NONLINEAR YOUNG INTEGRALS VIA FRACTIONAL CALCULUS

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Abstract. For Hölder continuous functions $W(t, x)$ and $\varphi_t$, we define nonlinear integral $\int_a^b W(dt, \varphi_t)$ via fractional calculus. This nonlinear integral arises naturally in the Feynman-Kac formula for stochastic heat equations with random coefficients [3]. We also define iterated nonlinear integrals.

1. Introduction

Let $\{\varphi_t, t \geq 0\}$ be a Hölder continuous function and let $\{W(t, x), t \geq 0, x \in \mathbb{R}^d\}$ be another jointly Hölder continuous function of several variables. In authors’ recent paper [3] the nonlinear Young integral $\int_a^b W(dt, \varphi_t)$ is introduced to establish the Feynman-Kac formula for general stochastic partial differential equations with random coefficients, namely, $\partial_t u(t, x) + Lu(t, x) + u(t, x) \partial_t W(t, x) = 0$, where $Lu(t, x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x, W) \partial_{x_i x_j} u(t, x) + \sum_{i=1}^d b_i(t, x, W) \partial_{x_i} u(t, x)$ with the coefficients $a_{ij}$ and $b_i$ depending on $W$. In that paper, the nonlinear Young integral $\int_a^b W(dt, \varphi_t)$ was defined by approximation, in particular, by the sewing lemma of [1]. In this paper, we study the nonlinear Young integral $\int_a^b W(dt, \varphi_t)$ by means of fractional calculus. This approach may provide more detailed properties of the solutions to the equations (see [4] and [5]).

To expand the solution of a (nonlinear) differential equation with explicit remainder term we need to define (iterated) multiple integrals (see [2]). We shall also give a definition of the iterated nonlinear Young integrals. Some elementary estimate is also obtained.

The paper is organized as follows. Section 2 briefly recall some preliminary material on fractional calculus that are needed lately. Section 3 deals with nonlinear Young integral and Section 4 is concerned with iterated nonlinear Young integral.

2. Fractional integrals and derivatives

In this section we recall some results from fractional calculus.

Let $-\infty < a < b < \infty$, $\alpha > 0$ and $p \geq 1$ be real numbers. Denote by $L^p(a, b)$ the space of all measurable functions on $(a, b)$ such that

$$\|f\|_p := \left( \int_a^b |f(t)|^p dt \right)^{1/p} < \infty.$$
The left-sided fractional Riemann-Liouville integral \( I_{a+}^\alpha \) is defined as
\[
I_{a+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \, ds, \quad t \in (a, b)
\]
and the right-sided fractional Riemann-Liouville integral \( I_{b-}^\alpha \) is defined as
\[
I_{b-}^\alpha f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) \, ds, \quad t \in (a, b)
\]
where \((-1)^{-\alpha} = e^{-i\pi \alpha}\) and \(\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} \, dr\) is the Euler gamma function. Let \( I_{a+}^\alpha (L^p) \) (resp. \( I_{b-}^\alpha (L^p) \)) be the image of \(L^p(a, b)\) by the operator \( I_{a+}^\alpha \) (resp. \( I_{b-}^\alpha \)). If \( f \in I_{a+}^\alpha (L^p) \) (resp. \( f \in I_{b-}^\alpha (L^p) \)) and \(0 < \alpha < 1\), then the (left-sided or right-sided) Weyl derivatives are defined (respectively) as
\[
D_{a+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(t)}{(t-a)^\alpha} + \frac{1}{\alpha} \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} \, ds \right)
\]
and
\[
D_{b-}^\alpha f(t) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left( \frac{f(t)}{(b-t)^\alpha} + \frac{1}{\alpha} \int_t^b \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} \, ds \right),
\]
where \(a \leq t \leq b\) (the convergence of the integrals at the singularity \(s = t\) holds point-wise for almost all \(t \in (a, b)\) if \(p = 1\) and moreover in \(L^p\)-sense if \(1 < p < \infty\)).

It is clear that if \(f\) is Hölder continuous of order \(\mu > \alpha\), then the two Weyl derivatives exist.

For any \(\beta \in (0, 1)\), we denote by \(C^\beta([a, b])\) the space of \(\beta\)-Hölder continuous functions on the interval \([a, b]\). We will make use of the notation
\[
\|f\|_{\infty; a, b} = \sup_{a \leq r \leq b} |f(r)|,
\]
where \(f : \mathbb{R} \to \mathbb{R}\) is a given continuous function.

It is well-known that \(C^\beta([a, b])\) with the Hölder norm \(\|f\|_{\beta; a, b} + \|f\|_{\infty; a, b}\) is a complete Banach space. But it is not separable.

Using the fractional calculus, we have

**Proposition 2.1.** Let \(0 < \alpha < 1\). If \(f\) and \(g\) are continuously differentiable functions on the interval \([a, b]\), then
\[
\int_a^b f(t) \, dg = (-1)^\alpha \int_a^b \left( D_{a+}^\alpha f(t) \right) \left( D_{b-}^{1-\alpha} g_b_{-}(t) \right) \, dt,
\]
where \(g_{b-}(t) = g(t) - g(b)\).

In what follows \(\kappa\) denotes a universal generic constant depending only on \(\lambda, \tau, \alpha\) and independent of \(W, \varphi\) and \(a, b\). The value of \(\kappa\) may vary from occurrence to occurrence.

For two function \(f, g : [a, b] \to \mathbb{R}\), we can define the Riemann-Stieltjes integral \(\int_a^b f(t) \, dg(t)\). Here we recall a result which is well-known (see for example [7], [2] or [4], [5]).
Lemma 2.2. Let $f$ and $g$ be Hölder continuous functions of orders $\alpha$ and $\beta$, respectively. Suppose that $\alpha + \beta > 1$. Then the Riemann-Stieltjes integral $\int_a^b f(t)dg(t)$ exists and for any $\gamma \in (1 - \beta, \alpha)$, we have

$$\int_a^b f(t)dg(t) = (-1)^\gamma \int_a^b D_a^\gamma f(t)D_b^1 - \gamma g_b(t)dt. \quad (2.6)$$

Moreover, there is a constant $\kappa$ such that

$$\left| \int_a^b f(t)dg(t) \right| \leq \kappa \|g\|_{\beta, a, b} \left( \|f\|_{\infty, a, b} |b - a|^{\beta} + \|f\|_{\alpha, a, b} |b - a|^{\alpha + \beta} \right). \quad (2.7)$$

Proof. We refer to [7] or [2] for a proof of (2.6). We shall outline a proof of (2.7). Let $\gamma$ be such that $\alpha > \gamma > 1 - \beta$. Applying fractional integration by parts formula (2.6), we obtain

$$\left| \int_a^b f(t)dg(t) \right| \leq \int_a^b |D_a^\gamma f(t)D_b^1 - \gamma g_b(t)|dt.$$ 

From (2.3) and (2.4) it is easy to see

$$|D_b^1 - \gamma g_b(t)| \leq \kappa \|f\|_{\infty, a, b} \leq |b - r|^{\beta + \gamma - 1}$$

and

$$|D_a^\gamma f(t)| \leq \kappa \|f\|_{\infty, a, b} (t - a)^{-\gamma} + \|f\|_{\alpha, a, b} (t - a)^{\alpha - \gamma}.$$ 

Therefore

$$\left| \int_a^b f(t)dg(t) \right| \leq \kappa \|g\|_{\beta, a, b} \left( \|f\|_{\infty, a, b} \int_a^b (t - a)^{-\gamma} |b - t|^{\beta + \gamma - 1} dt \right) \leq \kappa \|g\|_{\beta, a, b} \int_a^b (t - a)^{\alpha - \gamma} |b - t|^{\beta + \gamma - 1} dt.$$ 

The integrals on the right hand side can be computed by making the substitution $t = b - (b - a)s$, hence we derive (2.7).

We also need the following lemma in the proof of our main results.

Lemma 2.3. Let $f(s, t), a \leq s < t \leq b$ be a measurable function of $s$ and $t$ such that

$$\int_a^b \int_a^t \frac{|f(s, t)|}{(t - s)^{1-\alpha}} dsdt < \infty.$$ 

Then

$$\int_a^b \int_a^t f(s, t)\big|_{v=t} dsdt = (-1)^\alpha \int_a^b \int_a^t f(s, t)\big|_{v=t} dsdt.$$ 

Proof. An application of Fubini’s theorem yields

$$\int_a^b \int_a^t f(s, t)\big|_{v=t} dsdt = \frac{1}{\Gamma(\alpha)} \int_a^b \int_a^t \frac{f(s, t)}{(t - s)^{1-\alpha}} dsdt$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^b \int_s^b \frac{f(s, t)}{(t - s)^{1-\alpha}} dt ds$$

$$= (-1)^\alpha \int_a^b \int_s^b f(s, t)\big|_{v=t} dsdt$$

which is the lemma.
3. Nonlinear integral

In this section we shall use fractional calculus to define the (pathwise) nonlinear integral \( \int_a^b W(dt, \varphi_t) \). This method only relies on regularity of the sample paths of \( W \) and \( \varphi \). More precisely, it is applicable to stochastic processes with Hölder continuous sample paths.

Another advantage of this approach is that in the theory of stochastic processes it is usually difficult to obtain almost sure type of results. If the sample paths of the process is Hölder continuous, then one can apply this approach to each sample path and almost surely results are then automatic.

In what follows, we shall use \( W \) to denote a deterministic function \( W : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \). We make the following assumption on the regularity of \( W \):

There are constants \( \tau, \lambda \in (0, 1] \), \( \beta \geq 0 \) such that for all \( a < b \), the seminorm

\[
\|W\|_{\tau, \lambda; a,b} := \sup_{a \leq s < t \leq b} \frac{|W(s, x) - W(t, x) - W(s, y) + W(t, y)|}{|t - s|^\tau |x - y|^\lambda} + \sup_{a \leq s < t \leq b} \frac{|W(t, y) - W(t, x)|}{|t - s|^\tau |x - y|^\lambda} + \sup_{a \leq s \leq t \leq b} \frac{|W(t, y) - W(t, x)|}{|x - y|^\lambda},
\]

is finite.

About the function \( \varphi \), we assume

(\( \phi \)) \( \varphi \) is locally Hölder continuous of order \( \gamma \in (0, 1] \). That is the seminorm

\[
\|\varphi\|_{\gamma; a,b} = \sup_{a \leq s < t \leq b} \left| \frac{\varphi(t) - \varphi(s)}{|t - s|^\gamma} \right|
\]

is finite for every \( a < b \).

Among three terms appear in (\( W \)), we will pay special attention to the first term. Thus, we denote

\[
[W]_{\tau, \lambda; a,b} := \sup_{a \leq s < t \leq b} \sup_{x,y \in \mathbb{R}^d; x \neq y} \frac{|W(s, x) - W(t, x) - W(s, y) + W(t, y)|}{|t - s|^\tau |x - y|^\lambda}.
\]

If \( a, b \) is clear in the context, we frequently omit the dependence on \( a, b \). For instance, \( \|W\|_{\tau, \lambda} \) is an abbreviation for \( \|W\|_{\tau, \lambda, a,b} \), \( \|\varphi\|_\gamma \) is an abbreviation for \( \|\varphi\|_{\gamma; a,b} \) and so on. We shall assume that \( a \) and \( b \) are finite. Thus it is easy to see that for any \( c \in [a, b] \)

\[
\sup_{a \leq t \leq b} |\varphi(t)| = \sup_{a \leq t \leq b} |\varphi(c) + \varphi(t) - \varphi(c)| \leq |\varphi(c)| + \|\varphi\|_\gamma |b - a|^{\gamma} < \infty.
\]

Thus assumption (\( \phi \)) also implies that

\[
\|\varphi\|_\infty := \sup_{a \leq t \leq b} |\varphi(t)| < \infty.
\]

For the results presented in this section, the condition (\( W \)) can be relaxed to
There are constants $\tau, \lambda \in (0, 1]$, such that for all $a < b$ and compact set $K$ in $\mathbb{R}^d$, the seminorm
\[
\sup_{a \leq s < t \leq b, x, y \in K, x \neq y} \frac{|W(s, x) - W(t, x) - W(s, y) + W(t, y)|}{|t - s|^\tau |x - y|^{\lambda}} + \sup_{a \leq t \leq b, x, y \in K, x \neq y} \frac{|W(t, y) - W(t, x)|}{|x - y|^\lambda},
\]
is finite.

However, for simplicity, we only employ $(W)$.

One of our main results in this section is to define $\int_a^b W(dt, \varphi_t)$ under the condition $\lambda\gamma + \tau > 1$ through fractional integration by parts technique. The following definition is motivated from Lemma 2.2.

**Definition 3.1.** We define
\[
\int_a^b W(dt, \varphi_t) = (-1)^\alpha \int_a^b D_{a+}^{\alpha, t'} D_{b-}^{1-\alpha, t} W_{b-}(t, \varphi_t)|_{t' = t} dt
\]
whenever the right hand side makes sense.

**Remark 3.2.** Assume $d = 1$. Let $W(t, x) = g(t)x$ be of the product form and let $\varphi(t) = f(t)$, where $g$ is a Hölder continuous function of exponent $\tau$ and $f$ is a Hölder continuous function of exponent $\lambda$. If $1 - \tau < \alpha < \lambda$, then
\[
\int_a^b W(dt, \varphi_t) = (-1)^\alpha \int_a^b D_{a+}^{\alpha, t'} D_{b-}^{1-\alpha, t} W_{b-}(t, \varphi_t)|_{t' = t} dt
\]
\[
= (-1)^\alpha \int_a^b D_{b-}^{1-\alpha, t} g_b(t) D_{a+}^{\alpha, t} f(t) dt.
\]

Thus from (2.6), $\int_a^b W(dt, \varphi_t)$ is an extension of the classical Young’s integral $\int_a^b f(t)dg(t)$ (see [2], [6], [7]). For general $d$, if $W(t, x) = \sum_{i=1}^d g_i(t)x_i$ and $\varphi_i(t) = f_i(t)$, then it is easy to see that $\int_a^b W(dt, \varphi_t) = \sum_{i=1}^d \int_a^b f_i(t)dg_i(t)$.

The following result clarifies the context in which Definition 3.1 is justified.

**Theorem 3.3.** Assume the conditions $(W)$ and $(\phi)$ are satisfied. In addition, we suppose that $\lambda\gamma + \tau > 1$. Let $\alpha \in (1 - \tau, \lambda\tau)$. Then the right hand side of (3.2) is
finite and is independent of $\alpha \in (1 - \tau, \lambda)$. As a consequence, we have

\begin{equation}
\int_a^b W(dt, \varphi_t) \leq (1-a)^{\tau}(b-a) + a \int_a^b \frac{W(t, \varphi_t) - W(s, \varphi_t)}{(s-t)^{2-\alpha}} dsdt.
\end{equation}

Proof. We denote $\|W\| = \|W\|_{r, \lambda; a, b}$. First by the definitions of fractional derivatives (2.3) and (2.4), we have

\begin{equation}
D^{1-\alpha,t}_b W_b(t, \varphi) = -\frac{1}{\Gamma(\alpha)} \left( \frac{W_b(t, \varphi_t)}{(b-t)^{1-\alpha}} + (1-a) \int_t^b \frac{W(t, \varphi_t) - W(s, \varphi_t)}{(s-t)^{2-\alpha}} ds \right).
\end{equation}

and

\begin{equation}
D^{\alpha,t}_a D^{1-\alpha,t}_b W_b(t, \varphi) = \frac{1}{\Gamma(\alpha)(1-\alpha)} \left( \frac{W(t, \varphi_t)}{(t'-a)^{\alpha}} + \alpha \int_a^{t'} \frac{W(t, \varphi_t) - W(t, \varphi_{t'})}{(t'-r)^{\alpha+1}} dr \right)
\end{equation}

\begin{equation}
+ \frac{1-a}{(t'-a)^{\alpha}} \int_t^{t'} \frac{W(t, \varphi_t) - W(s, \varphi_t)}{(s-t)^{2-\alpha}} ds + (1-a) \int_a^{t'} \frac{\alpha}{(t'-r)^{\alpha+1}} \int_t^{t'} \frac{W(t, \varphi_t) - W(s, \varphi_t)}{(s-t)^{2-\alpha}} dsdr.
\end{equation}
Thus the right hand side of (3.2) is

\[
- \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \left\{ \int_a^b \frac{W_b-(t, \varphi_t)}{(b-t)^{1-\alpha}(t-a)^\alpha} dt + \alpha \int_a^b \int_t^b \frac{W_b-(t, \varphi_t) - W_b-(t, \varphi_r)}{(b-t)^{1-\alpha}(t-r)^{\alpha+1}} dr dt \right. \\
+ (1-\alpha) \int_t^b \int_t^b \frac{W(t, \varphi_t) - W(s, \varphi_t) - W(s, \varphi_r) + W(s, \varphi_r)}{(s-t)^2 - (t-r)^{\alpha+1}} ds dt \\
+ \alpha(1-\alpha) \int_a^b \int_t^b \int_t^b \frac{W(t, \varphi_t) - W(s, \varphi_t) - W(t, \varphi_r) + W(s, \varphi_r)}{(s-t)^2 - (t-r)^{\alpha+1}} ds dr dt \right\}
\]

(3.5)

\[ =: I_1 + I_2 + I_3 + I_4. \]

The condition (W) implies

\[
I_1 \leq \kappa \|W\| \int_a^b (b-t)^{\tau+\alpha-1}(t-a)^{-\alpha} dt \\
= \kappa \|W\|(b-a)^\tau.
\]

(3.6)

Similarly, we also have

\[
I_3 \leq \kappa \|W\| \int_a^b \int_t^b (s-t)^{\tau+\alpha-2}(t-a)^{-\alpha} ds dt \\
\leq \kappa \|W\|(b-a)^\tau.
\]

(3.7)

The assumptions (W) and (\phi) also imply

\[
|W_b-(t, \varphi_t) - W_b-(t, \varphi_r)| \leq \kappa \|W\||b-t|^{\tau}|\varphi_t - \varphi_r|^{\lambda} \\
\leq \kappa \|W\|\|\varphi\|_\lambda |b-t|^{\tau}|t-r|^{\lambda \gamma}.
\]

This implies

\[
I_2 \leq \kappa \|W\|\|\varphi\|_\lambda \int_a^b \int_a^t ((b-t)^{\tau+\alpha-1}(t-r)^{\lambda \gamma-\alpha-1} dr dt \\
\leq \kappa \|W\|\|\varphi\|_\lambda (b-a)^{\tau+\lambda \gamma}.
\]

(3.8)

Using

\[
|W(t, \varphi_t) - W(s, \varphi_t) - W(t, \varphi_r) + W(s, \varphi_r)| \leq \kappa \|W\|\|\varphi\|_\lambda |t-s|^{\tau}|t-r|^{\lambda \gamma},
\]

we can estimate I_4 as follows.

\[
I_4 \leq \kappa \|W\|\|\varphi\|_\lambda \int_a^b \int_a^t \int_t^b \frac{|t-s|^{\tau}|t-r|^{\lambda \gamma}}{(s-t)^2 - (t-r)^{\alpha+1}} ds dr dt \\
\leq \kappa \|W\|\|\varphi\|_\lambda (b-a)^{\tau+\lambda \gamma}.
\]

(3.9)

The inequalities (3.6)-(3.9) imply that for any \( \alpha \in (1-\tau, \gamma \lambda) \), the right hand side of (3.2) is well-defined. The inequalities (3.6)-(3.9) also yield (3.4).
To show (3.3) is independent of $\alpha$ we suppose $\alpha', \alpha \in (1 - \tau, \lambda\gamma)$, $\alpha' > \alpha$. Denote $\beta = \alpha' - \alpha$. Using Lemma 2.3, it is straightforward to see that

$$(-1)^{\alpha} \int_{a}^{b} D_{a+}^{\alpha} D_{b-}^{1 - \alpha, t'} W_{b-}(t, \varphi_{t'})|_{t' = t} dt = (-1)^{\alpha} \int_{a}^{b} D_{a+}^{\alpha} D_{b-}^{1 - \alpha, t'} W_{b-}(t, \varphi_{t'})|_{t' = t} dt$$

$$= (-1)^{\alpha + \beta} \int_{a}^{b} D_{a+}^{\alpha} D_{b-}^{1 - \alpha, t'} W_{b-}(t, \varphi_{t'})|_{t' = t} dt = (-1)^{\alpha'} \int_{a}^{b} D_{a+}^{\alpha'} D_{b-}^{1 - \alpha', t'} W_{b-}(t, \varphi_{t'})|_{t' = t} dt$$

$$= (-1)^{\alpha'} \int_{a}^{b} D_{a+}^{\alpha'} D_{b-}^{1 - \alpha', t'} W_{b-}(t, \varphi_{t'})|_{t' = t} dt .$$

This proves the theorem. ■

Now we can improve the equality (3.4) as in the following theorem

**Theorem 3.4.** Let the assumptions (W) and (\(\phi\)) be satisfied. Let $a, b, c$ be real numbers such that $a \leq c \leq b$. Then there is a constant $\kappa$ depending only on $\tau, \lambda$ and $\alpha$, but independent $W, \varphi$ and $a, b, c$ such that

$$\int_{a}^{b} W(dt, \varphi(t)) - W(b, \varphi(c)) + W(a, \varphi(c)) \leq \kappa \|W\|_{\tau, \lambda; a, b}\|\varphi\|_{\lambda, \gamma; a, b}(b - a)^{\tau + \lambda\gamma} .$$

**Proof** Let $a \leq c < d \leq b$ and let $\tilde{\varphi}(t) = \varphi(c)\chi_{[c, d)}(t)$, where $\chi_{[c, d)}$ is the indicate function on $[c, d)$. Then

$$W(t, \tilde{\varphi}(t')) = \begin{cases} W(t, \varphi(c)) & c \leq t' < d \\ W(t, 0) & \text{elsewhere} . \end{cases}$$

This means $W(t, \tilde{\varphi}(t')) = W(t, \varphi(c))\chi_{[c, d)}(t')$. Hence, from (2.6) we have

$$\int_{a}^{b} W(dt, \varphi(t)) = (-1)^{\alpha} \int_{a}^{b} D_{b-}^{1 - \alpha, t} W_{b-}(t, \varphi(c))D_{a+}^{\alpha} \chi_{[c, d)}(t')|_{t' = t} dt$$

$$= (-1)^{\alpha} \int_{a}^{b} D_{b-}^{1 - \alpha, t} W_{b-}(t, \varphi(c))D_{a+}^{\alpha} \chi_{[c, d]}(t) dt$$

$$= W(d, \varphi(c)) - W(c, \varphi(c)) .$$

Let $c$ be any point in $[a, b]$. Denote $\tilde{W}(t, x) = W(t, x) - W(t, \varphi_{c})$. Then $\tilde{W}$ satisfies (W). As in the equation (3.5), we have

$$\int_{a}^{b} W(dt, \varphi(t)) - W(b, \varphi_{c}) + W(a, \varphi_{c}) = \int_{a}^{b} \tilde{W}(dt, \varphi(t))$$

$$= \tilde{I}_{1} + \tilde{I}_{2} + \tilde{I}_{3} + \tilde{I}_{4} .$$
where \( \tilde{I}_2 = I_2 \) and \( \tilde{I}_4 = I_4 \) are the same as \( I_2 \) and \( I_4 \) in the proof of Theorem 3.3. But

\[
\tilde{I}_1 = - \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_a^b W(t, \varphi_t) - W(b, \varphi_t) - W(t, \varphi_c) + W(b, \varphi_c) dt \tag{3.11}
\]

\[
\tilde{I}_3 = - \frac{(1-\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_a^b \int_t^b W(t, \varphi_t) - W(s, \varphi_t) - W(t, \varphi_c) + W(s, \varphi_c) dsdt. \tag{3.12}
\]

From the assumptions \((W)\) and \((\phi)\) we see that

\[
|W(t, \varphi_t) - W(b, \varphi_t) - W(t, \varphi_c) + W(b, \varphi_c)| \leq \kappa \|W\|_{\tau, \alpha, b} \|\varphi\|_{\gamma, a, b}^\lambda |b - t|^\tau |t - c|^{\lambda \gamma}
\]

This implies that

\[
\tilde{I}_1 \leq \kappa \|W\|_{\tau, \alpha, b} \|\varphi\|_{\gamma, a, b}^\lambda (b - a)^{\tau + \lambda \gamma}. \tag{3.11}
\]

Similarly, we have

\[
\tilde{I}_3 \leq \kappa \|W\|_{\tau, \alpha, b} \|\varphi\|_{\gamma, a, b}^\lambda (b - a)^{\tau + \lambda \gamma}. \tag{3.12}
\]

Combining these two inequalities (3.11) and (3.12) with the inequalities (3.8) and (3.9) we have

\[
\int_a^b \tilde{W}(dt, \varphi_t) \leq \kappa \|W\|_{\tau, \alpha, b} \|\varphi\|_{\gamma, a, b}^\lambda (b - a)^{\tau + \lambda \gamma},
\]

which yields (3.10). ■

**Theorem 3.5.** Let the assumptions \((W)\) be satisfied. Let \( \varphi : [a, b] \to \mathbb{R}^d \) satisfy

\[
|\varphi(s) - \varphi(a)| \leq L |s - a|^{\ell} \quad \forall s \in [a, b] \quad \text{and} \quad \sup_{a \leq t, s \leq b} \frac{|\varphi(s) - \varphi(t)|}{(s - t)^{\gamma}} \leq L
\]

for some \( \ell \in (\gamma, \infty) \) and for some constant \( L \in (0, \infty) \). If \( \tau + \lambda \gamma > 1 \), then for any \( \beta < 1 + \frac{\lambda + \ell - 1}{\gamma} \) we have

\[
\int_a^b W(dt, \varphi_t) - W(b, \varphi_a) + W(a, \varphi_a) \leq C(b - a)^\beta, \tag{3.14}
\]

where the constant \( C \) does not depend on \( b - a \).

**Proof** As in the proof of Theorem 3.4 we express \( \int_a^b W(dt, \varphi_t) - W(b, \varphi_a) + W(a, \varphi_a) \) as the sum of the terms \( \tilde{I}_j, j = 1, 2, 3, 4 \) (we follow the notation there). First, we explain how to proceed with \( \tilde{I}_4 \). We shall use \( C \) to denote a generic constant independent of \( b - a \). Denote

\[
J := |W(t, \varphi_t) - W(s, \varphi_t) - W(t, \varphi_r) + W(s, \varphi_r)|
\]

First, we know that we have

\[
J \leq C |t - s| |t - r|^{\lambda \gamma}. \tag{3.15}
\]
On the other hand, we also have

\[ J \leq |W(t, \varphi_t) - W(s, \varphi_t) - W(t, \varphi_a) + W(s, \varphi_a)| \\
+ |W(t, \varphi_r) - W(s, \varphi_r) - W(t, \varphi_a) + W(s, \varphi_a)| \\
\leq C|t - s|^{\tau} |t - a|^{\lambda \ell} + |r - a|^{\lambda \ell} \]

(3.16)

when \( a \leq r < t < s \leq b \). Therefore, from (3.15) and (3.16) it follows that for any \( \beta_1 \geq 0 \) and \( \beta_2 \geq 0 \) with \( \beta_1 + \beta_2 = 1 \), we have

\[ J \leq C|t - s|^{\tau} |t - r|^{\beta_1 \lambda \gamma} |t - a|^{\beta_2 \lambda \ell} \]

If we choose \( \alpha \) and \( \beta_1 \) such that

\[ \tau + \alpha > 1, \quad \beta_1 \lambda \gamma - \alpha > 0 \]

then

\[ \tilde{I}_4 \leq C(b - a)^{\beta_1 \lambda \gamma + \beta_2 \lambda \ell + \tau} \]

For any \( \beta < 1 + \frac{\lambda \gamma + \tau - 1}{\gamma} \ell \) we can choose \( \alpha, \beta_1, \) and \( \beta_2 \) such that (3.17) is satisfied and

\[ \tilde{I}_4 \leq C(b - a)^{\beta} \]

The term \( \tilde{I}_2 \) can be handled in a similar but easier way and similar bound can be obtained.

Now, let us consider \( \tilde{I}_3 \). We have

\[ |W(t, \varphi_t) - W(s, \varphi_t) - W(t, \varphi_a) + W(s, \varphi_a)| \leq C|t - s|^{\tau} |t - a|^{\lambda \ell} \]

This easily yields

\[ \tilde{I}_3 \leq C(b - a)^{\tau + \lambda \ell} \]

Similar estimate holds true for \( \tilde{I}_1 \). However, it is easy to verify \( \tau + \lambda \ell > 1 + \frac{\lambda \gamma + \tau - 1}{\gamma} \ell \) if \( \ell > \gamma \). The theorem is proved. \( \blacksquare \)

For every \( s, t \) in \([a, b]\), we put \( \mu(s, t) = W(t, \varphi_s) - W(s, \varphi_s) \). Let \( \pi = \{a = t_0 < t_1 < \cdots < t_n = b\} \) be a partition of \([a, b]\) with mesh size \( |\pi| = \max_{1 \leq i \leq n} |t_i - t_{i-1}| \), one can consider the limit of the Riemann sums

\[ \lim_{|\pi| \to 0} \sum_{i=1}^{n} \mu(t_{i-1}, t_i) \]

whenever it exists. A sufficient condition for convergence of the Riemann sums is provided by following two results of [1].

**Lemma 3.6 (The sewing map).** Let \( \mu \) be a continuous function on \([0, T]^2\) with values in a Banach space \( B \) and \( \varepsilon > 0 \). Suppose that \( \mu \) satisfies

\[ |\mu(a, b) - \mu(a, c) - \mu(c, b)| \leq K|b - a|^{1 + \varepsilon} \quad \forall \ 0 \leq a \leq c \leq b \leq T. \]

Then there exists a function \( J\mu(t) \) unique up to an additive constant such that

\[ |J\mu(b) - J\mu(a) - \mu(a, b)| \leq K(1 - 2^{-\varepsilon})^{-1}|b - a|^{1 + \varepsilon} \quad \forall \ 0 \leq a \leq b \leq T. \]

We adopt the notation \( J^a\mu = J\mu(b) - J\mu(a) \).
**Lemma 3.7 (Abstract Riemann sum).** Let \( \pi = \{a = t_0 < t_1 < \cdots < t_m = b\} \) be an arbitrary partition of \([a, b]\) with \(|\pi| = \sup_{i=0,\ldots,m-1} |t_{i+1} - t_i|\). Define the Riemann sum
\[
J_\pi = \sum_{i=0}^{m-1} \mu(t_i, t_{i+1})
\]
then \(J_\pi\) converges to \(J_\mu\) as \(|\pi| \downarrow 0\).

Because \(\tau + \lambda \gamma\) is strictly greater than 1, the estimate (3.10) together with the previous two Lemmas implies

**Proposition 3.8.** As the mesh size \(|\pi|\) shrinks to 0, the Riemann sums
\[
\sum_{i=1}^{n} [W(t_i, \varphi_{t_{i-1}}) - W(t_{i-1}, \varphi_{t_{i-1}})]
\]
converges to \(\int_a^b W(dt, \varphi_t)\).

It is easy to see from here that
\[
\int_a^b W(dt, \varphi_t) = \int_a^c W(dt, \varphi_t) + \int_c^b W(dt, \varphi_t) \quad \forall a < c < b.
\]
This together with (3.4) imply easily the following.

**Proposition 3.9.** Assume that (W) and (\(\phi\)) hold with \(\lambda \gamma + \tau > 1\). As a function of \(t\), the indefinite integral \(\int_a^t W(ds, \varphi_s)\), \(t \leq a \leq b\) is Hölder continuous of exponent \(\tau\).

Further properties can be developed. For instance, we study the dependence of the nonlinear Young integration \(\int W(ds, \varphi_s)\) with respect to the medium \(W\) and the integrand \(\varphi\). We state the following two propositions whose proofs left for readers (see e.g. [3]).

**Proposition 3.10.** Let \(W_1\) and \(W_2\) be functions on \(\mathbb{R} \times \mathbb{R}^d\) satisfying the condition (W). Let \(\varphi\) be a function in \(C^\gamma(\mathbb{R}; \mathbb{R}^d)\) and assume that \(\tau + \lambda \gamma > 1\). Then
\[
|\int_a^b W_1(ds, \varphi_s) - \int_a^b W_2(ds, \varphi_s)| \leq |W_1(b, \varphi_b) - W_1(a, \varphi_a) - W_2(b, \varphi_b) + W_2(a, \varphi_a)|
+ c(\|\varphi\|_\infty)(W_1 - W_2)_{\beta, \tau, \lambda} \|\varphi\|_\gamma \|b - a\|^\tau + \lambda \gamma.
\]

**Proposition 3.11.** Let \(W\) be a function on \(\mathbb{R} \times \mathbb{R}^d\) satisfying the condition (W). Let \(\varphi^1\) and \(\varphi^2\) be two functions in \(C^\gamma(\mathbb{R}; \mathbb{R}^d)\) and assume that \(\tau + \lambda \gamma > 1\). Let \(\theta \in (0, 1)\) such that \(\tau + \theta \lambda \gamma > 1\). Then for any \(u < v\)
\[
|\int_u^v W(ds, \varphi_s^1) - \int_u^v W(ds, \varphi_s^2)|
\leq C_1 \|W\|_{\tau, \lambda} \|\varphi^1 - \varphi^2\|_\lambda^\gamma |v - u|^\tau
+ C_2 \|W\|_{\tau, \lambda} \|\varphi^1 - \varphi^2\|_\lambda^{(1-\theta)}|v - u|^\tau + \theta \lambda \gamma,
\]
where \(C_1\) is an absolute constant and \(C_2 = 2^{1-\theta} C_1 (\|\varphi^1\|_\gamma + \|\varphi^2\|_\gamma)^\theta\).
4. Iterated nonlinear integral

From Remark 3.2 we see that if \( W(t, x) = \sum_{i=1}^{d} g_i(t) x_i \) and \( \varphi_i(t) = f_i(t) \), then
\[
\int_{a}^{b} W(dt, \varphi_t) = \sum_{i=1}^{d} \int_{a}^{b} f_i(t) dg_i(t).
\]
We know that the multiple (iterated) integrals of the form
\[
\int_{a \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq b} \varphi(s_1, s_2, \cdots, s_n) dg(s_1)dg(s_2)\cdots dg(s_n)
\]
are well-defined and have applications in expanding the solutions of differential equations (see [2]). What is the extension of the above iterated integrals to the nonlinear integral? To simplify the presentation, we consider the case \( d = 1 \). General dimension can be considered in a similar way with more complex notations.

We introduce the following notation. Let
\[
\Delta_{n,a,b} := \{(s_1, \cdots, s_n) ; a \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq b\}
\]
be a simplex in \( \mathbb{R}^n \).

**Definition 4.1.** Let \( \varphi : \Delta_{n,a,b} \to \mathbb{R} \) be a continuous function. For a fixed \( s_n \in [a, b] \), we can consider \( \varphi(\cdot, s_n) \) as a function of \( n-1 \) variables. Assume we can define
\[
\int_{\Delta_{n-1,a,b}} \varphi(s_1, \cdots, s_{n-1}, s_n) W(ds_{n-1}, s_n) \int_{[a,b]} \cdots \int_{[a,b]} W(ds_1, \cdots, s_{n-1}) \]
which is a function of \( s_n \), denoted by \( \phi_{n-1}(s_n) \), then we define

(4.1)
\[
\int_{a \leq s_1 \leq \cdots \leq s_n \leq b} \varphi(s_1, \cdots, s_n) W(ds_1, \cdots, s_n) = \int_{a}^{b} W(ds_n, \phi_{n-1}(s_n)).
\]

In the case \( W(t, x) = f(t)x \), such iterated integrals have been studied in [2], where an important case is when \( \varphi(s_1, \cdots, s_n) = \rho(s_1) \) for some function \( \rho \) of one variable. This means that \( \varphi(s_1, \cdots, s_n) \) depends only on the first variable. This case appears in the remainder term when one expands the solution of a differential equation and can be dealt with in the following way.

Let \( F_1, F_2, \ldots, F_n \) be jointly Hölder continuous functions on \([a, b]^2\). More precisely, for each \( i = 1, \ldots, n \), \( F_i \) satisfies

(4.2)
\[
|F_i(s_1, t_1) - F_i(s_2, t_1) - F_i(s_1, t_2) + F_i(s_2, t_2)| \\
\leq \|F_i\|_{\tau, \lambda, a, b}|s_1 - s_2|^\tau|t_1 - t_2|\lambda, \quad \text{for all } s_1, s_2, t_1, t_2 \text{ in } [a, b].
\]

We assume that \( \tau + \lambda > 1 \).

Suppose that \( F \) is a function satisfying (4.2) with \( \tau + \lambda > 1 \). The nonlinear integral \( \int_{a}^{b} F(ds, s) \) can be defined analogously to Definition 3.1. Moreover, for a Hölder continuous function \( \rho \) of order \( \lambda \), we set \( G(s, t) = \rho(t) F(s, t) \), it is easy to see that
\[
|G(s_1, t_1) - G(s_2, t_1) - G(s_1, t_2) + G(t_1, t_2)| \\
\leq |\rho(t_1) - \rho(t_2)||F(s_1, t_1) - F(s_2, t_1)| \\
+ |\rho(t_2)||F(s_1, t_1) - F(s_1, t_2) + F(t_1, t_2)| \\
\leq (||\rho||_\tau + ||\rho||_\infty)||F||_{\tau, \lambda}|s_1 - s_2|^\tau|t_1 - t_2|\lambda.
\]
Hence, the integration \( \int_{a}^{b} \rho(s) F(ds, s) \) is well defined. In addition, it follows from Theorem 3.4 that the map \( t \mapsto \int_{a}^{t} \rho(s) F(ds, s) \) is Hölder continuous of order \( \tau \).

We have then easily
Proposition 4.2. Let \( \rho \) be a Hölder continuous function of order \( \lambda \). Under the condition (4.2) and \( \tau > 1/2 \), the iterated integral

\[
I_{a,b}(F_1, \ldots, F_n) = \int_{a \leq s_1 \leq \cdots \leq s_n \leq b} \rho(s_1)F_1(ds_1, s_1)F_2(ds_2, s_2) \cdots F_n(ds_n, s_n)
\]

is well defined.

In the simplest case when \( \rho(s) = 1 \) and \( F_i(s, t) = f(s) \) for all \( i = 1, \ldots, n \), the above integral becomes

\[
\int_{a \leq s_1 \leq \cdots \leq s_n \leq b} df(s_1) \cdots df(s_n) = \frac{(f(b) - f(a))^n}{n!}.
\]

Therefore, one would expect that

\[
|I_{a,b}(F_1, \ldots, F_n)| \leq \kappa \frac{|b-a|^\gamma}{n!}.
\]

This estimate turns out to be true for (4.3).

Theorem 4.3. Let \( F_1, \ldots, F_n \) satisfy (4.2) and \( \rho \) be Hölder continuous with exponent \( \lambda \). We assume that \( \rho(a) = 0 \). Denote \( \beta = \frac{\lambda + \tau - 1}{\lambda} \) and \( \ell_n = \frac{\beta^n - 1}{\beta - 1} + \beta^{n-1}(\tau + \lambda) \). Then, for any \( \gamma_n < \ell_n \), there is a constant \( C_n \), independent of \( a \) and \( b \) (but may depends on \( \gamma_n \)) such that

\[
|I_{a,b}(F_1, \ldots, F_n)| \leq C_n|b-a|^{\gamma_n}.
\]

Proof. Denote

\[
I_{a,s}^{(k)}(F_1, \ldots, F_k) = \int_{a \leq s_1 \leq \cdots \leq s_k \leq s} \rho(s_1)F_1(ds_1, s_1)F_2(ds_2, s_2) \cdots F_k(ds_k, s_k).
\]

Thus, we see by definition that

\[
I_{a,s}^{(k+1)}(F_1, \ldots, F_{k+1}) = \int_a^s F_{k+1}(dr, I_{a,r}^{(k)}(F_1, \ldots, F_k)).
\]

We prove this theorem by induction on \( n \). When \( n = 1 \), the theorem follows straightforward from (3.10) with the choice \( c = a \). Indeed, we have \( |I_{a,t}^{(1)}| \leq C|t - a|^{\lambda+\tau} \) and \( |I_{a,t}^{(1)} - I_{a,s}^{(1)}| \leq C|t - s|^\tau \).

The passage from \( n \) to \( n + 1 \) follows from the application of (3.14) to (4.6) and this concludes the proof of the theorem. \( \blacksquare \)

Remark 4.4. The estimate of Theorem 4.3 also holds true for the iterated nonlinear Young integral \( I_{a,b}^{(n)}(F_1, \ldots, F_n) = \int_{a \leq s_1 \leq \cdots \leq s_k \leq s} F_1(ds_1, \rho(s_1))F_2(ds_2, s_2) \cdots F_n(ds_n, s_n) \)

where \( I_{a,b}^{(k)}(F_1, \ldots, F_k) = \int_a^b F_k(ds, I_{a,s}^{(k-1)}(F_1, \ldots, F_{k-1})) \) and \( I_{a,b}^{(1)}(F_1) = \int_a^b F_1(ds, \rho(s)) \).

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