On Euler—Jaczewski sequence
and Remmert—Van de Ven problem for toric varieties

Dedicated to the memory of Krzysztof Jaczewski

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Introduction.

In [RV] Remmert and Van de Ven posed a problem concerning holomorphic maps from a complex projective space onto a smooth variety of the same dimension: the question was whether the target variety has to be the projective space as well. The problem has a positive answer provided by Lazarsfeld, [La].

**Theorem.** [Lazarsfeld] Suppose that $Y$ is a smooth projective variety of positive dimension. If $\varphi : \mathbb{P}^n \rightarrow Y$ is a surjective morphism then $Y \simeq \mathbb{P}^n$.

Lazarsfeld’s proof depends on a (somewhat technical) characterization of the projective space, by Mori, which was obtained as a by-product of his proof of Frankel-Hartshorne conjecture, [Mo]. We will explain the result in the section on rational curves.

Following Lazarsfeld, questions were raised concerning possible extensions of Remmert – Van de Ven problem for a broader class of varieties. That is, given an $X$ from a class of varieties and a morphism $\varphi : X \rightarrow Y$ onto a smooth projective variety (with possibly additional assumptions on $\varphi$ and $Y$, like $\rho(Y) = 1$, where $\rho$ denotes the Picard number, or the rank of Neron-Severi group of the variety), one would like to deduce the structure of $Y$, preferably to claim that $Y \simeq \mathbb{P}^n$, unless $\varphi$ is an isomorphism. In particular, the following cases have been considered: $X$ is a smooth quadric, see [PS] and [CS], $X$ is an irreducible symmetric Hermitian space, [Ts], $X$ is rational homogeneous with $\rho(X) = 1$, [HM] and $X$ is a Fano 3-fold, [Am] and [Sc].

The main result of the present note is the following.

**Theorem 1.** Suppose that $X$ is a complete toric variety and $Y$ a smooth projective variety with $\rho(Y) = 1$. If $\varphi : X \rightarrow Y$ is a surjective morphism then $Y \simeq \mathbb{P}^n$.

The proof of the main result follows the line of Lazarsfeld’s argument, in particular we apply Mori’s ideas of considering families of rational curves. A new ingredient of the proof is Euler-Jaczewski sequence for toric varieties which we explain in the subsequent section. As an application we obtain a result on varieties admitting two projective bundle structures (Theorem 2 in the last section of the paper).

In the present paper all varieties are defined over an algebraically closed field of characteristic zero.

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Toric varieties.

For generalities on toric varieties we refer the reader to [Fu] or [Od]. Let $X = X_\Sigma$ be a toric variety defined by a fan $\Sigma$ in the space $\mathbb{N}_\mathbb{R}$, with $\mathbb{N}$ denoting the lattice of 1-parameter subgroups of the big torus $T \simeq (\mathbb{C}^*)^n$ and $\mathbb{M}$ denoting its dual. By $\Sigma^1 = \{\rho\}$ let us denote the set of rays (1-dimensional cones) in the fan $\Sigma$. The equivariant divisor in $X$ (the closure of a codimension 1 orbit of the action of $T$) associated to a ray $\rho \in \Sigma^1$ we shall denote by $D_\rho$. The variety $X$ is decomposed into the union of the open orbit of $T$ and the divisor $\bigcup_{\rho \in \Sigma^1} D_\rho$.

Let us recall the following general fact due to Blanchard, [Bl].

**Theorem.** [Blanchard] Let $X$ be a normal complete variety and $G$ be a connected algebraic group acting on $X$ such that the induced action on $Pic(X)$ is trivial (assume, for example, that $H^1(X, \mathcal{O}_X) = 0$). Suppose that $\varphi : X \rightarrow Y$ is a morphism to a projective normal variety with connected fibers, so that $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$. Then there exists an action of $G$ on $Y$ such that $\varphi$ is equivariant.

**Proof.** Let $L = \varphi^* L_Y$, where $L_Y$ is an ample line bundle over $Y$. We have $X = \text{Proj} \bigoplus_m L^\otimes m$ and $Y = \text{Proj} \bigoplus_m H^0(X, L^\otimes m)$ and $\varphi : X \rightarrow Y$ is induced by the evaluation morphism $H^0(X, L^\otimes m) \rightarrow L^\otimes m$. The natural action of $G$ on the graded ring of sections $\bigoplus_m H^0(X, L^\otimes m)$ is clearly compatible with the evaluation.

**Corollary 1.** Let $X$ be a toric variety, $\varphi : X \rightarrow Y$ a morphism to a projective variety $Y$ and let $X \xrightarrow{f_0} X' \xrightarrow{f_1} Y$ be the Stein factorization of $\varphi$. Then $X'$ is a toric variety.

**Proof.** By Blanchard’s theorem the big torus of $X$ acts on $Y$ with an open orbit. Thus the quotient of the big torus of $X$ by the isotropy of a general point of $Y$ is a torus and it acts on $Y$ with an open orbit, hence $Y$ is toric by [Od], Theorem 1.5.

**Euler-Jaczewski sequence.** Let $X$ be a complete algebraic variety, let $H = H^1(X, \Omega_X)$ and $\mathcal{H}$ be the sheaf $H \otimes \mathcal{O}_X$.

**Definition.** The short exact sequence

$$0 \rightarrow \Omega_X \rightarrow \mathcal{P}_X^Y \rightarrow \mathcal{H} \rightarrow 0$$

corresponding to $Id_H \in \text{Hom}(H, H) = \text{Ext}^1(\mathcal{H}, \Omega_X)$ will be called the Euler-Jaczewski sequence of the variety $X$ whereas the sheaf $\mathcal{P}_X$ will be called the potential sheaf of $X$.

We have the following characterization of toric varieties in terms of the Euler-Jaczewski sequence, [Ja, 3.1]:

**Theorem.** [Jaczewski] A smooth complete and connected variety $X$ is a toric variety if and only if there exists an effective divisor $D = \bigcup_i \mathbb{D}_i$ with normal crossing such that

$$\mathcal{P}_X = \bigoplus_{i \in I} \mathcal{O}_X(D_i)$$

2
where \( D_i \) are the irreducible components of the divisor \( D \). The divisors \( D_i \) are then the closures of the codimension one orbits of the torus.

The existence of a generalized Euler sequence on a smooth toric variety was discovered by Batyrev and Melnikov in [BM]. The characterization of toric varieties in terms of this sequence was proved later by Jaczewski, [Ja], who apparently was not aware of the Batyrev and Melnikov’s work.

On a smooth toric variety we have a short exact sequence:

\[
0 \rightarrow M \rightarrow \text{Div}^T X \rightarrow \text{Pic}(X) \rightarrow 0
\]

where \( \text{Div}^T X \) denotes the group of \( T \) equivariant divisors \( \sum \rho a_\rho D_\rho \). The first map associates to a character its divisor of poles and zeroes while the second map associates to a \( T \)-equivariant divisor its class in \( \text{Pic}(X) \).

Dualizing the above sequence and tensoring it with \( \mathbb{C} \) we obtain

\[
0 \rightarrow N_1(X)_\mathbb{C} \rightarrow \bigoplus_{\rho \in \Sigma^1} \mathbb{C}[\rho] \rightarrow N_\mathbb{C} \rightarrow 0
\]

where \( N_1(X)_\mathbb{C} \) is the space of 1-cycles on \( X \).

The maps in the sequence are defined as follows: \( N_1(X)_\mathbb{C} \ni Z \rightarrow \sum_{\rho \in \Sigma^1} (Z \cdot D_\rho) \cdot [\rho] \) and \( \sum_\rho a_\rho \cdot [\rho] \rightarrow \sum_\rho a_\rho \cdot e_\rho \), where \( e_\rho \) is the generator of the semigroup \( \mathbb{N} \cap \rho \).

**Lemma 1.** Let \( X = X_\Sigma \) be a toric variety as above. Consider \( \varphi : X \rightarrow Y \) a surjective and generically finite morphism. By \( J(\varphi) \) let us denote the subset of \( \Sigma^1 \) corresponding to divisors \( D_\rho \) which are mapped to divisors in \( Y \), that is \( J(\varphi) = \{ \rho \in \Sigma^1 \mid \varphi_* D_\rho \neq 0 \} \). Then the restriction of the above map \( \bigoplus_{\rho \in J} \mathbb{C}[\rho] \rightarrow N_\mathbb{C} \) is surjective.

**Proof.** Consider the Stein factorization of \( \varphi : X \rightarrow Y \). By corollary 1 the connected part of this factorization is a birational toric morphism \( \varphi_0 : X_\Sigma \rightarrow X_{\Sigma'} \), where the fan \( \Sigma \) is a subdivision of a fan \( \Sigma' \) and the rays of \( \Sigma' \) are exactly the rays corresponding to the divisors \( D_\rho \), with \( \rho \in J(\varphi) \). Since the fan \( \Sigma' \) is complete, its rays span \( N_\mathbb{C} \) and we are done.

Note that, for a toric variety, we have \( H^1(X, \Omega_X) = \text{Pic}(X)_\mathbb{C} \). In particular \( N_1(X)_\mathbb{C} \otimes \mathcal{O}_X = \mathcal{H}^\vee \); denoting by \( \mathcal{O}_X[\rho] \) the sheaf \( \mathbb{C}[\rho] \otimes \mathcal{O}_X \) and by \( \mathcal{N} \) the sheaf \( N_\mathbb{C} \otimes \mathcal{O}_X \) we have a commutative diagram of sheaves over \( X \) with exact rows and columns, which contains both the Euler-Jaczewski sequence and the sequence we have just discussed (see [Ja], diagram...
Here $s = (s_\rho)$, where $s_\rho$ is the section of $\mathcal{O}_X(D_\rho)$ vanishing along $D_\rho$, and $ev$ associates to a 1-parameter group $\gamma \in \mathbb{N}$ its tangent field.

**Families of curves.**

Let $Y$ be a smooth projective variety, and $y \in Y$ a point. We consider schemes $\text{Hom}(\mathbb{P}^1, Y)$, parametrizing morphisms from $\mathbb{P}^1$ to $Y$, and $\text{Hom}(\mathbb{P}^1, Y; 0 \to y)$, parametrizing morphisms sending $0 \in \mathbb{P}^1$ to $y \in Y$. Let $V \subset \text{Hom}(\mathbb{P}^1, Y)$ be a closed irreducible subvariety; we will call $V$, by abuse, a family of rational curves on $Y$ and we will denote by $V_y$ the variety $V \cap \text{Hom}(\mathbb{P}^1, Y; 0 \to y)$. If the evaluation $F : \mathbb{P}^1 \times \text{Hom}(\mathbb{P}^1, Y) \to Y$ is a dominating morphism then we will call $V$ a dominating family of rational curves. Suppose that $\rho(Y) = 1$. Then among all dominating families of rational curves on $Y$ (if there exists any) we can choose a family $\mathcal{V}$ parametrizing curves of minimal degree with respect to a chosen ample divisor on $Y$; we will call such a family a minimal dominating family of rational curves on $Y$.

Let us note that in case of Theorem 1, that is when $Y$ is dominated by a toric variety and $\rho(Y) = 1$, there exists a minimal dominating family $V$ of rational curves for $Y$. For this family we will use the following version of Mori’s result, [Mo].

**Theorem.** [Mori] Assume that $Y$ is a smooth projective variety such that $\rho(Y) = 1$. Let $\mathcal{V} \subset \text{Hom}(\mathbb{P}^1, Y)$ be a minimal dominating family of rational curves on $Y$. If for a general point $y \in Y$ and for any $f \in \mathcal{V}_y$ the pull-back $f^*TY$ is ample then $Y \simeq \mathbb{P}^n$.

For the proof of Theorem 1 we will need to understand properties of the minimal dominating family of rational curves on $Y$ from the point of view of properties of the dominating variety $X \to Y$. The key is the following technical observation which we prove in a more general setup:

**Lemma 2.** Let $\varphi : X \to Y$ be a surjective morphism of smooth irreducible projective varieties. Assume that $\mathcal{M}$ is a dominating family of curves of $Y$, that is, there exists a
variety \( C_Y \) with morphisms \( p : C_Y \rightarrow M \) and \( q : C_Y \rightarrow Y \), the latter morphism is dominating, such that all fibers of \( p \) are 1-dimensional and mapped via \( q \) to curves in \( Y \).

Let \( C_X \) be an irreducible component of the fiber product \( X \times_Y C_Y \) which dominates \( C_Y \); by \( \bar{q} \) and \( \bar{\varphi} \) let us denote morphisms of \( C_X \) to \( X \) and \( C_Y \), respectively. Suppose that \( D \) is an irreducible effective Cartier divisor on \( X \) such that \( \varphi_*D \) is an ample Cartier divisor on \( Y \). Then, for every \( m \in M \), the pull-back \( \bar{p}^*D \) is non zero on every connected component of \((p \circ \bar{\varphi})^{-1}(m)\).

**Proof.** Let \( C_X \xrightarrow{p_0} M_0 \xrightarrow{p_1} M \) be the Stein factorization of the map \( p \circ \bar{\varphi} : C_X \rightarrow M \), so that we the following commutative diagram

\[
\begin{array}{ccc}
C_X & \xrightarrow{\varphi} & C_Y \\
\downarrow{q} & & \downarrow{p} \\
X & \xrightarrow{\varphi} & Y
\end{array}
\]

By our assumptions \( p_1 \circ p_0 : \bar{q}^*D \rightarrow M \) dominates \( M \), so its image in \( M_0 \) is an irreducible subvariety of maximal dimension, hence, since \( M_0 \) is irreducible, it coincides with \( M_0 \).

**Proof of Theorem 1.**

First of all, by toric Chow’s Lemma we can assume that \( X \) is projective. Then, in view of Corollary 1, considering the Stein factorization of \( \varphi \), we can assume that \( X \) and \( Y \) are of the same dimension. Finally, by taking a desingularization of \( X \), we can assume that \( X \) is smooth.

Let \( V \) be a minimal dominating family of rational curves on \( Y \). Let \( y \in Y \) be a general point which is not contained in the branch locus of \( \varphi \) nor it is contained in the image of \( \varphi(\bigcup D_\rho) \). Let \( f : \mathbf{P}^1 \rightarrow Y \) be a curve in \( V \) passing through \( y \); the pullback of the tangent bundle splits into a sum of line bundles: \( f^*TY \cong \bigoplus \mathcal{O}_{\mathbf{P}^1}(a_i) \). By Mori’s theorem explained above we shall be done if all numbers \( a_i \) are positive.

Suppose, by contradiction that \( a_l \leq 0 \) for some \( l \); in this case we have a surjection \( f^*TY \rightarrow \mathcal{O}_{\mathbf{P}^1}(a_l) \) and hence a surjection \( f^*TY \rightarrow \mathcal{O}_{\mathbf{P}^1} \). By Lemma 1 and the commutativity of diagram \((*)\) the map \( \bigoplus_{\rho \in J} \mathcal{O}_X(D_\rho) \rightarrow TX \) is generically surjective. Since we assume that \( \varphi \) is generically finite, we get a generically surjective map \( \bigoplus_{\rho \in J} \mathcal{O}(D_\rho) \rightarrow \varphi^*TY \) which, by the choice of \( y \), is surjective over \( \varphi^{-1}(y) \).

Let us take \( C_Y = V \times \mathbf{P}^1 \) with \( p : C_Y \rightarrow V \) the projection and \( q : C_Y \rightarrow Y \) the evaluation. Now consider the situation discussed in Lemma 2

\[
\begin{array}{ccc}
C_X & \xrightarrow{\varphi} & C_Y \\
\downarrow{q} & & \downarrow{p} \\
X & \xrightarrow{\varphi} & Y
\end{array}
\]
Let $X_f$ be a connected component of $(p \circ \varphi)^{-1}(f)$. We obtain a generically surjective map \( q^*(\bigoplus_{\rho \in J} \mathcal{O}(D_{\rho}))|_{X_f} \to ((q^* \circ \varphi^*)TY)|_{X_f} = (\varphi^*(f^*TY))|_{X_f} \) and therefore we have a generically surjective map \( q^*(\bigoplus_{\rho \in J} \mathcal{O}(D_{\rho}))|_{X_f} \to \mathcal{O}_{X_f} \).

Thus there exists a non-zero section in

\[
H^0(X_f, q^*(\bigoplus_{\rho \in J} \mathcal{O}(-D_{\rho}))) = \bigoplus_{\rho \in J} H^0(X_f, q^*\mathcal{O}(-D_{\rho})),
\]

but this is impossible, since \( X_f \) is connected and for any \( \rho \in J \) the divisor \( q^*D_{\rho} \) is effective and non-zero on \( X_f \) by Lemma 2.

**An application.**

Let \( X \) be a smooth variety endowed with two different \( \mathbb{P} \)-bundle structures \( \varphi : X \to Y \) and \( \psi : X \to Z \). Since fibers of different extremal ray contractions can meet only in points we have \( \dim X \geq \dim Y + \dim Z \); an easy corollary of Lazarsfeld’s theorem is that we have equality if and only if \( X = \mathbb{P}^r \times \mathbb{P}^s \).

Using Theorem 1, we are able to describe the next case; we have the following

**Theorem 2.** Let \( X \) be a smooth projective variety of dimension \( n \), endowed with two different \( \mathbb{P} \)-bundle structures \( \varphi : X \to Y \) and \( \psi : X \to Z \) such that \( \dim Y + \dim Z = n + 1 \). Then either \( n = 2m - 1 \), \( Y = \mathbb{P}^m \) and \( X = \mathbb{P}(\mathbb{T}\mathbb{P}^m) \) or \( Y \) and \( Z \) have a \( \mathbb{P} \)-bundle structure over a smooth curve \( C \) and \( X = Y \times_C Z \).

**Proof.** Let \([l_\varphi]\) and \([l_\psi]\) be the numerical equivalence classes of lines in the fibers of \( \varphi \) and \( \psi \). Let \( x \) be a point of \( X \) and let \( B_x \) be the set of points of \( X \) that can be joined to \( x \) by a chain of rational curves whose numerical class is either \([l_\varphi]\) or \([l_\psi]\).

If there exists a point \( x \) such that \( B_x = X \) then, by [Ko, IV.3.13.3] the Picard number of \( X \) is two, hence \( Y \) and \( Z \) are Fano varieties of Picard number one. We also note that every pair of points of \( Z \) is connected by a chain of rational curves whose members are images of rational curves in \( X \) whose numerical class is \([l_\varphi]\).

Let \( F_t \) and \( F_w \), with \( t, w \in Z \), be two fibers of \( \psi \) such that \( \varphi(F_t) \neq \varphi(F_w) \). Since points \( t \) and \( w \) are joined by a chain of rational curves as above thus there exists a rational curve \( \Gamma \) on \( Z \) such that \( \varphi : \psi^{-1}(\Gamma) \to Y \) is dominant.

Let \( \nu : \mathbb{P}^1 \to Z \) be the normalization of \( \Gamma \) and let \( X_\Gamma \to \mathbb{P}^1 \) be the pull-back of the projective bundle, that is \( X_\Gamma = \mathbb{P}^1 \times_\Gamma X \). The composition of the induced map \( \tilde{\nu} : X_\Gamma \to X \) with \( \varphi \) is a surjective morphism \( \varphi \circ \tilde{\nu} : X_\Gamma \to Y \). The variety \( X_\Gamma \) is a \( \mathbb{P} \)-bundle on \( \mathbb{P}^1 \), hence a toric variety, and \( Y \) is a smooth variety with \( \rho(Y) = 1 \), hence Theorem 1 applies to give \( Y \simeq \mathbb{P}^{\dim Y} \); in the same way we also get \( Z \simeq \mathbb{P}^{\dim Z} \).

We conclude this case by the main theorem of [Sa], that is we get \( \dim Y = \dim Z = m \) and \( X \simeq \mathbb{P}(\mathbb{T}\mathbb{P}^m) \).

If \( B_x \) is a divisor for every \( x \in X \) then, for general \((x_1, x_2)\) in \( X \times X \), the divisors \( B_{x_1} \) and \( B_{x_2} \) are disjoint and because they are numerically equivalent we have \( B_x^2 \equiv 0 \). We claim that there exists a regular morphism \( p : X \to C \) with connected fibers, onto a smooth curve \( C \), which contracts all divisors \( B_x \). If \( H^1(X, \mathcal{O}_X) \neq 0 \) then \( p \) is obtained from the Albanese map of \( X \) (note that \( B_x \) are rationally connected hence contracted by
Albanese). If \( H^1(X, O_X) = 0 \) then divisors \( B_x \) are linearly equivalent, the linear system \( |B_x| \) is base point free and defines \( p \). By construction \( p \) factors through \( \varphi \) and \( \psi \) to produce \( p_Y : Y \to C \) and \( p_Z : Z \to C \). Moreover, by construction \( \rho(X/C) = 2 \) hence \( \rho(Y/C) = 1 \) and \( \rho(Z/C) = 1 \).

A general \( B_x \) is smooth with two projective bundle structures hence, by what we have said at the beginning of this section, it is a product of projective spaces. Thus a general fiber of \( p_X \) as well as \( p_Z \) is a projective space and over an open Zariski \( U \) subset of \( C \) both morphisms are projective bundles. By taking the closure in \( Y \) and \( Z \) of a hyperplane section of \( p_Y \) and \( p_Z \), respectively, defined over the open set \( U \) we get a global relative hyperplane section divisor (we use \( \rho(Z/C) = \rho(Y/C) = 1 \) hence \( p_Y \) and \( p_Z \) are projective bundles globally. The conclusion \( X = Y \times_C Z \) is immediate.

References.

[Am] Amerik, E., Maps onto certain Fano threefolds, Doc. Math. 2 (1997), 195-211.
[Bl] Blanchard, A. Sur les variétés analytiques complexes, Ann. Sci. École Norm. Sup. 73, (1956), 157-202.
[BM] Batyrev, V. V.; Mel’nikov, D. A. A theorem on nonextendability of toric varieties. (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. 118 (1986), no. 3, 20–24. English translation: Moscow Univ. Math. Bull. 41 (1986), no. 3, 23–27.
[CS] Cho, K. and Sato, E., Smooth projective varieties dominated by smooth quadric hypersurfaces in any characteristic, Math. Z. 217, (1994), 553-565.
[Fu] Fulton, W., Introduction to toric varieties. Princeton University Press. Princeton NJ 1993.
[HM] Hwang, J.-M. and Mok, N., Holomorphic maps from rational homogeneous spaces of Picard number 1 onto projective manifolds, Invent. Math. 136, (1999), 208-236.
[Ja] Jaczewski, K., Generalized Euler sequence and toric varieties. Classification of algebraic varieties (L’Aquila, 1992), 227–247, Contemp. Math. 162, AMS 1994
[Ko] Kollár, J., Rational Curves on Algebraic Varieties, volume 32 of Ergebnisse der Math. Springer Verlag, Berlin, Heidelberg, New York, Tokio, 1996.
[La] Lazarsfeld, R., Some applications of the theory of positive vector bundles. Complete Intersections (Acireale 1983), Lecture Notes in Math., Vol. 1092, Springer, Berlin, (1984), 29-61.
[Mo] Mori, Sh., Projective manifolds with ample tangent bundles. Ann. of Math. 110 (1979), 593–606.
[Od] Oda, T., Convex bodies and algebraic geometry. An Introduction to the theory of toric varieties. volume 15 of Ergebnisse der Math. Springer Verlag, Berlin, Heidelberg, New York, Tokio, 1988.
[PS] Paranjape, K. H. and Srinivas, V., Self-maps of homogeneous spaces, Invent. Math. 98, (1989), 425-444.
[RV] Remmert, R. and Van de Ven, A., Über holomorphe Abbildung projektiv-algebraischer Mannigfaltigkeiten auf komplexe Räume, Math. Ann. 142, (1961), 453-486.
[Sa] Sato, E., Varieties which have two projective space bundle structures, J. Math. Kyoto Univ. 25, (1985), 445-457.
[Sc] Schuhmann, C., Morphisms between Fano threefolds. *J. Algebraic Geom.* 8, (1999), 221-244.

[Ts] Tsai, I H., Rigidity of holomorphic maps between compact Hermitian symmetric spaces, *J. Differential Geom.* 33, (1991), 717-729.

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