Rényi entropy and $C_T$ for higher derivative free scalars and spinors on even spheres

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General expressions for the Rényi entropies and central charges for higher derivative free spinors and scalars on even spheres are obtained using a direct spectral method on a compact lune division of the sphere. Formulae and numbers are rapidly obtained for any dimension and order of derivative.

The relation between the conformal anomaly and the hyperbolic free energy is briefly explored using standard expansions.

A field theoretic derivation of the central charge formula for higher derivative scalars in any (even) dimension, given by Osborn and Stergiou and by Gliozzi et al, is thereby provided. The extension to spinors is made.

Generalised Bernoulli polynomials play an important technical role.

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1. Introduction

Rényi entropy has recently appeared, as an intermediary, in the computation of the central charge, and Weyl anomaly, for higher derivative conformal fields, [1]. The particular manifold used was the hyperbolic cylinder, on which the propagation operators factorise. In the present work I will use the periodic spherical $q$–lune for the same purpose. This is a segment of the $d$–sphere with apex angles, $2\pi/q$. I have employed this on many previous occasions. The advantage is, partly, its compactness and the relative ease of higher dimensional calculation.

The Rényi entropy for standard fields has been evaluated by Klebanov et al, [2], some time ago. In this, and similar, works the $q$-lune appears as a ‘branched sphere’ usually interpreted as a covered sphere for which $q = 1/n$, with $n$ an integer. The present evaluations use $q \in \mathbb{Z}$ which I have always found more convenient. The results can be extended to continuous values of $q$.

2 Rényi entropy

I repeat a few standard facts for continuity. The Rényi entropy, $\mathcal{S}_n$, is defined by,

$$\mathcal{S}_n = \frac{nW(1) - W(1/n)}{1 - n},$$

(1)

where $W(q)$ here is the effective action on the periodic $q$–lune. In even dimensions, a universal component of $\mathcal{S}_n$ is extracted, usually from the coefficient of a logarithmic term, or a divergent pole, in the effective action. This is the heat–kernel expansion coefficient, $C_{d/2}$, or, equivalently, the value, $\zeta(0)$, of the relevant $\zeta$–function, up to zero modes. To a known factor, this is the conformal anomaly.

Considering $n$, or preferably $q$, as a real variable, the expansion of $\mathcal{S}_n$ about $n = 1$ yields particular information. For example, the first term gives the entanglement entropy and the second the central charge, [4]. This is the route to $C_T$, chosen in [1].

The technical aim then is to construct the $\zeta$–function of the chosen propagating operator and to evaluate $\zeta(0)$ and its second derivative with respect to $n$, [4]. I do this is a direct spectral fashion. The next two sections outline the dynamical situation and some spectral facts. The basic computation and results are in sections

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2 It is interesting to note that Yankielowicz and Zhou, [3], have recently reached the same conclusion.
5 and 6. Section 8 discusses a thermal aspect. The central charge results are in section 10 with the calculation relegated to Appendix A.

3 The operators

All the operators under investigation here take the rather particular Branson spherical GJMS intertwinor form,

\[ \Omega_k(d) = \frac{\Gamma(B + k + 1/2)}{\Gamma(B - k + 1/2)}, \]

where \( k \) is a real parameter, which will be either an integer or a half–integer. The (pseudo) operator \( B \) depends on the field theory. For scalars \( B = B_S = \sqrt{Y_d + 1/4} \) where \( Y_d \) is the conformally covariant Penrose–Yamabe Laplacian. For Dirac spinors \( B = B_D = (\nabla^2)^{1/2} = |\nabla| \).

When \( k \) is integral or half integral, the form (2) reduces to a finite product of operators. To accord with the ordinary field case, \( k \) is an integer for integer spin and a half–integer for half integer spin. These give, respectively, an even or an odd number of factors. More generally, \( k \) has a conformal field theory significance as a scaling dimension, cf [5–7].

However, since, it turns out, \( \zeta(0) \) is a rational function of \( k \), it is possible (curiously) to treat all cases by taking, initially, \( k \) integral. To get the spinor result one then just continues in \( k \). This saves work. The operator (2) for integer \( k \) reads,

\[ \Omega_k(d) = \prod_{j=0}^{k-1} (B^2 - (j + 1/2)^2) \]

\[ = \prod_{j=0}^{k-1} (B - (j + 1/2))(B + (j + 1/2)). \]

In order to find \( \zeta(0) \), the spectrum of \( \Omega_k \), i.e. that of \( B^2 \), is required for the \( q \)–lune \( d \)–manifold. On spherical domains, this spectrum is usually presented in the form of eigenlevels, depending on one integer, together with their degeneracies which first have to be determined (no easy task) and then expanded in order to compute the \( \zeta \)–function usually in terms of the Hurwitz \( \zeta \)–function. For the present set of fields this process can be bypassed, most easily for scalars and spinors. Gauge fields are more involved.

\[ ^3 \text{Gauge forms will be treated at another time.} \]
4. Scalar and spinor spectra on the lune

I give here a rather abbreviated, heuristic treatment.

The situation is described in [8] for scalars. The standard separation of variables for the eigenproblem takes on a recursive nature in accordance with the nested form of the spherical lune metric following from iteration of the geodesic polar coordinate system. The conical deformation of the lune $d$-manifold can thus be traced ultimately just to the deformation of the unit circle, $S^1_q$ coordinatised by the polar angle, $\phi$, which ranges from 0 to $2\pi/q$.

It was shown in [8] that the spectrum for a charged scalar undergoing a phase change of $2\pi \delta$ on circling $S^1_q$ is the union of two sets of eigenvalues,

$$\left[ \left( \frac{d-1}{2} + q|\delta| + \omega \cdot n \right)^2 - \alpha^2 \right] \cup \left[ \left( \frac{d-1}{2} + q - q|\delta| + \omega \cdot n \right)^2 - \alpha^2 \right], \quad (4)$$

where $\omega$ is the set of $d$ integers, $(q, 1)$, but can be considered, formally, as real non-negative and referred to as the parameters. $n$ is a set of $d$ integers, $(n_1, n_2, \ldots, n_d)$ each ranging from 0 to $\infty$. If the propagating operator is conformal, $\alpha = 1/2$. Degeneracies arise simply from coincidences between the eigenvalues as presented in (4).

In the present work there is no phase change, $\delta = 0$, and each set of eigenvalues in (4), corresponds, respectively, to Neumann and Dirichlet conditions on the boundaries of the two ‘hemilunes’, $0 \leq \phi \leq \pi/q$ and $\pi/q \leq \phi \leq 2\pi/q$, that make up the periodic lune.4 Also, in this case, one could have real scalars.

For the spinor case, I recall that the Dirac spectrum of $\nabla^2$ on the round sphere $S^d_1$ is usually expressed in the form of eigenlevels,

$$\lambda_n(a) = (n + a)^2, \quad n = 0, 1, 2, \ldots, \infty, \quad a = \frac{d}{2}, \quad (5)$$

and degeneracies,

$$g_d(n) = 2 \cdot 2^{[d/2]} \binom{n + d - 1}{n}. \quad (6)$$

I re-express these as follows firstly defining, for convenience,

$$\lambda(n, a, q) = (a + n \omega)^2, \quad \omega = (q, 1, \ldots, 1). \quad (7)$$

Then the eigenlevels, (5), are given by,

$$\lambda_n(a) = \{ \lambda(n, a, 1), n : \sum_i n_i = n \},$$

4 If $\delta = 1$, Dirichlet and Neumann are interchanged.
and the degeneracies of the eigenlevels, $\lambda_n$, arise through coincidences, i.e. $g_n = \{\# n_i : \sum_i n_i = n\}$. This gives the binomial coefficient. The $2$s are mostly spin factors.

To find the eigenvalues on the lune, one notes that the sign change in the spinor components on circumscribing the manifold (i.e. around the $S^1_q$) (occasioned by the rotation of the Zweibeine) can be allowed for by choosing $\delta = 1/2$ in (4), for scalars. The Dirac eigenvalues then equal,

$$\lambda(n,a,q) - \alpha^2, \quad a = \frac{d - 1 + q}{2},$$

since they give the round values when $q = 1$. There is a degeneracy factor of 2 which reflects the equality of the two sets in (4).

These results can be justified properly by an iterative eigenfunction separation of variables method detailed, for the round sphere, by Camporesi and Higuchi, [9], and on the lune by Apps, [10] who showed that the two, identical sets of eigenvalues correspond to the application of the two mixed (local) conditions at the boundary of a hemilune mentioned above.

The degeneracies on the lune can be calculated, but they become quite complicated, especially for higher dimensions. Some examples are given by Apps, [10]. Klebanov et al, [2], give the Dirac expressions in 3 dimensions for the covering case $q = 1/n$. However, there is no need to use them as the forms, (4) and (8), can be employed directly, to advantage.

It is worth recalling that the spinor and scalar expressions are related in a well known thermodynamic fashion that reflects the statistics sign changes. Here, from the specific forms (7), (8), simple rearrangement of the summation gives the relation between the $\zeta$–functions,

$$Z_k(s,q,a_S)|_{\text{spinor}} = 2S \left( Z_k(s,q/2,a_N) - Z_k(s,q,a_N) \right)|_{\text{scalar}}$$

and hence the same relation between derived spectral quantities

$$Q_{\text{spinor}}(q) = 2S( Q_{\text{scalar}}(q/2) - Q_{\text{scalar}}(q) ).$$

Invoking this can save effort or act as a check.
5. Conformal anomaly for product operators

The spectra outlined in the previous section show that the eigenvalues of the product $\Omega_k$ involve the quantities

$$\lambda(n, a, q) - \alpha^2,$$

arising from each factor with, explicitly, $\alpha \equiv \alpha_j = j + 1/2$, $j = 0, 1, \ldots k - 1$. There are two values of the parameter $a$ to take into account, the Neumann value $a \equiv a_N = (d - 1)/2$ and the Dirichlet one, $a \equiv a_D = a_N + q$ (see (4)). For spinors, there are also two values, but they are equal and given in (8), $a \equiv a_S = (d - 1 + q)/2$. I note that all three values of $a$ coincide when $q = 0$. This means that in this limit (discussed later) the scalar and spinor expressions coincide, as functions of $k$, up to spin factors.

The object used in the computation of the conformal anomaly is the spectral $\zeta$–function of the propagation operator, $\Omega_k$. I denote this by $Z(s)$ with,

$$Z_k(s, q) = \sum_a Z(s, q, a)$$

$$Z_k(s, q, a) = S \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} \frac{1}{(\lambda(n, a, q) - \alpha_j^2)^s},$$

(10)

where $S$ is the bundle dimension. For real scalars $S = 1$ and for Dirac spinors, $S = 2^{d/2}$.

The conformal anomaly for a product, $\Omega$, of $k$ second order operators is $kZ(0)$ where $Z(s)$ is the corresponding $\zeta$–function. This follows, firstly, because the scaling dimension of the effective action, essentially log det $\Omega$, is the sum of the scaling dimensions of the individual factors using linearity of log det up to a multiplicative anomaly which plays no part in scaling. Secondly, $Z(0)$, is the average of the $k$ individual second order $\zeta(0)\text{s}$, [11], equn.(9) using [12]. Hence the result.\(^5\)

The computation has already been done in [11] by the simple expedient of proceeding to the completely linearised product, as in (3). From the structure (7), only the value of the Barnes $\zeta$–function at $s = 0$ is required and this is just a generalised Bernoulli polynomial, easily evaluated. Barnes’ formula is (a residue),

$$\zeta_d(0, a | \omega) = \frac{(-1)^d}{q d!} B_d^{(d)}(a | \omega), \quad \omega = (q, 1, \ldots, 1),$$

\(^5\)This is used in [13–15] for example.
where the Barnes $\zeta$-function is the continuation of,

$$\zeta_d(s, a | \omega) = \sum_{n=0}^{\infty} \frac{1}{(a + n \omega)^s}, \quad \text{Re } s > d, \quad (11)$$

so that from (10), if $d$ is even,

$$kZ_k(0, q, a) = \frac{S}{2d!q} \sum_{j=0}^{k-1} \left( B_d^{(d)}(a + j + 1/2 | \omega) + B_d^{(d)}(a - j - 1/2 | \omega) \right)$$

$$= \frac{S}{2d!q} \sum_{j=0}^{k-1} \left( B_d^{(d)}(q + d - 1 - a - j - 1/2 | \omega) + B_d^{(d)}(q + d - 1 - a + j + 1/2 | \omega) \right), \quad (12)$$

using a symmetry property of the Bernoulli polynomials.

Hence we have the equality,

$$Z_k(0, q, a_1) = Z_k(0, q, a_2),$$

if

$$a_1 + a_2 = d - 1 + q,$$

which holds for Neumann and Dirichlet conditions and also for spinors (when $a_1 = a_2$, (8)). Trivially, the sum over $a$ in (10) gives a factor of 2 in both cases. Therefore, as a slight notational simplification, I can define

$$C_k(q, a) \equiv (-1)^{2s} 2k Z_k(0, q, a), \quad (13)$$

where $s$ is the spin.

The sum in (12) can be performed (see Appendix B). This produces the simpler expression in one higher dimension,

$$C_k(q, a) = (-1)^{2s} \frac{S}{(d+1)!q} \left( B_{d+1}^{(d+1)}(a+k+1/2 | \omega, 1) - B_{d+1}^{(d+1)}(a-k+1/2 | \omega, 1) \right), \quad (14)$$

for the product conformal anomaly.
6. The results

In accordance with the remarks in section 1, the universal part of the Rényi entropy, is determined by,

$$\mathcal{G}_d(q, k) = -\frac{1}{1-q} \left( -q C_k(q, a) + C_k(q, a) \right)_{q=1},$$

where the parameter $a$ is $a_N$ (or $a_D$) for scalars, and $a_S$ for spin–half.

The calculation of the Bernoulli polynomials in (14) is rapid. There is no practical difficulty in taking the sphere dimension, $d$, as large as desired.

Some polynomial results for a real scalar are,

$$\mathcal{G}_4(q, k) = \frac{k (q + 1) (q^2 - 10 k^2 + 11)}{360}$$

$$\mathcal{G}_6(q, k) = -\frac{k (q + 1) (2 q^4 - (14 k^2 - 23) q^2 + 42 k^4 - 224 k^2 + 191)}{30240}$$

$$\mathcal{G}_8(q, k) = \frac{k (q + 1)}{1814400} \left( 3 q^6 - (20 k^2 - 43) q^4 + (42 k^4 - 300 k^2 + 337) q^2 - 60 k^6 + 882 k^4 - 3240 k^2 + 2497 \right).$$

I remind that $n = 1/q$ is the Rényi ‘index’.

If $k$ exceeds $d/2$, the scalar GJMS operators cannot be constructed classically. However, the above evaluation proceeds without hindrance.

I now list some spinor expressions.

$$\mathcal{G}_2(q, k) = -\mathcal{S} \frac{k(q+1)}{12}$$

$$\mathcal{G}_4(q, k) = \mathcal{S} \frac{k (q + 1) (7 q^2 - 40 k^2 + 47)}{1440}$$

$$\mathcal{G}_6(q, k) = -\mathcal{S} \frac{k (q + 1) (31 q^4 - 196 k^2 q^2 + 325 q^2 + 336 k^4 - 1876 k^2 + 1669)}{241920}$$

$$\mathcal{G}_8(q, k) = \mathcal{S} \frac{k (q + 1)}{116121600} \left( 381 q^6 - (2480 k^2 - 5341) q^4 + (4704 k^4 - 33840 k^2 + 38269) q^2 - 3840 k^6 + 58464 k^4 - 222000 k^2 + 176509 \right).$$

As a check, continuing to $k = 1/2$ and $q = 1$ yields minus the ordinary Dirac conformal anomaly on the round sphere.

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6To be clear, this is the coefficient of $\log \epsilon$ where $1/\epsilon$ is an IR cutoff.
I also list some values for the universal part of the effective action (or free energy), \( F_d(q,k) \) (denoted \( F_q \) by Beccaria and Tseytlin, [1]), \(^7\). It equals the coefficient, \( C_k(q,a) \), (13). I use both notations.

For scalars,

\[
F_2(q,k) = \frac{kq^2 + 2k^3 - k}{6q} \\
F_4(q,k) = -\frac{kq^4 + k(1 - k^2)q^2 - 6k^5 + 20k^3 - 11k}{360q} \\
F_6(q,k) = \frac{1}{30240q}(2kq^6 + k(3 - 2k^2)q^4 + 42k(k^2 - 1)(k^2 - 4)q^2 + 12k^7 - 126k^5 + 336k^3 - 191k).
\]

(18)

For spinors,

\[
F_2(q,k) = \frac{S(kq^2 - 4k^3 + 2k)}{6q} \\
F_4(q,k) = -\frac{S(7kq^4 + 40k(1 - k^2)q^2 + 48k^5 - 160k^3 + 88k)}{1440q} \\
F_6(q,k) = \frac{S}{241920q}(31kq^6 + 98k(3 - 2k^2)q^4 + 336k(k^2 - 1)(k^4 - 4)q^2 - 192k^7 + 2016k^5 - 5376k^3 + 3056k).
\]

(19)

The relation (9) could be checked at this point.

For scalars, there is no term proportional to \( q \) at the subcritical, physical values \( k = 1,2,\ldots,d/2 - 1 \). This is a consequence of the conformal invariance. The same is not true for spinors at the Dirac physical values, \( k = 1/2,\ldots \).

7. Comparison

We can make contact with previous evaluations of some of these quantities. Beccaria and Tseytlin, [1], compute the \(|\nabla|^3 \) entropy (corresponding to \( k = 3/2 \) in (17)) in four and six dimensions.\(^8\) They employ the popular hyperbolic cylinder method advocated by Casini and Huerta, [16], in their evaluation of Rényi entropy for standard fields on spheres. The propagating operators again factorise on the

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\(^7\) Unfortunately my \( q \) is the inverse of that in this reference.

\(^8\) The expressions in this reference differ in sign from the ones here because they are the universal coefficients of \( \log(1/\epsilon) \).
cylinder. For standard fields, \( k = 1 \), the expressions (16) agree with those in [16] and [17]. An alternative derivation is in [8].

A comparison of the free energy expressions here and in [1] shows that they differ by a term proportional to \( 1/q \) which does not affect the Rényi entropy. As \( q \to 0 \), which is a low temperature limit, the coefficient of \( 1/q \), \( \sim \) the inverse temperature, is related to the vacuum energy on the odd \((d - 1)\)-sphere. This numerical statement is made a little more analytical in the next section.

8. \( q \to 0 \) limit. Thermal interpretation and vacuum energy

The value \( q C_k(q, a) \) at \( q = 0 \) is easily obtained from (13) with (12).

From their definition, the generalised Bernoulli polynomials \( B^{(n)}_\nu(x(q) \mid q, 1) \) reduce, at \( q = 0 \) to,

\[
B^{(n)}_\nu(x(0) \mid 0, 1) = B^{(n-1)}_\nu(x(0)),
\]

where \( B^{(n-1)}_\nu(x) \equiv B^{(n-1)}_\nu(x \mid 1) \), all the parameters equaling unity. (This is usually referred to as the generalised Bernoulli polynomial.) They are easily calculated.

Hence, making use of the simplification derived in Appendix B,

\[
q \mathcal{F}_d(q, k) \bigg|_{q=0} = (-1)^{2s} \frac{S}{(d + 1)!} \left( B^{(d)}_{d+1}(a_0 + k + 1/2) - B^{(d)}_{d+1}(a_0 - k + 1/2) \right),
\]

where \( a_0 = (d - 1)/2 \) for both scalars and spinors.

I now recall the expression for the vacuum energy for GJMS operators on the Einstein cylinder \( R \times S^{d-1} \), with \( d \) even, obtained and computed in [18].

In particular for scalars,

\[
E_0(d, k) = -\frac{1}{2(d+1)!} \left( B^{(d)}_{d+1}(d/2 + k) - B^{(d)}_{d+1}(d/2 - k) \right)
\]

\[
= -\frac{1}{(d+1)!} B^{(d)}_{d+1}(d/2 + k),
\]

One sees therefore that the coefficient of \( 1/q \) in the free energy, \( \mathcal{F}_d(q, k) \), is minus twice the vacuum energy, \( E_0 \), on the Einstein cylinder.

In order to give this result a hyperbolic thermal interpretation it is best to restate it as the coefficient of \( 2\pi/q \) in (the universal part of) the free energy is \( -\pi E_0 \) which can be taken as a (regularised) vacuum energy on the open Einstein cylinder \( R \times H^{d-1} \) since the volume of \( S^d \) is \( \pi \) times the regularised volume of \( H^d \), up to an alternating sign. This means that the free energy density on \( S^1_q \times H^{d-1} \) tends
to $-\beta|\eta_0|$ as $\beta = 2\pi/q \to \infty$ where $\eta_0$ is the vacuum energy density on $S^{d-1}$ which also alternates in sign.\(^9\) By contrast, a conventional direct field theory calculation of the local energy density on the open Einstein cylinder actually yields zero.

Spinors can be discussed likewise.

9. Entanglement entropy

The (universal part of) the entanglement entropy arises as the $q \to 1$ value of the Rényi entropy and is readily found. For standard fields it equals (minus) the conformal anomaly. This is a consequence of the stationarity of the heat–kernel coefficient at $q = 1$ as proved in [19]. A similar statement holds in the case of product operators, as I now show in a very similar fashion.

One needs to calculate the derivative $\partial C_k(q,a)/\partial q$ using (14). The necessary formula, due to Barnes, was used in [19,20]. I repeat it here,

$$\partial_q \frac{1}{q} B^{(d)}(x|q,1) \bigg|_{q=1} = -B^{(d+1)}(x+1|1). \quad (20)$$

This gives (I detail scalars only),

$$\frac{\partial C_k(q,a)}{\partial q} \bigg|_{q=1} = \frac{1}{(d+1)!} \left( -B^{(d+2)}(d/2+k+1|1) + B^{(d+2)}(d/2-k+1|1) \right)$$

$$= -\frac{2}{(d+1)!} B^{(d+2)}(d/2+k+1|1) \quad (21)$$

after using a symmetry of the Bernoulli polynomials ($d$ is even).

The simple product structure,

$$B^{(d+2)}_{d+1}(x|1) = (x-1)(x-2)\ldots(x-d-1), \quad (22)$$

then trivially shows that the derivative is zero if $k$ is an integer in the range $-d/2 \leq k \leq d/2$. The spinor result is that $k$ is a half–integer with $(1-d)/2 \leq k \leq (1+d)/2$. The positive values are the relevant ones.

\(^9\) The signs all stem from dimensions and the fact that a hyperboloid is a sphere of imaginary radius.
10. Derivatives and central charge

It is easy to compute, or read off, the derivatives of the Rényi entropy at $q = 1$. As in [21], giving the expressions as polynomials in $q$ is convenient for this. Unfortunately from the explicit forms, one has to proceed dimension by dimension but further work can produce a more compact result whose form can be compared with existing ones. The details appear in Appendix A where it is shown that the required derivative (I exhibit the free energy) is given by,

$$F''_d(k) \equiv \frac{\partial^2 F_d(q, k)}{\partial n^2} \bigg|_{n=1} = (-1)^{d/2+k} \frac{4k}{d+2} \frac{(d/2+k)!(d/2-k)!}{(d+1)!}, \quad (23)$$

for scalars.

From its derivation, this formula works only for $k$ an integer in the range $-d/2 \leq k \leq d/2$. At these values it agrees with the polynomial (in $k$) expressions obtained, for each dimension, from the results in (18). For example,

$$F''_2(k) = \frac{k}{3},$$
$$F''_4(k) = \frac{5k^3 - 8k}{90},$$
$$F''_6(k) = \frac{7k^5 - 49k^3 + 54k}{2520},$$

which can therefore be taken as some sort of continuation of the general, but restricted, formula, (23).\(^{10}\)

This formula also leads to an expression for the scalar central charge using the relations given by Perlmutter, [4]. I find,

$$C_T(d, k) = (-1)^{k+1} \frac{4k(d/2 + k)!(d/2 - k)!}{(d + 2)(d - 1)(d/2 - 1)!^2}. \quad (25)$$

Computation reveals agreement with values occurring in [1] \(^{11}\) which reference also details earlier calculations.

I list a few numerical values from (25)

$$C_T(6, k) = \frac{6}{5}, -6, 54$$
$$C_T(8, k) = \frac{8}{7}, -\frac{32}{7}, 24, -256$$
$$C_T(10, k) = \frac{10}{9}, \frac{35}{9}, \frac{140}{9}, -\frac{280}{3}, \frac{3500}{3}. \quad (26)$$

\(^{10}\) The values supplied by (23) could be used to fix the form of an unknown polynomial.

\(^{11}\) My use of a UV log cutoff accounts for any sign differences.
Osborn and Stergiou, [22], address higher derivative fields (restricted to \( k \leq 3 \)) using CFT techniques. My evaluation furnishes a self-contained proof of their conjecture for \( C_T \) for any \( k \) which employs only simple spectral methods.

Based on work by Guerrieri, Petkou and Wen, [23], a general demonstration was outlined in [22] which imported an expression for the CFT four-point function derived elsewhere with some work. A CFT proof was also provided by Gliozzi et al, [24].

I find the corresponding spinor general expression to be (briefly mentioned in Appendix A),

\[
C_T(d, k) = (-1)^l S \frac{(8l(l + 1) - (d + 2)(d - 1))(d/2 + l)!(d/2 - l - 1)!}{(d + 2)(d - 1)((d/2 - l)!)}^2. \tag{27}
\]

For uniformity I have used the half-integer (for spinors) \( k, = l + 1/2 \), as the argument.

Any numerical value is quickly generated from the previous explicit results, e.g. (19), and the Perlmutter factor, or from (27). As examples,

\[
\begin{align*}
C_T(4, k) &= 2S, -\frac{2S}{3} \\
C_T(6, k) &= 3S, -\frac{18S}{5}, -6 \\
C_T(8, k) &= 4S, -\frac{36S}{7}, \frac{44S}{7}, 52S \\
C_T(10, k) &= 5S, -\frac{115S}{18}, \frac{175S}{18}, -\frac{70S}{9}, -\frac{910S}{3} \\
C_T(12, k) &= 6S, -\frac{414S}{55}, \frac{636S}{55}, -\frac{1044S}{55}, -\frac{108S}{11}, 1548S,
\end{align*}
\]

where \( k \) runs from 1/2 to \((d - 1)/2\), \( 2k \) being the derivative power of the spinor.

In the notation of [1], \( C_T(d, k) = C_{T,d}(\chi^{(2k)}) \) where \( \chi \) is a generic field (scalar or spinor). In particular, one sees that \( C_{T,6}(\psi^{(3)}) = -18S/5 \), the value preferred in [1], \( (S = n_F) \). I am not aware of any significance to the higher values.

11. Comments

The present calculation is a compact version of that in [1], but one that allows more general expressions to be derived efficiently and numbers to be found rapidly.

The formula for the central charge relies on the relation with the derivatives of the free energy derived by Perlmutter, [4], in a non-CFT way.
The precise nature of the continuation provided by the polynomials, e.g. (24), in the number of quadratic factors, \( k \), remains to be elucidated in the light of the general expression, (2), which implies, *inter alia*, \( \Omega_{-k} = \Omega_k^{-1} \).

Gauge fields present extra technical difficulties connected with the ghost sums.

The thermal aspects in section 8 need clarification.

The termination of the heat–kernel expansion on odd \( d-1 \)–spheres, and pseudospheres, for conformality in \( d \), has been known and used for a very long time as have its implications for the finite temperature field theory. In particular for pseudospheres the high temperature expansion, of, say, the free energy is both finite and *exact*, [25]. For spheres there is a non–local exponentially small component reflecting the existence of closed geodesics, responsible for the Casimir energy.

In the present calculation, the pseudosphere exactness accounts for the finite nature of the free energy polynomials in \( q \), (18), by noting the relation of these with the hyperbolic expressions and that these last are derived by the same method given originally in [26], where general expansions can be found. Some further details are given in Appendix C.

The calculation here via the conically deformed \( d \)–sphere, \( S^d_q \), shows that the heat–kernel coefficients on the round \( (d-1) \)–sphere, \( S^{d-1}_1 \), (they are the same as on the hyperboloid, \( H^{d-1} \), up to signs) can be obtained from the conformal anomaly on \( S^d_q \). They are, of course, well known.

The corresponding calculation for odd spheres will require the calculation of the functional determinants of the operator, \( \Omega_k \).

**Appendix A**

I here calculate the second derivative of the conformal anomaly (equivalently the ‘free energy’), \( \partial^2 C_k(q, a) / \partial n^2 \) at \( n = 1 \). It is better to use \( q = 1/n \) so for scalar fields, *i.e.* \( a = (d - 1)/2 \), I require, from (14),

\[
q^2 \frac{\partial}{\partial q} \left( q \frac{1}{q} B_{d+1}^{(d+1)}(b \mid q, 1) \right) = -q^2 \frac{\partial}{\partial q} B_{d+1}^{(d+2)}(b + q \mid q, 1),
\]

at \( q = 1 \) with the integer \( b = d/2 \pm k \).

The algebra is performed in [20] and I find,

\[
\left. - \frac{\partial}{\partial q} B_{d+1}^{(d+2)}(b + q \mid q, 1) \right|_{q=1} = 2B_{d+1}^{(d+3)}(b + 2 \mid 1) - 2B_{d+1}^{(d+2)}(b + 1 \mid 1) - (d + 1) B_d^{(d+2)}(b + 1 \mid 1),
\]

\[
(30)
\]

13
where the second term is immediately zero in light of the product (22) and the allowed range of \( k \).

The expression can be reduced by using the recursion,

\[
B_{d+1}^{(d+3)}(b + 2) - B_{d+1}^{(d+3)}(b + 1) = (d + 1) B_d^{(d+2)}(b + 1),
\]

(31)
to obtain for (30),

\[
B_{d+1}^{(d+3)}(b + 2) + B_{d+1}^{(d+3)}(b + 1).
\]

(32)

I have dropped the reference to the parameters, which are now all unity so that use can be made of existing explicit expressions, [27], for these Bernoulli polynomials, e.g.

\[
B_{d+1}^{(d+3)}(x) = \frac{1}{d + 2} \sum_{i=1}^{d+2} (x - 1)(x - 2) \ldots (x - i) \ldots (x - d - 2),
\]

omitting the \( i \)th factor.

There is always a vanishing factor for the polynomials in (32), if \( k \) is in the allowed range. Substitution and combination produce the final answer for the second derivative of the scalar free-energy,

\[
\left. \frac{\partial^2 C_k(q, a)}{\partial n^2} \right|_{n=1} = 4(-1)^{d/2+k} \frac{k(d/2 + k)!(d/2 - k)!}{(d+2)(d+1)!}.
\]

The spinor calculation is a little more complicated because the parameter \( a_S \) depends on \( q \) from the start.

In place of (32) there results,

\[
-\frac{1}{2} B_{d+1}^{(d+3)}(b + 3) - 3B_{d+1}^{(d+3)}(b + 2) - \frac{1}{2} B_{d+1}^{(d+3)}(b + 1),
\]

(33)

and for the second derivative of the spinor free energy, \( l + 1 \) is the order of the Dirac operator), simple algebra reveals,

\[
\left. \frac{\partial^2 C_k(q, a)}{\partial n^2} \right|_{n=1} = (-1)^{d/2+l} S \frac{8l(l + 1) - (d + 2)(d - 1)(d/2 + l)!(d/2 - l - 1)!}{(d+2)(d+1)!}.
\]

This yields the formula for the spinor central charge given in section 10.
Appendix B

The summation over \( j \) in (12) is here performed. This has handy formal consequences. The Barnes \( \zeta \)–function has the Hankel contour representation,

\[
\zeta_d(s, a \mid \omega) = \frac{i\Gamma(1-s)}{2\pi} \int_L d\tau \frac{\exp(-a\tau)(-\tau)^{s-1}}{\prod_{i=1}^{d}(1 - \exp(-\omega_i\tau))}.
\]  

(34)

Set \( a \to a \pm (j + 1/2) \) and perform the geometric sum over \( j \). Simple algebra yields,

\[
\sum_{j=0}^{k-1} \left( \zeta_d(s, a + j + 1/2 \mid \omega) + \zeta_d(s, a - j - 1/2 \mid \omega) \right)
= \zeta_{d+1}(s, a + 1/2 - k \mid \omega, 1) - \zeta_{d+1}(s, a + 1/2 + k \mid \omega, 1).
\]

The \( s = 0 \) values provide the specific simplification,

\[
\sum_{j=0}^{k-1} \left( B_d^{(d)}(a + j + 1/2 \mid \omega) + B_d^{(d)}(a - j - 1/2 \mid \omega) \right)
= \frac{1}{d+1} \left( B_{d+1}^{(d+1)}(a + k + 1/2 \mid \omega, 1) - B_{d+1}^{(d+1)}(a - k + 1/2 \mid \omega, 1) \right).
\]  

(35)

Other values for \( s \) give other relations.

Appendix C

It is interesting to follow through the relation between the conformal anomaly on the \( q \)–deformed sphere and the conventional high temperature expansion of the free energy on the open Einstein cylinder. For simplicity, I consider only standard scalars so that many formulae will have appeared before e.g. in [19]. Some of the general polynomial structure behind the specific results like (18) will thereby appear.

The general high temperature expansions in an ultrastatic space–time, \( R \times \mathcal{M} \), were early derived in [26] in terms of the heat–kernel coefficients, \( c_m \), on \( \mathcal{M} \), and extended to any dimension in [28]. The relevant part of the series for the finite temperature free energy is (\( \mathcal{M} \) is supposed closed, and of odd dimension, \( d - 1 \)),

\[
\beta F \sim -\frac{1}{\pi^{1/2}} \sum_{m=0,1,...} \frac{c_m(d)}{2^{2m-d+1}} \zeta_R(d - 2m) \Gamma(d/2 - m) \beta^{2m-d+1} - \frac{1}{2} \zeta'_{\mathcal{M}}(0)
\]

\[
\sim \frac{2\pi}{\pi^{1/2}} \sum_{m=0,1,...} c_m(d) (-1)^{d/2-m} \frac{2^{d-2m-3} B_{d-2m}}{(d-2m)!} \Gamma(d/2 - m) q^{d-2m-1} - \frac{1}{2} \zeta'_{\mathcal{M}}(0)
\]  

(36)
where the dash on the summation indicates that the term \( m = d/2 \) is to be omitted and where \( q \equiv 2\pi/\beta \).

The final term is the formal effect of the zero mode on the thermal circle. In the hyperbolic calculations this mode is dropped as a renormalisation.

Apply (36) to \( M = S^{d-1} \). The transition to the pseudosphere, \( H^{d-1} \), is made shortly.

The heat–kernel coefficients, \( c_m \), have been given, in the form I need, in [29]. They are zero from at least \( m = d/2 \) onwards and

\[
c_m(d) = \frac{2 \Gamma((d-1)/2 - m)}{(d-2 - 2m)! (2m)!} B_{2m}^{(d-1)}(d/2) = \sqrt{\pi} \frac{2^{2m+3-d}}{(d/2 - 1 - m)! (2m)!} B_{2m}^{(d-1)}(d/2), \quad m = 0, 1, \ldots
\]  

(37)

Moreover, the product form of the Bernoulli polynomial \( B_{d-2}^{(d-1)} \) shows that the coefficients actually vanish from \( m = d/2 - 1 \) onwards. All this is well known.

Substituting (37) into (36),

\[
\beta F = 2\pi \sum_{m=0}^{d/2-2} (-1)^{d/2-m} \frac{B_{d-2m} B_{2m}^{(d-1)}(d/2) q^{d-2m-1}}{(2m)! (d-2m)!} - \frac{1}{2} \zeta'_{\mathcal{M}}(0). \quad (38)
\]

I now turn to the conformal anomaly which is, from (12),

\[
C_1(q, a) = \frac{1}{d! q} \left( B_d^{(d)}(d/2 | \omega) + B_d^{(d)}(d/2 - 1 | \omega) \right). \quad (39)
\]

To relate this to (38), I invoke the expansion ([27] p.165, (23)),

\[
B_d^{(d)}(x | \omega) = \sum_{\mu=0}^{d} \binom{d}{\mu} q^{d-\mu} B_{d-\mu} B_{d-\mu-1}^{(d-1)}(x)
\]

\[=-\frac{d}{2} q B_{d-1}^{(d-1)}(x) + \sum_{m=0}^{d/2} \binom{d}{2m} q^{d-2m} B_{d-2m} B_{2m}^{(d-1)}(x)
\]

which gives the structure of the \( q \)-polynomials.

Addition of the two terms in (39) and use of a symmetry of the Bernoulli polynomials produces

\[
\frac{1}{qd!} \left( B_d^{(d)}(d/2 | \omega) + B_d^{(d)}(d/2 - 1 | \omega) \right) = 2 \sum_{m=0}^{d/2} q^{d-2m-1} \frac{B_{d-2m} B_{2m}^{(d-1)}(d/2) (2m)! (d-2m)!}{(d-2m)!} - \frac{2}{d! q} B_d^{(d-1)}(d/2).
\]

(40)
There are various ways of organising the heat–kernel coefficients on the $q$–deformed sphere as functions of $q$, some more expressive than others, but the above is sufficient for my purpose, at the moment.

To finalise the equivalence, it is necessary to convert the sphere free energy, $F$, to the pseudosphere value. To do this, I first pass to densities by dividing the series extensive part of (38) by the volume of the $(d-1)$–sphere, then changing the signs of the resulting (local) $c_m$ coefficients by $(-1)^m$ to give those on the pseudosphere and finally multiplying by the (regularised) volume of $H^{d-1}$ to give the ‘global’ pseudosphere values. The ratio of volumes removes the $(-1)^{d/2} \pi$, leaving a factor of 2. Comparison of the result for $\beta F$ with the conformal anomaly, (40), then shows that the series terms in each are identical.

The final term in $\beta F$, (38) is global and non–extensive. Like the Casimir energy, it is occasioned by the existence of closed (finite) geodesics on the sphere and so would not carry over to the non–compact hyperboloid.

As mentioned earlier, because of the absence of exponentially small corrections to the asymptotic heat–kernel expansion, and also of the termination of this, the high temperature series for the hyperbolic case is actually exact.

The final term in the conformal anomaly, (40), does not appear in $\beta F$. This term is minus twice the Casimir energy on the closed Einstein cylinder, [29], as has been shown more generally in section 8.

References.

1. Beccaria,M. and Tseytlin,A.A. \textit{C}_T for higher derivative conformal fields and anomalies of (1,0) superconformal 6d theories, ArXiv:1705.00305.
2. Klebanov,I.R., Pufu,S.S., Sachdev,S. and Safdi,B.R. JHEP 1204 (2012) 074.
3. Yankielowicz, S. and Zhou,Y. Supersymmetric Rényi Entropy and Anomalies in Six–Dimensional (1,0) Superconformal Theories, ArXiv:1702.03518.
4. Perlmutter,E. A universal feature of CFT Rényi entropy JHEP 03 (2014) 117. ArXiv:1308.1083.
5. Diaz,D.E. Polyakov formulas for GJMS operators from AdS/CFT, JHEP 0807 (2008) 103.
6. Diaz,D.E. and Dorn,H. Partition functions and double trace deformations in AdS/CFT, JHEP 0705 (2007) 46.

\footnote{It does not contribute to the internal energy but adds to the entropy. It has been termed a ‘topological entropy’ by Asorey \textit{et al}, [30,31]}
7. Klebanov, I.R., Pufu, S.S. and Safdi, B.R. JHEP 1110 (2011) 038.

8. Dowker, J.S. Charged Rényi entropy for free scalar fields, J. Phys. A50 (2017) 165401, ArXiv:1512.01135.

9. Camporesi, R. and Higuchi, A. J. Geom. and Physics 15 (1994) 57.

10. Apps, J.S. The effective action on a curved space and its conformal properties PhD thesis (University of Manchester, 1996).

11. Dowker, J.S. Determinants and conformal anomalies of GJMS operators on spheres, J. Phys. A44 (2011) 115402.

12. Dowker, J.S. Effective action on spherical domains, Comm. Math. Phys. 162 (1994) 633.

13. Tseytlin, A.A. Nucl. Phys. B877 (2013) 598.

14. Tseytlin, A.A. Nucl. Phys. B877 (2013) 632.

15. Dowker, J.S. Effective action of conformal spins on spheres with multiplicative and conformal anomalies, J. Phys. A48 (2015) 225402, ArXiv:1501.04881.

16. Casini, H. and Huerta, M. Entanglement entropy for the n-sphere, Phys. Letts. B694 (2010) 167.

17. Dowker, J.S. Rényi entropy on spheres, J. Phys. A: Math. Theor. 46 (2013) 2254.

18. Dowker, J.S. Revivals and Casimir energy for a free Maxwell field (spin-1 singleton) on $R \times S^d$ for odd $d$, ArXiv:1605.01633.

19. Dowker, J.S. Hyperspherical entanglement entropy, J. Phys. A43 (2010) 445402, ArXiv:1007.3865.

20. Dowker, J.S. Spherical Casimir pistons, Class. Quant. Grav. 28 (2011) 155018, ArXiv:1102.1946.

21. Dowker, J.S. Expansion of Rényi entropy for free scalar fields, ArXiv:1412.0549.

22. Osborn, H. and Stergiou, A. C$_T$ for Non–unitary CFTs in higher dimensions, JHEP 06 (2016) 079, ArXiv:1603.07307.

23. Guerrieri, A.L., Petkou, A. C. and Wen, C. The free $\sigma$ CFTs, ArXiv:1604.07310.

24. Gliozzi, F., Guerrieri, A.L., Petkou, A.C. and Wen, C. The analytic structure of conformal blocks and the generalized Wilson–Fisher fixed points, JHEP 1704 (2017) 056, ArXiv:1702.03938.

25. Candelas, P. and Dowker, J.S. Phys. Rev. D19 (1979) 2902.

26. Dowker, J.S. and Kennedy, G. J. Phys. A11 (1978) 895.

27. Nörlund, N.E. Mémoire sur les polynomes de Bernoulli, Acta Mathematica 43 (1922) 121.

28. Dowker, J.S. Finite temperature and vacuum effects in higher dimensions. Class. Quant. Grav. 1 (1984) 359.
29. Chang, P. and Dowker, J. S. *Nucl. Phys.* **B395** (1993) 407.

30. Asorey, M, Beneventano, Calvero-Peláes, I, C.G., D’Ascanio, D. and Santangelo, E.M. *Topological entropy and renormalization group flow in 3-dimensional spherical spaces*, ArXiv:1406.6602.

31. Asorey, M, Beneventano, C.G., D’Ascanio, D. and Santangelo, E.M. *Thermodynamics of conformal fields in topologically non-trivial space-time backgrounds*, ArXiv:1212.6220.