SIMULTANEOUS EXTENSION OF CONTINUOUS AND UNIFORMLY CONTINUOUS FUNCTIONS

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Abstract. The first known continuous extension result was obtained by Lebesgue in 1907. In 1915, Tietze published his famous extension theorem generalizing Lebesgue’s result from the plane to general metric spaces. He constructed the extension by an explicit formula involving the distance function on the metric space. Thereafter, several authors contributed other explicit extension formulas. In the present paper, we show that all these extension constructions also preserve uniform continuity, which answers a question posed by St. Watson. In fact, such constructions are simultaneous for special bounded functions. Based on this, we also refine a result of Dugundji by constructing various continuous (nonlinear) extension operators which preserve uniform continuity as well.

1. Introduction

In his 1907 paper [25] on Dirichlet’s problem, Lebesgue showed that for a closed subset $A \subset \mathbb{R}^2$ and a continuous function $\varphi : A \to \mathbb{R}$, there exists a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f \upharpoonright A = \varphi$. Here, $f$ is commonly called a continuous extension of $\varphi$, and we also say that $\varphi$ can be extended continuously.

In 1915, Tietze [31] generalised Lebesgue’s result for all metric spaces.

Theorem 1.1 (Tietze, 1915). If $(X, d)$ is a metric space and $A \subset X$ is a closed set, then each bounded continuous function $\varphi : A \to \mathbb{R}$ can be extended to a continuous function $f : X \to \mathbb{R}$.

Tietze gave two proofs of Theorem 1.1, the second of which was based on the following explicit construction of the extension. For a nonempty closed set $A \subset X$ and a continuous function $\varphi : A \to [1, 2]$, he defined a continuous extension $f : X \to \mathbb{R}$ of $\varphi$ by

\[
(1.1) \quad f(p) = \sup_{a \in A} \frac{\varphi(a)}{(1 + [d(a, p)]^2)^{\varphi(a)/\varphi(A)}}, \quad \text{whenever } p \in X \setminus A.
\]

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Here, $d(p, A) = \inf \{d(p, a) : a \in A \}$ is the distance to the set $A$, and it is assumed from the context that $f \upharpoonright A = \varphi$.

Nowadays, Theorem 1.1 is commonly called Tietze’s extension theorem. The history of this theorem is fascinating. In 1916, in his book [13], de la Vallée Poussin gave a proof of this theorem for $X = \mathbb{R}^n$. The book of Carathéodory [11] contains another proof of Tietze’s extension theorem for Euclidean spaces, it was credited to H. Bohr and works for general metric spaces, see Section 4. In 1918, Brouwer [10] also gave an alternative proof of Tietze’s extension theorem for Euclidean spaces. In 1919, Hausdorff [21] gave a simple proof of Tietze’s extension theorem.

For a nonempty closed set $A \subset X$ and a continuous function $\varphi : A \to [0, 1]$, he defined a continuous extension $f : X \to \mathbb{R}$ of $\varphi$ by

\begin{equation}
(1.2) \quad f(p) = \inf_{a \in A} \left[ \varphi(a) + \frac{d(a, p)}{d(p, A)} - 1 \right], \quad \text{whenever } p \in X \setminus A.
\end{equation}

In 1923, in his book [24], Kerékjártó refers to a letter of Riesz which contains a simple proof of Tietze’s extension theorem, and presented this proof. The same proof was also presented in the book of Alexandroff and Hopf [1] and credited again to Riesz. In Riesz’s construction, $\varphi : A \to [1, 2]$ and the extension $f : X \to \mathbb{R}$ is given by the following very simple formula:

\begin{equation}
(1.3) \quad f(p) = \sup_{a \in A} \varphi(a) \cdot \frac{d(p, A)}{d(a, p)}, \quad \text{whenever } p \in X \setminus A.
\end{equation}

In his book [14], Dieudonné included a proof of Tietze’s extension theorem which is virtually the same as previously given by Riesz. In the same setting, he defined an extension $f : X \to \mathbb{R}$ of $\varphi$ by

\begin{equation}
(1.4) \quad f(p) = \inf_{a \in A} \varphi(a) \cdot \frac{d(a, p)}{d(p, A)}, \quad \text{whenever } p \in X \setminus A.
\end{equation}

In 1951, Dugundji [15] generalised Theorem 1.1 by replacing the range $\mathbb{R}$ with an arbitrary locally convex topological vector space using virtually the same method as Brouwer did. This was done at the cost of applying A. H. Stone’s theorem [30, Corollary 1] that each metrizable space is paracompact. On the other hand, generalising a result of Borsuk [8] and Kakutani [23], Dugundji obtained the following very interesting application. For a space $Z$, let $C^*(Z)$ be the Banach space of all bounded continuous functions on $Z$ equipped with the sup-norm $\| \cdot \|$.

**Theorem 1.2** (Dugundji, 1951). Let $A \subset X$ be a closed set in a metric space $(X, d)$. Then there exists a linear map $\Phi : C^*(A) \to C^*(X)$ such that

\[ \Phi[\varphi] \upharpoonright A = \varphi \quad \text{and} \quad \| \Phi[\varphi] \| = \| \varphi \|, \quad \text{for every } \varphi \in C^*(A). \]

Regarding Theorem 1.2, let us explicitly remark that all extension constructions mentioned above are actually “simultaneous” extensions from some subset of $C^*(A)$ to $C^*(X)$, see the next sections.
Extensions of uniformly continuous maps came naturally in the context of the completion problem of metric spaces, see Hausdorff’s 1914 monograph “Grundzüge der Mengenlehre” [20]. The development of such extensions was gradual, the interested reader is referred to [7, 22] for an interesting outline on the history of this extension problem. One of the first explicit solutions of the extension problem for uniformly continuous functions was obtained by McShane [28, Corollary 2].

**Theorem 1.3** (McShane, 1934). Let \((X, d)\) be a metric space and \(A \subset X\). Then each uniformly continuous bounded function \(\varphi : A \to \mathbb{R}\) can be extended to the whole of \(X\) preserving the uniform continuity and the bounds.

McShane obtained the above result as an application of his Lipschitz extension theorem [28, Theorem 1], see also Whitney [33, the footnote on p. 63]. His construction was implicit and based on moduli of continuity of bounded uniformly continuous functions. Regarding the relationship with Lipschitz extensions, let us explicitly remark that, recently, G. Beer [4] refined McShane’s construction to show that a bounded uniformly continuous function on \(A\) is actually Lipschitz with respect to a metric on \(X\) which is uniformly equivalent to \(d\).

In 1990, Mandelkern [27] gave the following explicit construction of the extension in Theorem 1.3, thus relating it to Tietze’s extension theorem.

**Theorem 1.4** (Mandelkern, 1990). Let \((X, d)\) be a metric space, \(A \subset X\) be a nonempty closed set and \(\varphi : A \to [1, 2]\) be uniformly continuous. Then the extension \(f : X \to \mathbb{R}\) defined as in (1.4) is also uniformly continuous.

In his review of Mandelkern’s paper [27], St. Watson [32] remarked that Hausdorff’s formula (1.2) is simpler and even included in Engelking’s General Topology in [16, Exercise 4.1.F], and questioned whether it also preserves uniform continuity. He further remarked that the study of the mechanics of Tietze’s extension theorem is worthwhile and proposed the following general formula of constructing an extension. He wrote that we can measure how close \(a \in A\) is to \(p \in X \setminus A\) by the function \(s(a, p) = \frac{d(a, p)}{d(p, A)}\). Then he suggested to consider a general function \(c(s, r)\) representing the extension by \(f(p) = \inf_{a \in A} c(s(a, p), \varphi(a))\). For instance, in Hausdorff’s approach, \(c(s, r) = r + s - 1\). Thus, he posed the following general question.

**Question 1.** Can those functions \(c(s, r)\) which preserve continuity and uniform continuity, respectively, be characterised?

We are now ready to state also the main purpose of this paper. In the next section, we show that Hausdorff’s extension construction (1.2) preserves the uniformly continuous functions (Theorem 2.1). In fact, we show that this construction defines an extension operator \(\Psi : C^*(A) \to C^*(X)\) which is an isometry and \(\Psi[\varphi] : X \to \mathbb{R}\) is uniformly continuous whenever \(\varphi \in C^*(A)\) is. Regarding this,
let us explicitly remark that \( \Psi \) is not linear. In fact, as shown in the Remark after Corollary D in Pelczyński [29, Notes and Remarks], see also Lindenstrauss [26], one cannot expect a linear extension operator from \( C^*(A) \) to \( C^*(X) \) to preserve uniform continuity. In Section 3, we deal with an alternative setting of Question 1. In contrast to Watson’s approach, we consider and abstract function \( F(s,t) \) of the variables \( s \geq t > 0 \), where \( s \) plays the role of \( d(a,p) \) and \( t \) — that of \( d(p,A) \), for \( a \in A \) and \( p \in X \setminus A \). Then we associate the function \( F^* : A \times (X \setminus A) \to \mathbb{R} \) defined by \( F^*(a,p) = F(d(a,p),d(p,A)), a \in A \) and \( p \in X \setminus A \). The extension of a bounded function \( \varphi : A \to [0, +\infty) \) is now defined by taking supremum or infimum on \( A \) of the product “\( \varphi(a) \cdot F^*(a,p) \)”. By considering the multiplicative inverse of \( F \), if necessary, the construction is reduced only to taking supremum. In this case, we require \( F \) to take values in \( (0,1] \) and place three conditions on it, see (3.2), (3.3) and (3.4), which are satisfied by all classical constructions of this type. In this general setting, we show that \( \Omega_F[\varphi](p) = \sup_{a \in A} \varphi(a) \cdot F^*(a,p), p \in X \setminus A, \), transforms each member \( \varphi \in C^*_+(A) \) of the positive cone of \( C^*(A) \) into a member \( \Omega_F[\varphi] \in C^*_+(X) \) which is uniformly continuous whenever \( \varphi \) is, Theorem 3.1. In fact, this defines a sublinear extension operator \( \Omega_F : C^*_+(A) \to C^*_+(X) \) which is an isotone isometry. Based on a classical construction, we extend \( \Omega_F \) to a positively homogeneous extension operator \( \Theta_F : C^*(A) \to C^*(X) \) which is 2-Lipschitz and still preserves uniform continuity, Theorem 3.8. Section 3 also contains several supporting examples and remarks. In the last Section 4, we consider Bohr’s extension construction which is somewhat different being based on an integral formula. This construction may look a bit artificial, but simple arguments show that it also preserves the uniformly continuous functions (Theorem 4.1). Furthermore, this construction gives at once a sublinear extension operator \( \Phi : C^*(A) \to C^*(X) \) which is an isotone isometry and preserves uniform continuity. Finally, let us remark that the extension operators constructed in this paper are complementary to a result for simultaneous extension of bounded uniformly continuous function obtained in [3, Theorem 3.1]. The method used in [3] is based on the original arguments of McShane for proving Theorem 1.3. Our extension operators can be also compared with a nonlinear extension operator constructed by Borsuk in [9].

2. Hausdorff’s Extension Operator

An ordered vector space is a real vector space \( E \) endowed with a partial order \( \leq \) such that for every \( u, v \in E \) with \( u \leq v \),

(i) \( u + x \leq v + x, \) for every \( x \in E, \)

(ii) \( \alpha \cdot u \leq \alpha \cdot v, \) for every \( \alpha > 0. \)

In such a space \( E, \) the set \( \{ v \in E : v \geq 0 \} \) is called the positive cone of \( E, \) and denoted by \( E^+ \) or \( E_+. \) An ordered vector space \( (E, \leq) \) is a Riesz space, or a vector lattice, if it is also a lattice, i.e. if each pair of elements \( u, v \in E \) has a supremum
is a Riesz space (normed Riesz space $E$) called isotone

$$\Psi(a)$$

everywhere $u, v \in E$ with $|u| \leq |v|$. A complete normed Riesz space is called a Banach lattice.

A map $\Phi : E \to V$ between Riesz spaces is positive if $\Phi(E^+) \subset V^+$, and $\Phi$ is called isotone, or order-preserving, if $\Phi(u) \leq \Phi(v)$ whenever $u, v \in E$ with $u \leq v$.

Each linear positive operator between Riesz spaces is isotone because for elements $u, v \in E$ we have that $u \leq v$ precisely when $v - u \in E^+$.

Let $C^*(Z)$ be the Banach space of all bounded, continuous, real-valued functions on a space $Z$ equipped with the sup-norm $\|f\| = \sup \{|f|, f \in C^*(Z)\}$. The vector space $C^*(Z)$ is partially ordered by defining that $f \leq g$ holds whenever $f(z) \leq g(z)$ for all $z \in Z$. This makes $C^*(Z)$ a Banach lattice. Namely, for $f, g \in C^*(Z)$, the elements $f \lor g$ and $f \land g$ are the pointwise maximum and minimum of $f$ and $g$. The positive cone of $C^*(Z)$ will be denoted by $C^*_+(Z)$; and for $t \in \mathbb{R}$, we will use $t_Z$ for the constant function $t_Z : Z \to \{t\}$. In case $(Z, \rho)$ is a metric space, we will also use $C^*_+(Z)$ for the uniformly continuous members of $C^*(Z)$.

In this section and the rest of the paper, $(X, d)$ is a fixed metric space and $A \subset X$ is a nonempty closed set. A map $\Phi : C^*(A) \to C^*(X)$ is called an extension operator if $\Phi[\varphi] \upharpoonright A = \varphi$, for every $\varphi \in C^*(A)$. A linear map $\Phi : C^*(A) \to C^*(X)$ is called regular if $\Phi[1_A] = 1_X$ and $\|\Phi[\varphi]\| = \|\varphi\|$, $\varphi \in C^*(A)$, see [29]. Evidently, a regular linear map $\Phi : C^*(A) \to C^*(X)$ is an isometry. Furthermore, according to [29, Proposition 1.2], a linear operator $\Phi : C^*(A) \to C^*(X)$ is regular if and only if $\Phi$ is positive and $\Phi[1_A] = 1_X$. In particular, each regular linear operator $\Phi : C^*(A) \to C^*(X)$ is also isotone.

In what follows, we shall say that a map $\Psi : C^*(A) \to C^*(X)$ preserves uniform continuity if $\Psi[C^*_+(A)] \subset C^*_+(X)$. Also, for convenience, we shall say that $\Psi$ preserves the constants if $\Psi[t_A] = t_X$ for every $t \in \mathbb{R}$. In this section, we consider Hausdorff’s extension construction (1.2) as an extension operator from $C^*(A)$ to $C^*(X)$. While this operator is not linear, it preserves uniform continuity and some of the properties of regular linear operators.

**Theorem 2.1.** For each $\varphi \in C^*(A)$, let $\Psi[\varphi] : X \to \mathbb{R}$ be the extension of $\varphi$ defined as in (1.2), i.e. by

$$\Psi[\varphi](p) = \inf_{a \in A} \left[ \varphi(a) + \frac{d(a, p)}{d(p, A)} - 1 \right], \quad \text{whenever } p \in X \setminus A.$$  

Then $\Psi : C^*(A) \to C^*(X)$ is an extension operator which is an isotone isometry. Moreover, $\Psi$ preserves both uniform continuity and the constants.
The proof of Theorem 2.1 is based on two general observations, the first of which is related to another construction of Hausdorff about Lipschitz functions. Let us recall that a map \( f : X \to Y \) in a metric space \((Y, \rho)\) is *Lipschitz* if there exists \( K \geq 0 \) such that \( \rho(f(p), f(q)) \leq Kd(p, q) \), for all \( p, q \in X \). In this case, to emphasise on the constant \( K \), we also say that \( f \) is \( K \)-Lipschitz.

In his 1919 paper [21], Hausdorff described a very interesting construction of Lipschitz functions. Namely, to each bounded function \( \varphi : X \to \mathbb{R} \) and \( \kappa > 0 \), he associated the function \( f_{\kappa} : X \to \mathbb{R} \) defined by
\[
 f_{\kappa}(p) = \inf_{x \in X} [\varphi(x) + \kappa d(x, p)], \quad p \in X.
\]
He showed that \( f_{\kappa} \) is \( \kappa \)-Lipschitz, see [21, page 293], and credited the construction to Moritz Pasch. In the literature, the functions \( f_{\kappa}, \kappa > 0 \), are often called the *Pasch-Hausdorff envelope* of \( \varphi \); the function \( f_{\kappa} \) is also called the \( \kappa \)-Lipschitz *regularisation* of \( \varphi \). It should be remarked that \( f_{\kappa} \) can be defined for each function \( \varphi : X \to \mathbb{R} \) for which there exists a \( \kappa \)-Lipschitz function \( f : X \to \mathbb{R} \) with \( f \leq \varphi \).

For an extended discussion on the Pasch-Hausdorff construction, the interested reader is referred to [19]. In the proof of Theorem 2.1, we will use the following slight modification of this construction.

**Proposition 2.2.** For a bounded function \( \varphi : A \to [0, +\infty) \), define a function \( f : X \to \mathbb{R} \) by
\[
(2.2) \quad f(p) = \inf_{a \in A} [\varphi(a) d(p, A) + d(a, p)], \quad p \in X.
\]

Then \( f \) is Lipschitz.

**Proof.** Suppose that \( \varphi : A \to [0, \kappa] \) for some \( \kappa > 0 \). If \( x, p \in X \) and \( a \in A \), then \( d(x, A) \leq d(p, A) + d(x, p) \) and \( d(a, x) \leq d(a, p) + d(x, p) \). Multiplying the first inequality with \( \varphi(a) \) and adding it to the second one, it follows from (2.2) that \( f(x) \leq f(p) + (\kappa + 1) d(x, p) \) because \( \varphi \) is bounded by \( \kappa \). Accordingly, \( |f(x) - f(p)| \leq (\kappa + 1) d(x, p) \). \( \square \)

For the other general observation, let us briefly review the difference between continuity “at a point” and continuity “on a subset”. In some sources, \( f : X \to \mathbb{R} \) is continuous on \( A \) if it is continuous at each point of \( A \). In other sources, \( f \) is continuous on \( A \) if its restriction \( f \restriction A : A \to \mathbb{R} \) is continuous. To avoid any misunderstanding, we will say that \( f \) is continuous at the points of \( A \) if it is continuous at each point of \( A \). Similarly, in most sources, \( f \) is assumed to be uniformly continuous on \( A \) if its restriction \( f \restriction A \) is uniformly continuous. The provision for the other interpretation was made in [5] where \( f \) was called “uniformly continuous on \( A \)” if for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |f(x) - f(a)| < \varepsilon \), for every \( a \in A \) and \( x \in X \) with \( d(a, x) < \delta \).
was further refined in [6, Definition 1.1], where this property was called *strongly uniformly continuous on A*. To avoid any confusion with the existing literature, we will retain this concept. However, for the proper understanding of our results, let us explicitly remark that \( f \) is strongly uniformly continuous on \( A \) precisely when it is continuous at the points of \( A \) with the same \( \delta \) for all points of \( A \).

If a function \( f : X \to \mathbb{R} \) is continuous at the points of the closed set \( A \) and its restriction on \( X \setminus A \) is also continuous, then \( f \) is itself continuous. This follows from the fact that \( X \setminus A \) is open, hence \( f \restriction_{X \setminus A} \) is continuous precisely when \( f \) is continuous at the points of \( X \setminus A \). In the case of strong uniform continuity, this property was summarised by M. Hušek [22, Corollary 16]. It is in good accord with our method of proving Theorem 2.1, and we briefly reproduce the arguments. To this end, for \( \varepsilon > 0 \) we will use \( O(p, \varepsilon) = \{ x \in X : d(x, p) < \varepsilon \} \) to denote the open \( \varepsilon \)-ball centred at a point \( p \in X \). Also, let \( O(S, \varepsilon) = \bigcup_{p \in S} O(p, \varepsilon) \) be the \( \varepsilon \)-enlargement of \( S \), whenever \( S \subseteq X \).

**Proposition 2.3.** If a function \( f : X \to \mathbb{R} \) is strongly uniformly continuous on \( A \) and for each \( \tau > 0 \), its restriction on \( X \setminus O(A, \tau) \) is uniformly continuous, then \( f \) is itself uniformly continuous.

**Proof.** Let \( \varepsilon > 0 \). Since \( f \) is strongly uniformly continuous on \( A \), there exists \( \tau > 0 \) such that \( |f(x) - f(a)| < \varepsilon \) for every \( a \in A \) and \( x \in X \) with \( d(a, x) < 2\tau \). For the same reason, since the restriction of \( f \) on \( X \setminus O(A, \tau) \) is uniformly continuous, there exists \( 0 < \delta \leq \tau \) such that \( |f(x) - f(p)| < \varepsilon \) for every \( x, p \notin O(A, \tau) \) with \( d(x, p) < \delta \). To show that this \( \delta \) works, take points \( x, p \in X \) with \( d(x, p) < \delta \). If \( x, p \notin O(A, \tau) \), then \( |f(x) - f(p)| < \varepsilon \). If \( d(x, A) < \tau \) or \( d(p, A) < \tau \), there exists a point \( a \in A \) with \( d(x, a) < 2\tau \) and \( d(p, a) < 2\tau \) because \( d(x, p) < \delta \leq \tau \). Accordingly, \( |f(x) - f(p)| \leq |f(x) - f(a)| + |f(a) - f(p)| < 2\varepsilon \).

In the proof of Theorem 2.1 and what follows, for convenience, we set
\[
\text{(2.3)} \quad O_A(p, \varepsilon) = O(p, \varepsilon) \cap A, \quad \text{for every } p \in X \text{ and } \varepsilon > 0.
\]

Evidently, the set \( O_A(p, \varepsilon) \) could be empty for some \( p \in X \setminus A \). However, \( O_A(p, \varepsilon) \neq \emptyset \) for every \( \varepsilon > d(p, A) \). In fact, \( d(p, A) = \inf_{a \in O_A(p, \varepsilon)} d(p, a) \), whenever \( \varepsilon > d(p, A) \). Below, we translate this property only in terms of the open balls centred at the points of \( A \). Namely, if \( p \in A \) and \( \tau > 0 \), then
\[
\text{(2.4)} \quad d(x, A) = \inf\{d(x, a) : a \in O_A(p, 2\tau)\}, \quad \text{for every } x \in O(p, \tau).
\]

This follows from the fact that \( \emptyset \neq O_A(x, \tau) \subseteq O_A(p, 2\tau) \).

Finally, we will also use the property of bounded functions \( g, h : Z \to \mathbb{R} \) that
\[
\text{(2.5)} \quad |\inf g - \inf h| \leq \sup |g - h| \quad \text{and} \quad |\sup g - \sup h| \leq \sup |g - h|.
\]

It is a simple consequence of the fact that \( \sup(B + C) = \sup B + \sup C \), whenever \( B, C \subseteq \mathbb{R} \) are nonempty bounded sets.
Proof of Theorem 2.1. Let \( \varphi : A \to \mathbb{R} \) be a bounded (uniformly) continuous function and \( \Psi[\varphi] : X \to \mathbb{R} \) be the extension of \( \varphi \) defined as in (2.1). Whenever \( \lambda \in \mathbb{R} \), it is evident from (2.1) that \( \Psi[\varphi + \lambda] = \Psi[\varphi] + \lambda \). Accordingly, we may assume that \( \varphi : A \to [0, \kappa] \) for some \( \kappa \geq 1 \). In this setting, we will first show that \( \Psi[\varphi] \) is continuous at the points of \( A \). Also, that it is strongly uniformly continuous on \( A \) provided \( \varphi \) is uniformly continuous. So, take a point \( p \in A \) and \( 0 < \varepsilon \leq \kappa \). Since \( \varphi : A \to [0, \kappa] \) is continuous at \( p \), there exists \( \delta > 0 \) such that \( |\varphi(a) - \varphi(p)| < \varepsilon \) for every \( a \in O_{A}(p, 4\kappa\delta) \). If \( \varphi \) is uniformly continuous, we may assume that this \( \delta \) is the same for all points of \( A \), namely that \( |\varphi(a) - \varphi(b)| < \varepsilon \), for every \( a, b \in A \) with \( d(a, b) < 4\kappa\delta \). Take \( x \in O(p, \delta) \setminus A \). If \( a \in O_{A}(p, 4\kappa\delta) \), then \( \varphi(p) - \varepsilon < \varphi(a) + \frac{d(a, x)}{d(x, A)} - 1 < [\varphi(p) + \varepsilon] + \frac{d(a, x)}{d(x, A)} - 1 \) because \( \frac{d(a, x)}{d(x, A)} - 1 \geq 0 \). Since \( \varphi(p) \leq \kappa \) and \( \varepsilon \leq \kappa \), it follows from (2.4) that

\[
\varphi(p) - \varepsilon \leq \inf_{a \in O_{A}(p, 4\kappa\delta)} \left[ \varphi(a) + \frac{d(a, x)}{d(x, A)} - 1 \right] \leq \varphi(p) + \varepsilon \leq 2\kappa.
\]

If \( a \in A \setminus O(p, 4\kappa\delta) \), then \( \varphi(a) + \frac{d(a, x)}{d(x, A)} - 1 > \varphi(a) + 3\kappa - 1 \geq 2\kappa \) because \( \varphi(a) \geq 0, \kappa \geq 1 \) and \( d(a, x) \geq 3\kappa\delta > 3\kappa d(p, x) \geq 3\kappa d(x, A) \). Accordingly, \( \Psi[\varphi](x) = \inf_{a \in O_{A}(p, 4\kappa\delta)} \left[ \varphi(a) + \frac{d(a, x)}{d(x, A)} - 1 \right] \) and, therefore, \( |\Psi[\varphi](x) - \varphi(p)| \leq \varepsilon \).

Next, take \( \tau > 0 \), and let us show that \( \Psi[\varphi] \upharpoonright X \setminus O(A, \tau) \) is uniformly continuous. To this end, let \( f : X \to \mathbb{R} \) be the function defined as in (2.2) with respect to the given function \( \varphi : X \to [0, \kappa] \). Then by Proposition 2.2, \( f \) is Lipschitz. Since \( \Psi[\varphi](x) = \frac{f(x)}{d(x, A)} - 1 \), \( x \in X \setminus A \), and the restriction of \( \frac{1}{d(x, A)} \) on \( X \setminus O(A, \tau) \) is Lipschitz, \( \Psi[\varphi] \upharpoonright X \setminus O(A, \tau) \) is also Lipschitz. Thus, \( \Psi[\varphi] \) is continuous and by Proposition 2.3, it is also uniformly continuous provided so is \( \varphi \). In other words, \( \Psi : C^{*}(A) \to C^{*}(X) \) is an extension operator which preserves uniform continuity. Moreover, by (2.1), \( \Psi \) is isotone and preserves the constants as well. Finally, take \( \varphi, \psi \in C^{*}(A) \). Then for \( p \in X \setminus A \), it follows from (2.1) and (2.5) that

\[
|\Psi[\varphi](p) - \Psi[\psi](p)| \leq \sup_{a \in A} |\varphi(a) - \psi(a)| = ||\varphi - \psi||.
\]

Hence, \( ||\Psi[\varphi] - \Psi[\psi]|| = ||\varphi - \psi|| \) because \( \Psi \) is an extension operator. \( \square \)

We conclude this section with two remarks regarding Theorem 2.1.

Remark 2.4. The extension operator \( \Psi : C^{*}(A) \to C^{*}(X) \) defined in (2.1) is not necessarily linear. For instance, take \( X = [-1, +\infty) \) and \( A = [-1, 0] \). Also, let \( \varphi(a) = a, a \in A \), be the identity function. If \( a \in A \) and \( p \in X \setminus A \), then \( d(a, p) = p - a \) and \( d(p, A) = p \). Hence, for \( k \in \mathbb{N} \) and \( p \in X \setminus A \), we get that

\[
\Psi[k\varphi](p) = \inf_{a \in A} \left[ ka + \frac{p - a}{p} - 1 \right] = \inf_{-1 \leq a \leq 0} a \left[ k - \frac{1}{p} \right] = \begin{cases} 0 & \text{if } p \leq \frac{1}{k}, \\ \frac{p}{k} - k & \text{if } p \geq \frac{1}{k}. \end{cases}
\]
Accordingly, $\Psi[2\varphi] = \Psi[\varphi + \varphi] \neq 2\Psi[\varphi]$. □

**Remark 2.5.** For a nonempty set $Z$, let $\ell_\infty(Z)$ be the Banach lattice of all bounded real-valued functions on $Z$ equipped with the sup-norm $\|\varphi\| = \sup |\varphi|$, $\varphi \in \ell_\infty(Z)$. The partial order on $\ell_\infty(Z)$ is defined precisely as that for $C^*(Z)$, in fact $C^*(Z)$ is a Banach sublattice of $\ell_\infty(Z)$. The positive cone of $\ell_\infty(Z)$ will be denoted by $\ell_\infty^+(Z)$. Evidently, in our setting of $(X,d)$ and $A \subset X$, (2.1) defines an extension operator $\Psi : \ell_\infty(A) \rightarrow \ell_\infty(X)$. Moreover, the same argument as in Theorem 2.1 shows that $\Psi : \ell_\infty(A) \rightarrow \ell_\infty(X)$ is an isotone isometry and, in particular, $\Psi[\ell_\infty^+(A)] \subset \ell_\infty^+(X)$. In this interpretation, $\Psi$ preserves continuity in the sense that $\Psi[C^*(A)] \subset C^*(X)$. □

### 3. Tietze-Like Extension Operators

If $a \in A$ and $p \in X \setminus A$, then $s = d(a,p) \geq d(p,A) = t > 0$. Motivated by this, we consider the set

$$\Delta = \{(s,t) \in \mathbb{R}^2 : s \geq t > 0\}.$$ 

Next, to each $F : \Delta \rightarrow \mathbb{R}$ we will associate the “composite” function

$$F^* : A \times (X \setminus A) \ni (a,p) \longrightarrow (d(a,p), d(p,A)) \in \Delta \xrightarrow{F} \mathbb{R}.$$ 

In other words, $F^* : A \times (X \setminus A) \rightarrow \mathbb{R}$ is defined by $F^*(a,p) = F(d(a,p), d(p,A))$, for every $a \in A$ and $p \in X \setminus A$. Finally, following Tietze’s construction (1.1), for $F : \Delta \rightarrow (0,1]$ and a bounded function $\varphi : A \rightarrow [0, +\infty)$, let $\Omega_F[\varphi] : X \rightarrow [0, +\infty)$ be the extension of $\varphi$ defined by

$$\Omega_F[\varphi](p) = \sup_{a \in A} \varphi(a) \cdot F^*(a,p), \quad \text{for every } p \in X \setminus A. \quad (3.1)$$

Since $0 < F \leq 1$, it follows that $\varphi(a) \cdot F^*(a,p) \leq \varphi(a)$, for every $a \in A$ and $p \in X \setminus A$. Accordingly, $\Omega_F[\varphi]$ is also a bounded function, in fact it is bounded above by $\|\varphi\| = \sup_{a \in A} \varphi(a)$. Thus, $\Omega_F : \ell_\infty^+(A) \rightarrow \ell_\infty^+(X)$ is an extension operator which is an isotone isometry; the latter follows easily from (2.5) and (3.1). Furthermore, $\Omega_F$ is sublinear, i.e. for every $\varphi, \psi \in \ell_\infty^+(A)$ and $\lambda \geq 0$ we have that $\Omega_F[\varphi + \psi] \leq \Omega_F[\varphi] + \Omega_F[\psi]$ ($\Omega_F$ is subadditive) and $\Omega_F[\lambda \varphi] = \lambda \Omega_F[\varphi]$ ($\Omega_F$ is positively homogeneous).

In this section, we will show that the extension operator $\Omega_F : \ell_\infty^+(A) \rightarrow \ell_\infty^+(X)$ preserves both continuity and uniform continuity based only on the following three general properties of $F : \Delta \rightarrow (0,1]$. 
In this setting, each function \( F : \Delta \to (0, 1] \) satisfying (3.2), (3.3) and (3.4) will be called a Tietze Extender.

**Theorem 3.1.** Let \( F : \Delta \to (0, 1] \) be a Tietze extender. Then the extension operator \( \Omega_F : \ell^+_{\infty}(A) \to \ell^+_{\infty}(X) \), defined as in (3.1), is a sublinear isotone isometry which preserves both continuity and uniform continuity.

Regarding the proper place of Theorem 3.1, here are the simplest examples of two different types of Tietze extenders.

**Example 3.2.** Following (3.1) and Tietze’s extension construction (1.1), let

\[
(3.5) \quad T(s, t) = \frac{1}{(1 + s^2)^{\frac{1}{2}}} \quad s \geq t > 0.
\]

Then \( T : \Delta \to (0, 1] \) is a Tietze extender. Indeed, (3.2) and (3.3) are evident for this function. To see that \( T(s, t) \) also has the property in (3.4), consider the function \( G(x, y) = \frac{1}{(1+x^2)^y} = e^{-y \ln(1+x^2)} \) for \( x \geq 0 \) and \( y \geq 0 \). Then it is uniformly continuous being a continuous function with \( \lim_{x \to +\infty} \lim_{y \to +\infty} G(x, y) = 0 \). Accordingly, so is \( T(s, t) = G(s, \frac{1}{t}) \) on the domain \( s \geq t \geq \tau > 0 \). Thus, Theorem 3.1 is valid for \( T(s, t) \). Since Tietze’s extension construction (1.1) is identical to that in (3.1), it preserves not only continuity, but also uniform continuity. On the other hand, this construction doesn’t preserves the constants (in particular, the bounds). For instance, \( 1_A : A \to [1, \mu] \) for every \( \mu \geq 1 \), but the extension \( \Omega_T[1_A] \) is not necessarily the constant function \( 1_X \) of \( X \). Namely, \( p \in X \setminus A \) implies that

\[
\Omega_T[1_A](p) = \sup_{a \in A} \frac{1}{\left(1 + [d(a, p)]^2\right)^{\frac{1}{2}[a, p, A]}} = \frac{1}{\left(1 + [d(p, A)]^2\right)^{\frac{1}{2}[a, p, A]}} < 1.
\]
Example 3.3. Following (3.1) and Riesz’s extension construction (1.3), let

\[(3.6) \quad R(s, t) = \frac{t}{s}, \quad s \geq t > 0.\]

It is obvious that \( R : \Delta \to (0, 1] \) is a Tietze extender. Hence, Theorem 3.1 is also valid for the Riesz function. In contrast to the Tietze function \( T(s, t) \) in (3.5) of Example 3.2, the Riesz function \( R(s, t) \) preserves the bounds because \( R(t, t) = 1 \) for every \( t > 0 \). Indeed, for \( \varphi : A \to [\lambda, \mu] \) and \( p \in X \setminus A \), where \( \mu \geq \lambda \geq 0 \), we have that

\[
\Omega_R[\varphi](p) = \sup_{a \in A} \varphi(a) \cdot R^*(a, p) \geq \lambda \sup_{a \in A} \frac{d(p, A)}{d(a, p)} = \lambda \frac{d(p, A)}{d(p, A)} = \lambda.
\]

Accordingly, Theorem 3.1 gives McShane’s result on uniformly continuous extensions at once. Let us remark that in Theorem 1.3, McShane first extended \( \varphi \) to a uniformly continuous function on \( X \), and next adjusted its bounds. □

Following Dieudonné’s extension construction (1.4), for \( G : \Delta \to [1, +\infty) \) and a bounded function \( \varphi : A \to [0, +\infty) \), we may define an alternative extension \( \tilde{U}_G[\varphi] : X \to [\inf \varphi, +\infty) \) of \( \varphi \) by

\[(3.7) \quad \tilde{U}_G[\varphi](p) = \inf_{a \in A} \varphi(a) \cdot G^*(a, p), \quad p \in X \setminus A.\]

Evidently if \( F : \Delta \to (0, 1] \), then \( G = \frac{1}{F} : \Delta \to [1, +\infty) \). Thus, for a bounded function \( \varphi : A \to (0, +\infty) \) we have that

\[(3.8) \quad \tilde{U}_{\frac{1}{F}}[\varphi] = \frac{1}{\Omega_F[\frac{1}{\varphi}]}.
\]

Example 3.4. Following Dieudonné’s extension construction (1.4), define a function \( D : \Delta \to [1, +\infty) \) by

\[(3.9) \quad D(s, t) = \frac{s}{t}, \quad s \geq t > 0.\]

Then for \( \mu > \lambda > 0 \) and \( \varphi : A \to [\lambda, \mu] \), Dieudonné’s extension is identical to that in (3.7), and is given by \( \tilde{U}_D[\varphi] : X \to [\lambda, \mu] \). However, the multiplicative inverse \( \frac{1}{\varphi} \) is the Riesz function \( R \) in (3.6). Hence, by (3.8), Example 3.3 and Theorem 3.1, \( \tilde{U}_D[\varphi] = \frac{1}{\Omega_R[\frac{1}{\varphi}]} : X \to [\lambda, \mu] \) is (uniformly) continuous whenever so is \( \frac{1}{\varphi} \), equivalently \( \varphi \). Accordingly, Dieudonné’s extension \( \tilde{U}_D[\varphi] \) is (uniformly) continuous precisely when so is \( \varphi \). Thus, Theorem 3.1 also contains Theorem 1.4 as a special case. □

The proof of Theorem 3.1 is based on three properties of Tietze extenders. The first one is the following relaxed representation of the extension \( \Omega_F[1_A] \) of the constant function \( 1_A \).
Proposition 3.5. Let $F : \Delta \to (0,1]$ be a continuous function as in (3.3), and
$\Omega_F[1_A] : X \to (0,1]$ be the extension of $1_A : A \to \{1\}$ defined as in (3.1). If
$p \in A$, $\tau > 0$ and $x \in O(p,\tau) \setminus A$, then
$$\Omega_F[1_A](x) = F(d(x,A), d(x,A)) = \sup_{a \in O(\tau)} F^*(a,x).$$

Proof. Since $F$ is continuous, it follows from (2.4) and (3.3) that
$$\Omega_F[1_A](x) = F\left(\inf_{a \in A} d(a,x), d(x,A)\right) = F(d(x,A), d(x,A))$$
$$= F\left(\inf_{a \in O(\tau)} d(a,x), d(x,A)\right) = \sup_{a \in O(\tau)} F^*(a,x). \quad \square$$

Tietze extenders also have the following property related to uniform continuity.

Proposition 3.6. Let $F : \Delta \to \mathbb{R}$ be a function satisfying (3.4), and $\varepsilon, \tau > 0$.
Then there exists $\delta > 0$ such that for every $x, p \in X \setminus O(A,\tau)$ with $d(x,p) < \delta$,
(3.10) $|F^*(a,x) - F^*(a,p)| < \varepsilon$, for all $a \in A$.

Proof. By (3.4), there is $\delta > 0$ such that $|F(s_1,t_1) - F(s_0,t_0)| < \varepsilon$, for every
$s_i \geq t_i \geq \tau$, $i = 0,1$, with $|s_1 - s_0| < \delta$ and $|t_1 - t_0| < \delta$. If $a \in A$ and
$x, p \in X \setminus O(A,\tau)$ with $d(x,p) < \delta$, then $s_0 = d(a,p) \geq d(p,A) = t_0 \geq t_\tau$ and
$s_1 = d(a,x) \geq d(x,A) = t_1 \geq \tau$. Moreover, $|d(x,a) - d(a,p)| \leq d(x,p) < \delta$ and
$|d(x,A) - d(p,A)| \leq d(x,p) < \delta$. Hence, $|F^*(a,x) - F^*(a,p)| < \varepsilon$. \quad \square

We conclude the preparation for the proof of Theorem 3.1 with a crucial observation
about continuity and uniform continuity of the extension defined in (3.1).

Proposition 3.7. Let $F : \Delta \to (0,1]$ be a continuous function satisfying (3.2) and
(3.3). If $\varphi \in \ell^+_\infty(A)$ is a continuous function, then the extension $\Omega_F[\varphi] \in \ell^+_\infty(X)$
defined as in (3.1) is continuous at the points of $A$. Moreover, $\Omega_F[\varphi]$ is strongly
uniformly continuous on $A$ provided $\varphi$ is uniformly continuous.

Proof. Take a point $p \in A$ and $\varepsilon > 0$. Since $\varphi : A \to [0, +\infty)$ is continuous at $p$,
there exists $\tau > 0$ such that $|\varphi(a) - \varphi(p)| < \varepsilon$ for every $a \in O(p,\tau)$, see (2.3).
If $\varphi$ is uniformly continuous, we may assume that this $\tau$ is the same for all points
of $A$, namely that $|\varphi(a) - \varphi(b)| < \varepsilon$, for every $a, b \in A$ with $d(a,b) < 2\tau$. Thus,
it will be now sufficient to show that there exists $\delta > 0$ depending only on $\tau$ and
the function $F$ such that
(3.11) $\varphi(p) - 2\varepsilon \leq \Omega_F[\varphi](x) \leq \varphi(p) + \varepsilon$, for every $x \in O(p,\delta) \setminus A$.

In the construction below, we treat each part of this inequality separately.

For convenience take $\lambda > \varepsilon$ such that $\varphi : A \to [0, \lambda]$. Then for $s \geq \tau \geq t > 0$,
it follows from (3.2) and (3.3) that $F(s,t) \leq F(\tau,t) \rightarrow 0 < \frac{\varepsilon}{\lambda}$. Hence, there is
$0 < r \leq \tau$ such that $F(t,x) < \frac{\varepsilon}{\lambda}$ for every $0 < t < r \leq \tau \leq s$. This implies the
right-hand side of (3.11) for every $0 < \delta \leq r$. Indeed, take $x \in O(p, r) \setminus A$ and $a \in A$. If $d(a, p) \geq 2\tau$, then $t = d(x, A) < r \leq \tau \leq d(a, x) = s$ and, accordingly,

$$\varphi(a) \cdot F^*(a, x) = \varphi(a) \cdot F(s, t) \leq \varphi(a) \cdot \frac{\varepsilon}{\lambda} \leq \varepsilon \leq \varphi(p) + \varepsilon.$$  

Otherwise, if $d(a, p) < 2\tau$, then $\varphi(a) \cdot F^*(a, x) \leq \varphi(a) \cdot 1 < \varphi(p) + \varepsilon$. Thus, $\Omega_F[\varphi](x) = \sup_{a \in A} \varphi(a) \cdot F^*(a, x) \leq \varphi(p) + \varepsilon$.

For the left-hand side of (3.11), we will use (3.2) that $F(t, t) \xrightarrow{t \rightarrow 0} 1$. Since $0 < \varepsilon < \lambda$, there exists $0 < \delta \leq r$ such that $F(t, t) > 1 - \frac{\varepsilon}{\lambda}$ for every $0 < t < \delta$. If $\varphi(p) - 2\varepsilon \leq 0$, then $\Omega_F[\varphi](x) \geq 0 \geq \varphi(p) - 2\varepsilon$ for every $x \in X$. Suppose that $\varphi(p) - 2\varepsilon > 0$. Then $1 - \frac{\varepsilon}{\lambda} > 1 - \frac{\varepsilon}{\varphi(p) - \varepsilon} = \frac{\varphi(p) - 2\varepsilon}{\varphi(p) - \varepsilon}$ because $0 < \varphi(p) - \varepsilon < \lambda$. Accordingly, for a point $x \in O(p, \delta) \setminus A \subset O(p, \tau)$ and $t = d(x, A) < \delta$, it follows from Proposition 3.5 that

$$\Omega_F[\varphi](x) \geq \sup_{a \in O_A(p, 2\tau)} \varphi(a) \cdot F^*(a, x) \geq [\varphi(p) - \varepsilon] \cdot \sup_{a \in O_A(p, 2\tau)} F^*(a, x)$$

$$= [\varphi(p) - \varepsilon] \cdot F(t, t) > [\varphi(p) - \varepsilon] \cdot \frac{\varphi(p) - 2\varepsilon}{\varphi(p) - \varepsilon} = \varphi(p) - 2\varepsilon.$$  

In other words, $\Omega_F[\varphi](x) \geq \varphi(p) - 2\varepsilon$ for every $x \in O(p, \delta) \setminus A$. Since $\delta \leq r$, this shows (3.11) and the proof is complete. \hfill \Box

**Proof of Theorem 3.1.** Let $F : \Delta \rightarrow (0, 1]$ be a Tietze extender, $\varphi \in C^+_\sigma(A)$ and $f = \Omega_F[\varphi] \in \ell^+_\infty(X)$ be the extension defined in (3.1). According to Propositions 2.3 and 3.7, it suffices to show that for each $\tau > 0$, the restriction of $f$ on $X \setminus O(A, \tau)$ is uniformly continuous. So, take $\tau > 0$ and $\varepsilon > 0$. Since $F$ satisfies (3.4), by Proposition 3.6, there exists $\delta > 0$ such that (3.10) holds for the function $F^*$. Accordingly, for $x, p \notin O(A, \tau)$ with $d(x, p) < \delta$, it follows from (2.5) and (3.10) that $|f(x) - f(p)| \leq \sup_{a \in A} \varphi(a) |F^*(a, x) - F^*(a, p)| \leq \varepsilon \cdot \|\varphi\|$. Thus, $f \upharpoonright X \setminus O(A, \tau)$ is uniformly continuous and the proof is complete. \hfill \Box

Let $E$ and $V$ be Riesz spaces, and $\Omega : E^+ \rightarrow V^+$ be a positively homogeneous additive map, i.e. a positively homogeneous map with the property that

$$\Omega(u + v) = \Omega(u) + \Omega(v), \text{ for every } u, v \in E^+.$$  

Then it has a unique extension to a positive linear map $\Lambda : E \rightarrow V$. This is a classical result and can be found in several books on Riesz spaces, see for instance [18, Extension Lemma on p. 51] and [12, Lemma 2.10], also [34, Lemma 20.1]. Briefly, each $u \in E$ can be represented in the form $u = u^+ - u^-$, where $u^+ = u \lor 0 \in E^+$ and $u^- = (-u) \lor 0 \in E^+$. This representation is minimal in the sense that $u = v - w$ for $v, w \in E^+$ implies that $u^+ \leq v$ and $u^- \leq w$. The extension $\Lambda$ of $\Omega$ is now simply defined by $\Lambda(u) = \Omega(u^+ - \Omega(u^-), u \in E$. Based on the same idea, we will extend the operator $\Omega_F : \ell^+_\infty(A) \rightarrow \ell^+_\infty(X)$ to the entire

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space \( \ell_\infty(A) \). Namely, if \( F : \Delta \to (0,1) \) and \( \Omega_F \) is as in (3.1), then we may define an extension operator \( \Theta_F : \ell_\infty(A) \to \ell_\infty(X) \) by

\[
\Theta_F[\varphi] = \Omega_F[\varphi^+] - \Omega_F[\varphi^-], \quad \varphi \in \ell_\infty(A).
\]

In the theorem below, we say that \( \Theta : \ell_\infty(A) \to \ell_\infty(X) \) preserves the norm if \( \|\Theta[\varphi]\| = \|\varphi\| \), for every \( \varphi \in \ell_\infty(A) \).

**Theorem 3.8.** Let \( F : \Delta \to (0,1) \) be a Tietze extender, and \( \Theta_F : \ell_\infty(A) \to \ell_\infty(X) \) be the extension operator defined in (3.12). Then \( \Theta_F \) is a 2-Lipschitz map which is positive and positively homogeneous. Moreover, it preserves the norm and both continuity and uniform continuity.

**Proof.** If \( \varphi \in \ell_\infty(A) \) is (uniformly) continuous, then so are \( \varphi^+ \) and \( \varphi^- \). Hence, by (3.12) and Theorem 3.1, \( \Theta_F \) preserves both continuity and uniform continuity. For the same reason, \( \Theta_F \) is positive and positively homogeneous. Take \( \varphi \in \ell_\infty(A) \) and \( p \in X \setminus A \). Then by (2.5), (3.1) and (3.12),

\[
|\Theta_F[\varphi](p)| = |\Omega_F[\varphi^+](p) - \Omega_F[\varphi^-](p)| \\
\leq \sup_{a \in A} |\varphi^+(a) - \varphi^-(a)| \cdot F^*(a,p) \\
= \sup_{a \in A} |\varphi(a)| \cdot F^*(a,p) \leq \sup_{a \in A} |\varphi(a)| = \|\varphi\|.
\]

Accordingly, \( \Theta_F \) preserves the norm as well. Finally, we show that \( \Theta_F \) is 2-Lipschitz. Indeed, if \( \varphi, \psi \in \ell_\infty(A) \) and \( a \in A \), then \( |\varphi^+(a) - \psi^+(a)| \leq |\varphi(a) - \psi(a)| \) and \( |\varphi^-(a) - \psi^-(a)| \leq |\varphi(a) - \psi(a)| \). Hence, by (3.12) and Theorem 3.1,

\[
\|\Theta_F[\varphi] - \Theta_F[\psi]\| \leq \|\varphi^+ - \psi^+\| + \|\varphi^- - \psi^-\| \leq 2\|\varphi - \psi\|.
\]

\( \square \)

We conclude with several remarks.

**Remark 3.9.** The extension operator \( \Theta_F \) defined in (3.12) is not necessarily an isometry. For instance, take \( X = [-1, +\infty) \subset \mathbb{R} \) and \( A = [-1, 1] \). Then for \( a \in A \) and \( p \in X \setminus A \) we have that \( d(p, a) = p - a \) and \( d(p, A) = p - 1 \). Let \( F(s, t) = R(s, t) = \frac{t}{s} \) be the Riesz function in (3.6), which is a Tietze extender, see Example 3.3. If \( \varphi : A \to \mathbb{R} \) is defined by \( \varphi(a) = \frac{3a+1}{4} \), then one can easily see that \( \Theta_R[\varphi](p) = \frac{p+3}{2(p+1)} \), whenever \( p \in X \setminus A \). Similarly, if \( \psi : A \to \mathbb{R} \) is defined by \( \psi(a) = \frac{3a-1}{4} \), then \( \Theta_R[\psi](p) = \frac{3-p}{2(p+1)} \), for \( p \in X \setminus A \). In fact, \( \varphi \) and \( \psi \) are uniformly continuous functions with \( \psi = \varphi - \frac{1}{2} \), so \( \|\varphi - \psi\| = \frac{1}{2} \). However, for \( p \in X \setminus A \), we have that \( \|\Theta_R[\varphi](p) - \Theta_R[\psi](p)\| = \frac{p}{p+1} \xrightarrow{p \to +\infty} 1 \). \( \square \)

**Remark 3.10.** Theorem 3.1 is not valid for bounded functions \( \varphi : A \to \mathbb{R} \) which may take negative values. For instance, let \( X = \mathbb{R} \) and \( A = (-\infty, 0] \). Also, let \( R(s, t) = \frac{t}{s}, s \geq t > 0 \), be the Riesz function in (3.6) of Example 3.3, and
−1_A : A → \{-1\} be the constant function −1. Then for \( x \in X \setminus A \) and \( a \in A \), we have that \( R^*(a, x) = \frac{x}{x-a} = \frac{x}{x+|a|} \) and, accordingly,

\[
\Omega_R[-1_A](x) = \sup_{a \in A} [-R^*(a, x)] = -\inf_{a \in A} R^*(a, x) = -\inf_{a \in A} \frac{x}{x+|a|} = 0. \tag{4.2} \]

**Remark 3.11.** In contrast to Tietze extenders, the extension construction in (3.7) doesn’t preserve continuity for bounded functions \( \varphi : A \to [0, +\infty) \) which may take the value 0. Indeed, let \( A = [-1, 0] \subset [-1, +\infty) = X \) and \( \varphi : A \to [0, +\infty) \) be defined by \( \varphi(a) = 1 + a, \ a \in A \). If \( G : \Delta \to [1, +\infty) \) is any function, then the extension \( \mathcal{U}_G[\varphi] : X \to [0, +\infty) \) is not continuous because \( \mathcal{U}_G[\varphi](0) = \varphi(0) = 1 \) and \( \mathcal{U}_G[\varphi](p) = 0 \) for every \( p \in X \setminus A \). \( \square \)

### 4. Bohr’s Extension Operator

As mentioned in the Introduction, the book of Carathéodory [11] contains another proof of Tietze’s extension theorem (Theorem 1.1), it is credited to Harald Bohr. In fact, Bohr’s extension construction gives at once an extension operator \( \Phi : \ell_\infty(A) \to \ell_\infty(X) \) which has the best properties comparing with the previous extension operators. In this construction, for convenience, let \( \rho = d(\cdot, A) \) be the distance function to the set \( A \). Whenever \( \varphi \in \ell_\infty(A) \) and \( x \in X \setminus A \), Bohr associated the bounded increasing function \( \eta_{x,\varphi} : (0, +\infty) \to \mathbb{R} \) defined by

\[
\eta_{x,\varphi}(t) = \begin{cases} 
\sup_{a \in O_{A}(x,t)} \varphi(a) & \text{if } t > \rho(x), \text{ and} \\
\inf \varphi & \text{if } t \leq \rho(x). 
\end{cases} \tag{4.1}
\]

Next, using these functions, he defined an extension \( \Phi[\varphi] \in \ell_\infty(X) \) of \( \varphi \) by the following explicit formula:

\[
\Phi[\varphi](x) = \frac{1}{\rho(x)} \int_{\rho(x)}^{2\rho(x)} \eta_{x,\varphi}(t) \, dt, \quad x \in X \setminus A. \tag{4.2}
\]

Since \( \eta_{x,\varphi} \) is bounded and increasing, it is Riemann integrable. Hence, \( \Phi[\varphi] \) is well defined. Also, by the properties of the integral, we get at once that Bohr’s extension operator \( \Phi : \ell_\infty(A) \to \ell_\infty(X) \) is isotone and, in particular, positive. Furthermore, \( \Phi \) is sublinear, i.e. both positively homogeneous and subadditive. Indeed, if \( \lambda \geq 0 \) and \( \varphi \in \ell_\infty(A) \), then by (4.2), \( \Phi[\lambda \varphi] = \lambda \Phi[\varphi] \) because \( \eta_{x,\lambda \varphi} = \lambda \eta_{x,\varphi} \) for every \( x \in X \setminus A \), see (4.1). Similarly, for \( \varphi, \psi \in \ell_\infty(A) \), we get that \( \Phi[\varphi + \psi] \leq \Phi[\varphi] + \Phi[\psi] \) because \( \eta_{x,\varphi+\psi}(t) \leq \eta_{x,\varphi}(t) + \eta_{x,\psi}(t) \), for every \( x \in X \setminus A \) and \( t > \rho(x) \). Finally, it is also easy to see that \( \Phi \) is an isometry. Namely, for \( \varphi, \psi \in \ell_\infty(A) \) and \( x \in X \setminus A \), it follows from (2.5) and (4.1) that \( |\eta_{x,\varphi}(t) - \eta_{x,\psi}(t)| \leq ||\varphi - \psi|| \) for every \( t \geq \rho(x) \). Accordingly,
\[
|\Phi[\varphi](x) - \Phi[\psi](x)| \leq \frac{1}{\rho(x)} \int_{\rho(x)}^{2\rho(x)} |\eta_{x,\varphi}(t) - \eta_{x,\psi}(t)| \, dt \\
\leq \frac{1}{\rho(x)} \int_{\rho(x)}^{2\rho(x)} \|\varphi - \psi\| \, dt = \|\varphi - \psi\|.
\]

Regarding other properties of \( \Phi : \ell_\infty(A) \to \ell_\infty(X) \), Bohr actually showed that it preserves continuity. The interested reader is also referred to [17] and [2], where these arguments were reproduced. Below we show that \( \Phi \) preserves uniform continuity as well.

**Theorem 4.1.** The extension operator \( \Phi : \ell_\infty(A) \to \ell_\infty(X) \) defined as in (4.2) is a sublinear isotone isometry which preserves both continuity and uniform continuity.

**Proof.** Let \( \varphi \in \ell_\infty(A) \) be a (uniformly) continuous function, and \( f = \Phi[\varphi] \) be the associated extension in (4.2). If \( \lambda > 0, \mu \in \mathbb{R} \) and \( x \in X \setminus A \), then it follows from (4.1) that \( \eta_{x,\lambda \varphi + \mu} = \lambda \eta_{x,\varphi} + \mu \). Hence, by (4.2), \( \Phi[\lambda \varphi + \mu] = \lambda f + \mu \). Accordingly, we may assume that \( \varphi : A \to [0, 1] \). Moreover, in the rest of this proof we will simply write \( \eta_x \) instead of \( \eta_{x,\varphi} \).

First, we will show that \( f \) is continuous at the points of \( A \), also that it is strongly uniformly continuous on \( A \) provided \( \varphi \) is itself uniformly continuous. To this end, take \( \varepsilon > 0 \) and \( p \in A \). As in the previous proofs, using that \( \varphi \) is continuous at \( p \), there is \( \delta > 0 \) (independent of \( p \) provided that \( \varphi \) is uniformly continuous) such that \( |\varphi(a) - \varphi(p)| < \varepsilon \) for every \( a \in O_A(p, 3\delta) \). If \( x \in O(p, \delta) \setminus A \), then \( O(x, 2\rho(x)) \subset O(p, 3\delta) \) because \( \rho(x) \leq d(x, p) < \delta \). Accordingly, see (4.1) and (4.2),

\[
\varphi(p) - \varepsilon \leq \inf_{a \in O_A(p, 3\delta)} \varphi(a) \leq \inf_{\rho(x) < t \leq 2\rho(x)} \eta_x(t) \leq f(x) \leq \eta_x(2\rho(x)) \leq \sup_{a \in O_A(p, 3\delta)} \varphi(a) \leq \varphi(p) + \varepsilon.
\]

Thus, \( |f(x) - \varphi(p)| \leq \varepsilon \).

Let \( \tau > 0 \). We finalise the proof by showing that \( f \mid X \setminus O(A, \tau) \) is uniformly continuous, in fact Lipschitz. So, let \( x, p \in X \setminus O(A, \tau) \) with \( 0 < d(x, p) \leq \frac{\tau}{3} \). Then \( |\rho(x) - \rho(p)| \leq d(x, p) = \delta \leq \frac{\rho(x)}{3} \) and, therefore,

\[
\rho(p) \leq \rho(x) + \delta \leq 2\rho(x) - 2\delta \leq 2\rho(p).
\]
Moreover, \( \eta_p(t) \geq \eta_x(t - \delta) \) for every \( t > \delta \) because \( O(x, t - \delta) \subset O(p, t) \), see (4.1). Thus, substituting in (4.2), we get that
\[
f(p) = \frac{1}{\rho(p)} \int_{\rho(p)}^{2\rho(p)} \eta_p(t) dt \geq \frac{1}{\rho(x) + \delta} \int_{\rho(x)+\delta}^{2\rho(x)-2\delta} \eta_x(t - \delta) dt
\]
\[
(s = t - \delta) = \frac{1}{\rho(x) + \delta} \int_{\rho(x)}^{2\rho(x)-3\delta} \eta_x(s) ds.
\]
Since \( f(x) = \frac{1}{\rho(x)} \int_{\rho(x)}^{2\rho(x)-3\delta} \eta_x(t) dt + \frac{1}{\rho(x)} \int_{2\rho(x)-3\delta}^{2\rho(x)} \eta_x(t) dt \) and \( \eta_x \leq 1 \), this implies that
\[
f(x) - f(p) \leq \left[ \frac{1}{\rho(x)} - \frac{1}{\rho(x) + \delta} \right] \int_{\rho(x)}^{2\rho(x)-3\delta} \eta_x(t) dt + \frac{1}{\rho(x)} \int_{2\rho(x)-3\delta}^{2\rho(x)} \eta_x(t) dt
\]
\[
= \frac{\delta}{\rho(x)[\rho(x) + \delta]} \int_{\rho(x)}^{2\rho(x)-3\delta} \eta_x(t) dt + \frac{1}{\rho(x)} \int_{2\rho(x)-3\delta}^{2\rho(x)} \eta_x(t) dt
\]
\[
\leq \frac{\delta}{\rho(x)} \cdot \frac{\rho(x) - 3\delta}{\rho(x) + \delta} + \frac{3\delta}{\rho(x)} \leq \frac{4\delta}{\rho(x)} \leq \frac{4}{\tau} d(x, p).
\]
Interchanging \( p \) and \( x \), this is equivalent to \( |f(p) - f(x)| \leq \frac{4}{\tau} d(x, p) \). In particular, \( f \) is continuous and by Proposition 2.3, it is also uniformly continuous whenever so is \( \varphi \).

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