THE RECURRENCE RATE AND HAUSDORFF DIMENSION
OF A NEIGHBOURHOOD OF SOME TYPICAL POINT
IN THE JULIA SET OF A RATIONAL MAP

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Abstract

In this article, we consider hyperbolic rational maps restricted on their Julia sets and study about the recurrence rate of typical orbits in arbitrarily small neighbourhoods around them and their relationship to the Hausdorff dimension of such small neighbourhoods.

Keywords

Complex dynamics of rational maps,
Recurrence rates, Hausdorff dimension.

AMS Subject Classifications 37F10, 37F15, 37F35.
1 Introduction

Let \( T : X \rightarrow X \) be a transformation that preserves an ergodic probability measure \( \mu \). Any neighborhood, say of radius \( r > 0 \), however small, around a generic point \( x \in X \) in \( X \), denoted by \( N_r(x) \) must return to itself infinitely often, according to the Poincaré recurrence theorem. In this paper, we study the relationship between the Poincaré recurrence rate of typical points \( y \in N_r(x) \subset X \) and the Hausdorff dimension of \( N_r(x) \). We shall also focus on the case when \( r \searrow 0 \). In recent papers, Boshernitzan studied the relationship between recurrence rates and \( \sigma \)-finite outer measures for measure preserving dynamical systems focusing on billiards flows, symbolic systems and interval maps, [2]. Barreira and Saussol studied the quantitative behaviour of the recurrence rates by imposing a condition on the measure, [1] and Saussol studied the recurrence rates in rapidly mixing dynamical systems, [8]. In yet another recent paper, the author studied the relationship between incidence rate at which the forward orbit of a generic point \( y \notin N_r(x) \) would reach \( N_r(x) \) and the Hausdorff dimension of \( N_r(x) \), [9]. This paper complements the study in [9].

In this paper, we prove analogous results in the setting of complex dynamics; hyperbolic rational maps restricted on their Julia sets and a non-atomic probability measure preserved by the rational map; for example, the Sinai - Ruelle - Bowen (SRB) measure.

In sections 2 and 3, we write the fundamental and involved definitions that constitute the skeleton on which this paper rests its studies on. We shall also encounter some simple properties, as much necessary of the terms defined therein. In section 4, we state the main results. In further sections, we prove the theorems stated in section 4.

2 Fundamental Definitions

Let \( \mathbb{C} \) denote the Riemann sphere and let \( T \) be a rational map defined on the Riemann sphere. By degree of the rational map, we mean the number of inverse images for a typical point \( z \in \mathbb{C} \) counted with multiplicity. In other words, the maximum among the degrees of the two relatively prime polynomials whose quotient yields the rational map is defined to be its degree denoted by \( d \). For our purpose of study in this paper, we shall only consider those rational maps whose degree is at least 2. One of the several possible definitions of the Julia set \( J \subset \mathbb{C} \) of \( T \) states that it is the closure of the set of all repelling periodic points, i.e.,

\[
J := \left\{ z \in \mathbb{C} : T^p z = z \text{ for some } p \in \mathbb{Z}_+ \text{ and } |(T^p)'(z)| > 1 \right\}. \tag{2.1}
\]

Elementary observations reveal that the rational map remains completely invariant on its Julia set, i.e., \( T^{-1}(J) = J \). For example, consider the polynomial map \( T(z) := z^d \) defined on \( \mathbb{C} \). The Julia set of this polynomial map is the unit circle in the complex plane;

\[
J = S^1 := \{ z \in \mathbb{C} : |z| = 1 \}.
\]

For more properties of Julia sets of rational maps, please refer [4]. We focus on hyperbolic rational maps restricted on their Julia sets in this paper, i.e., there exists \( C > 0 \) and \( \lambda > 1 \).
such that for all $z \in J$ and $n \geq 1$, we have $|(T^n)'(z)| \geq C \lambda^n$. Since the Julia set is compact and the transformation $T$ is continuous, the set of non-atomic $T$-invariant Borel probability measures defined on $J$, denoted by $\mathcal{M}_T(J)$ is non-empty. The Lyubich’s measure that equidistributes the pre-images of a typical point in $J$ and the periodic points of $T$ in $J$ is one such example. Observe that the Lyubich’s measure reduces to the Haar measure on $S^1$.

We now define the pressure of a real-valued continuous function in accordance with thermodynamic formalism. Consider a continuous function, $f : J \to \mathbb{R}$. Its pressure is defined by

$$\text{Pr}(f) := \sup \left\{ h_\mu(T) + \int f \, d\mu : \mu \in \mathcal{M}_T(J) \right\}. \quad (2.2)$$

Here, $h_\mu(T)$ denotes the entropy of the transformation with respect to the measure $\mu$, see [10] for more details. If $f$ is a real-valued, Hölder continuous function defined on the Julia set $J$ of some hyperbolic rational map $T$, then, by a result due to Denker and Urbanski in [3], there exists a unique equilibrium measure, called the Sinai - Ruelle - Bowen measure (the Gibbs’ state) denoted by $\mu_f \in \mathcal{M}_T(J)$ realising the supremum in the definition of pressure. We further remark that by adding a coboundary to the Hölder continuous function $f$, one can normalise pressure, as done by Haydn in [5] so that $\text{Pr}(f) = 0$.

It is merely an observation that such systems $T : J \to J$ along with $\mu \in \mathcal{M}_T(J)$ are ergodic. One of the several possible definitions of ergodicity states, given any real-valued function $f \in L^1(\mu)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j(z) - \int f \, d\mu = 0, \quad \mu\text{-a.e.} \quad (2.3)$$

Now let $f, g \in L^2(\mu)$. Consider the quantity called covariance defined by

$$\text{Cov}_T(f, g) := \lim_{n \to \infty} \int f \circ T^n g \, d\mu - \int f \, d\mu \int g \, d\mu. \quad (2.4)$$

$T$ is said to be mixing if $\text{Cov}_T(f, g) = 0$ for every $f, g \in L^2(\mu)$.

In general, $\text{Cov}_T(f, g)$ vanishes at an arbitrarily slow pace as $n \to \infty$. It requires a little more structure on the underlying space for us to say anything tangible about the vanishing rate of $\text{Cov}_T(f, g)$. Fortunately, we are only dealing with Julia sets of rational maps in this paper and they have the requisite structure being compact metric spaces. An upper bound for $\text{Cov}_T(f, g)$ is then provided by $\|f\| \|g\| \theta_n$, where

$$\|f\| := \sup_{z_1, z_2 \in J} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|},$$

and $\theta_n \to 0$, as $n \to \infty$.

Dependent on the rate at which $\theta_n$ vanishes as $n \to \infty$, we shall call $\text{Cov}_T(f, g)$ to vanish as $n \to \infty$. For example, if $(\theta_n)$ is a sequence of real numbers such that $n^{-p} \theta_n \to 0$ as $n \to \infty$ for some fixed $p > 0$, we shall say that $\text{Cov}_T(f, g)$ has a polynomial decay. In this paper we shall focus on systems whose covariance have super-polynomial decay, i.e., $n^{-p} \theta_n \to 0$ as $n \to \infty$ for every $p > 0$. 

Page 3
Suppose we partition $\mathcal{J}$ finitely into $\{N_i\}_{i=1}^M$ such that $\bigcup N_i = \mathcal{J}$ and $\mu(N_i \cap N_j) = 0$ whenever $i \neq j$. We call the decay as a local decay (say on $N_i$) if $\text{Cov}_T(f,g) \leq \|f\|\|g\|\theta_{i,n}$ where $\text{supp}(f) \subset N_i$, $\text{supp}(g) \subset N_i$ and $\theta_{i,n}$ decays at some rate. Observe that if the covariance has a super-polynomial rate of decay, then the system has a super-polynomial rate of local decay in every component $N_i$ of the partition of $\mathcal{J}$.

### 3 Recurrence Rate and Hausdorff Dimension

Fix $z \in \mathcal{J}$ and for any $r > 0$, consider the neighbourhood $N_r(z)$ in $\mathcal{J}$ centered at $z$ of radius $r$. Owing to the non-atomicity of the measure $\mu \in \mathcal{M}_T(\mathcal{J})$, it is clear that $\mu(N_r(z)) > 0$ whenever $r > 0$. By the Poincaré recurrence theorem, we then have that the orbit of $\mu$-a.e. $w \in N_r(z)$ returns to $N_r(z)$ infinitely often. We define the Poincaré recurrence time of the centre $z \in N_r(z)$ as follows.

$$\tau_r(z) := \inf \{n \in \mathbb{Z}_+ : T^n(z) \in N_r(z)\}.$$  \hspace{1cm} (3.1)

More generally, allowing a slight abuse of notation, the Poincaré recurrence time of any typical point $w \in N_r(z)$ in $N_r(z)$ is denoted and defined by

$$\tau_r(w,z) := \inf \{n \in \mathbb{Z}_+ : T^n(w) \in N_r(z)\}.$$  \hspace{1cm} (3.2)

Thus, $\tau_r(z) \equiv \tau_r(z,z)$. In fact, this definition is valid even for $w \notin N_r(z)$. Hence, for any point $w \in \mathcal{J}$, one can define the incidence time of $w$ in $N_r(z)$ as

$$\tau_r(w,z) := \inf \{n \in \mathbb{Z}_+ : T^n(w) \in N_r(z)\}.$$  \hspace{1cm} (3.3)

There is a possibility that $T^n(w) \notin N_r(z)$, $\forall n$. For example, consider a periodic point $w \in \mathcal{J}$ whose cycle remains away from $N_r(z)$. In such situations, we define

$$\tau_r(w,z) := \infty.$$  \hspace{1cm} (3.4)

Observe that in all these definitions, $\tau$ and $r$ are inversely proportional to each other. In particular,

$$\tau_{kr}(z) \leq \tau_r(z) \quad \forall k \geq 1.$$  \hspace{1cm} (3.5)

Similarly,

$$\tau_{kr}(w,z) \leq \tau_r(w,z) \quad \forall w \in \mathcal{J} \quad \text{and} \quad \forall k \geq 1.$$  \hspace{1cm} (3.6)

Further, for any $w \in N_r(z)$, we have

$$\tau_{kr}(z) \leq \tau_r(w,z) \leq \tau_{kr}(z) \quad \forall k \geq 1.$$  \hspace{1cm} (3.7)
In this paper, we shall use the above definitions of the recurrence times to understand the concept of recurrence rate of typical orbits as $r \searrow 0$. We first define the recurrence rate of the centre $z \in N_r(z)$ as follows.

$$R(z) := -\liminf_{r \to 0} \frac{\log \tau_r(z)}{\log r};$$

$$\overline{R}(z) := -\limsup_{r \to 0} \frac{\log \tau_r(z)}{\log r}.$$

If $R(z) = \overline{R}(z)$, then the recurrence rate is denoted and defined by,

$$R(z) := -\lim_{r \to 0} \frac{\log \tau_r(z)}{\log r}.$$  \hspace{1cm} (3.8)

Now consider $\mu \in \mathcal{M}_T(J)$. We now define a local version of the fractal dimension of $N_r(z)$ with respect to this chosen measure $\mu$ as follows.

$$d_{\mu}(z) := \liminf_{r \to 0} \frac{\log (\mu(N_r(z)))}{\log r};$$

$$\overline{d}_{\mu}(z) := \limsup_{r \to 0} \frac{\log (\mu(N_r(z)))}{\log r}.$$

Now consider the real-valued Hölder continuous function $f$ defined on $J$, given by $f = -s \log |T'|$. Then we know that $f$ could be suitably normalised in order that $Pr(-s \log |T'|) = 0$. In other words, there exists a unique $s \in \mathbb{R}$ such that $Pr(-s \log |T'|) = 0$. This unique value of $s$ is called the Hausdorff dimension of the Julia set, $J$. Furthermore, observe that the essential supremum of $d_{\mu}$ is nothing but the Hausdorff dimension of $J$, given by $s$.

### 4 Main Results

We state the main results of this paper in this section. The proofs of the results are given in the following sections.

**Theorem 1** Let $T$ be a hyperbolic rational map restricted on its Julia set, $J$. Let $N_r(z)$ be a neighbourhood of radius $r > 0$ about the point $z$ in $J$ and let $\mu \in \mathcal{M}_T(J)$. Then for $\mu$-a.e. $z \in J$, we have

1. $R(z) \leq d_{\mu}(z);$
2. $\overline{R}(z) \leq \overline{d}_{\mu}(z).$
Theorem 2 Let $T$ be a hyperbolic rational map restricted on its Julia set, $J$. Let $\text{Cov}_T(f, g)$ decay at a super-polynomial rate ($f, g \in L^2(\mu)$). Let $N_r(z)$ be a neighbourhood of radius $r > 0$ about the point $z$ in $J$ and let $\mu \in \mathcal{M}_T(J)$. Then for $\mu$-a.e. $z \in J$, we have

1. $R(z) \geq d_\mu(z)$;
2. $\overline{R}(z) \geq \overline{d}_\mu(z)$.

The following is an immediate corollary that follows from the statements of theorems 1 and 2.

Corollary 3 Let $T$ be a hyperbolic rational map restricted on its Julia set, $J$. Let $\text{Cov}_T(f, g)$ decay at a super-polynomial rate ($f, g \in L^2(\mu)$). Let $N_r(z)$ be a neighbourhood of radius $r > 0$ about the point $z$ in $J$ and let $\mu \in \mathcal{M}_T(J)$. Then for $\mu$-a.e. $z \in J$, we have

1. $R(z) = d_\mu(z)$;
2. $\overline{R}(z) = \overline{d}_\mu(z)$.

5 Proof of Theorem 1

We begin this section with two definitions; diametrically regular measures and weakly diametrically regular measures, as can be found in Federer, [4].

A measure $\mu$ is called diametrically regular if there exists $k > 1$ and $c > 0$ such that

$$\mu(N_{kr}(z)) \leq c\mu(N_r(z)), \quad \forall z \in J, \quad \text{and} \quad \forall r > 0. \quad (5.1)$$

A measure $\mu$ is called weakly diametrically regular on a set $B \subset J$ if there exists $k > 1$ such that for $\mu$-a.e. $z \in B$ and $\alpha > 0$, there exists $\delta > 0$ that satisfies,

$$\mu(N_{kr}(z)) \leq \frac{1}{r^\alpha}\mu(N_r(z)) \quad \text{whenever} \quad r < \delta. \quad (5.2)$$

Though the next lemma is obvious by the nomenclatures of the above defined two terms, it is imperative that we specify the constants that relate them.

**Lemma 4** Diametrically regular measures are weakly diametrically regular on $J$. 

Page 6
A $T$-invariant probability measure $\mu$ supported on $\mathcal{J}$ is weakly diametrically regular if for every fixed constant $k > 1$, there exists a $\delta \equiv \delta(z, \alpha) > 0$ so that for $\mu$-a.e. $z \in B \subset \mathcal{J}$ and every $\alpha > 0$, we have

$$
\mu \left( N_k r(z) \right) \leq \frac{1}{\mu^2} \mu \left( N_r(z) \right) \quad \text{whenever} \quad r < \delta.
$$

(5.3)

**Lemma 5** Any Borel probability measure on $\mathbb{C}$ is weakly diametrically regular on its support.

**Proof:** In order to prove the statement in lemma 5, we should show: For $\mu$-a.e. $z \in \mathbb{C},$

$$
\mu \left( N_{\frac{1}{2n}}(z) \right) \leq n^2 \mu \left( N_{\frac{1}{2n+1}}(z) \right),
$$

(5.4)

for sufficiently large $n \in \mathbb{Z}_+$.

For $n \in \mathbb{Z}_+$ and $\delta > 0$, define

$$
K_n(\delta) := \left\{ z \in \mathcal{J} : \mu \left( N_{\frac{1}{2n+1}}(z) \right) < \delta \mu \left( N_{\frac{1}{2n}}(z) \right) \right\}.
$$

Let $E \subset K_n(\delta)$ be a maximal $\frac{1}{2n+2}$-separated set. Then,

$$
\mu \left( K_n(\delta) \right) \leq \sum_{z \in E} \mu \left( N_{\frac{1}{2n+1}}(z) \right) \leq \sum_{z \in E} \delta \mu \left( N_{\frac{1}{2n}}(z) \right).
$$

Observe that $E$ can be written as a finite union of $\frac{1}{2n}$-separated sets, i.e., $E = \bigcup_{i=1}^M E_i$ such that each $E_i$ is $\frac{1}{2n}$-separated. Here, $M$ depends on $n$. Thus the sets, $\left\{ N_{\frac{1}{2n}}(z_i) \right\}_{z_i \in E_i}$, are pairwise disjoint. Hence,

$$
\mu \left( K_n(\delta) \right) \leq \sum_{z \in E} \delta \mu \left( N_{\frac{1}{2n}}(z) \right) \leq M \delta.
$$

Put $\delta = \frac{1}{n^2}$. Then we have obtained,

$$
\mu \left( K_n \left( \frac{1}{n^2} \right) \right) \leq \frac{M}{n^2} \quad \forall n.
$$

Thus,

$$
\sum_{n \geq 1} \mu \left( K_n \left( \frac{1}{n^2} \right) \right) \leq M \sum_{n \geq 1} \frac{1}{n^2} < \infty.
$$

(5.5)
Lemma 6 (Borel - Cantelli Lemma) For a sequence \( \{ B_n \} \) in the \( \sigma \)-algebra of \((X, \mathcal{B}, \mu)\) that satisfies \( \sum_{n \geq 1} \mu(B_n) < \infty \), we have \( \mu(\limsup_{n \to \infty} B_n) = 0 \).

An application of Borel - Cantelli lemma then says,

\[ \mu \left( \limsup_{n \to \infty} K_n \left( \frac{1}{n^2} \right) \right) = 0. \]

In other words, the set of all points that satisfy

\[ \mu \left( N_{\frac{1}{2n^2}}(z) \right) < \frac{1}{n^2} \mu \left( N_{\frac{1}{n^2}}(z) \right) \]

is of measure zero, for sufficiently large \( n \in \mathbb{Z}_+ \), whence our claim. \( \square \)

Proof: (of Theorem 1) Consider the function \( \delta(z, \cdot) \) in the definition of a weakly diametrically regular measure. Observe that for every fixed \( z \in B \subset \mathcal{J} \), \( \delta(z, \cdot) \) is a measurable function.

Fix \( \alpha > 0 \) and choose \( \rho > 0 \) such that

\[ \mu(B) - \mu(G) \leq \epsilon, \]

where \( G = \{ z \in B \subset \mathcal{J} : \delta(z, \alpha) > \rho \} \).

For any \( r > 0 \), \( \lambda > 0 \) and \( z \in \mathcal{J} \), consider the set

\[ A_{4r}(z) := \left\{ w \in N_{4r}(z) : \tau_{4r}(w, z) \geq \frac{1}{\lambda} \frac{1}{\mu(N_{4r}(z))} \right\}. \]

The following is the well-known Chebyshev's inequality.

**Theorem 7 (Chebyshev's inequality)** Let \((X, \mathcal{B}, \mu)\) be a probability measure space. Let \( f \) be a real-valued measurable function defined on \( X \). Then for any \( t > 0 \), we have

\[ \mu \left( \{ x \in X : f(x) \geq t \} \right) \leq \frac{1}{t} \int_X f d\mu. \]

Using Chebyshev's inequality on the set \( A_{4r}(z) \), we have

\[ \mu(A_{4r}(z)) \leq \lambda \mu(N_{4r}(z)) \int_{N_{4r}(z)} \tau_{4r}(w, z) d\mu(w) \]

\[ = \lambda \mu(N_{4r}(z)) \mu(\{ w \in \mathcal{J} : \tau_{4r}(w, z) < \infty \}) \leq \lambda \mu(N_{4r}(z)). \]

Since \( N_{2r}(z) \subset N_{4r}(z) \), we have

\[ \mu \left( A_{2r}(z) := \left\{ w \in N_{2r}(z) : \tau_{4r}(w, z) \geq \frac{1}{\lambda} \frac{1}{\mu(N_{4r}(z))} \right\} \right) \leq \lambda \mu(N_{4r}(z)). \]
Moreover, for \( w \in N_{2r}(z) \), we have
\[
\tau_{8r}(w) \mu (N_{2r}(w)) \leq \tau_{4r}(w, z) \mu (N_{4r}(z)).
\]
Hence,
\[
\mu \left( \left\{ w \in N_{2r}(z) : \tau_{8r}(w) \mu (N_{2r}(w)) \geq \frac{1}{\lambda} \right\} \right) \leq \lambda \mu (N_{4r}(z)).
\]

Now we state a lemma that is useful to complete the proof of theorem \(^1\). The proof of the lemma will be taken up after we complete the proof of the theorem.

**Lemma 8** Let \( \mu \in \mathcal{M}_{\mathcal{T}}(\mathcal{J}) \). Let \( E \subset \mathcal{J} \) be a measurable set. Given \( r > 0 \), there exists a countable set \( K \subset E \) such that

1. \( N_r(z) \cap N_r(w) = \varnothing \), for any two distinct points \( z, w \in K \).
2. \( \mu (E \setminus \bigcup_{z \in K} N_{2r}(z)) = 0 \).

Define a quantity \( D_\alpha(r) \) as,
\[
D_\alpha(r) := \mu \left( \left\{ w \in E : \tau_{8r}(w) \mu (N_{2r}(w)) \geq \left( \frac{1}{r} \right)^{2\alpha} \right\} \right). \tag{5.6}
\]

Then observe that one can obtain an upper bound for \( D_\alpha(r) \) as follows.
\[
D_\alpha(r) = \mu \left( \left\{ w \in E : \tau_{8r}(w) \mu (N_{2r}(w)) \geq \left( \frac{1}{r} \right)^{2\alpha} \right\} \right)
\leq \sum_{z \in K} \mu \left( \left\{ w \in N_{2r}(z) : \tau_{8r}(w) \mu (N_{2r}(w)) \geq \left( \frac{1}{r} \right)^{2\alpha} \right\} \right)
\leq r^{2\alpha} \sum_{z \in K} \mu (N_{4r}(z))
\leq r^{2\alpha} \sum_{z \in K} \mu (N_r(z))
\leq r^{\alpha}. \tag{5.7}
\]

If we choose \( r \) to vanish at an exponential rate, i.e., \( r = e^{-n} \), then by the inequality in \([5.7]\), we have
\[
D_\alpha \left( \frac{1}{e^n} \right) \leq \frac{1}{e^{n\alpha}}.
\]
Further,
\[
\sum_n D_\alpha \left(\frac{1}{e^n}\right) \leq \sum_n \frac{1}{e^{na}} < \infty,
\]
as \(n\) grows larger.

We complete the proof of theorem \(\square\) by invoking the Borel - Cantelli lemma again that asserts
\[
\mu \left( \limsup_{n \to \infty} D_\alpha \left(\frac{1}{e^n}\right) \right) = 0.
\]
In other words, for \(\mu\)-a.e. \(z \in E\), we must have for sufficiently large \(n\),
\[
\tau_{S_{r}}(z) \mu \left( N_{2r}(z) \right) \leq \left(\frac{1}{r}\right)^{2\alpha}, \quad \text{where} \quad r = \frac{1}{e^n}.
\]
Taking logarithms, we have for sufficiently large \(n\),
\[
\log \tau_{S_{r}}(z) + \log \mu \left( N_{2r}(z) \right) \leq 2n\alpha. \quad (5.8)
\]
Thus, we have
\[
\frac{1}{n} \log \tau_{S_{r}}(z) \leq 2\alpha - \frac{1}{n} \log \mu \left( N_{2r}(z) \right), \quad (5.9)
\]
for sufficiently large \(n\).

We now complete this section by writing the proof of lemma \(\square\). This is only an elementary exercise in basic set theory.

**Proof:** (of Lemma \(\square\)) Fix \(z \in E\) and consider the family of subsets of \(E\) around \(z\), ordered by inclusion;
\[
\mathcal{F}_z := \{ N_r(z) \subset E \text{ for various values of } r \}.
\]
Observe that \(\mathcal{F}_z\) is a totally ordered set with a maximal element in \(\mathcal{F}_z\). However, \(E\) is only a partially ordered set; i.e., given \(r > 0\), we can find a \(w \in E\) such that \(N_r(z) \cap N_r(w) = \phi\). Put all such \(z\) and \(w\) in \(K\). Then by construction, \(K\) is countable. We must still verify the second property that the lemma asserts.

Consider the totally ordered family \(\mathcal{F}_z\) of subsets of \(E\) for every \(z \in K\). By Zorn’s lemma, there exists a maximal element for every totally ordered chain in \(E\). Observe that with this maximal element, the second property is satisfied. \(\square\)
6 Proof of Theorem \[2\]

We begin this section with the statement of a proposition. The motivation behind the proposition, given after its statement, clinches the proof of theorem \[2\].

Proposition 9 Let \( T \) be a hyperbolic rational map restricted on its Julia set, \( J \). Let \( \text{Cov}_T(f,g) \ (f,g \in L^2(\mu)) \) decay at a super-polynomial rate. For \( a > 0 \), consider the set \( \{ z \in J : d_\mu(z) \geq a \} \). Given \( \delta, \epsilon > 0 \), there exists \( \rho > 0 \) (depending on \( z \)) such that

\[
T^n(z) \notin N_r(z) \quad \text{for any } r \in (0, \rho) \quad \text{and} \quad n \in \mathbb{Z}_+ \cap \left[ \frac{1}{r^\delta}, \frac{1}{(\mu(N_r(z)))^{1-\epsilon}} \right].
\]  

We shall now briefly look at the motivation behind this proposition. A rigorous proof of the same is written in subsection 6.1. The definition of mixing implies \( \mu(N \cap T^{-n}N) \to (\mu(N))^2 \) as \( n \to \infty \). Therefore, for large \( n \), we have \( \mu(N \cap T^{-n}N) \leq 2(\mu(N))^2 \). Extending this line of argument, one may observe that

\[
\mu(N \cap T^{-n}N \cap T^{-n-1}N \cap \cdots T^{-n-l}N) \leq 2l(\mu(N))^2.
\]  

Now suppose \( l \leq (\mu(N))^{\epsilon-1} \), then

\[
\mu(N \cap T^{-n}N \cap T^{-n-1}N \cap \cdots T^{-n-l}N) \leq 2(\mu(N))^\epsilon.
\]  

Making use of the fact that the covariance decays at a super-polynomial rate, one can then estimate the size of the connected neighbourhood \( N \subset J \). One may then use the Borel-Cantelli lemma to show that typical points exhibit this property. An appropriate condition on \( R \) should then help in completing the proof of the theorem.

Lemma 10 Let \( T \) be a hyperbolic rational map restricted on its Julia set, \( J \). Let \( \text{Cov}_T(f,g) \) decay at a super-polynomial rate locally \((f,g \in L^2(\mu))\). Then

1. \( R(z) \geq d_\mu(z) \);
2. \( \overline{R}(z) \geq d_\mu(z) \),

for \( \mu \)-a.e. \( z \in \{ z \in J : R(z) > 0 \} \).

Proof: Fix \( a > 0 \) and consider the set \( \{ z \in J : R(z) > a \} \). Then, by theorem \[9\] we have that

\[
\{ z \in J : R(z) > a \} \subset \{ z \in J : d_\mu(z) > a \}.
\]

By the definition of \( R \), we know that \( r^a \tau_r(z) \geq 1 \) for sufficiently small \( r \) where \( z \in \{ z \in J : \overline{R}(z) > a \} \).
Put $\delta = a$ in proposition 9. Then given $\epsilon > 0$, we have for $\mu$-a.e. $z \in \{z \in J : R(z) > a\}$,

$$\tau_r(z) \geq \frac{1}{(\mu(N_r(z)))^{1-\epsilon}},$$

provided $r$ is sufficiently small.

Thus,

$$R(z) \geq (1 - \epsilon)d_n(z). \quad (6.4)$$

The arbitrariness of $\epsilon$ completes the proof. The other inequality can be obtained in a similar fashion.

We now explore the validity of the set $\{z \in J : R(z) > a\}$, i.e., does there exist a subset of $J$ whose elements have a strictly positive recurrence rate. A definition and a lemma that is useful for this purpose has been studied by Ornstein and Weiss [7].

Let $\mathfrak{P}$ be a partition of $J$ that has finitely many subsets $\{N_i\}_{i=1}^M$ such that

$$\bigcup_{i=1}^M N_i = J; \quad \mu(N_i \cap N_j) = 0 \text{ whenever } i \neq j.$$

By $\mathfrak{P}(z)$, we denote that element $N_i$ in the partition $\mathfrak{P}$ that contains $z$. Further, a dynamically refined partition is defined as,

$$\mathfrak{P}^n := \mathfrak{P} \vee T^{-1}\mathfrak{P} \vee \cdots \vee T^{-(n-1)}\mathfrak{P}.$$

Suppose for $\mu$-a.e. $z \in J$, there exists a positive real number $\lambda$ (dependent on $z$) such that $N_{\frac{1}{\lambda^n}}(z) \subset \mathfrak{P}^n(z)$ for all $n$ sufficiently large, then we say that the partition $\mathfrak{P}$ has a large interior.

**Theorem 11 [7]** Let $\mathfrak{P}$ be a partition with large interior. Then $R(z) > 0$ for $\mu$-a.e. $z \in J$.

Thus, by theorem 11 due to Ornstein and Weiss, the existence of the set $\{z \in J : R(z) > 0\}$ depends on the existence of a partition $\mathfrak{P}$ of $J$ with a large interior. The next proposition sheds light on the existence of such partitions.

**Proposition 12** Let $T$ be a hyperbolic rational map restricted on its Julia set, $J$. If for $\mu$-a.e. $z \in J$ there exists positive constants $\alpha$, $\beta$ such that $T^n$ behaves like a Lipschitz function with Lipschitz constant $e^{\alpha n}$ on $N_{\frac{1}{\alpha^n}}(z)$ for all $n$ sufficiently large, then there exists a partition $\mathfrak{P}$ of $J$ that has a large interior.
We now make use of proposition \text{[12]} to prove theorem \text{[2]}. The proof of this proposition shall be given in subsection 6.2. We first prove that hyperbolic rational maps restricted on their Julia sets satisfy the hypothesis of proposition \text{[12]} and will then complete the proof of theorem \text{[2]}

\textbf{Proof:} (of Theorem \text{[2]}) As earlier, let $\mathcal{P}$ be a partition of $\mathcal{J}$ into finitely many subsets $\{N_i\}$ such that

$$\bigcup_{i=1}^{M} N_i = \mathcal{J}; \quad \mu(N_i \cap N_j) = 0 \quad \text{whenever} \quad i \neq j.$$ 

Observe that there exists a positive constant $\kappa(N_i)$ such that

$$|Tz_1 - Tz_2| \leq \kappa(N_i)|z_1 - z_2| \quad \forall z_1, z_2 \in N_i.$$ 

Define

$$\log K := \int \log^+ \kappa(\mathcal{P}(z)) \, d\mu(z) = \sum_{N_i \in \mathcal{P}} \log^+ \kappa(N_i) \mu(N_i).$$

Now choose $\alpha > \log K$. Observe that by the definition of ergodicity, as stated in section 2 the following statement is true. For $\mu$-a.e. $z \in \mathcal{J}$, there exists a $m \in \mathbb{Z}_+$ (dependent on $z$) that satisfies

$$\kappa(\mathcal{P}(z)) \times \kappa(\mathcal{P}(Tz)) \times \cdots \times \kappa(\mathcal{P}(T^{n-1}z)) \leq e^{n\alpha} \quad \forall n \geq m.$$ \quad (6.5)

Replace the upper bound in inequality 6.5 by $c(z)e^{n\alpha}$ for some constant $0 \leq c(z) \leq 1$ in order that the above inequality is valid for every positive integer $n \in \mathbb{Z}_+$.

Now choose $\beta > 0$ such that

$$N_{\frac{1}{c(z)\epsilon^2}}(Tz) \subset \mathcal{P}(Tz).$$

It is then a simple exercise to prove (by induction on $n$) that

$$N_{\frac{1}{c(z)\epsilon^2}}(T^n z) \subset \mathcal{P}(T^n z).$$ \quad (6.6)

Thus, we can apply proposition \text{[12]} to say that there exists a partition $\mathcal{P}$ of $\mathcal{J}$ that has a large interior. A result due to Ornstein and Weiss as stated in theorem \text{[11]} and lemma \text{[10]} then completes the proof of theorem \text{[2]} \hfill \Box

6.1 Proof of Proposition \text{[9]}

Consider the real-valued Hölder continuous function $f = -s \log |T'|$ defined on $\mathcal{J}$. Then we know that there exists a unique value of $s \in \mathbb{R}$ called the Hausdorff dimension of $\mathcal{J}$ so that $\Pr(-s \log |T'|) = 0$. Let $a > 0$ be as given in the statement of proposition \text{[9]} Fix $b > 0$ and let $c = \frac{a}{\epsilon}$ for some $\epsilon > 0$. Denote by $\mathcal{J}_a$ the set,

$$\mathcal{J}_a := \{z \in \mathcal{J} : d_{\mu}(z) \geq a\}.$$
For some $\rho > 0$, consider the following sets.

$$G_1 := \{z \in \mathcal{J}_a : \mu(N_r(z)) \leq r^a \ \forall r \leq \rho\}$$

$$G_2 := \{z \in \mathcal{J} : \mu(N_r(z)) \geq r^{b+s} \ \forall r \leq \rho\}$$

$$G_3 := \{z \in \mathcal{J} : \mu(N_r^z(z)) \geq r^s \mu(N_r(z)) \ \forall r \leq \rho\}$$

Observe that by the definition of lower point wise dimension, we have that $\mu(G_1) \to \mu(\mathcal{J}_a)$ as $\rho \to 0$. Further, $\mu(G_i) \to 1$ as $\rho \to 0$ for $i = 2, 3$, the former since $d_0(z) \leq s$ a.e. while the latter because the measure $\mu$ is diametrically regular. Thus, defining $G := G_1 \cap G_2 \cap G_3$, we claim $\mu(G) \to \mu(\mathcal{J}_a)$.

For $r \leq \rho$, define

$$B_r(r) := \left\{z \in \mathcal{J} : \exists n \in \mathbb{Z}_+ \cap \left[\frac{1}{r^3}, \frac{1}{\mu(N_r(z))} \right] \text{ satisfying } n \geq \tau_r(z) \right\}$$

(6.7)

For $z \in \mathcal{J}$, observe

$$N_r(z) \cap B_r(r) \subset \left\{w \in N_r(z) : \exists n \in \mathbb{Z}_+ \cap \left[\frac{1}{r^3}, \frac{1}{\mu(N_r(z))} \right] \text{ satisfying } n \geq \tau_r(w, z) \right\}$$

$$\subset \bigcup_{n \in \mathbb{Z}_+ \cap \left[\frac{1}{r^3}, \frac{1}{\mu(N_r(z))} \right]} N_r(z) \cap T^{-n}N_{2r}(z).$$

Let $\varphi_r : [0, \infty) \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $r^{-1}$ such that it is sandwiched between the characteristic functions of the intervals $[0, r]$ and $[0, 2r]$. Fix $z \in \mathcal{J}$ and define $\psi_{z,r} : \mathcal{J} \to \mathbb{R}$ by $\psi_{z,r}(w) := \varphi_r(|z - w|)$. It is a simple observation that $\psi_{z,r}$ is a Lipschitz function with the same Lipschitz constant as $\varphi_r$.

$$\mu \left( N_r(z) \cap T^{-n}N_{2r}(z) \right) \leq \int \psi_{z,2r} \psi_{z,2r} \circ T^n d\mu$$

$$\leq \|\psi_{z,2r}\| \theta_n + \left(\int \psi_{z,2r} d\mu\right)^2$$

$$\leq \frac{\theta_n}{r^2} + (\mu(N_{4r}(z)))^2.$$

Here, to obtain the second inequality, we have used the hypothesis of lemma 9 that the covariance decays at a super-polynomial rate and to obtain the third inequality, we have used the fact that $\psi_{z,r}$ is a Lipschitz function.

Choose $p > 1$ sufficiently large such that $\delta(p - 1) \geq s + 2b + 2$. Further, choose $\rho > 0$ so small in order that

$$\frac{1}{\rho^\delta} \leq n \implies \theta_n \leq \frac{p - 1}{n^p}.$$ 

Since

$$\sum_{n \geq q} \frac{1}{n^p} \leq \frac{1}{p - 1} q^{p-1},$$

Page 14
we obtain
\[\mu(N_r(z) \cap B_r(\epsilon)) \leq r^{\delta(p-1)} + \frac{(\mu(N_{4r}(z)))^2}{(\mu(N_{2r}(z)))^{1-\epsilon}} \leq \mu(N_{2r}(z)) \left(r^b + r^{a\epsilon-2c}\right).\] (6.8)

Let \(B \subset G\) be a maximal \(r\)-separated set, i.e., \(N_r(\zeta) \cap B = \{\zeta\}\) for every \(\zeta \in B\) and for every \(\omega \in G \setminus B\) there exists a \(\zeta \in B\) such that \(\zeta \in N_r(\omega)\). In other words, the family of neighbourhoods \(\{N_r(\zeta)\}_{\zeta \in B}\) is an open cover for \(G\). Thus,
\[\mu(G \cap B_r(\epsilon)) \leq \sum_{\zeta \in B} \mu(N_r(\zeta) \cap B_r(\epsilon)) \leq \sum_{\zeta \in B} \mu(N_{2r}(\zeta)) \left(r^b + r^{a\epsilon-2c}\right) \leq r^b + r^{a\epsilon-2c}.\] (6.9)

Choosing \(r\) to vanish at an exponential rate, i.e., \(r = e^{-n}\), we obtain
\[\sum_{n \geq 0} \mu\left(B_r\left(\frac{1}{e^n}\right)\right) < \infty.\] (6.10)

Applying the Borel-Cantelli lemma, we then obtain that for \(\mu\)-a.e. \(w \in G\) there exists a \(m(w) \in \mathbb{Z}_+\) such that for every \(m > m(w)\) there does not exist any\n\[n \in \mathbb{Z}_+ \cap \left[\frac{1}{e^{n\delta}}, \frac{1}{\mu(N_{\frac{3}{2r}}(w))^{1-\epsilon}}\right] \text{ such that } T^m w \in N_{\frac{3}{2r}}(w).\] (6.11)

The weak diametric regularity of \(\mu\) then completes the proof of proposition 9.

### 6.2 Proof of Proposition 12

Before we embark on proving proposition 12, we state and prove the following lemma.

**Lemma 13** For every \(z \in J\), and for any \(r > 0\), there exists a \(\rho \in (r, 2r)\) such that
\[
\mu\left(\left\{w \in J : \rho - \frac{r}{4n+1} < |z - w| < \rho + \frac{r}{4n+1}\right\}\right) \leq \frac{1}{2n}\mu(N_{2r}(z)).
\]
Proof: Define a measure $m$ on the interval $(0, 2)$ by $m((0, t)) := \mu(N_{rt}(z))$. Let $I_0 = (1, 2)$. Divide $I_0$ into four pieces of equal length and define $I_1$ to be that interval which satisfies, $m(I_1) \leq m(I_0)/2$. Proceeding on, define $I_{n+1}$ to that interval which satisfies, $m(I_{n+1}) \leq m(I_n)/2$. Thus, we have constructed a decreasing sequence of nested intervals $\{I_n\}_{n \geq 0}$, in whose intersection must lie a single point, say $R$. Thus, 

$$m\left(\left(R - \frac{1}{4n+1}, R + \frac{1}{4n+1}\right)\right) \leq m\left(I_n\right) \leq \frac{1}{2^n} m\left(I_0\right).$$

We now complete the proof of proposition 12 and thus the proof of theorem 2.

Proof: (of proposition 12) Fix $r > 0$. Let $\mathcal{P} = \{N_r(z)\}$ be some partition of $J$. Choose a maximal $r$-separated set $E \in \mathcal{P}$. For any $z \in E$, take $\rho_z \in (r, 2r)$ such that lemma 13 holds. Let $E = \{z_1, z_2, \ldots\}$ be an enumeration of the (at most) countable set $E$. Write $N_i = N_{\rho_{z_i}}(z_i)$ and define 

$$Q_i := N_i \setminus (Q_1 \cup Q_2 \cup \cdots \cup Q_{i-1}) \quad \text{starting with} \quad Q_1 = N_1.$$ 

Observe that by the maximality of the collection of sets, $\mathcal{Q} := \{Q_i\}_{i \geq 1}$ is a partition of $J$. Further, $\partial \mathcal{Q} \subset \bigcup_i \partial N_i$. Thus, 

$$\mu\left(\left\{z \in J : d(z, \partial \mathcal{Q}) < \frac{r}{4^{n+1}}\right\}\right) \leq \mu\left(\bigcup_i \left\{z \in J : \rho_{z_i} - \frac{1}{4^{n+1}} < |z - z_i| < \rho_{z_i} + \frac{1}{4^{n+1}}\right\}\right) \leq \frac{1}{2^n} \sum_i \mu(N_{2r}(z_i)).$$

Recall that $z_i$ were so chosen that they were $r$-separated. Hence, there exists at most $c = c(\dim(J) = s)$ balls of radius $2r$ that can intersect $J$. Thus for some constants $a, c > 0$ and for all $\epsilon > 0$, we have 

$$\mu\left(\left\{z \in J : d(z, \partial \mathcal{Q}) < \epsilon\right\}\right) < c\epsilon^a.$$ 

Thus for any $b > 0$ we have by the invariance of $\mu$, 

$$\sum_n \mu\left(\left\{z \in J : d(T^n z, \mathcal{Q}) < \frac{1}{\epsilon \delta^{bn}}\right\}\right) \leq \sum_n \frac{c}{\epsilon \delta^{bn}} < \infty.$$ 

Hence, by the Borel - Cantelli Lemma 

$$\exists n(z) < \infty \quad \text{such that} \quad d(T^n z, \partial \mathcal{Q}) \geq \frac{1}{\epsilon \delta^{bn}}, \quad \text{for} \quad \mu\text{-a.e. } z \in J. \quad \text{(6.13)}$$
This implies, \( N_{\frac{1}{e^n}}(T^n z) \subset \Omega(T^n z) \), for any \( n \geq n(z) \). We can always choose \( c(z) \in (0, 1) \) sufficiently small in order that
\[
N_{\frac{1}{e^n}}(T^n z) \subset \Omega(T^n z) \quad \forall n \in \mathbb{Z}_+.
\] (6.14)

Choose \( z \in J \) such that the hypothesis in proposition [12] holds, i.e., let \( z \in J \) be a point that assures the existence of positive constants \( \alpha, \beta \) such that \( T^n \) behaves like a Lipschitz function with Lipschitz constant \( e^{n \alpha} \) on \( N_{\frac{1}{e^n}}(z) \) for \( n \) sufficiently large. Observe that without of generality, we can change \( c(z) \) into a smaller constant (if necessary) so that \( T^n \) remains \( e^{n \alpha} \)-Lipschitz on \( N_{\frac{1}{e^n}}(z) \) for all integers \( n \) and that \( \alpha > b + \beta \).

The proof is then complete if we prove
\[
N_{\frac{1}{e^n}}(z) \subset \Omega^k(z) \quad \text{for any} \quad k \leq n.
\] (6.15)

We prove this by induction. Observe that the statement is trivially true for \( k = 1 \). Assuming the statement to be true for \( k \leq n - 1 \), observe that
\[
T^k \left( N_{\frac{1}{e^n}}(z) \right) \subset N_{\frac{1}{e^n}}(T^k z) \subset \Omega(T^k z).
\] (6.16)

Thus, \( N_{\frac{1}{e^n}}(z) \subset \Omega^{k+1}(z) \). \( \Box \)

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S. Sridharan

Recurrence rate and Hausdorff dimension

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