EXISTENCE THEOREM OF A WEAK SOLUTION FOR NAVIER-STOKES TYPE EQUATIONS ASSOCIATED WITH DE RHAM COMPLEX

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Abstract. Let \( \{d_q, \Lambda_q\} \) be de Rham complex on a smooth compact closed manifold \( X \) over \( \mathbb{R}^3 \) with Laplacians \( \Delta_q \). We consider operator equations, associated with the parabolic differential operators \( \partial_t + \Delta_2 + N^2 \) on the second step of complex with nonlinear bi-differential operator of zero order \( N^2 \). Using by projection on the next step of complex we show that the equation has unique solution in special Bochner-Sobolev type functional spaces for some (small enough) time \( T^* \).

Introduction

Consider the de Rham complex on a Riemannian \( n \)-dimensional smooth compact closed manifold \( X \) with vector bundles \( \Lambda_q \) of exterior forms of degree \( q \) over \( X \),

\[
0 \longrightarrow \Omega^0(X) \xrightarrow{d_0} \Omega^1(X) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Omega^n(X) \longrightarrow 0.
\] (0.1)

Here \( \Omega_q(X) \) denotes the space of all differential forms of degree \( q \) with smooth coefficients on \( X \). In this case the Laplacians \( \Delta_q = d_q^*d_q + d_{q-1}d_{q-1}^* \), \( q = 0, 1, \ldots, n \), of the complex are second order strongly elliptic differential operators on \( X \), where operator \( d_q^* \) is a formal adjoint to \( d_q \). As usual, for \( q < 0 \) and \( q \geq n \) we assume that \( d_q = 0 \).

We want to study the non-linear problems, associated with the complex. With this purpose, we denote by \( M_{i,j} \) two bilinear bi-differential operators of zero order (see [24] or [25]),

\[
\begin{align*}
M_{q,1}(\cdot, \cdot) &: (\Omega^{q+1}(X), \Omega^q(X)) \rightarrow \Omega^q(X), \\
M_{q,2}(\cdot, \cdot) &: (\Omega^q(X), \Omega^q(X)) \rightarrow \Omega^{q-1}(X).
\end{align*}
\] (0.2)

We set for a differential form \( u \) of the degree \( q \)

\[
N^q(u) =: M_{q,1}(d_qu, u) + d_{q-1}M_{q,2}(u, u).
\] (0.3)

Note, that operator \( N^q(u) \) is non-linear.

Let now time \( T > 0 \) is finite. Then for any fixed positive number \( \mu \) the operators \( \partial_t + \mu \Delta_q \) are parabolic on the cylinder \( X \times (0, T) \) (see [24]). Consider the following initial problem: given sufficiently regular differential forms \( f \) of the induced bundle \( \Lambda^q(t) \) (the variable \( t \) enters into this bundle as a parameter) and \( u_0 \) of the bundle

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\[ \Lambda^q, \text{find a differential forms } u \text{ of the induced bundle } \Lambda^q(t) \text{ and } p \text{ of the induced bundle } \Lambda^{q-1}(t) \text{ such that} \]

\[ \begin{cases} 
\partial_t u + \mu \Delta_q u + N^q(u) + d_{q-1}p = f & \text{in } X \times (0, T), \\
\ast_{q-1} u = 0 & \text{in } X \times [0, T], \\
\ast_{q-2} u = 0 & \text{in } X \times [0, T], \\
u(x, 0) = u_0 & \text{in } X, 
\end{cases} \quad (0.4) \]

For general elliptic complexes this problem was considered in the works [21] and [29], where the open mapping theorems were proved in the special spaces of Hölder (see [21]) and Sobolev (see [29]) types. It means that the range of the non-linear operator \( A_q \), related to the problem, is open in the constructed spaces. However, obtaining an existence theorem for the solution (even the so-called weak one) and closedness of the range of the related non-linear operator in such spaces appears to be a more difficult task.

For example, if we take \( q = 1 \) and a suitable nonlinear term we may treat (0.4) as the initial problem for the well known Navier-Stokes equations for incompressible fluid over the manifold \( X \) (see, for instance, [16] or [30]). Note that the equation with respect to \( p \) is actually missing in this case, because \( \ast_{q-1} = 0 \).

We consider problem (0.4) in the case \( n = 3 \), \( q = 2 \) and a special nonlinearity \( M_{q,1}(d_q u, u) = (d_q u) u \). It easy to see that in this case we can treat the de Rham differentials as \( d_2 = \text{div}, \ d_1 = \text{rot}, \ d_2^* = -\nabla, \ d_1^* = \text{rot} \) and then (0.4) transforms to

\[ \begin{cases} 
\partial_t u + \mu \Delta_2 u + N^2(u) + \text{rot} \ p = f & \text{in } X \times (0, T), \\
\text{rot} \ u = 0 & \text{in } X \times [0, T], \\
\text{div} \ p = 0 & \text{in } X \times [0, T], \\
u(x, 0) = u_0 & \text{in } X, 
\end{cases} \quad (0.5) \]

where

\[ N^2(u) = (\text{div} u) u + \text{rot} (M_{q,2}(u, u)), \quad (0.6) \]

and Laplacian

\[ \Delta_2 u = d_2^* d_2 + d_1 d_1^* = -\nabla \text{div} u + \text{rot rot} \ u = -\Delta u. \]

Here \( \Delta u \) is a standard Laplace operator applied componentwise to the differential form \( u \) in the space variable \( x \).

Using projection to the next step of complex (0.1), we prove an existence theorem of weak (distributional) solution in the constructed Bochner-Sobolev type spaces for some (small enough) time \( T^* \). Note, that considering general non-linear perturbations of linear parbolic equations one have to impose essential restrictions on the non-linear term \( N^2(u) \) in order to achieve existence of weak solutions. For example, one of such condition can be positiveness of non-linear operator \( N^2(u) \). However, we do not impose such strong conditions on the non-linear term, but still have an existence of weak solutions due to special properties of the de Rham complex.

1. Functional spaces

Denote by \( L^p_X \), \( 1 \leq p \leq \infty \), space of differential forms of the degree \( q \) with coefficients in the Lebesgue space \( L^p(X) \). In a similar way we designate the spaces of forms on \( X \) whose components are of Sobolev class or have continuous partial
derivatives. We denote it by $W^p_{q^s}$ and $C^s_{q^s}$ respectively with smoothness $s$. In particular case, for $p = 2$ we designate $H^2_{q^s} := W^2_{q^s}$.

For calculations, it is convenient to use the fractional powers of the Laplace operator. Namely, for differential form $u$ of degree $q$ we denote by

$$
\nabla_q^m u := \begin{cases} 
\Delta_q^{m/2} u, & m \text{ is even,} \\
(d_q \oplus d_{q-1}) \Delta_q^{(m-1)/2} u, & m \text{ is odd.}
\end{cases}
$$

(1.1)

It easy to see that integration by parts yields

$$
\sum_{|\alpha|=m} \|\partial^\alpha u\|^2_{L^2_{q^s}} = \|\nabla_q^m u\|^2_{L^2_{q^s}}.
$$

Now, we want to recall the standard Hodge theorem for elliptic complexes. For this purpose denote by $H^q$ the harmonic space of complex (1.1), i.e.

$$
H^q = \{u \in C^\infty_{q^s} : d_q u = 0 \text{ and } d^*_q u = 0 \text{ in } X\},
$$

(1.2)

and by $\Pi^q$ the orthogonal projection from $L^2_{q^s}$ onto $H^q$.

**Theorem 1.1.** Let $0 \leq q \leq n$, $s \in \mathbb{Z}_+$. Then operator

$$
\Delta_q : H^{s+2}_{q^s} \to H^s_{q^s}
$$

(1.3)

is Fredholm:

1. the kernel of operator (1.3) equals to the finite-dimensional space $H^q$;
2. given $v \in H^s_{q^s}$ there is a form $u \in H^{s+2}_{q^s}$ such that $\Delta_q u = v$ if and only if $(v, h)_{L^2_{q^s}} = 0$ for all $h \in H^q$;
3. there exists a pseudo-differential operator $\varphi^q$ on $X$ such that the operator

$$
\varphi^q : H^s_{q^s} \to H^{s+2}_{q^s},
$$

(1.4)

induced by $\varphi^q$, is linear bounded and with the identity $I$ we have

$$
\varphi^q \Delta_q = I - \Pi^q \text{ on } H^{s+2}_{q^s}, \quad \Delta_q \varphi^q = I - \Pi^q \text{ on } H^s_{q^s}
$$

(1.5)

**Proof.** See, for instance, [25, Theorem 2.2.2].

Denote by $V^s_{q^s} := H^s_{q^s} \cap S^s_{q^s-1}$, the space of all differential forms $u \in H^s_{q^s}$ satisfying $d^*_q u = 0$ in the sense of distributions in $X$. Let now $L^2(I, H^s_{q^s})$ be the Bochner space of $L^2$-mappings

$$
u(t) : I \to H^s_{q^s},
$$

where $I = [0, T]$, see, for instance, [13]. It is a Banach space with the norm

$$
\|u\|^2_{L^2(I, H^s_{q^s})} = \int_0^T \|u\|^2_{H^s_{q^s}} dt.
$$

We need to introduce suitable Bochner-Sobolev type spaces, see [24] or [30] for the de Rham complex and [29] for the general elliptic complexes. Namely, for $s \in \mathbb{Z}_+$ denote by $B_k^{q,2s} \sum_{X_T}$ the space of all differential forms of degree $q$ over $X_T := X \times [0, T]$ with variable $t \in [0, T]$ as a parameter, such that

$$
u \in C(I, V^{k+2s}_{q^s}) \cap L^2(I, V^{k+2s+1}_{q^s})
$$

and

$$
\nabla_q^m \partial_t^j u \in C(I, V^{k+2s-m-2j}_{q^s}) \cap L^2(I, V^{k+2s+1-m-2j}_{q^s})
$$

and
for all $m + 2j \leq 2s$. It is a Banach space with the norm

$$
\|u\|^2_{B^{k,2s}_{q,vel}} := \sum_{m+2j \leq 2s, 0 \leq l \leq k} \|\nabla^l_q \nabla^{m}_q \partial^j_k u\|_{C^l[I, L^2_{\Lambda^k}]}^2 + \|\nabla^{l+1}_q \nabla^{m}_q \partial^j_k u\|_{L^2[I, L^2_{\Lambda^k}]}^2.
$$

Similarly, for $s, k \in \mathbb{Z}_+$, we define the space $B^{k,2s}_{q,vel}(X_T)$ to consist of all differential forms

$$
f \in C(I, H^{2s+k}_A) \cap L^2(I, H^{2s+k+1}_A)
$$

with the property that

$$
\nabla^m \partial^j_k f \in C(I, H^{k+2s-m-2j}_A) \cap L^2(I, H^{k+2s-m-2j+1}_A)
$$

for all $m + 2j \leq 2s$. We endow the space $B^{k,2s}_{q,vel}(X_T)$ with the natural norm

$$
\|f\|^2_{B^{k,2s}_{q,vel}} := \sum_{m+2j \leq 2s, 0 \leq l \leq k} \|\nabla^l_q \nabla^{m}_q \partial^j_k f\|_{C^l[I, L^2_{\Lambda^k}]}^2 + \|\nabla^{l+1}_q \nabla^{m}_q \partial^j_k f\|_{L^2[I, L^2_{\Lambda^k}]}^2.
$$

Lastly, the space for the differential form $p$ we denote by $B^{k+1,2s}_{q,vel}(X_T)$. This space consists of all forms $p$ from the space $C(I, H^{2s+k+1}_A) \cap L^2(I, H^{2s+k+2}_A)$ such that $d_{q-1}p \in B^{k,2s}_{q,vel}(X_T)$, $d'_{q-2}p = 0$ and for all $h \in H^{-1}$

$$(p, h)_{L^2_{\Lambda^q-1}} = 0.$$  (1.6)

It is a Banach space with the norm

$$
\|p\|_{B^{k+1,2s}_{q-1,vel}} := \|d_{q-1}p\|_{B^{k,2s}_{q,vel}}.
$$

Define now for suitable forms $v$ and $w$ of degree $q$ a bi-differential operator

$$
B_q(w, v) = M_{q,1}(d_qw, v) + M_{q,1}(d_qv, w) + d_{q-1}(M_{q,2}(w, v) + M_{q,2}(v, w)), \quad (1.7)
$$

with the operators $M_{q,1}$ and $M_{q,2}$ satisfying

$$
|M_{q,1}(u, v)| \leq c_{q,1}|u||v|, \quad |M_{q,2}(u, v)| \leq c_{q,2}|u||v| \text{ on } X
$$

with some positive constants $c_{i,j}$. Following theorem allows us to see the correctness of operators in this spaces.

**Theorem 1.2.** Suppose that $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$ and $2s+k > \frac{q}{2} - 1$. Then the mappings

$$
\nabla^m_q : B^{k,2(s-1),s-1}_{q,vel}(X_T) \to B^{k-m,2(s-1),s-1}_{q,vel}(X_T), \quad m \leq k
$$

$$
\Delta_q : B^{k,2s}_{q,vel}(X_T) \to B^{k,2(s-1),s-1}_{q,vel}(X_T),
$$

$$
\partial_k : B^{k,2s}_{q,vel}(X_T) \to B^{k,2(s-1),s-1}_{q,vel}(X_T),
$$

are continuous. Besides, if $w, v \in B^{k+2,2(s-1),s-1}_{q,vel}(X_T)$ then the mappings

$$
B_q(w, \cdot) : B^{k+2,2(s-1),s-1}_{q,vel}(X_T) \to B^{k,2(s-1),s-1}_{q,vel}(X_T),
$$

$$
B_q(w, \cdot) : B^{k,2s}_{q,vel}(X_T) \to B^{k,2(s-1),s-1}_{q,vel}(X_T), \quad (1.9)
$$

are continuous, too. In particular, for all $w, v \in B^{k+2,2(s-1),s-1}_{q,vel}(X_T)$ there is positive constant $c_{s,k}$ independent on $v$ and $w$, such that

$$
\|B_q(w, v)\|_{B^{k,2(s-1),s-1}_{q,vel}} \leq c_{s,k}\|w\|_{B^{k+2,2(s-1),s-1}_{q,vel}} \|v\|_{B^{k+2,2(s-1),s-1}_{q,vel}}. \quad (1.10)
$$

**Proof.** See, for instance, [30] or [29]. \qed
Let us introduce now the Helmholtz type projection $P^q$ from $B_{k,\text{for}}^{k,2(s-1),s-1}(X_T)$ to the kernel of operator $d_q$.

**Lemma 1.3.** If $s, k \in \mathbb{Z}_+$, then for each $q$ the pseudo-differential operator $P^q = d_q^*d_q\phi^q + \Pi^q$ on $X$ induce continuous map

$$P^q : B_{k,\text{for}}^{k,2(s-1),s-1}(X_T) \to B_{q,\text{vel}}^{k,2(s-1),s-1}(X_T),$$

such that

$$P^q \circ P^q u = P^q u, \quad (P^q u, v)_{L^2_{\Lambda^2}(X)} = (u, P^q v)_{L^2_{\Lambda^2}(X)}, \quad (P^q u, (I - P^q) u)_{L^2_{\Lambda^2}(X)} = 0$$

for all $u, v \in B_{k,\text{for}}^{k,2(s-1),s-1}$.

**Proof.** See, for instance, [29]. \hfill \Box

The following Lemma is just a consequence of Hodge Theorem 1.1.

**Lemma 1.4.** Let $F \in B_{k,\text{for}}^{k,2(s-1),s-1}(X_T)$ satisfy $P^q F = 0$ in $X_T$. Then there is a unique section $p \in B_{k-1,\text{pre}}^{k+1,2(s-1),s-1}(X_T)$ such that (1.6) holds and

$$d_{q-1} p = F \text{ in } X \times [0,T].$$

Now we are ready to go to the main section of this paper.

2. Existence theorem

In order to get existence theorem to the Problem (1.5) we use a projection to the next step of complex (1.1). Namely, applying an operator $d_2 = \text{div}$ to the equation (0.5) we have

$$\begin{cases} 
\partial_t \text{div} u - \mu \text{div}(\nabla \text{div} u) + \text{div}(\text{div} u) u = \text{div} f & \text{in } X \times (0,T), \\
\text{div} u(x,0) = \text{div} u_0 & \text{in } X,
\end{cases}$$

(2.1)

because of rot $u = 0$ and $\text{div} \circ \text{rot} \equiv 0$. Now,

$$\text{div}(\text{div} u) u = (\text{div} u)^2 + \Delta u \cdot u = (\text{div} u)^2 + \nabla \text{div} u \cdot u.$$

By Theorem 1.1

$$u = \varphi^2 \Delta_2 u + \Pi^2 u = \varphi^2 \nabla \text{div} u + \Pi^2 u.$$

Denote

$$g = \text{div} u,$$

then we can rewrite (2.1) by the next way

$$\begin{cases} 
\partial_t g - \mu \text{div}(\nabla g) + g^2 + \nabla g \cdot (\varphi^2 \nabla g + \Pi^2 u) = \text{div} f & \text{in } X \times (0,T), \\
g(x,0) = \text{div} u_0 & \text{in } X.
\end{cases}$$

(2.2)

**Theorem 2.1.** Given any pair $(f, u_0) \in L^2(I, (V_{\Lambda^2}^0)' \times V_{\Lambda^2}^1)$. There is time $t_0 \in (0, T)$ such that for all $t \in [0, t_0]$ there exist a differential form $g \in C(I, L^2_{\Lambda^2}) \cap L^2(I, H^1_{\Lambda^2})$ with $0 \neq g \in L^2(I, (H^1_{\Lambda^2})')$, satisfying

$$\begin{cases} 
\frac{d}{dt} (g, v)_{L^2_{\Lambda^2}} + \mu (\nabla g, \nabla v)_{L^2_{\Lambda^2}} = \langle \text{div} f - g^2 - \nabla g \cdot (\varphi^2 \nabla g + \Pi^2 u), v \rangle, \\
g(x,0) = \text{div} u_0
\end{cases}$$

(2.3)

for all $v \in H^k_{\Lambda^2}$ with $k \geq 2$. 
Proof. Let \( \{u_m\} \) be the sequence of Faedo-Galerkin approximations, namely,

\[
  u_m = \sum_{j=1}^{M} c_j^{(m)}(t) b_j(x),
\]

then

\[
  g_m = \text{div} u_m = \sum_{j=1}^{M} c_j^{(m)}(t) \text{div} b_j(x),
\]

where the system \( \{b_j\}_{j \in \mathbb{N}} \) is a \( L^2_{\lambda}(X) \)-orthogonal in \( V_{\lambda}^1 \) and the functions \( u_m \) satisfy the following relations

\[
  \frac{d}{dt} (g_m, \text{div} b_j)_{L^2_{\lambda}} + \mu (\nabla g_m, \nabla \text{div} b_j)_{L^2_{\lambda}} = \langle \text{div} f - g_m^2 - \nabla g_m \cdot \varphi^2 \nabla g_m - \nabla g_m \cdot \Pi^2 u_m, \text{div} b_j \rangle,
\]

\[
  g_m(x, 0) = \text{div} u_{0,m}(x),
\]

for all \( 0 \leq j \leq m \) with the initial date \( u_{0,m} \) from the linear span \( \mathcal{L}(\{b_j\}_{j=1}^{m}) \) such that the sequence \( \{u_{0,m}\} \) converges to \( u_0 \) in \( V_{\lambda}^1 \). For instance, as \( \{u_{0,m}\} \) we may take the orthogonal projection onto the linear span \( \mathcal{L}(\{b_j\}_{j=1}^{m}) \).

Multiplying (2.6) by \( c_j^{(m)}(t) \) and summing by \( j \) we have

\[
  (\partial_t g_m, g_m)_{L^2_{\lambda}} + \mu (\nabla g_m, \nabla g_m)_{L^2_{\lambda}} = \langle \text{div} f - g_m^2 - \nabla g_m \cdot \varphi^2 \nabla g_m - \nabla g_m \cdot \Pi^2 u_m, g_m \rangle.
\]

(2.7)

It follows from Lemma by J.-L. Lions (see, for instance, [27, Ch. III, § 1, Lemma 1.2]) that

\[
  \frac{d}{dt} ||g_m(\cdot, t)||^2_{L^2_{\lambda}} = 2 \langle \partial_t g_m, g_m \rangle.
\]

Then integrating by \( t \in [0, T] \) we see that

\[
  ||g_m(\cdot, t)||^2_{L^2_{\lambda}} + 2 \mu \int_0^t \left| \nabla g_m \right|^2_{L^2_{\lambda}} dt = ||g_m(\cdot, 0)||^2_{L^2_{\lambda}} + 2 \mu \int_0^t \langle \text{div} f - g_m^2 - \nabla g_m \cdot \varphi^2 \nabla g_m - \nabla g_m \cdot \Pi^2 u_m, g_m \rangle dt.
\]

(2.8)

Since \( f \in L^2(I, L^2_{\lambda}) \) then \( \text{div} f \in L^2(I, (V_{\lambda}^1)'s) \) and

\[
  2 \left| \int_0^t \langle \text{div} f, g_m \rangle dt \right| \leq 2 \int_0^t \left| \text{div} f \right|_{(V_{\lambda}^1)'} \left| g_m \right|_{V_{\lambda}^1} dt \leq \frac{4}{\mu} \int_0^t \left| \nabla g_m \right|^2_{L^2_{\lambda}} dt + \frac{\mu}{4} \int_0^t \left| g_m \right|^2_{L^2_{\lambda}} dt.
\]

(2.9)

On the other hand

\[
  2 \left| \int_0^t \langle g_m^2, g_m \rangle dt \right| \leq 2 \int_0^t \left| g_m \right|^3_{L^3_{\lambda}} dt.
\]

(2.10)

Note that in our case \( \nabla_3 = -\nabla \) with \( n = 3 \). Then from the Gagliardo-Nirenberg inequality (see [20] or [4, Theorem 3.70]) we have

\[
  2 \int_0^t \left| g_m \right|^3_{L^3_{\lambda}} dt \leq \left( \left( \left| \nabla g_m \right|^2_{L^2_{\lambda}} + \left| g_m \right|^2_{L^2_{\lambda}} \right)^{\frac{1}{2}} \left| g_m \right|^\frac{3}{2}_{L^3_{\lambda}} + ||g_m||_{L^3_{\lambda}} \right)^3 dt \leq \]

(2.11)
\[ c_1 \int_0^t \left[ \| \nabla g_m \|_{L^2}^2, \| g_m \|_{L^3}^2 + \| g_m \|_{L^6}^2 \right]^3 dt \leq \]
\[ c_2 \int_0^t \left( \| \nabla g_m \|_{L^2}^2, \| g_m \|_{L^3}^2 + \| g_m \|_{L^6}^2 \right) dt \leq \]
\[ \frac{\mu}{2} \int_0^t \| \nabla g_m \|_{L^2}^2 dt + c_3 \int_0^t \left( \| g_m \|_{L^3}^3 + \| g_m \|_{L^6}^2 \right) dt \]

with positive constants \( c, c_1 \) and \( c_2 \). The last expression is consequence of standard Young’s inequality. Moreover, there are positive constants \( c \) and \( c_1 \) such that
\[ \int_0^t \left( \| g_m \|_{L^2}^3 + \| g_m \|_{L^6}^3 \right) dt \leq c \int_0^t \| g_m \|_{L^2}^2 \left( 1 + \| g_m \|_{L^3}^2 \right)^2 dt \leq c_1 \left( \int_0^t \| g_m \|_{L^2}^2 dt + \int_0^t \| g_m \|_{L^6}^3 dt \right). \]

Then we conclude that
\[ 2 \int_0^t \| g_m \|_{L^3}^3 dt \leq \frac{\mu}{2} \int_0^t \| \nabla g_m \|_{L^2}^2 dt + c \int_0^t \| g_m \|_{L^3}^2 dt + c \int_0^t \| g_m \|_{L^6}^2 dt \quad (2.12) \]

with some constant \( c > 0 \). Next,
\[ \int_0^t (\nabla g_m \cdot \varphi^2 \nabla g_m, g_m) dt = \sum_{j=1}^3 \int_0^t \int_X \partial_j g_m (\varphi^2 \partial_j g_m) g_m dx dt = \]
\[ - \sum_{j=1}^3 \int_0^t \int_X g_m (\varphi^2 \partial_j g_m) \partial_j g_m dx dt - \int_0^t \int_X g_m^3 dx dt, \]

because \( \varphi^2 \Delta g_m = g_m \). It means that
\[ \int_0^t (\nabla g_m \cdot \varphi^2 \nabla g_m, g_m) dt = - \frac{1}{2} \int_0^t \int_X g_m^3 dx dt, \]

and hence
\[ 2 \left| \int_0^t (\nabla g_m \cdot \varphi^2 \nabla g_m, g_m) dt \right| \leq \int_0^t \| g_m \|_{L^3}^3 dt. \quad (2.13) \]

Finally,
\[ \int_0^t (\nabla g_m \cdot \Pi^2 u_m, g_m) dt = \sum_{j=1}^3 \int_0^t \int_X \partial_j g_m (\Pi^2 u_m^j) g_m dx dt = \]
\[ - \sum_{j=1}^3 \int_0^t \int_X g_m (\Pi^2 u_m^j) \partial_j g_m dx dt - \sum_{j=1}^3 \int_0^t \int_X g_m^2 \partial_j (\Pi^2 u_m^j) dx dt, \]

and then
\[ \int_0^t (\nabla g_m \cdot \Pi^2 u_m, g_m) dt = 0, \quad (2.14) \]

because \( \text{div} \Pi^2 u_m = 0 \).

Now, inequalities \( (2.3) \) - \( (2.14) \) give
\[ \| g_m (\cdot, t) \|_{L^3}^3 + 2\mu \int_0^t \| \nabla g_m \|_{L^2}^2 dt \leq \| g_m (\cdot, 0) \|_{L^3}^3 + \]
\[ (2.15) \]
\[
\frac{4}{\mu} \int_0^t \|\text{div } f\|_{L^2(I, V^k_{\Lambda_2})}^2 \, dt + \mu \int_0^t \|\nabla g_m\|_{L^2(\Lambda_3)}^2 \, dt + \frac{\mu}{4} \int_0^t \|g_m\|_{L^2(\Lambda_3)}^2 \, dt + 
\]
\[
2c \int_0^t \|g_m\|_{L^2(\Lambda_3)}^2 \, dt + 2c \int_0^t \|g_m\|_{L^2(\Lambda_3)}^2 \, dt,
\]
and then
\[
\|g_m(\cdot, t)\|_{L^2(\Lambda_3)}^2 \leq \|g_m(\cdot, 0)\|_{L^2(\Lambda_3)}^2 + \frac{4}{\mu} \|\text{div } f\|_{L^2(I, V^k_{\Lambda_2})}^2(t, \cdot) + \left( \frac{\mu}{4} + 2c \right) \int_0^t \|g_m\|_{L^2(\Lambda_3)}^2 \, dt + 2c \int_0^t \|g_m\|_{L^2(\Lambda_3)}^2 \, dt.
\]

It follows from the Gronwall-Perov’s Lemma (see, for instance [18, p. 360]) that there exist a time \(t_0 \in (0, T]\) and a positive constant \(C_{t_0}\) such that
\[
\|g_m(\cdot, t)\|_{L^2(\Lambda_3)}^2 \leq C_{t_0}
\]
for all \(t \in [0, t_0]\). Then the sequence \(g_m\) is bounded in \(L^\infty(I_{t_0}, L^2_{\Lambda_3})\), where \(I_{t_0} = [0, t_0]\). Moreover it follows from (2.16) and (2.17) that \(\|\nabla g_m(\cdot, t)\|_{L^2(I_{t_0}, L^2_{\Lambda_3})}^2\) is bounded too. It means that there is a subsequence that converges weakly-\(^*\) in \(L^\infty(I_{t_0}, L^2_{\Lambda_3})\) and weakly in \(L^2(I_{t_0}, L^2_{\Lambda_3})\) to some \(g \in L^\infty(I_{t_0}, L^2_{\Lambda_3}) \cap L^2(I_{t_0}, L^2_{\Lambda_3})\).

We use the same designation \(g_m\) for such a subsequence. Then the standard argument show (see, for instance [15, 27] or [13]) that we can to pass to the limit in (2.18) with respect to \(m \to \infty\) and to conclude that the element \(g\) satisfies (2.19).

Let us now return to the Problem (1.5). Denoting again \(g = \text{div } u\) and multiplying (1.5) scalar in \(L^2_{\Lambda_2}\) by differential form \(v \in V^k_{\Lambda_2}\) we get
\[
\left\{ \begin{array}{l}
\frac{d}{dt}(u(x), v)_{L^2_{\Lambda_2}} + \mu (g, \text{div } v)_{L^2_{\Lambda_3}} = \langle f - gu, v \rangle, \\
u(x, 0) = u_0.
\end{array} \right.
\]

**Theorem 2.2.** Let \(g \in C(I, L^2_{\Lambda_3}) \cap L^2(I, H^1_{\Lambda_3})\) be the solution of (2.3) for given any pair \((f, u_0) \in L^2(I, V^k_{\Lambda_2}) \times V^k_{\Lambda_2}\). Then there exist a unique differential form \(u \in C(I, V^k_{\Lambda_2}) \cap L^2(I, V^k_{\Lambda_2})\) satisfying (2.18) for all \(v \in V^k_{\Lambda_2}\) with \(k \geq 2\).

**Proof.** Indeed, let \(\{u_m\}\) be the sequence of Faedo-Galerkin approximations (see (2.4)) such that the sequence \(\{g_m\} = \{\text{div } u_m\}\) converges to \(g \in C(I, L^2_{\Lambda_3}) \cap L^2(I, H^1_{\Lambda_3})\). Substituting \(u_m\) to (2.18) instead of \(u\), \(v\) and integrating by \(t \in [0, t_0]\) we have
\[
\|u_m(\cdot, t)\|_{L^2_{\Lambda_2}}^2 + 2\mu \|g_m\|_{L^2(I_{t_0}, L^2_{\Lambda_3})}^2 = \|u_m(x, 0)\|_{L^2_{\Lambda_2}}^2 + 2 \int_0^{t_0} \langle f - g u_m, u_m \rangle \, dt.
\]

As usual, we evaluate by Hölder inequality
\[
2 \left| \int_0^{t_0} \langle f, u_m \rangle \, dt \right| \leq \int_0^{t_0} \|u_m\|_{L^2_{\Lambda_2}}^2 \, dt + \int_0^{t_0} \|f\|_{L^2_{\Lambda_2}}^2 \, dt
\]
and by Gagliardo-Nirenberg inequality
\[
2 \left| \int_0^{t_0} \langle g u_m, u_m \rangle \, dt \right| \leq \int_0^{t_0} \|u_m\|_{L^2_{\Lambda_2}}^2 \|g_m\|_{L^2_{\Lambda_3}} \, dt \leq 
\]
\[
c \int_0^{t_0} \|g_m\|_{L^2_{\Lambda_3}} \left( \|g_m\|_{L^2_{\Lambda_3}}^{3/2} \|u_m\|_{L^2_{\Lambda_2}}^{1/2} + \|u_m\|_{L^2_{\Lambda_2}}^2 \right) \, dt \leq
\]
2\mu \int_0^{t_0} \|g_m\|_{L^2_{\Lambda^3}}^2 dt + c_1 \int_0^{t_0} \|g_m\|_{L^3_{\Lambda^3}}^4 \|u_m\|_{L^2_{\Lambda^3}}^2 dt + c_2 \int_0^{t_0} \|g_m\|_{L^2_{\Lambda^3}} \|u_m\|_{L^2_{\Lambda^3}}^2 dt}

with some positive constants \(c, c_1, c_2\). Then

\[\|u_m(\cdot, t)\|_{L^2_{\Lambda^3}}^2 \leq \|u_0\|_{L^2_{\Lambda^3}}^2 + \int_0^{t_0} \|f\|_{L^2_{\Lambda^3}}^2 dt + c \int_0^{t_0} \|u_m\|_{L^2_{\Lambda^3}}^2 dt\] (2.20)

with constant \(c > 0\), independent on \(m\). It follows from Gronwall’s Lemma that

\[\|u_m(\cdot, t)\|_{L^2_{\Lambda^3}}^2 \leq C,\] (2.21)

where constant \(C\) depends on norms \(\|f\|_{L^2(I_{t_0}, L^2_{\Lambda^3})}, \|u_0\|_{L^2_{\Lambda^3}}^2\) and \(\|g\|_{C(I_{t_0}, L^2_{\Lambda^3})}\), but not on \(m\).

It follows that the sequence \(u_m\) is bounded in \(L^\infty(I_{t_0}, L^2_{\Lambda^3})\) and there is a subsequence that converges weakly-* in \(L^\infty(I_{t_0}, L^2_{\Lambda^3})\) to some \(u \in L^\infty(I_{t_0}, L^2_{\Lambda^3})\). We again use the same designation \(u_m\) for such a subsequence. Under hypothesis of this Theorem the sequence \(g_m = \text{div} u_m\) converges to \(g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})\), then actually \(u \in C(I, V^{1, \infty}_{\Lambda^3}) \cap L^2(I, V^2_{\Lambda^3})\). Passing to the limit in (2.19) with respect to \(m \to \infty\) we conclude that the element \(u\) satisfies (2.18).

Let now \(u'\) and \(u''\) are two solutions of (2.18) such that \(\text{div} u' = \text{div} u'' = g\). Hence differential form \(u = u' - u''\) satisfies (2.18) with zero date \((f, u_0) = (0, 0)\). It follows from (2.20) and Gronwall-Perov’s Lemma that \(\|g(\cdot, t)\|_{L^2_{\Lambda^3}}^2 = 0\), then the Problem (2.18) has unique solution.

Moreover, if \(u_1, u_2\) are two solutions of (2.18), corresponding to the solutions \(g_1 = \text{div} u_1\) and \(g_2 = \text{div} u_2\) of (2.23), then differential form \(u = u_1 - u_2\) satisfies

\[
\begin{align*}
\frac{d}{dt} (u, v)_{L^2_{\Lambda^3}} + \mu (g, \text{div} v)_{L^2_{\Lambda^3}} &= \langle -gu, v \rangle, \\
u(x, 0) &= 0,
\end{align*}
\] (2.22)

where \(g = g_1 - g_2\).

\[\|u(\cdot, t)\|_{L^2_{\Lambda^3}}^2 + 2\mu \|g\|_{L^2(I_{t_0}, L^2_{\Lambda^3})}^2 = -2\int_0^{t_0} \langle gu, u \rangle dt.\] (2.23)

Applying the Gagliardo-Nirenberg inequality we have

\[2 \int_0^{t_0} \langle gu, u \rangle dt \leq c_1 \int_0^{t_0} \|g\|_{L^2_{\Lambda^3}} \left( \|\nabla u\|_{L^2_{\Lambda^3}}^{3/4} \|u\|_{L^2_{\Lambda^3}}^{1/4} + \|u\|_{L^2_{\Lambda^3}} \right)^2 dt \leq c_2 \left( \|g\|_{L^2_{\Lambda^3}}^{5/2} \|u\|_{L^2_{\Lambda^3}}^{1/2} + \|u\|_{L^2_{\Lambda^3}} \|g\|_{L^2_{\Lambda^3}} \right) dt \]

\[2\mu \|g\|_{L^2(I_{t_0}, L^2_{\Lambda^3})}^2 + c_3 \left( \|g\|_{C(I_{t_0}, L^2_{\Lambda^3})}^4 + \|g\|_{C(I_{t_0}, L^2_{\Lambda^3})} \right) \int_0^{t_0} \|u\|_{L^2_{\Lambda^3}} dt \]

with positive constants \(c_1, c_2, c_3\). The last inequality, (2.23) and Gronwall-Perov’s Lemma yields

\[\|u(\cdot, t)\|_{L^2_{\Lambda^3}}^2 \leq 0,\]

then \(u_1 = u_2\) and the Problem (2.18) has unique solution.

\[\square\]

**Corollary 2.3.** Under hypothesis of the Theorem (2.21) let \(g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})\) is a solution of (2.23) and \(u \in C(I, V^1_{\Lambda^3}) \cap L^2(I, V^2_{\Lambda^3})\) is a corresponding to \(g\) solution of (2.18). If moreover \(g \in C(I, H^1_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})\), then the solution \(g\) is unique.
Proof. Indeed, let \( g_1, g_2 \in C(I, H^1_{A^3}) \cap L^2(I, H^1_{A^3}) \) are two solutions of (2.3) with corresponding forms \( u_1, u_2 \in C(I, H^1_{A^3}) \cap L^2(I, H^1_{A^3}) \) satisfying (2.18). Hence differential form \( g = g_1 - g_2 \) satisfies
\[
\begin{aligned}
\frac{d}{dt} \| g \|_{L^2_{A^3}}^2 + \mu \| \nabla g \|_{L^2_{A^3}}^2 &= \langle -(g_1^2 - g_2^2) - (\nabla g_1 \cdot (\varphi^2 \nabla g_1 + \Pi^2 u_1) - \\
&- \nabla g_2 \cdot (\varphi^2 \nabla g_2 + \Pi^2 u_2)), v \rangle,
\end{aligned}
\] (2.24)
Integrating by \( t \in I_t \), we get
\[
\begin{aligned}
\| g(t, t) \|_{L^2_{A^3}}^2 + 2\mu \int_0^{I_t} \| \nabla g \|_{L^2_{A^3}}^2 \, dt &\leq 2 \int_0^{I_t} \| g(g_1 + g_2) + \\
(\nabla g_1 \cdot (\varphi^2 \nabla g_1 + \Pi^2 u_1) - \nabla g_2 \cdot (\varphi^2 \nabla g_2 + \Pi^2 u_2)), v \rangle \, dt,
\end{aligned}
\] (2.25)
We have to estimate right side of (2.25). First, it follows from Gagliardo-Nirenberg interpolation inequality that
\[
\begin{aligned}
2 \int_0^{I_t} \langle (g(g_1 + g_2)), g \rangle \, dt &\leq \\
c \| g_1 + g_2 \|_{C(I_t, L^2_{A^3})} \int_0^{I_t} \left( \| \nabla g \|_{L^2_{A^3}}^{3/2} \| g \|_{L^2_{A^3}}^{1/2} + \| g \|_{L^2_{A^3}}^2 \right) \, dt \\
&\leq \mu \int_0^{I_t} \| \nabla g \|_{L^2_{A^3}}^2 \, dt + c_1 \int_0^{I_t} \| g \|_{L^2_{A^3}}^2 \, dt
\end{aligned}
\] (2.26)
with positive constants \( c, c_1 \). Next,
\[
\begin{aligned}
2 \int_0^{I_t} \left| \langle \nabla g_1 \cdot \varphi^2 \nabla g_1 - \nabla g_2 \cdot \varphi^2 \nabla g_2 + \nabla g_1 \cdot \varphi^2 \nabla g_2 - \nabla g_1 \cdot \varphi^2 \nabla g_2, g \rangle \right| \, dt &\leq \\
2 \int_0^{I_t} \left| \langle \nabla g \cdot \varphi^2 \nabla g, g \rangle \right| \, dt + 2 \int_0^{I_t} \left| \langle \nabla g \cdot \varphi^2 \nabla g_2, g \rangle \right| \, dt \\
&\leq \mu \int_0^{I_t} \| g \|_{L^2_{A^3}}^2 \, dt + c \int_0^{I_t} \| g \|_{L^2_{A^3}}^2 \, dt
\end{aligned}
\] (2.27)
with some positive constants \( c, c_1 \) and \( c_2 \). Finally,
\[
\begin{aligned}
2 \int_0^{I_t} \left| \langle \nabla g_1 \cdot \Pi^2 u_1 - \nabla g_2 \cdot \Pi^2 u_2 + \nabla g_1 \cdot \Pi^2 u_1 - \nabla g_2 \cdot \Pi^2 u_1, g \rangle \right| \, dt &\leq \\
2 \int_0^{I_t} \left| \langle \nabla g \cdot \Pi^2 u_1, g \rangle \right| \, dt + 2 \int_0^{I_t} \left| \langle \nabla g_1 \cdot \Pi^2 u_2, g \rangle \right| \, dt
\end{aligned}
\] (2.28)
The Theorem (2.28) implies that \( u_1 = u_2 \). On the other hand, integrating by parts we easily see that
\[
2 \int_0^{I_t} \left| \langle \nabla g \cdot \Pi^2 u_1, g \rangle \right| \, dt = 0,
\] and then (2.28) equals to zero.
Finally, using by \((2.25)\) - \((2.28)\) we get
\[
\|g(\cdot, t)\|_{L^2_{\Lambda^3}}^2 \leq c \int_0^{t_0} \|g\|_{L^2_{\Lambda^3}}^2 \, dt
\]
with some constant \(c > 0\). Then it follows from Gronwall-Perov’s Lemma that \(\|g(\cdot, t)\|_{L^2_{\Lambda^3}} = 0\), then the Problem \((2.3)\) has unique solution. \(\square\)

**Theorem 2.4.** Let \(s \in \mathbb{N}\) and \(k \in \mathbb{Z}_+\) with \(k > 3/2\). Then for all
\[
(f, u_0) \in B^k_{2,\for}(X_T) \times V^{k+1}_{\Lambda^2}
\]
there exist a time \(T_k \in (0, T]\) such that the Problem \((2.22)\) has solution
\[
g \in B_{2,\for}^{k,2s}(X_{T_k}).
\]
Moreover, solution \(g\) is unique, if form \(u\) in \((2.1)\) satisfied \((2.18)\).

**Proof.** First of all, denote by
\[
\Lambda_r = \begin{cases} 
\Lambda^3, & r \text{ is even,} \\
\Lambda^2, & r \text{ is odd.}
\end{cases}
\]
As before, let \(g_m\) be the Faa’di-Galerkin approximations, see \((2.5)\). We start with the following priory estimates.

**Lemma 2.5.** Under hypothesis of the Theorem \((2.4)\) if \((f, u_0) \in B^k_{2,\for}(X_T) \times V^{k+1}_{\Lambda^2}\) with some \(k \in \mathbb{Z}_+\), then there exist a time \(T_k \in (0, T]\) such that
\[
\|\nabla_2^k g_m(t)\|_{L^2(I_{T_k}, T_{k+1}^2)} + \mu \|\nabla_3^{k+1} g_m\|_{L^2(I_{T_k}, T_{k+1}^2)} \leq C_k
\]
for any \(0 \leq k' \leq k + 2\), where \(I_{T_k} = [0, T_k]\) and the constants \(C_k' = C_k'(\mu, f, u_0) > 0\) depending on \(k', \mu\) and the norms \(\|f\|_{B^k_{2,\for}(X_T)}\), \(\|u_0\|_{V^k_{\Lambda^2}}\) but not on \(m\).

**Proof.** Indeed, if \(k' = 0\) then \((2.29)\) follows immediately from \((2.10)\) and Gronwall-Perov’s Lemma. Now, substituting \(g_m\) and \(\nabla_3^{k+1} g_m\) in \((2.2)\) instead of \(g\) and \(v\) respectively with some \(r \in \mathbb{N}\) and integrating by \(t \in [0, T]\) we get
\[
\left\|\nabla_3^r g_m(\cdot, t)\right\|_{L^2_{\Lambda^3}}^2 + 2\mu \int_0^t \left\|\nabla_3^{r+1} g_m\right\|_{L^2_{\Lambda^3}}^2 \, dt = \tag{2.30}
\]
\[
\left\|\nabla_3^{2r} g_m(\cdot, 0)\right\|_{L^2_{\Lambda^3}}^2 + 2 \int_0^t \langle \text{div} f - g_m^2 - \nabla g_m, v^2 \nabla g_m - \nabla g_m, \Pi^2 u_m, \nabla_3^{2r} g_m\rangle \, dt.
\]
We have to estimate the right side of \((2.30)\). First,
\[
2 \int_0^t \langle \text{div} f, \nabla_3^{2r} g_m \rangle \, dt \leq 2 \int_0^t \left\|\nabla_3^{r-1} \text{div} f\right\|_{L^2_{\Lambda^{r-1}}}, \left\|\nabla_3^{r+1} g_m\right\|_{L^2_{\Lambda^{r+1}}} \, dt \leq \tag{2.31}
\]
\[
\frac{4}{\mu} \int_0^t \left\|\nabla_3^{r-1} \text{div} f\right\|_{L^2_{\Lambda^{r-1}}}^2 \, dt + \frac{\mu}{4} \int_0^t \left\|\nabla_3^{r+1} g_m\right\|_{L^2_{\Lambda^{r+1}}}^2 \, dt.
\]
Further,
\[
2 \int_0^t \langle g_m^2, \nabla_3^{2r} g_m \rangle \, dt \leq 2 \int_0^t \left\|\nabla_3^{r-1} (g_m^2)\right\|_{L^2_{\Lambda^{r-1}}}, \left\|\nabla_3^{r+1} g_m\right\|_{L^2_{\Lambda^{r+1}}} \, dt. \tag{2.32}
\]
Let \(r \geq 2\), using by Hölder and Gagliardo-Nirenberg inequalities we get
\[
\left\|\nabla_3^{r-1} (g_m^2)\right\|_{L^2_{\Lambda^{r-1}}} \leq \sum_{|\alpha| + |\beta| = r-1} c_{\alpha \beta} \left\|\partial^{\alpha} g_m\right\|_{L^2_{\Lambda^{r-1}}} \left\|\partial^{\beta} g_m\right\|_{L^2_{\Lambda^{r}}} \leq \tag{2.33}
\]
\[
\sum_{|\alpha|+|\beta|=r-1} c_{\alpha\beta} \left( \left\| \nabla^{|\alpha|+1} g_m \right\|_{L^3_H(\partial^r_{\alpha})} + \left\| \nabla^{|\alpha|} g_m \right\|_{L^2_H(\partial_{\alpha})} \right)^{3/4} \left\| \nabla^{|\alpha|} g_m \right\|_{L^3_H(\partial_{\alpha})}^{1/4} + \\
+ \left\| \nabla^{|\alpha|} g_m \right\|_{L^3_H(\partial_{\alpha})}^{3/4} \left\| \nabla^{|\beta|+1} g_m \right\|_{L^3_H(\partial_{\beta})} \left\| \nabla^{|\beta|} g_m \right\|_{L^2_H(\partial_{\beta})}^{1/4} + \\
+ \left\| \nabla^{|\beta|} g_m \right\|_{L^3_H(\partial_{\beta})}^{3/4} \left\| \nabla^{|\alpha|} g_m \right\|_{L^2_H(\partial_{\alpha})}
\]
with some positive constants \(c\) and \(c_{\alpha\beta}\). For the exception case \(r = 1\) the last inequality take the form

\[
\left\| \nabla^3(g_m^2) \right\|_{L^r_H(\partial_{r-1})} \leq c \left( \left\| g_m \right\|_{L^3_H(\partial_{r-1})}^2 + \left\| g_m \right\|_{L^3_H(\partial_{r-1})}^{1/2} \left\| \nabla^3 g_m \right\|_{L^2_H(\partial_{r-1})}^{3/2} \right)
\]

because of in (2.33) arise a case when \(|\alpha| = |\beta| = r - 1 = 0\). It follows from (2.32), (2.33) and Young’s inequality that

\[
2 \left\| \int_0^t \langle g_m^2, \nabla^2 g_m \rangle dt \right\| \leq \frac{\mu}{4} \int_0^t \left\| \nabla^2 g_m \right\|_{L^2_H(\partial_{r-1})}^2 dt +
\]

(2.35)

\[
c \left\| g_m \right\|_{C(1,H^{-1}_{r-1})} + c \left\| g_m \right\|_{C(1,H^{-1}_{r-1})} \int_0^t \left\| \nabla^2 g_m \right\|_{L^2_H(\partial_{r-1})}^{1/2} dt
\]
for \(r \geq 2\) and

\[
2 \left\| \int_0^t \langle g_m^2, \nabla^2 g_m \rangle dt \right\| \leq \frac{\mu}{4} \int_0^t \left\| \nabla^2 g_m \right\|_{L^2_H(\partial_{r-1})}^2 dt +
\]

(2.36)

\[
c \left\| g_m \right\|_{C(1,L^2_H(\partial_{r-1}))} + c \left\| g_m \right\|_{C(1,L^2_H(\partial_{r-1}))} \int_0^t \left\| \nabla^2 g_m \right\|_{L^2_H(\partial_{r-1})}^2 dt
\]
for \(r = 1\) with some constant \(c > 0\).

Next,

\[
2 \left\| \int_0^t \langle \nabla g_m \cdot \varphi^2 \nabla g_m, \nabla^2 g_m \rangle dt \right\| \leq
\]

(2.37)

\[
2 \left\| \int_0^t \left\| \nabla^3 \left( \nabla g_m \cdot \varphi^2 \nabla g_m \right) \right\|_{L^r_H(\partial_{r-1})} \left\| \nabla^2 g_m \right\|_{L^3_H(\partial_{r-1})} dt.
\]

Analogous by (2.33) we have

\[
\left\| \nabla^3 \left( \nabla g_m \cdot \varphi^2 \nabla g_m \right) \right\|_{L^r_H(\partial_{r-1})} \leq
\]

(2.38)

\[
c \left( \left\| g_m \right\|_{H^{-1}_{r-1}} \left\| \varphi^2 g_m \right\|_{H^{r+1}_{r-1}} + \left\| g_m \right\|_{H^{-1}_{r-1}}^{1/4} \left\| \varphi^2 g_m \right\|_{H^{r+1}_{r-1}}^{1/4} \left\| \nabla^3 g_m \right\|_{L^2_H(\partial_{r-1})}^{3/4} + \\
\left\| \nabla^3 g_m \right\|_{L^2_H(\partial_{r-1})}^{1/4} \left\| \varphi^2 g_m \right\|_{H^{r+1}_{r-1}}^{1/4} \left\| \nabla^3 g_m \right\|_{L^2_H(\partial_{r-1})}^{3/4} \right)
\]
with \(r \in \mathbb{N}\) and some constant \(c > 0\). Theorem imply that \(\left\| \varphi^2 g_m \right\|_{H^{r+1}_{r-1}} \leq c \left\| g_m \right\|_{H^{-1}_{r-1}}^{1/4} \) with some positive constant \(c\), then

\[
2 \left\| \int_0^t \langle \nabla g_m \cdot \varphi^2 \nabla g_m, \nabla^2 g_m \rangle dt \right\| \leq \frac{\mu}{4} \int_0^t \left\| \nabla^2 g_m \right\|_{L^2_H(\partial_{r-1})}^2 dt +
\]

(2.39)

\[
c \left\| g_m \right\|_{C(1,H^{-1}_{r-1})} + c \left\| g_m \right\|_{C(1,H^{-1}_{r-1})} \int_0^t \left\| \nabla^3 g_m \right\|_{L^2_H(\partial_{r-1})}^{3/2} dt +
\]

\[
c \left\| g_m \right\|_{C(1,H^{-1}_{r-1})} \int_0^t \left\| \nabla^3 g_m \right\|_{L^2_H(\partial_{r-1})}^2 dt
\]
with $c > 0$.

Finally,

$$2 \left| \int_0^t \langle \nabla g_m \Pi^2 u_m, \nabla^2 g_m \rangle dt \right| \leq (2.40)$$

$$2 \int_0^t \| \nabla_{3r+1}^r g_m \|_{L_{\Lambda r+1}^2} \| \nabla_{3r}^{r-1} (\nabla g_m \cdot \Pi^2 u_m) \|_{L_{\Lambda r}^2} dt,$$

and we have again

$$\| \nabla_{3r}^{r-1} (\nabla g_m \cdot \Pi^2 u_m) \|_{L_{\Lambda r-1}^2} \leq (2.41)$$

with positive constant $c$. Operator $\Pi^2$ is bounded in $L_{\Lambda 2}^2$ by the Hodge Theorem 2.2. On the other hand, Theorem 2.2 yields that the sequence $\{u_m\}$ is bounded in $L_{\Lambda 2}^2$ (see (2.21)), then $\| \Pi^2 u_m \|_{H_{\Lambda r}^2} \leq c \| g_m \|_{H_{\Lambda r-1}^2}$ and we get

$$2 \left| \int_0^t \langle \nabla g_m \Pi^2 u_m, \nabla^2 g_m \rangle dt \right| \leq \frac{1}{4} \int_0^t \| \nabla_{3r+1}^r g_m \|_{L_{\Lambda r+1}^2}^2 dt + (2.42)$$

$$c \| g_m \|_{C(I, H_{\Lambda r-1}^2)}^2 + c \| g_m \|_{C(I, H_{\Lambda r-1}^2)}^{5/2} \int_0^t \| \nabla_{3r}^{r-1} g_m \|_{L_{\Lambda r}^2}^{3/2} dt +$$

$$c \| g_m \|_{C(I, H_{\Lambda r-1}^2)}^{10} \int_0^t \| \nabla_{3r}^r g_m \|_{L_{\Lambda r}^2}^2 dt$$

with $c > 0$.

It follows from (2.30) - (2.42) and Gronwall-Perov's Lemma that if $(f, u_0) \in B_{2, r+1, 0, 0}^k (X_T) \times V_{\Lambda + 2}^{k+3}$ and the norm $\| g_m \|_{C(I, H_{\Lambda r-1}^2)}$ is bounded for some $r \in \mathbb{N}$, $r \leq k + 2$, then there exist a time $t_r \in (0, t_0)$ and a positive constant $C_r$, depending on the norms $\| f \|_{B_{2, r+1, 0, 0}^k (X_{T_k})}$ and $\| u_0 \|_{V_{\Lambda + 2}^{k+3}}$, such that

$$\| \nabla_{3r}^r g_m (\cdot, t) \|_{L_{\Lambda r}^2}^2 + \mu \int_0^{t_r} \| \nabla_{3r}^{r+1} g_m \|_{L_{\Lambda r}^2}^2 dt \leq C_r (\mu, f, u_0). (2.43)$$

Using by (2.43) consistently for $r = 1, \ldots, k + 2$ we get family of times $t_r$. Denote $T_k = \min_{r \leq k+2} t_r$, then (2.43) yields that for any $k \in \mathbb{Z}$ there exist a time $T_k$ such that (2.3) is fulfilled.

Theorem 2.1 imply that there exist a solution $g \in C(I, L_{\Lambda 2}^2) \cap L^2(I, H_{\Lambda 2}^1)$ of (2.3). On the other hand, it follows from Lemma 2.3 that for each $(f, u_0) \in B_{2, for}^{k+1, 1, 0, 0} (X_T) \times V_{\Lambda 2}^{k+1}$ there exist a time $T_k \in (0, T]$ and a subsequence $\{g_{m'}\} = \text{div} u_{m'}$ such that $\{g_m\}$ converges weakly in $L^2(I_{T_k}, L_{\Lambda 2}^2)$ and *-weakly in $L^\infty(I_{T_k}, H_{\Lambda 2}^{k+3}) \cap L^2(I, H_{\Lambda 2}^{k+3})$ to an element $g$, then $g \in B_{2, for}^{k, 2, 0} (X_{T_k})$. Moreover, the uniqueness of $g$ immediately follows from Corollary 2.3.
Theorem 2.6. Let $s \in \mathbb{N}$ and $k \in \mathbb{Z}_+$ with $k \geq 2$. Then for all $(f, u_0) \in B^{k+1,2(s-1),s-1}_{\Lambda^2,\text{vel}}(X_T) \times V^{2s+k+1}_\Lambda$ there exist a time $T^* \in (0, T]$ such that the Problem (0.5) has unique solution $(u, p) \in B^{k+1,2s,s}_{\Lambda^2,\text{pre}}(X^k_T)$. 

Proof. Indeed, apply the projection $P^2$ (see Lemma 1.3 above) to the equation (0.5) we have

$$\begin{cases}
\partial_t u + \mu \Delta_2 u + P^2 N^2(u) = P^2 f & \text{in } X \times (0, T), \\
u(x, 0) = u_0 & \text{in } X,
\end{cases} \tag{2.44}$$

then the form $p$ actually has to satisfy the equation

$$\text{rot } p = (I - P^2)(f - N^2(u)) \text{ in } X \times (0, T). \tag{2.45}$$

Multiplying (2.44) by $v \in V^k_\Lambda$, we get the Problem (2.18), then the existence and regularity of solution $u$ follows immediately from the Theorems 2.2 and 2.4. On the other hand, it follows from Lemma 1.4 that there exist unique differential form $p \in B^{k+2,2(s-1),s-1}_{\Lambda^2,\text{pre}}(X^k_T)$, satisfying (2.45). \hfill \Box

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