EIGENVALUES OF THE STURM-LIOUVILLE PROBLEM WITH A FROZEN ARGUMENT ON TIME SCALES

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Abstract. In this study, we consider a boundary value problem generated by the Sturm-Liouville problem with a frozen argument and with non-separated boundary conditions on a time scale. Firstly, we present some solutions and characteristic function of the problem on an arbitrary bounded time scale. Secondly, we prove some properties of eigenvalues and obtain a formulation for the eigenvalues-number on a finite time scale. Finally, we give an asymptotic formula for eigenvalues of the problem on another special time scale: \( T = [\alpha, \delta_1] \cup [\delta_2, \beta] \).

1. Introduction

A Sturm-Liouville equation with a frozen argument has the form

\[-y''(t) + q(t)y(a) = \lambda y(t),\]

where \( q(t) \) is the potential function, \( a \) is the frozen argument and \( \lambda \) is the complex spectral parameter. The spectral analysis of boundary value problems generated with this equation is studied in several publications [3], [11], [12], [20], [27] and references therein. This kind problems are related strongly to non-local boundary value problems and appear in various applications [4], [8], [25] and [32].

A Sturm-Liouville equation with a frozen argument on a time scale \( T \) can be given as

\[-y^{\Delta\Delta}(t) + q(t)y(a) = \lambda y^{\sigma}(t), \quad t \in T^\kappa,\]

where \( y^{\Delta\Delta} \) and \( \sigma \) denote the second order \( \Delta \)-derivative of \( y \) and forward jump operator on \( T \), respectively, \( q(t) \) is a real-valued continuous function, \( a \in T^\kappa := T \setminus (\rho(\sup T), \sup T) \), \( y^{\sigma}(t) = y(\sigma(t)) \) and \( T^\kappa = (T^\kappa)^\kappa \).

Spectral properties the classical Sturm-Liouville problem on time scales were given in various publications (see e.g. [1], [2], [4], [6], [13]-[19], [21]-[23], [28]-[31] and references therein). However, there is no any publication about the Sturm-Liouville equation with a frozen argument on a time scale.

In the present paper, we consider a boundary value problem which is generated by equation (1) and the following boundary conditions

\[
U(y) : = a_{11}y(a) + a_{12}y^\Delta(a) + a_{21}y(\beta) + a_{22}y^\Delta(\beta)
\]
\[
V(y) : = b_{11}y(a) + b_{12}y^\Delta(a) + b_{21}y(\beta) + b_{22}y^\Delta(\beta)
\]

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where α = inf T, β = ρ(sup T), α ≠ β and a_ij, b_ij ∈ R for i, j = 1, 2. We aim to give some properties of some solutions and eigenvalues of (1)-(3) for two different cases of T.

For the basic notation and terminology of time scales theory, we recommend to see [7], [9], [10] and [26].

2. Preliminaries

Let S(t, λ) and C(t, λ) be the solutions of (1) under the initial conditions
\begin{equation}
S(a, λ) = 0, S^Δ(a, λ) = 1,
\end{equation}
\begin{equation}
C(a, λ) = 1, C^Δ(a, λ) = 0,
\end{equation}
respectively. Clearly, S(t, λ) and C(t, λ) satisfy
\begin{align*}
S^ΔΔ(t, λ) + λS^σ(t, λ) &= 0 \\
C^ΔΔ(t, λ) + λC^σ(t, λ) &= q(t),
\end{align*}
respectively and so these functions and their ∆-derivatives are entire on λ for each fixed t (see [28]).

Lemma 1. Let ϕ(t, λ) be the solution of (1) under the initial conditions ϕ(a, λ) = δ_1, ϕ^Δ(t, λ) = δ_2 for given numbers δ_1, δ_2. Then ϕ(t, λ) = δ_1C(t, λ) + δ_2S(t, λ) is valid on T.

Proof. It is clear that the function y(t, λ) = δ_1C(t, λ) + δ_2S(t, λ) is the solution of the initial value problem
\begin{align*}
y^ΔΔ(t) + λy^σ(t) &= q(t)δ_1 \\
y(a, λ) &= δ_1 \\
y^Δ(a, λ) &= δ_2.
\end{align*}
We obtain by taking into account uniqueness of the solution of an initial value problem that y(t, λ) = ϕ(t, λ).

Consider the function
\begin{equation}
Δ(λ) : \text{det} \begin{pmatrix} U(C) & V(C) \\ U(S) & V(S) \end{pmatrix}.
\end{equation}
It is obvious Δ(λ) is also entire.

Theorem 1. The zeros of the function Δ(λ) coincide with the eigenvalues of the problem (1)-(3).

Proof. Let λ₀ be an eigenvalue and y(t, λ₀) = δ_1C(t, λ₀) + δ_2S(t, λ₀) is the corresponding eigenfunction, then y(t, λ₀) satisfies (2) and (3). Therefore,
\begin{align*}
δ_1U(C(t, λ₀)) + δ_2U(S(t, λ₀)) &= 0, \\
δ_1V(C(t, λ₀)) + δ_2V(S(t, λ₀)) &= 0.
\end{align*}
It is obvious that y(t, λ₀) ≠ 0 iff the coefficients-determinant of the above system vanishes, i.e., Δ(λ₀) = 0.

Since Δ(λ) is an entire function, eigenvalues of the problem (1)-(3) are discrete.
3. Eigenvalues of (1)-(3) on a finite time scale

Let $T$ be a finite time scale such that there are $m$ (or $r$) many elements which are larger (or smaller) than $a$ in $T$. Assume $m \geq 1$, $r \geq 0$ and $r + m \geq 2$. It is clear that the number of elements of $T$ is $n = m + r + 1$. We can write $T$ as follows

$$T = \left\{ \rho^r (a), \rho^{r-1} (a), \ldots, \rho^2 (a), \rho (a), \sigma (a), \sigma^2 (a), \ldots, \sigma^{m-1} (a), \sigma^m (a) \right\},$$

where $\sigma^j = \sigma^{j-1} \circ \sigma$, $\rho^j = \rho^{j-1} \circ \rho$ for $j \geq 2$, $\rho^r (a) = \alpha$ and $\sigma^{m-1} (a) = \beta$.

**Lemma 2.** i) If $r \geq 3$ and $m \geq 2$, the following equalities hold for all $\lambda$

$$S (\alpha, \lambda) = (-1)^r \mu^r (a) \left[ \mu^r (a) \mu^3 (a) \ldots \mu^r (a) \right]^2 \lambda^{r-1} + O \left( \lambda^{r-2} \right)$$

$$S^\sigma (\alpha, \lambda) = (-1)^{r-1} \mu^r (a) \left[ \mu^r (a) \mu^3 (a) \ldots \mu^{r-1} (a) \right]^2 \lambda^{r-2} + O \left( \lambda^{r-3} \right)$$

$$S (\beta, \lambda) = S^{\sigma^{m-1}} (\alpha, \lambda) = (-1)^m \left[ \mu (a) \mu^\sigma (a) \ldots \mu^{\sigma^{m-3}} (a) \right] \lambda^{m-2} \mu^{\sigma^{m-2}} (a) + O \left( \lambda^{m-3} \right)$$

$$S^\sigma (\beta, \lambda) = S^{\sigma^{m-1}} (\alpha, \lambda) = (-1)^{m+1} \left[ \mu (a) \mu^\sigma (a) \ldots \mu^{\sigma^{m-2}} (a) \right] \lambda^{m-1} \mu^{\sigma^{m-1}} (a) + O \left( \lambda^{m-2} \right)$$

$$C (\alpha, \lambda) = (-1)^r \left[ \mu^\sigma (a) \mu^{\sigma^2} (a) \ldots \mu^r (a) \right] \lambda^r + O \left( \lambda^{r-1} \right)$$

$$C^\sigma (\alpha, \lambda) = (-1)^{r-1} \left[ \mu^\sigma (a) \mu^{\sigma^2} (a) \ldots \mu^{\sigma^{r-1}} (a) \right] \lambda^{r-1} + O \left( \lambda^{r-2} \right)$$

$$C (\beta, \lambda) = C^{\sigma^{m-1}} (\alpha, \lambda) = (-1)^m \left[ \mu (a) \mu^\sigma (a) \ldots \mu^{\sigma^{m-3}} (a) \right] \mu^{\sigma^{m-2}} (a) \lambda^{m-2} + O \left( \lambda^{m-3} \right)$$

$$C^\sigma (\beta, \lambda) = C^{\sigma^{m-1}} (\alpha, \lambda) = (-1)^{m+1} \mu (a) \left[ \mu^\sigma (a) \mu^2 (a) \ldots \mu^{\sigma^{m-2}} (a) \right] \mu^{\sigma^{m-1}} (a) \lambda^{m-1} + O \left( \lambda^{m-2} \right),$$

where $O(\lambda^l)$ denotes polynomials whose degrees are $l$.

ii) If $r \in \{0, 1, 2\}$ or $m \in \{0, 1\}$, degrees of all above functions are vanish.

**Proof.** It is clear from $f^\sigma (t) = f(t) + \mu(t)f^\lambda (t)$ that $S^\sigma (\alpha, \lambda) = \mu (a)$ and $C^\sigma (\alpha, \lambda) = 1$. On the other hand, since $S(t, \lambda)$ and $C(t, \lambda)$ satisfy (1) then the following equalities hold for each $t \in T^\kappa$ and for all $\lambda$.

$$S^\sigma (t, \lambda) = \left( 1 + \frac{\mu(t)}{\mu^\sigma(t)} - \lambda \mu(t) \mu^\sigma(t) - \lambda \mu^2(t) \mu^\sigma(t) \right) S^\sigma (t, \lambda)$$

$$-\mu^\sigma(t) S(t, \lambda)$$

$$C^\sigma (t, \lambda) = \left( -\mu(t) \mu^\sigma(t) \lambda + \frac{\mu(t) + \mu^\sigma(t)}{\mu(t)} \right) C^\sigma (t, \lambda)$$

$$+ \mu(t) \mu^\sigma(t) q(t)$$
Corollary 1. \( \deg C(\alpha, \lambda)S^\sigma(\beta, \lambda) = \begin{cases} r + m - 1, & r > 0 \text{ and } m > 1 \\ 1, & \text{the other cases} \end{cases} \)

Lemma 3. The following equalities hold for all \( \lambda \in \mathbb{C} \).
\[
S^\sigma(\alpha, \lambda)C(\alpha, \lambda) - S(\alpha, \lambda)C^\sigma(\alpha, \lambda) = A\lambda^\delta + O(\lambda^{\delta-1})
\]
\[
S^\sigma(\beta, \lambda)C(\beta, \lambda) - S(\beta, \lambda)C^\sigma(\beta, \lambda) = B\lambda^\gamma + O(\lambda^{\gamma-1})
\]
where \( A = (-1)^r \mu(\alpha) \mu^\rho(\alpha) \left[ \mu^{\sigma^2}(\alpha) \cdots \mu^{\sigma^r-1}(\alpha) \right]^2 \mu^{\sigma^r}(\alpha) q(\alpha) \),
\[
B = (-1)^m \mu(\beta) \left[ \mu(\alpha) \mu^\sigma(\alpha) \cdots \mu^{\sigma^m-2}(\alpha) \right]^2 q(\rho(\beta)),
\]
\[
\delta = \begin{cases} r - 2, & r \geq 3 \\ 0, & r < 3 \end{cases}
\]
\[
\gamma = \begin{cases} m - 2, & m \geq 3 \\ 0, & m < 3. \end{cases}
\]
Proof. Consider the function
\[
\varphi(t, \lambda) := \frac{1}{\mu(t)} \left[ S^\sigma(t, \lambda)C(t, \lambda) - S(t, \lambda)C^\sigma(t, \lambda) \right]
\]
It is clear that
\[
\varphi(t, \lambda) := \left[ S^\Delta(t, \lambda)C(t, \lambda) - S(t, \lambda)C^\Delta(t, \lambda) \right] = W[C(t, \lambda), S(t, \lambda)]
\]
and it is the solution of initial value problem
\[
\varphi^\Delta(t) = -q(t) S^\sigma(t, \lambda)
\]
\[
\varphi(a) = 1
\]
Therefore, we can obtain the following relations
\[
\varphi^\sigma(t, \lambda) = \varphi(t, \lambda) - \mu(t) q(t) S^\sigma(t, \lambda)
\]
\[
\varphi^\rho(t, \lambda) = \varphi(t, \lambda) + \mu^\rho(t) q(\rho(t)) S(t, \lambda).
\]
By using (9), (10), (14) and (15), the proof is completed. \( \square \)
Corollary 2. i) \( \deg (S^\sigma (\alpha, \lambda) C(\alpha, \lambda) - S (\alpha, \lambda) C^\sigma (\alpha, \lambda)) < \deg C(\alpha, \lambda) S^\sigma (\beta, \lambda) \),

ii) \( \deg (S^\sigma (\beta, \lambda) C (\beta, \lambda) - S (\beta, \lambda) C^\sigma (\beta, \lambda)) < \deg C(\alpha, \lambda) S^\sigma (\beta, \lambda) \).

The next theorem gives the number of eigenvalues of the problem (1)-(3) on \( T \). Recall \( n = m + r + 1 \) denotes the number of elements of \( T \) and put \( A = \begin{pmatrix} a_{11} \mu (\alpha) - a_{12} & b_{11} \mu (\alpha) - b_{12} \\ a_{22} & b_{22} \end{pmatrix} \).

Theorem 2. If \( \det A \neq 0 \), the problem (1)-(3) has exactly \( n - 2 \) many eigenvalues with multiplications, otherwise the eigenvalues-number of (1)-(3) is least than \( n - 2 \).

Proof. Since \( T \) is finite, \( \Delta(\lambda) \) is a polynomial and its degree gives the number eigenvalues of the problem. It can be calculated from (6)-(14) that

\[
\Delta(\lambda) = \frac{1}{\mu(\alpha) \mu(\beta)} \det \begin{pmatrix} a_{11} \mu(\alpha) - a_{12} & b_{11} \mu(\alpha) - b_{12} \\ a_{22} & b_{22} \end{pmatrix} C(\alpha, \lambda) S^\sigma (\beta, \lambda) + \\
+ \frac{1}{\mu(\alpha)} \det \begin{pmatrix} a_{11} & a_{12} \\ b_{11} & b_{12} \end{pmatrix} (S^\sigma (\alpha, \lambda) C(\alpha, \lambda) - S (\alpha, \lambda) C^\sigma (\alpha, \lambda)) + \\
+ \frac{1}{\mu(\beta)} \det \begin{pmatrix} a_{21} & a_{22} \\ b_{21} & b_{22} \end{pmatrix} (S^\sigma (\beta, \lambda) C (\beta, \lambda) - S (\beta, \lambda) C^\sigma (\beta, \lambda)) + O(\lambda^{n+m-2}).
\]

According to Corollary 1 and Corollary 2, if \( \det A \neq 0 \), \( \deg \Delta(\lambda) = \deg C(\alpha, \lambda) S^\sigma (\beta, \lambda) = m + r - 1 = n - 2 \).

Corollary 3. i) The eigenvalues-number of (1)-(3) depends only on the elements-number of \( T \) and the coefficients of the boundary conditions (2) and (3). On the other hand, it does not depend on \( q(t) \) and \( a \) (neither value nor location of \( a \) on \( T \)).

ii) If \( \det A \neq 0 \), the eigenvalues-number of (1)-(3) and the elements-number of \( T \) determine uniquely each other.

Example 1 (Separated boundary conditions). Let us consider the time scale \( T = \{0, 1, 2, ..., n - 1\} \) and the following boundary value problem which appears some applications.

(16) \(-y^\Delta (t) + q(t) y(a) = \lambda y (t + 1), t \in T^2 = \{0, 1, ..., n - 3\} \)

(17) \(y^\Delta (0) + h y(0) = 0 \)

(18) \(y^\Delta (n - 2) + H y(n - 2) = 0, \)

where \( 0 \leq a \leq n - 2 \) and \( h, H \in \mathbb{R} \). According to Theorem 2, if \( h \neq 1 \), the eigenvalues-number of this problem is exactly \( n - 2 \), otherwise less than \( n - 2 \).

Now, we want to give a theorem which includes some informations about the eigenvalues of (16)-(18).
Theorem 3. Let $Q = Q_1 + Q_2$,

$$Q_1 = \begin{pmatrix} 2 - \frac{1}{t_H} & -1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & -1 & 1 - H \end{pmatrix}_{(n-2) \times (n-2)}$$

and $Q_2 = (q_{ij})_{(n-2) \times (n-2)}$, where $q_{ij} = \begin{cases} q(i-1), & j = a \\ 0, & j \neq a \end{cases}$ and $h \in \mathbb{R} - \{1\}$.

The problem (16)-(18) and the matrix $Q$ have the same eigenvalues.

Proof. Let $y(t)$ be an eigenfunction of (16)-(18). Since $y^\Delta(t) = y(t+1) - y(t)$ for $t \in \mathbb{T}^c$ and $y^{\Delta\Delta}(t) = y(t+2) - 2y(t+1) + y(t)$ for $t \in \mathbb{T}^c$, we can write (16)-(18) as follows

(19) \( y(t+2) - 2y(t+1) + y(t) = q(t)y(a) - \lambda y(t+1), \ t \in \mathbb{T}^c, \)

(20) \( y(1) = (h-1)y(0), \)

(21) \( y(n) = (1-H)y(n-1). \)

We obtain from (19)-(21) a linear system whose coefficients-matrix is $\lambda I - Q$. Therefore it is concluded that the boundary value problem (16)-(18) and the matrix $Q$ have the same eigenvalues. \hfill \square

Remark 1. The result in the Theorem 3 can be generalized easily to (1)-(3) on the general discrete time scale.

Remark 2. As is known, all eigenvalues of the classical Sturm-Liouville problem with separated boundary conditions on time scales are real and algebraically simple [2]. However, the Sturm-Liouville problem with the frozen argument may have non-real or non-simple eigenvalues even if it is equipped with separated boundary conditions.

We end this section with two examples. The problem in the first example has non-real or non-simple eigenvalues, unlike, all eigenvalues of the latter problem are real and simple.

Example 2. Consider the following problem on $\mathbb{T} = \{0, 1, 2, 3, 4, 5\}$.

$$L_1 : \begin{cases} -y^{\Delta\Delta}(t) + q_1(t)y(3) = \lambda y^\sigma(t), \ t \in \{0, 1, 2, 3\} \\ y^{\Delta}(0) + \frac{1}{2}y(0) = 0 \\ y^{\Delta}(4) + y(4) = 0, \end{cases}$$
where \( q_1(t) = \begin{cases} 
-3 & t = 0 \\
10 & t = 1 \\
-5 & t = 2 \\
1 & t = 3 
\end{cases} \). According to Theorem 3, eigenvalues of \( L_1 \) coincide with eigenvalues of the matrix \( Q_1 = \begin{pmatrix} 
0 & -1 & -3 & 0 \\
-1 & 2 & 9 & 0 \\
0 & -1 & -3 & -1 \\
0 & 0 & 0 & 0 
\end{pmatrix} \) and they are \( \lambda_1 = \frac{1}{2} - \frac{1}{2}i\sqrt{7} \), \( \lambda_2 = 0 \), \( \lambda_3 = -\frac{1}{2} + \frac{1}{2}i\sqrt{7} \), \( \lambda_4 = -\frac{1}{2} - \frac{1}{2}i\sqrt{7} \).

**Example 3.** Consider the following problem on \( \mathbb{T} = \{0, 1, 2, 3, 4\} \).

\[
L_2 : \begin{cases} 
-y^\Delta(t) + q_2(t)y(4) = \lambda y^\sigma(t), & t \in \{0, 1, 2, 3\} \\
y^\Delta(0) = 0 & y^\Delta(4) = 0,
\end{cases}
\]

where \( q_2(t) = t \).

Clearly, eigenvalues of \( L_2 \) coincide with eigenvalues of \( Q_2 = \begin{pmatrix} 
1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 1 \\
0 & -1 & 2 & 1 \\
0 & 0 & -1 & 4 
\end{pmatrix} \) and they are \( \lambda_1 = 2 + \sqrt{3} \), \( \lambda_2 = 2 - \sqrt{3} \), \( \lambda_3 = 3 \), \( \lambda_4 = 2 \).

### 4. Eigenvalues of (1)-(3) on the time scale \( \mathbb{T} = [\alpha, \delta_1] \cup [\delta_2, \beta] \)

In this section, we investigate eigenvalues of the problem (1)-(3) on another special time scale: \( \mathbb{T} = [\alpha, \delta_1] \cup [\delta_2, \beta] \), where \( \alpha < a < \delta_1 < \delta_2 < \beta \). We assume that \( a \in (\alpha, \delta_1) \). The similar results can be obtained in the case when \( a \in (\delta_2, \beta) \).

The following relations are valid on \( [\alpha, \delta_1] \) (see [11]).

\[
S(t, \lambda) = \frac{\sin \sqrt{\lambda} (t - a)}{\sqrt{\lambda}} \\
C(t, \lambda) = \cos \sqrt{\lambda} (t - a) + \int_a^t \frac{\sin \sqrt{\lambda} (t - \xi)}{\sqrt{\lambda}} q(\xi) d\xi
\]

The following asymptotic relations for the solutions \( S(t, \lambda) \) and \( C(t, \lambda) \) can be proved by using a method similar to one in [29].

(22) \( S(t, \lambda) = \begin{cases} 
\frac{\sin \sqrt{\lambda} (t - a)}{\sqrt{\lambda}}, & t \in [\alpha, \delta_1], \\
\delta^2 \sqrt{\lambda} \cos \sqrt{\lambda} (\delta_1 - a) \sin \sqrt{\lambda} (\delta_2 - t) + O(\exp |\tau| (\delta + a - t)), & t \in [\delta_2, \beta],
\end{cases} \)

(23) \( S^\Delta(t, \lambda) = \begin{cases} 
\cos \sqrt{\lambda} (t - a), & t \in [\alpha, \delta_1], \\
-\delta^2 \sqrt{\lambda} \cos \sqrt{\lambda} (\delta_1 - a) \cos \sqrt{\lambda} (\delta_2 - t) + O \left( \sqrt{\lambda} \exp |\tau| (\delta + a - t) \right), & t \in [\delta_2, \beta],
\end{cases} \)
\( \lambda \) and \( \delta \) satisfies initial value problem

\[
\varphi (t, \lambda) := C_{\lambda}(t, \lambda)S(t, \lambda) - C(t, \lambda)S_{\lambda}(t, \lambda)
\]

satisfies initial value problem

\[
\varphi^{\Delta} (t) = q(t)S^{\sigma} (t, \lambda), \ t \in [\alpha, \delta_1]
\]

and

\[
\varphi (a) = 1
\]

Hence, we get proof by using (22). \( \square \)

**Theorem 4.**

i) The problem (1)-(3) on \( T = [\alpha, \delta_1] \cup [\delta_2, \beta] \) has countable many eigenvalues such as \( \{ \lambda_n \}_{n \geq 0} \).

ii) The numbers \( \{ \lambda_n \}_{n \geq 0} \) are real for sufficiently large \( n \).

iii) If \( a_{22}b_{12} - a_{12}b_{22} \neq 0 \) and \( \beta - \delta_2 = \delta_1 - \alpha \), the following asymptotic formula holds for \( n \to \infty \).

\[
\sqrt{\lambda_n} = \frac{(n-1)\pi}{2(\beta - \delta_2)} + O \left( \frac{1}{n} \right)
\]

**Proof.** The proof of (i) is obvious, since \( \Delta(\lambda) \) is entire on \( \lambda \).

By calculating directly, we get

\[
\Delta(\lambda) = \det \begin{pmatrix} U(C) & V(C) \\ U(S) & V(S) \end{pmatrix}
\]

\[
= \left[ C_{\lambda}(\beta, \lambda)S_{\lambda}(\alpha, \lambda) - C_{\lambda}(\alpha, \lambda)S_{\lambda}(\beta, \lambda) \right] + \left[ C_{\lambda}(\beta, \lambda)S_{\lambda}(\beta, \lambda) - C(\beta, \lambda)S_{\lambda}(\beta, \lambda) \right] + \left[ C_{\lambda}(\beta, \lambda)S(\alpha, \lambda) - C(\alpha, \lambda)S_{\lambda}(\alpha, \lambda) \right] + O(\lambda \exp |\tau| (\beta - \alpha - \delta)).
\]
It follows from (22)-(25) and Lemma 4 that
\[
\Delta(\lambda) = (a_{22}b_{12} - a_{12}b_{22})\delta^2 \lambda^{3/2} \sin \sqrt{\lambda}(\delta_1 - \alpha) \cos \sqrt{\lambda}(\beta - \delta_2) \\
+ O(\lambda \exp|\tau| (\beta - \alpha - \delta))
\]
is valid for $|\lambda| \to \infty$. Since $a_{22}b_{12} - a_{12}b_{22} \neq 0$ and $\beta - \delta_2 = \delta_1 - \alpha$, the numbers $\{\lambda_n\}_{n \geq 0}$ are roots of
\[
(27) \quad \lambda^2 \frac{\sin 2\sqrt{\lambda}(\beta - \delta_2)}{\sqrt{\lambda}} + O(\lambda \exp 2|\tau| (\beta - \delta_2)) = 0.
\]
Now, we consider the region
\[
G_n := \{\lambda \in \mathbb{C} : \lambda = \rho^2, |\rho| < \frac{n\pi}{2(\beta - \delta_2)} + \varepsilon\}
\]
where $\varepsilon$ is sufficiently small number. There exist some positive constants $C_\varepsilon$ such that,
\[
|\lambda^2 \frac{\sin 2\sqrt{\lambda}(\beta - \delta_2)}{\sqrt{\lambda}}| \geq C_\varepsilon |\lambda|^{3/2} \exp 2|\tau| (\beta - \delta_2)
\]
for sufficiently large $\lambda \in \partial G_n$. Thus, we establish the proof of (ii). On the other hand by using Rouche’s theorem to (27) on $G_n$, we can show clearly that (26) holds for sufficiently large $n$. \hfill \Box

**Remark 3.** Since $\mu(\alpha) = 0$ in the considered time scale, the term $a_{22}b_{12} - a_{12}b_{22}$ is not another than $\det A$ in section 3.

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