A COMBINATORIAL PROOF OF A SYMMETRY OF
(t, q)-EULERIAN NUMBERS OF TYPE B AND D

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ABSTRACT. A symmetry of (t, q)-Eulerian numbers of type B is combinatorially proved using permutation tableaux of type B; we define a bijection preserving many important statistics on the set of permutation tableaux of type B. This bijection proves a symmetry of the generating polynomial \( \hat{D}_{n,k}(p, q, r) \) of number of crossings, (two types of) alignments, and hence \( q \)-Eulerian numbers of type A (defined by L. Williams). By considering the restriction of our bijection on permutations of type D, we were lead to define a new statistic on the set of permutations of type D and (t, q)-Eulerian numbers of type D, which is proved to have a nice symmetry also. We conjecture that our new statistic is in the family of Eulerian statistics for permutations of type D.

1. Introduction

Permutation tableaux are in bijection with permutations, and therefore they provide another point of view to look at permutations. They were introduced by E. Steingrímsson and L. Williams as distinguished \( J \)-diagrams in [19], and have been extensively studied in many different context (see [19, 15, 8, 13, 12, 9]). A remarkable work was done by S. Corteel and L. Williams in [13] to reveal the relation between permutation tableaux and PASEP(Partially Asymmetric Exclusion Process).

The number of rows of a permutation tableau records the weak excedence, and the number of (certain) 1’s, for example, counts the number of crossings of the corresponding permutation. This enables one to define \( q \)-analogues of the Eulerian numbers, which have many nice properties [19].

Signed permutations form a Coxeter group of type B and the corresponding permutation tableaux (of type B) have also been defined in [14]. Permutation tableaux of type B also retain many useful combinatorial statistics in them; number of negative values, (certain weak) excedance, number of crossings of the corresponding signed permutation, for example (see [12, 10, 16]). There have been efforts to naturally extend the notion of weak excedance, descent, major index of (type A) permutations to the ones for signed permutations, that made many possible definitions of statistics for permutations of type B in [7, 11, 6, 2, 5]. Permutation tableaux

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provide another way to naturally extend the well known statistics for permutations of type $A$ to the ones for permutations of type $B$.

Our main interest in this article is on the following symmetry;

\begin{align}
B^*_n(t,q) &= B^*_{n,2n-1-k}(t,q), \quad \text{for } 1 \leq k \leq 2n , \\
B^*_n,k(t,q) &= \sum_{\sigma \in S_n^B} f_{\text{neg}}(\sigma) + \chi(\sigma(1) > 0) q^{\text{cr}}(\sigma).
\end{align}

In the definition of the $(t,q)$-polynomial $B^*_n,k(t,q)$, $S_n^B$ is the group of signed permutations, and $\text{neg}, \text{fwex}, \text{cr}$ are statistics defined on the signed permutations, whose definitions are given in Section 2. The polynomial $B^*_n,k(t,q)$ is a natural generalization of Eulerian numbers of type $B$, and hence the symmetry in (1.1) gives the symmetry of Eulerian numbers of type $B$ [16]. The symmetry (1.1) is proved in [16] using pignose diagrams, and S. Corteel et al. raised a problem to prove (1.1) using permutation tableaux of type $B$ in the same paper. Since the statistics $\text{fwex}, \text{neg}$ and $\text{cr}$ are very well understood in permutation tableaux, it is natural to ask if we can understand the symmetry (1.1) in terms of permutation tableaux of type $B$. In the present article we use permutation tableaux of type $B$ to prove the symmetry (1.1); an algorithm to implement a weight preserving bijection is defined and it is shown that the algorithm really gives a bijection in detail. As special cases, by restricting the bijection to permutations of type $A$ and $D$ we show two symmetries on the $q$-Eulerian numbers of type $A$ and $D$ respectively. For type $A$ case, we show that the polynomial $\hat{D}^*_{k,n}(p,q)$ enumerating permutations according to the number of weak excedances, crossings and two types of alignments has a nice symmetry, proving the symmetry of $q$-Eulerian number $\hat{E}^*_{k,n}(q)$ of Williams [20]. For type $D$ case, we were lead to define a new statistic on permutations of type $D$ and $(t,q)$-Eulerian numbers of type $D$, which are proved to have a nice symmetry also.

The present article is organized as follows: We set up the notations, introduce the basic definitions and known related results in Section 2. In Section 3 we define the algorithm to produce the corresponding permutation tableau of a given permutation tableau of type $B$, and show that our algorithm really defines an expected bijection in Section 4. In Section 4 we restrict our algorithm to type $A$ and $D$ permutations: For type $A$ permutations, we recover the symmetry of a $q$-Eulerian polynomial (of type $A$) shown by Williams [20], while our algorithm turns out to preserve extra statistics such as certain types of alignments. For type $D$, we define a new statistics $f_{\text{ex}} D$, a $(t,q)$-Eulerian number of type $D$ and prove that it has a nice symmetry.

2. Preliminaries

We introduce necessary notations, terms and known related results in this section. We use $[n]$ for the set $\{1, \ldots, n\}$ when $n$ is a positive integer.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_t)$, a non-increasing sequence of positive numbers, a (Young) diagram of shape $\lambda$ is a left justified array of boxes with $\lambda_i$ boxes in its
ith row. For two positive integers \( r \leq n \), if \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) is a partition such that \( \lambda_1 \leq n - r \) and \( \ell \leq r \) then an \((r,n)\)-(Young) diagram of shape \( \lambda \) is the diagram of shape \( \lambda \) inside the \( r \times (n-r) \) rectangle. An \((r,n)\)-shifted (Young) diagram of shape \( \lambda \) is an \((r,n)\)-diagram of shape \( \lambda \) together with stair shaped array of boxes added above, where the \( i \)th column has \((n-r-i+1)\) additional boxes for \( i = 1, 2, \ldots, n-r \). We call the \((n-r)\) topmost boxes in an \((r,n)\)-shifted diagram diagonals. (See Figure 1.)

![Figure 1. (4,8)-diagram and (4,8)-shifted diagram of shape \( \lambda = (4,3,3) \)](image)

A partial filling of an \((r,n)\)-shifted diagram \( \hat{D} \) is obtained by putting 0, 1, or * in some boxes of \( \hat{D} \), where every diagonal box is filled with either 0 or 1. A filling is a partial filling with no empty boxes in the diagram.

An \((r,n)\)-permutation tableau (of type \( A \)) of shape \( \lambda \) is a (0,1)-filling of the \((r,n)\)-diagram of shape \( \lambda \) that satisfies the following conditions: See Figure 2

1. Every column has at least one 1.
2. The box with a 1 above it in the same column and a 1 to the left in the same row must have a 1 (\( J \)-condition).

An \((r,n)\)-permutation tableau of type \( B \) of shape \( \lambda \) is a (0,1)-filling of the \((r,n)\)-shifted diagram of shape \( \lambda \) that satisfies the following conditions: See Figure 2

1. Every column has at least one 1.
2. The box with a 1 above it in the same column and a 1 to the left in the same row must have a 1 (\( J \)-condition).
3. If a diagonal box has a 0, then there is no 1 to the left of the diagonal box in the same row.

We say that an \((r,n)\)-permutation tableau (of type \( A \) and of type \( B \)) is of length \( n \), and the set of permutation tableaux of type \( A \) and \( B \) of length \( n \) is denoted by \( PT_n \) and \( PT_n^B \) respectively.
Definition 2.3. For a permutation tableau \( T \) of type \( A \), we define \( \mathcal{PT}_n \) as a subset of \( \mathcal{PT}_n^B \).

We label the rows and columns of an \((r,n)\)-diagram of shape \( \lambda \) as follows: From the northeast corner of the \( r \times (n - r) \) rectangle to the southwest corner, follow the southeast border edges of the diagram and give labels 1, 2, 3, \ldots, \( n \). If a south edge earned the label \( i \) then the corresponding column is labeled by \( i \), and if an east edge earned the label \( j \) then the corresponding row is labeled by \( j \). We usually put the labels on the left of each row and on the top of each column.

The labeling of rows (and columns) of an \((r,n)\)-shifted diagram \( \tilde{D} \) of shape \( \lambda \) is done in a similar way. Label the columns and rows of \( D \), where \( D \) is the \((r,n)\)-diagram from which \( \tilde{D} \) is obtained by adding boxes on the top. If the diagonal box of an added row is in the column labeled by \( i \) in \( D \), then the the row is labeled by \( -i \). We usually put the label on the left of each row. In each \((r,n)\)-(shifted) diagram, we use row \( i \) to denote the row labeled by \( i \). We use col \( j \) for the column labeled by \( j \) in a diagram and for the column whose diagonal is in row \( -j \) in a shifted diagram. We usually omit column labels in a shifted diagram.

For an \((r,n)\)-(partial) filling \( F \) with underlying shifted diagram \( \tilde{D} \), we use col \( j \) \( (\text{or} \ \text{row} \ i) \) of \( F \) for \( \text{col} \ j \ \text{(or} \ \text{row} \ i) \) of \( \tilde{D} \) and also for its content in the filling \( F \). We define labeling sets of \( F \) as subsets of \([n]\):

\[
\text{lab}_+ (F) = \{ i \in [n] \mid i \text{ is a label of a row in } \tilde{D} \},
\]

\[
\text{lab}_- (F) = \{ j \in [n] \mid -j \text{ is a label of a row in } \tilde{D} \}.
\]

\[
\text{lab}_d^\lambda (F) = \{ j \in \text{lab}_- (F) \mid \text{the diagonal box of } \text{row} (-j) \text{ is filled with 1} \}.
\]

It is easy to check that labeling sets of \( F \) completely determine the shape of \( F \). We use \((i,j)\) for the box in \text{row} \( i \) and \text{col} \( j \) of \( F \). Then the set of boxes in \( F \) is \( \{(i, j) \mid i \in \text{lab}_+ (F), j \in \text{lab}_- (F), j > i \} \cup \{(-i, j) \mid i \in \text{lab}_- (F), j \in \text{lab}_- (F), j \geq i \} \).

For a box \( c = (i, j) \) in \( F \), we let \text{row} \( c \) = \( i \), \text{col} \( c \) = \( j \).

The following lemma is straightforward from the definitions while it is very useful.

Lemma 2.2. For a \((\text{partial})\) filling \( F \), \text{lab}_+ (F), \text{lab}_- (F), \text{lab}_d^\lambda (F) \) determine the number of boxes in rows and columns of \( F \). More precisely, the number of boxes in row \( i \) of \( F \) is \(| \{ k \in \text{lab}_- (F) \mid k \geq |i| \} | \) and the number of boxes in col \( i \) of \( F \) is \( i \).

Definition 2.3. For a permutation tableau \( T \in \mathcal{PT}_n^B \) of type \( B \),

1. A non-topmost 1 is called a \textit{superfluous one}, and \( \text{so}(T) \) denote the number of superfluous ones in \( T \).
2. \( \text{diag}(T) \) is the number of 1’s on the diagonal of \( T \).
3. \( \text{row}(T) \) is the number of rows labeled by positive integers in \( T \).

The group of permutations of \([n]\) is denoted by \( \mathfrak{S}_n \), and the group of signed permutations on \([n]\) is denoted by \( \mathfrak{S}_n^\pm \). Recall that a signed permutation is a permutation \( \sigma \) of \([n]\) that satisfies \( \sigma(-i) = -\sigma(i) \) for all \( i = 1, 2, \ldots, n \).
The followings are well known and extensively studied statistics defined on the permutations in $\mathfrak{S}_n$: see [15 6] for the definitions.

**Eulerian:** descent $(\text{des})$, weak excedance $(\text{wex})$, excedance $(\text{exc})$, ascent $(\text{asc})$

**Mahonian:** inversion number $(\text{inv})$, length $(\ell)$, major index $(\text{maj})$

Remember that Eulerian numbers $A_{n,k}$ are the numbers of permutations $\sigma \in \mathfrak{S}_n$ with $\text{des}(\sigma) = k - 1$ (or equivalently $\text{wex}(\sigma) = k$), and the $q$-Eulerian numbers of Carlitz are $A_{n,k}(q) = \sum_{\sigma \in \mathfrak{S}_n, \text{wex}(\sigma) = k} q^{\text{maj}(\sigma)}$.

The crossing number and the alignment number are relatively new statistics on the permutations defined in [11]: For $\sigma$ alignment of $(2.4)$ introduced by Williams in terms of crossing numbers; $\sigma$ or $\sigma$ and alignments of $(2.5)$ recurrence relation and a symmetry, they enjoy nice properties as Carlitz’s polynomials do [20 19]; they satisfy nice properties as well.

**Definition 2.6.** For a signed permutation $\sigma \in \mathfrak{S}_n^B$,

1. $i \in [n]$ is called a weak excedance of $\sigma$ if $\sigma(i) \geq i$ and the number of weak excedances of $\sigma$ is denoted by $\text{wex}(\sigma)$.
2. $i \in [n]$ is called a negative of $\sigma$ if $\sigma(i) < 0$, and the number of negatives in $\sigma$ is denoted by $\text{neg}(\sigma)$.
3. $f\text{wex}(\sigma) := 2\text{wex}(\sigma) + \text{neg}(\sigma)$.
4. A pair $(i,j)$ with $i,j > 0$ is a crossing of $\sigma$ if $i < j \leq \sigma(i) < \sigma(j)$ or $-i < j \leq -\sigma(i) < \sigma(j)$ or $i > j > \sigma(i) > \sigma(j)$, and the number of crossings of $\sigma$ is denoted by $\text{cr}(\sigma)$.

The importance of permutation tableaux is in the fact that they are in bijection with Coxeter groups of corresponding types:

**Proposition 2.7.** [11 15 17 16 10] There is a bijection $\Phi$ between $\mathcal{PT}_n^B$ and $\mathfrak{S}_n^B$ that satisfies the following properties:

1. For $T \in \mathcal{PT}_n^B$, $\text{row}(T) = \text{wex}(\Phi(T))$, $\text{diag}(T) = \text{neg}(\Phi(T))$ and $\text{so}(T) = \text{cr}(\Phi(T))$.
2. $i \in \text{neg}(\sigma)$ if and only if $i \in \text{lab}^d_\Phi(\Phi^{-1}(\sigma))$, where $\Phi^{-1}$ is the inverse of $\Phi$. 


(3) \( \sigma(1) > 0 \) if and only if \( 1 \in \text{lab}_+\left(\Phi^{-1}(\sigma)\right) \),

(4) the restriction of \( \Phi \) to \( \mathcal{PT}_n \) is a bijection between \( \mathcal{PT}_n \) and \( \mathfrak{S}_n \) such that \( \text{row}(T) = \text{wex}(\Phi(T)) \), \( \text{col}(T) = \text{cr}(\Phi(T)) \) for \( T \in \mathcal{PT}_n \).

In particular, the number of permutation tableaux of length \( n \) is \( n! \), and the number of permutation tableaux of type \( B \) is \( 2^n n! \).

\( q \)-Eulerian numbers of type \( B \) can be defined in similar ways as they were defined for type \( A \) permutation groups:

The following is due to R. Adin, F. Brenti and Y. Roichman, see [1] for the definition of statistics \( f\text{maj} \) and \( n\text{maj} \).

\[ B_{n,k}(q) = \sum_{\sigma \in \mathfrak{S}_n \mid \text{wex}(\sigma) = k} q^{f\text{maj}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n \mid \text{ndes}(\sigma) = k-1} q^{n\text{maj}(\sigma)} . \]

\( q \)-Eulerian number \( E_{n,k} \) in (2.4) can be generalized as follows, and a symmetry similar to (2.5) holds:

Proposition 2.8. [16] Let \( E_{n,k}^B(q) = \sum_{\sigma \in \mathfrak{S}_n \mid \text{wex}(\sigma) = k} q^{\text{cr}(\sigma)} \) be the \( q \)-Eulerian number of type \( B \), then the following symmetry holds;

\[ E_{n,k}^B(q) = E_{n,n-k}^B(q) . \]

For a proof of Proposition 2.8 S. Corteel, M. Josuat-Vérès and J. Kim introduced another \((t,q)\)-polynomial \( B_{n,k}^*(t,q) \) and showed its symmetry, that proves the symmetry of \( q \)-Eulerian number of type \( B \) in Proposition 2.8 as its special case:

Proposition 2.9. [16] Let \( B_{n,k}^*(t,q) = \sum_{\sigma \in \mathfrak{S}_n \mid \text{wex}(\sigma) = k} t^{\text{neg}(\sigma)} + \chi(\sigma(1) > 0) q^{\text{cr}(\sigma)} \) be a \((t,q)\)-Eulerian number, then \( B_{n,k}^*(t,q) = B_{n,2n+1-k}^*(t,q) \), for \( 1 \leq k \leq 2n \).

The proof of Proposition 2.9 is done by using pignose diagram in [16] and it has been suggested as an open problem to do a combinatorial proof of Proposition 2.9 using permutation tableaux of type \( B \); Problem 2 in [16]. Due to Proposition 2.7, we can restate Proposition 2.9 in terms of permutation tableaux of type \( B \).

Theorem 2.10. Let

\[ B_{n,k}^*(t,q) = \sum_{T \in \mathcal{PT}_n^B \mid \text{row}(T) + \text{diag}(T) = k} t^{\text{diag}(T)} + \chi(1 \in \text{lab}_+\left(\Phi(T)\right)) q^{\text{so}(T)}, \]

then \( B_{n,k}^*(t,q) = B_{n,2n+1-k}^*(t,q) \), for \( 1 \leq k \leq 2n \).

The main purpose of this article is to solve the problem suggested in [16]; we give a combinatorial proof of Theorem 2.10 using permutation tableaux of type \( B \).
3. Algorithm

In this section, we define a map from $\mathcal{P}T^B_n$ to itself, which will give a proof of Theorem 2.10.

Con
cretion 3.1. (1) For certain (usually superfluous) 1’s in (partial) fillings, we use ‘S’ instead.
(2) Most of 0’s are omitted in a filling.
(3) Every filling in this section is a filling of a ‘shifted’ diagram.

We let $(\mathcal{P}T^B_n)^+ \subset (\mathcal{P}T^B_n)^-$ be two subsets of $\mathcal{P}T^B_n$ defined as follows:

$$(\mathcal{P}T^B_n)^+ = \{ T \in \mathcal{P}T^B_n \mid 1 \in \text{lab}_+(T) \},$$

$$(\mathcal{P}T^B_n)^- = \{ T \in \mathcal{P}T^B_n \mid 1 \in \text{lab}_d^-(T) \}.$$ 

When $1 \in \text{lab}_d(T)$, $\text{col} 1$ of $T$ has only one box and therefore it must be filled with “1” by the definition of permutation tableaux; and therefore $1 \in \text{lab}_d(T)$. This shows that $\mathcal{P}T^B_n = (\mathcal{P}T^B_n)^+ \cup (\mathcal{P}T^B_n)^-$. To define a bijection on $\mathcal{P}T^B_n$ with expected properties, we first define a map $\Xi$ from $(\mathcal{P}T^B_n)^+$ to $(\mathcal{P}T^B_n)^-$ satisfying

$$\text{diag}(\mathcal{T}) + 1 = \text{diag}(\Xi(T)),$$

$$\text{so}(T) = \text{so}(\Xi(T)),$$

and

$$[2\text{row}(T) + \text{diag}(T)] + [2\text{row}(\Xi(T)) + \text{diag}(\Xi(T))] = 2n + 1.$$ 

$\Xi$ can be naturally extended to $\mathcal{P}T^B_n$, as we will show in the latter part of this section.

Before we start construction of $\Xi$, we introduce some necessary terms.

Definition 3.2. For a partial filling $F$ with $\text{lab}_+(F)$, $\text{lab}_-(F)$, $\text{lab}_d^-(F)$ such that $1 \in \text{lab}_+(F) \cup \text{lab}_d^-(F)$,

(1) $\text{col} i$ is called a boundary column of $F$ if $i \in \text{lab}_d^-(F)$, and an interior column of $F$ if $i \in \text{lab}_-(F) \setminus \text{lab}_d^-(F)$,

(2) $\text{row} i$ is called a boundary row of $F$ if $|i| = 1$ or $|i| \in \text{lab}_d^-(F)$, an interior row of $F$ if $|i| \in \text{lab}_+(F) \setminus \{1\}$, and a special zero row of $F$ if $|i| \in \text{lab}_-(F) \setminus \text{lab}_d^-(F)$.

Example 3.3. Let $T$ be the (4, 8)-permutation tableau of type $B$ in Figure 3. Then $\text{lab}_-(T) = \{4, 6, 7, 8\}$, $\text{lab}_d^-(T) = \{4, 7\}$, and $\text{lab}_+(T) = \{1, 2, 3, 5\}$. Therefore, $\text{col} 4$, $\text{col} 7$ are boundary columns of $T$ and $\text{col} 16$, $\text{col} 18$ are interior columns of $T$.

![Figure 3. A (4, 8)-permutation tableau of type B](image)
Moreover, row1, row(−4), row(−7) are boundary rows of \( T \), row2, row3, row5 are interior rows of \( T \), and row(−6), row(−8) are special zero rows of \( T \).

**Definition 3.4.** For a partial filling \( F \) with \( \text{lab}_+(F), \text{lab}_-(F), \text{lab}_d^-(F) \) such that \( 1 \in \text{lab}_+(F) \cup \text{lab}_d^d(F), j \in \text{lab}_-(F) \), we define a subset \( S(F) \) of boxes in \( F \):

\[
S(F) = \{ (i, j) \mid |i| \in \text{lab}_+(F) \cup \text{lab}_d^d(F), j \in \text{lab}_-(F) \} \setminus \{(i_j, j) \mid i_j \text{ is the smallest among } i \text{'s such that } |i| \leq j \text{ and } |i| \in \text{lab}_+(F) \cup \text{lab}_d^d(F) \}.
\]

We use the same row and column labelings for \( S(F) \) as the ones for \( F \), for example row \( i \) of \( S(F) \) means the row \( i \) of \( F \) intersected with \( S(F) \).

For a row \( i \) in \( S(F) \), we define the row-numbering \( r_i \) from the set of boxes in row \( i \) of \( S(F) \) to \([n_i]\) as follows, where \( n_i \) is the number of boxes in row \( i \) of \( S(F) \).

1. First, number the boxes in row \( i \) of \( S(F) \) contained boundary columns from right to left.
2. Next, number the boxes contained interior columns from left to right.

**Figure 4.** \( S(F) \) and row-numbering \( r_{−3}, r_2 \)

**Remark 3.5.** \( S(F) \) and row numbering of each row of \( S(F) \) play important role in the proof of the main theorem. The subfilling \( S(F) \) of a permutation tableau \( T \) is the essential part of \( T \). It completely determines \( T \), and hence we only need to manipulate \( S(T) \) to obtain \( \Sigma(T) \).

In the rest of the section, we fix a permutation tableau \( T \in (PT^B_n)^+, \) in which we use ‘S’ for superfluous 1’s. We have two steps to define \( \Sigma(T) \in (PT^B_n)^−: \)

1. We first define a filling \( T_{pr} \) called **pre tableau** of \( T \) in subsection 3.1.
2. We apply sequence of rules to \( T_{pr} \) to make it a permutation tableau in subsection 3.2.

### 3.1. Pre tableau \( T_{pr} \)

We first define the labeling sets for \( T_{pr} \) and hence the shape. We then define the filling. Recall from Lemma 2.2 that \( \text{lab}_+(T_{pr}) \) (or \( \text{lab}_-(T_{pr}) \)) completely determines the shape of \( T_{pr} \).

**Step 1 (shape of \( T_{pr} \))**

(a) \( 1 \in \text{lab}_d^d(T_{pr}) \)
(b) \( i \in \text{lab}_d^d(T_{pr}) \setminus \{1\} \text{ if and only if } (n + 2 - i) \in \text{lab}_d^d(T) \)
(c) \( i \in \text{lab}_-(T_{pr}) \setminus \text{lab}_d^d(T_{pr}) \text{ if and only if } (n + 2 - i) \in \text{lab}_+(T) \setminus \{1\} \)
Let \( T \) be the \((4,8)\)-permutation tableau in Figure 3. Then \( \text{lab}_+(T_{pr}) = \{1,3,6\} \), \( \text{lab}_-(T_{pr}) = \{1,3,5,6,7,8\} \) and \( \text{lab}_-(T_{pr}) = \{2,4\} \), from which we obtain the partial filling in Figure 5.

**Example 3.6.** Let \( T \) be the \((4,8)\)-permutation tableau in Figure 3 again. We have the shape of \( T_{pr} \) and the partial filling \( S(T_{pr}) \) of \( T_{pr} \) in Figure 5. To complete \( T_{pr} \), we copy columns in \( S(T) \) to the corresponding rows of \( S(T_{pr}) \) as in Figure 6. The last filling is \( T_{pr} \). Remember that we use the row numbering of \( \text{row}_j \) of \( S(T_{pr}) \).

**Example 3.7.** Consider the \((4,8)\)-permutation tableau \( T \) in Figure 3 again. We have the shape of \( T_{pr} \) and the partial filling \( S(T_{pr}) \) of \( T_{pr} \) in Figure 5. To complete \( T_{pr} \), we copy columns in \( S(T) \) to the corresponding rows of \( S(T_{pr}) \) as in Figure 6. The last filling is \( T_{pr} \). Remember that we use the row numbering of \( \text{row}_j \) of \( S(T_{pr}) \) when we fill the \( j \)th row of \( S(T_{pr}) \).

**Lemma 3.8.** Step 3 is well-defined. That is,

1. the number of boundary columns of \( S(T) \) is equal to the number of boundary rows of \( S(T_{pr}) \),
2. the number of interior columns of \( S(T) \) is equal to the number of interior rows of \( S(T_{pr}) \), and
3. the number of boxes in each column of \( S(T) \) is equal to the number of boxes in the corresponding row in \( S(T_{pr}) \).

**Proof.** Let 
\[ \text{lab}_+(T) = \{a_1 = 1 < a_2 < \cdots < a_r\}, \]
\[ \text{lab}_- (\mathcal{T}) \setminus \text{lab}_d^d (\mathcal{T}) = \{ b_1 < \cdots < b_s \}, \quad \text{and} \]
\[ \text{lab}_d^d (\mathcal{T}) = \{ d_1 < \cdots < d_{n-r-s} \}. \]

Then,
\[ \text{lab}_+ (\mathcal{T}_{pr}) = \{ n + 2 - b_s < \cdots < n + 2 - b_1 \}, \]
\[ \text{lab}_- (\mathcal{T}_{pr}) \setminus \text{lab}_d^d (\mathcal{T}_{pr}) = \{ n + 2 - a_r < \cdots < n + 2 - a_2 \}, \quad \text{and} \]
\[ \text{lab}_d^d (\mathcal{T}_{pr}) = \{ a_1 = 1 < n + 2 - d_{n-r-s} < \cdots < n + 2 - d_1 \}. \]

Let us consider a column \( \text{col}_i \) in \( \mathcal{T} \). The number of boxes in \( \text{col}_i \) of \( S(\mathcal{T}) \) is
\[(3.9) \quad |\{ k \in \text{lab}_+ (\mathcal{T}) \cup \text{lab}_d^d (\mathcal{T}) \mid k \leq i \}| - 1, \]
since we exclude the boxes contained in special zero rows from \( \text{col}_i \) in \( \mathcal{T} \) and the topmost box among the rest. It is easy to see that
\[
\begin{align*}
\text{number of boundary columns of } S(\mathcal{T}) &= \text{number of boundary rows of } S(\mathcal{T}_{pr}) = n - r - s \quad \text{and} \\
\text{number of interior columns of } S(\mathcal{T}) &= \text{number of interior rows of } S(\mathcal{T}_{pr}) = s.
\end{align*}
\]
Hence the first part and the second part of the Lemma are proved.

If \( i \notin \text{lab}_d^d (\mathcal{T}) \), that is \( \text{col}_i \) is an interior column of \( \mathcal{T} \), then \( n + 2 - i \in \text{lab}_+ (\mathcal{T}_{pr}) \).
Let
\[ j_1 = |\{ k \in \text{lab}_+ (\mathcal{T}) \mid k \leq i \}| \quad \text{and} \quad j_2 = |\{ k \in \text{lab}_d^d (\mathcal{T}) \mid k \leq i \}|, \]
then the number of boxes in \( \text{col}_i \) of \( S(\mathcal{T}) \) is \( (j_1 + j_2 - 1) \) and
\[(3.10) \quad a_1 < \cdots < a_{j_1} < i < a_{j_1+1}, \quad d_1 < \cdots < d_{j_2} < i < d_{j_2+1}. \]
It is easy to check that for a partial filling \( \mathcal{F} \) and \( j \in \text{lab}_+ (\mathcal{F}) \), \( \text{row}_j \) has \( |\{ k \in \text{lab}_- (\mathcal{F}) \mid k > j \}| \) boxes and \( \text{row}_j \) of \( S(\mathcal{F}) \) is the same as \( \text{row}_j \) of \( \mathcal{F} \). We hence only have to show that \( \text{row}(n + 2 - i) \) in \( \mathcal{T}_{pr} \) has \( (j_1 + j_2 - 1) \) boxes, which is clear from the following:
\[
\begin{align*}
n + 2 - a_{j_1+1} &< n + 2 - i < n + 2 - a_{j_1} < \cdots < n + 2 - a_2, \\
n + 2 - d_{j_2+1} &< n + 2 - i < n + 2 - d_{j_2} < \cdots < n + 2 - d_1.
\end{align*}
\]
Therefore, \( \text{row}(n + 2 - i) \) in \( S(\mathcal{T}_{pr}) \) has \( (j_1 + j_2 - 1) \) boxes.
If \( i \in \text{lab}_d(T) \), that is \( \text{col}_i \) is a boundary column of \( T \), then \( \text{col}_i \) in \( S(T) \) is corresponding to \( \text{row}_j \) in \( S(T_{pr}) \) where \( j \) is the smallest among \( j \)'s such that \( j \in \text{lab}_d(T_{pr}) \) and \( j > -n - 2 + i \). Let
\[
j_3 = |\{ k \in \text{lab}_+(T) \mid k \leq i \}| \quad \text{and} \quad j_4 = |\{ k \in \text{lab}_d(T) \mid k \leq i \}|,
\]
then \( \text{col}_i \) of \( S(T) \) has \( (j_3 + j_4 - 1) \) boxes and
\[
(3.11) \quad a_1 = 1 < a_2 < \cdots < a_{j_3} < i < a_{j_3+1}, \quad d_1 < \cdots < d_{j_4} = i < d_{j_4+1}.
\]
\( \text{row}_j \) in \( S(T_{pr}) \) has boxes as much as the number of boxes of \( \text{row}(-n - 2 + i) \) in \( T_{pr} \) because the boxes of \( \text{row}_j \) in \( T_{pr} \) which are contained in interior columns between \( \text{col}(-j) \) and \( \text{col}(n + 2 - i) \) have *'s and the box in \( \text{row}_j \) which is contained in \( \text{col}(-j) \) has 1. So, \( \text{row}_j \) in \( S(T_{pr}) \) has \( |\{ k \in \text{lab}_-(T_{pr}) \mid k \geq n + 2 - i \}| \) boxes and we have to show that this is equal to \( (j_3 + j_4 - 1) \). However, (3.11) holds in \( T_{pr} \) as
\[
n + 2 - a_{j_3+1} < n + 2 - i < n + 2 - a_{j_3} < \cdots < n + 2 - a_2,
\]
\[
n + 2 - d_{j_4+1} < n + 2 - i = n + 2 - d_{j_4} < \cdots < n + 2 - d_1.
\]
Therefore, \( \text{row}_j \) in \( S(T_{pr}) \) has \( (j_3 + j_4 - 1) \) boxes.

In the following lemma, we show that \( T_{pr} \) has the required properties in terms of shape and diagonals to be the corresponding permutation tableau of \( T \) to show Theorem 2.10.

**Lemma 3.12.** Let \( T \in (PT_n^B)^+ \) and \( T_{pr} \) be the pre tableau of \( T \). Then
\[
\text{diag}(T_{pr}) = \text{diag}(T) + 1 \quad \text{and} \quad [2\text{row}(T) + \text{diag}(T)] + [2\text{row}(T_{pr}) + \text{diag}(T_{pr})] = 2n + 1.
\]

**Proof.** Note that \( 1 \in \text{lab}_+(T) \) and \( 1 \in \text{lab}_-(T_{pr}) \). Moreover, \( i \in \text{lab}_d(T_{pr}) \setminus \{ 1 \} \) if and only if \( (n + 2 - i) \in \text{lab}_d(T) \). Therefore we have \( \text{diag}(T_{pr}) = \text{diag}(T) + 1 \).

Since \( i \in \text{lab}_+(T_{pr}) \) if and only if \( (n + 2 - i) \in \text{lab}_-(T) \setminus \text{lab}_d(T) \) by the definition of \( T_{pr} \), we have \( \text{row}(T_{pr}) = n - \text{row}(T) - \text{diag}(T) \).

Therefore,
\[
2\text{row}(T) + \text{diag}(T) + 2\text{row}(T_{pr}) + \text{diag}(T_{pr})
\]
\[
= 2\text{row}(T) + \text{diag}(T) + 2\{ n - \text{row}(T) - \text{diag}(T) \} + \{ \text{diag}(T) + 1 \} = 2n + 1.
\]

**Example 3.13.** In Example 3.3 and 3.6 since \( n = 8 \),
\[
[2\text{row}(T) + \text{diag}(T)] + [2\text{row}(T_{pr}) + \text{diag}(T_{pr})] = 10 + 7 = 17 = 2n + 1.
\]

### 3.2. \( T_{pr} \) to \( T \).

The -condition is not guaranteed in \( T_{pr} \), so we are going to make \( T_{pr} \) become a permutation tableau indeed.

In a row of \( T \), because of the -condition if a box \( c \) has 1 or \( S \), then the boxes right to \( c \) and contained in boundary columns must have \( S \). The main idea of our algorithm is that \( S \) influenced by certain 1's must give back the \( S \) to the original box in \( T_{pr} \).

**Definition 3.14.** Let \( F \) be a partial filling.
For a box $c$ in $\text{row}_i$ of $S(F)$, we define the partial row-numbering for $c$ as the restriction of the row-numbering $r_i$ on the boxes which are on the left side of $c$ including $c$.

For a box $c$ in a boundary column of $F$, if there is an $S$ to the left of $c$, then we let $S_c$ be the first $S$ with respect to the row numbering for $c$ and $\text{left}(c)$ be the box where $S_c$ is located.

For two boxes $c = (i, j)$ and $c' = (i', j)$ in $S(F)$, we say that $c'$ is in relevant position to $c$ if $\text{row}_i'$ is a boundary row of $S(F)$ and $\text{length of } \text{row}_i' \text{ in } S(F) \leq \text{length of } \text{row}_i \text{ in } S(F)$.

A box $c$ is constrained if the downmost nonzero above $c$ is an $S$ and it is contained in a box in relevant position to $c$.

When $c$ is constrained, the downmost $S$ above $c$ is called the relevant $S$ of $c$ and denoted by $\text{rel}_cS$, and $\text{rel}_c(c)$ is the box where $\text{rel}_cS$ is located.

We define types for boxes in $S(F)$ of a partial filling $F$.

**Definition 3.15.** Let $c$ be a box in $S(F)$ of a partial filling $F$.

1. When a non-diagonal box $c$ has 1 with $S$ or 1 above it,
   - $c$ is of type 0 if it is unconstrained.
   - $c$ is of type 1 if it is constrained.
2. When a box $c$ in an interior column has 0,
   - $c$ is of type 2 if it is constrained and it has nonzero on the left in the same row.
3. When a box $c$ in a boundary column has 0,
   - $c$ is of type 3 if it has constrained $\text{left}(c)$.
   - $c$ is of type 4 if it has unconstrained $\text{left}(c)$.

Here are five rules which implement the main idea of our algorithm. For a pair $(F, c)$ of a partial filling $F$ and a box $c$ of $S(F)$, each rule changes contents of certain boxes to obtain a new filling.

(R1) if $c$ is of type 1, then convert 1 in $c$ into $S$ and $\text{rel}_cS$ into 0.

(R2) if $c$ is of type 2, then convert 0 in $c$ into $S$ and $\text{rel}_cS$ into 0.

(R3) if $c$ is of type 3, then convert 0 in $c$ into $S$ and $\text{rel}_{\text{left}(c)}S$ into 0.

(R4.1) if $c$ is of type 4 and $\text{left}(c)$ has no 1 or $S$ above it, then convert 0 in $c$ into $S$ and $S_c$ into 1.

(R4.2) if $c$ is of type 4 and $S_c$ has 1 or $S$ above it, then convert 0 in $c$ into $S$ and $S_c$ into 0.

**Remark 3.16.** $R2$, $R3$, $R4.1$ and $R4.2$ make a 0 which is against $J$-condition, into an $S$ while preserving the number of superfluous 1’s. These rules are necessary to make pre tableau $T_{pr}$ a permutation tableau (satisfying $J$-condition).

Before we define the algorithm to convert $T_{pr}$ to $\Sigma(T)$, we define a function operating on pairs of a partial filling and a box in $S(F)$:

**Function $\phi_c$.**

**Data:** a filling $F$, a box $c$ of $S(F)$

**Result:** $F' = \phi_c(F)$

if $c$ is of type 1 then do
\[ F' \leftarrow \text{resulting filling from applying } R1 \text{ to } (F,c); \]

elseif \( c \) is of type 2 then do
\[ F' \leftarrow \text{resulting filling from applying } R2 \text{ to } (F,c); \]

elseif \( c \) is of type 3 then do
\[ F' \leftarrow \text{resulting filling from applying } R3 \text{ to } (F,c); \]

elseif \( c \) is of type 4 then do
  if \( S_c \) is contained an interior column
    and has no 1 or \( S \) above it in the same column then do
    \[ F' \leftarrow \text{resulting filling from applying } R4.1 \text{ to } (F,c); \]
  else do
    \[ F' \leftarrow \text{resulting filling from applying } R4.2 \text{ to } (F,c); \]
  end
end
end

We also define functions \( \phi_o \) and \( \phi_\bullet \) from the set of fillings to itself.

Function \( \phi_o \)

Data: a filling \( F \)
Result: \( F' = \phi_o(F) \)
\[ F' \leftarrow \text{resulting filling} \]
from converting a 1’s in all boxes of type 0 of \( F \) into 0’s;

Function \( \phi_\bullet \)

Data: a filling \( F \) with \( lab_+(F), \, lab_-(F), \, lab_d^l(F) \)
such that each column of \( F \) has at most one *
Result: \( F' = \phi_\bullet(F) \)
for \( i \) from 1 to \( n \) do
  if \( i \in lab_-(F) \smallsetminus lab_d^l(F) \) do
    if \( col_i \) of \( F \) has 1 then do
      \[ F \leftarrow \text{resulting filling from replacing the } * \text{ in } col_i \text{ with } 0; \]
    else do
      if \( col_i \) has a 0 with 1 or \( S \) left to it in the same row then do
        \[ F \leftarrow \text{resulting filling from replacing the downmost such } 0 \text{ of } F \text{ with } 1; \]
      else do
        \[ F \leftarrow \text{resulting filling from replacing the } * \text{ in } col_i \text{ with } 1; \]
      end
    end
  end
end
\[ F' \leftarrow F; \]

\( \phi_o \) and \( \phi_\bullet \) will be applied at the first and the last step of our algorithm, respectively. \( \phi_\bullet \) will guarantee that \( S(T) \) satisfies the first condition for permutation tableaux; each column has at least one 1.

Definition 3.17 (Box numbering of \( S(F) \)). We number the boxes of \( S(T_{pr}) \) in order from downmost row to top row and from right to left in each row.
We now introduce our algorithm. For given $T \in (P T^B_n)^+$, let $T_{pr}$ be the pre tableau obtained from $T$.

**Algorithm:** $\mathcal{F}$

**Data:** $T \in P T^B_n$ and $T_{pr}$, numbered boxes $c_1, c_2, \ldots, c_m$ of $S(T_{pr})$

**Result:** $\mathcal{F} = \mathcal{F}(T)$

I. $j \leftarrow 0$;
   $\mathcal{F} \leftarrow \phi_o(T_{pr})$;

II. for $i \leftarrow 1$ to $m$ do
   if $c_i$ is of type 1 then do
      $\mathcal{F} \leftarrow \phi_{c_i}(\mathcal{F})$;
      $j \leftarrow j + 1$;
      $\text{inj} \leftarrow c_i$;
      $\text{outj} \leftarrow \text{rel}(c_i)$;
   end

III. for $i \leftarrow 1$ to $m$ do
   if $c_i$ is of type 2 then do
      $\mathcal{F} \leftarrow \phi_{c_i}(\mathcal{F})$;
      $j \leftarrow j + 1$;
      $\text{inj} \leftarrow c_i$;
      $\text{outj} \leftarrow \text{rel}(c_i)$;
   end
   elseif $c_i$ is of type 3 then do
      $\mathcal{F} \leftarrow \phi_{c_i}(\mathcal{F})$;
      $j \leftarrow j + 1$;
      $\text{inj} \leftarrow c_i$;
      $\text{outj} \leftarrow \text{rel}(\text{left}(c_i))$;
   end
   elseif $c_i$ is of type 4 then do
      $\mathcal{F} \leftarrow \phi_{c_i}(\mathcal{F})$;
      $j \leftarrow j + 1$;
      $\text{inj} \leftarrow c_i$;
      $\text{outj} \leftarrow \text{left}(c_i)$;
   end
   end

IV. $\mathcal{F} \leftarrow \phi_\ast(\mathcal{F})$;

It is easy to see that for each $T \in P T^B_n$, $\{\text{in1, out1}, \ldots, \text{ink, outk}\}$ and $\{\phi_{\text{in1}}, \ldots, \phi_{\text{ink}}\}$ are determined uniquely from Algorithm. We can consider them as properties of $T$.

The following lemma is immediate from Algorithm.
Lemma 3.18. For $\mathcal{T} \in (\mathcal{PT}_n^B)^+$, $\Xi(\mathcal{T}) = \phi_\bullet \phi_{\text{in}k} \cdots \phi_{\text{in}1} \phi_{\text{out}}(\mathcal{T}_{\text{pr}})$ is a permutation tableau of type $B$ with 

$$so(\mathcal{T}) = so(\Xi(\mathcal{T})).$$

Proof. First we prove that $\Xi(\mathcal{T})$ is a permutation tableau of type $B$. Each boundary column always has topmost 1 and each interior column also has topmost 1 in $\Xi(\mathcal{T})$ due to $\phi_\bullet$. It is clear that there is no 0 against J-condition in $\Xi(\mathcal{T})$ because all boxes against J-condition in $\mathcal{T}_{\text{pr}}$ get S’s from other boxes through the Algorithm. Moreover, Algorithm preserves the special zero rows of $\mathcal{T}_{\text{pr}}$. Hence, $\Xi(\mathcal{T})$ is a permutation tableau of type $B$.

Every rule in the Algorithm preserves the number of S’s and the set of S’s in $\Xi(\mathcal{T})$ is exactly the set of superfluous one’s of $\Xi(\mathcal{T})$. Thus, $so(\mathcal{T}) = so(\Xi(\mathcal{T}))$. □

Example 3.19. Consider $\mathcal{T}$ and $\mathcal{T}_{\text{pr}}$ in previous examples. There are 17 boxes in $\mathcal{S}(\mathcal{T}_{\text{pr}})$ and we have the box numbering in $\mathcal{S}(\mathcal{T}_{\text{pr}})$ as in the following figure.

![Figure 7. Box numbering in $\mathcal{S}(\mathcal{T}_{\text{pr}})$](image)

(1) In $\mathcal{S}(\mathcal{T}_{\text{pr}})$, there is only one box of type 0, $c_2$. So, in Step I of Algorithm, we replace ‘1’ in $c_2$ by ‘0’.

![Figure 8. Step I of Algorithm](image)

(2) In the resulting filling of (1), there is only one box of type 1, $c_6$. So, in Step II of Algorithm, we apply the rule $R1$. Here $\text{in1} = c_6$ and $\text{out1} = c_{11}$.

Now in Step III of Algorithm we look at $c_1, c_2, \ldots, c_{17}$ in order:

(3) In the resulting filling of (2), $c_2$ is of type 4 and has unconstrained left($c_2$) with S above it. Thus, we apply $R4.2$. Here $\text{in2} = c_2$ and $\text{out2} = c_4$. 

(4) In the resulting filling of (3), $c_5$ is of type 4 and has unconstrained left($c_5$) with no 1 or S above it. Thus, we apply $R4.1$. Here $\text{in3} = c_5$ and $\text{out3} = c_6$. 
(5) In the resulting filling of (4), $c_{10}$ is of type 3. Thus, we apply $R3$. Here $in_4 = c_{10}$ and $out_4 = c_{15}$.

(6) In the resulting filling of (5), $c_{13}$ is of type 2. Thus, we apply $R2$. Here $in_5 = c_{13}$ and $out_5 = c_{16}$.

(7) In the resulting filling of (6), $c_{15}$ is of type 4 and has unconstrained left$(c_{15})$ with no 1s or 8 above it. Thus, we apply $R4.1$. Here $in_6 = c_{15}$ and $out_6 = c_{17}$.

Now in Step IV of Algorithm,

(8) there is no 1 in $col_7$ of the resulting filling of (7) and $col_7$ has one 0 which has 1 left of it. So, replace the 0 with 1.

(9) In the resulting filling of (8), since all columns have own topmost 1, delete all *’s. Then, we obtain a new permutation tableau

$$\mathcal{I}(\mathcal{T}) = \phi_{in_5} \circ \phi_{in_4} \circ \phi_{in_3} \circ \phi_{in_2} \circ \phi_{in_1} \circ \phi_{in_6}(\mathcal{T}_{pr})$$

corresponding to $\mathcal{T}$.

As we previously mentioned, $\mathcal{I}$ can be naturally extended to $\mathcal{P}\mathcal{T}_n^B$. Let $\iota : \mathcal{P}\mathcal{T}_n^B \to \mathcal{P}\mathcal{T}_n^B$ be the map such that,
Then \( \iota \) works exactly same for \( T \in \mathcal{P} \mathcal{T}_n^B \), the composition \( \iota \circ \iota : (\mathcal{P} \mathcal{T}_n^B)^- \rightarrow (\mathcal{P} \mathcal{T}_n^B)^+ \) gives an extension of \( \iota : (\mathcal{P} \mathcal{T}_n^B)^+ \rightarrow (\mathcal{P} \mathcal{T}_n^B)^- \) as it does for \( T \in (\mathcal{P} \mathcal{T}_n^B)^+ \).

Lemma 3.20. For \( T \in \mathcal{P} \mathcal{T}_n^B \),

1. \( \text{diag}(\iota(T)) = \begin{cases} \text{diag}(T) + 1 & \text{if } T \in (\mathcal{P} \mathcal{T}_n^B)^+ \\ \text{diag}(T) - 1 & \text{if } T \in (\mathcal{P} \mathcal{T}_n^B)^- \\ \end{cases} \)

2. \( 2 \text{row}(T) + \text{diag}(T) + [2 \text{row}(\iota(T)) + \text{diag}(\iota(T))] = 2n + 1, \)

3. \( s \text{o}(T) = s \text{o}(\iota(T)) \).

The following theorem with Lemma 3.20 proves Theorem 2.10 whose proof will be done in Section 4.

Theorem 3.21. \( \iota \) is an involution, that is \( \iota^2 = \text{id}_{\mathcal{P} \mathcal{T}_n^B} \), and therefore a bijection from \( \mathcal{P} \mathcal{T}_n^B \) to itself.

4. Proof of Theorem 3.21

In this section, we give a proof of Theorem 3.21. Since the bijection from \( (\mathcal{P} \mathcal{T}_n^B)^- \) to \( (\mathcal{P} \mathcal{T}_n^B)^+ \) is completely determined by \( \iota \) and the bijection from \( (\mathcal{P} \mathcal{T}_n^B)^+ \) to \( (\mathcal{P} \mathcal{T}_n^B)^- \), we only need to show that \( \iota : (\mathcal{P} \mathcal{T}_n^B)^+ \rightarrow (\mathcal{P} \mathcal{T}_n^B)^- \) is a bijection. We therefore will show that \( \iota^2 \) is the identity map on \( (\mathcal{P} \mathcal{T}_n^B)^+ \).

We start with definitions of important notions for the proof. Remember from Definition 3.4 that for a partial filling \( F \), \( r_i \) is the row-numbering of the set of boxes in \( \text{row}_i \) of \( S(F) \) with numbers in \( [n_i] \) where \( n_i \) is the number of boxes in \( \text{row}_i \) of \( S(F) \).

Definition 4.1. (1) Let \( F \) be a partial filling, \( c_1, c_2, c_0 \) be boxes of \( S(F) \).

Suppose that \( c_1 = (i_1, j_1), c_2 = (i_2, j_2), c_0 = (i_1, j_2) \) and \( i_1 \geq i_2, j_1 \leq j_2 \), that is, \( c_2 \) is located northwest of \( c_1 \). Then \( h \text{set} \) and \( v \text{set} \) are defined as follows:

\[
\text{hset}_{c_1,c_2}(F) := \{ c \in \text{row}_1 \mid r_{i_1}(c) < r_{i_2}(c) < r_{i_1}(c_0), \ c \text{ is filled with } 0 \},
\]

\[
\text{vset}_{c_1,c_2}(F) := \{ c \in \text{col}_{j_2} \mid \text{row}(c) < i_1, \ c_2 \text{ is in relevant position to } c, \ c \text{ is filled with } 0 \}.
\]

(2) We define \( \iota \) from the set of partial fillings \( \{ F \mid \text{lab}_+(F), \text{lab}_-(F) \} \) to itself such that

(a) if \( 1 \in \text{lab}_+(F) \), then \( \iota(F) \) is the partial filling from \( F \) by converting the row label 1 to \( -1 \) and attaching the box \((-1,1)\) with content 1,

(b) if \( 1 \in \text{lab}_-(F) \), then \( \iota(F) \) is the partial filling from \( F \) by deleting the box \((-1,1)\) with its content and converting the row label 1 to \(-1\).
(3) For a partial filling $F$ with $\text{lab}_+(F)$, $\text{lab}_-(F)$, $\text{lab}^d(F)$, where $1 \in \text{lab}_+(F) \cup \text{lab}_-(F)$, we define $F_{pr}$ in the same way we define $F_{pr}$ from $F$ where $1 \in \text{lab}_+(F)$. For $F$ with $1 \in \text{lab}_+(F)$, we define $F_{pr}$ using the map $i$ on the set of partial fillings. Then $pr$ is an operator on the set of partial fillings with labeling sets. We let $pr(c)$ (or $pr(i,j)$) denote the corresponding box in $F_{pr}$ to the box $c = (i, j)$ in $F$.

From now on, $F$ is a partial filling $F$ with $\text{lab}_+(F)$, $\text{lab}_-(F)$, $\text{lab}^d(F)$, where $1 \in \text{lab}_+(F) \cup \text{lab}_-(F)$.

**Lemma 4.2.** For a partial filling $F$, $(F_{pr})_{pr} = F$.

**Proof.** Since $F$ and $(F_{pr})_{pr}$ have the same shape and labeling sets, we only need to show that the box $c = (x, y)$ of $S(F)$ and $S((F_{pr})_{pr})$ are filled with the same content. We assume that $1 \in \text{lab}_+(F)$; the other case can be shown in a similar way.

Let us first look at the corresponding box $pr(c)$ in $S(F_{pr})$ of $c = (x, y)$ in $S(F)$. Let

$$\text{lab}_+(F) = \{a_1 = 1 < a_2 < \cdots < a_r\},$$

$$\text{lab}_-(F) \setminus \text{lab}^d(F) = \{b_1 < \cdots < b_s\},$$

and

$$\text{lab}^d(F) = \{d_1 < \cdots < d_{n-r-s}\}.$$

Then,

$$\text{lab}_+(F_{pr}) = \{n + 2 - b_s < \cdots < n + 2 - b_1\},$$

$$\text{lab}_-(F_{pr}) \setminus \text{lab}^d(F_{pr}) = \{n + 2 - a_r < \cdots < n + 2 - a_2\},$$

and

$$\text{lab}^d(F_{pr}) = \{a_1 = 1 < n + 2 - d_{n-r-s} < \cdots < n + 2 - d_1\}.$$

There are four cases to be considered according to the properties of $x$ and $y$ respectively:

1. $\text{row}x$ is an interior row of $S(F)$ and $\text{col}y$ is an interior column of $S(F)$, i.e., $(x, y) = (a_i, b_j)$,

2. $\text{row}x$ is an interior row of $S(F)$ and $\text{col}y$ is a boundary column of $S(F)$, i.e., $(x, y) = (a_i, d_j)$,

3. $\text{row}x$ is a boundary row of $S(F)$ and $\text{col}y$ is an interior column of $S(F)$, i.e., $(x, y) = (-d_i, b_j)$,

4. $\text{row}x$ is a boundary row of $S(F)$ and $\text{col}y$ is a boundary column of $S(F)$, i.e., $(x, y) = (-d_i, d_j)$.

In case (1), $\text{col}b_j$ of $S(F)$ fills in the interior row $\text{row}(n + 2 - b_j)$ of $S(F_{pr})$ by the definition of $F_{pr}$. Since $1 \in \text{lab}_+(F)$, box $(a_i, b_j)$ is the $(i-1)$st interior row from the top in $S(F)$. However, since $1 \notin \text{lab}_-(F_{pr}) \setminus \text{lab}^d(F_{pr})$, the $(i-1)$st interior column from the left in $S(F_{pr})$ is $\text{col}(n + 2 - a_i)$. Thus, the content of box $(a_i, b_j)$ of $S(F)$ fills in box $(n + 2 - b_j, n + 2 - a_i)$ of $S(F_{pr})$.

In case (2), $\text{col}d_j$ of $S(F)$ fills in the boundary row $\text{row}(-n + 2 + d_{j+1})$ of $S(F_{pr})$ because

$$\cdots < d_{j-1} < d_j < d_{j+1} < \cdots$$

implies

$$\cdots < -n + 2 + d_{j-1} < -n + 2 + d_j < -n + 2 + d_{j+1} < \cdots.$$
As in case (1), box \((a_i, d_j)\) is the \((i-1)st\) interior row from the topmost interior row in \(S(F)\) and the \((i-1)st\) interior column from the leftmost interior column in \(S(F_{pr})\) is \(col(n + 2 - a_i)\). Thus, the content of box \((a_i, d_j)\) of \(S(F)\) fills in the box \((-n - 2 + d_{j+1}, n + 2 - a_i)\) of \(S(F_{pr})\).

In case (3), \(col_{bj} \) of \(S(F)\) fills in the interior row \(row(n + 2 - b_j)\) of \(S(F_{pr})\). Since \(row1\) is a boundary row of \(S(F)\), \(row(-d_i)\) is the \((i+1)st\) boundary row from the \(row1\) in \(S(F)\). However, since

\[ n + 2 - d_1 > n + 2 - d_2 > \cdots > 1 \]

holds among boundary column of \(S(F_{pr})\), the \((i+1)st\) boundary column from the leftmost boundary column in \(S(F_{pr})\) is \(col(n + 2 - d_{i+1})\). Thus, the content of box \((-d_i, b_j)\) of \(S(F)\) fills in the box \((-n + 2 - b_j, n + 2 - d_{i+1})\) of \(S(F_{pr})\).

In case (4), as in case (2) \(col_{d_j}\) of \(S(F)\) fills in the boundary row \(row(-n - 2 + d_{j+1})\) of \(S(F_{pr})\). As in case (3), \(row(-d_i)\) is the \((i+1)st\) boundary row from the \(row1\) in \(S(F)\) and the \((i+1)st\) boundary column from the leftmost boundary column in \(S(F_{pr})\) is \(col(n + 2 - d_{i+1})\). Thus, the content of box \((-d_i, d_j)\) of \(S(F)\) fills in the box \((-n - 2 + d_{j+1}, n + 2 - d_{i+1})\) of \(S(F_{pr})\).

We now compare the contents of \(S((F_{pr})_{pr})\) with the ones of \(S(F_{pr})\).

Let \(F' = i(F_{pr})\) then \(F'_{pr} = i((F_{pr})_{pr})\) and labeling sets of \(F'\) are

\[ lab_+(F') = \{1 < n + 2 - b_s < \cdots < n + 2 - b_1\} := \{a'_1 < \cdots < a'_{s+1}\}, \]

\[ lab_-(F') \setminus lab_+(F') = \{n + 2 - a_r < \cdots < n + 2 - a_2\} := \{b'_1 < \cdots < b'_{r-1}\}, \]

\[ lab_-(F') = \{n + 2 - d_{n-r-s} < \cdots < n + 2 - d_1\} := \{d'_1 < \cdots < d'_{n-r-s}\}. \]

We therefore have

\[ lab_+(F'_{pr}) = \{n + 2 - b'_{r-1} < \cdots < n + 2 - b'_1\} := \{a_2 < \cdots < a_r\}, \]

\[ lab_-(F'_{pr}) \setminus lab_+(F'_{pr}) = \{n + 2 - a'_{s+1} < \cdots < n + 2 - a'_2\} := \{b_1 < \cdots < b_s\}, \]

\[ lab_-(F'_{pr}) = \{1 < n + 2 - d'_{n-r-s} < \cdots < n + 2 - d'_1\} := \{a'_1 < d_1 < \cdots < d'_{n-r-s}\}. \]

Then, in similar ways we can show that each content of box \((x, y)\) of \(S(F)\) fills in the box \((x, y)\) of \((F_{pr})_{pr}\). Therefore, \((F_{pr})_{pr} = F\).

Since \(F_{pr}\) is also a partial filling and \((F_{pr})_{pr} = F\) by Lemma 4.2, the contents of boxes in one column of \(S(F_{pr})\) fill in the boxes in one row of \(S((F_{pr})_{pr})\) and \((F_{pr})_{pr} = F\). Therefore we have the following corollary.

**Corollary 4.3.** The contents of boxes in one row of \(S(F)\) fill in the boxes in one column of \(S(F_{pr})\). Hence, each column of \(S(F)\) is corresponding to a row of \(S(F_{pr})\) and each row of \(S(F)\) is corresponding to a column of \(S(F_{pr})\).

**Lemma 4.4.** For boxes \(c_1, c_2, c\) in \(S(F)\),

\[ (1) \ c \in vset(c_1, c_2)(F) \text{ if and only if } pr(c) \in hset(c_2, c_1)(F_{pr}), \]

\[ (2) \ c \in hset(c_1, c_2)(F) \text{ if and only if } pr(c) \in vset(c_2, c_1)(F_{pr}). \]

**Proof.** (1) is immediate from the definitions of \(hset\), \(vset\) and \(pr\), and (2) is equivalent to (1) by Lemma 4.2.

**Lemma 4.5.** For two boxes \(c_1 = (i, j_1), c_2 = (i, j_2)\) of \(S(F)\) where \(j_1 < j_2\),
Corollary 4.7. \[ \text{Proof.} \] Since \( j_1 < j_2 \), \( \text{col}_{j_2} \) is on the left of \( \text{col}_{j_1} \) in \( \mathcal{F} \). By Corollary \[ \text{4.3} \] \( \text{pr}(c_1) \) and \( \text{pr}(c_2) \) are on the same column of \( \mathcal{F}_{pr} \) for both cases.

1. Let \( j_1 \in \text{lab}^d(\mathcal{F}) \). If \( j_2 \in \text{lab}^d(\mathcal{F}) \), then \( \text{col}_{j_2} \) and \( \text{col}_{j_1} \) are boundary columns of \( \mathcal{F} \). Thus, by the definition of \( \text{pr} \), the boundary row containing \( \text{pr}(c_2) \) is below the boundary row containing \( \text{pr}(c_1) \) in \( \mathcal{F}_{pr} \). If \( j_2 \notin \text{lab}^d(\mathcal{F}) \), then \( \text{col}_{j_2} \) is an interior column of \( \mathcal{F} \) while \( \text{col}_{j_1} \) is a boundary column of \( \mathcal{F} \). Since an interior row is always below a boundary row, the interior row containing \( \text{pr}(c_2) \) is below the boundary row containing \( \text{pr}(c_1) \) in \( \mathcal{F}_{pr} \). Therefore, \( \text{pr}(c_2) \) precedes \( \text{pr}(c_1) \) according to the box numbering of \( \mathcal{S}(\mathcal{F}_{pr}) \).

2. Let \( j_1 \in \text{lab}^d(\mathcal{F}) \). If \( j_2 \in \text{lab}^d(\mathcal{F}) \), then \( c_2 \) is in a boundary column of \( \mathcal{F} \) while \( c_1 \) is in an interior column of \( \mathcal{F} \). Hence, the row containing \( \text{pr}(c_2) \) is above the interior row containing \( \text{pr}(c_1) \) in \( \mathcal{F}_{pr} \). If \( j_2 \notin \text{lab}^d(\mathcal{F}) \), then \( \text{col}_{j_2} \) and \( \text{col}_{j_1} \) are both interior columns of \( \mathcal{F} \). Thus, by definition of \( \text{pr} \), the interior row containing \( \text{pr}(c_2) \) is above the interior row containing \( \text{pr}(c_1) \) in \( \mathcal{F}_{pr} \). Therefore, \( \text{pr}(c_1) \) precedes \( \text{pr}(c_2) \) according to the box numbering of \( \mathcal{S}(\mathcal{F}_{pr}) \).

Corollary 4.6. Let \( c_1, c_2 \) be two boxes of \( \mathcal{S}(\mathcal{F}) \).

1. If \( \text{hset}_{(c_1,c_2)}(\mathcal{F}) \) is not empty, then \( \text{pr}(c) \) precedes \( \text{pr}(c_1) \) according to the box numbering of \( \mathcal{S}(\mathcal{F}_{pr}) \) for all \( c \in \text{hset}_{(c_1,c_2)}(\mathcal{F}) \).
2. If \( \text{vset}_{(c_1,c_2)}(\mathcal{F}) \) is not empty, then \( \text{pr}(c_2) \) precedes \( \text{pr}(c) \) according to the box numbering of \( \mathcal{S}(\mathcal{F}_{pr}) \) for all \( c \in \text{vset}_{(c_1,c_2)}(\mathcal{F}) \).

Corollary 4.7. When \( \phi_{\text{ini}} \) occurs in Algorithm to compute \( \mathcal{S}(\mathcal{T}) \) for \( \mathcal{T}_{pr} \), \( \text{pr}(\text{outi}) \) precedes \( \text{pr}(\text{ini}) \) according to the box numbering of \( \mathcal{S}((\mathcal{T}_{pr})_{pr}) \). Especially, if \( \text{left}(\text{ini}) \) exists, then \( \text{pr}(\text{outi}) \) precedes \( \text{pr}(\text{left}(\text{ini})) \) and \( \text{pr}(\text{left}(\text{ini})) \) precedes \( \text{pr}(\text{ini}) \) according to the box numbering of \( \mathcal{S}((\mathcal{T}_{pr})_{pr}) \).

Proof. If \( \text{hset}_{(\text{in}, \text{outi})}(\mathcal{T}_{pr}) \cup \text{vset}_{(\text{in}, \text{outi})}(\mathcal{T}_{pr}) \) is not empty, then it is immediate from Corollary \[ \text{4.6} \] that \( \text{pr}(\text{outi}) \) precedes \( \text{pr}(\text{ini}) \) according to the box numbering of \( \mathcal{S}((\mathcal{T}_{pr})_{pr}) \).

Now we assume that \( \text{hset}_{(\text{in}, \text{outi})}(\mathcal{T}_{pr}) \cup \text{vset}_{(\text{in}, \text{outi})}(\mathcal{T}_{pr}) \) is empty, although \( \phi_{\text{ini}} \) occurs. We consider only the case that \( \phi_{\text{ini}} \) is rule R3, since \( \text{ini} = \text{left}(\text{ini}) \) or \( \text{outi} = \text{left}(\text{ini}) \) for other cases.
If \( \phi_{in} \) is rule \( R3 \) hence \( left(\phi_{in}) \) exists, then \( \phi_{in} \) must be in a boundary column in \( T_{pr} \). Since \( left(\phi_{in}) \) is left to \( \phi_{in} \) in the same row in \( S(T_{pr}) \), \( pr(\phi_{in}) \) is above \( pr(left(\phi_{in})) \) in \( (T_{pr})_{pr} \), hence \( pr(\phi_{in}) \) precedes \( pr(left(\phi_{in})) \) according to the box numbering of \( S((T_{pr})_{pr}) \). Since \( out(\phi_{in}) \) is in relevant position to \( left(\phi_{in}) \) in \( T_{pr} \), \( pr(out(\phi_{in})) \) is left to \( pr(left(\phi_{in})) \) in the same row in \( S((T_{pr})_{pr}) \), hence \( pr(out(\phi_{in})) \) precedes \( pr(left(\phi_{in})) \) according to the box numbering of \( S((T_{pr})_{pr}) \). Therefore, \( pr(out(\phi_{in})) \) precedes \( pr(\phi_{in}) \) according to the box numbering of \( S((T_{pr})_{pr}) \).

**Example 4.8.** Consider the filling \( F \) in Figure 12 with \( lab_+(F) = \{1, 2, 4, 6\} \), \( lab_-(F) = \{3, 5, 7, 8, 9\} \), \( lab_d^d(F) = \{3, 5, 8\} \). Then, we have \( F_{pr} \) with \( lab_+(F_{pr}) = \{2, 4\} \), \( lab_-(F_{pr}) = \{1, 3, 5, 6, 7, 8, 9\} \), \( lab_d^d(F_{pr}) = \{1, 3, 5, 6, 8\} \) from \( F \). For two boxes \( c_1 = (4, 5) \) and \( c_2 = (-3, 7) \) in \( F \), we have \( hset_{(c_1, c_2)}(F) = \{(4, 8), (4, 9)\} \) and \( vset_{(c_1, c_2)}(F) = \{(2, 7), (1, 7)\} \) (Figure 13). In \( F_{pr} \), we have \( pr(c_1) = (-3, 7) \), \( pr(c_2) = (4, 5) \). Then, the corresponding boxes in \( F_{pr} \) to boxes in \( hset_{(c_1, c_2)}(F) \) and \( vset_{(c_1, c_2)}(F) \) are as in Figure 14.

**Figure 12.** \( F \) and \( F_{pr} \)

**Figure 13.** \( hset_{(c_1, c_2)}(F) \) and \( vset_{(c_1, c_2)}(F) \)

**Figure 14.** The correspondence via \( pr \)
In the rest of this section we fix a $T \in (PT_n^B)^+$ hence $\text{ini}$’s, $\text{outi}$’s; and

$$\Xi(T) = \phi_\bullet \phi_{\text{in}k} \phi_{\text{in}(k-1)} \cdots \phi_{\text{in}i} \phi_{\text{o}}(T_{pr}).$$

Remember that our aim is to prove $\Xi^2(T) = T$. We hence need to consider $\Xi(T')$ when $T' = \Xi(T)$; we first make $(T')_{pr}$ and apply Algorithm to $(T')_{pr}$. Note that $(\text{ini})$’s, $(\text{outi})$’s are the boxes of $T_{pr}$ which is of the same shape as $T'$, and we will need to consider $(\text{in}_{T'j})$’s, $(\text{out}_{T'j})$’s of $(T')_{pr}$ which has the same shape as $T$. One natural question can be on the relation between $(\text{in}_{T'j})$’s, $(\text{out}_{T'j})$’s and $(\text{in}_{Tj})$’s, $(\text{out}_{Tj})$’s: Is it true that $(\text{in}_{T'j}) = pr(\text{out}_{T}k+1-j)$, $(\text{out}_{T'j}) = pr(\text{in}_{T}k+1-j)$? The answer is negative but we can rearrange the boxes, hence change the order of functions $\phi_\circ$’s, so that the Algorithm remain the same, and this will prove that $\Xi(T') = T$.

**Definition 4.9.** Let $T' = \Xi(T) = \phi_\bullet \phi_{\text{in}k} \cdots \phi_{\text{in}i} \phi_{\text{o}}(T_{pr})$.

1. We say that $\text{outi}$ precedes $\text{outj}$ if one of the following conditions is satisfied;
   (a) the box $pr(\text{outi})$ of $T'_{pr}$ is of type 1 but the box $pr(\text{outj})$ of $T'_{pr}$ is not,
   (b) both $pr(\text{outi})$ and $pr(\text{outj})$ are of type 1 and $pr(\text{outi})$ precedes $pr(\text{outj})$
      according to the box-numbering of $T_{pr}$,
   (c) neither $pr(\text{outi})$ nor $pr(\text{outj})$ is of type 1 and $pr(\text{outi})$ precedes $pr(\text{outj})$
      according to the box-numbering of $T'_{pr}$.
2. For $i < j$, we say that a pair $(\text{ini}, \text{inj})$ is an inversion pair of $T$ if $\text{outi}$ precedes $\text{outj}$.
3. The set of all inversion pairs of $T$ is denoted by $\text{Inv}(T)$ and $\text{inv}(T) = |\text{Inv}(T)|$.
4. We say that a pair $(\text{ini}, \text{inj})$ is an inversion pair of $T$ if $\text{in} \phi_{\text{in}k} \cdots \phi_{\text{in}i}$ is denoted by $\phi_{\text{in}k} \cdots \phi_{\text{in}i}$ such that
   (a) $(\text{ini}, \text{inj}) \in \text{Inv}(T)$ and
   (b) $x < y$ when $i = \alpha_x$ and $j = \alpha_y$.

To define the inverse map of $\Xi$, we first find a rearrangement $\phi_{\text{in}k} \cdots \phi_{\text{in}i}$ of $\phi_{\text{in}k} \cdots \phi_{\text{in}i}$ such that

$$\phi_\bullet \phi_{\text{in}k} \cdots \phi_{\text{in}i} \phi_{\text{o}}(T_{pr}) = \phi_\bullet \phi_{\text{in}k} \cdots \phi_{\text{in}i} \phi_{\text{o}}(T_{pr})$$

and there is no inversion pair of $T$ in $\phi_{\text{in}k} \cdots \phi_{\text{in}i}$. We then find the inverse map $(\phi_{\text{in}k})^{-1}$ of $\phi_{\text{in}k}$ for each $x \in [k]$, $(\phi_{\text{o}})^{-1}$ and $(\phi_\circ)^{-1}$ to define $\Xi^{-1}$. Each inverse map will be corresponding to a certain rule. In fact, if $\phi_\circ$ is rule $R1$, $R2$, $R3$, $R4.1$, or $R4.2$, then $(\phi_{\text{o}})^{-1}$ is rule $R4.1$, $R4.2$, $R3$, $R1$, or $R2$, respectively. In particular, since the inverse map of rule $R2$ is corresponding to rule $R4.2$, rule $R4.2$ is applied only in an interior row. The following lemma can be proved by considering the original permutation tableau $T$ corresponding to $T_{pr}$, whose proof will not be covered.

**Lemma 4.10.** If $\phi_\circ$ is rule $R4.2$, then $\text{in}x$ and $\text{out}x$ are contained in an interior row and $\text{out}x$ has at least one 1 or $S$ above it, not in relevant position to it in $\phi_{\text{in}(x-1)} \cdots \phi_{\text{in}1} \phi_{\text{o}}(T_{pr})$.

**Definition 4.11.** For $T \in PT_n^B$, we recursively define $\psi_0, \psi_1, \ldots$ as follows:

1. $\psi_0 := \phi_{\text{in}k} \cdots \phi_{\text{in}1}$. 

(2) For $s \geq 1$ and $\psi_{s-1} = \phi_{i1} \cdots \phi_{in}$, if there are inversion pairs of $T$ in $\psi_{s-1}$ hence $\psi_{s-1}$ has $\alpha_x$ and $\alpha_y$ such that

(a) $i = \min(x \mid \{(x, y) \mid \text{is an inversion pair of } T \text{ in } \psi_{s-1} \text{ for some } y\}$,

(b) for $i$ in (a), $j = \min\{y \mid \{(x, y) \mid \text{is an inversion pair of } T \text{ in } \psi_{s-1}\}$,

then $\psi_s := \phi_{i1} \cdots \phi_{in} \circ \phi_{i1} \cdots \phi_{in}$.

It is easy to see that for each $T \in \mathcal{PT_T}^B$, $\psi_s$'s are determined uniquely from the Definition 1.11. We are going to prove that each $\psi_s = \phi_{i1} \cdots \phi_{in}$ satisfies the following properties:

(P1) $\phi_{i1} \circ \phi_{i2} (T_{pr}) = \mathcal{I}(T)$,

(P2) each $\phi_{i1}$ in $\phi_{i1} \circ \phi_{i2} (T_{pr})$ moves 'S' from the box out $x$ to the box in $x$, and

(P3) if $x > y$ exists such that $\alpha_x < \alpha_y$, then $(x, y) \in \text{Inv}(T)$, where $\psi_s = \phi_{i1} \cdots \phi_{in}$.

Note that $\psi_0 = \phi_{i1} \cdots \phi_{in}$ satisfies (P1), (P2), and (P3).

**Lemma 4.12.** Suppose that $\psi_{s-1} = \phi_{i1} \cdots \phi_{in}$ satisfies the property (P3) and let $i$ and $j$ be the indices defined in Definition 1.11. Then, $(x, y) \in \text{Inv}(T)$ for all $x \in \{i, \ldots, j-1\}$, hence the number of inversion pairs of $T$ in $\psi_s$ decreases by $(j-i)$ from the number of inversion pairs of $T$ in $\psi_{s-1}$. Moreover, $\psi_s$ also satisfies the property (P3).

**Proof.** It is immediate from the definition of $\alpha_i$ in $\psi_{s-1}$ in Definition 1.11 that $(x, y) \in \text{Inv}(T)$, hence out $x$ precedes out $y$ and $\alpha_i < \alpha_j$.

Let $x \in \{i+1, \ldots, j-1\}$. If $\alpha_i < \alpha_x$, then $(x, y) \in \text{Inv}(T)$ is not an inversion pair of $T$ in $\psi_{s-1}$ from the definition of $\alpha_j$. Besides, $\phi_{i1}$ is right to $\phi_{i1}$ in $\psi_{s-1}$. Thus, out $x$ precedes out $y$ hence out $x$ precedes out $y$.

If $x \in \{i+1, \ldots, j-1\}$ satisfies $\alpha_i > \alpha_x$ in $\psi_{s-1}$, then $(x, y) \in \text{Inv}(T)$ by the property (P3) of $\psi_{s-1}$. Thus, out $x$ precedes out $y$ hence out $x$ precedes out $y$.

To prove $(x, y) \in \text{Inv}(T)$ for $x \in \{i, \ldots, j-1\}$, we show that $\alpha_x < \alpha_j$ for all $x \in \{i, \ldots, j-1\}$. Suppose that there exists $x \in \{i, \ldots, j-1\}$ such that $\alpha_x > \alpha_j$, hence $\alpha_x > \alpha_j > \alpha_i$. We then know that $(x, y) \in \text{Inv}(T)$ by the property (P3) of $\psi_{s-1}$. Then there exists $y \leq s$ such that $\psi_y$ has $x$ as $j$ in Definition 1.11. On the other hand, $\phi_{i1}$ did not exchange $\phi_{i1}$ although $\alpha_x > \alpha_i$. This is a contradiction to minimality of $i$ in the definition of $\psi_{y+1}$. Therefore, $(x, y) \in \text{Inv}(T)$ for all $x \in \{i+1, \ldots, j-1\}$.

What order changes when we change from $\psi_{s-1}$ to $\psi_s$ are pairs $(x, j)$ for all $x \in \{i, \ldots, j-1\}$. We have to check that they satisfy the property (P3) in $\psi_s$. In $\psi_s$, each pair $(x, j)$ for $x \in \{i, \ldots, j-1\}$ satisfies $x > j$ and $\alpha_x < \alpha_j$. However, we already know that each $(x, y) \in \text{Inv}(T)$ for $x \in \{i, \ldots, j-1\}$, hence $(x, j)$ satisfies (P3) for all $x \in \{i, \ldots, j-1\}$. Besides, since other pairs such that $x > y$ and $\alpha_x < \alpha_y$ in $\psi_s$ maintained the order from $\psi_{s-1}$, therefore $\psi_s$ also satisfies the property (P3).

Since $\text{Inv}(T)$ is finite, using Lemma 4.12 we can always find $\psi_l = \phi_{i1} \cdots \phi_{in}$ for some $l \geq 0$ satisfying the property (P3) and that there is no inversion pair of
Lemma 4.13. For all $s \in \{0, 1, \ldots, l\}$, $\psi_s$ satisfies the properties (P1) and (P2).

Proof. We will prove this lemma by induction on $s$. $\psi_0$ has the properties (P1) and (P2). Let $\psi_s = \phi_{in_{\alpha_i}} \cdots \phi_{in_{\alpha_j}}$ have $i, j$ in Definition 4.11. From Lemma 4.12 we know $(in_{\alpha_i}, in_{\alpha_j}) \in Inv(\Phi)$ for all $x \in \{i, \ldots, j-1\}$. If we repeatedly exchange $\phi_{in_{\alpha_i}}$ with each $\phi_{in_{\alpha_j}}$, $x = j - 1, j - 2, \ldots, i$, then we can obtain $\psi_{s+1}$ from $\psi_s$. So, it is enough to prove the following:

When $(in_{\alpha_r}, in_{\alpha_{r+1}}) \in Inv(\Phi)$ and $\Phi_{in_{\alpha_{r-1}}, \cdots, \phi_{in_{1}}, \phi_{in_{\alpha} (T_{pr})}}$, if $\phi_{in_{\alpha_r}}$ moves $S$ from $out_{\alpha_r}$ to $in_{\alpha_r}$ in $\phi_{in_{\alpha_{r-1}}, \cdots, \phi_{in_{1}}, \phi_{in_{\alpha} (T_{r})}}$, then $\phi_{in_{\alpha_r}, \phi_{in_{\alpha_{r+1}}}}$ in $\phi_{in_{\alpha_{r-1}}, \cdots, \phi_{in_{1}}, \phi_{in_{\alpha} (T_{r+1})}}$ moves $S$ from $out_{\alpha_{r+1}}$ to $in_{\alpha_{r+1}}$ in $\phi_{in_{\alpha_{r-1}}, \cdots, \phi_{in_{1}}, \phi_{in_{\alpha} (T_{r+1})}}$.

Suppose that $\phi_{in_{\alpha_{r+1}}}$ cannot move $S$ from $out_{\alpha_{r+1}}$ to $in_{\alpha_{r+1}}$ in $\phi_{in_{\alpha_{r+1}}, \phi_{in_{\alpha} (T_{r+1})}}$. This must be because that $\phi_{in_{\alpha_r}}$ did not occur before $\phi_{in_{\alpha_r}}$ is applied. There are four possible cases according to the effect of changes in $in_{\alpha_r}$ and $out_{\alpha_r}$ by $\phi_{in_{\alpha_r}}$:

1. $in_{\alpha_r} = out_{\alpha_{r+1}}$ in $S(T_{pr})$.
2. $in_{\alpha_r} = left(in_{\alpha_{r+1}})$ in $S(T_{pr})$.
3. $out_{\alpha_r} = in_{\alpha_{r+1}}$ in $S(T_{pr})$.
4. $out_{\alpha_r} \in hset(in_{\alpha_{r+1}, out_{\alpha_{r+1}}}) \cup set(in_{\alpha_{r+1}, out_{\alpha_{r+1}}})$.

Let us suppose (1). Note that $\alpha_r < \alpha_{r+1}$. By considering the box numbering $S(T_{pr})$ the only possible case is that $\phi_{in_{\alpha_r}}$ is the rule R1 and $\phi_{in_{\alpha_{r+1}}}$ is not the rule R1 in $\phi_{in_{\alpha_{r+1}}, \phi_{in_{\alpha} (T_{r})}}$. We remark that $in_{\alpha_r}$ hence $out_{\alpha_{r+1}}$ is in an interior row in $S(T_{pr})$ because 1 came from an interior column of $S(T)$. Thus, $\phi_{in_{\alpha_{r+1}}}$ must be rule R4.1 or rule R4.2.

Suppose that $\phi_{in_{\alpha_{r+1}}}$ is rule R4.1 in $\phi_{in_{\alpha_{r+1}}, \phi_{in_{\alpha} (T_{r})}}$. Then, by the definition of rule R4.1 and R1, $pr(out_{\alpha_{r+1}})$ is of type 1 in $S((\Phi(T))_{pr})$. Since $pr(out_{\alpha_r})$ is not of type 1 but $pr(out_{\alpha_{r+1}})$ is of type 1 in $S((\Phi(T))_{pr})$, $out_{\alpha_{r+1}}$ precedes $out_{\alpha_r}$. However, this is a contradiction to $(in_{\alpha_r}, in_{\alpha_{r+1}}) \in Inv(\Phi)$.

We then assume that $\phi_{in_{\alpha_{r+1}}}$ is rule R4.2 in $\phi_{in_{\alpha_{r+1}}, \phi_{in_{\alpha} (T_{r})}}$ and there is another $S$ above $out_{\alpha_r}$ in the same column in $S(T_{pr})$ by Lemma 4.10. However, the closest $S$ above $out_{\alpha_r}$ is in relevant position to $in_{\alpha_r}$ hence $out_{\alpha_{r+1}}$. Thus $\phi_{in_{\alpha_{r+1}}}$ cannot move $S$ from $out_{\alpha_{r+1}}$ to $in_{\alpha_{r+1}}$ in $\phi_{in_{\alpha_{r+1}}, \phi_{in_{\alpha} (T_{r})}}$. This is a contradiction.

Suppose (2). Note that $\phi_{in_{\alpha_{r+1}}}$ is rule R3 because $left(in_{\alpha_{r+1}})$ exists. Since $in_{\alpha_{r+1}}$ precedes $left(in_{\alpha_{r+1}})$ hence $in_{\alpha_r}$ according to the box numbering of $S(T_{pr})$, the only possible case is that $\phi_{in_{\alpha_r}}$ is the rule R1 and $\phi_{in_{\alpha_{r+1}}}$ is not the rule R1.
in $\phi_{\text{in} r+1} \phi_{\text{in} r}(T_s)$ as in case (1). On the other hand, it is easy to check that

$$\text{out}_r \in vset(\text{in}_{r+1}, \text{out}_{r+1})(\phi_{\text{in} r}(T_s)),$$

hence $\text{out}_{r+1}$ precedes $\text{out}_r$ by Corollary 4.6 However, this is a contradiction to $(\text{in}_r, \text{in}_{r+1}) \in \text{Inv}(T)$.

Suppose (3). Then, $\text{pr} (\text{out}_{r+1})$ precedes $\text{pr} (\text{out}_r)$ according to the box numbering in $S((T(T))_{pr})$ by Corollary 4.7 Since $\phi_{\text{in} r}$ cannot be rule $R4.1$ in $\phi_{\text{in} r}(T_s)$, $\text{pr} (\text{out}_r)$ is not of type $1$ in $S((T(T))_{pr})$. Thus, $\text{out}_{r+1}$ precedes $\text{out}_r$. However, this is a contradiction to $(\text{in}_r, \text{in}_{r+1}) \in \text{Inv}(T)$.

Suppose (4). If $\text{out}_r \in vset(\text{in}_{r+1}, \text{out}_{r+1})(\phi_{\text{in} r}(T_s))$, then $\text{pr} (\text{out}_{r+1})$ precedes $\text{pr} (\text{out}_r)$ according to the box numbering in $S((T(T))_{pr})$ by Corollary 4.6. If $\text{out}_r \in hset(\text{in}_{r+1}, \text{out}_{r+1})(\phi_{\text{in} r}(T_s))$, then since $\text{out}_{r+1}$ is in a boundary row and the row containing $\text{in}_{r+1}$ is below the row containing $\text{out}_{r+1}$ in $S(T_{pr})$, $\text{pr} (\text{out}_{r+1})$ precedes $\text{pr} (\text{out}_r)$ hence $\text{pr} (\text{out}_r)$ according to the box numbering in $S((T(T))_{pr})$. However, this is a contradiction to $(\text{in}_r, \text{in}_{r+1}) \in \text{Inv}(T)$. We assume that $\text{pr} (\text{out}_r)$ is of type 1 and $\text{pr} (\text{out}_{r+1})$ is of type 2, 3, or 4 in $S((T(T))_{pr})$, this is impossible because $(\text{in}_r, \text{in}_{r+1}) \in \text{Inv}(T)$. Since $\text{out}_r$ has 1 in $\phi_{\text{in} r}(T_s)$, $\text{out}_r \notin hset(\text{in}_{r+1}, \text{out}_{r+1})(\phi_{\text{in} r}(T_s)) \cup vset(\text{in}_{r+1}, \text{out}_{r+1})(\phi_{\text{in} r}(T_s))$. This is a contradiction.

Therefore, $\phi_{\text{in} r+1}$ moves $S$ from $\text{out}_{r+1}$ to $\text{in}_{r+1}$ in $\phi_{\text{in} r+1}(T_s)$.

One more thing we need to check is that $\phi_{\text{in} r+1}$ in $\phi_{\text{in} r+1} \phi_{\text{in} r}(T_s)$ is the same rule to the one in $\phi_{\text{in} r} \phi_{\text{in} r+1}(T_s)$. Since the box from which $\text{in}_{r+1}$ obtains $S$ does not change in $\phi_{\text{in} r+1}(T_s)$, we only have to check for rules $R4.1$ or $R4.2$.

Suppose that while $\phi_{\text{in} r+1}$ is rule $R4.1$ in $\phi_{\text{in} r+1} \phi_{\text{in} r}(T_s)$, $\phi_{\text{in} r+1}$ is rule $R4.2$ in $\phi_{\text{in} r} \phi_{\text{in} r+1}(T_s)$. This means that $\text{out}_r$ with $S$ is above $\text{out}_{r+1}$ in the same column in $T_s$ but not in relevant position to $\text{out}_{r+1}$. On the other hand, since $(\text{in}_r, \text{in}_{r+1}) \in \text{Inv}(T)$, $\phi_{\text{in} r}$ is also rule $R4.1$ in $\phi_{\text{in} r+1} \phi_{\text{in} r}(T_s)$. Hence, the row containing $\text{in}_r$ is above the row containing $\text{in}_{r+1}$. However, it is a contradiction to $\alpha_r < \alpha_{r+1}$, since neither $\phi_{\text{in} r}$ nor $\phi_{\text{in} r+1}$ is rule $R1$.

Suppose that while $\phi_{\text{in} r+1}$ is rule $R4.2$ in $\phi_{\text{in} r+1} \phi_{\text{in} r}(T_s)$, $\phi_{\text{in} r+1}$ is rule $R4.1$ in $\phi_{\text{in} r+1} \phi_{\text{in} r+1}(T_s)$. This means that $\text{in}_r$ with 0 is above $\text{out}_{r+1}$ in the same column in $T_s$ but not in relevant position to $\text{out}_{r+1}$, hence $\text{in}_r$ is not of type 1 in $T_s$. However, this is impossible for the same reason as in the previous case.

Therefore, $\phi_{\text{in} r+1}$ in $\phi_{\text{in} r+1} \phi_{\text{in} r}(T_s)$ and the one in $\phi_{\text{in} r} \phi_{\text{in} r+1}(T_s)$ are the same rule.

We now prove that $\text{in}_r$ obtains $S$ from $\text{out}_r$ in $\phi_{\text{in} r} \phi_{\text{in} r+1}(T_s)$, and that $\phi_{\text{in} r}$ in $\phi_{\text{in} r} \phi_{\text{in} r+1}(T_s)$ is the same rule with one in $\phi_{\text{in} r+1} \phi_{\text{in} r}(T_s)$.

Suppose that $\phi_{\text{in} r}$ cannot move $S$ from $\text{out}_r$ to $\text{in}_r$ in $\phi_{\text{in} r} \phi_{\text{in} r+1}(T_s)$. This must be because that $\phi_{\text{in} r+1}$ does occur before $\phi_{\text{in} r}$. Note that $\text{out}_r$ is different from $\text{out}_{r+1}$. Therefore, we only have to consider the following cases:
(1) \( \text{in} \alpha_{r+1} \in \text{hset}(\text{in} \alpha_r, \text{out} \alpha_r)(T_r) \).

(2) \( \text{in} \alpha_{r+1} \in \text{vset}(\text{in} \alpha_r, \text{out} \alpha_r)(T_r) \).

(3) \( \text{out} \alpha_{r+1} = \text{left}(\text{in} \alpha_r) \in S(T_{pr}) \).

Suppose (1). Then, \( \text{out} \alpha_{r+1} \) is in the same column with \( \text{in} \alpha_{r+1} \) in \( S(T_{pr}) \). (Otherwise, \( \text{out} \alpha_{r+1} = \text{out} \alpha_r \).) However, since \( \alpha_r < \alpha_{r+1} \), \( \phi \text{in} \alpha_{r+1} \) cannot be rule R1, hence \( \phi \text{in} \alpha_{r+1} \) is rule R2 in \( \phi \text{in} \alpha_1 \cdots \phi \text{in} \alpha_1 \). Thus, there is a box \( c \) with nonzero on the left of \( \text{in} \alpha_{r+1} \) in the same row in \( T_r \). However, since \( \text{in} \alpha_{r+1} \) is in an interior column in \( S(T_{pr}) \), the box \( c \) is also in \( \text{hset}(\text{in} \alpha_{r+1}, \text{out} \alpha_{r+1})(T_r) \), hence \( c \) has 0. It is a contradiction.

Suppose (2). Since \( \text{vset}(\text{in} \alpha_r, \text{out} \alpha_r)(T_r) \) is not empty, \( \text{out} \alpha_r \) is in a boundary row in \( S(T_{pr}) \) and in relevant position to \( \text{in} \alpha_{r+1} \) by the definition of \( \text{vset} \). Thus, \( \phi \text{in} \alpha_{r+1} \) is the rule R3, R4.1, or R4.2 in \( \phi \text{in} \alpha_1 \phi \text{in} \alpha_1 \). (otherwise, \( \text{out} \alpha_{r+1} = \text{out} \alpha_r \).) Since \( \phi \text{in} \alpha_{r+1} \) is not rule R2, \( \text{in} \alpha_{r+1} \) hence \( \text{out} \alpha_r \) is in a boundary column in \( S(T_{pr}) \). Then, the boundary column with \( \text{out} \alpha_r \) is on the right of the column with \( \text{out} \alpha_{r+1} \) in \( S(T_{pr}) \). Besides, since \( \text{out} \alpha_r = \text{in} \alpha_{r+1} \), \( \text{pr}(\text{out} \alpha_r) \) cannot have 1, so it cannot be of type 1. Thus, \( \text{out} \alpha_{r+1} \) precedes \( \text{out} \alpha_r \) by Corollary 4.7. It is impossible that \( (\text{in} \alpha_r, \text{in} \alpha_{r+1}) \in \text{Inv}(T) \).

If we suppose (3), then \( \phi \text{in} \alpha_r \) must be rule R3. Then, \( \text{in} \alpha_r \) is right to \( \text{left}(\text{in} \alpha_r) \) hence \( \text{out} \alpha_{r+1} \) in the same row and \( \text{out} \alpha_r \) is above \( \text{left}(\text{in} \alpha_r) \) hence \( \text{out} \alpha_{r+1} \) in the same column in \( S(T_{pr}) \). If \( \text{in} \alpha_{r+1} \) is in the same row with \( \text{out} \alpha_{r+1} \), then \( \phi \text{in} \alpha_{r+1} \) moves \( S \) from \( \text{out} \alpha_r \) to \( \text{in} \alpha_{r+1} \) in \( T_r \), however this is impossible. Thus, we assume that the row containing \( \text{in} \alpha_{r+1} \) is below the row containing \( \text{out} \alpha_{r+1} \) in \( S(T_{pr}) \). However, this is a contradiction to \( \alpha_r < \alpha_{r+1} \) because \( \phi \text{in} \alpha_{r+1} \) is not rule R1.

We can show that \( \phi \text{in} \alpha_r \) in \( \phi \text{in} \alpha_{r+1} \phi \text{in} \alpha_r (T_r) \) is the same rule to the one in \( \phi \text{in} \alpha_r \phi \text{in} \alpha_{r+1} (T_r) \), whose proof will not be included.

Thus, we have \( \phi \text{in} \alpha_r \phi \text{in} \alpha_{r+1} (T_r) = \phi \text{in} \alpha_{r+1} \phi \text{in} \alpha_r (T_r) \) when \( (\text{in} \alpha_r, \text{in} \alpha_{r+1}) \in \text{Inv}(T) \), while \( \phi \text{in} \alpha_r \), \( \phi \text{in} \alpha_{r+1} \) move \( S \) from \( \text{out} \alpha_r \), \( \text{out} \alpha_{r+1} \) to \( \text{in} \alpha_r \), \( \text{in} \alpha_{r+1} \) respectively. Using this result and Lemma 4.12, we have

\[
\phi \psi_s \phi_0 (T_{pr}) = \phi \phi \phi \text{in} \alpha_1 \cdots \phi \text{in} \alpha_{r+1} \phi \text{in} \alpha_1 \phi \text{in} \alpha_{r+1} \cdots \phi \text{in} \alpha_1 (T_r) = \phi \phi \phi \text{in} \alpha_1 \cdots \phi \text{in} \alpha_{r+1} \phi \text{in} \alpha_1 \phi \text{in} \alpha_{r+1} \cdots \phi \text{in} \alpha_1 (T_r) = \phi \phi \phi \text{in} \alpha_1 \cdots \phi \text{in} \alpha_{r+1} \phi \text{in} \alpha_1 \phi \text{in} \alpha_{r+1} \cdots \phi \text{in} \alpha_1 (T_r)
\]

and \( \phi \text{in} \alpha_r \) in \( \psi_{s+1} \phi_0 (T_{pr}) \) moves \( S \) from \( \text{out} \alpha_r \) to \( \text{in} \alpha_r \) for each \( x \in [k] \). Therefore, we have

\[
\Sigma(T) = \phi \psi_0 \phi_0 (T_{pr}) = \phi \psi_1 \phi_0 (T_{pr}) = \cdots = \phi \psi_l \phi_0 (T_{pr})
\]

and for all \( s \in [l] \), \( \psi_s \) has the property (P2).

Lemma 4.14. For \( T \in \mathcal{P} T^E_n \), let \( \psi_1 = \phi \text{in} \alpha_1 \cdots \phi \text{in} \alpha_1 \), \( F_i = \phi \text{in} \alpha_1 \cdots \phi \text{in} \alpha_1 (T_{pr}) \) and \( F_{i+1} = \phi \text{in} \alpha_{r+1} \cdots \phi \text{in} \alpha_{r+1} (T_{pr}) \).

(1) If \( \text{in} \alpha_{r+1} \) of \( F_i \) is of type 1, then \( \text{pr}(\text{out} \alpha_{r+1}) \) in \( (F_{i+1})_{pr} \) is of type 4 and there is no 0 or \( S \) above \( \text{pr}(\text{out} \alpha_{r+1}) \) in the same column.
(2) If $\text{in}_{i+1}$ of $\mathcal{F}_i$ is of type 2, then $\text{pr}(\text{out}_{i+1})$ in $(\mathcal{F}_{i+1})_{pr}$ is of type 4 and there is 1 or $S$ above $\text{pr}(\text{out}_{i+1})$ in the same column.

(3) If $\text{in}_{i+1}$ of $\mathcal{F}_i$ is of type 3, then $\text{pr}(\text{out}_{i+1})$ in $(\mathcal{F}_{i+1})_{pr}$ is of type 3.

(4) If $\text{in}_{i+1}$ of $\mathcal{F}_i$ is of type 4 and there is no 1 or $S$ above it in the same column, then $\text{pr}(\text{out}_{i+1})$ in $(\mathcal{F}_{i+1})_{pr}$ is of type 1.

(5) If $\text{in}_{i+1}$ of $\mathcal{F}_i$ is of type 4 and there is 1 or $S$ above it in the same column, then $\text{pr}(\text{out}_{i+1})$ in $(\mathcal{F}_{i+1})_{pr}$ is of type 2.

Therefore, $(\phi_{pr}(\text{out}_{i+1}))((\mathcal{F}_{i+1})_{pr})_{pr} = \mathcal{F}_i$.

Proof. (1) In $\mathcal{F}_i$, $\text{in}_{i+1}$ has 1 in an interior row and $\text{out}_{i+1}$ has $S$ in a boundary row. Then, in $(\mathcal{F}_{i+1})_{pr}$, $\text{pr}(\text{in}_{i+1})$ has $S$ in an interior column and $\text{pr}(\text{out}_{i+1})$ has 0 in a boundary column. The 1 in $\text{pr}(\text{in}_{i+1})$ has no 1 or $S$ above it in the same column in $\mathcal{F}_i$, so does the $S$ in $\text{pr}(\text{in}_{i+1})$ in $(\mathcal{F}_{i+1})_{pr}$. Also, since $\text{hset}(\text{in}_{i+1}, \text{out}_{i+1})(\mathcal{F}_i)$ is empty, $\text{vset}(\text{pr}(\text{out}_{i+1}), \text{pr}(\text{in}_{i+1}))(\mathcal{F}_{i+1})_{pr}$ is empty by Lemma 4.4. Hence, $\text{pr}(\text{out}_{i+1})$ is of type 4 in $(\mathcal{F}_{i+1})_{pr}$ and there is no 1 or $S$ in the same column.

(2) In $\mathcal{F}_i$, $\text{in}_{i+1}$ has 0 and $\text{out}_{i+1}$ has $S$. Then, in $(\mathcal{F}_{i+1})_{pr}$, $\text{pr}(\text{in}_{i+1})$ has $S$ and $\text{pr}(\text{out}_{i+1})$ has 0. Since $\text{out}_{i+1}$ is in a boundary row of $\mathcal{F}_i$, $\text{pr}(\text{out}_{i+1})$ is in a boundary column $(\mathcal{F}_{i+1})_{pr}$. Moreover, since $\text{in}_{i+1}$ is in an interior column in $\mathcal{F}_i$ and the 0 in $\text{in}_{i+1}$ has 1 or $S$ left in the same row of $\mathcal{F}_i$, $\text{pr}(\text{in}_{i+1})$ has 1 or $S$ above it in the same column in $(\mathcal{F}_{i+1})_{pr}$. However, since $\text{in}_{i+1}$ is in an interior column in $\mathcal{F}_i$, none of the boxes left to $\text{in}_{i+1}$ of $\mathcal{F}_i$ is moved to a relevant position to $\text{pr}(\text{in}_{i+1})$ in $(\mathcal{F}_{i+1})_{pr}$. Besides, since $\text{hset}(\text{in}_{i+1}, \text{out}_{i+1})(\mathcal{F}_i)$ is empty, $\text{vset}(\text{pr}(\text{out}_{i+1}), \text{pr}(\text{in}_{i+1}))(\mathcal{F}_{i+1})_{pr}$ is also empty by Lemma 4.4. Hence, $\text{pr}(\text{out}_{i+1})$ is of type 4 in $(\mathcal{F}_{i+1})_{pr}$ and there is no 1 or $S$ in the same column.

(3) In $\mathcal{F}_i$, $\text{in}_{i+1}$ has 0 and $\text{out}_{i+1}$ has $S$. Then, in $(\mathcal{F}_{i+1})_{pr}$, $\text{pr}(\text{in}_{i+1})$ has $S$ and $\text{pr}(\text{out}_{i+1})$ has 0. Since $\text{in}_{i+1}$ is in a boundary column and $\text{out}_{i+1}$ is in a boundary row in $\mathcal{F}_i$, $\text{pr}(\text{in}_{i+1})$ is in a boundary row and $\text{pr}(\text{out}_{i+1})$ is a boundary column. Moreover, we know that the boxes in $\text{vset}(\text{in}_{i+1}, \text{out}_{i+1})(\mathcal{F}_{i+1})_{pr}$, $\text{hset}(\text{in}_{i+1}, \text{out}_{i+1})(\mathcal{F}_{i+1})_{pr}$ fill in the boxes in $\text{hset}(\text{pr}(\text{out}_{i+1}), \text{pr}(\text{in}_{i+1}))(\mathcal{F}_{i+1})_{pr}$, $\text{vset}(\text{pr}(\text{out}_{i+1}), \text{pr}(\text{in}_{i+1}))(\mathcal{F}_{i+1})_{pr}$ respectively by Lemma 4.4. Thus, $\text{pr}(\text{out}_{i+1})$ is of type 3 in $(\mathcal{F}_{i+1})_{pr}$.

(4) In $\mathcal{F}_i$, $\text{in}_{i+1}$ has 0 and $\text{out}_{i+1}$ has $S$. Then, in $(\mathcal{F}_{i+1})_{pr}$, $\text{pr}(\text{in}_{i+1})$ has $S$ and $\text{pr}(\text{out}_{i+1})$ has 1. Since the length of boundary column containing $\text{in}_{i+1}$ is less than and equal to the length of interior column containing $\text{out}_{i+1}$ in $\mathcal{S}(\mathcal{T}_{pr})$, $\text{pr}(\text{in}_{i+1})$ is in a relevant position to $\text{pr}(\text{out}_{i+1})$. Moreover, we know that $\text{hset}(\text{in}_{i+1}, \text{out}_{i+1})(\mathcal{F}_i) = \text{vset}(\text{pr}(\text{out}_{i+1}), \text{pr}(\text{in}_{i+1}))(\mathcal{F}_{i+1})_{pr}$ by Lemma 4.4. Thus, $\text{pr}(\text{out}_{i+1})$ is of type 1 in $(\mathcal{F}_{i+1})_{pr}$.

(5) In $\mathcal{F}_i$, $\text{in}_{i+1}$ has 0, $\text{out}_{i+1}$ has $S$ with nonzero above it in non-relevant positions by Lemma 4.10. Then, in $(\mathcal{F}_{i+1})_{pr}$, $\text{pr}(\text{in}_{i+1})$ has $S$ and $\text{pr}(\text{out}_{i+1})$ has 0 with some $S$'s left it in the same row. Moreover, we also know that $\text{out}_{i+1}$ is in an interior row of $\mathcal{F}_i$ by Lemma 4.10. Hence, $\text{hset}(\text{in}_{i+1}, \text{out}_{i+1})(\mathcal{F}_i) = \text{vset}(\text{pr}(\text{out}_{i+1}), \text{pr}(\text{in}_{i+1}))(\mathcal{F}_{i+1})_{pr}$ by Lemma 4.4. Thus, $\text{pr}(\text{out}_{i+1})$ is in an interior column of $(\mathcal{F}_{i+1})_{pr}$, hence $\text{pr}(\text{out}_{i+1})$ is of type 2 in $(\mathcal{F}_{i+1})_{pr}$. \qed
Lemma 4.15. Let $\mathcal{T}(T) = \phi_\bullet \psi_l \phi_o(T_{pr})$ and $\mathcal{F}_i = \phi_{in_{\alpha_i}} \cdots \phi_{in_{\alpha_1}} \phi_o(T_{pr})$. Then

$$\phi_o((\phi_\bullet(\mathcal{F}_k))_{pr}) = (\mathcal{F}_k)_{pr}$$

and

$$\phi_\bullet((\phi_o(T_{pr}))_{pr}) = (T_{pr})_{pr} = T.$$ 

Proof. Let $T' = \mathcal{T}(T) = \phi_\bullet \phi_{in_{\alpha_k}} \cdots \phi_{in_{\alpha_1}} \phi_o(T_{pr})$ and $\mathcal{F}_i = \phi_{in_{\alpha_i}} \cdots \phi_{in_{\alpha_1}} \phi_o(T_{pr})$. We first show that $\phi_o((\phi_\bullet(\mathcal{F}_k))_{pr}) = (\mathcal{F}_k)_{pr}$. In $S(T'_{pr})$, there are three kinds of boxes with 1:

1. a box $c_1$ which obtains a 1 while $\phi_\bullet$ is applied to $\psi_l \phi_o(T_{pr})$,
2. a box $c_2$ which obtains a 1 in the rule R4.1, when the rule R4.1 is applied in computing $\psi_l(\phi_o(T_{pr}))$,
3. a box $c_3$ which has a 1 in $S(T_{pr})$ and has not been changed by $\phi_\bullet \psi_l \phi_o$.

We have to check that $pr(c_1)$ is of type 0 and $pr(c_2)$, $pr(c_3)$ are not of type 0 in $S(T'_{pr})$.

$c_1$ is an interior column in $S(T_{pr})$ by the definition of $\phi_\bullet$. Thus $pr(c_1)$ has 1 or $S$ above it not in relevant position in the same column in $S(T'_{pr})$, hence $pr(c_1)$ is of type 0 in $S(T'_{pr})$.

Note that $c_2 = out_{\alpha_x}$ in $S(T_{pr})$ for some $x \in [k]$ and has no 1 or $S$ above it in the same column in $S(\mathcal{F}_y)$ for all $y = x, x+1, \ldots, k$ hence in $T'$. Then $pr(c_2)$ has no 1 or $S$ left it in the same row in $S(T'_{pr})$, therefore $pr(c_2)$ is not of type 0 in $S(T'_{pr})$.

We know that $c_3$ with 1 has no 1 or $S$ above it in the same (interior) column in $S(T_{pr})$. $c_3$ also has no 1 or $S$ left it in the same (interior) row in $S(T_{pr})$, otherwise $pr(c_3)$ with 1 has nonzero entry above it in the same column in $S(T)$, however this is a contradiction because that $T$ is a permutation tableau of type $B$. Thus, $c_3$ with 1 has no 1 or $S$ both ‘above it’ and ‘left it’ in $S(T_{pr})$. So, we do nothing to $c_3$ hence $pr(c_3)$ with 1 also has no 1 or $S$ both ‘above it’ and ‘left it’ in $T'$. (In fact, this is possible because $c_3$, $pr(c_3)$ are in interior columns of $S(T_{pr})$, $S(T'_{pr})$, respectively.) Thus $c_3$ is not of type 0 in $S(T'_{pr})$.

Therefore, $\phi_o((\phi_\bullet(\mathcal{F}_k))_{pr}) = (\mathcal{F}_k)_{pr}$. It is easy to check that $\phi_\bullet((\phi_o(T_{pr}))_{pr}) = (T_{pr})_{pr} = T$ in a similar way.

Now we show that $\mathcal{T}$ is indeed $\mathcal{T}^{-1}$.

Proposition 4.16. $\mathcal{T}(T') = T$, where $T' = \mathcal{T}(T)$.

Proof. Through the Algorithm, we always have $\psi_0, \psi_1, \ldots, \psi_l$ for $T \in \mathcal{PT}_n^B$ satisfying the properties (P1), (P2), and (P3) by Lemma 4.12 and Lemma 4.13. In particular, we know that there is no inversion pair of $T$ in $\psi_l = \phi_{in_{\alpha_k}} \cdots \phi_{in_{\alpha_1}}$. Also we know that $pr(out_{\alpha})$ of $S(T_{pr})$ becomes an ‘in’ box of $S(T'_{pr})$ by Lemma 4.14.

We claim that applying the Algorithm to $S(T'_{pr})$ in order $pr(out_{\alpha_k}), \ldots, pr(out_{\alpha_1})$ gives us $T$.

We have $\mathcal{T}(T') = \phi_\bullet \phi_{pr(out_{\alpha_k})} \cdots \phi_{pr(out_{\alpha_1})} \phi_o(T_{pr})$ by applying Algorithm to $S(T'_{pr})$. Let $\mathcal{F}_i = \phi_{in_{\alpha_i}} \cdots \phi_{in_{\alpha_1}} \phi_o(T_{pr})$ for $i \in [k]$ and $\mathcal{F}_0 = \phi_o(T_{pr})$. Then, by
Lemma 4.14 and Lemma 4.15.

\[\phi_o((\phi \circ \phi_{in_{\alpha_k}} \cdots \phi_{in_{\alpha_1}} \phi_o(T_{pr}))_{pr}) = (F_k)_{pr},\]

\[\phi_{pr(out_{\alpha_k}})((\phi_{in_{\alpha_k}} \cdots \phi_{in_{\alpha_1}} \phi_o(T_{pr}))_{pr}) = (F_{k-1})_{pr},\]

\[\phi_{pr(out_{\alpha_{k-1}}})((\phi_{in_{\alpha_{k-1}}} \cdots \phi_{in_{\alpha_1}} \phi_o(T_{pr}))_{pr}) = (F_{k-2})_{pr},\]

\[\vdots\]

\[\phi_{pr(out_{\alpha_1}})((\phi_{in_{\alpha_1}} \phi_o(T_{pr}))_{pr}) = (F_0)_{pr},\]

and

\[\phi_o((\phi_o(T_{pr}))_{pr}) = T.\]

Therefore, \(\Xi(T') = T.\)

\[\square\]

Example 4.17. In Example 3.19, we obtained

\[T' = \Xi(T) = \phi_o \phi_{in_6} \phi_{in_5} \phi_{in_4} \phi_{in_3} \phi_{in_2} \phi_{in_1} \phi_o(T_{pr}).\]

From the definition of inversion pairs we have

\[Inv(T) = \{(in_3, in_6), (in_1, in_5), (in_3, in_5), (in_1, in_4), (in_2, in_4), (in_3, in_4), (in_1, in_2)\}.

\[\text{Figure 15. } \Xi(T')\]

1. In \(\phi_{in_6} \phi_{in_5} \phi_{in_4} \phi_{in_3} \phi_{in_2} \phi_{in_1} \phi_o(T_{pr}),\) since \(\alpha_i = 1\) and \(\alpha_j = 2,\) by Lemma 4.13

\[\phi_{in_6} \phi_{in_5} \phi_{in_1} \phi_o(T_{pr}) = \phi_{in_1} \phi_{in_2} \phi_o(T_{pr}).\]

2. In \(\phi_{in_6} \phi_{in_5} \phi_{in_4} \phi_{in_3} \phi_{in_1} \phi_o(T_{pr}),\) since \(\alpha_i = 2\) and \(\alpha_j = 4,\) by Lemma 4.13

\[\phi_{in_6} \phi_{in_5} \phi_{in_1} \phi_o(T_{pr}) = \phi_{in_3} \phi_{in_4} \phi_{in_5} \phi_{in_2} \phi_o(T_{pr}).\]

3. In \(\phi_{in_6} \phi_{in_5} \phi_{in_3} \phi_{in_2} \phi_{in_1} \phi_{in_4} \phi_o(T_{pr}),\) since \(\alpha_i = 1\) and \(\alpha_j = 5,\) by Lemma 4.13

\[\phi_{in_6} \phi_{in_5} \phi_{in_3} \phi_{in_1} \phi_o(T_{pr}) = \phi_{in_3} \phi_{in_2} \phi_{in_1} \phi_{in_5} \phi_{in_4} \phi_o(T_{pr}).\]

where \(T_s = \phi_{in_2} \phi_{in_1} \phi_o(T_{pr}).\)
Finally, in \( \phi \text{in}_0 \phi \text{in}_1 \phi \text{in}_2 \phi \text{in}_3 \phi \text{in}_4 \phi \text{in}_5 \phi \text{in}_6(T_{pr}) \), since \( \alpha_i = 3 \) and \( \alpha_j = 6 \), by Lemma \[4.13\]

\[\phi \text{in}_0 \phi \text{in}_1 \phi \text{in}_2 \phi \text{in}_3 \phi \text{in}_4 \phi \text{in}_5 \phi \text{in}_6(T_s) = \phi \text{in}_5 \phi \text{in}_6(T_s),\]

where \( T_s = \phi \text{in}_1 \phi \text{in}_2 \phi \text{in}_3 \phi \text{in}_4 \phi \text{in}_5 \phi \text{in}_6(T_{pr}). \) So, we get

\[ T' = \Xi(T) = \phi \phi \text{in}_3 \phi \text{in}_6 \phi \text{in}_1 \phi \text{in}_2 \phi \text{in}_5 \phi \text{in}_4 \phi \text{in}_6(T_{pr}). \]

We have the results in Figure 15 by applying the algorithm to \( T' \). Indeed, the \('in'\) boxes \((5, 8), (2, 4), (5, 7), (3, 4), (2, 6), (1, 4)\) in \( T' \) are from \text{out}3, \text{out}6, \text{out}1, \text{out}5, \text{out}2, \text{out}4 in T_{pr} \) respectively:

\[ \Xi(T') = \phi \phi \text{pr(out}_4) \phi \text{pr(out}_2) \phi \text{pr(out}_5) \phi \text{pr(out}_1) \phi \text{pr(out}_6) \phi \text{pr(out}_3) \phi \text{pr}(T_{pr}) .\]

5. \( q \)-Eulerian Numbers of Type A and D

In this section, we restrict our algorithm to type A and D permutation tableaux, respectively. For type A case, it enables us not only to give a new proof of the symmetry of \( q \)-Eulerian number \( \hat{E}_{n,k}(q) \) introduced by Williams but also to prove a symmetry of the generating polynomial \( \hat{D}_{k,n}(p, q, r) \) of number of crossings and numbers of two types of alignments. For type D we were lead to define a new statistic and a \((t, q)\)-Eulerian number of type D, which has a nice symmetry.

5.1. Type A. Remember that the set \( \mathcal{PT}_n \) of permutation tableaux of type A can be understood as a subset of \( \mathcal{PT}_n^B \) by adding stair shaped boxes with 0’s on the top of permutation tableaux of type A. (Remark \[2.1\]) We hence can consider the map \( \Xi \) on \( \mathcal{PT}_n \), and since \( \Xi(T) \in (\mathcal{PT}_n^B)^- \) we can define a bijection \( \Xi_A \) on \( \mathcal{PT}_n \) by composing \( \iota \):

\[ \Xi_A := \iota \circ \Xi : \mathcal{PT}_n \to \mathcal{PT}_n .\]

The following lemma is easy to prove since \( \text{diag} (T) = 0 \) for \( T \in \mathcal{PT}_n \).

**Lemma 5.1.** For given \( T \in \mathcal{PT}_n \), \( \Xi_A \) changes label \( i \) into \(-(n+2-i)\) except 1 and \( \Xi_A \) can be understood as a reflection along the diagonal boxes \((i, n+2-i)\) of \( S(T) \) up to some 1’s.

**Example 5.2.** Consider a permutation tableau \( T \in \mathcal{PT}_n \) in Figure 16. Then, to have \( \Xi_A(T) \), we first apply \( \Xi \) to \( T \). Next, by applying \( \iota \) to \( \Xi(T) \) we have \( \Xi_A(T) \), see Figure 16.

\[ T = \begin{array}{cccc}
-8 & 0 & -6 & 0 \\
-5 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
2 & 5 & 0 & 1 \\
4 & 0 & 1 & 5 \\
7 & 5 & & \\
\end{array} \]

\[ \Xi(T) = \begin{array}{cccc}
-8 & 0 & -6 & 0 \\
-3 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 \\
2 & 5 & 0 & 1 \\
4 & 0 & 1 & 5 \\
7 & 5 & & \\
\end{array} \]

\[ \Xi_A(T) = \begin{array}{cccc}
-8 & 0 & -6 & 0 \\
-3 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 \\
2 & 5 & 0 & 1 \\
4 & 0 & 1 & 5 \\
7 & 5 & & \\
\end{array} \]

**Figure 16.**
Let us introduce some necessary statistics on permutation tableaux.

**Definition 5.3.** Given a \((k,n)\)-permutation tableau \(T\), we consider \(T\) as a 0,1,2-filling of the \(k \times (n-k)\) rectangle; fill in the boxes outside of the diagram of shape \(\lambda\) as in the following figure. We define \(\text{zero}(T)\) as the number of 0’s in \(T\) and \(\text{two}(T)\) as the number of 2’s in \(T\).

\[
T = \begin{array}{ccc}
0 & 0 & 1 & 1 \\
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 1 \\
\end{array} \quad \begin{array}{ccc}
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 2 \\
0 & 1 & 1 & 2 \\
1 & 1 & 2 & 2 \\
\end{array}
\]

We restate Proposition 2.7 for permutation tableaux of type \(A\) in more detail. Remember that \(al(\sigma)\) is the number of alignments of \(\sigma \in S_n\).

**Proposition 5.4.** [19] There is a bijection \(\Phi\) between \(PT_n\) and \(S_n\) that satisfies the following properties:

1. \(\text{row}(T) = \text{wex}(\Phi(T))\),
2. \(\text{so}(T) = \text{cr}(\Phi(T))\), and
3. \(\text{zero}(T) + \text{two}(T) = al(\Phi(T))\).

The following lemma is immediate from Lemma 3.20.

**Lemma 5.5.** Let \(T \in PT_n\) be a \((k,n)\)-permutation tableau and \(T' = T_A(T)\). Then

1. \(T'\) is a \((n-k+1,k-1)\)-permutation tableau,
2. \(\text{row}(T) + \text{row}(T') = n+1\),
3. \(\text{so}(T) = \text{so}(T')\),
4. \(\text{two}(T) = \text{two}(T')\), and
5. \(\text{zero}(T) = \text{zero}(T')\).

**Proof.** (1) and (2) are immediate from Lemma 3.20 and (3) is immediate from the definition of \(T_A\). (4) is obvious because \(T_A\) can be considered as a reflection by Lemma 5.1.

We prove (5). Since \(T\) is in a \(k \times (n-k)\) array and has \((n-k)\) topmost ones,

\[
k(n-k) = (n-k) + \text{so}(T) + \text{two}(T) + \text{zero}(T).
\]

Since \(T'\) is in a \((n-k+1) \times (k-1)\) array and has \((k-1)\) topmost ones,

\[
(n-k+1)(k-1) = (k-1) + \text{so}(T') + \text{two}(T') + \text{zero}(T').
\]

Then, since \(\text{so}(T) = \text{so}(T')\) and \(\text{two}(T) = \text{two}(T')\), (5.6) and (5.7) give us

\[
\text{zero}(T) = \text{zero}(T').
\]

Let

\[
\hat{D}_{k,n}(p,q,r) = \sum_{T \in PT_n \atop \text{row}(T) = k} p^{\text{zero}(T)} q^{\text{so}(T)} r^{\text{two}(T)}
\]
be the polynomial introduced in [19], which enumerates permutations according to the wex, crossings and two types of alignments. Then Lemma 5.5 proves the following theorem.

**Theorem 5.8.**

\[ \hat{D}_{k,n}(p,q,r) = \hat{D}_{n+1-k,n}(p,q,r). \]

Hence

\[ \hat{E}_{n,k}(q) = \hat{E}_{n,n+1-k}(q), \]

where \( \hat{E}_{n,k}(q) = \sum_{\sigma \in S_n, \text{wex}(\sigma) = k} q^{\text{cr}(\sigma)} \) is the \( q \)-Eulerian number introduced by Williams in [20].

**Remark 5.9.** The known proof of symmetry by Williams in [20] of \( q \)-Eulerian number \( \hat{E}_{n,k}(q) = \hat{E}_{n,n+1-k}(q) \) preserves the number of superfluous ones only, while our bijection preserves the number of superfluous ones, twos, and zeros respectively, proving a symmetry of \( \hat{D}_{k,n}(p,q,r) \).

### 5.2. Type D

We consider the restriction of our algorithm to type D permutation tableaux. A type D permutation \( \sigma \) is a type B permutation such that \( \text{neg}(\sigma) \) is even and we let \( S^D_n \) be the group of signed permutations on \( [n] \) with even negatives. Hence permutations of type D is in one-to-one correspondence with permutation tableaux with even diagonal 1's (boundary columns). Let \( \mathcal{PT}^D_n \) be the set of permutation tableaux of type D and \( \mathcal{PT}^D_n^+ = \{ T \in \mathcal{PT}^D_n \mid 1 \in \text{lab}^+(T) \} \) and \( \mathcal{PT}^D_n^- = \{ T \in \mathcal{PT}^D_n \mid 1 \in \text{lab}^-(T) \} \). We consider the set of permutations of type D as a subset of the set of permutations of type B as a set. Note however that the Coxeter group of type D has different feature from the Coxeter group of type B.

The bijection \( \varpi : \mathcal{PT}^B_n \to \mathcal{PT}^B_n \) either increases or decreases the statistic \( \text{diag} \) by 1, and we consider the following map.

\[ \varpi_D := \iota \circ \varpi : \mathcal{PT}^D_n \to \mathcal{PT}^D_n. \]

Then it is clear that \( \varpi_D \) defines a bijection on \( \mathcal{PT}^D_n \). Moreover \( \varpi_D \) sends \( \mathcal{PT}^D_n^+ \) to itself and \( \mathcal{PT}^D_n^- \) to itself.

Since

\[ \text{fwex}(T) + \text{fwex}(\varpi_D(T)) = 2n + 2 \quad \text{if} \quad T \in \mathcal{PT}^D_n^+ \]

and

\[ \text{fwex}(T) + \text{fwex}(\varpi_D(T)) = 2n \quad \text{if} \quad T \in \mathcal{PT}^D_n^- \]

the sum of \( \text{fwex} \) of \( T \) and \( \varpi_D(T) \) is not a constant anymore, and we define a new \( \text{fwex} \) (of type D) as follows.

**Definition 5.10.** For \( \sigma \in S^D_n \), let

\[ \text{fwex}_D(\sigma) := \text{fwex}(\sigma) + \chi(\sigma_1 < 0) \]

and \((t,q)\)-Eulerian number of type D be the polynomial

\[ E^D_{n,k}(t,q) := \sum_{\sigma \in S^D_n, \text{fwex}_D(\sigma) = k} t^{\text{neg}(\sigma)} q^{\text{cr}(\sigma)}. \]
Equivalently, by the correspondence between permutations of type $D$ and permutation tableaux of type $D$, we can also define $\text{fwex}_D$ and $(t, q)$-Eulerian numbers of type $D$ on $\mathcal{P}T^D_n$: For $T \in \mathcal{P}T^D_n$,

$$\text{fwex}_D(T) := \text{diag}(T) + 2 \text{row}(T) + \chi(1 \in \text{lab}^d(T)),$$

$$E^D_{n,k}(t,q) := \sum_{T \in \mathcal{P}T^D_n, \text{fwex}_D(T) = k} t^{\text{diag}(T)} q^{\text{so}(T)}.$$

Then, we obtain a symmetry of $E^D_{n,k}(t,q)$.

**Theorem 5.11.** For $1 \leq k \leq 2n + 1$,

$$E^D_{n,k}(t,q) = E^D_{n,2n+2-k}(t,q).$$

**Proof.** Let $T \in \mathcal{P}T^D_n$ and $T' = \Sigma_D(T)$. Then it is obvious that $\text{diag}(T') = \text{diag}(T)$ and $\text{so}(T') = \text{so}(T)$ by the definition of $\Sigma_D$. Since $T \in \mathcal{P}T^B_n$,

$$\text{diag}(T) + 2 \text{row}(T) + \text{diag}(\Sigma(D)) + 2 \text{row}(\Sigma(D)) = 2n + 1.$$

If $1 \in \text{lab}_+(T)$, then $1 \in \text{lab}_+(T')$ while $1 \in \text{lab}^d(\Sigma(D))$. Thus $\text{diag}(T') = \text{diag}(\Sigma(D)) - 1$ while $\text{row}(T') = \text{row}(\Sigma(D)) + 1$. Moreover, $\chi(1 \in \text{lab}^d(T)) = \chi(1 \in \text{lab}^d(T')) = 0$. Hence,

$$\text{fwex}_D(T) + \text{fwex}_D(T') = \text{diag}(T) + 2 \text{row}(T) + \text{diag}(T') + 2 \text{row}(T') = 2n + 2.$$

If $1 \in \text{lab}^d(T)$, then $1 \in \text{lab}^d(T')$ while $1 \in \text{lab}_+(\Sigma(D))$. Thus $\text{diag}(T') = \text{diag}(\Sigma(D)) + 1$ while $\text{row}(T') = \text{row}(\Sigma(D)) - 1$. Moreover, $\chi(1 \in \text{lab}^d(T)) = \chi(1 \in \text{lab}^d(T')) = 1$. Hence,

$$\text{fwex}_D(T) + \text{fwex}_D(T') = \text{diag}(T) + 2 \text{row}(T) + \text{diag}(T') + 2 \text{row}(T') + \chi(1 \in \text{lab}^d(T)) + \chi(1 \in \text{lab}^d(T')) = 2n + 2.$$

\[ \Box \]

There are known (Eulerian) statistics for permutations of type $D$ which are equidistributed: R. Biagioli defined $\text{ades}$ in [3], and R. Biagioli and F. Caselli defined $D\text{des}$ in [4]. For $\sigma \in \mathfrak{S}^D_n$,

1. $\text{ades}(\sigma) = \text{des}(\sigma) + \text{neg}(\sigma) - \chi(1 \not\in \sigma([n]))$, where $\text{des}(\sigma) = \text{des}(\sigma(1), \ldots, \sigma(n))$.
2. $D\text{des}(\sigma) = f\text{des}(\sigma(1), \ldots, \sigma(n-1), [\sigma(n)])$, where $f\text{des}(\pi) = 2\text{des}(\pi) + \chi(\pi(1) < 0)$.

$\text{fwex}_D$ is a new statistic that naturally came out of the work on $q$-Eulerian numbers of type $B$. Hence, it is reasonable to expect that $\text{fwex}_D$ is in the family of (Eulerian) statistics for the Coxeter group of type $D$, and we make a conjecture on the relation between $\text{fwex}_D$ and known Eulerian statistics for permutations of type $D$. The conjecture has been confirmed up to $n = 8$.

**Conjecture 5.12.** $\text{fwex}_D$ and $(D\text{des} + 2)$ hence $(\text{ades} + 2)$ are equidistributed.
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