On the primefulness of local cohomology modules

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Abstract
Let $R$ be a commutative Noetherian ring with identity $1 \neq 0$. For a non-zero $R$-module $M$. We prove that a multiplication primeful $R$-module $M$ and $H^k_I(M)$ are $I$-cofinite and primeful, for each $k > 0$ where $I$ is an ideal of $R$ with $I \subseteq \text{Ann}(M)$. As a consequence, we deduce that, if $M$ and $N$ are multiplication primeful $R$-modules, then $\text{Ext}^k_R(M, N)$ is primeful. Another result is, for a projective $R$-module $M$ over an integral domain, $M$ admits projective resolution $P^*$ such that each $P_i$ is primeful (faithfully flat).

Keywords:
Local cohomology modules, Minimax modules and Primeful modules

Introduction:
In this paper, $R$ is a commutative Noetherian ring with identity $1 \neq 0$ until otherwise stated and $M$ is a non-zero $R$-module. A submodule $N$ of an $R$-module $M$ is called prime, if $rm \in N$, for each $r \in R$ and $m \in M$ then $m \in N$ or $r \in (N:_RM)$, in this case $(N:_RM)$ is a prime ideal of $R$ and $N$ is called $p$-prime. Consider $\rho : \text{Spec}(M) \to \text{Spec}(R/\text{Ann}(M))$ such that $\rho(P) = (P:M)/\text{Ann}(M)$ for all $P \in \text{Spec}(M)$ is called the natural map of $\text{Spec}(M)$[5]. A non-zero $R$-module $M$ is called primeful if $\rho$ is surjective. Chin Pi Lu [5, Theorem 2.2] showed that every finitely generated $R$-module is primeful but the converse is not true in general, for example every infinite dimensional vector space is primeful.

A Primeful $R$-modules are generalization of finitely generated $R$-modules. Many results for finitely generated modules are generalized to primefuls, the most important one is the Nakayama's Lemma and the equality $\text{Supp}(M) = V(\text{Ann}(M))$ for $M$ [5].

It is well-known that, if $F = \{F^k, \alpha^k\}$ is a cochain complex, then $H^k_R(M) = \ker \alpha^k/\text{Im} \alpha^{k-1}$ is $k$-th cohomology module of $F$ [4]. The $k$-th local cohomology module of $M$ with respect to an ideal $I \subseteq \text{Ann}(M)$ is $\lim_k \text{Ext}^k_R(R/I^k, M)$ [2]. An $R$-module $M$ is called $I$-cofinite if $\text{Supp}(M) \subseteq V(I)$ and $\text{Ext}^k_R(R/I, M)$ is finitely generated for all $k$ [2].
On the other hand, Sean Sather-Wagstaff [6] proved that if $R$ is a commutative ring and $M$ is an $R$-module, then $M$ admits a free (hence projective) resolution $P^+$ over $R$. Also if $M$ is finitely generated then each $P_i$ of $P^+$ is finitely generated over $R$.

The main purpose of this article, is changing the direction of study by using cohomology facts. We prove that, if $M$ is a projective module over an integral domain, then $M$ admits a free (hence projective) resolution $P^+$ over $R$ such that each $P_i$ is primeful (faithfully flat) over $R$. In [5] it is shown that a submodule of primeful module need not be primeful. One of the results in this paper, is if we have PID then the following are equivalent for a projective module $P$:

1. $P$ is projective,
2. $P$ is primeful,
3. There exist a primeful module such that every submodule is primeful.

In section two we give a condition that help $\text{Ext}_R^i(R/I, \Gamma_I(M))$ and $\text{Tor}_i^R(R/I, \Gamma_I(M))$ for a primeful $R$-module to be primeful for all $i$.

**The Results**

In this section, we prove that if $M$ is an $R$-module over PID, then $M$ admits free (hence projective) resolution $P^+$ over $R$ such that each $P_i$ is primeful over $R$. Also find a primeful module that every submodule of it is primeful.

**Lemma 2.1.** Let $M$ be a projective module over an integral domain, then $M$ admits a free (hence projective) resolution $P^+$ over $R$ such that each $P_i$ is primeful over $R$.

**Proof.** It is well-known that if $R$ is a commutative ring and $M$ is an $R$-module, then $M$ admits a free (hence projective) resolution over $R$. A projective module over an integral domain is primeful [5, corollary 4.3]. In this case $M$ admits a free (hence projective) resolution $P^+$ over $R$ such that each $P_i$ is primeful (faithfully flat) over $R$.

An $R$-module $M$ is called multiplication if every submodule $N = IM$ where $I$ is an ideal of $R$.

In [5, Theorem 2.2] showed that every finitely generated module $M$ is primeful, consequently the quotient module $M/N$ for any submodule $N$ of $M$. For a multiplication module we have some other results, we start with Lemma 2.2.

**Lemma 2.2.** Let $M$ be a multiplication $R$-module and $0 \to L \to M \to N \to 0$ be a short exact sequence, then $M$ primeful if and only if $L$ and $N$ are primeful.

**Proof.** Suppose that $M$ is a multiplication primeful module, we consider $L$ as a submodule of $M$ and $N = M/L$, so by [1, proposition 3.8] $M$ is finitely generated and hence $L$ is also finitely generated which implies that $L$ and $M/L$ are finitely generated hence primeful [5, Theorem 2.2].

Conversely, suppose that $L$ and $M/L$ are primeful then they are finitely generated, so $M$ is also finitely generated which implies that $M$ is primeful.

It is proved that in [1] that a submodule of a primeful module need not be primeful. In Theorem 2.3 we give the condition under which a submodule of primeful module is primeful.
**Theorem 2.3.** For a projective $R$-module $P$ over PID $R$ the following are equivalent:

1. $P$ is projective
2. $P$ is primeful
3. There exist a primeful module such that every submodule is primeful.

**Proof.** Suppose that $P$ is a projective module, then by [1] projective modules over an integral domain is primeful. To prove (3), it is well known that, for a projective $R$-module there exist a free $R$-module $F$ such that $P$ is a direct sum of $F$ [6]. Now, free modules over PID are primeful and $F = X \oplus P$ which implies that $F$, $X$, $P$ and all other submodules of $F$ are primeful.

Recall (Schanuel’s Lemma [6]: Let $R$ be a commutative ring, and let $M$ be an $R$-module. Consider two exact sequences

\[
0 \to K \to P_t \to P_{t-1} \to \cdots \to P_1 \to P_0 \to M \to 0 \\
0 \to L \to Q_t \to Q_{t-1} \to \cdots \to Q_1 \to Q_0 \to N \to 0
\]

such that each $P_i$ and $Q_i$ is projective. Then $K$ is projective if and only if $L$ is projective.

Now by using (Schanuel’s Lemma) and applying Theorem 2.3 we can prove the following corollary.

**Corollary 2.4.** Let $R$ be an integral domain. Consider two exact sequence:

\[
0 \to K \to P_t \to P_{t-1} \to \cdots \to P_1 \to P_0 \to M \to 0 \\
0 \to L \to Q_t \to Q_{t-1} \to \cdots \to Q_1 \to Q_0 \to N \to 0
\]

Where each $P_i$ and $Q_i$ are projective then:

1. $K \oplus Q_0 \cong L \oplus P_0$
2. $K$ is primeful if and only if $L$ is primeful.

**Proof.** By (Schanuel’s Lemma) we have each $P_i$ and $Q_i$ are projective and $R$ be an integral domain. Hence 1 and 2 are satisfying.

**Proposition 2.5.** If $M$ is a multiplication primeful module, then $\text{Ext}_R^i(\frac{R}{I}, \Gamma_i(M))$ and $\text{Tor}_i^R(R/I, \Gamma_i(M))$ are primeful for all $i$.

**Proof.** Directly by Lemma 2.2.

In [5, Proposition 3.8] it is provide that for a non-zero $R$-module $M$ the following are equivalent:

1. $M$ is finitely generated
2. $M$ is primeful
3. $\text{Supp}(M) = V(\text{Ann}(M))$
4. $pM: M = p$ for every $p \in V(\text{Ann}(M))$
5. $pM \neq M$ for every $p \in V(\text{Ann}(M))$.

**Proposition 2.6.** Let $M$ and $N$ be two multiplication primeful modules, then $\text{Ext}_R^i(M, N)$ is primeful for each $i$.

**Proof.** Since we have $M$ and $N$ two multiplication primeful modules, hence by [5, proposition 3.8] they are finitely generated. On the other hand, [6, Proposition IV 3.9] shows that for a commutative Noetherian ring, if $M$ and $N$ are finitely generated, then $\text{Ext}_R^i(M, N)$ is finitely generated for each $i$. Thus by [5, Theorem 2.2] $\text{Ext}_R^i(M, N)$ is primeful for each $i$. 

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In the following result we provide a condition under which a primeful R-module \( M \) and the local cohomology \( H^i_R(M) \) are \( I \)-cofinite for each \( i \).

**Proposition 2.7.** Suppose that \( M \) is a multiplication primeful R-module, then \( M \) and the local cohomology \( H^i_R(M) \) are also \( I \)-cofinite for each \( i \).

**Proof.** By [5, proposition 3.8], \( \text{Supp}(M) = V(\text{Ann}(M)) \).

In [5] shown that, if \( M \) is a multiplication module then primeful and finitely generated modules are equivalent. Thus \( M \) is \( I \)-cofinite. Similar argument is true for \( H^i_R(M) \) [2].

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