Research Article

Stability and Stabilization for a Class of Semilinear Fractional Differential Systems

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This paper considers a class of semilinear fractional-order systems with Caputo derivative. New conditions ensuring asymptotic stability and stabilization of fractional systems with the fractional order between 0 and 2 are proposed. The analysis is based on a property of convolution and asymptotic properties of Mittag-Leffler functions. Some numerical examples are provided to illustrate the feasibility and validity of the proposed approach.

1. Introduction

Over the past several decades, fractional calculus has attracted much attention from scientists and engineers. This is because fractional differential equations have proven to be effective in modeling many physical phenomena and have been applied in different science and engineering fields. Significant contributions have been proposed in fractional differential equations both in theory and applications. For example, see [1–4] and references therein.

In recent years, the stability of fractional-order systems has gained increasing interest due to its importance in control theory. Also, stability theory is an important topic in the study of differential equations. In 1996, Matignon studied the stability of linear fractional differential equations in [5], which is regarded as the first work in this area. Li et al. investigated the Mittag-Leffler stability of nonlinear fractional dynamic systems in [6] and suggested the Lyapunov direct method for nonlinear fractional-order stability systems [7]. There has been more literature on stability of dynamic fractional-order systems, in which important and sufficient conditions were discussed for the stability of linear and linear time-delay fractional differential equations as stated in [8–10]. The modelling and stability of the water jet mixed-flow pump fractional-order shafting system has also been studied [11]. The stability of fractional-order nonlinear systems with \( 0 < \alpha < 1 \) was derived in [6, 12, 13], according to the Lyapunov approach. Based on the uncertain Takagi–Sugeno fuzzy model, the stability problems of nonlinear fractional-order systems were studied, whereas the sliding-mode control approach was used to investigate the stabilization and synchronization problems of the nonlinear fractional-order system (e.g., [14–18]). As noted, a growing number of scientists are dedicated to the stability of fractional systems, with most of the above findings concentrating only on nonlinear fractional systems of \( 0 < \alpha < 1 \).

In [19–24], the authors studied the stability of fractional nonlinear systems with order \( 0 < \alpha \leq 2 \). Various sufficient conditions for asymptotic stability (local or global) are obtained by using Mittag-Leffler function, Laplace transform, and the generalized Gronwall inequality. In summary, the authors of [19, 20, 23, 25] conducted studies on the stability relying on a class of commensurate and incommensurate fractional-order systems together with fractionally controlled systems with linear feedback inputs. By using Mittag-Leffler, Laplace transforms, and Gronwall–Bellman lemma, Zhang et al. [21] discussed the stability of n-dimensional nonlinear fractional order.
In this paper, we discuss the stability of a class of semilinear fractional differential systems with the fractional order between 0 and 2. Theoretically, a stability theorem is developed with the property of convolution and the asymptotic properties of Mittag-Leffler functions. Therefore, based on this theory of stability, a basic criterion for stabilizing a class of nonlinear fractional-order systems is derived, in which control parameters can be selected through the linear control theory pole placement technique. Our results give us a simple method for determining the stability of nonlinear fractional systems with Caputo derivative with order $0 < \alpha \leq 2$. Compared with the abovementioned stability method, the conditions we have proposed for the nonlinear component $f(t, x(t))$ are new and much simpler to test. Thus, there is no need to arrive at an exact solution if only the nonlinear term satisfies certain conditions. What is needed is to calculate the eigenvalues of the linear coefficient matrix $A$ and test that $||\arg(\lambda_i(A))|| > (\alpha\pi/2)$ is satisfied by its arguments. In addition, the results obtained can be used to stabilize the class of fractional-order nonlinear systems by means of a linear state feedback controller.

The remainder of this paper is organized as follows. In Section 2, we recall some definitions and lemmas that will be used in the analysis. In Section 3, the main results, together with the stability and stabilization of the equilibrium points are presented. Section 4 is devoted to some numerical simulation examples that illustrate the validation and effectiveness of the theory. We conclude our work in Section 5.

2. Preliminaries

In this section, we state some definitions and results that are going to be used in our investigations.

Definition 1 (see [3]). The Caputo fractional derivative of order $\alpha$ of function $x(t)$ is defined as

$$\frac{C}{t^\alpha} D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-1} x^{(n)}(\tau) d\tau,$$  

where $\Gamma(\cdot)$ is the gamma function, i.e., $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$, and $n$ is an integer satisfying $n-1 < \alpha \leq n$.

The Laplace transform of the Caputo fractional derivative $\frac{C}{t^\alpha} D^\alpha x(t)$ is

$$\int_0^\infty e^{-st} \frac{C}{t^\alpha} D^\alpha x(t) dt = s^{\alpha} X(s) - \sum_{k=0}^{n-1} \frac{1}{\Gamma(n-k)} x^{(k)}(t_0),$$

where $n$ is an integer such that $n-1 < \alpha \leq n$.

Similar to the exponential function, the function frequently used in the fractional differential equations is the Mittag-Leffler function. The definitions and properties are therefore given as follows.

Definition 2 (see [3]). The Mittag-Leffler function is defined as

$$E_{\alpha} (z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \text{Re}(\alpha) > 0, z \in \mathbb{C}.  \quad (3)$$

The Mittag-Leffler function with two parameters is defined as

$$E_{\alpha, \beta} (z) = \frac{\Gamma(\alpha \beta + 1)}{\Gamma(\alpha + 1)} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + 1)}, \quad \text{Re}(\alpha) > 0, \beta \in \mathbb{R}, z \in \mathbb{C}. \quad (4)$$

It is easy to see that $E_{\alpha}(z) = E_{\alpha,1}(z)$ and $E_{1}(z) = E_{1,1}(z) = e^z$.

The Laplace transform of Mittag-Leffler function is formulated as

$$\int_0^\infty e^{-st} E_{\alpha, \beta} (\pm at^\alpha) dt = \frac{k!^{\alpha-\beta}}{(s^{\alpha} + \alpha)^{\beta+1}}, \quad \text{Re}(s) > |\alpha|^{1/\alpha}. \quad (5)$$

Definition 3 (see [2]). For $A \in \mathbb{C}^{n \times n}$, the matrix Mittag-Leffler function is defined by

$$E_{\alpha, \beta} (A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(ak + \beta)}, \quad \beta \in \mathbb{C}, \text{Re}(\alpha) > 0. \quad (6)$$

Lemma 1 (see [26]). The following properties hold:

(i) There exist finite real constants $M_1 \geq 1$, $M_2 \geq 1$, $M_3 \geq 1$ such that for any $0 < \alpha < 1$,

$$E_{\alpha,1} (A^{\alpha}) \leq M_1 \|e^{At}\|,$$

$$E_{\alpha,\alpha} (A^{\alpha}) \leq M_2 \|e^{At}\|, \quad (7)$$

where $A$ denotes matrix and $\| \cdot \|$ denotes any vector or induced matrix norm.

(ii) If $\alpha \geq 1$, then for $\beta = 1, 2, \alpha$,

$$E_{\alpha, \beta} (A^{\alpha}) \leq M_3 \|e^{At}\|. \quad (8)$$

Definition 4 (see [7]). The constant $x_0 \in \mathbb{R}^n$ is an equilibrium point of the Caputo dynamic system $\frac{C}{t^\alpha} D^\alpha x(t) = f(t, x(t))$ if and only if $f(t, x_0) = 0$.

Without loss of generality, we may assume that the equilibrium point is $x_0 = 0$, representing the origin of $\mathbb{R}^n$ (See [7]). Hence, in the rest of this paper, we always assume that the nonlinear function $f$ satisfies $f(t, 0) = 0$.

Definition 5. The zero solution of $\frac{C}{t^\alpha} D^\alpha x(t) = f(t, x(t))$, with order $0 < \alpha \leq 1 (1 < \alpha < 2)$ is said to be stable if for any initial values $x_0(k)(0) = (x_0(k)(0), 1)$ and any $\varepsilon > 0$ there exists $t_0 > 0$ such that $\|x(t)\| < \varepsilon$ for all $t > t_0$. The zero solution is said to be asymptotically stable if $\lim_{t \to \infty} \|x(t)\| = 0$.

The following property of convolution plays a key role in the proof of the main results.

Lemma 2 (see [27]). Let $1/p, q < \infty$ satisfy $(1/p) + (1/q) = 1$. If $f \in L^p(\mathbb{R}, X)$ and $g \in L^q(\mathbb{R}, X)$, then $f \ast g \in C_0(\mathbb{R}, X)$. Here, $X$ is a Banach space, $f \ast g$ denotes the convolution of the functions $f$ and $g$. 

3. Existence Results

In this section, we consider the following system of fractional differential equation:

$$\frac{d^\alpha}{dt^\alpha} x(t) = A x(t) + f(t, x(t)),$$  \hspace{1cm} (9)

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ denotes the state vector of the system, $\alpha \in (0, 2)$ is the order of the fractional-order derivative, $f : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defines a nonlinear vector field in the $n$-dimensional vector space, and $A \in \mathbb{R}^{n \times n}$ is a constant matrix. Hereafter, we assume that $\lambda_i(i = 1, 2, \ldots, n)$ are the eigenvalues of the matrix $A$ and that $f(t, 0) = 0$, i.e., 0 is an equilibrium point of system (9).

3.1. Stability for the Case $0 < \alpha \leq 1$. We first present a stability result for the case $0 < \alpha \leq 1$. In this case, the solution to equation (9), with initial condition $x(0) = x_0$, can be expressed as

$$x(t) = E_{\alpha, 1}(At)x_0 + \int_0^t (t - \tau)^{\alpha - 1} E_{\alpha, \alpha}(A(t - \tau)E_{\alpha, 1}(At)x_0) d\tau.$$  \hspace{1cm} (10)

The existence and uniqueness of solutions to this problem are widely studied [1–3].

**Theorem 1.** The zero solution of system (9) is locally asymptotically stable if the following conditions are satisfied:

1. The matrix $A$ is stable.
2. There is a function $g \in L^q([0, +\infty))$ such that

$$\|f(t, x(t))\| \leq g(t),$$  \hspace{1cm} (11)

where $g$ is a positive constant satisfying $1/p + 1/q = 1$ and $p > (1/\alpha)$.

**Proof.** Let the initial condition be $x(0) = x_0$. Then, the solution of (9) is given by (10). This is obtained from condition (1) that $|\text{arg} \lambda_i(A)| > (\alpha \pi/2)$. It then follows from Lemma 1 that there exist constants $M_1 > 0$ and $M_2 > 0$ such that

$$\|x(t)\| \leq M_1 e^{\omega \tau} \|x_0\| + M_2 \int_0^t (t - \tau)^{\alpha - 1} e^{\alpha(t - \tau)\omega} \|f(\tau, x(\tau))\| d\tau.$$  \hspace{1cm} (12)

Because the matrix $A$ is stable, there exist constants $M > 0$ and $\omega > 0$ such that

$$\|e^{At}\| \leq Me^{-\omega t}.$$  \hspace{1cm} (13)

Substituting it into (12) and using condition (2), one has

$$\|x(t)\| \leq MM_1 e^{\omega \tau} \|x_0\| + MM_2 \int_0^t (t - \tau)^{\alpha - 1} e^{\alpha(t - \tau)\omega} \|f(\tau, x(\tau))\| d\tau \leq MM_1 e^{\omega \tau} \|x_0\| + MM_2 \int_0^t (t - \tau)^{\alpha - 1} e^{\alpha(t - \tau)\omega} g(\tau) d\tau.$$  \hspace{1cm} (14)

Now, we define a function $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ as

$$\varphi(u) = u^{\alpha - 1} e^{-\omega u}.$$  \hspace{1cm} (15)

Because $0 < \alpha \leq 1$ and $p > 1/\alpha$, a simple computation shows that $\varphi \in L^p([0, +\infty))$. On the other hand, from condition (2) we know that $g \in L^q([0, +\infty))$. Then, it follows from Lemma 2 that $\varphi \circ t \in C_0([0, +\infty))$, which implies that

$$\lim_{t \rightarrow +\infty} \int_0^t (t - \tau)^{\alpha - 1} e^{\alpha(t - \tau)\omega} g(\tau) d\tau = 0.$$  \hspace{1cm} (16)

This, combined with (14), shows that $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$; hence, Theorem 1 is proved. \qed

3.2. Stability for the Case $1 < \alpha \leq 2$. When $1 < \alpha \leq 2$, the solution to equation (9), with initial condition $x(0) = x_0$ and $x(0) = x_1$, satisfies the following integral equation:

$$x(t) = E_{\alpha, 1}(At)x_0 + t E_{\alpha, 2}(At)x_1 + \int_0^t \int_0^\tau (t - \tau)^{\alpha - 1} E_{\alpha, \alpha}(A(t - \tau)E_{\alpha, 1}(At)x_0) d\tau d\tau.$$  \hspace{1cm} (17)

Because $1 < \alpha \leq 2$, from Lemma 1 (ii), there exists $M_3 > 0$ such that

$$\|E_{\alpha, \beta}(At)\| \leq M_3 \|e^{\alpha t}\|.$$  \hspace{1cm} (18)

for $\beta = 1, 2, \text{or} \alpha$. We first prove an existence and uniqueness result via the principle of Banach contraction.

**Theorem 2.** Let $T > 0$ be arbitrary, the matrix $A$ be stable, and $f$ be Lipschitz w.r.t. the second variable, i.e., there exists a constant $L > 0$ such that

$$\|f(t, u) - f(t, v)\| \leq L\|u - v\|,$$  \hspace{1cm} (19)

for each $t \in [0, T]$ and $u, v \in \mathbb{R}^n$. If

$$L < \frac{\alpha \omega}{MM_3},$$  \hspace{1cm} (20)

where $M, M_3, \text{and} \omega$ are constants appeared in (13) and (8), then equation (9), with the initial condition $x(0) = x_0$ and $x(0) = x_1$, has a unique solution in $[0, T]$.

**Proof.** We transform the problem into a fixed-point problem. Let $X = C([0, T], \mathbb{R}^n)$ be the Banach space of all continuous functions from $[0, T]$ into $\mathbb{R}^n$, endowed with the norm $\|x\| = \sup_{t \in [0, T]} \|x(t)\|$. We define an operator $Q : X \rightarrow X$ as
Qx(t) = E_{α,1}(At^α)x_0 + tE_{α,2}(At^α)x_1 \\
+ \int_0^t (t-\tau)^{α-1}E_{α,α}(A(t-\tau)^α)f(\tau, x(\tau))d\tau, \tag{21}

\text{for } t \in [0, T]. \text{ Then, equation (9) has a unique solution if and only if } Q \text{ has a unique fixed point. Taking } x, y \in C([0, T], \mathbb{R}^n) \text{ arbitrarily and } t \in [0, T]. \text{ We obtain}

\|Qx(t) - Qy(t)\| \leq \int_0^t (t-\tau)^{α-1}\|E_{α,α}(A(t-\tau)^α)f(\tau, x(\tau))\|d\tau \\\n\leq M_3 \int_0^t (t-\tau)^{α-1}\|e^{A(t-\tau)^α}\|\cdot\|(f(\tau, x(\tau)) - f(\tau, y(\tau)))\|d\tau \\\n\leq MM_3L\|x - y\| \int_0^t (t-\tau)^{α-1}e^{-\omega(t-\tau)^α}d\tau \\\n\leq \frac{MM_3L}{\omega} \|x - y\|. \tag{22}

It follows that

\|Qx - Qy\| \leq \frac{MM_3L}{\omega} \|x - y\|. \tag{23}

Condition (20) shows that \( (MM_3\omega)L < 1 \), which implies that \( Q \) is a contraction. Hence, we deduce by the principle of Banach contraction that \( Q \) has a unique fixed point, which is the unique solution to equation (9). The proof is completed.

Similar to Theorem 1, we now prove the stability of equation (9) for \( 1 < α ≤ 2 \).

\textbf{Theorem 3.} Suppose that the conditions in Theorem 2 hold. Then, the zero solution of system (9) is locally asymptotically stable, if the following conditions are satisfied:

(1) \( |\arg(\lambda_i(A))| > (\alpha π)/2 \), \( i = 1, 2, \ldots, n \).

(2) There is a function \( g \in L^q(0, \infty) \) such that

\[ \|f(t, x(t))\| \leq g(t), \tag{24} \]

where \( g \) is a positive constant satisfying \( 1/p + 1/q = 1 \) and \( p > 1 \).

\textbf{Proof.} Suppose that the initial condition is \( x(0) = x_0 \) and \( x'(0) = x_1 \), then the solution of (9) has the following form:

\[ x(t) = E_{α,1}(At^α)x_0 + tE_{α,2}(At^α)x_1 \]

\[ + \int_0^t (t-\tau)^{α-1}E_{α,α}(A(t-\tau)^α)f(\tau, x(\tau))d\tau, \tag{25} \]

for \( t > 0 \). Because \( 1 < α ≤ 2 \), it follows from (25) and condition (2) that

\[ \|x(t)\| \leq M_3\|e^{At^α}\|\|x_0\| + M_3\|e^{At^α}\|\|x_1\|t \]

\[ + M_3 \int_0^t (t-\tau)^{α-1}\|e^{A(t-\tau)^α}\|\|f(\tau, x(\tau))\|d\tau \]

\[ \leq M_3\|e^{At^α}\|\|x_0\| + M_3\|e^{At^α}\|\|x_1\|t \]

\[ + M_3 \int_0^t (t-\tau)^{α-1}\|e^{A(t-\tau)^α}\|g(\tau)d\tau, \tag{26} \]

for all \( t > 0 \). From condition (1) we know that the matrix \( A \) is stable. Hence, the inequality (13) holds. Therefore, we obtain

\[ \|x(t)\| \leq MM_3e^{-\omega t}\|x_0\| + MM_3e^{-\omega t}\|x_1\|t \]

\[ + MM_3e^{-\omega t}\|x_0\| + MM_3e^{-\omega t}\|x_1\|t \]

\[ + MM_3 \int_0^t (t-\tau)^{α-1}e^{-\omega(t-\tau)^α}g(\tau)d\tau, \tag{27} \]

for all \( t > 0 \). Similar to the proof of Theorem 1, we define the function \( φ: [0, +\infty) \to [0, +\infty) \) as

\[ φ(u) = ue^{-\omega u}. \tag{28} \]

Because \( 1 < α ≤ 2 \), it is easy to see that \( φ \in L^p(0, +\infty) \) for any \( p > 1 \). Because of Lemma 2 and condition (2), we obtain \( φ \in C_0[0, +\infty) \). Therefore, \( \lim_{t \to +\infty} \|x(t)\| = 0 \). The proof is completed. \( \square \)

\textbf{Remark 1.} Theorems 1 and 3 give us a simple procedure for determining stability of the fractional-order nonlinear system with Caputo derivative of order \( 0 < α < 2 \). If the nonlinear term \( f(t, x(t)) \) fulfills condition (2), then the exact solution need not be reached. Importantly, it is required to calculate matrix \( A \)'s eigenvalues and test their arguments. If \( |\arg(\lambda_i(A))| > (\alpha π)/2 \), \( i = 1, 2, \ldots, n \), we conclude that the origin is stable asymptotically.

\subsection*{3.3. Stabilization of a Class of Fractional-Order Semilinear System.} In this subsection, we propose the stabilization theory of a class of fractional-order semilinear controlled systems. We consider the controlled systems of the following form:

\[ C_0^D_{\alpha}x(t) = Ax(t) + f(t, x(t)) + Bu(t), \tag{29} \]

where \( x, A \) and \( f \) are as in system (9), \( B \in \mathbb{R}^{m \times n} \) is the input matrix, and \( u \) is the control input. If \( u \) is chosen to be a linear state feedback control, i.e., \( u = Kx \) for some feedback gains \( K \), then system (29) becomes a closed-loop system:

\[ C_0^D_{\alpha}x(t) = Ax(t) + f(t, x(t)) + Bu(t) \]

\[ = Ax(t) + f(t, x(t)) + BKx(t) \]

\[ = (A + BK)x(t) + f(t, x(t)) \]

\[ = Ax(t) + f(t, x(t)). \tag{30} \]

Suppose that \( (A, B) \) is controllable, then the feedback gain \( K \) can be chosen such that system (30) is asymptotically stable.
Theorem 4. If $0 < \alpha < 2$, feedback gain $K$ is chosen such that the following conditions hold:

(i) $\|\lambda_i(\hat{A})\| > (\alpha \pi/2)$, $(i = 1, 2, \ldots, n)$, where $\lambda_i(\hat{A})$ is the eigenvalue of matrix $\hat{A}$.
(ii) $f(t, x(t))$ satisfies $\|f(t, x(t))\| \leq g(t)$, where $g \in L^q(0, \infty)$.
(iii) The matrix $\hat{A}$ is stable. Then, the controlled system \( \text{(29)} \) is locally asymptotically stable.

Proof. The proof of Theorem 4 is similar to that of Theorems 1 and 3.

Remark 2. The nonlinear term of Chaotic fractional-order systems satisfies $\|f(t, x(t))\| \leq g(t)$, where $g \in L^q(0, \infty)$, i.e., the hyperchaotic fractional-order novel system. Therefore, in a large class of generalized fractional-order chaotic or hyperchaotic systems, Theorem 4 can be applied to control chaotic. Of all control methods, linear feedback control is particularly attractive and has been widely extended to practical implementation due to its ease among configuration and implementation.

4. Applications

In this section, we apply the obtained results to some semilinear systems to illustrate the effectiveness of the theory.

Example 1. Consider the fractional nonlinear system with Caputo derivative:

\[
\begin{align*}
0 D_t^\alpha x_1(t) &= 3x_1 + 7x_2 + 2x_3 + e^{-w_1t} \sin x_2x_3, \\
0 D_t^\alpha x_2(t) &= -2x_1 - 4x_2 + 4x_3 + e^{-w_2t} \sin x_2^2, \\
0 D_t^\alpha x_3(t) &= e^{-w_3t} \sin x_1x_2 - x_3,
\end{align*}
\]

where $w_i > 0$, $i = 1, 2, 3$. This system can be rewritten as \( \text{(9)} \), where

\[
A = \begin{bmatrix} 3 & 7 & 2 \\ -2 & -4 & 4 \\ 0 & 0 & -1 \end{bmatrix},
\]

\[
f(t, x) = \begin{bmatrix} e^{-w_1t} \sin x_2x_3 \\ e^{-w_2t} \sin x_2^2 \\ e^{-w_3t} \sin x_1x_2 \end{bmatrix}.
\]

It is easy to verify that

\[
\|f(t, x)\| \leq \sqrt{3} e^{-w_3t} \in L^q(0, +\infty),
\]

where $w_0 = \max\{w_1, w_2, w_3\}$. So, condition (2) in Theorems 1 and 3 is satisfied. The eigenvalues of $A$ are $\lambda_{1,2} = -0.5000 \pm 1.3229i$ and $\lambda_3 = -1$. According to Theorems 1 and 3, if $\alpha < 1.2300$, then the zero solution of \( \text{(31)} \) is asymptotically stable. For the initial values $(x_1(0), x_2(0), x_3(0)) = (0.1, -0.2, 0.3)$, simulation results are displayed in Figures 1–3. Figures 1 and 2 show that the zero solution of system \( \text{(31)} \) is asymptotically stable with $\alpha = 1.21$ and $\alpha = 1.22$, respectively. Figure 3 also shows that the zero solution of system \( \text{(31)} \) is unstable with $\alpha = 1.23$.

Example 2. Consider the following fractional nonlinear system:

\[
\begin{align*}
0 D_t^\alpha x_1(t) &= -x_1 + e^{-w_1t} \sin x_2x_3, \\
0 D_t^\alpha x_2(t) &= x_3, \\
0 D_t^\alpha x_3(t) &= x_1 - x_2 - x_3 - e^{-w_3t} \sin x_1x_2.
\end{align*}
\]

The system can be rewritten as \( \text{(9)} \), where

\[
A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix},
\]

\[
f(t, x) = \begin{bmatrix} e^{-w_1t} \sin x_2x_3 \\ e^{-w_3t} \sin x_1x_2 \end{bmatrix}.
\]

Obviously,

\[
\|f(t, x(t))\| \leq e^{-w_0} := g(t) \in L^q.
\]

The eigenvalues of $A$ are $\lambda_{1,2} = (-1/2) \pm (\sqrt{3} i/2)$ and $\lambda_3 = -1$. According to Theorems 1 and 3, if $\alpha < (4/3)$, then the zero solution of \( \text{(32)} \) is asymptotically stable.

Simulation results with initial values $(x_1(0), x_2(0), x_3(0)) = (-0.01, -0.02, 0.03)$ are displayed in Figures 4–6. Figures 4 and 5 show that the zero solution of system \( \text{(34)} \) is asymptotically stable with $\alpha = 1.1$ and $\alpha = 1.3$, respectively. Figure 6 further shows that the zero solution of system \( \text{(34)} \) is unstable when $\alpha = 1.34$.

Example 3. The fractional-order novel hyperchaotic system can be written as

\[
\begin{align*}
0 D_t^\alpha x_1(t) &= a(x_2 - x_4), \\
0 D_t^\alpha x_2(t) &= -bx_1 + e^{-w_1t} \sin x_1x_3 + x_4, \\
0 D_t^\alpha x_3(t) &= -2e^{-w_1t} \sin x_2^2 - 2e^{-w_1t} \sin x_2^2 - cx_3 - x_4, \\
0 D_t^\alpha x_4(t) &= -dx_1,
\end{align*}
\]

where $a$, $b$, $c$, and $d$ are some parameters. System \( \text{(37)} \) can be rewritten as \( \text{(9)} \) if we write
Figure 1: System (31) is asymptotically stable with \( \alpha = 1.21 \).

Figure 2: System (31) is asymptotically stable with \( \alpha = 1.22 \).

Figure 3: System (31) is unstable with \( \alpha = 1.23 \).
Figure 4: System (34) is asymptotically stable with $\alpha = 1.1$.

Figure 5: System (34) is asymptotically stable with $\alpha = 1.3$.

Figure 6: System (34) is unstable with $\alpha = 1.34$. 
Figure 7: Asymptotical stabilization of the fractional-order novel hyperchaotic system (37) with $\alpha = 1.14$ and feedback gain $K = (2, -8, -3.5, -1)$.

Figure 8: Attractor of the fractional-order novel hyperchaotic system with order $\alpha = 1.14$ ($a = 10$, $b = 40$, $c = 2.5$, and $d = 10$).
Figure 9: Asymptotical stabilization of the fractional-order novel hyperchaotic system (37) with $\alpha = 1.15$ and feedback gain $K = (2, -8, -3.5, -1)$.

Figure 10: Attractor of the fractional-order novel hyperchaotic system with order $\alpha = 1.15$ ($a = 10$, $b = 40$, $c = 2.5$, and $d = 10$).
According to Theorem 4, feedback gain can be obtained by

\[ A = \begin{bmatrix} -a & a & 0 & 0 \\ -b & 0 & 0 & 1 \\ 0 & 0 & -c & -1 \\ -d & 0 & 0 & 0 \end{bmatrix}, \]

\[ f(t, x(t)) = \begin{bmatrix} e^{-\alpha t} \sin x_1 x_3 \\ -2e^{-\alpha t} \sin x_1^2 - 2e^{-\alpha t} \sin x_2^2 \\ 0 \end{bmatrix}. \]

It is easy to see that \( f(t, x(t)) \) satisfies

\[ \| f(t, x(t)) \| \leq \sqrt{e^{-2\alpha t} \sin^2 x_1 x_3 + 8e^{-2\alpha t} \sin^2 x_1^2 + 8e^{-2\alpha t} \sin^2 x_2^2} \]

\[ \leq \sqrt{\sin^2 x_1 x_3 + 8 \sin^2 x_1^2 + 8 \sin^2 x_2^2} e^{-\alpha t} \]

\[ \leq 5e^{-\alpha t} \in L^q. \]

When \( a = 10, b = 40, c = 2.5, \) and \( d = 10, \) system (37) displays a chaotic attractor, and we construct the linear state feedback controller as (9). For simplicity, let \( K = (1,1,1,1) \). According to Theorem 4, feedback gain can be obtained by \( \| \arg(\lambda_i(\hat{A})) \| > (\alpha/2). \) By a simple calculation, if we choose \( \alpha = (2, -3.5, -1, 1) \), then the eigenvalues of \( \hat{A} \) are \((-17.3028, -0.11769, -2.78989 \pm 10.7230i). \) Hence, \( \| \arg(\lambda_i(\hat{A})) \| > 0.57\alpha \) if \( \alpha < 1.6203. \) Thus, the conditions of Theorem 4 are satisfied. The zero solution of the controlled system is asymptotically stable with \( \alpha = 1.14 \) and \( \alpha = 1.15, \) respectively. The results of simulation with initial values \((x_1(0), x_2(0), x_3(0), x_4(0)) = (-0.01, -0.02, -0.03, 0.04)\) are shown in Figures 7–10.

5. Conclusions

The stability of nonlinear dynamical systems is important for scientists and engineers. Therefore, in this paper, we studied a class of semilinear fractional-order systems with Caputo derivative by using properties of convolution and Mittag-Leffler function methods. We introduced the fractional comparison principle for Caputo fractional-order systems, which enriched the knowledge of both system theory and fractional calculus. We established a new sufficient condition of the asymptotic stability of zero solution for a class of fractional-order semilinear systems with order \( 0 < \alpha < 2. \) Three illustrative examples were provided to demonstrate the applicability of the proposed approach.

Data Availability

The data used in the examples were originally taken from MATLAB, and they are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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