Thermodynamic properties of spherical nuclei *

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Abstract

The effect of the residual interaction between the nucleons (quasiparticles) on the thermodynamic behavior of spherical nuclei is considered. Thermodynamic relations are derived for the simplest model of the residual interaction that has the necessary macroscopic structure previously established. The range of applicability of the theory is discussed.

1 Introduction

The nuclear physics data accumulated up to now indicate that there is an essential difference between spherical and nonspherical nuclei. This difference does not reduce merely to “nuclear shape effects” as such, but intimately involves the internal structure of the nucleus. The shell oscillations of nuclear masses [1] can serve as an example of the phenomena associated with the difference between the structures of spherical and nonspherical nuclei. The shell oscillations are due to the ordered grouping of one-quasiparticle levels in spherical nuclei that arises because of the existence of the orbital angular momentum quantum number $l$ of the quasiparticle [1,2]. There is no such ordered grouping in nonspherical nuclei, and the trend of the mass of such nuclei as a function of the number of nucleons is entirely different. This difference is manifested on a plot of nuclear masses versus number of nucleons as a break in the curve corresponding to a phase transformation from the normal phase (nonspherical nuclei) to the “magic phase” (spherical nuclei) [3].

The difference between the internal structures of spherical and nonspherical nuclei does not affect only the ground states: the thermodynamic properties of the two phases are also different. This difference is manifested, for example, in the dependence of the form of the spectra of the $\gamma$-ray cascades emitted by heated nuclei on phase of the compound nucleus [4,5]. The energy spectrum of a nonspherical nucleus is not distorted by the residual interaction and is essentially of the Fermi-liquid type. Spherical nuclei, on the other hand, apparently cannot exist without the residual interaction (this was proved in [6] for a very simple nuclear model), and this interaction has an important effect on their thermodynamic behavior. As was explained in [2], the residual interaction must have a certain macroscopic

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structure, the only interactions between the quasiparticles that are compatible with the observed pattern of the shell oscillations of nuclear masses being those that lead to a quadratic dependence of the width of the diffuse region of the Fermi distribution on the orbital angular momentum $l$ of the quasiparticle:

$$\delta \varepsilon \propto l^2 \cong (l + 1/2)^2. \quad (1)$$

In this paper we consider the thermodynamics of spherical nuclei on the basis of the simplest model of the residual interaction that has the required macroscopic structure.

### 2 The Racah-Mottelson model

The Racah-Mottelson model was used in [2] to describe the residual interaction: the Hamiltonian for the interaction between quasiparticles on the same $j$ level is given by

$$H_{int}^f = -G_j \sum_{m,m'>0} a_{m'}^\dagger a_{-m'} a_m a_m,$$  \quad (2)

where $a_m^\dagger$ and $a_m$ are the creation and annihilation operators for a quasiparticle whose angular momentum has the $z$ projection $m$. The eigenvalues of this Hamiltonian (see, e.g., [7]) are

$$E_{int}^f = -G_j b_j (\Omega_j - b_j - s_j + 1), \quad (3)$$

where $2\Omega_j = 2j + 1$, $b_j$ is the number of interacting pairs on level $j$, and $s_j$ is the seniority quantum number.

To construct the thermodynamics of spherical nuclei, we need a formula for the degree of degeneracy $\Gamma(s, \Omega)$ (we shall drop the subscript $j$ in the rest of this section) of the Racah-Mottelson energy levels (3). The symmetry of the zero-seniority ground-state wave function (see, e.g., [8]) is such that the wave function can be constructed from single-particle wave functions in only one way. Hence the $s = 0$ ground state is not degenerate: $\Gamma(0, \Omega) = 1$. One more quasiparticle can be added to the $s = 0$ state in $C_{2\Omega}^1$ ways ($C_{2\Omega}^s = (2\Omega)!/(s!(2\Omega - s)!$ is the number of combinations of $2\Omega$ things taken $s$ at a time), and we thereby obtain an $s = 1$ state, so that $\Gamma(1, \Omega) = C_{2\Omega}^1$. Two quasiparticles can be added to the $s = 0$ state with $2b$ quasiparticles in $C_{2\Omega}^2$ ways, but among them there is one that describes the ground state of $2(b + 1)$ quasiparticles, so that $\Gamma(2, \Omega) = C_{2\Omega}^2 - C_{2\Omega}^0$. Arguing further in the same way we can prove the following general formula for the degree of degeneracy of the Racah-Mottelson energy levels by mathematical induction:

$$\Gamma(s, \Omega) = C_{2\Omega}^s - C_{2\Omega}^{s-2}, \quad 0 \leq s \leq \Omega. \quad (4)$$

For the case $\Omega \gg 1$, $s \gg 1$, we can use Stirling’s formula $n! \cong \sqrt{2\pi n} \exp(n \ln n - n)$ to rewrite Eq. (4) in the form

$$\Gamma(s, \Omega) \cong \frac{4\Omega(\Omega - s)}{(2\Omega - s)^2} \sqrt{\frac{\Omega}{\pi s(2\Omega - s)}} \exp[(2\Omega) \ln(2\Omega) - s \ln s - (2\Omega - s) \ln(2\Omega - s)]. \quad (5)$$
Now let us find the number \( \Gamma(M, s, \Omega) \) of states with seniority \( s \) and angular momentum projection \( M \). It is clear that the sum of \( \Gamma(M, s, \Omega) \) over all possible values of \( M \) is \( \Gamma(s, \Omega) \). In order to find a way of evaluating \( \Gamma(M, s, \Omega) \) let us consider the matrix of Hamiltonian (2) in more detail. The dimension of this matrix is the number \( C_{2\Omega}^N \) of ways in which \( N \) fermions can be distributed among \( 2\Omega \) cells. (It is easily seen that the sum of \( \Gamma(s, \Omega) \) over all seniorities \( s \) possible for the given \( N \) is \( C_{2\Omega}^N \).) The Hamiltonian matrix for the \( C_{2\Omega}^N \) basis functions breaks up into blocks corresponding to different values of the quantum number \( s \). However, the block for the states with a given seniority breaks up in turn into still smaller blocks, since the Hamiltonian conserves the angular momentum projection \( M \), so that different values correspond to different blocks. Finally, the block for states with given fixed values of \( s \) and \( M \) also consists of smaller blocks, since different sets of unpaired particles can give rise to the same angular momentum projection \( M \). (Here the term “pairing” is used in a purely kinematic sense, namely, as an expression of the fact that two quasiparticles lie in cells corresponding to the same absolute value but opposite signs of the \( z \) projection of the angular momentum.) For definiteness let is consider the case of even \( s \) (and hence even \( N \)). Then the “smallest” blocks in the matrix for Hamiltonian (2) describe states in which there occur some number \( 2k \) of unpaired quasiparticles giving a total angular momentum projection \( M \), and \( N - 2k \) paired quasiparticles, which have \( M = 0 \) and seniority \( s - 2k \). The number of such blocks is equal to the number \( C(M, 2k, \Omega) \) of ways in which the angular momentum projection \( M \) can be expressed as the sum of \( 2k \) single-particle projections taken from the set \( \{-j, -j + 1, \ldots, j - 1, j\} \) (no two single-particle projections equal in modulus but opposite in sign occurring in the sum), while the dimension of each block is equal to the degree of degeneracy \( \Gamma_0(s - 2k, \Omega - 2k) \) of a state of paired quasiparticles that are distributed among \( \Omega - 2k \) free cells and have seniority \( s - 2k \). Thus, the sum of the dimensions of all the blocks for states with angular momentum projection \( M \) and seniority \( s \) is

\[
\Gamma(M, s, \Omega) = \sum_{k=0}^{s/2} C(M, 2k, \Omega)\Gamma_0(s - 2k, \Omega - 2k).
\]  

(6)

A similar expression can also be written for odd \( s \).

Direct diagonalization of the Hamiltonian matrix for small values of \( \Omega \) and \( N \) leads to values of \( \Gamma_0(s, \Omega) \) that can be expressed by the formula

\[
\Gamma_0(s, \Omega) = C_s^{\Omega/2} - C_s^{s/2 - 1}.
\]  

(7)

By definition, \( \Gamma_0(s, \Omega) \) is meaningful only for even \( s \). If we assume that formula (7) is valid for all even seniorities \( \leq s \), we can prove that it is valid for seniority \( s + 2 \) in much the same way as Eq. (4) was proved: hence it follows by mathematical induction that (7) is valid for all even seniorities \( 0 \leq s \leq \Omega \).

Now let us seek the number \( C(M, s, n) \) of ways in which \( s \) unpaired particles can give a total angular momentum projection of \( M \). Since the formula for \( \Gamma(M, s, \Omega) \) will be used later to describe macroscopic phenomena for which it is essential that \( j \gg 1 \), we can set \( j \approx \Omega \) (also see [2]), so that the single-particle angular momentum projections are the elements of the set \( \{-\Omega, -\Omega + 1, \ldots, -1, 1, \ldots, \Omega - 1, \Omega\} \). Now we shall construct a generating function for the unknown numbers \( C(M, s, \Omega) \). To do this we consider the sum

\[
\sum (\zeta^1)^{\varepsilon_1}(\zeta^2)^{\varepsilon_2} \cdots (\zeta^\Omega)^{\varepsilon_\Omega}
\]  

(8)
taken over the values
\[ \varepsilon_1 = -1, 0, 1; \quad \varepsilon_2 = -1, 0, 1; \quad \ldots; \quad \varepsilon_{\Omega} = -1, 0, 1 \] (9)
under the condition that
\[ |\varepsilon_1| + |\varepsilon_2| + \ldots + |\varepsilon_{\Omega}| = s. \] (10)
The coefficient of \( \zeta^M \) in the sum (8) is \( C(M, s, \Omega) \). To take condition (10) into account, we make use of the identity
\[ \frac{1}{2\pi i} \oint \frac{dz}{z^{|\varepsilon_1|+|\varepsilon_2|+\ldots+|\varepsilon_{\Omega}|} - s + 1} = \begin{cases} 0, & |\varepsilon_1| + |\varepsilon_2| + \ldots + |\varepsilon_{\Omega}| \neq s, \\ 1, & |\varepsilon_1| + |\varepsilon_2| + \ldots + |\varepsilon_{\Omega}| = s, \end{cases} \] (11)
in which the integration contour goes around the origin. If we multiply (11) by (8), we can sum over all the values (9) independently, without regard for condition (10). As a result we obtain the following expression for the generating function:
\[ \sum_M C(M, s, \Omega) \zeta^M = \frac{1}{2\pi i} \oint \frac{dz}{z^s} \prod_{n=1}^{\Omega} \left( 1 + \frac{\zeta^n + \zeta^{-n}}{z} \right). \] (12)
If we make the substitution \( \zeta = e^{it} \) in (12), the sum over \( M \) becomes a finite Fourier series. Using the orthogonality relation
\[ \int_{-\pi}^{\pi} \cos Mt \cos Lt \, dt = \pi \delta_{ML}, \] (13)
we obtain the following expression for the coefficients:
\[ C(M, s, \Omega) = \frac{1}{(2\pi)^2} \oint \frac{dz}{z} \int_{-\pi}^{\pi} \cos Mt \prod_{n=1}^{\Omega} \left( 1 + \frac{2}{z} \cos nt \right) \sin t \] (14)
Now we expand the product of the expressions in parentheses in powers of \( t \) and extend the integration over the entire real axis, obtaining:
\[ C(M, s, \Omega) = \frac{1}{(2\pi)^2} \sqrt{\frac{3\pi}{\Omega^3}} \oint \frac{dz}{z} \sqrt{z + 2} z^s \exp \left( \Omega \ln \left( 1 + \frac{2}{z} \right) \right) \exp \left( -\frac{3(z + 2)^2}{4\Omega^3} M^2 \right). \] (15)
This operation is legitimate since the integral over \( t \) converges for \( t \sim \sqrt{(z + 2)/\Omega^3} \) and the characteristic values of \( z \) (see below) are such that \( z + 2 = 2\Omega/s \), so that \( t \sim 1/\Omega \sqrt{s} \). We evaluate the remaining integral over \( z \) by the saddle-point method, choosing the integration contour so that the convergence of the integral will be due mainly to the function
\[ \exp[f(t)] = z^s \exp[\Omega \ln(1 + 2/z)] = \exp[(s - \Omega) \ln z + \Omega \ln(z + 2)]. \] (16)
At the saddle-point we have
\[ z_0 = \frac{2(\Omega - s)}{s}, \quad \left. \frac{d^2 f}{dz^2} \right|_{z_0} = \frac{s^3}{4\Omega(\Omega - s)}, \] (17)
and we must therefore integrate over the contour \[ z = z_0 + iy \]. As a result we obtain

\[
C(M, s, \Omega) = \sqrt{\frac{3}{2\pi \Omega/s}} \sqrt{\frac{\Omega}{2\pi s(\Omega - s)}} \\
\times \exp[s \ln 2 + \Omega \ln \Omega - s \ln s - (\Omega - s) \ln(\Omega - s)] \exp(-3M^2/2s\Omega^2).
\] (18)

It is easy to show that the condition for the validity of Eq. (18) is that \( s \gg 1 \).

To calculate \( \Gamma(M, s, \Omega) \) it remains to substitute expressions (7) and (18) into (6) and perform the summation. Approximating the right-hand side of (7) with the aid of Stirling’s formula and replacing the summation by integration, we obtain

\[
\Gamma(M, s, \Omega) = \sqrt{\frac{3}{2\pi}} \frac{\Omega - s}{2\sqrt{\Omega}} \int_{0}^{s/2} \frac{\exp[\phi(k)] \exp(-3M^2/4k\Omega^2)dk}{k(\Omega - s/2 - k)^{\sqrt{(s/2 - k)(\Omega - s/2 - k)}}},
\] (19)

where

\[
\phi(k) = \Omega \ln \Omega - 2k \ln k - (s/2 - k) \ln(s/2 - k) - (\Omega - s/2 - k) \ln(\Omega - s/2 - k).
\] (20)

The function \( \phi(k) \) has a maximum at the point \( k_0 = s/2 - s^2/4\Omega \),

and at that point

\[
(d^2 \phi/dk^2) \bigg|_{k_0} = -16\Omega^3/s^2(2\Omega - s)^2.
\] (22)

Thus, if

\[
s^2(2\Omega - s)^2/\Omega^3 \ll s^4/\Omega^2,
\] (23)

then \( \phi(k) \) will have a sharp maximum within the integration range, so we can replace the other slowly varying functions by their values at \( k = k_0 \) and integrate over \( k \) from \(-\infty\) to \( \infty \). As a result we obtain the following final expression:

\[
\Gamma(M, s, \Omega) = \frac{4\sqrt{3}}{\pi} \frac{\Omega(\Omega - s)}{s(2\Omega - s)} \exp[(2\Omega) \ln(2\Omega) - s \ln s - (2\Omega - s) \ln(2\Omega - s)] \\
\times \exp[-3M^2/\Omega s(2\Omega - s)].
\] (24)

Condition (23) for the validity of Eq. (24) is satisfied when \( s \gg \sqrt{\Omega} \). It should be noted, however, that the exponential factors in (24), which are essential for the thermodynamic applications, are determined only by the position (21) of the maximum, so that the condition \( s \gg \sqrt{\Omega} \) is essentially just a condition for the validity of the preexponential factor in (24); the exponential factor, on the other hand, is valid for \( s \gg 1 \). On integrating (24) over \( M \) from \(-\infty\) to \( \infty \) we again obtain Eq. (5).

With that we conclude our study of an individual \( j \) level of the Racah-Mottelson model and turn to the application of the model to the nucleus as a whole.
3 The thermodynamics of spherical nuclei

Let us calculate the function describing the distribution of quasiparticles in a nucleus heated to the temperature $T$. The preexponential factor in (5) can be neglected in calculating the entropy of a single $j$ level with the formula $S_j = \ln \Gamma(s_j, \Omega_j)$. Then we obtain the following expression for the entropy of the nucleus as a whole:

$$S = \sum_j S_j = - \sum_j 2\Omega_j \left[ \frac{s_j}{2\Omega_j} \ln \frac{s_j}{2\Omega_j} + \left(1 - \frac{s_j}{2\Omega_j}\right) \ln \left(1 - \frac{s_j}{2\Omega_j}\right) \right].$$  \hspace{1cm} (25)

To find the function describing the distribution of the quasiparticles among the levels, we must minimize the free energy $F = E - TS$, $E = \sum_j \left\{(2b_j + s_j)\varepsilon_j - G_jb_j(\Omega_j - b_j - s_j)\right\}$ under the condition that the number of quasiparticles, $\tilde{N} = \sum_j (2b_j + s_j)$, remains constant. In view of the macroscopic nature of the problem under study we have neglected the “1” in Eq. (3) for the interaction energy (see [2]).

Condition (27) can be easily taken into account by the Lagrange multiplier method. By varying the function $F' = F - \varepsilon_f \tilde{N}$ ($\varepsilon_f$ is the chemical potential) with respect to $b_j$ and $s_j$ we obtain the set of equations

$$w_j^b = \frac{1}{2} - \frac{\varepsilon_j - \varepsilon_f}{G_j\Omega_j} - w_j^s,$$
$$w_j^s = \left[\exp\left(\frac{\varepsilon_j - \varepsilon_f - G_j\Omega_jq^b_j}{T}\right) + 1\right]^{-1},$$  \hspace{1cm} (28)

for distribution functions $w_j^b = b_j/\Omega_j$ and $w_j^s = s_j/2\Omega_j$ in the region $b_j + s_j < \Omega_j$, $b_j \neq 0$ (region II on Fig. 1). On substituting the expression for $w_j^b$ from the first of Eqs. (28) into the second we obtain the following equation for the distribution function $w_j^s$:

$$w_j^s = \left\{\exp\left[\frac{G_j\Omega_j}{2T}(1 - 2w_j^s)\right] + 1\right\}^{-1} \text{ (region II).}$$  \hspace{1cm} (29)

On increasing $\varepsilon_j$, we pass from region II to region III, where there are no paired quasiparticles, i.e., $b_j = 0$. On taking this additional condition into account in the $s_j$ variation we obtain the following expression for the distribution function $w_j^s$:

$$w_j^s = \left[\exp\left(\frac{\varepsilon_j - \varepsilon_f}{T}\right) + 1\right]^{-1} \text{ (region III),}$$  \hspace{1cm} (30)
i.e., an ordinary Fermi distribution. On decreasing \( \varepsilon_j \) we pass from region II to region I, where all the vacancies permitted by the Pauli principle are filled, i.e., \( b_j + s_j = \Omega_j \). This auxiliary condition gives a Fermi distribution for the holes:

\[
w_j^s = \left[ \exp \left( \frac{\varepsilon_f - \varepsilon_j}{T} \right) + 1 \right]^{-1} \quad \text{(region I)}.
\]

The equations for the boundaries between regions I, II, and III are obtained by equating the expressions for the distribution functions on the two sides of the boundary. This gives the equation

\[
\frac{\varepsilon_j - \varepsilon_f}{T} = \frac{G_j \Omega_j}{2T} \tanh \frac{\varepsilon_j - \varepsilon_f}{2T}
\]

for the boundary between regions II and III. We denote the solution of this equation by

\[
\varepsilon_j - \varepsilon_f = \gamma \left( \frac{G_j \Omega_j}{2T} \right).
\]

For the boundary between regions I and II we obtain

\[
\frac{\varepsilon_j - \varepsilon_f}{T} = -\gamma \left( \frac{G_j \Omega_j}{2T} \right).
\]

Since the solution of Eq. (29) is independent of \( \varepsilon_j \), its value within region II is equal to its value on the boundary (for constant \( \Omega_j \)) or to the values of functions (30) and (31) on the corresponding boundaries (33) and (34), so that

\[
w_j^s = \left\{ \exp \left[ \gamma \left( \frac{G_j \Omega_j}{2T} \right) \right] + 1 \right\}^{-1} \quad \text{(region II)}.
\]

A graph of the function \( \gamma(x) \), defined implicitly by the equation

\[
\gamma = x \tanh(\gamma/2),
\]

is shown in Fig. 2. We have \( \gamma(x) \cong \sqrt{6(x-2)} \) as \( x \to 2 + 0 \), and \( \gamma(x) \cong x \) as \( x \to \infty \).

Equations (30)-(36) completely determine the distribution function \( w_j^s \) and make it possible to calculate the entropy of a spherical nucleus from Eq. (25).

It is convenient to do the calculation in the variables \( \rho \) and \( \tilde{\beta} \) introduced in [6,2] \( (\rho = kR, \ k \) is the wave number of the quasiparticle, and \( R \) is the radius of the nucleus):

\[
\rho [\cos \tilde{\beta} - (\pi/2 - \tilde{\beta}) \sin \tilde{\beta}] = \pi(n + 3/4),
\]

\[
\tilde{\beta} = \arcsin(l/\rho), \quad l = l + 1/2, \quad dn d\rho = (\rho d\rho / \pi) \cos^2 \tilde{\beta} d\tilde{\beta},
\]

where \( n \) is the principal quantum number. To the macroscopic accuracy that we are confining ourselves to, we can replace the total angular momentum \( j \) of the quasiparticle by its orbital angular momentum \( l \), so that \( \Omega_j \cong l \cong \rho \sin \tilde{\beta} \). Taking also into account the fact that in our theory small values \( \tilde{\beta} \) and \( \rho \approx \rho_f \) are important at low temperatures \( (\rho_f = k_f R, \ where \ k_f \) is the wave number at the Fermi boundary) we can put

\[
\Omega_j \cong \rho_f \tilde{\beta}, \quad G_j = \frac{d\varepsilon}{d\rho_f} g_j, \quad T = \frac{d\varepsilon}{d\rho_f} \tau.
\]

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In order to maintain the required macroscopic structure (see the Introduction) in this same limit, it is necessary, as was shown in [2], that
\[ g_j = g_{\beta}. \]  
(39)

The distribution function \( w^* \) assumes the following form in the variables \( \rho \) and \( \beta \) (see Fig. 3):
\[
w^*(\rho, \beta) = \begin{cases} 
\exp\left(\frac{\rho - \rho_f}{\tau}\right) + 1 \>, & \rho - \rho_f < -\tau \gamma \left(\frac{g_{\rho_f} \beta^2}{2\tau}\right), \\
\exp\left(\frac{\gamma \left(\frac{g_{\rho_f} \beta^2}{2\tau}\right)}{2\tau}\right) + 1 \>, & -\tau \gamma \left(\frac{g_{\rho_f} \beta^2}{2\tau}\right) < \rho - \rho_f < \tau \gamma \left(\frac{g_{\rho_f} \beta^2}{2\tau}\right), \\
\exp\left(\frac{\rho - \rho_f}{\tau}\right) + 1 \>, & \rho - \rho_f > \tau \gamma \left(\frac{g_{\rho_f} \beta^2}{2\tau}\right).
\end{cases}
\]  
(40)

We also transform to the variables \( \rho \) and \( \beta \) in Eq. (25), and using the Jacobian \( \frac{dnl}{d\rho d\beta} \approx (\rho f/\pi) d\rho d\beta \) (see Eq. (37)) to replace the summation over the levels by an integration, we obtain
\[
S = -\int_{-\infty}^{\infty} \int_{0}^{\infty} 2 \rho f \beta \left[ w^* \ln w^* + (1 - w^*) \ln(1 - w^*) \right] \frac{\rho f d\rho d\beta}{\pi}.
\]  
(41)

Substituting (40) into (41) and integrating, we obtain the following expression for the entropy of one nucleon component of a spherical nucleus:
\[
S = \frac{\pi^3}{4} \frac{\rho f T^2}{g} = \frac{\pi^3}{4} \frac{\rho f}{g(dz/d\rho)} T^2 = \frac{\pi^3}{4} R^4 m^* g^2 \rho f T^2,
\]  
(42)

where \( m^* = P_f/v_f \) is the effective mass of a quasiparticle in the nucleus. Taking into account the fact that nuclear matter is a two-component system, we find
\[
S = \frac{b}{2} T^2, \quad b = \frac{\pi^3}{2} \frac{R^4 m^*}{h^2} \left( \frac{1}{\rho f^N g_N} + \frac{1}{\rho f^Z g_Z} \right).
\]  
(43)

Thus, in spherical nuclei (at low enough temperatures—see below) the entropy depends quadratically on the temperature, in contrast to the frequently assumed linear dependence. The latter is characteristic of the energy spectrum of a normal Fermi liquid [9,10], but in nuclear physics it can apparently be regarded as fairly reliably established only for nonspherical nuclei.

The thermodynamic relations
\[
E = \frac{b}{3} T^3, \quad C = \frac{dE}{dT} = b T^2, \quad S = \frac{3^{2/3}}{2} b^{1/2} E^{2/3},
\]  
(44)

follow from Eqs. (43), so that the level density in a spherical nucleus depends as follows on the excitation energy:
\[
\rho_{l.d.}(E) \propto \exp[S(E)] = \exp \left( \frac{3^{2/3}}{2} b^{1/2} E^{2/3} \right).
\]  
(45)
Now let us calculate the moment of inertia of a spherical nucleus. To do this we shall examine the Racah-Mottelson states that have not only a definite seniority \( s_j \), but also a fixed angular momentum projection \( M_j \). Then the entropy of level \( j \) will be \( S_j = \ln \Gamma(M_j, s_j, \Omega_j) \), where the number of states is given by Eq. (24) (as before, we neglect the preexponential factor). The distribution functions for the quasiparticles and the angular momentum \( M_j \) are determined from the condition that the free energy \( F' = E - TS - \varepsilon_j \tilde{N} - \omega M \) be minimum, where \( \omega \) is the angular velocity and \( M \) is the total angular momentum projection. Thus, the \( M_j \) dependent terms

\[
\sum_j \left[ \frac{3M_j^2T}{\Omega_j s_j}(2\Omega_j - s_j) - \omega M_j \right]
\]

are added to the terms determined by formulas (25)–(27). Varying \( M_j \) yields the angular momentum projection

\[
M_j = \frac{\Omega_j s_j(2\Omega_j - s_j)}{6T}
\]

of the \( j \) level, so that the total angular momentum projection is given by

\[
M = \omega \frac{2}{3T} \sum_j \Omega_j^3 w_j^s(1 - w_j^s).
\]

In this case the distribution function \( w_j^s \) depends on \( \omega \), but for small \( \omega \) we can neglect this dependence in Eq. (48) and use Eq. (40). The coefficient of \( \omega \) in (48) is the moment of inertia: in the usual units (\( M \) in erg·sec and \( \omega \) in sec\(^{-1}\)) it takes the form

\[
I = \frac{2\hbar^2}{3T} \sum_j \Omega_j^3 w_j^s(1 - w_j^s).
\]

Now we transform (49) to the variables \( \rho, \tilde{\beta} \) and substitute expression (40) for the distribution function; then after integrating over regions I–III we obtain

\[
I = \frac{\pi^3 \rho_f^3}{12 \ g^2 \ (d\varepsilon/d\rho_f)} h^2 T^2.
\]

In our calculation, however, we have not taken into account the effect of the second functional derivative of the energy with respect to the distribution function \([9,10]\), which leads to an additional constant factor in Eq. (50). This factor can be found by calculating the moment of inertia of a normal Fermi liquid by a method similar to that used above. A simple calculation shows that the degree of degeneracy of a state with \( s \) fermions on level \( j \) having the angular momentum projection \( M \) differs from (24) only in the preexponential factor, which is not important for thermodynamic applications. This means that in the thermodynamic sense the seniorities \( s \) are fermion excitations with a dispersion law leading to distribution function (40). The moment of inertia of a Fermi liquid is therefore given by Eq. (49) with the ordinary Fermi distribution substituted for \( w^s \).

Now we easily find

\[
I_{FL} = \frac{4}{45\pi} \frac{\hbar^2 \rho_f^4}{d\varepsilon/d\rho_f}.
\]
Here, too, we have neglected the second functional derivative. Requiring that taking this into account should lead to the rigid-body value \( I_{FL} = (2/5)N m_n R^2 = (2/5)(2\rho_f^3/9\pi)m_n R^2 \), where \( m_n \) is the nucleon mass, we find
\[
\frac{d\varepsilon}{d\rho}\bigg|_f = \frac{\hbar^2 \rho_f}{m_n R^2},
\]
i.e., \( d\varepsilon/d\rho\big|_f \) should have the Fermi-gas value. Hence one of the factors \( d\varepsilon/d\rho\big|_f \) in Eq. (50) should also have the Fermi-gas value. Taking this into account, as well as the fact that nuclear matter is a two-component system, we obtain the expression
\[
I = \frac{\pi^3 m^2 m_n R^6}{12 \hbar^4} \left( \frac{1}{\rho_j^N g_N} + \frac{1}{\rho_f^Z g_Z} \right) T^2
\]
for the moment of inertia of a spherical nucleus.

The function describing the distribution of the nuclear levels over the angular momentum \( J \) at fixed temperature has the form (see [11])
\[
w_J = \frac{\hbar^2}{IT} \tilde{J} \exp \left( -\frac{\hbar^2 \tilde{J}^2}{2IT} \right), \quad \tilde{J} = J + \frac{1}{2},
\]
so that the mean and mean square values of the angular momentum are
\[
\bar{J} = \sqrt{\frac{\pi IT}{2\hbar^2}}, \quad \bar{J}^2 = \frac{2IT}{\hbar^2}.
\]
Thus we have
\[
\bar{J}^2 = \frac{4}{\pi} \left( \bar{J} \right)^2,
\]
and this relation can be used to check the correctness and completeness of the experimental data on level spins. We also note the equation
\[
\bar{J}^2 = \frac{m_n R^2}{\hbar^2} \frac{\rho_j^N g_N^2 + \rho_f^Z g_Z^2}{g_N g_Z (\rho_j^N g_N + \rho_f^Z g_Z)} E,
\]
which follows from Eqs. (55), (53), and (44).

4 Discussion

As we already mentioned, the formulas derived here are valid only at low temperatures. In order to find out how low the temperature must be, we take account of the fact that Eq. (39) was established only for angles satisfying the condition
\[
\tilde{\beta} \lesssim \frac{1}{\sqrt{2\rho_f}}
\]
(see [2]). Hence the quantity
\[ \tilde{\beta} = \sqrt{\frac{4\tau}{g\rho f}}, \]  
which is characteristic of our theory, marking, as it does, the point at which region II begins to have a finite extent along the \( \varepsilon \) or \( \rho \) axis (see Figs. 1 and 3 and Eq. (40)), should satisfy condition (58). This gives the criterion
\[ \tau \ll \frac{g}{8} \text{ or } T \ll \delta \varepsilon, \]  
where \( \delta \varepsilon = \left. \frac{d\varepsilon}{d\rho} \right|_{g/8} g/8 \) is the characteristic width of the diffuse region of the Fermi distribution caused by the residual reaction (see [2]). The values of \( \delta \varepsilon \) found in [2] range from 1.9 MeV (heavy nuclei) to 5.2 MeV (light nuclei) for neutron magic nuclei and from 1.6 MeV (heavy nuclei) to 3.8 MeV (light nuclei) for proton magic nuclei. However, it is difficult at present to judge the part played by another criterion,
\[ T \ll T_C, \]  
implied by the theory, where \( T_C \) is the Curie temperature at which the spherical nucleus passes from the magic phase to the normal phase, i.e., becomes nonspherical (see [3]). We can only assert that condition (61) becomes more important, and may become dominant, for spherical nuclei that lie close as regards number of nucleons to the phase transformation point. The \( \gamma \)-ray spectra and radiative widths of nuclei that are spherical in the ground state confirm this (see [5,12]).

We also note that all the equations we have derived relate the average values of the thermodynamic quantities to one another, i.e., they are valid only when the fluctuations are small. However, the temperature fluctuations [10]
\[ \Delta T = \frac{T}{\sqrt{C}} = \frac{1}{\sqrt{b}} \]  
amount to \( \sim 0.6 \) MeV for \( ^{208}\text{Pb} \), so that the thermodynamic formulas can only serve as rough estimates at the excitation energies now being investigated for spherical nuclei. For example, it follows from Eq. (57) that \( \overline{J^2} \simeq 4 \) for \( ^{208}\text{Pb} \) at an excitation energy of 5 MeV, whereas Eqs. (55) with the rigid-body moment of inertia and temperature \( T \simeq 0.4 \) MeV (which is apparently characteristic of nonspherical nuclides; see [12]) give \( \overline{J^2} \simeq 160 \). As well as can be judged from the available data [13] on levels of \( ^{208}\text{Pb} \), however, the experimental value is \( \overline{J^2} \simeq 10 \), which is in qualitative agreement with the theory under discussion.

Although there is no thermal phase transformation in the present theory, the macroscopic structure of the residual interaction as a function of the orbital quantum number \( l \) together with the fact (a macroscopic quantum effect) that the spherical nuclear configuration is unstable in the absence of interaction between the quasiparticles (see [6]) enables us to suggest a possible reason for such a transformation. In fact, as the temperature rises region II shifts to the right on the \( l \) axis and more and more quasiparticles fall into regions I and III, where they conform to the ordinary Fermi distribution. Hence a phase transformation from the magic phase to the normal phase should take place when there are no longer enough quasiparticles left in region II to stabilize the spherical shape of the nucleus.

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