MULTI-FREQUENCY CALDERÓN-ZYGMUND ANALYSIS AND
CONNEXION TO BOCHNER-RIESZ MULTIPLIERS

FRÉDÉRIC BERNICOT

Abstract. In this work, we describe several results exhibited during a talk at the El Escorial 2012 conference. We aim to pursue the development of a multi-frequency Calderón-Zygmund analysis introduced in [10]. We set a definition of general multi-frequency Calderón-Zygmund operators. Unweighted estimates are obtained using the corresponding multi-frequency decomposition of [10]. Involving a new kind of maximal sharp function, weighted estimates are obtained.

The so-called Calderón-Zygmund theory and its ramifications have proved to be a powerful tool in many aspects of harmonic analysis and partial differential equations. The main thrust of the theory is provided by

• the Calderón-Zygmund decomposition, whose impact is deep and far-reaching. This decomposition is a crucial tool to obtain weak type (1, 1) estimates and consequently $L^p$ bounds for a variety of operators;

• the use of the “local” oscillation $f - \left( \frac{1}{|Q|} \int_Q f \right)$ (for $Q$ a ball). These oscillations appear in the elementary functions of the “bad part” coming from the Calderón-Zygmund decomposition and in the definition of the maximal sharp function, which allows to get weighted estimates.

The oscillation $f - \left( \frac{1}{|Q|} \int_Q f \right)$ can be seen as the distance between the function $f$ and the set of constant functions on the ball $Q$, indeed the average is the best way to locally approximate the function by a constant. By this way, the constant function being associated to the frequency 0, we understand how the classical Calderón-Zygmund theory is related to the frequency 0.

As for example, well-known Calderón-Zygmund operators are the Fourier multipliers associated to a symbol $m$ satisfying Hörmander’s condition

$$|\partial^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|} = d(\xi, 0)^{-|\alpha|},$$

which encodes regularity assumption of the symbol relatively to the frequency 0.

In this work, we are interested in the extension of this theory with respect to a collection of frequencies and we focus on sharp constants relatively to the number of the considered frequencies.

Such questions naturally arise as soon as we work on a multi-frequency problem:

• Uniform bounds for a Walsh model of the bilinear Hilbert transform (see [12] by Oberlin and Thiele);
• A variation norm variant of Carleson’s theorem (see [11] by Oberlin, Seeger, Tao, Thiele and Wright);
• Such a multi-frequency Calderón-Zygmund was introduced by Nazarov, Oberlin and Thiele in [10] for proving a variation norm variant of a Bourgain’s maximal inequality.

Similarly to the fact that a Fourier multiplier with a symbol satisfying Hörmander’s condition is a classical Calderón-Zygmund, we may extend this property to a collection of frequencies. More precisely, let \( \Theta := (\xi_1, ..., \xi_N) \) be a collection of frequencies and consider a symbol \( m \) verifying for all multi-indices \( \alpha \)

\[
|\partial^\alpha m(\xi)| \lesssim d(\xi, \Theta)^{-|\alpha|},
\]

with \( d(\xi, \Theta) := \min_{1 \leq i \leq N} |\xi - \xi_i| \). Such symbols give rise to Fourier multipliers, which should be the prototype of what we want to call \textit{multi-frequency Calderón-Zygmund operators}.

In the 1-dimensional setting with a collection of frequencies \( \Theta := (\xi_1, ..., \xi_N) \) (assumed to be indexed by the increasing order \( \xi_1 < \xi_2 < \cdots < \xi_N \)), an example is given by the multi-frequency Hilbert transform which corresponds to the symbol

\[
m(\xi) = \begin{cases} 
-1, & \xi < \xi_1 \\
(-1)^{j+1}, & \xi_j < \xi < \xi_{j+1} \\
(-1)^{N+1}, & \xi > \xi_N.
\end{cases}
\]

Let us now detail a definition of “multi-frequency Calderón-Zygmund” operator:

**Definition 0.1.** Let \( \Theta := (\xi_1, ..., \xi_N) \) be a collection of \( N \) frequencies of \( \mathbb{R}^n \). An \( L^2 \)-bounded linear operator \( T \) is said to be a Calderón-Zygmund operator relatively to \( \Theta \) if there exist operators \( (T_j)_{j=1}^N \) and kernels \( (K_j)_{j=1}^N \) verifying

- Decomposition: \( T = \sum_{j=1}^N T_j \);
- Integral representation of \( T_j \): for every function \( f \in L^2 \) compactly supported and \( x \in \text{supp}(f)^c \),
  \[
  T_j(f)(x) = \int K_j(x, y)f(y);
  \]
- Regularity of the modulated kernels: for every \( x \neq y \)
  \[
  \sum_{j=1}^N \left| \nabla_{(x, y)^c} e^{i\xi_j \cdot (x-y)} K_j(x, y) \right| \lesssim |x - y|^{-n-1}.
  \]

**Remark 0.1.** As usual, we can weaken the regularity assumption and just require an \( \epsilon \)-Hölder regularity on the modulated kernels.

**Remark 0.2.** If the decomposition is assumed to be orthogonal (which means that for \( i \neq j \), \( T_i T_j^* = 0 \)) then it follows that each operator \( T_j \) is a modulated Calderón-Zygmund operator. Such a multi-frequency Calderón-Zygmund operator can also be pointwisely bounded by a sum of \( N \) modulated (classical) Calderón-Zygmund operators and have the same boundedness properties with an implicit constant of order \( N \). The aim is to study how this order can be improved using sharp estimates.

We first obtain unweighted estimates for such operators:

**Theorem 0.1.** Let \( \Theta \) be a collection of \( N \) frequencies and \( T \) an associated multi-frequency Calderón-Zygmund operator. Then
• for $p \in (1, \infty)$, $T$ is bounded on $L^p$ with
  \[ \|T\|_{L^p \to L^p} \lesssim N^{\frac{1}{p} - \frac{1}{2}}. \]

• for $p = 1$, $T$ is of weak-type $(1, 1)$ with
  \[ \|T\|_{L^1 \to L^{1, \infty}} \lesssim N^{\frac{1}{2}}. \]

This theorem relies on an adapted Calderón-Zygmund decomposition introduced in [10] by Nazarov, Oberlin and Thiele. We point out that there the constant $N^{\frac{1}{2}}$ is shown to be optimal and this is the same for the previous weak-type estimate.

Concerning weighted estimates, it is well-known that linear Calderón-Zygmund operators are bounded on $L^p(\omega)$ for $p \in (1, \infty)$ and every weight $\omega$ belonging to the Muckenhoupt’s class $A_p$ (see Definitions 1.1 and 1.2 for more details about Muckenhoupt’s class $A_p$ and Reverse Hölder class $RH_s$). Similar properties are satisfied by the Hardy-Littlewood maximal operator and some other linear operators as Bochner-Riesz multipliers [15, 4] or non-integral operators (like Riesz transforms) [1]. All these boundedness, obtained by using suitable Fefferman-Stein inequalities related to maximal sharp functions, involve weights belonging to the class $W^p(p_0, q_0) := A_{p_0} \cap RH_{\left(\frac{q_0}{p_0}\right)}$ for some exponents $p_0 < q_0$.  

As a consequence, it seems that these classes of weights are well-adapted for proving boundedness of linear operators. Following this observation, we will consider a multi-frequency maximal sharp function, in order to prove weighted estimates for our multi-frequency operators:

**Theorem 0.2.** Let $\Theta$ be a collection of $N$ frequencies. For $p \in (1, \infty)$, $s \in (1, p)$ and $t \in (1, \infty)$, then every multi-frequency Calderón-Zygmund operator $T$ is bounded on $L^p(\omega)$ for every weight $\omega \in RH_t \cap A_s$ with
  \[ \|T\|_{L^p(\omega) \to L^p(\omega)} \lesssim N^\gamma \]
and
  \[ \gamma := \frac{tp}{s \min\{2, s\}} + \left| \frac{1}{2} - \frac{1}{s} \right|. \]

We emphasize that this result is only interesting when $\gamma < 1$.

The current paper is organized as follows: after some preliminaries about weights, examples of multi-frequency operators and the main lemma for the multi-frequency analysis, Theorem 0.1 is proved in Section 2. Then in Section 3, we develop the general approach for weighted estimates, based on a suitable maximal sharp function. In Section 4, we describe how this point of view could be used to Bochner-Riesz multipliers.

---

1From [8], we know that for $r, s > 1$,
  \[ A_r \cap RH_s = \{ \omega, \omega' \in A_{1+s(r-1)} \}, \]
so these classes of weights are equivalent to a class of powers of Muckenhoupt’s weights.
1. Notations and preliminaries

Let us consider the Euclidean space \( \mathbb{R}^n \) equipped with the Lebesgue measure \( dx \) and its Euclidean distance \( |x - y| \). Given a ball \( Q \subset \mathbb{R}^n \) we denote its center by \( c(Q) \) and its radius by \( r_Q \). For any \( \lambda > 1 \), we denote by \( \lambda Q := B(c(Q), \lambda r_Q) \). We write \( L^p \) for \( L^p(\mathbb{R}^n, \mathbb{R}) \) or \( L^p(\mathbb{R}^n, \mathbb{C}) \).

For a subset \( E \subset \mathbb{R}^n \) of finite and non-vanishing measure and \( f \) a locally integrable function, the average of \( f \) on \( E \) is defined by

\[
\int_E f \, dx := \frac{1}{|E|} \int_E f(x) \, dx.
\]

Let us denote by \( \mathcal{Q} \) the collection of all balls in \( \mathbb{R}^n \). We write \( \mathcal{M} \) for the maximal Hardy-Littlewood function:

\[
\mathcal{M}f(x) = \sup_{Q \in \mathcal{Q}} \int_Q |f| \, dx.
\]

For \( p \in (1, \infty) \), we set \( \mathcal{M}_p f(x) = \mathcal{M}(|f|^p)(x)^{1/p} \). The Fourier transform will be denoted by \( \mathcal{F} \) as an operator and we make use of the other usual notation \( \hat{f} \).

In the current work, we aim to develop a multi-frequency analysis, based on the following lemma:

**Lemma 1.1** ([2]). Let \( \Theta \subset \mathbb{R}^n \) be a finite collection of frequencies and \( Q \) be a ball. For every function \( \phi \) belonging to the subspace of \( L^2(3Q) \), spanned by \( (e^{i \xi \cdot x})_{\xi \in \Theta} \), we have for \( p \in [1, 2] \)

\[
\|\phi\|_{L^\infty(Q)} \lesssim (\# \Theta)^{1/p} \left( \int_{3Q} |\phi|^p \, dx \right)^{1/p}.
\]

**Remark 1.1.** In [2], this lemma is stated and proved in a one-dimensional setting. However, the proof only relies on the additive group structure of the ambient space by using translation operators. So the exact same proof can be extended to a multi-dimensional setting.

**Remark 1.2.** The question of extending the previous lemma for \( p \in (2, \infty) \) is still open in such a general situation. Of course, (1) is true for \( p = \infty \) and so it would be reasonable to expect the result for intermediate exponents \( p \in (2, \infty) \). Unfortunately, the well-known interpolation theory does not apply here.

However, in some specific situations, we may extend this lemma for \( p \geq 2 \). Indeed, if \( p = 2k \) is an even integer then applying (1) with \( p = 2 \) and \( \Theta^k := \{\theta_{i_1} + \ldots + \theta_{i_k}, \ i \in \Theta\} \) to \( \phi^k \) yields

\[
\|\phi\|_{L^\infty(Q)} \lesssim \|\phi^k\|_{L^\infty(Q)}^{1/k} \left( \int_{3Q} |\phi|^{2k} \, dx \right)^{1/2k} \lesssim (\# \Theta^k)^{1/p} \left( \int_{3Q} |\phi|^p \, dx \right)^{1/p}.
\]

By this way, we see that an extension of (1) for \( p \geq 2 \) may be related to sharp combinatorial arguments, to estimate \( \# \Theta^k \) (a trivial bound is \( \# \Theta^k \leq (\# \Theta)^k \) which does not improve (1)).

We aim to obtain weighted estimates, involving Muckenhoupt’s weights.
Definition 1.1. A weight \( \omega \) is a non-negative locally integrable function. We say that a weight \( \omega \in A_p, 1 < p < \infty \), if there exists a positive constant \( C \) such that for every ball \( Q \),
\[
\left( \int_Q \omega \, dx \right) \left( \int_Q \omega^{1-p'} \, dx \right)^{p-1} \leq C.
\]
For \( p = 1 \), we say that \( \omega \in A_1 \) if there is a positive constant \( C \) such that for every ball \( Q \),
\[
\int_Q \omega \, dx \leq C \omega(y), \quad \text{for a.e. } y \in Q.
\]
We write \( A_{\infty} = \cup_{p \geq 1} A_p \).

We just recall that for \( p \in (1, \infty) \), the maximal function \( M \) is bounded on \( L^p(\omega) \) if and only if \( \omega \in A_p \). We also need to introduce the reverse H"older classes.

Definition 1.2. A weight \( \omega \in RH_p, 1 < p < \infty \), if there is a constant \( C \) such that for every ball \( Q \),
\[
\left( \int_Q \omega^p \, dx \right)^{1/p} \leq C \left( \int_Q \omega \, dx \right).
\]
It is well known that \( A_{\infty} = \cup_{r>1} RH_r \). Thus, for \( p = 1 \) it is understood that \( RH_1 = A_{\infty} \).

1.1. Examples of multi-frequency Calderón-Zygmund operators. Let us detail particular situations where such multi-frequency operators appear.

The multi-frequency Hilbert transform. As explained in the introduction, an example of such multi-frequency operators in the 1-dimensional setting is the multi-frequency Hilbert transform. In \( \mathbb{R} \), consider an arbitrary collection of frequencies \( \Theta := (\xi_1, ..., \xi_N) \) (assumed to be indexed by the increasing order \( \xi_1 < \xi_2 < \cdots < \xi_N \)). The associated multi-frequency Hilbert transform is the Fourier multiplier corresponding to the symbol
\[
m(\xi) = \begin{cases} 
-1, & \xi < \xi_1 \\
(-1)^{j+1}, & \xi_j < \xi < \xi_{j+1} \\
(-1)^{N+1}, & \xi > \xi_N.
\end{cases}
\]
Associated to \( \Theta \), we have a collection of disjoint intervals \( \Delta := \{(-\infty, \xi_1), (\xi_1, \xi_2), ..., (\xi_N, \infty)\} \).

It is well-known by Rubio de Francia’s work [13] that for \( q \in (1, 2] \), the functional
\[
f \to \left( \sum_{\omega \in \Delta} \left| \mathcal{F}^{-1} [1_{\omega} \mathcal{F} f] \right|^q \right)^{1/q}
\]
is bounded on \( L^p \) for \( p \in (q', \infty) \).

The boundedness of the multi-frequency Hilbert transform is closely related to the understanding of (2) for \( q \to 1 \).

We point out that in Rubio de Francia’s result, the obtained estimates do not depend on the collection of intervals \( \Delta \). More precisely, excepted the end-point \( p = q' \), the range \( (q', \infty) \) is optimal for a uniform (with respect to the collection \( \Delta \)) \( L^p \)-boundedness of (2). So it is natural that for \( q \to 1 \) things are more difficult, which is illustrated by our multi-frequency Calderón-Zygmund analysis. Indeed, for example if one considers the particular case \( \Theta := (1, ..., N) \), then following the notations of Remark 1.2, we have \( \Theta^k = \{k, ..., kN\} \) and so \( \sharp \Theta^k = k(N-1)+1 \simeq kN \).
Hence, in this situation we have observed (see Remark 1.2) that we can extend Lemma 1.1 to exponents \( p \in [1, \infty] \) (the implicit constant appearing in (1) is only depending on \( p \)). By this way, Theorem 0.2 can be improved and we obtain a better exponent
\[
\gamma = \frac{tp}{s^2} + \left| \frac{1}{2} - \frac{1}{s} \right|.
\]
Consequently, it seems that for the \( L^p \)-boundedness of the multi-frequency Hilbert transform, the collection \( \Theta \) could play an important role (which was not the case for the \( \ell^q \)-functional (2) with \( q' < p \)).

**Multi-frequency operators coming from a covering of the frequency space.** Let \((Q_j)_{j=1,...,N}\) be a family of disjoint cubes and \( \phi_j \) a smooth function with \( \hat{\phi}_j \) supported and adapted to \( Q_j \). Then consider the linear operator given by
\[
T(f) = \sum_{j=1}^{N} \phi_j \ast f.
\]

It is easy to check that \( T \) is a multi-frequency Calderón-Zygmund operator, associated to the collection \( \Theta := (\xi_1, ..., \xi_N) \) where for every \( j \), \( \xi_j := c(Q_j) \) is the center of the ball \( Q_j \). With \( r_j \) the radius of \( Q_j \), we have the regularity estimate
\[
\sum_{j=1}^{N} \left| \nabla (x,y) e^{i \xi_j \cdot (x-y)} \phi_j (x-y) \right| \lesssim |x-y|^{n-1} \sum_{j=1}^{N} \frac{(r_j |x-y|)^{n+1}}{(1+r_j |x-y|)^{M}},
\]
for every integer \( M > 0 \).

So boundedness of \( T \) (Theorem 0.1) yields the inequality
\[
\left\| \sum_{j=1}^{N} \phi_j \ast f \right\|_{L^p} \lesssim C(r_1, ..., r_N)N^{\frac{1}{p} - \frac{1}{2}} \|f\|_{L^p},
\]
with
\[
C(r_1, ..., r_N) := \sup_{t > 0} \sum_{j=1}^{N} \frac{(r_j t)^{n+1}}{(1+r_j t)^M}.
\]

Let us examine some particular situations:

- If the cubes \((Q_j)_j\) have an equal side-length, then as for Proposition 4.1, simple arguments imply (3) for \( p \in [1, \infty] \) without the constant \( C(r_1, ..., r_N) \).
- If the collection \((Q_j)_j\) is dyadic: it exists a point \( \xi_0 \), \( d(Q_j, \xi_0) \simeq r Q_j \simeq 2^j \) then Littlewood-Paley theory implies (3) without the factor \( N^{\frac{1}{p} - \frac{1}{2}} \) (in this case \( C(r_1, ..., r_N) \simeq 1 \)).
- If the cubes \((Q_j)\) have only the dyadic scale: \( r Q_j \simeq 2^j \) (but no assumptions on the centers of the balls) then Littlewood-Paley theory cannot be used. However, our previous results can be applied in this situation and so (3) holds and \( C(r_1, ..., r_N) \simeq 1 \).

We aim to use the new multi-frequency Calderón-Zygmund analysis to extend these inequalities with replacing the convolution operators by more general Calderón-Zygmund operators, still satisfying some orthogonality properties.
Multi-frequency operators coming from variation norm estimates. As explained in the introduction, the multi-frequency Calderón-Zygmund analysis has been first developed for proving a variation norm variant of a Bourgain’s maximal inequality. So our results can be adapted in such a framework. For example, in [7] Grafakos, Martell and Soria have studied maximal inequalities of the form

$$\left\| \sup_{j=1,\ldots,N} |T(e^{i\theta_j} \cdot f)| \right\|_{L^p} \lesssim \|f\|_{L^p}$$

where \((\theta_j)_{j=1,\ldots,N}\) is a collection of frequencies and \(T\) a fixed Calderón-Zygmund operator.

We can ask the same question, for a variation norm variant: for \(q \in [1, \infty)\) consider

$$\left( \sum_{j=1}^N \left| T(e^{i\theta_j} \cdot f) \right|^q \right)^{\frac{1}{q}}$$

and study its boundedness on \(L^p\), with a sharp control of the behaviour with respect to \(N\). By a linearization argument (involving Rademacher’s functions), this \(\ell^q\)-functional can be realized as an average of modulated Calderón-Zygmund operators, associated to the collection \(\Theta := (\theta_j)_j\).

2. Unweighted estimates for multi-frequency Calderón-Zygmund operators

In this section, we aim to prove the weak \(L^1\)-estimate for a multi-frequency Calderón-Zygmund operator, then Theorem 0.1 will easily follow from interpolation and duality.

**Proposition 2.1.** Let \(\Theta = (\xi_1, \ldots, \xi_N)\) be a collection of \(N\) frequencies as above and \(T\) be a Calderón-Zygmund operator relatively to \(\Theta\). Then \(T\) is of weak type \((1, 1)\) with (uniformly with respect to \(N\))

$$\|T\|_{L^1 \to L^{1,\infty}} \lesssim N^{\frac{1}{2}}.$$  

**Proof.** Consider \(f\) a function in \(L^1\) and \(\lambda > 0\), we use the Calderón-Zygmund decomposition of [10] related to the collection of frequencies \(\Theta\). So the function \(f\) can be decomposed \(f = g + \sum_{J \in J} b_J\) with the following properties:

1. \(J\) is a collection of balls and \((3J)_{J \in J}\) has a bounded overlap;
2. for each \(J \in J\), \(b_J\) is supported in \(3J\);
3. we have

$$\sum_{J \in J} |J| \lesssim \sqrt{N} \|f\|_{L^1} \lambda^{-1};$$

• the “good part” \(g\) satisfies

$$\|g\|_{L^2}^2 \lesssim \|f\|_{L^1} \sqrt{N} \lambda;$$

• the cubes \(J\) satisfy

$$\|f\|_{L^1(J)} \lesssim |J| \lambda N^{-\frac{1}{2}}, \|f - b_J\|_{L^2(J)} \lesssim \sqrt{|J|} \lambda;$$

\(^2\)In [10], the multi-frequency Calderón-Zygmund decomposition is only described in \(\mathbb{R}\). The proof is a combination of Lemma 1.1 and the usual Calderón-Zygmund decomposition. Since both of them can be extended in a multi-dimensional framework, the multi-frequency Calderón-Zygmund decomposition performed in [10] still holds in \(\mathbb{R}^n\).
Then, we can use the integral representation and we have

\[ y \text{ projection of } J \] 

To each \( L \) it is sufficient to estimate the measure of the level-set

\[ \Upsilon_\lambda := \{ x, |T(f)(x)| > \lambda \}. \]

With \( b = \sum J b_J \), we have

\[
|\Upsilon_\lambda| \leq |\{ x, |T(g)(x)| > \lambda/2 \}| + |\{ x, |T(b)(x)| > \lambda/2 \}|
\]

\[
\lesssim \lambda^{-2} \|T(g)\|_{L^2}^2 + |\{ x, |T(b)(x)| > \lambda/2 \}|
\]

\[
\lesssim \lambda^{-1} \sqrt{N} \|f\|_{L^1} + |\{ x, |T(b)(x)| > \lambda/2 \}|
\]

where we used the \( L^2 \)-boundedness of \( T \). So it remains us to study the last term. Since (4), we get

\[
\left| \bigcup_{J \in J} 4J \right| \lesssim \sum J |J| \lesssim \sqrt{N} \|f\|_{L^1} \lambda^{-1}.
\]

Consequently, it only remains to estimate the measure of the set

\[
O_\lambda := \left\{ x \in \left( \bigcup_{J \in J} 4J \right)^c, \quad |T(b)(x)| > \lambda/2 \right\}.
\]

Since

\[ (7) \]

\[ |O_\lambda| \lesssim \lambda^{-1} \|T(b_J)\|_{L^1((2J)^c)}, \]

it is sufficient to estimate the \( L^1 \)-norms. Consider \( K \) the kernel of \( T \) and a point \( x_0 \in (\bigcup_{J \in J} 4J)^c \).

Then, we can use the integral representation and we have

\[
T(b)(x_0) = \int K(x_0, y)b(y)dy = \sum J \int_{3J} K(x_0, y)b_J(y)dy.
\]

To each \( J \), we aim to take advantage of the cancellation properties of \( b_J \), so we subtract the projection of \( [y \to K(x_0, y)] \) on the space, spanned by \((e^{iy\cdot\eta})_{\eta \in \Theta}\). So we have

\[
T(b)(x_0) = \sum_J \sum_{j=1}^N \int_{3J} \left[ K_j(x_0, y) - e^{i\xi_j\cdot c(J)} K_j(x_0, c(J)) e^{-i\xi_j\cdot y} \right] b_J(y)dy
\]

\[
= \sum_J \sum_{j=1}^N \int_{3J} \left[ \tilde{K}_j(x_0, y) - \tilde{K}_j(x_0, c(J)) \right] e^{i\xi_j\cdot (x_0-y)} b_J(y)dy
\]

where \( c(J) \) is the center of \( J \) and \( \tilde{K}_j(x, y) := K_j(x, y) e^{-i\xi_j\cdot (x-y)} \). We then write

\[
T_j(b)(x_0) := \int \left[ \tilde{K}_j(x_0, y) - \tilde{K}_j(x_0, c(J)) \right] e^{i\xi_j\cdot (x_0-y)} b(y)dy.
\]

such that \( T(b) = \sum_j T_j(b) \). Due to the regularity assumption on \( K \) (and so on \( \tilde{K}_j \)), it comes for \( y \in J \) and \( x_0 \in (2J)^c \)

\[ (8) \]

\[
\sum_{j=1}^N \left| \tilde{K}_j(x_0, y) - \tilde{K}_j(x_0, c(J)) \right| \lesssim \frac{r_j}{|x_0 - y|^{n+1}}.
\]
So we have
\[ \|T(b_J)\|_{L^1((2J)^c)} \lesssim \int \int_{|x-y| \geq r_J} \frac{r_J}{|x-y|^{n+1}} |b_J(y)| dxdy \lesssim \|b_J\|_{L^1} \lesssim |J| \lambda. \]

Finally, we obtain with (7) that
\[ |O_\lambda| \lesssim \sum J |J| \lesssim \sqrt{N} \|f\|_{L^1} \lambda^{-1}, \]
which concludes the proof. \( \square \)

**Remark 2.1.** Following [10], the bound of order \( N^{\frac{1}{2}} \) is optimal for the multi-frequency decomposition and for the weak-\( L^1 \) estimate.

### 3. Weighted estimates for multi-frequency Calderón-Zygmund operators

Aiming to obtain weighted estimates on such multi-frequency operators (using *Good-lambda inequalities*), we also have to define a suitable maximal sharp function, associated to a collection of frequencies.

**Definition 3.1 (Maximal sharp function).** Let \( \Theta \) be a collection of \( N \) frequencies and \( s \in [1, \infty) \). Consider a ball \( Q \), we denote by \( P_{\Theta,Q} \) the projection operator (in the \( L^s \)-sense) on the subspace of \( L^s(3Q) \), spanned by \( (\exp i\xi \cdot)_{\xi \in \Theta} \). Let us specify this projection operator: consider \( E \) the finite dimensional sub-space of \( L^s(3Q) \), spanned by \( (e^{i\xi \cdot})_{\xi \in \Theta} \) and equipped with the \( L^s(3Q) \)-norm. Since \( E \) is of finite dimension, then for every \( f \in L^s(Q) \) there exists \( v := P_{\Theta,Q}(f) \in E \) such that
\[ \|f - v\|_{L^s(3Q)} = \inf_{\phi \in E} \|f - \phi\|_{L^s(3Q)}. \]

This projection operator may depend on \( s \), which is not important for our purpose so this is implicit in the notation and we forget it.

Since \( 0 \in E \), we obviously have
\[ \|P_{\Theta,Q}(f)\|_{L^s(3Q)} \leq 2\|f\|_{L^s(Q)}. \]

Then, we may define the maximal sharp function
\[ M_{s,\Theta}^*(f)(x_0) := \sup_{x_0 \in Q} \left( \int_Q |f - P_{\Theta,Q}(f 1_Q)|^s dx \right)^{\frac{1}{s}}. \]

Note that the usual sharp maximal function is the one obtained for \( \Theta := \{0\} \) and in this situation it is well-known that the maximal sharp function satisfies a so-called Fefferman-Stein inequality (see [6]). We first prove an equivalent property for this generalised maximal sharp function:

**Proposition 3.1.** Let \( s \in (1, \infty) \), \( t \in [1, \infty) \) and \( p \in (s, \infty) \) be fixed. Then for every function \( f \in L^s \) and every weight \( \omega \in RH_t \), we have for every \( p \geq s \)
\[ \|f\|_{L^p(\omega)} \lesssim N^{\frac{s}{2p}} \max(\frac{1}{2}, \frac{s}{s+1}) \|M_{s,\Theta}^*(f)\|_{L^p(\omega)}. \]

The proof relies on a *Good-lambda inequality* and Lemma 1.1.
Proof. We make use on the abstract theory developed in [1] by Auscher and Martell. We also follow notations of [1, Theorem 3.1]. Indeed, for each ball $Q \subset \mathbb{R}^n$ we have the following

$$F(x) := |f(x)|^s \lesssim |f(x) - \mathbb{P}_{\Theta, Q}(f_{1Q})(x)|^s + |\mathbb{P}_{\Theta, Q}(f_{1Q})(x)|^s := G_Q(x) + H_Q(x).$$

By definition, it comes

$$\int_Q G_Qdx \leq \inf_Q \mathcal{M}_{s, \Theta}^2(f)^s$$

and following Lemma 1.1 (with (9))

$$\sup_{x \in Q} H_Q = \|\mathbb{P}_{\Theta, Q}(f_{1Q})\|_{L^\infty(Q)}^s \lesssim N^{s\max\left(\frac{1}{2}, \frac{1}{s}\right)} \left(\int_{3Q} |\mathbb{P}_{\Theta, Q}(f_{1Q})|^s dx\right) \lesssim N^{s\max\left(\frac{1}{2}, \frac{1}{s}\right)} \left(\int_Q |f|^s dx\right) \lesssim N^{s\max\left(\frac{1}{2}, \frac{1}{s}\right)} \inf_Q MF.$$

So we can apply [1, Theorem 3.1] (with $q = \infty$ and $a \simeq N^{s\max\left(\frac{1}{2}, \frac{1}{s}\right)}$) and by checking the behaviour of the constants with respect to “$a$” in its proof, we obtain for every $p \geq 1$

$$\|\mathcal{M}_s(f)^s\|_{L^p(\omega)} \lesssim N^{s\max\left(\frac{1}{2}, \frac{1}{s}\right)} \left\|\mathcal{M}_{s, \Theta}^2(f)^s\right\|_{L^p(\omega)},$$

which yields the desired result. \qed

Then, we evaluate a multi-frequency Calderón-Zygmund operator via this new maximal sharp function.

**Proposition 3.2.** Let $T$ be a Calderón-Zygmund operator relatively to $\Theta$ and $s \in (1, \infty)$. Then, we have the following pointwise estimate:

$$\mathcal{M}_{s, \Theta}^2(T(f)) \lesssim N^{\frac{1}{2} - \frac{1}{2s}} \mathcal{M}_s(f).$$

**Proof.** We follow the well-known proof for usual Calderón-Zygmund operators and adapt the arguments to the current situation. So consider a point $x_0$ and a ball $Q \subset \mathbb{R}^n$ containing $x_0$, we have to estimate

$$\left(\int_Q |T(f) - \mathbb{P}_{\Theta, Q}(T(f)1_Q)|^s dx\right)^{\frac{1}{s}}.$$

We split the function into a local part $f_0$ and an off-diagonal part $f_\infty$:

$$f = f_0 + f_\infty := f_{1_{10Q}} + f_{1_{(10Q)^c}}.$$

By definition of the projection operator, we know that

$$\left(\int_Q |T(f) - \mathbb{P}_{\Theta, Q}(T(f)1_Q)|^s dx\right)^{\frac{1}{s}} \leq \left(\int_Q |T(f) - \mathbb{P}_{\Theta, Q}(T(f_\infty)1_Q)|^s dx\right)^{\frac{1}{s}} \leq \left(\int_Q |T(f_0)|^s dx\right)^{\frac{1}{s}} + \left(\int_Q |T(f_\infty) - \mathbb{P}_{\Theta, Q}(T(f_\infty)1_Q)|^s dx\right)^{\frac{1}{s}}.$$
For the local part, we use boundedness in $L^s$ of the operator $T$ (Proposition 2.1), hence
\[
\left( \int_Q |T(f_0)|^s \ dx \right)^{\frac{1}{s}} \lesssim |Q|^{-\frac{1}{2}} \|T(f_0)|_{L^s(Q)} \lesssim N^{(\frac{1}{2} - \frac{1}{2})} \|f_0\|_{L^s} \]
\[
\lesssim N^{\frac{1}{2} - \frac{1}{2}} |M_s(f)(x_0)|.
\]
Then let us focus on the second part, involving $f_\infty$.
We use the decomposition (with an integral representation) since we are in the off-diagonal case:
for $x \in Q$
\[
T(f_\infty)(x) = \sum_{j=1}^N \int K_j(x, y)f_\infty(y) dy.
\]
Consider the following function, defined on $3Q$ by (where $c(Q)$ is the center of $Q$)
\[
\Phi := x \in 3Q \to \sum_{j=1}^N \int e^{i \xi_j \cdot (y - c(Q))} K_j(c(Q), y) f_\infty(y) dy.
\]
So $\Phi \in E$ (see Definition 3.1) and hence
\[
\left( \int_Q |T(f_\infty) - \Phi(f_\infty)|^s \ dx \right)^{\frac{1}{s}} \leq \left( \int_Q |T(f_\infty) - \Phi|^s \ dx \right)^{\frac{1}{s}}.
\]
If we set $\tilde{K}_j(x, z) := K_j(x, z)e^{-i \xi_j \cdot (x - z)}$, then
\[
T(f_\infty)(x) - \Phi(x) = \sum_j \int \left[ \tilde{K}_j(x, y) - \tilde{K}_j(c(Q), y) \right] e^{i \xi_j \cdot (y - c)} f_\infty(y) dy.
\]
From the regularity assumption on the kernels $K_j$’s, we have for $y \in (10Q)^c$
\[
\sum_j \left| \tilde{K}_j(x, y) - \tilde{K}_j(c(Q), y) \right| \lesssim r_Q^s \sup_{z \in Q} \sum_j \left| \nabla_x \tilde{K}_j(z, y) \right| \lesssim r_Q^{-n} \left( 1 + \frac{d(y, Q)}{r_Q} \right)^{-n-1}.
\]
We also have (since $y \in (10Q)^c$ and $x, c(Q) \in Q$)
\[
|T(f_\infty)(x) - \Phi(x)| \lesssim \int_{|z| \geq 10r_Q} r_Q^{-n} \left( 1 + \frac{|x - c(Q) - z|}{r_Q} \right)^{-n-1} |f(c(Q) + z)| dz
\]
\[
\lesssim \int_{|z| \geq 5r_Q} r_Q^{-n} \left( 1 + \frac{|z|}{r_Q} \right)^{-n-1} |f(x_0 + z)| dz
\]
\[
\lesssim |M(f)(x_0)|,
\]
which concludes the proof.
\[
\square
\]
We obtain the following corollary:

**Corollary 3.3.** Let $\Theta$ be a collection of $N$ frequencies. For $p \in (2, \infty)$, $s \in [2, p)$ and $t \in (1, \infty)$, a multi-frequency Calderón-Zygmund operator $T$ is bounded on $L^p(\omega)$ for every weight $\omega \in RH_\nu \cap A_p$ with
\[
\|T\|_{L^p(\omega) \to L^p(\omega)} \lesssim N^{\frac{1}{2} + \left( \frac{1}{2} - \frac{1}{p} \right)}.
\]
Proof. Using Propositions 3.1 and 3.2, it follows that for $p > s \geq 2$ (assuming $\omega \in A_p^s$)

$$\|T(f)\|_{L^p(\omega)} \lesssim N^{\frac{sp}{2s+1}} \|\mathcal{M}_{s,\Theta}^p[T(f)]\|_{L^p(\omega)}$$

$$\lesssim N^{\frac{sp}{2s+1} + \frac{1}{s} - \frac{1}{2}} \|M_s(f)\|_{L^p(\omega)}$$

$$\lesssim N^{\frac{sp}{2s+1} + \frac{1}{s} - \frac{1}{2}} \|f\|_{L^p(\omega)},$$

where we used weighted boundedness of the maximal function since $\omega \in A_p^s$. \qed

As explained in the introduction, this estimate is only interesting when the exponent $\frac{sp}{2s+1} + \frac{1}{s} - \frac{1}{2}$ is lower than 1.

4. ConneXion to Bochner-Riesz multipliers

In this section, we aim to describe how such arguments could be applied to generalized Bochner-Riesz multipliers. Weighted estimates for Bochner-Riesz multipliers has been initiated in [15, 5, 4]. We first emphasize that we do not pretend to obtain new weighted estimates for Bochner-Riesz multipliers. But we only want to describe here a new point of view and a new approach for such estimates, which will be the subject of a future investigation. Such an application is a great motivation for pursuing the study of a multi-frequency Calderón-Zygmund analysis.

Consider also $\Omega$ a bounded open subset of $\mathbb{R}^n$ such that its boundary $\Gamma := \overline{\Omega} \setminus \Omega$ is an hyper-manifold of Hausdorff dimension $n - 1$. For $\delta > 0$, we then define the generalized Bochner-Riesz multiplier, given by

$$R_{\Omega,\delta}(f)(x) := \int_{\Omega} e^{ix \cdot \xi} \hat{f}(\xi)m_{\delta}(\xi) d\xi,$$

where $m_{\delta}$ is a smooth symbol supported in $\overline{\Omega}$ and satisfying in $\Omega$

$$|\partial^\alpha m_{\delta}(\xi)| \lesssim d(\xi, \Gamma)^{d-|\alpha|}.$$

We first use a Whitney covering $(O_i)_i$ of $\Omega$. That is a collection of sub-balls such that

- the collection $(O_i)_i$ covers $\Omega$ and has a bounded overlap;
- the radius $r_{O_i}$ is equivalent to $d(O_i, \Gamma)$.

Associated to this collection, we build a partition of the unity $(\chi_i)_i$ of smooth functions such that $\chi_i$ is supported on $O_i$ with

$$\sum_i \chi_i(\xi) = 1_{\Omega}(\xi)$$

and $\|\partial^\alpha \chi_i\|_{\infty} \lesssim r_{O_i}^{-|\alpha|}$.

Then, $R_{\delta}$ may be written as

$$R_{\delta}(f)(x) = \sum_{j=-\infty}^{\infty} T_j(f)(x),$$
with

\[
T_j(f)(x) := \sum_{l \in \mathcal{I}} \int_{\Omega} e^{ix \cdot \xi} \mathcal{F}(\xi)m_\delta(\xi)\chi_l(\xi)d\xi
\]

\[= 2^{j\delta}U_j(f)(x),\]

(12)

where we set

\[
U_j(f)(x) := \sum_{l \in \mathcal{I}} \int_{\Omega} e^{ix \cdot \xi} \mathcal{F}(\xi)(2^{-j\delta}m_\delta(\xi))\chi_l(\xi)d\xi.
\]

Observation: The main idea is to observe that the operator \(U_j\) is a multi-frequency Calderón-Zygmund operator associated to the collection

\[\Theta_j := \{c(O_l), 2^j \leq r_{O_l} < 2^{j+1}\} \quad \text{with} \quad \#\Theta_j \simeq 2^{-j(n-1)}.
\]

However, these operators have specific properties, one of them is that the considered balls have equivalent radius, which means that these operators have only one scale \(2^j\). For example, this observation allows us to easily prove some boundedness:

**Proposition 4.1.** Uniformly with \(j \lesssim 0\), the multiplier \(U_j\) is a convolution operation with a kernel \(K_j\) satisfying

\[\|K_j\|_{L^1} \lesssim 2^{-j\frac{n-1}{2}}.
\]

Hence, it follows that \(U_j\) is bounded on Lebesgue space \(L^p\) for every \(p \in [1, \infty]\). Moreover for every \(s \in [1, 2]\), \(p \in (s, \infty)\) and every weight \(\omega \in A_p\), \(U_j\) is bounded on \(L^p(\omega)\) with

\[\|U_j\|_{L^p(\omega) \rightarrow L^p(\omega)} \lesssim 2^{-j\frac{n-1}{2}}.
\]

**Proof.** The operator \(U_j\) is a Fourier multiplier, associated to the symbol

\[\sigma_j(\xi) := \sum_{l \in \mathcal{I}} (2^{-j\delta}m_\delta(\xi))\chi_l(\xi).
\]

Since the considered balls \((O_l)_l\) are almost disjoint, it comes that

\[\|\sigma_j\|_{L^2} \lesssim \{|\xi, d(\xi, \partial\Omega) \simeq 2^j\}|^{\frac{1}{2}} \lesssim 2^{\frac{j}{2}}.
\]

Moreover, using regularity assumptions on \(m_\delta\), we deduce that for every \(\alpha\)

\[\|\partial^\alpha \sigma_j\|_{L^2} \lesssim 2^{-j|\alpha|}\{|\xi, d(\xi, \partial\Omega) \simeq 2^j\}|^{\frac{1}{2}} \lesssim 2^{j(\frac{1}{2}-|\alpha|)}.
\]

So with \(K_j := \mathcal{F}(\sigma_j)\), it follows that for any integer \(M\)

\[(1 + 2^j |\cdot|)^M K_j \|_{L^2} \lesssim 2^{\frac{j}{2}}.
\]

Hence

\[\|K_j\|_{L^1} \lesssim 2^{-j\frac{n-1}{2}}.
\]

Using Minkowski inequality, we deduce that for every \(p \in [1, \infty]\)

\[\|U_j\|_{L^p \rightarrow L^p} \lesssim \|K_j\|_{L^1} \lesssim 2^{-j\frac{n-1}{2}}.
\]
Let us now focus on the second claim about weighted estimates. Using integrations by parts for computing the kernel $K_j$, it comes for any integer $M$

\begin{equation}
\| (1 + 2^j \cdot | \cdot |^M K_j \|_{L^\infty} \lesssim 2^j. \tag{14}
\end{equation}

By interpolation with (13), for $s \in [1, 2]$ we get

\begin{equation}
\| (1 + 2^j \cdot | \cdot |^M K_j \|_{L^{s'}} \lesssim 2^j, \tag{15}
\end{equation}

which gives

$$U_j(f) \lesssim 2^{-j \frac{n-1}{s}} M_s(f).$$

Hence, for every $p > s$ and every weight $\omega \in A_p$

$$\| U_j \|_{L^p(\omega) \to L^p(\omega)} \lesssim 2^{-j \frac{n-1}{s}}.$$  \hfill \Box

In this context, $\sharp \Theta_j \simeq 2^{-j(n-1)}$, so the constant $2^{-j \frac{n-1}{s}}$ is equivalent to $(\sharp \Theta_j)^{\frac{1}{s}}$ and this is a better constant than the one obtained in Corollary 3.3 (for a subclass of $A_p^n$ weights).

So improving these “easy bounds” means to obtain inequalities such as

$$\| U_j \|_{L^p(\omega) \to L^p(\omega)} \lesssim (\sharp \Theta_j) \gamma$$

for some better exponent $\gamma < \frac{1}{s}$.

Let us finish by suggesting how could we get improvements of our approach to get interesting results for Bochner-Riesz multipliers:

**Question:** The general approach, developed in the previous section, only allows to get an exponent

$$\gamma = \frac{tp}{2s} + \left( \frac{1}{2} - \frac{1}{s} \right)$$

(with some $s \in [2, p]$) which is bigger than $\frac{1}{2}$ (since $p > s \geq 2$ and $t > 1$). So to improve this exponent $\gamma$, two things seem to be crucial:

- to extend the use of Lemma 1.1 for $p \geq 2$ which would allow us to get an exponent $\frac{tp}{2s}$ instead of $\frac{tp}{s}$;
- to use the geometry of the boundary $\Gamma$ to get better exponents, even for the unweighted estimates. Indeed, for example for the unit ball (using its non vanishing curvature), we know that (see [9, 14])

$$\| U_j \|_{L^p \to L^p} \lesssim 2^{-j \delta(p)}$$

with if $n = 2$

$$\delta(p) := \max \left\{ 2 \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{2}, 0 \right\}.$$  

and if $n \geq 3$ and $p \geq \frac{2(n+2)}{n}$ or $p \leq \frac{2(n+2)}{n+1}$

$$\delta(p) := \max \left\{ n \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{2}, 0 \right\}.$$
References

[1] P. Auscher and J.M. Martell, Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part I: General operator theory and weights. Adv. Math. 212 (2007), no. 1, 225–276.

[2] P. Borwein and T. Erdélyi, Nikoliskii-type inequalities for shift invariant function spaces, Proc. Amer. Math. Soc. 134 (2006), no. 11, 3243–3246.

[3] J. Bourgain, Estimates for cone multipliers, Operator Theory: Advances and Applications 77 (1995), 41–60.

[4] M. J. Carro, J. Duoandikoetxea and M. Lorente, Weighted estimates in a limited range with applications to the Bochner-Riesz operators, Indiana Univ. Math. J.

[5] M. Christ, On almost everywhere convergence of Bochner-Riesz means in higher dimensions, Proc. Amer. Math. Soc. 95 (1985), no. 1, 16–20.

[6] C. Fefferman and E.M. Stein, $H^p$ spaces in several variables, Acta Math. 129 (1972), 137–193.

[7] L. Grafakos, J.M. Martell and F. Soria, Weighted norm inequalities for maximally modulated singular integral operators, Math. Ann. 331 (2005), no. 2, 359–394.

[8] R. Johnson and C.J. Neugebauer, Change of variable results for $A_p$ and reverse Hölder $RH_p$ classes, Trans. Amer. Math. Soc. 328 (1991), 639–666.

[9] S. Lee, Improved bounds for Bochner-Riesz and maximal Bochner-Riesz operators, Duke Math. J. 122 (2004), 205–232.

[10] F. Nazarov and R. Oberlin and C. Thiele, A Calderón Zygmund decomposition for multiple frequencies and an application to an extension of a lemma of Bourgain, Math. Res. Lett. 17 (2010), no. 3, 529–545.

[11] R. Oberlin, A. Seeger, T. Tao, C. Thiele and J. Wright, A variation norm Carleson theorem, J. Eur. Math. Soc. 14 (2012), no. 2, 421–464.

[12] R. Oberlin and C. Thiele, New uniform bounds for a Walsh model of the bilinear Hilbert transform, Indiana Univ. Math. J. 60 (2011), 1693–1712.

[13] J.L. Rubio de Francia, A Littlewood-Paley inequality for arbitrary intervals, Rev. Mat. Iberoamericana 1 (1985), no. 2, 1–14.

[14] T. Tao, Recent progress on the restriction conjecture, ArXiv math.CA/0311181.

[15] A. Vargas, Weighted weak type $(1,1)$ bounds for rough operators, J. London Math. Soc. (2) 54 (1996), no. 2, 297–310.

Frédéric Bernicot, Laboratoire de Mathématiques Jean Leray, 2, Rue de la Houssinière F-44322 Nantes Cedex 03, France.

E-mail address: frederic.bernicot@univ-nantes.fr