Composite Anomaly Detection via Hierarchical Dynamic Search

Benjamin Wolff, Tomer Gafni, Guy Revach, Nir Shlezinger, and Kobi Cohen

Abstract—Anomaly detection among a large number of processes arises in many applications ranging from dynamic spectrum access to cybersecurity. In such problems one can often obtain noisy observations aggregated from a chosen subset of processes that conforms to a tree structure. The distribution of these observations, based on which the presence of anomalies is detected, may be only partially known. This gives rise to the need for a search strategy designed to account for both the sample complexity and the detection accuracy, as well as cope with statistical models that are known only up to some missing parameters. In this work we propose a sequential search strategy using two variations of the Generalized Log Likelihood Ratio statistic. Our proposed Hierarchical Dynamic Search (HDS) strategy is shown to be order-optimal with respect to the size of the search space and asymptotically optimal with respect to the detection accuracy. An explicit upper bound on the error probability of HDS is established for the finite sample regime. Extensive experiments are conducted, demonstrating the performance gains of HDS over existing methods.

I. INTRODUCTION

Dynamic search of rare events with unknown characteristics in an ambient noise has many potential applications, including transmission opportunities in dynamic spectrum access [1], attacks and intrusions in communication and computer networks [2], and anomalies in infrastructures that may indicate catastrophes [3]. Here, we consider the problem of detecting an anomalous process (target), for which there is uncertainty in the distribution of observations among M processes (cells), where we assume that we can get access to aggregated observations that are drawn from a general distribution that depends on a chosen subset of processes.

At each time, the decision maker observes a chosen subset of processes that conforms to a predetermined tree structure. The distribution of the aggregated observations depends on the size of the chosen subset and the presence of the anomaly, forming a composite hypothesis case, where measurements drawn when observing a subset of processes follow a common distribution parametrized by an unknown vector when containing the target. The objective is to determine a search strategy that adaptively selects which subset to observe and when to terminate the search in order to minimize a Bayes risk that accounts for sample complexity and detection accuracy.

The hierarchical structure model is relevant in settings where a massive number of data streams can be observed at different levels of granularity. For example, financial transactions can be aggregated at different temporal and geographic scales [4]. In visual monitoring applications, sequentially determining areas to zoom in or out can quickly locate anomalies by avoiding giving each pixel equal attention [5]. Another relevant application is heavy hitter detection in internet traffic monitoring, where a small number of flows accounts for most of the total traffic, and an efficient search for these heavy hitters involves a tree structure where each node represents an aggregated flow [6]. Other applications include direction of arrival estimation [7] and system control [8].

The key to utilizing the hierarchical structure of the search space to its full extent, is to determine the number of samples one should observe at each level of the tree, and when to zoom in or out on the hierarchy. It is of particular interest to explore whether accurate detection can be obtained by examining a diminishing fraction of the search space as the search space grows. In [9], the case where the distribution of the measurements is fully known was considered. For such settings, the Information-Directed Random Walk (IRW) algorithm was proposed and shown to be asymptotically optimal with respect to the detection accuracy and order-optimal with respect to the number of cells. Since the anomalous hypothesis in our case is composite, the IRW policy serves as a benchmark for the performance of our setting, as also demonstrated in the numerical experiments. The recent studies [10]–[12] considered hierarchical search under unknown observation models. The key difference is that the search strategies in [10], [11] are based on a sample mean statistic, which fails to detect a general anomalous distribution with a mean close to the mean of the normal distribution. The work in [12] does not assume a structure on the abnormal distribution, and uses the Kolmogorov-Smirnov statistic, which fails to utilize the parametric information considered in our setting.

This work considers for the first time the task of hierarchical anomaly detection over a general and known distribution model with unknown parameters. Here, the measurements can take continuous values and the decision maker is allowed to sample an aggregated subset of cells that conforms to a tree structure. To cope with this observation model in a dynamic search setting, we develop a novel sequential search strategy, coined Hierarchical Dynamic Search (HDS), which uses two
carefully chosen statistics to harness the information on the null hypothesis and the structure of the hierarchical samples, allowing it to achieve asymptotically optimal performance.

In particular, HDS uses the fixed sample size Generalized Log Likelihood Ratio (GLLR) statistic for the high level nodes test and the sequential Adaptive Log Likelihood Ratio (ALLR) statistic for the leaf nodes test. The ALLR statistic, introduced by Robbins and Siegmund [13, 14], builds upon the one-stage delayed estimator of the unknown parameter; i.e., the density of the n-th observation is estimated based on the previous n - 1 observations, while the current observation is not included in this estimate. As opposed to the GLLR, the ALLR preserves the martingale properties. This allows one to choose threshold s_0 so to ensure the desired asymptotic properties. The proposed policy is shown to be asymptotically optimal with respect to the detection accuracy and order-optimal with respect to the size of the search space. Extensive numerical experiments support the theoretical results. HDS with active local tests for the high level nodes is also analyzed numerically and is shown to outperform the fixed sample-size local test and approach the performance bound of IRW.

The rest of this paper is organized as follows: in Section II we present the system model and discuss its relationship with the existing literature. Section III designs the HDS policy and analyzes its performance. We numerically evaluate HDS in Section IV, and provide concluding remarks in Section V.

II. SYSTEM MODEL AND PRELIMINARIES

A. Problem Formulation

We consider the problem of locating an anomaly in a hierarchical data stream that comprises a large number M of processes. The observations \{y(i)\}, are drawn in an i.i.d. manner with probability density function f_0(y(i)|\theta) that is known up to a parameter \theta. A process is considered normal if \theta = \theta_0(0) and anomalous if \theta \in \Theta_1(0).

In addition to observing individual processes, the decision maker can measure aggregated processes that conform to a binary tree structure. Sampling an internal node of the tree gives a blurry image of the processes beneath it (Fig. 1). Consequently, the observations y(i) of an internal node on level l = 1, \ldots, \log_2 M of the tree also follow a model f_l(y(i)|\theta) that is known up to a parameter \theta. If a node at level l contains the anomaly, its associated parameter \theta is in \Theta_1(l). A node at level l is normal if \theta = \theta_0(l). The normal parameter \theta_0(l) and the anomaly parameter set \Theta_1(l) are known for all l, and we assume informative observations at all levels; i.e., for all 0 \leq l \leq \log_2 M there exists \Delta > 0 independent of M such that

\[ D_l(\theta_0(l)||\theta) \geq \Delta, \quad D_l(\theta||\theta_0(l)) \geq \Delta, \quad \forall \theta \in \Theta_1(l). \quad (1) \]

In (1), we use D_l(x||z) to denote the Kullback-Leibler (KL) divergence between two distributions, f_l(\cdot|x), f_l(\cdot|z).

An active search strategy \Gamma = (\phi, \tau, \delta) is given by a selection rule \phi, a stopping rule \tau, and a decision rule \delta. At every time step t a sample is drawn from the selected node \phi(t). The time at which the decision maker decides to end the search is \tau, and the decided anomaly is \delta \in \{1, \ldots, M\}.

Let H_m denote the hypothesis in which process m \in \{1, \ldots, M\} is anomalous. Further, let \pi_m be the prior probability of H_m, while \mathcal{P}_m and \mathcal{E}_m denote the probability measure and expectation under H_m, respectively. The error rate of \Gamma is

\[ P_{Err}(\Gamma) \triangleq \sum_{m=1}^{M} \pi_m \cdot \mathcal{P}_m[\delta \neq m], \quad (2) \]

and the sample complexity is

\[ Q(\Gamma) \triangleq \sum_{m=1}^{M} \mathcal{E}_m[\tau|\Gamma]. \quad (3) \]

Our aim is to find strategy \Gamma that minimizes the Bayes risk

\[ R(\Gamma) \triangleq P_{Err}(\Gamma) + c \cdot Q(\Gamma), \quad (4) \]

where c \in (0, 1) is a fixed coefficient balancing (2) and (3).

B. Related Literature

Target search problems have been widely studied under various scenarios. Optimal policies for target search with a fixed sample size were derived in [15]-[18] under restricted settings involving binary measurements and symmetry assumptions. Results under the sequential setting can be found in [19]-[22], all assuming single process observations. In this paper we address these questions under the asymptotic regime as the error probability approaches zero. Asymptotically optimal results for sequential anomaly detection in a linear search under various setting can be found in [23]-[26]. In this paper, however, we consider a composite hypothesis case, which was not addressed in the above. Results under the composite hypothesis case with linear (i.e., non-hierarchical) search can be found in [27]-[32]. Detecting anomalies or outlying sequences has also been studied under different formulations, assumptions, and objectives [33]-[36]; see survey in [37]. These studies, in general, do not address the optimal scaling in the detection accuracy or the size of the search space.

The problem considered here also falls into the general class of sequential design of experiments pioneered by Chernoff in 1959 [38]. Compared with the classical sequential hypothesis testing pioneered by Wald [39] where the observation model

![Fig. 1. A binary tree observation model with }M = 8\text{ processes, } \log_2 M = 3\text{ levels, and a single anomaly. The anomaly is measurable at the red nodes.](image-url)
under each hypothesis is fixed, active hypothesis testing has a control aspect that allows the decision maker to choose different experiments (associated with different observation models) at each time. The work [40] developed a variation of Chernoff’s randomized test that achieves the optimal logarithmic order of the sample complexity in the number of hypotheses under certain implicit assumptions on the KL divergence between the observation distributions under different hypotheses. These assumptions, however, do not always hold for general observation models as considered here. Finally, tree-based search in data structures is a classical problem in computer science (see, for example, [41], [42]). It is mostly studied in a deterministic setting; i.e., the observations are deterministic when the target location is fixed. The problem studied in this work is a statistical inference problem, where the observations taken from the tree nodes follow general statistical distributions. This problem also has intrinsic connections with several problems studied in different application domains, e.g., adaptive sampling [43], [44], noisy group testing [45], [46], and channel coding with feedback [47], [48].

III. HIERARCHICAL DYNAMIC SEARCH

In this section we present and analyze the proposed HDS active search strategy. We start by introducing the algorithm in Subsection III-A, after which we analyze its performance and provide a discussion in Subsections III-B and III-C, respectively.

A. Algorithm Design

Rationale: The anomaly is searched using a random walk on the process tree that starts at the root node. The individual steps of the walk are determined by local tests. On internal (i.e., high level) nodes, the outcome of the test can be moving to the left or right child, or returning to the parent node (where the parent of the root is itself). The internal test is constructed to create a bias in the walk towards the anomalous leaf. On a leaf node, say process \(m\), the possible outcomes are either terminating the search and declaring process \(m\) anomalous, or moving back to parent node. The leaf test is designed to terminate the walk on the true anomaly with sufficiently high probability. In the following, we specify the internal and leaf tests.

**Internal Test:** Suppose that the random walk arrives at a node on level \(l > 0\). A fixed number \(K_{l-1}\) of samples \(y(i)\) is drawn from both children, and are used to compute the GLLRs

\[
\tilde{S}_{\text{GLLR}}^{(l-1)}(K_{l-1}) \triangleq \sum_{i=1}^{K_{l-1}} \log \frac{f_{l-1}(y(i) | \hat{\theta}_1^{(l-1)})}{f_{l-1}(y(i) | \hat{\theta}_0^{(l-1)})},
\]

where \(\hat{\theta}_j^{(l-1)}\) is the maximum likelihood estimate of the anomaly parameter, given by

\[
\hat{\theta}_j^{(l-1)} = \arg\max_{\theta \in \Theta_j^{(l-1)}} \prod_{i=1}^{K_{l-1}} f_{l-1}(y(i) | \theta).
\]

The statistics (5) utilize the information on the normal distribution. If both children have a negative GLLR, the random walk moves to the parent. Otherwise, it moves to the child that has the higher GLLR. The sample size \(K_l\) for \(l = 0, \ldots, \log_2 M - 1\) is determined offline, such that the probability of moving in the direction of the anomaly is greater than \(\frac{1}{2}\). The sample size \(K_1\) is finite under assumption (1).

**Leaf Test:** When the random walk visits a leaf node, we perform an ALLR test. Here, samples \(y(i)\) are drawn sequentially from the process and the local ALLR

\[
\tilde{S}_{\text{ALLR}}(n) = \sum_{i=1}^{n} \log \frac{f_0(y(i) | \hat{\theta}_1^{(0)}(i-1))}{f_0(y(i) | \hat{\theta}_0^{(0)})},
\]

is continuously updated, where

\[
\hat{\theta}_j^{(0)}(i-1) = \arg\max_{\theta \in \Theta_j^{(0)}} \prod_{j=1}^{i-1} f_0(y(j) | \theta),
\]

is the delayed maximum likelihood estimate of \(\theta_1^{(0)}\). The initial estimate \(\hat{\theta}_1^{(0)}(0)\) can be chosen arbitrarily. As opposed to GLLR, the \(\tilde{S}_{\text{ALLR}}(n)\) is a viable likelihood ratio, so that the Wald likelihood ratio identity can still be applied to upper-bound the error probabilities of the sequential test [39].

At every time step \(n > 0\), the ALLR (7) is examined: if \(\tilde{S}_{\text{ALLR}}(n) > \log \frac{\log_2 M}{c}\), the random walk terminates and the tested process is declared anomalous, while a negative ALLR results in returning to the parent node. The resulting search policy is summarized in Algorithm 1. An additional mechanism ensures the theoretical guarantees established in section III-B (see appendix B).

**Lemma 1:** The statistics (5), (7), and (8) are non-negative and upper bound the error probabilities of the sequential test [39].

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**Algorithm 1:** Hierarchical Dynamic Search

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Input: Inspected node at level \(l\)

1. if \(l > 0\) (internal node) then
2. Measure \(K_{l-1}\) samples from each child node;
3. Compute GLLR for each child via (5);
4. if Both GLLRs are negative then
5. Invoke Algorithm 1 on parent node;
6. else
7. Invoke Algorithm 1 on child with larger GLLR;
else
9. Init \(\theta_1^{(0)}\) and \(n = 1\);
10. Draw \(y(i)\) and compute ALLR (7);
11. if \(\tilde{S}_{\text{ALLR}}(n) > \log \frac{\log_2 M}{c}\) then
12. Identify node as target and terminate;
else if \(\tilde{S}_{\text{ALLR}}(n) < 0\) then
14. Invoke Algorithm 1 on parent node;
15. Increment \(n\) and jump to step 9;
```

B. Performance Analysis

In this subsection we theoretically analyze the proposed HDS policy denoted \(\Gamma_{\text{HDS}}\). In particular, we establish that
the HDS policy $\Gamma_{\text{HDS}}$ is asymptotically optimal in $c$, i.e.,
\[ \lim_{c \to 0} \frac{R(\Gamma_{\text{HDS}})}{R^*} = 1, \]
and order optimal in $M$, namely,
\[ \lim_{M \to \infty} \frac{R(\Gamma_{\text{HDS}})}{R^*} = O(1) \]
where $R^*$ is a lower bound on the Bayes risk. This is stated in the following theorem:

**Theorem 1.** When (1) holds and the $\Theta_i^{(l)}$ are finite for $0 \leq l \leq \log_2 M - 1$, the Bayes risk of $\Gamma_{\text{HDS}}$ is bounded by
\[ R(\Gamma_{\text{HDS}}) \leq c \log_2 M + \frac{c \log_2 M}{D_0(\theta^{(0)}_1||\theta^{(0)}_0)} + O(c) \]
where $B$ is a constant independent of $M$ and $c$.

**Proof:** The complete proof is given in appendix B. Here, we only present the proof outline: to find an upper bound on the Bayes risk of HDS, we analyze the case where it is implemented indefinitely according to its selection rule, while the stopping rule is disregarded. We divide the trajectory of the random walk into two stages: search and target test.

In the **search stage** the random walk explores the high level nodes and eventually is expected to concentrate on the true anomaly. Based on this insight, we partition the tree $T$ into a sequence of sub-trees $T_0, T_1, \ldots, T_{\log_2 M}$ (Fig. 2). Sub-tree $T_{\log_2 M}$ is obtained by removing the half-tree that contains the target from $T$. Sub-tree $T_l$ is iteratively obtained by removing the half-tree that contains the target from $T \setminus T_{l+1}$. $T_0$ consists of only the target node. We then define the last passage time $\tau_l$ of the search phase from each sub-tree $T_l$. An upper bound on the end of this first stage is found by proving that the expected last passage time to each of the half trees that do not contain the target is bounded by a constant. Roughly speaking, this upper bound holds since the fixed size internal tests and the leaf tests have a greater probability of moving towards the anomaly than away from it. Summing the upper bound on the last passage times yields the first term in (11). The second stage is the **leaf target test**, which ends with the declaration of the target with expected time $E[\tau_0]$. To bound $E[\tau_0]$, we first define a random time $\tau_M$ to be the smallest integer such that the estimator of the target leaf’s parameter equals to $\theta_i^{(0)}$ for all $n > \tau_M$, and we show that $E[\tau_M]$ is bounded by a constant independent of $c$ and $M$. The bound holds by utilizing the properties of the maximum likelihood estimator and applying the Chernoff bound. We then bound $E[\tau_0]$ using Wald’s equation [39] and Lorden’s inequality [49], which yields the second and third terms in (11).

Finally, we show that the detection error is of order $O(c)$. The detection errors can only occur in the search stage, where the expected number of times a normal leaf is in the order of $\log_2 M$. By using the martingale properties of the ALLR statistic we prove that the false positive rate of the leaf test is bounded by $\frac{c}{\log_2 M}$. The resulting error rate $P_{\text{Err}}(\Gamma_{\text{HDS}})$ is therefore in the order of $c$ (third term in (11)).

The optimality properties of the Bayes risk of HDS in both $c$ and $M$ directly carry through to the sample complexity of HDS, as stated in the following corollary:

**Corollary 1.** The sample complexity of HDS is bounded via
\[ Q(\Gamma_{\text{HDS}}) \leq B \log_2 M + \frac{\log_2 M}{D_0(\theta^{(0)}_1||\theta^{(0)}_0)} + O(1), \]
\[ Q(\Gamma_{\text{HDS}}) \geq \frac{\log_2 M}{I_{\text{max}}} + \frac{\log((1 - c)/c)}{D_0(\theta^{(0)}_1||\theta^{(0)}_0)} + O(1), \]
where $I_{\text{max}}$ is the maximum mutual information between the true hypothesis and the observation under an optimal action.

**Proof:** The upper bound (12) follows directly from Theorem 1, while (13) is obtained using [40, Thm. 2].

Corollary 1 indicates that HDS is asymptotically optimal in $c$ and order optimal in $M$.

**C. Discussion**

The proposed HDS algorithm is designed to efficiently search in hierarchical data structures while coping with an unknown anomaly distribution. It can be viewed as an extension of the IRW method [9] to unknown anomaly parameters, while harnessing the existing knowledge regarding the distribution of the anomaly-free measurements. In contrast to existing hierarchical algorithms, HDS can incorporate general parameterized anomaly observation models, resulting in it being order-optimal with respect to the search space size and asymptotically optimal in detection accuracy.

The derivation of HDS motivates the exploration of several extensions. First, HDS is derived for hierarchical data that can be represented as a binary, while anomaly search with adaptive granularity may take the form of an arbitrary tree. Furthermore, we design HDS for detecting a single target, while often in practice one may have to cope with multiple anomaly processes. An additional extension would be to consider a composite model for both normal and anomalous distributions. We leave the extension of HDS to these settings for future work.
IV. NUMERICAL EVALUATIONS

In this section we empirically compare HDS (Algorithm 1) with existing search strategies of Deterministic Search (DS) [30], IRW [9], and the Confidence Bounds based Random Walk (CBRW) algorithm [10]. The IRW algorithm has access to the true anomaly parameter \( \Theta_{1}^{(l)} \), while the other algorithms only have access to \( \Theta_{1}^{(l)} \). Thus, IRW serves as a benchmark for the performance. IRW and HDS use fixed size internal tests that are not optimized for the specific simulation. Instead the sample sizes \( K_{i} \) are chosen as small as possible such that the desired drift towards the target is ensured. The performance of IRW should therefore be a best-case scenario for HDS. IRW, DS, and HDS use \( c = 10^{-2} \), and CBRW uses \( p_{0} = 0.2 \) and \( \epsilon = 10^{-2} \). The values are averaged over \( 10^{6} \) Monte Carlo runs. We first simulate a scenario where the decision maker observes the interoccurrence time of Poisson point processes with normal rate \( \lambda_0 = 1 \) and anomalous rate \( \lambda_1 = 10^{3} \). The rates at the internal nodes are equal to the sum of the rates of their children. The minimum rate that is considered anomalous is \( \lambda_{1, \min} = \frac{\lambda_0 + \lambda_1}{2} \) such that the anomaly parameter set is \( \Theta_{1}^{(0)} = (\lambda_{1, \min}, \infty) \). This scenario models the detection of heavy hitters among Poisson flows where the measurements are exponentially distributed packet inter-arrival times.

Fig. 3 depicts the risk \( R(\Gamma) \) as in (4) versus the number of processes \( M \). We can clearly observed that HDS outperforms CBRW and DS (for most values), and it is within a minor gap of that of IRW. While for \( M \geq 16 \), HDS is only slightly outperforms CBRW, it is notably outperforms DS. However, it is worth noting, that CBRW uses sequential internal tests, which should be more efficient than the fixed size internal tests of HDS. For this reason, in this scenario we also compare an alternative internal test for HDS. The results of this study, depicted in Fig. 4, show that switching to the GLLR statistic for the leaf test instead of the ALLR statistic yields a performance gain for all \( M \). An even greater jump in performance is achieved by using an active test for the internal nodes. The details of the active test are given in appendix A.

Next, we simulate our decision making algorithm when considering a set of Poisson point processes with rate \( \lambda_0 = 0.1 \). Here, the measurements of the nodes that contain the anomaly are corrupted by Bernoulli interference; i.e.,

\[
y(i) \sim \text{Exp}(2^{i} \lambda_0) + z \cdot [0 + (a + 6) \cdot \text{Bernoulli}(0.5)].
\]

In (14), \( z \in \{0, 1\} \) indicates whether the node is anomalous, and \( a \) is unknown. The node parameter \( \theta \) is given by the pair \( (z, a) \), where \( \theta_{0}^{(l)} = (0, 0), \theta_{1}^{(l)} = (1, 10) \), and \( \Theta_{1}^{(l)} = \{1\} \times \{1, 5, 10\} \) for all levels \( 0 \leq l \leq \log_{2} M \). In this case the mean values of the normal and abnormal distribution are close to each other, and the anomalous process is reflected by higher moments of the distributions. The results for this setting, depicted in Fig. 5, show that while CBRW achieves poor performance, HDS detects the anomaly quite efficiently, resulting in a larger gap between HDS and CBRW than in the first scenario.
V. Conclusions

In this work we developed a novel sequential search strategy for the composite hierarchical anomaly detection problem, dubbed HDS that uses two variations of the GLLR statistic to ensure a biased random walk for a quick and accurate detection of the anomaly process. HDS is shown to be order-optimal with respect to the size of the search space and asymptotically optimal with respect to the detection accuracy. The addition of the hierarchical search significantly improves the performance over the linear search algorithms in the common case of a large number $M$ of processes and heavy hitting anomalies. We also show that the empirical performance can be further improved by using different statistics and local tests.

APPENDIX A

ACTIVE INTERNAL TEST

Instead of the fixed size internal test described in section III-A, we can use an active internal test:

Let $S_L(t)$ and $S_R(t)$ be the GLLR of the left and right children respectively at time $t$ and initialize them with zero at $t = 0$. Similar to the IRW active test [9], we define the two thresholds

$$v_0 \triangleq - \log \frac{2p}{1-p}, \quad v_1 \triangleq \log \frac{2p}{1-p} \quad (15)$$

where $p > \frac{1}{2}$ is the confidence level. Let child

$$x(t-1) = \arg\max_{i \in \{L,R\}} S_i(t-1) \quad (16)$$

be the child with the higher GLLR at time $t-1$. Then, in every step $t$, we draw a sample from child $x(t-1)$ and update $S_{\hat{x}(t)}(t)$. The other child $\hat{x}(t) \neq x(t)$ keeps the previous GLLR i.e., $S_{\hat{x}(t)}(t) = S_{\hat{x}(t)}(t-1)$. The test terminates at the random time

$$k = \inf \{t \in \mathbb{N} : S_{\hat{x}}(t) \geq v_0 \text{ or } S_{x}(t) \leq v_0\} \quad (17)$$

If $S_{x(k)}(k) \geq v_1$, the random walk zooms into child $x$ and if $S_{x(k)}(k) \leq v_0$, the random walk zooms out to the parent.

In contrast to the IRW active tests, we do not claim that this test ensures the same desired drift behavior as the fixed size internal test (Fig. 4). Instead, it is a heuristic. Nevertheless, we observe a significant gain in empirical performance when compared to the fixed sample internal test (Fig. 4).

APPENDIX B

PROOF OF THEOREM 1

To find an upper bound on the Bayes risk of HDS, we analyze the case where it is implemented indefinitely, meaning that HDS probes the cells indefinitely according to its selection rule, while the stopping rule is disregarded. We divide the trajectory of indefinite HDS into discrete steps at times $t \in \mathbb{N}$. A step is not necessarily associated with every sample as will become clear later. Let $\tau_{\infty}$ mark the first time that indefinite HDS performs a leaf test on the true anomaly and $S_{\text{ALLR}}$ rises above the threshold. It is easy to see that regular HDS terminates no later than $\tau_{\infty}$. We divide the initial trajectory $t = 1, 2, \ldots, \tau_{\infty}$ of the indefinite random walk into two stages:

- In the search stage the random walk explores the high level nodes and eventually concentrates at the true anomaly. This stage ends at time $\tau_s$ which is the last time a leaf test is started on the true anomaly before $\tau_{\infty}$.
- The second stage is the target test which ends with the declaration of the target. The duration of this stage is $\tau_0 = \tau_{\infty} - \tau_s$.

Step 1: Bound the sample complexity of the search stage:

Similarly to [9], we partition the tree $T$ into a sequence of sub-trees $T_0, T_1, \ldots, T_{\log_2 M}$ (Fig. 2). Sub-tree $T_{\log_2 M}$ is obtained by removing the half-tree that contains the target from $T$. Sub-tree $T_l$ is iteratively obtained by removing the half-tree that contains the target from $T \setminus T_{l+1}$. $T_0$ consists of only the target node. We then define the last passage time $\tau_l$ to each sub-tree $T_l$ for $1 \leq l \leq \log_2 M$. Let $G(t)$ indicate the sub-tree of the node tested at time $t$. The last passage time to $T_{\log_2 M}$ is

$$\tau_{\log_2 M} = \sup \{t \in \mathbb{N} : G(t) = T_{\log_2 M}\} \quad (18)$$

For the smaller sub-trees $T_1, \ldots, T_{\log_2 M-1}$ the last passage times are defined recursively such that

$$\tau_l = \sup \{t \in \mathbb{N} : G(t) = T_l\} - \tau_{l+1} \quad (19)$$

Notice, that the search time is bounded by

$$\tau_s = \sup_{1 \leq l \leq \log_2 M} \tau_l \leq \sum_{l=1}^{\log_2 M} \tau_l \quad (20)$$

Next, we bound the expected last passage times $E[\tau_l]$ for $1 \leq l \leq \log_2 M$. Towards this end, we define a distance $L_t$ from the state of the indefinite random walk at time $t$ to the anomalous leaf. When an internal node is probed, $L_t$ is equal to the discrete distance to the anomaly on the tree. Since the walk starts at the root, we have $L_0 = \log_2 M$ when testing a normal leaf, $L_t$ is equal to the sum of the discrete distance on the tree and the accumulated $S_{\text{ALLR}}$ of the current leaf test. When the true anomaly is probed, the distance is negative i.e. $L_t = -S_{\text{ALLR}}$. Let the step $W_t$ be the random change in the distance at time $t$ such that $L_{t+1} = L_t + W_t$. Internal tests comprise only a single step either towards or away from the anomaly, i.e., $W_t \in \{-1, 1\}$. Because the sample sizes $K_i$ of the internal tests are constructed such that $P(W_t = 1) < \frac{1}{2}$, we have

$$E[W_t] = 2P(W_t = 1) - 1 < 0 \quad (21)$$

On leaf nodes, every single sample of the sequential test comprises a step. A step is therefore the change in $S_{\text{ALLR}}$. Using the assumption in (1) and the independence of $\hat{\theta}^{(0)}_i(t-1)$ and $y(i)$ we find that for normal leaves

$$E[W_t] = E_{\theta^{(0)}_i} \left[ \frac{\log f_0(y(t) | \theta^{(0)}_i(t-1))}{f_0(y(t) | \theta^{(0)}_i)} \right] \leq -\Delta < 0 \quad (22)$$

Similarly, we want to show that for the anomalous leaf that

$$E[W_t] = E_{\theta^{(0)}_i} \left[ -\log \frac{f_0(y(t) | \theta^{(0)}_i(t-1))}{f_0(y(t) | \theta^{(0)}_i)} \right] < 0 \quad (23)$$
Denoting $\hat{\theta} = \hat{\theta}_1(0)(t-1)$, we split the term and use the law of total expectation to find that
\[
\begin{align*}
E[W_i] &= \mathbb{E}_{\theta_0} \left[ - \log \frac{f_0(y(t) | \hat{\theta})}{f_0(y(t) | \theta_0)} + \log \frac{f_0(y(t) | \hat{\theta}_1(0))}{f_0(y(t) | \theta_1(0))} \right] \\
&= \mathbb{E}_{\theta_0} \left[ - \log \frac{f_0(y(t) | \theta_0)}{f_0(y(t) | \hat{\theta}_1(0))} + \log \frac{f_0(y(t) | \hat{\theta}_1(0))}{f_0(y(t) | \theta_0)} \right] \\
&= -D_0(\theta_1(0) || \theta_0) + \mathbb{P}_{\theta_1(0)}[\hat{\theta} \neq \theta_1(0)] D_0(\theta_1(0) || \hat{\theta})
\end{align*}
\]
where we used the fact that $D_0(\theta_1(0) || \theta_0) = 0$. For (23) to hold, it remains to be shown that
\[
\mathbb{P}_{\theta_1(0)}[\hat{\theta} \neq \theta_1(0)] \leq \lambda_{\theta_1(0)} = \frac{\log(1 + M)c}{\gamma n}.
\]
Notice, that the $\lambda_{\theta_1(0)}$ are strictly positive due to the assumption in (1) and assuming that
\[
\sup_{\theta_1(0), \hat{\theta} \in \Theta_1} D_0(\theta_1(0) || \hat{\theta}) < \infty.
\]
For this purpose, we first introduce the following Lemma:

**Lemma 1.** Let $\Theta_1(0)$ be finite, i.e., $R = |\Theta_1(0)| < \infty$ and let $\hat{\theta}_1(0)(n)$ be the ML estimate of $\theta_1(0)$ using $n$ samples. Let $\tau_{\text{ML}}$ be the smallest integer such that $\hat{\theta}_1(0)(n) = \theta_1(0)$ for all $n > \tau_{\text{ML}}$. Then, there exist constants $C > 0$ and $\gamma > 0$ independent of $M$ and $c$ such that
\[
\mathbb{P}_{\theta_1(0)}[\tau_{\text{ML}} > n] \leq C e^{-\gamma n}.
\]

**Proof:** The event $\tau_{\text{ML}} > n$ implies that there exists a time $t > n$ such that $\hat{\theta}_1(0)(t) \neq \theta_1(0)$ and therefore we have
\[
\mathbb{P}_{\theta_1(0)}[\tau_{\text{ML}} > n] \leq \sum_{i=1}^{t} \mathbb{P}_{\theta_1(0)}[\hat{\theta}_1(0)(t) \neq \theta_1(0)].
\]
By definition of the maximum likelihood estimate, the event $\hat{\theta}_1(0)(t) \neq \theta_1(0)$ implies
\[
\sum_{i=1}^{t} S_{\hat{\theta}}(i) \geq 0
\]
for some $\hat{\theta} \neq \theta_1(0)$, where
\[
S_{\hat{\theta}}(i) = \log \frac{f_0(y(i) | \hat{\theta})}{f_0(y(i) | \theta_1(0))}.
\]
Applying the Chernoff bound and using the i.i.d. property yields
\[
\mathbb{P}_{\theta_1(0)}[\sum_{i=1}^{t} S_{\hat{\theta}}(i) \geq 0] \leq \left( \mathbb{E}_{\theta_1(0)}[e^{S_{\hat{\theta}}(i)}] \right)^t
\]
for all $s \geq 0$. The moment generating function (MGF) $e^{S_{\hat{\theta}}(i)}$ is equal to one at $s = 0$. The derivative of the MGF at $s = 0$ is
\[
\mathbb{E}_{\theta_1(0)}[S_{\hat{\theta}}(i)] = -D_0(\theta_1(0) || \hat{\theta}) < 0.
\]
Because the derivative is negative and assuming that the distribution of $S_{\hat{\theta}}(i)$ is light-tailed $^1$, there exist $s > 0$ and $\gamma > 0$ such that $\mathbb{E}[e^{s S_{\hat{\theta}}(i)}] = e^{-\gamma s} < 1$ and the RHS of (31) decays exponentially with $t$. Summing over all $\hat{\theta} \neq \theta_1(0)$, we get
\[
\mathbb{P}_{\theta_1(0)}[\hat{\theta}(0)(t) \neq \theta_1(0)] \leq Re^{-\gamma t}.
\]
Using the formula for the partial sum of a geometric series we find that the RHS of (28) is bounded by
\[
\sum_{t=n}^{\infty} Re^{-\gamma t} = \frac{R}{1 - e^{-\gamma} e^{-\gamma n}}.
\]
In light of lemma 1, we propose the following mechanism to ensure that (25) holds: Whenever a leaf test is started, before beginning with the sequential test described in section III-A, a fixed number $N_{\text{leaf}} \geq 0$ of samples $\{y_i\}_{i=0}^{n - N_{\text{leaf}} - 1}$ is drawn from the leaf to initialize the estimate $\hat{\theta}_1(0)$, meaning, instead of (8) we write
\[
\hat{\theta}_1(0)(i-1) = \arg \max_{\theta \in \Theta_1(0)} \prod_{j=0}^{i-1} f_0(y(j) | \theta).
\]
This has the effect, that at every step of the subsequent sequential test, the estimate $\hat{\theta}_1(0)$ is based on at least $N_{\text{leaf}}$ samples. Since $\hat{\theta} \neq \theta_1(0)$ implies that $\tau_{\text{ML}} > N_{\text{leaf}}$, we have
\[
\mathbb{P}_{\theta_1(0)}[\hat{\theta} \neq \theta_1(0)] \leq \mathbb{P}_{\theta_1(0)}[\tau_{\text{ML}} > N_{\text{leaf}}].
\]
Using
\[
\lambda = \inf_{\theta \in \Theta_1(0)} \lambda_{\theta_1(0)}
\]
and lemma 1 we find that (25) is satisfied if
\[
N_{\text{leaf}} > -\frac{\log \hat{\Delta}}{\gamma}.
\]
Notice, that $N_{\text{leaf}}$ is chosen independent of the size of search space $M$ and the cost $c$.

With (21), (22) and (23) we established that HDS has the same drift behavior as IRW. Furthermore, we assume that the distribution of
\[
\log \frac{f_0(y(i) | \hat{\theta})}{f_0(y(i) | \theta_1(0))}
\]
is light-tailed for all $\hat{\theta} \in \Theta_1(0)$. Thus, we can apply [9, Lemma 1.2] and find that the expected last passage times $\mathbb{E}[\tau_i]$ for $1 \leq i \leq \log_2 M$ are

$^1$A distribution with density $f$ is light-tailed if $\int_{-\infty}^{\infty} e^{\lambda x} f(x) dx < \infty$ for some $\lambda > 0$ [50].
bounded by a constant $\beta$ independent of $M$ and $c$. Applying (20) yields
$$E[\tau_s] \leq \beta \log_2 M. \quad (40)$$
Let
$$K_{\max} = \sup_{0 \leq t \leq \log_2 M} K_t \quad (41)$$
be the maximum number of samples taken from a child during an internal test. Then every step $W_t$ takes at most $N_{\max} = \max \{2K_{\max}, N_{\text{leaf}} + 1\}$ samples and the complexity of the search stage $Q_s$ is bounded by
$$Q_s \leq N_{\max} E[\tau_s] \leq B \log_2 M \quad (42)$$
where $B = \beta N_{\max}$ is a constant independent of $M$ and $c$.

**Step 2: Bound the sample complexity of the target test:**
In the analysis of the target test we associate a time step $f$ sum for expectation we find
$$E[\tau_{ML}] = O(1). \quad (43)$$
At all times $n > \tau_{ML}$, we necessarily have $\hat{\theta}_i^{(0)} = \theta_i^{(0)}$. From the definition of $\hat{S}_{\text{ALLR}}$ in (7) it is easy to see, that after $n > \tau_{ML} + 1$, the leaf test is essentially a sequential likelihood ratio test. The expected number of times a normal leaf is tested $E$ is bounded by the number of steps in the search stage. Thus,
$$E[\tau_f] = \tau_0 - \tau_{ML} \text{ is bounded by}$$
$$E[\tau_f] \leq \frac{\log \frac{\log_2 M}{c}}{D_0 (\theta_1^{(0)}|\theta_0^{(0)})} + O(1) \quad (44)$$
where we used Wald’s equation [39] and Lorden’s inequality [49] and assumed that the first two moments of the log-likelihood ratio are finite. Combining (43) and (44) yields the sample complexity of the target test
$$Q_1 = E[\tau_0] \leq \frac{\log \frac{\log_2 M}{c}}{D_0 (\theta_1^{(0)}|\theta_0^{(0)})} + O(1). \quad (45)$$

**Step 3: Bound the error rate:**
Notice, that detection errors can only occur in the search stage. The expected number of times a normal leaf is tested $E[N]$ is bounded by the number of steps in the search stage. Thus, using (40) we get
$$E[N] \leq E[\tau_s] \leq \beta \log_2 M. \quad (46)$$
Let $Z(n) = e^{S_{\text{ALLR}}(n)}$ be adaptive likelihood ratio at time $n$. In the following, we use the properties of the ALLR to bound the false positive rate of the leaf test
$$\alpha = P_{\theta_0^{(0)}} \left[ Z(n) \geq \frac{\log_2 M}{c} \text{ for some } n \geq 1 \right]. \quad (47)$$
Note, that on normal leaves $Z(n)$ is a non-negative martingale i.e.
$$E_{\theta_0^{(0)}} [Z(n + 1)] \{ (y(i)) \}_{i=1}^n = Z(n) \quad (48)$$
$$E_{\theta_0^{(0)}} \left[ f \left( \frac{y(n + 1) | \theta_1^{(0)}(n)}{\theta_0^{(0)}} \right) \right] = Z(n) \quad (49)$$
where we used the independence of $\theta_1^{(0)}(n)$ and $y(n + 1)$ in the last step. Using a lemma for nonnegative supermartingales [51] we find
$$P_{\theta_0^{(0)}} \left[ Z(n) \geq \frac{\log_2 M}{c} \text{ for some } n \geq 1 \right] \leq \frac{e}{\log_2 M} E_{\theta_0^{(0)}} [Z(1)]. \quad (50)$$
Since
$$Z(1) = E_{\theta_0^{(0)}} \left[ f \left( \frac{y(1) | \theta_1^{(0)}(0)}{\theta_0^{(0)}} \right) \right] = 1, \quad (51)$$
the false positive rate is bounded by
$$\alpha \leq \frac{e}{\log_2 M}. \quad (52)$$
Finally, combining (46) and (54) yields the bound on the error rate
$$P_{\text{Err}}(\Gamma_{\text{HDS}}) = E[N] \alpha \leq \beta c = O(c). \quad (55)$$
Theorem 1 follows from (42), (45) and (55).

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