Experimental measurement-device-independent quantification of quantum steering

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Einstein-Podolsky-Rosen steering is operationally defined as an entanglement verification task between two parties when one of their measurement devices is untrusted. Recent progress shows that the trustness of the other device can even be removed by preparing a set of tomographically complete quantum states along with it, in which the scheme is dubbed a measurement-device-independent (MDI) scenario. A benefit of the MDI scheme is that the original trusted measurement device does not need to perform quantum state tomography to characterize the set of steerable resources. In this work, we theoretically construct quantitative MDI steering witnesses without prior knowledge about the resource. By using these witnesses, we experimentally, for the first time, quantify the degree of steerability of the underlying steerable resource merely based on the observed statistics. Moreover, our result is not affected by the detection bias between the detectors of the Bell-state measurement. Finally, as two by-products, our experimental data directly estimates, also for the first time in a MDI manner, the degree of entanglement of the underlying state as well as the degree of measurement incompatibility of the involved measurements.

Introduction.—Quantum steering is an intriguing phenomenon in quantum mechanics that enables one party, say Alice, to remotely prepare a collection of quantum states for the other one, say Bob [1,2], in a way that it cannot be produced by any classical resource, e.g., a local-hidden-state model [3–5]. This collection of states, collected by a trusted measurement device, forms a steerable resource [6] and provides advantages for some quantum information tasks, such as entanglement certification [4], quantum key distribution [7], verifying incompatible measurements [8,9], quantifying non-Markovianity with the temporal analogue of quantum steering [10–13], as well as subchannel discrimination problems [14]. There are many significant experimental realizations [15–23] and theoretical works [24–25] in quantum steering.

Apart from quantum steering, there is another quantum inseparability called Bell nonlocality [29,30]. To demonstrate Bell nonlocality, the involved measurement devices do not have to be trusted, leading to an emergent discipline called device-independent (DI) quantum information processing [31]. Due to the inequivalence between steerable and nonlocal states [31,32], only a subset of steerable states can be certified through this DI way. Recently, motivated by the seminal works [33,34], Cavalcanti et al. [35] generalized the standard Bell-type experiment by replacing the real numbers for the inputs of a measurement device with a set of tomographically complete quantum states. Such a generalization makes all steerable states detectable even when both measurement devices are untrusted. Remarkably, Ku et al. [36] recently showed that, not only any steerable resource can be certified in a measurement-device-independent (MDI) scenario, but also the degree of steerability can be measured. The main benefit of the proposed MDI measure of steerability in Ref. [36] is that it is merely based on the observed statistics, while previous measures [6,13,37–42] require the trusted measurement device to perform quantum state tomography in order to fully characterize the underlying steerable resource.

In this work, we theoretically construct MDI steering witnesses (MDI-SWs) without prior knowledge about the underlying steerable resource (known as an assemblage). This approach not only certifies but also estimates the degree of steerability of the underlying assemblage. If Bob’s measurement is the projection onto the maximally entangled states, the estimation of the degree of steerability will be tight and becomes the MDI measure of steerability proposed by Ku et al. [36]. We experimentally, for the first time, estimate the degree of steerability of the family of two-qubit Werner states in a MDI scenario. We consider that Alice performs three measurements in the mutually unbiased bases (MUBs), for they can be used to demonstrate the strongest steerability to Bob [37]. On the other hand, Bob performs Bell-state...
measurements (BSMs) on his part of the states and the quantum inputs. Based on the observed correlations, the steerability for the family of two-qubit Werner states are quantified by constructing the tailored MDI-SWs. Moreover, as biproducts, the experimental data also directly estimates the degree of entanglement of the underlying state, as well as the amount of measurement incompatibility of Alice’s measurements. Compared with the previous experimental works [43][45] in the MDI scenarios, our work not only certifies the existence of entanglement and measurement incompatibility, but also quantifies these quantities.

Quantifying quantum steering in a measurement-device-independent scenario. —We first introduce our new approach to directly construct MDI-SWs according to the observed statistics. A MDI scenario is composed of two parties, Alice and Bob, sharing a quantum state \( \rho_{AB} \) (see Fig. 1). During each round of the experiment, Alice chooses a measurement setting \( x \) to perform the measurement on her system, and obtains an outcome \( a \). On the other hand, Bob performs a joint measurement on his system jointly with an input quantum state \( \tau_y \), the set of which forms a tomographically complete set. Their joint probability distributions can be expressed as:

\[
p(a, b|x, \tau_y) = \text{Tr} \left[ (E_{a|x} \otimes E_b)(\rho_{AB} \otimes \tau_y) \right] \quad \forall a, b, x, y,
\]

where \( \{E_{a|x}\}_a \) and \( \{E_b\}_b \) are the positive-operator-valued measurements (POVM) (i.e., the general quantum measurements) describing Alice’s measurement \( x \) and Bob’s joint measurement with the corresponding sets of outcomes \( \{a\} \) and \( \{b\} \), respectively. In the resource theory of steering [6], the interested quantity is a collection of Bob’s reduced subnormalized quantum states corresponding to Alice’s measurements, i.e., \( \{\sigma_{a|x} = \text{Tr}_A(E_{a|x} \otimes \mathbb{1}_B \rho_{AB})\}_{a,x} \), called an assemblage [46]. Therefore, the correlation can be rewritten as:

\[
p(a, b|x, \tau_y) = \text{Tr} \left[ E_b \left( \sigma_{a|x} \otimes \tau_y \right) \right] \quad \forall a, b, x, y.
\]

Now, if an assemblage satisfies a local-hidden-state (LHS) model [4], described by a probability distribution \( D(a|x, \lambda) \) and preexists (subnormalized) quantum states \( \rho_{\lambda} \), i.e., \( \sigma_{\text{LHS}}^{a|x} = \sum_{\lambda} D(a|x, \lambda) \rho_{\lambda} \), Eq. (1) becomes

\[
p(a, b|x, \tau_y) = \sum_{\lambda} D(a|x, \lambda) \text{Tr} \left[ (E_{b|\lambda} \tau_y) \right] \quad \forall a, b, x, y,
\]

where \( E_{b|\lambda} := \text{Tr}_B \left[ E_b(\rho_\lambda \otimes \mathbb{1}) \right] \) is an effective POVM with \( \sum_{\lambda} E_{b|\lambda} = \mathbb{1} \). The partial trace is on the Hilbert space where \( \rho_{\lambda} \) is acting. The question that if a given assemblage \( \{\sigma_{a|x}\}_a \) admits a LHS model is equivalent to if a given correlation \( \{p(a, b|x, \tau_y)\} \) admits Eq. (2). In Sec. A of the Supplemental Material [47], we show that the question above can be verified by solving the feasibility problem with a semidefinite program.

In Ref. [36], a measure of steerability in a MDI scenario was proposed with the optimal measurement \( E_2 \) in Eq. (1). However, in a more general and practical MDI scheme, Bob’s measurements are uncharacterized. In the following, we will consider such a general case and provide a method to obtain a lower bound on the degree of steerability. First, we transform the formulation of the MDI measure proposed in Ref. [36] and define the following MDI steering witness (MDI-SW) (see Sec. B of the Supplemental Material [47] for the detailed derivation).

\[
W_1 = \sum_{a,x,y} \beta_{a,1}^{x,y} p(a, 1|x, \tau_y),
\]

where \( \beta_{a,1}^{x,y} \) are some real numbers. The witness \( W_1 \) satisfies two properties: (i) \( W_1 \leq 1 \) for any given unsteerable assemblage no matter what measurements Bob performs; (ii) For any given steerable assemblage, one can always choose some proper \( \{\beta_{a,1}^{x,y}\} \) and some suitable measurements on Bob’s side, such that \( W_1 > 1 \). The classical bound 1 is due to the structure of \( \{\beta_{a,1}^{x,y}\} \), a valid set can be computed by the semidefinite program (see Sec. B of the Supplemental Material [47]). With Eq. (3), we define the quantitative MDI-SW as the following optimization problem:

\[
S_1^{\text{MDI}}(P) = \max_{\beta} \left\{ \max_{W_1} W_1 - 1, 0 \right\},
\]

where \( \beta := \{\beta_{a,b}^{x,y}\}_{a,x,y} \) and \( P := \{p(a, b|x, \tau_y)\}_{a,x,y} \) is the correlation concerning only one of Bob’s outcomes \( b = 1 \) in this case). The quantitative MDI-SW can be computed by the semidefinite program (see Sec. B of the Supplemental Material [47]) and it is a lower bound on the MDI steering measure proposed in Ref. [36]. The bound becomes tight when Bob’s measurement is the projection onto the maximally entangled state. Note that even if Bob’s inputs do not form a complete set, Eq. (1) still provides a valid lower bound. This can be understood from the fact that the set of tomographically complete inputs is a resource for Bob to demonstrate steerability in a MDI scenario. The lack of a complete set of quantum inputs can only decrease the degree of steerability.
One can see a similar discussion on the quantification of entanglement in a MDI scenario in Ref. [48].

Furthermore, in Sec. C of the Supplemental Material [47], we prove that for the underlying assemblage \( \{ \sigma_{a|x} \} \) being a qubit state, all of the four measurement operators \( E_b \) of the BSM are optimal for Bob, i.e., the produced correlation for each \( b \) leads to the maximum value of \( S_{\text{MDI}}(P) \). Therefore, Eq. (4) can be modified into the following form:

\[
S_{\text{MDI}}(P) := \max \left\{ \frac{1}{4} \sum_{b=1}^{4} \left( \max_{\beta} W_b - 1 \right), 0 \right\}, \quad (5)
\]

As we will show later, this formulation overcomes the problem of the detection bias between the detectors of the BSM.

In the following, we will experimentally demonstrate how to estimate, in a MDI manner, the degree of steerability of the underlying steerable resource given by Alice’s three measurement settings with the two dimensional MUBs acting on the two-qubit Werner states, namely \( \rho_{AB} = v |\psi^{-}\rangle \langle \psi^{-}| + \left( \frac{1 - v}{4} \right) \mathbb{I} \), with visibility \( 0 \leq v \leq 1 \), \( |\psi^{-}\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \), and \( \mathbb{I} \) being the identity operator.

**Experimental setup.**—The schematic diagram of our experimental setup is given in Fig. 2. A 100 mW continuous laser beam passes through HWP@404 nm to make the horizontally polarized (H) component and vertically polarized (V) component balanced. The beam is focused on two type-I phase-matched \( \beta \)-barium borate crystals (0.5 mm \( \times \) 6 mm \( \times \) 6 mm), whose optical axes are normal to each other, to produce a pair of entangled photons with 808 nm. The photons are sent to Alice and Bob through the polarization-maintaining single mode fibers. The set of components marked as \( \Omega \) is where the photon is reflected by or transmitted through a 50:50 BS. When the photon is reflected, the two-qubit state will dephase to a completely mixed state by three 386\( \lambda \) quartz plates (QP) and a 22.5\( ^\circ \) rotated HWP [49, 50]. At last, the reflected part, combined with the transmission part, incoherently prepares the Werner state, and the visibility \( v \) can be tuned by the attenuators. In our experiment, the photons are filtered by 3 nm bandwidth interference filters (IF), creating a coherence length of about 260\( \lambda \), which is much smaller than the path difference, 0.15 m. Therefore, the prepared Werner state is an incoherent mixture, instead of a coherent superposition.

Before the photon arrives to Bob’s side, the quantum input \( \tau_y \) is encoded on the path degree of freedom of Bob’s particle. The blue box in Fig. 2 (the detailed struc-
tured is shown below) performs like a non-polarization beam splitter. Here, the main component is the designed beam displacer (BD), which can make the V light pass through it directly and make the H light pass through it with a 4 mm displacer at 808 nm parallel with V. Firstly, the photons are separated into two beams with the first BD, then a cut HWP is used to unify the polarization of the photons. The second BD splits the H(V) component of the input light once again into 0H and 1H (0V and 1V) with the ratio \(\cos^2 \theta / \sin^2 \theta\), where \(\theta\) is the rotation angle of the half-wave plate H2. At last, the third BD combines the 0H and 0V components into the output light 0, and 1H and 1V components into the output light 1. By slightly tilting the third BD, we can compensate the phase of the two-photon state \(|HV\rangle - |VH\rangle\). At the same time, the phase between 1H and 1V is controlled by tilting the HWP.

Now let us illustrate the way to implement Bob’s optimal joint measurement, i.e., the Bell-state measurement (BSM). The photons in path 1 undergo a bit-flip operation while the photons in path 0 undergo an identity operation. The two operations together are equivalent to a controlled NOT (CNOT) gate. Then the following are joint measurements of the control qubit and the target qubit of the CNOT gate. The ports D1, D2, D3 and D4 correspond to the measurements in the basis \(H \otimes (0 + 1), H \otimes (0 - 1), V \otimes (0 + 1), \) and \(V \otimes (0 - 1), \) respectively, implementing a completed BSM. Here we only need to measure the operator on two degrees of freedom of the same particle (the polarization and the path degree of freedom), similar to the former works [45, 51, 52]. This method avoids the entangled measurement on two particles, which is a tough task with 50% efficiency in linear optics [53, 54].

Experimental results.—To quantify the steerability, we use the above experimental setup. More specifically, after sending the two-qubit Werner state \(\rho_{AB}\) to Alice and Bob, we obtain the set of probability distributions \(\{p(a, b|x, \tau_y)\}\) (Eqs. (1)) by Alice performing measurements in the bases of \(\{X, Y, Z\}\) on her part of the system while Bob performs the joint measurement on his part of system and his quantum inputs \(\tau_y\). Bob’s tomographically complete set of quantum inputs is composed of eigenstates of the three Pauli matrices. The joint measurement performed by Bob is set to be the BSM [36] so that we can obtain the maximal value of MDI-SW. As mentioned before, for Bob’s part of system being a qubit, one can count all of the four outcomes of his BSM and still obtain the same value of \(S_{\text{MDI}}(P)\), therefore arriving at the formulation of Eq. (3). This overcomes the problem when there are some biases between the four detectors of the BSM. More specifically, consider that we have four detectors with the biased detection rates of \(\xi_1, \xi_2, \xi_3, \) and \(\xi_4, \) respectively, with \(\sum_b \xi_b = 4 \) and \(\xi_b \geq 0 \forall b\). For the ideal case, \(\xi_1 = 1\) for all \(b\). When there exists some bias, the observed correlation will be \(\xi_0 \cdot p(a, b|x, \tau_y)\). However,
this does not affect the degree of steerability:

\[
S^{\text{MDI}}(\mathcal{P}, \{\xi_b\}) := \max \left\{ \frac{1}{4} \sum_{b=1}^{4} \left( \max_\beta W_b(\mathcal{P}, \{\xi_b\}) - 1 \right), 0 \right\} \\
= \max \left\{ \frac{1}{4} \sum_{b=1}^{4} \max_\beta \frac{\xi_b}{b} \cdot \sum_{a, x, y} \beta^a_{a, b} p(a, b|x, \tau_y) - \xi_b, 0 \right\} \\
= \max \left\{ \frac{1}{4} \sum_{b=1}^{4} \xi_b \left( \max_\beta W_b(\mathcal{P}) - 1 \right), 0 \right\} \\
= S^{\text{MDI}}(\mathcal{P}).
\]

Our experimental results are plotted in Fig. 3 (a). As seen there, although the quantitative MDI-SW we propose in Eq. (6) may not perform the best among the other ones described by Eq. (4), it is the most suitable one in the sense that the variance to the theoretical prediction is the smallest. We note that the detection biases dominate the experimental errors instead of the photon losses. If one considers the photon losses, the effect merely shrinks the degree of steerability \[36\]. Therefore, the improved MDI-SWs are robust against not only detection biases but also losses.

In addition to quantifying the degree of steerability of the underlying assemblage in a MDI scenario, here we show also for the first time, that our experimental results directly estimate degree of the entanglement ER(\(\rho_{AB}\)) of the underlying state and the degree of measurement incompatibility IR(\(\{E_{a|x}\}\)) of Alice’s measurements in Fig. 3 (b). We give a brief introduction to these two quantities in Sec. D of the Supplemental Material \[47\]. Note that, in Ref. \[13\], it has been shown that the steering robustness of the assemblage SR(\(\{\sigma_{a|x}\}\)) is a lower bound both on ER(\(\rho_{AB}\)) and IR(\(\{E_{a|x}\}\)). Therefore, as a lower bound on SR(\(\{\sigma_{a|x}\}\)), the tailored MDI-SW \(S^{\text{MDI}}(\mathcal{P})\) is also a lower bound on ER(\(\rho_{AB}\)) and IR(\(\{E_{a|x}\}\)).

In summary, we experimentally estimate the degree of steerability in a MDI scenario. We also propose an improved MDI-SWs tailored to the case where Bob receives a qubit system. Furthermore, the improved MDI-SWs overcomes the problem that there exist some detection biases between Bob’s detectors. As bi-products, we also use our experimental data to estimate the degree of entanglement of the underlying state and the amount of incompatibility of the involved measurements.

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Note added. After finishing this work, a somewhat similar work concerning high-dimensional MDI quantum steering and randomness was recently proposed by Guo et al. \[75\].

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See Supplemental Material at [URL will be inserted by publisher], which contains also references to [56–63], for further details on (1) the semidefinite program of certifying if the given correlation is steerable, (2) the semidefinite program of computing the quantitative MDI-SW described by Eq. (5) in the main text, (3) the proof of the optimal two-qubit joint measurement for Bob, (4) a brief review of the quantities (the steering robustness, the entanglement robustness, and the incompatibility robustness) used in our work and their bound relations, and (5) the related experimental data of the quantum state tomography on the prepared Werner states.

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Appendix A: Certification of correlations compatible with assemblages admitting a LHS model

For a given correlation \( \{ p(a, 1|x, \tau_y) \} \) and Bob’s set of inputs \( \{ \tau_y \} \), the problem of checking whether the correlation is steerable or not, i.e., checking if it is compatible with a correlation generated from a steerable or unsteerable assemblage, is equivalent to the following problem:

\[
\text{given } \{ p(a, 1|x, \tau_y) \} \text{ and } \{ \tau_y \}, \quad \text{find } \{ E_1, \{ \sigma_\lambda \} \}
\]

\[
\text{s.t. } \begin{align*}
p(a, 1|x, \tau_y) &= \text{Tr} \left[ E_1 \left( \sum_\lambda D(a|x, \lambda) \sigma_\lambda \otimes \tau_y \right) \right] \forall a, x, y, \\
E_1 &\geq 0, \\
\text{Tr} \sum_\lambda \sigma_\lambda &= 1,
\end{align*}
\]

(A1)

where the notation \( E_1 \geq 0 \) denotes that \( E_1 \) is positive semidefinite. The first constraint is from Eq. (1), where the assemblage \( \{ \sigma_{a|x} \} \) is unsteerable, i.e., \( \sigma_{a|x} = \sum_\lambda D(a|x, \lambda) \sigma_\lambda \), \( \forall a, x \). The correlation \( \{ p(a, 1|x, \tau_y) \} \) in the first constraint can be written as

\[
p(a, 1|x, \tau_y) = \sum_\lambda D(a|x, \lambda) \text{Tr}[E_{1|\lambda} \tau_y] \quad \forall a, x, y.
\]

(A2)

Here, we define an effective POVM element

\[
E_{1|\lambda} := \text{Tr}_B E_1 (\sigma_\lambda^B \otimes \mathbb{1}^B_0).
\]

(A3)

Using Eq. (A2), Eq. (A1) can be formulated as the following semidefinite program (SDP)

\[
\text{given } \{ p(a, 1|x, \tau_y) \} \text{ and } \{ \tau_y \}, \quad \text{find } \{ E_{1|\lambda} \}
\]

\[
\text{s.t. } \begin{align*}
p(a, 1|x, \tau_y) &= \sum_\lambda D(a|x, \lambda) \text{Tr}[E_{1|\lambda} \tau_y] \quad \forall a, x, y, \\
E_{1|\lambda} &\geq 0 \quad \forall \lambda.
\end{align*}
\]

(A4)

If the above SDP is feasible, the given correlation is unsteerable; otherwise it is steerable. Note that Eq. (A2) [and Eq. (2) in the main text] can be seen as a LHS model written in the formulation of correlations in a MDI scenario. Also note that even if the underlying assemblage is steerable, if Bob inappropriately chooses a measurement, their correlation still admits Eq. (A2), i.e., a LHS model. Moreover, as the result shown in Ref. [36], if Bob’s measurement is the projection onto the maximally entangled state, the correlation \( \{ p(a, b|x, \tau_y) \} \) is steerable [i.e., not admit Eq. (A2)] if and only if the assemblage is steerable.

**Appendix B: Constructing quantitative MDI steering witnesses by semidefinite programming**

First, we would like to introduce the MDI steering measure (MDI-SM) proposed by Ku et al. [36] in the following form:

\[
S_{\text{measure}}^{\text{MDI}} := \max \left\{ \frac{I(\vec{\beta}, P)}{\max_{P \in \text{LHS}} I(\vec{\beta}, P)} - 1 \bigg| b = 1, 0 \right\},
\]

(B1)

where

\[
I(\vec{\beta}, P) := \sum_{a,x,y} \beta_{a,b}^{x,y} p(a, b|x, \tau_y)
\]

for a chosen \( b \),

(B2)

with \( \vec{\beta} := \{ \beta_{a,b}^{x,y} \}_{a,x,y} \) and \( P := \{ p(a, b|x, \tau_y) \}_{a,x,y} \) is the correlation concerning only one of Bob’s outcomes \( b = 1 \) in this case. Here, \( P \in \text{LHS} \) is the set of correlations obtained from the assemblages admitting a LHS model, i.e., Eq. (A2) [and Eq. (2) in the main text]. In Ref. [36], it has been proved that the optimal statistics \( P \) in Eq. (B1) is obtained whenever Bob performs the measurement projecting his two systems onto the maximally entangled state \( \frac{1}{\sqrt{2}} \sum_i |i \rangle \otimes |i \rangle \). Furthermore, it has been shown [36] that this measure is equivalent to the steering robustness [14], therefore it is a steering monotone [6].

An open question raised in Ref. [36] is: can we construct a valid set of the coefficients \( \{ \beta_{a,b}^{x,y} \} \) if Bob’s measurement is not the optimal one, i.e., not the projection onto the maximally entangled state. This is crucial in a MDI scenario because, in general, we do not make any assumption on the involved measurements. In other words, in the most general case, we are not able to optimize Bob’s measurements to obtain the optimal correlation. Therefore, the optimization over \( P \) in Eq. (B1) is removed, and the result is a lower bound on the MDI-SM:

\[
\max \left\{ \frac{I(\vec{\beta}, P)}{\max_{P \in \text{LHS}} I(\vec{\beta}, P)} - 1 \bigg| b = 1, 0 \right\} \leq S_{\text{measure}}^{\text{MDI}}
\]

(B3)

In what follows, we show that the lower bound can be computed by a semidefinite program.

The first step is to redefine the set of coefficients \( \{ \beta_{a,b}^{x,y} \} \) as

\[
\tilde{\beta}_{a,1}^{x,y} := \frac{\beta_{a,b}^{x,y}}{\max_{P \in \text{LHS}} I(\vec{\beta}, P)}.
\]

(B4)

The left-hand-side in Eq. (B3) can then be written as

\[
\max_{\vec{\beta}} \sum_{a,x,y} \tilde{\beta}_{a,1}^{x,y} p(a, 1|x, \tau_y) - 1,
\]

(B5)

which is exactly Eq. (1). Note that, we omit the trivial case where the given correlation admits a LHS model for simplicity. We assume the given correlation is steerable, therefore the outermost maximization in Eq. (B3) is the first term instead of 0. The above optimization problem in Eq. (B3) can be solved by the following semidefinite program:
Note that if one obtains an optimal leading to an unsteerable correlation, i.e., the assemblage \( \{ \beta_{a,1}^x, y \} \) is multiplied by a positive semidefinite operator \( \text{Tr}_B \{ |\psi\rangle \langle \psi| (\sigma_\lambda \otimes \mathbb{1}) \} \), where \( |\psi\rangle = 1/\sqrt{d} \sum_{i=1}^d |i\rangle \otimes |i\rangle \) and \( \text{Tr} \sum_\lambda \sigma_\lambda = 1 \) with \( \sigma_\lambda \geq 0 \) \( \forall \lambda \). After taking the trace and summing over all \( \lambda \), we obtain

\[
\text{Tr} \sum_\lambda \left\{ \left( d\mathbb{1} - \sum_{a,x,y} D(a|x,\lambda) \beta_{a,1}^{x,y} \tau_y \right) \cdot \text{Tr}_B \{ |\psi\rangle \langle \psi| (\sigma_\lambda \otimes \mathbb{1}) \} \right\} = \text{Tr} \sum_\lambda \left\{ |\psi\rangle \langle \sigma_\lambda \otimes \mathbb{1} | d \right\} - \sum_{a,x,y} \beta_{a,1}^{x,y} \sum_\lambda \text{Tr} \sum_\lambda \sum_\lambda D(a|x,\lambda) \text{Tr} \{ |\psi\rangle \langle \sigma_\lambda \otimes \tau_y | \}
\]

\[
\text{max}_{\beta_{a,1}^{x,y}} \sum_\lambda \sum_{a,x,y} \beta_{a,1}^{x,y} \text{LHS}(a,1|x,\tau_y) \leq \text{max}_{\beta_{a,1}^{x,y}} \sum_\lambda \sum_{a,x,y} \beta_{a,1}^{x,y} \text{LHS}(a,1|x,\tau_y) \geq 0.
\]

Therefore, the first constraint in Eq. (B6) holds. The second equality in the above equation comes from the fact that

\[
\text{Tr} \{ |\psi\rangle \langle (A \otimes B) | \} = \text{Tr} \{ A \cdot B^T | d \}
\]

and that the numerator of the second term in the second line can be treated as a correlation obtained by Bob applying his measurement (corresponding to \( |\psi\rangle \langle \psi| \)) on an unsteerable assemblage, i.e.,

\[
\sum_{a,x,y} \beta_{a,1}^{x,y} \text{Tr} \{ |\psi\rangle \langle \sigma_\lambda \otimes \tau_y | \}
\]

leading to an unsteerable correlation \( \{ p_{\text{LHS}}^{\sigma_\lambda}(a,1|x,\tau_y) \} \).

The last inequality holds because \( \text{Tr} \sum_\lambda \sigma_\lambda = 1 \) and

\[
\sum_{a,x,y} \beta_{a,1}^{x,y} \text{LHS}(a,1|x,\tau_y) \leq 1.
\]

The second constraint in Eq. (B6) is due to the relation

\[
F_{a|x} = \sum_y \beta_{a,1}^{x,y} \tau_y \geq 0
\]

between the coefficient \( \{ \beta_{a,1}^{x,y} \} \) and the standard steering witness \( \{ F_{a|x} \} \), which is chosen to be positive semidefinite when constructing the MDI-SM [36].

With the above semidefinite program, Eq. (B5) (or Eq. (A) in the main text) can be obtained, and it provides a lower bound on the MDI steering measure \( S_{\text{measure}}^{\text{MDI}} \).

Note that if one obtains an optimal \( \beta^* \) for a correlation \( \mathbf{P}_1 \), this set of coefficients \( \beta^* \) is still a valid set, although may not be optimal, for any other correlation \( \mathbf{P}_2 \). That is, \( \beta^* \) satisfies the constraints in Eq. (B6) for either \( \mathbf{P}_1 \) or \( \mathbf{P}_2 \). This means when one obtains an optimal set \( \{ \beta_{a,1}^{x,y} \} \) for a given correlation, this set is also a steering witness for some other steerable assemblages. Therefore in Eq. (3), we define MDI steering witnesses with the following general formulation:

\[
W_1 = \sum_{a,x,y} \beta_{a,1}^{x,y} p(a,1|x,\tau_y) \leq 1 \quad \forall \mathbf{P} \in \text{LHS}.
\]

**Appendix C: The optimal two-qubit joint measurements for Bob**

Recall that the original proposed MDI measure of steerability [36] is written as (see Eq. (C1) in the last section)

\[
S_{\text{measure}}^{\text{MDI}} := \text{max} \left\{ \max_{\beta} \right. \left. \frac{\sum_{a,x,y} \beta_{a,1}^{x,y} p^*(a,1|x,\tau_y)}{\text{sup}_\mathbf{P} \sum_{a,x,y} \beta_{a,1}^{x,y} p(a,1|x,\tau_y)} - 1,0 \right\}
\]

where the set of probability distributions

\[
p^*(a,1|x,\tau_y) = \text{Tr} \{ E_1^* (\sigma_{a|x} \otimes \tau_y) \} \quad \forall a, x, y
\]

is the optimal correlation obtained by performing the optimal projection \( E_1^* \) of Bob’s joint measurement on the assemblage \( \{ \sigma_{a|x} \} \) and the quantum inputs \( \{ \tau_y \} \). In Ref. [36], it has been proved that the projection onto the maximally entangled state \( \frac{1}{\sqrt{d}} \sum_i |i\rangle \otimes |i\rangle \) is the optimal one for Bob. In what follows, we show that for Bob’s
assemblage \( \{ \sigma_{a|x} \} \) being a qubit, the four projections of the Bell-state measurement, i.e.,

\[
\begin{align*}
|\phi_1\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\
|\phi_2\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\
|\phi_3\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\
|\phi_4\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).
\end{align*}
\]

(C3)

are all the optimal ones providing the optimal correlation \( \{ p^*(a, b|x, \tau_y) \} \) if the set of tomographically complete quantum inputs is composed of the eigenstates of the three Pauli matrices. That is, \( \{ \tau_y \} = \{ |0\rangle, |1\rangle, |V\rangle, |H\rangle, |L\rangle, |R\rangle \} \), where \( \{ |0\rangle, |1\rangle \} \), \( \{ |V\rangle, |H\rangle \} \), \( \{ |L\rangle, |R\rangle \} \), are, respectively, the eigenstates of the Pauli matrices \( Z \), \( X \), and \( Y \). Indeed, the four Bell states in Eq. (C3) can be transformed into each other by applying some Pauli gates on them, i.e.,

\[
|\phi_y\rangle = (\mathbb{1} \otimes U_b)|\phi_1\rangle(\mathbb{1} \otimes U_b) \quad \forall b,
\]

(C4)

where \( U_b \in \{ \mathbb{1}, X, Y, Z \} \). Therefore, when Bob’s measurement outcomes correspond to the other three projections (i.e., \( b \neq 1 \)), the obtained correlation becomes

\[
p^*(a, b|x, \tau_y) := \text{Tr} \left[ E_b(\sigma_{a|x} \otimes \tau_y) \right] = \frac{1}{2} \left| \sum_x \text{Tr} \left[ (\mathbb{1} \otimes U_b)|\phi_1\rangle(\mathbb{1} \otimes U_b) (\sigma_{a|x} \otimes \tau_y) \right] \right|.
\]

(C5)

It is easy to see that the elements of the set \( \{ \tau_y \} \) remain the same as that of the set \( \{ \tau_y \} \). Therefore, the components of the correlation \( \{ p^*(a, b \neq 1|x, \tau_y) \} \) are just a permutation of the components of \( \{ p^*(a, 1|x, \tau_y) \} \), which means that these correlations can all achieve the value of \( S_{\text{measure}} \) in the main text, including the steering robustness \([14]\), the entanglement robustness \([56][58]\), and the incompatibility robustness \([25]\). We also review their bound relations proposed in Ref. \([14][59][61]\).

Among several measures of steerability, we consider the steering robustness \([14]\) in order to provide bounds on entanglement and measurement incompatibility. The steering robustness for a given assemblage \( \{ \sigma_{a|x} \} \) is to minimize the ratio of a noisy assemblage that one has to mix with to destroy the steerability, i.e.,

\[
\text{SR}(\{ \sigma_{a|x} \}) = \min_{\{ \delta \}} \sum_{x} \text{Tr} (\sigma_{a|x}) - 1
\]

(D1)

\[
\delta \geq 0 \quad \forall x,
\]

(D2c)

where \( D(a|x, \lambda) := \delta_{a\lambda} \) is the deterministic probability distribution \([14][37]\). We note that one can further define the steering robustness of a given "quantum state" \( \rho_{AB} \), which is obtained by optimizing over all possible assemblages \( \{ \sigma_{a|x} \} \) Bob can obtain. It is equivalent with the optimization over all Alice’s possible measurements \( \{ E_{a|x} \} \) due to the relation \( \sigma_{a|x} = \text{Tr}_{A}(E_{a|x} \otimes \mathbb{1} \rho_{AB}) \) for all \( a, x \). Apparently, \( \text{SR}(\{ \sigma_{a|x} \}) \) is a lower bound on \( \text{SR}(\rho_{AB}) \).

The generalized robustness of entanglement (or the entanglement robustness in short) \([56][58]\) of a given quantum state \( \text{ER}(\rho_{AB}) \) is the minimum amount the noisy state one has to mix with, such that the mixture becomes a separable state. That is,

\[
\text{ER}(\rho_{AB}) = \min_{t} \quad \frac{\rho_{AB} + t\omega_{AB}}{1 + t} \quad \text{is separable,}
\]

(D3)

\[
\omega_{AB} \quad \text{is a quantum state.}
\]

In general, it is hard to characterize the set of separable states. However, one can still relax this set to the positive-partition-transposition states. Through this way, a lower bound on the above solution can be obtained by solving the following semidefinite program \([62]\):

\[
\min_{\omega_{AB}} \quad \text{Tr}(\omega_{AB}) - 1
\]

(D4)

\[
\omega_{AB} \geq 0, \quad \omega_{AB} \geq \rho_{AB},
\]

where \( \geq \) denotes a matrix being positive semidefinite and \( T_A \) for the partial transposition of the operator with respect to the Hilbert space of A. In particular, if the given

Appendix D: Bound relations between the steering robustness, the entanglement robustness, and the incompatibility robustness

For readers’ reference, in this section we briefly review the detailed formulation of the quantities mentioned in
state \( \rho_{AB} \) is a qubit-qubit or a qubit-qutrit state, which is also the case we consider in this work, it has been shown that this lower bound is tight \( \text{[57]} \).

In quantum theory, not all observables can be measure simultaneously. Such a property can be formulated as that there is no single POVM describing a nonjointly measurable measurement \( \text{[63]} \), i.e.,

\[
E_{a|x} \neq \sum_{\lambda} p(a|x, \lambda) G_{\lambda},
\]  

(D5)

for some \( a, x \), where \( \{E_{a|x}\}_a \) is the POVM representing the measurement input \( x \) and \( a \) is a measurement outcome. Note that \( G_{\lambda} \geq 0 \ \forall \lambda \) and \( \sum_{\lambda} G_{\lambda} = 1 \). Here, \( p(a|x, \lambda) \) is a probability distribution, and can be chosen, without loss of generality, to be \( p(a|x, \lambda) = D(a|x, \lambda) := \delta_{a, \lambda(x)} \). A way to quantify the incompatibility of given measurements is to minimize the ratio of noisy measurements one has to mix with, such that the mixture becomes jointly measurable. This so-called incompatibility robustness is formulated as \( \text{[25]} \).

\[
\text{IR}(\{E_{a|x}\}) = \min_{\{G_{\lambda}\}} \frac{r}{1 + r} \quad \text{s.t.} \quad E_{a|x} + r N_{a|x} = \sum_{\lambda} p(a|x, \lambda) G_{\lambda}, \quad \forall a, x,
\]

(D6)

\( \{N_{a|x}\}_a \) is a POVM \( \forall x \),

which can be solved by the following semidefinite program:

\[
\text{IR}(\{E_{a|x}\}) = \min_{\{\tilde{G}_{\lambda}\}} \frac{1}{d} \sum_{\lambda} \text{Tr}[\tilde{G}_{\lambda}] - 1
\]

s.t. \( \sum_{\lambda} D(a|x, \lambda) \tilde{G}_{\lambda} \geq E_{a|x} \quad \forall a, x, \)

\[
\tilde{G}_{\lambda} \succeq 0 \quad \forall \lambda,
\]

\[
\sum_{\lambda} \tilde{G}_{\lambda} = \frac{1}{d} \sum_{\lambda} \text{Tr}[\tilde{G}_{\lambda}],
\]

(D7)

where \( d \) is the dimension of \( E_{a|x} \).

Finally, let us review the bound relations used in our work. In Ref. \( \text{[14]} \), it has been shown that the steering robustness of the underlying quantum state is a lower bound on the entanglement robustness, i.e.,

\[
\text{ER}(\rho_{AB}) \geq \text{SR}(\rho_{AB}) \geq \text{SR}(\{\sigma_{a|x}\}).
\]

(D8)

On the other hand, it has been shown that the steering robustness of the assemblage is a lower bound on the incompatibility robustness of the involved measurements \( \text{[59-61]} \), i.e.,

\[
\text{IR}(\{E_{a|x}\}) \geq \text{SR}(\{\sigma_{a|x}\}).
\]

(D9)

In the main text, we use the bound relations Eqs. \( \text{(D8)} \) and \( \text{(D9)} \) to quantify the degree of entanglement of the underlying state and the incompatibility of the involved measurements based on our quantitative MDI-SM.

Appendix E: Quantum state tomography

In our experiment, the detailed forms of the prepared states are obtained by standard tomography, and the local measurements are realized by properly adjusting the configuration of the experimental setup in Fig. 2 in the main text. To be specific, Bob adjusts \( H2 \) to the angle of 0° to make the photon pass through Path-1 entirely, and then uses the QWP, HWP combined with the following polarizing beam splitter (PBS), to complete the standard polarization analysis, while Q1, H1 and the PBS are used on Alice’s side. In our experiment, we prepare the Werner states with the visibilities \( v = 0.9934(11), 0.8575(56), 0.7250(72), 0.5870(77) \) and \( 0.4689(72) \), and the corresponding fidelities are \( f = 0.996(1), 0.980(7), 0.958(6), 0.959(12) \) and \( 0.977(2) \) respectively. By the projection onto \( |HH\rangle \langle HH| \) and \( |VV\rangle \langle VV| \), the visibilities of the Werner states can be obtained through

\[
v = 1 - 2(\text{Tr}[(\rho_{AB})^2]|HH\rangle \langle HH| + |VV\rangle \langle VV|]).
\]

(E1)