On the structure of double complexes

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Abstract

We study consequences and applications of the folklore statement that every double complex over a field decomposes into so-called squares and zigzags. This result makes questions about the associated cohomology groups and spectral sequences easy to understand. We describe a notion of ‘universal’ quasi-isomorphism and the behaviour of the decomposition under tensor product and compute the Grothendieck ring of the category of bounded double complexes over a field with finite cohomologies up to such quasi-isomorphism (and some variants).

Applying the theory to the double complexes of smooth complex valued forms on compact complex manifolds, we obtain a Poincaré duality for higher pages of the Fröhlicher spectral sequence, construct a functorial three-space decomposition of the middle cohomology, give an example of a map between compact complex manifolds which does not respect the Hodge filtration strictly, compute the Bott-Chern and Aeppli cohomology for Calabi-Eckmann manifolds, introduce new numerical bimeromorphic invariants, show that the non-Kählerness degrees are not bimeromorphic invariants in dimensions higher than three and that the $\partial\bar{\partial}$-lemma and some related properties are bimeromorphic invariants if, and only if, they are stable under restriction to complex submanifolds.

Introduction

Double complexes are linear-algebraic objects which arise in many situations in algebra and geometry. Possibly the most prominent example is the double complex $A_X$ of $\mathbb{C}$-valued forms on a complex manifold $X$. Other examples include the space of forms on a manifold carrying a pair of transverse foliations (cf. Klingler’s recent work on the Chern conjecture [24]) and, more generally, double complexes appearing in the construction of injective (or acyclic) resolutions of simple complexes.

The present article is concerned with applications of the structure theory of double complexes $A$ over a field $K$ which are bounded, that is, have non-zero components only in finitely many bidegrees. This is the case in many of the examples mentioned before. The applications will roughly fall in two classes: algebraic ones, dealing with the general theory of double complexes and objects associated with them, and complex-geometric ones, treating the specific double complexes $A_X$.† We will borrow notation common in the latter context also for the general discussion.

Associated with a bounded double complex, there are several different cohomology theories (in the particular case of $A = A_X$ known as de Rham, Dolbeault, Bott-Chern and Aeppli) which are related by various maps and the Fröhlicher spectral sequences (cf. [3]):

†The case of a bifoliation will be treated in forthcoming work.
If this diagram degenerates (that is, the spectral sequences degenerate on the first page and for all \( p, q \in \mathbb{Z} \), the diagonal maps are isos and the vertical ones are injective, respectively, surjective), the double complex is said to satisfy the \( \partial_1 \partial_2 \)-lemma. A well-known result by Deligne, Griffiths, Morgan and Sullivan [17] states that for \( A \) a bounded double complex of \( K \)-vector spaces, this is the case if, and only if, \( A \) is isomorphic to a direct sum of double complexes of the following two kinds:

- squares

\[
\begin{array}{c}
K \\
\downarrow \text{id} \\
K
\end{array}
\begin{array}{c}
\downarrow \text{id} \\
K \overset{\text{id}}{\longrightarrow} K
\end{array}
\]

- and dots, that is, 1-dimensional complexes, being necessarily concentrated in a single bidgree, with all maps equal to zero.

A somewhat lesser known theorem, going back at least to Khovanov, is the starting point of this article. It states that a similar decomposition actually holds for any bounded double complexes over fields. More precisely, it is given in the following.

**Theorem A (Theorem 3).** For any bounded double complex \( A \) over a field \( K \), there exist unique cardinal numbers \( \text{mult}_S(A) \) and a (non-functorial) isomorphism \( A \cong \bigoplus_S S^\oplus \text{mult}_S(A) \), where \( S \) runs over

- Squares
- and zigzags

\[
\begin{array}{c}
K \\
\downarrow \text{id} \\
K
\end{array}
\begin{array}{c}
\downarrow \text{id} \\
K \overset{\text{id}}{\longrightarrow} K
\end{array}
\begin{array}{c}
\downarrow \text{id} \\
K \overset{\text{id}}{\longrightarrow} K
\end{array}
\begin{array}{c}
\downarrow \text{id} \\
K \overset{\text{id}}{\longrightarrow} K
\end{array}
\begin{array}{c}
\downarrow \text{id} \\
\cdots
\end{array}
\]

An elementary proof of this theorem is given in Section 1. In the first part of Section 2, we study the cohomology diagram \( H(A) \) from the point of view of Theorem A. Summarised briefly, squares do not contribute to any cohomology, even length zigzags correspond to differentials in the Frölicher spectral sequences, while odd length zigzags correspond to de Rham classes (the length of a zigzag being defined as the number of non-zero components). Bott–Chern and Aeppli cohomology count top right, respectively, bottom left, corners in the zigzags. In particular, one obtains the following.

**Corollary B (Proposition 6 and Lemma 8).** For any bounded double complex \( A \) over a field, the dimensions of \( H^d_{dR}(A), H^p_{\partial_1}(A), H^p_{\partial_2}(A), H^p_{BC}(A) \) and \( H^p_{A}(A) \) are linear combinations (with coefficients 1 or 0) of the numbers \( \text{mult}_Z(A) \) for zigzags \( Z \).

Conversely, if one is willing to take into account the two filtrations on the de Rham cohomology, one may reconstruct all multiplicities of zigzags from the de Rham cohomology and the Frölicher spectral sequences. In fact, the refined Betti numbers \( b_d^{p,q} := \dim \text{gr}_{F_1}^{p} \text{gr}_{F_2}^{q} H^d_{dR}(A) \) (which sum up to the \( d \)th Betti number \( b_d := \dim H^d_{dR}(A) \) and specialise to the Hodge numbers in case the double complex satisfies the \( \partial_1 \partial_2 \)-lemma) encode the multiplicities of individual odd
zigzags. In particular, one obtains the following characterisation of degeneration of the Frölicher spectral sequence(s) and Hodge-structures on de Rham cohomology.

**Theorem C (Corollary 7).** Let $A$ be a bounded double complex over a field.

1. The two Frölicher spectral sequences degenerate on the $r$th page if and only if the length of all even zigzags appearing in some (any) decomposition as above is smaller than $2r$.
2. There is a pure Hodge structure of weight $k$ on the total cohomology in degree $d$, that is, 

   $$H_{dR}^d(A) = \bigoplus_{p,q} (F^p_1 \cap F^q_2)H_{dR}^d(A)$$

if and only if all zigzags contributing to $H_{dR}^d(A)$ have length $2|d-k| + 1$ and are concentrated in total degree $d$ and $d + \text{sgn}(d-k)$.

If the involved quantities are finite, then the first point is equivalent to the equality

$$\sum_{p,q \in \mathbb{Z}} \dim E^{p,q}_r = \sum_{k \in \mathbb{Z}} b_k$$

and the second to $b_d = \sum_{p+q=k} b^{p,q}_d$.

Together, Theorems A and C generalise the result of [17] to arbitrary (bounded) double complexes. Apart maybe from the consideration of refined Betti numbers, they have been known to some experts [22, 23, 25, 27, 31, 36], but there used to be no reference available. They have also been used in non-Kähler geometry, albeit in lack of reference as heuristics only [3, 5, 6]. As we exemplify in Corollary 9, the exposition given here allows to turn these heuristics into actual proofs.

With this background at hand, let us survey the main applications: Firstly, the following refined notion of quasi-isomorphism is particularly well behaved (see also [15, 29] for studies of different notions of quasi-isomorphism from a rational homotopy-theoretic point of view).

**Definition D (Definition 10).** A morphism of bounded double complexes $A \rightarrow B$ is called an $E_r$-isomorphism ($r \in \mathbb{Z}_{>0} \cup \{\infty\}$) if it induces an isomorphism on the $r$th page of both Frölicher spectral sequences. Write $A \simeq_r B$ if such a morphism exists.

**Proposition E (Proposition 11, Proposition 12 and Corollary 13).** Let $A, B$ be bounded double complexes over a field $K$.

- $A \simeq_1 B$ if and only if $\text{mult}_S(A) = \text{mult}_S(B)$ for all zigzags.
- An $E_1$-isomorphism $A \rightarrow B$ also induces an isomorphism in Bott–Chern and Aeppli cohomology and more generally under any linear functor from bounded double complexes to $K$-vector spaces that maps squares to 0.

We also give a version for $E_r$-isomorphisms. In particular, $\simeq_r$ is an equivalence relation on bounded double complexes over fields and the equivalence classes of $\simeq_1$ contain all cohomological information.

In many situations, for example, when considering the product of two complex manifolds, one is lead to consider the tensor product of two double complexes. In Section 3, we therefore describe the behaviour of the decomposition under this operation. As an application of theoretical nature, we compute the rings $R_r$ of formal linear combinations of $E_r$-isomorphism classes of bounded double complexes with finite-dimensional $E_r$-page (forcing sum and product in the ring to be induced by direct sum and tensor product). The main result is given as follows.

**Theorem F (Theorem 17).** The ring $R_\infty \cong \mathbb{Z}[U^{\pm 1}, R^{\pm 1}, L^{\pm 1}]$ is a Laurent polynomial ring in three variables. The ring $R_1$ is a (still infinitely generated) quotient of $R_\infty[\{X_l\}_{l \geq 1}, \{Y_l\}_{l \geq 1}]$, where the two sets of generators satisfy $X_l \cdot Y_l = 0$ (and all further relations are given explicitly).
We also state variants where we only consider first-quadrant double complexes or only complexes satisfying the $\partial_1\partial_2$-lemma (see Theorem 18).

In Section 4, we apply the theory to our main example: The double complex $\mathcal{A}_X$ of $\mathbb{C}$-valued forms on a compact complex manifold $X$, which we call the Dolbeault double complex.

Firstly, we review some foundational results on this complex in the light of the theory sketched above, which already allows to strengthen some of them significantly. For example, one may read Serre duality as the statement that the map from $\mathcal{A}_X$ to its dual complex is an $E_1$-quasi isomorphism. From this one gets the following.

**Corollary G.** Let $X$ be a compact complex manifold of dimension $n$. There is a canonical isomorphism

$$E_r^{p,q}(X) \cong (E_r^{n-p,n-q}(X))^\vee,$$

where $E_r(X)$ denotes the $r$th page of the Frölicher spectral sequence. A similar formula holds if one replaces $E_r$ with any linear functor from bounded double complexes to vector spaces that maps squares to $0$.

Next, we compute $\mathcal{A}_X$ up to $E_1$-isomorphism in various cases: For compact Kähler manifolds, for most Calabi–Eckmann manifolds (building on a result by Borel) and for a particular nilmanifold. As a by-product, we obtain the following.

**Corollary H.** Let $M_{u,v}$ be $S^{2u+1} \times S^{2v+1}$ with one of the Calabi–Eckmann complex structures and assume $u < v$. Then

$$\dim H^{p,q}_{BC}(M_{u,v}) = \begin{cases} 2 & \text{if } u \geq 1 \text{ and } (p, q) \in \{(1,1), \ldots, (u,u)\} \\ 1 & \text{if } (p, q) \in \{(0,0), (u+1, u+1), (u+v+1, u+v+1)\} \\ & \text{or } (p, q) \in \{(v,v+1), \ldots, (u+v, u+v+1)\} \\ & \text{or } (p, q) \in \{(v+1,v), \ldots, (u+v+1, u+v)\} \\ 0 & \text{else}. \end{cases}$$

This also yields a formula for Aeppli-cohomology by duality.

**Proposition I.** There is a compact complex 3-fold $X$ and a holomorphic map $\varphi : X \to X$ s.t. $\varphi^*$ does not respect the Hodge filtration on the de Rham cohomology strictly.

Contrast the second statement with the case of Kähler (or $\partial\bar{\partial}$-)manifolds, where any geometrically induced morphism automatically respects the Hodge filtration strictly for linear algebraic reasons. Although likely the expected behaviour for maps of general compact complex manifolds, this appears to be the first example in the literature.

In the final part, we turn to questions about bimeromorphic invariants: In earlier work [38], $\mathcal{A}_X$ was computed up to $E_1$-isomorphism for projective bundles, modifications and blow-ups. Combined with the theory developed here, one may obtain statements about bimeromorphic invariants (see also [8, 33, 40] for other works in this direction).

**Proposition J** (Corollary 27). For $X$ a compact complex manifold of dimension $n$, the multiplicities in $\mathcal{A}_X$ of all zigzags which have a non-zero component in the region

$$\square := \{(p,q) \in \mathbb{Z}^2_{\geq 0} \mid p \in \{0,n\} \text{ or } q \in \{0,n\}\}$$

are bimeromorphic invariants. The same holds for the multiplicities of zigzags which are not dots and have a non-zero component in bidegree $(1,1)$, $(n-1,1)$, $(1,n-1)$ or $(n-1,n-1)$. 

It is known that for a compact complex manifold $X$, the dimensions of $H^p_{BC}(X)$, $H^0, p(X)$ and $H^p, 0(X)$ are bimeromorphic invariants. The invariants here are finer: the multiplicities of any zigzag hitting the boundary region (generalising the previously known invariants) and the multiplicities of non-dot zigzags in the vicinity of the corners of $A_X$. We stress that the latter are numerical invariants not concentrated in degrees $(0, p)$ or $(p, 0)$.

Let us say a property $(P)$ of compact complex manifolds is a bimeromorphic invariant if for two bimeromorphic compact complex manifolds, one satisfies $(P)$ if and only if the other one does. Similarly, $(P)$ is said to be stable under restriction if any complex submanifold of a compact complex manifold satisfying $(P)$ also satisfies $(P)$.

It has been proved that the $\partial\bar{\partial}$-lemma is a bimeromorphic invariant in dimensions up to three and degeneration of the Frölicher spectral sequence at the first page is a bimeromorphic invariant in dimensions up to four (see [33, 40] and also [8]). This admits the following generalisation.

**Theorem K** (Corollary 28). The following properties are bimeromorphic invariants of compact complex manifolds if and only if they are stable under restriction.

- The Frölicher spectral sequence degenerates at stage $\leq r$.
- The $k$th de Rham cohomology groups satisfy a Hodge decomposition
  $H^k_{dR} = \bigoplus_{p+q=k} (F^p \cap F^q) H^k_{dR}$
  for all $k$.
- The $\partial\bar{\partial}$-lemma holds.

In [9], the non-Kählerness-degrees
  $$\Delta^k(X) := \sum_{p+q=k} \dim H^p_{BC}(X) + \dim H^p_{\bar{A}}(X) - 2 \dim H^k_{dR}(X)$$
were introduced and shown to be non-negative with vanishing being equivalent to the $\partial\bar{\partial}$-lemma. We generalise this to arbitrary bounded double complexes for which the involved quantities are finite by a method building on Theorem A and sketched as a heuristic in [5]. In [40], it was shown that these are bimeromorphic invariants in dimensions up to three and asked whether this was true in higher dimensions. We settle this question by proving the following.

**Theorem L** (Corollary 29). Given a blowup $\tilde{X}$ of a compact complex manifold $X$ along a submanifold $Z$, the non-Kählerness degrees satisfy
  $$\Delta^k(\tilde{X}) \geq \Delta^k(X)$$
and equality holds for $k = 0, 1, 2, 2n - 2, 2n - 1, 2n$ (and $k = 3$ if $n = 3$). Equality holds for all $k$ if and only if $Z$ satisfies the $\partial\bar{\partial}$-lemma.

By considering any complex manifold that admits a non-$\partial\bar{\partial}$-submanifold, one sees the following.

**Corollary M.** The numbers $\Delta^k(X)$ are generally not bimeromorphic invariants in dimensions $n \geq 4$.

**Added in Proof:** After submission of the current manuscript, the article [23] appeared, which, following the unpublished note [22], contains a different proof of Theorem A and a discussion similar to the first part of Section 2.
1. Decomposing double complexes

Notations and conventions: The letter $K$ will always denote a field. By a double complex (sometimes also called bicomplex) over $K$, we mean a bigraded $K$-vector space $A = \bigoplus_{p,q \in \mathbb{Z}} A^{p,q}$ with two endomorphisms $\partial_1, \partial_2$ of bidegree $(1,0)$ and $(0,1)$ that satisfy the ‘boundary condition’ $\partial_i \circ \partial_i = 0$ for $i = 1, 2$ and anticommute, that is, $\partial_1 \circ \partial_2 + \partial_2 \circ \partial_1 = 0$.\(^\dagger\) We write $\partial_{p,q}^{i}$ for the map from $A^{p,q}$ to $A^{p+1,q}$ induced by restriction and similarly for $\partial_{p,q}^{j}$. We always assume double complexes to be bounded, that is, $A^{p,q} = 0$ for almost all $(p,q) \in \mathbb{Z}^2$ and denote by $\text{D}C_{\mathbb{B}}^b K$ the category of bounded double complexes over $K$ and $K$-linear maps respecting the grading and the $\partial_i$. If no confusion is likely to result, we say complex instead of double complex over a field $K$.

The following is a standard definition.

**Definition 1.** A (non-zero) double complex $A$ is called indecomposable if there is no non-trivial decomposition $A = A_1 \oplus A_2$.

**Example 2.** The following double complexes over $K$ are indecomposable. The drawn components are supposed to be one-dimensional and the drawn maps to be isomorphisms, while all components and maps not drawn are zero.

**Squares**

\[
\begin{array}{ccc}
A^{p-1,q} & \xrightarrow{\partial_1} & A^{p,q} \\
\downarrow{\partial_2} \downarrow & & \downarrow{\partial_2} \\
A^{p-1,q-1} & \xrightarrow{\partial_1} & A^{p,q-1}
\end{array}
\]

and zigzags

\[
\begin{array}{ccc}
A^{p,q} & \xrightarrow{\partial_1} & A^{p+1,q} \\
\downarrow{\partial_2} \downarrow & & \downarrow{\partial_2} \\
A^{p,q+1} & \xrightarrow{\partial_1} & A^{p,q+1}
\end{array}
\]

For a square or a zigzag $A$, the shape is defined to be the set $S(A) := \{(p,q) \in \mathbb{Z}^2 \mid A^{p,q} \neq 0\}$. The isomorphism class of a square or a zigzag $A$ is uniquely determined by $S(A)$. If we say shape in the following, we always mean the shape of a square or a zigzag. Let us choose a section $S \mapsto C(S)$ which associates to each shape a square (respectively, zigzag) of this shape. For concreteness, one may always choose all non-zero components to be $K$ and all non-zero differentials to be $\pm \text{Id}$.

**Theorem 3.** For every bounded double complex $A$ over $K$, there exist unique cardinal numbers $\text{mult}_S(A)$ and a (non-unique) isomorphism $A \cong \bigoplus C(S)^{\text{mult}_S(A)}$, where $S$ runs over the set of all shapes of squares and zigzags. In particular, every indecomposable complex is isomorphic to a square or a zigzag.

\(^\dagger\)The anti- not essential. In fact, replacing $\partial_1$ by $\partial'_1$ defined by $(\partial'_1)^{p,q} := (-1)^p \partial_1^{p,q}$ we can pass to a commutative double complex (satisfying $\partial_1 \circ \partial_2 = \partial_2 \circ \partial_1$) and vice versa.
It will be convenient to call elementary complex a complex $T$ which is a direct sum of squares or zigzags of a single isomorphism type (that is, a summand in the big sum above). The shape $S(T)$ (defined as before) coincides with the shape of any indecomposable component of $T$. Elementary complexes are intrinsically characterised as those complexes in which every map is an isomorphism or zero and whose undirected support graph is connected.

**Proof.** The main strategy is to define a filtration on an arbitrary double complex which behaves functorially and s.t. the associated graded pieces are less complicated than the original complex. Then one shows that the filtration splits (giving existence of a decomposition into the less complicated pieces) and uses that a filtered isomorphism of filtered double complexes induces an isomorphism of the associated graded pieces (giving uniqueness). This process is repeated several times until the associated graded pieces are elementary complexes.

Consider the functorial ascending filtration $W_\bullet$ on $A$ given in total degree $k$ by the subcomplex generated by all components in total degree $\leq k$, that is,

$$(W_k A)^{p,q} = \begin{cases} A^{p,q} & \text{if } p + q \leq k \\ (\operatorname{im} \partial_1 + \operatorname{im} \partial_2)^{p,q} & \text{if } p + q = k + 1 \\ (\operatorname{im} \partial_1 \circ \partial_2)^{p,q} & \text{if } p + q = k + 2 \\ \{0\} & \text{else.} \end{cases}$$

The filtration $W_\bullet$ splits: Choose (in each degree) complements $D_{\partial_1} \oplus \operatorname{im} \partial_1 \partial_2 = \operatorname{im} \partial_1$ and $D_{\partial_2} \oplus \operatorname{im} \partial_1 \partial_2 = \operatorname{im} \partial_2$ with $\operatorname{im} \partial_1 + \operatorname{im} \partial_2 = \operatorname{im} \partial_1 \partial_2 \oplus (D_{\partial_1} + D_{\partial_2})$. One verifies that $(\operatorname{im} \partial_1 + \operatorname{im} \partial_2) + (\partial_1^{-1} D_{\partial_1} \cap \partial_2^{-1} D_{\partial_2}) = A$ and so one may pick degreewise a complement $A = (\operatorname{im} \partial_1 + \operatorname{im} \partial_2) \oplus C$ with $C \subseteq (\partial_1^{-1} D_{\partial_1} + \partial_2^{-1} D_{\partial_2})$. Defining $B_k$ to be the subcomplex generated in total degree $k$ by $C$, that is

$$B_k^{p,q} := \begin{cases} C^{p,q} & \text{if } p + q = k \\ (D_{\partial_1} + D_{\partial_2})^{p,q} & \text{if } p + q = k + 1 \\ (\operatorname{im} \partial_1 \circ \partial_2)^{p,q} & \text{if } p + q = k + 2 \\ \{0\} & \text{else,} \end{cases}$$

one has $W_\bullet A = \bigoplus_{k \leq \bullet} B_k$.

Given any decomposition into elementary complexes $A \cong \bigoplus T_i$, it induces an isomorphism

$$\operatorname{gr}_k^{W_\bullet} A \cong \bigoplus_{T_i \text{ generated in degree } k} T_i.$$ 

We are thus reduced to the case that $A$ is generated in a single total degree $k$.

Given $A$ generated in a single total degree $k$, for all $p, q$ with $p + q = k$, set $K^{p,q} := (\ker \partial_1 \circ \partial_2)^{p,q}$ and define $K$ to be the subcomplex generated by the $K^{p,q}$. Thus, we have a two-step filtration $W'$:

$$W'_0 = \{0\} \subseteq K \subseteq A = W'_2$$

Given any decomposition $A \cong \bigoplus T_i$ into elementary complexes with distinct support, there are isomorphisms

$$\operatorname{gr}_1^{W'} A \cong \bigoplus_{S(T_i) \text{ zigzag shape}} T_i \quad \operatorname{gr}_2^{W'} A \cong \bigoplus_{S(T_i) \text{ square shape}} T_i.$$ 

Thus, we are reduced to check uniqueness in the two cases that the complex is generated in degree $k$ and either all the $T_i$ are direct sums of zigzags or all the $T_i$ are direct sums of squares. In this last case, the $T_i$ with base in $(p, q)$ has the intrinsic definition as the subcomplex generated by $A^{p,q}$, so the decomposition is unique. Also, $W'_\bullet$ splits: In fact, for every $p, q \in \mathbb{Z}$
with \( p + q = k \), choose a complement \( S^{p,q} \) s.t. \( A^{p,q} = K^{p,q} \oplus S^{p,q} \). Let \( S \) denote the subcomplex generated by the \( S^{p,q} \). By construction, \( S \) splits uniquely as a direct sum of squares generated in degree \( k \) (the subcomplexes generated by the \( S^{p,q} \)) and we have a direct sum decomposition

\[
A = S \oplus K.
\]

It remains to treat the case of a complex generated in a single total degree \( k \) and concentrated in total degrees \( k, k + 1 \). For such a complex the conditions \( \partial_i \circ \partial_i = 0 \) and \( \partial_1 \circ \partial_2 + \partial_2 \circ \partial_1 = 0 \) are vacuous and, after relabelling, it is nothing than a representation of a quiver of type \( \mathbb{A}_n \) (defined below), for which the statement needed follows from Lemma 4 below.

Recall that a quiver of type \( \mathbb{A}_n \) is a directed graph obtained from the following diagram:

\[
1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow n
\]

by assigning a direction to each dash (in the case we are interested in, in an alternating manner). A representation of such a quiver is given by assigning a vector space to each dot and a linear map to each arrow (in accordance with the specified direction).

A non-zero representation of a quiver of type \( \mathbb{A}_n \) is called indecomposable if there is no non-trivial decomposition into subrepresentations. A subset \( S \subseteq \mathbb{N} := \{1, \ldots, n\} \) is called connected if it is the intersection of \( \mathbb{N} \) with some connected real interval. Given a quiver \( Q \) of type \( \mathbb{A}_n \), one obtains an indecomposable representation \( I_S \) for every non-empty connected subset \( S \subset \mathbb{N} \) which is \( K \) on every dot in \( S \) and has all possible maps the identity. For example, for the (up to relabelling unique) quiver of type \( \mathbb{A}_2 \)

\[
1 \longrightarrow 2
\]

the indecomposables obtained in this way are

\[
\begin{align*}
K & \longrightarrow 0 \\
K & \xrightarrow{\text{Id}} K \\
0 & \longrightarrow K.
\end{align*}
\]

**Lemma 4.** Let \( Q \) be a quiver of type \( \mathbb{A}_n \) and \( A \) a representation of \( Q \). There are unique (cardinal) numbers \( \text{mult}_S(A) \) and a (non-unique) isomorphism

\[
A \cong \bigoplus S^{\text{mult}_S(A)},
\]

where \( S \) runs over all connected subsets of \( \mathbb{N} \). In particular, each indecomposable representation \( Q \) is of the form \( I_S \).

As above, we will call representations isomorphic to \( I_S^r \) for some (cardinal) number \( r \) elementary representations.

**Proof.** In the finite-dimensional case, this result is due to Gabriel [19]. It has also been studied in the context of persistent homology in [14], where references for the infinite-dimensional case are given: The decomposition is implied by a theorem of Auslander [10] and uniqueness follows from the Krull–Schmidt–Azumaya theorem [11]. Since [10, 11] contain more general and technical statements than needed here, we sketch an elementary proof here for completeness, which is a quite minor adaption to the infinite-dimensional case of the arguments in [19] and the presentation in [34]:

\[
\begin{align*}
A^{p,q} = K^{p,q} & \oplus S^{p,q} \\
\partial_i & \circ \partial_i = 0 \\
\partial_1 & \circ \partial_2 + \partial_2 \circ \partial_1 = 0
\end{align*}
\]

with \( p + q = k \), choose a complement \( S^{p,q} \) s.t. \( A^{p,q} = K^{p,q} \oplus S^{p,q} \). Let \( S \) denote the subcomplex generated by the \( S^{p,q} \). By construction, \( S \) splits uniquely as a direct sum of squares generated in degree \( k \) (the subcomplexes generated by the \( S^{p,q} \)) and we have a direct sum decomposition

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1 \longrightarrow 2
\]

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\[
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\end{align*}
\]

**Lemma 4.** Let \( Q \) be a quiver of type \( \mathbb{A}_n \) and \( A \) a representation of \( Q \). There are unique (cardinal) numbers \( \text{mult}_S(A) \) and a (non-unique) isomorphism

\[
A \cong \bigoplus S^{\text{mult}_S(A)},
\]

where \( S \) runs over all connected subsets of \( \mathbb{N} \). In particular, each indecomposable representation \( Q \) is of the form \( I_S \).

As above, we will call representations isomorphic to \( I_S^r \) for some (cardinal) number \( r \) elementary representations.

**Proof.** In the finite-dimensional case, this result is due to Gabriel [19]. It has also been studied in the context of persistent homology in [14], where references for the infinite-dimensional case are given: The decomposition is implied by a theorem of Auslander [10] and uniqueness follows from the Krull–Schmidt–Azumaya theorem [11]. Since [10, 11] contain more general and technical statements than needed here, we sketch an elementary proof here for completeness, which is a quite minor adaption to the infinite-dimensional case of the arguments in [19] and the presentation in [34]:
Firstly one proves the lemma ‘by hand’ for the cases $n = 1, 2, 3$: The case $n = 1$ is trivial and the cases $n = 2, 3$ can be handled in a similar manner to the proof of Theorem 3, by defining a canonical filtration and construct a splitting. For example, for $n = 2$, and a representation $A_1 \overset{\alpha}{\rightarrow} A_2$, the filtration is given by
\[(\ker \alpha \rightarrow 0) \subseteq (A_1 \rightarrow \text{im} \alpha) \rightarrow (A_1 \rightarrow A_2)\]
and the splitting is constructed by choosing complements of $\ker \alpha \subseteq A_1$ and $\text{im} \alpha \subseteq A_2$.

In the case $n = 3$, there are, up to isomorphism, three possible quivers:

- $1 \leftarrow 2 \rightarrow 3$ 'source'
- $1 \rightarrow 2 \leftarrow 3$ 'sink'
- $1 \rightarrow 2 \rightarrow 3$ 'river'

We indicate only the filtration in the first case, leaving the splitting and the other cases to the reader. Let
\[A \overset{\alpha}{\leftarrow} C \overset{\beta}{\rightarrow} B\]
a representation of the ‘source’ quiver. The first column of the following table is the promised filtration, whereas the second indicates the support of the associated graded in that step.

| Step | Filtration | Support |
|------|------------|---------|
| 0    | $0 \leftarrow \ker \alpha \cap \ker \beta \rightarrow 0$ | $0 \bullet 0$ |
|      | $0 \leftarrow \ker \alpha \rightarrow \beta(\ker \alpha)$ | $0 \bullet \bullet$ |
|      | $\alpha(\ker \beta) \leftarrow \ker \alpha + \ker \beta \rightarrow \beta(\ker \alpha)$ | $\bullet \bullet 0$ |
|      | $\text{im} \alpha \leftarrow C \rightarrow \text{im} \beta$ | $\bullet \bullet \bullet$ |
|      | $A \leftarrow C \rightarrow B$ | $\bullet 00$ |

For general $n > 3$, let $V$ be a representation of a quiver of type $\mathbb{A}_n$. Denote by $V|_{\{1, \ldots, n-1\}}$ its restriction to the first $n-1$ nodes. Inductively, we can assume $V|_{\{1, \ldots, n-1\}} \cong \bigoplus_{i=1} T_i$ for some (essentially unique) elementary representations $T_i$. Grouping together those $T_i$ with $T_i(n-1) = 0$ and those with $T_i(n-1) \neq 0$, we obtain a decomposition
\[V = V^- \oplus V',\]
where $V^-(n-1) = 0 = V^{-1} = 0$ and $V'$ is increasing up to degree $n-1$, that is, if an arrow goes from $V'(i-1)$ to $V'(i)$ with $1 < i \leq n-1$, it is injective, whereas it is surjective if goes from $V'(i)$ to $V'(i-1)$. Both summands are unique up to isomorphism.
Similarly, denote $V'|_{\{2, \ldots, n\}}$ the restriction of $V'$ to the subquiver given by the last $n - 1$ nodes. Again, this splits by induction as a sum of elementary representations and we obtain a splitting with summands unique up to isomorphism

$$V' = V'' \oplus V^+,$$

where $V^+(1) = 0 = V^+(2)$ and $V''$ is decreasing from degrees 2 to $n$, that is, for $2 \leq i < n$, if a morphism goes from $V''(i)$ to $V''(i + 1)$, it is surjective, and if it goes from $V''(i + 1)$ to $V''(i)$, it is injective. But it is also increasing, hence all morphisms between $V''(2)$ and $V''(n - 1)$ are isomorphisms and we may contract $V''$ to a representation of a quiver of type $A_3$, where we know the statement. □

**Remark 5.** • The proof shows that Theorem 3 remains true if we only assume $A$ to have bounded antidiagonals, that is, for any $k$, there are only finitely many pairs $(p, q)$ with $p + q = k$ s.t. $A^{p,q} \neq 0$. Without this condition, ‘infinite zigzags’ may occur.

• The consideration of cohomological invariants in the next section (Proposition 6) yields another proof for uniqueness of the numbers $\mu(A)$ in Theorem 3.

• Essentially the only thing used in the proof is that one can choose complements of subvector spaces. Therefore, one can adapt Theorem 3 to double complexes in other abelian categories $C$ which are semisimple in a suitable sense, for example, characteristic 0 representations of a finite group $G$. Isomorphism classes of indecomposable complexes are then described by pairs $(S, V)$, where $S$ is a shape and $V$ an isomorphism class of a simple object in $C$ (corresponding to any non-zero component).

• After replacing $\partial_{1}^{p,q}$ by $(-1)^p \partial_{1}^{p,q}$, a double complex is a complex of complexes. So one might hope to get a similar statement for complexes of complexes of complexes and so on. In particular, this would also treat maps between double complexes.

However, as was pointed out to me by L. Hille, there is no longer a discrete classification of elementary complexes. In fact, for any isomorphism $\alpha : K \cong K$ consider the following complex, where all arrows except $\alpha$ are the identity:

These are pairwise non-isomorphic for different $\alpha$.

Even worse, the indecomposables do not have to have the same dimension in every component: For example, a triple complex of the following form, where only non-zero arrows are drawn, cannot be decomposed:
This example also shows that there can be a map from a double complex consisting only of squares s.t. the kernel and image consist only of zigzags.

2. Cohomologies and multiplicities

The previous chapter showed that the isomorphism type of a double complex is uniquely determined by the (cardinal) numbers $\text{mult}_{\mathcal{S}}(A)$. In this section, we show how these numbers relate to more classical cohomological invariants.

In all of the following, $A$ denotes a bounded double complex. We briefly recall several standard constructions.

- The total complex is the simple complex given by summing up the antidiagonals:
  $$A_{\text{tot}}^\bullet := \bigoplus_{p+q=\bullet} A^{p,q}$$
  with differential $d := \partial_1 + \partial_2$.

- The total (or de Rham) cohomology is the cohomology of the total complex:
  $$H^k_{dR}(A) := H^k_{\text{tot}}(A) := H^k(A_{\text{tot}}, d).$$

- The row and column (or Dolbeault) cohomologies are given by taking cohomology with respect of one of the two differentials:
  $$H^p_{\partial_1}(A) := H^p(A^\bullet, q, \partial_1) \quad \text{and} \quad H^q_{\partial_2}(A) := H^q(p^\bullet, \partial_2).$$

- The filtrations by columns and rows
  $$F_1^\bullet := \bigoplus_{p \geq \bullet} A^{p,q} \quad \text{and} \quad F_2^\bullet := \bigoplus_{q \geq \bullet} A^{p,q}$$
  induce filtrations on the total complex on the total cohomology. We will still denote by $F_i$ these last filtrations and call them Hodge filtrations. If not explicitly mentioned otherwise, in the following we will always mean the Hodge filtrations if we write $F_i$.

The filtrations by columns and rows also induce the converging Fölscher spectral sequences, which compute the Hodge filtrations on the total cohomology from the column or row cohomology of the double complex:

$$S_1 : \quad E_1^{p,q} = H^p_{\partial_2}(A) \implies (H^{p+q}_{dR}(A), F_1)$$

$$S_2 : \quad E_1^{p,q} = H^p_{\partial_1}(A) \implies (H^{p+q}_{dR}(A), F_2).$$

- The Bott–Chern and Aeppli cohomologies:
  $$H_{BC}^{p,q}(A) := \left( \frac{\ker \partial_1 \cap \ker \partial_2}{\im \partial_1 \circ \partial_2} \right)^{p,q} \quad \text{and} \quad H_A^{p,q}(A) := \left( \frac{\ker \partial_1 \circ \partial_2}{\im \partial_1 + \im \partial_2} \right)^{p,q}.$$ 

The identity induces natural maps from the Bott–Chern cohomology to row, column and total cohomology and from those three to the Aeppli-cohomology. If the induced map from Bott–Chern to Aeppli cohomology is injective for all $(p, q) \in \mathbb{Z}^2$, $A$ is said to satisfy the $\partial_1 \partial_2$-lemma.

We now investigate these cohomologies in detail for indecomposable double complexes. To describe the results precisely, we will have to label the possible indecomposable complexes, or rather their shapes. Even though shapes are by definition just certain subsets of $\mathbb{Z}^2$, when drawing them we prefer to draw the entire labelled directed support graph of any complex with the given shape.
Zigzags
Given any zigzag $Z$, the length of $Z$ is defined to be the number of elements in its shape $l(Z) := \#S(Z)$. We will distinguish even and odd zigzags, according to their length. A zigzag length one will also be called a dot.

Even zigzags:
Given a zigzag of length $l = 2r$ for some integer $r \geq 1$, we denote its shape as $S_{p,q}^{r}$, where $(p,q)$ is the bidegree of the ‘starting point’, that is of the unique component which has one and only one outgoing arrow and $i$ is 1 or 2, depending on the direction of this outgoing arrow (that is, whether it is $\partial_1$ or $\partial_2$). For example:

The numbers $(p,q) \in \mathbb{Z}^2$, $i \in \{1,2\}$ and $r \in \mathbb{Z}_{\geq 1}$ determine the shape uniquely. Let us fix a zigzag $Z$ of shape $S_{p,q}^{r}$. The total complex is non-zero only in degree $p + q$ and $p + q + 1$ and one may check that the differential is an isomorphism (in fact, it can be described via a triangular matrix with isomorphisms on the diagonal). Therefore, the de Rham cohomology vanishes and the Frölicher spectral sequences have to degenerate. If $i = 1$, the row cohomology vanishes completely and therefore $S_2$ is zero on all pages. On the other hand, $1E^{r,s}_1 \neq 0$ exactly for $(r,s) \in \{(p,q),(p+r,q+r-1)\}$ and therefore $1d_r^{p,q}$, the differential starting in $1E^{p,q}_1$, has to be the only non-zero differential and $Z^{p+r,q+r-1} = \text{im} \ 1d_r^{p,q}$. Similarly, if $i = 2$, the first spectral sequence $S_1$ vanishes on all pages and $Z^{p-r+1,q+r} = \text{im} \ 2d_r^{p,q}$. The following diagram illustrates this for the shape $S_{1,2}^{p,q}$:

Odd zigzags:
Let $Z$ be a zigzag of length $2r + 1$ with $r \geq 0$ which has most components in total degree $d$ and top left component in bidegree $(a,b)$ and bottom right component $(a',b')$. $Z$ is concentrated at most in total degrees $d,d+1$ or $d,d-1$. In the first case, we denote its shape by $S_{d}^{a,b}$ and in the second by $S_{d}^{a',b'}$, for example,
The label \( S^{p,q}_d \) determines a unique shape for any triple \((p, q, d) \in \mathbb{Z}\). In fact, a corresponding zigzag has endpoints \((p, d - p)\) and \((d - q, q)\), length \(|2p + q - d| + 1\) and is concentrated in degrees \(d, d - \text{sgn}(p - q - d)\).

The Frölicher spectral sequences for \( Z \) degenerate on the first page: In fact, since there is an odd number of components but an even number of arrows in each direction, it turns out that the first page of \( S_1 \) and \( S_2 \) has only one non-zero component. If \( Z \) is concentrated in total degrees \(d, d + 1\), the non-zero components are \( E_{1,1}^{a,b} \) and \( E_{2,2}^{a,b} \). Similarly, if \( Z \) is concentrated in degrees \(d, d - 1\), the non-zero components are \( E_{1,1}^{a,b} \) and \( E_{2,2}^{a,b} \). Hence, in both cases \( H^d_{dR}(Z) = 0 \) unless \( \bullet = d \), in which case it is one-dimensional. Writing \( H^d := H_{dR}^d(Z) \), one has

\[
\text{F}^1_\bullet H^d = H^d
\]

for \( \bullet \leq a \) (and zero else) and \( \text{F}^2_\bullet H^d = H^d \) for \( \bullet \leq b' \) (and zero else) in the first case and \( \text{F}^1_\bullet H^d = H^d \) only for \( \bullet \leq a' \) and \( \text{F}^2_\bullet H^d = H^d \) only for \( \bullet \leq b \) in the second case. So for a zigzag of shape \( S^{p,q}_d \), the numbers \( p \) and \( q \) indicate where the Hodge filtrations on \( H_{dR}^d(Z) \) jump.

The de Rham cohomology can be described more explicitly: Let \( \omega^{a,b}, \ldots, \omega^{a',b'} \) be generators of the components of \( Z \) in degree \( d \). If \( Z \) is concentrated in degrees \(d, d + 1\), all \( \omega^{r,s} \) are closed and are, up to a constant, cohomologous, that is, there is a single class which has representatives of \( \left| p + q - d \right| + 1 \) pure types. On the other hand, if \( Z \) is concentrated in degrees \(d, d - 1\), a generator for the non-trivial class in \( H^d_{dR}(Z) \) is given by a linear combination of the \( \omega^{r,s} \) in which all \( \omega^{r,s} \) have non-zero coefficient, that is, one has a class of (generally) very ‘non-pure type’.

We illustrate the above discussion for a zigzag \( Z \) of shape \( S_{p+1,q+1}^{p,q} \):

**Squares:**
A square shape is defined by the position of any corner. We choose the top right one and define \( S^{p,q} := \{(p, q), (p - 1, q), (p, q - 1), (p - 1, q - 1)\} \). For any square \( A \) of shape \( S^{p,q} \), one sees that row and column cohomology vanishes in every degree. Therefore, all higher pages of the Frölicher spectral sequences and the de Rham cohomology have to vanish as well. However, one has

\[
\text{im} \left( \partial_1 \circ \partial_2 \right) = A^{p,q}.
\]

Note also that \( \partial_1 \circ \partial_2 \) vanishes on all zigzags.

Since ‘everything’ is compatible with direct sums, the above discussion of the individual indecomposable complexes yields several results for general \( A \):

**Proposition 6.** Let \( A \) be a bounded double complex over \( K \).

1. Even zigzags: There is an equality

\[
\text{mult}_{S^{p,q}_d}(A) = \dim \text{im} d_{E_1}^{p,q}.
\]

2. Odd zigzags: Given \( \varphi : \bigoplus T_i \to A \) a decomposition into elementary complexes with distinct shapes, denote \( H^{p,q}_{E,d} := H^d_{dR}(\varphi T_i) \subseteq H^d_{dR}(A) \) if \( S(T_i) = S^{p,q}_d \). These spaces split the
filtrations $F_1$ and $F_2$, that is,

$$F_1^d H^d_{dR}(A) = \bigoplus_{r \geq p} H^{r,s}_{\varphi,d}, \quad F_2^d H^d_{dR}(A) = \bigoplus_{s \geq q} H^{r,s}_{\varphi,d}. $$

In particular,

$$\text{mult}_{S^p,q} (A) = \dim_{\mathbb{F}} F_1^p \cap F_2^q H^d_{dR}(A).$$

3. Squares: There is an equality

$$\text{mult}_{S^p,q}(A) = \dim(\text{im} \partial_1 \circ \partial_2) \cap A^{p,q}.$$

It was shown in [17] that a double complex satisfies the $\partial_1 \partial_2$-lemma if and only if it has degenerate Frölicher spectral sequences and the $k$th total cohomology has a pure Hodge structure of weight $k$ if and only if it is a direct sum of squares and zigzags of length 1. The following corollary, together with Theorem 3, is a generalisation of this last equivalence.

**Corollary 7.** Let $A$ be a bounded double complex over a field $K$.

- The Frölicher spectral sequences degenerate at stage $r$ if and only if only shapes of even zigzags of length strictly less than $2r$ have non-zero multiplicity.
- Fix some degree $d \in \mathbb{Z}$ and set $H := H^d_{dR}(A)$ and for any $p, q \in \mathbb{Z}$ set $H^{p,q} := F^p_H \cap F^q_{dR}$. The space $H$ carries a pure Hodge structure of weight $k$ (that is, $H = \bigoplus_{p+q=k} H^{p,q}$) if and only if all odd length zigzag shapes of the form $S^p,q$ which have non-zero multiplicity in $A$ satisfy $p + q = k$. Further, the spaces $H^{p,q}$ admit the following description: If $p + q \geq d$:

$$H^{p,q} = \left\{ \text{classes that admit a representative } \omega \in A^{r,s} \right\},$$

If $p + q \leq d$:

$$H^{p,q} = \left\{ \text{classes that admit a representative } \omega = \sum_{j=p}^{d-q} \omega_j,d-j \text{ with } \omega_r,s \in A^{r,s} \right\}.$$

If the involved quantities are finite, the first point is equivalent to the equality

$$\sum_{p,q \in \mathbb{Z}} \dim E^{p,q}_k = \sum_{k \in \mathbb{Z}} b_k$$

and the second to $b_d = \sum_{p+q=k} b^{p,q}_d$.

Proposition 6.2 states in particular that the Betti numbers (that is, the dimensions of de Rham cohomology) count the odd zigzags contributing to a certain total degree. Similarly, the dimensions of Dolbeault cohomology count the zigzags starting or ending in a certain bidegree. The dimensions of Bott–Chern (respectively, Aeppli) cohomology count ‘corners’, that is, the zigzags meeting a certain bidegree with possibly incoming (respectively, outgoing) arrows. The following is a more technical formulation of these sentences.

**Lemma 8.** Let $A$ be a bounded double complex with a decomposition into elementary complexes with pairwise distinct support $\varphi : \bigoplus A_i \to A$.

- For every $(p, q) \in \mathbb{Z}^2$, the maps induced by $\varphi$

$$A_i \xrightarrow{\text{zigzag}} H^{p,q}_{\partial_1}(A_i) \to H^{p,q}_{\partial_1}(A)$$

$$A_i \xrightarrow{\text{zigzag}} H^{p,q}_{\partial_2}(A_i) \to H^{p,q}_{\partial_2}(A)$$

are isomorphisms.
For every \((p, q) \in \mathbb{Z}^2\), the maps induced by \(\varphi\)

\[
\bigoplus_{A_i \text{ zigzag}} H_{BC}^{p,q}(A_i) \rightarrow H_{BC}^{p,q}(A)
\]

\[
\bigoplus_{A_i \text{ zigzag}} H_A^{p,q}(A_i) \rightarrow H_A^{p,q}(A)
\]

are isomorphisms.

In particular, one sees that both \(H_{BC}^{p,q}(A)\) and \(H_A^{p,q}(A)\) are finite dimensional for all \((p, q) \in \mathbb{Z}^2\) if and only if this is true for \(H_{\partial_1 \partial_2}^{p,q}(A)\) and \(H_{\partial_2}^{p,q}(A)\).

We now illustrate how the theory developed so far can be used to turn certain heuristics (see, for example, [5, 6]) previously used into actual proofs, by reproving, and slightly generalising, the main result of [9] via ‘counting zigzags’ as follows.

For any double complex \(A\) such that all involved quantities are finite, the non-\(\partial_1\partial_2\)-degrees are defined as

\[
\Delta_k(A) := \sum_{p + q = k} (\dim H_{BC}^{p,q}(A) + \dim H_A^{p,q}(A)) - 2 \dim H_{dR}^k(A).
\]

These were studied in [9] for the double complex of forms on a compact complex manifold. By definition, the \(\Delta_k(A)\) are additive under direct sums. One also verifies that they vanish on squares and dots.

If \(Z\) is a zigzag of shape \(S_{d}^{p,q}\) with \(p + q \neq d\), one has

\[
\Delta_k(Z) = \begin{cases} 
|p + q - d| - 1 & \text{if } k = d \\
|p + q - d| & \text{if } k = d - \text{sgn}(p + q - d) \\
0 & \text{else}
\end{cases}
\]

Similarly, for a zigzag \(Z\) of even length with shape \(S_{l,l}^{p,q}\), one may verify that

\[
\Delta_k(Z) = \begin{cases} 
l & \text{if } k = p + q, p + q + 1 \\
0 & \text{else}
\end{cases}
\]

In particular, one obtains a generalisation to arbitrary bounded double complexes over field of the main result of [9], also explaining the name non-\(\partial_1\partial_2\)-degrees:

**Theorem 9.** For any bounded double complex \(A\) over a field with \(H_{BC}^{p,q}(A)\) and \(H_A^{p,q}(A)\) finite for all \(p, q \in \mathbb{Z}\), one has

\[
\Delta_k(A) \geq 0 \quad \text{for all } k \in \mathbb{Z}
\]

and equality holds if and only if \(A\) satisfies the \(\partial_1\partial_2\)-lemma.

Now we turn to a new notion of quasi-isomorphism: We saw that all the cohomological information is encoded in the zigzags and all the information about the zigzags is encoded in the Frölicher spectral sequences and the bifiltered de Rham cohomology. This motivates the following refined version of quasi-isomorphism (cf. [15, 29] for different but related notions).

**Definition 10.** Let \(r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}\). A morphism \(f : A \rightarrow B\) of bounded double complexes is called an \(E_r\)-isomorphism if it induces an isomorphism on the \(r\)th page of both Frölicher spectral sequences (where the 0th page is defined to be the complex itself).
So an $E_0$-isomorphism is just an ordinary isomorphism and we will usually reserve the name $E_r$-isomorphism for the cases $r \geq 1$. For example, a morphism is an $E_1$-isomorphism if and only if it induces a morphism in Dolbeault cohomology and an $E_\infty$-isomorphism if and only if it induces an isomorphism in de Rham cohomology which is strictly compatible with both filtrations (that is, it induces isomorphisms $F^p_i \to F^p_i$). Any $E_r$-isomorphism is also an $E_r$-isomorphism for any $r' > r$.

Our next goal are two results on $E_r$-isomorphisms. One is a characterisation in of the notion of $E_r$-isomorphism in terms of indecomposable summands and the other is an easy-to-check criterion for a functor to send $E_r$-quasi-isomorphisms to isomorphisms.

**Proposition 11.** Let $A$ and $B$ be bounded double complexes over $K$.

(i) A morphism $f : A \to B$ is an isomorphism (respectively, $E_r$-isomorphism for $r \geq 1$) if and only if for any decomposition into elementary complexes with pairwise distinct support $A \cong \bigoplus C(S)^{\oplus \text{mult}_S(A)}$, $B \cong \bigoplus C(S)^{\oplus \text{mult}_S(B)}$, the induced map $f_{S,S} : C(S)^{\oplus \text{mult}_S(A)} \to C(S)^{\oplus \text{mult}_S(B)}$ is an isomorphism for all $S$ (respectively, for all $S$ which are zigzag shapes of odd length or of even length $\geq 2r$).

(ii) There exists an isomorphism (respectively, $E_r$-isomorphism for $r \geq 1$) between $A$ and $B$ if and only if $\text{mult}_S(A) = \text{mult}_S(B)$ for all shapes (respectively, all zigzag shapes $S$ of odd length or of even length $\geq 2r$).

In particular, the relation

$$A \simeq_r B :\iff \text{there exists an } E_r\text{-isomorphism } A \to B$$

is an equivalence relation.

For example, two complexes are $E_1$-isomorphic if and only if all zigzags occur with the same multiplicity and $E_\infty$-isomorphic if and only if all odd zigzags occur with the same multiplicity.

**Proposition 12.** An $E_r$-isomorphism $f : A \to B$ ($r \geq 1$) induces an isomorphism after applying any linear functor from double complexes to vector spaces which sends squares and even-length zigzags of length $< 2r$ to the zero vector space.

In particular, one immediately recovers the following fact (see also [4, 7], for previous special cases).

**Corollary 13.** An $E_1$-isomorphism induces an isomorphism in Bott–Chern and Aeppli cohomology.

The proof of these results will rest on the fact that, after some simplification steps, one can bring $E_r$-isomorphisms to ‘triangular shape’. To make this precise, we need a few technical preparations:

For any shape $S$, denote by $\text{deg } S := \min\{p + q \mid (p,q) \in S\}$ the degree in which an elementary complex of this shape is generated. Let $S$ be a finite set of shapes. We are going to define a total ordering on $S$ as follows.

**Order by total degree:** Set $S < S'$ if $\text{ord } S < \text{ord } S'$.

**Order within a total degree:** Let $S_d \subseteq S$ the subset of all shapes of degree $d$. Set $p_d := \min\{p \mid (p,d-p) \in S\}$ and similarly $q_d := \max\{q \mid (d-q,q) \in S\}$.

Since $S_d$ is finite, a total order on $S_d$ is the same as writing down all elements in a list, with elements appearing later being declared greater than previous ones. We compile such a list as follows.
1. For each $n$, starting with $n = 0$, add, in arbitrary order, all the shapes with $2n + 1$ elements in $S_d$, $n + 1$ of which have total degree $n$ (that is, first all dots, then all length-three-shapes contributing to total degree $d$, etc.).

2. For $p$, starting at $p_d$ and going up to $d - q_d$: Add all even length shapes in $S_d$ containing $(p, d - p)$ and $(p, d - p + 1)$ but not $(p - 1, d - p + 1)$, ordered by increasing length (that is first $S_{2,1}^{d-p_d}$, then $S_{2,2}^{d-p_d-1}$ etc. After that $S_{2,d}^{d-p_d-1}$ etc.).

3. Analogously, for $q$ starting at $q_d$ and going down to $d - p_d$: Add all even length shapes in $S_d$ containing $(d - q, q)$ and $(d - q + 1, q)$, ordered by increasing length.

4. For each $n$, starting with $n$ as large as possible and decreasing, add, in arbitrary order, all the shapes with $2n + 1$ elements in $S_d$, $n + 1$ of which have total degree $n + 1$.

5. Add all square shapes in $S_d$, in arbitrary order.

One then verifies the following tedious but elementary lemma.

**Lemma 14.** For two elementary complexes $Z, Z' \in S$, if $S \subseteq S'$, then $\text{Hom}(Z, Z') = 0$.

Using this, let us prove the above results.

**Proof of Proposition 11.** Fix two decompositions $A \cong A^{<2r} \oplus A^{\geq r}$, $B \cong B^{<2r} \oplus B^{\geq r}$ in which the first summand consists of squares and even length zigzags of length $< 2r$ (it is trivial if $r = 0$) and the second one of odd zigzags and even ones of length $\geq 2r$. Denote by $S$ the set of all shapes occurring in $A^{\geq r}$ and $B^{\geq r}$ (a posteriori, it would be sufficient to consider just one of them).

For the ‘if’-part of (i), note that, since we only care for the behaviour after applying $E_r$, we may assume that $A = A^{\geq r}$ and $B = B^{\geq r}$. Now, ordering the shapes in $S$ as before and decomposing $A \cong \bigoplus_{S \in S} C(S)^{\oplus \mult_S(A)}$ and $B \cong \bigoplus_{S \in S} C(S)^{\oplus \mult_S(B)}$, the resulting ‘matrix’ $(f_{S,S'})_{S,S' \in S}$ of induced maps $f_{S,S'} : C(S)^{\oplus \mult_S(A)} \rightarrow C(S)^{\oplus \mult_S(B)}$ has triangular shape by Lemma 14 and the diagonal entries $f_{S,S}$ are invertible by assumption. So, $f$ induces an isomorphism $A^{\geq r} \cong B^{\geq r}$, and therefore necessarily one after applying $E_r$.

For the ‘only if’ part of (i), one uses that a map between two elementary complexes of the same shape is an isomorphism if and only if it is an isomorphism in some bidegree and that the map $f_{S,S}$ may, at least in a certain bidegree, canonically factored through the spaces occurring in Proposition 6. For example, let $A'$ and $B'$ be elementary complexes of shape $S^{p,q}$ in some decomposition of $A$ and $B$. The inclusion $A' \subseteq A$ induces an isomorphism $A^{p,q} \cong \text{im} \partial_1 \partial_2 \cap A^{p,q}$ and similarly the projection $B \rightarrow B'$ an isomorphism $\text{im} \partial_1 \partial_2 \cap B^{p,q} \cong B^{p,q}$. In particular, whenever $f$ induces an iso on $\text{im} \partial_1 \partial_2$ in degree $(p, q)$, it induces an iso $A' \cong B'$ and vice versa. The cases of even and odd zigzags may be treated similarly.

The ‘only if’ part of (ii) is a consequence of the first since any $E_r$-isomorphism induces an isomorphism on the spaces whose dimensions encode the relevant multiplicities. On the other hand, given decompositions $\varphi : A \cong \bigoplus_{S \in S} C(S)^{\oplus \mult_S(A)}$, $\psi : B \cong \bigoplus_{S \in S} C(S)^{\oplus \mult_S(B)}$ and $\mult_S(A) = \mult_S(B)$ for all $S$ in a certain set of shapes $S$, one may construct a map inducing an isomorphism on all elementary components with shapes in $S$ as $\varphi$ followed by the projection to $\bigoplus_{S \in S} C(S)^{\oplus \mult_S(A)}$, followed by the inclusion and $\psi^{-1}$. \[\Box\]

**Proof of Proposition 12.** Let $F$ be a functor as in the statement. Since it is linear, it automatically commutes with direct sums. Decompose source and target of $f$ into $A = A^{eq} \oplus A^{\geq r}$, $B = B^{eq} \oplus B^{\geq r}$. Since $f$ is an $E_1$-quasi-isomorphism, it sends $A^{eq}$ to $B^{eq}$, and therefore, it makes sense to consider the reduced map $\tilde{f} : A^{\geq r} \cong A^{eq} \rightarrow B^{eq} \cong B^{\geq r}$. As $F$ sends squares to 0, $F(f)$ may be identified with $F(\tilde{f})$ and we may assume that $A, B$ do not contain any squares.
Decomposing $A = A^{zig} \oplus C(S)^{\oplus \text{mult}_S(A)}$ and $B = B^{zig} = \oplus C(S)^{\oplus \text{mult}_S(B)}$ into zigzags, the induced map $F(f)$ can be written as a ‘matrix’ with rows and columns indexed by zigzag shapes. Again using the previous ordering, this matrix has triangular shape with invertible elements on the diagonal as a result of Proposition 11 and Lemma 14. □

3. Tensor product and Grothendieck rings

We are going to investigate the behaviour of the above decomposition under tensor product and compute the Grothendieck rings of several categories of double complexes.

For any $r \geq 0$ we consider the following categories.

- $\text{DC}_{fr}^{E_r-fin,b}$ bounded double complexes over $K$ s.t. the $E_r$-page of both Frölicher spectral sequences is finite dimensional, localised at $E_r$-isomorphisms.
- $\text{DC}_{\partial_1,\partial_2}^{fin,b}$ bounded double complex satisfying the $\partial_1,\partial_2$-lemma s.t. the $E_\infty$ page of both Frölicher spectral sequences is finite dimensional, localised at $E_\infty$-isomorphisms.

Note that $\text{DC}_{fr}^{E_0-fin,b}$ is just the subcategory of $\text{DC}_{fr}^{fin,b}$ consisting of finite dimensional complexes and that by Corollary 7 we could replace $E_\infty$ by $E_r$ for any $r \geq 1$ in the definition of $\text{DC}_{fr}^{\partial_1,\partial_2-fin,b}$ since the Frölicher spectral sequences degenerate.

The following lemma is a consequence of the Künneth formula and the compatibility of (co)homology with direct sums.

**Lemma 15.** Let $A, B$ be double complexes. For every $r \geq 0$, there are functorial isomorphisms

\[
\begin{align*}
\iota_{E_r}(A \oplus B) &\cong \iota_{E_r}(A) \oplus \iota_{E_r}(B), \\
\iota_{E_r}(A \otimes B) &\cong \iota_{E_r}(A) \otimes \iota_{E_r}(B).
\end{align*}
\]

As a consequence of this lemma, direct sum and tensor product are well defined on the categories defined above and we can define the following Grothendieck rings (that is, formal sums of isomorphism classes modulo the relation $[A] + [B] = [A \oplus B]$ and with multiplication induced by tensor product):

\[
\begin{align*}
\mathcal{R}_r := K_0(\text{DC}_{fr}^{E_r-fin,b}), \\
\mathcal{R}_{\partial_1,\partial_2} := K_0(\text{DC}_{fr}^{\partial_1,\partial_2-fin,b}).
\end{align*}
\]

Given a double complex $A$ with suitable finiteness conditions, write $[A]$ for its class in one of these rings. Abusing notation slightly, given a shape $S$ we write $[S]$ for the class of some elementary complex of rank 1 with shape $S$. By Theorem 3, equations of the form

\[
[A] = \sum_{S \text{ shape}} \text{mult}_S(A)[S]
\]

hold.

So, as an abelian group, $\mathcal{R}_0$ (respectively, $\mathcal{R}_\infty$, respectively, $\mathcal{R}_r$ for any $r \in \mathbb{Z}_{\geq 0}$) is free with basis given by all shapes (respectively, all odd zigzag shapes, respectively, all odd zigzag shapes and all even zigzag shapes of length $\geq 2r$). We now describe the multiplicative structure.
Proposition 16. For a square shape $S^{p,q}$ and any other shape $S$, there is an equality in $\mathcal{R}_0$: 
\[
[S^{p,q}] \cdot [S] = \sum_{(r,s) \in S} [S^{p+r,q+s}].
\]
In particular, the subgroup generated by square shapes is an ideal $I_{S^q}$ in $\mathcal{R}_0$. There are equalities in $\mathcal{R}_1 \cong \mathcal{R}_0/I_{S^q}$:
\[
[S_d^{p,q}] \cdot [S_d^{p',q'}] = [S_d^{p+p',q+q'}],
\]
\[
[S_{1,l}^{p,q}] \cdot [S_{1,l}^{p',q'}] = [S_{1,l}^{p+p',q+q'}] + [S_{1,\min(l,l')}^{p+p+\max(l,l'),q+q-\max(l,l')+1}],
\]
\[
[S_{2,l}^{p,q}] \cdot [S_{2,l}^{p',q'}] = [S_{2,l}^{p+p',q+q'}] + [S_{2,\min(l,l')}^{p+p'-\max(l,l')+1,q+q'+\max(l,l')}],
\]
\[
[S_d^{p,q}] \cdot [S_d^{p',q'}] = 0,
\]
\[
[S_d^{p,q}] \cdot [S_d^{p',q'}] = [S_d^{p+p',q+d-p}],
\]
\[
[S_d^{p,q}] \cdot [S_d^{p',q'}] = [S_d^{p+d-q,q+q'}].
\]
Proof. To see the equation for squares, let $Z$ be an indecomposable complex with shape $S^{p,q}$ and $Z'$ an indecomposable complex with shape $S$. Choose a basis element $s \in Z^{p-1,q-1}$. In particular, $\partial_1 \partial_2 s \neq 0$. Given a basis element $\alpha^{r,s}$ of any non-zero component $Z'^{r,s}$, the element 
\[
\partial_1 \partial_2 (s \otimes \alpha^{r,s}) = \partial_1 \partial_2 s \otimes \alpha^{r,s} + s \otimes \partial_1 \partial_2 \alpha^{r,s} \in Z^{p,q} \otimes Z'^{r,s} \otimes Z^{p-1,q-1} \otimes Z'^{r+1,s+1}
\]
is not zero, so one obtains 
\[
\text{mult}_{S^{p+r,q+s}} (Z \otimes Z') \geq 1
\]
whenever $Z'^{r,s} \neq 0$ and so one has 
\[
\dim(Z \otimes Z') \geq 4 \cdot \sum_{(r,s) \in S} \text{mult}_{S^{p+r,q+s}} (Z \otimes Z')
\]
\[
\geq 4 \cdot \dim Z'
\]
\[
\geq \dim(Z \otimes Z')
\]
and hence equality, which implies the formula.

The other equations all follow from a consideration of the Frölicher spectral sequences for two indecomposable complexes with the given shapes and using Proposition 6. In each case there are only very few (that is, $\leq 2$) non-zero entries on each page. We only do this for the most terrible looking formula, the others follow similarly.

Let $l \leq l'$ and $Z, Z'$ elementary double complexes of rank one with shapes $S_{1,l}^{p,q}$ and $S_{1,l'}^{p',q'}$. Then, $2E_r(Z) = 2E_r(Z') = 0$ for all $r \geq 1$. Therefore, $2E_r(Z \otimes Z') = 0$ for all $r \geq 1$ and so 
\[
\text{mult}_{S_{2,l}^{a,s}} (Z \otimes Z') = 0
\]
for all $a, b \in \mathbb{Z}, d \in \mathbb{Z}_{\geq 0}$ and 
\[
\text{mult}_{S} (Z \otimes Z') = 0
\]
for all odd length zigzag shapes $S$, since the total cohomology has to vanish.

Considering the other spectral sequence, one has $1E_r(Z) = 0$ for $r > l$ and if $r \leq l$, it is non-zero only in bidegrees $(p,q)$ and $(p+1,q-l+1)$, where it has dimension 1. Similarly,
1 \mathcal{E}_r(Z') = 0 \text{ for } r > l' \text{ and if } r \leq l', \text{ it is non-zero only in bidegrees } (p,q) \text{ and } (p + l', q - l' + 1), \text{ where it has dimension } 1.

In summary, \( 1 \mathcal{E}_r(Z \otimes Z') \) = 0 for all \( r > l = \min(l, l') \) and non-zero in bidegrees \( (p + l', q + q') \), \( (p + p' + l, q + q' - l + 1) \), \( (p + p' + l + l', q + q' - l - l' + 2) \), so the two necessary non-zero differentials of bidegree \( (l, -l + 1) \) on page \( 1 \mathcal{E}_l \) have to start at bidegrees \( (p + p', q + q') \) and \( (p + p' + l', q + q' - l' + 1) \) and all other differentials vanish. □

The last equalities can be memorised by the following rules, alluding to the parity of the zigzag length:

\[
\begin{align*}
ext & = \text{even}, & \text{odd} & = \text{odd} \\
\text{even} & \cdot \text{odd} & = \text{odd} \\
\text{odd} & \cdot \text{even} & = \text{even}.
\end{align*}
\]

These multiplication rules allow several immediate conclusions.

- The maps of abelian groups \( R_{\infty} \rightarrow R_r \) induced for any \( r \in \mathbb{Z} > 0 \) by the equations (*) are maps of rings, that is, \( R_r \) is a \( R_{\infty} \)-algebra.
- For \( r \in \mathbb{Z} > 0 \), the rings \( R_r \) are not finitely generated as \( \mathbb{Z} \)-algebras. In fact, for any hypothetical finite set of generators, there would be a natural number \( l_0 \) s.t. the length of all even length zigzag occurring as summands in the generators is bounded by \( l_0 \). By the multiplication rules, this would also be true for all sums of products of these hypothetical generators.

We will now describe \( R_r \) for all \( r \geq 1 \) using generators and relations. We invite the reader to draw pictures of all generators and relations to see what is going on.

**Theorem 17.** Let \( \mathcal{P} := \mathbb{Z}[R^\pm, U^\pm, L^\pm, \{X_l\}_{l \in \mathbb{Z} > 0}, \{Y_l\}_{l \in \mathbb{Z} > 0}] \) be the polynomial ring in infinitely many generators with inverses for three variables. The map

\[
\Phi : \mathcal{P} \rightarrow R_1
\]

\[
R \rightarrow [S_{0,1}^0],
U \rightarrow [S_{1,0}^1],
L \rightarrow [S_{1,1}^1],
X_l \rightarrow [S_{1,l}^0],
Y_l \rightarrow [S_{2,l}^0]
\]

is surjective and the kernel is described by enforcing the relations

\[
\begin{align*}
X_l \cdot Y_l & = 0 \\
R \cdot X_l & = L \cdot X_l \\
R \cdot Y_l & = L^{-1} \cdot Y_l \\
U \cdot X_l & = L^{-1} \cdot X_l \\
U \cdot Y_l & = L \cdot Y_l \\
X_l \cdot X_l & = X_l + R^{l'} \cdot U^{-l'+1} \cdot X_l \\
Y_l \cdot Y_l & = Y_l + R^{-l'+1} \cdot U^{l'} \cdot Y_l
\end{align*}
\]

for all \( l' \geq l \) in \( \mathbb{Z} > 0 \).
Via the ‘same’ map, \( \mathcal{R}_\infty \) and \( \mathcal{R}_{\partial_1, \partial_2} \) are identified with \( \mathbb{Z}[R^\pm 1, U^\pm 1, L^\pm 1] \) and \( \mathbb{Z}[R^\pm 1, U^\pm 1] \).

**Proof.** That the map \( \Phi \) is well defined and the indicated relations have to hold in the image follows from Proposition 16.

Let us show that the induced map \( \Phi \) from \( \Phi \) modulo the above relations is indeed an isomorphism: Given a polynomial \( P \in \mathcal{P} \) we can, using only the given relations, always arrange it to a sum

\[
P = P_\infty + P_X + P_Y,
\]

where

\[
P_\infty \in \mathbb{Z}[R^\pm 1, U^\pm 1, L^\pm 1],
\]

\[
P_X \in \bigoplus_{l>0} \mathbb{Z}[R^\pm 1, U^\pm 1]X_l,
\]

\[
P_Y \in \bigoplus_{l>0} \mathbb{Z}[R^\pm 1, U^\pm 1]Y_l.
\]

Then \( \Phi(P_\infty) \) consists only of odd zigzags, \( \Phi(P_X) \) (respectively, \( \Phi(P_Y) \)) only of even zigzags contributing to some page in \( S_1 \) (respectively, \( S_2 \)). In particular, the given representation is necessarily unique. One concludes using the following two observations.

- Multiplication by \( \Phi(R_{\pm 1}) \) and \( \Phi(U_{\pm 1}) \) acts as horizontal and vertical shifts on all other zigzag shapes.
- The powers of \( \Phi(L_{\pm 1}) \) (respectively, the images of \( X_l \) and \( Y_l \)) are precisely a set of representatives for the equivalence classes of odd zigzags (respectively, even zigzags contributing to \( S_1 \) or \( S_2 \)) up to shifts. \( \Box \)

In many ‘real-world’ cases, all double complexes one might be interested in are concentrated in the first quadrant, that is, \( A^{p,q} = 0 \) whenever \( p < 0 \) or \( q < 0 \). We denote by \( \text{DC}_{\mathcal{E}_r, \text{fin}, b^+} \) and \( \text{DC}_{\mathcal{E}_r, \partial_1, \partial_2, b^+} \) the analogues of the categories at the beginning of this sections where we assume in addition that all double complexes are (at least from the \( E_r \)-page on) concentrated in the first quadrant. The corresponding Grothendieck rings will be denoted by \( \mathcal{R}_{\pm}^+ \) and \( \mathcal{R}_{\partial_1, \partial_2}^+ \). The proof of the following corollary is very similar to that of the previous, so we omit it as follows.

**Corollary 18.** Let \( \mathcal{P}^+ := \mathbb{Z}[R, U, L, \widetilde{L}, \{\widetilde{X}_l\}_{l \in \mathbb{Z} > 0}, \{\widetilde{Y}_l\}_{l \in \mathbb{Z} > 0}] \) be a polynomial ring in infinitely many generators. The map

\[
\Phi^+: \mathcal{P}^+ \longrightarrow \mathcal{R}_{\pm}^+
\]

\[
\begin{align*}
R & \mapsto [S_1^{0,1}] \\
U & \mapsto [S_1^{1,0}] \\
L & \mapsto [S_1^{1,1}] \\
\widetilde{L} & \mapsto [S_1^{0,0}] \\
\widetilde{X}_l & \mapsto [S_{1,l}^{0,l-1}] \\
\widetilde{Y}_l & \mapsto [S_{2,l}^{0,l-1}] 
\end{align*}
\]
is surjective and the kernel is described by enforcing the relations

\[ R \cdot U = L \cdot \tilde{L} \]
\[ \tilde{X}_l \cdot \tilde{Y}_l' = 0 \]
\[ R \cdot \tilde{X}_l = L \cdot \tilde{X}_l \]
\[ R \cdot \tilde{Y}_l = \tilde{L} \cdot \tilde{Y}_l \]
\[ U \cdot \tilde{X}_l = \tilde{L} \cdot \tilde{X}_l \]
\[ U \cdot \tilde{Y}_l = L \cdot \tilde{Y}_l \]
\[ X_l' \cdot \tilde{X}_l = U^{l-1} \tilde{X}_l + R^{l'} \cdot \tilde{X}_l \]
\[ Y_l' \cdot \tilde{Y}_l = R^{l-1} \tilde{Y}_l + U^{l'} \cdot \tilde{Y}_l \]

for all \( l' \geq l \) in \( \mathbb{Z}_{>0} \).

Via the ‘same’ map, \( R^+ \) and \( R^+_{\partial_1, \partial_2} \) are identified with \( \mathbb{Z}[R, U, L, \tilde{L}]/(R \cdot U - L \cdot \tilde{L}) \) and \( \mathbb{Z}[R, U] \).

The inclusion \( R^+_1 \hookrightarrow R_1 \) is induced by the map

\[ p^+ \longrightarrow p \]

defined by

\[ R \mapsto R, \quad U \mapsto U, \quad L \mapsto L, \quad \tilde{L} \mapsto R U L^{-1}, \quad \tilde{X}_l \mapsto U^{l-1} X_l, \quad \tilde{Y}_l \mapsto R^{l-1} Y_l. \]

4. Complex manifolds

In this section we apply the theory developed so far to the double complex

\[ A_X = (A^\bullet \cdot, \partial, \bar{\partial}) \]

of \( \mathbb{C} \)-valued differential forms on a complex manifold \( X \) of pure dimension \( n \).

Firstly, we reinterpret several known results on \( A_X \) in terms of the decomposition into indecomposables. Some of them are also summarised briefly in [5]:

- **Real structure:** Consider the following involution on the category \( DC^\phi_{\mathbb{C}} \):

\[ A = \bigoplus_{p,q \in \mathbb{Z}} A^{p,q}, \partial_1, \partial_2 \big) \mapsto \bar{A} := \bigoplus_{p,q \in \mathbb{Z}} A^{q,p}, \partial_2, \partial_1 \big), \]

where for some \( \mathbb{C} \)-vector space \( V \), the conjugate space \( \overline{V} \) is the same space as a set, but with scalar multiplication twisted by complex conjugation, that is, \( \alpha \cdot v := \overline{\alpha} \cdot V \).

Complex conjugation of forms induces a canonical isomorphism \( \sigma : A_X \cong \overline{A}_X \), that is, \( A_X \) is a fixed point of this involution. In particular, by Proposition 11 (ii), for every shape \( S \) occurring with multiplicity \( m \) in \( A_X \), its reflection along the diagonal occurs with the same multiplicity. Note that for any double complex \( A \) of \( \mathbb{C} \)-vector spaces, the involution \( A \mapsto \overline{A} \) interchanges \( F_1 \) and \( F_2 \). Therefore, if \( A \) has a real structure (that is, is a fixed point of the involution), we will write \( F := F_1 \) and \( \overline{F} := F_2 \). In such a case, the Frölicher spectral sequence \( S_2 \) is completely determined by \( S_1 \). In particular, if one wants to check whether a map between double complexes with real structure is an \( E_r \)-isomorphism, it suffices to do so for one of the two Frölicher spectral sequences, provided that the map is compatible with the real structures.
• Dimension: As $X$ is of complex dimension $n$, the complex $\mathcal{A}_X$ is concentrated in degrees $(p, q)$ with $n \geq p, q \geq 0$. In particular, only shapes that lie in that region can have non-zero multiplicity in $\mathcal{A}_X$.

From now on, let us assume that $X$ is compact.

• Finite dimensional cohomology: The complex $\mathcal{A}_X$ itself need not be finite dimensional. However, Dolbeault cohomology (or alternatively Bott–Chern and Aeppli cohomology) can be shown to be finite dimensional by elliptic theory and thus all zigzags have finite multiplicity by Lemma 8. Note that it suffices to know finite dimensionality for Bott–Chern or Aeppli or Dolbeault cohomology, it is automatically implied for the others by the general theory.

• Duality: Let $\mathcal{D} \mathcal{A}_X$ denote the ‘dual complex’ of $\mathcal{A}_X$, given by $\mathcal{D} \mathcal{A}_X^{p,q} := (\mathcal{A}_X^{p,n-q})^\vee := \text{Hom}_{C}(\mathcal{A}_X^{p,n-q}, \mathbb{C})$ with differentials $\partial^\vee, \bar{\partial}^\vee$, defined by $(\partial^\vee)^{p,q} := (\varphi \mapsto (-1)^{p+q+1} \varphi \circ \bar{\partial}^{n-p, n-q})$ and similarly for $\bar{\partial}^\vee$.

By construction and as we know that all zigzag shapes have finite multiplicity, for a zigzag shape occurring with a certain multiplicity in $\mathcal{A}_X$, the shape obtained by reflection at the antidiagonal $p + q = n$ occurs with the same multiplicity in $\mathcal{D} \mathcal{A}_X$.

As $X$ is a complex manifold, it is automatically oriented. In particular, integration yields a pairing:

$$\mathcal{A}_X^{p,q} \otimes \mathcal{A}_X^{n-p,n-q} \longrightarrow \mathbb{C}$$

$$(\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta.$$ 

This induces maps $\Phi^{p,q} : \mathcal{A}_X^{p,q} \rightarrow \mathcal{D} \mathcal{A}_X^{p,q}$ and the signs are set up so that it yields a morphism of complexes $\Phi : \mathcal{A}_X \rightarrow \mathcal{D} \mathcal{A}_X$. Serre duality ([35, Theorem 4]) implies that this map is an $E_1$-isomorphism. Thus, by Proposition 11, every zigzag shape occurs with the same multiplicity in $\mathcal{A}_X$ and $\mathcal{D} \mathcal{A}_X$.

• Connectedness: The shape $\{(0, 0)\}$ (and therefore, by duality, also the shape $\{(n, n)\}$) has multiplicity $\#\pi_0(X)$, as functions satisfying $df = 0$ are constant on each connected component.

• Only dots and squares in the corners: For $\omega$ a function or an $(n-1, 0)$ form, $\bar{\partial} \partial \omega = 0$ implies $\bar{\partial} \omega = 0$ (this follows from the maximum principle for pluriharmonic functions and Stokes’ theorem, see, for example, [26, p. 7f].). Combining this with the two dualities, one sees that the implications

$$P \in S \text{ and } \text{mult}_S(\mathcal{A}_X) \neq 0 \implies S = \{P\}$$

hold for $P \in \{(0, 0), (n, 0), (0, n), (n, n)\}$ and zigzag shapes $S$.

• Disjoint unions and direct products: Given two compact complex manifolds $X, Y$, there is a canonical identification $\mathcal{A}_X \cup \mathcal{A}_Y = \mathcal{A}_X \oplus \mathcal{A}_Y$ and the natural map $\mathcal{A}_X \otimes \mathcal{A}_Y \rightarrow \mathcal{A}_{X \times Y}$ is an $E_1$-quasi isomorphism. The first statement is clear and the second is the Künneth-formula for Dolbeault-cohomology (see, for example, [20]). In particular, the multiplicities of zigzags in $\mathcal{A}_{X \times Y}$ can be computed via Proposition 16.

Remark 19. In dimension 2, more restrictions are known: In fact, for compact complex surfaces, the Frölicher spectral sequence degenerates (see [12, Theorem IV.2.7]). Thus, for $n = 2$, $\text{mult}_S(\mathcal{A}_X) = 0$ for all even zigzag shapes $S$. Also, as a consequence of [12, Theorem IV.2.6], a compact complex surface $S$ either satisfies the $\partial \bar{\partial}$-lemma or exactly two zigzags of length 3 occur in $\mathcal{A}_S$ (which have shapes $S_{1^2}$ and $S_{2^2}$).
Even though this was little more than a restatement of known results, we can now draw the first (apparently new) consequences:

The duality along the antidiagonal implies the following strengthened version of Serre-duality.

**Corollary 20.** Let $H$ be any linear functor $DC^b_T$ to vector spaces that maps squares to zero. Then the integration pairing induces a canonical isomorphism

$$H(A_X) \cong H(DA_X)$$

In particular,

$$E^{p,q}_r(X) \cong (E^{n-p,n-q}_r(X))^\vee$$

for all higher pages of the Frölicher spectral sequence.

**Proof.** The general statement is a direct consequence of Proposition 12. Note that $E^{p,q}(DA) \cong E^{n-p,n-q}(A_X)^\vee$, since taking duals is exact, so it commutes with taking cohomology. □

**Remark 21.** Corollary 20 also allows to rederive the duality between $H_{BC}(X)$ and $H_A(X)$ from Serre duality, without the need for Bott–Chern and Aeppli–Laplacians. If one is only interested in the higher pages of the Frölicher spectral sequence, one can use directly the second part of the argument and classical Serre duality for $E_1$ and does not need to resort to Proposition 12. The corresponding equality of the dimensions one gets for the spaces on the antidiagonals of the $r$th page, that is, $\dim E^k_r = \dim E^{2n-k}_r$ for $E^k_r := \bigoplus_{p+q=k} E^{p,q}_r$, has been shown by Dan Popovici in [30] using analytic techniques.

**Theorem 22.** Let $X$ be an $n$-dimensional compact complex manifold. There is a functorial $3$-space decomposition, orthogonal with respect to the intersection form,

$$H^n_{dR}(X) = H^{n,0}_\partial(X) \oplus H^{mid}(X) \oplus H^{0,n}_\partial(X),$$

where

$$H^{mid}(X) := \frac{\ker d \cap (A^{n-1,1}_X + \ldots + A^{1,n-1}_X)}{\text{im} d \cap (A^{n-1,1}_X + \ldots + A^{1,n-1}_X)} \subseteq H^n_{dR}(X).$$

**Proof.** First of all, if there is such a decomposition, it is by definition functorial, since the maps from the three spaces on the right to de Rham cohomology are induced by the identity and it is orthogonal for bidegree reasons. In order to see that the maps are injective and their image spans the whole space, one chooses a decomposition and counts zigzags: To the de Rham cohomology all odd-zigzags with most components on the $n$th antidiagonal contribute. On the other hand, to $H^{n,0}_\partial(X)$, only zigzags with non-zero component in $(n,0)$ and zero component in $(n,1)$ contribute. By the ‘only dots and squares in the corners’ statement, these are only dots. Similarly, the only contribution to $H^{0,n}_\partial(X)$ comes from dots in degree $(0,n)$. Furthermore, $H^{mid}$, considered as a functor from bounded double complexes to vector spaces which is compatible with direct sums, vanishes on squares and all zigzags except those of shape $S^{p,q}_n$ where either $p+q \leq k$ and $S^{p,q}_n \subseteq \{(n-1,1), \ldots, (1,n-1)\}$ or $p+q \geq k$ and $S^{p,q}_n \cap \{(n-1,1), \ldots, (1,n-1)\} \neq \emptyset$. In particular, the pairwise intersections of the three spaces are $\{0\}$. Finally, again by the dimension restrictions and the ‘only dots and squares in the corners’ statement, if a shape $S^{p,q}_n$ contains $(n,0)$ (respectively, $(0,n)$) and an element in $\{(n-1,1), \ldots, (1,n-1)\}$, it has multiplicity zero. So the three spaces also span $H^n_{dR}(X)$.
We now calculate $A_X$ up to $E_1$-isomorphism for several examples of compact complex manifolds:

$\partial\bar{\partial}$-manifolds:
Assume $A_X$ satisfies the $\partial\bar{\partial}$-lemma (for example, $X$ Kähler or more generally in class $\mathcal{C}$, that is, bimeromorphic to a Kähler manifold). By Proposition 7, $A_X$ is then a direct sum of squares and dots and for any $k$ there is the Hodge decomposition $H^k_{dR}(X) = F^p \cap F^q$, so we consider $H_{dR}(X) := \bigoplus_{k \in \mathbb{Z}} H^k_{dR}(X)$ as a bigraded vector space and we make it into a double complex by declaring the differentials to be 0. The multiplicities of the dots coincide with the Hodge numbers, which also coincide with the dimensions of the $F^p \cap F^q$.

Therefore,

$$ (A_X, \partial_1, \partial_2) \simeq_1 (H_{dR}(X), 0, 0), $$

for example, if $S_g$ denotes a Riemann surface of genus $g \geq 0$:

Calabi–Eckmann manifolds:
In [13], for any $\alpha \in \mathbb{C} \setminus \mathbb{R}$ and $u, v \in \mathbb{Z}_{\geq 0}$ a manifold $M^\alpha_{u,v}$, was defined by putting a complex structure on the product $S^{2u+1} \times S^{2v+1}$ such that the projection

$$ S^{2u+1} \times S^{2v+1} \longrightarrow \mathbb{P}^u \times \mathbb{P}^v $$

is a holomorphic fibre bundle with fibre $T = \mathbb{C}/(\mathbb{Z} + \alpha \mathbb{Z})$. Explicitly, they can be realised as the quotient

$$ M^\alpha_{u,v} = ((\mathbb{C}^{u+1}\setminus\{0\}) \times (\mathbb{C}^{v+1}\setminus\{0\}))/\sim $$

where

$$ (x, y) \sim (e^{tx}, e^{\alpha t} y) \quad \text{for any } t \in \mathbb{C}. $$

Since the following discussion does not depend on the choice of $\alpha$, we write $M_{u,v}$ for the product $S^{2u+1} \times S^{2v+1}$ equipped with any of these complex structures.

**Example 23.** $M_{0,0}$ is a complex torus and $M_{0,v}$ or $M_{u,0}$ are Hopf manifolds.

In the following, we assume $u < v$ for simplicity.

In [21], Borel computed the first page of the Frölicher spectral sequence for $M_{u,v}$.
Numerically, the result reads:

$$ h^{p,q}_{M_{u,v}} = \begin{cases} 
1 & \text{if } p \leq u \text{ and } q = p, p + 1 \\
1 & \text{if } p > v \text{ and } q = p, p - 1 \\
0 & \text{else.}
\end{cases} $$
The following is a picture of the $E_1$-page (without differentials) for $u = 1, v = 2$:

$$
\begin{array}{cccc}
q & 4 & \mathcal{C} & \mathcal{C} \\
3 & \mathcal{C} & \mathcal{C} \\
2 & \mathcal{C} & \mathcal{C} \\
1 & \mathcal{C} & \mathcal{C} \\
0 & 0 & 1 & 2 & 3 & 4 & p
\end{array}
$$

Since the underlying topological space of $M_{u,v}$ is a product of spheres, the Betti-numbers are given by

$$b_i^{M_{u,v}} = \begin{cases} 1 & \text{if } i = 0, 2u + 1, 2v + 1, 2(u + v) + 2 \\ 0 & \text{else.} \end{cases}$$

In particular, the Frölicher spectral sequence degenerates at the second stage and the multiplicities of all zigzags can be determined combinatorially. (For the odd-length zigzags, use that the de Rham cohomology is one-dimensional, so knowing the breakpoints of each filtration individually determines the non-zero bidegree of the associated bigraded.) This results in:

$$\text{mult}_S(A_{M_{u,v}}) = \begin{cases} 1 & \text{if } S = S_0^0, S_{2u+1}^w \\ 1 & \text{if } S = S_{1,1}^{p+1} \text{ for } 0 \leq p < u \\ 0 & \text{in all other cases not determined by duality and real structure.} \end{cases}$$

The result on Bott–Chern and Aeppli-cohomology stated in the introduction can be read off from this using Lemma 8.

Continuing with the case $u = 1, v = 2$, a picture of all the zigzags in $A_{M_{1,2}}$ looks as follows:

$$
\begin{array}{cccc}
q & 4 & \mathcal{C} & \mathcal{C} \\
3 & \mathcal{C} & \mathcal{C} & \mathcal{C} \\
2 & \mathcal{C} & \mathcal{C} & \mathcal{C} \\
1 & \mathcal{C} & \mathcal{C} & \mathcal{C} \\
0 & 0 & 1 & 2 & 3 & 4 & p
\end{array}
$$

**Remark 24.** For the cases $u$ or $v$ equal to 0, this coincides with a description of the zigzags in the Dolbeault double complex obtained (under a conjecture) by G. Kuperberg on MO.†

**Remark 25.** For $u = v > 0$, it seems that one cannot just argue numerically since in this case $b_{M_{u,v}}^{2u+1} = 2$, and so it does not suffice to know in which degree the Hodge filtration and its

†[https://mathoverflow.net/questions/25723/](https://mathoverflow.net/questions/25723/)
conjugate jump individually. More precisely, the methods used here only allow to say that there are either two zigzags with shapes $S_{2u+1}^{a,n,u}$, $S_{2u+1}^{a,n,u}$ or two zigzags with shapes $S_{2u+1}^{a,n,u}$, $S_{2u+1}^{a,n+1,u}$.

**Nilmanifolds**

A rich and well-studied class of examples of compact complex manifolds which are very accessible from the computational point of view are complex nilmanifolds: ¹ They are quotients $X = G / \Gamma$ of a nilpotent (real) Lie groups with left invariant complex structure by a lattice. They have the distinguished finite-dimensional subcomplex $\mathcal{A}_X^{(n)} \subseteq \mathcal{A}_X$ of invariant differential forms. In many (and conjecturally all) cases [16], the inclusion $\mathcal{A}_X^{(n)} \subseteq \mathcal{A}_X$ is an $E_1$-isomorphism. In particular, this holds in complex dimensions up to 3 [18].

One can construct an $n$-dimensional nilmanifold starting from the a tuple $(g, J, L)$, where

- $g$ is a nilpotent real Lie-algebra of real dimension $2n$ which has a $\mathbb{Q}$-structure, that is, there is a Lie-algebra $g_{\mathbb{Q}}$ over the rationals s.t. $g = g_{\mathbb{Q}} \otimes \mathbb{R}$. Concretely, this means that there exists a basis $(e_1, \ldots, e_{2n})$ with rational structure constants $c_{i,j}^k \in \mathbb{Q}$ (defined by $[e_i, e_j] = \sum_{k=1}^{2n} c_{i,j}^k e_k$).
- $J : g \to g$ is a linear map which is an almost complex structure (that is, $J^{-1} = -1$) and integrable ($[Jx, Jy] = J[Jx, y] + J[x, Jy] + [x, y]$ for all $x, y \in g$).
- $L$ is a lattice in $g_{\mathbb{Q}}$.

Via the exponential map, the pair $(g, L)$ is turned into a simply connected nilpotent Lie group $G$ with lattice $\Gamma$ (see, for example, [32, Theorem 2.12.]) and $J$ induces a left invariant complex structure on $G$. The space $G / \Gamma$ is then a nilmanifold. $J$ induces a decomposition $g_{\mathbb{C}} = g_{\mathbb{R}}^{1,0} \oplus g_{\mathbb{R}}^{0,1}$ of $g_{\mathbb{C}} = \text{Hom}(g, \mathbb{C})$ into $i$ and $-i$ eigenspaces. The left invariant $(p, q)$ forms are identified with the spaces $\Lambda^p g_{\mathbb{R}}^{p,0} \otimes \Lambda^q g_{\mathbb{R}}^{0,q}$ and the exterior differential with the one defined in degree one by $d \omega(X, Y) := -\omega([X, Y])$.

For a concrete example, let us consider the Lie-algebra $h_9$, that is, the 6-dimensional real vector space with basis $e_1, \ldots, e_6$ and Lie bracket given by:

$$[e_1, e_2] = e_4, \quad [e_1, e_3] = [e_2, e_4] = -e_6$$

where the non-mentioned brackets are defined by antisymmetry or 0. We endow this with an almost complex structure defined by:

$$Je_1 = e_2, \quad Je_3 = e_4, \quad Je_5 = e_6.$$ 

A $\mathbb{Q}$-structure is given by the $\mathbb{Q}$-span of the $e_i$ and a lattice $L$ by their $\mathbb{Z}$-span.

Let us denote the dual basis vectors to the $e_i$ by $e^i$. A basis of $g_{\mathbb{C}}^{1,0}$, the $i$ eigenspace of $J$ in the complexified Lie algebra $g_{\mathbb{C}}$, is obtained by setting:

$$\omega^1 := e^1 - ie^2, \quad \omega^2 := e^3 - ie^4, \quad \omega^3 := e^5 - ie^6.$$ 

Plugging in the definitions, one checks that the differential is given on $g_{\mathbb{R}}^{1,0}$ by the following rules:

$$d\omega^1 = 0, \quad d\omega^2 = \frac{1}{2} \bar{\omega}^1 \land \omega^1, \quad d\omega^3 = \frac{i}{2} (\omega^1 \land \bar{\omega}^2 + \bar{\omega}^1 \land \omega^2).$$

This and the proof of Theorem 3, or Proposition 6, allow the computation of the $E_1$-isomorphism type of the Lie-algebra double complex (and hence of $\mathcal{A}_X$ for the corresponding

¹ In addition to the references we give in the text, we refer to [3, Chapter 1.7.2 and Chapter 3] for a survey of the literature

¹ This Lie algebra and background information can be found, for example, in [2].
nilmanifold $X$). We spare the reader the calculation and just give the result. All zigzag shapes whose multiplicity is not determined via duality and real structure from the ones listed here have multiplicity zero.

| Shape of zigzag | Generators |
|-----------------|------------|
| $S^0,0$         | $1$        |
| $S^1,0$         | $\omega^1$ |
| $S^2,0$         | $\omega^1 \wedge \omega^2$ |
| $S^3,1$         | $\omega^2 \wedge \overline{\omega^1} - \omega^1 \wedge \overline{\omega^2}, i\omega^2 \wedge \overline{\omega^3} + \omega^1 \wedge \overline{\omega^3} - \omega^3 \wedge \overline{\omega^1}$ |
| $S^2,0$         | $\omega^2 \wedge \omega^3, \omega^3 \wedge \overline{\omega^2} + \omega^2 \wedge \overline{\omega^3}, \overline{\omega^2} \wedge \overline{\omega^3}$ |
| $S^3,0$         | $\omega^1 \wedge \omega^2 \wedge \omega^3$ |
| $S^3,1$         | $\omega^1 \wedge \omega^2 \wedge \overline{\omega^3}$ |
| $S^2,2$         | $\omega^3 \wedge \overline{\omega^1}, \omega^2 \wedge \overline{\omega^3} - \omega^3 \wedge \overline{\omega^2}$ |

The following encodes this information (without explicit generators) in a diagram. Again, zigzags determined by duality and real structure are not drawn.

We conclude this example by remarking that this particular nilmanifold is interesting because it admits an endomorphism which is not strictly compatible with the Hodge filtration on the de Rham cohomology.† In fact, if $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ is defined by

$$
e_i \mapsto \begin{cases} 
  e_3 - e_6 & i = 1 \\
  e_4 + e_5 & i = 2 \\
  0 & \text{else.}
\end{cases}$$

This is a map of Lie-algebras compatible with the complex structure and maps $L$ to $L$ and therefore induces a holomorphic map $\bar{\varphi} : G/\Gamma \rightarrow G/\Gamma$. The cohomology can be computed by left invariant forms and the induced morphism $\varphi^* : H^1_{dR}(G/\Gamma, \mathbb{C}) \rightarrow H^1_{dR}(G/\Gamma, \mathbb{C})$ is determined by

†Recall that a linear map of filtered vector spaces $\varphi : (V, F) \rightarrow (\tilde{V}, \tilde{F})$ is called strict if $\varphi F^p = \tilde{F}^p \cap \text{im} \varphi$ for all $p$. 


the dual map $\varphi: g^\vee_C \rightarrow g^\vee_C$, which in turn is determined by its values on $g^{1,0}$, namely,
\[
\begin{align*}
\omega^1 &\mapsto 0 \\
\omega^2 &\mapsto \omega^3 \\
\omega^3 &\mapsto i\omega^1.
\end{align*}
\]

Using this, one checks
\[
\varphi^* F^1 H^1_{dR}(G/\Gamma, \mathbb{C}) = \{0\} \neq \langle [\omega^1] \rangle = F^1 H^1_{dR}(G/\Gamma, \mathbb{C}) \cap \varphi^* H^1_{dR}(G/\Gamma, \mathbb{C}),
\]
that is, $\varphi^*$ is not strict. This phenomenon can be seen as an incarnation of the fact that the decomposition of double complexes into indecomposables is not functorial (although certainly this theory is not needed to give this example). While it is certainly the expected behaviour for a general morphism between general complex manifolds (in sharp contrast to $\partial\bar{\partial}$-manifolds, where morphisms are automatically strict), it seems that no example of a non-strict morphism between compact manifolds has appeared in the literature before.

Blowups:
For a double complex $A$ and an integer $i \in \mathbb{Z}$, denote by $A[i]$ the shifted double complex, which has underlying graded $A[i]_{p,q} = A_{p-i,q-i}$. In [38] the following results were shown (with explicit maps).

**Theorem 26.** Let $X$ be a compact, connected manifold of dimension $n$.

1. Projective bundles: Let $V \rightarrow X$ be a vector bundle of rank $m+1$ and $\mathbb{P}(V) \rightarrow X$ the associated projective bundle. Then
\[
A_{\mathbb{P}(V)} \simeq A^n_{\mathbb{C}} \otimes A_X \simeq \bigoplus_{i=0}^m A_X[i].
\]

2. Modifications: For a surjective holomorphic map $f: Y \rightarrow X$ with $Y$ compact connected of dimension $n$, one has
\[
A_Y \simeq A_X \oplus A_Y/f^* A_X.
\]

3. Blowups: For a submanifold $Z \subseteq X$ of codimension $r \geq 2$, let $\tilde{X}$ be the blowup of $X$ in $Z$. Then
\[
A_{\tilde{X}} \simeq A_X \oplus \bigoplus_{i=1}^{r-1} A_Z[i].
\]

Pictorially, these results can be understood as follows, where in the first case, the small square represents $A_X$ and there are $m$ copies of it, and in the second case, the big square represents $A_X$ and the smaller ones $A_Z$ and there are $r-1$ copies.

One can use the theory developed so far to refine several corollaries in [8, 33, 40]. A key point in the proofs is (as in the cited works) the fact that every bimeromorphic map can be
factored as a sequence of blowups and blowdowns \([1, 39]\) and can be deduced easily from the above pictorial description of the double complex of a blowup.

**Corollary 27.** The numbers \(\text{mult}_S(X)\) are bimeromorphic invariants whenever \(S\) is a zigzag shape and there is a point \((a, b) \in S\) s.t.

1. \(a \in \{0, n\}\) or \(b \in \{0, n\}\) or
2. \((a, b) \in \{(1, 1), (1, n - 1), (n - 1, 1), (n - 1, n - 1)\}\) and \(S\) is not a dot.

**Proof.** By the weak factorisation theorem, it is enough to show that these numbers do not change when passing from \(A_X\) to the double complex of a blowup \(A_\tilde{X}\). The statement is then a direct consequence of the pictorial description and the ‘only squares and dots in the corners’ for \(Z\) (the centre of the blowup). \(\square\)

In particular, this corollary gives a unified proof of the known fact that \(H^{p,0}_\partial(X), H^0_{\overline{\partial}}\) and \(H^{2n-2}_{B,C}\) are bimeromorphic invariants, but also shows that the same holds for \(E^{p,0}_r(X)\) and \(E^{0,p}_r\) for any \(r \geq 1\). Maybe somewhat surprisingly, it shows that bimeromorphic invariance also holds for certain numbers away from the boundary region, such as the refined Betti-numbers \(b^{2,2}_a\) or \(b^{n-1,p}_n\) for \(p > 1\) or the dimension of the image of differentials in the Frölicher spectral sequence ending in degree \((n - 1, 1)\).

Let us say a property \((P)\) of compact complex manifolds is stable under restriction if any complex submanifold of a compact complex manifold satisfying \((P)\) also satisfies \((P)\).

**Corollary 28.** The following properties are bimeromorphic invariants of compact complex manifolds if and only if they are stable under restriction.

- The Frölicher spectral sequence degenerates at stage \(\leq r\).
- The \(k\)th de Rham cohomology groups satisfy a Hodge decomposition
  \[ H^k_{dR} = \bigoplus_{p+q=k} F^p \cap \overline{F}^q \]
  for all \(k\).
- The \(\partial\overline{\partial}\)-lemma holds.

In particular, the degeneracy of the Frölicher spectral sequence on the first page is a bimeromorphic invariant in dimension \(\leq 4\), because it always holds for surfaces. In general, this corollary only implies that in dimension \(n\), degeneracy at page \(n - 2\) is bimeromorphically invariant. The second and third properties are implied to be bimeromorphic invariants in dimensions \(\leq 3\).

To my knowledge, it is in general an open problem whether or not all three properties hold for submanifolds.

**Proof.** Recall that by Corollary 7, the first statement is equivalent to only odd zigzags and even zigzags up to a certain length having non-zero multiplicity, the second to ‘the only odd zigzags are dots’ and the third to the first and second at the same time. By the weak factorisation theorem, it suffices to consider the situation of a blow up \(\tilde{X}\) of \(X\) along a submanifold \(Z \subset X\) of codimension \(\geq 2\). Looking at the pictorial description given above, it is clear that any of these three statements holds for \(A_\tilde{X}\) if and only if it holds for \(A_Z\) and \(A_X\). Since one can always increase the codimension of \(Z\) without changing any of the desired
properties of the ambient manifold by considering \( \mathbb{Z} \times \{ \text{pt} \} \subseteq X \times Y \) for appropriate \( Y \) (for example, satisfying the \( \partial \bar{\partial} \)-Lemma), this implies the statement.\(^1\)

In \([40]\), it was shown that the non-Kählerness degrees \( \Delta^k(X) := \Delta^k(A_X) \) are bimeromorphic invariants in dimension \( \leq 3 \) and it was asked whether the same holds in dimensions \( n \geq 4 \). We may answer this question negatively as follows.

Take a Hopf surface \( H \) and consider the blowup \( H' \) of \( H \times \{ \text{pt} \} \) in \( H \times \mathbb{P}^2 \). It is instructive to compute the double complexes of all spaces involved in this example up to \( E_1 \)-isomorphism, using the pictorial interpretation of Theorem 26 and the explicit description of the \( E_1 \)-isomorphism type of \( A_H \), so let us do this.

Taking the product with \( \mathbb{P}^2 \) yields a direct sum of three (shifted) copies:

\[
A_{H \times \mathbb{P}^2} \simeq_A A_H \oplus A_H[1] \oplus A_H[2].
\]

Taking the blowup along \( H \times \{ \text{pt} \} \) doubles the middle copy:

\[
A_{H'} \simeq_A A_H \oplus A_H[1]^{\oplus 2} \oplus A_H[2].
\]

The double complex of \( H \) has been computed as

\[
A_H \simeq_A \begin{array}{ccc}
  \mathbb{C} & \mathbb{C} & \mathbb{C} \\
  \mathbb{C} & \mathbb{C} & \mathbb{C} \\
  \mathbb{C} & \mathbb{C} & \mathbb{C}
\end{array}
\]

In particular, one may now count zigzags and see that

\[
\Delta^4(H \times \mathbb{P}^2) = \Delta^4(H) + \Delta^2(H) + \Delta^0(H) = \Delta^2(H) = 2 < 4 = 2\Delta^2(H) = \Delta^4(H').
\]

In fact, one has the following more general result, proven by similar means as a consequence of Theorem 26 and the calculation of the non-Kählerness degrees in Section 2.

**Corollary 29.** Given a blowup \( \tilde{X} \) of a compact complex manifold \( X \) along a submanifold \( Z \), the non-Kählerness degrees satisfy

\[
\Delta^k(\tilde{X}) \geq \Delta^k(X)
\]

and equality holds for \( k = 0, 1, 2, 2n - 2, 2n - 1, 2n \) (and \( k = 3 \) if \( n = 3 \)). Equality holds for all \( k \) if and only if \( Z \) satisfies the \( \partial \bar{\partial} \)-lemma. In particular, they are generally not bimeromorphic invariants in dimensions \( n \geq 4 \).

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\(^1\)In the preprint version of the article, the proof was left to the reader. Recently, a more detailed version of the proof for the \( \partial \bar{\partial} \)-case, using Corollary 29 below, has been spelled out in the expository note \([28]\).
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