An improved quantum algorithm for A-optimal projection

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Dimensionality reduction (DR) algorithms, which reduce the dimensionality of a given data set while preserving the information of original data set as well as possible, play an important role in machine learning and data mining. Duan et al. proposed a quantum version of A-optimal projection algorithm (AOP) for dimensionality reduction [Phys. Rev. A 99, 032311 (2019)] and claimed that the algorithm has exponential speedups on the dimensionality of the original feature space n and the dimensionality of the reduced feature space k over the classical algorithm. In this paper, we correct the complexity of Duan et al.’s algorithm to $O\left(\frac{n\kappa s}{\varepsilon^2} \text{polylog}\left(\frac{nm}{\varepsilon}\right)\right)$, where $\kappa$ is the condition number of a matrix that related to the original data set, $s$ is the number of iterations, $m$ is the number of data points and $\varepsilon$ is the desired precision of the output state. Since the complexity has an exponential dependence on $s$, the quantum algorithm can only be beneficial for high dimensional problems with a small number of iterations $s$. To get a further speedup, we proposed an improved quantum AOP algorithm with complexity $O\left(\frac{n\kappa s}{\varepsilon^2} \text{polylog}\left(\frac{nm}{\varepsilon}\right) + \frac{nm}{\varepsilon} \text{polylog}\left(\frac{\kappa}{\varepsilon}\right)\right)$. Our algorithm achieves a nearly exponential speedup if $s$ is large and a polynomial speedup even if $s$ is a small constant compared with Duan et al.’s algorithm. Also, our algorithm shows exponential speedups in $n$ and $m$ compared with the classical algorithm when both $\kappa$, $k$ and $1/\varepsilon$ are $O(\text{polylog}(nm))$.

I. INTRODUCTION

Quantum computing has been shown to be more computationally powerful than classical computing in solving specific problems, such as the factoring problem [1], unstructured data search problem [2] and matrix computation problems [3, 4]. In recent years, quantum machine learning (QML) has received wide attention as an emerging research area which successfully combines quantum physics and machine learning. An important part of the study of QML focuses on designing quantum algorithms to speedup the machine learning problems, such as data classification [5, 6], linear regression [10, 11], association rules mining [15] and anomaly detection [16].

In the big data era, most of the real world data are high-dimensional, which require high computational performance and usually cause a problem called curse of dimensionality [17]. Since the high-dimensional real world data are often confined to a region of the space having lower effective dimensionality [17], a technique called dimensionality reduction (DR) which reduces the dimensionality of the given data set while preserving the information of original data set as well as possible, was proposed. The DR algorithm often serves as a preprocessing step in data mining and machine learning.

Based on the feature space that the data lying on and the learning task that we want to handle, various DR algorithms have been developed. Generally, when the data lie on a linear embedded manifold, principal component analysis (PCA) [18], a DR algorithm maintaining the characteristics of the data set that contribute the most to the variance, is guaranteed to uncover the intrinsic dimensionality of the manifold. When the data lie on a non-linearly embedded manifold, the manifold learning techniques, such as Isomap [19], Locally Linear Embedding [20], and Laplacian Eigenmap [21] can be use to discover the nonlinear structure of the manifold. Since the DR algorithms mentioned above aim to discover the geometrical or cluster structure of the training data, these algorithms are not suitable for classification and regression tasks, which are the two most important tasks in machine learning and data mining. For classification task, a famous DR technique called linear discriminant analysis (LDA) was put forward, which maximize the between-class variance while minimizing the within-class variance of the training data [22]. For regression task, He et al. proposed a novel DR algorithm called A-Optimal Projection (AOP) that aims to minimize the prediction error of a regression model while reducing the dimensionality [23]. Their algorithm improves the regression performance in the reduced space.

In the context of quantum computing, the quantum PCA was proposed by Lloyd et al. to reveal in quantum form the eigenvectors corresponding to the large eigenvalues of an unknown low-rank density matrix [24]. Latter, Yu et al. proposed a quantum algorithm that compresses training data based on PCA [25]. When the dimensionality of the reduced space is polylogarithmic of the training data, their quantum algorithm achieves an exponential speedup compared with the classical algorithm. Cong et al. implemented a quantum LDA algorithm which has
an exponential speedup in the scales of the original data set compared with the classical algorithm [26]. In [27], Duan et al. studied the AOP algorithm, and proposed its quantum counterpart. Duan et al.’s algorithm is an iterative algorithm and was expected to have a complexity \(O(\text{polylog}(nk/\epsilon))\), where \(s\) is the number of iterations, \(n\) is the dimensionality of the original feature space, \(k\) is dimensionality of the reduced feature space and \(\epsilon\) is the desired precision of the output state.

In this paper, we reanalyze Duan et al.’s algorithm and correct the complexity to \(O(\frac{\kappa k}{\epsilon} \text{polylog}(\frac{n}{\epsilon}))\), where \(\kappa\) is the condition number of a matrix that related to the original data set, \(m\) is the number of data points. We find that in Duan et al.’s algorithm, multiple copies of the current candidate are consumed to improve the candidate by quantum phase estimation and post selection in each iteration, which results in the total complexity having exponential dependence on the number of iterations \(s\). Thus Duan et al.’s algorithm can only be beneficial for high dimensional problems with a small \(s\), which limits the practical application of the algorithm. To get a further speedup and reduce the dependence on \(s\), we propose an improved quantum AOP algorithm with complexity \(O(\frac{\kappa k}{\epsilon} \text{polylog}(\frac{n}{\epsilon}) + \frac{n}{\epsilon^4} \text{polylog}(\frac{m}{\epsilon}))\). Note that in Duan et al.’s algorithm, one only change the amplitude of the candidate in each iteration. In our algorithm, we process the amplitude information of the candidate in computational basis to reduce the consumption of the copies of the current candidate. The complexity of our algorithm has a quadratic dependence on \(s\) rather than an exponential dependence that achieves a significant speedup over Duan et al.’s algorithm. Also, it shows an exponential speedups over the classical algorithm on \(n\) and \(m\), when both \(\kappa\), \(k\) and \(1/\epsilon\) are \(O(\text{polylog}(nm))\).

The rest of this paper is organized as follows. In Sec. II, we review the classical AOP algorithm in part A and its quantum version in part B, and analyze the complexity of Duan et al.’s in part C. We then propose our quantum AOP algorithm and analyze the complexity in Sec. III. The conclusion is given in Sec. IV.

II. REVIEW OF THE CLASSICAL AND QUANTUM AOP ALGORITHM

In this section, we will briefly review the AOP algorithm in part A. Duan et al.’s algorithm will be introduce in part B and we will analyze its complexity in part C.

A. Review of AOP algorithm

Suppose \(X = (x_1, x_2, \ldots, x_m)\) is a data matrix with dimension \(n \times m\), where \(n\) is the number of the features and \(m\) is the number of data points. The objective of the AOP is to find the optimal projection matrix \(A \in n \times k\) which minimizes the trace of the covariance matrix of regression parameters to reduce the dimensionality of \(X\).

In He et al. [23], a graph regularized regression model was chosen and thus the optimal projection matrix \(A\) can be obtained by solving the following objective function:

\[
\min_A \text{Tr} \left[ (A^T X (I + \lambda_1 L) X^T A + \lambda_2 I)^{-1} \right],
\]

where \(\lambda_1\) and \(\lambda_2\) are the regularized coefficients, \(L = \text{diag}(S I) - S\) is graph Laplacian where \(S\) is the weight matrix of the data points and \(I\) is a vector of all ones. Let \(N_k(x)\) denotes the \(k\) nearest neighbors of \(x\), a simple definition of \(S\) is as follows:

\[
S_{i,j} = \begin{cases} 
1, & \text{if } x_i \in N_k(x_j) \text{ or } x_j \in N_k(x_i); \\
0, & \text{otherwise}.
\end{cases}
\]

To solve the optimization problem (1), He et al. introduced a variable \(B\) and proposed the following theorem [23]:

**Theorem 1.** (Theorem 4.3 in [23]) The optimization problem (1) is equivalent to the following optimization problem:

\[
\min_{A,B} \left\| I - A^T \tilde{X} B \right\|^2 + \lambda \| B \|^2,
\]

where \(\tilde{X} = X\Sigma\) and \(I + \lambda_1 L = \Sigma \Sigma^T\).

Then we can use the iterative method to find the optimal \(A\). The procedure of computing the projection matrix \(A\) can be summarized as follows:

1. Initialize the matrix \(A\) by computing the PCA of the data matrix \(X\);
2. Computing matrix \(B\) according to equation (4):

\[
B = \left( \tilde{X}^T A A^T \tilde{X} + \lambda_2 I \right)^{-1} \tilde{X}^T A.
\]
3. Computing matrix \(A\) according to equation (5):

\[
A = \left( \tilde{X} B B^T \tilde{X}^T \right)^{-1} \tilde{X} B.
\]

Normalize \(A\) to satisfy \(\|A\|_F \leq \rho\) (\(\rho\) is a constant and here we set it to 1).

4. Repeat steps 2 and 3 until convergence.

Since the AOP algorithm involves matrix multiplication and inversion, the complexity of the classical algorithm is \(\Omega(s\text{poly}(nm))\), where \(s\) is the number of iterations.

B. Review of Duan et al.’s algorithm

In [27], the authors reformulated the iterative method of AOP to make the algorithm suitable for quantum settings. They adjusted the initialization of matrix \(A\),
Duan et al.’s algorithm can be summarized as follows: 

1. Initialize the matrix $A(0)$ by computing the PCA of the data matrix $\tilde{X}$, 
   $$A(0) = \text{PCA}(\tilde{X}) = \sum_{j=0}^{k-1} |u_j\rangle \langle j|,$$ 
   where $A(i)$ is the matrix $A$ of iteration $i$, $k$ is the rank of $A(0)$ and $|j\rangle$ is the computational basis state.

2. Update the matrix $A$ according to the following equation
   $$A(i) = \sum_{j=0}^{k-1} \beta_j^{(i)} |u_j\rangle \langle j| = \sum_{j=0}^{k-1} \left( \sigma_j \beta_j^{(i-1)} + \lambda_2 \right) |u_j\rangle \langle j|,$$
   where $\beta_j^{(i)}$ is the singular value of $A(i)$ with corresponding left and right singular vectors $|u_j\rangle$ and $|j\rangle$, $c^{(i)}$ is a constant to ensure that $\|A(i)\|_F \leq 1$.

3. Repeat step (2) until convergence.

Duan et al.’s algorithm can be summarized as follows:

1. Initialize $i = 0$, and prepare the state $|\psi_{A(0)}\rangle$, where
   $$|\psi_{A(0)}\rangle = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} |u_j\rangle \langle j|.$$  

2. Suppose the quantum state $|\psi_{A(i-1)}\rangle$ is given, prepare the following state:
   $$|\psi_0^{(i-1)}\rangle = |0\rangle^D (|0\rangle...|0\rangle)^C (|0\rangle...|0\rangle)^B |\psi_{A(i-1)}\rangle^A$$
   $$= |0\rangle^D \sum_{j=0}^{k-1} \beta_j^{(i-1)} (|0\rangle...|0\rangle)^C (|0\rangle...|0\rangle)^B (|u_j\rangle \langle j|)^A,$$
   where the superscripts $D, C, B, A$ represent the register $D, C, B, A$ respectively.

3. Perform phase estimation on the $|\psi_0^{(i-1)}\rangle$ for the unitary $e^{i\tilde{X} \lambda t_0}$ and $e^{iA(i-1)\tilde{X} \lambda t_0}$,
   $$|\psi_1^{(i-1)}\rangle =$$
   $$|0\rangle^D \sum_{j=0}^{k-1} \beta_j^{(i-1)} \sigma_j^2 |\beta_j^{(i-1)}|^2 B(|u_j\rangle \langle j|)^A.$$  

4. Perform an appropriate controlled rotation on the register $B, C$ and $D$, transforms the system to:
   $$|\psi_2^{(i-1)}\rangle = \sum_{j=0}^{k-1} \beta_j^{(i-1)} |\sigma_j^2 C| (|\beta_j^{(i-1)}|^2 B (|u_j\rangle \langle j|)^A$$
   $$(\sqrt{1 - \rho^2 f(\sigma_j, \beta_j^{(i-1)})^2 |0\rangle + \rho f(\sigma_j, \beta_j^{(i-1)}) |1\rangle)^D,$$
   where $\rho$ is a constant to ensure $|\rho f(\sigma_j, \beta_j^{(i-1)})| \leq 1$.

5. Measure the register $D$, then uncompute the register $C, B$ and $A$, and remove the register $C, B$. Conditioned on seeing 1 in $D$, we have the state
   $$|\psi_3^{(i-1)}\rangle = \frac{1}{\sqrt{N^{(i)}}} \sum_{j=0}^{k-1} \frac{(\sigma_j \beta_j^{(i-1)})^2 + \lambda_2}{\sigma_j^2 \beta_j^{(i-1)}} |u_j\rangle \langle j|$$
   $$= \sum_{j=0}^{k-1} \beta_j^{(i)} |u_j\rangle \langle j| = |\psi_{A(i)}\rangle,$$
   where $N^{(i)} = \sum_{j=0}^{k-1} (\sigma_j \beta_j^{(i-1)})^2 + \lambda_2$. Thus $\beta_j^{(i)} \in [0, 1]$ for $j \in \{0, 1, 2, ..., k-1\}$ and $i \geq 0$.

6. For $i = 1$ to $s - 1$, repeat step 2 to 5.

C. Complexity Analysis of Duan et al.’s algorithm

In [23], the authors analyzed the time complexity of each iteration (step 2 to step 5 in this paper) and claimed that the total time complexity is the product of the number of iterations and the complexity of each iteration. Actually, in the $i$th iteration, the algorithm have to prepare the state $|\psi_{A(i-1)}\rangle$ several times to perform phase estimation in step 3 and do measurement to obtain a appropriate state in step 5, which means that the total time complexity is exponential on the number of iterations $s$. The complexity of each step can be see in TABLE I and the proof details can be see in appendix A.

| Steps | Time complexity |
|-------|-----------------|
| Step 1 | $O(\log(e^\epsilon) \log(nk))$ |
| Step 3 | $O((G^{(1-1)}/\epsilon_1) \log(1/\epsilon_1)) + O((1/\epsilon_1) \log(nm/\epsilon_1))$ |
| Step 4 | $O(\log(1/\epsilon))$ |
| Step 5 | $O(\kappa^2 \text{ repetitions})$ |

Here the step (3-5) is the steps of the $i$th iteration and we neglect the runtime of step 2, $G^{(1-1)}$ is the time complexity to prepare state $|\psi_{A(i-1)}\rangle$, $\kappa$ is the condition number of $\tilde{X}$, $\epsilon$ is the desired precision of the output state, $\epsilon_1 = O(\frac{\sqrt{\kappa}}{\kappa^2 \epsilon})$. 

TABLE I. The complexity of each step of Duan et al.’s algorithm.
Putting all together, the runtime of the $i$th iteration (i.e., preparing the state $|\psi_{A(i)}\rangle$) is
\[
G(i) = O\left(\frac{G(i-1)}{\epsilon} \log(\frac{1}{\epsilon}) + \frac{1}{\epsilon} \text{polylog}(\frac{mn}{\epsilon}) + \text{polylog}(\frac{1}{\epsilon})\right)^2
= O\left(\frac{k^4\sqrt{k}G(i-1)}{\epsilon} \text{polylog}(\frac{mn}{\epsilon})\right) = O(TG(i-1)),
\]
where $T = (k^4\sqrt{k}/\epsilon)\text{polylog}(mn/\epsilon)$. Since $G(0) = O(\log(1/\epsilon) \log(nk))$, the overall time complexity of the algorithm is
\[
G(s) = O(TG(s-1)) = O(T^s G(0))
= O\left(\frac{k^{4s}\sqrt{k}G(0)}{\epsilon^s} \text{polylog}^s(mn/\epsilon)\right).
\]

III. AN IMPROVED QUANTUM AOP ALGORITHM

In this section, we present an improved quantum AOP algorithm.

In Duan et al.’s algorithm, to get the state $|\psi_{A(i)}\rangle$ from state $|\psi_{A(i-1)}\rangle$, one performs phase estimation and post selection, which consumes multiple copies of $|\psi_{A(i-1)}\rangle$, thus the number of copies of the initial state $|\psi_{A(0)}\rangle$ required depends exponentially on the number of iteration $s$. Note that in each iteration of Duan et al.’s algorithm, one only changes the eigenvalue $\beta_j$ for $j = 0, 1, \ldots, k - 1$. In our algorithm, we put the calculation into the computational basis to reduce the consumption of the copies of $|\psi_{A(i-1)}\rangle$.

A. An improved quantum AOP algorithm

The specific process of our quantum algorithm is as follows.

(1) Initialization Initialize $i = 0$, and prepare the state $|\psi_{A(0)}\rangle = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} |u_j\rangle |j\rangle$.

(2) Prepare state $|\psi_0\rangle$ Perform phase estimation with precision parameters $\epsilon$ on the state $|\psi_{A(i)}\rangle$ for the unitary $e^{iX^1t_0}$, and then append state $|\frac{1}{\sqrt{k}}\rangle|0\rangle$, thus we obtain
\[
|\psi_{0}\rangle = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} (|u_j\rangle |j\rangle)^A \sigma_j^2 |B\rangle |\frac{1}{\sqrt{k}}\rangle C |0\rangle^D,
\]
where $\sigma_j^2 = \frac{1}{\sqrt{k}}$ for $j = 0, 1, \ldots, k - 1$, the superscript $A, B, C, D$ represent the register $A, B, C$ and $D$, respectively (in the absence of ambiguity, we omit these superscripts below for the sake of simplicity).

Assuming that we can prepare the state $|\psi_{1-1}\rangle$, where
\[
|\psi_{1-1}\rangle = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} |u_j\rangle |j\rangle |\sigma_j^2\rangle |\beta_{j(i-1)}\rangle |0\rangle.
\]
Thus we could perform quantum arithmetic operations to get
\[
|\phi_{1}(i)\rangle = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} |u_j\rangle |j\rangle |\sigma_j^2\rangle |\beta_{j(i-1)}\rangle |c(i)\rangle |\beta_j(i)\rangle,
\]
where $c(i)\beta_j(i) = (\sigma_j^2\beta_{j(i-1)}^2 + \lambda_2)\beta_{j(i)}^2 + \frac{k-1}{k}(\beta_j(i))^2 = 1$.

In order to obtain the information of $\beta_j(i)$, we will estimate $c(i)$ first, then we can prepare the state
\[
|\phi_{2}(i)\rangle = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} |u_j\rangle |j\rangle |\sigma_j^2\rangle |\beta_{j(i-1)}\rangle |\beta_j(i)\rangle := |\phi_3(i)\rangle
\]
from the state $|\phi_1(i)\rangle$.

(3) Estimate $c(i)$ Assuming that we can prepare the state $|\psi_{1-1}\rangle$ in time $G_{i-1}$.

(i) Prepare the state $|\phi_1(i)\rangle$ from the state $|\psi_{1-1}\rangle$.

(ii) Add an ancillary qubit (register $E$) and perform an appropriate controlled rotation on the registers $D$ and $E$, transforms the system to:
\[
|\phi_2(i)\rangle = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} (|u_j\rangle |j\rangle)^A |\sigma_j^2\rangle |\beta_{j(i-1)}\rangle |c(i)\rangle |\beta_j(i)\rangle^D
\[
= (1 - \frac{c(i)|\beta_j(i)|}{c(i)|\beta_{j(i-1)}|}) |0\rangle + \frac{c(i)|\beta_j(i)|}{c(i)|\beta_{j(i-1)}|} |1\rangle^E
\]
\[
:= \cos(\theta)|a\rangle|0\rangle^E + \sin(\theta)|b\rangle|1\rangle^E,
\]
where the parameter $c$ is a constant to ensure $c(i)\beta_j(i) = 1$,
\[
|a\rangle = \sum_{j=0}^{k-1} \sqrt{|\frac{c(i)|\beta_j(i)|}{c(i)|\beta_{j(i-1)}|} - (\frac{c(i)|\beta_j(i)|}{c(i)|\beta_{j(i-1)}|})^2} |u_j\rangle |j\rangle |\sigma_j^2\rangle |\beta_{j(i-1)}\rangle |c(i)\rangle |\beta_j(i)\rangle^E,
\]
\[
|b\rangle = \sum_{j=0}^{k-1} \beta_j(i) |u_j\rangle |j\rangle |\sigma_j^2\rangle |\beta_{j(i-1)}\rangle |c(i)\rangle |\beta_j(i)\rangle^E,
\]
\[
\sin(\theta) = \frac{c(i)}{c(i)\beta_j(i)},
\]

(iii) Perform quantum amplitude estimation to estimate $\sin(\theta)$. Then we can obtain the classical information of $c(i)$ by $c(i) = \sqrt{k}\sin(\theta)$.

(4) Prepare state $|\psi_3\rangle$

(i) Since we have the classical information of $c(i)$, we can perform quantum arithmetic operation to get
\[
|\phi_3(i)\rangle = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} |u_j\rangle |j\rangle |\sigma_j^2\rangle |\beta_{j(i-1)}\rangle |\beta_j(i)\rangle^D.
\]
Thus we obtain state $|\psi_{A(\alpha)}\rangle$. Then followed by controlled rotation, uncomputing and measurement, we could obtain the desired state $|\psi_{A(\alpha)}\rangle$. However, it requires much more space resource than algorithm. To reduce the space complexity, we transform the register $C$ to $|0\rangle$ and only keep $|\beta_j^{(i)}\rangle$ after iteration $i$, i.e., obtain state $|\psi_i\rangle = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} |u_j\rangle |j\rangle |\sigma_j^2| |\beta_j^{(i)}\rangle |D\rangle |0\rangle_C$.

(ii) Perform quantum arithmetic operation on register $B$, $C$, and $D$, to get $|\psi_i\rangle$.

Since $\sigma_j^2 |\beta_j^{(i)}\rangle^2 = \frac{\sigma_j^2 |\beta_j^{(i)}\rangle^2 + \lambda_2}{\lambda_2}$, we have

$$\sigma_j^2 |\beta_j^{(i-1)}\rangle^2 - c(i) \lambda_2 |\beta_j^{(i)}\rangle + \lambda_2 = 0,$$

which is a one-variable quadratic equation about the variable $|\beta_j^{(i-1)}\rangle$. The solutions of the equation is

$$|\beta_j^{(i-1)}\rangle = \frac{c(i) |\beta_j^{(i)}\rangle \pm \sqrt{c(i) |\beta_j^{(i)}\rangle^2 - 4 |\beta_j^{(i)}\rangle^2 \lambda_2}}{2 |\beta_j^{(i)}\rangle}.$$

Two cases are considered here. In case 1, when $\lambda_2 \geq 1$, then for $x \leq 1$, the function $f(x) = \frac{(\sigma_j^2 |\beta_j^{(i)}\rangle x^2 + \lambda_2}{\lambda_2}$ is a monotonic decreasing function, which means that $|\beta_j^{(i-1)}\rangle = |\beta_j^{(i)}\rangle$. In case 2, when $\lambda_2 < 1$, we add a qubit (register $F$) to store the magnitude relationship between $|\beta_j^{(i)}\rangle$ and $\sqrt{\frac{1}{\sigma_j^2}}$ on the state in equation (19), i.e.,

$$|\phi_i^F\rangle = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} |u_j\rangle |j\rangle |\sigma_j^2| |\beta_j^{(i-1)}\rangle |D\rangle |\gamma_j^{(i)}\rangle |F\rangle,$$

where

$$|\gamma_j^{(i)}\rangle = \begin{cases} |1\rangle, & \text{if } |\beta_j^{(i-1)}\rangle \geq \sqrt{\frac{1}{\sigma_j^2}}, \\ |0\rangle, & \text{if } |\beta_j^{(i-1)}\rangle < \sqrt{\frac{1}{\sigma_j^2}}. \end{cases}$$

Then, according to the information in register $F$, we could get

$$|\beta_j^{(i-1)}\rangle = \begin{cases} |\beta_j^{(i-1)}\rangle, & \text{if } |\gamma_j^{(i)}\rangle = |1\rangle, \\ |\beta_j^{(i-1)}\rangle, & \text{if } |\gamma_j^{(i)}\rangle = |0\rangle. \end{cases}$$

Thus we could transform the state of register $C$ to $|0\rangle$ by a simple quantum arithmetic operation on register $C$ and $D$. We should keep in mind that we need an ancillary qubit in each iteration for case 2.

(5) Iteration For $i = 1$ to $s - 2$, repeat step 2 to 4. Thus we obtain state

$$|\psi_{s-1}\rangle = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} |u_j\rangle |j\rangle |\sigma_j^2| |\beta_j^{(s-1)}\rangle |0\rangle.$$

(6) Control rotation Add an ancillary qubit (register $E$) and perform an appropriate controlled rotation on the state $|\psi_{s-1}\rangle |0\rangle_F$, transforms the system to

$$|\phi_2^{(s)}\rangle = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} |u_j\rangle |j\rangle |\sigma_j^2| |\beta_j^{(s-1)}\rangle |c(i) |\beta_j^{(s)}\rangle |0\rangle + \frac{\epsilon^{(s)} |\beta_j^{(s)}\rangle |1\rangle}.\tag{23}$$

(7) Uncomputing and measurement Uncompute register $B$, $C$, and $D$, and measure the register $E$ to seeing 1, thus obtain

$$|\psi_{A(\alpha)}\rangle = \sum_{j=0}^{k-1} \beta_j^{(s)} |u_j\rangle |j\rangle.$$

B. The complexity of the improved quantum AOP algorithm

We have described an improved quantum AOP algorithm above. In this subsection, we will analyze the time complexity of the algorithm.

The time complexity of step 1 is $O(\log(\epsilon^{-1}) \log(nk))$, the same as Duan et al.’s algorithm.

In step 2, similar to the complexity of the step (3) of Duan et al.’s algorithm, the phase estimation stage is of complexity $O(\frac{1}{\epsilon} \text{polylog}(\frac{1}{\epsilon}))$ with error $\epsilon_1$. The stage of appending registers $|\frac{1}{\sqrt{k}} |C\rangle |0\rangle_D$ is of complexity $O(\log(\frac{1}{\epsilon_1}))$, where the number of qubits in register $B$, $C$, and $D$ is $O(\log(\frac{1}{\epsilon_1}))$. Thus the complexity of this step is $O(\frac{1}{\epsilon} \text{polylog}(\frac{1}{\epsilon})).$

In step 3, since the complexity of preparing the state $|\psi_{i-1}\rangle$ is much greater than the complexity of stage (i) and stage (ii) (which is $O(\text{polylog}(1/\epsilon_1))$ and $O(\text{log}(1/\epsilon_1))$ respectively), we will neglect the complexity of these two stages. In stage (iii), define

$$U_{i-1} : U_{i-1} |0\rangle = |\phi_2^{(i)}\rangle = \sin(\theta) |a\rangle |0\rangle + \cos(\theta) |b\rangle |1\rangle,$$

$$S_0 : S_0 = I - 2 |0\rangle_A B C D E |0\rangle_A B C D E,$$

$$S_X : S_X = I - 2 |1\rangle_A B C D E |1\rangle_A B C D E.$$

According to quantum amplitude estimation algorithm \cite{28,29}, the unitary operator $Q = -U_{i-1} S_0 U_{i-1}^\dagger S_X$ act as a rotation on the two dimensional space $\text{Span}(|a\rangle |0\rangle, |b\rangle |1\rangle)$, with eigenvalues $e^{\pm 2i\theta}$ and corresponding eigenvectors $\frac{|a\rangle |0\rangle \pm |b\rangle |1\rangle}{2}$, thus the complexity of $U_{i-1}$ is $O(G_{i-1})$. The quantum amplitude estimation algorithm will generate $\theta$ within error $\epsilon_2$, the corresponding complexity is $O(\frac{1}{\epsilon_2} (2 + \frac{1}{\epsilon_2}) G_{i-1})$, where $1 - \eta$ is the probability to success (we could simply choose $\eta = O(1)$).

Finally, we obtain the classical information of $c(i)$ within relative error $O(\kappa^2 c_2)$ (see appendix [12]), here we use relative error to ensure that the estimation of $c(i)$ won’t be influence by the scale of $c(i)$.

In step 4, for the stage (i), similar to the analysis of Duan et al.’s algorithm, we want to bound the relative
error of $\beta^i_\delta$ (denote as $\hat{\epsilon}_\delta$) by $O(\epsilon)$. According to Appendix [13], $\hat{\epsilon}_\delta = O(\sqrt{\kappa} + \epsilon)$, thus we can choose $\epsilon_1 = O(\sqrt{\kappa})$ and $\epsilon_2 = O(\sqrt{\kappa} / \epsilon)$ to ensure $\hat{\epsilon}_\delta = O(\epsilon)$. Since the classical information of $c^{(i)}$ is given by step (3), this stage is of complexity $O(G_{i-1} + \text{polylog}(\frac{\kappa k}{\epsilon^2}))$. As for the stage (ii), the time complexity of the two cases are $O(\text{polylog}(\frac{\kappa k}{\epsilon^2}))$.

In step 6, the time complexity is $O(\text{polylog}(\epsilon_1)) = O(\text{polylog}(\frac{\kappa k}{\epsilon})), \text{ which is similar with the stage (ii) and stage (ii) of step 3.}$

In step 7, the time complexity of the uncomputing stage is just the same as the complexity to prepare state $|\phi^{(s)}_2\rangle$ (equation (23)) from state $|\psi_{A(0)}\rangle$. For $c^{(s)} = O(k)$ and $c = O(\sqrt{\kappa k^2})$, the probability of seeing 1 is

$$p(1) = \frac{1}{k} \sum_{j=0}^{k-1} (\frac{c^{(s)}}{c} |\beta^j_\delta|)^2 = O(\frac{1}{k^2}).$$

We can use the quantum amplitude amplification [28, 29] to reduce the repetition to $O(\kappa^2)$. Now we put all together. From the analysis of step 3, we know that the complexity to estimate $c^{(i)}$, $i = 1, ..., s$ is $O(\kappa^2 G_{i-1})$. Also, from step (4), given $c^{(i)}$, the complexity to prepare state $|\psi_i\rangle$ is

$$G_i = G_{i-1} + 2\text{polylog}(\frac{\kappa k}{\epsilon}).$$

Thus given $c^{(i)}$ for $i = 1, 2, ..., s - 1$, $G_{s-1} = G_0 + 2(s - 1)\text{polylog}(\frac{\kappa k}{\epsilon})$, (27)

where $G_0 = \frac{s^2 \sqrt{\kappa}}{\epsilon} \text{polylog}(\frac{mn}{\epsilon})$.

The complexity to obtain state $|\psi_{A(i)}\rangle$ is as follows:

$$O(|\psi_{A(i)}\rangle) = O(\kappa^2 G_{i-1})$$

$$= O(\kappa^2 (\text{polylog}(\frac{\kappa k}{\epsilon}) + G_{s-1} + \sum_{i=1}^{s-1} c^{(i)}))$$

$$= O\left(\frac{s^2 \sqrt{\kappa}}{\epsilon} \text{polylog}(\frac{mn}{\epsilon}) + \frac{s^2 \kappa^4}{\epsilon} \text{polylog}(\frac{\kappa k}{\epsilon})\right).$$

Compared with the complexity of Duan et al.'s algorithm, which is $O(\frac{s^2 \sqrt{\kappa}}{\epsilon} \text{polylog}(\frac{mn}{\epsilon}))$, the complexity of our algorithm has only a quadratic dependence on $s$, which achieves a nearly exponential speedup if $s$ is large and a polynomial speedup even if $s$ is a small constant. When the parameters $\kappa$, $k$ and $1/\epsilon$ are $O(\text{polylog}(mn))$, our algorithm has exponential speedups on $n$ and $m$ over the classical algorithm.

IV. CONCLUSION

In this paper, we reanalyzed Duan et al.'s algorithm in detail and corrected the complexity calculation. It was shown that Duan et al.'s algorithm has an exponential dependence on the number of iterations $s$, thus the quantum algorithm may lose its advantage as $s$ increases. To get a further speedup, we presented an improved quantum AOP algorithm with complexity quadratic on $s$. Our algorithm achieves at least a polynomial speedup over Duan et al.'s algorithm, especially achieves nearly an exponential speedup when $s$ is large. When both $k$, $1/\epsilon$ are $O(\text{polylog}(mn))$, our algorithm achieves exponential speedups compared with the classical algorithm on $n$ and $m$.

The speedups of our algorithm mainly comes from the idea of putting the information to be updated into computational basis, which saves the consumption of the current candidates. We hope this idea could inspire more iterative algorithms to get a quantum speedup. We will explore the possibility in the future.

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Appendix A: The complexity of each step of Duan et al.'s algorithm

In this appendix, we analyze the time complexity of each step of Duan et al.'s algorithm in detail. Different with the original paper [27], on the one hand, we use the best known results on Hamiltonian simulation to get a tight bound of the time complexity, on the other hand, we estimate the parameters which have not been estimate in the original paper or need to be correct.

In step 1, $|\psi_{A(0)}\rangle$ can be written in computational basis as $\sum_{x_1 x_2...x_l \in \{0, 1\}}^{\alpha x_1 x_2...x_l} |x_1 x_2...x_l\rangle$, where $l = k \log n$. By the assumption that the elements of $A(0)$ and $\omega^{(i)}$ are given and stored in QRAM, where $\omega^{(i)}$ is defined as

$$\cos^2(2\pi \omega^{(i)}) = \left(\frac{2x_1 x_2...x_{l-1}}{x_1 x_2...x_l}\right)^2 + O(\text{poly}(\epsilon)),$$

the complexity for preparing $|\psi_{A(0)}\rangle$ is $O(\log(e^{-1}) \log(nk))$ [27].

In step 3, Assume the condition number of $X$ and $A^{(i-1)}$ is $\kappa$ and $\kappa^{(i-1)}$, then $\kappa^{(i-1)} = O(\kappa^2)$ (the proof is given in appendix [4]). We should mention that in the original paper [27], the author gave a wrong estimate of $\kappa^{(i-1)}$, which influenced the total complexity). Note that

$$\text{tr}_2(|\psi^{(i-1)}_A\rangle \langle \psi^{(i-1)}_A|) = \sum_{j=1}^{k} \langle \beta_j^{(i-1)} | u_j \rangle | u_j \rangle,$$

According to Corollary [1] (Corollary 17 of [30]), the time complexity to simulate $e^{iA^{(i-1)}(t_0 + \log(1/\epsilon))}G^{(i-1)}$, where $G^{(i-1)}$ is the
time complexity to prepare state \(|\psi_A^{(i-1)}\rangle\). For the simulation of \(\tilde{X} \tilde{X}^\dagger\), assume there is a quantum circuit to prepare state \(|\psi_{\tilde{X}}\rangle = \frac{1}{\|\tilde{X}\|_F} \sum_{j=0}^{\tau_{\tilde{X}}} \sum_{j=0}^{m-1} \tilde{X}_j |i\rangle |j\rangle\) in time \(O(\text{polylog}(nm))\). Notice that \(\tilde{X} \tilde{X}^\dagger = \|\tilde{X}\|_F^2 \text{tr}_2(|\psi_{\tilde{X}}\rangle\langle\psi_{\tilde{X}}|)\), where \(\|\tilde{X}\|_F = \sqrt{\sum_{j=1}^{\tau_{\tilde{X}}} \sigma_j^2} \leq \sqrt{\tau} = O(\text{polylog}(nm))\). Thus the complexity to simulate \(e^{i\tilde{X} \tilde{X}^\dagger} t_0\) within \(\epsilon_0\) is \(O\left(\text{polylog}(nm)\left(\|\tilde{X}\|_F^2 t_0 + \log(1/\epsilon_0)\right)\right)\).

**Corollary 1.** \((30)\) Given access to the oracle \(\tilde{G}\) specifying a Hamiltonian \(\tilde{H} = \rho\) that is a density matrix \(\hat{\rho}\), where \(\tilde{G}(0) = |G\rangle = \sum_j \sqrt{\alpha_j} |j\rangle \langle j|\), \(\hat{\rho} = \text{tr}(G^\dagger G)|a_1\rangle \langle a_2|\), (A2)
time evolution by \(\hat{H}\) can be simulated for time \(t\) and error \(\epsilon\) with \(O(t + \log(1/\epsilon))\) queries.

According to \(28, 31, 32\), taking \(O(1/\epsilon_1)\) times of controlled-\(e^{i\tilde{A} \tilde{A}^\dagger} t_0\) to perform phase estimation ensures that the eigenvalues \(\beta_j^{\tilde{A}^{(i-1)}}\) being estimated with error \(O(\epsilon_1)\), so as the phase estimation on \(e^{i\tilde{X} \tilde{X}^\dagger} t_0\). Let \(\epsilon_0 = \epsilon_1^2\) and \(t_0 = O(1)\), the complexity of the two phase estimations are \(O\left((\tilde{G}(i-1)/\epsilon_1) \log(1/\epsilon_1)\right)\) and \(O\left((1/\epsilon_1)\text{polylog}(nm/\epsilon_1)\right)\) respectively.

The implementation of controlled rotation of step 4 can be divide into two stages \(27\). The first stage is a quantum circuit to compute \(y_j = \rho \left(\frac{\sigma_j \beta_j^{\tilde{A}^{(i-1)}} + \lambda_2}{\sigma_j \beta_j^{\tilde{A}^{(i-1)}}}\right)^2\) and store in an auxiliary register \(L\) with \(O(\log(1/\epsilon))\) qubits, where \(\rho = O(\frac{1}{\sqrt{k - \epsilon}})\) (the proof is given in appendix \(C\) we should mention that in the original paper \(27\), the authors did not analyze the parameter \(\rho\) which has a strong correlation with the total complexity).

Since \(\beta_j^{\tilde{A}^{(i-1)}}\) and \(\sigma_j\) is estimate with error \(O(\epsilon_1)\), the relative error of estimating \(y_j\) is \(\epsilon_j = O\left(\frac{\lambda_2 (\beta_j^{\tilde{A}^{(i-1)}} - \sigma_j) \beta_j^{\tilde{A}^{(i-1)}} + \lambda_2 (2 \beta_j^{\tilde{A}^{(i-1)}})}{\lambda_2 (\beta_j^{\tilde{A}^{(i-1)}} + \sigma_j) \beta_j^{\tilde{A}^{(i-1)}}}\right)^2 \epsilon_1\)
\(= O\left(\frac{\lambda_2 (\beta_j^{\tilde{A}^{(i-1)}} + \sigma_j) \beta_j^{\tilde{A}^{(i-1)}}}{\lambda_2 (\beta_j^{\tilde{A}^{(i-1)}} + \sigma_j)^2}\right)^2 \epsilon_1\)
\(= O\left(\frac{\lambda_2 (\beta_j^{\tilde{A}^{(i-1)}} + \sigma_j) \beta_j^{\tilde{A}^{(i-1)}}}{\lambda_2 (\beta_j^{\tilde{A}^{(i-1)}} + \sigma_j)^2}\right)^2 \epsilon_1\)
in the first equation we neglect the terms with the power of \(\epsilon_1\) greater than 1 and in the last equation we use the conclusion of equation \(C1\). To ensure that the final error of this iteration is within \(O(\epsilon)\), we could take \(\epsilon_j = O(\epsilon)\), which means \(\epsilon_1 = O(\frac{\epsilon}{\sqrt{k - \epsilon}})\). Following the result of \(33\), the complexity of this stage is \(O(\text{polylog}(1/\epsilon))\).

The second stage is performing control rotation \(CR\) on the state, where \(CR|y_j^{D} 0\rangle = |y_j^{D} (\sqrt{1 - y_j^{2}} 0) + y_j^{D} 1\rangle\). The complexity of this stage is \(O(\log(1/\epsilon))\) \(3, 9–11, 13, 14\).

In step 5, the probability to seeing 1 in register D is \(p(1) = \rho^2 \sum_j \left(\frac{\sigma_j \beta_j^{\tilde{A}^{(i-1)}} + \lambda_2}{\sigma_j \beta_j^{\tilde{A}^{(i-1)}}}\right)^2 = O\left(\frac{1}{\epsilon}\right)\) as shown in Appendix C (We should mention that in the original paper \(27\), the authors did not analyze this probability which is directly related to the total complexity). Using amplitude amplification \(29\), we find that \(\text{O}(\kappa^2)\) repetitions are sufficient.

**Appendix B: Estimate the parameter \(\kappa^{(i)}\)**

In this appendix, we analyze the condition number \(\kappa^{(i)}\) of \(A^{(i)}\). Firstly, we give the following theorem:

**Theorem 2.** If \(\beta_j^{(i-1)} > \beta_j^{(i-1)}\), then \(\beta_j^{(i)} > \beta_j^{(i)}\).

**Proof.** (1) Note that \(\beta_0^{(0)} = \beta_0^{(0)} = \ldots = \beta_{k-1}^{(0)}\) and \(1 = \sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_{k-1} = \frac{1}{\kappa}\). For \(j > j'\), the following inequalities hold:

\(\beta_j^{(i)} > \beta_j^{(i)}\),
\(\sigma_j^{2} \beta_j^{(i)} \leq \sigma_j^{2} \beta_j^{(i)}\).
(B2)

(2) Assuming that for \(i \geq 1\), the following inequalities hold:

\(\beta_j^{(i-1)} \geq \beta_j^{(i-1)}\),
\(\sigma_j^{2} \beta_j^{(i-1)} \leq \sigma_j^{2} \beta_j^{(i-1)}\).

Then

\(\sigma_j^{2} \beta_j^{(i)} = \sigma_j^{2} \left(\frac{\sigma_j \beta_j^{(i-1)}}{\sigma_j \beta_j^{(i-1)}}\right)^2 + \lambda_2\)
\(= \frac{1}{\epsilon^{(i-1)}} \sigma_j^{2} \beta_j^{(i-1)} + \lambda_2\)
\(\leq \frac{1}{\epsilon^{(i-1)}} \sigma_j^{2} \beta_j^{(i-1)} + \lambda_2\)
(B3)

Also,

\(\beta_j^{(i)} = \left(\frac{\sigma_j \beta_j^{(i-1)}}{\sigma_j \beta_j^{(i-1)}}\right)^2 + \lambda_2\)
\(= \frac{1}{\epsilon^{(i-1)}} \left(\beta_j^{(i-1)} + \frac{\lambda_2}{\sigma_j^{2} \beta_j^{(i-1)}}\right)\)
\(\geq \frac{1}{\epsilon^{(i-1)}} \left(\beta_j^{(i-1)} + \frac{\lambda_2}{\sigma_j^{2} \beta_j^{(i-1)}}\right)\)
(B4)

Thus the theorem holds.
Further, \( \kappa \)verges on \( \kappa \).

Appendix C: Estimate the parameter \( \rho \) of step 5 in Duan et al.’s algorithm

In this appendix, we analyze the value of parameter \( \rho \) which first appeared in step 4 of Duan et al.’s algorithm.

It is obvious that \( \sum_{j=1}^{k} (\beta_j^{(i-1)})^2 = 1 \), for \( \beta_j^{(i-1)} (j = 0, 1, \ldots, k - 1) \) is the amplitude of the quantum state \( |\psi_j^{(i-1)} \rangle \). Thus,

\[
\frac{1}{\sqrt{k}} \leq \max_j \beta_j^{(i-1)} \leq 1,
\]

Since \( \rho (\sigma_j \beta_j^{(i-1)})^2 + \lambda_2 (\sigma_j \beta_j^{(i-1)})^2 \leq 1, \rho \leq \min_j (\sigma_j \beta_j^{(i-1)})^2 \). Note that

\[
\min_j \beta_j^{(i-1)} = \frac{\min_j \beta_j^{(i-1)} - \lambda_2}{\sigma_j \beta_j^{(i-1)} + \lambda_2} \geq \frac{\min_j \beta_j^{(i-1)} - \lambda_2}{\sigma_j \beta_j^{(i-1)} + \lambda_2} \geq \frac{1}{(\kappa-1)k} \geq \frac{1}{2\lambda_2 \kappa^2 (\kappa-1)},
\]

for the \( \kappa \) is \( O(\kappa^2) \), we could choose the parameter \( \rho = O(2\lambda_2 \kappa^2) \). Then the probability to seeing 1 in register D in step 5 is

\[
p(1) = \rho^2 \sum_j \left( \frac{(\sigma_j \beta_j^{(i-1)})^2 + \lambda_2}{\sigma_j \beta_j^{(i-1)}} \right)^2 \leq \rho^2 k \left( \frac{1}{\kappa} \max_j \beta_j^{(i-1)} \right)^2 \leq \rho^2 k \left( \frac{1 + \lambda_2 \kappa^2}{\max_j \beta_j^{(i-1)}} \right)^2 \leq \rho^2 k^2 (1 + \lambda_2 \kappa^2)^2 \leq \left( \frac{1}{2\lambda_2 \kappa^4} \right) k^2 (1 + \lambda_2 \kappa^2)^2 = O\left( \frac{1}{\kappa^4} \right).
\]

Appendix D: Estimate the parameter \( \epsilon \) and analyze the relative error of \( \epsilon \) and \( \beta_j^{(i)} \)

In this appendix, we analyze the value of parameter \( \epsilon \) and \( \theta \) first, and then analyze the relative error of \( \epsilon \) and \( \beta_j^{(i)} \) of our algorithm.

Note that

\[
\epsilon \beta_j^{(i)} = \frac{(\sigma_j \beta_j^{(i-1)})^2 + \lambda_2}{\sigma_j \beta_j^{(i-1)}} = \beta_j^{(i-1)} + \frac{\lambda_2}{\sigma_j \beta_j^{(i-1)}} (D1)
\]
According to Theorem \[2\] we have

\[
\epsilon_j^{(i)} \leq \min_j \left( \beta_j^{(i-1)} + \frac{\lambda_2}{\sigma_j^2 \beta_j^{(i-1)}} \right)
= \min_j \beta_j^{(i-1)} + 2 \min_j \beta_j^{(i-1)}.
\]

Similarly, we have \(c_j^{(i)} \beta_j^{(i)} \leq \max_j \beta_j^{(i-1)} + \frac{\lambda_2 k^2}{\max_j \beta_j^{(i-1)}}\). Thus, with \(c_j^{(i)} \beta_j^{(i)} \geq \max_j \beta_j^{(i-1)} + \frac{\lambda_2 k^2}{\max_j \beta_j^{(i-1)}}\), we can choose \(c \geq \max_j \beta_j^{(i-1)} + \frac{\lambda_2 k^2}{\max_j \beta_j^{(i-1)}}\). Since

\[
\max_j \beta_j^{(i-1)} + \frac{\lambda_2 k^2}{\max_j \beta_j^{(i-1)}} \leq 1 + \lambda_2 \sqrt{k} k^2,
\]

we can take \(c = 1 + \lambda_2 \sqrt{k} k^2 = O(\sqrt{k} k^2)\).

Based on equation \(\ref{eq:18}\),

\[
(c(i))^2 = (c(i))^2 \sum_j (\beta_j^{(i)})^2
= \sum_j \left( \beta_j^{(i-1)} + \frac{\lambda_2}{\sigma_j^2 \beta_j^{(i-1)}} \right)^2
= \sum_j \left( \beta_j^{(i-1)} + \frac{2 \lambda_2}{\sigma_j^2 \beta_j^{(i-1)}} \left( \frac{\lambda_2}{\sigma_j^2 \beta_j^{(i-1)}} \right)^2 \right)
\geq 1 + 2 k \lambda_2 + \lambda_2^2 k^2 = O(\lambda^2),
\]

thus \(c(i) = \Omega(k)\). Then according to Eq \(\ref{eq:18}\),

\[
\sin(\theta) = \frac{c(i)}{c \sqrt{k}} = \Omega \left( \frac{k}{k^2} \right) = \Omega \left( \frac{1}{k^2} \right).
\]

The relative error of \(c(i)\) is

\[
\tilde{\epsilon}_c = \left| \frac{c(i) - \tilde{c}(i)}{c(i)} \right| = \left| \frac{c \sqrt{k} \sin(\theta) - c \sqrt{k} \sin(\tilde{\theta})}{c \sqrt{k} \sin(\theta)} \right|
\leq \frac{\Delta \theta}{\sin(\theta)} = O(\kappa^2 \epsilon_2).
\]

Thus the relative error of \(\beta_j^{(i)}\) of step (4) is

\[
\left| \frac{\beta_j^{(i)} - \tilde{\beta}_j^{(i)}}{\beta_j^{(i)}} \right| = \frac{\left( \frac{c(i) \beta_j^{(i-1)} + \lambda_2}{c(i) \sigma_j^2 \beta_j^{(i-1)}} - \frac{c(i) \beta_j^{(i-1)} + \lambda_2}{c(i) \sigma_j^2 \beta_j^{(i-1)}} \right)}{\left( \frac{c(i) \beta_j^{(i-1)} + \lambda_2}{c(i) \sigma_j^2 \beta_j^{(i-1)}} \right)^2 + \frac{c(i) \beta_j^{(i-1)} + \lambda_2}{c(i) \sigma_j^2 \beta_j^{(i-1)}}}
\leq \frac{\sigma_j \lambda_2 + 2 \sigma_j \beta_j^{(i-1)} \lambda_2 - \sigma_j \beta_j^{(i-1)} \beta_j^{(i-1)} \lambda_2}{\sigma_j \beta_j^{(i-1)} \lambda_2}
\leq O(\kappa^2 \sqrt{k} \epsilon_1 + \tilde{\epsilon}_c),
\]

where we neglect the quadratic terms of \(\epsilon_1\) in the second equation, \(\epsilon_1\) is the absolute error of \(\beta_j\) and \(\sigma_j\).

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