BRANCHING PROCESSES IN RANDOM ENVIRONMENT WITH SIBLING DEPENDENCE

V. A. Vatutin\textsuperscript{1} and E. E. Dyakonova\textsuperscript{1,2}

We consider a population of particles with unit lifetime. Dying, each particle produces offspring whose size depends on the random environment specifying the reproduction law of all particles of the given generation and on the number of relatives of the particle. We study the asymptotic behavior of the survival probability of the population up to a distant moment \( n \) under some restrictions on the properties of the environment and family ties and prove a limit theorem for the number of particles in such processes given the respective populations survive for a long time.

1. Introduction and main results

We consider a population of particles with unit lifetime. Dying, each particle produces offspring whose size depends on a random environment specifying the reproduction law of all particles of the given generation and on the number of relatives of the particle. It will be clear from the description to follow that the model we consider includes a class of Galton–Watson branching processes in random environment (BPRE’s).

First we give an informal description of the model. We fix a positive integer \( N \) and, for each \( i \in \{1, \ldots, N\} \), specify on the set of \( i \)-dimensional vectors \((k_1, \ldots, k_i)\) with non-negative integer components \( 0 \leq k_j \leq N \) a probability measure \( P(i; \cdot) \) with

\[
\sum_{k_1=0, \ldots, k_i=0}^N P(i; (k_1, k_2, \ldots, k_i)) = 1,
\]

such that

\[
P(i; (k_1, k_2, \ldots, k_i)) = P\left(i; (k_{\sigma(1)}, k_{\sigma(2)}, \ldots, k_{\sigma(i)})\right)
\]

for any transposition \((\sigma(1), \sigma(2), \ldots, \sigma(i))\) of elements of the set \( \{1, \ldots, i\} \).

It follows from this assumption that all the marginal distributions of the probability measure \( P(i; \cdot) \) coincide.

We say that a tuple

\[
P_N = \{P(i; \cdot), i \in \{1, \ldots, N\}\}
\]

of probability measures on \( \{1, \ldots, N\} \) constitutes an environment of order \( N \). The set of all environments \( \mathcal{P}_N = \{P_N\} \) of order \( N \) equipped with the metric generated by the distance of total variation of the respective components of \( P_N \) is a Polish space. Therefore we can consider probability measures on this space. Let \( \mathcal{P} \) be such a probability measure. A sequence

\[
P_N^{(n)} = \left(P^{(n)}(i; \cdot), i \in \{1, \ldots, N\}\right), n = 0, 1, 2, \ldots
\]

of elements of \( \mathcal{P}_N \), which are selected at random and independently according to the measure \( \mathcal{P} \), is said to form a random environment.

A detailed description of the restriction we impose on the properties of the random environment will be given later.

\textsuperscript{1}Steklov Mathematical Institute, Moscow, Russia, e-mail: vatutin@mi-ras.ru
\textsuperscript{2}Steklov Mathematical Institute, Moscow, Russia, e-mail: elena@mi-ras.ru

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Suppose that a random environment \( \{P^{(n)}_N, n \geq 0\} \) is fixed. With the environment on hand, we can describe the development of the BPRE \( \{\zeta(n), n \geq 0\} \) with sibling dependence as follows. The process is initiated at time \( n = 0 \) by \( \zeta(0) \geq 1 \) particles of generation zero. We call these particles a sibling group. Particles of the initial generation have a unit lifetime and dying produce children. All the direct descendants of a particle of the initial or subsequent generations will be called siblings (thus, siblings have one and the same parent-particle). Some particles may have no direct descendants. In this case the respective set of siblings is empty. The total number of particles in a sibling group is called the type of the sibling group or simply the type of siblings. If a sibling group is of type \( i \), we agree to consider that each particle in this group has type \( i \) as well.

If \( \zeta(0) = i \), then the initial particles die at moment \( n = 1 \) and produce in total \( K(i) = k_1 + \cdots + k_i \) particles of the first generation, where \( k_j \) is the number of descendants of the \( j \)th particle from the initial sibling group. The distribution of the vector \( (k_1, k_2, \ldots, k_i) \) is specified by the measure \( P^{(0)}(i; \cdot) \).

Thus, \( i \) new sibling groups are produced, each of which contains all descendants (and only them) of some particle of zero generation. If \( k_j \geq 1 \) for some \( j \), then all the particles from this sibling group die at moment \( n = 2 \) and, independently of the behavior of particles from the other sibling groups and the prehistory of the process, give birth to \( K(k_j) = i_1 + \cdots + i_{k_j} \) particles of the second generation, where \( i_j \) is the number of descendants of the \( r \)th particle from the \( j \)th sibling group of the first generation. Thus, we have \( k_j \) new sibling groups consisting of the second generation particles. The distribution of the vector \( (i_1, i_2, \ldots, i_{k_j}) \) is specified by the measure \( P^{(1)}(k_j; \cdot) \), and so on.

Thus, given the environment, the sibling groups existing at moment \( n \geq 1 \) evolve independently of each other. However, the interaction of siblings at this moment is described by (random) measures \( P^{(n)}(i; \cdot), i = 1, \ldots, N \), specifying the joint distribution of the number of direct descendants of a type \( i \) sibling group.

Let \( \zeta(n) \) denote the number of particles in generation \( n \) in such BPRE with sibling dependence. The aim of this note is to investigate the asymptotic behavior of the survival probability of the process as \( n \to \infty \) under different conditions on the properties of random environment.

We would like to note that there are several papers studying the behavior of the Galton–Watson branching processes with sibling dependencies evolving in a constant environment. We mention here only papers \([2]\) and \([13]\), which are the most significant for us. On the other hand, there exists a lot of papers devoted to investigations of BPRE (see, for example, \([1,5–9,11,12,14–18]\)). However, as far as we know, BPRE’s with sibling dependences have not been yet analyzed.

According to the condition \( (1) \) the marginal distributions of the measure \( P^{(n)}(i; \cdot) \) coincide for any \( i = 1, \ldots, N \). Therefore, for any \( j = 0, 1, \ldots, N \) we can correctly define the (random) variable

\[
P^{(n)}_{ij} := \sum_{k_2=0}^{N} \cdots \sum_{k_i=0}^{N} P^{(n)}(i; (j, k_2, \ldots, k_i)), \tag{3}
\]

which is equal to the probability of the event that a particle belonging at time \( n \) to some sibling group of type \( i \) begets just \( j \) children, i.e., generates a type \( j \) sibling group.

We associate with the random environment \( (2) \) two sequences of (random) vector-valued multivariate generating functions

\[
\Phi^{(n)}(s) = \left( \Phi^{(n)}_1(s), \ldots, \Phi^{(n)}_N(s) \right), n \geq 0, \tag{4}
\]

and

\[
F^{(n)}(s) = \left( F^{(n)}_1(s), \ldots, F^{(n)}_N(s) \right), n \geq 0,
\]
where \( s = (s_1, \ldots, s_N) \in [0, 1]^N \) and, for \( i = 1, \ldots, N, \)

\[
\Phi_i^{(n)}(s) = P^{(n)}(i; (0, \ldots, 0)) + \sum_{\substack{k_1=0, \ldots, k_i=0, \\ k_1+\ldots+k_i>0}} P^{(n)}(i; (k_1, k_2, \ldots, k_i)) s_{k_1} s_{k_2} \cdots s_{k_i},
\]

(we assume that \( s_0 \equiv 1 \)) and

\[
F_i^{(n)}(s) = p_{i0}^{(n)} + \sum_{j=1}^{N} p_{ij}^{(n)} s_j, \quad i = 1, \ldots, N.
\]

Thus, the component \( \Phi_i^{(n)}(s) \) of the vector-valued multivariate generating function \( \Phi^{(n)}(s) \) describes in detail the joint law of generating sibling groups at time \( n \) by all representatives of a type \( i \) sibling group while the component \( F_i^{(n)}(s) \) of the vector-valued multivariate generating function \( F^{(n)}(s) \) describes the distribution law of the number of children at time \( n \) by a representative of a type \( i \) sibling group.

Recall that the size of any sibling group (i.e., the number of children of any particle) in our settings does not exceed \( N \). Of course, this assumption is an essential restriction. However, this assumption is natural in the framework of applications in theoretical biology (see, for example, monograph [11]). Note that it is more difficult (if at all possible) to evaluate in practice parameters associated with the generating function \( \Phi_i^{(n)}(s) \) than to evaluate parameters associated with the generating function \( F_i^{(n)}(s) \). For this reason we formulate the statements of theorems describing the asymptotic behavior of the survival probability of BPRE’s with sibling dependence in terms of the vector-valued (random) generating function

\[
F(s) = (F_1(s), \ldots, F_N(s))
\]

with components

\[
F_i(s) = p_{i0} + \sum_{j=1}^{N} p_{ij} s_j, \quad i = 1, \ldots, N,
\]

having the same distribution as the functions specified by (6).

We need some notation for \( N \)-dimensional vectors and \( N \times N \) matrices. Let \( e_i, \ i = 1, 2, \ldots, N, \) be the \( N \)-dimensional vector whose \( i \)th component is equal to 1 and others are zeroes. For vectors \( x = (x_1, \ldots, x_N) \) and \( y = (y_1, \ldots, y_N) \) set

\[
(x, y) := \sum_{i=1}^{N} x_i y_i, \quad |x| := \sum_{i=1}^{N} |x_i| \quad \text{and} \quad x^y := \prod_{i=1}^{N} (x_i)^{y_i}.
\]

For a \( N \times N \) matrix \( m = (m(i,j))_{i,j=1}^{N} \) introduce its norm by the equality

\[
|m| = \sum_{i=1}^{N} \sum_{j=1}^{N} |m(i,j)|.
\]

Set \( \delta_{kl} \) for the Kroneker symbol and let \( 1 = (1, \ldots, 1) \in [0, 1]^N \).

Basic restrictions we impose on the properties of the BPRE with sibling dependence are related to the mean matrix

\[
M = M(F) = (M(i,j))_{i,j=1}^{N} := (jp_{ij})_{i,j=1}^{N} = \left( \frac{\partial F_i(1)}{\partial s_j} \right)_{i,j=1}^{N}
\]
and the Hessian matrices
\[ B_i = B_i(\mathbf{F}) = (B_i(k,l))_{k,l=1}^N := (k(k-1)p_{ik}\delta_{kl})_{k,l=1}^N = \left( \frac{\partial^2 F_i(1)}{\partial s_k \partial s_l} \right)_{k,l=1}^N \]
constructed by the vector-valued generating function \( \mathbf{F}(s) \). The set of matrices generate two important random variables
\[ B := \sum_{i=1}^P |B_i|, \quad T := \frac{B}{|M|^2}. \tag{8} \]

We define the cone
\[ C = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_i \geq 0 \text{ for each } i \in \{1, \ldots, N\} \}, \]
the sphere
\[ S^{N-1} = \{ x : x \in \mathbb{R}^N, |x| = 1 \}, \]
and the space \( \mathcal{X} = C \cap S^{N-1} \).

To go further we need to use the linear semigroup \( S^{++} \) of \( N \times N \) matrices all of whose elements are non-negative.

Assume that the distribution of the random matrix \( M \) meets the following restrictions:

**Condition H1.** There exists \( \theta > 0 \) such that
\[ \mathbb{E} \left[ |M|^{\theta} \right] = \int_S |M|^{\theta} P(dM) < \infty. \]

**Condition H2** (strong irreducibility). The support of \( P \) in \( S^{++} \) acts strongly irreducibly on \( \mathbb{R}^N \), i.e., no proper finite union of subspaces of \( \mathbb{R}^N \) is invariant with respect to all elements of the multiplicative semi-group generated by the support of \( P \).

**Condition H3.** Elements of the random matrix \( M = (M(i,j))_{i,j=1}^N \) are positive and there exists a real positive number \( \gamma > 1 \) such that \( P \)-a.s.
\[ \frac{1}{\gamma} \leq M(i,j) M(k,l) \leq \gamma \]
for any \( i,j,k,l \in \{1, \ldots, N\} \).

For \( n = 0, 1, 2, \ldots \) introduce random matrices
\[ M^{(n)} = M^{(n)}(\mathbf{F}^{(n)}) = \left( M^{(n)}(i,j) \right) := \left( j p^{(n)}_{ij} \right) = \left( \frac{\partial F_i^{(n)}(1)}{\partial s_j} \right)_{i,j=1}^N, \tag{9} \]
\[ B^{(n)}_i = B^{(n)}_i(\mathbf{F}^{(n)}) = \left( B^{(n)}_i(k,l) \right)_{k,l=1}^N := \left( k(k-1)p^{(n)}_{ik}\delta_{kl} \right)_{k,l=1}^N = \left( \frac{\partial^2 F_i^{(n)}(1)}{\partial s_k \partial s_l} \right)_{k,l=1}^N, \]
and denote by
\[ R^{(n)} = \left( R^{(n)}(i,j) \right)_{i,j=1}^N := M^{(0)} M^{(1)} \cdots M^{(n)} \tag{10} \]
the right product of the random mean matrices \( M^{(n)}, n \geq 0 \).

It is known (see [10]) that given
\[ \mathbb{E} \left[ \max(0, \log |M|) \right] < \infty, \tag{11} \]
the sequence
\[ \frac{1}{n} \log |R^{(n)}|, n = 1, 2, \ldots \]
converges P-a.s. as \( n \to \infty \) to a limit
\[ \Lambda := \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ \log |R^{(n)}| \right], \]
called the upper Lyapunov exponent.

We need to impose two more conditions on \( P \).

**Condition H4.** The upper Lyapunov exponent \( \Lambda \) of the distribution generated by \( P \) on \( S^{++} \) is equal to 0.

**Condition H5.** There exists \( \delta > 0 \) such that
\[ P \left( M \in S^{++} : \log |xM| \geq \delta \text{ for any } x \in X \right) > 0. \]

We are now ready to formulate the first main result of the paper.

**Theorem 1.** Assume Conditions H1–H5. If
\[ \sup_{x \in X} \mathbb{E} \left[ \frac{1}{|xM|} \right] < \infty \] (12)
and, for \( T \) given in (8) and an \( \varepsilon > 0 \)
\[ \mathbb{E} \left[ |\log T|^{1+\varepsilon} \right] < \infty, \] (13)
then, for any \( i = 1, 2, \ldots, N \) there exists a number \( \beta_i \in (0, \infty) \) such that
\[ \lim_{n \to \infty} \sqrt{n} P \left( \zeta(n) > 0 \mid \zeta(0) = i \right) = \beta_i. \] (14)

In the sequel we call a BPRE with sibling dependence critical if \( \Lambda = 0 \).

We now describe conditions under which the asymptotics of the survival probability of the process \( \zeta(n) \) has a form different from that stated in Theorem 1.

**Condition H6.** There exists an \( \varepsilon > 0 \) such that
\[ \mathbb{E} \left[ |\log T|^{1+\varepsilon} |M| \right] < \infty. \]

Let
\[ \Theta := \left\{ \theta > 0 : \mathbb{E} \left[ |M|^\theta \right] < \infty \right\}. \]
It is known (see, for example, [10]) that for any \( \theta \in \Theta \) the limit
\[ \lambda(\theta) := \lim_{n \to \infty} \left( \mathbb{E} \left[ |R^{(n)}|^{\theta} \right] \right)^{1/n} < \infty \]
is well defined. Set
\[ \Lambda(\theta) := \log \lambda(\theta), \quad \theta \in \Theta. \]

**Theorem 2.** Assume that Conditions H1–H3 and H6 are valid, the point \( \theta = 1 \) belongs to the interior of the set \( \Theta \), and \( \Lambda'(1) < 0 \). Then, for any \( i = 1, \ldots, N \)

(a) there exists a constant \( C_i > 0 \) such that
\[ P \left( \zeta(n) > 0 \mid \zeta(0) = i \right) \sim C_i \lambda^n(1), \quad n \to \infty; \] (15)

(b) \[ \lim_{n \to \infty} \mathbb{E} \left[ s^{\zeta(n)} \mid \zeta(n) > 0; \zeta(0) = i \right] = \Psi_i(s), \quad s \in [0, 1), \]
where \( \Psi_i(s) \) is the probability generating function of a proper distribution on \( \mathbb{Z}_+ \).
1.1. Proofs of Theorems 1, 2

The basic idea of the proof of Theorems 1 and 2 is to compare the BPRE \( \{ \zeta(n), n \geq 0 \} \) with sibling dependence with another branching process, a macro process. This macro process consists of sibling groups, to be called macro particles.

The type of a macro particle is the number of particles from the initial BPRE with sibling dependence which belong to the sibling group constituting the macro particle.

As we have mentioned, it will be convenient for us to assume that all the particles constituting the macro particle have the same type as the macro particle. Thus, we assign each sibling group to one of \( N \) possible types of macro particles. This allows us to associate the BPRE \( \{ \zeta(n), n \geq 0 \} \) with sibling dependence with the macro process

\[
\{ Z(n) = (Z_1(n), Z_2(n), \ldots, Z_N(n)), \quad n \geq 0 \},
\]

where \( Z_k(n), k = 1, 2, \ldots, N, \) is the number of macro particles of type \( k \) in the \( n \)th generation of the macro process, i.e., \( Z_k(n) \) is the number of such sibling groups in the \( n \)th generation of \( \zeta(n) \), each of which is a sibling group of size \( k \), generated by a parent-particle belonging to the \( (n-1) \)th generation.

Recall that an \( i \) type macro particle existing at time \( n \) in the macro process generates offspring according to the probability generating function \( \Phi_i^n(s) \) specified in (5), while the marginal distributions for the number of direct descendants of a particle belonging to a size \( i \) sibling group are defined by (3).

Clearly, the macro process \( \{ Z(n), n \geq 0 \} \) is an \( N \)-type Galton–Watson process in random environment.

The main difference between the processes \( \{ \zeta(n), n \geq 0 \} \) and \( \{ Z(n), n \geq 0 \} \) is easy to explain: given the environment, the individuals of the initial BPRE \( \zeta(n) \) with sibling dependence do not reproduce independently, while macro particles do, since the only dependencies are within the sibling groups.

It is not difficult to see that

\[
\zeta(n) = \sum_{k=1}^{N} k Z_k(n).
\]  \( \quad (16) \)

We use the symbols \( E_\Phi, D_\Phi, \) and \( P_\Phi \) for the expectations, variances, and probabilities given the vector-valued probability generating function \( \Phi \). Denote by

\[
M_{\text{macro}} = (M_{\text{macro}}(i,j))_{i,j=1}^{N} := (E_\Phi[Z_j(1);Z(0) = e_i])_{i,j=1}^{N}
\]

the (random) mean matrix and by

\[
B_{i,\text{macro}} = (B_{i,\text{macro}}(j,k))_{j,k=1}^{N} := (E_\Phi[Z_j(1)(Z_k(1) - \delta_{jk});Z(0) = e_i])_{j,k=1}^{N}
\]

the (random) Hessian matrices for the macro process. We put

\[
p_{2(jk)} := P(2; (j, k))
\]

and set, for \( i \geq 3 \)

\[
p_{i(jk)} := \sum_{(k_3, \ldots, k_i)} P(i; (j, k, k_3, \ldots, k_i)).
\]

Lemma 1. The elements of the matrices \( M_{\text{macro}} \) and \( B_{i,\text{macro}} \) are calculated by the formulas

\[
E_\Phi[Z_j(1);Z(0) = e_i] = M_{\text{macro}}(i,j) = i p_{ij} = \frac{i}{j} M (i, j).
\]

\[
B_{i,\text{macro}}(j,k) = E_\Phi[Z_j(1)(Z_k(1) - \delta_{jk});Z(0) = e_i] = i(i - 1) p_{i(jk)}.
\]
Proof. Let \( \xi_{i,r} \) be the number of direct descendants of the \( r \)th particle entering an \( i \) type macro particle of generation zero and let \( I \{ C \} \) be the indicator of the event \( C \).

Then

\[
E_\Phi [Z_j (1); Z(0) = e_i] = E_\Phi \left[ \sum_{r=1}^{i} I \{ \xi_{i,r} = j \} \right] = \sum_{r=1}^{i} P(i; (j, k_2, \ldots, k_i)) = ip_{ij}.
\]

Recalling condition (1) we have for \( r \neq t \)

\[
E_\Phi [I \{ \xi_{i,r} = j \} I \{ \xi_{i,t} = k \}] = \sum_{k_3, \ldots, k_i} P(i; (j, k_3, \ldots, k_i)) = p_{i(jk)},
\]

and for \( r = t \)

\[
E [I \{ \xi_{i,r} = j \} I \{ \xi_{i,t} = k \}] = \delta_{jk}p_{ij}.
\]

Therefore, the random Hessian matrices \( B_{i,\text{macro}} = (B_{i,\text{macro}} (j, k))_{j,k=1}^{N} \) have elements

\[
B_{i,\text{macro}} (j, k) = \frac{\partial^2 \Phi_i (s)}{\partial s_j \partial s_k} = E \left[ \left( \sum_{r=1}^{i} I \{ \xi_{i,r} = j \} \right) \left( \sum_{t=1}^{i} I \{ \xi_{i,t} = k \} - \delta_{jk} \right) \right] = \sum_{r \neq t} E [I \{ \xi_{i,r} = j \} I \{ \xi_{i,t} = k \}] + \sum_{r=1}^{i} E [I \{ \xi_{i,r} = j; \xi_{i,r} = k \}] - \delta_{jk} \sum_{r=1}^{i} E [I \{ \xi_{i,r} = j \}] = i(i - 1)p_{i(jk)} + i\delta_{jk}p_{ij} - i\delta_{jk}p_{ij} = i(i - 1)p_{i(jk)}.
\]

The lemma is proved.

Let \( \rho \) be the Perron root of the matrix \( M \) and let \( u = (u_1, \ldots, u_N) \) be the right eigenvector of \( M \) corresponding to \( \rho \) and satisfying the condition \( |u| = u_1 + \ldots + u_N = 1 \).

**Lemma 2.** The value \( \rho \) is the maximal in modulo eigenvalue of the matrix \( M_{\text{macro}} \), and the right eigenvector \( U = (U_1, \ldots, U_N) \) corresponding to \( \rho \) has components

\[
U_j = \frac{j u_j}{\sum_{k=1}^{N} k u_k}, \quad j = 1, \ldots, N.
\]

Proof. If

\[
\sum_{j=1}^{N} M(i, j) u_j = \sum_{j=1}^{N} j p_{ij} u_j = \rho u_i,
\]

then

\[
\sum_{j=1}^{N} M_{\text{macro}}(i, j) U_j = \sum_{j=1}^{N} \frac{j u_j}{\sum_{k=1}^{N} k u_k} \sum_{k=1}^{N} k u_k = \frac{i}{\sum_{k=1}^{N} k u_k} \sum_{k=1}^{N} M(i, j) u_j = \frac{i}{\sum_{k=1}^{N} k u_k} \rho u_i = \rho U_i.
\]

Thus, \( U \) is the right eigenvector corresponding to \( \rho \).
Let us show that $\rho$ is the maximal in modulo eigenvalue of the matrix $M_{\text{macro}}$. Indeed, assume that there exists a $\rho^* > \rho$ and the respective eigenvector $U^* = (U_1^*, \ldots, U_N^*)$ with strictly positive components such that

$$\sum_{j=1}^N M_{\text{macro}}(i,j) U_j^* = \rho^* U_i^*$$

for all $i = 1, 2, \ldots, N$. Therefore,

$$\sum_{j=1}^N \frac{1}{j} M(i,j) U_j^* = \rho^* \frac{U_i^*}{i}.$$ 

Hence, setting $U_{i^{**}} := i^{-1} U_i^*$, $i = 1, 2, \ldots, N$, we see that

$$\sum_{j=1}^N M(i,j) U_{i^{**}} = \rho^* U_{i^{**}}.$$ 

Thus, $\rho^* > \rho$ is an eigenvalue of $M$. This contradicts the fact that $\rho$ is the maximal in modulo eigenvalue of the matrix $M$.

The lemma is proved.

**Proof of Theorem 1.** We consider, along with the macro process, an auxiliary $N$-type branching process in a random environment, the so-called individual process. The reproduction of $i$-type particles at moment $n$ in the new process is specified by the probability generating function

$$F^{(n)}_i(s_1, \ldots, s_N) = p^{(n)}_{i0} + \sum_{j=1}^N p^{(n)}_{ij} s_j,$$

which is the same as in (6).

It follows from Lemma 1 that the mean matrix for the reproduction law of the particles of the auxiliary process at moment $n$ has the form

$$M^{(n)} = \left( M^{(n)}(i,j) \right)_{i,j=1}^N = \left( j p^{(n)}_{ij} \right).$$

Thus,

$$M_{\text{macro}}^{(n)}(i,j) = i p^{(n)}_{ij} = \frac{i}{j} j p^{(n)}_{ij} = \frac{i}{j} M^{(n)}(i,j).$$ (17)

Set

$$R^{(n)}_{\text{macro}} = \left( R^{(n)}_{\text{macro}}(i,j) \right)_{i,j=1}^N := M^{(0)}_{\text{macro}} M^{(1)}_{\text{macro}} \cdots M^{(n)}_{\text{macro}}.$$ 

It is easy to see that the elements of the matrix $R^{(1)}_{\text{macro}}$ satisfy the relation

$$R^{(1)}_{\text{macro}}(i,j) = \sum_{k=1}^N \frac{i}{k} M^{(0)}(i,k) \frac{k}{j} M^{(1)}(k,j) = \frac{i}{j} \sum_{k=1}^N M^{(0)}(i,k) M^{(1)}(k,j) = \frac{i}{j} R^{(1)}(i,j).$$

Hence we conclude by induction that the elements $R^{(n)}_{\text{macro}}(i,j)$ of the matrix $R^{(n)}_{\text{macro}}$ have the form

$$R^{(n)}_{\text{macro}}(i,j) = \frac{i}{j} R^{(n)}(i,j).$$ (18)

Condition $H4$ gives

$$\Lambda_{\text{macro}} := \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ \log \left| R^{(n)}_{\text{macro}}(1) \right| \right] = \Lambda = 0.$$
Using (18) it is easy to show that under the assumptions of Theorem 1 the macro process \( \{Z(n), n \geq 0\} \) satisfies all the conditions of Theorem 1 in [17], according to which

\[
P(Z(n) \neq 0 | Z(0) = e_i) \sim \frac{D_i}{\sqrt{n}}
\]

as \( n \to \infty \). Since \( \{Z(n) \neq 0\} \iff \left\{ \sum_{k=1}^{N} kZ_k(n) > 0 \right\} \), we conclude by (16) that, as \( n \to \infty \)

\[
P(\zeta(n) > 0 | \zeta(0) = i) = P \left( \sum_{k=1}^{N} kZ_k(n) > 0 | Z(0) = e_i \right) \sim \frac{D_i}{\sqrt{n}}.
\]

Theorem 1 is proved.

**Proof of Theorem 2.** Based on the reasonings used earlier in the proof of Theorem 1, it is easy to check by (18) that one may apply Theorem 1 in [18], proved for the so-called strongly subcritical multitype BPRE, to the branching macro process \( \{Z(n), n \geq 0\} \). This fact and the equality (16) give the needed statement.

2. Limit distribution of the number of particles in the critical BPRE’s with sibling dependence

To describe the limiting behavior of the distribution of the number of particles in a critical BPRE with sibling dependence, we need to impose stronger restrictions than in Theorem 1.

Given a random vector-valued generating function \( \Phi(s) = (\Phi_1(s), \ldots, \Phi_N(s)) = \Phi^{(0)}(s) \), from (4) and (5) we introduce a random vector \( (\eta_{i1}, \ldots, \eta_{iN}) \), \( i = 1, \ldots, N \), whose distribution is specified by the generating function \( \Phi_i(s) = E \sum_{k=1}^{N} s_{kj} \). Recall that \( D_{\Phi} \) is the symbol for the variance given the vector-valued probability generating function \( \Phi \), and \( \rho \) is the Perron root of the mean matrix \( M \), i.e., it is its maximal in absolute value eigenvalue.

Introduce the random variables

\[
\Delta := \max_{i,j} D_{\Phi} [\eta_{ij}].
\]

We need the following conditions.

**Condition A1.** The mean matrices \( M \) and \( M^{(n)} \), \( n \geq 0 \), are positive and with probability 1 have a common non-random right eigenvector \( u = (u_1, \ldots, u_p)' \), \( |u| = 1 \) with positive components corresponding to their Perron roots \( \rho \) and \( \rho^{(n)} \).

**Condition A2.** The following relation is valid:

\[
P \left( \max_{1 \leq i \leq N} (p_{i0} + p_{i1}) < 1 \right) = 1,
\]

where \( p_{i0} \) and \( p_{i1} \) are from (7).

**Condition A3.** The distribution of the random variable \( X := \log \rho \) belongs without centering to the domain of attraction of some stable law \( T \) with index \( \alpha \in (0, 2] \). The limit law \( T \) is not a one-sided law, that is, \( 0 < T(R^+) < 1 \).

Note that according to Lemma 2 the Perron root of the mean matrix \( M_{\text{macro}} \) is equal to the Perron root of \( M \).

**Condition A4.** With probability 1 \( \Delta < \infty \),

and there exists \( \varepsilon > 0 \) such that

\[
\mathcal{E}[\log \rho^{-2\Delta}]^{\alpha+\varepsilon} < \infty,
\]

(20)
Recall that the meander of a strictly stable Levy process is a strictly stable Levy process conditioned to stay positive on the time interval (0, 1] (see [3] and [4] for more detail).

**Theorem 3.** Let Conditions A1–A4 be valid. Then there exists a slowly varying at infinity sequence \( l(0), l(1), l(2), \ldots \) such that, for any \( i = 1, \ldots, N \) as \( n \to \infty \),

\[
\lim_{n \to \infty} n^{-1/\alpha} l(n) \mathbb{P}(\zeta(n) > 0 | \zeta(0) = i) = \beta_i \in (0, \infty), \tag{21}
\]

where \( \alpha \) is from Condition A3, and

\[
\mathcal{L}\left(\left\{ n^{-1/\alpha} l(n) \log \zeta([nt]), 0 \leq t \leq 1 \right\} | \zeta(n) > 0, \zeta(0) = i \right) \Rightarrow \mathcal{L}(L^+), \tag{22}
\]

where \([x]\) denotes the integer part of the number \( x \) and \( L^+ \) is the meander of a strictly stable Levy process with index \( \alpha \).

Here and in what follows the symbol \( \Rightarrow \) stands for weak convergence with respect to the Skorokhod topology in the space \( D[0, 1] \) of cadlag functions on the unit interval.

**Proof of Theorem 3.** We know that according to Lemma 2 the Perron root of the mean matrix \( M_{\text{macro}} \) is equal to the Perron root of \( M \). This and Condition A3 show that the macro process constructed by the initial BPRE with sibling dependence satisfies all the conditions of Lemma 9 and Theorem 1 in [9]. Therefore, there exists a slowly varying at infinity sequence \( l(0), l(1), l(2), \ldots \) such that, for each \( i = 1, \ldots, N \) as \( n \to \infty \),

\[
\lim_{n \to \infty} n^{-1/\alpha} l(n) \mathbb{P}(Z(n) > 0 | Z(0) = e_i) = \beta_i \in (0, \infty), \tag{23}
\]

where \( \alpha \) is from Condition A3, and

\[
\mathcal{L}\left(\left\{ n^{-1/\alpha} l(n) \log (Z(nt), u), 0 \leq t \leq 1 \right\} | Z(n) > 0, Z(0) = e_i \right) \Rightarrow \mathcal{L}(L^+). \tag{24}
\]

Combining (16), (23), and (24) we obtain (21) and (22). The theorem is proved.

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