IMPROVING BECKNER’S BOUND VIA HERMITE FUNCTIONS

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ABSTRACT. We obtain an improvement of the Beckner’s inequality \( \|f\|_2^2 - \|f\|_p^2 \leq \left(2 - p\right)\|\nabla f\|_2^2 \) valid for \( p \in [1, 2] \) and the Gaussian measure. Our improvement is essential for the intermediate case \( p \in (1, 2) \), and moreover, we find the natural extension of the inequality for any real \( p \).

1. INTRODUCTION

1.1. The history of the problem. The Poincaré inequality \[ \int_{\mathbb{R}^n} f^2 d\gamma_n - \left(\int_{\mathbb{R}^n} f d\gamma_n\right)^2 \leq \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n \] for any smooth bounded function \( f : \mathbb{R}^n \to \mathbb{R} \). Later William Beckner \cite{Beckner} generalized \[ \int_{\mathbb{R}^n} f^p d\gamma_n - \left(\int_{\mathbb{R}^n} f d\gamma_n\right)^p \leq \frac{p(p-1)}{2} \int_{\mathbb{R}^n} f^{p-2} |\nabla f|^2 d\gamma_n \] for any smooth bounded \( f : \mathbb{R}^n \to (0, \infty) \). We caution the reader that in \cite{Beckner} inequality \[ \int_{\mathbb{R}^n} f^p d\gamma_n - \left(\int_{\mathbb{R}^n} f d\gamma_n\right)^p \leq \frac{p(p-1)}{2} \int_{\mathbb{R}^n} f^{p-2} |\nabla f|^2 d\gamma_n \] was formulated in a slightly different but equivalent form (see Theorem 1, inequality \ref{eq:log-Sobolev} in \cite{Beckner}). It should be also mentioned that in case \( p = 2 \) inequality \[ \int_{\mathbb{R}^n} f^p d\gamma_n - \left(\int_{\mathbb{R}^n} f d\gamma_n\right)^p \leq \frac{p(p-1)}{2} \int_{\mathbb{R}^n} f^{p-2} |\nabla f|^2 d\gamma_n \] does coincide with \[ \int_{\mathbb{R}^n} f^2 d\gamma_n - \left(\int_{\mathbb{R}^n} f d\gamma_n\right)^2 \leq \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n \] for any smooth bounded \( f : \mathbb{R}^n \to (0, \infty) \). If \( p \to 1+ \) then \[ \int_{\mathbb{R}^n} f^p d\gamma_n - \left(\int_{\mathbb{R}^n} f d\gamma_n\right)^p \leq \frac{p(p-1)}{2} \int_{\mathbb{R}^n} f^{p-2} |\nabla f|^2 d\gamma_n \] provides us with log-Sobolev inequality (see \cite{Beckner}). In general, the constant \( \frac{p(p-1)}{2} \) is sharp in the right hand side of \[ \int_{\mathbb{R}^n} f^p d\gamma_n - \left(\int_{\mathbb{R}^n} f d\gamma_n\right)^p \leq \frac{p(p-1)}{2} \int_{\mathbb{R}^n} f^{p-2} |\nabla f|^2 d\gamma_n \] as it can be seen for \( n = 1 \) on the test functions \( f(x) = e^{\varepsilon x} \) by sending \( \varepsilon \to 0 \).
Later Beckner’s inequality (2) was studied by many mathematicians for different measures, in different settings and for different spaces as well. For possible references we refer the reader to [1, 2, 4, 5, 6, 8, 9, 10, 11, 26, 21, 20].

An analysis done in [18] indicates that the right hand side (RHS) of (2) can be improved. In the present paper we address this issue: what is the precise estimate of the difference given in the left hand side (LHS) of (2), and whether the requirement $p \in [1, 2]$ can be avoided by slightly changing the RHS of (2).

We give complete answers to these questions. For example, if $p = \frac{3}{2}$ we will obtain an improvement in Beckner’s inequality (2)

$$\int_{\mathbb{R}^n} f^{3/2} d\gamma_n - \left( \int_{\mathbb{R}^n} f d\gamma_n \right)^{3/2} \leq \int_{\mathbb{R}^n} \left( f^{3/2} - \frac{1}{\sqrt{2}} \left( 2 f - \sqrt{f^2 + |\nabla f|^2} \right) \sqrt{f + \sqrt{f^2 + |\nabla f|^2}} \right) d\gamma_n. \quad (3)$$

The LHS of (3) coincides with the LHS of (2) for $p = \frac{3}{2}$, but the RHS of (3) is strictly smaller than the RHS in (2). Indeed, notice that we have the following pointwise inequality

$$x^{3/2} - \frac{1}{\sqrt{2}} \left( 2x - \sqrt{x^2 + y^2} \right) \sqrt{x + \sqrt{x^2 + y^2}} \leq \frac{3}{8} x^{-1/2} y^2 \quad \text{for all} \quad x, y \geq 0, \quad (4)$$

which follows from the homogeneity, i.e., take $x = 1$. As one can see the improvement of Beckner’s inequality (2) is essential. Indeed, if $y \to \infty$ then the RHS of (4) increases as $y^2$ whereas the LHS of (4) increases as $y^{3/2}$. Also notice that if $x \to 0$ then the difference in (4) tends to infinity. The only place where the quantities in (4) are comparable is when $y/x \to 0$.

1.2. Main results. Let $k$ be a real parameter. Let $H_k(x)$ be the Hermite function such that it satisfies the Hermite differential equation

$$H_k'' - xH_k' + kH_k = 0, \quad x \in \mathbb{R}, \quad (5)$$

and which grows relatively slowly $H_k(x) = x^k + o(x^k)$ as $x \to +\infty$. If $k$ is a nonnegative integer then $H_k$ is the probabillists’ Hermite polynomial of degree $k$ with the leading coefficient 1, for example, $H_0(x) = 1, H_1(x) = x, H_2(x) = x^2 - 1$ etc. In general, for arbitrary $k \in \mathbb{R}$ one should think that $H_k$ is the analytic extension of the Hermite polynomials in $k$ (existence and many other properties will be mentioned in Section 2).

For $k \in \mathbb{R}$ let $R_k$ be the rightmost zero of $H_k(x)$ (see Lemma 1). If $k \leq 0$ then we set $R_k = -\infty$. Define $F_k(x)$ as follows

$$F_k \left( \frac{H_k'(q)}{H_k(q)} \right) = \frac{H_{k+1}(q)}{H_k^{1+1/k}(q)} \quad \text{for} \quad q \in (R_k, \infty). \quad (6)$$
We will see in the next section $F_k \in C^2([0, \infty))$ is well-defined and $F_k(0) = 1$. Moreover, if $k > -1$ then $F_k$ will be decreasing concave function, and if $k < -1$ then $F_k$ will be increasing convex function.

One may observe that

$$F_1(y) = 1 - y^2; \quad F_2(y) = \frac{1}{\sqrt{2}} \sqrt{2 - \sqrt{1 + y^2}} \sqrt{1 + \sqrt{1 + y^2}}.$$ 

If $k = 0$ then definition (6) should be understood in the limiting sense as follows

$$F_{\exp}(H_{-1}(q)) = q \exp \left( \alpha - \int_1^q H_{-1}(s) ds \right) \quad \text{for all} \quad q \in \mathbb{R},$$

where

$$\alpha = \int_1^\infty \left( H_{-1}(s) - \frac{1}{s} \right) ds \approx -0.266 \ldots.$$ 

**Theorem 1.** For any $p \in \mathbb{R} \setminus [0, 1]$ and any smooth bounded $f \geq 0$ with $\int_{\mathbb{R}^n} f^p d\gamma_n < \infty$ we have

$$\int_{\mathbb{R}^n} f^p F_{\frac{1}{p-1}}\left( \frac{\|\nabla f\|}{f} \right) d\gamma_n \leq \left( \int_{\mathbb{R}^n} f d\gamma_n \right)^p.$$ 

The inequality is reversed if $p \in (0, 1)$.

The theorem improves Beckner’s inequality (2). This will follow by taking the first two nonzero Taylor terms of $F_{\frac{1}{p-1}}(t)$ as its lower estimate.

**Proposition 1.** We have pointwise improvement in Beckner’s inequality (2), i.e.,

$$1 - \frac{p(p - 1)}{2} t^2 \leq F_{\frac{1}{p-1}}(t) \quad \text{for all} \quad t \geq 0, \ p \in (1, 2].$$

The improvement will be essential when $t \to \infty$. For example, it will become clear in the next section that as $t \to \infty$ we have

$$F_{\frac{1}{p-1}}(t) \sim -t^p \left( H_{p-1} \left( \frac{1}{p-1} \right) \right)^{1-p} \quad \text{for} \quad p > 1;$$

$$F_{\frac{1}{p-1}}(t) \sim \left( \frac{p}{1-p} \right) \left( \frac{e^{t^2/2} \sqrt{2\pi}}{t \Gamma(\frac{1}{1-p})} \right)^{1-p} \quad \text{for} \quad p < 1, \ p \neq 0.$$ 

Our theorem interpolates several inequalities. If $p \to 1+$ then (8) gives log-Sobolev inequality. If $p = 2$ then (8) provides us with Poincaré inequality. If $p \to \pm \infty$ then we obtain $e$-Sobolev inequality.

**Corollary 1.** For any smooth bounded $f$ we have

$$\int_{\mathbb{R}^n} \exp(f) F_{\exp}(\|\nabla f\|) d\gamma_n \leq \exp \left( \int_{\mathbb{R}^n} f d\gamma_n \right).$$
Finally if $p \to 0$ we obtain negative log-Sobolev inequality:

**Corollary 2.** For any smooth bounded $f \geq 0$ with $\int_{\mathbb{R}^n} \ln f \, d\gamma_n > -\infty$ we have

$$\int_{\mathbb{R}^n} - \ln f \, d\gamma_n + \ln \left( \int_{\mathbb{R}^n} f \, d\gamma_n \right) \leq \int_{\mathbb{R}^n} -F_{-\ln}(\frac{\nabla f}{f}) \, d\gamma_n$$

where $F_{-\ln}(t)$ is defined as follows

$$F_{-\ln}\left( \frac{H_{-2}(x)}{H_{-1}(x)} \right) = \int_1^x H_{-1}(s) \, ds - c + \ln H_{-1}(x), \quad x \in \mathbb{R}.$$

It is worth mentioning that the current paper provides with estimates of $\Phi$-entropy (see [11]):

$$\text{Ent}_{\gamma_n}^{\Phi}(f) \overset{\text{def}}{=} \int_{\mathbb{R}^n} \Phi(f) \, d\gamma_n - \Phi \left( \int_{\mathbb{R}^n} f \, d\gamma_n \right)$$

for the following fundamental examples:

- $\Phi(x) = x^p$ for $p \in \mathbb{R} \setminus [0, 1]$  \hspace{1cm} \text{Theorem [1]}
- $\Phi(x) = -x^p$ for $p \in (0, 1)$  \hspace{1cm} \text{Theorem [1]}
- $\Phi(x) = e^x$, \hspace{1cm} \text{Corollary [1] or $p \to \pm \infty$ in Theorem [1]}
- $\Phi(x) = -\ln x$, \hspace{1cm} \text{Corollary [2] or $p \to 0$ in Theorem [1]}
- $\Phi(x) = x \ln x$, \hspace{1cm} $p \to 1$ in Theorem [1]

### 2. The proof of the theorem

The proof of the theorem amounts to check that the real valued function

$$M(x, y) = x^p F_k \left( \frac{y}{x} \right)$$

defined on $[\varepsilon, \infty) \times [0, \infty)$ for any $\varepsilon > 0$ obeys necessary smoothness condition, it has a boundary condition $M(x, 0) = x^p$ and it satisfies the following partial differential inequality

$$\left( M_{xx} + \frac{M_y}{y} M_{xy} M_{yy} \right) \leq 0,$$

with reversed inequality in (13) if $p \in (0, 1)$. Then by Theorem 1 in [18] we obtain that

$$\int_{\mathbb{R}^n} f^p F_k \left( \frac{|\nabla f|}{f} \right) \, d\gamma_n = \int_{\mathbb{R}^n} M(f, |\nabla f|) \, d\gamma_n \leq M \left( \int_{\mathbb{R}^n} f \, d\gamma_n, 0 \right) = \left( \int_{\mathbb{R}^n} f \, d\gamma_n \right)^p$$

for any smooth bounded $f \geq \varepsilon$ which is the statement of the theorem we want to prove (except we need to justify the passage to the limit $\varepsilon \to 0$ and this will be done later). Notice that the inequality is reversed if $p \in (0, 1)$, indeed, in this case we should work with $-M(x, y)$ instead of $M(x, y)$. 
2.1. Properties of Hermite functions. $H_k$ can be defined (see [15]) by

\begin{equation}
H_k(x) = -\frac{2^{-k/2} \sin(\pi k) \Gamma(k+1)}{2\pi} \sum_{n=0}^{\infty} \frac{\Gamma((n-k)/2)}{n!} (-x\sqrt{2})^n,
\end{equation}

or in terms of the confluent hypergeometric functions (see [12]) by

\begin{equation}
H_k(x) = \sqrt{\frac{2}{\pi}} \left[ \cos \left( \frac{\pi k}{2} \right) \Gamma \left( \frac{k+1}{2} \right) \, _1F_1 \left( -\frac{k}{2}, \frac{1}{2}; x^2 \right) \\
+ t \sqrt{2} \sin \left( \frac{\pi k}{2} \right) \Gamma \left( \frac{k}{2} + 1 \right) \, _1F_1 \left( \frac{1-k}{2}, \frac{3}{2}; x^2 \right) \right].
\end{equation}

If $k$ is a nonnegative integer then one should understand (14) and (15) in the limiting sense. Notice the following recurrence properties:

\begin{equation}
H_k'(x) = kH_{k-1}(x);
\end{equation}

(16) \hspace{1cm} \begin{equation}
H_{k+1}(x) = xH_k(x) - H_k'(x).
\end{equation}

These properties follow from (14) and the fact that $\Gamma(z+1) = z\Gamma(z)$.

We also notice that

\[ H_k(x) := e^{x^2/4} D_k(x), \]

where $D_k(x)$ is the parabolic cylinder function, i.e., it is the solution of the equation

\[ D_k'' + \left( k + \frac{1}{2} - \frac{x^2}{4} \right) D_k = 0. \]

Since $H_k(x)$ is an entire function in $x$ and $k$ (see [25] for the parabolic cylinder function) sometimes it will be convenient to write $H(x, k)$ instead of $H_k(x)$. The precise asymptotic for $x \to +\infty$, $x > 0$ and any $k \in \mathbb{R}$ is given as follows

\begin{equation}
H_k(x) \sim x^k \sum_{n=0}^{\infty} (-1)^n \frac{(-k)_{2n}}{n!(2x^2)^n}.
\end{equation}

Here $(a)_n = 1$ if $n = 0$ and $(a)_n = a(a+1)\ldots(a+n-1)$ if $n > 0$. When $x \to -\infty$ we have

\begin{equation}
H_k(x) \sim |x|^k \cos(k\pi) \sum_{n=0}^{\infty} (-1)^n \frac{(-k)_{2n}}{n!(2x^2)^n} + \sqrt{2\pi} \frac{\Gamma(-k)}{(-k)^{1/2}} |x|^{-k-1} e^{-x^2/2} \sum_{n=0}^{\infty} \frac{(1+k)_{2n}}{n!(2x^2)^n}.
\end{equation}

We refer the reader to [25, 24]. For instance, for (18) we can use the asymptotic formula (12.9.1) in [24] for the parabolic cylinder function. To verify (19) we can express $H_k(-x)$ as a linear combination of two parabolic cylinder functions but having argument $x$ instead of $-x$ (see (12.2.15) in [24]), and then we can use (12.9.1) and (12.9.2) in [24].
Next we will need the result of Elbert–Muldoon [13] which describes the behavior of the real zeros of $H_k(x)$ for any real $k$.

**Lemma 1.** For $k \leq 0$, $H_k(x)$ has no real zeros, and it is positive on the real axis. For $n < k \leq n+1$, $n = 0, 1, \ldots$, $H_k(x)$ has $n+1$ real zeros. Each zero is increasing function of $k$ on its interval of definition.

The proof of the lemma is Theorem 3.1 in [13]. It is explained in the paper that as $k$ passes through each nonnegative integer $n$ a new leftmost zero appears at $-\infty$ while the right-most zero passes through the largest zero of $H_k(x)$. More precise information about the asymptotic behavior of the zeros as $k \to \infty$ can be found in [14].

Further we will need Turán’s inequality for $H_k(x)$ for any real $k$.

**Lemma 2.** We have the following Turán’s inequality:

\begin{equation}
H_k^2(x) - H_{k-1}(x)H_{k+1}(x) > 0 \quad \text{for all} \quad k \in \mathbb{R}, \ x \geq L_k
\end{equation}

where $L_k$ denotes the leftmost zero of $H_k$. If $k \leq 0$ then $L_k = -\infty$.

The lemma is known as Turán’s inequality when $k$ is a nonnegative integer. Unfortunately we could not find the reference in the case when $k$ is different from a positive integer therefore we decided to include the proof of the lemma.

The following is borrowed from [22].

**Proof.** Take $f(x) = e^{-x^2/2}(H_k^2(x) - H_{k-1}(x)H_{k+1}(x))$. Asymptotic formulas (18) and (19) imply that

\[
\lim_{x \to +\infty} f(x) = 0 \quad \text{for all} \quad k \in \mathbb{R};
\]

\[
f(x) \sim \sqrt{2\pi}|x| > 0 \quad \text{for} \quad x \to -\infty, \quad k = 0;
\]

\begin{equation}
f(x) \sim \frac{2\pi e^{x^2/2}}{\Gamma(-k)\Gamma(-k+1)}|x|^{-2k-2} \quad \text{for} \quad x \to -\infty, \quad k \notin \{0\} \cup \mathbb{N}.
\end{equation}

On the other hand notice that

\begin{equation}
f'(x) = -e^{-x^2/2}H_kH_{k-1}.
\end{equation}

If $k \leq 0$ then by Lemma[1] $f' < 0$, and because of the conditions $f(-\infty) = +\infty$ and $f(\infty) = 0$ we obtain that $f > 0$ on $\mathbb{R}$. To verify the statement for $k > 0$ we notice that

\begin{equation}
f''(x) = e^{-x^2/2}(H_k^2 - kH_k^2).
\end{equation}

Now we notice that if $H_k(c) = 0$ then $H_{k-1}(c) \neq 0$. Indeed, assume contrary $H_{k-1}(c) = 0$. Then by (16) we have $H_k'(c) = 0$ and by (5) we obtain $H_k''(c) = 0$, and again taking derivative in (16) we obtain that $H_{k-2}(c) = 0$. Repeating this process we obtain that $H_{k-N}(c) = 0$ for any large integer $N > 0$. But this contradicts to Lemma[1]
Thus by (22) and (23) we obtain that \( c \) is a point of the local minimum of \( f \) if and only if \( H_{k-1}(c) = 0 \). Then \( f(c) = e^{-x^2/2}H_k(c) > 0 \). Finally we obtain that \( f : [L_k, \infty) \rightarrow \mathbb{R} \) is positive on its local minimum points, \( f(\infty) = 0 \) and \( f(L_k) > 0 \) because \( H_{k-1}, H_{k+1} \) have opposite signs at zeros of \( H_k \) by (17). Therefore \( f > 0 \) on \([L_k, \infty) \rightarrow \mathbb{R}\) and the lemma is proved.

\[ \square \]

**Remark 1.** If \( k \in \mathbb{N} \) then \( H_k \) is the probabilists’ Hermite polynomial of degree \( k \), so \( f(x) \) will be even and inequality (20) will hold for all \( x \in \mathbb{R} \) which confirms the classical Turán’s inequality. However, if \( k > 0 \) but \( k \notin \mathbb{N} \) then (20) fails when \( x \rightarrow -\infty \) (see (21)).

Finally the next corollary together with Lemma 1 implies that \( \left| \frac{H_q'}{H_k} \right| = \text{sign}(k) \frac{H_q'(t)}{H_k(t)} \) is positive and decreasing for \( q \in (R_k, \infty) \) and \( k \in \mathbb{R} \setminus \{0\} \).

**Corollary 3.** For any \( x \geq L_k \) and any \( k \in \mathbb{R} \setminus \{0\} \) we have

\[ \text{sign}[(H_k')^2 - H_kH_k''] = \text{sign}(k). \]

**Proof.** The proof follows from Lemma 2 and the following identity

(24) \[ k(H_k^2 - H_{k-1}H_{k+1}) = (H_k')^2 - H_kH_k'' \]

from (5), (16) and (17). \[ \square \]

2.2. Checking the partial differential inequality. Let \( p = 1 + \frac{1}{k} \). Further we assume \( k \neq 0, -1 \). Define \( F = F_k \) as in the introduction:

(25) \[ F(t) = \frac{H_{k+1}(q)}{H_k^{1+1/k}(q)} \text{ where } \left| \frac{H_k'(q)}{H_k(q)} \right| = t, \quad q \in (R_k, \infty), \quad t \in (0, \infty). \]

Notice that by Corollary 3 function \( \left| \frac{H_k'(q)}{H_k(q)} \right| = \text{sign}(k) \frac{H_k'(q)}{H_k(q)} \) is positive decreasing in \( q \) for \( q \in (R_k, \infty) \), moreover by (18) we have \( \frac{H_k'(q)}{H_k(q)} \sim \frac{k}{q} \) when \( q \rightarrow +\infty \). From the same asymptotic formulas it follows that when \( t \rightarrow 0^+ \) we have

\[ F(t) = 1 - \frac{p(p-1)}{2} t^2 + O(t^4). \]

Therefore \( F \) is well-defined function and \( F \in C^2([0, \infty)) \).

Take a positive \( \varepsilon > 0 \) and define \( M(x, y) \) as in (12):

(26) \[ M(x, y) := x^p F \left( \frac{y}{x} \right) \text{ for } y \geq 0, \quad x > \varepsilon > 0. \]

Clearly \( M(x, \sqrt{y}) \in C^2((\varepsilon, \infty) \times [0, \infty)) \). By Theorem 1 in (18) we have inequality

(27) \[ \int_{\mathbb{R}^n} M(f, |\nabla f|) d\gamma_n \leq M \left( \int_{\mathbb{R}^n} f d\gamma_n, 0 \right) \]
for all smooth bounded $f \geq \varepsilon$ if \((13)\) holds. In terms of $F$ (see \((26)\)) condition \((13)\) takes the form
\[(28)\]
\[t F''(p-1) + F' F'' - t(p-1)^2(F')^2 \geq 0 \quad \text{i.e., the determinant of \((13)\) is nonnegative}\]
\[(29)\]
\[F''(t + t^3) + F'(2t^2 + 1 - 2pt^2) + F(p-1)t \leq 0 \quad \text{i.e., the trace of \((13)\) is nonpositive}\]
where $t = \frac{y}{\varepsilon}$ is the argument of $F$. In fact we will show that we have equality in \((28)\) instead of inequality therefore the sign of \((13)\) will depend on the sign of trace \((29)\). We will see that inequality \((29)\) will be reversed for $p \in (0, 1)$.

From \((25)\), \((24)\) and \((20)\) we obtain
\[(30)\]
\[F'(t) = -\frac{k+1}{|k|} \frac{H^{1/k}}{H_k};\]
\[(31)\]
\[F''(t) = \frac{F'}{|k|} \frac{H_k H_{k-1}}{H_k^2 - H_{k+1} H_{k-1}};\]
\[(32)\]
\[F(t) = -\frac{|k|}{k+1} \frac{H_k}{F'} H_{k+1} F'.\]

If we plug \((31)\) and \((32)\) into \((28)\) we obtain that the left hand side of \((28)\) is zero. If we plug \((31)\) and \((32)\) into \((29)\) we obtain
\[\text{LHS of \((29)\)} = \left[\frac{(kH_{k-1}^2 - H_k^2 + H_{k-1} H_{k+1})^2 + H_{k-1}^2 H_k^2}{H_k^2 (H_k^2 - H_{k+1} H_{k-1})} \right] F'.\]
Thus the sign of LHS of \((29)\) coincides with the sign of $F'$ which coincides with $\text{sign}(- (k+1))$. The condition $p \in \mathbb{R} \setminus [0, 1]$ implies that $k > -1$ and therefore \((13)\) holds. The condition $p \in (0, 1)$ implies that $k < -1$ and therefore inequality in \((13)\) is reversed.

Thus we have obtained \((27)\) for smooth bounded functions $f \geq \varepsilon$. Next we claim that for an arbitrary smooth bounded $f \geq 0$ with $\int_{\mathbb{R}^n} f^p d\gamma_n < \infty$ we can apply the inequality to $f_\varepsilon := f + \varepsilon$ and send $\varepsilon$ to 0 in \((8)\). Indeed, it follows from \((6)\) and \((18)\) that as $t \to \infty$ we have
\[F(t) \sim -t^{1+\frac{1}{p}} (H_k(R_k))^{-\frac{1}{p}} \quad \text{for} \quad k > 0;\]
\[F(t) \sim \text{sign}(-1-k) \left( \frac{e^{t^2/2} \sqrt{2\pi}}{t^\Gamma(-1-k)} \right)^{-\frac{1}{p}} |1+k|^{1+\frac{1}{p}} \quad \text{for} \quad k < 0, \quad k \neq -1.\]
Thus for $p > 1$ (i.e., $k > 0$) the claim about the limit follows from the estimate $|F(t)| \leq C_1 + C_2 t^p$ together with the Lebesgue dominated convergence theorem.

If $p < 0$ (i.e., $k \in (-1, 0)$) we rewrite \((8)\) in a standard way as follows
\[(33)\]
\[\int_{\mathbb{R}^n} f_{\varepsilon}^p d\gamma_n - \left( \int_{\mathbb{R}^n} f_{\varepsilon} d\gamma_n \right)^p \leq \int_{\mathbb{R}^n} f_{\varepsilon}^p \left( 1 - F \left( \frac{|\nabla f|}{f_{\varepsilon}} \right) \right) d\gamma_n.\]
Since \( f \) is bounded, \( f \geq 0 \) and \( \int_{\mathbb{R}^n} f^p d\gamma_n < \infty \) there is no issue with the left hand side of (33) when \( \varepsilon \to 0 \). For the right hand side of (33) we notice that the function \( x^p(1 - F(y/x)) \) is nonnegative and decreasing in \( x \) then the claim follows from the monotone convergence theorem. The non negativity follows from the observation that \( F(0) = 1 \) and \( F' < 0 \) (see (30) where we have \( k > -1 \)). The monotonicity follows from (6), (30), (16) and the straightforward computations

\[
\frac{\partial}{\partial x} (x^p(1 - F(y/x))) = x^{p-1} (p - pF(t) + tF'(t)) = x^{p-1} p \left[ 1 - \frac{q}{H_{k,q}^s} \right],
\]

where \( |k| \frac{H_{k-1,q}}{H_k(q)} = t = \frac{y}{x} \) and \( q \in (R_k, \infty) \). The last expression in (34) is negative because

\[
1 \geq F(t) = \frac{H_{k+1}}{H_k^{1+\frac{1}{k}}} = \frac{qH_k - kH_{k-1}}{H_k^{1+\frac{1}{k}}} > \frac{q}{H_k},
\]

Finally if \( p \in (0,1) \) (i.e., \( k < -1 \)) we have the opposite inequality in (33). In this case the situation is absolutely the same as for \( k \in (-1,0) \) except now we should consider the function \( x^p(F(y/x) - 1) \) which is nonnegative and decreasing in \( x \) (see (34)). This finishes the proof of the theorem.

Now let us show Proposition [1]. Since \( F(0) = 1 \) it is enough to show a stronger inequality, namely \( F' + p(p-1)t \geq 0 \). From (30) and the fact that \( k \geq 1 \) (since \( p \in [1,2] \)) we obtain that it is enough to show the following inequality

\[
-\frac{p}{H_{k,q}^{1/k}} + p(p-1) \frac{H_k'}{H_k} \geq 0 \quad \text{for all} \quad k \geq 1, \quad q \in (R_k, \infty).
\]

Using (16) and \( p = 1 + \frac{1}{k} \) we notice that the inequality can be rewritten as follows

\[
1 > \frac{H_k(q)}{H_{k-1}(q)} \quad \text{for all} \quad q \in (R_k, \infty).
\]

To verify the last inequality we remind that \( F(0) = 1 \) and \( F'(t) < 0 \). Therefore \( F(t) \leq 1 \). We recall the definition of \( F(t) \) (see (25)). It follows that \( 1 \geq F = \frac{H_{k+1}}{H_k} \) for all \( k > 0 \). The last inequality is the same as

\[
1 \geq \frac{H_k(q)}{H_{k-1}(q)} \quad \text{for all} \quad q \in (R_k, \infty), \quad k \geq 1.
\]

This finishes the proof of the theorem.

2.3. **Proof of Corollary [1] and Corollary [2].** Notice that as \( t \to 0 \) we have

\[
F_{\exp}(y) = 1 - \frac{y^2}{2} + O(y^4) \quad \text{and} \quad F_{-\ln}(y) = -\frac{y^2}{2} + O(y^4).
\]

There are two ways to obtain the corollaries.
2.3.1. The first way: One can check that

\[
M_{\exp}(x, y) = e^x F_{\exp}(y), \quad M_{\exp}(x, 0) = e^x, \quad M_{\exp}(x, \sqrt{y}) \in C^2(\mathbb{R} \times \mathbb{R}_+);
\]

\[
M_{-\ln}(x, y) = -\ln(x) + F_{-\ln}\left(\frac{y}{x}\right), \quad M_{-\ln}(x, 0) = -\ln x, \quad x > 0,
\]

and \(M_{-\ln}(x, \sqrt{y}) \in C^2(\varepsilon, \infty) \times \mathbb{R}_+)\) for any \(\varepsilon > 0\). By straightforward computations we notice that if we set \(\psi(q) = \alpha - \int_1^q H_{-1}(s)\,ds\) then using the identity \(1 = qH_{-1}(q) + H_{-2}(q)\) we obtain

\[
F_{\exp}(H_{-1}) = qe^\psi, \quad F'_{\exp}(H_{-1}) = -e^\psi \quad \text{and} \quad F''_{\exp}(H_{-1}) = -\frac{H_{-1}}{H_{-2}}.
\]

Similarly we compute that

\[
F'_{-\ln}\left(\frac{H_{-2}}{H_{-1}}\right) = -H_{-1} \quad \text{and} \quad F''_{-\ln}\left(\frac{H_{-2}}{H_{-1}}\right) = -\frac{H_{-2}H_{-3}^2}{H_{-1}^2 - H_{-2}}.
\]

Next one notices that \(M_{\exp}\) and \(M_{-\ln}\) satisfy (13) (in fact the determinant of (13) is zero). Then by Theorem 1 in [18] we obtain the corollaries. The passage to the limit for \(M_{-\ln}(x, y)\) when \(\varepsilon \to 0\) follows from the monotone convergence theorem. Indeed, we notice that \(-F_{-\ln}(y/x) \geq 0\) is decreasing in \(x\). We apply Corollary 2 to \(f_\varepsilon = f + \varepsilon\) and send \(\varepsilon \to 0\).

2.3.2. The second way: We will obtain the corollaries as a limiting case of Theorem 1. Indeed, to verify Corollary 1 let \(f^p = e^{\varphi}\) in (8). Then (8) takes the form

\[
\int_{\mathbb{R}^n} e^{\varphi^p} \left(\frac{|\nabla g|}{p}\right) d\gamma_n \leq \left(\int_{\mathbb{R}^n} e^{\varphi/p} d\gamma_n\right)^p.
\]

Now we take \(p \to \infty\). The RHS of (35) tends to \(\exp((\int_{\mathbb{R}^n} g d\gamma_n))\). For the LHS of (35) we should compute the limit

\[
F_{\exp}(t) := \lim_{p \to \infty} F_{\frac{t}{p-1}}\left(\frac{t}{p}\right) = \lim_{p \to \infty} F_{\frac{t}{p-1}}\left(\frac{t}{p}\right) = \lim_{k \to 0^+} F_k(tk).
\]

It is clear that \(F_{\exp}(0) = 1\). Next if we take \(k \to 0^+\) in (6) we obtain

\[
\lim_{k \to 0^+} F_k\left(\frac{H_k'}{H_k}\right) = \lim_{k \to 0^+} F_k\left(\frac{kH_{k-1}}{H_k}\right) = \lim_{k \to 0^+} F_k\left(\frac{H_{k-1}}{H_0}\right) = F_{\exp}(H_{-1})
\]

On the other hand for the RHS of (6) we have

\[
\lim_{k \to 0^+} \frac{H_{k+1}(q)}{H_k^{1+1/k}} = q \lim_{k \to 0^+} H_k^{-1/k}.
\]

Here we have used \(H_0(q) = 1\) and \(H_1(q) = q\). Thus it remains to find \(\lim_{k \to 0^+} H_k^{-1/k}\). Notice that \(H(x, k) := H_k(x)\) is an entire function in \(x\) and \(k\) (see [25] for the Parabolic cylinder function). If we take derivative in \(k\) of (16) we obtain \(H_{2k}(x, k) = H(x, k -
1) + kH_k(x, k) (here subindices denote partial derivatives). Now taking \( k = 0 \) we obtain 
\[ H_{xk}(x, 0) = H(x, -1) \]. Thus \( H_k(x, 0) \) is an antiderivative of \( H(x, -1) = H_{-1} \). So

\[
\lim_{k \to 0^+} H_k^{-1/k} = \lim_{k \to 0^+} \exp \left( -\frac{1}{k} \ln(1 + kH_k(x, 0) + o(k)) \right) = \exp \left( -\int H_{-1}(s)ds \right).
\]

Finally we obtain

\[
F \exp (H_{-1}(q)) = q \exp \left( C - \int_{1}^{q} H_{-1} \right)
\]

In order to satisfy the condition \( F \exp (0) = 1 \) the constant \( c \) must be chosen as follows
\[
C = \int_{1}^{\infty} (H_{-1} - \frac{1}{s}) ds \quad \text{indeed send } q \to \infty \quad \text{in (36)}. \]

This finishes the proof of Corollary 1.

It is worth mentioning that we have also obtained (see (7))
\[ H_k(x, 0) = \int_{1}^{x} H_{-1}(s)ds - \alpha. \]

To verify Corollary 2 let \( F(x, k) := F_k(x) \). Let \( F_k(x, k) \) denotes the partial derivative in \( k \) of \( F(x, k) \). If we send \( p \to 0, p < 0 \) in (8) and compare the terms of order \( p \) we obtain
\[
\int_{\mathbb{R}^n} \left( \ln f - F_k \left( \frac{1}{f}, -1 \right) \right) d\gamma_n \geq \ln \left( \int_{\mathbb{R}^n} f d\gamma_n \right)
\]

It remains to find the function \( F_k(x, -1) \). Let us equate terms of order \( (k + 1) \) as \( k \to -1, k < -1 \) in the following equality
\[ F \left( \frac{H_x(x, k)}{H(x, k)} , k \right) = \frac{H(x, k + 1)}{H(x, k)^{1 + \frac{1}{k}}}. \]

The straightforward computation shows that
\[
F_k \left( \frac{H_{-2}(x)}{H_{-1}(x)}, -1 \right) = H_k(x, 0) + \ln H_{-1}(x) = \int_{1}^{x} H_{-1}(s)ds - \alpha + \ln H_{-1}(x)
\]

where
\[
\alpha = \int_{1}^{\infty} \left( H_{-1}(s) - \frac{1}{s} \right) ds.
\]

3. CONCLUDING REMARKS

The reader may wander how we guessed the choice (12). Of course it was not a random guess. Function (12) is the best possible in the sense that the determinant of (13) is identically zero

\[
M_{yy} (M_{xx} + \frac{M_y}{y}) - M_{xy}^2 = 0,
\]

\[
M(x, 0) = x^p \quad \text{for } x \geq 0.
\]
Initially this was the way we started looking for \( M(x, y) \) as the solution of the Monge–Ampère equation with a drift (37). By a proper change of variables the equation reduces to the backwards heat equation (see [18] for more details where the connection with R. Bryant, Ph. Griffiths theory of exterior differential systems was exploited)

\[
\begin{align*}
  u_{xx} + u_t &= 0, \\
  u(x, 0) &= C x^p \quad \text{for} \quad x \geq 0.
\end{align*}
\]

(38) \hspace{1cm} (39)

One can notice that the Hermite polynomials do satisfy (38) and (39) when \( \frac{p}{p-1} \) is a positive integer. In general, one should invoke Hermite functions and this is the reason of appearance of these functions in our theorem.

Another possibility is to assume that \( M(x, y) \) should be homogeneous of degree \( p \) which enforces \( M \) to have the form (26) for some \( F \). Next setting \( h = \frac{F}{F'} \) and further by a subtle change of variables one obtains Hermite differential equation (5).

Nevertheless, for the formal proof of Theorem 1 we do not need to go through the details. We have \( M(x, y) \) defined by (12) and we just need to check that it satisfies the desired properties.

The fact that \( M(x, y) \) (see (12)) satisfies (13) makes it possible to extend Theorem 1 in a semigroup setting for uniformly log-concave measures. Indeed, let \( d\mu = e^{-U} dx \) where \( \text{Hess } U \geq R \cdot \text{Id} \), \( R > 0 \). Let \( L = \Delta - \nabla U \cdot \nabla \), and let \( P_t = e^{tL} \) be the semigroup with generator \( L \) (see [18, 3]).

**Corollary 4.** For any \( p \in \mathbb{R} \setminus [0, 1] \) and any smooth bounded \( f \geq 0 \) with \( \int_{\mathbb{R}^n} f^p d\mu < \infty \) we have

\[
P_t \left[ f^p \frac{F_1}{p-1} \left( \frac{|\nabla f|}{f^{\frac{1}{\sqrt{R}}} \sqrt{R}} \right) \right] \leq (P_t f)^p \frac{F_1}{p-1} \left( \frac{|\nabla P_t f|}{P_t f \sqrt{R}} \right).
\]

The inequality is reversed if \( p \in (0, 1) \).

**Proof.** Notice that \( \tilde{M}(x, y) = M(x, \frac{y}{\sqrt{R}}) \) satisfies (13). Now it remains to use inequality (2.3) from [18]. \qed

Next by taking \( t \to \infty \) and using the fact that \( |\nabla P_t f| \leq e^{-tR} |\nabla f| \) we obtain the following corollary

**Corollary 5.** Let \( d\mu = e^{-U} dx \) where \( \text{Hess } U \geq R \cdot \text{Id} \) for some \( R > 0 \). For any \( p \in \mathbb{R} \setminus [0, 1] \) and any smooth bounded \( f \geq 0 \) with \( \int_{\mathbb{R}^n} f^p d\mu < \infty \) we have

\[
\int_{\mathbb{R}^n} f^p \frac{F_1}{p-1} \left( \frac{|\nabla f|}{f^{\frac{1}{\sqrt{R}}} \sqrt{R}} \right) d\mu \leq \left( \int_{\mathbb{R}^n} f d\mu \right)^p.
\]

The inequality is reversed if \( p \in (0, 1) \).

**Proof.** See Corollary 1 in [18]. \qed
The limiting cases of these inequalities when $p \to \pm\infty$ and $p \to 0$ should be understood in the sense of functions $M_{\exp}$ and $M_{-\ln}$ as in Corollary 1 and Corollary 2.

Finally we would like to mention that having characterization of functional inequalities makes approach to the problem systematic. Very similar local estimates happen to rule some global inequalities. We refer the reader to our recent papers on this subject.

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REFERENCES

[1] A. Arnold, J. P. Bartier, J. Dolbeault, Interpolation between logarithmic Sobolev and Poincaré inequalities, Commun. Math. Sci. 5 (2007) 971–979.

[2] P. D. Pelo, A. Lanconelli, A. I. Stan, An extension of the Beckner's type Poincaré inequality to convolution measures on abstract Wiener spaces, arXiv: 1409.5861

[3] D. Bakry, I. Gentil, M. Ledoux, Analysis and Geometry of Markov Diffusion Operators, Grundlehren der Mathematischen Wissenschaften 348. Springer, Cham.

[4] F. Barthe, P. Cattiaux, C. Roberto, Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry, (to appear in Revista Mat. Iberoamericana), arXiv:0407219

[5] F. Barthe, P. Cattiaux, C. Roberto, Isoperimetry between exponential and Gaussian, arXiv:0601475

[6] F. Barthe, C. Roberto, Sobolev inequalities for probability measures on the real line, Studia Math., 159(3):481–497, 2003

[7] W. Beckner, A generalized Poincaré inequality for Gaussian measures, Proceedings of the American Mathematical Society 105, no. 2, 397–400 (1989)

[8] S. G. Bobkov, P. Tetali, Modified log-sobolev inequalities, mixing and hypercotractivity, In Proceedings of the thirty-fifth annual ACM symposium on Theory of computing, pages 287–296. ACM, 2003.

[9] S. G. Bobkov, F. Götze, Exponential integrability and transportation cost related to logarithmic Sobolev inequalities, J. Funct. Anal. 163:1–28, 1999

[10] S. Boucheron, O. Bousquet, G. Lugosi, P. Massart, Moment inequalities for functions of independent random variables, Ann. Probab., to appear, 2004

[11] D. Chafai, On $\Phi$-entropies and $\Phi$-Sobolev inequalities, preprint, 2002.

[12] L. Durand, Nicholson-type integrals for products of Gegenbauer functions and related topics, Theory and Applications of Special Functions, R. Askey, ed., Academic Press, New York and London, 1975, 353–374.

[13] Á. Elbert, M. E. Muldoon, Inequalities and monotonicity properties for zeros of Hermite functions, Proc. Roy. Soc. Edinburgh Sect. A 129 (1999), 57–75.

[14] Á. Elbert, M. E. Muldoon, Approximations for zeros of hermite functions, Contemporary Mathematics, 471: 117–126, 2008

[15] W. K. Hayman, E. L. Ortiz, An upper bound for the largest zero of Hermite's function with applications to subharmonic functions, Proc. Roya. Soc. Edinburgh 75A (1975-76), 183–197

[16] P. Ivanisvili, A. Volberg, Bellman partial differential equation and the hill property for classical isoperimetric problems, arXiv: 1506.03409

[17] P. Ivanisvili, A. Volberg, Hessian of Bellman functions and uniqueness of Brascamp–Lieb inequality, J. London Math. Soc. (2015) 92 (3): 657–674.

[18] P. Ivanisvili, A. Volberg, Isoperimetric functional inequalities via the maximum principle: the exterior differential systems approach, arXiv: 1511.06895
[19] P. Ivanisvili, *Boundary value problem and the Ehrhard inequality*, arXiv: 1605.04840
[20] A. Kolesnikov, *Modified Log-Sobolev inequalities and isoperimetry*, arXiv: 0608681
[21] R. Latala, K. Oleszkiewicz, *Between Sobolev and Poincaré*. Geometric Aspects of Functional Analysis. *Lect. Notes Math.*, 1745: 147–168, 2000
[22] B. S. Madhava Rao, V. R. Thiruvenkatachar, *On an inequality concerning orthogonal polynomials*, Proceedings of the Indian Academy of Sciences - Section A, June 1949, Volume 29, Issue 6, pp 391–393.
[23] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. 88 (1958), 931–954
[24] F. W. J. Oliver, D. W. Lozier, R. F. Boisvert and C. W. Clark, *NIST handbook of mathematical functions*, Cambridge University Press, Cambridge U.K. (2010).
[25] N. Temme, *Asymptotic Methods for Integrals*, World Scientific, Singapore, 2015.
[26] F. Y. Wang, *A generalization of Poincaré and log-Sobolev inequalities*, Potential Analysis 22 (2005) 1-15

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