Higher-order layer-adapted finite difference schemes for singularly perturbed Robin problems

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Abstract. In this paper, we construct the midpoint upwind scheme on the Shishkin mesh and on the Bakhvalov-Shishkin mesh respectively for solving the singularly perturbed Robin boundary value problem. The central divided difference is used to discretize the first derivative in the Robin boundary condition to achieve the higher-order uniform convergence. The elaborate $\varepsilon$-uniform pointwise error estimates $O(N^{-1}\ln N)$ for $1 \leq i \leq p_mN$ and $O(N^{-2})$ for $p_mN < i < N$ with $p_m = 1/(4\varepsilon) + 1/4$ on the Shishkin mesh and $O(N^{-1})$ for $1 \leq i \leq N/2$ and $O(N^{-2})$ for $N/2 < i < N$ on the Bakhvalov-Shishkin mesh are proved. We also construct the hybrid finite difference scheme that combines the midpoint upwind scheme on the coarse part with the central difference scheme on the fine part on the Shishkin mesh, and prove a better uniform convergence of orders $O(N^{-1} \ln^2 N)$ for $1 \leq i \leq p_hN$ and $O(N^{-2})$ for $p_hN < i < N$ with $p_h = 1/(2\varepsilon)$. Finally, a numerical experiment illustrates that these error estimates are sharp and the convergence is uniform with respect to the perturbation parameter.

1. Introduction
Singualr perturbation convection-diffusion problems arise in various branches of science and engineering such as modeling of water quality problems in river networks, fluid flow at high Reynolds numbers, drift-diffusion equation of semiconductor device modeling. A wide variety of numerical methods were established to solve singularly perturbed problems in the past few decades (see [1, 2]). Kellog and Tsan in [3] analyzed some three point difference schemes for a singular perturbation problem without turning points. Bounds for the discretization error were obtained which are uniformly valid for all $h$ and $\varepsilon > 0$. In order to achieve higher-order uniform convergence for solving the singularly perturbed two-point Dirichlet boundary value problem, the midpoint upwind scheme was constructed and proved the convergence $O\left(\max\left\{N^{-2}, N^{-5+4\varepsilon/N} \ln N\right\}\right)$ for $i = 1, ..., N$ on the Shishkin mesh in [4] and the convergence order $O(N^{-2})$ on the coarse part and $O(N^{-3})$ on the fine part on the Bakhvalov-Shishkin mesh in [5]. The hybrid finite difference scheme combining the midpoint upwind scheme on the coarse part with the central difference scheme on the fine part on the Shishkin mesh was proposed and proved the convergence order $O(N^{-2})$ on the coarse part and $O(N^{-2} \ln^2 N)$ on the fine part in [4].

Consider the singularly perturbed convection-diffusion Robin boundary value problem:
\[
\begin{align*}
Lu := & -\varepsilon u''(x) - b(x)u'(x) + c(x)u(x) = f(x), x \in (0,1), \\
B_i u := & u(0) - \varepsilon u'(0) = A, B_i u := u(1) = B,
\end{align*}
\]

where \(0 < \varepsilon \ll 1\) is a small perturbation parameter, \(A\) and \(B\) are given constants, and functions \(b(x)\), \(c(x)\) and \(f(x)\) are sufficiently smooth satisfying \(\beta > b(x) > \beta > 0\) and \(\gamma > c(x) \geq 0\), where \(\beta\), \(\beta\) and \(\gamma\) are constants. Under these conditions, singularly perturbed problem (1) has a unique solution that possesses a boundary layer at \(x = 0\), see Remark 2.100 in [2].

For singularly perturbed problem with \(c(x) = 0\) under Robin boundary conditions \(\gamma_1 u(0) - \gamma_2 \varepsilon u'(0) = A\) and \(\beta_1 u(1) + \beta_2 u'(1) = B\) where \(\gamma_1 > 0, \gamma_2 \geq 0, \beta_1 \geq 0, \beta_2 \geq 0\) and \(\beta_1 + \beta_2 > 0\), Ansari and Hegarty in [6] proved that the simple upwind scheme have the error estimate \(O\left(N^{-1}\right)\) on the coarse part and \(O\left(N^{-1} \ln N\right)\) on the fine part on the Shishkin mesh. Mohapatra and Natesan in [7] applied the upwind scheme on a more general nonuniform mesh and obtained the error estimate \(O\left(N^{-1}\right)\) on the whole interval with \(c(x) \geq 0\). For singularly perturbed problem (1), Andreev and Savin in [8] used a modified Samarskii’s monotone scheme on a Shishkin mesh to obtain the error estimate \(\|u - u^h\| \leq CN^{-2} \ln^2 N\). Natesan and Ramanujam in [9] used asymptotic-numerical method to obtain the error estimate \(O\left(N^{-1}\right)\) on the whole interval. By comparing with the higher-order uniform convergence obtained for the singularly perturbed two-point Dirichlet boundary value problem, the orders of the uniform convergence of the methods for solving Robin problem (1) need to be improved.

In this paper, the properties of the exact solution and the Shishkin mesh are introduced in Section 2. In Section 3, we construct the midpoint upwind scheme on the Shishkin mesh and the Bakhvalov-Shishkin mesh for solving problem (1), where the central divided difference is used to discretize the first derivative in the Robin boundary condition to achieve higher-order convergence. We also prove its \(\varepsilon\)-uniform pointwise convergence by using the comparison principle and constructing the barrier functions. In Section 4, the error estimate of a new hybrid difference scheme on the Shishkin mesh is analyzed. In Section 5, a numerical example supports the elaborate error estimates.

2. The solution and the Shishkin mesh

**Lemma 1** The solution \(u(x)\) of problem (1) can be decomposed as \(u(x) = S(x) + E(x)\) on \([0,1]\), where the smooth part \(S\) satisfies
\[
LS(x) = f(x) \quad \text{and} \quad \|S^{(i)}(x)\| \leq C_0, 0 \leq i \leq q,
\]
while the layer part \(E\) satisfies
\[
LE = 0, \quad \|E^{(i)}(x)\| \leq C_\varepsilon^{-i} \exp(-\beta x/\varepsilon), 0 \leq i \leq q,
\]
where the maximal order \(q\) depends on the smoothness of the data.

**Proof.** The proof of Lemma 1 is similar to the proofs in [2, 5].

Let \(N\) be a positive even integer and \(\tau = \min \{1/2, 4\varepsilon \ln N/\beta\}\). Choose \(\tau\) be the transition point. Divide \([0, \tau]\) and \([\tau, 1]\) uniformly into \(N/2\) subintervals, respectively. Then the Shishkin mesh is:
\[
x_i = \begin{cases} 
2\tau/N i, 0 \leq i \leq N/2, \\
\tau + 2(1-\tau)(i-N/2)/N, N/2 \leq i \leq N.
\end{cases}
\]

**Lemma 2** Denote \(h_i = x_i - x_{i-1}\) for (4), then \(h_i = 8\varepsilon \ln N/(\beta N)\) and \(N^{-1} h_{i/2^k} \leq 2N^{-1}\) for \(i = 1, \ldots, N/2\).
Throughout the paper, the nontrivial case $\varepsilon \leq CN^{-1}$ is considered, $C$ denotes a generic positive constant that is independent of both perturbation parameter $\varepsilon$ and mesh parameter $N$, and $C$ can take different values at each occurrence, even in the same argument.

3. Error estimates of the midpoint upwind scheme on the Shishkin mesh

We construct the following midpoint upwind difference scheme:

\[
\begin{align*}
L^N u^N_i & := -\varepsilon D^i_D D_i u^N_i - b_{i+1/2} D_i u^N_i + c_{i+1/2} (u^N_{i+1} + u^N_{i-1})/2 = f_{i+1/2}, i = 0, 1, \ldots, N - 1, \\
B^N u^N_i & := u^N_0 - \varepsilon Du^N_0 = A, B^N u^N_N := u^N_N = B,
\end{align*}
\]

where define $L^N_i$ as a discrete operator,

\[
D_i u^N_i = \frac{u^N_{i+1} - u^N_i}{h_i}, \quad D^i_D u^N_i = \frac{u^N_i - u^N_{i-1}}{h_i}, \quad D^i_D D_i u^N_i = 2\left(D^i_D u^N_i - D_i u^N_i\right), \quad Du^N_0 = \frac{u^N_0 - u^N_{-1}}{2h_1},
\]

the virtual point $x_{-1} = -x_i$, $b_{i+1/2} = b_i\left(\frac{x_{i+1} + x_i}{2}\right)$, $c_{i+1/2} = c_i\left(\frac{x_{i+1} + x_i}{2}\right)$, $f_{i+1/2} = f_i\left(\frac{x_{i+1} + x_i}{2}\right)$, $Du^N_0 = \frac{u^N_0 - u^N_{-1}}{2h_1}$ and so on. Here, we use the central divided difference by the help of a virtual node to approximate the first derivative quadratically in the Robin boundary condition.

**Lemma 3** (see [4]) (Discrete comparison principle) If $N > \gamma/\beta$, then the coefficient matrix associated with (5) on (4) is an M-matrix and (5) satisfies the discrete comparison principle, i.e., let \{$v_i$\} and \{w_i\} are mesh functions, if $B_i^N v_i \leq B_i^N w_i$, $B_i^N v_i \leq B_i^N w_i$ and $L^N v_i \leq L^N w_i$ for $i = 0, 1, \ldots, N - 1$, then $v_i \leq w_i$ for all $i$.

**Proof.** Scheme (3) is equivalent to

\[
\begin{align*}
L^N u^N_i & := \frac{2\varepsilon + h_i + h_{i+1}}{h_i} u^N_i - \frac{2\varepsilon + h_i + h_{i-1}}{h_i} u^N_i + \frac{2\varepsilon + h_i + h_{i+1}}{h_i} u^N_i - \frac{2\varepsilon + h_i + h_{i-1}}{h_i} u^N_i = f_i, i = 0, \\
L^N u^N_{i-1/2} & := \frac{2\varepsilon + h_i + h_{i+1}}{h_i} u^N_{i-1/2} - \frac{2\varepsilon + h_i + h_{i-1}}{h_i} u^N_{i-1/2} + \frac{2\varepsilon + h_i + h_{i+1}}{h_i} u^N_{i+1/2} - \frac{2\varepsilon + h_i + h_{i-1}}{h_i} u^N_{i+1/2} = f_{i-1/2}, i = 1, \ldots, N - 2, \\
L^N u^N_{N-1/2} & := \frac{2\varepsilon + h_i + h_{i-1}}{h_i} u^N_{N-1/2} - \frac{2\varepsilon + h_i + h_{i-1}}{h_i} u^N_{N-1/2} + \frac{2\varepsilon + h_i + h_{i+1}}{h_i} u^N_{N+1/2} - \frac{2\varepsilon + h_i + h_{i+1}}{h_i} u^N_{N+1/2} = f_{N-1/2}.
\end{align*}
\]

Its coefficient matrix is diagonally dominant and has non-positive off diagonal entries. Hence, the matrix is an irreducible $M$ matrix, so has a positive inverse. Hence, the solution $u_i, 0 \leq i \leq N - 1$, exists and, if the $w_i$ are as described in the lemma, $v_i \leq w_i, 0 \leq i \leq N$. \(\Box\)

The function $w_i$ is defined as a barrier function for $v_i$ by Lemma 3.

**Lemma 4** (see [4]) Set $Z_i = Z_0 = 1$, define the mesh function $Z_i = \prod_{j=i}^N \left(1 + \beta h_j / (2\varepsilon)\right)^{-1}$ for $i = 1, 2, \ldots, N$, then the operator $L^N_i$ of (5) satisfies $L^N Z_i \geq \frac{CZ_i}{\max \{\varepsilon, h_{i+1}\}}$ for $i = 0, 1, \ldots, N - 1$. \(\Box\)

**Lemma 5** (see [4]) Assuming that $u(x)$ be sufficiently smooth function defined on $[0,1]$, for the truncation error of (5) on the Shishkin mesh to solve problem (1), there exists a constant $C$ such that

\[
|L^N (u_i - u_i^N)| \leq C \left[ \int_{x_{i-1}}^{x_i} \varepsilon |u''(x)| dx + h_i \int_{x_{i-1}}^{x_i} |u''(x)| dx + h_{i+1} \int_{x_{i+1}}^{x_i} |u''(x)| dx \right]
\]

and

\[
|L^N (u_i - u_i^N)| \leq C \left[ \int_{x_{i-1}}^{x_i} \varepsilon |u''(x)| dx + \int_{x_{i+1}}^{x_i} |u''(x)| dx + h_{i+1} \int_{x_{i+1}}^{x_i} |u''(x)| dx \right].
\]

**Proof.** The results follow by noting that $c(x)u(x)$ contributes
\[ e_{i+1/2} = \left| u(x_{i+1}) - u(x_i) \right| / 2 - u(x_{i+1/2}) \leq C h_{i+1} \int_{x_i}^{x_{i+1}} |u^*(x)| \, dx \]  
\hspace{1cm} (9)

for \( i = 0, 1, \ldots, N \) to the truncation error in the Lemma 2.4 in [4].

Similarly, the numerical solution can also be split into the smooth part and the layer part by

\[ u_i^N = \bar{S}_i^N + E_i^N, \]

\[ \bar{S}_i^N \text{ satisfies } L_i^N S_i^N = f_i, \quad i = 0, 1, \ldots, N - 1, \]

\[ \bar{S}_i^N = \bar{S}_0^N - \varepsilon \left( \bar{S}_i^N - \bar{S}_{i-1}^N \right) / (2h_i) \]

\[ E_i^N = E_0^N - \varepsilon \left( E_i^N - E_0^N \right) / (2h_i) \]

\[ i = 0, 1, \ldots, N - 1. \]

\[ B_i^N \bar{S}_i^N = \bar{S}_0^N - \varepsilon \left( \bar{S}_i^N - \bar{S}_{i-1}^N \right) / (2h_i) = S(0) - \varepsilon S'(0), \]

\[ B_i^N E_i^N = E_0^N - \varepsilon \left( E_i^N - E_0^N \right) / (2h_i) = E(0) - \varepsilon E'(0), \]

\[ \text{for } i = 0, 1, \ldots, N - 1. \]

Therefore

\[ \left| u_i - u_i^N \right| \leq \left| \bar{S}_i - \bar{S}_i^N \right| + \left| E_i - E_i^N \right|. \]  
\hspace{1cm} (10)

**Lemma 6** (see [4]) If \( N > \gamma / \beta \), then for the smooth part of the solutions of problem (1) and (5), there exists constant \( C \) such that \( \left| \bar{S}_i - \bar{S}_i^N \right| \leq C N^{-2} \) for all \( i = 0, 1, \ldots, N - 1. \)

**Lemma 7** (see [4]) For the Shishkin mesh (5) there exists a constant \( C \) such that

\[ \prod_{j=1}^{i} \left( 1 + \frac{\beta h_j}{2\varepsilon} \right)^{-1} \leq C N^{-4i/N} \text{ for } 0 \leq i \leq N/2. \]  
\hspace{1cm} (11)

**Lemma 8** (see [4]) If \( N > \gamma / \beta \), then for the layer part of the numerical solution from (5) for problem (1), there exists a constant \( C \) such that

\[ |E_i^N| \leq C N^{-2} \text{ for } i = N/2, \ldots, N. \]  
\hspace{1cm} (12)

**Lemma 9** (see [4]) If \( N > \gamma / \beta \), then for the layer part of the solutions of problem (1) and (5), there exists a constant \( C \) such that

\[ |E_i - E_i^N| \leq |E_i| + |E_i^N| \leq C N^{-2} \text{ for } i = N/2, \ldots, N. \]  
\hspace{1cm} (13)

**Lemma 10** (see [4]) If \( N > \gamma / \beta \), then for the layer part of the solutions of problem (1) and (5) on the Shishkin mesh, there exists a constant \( C \) such that

\[ |E_i - E_i^N| \leq C \max \left\{ N^{-2}, N^{-4i/N} \right\} \text{ for } i = 0, 1, \ldots, N/2 - 1. \]  
\hspace{1cm} (14)

**Theorem 1** Assuming that \( N > \gamma / \beta \), the midpoint upwind scheme (5) on the Shishkin mesh (4) for problem (1) satisfies:

\[ \left| u_i - u_i^N \right| \leq C \max \left\{ N^{-2}, N^{-4i/N} \right\} \ln N \text{ for } i = 0, 1, \ldots, N. \]  
\hspace{1cm} (15)

Furthermore,

\[ \left| u_i - u_i^N \right| \leq \begin{cases} CN^{-1} \ln N, & 0 \leq i < p_m N, \\ CN^{-2}, & p_m N \leq i \leq N, \end{cases} \]  
\hspace{1cm} (16)

where \( p_m = 1/(4e) + 1/4 \approx 0.3420 \).

**Proof.** From (10) and Lemmas 6, 9, and 10, we have the error estimate (15).

Furthermore, since \( N^{-4i/N} \ln N = N^{-2} N^{-4i/N} \ln N \), we consider the function

\[ f(x) = x^{-4p_m} \ln x, x > 1. \]

From \( f'(x) = (-4p_m + 1)x^{-4p_m} \ln x + x^{-4p_m} = x^{-4p_m}((-4p_m + 1) \cdot \ln x + 1) = 0 \), we have \( x = e^{4p_m - 1}. \) So

\[ \max_{x > 1} \{ f(x) \} = e^{-1} \cdot \frac{1}{4p_m - 1}. \]

then
\[ N^{-4p_m+1} \ln N \leq e^{-1} \cdot \frac{1}{4p_m-1} = 1. \]

By using \( N^{-(1+i/N)} \ln N \leq N^{-1} \ln N \) for \( 0 \leq i < p_m h \) and \( N^{-4i/N+1} \ln N \leq N^{-4p_m+1} \ln N \leq 1 \) for \( p_m N \leq i \leq N \) in (15), (16) is proved. □

The Bakhvalov-Shishkin mesh is as follows:
\[
x(i) = \begin{cases} 
-4 \epsilon / \beta \cdot \ln \left[1 - 2i \left(1 - N^{-1}\right) / N\right], & 0 \leq i \leq N/2 - 1, \\ 
\tau + (1-\tau) (2i-N) / N, & N/2 \leq i \leq N.
\end{cases}
\]

We can also prove that the midpoint upwind scheme on the Bakhvalov-Shishkin mesh has the uniform convergence:
\[
|u_i - u_i^N| \leq \begin{cases} 
CN^{-1}, & 0 \leq i \leq N/2 - 1, \\ 
CN^{-2}, & N/2 \leq i \leq N.
\end{cases}
\]

Further, we construct the hybrid finite difference scheme that combines the midpoint upwind scheme on the coarse part with the central difference scheme on the fine part for problem (1):
\[
\begin{align*}
L_i^N u_i^N & := -\epsilon D_i^x D_i^x u_i^N - b_i D_i^x u_i^N + c_i u_i^N = f_i, & 0 < i \leq N/2 - 1, \\
-\epsilon D_i^x D_i^x u_i^N - b_i D_i^x u_i^N + c_i u_i^N + c_{i+1/2} u_{i+1/2}^N = f_{i+1/2}, & N/2 \leq i < N,
\end{align*}
\]
(19)

where \( D_i^0 u_i^N := u_i^N - u_i^{N-1} / \left(h_i + h_{i+1}\right) \).

Assuming that \( N > \gamma^* / \beta \) and \( N / \ln N > 4 \beta^* / \beta \), the coefficient matrix associated with (19) is also an M-matrix and a discrete comparison principle holds. By using barrier functions, we can obtain the higher-order error estimate of the hybrid scheme on the Shishkin mesh for problem (1) in the following:
\[
|u_i - u_i^N| \leq \begin{cases} 
CN^{-2} \ln N, & 0 \leq i < p_m N, \\ 
CN^{-2}, & p_m N \leq i \leq N,
\end{cases}
\]
(20)

where \( p_m = 1/(2e) \approx 0.1839 \).

4. Numerical example
The numerical results are illustrated in tables 1-2. The numerical convergence order is computed by
\[
\log_2 \left( \frac{\max_{0 \leq i \leq p_m n} |u_i - u_i^N|}{\max_{0 \leq i \leq p_m n} |u_i - u_i^{2n}|} \right) \quad \text{with} \quad p_m = 1/(4e) + 1/4 \quad \text{and the numerical convergence constant on the S mesh is computed by} \quad \max_{0 \leq i \leq p_m n} |u_i - u_i^N| \left/N^{-1} \ln N \right., \quad \text{with the corresponding formulas for} \quad i > p_m N.
\]

Similar formulas are used for the midpoint upwind scheme on the Bakhvalov-Shishkin mesh and the hybrid difference scheme on the Shishkin mesh.

**Example 1** Consider the following linear singular perturbation convection-diffusion Robin problem:
\[
\begin{align*}
-\epsilon u''(x) - \frac{1}{2-x} u'(x) + \frac{1}{3-x} u(x) &= f(x), \\
u(0) - \epsilon u'(0) &= (e+1)(e+1) + 2u(1) = 1 + 2^{-1/e},
\end{align*}
\]
(21)

where \( f(x) \) is chosen such that \( u(x) = e^{1-x} + 2^{-1/e} (2-x)^{1/e} \) is the exact solution.
The convergence orders of them both are superior to those of the simple central discretization. The numerical results of \( \beta = 0.9, \varepsilon = 10^{-10} \).

| \( N \) | \( i \leq p_{N} \) | order | const | \( i > p_{N} \) | order | const | \( i \leq N/2 \) | order | const | \( i > N/2 \) | order | const |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 0.4462 | —— | 2.575 | 0.0021 | —— | 0.5249 | 0.2661 | —— | 4.257 | 0.0021 | —— | 0.5249 |
| 32 | 0.3456 | 0.369 | 3.191 | 5.6535e−4 | 1.893 | 0.5791 | 0.1442 | 0.884 | 4.615 | 5.6535e−4 | 1.893 | 0.5791 |
| 64 | 0.2276 | 0.603 | 3.502 | 1.5338e−4 | 1.883 | 0.6282 | 0.0739 | 0.964 | 4.730 | 1.4825e−4 | 1.932 | 0.6072 |
| 128 | 0.1485 | 0.616 | 3.918 | 3.8790e−5 | 1.983 | 0.6355 | 0.0372 | 0.990 | 4.755 | 3.7939e−5 | 1.966 | 0.6216 |
| 256 | 0.0903 | 0.712 | 4.170 | 3.8790e−5 | 1.998 | 0.6361 | 0.0186 | 0.984 | 4.757 | 9.5951e−6 | 1.983 | 0.6288 |
| 512 | 0.0527 | 0.777 | 4.327 | 9.7056e−6 | 1.999 | 0.6361 | 0.0093 | 0.984 | 4.754 | 2.4126e−6 | 1.992 | 0.6325 |
| 1024 | 0.0300 | 0.813 | 4.430 | 6.0664e−7 | 2.000 | 0.6361 | 0.0046 | 1.016 | 4.751 | 6.0490e−7 | 1.996 | 0.6343 |
| 2048 | 0.0167 | 0.845 | 4.489 | 1.5166e−7 | 2.000 | 0.6361 | 0.0023 | 1.016 | 4.750 | 1.5144e−7 | 1.998 | 0.6352 |

5. Conclusion remarks

In this paper, we adopt a second-order central divided difference to discretize the first derivative in the Robin boundary condition to guarantee an associated M-matrix and the discrete maximum principle. Similar to the related results of the two-point Dirichlet boundary value problem, the midpoint upwind scheme on the B-S mesh achieves higher-order uniform convergence than on the S mesh for the Robin boundary value problem. The convergence orders of them both are superior to those of the simple upwind scheme on the corresponding meshes. Further, we also construct the new hybrid difference scheme combining the midpoint upwind scheme with central difference scheme to improve the convergence order on the S mesh for the Robin boundary value problem. The numerical experiment indicates that the error estimates of the proposed methods are sharp and the high accuracy is achieved for the boundary layer.

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