On Relevance Of Matrix Coordinates
For The Inside Of Baryons

Amir H. Fatollahi

Institute for Advanced Studies in Basic Sciences (IASBS),
P.O.Box 45195-159, Zanjan, Iran
fath@iasbs.ac.ir

Abstract

It is argued that one natural choice for coordinates of constituents of a baryonic state in a SU(N) gauge theory are \( N \times N \) hermitian matrices. It is discussed that the relevance of matrix coordinates is supported at least by the restricted form of the color symmetry. Based on the previous investigations in this direction, the consequences of the idea are reviewed. The model has been considered is originated by the D0-branes of String Theory.

PACS: 12.38.Aw, 02.40.Gh, 14.65.-q
Keywords: QCD, Noncommutative Geometry, Quark Dynamics
One of the main themes in quantum mechanics is to found our physical theories exclusively upon relationships between quantities which in principle are observable [1]. At the present status, it is commonly believed that a hadron has quarks as part of its ingredients, though they cannot be detected directly. From the pure theoretical point of view, one quark on its own is like the other particles, and has some observable quantities, such as position, momentum, spin or charge. In practice, seemingly we are always faced with hadrons that the properties of quarks are hidden inside them. Although, it does not seem natural to assume that quarks do not carry any of the usual degrees of freedom or their degrees of freedom can be completely ignored, it may be a desirable framework if it is possible that the degrees of freedom can become “unreachable” due to some kind of symmetry. In other words, due to a symmetry it would not be expected that, for example, the position of an individual quark can be measured, or even the question about “the position of an individual quark with a specific color” become meaningless.

In [2, 3, 4, 5], a model was considered which shares the feature we mentioned in above. The model has been considered is originated by the D0-branes [6, 7] of String Theory, for which it is known that their degrees of freedom are captured by matrices, rather than numbers [8]. The concerned model in [2–5] has shown its ability to reproduce or cover some features and expectations in hadron physics. Some of these features and expectations are: phenomenological inter-quark potentials, the behavior of total scattering amplitudes, rich pololgy of scattering amplitude, behavior in large-$N$ limit, and the whiteness of baryons with respect to the SU($N$) sector of the external fields.

As mentioned, the internal dynamics of the D0-brane bound state is described by a matrix model of coordinates, and if the matrix coordinates of D0-branes have something to do with hadron physics, it is very logical to ask “Is it possible to extract or derive these matrices from some first principles of quantum chromodynamics (QCD)?” In fact the answer to this question has been the motivation for the present work, and as we will see, it appears that the appearance of matrix coordinates in the theory of quarks is as natural as the appearance in the theory of D0-branes. Before presentation the much more formal derivation, let us present the heuristic argument. Reminding the procedure of reasoning in D0-brane theory, we recall that the matrix coordinates are the result of some states, to be specific some open string states, which are equipped with two more labels than the usual ones, the so-called Chan-Paton labels [8, 7, 9].

2
In the open string picture these labels are attached to the ends of string. On the other side, D0-branes are defined as point-like objects to whom the open strings end. By this picture, each D0-brane is accompanied with some more degrees of freedom than the usual ones of an ordinary particle. In other words, each D0-brane has some more degrees of freedom which express to which other D0-branes and in what places is connected, i.e., has made a bound state with which others. Eventually it appears that in a bound state of \( N \) D0-branes, the relevant degrees of freedom in each direction of space, rather than \( N \), are \( N^2 \) which may be represented by a matrix belonging to the \( U(N) \) algebra. Now as we shall recognize in a moment, this reasoning is applicable for the case of quarks, in which the states have the additional degrees of freedom as “color”.

In the constituent quark picture of a \( SU(N) \) gauge theory, a baryon is made by \( N \) quarks in different colors, and besides, to bring the baryon state as a singlet in color space, an anti-symmetrization in the color labels is understood. Let us for the moment forget about the rotation in the color space, and assume that the baryon is just made by \( N \) quarks in different colors, represented by the states and wave functions \( |\psi_a(t)\rangle \) and \( \langle x_a|\psi_a(t)\rangle = \psi_a(x_a, t), \ a = 1, \cdots, N, \) respectively; in this work we also do not care about the fermionic or bosonic nature of quarks. Though the wave functions depend on different arguments \( x_1, \cdots, x_N \), while keeping the right form of each function \( \psi_a \), we may present them by one argument \( x \), as \( \psi_a(x, t) \). By this we can define the \( N \times N \) matrix \( X \) via its elements \( X_{ab}(t) = \int dx \ \psi_b^*(x, t) x \psi_a(x, t) \). Here we assume that the states are normalized properly, to yield the length dimension for the elements of \( X \), accompanied with the value one for the total probability. It is easily seen that \( X \) is \( N \times N \) hermitian matrix, and its elements are characterized by the color labels \( a, b = 1, \cdots, N \).

Let us take the case for which we have well separated quarks, may be represented by wave functions \( \psi_a(x, t) \approx \delta(x - x_a) \) with \( |x_a - x_b| \gg \ell, \ (a \neq b) \), for some characteristic length \( \ell \). For this case, the matrix \( X(t) \) is almost, or even in the case, exactly diagonal. Suppose we take the length scale \( \ell \) to be the order of the baryon size. From our experience, we know that the situation we have considered above is never seen in practice! The most expected situation is that, due to confinement, the \( N \) quarks find considerable overlap between their wave-functions and form a baryon. Correspondingly, we have learnt to face always with permanently ‘connected’ quarks, for which the matrix \( X(t) \) appears always in non-diagonal form, and this may cause one believes
in the essence of more degrees of freedom as representatives and also to describe the permanent connectivity of quarks. We note that in fact the huge amount of information about the inside of a baryon is encoded in the wave functions of its constituents, or equivalently in the matrix coordinate $X$ and its generalizations to higher moments as $X^{(n)}_{ab}(t) \equiv \int d\mathbf{x} \, \psi^*_b(\mathbf{x}, t) \mathbf{x}^n \psi_a(\mathbf{x}, t)$, $n = 0, 2, 3, 4, \ldots$. The above simple observation may suggest that the matrix coordinate $X$ and its generalizations to higher moments can present the criteria for the identification the confined phase of a theory. Besides, the matrix coordinate $X$ and its higher moments may be taken as a set of very powerful tools for characterization and study of the observable states in a confined theory. Therefore, it will be very tempting to see that by considering the matrix $X(t)$ as the dynamical variable relevant for the inside of a baryon, what kind of information and conceptual insights come out.

Before to proceed further, it is useful to mention that the matrix coordinate can also be constructed from the original quark field in the Lagrangian. We take a SU($N$) gauge theory, consisting one kind of flavor in the fundamental representation as matter. We treat this example as a quantum mechanics, rather than a field theory. The states of matter in this quantum mechanical problem are represented by

$$
|\Psi(t)\rangle = \begin{pmatrix}
|\psi_1(t)\rangle \\
|\psi_2(t)\rangle \\
\vdots \\
|\psi_N(t)\rangle
\end{pmatrix}.
$$

So we have the expansion $|\Psi(t)\rangle = \int d\mathbf{x} \sum_{a=1}^{N} \psi_a(\mathbf{x}, t)|\mathbf{x}\rangle \otimes |a\rangle$, in which the index $a$ is labelling the isospin, and $\psi_a(\mathbf{x}, t) \equiv \langle \mathbf{x} |\psi_a(t)\rangle$. We define the density matrix operator $\hat{\rho}(t)$ as

$$
\hat{\rho}(t) \equiv |\Psi(t)\rangle \langle \Psi(t)|,
$$

which is an $N \times N$ matrix with the general element as $\hat{\rho}_{ab}(t) = |\psi_a(t)\rangle \langle \psi_b(t)|$. By the density operator $\hat{\rho}_{ab}(t)$, we can evaluate a particular expectation value for the position operator simply by:

$$
X(t) \equiv \text{tr}_X(\hat{x}\hat{\rho}(t)) = \begin{pmatrix}
\langle \psi_1(t)|\hat{x}|\psi_1(t)\rangle & \langle \psi_2(t)|\hat{x}|\psi_1(t)\rangle & \cdots & \langle \psi_N(t)|\hat{x}|\psi_1(t)\rangle \\
\langle \psi_1(t)|\hat{x}|\psi_2(t)\rangle & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
\langle \psi_1(t)|\hat{x}|\psi_N(t)\rangle & \cdots & \cdots & \langle \psi_N(t)|\hat{x}|\psi_N(t)\rangle
\end{pmatrix},
$$

which is an $N \times N$ matrix with the general element as $\hat{\rho}_{ab}(t) = |\psi_a(t)\rangle \langle \psi_b(t)|$. By the density operator $\hat{\rho}_{ab}(t)$, we can evaluate a particular expectation value for the position operator simply by:
in which $\hat{x}$ is the usual position operator, and $\text{tr}_x$ means the integration on the volume of space, yielding $\langle \psi_a(t)|\hat{x}|\psi_b(t)\rangle = \int d\mathbf{x} \; \psi_a^\dagger(\mathbf{x},t)\mathbf{x}\psi_b(\mathbf{x},t)$. The general element is defined by $X_{ab}(t) = \langle \psi_b(t)|\hat{x}|\psi_a(t)\rangle = X_{ba}(t)$, and so the matrix coordinate $X(t)$ is $N \times N$ hermitian matrix with the usual expansion in color (isospin) space as $X(t) = \sum_{a,b=1}^{N} X_{ab}(t)|a\rangle\langle b|$. Again as is recognized easily, the elements of the matrix coordinate $X$ are characterized by the color labels $a, b = 1, \cdots, N$.

As usual, it is natural to assume that the expectation values satisfy some classical equations of motion. Also, we expect that via the quantization of the outcome classical theory, we end up with the original quantum theory. In general case, one expects the classical equations can be derived from the quantized theory, in particular by the equations of motion for the states or wave functions. Since in the problem at hand the quantum theory, specially in the non-perturbative regime, is too hard to solve one may try to formulate the classical theory on some general grounds. In our specific case naturally we are faced with a matrix model. So a general classical action for the coordinates $X(t)$ may be taken as

$$S[X] = \int dt \; \text{Tr} \left( \frac{1}{2} \mathbf{X} \cdot \dot{\mathbf{X}} - \mathcal{V}(\mathbf{X}, \dot{\mathbf{X}}, X_{ab}, \dot{X}_{ab}) \right),$$

where $\text{Tr}$ acts on the matrix structure, and “$\mathcal{V}(\cdots)$” is for the possible potential term, depending on matrix coordinate or velocity, or probably some of their individual elements $X_{ab}$ and $\dot{X}_{ab}$. For the well separated quarks, as is mentioned the coordinate matrix $X(t)$ is almost, or even in the case, exactly diagonal and the action (4) becomes

$$S[X] \simeq S[x_1, \cdots, x_N] = \int dt \; \sum_{a=1}^{N} \left( \frac{1}{2} m \mathbf{x}_a \cdot \dot{\mathbf{x}}_a - \cdots \right),$$

in which $x_a = X_{aa}$. The kinetic term of the action (4) may be interpreted as the kinetic term of $N$ quarks. This shows that our new tool, matrix coordinates, consists the information we usually realize, in particular the positions and velocities of individual quarks.

The issue of gauge symmetry of original quantum mechanical problem should be considered. The theory we start with is invariant under the transformations:

$$|\Psi(t)\rangle \rightarrow |\Psi'(t)\rangle = \hat{V}(\hat{x}, t)|\Psi(t)\rangle,$$

$$\langle \Psi(t) | \rightarrow \langle \Psi'(t) | = \langle \Psi(t) | \hat{V}^\dagger(\hat{x}, t),$$

$$\hat{O}(\hat{x}, \hat{p}, t, \partial_t) \rightarrow \hat{O}'(\hat{x}, \hat{p}, t, \partial_t) = \hat{V}(\hat{x}, t)\hat{O}(\hat{x}, \hat{p}, t, \partial_t)\hat{V}^\dagger(\hat{x}, t),$$

(6)
for the Hamiltonian of the form $H = \langle \Psi(t)|\hat{O}(\hat{x}, \hat{p}, t, \partial_t)|\Psi(t)\rangle$, and $\hat{V}(\hat{x}, t)$ is an $N \times N$ unitary operator, \textit{i.e.}, $\hat{V}\hat{V}^\dagger = \hat{V}^\dagger\hat{V} = 1_N$. Due to integration on space $\int d\mathbf{x}$, it might not be expected that in a simple way all of the large symmetry above can be recovered in the theory for matrix coordinates. Instead we assume that the position dependence of the $\hat{V}$ matrix is in the form of $\hat{V}(\hat{x}, t) = \tilde{U}(\hat{x})U(t)$, where $U(t)$ is an $N \times N$ unitary matrix, and $\tilde{U}(\hat{x})$ is a phase depending on the position operator $\hat{x}$, \textit{i.e.}, $\tilde{U}\tilde{U}^* = 1$. By this kind of transformations we are treating the position dependence of matrix $\hat{V}$ as a $U(1)$ group, rather than a non-Abelian one. Later we try to present some kind of justification for the restriction on the transformations. It can be seen that the matrix coordinate transforms as $X(t) \rightarrow X'(t) = U(t)X(t)U(t)^\dagger$. So our matrix theory, at least, should be invariant under this kind of transformations, \textit{i.e.},

\begin{align}
S[a_t, X] &= \int dt \text{ Tr} \left( \frac{1}{2} m D_t X \cdot D_t X - \mathcal{V}(X, D_t X) \right),
\end{align}

in which $D_t X = \dot{X} + i[a_t, X]$, with $a_t(t)$ as the one dimensional $N \times N$ hermitian gauge field. We see that the action is now invariant under the transformations:

\begin{align}
X &\rightarrow X' = UXU^\dagger, \\
a_t &\rightarrow a'_t = Ua_tU^\dagger - iU\partial_t U^\dagger,
\end{align}

with $U \equiv U(t)$ as an arbitrary $N \times N$ unitary matrix; in fact under these transformations one obtains

\begin{align}
D_t X \rightarrow D'_t X' &= U(D_t X)U^\dagger, \\
D_t D_t X \rightarrow D'_t D'_t X' &= U(D_t D_t X)U^\dagger.
\end{align}

One may go a little more in details of the potential term. First, we assume that the potential is linear in velocity $D_t X$, appearing in the potential as $D_t X \cdot A(X, t)$. Second, since here we have matrices as coordinates, we can decompose the velocity independent term to completely symmetric and non-symmetric parts in components of $X = (X^1, X^2, \cdots, X^d)$. We note that each component $X^i$ is a matrix. The non-symmetric part can be expanded as

\begin{align}
\mathcal{V}_{\text{veloc. indep. non-symm.}}(X) &= X^i + [X^i, X^j] + X^i[X^j, X^k] - \frac{m}{4l^4} [X^i, X^j][X_i, X_j] + O(X^6),
\end{align}

\footnote{The invariance under the global transformations by $\hat{V}(\hat{x}, t) = V_0$, with $V_0$ as a constant $N \times N$ unitary matrix, requires that the action should not consist the individual elements of $X$, as we assumed in \ref{eq:4}, in the first step.}
in which the terms “· · ·” consist free space indices or traceless parts. So the first surviving term is “$-m[X^i, X^j]^2/4l^4$”, with $l$ as a proper length scale. Finally we require that the vector potential $A(X, t)$ is also symmetric in the components $X^i$’s. Putting these all for the potential term, we end up with the action

$$S[a_t, X] = \int dt \; \text{Tr} \left( \frac{1}{2} m D_t X \cdot D_t X + q D_t X \cdot A(X, t) - qA_0(X, t) + \frac{m}{4l^4} [X^i, X^j]^2 \right),$$

(11)
in which $A_0(X, t)$ is the symmetric part of velocity independent term of potential, and $q$ plays the role of the charge. We note that the fields $(A_0(X, t), A(X, t))$ appear as $N \times N$ hermitian matrices due to their functional dependence on the matrix coordinate $X$. It is interesting to study the gauge symmetry of this action. One can check easily that action (11) is invariant under the symmetry transformations [10, 11, 12]:

$$X \rightarrow X' = UXU^\dagger,
\quad a_t(t) \rightarrow a'_t(t) = UA_t(t)U^\dagger - U\frac{d}{dt}U^\dagger,
\quad A_i(X, t) \rightarrow A'_i(X', t) = UA_i(X, t)U^\dagger + iU\delta_iU^\dagger,
\quad A_0(X, t) \rightarrow A'_0(X', t) = UA_0(X, t)U^\dagger - iU\partial_tU^\dagger,$$

(12)

where $U \equiv U(X, t) = \exp(i\Lambda)$ is arbitrary up to the condition that $\Lambda(X, t)$ is hermitian and totally symmetrized in the $X^i$’s. In above, $\delta_i$ is the functional derivative $\delta \delta X_i$.

We recall that, in approving the invariance of the action, the symmetrization prescription on the matrix coordinates plays an essential role [10, 11]. It is by this symmetry transformation that we expect no distinguished role should be identified to the (diagonal or off-diagonal) elements of matrix coordinate. In other words, since each of the matrix elements are not gauge invariant quantities, they are not expected to appear as an observable final state.

The above transformations on the gauge potentials are similar to those of non-Abelian gauge theories, and we mention that it is just the consequence of enhancement of degrees of freedom from numbers ($x$) to matrices ($X$). In other words, we are faced with a situation in which “the rotation of fields” is generated by “the rotation of coordinates” [11]. In addition, the case we see here may be considered as finite-$N$ version of the relation between gauge symmetry transformations and transformations of matrix coordinates [13]. Despite the non-Abelian behavior of the gauge transformations,

\footnote{We note that though $U(X, t)$ depends on $X(t)$, due to the total derivative $\frac{d}{dt}$, $a'_t(t)$ still only depends on time. In this sense the transformations in [10, 11, 12] are interpreted incorrectly.}
we should note that the symmetry is still not equivalent to a non-Abelian one. To see this, we should recall that the symmetry transformations of, for example a U($N$) gauge theory, is generated by $N^2$ functions of space-time, say $\Lambda_\alpha(x, t)$ ($\alpha = 0, \ldots, N^2 - 1$), in the group element $\exp(i\Lambda_\alpha T^\alpha)$, where $T^\alpha$s are U($N$) generators. Now although $U(X, t) = \exp(i\Lambda(X, t))$ in (12) is a unitary matrix due to its dependence on matrix coordinate, it is constructed by just one function $\Lambda(x, t)$, after replacing coordinates by matrices i.e. $x \rightarrow X$, under the condition of symmetrization. After all, it is quite natural to interpret the fields $(A_0, A)$ as the external gauge fields that the constituents, whose degrees of freedom are included in the matrix coordinate, interact with them.

The action (11) is known to be the action of $N$ D0-branes of String Theory, in the background of (RR) gauge field $(A_0(x, t), A(x, t))$, for $x$ as the ordinary coordinates \[14\]. As mentioned before, from the String Theory point of view, D0-branes are point particles to which ends of strings are attached \[3\]. In a bound state of $N$ D0-branes, they are connected to each other by strings stretched between them, and it can be shown that, by counting the degrees of freedom for the oriented strings, the correct dynamical variables describing the positions of D0-branes are $N \times N$ hermitian matrices \[8\]. By comparison, we find out that $m$ is the mass of D0-branes and $l$ is the order of the string length. In \[2, 3, 4, 5\] the possibility for the identification of dynamics of D0-branes and quarks are investigated. Here we recall some of the aspects mentioned in these papers. First of all, we see that by the gauge transformation (12), the elements of the position matrix mix with each other, and so the interpretation of the positions for D0-branes remains obscure. Nevertheless, we note that the concept of center-of-mass (c.m.), here presented by the trace of the matrix coordinate is meaningful. So the ambiguity of the positions only remains for the degrees of freedom counting the relative positions of D0-branes and the strings stretched between them. The equations of motion for $X^i$’s and $a_t$ by action (11), ignoring the commutator potential $[X_i, X_j]^2$, is found to be \[10, 11, 12\]

\[
\begin{align*}
mD_tD_tX_i &= q\left(E_i(X, t) + D_tX_j B_{ji}(X, t)\right), \\
m[X_i, D_tX^i] &= q[A_i(X, t), X^i],
\end{align*}
\]

with the following definitions

\[
\begin{align*}
E_i(X, t) &\equiv -\delta_i A_0(X, t) - \partial_t A_i(X, t), \\
B_{ji}(X, t) &\equiv -\delta_j A_i(X, t) + \delta_i A_j(X, t).
\end{align*}
\]
In (13), the symbol \(D_t X^j B_{ji}(X, t)\) denotes the average over all of positions of \(D_t X^j\) between the \(X\)'s of \(B_{ji}(X, t)\). The above equations for the \(X\)'s are like the Lorentz equations of motion, with the exceptions that two sides are \(N \times N\) matrices, and the time derivative \(\partial_t\) is replaced by its covariant counterpart \(D_t\).

The behavior of eqs. (13) and (14) under gauge transformation (12) can be checked. Since the action is invariant under (12), it is expected that the equations of motion change covariantly. The left-hand side of (13) changes to \(U^\dagger D_t D_t X U\) by (9), and therefore we should find the same change for the right-hand side. One can check that in fact this is the case [10, 11, 12], and consequently one finds that Eq. (16) leads to

\[
E_i(X, t) \rightarrow E'_i(X', t) = U E_i(X, t) U^\dagger,
\]

\[
B_{ji}(X, t) \rightarrow B'_{ji}(X', t) = U B_{ji}(X, t) U^\dagger.
\]

This result is consistent with the fact that \(E_i\) and \(B_{ji}\) are functionals of \(X\)'s. We thus see that, in spite of the absence of the usual commutator term \([A_\mu, A_\nu]\) of non-Abelian gauge theories, in our case the field strengths transform like non-Abelian ones. We recall that these are all consequences of the matrix coordinates of D0-branes. Finally by the similar reason for vanishing the second term of (11), both sides of (14) transform identically.

An equation of motion similar to (13) is considered in [5, 4] as a part of similarities between the dynamics of D0-branes and bound states of quarks–QCD strings in a baryonic state [3, 4, 2]. The point is that, the dynamics of the bound state c.m. is not affected directly by the non-Abelian sector of the background, i.e., the c.m. is “white” with respect to SU\((N)\) sector of matrices. The c.m. coordinates and momenta are defined by:

\[
x_{c.m.} \equiv \frac{1}{N} \text{Tr} X, \quad p_{c.m.} \equiv \text{Tr} P,
\]

where we are using the convention \(\text{Tr} 1_N = N\). To specify the net charge of a bound state (which is an extended object) its dynamics should be studied in zero magnetic and uniform electric fields, i.e., \(B_{ji} = 0\) and \(E_i(X, t) = E_{0i}\). Since the fields are uniform, they do not involve the \(X^i\) matrices, and contain just the U(1) part. In other words, under gauge transformations \(E_{0i}\) and \(B_{ji} = 0\) transform to \(E'_i(X, t) = E_{0i}\) and \(B'_{ji}(X, t) = 0\), respectively.

---

\(3\)In a non-Abelian gauge theory a uniform electric field can be defined up to a gauge transformation, which sufficient for identification of white (singlet) states.
Figure 1: The net electric flux extracted from each quark is equivalent in a baryon (a) and a meson (b). The D0-brane–quark correspondence suggests the string-like shape for flux inside a baryon (a).

\[ U(X,t)E_{0i}U^\dagger(X,t) = E_{0i} \text{ and } B'_{ji} = 0. \]

Thus the action (11) yields the following equation of motion:

\[ (Nm)\ddot{x}_{\text{c.m.}} = NqE_{0(1)}; \]  

in which the subscript (1) emphasizes the U(1) electric field. So the c.m. interacts directly only with the U(1) of U(N). From the String Theory point of view, this observation is based on the simple fact that the SU(N) structure of D0-branes arises just from the internal degrees of freedom inside the bound state. In other words, the matrix behavior of the coordinates, and the resulted non-commutativity, is just restricted to the relative positions of D0-branes. By this picture, we may call this situation as ‘confined non-commutativity’ [12, 11, 4, 3]. This behavior of D0-brane bound states is the same as that of baryons. It means that each D0-brane feels the net effect of other D0-branes as the white-complement of its color. In other words, the field flux extracted from one D0-brane to the other ones are the same as the flux between a color and an anti-color, Fig.1. This shape for the electric flux are in agreement with the result of field theory correlator method [15]. It was pointed that the gauge symmetry associated to gauge field \((A_0(X,t), A(X,t))\), though looking similar to the non-Abelian gauge theories, is in intrinsic U(1). Based on the observation we have made here about whiteness of the bound state, we may argue in the phase that all of the observable states should have equivalent amount of U(N) sectors, the symmetry appears to be
restricted, and equivalently as U(1). In fact it is the case that we expect to see when we are faced with matrix coordinates as relevant degrees of freedom.

It is desirable to assign a net charge different from $Nq$ to the c.m. It can be done simply by modifying the action (11)

$$S'[a_t, X] = S[a_t, X] + \int dt \left( Nq' \dot{x}_{c.m.} \cdot A(x_{c.m.}, t) - Nq' A_0(x_{c.m.}, t) \right),$$

in which $S[a_t, X]$ is the action (11). By this action the charge of c.m. is equal to $N(q + q')$, rather than $Nq$.

Now, let us ignore for the moment the external gauge field ($A_0, A$). The equations of motion can be solved by diagonal configurations, such as:

$$X(t) = \text{diag.}(x_1(t), \cdots, x_N(t)),
\quad a_t(t) = \text{diag.}(a_{1t}(t), \cdots, a_{Nt}(t)),\quad (21)$$

with $x_a = x_{a0} + v_{at}, a = 1, \cdots, N$. By this configuration, we restrict the U(N) generators to the $N$ dimensional Cartan sub-algebra. This configuration describes the “classical” free motion of $N$ D0-branes, neglecting the effects of the strings (and the symmetry supported by them). Of course the situation is different when we consider quantum effects, and consequently it will be realized that the dynamics of the off-diagonal elements affect the dynamics of D0-branes significantly. Concerning the effect of the strings, one may try to extract the effective theory for D0-branes, i.e., for the diagonal configurations. In particular, it will be found out that the commutator potential is responsible for the formation of the bound state, and by a simple dimensional analysis we understand that the size of the bound state, $\ell$, is $\sim m^{-1/3}l^{2/3}$. As in [2] (see also [4, 5]), let us take the example of static D0-branes. For this configuration one can easily calculate one-loop effective potential between the quarks, getting [4, 5, 2]:

$$V_{\text{one-loop}} \sim 4\pi d^2 \frac{d - 1}{2} \sum_{a>b=1}^{N} \frac{|x_a - x_b|}{l^2}.$$

This result shows the linear potential between each pair of D0-branes. Previously we mentioned that, by qualitative considerations, what should be the shape of the electric flux (Fig.1). Now, by interpretation of (22) as the effective potential of a constituent quark model, we are enable to know something more about the bound state and more quantitative details. One can trace supports for the linear behavior of the potential in the literature, namely results by lattice calculations [16, 17], and things we expect.
from the spin-mass Regge trajectories. In [18] by taking the linear potential between quarks of a baryonic state in transverse direction of the light cone frame, the structure functions are obtained in good agreement with the observed ones. Since the original theory is invariant under the rotation among the color indices 1, · · · , N, we mention that only the states which are singlets under the (global) rotation among the indices can be accepted as the physical states of effective theory for diagonal elements.

The formulation we presented above is in the non-relativistic limit. Though it is expected this limit produces good results for heavy quarks, for light or massless quarks we should change our approach. One way can be starting by a covariant theory; treating time and space equivalently. In this way, although the terms responsible for kinetic energy and interaction with external gauge fields find reasonable forms (see [11, 12]), the main problem will appear to be with potentials as $[X^\mu, X^\nu]^2$. Instead one may follow another approach to say something about the covariant theory. The world-line formulation we have here is that of the M(atrix) model conjecture, accompanied with the interaction terms with external gauge fields. For the case of the dynamics of a massless charged particle with ordinary coordinates, we can see easily that the light-cone dynamics have a form similar to that we have in action (11); see Appendix of [4]. To approach the covariant formulation, following finite-\(N\) interpretation of [21], it is reasonable to interpret things in the DLCQ framework [3, 4, 5, 12]. In this way of interpretation, the mass parameter \(m\) is the longitudinal momentum, and the spatial directions present the transverse coordinates in the light-cone frame. In addition, according to the specific form of action (11) the rest mass of quarks is assumed to be zero (see [4, 4]).

In [3, 4] and [12] the problem of scattering of 1) a D0-brane off another one and, 2) a D0-brane bound state off an external gauge field probe, were considered. As we mentioned above, both of the scattering processes can be interpreted in the light-cone frame. For the case of scattering of a D0-brane off another one, the expectations for the well known Regge behavior are satisfied. As for the problem of interaction between D0-brane bound state and ‘photons’ of gauge field, the interesting observations is expected for the regime in which the details of the bound state can be probed. Here we just present the general expected features; see [12] for more details. As argued in before, the external field depend on the internal coordinates of the bound state under the symmetrization condition in matrix coordinates. One way to cover the symmetrization is to use the so-called ‘non-Abelian Fourier expansion’ [12]. For an arbitrary function
The non-Abelian Fourier expansion will be found to be:

\[ f(X, t) = \int d\mathbf{k} \tilde{f}(\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{X}}, \quad (23) \]

in which \( \tilde{f}(\mathbf{k}, t) \) are the Fourier components of the function \( f(x, t) \) (i.e., function by ordinary coordinates) which is defined by the known expression:

\[ \tilde{f}(\mathbf{k}, t) \equiv \frac{1}{(2\pi)^d} \int d\mathbf{x} f(x, t) e^{-i\mathbf{k} \cdot \mathbf{x}}. \quad (24) \]

Since the momentum numbers \( k_i \)'s are ordinary numbers, and so commute with each other, the symmetrization prescription is automatically recovered in the expansion of the momentum eigen-functions \( e^{i\mathbf{k} \cdot \mathbf{X}} \). Now, by using the symmetric expansion (23), we can imagine some general aspects of the interaction between D0-brane bound states and RR photons. As we mentioned in before the size of the bound state, for finite number \( N \) of D0-branes is finite and is of order of \( \ell \sim m^{-1/3} l^{2/3} \).

Before proceeding further, we should distinguish the dynamics of the c.m. from the internal degrees of freedom of the bound state. As mentioned in before, the c.m. position and momentum of the bound state are presented by the \( U(1) \) sector of the \( U(N) = SU(N) \times U(1) \), and thus the information related to the c.m. can be gained simply by the Tr operation. So, the internal degrees of freedom of the bound state, which consist the relative positions of \( N \) D0-branes together with the dynamics of strings stretched between D0-branes, are described by the \( SU(N) \) sector of the matrix coordinates. It is easy to see that the commutator potential in the action has some flat directions, along which the eigen-values can take arbitrary large values. But it is understood that, by considering the quantum effects and in the case that we expect formation of the bound state, we should expect suppression the large values of the internal degrees of freedom [22]. Consequently, it is expected that the \( SU(N) \) sector of matrix coordinates take mean values like \( \langle X_{\alpha}^i \rangle \sim \ell (\alpha = 1, \cdots, N^2 - 1, \text{not } \alpha = 0 \text{ as c.m.}) \), with \( \ell \) as the bound state size scale mentioned in above. We should mention that, though the c.m. is represented by the \( U(1) \) sector, but its dynamics is affected by the interaction of the ingredients of bound state with the \( SU(N) \) sector of external fields, similar to the situation we imagine in the case of the Van der Waals force.

The important question about the interaction of a bound state (as an extended object) with an external field, is about ‘the regime in which the substructure of bound state is probed’. As we mentioned in introduction, in our case the quanta of RR fields are the representatives of the external field. The quanta are coming from a ‘source’
Figure 2: Substructure is not seen by the long wave-length modes (a). Due to functional dependence on matrix coordinates, the short wave-length modes can probe inside the bound state (b). $\ell$ and $\bar{A}_\mu(k,t)$ represent the size of the bound state and the Fourier modes, respectively.

and so, as it makes easier things, we ignore its dynamics. The source is introduced to our problem by the gauge field $A_\mu(x,t)$. These fields appear in the action by functional dependence on matrix coordinates $X$’s. In fact this is the key of how we can probe the substructure of the bound state. According to the non-Abelian Fourier expansion we mentioned in above, we have

$$A_\mu(X,t) = \int d\mathbf{k} \, \bar{A}_\mu(k,t) e^{i \mathbf{k} \cdot \mathbf{X}},$$

(25)

in which $\bar{A}_\mu(k,t)$ is the Fourier components of the fields $A_\mu(x,t)$ (i.e., fields by ordinary coordinates). One can imagine the scattering processes which are designed to probe inside the bound state. Such as every other scattering process two limits of momentum modes, corresponding to long and short wave-lengths, behave differently.

In the limit $\ell|\mathbf{k}| \to 0$ (long wave-length regime), the field $A_\mu$ is not involved by $X$ matrices mainly. It means that the fields appear to be nearly constant inside the bound state, and in rough estimation we have

$$e^{i \mathbf{k} \cdot \mathbf{X}} \sim e^{i \mathbf{k} \cdot \mathbf{X}_{c.m.}}.$$

(26)

So in this limit we expect that the substructure and consequently non-commutativity will not be seen; Fig.-2a. As the consequence, after interaction with a long wave-length
mode, it is not expected that the bound state jump to another energy level different from the first one. It should be noted that the c.m. dynamics can be affected as well in this case.

In the limit \( |\ell| k = \text{finite} \) (short wave-length regime), the fields depend on coordinates \( X \) inside the bound state, and so the substructure responsible for non-commutativity should be probed; Fig.-2b. In fact, we know that the non-commutativity of D0-brane coordinates is the consequence of the strings which are stretched between D0-branes. In this case, it is completely expectable that the energy level of the incoming and outgoing bound states will be different, since the ingredients of bound state substructure can absorb quanta of energy from the incident wave. In this case the c.m. dynamics can be affected in a novel way by the interaction of the substructure with the external fields (the Van der Waals effect). In general case, one can gain more information about the substructure of a bound state by analysing the ‘recoil’ effect on the source. To do this, one should be able to include the dynamics of the source in the formulation. Considering the dynamics of source, in the terms of quantized field theory, means that we consider the processes in which the source and the target exchange ‘one quanta of gauge field’ with definite wave-length and frequency, though off-shell, as \( A_\mu(x, t) \sim e^{i k \cdot x - i \omega t} \).

Up to now, we have considered things for the theory with one kind of flavor. It is interesting to think about the case with more than one flavor. One suggestion can be as follows: assume the flavor \( A \) with mass \( m_A \) is represented by the state \( |\Psi_A(t)\rangle \). We may re-scale the states as \( |\Psi_A\rangle \rightarrow |\tilde{\Psi}_A\rangle = (m_A)^{1/4} |\Psi_A\rangle \). For a baryon consisting \( N \) heavy flavors we define the matrix coordinate as

\[
\tilde{X}(t) \equiv \begin{pmatrix}
\langle \tilde{\psi}_1(t)|\tilde{x}|\tilde{\psi}_1(t)\rangle & \langle \tilde{\psi}_2(t)|\tilde{x}|\tilde{\psi}_1(t)\rangle & \ldots & \langle \tilde{\psi}_N(t)|\tilde{x}|\tilde{\psi}_1(t)\rangle \\
\langle \tilde{\psi}_1(t)|\tilde{x}|\tilde{\psi}_2(t)\rangle & \ldots & \ldots & \ldots \\
\vdots & \ddots & \ddots & \vdots \\
\langle \tilde{\psi}_1(t)|\tilde{x}|\tilde{\psi}_N(t)\rangle & \ldots & \ldots & \langle \tilde{\psi}_N(t)|\tilde{x}|\tilde{\psi}_N(t)\rangle
\end{pmatrix}.
\] (27)

For this coordinate we take the action

\[
S[\tilde{X}] = \int dt Tr \left( \frac{1}{2} \dot{\tilde{X}} \cdot \tilde{X} - \cdots \right).
\] (28)

Now, for the well separated states, for which we have diagonal coordinates, the action in terms of original coordinates (i.e., before re-scaling) becomes

\[
S[x_A] = \int dt \sum_{A=1}^{N} \left( \frac{1}{2} m_A \dot{x}_A \cdot \dot{x}_A - \cdots \right).
\] (29)
in which we see that each flavor has the expected kinetic term. It is worth recalling that due to the color symmetry we expect, the coordinate to which the symmetry transformation should apply is $\tilde{X}$.

In [11] a conceptual relation between use of matrix coordinate for non-Abelian gauge theory purposes and the ideas concerned in special relativity is mentioned; see also [5, 4, 2]. According to an interpretation of the special relativity, it is meaningful if the ‘coordinates’ and the ‘fields’ in a theory have some kinds of similar characters. As an example, we observe that both the space-time coordinates $x^\mu$ and the electromagnetic potentials $A^\mu(x)$ transform equivalently (i.e., as a $(d + 1)$-vector) under the boost transformations. Also by this way of interpretation, the super-space formulations of supersymmetric field and superstring theories are the natural continuation of the special relativity program. In the case of use of matrix coordinates, it may be argued that the relation between ‘matrix coordinates’ and ‘matrix fields’ (gauge fields of a non-Abelian gauge theory) is one of the expectations which is supported by the spirit of the special relativity. From the previous discussion we recall, 1) the matrix character of gauge fields is the result of dependence of them on matrix coordinates [11], 2) the symmetry transformations of gauge fields is induced by the transformations of matrix coordinates [11], 3) the transformations of fields in the theory on matrix space appeared to be similar to those of non-Abelian gauge theories, relations (12) and (17). This way of interpretation leads us to conclude that the non-Abelian gauge fields in a confined theory do not have an independent character, and they are introduced to the formalism due to functional dependence on the matrix coordinates of ‘bounded quarks’. It seems very interesting when we note that by the present situation of experimental data, the existence of pure gluonic states, the so-called glueballs, is quite doubtful. This lack of detection may be taken as a support for the interpretation presented above.

Acknowledgement: The author is grateful to A. Shariati, and specially to M. Khorrami for helpful discussions. The comments on the manuscript by Gh. Exirifard, and specially by S. Parvizi and M.M. Sheikh-Jabbari are acknowledged.

References

[1] W. Heisenberg, “Quantum-Theoretical Re- Interpretation Of Kinematic And Mechanical Relations,” (in German) Zs. Phys. 33 (1925) 879.
[2] A.H. Fatollahi, “Do Quarks Obey D-Brane Dynamics?,” Europhys. Lett. 53(3) (2001) 317, hep-ph/9902414.

[3] A.H. Fatollahi, “Do Quarks Obey D-Brane Dynamics? II,” Europhys. Lett. 56(3) (2001) 523, hep-ph/9905484.

[4] A.H. Fatollahi, “D0-Branes As Light-Front Confined Quarks,” Eur. Phys. J. C19 (2001) 749, hep-th/0002021.

[5] A.H. Fatollahi, “D0-Branes As Confined Quarks,” talk given at “Isfahan String Workshop 2000, May 13-14, Iran,” hep-th/0005241.

[6] J. Polchinski, “Dirichlet-Branes And Ramond-Ramond Charges,” Phys. Rev. Lett. 75 (1995) 4724, hep-th/9510017.

[7] J. Polchinski, “TASI Lectures On D-Branes,” hep-th/9611050.

[8] E. Witten, “Bound States Of Strings And p-Branes,” Nucl. Phys. B460 (1996) 335, hep-th/9510135.

[9] J. Polchinski, “String Theory,” vol. I, Cambridge Univ. Press, pp. 184 and 268.

[10] A.H. Fatollahi, “On Non-Abelian Structure From Matrix Coordinates,” Phys. Lett. B512 (2001) 161, hep-th/0103262.

[11] A.H. Fatollahi, “Electrodynamics On Matrix Space: Non-Abelian By Coordinates,” Eur. Phys. J. C21 (2001) 717, hep-th/0104210.

[12] A.H. Fatollahi, “Interaction Of D0-Brane Bound States And Ramond-Ramond Photons,” Phys. Rev. D65 (2002) 046004, hep-th/0108198.

[13] A.H. Fatollahi, “Gauge Symmetry As Symmetry Of Matrix Coordinates,” Eur. Phys. J. C17 (2000) 535, hep-th/0007023.

[14] R.C. Myers, “Dielectric-Branes,” JHEP 9912 (1999) 022, hep-th/9910053. W. Taylor and M. Van Raamsdonk, “Multiple Dp-Branes In Weak Background Fields,” Nucl. Phys. B573 (2000) 703, hep-th/9910052.
[15] D.S. Kuzmenko and Y.A. Simonov, “QCD String In Mesons And Baryons,” Phys. Atom. Nucl. 64 (2001) 107; Yad. Fiz. 64 (2001) 110, [hep-ph/0010114]; A.D. Giacomo, H.G. Dosch, V.I. Shevchenko and Y.A. Simonov, “Field Correlators In QCD. Theory And Applications,” [hep-ph/0007223].

[16] G.S. Bali, “QCD Forces And Heavy Quark Bound States,” Phys. Rept. 343 (2001) 1, [hep-ph/0001312], page 73; C. Alexandrou, Ph. de Forcrand and A. Tsapalis, “The Static Baryon Potential,” nucl-th/0111046.

[17] J.M. Cornwall, “Baryon Wilson Loop Area Law In QCD,” Phys. Rev. D54 (1996) 6527.

[18] G.S. Kirishnaswami, “A Model Of Interacting Partons For Hadronic Structure Functions,” [hep-ph/9911538]. G.S. Kirishnaswami and S. G. Rajeev, Phys. Lett. B441 (1998) 449.

[19] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, “M Theory As A Matrix Model: A Conjecture,” Phys. Rev. D55 (1997) 5112, [hep-th/9610043].

[20] T. Banks, “Matrix Theory,” Nucl. Phys. Proc. Suppl. 67 (1998) 180, [hep-th/9710231]. “TASI Lectures On Matrix Theory,” [hep-th/9911068]. D. Bigatti and L. Susskind, “Review Of Matrix Theory,” [hep-th/9712072].

[21] L. Susskind, “Another Conjecture About M(atrix) Theory,” [hep-th/9704080].

[22] B. de Wit, “Supersymmetric Quantum Mechanics, Supermembranes And Dirichlet Particles,” Nucl. Phys. Proc. Suppl. B56 (1997) 76, [hep-th/9701169]; H. Nicolai and R. Helling, “Supermembranes And M(atrix) Theory,” [hep-th/9809103]; B. de Wit, “Supermembranes And Super Matrix Models,” [hep-th/9902051].