Generating cat states with fermionic atoms in a driven optical lattice

Mikhail Mamaev\textsuperscript{1,2} and Ana Maria Rey\textsuperscript{1,2}
\textsuperscript{1}JILA, NIST and Department of Physics, University of Colorado, Boulder, CO 80309, USA
\textsuperscript{2}Center for Theory of Quantum Matter, University of Colorado, Boulder, CO 80309, USA

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We propose a protocol for generating Schrödinger cat states of the Greenberger-Horne-Zeilinger (GHZ) variety using ultracold fermions in 3D optical lattices or optical tweezer arrays. The protocol uses the interplay between laser driving, onsite interactions and external trapping confinement to enforce energetic spin- and position-dependent constraints on the atomic motion. These constraints allow us to transform a local superposition into a cat state through a stepwise protocol that flips one site at a time. The protocol requires no site-resolved drives or spin-dependent potentials, exhibits robustness to slow global laser phase drift, and naturally makes use of the harmonic trap that would normally cause difficulties for entanglement-generating protocols in optical lattices. We also discuss an improved protocol that can compensate for holes in the loadout at the cost of increased generation time. The cat state can immediately be used for quantum enhanced metrology in 3D optical lattice atomic clocks, opening a window to push the sensitivity of state-of-the-art sensors beyond the standard quantum limit.

Introduction. Creating useful entanglement is one of the most important goals in modern quantum research. In recent years, there has been significant effort towards generating multi-body entangled states, which exhibit massive utility for quantum computation, simulation and metrology. In particular, for the latter application of metrology, an $N$-body fully entangled state can yield sensitivity improvement by a factor of $\sqrt{N}$ compared to experiments using unentangled atoms or modes \cite{1}. Such gains in precision are not only relevant for real-world applications such as time-keeping, magnetometry and navigation, but also for fundamental science including searches for dark matter and physics beyond the Standard Model \cite{2}.

While there has been progress on many-body entanglement generation in many fields, one of the most promising platforms is ultracold atoms. A variety of useful entangled states have been proposed and/or experimentally realized with such systems, including spin-squeezed states \cite{3}, W-states \cite{4}, and in particular cat states using trapped ions \cite{5,8} or Rydberg atoms in optical tweezers \cite{9}. However, the difficulty of combining single-site resolution with protocol scalability has limited the fidelity and size of the states that have been realized thus far, especially in systems where they can be directly used for metrological purposes.

In this work, we propose a method for generating $N$-particle spin cat states (also called generalized GHZ states) using ultracold fermionic atoms loaded into a 3D optical lattice. Our protocol uses onsite repulsive interactions, spin-orbit coupled (SOC) laser driving \cite{10,12}, and the harmonic trapping potential naturally generated by the curvature of the laser beams forming the lattice. While we focus on 3D lattices in our description, the setup may also be realized in optical tweezer arrays with an additional AC-Stark shift gradient to emulate the trap. We describe a step-by-step generation of a many-atom superposition by creating an initially-local two-body quantum state, and spatially changing the structure of one of its components while leaving the other component untouched due to energetic constraints.

Despite having site-resolved atomic motion, we do not require site-resolved focused lasers, instead only needing a collective driving laser with the ability to quench its Rabi frequency at various time steps. We also require no spin-dependent lattice potentials or lattice modulation. The drive, trap and interactions lead to energetic constraints that only allow tunneling between one lattice site pair at a time, while all other sites are effectively decoupled. Our protocol is also robust to slow global phase drifts of the drive, because the system adiabatically follows the drive’s single-particle eigenstates throughout the evolution. After state generation, we describe a method to observe the cat-enhanced phase sensitivity without needing many-body measurements such as parity, by instead implementing an effective reversal of the cat generation protocol after applying the small perturbation to be sensed. Finally, we give an augmentation to the protocol that compensates for holes in the loadout. All these features to...
gether with scalability make our proposal promising for massive entanglement generation and sensitivity improvements in state-of-the-art sensors.

**Model.** We consider a laser-driven 3D optical lattice populated by fermionic atoms in the lowest motional band, with two internal spin-like states $\sigma \in \{g, e\}$. We assume strong transverse confinement, restricting tunneling to an array of independent 1D chains each of length $L$ and containing $N$ atoms. Each chain operates in the Mott insulating regime with one atom per site ($N = L$). Similar configuration can be generated by using optical tweezer arrays loaded with single atoms. Fig.1 depicts the setup. The Hamiltonian is

$$
\hat{H} = \hat{H}_{\text{Hubbard}} + \hat{H}_{\text{Drive}} + \hat{H}_{\text{Trap}},
$$

(1)

where $\hat{H}_{\text{Hubbard}} = -J \sum_{\langle i,j \rangle,\sigma} (\hat{c}_{i,\sigma}^\dagger \hat{c}_{j,\sigma} + h.c.) + U \sum_j \hat{n}_{j,e} \hat{n}_{j,g}$ is the Fermi-Hubbard Hamiltonian with nearest-neighbour tunneling rate $J$, repulsion $U$, operator $\hat{c}_{j,\sigma}$ annihilating an atom of spin $\sigma$ on site $j$, and $\hat{n}_{j,\sigma} = \hat{c}_{j,\sigma}^\dagger \hat{c}_{j,\sigma}$. The laser $\hat{H}_{\text{Drive}} = \Omega \sum_{j} (\hat{e}^{ij}\pi \hat{c}_{j,e}^\dagger \hat{c}_{j,g} + h.c.)$ is a collective driving field inducing on-site spin-flips. The phase factor $\hat{e}^{ij}\pi$ is created by a mismatch between the driving and confining laser wavelengths, corresponding to an effective flux $\phi = \pi$ that induces spin-orbit coupling (SOC) \([13]\). We also include the trapping potential $\hat{H}_{\text{Trap}} = \eta_{\text{ext}} \sum_j (\hat{J} - j_0^2)(\hat{n}_{j,e} + \hat{n}_{j,g})$ with trap energy $\eta_{\text{ext}}$ from external harmonic confinement due to finite lattice laser beam waist (centered on site $j_0$), approximated as quadratic near the center of the lattice, yielding linear potential differences $\Delta \eta_j = -2\eta_{\text{ext}} (j - j_0 + 1/2)$ between neighboring sites $j$ and $j + 1$ \([SOM]\).

We assume that the drive frequency is much stronger than the atom tunneling rate, $\Omega \gg J$. Under this condition, the single-particle eigenstates of the system are set by the single-site eigenstates of the drive. We rotate into the basis of these eigenstates by defining new fermions $\hat{a}_{j,\uparrow} = (\hat{c}_{j,e} + e^{ij}\pi \hat{c}_{j,g}^\dagger) / \sqrt{2}$, $\hat{a}_{j,\downarrow} = (\hat{c}_{j,e} - e^{ij}\pi \hat{c}_{j,g}) / \sqrt{2}$. In this basis the Hubbard and drive Hamiltonians become

\begin{align*}
\hat{H}_{\text{Hubbard}} &= -J \sum_{\langle i,j \rangle} (\hat{a}_{i,\uparrow}^\dagger \hat{a}_{j,\uparrow} + h.c.) + U \sum_j \hat{n}_{j,\uparrow} \hat{n}_{j,\downarrow}, \\
\hat{H}_{\text{Drive}} &= \Omega \sum_{j} (\hat{n}_{j,\uparrow} - \hat{n}_{j,\downarrow}),
\end{align*}

(2)

with $\hat{a}_{j,\sigma} = \hat{a}_{j,\sigma}^\dagger \hat{a}_{j,\sigma}$ for drive eigenstates $\sigma \in \{\uparrow, \downarrow\}$. Note that the tunneling is now accompanied by a spin-flip, corresponding to the eigenstates alternating in sign due to the SOC phase. The trapping potential keeps the same form.

While the tunneling couples the drive eigenstates, actual transfer of atom population will depend on the energy differences between states. Some sample tunneling processes are depicted in Fig.1. A spin-$\uparrow$ atom tunneling down the trap gradient can incure an energy change $-\Delta \eta_j$ from the trap, a change $-\Omega$ from flipping to spin-$\downarrow$, and a change $U$ for creating a doublon (two atoms on one site). A spin-$\downarrow$ atom tunneling would instead have a change $+\Omega$ from the drive. If the total energy penalty is much larger than $J$, tunneling is suppressed. Furthermore, since the trap energy differences $\Delta \eta_j$ vary from site to site, by making the trap strong ($\eta_{\text{ext}} \gg J$) we can tune the drive frequency $\Omega$ to resonantly enable a single tunnel coupling of a chosen spin between two chosen lattice sites while keeping all other tunneling processes offresonant. This allows for selective control of lattice dynamics at a single-site level without needing a site-resolved laser.

**Generation protocol.** The control over tunneling allows us to generate a cat state. The scheme is depicted in Fig.2. We assume for simplicity that the populated lattice sites do not include the center of the quadratic trap potential ($j_0 > L$, with sites indexed $j = 1, 2, \ldots, L$ from left to right). This can be achieved for example by applying a superimposed linear potential; a more conventional trap centered at the middle will be discussed afterwards. We start with a product state $|\psi_0\rangle = \bigotimes_j |\downarrow\rangle_j$ \([panel (a)]\), which can be prepared with a pulse or an adiabatic ramp \([SOM]\). The first step is to generate a local two-atom superposition on two adjacent lattice sites, by resonantly enabling the tunneling of the $\downarrow$ atom at the edge site $j = 1$ to its neighbour $j = 2$. The required drive Rabi frequency must satisfy $\Omega + \Delta \eta_1 = 0$. We keep the laser on with this frequency for a time $t_J = \pi/4$, realizing a unitary operation $\hat{U}_1^{(\pi/2)}$ equivalent to a $\pi/2$ pulse creating an equal-weight superposition of the initial state and a state with a doublon on $j = 2$ \([panels (b),(c)]\). Analogous tunneling processes on other sites do not occur because other trap energies $\Delta \eta_j$ for $j > 1$ differ by at least $2\eta_{\text{ext}} \gg J$.

We next force the $j = 2$ site’s $\downarrow$ atom to tunnel to $j = 3$, but now, set the Rabi frequency to $\Omega - \Delta \eta_2 = 0$. The first component of the superposition \([panel (b)]\) will tunnel because it goes from one doublon configuration to another and suffers no energy penalty $U$. The second component \([panel (c)]\) will have an additional cost $U$, its tunneling will be off-resonant, and it will remain unaltered. We wait for a time $t_J = \pi/2$, realizing a unitary $\hat{U}_2^{(\pi)}$ corresponding to a $\pi$ pulse transferring the $\downarrow$ atom from $j = 2$ to $j = 3$, resulting in a new superposition \([panels (d),(e)]\). We then make the site $j = 3$ doublon have its $\downarrow$ atom tunnel to $j = 4$ with another coherent $\pi$ pulse (unitary $\hat{U}_3^{(\pi)}$), followed by $j = 4$ to $j = 5$, repeating all the way to the end of the chain. The superposition component corresponding to the initial state (bottom panels) will remain unaffected because its tunneling will always be offresonant. The final state will take the form

$$
|\psi_{\text{cat}}\rangle = \hat{U}_{L-2}^{(\pi)} \ldots \hat{U}_2^{(\pi)} \hat{U}_1^{(\pi/2)} |\psi_0\rangle,
$$

(3)

as shown in panels (f),(g), corresponding to a cat state involving $L$ sites, $L - 2$ of which will differ in spin projection between the superposition components (still assuming unit filling $N = L$). Here, $\theta_j$ is a relative phase picked up during the evolution \([SOM]\). The total evolution time is $t_J = \pi/4 + (L - 2)\pi/2$. While the protocol thus far assumed that the chain did not contain the center of the trap, we can also extend it to a symmetric version ($j_0 = L/2$). In this case, the
superposition will have four components instead of two because each side propagates independently. Such an outcome may be useful in its own right, e.g. to create compass-type generalized cat states. However, we can also prevent it from happening by disrupting the \( \hat{U}(\pi/2) \) step on one side. Following steps will then fail on that side, allowing the protocol to proceed as before [SOM].

An important advantage lies in the protocol’s piecewise nature. Some methods such as adiabatic dragging suffer from reduced fidelity for larger cats due to exponentially shrinking many-body energy gaps with system size. Here, the reduction of the system to an effective two-level configuration at every step allows for easier optimization of the individual steps, and is conceptually straightforward to scale up. Furthermore, the evolving state exhibits some robustness to collective phase-drift effects, e.g. unwanted phases \( e^{i\lambda(t)} \) in \( \hat{H}_{\text{Drive}} \) for some function \( \lambda(t) \). The system will follow the drift by adiabatically remaining in the drive’s eigenbasis (provided \( \Omega \gg J \) and \( \lambda(t) \) varies slowly on the timescale of \( J \)), preserving the superposition. The main source of error would be imperfect resonance matching \( \delta\Omega \) between the desired and actual Rabi frequency \( \Omega \) at each step, resulting in imperfect pulses. Fig. 3 shows a benchmark of the protocol fidelity, averaged over trajectories with random disorder \( \delta\Omega \). We see that cats of 10+ sites can be made with fidelities above 90%. Assuming a quadratic decay, we can extrapolate these results to larger cats of \( L = 20 \), finding expected fidelities of \( F \approx 83\% \) with \( \delta\Omega/J = 0.25 \) and \( F \approx 56\% \) with \( \delta\Omega/J = 0.5 \). This tolerance can be further improved with a deeper harmonic trap, for which the allowed \( J \) (and thus mismatch \( \delta\Omega \)) can be larger.

**Experimental implementation and measurement.** A feasible platform for implementing our protocol is a state-of-the-art 3D optical lattice [14] or optical tweezer array [15] loaded with quantum-degenerate fermionic alkaline earth or earthlike atoms such as Sr or Yb. The bare atomic states \( \{g, e\} \) can be represented by electronic clock states with optical frequency separation. For the lattice implementation the confinement should be made strong along the transverse directions \( (\hat{x}, \hat{y}) \) and intermediate along the cat direction \( \hat{z} \). For a lattice using spin-polarized fermionic \( ^{87}\text{Sr} \) at the magic wavelength, we can realize parameters of \( U/J \approx 400 \), \( \eta_{\text{ext}}/J \approx 20 \), \( J/(2\pi) \approx 10 \text{ Hz} \) with current-generation setups [SOM]; deeper traps can also be made by reducing beam waist. Cat generation time for these parameters is \( t \sim L \times 25 \text{ ms} \), which
is small compared to coherence times $\sim 10 \text{s}$ [16] for cats on the order of $\sim 10$ sites. While the respective 1D geometries will be at different transverse trap energies in a 3D lattice, the relative shifts between lattice sites along the cat direction will be the same for a separable trap, allowing for simultaneous creation of an array of cats from which a constructive measurement signal can be obtained as described below.

To use the cat state for enhanced sensing, we allow it to pick up a relative phase from laser detuning. The scheme is depicted in Fig. 4(a). After generation, a pulse $\hat{P}$ rotates the cat into a form where its superposition components will acquire a relative phase $\theta_\delta = \delta(N - 1)t_\delta$ in the lab frame if they precess for a time $t_\delta$ [N−1 because of the edge sites, SOM for details]. Conventionally, this $N$-proportional enhancement is observed using a standard Ramsey sequence followed by a parity measurement [17-19], requiring measurement of $N$-body correlators which can be challenging for standard clock setups, although it can be done in optical tweezer arrays [15].

As an alternative approach, we can instead undo the generation sequence, as shown in Fig. 4(a). After precession, we rotate the cat back into the gauged frame with another pulse $\hat{P}^{-1}$ [SOM]. We then do the $\pi$-pulse steps in reverse order, $\hat{U}_{L-2}^{(\pi/2)} \ldots \hat{U}_{2}^{(\pi)} |\psi_{\text{cat}, \delta}\rangle$ (with $|\psi_{\text{cat}, \delta}\rangle$ the cat after precession and applying $\hat{P}$). These steps reduce the state to the form $(|\downarrow, \downarrow\rangle + e^{i(\theta_\delta + \theta_\delta)} |0, \uparrow\rangle) / \sqrt{2} \otimes |\downarrow, \downarrow\rangle, |\downarrow, \uparrow\rangle$, where the superposition is back on the first two sites $j = 1, 2$ and $\theta_\delta$ is a constant phase depending on the cat size and parameters. Reapplying unitary $\hat{U}_{L}^{(\pi/2)}$ will rotate this state into a form where the relative phase may be measured from doubling number $\langle \hat{n}_d \rangle = \sum_j \langle \hat{n}_j \rangle \rho_j$ in the vicinity of $j = 1, 2$, without needing $N$-body correlators. The doubling number will oscillate as a function of precession time $t_\delta$, allowing the detunning to be obtained from the oscillation period.

Thus far, we have assumed unit filling. While unwanted holes will be confined by the energy gaps, they will interrupt the state generation, leading to shorter-length cats. However, the above measurement protocol can still work for low hole fraction. A 3D optical lattice away from unit filling will generate cats of different sizes, but sufficiently high filling will allow the maximum-length cats to dominate the signal and give clear oscillation fringes. We benchmark the measurement protocol in Fig. 4(b-e) by randomly sprinkling holes into a 3D lattice, and computing how many cats of each length we get. Panel (b) shows the distribution of cat number $n_{\text{tot}}$ for cat size $l \in [0, 1, \ldots, L]$ while panels (c-e) give sample oscillation trajectories of total doubling number $\langle \hat{n}_d \rangle$ summed over the array of cats, SOM. For $L = 10$, fillings above $N/L \gtrsim 0.9$ yield a clear oscillatory signal $10 - 1 = \times$ faster than a single unentangled atom, leading to $\sqrt{10 - 1}$ faster clock protocols [20]. One may also employ Fourier analysis to discern the contributions of different-size cats, provided the precession time is long enough to see multiple periods of oscillation.

**Hole correction protocol.** Our protocol can be modified to compensate for small numbers of holes at the cost of longer generation time. We can augment every primary step of the original protocol except the first with two auxiliary steps, which enable the next primary step should a hole be present. We first attempt to run the protocol as normal, moving a $\downarrow$ atom to make a doublon on the next lattice site ($|\uparrow\downarrow, \downarrow\rangle \rightarrow |\uparrow\uparrow\rangle$). If the second lattice site is missing an atom, $|\uparrow\downarrow, 0\rangle$, then this primary step will fail. We then apply an auxiliary step that repeats the same tunneling process, but now assuming the target site to have no atom, allowing the transfer $|\uparrow\downarrow, 0\rangle \rightarrow |\uparrow, \uparrow\rangle$. A second auxiliary step moves the remaining atom over, $|\uparrow, \uparrow\rangle \rightarrow |0, \uparrow\rangle$. The hole is effectively jumped over, and the protocol may continue as normal. If no holes were present, neither of the auxiliary steps would have an effect because they would be off-resonant [SOM]. Note that the all-spin-$\downarrow$ superposition component will also suffer local changes in the vicinity of the hole, but these changes will not propagate further, maintaining a significant difference in spin projection between the two components for low hole numbers [SOM]. While this protocol is not as useful to 3D lattice setups whose measurement signal comes from the largest-size cat only (as described above), it is useful for optical tweezer systems that can control the cats independently.

**Conclusions.** We have proposed a method for generating cat states with ultracold fermions that can be directly implemented with state-of-the-art 3D lattice systems or tweezer arrays. Our protocol does not suffer from the slow-down caused
by small many-body gaps, and enjoys straightforward troubleshooting and benchmarking because of its stepwise nature. The cat state can be immediately used in situ for metrological purposes through a Ramsey-like sequence combined with protocol reversal. Tweezer systems can also realize the protocol and make use of the cats via parity measurements. A 2D tweezer array could even generate a single cat along one 1D tube, then repeat the protocol along the transverse axis, leading to a 2D cat. The cat fidelity requires good control over drive frequency noise, but this requirement can be made less stringent with a stronger harmonic confinement. The latter not only relaxes the required noise stability, but also allows for larger tunneling rates and faster generation time. One may also use a purification scheme to convert many bad cat states into a smaller number of good ones [21]. Altogether, this scheme offers a promising way to both generate and use strongly entangled states in metrologically relevant systems.

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Supplementary Material

A. STATE ROTATIONS AND PREPARATION

The cat-generating protocol requires an initial state of \( |\psi_0\rangle = \bigotimes_j |\downarrow\rangle_j \) in the basis of the drive eigenstates. Such a state has a nontrivial spin structure in the bare atomic state basis \( \{g,e\} \) due to the alternating sign of the drive; if we enumerate the lattice sites as \( j = 1, 2, \ldots \), the state would be written in the lab frame as,

\[
|\psi_0\rangle = |+x,-x,+x,-x,\ldots\rangle, \tag{A1}
\]

where \( |\pm x\rangle = (|e\rangle \pm |g\rangle)/\sqrt{2} \). Preparing this state can be done in two ways. The first is to use a pulse from a laser with the same SOC spatially-varying phase \( e^{ij\pi} \), but with an overall phase shift from the drive laser of main text Eq. (1) by \( \pi/2 \). Recalling that the drive laser Hamiltonian is,

\[
\hat{H}_{\text{Drive}} = \frac{\Omega}{2} \sum_j \left( e^{ij\pi} \hat{c}_{j,e} \hat{c}_{j,g} + h.c. \right), \tag{A2}
\]

the pulse laser would need to be of the form,

\[
\hat{H}_P = \frac{\Omega_P}{2} \sum_j \left( e^{ij\pi-2i\pi/2} \hat{c}_{j,e} \hat{c}_{j,g} + h.c. \right). \tag{A3}
\]

Note that the drive laser’s overall phase besides the SOC does not matter as the system will follow the drive’s eigenstates; the pulse laser only needs to have its phase behind that of the drive laser by \( \pi/2 \). One can do both the pulse and driving with the same laser setup since only one beam needs to be active at a time; a mirror and switching configuration can first enable the pulse, followed by the drive for the main steps of the protocol.

The initial state can be prepared by first loading the atoms into their natural ground-state \( \bigotimes_j |g\rangle_j \) in the lab frame by standard cooling techniques, then making a fast pulse,

\[
\hat{P} = e^{-i\pi\Omega_P \hat{H}_P},
\]

\[
|\psi_0\rangle = \hat{P} \left( \bigotimes_j |g\rangle_j \right). \tag{A4}
\]

assuming that \( \Omega_P \gg J \) to avoid unwanted lattice dynamics. Once this is done the pulse laser is turned off, the drive laser enabled, and the generation protocol may proceed. The same pulse \( \hat{P} \) may be used to rotate the final cat state into a form where its components will accrue opposite phases from any laser detuning, as described in the measurement protocol of the main text (\( d \) is a doublon):

\[
\hat{P} |\psi_{\text{cat}}\rangle = (|g,g,\ldots,g,g\rangle + e^{-i\theta_f} |0,e,\ldots,e,d\rangle)/\sqrt{2}. \tag{A5}
\]

An alternative method for preparing the initial state is to instead use an adiabatic ramp. For this, we only use the drive laser with no need for a pulse. Recall that the drive may have a detuning,

\[
\hat{H}_\delta = \frac{\delta}{2} \sum_j (\hat{n}_{j,e} - \hat{n}_{j,g}). \tag{A6}
\]

If the detuning is much larger than the drive frequency, \( \delta \gg \Omega \), then the ground-state of the system will be \( \bigotimes_j |g\rangle_j \) even in the presence of the drive. We slowly reduce the detuning from \( \delta_0 \gg \Omega \) to zero over a time \( t_{\text{ramp}} \), as depicted in Fig. [A1]

\[
\delta(t) = \delta_0 \left[ \tanh \left( \frac{t_{\text{ramp}}}{2} - t \right) J \right] - 1. \tag{A7}
\]

The system will adiabatically remain in the ground-state, which will transition from \( \bigotimes_j |g\rangle_j \) to \( |\psi_0\rangle \), provided that the rate \( d\delta/dt \) is smaller than the gap to the next-lowest energy state proportional to \( \Omega \), and \( \Omega \) is chosen to avoid any tunneling resonances.
B. SYSTEM PARAMETERS

In this section we give an overview of the sample parameters used throughout the main text, assuming realistic experimental setups. We consider a 3D optical lattice loaded with nuclear-spin polarized fermionic $^{87}\text{Sr}$ at the magic wavelength $\lambda_\text{L} \approx 813$ nm (lattice constant $a = \lambda_\text{L}/2$), with the clock states $^1S_0, ^3P_0$ acting as the bare spin states $g, e$. The 3D lattice confinement strengths are set to $(V_x, V_y, V_z) = (200, 200, 19)E_r$, with $E_r = \hbar^2/(2ma^2)$ the recoil energy ($m \approx 87$ amu). The desired parameter regimes are $U, \eta_\text{ext} \gg J$.

The cat state is generated along the $\hat{z}$ direction. While there will be a gravitational potential shift, its only effect will be to move the center of the trap by a few lattice sites, which may be accounted for when choosing $\delta(t)$ to prevent unwanted tunneling during this step.

As seen from Fig. B1, this approximation works well for $\sim 40$ sites nearest to the center of the trap. The first term creates the lattice potential built into our Fermi-Hubbard model. The last term’s prefactor sets the trap energy (normalizing by the lattice constant).

$$V(z) \approx V_z \sin^2 \left( \frac{\pi z}{a} \right) + mgz - \tilde{V}_x e^{-\frac{x^2}{\nu_x^2}} - \tilde{V}_y e^{-\frac{y^2}{\nu_y^2}},$$

where $\tilde{V}_x = V_x - \sqrt{V_x E_r/2}$, $\tilde{V}_y = V_y - \sqrt{V_y E_r/2}$ are renormalized lattice depths, and $g$ is gravitational acceleration. Fig. B1 shows this potential as a function of lattice site number $j$ (i.e. in units of $z/a$). The Gaussian profile can be approximated by a quadratic function near the bottom,

$$V(z) \approx V_z \sin^2 \left( \frac{\pi z}{a} \right) + mgz - (\tilde{V}_x + \tilde{V}_y) + \left( \frac{2\tilde{V}_x}{\nu_x^2} + \frac{2\tilde{V}_y}{\nu_y^2} \right) z^2.$$

As seen from Fig. B1 this approximation works well for $\sim 40$ sites nearest to the center of the trap. The first term creates the lattice potential built into our Fermi-Hubbard model. The last term’s prefactor sets the trap energy (normalizing by the lattice constant),

$$\eta_\text{ext}/(2\pi) = \frac{2\tilde{V}_x}{(\nu_x/a)^2} + \frac{2\tilde{V}_y}{(\nu_y/a)^2} \approx 219 \text{ Hz}.$$  

The gravitational potential $mgz$ creates a shift, $j_0 = -\eta_\text{ext}/(2mg) \approx -2$ sites, which may be accounted for when choosing the Rabi frequency for step $\tilde{U}_1^{(\pi/2)}$.

We also evaluate the tunneling overlap integral and onsite s-wave interaction strength (for scattering length $a_{-g} = 69.1a_0$ with $a_0$ the Bohr radius) via standard Wannier orbital calculations, yielding.

$$J/(2\pi) \approx 10.4 \text{ Hz}, \quad U/(2\pi) \approx 4212 \text{ Hz}.$$  

From the above, we conclude that our system parameters in units of $J$ are given by,

$$U/J \approx 405, \quad \eta_\text{ext}/J \approx 21.$$
FIG. B1. Schematic of lattice trapping potential along the $\hat{z}$ direction which we use to make cat states, in units of site number $j = z/a$. The full Gaussian profile [Eq. (B2)] and its approximate quadratic form [Eq. (B3)] are shown. The center is shifted by gravity, but only by a few sites $j_0 \approx -2$. We see that the quadratic approximation remains valid for $\sim 40$ sites.

FIG. C1. Schematic for the hole-correcting protocol. Panel (a) shows the primary step of the regular protocol, which will fail if site $j$ has a hole. Panels (b),(c) give two auxiliary steps (i) and (ii), which compensate for the hole by manually moving the doublon on site $j - 1$ into the hole’s location so that the protocol can keep going. If the primary step in panel (a) had succeeded, the auxiliary steps would have no effect. Panel (d) shows the fidelity of obtaining the desired state after every primary-auxiliary-auxiliary sequence of the protocol, using a numerical evolution of size $L = 8$ with $N = 7$ and a single hole at $j = 4$. The other component of the superposition will see local changes near the hole, but will otherwise be unaffected [SOM].

The drive Rabi frequency can be made on the order of kHz, yielding possible values $\Omega/J \sim 1 - 1000$.

As a side note, in the above parameters we have ensured that $U \gg \eta_{\text{ext}}$. This is not strictly necessary, and is done to ensure that no accidental resonances occur with lattice sites not involved in the current active step of the protocol (many such unwanted resonances are shifted by $U$, and can thus be enabled by accident if $U \approx \Delta \eta_j$ for some $j$ uninvolved in the current step). For larger cats where the trap energy differences $\Delta \eta_j$ grow large, one can instead dodge unwanted resonances by tuning $U$ between them. It is straightforward to analytically compute all possible resonant drive frequencies for all tunneling events at every step, and determine experimentally-appropriate values of $U$, $\eta_{\text{ext}}$ for which the drive frequency can isolate the desired resonance while being sufficiently far from all others.

C. HOLE CORRECTION PROTOCOL

In this section we provide details for the hole-correcting protocol described in the main text. Fig. C1(a-c) shows a schematic diagram. Every primary step [panel (a), moving a doublon over one site by making its ↓ atom tunnel] is followed by two auxiliary steps [panels (b),(c)], whose combined effect is to manually move the doublon over if there was a hole present and the primary step failed. Fig. C1(d) shows fidelities of generating the desired state after every primary-auxiliary-auxiliary sequence of the protocol, using a numerical evolution of size $L = 8$ with $N = 7$ and a single hole at $j = 4$. The other component of the superposition will see local changes near the hole, but will otherwise be unaffected [SOM].

Fig. C2 depicts the protocol in more detail. We compare the situation where a hole is present on the site we want to move the doublon into (top half), with the situation of unit filling where only the primary step should take effect (bottom half). For
FIG. C2. Schematic for the hole correction protocol. The primary step of the original protocol is followed by two auxiliary steps (i), (ii). The top panel depicts the superposition that would exist if a vacancy was present (Hole), while the bottom panel shows unit filling (Full). For every step, a green arrow means the drive is resonant with the tunneling process in question, and an atom is moved over with a \( \pi \) pulse. For an orange dashed arrow, the process is either offresonant or otherwise inhibited. For unit filling, only the primary step succeeds, moving a doublon over one site. For the hole, the primary step fails, but the two auxiliary steps move the doublon over manually so the protocol may continue. Note that in the case of the hole, there is also a spin-flip on the all-\( \downarrow \) component which we do not want to affect, reducing the relative difference in spin projection. However, this change is not propagated further.

For each step, the drive frequency is shown, as well as the two components of the superposition. Steps where the atom tunneling will succeed are shown in green; for those, the Rabi frequency satisfies the respective resonance condition, and the total energy difference before/after tunneling is \( \Delta E = 0 \). Steps where tunneling fails are shown in orange; for these, either \( |\Delta E| \sim U \gg J \), there are no atoms in the coupled levels, or atoms populate both levels and are Pauli blocked.

The price we pay aside from increased evolution time is that the other (no-doublon) component of the superposition will now have an additional \( \uparrow \) atom, whereas we want it to be all \( \downarrow \). However, this will not propagate further, and assuming a small density of holes we should still have mostly \( \downarrow \) atoms in the no-doublon component, and mostly \( \uparrow \) atoms in the evolving one.

D. RELATIVE PHASE FROM UNPERTURBED GENERATION

In this section, we give the relative phase that the cat state picks up during the generation protocol. The first step only yields a phase of \( e^{i\pi/2} \) between the superposition components due to the \( \pi/2 \) pulse, because the relative energies of the coupled states are manually set to be equal by choice of drive frequency. For all subsequent steps, while the energies of the two sites tunneling are still matched, the superposition components will have other uninvolved lattice sites with different spin structure (as part of our entanglement-building process), thus picking up a phase from the drive at different rates. This relative phase will vary from step to step, because both the Rabi frequency and the number of misaligned spins between the two superposition components will change.

For a system of \( L \) sites and \( N = L \) atoms, we have 1 superposition-generating \( \pi/2 \) pulse, followed by \( L - 2 \) atom-transferring \( \pi \) pulses. The total relative phase for the cat,

\[
|\psi_{\text{cat}}\rangle = (|\downarrow, \downarrow, \downarrow, \downarrow\rangle + e^{i\theta_f} |0, \uparrow, \ldots, \uparrow, \uparrow\rangle) / \sqrt{2},
\]

may be found after some algebra to be,

\[
\theta_f = \frac{\pi}{2} \left[ (L - 1) - \frac{U}{J}(L - 2) + \frac{\eta_{\text{ext}}}{J} \frac{L}{3} (L - 1)(L - 2) \right].
\]
Any measurement protocol would create an additional shift to this overall phase. Note that this result is only exact in the limit where the energy gaps to all unwanted resonances are infinite. For realistic experimental parameters, there may be some deviation to the above with larger cats. However, for a measurement protocol such as the reversal described in the main text, this relative phase is unimportant anyways; we only provide it for completeness.

E. TWO-SIDED TRAP

Our cat generation protocol can be generalized to include both halves of the harmonic trap. With the right half included the first step will generate a four-component superposition instead of two, because assuming the center of the trap \( j_0 \) is an integer, the left and right sides will have identical resonant tunneling and generate independent two-component superpositions (a four-component tensor product overall). The following steps will propagate these superpositions down the lattice on their respective sides independently, at least until we reach the very bottom. We can prevent this from happening by modifying the state preparation. One simple way is to shift the trap potential so that its center is closer to the edge of the atomic cloud (i.e. \(|j_0| \gg 1\)). Another way is to use a narrow beam-waist laser to effect a \( \pi \) pulse on the atoms in the upper-right half of the trap after preparing the \( \downarrow \) product state, shown in Fig. E1. This does not need to be single-site focused or fully coherent; we simply need to disrupt the state of the right-side lattice site at the height of the left-side starting point, so that it cannot participate in the protocol’s first step. Collateral changes to neighbouring sites on the right are also acceptable, so long as they do not stretch across the whole lattice. With this done, the protocol will fail to start on the right side. Further steps will also fail as they are contingent upon one another. We can then enact the protocol from the left side as before. In principle, we can even continue through the center and out to any unchanged sites on the right.

F. CAT MEASUREMENT THROUGH UNITARY REVERSAL

In this section we detail the way to measure the relative phase between the components of the cat state through time-reversal. We assume that the cat is generated, and allow it to accrue a phase during precession time \( t_\delta \) from detuning. The drive frequency \( \Omega \) is either turned off or tuned to some value far from any resonances during this time, to help prevent the atoms from tunneling. We use the pulse described in Section A to put the cat into the lab frame before the precession starts, and convert it back into the drive frame after the precession, so that its components can accrue the maximum possible phase. This precession may be written as,

\[
|\psi_{\text{cat},\delta}\rangle = \hat{P}^\dagger e^{-i(\hat{H} + \hat{H}_\delta) t_\delta} \hat{P} |\psi_{\text{cat}}\rangle,
\]

which will yield a state of the form,

\[
|\psi_{\text{cat},\delta}\rangle = (|\downarrow, \downarrow, \ldots, \downarrow, \downarrow\rangle + e^{i(\theta_p + \theta_\delta)} |0, \uparrow, \ldots, \uparrow, \downarrow\rangle) / \sqrt{2},
\]

where \( \theta_\delta \) is a bare phase coming from precession under the drive and interactions, and \( \theta_\delta = \delta(L - 1) t_\delta \) is the additional phase from the detuning (minus one because of the edge sites). Note that we have not provided an explicit expression for \( \theta_\delta \), which will depend on the system parameters and precession time. However if we emulate a Ramsey-type sequence and the filling fraction
is sufficiently high, this bare phase will be irrelevant so long as it is the same for all 1D chains, because we only care about the period of resulting oscillations.

We now run the cat-generating protocol on \( |\psi_{\text{cat},\delta}\rangle \) in reverse. All of the \( \pi \) -pulse transfer steps (i.e. all except the first step) are done the same way as the original protocol, only in opposite order. After doing all the steps except \( \hat{U}^{(\pi/2)}_1 \), the result will be,

\[
|\psi_{\text{cat},r}\rangle = \hat{U}_1^{(\pi/2)} \cdots \hat{U}_{L-2}^{(\pi/2)} |\psi_{\text{cat},\delta}\rangle = \frac{1}{\sqrt{2}} \left[ |\downarrow, \downarrow\rangle + e^{i(\theta_r + \theta_\delta)} |0, \uparrow\rangle \right] \otimes |\downarrow, \ldots, \downarrow\rangle_{j=3 \ldots L} .
\]  

Again, the bare phase \( \theta_r \) after reversing will be nontrivial even for no precession \( t_\delta = 0 \) because applying the steps in reverse does not constitute a true many-body unitary reversal. However, for a given cat length and set of parameters it should be the same for every experiment shot.

At this point, we have reduced the system back into a two-state configuration where the relative phase can be measured through a direct laser coupling. The final step is to reapply \( \hat{U}^{(\pi/2)}_1 \),

\[
\hat{U}_1^{(\pi/2)} |\psi_{\text{cat},r}\rangle = \frac{1}{2} \left[ (1 - ie^{i(\theta_r + \theta_\delta)}) |\downarrow, \downarrow\rangle - i(1 + ie^{i(\theta_r + \theta_\delta)}) |0, \uparrow\rangle \right] \otimes |\downarrow, \ldots, \downarrow\rangle_{j=3 \ldots L} ,
\]

for which the relative phase can be obtained from the overlap with the second state of the superposition, equivalent to measuring the doublon number. We only count doublons in the vicinity of the initial site where the original protocol was started, but do not need single-site resolution; a few sites’ width is fine, as if the protocol started at \( j \neq 1 \).

Provided any unwanted resonances are avoided and the lattice is sufficiently deep, the only way a doublon could be created in this vicinity is if the cat-generating protocol reversed itself as described. Measuring the doublon number for this state yields,

\[
\langle \hat{n}_d \rangle = \sum_j \langle \hat{n}_{j,\uparrow}\hat{n}_{j,\downarrow} \rangle = \frac{1}{2} [1 - \sin(\theta_r + \theta_\delta)],
\]  

which contains full phase information about the cat, including the \( L \) -proportional phase accrued from detuning.

Of course, when holes are present, not all cat states will be of the desired length \( L \). To estimate the signal, we randomly sample a 3D lattice of dimensions \( L \times L \times L \) by sprinkling holes to a desired filling fraction \( N/L \). We then compute a histogram of the distribution of the number of cats \( m_l \) with length \( l \in [0, 1, \ldots, L] \) that one can realize with this lattice along a given direction \( \hat{z} \). For example, if a given 1D tube has a hole at the 8th site from the bottom, that tube adds one to \( m_7 \). Main text Fig 4(b) plots this distribution for different filling fractions, finding that for \( N/L \gtrsim 0.9 \) the majority of the tubes should yield full-length cats. We then sum the doublon number from Eq. (F5),

\[
\langle \hat{n}_{d,\text{tot}} \rangle = \sum_{l=0}^{L} m_l \times \frac{1}{2} [1 - \sin(\theta_r^{(l)}) + \delta(l - 1)t]].
\]  

The bare phase \( \theta_r^{(l)} \) is equal for all cats of a given size \( l \) for fixed system parameters, which allows equal-length cats to contribute to the signal constructively. For simplicity, we select a random \( \theta_r^{(l)} \) for every \( l \). The total resulting signal is plotted as a function of \( t \) in main text Fig 4(c)-(e). For sufficiently high filling the longest-length cat is dominant, and a clear oscillation period \( T = 2\pi/|\delta(L - 1)| \) may be extracted, from which \( \delta \) is obtained.