Collective excitations of low density fermion-boson quantum-liquid mixtures

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We investigate the collective excitations of a low temperature dilute gas mixture that consists of a Bose-Einstein condensate and a Fermi-gas that is a normal (i.e. non-superfluid) Fermi-liquid. We find that the BEC-mediated fermion-fermion interactions, as a consequence of retardation, can become repulsive and support a zero-sound mode that is essentially undamped. In addition, we find a damped zero-sound mode that can be described as a BEC-sound mode modified by fermion mediated boson-boson interactions, and we derive its decay-rate caused by Landau damping. We study the mode structure of these excitations and find avoided crossing behavior as well as a termination point. The collective mode dynamics also reveals that phase separation sets in when the fermion-mediated boson-boson interaction destroys the stability of the homogeneous BEC. We estimate the time and length scales of the onset of the phase separation, and we discuss the feasibility of experimentally probing these consequences of mediated interactions.

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I. INTRODUCTION

Current atom trap experiments create quantum degenerate Fermi-gas systems$^{1,2}$ and Fermi-Bose gas mixtures in sympathetic cooling experiments$^2$. The sympathetic cooling of fermion atoms utilizes bosonic atoms at lower temperature, often condensed into a Bose-Einstein condensate (BEC). At the time of writing, the fermions do not quite enter the Fermi-liquid regime before the BEC is separated out and the fermions are further cooled evaporatively$^3$; however, unambiguous Fermi-liquid/BEC mixtures are attainable$^4$. Fermi-liquid behavior sets in when the energy can be expanded up to second order in the (quasi-) particle occupation numbers, the heat capacity varies linearly with temperature, and the Pauli exclusion principle sufficiently inhibits incoherent fermion scattering to validate the assumption of collisionless dynamics. A dilute two-component Fermi-gas enters this regime when the temperature drops below 10% of the Fermi temperature$^5$. Strong interaction effects further lower the Fermi-liquid range: experiments reported an upper bound of 2% of the Fermi-temperature in the condensed$^3$He liquid.

With the creation of cold atom Fermi–Bose quantum liquid mixtures, atom traps can access novel regimes of low temperature physics. At present, table-top explorations of Fermi–Bose quantum-liquid mixtures are confined to the condensed$^3$He–$^4$He mixtures. These experiments revealed an intricate phase diagram, even if the$^3$He component did not reach superfluidity. The critical temperature for$^4$He superfluidity drops in the$^3$He ‘rich’ regime, and depending on the temperature and fraction of the$^3$He isotope, the mixture spontaneously breaks translational symmetry by forming ‘$^3$He only’ regions. The microscopic understanding of this phase separation is, however, complicated due to the strong interaction which masks or, at least, competes with the quantum statistical effects$^6$. General arguments based on Fermi-liquid theory indicated the importance of mediated interactions$^7$: they are the modification of the inter-particle interactions in one liquid due to the presence of the second liquid and directly causes the phase separation by destroying the stability of the homogeneous state. In the static limit, applicable to the description of the helium mixtures in which the sound velocity $c$ of the boson superfluid greatly exceeds the Fermi velocity $v_F$, the boson-mediated fermion-fermion interactions are attractive and can Cooper-pair the fermions$^8,9,10$. From the experimental measurements in helium mixtures, it appears that the strong interactions greatly weaken the mediated interactions$^7$.

In contrast, the phase separation of cold atom Fermi-mixtures$^11$ would be amenable to a transparent first-principles description$^12$ and the boson-mediated fermion-fermion interactions can significantly increase the critical temperature for Cooper-pairing$^{13}$. In addition, the relative densities of the cold atom mixtures can reverse the Fermi-velocity $v_F$ to Bose sound velocity $c$ ratio to yield $c/v_F \leq 1$, which is not attainable in the helium mixtures. Under those conditions, satisfied in most sympathetic cooling experiments, the static description breaks down and we show that retardation radically alters the nature of the boson-mediated fermion-fermion interactions.

There are other aspects that motivate cold atom Fermi-liquid BEC studies. For example, the range of the BEC-mediated fermion-fermion interactions, which is the BEC-healing length, can exceed the inter-particle distances. Cold atom experiments with fermion-BEC mixtures can then go beyond the usual cold atom paradigm of short-range,
contact-like inter-particle potentials while examining similarities with other Fermi-liquids. In particular, the form of the static BEC-mediated interaction suggests an intriguing connection between the anticipated cold atom Cooper-paired Fermi–Bose mixtures and the BEC-exciton superfluid that has been pursued in condensed matter for more than two decades [14]. The BEC-mediated interaction potential is of the Yukawa form [15], which resembles the electron-hole Coulomb attraction of an exciton at distances smaller than the Yukawa range.

The realization of s-wave Cooper-pairing in fermion-BEC mixtures, will, however, require much experimental skill. This pairing scheme requires the simultaneous trapping of fermion atoms in two distinct hyperfine states at nearly equal densities, a considerable challenge in the presence of a BEC: A dependence of the boson-fermion interactions on the fermion spin introduces a difference in the local spin densities even if the total number of fermion atoms is evenly divided over the two spin states and both spins experience identical trapping potentials.

In this paper, we focus on a dilute mixture of BEC and one-component Fermi gas that remains normal. We present a theoretical study of the small amplitude oscillations around a spatially uniform ground state. First, we show that the resonant dynamics of the mediated interactions introduces novel features. If $c < v_F$, the mediated interaction supports a long-lived essentially undamped zero sound mode as in a two-component Fermi-liquid case, which is incompatible with the effective attraction predicted by the static treatment. On the other hand, if $v_F < c$, the mixture supports an additional mode that is the remnant of the BEC-sound mode modified by a cloud of fermion excitations that introduce fermion mediated boson-boson interactions. This second mode is damped. Both oscillations have been discussed by Yip [16] from a calculation of many-body response functions. Several calculations of zero-sound modes in trapped fermions have been also reported in [17]. We analyze the homogeneous BEC–fermion mixture using the semi-classical description, and obtain a single self-consistent Fermi-liquid-like collective mode equation. The semiclassical description in terms of the spatially oscillating deformation of the local Fermi-surface elucidates the zero-sound nature of the fermion oscillations in both modes, revealing that the dominant decay mechanism of the BEC-like sound mode at long wavelengths is Landau damping.

The dispersion relations of the two collective modes, the undamped and damped modes, avoid crossing each other in the wavenumber interval where the undamped mode gradually changes its character from a pure Fermi-liquid zero sound mode to a Bogoliubov excitation of the BEC. The damped mode terminates at a certain wavelength. The damping rate at the termination point remains significantly smaller than the excitation energy so that the mode represents a good quasi-particle for all wavenumbers and the termination point should be observable.

The phase separation is manifested by the collective mode dynamics. The instability corresponds to collective mode eigenvalues of purely imaginary value. Our analysis reveals the mechanism of the instability, and gives the length and time scales on which the phase separation sets in.

First, we discuss the mediated interaction of the fermion–BEC in a static limit in Sec. II. Then, the dynamical effects are obtained in Sec. III, followed by the mode analysis in Sec. IV. Finally, experimental aspects are examined in Sec. V. This is an extension of the work previously presented in Ref. [13].

II. MEDIATED INTERACTIONS IN THE STATIC LIMIT ($\omega \to 0$)

In most mixtures, the mediated interaction shifts the frequency of the existing mode. In the single-component fermion BEC mixture, the BEC-mediated interactions actually support collective Fermi-liquid like oscillations. Interactions amongst identical Fermions are prevented by Pauli-exclusion, so that any fermion-fermion interactions in the single component fermion BEC mixture are necessarily BEC mediated.

In this section we illustrate BEC-mediated interactions by considering the simplest system to exhibit this phenomenon: two indistinguishable, stationary fermions of mass $m_F$ are embedded in a dilute BEC of the particle density $\rho_B^0$ and particle mass $m_B$ in equilibrium located at position $x_1$, and $x_2$. The mediated interaction obtained in a static system highlights the dynamical effects that we discuss in Sec. III. Without the impurity atoms, the BEC would be homogeneous. The bosons interact by short range interactions, which we describe by a pseudo-potential $\lambda_{BB}/(r-r')$, with interaction strength, $\lambda_{BB} = 4\pi\hbar^2 a_{BB}/m_B$, that is proportional to the boson-boson scattering length $a_{BB}$. Similarly, we assume that the interaction between the fermion and BEC-boson atoms is well described by $\sum_{j} \lambda_{BF}(x_j - r)$, with an interaction strength proportional to the impurity-boson scattering length $a_{BF}$ and inversely proportional to the effective mass of the fermion-boson system $m_{BF}$, where $m_{BF}^{-1} = m_F^{-1} + m_B^{-1}$, $\lambda_{BF} = 2\pi\hbar^2 a_{BF}/m_{BF}$. The pseudo-potential description presupposes that the scattering lengths are shorter than the average inter-particle distances, $a_{BB}, a_{BF} \ll (\rho_B^0)^{-1/3}$.

This system lends itself to a particularly transparent description of the mediated interaction. The BEC-mediated interaction is the modification of the mean-field energy experienced by the fermion at $x_1$ due to the other fermion at position $x_2$. The energy shift is caused by the change in BEC-density $\rho_B(x_1)$ due to the interaction of the other atom with the BEC. Within the mean-field description, the equilibrium BEC field in the presence of the $x_2$ atom follows...
from the time-independent Gross-Pitaevskii equation

\[ \mu_B \phi(r) = \left[ -\frac{\hbar^2 \nabla^2}{2m_B} + \lambda_{BB}\phi(r)^2 \right] \phi(r) + \lambda_{BF}\delta(r - x_2)\phi(r), \] (1)

where \( \phi(r) \) represents the condensate field \( \phi = \langle \hat{\psi} \rangle \) with \( \rho_B(r) = |\phi(r)|^2 \) and \( \mu_B = \lambda_{BB}\rho_B^0 \) is the chemical potential of the BEC ensuring that the density tends to \( \rho_B^0 \) at large distances from \( x_2 \). Since the BEC is in equilibrium, the field \( \phi \) can be chosen to be real-valued.

Linearizing the condensate wavefunction by setting \( \phi \approx \sqrt{\rho_B} + \delta\phi \), Eq. (1) becomes

\[ \left[ \nabla^2 - \xi^{-2} \right] \delta\phi(r) = -\Xi\delta(r - x_2), \] (2)

where \( \xi \) is the healing length of the BEC (also referred to as the coherence length), \( \xi = 1/\sqrt{16\pi\rho_B^0 a_{FB}} \) and \( \Xi \) is the strength of the source term, \( \Xi = -4\pi(1 + m_B/m_F) a_{FB}\sqrt{\rho_B^0} \). The solution to Eq. (2) is proportional to the Green function of the modified Helmholtz equation,

\[ \delta\phi(r) = -\sqrt{\rho_B^0} \left( 1 + \frac{m_B}{m_F} \right) a_{FB} \exp\left( \frac{|x_2 - x|}{\xi} \right). \] (3)

The condition, \( |\delta\phi(r)|/\sqrt{\rho_B^0} \equiv \xi \ll 1 \) suggests that the Yukawa profile is accurate if \( |x_1 - x_2| > r_s \), where \( r_s \equiv a_{BB}(1 + m_B/m_F)/\xi \), and we have assumed that \( \xi > r_s \). The BEC-mediated fermion–fermion interaction potential in the static limit is

\[ V_F^{med}(|x_1 - x_2|) = 2\lambda_{FB}\sqrt{\rho_B^0}\delta\phi(x_1) \]

\[ = -2 \left( 1 + \frac{m_B}{m_F} \right) \lambda_{FB}\rho_B^0 a_{FB} \exp\left( \frac{-|x_1 - x_2|}{\xi} \right). \] (4)

The result is an attractive Yukawa potential, familiar from the description of effective interactions mediated by massive scalar fields in relativistic field theory \[15\]. For BEC’s this result was obtained as well using perturbation theory \[11,12,21\].

From Eq. (4) we find that the average value of the static mediated interaction for a fermion particle in a homogeneous BEC-fermion mixture is only a fraction of the mean-field fermion–boson interaction energy experienced by a fermion, \( \lambda_{FB}\rho_B^0 \), with the fraction being of the order of \( 2(1 + m_B/m_F)(\rho_F^0)^{1/3} a_{FB}\exp\left(-1/(\xi\rho_F^0)^{1/3}\right) \). However, in the long wavelength limit, the Fourier transformed potential, \( V_{FF}^{med}(k) \), is

\[ V_{FF}^{med}(k) = -\frac{\lambda_{FB}^2}{\lambda_{BB}} \frac{1}{1 + (k\xi)^2}, \] (5)

and comparable to or possibly larger than that of the effective contact potential, \( \lambda_{FB}, \lim_{k \to 0} V_{FF}^{med}(k) = -\lambda_{FB}^2/\lambda_{BB} \). We note that Eq. (5) agrees with the long wavelength limit derived in Ref. \[19\] using the Landau Fermi-liquid approach (which is also valid in describing the strongly interacting systems),

\[ V_{FF}^{med}(k = 0) = \lim_{p \to \rho_F^0} \frac{\partial E_F}{\partial p_B^0} \frac{\partial E_F}{\partial p_B^0}, \] (6)

where \( E_F \) is the fermion quasi-particle dispersion, and the derivatives are taken while keeping the occupation numbers constant. For the weak-interaction, \( E_F \approx \lambda_{FB}\rho_F^0 + p^2/2m_F \), and we recover the long-wavelength limit of Eq. (4).

Similarly, we can determine the fermion-mediated interactions in the static limit by calculating the density variation caused by a single impurity atom located at \( x \), interacting with the fermions via a short-range interactions. The resulting density variation oscillates on the length scale of the average fermion-fermion distance, yielding a density profile \( \rho_F(r) \) with features known as Friedel-oscillations. At large distances, the density variation \( \delta\rho_F(r) \) of a three-dimensional weakly interacting fermion system oscillates as

\[ \delta\rho_F(r) \propto \frac{\cos(k_F|x - r| + \kappa)}{|x - r|^3}, \] (7)

where \( k_F \) is the Fermi momentum, \( k_F = (6\pi^2\rho_F^0)^{1/3} \), and \( \kappa \) is the phase shift. The resulting mediated interaction is known as the RKKY-interaction.
III. COLLECTIVE MODE DYNAMICS

While the static limit of the mediated interactions presents a picture of appealing simplicity, most sympathetic cooling experiments operate outside the boundary of its validity (which amounts to \( v_F \ll c \)). In this section, we investigate the effects of the dynamics of the BEC-response in collective modes. The BEC density fluctuation propagates at a finite velocity and the consequent retardation of its response to a fermion density fluctuation can alter the physics of the collective oscillations radically as we show below.

We assume that the boson-boson interactions are repulsive, \( a_{BB} > 0 \), and the temperature is sufficiently low so that almost all boson particles are condensed into the BEC, thus the fermion dynamics can be described as collisionless. These assumptions will lead to a description of the dynamics that is effectively temperature independent even though it is tacitly understood that the temperature exceeds the critical temperature for fermion pairing.

A. Fermion dynamics

The inhibition of incoherent fermion-fermion scattering by Pauli-exclusion principle does not eliminate the effects of fermion-fermion interactions. As shown by Landau [20], collective modes can be driven by the mean-field variations that accompany the propagating density and current variations. Zero-sound, which corresponds to a periodic oscillation of fermion-fermion interactions. As shown by Landau [20], collective modes can be driven by the mean-field variations. The dynamics of the BEC-density fluctuation propagates in the plane wave of the BEC fluctuation, the complex phase of the BEC-field varies as \( e^{i(k \cdot r - \omega t)} \), and, in Cartesian coordinates, \( \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \).

Let us define the single particle fermion distribution function as \( n(r, p; t) \), where \( r \) is the location and \( p \) is the momentum. The time evolution of this function is governed by the collisionless transport equation

\[
\frac{\partial n(r, p; t)}{\partial t} + v \cdot \frac{\partial n(r, p; t)}{\partial r} + F(r, p; t) \cdot \frac{\partial n(r, p; t)}{\partial p} = 0,
\]

(8)

where we have approximated the velocity of the dilute fermion gas as \( v \approx p/m_F \), and \( F \) is the gradient of the BEC-induced mean-field interaction energy, \( F = -\nabla \left[ \lambda_{FB}\rho_B(r; t) \right] \). Then, we linearize \( n \) assuming that \( n(r, p; t) \) fluctuates around the equilibrium distribution \( n^0(p) \) and the fluctuation, which is confined in momentum space to a thin shell in the vicinity of the equilibrium Fermi-surface, propagates in real space as a plane wave with wavevector \( k \). We write \( n(r, p; t) \) as

\[
n(r, p; t) = n^0(p) + u_p \delta(p) (|p| - p_F) e^{i(k \cdot r - \omega t)},
\]

(9)

where \( p_F \) is the equilibrium Fermi momentum and \( u_p \) denotes the amplitude of deformation. Similarly, assuming that the BEC fluctuation also propagates as a plane wave, \( \rho_B(r, t) = \rho_B^0 + \delta\rho_B(r, t) \) with \( \delta\rho_B(r, t) = \eta_k \rho_B^0 \exp \left[ i (k \cdot r - \omega t) \right] \), we obtain from Eq. (8) with Eq. (9) that

\[
-i\omega u_p + i \frac{k \cdot p}{m_F} u_p = -i \frac{k \cdot p}{p_F^2} \lambda_{FB} \rho_B^0 \eta_k.
\]

(10)

In the plane wave propagation of the BEC fluctuation, the complex phase of the BEC-field varies as \( \delta\theta_B = \eta_k \exp \left[ i (k \cdot r - \omega t) \right] \). The assumption of small amplitude oscillations implies that the fluctuation amplitudes \( \theta_k \) and \( \eta_k \ll 1 \). Below, we describe their time evolution.

B. BEC dynamics

In general, the BEC dynamics is conveniently described in terms of the real valued density and phase fields, \( \rho_B(r; t) \) and \( \theta_B(r; t) \), the fluctuations of which we introduced above [21]. These quantities relate to the complex valued condensate field as \( \phi(r; t) = \langle \hat{\psi}(r; t) \rangle = \sqrt{\rho_B(r; t)} e^{i\theta_B(r; t)} \). The time-dependent Gross-Pitaevskii equation (Eq. II with \( \mu_B \rightarrow -i\hbar\partial/\partial t \)) leads to

\[
\frac{\partial |\phi|^2}{\partial t} + \nabla \cdot \left[ \frac{\hbar}{2m_B} (\phi^* \nabla \phi - \phi \nabla \phi^*) \right] = 0.
\]

(11)

With the density and phase fields, Eq. (11) takes the form of a classical continuity equation,

\[
\frac{\partial \rho_B(r; t)}{\partial t} = -\nabla \cdot [\rho_B(r; t) v_s],
\]

(12)
where \( \mathbf{v}_s = (\hbar/\lambda_B) \nabla \theta_B(\mathbf{r}; t) \) is a superfluid velocity. The real part of the Gross-Pitaevskii equation gives

\[
-\hbar \frac{\partial \theta_B(\mathbf{r}, t)}{\partial t} = \frac{-\hbar^2}{2m_B \sqrt{\rho_B(\mathbf{r}, t)}} \nabla^2 \sqrt{\rho_B(\mathbf{r}, t)} + \lambda_B \rho_B(\mathbf{r}, t) + \lambda_F \rho_F(\mathbf{r}, t) + \frac{1}{2} m_B \mathbf{v}_s^2(\mathbf{r}; t),
\]

which is similar to the Landau superfluid velocity equation with the Bohm-potential \((\propto \nabla^2 \sqrt{\rho_B})\) added to the usual mean-field contributions \(\lambda_B \rho_B + \lambda_F \rho_F\) and the superfluid kinetic energy, \(m_B \mathbf{v}_s^2/2\).

Linearizing the BEC-fields as in Sec. IIIA, then substituting them into Eqs. (12) and (13) give

\[
-i \omega \eta_k = \frac{\hbar k^2}{m_B} \theta_k,
\]

\[
i \hbar \omega \theta_k = \left(\frac{\hbar^2 k^2}{4m_B} + \lambda_B \rho_B^0 \right) \eta_k + \lambda_F \delta \rho_{F,k},
\]

where \(\delta \rho_{F,k}\) is the Fourier transform of the fermion density fluctuation that corresponds to Eq. (9). The fluctuation amplitude of the boson density is then proportional to the fermion density fluctuation and can be written as

\[
\eta_k = \lambda_F \rho_B^0 \frac{\delta \rho_{F,k}}{\lambda_B \omega/kc^2 - [1 + (k\xi)^2]},
\]

where \(c\) is the sound velocity of the pure BEC, \(c = (\lambda_B \rho_B^0/m_B)^{1/2}\). Note that the sign of the ratio of the boson to fermion density fluctuations depends on the frequency \(\omega\).

C. Dispersion relation of the collective modes

Returning to the Fermi-liquid dynamics of Sec. IIIB we write Eq. (10) as

\[
\left[ \frac{\omega}{k_F} - \cos (\varphi) \right] u_p - \eta_k \cos (\varphi) \left[ \frac{\lambda_F \rho_B^0}{2\xi_F} \right] = 0,
\]

where \(\varphi\) is the angle between the \(\mathbf{k}\) and \(\mathbf{p}\) vectors, and where \(\xi_F\) represents the equilibrium Fermi-energy.

From the definition of \(n\), which implies \(\rho_F(\mathbf{r}, t) = \hbar^{-3} \int d^3 \rho \; n(\mathbf{r}, \mathbf{p}, t)\), and with Eq. (9), we find that the Fermi-density fluctuation \(\delta \rho_{F,k}\) in Eq. (10) is proportional to the angular average of the deformation amplitude,

\[
\delta \rho_{F,k} = 3 \rho_F^0 \langle u \rangle,
\]

where \(\langle u \rangle = (4\pi)^{-1} \int d\Omega \rho u_p\). Combining Eqs. (10) and (17), we obtain

\[
\left[ \frac{\omega}{k_F} - \cos (\varphi) \right] u_p = \cos (\varphi) \left[ \frac{F}{\omega/kc^2 - [1 + (k\xi)^2]} \right] \langle u \rangle,
\]

where \(F = (\lambda_F/\lambda_B)(3\lambda_B \rho_B^0/2\xi_F)\) is the interaction parameter of the system that embodies all interaction information necessary for understanding the zero-sound mode behavior. Alternatively, we can write

\[
F = \frac{k_F a_F a_B}{2\pi} \frac{m_F}{m_B} \left(1 + \frac{m_F}{m_B}\right),
\]

to express the interaction dependence in terms of the scattering lengths.

We find that the angular dependence of the mode fluctuation is restricted by the density-density coupling of the liquids in the dilute fermion-BEC mixture. Expressing \(u_p\) in a spherical harmonics form, \(u_p = \sum_{l,m} u_{lm} Y_{lm}\), we obtain that \(u_{lm} = 0\) for \(m \neq 0\). In the collective excitation, the local Fermi momentum distribution fluctuates as a longitudinal mode with an azimuthal symmetry around the direction of propagation with the deformation amplitude \(u_p\), so that the sole momentum dependence is upon the angle between \(\mathbf{k}\) and \(\mathbf{p}\). Rather than writing \(u_p = \sum_l u_{l0} Y_{l0}\), we simply determine the dependence upon \(\varphi\) by solving for \(u_p\) from Eq. (14) and take the angular average. Defining the scaled phase velocity as \(s = \omega/k_F\), we obtain

\[
\frac{s^2 (v_F/c)^2 - [1 + (k\xi)^2]}{F} = \frac{1}{2} \int_{-1}^{1} \frac{x}{s^2 - x} dx,
\]
where we set \( x = \cos \varphi \). From Eq. (21) we can determine the dispersion relation of the collective modes. The complex conjugate, \( s^* \), follows from the appropriate choice of boundary conditions as we explain in the next section.

The structure of the excited state is apparent from neither Eq. (21) nor Landau’s semiclassical description (see 24). We believe that the actual excited state can be written a linear superposition of two states: One is a product state of the ground state BEC and an excited fermion state, which is itself a linear superposition of particle-hole excitations as is the case in the pure fermi-liquid zero-sound mode. The other is the product of the ground state of the Fermi-liquid and a BEC-state that contains a single Bogoliubov quasi-particle excitation.

IV. MODE ANALYSIS

Equation (21) is a close relative of the one that describes the zero sound excitation of a two-component pure Fermi-liquid whose inter-particle interactions are restricted to s-wave partial waves 23.

\[
\frac{1}{F_0} = -1 + \frac{s}{2} \ln \left( \frac{s + 1}{s - 1} \right),
\]

(22)

where \( F_0 \) is the Landau interaction parameter.

When \( s > 1 \), the solution \( s \) to Eq. (21) is real and the right-hand side of Eq. (21) reduces to a simple expression

\[
\frac{s^2(v_F/c)^2 - (1 + (k\xi)^2)}{F} = -1 + \frac{s}{2} \ln \left( \frac{s + 1}{s - 1} \right),
\]

(23)

of the form of Eq. (22). The similarity suggests identifying \( F/ \left[ s^2(v_F/c)^2 - (1 + (k\xi)^2) \right] \) as an effective Landau-liquid parameter. However, a true Landau-liquid parameter does not depend on the wavenumber \( k \). On the other hand, the appearance of the wavenumber leads to the rich mode structure that we discuss in the following subsections.

As mentioned before, \( F \) parameterizes all the dependence on the interaction parameters, and zero-sound modes exist for \( 0 < F < 1 \). The eigenvalues of Eq. (22) depend on the velocity ratio \( v_F/c \), which quantifies the ratio of the time scales of the boson and fermion dynamics. In fact, in the long wavelength limit, the oscillation periods of both the BEC and fermion modes of wavenumber \( k \) are order of \( (kc)^{-1} \) and \( (kv_F)^{-1} \). These time-scales are also associated with the response time to density perturbations of spatial variation of \( k^{-1} \).

A. Collective mode excitations \( 0 < F < 1 \)

If \( v_F \gg c \) and \( 0 < F < 1 \), two collective modes exist: an undamped mode with a real \( s \) and a damped mode with a complex \( s \). The imaginary part of \( s \) describes the decay rate of the damped mode and the zero-sound analogy reveals that the decay mechanism is indeed Landau damping: the creation of particle-hole pairs in the Fermi-system. Landau damping only occurs when \( s \leq 1 \) due to energy conservation: a particle-hole excitation with momenta \( k + k/2 \) (particle) and \( K - k/2 \) (hole) near the Fermi-surface of momentum \( K \) has an oscillation frequency that is smaller than \( \omega \). In other words, \( iK \cdot k/m_F < v_F k \), so that a single particle hole excitation cannot absorb all the energy of the collective excitation.

1. Undamped mode

The scaled dispersion relations of the undamped modes, \( \omega \xi/v_F \) vs. \( k\xi \), at various velocity ratios of \( v_F/c \) with a fixed \( F = 0.5 \) are shown in Figs. 1, 2 and 3. For \( v_F \geq c \) the dispersion is linear with the phase velocity \( v \approx v_F \) at small \( k \), similar to the long wavelength dispersion of the zero-sound mode in a weakly interacting pure Fermi-liquid. At larger \( k \), the dispersion becomes identical to that of an elementary excitation of a single homogeneous BEC (i.e., Bogoliubov mode) with \( \omega = kc\sqrt{1 + (k\xi)^2} \). The transition from linear dispersion to Bogoliubov dispersion occurs gradually around \( k_{\text{tran}} = \xi^{-1}\sqrt{(v_F/c)^2 - 1} \) as a result of the avoided crossing of the two dispersions (See Figs. 1 and 2). On the other hand, when \( v_F < c \), the undamped mode dispersion is the Bogoliubov dispersion for the entire range of \( k \)-values, as seen in Fig. 3.

Since our treatment of the Fermi-liquid dynamics assumes \( k \ll k_F \), we need to check the wavenumber regime of validity. However, for mixtures that consist of light boson and heavy fermion atoms (for example, \(^1\text{H} \) and \(^{40}\text{K} \), or \(^6\text{Li} \) and \(^{87}\text{Rb} \)), this range can extend to wavenumbers comparable to or greater than \( \xi^{-1} \), as implied by \( k_F \xi = (m_F/2m_{1B})(v_F/c) \).
FIG. 1: The dispersion relation ($\omega \xi / v_F$ vs $k \xi$) for a fixed $F = 0.5$ and a fixed $v_F/c = 2$. The dispersion changes from linear to Bogoliubov dispersion.

FIG. 2: The dispersion relation ($\omega \xi / v_F$ vs $k \xi$) for a fixed $F = 0.5$ and a fixed $v_F/c = 3$. The dispersion changes from linear to Bogoliubov dispersion.

If $v_F < c$, the BEC responds sufficiently fast to follow the fermion oscillations adiabatically and the static description of BEC-mediated interaction is valid. In this case, the BEC-mediated interaction is attractive and there is no Fermi-liquid like zero-sound mode. Instead, there is a collective oscillation that resembles the elementary excitation of a pure BEC with $v \approx c$ for all range of $k$ (Fig. 3). On the other hand, if $v_F \gg c$, the BEC response is significantly slower than the fermion frequency and the BEC oscillates out of phase with the fermions. The overall sign of the left hand side in Eq. (21) is positive for small $k$, giving an effective repulsive fermion-fermion interaction. In this regime, the collective fermion mode with zero-sound characteristics occurs with phase velocity velocity $v \approx v_F$ (see Figs. 1 and 2).

2. Damped mode

In addition to the undamped mode, a damped collective excitation exists in $0 < F < 1$. As stated before, Landau damping $\gamma$ is the dominant damping mechanism of zero-sound in the regime of collisionless dynamics. The imaginary part of the complex eigenvalues for Eq. (21) is proportional to the damping rate. Assigning the correct sign to the imaginary part of $s$ in the eigenvalue equation Eq. (21) is rather delicate: While the damped mode corresponds to $s = r - i \gamma$ with $r, \gamma > 0$, causality requires that the integral on the right-hand side, which describes
the retarded fermion-mediated boson-boson interaction, acquires \( s^* = s + i\gamma \). The damped excitation evolves in time as \( \exp[-i(\omega - i\Gamma/2)t] \) with \( \Gamma = 2kv_F\gamma \). Independent from the collective mode equation, we estimate the damping rate \( \Gamma(k) \) using the Fermi-golden rule. The initial state \( |i\rangle \) is the product of the single quasi-particle BEC state with momentum \( k \) and a fermion state that consists of a completely filled Fermi sphere. The final state \( |f\rangle \) is the product of the BEC-ground state and an excited fermion state with a particle-hole pair of momenta \( K + k/2 \) for the particle and \( K - k/2 \) for the hole. The quasi-particle-hole pair is created near the Fermi-surface of momentum \( K \). Treating the fermion-boson density-density interaction as a perturbation \( \hat{H}_F \) and summing over the final fermion states, we obtain the damping rate

\[
\Gamma_{FG}(k) = (2\pi/\hbar) \sum_{f} |\langle f|\hat{H}_F|i\rangle|^2 \delta(E_f - E_i) = \frac{\pi F}{2} \left( \frac{c}{v_F} \right) k c ,
\]

(24)

where the energy \( E_i \) and \( E_f \) represent the energies of the initial and final state, respectively. The damping rate is proportional to \( kc \) as well as \( F \) and \( (v_F/c)^{-1} \).

Note that Eq. (24) is valid for \( k \ll k_F \). Figures 3 and 4 show \( \text{Im}[\omega]\xi/v_F \) obtained both by numerics from Eq. (21) and by the Fermi-golden rule estimate for \( v_F/c = 2 \), and \( F = 0.01 \) and \( F = 0.5 \), respectively. For \( 0 < F \ll 1 \) with \( v_F/c > 1 \), the rate of damping is orders of magnitude smaller than the excitation frequency, the phase velocity of the damped mode is almost equal to that of the BEC sound mode, and the Fermi-Golden rule approximation is in good agreement with the numerical solutions to Eq. (21).

The damped mode exists for small wavenumbers \( k \) and ends at the ‘termination point’, \( k_{\text{end}} \), as seen in Figs. 3 and 4. This behavior is reminiscent of the excitations in \(^4\text{He} \) [24]. In the wavenumber interval that the mode exists, the rate for Landau-damping does not increase monotonically with \( k \). This behavior is somewhat counter-intuitive as an increase in \( k \) generally increases the phase space volume of the available final states. The termination point \( k_{\text{end}} \xi \) expressed in terms of \( \gamma \) is

\[
k_{\text{end}}\xi = (1 - \gamma^2)^{1/2} \left\{ \left( \frac{v_F}{c} \right)^2 - 1 + F + \frac{F}{4} \left[ \pi \gamma - \frac{3\gamma^2}{4} - 2 \ln \left( \frac{\gamma}{2} \right) \right] \right\}^{1/2} ,
\]

(25)

which can be tested experimentally.

The magnitude of \( \text{Im}[\omega] \) is significantly smaller than \( \text{Re}[\omega] \) even for a system of relatively strong interactions such as \( F = 0.5 \). Since we expect the collective excitations to be good quasi-particles, if the lifetime of the mode is short enough, the damping of the mode can be measured in atom trap experiments. The time scale of the mode damping is set by

\[
\tau_0 = \frac{\xi}{2v_F} = \frac{m_F}{m_B} \frac{1}{ck_F} .
\]

(26)

\( \tau_0 \) is the time that takes a particle with velocity \( v_F \) to travel half the distance of the BEC healing length. Alternatively, we write \( k_F \) in terms of the average inter-fermion distance as \( r_F = \left( \rho_F^0 \right)^{-1/3} \), \( k_F = (6\pi^2)^{1/3} \left( \rho_F^0 \right)^{1/3} \approx 3.9/r_F \), and
find that $\tau_0$ is proportional to the time a particle requires to travel at the velocity of the BEC-sound in order to cover the average inter-fermion distance $\tau_0 \approx (m_F/3.9m_B) \times (v_F/c)$. Under typical experimental conditions $\tau_0$ can be a few milliseconds, which is sufficiently long to measure the damping and sufficiently short to keep a propagating wavepacket from leaving the trap region where the system is approximately homogeneous.

Since the damped mode can be viewed as an elementary BEC-excitation that has been modified by the fermion-mediated boson-boson interactions, the decay can also occur by Belyaev damping where pairs of boson quasi-particles are created. While our treatment does not describe this damping, the decay rate due to Belyaev damping tends to be slow in the long wavelength limit $k<\xi^{-1}$, \( \Gamma_{\text{Bel}} \approx (6\sqrt{\pi}/5)\sqrt{\rho_B a_{BB}(k\xi)^4ck} \), thus can be neglected in the range of $k$ of our interest.

Finally, it is instructive to express $s$ as a function of the velocity ratio, $v_F/c$, in the limit $k \to 0$ (see Fig. 6).

Similar to the behavior of the dispersion relations, there is a transition of the long wavelength phase velocity $v$ from its undamped mode $v \approx v_F$ to the BEC-sound velocity $v \approx c$. The smoothness of this transition depends on the value of $F$: the smaller the value of $F$ the more abrupt the transition occurs with varying velocity ratio. If $v_F < c$, the static description of the BEC-mediated fermion-fermion interaction is valid and the fermions experience an effective mutual attraction that does not support a long lived zero-sound excitation in a two-component pure Fermi-liquid system. We can determine the transition point (or region), $(v_F/c)_{\text{trans}}$. For $k \to 0$, the BEC and zero sound modes

---

**FIG. 4:** $v_F/c = 2$, $F = 0.01$. The dashed line shows the numerically evaluated $\text{Im}[\omega \xi / v_F]$, while the dotted line plots $\text{Im}[\omega \xi / v_F]$ evaluated from the Fermi-Golden Rule.

**FIG. 5:** $v_F/c = 2$, $F = 0.5$. The solid line shows the numerically evaluated values of $\text{Re}[\omega \xi / v_F]$, the dashed line plots the numerically evaluated curve of $\text{Im}[\omega \xi / v_F]$, whereas the dotted line gives the values of $\text{Im}[\omega \xi / v_F]$ determined from the Fermi-Golden Rule calculation.
FIG. 6: Plot showing the solution of Eq. (21) for $s$ at $k\xi = 0$ as a function of $c/v_F$. The solid line shows the undamped mode phase velocity with $F = 0.01$, the dash-dotted line plots its value for the undamped mode with $F = 0.5$, whereas the dashed line shows the values for the damped mode with $F = 0.5$ and the dotted line for the damped mode with $F = 0.01$.

mix strongly near $v_F/c \simeq 1$, similar to the region where the linear dispersion merges into a Bogoliubov dispersion as we have seen in the dispersion relations. Quantifying the deviation of $s$ from unity by setting $s = 1 + \epsilon$, where $0 < \epsilon \ll 1$ in the Eq. (21) we find

$$\frac{v_F}{c} \bigg|_{\text{trans}} \simeq \left[ 1 + F \left( \frac{1}{2} \ln \frac{2}{\epsilon} - 1 \right) \right]^{-1/2}. \quad (27)$$

The avoided crossing, termination point and strong mode mixing occur in the velocity ratio interval between $(v_F/c)_{\text{trans}}$ and 1.

### B. Instability and phase transition

Equation (21) provides insight into the stability, or lack thereof, of a homogeneous fermion-boson mixture. If $F > 1$, $s$ the eigenfrequency of the damped modes of longest wavelengths take on purely imaginary values. The purely imaginary value, and not just the acquisition of an imaginary part as in [26] (a claim that was also corrected in [27]), signals the instability of the homogeneous mixture. Then Eq. (21) is reduced to

$$\frac{\gamma^2 (v_F/c)^2 + \left[ 1 + (k\xi)^2 \right]}{F} = 1 + \gamma \arctan \left( \frac{1}{\gamma} \right). \quad (28)$$

Since the unstable mode is the lowest frequency mode, i.e., the damped or BEC-sound like mode in the stable mixture, we can say that it is the fermion-mediated boson-boson interaction that triggers the instability of the homogeneous BEC in the mixture. Since $F$ depends on $k_F$, the instability condition $F > 1$ can be expressed as a condition on the fermion density: $\rho_F^0 > \rho_{F,\text{crit}}$, where

$$\rho_{F,\text{crit}} = \frac{4\pi}{3a_{FB}^3} \left[ \frac{a_{BB}/a_{FB}}{(1 + m_B/m_F)(1 + m_F/m_B)} \right]^3. \quad (29)$$

If the equilibrium fermion density exceeds $\rho_{F,\text{crit}}$, the Fermi-liquid becomes immiscible to the BEC and spontaneously breaks the translational symmetry by undergoing phase separation into either (a) pure fermion and pure BEC or (b) pure fermion and mixed fermion-BEC phases [11].

Equation (28) also provides insight into the dynamics of the onset (the early stages) of the phase separation instability. Consider an experiment that brings $\rho_F$ in a fermion-BEC mixture to above $\rho_{F,\text{crit}}$ by an abrupt change of $a_{BB}$ (29). In response, the amplitude of the longest wavelength modes of the damped zero-sound excitations grow exponentially at a rate $\gamma(k)k\xi/\tau_0$ nucleating spatial regions of pure fermion matter when the dynamics becomes
non-linear. Those modes that grow fastest (with wavenumber \( k_d \)) dominate the dynamics and we expect them to determine the size of the single phase matter clusters and the rate of the phase separation \(^{28}\).

When \( \gamma \) is sufficiently small, we can expand the right-hand side of Eq. \(^{28}\) and determine \( k_d \) analytically. The modes with wavenumber in the range, \( k\xi \in [0, \sqrt{F-1}] \) are unstable and grow exponentially at the rate \( R = \gamma k\xi \tau_0^{-1} \) with

\[
R\tau_0 = \left\{ \frac{\pi F}{4} + \sqrt{\left(\frac{\pi F}{4}\right)^2 - \left(\frac{v_F}{c}\right)^2 (1 - F + q^2)} \right\} k\xi. 
\]  

The maximum growth rate occurs at \( k_d \), where

\[
(k_d\xi)^2 = \frac{F - 1}{2} + \frac{\pi F}{128} \left(\frac{v_F}{c}\right)^{-2} \left[ 3\pi F - \sqrt{(3\pi F)^2 + 128 \left(\frac{v_F}{c}\right)^2 (F - 1)} \right].
\]

with the average cluster size \( \sim 2\pi/k_d \).

V. EXPERIMENTAL ASPECTS

In this section, we mention relevant atom trap techniques and estimate magnitudes of quantities to gauge the feasibility of experimentally observing the collective mode physics. The important parameter for the boson-boson interactions in a dilute BEC is the dimensionless gas parameter, \( D = \sqrt{\rho_B a_{BB}} \). We find it useful to express the density in units of a reference density \( \rho_R \) that corresponds to the lower range obtained in atom traps, \( \rho_R = 10^{12} \text{cm}^{-3} \). Then, the gas parameter is

\[
D = \sqrt{\frac{\rho_B a_{BB}}{\rho_R 1\text{nm}}} 3.2 \times 10^{-5}.
\]  

We introduce the atomic mass number of the bosonic and fermionic isotopes, \( A_B \) and \( A_F \) (\( A_F = 6 \) for \(^6\text{Li}\), for instance) and express \( \lambda_{BB} \rho_B^0 \) and Fermi-energies \( \epsilon_F \) as

\[
\lambda_{BB} \rho_B^0 = \frac{6nK \sqrt{a_{BB}}}{A_B \rho_R 1\text{nm}},
\]

\[
\epsilon_F = \frac{3.65\mu K}{A_F} \left( \frac{\rho_F^0}{\rho_R} \right)^{2/3}.
\]

The time scale for the fermion dynamics is equal to

\[
\tau_F = \frac{\hbar}{\epsilon_F} \approx 2.09\mu\text{sec} \times A_F \left( \frac{\rho_F^0}{\rho_R} \right)^{-2/3},
\]

from which it follows that the Fermi velocity can be of the order of several cm/sec,

\[
v_F = \frac{2}{\tau_F k_F} \approx 24.5 \text{cm/sec} \times \left( \frac{\rho_F^0}{\rho_R} \right)^{1/3}.
\]

Most importantly, the velocity ratio \( v_F/c \) which quantifies the ratio of the relevant fermion to boson time scales and which determines the validity of the static approximation of the BEC-mediated interaction, can be written as

\[
\frac{v_F}{c} = \frac{A_B}{A_F} \frac{0.55}{D^{1/3}} \left( \frac{\rho_B^0}{\rho_R^0} \right)^{1/3}.
\]

Clearly experiments can access both the \( v_F/c < 1 \) and \( v_F/c > 1 \) regimes. For example, \(^6\text{Li}\) with \( \rho_F^0 = 10^{13} \text{cm}^{-3} \) immersed in \(^{23}\text{Na}\) BEC with \( \rho_B^0 = 3 \times 10^{14} \text{cm}^{-3} \) and \( a_{BB} \sim 1.5\text{nm} \) gives \( v_F/c \approx 0.46 \), whereas changing the density of the BEC to \( \rho_B^0 = 3 \times 10^{12} \text{cm}^{-3} \) increase the velocity ratio to \( v_F/c \approx 4.6 \).

To observe the predicted collective modes an experimentalist could measure the group velocities in a fermion-boson mixture that is contained in a cigar shaped trap. We assume that the transverse confinement is not so tight as to
bring the system into the regime where it becomes effectively one-dimensional. A sudden perturbation near the trap middle created by a focused laser beam can excite both collective modes. In response, two wavepackets of different densities propagate outward with different group velocities. The density variations (fermion and boson) can be imaged directly, as in the first observation of the BEC-sound mode \[30\]. The slower wavepacket would correspond to the Landau damped mode with

\[
\frac{1}{\tau_0} \approx 110 \text{ A}_F \text{msec}^{-1} \times \left( \frac{\rho_0^F}{\rho_R^F} \right)^{1/3} \sqrt{\frac{a_{BB}^{1nm}}{\lambda_{BB}}} \left( \frac{\rho_0^B}{\rho_R^B} \right)^{1/3},
\]

which can be slow enough to observe significant propagation while sufficiently fast to measure damping before the wavepacket reaches equilibrium densities that are significantly different from trap middle.

In more sophisticated experiments, the initial perturbation can be controlled, and an analysis of the time evolution of the shape of the propagating wavepackets can be used to infer the dispersion relations and damping rates. In the past, the greatest control was achieved by accessing a two-photon resonance to cause low intensity Bragg scattering. In such an experiment the momentum of the excitation can be varied by changing the angle of the crossed laser beams. The laser beams need to be focused near the middle of the trap to probe the nearly uniform region of the fermion-boson mixture. Similar techniques have been used in 'Bogoliubov spectroscopy experiments' \[31\].

Finally, we remark that by tuning the boson-boson scattering length close to its zero-point \(a_{BB} \approx 0\), thereby lowers the critical fermion density can always trigger the phase separation phenomenon. By triggering the phase separation in a cigar shaped trap, the surface tension can prevent the single phase domains to move past each other, the experimenter can obtain a string of single phase ‘droplets’. The size of these domains reflect the dominant mode wave vector. Such experimental study was demonstrated in the phase separation of multi-component BEC’s \[32\].

\[\text{VI. CONCLUSIONS}\]

We have analyzed the collective modes of a single component fermion–BEC mixture in ultra-low temperature where the fermions behave as normal Fermi-liquid.

If \(v_F > c\) and \(0 < F < 1\), the homogeneous mixture is mechanically stable and two collective modes with zero-sound character exist. One is an undamped mode that resembles the zero-sound mode of a pure Fermi-liquid system. The dispersion of this mode is linear with phase velocity \(\omega/k \approx v_F\) in the long wavelength limit and then merges into the Bogoliubov dispersion of an elementary BEC excitation near \(k_{\text{trans}} = \xi^{-1}\sqrt{(v_F/c)^2 - 1}\). The other mode resembles the sound mode of a pure BEC system in the long wavelength limit. The mode undergoes Landau damping and decays at the rate \(\Gamma(k) = \gamma k \xi / \tau_0\). The dispersion of this mode terminates at a well-defined wavenumber \(k_{\text{end}}\).

In the long wavelength limit, the rate for Landau damping can be approximated using the Fermi-golden rule and is \(\Gamma_{\text{FG}}(k) = \left( \pi F/2 \right) / (c/v_F) \left( k \right)\).

On the other hand, if \(v_F < c\) and \(0 < F < 1\), the BEC response is sufficiently fast to follow the fermion oscillations adiabatically, thus the dispersion is the Bogoliubov dispersion for all \(k\).

When \(F > 1\), we have \(\rho_0^F > \rho_{F,\text{crit}}\) the homogenous mixture is unstable and undergoes phase separation by forming clusters of pure fermions. Our analysis shows that the growth rate is \(R = \gamma k \xi / \tau_0\) and the average cluster size is \(\sim 2\pi/k_d\).

Finally, our discussion of current atom trap technology and relevant observables indicate that the collective modes can be observed and the dispersions and damping rates may be measured. In addition, the phase separation can be triggered, enabling the phase separation dynamics characterization as soon as the fermions are cooled into the Fermi-liquid regime in the BEC–fermion mixture.

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