STOCHASTIC HOMOGENIZATION IN AMORPHOUS MEDIA 
AND APPLICATIONS TO EXCLUSION PROCESSES

ALESSANDRA FAGGIONATO

Abstract. We consider random walks on marked simple point processes with symmetric jump rates and unbounded jump range. We prove homogenization properties of the associated Markov generators. As an application, we derive the hydrodynamic limit of the simple exclusion process given by multiple random walks as above, with hard–core interaction, on a marked Poisson point process. The above results cover Mott variable range hopping, which is a fundamental mechanism of phonon–induced electron conduction in amorphous solids as doped semiconductors. Our techniques, based on an extension of two-scale convergence, can be adapted to other models, as e.g. the random conductance model.

Keywords: marked simple point process, Palm distribution, Mott variable range hopping, stochastic homogenization, two-scale convergence, exclusion process, hydrodynamic limit.

AMS 2010 Subject Classification: 60G55, 60H25, 60K37, 35B27.

1. Introduction

We consider stochastic jump dynamics in a random environment, not necessarily with an underlying lattice structure, where jumps can be arbitrarily long. A fundamental example comes from Mott variable range hopping (v.r.h.), which is at the basis of electron transport in disordered solids, as doped semiconductors, in the regime of strong Anderson localization [24, 25, 27, 29, 36]. Starting from a quantum modelization, in the regime of low density of impurities one arrives to a classical model given by a family of non–interacting random walkers hopping on the sites of a marked simple point process (cf. [2, 3, 16, 23], [14, Section 1]). The latter is given by a random subset \( \{(x_i, E_i)\} \subset \mathbb{R}^d \times \mathbb{R} \) sampled as follows. The locally finite subset \( \{x_i\} \subset \mathbb{R}^d \) is given by a simple point process with stationary and ergodic (w.r.t. to spatial translations) law. For example, \( \{x_i\} \) can be a Poisson point process on \( \mathbb{R}^d \). Given \( \{x_i\} \), to each point \( x_i \) one associates independently a random variable \( E_i \) (called mark) according to a fixed distribution \( \nu \). In doped semiconductors, the \( x_i \)'s correspond to the locations of the impurities, while \( E_i \) is the fundamental energy of an electron with quantum wavefunction localized around \( x_i \).

Given a realization \( \{(x_i, E_i)\} \) of the environment, Mott v.r.h. can be described in terms of a time–continuous random walk with state space \( \{x_i\} \). The probability rate for a jump from \( x_i \) to \( x_j \) is then given by 
\[
c_{x_i,x_j}(\omega) = \exp \left\{ -\gamma |x_i - x_j| - \beta (|E_i| + |E_j| + |E_i - E_j|) \right\},
\]
where \( \beta \) denotes the inverse

This work has been supported by PRIN 20155PAWZB “Large Scale Random Structures”.

1
temperature and $\gamma > 0$ is a fixed constant. The presence of long jumps and the special energy dependence in the jump rates is fundamental to explain the anomalous decay of conductivity in strongly disordered amorphous solids, which follows the so called Mott’s law (cf. 2, 3, 15, 16, 23 and references therein). In general, we will refer to Mott v.r.h. when the jump rates have the form

$$c_{x_i, x_j}(\omega) = \exp \{ -\gamma |x_i - x_j| - u(E_i, E_j) \},$$

(1)

for some symmetric bounded function $u$.

Stochastic jump dynamics, where hopping takes place on marked simple point processes, is relevant e.g. also in population dynamics. If sites are given for example by a Poisson point process, then the medium is genuinely amorphous. However, we include in our analysis also hybrid environments given by diluted lattices as e.g. site percolation. Indeed, the random set $\{x_i\}$, where $x_i := x + y_i$, $x$ is chosen with uniform probability among $[0,1]^d$ and $\{y_i\}$ is the realization of the site percolation on $\mathbb{Z}^d$, has stationary and ergodic law.

Of course, there are also other relevant models of stochastic jump dynamics with jump rates having unbounded range, as for example the conductance model on $\mathbb{Z}^d$ where the random walk hops among sites of $\mathbb{Z}^d$ and the probability rate for a jump from $x$ to $y$ is given by a random number, called conductance [17]. Another example is given by random walks on Delaunay triangulations [34]. We focus here on hopping on marked simple point processes with jump rates not necessarily of the form (1), but our proofs and results can be adapted to other models as the above ones.

The main part of our work is devoted to prove homogenization results for the random walk (cf. Theorems 1 and 2). We denote by $L_\omega$ the generator of the random walk with environment $\omega$ and by $L_\omega^\varepsilon$ its version under an $\varepsilon$-parametrized space rescaling. We consider the Poisson equation $\lambda u - L_\omega^\varepsilon u = f_\varepsilon$ with $\lambda > 0$ and show the convergence of $u_\varepsilon$ to the solution $u$ of the effective equation $\lambda u - \nabla \cdot D \nabla u = f$ if $f_\varepsilon$ converges to $f$. Above $D$ is the effective diffusion matrix, having a variational characterization. We also prove the convergence of the associated gradients, energies and semigroups.

The proof of Theorems 1 and 2 is based on two-scale convergence, a notion introduced by G. Nguetseng [30] and developed by G. Allaire [1]. In particular, our proof is inspired by the method developed in [38] for random differential operators on singular structures. Two–scale convergence has already been applied in random walks in random environment in [12, 17, 20]. Due to long jumps, the standard discrete gradients have to be replaced by amorphous gradients. Roughly, given a random function $v(\omega)$, its amorphous gradient keeps knowledge of all the differences $v(\tau_x \omega) - v(\tau_y \omega)$ as the sites $x, y$ vary among the sites of the marked simple point process, where $\tau_z \omega$ denotes the environment obtained from $\omega$ by a translation along the vector $z$ (an analogous version holds for functions on $\mathbb{R}^d$). The two–scale convergence is a genuine $L^2$–concept but due to the presence of infinite jumps a key technical obstruction to the analysis in [38] appears. Even when the gradient is square integrable, its contraction (obtained by a weighted averaging with weights given by the jump rates, cf.
Sec. [7] is not necessarily square integrable. We have been able to overcome this difficulty by means of a cut-off procedure developed in Sections [11] and [13]. We stress that the presence of long jumps leads to new technical problems also in other spots when trying to adapt the strategy in [38] to the present context.

We point out that [17, Thm. 2.1], referred to the random conductance model, is somehow similar to Item (i) in Theorem [16] (there the authors consider Dirichlet boundary conditions on a finite box). There are however two fundamental differences. Our homogenization results (the above mentioned convergences concerning Poisson equation) hold for almost any environment, whatever the choice of the known functions \( f_\varepsilon, f \) with \( f_\varepsilon \) converging to \( f \). On the contrary, in [17, Thm. 2.1] the class of environments for which one has homogenization depends on the functions \( f_\varepsilon, f \). In addition, the method developed in [17] is based on stronger additional assumptions, concerning the existence of special paths with suitable small resistance which implies between others the Poincaré inequality. Our method avoids this kind of technical assumptions (cf. [5] for isoperimetric and Poincaré inequalities for Mott v.r.h.). Finally, we mention also [31] for other results on homogenization for non–local operators.

As an application of the homogenization results presented in Theorems [1] and [2] we prove the hydrodynamic limit of the exclusion process obtained by taking multiple random walks as above with the addition of the hard-core interaction, when the sites \( \{x_i\} \) are given by a Poisson point process on \( \mathbb{R}^d \). It is known (cf. e.g. [11, 12] and references therein) that the proof of the exclusion process with symmetric jumps rates (possibly in a random environment) can be obtained using homogenization properties of the Markov generator of a single random walk (see also [32] for recent progresses on hydrodynamic limits in random environments). We point out that, as a further application of Theorems [1] and [2] in interacting particle systems, one could prove the hydrodynamic limit of zero–range processes on marked simple point processes similarly to what done in [13] using - between others - homogenization. This further application will be presented in a separate work [35].

2. Notation and setting

In this section we fix our notation concerning point processes and state our main assumptions.

We fix a Polish space \( S \) (e.g. \( S \subset \mathbb{R} \)) and we denote by \( \Omega \) the space of locally finite subsets \( \omega \subset \mathbb{R}^d \times S \) such that for each \( x \in \mathbb{R}^d \) there exists at most one element \( s \in S \) with \( (x, s) \in \omega \). We write a generic \( \omega \in \Omega \) as \( \omega = \{(x_i, s_i)\} \) (\( s_i \) is called the mark at the point \( x_i \)) and set \( \hat{\omega} := \{x_i\} \). We will identify the sets \( \omega = \{(x_i, s_i)\} \) and \( \hat{\omega} = \{x_i\} \) with with the counting measures \( \sum_i \delta_{(x_i, s_i)} \) and \( \sum_i \delta_{x_i} \), respectively. On \( \Omega \) one defines a special metric \( d \) such that the following facts are equivalent: (i) a sequence \( (\omega_n) \) converges to \( \omega \) in \( (\Omega, d) \), (ii) \( \lim_{n \to \infty} \int_{\mathbb{R}^d \times S} f(x, s)d\omega_n(x, s) = \int_{\mathbb{R}^d \times S} f(x, s)d\omega(x, s) \), for any bounded continuous function \( f : \mathbb{R}^d \times S \to \mathbb{R} \) vanishing outside a bounded set and (iii) \( \lim_{n \to \infty} \omega_n(A) = \omega(A) \) for any bounded Borel set \( A \subset \mathbb{R}^d \times S \) with \( \omega(\partial A) = 0 \).
(see [9 App. A2.6 and Sect. 7.1], [18 Sect. 1.1.5], [21 Sect. 1.15]). Moreover, the \( \sigma \)-algebra \( \mathcal{B}(\Omega) \) of Borel sets of \( (\Omega, d) \) is generated by the sets \( \{ \omega(A) = k \} \) with \( A \) and \( k \) varying respectively among the Borel sets of \( \mathbb{R}^d \times S \) and the nonnegative integers. In addition, \( (\Omega, d) \) is a separable metric space. Indeed, the above distance \( d \) is defined on the larger space \( N \) of counting measures \( \mu = \sum_{i} k_i \delta_{(x_i, s_i)} \), where \( k_i \in \mathbb{N} \) and \( \{(x_i, s_i)\} \) is a locally finite subset of \( \mathbb{R}^d \times S \), and one can prove that \( (N, d) \) is a Polish space having \( \Omega \) as Borel subset [9 Cor. 7.1.IV, App. A2.6.I]. Finally, given \( x \in \mathbb{R}^d \) we define the translation \( \tau_x : \Omega \rightarrow \Omega \) as 

\[
\tau_x \omega = \{(x_i - x, s_i)\} \quad \text{if} \quad \omega = \{(x_i, s_i)\}.
\]

We consider now a marked simple point process, which is a measurable function from a probability space to the measurable space \( (\Omega, \mathcal{B}(\Omega)) \). We denote by \( P \) its law and by \( E[\cdot] \) the associated expectation. \( P \) is therefore a probability measure on \( \Omega \). We assume that \( P \) is stationary and ergodic w.r.t. translations. Stationarity means that \( P(\tau_x A) = A \) for any Borel set \( A \subset \Omega \), while ergodicity means that \( P(A) \in \{0, 1\} \) for any Borel subset \( A \subset \Omega \) such that \( \tau_x A = A \). Due to our main assumptions stated below, \( P \) will have finite positive intensity \( m \), i.e.

\[
m := \mathbb{E}[\hat{\omega} \cap [0, 1]^d] \in (0, +\infty). \tag{2}
\]

As a consequence, the Palm distribution \( P_0 \) associated to \( P \) is well defined [9, Chp. 12]. Roughly, \( P_0 \) can be thought as \( P \) conditioned to the event \( \Omega_0 \), where

\[
\Omega_0 := \{\omega \in \Omega : 0 \in \hat{\omega}\}. \tag{3}
\]

\( P_0 \) is a probability measure with support inside \( \Omega_0 \) and it can be characterized by the identity

\[
P_0(A) = \frac{1}{m} \int_{\Omega} P(d\omega) \int_{[0,1]^d} d\hat{\omega}(x) 1_A(\tau_x \omega), \quad \forall A \subset \Omega_0 \text{ Borel}. \tag{4}
\]

The above identity [4] is a special case of the so-called Campbell’s formula (cf. [18 Thm. 1.2.8], [9 Eq. (12.2.4)]): for any nonnegative Borel function \( f : \mathbb{R}^d \times \Omega \rightarrow [0, \infty) \) it holds

\[
\int_{\mathbb{R}^d} dx \int_{\Omega} P_0(d\omega) f(x, \omega) = \frac{1}{m} \int_{\Omega} P(d\omega) \int_{\mathbb{R}^d} d\hat{\omega}(x) f(x, \tau_x \omega). \tag{5}
\]

An alternative characterization of \( P_0 \), described in [38 Section 1.2].

Below, we write \( \mathbb{E}_0[\cdot] \) for the expectation w.r.t. \( P_0 \).

In addition to the marked simple point process with law \( P \) we fix a nonnegative Borel function

\[
\mathbb{R}^d \times \mathbb{R}^d \times \Omega \ni (x, y, \omega) \mapsto c_{x,y}(\omega) \in [0, +\infty).
\]

The value of \( c_{x,y}(\omega) \) will be relevant only when \( x, y \in \hat{\omega} \).

**Assumptions.** We make the following assumptions:

(A1) the law \( P \) of the marked point process is stationary and ergodic w.r.t. spatial translations;
(A2) \( \mathcal{P} \) has finite positive intensity, i.e.
\[
0 < \mathbb{E}[\hat{\omega} \cap [0,1]^d] < +\infty ;
\]  
(6)

(A3) \( \mathcal{P}(\omega \in \Omega : \tau_x \omega \neq \tau_y \omega \ \forall x \neq y \in \hat{\omega}) = 1; \)

(A4) the weights \( c_{x,y}(\omega) \) are symmetric, i.e. \( c_{x,y}(\omega) = c_{y,x}(\omega) \) for all \( x, y \in \hat{\omega} \),
and covariant, i.e. \( c_{x,y}(\omega) = c_{x-a,y-a}(\tau_a \omega) \) for all \( x, y \in \hat{\omega} \) and \( a \in \mathbb{R}^d \);

(A5) it holds
\[
\lambda_0 \in L^2(\mathcal{P}_0), \ \lambda_1 \in L^2(\mathcal{P}_0), \ \lambda_2 \in L^1(\mathcal{P}_0),
\]  
(7)
where
\[
\lambda_k(\omega) := \int_{\mathbb{R}^d} d\hat{\omega}(x)c_{0,x}(\omega)|x|^k
\]  
(8)
and \( |x| \) denotes the norm of \( x \in \mathbb{R}^d \);

(A6) the function
\[
F_s(\omega) := \int d\hat{\omega}(y)\int d\hat{\omega}(z)c_{0,y}(\omega)c_{y,z}(\omega)
\]  
(9)
belongs to \( L^1(\mathcal{P}_0) \).

(A7) the weights \( c_{x,y}(\omega) \) induce irreducibility: for \( \mathcal{P} \)–almost all \( \omega \in \Omega \), given
any \( x, y \in \hat{\omega} \) there exists a path \( x = x_0, x_1, \ldots, x_{n-1}, x_n \) such that \( x_i \in \hat{\omega} \) and \( c_{x_i,x_{i+1}}(\omega) > 0 \) for all \( i = 0, 1, \ldots, n - 1 \);

(A8) the \( d \times d \) symmetric matrix \( D \) such that
\[
a \cdot Da = \inf_{f \in L^\infty(\mathcal{P}_0)} \frac{1}{2} \int d\mathcal{P}_0(\omega) \int d\hat{\omega}(x)c_{0,x}(\omega) (a \cdot x - \nabla f(\omega, x))^2 , \]  
(10)
where \( \nabla f(\omega, x) := f(\tau_x \omega) - f(\omega) \), is strictly positive–definite.

Above, and in what follows, we will denote by \( a \cdot b \) the scalar product of the
vectors \( a \) and \( b \).

2.1. Comments on the assumptions.

**Lemma 2.1.** The following holds:

(i) Having that \( \mathcal{P} \) is stationary, the ergodicity of \( \mathcal{P} \) is equivalent to the
ergodicity of \( \mathcal{P}_0 \) w.r.t. point–shifts, i.e. to the fact that \( \mathcal{P}_0(A) \in \{0,1\} \)
for any Borel subset \( A \subset \Omega_0 \) such that \( \omega \in A \) if and only if \( \tau_x \omega \in A \)
for all \( x \in \hat{\omega} \).

(ii) Assumption (A2) is equivalent to the following fact: \( \mathbb{E}[\hat{\omega} \cap U] < +\infty \)
for any bounded Borel set \( U \subset \mathbb{R}^d \).

(iii) Assumption (A3) is equivalent to the identity
\[
\mathcal{P}_0(\omega \in \Omega_0 : \tau_x \omega \neq \tau_y \omega \ \forall x \in \hat{\omega} \ \{0\}) = 1 .
\]  
(11)

(iv) Suppose that, for some function \( h : \mathbb{R}_+ \to \mathbb{R}_+ \), \( c_{x,y}(\omega) \leq h(|u - v|) \) for
any \( u, v \in \mathbb{Z}^d \) and any \( x, y \in \hat{\omega} \) with \( x \in [0,1]^d \), \( y \in [0,1]^d \) (this
is fulfilled e.g. by Mott v.r.h.). Then Assumption (A6) is implied by
the bounds
\[
\mathbb{E}[\hat{\omega} \cap [0,1]^d]^3 < +\infty , \sum_{u \in \mathbb{Z}^d} h(|u|) < +\infty . \]  
(12)
(v) For Mott v.r.h. Assumptions (A4) and (A7) are always satisfied, (A5) is equivalent to the bound \( E[\hat{\omega} \cap [0,1]^d] < +\infty \), which implies (A6).

We postpone the proof of Lemma 2.1 to Appendix A. We point out that the proof of the above Item (iii) is based on Lemmas 1 and 2 in [16]. The same arguments can be adapted to treat more general jumps rates.

One can verify Assumption (A8) by means of random resistor networks as done in [16] for Mott v.r.h. (see also [8] for an alternative derivation).

3. Main results

3.1. The microscopic measure \( \mu_\omega^\varepsilon \). Given \( \varepsilon > 0 \) and \( \omega \in \Omega \) we define \( \mu_\omega^\varepsilon \) as the Radon measure on \( \mathbb{R}^d \)

\[
\mu_\omega^\varepsilon := \varepsilon^d \sum_{a \in \hat{\omega}} \delta_{\varepsilon a}.
\]

For \( \mathcal{P} \)-a.a. \( \omega \) the measure \( \mu_\omega^\varepsilon \) converges vaguely to \( m dx \) (cf. (8)), i.e. \( \mathcal{P} \)-a.s. it holds

\[
\lim_{\varepsilon \downarrow 0} \int \mu_\omega^\varepsilon(x) \varphi(x) = \int dx m(x) \quad \forall \varphi \in C_c(\mathbb{R}^d) .
\]

The above convergence indeed follows from a stronger result which is at the basis of 2–scale convergence:

**Proposition 3.1.** Let \( g : \Omega_0 \rightarrow \mathbb{R} \) be a Borel function with \( \|g\|_{L^1(\mathcal{P}_0)} < +\infty \). Then there exists a Borel set \( \mathcal{A}[g] \) with the following properties: (i) \( \mathcal{P}(\mathcal{A}[g]) = 1 \), (ii) if \( \omega \in \mathcal{A}[g] \) and \( x \in \mathbb{R}^d \) then \( \tau_x \omega \in \mathcal{A}[g] \); (iii) for any \( \omega \in \mathcal{A}[g] \) and any \( \varphi \in C_c(\mathbb{R}^d) \) it holds

\[
\lim_{\varepsilon \downarrow 0} \int \mu_\omega^\varepsilon(x) \varphi(x) g(\tau_x/\varepsilon \omega) = \int \varphi(x) m dx \cdot E_0[g] .
\]

The above fact can be derived by Tempel’man’s ergodic theorem for weighted means as discussed in Appendix A.

We recall the definition of weak and strong convergence for functions belonging to different functional spaces:

**Definition 3.2.** Fix \( \omega \in \Omega \) and a family of \( \varepsilon \)-parametrized functions \( v_\varepsilon \in L^2(\mu_\omega^\varepsilon) \). We say that the family \( \{v_\varepsilon\} \) converges weakly to the function \( v \in L^2(\mu(dx)) \), and write \( v_\varepsilon \rightharpoonup v \), if

\[
\limsup_{\varepsilon \downarrow 0} \|v_\varepsilon\|_{L^2(\mu_\omega^\varepsilon)} < +\infty
\]

and

\[
\lim_{\varepsilon \downarrow 0} \int \mu_\omega^\varepsilon(x) v_\varepsilon(x) \varphi(x) = \int dx m(x) \varphi(x)
\]

for all \( \varphi \in C_c(\mathbb{R}^d) \). We say that the family \( \{v_\varepsilon\} \) converges strongly to \( v \in L^2(\mu(dx)) \), and write \( v_\varepsilon \rightarrow v \), if in addition to (16) it holds

\[
\lim_{\varepsilon \downarrow 0} \int \mu_\omega^\varepsilon(x) v_\varepsilon(x) g_\varepsilon(x) = \int dx m(x) g(x),
\]

for any family of functions \( g_\varepsilon \in L^2(\mu_\omega^\varepsilon) \) weakly converging to \( g \in L^2(\mu(dx)) \).
In general, when (11A) is satisfied, one simply says that the family \( \{v_\epsilon\} \) is bounded.

**Remark 3.3.** One can prove (cf. [39, Prop. 1.1]) that \( v_\epsilon \to v \) if and only if \( v_\epsilon \to v \) and \( \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^d} v_\epsilon(x)^2 d\mu_\epsilon(x) = \int_{\mathbb{R}^d} v(x)^2 m dx \).

### 3.2. The microscopic measure \( \nu_{\omega}^\epsilon \) and microscopic gradients.

We define \( \nu_{\omega}^\epsilon \) as the Radon measure on \( \mathbb{R}^d \times \mathbb{R}^d \) given by

\[
\nu_{\omega}^\epsilon := \epsilon^d \int d\omega(a) \int d(\tau_{a,\omega}) (z) c_{a,a+z}(\omega) \delta_{\omega(\epsilon a, z)}.
\]

(19)

Given \( \omega \in \Omega \) and a real function \( v \) whose domain contains \( \epsilon \hat{\omega} \), we define the *microscopic gradient* \( \nabla_{\epsilon} v \) as the function

\[
\nabla_{\epsilon} v(x, z) = \frac{v(x + \epsilon z) - v(x)}{\epsilon}, \quad x \in \epsilon \hat{\omega}, \ z \in \tau_{a,\omega}.
\]

(20)

Note that if \( v : \mathbb{R}^d \to \mathbb{R} \) is defined \( \mu_\omega^\epsilon \)-a.e., then \( \nabla_{\epsilon} v \) is defined \( \nu_{\omega}^\epsilon \)-a.e.

We say that \( v \in H^1_\omega,\epsilon \) if \( v \in L^2(\mu_\omega^\epsilon) \) and \( \nabla_{\epsilon} v \in L^2(\nu_{\omega}^\epsilon) \). Moreover, we endow the space \( H^1_\omega,\epsilon \) with the norm

\[
\|v\|_{H^1_\omega,\epsilon} := \|v\|_{L^2(\mu_\omega^\epsilon)} + \|\nabla_{\epsilon} v\|_{L^2(\nu_{\omega}^\epsilon)}.
\]

Equivalently, we can identify \( H^1_\omega,\epsilon \) with the subspace

\[
H_{\omega,\epsilon} := \{ (v, \nabla_{\epsilon} v) : v \in L^2(\mu_\omega^\epsilon), \ \nabla_{\epsilon} v \in L^2(\nu_{\omega}^\epsilon) \}
\]

of the product Hilbert space \( L^2(\mu_\omega^\epsilon) \times L^2(\nu_{\omega}^\epsilon) \).

**Lemma 3.4.** The space \( H_{\omega,\epsilon} \) is a closed subspace of \( L^2(\mu_\omega^\epsilon) \times L^2(\nu_{\omega}^\epsilon) \). In particular, \( H_{\omega,\epsilon} \) and \( H^1_{\omega,\epsilon} \) are Hilbert spaces.

The proof is simple and given in Appendix A for completeness.

We introduce a notion of weak and strong convergence for microscopic gradients. Consider the standard space \( H^1(dx) \) given by functions \( f \) in \( L^2(dx) \) whose weak derivatives belong to \( L^2(dx) \). Recall that \( C^\infty_c(\mathbb{R}^d) \) forms a dense subset of \( H^1(dx) \) (cf. [4, Thm. 9.2]), in particular standard gradients \( \nabla \varphi \) with \( \varphi \in C^\infty_c(\mathbb{R}^d) \) can be used to approximate in \( L^2(dx) \) the weak gradient \( \nabla f \) when \( f \in H^1(dx) \).

**Definition 3.5.** Fix \( \omega \in \Omega \) and a family of \( \epsilon \)-parametrized functions \( v_\epsilon \in L^2(\mu_\omega^\epsilon) \). We say that the family \( \{\nabla_{\epsilon} v_\epsilon\} \) converges weakly to the vector-valued function \( w \in L^2(m dx)^d \), and write \( \nabla_{\epsilon} v_\epsilon \rightharpoonup w \), if

\[
\limsup_{\epsilon \searrow 0} \|\nabla_{\epsilon} v_\epsilon\|_{L^2(\nu_{\omega}^\epsilon)} < +\infty
\]

(21)

and

\[
\lim_{\epsilon \searrow 0} \int d\nu_{\omega}^\epsilon(x, z) \nabla_{\epsilon} v_\epsilon(x, z) \nabla_{\epsilon} \varphi(x, z) = \int dx m D w(x) \cdot \nabla \varphi(x)
\]

(22)
for all \( \varphi \in C^1_c(\mathbb{R}^d) \). We say that family \( \{ \nabla_\varepsilon v_\varepsilon \} \) converges strongly to \( w \in L^2(mdx)^d \), and write \( \nabla_\varepsilon v_\varepsilon \rightarrow w \), if in addition to \( (21) \) it holds

\[
\lim_{\varepsilon \downarrow 0} \int dv_\varepsilon(x, z) \nabla_\varepsilon v_\varepsilon(x, z) \cdot \nabla g(x) = \int dx mDw(x) \cdot \nabla g(x),
\]

for any family of functions \( g_\varepsilon \in L^2(\mu^\varepsilon_\omega) \) with \( g_\varepsilon \rightarrow g \in L^2(dx) \) such that \( g_\varepsilon \in H^1_{\omega, \varepsilon} \) and \( g \in H^1(dx) \).

Due to Lemma 15.1 in Section 15 for \( \mathcal{P}_0 \)-a.a. \( \omega \), and in particular for all \( \omega \in \Omega_{\text{typ}} \) defined below, any function \( \varphi \in C^1_c(\mathbb{R}^d) \) has the property that \( \varphi_\varepsilon \in H^1_{\omega, \varepsilon} \) and \( L^2(\mu^\varepsilon_\omega) \ni \varphi_\varepsilon \rightarrow \varphi \in L^2(mdx) \), where \( \varphi_\varepsilon \) denotes the restriction of \( \varphi \) to \( \varepsilon \omega \). In particular, for such environments \( \omega \in \Omega_{\text{typ}} \), if \( \nabla_\varepsilon v_\varepsilon \rightarrow w \) then \( \nabla_\varepsilon v_\varepsilon \rightarrow w \).

### 3.3. Difference operators

We consider the operator \( L^\varepsilon_\omega \) defined as

\[
L^\varepsilon_\omega f(\varepsilon a) := \varepsilon^{-2} \int d\omega(y) c_{a,y}(\omega) (f(\varepsilon y) - f(\varepsilon a)), \quad \varepsilon a \in \varepsilon \omega,
\]

for functions \( f : \varepsilon \omega \rightarrow \mathbb{R} \) for which the series in the r.h.s. is absolutely convergent for each \( a \in \varepsilon \omega \). Note that this property is fulfilled if e.g. \( f \) has compact support. Moreover, if \( f, g \) have compact support, then the scalar product \( \langle -L^\varepsilon_\omega f, g \rangle_{\mu^\varepsilon_\omega} \) in \( L^2(\mu^\varepsilon_\omega) \) is well defined (indeed, one deals only with finite sums) and it holds

\[
\langle -L^\varepsilon_\omega f, g \rangle_{\mu^\varepsilon_\omega} = \frac{\varepsilon^{d-2}}{2} \int d\hat{\omega}(da) \int d\hat{\omega}(dy) c_{a,y}(\omega) [f(\varepsilon y) - f(\varepsilon a)] [g(\varepsilon y) - g(\varepsilon a)]
\]

\[
= \frac{1}{2} \langle \nabla_\varepsilon f, \nabla_\varepsilon g \rangle_{\nu^\varepsilon_\omega}.
\]

The above identity suggests a weak formulation of the equation \(-L^\varepsilon_\omega u + \lambda u = f\):

**Definition 3.6.** Given \( f \in L^2(\mu^\varepsilon_\omega) \) and \( \lambda > 0 \), a weak solution \( u \) of the equation

\[
-L^\varepsilon_\omega u + \lambda u = f
\]

is a function \( u \in H^1_{\omega, \varepsilon} \), such that

\[
\frac{1}{2} \langle \nabla_\varepsilon v, \nabla_\varepsilon u \rangle_{\mu^\varepsilon_\omega} + \lambda \langle v, u \rangle_{\mu^\varepsilon_\omega} = \langle v, f \rangle_{\mu^\varepsilon_\omega} \quad \forall v \in H^1_{\omega, \varepsilon}.
\]

By the Lax–Milgram theorem [4], given \( f \in L^2(\mu^\varepsilon_\omega) \) the weak solution \( u \) of \( (25) \) exists and is unique.

We now move to the effective equation, where \( D \) denotes the diffusion matrix introduced in [10].

**Definition 3.7.** Given \( f \in L^2(dx) \) and \( \lambda > 0 \), a weak solution \( u \) of the equation

\[
-\text{div} \nabla_\omega u + \lambda u = f
\]
is a function $u \in H^1(dx)$ such that
\[
\int D\nabla v(x) \cdot \nabla u(x) dx + \lambda \int v(x)u(x) dx = \int v(x)f(x) dx, \quad \forall v \in H^1(dx).
\]

(28)

We point out that the gradient $\nabla$ in (28) is the usual weak gradient. Again, by the Lax–Milgram theorem, given $f \in L^2(dx)$ the weak solution $u$ of (25) exists and is unique.

We can now state our first main results on homogenization:

**Theorem 1.** There exists a Borel subset $\Omega_{typ} \subset \Omega$, of so called typical environments, fulfilling the following properties. $\Omega_{typ}$ is translation invariant and $\mathcal{P}(\Omega_{typ}) = 1$. Moreover, given any $\lambda > 0$, $f_\varepsilon \in L^2(\mu_\varepsilon)$ and $f \in L^2(dx)$, let $u_\varepsilon$ and $u$ be defined as the weak solutions, respectively in $H^1_{\omega,\varepsilon}$ and $H^1(dx)$, of the equations
\[
-\mathbb{L}_\omega u_\varepsilon + \lambda u_\varepsilon = f_\varepsilon, \quad (29)
\]
\[
-\text{div} D\nabla u + \lambda u = f. \quad (30)
\]

Then, for any $\omega \in \Omega_{typ}$, we have:

(i) **Convergence of solutions** (cf. Def. 3.2):
\[
f_\varepsilon \rightharpoonup f \implies u_\varepsilon \to u, \quad (31)
\]
\[
f_\varepsilon \to f \implies u_\varepsilon \to u. \quad (32)
\]

(ii) **Convergence of flows** (cf. Def. 3.5):
\[
f_\varepsilon \to f \implies \nabla_\varepsilon u_\varepsilon \to \nabla u, \quad (33)
\]
\[
f_\varepsilon \to f \implies \nabla_\varepsilon u_\varepsilon \to \nabla u. \quad (34)
\]

(iii) **Convergence of energies**:
\[
f_\varepsilon \to f \implies \langle \nabla u_\varepsilon, \nabla u \rangle_{\mu_\varepsilon} \to \int dx m \nabla u(x) \cdot D\nabla u(x). \quad (35)
\]

**Remark 3.8.** Let $\omega \in \Omega_{typ}$. Then it is trivial to check that, for any $f \in C_c(\mathbb{R}^d)$, $L^2(\mu_\omega) \ni f \to f \in L^2(dx)$. By taking $f_\varepsilon := f$ and using (32), we get that $u_\varepsilon \to u$, where $u_\varepsilon$ and $u$ are defined as the weak solutions of (29) and (30), respectively.

Given $\omega \in \Omega_{typ}$, we write $(P_{\omega,t}^\varepsilon)_{t \geq 0}$ for the $L^2(\mu_\omega)$–Markov semigroup associated to the random walk on $\varepsilon \hat{\omega}$ with probability rate for a jump from $\varepsilon x$ to $\varepsilon y$ given by $e^{-2c_{\varepsilon,x,y}(\omega)}$. In particular, $P_{\omega,t}^\varepsilon = e^{t\varepsilon D}$. Similarly we write $(P_t)_{t \geq 0}$ for the Markov semigroup on $L^2(dx)$ associated to the Brownian motion on $\mathbb{R}^d$ with diffusion matrix $\mathcal{D}$.

**Theorem 2.** Take $\omega \in \Omega_{typ}$ and $f \in C_c(\mathbb{R}^d)$. Then it holds
\[
L^2(\mu_\omega) \ni P_{\omega,t}^\varepsilon f \to P_t f \in L^2(dx). \quad (36)
\]

For each $k \in \mathbb{Z}^d$ define the random variable $N_k(\omega)$ as $N_k(\omega) := \hat{\omega}(k + [0,1)^d)$. Suppose that at least one of the following conditions is satisfied:
\[\begin{align*}
\text{(i) for } P\text{-a.a. } \omega \exists C(\omega) > 0 \text{ such that } \sup_{k \in \mathbb{Z}^d} N_k(\omega) \leq C(\omega); \\
\text{(ii) there exists } C_0 \geq 0, \text{ and there exists } \alpha > 0 \text{ if } d = 1, \text{ such that} \\
\text{Cov}(N_k, N'_k) \leq \begin{cases} 
C_0|k-k'|^{-1} & \text{for } d \geq 2 \\
C_0|k-k'|^{-1-\alpha} & \text{for } d = 1
\end{cases}
\end{align*}\]

for any \( k \neq k' \in \mathbb{Z}^d \).

Then there exists a Borel set \( \Omega_d \subset \Omega \) with \( P(\Omega_d) = 1 \) such that for any \( \omega \in \Omega_d \cap \Omega_{\text{yp}} \) and any \( f \in C_c(\mathbb{R}^d) \) it holds:

\[
\lim_{\varepsilon \downarrow 0} \int \left| P_{e,t}^\varepsilon f(x) - P_t f(x) \right|^2 d\mu_\omega(x) = 0, \tag{37}
\]

\[
\lim_{\varepsilon \downarrow 0} \int \left| P_{e,t}^\varepsilon f(x) - P_t f(x) \right| d\mu_\omega(x) = 0. \tag{38}
\]

Note that any marked Poisson point process satisfies the above Condition (ii), while any marked diluted lattice satisfies Condition (i).

Theorem 2 is obtained from Theorem 1 by adapting some arguments from [12]. The proof is given in Section 16.

3.4. Hydrodynamic limit of simple exclusion processes. We discuss here how the above homogenization results together with the strategy in [11, 12] can be applied to obtain the hydrodynamic limit of exclusion processes [19] on marked simple point processes. The method used to get Theorem 3 is rather general and can be used to treat other exclusion processes.

Given \( \omega \in \Omega \) we consider the exclusion process on \( \hat{\omega} \) with formal generator

\[
L_{\omega} f(\eta) = \sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega}} c_{x,y}(\omega) \eta_x (1 - \eta_y) \left[ f(\eta^{x,y}) - f(\eta) \right], \quad \eta \in \{0,1\}^{\hat{\omega}}, \tag{39}
\]

where

\[
\eta^{x,y}_z = \begin{cases} 
\eta_y & \text{if } z = x, \\
\eta_x & \text{if } z = y, \\
\eta_z & \text{otherwise}.
\end{cases}
\]

Given a probability measure \( m \) on \( \{0,1\}^{\hat{\omega}} \), we write \( P_{\omega,m} \) for the above exclusion process with initial distribution \( m \) and we write \( \eta(t) \) for the particle configuration at time \( t \).

**Theorem 3.** Consider an ergodic stationary marked simple point process \( P \) on \( \mathbb{R}^d \) such that the law of its spatial support \( \hat{\omega} \) is a Poisson point process with intensity \( m > 0 \). Take jump rates satisfying assumptions (A4)–(A8) and such that, for \( P\text{-a.a. } \omega \), \( c_{x,y}(\omega) \leq g(|x - y|) \) for any \( x, y \in \hat{\omega} \), where \( g(|x|) \) is a fixed bounded function in \( L^1(dx) \) (for example take Mott v.r.h.). Then for \( P\text{-a.a. } \omega \) the exclusion process is well defined for any initial distribution and the following holds:

Let \( \rho_0 : \mathbb{R}^d \to [0,1] \) be a Borel function and let \( \{m_\varepsilon\} \) be an \( \varepsilon \)-parametrized family of probability measures on \( \{0,1\}^{\hat{\omega}} \) such that, for all \( \delta > 0 \) and all \( \varphi \in \mathbb{R}^d \),
$C_c(\mathbb{R}^d)$, it holds
\[
\lim_{\epsilon \downarrow 0} m_x \left( \left| \epsilon^d \sum_{x \in \hat{\omega}} \varphi(\epsilon x) \eta_x - \int_{\mathbb{R}^d} \varphi(x) \rho_0(x) dx \right| > \epsilon \right) = 0. \tag{40}
\]
Then for all $t > 0$, $\varphi \in C_c(\mathbb{R}^d)$ and $\delta > 0$ we have
\[
\lim_{\epsilon \downarrow 0} \mathbb{P}_{\omega,m_x} \left( \left| \epsilon^d \sum_{x \in \hat{\omega}} \varphi(\epsilon x) \eta_x (\epsilon^{-2} t) - \int_{\mathbb{R}^d} \varphi(x) \rho(x,t) dx \right| > \delta \right) = 0, \tag{41}
\]
where $\rho : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ solves the heat equation $\partial_t \rho = \nabla \cdot (D \nabla \rho)$ with boundary condition $\rho_0$ at $t = 0$ and where $D$ is the effective diffusion matrix given by (10), which is symmetric and strictly positive definite.

The proof of the above theorem is given in Section 17.

4. Preliminary facts on the Palm distribution $\mathcal{P}_0$

Lemma 4.1. Given a Borel subset $A \subset \Omega_0$, the following facts are equivalent:

(i) $\mathcal{P}_0(A) = 1$;
(ii) $\mathcal{P}(\omega \in \Omega : \tau_x \omega \in A \forall x \in \hat{\omega}) = 1$;
(iii) $\mathcal{P}_0(\omega \in \Omega_0 : \tau_y \omega \in A \forall x \in \hat{\omega}) = 1$.

Proof. By (11) (ii) implies (i). If (i) holds, by Campbell’s identity (3) with $f(x, \omega) := (2\ell)^{-d} 1_{[-\ell,\ell]^d}(x) 1_A(\omega)$ and $\ell > 0$, we get
\[
1 = \mathcal{P}_0(A) = \frac{1}{m(2\ell)^d} \int_{\Omega} \mathcal{P}(d\omega) \int_{[-\ell,\ell]^d} d\hat{\omega}(x) 1_A(\tau_x \omega). \tag{42}
\]
Hence, we obtain
\[
\int_{\Omega} \mathcal{P}(d\omega) \int_{[-\ell,\ell]^d} d\hat{\omega}(x) (1 - 1_A(\tau_x \omega)) = 0 \quad \forall \ell > 0, \tag{43}
\]
which implies (ii). This proves that (i) implies (ii).

Since $0 \in \hat{\omega}$ for all $\omega \in \Omega_0$, we have $\tilde{A} := \{ \omega \in \Omega : \tau_y \omega \in A \forall x \in \hat{\omega} \} \subset A$ and therefore (iii) implies (i). Suppose now that (i) is satisfied, equivalently that (ii) is satisfied, i.e. $\mathcal{P}(\tilde{A}) = 1$. We want to prove that (iii) holds. Trivially, if $\omega \in \tilde{A}$, then $\tau_y \omega \in \tilde{A}$ for any $y \in \hat{\omega}$. Due to this observation and (ii), we conclude that $\mathcal{P}(\omega \in \Omega : \tau_y \omega \in \tilde{A} \forall x \in \hat{\omega}) = 1$. Due to the already proved equivalence between (i) and (ii) (applied now with $\tilde{A}$ instead of $A$), we conclude that $\mathcal{P}_0(\tilde{A}) = 1$, which corresponds to Item (iii). \hfill \Box

Lemma 4.2. Let $f \in L^1(\mathcal{P}_0)$. Let $B := \{ \omega \in \Omega : |f(\tau_x \omega)| < +\infty \forall x \in \hat{\omega} \}$. Then $B$ is translation invariant, $\mathcal{P}(B) = 1$ and $\mathcal{P}_0(B) = 1$.

Proof. We define $A := \{ \omega \in \Omega_0 : |f(\omega)| < +\infty \}$. Since $f \in L^1(\mathcal{P}_0)$, we have $\mathcal{P}_0(A) = 1$. By applying Lemma 4.1, we get that $\mathcal{P}(B) = 1$ and $\mathcal{P}_0(B) = 1$. The translation invariance of $B$ follows immediately from the definition of $B$. \hfill \Box

In what follows we will use the following properties of the Palm distribution $\mathcal{P}_0$ obtained by extending [16] Lemma 1–(i):
Lemma 4.3. Let $k : \Omega_0 \times \Omega_0 \to \mathbb{R}$ be a Borel function such that (i) at least one of the functions $\int d\tilde{\omega}(x)|k(\omega, \tau_x \omega)|$ and $\int d\tilde{\omega}(x)|k(\tau_x \omega, \omega)|$ is in $L^1(\mathcal{P}_0)$, or (ii) $k(\omega, \omega') \geq 0$. Then

$$\int d\mathcal{P}_0(\omega) \int d\tilde{\omega}(x) k(\omega, \tau_x \omega) = \int d\mathcal{P}_0(\omega) \int d\tilde{\omega}(x) k(\tau_x \omega, \omega).$$

Proof. Case (i) with both functions in $L^1(\mathcal{P}_0)$ corresponds to [16] Lemma 1–(i). We now consider case (ii). Given $n \in \mathbb{N}$ we define $k_n(\omega, \omega')$ as

$$k_n(\omega, \omega') := \begin{cases} k(\omega, \omega') & \text{if } \omega' = \tau_x \omega \text{ for some } x \text{ with } |x| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, given $\omega$, $0 \leq k_n(\omega, \tau_x \omega) \not\supset k(\omega, \tau_x \omega)$ and $0 \leq k_n(\tau_x \omega, \omega) \not\supset k(\tau_x \omega, \omega)$. By monotone convergence to get (14) it is enough to prove the same identity with $k_n$ instead of $k$. To this aim we observe that, due to Assumption (A3) and Lemma 2.1 (cf. (11)), for $\mathcal{P}_0$–a.a. $\omega$ we have $k_n(\omega, \tau_x \omega) = 0 = k(\tau_x \omega, \omega)$ if $|x| > n$. Hence, we can bound both $\mathbb{E}_0[\int d\tilde{\omega}(x)k_n(\omega, \tau_x \omega)]$ and $\mathbb{E}_0[\int d\tilde{\omega}(x)k_n(\tau_x \omega, \omega)]$ by $n\mathbb{E}_0[\int d\tilde{\omega}(x)1_{|x| \leq n}] < +\infty$. At this point, (11) with $k_n$ instead of $k$ follows from (14) proved in case (i) under the condition that both the functions considered in case (i) belong to $L^1(\mathcal{P}_0)$. This concludes the proof of (14) in case (ii). We can now prove the thesis in case (i) in full generality. Indeed, by (14) proved in case (ii), for any function $k(\omega, \omega')$ we have that $\int d\tilde{\omega}(x)|k(\omega, \tau_x \omega)| = \int d\tilde{\omega}(x)|k(\tau_x \omega, \omega)|$. Hence, if one of these two integrals is finite, then both are finite since equal. \qed

5. Space of square integrable forms

We define $\nu$ as the Radon measure on $\Omega \times \mathbb{R}^d$ such that

$$\int d\nu(\omega, z) g(\omega, z) = \int d\mathcal{P}_0(\omega) \int d\tilde{\omega}(z) c_{0,z}(\omega) g(\omega, z)$$

for any nonnegative Borel function $g(\omega, z)$. We point out that, by Assumption (A5), $\nu$ has finite total mass: $\nu(\Omega \times \mathbb{R}^d) = \mathbb{E}_0[\lambda_0] < +\infty$. Elements of $L^2(\nu)$ are called square integrable forms.

Given a function $u : \Omega_0 \to \mathbb{R}$ we define the function $\nabla u : \Omega \times \mathbb{R}^d \to \mathbb{R}$ as

$$\nabla u(\omega, z) := u(\tau_z \omega) - u(\omega).$$

Note that if $u, f : \Omega_0 \to \mathbb{R}$ are such that $u = f \; \mathcal{P}_0$–a.s., then $\nabla u = \nabla f \; \nu$–a.s. Indeed, by Lemma 4.1 setting $A := \{ \omega \in \Omega_0 : u(\omega) = f(\omega) \}$ and $\hat{A} := \{ \omega \in \Omega_0 : u(\tau_z \omega) = f(\tau_z \omega) \; \forall z \in \hat{\omega} \}$, we have that $\mathcal{P}_0(A) = 1$ and therefore $\mathcal{P}_0(\hat{A}) = 1$, thus implying that $\nabla u = \nabla f \; \nu$–a.s.. In particular, if $u$ is defined only $\mathcal{P}_0$–a.s., then $\nabla u$ is well defined $\nu$–a.s.

If $u$ is bounded and measurable, then $\nabla u \in L^2(\nu)$. The subspace of potential forms $L^2_{\text{pot}}(\nu)$ is defined as the following closure in $L^2(\nu)$:

$$L^2_{\text{pot}}(\nu) := \{ \nabla u : u \text{ is bounded and measurable} \}.$$
The subspace of solenoidal forms $L^2_{\text{sol}}(\nu)$ is defined as the orthogonal complement of $L^2_{\text{pot}}(\nu)$ in $L^2(\nu)$.

### 5.1. The subspace $H^1_{\text{env}}$

We define

$$H^1_{\text{env}} := \{u \in L^2(\mathcal{P}_0) : \nabla u \in L^2(\nu)\}.$$  \hfill (47)

We endow $H^1_{\text{env}}$ with the norm

$$\|u\|_{H^1_{\text{env}}} := \|u\|_{L^2(\mathcal{P}_0)} + \|\nabla u\|_{L^2(\nu)}.$$  

It is convenient to introduce also the space

$$H_{\text{env}} := \{(u, \nabla u) : u \in L^2(\mathcal{P}_0) \text{ and } \nabla u \in L^2(\nu)\} \subset L^2(\mathcal{P}_0) \times L^2(\nu)$$

endowed with the norm $\|(u, \nabla u)\|_{H_{\text{env}}} := \|u\|_{L^2(\mathcal{P}_0)} + \|\nabla u\|_{L^2(\nu)}$. In particular, $H^1_{\text{env}}$ and $H_{\text{env}}$ are isomorphic spaces.

**Lemma 5.1.** The space $H_{\text{env}}$ is a closed subspace of $L^2(\mathcal{P}_0) \times L^2(\nu)$, hence $H^1_{\text{env}}$ and $H_{\text{env}}$ are Hilbert spaces.

**Proof.** Suppose that $(u_n, \nabla u_n)$ converges to $(u, g)$ in $L^2(\mathcal{P}_0) \times L^2(\nu)$. At cost to extract a subsequence, there exists a Borel $A \subset \Omega$ with $\mathcal{P}_0(A) = 1$ such that the following holds for any $\omega \in A$: $u_n(\omega) \to u(\omega)$ and $u_n(\tau_z \omega) - u_n(\omega) \to g(\omega, z)$ for all $z \in \hat{\omega}$. By Lemma 3.1, since $u_n(\omega) \to u(\omega)$ for all $\omega \in A$, we conclude that $u_n(\tau_z \omega) \to u(\tau_z \omega)$ for all $z \in \hat{\omega}$ for $\mathcal{P}_0$-a.a. $\omega$. As a consequence it must be $g(\omega, z) = u(\tau_z \omega) - u(\omega)$ for all $z \in \hat{\omega}$, for $\mathcal{P}_0$-a.a. $\omega$. This proves that $\nabla u \in L^2(\nu)$ and that $(u_n, \nabla u_n) \to (u, \nabla u)$.

We fix some simple notation which will be useful also later. Given $M > 0$ and $a \in \mathbb{R}$, we define $[a]_M$ as

$$[a]_M = M \mathbb{1}_{\{a > M\}} + a \mathbb{1}_{\{|a| \leq M\}} - M \mathbb{1}_{\{a < -M\}}.$$  \hfill (48)

Given $a \geq b$, it holds $a - b \geq [a]_M - [b]_M \geq 0$. Hence, for any $a, b \in \mathbb{R}$, it holds

$$\|[a]_M - [b]_M\| \leq |a - b|,$$  \hfill (49)

$$|[a - b] - ([a]_M - [b]_M)| \leq |a - b|.$$  \hfill (50)

**Lemma 5.2.** The subspace $\{(u, \nabla u) : u \text{ is bounded and measurable}\}$ is a dense subspace of $H_{\text{env}}$.

**Proof.** Let $u$ be bounded and measurable. We have $u \in H^1_{\text{env}}$. Let us take now a generic $u \in H^1_{\text{env}}$ and show that $|u|_M \to u$ in $H^1_{\text{env}}$. Since $|u - |u|_M| \leq |u|$, by dominated convergence we have that $\|u - |u|_M\|_{L^2(\mathcal{P}_0)} \to 0$. Due to (51) and dominated convergence we get $\|\nabla u - \nabla |u|_M\|_{L^2(\nu)} \to 0$.

Given $f \in L^2(\mathcal{P}_0)$ we consider the equation

$$-\text{div}\nabla u + u = f$$  \hfill (51)

in its weak form on the Hilbert space $H^1_{\text{env}}$: an element $u \in H^1_{\text{env}}$ is a weak solution of (51) if for any $v \in H^1_{\text{env}}$ it holds

$$\int d\nu \nabla u \nabla v + \int d\mathcal{P}_0 uv = \int d\mathcal{P}_0 fv.$$  \hfill (52)
Since $H_{\text{env}}^1$ is a Hilbert space, by the Lax–Milgram theorem, equation (51) has a unique solution $u \in H_{\text{env}}^1$.

5.2. Divergence.

**Definition 5.3.** Given a square integrable form $v \in L^2(\nu)$ we define its divergence $\text{div} v \in L^1(\mathcal{P}_0)$ as

$$\text{div} v(\omega) = \int d\hat{\omega}(z)c_{0,z}(\omega)(v(\omega, z) - v(\tau_z\omega, -z)) \quad (53)$$

By applying Lemma 4.3 with $k(\omega, \tau_z\omega) := c_{0,z}(\omega)|v(\omega, z)|$, Schwarz inequality and (7), one gets for any $v \in L^2(\nu)$ that

$$\int d\mathcal{P}_0(\omega) \int d\hat{\omega}(z)c_{0,z}(\omega)(|v(\omega, z)| + |v(\tau_z\omega, -z)|)$$

$$= 2\| v \|_{L^1(\nu)} \leq 2\mathbb{E}_0[\lambda_0]^{1/2}\| v \|_{L^2(\nu)} < +\infty \quad (54)$$

In particular, the definition of divergence is well posed and the map $L^2(\nu) \ni v \mapsto \text{div} v \in L^1(\mathcal{P}_0)$ is continuous.

By applying Lemma 4.3 with $k(\omega, \tau_z\omega) := c_{0,z}(\omega)v(\omega, x)u(\tau_z\omega)$ (and therefore $k(\tau_z\omega, \omega) = c_{0,z}(\omega)v(\tau_z\omega, -x)u(\omega)$ by (A3)), one easily gets the following:

**Lemma 5.4.** For any $v(\omega, z) \in L^2(\nu)$ and any bounded and measurable function $u : \Omega_0 \to \mathbb{R}$ it holds

$$\int d\mathcal{P}_0(\omega)\text{div} v(\omega)u(\omega) = -\int dv(\omega, z)v(\omega, z)\nabla u(\omega, z) \quad (55)$$

Trivially, the above result implies the following:

**Corollary 5.5.** Given a square integrable form $v \in L^2(\nu)$, we have $v \in L^2_{\text{sol}}(\nu)$ if and only if $\text{div} v = 0 : \mathcal{P}_0$–a.s.

We recall that, since $\mathcal{P}$ is ergodic, then $\mathcal{P}_0$ is ergodic w.r.t. point–shifts. As a consequence, $u = \text{constant} : \mathcal{P}_0$–a.s. if $u : \Omega_0 \to \mathbb{R}$ is a Borel function such that for $\mathcal{P}_0$–a.a. $\omega$ it holds $u(\omega) = u(\tau_z\omega)$ for all $x \in \hat{\omega}$.

**Remark 5.6.** Due to Assumption (A7), given $u$ with $\nabla u = 0 \nu$–a.s., it holds $u(\omega) = u(\tau_z\omega)$ for all $x \in \hat{\omega}$, for $\mathcal{P}_0$–a.a. $\omega$. Due to the ergodicity of $\mathcal{P}_0$ we conclude that $u = \text{constant} : \mathcal{P}_0$–a.s. if $\nabla u = 0 \nu$–a.s.

The proof of the following fact uses ideas from the proof of [38, Lemma 2.5] and is given in Appendix A for completeness (recall (47)):

**Lemma 5.7.** Let $\zeta \in L^2(\mathcal{P}_0)$ be orthogonal to all functions $g \in L^2(\mathcal{P}_0)$ with $g = \text{div} (\nabla u)$ for some $u \in H_{\text{env}}^1$. Then $\zeta \in H_{\text{env}}^1$ and $\nabla \zeta = 0$ in $L^2(\nu)$.

By combining Remark 5.6 and Lemma 5.7 we get:

**Lemma 5.8.** The functions $g \in L^2(\mathcal{P}_0)$ of the form $g = \text{div} v$ with $v \in L^2(\nu)$ are dense in $\{ w \in L^2(\mathcal{P}_0) : \mathbb{E}_0[w] = 0 \}$. 
Proof. Lemma 5.4 implies that $\mathbb{E}_0[g] = 0$ if $g = \text{div} \, v$, $v \in L^2(\nu)$. Suppose the density fails. Then there exists $\zeta \in L^2(P_0)$ different from zero with $\mathbb{E}_0[\zeta] = 0$ and such that $\mathbb{E}_0[\zeta g] = 0$ for any $g \in L^2(P_0)$ of the form $g = \text{div} \, v$ with $v \in L^2(\nu)$. By Lemma 5.4 we know that $\zeta \in H^1_{\text{div}}$ and $\nabla \zeta = 0 \, \nu$–a.s. By Remark 5.6 we get that $\zeta$ is constant $P_0$–a.s. Since $\mathbb{E}_0[\zeta] = 0$ it must be $\zeta = 0$ $P_0$–a.s., which is absurd. \hfill \Box

6. The Diffusion Matrix $D$ and the Quadratic Form $q$

Since $\lambda_2 \in L^1(P_0)$ (see Assumption (A5)), given $a \in \mathbb{R}^d$ the form

$$u_a(\omega, z) := a \cdot z$$

(56)

is square integrable (i.e. it belongs to $L^2(\nu)$). We note that the symmetric diffusion matrix $D$ defined in (11) satisfies, for any $a \in \mathbb{R}^d$,

$$q(a) := a \cdot Da = \inf_{v \in L^2_{\text{pot}}(\nu)} \frac{1}{2} \int dv(\omega, x) \left( u_a(x) + v(\omega, x) \right)^2$$

$$= \inf_{v \in L^2_{\text{pot}}(\nu)} \frac{1}{2} \left\| u_a + v \right\|_{L^2(\nu)}^2 = \frac{1}{2} \left\| u_a + v^a \right\|_{L^2(\nu)}^2,$$

(57)

where $v^a = -\Pi u_a$ and $\Pi : L^2(\nu) \to L^2_{\text{pot}}(\nu)$ denotes the orthogonal projection of $L^2(\nu)$ on $L^2_{\text{pot}}(\nu)$. Note that, as a consequence, the map $\mathbb{R}^d \ni a \mapsto v^a \in L^2_{\text{pot}}(\nu)$ is linear. Moreover, $v^a$ is characterized by the property

$$v^a \in L^2_{\text{pot}}(\nu), \quad v^a + u_a \in L^2_{\text{sol}}(\nu).$$

(58)

As a consequence we can write

$$a \cdot Da = \frac{1}{2} \left\| u_a + v^a \right\|_{L^2(\nu)}^2 = \frac{1}{2} \langle u_a, u_a + v^a \rangle_{L^2(\nu)}.$$

(59)

The above identity can be rewritten as

$$a \cdot Da = \frac{1}{2} \int dv(\omega, z)a \cdot z \left( a \cdot z + v^a(\omega, z) \right).$$

(60)

Since the two symmetric bilinear forms $(a, b) \mapsto a \cdot Db$ and

$$(a, b) \mapsto \frac{1}{2} \int dv(\omega, z) a \cdot z \left( b \cdot z + v^b(\omega, z) \right) = \frac{1}{2} \int dv(u_a + v^a)(u_b + v^b)$$

coincide on diagonal terms by (60), we conclude that

$$Da = \frac{1}{2} \int dv(\omega, z) z \left( a \cdot z + v^a(\omega, z) \right) \quad \forall a \in \mathbb{R}^d.\,$$

(61)

Let us come back to the quadratic form $q$ on $\mathbb{R}^d$ defined in (57). By (57) its kernel $\text{Ker}(q)$ is given by

$$\text{Ker}(q) := \{ a \in \mathbb{R}^d : q(a) = 0 \} = \{ a \in \mathbb{R}^d : u_a \in L^2_{\text{pot}}(\nu) \}.$$

(62)

Lemma 6.1. It holds

$$\text{Ker}(q)^\perp = \{ \omega \mapsto \int dv(\omega, z)b(\omega, z)z : b \in L^2_{\text{sol}}(\nu) \}.$$

(63)
Lemma 7.2. Let $\lambda_2 \in L^1(\mathcal{P}_0)$ by (A5), the integral in the r.h.s. of (55) is well defined. The above lemma corresponds to 38, Prop. 5.1].

Proof. Let $b \in L^2_{sol}(\nu)$ and $\eta_b := \int d\nu(\omega,z) b(\omega, z)$. Then, given $a \in \mathbb{R}^d$, $a \cdot \eta_b = \langle u_a, b \rangle_{L^2(\nu)}$. As a consequence, we have that $a \in \text{Ker}(q)$ if and only if $u_a \in L^2_{sol}(\nu) = L^2_{sol}(\nu)^\perp$ if and only if $a \cdot \eta_b = 0$ for any $b \in L^2_{sol}(\nu)$. \hfill $\square$

7. The contraction $b(\omega, z) \mapsto \hat{b}(\omega)$ and the set $\mathcal{A}_1[b]$

Definition 7.1. Let $b(\omega, z) : \Omega_0 \times \mathbb{R}^d \to \mathbb{R}$ be a Borel function with $\|b\|_{L^1(\nu)} < +\infty$. We define the Borel function $c_b : \Omega_0 \to [0, +\infty]$ as
\[
c_b(\omega) := \int d\hat{\omega}(z)c_{0,\hat{z}}(\omega)|b(\omega, z)|, \tag{64}
\]
the Borel function $\hat{b} : \Omega_0 \to \mathbb{R}$ as
\[
\hat{b}(\omega) := \begin{cases} 
\int d\hat{\omega}(z)c_{0,\hat{z}}(\omega)b(\omega, z) & \text{if } c_b(\omega) < +\infty, \\
0 & \text{if } c_b(\omega) = +\infty,
\end{cases} \tag{65}
\]
and the Borel set $\mathcal{A}_1[b] := \{\omega \in \Omega_0 : c_b(\tau_x \omega) < +\infty \forall x \in \hat{\omega}\}$.

Lemma 7.2. Let $b(\omega, z) : \Omega_0 \times \mathbb{R}^d \to \mathbb{R}$ be a Borel function with $\|b\|_{L^1(\nu)} < +\infty$. Then
\begin{enumerate}
  \item $\|\hat{b}\|_{L^1(\mathcal{P}_0)} \leq \|b\|_{L^1(\nu)} = \|c_b\|_{L^1(\mathcal{P}_0)}$ and $\mathbb{E}_0[\hat{b}] = \nu(b)$;
  \item given $\omega \in \mathcal{A}_1[b]$ and $\varphi \in C_c(\mathbb{R}^d)$, it holds
\[
\int d\mu_\omega^\omega(\tau_x, \omega)\varphi(x)\hat{b}(\tau_x, \omega) = \int d\nu_\omega^\omega(x, z)\varphi(x)b(\tau_x, \omega, z) \tag{66}
\]
(the series in the l.h.s. and in the r.h.s. are absolutely convergent);
  \item $\mathcal{P}_0(\mathcal{A}_1[b]) = 1$.
\end{enumerate}

Proof. It is trivial to check Item (i). Let us prove Item (ii). Since the integral in the l.h.s. of (66) is a finite sum, it is absolutely convergent. Let us prove that the integral in the r.h.s. of (66) corresponds to an absolutely convergent series. We have
\[
\int d\nu_\omega^\omega(x, z)|\varphi(x)b(\tau_x, \omega, z)| = \int d\mu_\omega^\omega(x)|\varphi(x)|c_b(\tau_x \omega).
\]
Since $\omega \in \mathcal{A}_1[b]$, the last term is a finite sum of (finite) positive numbers, thus implying that the r.h.s. of (66) is an absolutely convergent series. As a consequence we can arbitrarily arrange the terms in the series, without changing the final value. Hence, as above, we get (66).

We move to Item (iii). We have $\mathbb{E}_0[c_b] = \|b\|_{L^1(\nu)} < \infty$. This implies that $\mathcal{P}_0(\{\omega : c_b(\omega) < +\infty\}) = 1$ and therefore $\mathcal{P}_0(\mathcal{A}_1[b]) = 1$ by Lemma 7.2. \hfill $\square$

We point out that, since $\nu$ has finite mass, $L^2(\nu) \subset L^1(\nu)$ and therefore Lemma [7.2] can be applied to $b$ with $\|b\|_{L^2(\nu)} < +\infty$. 

Note that, since $\lambda_2 \in L^1(\mathcal{P}_0)$ by (A5), the integral in the r.h.s. of (66) is well defined. The above lemma corresponds to 38, Prop. 5.1].
8. The transformation \( b(\omega, z) \mapsto \tilde{b}(\omega, z) \)

**Lemma 8.1.** Given a Borel function \( b: \Omega_0 \times \mathbb{R}^d \to \mathbb{R} \) we set

\[
\tilde{b}(\omega, z) := \begin{cases} 
    b(\tau_z \omega, -z) & \text{if } z \in \hat{\omega}, \\
    0 & \text{otherwise}.
\end{cases}
\]  

(67)

Then \( \tilde{b}(\omega, z) = b(\omega, z) \) if \( z \in \hat{\omega} \). If \( b \in L^2(\nu) \), then \( \| b \|_{L^2(\nu)} = \| \tilde{b} \|_{L^2(\nu)} \) and \( \text{div} \tilde{b} = -\text{div} b \).

**Proof.** Let \( b \in L^2(\nu) \). We apply Lemma 4.3 with

\[
k(\omega, \omega') := \begin{cases} 
    c_{0, z}(\omega) b^2(\omega, z) & \text{if } \omega' = \tau_z \omega \text{ and } z \in \hat{\omega}, \\
    0 & \text{otherwise}.
\end{cases}
\]  

(68)

Due to Assumption (A3) the above function is well defined for \( \mathcal{P}_0 \)-a.a. \( \omega \). If \( z \in \hat{\omega} \) we have \( k(\tau_z \omega, \omega) = c_{0, -z}(\tau_z \omega) b^2(\tau_z \omega, -z) = c_{0, z}(\omega) \tilde{b}(\omega, z)^2 \). Then Lemma 4.3 implies that \( \| b \|_{L^2(\nu)} = \| \tilde{b} \|_{L^2(\nu)} \). The other identities follow from the definitions. \(\square\)

Recall the set \( A[g] \subset \Omega_0 \) introduced in Prop. 3.1 and recall Def. 7.1.

**Lemma 8.2.**

(i) Let \( b: \Omega_0 \times \mathbb{R}^d \to [0, +\infty) \) and \( \varphi, \psi: \mathbb{R}^d \to [0, +\infty) \) be nonnegative Borel functions. Then it holds

\[
\int dv^\varepsilon_\omega(x, z) \varphi(x) \psi(x + \varepsilon z) b(\tau_{x/\varepsilon} \omega, z) = \int dv^\varepsilon_\omega(x, z) \varphi(x) \psi(x + \varepsilon z) \tilde{b}(\tau_{x/\varepsilon} \omega, z) .
\]

(69)

(ii) Let \( b: \Omega_0 \times \mathbb{R}^d \to \mathbb{R} \) be a Borel function with \( \| b \|_{L^1(\nu)} < +\infty \). Take \( \omega \in A_1[b] \cap A_1[\tilde{b}] \). Given functions \( \varphi, \psi: \mathbb{R}^d \to \mathbb{R} \) such that at least one between \( \varphi, \psi \) has compact support and the other is bounded, identity (69) is still valid. Given now \( \varphi \) with compact support and \( \psi \) bounded, it holds

\[
\int dv^\varepsilon_\omega(x, z) \nabla \psi(x, z) \psi(x + \varepsilon z) b(\tau_{x/\varepsilon} \omega, z) = - \int dv^\varepsilon_\omega(x, z) \nabla \psi(x, z) \psi(x) \tilde{b}(\tau_{x/\varepsilon} \omega, z) .
\]

(70)

Moreover, the above integrals in (69), (70) (for Item (ii)) correspond to absolutely convergent series and are therefore well defined.

**Proof.** We check (69) for Item (ii) (the proof for Item (i) uses similar computations). Since \( c_{a, a'}(\omega) = c_{a', a}(\omega) \) and \( b(\tau_\omega a, a' - a) = \tilde{b}(\tau_{a'} \omega, a - a') \) for all
Since we deal with infinite sums, the above rearrangements have to be justified. We recall that \( \varphi \) has compact support and \( \psi \) is bounded, or viceversa. The same computations as above hold when taking the modulus of all involved functions. To conclude that the series are absolutely convergent we observe that, if \( \varphi \) has compact support, we can bound

\[
\int d\nu^\varepsilon_{\omega}(x, z) |\varphi(x)||b(\tau_{x/\varepsilon} \omega, z)| \leq \int d\mu^\varepsilon_{\omega}(x)|\varphi(x)|c_b(\tau_{x/\varepsilon} \omega) .
\]  

(72)

Since \( \omega \in A_1[b] \) the integral in the r.h.s. corresponds to a finite sum of finite terms, hence the r.h.s of (72) is finite and all the rearrangements in (71) are legal (recall that \( \psi \) is bounded). If \( \psi \) has compact support and \( \varphi \) is bounded, we do similar computations for \( \int d\nu^\varepsilon_{\omega}(x, z) |\psi(x)||\tilde{b}(\tau_{x/\varepsilon} \omega, z)| \) and use that \( \omega \in A_1[\tilde{b}] \).

We now prove (70). We have

\[
\int d\nu^\varepsilon_{\omega}(x, z) \nabla_x \varphi(x, z) \psi(x + \varepsilon z) b(\tau_{x/\varepsilon} \omega, z)
\]

\[
= \varepsilon^d \sum_{a \in \hat{\omega}} \sum_{a' \in \hat{\omega}} c_{a,a'}(\omega) \frac{\varphi(\varepsilon a') - \varphi(\varepsilon a)}{\varepsilon} \psi(\varepsilon a') b(\tau_{a \omega}, a' - a) \]

\[
= -\varepsilon^d \sum_{a' \in \hat{\omega}} \sum_{a \in \hat{\omega}} c_{a',a}(\omega) \frac{\varphi(\varepsilon a) - \varphi(\varepsilon a')}{\varepsilon} \psi(\varepsilon a') \tilde{b}(\tau_{a' \omega}, a - a') \]  

(73)

Since we deal with infinite sums, the above arrangements have to be justified. Indeed, the same computations as above still hold when taking the modulus of the involved functions. To show that all series are absolutely convergent. As \( \psi \) is bounded it is enough to show that

\[
\int d\nu^\varepsilon_{\omega}(x, z) |\varphi(x)||b(\tau_{x/\varepsilon} \omega, z)| < +\infty ,
\]  

(74)

\[
\int d\nu^\varepsilon_{\omega}(x, z) |\varphi(x + \varepsilon z)||b(\tau_{x/\varepsilon} \omega, z)| < +\infty .
\]  

(75)
The term (74) can be treated as in (72) and lines after. Due Item (i), the term (75) equals $\int dv^*_{\omega}(x,z)|\varphi(x)||\hat{b}(\tau_{x/\varepsilon},z)|$ and we are back to the previous case with $\tilde{b}$ instead of $b$. \hfill $\square$

**Definition 8.3.** Let $b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R}$ be a Borel function. If $\omega \in A_1[b] \cap A_1[\tilde{b}]$, we set $\text{div}_b(\omega) := \tilde{b}(\omega) - \hat{b}(\omega)$.

**Lemma 8.4.** Let $b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R}$ be a Borel function with $||b||_{L^2(\nu)} < +\infty$. Then $P_0(A_1[b] \cap A_1[\tilde{b}]) = 1$ and $\text{div}_b = \text{div} \hat{b}$ in $L^1(P_0)$.

**Proof.** By Lemma 7.2–(iii) we have $||\hat{b}||_{L^2(\nu)} < \infty$. Hence, both $b$ and $\tilde{b}$ are $\nu$–integrable. By Lemma 7.2–(iii) we get that $P_0(A_1[b] \cap A_1[\tilde{b}]) = 1$ and $\text{div}_b$ is defined $P_0$–a.s. The identity $\text{div}_b(\omega) := \hat{b}(\omega) - \tilde{b}(\omega)$ follows from the definitions of $\text{div} b$ and $\text{div}_b$.

**Lemma 8.5.** Let $b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R}$ be a Borel function with $||b||_{L^2(\nu)} < +\infty$ and such that its class of equivalence in $L^2(\nu)$ belongs to $L^2_{\text{ad}}(\nu)$. Let

$$A_1[b] := \{\omega \in A_1[b] \cap A_1[\tilde{b}] : \text{div}_b(\tau_{x,z}) = 0 \forall z \in \hat{\omega}\}.$$  

Then $P_0(A_1[b]) = 1$ and $\tau_{x,z} \omega \in A_1[b]$ whenever $z \in \hat{\omega}$ and $\omega \in A_0[b]$.

**Proof.** By Cor. 5.5 and Lemma 8.4 the set $A := \{\omega \in A_1[b] \cap A_1[\tilde{b}] : \text{div}_b(\omega) = 0\}$ has $P_0$–probability equal to 1. By Lemma 4.4 $P_0(A) = 1$ where $A := \{\omega \in \Omega_0 : \tau_{x,z} \omega \in A \forall z \in \hat{\omega}\}$. To get the thesis it is enough to observe that $A = A_1[b]$. \hfill $\square$

**Lemma 8.6.** Suppose that $b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R}$ is a Borel function with $||b||_{L^2(\nu)} < +\infty$. Take $\omega \in A_1[b] \cap A_1[\tilde{b}]$. Then for any $\varepsilon > 0$ and any $u : \mathbb{R}^d \to \mathbb{R}$ with compact support it holds

$$\int d\mu^*_{\omega}(x)u(x)\text{div}_b(\tau_{x/\varepsilon}) = -\varepsilon \int dv^*_{\omega}(x,z)\nabla_z u(x,z)\tilde{b}(\tau_{x/\varepsilon},z).$$  

**Proof.** We can write the l.h.s. of (77) as

$$\int dv^*_{\omega}(x,z)u(x)b(\tau_{x,z}) - \int dv^*_{\omega}(x,z)u(x)\tilde{b}(\tau_{x,z}).$$  

Due to our assumptions we are dealing with absolutely convergent series, hence rearrangements are free. By applying (69) to the r.h.s. of (78) we can rewrite (78) as $\int dv^*_{\omega}(x,z)b(\tau_{x,z})[u(x) - u(x + \varepsilon z)]$ and this allows to conclude. \hfill $\square$

9. **Typical environments**

Consider Prop. 3.1. We stress that the function $g$ appearing there is a given function and not an element of $L^1(P_0)$ (which would be an equivalence class of functions equal $P_0$–a.s.). Indeed, the set $A[g]$ is defined in terms of $g$ and not of its equivalence class in $L^1(P_0)$.

Recall that the space $(\mathcal{N}, d)$ is a Polish space, where $\mathcal{N}$ is given by the counting measures $\mu$ on $\mathbb{R}^d \times S$ (i.e. $\mu$ is an integer–valued measure on the
measurable space \((\mathbb{R}^d \times S, \mathcal{B}(\mathbb{R}^d \times S))\) such that \(\mu\) is bounded on bounded sets, where \(\mathcal{B}(\mathbb{R}^d \times S)\) denotes the family of Borel subsets of \((\mathbb{R}^d \times S)\).

**Remark 9.1.** Since \((\Omega, d)\) is a separable metric space, the same holds for \((\Omega_0, d)\) and \((\Omega_0, d)\). By [4] Theorem 4.13 we then get that the spaces \(L^p(\mathcal{P})\), \(L^p(\mathcal{P}_0)\) are separable for \(1 \leq p < +\infty\). Since the functions \(f(\omega)\varphi(x)\), with \(f, \varphi\) Borel and bounded, span a dense subset of \(L^p(\nu)\), we have that \(L^p(\nu)\) is separable for \(1 \leq p < +\infty\).

Recall the set \(\mathcal{A}[g]\) introduced in Prop. 3.1 and the set \(\mathcal{A}[b]\) introduced in Def. 7.1. Recall definition (48) of measurable space.

- **The functional sets** \(\mathcal{G}_1, \mathcal{H}_1\). We fix a countable set \(\mathcal{H}_1\) of Borel functions \(b : \Omega_0 \times \mathbb{R}^d \rightarrow \mathbb{R}\) such that \(\|b\|_{L^2(\nu)} < +\infty\) for any \(b \in \mathcal{H}_1\) and such that \(\{\text{div } b : b \in \mathcal{H}_1\}\) is a dense subset of \(\{w \in L^2(\mathcal{P}_0) : \mathbb{E}_0[w] = 0\}\) when thought of as set of \(L^2\)-functions (recall Lemma 5.8). For each \(b \in \mathcal{H}_1\) we define the Borel function \(g_b : \Omega_0 \rightarrow \mathbb{R}\) as

\[
g_b(\omega) := \begin{cases} 
\text{div } b(\omega) & \text{if } \omega \in \mathcal{A}_1[b] \cap \mathcal{A}_1[\hat{b}], \\
0 & \text{otherwise.}
\end{cases}
\]

(79)

Note that by Lemma 5.4 \(g_b = \text{div } b \mathcal{P}_0\)-a.s. Finally we set \(\mathcal{G}_1 := \{g_b : b \in \mathcal{H}_1\}\).

- **The functional sets** \(\mathcal{G}_2, \mathcal{H}_2, \mathcal{H}_3\). We fix a countable set \(\mathcal{G}_2\) of bounded Borel functions \(g : \Omega_0 \rightarrow \mathbb{R}\) such that the set \(\{\nabla g : g \in \mathcal{G}_2\}\), thought in \(L^2(\nu)\), is dense in \(L^2_{\text{pot}}(\nu)\) (this is possible by the definition of \(L^2_{\text{pot}}(\nu)\)). We define \(\mathcal{H}_2\) as the set of Borel functions \(h : \Omega_0 \times \mathbb{R}^d \rightarrow \mathbb{R}\) such that \(h = \nabla g\) for some \(g \in \mathcal{G}_2\). We define \(\mathcal{H}_3\) as the set of Borel functions \(h : \Omega_0 \times \mathbb{R}^d \rightarrow \mathbb{R}\) such that \(h(\omega, z) = g(\tau_i \omega)z_i\) for some \(g \in \mathcal{G}_2\) and some direction \(i = 1, \ldots, d\). Note that, since \(\mathbb{E}_0[\lambda_2] < +\infty\) by (A5) and since \(g\) is bounded, \(\|h\|_{L^2(\nu)} < +\infty\) for all \(h \in \mathcal{H}_3\).

- **The functional set** \(\mathcal{W}\). We fix a countable set \(\mathcal{W}\) of Borel functions \(b : \Omega_0 \times \mathbb{R}^d \rightarrow \mathbb{R}\) such that, thought of as subset of \(L^2(\nu)\), \(\mathcal{W}\) is dense in \(L^2_{\text{sol}}(\nu)\). By Cor. 5.3 and Lemma 8.1 \(\hat{b} \in L^2_{\text{sol}}(\nu)\) for any \(b \in L^2_{\text{sol}}(\nu)\). Hence, at cost to enlarge \(\mathcal{W}\), we assume that \(\hat{b} \in \mathcal{W}\) for any \(b \in \mathcal{W}\). Since \(L^2(\nu)\) is separable, such a set \(\mathcal{W}\) exists.

- **The functional set** \(\mathcal{G}\). We fix a countable set \(\mathcal{G}\) of Borel functions \(g : \Omega_0 \rightarrow \mathbb{R}\) such that:

- \(\|g\|_{L^2(\mathcal{P}_0)} < +\infty\) for any \(g \in \mathcal{G}\).
- \(1 \in \mathcal{G}, \mathcal{G}_1 \subset \mathcal{G}, \mathcal{G}_2 \subset \mathcal{G}\).
- \(\mathcal{G}\), thought as a subset of \(L^2(\mathcal{P}_0)\), is dense in \(L^2(\mathcal{P}_0)\).
- For each \(b \in \mathcal{W}\), \(M \in \mathbb{N}\) and coordinate \(i = 1, \ldots, d\), the function \(f : \Omega_0 \rightarrow \mathbb{R}\) defined as

\[
f(\omega) := \begin{cases} 
\int d\hat{\omega}(z)\nu_0(z)z_i[b](\omega, z) & \text{if } \int d\hat{\omega}(z)\nu_0(z)|z_i| < +\infty, \\
0 & \text{otherwise}
\end{cases}
\]

(80)
belongs to $\mathcal{G}$. Since $|f(\omega)| \leq MA_1(\omega), \|f\|_{L^2(P_0)} < +\infty$ by (A5).

- At cost to enlarge $\mathcal{G}$ we assume that $[g]_M \in \mathcal{G}$ for any $g \in \mathcal{G}$ and $M \in \mathbb{N}$.

$\mathcal{G}$ exists because of Remark 9.1.

- **The functional set $\mathcal{H}$.** We fix a countable set of Borel functions $b : \Omega_0 \times \mathbb{R}^d \to \mathbb{R}$ such that
  
  - \( \|b\|_{L^2(\omega)} < +\infty \) for any $b \in \mathcal{H}$.
  - $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3 \cup \mathcal{W} \subset \mathcal{H}$ and $1 \in \mathcal{H}$.
  - for any $i = 1, \ldots, d$ the map $(\omega, z) \mapsto z_i$ is in $\mathcal{H}$ (recall (A5)).
  - $\mathcal{H}$, thought as a subset of $L^2(\nu)$, is dense in $L^2(\nu)$.
  - At cost to enlarge $\mathcal{H}$ we assume that $[b]_M \in \mathcal{H}$ for any $b \in \mathcal{H}$ and $M \in \mathbb{N}$.

Recall Def. 7.1. Given a function $f : \Omega_0 \to [0, +\infty]$ such that $P_0(f = +\infty) = 0$, we define $A[f]$ as $A[f_*]$, where $f_* : \Omega_0 \to \mathbb{R}$ is defined as $f$ on $\{f < +\infty\}$ and as 0 on $\{f = +\infty\}$.

**Definition 9.2.** We define $\Omega_{\text{typ}}$ as the intersection of the following sets:

1. $A[gg']$ as $g, g'$ vary among $\mathcal{G}$. Note that $\|gg'\|_{L^1(P_0)} < +\infty$ by Schwarz inequality, hence $A[gg']$ is well defined. Note also that $A[g] = A[g \cdot 1]$.
2. $A[b\beta]$ as $b, \beta$ vary among $\mathcal{H}$. Note that $\|b\beta\|_{L^1(\nu)} < +\infty$ for all $b, \beta \in \mathcal{H}$. By Lemma 7.2 $P_0(A[b\beta]) = 1$.
3. $A[b\beta']$ as $b, \beta'$ vary among $\mathcal{H}$.
4. $A[[b - [b]_M]]$, as $b$ varies in $\mathcal{W}$ and $M$ varies in $\mathbb{N}$. Since $\|b\|_{L^2(\nu)} < +\infty$ for any $b \in \mathcal{W}$, by Lemma 7.2 $P_0(A[[b - [b]_M]]) = 1$.
5. $A[[d]] \cap A[[\hat{d}^2]]$, where $d := |b - [b]_M|$, as $b$ varies in $\mathcal{W}$ and $M$ varies in $\mathbb{N}$. Since $\|b\|_{L^2(\nu)} < +\infty$, it holds $\|d\|_{L^1(P_0)}, \|\hat{d}^2\|_{L^1(P_0)} < +\infty$ (cf. Lemma 7.2).
6. $A[[\beta]] \cap A[[\hat{b}]]$ as $b$ varies in $\mathcal{H}_1$.
7. $A[[\lambda]] = \{\omega \in \Omega_0 : \lambda \nu(z) \omega < +\infty \forall z \in \hat{\omega}\}$. The definition is well posed since $\|\lambda\|_{L^1(P_0)} < +\infty$ by (A5).
8. $A[[\beta^2]] \cap A[[\hat{b}^2]]$ as $b$ varies in $\mathcal{H}$. Note that $\|\beta^2\|_{L^1(\nu)}, \|\hat{b}^2\|_{L^1(\nu)} < +\infty$ by Lemma 8.1.
9. $A[[\lambda^2]]$ as $\lambda$ varies in $\mathcal{H}_1$.
10. $A[[\beta^2 + \hat{b}^2]]$ as $b$ varies in $\mathcal{H}_1$. Note that $E_0(\beta^2 + \hat{b}^2) = \nu(\hat{b}^2) + \nu(b^2) < +\infty$ (cf. Lemma 8.1).
11. $A[[\lambda^2]]$. By (A5), $\|\lambda^2\|_{L^1(P_0)} < +\infty$, hence the definition is well posed.
12. $A[[1]] = \{\omega \in \Omega_0 : \lambda_1 \nu(z) \omega < +\infty \forall z \in \hat{\omega}\}$. By (A5) $\|\lambda_1\|_{L^1(P_0)} < +\infty$.
13. $A[[F_*]]$ with $F_*$ as in (A6). By (A6), $\|F_*\|_{L^1(P_0)} < +\infty$.
14. $A[[b]]$ as $b$ varies in $\mathcal{W}$ (recall (7.1) in Lemma 8.3). Note that by Lemma 8.3 $P_0(A[[b]]) = 1$ and by definition $A[[b]] \subset A[[b]] \cap A[[\hat{b}]]$.
15. $A[[[b - [b]_M]^2]] \cap A[[[\hat{b} - [\hat{b}]_M]^2]]$, as $b$ varies in $\mathcal{W}$ and $M$ varies in $\mathbb{N}$. Note that both $(b - [b]_M)^2$ and $(\hat{b} - [\hat{b}]_M)^2$ have bounded $L^1(\nu)$-norm.
16. $A[[\hat{h}_t]]$ as $t$ varies in $\mathbb{N}_+$, where $h_t(\omega) := \int d\omega(z)c(z, \omega)|z|^21_{\{|z| \geq t\}}$. 

STOCHASTIC HOMOGENIZATION IN AMORPHOUS MEDIA 21
By immediate consequence of Prop. 3.1, Lemma 7.2 and Lemma 8.5, we get the following property:

**Proposition 9.3.** The above set \( \Omega_{\text{typ}} \) is a Borel subset of \( \Omega \) such that \( P_0(\Omega_{\text{typ}}) = 1 \) and \( \tau_z \omega \in \Omega_{\text{typ}} \) for any \( \omega \in \Omega_{\text{typ}} \) and \( z \in \hat{\omega} \).

10. 2-scale convergence of \( v_\varepsilon \in L^2(\mu_\varepsilon^x) \) and of \( w_\varepsilon \in L^2(\nu_\varepsilon^x) \)

**Definition 10.1.** Fix \( \hat{\omega} \in \Omega_{\text{typ}} \), an \( \varepsilon \)-parametrized family \( v_\varepsilon \in L^2(\mu_\varepsilon^x) \) and a function \( v \in L^2(dx \times P_0) \).

- We say that \( v_\varepsilon \) is weakly 2-scale convergent to \( v \), and write \( v_\varepsilon \xrightarrow{\text{w}} v \), if the family \( \{v_\varepsilon\} \) is bounded, i.e. \( \limsup_{\varepsilon \downarrow 0} \|v_\varepsilon\|_{L^2(\nu_\varepsilon^x)} < +\infty \), and

\[
\lim_{\varepsilon \downarrow 0} \int d\mu_\varepsilon^x v_\varepsilon(x) \varphi(x) b(\tau_{x/\varepsilon} \hat{\omega}) = \int dP_0(\omega) \int dx \, mv(x, \omega) \varphi(x) b(\omega),
\]

for any \( \varphi \in C_c(\mathbb{R}^d) \) and any \( b \in \mathcal{G} \).

- We say that \( v_\varepsilon \) is strongly 2-scale convergent to \( v \), and write \( v_\varepsilon \xrightarrow{s} v \), if the family \( \{v_\varepsilon\} \) is bounded and

\[
\lim_{\varepsilon \downarrow 0} \int d\mu_\varepsilon^x v_\varepsilon(x) u_\varepsilon(x) = \int dP_0(\omega) \int dx \, mv(x, \omega) u(x, \omega)
\]

whenever \( u_\varepsilon \xrightarrow{s} u \).

**Lemma 10.2.** Given \( \hat{\omega} \in \Omega_{\text{typ}} \), if \( v_\varepsilon \xrightarrow{\text{w}} v \) then

\[
\liminf_{\varepsilon \downarrow 0} \int d\mu_\varepsilon^x v_\varepsilon^2(x) \geq \int dP_0(\omega) \int dx \, mv(x, \omega)^2.
\]

The proof is similar to the proof of [39] Item (iii), p. 984. We give it for completeness since our definition of 2-scale convergence is different.

**Proof.** Since \( \mathcal{G} \) is dense in \( L^2(P_0) \) and \( C_c(\mathbb{R}^d) \) is dense in \( L^2(dx) \), given \( \delta > 0 \) we can find functions \( g_1, \ldots, g_k \in \mathcal{G}, \varphi_1, \ldots, \varphi_k \in C_c(\mathbb{R}^d) \) and coefficients \( a_1, \ldots, a_k \in \mathbb{R} \) such that the norm of \( v - \Phi \in L^2(dx \times P_0) \) is bounded by \( \delta \), where \( \Phi(x, \omega) := \sum_{i=1}^k a_i \varphi_i(x) g_i(\omega) \). We have

\[
\int d\mu_\varepsilon^x v_\varepsilon^2(x) \geq 2 \int d\mu_\varepsilon^x v_\varepsilon(x) \Phi(x, \tau_{x/\varepsilon} \hat{\omega}) - \int d\mu_\varepsilon^x(\Phi(x, \tau_{x/\varepsilon} \hat{\omega}))^2,
\]

\[
\int d\mu_\varepsilon^x v_\varepsilon(x) \Phi(x, \tau_{x/\varepsilon} \hat{\omega}) = \sum_{i=1}^k a_i \int d\mu_\varepsilon^x v_\varepsilon(x) \varphi_i(x) g_i(\tau_{x/\varepsilon} \hat{\omega}),
\]

\[
\int d\mu_\varepsilon^x(\Phi(x, \tau_{x/\varepsilon} \hat{\omega}))^2 = \sum_{i=1}^k \sum_{j=1}^k a_i a_j \int d\mu_\varepsilon^x((\varphi_i \varphi_j)(x)(g_i g_j)(\tau_{x/\varepsilon} \hat{\omega})).
\]
We take the limit $\varepsilon \downarrow 0$ in (85), We use that $v_\varepsilon \overset{2}{\to} v$ and $\tilde{\omega} \in \Omega_{\text{typ}}$ to deal with (85) and we use that $\omega \in A[g_i g_j]$ (cf. (S1) in Def. 9.2 and Prop. 3.1) to deal with (86). We then get

$$\lim_{\varepsilon \downarrow 0} \int d\mu^\varepsilon_\omega(x)v^2_\varepsilon(x) \geq 2 \int dP_0(\omega) \int dx \, m(x,\omega)\Phi(x,\omega) - \int dP_0(\omega) \int dx \, m(\Phi(x,\omega))^2 \quad (87)$$

$$\geq -\delta + \int dP_0(\omega) \int dx \, m(\varepsilon)\omega = 0.$$  

By the arbitrariness of $\delta$, we get (93). \square

**Lemma 10.3.** Given $\tilde{\omega} \in \Omega_{\text{typ}}$, we have $v_\varepsilon \overset{2}{\rightharpoonup} v$ if $v_\varepsilon \overset{2}{\to} v$ and

$$\lim_{\varepsilon \downarrow 0} \int d\mu^\varepsilon_\omega(x)v^2_\varepsilon(x) = \int dP_0(\omega) \int dx \, m(\varepsilon)\omega. \quad (88)$$

An analogous implication is contained [39, Item (iv), p. 984] and the proof there can be easily adapted to our setting. For completeness, we give the proof in Appendix A.

**Lemma 10.4.** Let $\tilde{\omega} \in \Omega_{\text{typ}}$. Then, given a bounded family of functions $v_\varepsilon \in L^2(\mu^\varepsilon_\omega)$, there exists a subsequence $\{v_{\varepsilon_k}\}$ such that $v_{\varepsilon_k} \overset{2}{\to} v$ for some $v \in L^2(md\times P_0)$ with $\|v\|_{L^2(\mu^\varepsilon_\omega)} \leq \lim sup_{\varepsilon \downarrow 0} \|v_\varepsilon\|_{L^2(\mu^\varepsilon_\omega)}$.

The proof is similar to the proof of [39 Prop. 2.2]. We give it for completeness since our definition of 2-scale convergence is different.

**Proof.** Since $\{v_\varepsilon\}$ is bounded in $L^2(\mu^\varepsilon_\omega)$, there exist $C, \varepsilon_0$ such that $\|v_\varepsilon\|_{L^2(\mu^\varepsilon_\omega)} \leq C$ for $\varepsilon \leq \varepsilon_0$. We fix a countable set $V \subset C_c(\mathbb{R}^d)$ such that $V$ is dense in $L^2(md\times P_0)$. We call $\mathcal{L}$ the family of functions $\Phi$ of the form $\Phi(x,\omega) = \sum_{i=1}^r a_i \varphi_i(x)g_i(\omega)$, where $r \in \mathbb{N}_+$, $g_i \in G$, $\varphi_i \in V$ and $a_i \in \mathbb{Q}$. Note that $\mathcal{L}$ is a dense subset of $L^2(md\times P_0)$. By Schwarz inequality we have

$$\left| \int d\mu^\varepsilon_\omega(x)v_\varepsilon(x)\Phi(x,\tau_x/\varepsilon\tilde{\omega}) \right| \leq C \left[ \int d\mu^\varepsilon_\omega(x)\Phi(x,\tau_x/\varepsilon\tilde{\omega})^2 \right]^{1/2}. \quad (89)$$

By expanding the square in the r.h.s., since $\tilde{\omega} \in \Omega_{\text{typ}}$ (cf. (S1) in Def. 9.2 and Prop. 3.1), we have

$$\lim_{\varepsilon \downarrow 0} \int d\mu^\varepsilon_\omega(x)\Phi(x,\tau_x/\varepsilon\tilde{\omega})^2 = \sum_i \sum_j a_i a_j \int dx \, m(\varphi_i(x)\varphi_j(x)\varepsilon_0[g_i g_j] = \|\Phi\|_{L^2(md\times P_0)}^2. \quad (90)$$

As a first application of (90) we get that, for $\varepsilon$ small, the l.h.s. of (93) is bounded uniformly in $\varepsilon$, hence it admits a convergent subsequence. Since $\mathcal{L}$
is a countable family, by a diagonal procedure we can extract a subsequence \( \varepsilon_k \downarrow 0 \) such that the limit

\[
F(\Phi) := \lim_{k \to \infty} \int d\mu_{\omega_k}^\varepsilon(x)v_{\varepsilon_k}(x)\Phi(x, \tau_{x/\varepsilon_k}\omega)
\]

exists for any \( \Phi \in \mathcal{L} \) and it satisfies \( |F(\Phi)| \leq C\|\Phi\|_{L^2(\text{mdx} \times \mathcal{P}_0)} \) by (89) and (90). Since \( \mathcal{L} \) is a dense subset of \( L^2(\text{mdx} \times \mathcal{P}_0) \), by Riesz's representation theorem we get that there exists a unique \( v \in L^2(\text{mdx} \times \mathcal{P}_0) \) such that

\[
F(\Phi) = \int d\mathcal{P}_0(\omega) \int dx \, m(\Phi(x, \omega)v(x, \omega))
\]

for any \( \Phi \in \mathcal{L} \). Moreover, we have \( \|v\|_{L^2(\text{mdx} \times \mathcal{P}_0)} \leq C \). As \( \Phi(x, \omega) := \varphi(x)g(\omega) \) - with \( \varphi \in \mathcal{V} \) and \( b \in \mathcal{G} \) - belongs to \( \mathcal{L} \), we get that (S1) is satisfied along the subsequence \( \{\varepsilon_k\} \) for any \( \varphi \in \mathcal{V} \), \( b \in \mathcal{G} \). Take now a generic \( \varphi \in C_c(\mathbb{R}^d) \). Trivially, \( \mathcal{V} \) can be chosen such that, if the generic \( \varphi \in C_c(\mathbb{R}^d) \) has support inside \([0,1]^d\) and uniform norm bounded by \( L \), then \( \varphi \) can by approximated in uniform norm by functions \( \psi_n \in \mathcal{V} \) with support inside \([0,1]^d\) and uniform norm bounded by \( L \). This also implies that \( \delta_n := \|\varphi - \psi_n\|_{L^2(\text{mdx})} \to 0 \). Moreover, for each \( N \) we fix a function \( \phi_N \in C_c(\mathbb{R}^d) \) with values in \([0,1]\) and equal to 1 on the ball \( \{|x| \leq N\} \). We take \( \mathcal{V} \) such that \( \phi_N \in \mathcal{V} \) for all \( N \in \mathbb{N} \). We bound

\[
\left| \int d\mu_{\omega_k}^\varepsilon(x)v_{\varepsilon_k}(x)[\varphi(x) - \psi_n]g(\tau_{x/\varepsilon_k}\omega) \right| \leq \|\varphi - \psi_n\| \int d\mu_{\omega_k}^\varepsilon(x)g(\tau_{x/\varepsilon_k}\omega).
\]

Since \( \omega_k \in \mathcal{A}[g] \) for all \( g \in \mathcal{G} \) (cf. (S1) in Def. [17]), by Prop. [3.1] the last integral converges as \( \varepsilon \to \infty \) to \( C' := \int dx m\phi_N\mathbb{E}_0[g] \). In particular, using also that \( \psi_n \in \mathcal{V} \), along the subsequence \( \{\varepsilon_k\} \) we have

\[
\underline{\lim}_{\varepsilon \downarrow 0} \int d\mu_{\omega_k}^\varepsilon(x)v_{\varepsilon_k}(x)\varphi(x)g(\tau_{x/\varepsilon_k}\omega)
\leq C'\|\varphi - \psi_n\| + \underline{\lim}_{\varepsilon \downarrow 0} \int d\mu_{\omega_k}^\varepsilon(x)v_{\varepsilon_k}(x)\psi_n(x)g(\tau_{x/\varepsilon_k}\omega)
\]

(91)

\[
= C'\|\varphi - \psi_n\| \int dx m v(x, \omega)\psi_n(x)(g)(\omega).
\]

We now take the limit \( n \to \infty \). Since \( \psi_n(x) \to \varphi(x) \) for any \( x \) and

\[
|v(x, \omega)\psi_n(x)b(\omega)| \leq L \mathbb{1}_{\{|x| \leq N\}}|b(\omega)| |v(x, \omega)| \in L^1(\text{mdx} \times \mathcal{P}_0),
\]

by dominated convergence we conclude that, along the subsequence \( \{\varepsilon_k\} \),

\[
\underline{\lim}_{\varepsilon \downarrow 0} \int d\mu_{\omega_k}^\varepsilon(x)v_{\varepsilon_k}(x)\varphi(x)g(\tau_{x/\varepsilon_k}\omega) \leq \int d\mathcal{P}_0(\omega) \int dx \, m v(x, \omega)\varphi(x)g(\omega).
\]

(92)

A similar result holds with the liminf, thus implying that (S1) holds along the subsequence \( \{\varepsilon_k\} \) for any \( \varphi \in C_c(\mathbb{R}^d) \) and \( b \in \mathcal{G} \).

**Definition 10.5.** Given \( \omega \in \Omega_{\text{typ}} \), a family \( w_{\varepsilon} \in L^2(\nu_{\varepsilon}^\omega) \) and a function \( w \in L^2(\text{mdx} \times dv) \), we say that \( w_{\varepsilon} \) is weakly 2-scale convergent to \( v \), and write \( w_{\varepsilon} \overset{2}{\rightharpoonup} v \), if the family \( \{w_{\varepsilon}\} \) is bounded in \( L^2(\nu_{\varepsilon}^\omega) \), i.e. \( \underline{\lim}_{\varepsilon \downarrow 0} \|w_{\varepsilon}\|_{L^2(\nu_{\varepsilon}^\omega)} < +\infty \),
and
\[
\lim_{\varepsilon \downarrow 0} \int dv_\varepsilon(x, z) w_\varepsilon(x, z) \varphi(x) b(\tau_{x/e} \tilde{\omega}, z)
= \int dx m \int dv(\omega, z) w(x, \omega, z) \varphi(x) b(\omega, z), \quad (93)
\]
for any \(\varphi \in C_c(\mathbb{R}^d)\) and any \(b \in \mathcal{H}\).

**Lemma 10.6.** Let \(\tilde{\omega} \in \Omega_{\text{typ}}\). Then, given a bounded family of functions \(w_\varepsilon \in L^2(\nu_\varepsilon^x)\), there exists a subsequence \(\{w_{\varepsilon_k}\}\) such that \(w_{\varepsilon_k} \xrightarrow{\mathcal{L}} w\) for some \(w \in L^2(m dx \times \nu)\) with \(\|w\|_{L^2(m dx \times \nu)} \leq \limsup_{\varepsilon \downarrow 0} \|w_\varepsilon\|_{L^2(\nu_\varepsilon^x)}\).

**Proof.** The proof of Lemma 10.6 is similar to the proof of Lemma 10.4. We only give some comments on some new steps. One has to replace \(E\) (cf. Lemma 7.2).

**Lemma 11.1.** Let \(\dot{\omega} \in \Omega_{\text{typ}}\). Then, given a bounded family of functions \(w_\varepsilon \in L^2(\nu_\varepsilon^x)\), there exists a subsequence \(\{w_{\varepsilon_k}\}\) such that \(w_{\varepsilon_k} \xrightarrow{\mathcal{L}} w\) for some \(w \in L^2(m dx \times \nu)\) with \(\|w\|_{L^2(m dx \times \nu)} \leq \limsup_{\varepsilon \downarrow 0} \|w_\varepsilon\|_{L^2(\nu_\varepsilon^x)}\).

**Proof.** The proof of Lemma 10.6 is similar to the proof of Lemma 10.4. We only give some comments on some new steps. One has to replace \(E\) (cf. Lemma 7.2).

11. **Cut-off for functions \(v_\varepsilon \in L^2(\mu_\varepsilon^x)\)**

We write \(\mathbb{N}_+\) for the set of positive integers. Recall (48).

**Lemma 11.1.** Let \(\dot{\omega} \in \Omega_{\text{typ}}\) and let \(\{v_\varepsilon\}\) be a family of functions such that \(v_\varepsilon \in L^2(\mu_\varepsilon^x)\) and \(\lim_{\varepsilon \downarrow 0} \|v_\varepsilon\|_{L^2(\mu_\varepsilon^x)} < +\infty\). Then there exist functions \(v, v_M \in L^2(m dx \times \mathcal{P}_0)\) with \(M\) varying in \(\mathbb{N}_+\) such that

(i) \(v_\varepsilon \xrightarrow{\mathcal{L}} v\) and \([v_\varepsilon]_M \xrightarrow{\mathcal{L}} v_M\) for all \(M \in \mathbb{N}_+\), along a subsequence \(\{\varepsilon_k\}\);
(ii) for any $\varphi \in C_c(\mathbb{R}^d)$ and $u \in \mathcal{G}$ it holds
\[
\lim_{M \to \infty} \int dx \, m \int d\mathcal{P}_0(\omega) v_M(x, \omega) \varphi(x) u(\omega)
= \int dx \, m \int d\mathcal{P}_0(\omega) v(x, \omega) \varphi(x) u(\omega). \tag{96}
\]

Proof. Without loss, we assume that $\|v_{\varepsilon}\|_{L^2(\mu_{\varepsilon}^1)} \leq C_0 < +\infty$ for all $\varepsilon$. We set $v_{M}^\varepsilon := [v_{\varepsilon}]_M$. Since $\|v_{M}^\varepsilon\|_{L^2(\mu_{\varepsilon}^1)} \leq \|v_{\varepsilon}\|_{L^2(\mu_{\varepsilon}^1)} \leq C_0$, Item (i) follows from Lemma 10.4 and a diagonal procedure.

Just to simplify the notation, we assume that the 2-scale convergence in Item (i) takes place for $\varepsilon \downarrow 0$ (avoiding in this way to specify continuously the subsequence $\{\varepsilon_k\}$). Let us define $F(\bar{v}, \bar{\varphi}, \bar{u}) := \int dx \, m \int \mathcal{P}_0(d\omega) \bar{v}(x, \omega) \bar{\varphi}(x) \bar{u}(\omega)$. Then Item (ii) corresponds to the limit
\[
\lim_{M \to \infty} F(v_M, \varphi, u) = F(v, \varphi, u) \quad \forall \varphi \in C_c(\mathbb{R}^d), \forall u \in \mathcal{G}. \tag{97}
\]
We fix such functions $\varphi, u$ and set $u_k := [u]_k$ for all $k \in \mathbb{N}^+$. By definition of $\mathcal{G}$, we have $u_k \in \mathcal{G}$ for all $k$.

Claim 11.2. For each $k, M \in \mathbb{N}^+$ it holds
\[
|F(v, \varphi, u) - F(v, \varphi, u_k)| \leq C_0 \|\varphi\|_{L^2(\mu_{\varepsilon}^1)} \|u - u_k\|_{L^2(\mathcal{P}_0)}, \tag{98}
\]
\[
|F(v_M, \varphi, u) - F(v_M, \varphi, u_k)| \leq C_0 \|\varphi\|_{L^2(\mu_{\varepsilon}^1)} \|u - u_k\|_{L^2(\mathcal{P}_0)}. \tag{99}
\]

Proof. By Schwarz inequality
\[
|F(v, \varphi, u) - F(v, \varphi, u_k)| = \left| \int dx \, m \int d\mathcal{P}_0(\omega) v(x, \omega) \varphi(x)(u - u_k)(\omega) \right| \\
\leq \|v\|_{L^2(\mu_{\varepsilon}^1 \times \mathcal{P}_0)} \|\varphi(u - u_k)\|_{L^2(\mu_{\varepsilon}^1 \times \mathcal{P}_0)}.
\]

To get (98) it is then enough to apply Lemma 10.2 (or Lemma 10.3) to bound $\|v\|_{L^2(\mu_{\varepsilon}^1 \times \mathcal{P}_0)}$ by $C_0$. The proof of (99) is identical. \qed

Claim 11.3. For each $k, M \in \mathbb{N}^+$ it holds
\[
|F(v, \varphi, u_k) - F(v_M, \varphi, u_k)| \leq (k/M) \|\varphi\|_\infty C_0^2. \tag{100}
\]

Proof. We note that $(v_{\varepsilon} - v_{M}^\varepsilon)(x) = 0$ if $|v_{\varepsilon}(x)| \leq M$. Hence we can bound
\[
|v_{\varepsilon} - v_{M}^\varepsilon|(x) = |v_{\varepsilon} - v_{M}^\varepsilon|(x) 1_{|v_{\varepsilon}(x)| > M} \leq |v_{\varepsilon} - v_{M}^\varepsilon|(x) \frac{|v_{\varepsilon}(x)|}{M} \leq \frac{v_{\varepsilon}(x)^2}{M}. \tag{101}
\]
We observe that $F(v, \varphi, u_k) = \lim_{\varepsilon \downarrow 0} \int d\mu_{\varepsilon}^\varepsilon(x) v_{\varepsilon}(x) \varphi(x) u_k(\tau_{x/\varepsilon} \hat{\omega})$, since $u_k \in \mathcal{G}$ and $v_{\varepsilon} \overset{\mathcal{P}_0}{\to} v$. A similar representation holds for $F(v_M, \varphi, u_k)$. As a consequence, and using (101), we get
\[
|F(v, \varphi, u_k) - F(v_M, \varphi, u_k)| \leq \lim_{\varepsilon \downarrow 0} \int d\mu_{\varepsilon}^\varepsilon(x) |v_{\varepsilon} - v_{M}^\varepsilon|(x) \varphi(x) u_k(\tau_{x/\varepsilon} \hat{\omega}) | \\
\leq (k/M) \|\varphi\|_\infty \lim_{\varepsilon \downarrow 0} \int d\mu_{\varepsilon}^\varepsilon(x) v_{\varepsilon}(x)^2 \leq (k/M) \|\varphi\|_\infty C_0^2.
\]
\qed
We can finally conclude the proof of Lemma 11.1. Given \( \varphi \in C_c(\mathbb{R}^d) \) and \( u \in \mathcal{G} \), by applying Claims 11.2 and 11.3, we can bound
\[
|F(v_M, \varphi, u) - F(v, \varphi, u)| \leq \|F(v_M, \varphi, u) - F(v_M, \varphi, u_k)\|
+ |F(v_M, \varphi, u_k) - F(v, \varphi, u_k)| + |F(v, \varphi, u_k) - F(v, \varphi, u)|
\leq 2C_0\|\varphi\|_{L^2(md\nu)}\|u - u_k\|_{L^2(\mathcal{P}_0)} + (k/M)\|\varphi\|_{C_2^0}.
\] (102)
The thesis then follows by taking first the limit \( M \to \infty \) and afterwards the limit \( k \to \infty \) together with the property that \( \lim_{k \to \infty} \|u - u_k\|_{L^2(\mathcal{P}_0)} = 0 \) by dominated convergence. \( \square \)

12. STRUCTURE OF THE 2-scale WEAK LIMIT OF A BOUNDED FAMILY IN \( H_{1,\omega}^1 \): PART I

It is simple to check the following Leibniz rule for discrete gradient:
\[
\nabla \varepsilon (fg)(x, z) = \nabla \varepsilon f(x, z)g(x) + f(x + \varepsilon z)\nabla \varepsilon g(x, z)
\] (103)
where \( f, g : \varepsilon \omega \to \mathbb{R} \).

The following Proposition 12.1 is the analogous of \cite[Lemma 5.3]{9}.

**Proposition 12.1.** Let \( \tilde{\omega} \in \Omega_{\text{typ}} \). Let \( \{v_{\varepsilon}\} \) be a family of functions \( v_{\varepsilon} \in H_{1,\varepsilon}^1 \) satisfying
\[
\limsup_{\varepsilon \downarrow 0} \|v_{\varepsilon}\|_{L^2(\nu^2_{\varepsilon})} < +\infty, \quad \limsup_{\varepsilon \downarrow 0} \|\nabla \varepsilon v_{\varepsilon}\|_{L^2(\nu^2_{\varepsilon})} < +\infty.
\] (104)
Then, along a subsequence, we have \( v_{\varepsilon} \overset{\text{w}}{\rightharpoonup} v \), where \( v \in L^2(md\nu \times \mathcal{P}_0) \) does not depend on \( \omega \): for \( dx \)-a.e. \( x \in \mathbb{R}^d \) the function \( \omega \mapsto v(x, \omega) \) is constant.

**Proof.** Recall the definition of the functional sets \( \mathcal{G}_1, \mathcal{H}_1 \) given in Section 9. We claim that \( \forall \varphi \in C_c^1(\mathbb{R}^d) \) and \( \forall \psi \in \mathcal{G}_1 \) it holds
\[
\int dx m \int \mathcal{P}_0(\omega)v(x, \omega)\varphi(x)\psi(\omega) = 0.
\] (105)
Before proving our claim, let us explain how it leads to the thesis. Since \( \varphi \) varies among \( C_c^1(\mathbb{R}^d) \) while \( \psi \) varies in a countable set, (105) implies that, \( dx \)-a.e., \( \int \mathcal{P}_0(\omega)v(x, \omega)\psi(\omega) = 0 \) for any \( \psi \in \mathcal{G}_1 \). Due to Lemma 5.8 we conclude that, \( dx \)-a.e., \( v(x, \cdot) \) is orthogonal in \( L^2(\mathcal{P}_0) \) to \( \{w \in L^2(\mathcal{P}_0) : \mathbb{E}_0[w] = 0\} \), which is equivalent to the fact that \( v(x, \omega) = \mathbb{E}_0[v(x, \cdot)] \) for \( \mathcal{P}_0 \)-a.a. \( \omega \).

It now remains to prove (105). Since \( \tilde{\omega} \in \Omega_{\text{typ}} \), along a subsequence Items (i) and (ii) of Lemma 11.1 hold (we keep the same notation of Lemma 11.1). Hence, in order to prove (105), it is enough to prove for any \( M \) that, given \( \varphi \in C_c^1(\mathbb{R}^d) \) and \( \psi \in \mathcal{G}_1 \),
\[
\int dx m \int \mathcal{P}_0(\omega)v_M(x, \omega)\varphi(x)\psi(\omega) = 0.
\] (106)
We write \( v_{\varepsilon}^M := [v_{\varepsilon}]_M \). Since \( \nabla \varepsilon v_{\varepsilon}^M \leq \nabla \varepsilon v_{\varepsilon} \) (cf. (19)), by Lemma 10.6 and a diagonal procedure, at cost to refine the subsequence \( \{\varepsilon_k\} \) we have for any \( M \) that \( \nabla \varepsilon v_{\varepsilon}^M \overset{\text{w}}{\rightharpoonup} w_M \in L^2(md\nu \times \nu) \), along the subsequence \( \{\varepsilon_k\} \). In what
that, by (S1) and since \( \tilde{\omega} \in \Omega_{\text{typ}} \) and \( \psi \in \mathcal{G}_1 \subset \mathcal{G} \),

\[
\text{l.h.s. of (105)} = \lim_{\varepsilon \downarrow 0} \int d\mu_{\varepsilon}^\omega(x)v_\varepsilon^\delta(x)\varphi(x)\psi(\tau_{x/\varepsilon}\tilde{\omega}) .
\] (107)

Let us write \( \psi = g_b \) with \( b \in \mathcal{H}_1 \) (recall (79)). By Lemma 8.6 and since \( \tilde{\omega} \in \Omega_{\text{typ}} \) (recall (S6)) for any \( b \in \mathcal{H}_1 \) we have (recall (110))

\[
\text{r.h.s. of (107)} = -\varepsilon \int dv_{\varepsilon}^\omega(x,z)\nabla_\varepsilon (v_\varepsilon^M \varphi)(x,z)b(\tau_{x/\varepsilon}\tilde{\omega}, z) = -\varepsilon C_1(\varepsilon) + \varepsilon C_2(\varepsilon),
\] (108)

where

\[
C_1(\varepsilon) := \int dv_{\varepsilon}^\omega(x,z)\nabla_\varepsilon v_\varepsilon^M(x,z)\varphi(\varepsilon x)b(\tau_{x/\varepsilon}\tilde{\omega}, z),
\]

\[
C_2(\varepsilon) := \int dv_{\varepsilon}^\omega(x,z)v_\varepsilon^M(x + \varepsilon z)\nabla_\varepsilon \varphi(x,z)b(\tau_{x/\varepsilon}\tilde{\omega}, z).
\]

Due to (107) and (108), to get (106) we only need to show that \( \lim_{\varepsilon \downarrow 0} \varepsilon C_1(\varepsilon) = 0 \) and \( \lim_{\varepsilon \downarrow 0} \varepsilon C_2(\varepsilon) = 0 \).

We move to \( C_2(\varepsilon) \). Let \( \ell \) be such that \( \varphi(x) = 0 \) if \( |x| \geq \ell \). Fix \( \phi \in C_c(\mathbb{R}^d) \) with values in \([0, 1]\), such that \( \phi(x) = 1 \) for \( |x| \leq \ell \) and \( \phi(x) = 0 \) for \( |x| \geq \ell + 1 \). Since \( \nabla_\varepsilon \varphi(x, z) = 0 \) if \( |x| \geq \ell \) and \( |x + \varepsilon z| \geq \ell \), by the mean value theorem we conclude that

\[
|\nabla_\varepsilon \varphi(x, z)| \leq \|\nabla \varphi\|_\infty |z| (\phi(x) + \phi(x + \varepsilon z)).
\] (110)

We apply the above bound and Schwarz inequality to \( C_2(\varepsilon) \) getting

\[
|C_2(\varepsilon)| \leq M\|\nabla \varphi\|_\infty \int dv_{\varepsilon}^\omega(x,z)|z| \big| b(\tau_{x/\varepsilon}\tilde{\omega}, z) \big| (\phi(x) + \phi(x + \varepsilon z)),
\] (111)

where (see below)

\[
A_1(\varepsilon) := \int v_{\varepsilon}^\omega(x,z)|z|^2(\phi(x) + \phi(x + \varepsilon z)) = 2 \int v_{\varepsilon}^\omega(x,z)|z|^2\phi(x)^2,
\]

\[
A_2(\varepsilon) := \int v_{\varepsilon}^\omega(x,z) b(\tau_{x/\varepsilon}\tilde{\omega}, z)^2 (\phi(x) + \phi(x + \varepsilon z))
\]

\[
= 2 \int v_{\varepsilon}^\omega(x,z)(\tilde{b}^2 + b^2)(\tau_{x/\varepsilon}\tilde{\omega}, z)\phi(x)^2.
\]

To get the second identities in the above formulas for \( A_1(\varepsilon) \) and \( A_2(\varepsilon) \) we have applied (69) to \((\omega, z) \mapsto |z|^2\) and to \((\omega, z) \mapsto b^2(\omega, z)\) and used that \( \tilde{\omega} \in \Omega_{\text{typ}} \) (recall (S7) and (S8) in Def. 9.2 and Assumption (A5)).
We now write
\[ A_1(\varepsilon) = 2 \int d\mu_x(\varepsilon)(\tau_x/\varepsilon)\phi(x)^2, \quad A_2(\varepsilon) = 2 \int d\mu_x(\varepsilon)(\tilde{b}^2 + b^2)(\tau_x/\varepsilon)\phi(x)^2. \]

At this point we use again that \(\tilde{\omega} \in \Omega_{\text{typ}}\) (cf. (S9) and (S10)). Due to Prop. 3.1 we conclude that \(A_1(\varepsilon), A_2(\varepsilon)\) have finite limits as \(\varepsilon \downarrow 0\), thus implying (cf. (111)) that \(\lim_{\varepsilon \downarrow 0} \varepsilon C_2(\varepsilon) = 0\). This concludes the proof of (106). \(\square\)

13. Cut-off for gradients \(\nabla_x v_\varepsilon\)

**Lemma 13.1.** Let \(\tilde{\omega} \in \Omega_{\text{typ}}\) and let \(\{v_\varepsilon\}\) be a family of functions with \(v_\varepsilon \in H^1_{\tilde{\omega},\varepsilon}\), satisfying (104). Then there exist functions \(w, w_M \in L^2(mdx \times \nu)\), with \(M\) varying in \(N_+\), such that

(i) \(\nabla_x v_\varepsilon \overset{2}{\to} w\) and \(\nabla_x [v_\varepsilon]_M \overset{2}{\to} v_M\) for all \(M \in N_+\);

(ii) for any \(\varphi \in C^1_{\text{lip}}(\mathbb{R}^d)\) and \(b \in \mathcal{H}\) it holds

\[
\lim_{M \to \infty} \int dx m \int d\nu(\omega, z) w_M(x, \omega, z) \varphi(x) b(\omega, z) = \int dx m \int dP_0(\omega) w(x, \omega, z) \varphi(x) b(\omega, z). \quad (112)
\]

**Proof.** At cost to restrict to \(\varepsilon\) small enough, we can assume that \(\|v_\varepsilon\|_{L^2(\mu_\varepsilon^0)} \leq C_0\) and \(\|\nabla_x v_\varepsilon\|_{L^2(\mu_\varepsilon^0)} \leq C_0\) for some \(C_0 < +\infty\) and all \(\varepsilon > 0\). Due to (19), the same holds respectively for \(v_M^\varepsilon\) and \(\nabla_x v_M^\varepsilon\), for all \(M \in N_+\), where we have set \(v_M^\varepsilon := [v_\varepsilon]_M\). In particular, by a diagonal procedure, due to Lemmas 10.4 and 10.6 along a subsequence we have that \(v_M^\varepsilon \overset{2}{\to} v_M\), \(v_\varepsilon \overset{2}{\to} v\), \(\nabla_x v_M^\varepsilon \overset{2}{\to} w_M\) and \(\nabla_x v_\varepsilon \overset{2}{\to} w\), where \(v_M, v \in L^2(mdx \times P_0)\), \(w_M, w \in L^2(mdx \times \nu)\), simultaneously for all \(M \in N_+\). This proves in particular Item (i). We point out that we are not claiming that \(v_M = [v_\varepsilon]_M\), \(w_M = [w_\varepsilon]_M\). Moreover, from now on we restrict to \(\varepsilon\) belonging to the above special subsequence without further mention.

We set \(H(\bar{\omega}, \varphi, \bar{b}) := \int dx m \int d\nu(\omega, z) \bar{\omega}(x, \omega, z) \varphi(x) \bar{b}(\omega, z)\). Then (112) corresponds to the limit \(\lim_{M \to \infty} H(w_M, \varphi, b) = H(w, \varphi, b)\). Here and below \(b \in \mathcal{H}\) and \(\varphi \in C^1_{\text{lip}}(\mathbb{R}^d)\). Recall that \(b_k := [b]_k \in \mathcal{H}\) for any \(k \in N_+\).

Reasoning exactly as in the proof of Claim 11.2 we get the following bounds:

**Claim 13.2.** For each \(k, M \in N_+\) it holds

\[
|H(w, \varphi, b) - H(w, \varphi, b_k)| \leq C_0 \|\varphi\|_{L^2(mdx)} \|b - b_k\|_{L^2(\nu)}, \quad \text{(113)}
\]

\[
|H(w_M, \varphi, b) - H(w_M, \varphi, b_k)| \leq C_0 \|\varphi\|_{L^2(mdx)} \|b - b_k\|_{L^2(\nu)}. \quad \text{(114)}
\]

**Claim 13.3.** For any \(k \in N_+\), it holds

\[
|H(w, \varphi, b_k) - H(w_M, \varphi, b_k)| \leq \frac{k}{\sqrt{M}} C_{3/2} C(\varphi), \quad \text{(115)}
\]

where \(C(\varphi)\) is a positive constant depending only on \(\varphi\). 

Proof. In what follows $C(\varphi)$ is a positive constant, depending at most on $\varphi$, which can change from line to line. We note that $\nabla_v v_\varepsilon(x, z) = \nabla_v v_\varepsilon^M(x, z)$ if $|v_\varepsilon(x)| \leq M$ and $|v_\varepsilon(x + z)| \leq M$. Moreover, by (50), we have $|\nabla_v v_\varepsilon - \nabla_v v_\varepsilon^M| \leq |\nabla_v v_\varepsilon|$. Hence we can bound

$$|\nabla_v v_\varepsilon - \nabla_v v_\varepsilon^M|(x, z) \leq |\nabla_v v_\varepsilon|(x, z)(\mathbb{1}_{\{|v_\varepsilon(x)| \geq M\}} + \mathbb{1}_{\{|v_\varepsilon(x + z)| \geq M\}}).$$

Due to the above bound we can estimate (see comments below)

$$|H(w, \varphi, b_k) - H(w_M, \varphi, b_k)| = \lim_{\varepsilon \downarrow 0} \int d\nu_\varepsilon(x, z)(\nabla_v v_\varepsilon - \nabla_v v_\varepsilon^M)(x, z) \varphi(x) b_k(\tau_x/\varepsilon \hat{\omega}, z) \leq k \lim_{\varepsilon \downarrow 0} \int d\nu_\varepsilon(x, z)|\nabla_v v_\varepsilon|(x, z)(\mathbb{1}_{\{|v_\varepsilon(x)| \geq M\}} + \mathbb{1}_{\{|v_\varepsilon(x + z)| \geq M\}})|\varphi(x)|.$$  

(117)

Note that the identity in (117) follows from (93) since $b_k \in \mathcal{H}$ (recall that $\hat{\omega} \in \Omega_{\text{typ}}$, $\nabla_v v_\varepsilon^M \overset{\text{a}}{\to} w_M$, $\nabla_v v_\varepsilon \overset{\text{a}}{\to} w$). By Schwarz inequality we have

$$\int d\nu_\varepsilon(x, z)|\nabla_v v_\varepsilon|(x, z)(\mathbb{1}_{\{|v_\varepsilon(x)| \geq M\}})|\varphi(x)| \leq C_0 A(\varepsilon)^{1/2},$$

(118)

where, by applying a Chebyshev-like estimate and Schwarz inequality,

$$A(\varepsilon) := \int d\nu_\varepsilon(x, z)\mathbb{1}_{\{|v_\varepsilon(x)| \geq M\}}\varphi(x)^2 \leq M^{-1} \int d\mu_\varepsilon(x)|v_\varepsilon(x)|\varphi(x)^2 \lambda_0(\tau_x/\varepsilon \hat{\omega}) \leq M^{-1}\|v_\varepsilon\|_{L_2(\mu_\varepsilon)} \left[ \int d\mu_\varepsilon(x)\varphi(x)^4 \lambda_0^2(\tau_x/\varepsilon \hat{\omega}) \right]^{1/2}.$$  

(119)

Since $\hat{\omega} \in \Omega_{\text{typ}}$ (cf. (S11) and Prop. 3.1), $\int d\mu_\varepsilon(x)\varphi(x)^4 \lambda_0^2(\tau_x/\varepsilon \hat{\omega})$ has finite limit as $\varepsilon \downarrow 0$. As a consequence, we get

$$\lim_{\varepsilon \downarrow 0} A(\varepsilon) \leq (C_0/M)C(\varphi).$$

Reasoning as above we have

$$\int d\nu_\varepsilon(x, z)|\nabla_v v_\varepsilon|(x, z)(\mathbb{1}_{\{|v_\varepsilon(x + \varepsilon z)| \geq M\}})|\varphi(x)| \leq C_0 B(\varepsilon)^{1/2},$$

(120)

where (applying also (69) for the map $(\omega, z) \mapsto 1$)

$$B(\varepsilon) := \int d\nu_\varepsilon(x, z)\mathbb{1}_{\{|v_\varepsilon(x + \varepsilon z)| \geq M\}}\varphi(x)^2 \leq \frac{1}{M} \int d\nu_\varepsilon(x, z)|v_\varepsilon(x + \varepsilon z)|\varphi(x)^2 = \frac{1}{M} \int d\nu_\varepsilon(x, z)|v_\varepsilon(x)|\varphi(x + \varepsilon z)^2 = \frac{\varepsilon^d}{M} \sum_{y \in \widehat{\omega}} |v_\varepsilon(\varepsilon y)| \sum_{a \in \widehat{\omega}} c_{y, a}(\hat{\omega}) \varphi(\varepsilon a)^2.$$
Due to Schwarz inequality, we have therefore that \( B(\varepsilon) \leq (C_0/M)C(\varepsilon)^{1/2} \) where

\[
C(\varepsilon) := \varepsilon^d \sum_{y \in \hat{\omega}} \left[ \sum_{a \in \hat{\omega}} c_{y,a}(\hat{\omega}) \varphi(\varepsilon a) \right]^2
\]

\[
\leq \| \varphi \|_{\infty}^2 \varepsilon^d \sum_{y \in \hat{\omega}} \sum_{a \in \hat{\omega}} c_{y,a}(\hat{\omega}) \varphi(\varepsilon a)^2 c_{y,a}(\hat{\omega}) = \| \varphi \|_{\infty}^2 \varepsilon^d \sum_{a \in \hat{\omega}} F_*(\tau_a \hat{\omega}) \varphi(\varepsilon a)^2,
\]

where \( F_*(\omega) := \int d\hat{\omega}(y) \int d\hat{\omega}(z) c_{0,y}(\omega) c_{y,z}(\omega) \) as in (A6). The r.h.s. converges to a finite constant as \( \varepsilon \downarrow 0 \) since \( \hat{\omega} \in \Omega_{\text{typ}} \) (recall (S13) and Prop. 3.1). We therefore conclude that \( \lim_{\varepsilon \to 0} C_\varepsilon \leq C(\varphi) \). Since \( B(\varepsilon) \leq (C_0/M)C(\varepsilon)^{1/2} \), we get that \( \lim_{\varepsilon \to 0} B(\varepsilon) \leq (C_0/M)C(\varphi) \). Since the same holds for \( A(\varepsilon) \) (cf. (119)), due to (117), (118) and (120) we get the claim. \( \square \)

We can finally derive (112), i.e. that \( \lim_{M \to \infty} H(w_M, \varphi, b) = H(w, \varphi, b) \). By using Claims (13.2) and (13.3) we have

\[
|H(w_M, \varphi, b) - H(w, \varphi, b)| \leq |H(w_M, \varphi, b) - H(w_M, \varphi, b_k)| + |H(w_M, \varphi, b_k) - H(w, \varphi, b_k)| + |H(w, \varphi, b_k) - H(w, \varphi, b)|
\]

\[
\leq C_0 C(\varphi) \| b - b_k \|_{L^2(\nu)} + C_0^{3/2} C(\varphi) (k/\sqrt{M}).
\]

At this point it is enough to take first the limit \( M \to \infty \) and afterwards the limit \( k \to \infty \) and to use that \( \lim_{k \to \infty} \| b - b_k \|_{L^2(\nu)} = 0 \). \( \square \)

### 14. Structure of the 2-scale weak limit of a bounded family in \( H_{\varepsilon,\omega} \): Part II

Differently from the previous results, for the following proposition we need that the form \( q \) is non–degenerate and in particular Assumption (A8). We point out the next result is the analogous of [38, Lemma 5.4].

**Proposition 14.1.** Let \( \hat{\omega} \in \Omega_{\text{typ}} \) and let \( \{ v_\varepsilon \} \) be a family of functions \( v_\varepsilon \in H^1_{\hat{\omega},\varepsilon} \) uniformly bounded in \( H^1_{\hat{\omega},\varepsilon} \), i.e. satisfying (101). Then, along a subsequence \( \{ \varepsilon_k \} \), we have:

(i) \( v_\varepsilon \to v \), where \( v \in L^2(\rho dx \times P_0) \) does not depend on \( \omega \). Writing \( v \) simply as \( v(x) \) we have that \( v \in H^1(\rho dx) \), i.e. the standard weak gradient \( \nabla v \) of \( v \) is in \( (L^2(\rho dx))^d \);

(ii) \( \nabla v_\varepsilon(x,z) \to \nabla v(x) \cdot z + v_1(x,\omega, z) \) where \( v_1 \in L^2(\mathbb{R}^d, L^2_{\text{pot}}(\nu)) \).

We point out that the property \( v_1 \in L^2(\mathbb{R}^d, L^2_{\text{pot}}(\nu)) \) means that for \( dx \)-almost every \( x \) in \( \mathbb{R}^d \) the map \( (\omega, z) \mapsto v_1(x, \omega, z) \in \mathbb{R}^d \) is a potential form, hence in \( L^2_{\text{pot}}(\nu) \), moreover \( \mathbb{R}^d \ni x \mapsto v_1(x, \cdot, \cdot) \in L^2_{\text{pot}}(\nu) \) is measurable and

\[
\int dx \| v_1(x, \cdot, \cdot) \|_{L^2_{\text{pot}}(\nu)}^2 = \int dx \int d\omega \int d\hat{\omega}(z) v^{(1)}(x, \omega, z)^2 < +\infty. \quad (121)
\]

**Proof of Prop. 14.1.** At cost to restrict to \( \varepsilon \) small enough, we can assume that \( \| v_\varepsilon \|_{L^2(\mu_\varepsilon^\omega)} \leq C_0 \) and \( \| \nabla v_\varepsilon \|_{L^2(\mu_\varepsilon^\omega)} \leq C_0 \) for some \( C_0 < +\infty \) and all \( \varepsilon > 0 \). We
can assume the same bounds for \( v_M^k := [v_c]_M \). Along a subsequence the 2-scale convergences in Item (i) of Lemma 11.1 and in Item (i) of Lemma 13.1 take place. By Lemmas 10.4 and 10.6 the norms \( \|v_M\|_{L^2(\mathcal{P}_0)} \), \( \|v\|_{L^2(\mathcal{P}_0)} \), \( \|w_M\|_{L^2(\nu)} \) and \( \|w\|_{L^2(\nu)} \) are upper bounded by \( C_0 \).

Due to Prop. 12.2, \( v = v(x) \) and \( v_M = v_M(x) \). We claim that for each solenoidal form \( b \in L^2_{\text{sol}}(\nu) \) and each function \( \varphi \in C^1_c(\mathbb{R}^d) \), it holds

\[
\int dx \varphi(x) \int dv(\omega, z) w(x, \omega, z) b(\omega, z) = - \int dx v(x) \nabla \varphi(x) \cdot \eta_b ,
\]

where \( \eta_b := \int dv(\omega, z) z b(\omega, z) \). Note that \( \eta_b \) is well defined since both \( b \) and the map \( (\omega, z) \mapsto z \) are in \( L^2(\nu) \). Moreover, by applying Lemma 13.3 with \( k(\omega, \tau, \omega) := c_{0, z}(\omega) z b(\omega, z) \), we get that \( \eta_b = -\bar{\eta}_b \). Before proving our Claim (122), we show how to conclude the proof of Prop. 14.1. Since the quadratic form \( q \) is not degenerate, we have that \( \{\eta_b : b \in L^2_{\text{sol}}(\nu)\} \) equals all \( \mathbb{R}^d \) by Lemma 6.1. For each direction \( i = 1, 2, \ldots, d \) we call \( b_i \) the solenoidal form such that \( \eta_b = e_i, e_i \) being the \( i \)-th vector of the canonical basis. Consider the measurable function

\[
g_i(x) := \int dv(\omega, z) w(x, \omega, z) b_i(\omega, z) .
\]

We have that \( g_i \in L^2(dx) \) since

\[
\int g_i(x)^2 dx \leq \int dx \left[ \int dv(\omega, z) w(x, \omega, z) b_i(\omega, z) \right]^2 \\
\leq \|b_i\|_{L^2(\nu)}^2 \int dx \int dv(\omega, z) w(x, \omega, z)^2 < \infty .
\]

Above we have used Schwarz inequality and the fact that \( w(x, \omega, z) \in L^2(md\times dv) \). Moreover, by (122) we have that \( \int dx \varphi(x) g_i(x) = -\int dx v(x) \partial_i \varphi(x) \). This proves that \( v(x) \in H^1(dx) \) and \( \partial_i v(x) = -g_i(x) \), \( \partial_i v \) being the weak derivative of \( v \) w.r.t. the \( i \)-th coordinate. This concludes the proof of Item (i).

We move to Item (ii). By Item (i) we can replace the r.h.s. of (122) by \( \int dx(\nabla v(x) \cdot \eta_b) \varphi(x) \). Hence (122) can be rewritten as

\[
\int dx \varphi(x) \int dv(\omega, z) [w(x, \omega, z) - \nabla v(x) \cdot z] b(\omega, z) = 0 .
\]

By the arbitrariness of \( \varphi \) we conclude that \( dx \)-a.s.

\[
\int dv(\omega, z) [w(x, \omega, z) - \nabla v(x) \cdot z] b(\omega, z) = 0 , \quad \forall b \in L^2_{\text{sol}}(\nu) .
\]

Let us now show that the map \( x \mapsto w(x, \omega, z) - \nabla v(x) \cdot z \) belongs to \( L^2(dx, L^2(\nu)) \). Indeed, we have

\[
\int dx \|w(x, \cdot, \cdot)\|_{L^2(\nu)}^2 = \int dx \int dv(\omega, z) w(x, \omega, z)^2 < +\infty ,
\]
since \( w(x, \omega, z) \in L^2(\text{d}x \times \text{d}v) \). Moreover, by Schwarz inequality, we have
\[
\int \text{d}x \| \nabla v(x) \cdot z \|^2 \leq \int \text{d}x |\nabla v(x)|^2 \int \text{d}v(\omega, z) |z|^2 < \infty, \tag{128}
\]
since \( \nabla v \in L^2(\text{d}x) \) and \( E_0[\lambda_k] < \infty \) by (A5).

As the map \((\omega, z) \mapsto w(x, \omega, z) - \nabla v(x) \cdot z \) belongs to \( L^2(\text{d}x, L^2(\nu)) \), for \( d \)–a.a. \( x \) we have that the map \((\omega, z) \mapsto w(x, \omega, z) - \nabla v(x) \cdot z \) belongs to \( L^2(\nu) \) and therefore, by (126), to \( L^2_{\text{pot}}(\nu) \). This concludes the proof of Item (ii).

It remains to prove (122). Since both sides of (122) are continuous as functions of \( b \in L^2_{\text{sol}}(\nu) \), it is enough to prove it for \( b \in \mathcal{W} \) (see Section 7). Since \( 1 \in \mathcal{G} \) and \( b \in \mathcal{W} \subset \mathcal{H} \), by Lemma 11.3 (applied to \( \nabla \varphi \) instead of \( \varphi \)) and Lemma 13.1 it is enough to show
\[
\int \text{d}x \varphi(x) \int \text{d}v(\omega, z) w_M(x, \omega, z) b(\omega, z) = -\int \text{d}x v_M(x) \nabla \varphi(x) \cdot \eta_b, \tag{129}
\]
for any \( \varphi \in C_c^1(\mathbb{R}^d) \) and \( b \in \mathcal{W} \). From now on \( M \) is fixed.

Since \( \nabla \epsilon v_M^\epsilon \xrightarrow{\epsilon} w_M \) and \( b \in \mathcal{W} \subset \mathcal{H} \) (cf. (93)) we can write
\[
\text{l.h.s. of } (129) = \lim_{\epsilon \downarrow 0} \int \text{d}v^\epsilon_M(x, z) \nabla \epsilon v_M^\epsilon(x, z) \varphi(x) b(\tau_{x/\epsilon} \hat{\omega}, z). \tag{130}
\]
Since \( b \in L^2(\nu) \) and \( \hat{\omega} \in \Omega_{\text{typ}} \) (cf. (S14), Lemmata 8.5 and 8.6), we get
\[
\int \text{d}v^\epsilon_M(x, z) \nabla \epsilon v_M^\epsilon(x, z) b(\tau_{x/\epsilon} \hat{\omega}, z) = 0.
\]
Using the above identity, that \( \nabla \epsilon (v_M^\epsilon \varphi)(x, z) = \nabla \epsilon v_M^\epsilon(x, z) \varphi(x) + v_M^\epsilon(x + \epsilon z) \nabla \varphi(x, z) \) and finally (70) in Lemma 8.2 (as \( \hat{\omega} \in \Omega_{\text{typ}} \) and due to (S6)), we conclude that
\[
\int \text{d}v^\epsilon_M(x, z) \nabla \epsilon v_M^\epsilon(x, z) \varphi(x) b(\tau_{x/\epsilon} \hat{\omega}, z)
\]
\[
= -\int \text{d}v^\epsilon_M(x, z) v_M^\epsilon(x + \epsilon z) \nabla \varphi(x, z) b(\tau_{x/\epsilon} \hat{\omega}, z) \tag{131}
\]
\[
= \int \text{d}v^\epsilon_M(x, z) v_M^\epsilon(x) \nabla \varphi(x, z) \hat{b}(\tau_{x/\epsilon} \hat{\omega}, z).
\]
Up to now we have obtained that
\[
\text{l.h.s. of } (129) = \lim_{\epsilon \downarrow 0} \int \text{d}v^\epsilon_M(x, z) v_M^\epsilon(x) \nabla \varphi(x, z) \hat{b}(\tau_{x/\epsilon} \hat{\omega}, z). \tag{132}
\]
We now set \( \hat{b}_k := [\hat{b}]_k = \tilde{b}_k \). We want to prove that
\[
\lim_{k \uparrow \infty} \lim_{\epsilon \downarrow 0} \int \text{d}v^\epsilon_M(x, z) v_M^\epsilon(x) \nabla \varphi(x, z) (\hat{b} - \hat{b}_k)(\tau_{x/\epsilon} \hat{\omega}, z) = 0. \tag{133}
\]
Let \( \ell \) be such that \( \varphi(x) = 0 \) if \( |x| \geq \ell \). Fix \( \phi \in C_c(\mathbb{R}^d) \) with values in \([0, 1]\), such that \( \phi(x) = 1 \) for \( |x| \leq \ell \) and \( \phi(x) = 0 \) for \( |x| \geq \ell + 1 \). Using (110) and
Schwarz inequality we can bound
\[
| \int d\nu_{\omega}^\varepsilon(x, z)v_M^\varepsilon(x)\nabla_x \varphi(x, z)(\tilde{b} - \tilde{b}_k)(\tau_{x/\varepsilon}\tilde{\omega}, z) |
\leq M\|\nabla\varphi\|_\infty \int d\nu_{\omega}^\varepsilon(x, z)|z(\phi(x) + \phi(x + \varepsilon z))| |\tilde{b} - \tilde{b}_k| (\tau_{x/\varepsilon}\tilde{\omega}, z)
\leq M\|\nabla\varphi\|_\infty [2A(\varepsilon)]^{1/2}[B(\varepsilon, k) + C(\varepsilon, k)]^{1/2}
\]  
(134)
where (using for \(A(\varepsilon)\) and \(C(\varepsilon)\)) in Lemma 8.2, \(\tilde{\omega} \in \Omega_{\text{typ}}, (S7)\) and \(S15))
\[
A(\varepsilon) := \int d\nu_{\omega}^\varepsilon(x, z)|z^2\phi(x) = \int d\nu_{\omega}^\varepsilon(x, z)|z^2\phi(x + \varepsilon z),
\]
\[
B(\varepsilon, k) := \int d\nu_{\omega}^\varepsilon(x, z)(\tilde{b} - \tilde{b}_k)^2(\tau_{x/\varepsilon}\tilde{\omega}, z)\phi(x),
\]
\[
C(\varepsilon, k) := \int d\nu_{\omega}^\varepsilon(x, z)(\tilde{b} - \tilde{b}_k)^2(\tau_{x/\varepsilon}\tilde{\omega}, z)\phi(x + \varepsilon z)
\]
\[
= \int d\nu_{\omega}^\varepsilon(x, z)(b - b_k)^2(\tau_{x/\varepsilon}\tilde{\omega}, z)\phi(x).
\]

Due to (S9) and Prop. 3.1 \(A(\varepsilon) = \int d\mu_\omega^\varepsilon(x)\phi(x)\lambda_2(\tau_{x/\varepsilon}\tilde{\omega})\) has finite limit as \(\varepsilon \downarrow 0\). Hence to get (133) we only need to show that \(\lim_{\varepsilon \uparrow \infty, \varepsilon \downarrow 0} B(\varepsilon, k) = \lim_{k \uparrow \infty, \varepsilon \downarrow 0} C(\varepsilon, k) = 0\). We can write \(B(\varepsilon, k) = \int d\mu_\omega^\varepsilon(x)\phi(x)d^2(\tau_{x/\varepsilon}\tilde{\omega})\) where \(d := |\tilde{b} - \tilde{b}_k|\). Due to (S5), \(\tilde{\omega} \in \Omega_{\text{typ}}, \text{Prop. 3.1}\) we conclude that \(\lim_{\varepsilon \downarrow 0} B(\varepsilon, k) = \int dx m(\phi(x)|b - b_k|^2_{L^2(\nu)}\). Similarly we get that \(\lim_{k \uparrow \infty} C(\varepsilon, k) = \int dx m(\phi(x)|b - b_k|^2_{L^2(\nu)}\). The above limits go to zero as \(k \to \infty\), thus implying (133).

Due to (132), (133) and since, by Schwarz inequality, \(\lim_{k \to \infty} \eta_k = \eta_b = -\eta_b\), to prove (135) we only need to show, for fixed \(M, k\), that
\[
\lim_{\varepsilon \downarrow 0} \int d\nu_{\omega}^\varepsilon(x, z)v_M^\varepsilon(x)\nabla_x \varphi(x, z)\tilde{b}_k(\tau_{x/\varepsilon}\tilde{\omega}, z) = \int dx v_M(x) \nabla \varphi(x) \cdot \eta_b.
\]  
(135)
To prove (135) we first show that
\[
\lim_{\varepsilon \downarrow 0} \left| \int d\nu_{\omega}^\varepsilon(x, z)v_M^\varepsilon(x)\left[\nabla_x \varphi(x, z) - \nabla \varphi(x) \cdot z\right]\tilde{b}_k(\tau_{x/\varepsilon}\tilde{\omega}, z) \right| = 0.
\]  
(136)
Since \(\|v_M^\varepsilon\|_\infty \leq M\) and \(\|\tilde{b}_k\|_\infty \leq k\), it is enough to show that
\[
\lim_{\varepsilon \downarrow 0} \int d\nu_{\omega}^\varepsilon(x, z)|\nabla_x \varphi(x, z) - \nabla \varphi(x) \cdot z| = 0.
\]  
(137)
By Taylor expansion we have \(\nabla_x \varphi(x, z) - \nabla \varphi(x) \cdot z = \frac{\lambda}{2} \sum_{i,j} \partial_{ij}^2 \varphi(\zeta_{\varepsilon}(x, z)) \cdot z_i z_j \varepsilon\), where \(\zeta_{\varepsilon}(x, z)\) is a point between \(x\) and \(x + \varepsilon z\). Moreover we note that \(\nabla_x \varphi(x, z) - \nabla \varphi(x) \cdot z = 0\) if \(|x| \geq \ell\) and \(|x + \varepsilon z| \geq \ell\). All these observations imply that
\[
|\nabla_x \varphi(x, z) - \nabla \varphi(x) \cdot z| \leq \varepsilon C(\varphi)|z|^2(\phi(x) + \phi(x + \varepsilon z))\].
\]  
(138)
Due to (69) and (S7) we can write
\[
\int d\nu_\omega^\varepsilon(x, z)|z|^2\phi(x + \varepsilon z) = \int d\nu_\omega^\varepsilon(x, z)|z|^2\phi(x) = \int d\mu_\omega^\varepsilon(x)\phi(x)\lambda_2(\tau_{x/\varepsilon}\hat{\omega}).
\]
(139)

Due to (S9) we conclude that the above r.h.s. has a finite limite as \(\varepsilon \downarrow 0\). Due to (138), we finally get (137) and hence (136).

Having (136), to get (135) it is enough to show that
\[
\lim_{\varepsilon \downarrow 0} \int d\nu_\omega^\varepsilon(x, z)v_M^\varepsilon(x)\nabla \varphi(x) \cdot z\tilde{b}_k(\tau_{x/\varepsilon}\hat{\omega}, z) = \int dxv_M(x)\nabla \varphi(x) \cdot \eta_k, \quad \text{for any } \eta_k \in \Omega_{\text{typ}},
\]
where \(u_k(\omega) := \int d\hat{\omega}(z)z_i\tilde{b}_k(\tau_{x/\varepsilon}\hat{\omega}, z).\) Since \(\hat{\omega} \in \Omega_{\text{typ}}\), \(v_M^\varepsilon \overset{\text{a.e.}}{\rightarrow} v_M\) and \(u_k \in \mathcal{G}\) (cf. (80)), by (81) we conclude that
\[
\lim_{\varepsilon \downarrow 0} \int d\mu_\omega^\varepsilon(x)\nabla \varphi(x) u_k(\tau_{x/\varepsilon}\hat{\omega}) = \int dxmv_M(x)\partial_i\varphi(x) \int dP_0(\omega)u_k(\omega) = \int dxmv_M(x)\partial_i\varphi(x)(\eta_k \cdot e_i),
\]
(142)
e_1, \ldots, e_d being the canonical basis of \(\mathbb{R}^d\). Our target (140) then follows as a byproduct of (141) and (142).

15. PROOF OF THEOREM 1

Without loss of generality, we prove Theorem 1 with \(\lambda = 1\) to simplify the notation. Some arguments are taken from (38), others are intrinsic to long jumps. We start with two results (Lemmas 15.1 and 15.2) concerning the amorphous gradient \(\nabla_{\varepsilon} \varphi\) for \(\varphi \in C_c(\mathbb{R}^d)\).

**Lemma 15.1.** Let \(\omega \in \Omega_{\text{typ}}\). Then \(\lim_{\varepsilon \downarrow 0} \|\nabla_{\varepsilon} \varphi\|_{L^2(\nu_{\omega})} < \infty\) for any \(\varphi \in C_c(\mathbb{R}^d)\).

**Proof.** Let \(\phi\) be as in (111). By (111) and since \(\omega \in \Omega_{\text{typ}}\) (apply (69) with \(b(\omega, z) := |z|^2\) and recall (S7)), we get
\[
\|\nabla_{\varepsilon} \varphi\|_{L^2(\nu_{\omega})}^2 \leq C(\varphi) \int d\nu_\omega^\varepsilon(x, z)|z|^2(\phi(x) + \phi(x + \varepsilon z))
\]
= \(2C(\varphi) \int d\nu_\omega^\varepsilon(x, z)|z|^2\phi(x) = 2C(\varphi) \int d\mu_\omega^\varepsilon(x)\phi(x)\lambda_2(\tau_{x/\varepsilon}\omega).\)
The thesis then follows from Prop. 3.1 (recall (S9)).

**Lemma 15.2.** Given \(\hat{\omega} \in \Omega_{\text{typ}}\) and \(\varphi \in C_c(\mathbb{R}^d)\) it holds
\[
\lim_{\varepsilon \downarrow 0} \int d\nu_\omega^\varepsilon(x, z)\left[\nabla_{\varepsilon} \varphi(x, z) - \nabla \varphi(x) \cdot z\right]^2 = 0.
\]
(143)
Proof. Let $\ell$ be as such that $\varphi(x) = 0$ if $|x| \geq \ell$. Fix $\phi \in C^c_0(\mathbb{R}^d)$ with values in $[0, 1]$, such that $\phi(x) = 1$ for $|x| \leq \ell$ and $\phi(x) = 0$ for $|x| \geq \ell + 1$. Recall (110). The upper bound given by (110) with $\nabla_x \varphi(x, z)$ replaced by $\nabla \varphi(x) \cdot z$ is also true. We will apply the above bounds for $|z| \geq \ell$.

By Taylor expansion we have $\nabla_x \varphi(x, z) - \nabla \varphi(x) \cdot z = \frac{1}{2} \sum_{i,j} \partial_{ij}^2 \varphi(x, z) z_i z_j \varepsilon$, where $\zeta(x, z)$ is a point between $x$ and $x + \varepsilon z$. Moreover we note that $\nabla_x \varphi(x, z) - \nabla \varphi(x) \cdot z = 0$ if $|x| \geq \ell$ and $|x + \varepsilon z| \geq \ell$. All these observations imply that

$$
|\nabla_x \varphi(x, z) - \nabla \varphi(x) \cdot z| \leq \varepsilon C |z|^2 \left( \phi(x) + \phi(x + \varepsilon z) \right).
$$

(144)

We will apply (144) for $|z| < \ell$.

By the above considerations, the integral in (143) can be bounded as

$$
\int d\nu^\varepsilon(x, z) \left| \nabla_x \varphi(x, z) - \nabla \varphi(x) \cdot z \right|^2 \leq C(\varphi) [A(\varepsilon, \ell) + B(\varepsilon, \ell)],
$$

(145)

where (cf. (69), (S7), (S12) and use that $\varphi \leq 1$)

$$
A(\varepsilon, \ell) : = \int d\nu^\varepsilon(x, z) |z|^2 \phi(x + \varepsilon z) 1_{\{|z| \geq \ell\}}
$$

$$
= 2 \int d\nu^\varepsilon(x, z) |z|^2 \phi(x) 1_{\{|z| \geq \ell\}} = 2 \int d\mu^\varepsilon(x) \phi(x) h_\ell(t_{x/\varepsilon} \omega),
$$

$$
h_\ell(\omega) : = \int d\tilde{\omega}(z) c_{0,z}(\omega) |z|^2 1_{\{|z| \geq \ell\}} ,
$$

$$
B(\varepsilon, \ell) : = \varepsilon^2 \ell^4 \int d\nu^\varepsilon(x, z) (\phi(x) + \phi(x + \varepsilon z))
$$

$$
= 2 \varepsilon^2 \ell^4 \int d\nu^\varepsilon(x, z) \phi(x) = 2 \varepsilon^2 \ell^4 \int d\mu^\varepsilon(x) \phi(x) \lambda_0(t_{x/\varepsilon} \tilde{\omega}).
$$

Due to (S16) $\lim_{\varepsilon \downarrow 0} \int d\mu^\varepsilon(x) \phi(x) h_\ell(t_{x/\varepsilon} \tilde{\omega}) = \int dx m\phi(x) \mathbb{E}_0 [h_\ell]$. By dominated convergence, we get that $\lim_{\ell \uparrow \infty} \lim_{\varepsilon \downarrow 0} A(\varepsilon, \ell) = 0$. Due to (S7) the integral $\int d\mu^\varepsilon(x) \phi(x) \lambda_0(t_{x/\varepsilon} \tilde{\omega})$ converges to $\int dx m\phi(x) \mathbb{E}_0 [\lambda_0]$ as $\varepsilon \downarrow 0$. As a consequence, $\lim_{\ell \uparrow \infty} \lim_{\varepsilon \downarrow 0} B(\varepsilon, \ell) = 0$. Coming back to (145) we finally get (143).

\section*{Convergence of solutions.}

We start by proving Item (i).

We consider (29), i.e. the equation $-\text{I}^\varepsilon \omega u_\varepsilon + u_\varepsilon = f_\varepsilon$. We recall that the weak solution $u_\varepsilon$ of this equation satisfies (cf. (26))

$$
\frac{1}{2} \langle \nabla_x v, \nabla_x u_\varepsilon \rangle_{\nu_\varepsilon} + \langle v, u_\varepsilon \rangle_{\mu_\varepsilon} = \langle v, f_\varepsilon \rangle_{\mu_\varepsilon} \quad \forall v \in H^1_{\omega, \varepsilon}.
$$

(146)

Moreover, $u_\varepsilon$ exists and is unique (by Lax–Milgram theorem). Due to (146) with $v := u_\varepsilon$ we get that $\|u_\varepsilon\|^2_{L^2(\mu_\varepsilon)} \leq \langle u_\varepsilon, f_\varepsilon \rangle_{\mu_\varepsilon}$ and therefore $\|u_\varepsilon\|^2_{L^2(\mu_\varepsilon)} \leq \|f_\varepsilon\|^2_{L^2(\mu_\varepsilon)}$ by Schwarz inequality. Hence, it holds (cf. (146)) $\frac{1}{2} \|\nabla_x u_\varepsilon\|^2_{L^2(\nu_\varepsilon)} \leq \|f_\varepsilon\|^2_{L^2(\mu_\varepsilon)}$. Since $f_\varepsilon \rightharpoonup f$, the family $\{f_\varepsilon\}$ is bounded and therefore there exists $C > 0$ such that, for $\varepsilon$ small enough as we assume below,

$$
\|u_\varepsilon\|_{L^2(\mu_\varepsilon)} \leq C, \quad \|\nabla_x u_\varepsilon\|_{L^2(\nu_\varepsilon)} \leq C.
$$

(147)
Then, by Lemma 14.1, when taking above \( \omega = \tilde{\omega} \in \Omega_{\text{typ}} \) along a subsequence we have:

(i) \( u_{\varepsilon} \xrightarrow{\ast} u \), where \( u \) is of the form \( u = u(x) \) and \( u \in H^1(\mathbb{R}^d \text{d}x) \);

(ii) \( \nabla_x u_{\varepsilon}(x, z) \xrightarrow{\ast} w(x, \omega, z) := \nabla u(x) \cdot z + u^{(1)}(x, \omega, z) \), where \( u^{(1)} \) belongs to \( L^2(\mathbb{R}^d, L^2_{\text{pot}}(\nu)) \).

Claim 15.3. For \( dx \)-a.e. \( x \in \mathbb{R}^d \) it holds

\[
\int d\nu(\omega, z) w(x, \omega, z) z = 2D\nabla u(x). 
\] (148)

Proof of Claim 15.3. We apply (146) to the test function \( v(x) := \varepsilon \varphi(x)g(\tau_{x/\varepsilon}\tilde{\omega}) \), where \( \varphi \in C^1_0(\mathbb{R}^d) \) and \( g \in G_2 \) (cf. Section 3). Recall that \( G_2 \) is given by bounded functions. We claim that \( v \in H^1_{\tilde{\omega}, \varepsilon} \). Indeed, being bounded, \( v \in L^2(\mu^\varepsilon) \). Let us bound \( \|\nabla_x v\|_{L^2(\nu^\varepsilon)} \). We can write

\[
\nabla_x v(x, z) = \varepsilon \nabla_x \varphi(x)g(\tau_{x/\varepsilon}\tilde{\omega}) + \varphi(x)\nabla g(\tau_{x/\varepsilon}\tilde{\omega}, z). 
\] (149)

In the above formula, the gradient \( \nabla g \) is the one defined in (146). By Lemma 15.1 since \( \tilde{\omega} \in \Omega_{\text{typ}} \) and \( g \) is bounded, the map \( (x, z) \mapsto \varepsilon \nabla_x \varphi(x)g(\tau_{x/\varepsilon}\tilde{\omega}) \) is in \( L^2(\nu^\varepsilon) \). On the other hand, \( \|\varphi(x)\nabla g(\tau_{x/\varepsilon}\tilde{\omega}, z)\|_{L^2(\nu^\varepsilon)} \) is bounded by

\[
\frac{2}{\varepsilon} \int d\mu^\varepsilon(x) \varphi(x)^2 \lambda_0(\tau_{x/\varepsilon}\tilde{\omega}),
\] (150)

which converges to a finite number since \( \tilde{\omega} \in \Omega_{\text{typ}} \) (cf. (S17) and Prop. 3.1). This complete the proof that \( v \in H^1_{\tilde{\omega}, \varepsilon} \).

Due to (149), (146) can be rewritten as

\[
\frac{\varepsilon}{2} \int d\nu^\varepsilon(x, z) \nabla_x u_{\varepsilon}(x, z) \nabla_x \varphi(x, z)g(\tau_{x/\varepsilon}\tilde{\omega}) + \\
\frac{1}{2} \int d\nu^\varepsilon(x, z) \nabla_x u_{\varepsilon}(x, z) \varphi(x) \nabla g(\tau_{x/\varepsilon}\tilde{\omega}, z) + \\
\varepsilon \int d\mu^\varepsilon(x) \varphi(x)g(\tau_{x/\varepsilon}\tilde{\omega}) u_{\varepsilon}(x) = \varepsilon \int d\mu^\varepsilon(x) \varphi(x)g(\tau_{x/\varepsilon}\tilde{\omega}) f_{\varepsilon}(x).
\] (150)

Since the families of functions \( \{u_{\varepsilon}(x)\} \), \( \{f_{\varepsilon}(x)\} \), \( \{\varphi(x)g(\tau_{x/\varepsilon}\tilde{\omega})\} \) are bounded families in \( L^2(\mu^\varepsilon) \), the expressions in the third line of (150) go to zero as \( \varepsilon \downarrow 0 \).

We now claim that

\[
\lim_{\varepsilon \downarrow 0} \int d\nu^\varepsilon(x, z) \nabla_x u_{\varepsilon}(x, z) \left[ \nabla_x \varphi(x, z) - \nabla \varphi(x) \cdot z \right] g(\tau_{x/\varepsilon}\tilde{\omega}) = 0. 
\] (151)

This follows by using that \( \|g\|_\infty < +\infty \), applying Schwarz inequality and afterwards Lemma 15.2 (recall that \( \|\nabla_x u_{\varepsilon}\|_{L^2(\nu^\varepsilon)} \leq C \)). The above limit (151), the 2-scale convergence \( \nabla_x u_{\varepsilon} \xrightarrow{\ast} w \) and the fact that (83) holds for all functions
in \( \mathcal{H}_3 \subset \mathcal{H} \) (cf. Section \([\text{I}]\)), imply that

\[
\lim_{\varepsilon \downarrow 0} \int d\nu_\varepsilon(x, z) \nabla_x u_\varepsilon(x, z) \nabla_x \varphi(x, z) g(\tau_{x/\varepsilon} \hat{\omega}) = \lim_{\varepsilon \downarrow 0} \int d\nu_\varepsilon(x, z) \nabla_x u_\varepsilon(x, z) \nabla_x \varphi(x) \cdot z g(\tau_{x/\varepsilon} \hat{\omega}) = \int dx \, m \int d\nu(\omega, z) w(x, \omega, z) \nabla \varphi(x) \cdot z g(\tau_{x/\varepsilon} \omega) \tag{152}
\]

Due to \([152]\) also the expression in the first line of \((150)\) goes to zero as \(\varepsilon \downarrow 0\). We conclude therefore that also the expression in the second line of \((150)\) goes to zero as \(\varepsilon \downarrow 0\). Hence

\[
\lim_{\varepsilon \downarrow 0} \int d\nu_\varepsilon(x, z) \nabla_x u_\varepsilon(x, z) \varphi(x) \nabla g(\tau_{x/\varepsilon} \hat{\omega}, z) = 0 .
\]

Due to the 2-scale convergence \( \nabla_x u_\varepsilon \overset{2}{\to} w \) and since \([93]\) holds for all gradients \( \nabla g, g \in \mathcal{G}_2 \) (since \( \mathcal{H}_2 \subset \mathcal{H} \)), we conclude that

\[
\int dx \, m \varphi(x) \int d\nu(\omega, z) w(x, \omega, z) \nabla g(\omega, z) = 0 .
\]

Since \( \{ \nabla g : g \in \mathcal{G}_2 \} \) is dense in \( L^2_{\text{pot}}(\nu) \), the above identity implies that, for \( dx \text{-a.e. } x \), the map \((\omega, z) \mapsto w(x, \omega, z)\) belongs to \( L^2_{\text{loc}}(\nu) \). On the other hand, we know that \( w(x, \omega, z) = \nabla u(x) \cdot z + u^{(1)}(x, \omega, z) \) where \( u^{(1)} \in L^2(\mathbb{R}^d, L^2_{\text{pot}}(\nu)) \).

Hence, by \((58)\), for \( dx \text{-a.e. } x \) we have that

\[
u^{(1)}(x, \cdot, \cdot) = v^a, \quad a := \nabla u(x).
\]

As a consequence (using also \((61)\)), for \( dx \text{-a.e. } x \), we have

\[
\int d\nu(\omega, z) w(x, \omega, z) z = \int d\nu(\omega, z) [\nabla u(x) \cdot z + v^{\nabla u(x)}(\omega, z)] = 2D \nabla u(x) .
\]

This concludes the proof of Claim \([153]\) \(\square\)

We now reapply \((146)\) but with \( \nu(x) := \varphi(x) \). We get

\[
\frac{1}{2} \int d\nu_\varepsilon(x, z) \nabla_x \varphi(x, z) \nabla_x u_\varepsilon(x, z) + \int d\mu_\varepsilon(x) \varphi(x) u_\varepsilon(x) = \int d\mu_\varepsilon(x) \varphi(x) f_\varepsilon(x) .
\]

Let us analyze the first term in \((153)\). By \((151)\) with \( g \equiv 1 \in \mathcal{G}_2 \), the expression

\[
\int d\nu_\varepsilon(x, z) \nabla_x \varphi(x, z) \nabla_x u_\varepsilon(x, z)
\]

equals \( \int d\nu_\varepsilon(x, z) \nabla_x u_\varepsilon(x, z) \nabla \varphi(x) \cdot z + o(1) \) as \( \varepsilon \downarrow 0 \). Since the function \((\omega, z) \mapsto z_i \) is in \( \mathcal{H} \) and since \( \omega \in \Omega_{\text{typ}} \), by the 2-scale convergence \( \nabla_x u_\varepsilon \overset{2}{\to} w \) we obtain that

\[
\lim_{\varepsilon \downarrow 0} \int d\nu_\varepsilon(x, z) \nabla_x \varphi(x, z) \nabla_x u_\varepsilon(x, z) = \int dx \, m \int d\nu(\omega, z) w(x, \omega, z) \nabla \varphi(x) \cdot z .
\]

To treat the second and third terms in \((153)\) we use that \( u_\varepsilon \overset{2}{\to} u \) with \( u = u(x) \), \( 1 \in \mathcal{G} \), and that \( f_\varepsilon \to f \), respectively. Due to the above observations, by taking
the limit \( \varepsilon \downarrow 0 \) in (153), we get

\[
\frac{1}{2} \int dx \, m \nabla \varphi(x) \cdot \int dv(\omega, z) w(x, \omega, z) + \int dx \, m \varphi(x) u(x) = \int dx \, m \varphi(x) f(x). \tag{155}
\]

Due to (148) the above identity reads

\[
\int dx \nabla \varphi(x) \cdot D \nabla u(x) + \int dx \varphi(x) u(x) = \int dx \varphi(x) f(x), \tag{156}
\]

i.e. \( u \) is a weak solution of (30). This concludes the proof of (31).

It remains to prove (32). It is enough to apply the same arguments of [48, Proof of Thm. 6.1]. Since \( f_\varepsilon \to f \), we have \( f_\varepsilon \to f \) and therefore, by (31), we have \( u_\varepsilon \to u \). This implies that \( v_\varepsilon \to v \) (again by (31)) where \( v_\varepsilon \) and \( v \) are respectively the weak solution of \(-L^\varepsilon_\omega v_\varepsilon + v_\varepsilon = u_\varepsilon\) and \(-\text{div} D \nabla v + v = u\).

By taking the scalar product of the weak version of (29) with \( v_\varepsilon \) (as in (28)), the scalar product of the weak version of (30) with \( v \) (as in (28)), the scalar product of the weak version of \(-L^\varepsilon_\omega v_\varepsilon + v_\varepsilon = u_\varepsilon\) with \( u_\varepsilon \) and the scalar product of the weak version of \(-\text{div} D \nabla v + v = u\) with \( u \) and comparing the resulting expressions, we get

\[
\langle u_\varepsilon, u_\varepsilon \rangle_{\mu_\varepsilon^\omega} = \langle v_\varepsilon, f_\varepsilon \rangle_{\mu_\varepsilon^\omega}, \quad \int u(x)^2 \, dx = \int f(x) v(x) \, dx. \tag{157}
\]

Since \( f_\varepsilon \to f \) and \( v_\varepsilon \to v \) we get that \( \langle v_\varepsilon, f_\varepsilon \rangle_{\mu_\varepsilon^\omega} \to \int v(x) f(x) \, dx \). Hence, by (154), we conclude that \( \lim_{\varepsilon \downarrow 0} \langle u_\varepsilon, u_\varepsilon \rangle_{\mu_\varepsilon^\omega} = \int u(x)^2 \, dx \). The last limit and the weak convergence \( u_\varepsilon \to u \) imply the strong convergence \( u_\varepsilon \to u \) by Remark 33. This concludes the proof of (32) and therefore of Theorem 1 (i).

- **Convergence of flows.** We prove now (33) in Item (ii), i.e. \( \nabla v_\varepsilon \to \nabla u \). By (147) the bound (21) is satisfied. Suppose that \( f_\varepsilon \to f \). Take \( \varphi \in C^1_c(\mathbb{R}^d) \), then

\[
\langle \varphi, f_\varepsilon \rangle_{L^2(\mu_\varepsilon^\omega)} \to \langle \varphi, f \rangle_{L^2(\mu^\omega)}. \tag{146}
\]

By Item (i) we know that \( u_\varepsilon \to u \) and therefore

\[
\langle \varphi, u_\varepsilon \rangle_{L^2(\mu_\varepsilon^\omega)} \to \langle \varphi, u \rangle_{L^2(\mu^\omega)}. \tag{160}
\]

The above convergences and (146) with \( v \) given by the restriction of \( \varphi \) to \( \mathbb{R}^d \) (by Lemma 15.1, \( v \in H^1_{\omega, \varepsilon} \)), we conclude that

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{2} \langle \nabla \varphi, \nabla u_\varepsilon \rangle_{L^2(\mu_\varepsilon^\omega)} = \lim_{\varepsilon \downarrow 0} \left[ \langle \varphi, f_\varepsilon \rangle_{L^2(\mu_\varepsilon^\omega)} - \langle \varphi, u_\varepsilon \rangle_{L^2(\mu_\varepsilon^\omega)} \right] = \langle \varphi, f - u \rangle_{L^2(\mu^\omega)}. \tag{33}
\]

Due to (30) and (28), the r.h.s. equals \( \int dx \, mD(x) \nabla \varphi(x) \nabla u(x) \). This proves (33).

Suppose now that \( f_\varepsilon \to f \). Then, by (32), \( u_\varepsilon \to u \). Reasoning as above we conclude that, given \( g_\varepsilon \in H^1_{\omega, \varepsilon} \) and \( g \in H^1(dx) \), then

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{2} \langle \nabla g_\varepsilon, \nabla u_\varepsilon \rangle_{L^2(\mu_\varepsilon^\omega)} = \lim_{\varepsilon \downarrow 0} \left[ \langle g_\varepsilon, f_\varepsilon \rangle_{L^2(\mu_\varepsilon^\omega)} - \langle \varphi, u_\varepsilon \rangle_{L^2(\mu_\varepsilon^\omega)} \right] = \langle g, f - u \rangle_{L^2(\mu^\omega)}. \tag{34}
\]

Since \( g \in H^1(dx) \), due to (30), the r.h.s. equals \( \int dx \, mD(x) \nabla g(x) \nabla u(x) \). This proves (34).
• **Convergence of energies.** We prove Item (iii). Since \(f_k \to f\), we have \(u_k \to u\) by (32) and \(\nabla u_k \to \nabla u\) by (34). It is enough to apply (23) with \(g_k := u_k\) and \(g := u\) and one gets (35).

16. **Proof of Theorem 2**

The limit (36) follows from Remark 3.8 and [38, Thm. 9.2]. To treat (37) and (38) we need the following fact:

**Lemma 16.1.** Suppose that Condition (i) or Condition (ii) in Theorem 2 is satisfied. Fix a weakly decreasing function \(\psi : [0, +\infty) \to [0, +\infty)\), such that \(Rd \ni x \mapsto \psi(|x|) \in [0, +\infty)\) is Riemann integrable. Then there exists a Borel set \(\Omega\)' such that \(P(\Omega) = 1\) and the following holds for each \(\omega \in \Omega\):

\[
\lim_{\varepsilon \downarrow 0} \int d\mu_\varepsilon(x) \psi(|x|) < \infty, \tag{158}
\]

\[
\lim_{\ell \uparrow \infty} \lim_{\varepsilon \downarrow 0} \int d\mu_\varepsilon(x) \psi(|x|) 1_{\{|x| \geq \ell\}} = 0. \tag{159}
\]

Thanks to Lemma 16.1 applied to the function \(\psi(x) := \frac{1}{1 + |x|^{a\varepsilon}}\), the limits (37) and (38) follow from Theorem 1 by the same arguments used in the proofs of Corollary 2.5 and Lemma 6.1 in [12, Sections 6,7] (which can be read disregarding the rest of [12] since the notation is essentially the same used here). Note that in [12] \(\hat{\omega}\) is a subset of \(\mathbb{Z}^d\) but indeed the proofs in [12] rely on (158) and (159).

**Proof of Lemma 16.1.** The limits (14) and (159) imply (158). Let us prove (159). To simplify the notation we prove a slightly different version of (159), the method can be easily adapted to (159). In particular, we want to prove that \(P\)-a.s. it holds

\[
\lim_{\ell \uparrow \infty} \lim_{\varepsilon \downarrow 0} X_{\varepsilon,\ell} = 0, \quad X_{\varepsilon,\ell}(\omega) := \varepsilon^d \sum_{k \in \mathbb{Z}^d: |k| \geq \ell/\varepsilon} \psi(|k|)N_k. \tag{160}
\]

Trivially Condition (i) implies (160). Let us suppose that Condition (ii) is satisfied. Given \(\varepsilon \in (0,1)\) let \(r = r(\varepsilon)\) be the positive integer of the form \(2^a, a \in \mathbb{N}\), such that \(r^{-1} \leq \varepsilon < 2r^{-1}\). Then, since \(\psi\) is weakly decreasing,

\[
X_{\varepsilon,\ell}(\omega) \leq 2^d Y_{r,\ell}(\omega), \quad Y_{r,\ell}(\omega) := r^{-d} \sum_{k \in \mathbb{Z}^d: |k| \geq r\ell/2} \psi(|k/r|)N_k. \tag{161}
\]

In particular, to get (160) it is enough to show that, \(P\)-a.s. \(\lim_{\ell \uparrow \infty} \lim_{r \uparrow \infty} Y_{r,\ell} = 0\), where \(r \in \Gamma := \{2^0, 2^1, 2^2, \ldots\}\). From now on we understand that \(r \in \Gamma\). Since \(E[N_k] = m\) and since \(\psi(|x|)\) is Riemann integrable, we have

\[
\lim_{r \uparrow \infty} E[Y_{r,\ell}] = z_\ell := m \int \psi(|x|) 1_{\{|x| \geq \ell/2\}} dx < \infty. \tag{162}
\]
We now estimate the variance of \( Y_{r,\ell} \). We let \( \gamma := \alpha \) if \( d = 1 \) and \( \gamma := 0 \) if \( d \geq 2 \). By Condition (ii) we have, for some fixed constant \( C_1 > 0 \),
\[
\Var(Y_{r,\ell}) \leq C_1 r^{-2d} \sum_{k \in \mathbb{Z}^d} \sum_{k' \in \mathbb{Z}^d \ : \ |k| \geq rt/2 \ , \ |k'| \geq rt/2} |k - k'|^{-1-\gamma} \psi(|k/r|) \psi(|k'/r|) =: I_0(r, \ell) + I_1(r, \ell) + I_2(r, \ell),
\]
where \( I_0(r, \ell), I_1(r, \ell) \) and \( I_2(r, \ell) \) denote the contribution from addenda as above respectively with (a) \( k = k' \), (b) \( |k - k'| \geq r \) and (c) \( 1 \leq |k - k'| < r \). Then we have
\[
\lim_{r \uparrow \infty} r^d I_0(r, \ell) = C_1 \int_{|x| \geq \ell/2} \psi(|x|)^2 dx < +\infty, \tag{163}
\]
\[
\lim_{r \uparrow \infty} r^{1+\gamma} I_1(r, \ell) = C_1 \int_{|x| \geq \ell/2} dx \int_{|y| \geq \ell/2} dy |y|^{1+\gamma} \psi(|x|) \psi(|y|) < +\infty. \tag{164}
\]
To control \( I_2(r, \ell) \) we observe that
\[
\sum_{v \in \mathbb{Z}^d \ : \ |v|_\infty \geq r} |v|^{-1-\gamma} \leq C' \sum_{n=1}^{cr} n^{d-2-\gamma} \leq \begin{cases} C_n r^{d-1} & \text{if } d \geq 2, \\ C_n & \text{if } d = 1. \end{cases}
\]
The above bound implies for \( r \) large that
\[
I_2(r, \ell) \leq C_1 \| \psi \|_\infty r^{-2d} \sum_{k \in \mathbb{Z}^d \ : \ |k| \geq rt/2} \psi(|k/r|) \sum_{k' \in \mathbb{Z}^d \ : \ 1 \leq |k - k'| \leq r} |k - k'|^{-1-\gamma} \leq C_2 r^{-1} \int_{|x| \geq \ell/2} \psi(|x|) dx. \tag{165}
\]
Due to (163), (164) and (165), \( \Var(Y_{r,\ell}) \leq C_3(\ell) r^{-1} \) for \( r \geq C_4(\ell) \). Now we write explicitly \( r = 2^j \). By Markov's inequality, we have for \( j \geq C_5(\ell) \) that
\[
\mathcal{P}(|Y_{2j,\ell} - \mathbb{E}[Y_{2j,\ell}]| \geq 1/j) \leq j^2 \Var(Y_{2j,\ell}) \leq C_3(\ell) j^{2^j - j}.
\]
Since the last term is summable among \( j \), by Borel–Cantelli lemma we conclude that, \( \mathcal{P} \)-a.s., \( |Y_{2j,\ell} - \mathbb{E}[Y_{2j,\ell}]| \leq 1/j \) for all \( j \geq 1 \) and \( j \geq C_6(\ell, \omega) \). This proves that, \( \mathcal{P} \)-a.s., \( \lim_{\ell \uparrow \infty, r \in \mathbb{R}} Y_{r,\ell} = z_\ell \) (cf. (162)). Since \( \lim_{\ell \uparrow \infty} z_\ell = 0 \), we get that \( \lim_{\ell \uparrow \infty, r \in \mathbb{R}} Y_{r,\ell} = 0, \mathcal{P} \)-a.s.

\[\square\]

17. Proof of Theorem 3

Note that Assumptions (A1), (A2), (A3) are automatically satisfied. By extending the probability space, given \( \omega \) we associate to each unordered pair of site \( \{x, y\} \in \hat{\omega} \) a Poisson process \( \{N_{x,y}(t)\}_{t \geq 0} \) with intensity \( c_{x,y}(\omega) \), such that \( N_{x,y}(.\)\) are independent processes when varying the pair \( \{x, y\} \). Note that \( N_{x,y}(t) = N_{y,x}(t) \). We write \( K := (N_{x,y}(\cdot)) \) for the above family of Poisson processes and denote by \( \mathbb{P}_\omega \) the associated law. We denote by \( \mathbb{P} \) the annealed law of the pair \( (\omega, K) \), defined as \( \mathbb{P} := \int d\mathbb{P}(\omega) \mathbb{P}_\omega \).
Lemma 17.1. There exists $t_0 > 0$ such that for $\mathbb{P}$–a.a. $(\omega, K)$ the undirected graph $G_{t_0}(\omega, K)$ with vertex set $\hat{\omega}$ and edges $\{\{x, y\} : x \neq y \in \hat{\omega}, N_{x,y}(t) > 1\}$ has only connected components with finite cardinality.

Proof. Note that $\mathbb{P}_\omega(N_{x,y}(t) > 1) = 1 - e^{-c_{x,y}(\omega)t} \leq 1 - \exp\{-g(|x - y|)t\} \leq C_1g(|x - y|)t$ for some fixed $C_1 > 0$ if we take $t \leq 1$ (since $g$ is bounded). We restrict to $t$ small enough such that $C_1\|g\|_{\infty}t < 1$ and $t \leq 1$. Consider the random connection model \cite{22} where first $\hat{\omega}$ is created with probability $C_1g(|x - y|)t$. One can couple the above random connection model with the previous process $(\omega, K)$ with law $\mathbb{P}$ in a way that the graph in the random connection model contains the graph $G_{t_0}(\omega, K)$. We choose $t = t_0$ small enough to have $mC_1t_0 \int_{\mathbb{R}^d} dxg(|x|) < 1$. The above bound and the branching process argument in the proof of \cite{22} Theorem 6.1 (cf. (6.3) there) imply that a.s. the random connection model has only connected components with finite cardinality. \hfill \Box

From now on $t_0$ will be as in Lemma 17.1. Due to the loss of memory of Poisson point processes, from Lemma 17.1 we get the following:

Corollary 17.2. For $\mathbb{P}$–a.a. $(\omega, K)$ and for each $r \in \mathbb{N}$ the undirected graph $G_{t_0}^r(\omega, K)$ with vertex set $\hat{\omega}$ and edges $\{\{x, y\} : x \neq y \in \hat{\omega}, N_{x,y}((r + 1)t_0) > N_{x,y}(rt_0)\}$ has only connected components with finite cardinality.

By the graphical representation of the exclusion process and Harris’ percolation argument \cite{10}, we conclude that, for $\mathbb{P}$–a.a. $\omega$, the exclusion process is well defined a.s. for all times $t \geq 0$. We explain in detail this issue. Take such a good $(\omega, K)$ fulfilling the property stated in Cor. 17.3. Given a particle configuration $\eta(0) \in \{0, 1\}^\omega$ we define the deterministic trajectory $(\eta(t)|\omega, K)_{t \geq 0}$ starting at $\eta(0)$ by an iterative procedure. Suppose the trajectory has been defined up to time $rt_0$. Let $C$ be any connected component of $G_{t_0}^r(\omega, K)$ and let

$$\{s_1 < s_2 < \cdots < s_k\} = \left\{s : N_{x,y}(s) = N_{x,y}(s-) + 1, \{x, y\} \text{ bond in } C, rt_0 < s \leq (r + 1)t_0\right\}.$$

Since $C$ is finite, the l.h.s. is indeed a finite set. The local evolution $\eta(t)|_{(\omega, K)}$ with $z \in C$ and $rt_0 < t \leq (r + 1)t_0$ is described as follows. Start with $\eta(rt_0)|_{(\omega, K)}$ as configuration at time $rt_0$ in $C$. At time $s_1$ exchange the values between $\eta_x$ and $\eta_y$ if $N_{x,y}(s_1) = N_{x,y}(s_1-1) + 1$ and $\{x, y\}$ is an edge in $C$ (there is exactly one such edge, a.s.). Repeat the same operation orderly for times $s_2, s_3, \ldots, s_k$. Then move to another connected component of $G_{t_0}^r(\omega, K)$ and so on. This procedure defines $\tilde{\eta}(t)|_{(\omega, K)}_{rt_0 < t \leq (r + 1)t_0}$. It is standard to check that for $\mathbb{P}$–a.a. $\omega$ the random trajectory $(\tilde{\eta}(t)|_{(\omega, K)})_{t \geq 0}$ (where the randomness comes from $K$) is an exclusion process on $\hat{\omega}$ with initial configuration $\eta(0)$ and formal generator (39).
Due to Lemma 16.1 (Condition (ii) is satisfied in the present context), Theorem 2 follows from (38) and Lemma 17.3 below. This derivation follows the same steps of [12, Section 3] with a unique exception: the \( \lim_{t \to \infty} \lim_{\varepsilon \to 0} \) of the l.h.s. of (3.4) in [12] goes to zero due to (159).

**Lemma 17.3.** For \( \mathcal{P} \)-a.a. \( \varphi \) the following holds. Fix \( \delta, t > 0 \) and \( \varphi \in C_c(\mathbb{R}^d) \) and let \( n_\varepsilon \) be an \( \varepsilon \)-parametrized family of probability measures on \( \{0,1\}^\omega \). Then

\[
\lim_{\varepsilon \to 0} \mathbb{P}_{\omega, n_\varepsilon} \left( \left| \varepsilon^d \sum_{x \in \omega} \varphi(\varepsilon x) \eta_x(\varepsilon^{-2}t) - \varepsilon^d \sum_{x \in \omega} \eta_x(0) P_{\omega,t}(\varepsilon x) \right| > \delta \right) = 0 .
\]

**Proof.** Without loss of generality we restrict to positive \( \varphi \). We think the exclusion process as built according to the graphical construction described above, after sampling \( \eta(0) \) with distribution \( n_\varepsilon \). We fix \( \omega \in \Omega_{\text{typ}} \cap \Omega_2 \) such that \((\omega, K)\) fulfills the property in Cor. 17.3 for \( \mathbb{P}_{\omega, A} \) \( K \) (this takes place for \( \mathcal{P} \)-a.a. \( \omega \)). Given \( x \in \omega, r \in \mathbb{N} \), we denote by \( C_r(x) \) the connected component of \( x \) in the graph \( G_0(\omega, K) \). Fix \( t \in (rt_0, (r + 1)t_0] \). Due to the above graphical construction, if we know \((\omega,K)\), then to determine \( \eta_x(t) [\omega,K] \) we only need to know \( \eta_x(rt_0) [\omega,K] \) with \( z \in C_r(x) \). By iterating the above argument we conclude that, knowing \((\omega,K)\), the value of \( \eta_x(t) [\omega,K] \) is determined by \( \eta_x(0) \) as \( z \) varies in the finite set

\[
Q_r(x) := \bigcup_{z \in C_r(x)} \bigcup_{z \in C_{r-1}(z_r)} \cdots \bigcup_{z \in C_1(z_1)} C_0(z_1) .
\]

Suppose that \( \varphi \) has support in the ball \( B(\ell) \) of radius \( \ell \) centered at the origin. Then, by the above considerations, for \( \ell_\varepsilon \) large enough

\[
\mathbb{P}_{\omega}(A) \leq \delta/4 , \quad A := \bigcup_{r \in \mathbb{N} \setminus [0,\varepsilon^{-2}] \cup x \in \omega : \exists \varepsilon n \leq \ell} Q_r(x) \subset B(\ell_\varepsilon) .
\]

Note that, when the event \( A \) takes place, the value \( \varepsilon^d \sum_{x \in \omega} \varphi(\varepsilon x) \eta_z(\varepsilon^{-2}t) \) depends on \( \eta(0) \) only through \( \eta_z(0) \) with \( z \in \omega \cap B(\ell_\varepsilon) \). On the other hand, due to Theorem 2 \( \varepsilon^d \sum_{x \in \omega} P_{\omega,t}^\varepsilon(\varepsilon x) \) converges to \( \int dx \mu P_t(\varphi(x)) \) as \( \varepsilon \downarrow 0 \) (use (159) to show that \( \lim_{t \to \infty, \varepsilon \to 0} \varepsilon^d \sum_{x \in \omega} P_t \varphi(x) 1_{\{|x| \leq \ell\}} = 0 \)). In particular, by taking \( \ell_\varepsilon \) large enough, it holds \( \varepsilon^d \sum_{x \in \omega: |x| \leq \ell} \eta_z(0) P_{\omega,t}^\varepsilon(\varepsilon x) \leq \delta/4 \) for any initial configuration \( \eta(0) \).

Call \( \pi_\varepsilon \) the probability measure on \( \{0,1\}^\omega \) obtained as follows: sample \( \eta(0) \) with law \( n_\varepsilon \), afterwards set the particle number equal to zero at any site \( x \in \omega \) with \( |x| \leq \ell_\varepsilon \). By the above considerations, to get (166) it is enough to prove the same limit with \( n_\varepsilon \) replaced by \( \pi_\varepsilon \) and with \( \delta \) replaced by \( \delta/2 \). This implies that, in order to prove Lemma 17.3, we can restrict (as we do) to probability measures \( n_\varepsilon \) such that \( n_\varepsilon(\eta_x(0) = 0 \forall x \in \omega \setminus B(\ell_\varepsilon)) \).

The key observation now, going back to [28], is that the symmetry of the jump rates implies the following representation for each \( x \in \omega \):

\[
\eta_x(t) = \sum_{y \in \omega} p_\omega(t, x, y) \eta_y(0) + \sum_{y \in \omega} \int_0^t p_\omega(t - s, x, y) dM_y(s) ,
\]

(167)
where \( p_\omega(t, x, y) \) is the probability to be at \( y \) for a random walk on \( \hat{\omega} \) with jump probability rates \( c_{a,b}(\omega) \) and starting at \( x \), and \( M_x(\cdot) \)'s are martingales defined as follows as

\[
\text{d}M_x(t) := \sum_{y \in \hat{\omega}} (\eta_y - \eta_x)(t-) \text{d}A_{x,y}(t), \quad A_{x,y}(t) := N_{x,y}(t) - c_{x,y}(\omega)t. \quad (168)
\]

The above identity (167) is well posed since we start with a configuration \( \eta(0) \) having a finite number of particles.

Due to (167), in order to conclude the proof of Lemma 17.3 it is enough to show that

\[
\lim_{\varepsilon \to 0} \mathbb{E}_{\omega, \eta_\varepsilon} \left[ \left( \varepsilon^d \sum_{x \in \hat{\omega}} \varphi(\varepsilon x) \sum_{y \in \hat{\omega}} \int_0^{\varepsilon^{-2}t} p_\omega(t-s, x, y) \text{d}M_y(s) \right)^2 \right] = 0. \quad (169)
\]

We can rewrite the expression inside \((\cdot)\)-brackets as

\[
\mathcal{R}_\varepsilon := \frac{\varepsilon^d}{2} \sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega} \cap B(t_\varepsilon)} \sum_{z \in \hat{\omega} \cap B(t_\varepsilon)} \varphi(\varepsilon x)(\eta_z - \eta_y)(t-) \int_0^{\varepsilon^{-2}t} (p_\omega(t-s, x, y) - p_\omega(t-s, x, z)) \text{d}A_{y,z}(s).
\]

Hence, similarly to [28], we get (using the symmetry of \( p_\omega(s, \cdot, \cdot) \))

\[
\mathbb{E}_{\omega, \eta_\varepsilon} \left[ \mathcal{R}_\varepsilon^2 \right] = \frac{\varepsilon^{2d}}{4} \sum_{y \in \hat{\omega} \cap B(t_\varepsilon)} \sum_{z \in \hat{\omega} \cap B(t_\varepsilon)} \int_0^{\varepsilon^{-2}t} c_{y,z}(\omega)(P_{\omega,s}\varphi(y) - P_{\omega,s}\varphi(z))^2 ds \\
\leq \frac{\varepsilon^d}{2} \int_0^{t} \left\langle P_{\omega,s}^e \varphi, -\mathbb{L}_\omega^s P_{\omega,s}^e \varphi \right\rangle_{L^2(\mu^s_{\omega})} ds = -\varepsilon^d \int_0^{t} \frac{d}{ds} \left\| P_{\omega,s}^e \varphi \right\|_{L^2(\mu^s_{\omega})}^2 ds \\
= \varepsilon^d \left\| P_{\omega,0}^e \varphi \right\|_{L^2(\mu^0_{\omega})}^2 - \varepsilon^d \left\| P_{\omega,t}^e \varphi \right\|_{L^2(\mu^t_{\omega})}^2 \leq \varepsilon^d \left\| \varphi \right\|_{L^2(\mu^t_{\omega})}^2 \varepsilon^{-\frac{t}{\varepsilon}} \to 0.
\]

\[
\square
\]

**Appendix A. Supplementary proofs**

**A.1. Proof of Lemma 2.1** For Item (i) see [9] Exercise 12.4.2], Item (ii) follows from stationarity and it is standard [9]. We prove Item (iii). Call the event appearing in (11), i.e. \( A = \{ \omega \in \Omega_0 : \tau_{\omega} \neq \omega \forall \tau \in \hat{\tau} \} \). Call \( \hat{A} := \{ \omega \in \Omega : \tau_{\omega} \neq \omega \forall \tau \in \hat{\tau} \} \) and observe that \( \hat{A} \) equals the event appearing in (A3), i.e. \( \hat{A} = \{ \omega : \tau_{\omega} \neq \tau_{\omega} \forall x \neq y \text{ in } \hat{\omega} \} \). Then the equivalence of (A3) with (11) follows from Lemma 4.4.

We consider now Item (iv). Let (12) be verified. Calling \( N_a := \hat{\omega}(a + [0, 1]^d) \), it holds \( F_a(\omega) \leq \sum_{u \in [0, 1]^d} \sum_{v \in [0, 1]^d} \sum_{n \neq 0} N_u N_v h(|u|) h(|u - v|) \). If \( x \in [0, 1]^d \), then \( F_a(\tau_{\omega} x) \leq \sum_{u \in [0, 1]^d} \sum_{v \in [0, 1]^d} M_a M_v h(|u|) h(|u - v|) \), where \( M_a := \hat{\omega}(a + [-1, 1]^d) \). By applying Campbell's identity [13] with \( f(x, \omega) := 1_{[0,1]^d}(x) F_a(\omega) \), we get

\[
\mathbb{E}_0[F_a] \leq m^{-1} \sum_{u \in [0, 1]^d} \sum_{v \in [0, 1]^d} h(|u|) h(|u - v|) \mathbb{E}[N_a M_u M_v]. \quad (170)
\]
To conclude we use that \( abc \leq C(a^3 + b^3 + c^3) \) for some \( C > 0 \) and for any \( a, b, c \geq 0 \) and apply (12) to get that \( \mathbb{E}[N_0^3], \mathbb{E}[M_0^3], \mathbb{E}[M_1^3] \) are finite.

We prove Item (v) for Mott v.r.h. By [16] Lemma 2, \( \lambda_0 \in L^k(P_0) \) if and only if \( \mathbb{E}[\hat{\omega}([0,1]^d)^{k+1}] < +\infty \). The proof provided there remains true when substituting \( \lambda_0 \) by any function \( f \) such that \( |f(\omega)| \leq C \int d\hat{\omega}(x) e^{-c|x|} \) with \( C, c > 0 \). As \( f \) we can take also \( f = \lambda_1 + f = \lambda_2 \). We therefore conclude that for Mott v.r.h. Assumption (A5) is equivalent to the bound \( \mathbb{E}[\hat{\omega}([0,1]^d)^3] < +\infty \). The check of other statements in Item (v) is trivial.

A.2. Proof of Prop. 3.1. It is enough to consider the case \( g \geq 0 \). Due to [37] Cor. 7.2 (see also [9] Sec. 10.2), given \( g \) and \( \varphi \) as in Prop. 3.1 (15) holds for any \( \omega \) in a Borel set \( A_{g,\varphi} \subset \Omega \) with \( \mathcal{P}(A_{g,\varphi}) = 1 \). We define \( A[g] := \cap_{\varphi \in C_1(\mathbb{R}^d)} A_{g,\varphi} \), where \( A_{g,\varphi} := \{ \omega \in \Omega \mid [15] \) is fulfilled \}. We fix a countable subset \( K \subset C_1(\mathbb{R}^d) \), dense in \( C_1(\mathbb{R}^d) \) w.r.t. the uniform norm. For any \( n \in \mathbb{N} \) we fix a continuous function \( \varphi_n \) with values in \([0,1]\) such that \( \varphi_n(x) = 1 \) for \( x \in [-n, n]^d \) and \( \varphi(x) = 0 \) for \( x \not\in [-n - 1, n + 1]^d \). Given \( \varphi \in C_1(\mathbb{R}^d) \) we let \( N \) be the smallest integer \( n \) such that the support of \( \varphi \) is contained in \([-n, n]^d \) and, fixed \( \varepsilon > 0 \), we take \( h \in K \) with \( \| \varphi - h \|_{\infty} \leq \delta \). Then

\[
(h(x) - \delta)\psi_N(x) \leq \varphi(x) \leq (h(x) + \delta)\psi_N(x).
\]

Since \( \int (h(x) \pm \delta)\psi_N(x)dx = \int \varphi(x)dx + O(\delta N^d) \), (171) and the fact that \( g \geq 0 \) imply that \( A[g] = \cap_{h \in K} \cap_{n=1}^{\infty} (A_{g,h} \cap A_{h,\varphi_n}) \). Being a countable intersection of Borel sets with \( \mathcal{P} \)-probability equal to 1, \( A[g] \) is Borel and \( \mathcal{P}(A[g]) = 1 \).

It remains to show that \( \tau_{y,\omega} \in A[g] \) if \( \omega \in A[g] \) and \( y \in \mathbb{R}^d \). Fix \( \varphi \in C_1(\mathbb{R}^d) \). We have

\[
\int d\mu_{\tau_{y,\omega}}^\varepsilon(x) \varphi(x)g(\tau_{x/\varepsilon}\tau_{y,\omega}) = \varepsilon^d \sum_{a \in \omega} \varphi(\varepsilon a - \varepsilon y)g(\tau_{a,\omega}).
\]

Given \( \delta > 0 \) we take \( \rho \in (0,1) \) such that the oscillation of \( \varphi \) is bounded by \( \delta \) in any box with sides of length at most \( \rho \). We can suppose \( \varepsilon \) small enough such that \( |\varepsilon y| < \rho \). Then we can bound

\[
(\varphi(\varepsilon a) - \delta)\psi_{N+1}(\varepsilon a) \leq \varphi(\varepsilon a - \varepsilon y) \leq (\varphi(\varepsilon a) + \delta)\psi_{N+1}(\varepsilon a).
\]

As a byproduct of (172) and (173) we have

\[
\int d\mu_{\tau_{y,\omega}}^\varepsilon(x)(\varphi(x) - \delta)\psi_{N+1}(x)g(\tau_{x/\varepsilon}\omega) \leq \int d\mu_{\tau_{y,\omega}}^\varepsilon(x)(\varphi(x) + \delta)\psi_{N+1}(x)g(\tau_{x/\varepsilon}\omega) \leq \int d\mu_{\tau_{y,\omega}}^\varepsilon(x)(\varphi(x) + \delta)\psi_{N+1}(x)g(\tau_{x/\varepsilon}\omega) \leq \int d\mu_{\tau_{y,\omega}}^\varepsilon(x)(\varphi(x) + \delta)\psi_{N+1}(x)g(\tau_{x/\varepsilon}\omega).
\]

By taking the limit \( \varepsilon \downarrow 0 \) and using that \( \omega \in A[g] \) to treat the first and third limit, and afterwards taking the limit \( \delta \downarrow 0 \), we get that the second term converges as \( \varepsilon \downarrow 0 \) to \( m \int \varphi(x)dx\mathbb{E}_0[g] \). This concludes the proof that \( \tau_{y,\omega} \in A[g] \).
A.3. Proof of Lemma 10.3 Since $H_{\omega,\epsilon}$ and $H^1_{\omega,\epsilon}$ are isomorphic, it is enough to focus on $H_{\omega,\epsilon}$. Take a sequence $(v_n, \nabla v_n)$ in $H_{\omega,\epsilon}^1$ converging to $(v, g)$ in $L^2(\mu_\omega^\epsilon) \times L^2(\nu_\omega^\epsilon)$. Since $\mu_\omega^\epsilon$ is an atomic measure, we have that $v_n(\epsilon x) \to v(\epsilon x)$ for any $x \in \hat{\omega}$. This implies that $\nabla v_n(\epsilon x,z) \to \nabla v(\epsilon x,z)$ for any $x \in \hat{\omega}$ and $z \in \tau_{\omega,\hat{\omega}}$, i.e. $\nabla v_n \to \nabla v$ $\nu_\omega^\epsilon$-a.s. On the other hand, since $\nabla v_n \to g$ in $L^2(\nu_\omega^\epsilon)$, at cost to extract a subsequence we have that $\nabla v_n \to g$ in $\nu_\omega^\epsilon$-a.s. By the uniqueness of the a.s. limit it must be $g = \nabla v$ $\nu_\omega^\epsilon$-a.s.

A.4. Proof of Lemma 5.7 Let $u \in H^1_{\text{env}}$ be the weak solution of equation (51) with $f := \zeta$. By taking $v := u$ in (52) we get

$$\int |\nabla u|^2 dv + \int |u|^2 d\mathcal{P}_0 = \int \zeta ud\mathcal{P}_0. \quad (175)$$

Hence it holds $\mathbb{E}[u^2] \leq \mathbb{E}[u\zeta]$, which implies by Schwarz inequality that $\mathbb{E}[u^2] \leq \mathbb{E}[\zeta^2]$. We also observe that Eq. (52) with $f := \zeta$ can be written as

$$\int (u - \zeta)vd\mathcal{P}_0 = - \int \nabla u \nabla vd\nu \quad \forall v \in H^1_{\text{env}}. \quad (176)$$

Take $v$ bounded and measurable. Then $v \in H^1_{\text{env}}$. By Lemma 5.4 we get

$$- \int \nabla u \nabla vd\nu = \int \text{div}(\nabla u)vd\mathcal{P}_0.$$ 

Hence, by (176), for each $v$ bounded and measurable it must be $\int (u - \zeta)vd\mathcal{P}_0 = \int \text{div}(\nabla u)vd\mathcal{P}_0$. This implies that $\text{div}(\nabla u) = u - \zeta \in L^2(\mathcal{P}_0)$. Since $\nabla u \in L^2(\nu)$, by the assumptions on $\zeta$ we get that $\mathbb{E}[u - \zeta]^2 = 0$. The last identity implies that $\mathbb{E}[u^2] = \mathbb{E}[\zeta^2]$ and that $0 \leq \mathbb{E}[(\zeta - u)^2] = -\mathbb{E}[u(\zeta - u)]$, i.e. $\mathbb{E}[u^2] = \mathbb{E}[\zeta^2]$ and $\mathbb{E}[u^2] \geq \mathbb{E}[u\zeta]$. We conclude that $\mathbb{E}[u^2] \geq \mathbb{E}[u\zeta] = \mathbb{E}[\zeta^2]$. We have already proved that $\mathbb{E}[u^2] \leq \mathbb{E}[\zeta^2]$. Hence, we get that $\mathbb{E}[u^2] = \mathbb{E}[\zeta^2] = \mathbb{E}[u\zeta]$. Using this last identity in (175) we conclude that $\int |\nabla u|^2 dv = 0$. By (176) and using that $\nabla u = 0 \nu$-a.s., for any $v \in H^1_{\text{env}}$, it holds $\int uvd\mathcal{P}_0 = \int \zeta vd\mathcal{P}_0$. By taking $v$ varying among the bounded Borel functions, we conclude that $u = \zeta$ in $L^2(\mathcal{P}_0)$. As $u = f \mathcal{P}_0$-a.s. and $u \in H^1_{\text{env}}$ we conclude that $\zeta = u$ in $H^1(\nu)$ and $\|\nabla \zeta\|_{L^2(\nu)} = \|\nabla u\|_{L^2(\nu)} = 0$.

A.5. Proof of Lemma 10.2 We need to prove (52) whenever $u_\epsilon \nrightarrow u$. It is enough to show that, for any sequence $\epsilon_n \downarrow 0$, there exists a subsequence $\epsilon_{k_n}$ such that (52) holds for $\epsilon$ varying in $\{\epsilon_{k_n}\}$. Since $\{u_\epsilon\}$ and $\{v_\epsilon\}$ are bounded families in $L^2(\mu_\omega^\epsilon)$, there exists $C > 0$ such that $\int_{\mathbb{R}^d} d\mu_\omega^\epsilon(x) v_\epsilon(x) u_\epsilon(x)$ and $\int_{\mathbb{R}^d} d\mu_\omega^\epsilon(x) u_\epsilon(x)^2$ are in $[-C,C]$ for $\epsilon$ small enough. By compactness, at cost to take a subsequence $\epsilon_{k_n}$, we can suppose that

$$\lim_{k \to \infty} \int d\mu_\omega^\epsilon(x) v_{\epsilon_k}(x) u_{\epsilon_k}(x) = \alpha, \quad \lim_{k \to \infty} \int d\mu_\omega^\epsilon(x) u_{\epsilon_k}(x)^2 = \beta, \quad (177)$$

for suitable $\alpha, \beta \in [-C,C]$. Given $t \in \mathbb{R}$, since $v_\epsilon + tu_\epsilon \nrightarrow v + tu$, by Lemma 10.2 we have

$$\lim_{k \to \infty} \int d\mu_\omega^\epsilon(x) (v_{\epsilon_k}(x) + tu_{\epsilon_k}(x))^2 \geq \int d\mathcal{P}_0(\omega) \int dx \; m(v + tu)^2(x,\omega). \quad (178)$$
By expanding the square in the l.h.s. and using (88) and (177), we get
\[ 2t\alpha + t^2\beta \geq 2t \int d\mathcal{P}_0(\omega) \int dx \, mv(x, \omega) u(x, \omega) + t^2 \int d\mathcal{P}_0(\omega) \int dx \, mu(x, \omega)^2. \]
Dividing by \( t \) and afterwards taking the limits \( t \to 0^+ \) and \( t \to 0^- \), we get that
\[ \alpha = \int d\mathcal{P}_0(\omega) \int dx \, mv(x, \omega) u(x, \omega), \]
which corresponds to (82).

Acknowledgements: I thank Mauro Mariani and Andrey Piatnitski for usefull discussions. I warmly thank Annibale Faggionato and Bruna Tecchio for their nice hospitality in Codroipo (Italy), where part of this work has been completed.

References

[1] G. Allaire; *Homogenization and two-scale convergence*. SIAM J. Math. Anal. 23, 1482–1518 (1992).
[2] V. Ambegaokar, B.I. Halperin, J.S. Langer; *Hopping conductivity in disordered systems*. Phys. Rev. B 4, 2612–2620 (1971).
[3] G. Androulakis, J. Bellissard, C. Sadel *Dissipative Dynamics in Semiconductors at Low Temperature*. J. Stat. Phys. 147, Issue 2, 448–486 (2012).
[4] H. Brezis; *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York, Springer Verlag, 2010.
[5] P. Caputo, A. Faggionato, *Isoperimetric inequalities and mixing time for a random walk on a random point process*. Ann. Appl. Probab. 17, 1707–1744 (2007).
[6] P. Caputo, A. Faggionato, *Diffusivity of 1-dimensional generalized Mott variable range hopping*. Ann. Appl. Probab. 19, 1459–1494 (2009).
[7] P. Caputo, A. Faggionato, A. Gaudilli`ere, *Recurrence and transience for a random walk on a random point process*. Electron. J. Probab. 14, 2580–2616 (2009).
[8] P. Caputo, A. Faggionato, T. Prescott, *Invariance principle for Mott variable range hopping and other walks on point processes*. Ann. Inst. H. Poincaré Probab. Statist. 49, 654–697 (2013).
[9] D.J. Daley, D. Vere-Jones; *An Introduction to the Theory of Point Processes*. New York, Springer Verlag, 1988.
[10] R. Durrett. Ten lectures on particle systems. In *Ecole d’Eté de Probabilités de Saint-Flour XXIII–1993*, Lect. Notes Math. 1608, 97–201 (1995).
[11] A. Faggionato; *Bulk diffusion of 1D exclusion process with bond disorder*. Markov Processes and Related Fields 13, 519–542 (2007).
[12] A. Faggionato; *Random walks and exclusion processes among random conductances on random infinite clusters: homogenization and hydrodynamic limit*. Electron. J. Probab. 13, 2217–2247 (2008).
[13] A. Faggionato; *Hydrodynamic limit of zero range processes among random conductances on the supercritical percolation cluster*. Electron. J. Probab. 15, 259–291 (2010).
[14] A. Faggionato; *1D Mott variable-range hopping with external field*. To appear as proceeding for the thematic quarter “Stochastic dynamics out of equilibrium”, IHP 2017. Available online at arXiv:1803.05166 (2018).
[15] A. Faggionato, P. Mathieu; *Mott law as upper bound for a random walk in a random environment*. Comm. Math. Phys. 281, 263–286 (2008).
[16] A. Faggionato, H. Schulz-Baldes, D. Spehner; *Mott law as lower bound for a random walk in a random environment*. Comm. Math. Phys., 263, 21–64 (2006).
[17] F. Flegel, M. Heida, M. Slowik; *Homogenization theory for the random conductance model with degenerate ergodic weights and unbounded-range jumps*. To appear on Ann. Inst. H. Poincaré Probab. Statist. (2019).
[18] P. Franken, D. König, U. Arndt, V. Schmidt; *Queues and Point Processes*. Berlin, Akademie Verlag, 1981.

[19] T. M. Liggett, *Interacting particle systems*. Grundlehren der Mathematischen Wissenschaften 276, Springer, New York (1985).

[20] P. Mathieu, A. Piatnitski. *Quenched invariance principles for random walks on percolation clusters*. Proceedings of the Royal Society A. 463 (2007) 2287–2307.

[21] K. Matthes, J. Kerstan, J. Mecke; *Infinitely Divisible Point Processes*. Wiley Series in Probability and Mathematical Physics, New York, Wiley, 1978.

[22] R. Meester, R. Roy. *Continuum percolation*. Cambridge Tracts in Mathematics 119. First edition, Cambridge University Press, Cambridge, 1996.

[23] A. Miller, E. Abrahams; *Impurity Conduction at Low Concentrations*. Phys. Rev. 120, 745–755 (1960).

[24] N.F. Mott; *On the transition to metallic conduction in semiconductors*. Can. J. Phys. 34, 1356–1368 (1956).

[25] N.F. Mott; *Conduction in glasses containing transition metal ions*. J. Non-Crystal. Solids 1, 1–17 (1968).

[26] N.F. Mott; *Conduction in non-crystalline materials III. Localized states in a pseudogap and near extremities of conduction and valence bands*. Phil. Mag. 19, 835–852 (1969).

[27] N.F. Mott, E.A. Davis; *Electronic Processes in Non-Crystalline Materials*. New York, Oxford University Press, 1979.

[28] K. Nagy; *Symmetric random walk in random environment*. Period. Math. Hung. 45, 101–120 (2002).

[29] M. Pollak, M. Ortuno, A. Frydman; *The electron glass*. First edition, Cambridge University Press, United Kingdom, 2013.

[30] G. Nguetseng; *A general convergence result for a functional related to the theory of homogenization*. SIAM J. Math. Anal. 20, 608–623 (1989).

[31] A. Piatnitski, E. Zhizhina. *Periodic homogenization of nonlocal operators with a convolution-type kernel*. SIAM J. Math. Anal. 49, 64–81, 2017.

[32] F. Redig, E. Saada, F. Sau; *Symmetric simple exclusion process in dynamic environment: hydrodynamics*. Preprint [arXiv:1811.01366] (2018).

[33] M. Reed, B. Simon; *Methods of modern mathematical physics*. Functional Analysis. Vol. I. Academic Press, San Diego, 1980.

[34] A. Rousselle; *Quenched invariance principle for random walks on Delaunay triangulations*. Electron. J. Probab. 20, p. 33 (2015)

[35] V. Bansaye, M. Salvi, V.-C. Tran. In preparation.

[36] S. Shklovskii, A.L. Efros; *Electronic Properties of Doped Semiconductors*. Springer Verlag, Berlin, 1984.

[37] A.A. Tempel’man; *Ergodic theorems for general dynamical systems*. Trudy Moskov. Mat. Obsc. 26, 95–132 (1972) [Translation in Trans. Moscow Math. Soc. 26, 94–132, (1972)]

[38] V.V. Zhikov, A.L. Pyatnitskii; *Homogenization of random singular structures and random measures*. (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 70, no. 1, 23–74 (2006); translation in Izv. Math. 70, no. 1, 19–67 (2006).

[39] V.V. Zhikov; *On an extension of the method of two-scale convergence and its applications*. (Russian) Mat. Sb. 191, no. 7, 31–72 (2000); translation in Sb. Math. 191, no. 7-8, 973–1014 (2000).

Alessandra Faggionato. Dipartimento di Matematica, Università di Roma ‘La Sapienza’ P.le Aldo Moro 2, 00185 Roma, Italy
E-mail address: faggiona@mat.uniroma1.it