the trajectories of Systems (2) and (3) achieve synchronization. And the unknown parameter \( \hat{\mu} \) can be identified.

**IV. SIMULATIONS**

In this section, we followed the methods of Liu [19], Zhang [20] and Wang [21].

In the light of Julia sets definition, the spatial Julia sets are four-dimensional graphics. In order to plot the graphics of the spatial Julia sets, we fix one of four real coordinates and use a three-dimensional graphic to simulate the spatial Julia sets. In this paper, the same real and imaginary parts of \( y \) in System (1) are taken to make a graph.

**Case 1: Coupled Synchronization**

\[ u_n = k \left[ \lambda w_n (1-w_{n-1}) - \mu x_n (1-x_{n-1}) \right] \]
\[ v_n = k \left[ \mu x_n (1-x_{n-1}) - \lambda w_n (1-w_{n-1}) \right] \]

are taken as the coupled control items in Systems (2) and (3). Therefore, Systems (2) and (3) become

\[ x_{n+1} = \mu x_n (1-x_{n-1}) + k \left[ \lambda w_n (1-w_{n-1}) - \mu x_n (1-x_{n-1}) \right], \]
\[ w_{n+1} = \lambda w_n (1-w_{n-1}) + k \left[ \mu x_n (1-x_{n-1}) - \lambda w_n (1-w_{n-1}) \right]. \]

For the same initial value \( x_0 = w_0 \), then we have

\[ |x_{n+1} - w_{n+1}| = |\mu x_n (1-x_{n-1}) - \lambda w_n (1-w_{n-1})| \]
\[ = |\mu x_n - \lambda w_n - (\mu x_n - \lambda w_n)| \]
\[ = |\mu x_n - \lambda w_n + \lambda w_n - (\mu x_n - \lambda w_n)|. \]

According to the definition of Julia set, Julia set is a set of bounded points, if \( M \) is a sufficiently large positive number, so there are \( |x_n| < M \) and \( |w_n| < M \). Then, we have

\[ |x_{n+1} - w_{n+1}| = |\mu (x_n - w_n) + (\mu - \lambda) w_n - (\mu x_n - \lambda w_n)| \]
\[ \leq |\mu x_n - w_n| + |(\mu - \lambda) w_n| + |\mu x_n - \lambda w_n| \]
\[ \leq |\mu x_n - w_n| + |\mu - \lambda| (M + M^2) \]
\[ \leq |\mu x_n - w_n| + |\mu - \lambda| (M + M^2) \]
\[ \leq |\mu x_n - w_n| + |\mu - \lambda| (M + M^2) \]
\[ \leq |\mu x_n - w_n| + |\mu - \lambda| (M + M^2) \]
\[ \leq |\mu x_n - w_n| + |\mu - \lambda| (M + M^2) \]

Combining Expressions (13) and (14), for the given \( \mu \) and \( \lambda \), there exist \( |x_{n+1} - w_{n+1}| \to 0 \) when \( |1 - 2k| \to 0 \), that is \( k \to 0.5 \). Therefore, when \( k \to 0.5 \), the Julia sets of Systems (11) and (12) achieve coupled synchronization.

Taking \( \mu = 2.5 \) and \( \lambda = 3.8 \) as example. Figure 1 illustrates the original Julia sets of Systems (2) and (3) with \( u_0 = 0 \) and \( v_0 = 0 \).

**FIGURE 1.** The original Julia set of Systems (2) and (3) without control.

The synchronous cases between different Logistic maps with different control parameter \( k \) are shown in Figure 2.
FIGURE 2. The coupled synchronization of Julia set between Systems (11) and (12) with different $k$.

**Case 2: Completed Synchronization**

\[ u_n = -k \left[ \frac{1}{1+k} \left( 2 \mu x_n (1-x_{n-1}) - \lambda w_n (1-w_{n-1}) \right) \right] \] and \( v_n = 0 \) are taken as example in Systems (2) and (3). Therefore, Systems (2) becomes

\[ x_{n+1} = \frac{1}{1+k} \left( \mu x_n (1-x_{n-1}) + \frac{k}{1+k} \lambda w_n (1-w_{n-1}) \right). \]  

(15)

For the same initial value \( x_0 = w_0 \), then we have

\[ |x_{n+1} - w_{n+1}| = \frac{1}{|1+k|} \left[ \mu x_n (1-x_{n-1}) - \lambda w_n (1-w_{n-1}) \right] \]

\[ = \frac{1}{|1+k|} \left[ \mu x_n - \lambda w_n - (\mu x_{n-1} - \lambda w_{n-1}) \right] \]

\[ \leq \frac{1}{|1+k|} \left[ \mu x_n - \lambda w_n + |\mu - \lambda| (M^2) \right] \]

\[ \leq \frac{1}{|1+k|} \left[ |x_n - w_n| + \left( 1 + \frac{1}{|1+k|} + \cdots + \frac{1}{|1+k|^n} \right) |\mu - \lambda| (M^2) \right] \]

\[ \leq \frac{1}{|1+k|} \left[ |x_n - w_n| + \left( 1 + \frac{1}{|1+k|} + \cdots + \frac{1}{|1+k|^n} \right) |\mu - \lambda| (M^2) \right] \]

\[ = \frac{1}{|1+k|} \left[ |x_n - w_n| + \left( 1 + \frac{1}{|1+k|} + \cdots + \frac{1}{|1+k|^n} \right) |\mu - \lambda| (M^2) \right] \]

\[ \leq \frac{1}{|1+k|} \left[ |x_n - w_n| + \left( 1 + \frac{1}{|1+k|} + \cdots + \frac{1}{|1+k|^n} \right) |\mu - \lambda| (M^2) \right] \]

\[ = \frac{1}{|1+k|} \left[ |x_n - w_n| + \left( 1 + \frac{1}{|1+k|} + \cdots + \frac{1}{|1+k|^n} \right) |\mu - \lambda| (M^2) \right] \]

\[ = \frac{1}{|1+k|} \left[ |x_n - w_n| + \left( 1 + \frac{1}{|1+k|} + \cdots + \frac{1}{|1+k|^n} \right) |\mu - \lambda| (M^2) \right] \]

\[ = \frac{1}{|1+k|} \left[ |x_n - w_n| + \left( 1 + \frac{1}{|1+k|} + \cdots + \frac{1}{|1+k|^n} \right) |\mu - \lambda| (M^2) \right] \]

(16)

From Expression (16), if \( \frac{1}{|1+k|} < 1 \), we have

\[ \lim_{n \to \infty} \left[ \frac{1}{|1+k|} \times |\mu - \lambda| (M^2) \right] \]

(17)

Combining Expressions (16) and (17), for the given $\mu$ and $\lambda$, there exist \( |x_{n+1} - w_{n+1}| \to 0 \) when \( |1+k| \to \infty \), that is $k \to \infty$. Therefore, when $k \to \infty$, the Julia sets of Systems (15) and (3) achieve completed synchronization.

FIGURE 3. The completed synchronization between Systems (3) and (15) with different $k$.

From Figure 3, the Julia set of System (15) changes gradually toward to the Julia set of System (3) with the
increasing of $k$, and it becomes the same with the Julia set of System (3) as $k \to \infty$. In other words, the synchronization of the Julia set is accomplished, which approves the above conclusion.

**Case 3: Adaptive Synchronization**

The adaptive controller $v_n = \hat{\mu}_n x_n (1-x_{n-1}) - \lambda w_n (1-w_{n-1}) - ke_e$ is taken in System (3) according to **Definition 4**, then System (3) becomes

$$w_{n+1} = \hat{\mu}_n x_n (1-x_{n-1}) - ke_e,$$

where $\hat{\mu}_{n+1} = \hat{\mu}_n - k \left[ \frac{e_{n+1}}{x_{n+1} (1-x_n)} - \frac{e_n}{x_n (1-x_{n-1})} \right]$, and

$$e_n = w_n - x_n.$$

The initial value of adaptive synchronization error $e_0 = 4.5$ is taken.

Figure 4 illustrates the adaptive synchronization between Systems (2) and (18) with adaptive control parameter $k = 0.5$ and the parameter of update law $k_0 = 0.4$.

![Figure 4](image1)

**FIGURE 4.** The adaptive synchronization between Systems (2) and (18).

The relationship between the convergence speed of the synchronous error $e_n$ and the parameter $|k + k_0|$ is shown in Figure 5 with $e_0 = 4.5$.

![Figure 5](image2)

**FIGURE 5.** The synchronous error $e_n$ with different $|k + k_0|$.

Obviously, it can be seen from Figure 5, for the given $e_0 = 4.5$, the value of parameter $|k + k_0|$ determines the convergence speed of the synchronous error $e_n$. The smaller the value of $|k + k_0|$, the faster the error will converge to 0. Figure 6 shows the estimation of the unknown parameter $\mu$.

![Figure 6](image3)

**FIGURE 6.** The estimation of unknown parameter $\mu$.

**V. CONCLUSION**

In this paper, some definitions of synchronization of Julia sets between different 2-D Logistic maps are discussed. Moreover, we discuss the adaptive synchronization between two different 2-D Logistic maps with the unknown parameters. The parameter estimator is designed, and the identification of unknown parameters is completed based on the known conditions. Some examples are taken to certificate the effectiveness of these three types of synchronization. These control methods and their theories are successfully applied to other aspects of fractal theory, which can help us to explain the corresponding complicated phenomena better.

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PEI WANG received the Ph.D. degree in control theory and control engineering from Shandong University, in 2016. She is currently a Lecturer with the School of Electrical Engineering and Automation, Qilu University of Technology (Shandong Academy of Sciences), China. Her research interest includes the control and applications of fractal.