\( \mathbb{F}_p((X)) \) IS DECIDABLE AS A MODULE OVER THE ADDITIVE POLYNOMIALS

GÖNENÇ ONAY

Abstract. Let \( R \) be the (non commutative-) ring of additive polynomials over the field \( K := \mathbb{F}_p((X)) \), the henselization of the field \( \mathbb{F}_p(X) \). We show that the (right-) \( R \)-module theory of the field \( \mathbb{F}_p((X)) \) is decidable. Moreover, we provide a recursively enumerable axiom system \( T_1 \) (satisfied by \( \mathbb{F}_p((X)) \)) in the language \( L_{O} \), the language of \( R \)-modules together with a predicate \( O \) for the valuation ring \( \mathbb{F}_p[[X]] \) and show that every primitive positive formula is equivalent to a universal formula modulo \( T_1 \). As an \( L_{O} \)-structure, \( \mathbb{F}_p((X)) \) is also decidable and the \( L_{O} \)-theory of \( \mathbb{F}_p((X)) \) is model-complete admitting \( K \) as its prime model.

1. Introduction

The decidability and the axiomatization of the field of Laurent series over the finite field \( \mathbb{F}_p \) is a longstanding problem. The axiomatization and decidability of the field of \( p \)-adic numbers, follows by the famous Ax-Kochen and Ershov Theorem. As a consequence the similarities between \( \mathbb{Q}_p \) and \( \mathbb{F}_p((X)) \) can be expressed in terms of ultralimits when \( p \to \infty \). The first and maybe the most famous application stands for the corrected form of a conjecture by Artin, which states that given any integer \( d > 0 \) there is some integer \( m \), such that for any prime \( p > m \), every homogeneous polynomial of degree \( d \) over \( \mathbb{Q}_p \) with \( > d^2 \) variables has a non-trivial zero in \( \mathbb{Q}_p \). This application uses simply that fact that for every prime \( p \) the field \( \mathbb{F}_p((X)) \) is \( C_2 \) (i.e. every homogeneous polynomial with strictly more variables than the square of its degree has a non-trivial zero).

Despite the (asymptotic-) analogy between \( \mathbb{Q}_p \) and \( \mathbb{F}_p((X)) \), for fixed prime \( p \) the field theory of \( \mathbb{F}_p((X)) \) stays unknown. Denef and Schoutens showed that its existential theory in the language of rings with a constant symbol for \( X \) is decidable assuming the resolution of singularities in positive characteristic (see [DS03]). Recently Anscombe and Fehm showed unconditionally that its existential theory in the language of rings is decidable [AF16]. However Kuhlmann proved that the naive translation into positive characteristic of the complete theory of \( \mathbb{Q}_p \) is incomplete [Kuh01].

After the works of Kuhlmann and van den Dries, incompleteness of this naive theory can be expressed using only properties of additive polynomials, hence in the language of \( S \)-modules where \( S \) is the ring of additive polynomials over \( \mathbb{F}_p(X) \) (we will explain in details in the following lines). We believe that Rohwer, in his thesis (see [Roh03]), shows that the complete theory of \( \mathbb{F}_p((X)) \) as an \( S \)-module is model-complete in the language of \( S \)-modules together with a predicate for the valuation ring. However, he does not provide an axiom system for this theory. We think that our philosophy is similar to Rohwer’s one but for our results we had been rather inspired from the articles [DDP02], [vdDK02] and [AF16]. Notice that in the Bélaïr and Point also studied the module theory some (valued-) fields satisfying strong divisibility conditions (see [BP10] and [BP15]).
Main results. Let $K$ be the henselization of the field of rational functions over $\mathbb{F}_p$ and
\[ \varphi : K \to K, \ x \mapsto x^p \]
be some (fixed-) power of the Frobenius map. We set $R$ to be the ring of $\varphi$-polynomials, that is, additive polynomials whose monomials are of the form $aT^{p^k}$ equipped with the composition and the usual addition. Let $L := \mathbb{F}_p((X))$ be the field of Laurent series over $\mathbb{F}_p$, let $L$ be the language of (right-) $R$-modules and finally let $L_\mathcal{O}$ be the language $L$ together with the unary predicate $\mathcal{O}$ (for the valuation ring $\mathbb{F}[[X]]$).

We prove in particular the following results in the present article.

1. There is a recursively enumerable $L_\mathcal{O}$-theory $T_1$, $M \models T_1$, such that any completion is model complete. Moreover, $K$ is the prime model of the complete $L_\mathcal{O}$-theory of $M$ (see Theorem 5.5 and following corollaries).

2. Both the complete $L$- and $L_\mathcal{O}$-theories of $M$ are decidable. (see Theorem 5.10).

Strategy and organization of the article. Following Kuhlmann, let us first explain why the naive adaptation of the theory of $p$-adics into positive characteristic is incomplete.

In [Kuh01] it is shown that the following theory $T_{\text{naive}}$, written in language of rings together with a unary predicate for the valuation ring and a constant for the uniformizer $X$, is incomplete. We recall that $(U,v) \models T_{\text{naive}}$ if and only if
- $(U,v)$ is a henselian, non trivially valued field of characteristic $p > 0$,
- the residue field $U/v$ is $\mathbb{F}_p$,
- the value group $\Gamma$ is a $\mathbb{Z}$-group,
- $(U,v)$ is defectless,
- $v(X) = \min \Gamma > 0$.

Set $M = \mathbb{F}_p((X))$. We know that we have:

\[ (*) \quad M = \bigoplus_{i=1}^{p-1} M^p X^i + \varphi(M) + \mathcal{O}_M, \]

where $\varphi$ is the Artin-Scherier map $x \mapsto x^p - x$ and $\mathcal{O}_M$ is the valuation ring of $M$. But the equality (*) fails to hold for some extension $N \supset M$, which is a model of $T_{\text{naive}}$. Hence $T_{\text{naive}}$ is incomplete.

Consider the polynomial $F(z_0, \ldots, z_{p-1}) := z_0^p - z_0 + \sum_{i=1}^{p-1} z_i^p X^i$. From the equality above and using Hensel’s Lemma one can deduce that for any $a$, the set $\{v(F(x) - a) \mid x \in M\}$ has a maximum in $\mathbb{Z} \cup \{\infty\}$: we say that image of $F$ has the optimal approximation property (see [vdDK02]). From this observation Kuhlmann suggests a candidate for a complete axiomatization of the theory of $M$ in the language of rings, which is essentially $T_{\text{naive}}$ together with sentences saying that the image of every multi-variable additive polynomial has the optimal approximation property (see [Kuh16]).

Now we present our approach. We work in the language $L_\mathcal{O}$.

Let us illustrate in an example the main ideas of the proof of Theorem 5.5 which states that modulo the theory $T_1$ every $p,p.$ formula is equivalent to a universal one. This theorem immediately shows every completion of $T_1$ is model-complete.

Instead of considering the multivariable polynomial $F$ above, we use the equation (*) to study the image of the Artin-Scherier map $\varphi$, and see the sum
\[ C(M) := \sum_{i=1}^{p-1} M^p X^i \]
as the **pseudo-complement** to the image \( \varphi(M) \). Note that

\[
\varphi(M) \cap C(M) \subseteq O_M,
\]

hence this intersection is **small** with respect to the valuation metric; that is why we call \( C \) the pseudo-complement. In addition, using Hensel’s lemma we have that

\[
O_M \cap \varphi(M) = m
\]

where \( m \) is the maximal ideal \( X \mathbb{F}_p[[X]] \).

At this point, we are able to describe the set \( \varphi(M) \), definable a priori by an existential formula, by the following formula:

\[
(1) \quad \phi(x) : \forall y \ ((\exists z \ \varphi(z) = x - y \land y \in C + O) \rightarrow y \in m).
\]

It is easy to see that this formula is equivalent to a universal formula in the language \( L_O \) (hence also in the language of rings with the parameter \( X \)) since \( C \) is existentially definable.

Motivated by the above example, let us consider the general case. We introduce the notion of a ball (in this section) as a \( L_O \)-formula of the form

\[
B(x) : x.X^\gamma \in O
\]

for some fixed \( \gamma \in \mathbb{Z} \). Let \( Q \) be a finite set of polynomials (scalars from \( R \)). Abusively, we also denote by \( Q \) the formula which defines the sum of the images of \( q \in Q \). Our strategy is to assign to \( Q \) a positive existential formula \( D \) and balls \( B_1, B_2 \) in a computable way, such that in every model \( N \models T_1 \),

\[
(1) \quad D(N) + Q(N) = N, \quad \text{and}
\]

\[
(2) \quad B_1(N) \subseteq B_2(N), \quad \text{such that } Q(N) \cap D(N) \subseteq B_2(N) \quad \text{and } B_1(N) \cap Q(N) \text{ is definable by a universal formula which depends only on } T_1.
\]

For instance, the above example suggests that for \( Q = \{ \varphi \} \), \( B_1 = O \), \( B_2 = O.X \) and \( D = C + O \) suit.

However, in the general case, for the task (2), we cannot always ensure that \( B_1(N) \cap Q(N) \) is a ball, or more generally quantifier free definable in \( L_O \): to see this it is enough to consider the image of \( x \mapsto x^p \). To handle this fact, we introduce \( p \)-th roots of coordinate functions, called \( \lambda \)-functions. For instance, for \( x \in \mathbb{F}_p((X)) \) we write

\[
x = x_0^p + x_1^pX + \cdots + x_{p-1}^pX^{p-1}
\]

and define

\[
\lambda_i(x) := x_i \quad (i = 0, \ldots, p - 1).
\]

We prove that \( B_1(N) \cap Q(N) \) is positive quantifier-free definable in the language \( L \) together with \( \lambda \)-functions (see Theorem \[3.16\]). We also notice that \( \lambda \)-functions are both universally and existentially definable in \( L \). Hence we can achieve our aim. At the end the effectiveness of our proof yields the desired decidability results.

This article is organized as follows: generalities and definitions about the non-commutative ring \( R \), its matrix ring and \( R \)-modules are given in the Section 2. In the third section we prove an equivalent of Hensel’s lemma (see Theorem \[3.2\]), we don’t know if it is really new- and use it to axiomatize what we call **henselian filtered \( R \)-modules**. This part consists of finding a ball (like \( m \)) above where the trace of the definable set considered can be defined by positive quantifier-free formula in the language \( L \) together with the \( \lambda \)-functions. In the fourth section, we introduce the notion of a valued module, and study pseudo-complements. In the final section we apply the previous results to obtain the main theorems. Here we use some general model theory of modules (e.g. Baur-Monk elimination) and some recent facts from \[AK14\] and \[AF17\].
In the course of this article we introduce several languages and theories. For the sections 2-4, (i.e. except the last section), it is worth noting that $K$ and $\mathbb{F}_p((X))$, realized with the natural interpretation of these languages are models of all of these theories. When we say that something is computable we mean that there is a Turing machine which can compute it.

Acknowledgements. This is a long-standing work, began with the my Ph.D thesis in early 2007. I would like to thank my advisor Françoise Point for suggesting me the topic and for giving the idea to introduce $\lambda$-functions. I thank Françoise Delon, also my advisor; without her help, this work could not be finished. I want to thank Franz-Viktor Kuhlmann to have invited me, as early as in 2008 and shared his ideas about the topic. Lou van den dries and Matthias Aschenbrenner have kindly discussed with me about some parts of the present article in last times. Finally, I want to thank Silvy Anscombe and Arno Fehm for their encouragements to write this manuscript.

2. Preliminaries

We set $d$ to be some fixed positive power of the prime $p$. Let $K$ be the henselization of the field of rational functions over $\mathbb{F}_d$, with respect to the $X$-adic valuation; denoted as $K = \mathbb{F}_d(X)^h$.

Let

$$\varphi : x \mapsto x^d$$

be the $k$-th power of the Frobenius endomorphism of $K$$. We fix the notation $v_K$ for the $X$-adic valuation on $K$.

$K$ is a finite extension of $K^\varphi$ of dimension $d$.

Let

$$\alpha := \alpha(1) = (\alpha_0, \alpha_1, \ldots, \alpha_{d-1})$$

be a basis of the $K^\varphi$-vector space of $K$. One can think of the basis $(1, X, \ldots, X^{d-1})$. Set $\alpha(0) := \{1\}$.

It is easy to see that for all $n \geq 1$, $\alpha$ induces canonically the basis $\alpha(n)$ of the $K^\varphi$-vector space $K$, defined by the following formula

$$\alpha(n) = \left( (\alpha_i^{n-1} \alpha_j(n-1)) \right)_{i,j \in d,n-1}$$

where $\alpha_j(n-1)$ is the $j$-th element of the basis $\alpha(n-1)$ and $d^{n-1}$ is identified with the set of functions $\{0, \ldots, n-1\} \rightarrow \{0, \ldots, d-1\}$. Hence for all $(i, j) \in d \times d^{n-1}$ as above there exists a unique $k \in d^n$ such that the $k$-th element of the basis $\alpha(n)$ is

$$\alpha(n) \ni \alpha_k = \alpha_i^{n-1} \alpha_j(n-1)$$

We define the ring

$$R := K(t \mid ta^\varphi = at).$$

the (right-) $K$-algebra with the indeterminate $t$, subject to the commutation rule $ta^\varphi = at$ for all $a \in K$. It is called the ring of $\varphi$-twisted polynomials. When $k = 1$, $R$ is isomorphic to the ring of additive polynomials over $K$, equipped with addition and composition. Every non-zero $q \in R$ can be written as

$$q = t^n a_n + t^{n-1} a_{n-1} + \cdots + a_0$$

Note that the content of this section can be generalized to any field and to any endomorphism satisfying similar hypotheses. In particular one can consider $\mathbb{F}_q(X), \mathbb{F}_q((X)), \ldots$ etc. For our main interest and for readability reasons, we preferred to stick to one case.
where the $a_i$ are from $K$ and $a_n \neq 0$. Since the image of $q$ in $K[T]$ is

$$Q(T) = a_n T^n + \cdots + a_0 T$$

under the aforementioned isomorphism, “the constant term $a_0$ of $q$” gives rise to the linear term $a_0 T$ of $Q(T)$.

**Definition 2.1.** An element $q$ as in (5) is said to be separable is $a_0 \neq 0$.

**Remark 2.2.** Note that $q$ is separable if and only if its image $Q(T) \in K[T]$ is separable.

The integer $n$ in the expression (5) is called the degree of $q$ and we set the degree of 0 as $-1$. With this degree function $R$ is right Euclidean: for every non-zero $r, q \in R$ there exists $q'$ and a unique $r'$ of degree $< \deg(q)$ such that $r = qq' + r'$. As a consequence least common multiple lcm($q, r$) and greatest common divisor gcd($q, r$) are well defined. In particular $R$ is right Ore. Note that if $\varphi$ is onto then $R$ is also left Euclidean and if it is not (which is the case that we consider here), $R$ is not even left Ore. The reader can see [Coh95], [Ona] Section 2, for more details.

For our quantifier elimination Theorem 3.16 we will import some definitions and results from [DDP02] on the ring $R$ and on $R$-modules. We give the proofs of some results not to be self-contained but rather, to initiate the reader with the style of the computations that will permit us easily refer to [DDP02].

**Fact 2.3.** The field $K$ is recursively enumerable since $\mathbb{F}_d(X)$ is, and $K$ is formed, by adding to $\mathbb{F}_d(X)$ for each polynomial satisfying the hypothesis of the Hensel’s Lemma the unique root of it with the concluding property. Consequently the ring $R$ and its matrix rings are recursively enumerable.

**Lemma 2.4.** Let $q \in R$ and $n$ be a positive integer. Let $\alpha$ be a basis of the $K^n$-vector space $K$. Then

1. $q$ can be uniquely written as:

$$q = \sum_{i \in d^n} q_i \alpha_i \quad (q_i \in R),$$

2. for all $n > 0$ there is an endomorphism $\sqrt[n]{\cdot} : (R, +) \to (R, +)$ such that if $q = \sum_{i \in d^n} q_i \alpha_i$ then

$$t^n q = \sum_{i \in d^n} \sqrt[n]{q_i} t^n \alpha_i.$$

In addition, if $q$ is separable there exists $i \in d^n$ such that $\sqrt[n]{q_i}$ is separable.

**Proof.**

1. Let $0 \leq k \leq s = \deg q$. For each monomial $t^k a_k$, by expressing $a_k$ with respect to the basis $\alpha(n)$, $q$ can be written as

$$t^k \left( \sum_{i \in d^n} a_k^{\alpha_i} \right).$$

Set $q_{k,i} = t^k a_k^{\alpha_i}$ and $q_i = \sum_{0 \leq k \leq s} q_{k,i}$. So we get

$$q = \sum_{i \in d^n} q_i \alpha_i.$$

2. Let $a \in K$. Write

$$a = \sum_{i \in d^n} a_i^{\alpha_i}.$$
We define
\[ \sqrt[n]{a} := \sum_{i \in I} a_i \alpha_i \]
and extend it to \( r \in R \) by applying it to its coefficients. By definition \( \sqrt[n]{\cdot} \) preserves the addition.

To see that
\[ t^n q = \sum_{i \in I} \sqrt[n]{q_i t^n} \alpha_i \]
(8)
it is enough to show that
\[ t^n q_i = \sqrt[n]{q_i t^n} \]

Note that \( \sqrt[n]{q_i k,} = t^k a_{k,i} \) and hence
\[ \sqrt[n]{q_i k, t^n} = t^{n+k} a_{k,i} = t^n q_{k,i} \]

Using additivity, we get \( \sqrt[n]{q_i l^n} = t^n q_i \).

Suppose now \( q \) is separable. If none of the \( \sqrt[n]{q_i} \) is separable, then for all \( i \), we have \( \sqrt[n]{q_i} = t_q i \). But then \( t^n q \in t^{n+1} R \) by the equality (8). Hence \( q \in t R \); which is a contradiction.

□

**Definition 2.5.** An \( m \times n \) matrix \( A = (q_{i,j}) \) over \( R \) is said to be
1. lower triangular if, \( j > i \) implies \( q_{i,j} = 0 \),
2. lower triangular diagonally separable if it is lower triangular, \( n \leq m \) and the \( q_{ii} \) \((i \leq n)\) are separable,
3. lower triangular separable if \( A = (A_1, 0) \) where \( A_1 \) is an \( m \times k \) lower triangular diagonally separable matrix and 0, is the \( m \times l \) null matrix with \( k + l = n \).

**Remark 2.6.** The definition of a lower triangular diagonally separable matrix is introduced by our self and it is not present in [DDP02].

**Proposition 2.7.** For any matrix \( A \) there exists an invertible matrix \( P \) with coefficients in \( \{0, 1\} \) and an invertible matrix \( Q \) such that \( PAQ \) is lower triangular.

**Proof.** See the Proposition 6.1 in [DDP02].

□

2.1. **R-modules.** For the rest of the article we will always understand the expression \( R \)-module as right \( R \)-module. In an module \( R \)-module \( M \), scalar multiplication will be denoted as \( x.r \), for \( x \in M \) and \( r \in R \).

**Definition 2.8.** Let \( M \) be an \( R \)-module and \( \beta \) a basis of the \( K^e \)-vector-space \( K \).
1. \( M \) is said to be \( t\beta \)-decomposable if \( x \mapsto x.t \) is injective and we have
\[ M = \bigoplus_{i \in I} M.t \beta_i, \]
(9)

(\( \bigoplus \) indicates the direct sum as abelian subgroups).

2. \( M \) is said to be \( t \)-decomposable if it is \( t \alpha \)-decomposable with \( \alpha = (1, X, \ldots, X^{d-1}) \). Furthermore, for \( i \in d \) we then define \( \lambda_i(x) = x \) where
\[ x = x_0.t \alpha_0 + \ldots + x_i.t \alpha_i + \ldots x_{d-1}.t \alpha_{d-1}. \]
Remark 2.9. The direct sum
\[ M = \bigoplus_{i \in d} M.t\beta_i \]
induces the direct sum below for every positive \( s \)
\[ M = \bigoplus_{i \in d^s} M.t^s\beta_i. \]

Notation 2.10. For the rest of the article we set \( \alpha := \{1, X, \ldots, X^{d-1}\} \).

Remark 2.11. The functions \( \lambda_i \) are both existentially and universally definable in the language of right \( R \)-modules:
\[ y = \lambda_i(x) \leftrightarrow \forall x_1, \ldots, \forall x_i, \ldots, \forall x_d \quad x = \sum_{j=1}^d x_i.t\alpha_j \rightarrow x_i = y \]
and
\[ y = \lambda_i(x) \leftrightarrow \exists x_1, \ldots, \exists x_{i-1}, \exists x_{i+1}, \ldots, \exists x_d \quad x = \sum_{j=1}^{i-1} x_j.t\alpha_j + y.t\alpha_i + \sum_{j=i+1}^d x_j.t\alpha_j. \]

For a positive \( s \), by Remark 2.9 above, we get canonically the \( \lambda \) functions of level \( s \), defined for all \( i \in d^s \), in an obvious way.

The language \( L(\lambda) \). For the rest of the article we let \( L(\lambda) \) be the language of \( R \)-modules together with the functions \( \lambda \)-symbols \( \lambda_i \).

Definition 2.12. We denote by \( T_\lambda \) the \( L(\lambda) \)-theory of \( t \)-decomposable \( R \)-modules, that is, the theory of \( R \)-modules together with the axioms
\[ \lambda \text{-decomposition :} \]
\[ \forall x \quad x = \sum_{i \in d} \lambda_i(x).t\alpha_i. \]

\[ \forall x \forall (x_i)_{i \in d} \quad x = \sum_{i \in d} x_i.t\alpha_i \rightarrow \bigwedge_{i \in d} x_i = \lambda_i(x). \]

Lemma 2.13. In any \( t \)-decomposable \( R \)-module \( M \), any \( L(\lambda) \)-term can be evaluated on the tuple \( (x_i)_i \) from \( M \), as
\[ \sum_i \sum_j \lambda_j(x_i).r_{ij} \]
where \( r_{ij} \in R \).
Proof. This is Corollary 3.3 in [DDP02]. \( \square \)

Lemma 2.14. Let \( m > 0 \), and \( q_j, q'_j \in R \) such that \( q_j = t^m.q'_j \). Then the equation \( \sum y_j.q_j = u \) is equivalent to
\[ \bigwedge_{i \in d^m} \sum_j y_i. \sqrt[m]{q'_j} = \lambda_i(u). \]
in any \( t \)-decomposable \( R \)-module.
Proof. This is Lemma 3.4 in [DDP02]. \( \square \)
Observation 2.15. Let \( q = (q_0, \ldots, q_{n-1}) \) be a non-zero tuple from \( R \). We set \( e := \min \{ k \mid q_i \in t^k R, \text{ for all } i \} \).

So, \( e = 0 \) means that at least one of the \( q_i \) is separable.

Suppose \( e > 0 \), one can write
\[
q_i = t^e q_i' = \sum_{k \in d^e} e_k \sqrt{q_{ik}} t^e \alpha_k
\]

Since at least one of the \( q_i' \) is separable, for some \((i, k)\), the polynomial \( \sqrt{q_{ik}} \) is separable by Lemma 2.4 (2.). Set \( q_i = \sqrt{q_{ik}} \) and let \( q_{ik} \) be the \( n \times d^e \)-matrix \((q_{i,k})\) whose \( i \)-th line consists of the sequence \((q_{ik})_{k \in d^e}\). By this process we have replaced a tuple by a matrix which has a separable coefficient and iterating this process and using Lemma 2.14 above we get the following result.

Lemma 2.16. Let \( A \) be a non-zero \( n \times k \) matrix over \( R \). Then, the system \( y.A = u \) is equivalent to \( y.PQ = w \) (modulo \( T_\lambda \), where \( P \) is a permutation matrix (i.e. invertible with coefficient in \{0, 1\}), \( Q \) is lower triangular separable and \( w \) is a tuple consisting of \( L(\lambda) \)-terms).

Proof. This is a reformulation of Lemma 6.4.4 in [DDP02]. \( \square \)

2.2. Baur-Monk Elimination. The following is a reminder of Theorem A.1.1, Corollary A.1.2 and the discussion which follows in [Hod93], p. 653-656.

Let \( \mathcal{L} \) be any language which contains the language \{+,-,0\} of abelian groups. A positive primitive formula (p.p.) \( \phi \) of \( \mathcal{L} \), is the one of the form
\[
\exists \bar{y} \left( \bigwedge_i \psi_i(\bar{x}, \bar{y}) \right)
\]
where the \( \psi_i \) are atomic.

A group like \( \mathcal{L} \)-structure is the one such that its base set is a group with respect to \{+,-,0\}. A basic formula, is a p.p. formula which defines a subgroup of the corresponding cartesian power of \( A \).

Note that if \( S \) is any ring and \( \mathcal{L} \) is the language of \( S \)-modules then in any \( S \)-module \( N \) any p.p. formula defines a subgroup of the corresponding cartesian power of \( N \), hence any p.p. formula is basic.

Let \( T \) be an \( \mathcal{L} \)-theory such that every model of \( T \) is group-like, and every p.p. formula is a basic formula for all models of \( T \). An invariant \( \mathcal{L} \)-sentence of \( N \models T \) is an \( \mathcal{L} \)-sentence \( \Theta \) satisfied by \( N \), such that for some p.p. formulas \( G(x) \) and \( H(x) \) of one variable \( x_i \) for some \( m \in \mathbb{N} \),
\[
T \models \Theta \iff |G/G \land H| = m
\]
where the right-hand side of the equivalence is an abbreviation of the formula:
\[
(12) \exists(x_i)_{i=1..m}\left(\bigwedge_i G(x_i) \land \bigwedge_{i \neq j} \neg(G \land H)(x_i - x_j) \land \forall z (G(z) \rightarrow \bigvee_i (G \land H)(z - x_i))) \right).
\]
An invariant sentence is an invariant sentence of some \( N \).

Theorem 2.17 (Baur-Monk). Every \( \mathcal{L} \)-formula is equivalent modulo \( T \) to a boolean combination of p.p. formulas and invariant sentences. Hence for all models \( N, M \models T \), \( N \equiv M \) if and only if \( N \) and \( M \) have same invariant sentences.
**Corollary 2.18.** A completion of $T$ is model-complete if and only if every p.p. formula is equivalent to a universal formula modulo $T$.

3. **The tropical action of $R$ on $\mathbb{Z}$ and Filtration**

We recall some elementary facts about henselian valued fields. Let $(F, v)$ be a valued field with value group $\mathcal{G}$ and valuation ring $\mathcal{O}$. We set $\Gamma = \mathcal{G} \cup \{\infty\}$, extend the usual addition of $\mathcal{G}$ to $\Gamma$ by letting
\[ \infty + \infty = \infty + a = a + \infty = \infty \]
for all $a \in \mathcal{G}$. We recall the tropicalisation of a one variable polynomial $Q(T)$ over $F$: Write
\[ Q(T) = \sum a_i T^i. \]
Then tropicalisation of $Q$ is the map
\[ Q_v : \Gamma \to \Gamma \\
\gamma \mapsto \min \{i \gamma + v(a_i)\}. \]
A jump value (or a tropical zero) of $Q$ is some $\gamma \in \Gamma$ such that
\[ |\{i \mid i \gamma + v(a_i) = Q_v(\gamma)\}| \geq 2. \]
We denote by $\text{Jump}(Q)$ the set of jump values of $Q$. Note that this set is finite and has at most $n - 1$ element if $Q$ is of degree $n$.

**Fact 3.1** (Newton’s Lemma). Let $(F, v)$ be a valued field and $f$ be a polynomial with coefficients from the valuation ring $\mathcal{O}$. Consider the following property $h(f)$ of $f$,
\[ h(f) : v(f(0)) > 2v(f'(0)) \Rightarrow (\exists b \quad f(b) = 0 \quad \text{and} \quad v(b) = v(f((0) - v(f'(0))). \]
Then $(F, v)$ is henselian if and only if $h(f)$ holds for every $f$ over the valuation ring $\mathcal{O}$ of $F$.

**Proof.** See [EP05] Theorem 4.1.3. \hfill \Box

Consider a polynomial $G$, such that $G(0) = 0$ and $G'(0) \neq 0$. Then $G$ is of the form
\[ G(T) = aT + \text{sum of monomials of higher degree}. \]
Consider the set $A_1 := \{ \gamma \mid G_v(\gamma) < (G - aT)_v(\gamma) \}$. This is a non-empty final segment of $\Gamma$. Also let
\[ A_2 := \{ \gamma + v(a) \mid \gamma \in A_1 \}. \]
We set $B_1(G) := v^{-1}(A_1)$ and $B_2(G) := v^{-1}(A_2)$.

**Theorem 3.2.** $(F, v)$ is henselian if and only if for every polynomial $G$ with coefficients in the valuation ring $\mathcal{O}$ such that $G(0) = 0$ and $G'(0) \neq 0$, $G$ induces a bijection $B_1(G) \to B_2(G)$.

**Proof.** Suppose $(F, v)$ is henselian. Let $z \in B_2$ and set $f = G - z$. We have $v f(0) = v(z) = \gamma + v(a)$ where $\gamma \in A_1$ hence $G_v(\gamma) = \gamma + v(a) < (G - aT)_v(\gamma)$. Since a monomial of $G - aT$ has at least degree 2 and $G$ is over $\mathcal{O}$, we have
\[ \gamma + v(a) < 2 \gamma. \]
It follows that $\gamma > v(a)$ and $v(z) > 2v(a)$. By the above fact, there is $b$ root $f$, of valuation $\gamma$. In other words, $G(b) = z$ and hence $G(B_1) \supseteq B_2$.

Now we show that $G \mid B_1$ is 1-1. By the definitions of $B_1$ and $B_2$ we have $G(B_1) \subseteq B_2$. Let $z$ and $f$ be as above and $G(x_1) = G(x_2) = z$ with $x_1, x_2 \in B_1$. Then $x_1, x_2$ have the same valuation $\gamma$, but it is well-known that $f$ has at most one root of valuation $\gamma$ (or it can be proven by using Taylor
expansion, that \( v(G(x_1 - x_2)) > G_v(v(x_1 - x_2)) \) but this is not possible since \( v(x_1 - x_2) \) is bigger than then all the jump values of \( G \).

The converse is similar and easy (and we won’t need). \( \square \)

**Corollary 3.3.** Let \( G \) be any polynomial over \( (F, v) \) such that \( G(0) = 0 \) and \( G'(0) \neq 0 \) then there exist convex subsets \( B_1, B_2 \) such that \( G \upharpoonright B_1 : B_1 \to B_2 \) is a bijection.

**Proof.** Divide \( G \) by the coefficient, say \( a \), which has the minimal valuation. Let \( H \) be the obtained polynomial. Then \( B_1(G) := B_1(H) \) and \( B_2(G) := v_K(a) + B_2(H) \) suits. \( \square \)

**Notation 3.4.** Let \( G \) be as above. If the value group of \( F \) is discrete of rank 1, then \( A_1, A_2 \) are closed intervals; in this case we set

\[
h(G) := \min A_1 \quad \text{and} \quad \text{hens}(G) := \min A_2.\]

**The tropical action of \( R \).** From now on we set

\[
\Gamma := \mathbb{Z} \cup \{\infty\}
\]

and we equip \( \Gamma \) with a right action of \( R \) using tropicalisations:

Let \( q = \sum_i t^i a_i \in R \). We define the tropicalisation of \( q \), as the tropicalisation of the \( Q(T) = \sum a_i T^{d_i} \), that is, as the map

\[
(14) \quad \cdot q : \Gamma \to \Gamma; \quad \gamma \mapsto \gamma \cdot q = \min_i \{d^i \gamma + v(a_i)\}.
\]

In particular \( \cdot q \) is strictly increasing if \( q \neq 0 \). Note that \( \gamma \cdot a = (aT)_v(\gamma) = \gamma + v(a) \) for \( a \in K \).

We set \( \text{Jump}(q) := \text{Jump}(Q), \quad h(q) := h(Q) \) and \( hens(q) := hens(Q) \). For if \( q = t - 1 \) then \( A_1 = A_2 := \{\gamma \in \Gamma \mid \gamma > 0\} \). Hence \( h(q) = hens(q) = 1 \).

For our interests, we also introduce the tropicalisations of the \( \lambda \)-functions. Remark that, we have for all \( x \in K \), since \( x = \sum_{i \in d} \lambda_i(x)^d X^i \),

\[
v_K(x) = \min_i \{v_K(\lambda_i(x)^d) + i\},
\]

and the minimum is attained for a unique \( i \in d \). We define

\[
\lambda_i(\gamma) := \begin{cases} \frac{2d - 1}{d} & \text{if } \gamma \in d\mathbb{Z} + i \\ 0 & \text{else} \end{cases}
\]

and we set

\[
\lambda(\gamma) := \sum_i \lambda_i(\gamma).
\]

**Definition 3.5.** The theory of \( \Gamma \) together with the tropical action of \( R \) and \( \lambda \), is decidable since this structure is definable in the ordered abelian group structure of \( \mathbb{Z} \), together with \( \infty \) and constants for elements of \( \mathbb{Z} \). We call this structure the tropical structure of \( \Gamma \).

**Remark 3.6.** For \( x \in K \)

\[
v_K(x) \geq \gamma \iff \text{for all } i \in d \quad v_K(\lambda_i(x)) \geq \lambda(\gamma).
\]

**Proof.** Note that \( \Leftarrow \) is clear and if \( \gamma = v_K(x) \) then the assertion is trivial. Suppose \( v_K(x) > \gamma \). Let \( j \) be such that \( v_K(\lambda_j(x), t^d \alpha_j) = d v_K(\lambda_j(x)) + j = v_K(x) \). Let \( i \) be such that \( \gamma = d \lambda(\gamma) + i \). It follows that

\[
d(v_K(\lambda_j(x)) - \lambda(\gamma)) > j - i.
\]

Hence \( v_K(\lambda_j(x)) - \lambda(\gamma) > -1 \) and \( v_K(\lambda_j(x)) - \lambda(\gamma) \geq 0 \). \( \square \)
Corollary 3.7. For all non-zero \( x \in K \) and \( \gamma \in \Gamma \),
\[
v_K(x) \geq \gamma \iff \text{for all } i \in d^\ast \quad v_K(\lambda_i(x)) \geq \lambda^\ast(\gamma)
\]
where \( \lambda^\ast = \lambda \circ \ldots \circ \lambda \) \( s \)-times.

Remark 3.8. Let \( \gamma \in \Gamma \) and \( r, q \in R \). Then
1. \( \gamma \cdot rq = (\gamma \cdot r) \cdot q \),
2. \( \gamma \cdot (r + q) \geq \min\{\gamma \cdot r, \gamma \cdot q\} \),
3. \( r \) is strictly increasing for all non-zero \( r \),
4. \( \infty \cdot r = \gamma \cdot 0 = \infty \) for all \( r \) and \( \gamma \).

Proof. This follows by direct computations from the definition. Note that (1) follows more generally from [Ona11], Corollary 4.1.14. \( \square \)

3.1. Quantifier Elimination Near 0.

The languages \( L_\mathcal{O} \) and \( L_\mathcal{O}(\lambda) \). The language \( L_\mathcal{O} \) is obtained by adding to \( L \) a unary predicate \( \mathcal{O} \), and the language \( L_\mathcal{O}(\lambda) \) is the language \( L_\mathcal{O} \) together with the functions \( \lambda_i \) added to \( L_\mathcal{O} \). We want to study the divisibility conditions for \( t \)-decomposable \( R \)-modules which can be seen as properties reflecting a kind of henselianity, analogue to Theorem 3.2 just above.

Notation 3.9. For the rest of this article, for \( \gamma \in \Gamma \), we write \( P_\gamma \) for the predicate \( \mathcal{O}.X^\gamma \), i.e. in any \( L_\mathcal{O} \)-structure \( M \), \( x \in P_\gamma \) if and only if \( x.X^\gamma \in \mathcal{O} \).

Definition 3.10. An henseliant filtered module, is a \( t \)-decomposable \( R \)-module which is an \( L_\mathcal{O}(\lambda) \)-structure satisfying the following axioms:

0. **Balls:** \( P_\infty = \{0\} \) and the \( P_\gamma \) form a chain of subgroups decreasing with \( \gamma \) such that the inclusions are proper.
1. **Ultrametric:** \( \forall x \forall y. x \in P_\gamma \wedge y \in P_\delta \rightarrow (x + y) \in P_{\min(\gamma, \delta)} \) for all \( \gamma, \delta \in \Gamma \) and \( r \in R \).
2. **Regularity:** \( \forall x \in P_\gamma \iff x.r \in P_{\gamma.r} \), for all \( \gamma \) and \( r \neq 0 \) such that \( \gamma \notin \text{Jump}(r) \).
3. **\( \lambda \)-regularity:** \( \forall x \in P_\gamma \iff \lambda_i(x) \in P_{\lambda(\gamma)} \), for all \( \gamma \).
4. **Henselianity:** \( \forall x \in P_{\hens(s)} \setminus \{0\} \exists ! y \in P_{\hens(s)} y.s = x \) for all separable \( s \).

Note that axiom 3 implies
\[
\forall x \in P_\gamma \iff \bigwedge_{i \in d^\ast} \lambda_i(x) \in P_{\lambda(\gamma)},
\]
a consequence analogue to the one expressed in Corollary 3.7.

Notation 3.11. We denote by \( T_{\hens} \) the theory of henselian filtered modules.

We also isolate some theories of the \( R \)-modules that are already considered in [DDP02]:

- we denote by \( T_{\hens} \) the theory \( T_\lambda \) together with the following (scheme of-) axioms:
  \[
  \forall x \exists y. x = y.s,
  \]
  for all separable \( s \) and finally,
- we denote by \( T_{\hens}^0 \), the \( L_\lambda \)-theory of torsion-free non-zero models of \( T_{\hens} \).

Remark 3.12. In a henselian filtered module \( M \), we denote by \( P_\gamma \) the subgroup defined by the intersection
\[
P_\gamma := \bigcap_{\gamma \neq \infty} P_\gamma(M).
\]
Note that by axioms of ultrametric and regularity \( P_M \) is a \( L(\lambda) \)-substructure of \( M \) and it is torsion-free as an \( R \)-module. Moreover it is straightforward to check that by Hensel’s axioms
\[
P_M \models T^0_{\text{free}}.
\]

**Proposition 3.13.** Given \( L \)-p.p. formulas \( a(x) \) and \( b(x) \) of one variable \( x \), the quotient \( a / a \wedge b \) is either trivial or infinite in every model of \( T^0_{\text{free}} \). In particular \( T^0_{\text{free}} \) is complete and decidable.

**Proof.** This follows from Lemma 6.8 in [DDP02]. \( \square \)

**Definition 3.14.** A ball of \( L_\mathcal{O} \), is an atomic formula \( W(x_1, \ldots, x_k) \) of the form
\[
W(x_1, \ldots, x_k) : \bigwedge_{i=1}^k x_i \in P_{\gamma_i}.
\]
We will write it rather as a product
\[
W = P_{\gamma_1} \times \cdots \times P_{\gamma_k}
\]
of predicates. \( W \) is said to be proper if none of the \( \gamma_i \) is equal to \( \infty \).

**Remark 3.15.** A primitive positive formula \( \phi(\bar{x}) \) of \( L_\mathcal{O} \) is equivalent to one in the form
\[
\exists \bar{y} \bar{x}. B - \bar{y}. A \in O^k \times \{0\}^n
\]
which it self is equivalent to a formula
\[
\exists \bar{y} \bar{x}. B' - \bar{y}. A' \in W
\]
where \( A, B, A', B' \) are matrices over \( R \) and \( W \) is a ball. Hence any \( \text{p.p.} \) formula is a basic formula. For our purposes will rather use the latter equivalence.

Note that modulo \( T_{\text{hens}} \), the set of definable sets by an \( L_\mathcal{O} \)-p.p. formula contains the set of definable sets by an \( L \)-p.p. formula since \( W \) can be chosen equal to \( P_k \infty \) for some \( k \).

In this section, we will prove the following:

**Theorem 3.16.** Let \( \phi(x_1, \ldots, x_m) \) be a \( \text{p.p.} \) formula of \( L_\mathcal{O} \). Then there is some computable \( \delta \neq \infty \) and some positive quantifier free \( L(\lambda) \)-formula \( \psi \) which depends only to the theory \( T_\lambda \), such that, with \( V = P^m_\delta \), we have
\[
\phi \wedge V \leftrightarrow \psi \wedge V
\]
modulo \( T_{\text{hens}} \). Moreover, If \( \phi_1 \) is another \( \text{p.p.} \) formula with the same arity, and if \( \phi \) and \( \phi_1 \) are equivalent modulo \( T^0_{\text{free}} \), then for some computable \( \gamma \) and with \( W := P^m_\gamma \)
\[
T_{\text{hens}} \models \phi \wedge W \leftrightarrow \phi_1 \wedge W.
\]

**Lemma 3.17.** Given \( \delta \neq \infty \) and an \( L(\lambda) \)-term \( u(\bar{x}) \) there exists \( \gamma \neq \infty \) such that in every henselian filtered module the following holds:
\[
(15) \quad \forall \bar{x} \bar{x} \in P^{|\bar{x}|}_\gamma \rightarrow u(\bar{x}) \in P_\delta.
\]

**Proof.** Put \( u(\bar{x}) \) in the form \( \sum_{i,j} \lambda_j(x_i).r_{ij} \) using Lemma 2.13 By regularity, \( \lambda \)-regularity and ultrametric axioms for any \( \rho \),
\[
\bar{x} \in P^{|\bar{x}|}_\rho \rightarrow u(\bar{x}) \in P_{\min\{\lambda(\rho).r_{ij}\}}
\]
holds. Since all the tropical functions \( \lambda \) and \( r_{ij} \) are unbounded increasing one can choose \( \gamma \) such that \( \gamma \cdot r_{ij} \geq \delta \) for all \( i \). \( \square \)

**Remark 3.18.** The value \( \gamma \) is computable from \( \delta \) and \( u(\bar{x}) \) in the tropical structure of \( \Gamma \).
As a consequence of the henselianity axioms we observe the following.

**Lemma 3.19.** Given \( \gamma \neq \infty \) and a separable \( s \) there exists \( \delta \neq \infty \) such that

\[
T_{\text{hens}} \models \forall y \ [(y \in P_\delta) \rightarrow (\exists x \in P_\gamma \land x.s = y)]
\]

for every henselian module \( M \).

**Proof.** If \( \gamma \leq \text{hens}(s) \) then set \( \delta = \text{hens}(s) \). Otherwise \( \gamma \notin \text{Jump}(s) \) and by the regularity axioms \( x \in P_\gamma \) if and only if \( x.s \in P_\gamma s \). Since \( P_\gamma \subseteq P_{\text{hens}(s)} \), \( P_\gamma s \subseteq P_{\text{hens}(s)} \). Hence if \( y \in P_\gamma s \) then the unique solution \( x \) such that \( x.s = y \) lies in \( P_\gamma \). So \( \delta = \gamma \cdot s \) fits for our requirements. \( \square \)

**Remark 3.20.** As above, \( \delta \) is computable in the tropical structure \( \Gamma \).

**Corollary 3.21.** Let \( A = (a_{ij}) \) be an \( m \times k \) lower triangular diagonally separable matrix (hence \( k \leq m \)). Then, for every proper ball \( W \), there exists a proper computable ball \( W_1 \) such that

\[
T_{\text{hens}} \models \forall \bar{x} \ (\bar{x} \in W_1 \rightarrow \bar{x} \in W.A).
\]

**Proof.** Write \( W = \prod_{i=1}^m P_{\delta_i} \) and by Remark 3.17 chose \( \gamma_i \in \Gamma \setminus \{\infty\} \) such that, for fixed \( i \),

\[
\sum_{j \neq i} P_{\gamma_i} a_{ji} \subset P_{\delta_i}.
\]

Since the \( a_{ii} \) are separable by the above lemma there exist proper balls \( U_i \) such that whenever \( z_i \in U_i \), there exists \( y_i \in P_{\gamma_i} \) such that \( z_i = y_i.a_{ii} \) and hence

\[
z_i = \sum_{j=1}^n y_i.a_{ji} = \sum_{j \neq i} y_i.a_{ji} \in P_{\delta_i}
\]

for all \( 1 \leq j \leq k \). Take \( W_1 = \prod_i U_i \). \( \square \)

**Proof Theorem 3.16** Let \( \bar{y} := (y_1, \ldots, y_k) \), \( \bar{x} := (x_1, \ldots, x_m) \), \( \phi(\bar{x}) \) and \( W := \prod P_{\gamma_i} \) be such that

\[
\phi(\bar{x}) : \exists \bar{y} \ \bar{x}.B - \bar{y}.A \in W.
\]

Let \( I := \{ i \mid \gamma_i = \infty \} \) and \( J := \{ j \mid \gamma_j \neq \infty \} \), and \( A_I \) be the matrix formed by the columns \( C_{i \in I} \) (resp \( A_J \) be the matrix formed by the columns \( C_{j \in J} \) of \( A \)). We also note \( u(\bar{x}) := \bar{x}.B \) and \( u_I(\bar{x}) \) (resp \( u_J(\bar{x}) \)) denote the tuple formed by \( I \)-coordinates (resp. \( J \)-coordinates) of \( u(\bar{x}) \). Now

\[
\phi(\bar{x}) \leftarrow \exists \bar{y} \ (u_I(\bar{x}) = \bar{y}.A_I \land u_J(\bar{x}) - \bar{y}.A_J \in W_J)
\]

where \( W_J \) is the obvious projection of \( W \) to its non-zero coordinates. By Lemma 2.16 there exists a lower triangular separable matrix \( \tilde{A}_I = (S,0) \) such that the formula

\[
u(\bar{x}) = \bar{y}.A_I
\]

is equivalent, modulo \( T_{\lambda} \), to

\[
(t_1(u_I(x)), \ldots, t_{n-1}(u_I(x))) = \tilde{y}.P.S \land (t_{n-I}(u_I(x)), \ldots, t_n(u_I(x))) = 0,
\]

where \( P \) is a permutation matrix and the \( t_1(u_I(x)), \ldots, t_n(u_I(\bar{x})) \) are some \( L(\lambda) \)-terms.

By Remark 3.17 chose a proper ball \( U' \) such that \( U'.A_J \subseteq W_J \) and by Corollary 3.21 chose a proper ball \( V' \) such that \( V' \subseteq U'.P.S \).

Now by remark 3.17 we choose \( V \) such that for all \( \bar{a} \in V \) we have

\[
(t_1(u_I(\bar{a})), \ldots, t_{n-1}(u_I(\bar{x}))) \in V'
\]

and \( u_J(\bar{a}) \in W_J \).
Hence:

\[(17) \quad \forall \bar{x} \quad (\phi(\bar{x}) \land \bar{x} \in V) \leftrightarrow ((t_{n-l+1}(u_l(x)), \ldots, t_n(u_l(x))) = 0 \land \bar{x} \in V).\]

Denote by \(\psi\) the formula

\[(t_{n-l+1}(u_l(x)), \ldots, t_n(u_l(x))) = 0.\]

The first statement is now proved.

Now if \(\phi_1\) is another \(p.p.\) formula, modulo \(T_{\text{free}}^0\), \(\psi\) is equivalent to some positive quantifier free formula \(\psi_1\). Since \(T_{\text{free}}^0\) is decidable, we have an algorithm which checks if \(T_{\text{free}}^0 \models \psi \leftrightarrow \psi_1\). Modifying this algorithm we can remember the finitely many non zero \(r\) such that algorithm uses the axiom

\[\forall x \quad x \neq 0 \rightarrow x.r \neq 0,\]

and the finitely many separable \(s\), such that, the algorithm uses the axiom

\[\forall x \exists y \quad y.s = x.\]

Choose \(\gamma\) bigger than all the \(\max \text{Jump}(r)\) and \(\max \{h(s), \text{hens}(s)\}\) for the \(r\) and \(s\) as above. Set \(W := P_\gamma^n\). Hence for any non zero \(x \in P_\gamma\), \(x.r \neq 0\) and there exists \(y \in P_\gamma, y.s = x\). It follows that the same algorithm checks also if \(T_{\text{hens}} \models W \land \psi(\bar{x}) \leftrightarrow W \land \psi_1(\bar{x})\), in which case we have \(\phi_1 \land W \leftrightarrow W \land \phi(W)\) modulo \(T_{\text{hens}}\).

**Remark 3.22.** By passing to an \(\omega_1\)-saturated model \(M\), since \(P_M \models T_{\text{free}}^0\), it is trivial that if \(T_{\text{free}}^0 \models \phi \leftrightarrow \phi_1\) then for some proper \(W\)

\[\phi(W) = \phi_1(M).\]

What we show above is that the decidability of \(T_{\text{free}}^0\) yields the computability of \(W\).

**Corollary 3.23.** Let \(\phi\) and \(\phi_1\) be \(p.p.\) formulas with one free variable then, either there is a computable ball \(W\) such that the index \((\phi \land W)/(\phi \land \phi_1 \land W)\) is equal to 1, either for all \(W\) this index is infinite.

**Proof.** Follows by Proposition 3.13 and by the theorem above. \(\square\)

4. PSEUDO-COMPLEMENTS

We will introduce valued modules, \(R\)-modules \(M\), equipped with a function \(v : M \rightarrow \Gamma\) inducing the ultrametric topology, suitable to study henselian filtered modules. After investigating elementary properties of valued modules, we will get its consequences that can be expressible in the language \(L_\mathcal{O}\).

**Definition 4.1.** A valued module is a \(t\)-decomposable \(R\)-module together with a surjective map \(v : M \rightarrow \Gamma\) such that for all \(x, y \in M\),

1. \(v(x + y) \geq \min\{v(x), v(y)\}\)
2. \(v(x) = \infty \leftrightarrow x = 0\)
3. \(v(x.r) = v(x) \cdot r\) whenever \(v(x) \notin \text{Jump}(r)\), for all \(r \in R\).

**Remark 4.2.** Let \(r = t^a + \cdots + t^k a_k \in R\) where monomials are written following decreasing degrees, then

1. \(v(x.t^a_i) = v(x) \cdot t^a_i\), for all \(x \in M\),
2. \(v(x.r) = v(x) \cdot t^k a_k < v(x.(r-t^k a_k))\) whenever \(v(x) > \text{max Jump}(r)\),
3. \(v(x.r) = v(x) \cdot t^n a_n < v(x.(r-t^n a_n))\) whenever \(v(x) < \text{min Jump}(r)\).
Proof. 1. Follows from Definition 4.1 (3.) since a monomial has no jump value.

2. By Definition 4.1 (3.), if \( v(x) > \max\text{Jump}(r) \) then \( v(x,r) = v(x) \cdot r \). Let \( \gamma > \max\text{Jump}(r) \). Then for some \( i, \gamma \cdot r = \gamma \cdot t^i a_i < \gamma \cdot t^j a_j \) for all \( j \neq i \). In other words the line \( \{ (\delta, \delta \gamma + v(a_j)) \delta \} \) does not intersect any other line \( \{ (\delta, \delta \gamma + v(a_j)) \delta \} \) in the area \( (\max\text{Jump}(r), \infty) \times \Gamma \). This can happen only if \( i < j \) for all \( j \neq i \). Hence \( i = k \).

3. The proof is very similar to (2.) \( \Box \)

Now we will study the behavior near \(-\infty\) of \( p.p. \) definable sets in valued modules. Let \((M, v)\) be a valued module. We define the equivalence relation \( \text{RV} \) on \( M \) by

\[ x\text{RV}y \iff v(x) = v(y) < v(x - y). \]

We denote the \( \text{RV} \)-class of an element \( x \) by \( \text{rv}(x) \) whereas \( \text{rv}(A) \) denotes the set \( \{ \text{rv}(x) \mid x \in A \} \) for \( A \subseteq M \). We also note \( P_\gamma \) the closed ball of radius \( \gamma \) centered at 0.

**Definition 4.3.** For subgroups \( A \) and \( B \) of \( M \), we say that \( A \) and \( B \)

1. are \( m \)-immediate (\( m \) stands for mutually) if \( \text{rv}(A \smallsetminus P_\gamma) = \text{rv}(B \smallsetminus P_\gamma) \) for some \( \gamma \in \Gamma \), and we write \( A \approx B \),

2. are pseudo-orthogonal if \( vA \cap vB \subseteq [\gamma, \infty] \) for some \( \gamma \in \Gamma \), and we write \( A \parallel B \).

**Remark 4.4.** \( A \cap C \subseteq P_\gamma \) for some \( \gamma \), whenever \( A \parallel C \).

**Remark 4.5.** If \( A \approx B \) and \( B \subseteq A \) then for some \( \gamma \), \( A + P_\gamma = B + P_\gamma \).

**Proof.** Let \( \gamma \) be such that \( \text{rv}(A \smallsetminus P_\gamma) = \text{rv}(B \smallsetminus P_\gamma) \). It is enough to show \( \{ x \in A \mid v(x) \leq \gamma \} \subseteq B + P_\gamma \). We proceed by induction w.r.t. the dual order on the initial segment \( (-\infty, \gamma] \): Let \( a \in A \). If \( v(a) = \gamma \) there is nothing to do. Suppose \( v(a) < \gamma \) and for all \( a' \) of valuation \( > v(a') \) there is some \( b' \) such that \( a - b' \in P_\gamma \). Since \( A \approx B \) there is \( b \in B \) such that \( v(a - b) > v(a) \). Since \( b \in B \) and \( B \subseteq A \), \( a - b \in A \). Now by applying the induction hypothesis to \( a - b \) we have \( a - b - b' \in P_\gamma \) for some \( b' \in B \). \( \Box \)

**Definition 4.6.** A pseudo-complement of a subgroup \( A \) is a subgroup \( C \) such that, for some \( \gamma \)

\[ M = A + C + P_\gamma \quad \text{and} \quad A \parallel C. \]

**Remark 4.7.** If \( A, C \) and \( P_\gamma \) are as above and \( f : M \to M \) is an additive morphism such that \( f^{-1}(P_\gamma) \subseteq P_\delta \) for some \( \delta \), then \( f^{-1}(C) \) is a pseudo-complement to \( f^{-1}(A) \). In particular this is the case when \( f \) is given by a scalar multiplication.

**Remark 4.8.** If \( C \) is a pseudo-complement of \( A \) then \( C \) is a pseudo-complement of \( A + P_\gamma \) for any \( \gamma \).

**Remark 4.9.** It is straightforward to see that if \( A \) and \( B \) have the same pseudo-complement then \( A \approx B \). The following lemma establishes the converse using that the value set is \( \mathbb{Z} \cup \{ \infty \} \).

**Lemma 4.10.** If \( A \approx B \) then \( C \) is a pseudo-complement of \( A \) if and only if it is a pseudo-complement of \( B \).

**Proof.** Suppose \( C \) is a pseudo-complement of \( A \) satisfying

\[ M = A + C + P_\gamma \]

and \( A \approx B \). Let \( \delta \) be such that \( \text{rv}(A \smallsetminus P_\delta) = \text{rv}(B \smallsetminus P_\delta) \). We may assume that \( \delta \leq \gamma \). We claim that \( M = B + C + P_\delta \).
Remark 4.11. The above proof also shows that if 

\[ v(z) = \delta \]

then trivially \( z \in B + C + P_\delta \). Let \( z \in M \) and suppose that for all \( x \) with \( v(x) > v(z) \) there exist \( (b, c, x_\delta) \in B \times C \times P_\delta \) such that

\[ x = b + c + x_\delta. \]

Write \( z = a + c + z_\delta \) for some \( (a, c, z_\delta) \in A \times C \times P_\delta \) (in fact \( z_\delta \) can be chosen in \( P_\gamma \subseteq P_\delta \)). If \( v(a) \geq \delta \) there is nothing to do. If \( \delta > v(a) > v(z) \) then by induction hypothesis \( a = b + c' + z'_\delta \) with \( b, c', z_\delta \in B \times C \times P_\delta \), and hence

\[ z = b + c' + z_\delta + z'_\delta. \]

The only possibility which remains to be considered is \( v(z) = v(a) < \delta \) since \( A||C \). Pick \( b \in B \) such that \( rv(b) = rv(a) \). Then, since \( v(a - b) > v(a) \), by induction hypothesis the equality

\[ a - b = b' + c' + z'_\delta \]

holds for some \( (b', c', z'_\delta) \in B \times C \times P_\delta \). Hence \( z \in B + C + P_\delta \).

Now to see that \( B||C \) we claim that \( vB \cap vC \in [\delta, \infty] \). Suppose for a contradiction that for some \( b \in B \) and \( c \in C \), \( v(b) = v(c) < \delta \). Then we can choose \( a \in A \) such that \( v(a) = v(b) \) hence \( v(a) = v(c) \in vA \cap vC \) hence \( v(a) \in [\gamma, \infty] \). But \( \gamma \geq \delta \). This is a contradiction.

**Remark 4.11.** The above proof also shows that \( C \) is a pseudo-complement of \( A \) if and only if

\[
A||C \quad \text{and} \quad M \approx (A + C).
\]

**Lemma 4.12.** Let \( A, A', B, B' \) be subgroups such that \( A \approx A', \ B \approx B' \) and \( A||B \). Then the following holds:

1. \( A'||B' \),
2. every pseudo-complement of \( A + B \) is a pseudo-complement of \( A' + B' \).

**Proof.** 1. Let \( \gamma = v(a') = v(b') \) with \( a' \in A \) and \( b' \in B \). Let \( \delta \) be such that \( vA \cap vB \subseteq [\delta, \infty] \) and \( \gamma_1, \gamma_2 \) are respectively the values yielding \( A \approx A' \) and \( B \approx B' \). We claim that \( \gamma \geq \min\{\delta, \gamma_1, \gamma_2\} \). Let \( a \in A \) and \( b \in B \) be such that \( rv(a) = rv(a') \) and \( rv(b) = rv(b') \). Then \( \gamma = v(a) = v(b) \) hence \( \gamma \) must be \( \geq \delta \).

2. Let \( C \) be a pseudo-complement of \( A + B \). Let \( \delta, \gamma_1, \gamma_2 \) be as above. Let \( z = a + b \) with \( a \in A \) and \( b \in B \) such that \( v(z) < \min\{\delta, \gamma_1, \gamma_2\} \). Then \( v(a) \neq v(b) \) and \( v(z) = \min\{v(a), v(b)\} \) necessarily.

Suppose \( v(a) = v(z) \) then \( rv(a) = rv(z) \) hence for some \( a' \in A' \), \( rv(z) = rv(a') \). If \( v(z) = v(b) \) we can choose in the same way \( b' \) such that \( rv(b') = rv(z) \). Hence \( A + B \approx A' + B' \). We apply now Lemma 4.11.

**Definition 4.13.** A valuation independent basis \( \beta \) of the \( K^{\psi} \)-vector space \( K \), is the one such that the members of \( \beta \) have all different valuations in the finite set \( \{0, \ldots, d^n - 1\} \).

For example, the basis \( \alpha \) of the \( K^{\psi} \)-vector space \( K \) is valuation independent.

**Remark 4.14.** If \( \beta \) is a valuation independent basis of the \( K^{\psi} \)-vector space \( K \) then for all \( s > 0 \), \( \beta(n) \) is a valuation independent basis of \( K^{\psi} \)-vector space \( K \). Moreover, any valued module is \( t\beta \)-decomposable for all valuation independent \( \beta \).

**Lemma 4.15.** Let \( g_1, \ldots, g_m \) be all of the same degree \( s \) such that the leading coefficients \( b_1, \ldots, b_m \) have distinct valuations in \( \{0, \ldots, d^n - 1\} \) in \( \Gamma \). Then \( M.g_i||M.g_j \) whenever \( i \neq j \).
Proof. By 4.2, we have 
\[ v(x.g_i) = v(x) \cdot g_i = v(x) \cdot t^ib_i = p^{e_i}v(x) + v_K(b_i) \] 
whenever \( v(x) < \text{Jump}(g_i) \) for \( i = 1 \ldots m \). In particular if \( v(x) < \min\{\text{Jump}(g_i), \text{Jump}(g_j)\} \) for \( j \neq i \) then \( v(x.g_i) \neq v(x.g_j) \). \( \Box \)

Lemma 4.16. Let \( \beta \) be a valuation independent basis of the \( K^x \)-vector space \( K \). Let \( q = t^a + \cdots \in R \) be of degree \( s \). Then there is a unique \( j = j(q) \) such that \( \beta_j \in \beta(s) \) and \( M.q \approx M.t^\beta_j \). As a consequence \( C := \sum_{j \neq j(q)} M.t^\beta_j \) is a pseudo-complement for \( M.q \).

Proof. By Lemma 2.4 we have 
\[ a = \sum_{j \in d} a^s_j \beta_j. \]

Let \( j \) be such that \( v_K(a) = v_K(a^s_j \beta_j) < v_K(a^s_{j'} \alpha_{j'}) \) for all \( j' \neq j \). Then 
\[ v(x.t^s.a) = v(x) \cdot t^s(a^s_j \beta_j) < v(x) \cdot t^s(a^s_{j'} \beta_{j'}) \]
for all \( x \in M \) and \( j' \neq j \).

Let \( \gamma < \min v(x) \). Then by Remark 4.2 for all \( x \) of such that \( v(x) < \gamma \), we have \( \text{rv}(x.q) = \text{rv}(x.t^s.a) \) hence 
\[ \text{rv}(x.q) = \text{rv}(x.t^s.a^s_j \beta_j) = \text{rv}((x.a_j).t^\beta_j). \]

Since \( x \mapsto x.a_j \) is a bijection it follows that \( M.q \approx M.t^\beta_j \). The consequence is now given by Lemma 4.10 \( \Box \)

Lemma 4.17. Given a non-zero matrix \( Q \) with coefficient over \( R \), say with \( k \) rows, there exists a matrix \( Q' \) such that the first column of \( Q' \) consists of polynomials which have all the same degree \( s \) such that the leading coefficients in this column have distinct valuations in \( \{0, \ldots, d^s - 1\} \) and 
\[ M^k.Q = M^k.Q'. \]

Proof. This proof is essentially a slight generalization of the proofs of Lemma 3 and Lemma 4 in \cite{vdDK02}.

Let \( Q = (q_{ij}) \) be a matrix over \( R \) with \( k \)-many rows. We will proceed by induction on \( f = \sum_{\{(i,j) \mid q_{ij} \neq 0\}} (\deg(q_{ij}) + 1) \).

Since \( Q \) is non-zero, \( f > 0 \). Suppose \( f = 1 \). We may assume \( q_{11} = c \in K^x \) and all other entries of \( Q \) are zero. Then \( x = y.Q \) for some \( y \) if and only if, all the coordinates of \( x \) except possibly the first one, are 0. Hence we can take \( Q' \) the matrix which has 1 at the position \((1,1)\) and has all other entries equal to zero.

Now we suppose \( f > 1 \). Let \( e := \max\{\deg q_{ij} \} \geq 0 \). We may suppose that \( q_{11} \) has degree \( e \). Set 
\[ e_{ij} := \deg q_{ij}, \quad e_i := e_{11} \]
and 
\[ c_{ij} := \text{leading coefficient of } q_{ij}, \quad c_i = c_{11}. \]

Claim 1: We may assume that for all \( a_1, \ldots, a_k \in K \), not all are 0. \( \sum a^e_{1i}c_i \neq 0 \).

Suppose \( \sum a^e_{1i}c_i = 0 \). We may also suppose that \( a_1 = 1 \). We define for all \( j \),
\[ \tilde{q}_{1j} = \sum_{i=1}^{k} t^{e-e_i}a_i q_{ij}. \]

Since \( a_1 = 1 \) and \( e = e_1 \)
\[ \tilde{q}_{1j} = q_{1j} + t^{e-e_2}a_2 q_{2j} + \cdots + t^{e-e_k}a_k q_{kj}. \]
We also define $q'_{ij}$ by the equality

$$q_{ij} = t^{e_i} c_{ij} + q'_{ij}.$$  

We claim that $\tilde{q}_{11}$ has degree $< e$: We have

$$\tilde{q}_{11} = \sum_{i=1}^{k} t^{e_i - e_j} a_{ij} c_i + t^{e_i} a_i q'_{11} = \sum_{i=1}^{k} t^{e_i - e_j} a_{ij} c_i + t^{e_i} a_i q'_{11}.\tag{21}$$

Since each $q'_{11}$ has degree $< e$, each $t^{e_i} a_i q'_{11}$ has degree $< e$. Hence the coefficient $t^e$ in $\tilde{q}_{11}$ is $
abla_i a_i^\infty c_i = 0$ and $\deg(\tilde{q}_{11}) < e$.

Let $Q$ be the matrix where we have replaced $q_{11}$ by $\tilde{q}_{11}$. Now the sum of degrees of the non-zero entries of $Q$ is less than the sum of degrees of the non-zero entries of $Q$. Hence in order apply induction it is enough to prove that the solvability of the system $\bar{x} = \bar{y}$. $Q$ is equivalent to the solvability of $\bar{x} = \bar{z} Q$. But this follows by expressing the equations (20) by the equality

$$\bar{Q} = P \bar{Q}$$

where

$$P = \begin{pmatrix} 1 & t^{e_2} a_2 & t^{e_3} a_3 & \cdots & t^{e_k} a_k \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

is clearly invertible in $R$.

Claim 2: Assume Claim 1. We may assume that the polynomials $(q_{11})_t$ of the first column have same degree and leading coefficients of the $(q_{11})_t$ are $K^{e_t}$ linearly independent.

We will show that we can change $Q$ to some $S$, possibly having more rows, such that the system $\bar{x} = \bar{y}$. $Q$ is equivalent to $\bar{x} = \bar{z} S$ with first column of $S$ has required properties.

Recall that $q_{11}$ has degree $e_i$ with $e_1 = e$. For all $1 \leq i \leq k$, using the basis $\alpha(e - e_i)$, we write the equality of terms

$$y_i = \sum_{u \in e^{e_i} \alpha_i} \lambda_u(y_i) t^{e_i} \alpha_u.$$

Now

$$y_i q_{11} = y_i t^{e_i} c_i + y_i q'_{11} = \sum_{u} \lambda_u(y_i) (t^{e_i} \alpha_u c_i + r_{11}(u))$$

where $r_{11}(u) = t^{e_i} \alpha_u q'_{11}$ which has degree $< e$. Set $z_i(u) = \lambda_u(y_i)$ and

$$s_{11}(u) := t^{e_i} \alpha_u c_i + r_{11}(u).$$

Since the leading coefficient of $s_{11}(u)$ is $\alpha_u^{e_i} c_i$, for any $i$ and $u$, $s_{11}(u)$ has the degree $e$.

For $j > 1$ and $u \in e^{e_i}$, set $s_{ij}(u) := t^{e_i} \alpha_u q_{ij}$. Note that we keep $e_i = e_{11}$ but $j$ varies. Let $S$ be the matrix obtained from $Q$, by replacing $i$-th row by the matrix $(s_{ij}(u))_{u,j}$ where $u$ is the row-index and $j$ is the column index. Then the system $(x_j = \sum_i y_i q_{ij})_j$ is equivalent to the system $(x_j = \sum_i \sum_u \lambda_u(y_i). s_{ij}(u))_j$, which can be written as

$$\bar{x} = ((\lambda_u(y_1))_u, \ldots, (\lambda_u(y_k))_u). S.$$

Now we will show that the leading coefficients $\alpha_u^{e_i} c_i$’s are $K^{e_t}$-linearly independent.

Suppose

$$\sum_i \sum_u \alpha_u^{e_i} \alpha_u^{e_i} c_i = 0$$
for some tuple \((a_u)_{u,t}\) from \(K\). It follows by Claim 1 that, \(\sum_u (a^n_u)^{e_i}_u ^{x_i} \alpha_u) = 0\) for each \(i\). Since the \(\alpha_u\) are \(K^{x'}\)-linearly independent and \(\varphi\) is injective \(a_u = 0\) for all \(i,u\). The Claim 2 is proved.

We assume now that the first column of \(Q\) consists of polynomials of degree \(e\) with leading coefficients being \(K^{x'}\)-linearly independent. By section 3 and by the last paragraph of Lemma 4 of \cite{vdDK02}, there exists an invertible matrix over \(P\) over \(K\), where \(Q_1\) the first column of \(Q\), consists of polynomials with leading coefficients having all different valuations in \(\{0, \ldots, d^e - 1\}\). Hence considering \(P,Q\) finishes the proof.

For the following lemma, we will use the above result with \(Q\) a column matrix and then in the following corollary we will use it in whole generality.

**Lemma 4.18.** Let \(A \subseteq M\) of the form \(A = \sum_i M.t^s b_i\) then for some integer \(s\), \(A\) has a pseudo-complement of the form \(\oplus_{i \in I} M.t^s b_i\) where \(I \subseteq d^s\), and the \(b_i\) are valuation independent.

**Proof.** Use the above lemma to chose \(g_i\) such that \(\sum_j M.t^s b_j = A\), all of degree \(s\) with leading coefficients \(b_j\)'s have different valuations in \(\{0, \ldots, d^s - 1\}\). Then by the lemma \[4.16\] \(M.g_j = M.t^s b_j\) for all \(j\). Complete the \(b_j\) to a valuation independent basis of \(K^{x'}\)-vector space \(K\). We write the new elements of this basis as the \(b_i\). Let \(C := \sum_j M.t^s b_j\). Since \(C \oplus \sum_j M.t^s b_j = M\), \(C\) is in particular a pseudo-complement for \(\sum_j M.t^s b_j\). Hence by Lemma \[4.10\] \(C\) is a pseudo-complement for \(A\). \(\square\)

**Remark 4.19.** The pseudo-complement \(C\) is p.p. definable subgroup by a \(L\)-formula \(C(x)\), which does not depend on \(M\). In other words, if \(\phi(x)\) is the formula
\[
\exists y_1, \ldots, y_m\ x = \sum_i y_i . q_j
\]
then in any valued module \((M,v)\), \(C(M)\) is a pseudo-complement to \(\phi(M)\).

**Theorem 4.20.** Let \(\phi(\bar{x})\) be a p.p. formula of \(L_O\) of the form
\[
\bar{x} - y . Q \in W,
\]
where \(Q\) a matrix with coefficients from \(R\) and \(W\) a ball. Set \(k := |\bar{x}|\). Then there exists computable \(\gamma\) and an existential \(L\)-p.p. formula \(D(\bar{x})\) such that
\[
M^k = \phi(M^k) + D(M^k) + P^k_{\gamma} \\
&\text{&} D(M^k) \cap \phi(M^k) \subseteq P^k_{\gamma}
\]
for all valued module \((M,v)\).

**Proof.** By Lemma \[4.17\] we may assume that the first column \(Q_1\) of \(Q\) consists of polynomials having same degree \(s\) with leading coefficients having distinct valuations in \(\{1, \ldots, d^s - 1\}\). In addition by Remark \[4.8\] we may suppose that \(W = \{0\}^k\).

Take \(C\) a pseudo-complement to \(M.Q_1\) as in the above lemma. Let \(x = (x_1, \ldots, x_k)\) such that \(x_1 \in C \cap M.Q_1\). Then \(x_1 \in P_3\) for some \(\delta\). Write \(x_1\) also as
\[
x_1 = y_1 . q_{11} + y_2 . q_{21} + \ldots y_m q_{m1}.
\]
Since the leading coefficients of the \(q_{ij}\) have different valuations in \(\{0, \ldots, d^s - 1\}\), if
\[
v(y_j) \leq \rho := \min \{ \min_j \{ \text{Jump}(q_{ij}) \} \} - 1;
\]
we have \(v(y_j . q_{ij}) \neq v(y_j . q_{ij'})\) for \(j \neq j'\). Since \(v(x_1) \geq \delta\) the \(y_j\) can not have indefinitely small valuations. Now for \(i > 1\), since \(x_i = \sum_j y_j . q_{ij}\) we have \(v(x_i) \geq \min_j \{ \rho . q_{ij}, \delta \}\). Setting \(\gamma :=

\[ \min_{ij} \{ \rho \cdot q_{ij} \} \text{ and } D := C \times M^{k-1} \text{ yields our claim since } D \text{ and } \gamma \text{ depends only to } Q \text{ and to the theory of valued modules.} \]

**Remark 4.21.** The above proof shows that whenever a matrix \( Q \) with the first column consists of polynomials whose dominant coefficients are valuation independents and \( \gamma \in \Gamma \), as above are given, there is a computable \( \delta \in \Gamma \), such that

\[ \bar{y} \cdot Q \in P^m_\gamma \Rightarrow \bar{y} \in P^k_\delta. \]

The theory \( T_\Psi \). Let \((M,v)\) be a valued module,

\[ A := \sum M.q_i \]

where the \( q_i \) are all of degree \( s \), and for a valuation independent basis \( \beta \) of \( K^s \)-vector space \( K \) the leading coefficients of the \( q_i \) are from the basis \( \beta(s) \). Let \( A(\beta) \) be the tree whose nodes are subgroups \( M.\beta_i(n) \ (i, n \in \omega) \), ordered by inclusion. At the level \( n \) we have the subgroups \( M.\beta_i(n) \ (i \in d^n) \).

**Definition 4.22.** We call the the pseudo-complement for \( \sum M.q_i \) the unique pseudo-complement which can be written as the sum of some elements of level \( \deg(q_i) \) of the tree \( A(\beta) \).

This definition only depends in the decomposition of the leading coefficients of the \((q_i)_i\) in the basis \( \beta(n) \). Hence, given \( A \) as above, by Lemmas 4.17 and 4.18 there is a recursive function

\[ f : \bigcup_n K^n \rightarrow \bigcup_n K^n \]

which computes the basis \( \beta \) and the pseudo-complement \( C \) of in every valued \( t\beta \)-decomposable \( R \)-module \( M \). We permit ourself to write \( C_{f(q_1,\ldots,q_n)} \) for the pseudo-complement computed by \( f \).

On the other hand, the ball \( P_\beta \) such that \( M = A + C + P_\beta \), can be chosen by letting \( \gamma := \min\{\min_i \{\text{Jump}(q_i)\}\} - 1 \). This yields another recursive function

\[ j : \bigcup_n K^n \rightarrow \Gamma. \]

Hence we can express the statement of Corollary 4.20 by introducing an axiom \( \psi(q_1,\ldots,q_n) \) for any matrix \( Q \) which has first column \((q_1,\ldots,q_n)\), in language \( L_\omega \), which says that

\[ M^k = M^k.Q + C_{f(q_1,\ldots,q_n)} \times M^{k-1} + P^k_{f(q_1,\ldots,q_n)} \]

and

\[ M^k.Q \cap C_{f(q_1,\ldots,q_n)} \times M^{k-1} \subseteq P^k_{f(q_1,\ldots,q_n)}. \]

Let \( T_\Psi \) be the \( L_\omega \)-theory of \( R \)-modules together with the sentences \( \psi(q_1,\ldots,q_n) \). Hence \( T_\Psi \) is recursively enumerable.

**Theorem 4.23.** Let \( Q \) be a \( m \times k \) matrix over \( R \), \( W = \prod_{i=1}^k P_{\gamma_i} \) and \( \phi \) be the \( L_\omega \)-formula

\[ \phi(x_1,\ldots,x_k) : \exists y_1\ldots y_m \ (x_1,\ldots,x_k) - (y_1,\ldots,y_m).Q \in W. \]

Then for some computable \( \gamma \), and some positive primitive \( L_\omega \)-formula \( D(x) \) we have

\[ M^k = \phi(M) + D(M) \quad \& \quad D(M) \cap X \subseteq \prod_{i=1}^k P_{\gamma_i}, \]

for all \( M \models T_\Psi \).
5. Decidability and model completeness of $\mathbb{F}_d((X))$

We will introduce a new theory $T_1$, containing $T_\psi$, augmented by sentences counting the number of solutions in $B_0$ modulo $B_1$ for proper balls $B_1 \subseteq B_0$, of the p.p. formula $\bar{x} - \bar{y}.Q \in B$ where $Q$ is a matrix over $R$ and $B$ a ball.

For $Q$, a $m \times n$ matrix over $R$, such that its first column consists of polynomials whose leading coefficients are valuation independents, and $B$ a ball, set

$$A(\bar{x}) := \exists \bar{y}.\bar{x} - \bar{y}.Q \in B.$$  

Let $\delta$ be the value computed by Theorem 3.16 such that $A \land P_\delta$ is quantifier-free definable in language $L(\lambda)$.

Let $B_0 := P_\delta^m$, where $B_0$ is computed by Theorem 4.23 $B_1 := P_\delta^m$, and $k := \left| (A(K) \cap B_0(K))/A(K) \cap B_1(K) \right|$, if $\gamma \leq \delta$ we set the sentence $\theta(A,\gamma,\delta)$ expressing

$$k = \frac{(A \land B_0)(K)}{(A \land B_1)(K)}.$$  

Let $\Theta$ the be set of sentences $\theta(A,\gamma,\delta)$ and we set

$$T_1 := T'_{\text{hens}} \cup T_\emptyset \cup \Theta$$

where $T'_{\text{hens}}$ is the $L_\Theta$-theory composed by the axioms of $T_{\text{hens}}$, written by remplacing $\lambda$ functions by its equivalents modulo the theory of $R$-modules in language $L$ (recall Remark 2.11).

Note that $T_1$ implies that $P_{\gamma+1}/P_{\gamma}$ has exactly $d$ elements for all $\gamma \neq \infty$.

**Remark 5.1.** $T_1$ is a recursively enumerable theory. In fact, by Remark 4.21 if $x, y, \gamma$ are such that $\bar{y}.Q = \bar{x} \in P_\gamma^n$, then $\bar{y}$ is in some $P_\delta^n$ for a computable $\delta$, hence searching the solutions $\bar{y}$, of $\bar{y}.Q = \bar{x}$ with $\bar{x} \in P_\gamma^n$ can be bounded to searching $y$’s in some ball. Hence searching such solutions modulo another ball can be done in some finite $\mathbb{F}_d$-vector space by an algorithm.

**Proposition 5.2.** Let $M := \mathbb{F}_d((X))$ then as an $L_\Theta$-structure $M \models \Theta$ and hence $M \models T_1$.

**Proof.** By [Kuh16], Theorem 5.14, $K$ is existentially closed as a ring in $M$. In addition, by [AF17] Corollary 6.18, there exists an existential ring-formula without parameters which defines the maximal ideal both in $K$ and in $M$. Since the valuation ring is the complement of the set of inverses of the elements in the maximal ideal, we have a universal ring-formula which defines uniformly the valuation ring in $K$ and in $M$. Hence the balls centered at 0 are definable universally with the parameter $X$.

Suppose $|(A(K) \cap B_0(K))/(A \land B_1(K))| = k$. Consider the sentence

$$\sigma := |(A \land B_0)/(A \land B_1)| \geq k.$$  

Since by theorem 3.16 $(A(K) \cap B_1(K))$ is definable both universally and existentially in $L_\Theta$, and using the universal definition of the valuation ring both in $K$ and $M$, it is clear that $\sigma$ is equivalent to an existential ring-formula with parameters in $K$. Since this quotient is finite, we must have $|(A(M) \cap B_0(M))/(A(M) \cap B_1(M))| = k$. \hfill $\square$

To prove Theorem 5.5 we will use a lemma from Rohwer’s thesis ([Roh03] Lemma 8.2). This lemma is a generalization of the following fact:

**Observation 5.3.** In an abelian group $G$ with existentially definable subgroups $A, B$ such that $A + B = G$, if $A \land B$ is definable by a universal formula then $A$ is definable by the following universal formula $\psi(x)$:
(26) $\psi(x) : \forall y \ (x - y \in A \land y \in B) \rightarrow y \in A \land B$.

By iterating this observation, we have the following lemma.

Lemma 5.4 (Rohwer). Let $T$ be a theory expanding the theory of abelian groups. For each $M \models T$ and for $A, A_c, A_m, A_s, B_0, B_1$ definable subgroups of $M$ satisfying the following configuration,

1. $A + A_c = M$,
2. $A \cap A_c \subseteq B_0$,
3. $A \cap B_1 = A_s \cap B_1$,
4. $A \cap B_0 \subseteq A_m \subseteq A + B_1$.

where $A, A_c, B_1$ are definable by existential formulas, and $A_m, A_s$ by universal formulas (where all formulas in question do not depend on $M$), $A$ is definable by an universal formula (which does not depend on $M$).

Proof. See [Roh03] Lemma 8.2. \qed

Theorem 5.5. Any p.p. formula of $L_\mathcal{O}$ is equivalent modulo $T_1$ to an universal $L_\mathcal{O}$-formula.

Proof. Let

$$A(x) : \exists \bar{y} \bar{x}.S - \bar{y}.Q \in W$$

be a p.p. formula. For our purposes, we may assume that $S$ is identity by replacing $A(x)$ by

$$\forall \bar{z} \ (\bar{z} = \bar{x}.S \rightarrow \exists \bar{y} \bar{z}.-\bar{y}.Q \in W).$$

By Lemma 4.17, we may suppose that the first column of $Q$ consists of polynomials which have coefficients in a valuation independent basis. By Theorem 3.16, there is some proper ball $B_1$ determined by $T_1$ such that, $A_s := A \cap B_1$ is equivalent to a quantifier free $L(\lambda)$-formula and hence by Remark 2.11, to an universal $L$-formula. Now, let $B_0$ given by Theorem 4.23 such that

$$A \cap A_c \rightarrow B_0$$

where $A_c$ is of the form $A_c = C + P^l_\gamma$ with $C$ the pseudo-complement of $A$.

Claim: Set $A_m := (B_0 \land A) + A_s$. Then $A_m$ is equivalent to an universal $L_\mathcal{O}$-formula modulo $T_1$.

Proof of the claim: Let $M \models T_1$. Let $k$ be the cardinality of

$$(B_0(M) \land A(M))/A_s(M)$$

and $y_1, \ldots, y_k$ be representatives of the classes. Note that $k$ is determined by a sentence in $\Theta$, hence depends only on the theory $T_1$. Then

$$\{y_1, \ldots y_k\} + A_s(M) = A(M) \cap B_0(M) + A_s(M).$$

Moreover, for all $z_1, \ldots, z_k \in B_0$ satisfying $z_i - z_j \notin A_s(M)$ for $i \neq j$,

$$\{z_1, \ldots z_k\} + A_s(M) = A(M) \cap B_0(M) + A_s(M)$$

if and only if,

$$\{z_1, \ldots z_k\} + A_s(M) \subseteq A(M) \cap B_0(M) + A_s(M).$$
Hence the formula
\begin{equation}
(27) \quad x \in B_0 \land \forall y_1, \ldots, \forall y_k \\
\left( \bigwedge_{i=1}^{k} y_i \in B_0 \land (\bigwedge_{i \neq j} y_i - y_j \notin A_s) \land \right)
\left[ \exists z_1 \left( \bigwedge_{i=1}^{k} z_i \in A \land (\bigwedge_{i=1}^{k} y_i - z_i \in A_s) \right) \right] \rightarrow \bigvee_{i=1}^{k} y_i - x \in A_s.
\end{equation}
is equivalent to $A_m$, which is equivalent to a universal $L_\varnothing$-formula, thus the claim is proved.

Now $A, A_c, A_m, A_s, B_0, B_1$ are in the Rohwer’s configuration. 

\textbf{Corollary 5.6.} Every completion of $T_1$ is model-complete in language $L_\varnothing$. In particular the complete $L_\varnothing$-theory of $\mathbb{F}_d((X))$ is model-complete.

Note that $K$ embeds (via an $L_\varnothing$-embedding) to any model $N \models T_1$. In fact, choose any $s \in N \setminus \{0\}$ such that $\sigma t = s$. Consider the $L$-embedding $k \mapsto s.k$. We show that it is an $L_\varnothing$-embedding: It is easy see that the assertion
\[
\mathcal{O} = A \oplus \mathcal{O}.X
\]
where $A$ is the formula $x.t = x$, is a consequence of $T_1$ and it is clear that if $a \in \mathcal{O}_N$ than $s.a \in \mathcal{O}_N$. Suppose now that $s.a \in \mathcal{O}_N$ for some $a \in K$. Then $s.a = y + m$ where $y.t = y$ and $m \in \mathcal{O}_N.X$. Then $(s.a).t - 1 = s.(t - 1) \in \mathcal{O}_N.X$. Now note that $(s.a).t - 1 = s.ta^2 - sk = s.(a^2 - a) \in \mathcal{O}.X$ since $s.t = s$. This can only happen if $(a^2 - a) \in \mathcal{O}_K$ hence only if $a \in \mathcal{O}_K$.

By Corollary 4.2.18 we have:

\textbf{Corollary 5.7.} The models of $T_1$ in which $K$ is existentially closed as an $L_\varnothing$-structure are elementary equivalent to $K$.

\textbf{Proof.} Let $N \models T_1$ in which $K$ is existentially closed. Let $A, B$ be p.p. formulas with one free variable such that
\[
T_1 \models \forall x \ A_1(x) \rightarrow A(x).
\]
Set $k := |A(K)/A_1(K)|$. It is enough to show that we have
\[
|A(N)/A_1(N)| = k.
\]
Let $l > k$ and suppose
\[
M \models \exists x_1, \ldots, \exists x_l \left( \bigwedge_{i=1}^{l} x_i \in A \land (\bigwedge_{i \neq j} x_i - x_j \notin A_1) \right).
\]
Since modulo $T_1$ the formula $x \in A_1$ is equivalent to an universal formula, its negation is equivalent to an existential $L_\varnothing$-formula. Since $K$ is existentially closed in $N$, there exists at least $l$ element in $A(K)/A_1(K)$. Contradiction.

In particular we have:

\textbf{Corollary 5.8.} $K$ is the prime model of the complete theory of $\mathbb{F}_d((X))$.

\textbf{Proof.} $K$ is existentially closed in $\mathbb{F}_d((X))$ as a $L_\varnothing$-structure since it is existentially closed in $\mathbb{F}_d((X))$ as a ring. Hence
\[
K \equiv \mathbb{F}_d((X)).
\]
Since any completion of $T_1$ is model-complete $K$ is an $L_\varnothing$ elementary substructure of $\mathbb{F}_d((X))$. 

\clearpage
Decidability of $\mathbb{F}_d((X))$. The following fact is an easy exercise using Hensel’s lemma:

**Fact 5.9.** $\mathbb{F}_d[[X]]$ is definable by the $L$-p.p. formula

$$\exists y \ x.tX = y.(t - 1)$$

inside $\mathbb{F}_d((X))$.

Hence, the decidability of $\mathbb{F}_d((X))$ as an $L_O$-structure and as an $L$-structure are equivalent. Recall that the $L$-theory of $K$ (hence of $\mathbb{F}_d((X))$), are given by the sentences stating that

$$|A(K)/(A(K) \land B(K))| = k \quad (k \in \mathbb{N} \cup \{\infty\})$$

where $A, B$ are $L$-p.p. formulas of with one free variable.

We will show that a recursively enumerable subset of these sentences forms a complete axiom system which implies all of them. Hence the $L$-theory of $K$ is decidable.

We set $D := A \land B$. Let $A$ be given by $\exists y \ x.p = \bar{y}q$ and $D$ by $\exists y \ x.r = \bar{y}s$. We may suppose that both $p$ and $r$ are unitary. Hence, by 4.16 for $k = \deg(p)$ and $s = \deg(r)$ we have

$$K.p \approx K.t^k$$

and

$$K.r \approx K.t^s.$$  

Now by Remark 4.9 4.7 and Lemma 4.10 $A \approx D$ if and only if the preimage by $t^k$ of $C_{(q)}$ (which is a pseudo-complement of $A$) is equal to the preimage of $C_{(s)}$ by $t^s$ (which is a pseudo-complement of $D$). Hence it is decidable if $A \approx D$. If $A \neq D$ then $A/D$ is infinite: Let $a \in A \setminus D$. Then for $a' \in A$ with $v(a') < v(a)$ either $a'$ or $a - a'$ is not in $D$.

Now suppose $A \approx D$. It follows by Remark 4.5 that

$$A + P_\alpha = D + P_\alpha$$

for some computable $\alpha$: $\alpha$ can be chosen less than every jump values of all polynomials appearing in the definition of $A$ and $B$. Hence $|A/D| = |A \cap P_\alpha/D \cap P_\alpha|$.

Now by Corollary 3.23 there is a computable $\gamma$ such that $A/D$ is finite if and only if

$$A \cap P_\gamma = D \cap P_\gamma$$

Consider the following algorithm: given $A$ and $B$ setting $D = A \land B$, the algorithm check if $A \approx B$, if not it sets $|A/D| = \infty$. Otherwise computes $\alpha$ such that $A + P_\alpha = B + P_\alpha$. Then it computes the value $\gamma$, so that it can check whether $A \cap P_\gamma/D \cap P_\gamma$ is trivial or infinite. If it is infinite it sets $|A/D| = \infty$. If not, by Remark 1.21 the algorithm can compute the number of the elements of

$$A \cap P_\alpha/P_\gamma \quad D \cap P_\alpha/P_\gamma.$$  

(28)

Hence the $L$-theory consisting of sentences of type 28 can be recursively enumerable and implies the $L$-theory of $K$, hence of $\mathbb{F}_d((X))$. We have proved:

**Theorem 5.10.** Both the $L$- and the $L_O$-theories of $\mathbb{F}_d((X))$ are decidable.
$\mathbb{F}_p((X))$ is decidable as a module over the additive polynomials

References

[AF16] Sylvy Anscombe and Arno Fehm, The existential theory of equicharacteristic henselian valued fields, Algebra Number Theory 10 (2016), no. 3, 665–683.

[AF17] Will Anscombe and Jochen Koenigsmann, An existential $\emptyset$-definition of $\mathbb{F}_p[[t]]$ in $\mathbb{F}_p((t))$, The Journal of Symbolic Logic 79 (2014), no. 4, 1336–1343.

[BP10] Luc Bélair and Françoise Point, Quantifier elimination in valued Ore modules, J. Symbolic Logic 75 (2010), no. 3, 1007–1034. MR 2723780

[BP15] ____, Separably closed fields and contractive Ore modules, J. Symb. Log. 80 (2015), no. 4, 1315–1338. MR 3436370

[Coh95] Paul M Cohn, Skew fields, volume 57 of encyclopedia of mathematics and its applications, Cambridge University Press, Cambridge, 1995.

[DDP02] Pilar Dellunde, Françoise Delon, and Françoise Point, The theory of modules of separably closed fields 1, The Journal of Symbolic Logic 67 (2002), no. 03, 997–1015.

[DS03] Jan Denef and Hans Schoutens, On the decidability of the existential theory of $\mathbb{F}_p[[t]]$, Valuation theory and its applications 2 (2003), 43–60.

[EP05] Antonio J. Engler and Alexander Prestel, Valued fields, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005.

[Hod93] Wilfrid Hodges, Model theory, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993.

[Kuh01] Franz-Viktor Kuhlmann, Elementary properties of power series fields over finite fields, J. Symbolic Logic 66 (2001), no. 2, 771–791.

[Kuh16] ____, The algebra and model theory of tame valued fields, Journal für die reine und angewandte Mathematik (Crelles Journal) 2016 (2016), no. 719, 1–43.

[Ona] Gönenc Onay, Valued modules over skew polynomial rings 1, arXiv:1605.01221.

[Ona11] Gönenc Onay, Modules valués: en vue d’applications à la théorie des corps valués de caractéristique positive, Ph.D. thesis, Paris 7, 2011.

[Roh03] Thomas Rohwer, Valued difference fields as modules over twisted polynomial rings, Ph.D. thesis, University of Illinois at Urbana-Champaign, 2003.

[vdDK02] Lou van den Dries and Franz-Viktor Kuhlmann, Images of additive polynomials in $\mathbb{F}_q((t))$ have the optimal approximation property, Can. Math. Bulletin 45 (2002), 71–79.

Institut für Mathematische Logik und Grundlagenforschung Fachbereich und Informatik Einsteinstrasse 62, 48149 Münster, Deutschland

E-mail address, Gönenc Onay onay@uni-muenster.de