Critical points in edge tunneling between generic FQH states

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A general description of weak and strong tunneling fixed points is developed in the chiral-Luttinger-liquid model of quantum Hall edge states. Tunneling fixed points are a subset of “termination” fixed points, which describe boundary conditions on a multicomponent edge. The requirement of unitary time evolution at the boundary gives a nontrivial consistency condition for possible low-energy boundary conditions. The effect of interactions and random hopping on fixed points is studied through a perturbative RG approach which generalizes the Giamarchi-Schulz RG for disordered Luttinger liquids to broken left-right symmetry and multiple modes. The allowed termination points of a multicomponent edge are classified by a $B$-matrix with rational matrix elements. We apply our approach to a number of examples, such as tunneling between a quantum Hall edge and a superconductor and tunneling between two quantum Hall edges in the presence of interactions. Interactions are shown to induce a continuous renormalization of effective tunneling charge for the integrable case of tunneling between two Laughlin states. The correlation functions of electron-like operators across a junction are found from the $B$ matrix using a simple image-charge description, along with the induced lattice of boundary operators. Many of the results obtained are also relevant to ordinary Luttinger liquids.

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1. INTRODUCTION

Edge states of quantum Hall liquids have attracted continuous attention since their importance was pointed out by Halperin. The only gapless excitations of a two-dimensional electron gas on a quantum Hall plateau are the edge excitations, since the bulk is an incompressible quantum liquid. A quantum Hall (QH) state with a single condensate, such as $\nu = 1$ or $\nu = 1/3$, has a single bosonic mode of edge excitations, which can be thought of as hydrodynamic perturbations of the Hall droplet. For these states all excitations propagate in a single direction (the edge is ‘chiral’), the direction of the classical $E \times B$ drift at the edge due to the confining field $E$ producing the edge. The chiral-Luttinger-liquid (cLL) model of edge structure predicts that more complicated states, such as the integer QHE states with $\nu \geq 2$, have multiple modes of edge excitations with generally different velocities of propagation, and that some FQHE states have modes traveling in both directions along the edge, as verified for the $\nu = 2/3$ edge by numerical calculations.

This article studies the possible ways of tunneling electrons or quasiparticles between one FQHE edge and another or between an FQHE edge and a metallic or superconducting contact. Such tunneling is of interest because, at weak coupling (junction conductance much less than $e^2/h$), tunneling experiments provide the most sensitive probe available of edge properties; at strong coupling (junction conductance of order $e^2/h$) tunneling between simple edges is one of the few examples of a solvable, experimentally accessible nonequilibrium interacting system. In the remainder of the introduction we outline our results and then briefly review the comparison of the chiral-Luttinger-liquid description to existing tunneling experiments.

The first part of this article sets up a framework to describe possible tunneling fixed points in chiral Luttinger liquids and applies it to a number of examples. The best-known example of this type of tunneling is between two one-component edges such as $\nu = 1/3$. Quasiparticle tunneling at finite temperature across a slight constriction in a single $\nu = 1/3$ quantum Hall bar becomes stronger and stronger as the temperature is lowered (quasiparticle tunneling is “relevant” in renormalization-group language) until the constriction becomes large and the system can be described as weak electron tunneling between two separated $\nu = 1/3$ edges. The crossover between these two fixed points can also be driven by applying a voltage across the junction at zero temperature. The fascinating, experimentally accessible physics of tunneling between quantum Hall edges motivates a general study of what fixed points are possible in this system.

Our approach starts by considering the different ways in which two edges can be joined, or one edge can be terminated. The class of bosonic boundary conditions we consider is not formally complete (the set of all allowed conformal boundary conditions for even a single boson is unknown) but includes all previously treated cases plus many others. The requirement of unitary time evolution gives a physical restriction on possible tunneling junctions or edge terminations. As an example of the results, it is possible for a $\nu = 1$ quantum Hall edge to join smoothly to a $\nu = 1/9$ edge in the presence of
a superconductor (acting as a charge reservoir), but not to a $\nu = 1/3$ edge: joining a $\nu = 1$ and $\nu = 1/3$ edge requires an entropy flow to the surroundings. The experimentally relevant correlation functions for electron-like operators across the junction can be calculated from this framework, and we have a simple “image-charge” description when the tunneling problem is folded onto the half-line. We present a solvable model in which interactions between edges give a continuous variation with interaction strength of effective tunneling charge and the $I-V$ curve. This suggests that once interactions between edges are considered, nonuniversal features can appear in tunneling properties.

The last section develops a perturbative RG analysis of the effect on the edge interaction matrix $V$, defined below, of random hopping of quasiparticles between edge modes. The tunneling properties of an edge are affected by $V$ for both weak and strong tunneling, and for weak tunneling such random hopping (introduced to model impurity scattering) is required to obtain universal behavior.\cite{4,5} The agreement discussed below between the chiral-Luttinger-liquid and composite-fermion approaches requires such hopping to drive the edge from a nonuniversal starting point to a universal strong-coupling fixed point. The importance of $V$ for tunneling motivates a study of what form $V$ should take in real systems, where some degree of impurity scattering is always present.

We find that in all principal hierarchy states, random hopping of quasiparticles drives the $V$ matrix to fixed points where charge and neutral excitations are decoupled. Without random hopping, in general all eigenmodes carry charge. When the neutral modes have the same velocity, as in IQHE and main-sequence FQHE $\nu = n/(2n \pm 1)$ edges, each edge has one fixed point and these fixed points are exactly solvable, as shown by Kane, Fisher, and Polchinski (KFP)\cite{6,7,8} and have higher symmetry than the generic clean system (“symmetry restoration by disorder”). Edges with neutral modes in both directions such as $\nu = 5/7$ can have several charge-neutral separated fixed points, some of the solvable type found by KFP and some unsolvable at present. The RG flows have a simple description in terms of the “boost” and rotation coordinates for multicomponent $\chi$LLs. The RG flow equations can also be applied to the formation of the “chiral metal” phase\cite{9} and other coupled Luttinger-liquid problems.

We review one class of experiments to illustrate the current status of different descriptions of tunneling into edge states. The tunneling $I-V$ exponent from a Fermi liquid to a phase edge state is a sensitive measurement of edge structure\cite{10,11} (Fig. 1), which has received much attention since the results of Grayson et al.\cite{12} showing deviations from the predicted curve for filling fractions $1/3 < \nu < 1$. Other measurements\cite{13,14} do show a broad plateau in rough agreement with theory, although the plateau is displaced from its expected location, perhaps because the density profile at the edge is not simply a sudden decline.\cite{14}

In general, an edge can have multiple edge phases, leading to different tunneling exponents. Thus we can use tunneling to study transitions between those edge phases. Note that for each edge with multiple phases, exactly one phase has an electron tunneling exponent lies on the curve predicted by the composite-fermion (CF) approach of Shytov, Levitov, and Halperin\cite{15} in the limit of an infinite number of channels. Assuming this phase is the stable one at finite temperature, the $\chi$LL and CF approaches appear to include the same essential physics, even though the former assumes an incompressible bulk state and the latter a compressible bulk state. The currently unsolvable fixed points are relevant for experiments (e.g., the phase on the CF line at $\nu = 5/7$) and are distinguished by having relevant disorder operators coupling neutral modes in opposite directions. This suggests that the solvability of some fixed points is a consequence of their chiral nature. It is worth pointing out that Fig. 1 shows that smoothly varying physical properties (in this case, the tunneling exponent) can emerge even though the number of edge modes changes discretely.

The outline of the paper is as follows. Section II sets up a framework for describing the fixed points of a general quantum Hall edge with a junction or terminus. The description can be carried out without reference to the details of the microscopic action except for two essential formulas, which we derive in section V. Section III applies the framework to a number of cases of interest, obtaining the critical points for a quantum Hall edge coupled to
a superconductor and two coupled quantum Hall edges. Section IV studies the effects of interactions on tunneling in a solvable model, where interactions lead to a continuous change in the “effective” (i.e., observed in tunneling experiments) charge and filling fraction. Section V derives the correlation functions and selection rules for boundary operators in a chiral Luttinger liquid, which can be understood from an “image-charge” picture as in electrostatics. Section VI examines the effect of random hopping on the interaction matrix of a chiral Luttinger liquid, including charge-neutral separation at the edge, equilibration of velocities, and the basin of attraction of the fixed points of Fig. Section VII contains a brief summary of our main conclusions.

II. GENERAL DESCRIPTION OF CRITICAL POINTS

A. Characterization of an Abelian edge

In the following we will give a general description of edges of Abelian quantum Hall states, emphasizing the data needed to characterize such edges. An Abelian edge state with \( k \) branches is described by \( k \) bosonic fields \( \phi_a \), \( a = 1, \ldots, k \). The edge is characterized by a \( k \)-dimensional symmetric real matrix \( K \), a \( k \)-component charge vector \( q \), a lattice \( \Gamma_c \) that determines the allowed charge excitations, and a \( k \)-dimensional symmetric real matrix \( \Delta \) that describes the interaction between edge branches. The total electric charge density \( \rho_c \) and current \( j_c \) can be expressed through \( \phi_a \) and \( q \):

\[
\rho_c = q_a \rho_a, \quad \rho_a = \frac{1}{2\pi} \partial_x \phi_a \\
j_c = q_a j_a, \quad j_a = -\frac{1}{2\pi} \partial_x \phi_a.
\]

The allowed charge excitations are created by the vertex operators

\[
V_n = e^{in\cdot\phi}, \quad n \in \Gamma_c.
\]

Some \( V_n \) add charge to the edge and others transfer electrons or quasiparticles between different edge branches but are neutral overall. The charge and the statistics of \( V_n \) are given by

\[
Q = n^T K^{-1} q, \quad \theta = \pi n^T K^{-1} n.
\]

The vectors in \( \Gamma_c \) satisfy

\[
n^T K^{-1} n' = \text{integer}, \quad n, n' \in \Gamma_c.
\]

Thus the charge excitations in \( \Gamma_c \) are bosons or fermions with trivial mutual statistics. These vertex operators can appear in the Hamiltonian if they have bosonic statistics.

In many cases, the lattice \( \Gamma_c \) is generated by electron operators, so we will refer to \( \Gamma_c \) as the E-lattice (although \( \Gamma_c \) sometimes contains operators that transfer quasiparticles between different branches).

The quasiparticle operators are also labeled by points in a lattice, \( \Gamma_q \):

\[
\Gamma_q = \{ n | m^T K^{-1} n = \text{integer}, \text{for all } m \in \Gamma_c \}
\]

We will call this lattice the quasiparticle lattice or Q-lattice. Note that the Q-lattice \( \Gamma_q \) is the dual lattice of the E-lattice \( \Gamma_c \). From the definition of the E-lattice, we see that \( \Gamma_c \subseteq \Gamma_q \). Since we are going to discuss many lattices in this paper, we find it is convenient to use a matrix to describe a lattice. We will say a lattice \( \Gamma \) is described by a matrix \( M \) if the column vectors of the matrix generate the lattice. We will denote such a lattice as \( \Gamma = \text{Latt}(M) \). Also we will use \( W \) to denote the transformed lattice of \( \Gamma \) by \( W: WT = \text{Latt}(WM) \). Under this notation, we can write the E-lattice \( \Gamma_c = \text{Latt}(C) \), where the \( k \) by \( k \) matrix \( C \) satisfies \( C^T K^{-1} C = \text{integral matrix} \). The Q-lattice can be found to be

\[
\Gamma_q = \text{Latt}(K(C^T)^{-1}).
\]

The scaling dimensions of all vertex operators (electron or quasiparticle operators) \( V_n \) can be determined from a single matrix \( \Delta \), which depends on \( K \) and interaction strengths between edge branches:

\[
h(n) = \frac{1}{2} n^T \Delta n.
\]

Our reason for emphasizing \( \Delta \) here rather than the “velocity” matrix \( V \) appearing in the \( \chi \)LL action

\[
S_0 = \frac{1}{4\pi} \int dx dt \left[ K_{ij} \partial_x \phi_i \partial_t \phi_j + V_{ij} \partial_x \phi_i \partial_x \phi_j \right]
\]

is that the universal properties of the junction depend on scaling dimensions, determined by \( \Delta \); \( V \) contains additional information (e.g. the velocities of the eigenmodes) which is unnecessary for this section. For example, the (equal-space) correlation of \( V_n \) has a form

\[
\langle V_n(x, t)V_n^\dagger(x, 0) \rangle \sim t^{-2h(n)}
\]

The matrix \( \Delta \) is not an arbitrary real symmetric matrix. If we introduce \( K_{1/2} \) through:

\[
K = K_{1/2}^T \left( \begin{array}{cc} I_{k_+} & 0 \\ 0 & -I_{k_-} \end{array} \right) K_{1/2},
\]

then \( \Delta \) can be expressed as

\[
\Delta = K_{1/2}^{-1} B_{st}^2 (K_{1/2})^{-1}
\]

where \( B_{st} \) is the boost matrix introduced in section VI:

\[
B_{st} = \exp \left( \begin{array}{cc} 0 & b \\ b^T & 0 \end{array} \right)
\]

and \( b \) is a real \( k_+ \times k_- \) matrix.
B. Termination of an Abelian edge

First, let us consider the general problem of termination of an Abelian edge at a point. We know that a 1D electron gas contains two branches, one right-moving and one left-moving. Such a 1D system can be terminated at a point (say $x = 0$) and we can have a system on a half-line $0 < x < \infty$. In contrast, the edge state of a $\nu = 1/m$ QH state contains only one right-moving branch. Such an edge state cannot be terminated at any point without violating unitarity (alternately, the Hamiltonian would be non-Hermitian). Now the question is when a generic Abelian edge described by $(\bar{K}, \bar{q}, \bar{\Gamma}_c, \bar{\Delta})$ can be terminated at a point, and if the edge $(\bar{K}, \bar{q}, \bar{\Gamma}_c, \bar{\Delta})$ can be terminated, how to characterize the different ways in which the edge terminates. Our motivation for considering termination of an edge is that a tunneling junction in an edge can be considered as a special kind of termination, as in the next subsection.

We find that the edge $(\bar{K}, \bar{q}, \bar{\Gamma}_c, \bar{\Delta})$ can be terminated consistently (section V) if there is a $2k$ by $k$ matrix $B$ that satisfies

$$B^T \bar{K}^{-1} B = 0,$$

$$\det(B^T B) \neq 0$$

$$\Gamma_B \equiv \text{Latt}(B) \subset \bar{\Gamma}_c$$

where $k = \dim(\bar{K})/2$ and

$$\bar{\Gamma}_c = \{ n | n \in \bar{\Gamma}_c, n^T \bar{K}^{-1} n = \text{even} \}$$

(i.e., the points in $\bar{\Gamma}_c$ describe the bosonic vertex operators). The physical meaning of the first two conditions is that there are $k$ vectors of length $2k$ (the columns of the matrix $B$) which are null in the indefinite quadratic form $\bar{K}^{-1}$, orthogonal in $\bar{K}^{-1}$, and linearly independent. Previously Haldane [14] described charge-neutral null vectors of $\bar{K}^{-1}$ as “topological instabilities,” which allow oppositely directed edge modes to localize each other and drop out of the low-energy theory; thus condition [14] is that there be $k$ independent topological instabilities orthogonal in $\bar{K}^{-1}$. (We have relaxed the condition of charge-neutrality in order to include situations involving coupling to a superconductor.) In order for $B$ to exist, $\bar{K}$ must have the same number of positive and negative eigenvalues (the edge has the same number of right-moving and left-moving branches).

If more than one $B$ exists, then the edge $(\bar{K}, \bar{q}, \bar{\Gamma}_c, \bar{\Delta})$ can be terminated in more than one way. In other words, the boundary at $x = 0$ can have more than one fixed point. These different fixed points will be referred as different terminations of the edge.

For a termination labeled by the matrix $B$, the fields $\phi$ satisfy the following boundary conditions at the termination point

$$B^T \phi = 2\pi n, \quad n = \text{integral vectors}. \quad (15)$$

Note that $\phi$ can satisfy different boundary conditions (labeled by $n$) even for a single type of terminate labeled by $B$.

The allowed charge excitations (vertex operators) at the boundary are labeled by points in a $k$-dimensional lattice $\Gamma_{QB}$ (called the boundary Q-lattice):

$$\Gamma_{QB} = \text{Latt}\left(\bar{K}B(B^T B)^{-1} - \frac{1}{2}B(B^T B)^{-1}B^T \bar{K}B(B^T B)^{-1}\right)$$

The boundary quasiparticle operators have a form $V^b_i = e^{i\phi} \phi, \quad I \in \Gamma_{QB}$. The scaling dimension of $V^b_i$ is given by

$$h^b(I) = I^T \bar{K}^{-1} B(B^T \bar{\Delta} B)^{-1} B^T \bar{K}^{-1} I$$

(17)

This is one of the main results of this paper.

In the above discussion of termination points, we have ignored any symmetry properties and the related selection rules. In particular, the boundary condition characterized by $B$ may not conserve electric charge. As a result, a boundary vertex operator may not carry a definite electric charge. In order for the termination labeled by $B$ to conserve the electric charge, we must require the $B$ matrix to satisfy

$$B^T \bar{K}^{-1} \bar{q} = 0$$

(18)

For the charge conserving termination points, the electric charge of a boundary vertex operator $V^b_i$ is found to be

$$Q = \bar{q}^T \bar{K}^{-1} I$$

(19)

For a general termination described by $B$, there are $k$ combined charges that are conserved near the boundary. Their densities are given by

$$\rho_a = B_{ba} \delta_k \phi_a / 2\pi$$

(20)

The boundary operator $V^b_i$ carries definite values of these $k$ combined charges:

$$Q = B^T \bar{K}^{-1} I$$

(21)

C. Tunneling junction in an Abelian edge

Now let us consider a more practical problem – a tunneling junction in an Abelian edge. Assume the edge is described by $(K, \bar{q}, \bar{\Gamma}_c, \bar{\Delta})$. The tunneling is described by vertex operators in the Hamiltonian, $V_a, \quad n \in \Gamma_c$ (the lattice $\Gamma_c$ determines which vertex operators are allowed). If the charge is conserved, only the neutral vertex operators in $\Gamma_c$ can be added to the Hamiltonian to describe tunneling. These neutral vertex operators describe the different types of electron/quasiparticle tunneling between different edge branches. We would like to remark that the
The resulting edge is described by \((\tilde{\Gamma}, \tilde{q}, \tilde{\Delta})\):
\[
\tilde{K} = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}, \quad \tilde{\Delta} = \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix}, \\
\tilde{\Gamma}_c = \Gamma_c \oplus \Gamma_c, \quad \tilde{q} = \begin{pmatrix} q \\ q \end{pmatrix}.
\]
(23)

The edge is terminated at \(x = 0\). Now the problem of the different fixed points of a tunneling junction becomes a problem of different ways that the edge \((\tilde{K}, \tilde{q}, \tilde{\Gamma}_c, \tilde{\Delta})\) can terminate at \(x = 0\). More precisely, each fixed point of the tunneling junction correspond to a way in which the edge \((\tilde{K}, \tilde{q}, \tilde{\Gamma}_c, \tilde{\Delta})\) terminates.

### D. Joining of two Abelian edges

Next, let us consider the problem of possible different edge states for a given QH liquid. There is the possibility that different edge potentials and/or electron interactions can lead to different edge states without changing the bulk QH liquid. Indeed, it has been shown that at zero temperature there are multiple stable edge phases of some disordered FQH states with neutral modes in both directions, such as \(\nu = 5/7\). If a given QH liquid has different edge states, then we can put different edge potentials on different segments of the edge, leading to different edge states on different segments. Thus two different edge states of a given QH liquid can always be connected together. This motivates us to ask the following question: when can we connect two Abelian edges \((K_1, q_1, \Gamma_{c1}, \Delta_1)\) and \((K_2, q_2, \Gamma_{c2}, \Delta_2)\) at a point \(x = 0\)?

Again we can fold the edge \((K_2, q_2, \Gamma_{c2}, \Delta_2)\) in \((-\infty, 0)\) on top of the edge \((K_1, q_1, \Gamma_{c1}, \Delta_1)\) in \((0, \infty)\). The resulting edge is described by \((\tilde{K}, \tilde{q}, \tilde{\Gamma}_c, \tilde{\Delta})\):
\[
\tilde{K} = \begin{pmatrix} K_1 & 0 \\ 0 & -K_2 \end{pmatrix}, \quad \tilde{\Delta} = \begin{pmatrix} \Delta_1 & 0 \\ 0 & -\Delta_2 \end{pmatrix}, \\
\tilde{\Gamma}_c = \Gamma_{c1} \oplus \Gamma_{c2}, \quad \tilde{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.
\]
(22)

The two edges \((K_1, q_1, \Gamma_{c1}, \Delta_1)\) and \((K_2, q_2, \Gamma_{c2}, \Delta_2)\) can be joined together only if the edge \((\tilde{K}, \tilde{q}, \tilde{\Gamma}_c, \tilde{\Delta})\) can be terminated. The different ways to join the two edges correspond to the different ways to terminate the edge \((\tilde{K}, \tilde{q}, \tilde{\Gamma}_c, \tilde{\Delta})\).

### III. SOME SIMPLE EXAMPLES

To gain a more intuitive understanding of the results summarized above, we would like to discuss a few simple examples.

#### A. \(\nu = 1/m\) edge state coupled to superconductor

First let us consider an edge of a \(\nu = 1/m\) Laughlin state. We place a tunneling junction to a superconducting state at \(x = 0\). After the folding, we get a two-branch edge:
\[
\tilde{K} = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \quad \tilde{q} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]
(24)

\[
\tilde{\Delta} = \begin{pmatrix} 1/m & 0 \\ 0 & 1/m \end{pmatrix}.
\]

The E-lattice \(\tilde{\Gamma}_c\) is generated by \(n^T = (m, 0)\) (which creates an electron) and \(n^b = (1, 1)\) (which transfers a quasiparticle of charge \(1/m\) from one edge to the other). That is, \(\tilde{\Gamma}_c = \text{Latt}(C), \ C = \begin{pmatrix} m & 1 \\ 0 & 1 \end{pmatrix}\). To understand the fixed points of the tunneling junction, we first study the termination of the two-branch edge.

One termination (called fixed point A) is described by \(B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\). Such a termination conserves electric charge, since \(B\) satisfies (25). To obtain the boundary quasiparticle operators, we need to find the boundary Q-lattice \(\Gamma_{qa}\), which is given by (17). We find
\[
\Gamma_{qa} = \text{Latt}(\begin{pmatrix} m/2 \\ -m/2 \end{pmatrix}).
\]
(25)

An element \(l\) of the lattice \(\Gamma_{qa}\) has the form
\[
1 = l \begin{pmatrix} m/2 \\ -m/2 \end{pmatrix}
\]
(26)

where \(l \in Z\). Thus the boundary quasiparticle operator is labeled by a single integer \(l\). We denote such an operator by \(V_l^b\). The scaling dimension of \(V_l^b\) is
\[
h^b(l) = l^2 \tilde{K}^{-1} B (B^T \Delta B)^{-1} B^T \tilde{K}^{-1} = ml^2/2,
\]
(27)

and the charge is \(Q = \tilde{q}^T \tilde{K}^{-1} l = l\). It is clear that the boundary vertex operator \(V_l^b\) just creates an electron. It is also clear that \(V_l^b\) cannot appear in the boundary
Hamiltonian, since only pairs of electrons can be added or subtracted from the superconductor. To determine the boundary operators that can appear in the Hamiltonian, we need to consider charge conservation. In this problem the charge \( (Q \mod 2) \) is conserved. Thus only \( V_l^b \) with even \( l \) can appear in the boundary Hamiltonian. The leading operator that can appear in the boundary Hamiltonian is \( V_2^b \), which has scaling dimension \( h = 2m \) and a charge \( Q = 2 \). \( V_2^b \) adds two electrons to the boundary and describes the tunneling to the superconductor. The scaling dimension of \( V_2^b \) determines the stability of the fixed point A. Since the scaling dimension \( 2m > 1 \), the fixed point is stable. Physically, the fixed point A corresponds to a junction in which the tunneling to the superconductor vanishes at low energies.

Another termination (called fixed point B) is described by \( B = \begin{pmatrix} m \\ -m \end{pmatrix} \). Such a fixed point does not conserve the charge. The boundary Q-lattice \( \Gamma_{qh} \) is

\[
\Gamma_{qh} = \text{Latt}(\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}).
\] (28)

The elements \( I \) in \( \Gamma_{qh} \) have the form

\[
I = t \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.
\] (29)

and the boundary quasiparticle operator is labeled by an integer \( t \). The scaling dimension of \( V_l^b \) is

\[
h^b = t^2/2m.
\] (30)

Note that although the charge \( Q = Q_1 + Q_2 \) is not conserved, the charge difference \( Q_d = Q_1 - Q_2 \) is conserved at low energies. The \( Q_d \) charge of \( V_l^b \) is given by

\[
Q_d = q^T_d \tilde{K}^{-1} I = t/m
\] (31)

where \( q^T_d = (1, -1) \). This means that the boundary operator \( V_l^b \) transfers \( t/m \) charges between the two branches.

The leading operator that can appear in the boundary Hamiltonian is \( V_1^b \), since we allow any amount of charge to be transferred between the two branches. Its scaling dimension is \( h^b = 1/2m \). Since \( 1/2m < 1 \) the fixed point B is unstable. Physically the fixed point B corresponds to a junction with strong tunneling to the superconductor. A low energy incoming electron will be scattered into an outgoing hole by the junction.

Let us start with a \( \nu = 1/m \) edge state coupled strongly to a superconductor at a point. At high energy scales (temperature or applied voltage), the junction is close to the fixed point B. As the energy is lowered, the coupling strength flows to zero and the junction flows from the unstable fixed point B to the stable fixed point A. Note that the scaling dimensions of leading boundary operators at the fixed points A and B are given by \( 2m \) and \( 1/2m \), which are inverse of each other. Thus the fixed point A and B form a duality pair. The crossover between fixed points A and B can be solved exactly and has been studied extensively in tunneling between FQH edge states. In section IV we show how a simple model incorporating interactions between electrons on different edges can be mapped onto this exact solution.

B. \((\nu_1,\nu_2) = (1/m_1,1/m_2)\) edge state coupled to superconductor

Second, let us consider a two-branch edge described by

\[
\tilde{K} = \begin{pmatrix} m_1 & 0 \\ 0 & -m_2 \end{pmatrix}, \quad \tilde{q} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\] (32)

A superconductor covers and couples to the edge in the region \((-\infty,0)\). We would like to ask: can the coupling produce a gap in the \((-\infty,0)\) region, or equivalently: can the edge terminate at \( x = 0 \)? We find that if \( m_1/m_2 \) is a square of a rational number, then the \( B \) matrices exist and the edge can be terminated. These edges include \((m_1,m_2) = (m,m), (1,9), (3,27), (9,25), \) etc. Note that the \((m_1,m_2) = (1,9)\) edge is the edge of the \( \nu = 8/9 \) QH state. Thus the gapless edge excitations of a \( \nu = 8/9 \) QH state can disappear when coupled to a superconductor.

First, we discuss the case \((m_1,m_2) = (m,m)\). This case was discussed above, except that the lattice \( \Gamma_c \) is different for the present setup. Even here there are two different possibilities. In the first setup where the two edges belong to the same QH liquid (Fig. 3a), \( \Gamma_c \) is given by

\[
\Gamma_c = \text{Latt}(1 \begin{pmatrix} 1 & m \\ m & 0 \end{pmatrix}),(33)
\]

since we can transfer quasiparticles of charge \( 1/m \) between the two branches of the edge. This setup is identical to the case discussed in the last subsection, and here we just repeat the results obtained before. The fixed point A is described by \( B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). The leading boundary operator that can appear in the Hamiltonian has scaling dimension \( 2m \). The fixed point B is described by \( B = \begin{pmatrix} m \\ -m \end{pmatrix} \). The leading boundary operator that can appear in the Hamiltonian has scaling dimension \( 2m \).

In the second setup where the two edges belong to two different QH liquids (Fig. 3b), \( \Gamma_c \) is given by

\[
\Gamma_c = \text{Latt}(m \begin{pmatrix} m & m \\ m & 0 \end{pmatrix}).(34)
\]

since we can only transfer electrons between the two edge branches, not quasiparticles. Here we can define two charges. The first one is the total electric charge \( Q \) described by the charge vector \( q \). Only \( (Q \mod 2) \) is conserved. The second one is the difference of the electric charge on the two branches \( Q_d = Q_1 - Q_2 \). \( Q_d \) is
described by a second charge vector $\mathbf{q}_d = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Only $(Q_d \mod 2)$ is conserved here.

\[
\begin{array}{c}
\text{(a)} \\
\begin{array}{c}
V = 1/m \\
\text{SC}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{(b)} \\
\begin{array}{c}
V = 1/m \\
\text{SC}
\end{array}
\end{array}
\]

FIG. 2. Two schematic setups for a superconducting contact at the termination of two edge states. In (a) the edge states belong to the same droplet, in (b) to separate droplets.

The fixed point A is described by $B = \begin{pmatrix} m \\ m \end{pmatrix}$. The boundary Q-lattice $\Gamma_{qa}$ is

\[
\Gamma_{qa} = \text{Latt}( \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} ).
\]

(35)

The boundary quasiparticle operator is labeled by $l$ with $1 = l \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$. The scaling dimension of $V_l^b$ is

\[
h^b = l^2 / 2m,
\]

(36)

and total charge $Q$ is conserved at low energies. The boundary quasiparticle operator $V_l^b$ carries definite charge $Q = \mathbf{q}_d^T K^{-1} l = 1/m$. The leading boundary quasiparticle operator that can appear in the boundary hamiltonian is $V_{2m}^b$ which carries charge $Q \mod 2 = 0 \mod 2$, and has scaling dimension $2m$.

In the above, we see that not every boundary operator in $\Gamma_b$ can appear in the Hamiltonian. One may wonder what is the meaning of the operators which cannot appear in the Hamiltonian. We notice that although there are no gapless excitations in the region ($-\infty, 0$), there are degenerate ground states in that region. Thus there are domain-wall-like excitations in ($-\infty, 0$) region. The ground state changes from one to another as we go across a domain wall. A domain-wall-like excitations is labeled by a vector $l$ in the boundary Q-lattice $\Gamma_{qb}$. It carries the same quantum numbers as the boundary vertex operator $V_l^b$ labeled by the same vector $l$. As we move a domain-wall like excitation to the boundary, it becomes a boundary quasiparticle excitation described by $V_l^b$.

\[
\begin{array}{c}
\text{(a)} \\
\begin{array}{c}
V = 1/m \\
\text{SC}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{(b)} \\
\begin{array}{c}
V = 1/m \\
\text{SC}
\end{array}
\end{array}
\]

FIG. 3. Two schematic setups for a superconducting contact to two edge states. In (a) the edge states belong to the same droplet, in (b) to separate droplets.

To physically measure the effect of the boundary quasiparticle excitation, let us consider the two setups in Fig. 3 where only a segment of edge is covered by superconductor (The two setups in Fig. 3 are related to the two setups discussed above). In those setups, we can create a pair of boundary excitations: the first one is created by $V_l^b$, $l \in \Gamma_b$ at one boundary and the second one is created by $V_{-l}^b$ at the other boundary. Such an operation transfers a domain-wall-like excitation from one boundary to the other, and is an allowed process. For the fixed point A in the first setup, the lowest scaling dimension for the above pair-creation operators is $m$. For the fixed point B in the first setup, the lowest scaling dimension for the pair-creation operators is $1/m$. For $m > 1$, the fixed point A is stable while the fixed point B is unstable. For both fixed point A and B in the second setup, the lowest scaling dimension for the pair-creation operators is $1/m$. For $m > 1$, the operator is relevant (since $1/m < 1$), and the two edge states separated by the superconductor will join together at low energies.

Next, we would like to discuss the $(m_1, m_2) = (1, 9)$ case. We will also consider the possibility of short-ranged density-density interactions between modes on the $m_1 = 1$ and on the $m_2 = 2$ edge. The interaction effects being discussed are described in detail in the next section for a different case; to avoid duplication we will simply start here from the boost form $[\Pi]$ of the matrix $\Delta$: 
\[ \Delta = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} \cosh \tau & \sinh \tau \\ \sinh \tau & \cosh \tau \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}. \] (39)

The boost parameter \( \tau \) measures the interaction strength: \( \tau = 0 \) corresponds to unmixed \( \nu = 1 \) and \( \nu = 1/9 \) states. The lattice for charge excitations \( \Gamma_c \) is given by

\[ \tilde{\Gamma}_c = \text{Latt}(\begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}). \] (40)

The first termination (called fixed point A) is described by \( B = \begin{pmatrix} 3 \\ 9 \end{pmatrix} \). The boundary Q-lattice is given by (17):

\[ \Gamma_{qA} = \text{Latt}(\begin{pmatrix} 1/6 \\ -1/2 \end{pmatrix}). \] (41)

The boundary quasiparticle operators are labeled by an integer \( l \) with \( 1 \leq l \leq 1/6 \). The boundary operator \( V^b_l \) has scaling dimension \( h^b = \frac{4}{9\nu}(\cosh \tau - \sinh \tau) \). The combined charge \( Q_{(1,3)} = Q_1 + 3Q_2 \) described by the vector \( q^T_{(1,3)} = (1,3) \) is conserved at low energies. The \( Q_{(1,3)} \) charge of \( V^b_l \) is \( Q_{(1,3)} = q^T_{(1,3)}K^{-1}l/3 \). The leading operator that can appear in the Hamiltonian is \( V^b_0 \) which has an even number of \( Q_{(1,3)} \) charges and a scaling dimension \( h^b = \cosh \tau - \sinh \tau \).

The second termination (called fixed point B) is described by a lattice \( B = \begin{pmatrix} 3 \\ 9 \end{pmatrix} \). The boundary Q-lattice is given by

\[ \Gamma_{qB} = \text{Latt}(\begin{pmatrix} 1/6 \\ 1/2 \end{pmatrix}). \] (42)

The boundary quasiparticle is labeled by an integer \( l \) with \( 1 \leq l \leq 1/6 \). The boundary operator \( V^b_l \) has scaling dimension \( h^b = \frac{4}{9\nu}(\cosh \tau + \sinh \tau) \). The combined charge \( Q_{(1,-3)} = Q_1' + 3Q_2' \) described by the vector \( q^T_{(1,-3)} = (1,-3) \) is conserved at low energies. The \( Q_{(1,-3)} \) charge of \( V^b_l \) is \( Q_{(1,-3)} = q^T_{(1,-3)}K^{-1}l/3 \). The leading operator that can appear in the Hamiltonian is \( V^b_0 \) which has an even number of \( Q_{(1,-3)} \) charges and a scaling dimension \( h^b = \cosh \tau + \sinh \tau \). Note that since \((\cosh \tau - \sinh \tau)(\cosh \tau + \sinh \tau) = 1\), fixed points A and B form a duality pair.

### C. Junction between \( \nu_1 = 1/m_1, \nu_2 = 1/m_2 \) edge states

This case has been treated previously in the absence of intermode interactions by several authors [22,23]. Here we will show how the known results for the two fixed points are recovered in our framework, and in the next section consider interaction effects. A new result even in the absence of interactions is the information found below about the lattice of charged boundary operators. For definiteness we consider the case of tunneling between \( \nu = 1 \) and \( \nu = 1/3 \) states which has attracted the most attention. At weak coupling between the two edges, the most relevant neutral operator tunnels an electron between the two edges, with scaling dimension \( \frac{1}{2} \); at strong coupling the most relevant neutral operator has scaling dimension \( \frac{1}{2} \). This neutral operator can be interpreted as tunneling between different minima of the boundary cosine interaction in the sine-Gordon model, or as quasi-particle tunneling in an effective model of two \( \nu = \frac{1}{2} \) edges.

With no intermode interactions, we have

\[ \tilde{K} = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & -m_1 & 0 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & -m_2 \end{pmatrix}, \quad \tilde{q} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \] (43)

Fixed point A (weak coupling) is described by

\[ B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 3 \\ 0 & 3 \end{pmatrix}. \] (44)

The meaning of this \( B \) matrix is that the incoming and outgoing \( \nu = 1 \) edges are joined continuously to each other, and similarly for the incoming and outgoing \( \nu = 1/3 \) edges. The boundary Q-lattice is

\[ \Gamma_{qA} = \text{Latt}(\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}). \] (45)

A general boundary operator \( O_l \) with \( l = l_1(\frac{1}{2}, \frac{1}{2}, 0, 0) + l_2(0, 0, \frac{1}{2}, -\frac{1}{2}) \), \( l_1 \) and \( l_2 \) integers, has electric charge \( Q_l = l_1 + l_3/2 \) and scaling dimension \( l_1^2/2 + l_2^2/6 \). The most relevant neutral boundary operator has \( (l_1, l_2) = (1, -3) \) and scaling dimension 2, as expected for electron tunneling at weak coupling.

Fixed point B (strong coupling) is described by

\[ B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \\ 3 & 1 \end{pmatrix}. \] (46)

The meaning of the first column of \( B \) is that two incoming electrons become one outgoing electron and three outgoing \( q = \frac{1}{3} \) quasiparticles. In the boundary sine-Gordon
language, this corresponds to pinned (Dirichlet) boundary condition on the neutral mode. The second column is simply the charge vector, indicating that overall charge is conserved across the junction. The boundary Q-lattice is

$$\Gamma_{gb} = \text{Latt} \left( \begin{pmatrix} \frac{3}{m} & -\frac{1}{m} \\ -\frac{1}{m} & \frac{3}{m} \end{pmatrix} \right).$$  \hspace{1cm} (47)$$

A general boundary operator $O_1$ with $\Gamma = l_1(\frac{3}{m}, \frac{3}{m}, -\frac{1}{m}, -\frac{1}{m}) + l_2(-\frac{1}{m}, -\frac{1}{m}, \frac{3}{m}, \frac{3}{m})$ has charge $Q = l_2$ and scaling dimension $l_1^2/2 - 3l_1l_2/2 + 3l_2^2/2$. The most relevant neutral boundary operator has $(l_1, l_2) = (1, 0)$ and scaling dimension $\frac{1}{2}$, as expected.

**IV. EFFECTS OF SHORT-RANGE INTERACTIONS**

In this section we consider the effect of short-range interactions on tunneling through a point contact between two Laughlin states. In the absence of interactions, the nonlinear $I-V$ curve was found exactly by Fendel, Ludwig, and Saleur via a mapping onto the integrable boundary sine-Gordon (BSG) model. First we show in the BSG formalism that a simple solvable model incorporating interactions gives a continuous renormalization of the effective fractional charge appearing in the $I-V$ characteristic. We use the BSG formalism since we will eventually be interested not only in the fixed points, which can equally well be described in the $B$-matrix formalism of section III, but also in the crossover. The $I-V$ curve measured in tunneling experiments in real systems, where the screened Coulomb interaction is present, will thus be sensitive in some geometries (discussed below) to nonuniversal electron-electron interactions. Our model is different from that of Pryadko et al., which uses a long-range Coulomb interaction regularized by an opening angle at the junction.

The effective action describing tunneling between edges of two Laughlin states with filling fractions $\nu_1 = 1/m_1$, $\nu_2 = 1/m_2$ is

$$S = S_{\text{free}} + S_{\text{tun}},$$

$$S_{\text{free}} = \frac{1}{4\pi} \int dx \, dt \, \sum_{ij} [K_{ij} \partial_x \phi_i \partial_x \phi_j - V_{ij} \partial_x \phi_i \partial_x \phi_j],$$

$$S_{\text{tun}} = \Gamma \delta(x) (e^{im_1 \phi_1} - e^{-im_2 \phi_2}).$$  \hspace{1cm} (48)$$

Here the matrix $K$, which describes the statistics of the vertex operators $e^{im_i \phi_i}$ created from the bosonic fields, is

$$K = \begin{pmatrix} m_1 & 0 \\ 0 & -m_2 \end{pmatrix}$$  \hspace{1cm} (49)$$

where we have taken the two edges to propagate in opposite directions. If $V$ is diagonal the physics is independent of whether the modes are copropagating or counterpropagating, but we are interested in the case of general $V$ in which case there are differences, as seen below. If the positive definite matrix $V$ is diagonal, its two entries $V_{11}$ and $V_{22}$ are the velocities of the two modes. The off-diagonal elements of $V$ correspond to a density-density interaction across the two edges, since the electron density is proportional to $\partial_x \phi_i$ for each mode.

The above action maps onto a boundary sine-Gordon model, with boson radius determined by the filling fractions of the original states and by the matrix $V$. The boundary sine-Gordon model contains one nonchiral boson (i.e., with both left and right components) on the half-line. The mapping consists of rotating the fields $\phi_1, \phi_2$ so that one new combination $\tilde{\phi}_1$ is proportional to the exponent $m_1 \phi_1 - m_2 \phi_2$ in $S_{\text{tun}}$, while $\phi_2$ does not appear in $S_{\text{tun}}$ and hence is free. Then folding the field $\phi_1$ onto the half-line and rescaling gives the action

$$S_{\text{BSG}} = \int dt \int_{-\infty}^{0} dx \left[ \frac{(\partial_x \phi_1)^2}{2} + \frac{(\partial_x \phi_2)^2}{2} + \cos(\beta \Phi/2) \right].$$  \hspace{1cm} (50)$$

The constant $\beta$, given for diagonal $V$ by $\beta = \sqrt{4\pi/(\nu_1 + \nu_2)}$, gives the tunneling term in (50) the same scaling dimension $\Delta = (m_1^2 + m_2^2)/2$ as in (18). The velocities of the edge modes, defined as the velocities in a basis where $V$ is diagonal, should strictly speaking be equal for this rotation of fields to be valid, but since the tunneling takes place at a point and there is no coherence along the edge, a difference in velocities should not have much effect.

To calculate the conductance across the tunneling junction, the effective $\beta$ which appears in $S_{\text{BSG}}$ needs to be determined, as well as the contribution $q_{\text{eff}}$ to the current from each tunneling event. Previously only certain discrete values of $\beta$, corresponding to tunneling between Laughlin states, were thought to be physically relevant for edge tunneling. This is because a general $\beta$ describes tunneling between two chiral Luttinger liquids with continuous Luttinger parameter, but only specific values of the Luttinger parameter correspond to quantum Hall states $\nu = 1/m$. The main result of this section is that tunneling between Laughlin states with non-diagonal $V$ is described by the boundary sine-Gordon model with continuously varying $\beta$ and $q_{\text{eff}}$.

The model which we solve exactly has a region of constant interaction strength (between contacts $V_1$ and $V_2$ in Fig. 4) and zero interaction elsewhere. It is essential that the two modes in the interaction region be oppositely directed, so that the scaling dimension of the tunneling operator is affected by $V$. The first step is to write the positive definite matrices $V$ and $\Delta$ in terms of a “boost” parameter $\tau$. The advantage of doing so is that $\Delta$ is only a function of $\tau$ and not of the eigenmode velocities $v_i$ which affect $V$; the boost decomposition isolates the dependence of $\Delta$ on as few parameters as possible.
\[ V = K^{1/2}B \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} BK^{1/2}, \]
\[ \Delta = K^{1/2}B \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} BK^{1/2}, \]
\[ K^{1/2} = \begin{pmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{pmatrix}, \]
\[ B = \begin{pmatrix} \cosh \tau & \sinh \tau \\ \sinh \tau & \cosh \tau \end{pmatrix}. \]

(51)

Now the scaling dimension and transferred charge of the tunneling operator can be simply expressed in terms of \( \Delta(\tau) \). In the following we will specialize to the case \( m_1 = 1, m_2 = 3 \), i.e., a \( v_1 = 1 \) edge and \( v_2 = \frac{1}{3} \) edge propagating in opposite directions. The results generalize simply to other values of \( m_1 \) and \( m_2 \), although the boundary sine-Gordon crossover result used below requires that at most one relevant operator be present, restricting \( m_1 \) and \( m_2 \) somewhat.

The intuition behind the effective charge transfer can be understood by considering the “charge-unmixed” point \( \tau = 0 \) where one of the two eigenmodes (i.e., modes which diagonalize \( K = V \)) is neutral, and one charged. Then in this basis the tunneling operator, which is neutral, is of the form \( \exp(iC\phi_n) + \text{h.c.} \) for some constant \( C \). The tunneling operator for this value of \( \tau \) does not transfer any charge from one eigenmode to the other, since only one eigenmode carries charge. Hence the conductance measured across the junction is independent of the rate of tunneling events determined by the coefficient of the tunneling operator. The value \( \tau = \log(1 + \sqrt{3}/\sqrt{2}) \) corresponds to decoupled \( \nu = 1 \) and \( \nu = 1/3 \) edges in the interaction region, which has been previously studied in a number of works.

The scaling dimension of the tunneling operator \( \exp(im_i\phi_i) \) is \( \frac{1}{2}m\Delta(\tau)m \). The effective tunneling charge is determined by how far the neutral tunneling operator \( m \) is from being an eigenmode of the system: \( q_{\text{eff}} = \frac{e}{\hbar}\frac{1}{m}\Delta(m) \), which is zero if \( m \) is an eigenmode and grows as \( \Delta \) moves away from the charge-unmixed point. The current measured across the two contacts of Fig. 4 if there is no electron tunneling across the junction (if \( V_1 - V_2 \) is small) is \( I = (V_1 - V_2)(\frac{1}{3} + \frac{1}{3}(\sigma_1 + \sigma_2)) \), where \( \sigma_1 \) and \( \sigma_2 \) are the conductances along the edges in the interaction region of Fig. 4. We use the identity \( \sigma_1 + \sigma_2 = t\Delta \) in what follows.

The total change in conductance from small \( V \) to large \( V \) is fixed by the scaling dimension of the electron tunneling operator and the effective tunneling charge. From previous work, it is known that if the effective tunneling charge is \( \frac{e^2}{\hbar\Delta} \), where \( \Delta \) is the scaling dimension of the electron tunneling operator. For instance, in tunneling between two \( \nu = 1/3 \) states, the conductance change between no tunneling (two Hall droplets) and no backscattering (one Hall bar) is \( \frac{e^2}{\hbar\Delta} \). In the case of interactions, \( e \) must be replaced by the effective charge transfer per tunneling event \( q_{\text{eff}} \). Hence for the system of Fig. 4.

\[
\begin{align*}
\frac{\hbar\sigma_{\text{max}}}{e^2} &= \frac{2}{3} + \frac{t\Delta t}{2} = \frac{2 + \cosh(2\tau)}{3}, \\
\frac{\hbar\sigma_{\text{min}}}{e^2} &= \frac{3}{3} + \frac{t\Delta t}{2} = \frac{2 + \text{sech}(2\tau)}{3}.
\end{align*}
\]

(52)

In fact the whole conductance curve between these two values can be calculated from the mapping to \( S_{\text{BSG}} \). Before doing so, there is a simple check on our results which gives some insight into why the above values are natural. Assuming conservation of energy (i.e., no dissipation at the junction) gives two possible values of the current: \( I = (V_1 - V_2)(\frac{1}{3} + \frac{1}{3}(\sigma_1 + \sigma_2)) \), corresponding to no tunneling current, and \( I = (V_1 - V_2)(\frac{1}{3} + \frac{2}{3}\sigma_1 + \sigma_2) \). The two corresponding values of the conductance are exactly those in (52). Thus our calculation reproduces the asymptotic values of the conductance consistent with zero-dissipation fixed points.

The current-voltage characteristic can be calculated (Fig. 4) up to one overall constant in the energy scale, which corresponds to the initial strength of tunneling. The result is, with \( V = V_1 - V_2 \) and \( \sigma_{\text{max}} \) as in (53),

\[
I = \sigma_{\text{max}}V - I_{\text{tun}},
\]
\[
I_{\text{tun}} = \begin{cases} 
I^{(1)} & \text{if } V < T_B\Delta^{-1/(2-2/\Delta)}\sqrt{\Delta - 1}, \\
I^{(2)} & \text{if } V \geq T_B\Delta^{-1/(2-2/\Delta)}\sqrt{\Delta - 1},
\end{cases}
\]
\[
I^{(1)} = \frac{q_{\text{eff}}^2 V}{\hbar} \sum_{n=1}^{\infty} f_n(\Delta),
\]
\[
I^{(2)} = \frac{q_{\text{eff}}^2 V}{\hbar\Delta} \left( 1 - \frac{1}{\Delta} \sum_{n=1}^{\Delta} f_n(1/\Delta) \right),
\]
\[
f_n(g) = \frac{(-1)^{n+1} \sqrt{\pi\gamma(n)}(ng)^{n-1/2}}{2T_B\Gamma(3/2 + n(g-1))} \left( eV \right)^{2n(g-1)}.
\]

(53)
Here $T_B$ is some cutoff-dependent constant which may vary with the boost parameter $\tau$. This calculated current should be relevant as long as the interaction strength is nearly constant in the region around the tunneling junction. The details of the interaction far away from the tunneling junction should not matter as long as current is conserved in the incoming and outgoing edge branches.

\[
\mathcal{L} = -\frac{1}{4\pi} (K^{ab}_{\phi_a} \partial_\phi \phi_b - V^{ab}_{\phi_a} \partial_\phi^2 \phi_b)
\]

where $\partial_\phi \phi_a/2\pi = \rho_a$.

For a circular edge (of length $L$), if we assume $\phi_a$ to satisfy the periodic boundary condition $\phi_a(0) = \phi_a(L)$, then the $\phi$ only describe the neutral fluctuations. To go beyond the neutral fluctuations, let us first consider the charge fluctuations created by electron operators $\psi_e$. There are $\dim(K)$ different electron operators

\[
\psi_e^b(x) = e^{in^b_e \phi_e(x)}
\]

where $n^b_e$ is the $b$th column of $K$: $n_e^{ab} = K^{ab}$. An multi-electron operator has a form $e^{in_e^c \phi_e(x)}$ with $n_e$ belonging to a lattice $\Gamma_e$, which will be called the electron lattice. The lattice $\Gamma_e = \text{Latt}(K)$, since $\Gamma_e$ is generated by the column vectors of $K$.

The charge fluctuations created by the electron operators can also be described by the $\phi$ fields in the Lagrangian. But now the $\phi$ fields no longer satisfy the periodic boundary condition. Instead they satisfy

\[
\phi_e(L) = \phi_e(0) + 2\pi N_a
\]

where $N_a$ are integers. In fact, the above $\phi$ describe excitations created by $\prod_b (\psi_e^b)^{N_a}$ from the neutral ground state.

In general, the (multi-)electron operators do not represent all possible “charge” excitations. The most general charge excitations are created by

\[
\psi_e(x) = e^{in^c_e \phi_e(x)}
\]

where $n_e$ are integral vectors belonging to a lattice $\Gamma_e$ (which will be called the E-lattice). The general charge excitations contain all excitations created by the electron operators, and thus $\Gamma_e \subseteq \Gamma_c$. The general charge excitations can also include excitations that transfer fractional charges between different edge branches, but have integer overall charge.

The $\Gamma_e$ must satisfy certain conditions. Since $\psi_e(x)$ all carry integral charges (which can be zero), the vectors in $\Gamma_e$ satisfy

\[
q^T K^{-1} n_e = \text{integer}
\]

The operators $\psi_e(x)$ are also mutually local; that is

\[
(n_e')^T K^{-1} n_e = \text{integer}
\]

for any $n_e$ and $n_e'$ in $\Gamma_e$. This condition implies that $\Gamma_e \subseteq \text{Latt}(K)$. Just like the electron operators, the excitations created by $\psi_e = e^{in^c_e \phi_e}$ are described by $\phi$ that satisfy the boundary condition

\[
\phi(L) = \phi(0) + 2\pi K^{-1} n_e
\]

We can also use the above to define the periodicity of the $\phi$ field. That is $\phi$ and $\phi'$ are equivalent if

\[
\phi = \phi' + 2\pi K^{-1} n_e
\]

for some $n_e \in \Gamma_e$. 

---

**V. CORRELATION FUNCTIONS IN BOUNDED LUTTINGER LIQUIDS**

**A. Review of Abelian edge states**

First we review the $\chi$LL description of the edge of an Abelian QH state. The collective modes that propagate along the edge can be described by several density operators $\rho_a(x)$, which satisfy the algebra

\[
[\rho_a(x), \rho_b(y)] = \frac{i}{2\pi} (K^{-1})_{ab} \delta'(x - y)
\]

where $K$ is a symmetric integral matrix that characterizes an Abelian edge. The total electric density is a sum of $\rho_a(x)$ weighted by a charge vector $q^a$:

\[
\rho_e = q^a \rho_a.
\]

If there is no binding between electrons, we can choose the symmetric basis in which $q^a = 1$. In the following we will work in this symmetric basis.

The edge dynamics is described by the Hamiltonian

\[
H = \int dx \pi V_{ab} \rho_a(x) \rho_b(x)
\]

at low energies, where $V$ is positive definite symmetric matrix. The corresponding Lagrangian density is given by

\[\text{FIG. 5. Differential conductance } dI/dV \text{ in units of } e^2/h \text{ versus scaled voltage } C(\tau)V \text{ for three different values of interaction strength (boost parameter) } \tau. \text{ The horizontal axis is expected to scale by a different cutoff-dependent constant } C(\tau) \text{ for each value of } \tau.\]
B. Correlations with boundary

Now we add a tunnel junction or termination to the system. In what follows it is assumed that in the junction case the edge has been folded onto the half-line as in section II, so that one set \((\tilde{K}, \tilde{\phi}, \Gamma_c, \Delta)\) describes both incoming and outgoing edges, with a total of \(2k\) bosonic fields. It will be shown below that the number of incoming fields must equal the number of outgoing fields. In the presence of tunneling between different edge branches at \(x = 0\), the Lagrangian also contains terms of type

\[
\mathcal{L}_t = \sum_I t_I \delta(x) e^{\mathbf{n}^I \cdot \phi} + \text{h.c.} \quad (65)
\]

where \(t_I\) is the (real) tunneling strength, and the integral vectors \(\mathbf{n}^I\) are points in the \(\Gamma_c\) lattice which also satisfy

\[
(\mathbf{n}^I)^T \tilde{K}^{-1} \mathbf{n}^I = \text{even.} \quad (66)
\]

(Thus \(e^{\mathbf{n}^I \cdot \phi}\) are bosonic operators.) If electric charge is conserved at the boundary, \(\mathbf{n}^I\) should also satisfy

\[
\mathbf{q}^T \tilde{K}^{-1} \mathbf{n}^I = 0 \quad (67)
\]

so that \(e^{\mathbf{n}^I \cdot \phi}\) represents a neutral operator. For the time being, we will not impose the above charge-conservation condition.

The Lagrangian density on the half-line is given by

\[
\mathcal{L} = -\frac{1}{4\pi} \left( (\tilde{K})^{ab} \partial_x \tilde{\phi}_a \partial_x \tilde{\phi}_b - (\tilde{V})^{ab} \tilde{\phi}_a \tilde{\phi}_b \right) \quad (68)
\]

We also need to specify what are the allowed charge excitations, in addition to the Lagrangian (68). This can be achieved by specifying the periodicity conditions of the \(\phi\) fields:

\[
\tilde{\phi} \sim \tilde{\phi}' = \tilde{\phi} + 2\pi \tilde{K}^{-1} \mathbf{n}_c \quad (69)
\]

where \(\mathbf{n}_c\) are vectors in the lattice \(\tilde{\Gamma}_c\).

To completely define our system on the half-line, we need to include boundary conditions at \(x = 0\). First let us consider the boundary condition in Hamiltonian language. We will start with the following type of boundary conditions, specified by a collection of vectors \(B_I^0\) with \(I = 1, \ldots, k:\)

\[
B_I^0 \tilde{\phi}_a(0) = 0, \quad \text{for all } I \quad (70)
\]

Since we are looking for critical points, we would like to study boundary conditions that are invariant under scaling, suggesting the form (71). However, the above form is inconsistent with the periodicity conditions on \(\phi\). It turns out that (71), although sufficient to determine the correlation functions of vertex operators, must be improved (later in this section) in order to determine the lattice of allowed boundary quasiparticle operators.

Since \(\tilde{\phi}_a(x)\)'s do not commute with each other, the above boundary conditions are self-consistent only if

\[
[B_I^0 \tilde{\phi}_a(x), B_J^0 \tilde{\phi}_b(y)] = 0, \quad \text{for all } I, J. \quad (71)
\]

This implies that the vectors that characterize the boundary condition must satisfy

\[
B^T \tilde{K}^{-1} B = 0, \quad \text{det}(B^T B) \neq 0 \quad (72)
\]

where \(B\) is the \(2k\) by \(k\) matrix formed by \(B_I^0\). Note that two different \(B\) matrices, \(B_1\) and \(B_2\), specify the same boundary condition in the sense of (70) if they are related by

\[
B_1 = B_2 U, \quad U \in L(k) \quad (73)
\]

where \(U\) is an \(k \times k\) invertible real matrix. In this case we say \(B_1\) and \(B_2\) are “weakly” equivalent

\[
B_1 \sim B_2. \quad (74)
\]

In the Lagrangian language, the above boundary condition corresponds to \(B^T \tilde{\phi}(0) = 0\) on the fields \(\tilde{\phi}\). Only for certain choices of \(B\) can the operator \(K = (\tilde{K})^{ab} \partial_x \tilde{\phi}_a - (\tilde{V})^{ab} \tilde{\phi}_a \tilde{\phi}_b\) be hermitian. We will show that the condition on \(B\) that makes \(K\) hermitian is nothing but (72). To see this, we first note that

\[
\int dx dt \tilde{\phi}_2^T K \tilde{\phi}_1 = \int dx dt \tilde{\phi}_2^T K \tilde{\phi}_2 + \int dt \partial_t \tilde{\phi}_2^T K \tilde{\phi}_1 |_{x=0} = 0 \quad (75)
\]

Thus \(K\) is hermitian if \(B\) is such that

\[
\tilde{\phi}_a \tilde{\phi}_b^T = 0 \quad (76)
\]

for all \(\tilde{\phi}, \tilde{\phi}'\) that satisfy \(B^T \tilde{\phi} = B^T \tilde{\phi}' = 0\). Let us choose a basis that the vectors in the null space of \(B\) have the form \(\tilde{\phi}_n^T = (0, \ldots, 0, a, \ldots, b)\), i.e., the first \(k\) elements are zero. In this basis, by (70) \(K\) has the form

\[
\tilde{K} = \begin{pmatrix} K_0 & K_1 \\ K_1^T & 0 \end{pmatrix}, \quad (77)
\]

and \(B\) has the form

\[
B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \quad (78)
\]

where \(K_1\) and \(B_1\) are invertible. Since \(\tilde{K}^{-1}\) has the form

\[
\tilde{K}^{-1} = \begin{pmatrix} 0 & (K_1^{-1})^T K_0 K_1^{-1} \\ (K_1^{-1}) & -K_1^{-1} K_0 K_1^{-1} \end{pmatrix}, \quad (79)
\]

thus the condition (70) implies the condition (72). We see that the hermiticity of the operator \((\tilde{K})^{ab} \partial_x \tilde{\phi}_a - (\tilde{V})^{ab} \tilde{\phi}_a \tilde{\phi}_b\) also requires \(B\) to satisfy (72).

Let us summarize our results so far. The critical boundary conditions of an Abelian edge described by \(\tilde{K}\) are characterized by a matrix \(B\) that satisfies (72). If such a \(B\) does not exist, then the edge state described by \(\tilde{K}\) cannot be terminated at a point. If such a \(B\) does
exist, then we can consistently impose a boundary condition $B^T\tilde{\phi} = 0$ and terminate the edge state at $x = 0$ (provided that $B$ satisfies some other conditions that will be discussed later). If more then one inequivalent $B$ exists, then the edge can be terminated in more than one way (the tunneling junction has more than one fixed point).

Now the question is when $B$ exists. First, from (71) we see that the signature of $\tilde{K}$ must be zero in order for $B$ to exist. Since $\tilde{K}$ is invertible, we can write $\tilde{K}$ as

$$\tilde{K} = \tilde{K}_{1/2}^T \begin{pmatrix} I_{k \times k} & 0 \\ 0 & -I_{k \times k} \end{pmatrix} \tilde{K}_{1/2}.$$  (80)

We see that $B$ always exists for the above $\tilde{K}$. A generic $B$ that satisfies (72) can be written as

$$B = \left( \begin{array}{c} 1 \\ T_0^T \end{array} \right) \tilde{K}_{1/2}, \quad T_0 T_0^T = I.$$  (81)

Thus $B$ exists if and only if $\tilde{K}$ has a vanishing signature (there are as many incoming modes as outgoing modes). In this case the different boundary conditions are labeled by an element in the $O(k)$ group.

Next we would like to calculate the correlation functions of the fields $\tilde{\phi}$ in the presence of the boundary. First we choose a basis in which both $\tilde{K}$ and $\tilde{V}$ are diagonal (this can always be done):  

$$\tilde{K} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \Sigma_3, \quad \tilde{V} = \begin{pmatrix} v_R & 0 \\ 0 & -v_L \end{pmatrix} = \tilde{V}_{\text{diag}}$$  (82)

The block $v_R$ has positive diagonal elements which are the velocities of right movers, while $v_L$ has negative diagonal elements which are the velocities of left movers. Thus the velocity for each branch is given by $v_a = (\Sigma_3 \tilde{V}_{\text{diag}})_{aa}$. In this basis a generic $B$ is equivalent to a simple form:

$$B \sim \left( \begin{array}{c} I \\ T_0^T \end{array} \right), \quad T_0 T_0^T = I.$$  (83)

Let $G_{0,ab}(x,t)$ be the correlation of $\tilde{\phi}_a(x,t)$ and $\tilde{\phi}_b(0)$ in the absence of the boundary:  

$$G_{0,ab}(x,t) = \left\langle \tilde{\phi}_a(x,t) \tilde{\phi}_b(0) \right\rangle_0.$$  

The correlation satisfies a linear equation

$$\frac{1}{2\pi} \left( -\tilde{K}^{ac} \partial_x \partial_t + \tilde{V}^{ac} \partial_x^2 \right) G_{0,cb}(x,t) = \delta_{ab} \delta(x) \delta(t)$$  (84)

Since $\tilde{K}$ and $\tilde{V}$ are diagonal in the present basis, the above can be rewritten as

$$\frac{1}{2\pi} \left( -\tilde{K}^{aa} \partial_x \partial_t + \tilde{V}^{aa} \partial_x^2 \right) G_{0,aa}(x,t) = \delta(x) \delta(t)$$  (85)

$$a = 1, \ldots, \dim \tilde{K}.$$  

We notice that the pair $(a, x)$ is simply a label of the $\tilde{\phi}$ field. We can choose another label $(a, \tilde{x})$ to eliminate the velocities and further simplify the above equation. The two sets of labels are related by

$$(a, x) = (a, \tilde{V}^{aa} \tilde{x})$$  (86)

Now the equations for $G_{0,aa}$ become

$$\frac{1}{2\pi} \left( -\Sigma_3^{aa} \partial_x \partial_t + \tilde{V}^{aa} \partial_x^2 \right) G_{0,aa}(\tilde{x}, t) = \delta(\tilde{x}) \delta(t)$$  (87)

$$a = 1, \ldots, \dim \tilde{K}.$$  

The correlation function in the presence of the boundary,

$$G_{ab}(x_1, x_2, t) = \left\langle \tilde{\phi}_a(x_1, t) \tilde{\phi}_b(x_2, 0) \right\rangle,$$  

satisfies according to (70)

$$B^T G(0, x_2, t) = G(x_1, 0, t) B = 0,$$  (88)

and

$$\frac{1}{2\pi} \left( -\tilde{K}^{ac} \partial_x \partial_t + \tilde{V}^{ac} \partial_x^2 \right) G_{cb}(x_1, x_2, t) = \delta(x_1 - x_2) \delta(t)$$  (89)

for $x_1, x_2 > 0$. In the diagonal basis and in terms of the new label $(a, \tilde{x})$, the above equation becomes

$$\frac{1}{2\pi} \left( -\Sigma_3^{aa} \partial_{\tilde{x}} \partial_t + \tilde{V}^{aa} \partial_{\tilde{x}}^2 \right) G(\tilde{x}_1, \tilde{x}_2, t) = \delta(\tilde{x}_1 - \tilde{x}_2) \delta(t)$$  (90)

for $\tilde{x}_1, \tilde{x}_2 > 0$.

The boundary condition $B^T \tilde{\phi} = 0$ can be rewritten for $B$ in the form (83) as

$$\tilde{\phi}_a(0^+, t) = - \sum_{k=1}^k (T_0)_{ab} \tilde{\phi}_{b+k}(0^+, t), \quad a = 1, \ldots, k.$$  (91)

It simply connects the right moving fields $\tilde{\phi}_a$ to the left moving fields $\tilde{\phi}_{a+k}, a = 1, \ldots, k$, through an orthogonal matrix $T_0$. With this understanding, we find that $G_{ab}$ is given by

$$G_{ab}(\tilde{x}_1, \tilde{x}_2, t) = \left\langle \tilde{\phi}_a(\tilde{x}_1, t) \tilde{\phi}_b(\tilde{x}_2, 0) \right\rangle$$

$$= \left( G_0(\tilde{x}_1 - \tilde{x}_2, t) - G_0(\tilde{x}_1 + \tilde{x}_2, t) T \right)_{ab}$$  (92)

where $T = \left( \begin{array}{c} 0 \\ T_0^T \end{array} \right)$.

Note that by restricting to boundary conditions described in terms of the real bosonic fields, we are ignoring possible symmetries not present at the Abelian $K$-matrix level. For instance, if there are two incoming $\nu = 1$ edges described by Fermi fields $\psi_i$, there could be a unitary $U(2)$ rotation at $x = 0$ rather than the orthogonal rotation described above. As an example of the meaning of the orthogonal matrix $T_0$, consider the case of tunneling between $\nu = 1/3$ and $\nu = 1$ states discussed in section IV. The rescaled basis is $K = \text{diag}(1, 1, -1, -1)$, and the tunneling operator is $\exp(i(\partial_1 - \sqrt{3}\partial_2)) + \text{h.c.}$.

The matrix $T_0$ for strong tunneling is
\[ T_0 = \left( \begin{array}{cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{array} \right), \] (93)

which is the same as the matrix mapping incoming quasiparticles to outgoing quasiparticles in Sandler et al.\cite{12}. For instance, two incoming electrons on the \( \nu = 1 \) edge become one outgoing electron on the \( \nu = 1 \) edge and three charge e/3 quasiparticles on the \( \nu = 1/3 \) edge.

From the symmetry of the equation for \( G_0 \), we see that, as a function of \( \tilde{x} \), the matrix function \( G_0(\tilde{x}, t) \) satisfies

\[ TG_0(\tilde{x}, t)T = G_0(-\tilde{x}, t) \] (94)

Using (94) and \( B^T T = B^T \), we can check that the above \( G \) satisfies (88). Certainly, \( G \) also satisfies the equation (90).

To obtain the correlation function in the original basis and in terms of the original labeling \( (a, x) \), we need to start with the explicit form of \( G_0 \): \( G_0(\tilde{x}, t) = -\ln(\Sigma_3 \tilde{x} - t) \). After replacing the label \( (a, \tilde{x}) \) by \( (a, x/|v_a|) \), we find (no summation over repeated indices)

\[
G_{ab}(x_1, x_2, t) = \langle \hat{\phi}_a(x_1, t) \hat{\phi}_b(x_2, 0) \rangle 
= -\ln((\Sigma_3)_{aa}(x_1/|v_a| - x_2/|v_b|) - t)\delta_{ab} 
+ \ln((\Sigma_3)_{aa}(x_1/|v_a| + x_2/|v_b|) - t)(T^T)_{ab}. \] (95)

If \( \tilde{K}, \tilde{V} \) and the boundary condition \( B \) in the original basis are given by

\[
\tilde{K} = W^T \Sigma_3 W, \quad \tilde{V} = W^T \tilde{V}_{\text{diag}} W, \\
B \sim W^T \left( \begin{array}{c} I \\ T_0^T \end{array} \right) \] (96)

then the correlation function in the original basis can be obtained from the transformation \( \hat{\phi} \rightarrow W^{-1} \hat{\phi} \), \( \tilde{K} = \Sigma_3 \rightarrow (W^T) \Sigma_3 W, \tilde{V} = \tilde{V}_{\text{diag}} \rightarrow (W^T) \tilde{V}_{\text{diag}} W, B = \left( \begin{array}{c} I \\ T_0^T \end{array} \right) \rightarrow W^T \left( \begin{array}{c} I \\ T_0^T \end{array} \right), \) and \( G \rightarrow W^{-1}G(W^T)^{-1} \):

\[
G_{ab}(x_1, x_2, t) = \sum_{cd}(W^{-1})_{ac} \ln((\Sigma_3)_{cc}(x_1/|v_c| - x_2/|v_d|) - t)\delta_{cd}(W^T)_{db}^{-1} 
+ \sum_{cd}(W^{-1})_{ac} \ln((\Sigma_3)_{cc}(x_1/|v_c| + x_2/|v_d|) - t T_{cd}(W^T)^{-1})_{db}. \] (97)

The above result for the correlations of \( \hat{\phi} \) allows us to calculate the correlation functions of vertex operators \( O_1 \) by exponentiation. Consider an operator \( V_n = e^{i n x} \hat{\phi}_a \). Far away from the boundary \( (x \gg |v| t) \), the operator has a correlation which is determined by \( G_0 \) only:

\[
\langle V_n(x, t) V_n(x, 0) \rangle \sim 1/t^{g_n} \] (98)

where

\[
g_n = n^T \tilde{\Delta} n \] (99)

and \( \tilde{\Delta} = W^{-1}(W^T)^{-1} \). We can write \( W \) in a form

\[
W = \left( \begin{array}{cc} \tilde{R}_R & 0 \\ 0 & \tilde{R}_L \end{array} \right) \tilde{B}_{st} \tilde{K}_{1/2}, \] (100)

where \( \tilde{R}_{R,L} \in O(k) \), and \( \tilde{B}_{st} \) is the boost matrix of form

\[
\tilde{B}_{st} = \exp \left( \begin{array}{cc} 0 & \tilde{b} \\ \tilde{b}^T & 0 \end{array} \right). \] (101)

In this parameterization of \( W \), \( \tilde{\Delta} \) depends only on \( \tilde{B}_{st} \) (or the \( k \)-dimensional matrix \( b \)):

\[
\tilde{\Delta} = \tilde{B}_{1/2}^{-2}(\tilde{K}_{1/2}^T)^{-1}. \] (102)

Near the boundary \( (x \ll |v| t) \), the correlation has a different algebraic decay

\[
\langle V_n(0, t) V_n(0, 0) \rangle \sim 1/t^{g_n} \] (103)

with

\[
g_n = n^T \tilde{\Delta} n \] (104)

where from equation (97)

\[
\tilde{\Delta} = W^{-1}(I - T)(W^T)^{-1}. \] (105)

The above can be rewritten as

\[
\tilde{\Delta} = 2K^{-1}B(B^T \tilde{\Delta} B)^{-1}B^T K^{-1} \] (106)

Since it is invariant under \( B \rightarrow BU \). (106) is valid for all \( B \), not just the form \( B = W^T \left( \begin{array}{c} I \\ T_0^T \end{array} \right) \).

C. Boundary conditions compatible with periodicity conditions

In the above discussion of the boundary condition (90), we have not considered the problem that this condition violates the periodicity of the fields \( \hat{\phi} \). We need to take into account the periodic nature of \( \hat{\phi} \) field:

\[
\hat{\phi} \sim \hat{\phi} = \hat{\phi} + 2\pi \tilde{K}^{-1} n_c. \] (107)

It is clear that the boundary condition \( B^T \hat{\phi} = 0 \) is not consistent with all the periodic conditions of the \( \hat{\phi} \) field. Since \( \hat{\phi} \) and \( \hat{\phi} + 2\pi \tilde{K}^{-1} n_c \) are equivalent, if \( B^T \hat{\phi} = 0 \) is allowed, then \( B^T (\hat{\phi} + 2\pi \tilde{K}^{-1} n_c) = 0 \) should also be allowed. That is, we need to generalize the boundary condition to at least

\[
B^T \hat{\phi} = 2\pi B^T K^{-1} n_c, \quad n_c \in \tilde{\Gamma}_c. \] (108)

One technical way to understand what has been done in the previous section is that we have only considered boundary conditions for the neutral excitations created
by $\partial_\alpha \tilde{\phi}$. (Here “neutral” does not mean electrically neutral, but rather conserving the zero mode of the bosonic theory.) In addition to these neutral excitations, there are also charged excitations created by vertex operators $V_\mathbf{n} = e^{i \mathbf{n} \cdot \tilde{\phi}}$, where $\mathbf{n}$ is a vector in the E-lattice $\tilde{\Gamma}_c$. Since the vertex operators are the primary fields of the theory, it is the vertex operators which we expect to have scale-invariant boundary conditions, rather than the bosonic fields $\phi$. The generalized boundary condition (113) can also be written as
\[ e^{i \mathbf{n}^T \tilde{\phi}} = 1, \quad \mathbf{n} \in \Gamma_B, \quad (109) \]
where the rows of $B$ are basis vectors of $\Gamma_B$, or
\[ \Gamma_B = \text{Latt}(B), \quad (110) \]
Strictly speaking, it is the boundary condition of the normal-ordered exponential which is conformally invariant, and the normal-ordered version of (109) has $\infty$ rather than 1 on the right-hand side.

To gain a better understanding of the generalized boundary condition (108), let us consider a physical realization of the termination of the edge. We start with an edge described by $\tilde{K}$ on $(-\infty, \infty)$. We then add the following potential term on $(-\infty, 0)$:
\[ - \sum_{\mathbf{n} \in \Gamma_B} C_\mathbf{n} \cos(\mathbf{n} \cdot \tilde{\phi}) \quad (111) \]
where the $k$-dimensional lattice $\Gamma_B$ is a sublattice of $\tilde{\Gamma}_c$. The vectors in $\Gamma_B$ satisfy
\[ \mathbf{n}^T \tilde{K}^{-1} \mathbf{n}' = 0, \quad \mathbf{n}, \mathbf{n}' \in \Gamma_B \quad (112) \]
and $C_\mathbf{n} > 0$ are very large, so that the potential consistently pins $\tilde{\phi}$ to the potential minima, and opens an energy gap in the region $(-\infty, 0)$. Such a potential leads to the boundary condition
\[ B^T \tilde{\phi} = 2\pi \mathbf{n}, \quad \mathbf{n} \in \text{Latt}(I_{k \times k}) \quad (113) \]

From the above discussions, we can draw two conclusions. First not all $B$ matrices are consistent with the periodicity properties of $\tilde{\phi}$. To specify a valid termination of an edge, $B$ not only must satisfy (72), the rows of $B$ must also be in the $\tilde{\Gamma}_c$ lattice, or
\[ \text{Latt}(B) \subset \tilde{\Gamma}_c \quad (114) \]
Second, two $B$ matrices, $B_1$ and $B_2$, give rise to the same generalized boundary condition if
\[ B_1 = B_2 M, \quad M \in \text{GL}(k, Z) \quad (115) \]
Such a pair of $B$ matrices are regarded as equivalent:
\[ B_1 \cong B_2 \quad (116) \]
Note that the above equivalence relation for generalized boundary condition is stronger (i.e., has smaller equivalence classes) than the equivalence relation $B_1 \sim B_2$ defined in (73) for the simple boundary condition $B^T \tilde{\phi} = 0$. The equivalence classes (defined by (115)) of the $B$ matrices that satisfy (72) and (114) label different terminations (or fixed points) of the edge.

Now the question is what are the allowed vertex operators on the boundary. A boundary vertex operator has the form $V_1^b = e^{i \tilde{\phi}}$, where $\mathbf{l}$ is a vector in a $k$ dimensional lattice $\Gamma_{qB}$ (called the boundary quasiparticle lattice). To determine $\Gamma_{qB}$ we note that $V_1^b$ changes one boundary condition $B^T \tilde{\phi} = 2\pi \mathbf{n}$ to another $B^T = 2\pi(\mathbf{n} + B^T \tilde{K}^{-1} \mathbf{l})$. Thus in order for $B^T \tilde{\phi} = 2\pi(\mathbf{n} + B^T \tilde{K}^{-1} \mathbf{l})$ to be an allowed boundary condition, $B^T \tilde{K}^{-1} \mathbf{l}$ must be an integer vector. Also, we require that $V_1^b$ only shift the combination $B^T \tilde{\phi}$. In particular, $V_1^b$ does not shift the combination $\Gamma^T \tilde{K} \mathbf{l}' = 0$ for any $\mathbf{l}', \mathbf{l} \in \Gamma_{qB}$. The above two conditions allows us to determine $\Gamma_{qB}$:
\[ \Gamma_{qB} = \text{Latt}(\tilde{K}B(B^T B)^{-1} - \frac{1}{2} B(B^T B)^{-1} B^T \tilde{K}B(B^T B)^{-1}). \quad (117) \]
The scaling dimension of $V_1^b$ is given from (106) by
\[ h^b(1) = \tilde{T}^T K^{-1} B(B^T \tilde{\Delta} B)^{-1} B^T K^{-1} \mathbf{l}. \quad (118) \]
In the above discussion of termination points, we have ignored any symmetry properties and the related selection rules. In particular, the boundary condition characterized by $B$ may not conserve the electric charge. As a result, a boundary vertex operator may not carry a definite electric charge. In order for the termination labeled by $B$ to conserve the electric charge, we must require the $B$ matrix to satisfy
\[ B^T \tilde{K}^{-1} \mathbf{q} = 0. \quad (119) \]
In this case, we find that $B^T \tilde{\phi}$, the fields that are about to be set to a constant, commute with the electric charge density operator $\rho_c$. For charge-conserving termination points, the electric charge of a boundary vertex operator $V_1^k$ is found to be
\[ Q = \mathbf{q}^T \tilde{K}^{-1} \mathbf{l}. \quad (120) \]
The condition (115) ensures the vertex operators of form $e^{i \mathbf{n} \cdot \tilde{\phi}}|_{\mathbf{n} \in \Gamma_B}$ are all neutral, so that they can be set to one without violating the charge conservation. For a general $B$, the above vertex operators carry nonzero charges and setting them to one violates the charge conservation.

For a general boundary condition $B$, the charge is not conserved, but some other quantities may be conserved. On the edge there are $2k$ conserved currents (at low energies) $j_a = \partial_\alpha \phi_a$, one for each branch. Near the boundary $k$ combinations of the $2k$ conserved currents remain
conserved. These \( k \) combinations are given by \( B^T \partial_0 \hat{\phi} \). Thus a boundary operator \( V^b \) carries \( k \) definite combined charges:

\[
Q = B^T \hat{K}^{-1} l. \tag{121}
\]

To determine the stability of a fixed point, we also need to know which boundary operator \( V^b \) can appear in the boundary Hamiltonian. First let us discuss the corresponding issue along the edge. Along the edge, the lattice \( \Gamma_e \) label all the mutually local operators. Some carry fermionic statistics, and thus are not allowed in the edge Hamiltonian. Only the subset described by \( \Gamma_c,0 \) can appear in the edge Hamiltonian. \( (\Gamma_c,0) \) is formed by all the bosonic operators in \( \Gamma_c \). If the charge is conserved, we further require the operators in \( \Gamma_c,0 \) to be neutral.

On the boundary, only a subset of the boundary operators can appear in the boundary Hamiltonian. Since there is no statistics within the 0 + 1 dimensional boundary, we only need to check the conservation of the \( k \) combined charges. The values of the combined charge \( Q_{\alpha}, \alpha = 1, ... , k \) allows us to determine which boundary operators can appear in the boundary Hamiltonian, as in the examples of the previous section.

D. Image-charge picture and nonchiral fields

This section shows how the correlation functions can be calculated from a simple image-charge picture when the chiral bosonic fields are unified into nonchiral bosons, as is important for a number of applications. In particular, we find the falloff of the expectation value away from the boundary of a vertex operator \( e^{i\Phi} \) pinned to 1 at the boundary, and how the two-body correlations are affected by the boundary. The correlation functions of vertex operators found in subsection B are essentially quite simple: any correlation function of a vertex operator can be written as a product of exponentials of correlation functions of free chiral bosons. One subtlety is that after rescaling there may be more terms in these correlation functions than experimental points in the original problem, since fields at the same physical point become different points in the rescaled coordinates. Some additional structure appears in the correlations when the chiral fields are combined into nonchiral fields on the half-line, as in the boundary sine-Gordon model.

In practice it is useful to combine the chiral fields \( \phi_i \) on the whole line into nonchiral bosons \( \Phi_j \) defined on the half-line \( x < 0 \), in cases where the fields for \( x > 0 \) are the same as those for \( x < 0 \). As an example, the applicability of the integrable boundary sine-Gordon model used in to determine tunneling behavior depends on the mapping to the half-line. The system on the half-line can be understood as a classical system on the \( x > 0 \) half of the \( (x,t) \) plane, so that the physics known about such statistical-mechanical systems with boundaries is applicable. The technical motivation is that the theory on the half-line will be invariant at the fixed points under all the conformal generators which preserve the line \( x = 0 \). In what follows we show that the correlation functions of the nonchiral fields can be understood from an “image charge” picture (similar to electrostatics), and that the tunneling fixed points can be understood as “ordinary” and “extraordinary” transitions on the half-plane.

For each chiral boson field on the whole line \( \phi_j \), define the nonchiral field \( \Phi_j \) on the half-plane \( x < 0 \) by \( \Phi_j(z) = \phi_j(z) + \phi_j(\bar{z}) \), where \( z = t + ix \). (We use imaginary time \( t \) so that conformal invariance is manifest.) Note that if \( z \) has \( x > 0 \), \( \bar{z} \) has \( x < 0 \) and that \( \Phi_j \) has both left-moving and right-moving parts. The vertex operators \( \exp(i\alpha \Phi_j) \) will have different behavior depending on whether \( \phi \) changes sign at \( x = 0 \). First, with \( \eta_j = \pm 1 \) the sign gained by \( \phi_j \) across \( x = 0 \), for nonzero \( \alpha \)

\[
\langle e^{i\alpha \Phi_j(x,t)} \rangle = \begin{cases} 0 & \eta = 1 \\ (2x)^{-\alpha^2} & \eta = -1 \end{cases}. \tag{122}
\]

Here and in the sequel we use the normal-ordered exponential, which has maximum value \( \infty \) rather than 1. Also, below we will consider the case where \( \Phi_j(0,t) \) is not pinned simply to 0 but to some set of values. The profile of the order parameter near the boundary can be calculated simply:

\[
\langle e^{i\alpha \Phi_j(x,t)} \rangle = \begin{cases} 0 & \eta = 1 \\ (2x)^{-\alpha^2} & \eta = -1 \end{cases}. \tag{123}
\]

The above is the simplest case of the image-charge idea of Cardy [34]: a correlation function of \( n \) nonchiral fields on the half-plane is expressed as a correlation of \( 2n \) chiral fields on the full plane. For the boundary conditions we are considering, the full-plane correlation functions are known from (87), so the half-plane correlation functions of \( \Phi_j \) can be determined. The two-body function shows different scaling along the boundary from that in the bulk: (here index \( j \) suppressed)

\[
\langle e^{i\alpha \Phi_j(z_1)} e^{-i\beta \Phi_j(z_2)} \rangle = \langle e^{i\alpha(\phi_j(z_1)+\phi_j(z_2))} e^{-i\beta(\phi_j(z_2)+\phi_j(z_1))} \rangle = \delta(\alpha - \beta) \left( \frac{4\bar{z}_1 z_2}{|z_1 - z_2|^4 |z_1 - z_2|} \right)^{\alpha^2} \eta = 1
\]

\[
\left( \frac{2\bar{z}_1 z_2}{|z_1 - z_2|^2 |z_1 - z_2|} \right)^{\alpha^2} \eta = -1. \tag{124}
\]

For example, in the \( \eta = 1 \) case, the equal-\( x \) correlation falls off as \((t_1 - t_2)^{-4\alpha^2} \alpha \) for \( t_1 - t_2 \gg x \), while far from the boundary \((t_1 - t_2 \ll x) \) the falloff is only as \((t_1 - t_2)^{-2\alpha^2} \), i.e., with the bulk scaling dimension. For \( \eta = -1 \) the correlation along the boundary is constant at long distances, with leading correction \((t_1 - t_2)^{-\alpha^2} \). The critical theory with \( \eta = -1 \) corresponds to the “extraordinary” transition in statistical mechanics, where the boundary is ordered (the order parameter \( \exp(i\Phi) \) has nonzero expectation value) but the bulk is not, while the \( \eta = 1 \) theory corresponds to the “ordinary” transition.
Some aspects of the above picture change when the field $\Phi(0, t)$ is pinned to more than one value, e.g., to the minima $\Phi = 2\pi n$, $n \in \mathbb{Z}$ of $\cos(\Phi/r)$. Now there is an additional average over $\Phi = 0, \pm 2\pi, \pm 4\pi, \ldots$ in the correlation functions. The two-body correlation is unchanged, but the one-body correlation for $\eta = -1$ is now

$$\langle e^{i\alpha\Phi_j(x,t)} \rangle = \begin{cases} (2x)^{-\alpha^2} & \alpha = n/r, n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}, \quad (125)$$

which is natural as only those operators invariant under the symmetry transformation $\Phi \rightarrow \Phi + 2\pi r$ can have nonzero expectation values.

VI. EDGES WITH RANDOM HOPPING

In the $\chi LL$ theory, the edge of a bulk QH state with $n$ condensates is described by two symmetric $n \times n$ matrices, $K$ and $V$. The integer matrix $K$ is determined by the bulk QH state and is the same for all samples of a given edge. The positive matrix $V$ contains non-universal velocities and interaction strengths which are expected to vary from sample to sample. In this section, we study the RG flow of the $V$ matrix in the presence of impurity scattering toward fixed points, which describe an equilibrated edge.

The RG calculation is given in some detail in an appendix because there are several new features not present in similar treatments of the 2D classical XY model and 2D meltings as well as 1D disordered quantum electrons. Calculations on 1D quantum disordered systems differ from those on classical 2D systems in that quenched disorder is random in space but constant in time, so the two spacetime dimensions enter asymmetrically. The chirality of the $\chi LL$ is responsible for the differences between our results and previous results on electrons in 1D: correlation functions in a $\chi LL$ depend on $x + ivt$ rather than just the magnitude $x^2 + v^2 t^2$, and the operators of interest can have nonzero “conformal spin” (difference of right- and left-moving dimensions). One of the resulting RG equations disagrees with a result previously obtained by Kane, Fisher, and Polchinski. We outline our results before proceeding to the calculation.

In a maximally chiral edge, such as IQHE edges or $\nu = 2/5$, whether a given impurity operator (i.e., type of impurity scattering) is relevant depends only on $K$, not on $V$. For IQHE edges and also for the main-sequence chiral FQHE edges $\nu = 2/5, \psi = 3/7, \ldots$, there are relevant impurity operators which decouple the charge mode from the neutral mode(s). The charge mode must decouple and the neutral mode velocities must equilibrate for the system to flow to the $U(1) \times SU(n)$ fixed point ($n = \dim K$), where the impurity scattering can be “gauged away.” The $U(1) \times SU(n)$ symmetry possessed by $K$ for these edges is generically broken by $V$, but restored if $V$ flows to a decoupled charge mode (the $U(1)$) and $n - 1$ neutral modes with identical velocities (the $SU(n)$). To our knowledge it has not previously been shown that general initial conditions flow toward this fixed point for chiral edges. The charge mode velocity is not required to equal the neutral mode velocity, since the disorder drives $V$ to be diagonal in a basis where no disorder operator couples charged and neutral modes. There remain operators which couple the neutral modes to each other; thus although to leading order in the disorder strength the neutral mode velocities do not flow together, it seems clear that the eventual strong-disorder fixed point will have equal neutral mode velocities but a possibly different charge mode velocity. We find that the fixed point is only stable if the charge mode has greater velocity than the neutral modes.

The main-sequence nonchiral FQHE edges $\nu = 2/3, 3/5, \ldots$ have similar $U(1) \times SU(n)$ fixed points. Impurity scattering now can be either relevant or irrelevant, depending on $V$, and if it is irrelevant the system will not flow to the fixed point. For $\nu = 2/3$ KFP used a perturbative calculation for weak disorder to find the basin of attraction of the fixed point; this calculation is similar to ours, although we find a slight disagreement (Appendix A). The flow to the fixed point has a much more pronounced effect on some observable quantities than in the chiral case: away from the fixed point, conductance and tunneling properties are nonuniversal. The differences between chiral and nonchiral edges result because scaling dimensions of vertex operators are independent of $V$ (fixed by $K$) in chiral edges but depend on $V$ in nonchiral edges.

The edge theory of each daughter state of $\nu = 1$ in the hierarchy is essentially the same as that of the corresponding daughter state of $\nu = 1/3, 1/5, \ldots$. Every principal hierarchy state, chiral or not, with neutral modes parallel to each other has a single solvable charge-decoupled fixed point in the presence of disorder. Edges with neutral modes traveling in both directions, such as $\nu = 5/3$ and $\nu = 5/7$, can have infinitely many fixed points of different types. The RG shows how for all of these fixed points the charge mode decouples, while the neutral modes can reach different equilibria, with consequences for tunneling experiments. The fixed points not solvable by the KFP method have disorder operators which frustrate each other at the fixed point.

The $\chi LL$ action in imaginary time for a clean edge of a QH state characterized by the matrix $K$ contains $n = \dim K$ bosonic fields $\phi_i$:

$$S_0 = \frac{1}{4\pi} \int dx dt \left[ K_{ij} \partial_x \phi_i \partial_x \phi_j + V_{ij} \partial_x \phi_i \partial_x \phi_j \right], \quad (126)$$

where the sum over repeated indices is assumed. $K$ is a symmetric integer matrix and $V$ a symmetric positive matrix. $K$ gives the topological properties of the edge: the types of quasiparticles and their relative statistics. $V$, the velocity matrix, is positive definite so that the Hamiltonian is bounded below. The electromagnetic charges of
quasiparticles are specified by an integer vector \( \mathbf{t} \) and the filling factor is \( \nu \) = \( t_1(K^{-1})_{ij} t_j \).

Now a term representing quenched random impurity scattering is added to the action:

\[
S_1 = \int dx \, dt \left[ \xi(x)e^{im_j \phi_j} + \xi^*(x)e^{-im_j \phi_j} \right]
\]

(127)

Here \( \xi \) is a complex random variable and \( \langle \xi(x)\xi^*(x') \rangle = D\delta(x-x') \), with \( D \) the (real) disorder strength. The integer vector \( \mathbf{m} \) describes how many of each type of quasiparticle are annihilated or created by the operator \( O_{\mathbf{m}} = \exp(i m_j \phi_j) \). For a real system all charge-neutral scattering operators \( m_j \) are expected to appear, but most of these will be irrelevant in the RG sense. The condition for charge-neutrality is \( t_j(K^{-1})_{ij} m_j = 0 \). The random variables \( \xi_{\mathbf{m}} \) for different scattering operators \( O_{\mathbf{m}} \) may be uncorrelated or correlated depending on the nature of the physical impurities causing the scattering.

The clean action (126) is quadratic and hence does not flow under RG transformations. Adding impurities (127) causes the \( V \) matrix in (126) to flow, and in some cases the flow is to a new type of strong-disorder fixed point. Here we will quickly review the diagonalization of the clean action to find the eigenmodes \( \mathbf{a} \), and their velocities \( v_i \), and then find the RG flows for two-mode edges with a single impurity operator. Then a general edge with several modes and impurity operators is considered.

Let \( M_1 \) be some matrix which brings \( K \) to the pseudo-identity \( I_{n^+, n^-} : K^{-1} = M_1 I_{n^+, n^-} M_1^T \). Then \( V \) can be brought to a diagonal matrix \( V_D = M_2^T M_1^T V M_1 M_2 \), where \( M_2 \) is an element of the group \( SO(n^+, n^-) \) so that \( M_2 M_1 \) still takes \( K \) to the pseudo-identity. The point of these transformations is that the action is now diagonal in the basis \( \phi = (M_1 M_2)^{-1} \phi \), so the correlation functions are simple:

\[
\langle e^{i \phi_j(x,t)} e^{-im_j \phi_j(0,0)} \rangle = \langle e^{i \phi_j(x,t) \phi_j(0,0)}-i \phi_j(0,0) \phi_j(0,0) \rangle \\
\propto \langle x + iv_j t \rangle^{-1}
\]

(128)

where the sign depends on whether \( \phi_j \) appears with \(-1\) or \(+1\) in \( I(n^+, n^-) \). The vertex operator \( O_{\mathbf{m}} \) described by the integer vector \( \mathbf{m} \) has correlation function

\[
\langle e^{im_j \phi_j(x,t)} e^{-im_j \phi_j(0,0)} \rangle = \prod_{j=1}^{n} \langle x + iv_j t \rangle^{-c_j}
\]

(129)

with \( m_j \phi_j = c_j \phi_j \). The total scaling dimension of \( O_{\mathbf{m}} \) is \( \Delta(\mathbf{m}) = \sum c_j^2/2 \), which is bounded below by \( K(\mathbf{m})/2 = \mathbf{m}K^{-1} \mathbf{m}/2 \). The impurity term \( S_1 \) containing \( O_{\mathbf{m}} \) with a random coefficient is relevant if \( \Delta(\mathbf{m}) < 3/2 \); the corresponding marginal value for a uniform coefficient is 2, and for a \( \delta \)-function coefficient 1.

Appendix A calculates the change in the correlation function of \( O_{\mathbf{m}} \) under an infinitesimal RG transformation induced by the impurity term \( S_1 \). Here we find the change in the underlying \( V \) matrix required to produce the new correlation function. The \( K \) matrix is unchanged as it is “topological” (it does not enter the Hamiltonian). The \( V \) matrix flow has a simple interpretation, valid for any number of edge modes traveling in either direction. Each impurity operator \( \mathbf{m} \) drives \( V \) to become diagonal in the basis with \( \mathbf{m} \) an eigenvector. This automatically minimizes the scaling dimension of \( O_{\mathbf{m}} \) in a nonchiral edge. In cases where there are more impurity operators than independent eigenvectors, so that not all impurity operators can simulanelously be eigenvectors, the impurity operators frustrate each other.

Both \( \nu = 2 \) and \( \nu = 2/5 \) have a single \( K(\mathbf{m}) = 2 \) operator which is always relevant. In the basis \( \mathbf{e}_1 = \mathbf{t} \), \( \mathbf{e}_2 = \mathbf{m} \),

\[
K^{-1} = \begin{pmatrix} \nu & 0 \\ 0 & 2 \end{pmatrix}, \quad K^{1/2} = \begin{pmatrix} 1/\sqrt{\nu} & 0 \\ 0 & 1/2 \end{pmatrix}
\]

\[
V = K^{1/2} R \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} R^{-1} K^{-1/2}.
\]

(130)

with \( R \) a two-by-two rotation matrix by some angle \( \theta \). Note that all \( V \) are obtained by considering \( \theta \) in the interval \([0, \pi]\). Now \( \alpha = 2 \sin^2 \theta \) and \( \beta = 2 \cos^2 \theta \) are the exponents appearing in the correlation function of the impurity operator:

\[
\langle O_{\mathbf{m}}(x,t) O_{\mathbf{m}}^\dagger(0,0) \rangle = (x + iv_1 t)^{-\alpha} (x + iv_2 t)^{-\beta}, \quad \alpha + \beta = 2.
\]

Then to first order in disorder strength, the diagonal velocities \( v_1 \) and \( v_2 \) are unchanged, and (Appendix A)

\[
\frac{d\alpha}{dt} = -\frac{8\pi D\alpha\beta}{(v_1 - v_2)^2 v_1^{\alpha-1} v_2^{\beta-1}}.
\]

(131)

Since \( d\alpha = 2 \sin(\theta) d\theta \) = \( 2\sqrt{\alpha\beta} d\theta \),

\[
\frac{d\theta}{dt} = -\frac{4\pi D \sin(\theta) v_1 v_2}{(v_1 - v_2)^2 v_1^{2\sin^2(\theta)} v_2^{2\cos^2(\theta)}}.
\]

(132)

There are two fixed points of this equation, with \( \theta = 0 \) stable and \( \theta = \pi/2 \) unstable for \( v_1 > v_2 \), and vice versa for \( v_1 < v_2 \). The stable fixed point always corresponds to neutral mode velocity less than charge velocity (Fig. [F]).

We can summarize the effect of the disorder operator in the comoving case simply: it rotates \( V \) so that \( \mathbf{m} \) becomes an eigenvector. Since \( \mathbf{m} \) is neutral \( (\mathbf{m}K^{-1} \mathbf{m} = 0) \) the other eigenvector is driven to the charge vector. The idea that impurity operators drive \( V \) to make themselves eigenvectors is quite general. The case of two countermoving modes (e.g., \( \nu = 2/3 \)) with a \( K(\mathbf{m}) = -2 \) disorder operator is similar in form. The rotation matrix \( R \) in (130) is replaced by a boost matrix \( B \),

\[
B = \begin{pmatrix} \cosh \tau & \sinh \tau \\ \sinh \tau & \cosh \tau \end{pmatrix},
\]

(133)

and the exponents in the correlator are \( \alpha = 2 \sinh^2 \tau \), \( \beta = 2 \cosh^2 \tau \), \( \alpha - \beta = -2 \), \( \alpha + \beta = 2\Delta(\mathbf{m}) \). The flow equation for \( \tau \) is then
Here $v_+$ and $v_-$ are the (positive) velocities of the right- and left-moving modes. Now there is only one fixed point, at $\tau = 0$ (Fig. 6b), which is the solvable fixed point found by KFP.\[\text{(134)}\]

\[
\frac{d\tau}{d\ell} = -\frac{4\pi D \sinh(2\tau)v_+v_-}{(v_+ + v_-)^2 \sinh^2 \tau \cosh^2 \tau}.
\]

Introducing the matrix $\Delta_{ij} \text{ via } 2\Delta(m) = m_i\Delta_{ij}m_j,$

\[
\Delta = M_1BR^T B^T M_1^T = M_1B^2 M_1^T.
\] \[\text{(135)}\]

For each pair of comoving modes appearing in the correlation function, there is an infinitesimal change in the rotational part $R$, and for each pair of countermoving modes, there is an infinitesimal change in the boost part $B$.

Given an impurity operator $O_m$ and initial $V(\ell)$, we need to find $V(\ell + d\ell)$ which gives the changes to the correlation function calculated in Appendix A. However, there is an important issue not present in the two-mode case: there are more free parameters in $V$ than exponents in the correlation function, so $V$ is not uniquely determined without additional assumptions. We assume that each term in Appendix A coupling two modes affects only the components of $V$ between those two modes. We will also write the flow equations for the components of $V$ directly, rather than introducing a parametrization as we did above in terms of $\theta$ or $\tau$, because for multi-mode edges such parameterizations become quite complicated.

Suppose that under an infinitesimal change $d\ell$ the $V$ matrix changes from $(M_1M_2)^{-1}TV_D(M_1M_2)^{-1}$ to

\[
V(\ell + d\ell) = (M_1M_2N)^{-1}TV_D(M_1M_2N)^{-1}.
\] \[\text{(136)}\]

Note that $V$ does not flow if the eigenmodes have the same velocity, so that $V_D$ is a multiple of the identity. Here $N$ is an element of $SO(n_+, n_-)$ differing only by order $d\ell$ from the identity. The fields which diagonalize $V(\ell + d\ell)$ are $\phi' = (M_1M_2N)^{-1}\phi = N^{-1}\phi$. Now let $c_i$ be the components of the disorder operator in the $\phi$ basis which diagonalizes $V(\ell)$: $m_i\phi_i = c_j\phi_j$. The exponents appearing in the correlation function, whose flow is calculated in Appendix A, are $\alpha = c_i^2$, $\beta = c_2^2$, etc.

Now we are ready to construct $N$. For each pair of countermoving modes $i$ and $j$, $N$ has an infinitesimal rotation angle $d\theta_{ij} = -d\theta_{ji}$, and for each pair of comoving modes $i$ and $j$, a boost $d\tau_{ij} = d\tau_{ji}$. For an edge with two right-movers and one left-mover, e.g.,

\[
N = \begin{pmatrix}
1 & -d\theta_{12} & d\tau_{13} \\
d\theta_{12} & 1 & d\tau_{23} \\
d\tau_{13} & d\tau_{23} & 1
\end{pmatrix}.
\] \[\text{(137)}\]

Then for a disorder operator $O_m$ with projections $c_i$ and strength $D$,

\[
d\theta_{ij} = -\frac{4\pi c_i c_j v_i v_j D d\ell}{(v_i - v_j) \prod_i v_i c_i^2};
\]

\[
d\tau_{ij} = -\frac{4\pi c_i c_j v_i v_j D d\ell}{(v_i + v_j) \prod_i v_i c_i}.\] \[\text{(138)}\]

Note that the $c_i$ can have either sign and must change sign in the vicinity of a fixed point, as above in the two-mode case. Substituting $N$ in (136) gives the desired RG flow equation for the components of $V$. The condition for a fixed point is simple: $NTV_DN = V_D$. Hence for
any chiral edge (N a pure rotation), if all velocities are the same the system is at a fixed point.

The simplest multicomponent edges are the IQHE edges ν = n and chiral main-sequence edges ν = n/(2n + 1). The behavior of these is similar to the ν = 2 case discussed in detail above; the stable fixed point has charge and neutral mode velocities v_C > v_n. The neutral mode velocities are expected to equalize while the charge mode remains different, because there are always hopping operators connecting the different neutral modes, while there is no hopping connecting the charge mode to the neutral modes. As a result there is no stable fixed point unless the neutral velocities are the same.

For the ν = 3 case we can demonstrate the similarity to the ν = 2 case by an explicit calculation. Parametrize the rotation part R of V as R = R_{23}(θ_1)R_{12}(θ_2)R_{23}(θ_3), where R_{ij}(θ) is the rotation by θ in the i-j plane; then in a basis with e_1 = t, the charge mode is decoupled if θ_2 = 0. The flow equation for θ_2, with v_2 = v_3 = v_n, is

\[ \frac{dθ_2}{dℓ} = -\frac{4πv_cv_n \sin(2θ_2)}{(v_c - v_n)} \times \]
\[ \left( \cos^2 θ_1(4D_1 + D_2 + D_3) + \frac{\cos^2 θ_1 \sin^2 θ_2 v_n 2(\sin^2 θ_1 + \cos^2 θ_1 \cos^2 θ_2)}{4v_c^2 \cos^2 θ_1 \sin^2 θ_2 v_n 2(\cos^2 θ_1 + \sin^2 θ_1 \cos^2 θ_2)} + \right. \]
\[ \left. \left. \left. \left. \left. \frac{3 \sin^2 θ_1 (D_2 + D_3)}{4v_c^2 \sin^2 θ_1 \sin^2 θ_2 v_n 2(\cos^2 θ_1 + \sin^2 θ_1 \cos^2 θ_2)} \right) \right) \right) \right) \right) \] (139)

where D_{1,3} are the strengths of the 3 hopping operators. This is of the same form as (132) since the quantity in brackets is clearly positive. Henceforth we will not write out the flow equations but just discuss the qualitative behavior.

The nonchiral main-sequence edges ν = 2/3, 3/5, ... have solvable stable U(1) × SU(n) fixed points if all the neutral mode velocities are equal and the charge mode is decoupled. The charge mode does indeed decouple in the perturbative RG equations since each impurity operator reduces the scaling dimension of t toward its minimum ν, which is only attained when the charge mode is decoupled. Once the charge mode is decoupled, the neutral modes behave exactly as in the chiral case, except that the fixed point is stable even if the neutral mode velocity is greater than the charge mode velocity.

In the edges ν = 5/3 or ν = 5/7, which have neutral modes in both directions, there are an infinite number of possibly relevant impurity operators. However, near each of the possible fixed points there are only three relevant operators. Unlike the case with all neutral modes in the same direction, to first order in disorder strength there is no stable fixed point for V, even if the neutral mode velocities are equal. The mathematical difference is that, now that the modes move in opposite directions, the impurity operators cause infinitesimal boosts which do not disappear when the velocities are equal, unlike infinitesimal rotations. The fixed points in these edges have marginal operators not present for the KFP-type fixed points and seem to be of a different type, and their strong-disorder behavior and stability is not well understood.

There are several four-component edges, such as ν = 12/17 and ν = 12/31, which have solvable, SU(n) symmetric fixed points of the KFP type as well as fixed points with marginal operators similar to the ν = 5/3 and ν = 5/7 edges. The picture of equilibration differs depending on the fixed point. The solvable SU(2) × SU(2) fixed points found in these edges can have all three neutral mode velocities different (since there is a basis where no relevant hopping operator couples neutral modes), while the SU(3) fixed points have to have two neutral mode velocities equal in order to be solvable. However, as discussed in the introduction the fixed point accessible by the composite-fermion approach is the one with marginal operators present, even though this is the only one of the three fixed points whose stability is dubious.

VII. CONCLUSIONS

The main result of this paper is the analysis for a general Abelian edge of a large class of tunneling/termination fixed points. We find the conditions on allowable boundary conditions and the resulting correlation functions for vertex operators at the fixed points. We find that the requirement of unitary time evolution imposes conditions on when a quantum Hall edge can terminate at a point or join smoothly to another edge. Two edges can join smoothly only if they have the same number of modes, and there are additional statistical restrictions: for example, ν = 1 can join smoothly to ν = 1/9 (provided a source of charge is present), but not to ν = 1/3. We find the lattice of allowed boundary operators, both charged and neutral, for a given boundary condition.

The correlation functions in the presence of a boundary can be understood through an “image-charge” picture, and take a simple form which becomes more complex when the original chiral bosons are combined into nonchiral fields as for the boundary sine-Gordon model. The two-point correlation function of a vertex operator can show different behavior (a change in scaling dimension) as the two locations are moved toward the boundary.

We solve a simplified model of how interactions between electrons on different edges affect tunneling through a point junction. Most current tunneling experiments involve multiple contacts between the two edges and variable interaction strengths along the edge, but the basic result that interactions can complicate the identification of the effective tunneling charge should still apply. The model we consider maps onto the solvable model of Fendley et al. with a continuous effective filling fraction, giving a physical realization of this result beyond the discrete cases ν_{eff} = 1, 1/2, 1/3 known previously.
We have used a perturbative RG approach to find the flows of random hopping operators in a general quantum Hall edge. Two broad conclusions from this treatment are that with no initial assumptions favoring one type of state over another, significant differences emerge between non-principal states and principal states, and between states with all neutral modes in one direction and states with neutral modes in both directions. In states with neutral modes in both directions, different hopping operators compete to drive the system to different fixed points. This may explain why hierarchy quantum Hall states with neutral modes in both directions are much more difficult to observe than those at the same level of the hierarchy with all neutral modes in one direction, and why non-principal states are also rarely seen. We find that charge-neutral separation is a general feature when disorder is relevant, in both chiral and nonchiral edges. A specific result for the $\nu = n$ IQHE states and chiral main-sequence FQHE states is that the charge-neutral separated fixed point is only perturbatively stable when the charge mode is faster than the neutral mode, as for strong Coulomb interactions.

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APPENDIX A: PERTURBATIVE RENORMALIZATION GROUP

This appendix uses the perturbative RG technique to study the effect of impurity scattering operators on the velocity matrix $V$ in the chiral-Luttinger-liquid action. The $K$ matrix does not flow and thus remains an integer matrix, which follows directly from the fact that $K$ does not enter the Hamiltonian (i.e., it is “purely topological”). The constancy of $K$ to leading order in $D$ will be explicit in the results obtained below. In order to calculate the changes in $V$ under a change in the cutoff, we expand a correlation function to first order in the disorder strength, then show that the terms proportional to the disorder operator itself $(m = n)$. The symbol $P_{12}$ denotes $(x + iv_+t)^{-\alpha}(x - iv_-t)^{-\beta}$, $x = x_2 - x_1$, $t = t_2 - t_1$. Because $\alpha - \beta = K(m)$ is even, $P_{12} = P_{21}$. The last term in (A3) is now

$$D \int dX dT dt \left( \frac{P_{12} P_{13} P_{14} P_{23}}{P_{12} P_{24}} \right).$$

The integrand is $P_{12} P_{13} \exp(g_{13} + g_{24} - g_{14} - g_{23}) \approx P_{12} P_{23} \exp(r \cdot \nabla_R (g(r_1 - R) - g(r_2 - R)))$. Here $g_{ab} = -\log(P_{ab}) = \alpha \log(x + iv_+t) + \beta \log(x - iv_-t)$. Disorder fixes $x = 0$ in $r = (x,t)$ so the exponential is $\exp(x \partial_T g(r_1 - R) - g(r_2 - R))) \approx 1 + t^2(\partial_T g(r_1 - R) - \partial_T g(r_2 - R))^2$. Hence we need to evaluate the following integral:

$$D \int dX dT dt \left( \frac{1}{X - x_1 + iv_+T - t_1} \frac{i\alpha v_+ \beta v_-}{(X - x_1 - iv_-T - t_1)(X - x_2 + iv_+T - t_2) - (X - x_2 - iv_-T - t_2)^2} \right).$$

The constant term in the integrand cancels the leading term in the partition function in (A2). When the square is expanded, products of denominators at the same point will cancel infinities arising in the calculation of products of denominators at different points, leaving a finite answer. The change of cutoff in the $t$ integral will yield the RG equations at the end.

First consider the integrals of the $\alpha^2, \beta^2$ terms. After rescaling the time variables by $v_+$, the $\alpha^2$ integral is

$$\int dX dT \left( \frac{1}{X + iT - x_1 - it_1} \frac{1}{X + iT - x_2 - it_2} \right) = \int dX dT \left( \frac{1}{z_2 - w_2} \right),$$

with $z_2 = v_+ = v_-$, the correlation function has a phase as well as a magnitude.
with \( z = X + iT; 2w = x_2 - x_1 + ivt - it_1 \). But this integral is not uniformly convergent at \( \infty \) and thus not well-defined: for example, if \( w = i \) the integral is \( 2\pi \) if the \( X \) integration is done first, 0 if the \( T \) integration is done first, and \( \pi \) if the integration is done in radial coordinates. We believe that the appropriate value of the integral \([\text{A5}]\) is 0, because in Minkowski space (real rather than imaginary time) the corresponding integral has integrand \((x + t - x_1 - t_1)^{-1}(x + t - x_2 - t_2)^{-1}\) and is unambiguously zero. Also, zero is the only value consistent with the fact that the randomness can be rotated away at the KFP fixed point, since at that point the RG flow of the velocity should be independent of the disorder strength.

The \( \alpha \beta \) terms, give the renormalization of the scaling dimension \( \Delta \), are proportional to

\[
I = \int dX dT \frac{1}{c_+ d_- + 1 c_- d_+},
\]

\( c_\pm = X - x_1 \pm iv \pm (T - t_1) \),

\[ d_\pm = x_2 - X \pm iv \pm (t_2 - T) \].

(A6)

First do the \( dX \) as a contour integral. The poles of the first term are at \( w_1 = x_1 + iv \pm (T - t_1) \) and \( w_2 = x_2 + iv \pm (t_2 - T) \), and the integral vanishes unless the poles are on different sides of the real axis (likewise for the second term). Thus \( T \notin [t_1, t_2] \) and we are left with

\[
I = \left( \int_{-\infty}^{t_1} dT + \int_{t_2}^{\infty} dT \right) \frac{2\pi i}{y_1 - y_2} [2\pi i \frac{y_1}{y_2}],
\]

\[ y_1 = x_2 - x_1 + iv \pm t_2 + iv \pm t_1 - i(v_+ + v_-)T, \]

\[ y_2 = x_2 - x_1 - iv \pm t_1 - iv \pm t_2 + i(v_+ + v_-)T. \]

(A7)

Hence

\[
I = \frac{2\pi}{v_+ + v_-} \log \left( \frac{a_1^+ a_2^+ a_2^-}{b_+^+ b_-^+ b_-^+} \right),
\]

\[ a_1^\pm = x_2 - x_1 \pm iv \pm (v_+ + v_-) \infty + iv \pm t_2 + iv \pm t_1 \]

\[ a_2^\pm = x_2 - x_1 \pm iv \pm (v_+ + v_-) \infty - iv \pm t_1 - iv \pm t_2, \]

\[ b_\pm = x_2 - x_1 \pm iv \pm (t_2 - t_1). \]

(A8)

The finite part of the result is independent of the order of integration, even though \([\text{A4}]\) is superficially even less well-defined than \([\text{A3}]\). The infinite part of the result is canceled by the \( \alpha \beta \) terms in \([\text{A4}]\) with denominators at the same point. The \( dt \) integral is

\[
\int_{-\infty}^{\infty} dt^2 P_{34} = 2 \int_{-\infty}^{\infty} dt \left[ \left( \frac{1}{iv \pm t} \right)^\alpha \left( \frac{1}{-iv \pm t} \right)^\beta \right] = K(m) \frac{2}{v_+ \alpha v_- \beta} \int_{-\infty}^{\infty} dt t^{2-(\alpha + \beta)}. \]

(A9)

The dependence on \( K(m) \) here is an artifact of our using the unfortunate convention \([\text{A1}]\), when in fact the proportionality constant in front alternates sign as \( |K(m)| = 0, 2, 4, \ldots \) to keep the correlation function positive when its argument is on the time axis. The effective scaling dimension \( \Delta_{\text{eff}} \) after re-exponentiating the perturbation to the correlation function is

\[
2\Delta_{\text{eff}} = 2\Delta - \frac{4(4\pi)\alpha \beta D}{(v_+ + v_-)\alpha \beta - 1} \int_{-\infty}^{\infty} dt t^{4\alpha \beta - 2}. \]

(A10)

By the usual process of changing the cutoff \( a \to a \exp(\ell) \) we obtain the RG equations

\[
\frac{dD}{d\ell} = (3 - 2\Delta)D \quad \frac{d\Delta}{d\ell} = -8\pi D(\Delta^2 - K(m)^2/4) \quad \frac{d\alpha}{d\ell} = \frac{\pi Dm \alpha}{(v_1 - v_2) m_1 \alpha - 1 \beta^2 - 1}. \]

(A11)

These equations match those found by Kane, Fisher, and Polchinski for the \( |K(m)| = 2 \) state. However, as mentioned above we find no term which renormalizes the velocities to first order in \( D \).

Now consider the case of two comoving modes. The simplest example is the IQHE state \( v = 2 \), which has a relevant impurity operator hopping electrons from one mode to the other. The correlation function \( P_{12} \) of an impurity operator \( O_n = \exp(i in \phi) \) has the form \( (x + iv_1 t)^{-\alpha} (x + iv_2 t)^{-\beta} \), with \( v_1, v_2 \) the velocities of the two eigenmodes and \( \alpha + \beta = K(m) \) an even integer. Expanding the correlation function in the disorder strength and then evaluating the disorder average as before gives the correction to the correlation function:

\[
D \int dX dT dt P_{12} P_{34} \left[ 1 + (\Delta - 1) \frac{i\alpha v_1}{X - x_1 + iv_1(T - t_1)} \right. \]

\[ + \frac{i\beta v_2}{X - x_1 + iv_2(T - t_1) - X - x_2 + iv_1(T - t_2)} \]

\[ \left. - \frac{i\beta v_2}{X - x_2 + iv_2(T - t_2)} \right] \].

(A12)

Expanding the square gives terms with both denominators at the same point, which cancel infinites appearing elsewhere in the calculation, and terms with both denominators having the same velocity, which were previously argued to be zero (and in any event cannot cause the two velocities to flow toward each other, since each term only involves one velocity). The result is

\[
\alpha \beta v_1 v_2 P_{12} \int dX dT [ \frac{1}{X - x_1 + iv_1(T - t_1)} \times \frac{1}{X - x_2 + iv_2(T - t_2)} + (v_1 \leftrightarrow v_2)] \int dt t^2 P_{34} \]

\[ = \frac{4\pi \alpha \beta v_1 v_2 P_{12} (v_1 - v_2) t P_{34} \log |x_2 - x_1 + iv_1(t_2 - t_1) - x_2 - x_1 + iv_2(t_2 - t_1)|}. \]

(A13)

Thus \( \alpha \) and \( \beta \) are changed but not \( \alpha + \beta = K(m) \). The velocities of the eigenmodes are unaltered, and the RG flows are

\[
\frac{dD}{d\alpha} = (3 - \alpha - \beta)D \quad \frac{d\alpha}{d\ell} = \frac{d\beta}{d\ell} = \frac{8\pi D \alpha \beta}{(v_1 - v_2) m_1 \alpha - 1 \beta^2 - 1}. \]

(A14)

The singular denominator when \( v_1 = v_2 \) is acceptable because at \( v_1 = v_2 \), only \( \alpha + \beta \) is well-defined, not \( \alpha \) and \( \beta \) separately.
The generalization to a case with two modes and more than one impurity operator is simple: the contributions to the RG flow equations for the velocity matrix from each impurity operator add, since to leading order the impurity operators are independent.

Extending the calculation to an edge with more than two modes is quite simple. In the expansion of the square term in (A3), each pair of modes gives one term. If the two modes move in opposite directions, the term lowers the total scaling dimension as in (A11); if the two modes move in the same direction, the term maintains the total scaling dimension as in (A14). Each term preserves $K(m)$ separately. For explicitness, consider the case of an edge with two right-movers and one left-mover, which is relevant to the $\nu = 3/5$ edge. Then the correlation function has the form $(x + iv_1 t)^{-\alpha}(x + iv_2 t)^{-\beta}(x - iv_3 t)^{-\gamma}$, and the RG flows for the exponents are

$$\begin{align*}
d\alpha &= -\frac{8\pi D\alpha\beta v_1 v_2}{(v_1 - v_2)v_1^\alpha v_2^\beta v_3^{-\gamma}} - \frac{8\pi D\alpha\gamma v_1 v_2}{(v_1 + v_3)v_1^\alpha v_3^\gamma v_2^{-\beta}} - \frac{8\pi D\beta\gamma v_1 v_2}{(v_1 + v_3)v_1^\beta v_3^\gamma v_2^{-\alpha}} \\
d\beta &= \frac{8\pi D\alpha\gamma v_1 v_2}{(v_1 - v_2)v_1^\alpha v_2^\beta v_3^{-\gamma}} - \frac{8\pi D\beta\gamma v_1 v_2}{(v_2 + v_3)v_1^\beta v_3^\gamma v_2^{-\alpha}} - \frac{8\pi D\gamma v_1 v_2}{(v_2 + v_3)v_1^\gamma v_2^{-\alpha} v_3^{-\beta}}. \\
\end{align*}$$

(A15)

Section VI shows how the matrix flow in the $\chi$LL action determined by the correlation function flows found in this appendix has a natural interpretation in terms of the boost and rotation parts of the $V$ matrix.

\begin{itemize}
\item[\textsuperscript{1}]
B. I. Halperin, Phys. Rev. B \textbf{25}, 2185 (1982).
\item[\textsuperscript{2}]
R. B. Laughlin, Phys. Rev. Lett. \textbf{50}, 1395 (1983).
\item[\textsuperscript{3}]
X.-G. Wen, Phys. Rev. B \textbf{43}, 11025 (1991); Phys. Rev. Lett. \textbf{64}, 2206 (1990).
\item[\textsuperscript{4}]
X.-G. Wen, Adv. in Phys. \textbf{44}, 405 (1995).
\item[\textsuperscript{5}]
M. D. Johnson and A. H. MacDonald, Phys. Rev. Lett. \textbf{67}, 2060 (1991).
\item[\textsuperscript{6}]
X.-G. Wen, Int. J. Mod. Phys. B \textbf{6}, 1711 (1992).
\item[\textsuperscript{7}]
J. E. Moore and F. D. M. Haldane, Phys. Rev. B \textbf{55}, 7818 (1997).
\item[\textsuperscript{9}]
P. Fendley, A. W. W. Ludwig, and H. Saleur, Phys. Rev. B \textbf{12}, 8934 (1995).
\item[\textsuperscript{10}]
C. L. Kane, M. P. A. Fisher, and J. Polchinski, Phys. Rev. Lett. \textbf{72}, 4129 (1994).
\item[\textsuperscript{11}]
M. D. Johnson and A. H. MacDonald, Phys. Rev. B \textbf{55}, 13449 (1994).
\item[\textsuperscript{12}]
J. E. Moore and X.-G. Wen, Phys. Rev. B \textbf{57}, 10138 (1998).
\item[\textsuperscript{13}]
J. T. Chalker and A. Dohmen, Phys. Rev. Lett. \textbf{75}, 4496 (1995).
\item[\textsuperscript{14}]
L. Balents and M. P. A. Fisher, Phys. Rev. Lett. \textbf{76}, 2782 (1996).
\item[\textsuperscript{15}]
F. Milikien, C. Umbach and R. Webb, Solid State Comm., \textbf{97}, 309 (1995); A.M. Chang, L.N. Pfeiffer and K.W. West, Phys. Rev. Lett, \textbf{77}, 2538 (1996).
\item[\textsuperscript{16}]
A. M. Chang et al., Phys. Rev. Lett. \textbf{86}, 143 (2001).
\item[\textsuperscript{17}]
L. S. Levitov, A. V. Shytov, and B. I. Halperin, cond-mat/0005016 (2000).
\item[\textsuperscript{18}]
F. D. M. Haldane, Phys. Rev. Lett. \textbf{74}, 2090 (1995).
\item[\textsuperscript{19}]
C. L. Kane and M. P. A. Fisher, Phys. Rev. B \textbf{46}, 15233 (1992).
\item[\textsuperscript{20}]
D. B. Chklovskii and B. I. Halperin, Physica E \textbf{1}, 75 (1997).
\item[\textsuperscript{21}]
C. Chamon and E. Fradkin, Phys. Rev. B \textbf{56}, 2012 (1997).
\item[\textsuperscript{22}]
L. P. Pryadko, E. Shimshoni, A. Auerbach, Phys. Rev. B \textbf{61}, 10929 (2000).
\item[\textsuperscript{23}]
A. Schmid, Phys. Rev. Lett. \textbf{51}, 1506 (1983).
\item[\textsuperscript{24}]
J. E. Moore, P. Sharma, and C. Chamon, Phys. Rev. B \textbf{62}, 7298 (2000).
\item[\textsuperscript{25}]
N. P. Sandler, C. Chamon, and E. Fradkin, Phys. Rev. B \textbf{59}, 12521 (1999).
\item[\textsuperscript{26}]
J. L. Cardy, Nucl. Phys. B \textbf{270}, 186 (1986).
\item[\textsuperscript{27}]
P. Di Francesco, P. Mathieu, and D. Senechal, \textit{Conformal Field Theory}, Springer-Verlag (1998).
\item[\textsuperscript{28}]
J. M. Kosterlitz and D. J. Thouless, J. Phys. C \textbf{6}, 1181 (1973).
\item[\textsuperscript{29}]
J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B \textbf{16}, 1217 (1977).
\item[\textsuperscript{30}]
D. R. Nelson, Phys. Rev. B \textbf{18}, 2318 (1978); D. R. Nelson and B. I. Halperin, Phys. Rev. B \textbf{19}, 2457 (1979).
\item[\textsuperscript{31}]
T. Giamarchi and H. J. Schulz, Phys. Rev. B \textbf{37}, 325 (1988).
\item[\textsuperscript{32}]
N. Read, Phys. Rev. Lett. \textbf{65}, 1502 (1990).
\end{itemize}