Explicit inversion formulas for the spherical mean
Radon transform

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Abstract

We derive explicit formulas for the reconstruction of a function from its integrals
over a family of spheres, or for the inversion of the spherical mean Radon trans-
form. Such formulas are important for problems of thermo- and photo- acoustic
tomography. A closed-form inversion formula of a filtration-backprojection type is
found for the case when the centers of the integration spheres lie on a sphere in
\( \mathbb{R}^n \) surrounding the support of the unknown function. An explicit series solution is
presented for the case when the centers of the integration spheres lie on a general
closed surface.

Introduction

The problem of the reconstruction of a function from its spherical integrals (or means)
have recently attracted attention of researchers due to its connection to the thermo-
acoustic and photo-acoustic tomography [12, 13, 21, 22]. In these imaging modalities, the
object of interest is illuminated by a short electromagnetic pulse which causes a fast
expansion of the tissue. The intensity of the resulting ultrasound wave is recorded by
a set of detectors surrounding the object. The local intensity of such expansion is of
significant medical interest: it depends on the physical properties of the tissue (such as,
for example, water content) and its anomaly can be indicative of tumors. Under certain
simplifying assumptions the measurements can be represented by the integrals of the
expansion intensity over the spheres with the centers at the detectors locations. The
reconstruction of the local properties from these integrals is equivalent to the inversion of
the spherical mean Radon transform.

An introduction to the subject can be found in [12–14]; for the important results on
the injectivity of the spherical Radon transform and the corresponding range conditions
we refer the reader to [2–4, 9, 10]. In the present paper we concentrate on explicit inversion
formulas which are important from both theoretical and practical points of view. Most of
the known formulas of this sort pertain to the spherical acquisition geometry, i.e. to the
situation when centers of the integration spheres (the positions of the detectors) lie on
a sphere surrounding the body. Such are the series solutions for 2-D and 3-D presented
in [15, 16, 21]. More desirable backprojection-type formulas were derived in [9] for odd-
dimensional spaces, and implemented in [6]. A different explicit formula for the spherical
acquisition geometry valid in 3-D was found in [22] (together with formulas for certain
unbounded acquisition surfaces).

In this paper we present a set of closed-form inversion formulas for the spherical ge-
ometry and a series solution for certain other measuring surfaces. Our formulas for the
spherical case are of the filtration-backprojection type; they are valid in \( \mathbb{R}^n, n \geq 2 \).
Such formulas for the even-dimensional cases were not known previously. (A set of different inversion formulas for the even-dimensional case was announced by D. Finch [11] during tomography meeting in Oberwolfach in August, 2006, where our results were also presented for the first time.)

The spherical case is discussed in Sections 1 through 3. A series solution for a general acquisition geometry is presented in Section 4.

1 Formulation of the problem

Suppose that $C^1_0$ function $f(x) \in \mathbb{R}^n$, $n \geq 2$ is compactly supported within the closed ball $B$ of radius $R$ centered at the origin. We will denote the boundary of the ball by $\partial B$. Our goal is to reconstruct $f(x)$ from its projections $g(z, r)$ defined as the integrals of $f(x)$ over the spheres of radius $r$ centered at $z$:

$$g(z, r) = \int_{S^{n-1}} f(z + r\hat{t}) r^{n-1} ds(\hat{t})$$

where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$, $\hat{t}$ is a unit vector, and $ds$ is the normalized measure in $\mathbb{R}^n$. Projections are assumed to be known for all $z \in \partial B$, $0 \leq r \leq 2R$ (integrals for $r > 2R$ automatically equal zero, since the corresponding integration spheres do not intersect the support of the function). In the following two sections we will present an explicit formula of backprojection type that solves this reconstruction problem.

2 Derivation

Our derivation is based on certain properties of the solutions of the Helmholtz equation in $\mathbb{R}^n$

$$\Delta h(x) + \lambda^2 h(x) = 0.$$

For this equation the free space Green’s function $\Phi(x, y, \lambda)$ satisfying radiation boundary condition is described (see, for example [1]) by the formula

$$\Phi(x, y, \lambda) = \frac{i}{4} \left( \frac{\lambda}{2\pi|x - y|} \right)^{n/2-1} H_{n/2-1}^{(1)}(\lambda|x - y|),$$

where $H_{n/2-1}^{(1)}(t)$ is the Hankel function of the first kind and of order $n/2 - 1$. To simplify the notation, we introduce functions $J(t)$, $N(t)$, and $H(t)$, defined by the following formulas

$$J(t) = \frac{J_{n/2-1}(t)}{t^{n/2-1}},$$

$$N(t) = \frac{N_{n/2-1}(t)}{t^{n/2-1}},$$

$$H(t) = \frac{H_{n/2-1}^{(1)}(t)}{t^{n/2-1}} = J(t) + iN(t),$$

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where \( J_{n/2-1}(t) \) and \( N_{n/2-1}(t) \) are respectively the Bessel and Neumann functions of order \( n/2 - 1 \). In this notation Green’s function \( \Phi(x, y, \lambda) \) can be re-written in a simpler form:

\[
\Phi(x, y, \lambda) = ic(\lambda, n)H(\lambda|x - y|)
= c(\lambda, n) [iJ(\lambda|x - y|) - N(\lambda|x - y|)]
\]

where \( c(\lambda, n) \) is a constant for a fixed value of \( \lambda \):

\[
c(\lambda, n) = \frac{\lambda^{n-2}}{4(2\pi)^{n/2-1}}.
\]

We note that function \( J(\lambda|x|) \) is a solution of the Helmholtz equation for all \( x \in \mathbb{R}^n \), while \( N(\lambda|x|) \) solves this equation in \( \mathbb{R}^n \setminus \{0\} \).

In order to derive the inversion formula, we utilize the following integral representation of \( f \) in the form of a convolution with \( J(\lambda|y - x|) \):

\[
f(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^+} \left( \int_{\mathbb{R}^n} f(x)J(\lambda|x - y|)dx \right) \lambda^{n-1}d\lambda. \tag{1}
\]

The above equation easily follows from the Fourier representation of \( f \)

\[
f(0) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \xi} f(x) dxd\xi
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^+} \int_{\mathbb{R}^n} f(x) \left[ \int_{\mathbb{S}^{n-1}} e^{-i\lambda x \hat{\xi}} d\hat{\xi} \right] dx \lambda^{n-1}d\lambda
\]

and from the well-known integral representation for \( J(|u|) \) [17]

\[
J(|u|) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{S}^{n-1}} e^{iu \hat{\xi}} d\hat{\xi}.
\]

Let us denote the inner integral in (1) by \( G_J(y, \lambda) \)

\[
G_J(y, \lambda) = \int_{\mathbb{R}^n} f(x)J(\lambda|x - y|)dx. \tag{2}
\]

Similarly to the kernel \( J(\lambda|y - x|) \) of this convolution, function \( G_J(y, \lambda) \) is an entire solution of the Helmholtz equation. Boundary values of this function \( G_J(z, \lambda), z \in \partial B \) are easily computable from projections:

\[
G_J(z, \lambda) = \int_B f(x)J_0(\lambda|z - x|)dx = \int_0^{2R} J_0(\lambda r)g(z, r)dr.
\]

If \( \lambda \) is not in the spectrum of the Dirichlet Laplacian on \( B \), \( G_J(y, \lambda) \) is completely determined by its boundary values and can be found by solving numerically the corresponding
Dirichlet problem for the Helmholtz equation. Then \( f(x) \) can be reconstructed from equation (1). Unfortunately, such a solution would not have an explicit form.

In order to obtain an explicit formula for \( G_J(y, \lambda) \) we will utilize a Helmholtz representation for \( J(\lambda|y - x|) \); it results from an application of Green’s formula and has the following form:

\[
J(\lambda|y - x|) = \int_{\partial B} \left[ J(\lambda|z - x|) \frac{\partial}{\partial n_z} \Phi(y, z, \lambda) - \Phi(y, z, \lambda) \frac{\partial}{\partial n_z} J(\lambda|z - x|) \right] ds(z),
\]

or

\[
J(\lambda|y - x|) = -c(\lambda, n) \int_{\partial B} \left[ J(\lambda|z - x|) \frac{\partial}{\partial n_z} N(\lambda|y - z|) - N(\lambda|y - z|) \frac{\partial}{\partial n_z} J(\lambda|z - x|) \right] ds(z).
\]

(3)

Such a representation is valid for any bounded single-connected domain with sufficiently regular boundary. A straightforward substitution of equation (3) into (2) leads to a boundary value representation for \( G_J(y, \lambda) \) involving the normal derivative the latter function. Unlike the boundary values of \( G_J(z, \lambda) \), the normal derivative \( \frac{\partial}{\partial n_z} G_J(z, \lambda) \) cannot be explicitly computed from projections \( g(z, r) \).

This difficulty can be circumvented by modifying the Helmholtz representation as described below. We notice that in the special case of a spherical domain the second integral in the formula (3)

\[
I(x, y) = \int_{\partial B} N(\lambda|y - z|) \frac{\partial}{\partial n_z} J(\lambda|z - x|) ds(z)
\]

(4)

is a symmetric function of its arguments i.e. that

\[
I(x, y) = I(y, x).
\]

The proof of this fact is presented in the Appendix. Using this symmetry we obtain a modified Helmholtz representation for \( J(\lambda|y - x|) \):

\[
J(\lambda|y - x|) = -c(\lambda, n) \int_{\partial B} \left[ J(\lambda|z - x|) \frac{\partial}{\partial n_z} N(\lambda|y - z|) - N(\lambda|y - z|) \frac{\partial}{\partial n_z} J(\lambda|y - x|) \right] ds(z).
\]

(5)

Now the substitution of equation (5) into (2) yields
$$\int_B f(x)J(\lambda|y - x|)dx = -c(\lambda, n) \int_{\partial B} \left[ \left( \int_B f(x)J(\lambda|z - x|)dx \right) \frac{\partial}{\partial n_z} N(\lambda|y - z|) \right] \frac{\partial}{\partial n_z} J(\lambda|y - z|) ds(z),$$

where the inner integrals are easily computable from the projections $g(z, t)$:

$$\int_B f(x)J(\lambda|z - x|)dx = \int_0^{2R} J(\lambda t)g(z, t)dt,$$

$$\int_B f(x)N(\lambda|z - x|)dx = \int_0^{2R} N(\lambda t)g(z, t)dt.$$

Thus, the convolution of $f$ and $J$ can be reconstructed from the projections as follows:

$$\int_B f(x)J(\lambda|y - x|)dx = -c(\lambda, n) \int_{\partial B} \left[ \left( \int_0^{2R} J(\lambda t)g(z, t)dt \right) \frac{\partial}{\partial n_z} N(\lambda|y - z|) \right] \frac{\partial}{\partial n_z} J(\lambda|y - z|) ds(z)$$

$$= c(\lambda, n) \text{div} \int_{\partial B} \left[ \left( \int_0^{2R} J(\lambda t)g(z, t)dt \right) N(\lambda|y - z|) \right] \frac{\partial}{\partial n_z} J(\lambda|y - z|) ds(z).$$

Finally, by combining equations (1) and (6), one arrives at the following inversion formula

$$f(y) = \frac{1}{4(2\pi)^{n-1}} \text{div} \int_{\partial B} n(z) h(z, |y - z|)ds(z),$$

where

$$h(z, t) = \int_{\mathbb{R}^+} \left[ N(\lambda t) \left( \int_0^{2R} J(\lambda t')g(z, t')dt' \right) - J(\lambda t) \left( \int_0^{2R} N(\lambda t')g(z, t')dt' \right) \right] \lambda^{2n-3} d\lambda.$$
3 Particular cases

3.1 2-D case

From the point of view of practical applications the two- and three-dimensional cases are the most important ones. In 2-D, $J(t) = J_0(t)$, $N(t) = N_0(t)$, and the inversion formula has the following form

$$f(y) = \frac{1}{8\pi} \text{div} \int_{\partial B} n(z) h(z, |y - z|) dl(z), \quad (7)$$

where

$$h(z, t) = \int_{\mathbb{R}^+} \left[ N_0(\lambda t) \left( \int_0^{2R} J_0(\lambda t') g(z, t') dt' \right) 
- J_0(\lambda t) \left( \int_0^{2R} N_0(\lambda t') g(z, t') dt' \right) \right] \lambda d\lambda. \quad (8)$$

Equation (7) is a backprojection followed by the divergence operator. It is worth noticing that such a divergence form of a reconstruction formula is not unusual; it also naturally occurs in reconstruction formulas for the attenuated Radon transform (see, for instance [20]).

Equation (8) represents the filtration step of the algorithm. In order to better understand the nature of this operator we rewrite (8) in the form

$$h(z, t) = -\frac{i}{2} \int_{\mathbb{R}^+} \left[ H_0^{(1)}(\lambda t) \left( \int_0^{2R} H_0^{(1)}(\lambda t') g(z, t') dt' \right) 
- H_0^{(1)}(\lambda t) \left( \int_0^{2R} H_0^{(1)}(\lambda t') g(z, t') dt' \right) \right] \lambda d\lambda. \quad (9)$$

and recall that for large values of the argument $t$ the Hankel function $H_0^{(1)}(t)$ has the following asymptotic expansion ([19])

$$H_0^{(1)}(t) = \left( \frac{2}{\pi t} \right)^\frac{1}{2} e^{-\frac{i}{2} \pi} e^{it}. \quad (9)$$

In the situation when the support of the function $f(x)$ remains bounded and the radius $R$ of the ball $B$ becomes large, the inner and outer integrals in (9) reduce to the direct and inverse Fourier transforms, with one of the terms corresponding to the positive frequencies and the other to the negative ones. Thus, since the two terms have opposite signs, in the asymptotic limit of large $R$ operator (9) equals (up to a constant factor) to the Hilbert transform of $g(z, \cdot)$.
3.2 3-D case

In the three-dimensional case

\[ J(t) = \frac{J_{1/2}(\lambda r)}{\sqrt{\lambda r}} = \sqrt{\frac{2}{\pi}} j_0(\lambda r), \]
\[ N(t) = \frac{N_{1/2}(\lambda r)}{\sqrt{\lambda r}} = \sqrt{\frac{2}{\pi}} n_0(\lambda r), \]

where \( j_0(t) \) and \( n_0(t) \) are the spherical Bessel and Neumann functions respectively. The formula takes the following form

\[
f(y) = \frac{1}{16\pi^2} \text{div} \int |z| = R n(z) h(z, |y - z|) ds(z), \tag{10}\]

with

\[
h(z, t) = \frac{2}{\pi} \int_{R^+} \left[ n_0(\lambda t) \left( \int_0^{2R} j_0(\lambda t') g(z, t') dt' \right) \right. \\
- j_0(\lambda t) \left( \int_0^{2R} \lambda n_0(\lambda t') g(z, t') dt' \right) \left] \lambda^3 d\lambda. \right.
\]

In this case, however, a further simplification is possible since \( j_0(t) \) and \( n_0(t) \) have a simple representation in terms of trigonometric functions:

\[
j_0(t) = \frac{\sin t}{t}, \quad n_0(t) = -\frac{\cos t}{t}. \]

The substitution of these trigonometric expressions into the inversion formula leads to a significantly simpler formula:

\[
h(z, t) = -\frac{2}{\pi} \int_{R^+} \cos(\lambda t) \left[ \int_0^{2R} \sin(\lambda t') \frac{g(z, t')}{t'} dt' \right] \lambda d\lambda \\
+ \frac{2}{\pi} \int_{R^+} \sin(\lambda t) \left[ \int_0^{2R} \cos(\lambda t') \frac{g(z, t')}{t'} dt' \right] \lambda d\lambda \\
= -\frac{2}{\pi t} \int_{R^+} \sin(\lambda t) \left[ \int_0^{2R} \sin(\lambda t') \frac{g(z, t')}{t'} dt' \right] d\lambda \\
- \frac{2}{\pi t} \int_{R^+} \cos(\lambda t) \left[ \int_0^{2R} \cos(\lambda t') \frac{g(z, t')}{t'} dt' \right] d\lambda \\
= -\frac{2}{t} \frac{d}{dt} \frac{g(z, t)}{t}, \tag{11}\]
where we took into account the fact that the Fourier sine and cosine transforms are self-invertible. By combining (11) and (10) our inversion formula can be re-written in the form

\[
f(y) = -\frac{1}{8\pi^2} \text{div} \int_{\partial B} n(z) \left( \frac{1}{t} \frac{d}{dt} \frac{g(z,t)}{t} \right) \bigg|_{t=|z-y|} ds(z).
\]

This expression is equivalent to one of the formulas derived in [22] for the 3-D case.

4 Inversion of the spherical mean Radon transform in other geometries

In such applications as photo- and thermo-acoustic tomography, the designer of the measuring system has a freedom in selecting the detectors’ locations. The detectors (the centers of the integration spheres) do not have to lie on a sphere. In this section we present reconstruction formulas for the case when the measuring surface is a boundary of certain other domains. Namely, our method works for the domains whose eigenfunctions \( u_m(x) \) of the (zero) Dirichlet Laplacian are explicitly known. Such are, for example, the domains for which the eigenfunctions can be found by the separation of variables, i.e. sphere, annulus, cube, and certain subsets of those, and crystallographic domains (see [7, 8]).

The proof of the range theorem in [2] involves implicitly a reconstruction procedure also based on eigenfunction expansions. Unlike the present method, that procedure would involve division of analytic functions that have countable number of zeros. While the range theorem guarantees cancellation of these zeros when the data are in the range of the direct transform, a stable numerical implementation of such division would be complicated if not impossible. (Similarly, the series solution of [15] for 2-D circular geometry involves division by Bessel functions.) The technique we present below does not require such divisions.

Suppose \( \lambda_m^2, u_m(x) \) are the eigenvalues and eigenfunctions of the (negative) Dirichlet Laplacian on a bounded domain \( \Omega \) with zero boundary conditions, i.e.

\[
\Delta u_m(x) + \lambda_m^2 u_m(x) = 0, \quad x \in \Omega, \quad \Omega \subseteq \mathbb{R}^n, \\
u_m(x) = 0, \quad x \in \partial \Omega.
\]

As in the previous sections, we would like to reconstruct a function \( f(x) \in L^2(\Omega) \) from the known values of its spherical integrals \( g(z, r) \) with the centers on \( \partial \Omega \):

\[
g(z, r) = \int_{S^{n-1}} f(z + r\hat{s}) r^{n-1} d\hat{s}, \quad z \in \partial \Omega.
\]

We notice that \( u_m(x) \) is a solution of the Dirichlet problem for the Helmholtz equation
and thus admits the Helmholtz representation

$$u_m(x) = \int_{\partial \Omega} \Phi(x, z, \lambda_m) \frac{\partial}{\partial n} u_m(z) ds(z)$$

$$= ic(\lambda_m, n) \int_{\partial \Omega} H(\lambda_m |x - z|) \frac{\partial}{\partial n} u_m(z) ds(z) \quad x \in \Omega. \quad (12)$$

On the other hand, eigenfunctions \( \{u_m(x)\}_{0}^{\infty} \) form an orthonormal basis in \( L_2(\Omega) \). Therefore \( f(x) \) can be represented (in \( L^2 \) sense) by the series

$$f(x) = \sum_{m=0}^{\infty} \alpha_m u_m(x) \quad (13)$$

with

$$\alpha_m = \int_{\Omega} u_m(x) f(x) dx. \quad (14)$$

The reconstruction formula will result if we substitute representation (12) into (14) and change the order of integrations

$$\alpha_m = \int_{\Omega} u_m(x) f(x) dx$$

$$= ic(\lambda_m, n) \int_{\partial \Omega} \left( \int_{\Omega} H(\lambda_m |x - z|) \frac{\partial}{\partial n} u_m(z) ds(z) \right) f(x) dx$$

$$= ic(\lambda_m, n) \int_{\partial \Omega} \left( \int_{\Omega} H(\lambda_m |x - z|) f(x) dx \right) \frac{\partial}{\partial n} u_m(z) ds(z). \quad (15)$$

The change of the integration order is justified by the fact that eigenfunctions \( u_m(x) \) are continuous. The inner integral in (15) is easily computed from projections

$$\int_{\Omega} H(\lambda_m |x - z|) f(x) dx = \int_{\mathbb{R}^+} g(z, r) H(\lambda_m r) dr,$$

so that

$$\alpha_m = ic(\lambda_m, n) \int_{\partial \Omega} \left( \int_{\mathbb{R}^+} g(z, r) H(\lambda_m r) dr \right) \frac{\partial}{\partial n} u_m(z) ds(z). \quad (16)$$

With Fourier coefficients \( \alpha_m \) now known \( f(x) \) is reconstructed by summing series (13).

If desired, this solution can be re-written in the form of a backprojection-type formula:

$$f(x) = \sum_{m=0}^{\infty} \alpha_m u_m(x) = \int_{\partial \Omega} \left( \sum_{m=0}^{\infty} \alpha_m c(\lambda_m, n) i H(\lambda_m |x - z|) \frac{\partial}{\partial n} u_m(z) \right) ds(z)$$

$$= \int_{\partial \Omega} h(z, |x - z|) ds(z), \quad (17)$$

where

$$h(z, t) = i \sum_{m=0}^{\infty} c(\lambda_m, n) \alpha_m H(\lambda_m t) \frac{\partial}{\partial n} u_m(z), \quad (18)$$
and coefficients $a_m$ are computed using equation (16). In the above formula equation (17) is clearly a backprojection operator, and (18) is a filtration. However, the latter operator is now represented by a series rather than by a closed form expression.

The series solution described above have an interesting property not possessed (to the best of our knowledge) by any other explicit reconstruction technique. Let us consider a slightly more general problem. Suppose that region $\Omega$ is a proper subset of a larger region $\Omega_1$ ($\Omega \subset \Omega_1$) and that a $L^2$ function $F$ is defined on $\Omega_1$. We will denote the restriction of $F$ on $\Omega$ by $f$, i.e.

$$f(x) = \begin{cases} F(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^n \setminus \Omega \end{cases}.$$ 

We would like to reconstruct $f(x)$ from the integrals $g(z, r)$ of $F$ over spheres with the centers on $\partial \Omega$:

$$g(z, r) = \int_{S^{n-1}} F(z + r\hat{s}) r^{n-1} d\hat{s}, \quad z \in \partial \Omega.$$ 

The difference with the previously considered problem is in that the centers of the integration spheres are know lying on a surface which is inside the support $\Omega_1$ of the function $F$. While we are still trying to reconstruct the restriction $f$ of $F$ to $\Omega$, the integrals we know are those of $F$ and not of $f$.

It turns out that the solution to this problem is still given by formulas (13) and (16) (or equivalently by (17), (18), and (16)). Indeed, if we extend functions $u_m(x)$ by 0 to $\mathbb{R}^n \setminus \Omega$, formula (12) holds for all $x \in \mathbb{R}^n$, and (13) remains unchanged. In formula (15) $f$ can be replaced by $F$ as follows:

$$\alpha_m = \int_\Omega u_m(x)f(x)dx = \int_\Omega u_m(x)F(x)dx = \int_\Omega \left( \int_{\Omega_1} H(\lambda_m |x - z|) F(x)dx \right) \frac{\partial}{\partial n} u_m(z)ds(z) = \int_{\mathbb{R}^+} g(z, r) H(\lambda_m r)dr.$$ 

and the inner integral can be computed from projections as before:

$$\int_{\Omega_1} H(\lambda_m |x - z|) F(x)dx = \int_{\mathbb{R}^+} g(z, r) H(\lambda_m r)dr.$$ 

By combining the two above equations we again arrive at the formula (16).

To summarize, if the centers of the integration spheres lie on a closed surface inside of the support of the function, the values of the function corresponding to the interior of that surface can be stably reconstructed by a formula containing the normal derivatives of the eigenfunctions of the Dirichlet Laplacian. If these eigenfunctions are given by an explicit formula, our technique yields an explicit series solution to this problem.

**Appendix**

In this section we prove that for arbitrary $x, y \in B$ function $I(x, y)$ defined by equation (4) is a symmetric function of its arguments, i.e. that $I(x, y) = I(y, x)$. 
Functions \( u^{k,l}_\lambda(x) \) and \( v^{k,l}_\lambda(x) \) defined as follows:

\[
\begin{align*}
u^{k,l}_\lambda(x) &= Y^{(k)}_l(\hat{x}) J_\lambda(\lambda|x|), \\
u^{k,l}_\lambda(x) &= Y^{(k)}_l(\hat{x}) H_\lambda(\lambda|x|), \\
\end{align*}
\]

where we, as before, utilize the notation:

\[
\begin{align*}
J_\lambda(t) &= \frac{J_{n/2+k-1}(t)}{t^{n/2-1}}, \\
N_\lambda(t) &= \frac{N_{n/2+k-1}(t)}{t^{n/2-1}}, \\
H_\lambda(t) &= \frac{H_{n/2+k-1}(t)}{t^{n/2-1}}.
\end{align*}
\]

Functions \( u^{k,l}_\lambda(x) \) are ([18]) entire solutions of the Helmholtz equation for all \( x \in \mathbb{R}^n \), while \( v^{k,l}_\lambda(x) \) are radiating solutions of this equation in \( \mathbb{R}^n \setminus \{0\} \). If we apply the Green’s theorem in an exterior of a sphere of radius \( r_0 \) to functions \( u^{k,l}_\lambda(x) \) and \( v^{k,l}_\lambda(x) \), we will obtain (for \( |x| > r_0 \))

\[
0 = \int_{|z|=r_0} \left[ \frac{u^{k,l}_\lambda(z)}{\partial n_z} \Phi(x, z, \lambda) - \Phi(x, z, \lambda) \frac{\partial u^{k,l}_\lambda(z)}{\partial n_z} \right] dz
\]

and

\[
\begin{align*}
v^{k,l}_\lambda(x) &= \int_{|z|=r_0} \left[ \frac{v^{k,l}_\lambda(z)}{\partial n_z} \Phi(x, z, \lambda) - \Phi(x, z, \lambda) \frac{\partial v^{k,l}_\lambda(z)}{\partial n_z} \right] dz \\
&= \int_{|z|=r_0} \left[ Y^{(k)}_l(\hat{z}) \left[ H_\lambda(\lambda r_0) \frac{\partial}{\partial n_z} \Phi(x, z, \lambda) - \Phi(x, z, \lambda) \lambda J'_{(k)}(\lambda r_0) \right] \right] dz
\end{align*}
\]

where \( \hat{z} = z/|z| \). By combining the above two equations one can eliminate the terms with \( \frac{\partial}{\partial n_z} \):

\[
J_\lambda(\lambda r_0) v^{k,l}_\lambda(x) = \lambda \left[ H_\lambda(\lambda r_0) J'_{(k)}(\lambda r_0) - J_\lambda(\lambda r_0) H'_{(k)}(\lambda r_0) \right] \times \int_{|z|=r_0} \left[ Y^{(k)}_l(\hat{z}) \Phi(x, z, \lambda) \right] dz.
\]  

(19)

The expression in the brackets can be simplified using the well known formula ([19]) for the Wronskian of \( H^{(0)}_\alpha(t) \) and \( J_\alpha(t) \):

\[
H^{(0)}_\alpha(t) J'_\alpha(t) - J_\alpha(t) H^{(0)}_\alpha(t)'(t) = -\frac{i}{2\pi t}.
\]  

11
Formula (19) then takes form

\[ J_{(k)}(\lambda r_0)u_{(k)}^{(l)}(x) = -\frac{i}{2\pi r_0(\lambda r_0)^{n-2}} \int_{|z|=r_0} Y_{l}^{(k)}(\hat{z})\Phi(x, z, \lambda)dz \]

so that the single layer potential we consider is described by the following equation

\[ \int_{|z|=r_0} Y_{l}^{(k)}(\hat{z})\Phi(x, z, \lambda)dz = i2\pi r_0(\lambda r_0)^{n-2}J_{(k)}(\lambda r_0)H_{(k)}(\lambda |x|)Y_{l}^{(k)}(\hat{x}). \]  

(20)

By substituting into (20) the expression for the Green’s function \( \Phi(x, z, \lambda) \) in the form

\[ \Phi(x, z, \lambda) = ic(\lambda, n)H(\lambda |x - z|) \]

one obtains

\[ \int_{|z|=r_0} Y_{l}^{(k)}(\hat{z})H(\lambda |x - z|)dz = \frac{2\pi r_0(\lambda r_0)^{n-2}}{c(\lambda, n)}J_{(k)}(\lambda r_0)H_{(k)}(\lambda |x|)Y_{l}^{(k)}(\hat{x}). \]

Finally, by separating the real and imaginary parts of the above equation we arrive at the following two formulas:

\[ \int_{|z|=r_0} Y_{l}^{(k)}(\hat{z})J(\lambda |x - z|)dz = \frac{2\pi r_0(\lambda r_0)^{n-2}}{c(\lambda, n)}J_{(k)}(\lambda r_0)J_{(k)}(\lambda |x|)Y_{l}^{(k)}(\hat{x}), \]  

(21)

\[ \int_{|z|=r_0} Y_{l}^{(k)}(\hat{z})N(\lambda |x - z|)dz = \frac{2\pi r_0(\lambda r_0)^{n-2}}{c(\lambda, n)}J_{(k)}(\lambda r_0)N_{(k)}(\lambda |x|)Y_{l}^{(k)}(\hat{x}), \]  

(22)

valid for \( |x| > r_0 \). This completes the preparation for the proof of the symmetry of \( I(x, y) \).

Let us consider function \( I(\alpha \hat{x}, \beta \hat{y}) \). For fixed values of \( \alpha \) and \( \beta \) this is an infinitely smooth function of \( \hat{x} \) and \( \hat{y} \) defined on \( S^{n-1} \times S^{n-1} \). Consider the Fourier expansion of \( I(\alpha \hat{x}, \beta \hat{y}) \) in the spherical harmonics in both variables \( \hat{x} \) and \( \hat{y} \). Coefficients of such series are given by the formula

\[ a_{(k,l)}^{(l')}(\alpha, \beta) = \int_{S^{n-1}} \int_{S^{n-1}} Y_{l}^{(k)}(\hat{x})Y_{l'}^{(k)}(\hat{y})I(\alpha \hat{x}, \beta \hat{y})d\hat{x}d\hat{y} \]

(23)

with \( k = 0, 1, 2, ..., \) \( 0 \leq l \leq d_k \),

\[ d_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}, k \geq 2, \]

\[ d_0 = n, d_1 = 1. \]

In order to prove the symmetry \( I(x, y) = I(y, x) \) it is enough to prove that

\[ a_{(k,l)}^{(l')}(\alpha, \beta) = a_{(k,l')}(\beta, \alpha) \]

(24)
for all relevant values of $k, k', l, l'$. By substituting the expression for $I(\alpha \hat{x}, \beta \hat{y})$ into (23) one obtains

\[
a_{k,l}^{k',l'}(\alpha, \beta) = \int_{S^{n-1}} \int_{S^{n-1}} Y_k^l(\hat{x})Y_{k'}^{l'}(\hat{y}) \left[ \int_{|z|=R} N(\lambda |z - \alpha \hat{x}|) \frac{\partial}{\partial n_z} J(\beta \hat{y} - z)|dz\right] d\hat{x}d\hat{y}
\]

\[
= \int_{S^{n-1}} Y_{k'}^{l'}(\hat{y}) \left[ \int_{S^{n-1}} \left( \int_{|z|=R} Y_k^l(\hat{x})N(\lambda |z - \alpha \hat{x}|) d\hat{x} \right) \frac{\partial}{\partial n_z} J(\lambda |\beta \hat{y} - z|)|dz\right] d\hat{y}
\]

\[
= \frac{1}{\alpha^{n-1}} \int_{S^{n-1}} Y_{k'}^{l'}(\hat{y}) \left[ \int_{|z|=R} \left( \int_{|u|=|z|} Y_k^l(u/|u|)N(\lambda |z - u|) du \right) \frac{\partial}{\partial n_z} J(\lambda |\beta \hat{y} - z|)|dz\right] d\hat{y}.
\]

Utilizing formulas (21), (22) and the orthonormality of the spherical harmonics we find that $a_{k',l'}^{k,l}(\alpha, \beta) = 0$ if $k \neq k'$ or $l \neq l'$. Otherwise

\[
a_{k,l}^{k',l'}(\alpha, \beta) = \lambda \left( \frac{2\pi \lambda^{n-2}}{c(\lambda, n)} \right)^{2} R^{n-1} J_{(k)}(\lambda \alpha)J_{(k)}(\lambda \beta)N_{(k)}(\lambda R)J'_{(k)}(\lambda R).
\]

Inspection of the above formula shows that coefficients $a_{k',l'}^{k,l}(\alpha, \beta)$ indeed satisfy (24) and, thus, that $I(x, y) = I(y, x)$.

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