Modular Theory and Symmetry in QFT

B. Schroer
Institut für Theoretische Physik
Freie Universität Berlin
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Abstract

The application of the Tomita-Takesaki modular theory to the Haag-Kastler net approach in QFT yields external (space-time) symmetries as well as internal ones (internal “gauge para-groups”) and their dual counterparts (the “super selection para-group”). An attempt is made to develop a (speculative) picture on “quantum symmetry” which links space-time symmetries in an inexorable way with internal symmetries. In the course of this attempt, we present several theorems and in particular derive the Kac-Wakimoto formula which links Jones inclusion indices with the asymptotics of expectation values in physical temperature states. This formula is a special case of a new asymptotic Gibbs-state representation of mapping class group matrices (in a Haag-Kastler net indexed by intervals on the circle!) as well as braid group matrices.
I Introduction

The topics to be discussed in this paper belong to the so-called algebraic QFT. This terminology is referring to the mathematical methods used in the approach.

But the methods are not the most characteristic feature of this theory. It could incorporate any mathematical methods consistent with relativistic quantum physics.

A more intrinsic characterization would be to say that it is founded on mathematically formulated physical principles [1].

There is another better known approach based on functional integrals, which resulted from a formal extension of perturbatively successful quantization ideas (founded on the presumed existence of a classical-quantum parallelism i.e. a kind of inverse of the Bohr correspondence principle).

Algebraic QFT is in some physical sense very closely linked with the older “dispersion theoretical approach” whose modest success culminated in the theoretical derivation of Kramers-Kronig dispersion relation for the scattering amplitudes of relativistic particles from the two main principles of relativistic local quantum physics [1]: Einstein causality and spectral properties (Dirac stability by filling negative energy states.) as well as their subsequent experimental verification in the case of p-p forward scattering.

Algebraic QFT uses the same physical principles, but employs a vastly more subtle mathematical framework (von Neumann Algebras, Jones Inclusion Theory etc.) than just Cauchy’s theorem in dispersion relation.

A rough look at those old principles reveals that they can be divided into two groups as mentioned above: causality properties expressed in terms of commutativity properties of local algebras and stability properties expressed in terms of states on observable algebras (or to be a little bit more precise, in terms of spectral or modular (KMS) properties of representation of algebras canonically (GNS) affiliated with these states). There are also related properties which join algebras and states as e.g. the notion of “statistical independence” of quantum physics in space-like seperated regions leading, among other things, to limitation on multiplicities of high energy states and yielding the existence of temperature states from the knowledge of the vacuum states [2].

Since in no other area of theoretical physics (e.g. string theory, 2-d quantum gravity) anybody explains in detail the principles on which his approach is founded, I will follow this general trend. In my case, these are however not representing an “industrial secret”
but they have been exposed nicely in the literature [3].

The main idea is to avoid making assumptions on charge-carrying fields but rather to define them and derive their properties from observables. (In that respect algebraic QFT is even different from QFT in the spirit of Wightman).

Algebraic QFT has been very successful at that. It is e.g. quite easy to define a composition of charged localized states on the observable algebra $\mathcal{A}$ (which is really a net of algebras):

$$\omega_1, \omega_2 \rightarrow \omega_1 \times \omega_2(\mathcal{A}) \neq \omega_2 \times \omega_1(\mathcal{A}) \quad A \epsilon \mathcal{A}$$

which commutes (the equality sign holds) if the localization is space-like i.e. $\text{loc } \omega_1 \times \text{loc } \omega_2$. The statistics operator $\epsilon$ shows up if one tries to lift states to state vectors. For spacelike separations on encounters the so called statistics operator $\epsilon$ as an obstruction [4].

$$\psi_{\omega_2 \times \omega_1} = \epsilon \psi_{\omega_1 \times \omega_2}$$

One immediately realizes the analogy to the Wigner phenomenon of ray representations of symmetry groups which arise if one lifts the algebraic automorphisms to the space of vector states or of Berry’s phase (the latter being of a more phenomenological nature) which is a well-known measurable obstruction of atomic physics. Statistics is (with the exception of the abelian case the so called anyons) not described by a phase, but a unitary operator.

With a bit of additional work one finds that the operator $\epsilon$ generates the braid group (special case is the permutation group) in low dimensional QFT [5] and the permutation group otherwise [6]. Furthermore there are Markov-traces leading (after rescaling) to knot invariants, and by considering “$\infty$” knots (i.e. suitable limits resembling infinite order perturbation theory for vacuum polarization in QFT), one obtains invariants of 3-manifolds [7].

There are however two physically important areas which have remained rather difficult territory even for algebraic QFT. Unfortunatly these are just the areas in which the new low-dimensional theoretical discoveries could be confronted with experimental reality (e.g. of condensed matter physics).

One is the problem of constructing and classifying the simplest fields which obey the new statistics in the “freest” way as it is allowed by braid group statistics i.e. the analogon of free bosons and fermions (where the statistics refers to the permutation
group) which may be called “free anyons” (abelian) and more generally “free plektons” (nonabelian).

In conformal field theory, according to our present best understanding, only free objects (but they do not describe particles but rather “infraparticles”) exist.

The second important physical problem is symmetry. In the old theories of fermions and bosons one had a sharp distinction between external (space-time) and internal (isospin, flavour) symmetries. Any attempt to “marry” them in a nontrivial way was doomed by failure [8].

In the new theory to the contrary there are obstructions against a complete divorce e.g. in conformal field theory some global conformal transformations ($2\pi n$ rotations on $\mathbb{S}^1$) are inexorably linked with internal charge transformations [7].

According to our present best understanding of 3-d theories with braid group statistics, it is perfectly conceivable that those already mentioned limiting invariants of 3-manifolds show up in some new perturbation theories of plektons in infinite order [7]. With other words curved 3-manifolds could appear in ordinary (infinite order) plekton scattering (i.e. even though the “living space” of the 3 dim. theory remains Minkowskian). In this case the 3-manifolds are expected to have something to do with a “space” formed by external charge-type variables together with (subgroups of ) the Möbius-group, resp. the 3-d Poincaré-group: Transl. $\rtimes 0(2,1)$.

The so called Topological Field Theory [10] in itself offers no concept of a physical “living space” of fields i.e. the localization necessary for a physical interpretation is absent and therefore the physical interpretation of its topological content is difficult and ambiguous which, however in no way limits its mathematical usefulness.

These problems with the physical interpretation can be traced back to the global nature of the Feynman-Kac functional integral representation which (from the point of view of general QFT) constitutes an attempt to simultaneously guess a state and an operator algebra (e.g. its vacuum-representation) on the basis of perturbative experience. Causality- and locality-properties can only be checked at the end after a complete GNS reconstruction to the real-time analytically continued expectation values has been carried out. The formal locality properties of the classical action are necessary (for quantum locality), but by no means sufficient. This is exemplified by the use of non-regular states on nets of Weyl algebras, which convert the “would be”

\footnote{The formal canonical quantization of Chern-Simons actions yields Weyl-like algebras on 1-forms as their operator-algebraic content.}
translationally covariant $\text{III}_1$ factor into $\text{II}_1$ hyperfinite tracial factors as used by V. Jones (and the model becomes in this way “topological”); all continuous space-time covariances become “knocked out” by the singular (local gauge invariant) states [11], i.e. are not unitarily implementable in a continuous fashion.\footnote{This singular state description is presently only known for the abelian Chern-Simons action [11]. For illustrating the above warning, this is sufficient. In quantum mechanics such singular states destroy the continuous translation and violate the uniqueness theorem of the Schrödinger representation.}

A study of singular states on suitably defined nonabelian Weyl algebras over 1-forms in 3d which could explain the tantalizing numerical coincidence [62] of the semiclassical approximations in the continuous Witten approach with the exact Turaev Viro combinatorical theory (which is identical to that of algebraic QFT if one specializes to models with 2-channel (Hecke-Jones) and 3-channel (Bierman-Wenzl) statistics) and yield also the correct pair \{algebra, state\} behind Topological Field Theory is still missing. However I am not aware of any alternative idea which captures the intuitive picture of averaging over “gauge copies” in operator language (i.e. without resorting to semiclassical or quantization ideas as the BRST formalism).

Singular states have first been applied in QFT by Buchholz-Fredenhagen [see 11] and by Haag [1, page 180].

The geometrical approach based on functional integrals and formal (Verma modula) representation theory is in no way limited by positivity requirements since it only explores free-field-like (integrable) system. In physical “real life” the von Neumann algebra methods of algebraic QFT are however indispensable.

In contrast to the functional integral approach, the locality principles in algebraic QFT is taken care of in every step of the arguments viz. the title and the content of a recent book [1].

In this paper we will only use regular states on nets and then topological data will only appear in suitably defined intertwiner subalgebras (centralizer algebras) which are of the $\text{II}_1$ Jones type and contain mapping class group matrices. The numerical invariants of 3-manifolds are from our viewpoint computable by suitable limits in the spirit of vacuum polarization effects, which amounts to the same as “$\rho_{\text{reg}}$.” in [7].

If one adds spinor-matter to 3-d pure Chern-Simons actions, one expects a remaining genuine $\text{III}_1$ net as the “gauge invariant” physical part of the model [12]. In fact one expects the principle role of the formal local gauge principle in low-dimensional QFT’s to be just this: a method to find interesting observable (plektonic) subalgebras
inside the well-known algebras of Bosons and Fermions (but unfortunately not for the construction of the very non-classical and yet unknown field algebra). In the 4-d gauge theories, there is of course the additional problem of understanding the true quantum nature of “Magnetic” fields and “Stokes theorem” (on the same conceptual level as the notion of charges in the theory of superselection sectors) and the deeper meaning of “quark confinement”.

The present picture of 4-d QFT from the algebraic viewpoint is less rich than that of low-dimensional QFT. The reason for this is that the standard spectral gap assumption for the energy-momentum spectrum leads to the Buchholz-Fredenhagen semiinfinite space-like cone localization (the cones can be made arbitrarily thin) [63], which in turn yields permutation group statistics and compact group symmetry. The situation would change dramatically if one finds a physical argument in favor of infinite energy objects which allow no better localization than that around a core being a space-like hypersurface. The quark idea (outside the short-distance regime) suggests that it may be advantageous to think about quantum states (i.e. no gauge artifacts) with infinite energy. Quarks as Wigner-states with a finite mass and spin $\frac{1}{2}$ are too naive and not very well digestable by algebraic QFT. With the exception of this short interlude the paper will be focussed on low-dimensional QFT only.

While at the issue of physical interpretation, we find it worthwhile to mention the difficulties, which algebraic QFT encounters with the notion of chiral conformal QFT “on Riemann surfaces”. In the special case of genus $g = 1$, the algebraic approach (as well as the Wightman approach based on the use of point-like covariant fields) yields two nets: one “living” (in the sense of localizability) on the Minkowskian cycle and the other (being also non-commutative and lacking a Feynman-Kac representation) on the Euclidean cycle. Between these two cycles there exists (as in standard QFT’s) the analogue of the Bargman-Hall-Wightman region [13] of analyticity of correlation functions.\[3] which, at least a priori, has nothing to do with the localizability implicitly contained in the word “living on...”. For this reason, it is the more surprising that we find the presence of mapping class matrices belonging to all geni, even if we presently do not know the physical interpretation of this finding. How do nets on $S^1$ know about mapping class groups for all geni? Does this observation mean that plektonic

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\[3\] $g = 1$ belongs to the $L_\sigma$-Gibbs state correlation functions. A conceptually acceptable description of $g \geq 2$ in terms of the notion of algebra and states is presently not available although some formal observations on avering conformal correlations by Fuchsian groups exist [64].
physics is better prepared (than fermionic or bosonic) to accept ideas about quantum

gavity? Is it related to the absence of degrees of freedom of gauge- and metric-fields

in a quantization approach to low-dimensional QFT’s?

Concerning the aforementioned first problem of anyonic or plektonic free fields one

has an extremely useful physical picture [14], as yet without sufficiently general explicit

constructions. The picture emerges from a combination of the old Coleman-Mandula

[8] ideas with the results from the bootstrap program of introducing 2-d integrable

models via factorizing S-matrices and formfactors of the would be fields [15].

In S-matrix language the important observation (for our purpose) may be phrased

in the following manner. If, in a 4-d local QFT one performs the “complete cluster

limit” on the S-matrix i.e. the limit in which all the centers of individual particle

wave packets become infinitely space-like separated, then all scattering effects, (first the

inelastic ones and then also the particle conserving processes) die out and only $S = 1$

remains in the limit.

A trivial S-matrix however is believed to belong to the local equivalence class

(Borchers class) of a free field, although a mathematical proof exists only under special

additional conditions [17]. The complete clustering idea of S is used here to produce the

“freest” permutation group statistics QFT which turn out to be identical with a free

field theory in the usual technical sense (i.e. the higher point functions are products

two-point function, with the appropriate Wick-combinatories).

Let us now jump from 4 down to 2 space-time dimensions. In that case the same

extreme cluster limit of the S-matrix is also expected to lead to an asymptotic vanishing

of all inelastic processes and part of the elastic ones. Only the elastic two particle scat-

tering $S_{\text{limit}}^{(2)}(\theta)$ ($\theta=$rapidity) should survive\footnote{The higher elastic processes have a smoother threshold behaviour leading to a faster decrease in the spatial separation of wave packets.} and then all the higher elastic processes

for consistency reason have to go through 2-particle scattering.

$S_{\text{limit}}^{(2)}(\theta)$ allows no further asymptotic space-time simplification since a separation

of an interaction T-part from a noninteracting part in the decomposition $S = 1 + iT$

is physically meaningless (both parts contain the same momentum space $\delta$-functions).

The Yang Baxter equation emerges as a consistency condition from this picture.

Therefore this picture is “quantum integrability” par excellence without recourse to a

classical parallelism, conserved currents etc. The nontrivial part of this picture is of

course the statement that the limiting S-matrix belongs to a fullfledged localizable

QFT in its own right (partially obtained via the so-called form-factor program [18]).

So quantum integrability is conceptually much easier than classical integrability, which in the spirit of Bohr’s correspondence principle should be derivable in an appropriate semiclassical approximation (not an easy matter!).

The particles to which the limiting S-matrix and its affiliated localizable QFT belong to plektons in special cases (i.e. the internal structure of S may be complicated than that describable by symmetry groups) but generically they are “kinks” (which are not DHR-localizable but rather belong to half space localized solitons) for which statistics notions are not applicable (statistical “schizons”, whose operator commutation relations can be changed by neutral operators i.e. without changing charges [19]). In the calculational scheme of Smirnov [18] this last observation is turned into a virtue by selecting for each interpolating field a bosonic representative for which the formfactor computations seem to be easier.

Now, finally, let us apply the picture to 3-d QFT. In that case we know that the generically admissable statistics is braid group statistics. On the other hand the complete cluster limit does not allow for an energy dependent limiting S-matrix as it was the case in d=4. This leaves only the possibility of a piecewise constant S-matrix (in momentum space) with discontinuous jumps if asymptotic directions (directions of momenta) go through degenerate situations where at least two direction become parallel. The properly defined cross-section for such a S-matrix would vanish. The values in the different nondegenerate components are given in terms of constant braid-R-matrices. This picture gets additional support by looking at the structure of scattering states. [20]

Insufficient knowledge of low energy analytic momentum space properties in the presence of braid group statistics has up to now prevented a proof a là Coleman-Mandula which is based on the usual (permutation group) statistics analytic properties of dispersion theory.

An explicite construction of anyonic or plektonic free fields (i.e. their particle formfactors) belonging to that limiting piecewiese constant S-matrix is presently even more remote. Without doubt such objects, if they exist, should be considered as the 3-d version of quantum integrability (i.e. if those piecewiese constant S-matrices admit localized interpolating fields).

Note that such a picture leads to a new class division of theories according to their long distance properties: each interacting theory, as complicated is it may be, has
precisely one integrable companion namely that belonging to $S_{\text{limit}}$. Up to now one only knows a short-distance universality principle formulated in terms of "conformal companions" of quantum field theoretical models.

The second problem, namely (internal) symmetry is closely related to the previous one. In order to see this, I remind the reader of a well-understood (but unfortunately not so well-known) fact that in addition to the standard description of QFT with permutation group statistics in terms of local boson or fermion fields with tensorial compact group transformation, there exist a physically equivalent (regarding local observables and the scattering matrix of charged particles) formalism based on nonlocal para-statistics fields (the exchange-algebra fields for the permutation group) which have no group transformation properties and fulfill R-matrix commutation relations with $R^2 = 1$ [43].

If one wants to prove e.g. that the free field equation (considered as a defining 2-sided ideal) or $S = 1$ leads to the analytically correct correlation functions of a free field theory, it would not be advisable to use para-fields. But with local tensorial fields, the result follows easily from the field equation and the well-known analytic properties of correlation functions which follow from causality and spectral properties of tensorial fields.

The analytic structure of $d = 2, 3$ exchange algebra fields is much more complicated than that of para-fermions or bosons.

So if one would find a symmetry concept which could replace the tensorial formalism in the presence of braid group statistics, then this could be of great practical value. Nonrelativistic plektons and their (non understood) "quantum" symmetries could allow to obtain a new symmetry breaking which may be responsible for the fractional Hall effect and High-$T_c$ superconductivity. Braid group statistics, even at its present state of poor physical understanding has already produced an extremely rich collection of numbers: rational statistical phases (related to electromagnetic properties?) and algebraic integers in the form of statistical dimensions (related to amplification factors in cross section or thermodynamical properties?) which are generalized Casimir factors counting degrees of freedom. But one lacks a good understanding of their localized

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3By a trivial extension of the Jost-Schroer theorem [13].
4Statistics of (quasi-)particles and the ensuing change of composition laws of spins and charges is the only difference between 4-d and low-dimensional QFT. This is expected to lead to significant differences of plektonic phases from the Fermi-liquid phases in condensed matter physics. The problem of classifying these phases is similar (but in its analytic aspects more difficult) to the classification of
field carriers, which would be necessary to work out experimental consequences.

Present attempts using quantum group ideas do not seem to go into this direction. They often contain artificial integer numbers (the representation dimensionality of the $q$-deformed group), treat the notion of conjugates (antiparticles) in an extremely unsymmetric way, and generally lead to non-unique systems of fields not relatable by Klein-Transformations [22].

There are rather convincing arguments that the quantum symmetry aspect should be discussed together with space-time symmetries including the TCP symmetry and also with modular Tomita-Takesaki theory and not as an isolated kinematical (combinatorial) property being apart from the rest of QFT.

In the following I will present some recent results on symmetries obtained by modular theory (in most parts within the framework of algebraic 2-d conformal QFT) and forget the presently inaccessible problem of free plektons. But it should be clear that my motivation (for presenting these partial results on symmetry) is entirely physical.

The paper is organized as follows.

I start in section II with a net indexed by intervals on a line and build up a conformal net on the circle. The modular Tomita-Takesaki structure, which is heavily used in this construction, also leads to a Euclidean net whose construction is only sketched very briefly.

Section III recalls the derivation of the universal observable algebra with its non-trivial center. It contains an “invariant symmetry algebra” which lends itself to a generalization of the Kac-Wakimoto [23] formula.

In section IV we recall the proof of the TCP theorem in the framework of the exchange algebra. In the case of the $\mathbb{Z}_{2N}$-anyon algebra we show the coalescence of this theorem with the standard Tomita-Takesaki modular theory on the half-circles and argue that this state of affairs prevails for all models with vanishing “defect projector” which measures the deviation of statistical dimensions from integers.

In section V we study the “selfduality”, observed globally as the symmetry of Verlinde’s matrix $S$, on the local net level. Here another tool of modular theory is used: Longo’s “canonical endomorphism”. Again we employ the $\mathbb{Z}_{2N}$ anyonic model as an illustration of perfect selfduality.

Finally in the last section we use an old, but still mysterious observation of Vaughn Jones in order to make some speculative remarks on the possible nature of the non-2-d conformal QFT’s.
invariant aspect (i.e. beyond the symmetry algebra of section III) of a non-commutative
link between external and internal symmetry. This section belongs to the realm of
fantasy and only serves the purpose to open the eyes for radical new possibilities of
field algebras. done by space-time regions in Minkowski-space, but by intervals on a line
(a right or left light ray of a 2-d. conformal QFT). We assume a translation-invariant
vacuum state \( \omega_0 \) and construct via the GNS construction the vacuum-representa-
tion \( \pi_0(A) \) and the affiliated von Neumann algebra \( M = \pi_0(A)'' \) as well as the unitary
positive energy translation \( U(a) \) implementing the assumed translation automorphism
and leaving the GNS vacuum vector \( \Omega \) invariant.

We consider the half-line algebra \( M_+ = \pi_0(A(0, \infty))'' \) on which the right translation
\( U(a), a > 0 \) acts like a one-sided compression. Under these circumstances Borchers
proved the following theorem [24]:

**Theorem (Borchers):** The modular reflection \( J \) and the modular operator \( \Delta^it \) of the
pair \( (M_+, \Omega) \) have the following commutation relation with \( U(a) \):

\[
JU(a)J = U(-a) \tag{3}
\]
\[
\Delta^it U(a) \Delta^{-it} = U(e^{-2\pi t a}), \quad t, a \in \mathbb{R}
\]

It was already known before, that the abstract Tomita-Takesaki modular theory
(which affiliates a reflection \( J \) and a one parameter group \( \Delta^it \) with a von Neumann
algebra and a cyclic and separating state vector) has a deep physical and geometric
significance in ordinary Wightman QFT if one looks at algebras belonging to specific
regions ("wedges"): the Tomita reflection \( J \) becomes (up to a \( \pi \)-rotation) the TCP
operator related to the appearance of antiparticles and \( \Delta^it \) a special \( L \)-subgroup leaving
that wedge region invariant. The operator \( \Delta^it \) in Borchers theorem is clearly the
dilation.

From the translations and \( J \) one can build up a new net (as Borchers did) by starting
with \( M_+ \) which is by construction covariant under the Möbius subgroup leaving \( \infty \)
invariant. If the new net has the self-reproducing property, i.e., is equal to the original
net, the original net is admissible for the subsequent consideration. If, however, the
original net is perverse and fails to have this property, we work with the new net
(which is canonically affiliated with the old one and has the self-reproducing property
by construction) and forget the old one.

For the full Möbius-group including the change of \( \infty \) one needs the following
“quarter-circle assumption” (we use \( S^1 \) instead of \( R \cup \{ \infty \} \)):

**Assumption:** Essential duality holds for the interval \( (-\frac{\pi}{2}, \frac{\pi}{2}) \):

i.e.,

\[
\mathcal{L}\left( -\frac{\pi}{2}, \frac{\pi}{2} \right) = \mathcal{L}\left( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)' \right),
\]

with \( \mathcal{L}(I) := M(I)' \),

and the modular conjugation \( \tilde{J} \) of \( \mathcal{L} \) permutes the quarter circles, which one obtains by intersection the old half circle with the new one, so that we obtain

\[
\tilde{J}M_+\tilde{J} = M_+.
\]

Then one finds [25]:

**Theorem:** The group generated by \( U(a) \), \( \Delta^\xi \) and \( J \cdot \tilde{J} \) is the full Möbius group and the net generated from the four quadrant algebras is Möbius covariant.

Again we dismiss the input net if it is not identical to this final fully covariant net.

This theorem has been recently derived from a much wider and natural mathematical scheme than in [25] called “half-side modular inclusion” by H.W.-Wiesbrock [26].

Note that essential duality is weaker than Haag duality. It also holds for nets generated from the energy momentum tensor for \( c_{\text{Virasoro}} > 1 \) where, according to Buchholz and Schulz-Mirbach [27], Haag duality breaks down for the net on \( R \) generated by the energy-momentum tensor.

The above theorem suggests that the B.-S.M. mechanism of violating Haag duality and full conformal invariance by “drilling a hole” into \( S^1 \) (i.e., going from \( S^1 \) to \( R \)) is the only one, by which translation covariant nets indexed by intervals on the line \( R \) can fail to be a chiral conformal field theories. But then there always exists a canonically affiliated Möbius covariant \( S^1 \)-net. For such a covariant net one can directly show that the stronger Haag duality hold for each interval [28,29].

Replacing the assumed geometric translation by the product of two abstract Tomita-reflections one could even remove the last vestiges of geometry from the quantum-physical assumptions [26].

Since the T.-T. modular theory was shown to build up Möbius-invariance from pure algebraic inclusion data (indexed by intervals on \( S^1 \)), it is tempting to ask whether diffeomorphisms beyond the Möbius transformation can also be generated from algebraic net inclusions and suitably chosen states.
Let us look at the simplest such transformation (which together with the Möbius transformation generates the diffeomorphism group):

\[
Dil(t)^{(2)} z := \sqrt{Dil(t)z^2}.
\]  

(6)

Here we use the complex coordinate \( z \) for the circle, \( Dil(t) \) is the usual dilation, i.e., the modular group of the pair \( (\mathcal{A}(S_+), \Omega) \) and the resulting \( Dil^{(2)} \) denotes the “dilation” in the “second Virasoro sheet”, i.e., a geometric transformation built from infinitesimal generators \( L_{\pm 2}, L_0 \) which “dilates” with 4 fixed points: \( z = 1, i; -1 - i \) instead of the \( \pm 1 \) fixed points of \( Dil \).

This suggest to consider a doubly localized algebra

\[
\mathcal{A}(S_1) \vee \mathcal{A}(S_3)
\]

(7)

where \( S_1 \) and \( S_3 \) are the first and third quadrants of the circle (counterclockwise).

The corresponding state vector candidate for a modular pair cannot be the vacuum state-vector \( \Omega \). Here the split property [30] gives us clue. In conformal field theory this property follows from a controll of the asymptotic spectral density of the rigid rotation operator \( L_0 \) [28], and holds in all known conformal models [31]. It has the consequence that algebras like the previous one are spatially isomorphic to tensor products.

\[
\mathcal{A}(S_1) \vee \mathcal{A}(S_3) \sim \mathcal{A}(S_1) \otimes \mathcal{A}(S_3) \sim \mathcal{A}_I(S_+) \otimes \mathcal{A}_{II}(S_+).
\]

(8)

where \( \mathcal{A}_I, \mathcal{A}_{II} \) are two copies of the halfcircle algebra.

It is well-known that tensor-product algebras together with tensor product state vectors have also a tensor product modular operator. But our state vector is prevented to have this form as a result of the presence of a unitary equivalence transformation. We define the state on the doubly localized algebra by the vacuum product state on the single components. The use of the “natural cone” of the doubly localized algebra allows for a unique representation of that state in terms of a state vector \( \phi \). In order to understand the modular theory for this situation we investigate the KMS property of

\[
(\phi, A_{I} A_{II} B_{I} B_{II} \phi) := (\Omega, \hat{A}_{I} \hat{B}_{I} \Omega)(\Omega, \hat{A}_{II} \hat{B}_{II} \Omega)
\]

(9)

with: \( A_{I}, B_{I} \in \mathcal{A}(S_1), \quad A_{II}, B_{II} \in \mathcal{A}(S_3), \quad \hat{A}_{I}, \hat{B}_{I} \in \mathcal{A}_I(S_+), \quad \hat{A}_{II}, \hat{B}_{II} \in \mathcal{A}_{II}(S_+) \)

The modular operator \( \triangle \) appearing in the KMS relation on the left hand side:

\[
(\phi, A_{I} A_{II} B_{I} B_{II} \phi) = (\phi, \triangle B_{I} B_{II} \triangle^{-1} A_{I} A_{II} \phi)
\]

(10)
can easily be expressed in terms of the U-transformed modular operator $\hat{\Delta} \otimes \hat{\Delta}$ belonging to the right hand side where ($U$ is the unitary equivalence transformation).

$$\phi = U(\Omega \otimes \Omega) \quad (11)$$

The problem of finding the correct state vector has therefore been reduced to the question of whether one can construct a unitary operator with a geometric interpretation. Imagine that we start with the two copies on $S_+$ i.e. $\hat{A}_{I,II}(S_+)$ and the vacuum state vector $\Omega$. These algebras are certainly isomorphic to $A(S_1)$ resp. $A(S_3)$. Then the covering transformation $z \to z^2$ maps $S_1$ and $S_3$ onto $S_+$. As an automorphism of $A(S)$ this is meaningless; it is not a diffeomorphism. However, restricted to the doubly localized algebras it makes sense. So, there could exist a “partial automorphism” (i.e., not extendable by the geometric formula to $A(S^1)$) of the doubly localized von Neumann algebra which can be implemented in the vacuum representation by a unitary operator $U_{\text{cov.}}$ transforming from the tensor-product space to the original Hilbert-space, such that (12) with $U^{-1} = U_{\text{cov.}}$ is the state vector yielding $\Delta^it = \text{Dil}^{(2)}(t)$ as a modular object. Conditions for the uniqueness of the implementing unitary are known by the “natural cone” construction [1, Thm. 2.2.4]

The only piece which is lacking in order to have a proof, is the existence of the partial automorphism entailing that of $U_{\text{cov.}}$ with the geometric action on fields. Unfortunately we have not been able to find a structural analytic argument similar to [27]. However in models which have a “first” quantization like the Weyl algebra one has a control of the partial automorphism. We will treat this problem in a future publication.

Note that this method which is based on the use of the KMS property only deals with the modular group and does not determine the Tomita reflection and its possible geometric manifestation in the form of Haag duality.

The rigorous clarification of the general situation would be important because it allows for a modular (i.e., pure quantum physical) understanding of the Virasoro charge, the origin of diffeomorphisms and the intrinsic meaning of energy-momentum tensors in the framework of the algebraic net theory.

The way in which the Tomita-Takesaki modular theory converts formal net indexing by intervals into geometry and physics is really surprising. There is more to come later on.

For later use we also explain schematically the construction fo Euclidean chiral theories. The starting point is the observation, that the existence of a positive natural
cone (Araki, Connes [32]) in the Tomita-Takasaki modular theory entails the existence of another scalar product between state vectors of the form

$$\psi_A = \Delta^{1/4} A \Omega \quad (12)$$

$$\langle A, B \rangle := (\psi_A, \psi_B) = (J A^* J \Omega, B \Omega) \quad (13)$$

In our field-theoretic application $A \in M_+, \Delta^{it}$ is the dilation and $\Omega$ is the (with respect to $M_+$) cyclic and separating vacuum state vector. This yields another state-space $\mathcal{H}_{eucl.}$ with a different involutive and positive star-operation on which the old translation in positive direction becomes a contraction operator on $\mathcal{H}_{eucl}$. This in turn allows for a unitary analytic continuation which should be thought of as a translation on the imaginary axis. Using this euclidean translation, one can build up another net, whose localization intervals are to be placed on the imaginary axis and which is very non-local relative to the original real-time net. The only common elements are the non-local analytic (with respect to the modular operator $\Delta^{it}$) elements.

We call this euclidean theory the cartesian euclidean theory. A richer and more interesting euclidean theory is the radial euclidean theory. Its construction also starts from the upper half circle, but instead of the one-sided translation one uses a two-sided compression defined in terms of the modular dilation $\tilde{\Delta}^{it}$ of the right-hand half circle. In this case the analytic continuation of the two-sided contraction yields the euclidean rigid rotation, i.e., a rotation whose localization properties should be pictured in terms of a radial periodicized coordinate. It is remarkable that the Tomita reflection of the euclidean theory is identical to the original star-operation which now acts like a charge conjugation i.e. as a bounded operator on $\mathcal{H}_{eucl.}$ (and in turn the euclidean-star becomes related to the real-time Tomita reflection).

II Superselection Sectors and the Invariant Symmetry Algebra.

For a net indexed by intervals on a circle (rather then on a line), the previously constructed vacuum representation does not admit sectors, i.e., one cannot find interesting endomorphism [7]. On the line $\mathbb{R}$ one could find such endomorphism on the quasilocal

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7 The inner product is the same as that which appears in the Connes bimodule theory if one restricts the bimodule to the diagonal action.
\( C^* \) algebra \( A_{\text{ quasi}} \) (the DHR algebra) which is directed towards infinity [6], but then one would have wrecked conformal invariance.

It turns out that the correct observable algebra on \( S^1 \) which does not have these faults can be obtained by a universal construction: it is the free \( C^* \)-algebra generated from the vacuum-represented net amalgamated over all local relations [34]. Using the information one has from superselection theory, one finds that this universal algebra \( A_{\text{ univ.}} \) has global self-intertwiners which lead to a nontrivial center generated by invariant (abelian) charges [7].

As the old (DHR) quasilocal algebra \( A_{\text{ quasi}} \), it is uniquely determined from the net data in the vacuum representation. The universal observable algebra \( A_{\text{ univ.}} \) also has an analog representation theory in terms of localized endomorphism as the DHR theory, only the meaning of “localized” and “transportable” is somewhat more restricted [7].

The derivation of the braid group structure is also similar to the standard theory [7]: there is, however, one additional item as already mentioned: the appearance of a global self-intertwiner \( V_\rho \) which “flips” the statistics operator \( \epsilon(\rho, \sigma) \) to \( \epsilon^*(\sigma, \rho) \).

Together with the change of localization this fact accounts for the different properties of the exchange algebra, which, as in the standard theory, is constructed in a canonical fashion from the \( A_{\text{ vac.}} \) data together with the endomorphisms and constitutes a kind of “nonlocal” extension of \( A_{\text{ univ.}} \).

**Theorem (FRSII [7]):** In theories with a finite number of irreducible endomorphism (finite depth in the Jones theory, rational theory in conformal QFT) the exchange algebra is localizable on an \( N \)-fold covering (related to the depth) of \( S^1 \) and its \( R \)-matrices represent the braid group of the cylinder (generated by applying endomorphisms to the generators \( \epsilon(\rho, \rho) \) and \( V_\rho \)).

In this formulation we adapted the QFT to chirally conformal field theory, a generalization to the \( 3 - d \) case is straightforward [7].

In the framework of the DHR theory the exchange algebra (see also the beginning of the next section) is the reduced “field bundle” [5] (a bimodule over \( \mathcal{A} \) with a \( C^* \) algebra structure), whereas in the Wightman-framework it was first abstracted from nontrivial conformally invariant models [35].

It is profitable to define a subalgebra \( \mathcal{G}_{\text{ inv.}} \) of \( A_{\text{ univ.}} \) generated by the invariant charges (which depend only on equivalence classes of endomorphism), and their images under application of \( \rho \)'s. We do this as follows. We first pick one endomorphism \( \rho_0 \),
for each sector (the “rational” situation being always assumed) where all $\rho_i$’s are localized in one proper intervall on $S^1$. The theory then assures the existence of localized isometric intertwiners $V_i$ such that
\[
\rho := \sum_i V_i \rho_i V_i^* \tag{14}
\]
with $\rho(\mathcal{A}_{\text{univ.}}) \subset \mathcal{A}_{\text{univ.}}$ and $V_i \in \mathcal{A}_{\text{univ.}}$.

The centralizer algebra
\[
M = \lim \bigcup_n (\rho^n, \rho^n) \tag{15}
\]
with $(\rho^n, \rho^n) =$ space of intertwiners $\rho^n \rightarrow \rho^n$, is the inductive limit of finite matrix algebras.

$M$ allows for a restriction to the braid-group algebra $CB_\infty$, whereupon the tracial state on $M$ (obtained by the field-theoretic left inverse of $\rho$) acquires the Markov property.

The above reducible $\rho$ and the intertwiner-spaces appear in the work of Rehren [40]. Inductive limits of this kind are known to be the hyperfinite II$_1$ factor $R$, following the well-known arguments based on “commuting squares” [55].

Let us now consider the subalgebra generated by elements of the form
\[
\rho(C_{i_1}\rho(C_{i_2})\ldots\rho(C_{i_n})\ldots) \tag{16}
\]
with $C_i, i = 1 \ldots N$, the invariant global charges of $\mathcal{A}_{\text{univ.}}$. Clearly, these operators are in $(\rho^n, \rho^n)$, and their representation in terms of the path-space of intertwiners [37] shows that they are diagonal, in fact they are linear combinations of minimal projectors. The general intertwiners in $(\rho^n, \rho^n)$ are “strings” [6], i.e., pairs of intertwiner-path with equal length which correspond to matrix units. Clearly, these diagonal elements form a subalgebra inside the centralizer algebra.

In the following discussions we will call this subalgebra (for obvious reasons)\textsuperscript{8} the invariant symmetry algebra $\mathcal{G}^{\text{inv}}$. Its lowest nontrivial element (again restricting to the rational nondegenerate situation [36]) evaluated in the vacuum representation is the numerical matrix $S[7]$.
\[
\pi_0 \rho_i(C_j) = S_{ij}. \tag{17}
\]

\textsuperscript{8}Strictly speaking only the GNS representation of $\mathcal{G}^{\text{inv}}$ with class invariant states (Gibbs or tracial states only depend on the sector $[\rho]$) deserve this name.
This matrix, (together with a diagonal matrix $T$) first abstracted from characters of chiral conformal models by Cardy, Capelli, Itzykson and Zuber, then related to fusion rules by Verlinde [38], and generalized to a wider geometrical picture of chiral conformal field theory by Moore and Seiberg [39], was finally shown to relate fusion rules and monodromy properties (statistics character) independent of conformal QFT by Rehren [40]. By the present work we may now add the remark that the universal algebra on the circle (as well as the 3-d universal algebra) contains mapping class group representations for all geni.

Let us look now at more general elements of $G^{inv}$ evaluated in the vacuum representation. These are not just numerical-valued, since the multiple application of $\rho$ lead inside the non-central parts of the original algebra $A_{univ}$ (except for the abelian case). In order to get to numerical quantities, one can use the tracial state. It leads to numerical matrices which generalize $S$ and are made up from $S$ and traces on minimal projectors. These multi-indexed matrices will be shortly identified with the known mapping class matrices [41].

On the other hand $A_{univ}$, and therefore $G^{inv}$, allows for a Gibbs temperature-state in the vacuum sector which is quasiequivalent to the vacuum state. Formally this state becomes (up to a possibly infinite normalization factor) tracial in the infinite temperature limit and therefore, restricted to $M$ and $G^{inv}$ is proportional to the Markov tracial state (which the left inverse produces on the right-hand centralizer algebra.) with a possibly divergent normalization factor.

This observation explains why inclusion data like Jones indices, which are to be found in the centralizer algebra, can also be expressed (à la Kac Wakimoto [23]) in terms of infinite temperature limits of characters (partition function) of $G^{inv} \subset A_{univ}$.

The most surprising fact is that the tracial state on $G^{inv}$ leads to (as a generalization of the matrix $S$) multiindexed matrices which represent higher genus Riemann surfaces mapping class group. This is seen most easily from the known graphical representations of these matrices [41]. The crossings in those pictures correspond to the matrices $S$ and the trilinear vertices to the intertwiners. They occur always in pairs and these pairs correspond to projectors whose graphical representation is of the form of parallel path.

This observation on the presence of mapping class matrices for all geni in a chiral conformal field theory on a circle is very startling. It should be take as a hint of a new yet unknown quantum symmetry of which we presently only see its invariant part.
Note also that each operator in $G^{\text{inv.}}$ (and M) in the temperature state has a (generalized) Kac-Wakimoto representation in terms of an infinite temperature limit. This is so since $G^{\text{inv.}}$ permits two types of states: the temperature-states coming from the physics of $A_{\text{univ.}}$ restricted to $G^{\text{inv.}}$, and the more algebraically motivated states on the intertwiner spaces defined by the iteration of left-inverses [5,6]. They also appear in topological field theory [10], if one interprets a functional integral together with the rules for its calculation as the assignment of a state to a global algebra which is similar to $M$, i.e., lacks the localization property of $A_{\text{univ.}}$. What is the geometric meaning (if any) of the class invariant (referring to the dependence on individual endomorphisms) thermal expectation values of operators in $G^{\text{inv.}}$ before one takes the infinite temperature limit? I do not know the answer.

III The TCP Theorem of the Exchange Algebra and its Modular Interpretation for Anyons.

We first recall our findings [7] on the TCP operator of the exchange algebra. The latter is generated by operators $F$ which act on the Hilbert-space consisting of $N$ copies of the vacuum space:

$$\mathcal{H} = \bigoplus_{\alpha}(\alpha, \mathcal{H}_0)$$

according to:

$$F(e, A)(\alpha, \mathcal{H}_0) = (\beta, \pi_0(T_e^* \rho_\alpha(A))\psi)$$

with $A \in A_{\text{univ.}}, T_e \in A_{\text{univ.}} : \rho_\beta \to \rho_\alpha \cdot \rho$ and $\rho_\alpha, \rho_\beta, \rho$ taken from an arbitrary but fixed finite set of reference-endomorphisms.

If $e$ runs through the finite set of all “edges” $e$ of the form $(\rho_\alpha, \rho, \rho_\beta)$ and $A$ through $A_{\text{univ.}}$, then the $F$ generate a $C^*$ algebra (exchange algebra or reduced field bundle) which, since it is an operator algebra in a concrete $H$-space, also allows for the von Neumann closure.

In contrast to Boson-(CCR) or Fermion-(CAR) algebras, the charge conjugation in this algebra differs significantly from the *-operation.

The *-operation has the form:

$$F(e, A)^* = \frac{d_\rho}{\chi_\rho} \sum_{e^*} \eta_{ee^*} F(e^*, \bar{\rho}(A^*)R_\rho)$$
where $e^*$ is of the adjoint form $(\beta, \bar{\rho}, \alpha)$ and the coefficients $(d_{\rho}$ statistical, dimension, $\chi_{\rho}$ a phase and $\eta$ a structure constant matrix) are as in [7].

On the other hand the antilinear charge conjugation of the exchange algebra reads as:

$$\bar{F}(e, A) := \sqrt{d_{\alpha} d_{\beta} \chi_{\beta}} \sum_{\bar{e}} \zeta_{\bar{e}e} F(\bar{e}, \bar{\rho}(A^*)R_{\rho})$$

(21)

with $\bar{e}$ of the form $(\bar{\alpha}, \bar{\rho}, \bar{\beta})$ (and $\zeta$ another structure matrix). It fulfills

$$\bar{F} = \frac{\chi_{\beta}}{\chi_{\alpha}} \cdot F$$

(22)

where the phase can be adjusted to be $\pm 1$ (+1 in the non-selfconjugate case).

In analogy to the standard case, one may define an operator:

$$S_\pm : F(e, A)\Omega \to \overline{F(e, A)\Omega}, \quad F \in \mathcal{F}_{\text{red}}(R_\pm)$$

(23)

which together with its adjoint turns out to be densely defined, hence closable and therefore permits a polar decomposition [9]. Some simple considerations using “coordinates” of point-like fields [7] reveal that this polar decomposition is geometric

$$S_\pm = \kappa^{\pm \frac{1}{2}} \theta V(\pm i\pi)$$

(24)

where $\kappa$ is a statistics phase, $\theta$ the TCT operator and $V(\pm i\pi)$ the analytic continuation of the dilation on $\mathcal{H}$.

This geometric result parallels the previously obtained result based on the use of the Tomita-Takesaki modular theory for the observable algebra [27].

However, these formulae in the case of our exchange algebra cannot be backed up by Tomita-Takesaki modular theory, since this theory requires the (positive-definite) $*$-operation (and cannot be done with the involutive antilinear charge-conjugation) in formula (19).

In our paper [7] we speculated about the possible existence of a “twisted” Tomita-Takesaki theory relating a twisted involution with a twisted commutator.

But perhaps a more conservative idea is to follow the mechanism by which the “would be modular theory” in the case of trivial monodromy $\epsilon^2 = 1$, (i.e., the permutation group exchange algebra) can be converted into a true modular theory.

In that case, we know, according to Doplicher and Roberts [42] that there exists a genuine field algebra of Bosons and Fermions from which the exchange algebra can be obtained by removing multiplicities [43], i.e., making the Hilbert-space smaller.
The prize one pays for converting the Bosons and Fermions with a non-abelian symmetry group into Para-Bosons and -Fermions (on which no nonabelian compact group can act since the $H$-space lacks the necessary multiplicities), is a certain amount of non-locality, now expressed in terms of $R$-matrix commutation relation ($R^2 = 1$) instead of local Boson or Fermion (anti)commutation relations for space-like distances [43].

In fact, one could contemplate that the insistence in a geometric (local) version of the standard Tomita-Takesaki modular theory would determine the smallest enlargement of the $H$-space of the permutation statistics exchange algebra and in this way could yield a direct operator-algebraic proof of the D.R. theorem (which, in its present form contains a significant amount of $C^*$ category theory).

In any case, the previous considerations strongly suggest that the search for a new “quantum symmetry” behind braid group statistics should be pursed in conjunction with the modular Tomita-Takesaki theory and the TCP properties of QFT.

It is interesting to note that the special abelian case of $Z_{2N}$ anyons can still be covered by the standard Tomita-Takesaki theory. As in the case of Fermions [3], one has to introduce a suitably “twisted”or “quasi”-commutant (and no “twisted star” which would lead away from the T.-T. theory).

The starting point is the field algebra $F_N$ associated to the $\mathcal{A}_N^{\text{univ.}}$-observable algebra. The latter is the $Z_{2N}$ maximal extension of the Weyl-algebra on $S^1$ as constructed in [44].

This algebra has $2N$ central charges $C_i$ and its faithful representation requires a Hilbert space:

$$\mathcal{H} = \bigoplus_{n=0}^{2N-1} \mathcal{H}_n, \quad C\mathcal{H}_n = e^{i\pi n} \mathcal{H}_n$$

$$C = e^{2\pi i M}, \quad C_i = C^i.$$  

The field algebra $F_N$ is generated by sector-intertwining operators which “live” on a finite covering of $S^1$ and obey the cylinder-braid-group commutation relations of section III, which in this anyonic case reads as:

$$\phi_{\rho_1} \phi_{\rho_2} = e^{i \pi n_{\text{mon}}^M} e_{\text{mon}}^M \phi_{\rho_2} \phi_{\rho_1}$$

with $e_{\text{mon}} = \text{monodromy phase-factor}$, $n_i$ charge-values and

$$\text{loc}\phi_{\rho_i} \subset J_i, \quad \text{proj}\ J_1 \cap \text{proj}\ J_2 = \emptyset$$

$$J_1 + 2\pi M < J_2 < J_1 + 2\pi(M + 1).$$
Here *loc* refers to the localization region.

We restrict our discussion to the first sheet and choose

\[ \phi_{\rho_1} \in \mathcal{F}_1 = \mathcal{F}(S_+), \]
\[ \phi_{\rho_2} \in \mathcal{F}_2 = \mathcal{F}(S_-). \]

In analogy to Fermions, we construct a twist operator \( Z \) as a suitable square root of the \( 2\pi \)-rotation:

\[ e^{-2\pi i L_0} = e^{-i\pi Q^2} \]
\[ Z := e^{-i\pi Q^2} = \sum_{n=0}^{2N-1} e^{-i\pi n^2/n} P_n \]
\[ \text{with } P_n = \text{ central projector onto } \mathcal{H}_n. \]

The twist transforms \( \phi_{\rho_2} \) according to

\[ Z\phi_{\rho_2}Z^{-i} = \phi_{\rho_2} \sum_n e^{-i\pi n(n+n^2)/n} P_n \sum_n e^{i\pi n^2/n} P_n \]
\[ = e^{-i\pi n^2/n} \phi_{\rho_2} V_{n_2} \]
\[ \text{with } V_{n_2} := \sum_n e^{-i\pi n^2/2n} P_n. \]

As a consequence of

\[ V_{n_2}\phi_{\rho_1} = e^{i\pi n^2/n} \phi_{\rho_1} V_{n_2} \]

this \( V_{n_2} \) commutation phase precisely compensates the statistics phase, and we obtain for the \( \mathcal{F}_N \)-generators

\[ \phi_{\rho_1} Z\phi_{\rho_2}Z^{-1} = Z\phi_{\rho_2}Z^{-1}\phi_{\rho_1}. \]

This result does not change, if the localizations of the \( \phi_{\rho_i}, i = 1, 2 \) (each in one sheet) are in general relative position.

From the generator commutation relation we abstract the so-called “twisted locality”, i.e., the statement:

\[ \mathcal{F}(S_+) \subset \mathcal{F}(S_-)^q \]
\[ S_{\pm} : \text{ half circles} \]
\[ \text{with } \mathcal{F}(S_-)^q := (Z\mathcal{F}(S_-)Z^{-1})'. \]

In addition we have the involution property

\[ \mathcal{F}(S_-)^{qq} = \mathcal{F}(S_-) \]
since $Z^2 = 2\pi$ rotation.

The derivation of the twisted modularity from the geometrically twisted form of causality for $\mathcal{F}(S_-)$ is well-known procedure [3] and identifies the dilation as the modular group.

The KMS property of the dilation manifests itself in terms of point-like covariant fields (and only in terms of these) as a quasiperiodicity in the analytic continuation of their correlation functions, thus paralleling the well-known $Dil(2\pi i)$ periodicity properties of local generators in $\mathcal{A}_N^{univ}$. The quasiperiodicity (as opposed to periodicity) accounts for the braid-group commutation relations (as opposed to bosonic commutation relations).

The modular reflection $J$ contains, as for fermions, the twist $Z$ in addition to the TCP operator. The well-known modular theory of the observable algebra $\mathcal{A}_N(S_\pm)$ in the vacuum state, and the existence of a state-preserving conditional expectation $\mathcal{F} \to \mathcal{A}$ follows then by the “Takesaki devissage theorem” [31].

In $\mathcal{F}_N$ the charge conjugation and the star–operation coalesce. This is the main difference of this special model as compared to the general exchange algebra with noninteger statistical dimensions, where a geometric Tomita-Takesaki modular theory which is consistent with the geometric TCP theorem is still lacking.

It is however my conviction, that all models with integer statistical dimension or (using a concept recently developed by Rehren [45]) with vanishing “defect projector” can be treated with the above method.

There are arguments [46] that Hopf algebras exhaust the integer statistical dimension sector category of properly infinite von Neumann algebras. It has been known for some time that Hopf $C^*$-algebras in the form of Drinfeld “doubles” [47] also fulfil braid-group requirements (perhaps even the complete list of the braid-group-category axioms?). So, such doubles may have field theoretic models with a similar twisted geometric modular theory as above.

In this connection it is interesting to point out, that a framework for a systematic study of $C^*$-Hopf-algebra categories with the additional braid-statistics category requirements has been formulated recently [48].
IV Selfduality and QFT-Inclusions

For our purpose the most remarkable property of the matrix $S$ (i.e., the numerical values of the various invariant charges $C_{\rho_i}$ in the irreducible representations $\pi_0 \cdot \rho_j(\cdot)$) for our purpose is its symmetry.

In the case of finite nonabelian group, the invariant charges are obtained by averaging the group representers in the physical Hilbert-space over the various conjugacy classes, i.e., the index on $C_\rho$ refers to a conjugacy class. Whereas the number of irreducible representations equals the number of conjugacy classes, thus leading to a square matrix $S$ which is the same as the character table, the two indices remain conceptually different (e.g., the fusion of irreducible sectors is not the same as the fusion of conjugacy classes).

So, already in a very early state of development of low dimensional QFT it became clear, that the new symmetry must be more symmetric than e.g. finite non-abelian groups. It should be analogous to abelian symmetry groups even in cases where the statistical dimensions $d_\rho$ are bigger than one.

There are some interesting lessons which one can learn from the abelian $Z_{2N}$ model of section 3, if one analyses its duality structure from the modular point of view as proposed by Fredenhagen, Longo, Rehren and Roberts [45]. From a field algebra $\mathcal{F}(O)$ with a known geometric T.T.-modular theory and a smaller observable algebra $\mathcal{A}(O)$ which is assumed to be irreducibly contained in $\mathcal{F}(O)$ with finite Jones index, one may construct the beginning of a Jones tower (tunnel):

$$\mathcal{F}(O) \supset \mathcal{A}(O) \supset \gamma(\mathcal{F}(O)) \supset \gamma(\mathcal{A}(O)) \ldots$$

Here $\mu$ and $\nu$ are (unique) conditional expectations which exist according to the finite index assumption, and $\gamma$ is Longo’s canonical endomorphism which is only unique as a sector [49] and whose construction requires modular theory (it may be represented as the product of two Tomita reflections belonging to $\mathcal{F}(O)$ and $\mathcal{A}(O)$). The inclusion:

$$\mathcal{A}(O) \supset \gamma(\mathcal{A}(O)) \equiv \rho(\mathcal{A}(O))$$

is that inclusion, on which the superselection theory is based (to see this, one has to decompose $\rho$ into its irreducible pieces). It leads to the superselection-“paragroup”

\[\text{(using the language of Ocneanu who first analysed such inclusions in the case of)}\]

\[\text{Ocneanu would also introduce endomorphisms for the single step inclusion i.e. } \gamma = \sigma\bar{\sigma}, \bar{\sigma} \text{ conjugate}\]
hyperfinite $II_1$ algebras from the point of view of an analogue of group theory), whereas
the paragroup going with the inclusion

$$\mathcal{F}(\mathcal{O}) \supset \gamma(\mathcal{F}(\mathcal{O}))$$

(40)
deserves the name “symmetry” paragroup [45]. Selfduality means that the two para-
groups behind (40) and (41) are isomorphic.

Guido and Longo have shown that $\gamma$ allows to transfer the “Mackey reduction-
induction machine” to the sector theory of von Neumann algebras (by using Connes
bimodules) thus generalizing earlier observation of M. Rieffel

Instead of giving a more detailed explanation of the various concepts used in these
inclusions, let us specialize to the abelian $Z_n$-model. In that case $\mathcal{F}(\mathcal{O})$ and $\mathcal{A}(\mathcal{O})$ are
subalgebras of well-known global algebras $\mathcal{F}_{\text{univ.}}$ and $\mathcal{A}_{\text{univ.}}$. In addition to localized
anyonic or bosonic operators, they also contain global operators $\Gamma$ (in case of $\mathcal{F}_{\text{univ.}}$)
and $Q$ (in the case of $\mathcal{A}_{\text{univ.}}$) [43].

$Q$ is the “global $Z_{2N}$ charge-measurer” and $\Gamma$ the global “charge-creator”:

$$Q\Gamma Q^* = e^{\frac{2\pi i}{N}}\Gamma$$

$$[Q, e^{iyL_0}] = 0, \quad e^{iLy\Gamma}e^{-iy\Gamma} = e^{\frac{2\pi i}{N}}\Gamma.$$ (41)

The last relation is the transformation property of $\Gamma$ under rigid rotations.

The conditional expectations on the von Neumann algebras $\mu$ and $\nu$ may be directly
obtained from those of the global $C^*$ algebras:

$$\mu(b) = \frac{1}{|G|} \sum_n Q^n b Q^{-n} \quad b \in \mathcal{F}_{\text{univ.}}.$$ (42)

$$\nu(a) = \frac{1}{|G|} \sum \Gamma^n a \Gamma^{-n} \quad a \in \mathcal{A}_{\text{univ.}}.$$ (43)

From this one obtains two global fixed point algebras

$$\mathcal{F}_{\text{univ.}} \xrightarrow{\mu} \mathcal{A}_{\text{univ.}}, \quad \mathcal{A}_{\text{univ.}} \xrightarrow{\nu} \gamma(\mathcal{F}_{\text{univ.}}).$$

The image under $\nu$ can be easily described as the extended Weyl algebra with the
$Q$ charge removed (it is still bigger than the ordinary Weyl algebra on the circle!).
However, its interpretation in terms of $\gamma$ fails (the reason for the parenthesis), since $\gamma$
to $\sigma$. For a single algebra (but not coherently for the whole net) this is possible whenever $\mathcal{F}(0) \sim \mathcal{A}(0)$. But an extension of $\sigma$ to the net is generally not possible.
can only be constructed via the T.T.-modular theory in case of properly infinite (e.g. III) algebras whereas the global algebra are type I.

To be more precise: whereas it is possible to compute the Jones projector of the global (type I) inclusion \( \mathcal{A}_{\text{univ.}} \supset \nu(\mathcal{A}_{\text{univ.}}) \) as an operator in \( \mathcal{F}_{\text{univ.}} \), as

\[
E_0 = \sum_n \frac{1}{|G|} \Gamma^n, \quad E_i = Q^i E_0 Q^{-i}, \quad |G| = 2N \tag{44}
\]

with

\[
\mu(E_0) = \frac{1}{|G|}, \tag{45}
\]

one cannot split \( E_0 \) into an isometry

\[
E_0 \neq VV^* \tag{46}
\]

which is the “would be” generator of the Popa-Pimsner basis in QFT [50]

\[
\mu(bV^*) V = V^* \mu(Vb) = \lambda^{-1} b \tag{47}
\]

with \( \lambda = |G| \) in our case.

In order to obtain a projector \( E_0 \) as part of a local (type \( III_1 \)) algebra \( \mathcal{F}(O) \) with the split into Popa-Pimsner isometries \( V \), one must replace the \( \Gamma \)'s by anyonic fields localized in say (without loss of generality) a half-circle \( S_\pm \).

But then the isometry \( V \) (whose existence is secured by the properties of type \( III \) von Neumann algebras) does not act in a completely local way (i.e., does not transform all subalgebras of \( \mathcal{A}(A_\pm) \) into themselves as \( \mu \) does).

Once the \( V \) is known, one can define a corresponding (reducible, with index \( |G| \)) canonical endomorphism \( \gamma \) and a condition expectation \( \nu \) [50]:

\[
\gamma(b) : = \lambda \mu(VbV^*) \quad b \in \mathcal{F} \tag{48}
\]

\[
\nu(a) : = \lambda \mu(E_0 a E_0), \quad \lambda = |G|, \quad E_0 = VV^*. \tag{49}
\]

It is easy to see that the invariance requirement \( \nu(a) = a \) again imposes a linear condition on the generating charge distribution of the extended Weyl algebras \( \mathcal{A}(S_\pm) \), but this condition is less homogeneous as compared to the global algebra, i.e., cannot be simply expressed as the absence of certain zero modes \( Q \).

So, the self-duality picture, although by the choice of the model true for the group theory of the system charge-measures \( Q \) and charge-creators \( \Gamma \), apparently cannot be pushed to the local level: \( \mu \) remains completely local whereas \( \nu \) is only partially local.
In contradistinction to the known ergodic behaviour of the canonical endomorphisms constructed from double reflections of the geometric inclusions of the second section, the inclusions of this section lead to a Jones tunnel with a nontrivial limiting $\text{III}_1$ algebra which in the case of the present $\mathbb{Z}_{2N}$ model turns out to be a subalgebra of the circular Weyl algebra. This remarkable distinction between “deep inclusions” (leading to external symmetries) and “shallow inclusions” (yielding inner symmetries) clearly asks for a more profound understanding.

It is a curious fact that the euclidean theory, briefly sketched in section II, allows for linearly rising charge distributions which are vanishing outside a half circle (similar to “blips” [51]): such discontinuous charge distributions become finite operators with the euclidean star-operation.

We take this as an indication that the euclidean quantum field theory (i.e., the operator algebra formulation of the euclidean analytic continuation of correlation functions) may play an important role in the quest for a perfect (local) self-duality.

We also think of the analytic modular relation for temperature correlation functions found in conformal models (generalizing the modular relation between characters from which Verlinde [38] found the relation between $S$ and the fusion matrices) as an analytic consequence of an algebraic local self-duality obtained by using class-invariant temperature states. The validity of this conjecture would then entail that those mysterious modular identities from Jacobi to Ramanujan, which presently are (in most cases) derived by Poisson-resummation techniques, have their ultimate conceptual understanding via T.T. modular theory in terms of the two physical principles mentioned in the introduction: Einstein causality and Dirac stability.10

The reader finds further speculative remarks resulting from our formula (34) in a recent paper by Rehren. In particular Rehren shows that nonconfined nonabelian group-symmetries (never observed as exact symmetries in nature!) would destroy the pretty picture of selfduality.

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10Then the findings of Verlinde and others would return from their present algebra-geometric setting (not understandable by physicists) to their quantum-physical roots (understandable by some mathematicians).
Speculative Remarks on “Quantum Symmetry”

The observations about $G^\text{inv.}$ in section III and the subsequent discussions of selfduality invite to speculate about a full symmetry algebra $G$.

This problem has been previously discussed in the work of Mack and Schomerus [52] as well as Szlachányi [53] and Vecsernyès [54]. For the conformed Ising model, the nontrivial $d = \frac{1}{16}$ sector appears in their global approach with (assumed) multiplicity 2 in their field algebra, which then has an underlying weak quasi-Hopf algebra structure. Apart from the actual physical problem of whether such quasi-Hopf algebras, which lack strict associativity, can be used in practical calculation (symmetry breaking, Hartree-Fock approximations etc.), these proposals deviate in other aspects significantly from our picture obtained on the basis of algebraic QFT emphasizing “localization” in every step. Actually a detailed classification of global “rational” symmetry operations and an understanding of their possible local shortcomings may be quite helpful.

To get an idea about $G$, we contemplate that the overlap of the spatial abelian $2\pi$-rotations (only nontrivial as a covering transformation) with the invariant charges [7] is only the invariant projection of a bigger non-commutation overlap between space-time and internal symmetries. The completely unexpected appearance of mapping class matrices of all geni in the $A_{\text{univ.}}(S^1)$ observable algebra lends credibility to such a conjecture.

In that case there should exist a non-commutative subgroup of the Möbiusgroup (and not just of the covering) which contains the TCP reflection and whose relative size of conjugacy classes for the remaining generators should be measured by the Jones numbers (thinking about minimal models).

Surprisingly, Vaughn Jones found such a purely group theoretical quantization in which those numbers appear based on the following observation:

(i) any subgroup generated by the reflection $\tilde{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and a parabolic subgroup $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ leads to quantization $\lambda = 2 \cos \frac{\pi}{q}$ for $q \geq 3$ for $\lambda < 2$ if the group is supposed to act in a discrete fashion in the sense of hyperbolic geometry on $\mathbb{R} \cup \{\infty\}$

(ii) such groups $\Gamma$ are free group on two cyclic generators $\tilde{J}$ and
\[
\begin{pmatrix}
1 & \lambda \\
0 & 1
\end{pmatrix} \cdot \tilde{J} = \begin{pmatrix}
-\lambda & 1 \\
-1 & 0
\end{pmatrix} = K, \quad \text{with} \quad K^q = 1
\]

with \( K \) a rotation in \( PSL(2, R) \) and \( \Gamma \) the non-amenable free product \( \Gamma = \tilde{J} \ast K \).

The group (von Neumann)-algebras of such \( \Gamma \)'s are nonhyperfinite factors of type II\(_1\) [55]. This means that the standard representation theory via block-decomposition of the regular representation does not work. So, if \( \Gamma \) appears as a part of the unknown \( \mathcal{G} \), it should come together with some yet unknown representation theory.

From the viewpoint of physical principles there is nothing against nonhyperfinite field algebras. In fact, the proof of hyperfiniteness only applies to local observable algebras \( \mathcal{A}(O) \) [56].

The fact that the field algebras of the DR construction are also hyperfinite III\(_1\) algebras is just a fringe benefit of the permutation group statistics which in turn results from the assumption of \( 4-d \) finite energy particle states with mass gaps [58] and has no other physical principle behind it. Statistical mechanics also lead to infinite hyperfinite algebras in the thermodynamic limit (this is true by construction).

So, the construction of physically viable nonhyperfinite field algebras does not seem to be an easy matter.

In this context, one should mention an idea of Voiculescu [57]. In his approach to noncommutative probability theory based on the concept of “freeness”, the Bose field (at a fixed time) belongs to the standard commutative measure theoretical situation (type I), whereas the CAR fermionic algebra, which admits a tracial state (not describable in measure theoretic terms), is a hyperfinite type II\(_1\) factor. Beyond this he expects an ever-increasing wealth of more and more non-hyperfinite algebras with more complicated Fock-spaces (the plektons of algebraic QFT?).

In the spirit of Voiculescu one expects that the localizability and statistical independence for space-like separation in the exchange-algebra description ought to be replaced by space-like “freeness” for the unknown field algebras, similarly to the transition from statistical independence to “freeness” in his noncommutative probability theory. I believe that the startling selfduality and finiteness of the new symmetry (outside abelian groups) can only be properly understood with the help of very big (nonhyperfinite) von Neumann algebras: the simpler the properties, the bigger the objects carrying them. For the physical use of such ideas one must of course insure that the finite region
observable algebras (expected to be the gauge invariant subalgebras of a generalized unknown gauge principle) remain hyperfinite.

Last not least, the wealth of semiclassical observations made about the new invariants of 3-manifold on the basis of Witten’s topological QFT [10], Atiyah’s axiomatic approach to this problem [59] and the Reshtkhin-Turaev-Viro combinatorial approach [60] as well as some speculative ideas of Ocneanu [61], all point into the same direction: an inexorable new link between space-time and inner symmetries outwitting all the old NoGo theorems [8] based on analytic and algebraic properties of $4 - d$ QFT’s!

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References

1. R. Haag, Local Quantum Physics, Springer 1992.

2. D. Buchholz and P. Junglas “Commun.Math.Phys.121, 255 (1989).

3. Articles of J.E. Roberts, D. Kastler, and K.H. Rehren in: The Algebraic Theory of Superselection Sectors. D. Kastler (ed.), Singapore, World Scientific 1990, as well as ref. [1].

4. J.E. Roberts in $C^*$-algebras and their applications to statistical mechanics and quantum field theory, ed. D. Kastler, North Holland (1976).

K. Fredenhagen, to appear in the Proceedings of the Cargèse Summer Institute (1991).

5. K. Fredenhagen, K.H. Rehren, and B. Schroer, Commun.Math.Phys.125 (1983),201.

6. S. Doplicher, R. Haag, and J.E. Roberts, Commun.Math.Phys.23(1971),149 and 35(1974)49.

7. K. Fredenhagen, K.H. Rehren, and B. Schroer, Reviews in Mathematical Physics, Special Issue (1992) 113.

8. S. Coleman and J. Mandula, Phys.Rev.159, 1251(1967).

L. O’Raifeartaigh, Phys.Rev.Lett14,575(1965).

W.D. Garber and H. Reeh, J.Math.Phys.19,985(1978).

9. Remarks on the construction 3-manifolds in the frameworks of algebraic QFT, their relations with the Reshetikhin-Turaev resp. Turaev-Viro combinatorical approach, as well as and their possible physical interpretation in terms of properties of plektons can be found in the appendix of reference [7].

10. E. Witten, Commun.Math.Phys.121(1989)351.

11. F.Nill, International Journal of Mod.Phys. B6(1992)2159.

In order to obtain the topological field theory of the $Z_{2N}$ model i.e. the centralizer algebra (14), one has to extend the Weyl-algebra of 1-forms by the universal construction which leads to closed but not necessarily exact forms (in the angular
variable). The latter construction is therefore a kind of algebraic compactification.

12. H. Narnhofer and W. Thirring, Rev. in Math. Phys., Special Issue (1992). Here the general idea of using singular states for obtaining gauge invariant subalgebras is developed.

13. For the Bargman Hall-Wightman theorem on the analytic extension of vacuum expectation values of point-like covariant fields we refer to R.F. Streater and A.W. Wightman: PCT, Spin and Statistics and all That, Benjamin, New York (1964).

14. B. Schroer, Nucl. Phys. B369 (1992), 478.

15. M. Karowski, H.-J. Thun, T.T. Truong and P.H. Weisz, Phys. Lett. 67B, 321 (1977).

16. A.B. Zamolodchikov, JETP Lett. 499 (1977).

17. D. Buchholz and K. Fredenhagen, J. Math. Phys. 18, 5 (1977).

18. M. Karowski and P. Weisz, Nucl. Phys. B139 (1979) 209107. 
F.A. Smirnov, Nucl. Phys. B337 (1990)
For a recent account with an interesting physical application see also J. Cardy and G. Mussardo “Universal Properties of Self-Avoiding Walks from Two-Dimensional Field Theory” ISAS preprint 1993.

19. B. Schroer and J.A. Swieca, Nucl. Phys. B121 (1977)
For a formulation in the setting of algebraic QFT see K. Fredenhagen [34] and K. Fredenhagen, work in progress.

20. K. Fredenhagen, M. Gaberdiel and S.M. Rüger, “Scattering States of Plektons in (2+1) Dimensional Quantum Field Theory” in preparation.

21. See Theorem 4.15 of [13].

22. V. Schomerus, Quantum symmetry in quantum theory, DESY preprint 1993 and references therein.

23. V. Kac and M. Wakimoto, Adv. Math. 70 (1986).
24. H.J. Borchers, Commun.Math.Phys. 143 (1992) 315.

25. B. Schroer, Int.J. of Mod.Phys. B, 6 (1992) 2041.
Formula (11) and (12) in that paper contain an unfortunate misprint. \( \mathcal{A}(I) \)
should be replaced by \( \mathcal{L}(I) \).

26. H.W. Wiesbrock, Lett. in Mod.Phys. 28, 107 (1993).
Halvesided modular inclusions of von Neumann algebra, Commun.Math.Phys., in press.

27. D. Buchholz and H. Schulz-Mirbach, Reviews in Math.Phys. 2, 1 (1990) 105.

28. J. Fröhlich and F. Gabbiani, Operator algebra and conformal field theory ETH Zürich preprint (1992).

29. R. Brunetti, D. Guido and R. Longo, Modular structure and duality in conformal field theory, University of Rome II, preprint (1992).

30. The “split property” goes back to Borchers and Doplicher-Longo, see [1].

31. A. Wassermann, “Subfactors arising from positiv-energy representation of some infinite dimensional groups”, unpublished notes 1990.

32. See the contribution of H. Araki and A. Connes to “\( C^* \)-Algebra and their Applications to Statistical Mechanics and Quantum Field Theory”, Proceedings of the International School of Physics “Enrico Fermi”, Course IX, Varenna 1973, Intalian Physical Society.

33. Joint work with H.W. Wiesbrock, in preparation. In incomplete account of this work can be found in [25].

34. K. Fredenhagen in “The Algebraic Theory of Superselection Sectors”, D. Kastler (ed.), Singapore, World Scientific 1980.

35. K.H. Rehren and B. Schroer, Phys.Lett 198B, 84 (1987).
Nucl.Phys. B321, 3 (1989).

36. K.H. Rehren, Commun.Math.Phys. 145 (1992) 123.
37. V.S. Sunder, J. Operator Theory 18 (1987) 289.
   These graphical methods for inclusion theory were first used in
   A. Ocneanu, unpublished, Warwick lecture notes (1987).
   Within the DHR theory they were used in [5].

38. E. Verlinde, Nucl.Phys.300, 360 (1988).

39. G. Moore and N. Seiberg, Phys.Lett.B212, 451 (1988).

40. K.H. Rehren in: “The Algebraic Theory of Superselection Sectors, D. Kastler
    (ed.), Singapore, World Scientific 1990.

41. M. Karowski and R. Schrader, Commun.Math.Phys.151, 355 (1993), especially
    Fig. 25.

42. S. Doplicher and J.E. Roberts, Commun.Math.Phys.131 (1990) 51.

43. K. Drühl, R. Haag and J.E. Roberts, Commun.Math.Phys.18 (1970) 204.
   The step from parafields to the exchange algebra (reduced field bundle) with
   R-matrix commutation relation ($R^2 = 1$) is sketched in my contribution to The
   Algebraic Theory of Superselection Sectors, D. Kastler (ed.), Singapore, World
   Scientific 1990.

44. D. Buchholz, G. Mack, and I. Todorov, Nucl.Phys. B (Proc. Suppl.) 53 (1988)
    20.

45. K.H. Rehren, Subfactors and Coset Models, DESY preprint August 1993.
    K. Fredenhagen, R. Longo, K.H. Rehren, J.E. Roberts, work in progress.

46. R. Longo, a duality for Hopf algebras and for subfactors I, Univ. of Rome II,
    preprint 1992.

47. S. Baaj and G. Skandalis, Unitaires multiplicatifs et dualité pour les produits
    croisés de $C^*$ algèbres, preprint 1991.
    J. Cuntz, Regular actions of Hopf algebras on the $C^*$-algebra generated by a
    Hilbert space, preprint 1991.

48. T. Ceccherini, S. Doplicher, C. Pinzari and J.E. Roberts, A generalization of the
    Cuntz algebras and model action, University of Rome I, preprint 1993.
49. D. Guido and R. Longo, Commun.Math.Phys. \textbf{150}, 521 (1992),
and Longo’s previous work on the canonical endomorphism quoted therein.

50. The field-theoretic formulation as well as the reference to the general construction
by Popa and Pimsner is contained in [45].

51. A. Pressley and G. Segal, Loop Groups, Oxford University Press (1986).

52. G. Mack and V. Schomerus, Commun.Math.Phys. \textbf{137} (1990) 57.
V. Schomerus, Quantum symmetry in quantum theory, DESY preprint 1993.

53. K. Szlachányi, Chiral decompositon as a source of quantum symmetry in the
Ising model, University of Budapest preprint 1993.

54. P. Veccernyèes, On the quantum symmetry of the chiral Ising Model, Princeton
University, preprint 1993.

55. An account of this Jones quantization which embeds the Hecke-groups inside the
Möbius group can be found in F.M. Goodman, P. de la Harpe and V.F.R. Jones,
Coxeter Graphs and Towers of Algebras, Springer 1989. Appendix III.

56. D. Buchholz, C. D’Antoni and K. Fredenhagen, Commun.Math.Phys. \textbf{111} (1987)
123.

57. See the introduction of a new book by D. Voiculesu, Free Probability Theory,
Bures Sur Yvette 1992.

58. D. Buchholz and K. Fredenhagen, Commun.Math.Phys. \textbf{84}, 1 (1982).

59. M. Atiyah, Publ.Math.Inst. Hautes Etudes Sci. Paris \textbf{68}, 175-186 (1989).

60. N. Yu. Reshetikhin and V.G. Turaev, Inv.Math. \textbf{103}, 547 (1991).
V.G. Turaev and O.Y. Viro, to appear in Topology.

61. A. Ocneanu, unpublished, private communication.

62. M. Karowski, R. Schrader and E. Vogt, “Unitary representations of the mapping
class group and numerical calculations of invariants of hyperbolic three mani-
folds”, in preparation.

63. D. Buchholz and K. Fredenhagen, Commun.Math.Phys. \textbf{84}, 1 (1982).
64. B. Schroer, Phys.Lett.B199, 183 (1987).