Massive Analogue of Ashtekar-CJD Action

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abstract

The action of Ashtekar gravity have been found by Cappovilla, Jacobson and Dell. It does not depend on the metric nor the signature of the space-time. The action has a similar structure as that of a massless relativistic particle. The former is naturally generalized by adding a term analogous to a mass term of the relativistic particle. The new action possesses a constant parameter regarded as a kind of a cosmological constant. It is interesting to find a covariant Einstein equation from the action. In order to do it we will examine how the geometrical quantities are determined from the non-metric action and how the Einstein equation follows from it.
A new form of canonical gravity developed by Ashtekar (1986, 1987, 1991) has various nice features. It is formulated in a form of a gauge theory with the gauge group SO(3). The constraints generating the general coordinate and the gauge transformations are of polynomial form and are suitable for canonical quantization. However, the canonical variables are complex valued and must satisfy the so called reality conditions. It is nicely formulated starting from an action which is a sum of Einstein and complex total divergence terms and the significance of reality conditions has been discussed (Ashtekar (1986, 1987), Fukuyama and Kamimura (1990)).

The action in terms of SO(3) gauge fields is found by making a Legendre transformation backward from the Hamiltonian to Lagrangian formalism. It has been first given by Capovilla, Jacobson and Dell (CJD) (1989). The CJD action for the pure gravity theory is

\[ \mathcal{L}_0 = -\frac{1}{4\eta}[G^{ab}G_{ab} - \frac{1}{2}G^a_aG_b_b], \]

where \(G_{ab}\) is a building block of the action defined in terms of SO(3) gauge field strength,

\[ G_{ab} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu a}F_{\rho\sigma b}, \quad F_{\mu\nu a} = \partial_{[\mu}A_{\nu]a} + g\epsilon^{abc}A_\mu^bA_\nu^c. \]

\(\eta\) in the Lagrangian is a scalar density multiplier field. The action is invariant under general coordinate transformations as well as SO(3) gauge transformations. A characteristic property of the action is that it does not depend on any space-time metric. Especially there is no a priori sign of signature of the metric (Euclidian or Lorentzian). \(\epsilon^{\mu\nu\rho\sigma}\) is simply a Levi-Civita symbol, \(\epsilon^{0123} = 1\).

Another is absence of the reality conditions which are imposed in the Ashtekar formalism. In the previous paper (Kamimura, Makita and Fukuyama, 1992) we have discussed how the space-time metric arises from the algebraic properties of the constraints and the Hamiltonian. It is accomplished by finding a set of first class constraints satisfying the same algebra as diffeomorphism constraints. The metric of equal time space appears in the structure function. The temporal components of the metric are found in the coefficients of the diffeomorphism constraints in the Hamiltonian. There is one ambiguity in determining the weight factor of deformation generator of normal direction. The signature of the metric is determined by the choice of the reality conditions. Possible forms of reality conditions on the canonical variables are determined by a requirement that the derived tetrad and spin connections are real quantities. The SO(3) gauge connection is shown to be a self-dual spin connection both in Euclidian and Lorentzian signatures. It is complex valued in the Lorentzian case. In the quantum theory the signature factor may be taken into a choice of the inner product (Fukuyama and Kamimura 1990).

The CJD action is generalized to those including cosmological constants (Bengtsson (1990) Perdan (1991)). They discussed a Hamiltonian constraint

\[ H_\Lambda = \frac{1}{2}\epsilon_{abc}\epsilon_{ijk}(\pi^{ia}\pi^{jb}B^{kc} - \Lambda\pi^{ia}\pi^{jb}\pi^{kc}) = 0, \]

which is obtained by adding a cosmological constant term to the Ashtekar constraint for the pure Einstein theory. Since the constraint is third order polynomial with respect to the momenta the inverse Legendre transformation results in a Lagrangian of non simple form. It does not seem to be a fundamental Lagrangian of the gravity.

An additional constant is introduced into the CJD action by referring to the case of a relativistic particle. The action of a relativistic free particle is given by \(L_2 = -\mu\sqrt{x^2}\), where \(\mu\) is a constant having the meaning of mass of the particle. It is rewritten using a scalar density
Lagrange multiplier $\eta$ as $L_1 = -\frac{1}{4\eta}\dot{x}^2 - \eta \mu^2$. In this form we can take the smooth massless limit and obtain an action for the massless particle $L_0 = -\frac{1}{4\eta}\dot{x}^2$.

The CJD action (1) has a similar structure to the massless particle action. It is naturally generalized by adding a term, which is analogous to the mass term in the case of a relativistic particle, as

$$L_1 = -\frac{1}{4\eta}[G^{ab}G_{ab} - \frac{1}{2}G_a^a G_b^b] - \eta \mu^2.$$  \hspace{1cm} (4)

Here the constant $\mu$ is dimensionless for the conventional assignments of the gauge fields. For non-vanishing value of $\mu$ we have an action of square root form through the elimination of $\eta$

$$L_2 = -\mu \sqrt{G^{ab}G_{ab} - \frac{1}{2}G_a^a G_b^b}.$$  \hspace{1cm} (5)

The actions in (4) and (5) may be classically equivalent though the branch of square root in the latter must be examined carefully. In this way we can introduce an additional constant without violating any local symmetries of the original action. The action (4) has a smooth $\mu \to 0$ limit to (1) so it is a neighbour of the pure Einstein theory (Bengtsson 1990). In order to examine roles of the constant $\mu$ and find out how the Einstein equation is modified by it, we must find out the geometrical quantities as functions of the canonical variables. In this paper we present an explicit derivation of Einstein Equation from the CJD action (1) in a form applicable to general case including the action of present interest (4).

In the Hamiltonian formalism of the action (4) there appears a set of first class constraints,

$$\pi_\eta = 0, \quad \pi^{0a} = 0,$$  \hspace{1cm} (6)

$$J^a = D_i \pi^i a = 0,$$  \hspace{1cm} (7)

$$T_j = \pi^{ka} F_{jka} = 0,$$  \hspace{1cm} (8)

and

$$H_0 = \frac{1}{2}\epsilon_{abc}\epsilon_{ijk}(\pi^{ia} \pi^{jb} B^{kc} + \frac{2\mu^2}{3} B^{ia} B^{jb} B^{kc}) = 0,$$  \hspace{1cm} (9)

where $\pi_\eta$ and $\pi^{0a}$ are momenta conjugate to $\eta$ and $A_{0a}$ respectively, and $D_\mu$ is $SO(3)$ covariant derivative. The magnetic field is defined by $B^{ia} = \frac{1}{2}\epsilon^{ijk} F_{jka}$. The constraints in (6) tell that $\eta$ and $A_{0a}$ are arbitrary multipliers. Equation (7) is the Gauss law constraint for $SO(3)$. The last two of them, (8) and (9), are reflecting the general covariance of the Lagrangian. The $\mu^2$ dependent term appears only in the Hamiltonian constraint $H_0$. It has a smooth limit to the Ashtekar constraint.

The same set of constraints (7-9) are obtained starting from the action (5). In this case the Hamiltonian constraint $H_0 = 0$ arises as the primary constraint rather than the secondary one. It is more natural that both the Hamiltonian and the momentum constraints appear in the same stage of the canonical formalism.

The Hamiltonian constraint (9) is compared with the one in (3). The $\text{det} \pi$ term in (3) is replaced by $\text{det}B$ in (9). At first glance they seem to describe quite different systems. This point must be carefully examined since identification of the dynamical variables to the geometrical variables is not same for these cases. In the case of (3), the momentum is identified with the densitized triad and the gauge field is regarded as the self-dual spin connection as in the zero cosmological constant case. On the other hand the correspondence is modified in the case of (9).
The canonical Hamiltonian $\mathcal{H} = p\dot{q} - \mathcal{L}$ becomes
\[ \mathcal{H} = \pi^{0a}A_{0a} - A_{0a}J^a + \frac{1}{2} \epsilon^{ijk}E_{ia}B_j^aT_k + \frac{\eta}{2\det B}H_0, \]
where the electric field is $E_{ia} \equiv F_{0a}$ and $B_{ia}$ is the inverse of $B^a$ and $\det B(= \det B^a)$ is assumed to be non vanishing. In (10) we have used only $(p - \frac{\partial\mathcal{L}}{\partial\pi})^2 = 0$ as strong equality but $(p - \frac{\partial\mathcal{L}}{\partial\pi}) = 0$ has never been used. By this prescription we can obtain correct forms of multipliers on constraints in terms of $p, q$ and $\dot{q}$ (Kamimura, 1982).

The space-time metric is introduced if we can identify the constraints with diffeomorphism generators $H_\perp$ and $H_j (j = 1, 2, 3)$. In a metric space $H_j$'s generate transformations of three coordinates of a space-like hyper-surface. $H_\perp$, on the other hand, deforms the hyper-surface in its normal direction $n_\mu$. They satisfy the following algebra (Teitelboim, 1973).

\[ \{H_i(x), H_j(y)\} = H_j(y)\frac{\partial}{\partial x^i}\delta(x-y) - H_i(x)\frac{\partial}{\partial y^i}\delta(x-y), \quad (11) \]
\[ \{H_j(x), H_\perp(y)\} = H_\perp(y)\frac{\partial}{\partial x^j}\delta(x-y) \]
and
\[ \{H_\perp(x), H_\perp(y)\} = -\epsilon(\gamma^{ij}(x)H_j(x) + \gamma^{ij}(y)H_j(y))\frac{\partial}{\partial x^i}\delta(x-y), \quad (13) \]
where $\epsilon = n_\mu n^\mu$ is a signature of the metric i.e. $\epsilon = +1$ for Euclidian signature and $-1$ for Lorentzian one. $\gamma^{ij}$ is the induced metric of the space-like hyper-surface and appears in the Poisson bracket (13) as the structure function.

The constraints of the present system satisfy the same form of algebra
\[ \{T_i(x), T_j(y)\} = T_j(y)\frac{\partial}{\partial x^i}\delta(x-y) - T_i(x)\frac{\partial}{\partial y^i}\delta(x-y), \quad (14) \]
\[ \{T_j(x), \frac{1}{h^{1/2}}H_0(y)\} = \frac{1}{h^{1/2}}H_0(x)\frac{\partial}{\partial x^j}\delta(x-y) \quad (15) \]
and
\[ \{\frac{1}{h^{1/2}}H_0(x), \frac{1}{h^{1/2}}H_0(y)\} = -\frac{\epsilon}{h}[\pi^a_\pi_\pi_j - 2\mu^2B^aB_j^i]T_j(x) + \frac{\pi^a_\pi_\pi_j - 2\mu^2B^aB_j^i}{h}T_j(y)\frac{\partial}{\partial x^j}\delta(x-y). \quad (16) \]

Here and thereafter equalities are satisfied up to the $SO(3)$ gauge constraint; $J^a = 0$. It means that additional $SO(3)$ transformations are associated in the commutators. The weight factor $h$ multiplied on $H_0$ is a first order homogeneous function of $det\pi, det B$ and $\epsilon_{ijk}\epsilon_{abc}\pi^a B^jb^c$ and is assumed to be non vanishing. The case of $h = 0$ corresponds to a degenerate metric, which is not prohibited in the system of action. When $h$ vanishes, in a certain region of parameter space of $x^\mu$, the metric (of weight zero) is not well defined there. The action may describe such a generalization of the Einstein theory.

By comparing (11-13) with (14-16) we find
\[ H_j = T_j, \quad H_\perp = \frac{1}{h^{1/2}}H_0, \quad \gamma^{ij} = \epsilon \frac{\pi^a_\pi_\pi_j - 2\mu^2B^aB_j^i}{h}. \quad (17) \]
The identification of $H_\perp$ and $\gamma^{ij}$ is not unique since $h$ has not been determined.
The remaining components of the metric are found by noting that the Hamiltonian is the generator of surface deformation along the time axis. By projecting a vector along the time axis into normal and tangential components of the equal time surface the lapse and shift functions are determined as

\[ \alpha \equiv \frac{1}{\sqrt{\epsilon g^{00}}} = \frac{\epsilon h^{\frac{1}{2}}}{2 \det B} \quad \beta^k \equiv -\frac{g^{0k}}{g^{00}} = \frac{1}{2} \epsilon^{ijk} E_{ia} B^a_j, \]  

(18)

and

\[ \gamma^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}}. \]  

(19)

We have written all the components of the metric in terms of the dynamical variables of the action, the signature factor \( \epsilon \) and the density weight factor \( h \). We will examine whether the metric \( g_{\mu\nu} \) determined from (17-19) acquires a consistent interpretation. The requirements for the metric are the following: 1) all the components are real quantities and 2) \( \epsilon g^{00} \) must be positive. 3) \( \gamma^{ij} \) is positive definite so that the equal-time space spans a space-like hyper-surface.

The lapse function \( \alpha \) in (18) is a multiplier of secondary first class constraint \( H_\perp \) and is taken to be positive as a result of the gauge fixing condition on \( \eta \). The shift vector \( \beta^j \) is a multiplier of primary first class constraint \( T_j \) and is taken to be real as the gauge choice. \( \gamma^{ij} \), on the other hand, is determined dynamically by (17).

In the following we will show how the Einstein equation comes out explicitly. First we introduce an (inverse) triad variable \( e^{ia} \) by taking a square root of \( \gamma^{ij} \). Next we introduce a spin connection \( \omega_{\mu AB} \) \((\mu = 0,1,2,3)\) as an auxiliary function. It is defined using a torsion free condition

\[ \partial_{[\mu} e_{\nu]} A + \omega_{[\mu A} B e_{\nu]B} = 0 \]  

(20)

with the tetrad variables \( e_{\mu A} \),

\[ e_{ia} = (e^{ia})^{-1}, \quad e_{00} = \alpha, \quad e_{0a} = \beta^j e_{ja}, \quad e_{i0} = 0. \]  

(21)

From the torsion free condition the components of the spin connection are solved as functions of \( \pi^{ia}, B^{ia}, \alpha, \beta^j, \pi^a \) and \( B^a \). Among those, \( \omega_{iab} \) is given as a function of \( \pi^{ia} \) and \( B^{ia} \) only,

\[ \omega_{iab} = -\frac{1}{2} \epsilon^{c} (A_{abc} - A_{bca} - A_{cab}); \quad A_{abc} \equiv \epsilon^d e^j e_{k\epsilon} \partial_{ij} e_{k\epsilon c}. \]

Other components, \( \omega_{0ia}, \omega_{0ab} \) and \( \omega_{00a} \), depend on the time derivative of \( \pi^{ia} \) and \( B^{ia} \). They are evaluated using Hamilton’s equation of motion with the Hamiltonian (10)

\[ \mathcal{H} = \pi^{0a} \dot{A}_{0a} - A_{0a} J^a + \beta^j T_k + \epsilon \alpha H_\perp. \]  

(21)

The Hamilton’s equations are

\[ \dot{\pi}^{ia} = D_j [\frac{\epsilon \alpha}{h^{\frac{1}{2}}} \epsilon^{abc} (\pi^b \pi^c + 2 \mu^2 B^b B^c)] + D_j [\beta^j \pi^{ia} - \beta^i \pi^a] + g \epsilon^{abc} \pi^b A_{ac}, \]

(22)

\[ \dot{A}_{ia} = \frac{\epsilon \alpha}{h^{\frac{1}{2}}} \epsilon_{abc} \pi^b A_{kc} - \epsilon \epsilon_{ijk} \beta^j B^k_a + D_i A_{0a}. \]  

(23)

Thus the components of the spin connection are given as functions of \( \pi^{ia}, \alpha, \beta^j \) and \( A_{\mu a} \) (and their spatial derivatives).
In the case of $\mu = 0$, the first equation of motion (22) means that the gauge connection is the self dual spin connection. In showing it, it is crucial that the density weight factor $h$ is chosen to satisfy
\[ D_i \frac{e_{ia}}{\det e_{jb}} = 0. \] (24)

It is satisfied from the Gauss law constraint (7) if $h$ is proportional to $\det \pi$. The proportionality constant is chosen so that the tetrad becomes a real quantity,
\[ h = \frac{1}{I(\kappa \det \pi)^\frac{1}{2}} \rightarrow e_{ia} = \frac{\pi_{ia}}{(\kappa \det \pi)^\frac{1}{2}}. \] (25)

where $I = 1$ for the Euclidean and $I = i$ for the Lorentzian metric. By using (22) the components of the spin connection are explicitly evaluated and expressed as
\[ \frac{I g}{2} A_\mu a = \frac{1}{2} \left( \omega_\mu 0 a - \frac{I}{2} \epsilon_{abc} \omega_\mu bc \right) \equiv \omega_\mu^{(+)} 0 a. \] (26)

The self dual Riemannian tensor is then related to the gauge field strength by
\[ \frac{I g}{2} F_{ij}a = R_{ij}^{(+)} 0 a. \] (27)

Using this the constraints (8) and (9) are written as
\[ \epsilon_{ijk} e_{ab} R_{ij}^{(+)} 0 a k = 0, \quad \epsilon_{ijk} R_{ij}^{(+)} 0 a k = 0. \] (28)

The second equation of motion (23) is expressed in terms of self dual Riemannian tensors as
\[ R_{ij}^{(+)} 0 a + I \alpha \epsilon_{abc} e_{ab} R_{ij}^{(+)} 0 c a - \beta^j R_{ij}^{(+)} 0 a = 0. \] (29)

The equations (28) and (29) are exactly the set of Einstein equations for pure gravity theory written in terms of self dual Riemannian tensor. Thus in the case of $\mu = 0$ the pure Einstein gravity is reproduced from the action (1). The density weight $h$ has been determined under the criteria (24) and the equivalence of this system with that given by Ashtekar is completed.

For a small value of $\mu^2$ we can treat terms of $\mu^2$ perturbatively. For example the triad becomes
\[ e_{ia} = \frac{1}{h^2} (\pi_{ia} - \mu^2 B^ib \pi_j b B^ja), \]
where $\pi_{ia}$ is the inverse of $\pi_{ia}$. Equations (24-29) will be modified by terms of order $\mu^2$. The preliminary result shows that the additional terms do not mean presence of a standard cosmological constant but suggest a higher order curvature and/or torsion theory.

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