Quantum numbers of heavy neutrinos, tri-bi-maximal mixing through double seesaw with permutation symmetry, and comment on
\[ \theta_{\text{sol}} + \theta_c \simeq \frac{\pi}{4} \]

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ABSTRACT: Using the family symmetry, in the neutrino mass matrix we remove the Yukawa coupling (arising in the Dirac type mass between the heavy neutrinos and light lepton doublets) dependence in the double seesaw mechanism so that it is directly proportional to the mass matrix \( m_{nn} \) of heavy Majorana neutrinos. The family symmetry is supposed to be broken spontaneously at high energy scale so that the neutrino mass matrix is given by the family symmetry at high energy scale. With the permutation symmetry \( S_3 \), we note a variety of possible mass hierarchies arising distinctly in neutrinos, charged leptons, \( Q_{em} = -\frac{1}{3} \) quarks, and \( Q_{em} = \frac{2}{3} \) quarks. Distinguishing these hierarchies, we obtain a relation between the CKM angles and the MNS angles. Finally, we comment on the approximate relation \( \theta_{\text{sol}} + \theta_c \simeq \frac{\pi}{4} \).

KEYWORDS: Permutation symmetry, Double seesaw, Tri-bi-maximal mixing, CKM angles, MNS angles.
1. Motivation

Neutrino oscillations are parametrized by the MNS unitary matrix

\[
U_{\text{MNS}} = \begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{pmatrix}
\]  

(1.1)

where \( \alpha = \{e, \mu, \tau\} \) is the weak eigenstate index and \( i = \{1, 2, 3\} \) is the mass eigenstate index. The \( |\alpha\rangle \rightarrow |\beta\rangle \) transition amplitude in the time interval \( t \) is [1],

\[
\langle \nu_\beta | \nu_\alpha \rangle_t = \sum_j U_{\alpha j} U_{j\beta}^* e^{-iE_j t}.
\]  

(1.2)
Then, the survival probability of flavor $\nu_\alpha$ at high energy $E$ is given by

$$P_{\nu_\alpha \to \nu_\alpha} = 1 - \sum_{i,j} 4|U_{ai}|^2|U_{aj}|^2 \sin^2 \left( \frac{\Delta m^2_{ij} E}{4} \right)$$  \hspace{1cm} (1.3)

where $\Delta m^2_{ij} = m_i^2 - m_j^2$. Currently, the disappearance data from atmospheric and solar neutrinos point toward the following form,

$$U_{\text{MNS}} = \left( \begin{array}{ccc} \sqrt{\frac{2}{3}} e^{i\delta_3} & 0 & \frac{1}{\sqrt{2}} e^{i\delta_2} \\ \times & \times & \frac{1}{\sqrt{2}} e^{i\delta_1} \end{array} \right)$$  \hspace{1cm} (1.4)

where $\times$ is unspecified. Motivated by this observation, recently a tri-bi-maximal mixing form has been suggested $[2,3,4],$

$$U_{\text{MNS}} \simeq \left( \begin{array}{ccc} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{array} \right)$$  \hspace{1cm} (1.5)

which is possible starting from some discrete symmetries such as the permutation symmetry $S_3$ $[3]$ and tetrahedral symmetry $A_4$ $[4]$. Note that there does not exist a measurable CP phase in this form. Since the neutrino mixing matrix involves the unitary matrices diagonalizing charged lepton and neutrino mass matrices, one must consider both of these unitary matrices. With the $S_3$ symmetry, for example, the charged lepton and neutrino mass matrices are assumed to take different representations under $S_3$. One simple choice is assuming that charged leptons are singlets under the discrete group and the form $[1.3]$ is obtained purely from the neutrino mass matrix. Another possibility is to assume a bi-maximal form for the neutrino mass and a tri-maximal form for the charged lepton mass as done in Ref. $[3]$. Certainly, the latter choice is very appealing in the simplicity of explaining both the bi-maximal $[1]$ and tri-maximal $[3]$ structures in a single mixing matrix of $[1.5]$.

But there exists another complication due to the hypothetical mechanism for generating neutrino masses. In the standard model(SM), there exist renormalizable couplings for charged lepton masses.$^1$ But to generate neutrino masses at the SM level, non-renormalizable dimension-5 couplings are needed. To obtain these through the seesaw mechanism, one needs heavier neutrinos, collectively represented as $n$.

$^1$Renormalizable and nonrenormalizable couplings are the effective ones at low energy.
Thus, the above attractive proposal for the bi- and tri-structure has to be carefully addressed.

It is of utmost importance to relate the neutrino mass matrix and the $n$ mass matrix, or there are too many parameters to be assumed to specific values. In the seesaw scenario, there appear Yukawa couplings between singlet neutrinos and doublet neutrinos, which complicate a direct application of symmetry idea. In this regard, earlier Lindner et al. [7] studied the possibility of removing this Yukawa coupling dependence as ‘screening of Dirac flavor structure’.

For this purpose, we introduce a family symmetry and use the double seesaw mechanism to relate neutrino and $n$ mass matrices. In this process, we need two types of heavy neutrinos, collectively represented as $n$ and $N$ types and two continuous symmetries $F_1$ and $F_2$. Specifically, the dependence of neutrino mass matrix on the Yukawa couplings involving $N$ and lepton doublets are removed, which will be shown to be possible by a hierarchy of singlet vacuum expectation values (VEVs).

Another appealing phenomenological relation is the sum rule of the solar neutrino mixing, $\theta_{\text{sol}}$, and the Cabibbo angle, $\theta_c$,

$$\theta_{\text{sol}}^{\text{exp}} + \theta_c^{\text{exp}} \simeq 33^o + 13^o \rightarrow \frac{\pi}{4}. \quad (1.6)$$

Already there exist many ideas trying to explain the above relation [5, 6] with GUTs and with the quark-lepton complementarity idea, but most of them do not remove the Yukawa coupling dependence in the neutrino mass matrix. We find that Ref. [7] independently observes the same kind of the removal of Yukawa coupling dependence under the phrase, ‘screening of Dirac flavor structure’. In Eq. (1.6), $\theta_{\text{sol}}$ appears in the MNS matrix which is given by diagonalizing neutrino and charged lepton mass matrices,

$$U_{\text{MNS}} = U_l^\dagger U_\nu, \quad (1.7)$$

and $\theta_c$ appears in the CKM matrix which is obtained by diagonalizing $Q = \frac{2}{3}$ and $Q = -\frac{1}{3}$ quark mass matrices,

$$U_{\text{CKM}} = U_u^\dagger U_d. \quad (1.8)$$

To relate the mixing angles of the leptonic sector and the quark sector, one must unify leptons and quarks, or go beyond the SM to grand unified theories (GUTs) or

\[\text{\footnotesize Some think that the quark-lepton complementary relation is just a numerical accident.}\]
the quark-lepton complementarity. Here, we will be interested in the above sum rule also, and employ the quark-lepton complementarity idea. But we will not discuss any specific model in detail.

At the SM level, there are four types of mass matrices: $Q_{em} = -\frac{1}{3}$ quark mass matrix $m^{(d)}$, $Q_{em} = \frac{2}{3}$ quark mass matrix $m^{(u)}$, $Q_{em} = -1$ charged lepton mass matrix $m_l$, and $Q_{em} = 0$ neutrino mass matrix $m_\nu$. GUTs relate some of these mass matrices. The well-known one is the SU(5) relation with a Higgs quintet generating both $Q_{em} = -\frac{1}{3}$ quark and $Q_{em} = -1$ lepton mass matrices [9]. Then, we obtain a relation between four unitary matrices, $U_{\text{CKM}}, U_{\text{MNS}}$, and two unitary matrices $U_\nu$ and $U^{(u)}$ which diagonalize $m_\nu$ and $m^{(u)}$, respectively. Here, usually $U_{\text{CKM}}$ and $U_{\text{MNS}}$ are phenomenologically determined and $U_\nu$ and $U^{(u)}$ are given theoretically. Thus, in addressing the above questions, it is suggested that the relation arises naturally from a proposed discrete symmetry.

The four types of fermions have distinct mass hierarchies. The charged leptons and down type quarks have the similar pattern for masses, $m_{e,d} \ll m_{\mu,s} \sim \frac{1}{20}m_{\tau,b}$. The neutrino mass hierarchy is quite different from this,

$$\Delta m^2_{\nu ij} \ll \Delta m^2_{\nu jk}$$

where the LHS is for the solar neutrino oscillation and the RHS is for the atmospheric neutrino oscillation. Finally, the up type quark masses are distinct from any of the above patterns,

$$m_u \ll m_c \sim \frac{1}{150}m_t, \quad \text{or} \quad m_u, m_c \ll m_t.$$  \hfill (1.10)

The pattern (1.9) hints two almost degenerate neutrinos compared to the other neutrino, and the pattern (1.10) hints two almost massless quarks compared to top quark. These observations can be used as an input in the mixing angle relation.

In the SM without a family symmetry, there results the CKM mixing since the Yukawa couplings for the up-type quark masses are given differently from those for the down-type quark masses. With a family symmetry, nonzero mixing angles can arise only after breaking the imposed family symmetry. Since low energy Yukawa couplings are given in terms of dimension 4 renormalizable couplings, we assume that the original high energy couplings are non-renormalizable. If we consider an $S_3$ family symmetry, the quark mixing can arise only after the $S_3$ symmetry is spontaneously
broken differently for the up-type and down-type quark sectors. The same argument applies to the leptonic sectors also.

In Sec. 2, we introduce two continuous quantum numbers $F_1$ and $F_2$ to relate the neutrino masses and the $n$ masses via the double seesaw mechanism. In Sec. 4, distinct patterns of neutrinos, charged leptons, $Q_{em} = -\frac{1}{3}$ and $Q_{em} = \frac{2}{3}$ masses are used to obtain the tri-bi-maximal MNS unitary matrix. In Sec. 5, we relate $U_{CKM}$ and $U_{MNS}$ and obtain an approximate relation $\theta_{sol} + \theta_c \simeq \frac{\pi}{4}$. Sec. 6 is a conclusion.

2. Neutrino masses induced by heavy neutrinos

The charged lepton and quark masses arise from the dimension four Yukawa couplings

$$-\mathcal{L}_Y = f^{(u)}_{IJ} u^c_I H_2 q^J + f^{(d)}_{IJ} d^c_I H_1 q^J + f^{(e)}_{IJ} e^c_I H_1 l^J + \text{h.c.} \tag{2.1}$$

where all the fermions are represented in terms of left-handed Weyl fields, $q$ and $l$ are the quark and lepton doublets, and we used the two Higgs doublet notation with hypercharges $Y(H_1) = -\frac{1}{2}$ and $Y(H_2) = \frac{1}{2}$. If we introduce only one Higgs doublet $H_1$, then we replace $H_2$ by $-i\sigma_2 H_1^*$. In Eq. (2.1), the roman characters $I,J$ represent the family indices. The quark and lepton masses between families are distinguished by the difference of their Yukawa coupling strengths. The smallness of Cabibbo angle in the two family case is due to the hierarchy $f_{12}, f_{21}, f_{11} \ll f_{22}$. For the three family case, we have $f_{22}, f_{i3}(i = 1,2) \ll f_{33}$.

The gauge symmetry does not allow masses of neutrinos. To obtain neutrino mass, we must introduce more field(s). We adopt the seesaw idea of introducing SM singlet (neutral) heavy fermion(s), $n$. Then neutrino masses can arise through the seesaw mechanism, symbolically written as

$$m_\nu \sim \frac{(f v)^2}{\tilde{M}} \tag{2.2}$$

where $v$ is the vacuum expectation value of the Higgs doublet $H_2$, and $\tilde{M}$ is the Majorana mass of the heavy neutrino(s). Because the Yukawa coupling appears as $f^2$ in the numerator, it is not expected that the single seesaw would remove the $f^2$ dependence. To remove the $f^2$ dependence, we must have the same $f^2$ appearing in the denominator also. For this purpose, a double seesaw is needed as depicted in Fig. 1. In another context, the double seesaw was considered in Ref. [1], where however
Figure 1: A double seesaw diagram with generic eigenvalues of \( f_v, m \) and \( M \) have a hierarchy \( M \gg m \gg f_v \).

our attempt of removing the Yukawa coupling dependence was not tried. Also, a kind of \( U(2) \) symmetry for dimension-5 neutrino mass operator was considered [12], which does not belong to our scheme either. To relate Yukawa couplings appearing in the Dirac masses \( f_v \) and \( M \), we must introduce some symmetry. So let us introduce family quantum number \( Q_F \). For each family, let us introduce the following chiral fermions

\[
l_I \equiv \begin{pmatrix} \nu_I \\ l_I \end{pmatrix}, \quad N_I, \quad n_I, \quad q_I, \quad u^c_I, \quad d^c_I
\]

(2.3)

where \( N_I \) and \( n_I \) are neutral \( SU(2) \times U(1)_Y \) singlet heavy neutrinos and \( l \) and \( q \) are the lepton and quark doublets of the SM. One family is composed of 17 chiral fields, which together with Higgs multiplets can arise from the \( E_6 \) GUT with 27 [13] and trinification with \((3, 3^*, 1) + (1, 3, 3^*) + (3^*, 1, 3)\) [14, 15].

The SM singlet neutral leptons can have bare masses unless they are forbidden by a symmetry. The lepton number is a good symmetry forbidding their bare masses. We assign the opposite lepton numbers to \( l_I \) and \( N_I \). For \( n_I \), we introduce another independent quantum number, say \( n \)-number. \( l_I \) and \( N_I \) do not carry the \( n \) number. The symmetry of leptons is \( SU(2)_L \times U(1)_Y \times U(1)_{F_1} \times U(1)_{F_2} \) where \( U(1)_{F_1} \times U(1)_{F_2} \) is a continuous symmetry. To generate fermion masses, let us introduce the usual \( SU(2) \times U(1)_Y \) doublet Higgs field \( H_2 \) and \( SU(2) \times U(1)_Y \) singlet but \( U(1)_{F_2} \) nonsinglet
The lepton quantum numbers, $F_1$ and $F_2$, are

\[
\begin{array}{cccccccc}
l & e^c & N & n & S_1 & S_2 & H_2 \\
F_1 & 1 & -1 & -1 & 1 & -2 & 0 & 0 \\
F_2 & 0 & 0 & 0 & 1 & -2 & -1 & 0 \\
\end{array}
\]  \tag{2.4}

Consistently with the quantum numbers of (2.4), we can write the renormalizable Yukawa couplings involving singlet leptons as

\[
-\mathcal{L} = f_{lJ}^{(ln)} N^l H_2 l^J + f_{lJ}^{(nn)} N^l n^J S_2 + f_{lJ}^{(mn)} n^l n^J S_1 + \text{h.c.}
\]  \tag{2.5}

For a family symmetry, we require that $f_{lJ}$ are the same if $l, e^c, N$ and $n$ belong to the same family, i.e.

\[
f_{lJ}^{(ln)} = f_{lJ}^{(nn)} \rightarrow \ ? = f_{lJ}^{(nn)}.
\]  \tag{2.6}

The relation (2.6) for complex Dirac masses can be differentiated in principle from the relation (2.7) for real Majorana masses. But the family symmetry can be achieved by assigning $l$ and $n$ in the same multiplet. Note that the $F_2$ quantum numbers of $l$ and $n$ in (2.4) are different; thus we interpret $F_2$ as a $U(1)$ subgroup of a unifying group so that $l$ and $n$ can be put in the same representation of the unifying group.

The double seesaw diagram of Fig. 1 gives neutrino masses. We can see immediately that for $f^2$ to appear in the denominator, $M$ of Fig. 1 can be taken to be much larger than those of $m$. However, it is known that even without this restriction the Yukawa coupling dependence disappears [7]. However, an intuitive understanding of this phenomenon is most transparent in the limit

\[
V_2 \gg V_1 \gg v,
\]  \tag{2.8}

where $V_1 = \langle S_1 \rangle$, $V_2 = \langle S_2 \rangle$, and $v = \sqrt{2\langle H_2^0 \rangle}$. In this case, Fig. 1 gives the $f^2$ independent neutrino mass

\[
m^\nu_{IJ} = (v f_{IK})(M^{-1})_{KP}(m_{PQ})(M^{-1})_{QL}(v f_{LJ}) = \frac{v^2 V_1}{2 V_2^2} f_{IK}^{(ln)}(f_{lJ}^{(nn)})^{-1}_{KP} f_{PQ}^{(mn)}(f_{lJ}^{(nn)})^{-1}_{QL} f_{LJ}^{(ln)}
\]

where we used (2.6).
3. Some properties of $S_3$

If the family symmetry applied to the up-type and down-type quark s are identical, the CKM matrix would be diagonal. Therefore, it is necessary that the family symmetry is spontaneously broken differently for the up-type and down-type quark sectors. Let us briefly review how the $S_3$ symmetry can be broken differently for the up and down type quarks. The same strategy is applied to the leptons also. There is a long list of references on $S_3$, some of which are given in [17, 18].

3.1 Representations

$S_3$ is a permutation symmetry of three objects, which can be conveniently represented as permutations of three vertical points of equilateral triangle, $A \sim (1,0), B \sim (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $C \sim (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. In the complexified coordinate, $(x + iy, x - iy)$, these points are represented as

$$A \sim \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad B \sim \left( \begin{array}{c} \omega \\ \omega^2 \end{array} \right), \quad C \sim \left( \begin{array}{c} \omega^2 \\ \omega \end{array} \right)$$

where $\omega$ is a cube root of unity, $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. The permutation operation of three objects is

$$\begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ i & j & k \end{array}$$

(3.2)

where $\{ijk\}$ is a permutation of $\{123\}$. The operation (3.2) is simply written as $(ijk)$. Then, the six operations of $S_3$ are represented as

$$\begin{align*}
(123) & \sim \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad (231) \sim \left( \begin{array}{cc} \omega & 0 \\ 0 & \omega^2 \end{array} \right), \quad (312) \sim \left( \begin{array}{cc} \omega^2 & 0 \\ 0 & \omega \end{array} \right) \\
(132) & \sim \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad (321) \sim \left( \begin{array}{cc} 0 & \omega^2 \\ \omega & 0 \end{array} \right), \quad (213) \sim \left( \begin{array}{cc} 0 & \omega \\ \omega^2 & 0 \end{array} \right)
\end{align*}$$

(3.3)

(3.4)

From three objects $A, B, C$, we can construct a singlet $S \sim (A + B + C)$. The other remaining combinations form a doublet with components $D^\uparrow \sim (A + \omega^2 B + \omega C)$ and $D^\downarrow \sim (A + \omega B + \omega^2 C)$. Explicitly, we can show that

$$D^\uparrow \sim \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad D^\downarrow \sim \left( \begin{array}{c} 0 \\ 1 \end{array} \right).$$
3.2 Tensor products

Consider a tensor product from two doublets of $S_3$,

$$\Psi_X = \begin{pmatrix} \psi_1^1 \\ \psi_2^1 \end{pmatrix}, \quad \Psi_Y = \begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix}.$$  \hfill (3.5)

The product representation $\Psi_X \otimes \Psi_Y$ has four elements which have the transformation following properties under $S_3$

$$1 \sim (\psi_X^1 \psi_Y^2 + \psi_X^2 \psi_Y^1), \quad 1' \sim (\psi_X^1 \psi_Y^2 - \psi_X^2 \psi_Y^1),$$  \hfill (3.6)

and

$$2 \sim \begin{pmatrix} \psi_X^2 \psi_Y^2 \\ \psi_X^1 \psi_Y^1 \end{pmatrix}.$$  \hfill (3.7)

Repeating the multiplication rule (3.7), one can construct a singlet from three doublets of $S_3$ as

$$\psi_X^1 \psi_Y^1 \psi_Z^1 \pm \psi_X^2 \psi_Y^2 \psi_Z^2.$$  \hfill (3.8)

Including the above $2 \times 2$ tensor product, a dyadic is constructed from two triplets($3=S+D$), $S_1 + D_1$ and $S_2 + D_2$ where

$$3_1 : \quad S_1 = \frac{1}{3}(f_1 + f_2 + f_3), \quad D_1 = \frac{1}{3} \begin{pmatrix} f_1 + \omega^2 f_2 + \omega f_3 \\ f_1 + \omega f_2 + \omega^2 f_3 \end{pmatrix}$$  \hfill (3.9)

$$3_2 : \quad S_2 = \frac{1}{3}(f'_1 + f'_2 + f'_3), \quad D_2 = \frac{1}{3} \begin{pmatrix} f'_1 + \omega^2 f'_2 + \omega f'_3 \\ f'_1 + \omega f'_2 + \omega^2 f'_3 \end{pmatrix}.$$  \hfill (3.10)

Let the two index $S_3$ representation be $\Phi_{ij}$ which transforms as a diadic

$$\Phi_{ij} \sim \phi_i \otimes \phi'_j.$$  \hfill (3.11)

Similarly with (3.9,3.10), we define singlets and doublets with $\phi$s,

$$1_\phi : \quad \frac{1}{3}(\phi_1 + \phi_2 + \phi_3) \rightarrow \xi_1,$n

$$2_\phi : \quad \frac{1}{3} \begin{pmatrix} \phi_1 + \omega^2 \phi_2 + \omega \phi_3 \\ \phi_1 + \omega \phi_2 + \omega^2 \phi_3 \end{pmatrix} \rightarrow \begin{pmatrix} \xi_2 \\ \xi_3 \end{pmatrix}$$  \hfill (3.12)

$$1_{\phi'} : \quad \frac{1}{3}(\phi'_1 + \phi'_2 + \phi'_3) \rightarrow \xi'_1,$n

$$2_{\phi'} : \quad \frac{1}{3} \begin{pmatrix} \phi'_1 + \omega^2 \phi'_2 + \omega \phi'_3 \\ \phi'_1 + \omega \phi'_2 + \omega^2 \phi'_3 \end{pmatrix} \rightarrow \begin{pmatrix} \xi'_2 \\ \xi'_3 \end{pmatrix}$$  \hfill (3.13)

We can introduce $S_3$ representations having two indices following the transformation rules of dyadic (3.12,3.13) for two-index singlets and two-index doublets. The nine
components of the dyadic made of (3.12,3.13) split into the following $S_3$ multiplets, which constitute the representations of $\Phi$,

\[ S_3 = \xi_1 \xi'_1, \quad S_4 = \xi_2 \xi'_2 + \xi_3 \xi'_3, \quad S' = \xi_2 \xi'_3 - \xi_3 \xi'_2, \quad (3.14) \]

\[ D_3 = \xi_1 \begin{pmatrix} \xi'_2 \\ \xi'_3 \end{pmatrix}, \quad D_4 = \begin{pmatrix} \xi_2 \\ \xi_3 \end{pmatrix} \xi'_1, \quad D_5 = \begin{pmatrix} \xi_3 \xi'_3 \\ \xi'_2 \end{pmatrix}. \quad (3.15) \]

Below, we will use predominantly the dyadic symbols written with $\xi$s.

The renormalizable Yukawa coupling for quark masses are assumed to arise from nonrenormalizable dimension 5 operators at the Planck scale

\[ \sim \frac{1}{M_{Pl}} f_i f'_j H \langle \Phi_{ij} \rangle \quad (3.16) \]

where $f_i$ is the symbol for a fermion, $i, j$ are the labels of the permutation symmetry, $H$ is a Higgs doublet which does not carry a family index, and $\Phi_{ij}$ is the two-index scalar field. Since $H$ is a singlet of the family group, we can consider the relevant couplings presented in Table I with coupling constants $\lambda$s, which are interpreted as coupling times $\langle H^0 \rangle / M_{Pl}$. There are fifteen couplings.

Depending on the direction of VEVs, the permutation symmetry is broken.

If the couplings are the same $\lambda_1 = \cdots = \lambda_{15} = \lambda$, the mass matrix takes the following form for the $1 + 1'$ coupling

\[
\begin{array}{c|c}
\mathrm{S}_1\mathrm{S}_2\mathrm{S}_3 & 1 \\
\hline
\lambda_1(f_1 + f_2 + f_3)(f_1^* + f_2^* + f_3^*) & \lambda_3(f_1 + f_2 + f_3)(f_1^* + f_2^* + f_3^*)(\xi_2 \xi_3' - \xi_3 \xi_2') \\
\lambda_2(f_1 + f_2 + f_3)(f_1^* + f_2^* + f_3^*) & \lambda_4(f_1 + f_2 + f_3)(f_1^* + f_2^* + f_3^*)(\xi_1 \xi_3' - \xi_3 \xi_1') \\
\lambda_3(f_1 + f_2 + f_3)(f_1^* + f_2^* + f_3^*) & \lambda_5(f_1 + f_2 + f_3)(f_1^* + f_2^* + f_3^*)(\xi_1 \xi_2' - \xi_2 \xi_1') \\
\end{array}
\]
Then, for the vacuum direction

$$\langle \Phi_{11} \rangle = \langle \Phi_{22} \rangle = \langle \Phi_{33} \rangle,$$

we obtain a $C_3$ symmetric mass matrix,

$$\frac{\lambda}{9} \begin{pmatrix} a & c^* & b \\ b & a & c^* \\ c^* & b & a \end{pmatrix},$$

where

$$a = 3\xi_1^\prime \xi_1^\prime + 6\xi_2 \xi_3^\prime, \quad b = (1 + 2\omega^2)(\xi_1^\prime + 2\xi_2 \xi_3^\prime), \quad c^* = (1 + 2\omega)(\xi_1^\prime + 2\xi_2 \xi_3^\prime).$$

This is one example how a specific direction of the permutation group is chosen by spontaneous symmetry breaking. In gauge theories, such idea has been extensively studied \cite{16}. For other directions, the relations are not so simple and we do not present them here in detail. Below, we take the viewpoint that the vacuum chooses such directions when we assume a specific form of mass matrix.

4. Mixing matrix of light and heavy neutrinos

To fix the MNS mixing matrix $U_{\text{MNS}}$ of (1.7), one needs the neutrino mixing matrix $U_\nu$ obtained from (2.9) and charged lepton mixing matrix $U_l$ obtained from (2.1). If we identify the family symmetry ansatz even to Majorana neutrinos, (2.6) = (2.7), the MNS mixing matrix would be identity, leading to no mixing angle between different families. But the ‘family’ structure defined for Majorana neutrino masses can be in principle different from the family structure defined for Dirac masses. In this spirit, let us assume

$$f^{(\text{IN})} \neq f^{(\text{en})}$$

(4.1)
### 4.1 \( n \)-tuple maximal mixing

The maximal mixing angle in the fit of the atmospheric neutrino data suggests some kind of symmetry. The simple form of mass matrix with the flavor democracy [10] is

\[
m \propto \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\]

(4.2)

which has one heavy and two massless neutrinos. The above flavor democratic form belongs to a special case of permutation symmetry \( S_3 \) which has been extensively studied for neutrino masses [17].

In general, the permutation symmetry of \( n \) Majorana neutrinos dictates the following type mass matrix,

\[
m_n \propto \begin{pmatrix} 1 & r & r & \cdots & r \\ r & 1 & r & \cdots & r \\ r & r & 1 & \cdots & r \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r & r & r & \cdots & 1 \end{pmatrix}.
\]

(4.3)

With a real \( r \), we obtain for example the eigenvalues of \((1 \pm r)\) for \( n = 2 \), two \((1 - r)\) and one \((1 + 2r)\) for \( n = 3 \), and two \((1 - r)\) and \(1 + r(1 \pm \sqrt{2}i)\) for \( n = 4 \). Since we will be interested in three families, we do not consider the complication arising from \( n \geq 4 \). For \( n = 3 \), there exists a hierarchy of \( \Delta m^2_{ij} \), which is useful in explaining both atmospheric and solar neutrino data. For charged leptons, in general the mass matrix is complex and does not take the form (4.3).

Diagonalizing (4.3), we obtain a bi-maximal unitary transformation for the case of \( n = 2 \) [19],

\[
U_{\nu,2x2} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.
\]

(4.4)

\( m_2 \) is diagonalized as

\[
U_{\nu,2x2}^\dagger m_2 U_{\nu,2x2} \propto \begin{pmatrix} 1 - r & 0 \\ 0 & 1 + r \end{pmatrix}.
\]

(4.5)

For \( n = 3 \), we obtain a tri-maximal (third column) unitary transformation,

\[
U_{\nu,3x3} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}
\]

(4.6)
which diagonalizes $m_3$,

$$U_{\nu,3\times3}^\dagger m_3 U_{\nu,3\times3} \propto \begin{pmatrix} 1 - r & 0 & 0 \\ 0 & 1 - r & 0 \\ 0 & 0 & 1 + 2r \end{pmatrix}. \quad (4.7)$$

From (4.4) and (4.6), we try to make a tri-bi-maximal matrix, with one more row and column to be added to (4.4) at our disposal. But with the form (4.6), one cannot obtain a tri-bi-maximal mixing. Using the form (4.6) directly for charged leptons is not correct anyway since the mass matrix $m_l$ itself for charged leptons is not Hermitian. We can consider a Hermitian matrix $m_l^\dagger m_l$ for the diagonalization of which one uses the unitary transformation of left-handed charged leptons $U_l$.

### 4.2 Light neutrino mass matrix from heavy neutrino mass matrix

Following the scheme of the previous section, we investigate the mass matrix $m^{(nn)}$ which is proportional to $m_\nu$. The diagonalizing matrix of $m^{(nn)}$ will appear in the neutrino mixing matrix. The flavor democratic form for $m^{(nn)}$ (hence also for $m_\nu$ via the double seesaw) introduces one heavy and two massless neutrinos. Therefore, the $S_3$ symmetric form (4.3) introduces one heavy neutrino and two massive degenerate neutrinos. If the $S_3$ symmetry is slightly broken by $\epsilon_n$ in the degenerate subspace in the following way

$$m^{(nn)} = c \begin{pmatrix} 1 - r & 0 & \epsilon_n \\ 0 & 1 + 2r & 0 \\ \epsilon_n & 0 & 1 - r \end{pmatrix}, \quad (4.8)$$

the degenerate mass eigenvalues are split into $(1 - r \pm \epsilon_n)$ and $1 + 2r$. Here, the bi-maximal mixing matrix for diagonalizing $m^{(\nu)}$ is [19],

$$U_\nu = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (4.9)$$

But the form (4.9) does not depend on the strength of $\epsilon_n$ [19]. In fact, the mass matrix form (4.8) indicates that the neutrino triplet transforms as a singlet($n_2$) plus a doublet($n_1$ and $n_3$) since the $2 \times 2$ subspace of $m^{(nn)}$ has the structure of the form (4.8) in this subspace. However, we will treat $|\epsilon_n| \ll 1$ so that two scales of $\Delta m^2_{IJ}$ is obtained such that the atmospheric(2-3 subspace) and solar(1-3 subspace) neutrino data are explained.
So far we discussed the detailed structure of $f^{(nm)}$ in terms of the $S_3$ permutation symmetry. The form (4.9) can arise from a tiny breaking of the $S_3$ permutation symmetry. Note that with $\epsilon_n = 0$, we recover the $S_3$ symmetry in the original basis (4.3).

### 4.3 Charged leptons

On the other hand, the mass hierarchy of charged leptons is quite different from neutrino masses, $m_e \ll m_\mu \ll m_\tau$. Therefore, if $S_3$ is a useful symmetry, the mass matrix of charged leptons must break $S_3$ form (4.3) further to have three different masses for charged leptons. Since an $S_3$ symmetric real mass matrix form (4.3) has a degenerate pair which we try to avoid, we must use a subset of $S_3$ generators, leading to tri-maximal mixing. One obvious try is the cyclic permutation, i.e. $\{ijk\}$ of Eq. (3.2) is a cyclic permutation of (123). Thus, we choose only three elements among six $S_3$ generators, which is a cyclic permutation in one direction, $C_3$,

$$P_{123} \equiv I, \quad P_{231} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix}, \quad P_{312} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{pmatrix}. \quad (4.10)$$

Namely, we violate the exchange symmetry among any two indices. Choosing a subset of generators is achieved by the Higgs mechanism, which we will explore in a future communication. Still we have a subset permutation among three in $C_3$, there is a possibility of tri-maximal mixing. Taking the following mass matrix for charged leptons, consistently with the cyclic permutation $C_3$,

$$m^{(l)} = \begin{pmatrix} a & c^* & b \\ b & a & c^* \\ c^* & b & a \end{pmatrix}, \quad \text{we have}$$

$$M_l^2 \equiv (m^{(l)})^\dagger m^{(l)} = \begin{pmatrix} A & B^* & B \\ B & A & B^* \\ B^* & B & A \end{pmatrix}, \quad (4.12)$$

where

$$A = |a|^2 + |b|^2 + |c|^2, \quad B = a^*b + b^*c + ac.$$ 

Indeed, the diagonalizing matrix $U_l$ turns out to be tri-maximal [3],

$$U_l = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\omega^2}{\sqrt{3}} \\ \frac{\omega}{\sqrt{3}} & \frac{\omega^2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad (4.13)$$
where $\omega$ is a square root of unity, $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. $U_l$ diagonalizes $M_l^2$ to

$$U_l^\dagger M_l^2 U_l = \begin{pmatrix} A + B + B^* & 0 & 0 \\ 0 & A + B\omega + B^*\omega^2 & 0 \\ 0 & 0 & A + B\omega^2 + B^*\omega \end{pmatrix}.$$  

(4.14)

Thus, we identify $A, B$ and $B^*$ as

$$A = \frac{1}{3}(m_e^2 + m_\mu^2 + m_\tau^2)$$  

(4.15)

$$B = \frac{1}{3}(m_e^2 + m_\mu^2\omega^2 + m_\tau^2\omega)$$  

(4.16)

$$B^* = \frac{1}{3}(m_e^2 + m_\mu^2\omega + m_\tau^2\omega^2)$$  

(4.17)

### 4.4 Tri-bi-maximal mixing

Now, the MNS mixing matrix can be expressed in terms of $U_\nu$ and $U_l$,

$$U_{\text{MNS}} = U_l^\dagger U_\nu, \quad \text{or} \quad U_\nu = U_l U_{\text{MNS}}.$$  

(4.18)

From the neutrino mixing (4.19) and the charged lepton mixing (4.13), thus we obtain

$$U_{\text{MNS}} = U_l^\dagger U_\nu = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{\omega^2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} \\ \frac{\omega}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{7}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{2}} \end{pmatrix}.$$  

(4.19)

Defining $i\nu_3$ as a new mass eigenstate, the desired tri-bi-maximal mixing matrix results

$$U_{\text{MNS}} = \begin{pmatrix} \frac{\sqrt{7}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \end{pmatrix}.$$  

(4.20)

### 5. Relation $\theta_{\text{sol}} + \theta_c \simeq \frac{\pi}{4}$

#### 5.1 Up type quark masses

The intriguing phenomenological relation $\theta_{\text{sol}}^{\text{exp}} + \theta_c^{\text{exp}} \simeq \frac{\pi}{4}$ can be explained only if one relates the lepton and quark sectors, which is the basic principle of GUTs. In the quark sector, both the up and the down type mass matrices are complex. We observe the similarity in the hierarchies of charged lepton masses and $Q_{\text{em}} = -\frac{1}{3}$ quark masses [20]

$$m_e \simeq \frac{1}{200} m_\mu, \quad m_\mu \simeq \frac{1}{17} m_\tau$$  

(5.1)

$$m_d \simeq \frac{1}{20} m_s, \quad m_s \simeq \frac{1}{35} m_b.$$  

(5.2)
Along with charged leptons, we propose that the complex down type quark mass matrix is $C_3$ symmetric, leading to three hierarchical masses of $0, \sim \frac{1}{20} - \frac{1}{35}$ and $1$. In GUTs, the small discrepancy between charged lepton and $Q_{em} = -\frac{1}{3}$ quark masses is explained in various ways, for example by introducing the Georgi-Jarlskog type term [21].

But for the up type quarks there is a huge hierarchy of $0, \frac{1}{150}$ and $1$, due to the very large top quark mass

$$m_u \simeq \frac{1}{200} m_c \sim 0, \quad m_c \simeq \frac{1}{150} m_t.$$  \hspace{1cm} (5.3)

Here arises a question, “Should we treat the up type quark mass matrix $m^{(u)}$ as interpreting one heavy and two degenerate zero masses or three nondegenerate masses?” Since any perturbation can add a small addition, it is better to treat the up type matrix $m^{(u)}$ as the first case, namely having one heavy and two zero mass eigenvalues.

In addition to this phenomenological observation, treating $Q_{em} = -\frac{1}{3}$ quarks and $Q_{em} = \frac{2}{3}$ quarks differently is required to obtain a nontrivial CKM matrix. Then, it is of the same form as $m^{(nn)}$, but not quite because one is complex and the other is real. The matrix $m^{(u)\dagger}m^{(u)}$ is required to have two zero eigenvalues, which must be done with the $S_3$ symmetry.

Here we emphasize two aspects: one that the quark mass matrix is complex and another that $u$ quark is almost massless from the outset. Thus we introduce a flavor democratic form or an $S_2$ symmetric form with $r = \pm 1$ in the $2 \times 2$ subspace with zero entries at the other row and column. This type of mass matrix is

$$m^{(u)\dagger}m^{(u)} \propto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & \pm 1 & 1 \end{pmatrix}$$  \hspace{1cm} (5.4)

For an explicit demonstration, we choose the minus sign in Eq. (5.4) and obtain the third eigenvalue as $m_t$ with the following diagonalizing unitary matrix

$$U^{(u)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$  \hspace{1cm} (5.5)

where $U^{(u)\dagger}m^{(u)\dagger}m^{(u)}U^{(u)} = (m^{(u)\dagger}m^{(u)})_{\text{diag}}$. 

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5.2 Relating MNS and CKM angles

The MNS mixing matrix and the CKM mixing matrix are given by

$$U_{\text{MNS}} = U^\dagger_l U_\nu, \quad \text{or} \quad U_l = U_\nu U^\dagger_{\text{MNS}},$$  \hspace{1cm} (5.6)

$$U_{\text{CKM}} = U^{(u)\dagger} U^{(d)}, \quad \text{or} \quad U^{(d)} = U^{(u)} U_{\text{CKM}}.$$  \hspace{1cm}

In unifying models, $U^{(d)}$ and $U_l$ are usually related,

$$U^{(d)} = U_l : \text{quark-lepton complementarity} \hspace{1cm} (5.7)$$

$$U^{(d)} = U^\dagger_l : \text{SU}(5) \ GUT \hspace{1cm} (5.8)$$

If we choose the quark-lepton complementarity relation, the MNS and CKM angles are related by

$$U_{\text{CKM}} U_{\text{MNS}} \simeq U^{(u)\dagger} U_\nu : \text{quark-lepton complementarity} \hspace{1cm} (5.9)$$

On the other hand, the SU(5) GUT relation gives

$$U^{(u)} U_{\text{CKM}} = U_{\text{MNS}} U^\dagger_\nu : \text{SU}(5) \ GUT. \hspace{1cm} (5.10)$$

The SU(5) GUT relation can be studied with a specific form of $U^{(u)}$ and/or $U_\nu$. Here, we illustrate our idea with the quark-lepton complementarity, (5.9). The LHS of (5.9) relates $\theta_c$ and $\theta_{\text{sol}}$. Note that $\sin \theta_c^{\text{exp}} \simeq 0.22$ which leads to $\theta_c^{\text{exp}} \simeq 0.071 \pi$, and from $\cos \theta_{\text{sol}} = \frac{\sqrt{2}}{\sqrt{3}}$ we have $\theta_{\text{sol}}^{\text{th}} \simeq 0.196 \pi$; thus $\theta_c^{\text{exp}} + \theta_{\text{sol}}^{\text{th}} \simeq 0.267 \pi$. Basically, $\theta_c$ and $\theta_{\text{sol}}$ are related to $U_{11}$ elements of $U_{\text{CKM}}$ and $U_{\text{MNS}}$. Thus, we are interested in the first row of a real form of $U_{\text{CKM}}$ which are parametrized by two angles $\theta_c$ and $\varphi_q$

$$U_{\text{CKM}}^{\text{th}} \simeq \begin{pmatrix} \cos \theta_c^{\text{th}} & \sin \theta_c^{\text{th}} \cos \varphi_q & \sin \theta_c^{\text{th}} \sin \varphi_q \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \hspace{1cm} (5.11)$$

In the same vein, we are interested in the first column of a real form of $U_{\text{MNS}}$ which are parametrized by two angles $\theta_{\text{sol}}$ and $\varphi_l$,

$$U_{\text{MNS}}^{\text{th}} \simeq \begin{pmatrix} \cos \theta_{\text{sol}}^{\text{th}} & \times_{12} & 0 \\ \sin \theta_{\text{sol}}^{\text{th}} \cos \varphi_l & \times_{22} & \times_{23} \\ \sin \theta_{\text{sol}}^{\text{th}} \sin \varphi_l & \times_{32} & \times_{33} \end{pmatrix}. \hspace{1cm} (5.12)$$

from which we obtain

$$(U_{\text{CKM}} U_{\text{MNS}}^{\text{th}})_{11} = \cos \theta_c^{\text{th}} \cos \theta_{\text{sol}}^{\text{th}} + \sin \theta_c^{\text{th}} \sin \theta_{\text{sol}}^{\text{th}} (\cos \varphi_q \cos \varphi_l + \sin \varphi_q \sin \varphi_l).$$
From the tri-bi-maximal form (1.5), we identify $\cos \varphi_l = -\frac{1}{\sqrt{2}}$ and $\sin \varphi_l = -\frac{1}{\sqrt{2}}$, giving $\varphi_l = \frac{5}{4} \pi$. So, we obtain $\cos \varphi_q \cos \varphi_l + \sin \varphi_q \sin \varphi_l = \cos (\varphi_l - \varphi_q) = -\frac{1}{\sqrt{2}} (\cos \varphi_q + \sin \varphi_q)$.

The particle data book [20] gives $U^{\text{CKM} \, 11} = (0.9739 \text{ to } 0.9751)$ and $U^{\text{CKM} \, 13} = (0.0029 \text{ to } 0.0045)$, which gives $\varphi_q \simeq (0.0049 - 0.0065) \pi$. Thus, we have

\[
(U^{\text{th}}_{\text{CKM}} U^{\text{th}}_{\text{MNS}})_{11} = \cos(\theta^\text{th}_c + \theta^\text{th}_s) + \sin \theta^\text{th}_c \sin \theta^\text{th}_s \left[ 1 + \cos (\varphi_l - \varphi_q) \right] \\
= \cos(\theta^\text{th}_c + \theta^\text{th}_s) + \sin \theta^\text{th}_c \sin \theta^\text{th}_s \left[ 1 - \cos \left( \frac{\pi}{4} - \varphi_q \right) \right] \\
\rightarrow \cos(\theta^\text{th}_c + \theta^\text{th}_s) + \sin \theta^\text{exp}_c \sin \theta^\text{exp}_s \left[ 1 - \cos \left( \frac{0.25 - 0.0049}{\pi} \right) \right] \\
= \cos(\theta^\text{th}_c + \theta^\text{th}_s) + \sin \theta^\text{exp}_c \sin \theta^\text{exp}_s \left[ 1 - \cos \left( \frac{0.25 - 0.0065}{\pi} \right) \right] \\
\simeq \cos(\theta^\text{th}_c + \theta^\text{th}_s) + 0.034. \tag{5.13}
\]

where in the third row we used the experimental value for $\sin \theta^\text{th}_c \sin \theta^\text{th}_s$ since the replacement $\theta^\text{th}_c \rightarrow \theta^\text{exp}_c$ would introduce a small extra piece due to the smallness of $\sin \theta_c$.

On the other hand, the RHS of (5.9) with (5.5) is

\[
U^{(u)\dagger} U^\nu = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2}
\end{pmatrix}. \tag{5.14}
\]

Then, from Eqs. (5.13) and (5.14), we obtain

\[
\cos(\theta^\text{th}_c + \theta^\text{th}_s) \simeq \frac{1}{\sqrt{2}} \tag{5.15}
\]

where the accuracy of the sum is 5% in view of Eq. (5.13), which is a pretty good approximation. Thus, with the quark-lepton complementarity ansatz we obtain the following approximate relation,

\[
\theta^\text{th}_c + \theta^\text{th}_s \simeq \frac{\pi}{4}. \tag{5.16}
\]

Let us note that we obtained (5.16) with the following understanding:

(a) The possible dependence on the heavy neutrino Yukawa couplings is removed by the family symmetry. Here, the double seesaw mechanism is used.
(b) We used the $S_3$ symmetry categorically differently for $Q_{em} = 0, -1, -\frac{1}{3}$, and $\frac{2}{3}$ fermions. In particular, for the up type quarks, we use the mass matrix of the form (5.4), leading to one heavy and two zero masses.

(c) We obtain the relation $\theta_c^{\text{th}} + \theta_{\text{sol}}^{\text{th}} \simeq \frac{1}{4}\pi$ only approximately.

(d) The above relation is the one given at the GUT scale.

**Corrections** Running the Yukawa couplings from the GUT scale down to the electroweak scale can change the relation (5.10) significantly. But there exist ideas that this relation is not renormalized very much [22]. The relation is expected to be renormalized by large Yukawa couplings and the QCD coupling. For the Dirac type Yukawa couplings involving heavy leptons $N$ and $n$, they are independent from the top quark Yukawa coupling and hence can be taken as small values. So only the top quark Yukawa coupling is important. To use the relation (3.9), the RHS is evaluated at the unification scale and the LHS uses the experimental values at the electroweak scale. So we do not worry about the renormalization of the RHS.

The QCD coupling is flavor blind and hence the correction to quark masses is universal, leading to a factor of 3 modification [9] to quark masses down to 5 GeV, except that for top quark. For top quark, the difference from 3 is $\ln(5/175)/\ln(5/10^{16}) \sim 0.1$. But this correction is not what we are interested in since we use $m^{(u)}$ which has two zero eigenvalues. This structure of $m^{(u)}$ is not changed. For $m^{(d)}$, both $s$ and $b$ quarks are renormalized by the same factor 3, and hence we expect that $U_{\text{CKM}}$ is not changed very much by $\alpha_s$. In particular, we use only $U_{\text{CKM}11}$ which is close to 1 before and after the $\alpha_s$ correction [23].

For $m^{(d)}$, the most significant change due to the large top quark Yukawa coupling (symbolically represented as $Y_t$) is expected to arise from the violation of $m_t = m^{(d)}$ where $m^{(d)}$ is corrected by large $Y_t$. Both $m_t$ and $m^{(d)}$ are expected to take the form (4.11) at a quark-lepton complementary scale. The extra correction to $m^{(d)}$ due to $Y_t$ is expected to arise from the diagrams of the form given in Fig. 2, whose strength is estimated roughly as

$$\frac{|Y_t|^3 g_s^2}{(4\pi^2)^2} \cdot \text{(kinematic factor)} \sim \frac{|Y_t|^3 \alpha_s}{4\pi^3} \sim 2.5 \times 10^{-4} \quad (5.17)$$

which is smaller than $\frac{1}{20}$ of Eq. (5.2). Hence the down type mass category is not drastically changed, and hence the CKM angles are not changed drastically.
Most studies on the correction of MNS angles in GUTs have been performed by studying the running of neutrino masses arising through the dimension-5 operators $llH_2H_2$ \[22\]. Here, one usually assumes a large Yukawa coupling in view of the large top mass. Then, for non-hierarchical neutrino masses the MNS angle is known to go a drastic change, and for hierarchical neutrino masses the correction remains negligible. But in our double seesaw, the needed Yukawa couplings $f_{lIJ}^{(lN)}$ and $f_{lJJ}^{(Nn)}$ (viz. Eq. \((2.9)\)) can be taken to be small, and running of the MNS angle can be made negligible by taking $|f_{lIJ}^{(lN)}| \ll 1$.

Therefore, we expect that the LHS of \((5.9)\) is not corrected very much by going to the electroweak scale, and hence the sum \((5.16)\) is still valid.

**$Z_3$ orbifolds** The $S_3$ symmetry we discuss is expected to arise from a more fundamental theory. In the framework of quantum field theory, we can dictate relevant couplings corresponding to the presumed family symmetry. But it is our hope to obtain the couplings from a more fundamental theory. One example is string theory. From string theory, a good example allowing the permutation symmetry of 3 objects is the $Z_3$ orbifold compactification of $E_8 \times E_8'$ heterotic string \[24\]. The Yukawa couplings resulting from a $Z_3$ orbifold do not know how to distinguish the difference of three objects, leading to the discrete symmetry. The specific forms for Yukawa couplings are the result of spontaneous symmetry breaking of $S_3$ symmetry. We note that in $Z_3$ orbifolds, the $\sin^2 \theta_W$ problem hints toward a trinification model \[13\], in which we will explore a realization of the mass matrices discussed here in a future
communication.

6. Conclusion

In this paper, we use the family symmetry $S_3$ which is dictated to be realized differently for $Q_{em} = 0, -1, -\frac{1}{3},$ and $\frac{2}{3}$ fermions. We introduce two types of heavy neutral leptons $n$ and $N$ with two additional continuous symmetries $F_1$ and $F_2$. To discuss neutrino masses just from the symmetry principle, it is suggested to use the double seesaw mechanism so that the Yukawa couplings of $N$ and lepton doublets are removed. The double seesaw diagram of Fig. 1 removes the Yukawa coupling dependence if VEVs of singlets have a hierarchy $\langle S_2 \rangle \gg \langle S_1 \rangle \gg \langle H_2 \rangle$. Then one obtains a direct proportionality between the neutrino mass matrix and the $n$ type Majorana mass matrix, viz. Eq. (2.9). Now it becomes possible to discuss just the mass matrices, and we note that neutrinos, charged leptons, and $Q_{em} = -\frac{1}{3}, \frac{2}{3}$ quarks have distinct patterns of mass hierarchy. In the $S_3$ symmetric scheme, the mass matrix forms of $n$, charged leptons and $Q_{em} = -\frac{1}{3}, \frac{2}{3}$ quarks are dictated to be realized to conform with the observed mass patterns. These different patterns are the source of nontrivial MNS and CKM angles. A realization of these different patterns is expected to result from spontaneous symmetry breaking of family symmetry. In quark and lepton unification models, some of these angles can be related. In this paper, we studied the quark-lepton complementarity to relate the charged lepton type mixing matrix and the down type quark mixing matrix. The $SU(5)$ GUT type relation is also possible, for which however the resulting relation is not so simple. Finally, we also suggested a way to understand the approximate relation $\theta_{c}^{th} + \theta_{sol}^{th} \simeq \frac{\pi}{4}$.

Acknowledgments

We thank E. J. Chun, S. K. Kang and S. Raby for useful discussions. This work is supported in part by the KRF Sundo Grant No. R02-2004-000-10149-0, the KRF ABRL Grant No. R14-2003-012-01001-0, and the BK21 program of Ministry of Education.
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