Superpolynomial lower bounds for decision tree learning and testing

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Abstract

We establish new hardness results for decision tree optimization problems, adding to a line of work that dates back to Hyafil and Rivest in 1976. We prove, under the randomized exponential time hypothesis, superpolynomial runtime lower bounds for two basic problems: given an explicit representation of a function $f$ and a generator for a distribution $D$,

- construct a small decision tree approximator for $f$ under $D$, and
- decide if there is a small decision tree approximator for $f$ under $D$.

Our results imply new lower bounds for distribution-free PAC learning and testing of decision trees, settings in which the algorithm only has restricted access to $f$ and $D$. Specifically, we get that:

- $n$-variable size-$s$ decision trees cannot be properly PAC learned in time $n^{O(\log \log s)}$, and
- depth-$d$ decision trees cannot be tested in time $\exp(d^{O(1)})$.

For learning, the previous best lower bound only ruled out poly($n$)-time algorithms (Alekhnovich, Braverman, Feldman, Klivans, and Pitassi, 2009). For testing, recent work gives similar though incomparable lower bounds in the setting where $f$ is random and $D$ is nonexplicit (Blais, Ferreira Pinto Jr., and Harms, 2021).

Assuming a plausible conjecture on the hardness of SET-COVER, we show that our lower bound for properly PAC learning decision trees can be improved to $n^{\Omega(\log s)}$, matching the best known upper bound of $n^{\Theta(\log s)}$ due to Ehrenfeucht and Haussler (1989).

We obtain our results within a unified framework that leverages recent progress in two different lines of work: the inapproximability of SET-COVER and XOR lemmas for query complexity. Our framework is versatile and yields results for related concept classes such as juntas and DNF formulas.
1 Introduction

The algorithmic problem of constructing decision tree representations of functions is one of the most basic and well-studied problems of computer science. Greedy decision tree learning heuristics such as ID3, C4.5, and CART, developed in the 1980s, continue to be indispensable to everyday machine learning and enjoy empirical success. The data mining textbook [WFHP16] describes C4.5 as “a landmark decision tree program that is probably the machine learning workhorse most widely used in practice to date”. In addition to being extremely fast to evaluate, a key advantage of decision trees is their simple and easy-to-understand structure, making them the most canonical example of an interpretable model. The recent survey [RCC⁺22] lists decision tree learning as the very first of “10 grand challenges” in the emerging field of interpretable machine learning.

In terms of algorithms with theoretical guarantees, a classic result of Ehrenfeucht and Haussler [EH89] gives a quasipolynomial time algorithm for properly PAC learning decision trees: Given labeled examples \((x, f(x))\) where \(f : \{0,1\}^n \to \{0,1\}\) can be computed by a size-\(s\) decision tree and \(x\) is drawn from a distribution \(D\) over \(\{0,1\}^n\), their algorithm runs in \(n^{O(\log s)}\) time and returns a decision tree hypothesis that is close to \(f\) under \(D\). Numerous alternative algorithms have since been designed within restricted variants of the PAC model (e.g. where \(D\) is assumed to be uniform) and by relaxing the problem (e.g. allowing hypotheses that are not themselves decision trees\(^1\)) [Riv87, Blu92, Han93, KM93, KM96, Bsh93, GLR99, BM02, MR02, JS05, KS06, OS07, GKK08, KST09, HKY18, CM19, BLQT21], but Ehrenfeucht and Haussler’s algorithm remains state of the art in the standard PAC model.

Another interesting setting is when an explicit representation of the function \(f\), and possibly also the distribution \(D\), are given to the algorithm. This easier setting, where the algorithm can “inspect” \(f\), models a popular approach in explainable machine learning known as post-hoc explanations. The goal here is not to train a decision tree model for an unknown function \(f\), but instead to turn a complicated trained model \(f\) (e.g. a neural net) into its decision tree representation. While numerous algorithms for this task have been proposed in the empirical literature [CS95, BS96, VAB07, ZH16, BKB17, VLJ⁺17, FH17, VS20], among those with theoretical guarantees, the fastest one remains that of Ehrenfeucht and Haussler.

In parallel with these lines of algorithmic work, there has also been a similarly large body of work on the hardness of decision tree learning [HR76, GJ79, BFJ⁺94, HJLT96, KPB99, ZB00, LN04, CPR⁺07, RRV07, Sie08, ABF⁺09, AH12, Rav13, BLT20]. It is interesting to note that the earliest paper here, by Hyafil and Rivest in 1976, predates Ehrenfeucht and Haussler’s algorithm by more than a decade; indeed, it even predates the PAC model. Their paper, which established the NP-completeness of a certain formulation of decision tree learning with perfect accuracy, reveals that the problem was already intensively studied and recognized as central in the 1970s. Quoting the authors, “the importance of this result can be measured in terms of the large amount of effort that has been put into finding efficient algorithms for constructing optimal binary decision trees”.

A closely related problem is that of testing decision trees: while in learning one is interested in constructing small decision trees, here the goal is simply to decide if one such tree exists. The distribution-free model of property testing, introduced by Goldreich, Goldwasser, and Ron [GGR98]...
to parallel distribution-free PAC learning, has received increasing attention in recent years [CX16, LCS+18, Bsh19, Bel19, Har19, FY20, RR20, BFPJH21, Bsh22, BHZ22, ABF+22, CP22, HY22].

2 Our results

We establish new hardness results for distribution-free learning and testing of decision trees. For both problems, our lower bounds hold even when explicit representations of both the function \(f\) and distribution \(D\) are given to the algorithm; lower bounds in this setting imply lower bounds for learning and testing.

We obtain our results within a unified framework that brings together two active lines of research: the inapproximability of \(\text{Set-Cover}\) [LY94, Fei98, CHKX06, DS14, Mos15, KLM18, CL19, Lin19, CHK20, KI21] and XOR lemmas for query complexity [Dru12, BB19, BFPJH21]. Connections between \(\text{Set-Cover}\) and decision tree optimization problems, both in terms of algorithms and hardness, date back to [HR76] and are present in numerous prior works; we leverage recent progress in both the parameterized and nonparameterized settings. The connection to XOR lemmas, on the other hand, is new to this work. All our lower bounds, being computational in nature, are conditioned on the randomized Exponential Time Hypothesis (ETH). As a byproduct, our lower bounds hold even against randomized algorithms.

We now give a detailed overview of our results, in tandem with a discussion of how they compare with prior work.

2.1 Lower bounds for DT-Construction

The DT-Construction problem is the variant of decision tree learning where \(f\) and \(D\) are both given to the algorithm:

\[
\text{DT-Construction}(s, \varepsilon): \text{Given as input a circuit representation of a function } f : \{0, 1\}^n \rightarrow \{0, 1\}, \text{ a generator for a distribution } D \text{ over } \{0, 1\}^n, \text{ parameters } s \in \mathbb{N} \text{ and } \varepsilon \in (0, 1), \text{ and the promise that } f \text{ is a size-}s \text{ decision tree under } D, \text{ construct a decision tree } T \text{ that is } \varepsilon\text{-close to } f \text{ under } D.
\]

Our first result is a superpolynomial runtime lower bound for DT Construction:

**Theorem 1.** Under randomized ETH, for \(s = n\) and \(\varepsilon = \frac{1}{n}\), any algorithm for DT-Construction\((s, \varepsilon)\) must take \(n^{\Omega(\log \log s)}\) time.

Prior works also focused on the parameter settings \(s = n\) and \(\varepsilon = \frac{1}{n}\), corresponding to strong learning of linear-size decision trees. Most recently, Alekhnovich, Braverman, Feldman, Klivans, and Pitassi [ABF+09] ruled out poly\((n)\) time algorithms under the assumption that SAT cannot be solved in randomized subexponential time. Before that, Hancock, Jiang, Li, and Tromp [HJLT96] ruled out poly\((n)\) time algorithms that return a decision tree hypothesis of size \(n^{1+o(1)}\), under the assumption that SAT cannot be solved in randomized quasipolynomial time.

Our proof of Theorem 1 opens up a concrete route towards obtaining the optimal \(n^{\Omega(\log s)}\) lower bound. We can also show an \(n^{\Omega(\log s)}\) lower bound for the stricter version of DT-Construction where the algorithm has to return a decision tree of size \(s\) (instead of one of any size). We elaborate on both of these in Section 2.3.
Hardness of learning juntas with DNF hypotheses. We obtain Theorem 1 as a corollary of our first main result, which simultaneously allows for a stronger promise on the simplicity of the target function \( f \) and for the algorithm to return a more expressive hypothesis:

**Theorem 2.** Under randomized ETH, for \( s = n \) and \( \varepsilon = \frac{1}{n} \) any algorithm for DT-Construction\((s, \varepsilon)\) must take \( n^{\Omega(\log \log s)} \) time, even if \( f \) is further promised to be a \((\log s)\)-junta under \( D \) and the algorithm is allowed to return a DNF hypothesis.

We recall the strict inclusions

\[
\{(\log s)\text{-juntas}\} \subset \{\text{size-}s \text{ decision trees}\} \subset \{\text{size-}s \text{ DNFs}\}.
\]

Each class is exponentially more expressive than the previous one: a size-\( s \) decision tree can depend on as many as \( s \) variables, and a size-\( s \) DNF can require a decision tree of size \( 2^{\Omega(s)} \).

The results of [ABF+09, HJLT96] are not known to be amenable to such a strengthening. [ABF+09] did give lower bounds for DNF-Construction, the analogue of DT-Construction where the target \( f \) is promised to be a DNF under \( D \) and the algorithm is expected to construct a DNF hypothesis. They ruled out poly\((n)\) time algorithms for \( s = n \) and \( \varepsilon = \frac{1}{n} \). [ABF+09] gave two separate proofs of hardness for DT-Construction and DNF-Construction, reducing from SET-COVER for the former and from CHROMATIC-NUMBER for the latter. Theorem 2, on the other hand, yields new lower bounds for both problems via a single proof.

**Hardness of properly learning juntas.** Implicit in the proofs of Theorems 1 and 2 is a tight connection between algorithms for SET-COVER and algorithms for properly learning juntas. By making this connection explicit, we obtain strong lower bounds for the latter problem that hold even under the promise that the target is a monotone disjunction:

**Theorem 3.** Under randomized ETH, for any \( k \leq n^c \) where \( c < 1 \) is any constant and \( \varepsilon = O\left(\frac{1}{n}\right) \), there is no algorithm that, given as input a circuit representation of a function \( f : \{0,1\}^n \rightarrow \{0,1\} \), a generator for a distribution \( D \) and the promise that \( f \) is a monotone \( k \)-disjunction under \( D \), runs in \( n^{o(k)} \) time and constructs a \( k \)-junta that is \( \varepsilon \)-close to \( f \) under \( D \). Under randomized SETH, we get a lower bound of \( O(n^{k-\lambda}) \) for any constant \( \lambda > 0 \).

These lower bounds nearly match the \( O(n^k/\varepsilon) \) runtime algorithm of the trivial algorithm that iterates over all possible \( k \)-junta hypotheses. Previously, [ABF+09] ruled out poly\((n)\)-time algorithms for \( k \leq O(\log n) \).

### 2.2 Lower bounds for DT-Estimation

The second problem that we consider, DT-Estimation, is a variant of distribution-free decision tree testing where \( f \) and \( D \) are both given to the algorithm:\footnote{It will be more convenient for us to measure the complexity of decision trees by their depth in this section, though there are direct analogues of our results for size instead of depth.}
DT-Estimation\((d, \varepsilon)\): Given as input a circuit representation of a function \(f : \{0,1\}^n \to \{0,1\}\), a generator for a distribution \(D\) over \(\{0,1\}^n\), and parameters \(d \in \mathbb{N}, \varepsilon \in (0,1)\), distinguish between the following cases:

- **Yes**: \(f\) is a depth-\(d\) decision tree under \(D\).
- **No**: \(f\) is \(\varepsilon\)-far from every depth-\(d\) decision tree under \(D\).

Our second main result is an exponential lower bound for DT-Estimation:

**Theorem 4.** Under randomized ETH, any algorithm for DT-Estimation\((d, \varepsilon)\) must take \(\exp(d^{\Omega(1)})\) time. This holds even if \(\varepsilon = \frac{1}{2} - \exp(-d^{\Omega(1)})\) and the No case satisfies the stronger promise that \(f\) is \(\varepsilon\)-far from every decision tree of depth \(\Omega(d \log d)\) under \(D\).

Recent work of Blais, Ferreira Pinto Jr., and Harms [BFPJH21] gives an \(\tilde{\Omega}(2^d)\) lower bound on the query complexity testing of depth-\(d\) decision trees. This lower bound, however, only applies in the setting where both \(f\) and \(D\) are unknown to the algorithm, since it is based on a random function \(f\) and a nonexplicit distribution \(D\). In contrast, our proof of Theorem 4 is constructive: it is based on an \(f\) that is a depth-3 circuit (with \(\oplus, \lor\) gates) and a similarly simple generator for \(D\). Furthermore, [BFPJH21]'s lower bound only holds when \(\varepsilon\) is a sufficiently small constant, whereas ours holds for \(\varepsilon\) being exponentially close to \(\frac{1}{2}\), and with a gap between the decision tree depths of the Yes and No cases.

As for upper bounds, Bshouty and Haddad-Zaknoon [BHZ22] give a distribution-free tester that runs in \(2^{O(d)} n\) time and distinguishes depth-\(d\) decision trees from those that are \(\varepsilon\)-far from decision trees of depth \(d^2\). Under the uniform distribution, Blanc, Lange, and Tan [BLT22] give an algorithm that runs in \(\text{poly}(d, 1/\varepsilon) \cdot n \log n\) time and distinguishes depth-\(d\) decision trees from those that are \(\varepsilon\)-far from decision trees of depth \(O(d^3/\varepsilon^3)\).

### 2.3 Towards stronger lower bounds for DT-Construction

We show two ways in which the lower bounds of Theorems 1 and 2 can be further improved to \(n^{\Omega(\log s)}\). First, we consider the stricter version of DT-Construction where the algorithm has to return a size-\(s\) decision tree:

**Theorem 5.** Under randomized ETH, for \(s = \exp(\tilde{O}(\log \log n))\) and \(\varepsilon = \frac{1}{n}\) any algorithm for DT-Construction\((s, \varepsilon)\) must take \(n^{\Omega(\log s)}\) time if the algorithm has to return a size-\(s\) decision tree. As in Theorem 2, this holds even if \(f\) is further promised to be a \((\log s)\)-junta under \(D\) and the algorithm is allowed to return a size-\(s\) DNF hypothesis.

This more stringent version of DT-Construction corresponds to the notion of strictly proper learning of size-\(s\) decision trees, where the algorithm has to return a hypothesis that falls within the concept class. Ehrenfeucht and Haussler's algorithm is not strictly proper. On the other hand, for size-\(s\) decision trees of depth \(O(\log s)\), there is a simple dynamic programming algorithm that runs in \(n^{O(\log s)}\) time and is strictly proper [GLR99, MR02]. Since every \((\log s)\)-junta is a decision tree of depth \(\log s\), this matches the lower bound of Theorem 5.

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3This nonexplicit distribution is derived from lower bounds on the sample complexity of estimating distribution support size [WY19].
Finally, we show how an optimal lower bound of $n^{\Omega(\log s)}$ for the original version of DT-Construction, matching the runtime of Ehrenfeucht and Haussler’s algorithm, would follow from a natural and well-studied conjecture about Set-Cover:

**Conjecture 1** (Optimal inapproximability of parameterized Set-Cover). There exists constants $\alpha, \beta < 1$ such that for $k \leq N^\alpha$, there is no $N^{o(k)}$ time algorithm that, given a size-$N$ set cover instance, distinguishes between:

- **Yes**: There is a set cover of size $k$.
- **No**: Every set cover has size at least $k \cdot (1 - \beta) \ln N$.

There is a simple and efficient $\ln N$-approximation algorithm for Set-Cover, and various hardness results are known for the problem of achieving a better approximation ratio [LY94, Fei98, DS14, Mos15, CHK20]. Conjecture 1 states that this hardness carries over to the parameterized setting. Existing ETH-based lower bounds for parameterized Set-Cover [CHKX06, KLM18, CL19, Lin19, KI21] are evidence in favor of it, and it is plausible that Conjecture 1 can be shown to hold under ETH.\footnote{See [MPW19, GKMP20] for further discussions of this conjecture and its implications for proof complexity.}

**Theorem 6.** Under Conjecture 1, for $s = n$ and $\varepsilon = \frac{1}{n}$ any algorithm for DT-Construction$(s, \varepsilon)$ must take $n^{\Omega(\log s)}$ time. As in Theorem 2, this holds even if $f$ is further promised to be a $(\log s)$-junta under $\mathcal{D}$ and the algorithm is allowed to return a DNF hypothesis.

Table 1 summarizes our results for DT-Construction and shows how they compare with the prior state of the art.

Table 1: Algorithms and lower bounds for DT-Construction. All our results are conditioned on randomized ETH, the lower bounds of Theorems 5 and 6 are optimal.

| Reference | Target | Hypothesis | Time complexity |
|-----------|--------|------------|----------------|
| [ABF+09] | size-$s$ DT | DT | $n^{\omega(1)}$ lower bound |
| [ABF+09] | size-$s$ DNF | DNF | $n^{\omega(1)}$ lower bound |
| [EH89]   | size-$s$ DT | DT | $n^{O(\log s)}$ upper bound |
| Theorem 2 | $(\log s)$-junta | DNF | $n^{\tilde{O}(\log \log s)}$ lower bound |
| Theorem 5 | $(\log s)$-junta | size-$s$ DNF | $n^{\Omega(\log s)}$ lower bound |
| Theorem 6 | $(\log s)$-junta | DNF | $n^{\Omega(\log s)}$ lower bound under Conjecture 1 |
3 Our techniques

The starting point of all our reductions is the parameterized version of Set-Cover. For a set cover instance \( S \), we write \( \text{opt}(S) \) to denote the size of the smallest set cover.

**Definition 1.** The \((k,k')\)-Set-Cover problem is the following. Given as input a set cover instance \( S \) and parameters \( k,k' \in \mathbb{N} \), output Yes if \( \text{opt}(S) \leq k \) and No if \( \text{opt}(S) > k' \).

**Reducing from Set-Cover to juntas vs. DNFs.** Our key lemma, which is the crux of our lower bounds for both DT-Construction and DT-Estimation, is a reduction from \((k,k')\)-Set-Cover to the problem of distinguishing small juntas from large DNF formulas, where “small” and “large” are functions of \( k \) and \( k' \) respectively:

**Lemma 3.1.** There is an algorithm that, given a size-\( N \) instance \( S \) of \((k,k')\)-Set-Cover with \( n \) sets and a parameter \( \ell \leq N \), runs in \( \text{poly}(N) \) time and outputs a circuit representation of a function \( f : (\{0,1\}^\ell)^n \rightarrow \{0,1\} \) and a generator for a distribution \( \mathcal{D} \) over \((\{0,1\}^\ell)^n\) satisfying:

- If \( \text{opt}(S) \leq k \), then \( f \) is a \( k\ell \)-junta under \( \mathcal{D} \).
- If \( \text{opt}(S) > k' \), then any DNF of size \( \leq \exp(\Omega(k'\ell)) \) is \( \Omega\left(\frac{1}{N}\right)\)-far from \( f \) under \( \mathcal{D} \).

We obtain Theorems 1, 2 and 5 by combining Lemma 3.1 with a recent result on the ETH-hardness of \((k,k')\)-Set-Cover for \( k' = \frac{1}{2} \left( \log \frac{N}{\log \log N} \right)^{1/k} \), where \( N \) is the size of the instance [Lin19]. Similarly, we obtain Theorem 6 by combining Lemma 3.1 with Conjecture 1. For Theorem 3, we only need a simpler special case of Lemma 3.1, which we combine with the ETH- and SETH-hardness of \((k,k+1)\)-Set-Cover (i.e. the hardness of solving parameterized Set-Cover exactly) [CHKX06, PW10].

**Gap amplification.** We view Lemma 3.1 as a gap amplification procedure. Specifically, given a \((k,k')\)-Set-Cover instance it is straightforward to construct an instance of DT-Construction, a target function \( f \) and distribution \( \mathcal{D} \), where the decision tree complexity of \( f \) under \( \mathcal{D} \) exactly reflects the gap \((k,k')\): if \( \text{opt}(S) \leq k \) then \( f \) is a size-\( k \) decision tree under \( \mathcal{D} \), and otherwise \( f \) requires decision trees of size \( \geq k' \). To obtain stronger lower bounds we amplify this gap into a much larger gap in the complexity of \( f \) under \( \mathcal{D} \): if \( \text{opt}(S) \leq k \) then \( f \) is a small junta under \( \mathcal{D} \), and if \( \text{opt}(S) > k' \) then \( f \) is a large-size DNF under \( \mathcal{D} \). This reduction enables us to translate lower bounds for \((k,k')\)-Set-Cover into strong lower bounds for DT-Construction. See Figure 1 for an illustration of this gap amplification.

**Building hard instances of DT-Construction.** Our construction of \( f \) and \( \mathcal{D} \) in Lemma 3.1 is based on the one in [ABF+09], which in turn builds on [Hau88, HJLT96]. [ABF+09] also gave a gap amplifying reduction from \((k,k')\)-Set-Cover to the problem of distinguishing whether \( f \) has small or large decision tree complexity under \( \mathcal{D} \). Lemma 3.1 is a strengthening of their reduction where the same gap in set cover sizes leads to a more dramatic gap in \( f \)’s complexity under \( \mathcal{D} \). While the construction of \( f \) and \( \mathcal{D} \) is similar to the one in [ABF+09], our analysis is entirely different and is, in our opinion, simpler. Notably, our analysis enables us to obtain lower bounds even against DNF hypotheses whereas previous works relied crucially on the hypothesis being a decision tree. In addition to yielding our stronger conclusion, our analysis overcomes technical challenges that arise when we have to modify Lemma 3.1 in the context of DT-Estimation, which we now discuss.
(Easy reduction) (Gap amplification)

\begin{itemize}
\item \textbf{Yes}: \( f \) is a depth-\( d \) decision tree under \( D \).
\item \textbf{No}: \( f \) is \( \Omega(\frac{1}{n}) \)-far from every decision tree of depth \( \Omega(d \log d) \) under \( D \).
\end{itemize}

We amplify this mild hardness \( (\varepsilon = O(\frac{1}{n})) \) to very strong hardness \( (\varepsilon = \text{exponentially close to } \frac{1}{2}) \) by considering \( f^{\oplus m} : (\{0, 1\}^n)^m \to \{0, 1\} \), the \( m \)-fold XOR composition of \( f \):

\[ f^{\oplus m}(x^{(1)}, \ldots, x^{(m)}) := f(x^{(1)}) \oplus \cdots \oplus f(x^{(m)}) \]

and the corresponding distribution \( D^m \) over \( (\{0, 1\}^n)^m \). In the \textbf{Yes} case of Corollary 3.2, it is easy to see that \( f^{\oplus m} \) is a decision tree of depth \( \leq dm \) under \( D^m \). To analyze the \textbf{No} case, we prove the following lemma:
Lemma 3.3 (Hardness amplification for DT Estimation). Let \( f : \{0,1\}^n \rightarrow \{0,1\} \) and \( \mathcal{D} \) be such that \( f \) is \( \varepsilon \)-far from every depth-\( d \) decision tree under \( \mathcal{D} \). For any \( \gamma > 0 \), by taking \( m = \Theta(\log(1/\gamma) / \varepsilon) \), we get that \( f \oplus m \) is \( (\frac{1}{2} - \gamma) \)-far from every decision tree of depth \( \Omega(dm) \) under \( \mathcal{D}^m \).

Our proof of Lemma 3.3 combines existing XOR lemmas for distributional query complexity [Dru12, BB19, BKLS20]. Specifically, we first use one due to Brody, Kim, Lerdputtipongpoom, and Srinivasulu [BKLS20] to amplify from \( \varepsilon = O(\frac{1}{n}) \) to \( \Theta(1) \), and then one due to Drucker [Dru12] to amplify from \( \Theta(1) \) to exponentially close to \( \frac{1}{2} \). The quantitative parameters of these lemmas are incomparable, and we show how they can be applied in tandem in our setting.

Handling aborts. Lemma 3.3 as stated is actually not quite what we prove; see Lemma 7.4 for the actual version. For technical reasons, the XOR lemma of [BKLS20] (and hence Lemma 3.3) requires a stronger assumption, that \( f \) is \( \varepsilon \)-far from every depth-\( d \) decision tree that is allowed to abort with probability \( \delta \), and distance is measured with respect to non-aborts. [BKLS20]'s lemma requires \( \delta = \Theta(1) \) whereas \( \varepsilon = O(\frac{1}{n}) \) in our setting, so this is a significantly stronger assumption. To satisfy this stronger assumption, we have to prove a strengthening of Corollary 3.2 where the No case maps to an \( f \) that is \( \Omega(\frac{1}{n}) \)-far from decision trees that are allowed to abort with constant probability; this in turn necessitates a corresponding strengthening of Lemma 3.1. With these in hand, Theorem 4 then follows fairly easily.

4 Discussion and future work

Our work makes new progress on the longstanding open problem of determining the complexity of properly PAC learning decision trees. A natural avenue for future work is to close the remaining gap between our lower bound of \( n^{\tilde{O}(\log \log s)} \) and the \( n^{O(\log s)} \) runtime of Ehrenfeucht and Haussler’s algorithm. Our techniques point to an approach towards an \( n^{\Omega(\log s)} \) lower bound via Conjecture 1, which adds further motivation to the study of parameterized SET-COVER.

As for our testing lower bounds, a notable feature is that they hold in the regime where \( \varepsilon = \frac{1}{2} - o(1) \), which we obtain from an initial hardness for \( \varepsilon = \Omega(\frac{1}{n}) \) via XOR lemmas for query complexity. It would be interesting to further develop such hardness amplification techniques in property testing. For example, can the communication-complexity-based lower bound technique of Blais, Brody, and Matulef [BBM12] be fruitfully combined with the large body of work on XOR lemmas, and direct-product-type results more generally, for communication complexity?

More broadly, there is a growing and concerted effort within the machine learning community to design algorithms that produce simple hypotheses, such as decision trees, especially in the context of high-stakes applications where interpretability is paramount; see e.g. the position paper [Rud19]. Our lower bounds show that interpretability can come at the price of computational intractability, even under strong assumptions on the target function. There is substantial practical motivation for the development of a theoretical understanding of such tradeoffs and how they can be mitigated. For example, a concrete next step from our work is to identify reasonable assumptions under which our lower bounds can be circumvented; one could consider monotone target functions, a common assumption in both theory and practice.
5 Preliminaries

**Set Cover.** Given a bipartite graph $S = (S, U, E)$ on $N$-vertices, the SET-COVER problem is to find a minimum size subset $C \subseteq S$ such that every vertex in $U$ is adjacent to some vertex in $C$.\(^5\) We write $\text{opt}(S) \in \mathbb{N}$ to denote the size of the smallest set cover for $S$. We will often write $n$ to denote the size of $|S| \leq N$. The set of neighbors of a vertex $u \in U$ is $N_S(u) = \{s \in S : (s, u) \in E\}$. We identify a vertex $u \in U$ with its neighborhood set $N_S(u)$. Each set $N_S(u)$ can be viewed as a string in $\{0, 1\}^{|S|}$ where a 1 in the string indicates an edge between $u$ and the corresponding vertex $s \in S$. Hence, each vertex $u \in U$ can likewise be encoded as a string in $\{0, 1\}^{|S|}$.

**Hitting Set.** Given a bipartite graph $H = (S, U, E)$, the HITTING-SET problem is to find a minimum size subset $I \subseteq U$ which “hits” every vertex $s \in S$: $N_S(s) \cap I \neq \emptyset$ for all $s \in S$. We write $\text{opt}(H)$ for the size of the smallest hitting set.

An instance $H = (S, U, E)$ of HITTING-SET can equivalently be viewed as an instance $H = (U, S, E)$ of SET-COVER.

**Fact 5.1 (Set-Cover and Hitting-Set are equivalent).** SET-COVER and HITTING-SET are equivalent to each other under approximation-preserving reductions. In particular, any instance $S$ of SET-COVER can be transformed in linear-time into an instance $H$ of hitting set such that $\text{opt}(S) = \text{opt}(H)$ and vice versa.

The results of [ABF+09] are formulated in terms of hitting set. Though for consistency, in this work we will only refer to SET-COVER. See Figure 2 for an illustration of a set cover instance and a hitting set instance on a single bipartite graph.

![Figure 2: A bipartite graph $G = (S, U, E)$ viewed on the left as a set cover instance and on the right as a hitting set instance.](image)

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\(^5\)Typically, the set cover problem is cast as a combinatorial problem: given subsets $S_1, \ldots, S_m \subseteq [n]$ of some universe $[n]$, find the minimum size subcollection $S_{i_1}, \ldots, S_{i_k}$ whose union is $[n]$. We consider the graph theoretic formulation because it makes the connection to the hitting set problem more transparent.

\(^6\)We assume without loss of generality that each $N_S(u)$ is unique so that a vertex $u$ can be identified by its neighborhood set $N_S(u)$ (if $N_S(u) = N_S(u')$ for $u \neq u'$ we can simply delete $u'$ without affecting the set cover complexity).
Decision trees. For a decision tree $T : \{0,1\}^n \rightarrow \{0,1\}$, we write $L \in T$ to denote that $L$ is a leaf of $T$. The size of $T$ is its number of leaves and is denoted $|T|$. For an input $x \in \{0,1\}^n$, we write $\text{depth}_T(x) \in \mathbb{N}$ to denote the depth of $x$ in $T$, the number of variables queried on the root-to-leaf path consistent with $x$.

DNF formulas. A literal is a variable or its negation. A term is a conjunction ($\land$) of literals. A DNF formula $F : \{0,1\}^n \rightarrow \{0,1\}$ is a disjunction ($\lor$) of terms, denoted $F = t_1 \lor \cdots \lor t_s$. The size of the DNF formula is $|F| = s$, the number of terms. The width of a term $|t_i|$ is the number of literals in it. The width of an input $x \in \{0,1\}^n$ is defined as the width of the smallest width term accepting $x$ and 0 if no term accepts $x$:

$$\text{width}_F(x) := \begin{cases} \min_{t_i(x) = 1} |t_i| & F(x) = 1 \\ 0 & F(x) = 0. \end{cases}$$

Circuits. We consider Boolean circuits $C : \{0,1\}^n \rightarrow \{0,1\}$ with AND, OR, NOT, and PARITY gates: $\{\land, \lor, \neg, \oplus\}$. The size of a circuit $|C|$ is the number of gates in it. The depth of a circuit is the longest directed path from an input node to an output node.

$k$-juntas. A function $f : \{0,1\}^n \rightarrow \{0,1\}$ is a $k$-junta if its output depends on $\leq k$ bits. Hence, if $f$ is a $k$-junta it can be completely specified by a table of size $2^k$ corresponding to all possible assignments to the $k$ relevant variables. In particular, every $k$-junta is a size-$2^k$ decision tree and every size-$s$ decision tree is an $s$-junta.

Distributions. We use boldface letters e.g. $x, y$ to denote random variables. For a distribution $\mathcal{D}$, we write $\text{dist}_\mathcal{D}(f, g) = \Pr_{x \sim \mathcal{D}}[f(x) \neq g(x)]$. A function $f$ is $\varepsilon$-close to $g$ if $\text{dist}_\mathcal{D}(f, g) \leq \varepsilon$. When $\varepsilon = 0$, we drop the $\varepsilon$ and simply say $f$ computes $g$ over $\mathcal{D}$. Often $f$ is viewed as one of the combinatorial objects above and $g$ is a generic function, e.g. a decision tree $T : \{0,1\}^n \rightarrow \{0,1\}$ computes $g$ over $\mathcal{D}$ if $\text{dist}_\mathcal{D}(T, g) = 0$. Similarly, $f$ is $\varepsilon$-far from $g$ if $\text{dist}_\mathcal{D}(f, g) > \varepsilon$. We write $\mathcal{U}_b$ for the uniform distribution on $b$ bits. A generator for a distribution $\mathcal{D}$ over $\{0,1\}^n$ is an algorithm $G : \{0,1\}^n \rightarrow \{0,1\}^n$ which takes $n$ uniform random bits as input and outputs $n$ bits distributed according to $\mathcal{D}$: $
abla_{x \sim \mathcal{U}_n}[G(x) = x] = \Pr_{x \sim \mathcal{D}}[x = x]$ for all $x \in \{0,1\}^n$.

Learning. See Appendix C for the definitions of learning that we use. All learning algorithms we consider are proper learning algorithms. When referring to “learning decision trees” we mean properly learning the concept class $\mathcal{T} = \{T : \{0,1\}^n \rightarrow \{0,1\} \ | \ T \text{ is a decision tree}\}$. Likewise, when referring to “learning size-$s$ decision trees”, we mean properly learning the concept class $\mathcal{T}_s = \{T : \{0,1\}^n \rightarrow \{0,1\} \ | \ T \text{ is a size-$s$ decision tree}\}$. When discussing algorithms for learning $k$-juntas, we assume the output of the learning algorithm is a table of size $2^k$ (as in e.g. [MOS04]).

Complexity-theoretic assumptions. Many results on the hardness of SET-COVER are conditioned on the exponential time hypothesis.

Hypothesis 1 (Exponential time hypothesis (ETH) [Tov84, IP01, IPZ01]). There exists a constant $\delta > 0$ such that 3-SAT on $n$ variables cannot be solved in $O(2^{\delta n})$ time.
Since we are proving hardness against randomized algorithms, we will use a randomized variant of ETH.

**Hypothesis 2** (Randomized ETH, see [CIKP08, DHM+14]). There exists a constant $\delta > 0$ such that 3-SAT on $n$ variables cannot be solved by a randomized algorithm in $O(2^{n^\delta})$ time with error probability at most $1/3$.

We will also use two additional hypotheses.

**Hypothesis 3** (Strong exponential time hypothesis (SETH) [IP01, IPZ01]). For every $\delta > 0$, there exists a $k \in \mathbb{N}$ such that $k$-CNF-SAT on $n$ variables cannot be solved in time $O(2^{n(1-\delta)})$.

**Hypothesis 4** ($W[1] \neq FPT$, see [DF13, CFK+15]). For any computable function $f : \mathbb{N} \rightarrow \mathbb{N}$, no algorithm can decide if a graph $G = (V, E)$ contains a $k$-clique in $f(k) \cdot \text{poly}(|V|)$ time.

As with randomized ETH, randomized SETH and randomized $W[1] \neq FPT$ are the respective versions of these hypotheses against randomized algorithms. Also, we remark that $W[1] = FPT$ is a weaker assumption than ETH which itself is weaker than SETH. If $W[1] = FPT$, then SAT is solvable in subexponential time.

### 5.1 Existing results on the hardness of Set-Cover

Throughout, we use several different hardness results for Set-Cover and approximating Set-Cover. We start with the following theorem due to [Lin19] about the hardness of approximating set cover. We have slightly modified the theorem from its original form to fit our setting. We discuss Lin’s original theorem and our modifications in Appendix A.

**Theorem 7** ([Lin19]). Assuming randomized ETH, there is a constant $c \in (0, 1)$ such that for any $k \in \mathbb{N}$ with $k \leq \frac{1}{2} \cdot \frac{\log \log N}{\log \log \log N}$, there is no randomized $N^{ck}$ time algorithm that can solve \( (k, \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k}) \)-Set-Cover on $N$ vertices with high probability.

We will also use results on the inapproximability of unparameterized Set-Cover:

**Theorem 8** ([DS14, Mos15]). Under randomized ETH, for every $0 < \beta < 1$, any algorithm that approximates size-$N$ instances of Set-Cover to within $(1 - \beta) \ln N$ w.h.p. requires $2^{N^{O(\beta)}}$ time.

By a standard search-to-decision reduction, Theorem 8 implies the following lower bound for $(k, k')$-Set-Cover where, unlike in the parameterized setting, $k$ is no longer guaranteed to be “small”:

**Theorem 9.** Under randomized ETH, for every $0 < \beta < 1$, there exists $k \leq N$ such that any algorithm that solves size-$N$ instances of $(k, k')$-Set-Cover where $k' = k(1-\beta) \ln N$ w.h.p. requires $2^{N^{O(\beta)}}$ time.

Finally, we will also use existing lower bounds in the ungapped setting:

**Theorem 10** (Ungapped hardness of Set-Cover from $W[1] \neq FPT$ [CHKX06, Theorem 5.6]). Assuming $W[1] \neq FPT$, for all constants $c \in (0, 1)$ and for all $k \leq n^c$, any $(k, k + 1)$-Set-Cover instance $S = (S, U, E)$ cannot be solved in time $|S|^{o(k)}$. 
Furthermore, there are even stronger set cover lower bounds assuming SETH.

**Theorem 11** (Ungapped hardness of Set-Cover from SETH [PW10, Theorem 2.3]). Assuming SETH, for all constants \( c, \delta \in (0,1) \) and for all \( k \leq n^c \), any \((k,k+1)\)-Set-Cover instance \( S = (S,U,E) \) cannot be solved in time \( O(|S|^{k-\delta}) \).

### 6 Lower bounds for DT-Construction

In this section we prove Lemma 3.1 and use it to derive Theorems 2 and 3. The high-level idea behind Lemma 3.1 is to show how, given a set cover instance \( S \), we can construct a function \( f \) and a distribution \( D \) such that the optimal set cover size for \( S \) is reflected in the complexity of \( f \) under \( D \).

**Definition 2** (\( \Gamma_S \) and \( D_S \)). Let \( S = (S,U,E) \) be a set cover instance with \( |S| = n \). We identify each universe element \( u \in U \) with a vector \( \{0,1\}^n \), the indicator vector of its neighborhood set \( N_S(u) \) (i.e. the indicator vector of the sets that contain \( u \)). We define the partial function \( \Gamma_S : \{0,1\}^n \rightarrow \{0,1\} \) as follows:

\[
\Gamma_S(x) = \begin{cases} 
0 & x = 0^n \\
1 & x = u, \ u \in U. 
\end{cases}
\]

The distribution \( D_S \) over the support of \( \Gamma_S \) is given by the pmf

\[
D_S(x) = \begin{cases} 
\frac{1}{2} & x = 0^n \\
\frac{1}{2|U|} & x = u, \ u \in U. 
\end{cases}
\]

When \( S \) is clear from context we will drop the subscript and simply write \( \Gamma \) and \( D \). We observe that given any set cover \( C \subseteq S \), the monotone disjunction of the variables in \( C \) computes \( \Gamma \) over \( D \). In particular, we have:

**Fact 6.1.** If \( \text{opt}(S) \leq k \) then \( \Gamma \) is a monotone disjunction of \( k \) variables under \( D \).

We now define a “parity-amplified” version of \( \Gamma \). While \( \Gamma \) is a function over the domain \( \{0,1\}^n \), this new function will be over the domain \( \{0,1\}^\ell \) for some parameter \( \ell \in \mathbb{N} \).

**Notation.** For a string \( y \in \{0,1\}^n \), we write \( y_i \in \{0,1\}^\ell \) to denote the \( i \)th block of \( y \), and \( (y_i)_j \) to denote the \( j \)th entry of the \( i \)th block. We define the function BlockwisePar : \( \{0,1\}^\ell \rightarrow \{0,1\}^n \):

\[
\text{BlockwisePar}(y) := (\oplus y_1, \ldots, \oplus y_n),
\]

where \( \oplus y_i \) denotes the parity of the bits in \( y_i \).

**Definition 3** (\( \Gamma_{\oplus \ell} \) and \( D_{\oplus \ell} \)). For \( \Gamma \) and \( D \) as defined in Definition 2 and an integer \( \ell \in \mathbb{N} \), we define the partial function \( \Gamma_{\oplus \ell} : \{0,1\}^\ell \rightarrow \{0,1\} \),

\[
\Gamma_{\oplus \ell}(y) = \Gamma(\text{BlockwisePar}(y)).
\]

The distribution \( D_{\oplus \ell} \) over the support of \( \Gamma_{\oplus \ell} \) is defined as follows: to sample from \( D_{\oplus \ell} \),

1. First sample \( x \sim D \).
2. For each $i \in [n]$, sample $y_i \sim \{0,1\}^\ell$ u.a.r. among all strings satisfying $\oplus y_i = x_i$. Equivalently, sample $y \sim (\{0,1\}^\ell)^n$ u.a.r. among all strings satisfying $\text{BlockwisePar}(y) = x$.

**Fact 6.2** (Blockwise parity of $D_{\oplus \ell}$ induces $D$). For $y \sim D_{\oplus \ell}$, we have that $\text{BlockwisePar}(y)$ is distributed according to $D$.

We have the following analogue of Fact 6.1:

**Fact 6.3.** If $\text{opt}(S) \leq k$ then $\Gamma_{\oplus \ell}$ is a $k\ell$-junta (a disjunction of $k$ many parities, each over $\ell$ variables) under $D_{\oplus \ell}$.

**An equivalent way of sampling from $D_{\oplus \ell}$.** For our proof of Lemma 3.1, it will be useful for us consider a different, but equivalent, way of sampling from $D_{\oplus \ell}$. For $z \in (\{0,1\}^\ell)^n$, $x \in \{0,1\}^n$, and $j \in [\ell]$, we write $\text{ParComplete}_j(z, x)$ to denote the string $y \in (\{0,1\}^\ell)^n$ where for each block $i \in [n]$,

- All except the $j$th coordinate of $y_i \in \{0,1\}^\ell$ are filled in according to $z_i \in \{0,1\}^{\ell-1}$.

\[(y_i)_1, \ldots, (y_i)_{j-1}, (y_i)_{j+1}, \ldots, (y_i)_{\ell} = ((z_i)_1, \ldots, (z_i)_{\ell-1}).\]

- The $j$th coordinate of $y_i$ is filled in with the unique bit so that $\oplus y_i = x_i$.

**Example.** Consider $n = 4$ and $\ell = 3$ and $j = 2$. Then, we can view $z = (z_1, \ldots, z_4) \in (\{0,1\}^2)^4$ as a $4 \times 2$ matrix where the $i$th row is $z_i$. In this case, we may have for example:

\[
z = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
1 & 1 \\
1 & 0 \\
\end{bmatrix} \quad x = \begin{bmatrix}
1 \\
1 \\
0 \\
1 \\
\end{bmatrix} \quad \rightarrow \quad \text{ParComplete}_2(z, x) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}.
\]

Note that the first and third columns of $\text{ParComplete}_2(z, x)$, colored teal, are exactly the first and second columns of $z$ respectively, and that the second column of $\text{ParComplete}_2(z, x)$, colored purple, is filled in so that parity of each row of matches the corresponding row of $x$.

**Definition 4** (The distribution $D_{\oplus \ell}^j$). For $j \in [\ell]$, the distribution $D_{\oplus \ell}^j$ is obtained via the following sampling procedure: sample $x \sim D$, $z \sim (\{0,1\}^\ell)^n$ u.a.r., and output $\text{ParComplete}_j(z, x)$.

The following proposition on the equivalence between $D_{\oplus \ell}$ and $D_{\oplus \ell}^j$ can be easily verified. We defer the calculation to Appendix B.

**Proposition 6.4** ($D_{\oplus \ell}^j$ is equivalent to $D_{\oplus \ell}$). For all $j \in [\ell]$ and $y \in (\{0,1\}^\ell)^n$,

\[
\Pr_{y \sim D_{\oplus \ell}} [y = y] = \Pr_{y \sim D_{\oplus \ell}^j} [y = y].
\]
Constructiveness of $\Gamma_{\oplus \ell}$ and $D_{\oplus \ell}$

We can efficiently compute both a circuit representation of $\Gamma_{\oplus \ell}$ and a generator for the distribution $D_{\oplus \ell}$ from a given set cover instance.

**Lemma 6.5** (Constructiveness of $\Gamma_{\oplus \ell}$ and $D_{\oplus \ell}$). Let $S = (S, U, E)$ be an $N$-vertex set cover instance with $|S| = n$ and let $\ell \leq N$ be a parameter. Then there is an algorithm that runs in $\text{poly}(N)$ time and outputs a circuit representation of $\Gamma_{\oplus \ell}$ over $D_{\oplus \ell}$ and a generator for the distribution $D_{\oplus \ell}$.

**Proof.** We separate the proof into two parts. First, we give a circuit representation of $\Gamma_{\oplus \ell}$, then we give a generator for $D_{\oplus \ell}$.

**A circuit for $\Gamma_{\oplus \ell}$**. Recall that a circuit $C : (\{0, 1\}^\ell)^n \to \{0, 1\}$ represents $\Gamma_{\oplus \ell} : (\{0, 1\}^\ell)^n \to \{0, 1\}$ over $D_{\oplus \ell}$ if $\text{dist}_{D_{\oplus \ell}}(C, \Gamma_{\oplus \ell}) = 0$. The function $\Gamma : \{0, 1\}^n \to \{0, 1\}$ is computed over $D$ by the disjunction of all $n$ variables. That is, $\text{dist}_D(\Gamma, x_1 \lor \cdots \lor x_n) = 0$. Therefore, for $y = (y_1, \ldots, y_n) \in \text{supp}(D_{\oplus \ell})$,

$$
\Gamma_{\oplus \ell}(y) = \Gamma(\oplus y_1, \ldots, \oplus y_n) = (\oplus y_1) \lor \cdots \lor (\oplus y_n)
$$

(Definition of $\Gamma_{\oplus \ell}$)

(BlockwisePar($y \in \text{supp}(D)$))

It follows that the circuit given by

$$
C(y) := \bigvee_{i \in [n]} \bigoplus_{j \in [\ell]} (y_i)_j
$$

computes $\Gamma_{\oplus \ell}$ over $D_{\oplus \ell}$. See Figure 3 for an illustration of $C$. Since this circuit has size $n \cdot \ell$ and depth 3, the first part of the lemma statement follows.

---

This observation can equivalently be viewed as an application of Fact 6.1 plus the fact that $\text{opt}(S) \leq |S| = n$ holds for all $S$. 

---

Figure 3: A depth-2 circuit for $\Gamma_{\oplus \ell}$ consisting of one top gate that is an OR connected to $n$ PARITY gates, each of which is connected to a disjoint block of $\ell$ input variables.
**A generator for $D_{\oplus \ell}$.** Recall that a generator for a distribution takes uniform random bits as input and outputs bits distributed according to the desired distribution. First, we observe that there is an efficient generator for $D$ using $1 + \log |U|$ uniform random bits. Specifically, use 1 uniform random bit to decide between the two cases:

1. output $0^n$
2. output $u \in U$ uniformly at random.

The second case can be accomplished with $\log |U|$ uniform random bits. Then the following procedure generates the distribution $D_{\oplus \ell}$:

1. use $n(\ell - 1)$ uniform random bits to select $z \in \{0, 1\}^{\ell-1}$
2. use $1 + \log |U|$ bits to sample $x \sim D$
3. output $\text{ParComplete}_1(z, x)$.

By Proposition 6.4, this procedure equivalently generates the distribution $D_{\oplus \ell}$. The procedure uses $n(\ell - 1) + 1 + \log |U|$ bits. We can assume without loss of generality that $1 + \log |U| \leq |S| = n^8$ so that $n(\ell - 1) + 1 + \log |U| \leq n\ell$. It follows that this procedure efficiently generates $D_{\oplus \ell}$ from $n\ell$ uniform random bits.

---

### 6.1 Warmup for Lemma 3.1: Lower bounds against decision tree hypotheses

We will prove Lemma 3.1 with the function being $\Gamma_{\oplus \ell}$ and the distribution being $D_{\oplus \ell}$. The first bullet of the lemma statement is given by Fact 6.3, and so the bulk of the remaining work goes into establishing the second bullet of the lemma statement.

We begin with a warmup, showing the weaker statement that $\Gamma_{\oplus \ell}$ is far from any small decision tree under $D_{\oplus \ell}$. This proof will illustrate many of the key ideas in the actual proof for DNFs, which we give in the next subsection. Furthermore, this lower bound is already sufficient to establish Theorem 1, and will be the starting point of our lower bounds for DT-Estimation that we prove in the next section.

**Lemma 6.6.** Let $S = (S, U, E)$ be an $N$-vertex set cover instance and let $\ell \geq 2$. If $T : (\{0, 1\}^\ell)^n \rightarrow \{0, 1\}$ is a decision tree of size $|T| < 2^{\text{opt}(S)\ell/8}$, then $\text{dist}_{D_{\oplus \ell}}(T, \Gamma_{\oplus \ell}) \geq 1/(4N)$.

**High level idea.** There are three main steps:

1. No decision tree with small average depth can approximate $\Gamma$ under $D$ (Claim 6.7).
2. Any decision tree with small average depth that approximates $\Gamma_{\oplus \ell}$ under $D_{\oplus \ell}$ can be used to construct decision tree of much smaller average depth that approximates $\Gamma$ under $D$ (Claim 6.8). This is the key claim.
3. Any small size decision tree must have small average depth with respect to $D_{\oplus \ell}$ (Claim 6.10).

Together, these three claims imply that no small size decision tree can approximate $\Gamma_{\oplus \ell}$ under $D_{\oplus \ell}$, thereby yielding Lemma 6.6.

---

*If $|S| < 1 + \log |U|$, we just replicate sets until $|U| \leq |S|$. This change at most doubles $N$ and does not affect $\text{opt}(S)$. 

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Figure 4: Any decision tree for $\Gamma$ implicitly defines a set cover of $S$ consisting of the variables highlighted in red.

**Claim 6.7** (Good approximators for $\Gamma$ require large depth). Let $T : \{0, 1\}^n \rightarrow \{0, 1\}$ be a decision tree and $S = (S, U, E)$ be an $N$-vertex set cover instance with $|S| = n$. If $\mathbb{E}_{x \sim D}[\text{depth}_T(x)] < \text{opt}(S)/2$ then $\text{dist}_D(T, \Gamma) \geq 1/(2N)$.

*Proof.* Let $T$ be a decision tree satisfying $\mathbb{E}_{x \sim D}[\text{depth}_T(x)] < \text{opt}(S)/2$. We actually prove the stronger claim that $\text{dist}_D(T, \Gamma) \geq 1/(2|U|)$. Suppose for contradiction that $\text{dist}_D(T, \Gamma) < 1/(2|U|)$. Each $x \in \text{supp}(D)$ has mass $\geq 1/(2|U|)$ under $D$ and so we must have $\text{dist}_D(T, \Gamma) = 0$. Let $C \subseteq [n] = S$ be the set of vertices that $T$ queries in the computation of $0^n$ (equivalently, $C$ is the leftmost root-to-leaf path in $T$). See Figure 4 for an illustration of $C$. Since $\text{dist}_D(T, \Gamma) = 0$, we have that $T(0^n) = \Gamma(0^n) = 0$.

We claim $C$ is a valid set cover for $S$. Indeed, if some $u \in U$ is not covered by $C$, then $N_S(u) \cap C = \emptyset$, and $u$ would follow this same path $C$ as $0^n$ in $T$. This would imply that $0 = T(u) \neq \Gamma(u) = 1$, contradicting the fact that $\text{dist}_D(T, \Gamma) = 0$.

Since $C$ is a valid set cover, it follows that $|C| \geq \text{opt}(S)$ and so:

$$\mathbb{E}_{x \sim D}[\text{depth}_T(x)] \geq \frac{\text{Pr}_{x \sim D}[x = 0^n] \cdot |C|}{2} \geq \frac{|C|}{2} \geq \frac{\text{opt}(S)}{2}$$

which contradicts our original assumption on the average depth of $T$. \hfill $\square$

Our high-level proof strategy for the next claim is loosely inspired by [BKLS20] (which itself built on [BB19]). This proof also crucially relies on Proposition 6.4.
Claim 6.8 (Good approximators for \( \Gamma_{\oplus \ell} \) yield good approximators for \( \Gamma \)). Let \( T : (\{0,1\}^\ell)^n \to \{0,1\} \) be a decision tree such that

\[
\text{dist}_{\oplus \ell}(T, \Gamma_{\oplus \ell}) \leq \varepsilon \quad \text{and} \quad \mathbb{E}_{y \sim \text{dist}_{\oplus \ell}}[\text{depth}_T(y)] \leq d.
\]

Then there is a restriction \( T^* : \{0,1\}^n \to \{0,1\} \) of \( T \) satisfying

\[
\text{dist}_D(T^*, \Gamma) \leq 2\varepsilon \quad \text{and} \quad \mathbb{E}_{x \sim D}[\text{depth}_{T^*}(x)] \leq \frac{2d}{\ell}.
\]

Proof. Recalling the notation from Definition 4, when \( \underline{z} \in (\{0,1\}^{\ell-1})^n \) and \( j \in [\ell] \) are fixed, the function \( x \mapsto \text{ParComplete}_j(\underline{z}, x) \) is a function from \( \{0,1\}^n \) to \((\{0,1\}^\ell)^n\). Our proof proceeds by finding a suitable \( \underline{z} \) and \( j \) so that \( x \mapsto T(\text{ParComplete}_j(\underline{z}, x)) \) is a tree of much smaller average depth and computes \( \Gamma \) accurately over \( D \). Restricting \( T \) according to the values specified by \( \underline{z} \) and \( j \) yields the desired decision tree.

For \( j \in [\ell] \) and \( y \in (\{0,1\}^\ell)^n \), write \( q_j(y) \) for the number of times that \( T \), on the input \( y \), queries \( (y_i)_j \) for some \( i \in [n] \). Thus, \( \text{depth}_T(y) = \sum_{j \in [\ell]} q_j(y) \) and likewise

\[
\sum_{j \in [\ell]} \mathbb{E}_{y \sim \text{dist}_{\oplus \ell}}[q_j(y)] = \mathbb{E}_{y \sim \text{dist}_{\oplus \ell}}[\text{depth}_{T^*}(y)] \leq d.
\]

Let \( j \in [\ell] \) be the index that minimizes \( \mathbb{E}_{y \sim \text{dist}_{\oplus \ell}}[q_j(y)] \). By averaging, this \( j \) must satisfy \( \mathbb{E}_{y \sim \text{dist}_{\oplus \ell}}[q_j(y)] \leq d/\ell \). By Proposition 6.4, we can write

\[
\frac{d}{\ell} \geq \mathbb{E}_{y \sim \text{dist}_{\oplus \ell}}[q_j(y)] = \mathbb{E}_{j \sim \text{dist}_{\oplus \ell}}[q_j(y)] = \mathbb{E}_{z \sim \text{dist}_{\oplus \ell}}\mathbb{E}_{x \sim D}[q_j(\text{ParComplete}_j(z, x))].
\]

Similarly, we also have:

\[
\varepsilon \geq \mathbb{E}_{y \sim \text{dist}_{\oplus \ell}}[T(y) \neq \Gamma_{\oplus \ell}(y)] = \mathbb{E}_{j \sim \text{dist}_{\oplus \ell}}[T(y) \neq \Gamma_{\oplus \ell}(y)] = \mathbb{E}_{z \sim \text{dist}_{\oplus \ell}}\mathbb{E}_{x \sim D}[T(\text{ParComplete}_j(z, x)) \neq \Gamma(x)].
\]

Applying Markov’s inequality twice, we have

\[
\mathbb{E}_{z \sim \text{dist}_{\oplus \ell}}\mathbb{E}_{x \sim D}[T(\text{ParComplete}_j(z, x)) \neq \Gamma(x)] > 2\varepsilon \]

and

\[
\mathbb{E}_{z \sim \text{dist}_{\oplus \ell}}\mathbb{E}_{x \sim D}[q_j(\text{ParComplete}_j(z, x))] > \frac{2d}{\ell}.
\]
And thus by a union bound, there is some fixed $z \in \{0, 1\}^{n(\ell-1)}$ satisfying
\[ \Pr_{x \sim D}[T(\text{ParComplete}_y(z, x)) \neq \Gamma(x)] \leq 2\varepsilon \quad \text{and} \quad \E_{x \sim D}[q_D(\text{ParComplete}_y(z, x))] \leq \frac{2d}{T}. \]

The tree $T^*$ is formed by restricting $T$ according to $z$ and $j$. Also, this tree $T^*$ satisfies $\text{depth}_{T^*}(x) = q_D(\text{ParComplete}_y(z, x))$ by construction. The claim then follows. \qed

To prove Claim 6.10, we first need a simple proposition stating that the probability a string $y \sim D_{\oplus \ell}$ matches some fixed substring decays exponentially with the length of the substring.

**Proposition 6.9** ($D_{\oplus \ell}$ is uniform-like). Let $\ell \geq 2$. For all $R \subseteq [\ell], r \in \{0, 1\}^{|R|}, i \in [n]$, and $b \in \{0, 1\}$, we have
\[ \Pr_{y \sim D_{\oplus \ell}}[(y)_R = r \mid \oplus y_i = b] \leq 2^{-|R|/2} \]
where $(y)_R \in \{0, 1\}^{|R|}$ is the substring of $y_i \in \{0, 1\}^\ell$ consisting of the coordinates specified by $R$.

**Proof.** We first consider the case when $|R| < \ell$. By the definition of $D_{\oplus \ell}$, the conditional distribution in question is the uniform distribution over all strings in $\{0, 1\}^\ell$ whose parity is $b$. The marginal distribution of this distribution over any set of $|R| < \ell$ coordinates is uniform, and therefore:
\[ \Pr_{y \sim D_{\oplus \ell}}[(y)_R = r \mid \oplus y_i = b] = 2^{-|R|}. \]

If $|R| = \ell$, then depending on whether the parity of the bits in $r$ match $b$, we have:
\[ \Pr_{y \sim D_{\oplus \ell}}[y_i = r \mid \oplus y_i = b] = \begin{cases} 0 & \text{if } \oplus r \neq b \\ 2^{-|R|+1} & \text{if } \oplus r = b. \end{cases} \]

In either case, we have the desired probability bound. \qed

**Claim 6.10** (Small trees have small average depth). Let $T$ be a size-$s$ decision tree, then
\[ \E_{y \sim D_{\oplus \ell}}[\text{depth}_T(y)] \leq 2\log s. \]

**Proof.** We start by upper bounding $\Pr[y \text{ reaches } L]$ for any fixed leaf $L$ of $T$. For each block $i \in [n]$, we write $R_i(L)$ to denote the variables from the $i$th block queried on the root-to-$L$ path, and $r_i(L) \in \{0, 1\}^{R_i(L)}$ to denote the values that the path assigns to these variables. Note that $\sum_{i \in [n]} |R_i(L)| = |L|$, the depth of $L$ in $T$. With this notation in hand, for any fixed $x \in \{0, 1\}^n$, we have
\[ \Pr_{y \sim D_{\oplus \ell}}[y \text{ reaches } L \mid \text{BlockwisePar}(y) = x] \]
\[ = \prod_{i \in [n]} \Pr_{y \sim D_{\oplus \ell}}[(y)_R = r_i(L) \mid \text{BlockwisePar}(y) = x] \quad \text{(Independence of the $y_i$'s for fixed $x$)} \]
\[ = \prod_{i \in [n]} \Pr_{y \sim D_{\oplus \ell}}[(y)_R = r_i(L) \mid \oplus y_i = x_i] \]
\[ \leq \prod_{i \in [n]} 2^{-|R_i(L)|/2} \quad \text{(Proposition 6.9)} \]
\[ = 2^{-|L|/2}. \]
Since this holds for every $x$, it follows that
\[
\Pr_{y \sim \mathcal{D}_G} [y \text{ reaches } L] \leq 2^{-|L|/2}. \tag{1}
\]

We therefore conclude that
\[
\frac{1}{2} \cdot \mathbb{E}_{y \sim \mathcal{D}_G} [\text{depth}_T(y)] = \mathbb{E}_{y \sim \mathcal{D}_G} \left[ \log \left( 2^{\text{depth}_T(y)/2} \right) \right] \\
\leq \log \left( \mathbb{E}_{y \sim \mathcal{D}_G} \left[ 2^{\text{depth}_T(y)/2} \right] \right) \quad \text{(Concavity of log(·))} \\
= \log \left( \sum_{L \in T} \Pr_{y \sim \mathcal{D}_G} [y \text{ reaches } L] \cdot 2^{|L|/2} \right) \\
\leq \log \left( \sum_{L \in T} 2^{-|L|/2} \cdot 2^{|L|/2} \right) \quad \text{(Equation (1))} \\
= \log s.
\]

Rearranging completes the proof. \hfill \Box

**Putting things together: Proof of Lemma 6.6.** Suppose there is some tree $T$ computing $\Gamma \oplus \ell$ with $|T| \leq 2^{\text{opt}(S)\ell/8}$. We show that $\text{dist}(T, \Gamma \oplus \ell) \geq 1/(4N)$. Suppose for contradiction that $\text{dist}(T, \Gamma \oplus \ell) < 1/(4N)$. By Claim 6.10, we have $\mathbb{E}_{y \sim \mathcal{D}_G} [\text{depth}_T(y)] < 2 \cdot \log \left( 2^{\text{opt}(S)\ell/8} \right) = \text{opt}(S)\ell/4$. Then by Claim 6.8 there is a decision tree $T^*$ satisfying
\[
\text{dist}_D(T^*, \Gamma) < \frac{1}{2N} \quad \mathbb{E}_{x \sim \mathcal{D}} [\text{depth}_{T^*}(x)] < \frac{\text{opt}(S)}{2}.
\]

But this contradicts Claim 6.7. \hfill \Box

### 6.2 Proof of Lemma 3.1: Lower bounds against DNF hypotheses

We extend Lemma 6.6 to show that $\Gamma \oplus \ell$ cannot even be approximated by small DNFs. This extension will allow us to complete the proof of Lemma 3.1. For this section, we use the negation of $\Gamma$:
\[
\mathbf{\Gamma}(x) = \begin{cases} 
1 & x = 0^n \\
0 & x = u, u \in U 
\end{cases}.
\]

Analogous to Fact 6.1, any set cover $C \subseteq S$ yields a conjunction of $k$ literals which computes $\Gamma$ under $\mathcal{D}$.

**Fact 6.11.** If $\text{opt}(S) \leq k$, then $\mathbf{\Gamma}$ is a conjunction of $k$ literals under $\mathcal{D}$.

The literals in this case are the negation of the variables in the set cover $C \subseteq S$. We will likewise use the negation of $\Gamma \oplus \ell$:
\[
\mathbf{\Gamma} \oplus \ell(y) = \mathbf{\Gamma}(\text{BlockwisePar}(y)).
\]

The analogue of Fact 6.3 becomes:
Fact 6.12. If $\text{opt}(S) \leq k$ then $\overline{\Gamma}_{\oplus \ell}$ is a $k\ell$-junta (a conjunction of $k$ many parities, each over $\ell$ variables) under $D_{\oplus \ell}$.

Ultimately, this change allows us to prove that $\overline{\Gamma}_{\oplus \ell}$ cannot be approximated by small-size DNF formulas. If instead, one were interested in proving hardness against CNF formulas, one could work directly with the unnegated $\Gamma_{\oplus \ell}$. We find that working with DNFs is slightly less cumbersome than with CNFs which is why we focus on the negated function in this section. Specifically, we prove the following extension of Lemma 6.6.

Lemma 6.13. Let $S = (S, U, E)$ be an $N$-vertex set cover instance and let $\ell \geq 2$. If $F : \{0,1\}^{\ell} \rightarrow \{0,1\}$ is a DNF of size $|F| < 2^{\text{opt}(S)/16}$, then $\text{dist}_{D_{\oplus \ell}}(\overline{\Gamma}_{\oplus \ell}, F) \geq 1/(4N)$.

The high level proof strategy follows that of Lemma 6.6 and can be divided into the same three steps outlined in Section 6.1. The only difference is that “average depth” is no longer a well-defined quantity with DNF formulas. Instead, we consider “average width” which is a generalization of average depth suited to our purposes.

Claim 6.14 (Good approximators for $\overline{\Gamma}$ require large width). Let $F : \{0,1\}^{n} \rightarrow \{0,1\}$ be a DNF formula and $S = (S, U, E)$ be an $N$-vertex set cover instance with $|S| = n$. If $\mathbb{E}_{x \sim D}[\text{width}_F(x)] < \text{opt}(S)/2$, then $\text{dist}_{D}(F, \overline{\Gamma}) \geq 1/(2N)$.

Proof. Let $F = t_1 \lor \cdots \lor t_s$ be a DNF formula. If $F(0^n) = 0$, then $\text{dist}_{D}(F, \overline{\Gamma}) \geq 1/2$ since $\overline{\Gamma}(0^n) = 1$. Otherwise, let $t_i$ be the smallest width term such that $t_i(0^n) = 1$ so that $|t_i| = \text{width}_F(0^n)$. Since $t_i$ accepts the all $0$s input, it is a conjunction of $|t_i|$ negated variables. Let $C \subseteq S$ be the set of variables in $t_i$. Since

$$\frac{|t_i|}{2} = \Pr_{x \sim D}[x = 0^n] \cdot \text{width}_F(0^n) \leq \mathbb{E}_{x \sim D}[\text{width}_F(x)] \leq \frac{\text{opt}(S)}{2},$$

$C$ is not a set cover. Let $u \in U$ be some vertex not covered by $C$: $N_S(u) \cap C = \emptyset$. Then, $u$ is encoded with $0$s for all variables in $C$. It follows that $t_i(u) = 1$ and $F(u) = 1 \neq 0 = \overline{\Gamma}(u)$. Therefore:

$$\text{dist}_{D}(F, \overline{\Gamma}) \geq \Pr_{x \sim D}[x = u] = \frac{1}{2|U|} \geq \frac{1}{2N}.$$

Claim 6.15 (Good approximators for $\overline{\Gamma}_{\oplus \ell}$ yield good approximators for $\overline{\Gamma}$). Let $F : \{0,1\}^{\ell} \rightarrow \{0,1\}$ be a DNF formula such that

$$\text{dist}_{D_{\oplus \ell}}(F, \overline{\Gamma}_{\oplus \ell}) \leq \varepsilon \quad \text{and} \quad \mathbb{E}_{y \sim D_{\oplus \ell}}[\text{width}_F(y)] \leq w.$$

Then there is a restriction $F^* : \{0,1\}^{n} \rightarrow \{0,1\}$ of $F$ satisfying

$$\text{dist}_{D}(F^*, \overline{\Gamma}) \leq 2\varepsilon \quad \text{and} \quad \mathbb{E}_{x \sim D}[\text{width}_{F^*}(x)] \leq \frac{2w}{\ell}.$$

---

The lemma is indeed an “extension” because any size-$s$ decision tree computing $\Gamma_{\oplus \ell}$ yields a size-$s$ decision tree computing $\overline{\Gamma}_{\oplus \ell}$ simply by flipping leaf labels, and so Lemma 6.6 can equivalently be viewed as a statement about $\overline{\Gamma}_{\oplus \ell}$.
Proof. The proof is similar to that of Claim 6.8. First, let \( q_j(y) \) denote the number of variables of the form \((y_i)_j\) for some \( i \in [n] \) appearing in the smallest width term that accepts \( y \) and 0 if no term accepts \( y \). Then, \( \text{width}_F(y) = \sum_{j \in [\ell]} q_j(y) \) for all \( y \in \text{supp}(D_G) \). Therefore:

\[
\sum_{j \in [\ell]} E_{y \sim D_G} [q_j(y)] \leq w.
\]

Let \( \underline{j} \in [\ell] \) be the index that minimizes \( E_{y \sim D_G} [q_j(y)] \). By averaging, \( j \) satisfies \( E_{y \sim D_G} [q_j(y)] \leq w/\ell \).

Using Proposition 6.4:

\[
\frac{w}{\ell} \geq E_{y \sim D_G} [q_{\underline{j}}(y)] = E_{y \sim D_G} [q_{\underline{j}}(y)] \quad \text{(Proposition 6.4)}
\]

Similarly:

\[
\varepsilon \geq \Pr_{y \sim D_G} [F(y) \neq \overline{\Gamma} \oplus \ell(y)] = \Pr_{y \sim D_G} [F(y) \neq \overline{\Gamma} \oplus \ell(y)] \quad \text{(Proposition 6.4)}
\]

Applying Markov’s inequality twice, we have

\[
\Pr_{z \sim \mathcal{D}_n(\ell-1)} \left[ \Pr_{x \sim D} \left[ F(\text{ParComplete}_{\underline{j}}(z, x)) \neq \overline{\Gamma}(x) \right] > 2\varepsilon \right] < \frac{1}{2}
\]

and

\[
\Pr_{z \sim \mathcal{D}_n(\ell-1)} \left[ E_{x \sim D} \left[ q_{\underline{j}}(\text{ParComplete}_{\underline{j}}(z, x)) \right] > \frac{2w}{\ell} \right] < \frac{1}{2}
\]

And thus by a union bound, there is some fixed \( \underline{z} \in \{0, 1\}^{n(\ell-1)} \) satisfying

\[
\Pr_{x \sim D} \left[ F(\text{ParComplete}_{\underline{j}}(\underline{z}, x)) \neq \overline{\Gamma}(x) \right] \leq 2\varepsilon \quad \text{and} \quad E_{x \sim D} \left[ q_{\underline{j}}(\text{ParComplete}_{\underline{j}}(\underline{z}, x)) \right] \leq \frac{2w}{\ell}.
\]

The DNF formula \( F^* \) is formed by restricting \( F \) according to the string \( \underline{z} \). Also, this \( F^* \) satisfies \( \text{width}_{F^*}(x) = q_{\underline{j}}(\text{ParComplete}_{\underline{j}}(\underline{z}, x)) \) by construction. The claim then follows.

Claim 6.16 (Small DNFs have small average width). Let \( F \) be a size-\( s \) DNF formula for \( s \geq 4 \) such that \( \text{dist}_{D_G}(F, \overline{\Gamma} \oplus \ell) \leq 1/4 \), then

\[
E_{y \sim D_G} [\text{width}_F(y)] \leq 4\log(s).
\]

Proof. Let \( F = t_1 \lor \cdots \lor t_s \) be a DNF formula with \( s \) terms satisfying \( \text{dist}_{D_G}(F, \overline{\Gamma} \oplus \ell) \leq 1/4 \). We start by upper bounding the conditional probability \( \Pr[ t(y) = 1 \mid F(y) = 1] \) for any fixed term \( t \in \{t_1, \ldots, t_s\} \). We bound the probabilities \( \Pr[t(y) = 1] \) and \( \Pr[F(y) = 1] \) separately.
(1) \( \Pr[F(y) = 1] \geq 1/4 \). We write
\[
\frac{1}{4} \geq \text{dist}_{\mathcal{D} \oplus \ell}(F, \mathcal{T} \oplus \ell)
\]
\[
\geq \left| \Pr_{y \sim \mathcal{D} \oplus \ell} [F(y) = 1] - \Pr_{y \sim \mathcal{D} \oplus \ell} [\mathcal{T} \oplus \ell(y) = 1] \right|
\]
\[
= \left| \Pr_{y \sim \mathcal{D} \oplus \ell} [F(y) = 1] - \frac{1}{2} \right|
\]
which implies \( \Pr[F(y) = 1] \geq 1/4 \).

(2) \( \Pr[t(y) = 1] \leq 2^{-|t|/2} \). For each \( i \in [n] \), let \( R_i(t) \) denote the variables from the \( i \)th block which appear in the term \( t \) and let \( r_i(t) \in \{0, 1\}^{R_i(t)} \) denote the values assigned by those variables (i.e. 1 if the variable is unnegated in \( t \) and 0 if the variable is negated in \( t \)). Then \( \sum_{i \in [n]} |R_i(t)| = |t| \), the width of \( t \). Using this notation, for any fixed \( x \in \text{supp}(\mathcal{D}) \):
\[
\Pr_{y \sim \mathcal{D} \oplus \ell} [t(y) = 1 \mid \text{BlockwisePar}(y) = x]
\]
\[
= \prod_{i \in [n]} \Pr_{y \sim \mathcal{D} \oplus \ell} [(y_i)_{R_i(t)} = r_i(t) \mid \text{BlockwisePar}(y) = x]
\]
\[(\text{Independence of the } y_i \text{'s for fixed } x)\]
\[
= \prod_{i \in [n]} \Pr_{y \sim \mathcal{D} \oplus \ell} [(y_i)_{R_i(t)} = r_i(t) \mid \oplus y_i = x_i]
\]
\[
\leq \prod_{i \in [n]} 2^{-|R_i(t)|/2}
\]
\[(\text{Proposition 6.9})\]
\[
= 2^{-|t|/2}.
\]
Since this holds for any fixed \( x \), it follows that
\[
\Pr_{y \sim \mathcal{D} \oplus \ell} [t(y) = 1] \leq 2^{-|t|/2}.
\]

Together, these two points imply
\[
\Pr_{y \sim \mathcal{D} \oplus \ell} [t(y) = 1 \mid F(y) = 1] = \frac{\Pr[t(y) = 1]}{\Pr[F(y) = 1]} \leq 2^{-|t|/2 + 2}.
\]

(2)
Lastly:
\[
\frac{1}{2} \sum_{y \sim \mathcal{D}_{\oplus \ell}} \left[ \text{width}_F(y) \right] - 2 = \sum_{y \sim \mathcal{D}_{\oplus \ell}} \left[ \log \left( 2^{\text{width}_F(y)/2-2} \right) \right] \\
\leq \log \left( \sum_{y \sim \mathcal{D}_{\oplus \ell}} \left[ \log \left( 2^{\text{width}_F(y)/2-2} \right) \right] \right) \\
= \log \left( \sum_{y \sim \mathcal{D}_{\oplus \ell}} \left[ \log \left( 2^{\text{width}_F(y)/2-2} \right) \right] \right) \\
\leq \log \left( \sum_{y \sim \mathcal{D}_{\oplus \ell}} \left[ \log \left( 2^{\text{width}_F(y)/2-2} \right) \right] \right) \\
\leq \log \left( \sum_{i \in [s]} 2^{|t_i|/2} \cdot \Pr_{y \sim \mathcal{D}_{\oplus \ell}} \left[ t_i(y) = 1 \mid F(y) = 1 \right] \right) \\
= \log \left( \sum_{i \in [s]} 2^{|t_i|/2} \cdot 2^{-|t_i|/2+2} \right) \\
= \log s.
\]

Rearranging and applying the assumption that \(2 \leq \log(s)\) completes the proof. \(\square\)

**Putting things together: Proof of Lemma 6.13** Suppose there is some DNF formula \(F\) computing \(\Gamma_{\oplus \ell}\) with \(|F| \leq 2^{\text{opt}(S)\ell/16}\). We show that \(\text{dist}(F, \Gamma_{\oplus \ell}) \geq 1/(4N)\). Suppose for contradiction that \(\text{dist}(T, \Gamma_{\oplus \ell}) < 1/(4N)\) \(\leq 1/4\). If \(|F| < 4\), we add dummy terms (e.g. by replicating the terms already in \(F\)) so that \(|F| \geq 4\). We can then apply Claim 6.16: \(\sum_{y \sim \mathcal{D}_{\oplus \ell}} \left[ \text{width}_F(y) \right] < 4 \cdot \log \left( 2^{\text{opt}(S)\ell/16} \right) = \text{opt}(S)\ell/4\). Then by Claim 6.15, there is a DNF formula \(F^*\) satisfying
\[
\text{dist}_D(F^*, \Gamma) < \frac{1}{2N} \quad \sum_{x \sim D^*} \left[ \text{depth}_{F^*}(x) \right] < \frac{\text{opt}(S)}{2}.
\]
But such an \(F^*\) contradicts Claim 6.14. \(\square\)

**The last steps: finishing the proof of Lemma 3.1.** We prove the following lemma which immediately implies Lemma 3.1.

**Lemma 6.17** (\(\Gamma_{\oplus \ell}\) proves Lemma 3.1). Let \(S = (S, U, E)\) be an \(N\)-vertex instance of \((k, k')\)-Set-Cover and \(\ell \leq N\). Then there is an algorithm that runs in \(\text{poly}(N)\) time and outputs a circuit representation of \(\Gamma_{\oplus \ell}\) under \(D_{\oplus \ell}\) and a generator for \(D_{\oplus \ell}\) which satisfies:

- If \(\text{opt}(S) \leq k\), then \(\Gamma_{\oplus \ell}\) is a \(k\ell\)-junta under \(D_{\oplus \ell}\).
- If \(\text{opt}(S) > k'\), then any DNF of size \(\leq 2^{k'\ell/16}\) is \(\frac{1}{4N}\)-far from \(\Gamma_{\oplus \ell}\) under \(D_{\oplus \ell}\).

**Proof.** By Lemma 6.5, there is an algorithm that runs in \(\text{poly}(N)\) time and outputs a circuit representation of \(\Gamma_{\oplus \ell}\) and a generator for \(D_{\oplus \ell}\). Augmenting the circuit for \(\Gamma_{\oplus \ell}\) with a single NOT gate yields a circuit for \(\Gamma_{\oplus \ell}\). Moreover, we have shown:
if \( \text{opt}(S) \leq k \), then \( \Gamma_{\oplus \ell} \) is a \( k\ell \)-junta under \( D_{\oplus \ell} \); \hspace{1cm} \text{(Fact 6.12)}

if \( \text{opt}(S) > k' \), then any DNF of size \( \leq 2^{k' \ell/16} \) is \( \frac{1}{4N} \)-far from \( \Gamma_{\oplus \ell} \) under \( D_{\oplus \ell} \); \hspace{1cm} \text{(Lemma 6.13)}

which completes the proof of the lemma.

\[\square\]

6.3 Implications of Lemma 3.1

6.3.1 Proofs of Theorem 1 and Theorem 2

In this section, we prove the following theorem.

**Theorem 12.** Let \( \mu : \mathbb{N} \rightarrow \mathbb{N} \) be any computable, non-decreasing function satisfying \( \mu(n) = o \left( \frac{\log \log n}{\log \log \log n} \right) \). Assuming randomized ETH, there is some constant \( \lambda \in (0,1) \), a function \( f : \{0,1\}^n \rightarrow \{0,1\} \), and distribution \( D \) over \( \{0,1\}^n \) such that DT-Construction\((s,1/n)\) cannot be solved in time

\[s^\lambda \left( \frac{\log \log s}{\mu(n) \log \log \log s} \right) \]

for \( f \) and for any \( s \leq n^{\mu(n)} \), even if \( f \) is promised to be a \((\log n)\)-junta over \( D \) and the algorithm returns a DNF hypothesis.

Theorems 1 and 2 immediately follow as a consequence of this theorem by choosing \( \mu(n) = 1 \).

**Proof of Theorem 12.** We give a reduction from gapped set cover. Let \( S = (S, U, E) \) be an \( N \)-vertex \( (k,1/2) \left( \frac{\log N}{\log \log N} \right)^{1/k} \)-Set-Cover instance where \( k \) is taken to be

\[k = \frac{1}{2} \cdot \frac{\log \log N}{\log \log \log N}.\]

Using Lemma 6.17 with \( \ell = \log(N)/k \), we obtain the target function \( \Gamma_{\oplus \ell} : \{0,1\}^{N\ell} \rightarrow \{0,1\} \) and the distribution \( D_{\oplus \ell}. \)

Let \( \mu : \mathbb{N} \rightarrow \mathbb{N} \) be as in the theorem statement. Set \( s := (N\ell)^{\mu(N\ell)} \). We show that any algorithm for DT-Construction\((s,1/(4N))\) running in time \( s^\lambda \left( \frac{\log \log s}{\mu(N\ell) \log \log \log s} \right) \) for \( 0 < \lambda \leq 1/128 \) can be used to solve \( S \) in time \( N^{8\lambda k} \) even if the output of the algorithm is a DNF formula.

We run the algorithm for DT-Construction\((s,1/(4N))\) on \( \Gamma_{\oplus \ell} \) and \( D_{\oplus \ell} \) and terminate it after \( N^{4\lambda} \left( \frac{\log \log N}{\log \log \log N} \right) = N^{8\lambda k} \) times steps. The algorithm outputs some DNF formula \( F \). We estimate the error of \( F \) and \( \Gamma_{\oplus \ell} \) over the distribution \( D_{\oplus \ell} \) and output “Yes” if the error is \( \leq 1/(4N) \) and “No” otherwise.

**Runtime.** Constructing the circuit for \( \Gamma_{\oplus \ell} \) and the generator for \( D_{\oplus \ell} \) requires \( \text{poly}(N) \) time by Lemma 6.17. We can efficiently sample from the distribution \( D_{\oplus \ell} \) to efficiently estimate the error of the output decision tree via random samples. So the overall runtime of our algorithm is \( \leq N^{8\lambda k} \).

---

\(^{10}\)Technically, \( \Gamma_{\oplus \ell} \) is a function defined on \( |S| \ell \) bits, but as \( |S| \leq N \) we can pad the inputs to be \( N\ell \) bits long.
Correctness. To prove the reduction is correct, we show that if there is a size $k$ set cover for $S$ then we output YES with high probability and otherwise if $S$ requires a set cover of size at least
\[
\frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k}
\]
then we output NO with high probability.

Yes case: $\text{opt}(S) \leq k$. In this case, by Lemma 6.17, $\Gamma_{\oplus \ell}$ is computed exactly by a $\text{opt}(S)\ell \leq k\ell = \log N$-junta over $D_{\oplus \ell}$. Hence, it is computed by a DNF of width $k\ell$. The size of this DNF is at most $2^{k\ell} = N \leq (N\ell)^{\mu(N\ell)} = s$. To upper bound the runtime, we start by calculating
\[
\frac{\log s}{\log \log s} \leq \frac{\log (2\mu(N^2) \log N)}{\log \log \log N}
\]
\[
\leq \frac{\log ((\log N)^2)}{\log \log \log N}
\]
\[
= 4k.
\]
By our assumption on DT-CONSTRUCTION($s, 1/(4N)$), in the yes case, the algorithm runs for
\[
s^\lambda \left( \frac{\log s}{\mu(N\ell) \log \log \log s} \right) \leq s^{4\lambda k/\mu(N\ell)}
\]
\[
= (N\ell)^{4\lambda k}
\]
\[
\leq N^{8\lambda k}
\]  
(Equation (3))

By our assumption on DT-CONSTRUCTION($s, 1/(4N)$), the algorithm runs for
\[
s^\lambda \left( \frac{\log s}{\mu(N\ell) \log \log \log s} \right) \leq s^{4\lambda k/\mu(N\ell)}
\]
\[
= (N\ell)^{4\lambda k}
\]
\[
\leq N^{8\lambda k}
\]  
(Equation (3))

No case: $\text{opt}(S) > \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k}$. By Lemma 6.17 any DNF for $\Gamma_{\oplus \ell}$ with size at most $2^{\text{opt}(S)\ell/16}$ has error at least $1/(4N)$. The runtime bound on our algorithm serves as an upper bound on the size of the DNF built by the DT-CONSTRUCTION algorithm. Therefore, it is sufficient to show that
\[
N^{8\lambda k} < 2^{\text{opt}(S)\ell/16}
\]  
(Equation (4))

because this bound shows that our DNF must have error at least $1/(4N)$. Recalling that $k = \frac{1}{2} \cdot \frac{\log N}{\log \log N}$, we have
\[
(2k^2)^k < \frac{\log N}{\log \log N}
\]  
We observe
\[
\text{opt}(S) > \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k}
\]
\[
> k^2
\]
\[
\geq 128\lambda k^2
\]  
(128\lambda \leq 1)

which shows $k\ell(8\lambda k) < \text{opt}(S)\ell/16$. Exponentiating both sides and using the fact that $N = 2^{k\ell}$ completes the calculation and establishes Equation (4). It follows that our algorithm finds the error to be $> 1/(4N)$ and outputs NO with high probability.
Refuting randomized ETH. We now have an algorithm for solving \( (k, \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k}) \)-SET-COVER in time \( N^{8\lambda k} \) with high probability. By Theorem 7, there is a constant \( c \in (0, 1) \) such that \( (k, \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k}) \)-SET-COVER cannot be solved with high probability in time \( N^{ck} \). Therefore, we derive a contradiction for any \( \lambda \leq \min\{c/8, 1/128\} \).

6.3.2 PAC learning hardness

In this section, we discuss corollaries of Theorem 12. For a brief background on PAC learning and the definitions that we use, see Appendix C.

Corollary 6.18 (Hardness of learning decision trees, DNFs, and CNFs). Assuming randomized ETH, there is a constant \( \lambda \in (0, 1) \) such that decision trees cannot be distribution-free, properly PAC learned to accuracy \( \varepsilon = 1/n \) in time \( s^{\lambda \log \log s} \) where \( s \) is the size of the target. The same result also holds for properly learning DNFs and CNFs with size-\( s \) targets.

Proof. Let \( \mathcal{L} \) be a distribution-free, proper learning algorithm for the class \( \mathcal{T} \) of decision trees. We claim \( \mathcal{L} \) can be used to solve DT-CONSTRUCTION. In particular, let \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) and \( D \) be an instance of DT-CONSTRUCTION(\( s, 1/n \)). We run the learning algorithm on \( f \) and \( D \) and \( \varepsilon = 1/n \). If \( \mathcal{L} \) requests a random sample, we generate \( x \sim D \) using the generator for \( D \) and evaluate \( f(x) \) using the circuit for \( f \) and return \((x, f(x))\) to \( \mathcal{L} \). Since generating a sample from \( D \) and evaluating the circuit for \( f \) are both poly(\( n \))-time operations the overall runtime is dominated by the runtime of \( \mathcal{L} \). Theorem 12 then implies the desired time bound by setting \( \mu(n) = 1 \).

If \( \mathcal{L} \) is a learning algorithm for DNFs, we obtain the same hardness as in the decision tree case since any size-\( s \) decision tree target is equivalently a size-\( s \) DNF target. Moreover, Theorem 12 also applies when the output of the DT-CONSTRUCTION algorithm is a DNF formula. A symmetric argument works similarly for CNFs.

6.3.3 Proof of Theorem 3

In this section, we observe that the number of relevant inputs to \( \Gamma_S \) exactly characterizes the set cover complexity of \( S \). As a result, hardness of approximating set cover can be directly translated into hardness of distribution-free, proper PAC learning \( k \)-juntas. The next theorem formalizes this observation and was already implicit in [ABF+09].

Theorem 13 (Learning \( k \)-juntas is as hard as Set-Cover). Suppose there is a distribution-free PAC learning algorithm that runs in time \( t(n, k) \) and learns the class of \( k \)-juntas over \( \{0, 1\}^n \) to accuracy \( \varepsilon = O(1/n) \) by hypotheses which are \( g(k, n) \)-juntas for some function \( g : \mathbb{N}^2 \rightarrow \mathbb{N} \) satisfying \( k \leq g(k, n) \). Then \( (k, g(k, n)) \)-SET-COVER can be solved with high probability in time \( t(n, k) \).

Proof. Let \( S = (S, U, E) \) be an instance of \( (k, g(k, n)) \)-SET-COVER. We construct the function \( \Gamma : \{0, 1\}^{|S|} \rightarrow \{0, 1\} \) and the distribution \( D \) over \( \{0, 1\}^{|S|} \). Run the learning algorithm on \( \Gamma \) and \( D \) with \( \varepsilon = 1/(4|S|) \) for \( t(k, |S|) \) time steps. It outputs some truth table representation of a junta. We output \( \text{YES} \) if and only if this truth table has size at most \( g(k, n) \) and has error at most \( 1/(4|S|) \). The correctness of the reduction follows from Fact 6.1.

Corollary 6.19. There is no distribution-free PAC learning algorithm for properly learning \( k \)-juntas to accuracy \( \varepsilon = O(1/n) \) over \( \{0, 1\}^n \) that runs in time:
○ \(n^{o(k)}\), assuming randomized \(W[1] \neq \text{FPT}\);

○ \(O(n^{k-\lambda})\), for all \(\lambda > 0\), assuming randomized SETH.

These results hold in the regime where \(k \leq n^c\) for some absolute constant \(c < 1\).

Proof. By Theorem 13, distribution-free properly PAC learning \(k\)-juntas is equivalent to \((k, k+1)\)-Set-Cover. The first bullet follows by combining Theorems 10 and 13. The second bullet follows by combining Theorems 11 and 13. \(\square\)

7 Lower bounds for DT-Estimation

For our lower bounds for DT-Estimation, we have to consider decision trees that are allowed to abort:

Definition 5. A \(\delta\)-abort decision tree \(T\) under a distribution \(\mathcal{D}\) is a decision tree with leaves labeled \(\{0, 1, \perp\}\) satisfying \(\Pr_{x \sim \mathcal{D}}[T(x) = \perp] \leq \delta\). The distance between such a tree \(T : \{0, 1\}^n \to \{0, 1, \perp\}\) and a function \(f : \{0, 1\}^n \to \{0, 1\}\) under \(\mathcal{D}\) is

\[
\text{dist}_{\mathcal{D}}(T, f) := \Pr_{x \sim \mathcal{D}}[T(x) \neq f(x) \text{ and } T(x) \neq \perp].
\]

7.1 Lemma 6.6 for decision trees that abort

In this section we generalize Lemma 6.6 to \(\delta\)-abort decision trees:

Lemma 7.1 (Lemma 6.6 with aborts). Let \(S = (S, U, E)\) be an \(N\)-vertex set cover instance and let \(\ell \in \mathbb{N}\). If \(T : (\{0, 1\}^\ell)^n \to \{0, 1\}\) is a decision tree of size \(|T| < 2^{\text{opt}(S)\ell/40}\) that can abort with probability \(\delta < 0.4\), then \(\text{dist}_{\mathcal{D}^{\oplus \ell}}(T, \Gamma^{\oplus \ell}) \geq 1/(20N)\).

Since every depth-\(d\) tree is a tree of size \(2^d\), Lemma 7.1 also holds for decision trees \(T\) of depth \(< \text{opt}(S)\ell/40\).

Outline of Proof. As in the non-abort case, there are three main components to the proof of the above lemma:

1. No \(\delta\)-abort decision tree with small average depth can approximate \(\Gamma\) under \(\mathcal{D}\) where \(\delta < 1/2\) (Claim 7.2).

2. Any \(\delta\)-abort decision tree with small average depth that approximates \(\Gamma^{\oplus \ell}\) under \(\mathcal{D}^{\oplus \ell}\) can be used to construct a \(\delta\)-abort decision tree of much smaller average depth that approximates \(\Gamma\) under \(\mathcal{D}\) at the cost of a modest blowup in the size of \(\delta\) (Claim 7.3). This is the key claim.

3. Any small size decision tree must have small average depth with respect to \(\mathcal{D}^{\oplus \ell}\). This claim is unchanged from the non-abort version.

Analogous to the non-abort case, these claims together imply that no \(\delta\)-abort decision tree with small average depth can approximate \(\Gamma^{\oplus \ell}\) under \(\mathcal{D}^{\oplus \ell}\). We need to provide slightly different claims and proofs for the first two items, but the last claim is completely independent of aborts, so we need not reprove it.
Claim 7.2 (Abort version of Claim 6.7). Let \( T : \{0,1\}^n \rightarrow \{0,1\} \) be a \( \delta \)-abort decision tree with \( \delta < 1/2 \) and \( S = (S,U,E) \) be an \( N \)-vertex set cover instance with \( |S| = n \). If \( E \mathbb{E}_{x \sim D} [\text{depth}_T(x)] < \text{opt}(S)/2 \) then \( \text{dist}_D(T,\Gamma) \geq 1/(2N) \).

**Proof.** This proof is almost identical to that of Claim 6.7. We provide the start of the proof and then refer the reader back to Claim 6.7 for the rest.

Suppose that \( \text{dist}_D(T,\Gamma) < 1/(2N) \leq 1/(2|U|) \). We note as before that each \( x \in \text{supp}(D) \) has mass \( \geq 1/(2|U|) \) under \( D \), so it must be that \( \text{dist}_D(T,\Gamma) = 0 \). Since \( 0^n \) has weight 1/2 under \( D \), \( T(0^n) \neq \perp \) because \( T \) can only abort with probability \( < 1/2 \). It follows that \( T(0^n) = \Gamma(0^n) = 0 \).

The rest of the proof is identical to that of Claim 6.7. \( \square \)

For the next claim, we reuse the portions of Claim 6.8 that tell us that the restriction \( T^* \) of \( T \) is distance preserving and has small depth. We must show that \( T^* \) also has a small abort probability.

Claim 7.3 (Abort version of Claim 6.8). Let \( T : \{0,1\}^\ell \rightarrow \{0,1\} \) be a decision tree such that

\[
\text{dist}_{D^{\oplus \ell}}(T,\Gamma^{\oplus \ell}) \leq \varepsilon \quad \text{and} \quad E_{y \sim D^{\oplus \ell}} [\text{depth}_{T}(y)] \leq d \quad \text{and} \quad \Pr_{y \sim D^{\oplus \ell}} [T(y) = \perp] \leq \delta.
\]

Then there is a restriction \( T^* : \{0,1\}^n \rightarrow \{0,1\} \) of \( T \) satisfying

\[
\text{dist}_D(T^*,\Gamma) \leq 10\varepsilon \quad \text{and} \quad E_{x \sim D} [\text{depth}_{T^*}(x)] \leq \frac{10d}{\ell} \quad \text{and} \quad \Pr_{x \sim D} [T^*(x) = \perp] \leq \frac{5}{4} \delta.
\]

**Proof.** Recall the definition of \( q_j(y) \) in the proof of Claim 6.8. For \( j \in [\ell] \), \( q_j(y) \) is the number of times that \( T \), on input \( y \), queries \( (y_i)_j \) for some \( i \in [n] \). Refer to Definition 4 for the definitions of \( D^{\oplus \ell}_j \) and \( \text{ParComplete}_j(z,x) \). The proof of Claim 6.8 bounds the probabilities that \( \text{dist}_D(T^*,\Gamma) \) or \( E_{x \sim D} [\text{depth}_{T^*}(x)] \) are too large using Markov’s inequality. More concretely, we already know that for a particular \( j \in [\ell] \),

\[
\Pr_{z \sim \mathcal{M}_{n(\ell-1)}} \left[ \Pr_{x \sim D} \left[ T(\text{ParComplete}_j(z,x)) \neq \Gamma(x) \text{ and } T(\text{ParComplete}_j(z,x)) \neq \perp \right] > 10\varepsilon \right] < \frac{1}{10}
\]

and

\[
\Pr_{z \sim \mathcal{M}_{n(\ell-1)}} \left[ E_{x \sim D} [q_j(\text{ParComplete}_j(z,x))] > \frac{10d}{\ell} \right] < \frac{1}{10}
\]

where all we have done is change the constant used in the application of Markov’s inequality.

It remains for us to bound the probability that the tree aborts. We compute

\[
\delta \geq \Pr_{y \sim D^{\oplus \ell}} [T(y) = \perp] = \Pr_{y \sim D^{\oplus \ell}} [T(y) = \perp] \quad \text{(Proposition 6.4)}
\]

\[
= \mathbb{E}_{z \sim \mathcal{M}_{n(\ell-1)}} \left[ \Pr_{x \sim D} \left[ T(\text{ParComplete}_j(z,x)) = \perp \right] \right] . \quad \text{(Definition of } D^{\oplus \ell}_j \text{)}
\]

Again, we can apply Markov’s inequality to deduce

\[
\Pr_{z \sim \mathcal{M}_{n(\ell-1)}} \left[ \Pr_{x \sim D} \left[ T(\text{ParComplete}_j(z,x)) = \perp \right] \right] > \frac{10}{8} \delta \text{ } < \frac{8}{10}.
\]

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Thus, applying a union bound to all three of our Markov inequalities, we conclude that there exists a fixed \( \underline{z} \in \{0, 1\}^{n(\ell - 1)} \) such that

\[
\Pr_{x \sim \mathcal{D}} \left[ T(\text{ParComplete}_j(\underline{z}, x)) \neq \Gamma(x) \text{ and } T(\text{ParComplete}_j(\underline{z}, x)) \neq \perp \right] \leq 10\varepsilon
\]

and

\[
\mathbb{E}_{x \sim \mathcal{D}} \left[ q_j(\text{ParComplete}_j(\underline{z}, x)) \right] \leq \frac{10d}{\ell}
\]

and

\[
\Pr_{x \sim \mathcal{D}} \left[ T(\text{ParComplete}_j(\underline{z}, x)) = \perp \right] \leq \frac{10}{8}\delta
\]

The tree \( T^* \) is formed by restricting \( T \) according to \( \underline{z} \) and \( j \). As before, the depth of an input \( x \) is \( \text{depth}_{T^*}(x) = q_j(\text{ParComplete}_j(\underline{z}, x)) \). Thus, the claim follows. \( \square \)

Finally, we can directly apply Claim 6.10 without needing a special version for aborts. These three claims together allow us to complete the proof.

Putting things together: Proof of Lemma 7.1. Suppose there is some \( \delta \)-abort tree \( T \) computing \( \Gamma_{\oplus \ell} \) with \( |T| \leq \frac{2^{\text{opt}(S)}\ell}{40} \) and with \( \delta < 0.4 \). We show that \( \text{dist}(T, \Gamma_{\oplus \ell}) \geq \frac{1}{20N} \). Suppose for contradiction that \( \text{dist}(T, \Gamma_{\oplus \ell}) < \frac{1}{20N} \). By Claim 6.10, we have \( \mathbb{E}_{y \sim \mathcal{D}_{\oplus \ell}} \left[ \text{depth}_T(y) \right] < 2 \cdot \log \left( \frac{2^{\text{opt}(S)}\ell}{40} \right) = \text{opt}(S)\ell/20 \). Then by Claim 7.3 there is a decision tree \( T^* \) satisfying

\[
\text{dist}_{\mathcal{D}}(T^*, \Gamma) < \frac{1}{2N} \quad \mathbb{E}_{x \sim \mathcal{D}^m} \left[ \text{depth}_{T^*}(x) \right] < \frac{\text{opt}(S)}{2} \quad \Pr_{x \sim \mathcal{D}^m} \left[ T(\text{ParComplete}_j(\underline{z}, x)) = \perp \right] \leq \frac{5\delta}{4} < \frac{1}{2}.
\]

But this contradicts Claim 7.2. \( \square \)

7.2 Hardness amplification for DT-Estimation

Next, we amplify the distance given by Lemma 7.1 using the following hardness amplification lemma:

Lemma 7.4 (Precise restatement of Lemma 3.3). Let \( f : \{0, 1\}^n \to \{0, 1\} \) and \( \mathcal{D} \) be such that \( f \) is \( \varepsilon \)-far from every depth-d \( \delta \)-abort decision tree where \( \delta \geq 0.34 \) under \( \mathcal{D} \). Consider

\[
f^{\oplus m}(x^{(1)}, \ldots, x^{(m)}) := f(x^{(1)}) \oplus \cdots \oplus f(x^{(m)}),
\]

the \( m \)-fold XOR composition of \( f \) and \( \mathcal{D}^m \) be the corresponding distribution over \( (\{0, 1\}^n)^m \). For any \( \gamma > 0 \), by taking \( m = \Theta(\log(1/\gamma)/\varepsilon) \), we get that \( f^{\oplus m} \) is \( (\frac{1}{2} - \gamma) \)-far from every decision tree of depth \( \Omega(dm) \) under \( \mathcal{D}^m \).

Note that the probability of error \( \varepsilon \) is taken over inputs that do not abort. Thus, this statement is weaker than that of Lemma 3.3. We need to allow the possibility of aborting in order to apply [BKLS20]’s lemma. The proof of Lemma 7.4 consists of two parts, each of which introduces another layer of XOR composition in order to amplify the error. First, we amplify the error from \( \varepsilon \) to a constant \( O(1) \) and then from \( O(1) \) to exponentially close to \( \frac{1}{2} \). Each of these two steps uses an XOR lemma from [BKLS20] and [Dru12] respectively. We now state these lemmas and then proceed with the proof of Lemma 7.4.
Lemma 7.5 (Lemma 1 of [BKLS20]). Let \( f : \{0, 1\}^n \to \{0, 1\} \) and \( \mathcal{D} \) be such that \( f \) is \( \varepsilon \)-far from every depth-\( d \) \( \delta \)-abort tree where \( \delta \geq 0.34 \) under \( \mathcal{D} \). By taking \( m = \Theta(1/\varepsilon) \), we get that \( f^{\oplus m} \) is \( \frac{1}{800} \)-far from every decision tree of depth \( \Omega(dm) \) under \( \mathcal{D}^m \).

Lemma 7.6 (Theorem 1.3 of [Dru12]). Let \( f : \{0, 1\}^n \to \{0, 1\} \) and \( \mathcal{D} \) be such that \( f \) is \( \varepsilon \)-far from every depth-\( d \) decision tree under \( \mathcal{D} \). For every \( m \in \mathbb{N} \) and \( \alpha \in [0, 1] \), we get that \( f^{\oplus m} \) is
\[
\frac{1}{2} \left( 1 - (1 - 2\varepsilon + 6\alpha \ln(2/\alpha)\varepsilon)^n \right)
\]
far from every decision tree of depth \( \alpha \varepsilon d m \) under \( \mathcal{D}^m \).

Note that the original version of Lemma 7.6 in [Dru12] holds for randomized decision trees rather than deterministic ones; however, the above version is equivalent. If \( f \) is \( \varepsilon \)-far from all depth-\( d \) randomized decision trees, then clearly it is \( \varepsilon \)-far from all deterministic ones since randomness can only add power. On the other hand, suppose \( f \) is \( \varepsilon \)-far from all depth-\( d \) deterministic decision trees. Consider a depth-\( d \) randomized decision tree \( T(x, r) \) that in addition to \( x \) takes in a random string \( r \). The distance between \( T \) and \( f \) is given by
\[
E_{r \sim \mathcal{D}}[\Pr[T(x, r) \neq f(x)]].
\]
For each fixed \( r \), we have that \( \Pr[T(x, r) \neq f(x)] \) must be at least an \( \varepsilon \) fraction of total inputs. Thus, by linearity of expectation, the randomized decision tree must also be \( \varepsilon \)-far from \( f \).

Proof of Lemma 7.4. Consider \( f \) and \( \mathcal{D} \) as in the statement of Lemma 7.4, and simply apply Lemma 7.5 with \( m_1 = \Theta(1/\varepsilon) \). What results is a function \( f^{\oplus m_1} \) that is \( \frac{1}{800} \)-far from every decision tree of depth \( \Omega(dm_1) \) under \( \mathcal{D}^{m_1} \). Next, apply Lemma 7.6 to \( f^{\oplus m_1} \) with \( m_2 = \Theta(\log(1/\gamma)) \), \( \varepsilon = \frac{1}{800} \), and by choosing \( \alpha \) such that \( 6\alpha \ln(2/\alpha) = 1 \). Then, simplifying the expression in Lemma 7.6, we get that \( (f^{\oplus m_1})^{\oplus m_2} \) is
\[
\frac{1}{2} \left( 1 - (1 - \frac{1}{800})^{m_2} \right) = \frac{1}{2} - \frac{1}{2} \left( \frac{799}{800} \right)^{\Theta(\log(1/\gamma))} \\
\geq \frac{1}{2} - 2^{-\log(1/\gamma)} \\
= \frac{1}{2} - \gamma
\]
far from every decision tree of depth \( \Omega(dm_1 m_2) \) under \( \mathcal{D}^{m_1 m_2} \).

Globally, we define \( m = m_1 \cdot m_2 = \Theta(\log(1/\gamma)/\varepsilon) \) so that \( (f^{\oplus m_1})^{\oplus m_2} = f^{\oplus m} \). Then, \( f^{\oplus m} \) is \( \frac{1}{2} - \gamma \) far from decision trees of depth \( \Omega(dm) = \Omega(d \log(1/\gamma)/\varepsilon) \) under \( \mathcal{D}^m \) as desired. \( \square \)

7.3 Proof of Theorem 4

With Lemmas 7.1 and 7.4 in hand, we are now ready to prove Theorem 4. Given a size-\( N \) instance \( S \) of \((k, k')\)-\textsc{Set-Cover} with \( n \) sets, we apply Lemmas 3.1 and 7.1 with \( \ell = 2 \) to obtain a \( \text{poly}(N) \)-time reduction that produces a function \( \Gamma_{\oplus 2} : \{0, 1\}^{2n} \to \{0, 1\} \) and the generator for a distribution \( \mathcal{D}_{\oplus 2} \) over \( \{0, 1\}^{2n} \) satisfying:
- If \( \text{opt}(S) \leq k \), then \( \Gamma_{\oplus 2} \) is a \( 2k \)-junta under \( \mathcal{D}_{\oplus 2} \).
Figure 5: A depth-3 circuit for \((\Gamma \oplus 2)^{\oplus m}\) consisting of one top gate that is a PARITY connected to \(m\) independent copies of the circuit for \(\Gamma \oplus 2\).

- If \(\text{opt}(\mathcal{S}) > k'\), then for any \(\delta < 0.4\), any \(\delta\)-abort decision tree of depth \(k'/20\) is \(\Omega(1/\delta)\)-far from \(\Gamma \oplus 2\) under \(\mathcal{D} \oplus 2\).

Next, we consider \((\Gamma \oplus 2)^{\oplus m}\) and \((\mathcal{D} \oplus 2)^{\oplus m}\) where \(m = \Theta(N^2)\):

- If \(\text{opt}(\mathcal{S}) \leq k\), then \((\Gamma \oplus 2)^{\oplus m}\) is a \(2km\)-junta under \((\mathcal{D} \oplus 2)^{\oplus m}\). Such a junta can be computed by a decision tree of depth \(d := 2km\).

- If \(\text{opt}(\mathcal{S}) > k'\), then by Lemma 7.4, \((\Gamma \oplus 2)^{\oplus m}\) is \((\frac{1}{2} - 2^{-N})\)-far from decision trees of depth \(d' := \Omega(k'm)\) under \((\mathcal{D} \oplus 2)^{\oplus m}\).

Note that the circuit representation for \((\Gamma \oplus 2)^{\oplus m}\) and generator for \((\mathcal{D} \oplus 2)^{\oplus m}\) can be constructed in \(\text{poly}(N)\) time from those for \(\Gamma \oplus 2\) and \(\mathcal{D} \oplus 2\) by simply feeding \(m\) copies of \(f\) into an XOR gate and by using \(m\) copies of \(\mathcal{D}\). See Figure 5 for an illustration of this circuit. Since \((\Gamma \oplus 2)^{\oplus m}\) is a function over \(2nm \leq O(N^3)\) variables and \(d' \geq \Omega(d \log N) \geq \Omega(d \log d)\), Theorem 4 now follows by applying Theorem 9 with \(\beta \in (0, 1)\) being any constant.

8 Proof of Theorem 5

The PAC learning lower bound from Section 6.3.2 applies to properly learning decision trees. In this setting, the concept class is \(\mathcal{T} = \{T : \{0, 1\}^n \rightarrow \{0, 1\} \mid T\text{ is a decision tree}\}\). So the learner is allowed to output a decision tree hypothesis that may be much larger than the target. One could instead consider the problem of properly learning the class of size-\(s\) decision trees: \(\mathcal{T}_s = \{T : \{0, 1\}^n \rightarrow \{0, 1\} \mid T\text{ is a size-}s\text{ decision tree}\}\). This problem is strictly harder than learning decision trees since the output must satisfy a size constraint. Indeed, against this class, we are able to adapt the proof of Theorem 12 to obtain a stronger lower bound.

**Theorem 14.** Assuming randomized ETH, there is a constant \(\lambda \in (0, 1)\) such that \(\text{DT-Construction}(s, \frac{1}{n})\) cannot be solved in time \(n^{\lambda s}\) if the algorithm has to return a size-\(s\) DNF hypothesis, even when the function is promised to be a \(\log s\)-junta.
Proof. This proof is a combination of the proofs of Theorems 12 and 15. The analysis is similar so we only outline the important details here. In particular, let $\mathcal{S} = (S, U, E)$ be an $N$-vertex $\left( k, \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k} \right)$-SET-COVER instance where $k$ is taken to be $k = \frac{1}{2} \cdot \frac{\log \log N}{\log \log \log N}$ for $N$ large enough so that $32k < \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k}$. Using Theorem 7, there is a constant $c \in (0, 1)$ such that $\mathcal{S}$ cannot be solved in time $N^{ck}$. We derive a contradiction for any algorithm for DT-CONSTRUCTION$(s, \frac{1}{n})$ that returns a size-$s$ DNF and runs in time $n^{\lambda \log s}$ for $\lambda \leq c/5$.

Use Lemma 6.13 with $\ell = 4$ (as in Theorem 15) to obtain the target function $\overline{T}_{\oplus 4} : \{0, 1\}^{4N} \to \{0, 1\}$ and the distribution $D_{\oplus 4}$. Run DT-CONSTRUCTION$(s, \frac{1}{n^{\lambda}})$ with $s := 2^{4k}$ on $\overline{T}_{\oplus 4} : \{0, 1\}^{4N}$ and $D_{\oplus 4}$ for $N^{5\lambda k}$ time steps where $\lambda \leq c/5$. Output yes if and only if the DNF formula returned by the algorithm as size at most $2^{4k}$ and error less than $1/(4N)$. The correctness of the no case follows from the fact that $32k < \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k}$ and so the DNF lower bound from Lemma 6.13 ensures $2^{\text{opt}(S)\ell/16} > 2^{4k}$. Since the algorithm returns a size-$2^{4k}$ DNF formula if one exists, this separation between the DNF sizes is sufficient to establish correctness.

As in the case of Corollary 6.18, this theorem yields hardness of properly PAC learning the class of size-$s$ decision trees.

Corollary 8.1. Assuming randomized ETH, there is a constant $\lambda \in (0, 1)$ such that the class $\mathcal{T}_s$ of size-$s$ decision trees cannot be distribution-free, properly PAC learned to accuracy $\varepsilon = 1/n$ in time $n^{\lambda \log s}$. The same result also holds for properly learning size-$s$ DNFs and CNFs.

9 Proof of Theorem 6

In this section, we outline a concrete path towards proving optimal lower bounds for DT-CONSTRUCTION. In particular, we show that better lower bounds for gapped set cover yields better lower bounds for DT-CONSTRUCTION. Specifically, the main theorem assumes Conjecture 1 and proves an $n^{\Omega(\log s)}$ lower bound for DT-CONSTRUCTION$(s, \frac{1}{n})$.

Theorem 15. Assume Conjecture 1, then there is a constant $\lambda \in (0, 1)$ such that DT-CONSTRUCTION$(s, \frac{1}{n})$ cannot be solved in time $n^{\lambda \log s}$, even when the target is a log $s$-junta and the algorithm is allowed to return a DNF hypothesis.

Proof. Let $\beta \in (0, 1)$ be as in the statement of the Conjecture 1. Assume there is an algorithm for DT-CONSTRUCTION$(s, \frac{1}{n})$ running in time $n^{\lambda \log s}$ for any $\lambda \leq (1 - \beta)/(40 \log e)$. Then, following the proof strategy of Theorem 12, we derive a contradiction by solving $(k, k \cdot (1 - \beta) \ln N)$-SET-COVER over $N$ vertices in time $N^{5\lambda k}$. Let $\mathcal{S} = (S, U, E)$ be an $N$-vertex $(k, k \cdot (1 - \beta) \ln N)$-SET-COVER instance for $k \in \mathbb{N}$. Using Lemma 6.17 with $\ell = 4$, we obtain the target function $\overline{T}_{\oplus 4} : \{0, 1\}^{4N} \to \{0, 1\}$ and the distribution $D_{\oplus 4}$. We run the algorithm for DT-CONSTRUCTION$(s, \frac{1}{n^{\lambda}})$ on $\overline{T}_{\oplus 4}$ and $D_{\oplus 4}$ with $s := 2^{4k}$ and terminate it after $N^{5\lambda k}$ time steps. The output is some DNF formula $F$. We estimate the error of $F$ over $D_{\oplus 4}$ and output YES if it’s less than $1/(4N)$ and NO otherwise.

Runtime. By Lemma 6.17, we can construct the circuit for $\overline{T}_{\oplus 4} : \{0, 1\}^{4N} \to \{0, 1\}$ and generator for $D_{\oplus 4}$ in $\text{poly}(N)$-time. Moreover, we can use random sampling to efficiently estimate the error of $F$ over $D_{\oplus 4}$. Therefore, the runtime of the reduction is dominated by $N^{5\lambda k}$. 

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Correctness. We handle the yes case and the no case separately.

Yes case: \( \text{opt}(S) \leq k \). By Lemma 6.17, \( \Gamma \oplus 4 \) is a \( 4k \)-junta over \( D \oplus 4 \). Therefore, it is a decision tree of size \( s = 2^{4k} \) and \( \text{DT-Construction}(s, \frac{1}{4N}) \) runs in time
\[
(4N)^{\lambda \log s} = (2N)^{4\lambda k} \leq N^{5\lambda k}.
\]
The output is DNF formula with error at most \( 1/(4N) \). It follows that our algorithm outputs Yes with high probability.

No case: \( \text{opt}(S) > k \cdot (1 - \beta) \ln N \). By Lemma 6.17, any DNF for \( \Gamma \oplus 4 \) with size at most \( 2^{\text{opt}(S)/8} \) has error at least \( 1/(4N) \). Using the assumption on \( \text{opt}(S) \):
\[
2^{\text{opt}(S)/8} > N^{k(1-\beta)/(8 \log e)}
\geq N^{5\lambda k}
\]
\[
((1 - \beta)/(40 \log e) \geq \lambda)
\]
which shows that the DNF output by the algorithm must have error at least \( 1/(4N) \). It follows that our algorithm outputs No with high probability.

As discussed in Section 6.3.2, this lower bound for \( \text{DT-Construction} \) implies a lower bound for PAC learning decision trees.

Corollary 9.1. Assume Conjecture 1, then there is a constant \( \lambda \in (0, 1) \) such that decision trees cannot be distribution-free, properly PAC learned to accuracy \( \varepsilon = 1/n \) in time \( n^{\lambda \log s} \) where \( s \) is the size of the decision tree target. The same result also holds for properly learning DNFs and CNFs.

The proof of this corollary is identical to that of Corollary 6.18.

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A Hardness of Approximating Set Cover

We first state a lemma due to [Lin19], translated into our notation.

**Lemma A.1** (Lin’s lemma [Lin19, Lemma 3.6]). There is an algorithm which given \( k \in \mathbb{N}, \delta > 0 \) with \( (1 + 1/k^3)^{1/k} \leq (1 + \delta)/(1 + \delta/2) \) and \( (1 + \delta/2)^k \geq 2k^4 \) and a SAT instance \( \varphi \) with \( n \) variables and \( Cn \) clauses, where \( n \) is much larger than \( k \) and \( C \), outputs an integer \( N \leq 2^{n/k} + n/k^3 \) and a set cover instance \( S = (S, U, E) \) satisfying

- \( |S| + |U| \leq N; \)
- if \( \varphi \) is satisfiable, then \( \text{opt}(S) \leq k; \)
- if \( \varphi \) is unsatisfiable, then \( \text{opt}(S) > \frac{1}{1+\delta} \left( \frac{\log N}{\log \log N} \right)^{1/k} \).

The exact version of this lemma we use is the following.

**Lemma A.2** (Reducing SAT to Set-Cover). There is an algorithm that takes an \( n \)-variate SAT instance \( \varphi \) of size \(|\varphi|\) and an integer \( k \geq 100 \) with \( k^2 \leq n/\log n \) and produces a set cover instance \( S \) of size \( N \leq 2^{2|\varphi|/k} \) in time \( \leq 2^{5|\varphi|/k} \) such that

1. if \( \varphi \) is satisfiable then \( \text{opt}(S) \leq k; \)
2. if \( \varphi \) is unsatisfiable then \( \text{opt}(S) > \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k} \).

**Proof.** We use Lemma A.1 with \( \delta = 1/2 \). For this value of \( \delta \), if \( k \geq 100 \), then both conditions \( (1 + 1/k^3)^{1/k} \leq (1 + \delta)/(1 + \delta/2) \) and \( (1 + \delta/2)^k \geq 2k^4 \) of Lemma A.1 are satisfied. Moreover, an inspection of the proof of Lemma A.1 shows that the condition “\( n \) is much larger than \( k \)” in the lemma statement means \( k^2 \leq n/\log n \).

Therefore, Lemma A.1 returns a set cover instance \( S \) satisfying

1. if \( \varphi \) is satisfiable then \( \text{opt}(S) \leq k; \)
2. if \( \varphi \) is unsatisfiable then \( \text{opt}(S) > \frac{1}{1+\delta} \left( \frac{\log N}{\log \log N} \right)^{1/k} \).

By our choice of \( \delta, 1/(1 + \delta) = 2/3 > 1/2 \) as desired. Since \( n \leq |\varphi| \), the size of the set cover instance is \( N \leq 2|\varphi|/k + |\varphi|/k^3 \leq 2^{|\varphi|/k} \). The runtime of the reduction is \( \leq 2^{5|\varphi|/k} \). \(\square\)
We can now prove the main theorem from Section 5.1.

**Proof of Theorem 7.** Suppose there exists an algorithm that can solve \( (k, \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k} ) \)-SET-COVER on \( N \) vertices with high probability in time \( N^{ck} \). Then we show how to solve SAT with high probability for SAT formulas with \( n \) variables in time \( 2^{3cn} \).

Let \( \varphi \) be a SAT instance with \( n \) variables. Choose \( k \in \mathbb{N} \) so that
\[
k = \frac{1}{2} \cdot \log \log \left( \frac{2^{2n/k}}{N^{ck}} \right) = \frac{1}{2} \cdot \log \left( \frac{2n/k}{\log \log \left( \frac{2^{2n/k}}{N^{ck}} \right)} \right).
\]
Given \( n \) this equation can be numerically solved efficiently, and \( k \) will be some value between \( \log \log n \) and \( \log n \). We then apply Lemma A.2 with this value of \( k \) to obtain a set cover instance \( S \) of size \( N \leq 2^{2n/k} \) in time \( \leq 2^{5n/k} \). If \( N < 2^{2n/k} \) then we add dummy items/dummy sets to the universe so that \( N = 2^{2n/k} \). Note this padding will not affect the optimal set cover for the optimal set cover size. Hence, by construction, we have an instance of \( (k, \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k} ) \)-SET-COVER of size \( N \) where
\[
k = \frac{1}{2} \cdot \log \log \left( \frac{N}{\log \log \left( \frac{2^{2n/k}}{N^{ck}} \right)} \right).
\]
We can therefore run our algorithm for set cover on this instance \( S \) and output “Yes” if the algorithm outputs Yes and “No” if the algorithm outputs No.

**Runtime.** Our reduction runs in time
\[
2^{5n/k} + N^{ck} \leq 2^{5n/k} + (2^{2n/k})^{ck} = 2^{5n/k} + 2^{2cn} \leq 2^{3cn}.
\]

**Correctness.** By assumption, the set cover algorithm solves \( (k, \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k} ) \)-SET-COVER with high probability and therefore by Lemma A.2 our algorithm solves SAT with high probability. We also note that \( 2k^k < \frac{\log N}{\log \log N} \) and so \( k < \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k} \) for our choice of \( k \). If instead, one were to choose e.g. \( k = \log \log n \), then \( 2k^k > \frac{\log N}{\log \log N} \) and so the set cover instance would fail to determine the satisfiability of \( \varphi \).

It follows that if SAT cannot be solved in randomized time \( O(2^{\delta n}) \) for some \( \delta \in (0, 1) \) then \( (k, \frac{1}{2} \left( \frac{\log N}{\log \log N} \right)^{1/k} ) \)-SET-COVER cannot be solved in randomized time \( N^{\delta/3-k} \). \( \square \)
### B Proof of Proposition 6.4

We first compute:

$$\Pr_{y \sim D_{\oplus \ell}} [y = y] = \Pr_{y \sim D_{\oplus \ell}} \left[ \text{BlockwisePar}(y) = \text{BlockwisePar}(y) \right]$$

$$\cdot \Pr_{y \sim D_{\oplus \ell}} [y = y \mid \text{BlockwisePar}(y) = \text{BlockwisePar}(y)] \quad \text{(Law of total probability)}$$

$$= \Pr_{x \sim D} [x = \text{BlockwisePar}(y)] \cdot \Pr_{y \sim \mathcal{U}_{n, \ell}} [y = y \mid \text{BlockwisePar}(y) = \text{BlockwisePar}(y)] \quad \text{(Definition of } D_{\oplus \ell})$$

$$= \Pr_{x \sim D} [x = \text{BlockwisePar}(y)] \cdot \prod_{i \in [n]} \Pr_{y_i \sim \mathcal{U}_{\ell}} [y_i = y_i \mid \oplus y_i = \oplus y_i] \quad \text{(Independence of } y_i \text{'s)}$$

where the last step follows from the fact that conditioning on the parity of $y_i$ being a specific bit removes 1 out of $\ell$ degrees of freedom. With an analogous calculation for $D_{\oplus \ell}$, we obtain

$$\Pr_{y \sim D_{\oplus \ell}} [y = y] = \Pr_{y \sim D_{\oplus \ell}} \left[ \text{BlockwisePar}(y) = \text{BlockwisePar}(y) \right]$$

$$\cdot \Pr_{y \sim D_{\oplus \ell}} [y = y \mid \text{BlockwisePar}(y) = \text{BlockwisePar}(y)] \quad \text{(Law of total probability)}$$

$$= \Pr_{x \sim D} [x = \text{BlockwisePar}(y)] \cdot \prod_{i \in [n]} \Pr_{y_i \sim \mathcal{U}_{\ell-1}} [y_i = y_i - j \mid \oplus y_i = \oplus y_i] \quad \text{(Definition of } D_{\oplus \ell})$$

$$= \Pr_{x \sim D} [x = \text{BlockwisePar}(y)] \cdot \prod_{i \in [n]} 2^{-(\ell-1)}$$

where $y_i - j \in \{0, 1\}^{\ell-1}$ is the string $y_i$ with its $j$th bit removed.

### C PAC learning

In the realizable PAC learning model [Val84], there is an unknown distribution $D$ and some unknown target function $f \in \mathcal{H}$ from a fixed concept class $\mathcal{H}$ of functions over a fixed domain. An algorithm for learning $\mathcal{H}$ over $D$ takes as input $\epsilon \in (0, 1)$ and has oracle access to an example oracle $\text{EX}(f, D)$. The algorithm can query the example oracle to receive a pair $(x, f(x))$ where $x \sim D$ is drawn independently at random. The goal is to output a hypothesis $h$ such that dist$_D(f, h) \leq \epsilon$. Since the example oracle is inherently randomized, any learning algorithm is necessarily randomized. So we require the algorithm succeed with some fixed probability e.g. $2/3$. A learning algorithm is proper if it always outputs a hypothesis $h \in \mathcal{H}$.

Formally, we use the following definition for PAC learning decision trees.

**Definition 6** (PAC learning decision trees). Let $T = \{T : \{0, 1\}^n \rightarrow \{0, 1\} \mid T \text{ is a decision tree}\}$ be the class of decision trees over a fixed domain $\{0, 1\}^n$. A distribution-free learning algorithm $\mathcal{L}$ learns $T$ in time $t(n, s, \epsilon)$ if for all distributions $D$ and for all $T \in T, \epsilon \in (0, 1), \mathcal{L}$ with oracle
access to $\text{EX}(T, \mathcal{D})$ runs in time $t(n, |T|, \varepsilon)$ and with probability $2/3$ outputs $h : \{0, 1\}^n \rightarrow \{0, 1\}$ such that $\text{dist}_\mathcal{D}(T, h) \leq \varepsilon$. Furthermore, $\mathcal{L}$ is proper if $h \in \mathcal{T}$. 