WONG-ZAKAI APPROXIMATIONS AND ATTRACTORS FOR STOCHASTIC WAVE EQUATIONS DRIVEN BY ADDITIVE NOISE

XIANGHU WANG
Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064, China

DINGSHI LI
School of Mathematics
Southwest Jiaotong University
Chengdu, Sichuan 61003, China

JUN SHEN*
Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064, China

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Abstract. In this paper, we study the Wong-Zakai approximations given by a stationary process via Euler approximation of Brownian motion and the associated long term behavior of the stochastic wave equation driven by an additive white noise on unbounded domains. We first prove the existence and uniqueness of tempered pullback attractors for stochastic wave equation and its Wong-Zakai approximation. Then, we show that the attractor of the Wong-Zakai approximate equation converges to the one of the stochastic wave equation driven by additive noise as the correlation time of noise approaches zero.

1. Introduction. The idea of using deterministic differential equations to approximate the solutions of the stochastic differential equations was introduced by Wong and Zakai in their pioneer works [39, 40], where they studied both piecewise linear approximations and piecewise smooth approximations for one-dimensional Brownian motions. It was proved that the solution of a scalar random differential equation with approximations of Brownian motion converges to that of a stochastic differential equation (SDE) with white noise in Stratonovich sense. Their work was later extended by many authors, see, e.g., [27, 28, 5, 26, 16, 18] and the references therein.

A very natural question is about the dynamical behavior. Recently, the idea of the Wong-Zakai approximations has been used to investigate the dynamics of...
stochastic equations, including random attractors, invariant manifolds and foliations for stochastic differential equations, see, e.g., [15, 11, 17, 41, 13, 14, 21, 24, 22, 20, 36]. There are two processes used to approximate the Brownian motion: Euler approximation and colored noise. For the details of Euler approximation please see Section 2. The colored noise is given by \( \int_0^t y_\delta(\theta, \omega) \, d \omega \) for any \( \delta \neq 0 \), where

\[
y_\delta(\omega) = \frac{1}{\delta} \int_{-\infty}^{0} e^s \, d w = -\frac{1}{\delta^2} \int_{-\infty}^{0} e^s \omega(s) \, d s.
\]

The colored noise can be considered as an approximation of the Brownian motion, see Lemma 2.2 of [15, 13], also [23].

To investigate the dynamics of SDEs for almost all sample paths, including the random attractors, invariant manifolds and foliations. First of all, we must define a random dynamical system (or cocycle) based on the solution operator of the equations. Although it is well-known that a large class of partial differential equations with stationary random coefficients and Itô SDEs (for details see Arnold [1]), this problem is still open for stochastic partial differential equations (SPDEs) (see, e.g., [9]). Because it is not known how to extend Kolmogorov’s theorem to an infinite dimensional parameter range, which would be appropriate for dealing with SPDEs [12, 8]. More precisely, the stochastic integral is only defined almost surely where the exceptional set may depend on the initial condition. But it is complicated to generate a random dynamical system (or cocycle) if more than countable many exceptional sets may appear. Nevertheless, SPDEs can generate a random dynamical system (or cocycle) for very special noise terms, either additive noise or linear multiplicative noise[7, 6, 10, 9]. These special noises make it possible to transform such SPDEs into a random partial differential equation. For SPDEs driven by nonlinear multiplicative noises, Wong-Zakai type approximation noises are recently used. The approximate equation can generate a random dynamical system (or cocycle) for a wide class of nonlinearity, which is in sharp contrast with the original SPDEs. The solutions, random attractors, invariant manifolds and foliations of the approximate equation converge to the original one in some sense [15, 17, 41, 13, 23, 14, 21, 24, 11, 22, 20, 36].

The random attractors of the wave equation have been studied extensively in the literature (see, e.g., [35, 30, 42, 37] for additive noise and [33, 34, 38, 19] for linear multiplicative noise). In this paper, we will consider the Wong-Zakai approximations of white noise by a stationary stochastic process, and study the Wong-Zakai approximations of the following stochastic wave equation:

\[
u_{tt} + \alpha u_t - \Delta u + \nu u + f(x, u) = g(t, x) + h(x) \frac{d w(t)}{d t}, \quad t > \tau, \ x \in \mathbb{R}^n,
\]

with initial condition

\[
u(\tau, x) = u_\tau(x), \quad u_t(\tau, x) = u_{1, \tau}(x), \quad x \in \mathbb{R}^n,
\]

where \( 1 \leq n \leq 3, \ \tau \in \mathbb{R} \) is the initial time, \( \alpha, \nu \) are positive constants, \( f \) is a nonlinear function satisfying certain conditions, \( g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n)) \), \( h \in H^1(\mathbb{R}^n) \) and \( w = w(t, \omega) \) is a one-dimensional two-sided real-valued Brownian motion. When \( n = 3 \), we will consider a nonlinearity \( f \) with cubic growth in its second argument which is referred to as the critical exponent.

The main purpose of the present paper is to investigate the asymptotic behavior of the solutions of (1) and its Wong-Zakai approximation. Since the similar
properties of Euler approximation and colored noise, we will focus on the Euler approximation. We remark that our results also hold for colored noise. More precisely, we will show that both (1) and its Wong-Zakai approximation have unique tempered pullback attractors in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. We will further prove the convergence of pullback attractors of Wong-Zakai approximation equation as the correlation time of noise approaches zero, and show the limit is the attractor of (1) in the sense of Hausdorff semidistance. Note that the considered equation is defined on the entire space $\mathbb{R}^n$. This introduces some obstacles for proving the asymptotic compactness of the solutions since the Sobolev embeddings on unbounded domains are not compact. On the other hand, we will deal with the critical nonlinear drift term $f$ when $n = 3$. Such a critical $f$ introduces further difficulty for proving the asymptotic compactness of the solution operator even in the case of bounded domains (see, e.g., [2]). In the present paper, we will employ Ball’s idea [2] of energy equations as well as their values from line to line.

The letters $\{\theta\}$ are generic positive constants which may change their values from line to line.}

2. Pullback attractor. In this section, we first define continuous cocycles for (1) as well as its Wong-Zakai approximation, and then prove the existence of pullback attractors.

Denote by $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$. Let $\mathcal{F}$ be the Borel $\sigma$-algebra induced by the compact-open topology of $\Omega$, and $\mathbb{P}$ the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Let $\{\theta_t\}_{t \in \mathbb{R}}$ be the group on $(\Omega, \mathcal{F}, \mathbb{P})$ given by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}.$$  

Then $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ is a metric dynamical system (see [1]) and there exists a $\{\theta_t\}_{t \in \mathbb{R}}$-invariant subset $\tilde{\Omega} \subseteq \Omega$ of of full measure such that for each $\omega \in \tilde{\Omega}$,

$$\frac{\omega(t)}{t} \to 0 \quad \text{as} \quad t \to \pm \infty. \quad (3)$$

For the sake of convenience, from now on, we will write the space $\tilde{\Omega}$ as $\Omega$.

Let $F(x,u) = \int_0^u f(x,s) ds$. We assume $f$ and $F$ satisfy the following conditions, for every $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$

$$|f(x,s)| \leq \alpha_1 |s|^p + \varphi_1(x), \quad (4)$$

$$f(x,s) s - \alpha_2 F(x,s) \geq \varphi_2(x), \quad (5)$$

$$F(x,s) \geq \alpha_3 |s|^{p+1} - \varphi_3(x), \quad (6)$$

$$|\partial_s f(x,s)| \leq \alpha_4 |s|^{p-1} + \varphi_4(x), \quad (7)$$

where $p > 1$ for $n = 1, 2$ and $p \in (1,3]$ for $n = 3$, $\alpha_i$ are positive constants for $i = 1, 2, 3, 4$, $\varphi_1 \in L^2(\mathbb{R}^n)$ and $\varphi_2, \varphi_3 \in L^1(\mathbb{R}^n)$, and $\varphi_4 \in H^1(\mathbb{R}^n)$.

We remark that $p = 3$ is the so-called critical exponent of the wave equation when $n = 3$. Notice that (4) and (5) imply that there exists a positive constant $c$ such that

$$F(x,s) \leq c(|s|^2 + |s|^{p+1} + \varphi_1^2(x) + \varphi_2(x)), \quad (8)$$

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which is useful when deriving uniform estimates of solutions. Let $\beta$ be a fixed positive constant such that
\[ \alpha - \beta > 0, \quad \nu + \beta^2 - \alpha \beta > 0. \] (9)

For convenience, we let
\[ \lambda = \min\{\frac{\beta}{4}, \alpha - \beta, \alpha_2 \beta \}. \] (10)

The following condition will be needed for $g$ when deriving uniform estimates of solutions:
\[ \int_{-\infty}^{0} e^{p^{\frac{\beta}{2}}s} \|g(s + \tau, \cdot)\|^2 ds < \infty, \quad \forall \tau \in \mathbb{R}. \] (11)
Since $p > 1$, by (11) we find that for every $\tau \in \mathbb{R}$,
\[ \int_{-\infty}^{0} e^{\frac{\beta}{p}s} \|g(s + \tau, \cdot)\|^2 ds < \infty, \quad \forall \tau \in \mathbb{R}. \] (12)
and
\[ \lim_{k \to +\infty} \int_{-\infty}^{0} e^{\frac{\beta}{p}s} \int_{|x| \geq k} |g(s + \tau, x)|^2 dx ds = 0. \] (13)
Given a bounded nonempty subset $D$ of $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, we write $\|D\| = \sup_{\psi \in D} \|\psi\|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}$. Let $D$ be a family of nonempty subsets of $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,
\[ \lim_{s \to +\infty} e^{-\lambda s} \|D(\tau - s, \theta_{-s}\omega)\|^{p+1} = 0. \] (14)
Let $D$ be the collection of all such families, that is,
\[ D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies (14)}. \] (15)
It is clear that $D$ is inclusion-closed.

2.1. **Stochastic wave equations driven by an additive noise.** Let $z = u + \beta u$, where $\beta$ is a nonnegative number satisfies (9). By (1) we get
\[ \frac{du}{dt} + \beta u = z, \] (16)
\[ \frac{dz}{dt} + (\alpha - \beta) z + (\nu + \beta^2 - \alpha \beta) u - \Delta u + f(x, u) = g(t, x) + h(x) \frac{dw(t)}{dt}, \] (17)
with initial conditions
\[ u(\tau, x) = u_\tau(x), \quad z(\tau, x) = z_\tau(x), \quad x \in \mathbb{R}^n, \] (18)
where $z_\tau = u_{1, \tau} + \beta u_{\tau}$. Let $v(t, \tau, \omega) = z(t, \tau, \omega) - h(x) \omega(t)$. By (16)-(18) we have
\[ \frac{du}{dt} + \beta u - v = h(x) \omega(t), \] (19)
\[ \frac{dv}{dt} + (\alpha - \beta) v + (\nu + \beta^2 - \alpha \beta) u - \Delta u + f(x, u) = g(t, x) + (\beta - \alpha) h(x) \omega(t), \] (20)
with initial conditions
\[ u(\tau, x) = u_\tau(x), \quad v(\tau, x) = v_\tau(x), \quad x \in \mathbb{R}^n, \] (21)
where $v_\tau = z_\tau - h(x) \omega(\tau)$. Given $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $(u_\tau, z_\tau) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, by (4)-(7), one can show that system (19)-(21) has a unique solution $(u(\tau, \tau, \omega, u_\tau), v(\tau, \tau, \omega, v_\tau)) \in C([\tau, \infty), H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$. In addition, it is $(\mathcal{F}, \mathcal{B}(H^1(\mathbb{R}^n)) \times \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable with respect to $\omega \in \Omega$ and continuous in
with system (16)-(18), where for $t$ in continuous cocycle $\Phi$ associated with equation Lemma 2.1. Suppose $v$ where $L$ is given by for every $\tau \in \Omega$ and $\omega \in \Omega$

$$
\Phi_0(t, \tau, \omega, \langle u, z \rangle) = \langle u(t + \tau, \tau, \theta_{-\tau} \omega, u\tau), z(t + \tau, \tau, \theta_{-\tau} \omega, z\tau) \rangle
$$

$$
= \langle u(t + \tau, \tau, \theta_{-\tau} \omega, u\tau), v(t + \tau, \tau, \theta_{-\tau} \omega, v\tau) + h(x)(\omega(t) - \omega(-\tau)) \rangle,
$$

(22)

where $v_{\tau} = z_{\tau} + h(x)\omega(-\tau)$.

**Lemma 2.1.** Suppose (4)-(7) and (11) hold. Then the continuous cocycle $\Phi_0$ associated with equation (16)-(18) has a closed measurable $D$-pullback absorbing set $K_0 = \{K_0(\tau, \omega) : \tau \in \Omega, \omega \in \Omega \} \in D$, which is given by for every $\tau \in \Omega$ and $\omega \in \Omega$

$$
K_0(\tau, \omega) = \{(u, z) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) : \|u\|^2_{H^1(\mathbb{R}^n)} + \|z\|^2 \leq L_0(\tau, \omega)\},
$$

(23)

where $L_0(\tau, \omega)$ is given by

$$
L_0(\tau, \omega) = c \int_{-\infty}^{\theta} e^{2\lambda t} (1 + \|g(r + \tau, \cdot)\|^2 + \|\omega(r) - \omega(-\tau)\|^2 + \|\omega(r) - \omega(-\tau)\|^{p+1}) dr + c\|\omega(-\tau)\|^2
$$

(24)

with $c$ being a positive number independent of $\tau$ and $\omega$.

**Proof.** Taking the inner product of (20) with $v$ in $L^2(\mathbb{R}^n)$, we get

$$
\frac{d}{dt}\left(\|v\|^2 + (\nu + \beta^2 - \alpha\beta)\|u\|^2 + \|\nabla u\|^2 + 2 \int_{\mathbb{R}^n} F(x, u) dx \right)
$$

$$
+ 2\lambda \left(\|v\|^2 + (\nu + \beta^2 - \alpha\beta)\|u\|^2 + \|\nabla u\|^2 + 2 \int_{\mathbb{R}^n} F(x, u) dx \right)
$$

$$
+ 2\lambda \left(\|v\|^2 + (\nu + \beta^2 - \alpha\beta)\|u\|^2 + \|\nabla u\|^2 \right)
$$

$$
\leq c(1 + \|g(t, \cdot)\|^2 + |\omega(t)|^2 + |\omega(t)|^{p+1}).
$$

(25)

Multiplying (25) by $e^{2\lambda t}$, replacing $\omega$ by $\theta_{-\tau} \omega$ and then integrating over $(\tau - t, \tau)$ with $t \in \mathbb{R}^+$, we get for every $\omega \in \Omega$

$$
\|v(\tau, \tau - t, \theta_{-\tau} \omega, v_{\tau-t})\|^2 + (\nu + \beta^2 - \alpha\beta)\|u(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2
$$

$$
+ \|\nabla u(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2 + 2 \int_{\mathbb{R}^n} F(x, u(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})) dx
$$

$$
+ 2\lambda \int_{\tau-t}^{\tau} e^{2\lambda(r-\tau)} \left(\|v(r, \tau - t, \theta_{-\tau} \omega, v_{\tau-t})\|^2 + \|\nabla u(r, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2
$$

$$
+ (\nu + \beta^2 - \alpha\beta)\|u(r, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2 \right) dr
$$

$$
\leq e^{-2\lambda t}(\|v_{\tau-t}\|^2 + (\nu + \beta^2 - \alpha\beta)\|u_{\tau-t}\|^2 + \|\nabla u_{\tau-t}\|^2 + 2 \int_{\mathbb{R}^n} F(x, u_{\tau-t}) dx
$$

$$
+ c \int_{\tau-t}^{\tau} e^{2\lambda(r-\tau)}(1 + \|g(r, \cdot)\|^2 + |\omega(r-\tau) - \omega(-\tau)|^2 + |\omega(r-\tau) - \omega(-\tau)|^{p+1}) dr.
$$

(26)
For every \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \subset \mathcal{D} \) such that \((u_{\tau-t}, v_{\tau-t}) \in D(\tau-t, \theta^{-t} \omega)\), we find from (8)
\[
e^{-2M}((v_{\tau-t})^2 + (\nu + \beta^2 - \alpha \beta)\|u_{\tau-t}\|^2 + ||\nabla u_{\tau-t}||^2 + 2 \int_{\mathbb{R}^n} F(x, u_{\tau-t})dx)
\leq e^{-2M}(1 + ||v_{\tau-t}||^2 + ||u_{\tau-t}||_{H^1(\mathbb{R}^n)}^2 + ||u_{\tau-t}||^2_{H^1(\mathbb{R}^n)})
\leq e^{-2M}(1 + ||D(\tau-t, \theta^{-t} \omega)||^2 + ||D(\tau-t, \theta^{-t} \omega)||^2_{p+1}) \rightarrow 0 \quad \text{as} \quad t \to +\infty. \quad (27)
\]
From (26) and (27), there exists \( T_1 = T_1(\tau, \omega, D) > 0 \) such that for all \( t \geq T_1 \),
\[
\|v(\tau-t, \omega, v_{\tau-t})\|^2 + (\nu + \beta^2 - \alpha \beta)\|u(\tau-t, \theta^{-t} \omega, u_{\tau-t})\|^2
+ ||\nabla u(\tau-t, \theta^{-t} \omega, u_{\tau-t})||^2 + 2 \int_{\mathbb{R}^n} F(x, u(\tau-t, \theta^{-t} \omega, u_{\tau-t}))dx
+ 2\lambda \int_{\tau-t}^\tau e^{2\lambda(\tau-t)}(||v(r, \tau-t, \theta^{-t} \omega, v_{\tau-t})||^2 + ||\nabla u(r, \tau-t, \theta^{-t} \omega, u_{\tau-t})||^2)
+ (\nu + \beta^2 - \alpha \beta)\|u(r, \tau-t, \theta^{-t} \omega, u_{\tau-t})||^2)dr
\leq c(1 + R_0(\tau, \omega)),
\]
where
\[
R_0(\tau, \omega) = \int_{-\infty}^0 e^{2\lambda t} \big(1 + ||g(t, \cdot)||^2 + ||\omega(t)||^2 \big)dt \quad (29)
\]
Note that the integral in (29) is convergent due to (11) and (3). By (6) and (28), we have for all \( t \geq T_1 \)
\[
\|u(\tau-t, \theta^{-t} \omega, u_{\tau-t})\|^2 \leq c(1 + R_0(\tau, \omega)).
\]
On the other hand, (22) yields that
\[
\|\Phi_0(t, \tau-t, \theta^{-t} \omega, (u_{\tau-t}, z_{\tau-t}))\|^2_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}
\leq \|u(\tau-t, \theta^{-t} \omega, u_{\tau-t})\|^2_{H^1(\mathbb{R}^n)} + 2\|v(\tau-t, \theta^{-t} \omega, v_{\tau-t})\|^2
+ 2\|h\|^2 ||\omega(\cdot)||^2.
\]
For any \( \hat{D}(\tau-t, \theta^{-t} \omega) \in \mathcal{D} \), there is \( D(\tau-t, \theta^{-t} \omega) \in \mathcal{D} \) such that \((u_{\tau-t}, v_{\tau-t}) \in D(\tau-t, \theta^{-t} \omega)\) if \((u_{\tau-t}, z_{\tau-t}) \in \hat{D}(\tau-t, \theta^{-t} \omega)\). Then, by (30) and (31), we have for all \( t \geq T_1 \)
\[
\Phi_0(t, \tau-t, \theta^{-t} \omega, \hat{D}(\tau-t, \theta^{-t} \omega)) \subseteq K_0(\tau, \omega),
\]
where \( K_0(\tau, \omega) \) is given by (23). By using (3) and (13), one can easily check that \( K_0 \) is tempered, which along with (32) completes the proof.
where \( (u_{\tau-t}, v_{\tau-t}) \in D(\tau-t, \theta_{-\tau} \omega) \) and \( c \) is a positive constant independent of \( \varepsilon \).

**Proof.** The proof is similar as the autonomous case in Lemma 2.4 of [30]. We just sketch the proof. Let \( \rho \) be a smooth function defined on \( \mathbb{R}^+ \) such that \( 0 \leq \rho(s) \leq 1 \) for all \( s \in \mathbb{R}^+ \), and

\[
\rho(s) = \begin{cases} 
0, & \text{for } 0 \leq s \leq 1, \\
1, & \text{for } s \geq 2.
\end{cases}
\]  

(34)

Then there exists a constant \( C_0 \) such that \( |\rho'(s)| \leq C_0 \) for \( s \in \mathbb{R}^+ \). Taking the inner product of (20) with \( \rho(\frac{|x|^2}{k^2})v \) in \( L^2(\mathbb{R}^n) \), we find that there exists \( N_1 = N_1(\varepsilon) \geq 1 \) such that for all \( k \geq N_1 \)

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v|^2 + (\nu + \beta^2 - \alpha \beta)|u|^2 + |\nabla u|^2 + 2F(x, u))dx \\
+ 2\lambda \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v|^2 + (\nu + \beta^2 - \alpha \beta)||u||^2 + \|\nabla u\|^2 + 2F(x, u))dx \\
\leq \varepsilon (\|\nabla u\|^2 + \|v\|^2) + c \int_{|x| \geq k} |g(t, x)|^2 dx + c\varepsilon (1 + |\omega(t)|^2 + |\omega(t)|^{p+1}).
\]  

(35)

Then, it follows from (35) that

\[
\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v(\tau + s, \tau - t, \theta_{-\tau} \omega, v_{\tau-t})|^2 + (\nu + \beta^2 - \alpha \beta) \\
\times |u(\tau + s, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})|^2)dx \\
+ \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|\nabla u(\tau + s, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})|^2 + 2F(x, u(\tau + s, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})))dx \\
\leq e^{-2\lambda(t+s)} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v_{\tau-t}|^2 + (\nu + \beta^2 - \alpha \beta)|u_{\tau-t}|^2 + |\nabla u_{\tau-t}|^2 + 2F(x, u_{\tau-t}))dx \\
+ \varepsilon \int_{\tau-t}^{\tau+t} e^{2\lambda(r-s)} (\|\nabla u(r, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2 + \|v(r, \tau - t, \theta_{-\tau} \omega, v_{\tau-t})\|^2)dr \\
+ c \int_{-\infty}^{0} e^{2\lambda r} \int_{|x| \geq k} |g(r + \tau, x)|^2 dx dr \\
+ c \varepsilon \int_{-\infty}^{0} e^{2\lambda r} (1 + |\omega(r) - \omega(-\tau)|^2 + |\omega(r) - \omega(-\tau)|^{p+1})dr.
\]  

(36)

Since \( (u_{\tau-t}, v_{\tau-t}) \in D(\tau-t, \theta_{-\tau} \omega) \), we find from (27) there exists \( T_1 = T_1(\tau, \omega, D, \varepsilon) > 0 \) such that for all \( t \geq T_1 \),

\[
e^{-2\lambda t} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v_{\tau-t}|^2 + (\nu + \beta^2 - \alpha \beta)|u_{\tau-t}|^2 + |\nabla u_{\tau-t}|^2 + 2F(x, u_{\tau-t}))dx \\
\leq \varepsilon.
\]  

(37)

By (13), there is a \( N_2 = N_2(\tau, \varepsilon) \geq N_1 \) such that for all \( k \geq N_2 \)

\[
\int_{-\infty}^{0} e^{2\lambda r} \int_{|x| \geq k} |g(r + \tau, x)|^2 dx dr \leq \varepsilon.
\]  

(38)
Multiplying (25) by $e^{2\lambda t}$ and then integration over $(\tau - t, \tau + s)$ with $t \in \mathbb{R}^+$ and $s \in [-t, 0]$, we get for every $\omega \in \Omega$

$$
\int_{\tau-t}^{\tau+s} e^{2\lambda(r-\tau-s)}(\|u(r, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2_{H^1(\mathbb{R}^n)} + \|v(r, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2)dr \\
\leq M + Me^{-2\lambda s}R_0(\tau, \omega).
$$

(39)

Then we can get from (36)-(39) that there exists $T_2 = T_2(\tau, \omega, D, \varepsilon) \geq T_1$ such that for $t \geq T_2$, $s \in [-t, 0]$ and $k \geq N_2$

$$
\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) (|v(\tau + s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})|^2 + (\nu + \beta^2 - \alpha \beta)|u(\tau + s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2 \\
+ |\nabla u(\tau + s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2) dx \\
\leq -2\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) F(x, u(\tau + s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})) dx + c\varepsilon\varepsilon
$$

$$
+ c\varepsilon(1 + e^{-2\lambda s}R_0(\tau, \omega)).
$$

(40)

By (6) we have

$$
-\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) F(x, u(\tau + s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})) dx \leq \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \varphi_3(x) dx.
$$

Since $\varphi_3 \in L^1(\mathbb{R}^n)$, there is $N_4 = N_4(\tau, \omega, \varepsilon) \geq N_3$ such that for $k \geq N_4$

$$
-\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) F(x, u(\tau + s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})) dx \leq \varepsilon.
$$

(41)

Therefore it follows from (40)-(41) that for all $t \geq T_2$, $s \in [-t, 0]$ and $k \geq N_4$

$$
\int_{|x| \geq \sqrt{k}} \left(|v(\tau + s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})|^2 + |u(\tau + s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2 \\
+ |\nabla u(\tau + s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2\right) dx \leq c\varepsilon + c\varepsilon e^{-2\lambda s}(1 + R_0(\tau, \omega)),
$$

which concludes the proof.

We now derive an energy equation for problem (19)-(21), which will be used to prove the pullback asymptotic compactness of solutions. To this end, denote by,

$$
E(u, v) = \|v\|^2 + (\nu + \beta^2 - \alpha \beta)\|u\|^2 + \|\nabla u\|^2 + 2 \int_{\mathbb{R}^n} F(x, u) dx
$$

(42)

and

$$
\Gamma_0(u, v) = -2(\alpha - \beta - 2\lambda)\|v\|^2 - 2(\beta - 2\lambda)(\nu + \beta^2 - \alpha \beta)\|u\|^2 \\
- 2(\beta - 2\lambda)\|\nabla u\|^2 - 2\beta(f(x, u), u) + 8\lambda \int_{\mathbb{R}^n} F(x, u) dx \\
+ 2(f(x, u), h)\omega(t) + 2(\nu + \beta^2 - \alpha \beta)(h, u)\omega(t) \\
+ 2(\nabla h, \nabla u)\omega(t) + 2(\beta - \alpha)(h(x), v)\omega(t) + 2(g(t, x), v).
$$

(43)

Then it follows from (19) and (20) that

$$
\frac{d}{dt} E(u, v) + 4\lambda E(u, v) = \Gamma_0(u, v).
$$

(44)
Multiplying (44) by $e^{4\lambda t}$ and then integrating on $(\tau, t)$, we can get

$$E(u(t, \tau, \omega, u_\tau), v(t, \tau, \omega, v_\tau)) = e^{4\lambda(t-\tau)}E(u_\tau, v_\tau) + \int_{\tau-t}^{0} e^{4\lambda s} \Gamma_0(u(s + t, \tau, \omega, u_\tau), v(s + t, \tau, \omega, v_\tau)) ds. \quad (45)$$

Using the energy equation (45), we can get the pullback asymptotic compactness of solutions in the following. Please see [30] and [34] for the details. We have the existence of $D$-pullback attractors for $\Phi_0$.

**Theorem 2.3.** Suppose (4)-(7) and (11) hold. Then the continuous cocycle $\Phi_0$ associated with equation (16)-(18) has a unique $D$-pullback attractor $A_0 = \{A_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

### 2.2. Wong-Zakai approximation of wave equations

Given $\delta \neq 0$, define random variables $G_\delta$ and $G_{\tilde{\delta}}$ by

$$G_\delta(\omega) = \frac{\omega(\delta)}{\delta}, \quad G_\delta(t, \omega) = \int_0^t G_\delta(\theta_s \omega) ds, \quad \text{for all } \omega \in \Omega. \quad (46)$$

From (46) we find

$$G_\delta(\theta_t \omega) = \frac{\omega(t + \delta) - \omega(t)}{\delta} \quad \text{and} \quad G_\delta(t, \omega) = \int_t^{t+\delta} \frac{\omega(s)}{\delta} ds + \int_0^\delta \frac{\omega(s)}{\delta} ds. \quad (47)$$

By (47) and the continuity of $\omega$ we get for all $t \in \mathbb{R}$,

$$\lim_{\delta \to 0} G_\delta(t, \omega) = \omega(t). \quad (48)$$

Note that this convergence is uniform on a finite interval as stated below.

**Lemma 2.4 ([20]).** Let $\tau \in \mathbb{R}, \omega \in \Omega$ and $T > 0$. Then for every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon, \tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta_0$ and $t \in [\tau, \tau + T]$,

$$|G_\delta(t, \omega) - \omega(t)| < \varepsilon.$$

By Lemma 2.4, we find that there exist $\tilde{\delta} = \tilde{\delta}(\tau, \omega, T) > 0$ and $c = c(\tau, \omega, T) > 0$ such that for all $0 < |\delta| < \tilde{\delta}$ and $t \in [\tau, \tau + T]$,

$$|G_\delta(t, \omega)| \leq |G_\delta(t, \omega) - \omega(t)| + |\omega(t)| \leq c. \quad (49)$$

The approximation we use here was first introduced in [21] and later [24] where the authors studied the chaotic behavior of random differential equations driven by a multiplicative noise of $G_\delta(\theta_t \omega)$. It has been used to approximate the long term behavior of solutions of stochastic reaction diffusion equations and stochastic Navier-Stokes equations [20, 36, 14]. We will compare the solutions and dynamics of (1) with pathwise deterministic equation given by

$$\frac{\partial^2}{\partial t^2} u_\delta + \alpha \frac{\partial}{\partial t} u_\delta - \Delta u_\delta + \nu u_\delta + f(x, u_\delta) = g(t, x) + h(x)G_\delta(\theta_t \omega), \quad t > \tau, x \in \mathbb{R}^n, \quad (50)$$

along with initial condition

$$u_\delta(\tau, x) = u_{\delta, \tau}(x), \quad \frac{\partial}{\partial t} u_\delta(\tau, x) = u_{\delta, 1, \tau}(x) \quad x \in \mathbb{R}^n. \quad (51)$$
Let \( z_{\delta} = \frac{d}{d\tau} u_{\delta} + \beta u_{\delta} \). By (50) we get

\[
\frac{du_{\delta}}{dt} + \beta u_{\delta} = z_{\delta},
\]

\[
(52)
\]

\[
\frac{dz_{\delta}}{dt} + (\alpha - \beta)z_{\delta} + (\nu + \beta^2 - \alpha\beta)u_{\delta} - \Delta u_{\delta} + f(x, u_{\delta}) = g(t, x) + h(x)G_{\delta}(\theta, \omega),
\]

\[
(53)
\]

with initial conditions

\[
u_{\delta}(\tau, x) = u_{\delta, \tau}(x), \quad z_{\delta}(\tau, x) = z_{\delta, \tau}(x), \quad x \in \mathbb{R}^n,
\]

\[
(54)
\]

where \( z_{\delta, \tau} = u_{\delta, 1, \tau} + \beta u_{\delta, \tau} \). Let \( v_{\delta}(t, \tau, \omega) = z_{\delta}(t, \tau, \omega) - h(x)G_{\delta}(t, \omega) \). By (52)-(54) we have

\[
\frac{du_{\delta}}{dt} + \beta u_{\delta} - v_{\delta} = h(x)G_{\delta}(t, \omega),
\]

\[
(55)
\]

\[
\frac{dv_{\delta}}{dt} + (\alpha - \beta)v_{\delta} + (\nu + \beta^2 - \alpha\beta)u_{\delta} - \Delta u_{\delta} + f(x, u_{\delta}) = g(t, x) + (\beta - \alpha)h(x)G_{\delta}(t, \omega),
\]

\[
(56)
\]

with initial conditions

\[
u_{\delta}(\tau, x) = u_{\delta, \tau}(x), \quad v_{\delta}(\tau, x) = v_{\delta, \tau}(x), \quad x \in \mathbb{R}^n,
\]

\[
(57)
\]

where \( v_{\delta, \tau} = z_{\delta, \tau} - h(x)G_{\delta}(\tau, \omega) \).

The well-posedness of the deterministic problem (55)-(57) in \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) can be established by the standard methods as in [29]. More precisely, under assumptions (4)-(7), we can show that for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( (u_{\delta, \tau}, v_{\delta, \tau}) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \), problem (55)-(57) has a unique solution

\[
u_{\delta}(\cdot, \tau, \omega, u_{\delta, \tau}, v_{\delta}(\cdot, \tau, \omega, v_{\delta, \tau})) \in C([\tau, \infty), H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)).
\]

In addition, it is \( (\mathcal{F}, B(H^1(\mathbb{R}^n)) \times B(L^2(\mathbb{R}^n))) \)-measurable with respect to \( \omega \in \Omega \) and continuous in \( (u_{\delta, \tau}, v_{\delta, \tau}) \) with respect to the norm of \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \). Based on this fact, one can define a continuous cocycle for system (52)-(54). Let \( \Phi_{\delta} : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{F}(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) be a mapping given by

\[
\Phi_{\delta}(t, \tau, \omega, (u_{\delta, \tau}, z_{\delta, \tau})) = (u_{\delta}(t + \tau, \tau, \theta_{-\tau} \omega, u_{\delta, \tau}), z_{\delta}(t + \tau, \tau, \theta_{-\tau} \omega, z_{\delta, \tau}))
\]

\[
(58)
\]

for all \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega, \) \( (u_{\delta, \tau}, z_{\delta, \tau}) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) and in addition,

\[
z_{\delta}(t + \tau, \tau, \theta_{-\tau} \omega, z_{\delta, \tau}) = v_{\delta}(t + \tau, \tau, \theta_{-\tau} \omega) + h(x)G_{\delta}(t + \tau, \theta_{-\tau} \omega).
\]

Then it is easy to check that \( \Phi_{\delta} \) is a continuous cocycle on \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) over \( (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_{t}\}_{t \in \mathbb{R}}) \).

By the similar arguments of [2], one can verify the following weak continuity of solutions on initial data, which is useful when proving the asymptotic compactness of solutions.

**Lemma 2.5.** Suppose (4)-(7) hold. For every \( \delta \neq 0, \tau \in \mathbb{R} \) and \( \omega \in \Omega \), let \( (u_{\delta}(\cdot, \tau, \omega, u_{\delta, \tau, n}), v_{\delta}(\cdot, \tau, \omega, v_{\delta, \tau, n})) \) and \( (u_{\delta}(\cdot, \tau, \omega, u_{\delta, \tau}), v_{\delta}(\cdot, \tau, \omega, v_{\delta, \tau})) \) be the solutions of (55)-(56) with initial data \( (u_{\delta, \tau, n}, v_{\delta, \tau, n}) \) and \( (u_{\delta, \tau}, v_{\delta, \tau}) \) at initial time \( \tau \), respectively. If \( (u_{\delta, \tau, n}, v_{\delta, \tau, n}) \to (u_{\delta, \tau}, v_{\delta, \tau}) \) in \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \), then for all \( t \geq \tau \),

\[
(u_{\delta}(t, \tau, \omega, u_{\delta, \tau, n}), v_{\delta}(t, \tau, \omega, v_{\delta, \tau, n})) \to (u_{\delta}(t, \tau, \omega, u_{\delta, \tau}), v_{\delta}(t, \tau, \omega, v_{\delta, \tau}))
\]

in \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \).

\[
(59)
\]
Lemma 2.6. Suppose (4)-(6) and (11) hold. Then for every $\delta \neq 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in D$, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$ and $s \in [-t, 0]$, the solution $(u_{3}, v_{3})$ of system (55)-(57) satisfies

$$\begin{align*}
&\|u_{3}(\tau + s, \tau - t, \theta_{-\tau} \omega, u_{3, \tau - t})\|^{2}_{H^{1}(\mathbb{R}^{n})} + \|v_{3}(\tau + s, \tau - t, \theta_{-\tau} \omega, v_{3, \tau - t})\|^{2} \\
&+ \int_{-t}^{\tau} e^{2\lambda(r - \tau - s)}\|u_{3}(r, \tau - t, \theta_{-\tau} \omega, u_{3, \tau - t})\|^{2}_{H^{1}(\mathbb{R}^{n})}dr \\
&+ \int_{-t}^{\tau} e^{2\lambda(r - \tau - s)}\|v_{3}(r, \tau - t, \theta_{-\tau} \omega, v_{3, \tau - t})\|^{2}_{H^{1}(\mathbb{R}^{n})}dr \\
&\leq M_{1} + M_{1} e^{-2\lambda s} R_{3}(\tau, \omega),
\end{align*}$$

(60)

where $(u_{3, \tau - t}, v_{3, \tau - t}) \in D(\tau - t, \theta_{-\tau} \omega)$, $M_{1}$ is a positive constant independent of $\tau, \omega$, $D$ and $\delta$, and $R_{3}(\tau, \omega)$ is a random variable given by

$$R_{3}(\tau, \omega) = \int_{-\infty}^{0} e^{2\lambda r}(1 + \|g(r + \tau, \cdot)\|^{2} + |G_{3}(r + \tau, \theta_{-\tau} \omega)|^{2} + |G_{3}(r + \tau, \theta_{-\tau} \omega)|^{p+1})dr.$$

(61)

Proof. It follows from (55) and (56) that

$$\begin{align*}
\frac{d}{dt}\|v_{3}\|^{2} + (\nu + \beta^{2} - \alpha \beta)\|u_{3}\|^{2} + \|\nabla u_{3}\|^{2} + 2 \int_{\mathbb{R}^{n}} F(x, u_{3}) dx \\
+ 2(\alpha - \beta)\|v_{3}\|^{2} + 2\beta(\nu + \beta^{2} - \alpha \beta)\|u_{3}\|^{2} + 2\beta\|\nabla u_{3}\|^{2} + 2\beta(f(x, u_{3}), u_{3}) \\
= 2(f(x, u_{3}), h)G_{3}(t, \omega) + 2(\nu + \beta^{2} - \alpha \beta)(h, u_{3})G_{3}(t, \omega) \\
+ 2(\nabla h, \nabla u_{3})G_{3}(t, \omega) + 2(\beta - \alpha)(h(x), v_{3})G_{3}(t, \omega) + 2(g(t, x), v_{3}).
\end{align*}$$

(62)

We now estimate each term on the right-hand side of (62). For the first term on the right-hand side of (62), by (4) and (6), we obtain

$$\begin{align*}
2(f(x, u_{3}), h)G_{3}(t, \omega) &\leq 2\|\varphi_{1}\|\|h\| + c(\int_{\mathbb{R}^{n}} |u_{3}|^{p+1} dx) \frac{E^{\tau}}{\tau} \|h\|_{L^{p+1}} \|G_{3}(t, \omega)\| \\
&\leq (2\|\varphi_{1}\|\|h\| + c(\int_{\mathbb{R}^{n}} (F(x, u_{3}) + \varphi_{3}) dx) \frac{E^{\tau}}{\tau} \|h\|_{L^{p+1}} \|G_{3}(t, \omega)\| \\
&\leq c\|G_{3}(t, \omega)\| + \alpha_{2}\beta(\int_{\mathbb{R}^{n}} F(x, u_{3}) dx + \|\varphi_{3}\|_{L^{1}}) + c\|G_{3}(t, \omega)\|^{p+1}.
\end{align*}$$

(63)

The second and third terms on the right-hand side of (62) satisfy

$$2(\nu + \beta^{2} - \alpha \beta)(h, u_{3})G_{3}(t, \omega) \leq (\nu + \beta^{2} - \alpha \beta)\|u_{3}\|^{2} + c\|h\|^{2}|G_{3}(t, \omega)|^{2}$$

(64)

and

$$2(\nabla h, \nabla u_{3})G_{3}(t, \omega) \leq \beta\|\nabla u_{3}\|^{2} + c\|\nabla h\|^{2}|G_{3}(t, \omega)|^{2}.$$  

(65)

By Young’s inequality, the last two terms on the right-hand side of (62) are bounded by

$$2(\beta - \alpha)(h(x), v_{3})G_{3}(t, \omega) + 2(g(t, x), v_{3}) \leq (\alpha - \beta)\|v_{3}\|^{2} + c\|g(t, \cdot)\|^{2} + c\|G_{3}(t, \omega)\|^{2}.$$  

(66)
On the other hand, the condition (5) implies that
\[
(f(x, u_\delta), u_\delta) \geq \alpha_2 \int_{\mathbb{R}^n} F(x, u_\delta) dx + \int_{\mathbb{R}^n} \varphi_2(x) dx. 
\] (67)

By (62)-(67), we get for every \(\omega \in \Omega\)
\[
\frac{d}{dt} \left( \|v_\delta\|^2 + (\nu + \beta^2 - \alpha\beta)\|u_\delta\|^2 + \|\nabla u_\delta\|^2 + 2 \int_{\mathbb{R}^n} F(x, u_\delta) dx \right)
+ (\alpha - \beta)\|v_\delta\|^2 + (\nu + \beta^2 - \alpha\beta)\|u_\delta\|^2 + \|\nabla u_\delta\|^2 + \alpha_2 \beta \int_{\mathbb{R}^n} F(x, u_\delta) dx
\leq c(1 + \|g(t, \cdot)\|^2 + |G_\delta(t, \omega)|^2 + |G_\delta(t, \omega)|^{p+1}).
\] (68)

Moreover, it follows from (6) and (10)
\[
\alpha_2 \beta \int_{\mathbb{R}^n} F(x, u_\delta) dx \geq 4\lambda \int_{\mathbb{R}^n} F(x, u_\delta) dx + (4\lambda - \alpha_2 \beta) \int_{\mathbb{R}^n} \varphi_3(x) dx.
\] (69)

By (10), (68) and (69), we get for every \(\omega \in \Omega\)
\[
\frac{d}{dt} \left( \|v_\delta\|^2 + (\nu + \beta^2 - \alpha\beta)\|u_\delta\|^2 + \|\nabla u_\delta\|^2 + 2 \int_{\mathbb{R}^n} F(x, u_\delta) dx \right)
+ 2\lambda \left( \|v_\delta\|^2 + (\nu + \beta^2 - \alpha\beta)\|u_\delta\|^2 + \|\nabla u_\delta\|^2 + 2 \int_{\mathbb{R}^n} F(x, u_\delta) dx \right)
+ 2\lambda \left( \|v_\delta\|^2 + (\nu + \beta^2 - \alpha\beta)\|u_\delta\|^2 + \|\nabla u_\delta\|^2 \right)
\leq c(1 + \|g(t, \cdot)\|^2 + |G_\delta(t, \omega)|^2 + |G_\delta(t, \omega)|^{p+1}).
\] (70)

Multiplying (70) by \(e^{2\lambda t}\) and then integrating over \((\tau - t, \tau + s)\) with \(t \in \mathbb{R}^+\) and \(s \in [-t, 0]\), we get for every \(\omega \in \Omega\)
\[
\|v_\delta(\tau + s, \tau - t, \theta_{\tau \omega}, v_{\delta, \tau - t})\|^2 + (\nu + \beta^2 - \alpha\beta)\|u_\delta(\tau + s, \tau - t, \theta_{\tau \omega}, u_{\delta, \tau - t})\|^2
+ \|\nabla u_\delta(\tau + s, \tau - t, \theta_{\tau \omega}, u_{\delta, \tau - t})\|^2 + 2 \int_{\mathbb{R}^n} F(x, u_\delta(\tau + s, \tau - t, \theta_{\tau \omega}, u_{\delta, \tau - t})) dx
+ 2\lambda \int_{\tau - t}^{\tau + s} e^{2\lambda(r - \tau - t)}\|v_{\delta}(r, \tau - t, \theta_{\tau \omega}, v_{\delta, \tau - t})\|^2 dr
+ 2\lambda \int_{\tau - t}^{\tau + s} e^{2\lambda(r - \tau - t)}\|u_{\delta}(r, \tau - t, \theta_{\tau \omega}, u_{\delta, \tau - t})\|^2 dr
+ 2\lambda \int_{\tau - t}^{\tau + s} e^{2\lambda(r - \tau - t)}\|\nabla u_{\delta}(r, \tau - t, \theta_{\tau \omega}, u_{\delta, \tau - t})\|^2 dr
\leq e^{-2\lambda(t+s)}(\|v_{\delta, \tau - t}\|^2 + (\nu + \beta^2 - \alpha\beta)\|u_{\delta, \tau - t}\|^2
+ \|\nabla u_{\delta, \tau - t}\|^2 + 2 \int_{\mathbb{R}^n} F(x, u_{\delta, \tau - t}) dx)
+ c \int_{\tau - t}^{\tau + s} e^{2\lambda(r - \tau - s)} (1 + \|g(r, \cdot)\|^2 + |G_\delta(r, \theta_{\tau \omega})|^2 + |G_{\delta}(r, \theta_{\tau \omega})|^{p+1}) dr
\leq e^{-2\lambda(t+s)}(\|v_{\delta, \tau - t}\|^2 + (\nu + \beta^2 - \alpha\beta)\|u_{\delta, \tau - t}\|^2 + \|\nabla u_{\delta, \tau - t}\|^2)
\[ + 2 \int_{\mathbb{R}^n} F(x, u_{\delta,\tau-t})dx \]
\[ + c \int_{-\infty}^{0} e^{2\lambda(r-s)}(1 + \|g(r + \tau, \cdot)\|^2 + |G_\delta(r + \tau, \theta_{-\tau}\omega)|^2 \]
\[ + |G_\delta(r + \tau, \theta_{-\tau}\omega)|^{p+1} dr. \]  

Due to (3) and (11), the last integral on the right-hand side of (71) is well defined. Note that (8) implies that
\[ \int_{\mathbb{R}^n} F(x, u_{\delta,\tau-t})dx \leq c(1 + \|u_{\delta,\tau-t}\|^2 + \|u_{\delta,\tau-t}\|_{H^1(\mathbb{R}^n)}^{p+1}), \]
where we use the fact that \( H^1(\mathbb{R}^n) \) is continuously embedded in \( L^r(\mathbb{R}^n) \) for \( 1 \leq r \leq 6 \). Since \( (u_{\delta,\tau-t}, v_{\delta,\tau-t}) \in D(\tau-t, \theta_{-\tau}\omega) \) and \( D \) is tempered, we find from (72) that
\[ e^{-2\lambda t}(\|u_{\delta,\tau-t}\|^2 + (\nu + \beta^2 - \alpha\beta)\|u_{\delta,\tau-t}\|^2 + \|\nabla u_{\delta,\tau-t}\|^2 + 2 \int_{\mathbb{R}^n} F(x, u_{\delta,\tau-t})dx \]
\[ \leq ce^{-2\lambda t}(1 + \|u_{\delta,\tau-t}\|^2 + \|u_{\delta,\tau-t}\|_{H^1(\mathbb{R}^n)} + \|u_{\delta,\tau-t}\|_{H^1(\mathbb{R}^n)}^{p+1}) \]
\[ \leq ce^{-2\lambda t}(1 + \|D(\tau-t, \theta_{-\tau}\omega)\|^2 + \|D(\tau-t, \theta_{-\tau}\omega)\|^{p+1}) \to 0 \quad \text{as} \quad t \to +\infty. \]  

It follows from (6) that
\[ - \int_{\mathbb{R}^n} F(x, u_{\delta}(r, \tau-t, \theta_{-\tau}\omega, u_{\delta,\tau-t}))dx \leq \|\varphi_\delta\|_{L^1}. \]

Thus the desired estimates (60) follows from (71), (73) and (74).

\[ \square \]

**Corollary 2.1.** Suppose (4)-(7) and (11) hold. Then the continuous cocycle \( \Phi_\delta \) associated with system (52)-(54) has a closed measurable \( D \)-pullback absorbing set \( K_\delta = \{K_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \), which is given by for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \)
\[ K_\delta(\tau, \omega) = \{(u_\delta, z_\delta) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) : \|u_\delta\|_{H^1(\mathbb{R}^n)}^2 + \|z_\delta\|^2 \leq L_\delta(\tau, \omega)\}, \]
where \( L_\delta(\tau, \omega) \) is given by
\[ L_\delta(\tau, \omega) = c + c \int_{-\infty}^{0} e^{2\lambda(t)}(1 + \|g(t + \tau, \cdot)\|^2 + |G_\delta(t + \tau, \theta_{-\tau}\omega)|^2 \]
\[ + |G_\delta(t + \tau, \theta_{-\tau}\omega)|^{p+1} dt + c|G_\delta(t, \theta_{-\tau}\omega)|^2. \]

Here \( c \) is a positive constant independent of \( \tau, \omega \) and \( \delta \). In addition, we have for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \)
\[ \lim_{\delta \to 0} L_\delta(\tau, \omega) = L_0(\tau, \omega), \]
where \( L_0(\tau, \omega) \) is defined in (24) with a different constant \( c \).

**Proof.** Given \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( D \in D \), it follows from Lemma 2.5 with \( s = 0 \) that there exists \( T = T(\tau, \omega; D) > 0 \) such that for all \( t \geq T \),
\[ \|u_\delta(\tau-t, \theta_{-\tau}\omega, u_{\delta,\tau-t})\|_{H^1(\mathbb{R}^n)}^2 + \|v_\delta(\tau-t, \theta_{-\tau}\omega, v_{\delta,\tau-t})\|^2 \]
\[ \leq c(1 + R_\delta(\tau, \omega)), \]
where \( (u_{\delta,\tau-t}, v_{\delta,\tau-t}) \in D(\tau-t, \theta_{-\tau}\omega) \). Furthermore, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \)
\[ \lim_{t \to +\infty} e^{-\frac{\lambda t}{p+1}} R_\delta(\tau-t, \theta_{-\tau}\omega) = 0. \]
In fact, by (61) we get, for every $t \geq T$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,
\[
R_\delta(t, \tau - t, \theta_{-t}\omega) = \int_{-\infty}^{-t} e^{2\lambda(s+t)}(1 + \|g(s + \tau, \cdot)\|^2 + |G_\delta(s + \tau, \theta_{-t}\omega)|^p + |G_\delta(s + \tau, \theta_{-t}\omega)|^{p+1}) ds
\]
\[
\leq e^{\frac{\lambda t}{\tau}}\int_{-\infty}^{-t} e^{\frac{\lambda t}{\tau}}(1 + \|g(s + \tau, \cdot)\|^2 + |\int_{-\tau}^{\tau} \hat{G}_\delta(t, \omega) d\tau|^{p+1}) ds.
\] (80)

First of all, it follows from (11) that
\[
\lim_{t \to +\infty} \int_{-\infty}^{-t} e^{\frac{\lambda t}{\tau}}(1 + \|g(s + \tau, \cdot)\|^2) ds = 0.
\] (81)

On the other hand, by (3) we have
\[
\lim_{t \to +\infty} \int_{-\infty}^{-t} e^{\frac{\lambda t}{\tau}}(1 + \|g(s + \tau, \cdot)\|^2 + |\int_{-\tau}^{\tau} \hat{G}_\delta(t, \omega) d\tau|^{p+1}) ds = 0.
\] (82)

Therefore (79) follows from (80)-(82). Note that
\[
\Phi_\delta(t, \tau - t, \theta_{-t}\omega, (u_\delta, \tau - t, z_\delta, \tau - t))
\]
\[
= (u_\delta(\tau, \tau - t, \theta_{-t}\omega; u_\delta, \tau - t, z_\delta, \tau - t), z_\delta(\tau, \tau - t, \theta_{-t}\omega; z_\delta, \tau - t))
\]
\[
= (u_\delta(\tau, \tau - t, \theta_{-t}\omega; u_\delta, \tau - t, v_\delta, \theta_{-t}\omega; v_\delta, \tau - t) + hG_\delta(\tau, \theta_{-t}\omega)).
\] (83)

Therefore, we have
\[
\|\Phi_\delta(t, \tau - t, \theta_{-t}\omega, (u_\delta, \tau - t, z_\delta, \tau - t))\|^2_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}
\]
\[
\leq \|u_\delta(\tau, \tau - t, \theta_{-t}\omega, u_\delta, \tau - t)\|^2_{H^1(\mathbb{R}^n)} + 2\|v_\delta(\tau, \tau - t, \theta_{-t}\omega, v_\delta, \tau - t)\|^2 + cG_\delta(\tau, \theta_{-t}\omega).
\] (84)

For any $\tilde{D}(\tau - t, \theta_{-t}\omega) \in D$, there is $D(\tau - t, \theta_{-t}\omega) \in D$ such that $(u_\delta, \tau - t, z_\delta, \tau - t) \in D(\tau - t, \theta_{-t}\omega)$. Then by Lemma 2.5 and (84), there exists $T > 0$ such that for all $t \geq T$ and $\delta \neq 0$
\[
\|\Phi_\delta(t, \tau - t, \theta_{-t}\omega, (u_\delta, \tau - t, z_\delta, \tau - t))\|^2_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \leq L_\delta(\tau, \omega),
\]
where $L_\delta(\tau, \omega)$ is given by (76). This shows that for all $t \geq T$ and $\delta \neq 0$,
\[
\Phi_\delta(t, \tau - t, \theta_{-t}\omega, \tilde{D}(\tau - t, \theta_{-t}\omega)) \subseteq K_\delta(\tau, \omega).
\] (85)

Moreover, we have
\[
e^{-\lambda t}\|K_\delta(t, \tau - t, \theta_{-t}\omega)\|^{p+1} \leq e^{-\lambda t}|L_\delta(\tau - t, \theta_{-t}\omega)|^{p+1}
\]
\[
\leq c(e^{-\frac{\lambda t}{\tau}} R_\delta(t, \tau - t, \theta_{-t}\omega))^{p+1}
\]
\[
+ c e^{-\lambda t}|G_\delta(\tau, \theta_{-t}\omega)|^{2p+2}.
\]

By (3), (46) and (79), we get
\[
\lim_{t \to +\infty} e^{-\lambda t}\|K_\delta(t, \tau - t, \theta_{-t}\omega)\|^{p+1} = 0.
\] (86)

This shows that $K_\delta \in D$. On the other hand, for each fixed $\tau \in \mathbb{R}$, by (76) we find that $L_\delta(\tau, \cdot)$ is measurable, and so is the set-valued map $K_\delta(\tau, \cdot)$, which along with (85) and (86) implies that $K_\delta$ is a closed measurable $D$-pullback absorbing set for the continuous cocycle $\Phi_\delta$. 
The convergence (77) can be obtained by the Lebesgue’s dominated convergence theorem as in [20]. The details are omitted here. □

Next, we derive uniform estimates on the tails of solutions when \( t \to +\infty \), which will play an important role for proving the asymptotic compactness of solutions.

**Lemma 2.7.** Suppose (4)-(6) and (11) hold. Then for every \( \delta \neq 0 \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \), \( \varepsilon > 0 \) and \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \), there exist \( T = T(\tau, \omega, D, \varepsilon) > 0 \) and \( k = k(\tau, \omega, \varepsilon) > 0 \) such that for all \( t \geq T \) and \( s \in [-t, 0] \), the solution \((u_\delta, v_\delta)\) of (52)-(54) satisfies

\[
\int_{|x| \geq k} |v_\delta(\tau + s, \tau - t, \theta^{-t} \omega, v_{\delta, t, \tau})|^2 + |u_\delta(\tau + s, \tau - t, \theta^{-t} \omega, u_{\delta, t, \tau})|^2 \leq c(1 + e^{-2\lambda s})\varepsilon, \tag{87}
\]

where \((u_{\delta, t, \tau}, v_{\delta, t, \tau}) \in D(\tau - t, \theta^{-t} \omega)\) and \( c > 0 \) is a constant independent of \( \varepsilon \).

**Proof.** We here just sketch the idea of the proof since the details are similar to that of Lemma 2.2. Let \( \rho \) be the smooth function defined by Lemma 2.2. Let \( k \) be a fixed positive integer which will be specified later. First multiplying (56) by \( \frac{\partial (u_\delta)}{\partial t} \) and \( v_\delta \), taking the integral over \( \mathbb{R}^n \), and then following the process of (35), there exists \( N_1 = N_1(\varepsilon) \geq 1 \) such that for all \( k \geq N_1 \)

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) \left( |v_\delta| + \nu + \beta^2 - \alpha \beta \right) |u_\delta|^2 + |\nabla u_\delta|^2 + 2F(x, u_\delta) dx \\
+ 2\lambda \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) \left( |v_\delta|^2 + \nu + \beta^2 - \alpha \beta \right) |u_\delta|^2 + |\nabla u_\delta|^2 + 2F(x, u_\delta) dx \\
\leq \varepsilon (|\nabla u_\delta|^2 + ||u_\delta||^2) + c \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |g(t, x)|^2 dx \\
+ c\varepsilon (|G_\delta(t, \omega)|^{p+1} + |G_\delta(t, \omega)|^2 + 1). \tag{88}
\]

Integration (88) over \((\tau - t, \tau + s)\) with \( t \in \mathbb{R}^+ \) and \( s \in [-t, 0] \), we get for every \( \omega \in \Omega \)

\[
\int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) \left( |v_\delta(\tau + s, \tau - t, \theta^{-t} \omega, v_{\delta, t, \tau})|^2 \\
+ (\nu + \beta^2 - \alpha \beta) |u_\delta(\tau + s, \tau - t, \theta^{-t} \omega, u_{\delta, t, \tau})|^2 \\
+ |\nabla u_\delta(\tau + s, \tau - t, \theta^{-t} \omega, u_{\delta, t, \tau})|^2 + 2F(x, u_{\delta}(\tau + s, \tau - t, \theta^{-t} \omega, u_{\delta, t, \tau})) \right) dx \\
\leq e^{-2\lambda(t+s)} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) \left( |v_{\delta, t, \tau}|^2 + (\nu + \beta^2 - \alpha \beta) |u_{\delta, t, \tau}|^2 \\
+ |\nabla u_{\delta, t, \tau}|^2 + 2F(x, u_{\delta, t, \tau}) \right) dx \\
+ \varepsilon \int_{\tau-t}^{\tau+s} e^{2\lambda(r-s-t)} \left( ||\nabla u_\delta(r, \tau - t, \theta^{-t} \omega, v_{\delta, t, \tau})||^2 \\
+ ||v_\delta(r, \tau - t, \theta^{-t} \omega, v_{\delta, t, \tau})||^2 \right) dr \\
+ c e^{-2\lambda s} \int_0^0 e^{2\lambda r} \int_{|x| \geq k} |g(r + \tau, x)|^2 dx dr \\
+ c\varepsilon e^{-2\lambda s} \int_{-\infty}^0 e^{2\lambda r} \left( |G_\delta(r + \tau, \theta^{-t} \omega)|^{p+1} + |G_\delta(r + \tau, \theta^{-t} \omega)|^2 + 1 \right) dr. \tag{89}
\]
Since \((u_{\delta,\tau-t}, v_{\delta,\tau-t}) \in D(\tau-t, \theta_{-t}\omega)\), there exists \(T_1 = T_1(\tau, \omega, D, \varepsilon) > 0\) such that for all \(t \geq T_1\),
\[
e^{-2\lambda(t+s)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \left(|v_{\delta,\tau-t}|^2 + (\nu + \beta^2 - \alpha\beta)|u_{\delta,\tau-t}|^2 + \|\nabla u_{\delta,\tau-t}\|^2 + 2F(x, u_{\delta,\tau-t})\right) dx \leq \varepsilon e^{-2\lambda s}. \tag{90}
\]

It follows from (13) that
\[
\lim_{k \to +\infty} \int_{-\infty}^{0} e^{2\lambda r} \int_{|x| \geq k} |g(r + \tau, x)|^2 dx dr = 0. \tag{91}
\]
By (91), there is a \(N_2 = N_2(\tau, \varepsilon) \geq N_1\) such that for all \(k \geq N_2\)
\[
\int_{-\infty}^{0} e^{2\lambda r} \int_{|x| \geq k} |g(r + \tau, x)|^2 dx dr \leq \varepsilon. \tag{92}
\]
By (47), we find
\[
\int_{-\infty}^{0} e^{2\lambda r} \left(|G_\delta(r + \tau, \theta_{-\tau}\omega)|^{p+1} + |G_\delta(r + \tau, \theta_{-\tau}\omega)|^2 + 1\right) dr < c. \tag{93}
\]
Then we can get from Lemma 2.5 and (89)-(93) that there exists \(T_2 = T_2(\tau, \omega, D) \geq T_1\) such that for \(t \geq T_2\), \(s \in [-t, 0]\) and \(k \geq N_2\)
\[
\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \left(|v_\delta(\tau + s, \tau - t, \theta_{-\tau}\omega, v_{\delta,\tau-t})|^2 + |\nabla v_\delta(\tau + s, \tau - t, \theta_{-\tau}\omega, v_{\delta,\tau-t})|^2 \right. \\
+ \left. (\nu + \beta^2 - \alpha\beta)|u_\delta(\tau + s, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})|^2 \right) \right) dx \\
\leq -2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) F(x, u_\delta(\tau + s, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})) dx \\
+ c e^{-2\lambda s} + c\varepsilon(1 + e^{-2\lambda s} R_\delta(\tau, \omega)). \tag{94}
\]
By (6) and \(\varphi_3 \in L^1(\mathbb{R}^n)\), there is \(N_3 = N_3(\varepsilon) \geq N_2\) such that for \(k \geq N_3\)
\[
- \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) F(x, u_\delta(\tau + s, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})) dx \leq \varepsilon. \tag{95}
\]
Therefore it follows from (93)-(95) that for all \(t \geq T_2\), \(s \in [-t, 0]\) and \(k \geq N_3\)
\[
\int_{|x| \geq \sqrt{2}k} \left(|v_\delta(\tau + s, \tau - t, \theta_{-\tau}\omega, v_{\delta,\tau-t})|^2 + |u_\delta(\tau + s, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})|^2 \right) \right) dx \\
\leq c\varepsilon + c\varepsilon e^{-2\lambda s}(1 + R_\delta(\tau, \omega)),
\]
which concludes the proof.

We now derive an energy equation for problem (55)-(57), which will be used to prove the pullback asymptotic compactness of solutions. To this end, denote by,
\[
E(u_\delta, v_\delta) = ||v_\delta||^2 + (\nu + \beta^2 - \alpha\beta)||u_\delta||^2 + ||\nabla u_\delta||^2 + 2 \int_{\mathbb{R}^n} F(x, u_\delta) dx \tag{96}
\]
which are uniform with respect to small \( \varepsilon > 0 \). Suppose Lemma 3.1.

\[ \text{Theorem 2.8. Suppose } \tau \in \mathcal{S} \text{ associated with equation (52)} \]

We obtain the existence of \( \mathcal{D} \)-pullback asymptotic compactness of \( \Phi_\delta \) as stated below.

\[ \text{Theorem 2.8. Suppose (4)-(7) and (11) hold. Then the continuous cocycle } \Phi_\delta \text{ associated with equation (52)-(54) has a unique } \mathcal{D} \text{-pullback attractor } \mathcal{A}_\delta = \{ A_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \text{ in } H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n). \]

We prove Corollary 2.1, \( \Phi_\delta \) has a closed \( \mathcal{D} \)-pullback absorbing set. Moreover, we can get the pullback asymptotic compactness of \( \Phi_\delta \), which can be proven as that of Lemma 5.1 in [30] or Lemma 5.1 in [34]. As a result, the existence and uniqueness of \( \mathcal{D} \)-pullback attractors of \( \Phi_\delta \) follows immediately. \( \square \)

3. Upper semicontinuity of pullback attractors. In this section, we will study the limiting behavior of solutions of (55)-(57) as \( \delta \to 0 \). We first prove the convergence of solutions of (55)-(57) to that of the stochastic equation (19)-(21) in \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) as \( \delta \to 0 \). We then prove the upper semicontinuity of attractors \( \mathcal{A}_\delta \) as \( \delta \to 0 \). For later purpose, we now derive estimates on the tails of the solutions which are uniform with respect to small \( \delta \).

\[ \text{Lemma 3.1. Suppose (4)-(6) and (11) hold. Then for every } \tau \in \mathbb{R}, \omega \in \Omega, \varepsilon > 0 \text{ and } D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}, \text{ there exist } \delta_0 = \delta_0(\tau, \omega) > 0, T = T(\tau, \omega, D, \varepsilon) > 0 \text{ and } N = N(\tau, \omega, \varepsilon) > 0 \text{ such that for all } t \geq T \text{ and } s \in [-t, 0) \text{ and } 0 < |\delta| < \delta_0, \text{ the solution } (u_\delta, v_\delta) \text{ of (55)-(57) satisfies} \]

\[ \int_{|x| \geq k} |v_\delta(t + s, \tau - t, \theta_{-r} \omega, u_\delta, \tau - t)|^2 + |u_\delta(t + s, \tau - t, \theta_{-r} \omega, u_\delta, \tau - t)|^2 dx \leq (1 + e^{-2\lambda})\varepsilon, \]

where \( (u_\delta, v_\delta) \in D(\tau - t, \theta_{-r} \omega) \).

\[ \text{Proof. By (47) we see that for all } r, \tau \in \mathbb{R} \text{ and } \delta \neq 0, \]

\[ G_\delta(r + \tau, \theta_{-r} \omega) = \int_{-r + \delta}^{r + \delta} \frac{\omega(s)}{\delta} ds + \int_{r + \delta}^{r + \delta} \frac{\omega(s)}{\delta} ds. \]
By (3) we find that there exists $T_1 = T_1(\omega) < 0$ such that
$$\vert \omega(t) \vert \leq -ct \quad \text{for all} \ t \leq T_1.$$ (102)

By the continuity of $\omega$ we have $\lim_{\delta \to 0} \int_{-\tau+\delta}^{\tau} \frac{\omega(s)}{\delta} ds = 0$, and hence there exists $\delta_1 = \delta_1(\omega) > 0$ such that for all $0 < \vert \delta \vert < \delta_1$,
$$\vert \int_{-\tau+\delta}^{\tau} \frac{\omega(s)}{\delta} ds \vert \leq 1.$$ (103)

By the mean value theorem, there exists $r_1$ between $r$ and $r+\delta$ such that $\int_{r}^{r+\delta} \frac{\omega(s)}{\delta} ds = \omega(r_1)$. So if $\vert \delta \vert \leq 1$ and $r \leq T_1 - 1$, then we have $r_1 \leq r + \vert \delta \vert \leq T_1$. Consequently, by (102) with $c \leq \frac{\lambda}{2}$ we get
$$\vert \omega(r_1) \vert \leq -\frac{\lambda}{2} r_1 \leq \frac{\lambda}{2} \vert \delta \vert - \frac{\lambda}{2} r \leq \frac{\lambda}{2} r + 1.$$ (104)

Let $\delta_2 = \min\{\delta_1, 1\}$. By (101), (103) and (104) we get for all $0 < \vert \delta \vert < \delta_2$ and $r \leq T_1 - 1$,
$$\vert G_\delta(r + \tau, \theta, -\tau\omega) \vert \leq \frac{\lambda}{2} r + 1.$$ (105)

By (105) we have, for all $0 < \vert \delta \vert < \delta_2$,
$$\int_{-\infty}^{0} e^{2\lambda(r-s)} (\vert \omega_\delta(r + \tau, \theta, -\tau\omega) \vert^{p+1} + \vert G_\delta(r + \tau, \theta, -\tau\omega) \vert^2 + 1) dr$$ \begin{align*}
&\leq \int_{-\infty}^{0} e^{2\lambda(r-s)} (\vert \frac{\lambda}{2} r + 1 \vert^{p+1} + \vert \frac{\lambda}{2} r + 1 \vert^2 + 1) dr \leq c. (106)
\end{align*}

The proof can be completed by (89), (106) and the process of Lemma 2.7. \hfill \square

In the sequel, we also need the following weak convergence of solutions. Please see Lemma 3.5 in [34] for more details.

**Lemma 3.2.** Suppose (4)-(7) hold, and $\{\delta_n\}_{n=1}^\infty$ is a sequence such that $\delta_n \to 0$ as $n \to \infty$. For any $r \in \mathbb{R}$ and $\omega \in \Omega$, let $(u_{\delta_n}(t, \tau, \omega, u_{\delta_n, \tau}), v_{\delta_n}(t, \tau, \omega, v_{\delta_n, \tau}))$ and $(u(t, \tau, \omega, u_\tau), v(t, \tau, \omega, v_\tau))$ be the solutions of (55)-(56) and (19)-(20) with initial data $(u_{\delta_n, \tau}, v_{\delta_n, \tau})$ and $(u_\tau, v_\tau)$ at initial time $\tau$, respectively. If $(u_{\delta_n, \tau}, v_{\delta_n, \tau}) \rightharpoonup (u_\tau, v_\tau)$ in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ as $n \to \infty$, then for all $t \geq \tau$, as $n \to \infty$
$$(u_{\delta_n}(t, \tau, \omega, u_{\delta_n, \tau}), v_{\delta_n}(t, \tau, \omega, v_{\delta_n, \tau})) \rightharpoonup (u(t, \tau, \omega, u_\tau), v(t, \tau, \omega, v_\tau))$$ in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

For the attractor $A_\delta$ of $\Phi_\delta$, we have the following precompactness.

**Lemma 3.3.** Suppose (4)-(6) and (11) hold. If $\delta_n \to 0$ and $\{(u_n, z_n)\}_{n=1}^\infty \in A_{\delta_n}(\tau, \omega)$, then the sequence $\{(u_n, z_n)\}_{n=1}^\infty$ has a convergent subsequence in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

**Proof.** Since $\delta_n \to 0$, by (77) we find that for every $r \in \mathbb{R}$ and $\omega \in \Omega$, there exists $N_1 = N_1(\tau, \omega)$ such that for all $n \geq N_1$,
$$L_{\delta_n}(\tau, \omega) \leq 2L_0(\tau, \omega).$$ (107)

Let $(u_n, z_n) \in A_{\delta_n}(\tau, \omega)$, then we have $A_{\delta_n}(\tau, \omega) \subseteq K_{\delta_n}(\tau, \omega) \subseteq D$. It follows from (75) and (107) that, for all $n \geq N_1$,
$$\|u_n\|_{H^1(\mathbb{R}^n)}^2 + \|z_n\|^2 \leq 2L_0(\tau, \omega),$$
which implies that \( \{(u_n, z_n)\}_{n=1}^\infty \) is bounded in \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) and hence, there exists \((\bar{u}, \bar{z}) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) such that, up to a subsequence\[
(u_n, z_n) \to (\bar{u}, \bar{z}) \quad \text{in} \quad H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n). \tag{108}
\]
In what follows, we prove that the weak convergence in (108) is actually a strong one in \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \). Fix a sequence \( \{t_n\} \) such that \( t_n \to +\infty \). By the invariance of \( A_{\delta_n} \), there exists \((\hat{u}_n, \hat{z}_n) \in A_{\delta_n}(\tau - t_n, \theta_{-t_n} \omega) \) such that\[
(u_n, z_n) = \Phi_{\delta_n}(t_n, \tau - t_n, \theta_{-t_n} \omega, (\hat{u}_n, \hat{z}_n)) = (u_{\delta_n}(\tau - t_n, \theta_{-\tau} \omega, \hat{u}_n), z_{\delta_n}(\tau - t_n, \theta_{-\tau} \omega, \hat{z}_n)). \tag{109}
\]
Since \((\hat{u}_n, \hat{z}_n) \in A_{\delta_n}(\tau - t_n, \theta_{-t_n} \omega) \subseteq K_{\delta_n}(\tau - t_n, \theta_{-t_n} \omega) \), by (75), (107) and (109) we infer that for all \( n \geq N_1 \),\[
\|u_n\|^2_{H^1(\mathbb{R}^n)} + \|z_n\|^2 \leq 2L_0(\tau - t_n, \theta_{-t_n} \omega). \tag{110}
\]
Note that\[
v_{\delta_n}(\tau, \tau - t_n, \theta_{-\tau} \omega, \hat{v}_n) = z_{\delta_n}(\tau, \tau - t_n, \theta_{-\tau} \omega, \hat{z}_n) - h(x)G_{\delta_n}(\tau, \theta_{-\tau} \omega), \tag{111}
\]
where\[
\hat{v}_n = \hat{z}_n - h(x)G_{\delta_n}(\tau - t_n, \theta_{-\tau} \omega). \tag{112}
\]
By (109) and (111) we get\[
u_n = u_{\delta_n}(\tau, \tau - t_n, \theta_{-\tau} \omega, \hat{u}_n),
\]
\[
z_n = v_{\delta_n}(\tau, \tau - t_n, \theta_{-\tau} \omega, \hat{v}_n) + h(x)G_{\delta_n}(\tau, \theta_{-\tau} \omega). \tag{113}
\]
By (110), there is a \( D(\tau, \omega) \in D \) such that \((\hat{u}_n, \hat{v}_n) \in D(\tau - t_n, \theta_{-t_n} \omega) \). Let \( m \) be a positive integer and \( N_2 = N_2(\tau, \omega, D, m, \delta) \geq N_1 \) such that \( t_n \geq m \) for all \( n \geq N_2 \). By Lemma 2.5 we have, for \( n \geq N_2 \),\[
\|u_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \hat{u}_n)\|^2_{H^1(\mathbb{R}^n)} + \|v_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \hat{v}_n)\|^2_{L^{p+1}(\mathbb{R}^n)}
\]
\[
+ \|u_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \hat{u}_n)\|^2_{L^{p+1}(\mathbb{R}^n)} \leq c(1 + e^{2\lambda m}R_{\delta_n}(\tau, \omega))
\]
\[
\leq c(1 + e^{2\lambda m}L_{\delta_n}(\tau, \omega)) \leq c(1 + e^{2\lambda m}L_0(\tau, \omega)). \tag{114}
\]
By (114) for each fixed \( m \geq 1 \), the sequence \( \{(u_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \hat{u}_n), v_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \hat{v}_n))\}_{n=1}^\infty \) is bounded in \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \), and hence, by a diagonal process, we can find a subsequence (not relabeled) such that for every \( m \geq 1 \), there exists \((\bar{u}_m, \bar{v}_m) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) such that as \( n \to \infty \)
\[
(u_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \hat{u}_n), v_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \hat{v}_n)) \to (\bar{u}_m, \bar{v}_m)
\]
in \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \). \tag{115}

In addition, note that for \( s \in [-m, 0] \) and \( k \leq m \),
\[
u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \hat{u}_n)
\]
\[
= u_{\delta_n}(\tau + s, \tau - m, \theta_{-\tau} \omega, u_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \hat{u}_n)), \tag{116}
\]
and\[
u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \hat{v}_n)
\]
\[
= v_{\delta_n}(\tau + s, \tau - m, \theta_{-\tau} \omega, v_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \hat{v}_n)). \tag{117}
\]
By (115)-(117) and Lemma 3.2 we find that for all \( m \in \mathbb{N} \) and \( s \in [-m, 0] \), when \( n \to \infty \),

\[
(u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau}\omega, \hat{u}_n), v_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau}\omega, \hat{v}_n)) \\
\to (u(\tau + s, \tau - m, \theta_{-\tau}\omega, \bar{u}_m), v(\tau + s, \tau - m, \theta_{-\tau}\omega, \bar{v}_m))
\]

in \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \). \hfill (118)

On the other hand, we find by (108), (109) and (111) that

\[
(u_{\delta_n}(\tau, \tau - t_n, \theta_{-\tau}\omega, \hat{u}_n), v_{\delta_n}(\tau, \tau - t_n, \theta_{-\tau}\omega, \hat{v}_n)) \to (\bar{u}, \bar{v})
\]

in \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \), \hfill (119)

where

\[
\bar{v} = \bar{z} + h(x)\omega(-\tau).
\hfill (120)

As a result of (118) with \( s = 0 \) and (119) we get

\[
\bar{u} = u(\tau, \tau - m, \theta_{-\tau}\omega, \bar{u}_m) \text{ and } \bar{v} = v(\tau, \tau - m, \theta_{-\rho}\omega, \bar{v}_m).
\hfill (121)

Applying (45) to \((u(\tau, \tau - m, \theta_{-\tau}\omega, \bar{u}_m), v(\tau, \tau - m, \theta_{-\tau}\omega, \bar{v}_m))\), by (43) and (121) we get

\[
E(\bar{u}, \bar{v}) = e^{-4\lambda m}E(\bar{u}_m, \bar{v}_m)
\]

\[
+ \int_{\tau - m}^{\tau} e^{4s(\tau - s)} \Gamma_0(u(s, \tau - m, \theta_{-\tau}\omega, \bar{u}_m), v(s, \tau - m, \theta_{-\tau}\omega, \bar{v}_m))ds
\]

\[
= e^{-4\lambda m}E(\bar{u}_m, \bar{v}_m) - 2(\alpha - \beta - 2\lambda) \int_{-m}^{0} e^{4s\lambda} \|v(s, \tau - s, \theta_{-\tau}\omega, \bar{v}_m)\|^2 ds
\]

\[
- 2(\beta - 2\lambda)(\nu + \beta^2 - \alpha \beta) \int_{-m}^{0} e^{4s\lambda} \|u(s, \tau - m, \theta_{-\tau}\omega, \bar{u}_m)\|^2 ds
\]

\[
- 2(\beta - 2\lambda) \int_{-m}^{0} e^{4s\lambda} \|\nabla u(s, \tau - s, \theta_{-\tau}\omega, \bar{u}_m)\|^2 ds
\]

\[
- 2\beta \int_{-m}^{0} e^{4s\lambda} (f(x, u(s, \tau - s, \theta_{-\tau}\omega, \bar{u}_m)), u(s, \tau - s, \theta_{-\tau}\omega, \bar{u}_m)) ds
\]

\[
+ 8\lambda \int_{-m}^{0} e^{4s\lambda} \int_{\mathbb{R}^n} F(x, u(s, \tau - s, \theta_{-\tau}\omega, \bar{u}_m)) dx ds
\]

\[
+ 2 \int_{-m}^{0} e^{4s\lambda} \partial_{\theta_{-\tau}\omega}(\tau + s)(f(x, u(s, \tau - s, \theta_{-\tau}\omega, \bar{u}_m)), h) ds
\]

\[
+ 2(\nu + \beta^2 - \alpha \beta) \int_{-m}^{0} e^{4s\lambda} (h, u(s, \tau - s, \theta_{-\tau}\omega, \bar{u}_m)) \theta_{-\tau}\omega(s) ds
\]

\[
+ 2 \int_{-m}^{0} e^{4s\lambda} (\nabla h, \nabla u(s, \tau - s, \theta_{-\tau}\omega, \bar{u}_m)) \theta_{-\tau}\omega(s) ds
\]

\[
+ 2(\beta - \alpha) \int_{-m}^{0} e^{4s\lambda} (h, v(s, \tau - s, \theta_{-\tau}\omega, \bar{v}_m)) \theta_{-\tau}\omega(s) ds
\]

\[
+ 2 \int_{-m}^{0} e^{4s\lambda} (g(s, x), v(s, \tau - s, \theta_{-\tau}\omega, \bar{v}_m)) ds.
\hfill (122)
Similarly, applying (99) to \((u_{\delta_n}(\tau, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n), v_{\delta_n}(\tau, \tau - t_n, \theta_{-\tau} \omega, \tilde{v}_n))\), we get from (116)-(117),

\[
E(u_{\delta_n}(\tau, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n), v_{\delta_n}(\tau, \tau - t_n, \theta_{-\tau} \omega, \tilde{v}_n)) \\
= e^{-\lambda m} E(u_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n), v_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \tilde{v}_n)) \\
+ \int_{-m}^{\tau} e^{\lambda (s-r)} \Gamma_{\delta_n}(u_{\delta_n}(s, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n), v_{\delta_n}(s, \tau - t_n, \theta_{-\tau} \omega, \tilde{v}_n)) ds \\
= I + J, \tag{123}
\]

where \(I\) and \(J\) are the first and the second terms in the above inequality, respectively. By (8), (96) and (114) we get

\[
I \leq ce^{-4\lambda m} \left( \|u_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n)\|^2 \right)_1 \\
+ \|u_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n)\| + \|v_{\delta_n}(\tau - m, \tau - t_n, \theta_{-\tau} \omega, \tilde{v}_n)\|^2 + 1 \\
\leq ce^{-4\lambda m} (1 + e^{2\lambda m} L_0(\tau, \omega)) \leq ce^{-2\lambda m} (1 + L_0(\tau, \omega)). \tag{124}
\]

Now we deal with the term \(J\). It follows from (97) and (123) that

\[
J = -2(\alpha - \beta - 2\lambda) \int_{-m}^{0} e^{4\lambda s} \|v_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \tilde{v}_n)\|^2 ds \\
- 2(\beta - 2\lambda)(\nu + \beta^2 - \alpha\beta) \int_{-m}^{0} e^{4\lambda s} \|u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n\|^2 ds \\
- 2(\beta - 2\lambda) \int_{-m}^{0} e^{4\lambda s} \|\nabla u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n)\|^2 ds \\
+ 8\lambda \int_{-m}^{0} e^{4\lambda s} \int_{\mathbb{R}^n} F(x, u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n)) dx ds \\
- 2\beta \int_{-m}^{0} e^{4\lambda s} (f(x, u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n)), u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n)) ds \\
+ 2 \int_{-m}^{0} e^{4\lambda s} (g(\tau + s, x), v_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \tilde{v}_n)) ds \\
+ 2 \int_{-m}^{0} e^{4\lambda s} G_{\delta_n}(\tau + s, \theta_{-\tau} \omega) \left( f(x, u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n)), h ) ds \\
+ 2(\nu + \beta^2 - \alpha\beta) \int_{-m}^{0} e^{4\lambda s} G_{\delta_n}(\tau + s, \theta_{-\tau} \omega) (u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n), h) ds \\
+ 2 \int_{-m}^{0} e^{4\lambda s} G_{\delta_n}(\tau + s, \theta_{-\tau} \omega) (\nabla u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \tilde{u}_n), \nabla h) ds \\
+ 2(\beta - \alpha) \int_{-m}^{0} e^{4\lambda s} G_{\delta_n}(\tau + s, \theta_{-\tau} \omega) (v_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \tilde{v}_n), h) ds \\
= \sum_{i=1}^{10} J_i, \tag{125}
\]

where \(J_i\) is the \(i\)-th term on the right-hand side of (125). We now deal with the limit superior of each term of (125). The weak convergence of (118) implies that for all \(m \in \mathbb{N}\) and \(s \in [-m, 0] \)

\[
\liminf_{n \to +\infty} \|v_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \tilde{v}_n)\|^2 \geq \|v(\tau + s, \tau - m, \theta_{-\tau} \omega, \tilde{v}_m)\|^2. \tag{126}
\]
By (126) and Fatou’s lemma we obtain
\[
\liminf_{n \to +\infty} \int_{-m}^{0} e^{4\lambda s} \|u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau \omega}, \tau_n)\|^2 ds \\
\geq \int_{-m}^{0} e^{4\lambda s} \|v(\tau + s, \tau - m, \theta_{-\tau \omega}, \tau_m)\|^2 ds.
\]
(127)

Therefore, we get from (127) that
\[
\limsup_{n \to +\infty} J_1 \leq -2(\alpha - \beta - 2\lambda) \int_{-m}^{0} e^{4\lambda s} \|v(\tau + s, \tau - m, \theta_{-\tau \omega}, \tau_m)\|^2 ds.
\]
(128)

Similarly, we can also prove
\[
\limsup_{n \to +\infty} J_2 \leq -2(\beta - 2\lambda)(\nu + \beta^2 - \alpha\beta) \int_{-m}^{0} e^{4\lambda s} \|u(\tau + s, \tau - m, \theta_{-\tau \omega}, \tau_m)\|^2 ds,
\]
(129)

and
\[
\limsup_{n \to +\infty} J_3 \leq -2(\beta - 2\lambda) \int_{-m}^{0} e^{4\lambda s} \|\nabla u(\tau + s, \tau - m, \theta_{-\tau \omega}, \tau_m)\|^2 ds.
\]
(130)

For the forth terms on the right-hand side of (125), we claim that
\[
\lim_{n \to +\infty} J_4 = 8\lambda \int_{-m}^{0} e^{4\lambda s} \int_{\mathbb{R}^n} F(x, u(\tau + s, \tau - m, \theta_{-\tau \omega}, \tau_m)) dx ds.
\]
(131)

To prove (131), we first note that by Lemma 3.1, for every \(\varepsilon > 0\), there exists \(K_1 = K_1(m, \tau, \omega, \varepsilon) \geq 1\) and \(N = N(m, \tau, \omega, \varepsilon) \geq 1\) such that for all \(k \geq K_1, n \geq N\) and \(s \in [-m, 0]\),
\[
\int_{|x| \geq k} |u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau \omega}, \tau_n)|^2 dx \leq (e^{-2\lambda s} + 1)\varepsilon.
\]
(132)

By (118), (132) and the compact embedding \(H^1 \hookrightarrow L^2\) in bounded domains, we get
\[
u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau \omega}, \tau_n) \to u(\tau + s, \tau - m, \theta_{-\tau \omega}, \tau_m)\quad \text{strongly in } L^2(\mathbb{R}^n).
\]
(133)

On the other hand, we have
\[
\int_{-m}^{0} e^{4\lambda s} \int_{\mathbb{R}^n} \left|F(x, u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau \omega}, \tau_n)) - F(x, u(\tau + s, \tau - m, \theta_{-\tau \omega}, \tau_m))\right| dx ds
\]
\[
\leq \int_{-m}^{0} \int_{\mathbb{R}^n} \left|F(x, u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau \omega}, \tau_n)) - F(x, u(\tau + s, \tau - m, \theta_{-\tau \omega}, \tau_m))\right| dx ds
\]
\[
\leq \int_{-m}^{0} \int_{\mathbb{R}^n} |f(x, \xi)| u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau \omega}, \tau_n) - u(\tau + s, \tau - m, \theta_{-\tau \omega}, \tau_m)| dx ds
\]
\[
\leq \left(\int_{-m}^{0} \int_{\mathbb{R}^n} |f(x, \xi)|^2 dx ds\right)^\frac{1}{2} \left(\int_{-m}^{0} u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-\tau \omega}, \tau_n) - u(\tau + s, \tau - m, \theta_{-\tau \omega}, \tau_m)\|^2 dx ds\right)^\frac{1}{2}.
\]
(134)
It follows from (4) that
\[
\int_{\mathbb{R}^n} |f(x, \xi)|^2 dx \leq c \left( \|u_{\delta_n}(\tau + s, \tau - t_n, \theta_{-r}\omega, \hat{u}_n)\|_{H^1}^{2p} + \|u(\tau + s, \tau - m, \theta_{-r}\omega, \bar{u}_m)\|_{H^1}^{2p} + 1 \right),
\]
which along with (133) and (134) yields (131). By an argument similar to the proof of (131), we can also show the convergence of the remaining terms on the right-hand side of (125). More precisely, we have
\[
\lim_{n \to +\infty} J_5 = -2\beta \int_{-m}^{0} e^{4\lambda s} (f(x, u(\tau + s, \tau - m, \theta_{-r}\omega, \bar{u}_m)), \bar{u}(\tau + s, \tau - m, \theta_{-r}\omega, \bar{u}_m)) ds,
\]
\[
\lim_{n \to +\infty} J_6 = 2 \int_{-m}^{0} e^{4\lambda s} (g(\tau + s, x), v(\tau + s, \tau - m, \theta_{-r}\omega, \bar{v}_m)) ds,
\]
\[
\lim_{n \to +\infty} J_7 = 2 \int_{-m}^{0} e^{4\lambda s} \theta_{-r}\omega(\tau + s)(f(x, u(\tau + s, \tau - m, \theta_{-r}\omega, \bar{u}_m)), h) ds,
\]
\[
\lim_{n \to +\infty} J_8 = 2(\nu + \beta^2 - \alpha\beta) \int_{-m}^{0} e^{4\lambda s} \theta_{-r}\omega(\tau + s)(u(\tau + s, \tau - m, \theta_{-r}\omega, \bar{u}_m), h) ds,
\]
\[
\lim_{n \to +\infty} J_9 = 2 \int_{-m}^{0} e^{4\lambda s} \theta_{-r}\omega(\tau + s) (\nabla u(\tau + s, \tau - m, \theta_{-r}\omega, \bar{u}_m), \nabla h) ds
\]
and
\[
\lim_{n \to +\infty} J_{10} = 2(\beta - \alpha) \int_{-m}^{0} e^{4\lambda s} \theta_{-r}\omega(\tau + s)(v(\tau + s, \tau - m, \theta_{-r}\omega, \bar{v}_m), h) ds.
\]
Taking the limit of (123) as \(n \to \infty\), by (122) we get
\[
\limsup_{n \to +\infty} E(u_{\delta_n}(\tau, \tau - t_n, \theta_{-r}\omega, \bar{u}_n), v_{\delta_n}(\tau, \tau - t_n, \theta_{-r}\omega, \hat{v}_n))
\leq c \epsilon^{-2\lambda m} (1 + L_0(\tau, \omega)) - e^{-4\lambda m} E(\bar{u}_m, \bar{v}_m) + E(\bar{u}, \bar{v})
\leq c \epsilon^{-2\lambda m} (1 + L_0(\tau, \omega)) + e^{-4\lambda m} \|\varphi_3\|_{L^1} + E(\bar{u}, \bar{v}).
\]
Taking the limit of (141) as \(m \to +\infty\), by (6) and (24) we get
\[
\limsup_{n \to +\infty} E(u_{\delta_n}(\tau, \tau - t_n, \theta_{-r}\omega, \bar{u}_n), v_{\delta_n}(\tau, \tau - t_n, \theta_{-r}\omega, \hat{v}_n)) \leq E(\bar{u}, \bar{v}).
\]
Similar to the proof of (131), by (133) one can verify
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^n} F(x, u(\tau, \tau - t_n, \theta_{-r}\omega, \hat{u}_n)) dx = \int_{\mathbb{R}^n} F(x, \bar{u}) dx
\]
which together with (96) and (142) yields
\[
\limsup_{n \to +\infty} \left( \|u_{\delta_n}(\tau, \tau - t_n, \theta_{-r}\omega, \hat{v}_n)\|^2 + (\nu + \beta^2 - \alpha\beta) ||u_{\delta_n}(\tau, \tau - t_n, \theta_{-r}\omega, \hat{u}_n)||^2 + \|\nabla u_{\delta_n}(\tau, \tau - t_n, \theta_{-r}\omega, \hat{u}_n)\|^2 \right) \leq ||\bar{v}||^2 + (\nu + \beta^2 - \alpha\beta) ||\bar{u}||^2 + ||\nabla \bar{u}||^2.
\]
By (119) and (143) we get
\[(u_{\delta_n}, r - m, \theta - \tau \omega, u_{n,m}), v_{\delta_n}(r, r - m, \theta - \tau \omega, v_{n,m}) \rightarrow (\bar{u}, \bar{v})\]
strongly in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. \hspace{1cm} (144)

By Lemma 2.3, (113), (120) and (144) we obtain
\[(u_n, z_n) \rightarrow (\bar{u}, \bar{v} - h\omega(-\tau)) = (\bar{u}, \bar{z})\]
strongly in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ as desired. \hspace{1cm} \Box

Next, we establish the convergence of solutions of (55)-(57) as $\delta \rightarrow 0$.

**Lemma 3.4.** Suppose (4)-(7) hold. Let $(u, v)$ and $(u_\delta, v_\delta)$ are the solutions of (19)-(21) and (55)-(57), respectively. Then for every $r \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $\varepsilon \in (0, 1)$, there exists $\delta_0 = \delta_0(\tau, \omega, T, \varepsilon) > 0$ and $c = c(\tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta_0$ and $t \in [r, \tau + T],
\[
\|u_\delta(t, \tau, \omega, u_{\delta, r}) - u(t, \tau, \omega, u_{r})\|_{H^1(\mathbb{R}^n)}^2 + \|v_\delta(t, \tau, \omega, v_{\delta, r}) - v(t, \tau, \omega, v_r)\|^2 
\leq ce^{c(t - \tau)}(\|u_{\delta,r} - u_r\|_{H^1(\mathbb{R}^n)}^2 + \|v_{\delta,r} - v_r\|^2 + \varepsilon\varepsilon). \hspace{1cm} (145)
\]

**Proof.** Let $\tilde{v} = v_\delta - v$ and $\tilde{u} = u_{\delta} - u$. We get
\[
\frac{d\tilde{u}}{dt} + \beta \tilde{u} - \tilde{v} = h(x)(G_\delta(t, \omega) - \omega(t)), \hspace{1cm} (146)
\]
\[
\frac{d\tilde{v}}{dt} + (\alpha - \beta)\tilde{v} + (\nu + \beta^2 - \alpha\beta)\tilde{u} - \Delta \tilde{u} + f(x, u_\delta) - f(x, u)
= (\beta - \alpha)h(G_\delta(t, \omega) - \omega(t)). \hspace{1cm} (147)
\]
First taking the inner product of (147) with $\tilde{v}$ in $L^2(\mathbb{R}^n)$, and then using (146) to simplify the resulting equality, we obtain
\[
\frac{1}{2} \frac{d}{dt}(\|\tilde{v}\|^2 + (\nu + \beta^2 - \alpha\beta)\|\tilde{u}\|^2 + \|\nabla \tilde{u}\|^2) + (\alpha - \beta)\|\tilde{v}\|^2 + \beta(\nu + \beta^2 - \alpha\beta)\|\tilde{u}\|^2 
+ \beta\|\nabla \tilde{u}\|^2
= (f(x, u) - f(x, u_\delta), \tilde{v}) - (\nu + \beta^2 - \alpha\beta)(G_\delta(t, \omega) - \omega(t))(\tilde{u}, \tilde{h})
+ (G_\delta(t, \omega) - \omega(t))(\Delta \tilde{u}, \tilde{h}) + (\beta - \alpha)(G_\delta(t, \omega) - \omega(t))(\tilde{v}, \tilde{h}). \hspace{1cm} (148)
\]
By (7), the nonlinear term in (148) satisfies
\[
|(f(x, u) - f(x, u_\delta), \tilde{v})| \leq c_1 \int_{\mathbb{R}^n} |(u|^{p-1} + |u_\delta|^{p-1})|\tilde{u}|\|\tilde{v}\|dx + \int_{\mathbb{R}^n} \varphi_4|\tilde{u}|\|\tilde{v}\|dx 
\leq c_2 \|\tilde{u}\|_{L^6} \|\tilde{v}\|((\|u\|_{L^{p-1}}^{p-1} + \|u_\delta\|_{L^{p-1}}^{p-1}) + \|\varphi_4\|_{L^6}) \|\tilde{u}\|_{L^3} \|\tilde{v}\| 
\leq c_3((\|\tilde{u}\|_{H^1(\mathbb{R}^n)}^2 + \|\tilde{v}\|^2)(1 + \|u\|_{H^1(\mathbb{R}^n)} + \|u_\delta\|_{H^1(\mathbb{R}^n)}^2) \hspace{1cm} (149)
\]
By Lemma 2.3, for every $\varepsilon \in (0, 1)$, there exists $\delta_0 = \delta_0(\tau, \omega, T, \varepsilon) > 0$ such that for all $0 < |\delta| < \delta_0$ and $t \in [r, \tau + T],
\|G_\delta(t, \omega) - \omega(t)\| \leq \varepsilon.$

By Young’s inequality, we find the remaining terms on the right-hand side of (148) are bounded by
\[
c_4(\|\nabla \tilde{u}\|^2 + \|\tilde{u}\|^2 + \|\tilde{v}\|^2) + c_4\varepsilon.
\]
On the other hand, by (25) and (68) one can show that there exists $c_5 = c_5(\tau, \omega, T) > 0$ such that for all for all $0 < |\delta| < \delta_0$ and $t \in [\tau, \tau + T]$

\[
\|u(t, \tau, \omega, u_\tau)\|^2_{H^1(\mathbb{R}^n)} + \|v(t, \tau, \omega, v_\tau)\|^2 + \|u_\delta(t, \tau, \omega, u_\delta, \tau)\|^2_{H^1(\mathbb{R}^n)} + \|v_\delta(t, \tau, \omega, v_\delta, \tau)\|^2 \leq c_5.
\]

Therefore, by (148)-(150), we get for all $0 < |\delta| < \delta_0$ and $t \in [\tau, \tau + T]$

\[
\frac{d}{dt} (\|\tilde{v}\|^2 + (\nu + \beta^2 - \alpha\beta)\|\tilde{u}\|^2 + \|\nabla\tilde{u}\|^2) \leq c(\|\tilde{v}\|^2 + (\nu + \beta^2 - \alpha\beta)\|\tilde{u}\|^2 + \|\nabla\tilde{u}\|^2) + \epsilon_\varepsilon,
\]

which together with Gronwall’s inequality implies the desired estimates.

Finally, we present the upper semicontinuity of random attractors as $\delta \to 0$.

**Theorem 3.5.** Suppose (4)-(7) and (11) hold. Then for every fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

\[
\lim_{\delta \to 0} d_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}(A_\delta(\tau, \omega), A_0(\tau, \omega)) = 0.
\]

**Proof.** Let $K_\delta$ and $K_0$ be the $D$-pullback absorbing sets of $\Phi_\delta$ and $\Phi_0$ whose radii $L_\delta$ and $L_0$ are determined by (76) and (24), respectively. Then it follows from Corollary 2.1 that for all for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

\[
\limsup_{\delta \to 0} \|K_\delta(\tau, \omega)\| = \limsup_{\delta \to 0} \sqrt{L_\delta(\tau, \omega)} = \sqrt{L_0(\tau, \omega)} = \|K_0(\tau, \omega)\|.
\]

Let $\delta_n \to 0$ and $(u_{\delta_n, \tau}, z_{\delta_n, \tau}) \to (u_\tau, z_\tau)$ in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Then by Lemma 3.4, we find that for all $\tau \in \mathbb{R}$, $t \geq 0$ and $\omega \in \Omega$,

\[
\Phi_{\delta_n}(t, \tau, \omega, (u_{\delta_n, \tau}, z_{\delta_n, \tau})) \to \Phi_0(t, \tau, \omega, (u_\tau, z_\tau)) \quad \text{in} \quad H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n).
\]

Then by (152), (153) and Lemma 3.3, we obtain (151) from Theorem 3.1 in [32] immediately.

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*E-mail address*: wangxiaohu@scu.edu.cn
*E-mail address*: lidingshi2006@163.com
*E-mail address*: junshen85@163.com