Asymptotic adaptive threshold for connectivity in a random geometric social network

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Abstract
Consider a dynamic random geometric social network identified by $s_t$ independent points $x_i^t, \ldots, x_s^t$ in the unit square $[0,1]^2$ that interact in continuous time $t \geq 0$. The generative model of the random points is a Poisson point measures. Each point $x_i^t$ can be active or not in the network with a Bernoulli probability $p$. Each pair being connected by affinity thanks to a step connection function if the interpoint distance $\|x_i^t - x_j^t\| \leq a^*_f$ for any $i \neq j$. We prove that when $a^*_f = \sqrt{(s_t)^{l-1} p \pi}$ for $l \in (0,1)$, the number of isolated points is governed by a Poisson approximation as $s_t \to \infty$. This offers a natural threshold for the construction of a $a^*_f$-neighborhood procedure tailored to the dynamic clustering of the network adaptively from the data.

Keywords: Interacting particles; Complex networks; Isolated points; Discretization; Concentration bound; Monte Carlo; Clustering procedure.

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1 Introduction

A problem common to many disciplines is that of adequately studying complex networks. Research on this problem occurs in applied mathematics (dynamic systems governed by ODEs and PDEs), probability and statistics (random graph models) and in computer science and engineering (statistical learning networks). A less familiar studying in this area is stochastic individual based models (IBM). The theory of IBM is a quickly growing interdisciplinary area with a very broad spectrum of motivations and applications. In consideration of complex networks, we characterized recently in [Sid-Ali & Khadraoui (2018)] social network systems by individuality of components and localized interaction mechanisms among components realized via densities dependent spatial state which leads to a globally regular behavior of the system. In modern stochastic (with diffusion or point processes) and infinite dimensional modeling, interacting particle models for the study of complex systems forms a rich and powerful direction. For instance,
interacting particle systems are widely used to model population biology (Khadraoui, 2015), ecology (Finkelshtein et al., 2009), condensed matter physics, chemical kinetics, sociology and economics (agent based models). For an account on the subject of interacting particle systems, we refer the reader to the book Liggett (1985).

We study the following random geometric social network \((r_t)_{t \geq 0}\) presumed to be described by the punctual measure

\[
r_t(dx) = \sum_{i=1}^{s_t} \delta_{x_i}(dx),
\]

where \(\delta_x\) denotes the Dirac measure centered on \(x\) with \(x\) in some measurable space (defined in a rigorous way afterwards), \(s_t\) denotes the size of the network at time \(t\) and the independent points \(x_1^t, \ldots, x_{s_t}^t\) denote the spatial positions of the network members at time \(t\). The generative dynamic model of the random points is a Poisson point measures that we describe here its infinitesimal construction for completeness. As known, the main issue in this kind of study is the connection between points. In particular, we choose some deterministic rule where two members interact if and only if their distance does not exceed some threshold which is precisely what has been done in random geometric graphs (Gilbert, 1961). We refer the reader to Penrose (2003) for a thorough presentation of the many properties of Gilbert’s graph. However, the two main differences in our network compared to Gilbert’s graph are: \(i\) the state of our network is dynamic in continuous time and not static; \(ii\) the density of the random points is not uniform in \([0, 1]^d\) with \(d \geq 2\) as usually assumed by authors (see for instance Broutin et al. (2016, 2014) and references therein).

It is known that the main obstacle to connectivity inside certain networks is the existence of isolated members. In the present paper we prove that the number of isolated points has asymptotic Poisson distribution by employing a concentration bound on Poisson random variables together with a tailored discretization method of the unit square \([0, 1]^2\). We establish the optimal threshold associated with this approximation and show that this result holds over connection functions that are zero beyond the threshold. This was previously known for the random geometric graph and its variants where points are independently and uniformly distributed (Dette & Henze, 1989). Hence, our result may be seen as a generalization of this Poisson approximation for the number of isolated points to random points generated from Poisson point processes which encompass the uniform case. To our best knowledge, this is the first result in this direction and it open a door to other complex problems such as the extension to higher dimensions or other connection functions (Rayleigh fading functions, functions that decay exponentially in some power of distance, etc). Moreover, we take advantage of the asymptotic distribution of isolated points to built a data adaptive dynamic statistical cluster method tailored to our network. This conceivable strategy enables us to detect groups and isolated members at each time from the dynamic of the network. In particular, an important question investigated recently which is the community detection inside the network (Zhao et al., 2012 Arias-Castro & Verzelen, 2014; Jin, 2015; Bickel et al., 2015 and the references therein). In the Erdős-Rényi graph context, the
stochastic block model (SBM) \cite{Holland1983} is usually used to model communities where the probability of a connection occurring between two members depends solely on their community membership. There are many extensions of the SBM for various applications, including the biological, communication and social networks, for instance in \cite{Bickel2009,Snijders1997, Park2012}.

The outline of this paper is as follows: In Section 2, we describe in detail the generative model of the network data by giving an explicit representation and the exact Monte Carlo scheme for computation. In Section 3, we establish the asymptotic adaptive threshold needed for the Poisson approximation law of the number of isolated members inside the network. Section 4 is devoted to the tailored dynamic clustering procedure using the optimal threshold established in this paper in order to detect communities and isolated members in the network. Moreover, we present some numerical simulations. We then discuss our results and the outlook in Section 5. Section 6 contains the proofs of the main results of the paper.

2 Preliminaries: Generation of the point sets

We introduce the notation and the generative model for the random social network as an interacting particle system. The spatio-temporal paradigm retained here is represented by a random dynamic for the network in terms of its instantaneous size together with the spatial (geometric) patterns of the members. In a rigorous sense, the system of particles considered is a Markov process with values in a space of punctual measures and where each member of the network is tracked through time. Basically, we construct a geometric virtual space that is a closure \( \bar{D} \) of an open connected subset \( D \) of \( \mathbb{R}^d \), for some \( d \geq 1 \). We represent each member of the network located at a virtual state \( x \) as a Dirac measure \( \delta_x \). The idea behind the use of virtual space is for simple managing of interactions between members which is nothing beyond some distance. Roughly speaking, closer the members are in the virtual space bigger is their affinity (friendship) mechanism. We assume that each member may invite another individual at a given rate. When a new individual is arrived, it immediately disperses from the member who invited and becomes a member of the network. We also assume that members are subject to departures. That is, each member quits at a rate that depends on the local network state. Thus, in this social network new members could arrive at continuous time and become members as well as the members can leave the network at any time.

Formally, let denote by \( S_F(\bar{D}) \) the set of finite nonnegative measures on \( \bar{D} \) and \( S \subseteq S_F(\bar{D}) \) that consists of all finite point measures on \( \bar{D} \):

\[
S = \left\{ \sum_{i=1}^{s} \delta_{x^i}, \ s \geq 0, \ x^i \in \bar{D} \right\},
\]

where the states \( x^1, \ldots, x^s \) of members represent their spatial locations in the space \( D \) and \( s \in \mathbb{N} \) stands for the size of the network. According to our previous description the network that we
denote by \( r_t \) at time \( t \geq 0 \) is characterized by the distribution of the members at any given time inside the virtual space and is given by

\[
r_t(dx) = \sum_{i=1}^{s_t} \delta_{x_i}(dx),
\]

where the stochastic process \( r_t \) take its values in \( S \) and the indexes \( i \) in the positions \( \{x^i_t\}_{i=1}^{s_t} \) are ordered here from an arbitrary order point of view. Before giving an explicit description of the process \( (r_t)_{t \geq 0} \) we introduce now the heuristics of the network.

### 2.1 Heuristics of the network

The dynamic of the random social network considered here can be roughly summarized by endowing the system with three events as follows:

(i) a recruitment by invitation event: A member located at the virtual position \( x \) in the network could send an invitation to another individual in the outside of the network in order to join the network community. Then, the individual can accept the invitation and joins the network at the virtual position \( y = x + z \) where \( z \) is chosen randomly following a given dispersion kernel;

(ii) a departure from the network event: Each member can leave the network at any moment and its position inside the virtual space becomes empty immediately;

(iii) a recruitment by affinity with the network event: An individual outside the network can be interesting to join the network thanks to a certain affinity with the network. Then, this individual choose a position in the network following a given dispersion kernel.

We shall describe the system by the evolution in time of the measure \( r_t \). For this, let define the parameters of the previous events:

(i) \( v_r \in [0, \infty) \) denotes the invitation rate for each member in the network;

(ii) \( K(x, dz) \) denotes the dispersion law for the new individual invited by a member located at \( x \) and it is assumed to satisfy, for each \( x \in \bar{D} \),

\[
\int_{z \in \mathbb{R}^d, x+z \in \bar{D}} K(x, dz) = 1 \quad \text{and} \quad \int_{z \in \mathbb{R}, x+z \not\in \bar{D}} K(x, dz) = 0;
\]

(iii) \( d_r \in [0, \infty) \) denotes a departure rate for each individual at some \( x \in \bar{D} \);

(iv) \( K^{af}(dy) \) denotes the affinity dispersion law for the new arrived individual at some \( y \in \bar{D} \) and it is assumed to satisfy,

\[
\int_{y \in \bar{D}} K^{af}(dy) = 1 \quad \text{and} \quad \int_{y \not\in \bar{D}} K^{af}(dy) = 0;
\]

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(v) for all $x, y \in \bar{D}$, $\text{aff}(x, y) = \text{aff}(y, x) \in [0, \infty)$ is the affinity kernel which describes the strength of affinity between members located at $x$ and $y$.

To explain in more details the concept of affinity and its spatial dependence, we introduce another function $w^{\text{af}}$ which describes the affinity that may have an individual in the outside with the network; This individual may choose randomly a position $y$ for its recruitment by the network in function of the current member localizations in the neighborhood of this position $y$. To this end and for the sake of simplicity, we propose to sum the local affinities between the position $y$ and all the members around the future position $y$ such that:

$$w^{\text{af}}(y, r) = \sum_{x \in r} \text{aff}(x, y) = \int_{\bar{D}} \text{aff}(x, y)r(dx),$$

for all $x, y \in \bar{D}$ which leads to $w^{\text{af}}(y, r) \in [0, \infty)$. Concerning the local affinity we consider the following indicator function:

$$\text{aff}(x, y) = A_f \mathbb{I}_{\{\|x-y\| \leq a_f\}}(x, y), \quad (2.2)$$

where $\mathbb{I}_A$ denotes the indicator function of the set $A$, $A_f \in [0, \infty)$ denotes the maximum that interaction by affinity between two members can reach and $a_f \in [0, \infty)$ denotes the radius (affinity threshold) of the zone of interaction by affinity. It is easily seen that the affinity between $x$ and $y$ is equal to zero when the Euclidean distance $\|x - y\| > a_f$. The affinity presented here generalizes without difficulty to more complex connection scenarios provided that the affinity rate investigated is bounded.

We assume that all of these basic mechanisms are independent. Furthermore, to avoid explosion phenomenon, we suppose that the spatial dependence of the introduced kernels and rates is bounded. Indeed, we assume that the dispersion kernels induce densities with respect to the Lebesgue measure such that $K(x, dz) = k(x, z)dz$ and $K^{\text{af}}(dy) = k^{\text{af}}(y)dy$. We assume that there exists some positive reals $\gamma_1 > 0$ and $\gamma_2 > 0$ and two probability densities $\tilde{k}$ on $\mathbb{R}^d$ and $\tilde{k}^{\text{af}}$ on $\bar{D}$ such that, for all $x \in \bar{D}$,

$$k(x, z) \leq \gamma_1 \tilde{k}(z) \quad \text{and} \quad k^{\text{af}}(y) \leq \gamma_2 \tilde{k}^{\text{af}}(y).$$

We assume that there exists a constant $A_f$ such that, for all $x, y \in \bar{D}$ and for all $r \in \mathcal{S}$,

$$\text{aff}(x, y) \leq A_f \quad \text{then} \quad w^{\text{af}}(y, r) \leq A_f r.$$
2.2 Explicit representation of the network

First we derive an explicit representation of a function of the random network for a large class of functions. Second we deduce the explicit expression of the infinitesimal generator on that class. To obtain this representation for the random social network \((r_t)_{t \geq 0}\) we introduce some Poisson random measures which manage the recruitment of new members (by invitation and affinity) and the departure of members. To this end, let \((\Omega, \mathcal{A}, \mathbb{P})\) be a sufficiently large probability space and let consider three punctual Poisson random measures \(N_{v}, (d\tau, di, dz, d\alpha)\) defined on \([0, \infty) \times \mathbb{N}^* \times \mathbb{R}^d \times [0, 1]\), \(N_{d}, (d\tau, di, da)\) defined on \([0, \infty) \times \mathbb{N}^* \times [0, 1]\) and \(N_{af}(d\tau, di, dy, da)\) defined on \([0, \infty) \times \mathbb{N}^* \times \bar{D} \times [0, 1]\) with respective intensity measures:

\[
\begin{align*}
&n_{v}(d\tau, di, dz, d\alpha) = v_{\gamma_1}(z)d\tau d\alpha, \\
&n_{d}(d\tau, di, da) = d_{r}d\tau d\alpha, \\
&n_{af}(d\tau, di, dy, da) = A_r\gamma_2\tilde{k}(y)d\tau d\alpha d\alpha,
\end{align*}
\]

where \(d\tau, dz, dy\) and \(da \) are the Lebesgue measures on \([0, \infty), \mathbb{R}^d, \bar{D}\) and \([0, 1]\) and \(di\) is the counting measure on \(\mathbb{N}^*\).

Let denote by \(r_0\) the initial condition of the process, it is a random variable with values in \(S\). Suppose that \(N_{v}, N_{d}, N_{af}\) and \(r_0\) are mutually independent. We also consider the canonical filtration \((\mathcal{F}_t)_{t \geq 0}\) generated by the random objects \(N_{v}, N_{d}, N_{af}\) and \(r_0\). In the following, we denote by \(r_t\) the network process at time \(t\) before any possible event. The stochastic process \((r_t)_{t \geq 0}\) features a jump dynamics (recruitments and departures). We can therefore recall a well-known formula for the pure jump processes:

\[
\Phi(r_t) = \Phi(r_0) + \sum_{\tau \leq t} [\Phi(r_{\tau^-} + \{r_{\tau} - r_{\tau^-}\}) - \Phi(r_{\tau^-})], \quad \text{a.s. for } t \geq 0,
\]

(2.3)

for any function \(\Phi\) defined on \(\mathcal{S}_F(\bar{D})\) for all \(r \in S\). We remark that in equation (2.3) the sum \(\sum_{\tau \leq t}\) contains only a finite number of terms as the network process \((r_t)_{t \in [0,T]}\) admits only a finite number of jumps for any \(T < \infty\). Note that the number of jumps in the network \(r_t\) is bounded by a linear recruitment and departure process with arrival rate \(v_r\) and departure rate \(d_r\) (Allen 2003). According to the formula (2.3), for any function \(\Phi\) on \(\mathcal{S}_F(\bar{D})\), we can write

\[
\Phi(r_t) = \Phi(r_0) + \int_0^t \int_{\mathbb{N}^*} \int_{\mathbb{R}^d} \int_0^1 \mathbb{I}_{\{i \leq s_{\tau^-}\}} \mathbb{I}_{\{\alpha \leq (k(x_{\tau^-} + z))/(\gamma_1 k(z))\}} \\
\times \{\Phi(r_{\tau^-} + \{\delta(x_{\tau^-} + z)\}) - \Phi(r_{\tau^-})\} N_{v}(d\tau, di, dz, d\alpha) \\
+ \int_0^t \int_{\mathbb{N}^*} \int_{\mathbb{R}^d} \int_0^1 \mathbb{I}_{\{i \leq s_{\tau^-}\}} \mathbb{I}_{\{\alpha \leq (d_r/d_{\tau})\}} \\
\times \{\Phi(r_{\tau^-} - \{\delta(x_{\tau^-})\}) - \Phi(r_{\tau^-})\} N_{d}(d\tau, di, da) \\
+ \int_0^t \int_{\mathbb{N}^*} \int_{\mathbb{D}} \int_0^1 \mathbb{I}_{\{i \leq s_{\tau^-}\}} \mathbb{I}_{\{\alpha \leq (\alpha f(x_{\tau^-} + y)k(\gamma_2 k(y)))\}} \\
\times \{\Phi(r_{\tau^-} + \{\delta(y)\}) - \Phi(r_{\tau^-})\} N_{af}(d\tau, di, dy, d\alpha),
\]

(2.4)
where the three integral terms are associated with the three basic independent mechanisms and 
\(\alpha \in [0, 1]\) (for Monte Carlo acceptance-rejection). In this context, let us do a small remark on the
Monte Carlo method. Parameter \(\alpha\) corresponds to a decisional value for the type of the event
that will occur by acceptance-rejection, which is usually defined as a random uniform realization
in \([0, 1]\). Let us explain the foundation of the Monte Carlo simulation scheme in detail. In
particular, for all \(t \geq 0\), the explicit representation of the process \(r_t\) is given a.s. by
\[
\begin{align*}
 r_t &= r_0 + \int_0^t \int_{\mathbb{R}^d} \int_0^1 \mathbb{1}_{\{i \leq s_{t-}\}} \mathbb{1}_{\{\alpha \leq \frac{k(x_i^z_\tau)}{(\gamma_1 \tilde{k}(z))}\}} \delta(x_i^z_\tau + z) N_{v}(dr, di, dz, d\alpha) \\
 &- \int_0^t \int_{\mathbb{R}^d} \int_0^1 \mathbb{1}_{\{i \leq s_{t-}\}} \mathbb{1}_{\{\alpha \leq (dr/dv)\}} \delta(x_i^z) N_{dr}(dr, di, d\alpha) \\
 &+ \int_0^t \int_{\mathbb{R}^d} \int_0^1 \mathbb{1}_{\{i \leq s_{t-}\}} \mathbb{1}_{\{\alpha \leq \text{aff}(x_i^z, y)k(x, y)/\gamma_2 \tilde{k}(y)\}} \delta(y) N_{af}(dr, di, dy, d\alpha).
\end{align*}
\]
Thus, the network is expressed by (2.5) in terms of the stochastic objects introduced above. Even
the explicit expression looks somewhat complicated, the interpretation is easy to discuss. The
indicator functions that involve \(\alpha\) are related to the rates and the dispersion kernels. Indeed, the
first term describes the recruitment by invitation event, the second term describes the departure
from the network event and the last term describes the mechanism of recruitment by affinity.

The \((\mathcal{F}_t)_{t \geq 0}\)-adapted stochastic process \((r_t)_{t \geq 0}\) given by (2.5) is Markovian with values in
\(S\). For all bounded and measurable maps \(\Phi : \mathcal{S}_F(\mathcal{D}) \rightarrow \mathbb{R}\) and for all \(r \in S\), the infinitesimal generator \(G\) of the process \(r_t\) is defined by,
\[
G\Phi(r) = v_r \int_{\mathcal{D}} r(dx) \int_{\mathbb{R}^d} \left\{ \Phi(r + \delta_z) - \Phi(r) \right\} k(x, z) dz + d_r \int_{\mathcal{D}} \left\{ \Phi(r - \delta_x) - \Phi(r) \right\} r(dx) \\
+ \int_{\mathcal{D}} \left\{ \int_{\mathcal{D}} \left( \Phi(r + \delta_y) - \Phi(r) \right) \text{aff}(x, y)k(y) dy \right\} r(dx).
\]
It is not hard to see that \(r_t\) is a Markov process by classic arguments and to establish the
expression (2.6) by differentiating the expectation of (2.4) at \(t = 0\). Note that if we define the
extinction time as the stopping time:
\[
t_0 = \inf\{t \geq 0, s_t = 0\}
\]
with the convention \(\inf \emptyset = \infty\) then before \(t_0\) the infinitesimal generator is given by (2.6). After
the extinction time \(r_t\) is the null measure, i.e. the network does not contain any individual
and the infinitesimal generator is simply reduced to null measure. Furthermore, note that the
representation (2.5) allows obtaining an easy simulation scheme for the numerical computation
of the network which will be explained further on below.
2.3 Monte Carlo algorithm

The distribution (law) of the network process is characterized by its infinitesimal generator \((2.6)\). This characterization is relatively abstract, so we subsequently propose now an exact Monte Carlo algorithm that simulates the network and provides an empirical representation of its law. The method is exact as, up to the pseudo-random numbers generator approximation, it generates a network which has the same distribution as the considered Markov process \((r_t)_{t\geq 0}\). To give a computational representation of the network \((2.5)\), we shall detail the associated algorithm. We suppose that the members in the network are independent and, as we have seen previously, the network has three possible events. To describe these events, we endow each member with three independent exponential clocks:

- a recruitment exponential affinity clock with rate \(v_r\);
- an invitation exponential clock with rate \(A_f\);
- a departure exponential clock with rate \(d_r\).

So, endowing each member with 3 exponential clocks leads to a very high total number of clocks \((3 \times s_t)\) clocks at time \(t\) which is computationally intensive. Instead, a more efficient strategy consist in reducing the number of exponential realizations by considering only 3 "fast" clocks: A global clock for recruitment by affinity, a global clock for recruitment by invitation and a global clock for departure. Another more efficient strategy consist in considering one clock that control all the events thanks to the properties of the exponential law. We define one global clock that control all punctual mechanisms:

\[
H_t = h_v^r + h^{af} + h_d^r,
\]

where

\[
h_v^r = v_r s_t, \quad \text{and} \quad h^{af} = A_f s_t, \quad \text{and} \quad h_d^r = d_r s_t.
\]

To choose which event occurs, at each time step, we calculate the probabilities of each event. Let the time of the last event \(T_{k-1}\) and the corresponding random network \(r_{T_{k-1}}\), we simulate \(t_k\) and \(r_{t_k}\) as follows: We set

\[
T_k = T_{k-1} + S_k,
\]

with \(S_k \sim \text{Exp}(H_{T_{k-1}})\) and

\[
r_t = r_{T_{k-1}}, \quad \text{for} \ t \in [T_{k-1}, T_k).
\]

We calculate the probabilities of each event using the following objects:

\[
\alpha_{k}^{v_r} = \frac{h_v^{r}}{H_{T_{k-1}}}, \quad \alpha_{k}^{af} = \frac{h^{af}}{H_{T_{k-1}}}, \quad \text{and} \quad \alpha_{k}^{d_r} = \frac{h_d^{r}}{H_{T_{k-1}}}.
\]

We draw the events as follows:
(i) With probability $\alpha_{iv}^{k}$ an invitation event occurs. We draw a member $x_{T_k-1}^i$ where the index $i \sim U\{1, \ldots, s_{T_k-1}\}$. We draw $z \in \mathbb{R}^d$ with the dispersal kernel $K(x_{T_k-1}^i, dz)$. We add, with probability $\frac{k(x_{T_k-1}^i, z)}{\gamma_1 k(z)}$, a new individual to the virtual position $y = x_{T_k-1}^i + z$ and we set

$$r_{T_k} = r_{T_k-1} + \delta_{x_{T_k-1}^i} + z.$$ 

(ii) With probability $\alpha_{iv}^{dr}$ a departure event occurs. We draw a member $x_{T_k-1}^i$ where the index $i \sim U\{1, \ldots, s_{T_k-1}\}$ and we set

$$r_{T_k} = r_{T_k-1} - \delta_{x_{T_k-1}^i}.$$ 

(iii) With probability $\alpha_{iv}^{af}$ an affinity recruitment event occurs. We draw a state $y$ with the affinity kernel $K_{af}(dy)$ and a member $x_{T_k-1}^i$ where the index $i \sim U\{1, \ldots, s_{T_k-1}\}$. We reject the event of affinity recruitment with probability $1 - \frac{\text{aff}(x_{T_k-1}^i, y)k_{af}(y)}{A_{1}\gamma_2 k_{af}(y)}$, otherwise we add a new member to the virtual position $y$ and we set

$$r_{T_k} = r_{T_k-1} + \delta_{y}.$$ 

The exact Monte Carlo scheme is detailed in Algorithm 1. We are now in position to compute the network using a Monte Carlo strategy. Let us give a brief exposition of the use of the exact scheme to simulate some example of dynamics since an initial network (first members) through continuous time. Let the virtual space $\mathcal{D}$ be the unit square $[0,1]^2$. Then, each member of the network is characterized by a point with two coordinates inside the square. We let the algorithm run three times for the same number of iterations ($10^5$ updates) starting from the state $r_0 \sim U[0,1]^2$ with $s_0 = 100$. Following the description of the function $\text{aff}(x,y)$, the friendship between two members inside the network is function to their Euclidean distance in the virtual space. Furthermore, the dynamic of the network depends heavily of the chosen parameters and dispersion kernels. We run the algorithm with $v_r = 3$, $d_r = 1.6$, $A_{1} = 2$, $a_{1} = 0.1$ and we consider a normal kernel for the recruitments (by invitation and by affinity) with a certain dispersion parameter $\sigma$. We plot in Figure 1 the final state of the three simulated dynamics (with $\sigma = 0.01$, $\sigma = 0.005$ and $\sigma = 0.001$) from the initial state $r_0$. Our aim is to understand the behaviour of the network dynamics in function for instance of the dispersion. Thus, we are interested in understanding the influence of spatial dispersion on the formation of patterns and other aspects of social network dynamics. The analysis of Figures 1(b-d) shows that with high dispersion new members tend to occupy almost all the space and their dispersion is uniformly in the vicinity of already present members. With low dispersion new members tend to occupy
Algorithm 1 Exact Monte Carlo algorithm of the random social network

draw $r_0$ and set $T_0 = 0$

for $k = 1, 2, ..., N$ do
  % Setting clocks
  $h_{r_{T_{k-1}}}^r = v_{r_{T_{k-1}}}$, $h_{T_{k-1}}^f = A_f s_{T_{k-1}}$, $h_{T_{k-1}}^d = d_r s_{T_{k-1}}$
  $H_{T_{k-1}} = h_{T_{k-1}}^r + h_{T_{k-1}}^f + h_{T_{k-1}}^d$
  $S_k \sim \exp(H_{T_{k-1}})$
  $T_k = T_{k-1} + S_k$

  % We calculate the probabilities:
  $\alpha_k^v = \frac{h_{T_{k-1}}^r}{H_{T_{k-1}}}$; $\alpha_k^f = \frac{h_{T_{k-1}}^f}{H_{T_{k-1}}}$; $\alpha_k^d = \frac{h_{T_{k-1}}^d}{H_{T_{k-1}}}$.
  $u \sim \mathcal{U}[0, 1]$
  if $u \in [0, \alpha_k^v]$ then
    $i \sim \mathcal{U}\{1, ..., s_{T_{k-1}}\}$
    $z \sim K(x_{T_{k-1}}^i, dz)$
    $u' \sim \mathcal{U}[0, 1]$
    if $u' \leq k(x_{T_{k-1}}^i, z)$ then
      $y = x_{T_{k-1}}^i + z$
      $r_{T_k} = r_{T_{k-1}} + \delta_y$ % Add the new invited member
    end if
  end if
  if $u \in [\alpha_k^v, \alpha_k^v + h_{T_{k-1}}^d]$ then
    $i \sim \mathcal{U}\{1, ..., s_{T_{k-1}}\}$
    $r_{T_k} = r_{T_{k-1}} - \delta x_{T_{k-1}}^i$ % Departure of the member
  else
    $y \sim k^f(y)$
    $i \sim \mathcal{U}\{1, ..., s_{T_{k-1}}\}$
    $u' \sim \mathcal{U}[0, 1]$
    if $u' \leq \frac{k^f(x_{T_{k-1}}^i, y)}{A_f y z^d(y)}$ then
      $r_{T_k} = r_{T_{k-1}} + \delta_y$
    end if
  end if
end for
Figure 1: Figures shown the initial \((s_0 = 100)\) and the final states of the random network \(r_t\) (after \(10^5\) iterations) using different dispersion levels \(\sigma = 0.01, \sigma = 0.005\) and \(\sigma = 0.001\).
only the space located in the vicinity of their friend members. This favours the formation of clusters. Unsurprisingly with very small dispersion ($\sigma = 0.001$), the network dynamic evolves again through a process of clustering. On the basis of the numerical study evidence, it appears that the impact of the dispersion level is strongest.

3 Asymptotic adaptive threshold

As we have seen in the numerical tests, the model proposed in this framework contains several parameters that have significant effect on the dynamic and on the spatial patterns. Thus, a thorough analysis of the clusters (communities) seems crucial to ensure a well understanding of the different connections inside the network. Concerning the connectivity, it is well known that a fundamental question about any network is whether or not it is connected. We study from now on the clustering problem of the set of random points $\{x^t_1, \ldots, x^t_s\}$ in $[0,1]^2$. To avoid technicalities arising from irregularities around the borders of $[0,1]^2$, we consider the unit square as a torus. As in the random geometric graph of Gilbert, given the radius of affinity $a_f > 0$, we may consider our network $r_t$ in which members $i$ and $j$ are connected if and only if the distance of $x^t_i$ and $x^t_j$ does not exceed the threshold $a_f$. We shall consider an extension to this threshold by introducing a soft assumption that all members are active independently with a probability $p$ (be active $\sim$ Bernoulli($p$), $0 < p \leq 1$). Such soft assumption is motivated by ad-hoc social networks. From a practical viewpoint, a member may be inactive in the network for different reasons and will not take part in a community membership. Hence, taking into account the activity of the members, our social network is said to be connected if each inactive member is adjacent to at least one active member together with all active members form a connected network.

In the exact Algorithm [4], the generative model for the data $r^T_k = \{x^T_k_1, \ldots, x^T_k_s\}$, for $1 \leq k \leq N$, is an inhomogeneous Poisson point process with a spatial dependence intensity $\lambda^*_k$ (such that $\lambda^*_k(x) = v_k(y,x) + d_r + \sum_{y \in r^T_k} a_{xy}(x)k(x)$, for $x \in [0,1]^2$). We note that the intensity $\lambda^*_k$ is a locally integrable positive function. Now consider the sequences $(r^T_k)_{k \geq 1}$ and $(s^T_k)_{k \geq 1}$ of random variables respectively in $[0,1]^2$ and $N$, both adapted to a filtration $(F_k)_{k \geq 1}$. We study the problem of the clustering (known as community detection in network literature) of the random points $r^T_k$ at each time $T_k$ based on the observation of $(r^T_1, \ldots, r^T_N)$ and $(s^T_1, \ldots, s^T_N)$, where $N \geq 1$ is a finite $(F_k)$-stopping time.

Remark 1 The results given from now on are stated in a setting where one observes $(r^T_k, s^T_k)_{k=1}^N$ with $N$ a stopping time. It is worth pointing out that this contains the usual case $N \equiv 1$ and $s^T_1 \equiv n$, where $n$ is a fixed sample size. This strategy includes situations where the statistician decides to stop the recording process of network data according to some design of experiment rule.

The analysis in this section is conducted under the following assumption.

Hypothesis 1 ($H_1$) There is a $(F_k)$-adapted sequence of functions $\{f_k(s^T_k)\}_{k \geq 1}$ of positive random variables. This sequence has a limit assumed to be known such that for $f_k : N \mapsto \mathbb{R}_+$
(i) For all \( k \geq 1 \),
\[
\psi_k(s_{T_k}) := \log \left( \frac{s_{T_k} \pi}{4(s_{T_k} - 1)!} \right) + f_k(s_{T_k}) + (s_{T_k} - 1) \log(\alpha_k^+(s_{T_k})) - \alpha_k^+(s_{T_k}) > 0, \tag{3.1}
\]
where the sequence of functions \((\alpha_k^+)_{k \geq 1}\) will be stated later and depends on, among other, the constants \( v_r, d_r, A_f, \sigma, a_f \).

(ii) For all \( k \geq 1 \),
\[
\lim_{s_{T_k} \to \infty} \frac{\psi_k(s_{T_k})}{s_{T_k}} = 0, \quad \text{and} \quad \lim_{s_{T_k} \to \infty} \psi_k(s_{T_k}) = +\infty. \tag{3.2}
\]

The first condition of assumption I allows us to overcome the threshold \( a_f \) problem in the clustering whereas the second condition means that \( \psi_k \) (a function that depends on the current size of the network) does not grow to \( \infty \) more faster than \( s_{T_k} \).

We now come to our main results. Define \( \rho(x) = |B(x, a_f) \cap [0,1]^2| \) to be the number of points (active and inactive) in affinity with \( x \), where \( B(x, a_f) = \{ y \in [0,1]^2 : \|x - y\| \leq a_f \} \). Furthermore, we consider that a member \( x \) is isolated in the network when \( \rho(x) \) contains only the number of inactive members adjacent to \( x \) (which means that \( x \) is considered isolated if there is no active members in the ball \( B(x, a_f) \) except at least \( x \)). Let \( B(x) \) denotes the intersection of the ball \( B(x, a_f) \) with \([0,1]^2\). Next, with vol(\( x \)) the volume of \( B(x) \) (in other words the Lebesgue measure of the measurable set \( B(x) \)), for all \( x \in [0,1]^2 \) we have vol(\( x \)) \( \leq \pi a_f^2 \). For a purely notational reason, let \( E_{ik} \) be the event that \( x_{T_k}^i \) is isolated and \( F_{ik} \) be the event that \( x_{T_k}^i \) is an active isolated member for \( i \in \{1, \ldots, s_{T_k}\} \) and \( k \in \{1, \ldots, N\} \).

The following lemma gathers some standard deviation estimate (a concentration bound) on Poisson random variables; see Boucheron et al. (2013) for more details.

**Lemma 1** Let \( P(\lambda_0) \) be a Poisson random distribution with mean \( \lambda_0 \). Then, there exists a \( \delta_0 > 0 \) such that for all \( \delta \in [0, \delta_0] \) we have
\[
\mathbb{P}(|P(\lambda_0) - \lambda_0| \geq \delta \lambda_0) \leq 2e^{-\lambda_0 \delta^2/3}. \tag{3.3}
\]

**Proof** For \( \delta \geq \lambda_0 \) set \( \mu = \delta/\lambda_0 \) in the inequality \( \mu^2 \mathbb{P}(P(\lambda_0) \geq \delta) \leq \mathbb{E}[\mu^X] = e^{\lambda_0(\mu - 1)} \) where \( X \sim P(\lambda_0) \), to obtain
\[
\mathbb{P}(P(\lambda_0) \geq \delta) \leq e^{-\delta \log(\mu) + \lambda_0 \mu - \lambda_0} = e^{-\lambda_0 \left( \frac{\delta}{\lambda_0} \log(\frac{\delta}{\lambda_0}) - \frac{\delta}{\lambda_0} + 1 \right)}. \]
Hence, we get using a second order Taylor approximation that
\[
P\left(\mathcal{P}(\lambda_0) - \lambda_0 \geq \delta \lambda_0\right) \leq e^{-\lambda_0 \left(1\delta \log\left(1+\delta\right) - \delta\right)} \leq e^{-\lambda_0 \delta^2 / 3}.
\]

Now, if \(0 < \delta \leq \lambda_0\), one can set \(\mu = \delta / \lambda_0\) in the inequality \(P(\lambda_0) \leq \delta\) to establish
\[
P(\lambda_0) \leq \frac{e^{-\lambda_0 \left(\delta / \lambda_0 \log\left(\frac{\delta}{\lambda_0}\right) - \delta + 1\right)}}{e^{-\lambda_0 \delta^2 / 3}}.
\]

A similar bound for \(P(\lambda_0) - \lambda_0 \leq \delta \lambda_0\) follows similarly.

For the random geometric social network, the number of isolated members (denoted from now on \(N^k_0\)) enjoys a Poisson approximation when the size of the network tends to \(\infty\). So, for all \(k \geq 1\) and for \(m \in \mathbb{N}\), we have
\[
P\left(N^k_0 = m\right) \xrightarrow{s_{Tk} \to +\infty} \frac{e^{-\mathbb{E}[N^k_0]} \left(\mathbb{E}[N^k_0]\right)^m}{m!}.
\]

In the present section we prove result of this kind for the class of random network model described in Section 2 when we connect each pair of members with an indicator function \(\text{aff}(\cdot, \cdot)\) of the distance between them. We show that the approximation (3.4) holds for the network \(r_{Tk}\) (for all \(k \geq 1\)) for large network size, uniformly over affinity functions that are zero beyond a given distance. The proof relies heavily on some levels of discretization of the unit square into smaller subsquares (open boxes). Assume that \(\mathcal{D} = [0, 1]^2\) is compact in \(\mathbb{R}^2\) and consider the partitioning of \([0, 1]^2\) into a family \(\{C_\ell\}_{\ell=1,\ldots,L_k}\) of disjoint boxes of \(\mathbb{R}^2\) with side \(a_f / \sqrt{2}\) that we need to cover the state space \(\mathcal{D}\) where \(a'_f \leq a_f\). In the course of the proofs, for all \(k \geq 1\), we condition on the locations of the points \(x^{1}_{Tk}, \ldots, x^{s_{Tk}}_{Tk}\) and assume that they are sufficiently regularly distributed. The probability that this holds is proved in the Lemma 2 that relies on concentration bound of large deviations for Poisson random variables presented in Lemma 1. For a box \(C_\ell\), we have
\[
\mathbb{E}\left[|C_\ell \cap [0, 1]^2|\right] = \frac{s_{Tk}}{L_k} = \frac{s_{Tk} \times (a'_f)^2}{2}.
\]

Here, we need a definition for the regularity of boxes \(\{C_\ell\}_{\ell=1,\ldots,L_k}\).

**Definition 1** Fix \(\nu \in (0, 1)\). A box \(C_\ell\) is called \(\nu\)-regular if one has
\[
\frac{(1 - \nu)s_{Tk}(a'_f)^2}{2} \leq |C_\ell \cap [0, 1]^2| \leq \frac{(1 + \nu)s_{Tk}(a'_f)^2}{2}.
\]

In the following simple lemma we estimate the probability that all boxes \(C_\ell\) are \(\nu\)-regular for \(s_{Tk}\) large enough.
Lemma 2 There exists $\delta_k' > 0$ and $\gamma_k > 0$ such that for all $\delta_k \in [0, \delta_k']$ and if $a'_t(s_{T_k}) = \gamma_k \sqrt{\log(s_{T_k})/s_{T_k}}$ for all $k \geq 1$, then for all $s_{T_k}$ large enough,

$$\inf_{\ell = 1, \ldots, L_k} \mathbb{P}(C_{\ell} \text{ is } \nu \text{-regular}) \geq 1 - 2L_k s_{T_k}^{-\gamma^2 \delta_k^2/6}.$$ 

In particular, if $\gamma_k^2 > 6/\delta_k^2$ then

$$\mathbb{P}(\text{every box } C_{\ell} \text{ is } \nu \text{-regular}) \xrightarrow{s_{T_k} \to \infty} 1.$$

Proof For any box $C_{\ell}$, the number of points $|C_{\ell} \cap [0, 1]^2|$ is distributed like a Poisson random variable with mean $s_{T_k} \times (a'_t)^2/2$. By Lemma 1 we have for $\delta_k' > 0$ and $\delta_k \in [0, \delta_k']$,

$$\mathbb{P} \left( \left| \mathcal{P}(s_{T_k} (a'_t)^2) - s_{T_k} (a'_t)^2 \right| \geq \delta_k s_{T_k} (a'_t)^2 \right) \leq 2e^{-s_{T_k} (a'_t)^2 \delta_k^2/23}.$$  (3.5)

Now, for every box and for all $s_{T_k}$ large enough,

$$\mathbb{P}(C_{\ell} \text{ is not } \nu \text{-regular}) \leq 2e^{-s_{T_k} (a'_t)^2 \delta_k^2/6} \leq 2s_{T_k}^{-\gamma^2 \delta_k^2/6},$$

since $a'_t(s_{T_k}) = \gamma_k \sqrt{\log(s_{T_k})/s_{T_k}}$ for any $k \geq 1$. In addition, if there exists one box that is not $\nu$-regular, then one of the $L_k$ boxes has a number of points that is out of range, so that as $s_{T_k} \to \infty$,

$$\mathbb{P}(\exists C_{\ell} : C_{\ell} \text{ is not } \nu \text{-regular}) \leq 2 \times L_k \times s_{T_k}^{-\gamma^2 \delta_k^2/6} = 2 \times \frac{2s_{T_k}}{\gamma_k^2 \log(s_{T_k})} \times s_{T_k}^{-\gamma^2 \delta_k^2/6} \leq s_{T_k}^{1-\gamma^2 \delta_k^2/6 + o(1)},$$

which tends to zero provided that $\gamma_k^2 > 6/\delta_k^2$. \hfill \square

For simplicity and from now on we make the following technical assumption.

Hypothesis 2 (H2) (i) Assume that $a_t(s_{T_k}) \geq a'_t(s_{T_k})$ and $a'_t(s_{T_k}) \sim \gamma_k \sqrt{\log(s_{T_k})/s_{T_k}}$ for all $k \geq 1$ with $\gamma_k > \sqrt{6}/\delta_k$ and $\delta_k \in [0, \delta_k']$ and $\delta_k' > 0$. Assume also that there exists a $(\mathcal{F}_k)$-adapted sequence of functions $(g_k(x, s_{T_k}))_{k \geq 1}$, with $g_k : [0, 1]^2 \times \mathbb{N} \mapsto \mathbb{R}_+$, such that, for all $x, x^1_{T_k}, \ldots, x^L_{T_k} \in [0, 1]^2$ and all $k \geq 1$, we have

$$\sum_{i=1}^{s_{T_k}} \mathbb{1}_{\{x-x^i_{T_k} \leq a_t\}} = g_{s_{T_k}}(x) \sum_{i=1}^{s_{T_k}} \sum_{\ell=1}^{L_k} \mathbb{1}_{\{(C_{\ell} \cap B(x)) \neq \emptyset\}} \mathbb{1}_{[0,1]^2}(x) \mathbb{1}_{C_{\ell}(x^i_{T_k})},$$

with $g_k(x, s_{T_k})^{s_{T_k} \to \infty} 1$,

where $\{C_{\ell}\}_{\ell=1, \ldots, L_k}$ is a family of disjoint boxes with side $a'_t(s_{T_k})/\sqrt{2}$ that covers $[0, 1]^2$.  

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(ii) The dispersion kernel $k^{af}(\cdot) : \mathbb{R}^2 \mapsto \mathbb{R}_+$ is an integrable function that does not change sign in $[0, 1]^2$.

The assumption $H_2$ enables us to control the accuracy of the approximation of the number of points in the neighborhood of each $x \in [0, 1]^2$. Note that $L_k$ is clearly finite (for all $k \geq 1$) together with for each $x, y \in C^1, |x - y| \leq a'_k$. Note also that the function $g_k$ assesses the proportion of points inside the ball $B(x)$ from the number of points inside the boxes that intersect with $B(x)$. Such an assumption is realistic because, when $s_{T_k}$ is large enough, the side of each box $a'_k(s_{T_k})/\sqrt{2}$ is small enough which guarantee a fine mesh and hence $g_k \sim 1$ for all $x \in [0, 1]^2$. More rigorously, we use the fact that all open of $\mathbb{R}^d$ is a countable reunion of open pavers. Furthermore, the assumption is realistic since asymptotically all the boxes contains points (proved in Lemma 2) and is needed in practice to discard regions with strong variations. Concerning the point (ii) in assumption $H_2$ it’s mainly needed for computational issues.

In order to use $H_2$ for estimating the probability of isolated members, we need to make sure that the global affinity rate stays under control.

**Lemma 3** Admit the hypothesis $H_2$. For any $x \in [0, 1]^2$ and any bounded region $B(x)$, let consider the (finite) integral of $\lambda^*_k$ over region $B(x)$

$$
\Lambda^*_k(x) = \int_{B(x)} \lambda^*_k(z)dz = \int_{B(x)} \left(v_r k(y, z) + d_r + \sum_{y \in r_{T_k}} \text{aff}(z, y)k^{af}(z)\right)dz, \quad (3.6)
$$

with $v_r, d_r, A_t > 0, \sigma, a_t > 0$. Then, there exists two constants $\tilde{c}_0$ and $\tilde{c}_1$ in $B(x)$ such that for $s_{T_k}$ large enough

$$
\Lambda^*_k(x) = \left(v_r k(y, \tilde{c}_0) + d_r + A_t L'_k \frac{\gamma_k^2 \log(s_{T_k})}{2} \right) \text{vol}(x), \quad (3.7)
$$

where the integer $L'_k \in S_0 := \left\{ \left\lfloor \frac{\pi a^2}{2(a_t)^2} \right\rfloor, \ldots, \left\lfloor \frac{2\pi a^2}{(a_t)^2} \right\rfloor \right\}$ and $\gamma_k > \sqrt{6}/\delta_k$ with $\delta_k \in [0, \delta'_{k}]$ and $\delta'_{k} > 0$.

**Proof** We use the assumption $H_2$ to compute the integral of $\lambda^*_k$ over region $B(x)$. First, for any $z \in B(x)$ we easily remark that the number of boxes that intersect with the bounded region $B(z)$ is between $\left\lfloor \frac{\pi a^2}{(a_t)^2} \right\rfloor$ (when $z$ is located at the corners of $[0, 1]^2$) and $\left\lfloor \frac{2\pi a^2}{(a_t)^2} \right\rfloor$ (when $B(z)$ is fully contained in $[0, 1]^2$). For all $k \geq 1$, let $L'_k$ denote the number of boxes whose intersects with $B(z)$. If $s_{T_k}$ is sufficiently large, every square that intersect with $B(z)$ is fully contained in $B(z)$ and the large number of boxes allows us to take advantage of the approximation $g_k(z, s_{T_k}) \sim 1$.
for all $z \in [0,1]^2$. Now, since every square is $\nu$-regular for every $z \in [0,1]^2$, it follows that

$$\int_{B(x)} A_f \sum_{i=1}^{s_{Tk}} 1_{|z-x_i^{Tk}|\leq a_f} dz = A_f \int_{B(x)} \sum_{i=1}^{s_{Tk}} \sum_{\ell=1}^{L_k} 1_{(C_{\ell}\cap B(z))\neq \emptyset} 1_{C_{\ell}(x_i^{Tk})} 1_{B(x)}(z) dz$$

$$= A_f \int_{B(x)} \sum_{i=1}^{s_{Tk}} \sum_{\ell=1}^{L_k} 1_{(C_{\ell}\cap B(z))\neq \emptyset} 1_{C_{\ell}(x_i^{Tk})} 1_{B(x)}(z) dz$$

$$= \left( A_f L_k \frac{s_{Tk} \times (a_f')^2}{2} \right) \int_{B(x)} 1_{B(x)}(z) dz$$

$$= \left( A_f L_k \frac{s_{Tk} \times \gamma_k^2 \log(s_{Tk})/s_{Tk}}{2} \right) \text{vol}(x),$$

and the claim follows easily by application of the mean-value theorem. \hfill \square

We are now in position to assess the probability of an isolated member inside the network.

**Proposition 1** Under the assumption $H_2$ and for any $x_i^{Tk} \in [0,1]^2$ with $v_r, d_r, A_f, \sigma, a_f > 0$ and $0 < p \leq 1$, we have for $s_{Tk}$ sufficiently large and $i \in \{1, \ldots, s_{Tk}\}$ with $k \in \{1, \ldots, N\}$

$$P(E_{ik}) = \frac{(v_r k(y, \tilde{c}_0) + d_r + A_f L_k' \frac{\gamma_k^2 \log(s_{Tk})k^\sigma(\tilde{c}_1)}{2})^{s_{Tk}-1}}{(s_{Tk} - 1)!} e^{-\left(v_r k(y, \tilde{c}_0) + d_r + A_f L_k' \frac{\gamma_k^2 \log(s_{Tk})k^\sigma(\tilde{c}_1)}{2}\right)} \int_{[0,1]^2} \left(1 - p\text{vol}(x_i^{Tk})\right)^{s_{Tk}-1} dx_i^{Tk},$$

where $L_k' \in S_0$, $k(y, \tilde{c}_0) > 0$, $k^\sigma(\tilde{c}_1) > 0$ and $\gamma_k > \sqrt{6}/\delta_k$ with $\delta_k \in [0, \delta_k']$ and $\delta_k' > 0$. 

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At first sight and for simplicity, let $\Omega^k$ be the event that, for any $x_{i_k}^i \in [0, 1]^2$ and for any $j \neq i$, $x_{i_j}^j \notin B(x_{i_k}^i)$ or $x_{i_i}^i \in B(x_{i_k}^i)$ but inactive. By direct calculation we have

$$P(E_{ik}) = \int_{[0,1]^2} \mathbb{P}(E_{ik}) dx_{i_k}^i = \int_{[0,1]^2} \mathbb{P}(\Omega^k) dx_{i_k}^i$$

$$= \int_{[0,1]^2} \sum_{\ell=0}^{s_{T_k}} (1-p)^{\ell} \frac{\left(\int_{B(x_{i_k}^i)} \lambda_k^1(x) dx\right)^\ell}{\ell!} e^{-\int_{B(x_{i_k}^i)} \lambda_k^1(x) dx}$$

$$\times \left(\frac{\left(\int_{[0,1]^2 \setminus B(x_{i_k}^i)} \lambda_k^1(x) dx\right)^{s_{T_k}-\ell}}{(s_{T_k}-1)!} e^{-\int_{[0,1]^2 \setminus B(x_{i_k}^i)} \lambda_k^1(x) dx}\right) dx_{i_k}^i$$

$$= \int_{[0,1]^2} \sum_{\ell=0}^{s_{T_k}-1} (1-p)^{\ell} \left(\frac{(v_r k(y, \tilde{c}_0) + d_r + A_t L_k \gamma_k^2 \log(s_{T_k}) k_{\text{eff}}(\tilde{c}_1))}{\ell!} \right)^{-1}$$

$$\times \left(\frac{(1-p)(v_r k(y, \tilde{c}_0) + d_r + A_t L_k \gamma_k^2 \log(s_{T_k}) k_{\text{eff}}(\tilde{c}_1))}{(s_{T_k}-1)!} \right)$$

$$\times \left(\frac{(v_r k(y, \tilde{c}_0) + d_r + A_t L_k \gamma_k^2 \log(s_{T_k}) k_{\text{eff}}(\tilde{c}_1))}{(1 - \text{vol}(x_{i_k}^i))} \right)^{s_{T_k}-\ell} dx_{i_k}^i$$

$$= \frac{(v_r k(y, \tilde{c}_0) + d_r + A_t L_k \gamma_k^2 \log(s_{T_k}) k_{\text{eff}}(\tilde{c}_1))}{(s_{T_k}-1)!} \int_{[0,1]^2} \sum_{\ell=0}^{s_{T_k}-1} \frac{(s_{T_k}-1)!}{\ell!(s_{T_k}-\ell)!}$$

$$\times \left(\frac{(v_r k(y, \tilde{c}_0) + d_r + A_t L_k \gamma_k^2 \log(s_{T_k}) k_{\text{eff}}(\tilde{c}_1))}{(1 - \text{vol}(x_{i_k}^i))} \right)^{s_{T_k}-\ell} dx_{i_k}^i$$

$$= \frac{(v_r k(y, \tilde{c}_0) + d_r + A_t L_k \gamma_k^2 \log(s_{T_k}) k_{\text{eff}}(\tilde{c}_1))}{(s_{T_k}-1)!} \int_{[0,1]^2} \left(1 - p\text{vol}(x_{i_k}^i)\right)^{s_{T_k}-1} dx_{i_k}^i,$$

where the last line is obtained from the binomial theorem. This completes the proof. \qed

It is interesting to remark that the result cited in Proposition 4 suggests that the probability of being a member isolated in the network is inversely proportional to the size of the network and at the same time to the volume of the ball portion (around the member) that intersect with $[0, 1]^2$. Thus, this result seems intuitive. Now, we shall assess the probability that more than one
member (say $\kappa \geq 2$ members) are isolated inside the network. For any $(x^1, \ldots, x^\kappa) \in [0, 1]^{2\kappa}$ and to shorten notation, we use from now on $B(x^1, \ldots, x^\kappa) = B(x^1) \cup \cdots \cup B(x^\kappa)$ and $\text{vol}(x^1, \ldots, x^\kappa)$ for the volume of $B(x^1, \ldots, x^\kappa)$.

**Proposition 2** Under the assumption $H_2$ and for any $\kappa \geq 2$ and $(x^1_{T_k}, \ldots, x^\kappa_{T_k}) \in [0, 1]^{2\kappa}$ with $v_r, d_r, A_f, \sigma, a_f > 0$ and $0 < p \leq 1$, we have for $s_{T_k}$ sufficiently large and $k \in \{1, \ldots, N\}$

$$
\mathbb{P}(E_{1k} \cap \cdots \cap E_{nk}) \leq \frac{(v_r k(y, \tilde{c}_0) + d_r + A_f L_k \gamma \log(s_{T_k}) \delta \tilde{c}_1)}{(s_{T_k} - \kappa)!} e^{- (v_r k(y, \tilde{c}_0) + d_r + A_f L_k \gamma \log(s_{T_k}) \delta \tilde{c}_1)} 
\times \int_{(0,1)^{2\kappa}} \left(1 - p \text{vol}(x^1_{T_k}, \ldots, x^\kappa_{T_k})\right)^{s_{T_k} - \kappa} \, dx^1_{T_k} \cdots dx^\kappa_{T_k},
$$

where $L_k' \in S_0$, $k(y, \tilde{c}_0) > 0$, $\gamma \delta_k > 0$ and $\gamma_k > \sqrt{\delta_k}$ with $\delta_k \in [0, \delta_k']$ and $\delta_k' > 0$.

**Proof** For any $\kappa \geq 2$ and $(x^1_{T_k}, \ldots, x^\kappa_{T_k}) \in [0, 1]^{2\kappa}$, let $\Omega_0^{\kappa}$ be the event that $B(x^1_{T_k}, \ldots, x^\kappa_{T_k})$ contains no active members in $x^1_{T_k}, \ldots, x^\kappa_{T_k}$. Then we have

$$
\mathbb{P}(E_{1k} \cap \cdots \cap E_{nk}) = \int_{(0,1)^{2\kappa}} \mathbb{P}(E_{1k} \cap \cdots \cap E_{nk} \mid (x^1_{T_k}, \ldots, x^\kappa_{T_k}) \in [0, 1]^{2\kappa}) \, dx^1_{T_k} \cdots dx^\kappa_{T_k}
\leq \int_{(0,1)^{2\kappa}} \mathbb{P}(\Omega_0^{\kappa}) \, dx^1_{T_k} \cdots dx^\kappa_{T_k}
= \int_{(0,1)^{2\kappa}} \sum_{\ell=0}^{s_{T_k} - \kappa} \left(\frac{\int_{B(x^1_{T_k}, \ldots, x^\kappa_{T_k})} \lambda^\ell(x) \, dx}{\ell!} \right) e^{- \int_{B(x^1_{T_k}, \ldots, x^\kappa_{T_k})} \lambda_k(x) \, dx} \times \left(\frac{\int_{[0,1)^{2\kappa} \setminus B(x^1_{T_k}, \ldots, x^\kappa_{T_k})} \lambda^\ell(x) \, dx}{s_{T_k} - \kappa - \ell} \right) e^{- \int_{[0,1)^{2\kappa} \setminus B(x^1_{T_k}, \ldots, x^\kappa_{T_k})} \lambda_k(x) \, dx} \, dx^1_{T_k} \cdots dx^\kappa_{T_k},
$$

and the claim follows easily by similar arguments as the end of proof of Proposition 1. □

The elicitation of the probability of several isolated members is somewhat more difficult than the case of one isolated member. Then, attention shows that the elicitation of the event $\Omega_0^{\kappa}$ used in the proof of Proposition 2 gives only an upper bound. Unfortunately, this seems to be difficult to prove in a general setting. To establish an asymptotic expression for this probability we need a deeper development and more notations. Let $S_{\alpha_l}(x^1, \ldots, x^\kappa)$ denotes the sub-network over $(x^1, \ldots, x^\kappa)$ in which two members are connected (by affinity) if and only if their distance is at most $\alpha_l$. For any integer $n$ satisfying $1 \leq n \leq \kappa$, we denote by $C_{\kappa,n}$ the set of $\kappa$-tuples $(x^1, \ldots, x^\kappa) \in [0, 1]^{2\kappa}$ satisfying $S_{2\alpha_l}(x^1, \ldots, x^\kappa)$ has exactly $n$ connected components. Note that the set $C_{\kappa,n}$ consists of those tuples of $\kappa$ points which satisfies, for $i = 1, \ldots, \kappa$, $B(x^i)$ contains none of the other points of the tuple. In the following result, we derive an interesting formula for the computation of the probability of several isolated members.
Proposition 3  Under the assumption $H_2$ and for any $\kappa \geq 2$ and $(x_{T_1}^i, \ldots, x_{T_k}^i) \in \mathcal{C}_{\kappa \kappa}$ with $v_r, d_r, A_t, \sigma, \alpha_t > 0$ and $0 < p \leq 1$, we have for $s_{T_k}$ sufficiently large and $k \in \{1, \ldots, N\}$

$$
\mathbb{P}(E_{1k} \cap \cdots \cap E_{nk}) = \frac{(v_r k(y, \tilde{c}_0) + d_r + A_t L_k \frac{\gamma^2 \log(s_{T_k}) k^\alpha (\tilde{c}_1)}{2} s_{T_k} - \kappa)}{(s_{T_k} - \kappa)!} e^{-(v_r k(y, \tilde{c}_0) + d_r + A_t L_k \frac{\gamma^2 \log(s_{T_k}) k^\alpha (\tilde{c}_1)}{2})} \times \int_{\mathcal{C}_{\kappa \kappa}} \left( 1 - \operatorname{vol}(x_{T_1}^i, \ldots, x_{T_k}^i) \right) ^{s_{T_k} - \kappa} dx_{T_1} \cdots dx_{T_k},
$$

where $L_k \in S_0$, $k(y, \tilde{c}_0) > 0$, $k^\alpha (\tilde{c}_1) > 0$ and $\gamma_k > \sqrt{9}/\delta_k$ with $\delta_k \in [0, \delta_k']$ and $\delta_k' > 0$.

Proof  For any $\kappa \geq 2$ and $(x_{T_1}^i, \ldots, x_{T_k}^i) \in \mathcal{C}_{\kappa \kappa}$, let $\Omega_1^{\kappa}$ be the event that, for all $1 \leq i \leq \kappa$, $B(x_{T_i}^i)$ contains no active members in $x_{T_k}^{i+1}, \ldots, x_{T_k}^\kappa$. In addition, let $\Omega_2^{\kappa}$ be the event that, for all $1 \leq i \leq \kappa$, $B(x_{T_i}^i)$ contains $n_i$ inactive members and no active members in $x_{T_k}^{i+1}, \ldots, x_{T_k}^\kappa$. Thanks to the previous introducing two events, it follows that

$$
\mathbb{P}(E_{1k} \cap \cdots \cap E_{nk}) = \int_{\mathcal{C}_{\kappa \kappa}} \mathbb{P} \left( E_{1k} \cap \cdots \cap E_{nk} \bigg| (x_{T_1}^i, \ldots, x_{T_k}^i) \in \mathcal{C}_{\kappa \kappa} \right) dx_{T_1}^i \cdots dx_{T_k}^i
$$

$$
= \int_{\mathcal{C}_{\kappa \kappa}} \mathbb{P}(\Omega_1^{\kappa}) dx_{T_1}^i \cdots dx_{T_k}^i = \int_{\mathcal{C}_{\kappa \kappa}} \sum_{n_1 + \cdots + n_\kappa = 0}^{s_{T_k} - \kappa} \mathbb{P}(\Omega_2^{\kappa}) dx_{T_1}^i \cdots dx_{T_k}^i
$$

$$
= \int_{\mathcal{C}_{\kappa \kappa}} \sum_{n_1 + \cdots + n_\kappa = 0}^{s_{T_k} - \kappa} \left\{ \prod_{i=1}^{\kappa} \left( (1 - p)^{n_i} \frac{\left( \int_{B(x_{T_i}^i)} \lambda_k(x) dx \right)^{n_i}}{n_i!} - \int_{[0,1] \setminus B(x_{T_i}^i)} \lambda_k(x) dx \right) \right\} \times \left( \int_{[0,1]^2 \setminus B(x_{T_1}^i, \ldots, x_{T_k}^i)} \lambda_k(x) dx \right)^{s_{T_k} - \kappa - \sum_{i=1}^{\kappa} n_i} e^{-(\int_{[0,1]^2 \setminus B(x_{T_1}^i, \ldots, x_{T_k}^i)} \lambda_k(x) dx)} dx_{T_1}^i \cdots dx_{T_k}^i
$$
We have to constrain $\alpha$ and we know from Proposition 3 that which may be infinite. From Proposition 1, we have asymptotically

$$(\sum_{i=1}^n \text{vol}(x_{T_k}^i) + 1 - \text{vol}(x_{T_k}^1, \ldots, x_{T_k}^n))$$

where the last line is obtained from the multinomial theorem and by remarking that, for any set $(x_{T_k}^1, \ldots, x_{T_k}^n) \in C_{\kappa}$, we have $\sum_{i=1}^n \text{vol}(x_{T_k}^i) = \text{vol}(x_{T_k}^1, \ldots, x_{T_k}^n)$. This completes the proof.

Our previous result (Proposition 3) gives an asymptotic equivalence for the probability of several isolated members not in the whole domain of the torus but in a more restricted domain given by $(x_{T_k}^1, \ldots, x_{T_k}^n) \in C_{\kappa}$ where $\kappa \geq 2$. Even if this result appears somewhat restrictive, we will show in the sequel that the probability in the domain $[0, 1]^{2\kappa} \setminus C_{\kappa}$ (which is not covered by our result) converges asymptotically to nothing.

Let us highlight the major formulas of the previous results. Let denote

$$\alpha_k^*(s_{T_k}) = \left( v_r k(y, \tilde{c}_0) + d_r + A_T L_k \frac{\gamma^2 \log(s_{T_k}) (k \tilde{c}_1)}{2} \right),$$

which may be infinite. From Proposition 3 we have asymptotically

$$\mathbb{P}(E_{1k}) \propto \frac{(\alpha_k^*(s_{T_k}))^{s_{T_k} - 1}}{(s_{T_k} - 1)!} e^{-\left(\alpha_k^*(s_{T_k})\right)},$$

and we know from Proposition 3 that

$$\mathbb{P}(E_{1k} \cap \cdots \cap E_{nk}) \propto \frac{(\alpha_k^*(s_{T_k}))^{s_{T_k} - \kappa}}{(s_{T_k} - \kappa)!} e^{-\left(\alpha_k^*(s_{T_k})\right)}.$$  

The numerical problem that we aim to state now (to establish the distribution of isolated members) is the following; We have to constrain $\alpha^*$ to fulfill the following equation (needed for
the proof of the law of \( N_0^k \) based on a version of Brun’s sieve theorem and Bonferroni inequalities)

\[
\left( \alpha_k^*(s_{T_k}) \right)^{s_{T_k} - \kappa} / (s_{T_k} - \kappa)! \cdot e^{-\left( \alpha_k^*(s_{T_k}) \right)^{s_{T_k} - 1} / (s_{T_k} - 1)! \cdot e^{-\left( \alpha_k^*(s_{T_k}) \right)}} = \left( \left( \alpha_k^*(s_{T_k}) \right)^{s_{T_k} - 1} / (s_{T_k} - 1)! \cdot e^{-\left( \alpha_k^*(s_{T_k}) \right)} \right)^{\kappa},
\]

(3.11)

which is straightforwardly solved by

\[
\alpha_k^*(s_{T_k}) = -s_{T_k} W\left( -\left( (s_{T_k} - 1)! \left( (s_{T_k} - \kappa)! \right) \right) \right) \frac{1}{(s_{T_k} - \kappa)^{s_{T_k} - 1} \left( (s_{T_k} - \kappa)! \right)^{\kappa - 1} \left( (s_{T_k} - \kappa)! \right)^{\kappa - 1}},
\]

(3.12)

where \( W(\cdot) \) is the Lambert function. From now on we impose an additional constraint on the constants \( v_r, d_r, A_f, \sigma, \gamma_k \) in order that \( \alpha_k^* \) fulfills (3.11) together with of course the fact that \( v_r, d_r, A_f, \sigma > 0 \) and \( \gamma_k > \sqrt{6}/\delta_k \) with \( \delta_k \in [0, \delta'_k] \) and \( \delta'_k > 0 \). One may easily verify the constraint (3.11) if we fix the intrinsic network parameters \( v_r, d_r, A_f, \sigma \) and we compute the needed value of \( \gamma_k \) from formula (3.12). We state this more precisely in the next Theorem 1 below, which requires also the assumption \( H_1 \). Our main result (its proof is delayed until Section 6) is as follows.

**Theorem 1** Assume that hypothesis \( H_1 \) and \( H_2 \) are satisfied. Let

\[
a^*_k(s_{T_k}) = \left( \log \left( \frac{s_{T_k} \pi}{4(s_{T_k} - 1)!} \right) + f_k(s_{T_k}) + (s_{T_k} - 1) \log(\alpha_k^*(s_{T_k})) - \alpha_k^*(s_{T_k}) \right)^{1/2},
\]

(3.13)

for any \( p \in (0, 1] \) and where \( \alpha_k^* \) is given by (3.8) and satisfying (3.11). Then, we have

\[
s_{T_k} \left( \frac{\left( \alpha_k^*(s_{T_k}) \right)^{s_{T_k} - 1} / (s_{T_k} - 1)! \cdot e^{-\left( \alpha_k^*(s_{T_k}) \right)}} {e^{-s_{T_k} p \text{vol}(x) dx} \rightarrow \beta_k},
\]

(3.14)

as \( s_{T_k} \rightarrow \infty \) and where

\[
\beta_k = e^{-\lim_{s_{T_k} \rightarrow \infty} f_k(s_{T_k})}.
\]

If \( \beta_k \in (0, \infty) \), then as \( s_{T_k} \rightarrow \infty \), we have for \( m \in \mathbb{N} \) and \( k \geq 1 \) that

\[
\mathbb{P}(N_0^k = m) \rightarrow e^{-\beta_k \beta_k^m / m!}.
\]

(3.15)

If \( \beta_k = 0 \), then \( \mathbb{P}(N_0^k = 0) \rightarrow 1 \), and if \( \beta_k = \infty \), then \( \mathbb{P}(N_0^k = m) \rightarrow 0 \) for all \( m \in \mathbb{N} \).

The affinity threshold (3.13) established for the Poisson point process (resultant from the three processes) that generates the data \( r_{T_k} = \{x_1^{T_k}, \ldots, x_{s_{T_k}}^{T_k} \} \), for \( 1 \leq k \leq N \), looks somewhat complicated than the threshold obtained with uniform random points. The particularity of the
threshold (3.13) is that, in addition to the size of the data, it depends on the parameters of the Poisson process \((v, d, A, \sigma)\) and the parameters of the discretization \((\gamma, L)\).

We now discuss related work and open problems. Note that the Poisson distribution (3.16) [but not with the same threshold (3.13)] was already proved by Penrose (2016) and Yi et al. (2006) in the special case of points uniformly distributed (respectively in the unit square and in a disk of unit area). Here we are considering a much more general class of random point processes. The results of this paper goes a step beyond the literature in that it considers a Poisson point process with connection (affinity) function that is zero beyond the optimal threshold (3.13) in the same model. To our best knowledge, this is the first work where results about the optimal threshold and the distribution of isolated members are shown for geometric networks with points generated from Poisson point measures (rather than the uniform random points considered usually in literature). Other choices for the threshold of connection functions are proposed in the literature that deals with the subject of the connectivity of random geometric graphs initiated by Dette & Henze (1989). For instance, using a step connection function \(1_{[0,a]}(\|x−y\|)\), Dette & Henze (1989) showed that for any constant \(C > 0\), the disk graph on \(s_{T_k}\) uniform random points with connectivity threshold \(\sqrt{\ln(s_{T_k}) + C} / (\pi s_{T_k})\) has no isolated nodes with probability \(\exp(-e^{-C})\) when \(s_{T_k}\) tends to infinity. The theory has been generalized after in the unit square \([0,1]^d\) (with \(d \geq 2\)) by Penrose (2016) for a class of connection functions that decay exponentially in some fixed positive power of distance. In some applications, it is desirable to use the Rayleigh fading connection function given by \(\exp(-\bar{\beta}(\|x−y\|/\bar{\rho})^{\bar{\gamma}})\) for some fixed positive \(\bar{\beta}, \bar{\rho}, \bar{\gamma} > 0\) (typically \(\bar{\gamma} = 2\)). It would be interesting to try to extend our results to these connection functions but this would be a nontrivial task because the discretization method developed here can be quite hard to adaptation.

Another related problem is the connectivity of the network. It is known that the main obstacle to connectivity is the existence of isolated members. More clearly, for the geometric (Gilbert) graph \(G(\mathcal{X}_{s_{T_k}}, a_t(s_{T_k}))\) with vertex set \(\mathcal{X}_{s_{T_k}}\) given by a set of \(s_{T_k}\) independently uniformly distributed points in \([0,1]^d\) with \(d \geq 2\), and with an edge included between each pair of vertices at distance at most \(a_t(s_{T_k})\), Penrose (1997) showed that the probability that the graph is disconnected but free of isolated vertices tends to zero as \(s_{T_k}\) tends to infinity, for any choice of \((a_t(s_{T_k}))_{s_{T_k} \in \mathbb{N}}\). The same result happens with the Erdős-Rényi graph but the proof for the geometric graph is much harder as pointed out by Bollobás (2001). More formally, for random graphs \(G\), the number of isolated vertices (denoted \(N_0(G)\)) has (asymptotically) a Poisson distribution, hence with \(\mathcal{K}\) denoting the class of connected graphs we have \(\mathbb{P}(G \in \mathcal{K}) \sim \mathbb{P}(N_0(G) = 0) \sim \exp(-\mathbb{E}[N_0])\) as \(s_{T_k} \to \infty\). We would expect something similar to hold for our social network \((r_t)_{t \geq 0}\). Under additional assumption on \(f\) the connectivity result of graphs \(G\) presented here might naturally be conjectured for \((r_t)_{t \geq 0}\) as follows

\[
\forall k \geq 1, \quad \mathbb{P}(r_{T_k} \in \mathcal{K}) \to e^{-\beta_k} \quad \text{as} \quad s_{T_k} \to \infty,
\]

with \(e^{-\beta_k}\) interpreted as 0 for \(\beta_k = \infty\). More generally, it would be of interest (in its own right) to extend this to the case of \((r_t)_{t \geq 0}\).
We conclude our results by checking a statement shown essentially that the active isolated members of the social network enjoys also a Poisson approximation at each time but with a slightly different mean. The following theorem states this.

**Theorem 2** Admit assumptions of Theorem 1 and consider the affinity threshold (3.13). If \( \beta_k \in (0, \infty) \), then as \( s_{T_k} \to \infty \), we have for \( m \in \mathbb{N} \) and \( k \geq 1 \) that

\[
P(N^k_a = m) \to e^{-p\beta_k}(p\beta_k)^m/m!,
\]

where \( p \in (0, 1] \) and \( N^k_a \) denoting the number of active isolated members at time \( T_k \). If \( \beta_k = 0 \), then \( P(N^k_a = 0) \to 1 \), and if \( \beta_k = \infty \), then \( P(N^k_a = m) \to 0 \) for all \( m \in \mathbb{N} \).

In section 6, we prove Theorems 1 and 2. We finish this section with a short discussion about the limit of \( f_k(\cdot) \). As we have seen this limit plays a crucial role in determining the Poisson distribution of isolated members in the network. We fix \( k \geq 1 \). From the formula (3.13) of optimal threshold, we deduce the following limits:

\[
\lim_{s_{T_k} \to \infty} \log \left( \frac{s_{T_k} \pi}{4(s_{T_k} - 1)!} \right) = -\infty, \quad \lim_{s_{T_k} \to \infty} (s_{T_k} - 1) \log(\alpha^*_k(s_{T_k})) = +\infty \quad \text{and} \quad \lim_{s_{T_k} \to \infty} -\alpha^*_k(s_{T_k}) = -\infty.
\]

Furthermore, by series expansion at \( s_{T_k} = +\infty \), we find

\[
\lim_{s_{T_k} \to \infty} \log \left( \frac{s_{T_k} \pi}{4(s_{T_k} - 1)!} \right) + (s_{T_k} - 1) \log(\alpha^*_k(s_{T_k})) - \alpha^*_k(s_{T_k}) = -\infty,
\]

and

\[
\lim_{s_{T_k} \to \infty} \frac{\log \left( \frac{s_{T_k} \pi}{4(s_{T_k} - 1)!} \right) + (s_{T_k} - 1) \log(\alpha^*_k(s_{T_k})) - \alpha^*_k(s_{T_k})}{\pi p s_{T_k}} = -\infty.
\]

Then, in order that the function \( \psi_k \) satisfy the three conditions of assumption \( H_1 \), there exists only function \( f_k(s_{T_k}) \) such that its limit is \(+\infty\). The two others cases \( \lim_{s_{T_k} \to \infty} f_k(s_{T_k}) = \{-\infty, \in (0, \infty)\} \) discussed in Theorems 1 and 2 are in practice impossible since there exists no functions with these limits that satisfy at the same time \( H_1 \). Hence, \( \beta_k \) takes only the convenient value \( 0 \) and we conclude that there is no isolated members in the network with probability one as \( s_{T_k} \to \infty \).

### 4 Dynamic clustering with \( a^*_f \)

We apply in this section our main result, Theorem 1 of Section 3 to the problem of members clustering is that of grouping similar communities (components) of the social network, and
estimating these groups from the random points $r_{T_k}$ at each time $T_k$. When cluster similarity is defined via latent models, in which groups are relative to a partition of the index set $\{1, \ldots, s_{T_k}\}$, the most natural clustering strategy is K-means. We explain why this strategy cannot lead to perfect cluster in our context (especially towards the detection of isolated members) and offer another strategy, based on the optimal threshold $a^*_k$, that can be viewed as a density-based spatial clustering of applications with noise [Ester et al. 1996]. We introduce a cluster separation method tailored to our random network. The clusters estimated by this method are shown to be adaptively from the data. We compare this method with appropriate K-means-type procedure, and show that the former outperforms the latter for cluster with detection of isolated members.

The solutions to the problem of clustering are typically algorithmic and entirely data based. They include applications of K-means, spectral clustering, density-based spatial clustering or versions of them. The statistical properties of these procedures have received a very limited amount of investigation. It is not currently known what statistical cluster method can be estimated by these popular techniques, or by their modifications. We try here to offer an answer to this question for the case of random points data issued from Poisson point process.

To describe our procedure, we begin by defining the function $f_k(\cdot)$ for all $k \geq 1$, for instance, by

$$f_k(s_{T_k}) := (s_{T_k})^l - \log \left( \frac{s_{T_k} \pi}{4(s_{T_k} - 1)!} \right) - (s_{T_k} - 1) \log (\alpha_k^*(s_{T_k})) + \alpha_k^*(s_{T_k}),$$

where $l \in (0, 1)$. This gives $a^*_k(s_{T_k}) = \sqrt{\frac{s_{T_k} \pi}{4\pi s_{T_k}}}$. and, with this particular choice, the threshold $a^*_k$ is not affected by the intrinsic parameters of the network $(v_r, d_r, A_f, \sigma)$ and the parameters of the discretization $(\gamma_k, L'_k)$. The clustering algorithm has three steps, and the main step 2 produces an estimator of one cluster from which we derive the estimated members (active and inactive) of this cluster. The three steps of the procedure are:

(i) Start with an arbitrary starting random point $x_{T_k}^i$ where $i \sim U_{\{1, \ldots, s_{T_k}\}}$.

(ii) Compute $\rho(x_{T_k}^i)$ and this point’s $a^*_k$-neighborhood is retrieved, and if it contains at least one active point, except $x_{T_k}^i$ itself, a cluster is started. Otherwise, the point is labeled as isolated. Note that we test if a point is active or not thanks to one realization of Bernoulli($p$). Note also that this point might later be found in a sufficiently sized $a^*_k$-neighborhood of a different point and hence be made part of a cluster. If a point is found to be a dense part of a cluster, its $a^*_k$-neighborhood is also part of that cluster. Hence, all points that are found within the $a^*_k$-neighborhood are added, as is their own $a^*_k$-neighborhood when they are also dense. This process continues until the cluster is completely found.

(iii) A new unvisited point is retrieved and processed, leading to the discovery of a further cluster or isolated point.
The construction of an accurate function $f_k(\cdot)$ is a crucial step for guaranteeing the statistical optimality of the clustering procedure and for which the results of Theorems 1 and 2 hold. Estimating $p$ (if it is unknown) before estimating the partition itself is a non-trivial task, and needs to be done with extreme care. The required inputs for Step 2 of the procedure are: (i) $s_{T_k}$, the current size of the network; (ii) $p$, the probability to be active in the network. Hence, this step is done at no additional accuracy cost. Remark that, unlike a K-means procedure, the $a^*_T$-neighborhood procedure does not require the number of groups, which need an approach for selecting it in a data adaptive fashion.

Figure 2 shows a cluster analysis of a network at some time. The size of this network at $10^5$ updates is 14102 members. For the cluster analysis shown in Figure 2(b), the function (4.1) is used with $l = \frac{\log((a_1)^2 p \pi r_1^2)}{\log(k \pi r_1^2)}$ and the threshold $a^*_T$ is computed with $p = 1$ (all members are active) and $a_T = 0.1$. These gave $l = 0.6378047$. The number of isolated members found by the $a^*_T$-neighborhood clustering procedure is 71 and the number of groups is 54. To compare this clustering result, we plot in Figure 2(c) a K-means procedure applied to the same network state by calibrating the number of groups to 54 (to assess the difference between the two procedures at least by visual inspection). As seen, the clustering obtained now does not detect isolated members. As known, the strategy of K-means is not based on the point’s neighborhood and this has important repercussions on the analysis and detection of isolated points in cluster estimates. The analysis of these geometric networks is non-standard, and needs to be done with care, as illustrated by the proof of our Theorem 1. Moreover, in contrast to the $a^*_T$-neighborhood procedure tailored to our spatial-temporal underlying model, K-means and spectral methods for this kind of models need to be corrected in a non-trivial fashion.

Now, we consider another configuration when we increase the dispersion ($\uparrow \sigma$) in order that the members of the network tend to occupy almost all the space $[0, 1]^2$. Figure 3 shows a
cluster analysis of a network with high dispersion ($\sigma = 0.15$). A cluster analysis using the $a_t^*$-neighborhood procedure (with again $p = 1$) has found 3 isolated members and 3 groups (one very large and two very small), here $s_{T_k} = 10125$ members. We report the clustering result obtained from the K-means procedure to highlight again that this type of clustering is not perfect for our kind of network modeling. Finally, we increase a little more the dispersion to $\sigma = 0.21$ and the invited rate to $v_r = 4$ in order to strengthen the occupation of the space by the members of the network. We conclude by observing that in Figure 4, the isolated members have disappeared and the network forms one connected group which confirms, even numerically, our previous conjecture on the connectivity of the network ($\mathbb{P}(r_{T_k} \in \mathcal{K}) \rightarrow e^{-\beta_k} = 1$ as $s_{T_k} \rightarrow \infty$).

5 Discussion

In this framework, a geometric and dynamic social network constructed from Poisson point measures is investigated by the distribution of isolated members in the network. We presume a social network is composed of particles in interaction in continuous time represented by a set of points over the unit square. We assume a Bernoulli probability on the activity state of each point in the network. We prove that if all points have the same radius of affinity $a_t^* = \sqrt{\frac{\lambda s_t^3}{p n s_t}}$ for $l \in (0, 1)$, the number of isolated points is asymptotically Poisson with mean $\beta_k$ where $\beta_k \sim 0$ (at each time $T_k$ for $k \geq 1$) as the current size $s_{T_k}$ of the network tends to infinity. This offers a natural threshold for the construction of an $a_t^*$-neighborhood procedure tailored to the dynamic clustering of the network adaptively from the data. The question whether vanishment of isolated members almost surely ensures connectivity of the network or not remains open.
Figure 4: Figures shown a state of the network generated by Algorithm 1 with parameters $v_r = 3.5$, $d_r = 2$, $A_t = 2$, $a_t = 0.1$ and $\sigma = 0.21$ and it’s cluster analysis by the $a_t^2$-neighborhood procedure.

6 Proofs

We split the proof of Theorem 1 into several lemmas. Before presenting the following statements, we partition the unit square $[0, 1]^2$ into four regions $D_1, \ldots, D_4$ as explained in Figure 5. From this partition, it is easy to see that for any $x$ in the torus we have $\pi a_t^2 / 4 \leq \text{vol}(x) \leq \pi a_t^2$ (in particular $\text{vol}(x) = \pi a_t^2 / 4$ when $x$ is at the corners and $\text{vol}(x) = \pi a_t^2$ for all $x \in D_1$).

In the following lemma, we shall study a lower bound on $\text{vol}(x)$ for all $x \in D_2$.

**Lemma 4** Under the previous partition of the torus $[0, 1]^2$ given in Figure 5 then, for all $x \in D_2$, we have

$$\text{vol}(x) \geq \frac{\pi a_t^2}{2} + a_t \left( \frac{1}{2} - \eta \|x\| \right), \quad \text{with} \quad \eta \geq \frac{1}{\sqrt{1/2 - 2a_t^2}} - 1,$$

where $0 < a_t \leq 1/2$.

**Proof** Without loss of generality, let consider two points $x, y \in [0, 1]^2$ such that $y$ be a point in $\partial D$ (the boundary of the torus) with coordinates $(0.5, 0)$ or $(1, 0.5)$ or $(0.5, 1)$ or $(0, 0.5)$. Straightforwardly, at these fourth coordinates the distance between the circle $C_2$ and the boundary of the torus achieves its minimum. It is sufficient to prove the lemma for one of these coordinates. The point $x$ be in $\partial C_2$ together with $\|y - x\| = 1/2 - \|x - \emptyset\|$ and $ab$ be the diameter of $B(x, a_t)$ perpendicular to $xy$ (see Figure 6).

Thus, for all $x \in D_2$, we can write

$$\text{vol}(x) \geq \frac{\pi a_t^2}{2} + a_t \|y - x\| = \frac{\pi a_t^2}{2} + a_t \left( \frac{1}{2} - \|x - \emptyset\| \right).$$

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Figure 5: The different levels of partitioning of the torus \([0, 1]^2\) are shown here with three circles \(C_1\) (–), \(C_2\) (· · ·) and \(C_3\) (- -). The torus is subdivided into 4 regions; \(D_1\) (given by the disk of radius \(1/2 - a_\ell\) centred at \(O = (1/2, 1/2)\)), \(D_2\) (given by the annulus of radii \(1/2 - a_\ell\) and \(\sqrt{1/4 - a_\ell^2}\) centered at \(O\)), \(D_3\) (given by the annulus of radii \(\sqrt{1/4 - a_\ell^2}\) and \(1/2\) centered at \(O\)) and \(D_4\) (given by the rest of the torus).

Now, for all \(x \in \partial C_2\) and from Figure 5 we have

\[
\inf_{x \in \partial C_2} \|x\| = \frac{\sqrt{2}}{2} - \sqrt{1/4 - a_\ell^2} = \frac{\sqrt{2}}{2} - \|x - O\| \leq \|x\|.
\]

One may easily find an upper bound for \(\|x - O\|\) in function of \(\|x\|\) by assuming that exist \(\eta \in (0, \infty)\) such that \(\|x - O\| \leq \eta \inf_{x \in \partial C_2} \|x\|\) and by easy calculation we find

\[
\|x - O\| \leq \eta \left(\frac{\sqrt{2}}{2} - \|x - O\|\right) \iff \sqrt{1/4 - a_\ell^2} \leq \eta \left(\frac{\sqrt{2}}{2} - \sqrt{1/4 - a_\ell^2}\right)
\]

\[
\iff \eta \geq \frac{1}{\sqrt{1/2 - 2a_\ell^2} - 1}.
\]

Hence, we complete the proof by using \(\|x - O\| \leq \eta \inf_{x \in \partial C_2} \|x\| \leq \eta \|x\|\) in (6.2) which leads to (6.1).

We may now formulate a lower bound on the volume of more than one member inside the network, for which the proof follows the same geometric spirit than Lemma 4.

**Lemma 5** Admit the partition of the torus \([0, 1]^2\) given in Figure 5. For any sequence \((x^1, \ldots, x^\kappa)\) ∈ \([0, 1]^{2\kappa}\) where \(\kappa \geq 2\) and such that \(x^1\) has the largest norm with \(\|x^i - x^j\| \leq 2a_\ell\) if and only if
Figure 6: A partial configuration of the torus boundary and the ball $B(x, a_\ell)$ that contains the point $y = (1, 0.5)$ and centered at $x = (0.5 + \sqrt{1/4 - a_\ell^2}, 0.5)$. The volume $\text{vol}(x)$ is higher than the volume of the half of $B(x, a_\ell)$ plus the area of the triangle $aby$.

\[ |i - j| \leq 1, \text{ we have} \]
\[ \text{vol}(x^1, \ldots, x^\kappa) \geq \text{vol}(x^1) + \frac{\pi a_\ell}{16} \sum_{i=1}^{\kappa-1} \Delta x^i, \quad (6.3) \]

where $\Delta x^i = \|x^{i+1} - x^i\|$.

**Proof** We show the result (6.5) for $\kappa = 2$ at first. Then, we prove by induction that the result holds for any $\kappa \geq 2$. For simplicity, let fix $\kappa = 2$ and consider the function $\varphi(\Delta x^1) = \text{vol}(B(x^2, a_\ell) \setminus B(x^1, a_\ell))$. Our aim is to prove that $\varphi(\Delta x^1) \geq \frac{\pi a_\ell}{2} \Delta x^1$ for any $x^1, x^2 \in [0, 1]^2$ with $\|x^1\| \geq \|x^2\|$. To do this, let $y_1 y_2$ be the common chord of $\partial B(x^1, a_\ell)$ and $\partial B(x^2, a_\ell)$. Let also $z_1 z_2$ be another chord of $\partial B(x^2, a_\ell)$ that is parallel to $y_1 y_2$ and has the same length as $y_1 y_2$ as explained in Figure 7.

From Figure 7, it’s clear that $\varphi(\Delta x^1)$ is equal to the volume of the portion of $B(x^2, a_\ell)$ between the chords $y_1 y_2$ and $z_1 z_2$ and immediately we deduce that the second derivative $\varphi''(\Delta x^1) \leq 0$ (since $\varphi'(\Delta x^1) = \|y_1 - y_2\|$ which is decreasing). Hence, the function $\varphi$ is concave with $\varphi(0) = 0$ and $\varphi(2a_\ell) = \pi a_\ell^2$ which enables us to write

\[ \forall \Delta x, \Delta x' \in [0, 2a_\ell] \quad \text{and} \quad \forall \ell^0 \in [0, 1], \quad \varphi(\ell^0 \Delta x + (1 - \ell^0)\Delta x') \geq \ell^0 \varphi(\Delta x) + (1 - \ell^0)\varphi(\Delta x'), \]

and by taking $\Delta x = 0$ and $\Delta x' = 2a_\ell$,

\[ \varphi((1 - \ell^0)2a_\ell) \geq (1 - \ell^0)\pi a_\ell^2, \quad (6.4) \]
Figure 7: Three examples of configuration for the chords $y_1y_2$ and $z_1z_2$ (of two intersecting balls $B(x_1, a_\ell)$ and $B(x_2, a_\ell)$), where we see that $\|y_1 - y_2\|$ is decreasing over $[0, 2a_\ell]$.

where $\varphi(\Delta x^1) \geq \frac{\pi a_\ell}{2} \Delta x^1$ holds by choosing $\ell^0$ in (6.4) such that $\Delta x^1 = (1 - \ell^0)2a_\ell$. We discuss now the lower bound of $\text{vol}(x_1, x_2) - \text{vol}(x_1)$ following the position of $x_1$ in the partition of $[0, 1]^2$. Indeed, if $x_1 \in D_1$, then the two balls $B(x_1, a_\ell)$ and $B(x_2, a_\ell)$ are completely contained in $[0, 1]^2$ which enables us to obtain

$$
\text{vol}(x_1, x_2) - \text{vol}(x_1) = \text{vol}(B(x_2, a_\ell) \setminus B(x_1, a_\ell)) = \varphi(\Delta x^1) \geq \frac{\pi a_\ell}{2} \Delta x^1,
$$

and the lemma is proved for $\kappa = 2$ and if $x_1 \in D_1$. If $x_1 \notin D_1$, we remark that for the same distance $\Delta x^1$, the value of $\text{vol}(x_1, x_2) - \text{vol}(x_1)$ achieves its minimum if $x_1, x_2 \in \partial D$ together with $x_2$ is at the corner. It suffice to show the lemma in this case. For the ease of exposition, let consider again the previous chords such that $y_2 \in [0, 1]^2$ as shown in Figure 8.

From Figure 8, it appeared that $\text{vol}(x_1, x^2) - \text{vol}(x^1) = \frac{\varphi(\Delta x^1)}{4} \geq \frac{\pi a_\ell}{8} \Delta x^1$ and the lemma follows also in this case. Now, let consider $\kappa = 3$ and write

$$
\text{vol}(x_1, x_2, x_3) \geq \text{vol}(x_1) + \text{vol}(x_3) \geq \text{vol}(x_1) + \frac{\pi a_\ell^2}{4} = \text{vol}(x^1) + \frac{\pi a_\ell}{16} 4a_\ell \geq \text{vol}(x^1) + \frac{\pi a_\ell}{16} \sum_{i=1}^{2} \Delta x^i,
$$

since $\Delta x^i \leq 2a_\ell$. Finally, we generalize for $\kappa > 3$ by induction as follows

$$
\text{vol}(x^1, \ldots, x^\kappa) \geq \text{vol}(x^1, \ldots, x^{\kappa-2}) + \text{vol}(x^{\kappa})
\geq \text{vol}(x^1) + \left( \frac{\pi a_\ell}{16} \sum_{i=1}^{\kappa-3} \Delta x^i \right) + \frac{\pi a_\ell^2}{4}
\geq \text{vol}(x^1) + \frac{\pi a_\ell}{16} \sum_{i=1}^{\kappa-1} \Delta x^i,
$$
Figure 8: An example of configuration for the chords $y_1y_2$ and $z_1z_2$ (of two intersecting balls $B(x^1,a_f)$ and $B(x^2,a_f)$), where $x^2$ is at the corner of $[0,1]^2$.

which completes the proof. □

We prove now a lower bound on the volume of more than one member inside the sub-network $S_{2a_f}(x^1,\ldots,x^\kappa)$ over $(x^1,\ldots,x^\kappa) \in C_{\kappa,1}$ in which two members are connected (by affinity) if and only if their distance is at most $2a_f$.

**Lemma 6** For any sequence $(x^1,\ldots,x^\kappa) \in C_{\kappa,1}$ where $\kappa \geq 2$ and such that $x^1$ being the one of the largest norm among $x^1,\ldots,x^\kappa$, we have

$$\vol(x^1,\ldots,x^\kappa) \geq \vol(x^1) + \frac{\pi a_f}{16} \sup_{2 \leq i \leq \kappa} \|x^i - x^1\|. \quad (6.5)$$

**Proof** For the easy of the proof, we assume that $\|x^\kappa - x^1\| = \sup_{2 \leq i \leq \kappa} \|x^i - x^1\|$ and let $\mathcal{P}$ be a minimum-hop path between $x^1$ and $x^\kappa$ in $S_{2a_f}(x^1,\ldots,x^\kappa)$ with $\Delta^x x$ be the total length of $\mathcal{P}$. Thus, every pair of members in $\mathcal{P}$ that are not adjacent members in $\mathcal{P}$ are distant by more than $2a_f$ and by application of Lemma 5 to the members in the path $\mathcal{P}$ we find that

$$\vol(\{x^i : x^i \in \mathcal{P}\}) \geq \vol(x^1) + \frac{\pi a_f}{16} \Delta^x x.$$ 

We conclude by remarking that $\vol(x^1,\ldots,x^\kappa) \geq \vol(\{x^i : x^i \in \mathcal{P}\})$ and $\Delta^x x \geq \|x^\kappa - x^1\|$. This completes the proof. □

In proving (3.14), we shall use the following lemma.
Lemma 7 Assume that hypothesis \( H_1 \) and \( H_2 \) are satisfied. Let

\[
a^*_k(s_{Tk}) = \left( \log \left( \frac{s_{Tk} \pi}{4(s_{Tk} - 1)} \right) + f_k(s_{Tk}) + (s_{Tk} - 1) \log(\alpha^*_k(s_{Tk})) - \alpha^*_k(s_{Tk}) \right)^{1/2},
\]

for any \( p \in (0, 1) \) and where \( \alpha^*_k \) is given by (3.8). Then, we have

\[
s_{Tk} \left( \frac{(\alpha^*_k(s_{Tk}))^{s_{Tk} - 1}}{(s_{Tk} - 1)!} e^{-\alpha^*_k(s_{Tk})} \right) \int_{[0,1]^2} e^{-s_{Tk} \text{vol}(x)} dx = \lim_{s_{Tk} \to \infty} f_k(s_{Tk})
\]

as \( s_{Tk} \to \infty \).

Proof We shall proceed by approximating the integral in the left-hand side using the four regions \( D_1, \ldots, D_4 \) of the unit square \([0,1]^2\). If \( x \in D_1 \), we know that \( \text{vol}(x) = \pi(a^*_t)^2 \) and it’s straightforward that

\[
s_{Tk} \left( \frac{(\alpha^*_k(s_{Tk}))^{s_{Tk} - 1}}{(s_{Tk} - 1)!} e^{-\alpha^*_k(s_{Tk})} \right) \int_{D_1} e^{-s_{Tk} \text{vol}(x)} dx = s_{Tk} e^{-\log(\pi/a^*_t) - f_k(s_{Tk})} \int_{D_1} dx
\]

\[
= \frac{4}{\pi} e^{-f_k(s_{Tk})} \pi \left( \frac{1}{2} - a^*_t \right)^2
\]

\[
= e^{-f_k(s_{Tk})} \left( 1 - 2a^*_t \right)^2
\]

\[
\xrightarrow{s_{Tk} \to \infty} e^{-f_k(s_{Tk})},
\]

where we note that for sufficiently large \( s_{Tk} \) we have \( a^*_t \to 0 \) (by hypothesis \( H_1 \)). Therefore, if \( x \in D_3 \), we know that \( \text{vol}(x) \geq \frac{1}{2} \pi(a^*_t)^2 \) and by using the upper bound

\[
\forall k \geq 1, \quad \left( \frac{(\alpha^*_k(s_{Tk}))^{s_{Tk} - 1}}{(s_{Tk} - 1)!} e^{-\alpha^*_k(s_{Tk})} \right) \leq 1,
\]

we find

\[
s_{Tk} \left( \frac{(\alpha^*_k(s_{Tk}))^{s_{Tk} - 1}}{(s_{Tk} - 1)!} e^{-\alpha^*_k(s_{Tk})} \right) \int_{D_3} e^{-s_{Tk} \text{vol}(x)} dx \leq s_{Tk} e^{-\frac{1}{2}s_{Tk} \pi \alpha^*_t} \int_{D_3} dx
\]

\[
= s_{Tk} \pi(a^*_t)^2 e^{-\frac{1}{2}s_{Tk} \pi \alpha^*_t} e^{-\frac{1}{2}s_{Tk} \pi \alpha^*_t} \xrightarrow{s_{Tk} \to \infty} 0.
\]

The same think happens if \( x \in D_4 \) where \( \text{vol}(x) \geq \frac{1}{4} \pi(a^*_t)^2 \) but we need more notation. Let consider the points \( A = (0,0), B = (\frac{1}{2}, 0), C = (0, \frac{1}{2}) \) and the triangle \( \mathcal{T} \) formed by \( ABC \). From Figure 5 it’s clear that \( \text{vol}(\mathcal{D}_4) \leq 4 \text{vol}(\mathcal{T}) = \frac{4}{3} \) and we find

\[
s_{Tk} \left( \frac{(\alpha^*_k(s_{Tk}))^{s_{Tk} - 1}}{(s_{Tk} - 1)!} e^{-\alpha^*_k(s_{Tk})} \right) \int_{D_4} e^{-s_{Tk} \text{vol}(x)} dx \leq s_{Tk} e^{-\frac{1}{2}s_{Tk} \pi \alpha^*_t} \int_{D_4} dx
\]

\[
\lesssim \frac{s_{Tk}}{2} e^{-\frac{1}{2}s_{Tk} \pi \alpha^*_t} e^{-\frac{1}{2}s_{Tk} \pi \alpha^*_t} \xrightarrow{s_{Tk} \to \infty} 0.
\]
Let us now turn out to the region $x \in D_2$. Hence, by application of Lemma 4 and the polar coordinate system we have

$$
\begin{align*}
&\int_{D_2} e^{-sT_k \text{vol}(x)} \, dx \\
&\leq sT_k e^{-\frac{1}{2} sT_k \text{val}(a^*_t)^2} \int_{D_2} e^{-sT_k \text{vol}(\frac{1}{2} - \eta \|x\|)} \, dx \\
&= 2\pi sT_k e^{-\frac{1}{2} sT_k \text{val}(a^*_t)^2} \int_{\frac{1}{2} - a^*_t}^{\frac{1}{2}} \rho_x e^{-sT_k \text{val}(\frac{1}{2} - \eta \rho_x)} \, d\rho_x \\
&\leq 2\sqrt{2}\pi sT_k e^{-\frac{1}{2} sT_k \text{val}(a^*_t)^2} \int_{\frac{1}{2} - \eta}^{\frac{1}{2}} e^{-sT_k \text{val}(\frac{1}{2} - \eta)} \, dz \\
&= \frac{2\sqrt{2}\pi}{\eta a^*_t} e^{-\frac{1}{2} sT_k \text{val}(a^*_t)^2} \left(1 - e^{-sT_k \text{val}(a^*_t)^2}\right) \\
&\leq \frac{2\sqrt{2}\pi}{p} \left(\frac{1}{2} - 2(a^*_t)^2\right)^2 - 1 e^{-\frac{1}{2} sT_k \text{val}(a^*_t)^2} = \frac{O(1)}{\sqrt{\psi_k(sT_k)}} = o(1),
\end{align*}
$$

by our choice of $f_k$, this shows $\psi_k$ tends to infinity, completing the proof. \hfill \Box

**Lemma 8** Assume that hypothesis $H_1$ and $H_2$ are satisfied. Let

$$
a^*_t(sT_k) = \left(\frac{\log\left(\frac{sT_k \pi}{a^*_k(sT_k)}\right) + f_k(sT_k) + (sT_k - 1) \log(a^*_k(sT_k)) - a^*_k(sT_k)}{\pi psT_k}\right)^{1/2},
$$

for any $p \in (0, 1)$ and where $\alpha^*_k$ is given by (3.8) and satisfying (3.11). Then, for any $\kappa \geq 2$ and $(x^1, \ldots, x^\kappa) \in C_{\kappa n}$ with $1 \leq n < \kappa$, we have

$$
(sT_k)^\kappa \left(\frac{(\alpha^*_k(sT_k))^{sT_k - \kappa}}{(sT_k - \kappa)!} e^{-\alpha^*_k(sT_k)}\right) \int_{C_{\kappa n}} e^{-sT_k \text{vol}(x^1, \ldots, x^\kappa)} \, dx^1 \cdots dx^\kappa \to 0, \quad \text{as } sT_k \to \infty.
$$

**Proof** We divide the proof into two steps.
Step 1. Let us first show the lemma for \( n = 1 \). For simplicity and without loss of generality, let consider the subset \( C_{k}^{0} \) denoting the set of \( (x^{1}, \ldots, x^{\kappa}) \) satisfying that \( x^{1} \) being the one of the largest norm among \( x^{1}, \ldots, x^{\kappa} \) and \( x^{\kappa} \) being the one with longest distance from \( x^{1} \) among \( x^{1}, \ldots, x^{\kappa} \) which enables us to write

\[
(s_{T})^{\kappa} \left( \frac{(\alpha_{k}^{s}(s_{T}))^{s_{T} - \kappa}}{(s_{T} - \kappa)!} \right) \int_{C_{k}^{0}} e^{-s_{T}^{\kappa} \text{vol}(x^{1}, \ldots, x^{\kappa})} dx^{1} \ldots dx^{\kappa}
\]

\[
\leq \kappa(\kappa - 1)(s_{T})^{\kappa} \left( \frac{(\alpha_{k}^{s}(s_{T}))^{s_{T} - \kappa}}{(s_{T} - \kappa)!} \right) \int_{C_{k}^{0}} e^{-s_{T}^{\kappa} \text{vol}(x^{1}, \ldots, x^{\kappa})} dx^{1} \ldots dx^{\kappa}.
\]

We claim next that (6.5) (in Lemma 6) holds also for \( (x^{1}, \ldots, x^{\kappa}) \in C_{k}^{0} \) with \( \frac{\pi}{\theta} \) replaced by some constant \( \tilde{C} \), that is,

\[
\text{vol}(x^{1}, \ldots, x^{\kappa}) \geq \text{vol}(x^{1}) + \tilde{C}a_{1}^{s} ||x^{\kappa} - x^{1}||, \quad (6.6)
\]

with \( x^{\kappa} \in B(x^{1}, 2(\kappa - 1)a_{1}^{s}) \) and, for \( i \in \{2, \ldots, \kappa - 1\}, x^{i} \in B(x^{1}, ||x^{\kappa} - x^{1}||) \). Indeed, by the constraint (3.11) imposed on \( \alpha^{s} \) and the fact that

\[
\forall k \geq 1, \quad \left( \frac{(\alpha_{k}^{s}(s_{T}))^{s_{T} - 1}}{(s_{T} - 1)!} \right) e^{-\left( \frac{\alpha_{k}^{s}(s_{T})}{s_{T} - 1} \right)}^{(\kappa - 1)} \leq 1,
\]

we obtain

\[
(s_{T})^{\kappa} \left( \frac{(\alpha_{k}^{s}(s_{T}))^{s_{T} - \kappa}}{(s_{T} - \kappa)!} \right) \int_{C_{k}^{0}} e^{-s_{T}^{\kappa} \text{vol}(x^{1}, \ldots, x^{\kappa})} dx^{1} \ldots dx^{\kappa}
\]

\[
\leq (s_{T})^{\kappa} \left( \frac{(\alpha_{k}^{s}(s_{T}))^{s_{T} - \kappa}}{(s_{T} - \kappa)!} \right) \int_{C_{k}^{0}} e^{-s_{T}^{\kappa} \text{vol}(x^{1}, \ldots, x^{\kappa}) + \tilde{C}a_{1}^{s} ||x^{\kappa} - x^{1}||} dx^{1} \ldots dx^{\kappa}
\]

\[
\leq (s_{T})^{\kappa} \left( \frac{(\alpha_{k}^{s}(s_{T}))^{s_{T} - \kappa}}{(s_{T} - \kappa)!} \right) \int_{[0,1]} e^{-s_{T}^{\kappa} \text{vol}(x^{1})} dx^{1}
\]

\[
\times \int_{B(x^{1}, 2(\kappa - 1)a_{1}^{s})} e^{-s_{T}^{\kappa} \tilde{C}a_{1}^{s} ||x^{\kappa} - x^{1}||} dx^{1} \prod_{i=2}^{\kappa - 1} \int_{B(x^{1}, ||x^{\kappa} - x^{1}||)} dx^{i}
\]

\[
= 2(\pi s_{T})^{\kappa - 1} \left( \frac{(\alpha_{k}^{s}(s_{T}))^{s_{T} - \kappa}}{(s_{T} - \kappa)!} \right) \int_{[0,1]} e^{-s_{T}^{\kappa} \text{vol}(x^{1})} dx^{1}
\]

\[
\times \left( \int_{0}^{2(\kappa - 1)a_{1}^{s}} (\rho_{x})^{2\kappa - 3} e^{-s_{T}^{\kappa} \tilde{C}a_{1}^{s} \rho_{x}} d\rho_{x} \right)
\]

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\[
< 2(\pi s_{T_k})^{\kappa - 1} \left( s_{T_k} \frac{(\alpha_k^*(s_{T_k}))^{s_{T_k}}}{(s_{T_k} - 1)!} e^{-\alpha_k^*(s_{T_k})} \right) \int_{[0,1]^2} e^{-s_{T_k} \text{vol}(x)} dx^1 \]
\[
\times \left( \int_0^\infty (\rho_x)^{2\kappa - 3} e^{-s_{T_k} p\tilde{C}_a^* \rho_x} d\rho_x \right) \]
\[
= \frac{\Gamma(2\kappa - 2)}{(s_{T_k} p\tilde{C}_a^*)^{2\kappa - 2}} \left( s_{T_k} \frac{(\alpha_k^*(s_{T_k}))^{s_{T_k}}}{(s_{T_k} - 1)!} e^{-\alpha_k^*(s_{T_k})} \right) \int_{[0,1]^2} e^{-s_{T_k} \text{vol}(x)} dx^1 \]
\[
= O(1) \frac{(\psi_k(s_{T_k}))^{\kappa - 1} \left( (\alpha_k^*(s_{T_k}))^{s_{T_k} - \kappa} \right) \int_{\bar{D}^\kappa(\Pi_\kappa)} e^{-s_{T_k} \text{vol}(x^1, \ldots, x^\kappa)} dx^1 \cdots dx^\kappa \rightarrow 0, \quad \text{as } s_{T_k} \rightarrow \infty, \]
by application of Lemma 7 and where \( \Gamma(\kappa) = \int_0^\infty x^{\kappa - 1} e^{-x} dx \) denoting the gamma function.

**Step 2.** We show now that the same result holds for \((x^1, \ldots, x^\kappa) \in \mathcal{C}_{\kappa n}\) for any \(2 \leq n < \kappa\). For any random \(n\)-partition

\[
\Pi_\kappa = \{P_1, \ldots, P_n\} \quad \text{of the subset } [\kappa] := \{1, \ldots, \kappa\},
\]
where each component \(P_j, j = 1, \ldots, n\), is of cardinal \(|P_j|\) and let denote by \(\bar{D}^\kappa(\Pi_\kappa)\) the set of \((x^1, \ldots, x^\kappa) \in \mathcal{D}^\kappa\) such that the points \(\{x^i : i \in P_j\}\) formed a connected component of \(S_{2a_f}(x^1, \ldots, x^\kappa)\). Hence,

\[
\mathcal{C}_{\kappa n} = \bigcup_{\text{all } n\text{–partitions } \Pi_\kappa} \bar{D}^\kappa(\Pi_\kappa),
\]
and it suffice to prove that for any \(n\)-partition \(\Pi_\kappa\)

\[
(s_{T_k})^\kappa \left( \frac{(\alpha_k^*(s_{T_k}))^{s_{T_k} - \kappa}}{(s_{T_k} - \kappa)!} e^{-\alpha_k^*(s_{T_k})} \right) \int_{\bar{D}^\kappa(\Pi_\kappa)} e^{-s_{T_k} \text{vol}(x^1, \ldots, x^\kappa)} dx^1 \cdots dx^\kappa \rightarrow 0, \quad \text{as } s_{T_k} \rightarrow \infty.
\]
Second, without loss of generality, let now fix one arbitrary \(n\)-partition \(\Pi_\kappa\) and observe that for any \((x^1, \ldots, x^\kappa) \in \bar{D}^\kappa(\Pi_\kappa)\), we have

\[
\bar{D}^\kappa(\Pi_\kappa) \subseteq \prod_{j=1}^n \mathcal{C}_{|P_j|} \quad \text{and} \quad \text{vol}(x^1, \ldots, x^\kappa) = \sum_{j=1}^n \text{vol}(\{x^i : i \in P_j\}). \quad (6.7)
\]
It follows from (6.7) that
\[
\begin{align*}
(s_T)^\kappa \left( \frac{(\alpha_k^*(s_{T_k})))^{s_{T_k} - \kappa}}{(s_{T_k} - \kappa)!} \right) & \int_{\mathcal{D}^\kappa(\Pi_k)} e^{-s_{T_k}\text{pvol}(x^1, \ldots, x^\kappa)} dx^1 \cdots dx^\kappa \\
& = \left( \frac{(\alpha_k^*(s_{T_k})))^{s_{T_k} - 1}}{(s_{T_k} - 1)!} \right) \int_{\mathcal{D}^\kappa(\Pi_k)} e^{-s_{T_k}\text{pvol}(x^1, \ldots, x^\kappa)} dx^1 \cdots dx^\kappa \\
& \leq \left( \frac{(\alpha_k^*(s_{T_k})))^{s_{T_k} - 1}}{(s_{T_k} - 1)!} \right) \prod_{j=1}^{n} \left( \int_{\mathcal{C}_{|P_j|}} e^{-s_{T_k}\text{pvol}(x^1, \ldots, x^\kappa)} dx^1 \cdots dx^\kappa \right),
\end{align*}
\]
which tends to zero as shown in Step 1, completing the proof.

Next we study the limit in $\mathcal{C}_{\kappa\kappa}$.

**Lemma 9** Assume that hypothesis $H_1$ and $H_2$ are satisfied. Let
\[
a_k^*(s_{T_k}) = \left( \log\left( \frac{s_{T_k}^\kappa}{(s_{T_k} - 1)!} \right) + f_k(s_{T_k}) + (s_{T_k} - 1) \log(\alpha_k^*(s_{T_k}))) - \alpha_k^*(s_{T_k}) \right)^{1/2},
\]
for any $p \in (0, 1)$ and where $\alpha_k^*$ is given by (3.8) and satisfying (3.11). Then, for any $\kappa \geq 2$ and $(x^1, \ldots, x^\kappa) \in \mathcal{C}_{\kappa\kappa}$, we have
\[
(s_T)^\kappa \left( \frac{(\alpha_k^*(s_{T_k})))^{s_{T_k} - \kappa}}{(s_{T_k} - \kappa)!} \right) \int_{\mathcal{C}_{\kappa\kappa}} e^{-s_{T_k}\text{pvol}(x^1, \ldots, x^\kappa)} dx^1 \cdots dx^\kappa \xrightarrow{s_{T_k} \to \infty} e^{-\kappa \lim_{s_{T_k} \to \infty} f_k(s_{T_k})}.
\]
Proof. Recall that for any \((x^1, \ldots, x^\kappa) \in C_\kappa\) we observe \(\text{vol}(x^1, \ldots, x^\kappa) = \sum_{i=1}^\kappa \text{vol}(x^i)\). This enables to write
\[
(s_T_k)^\kappa \left( \frac{(\alpha_k'(s_T_k))^{s_T_k - \kappa}}{(s_T_k - \kappa)!} e^{-\alpha_k'(s_T_k)} \right) \int_{C_\kappa} e^{-s_T_k p \text{vol}(x^1, \ldots, x^\kappa)} \, dx^1 \cdots dx^\kappa
= (s_T_k)^\kappa \left( \frac{(\alpha_k'(s_T_k))^{s_T_k - \kappa}}{(s_T_k - \kappa)!} e^{-\alpha_k'(s_T_k)} \right) \left\{ \int_{\mathcal{D}^*} e^{-s_T_k p \sum_{i=1}^\kappa \text{vol}(x^i)} \, dx^1 \cdots dx^\kappa \right\}
- \int_{\mathcal{D}^* \setminus C_\kappa} e^{-s_T_k p \sum_{i=1}^\kappa \text{vol}(x^i)} \, dx^1 \cdots dx^\kappa
= \left\{ \prod_{i=1}^\kappa \left( \frac{(\alpha_k'(s_T_k))^{s_T_k - 1}}{(s_T_k - 1)!} e^{-\alpha_k'(s_T_k)} \right) \int_{[0,1]^2} e^{-s_T_k p \text{vol}(x^i)} \, dx^i \right\}
- (s_T_k)^\kappa \left( \frac{(\alpha_k'(s_T_k))^{s_T_k - \kappa}}{(s_T_k - \kappa)!} e^{-\alpha_k'(s_T_k)} \right) \int_{\mathcal{D}^* \setminus C_\kappa} e^{-s_T_k p \sum_{i=1}^\kappa \text{vol}(x^i)} \, dx^1 \cdots dx^\kappa \quad (6.8)
\]
\[
\frac{s_T_k \to +\infty}{\longrightarrow} \left( e^{-\lim_{s_T_k \to \infty} f_k(s_T_k)} \right)^\kappa = 0.
\]
There may be some doubt as to why the term \((6.8)\) tends to zero. Let us verify this by observing that for any \((x^1, \ldots, x^\kappa) \in \mathcal{D}^* \setminus C_\kappa\) we have \(\text{vol}(x^1, \ldots, x^\kappa) \leq \sum_{i=1}^\kappa \text{vol}(x^i)\) which enables us to find
\[
(s_T_k)^\kappa \left( \frac{(\alpha_k'(s_T_k))^{s_T_k - \kappa}}{(s_T_k - \kappa)!} e^{-\alpha_k'(s_T_k)} \right) \int_{\mathcal{D}^* \setminus C_\kappa} e^{-s_T_k p \sum_{i=1}^\kappa \text{vol}(x^i)} \, dx^1 \cdots dx^\kappa
\leq (s_T_k)^\kappa \left( \frac{(\alpha_k'(s_T_k))^{s_T_k - \kappa}}{(s_T_k - \kappa)!} e^{-\alpha_k'(s_T_k)} \right) \int_{\mathcal{D}^* \setminus C_\kappa} e^{-s_T_k p \text{vol}(x^1, \ldots, x^\kappa)} \, dx^1 \cdots dx^\kappa
\leq \sum_{n=1}^{\kappa-1} \left\{ (s_T_k)^\kappa \left( \frac{(\alpha_k'(s_T_k))^{s_T_k - \kappa}}{(s_T_k - \kappa)!} e^{-\alpha_k'(s_T_k)} \right) \int_{C_\kappa} e^{-s_T_k p \sum_{i=1}^\kappa \text{vol}(x^i)} \, dx^1 \cdots dx^\kappa \right\},
\]
where it’s straightforward that the sum tends to zero thanks to Lemma \(8\). \(\square\)

A key result needed in the sequel for the proofs of Theorems \(1\) and \(2\) is as follows:

**Lemma 10** Given a sequence \(E_{1k}, E_{2k}, \ldots, E_{s_T_k k}\) of events such that \(E_{ik}\) be the event that the point \(x^i_{T_k}\) is isolated, define \(U_{s_T_k}\) to be the random number of \(E_{ik}\) that hold. If for any set \(\{i_1, \ldots, i_\kappa\}\) it is true that
\[
\mathbb{P} \left( \bigcap_{j=1}^\kappa E_{jk} \right) = \mathbb{P} \left( \bigcap_{j=1}^\kappa E_{i_j k} \right), \quad (6.9)
\]

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and there is a constant $u$ such that for any fixed $\kappa$
\[
(s_{T_k})^{\kappa} \times \mathbb{P}\left( \bigcap_{j=1}^{\kappa} E_{jk} \right)^{s_{T_k} \to +\infty} \to u^\kappa,
\] (6.10)
hence the sequence $U_{s_{T_k}}$ converges in distribution to a Poisson random variable with mean $u$.

We do not claim originality of the Lemma 10 and, in fact, similar result have been proved in Alon & Spencer (2000) using a probabilistic version of Brun’s sieve theorem and the Bonferroni inequalities. Since we have found this particular result in the literature, we not provide a detailed proof.

We now proceed to conclude the proof of Theorem 1.

Proof. For $(x^1_{T_k}, \ldots, x^\kappa_{T_k}) \in C_{\kappa\kappa}$, we proved in Proposition 3 that
\[
\mathbb{P}(E_1 \cap \cdots \cap E_{nk}) = \left( \frac{(\alpha_k^{\star}(s_{T_k}))^{s_{T_k} - \kappa}}{(s_{T_k} - \kappa)!} e^{-(\alpha_k^{\star}(s_{T_k}))} \right) \times \int_{C_{\kappa\kappa}} \left( 1 - p\text{vol}(x^1_{T_k}, \ldots, x^\kappa_{T_k}) \right)^{s_{T_k} - \kappa} dx^1_{T_k} \cdots dx^\kappa_{T_k}.
\]

Or, for $s_{T_k}$ sufficiently large we find
\[
\left( 1 - p\text{vol}(x^1_{T_k}, \ldots, x^\kappa_{T_k}) \right)^{s_{T_k} - \kappa} \leq \frac{\left( 1 - p\sum_{i=1}^{\kappa} \text{vol}(x^i_{T_k}) \right)^{s_{T_k}}}{\left( 1 - pk\pi(a^\star_f)^2 \right)^{\kappa}} \leq e^{-s_{T_k}p\text{vol}(x^1_{T_k}, \ldots, x^\kappa_{T_k})}.
\]

Thus, by application of Lemma 9,
\[
(s_{T_k})^{\kappa} \times \mathbb{P}(E_1 \cap \cdots \cap E_{nk})^{s_{T_k} \to +\infty} \left( e^{-s_{T_k} \lim_{s_{T_k} \to \infty} f_k(s_{T_k})} \right)^{\kappa} = (\beta_k)^{\kappa} < \infty,
\]
and immediately the condition (6.10) in Lemma 10 is verified. It is easily seen that the condition (6.9) is also verified (its proof is left to the reader) and hence Theorem 1 follows for $(x^1_{T_k}, \ldots, x^\kappa_{T_k}) \in C_{\kappa\kappa}$ and generally for $(x^1_{T_k}, \ldots, x^\kappa_{T_k}) \in \bar{D}^{\kappa}$ since for $(x^1_{T_k}, \ldots, x^\kappa_{T_k}) \in \bar{D}^{\kappa} \setminus C_{\kappa\kappa}$ and for $s_{T_k}$ sufficiently large
\[
\left( 1 - p\text{vol}(x^1_{T_k}, \ldots, x^\kappa_{T_k}) \right)^{s_{T_k} - \kappa} \leq \frac{\left( 1 - p\text{vol}(x^1_{T_k}, \ldots, x^\kappa_{T_k}) \right)^{s_{T_k}}}{\left( 1 - pk\pi(a^\star_f)^2 \right)^{\kappa}} \leq e^{-s_{T_k}p\text{vol}(x^1_{T_k}, \ldots, x^\kappa_{T_k})}.
\]

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Then, the probability of the event $(E_{1k} \cap E_{2k} \cap \cdots \cap E_{nk})$ times $(s_{T_k})^k$ tends to zero as $s_{T_k} \to \infty$ by Lemma 8 this completes the proof.

Finally, we conclude the proof of Theorem 2.

**Proof** Note that

$$
P(F_{1k} \cap \cdots \cap F_{nk}) = p^k P(E_{1k} \cap \cdots \cap E_{nk}),$$

and Theorem 2 follows using the same arguments as for Theorem 1.

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