Unifying Geometrical Representations of Gauge Theory

Scott Alsid∗ and Mario Serna†

2354 Fairchild Drive, Department of Physics
United States Air Force Academy, CO 80840

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Abstract

We unify three camps that have independently developed distinct geometric representations of gauge theory over the last century: those who use the compactified dimensions of Kaluza-Klein theory, those who use Grassmanian models (also called gauge theory embedding or $CP^{N-1}$ models) to represent gauge fields, and those who use a hidden spatial metric to replace the gauge fields. In this paper we identify a correspondence between the geometrical representations of the three camps. Each camp attempts to isolate the gauge-invariant core responsible for the resulting physics. By providing a mapping between geometrical representations, we hope physicists can now isolate representation-dependent physics from gauge-invariant physical results and share results between each camp. We provide visual examples of the geometrical relationships between each camp for $U(1)$ electric and magnetic fields. In all three representations, static electric fields (electric field from a fixed ring of charge or a sphere of charge) have a hidden time dependence ignored by modern notation.

Keywords: Kaluza Klein; Gauge field theory: Composite; Field theoretical model: $CP^{N-1}$; Gauge Geometry Embedding; Grassmannian Models, Hidden-spatial geometry; Visualization

1 Introduction

In this study, we unify three camps of gauge-invariant geometrical representations. Each camp has been employed by Fields Medalist and Nobel Prize winners to extract gauge-invariant key physics from gauge-dependent artifacts. Each camp has been largely independently invented and has often had distinct strands of papers that do not cite the relation to other reinventions within the same camp and do not cite the relationship to the other camps. We highlight the commonalities of the different strands within each camp, and then we tie the geometric representations of each camp onto a common representation. We will show that in all three representations, a static electric field has a hidden time dependence that is not captured by our normal notation.

The first ‘camp’ was started by Theodor Kaluza and Oskar Klein [1, 2] and uses a $4 + n$ dimensional space-time where extra dimensions are curled up and result in a gauge theory (see [3, 4] for reviews). This well-known camp has over 1600 papers. The second camp uses a Grassmannian manifold to represent gauge fields using a type of gauge-theory embedding [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. The third camp introduces alternative variables for gauge theory that uncover a hidden spatial metric which reproduces the gauge fields [32, 33, 34, 35, 36, 37, 38, 39, 40]. Each of the camps start with a different geometrical representation which then faithfully maps onto the traditional gauge fields. This paper unifies the geometrical representations of the three camps. A unified geometrical representation will allow physicists to better identify gauge-invariant physics. As an example, we will discuss the hidden time dependence in static electric fields which is evident in all three camps. Our unification will also help physicists to translate results such as instanton solutions, between each independent camp.

∗c15scott.alsid@usafa.edu
†mario.serna@usafa.edu
Figure 1: The angular change in the phase of a wave function after parallel transport around a closed loop in space-time yields electric and magnetic fields multiplied by the area of the loop enclosed. This parallels Riemannian geometry where the Riemann tensor gives the rotation matrix that results from parallel transport around a loop.

Mathematicians describe both gauge theory and Riemannian manifolds with the language of fiber bundles. Fiber bundles are not a geometrical representation, but rather a rigorous lexicon used to describe a wide array of geometrical structure. This language enables descriptions of gauge theories on topologically non-trivial spaces. However, the power gained by abstraction to the fiber bundle language often leaves out insight that may be gained from explicit examples. Here we concern ourself only with local descriptions of gauge fields on topologically trivial flat space-time; therefore, we will avoid extensive use of the bundle language in favor of explicit examples.

This paper is organized as follows. In section 2 we review the development and research activity of each camp. Sections 3 and 4 contain our new research results; they present the connections between the Kaluza-Klein and Grassmannian camp, and the Hidden-Spatial-Geometry and the Grassmannian camp respectively. Finally in section 5 we provide examples with familiar electric and magnetic fields. In our conclusion, we discuss the hidden time dependence in static electric fields.

2 Literature Survey of the Three Approaches

All the geometrized representations that we discuss emphasize non-integrable phase factors to define the internal curvature \[ \int_A \cdot d\mathbf{r} \]. The Wilson loop \( \Delta \theta = \oint A \cdot d\mathbf{r} \) gives the phase-angle shift resulting from parallel transport of the wave function around an infinitesimal loop. The non-integrable phase is similar to how curvature is found in Riemannian geometry.

For example, the magnetic field is equal to the phase-angle change in the wave function after parallel transporting the wave function around a closed loop on a spatial slice of space-time. In the limit of an

\[ ^{1}\text{We have chosen to work in units where } \hbar = c = 1 \text{ and where we absorb the electron's charge } e \text{ into the definition of } A_\mu. \]
infinitesimal loop, the magnetic field is given in terms of the phase-shift $\Delta \theta$ as

\[
B_z = \frac{\Delta \theta}{\Delta x \Delta y} = \oint \vec{A} \cdot d\vec{r} = \frac{\int \int (\vec{\nabla} \times \vec{A}) \cdot (d\vec{x} \times d\vec{y})}{|d\vec{x} \times d\vec{y}|}
\]

(1)

where we have used the Wilson loop and classic vector identities.

Figure 1a and 1b show this non-integrable phase angle using the tools of the Grassmannian camp described in section 2.2. In this figure, the complex plane on which the wave function lives is represented by the plane spanned by the two red basis vectors. The complex plane is inserted into a trivial internal space at each space-time point and is represented by a disk. The wave function is shown as a vector (black or yellow) on the disk inserted at each space-time point. We parallel transport the wave function along two paths (A and B) represented by the black and yellow vectors. Comparing path A with path B gives the non-integrable phase shift $\Delta \theta$. Because of this phase shift, the wave front is pulled in a direction and the plane wave changes directions.

Figure 1a shows the case of the magnetic field where the loop is all spatial. Likewise, figure 1b shows the electric field is equal to parallel transporting the wave function or matter field around a closed loop on a part spatial and part temporal slice of space-time.

2.1 The Kaluza-Klein Camp

Kaluza Klein theories unify classical electromagnetism with Einstein’s general relativity [1] [2]. They posit extra spatial dimensions that are compactified within ordinary space-time along a very small radius $R$. All tensor quantities are independent of this fifth coordinate (cylinder condition).

In traditional Kaluza-Klein theory the line element of the five-dimensional space is

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu + (RA_\mu dx^\mu + dx^5)^2,
\]

(2)

where we omit the dilaton field for simplicity of presentation. Here $g_{\mu\nu}$ is the familiar four-dimensional metric from general relativity, $A_\mu$ is the four-vector potential, $x^5$ is the fifth dimension’s coordinate, and $R$ is the radius of the curled up fifth dimension.

In Kaluza-Klein theory charge is explained as motion of a neutral particle along the fifth dimension, where the two directions it can go in $x^5$ explain the two different types of charge. Electric fields are four-dimensional manifestations of the inertial-dragging effect in the fifth dimension [42] [43]. Furthermore coordinate transformations of the fifth dimension are shown to be $U(1)$ gauge transformations.

The main pitfall of the classical theory is that there are no measurable new predictions. Another pitfall occurs with quantum mechanics. The wave function around the fifth dimension gives particles a mass-spectrum tower of $m^2 = (n/R)^2$ where $n$ is an arbitrary integer. For an $R$ near the Planck scale, particles would be either massless or have Planck-scale masses. Kaluza-Klein theories play a large role in string theory. For a further review of Kaluza-Klein theory see references [4, 3] and the references therein.

2.2 The Grassmannian Camp

Grassmannian representations of gauge fields started in 1961 when Narasimhan and Ramanan showed that every $U(n)$ gauge theory could be represented a section of a Grassmannian manifold $G(n,N)$ [5] [6]. The Grassmannian manifold is the set of orientations an $n$-plane can take in a larger $N$-dimensional space with a fixed origin. Another way to view the setup is one has a vector space embedded as a plane into a higher-dimensional Euclidean space that is inserted at each point in space-time. The Grassmannian camp is essentially a gauge theory version of the 1956 Nash embedding theorem which proved that every Riemannian manifold could be embedded in a higher-dimensional Euclidean space [44].

Throughout this paper we use the convention that lower-case Latin letters near the beginning of the alphabet $a, b, ...$ will be gauge-theory color indices, Greek letters $\mu, \nu, ...$ will be spacetime coordinates, upper-case Latin letters $A, B, ...$ will be used for Kaluza-Klein metric indices, and lower-case Latin letters towards the middle of the alphabet $i, j, ...$ will be used for the variables corresponding to subspaces of space-time and the embedding dimensions, where context will keep them distinct. The Kaluza-Klein index values 0 through 3 are the usual space-time coordinates $t, x, y, z$ and the index value 5 is the fifth dimension coordinate $x^5$, which is used to parameterize the tiny compact dimension.
In the language of bundles, Narasimhan and Ramanan proved for any $U(n)$ gauge field and $d$ space-time dimensions, the gauge field can be constructed by inserting a $\mathbb{C}^n$ vector bundle into a trivial $\mathbb{C}^N$ vector bundle if $N \geq (d+1)(2d+1)n^3$. Narasimhan’s condition guarantees us an embedding for this $N$, but we can sometimes represent the embedding for specific field configurations for smaller $N$ as we will do in section $3$. For an $O(n)$ gauge field $\mathbb{R}^n$ vector bundles are embedded in a trivial $\mathbb{R}^N$ vector-bundle.

In the Grassmannian camp, wave functions are sections of the $n$-dimensional vector bundle. That is, they are a vector on the $\mathbb{C}^n$ or $\mathbb{R}^n$ vector space. By definition the vector bundles have a fixed origin. The embedding-camp approaches have a set of $n$ orthonormal gauge basis vectors $\vec{e}_a$ that span the gauge fiber internal to each space-time point. The dual basis vectors $\vec{e}^a$ satisfy $\vec{e}^a \cdot \vec{e}_b = \delta^a_b$. There are $n$ vectors $\vec{e}_a$ in a real or complex Euclidean $N$-dimensional embedding space. The matter field (wave function) exists as a vector on the gauge fiber spanned by the gauge-fiber basis vectors:

$$\vec{\phi} = \phi_a \vec{e}_a.$$  

(3)

The projection operator is the outer product $P^j_k = e^j_a e^a_k$. The gauge field is then

$$(A_\mu)^a_b = ie^a_a \cdot \partial_\mu \vec{e}_b.$$  

(4)

Figure 2 show $G(2, 3)$ Grassmannian model visually. The bubbles show the $N = 3$ trivial vector space inside each space-time point. The red vectors are the gauge-fiber basis vectors $\vec{e}_a$ which span the displayed disk. The gauge fields depend on all the ways one can orient the $R^2$ space within the trivial $R^3$ space. The wave function or matter field $\vec{\phi}$ is the black vector that lives on the disks.

A gauge transformation is a rotation of the basis vectors $\vec{e}_a$ that preserves their inner product and leaves the wave function $\vec{\phi} = \phi^a \vec{e}_a$ and the projection operator $P^j_k = e^j_a e^a_k$ invariant.

A long list of notable physicists have employed the embedding approach to gauge theory. Atiyah in 1979 $7$ defined the linear maps $u_x : \mathbb{R}^n \to \mathbb{R}^N$, whose image was in the trivial space $\mathbb{R}^N$. Atiyah’s $u$’s play the role of the gauge-fiber basis vectors $\vec{e}_a$. The projection operator is written as $P = uu^*$, with $u^* u = 1$, and the gauge potential is $A_\mu = u^* \partial_\mu u$, where $u^*$ is the dual to $u$. Corrigan and followers $8, 45$ used the embedding representation in finding Green’s functions for self-dual gauge fields. Dubois-Violette $9$ created a formulation of gauge theory using only globally defined complex $N \times n$ matrices $V$ (analogous to $e^a_a$) such that $V^\dagger V = I$ and $VV^\dagger = P$, and $A_\mu = V^\dagger(x) \partial_\mu V(x)$.

A independent research line refers to the Grassmannian camp as $CP^N$ models $23, 24, 25, 26, 27, 28, 29, 30, 31$. 

Figure 2: A set of two basis vectors span the internal vector space attached to every point in space-time. How they vary determines the electromagnetic field.
In these models a setup is created with $z^1 \cdot z = 1$ where $z$, which is sometimes called a zweibein, is a complex $N$-vector. The gauge field $A_\mu = z^1 \partial_\mu \cdot z$ is discovered in the equations of motion.

Felsager, Leinaas, and Gliozzi \cite{22,10} had a similar approach. In a manner very similar to figure 2 and section 5 they geometrically represented magnetic fields by use of plane bundles in $\mathbb{R}^3$, where the distribution of the planes in each point was characterized by a curvature related to the magnetic field strength. For two vectors $\vec{e}_1, \vec{e}_2$ orthonormal to each other and to the normal vector of the plane, the vector potential is $A_j = \lambda \vec{e}_1 \cdot \nabla_j \vec{e}_2$, where the $e$s play the same role as $\vec{e}_i$ introduced in the beginning of this section, and $\lambda$ is a constant for dimensionality. In the nineties, Cahill \cite{11,13,12} used gauge basis vectors $\vec{e}_a$ in lattice simulations.

In finding projectors for the fuzzy sphere, Valtancoli \cite{14} used the connection $A^a_\mu = \langle \psi_a, d\psi_a \rangle$ for $n$-monopoles. Here $|\psi\rangle$ plays the role of $\vec{e}_a$.

Bars \cite{15,16} used a separate embedding for each of the spatial dimensions of the gauge field (corner variables), as opposed to using a single embedding for all the gauge-fiber basis vectors. He used $n \times n$ unitary matrices $B_{13}, B_{23}$ to rewrite the canonical variables $A^a_\mu$ and $E^a_\mu$. For example, $A^a_\mu$ was written as $T^a A^\mu_1 = iB_{13}^J \partial_1 B_{13}$, where $T^a$ is a generator of $SU(n)$. Stoll \cite{18,17} introduced angle variables in the Hamiltonian formulation of QCD to investigate the low-energy properties in terms of gauge invariant degrees of freedom. The angle variables are similar to corner variables and are the exponents of $SU(n)$ matrices, and the gauge fields are defined as $A_j(x) = \frac{i}{4} V_j(x) \partial_j V_j(x)$ (no summation). Zee and Wilczek, building on work by Simon, also independently developed a Yang-Mills structure associated with Barry’s phase and degenerate spaces (see \cite{16,19,10,47}). A given wave function is expanded in terms of eigenfunctions spanning a degenerate subspace $\Psi(t) = c_\mu \psi_\mu(t)$. One finds in the adiabatic limit that $\frac{dc_\mu}{dt} = -A_{ba} c_\mu$, where $A_{ba}(t) = i\langle \psi_b(t), \psi_a \rangle \frac{d\psi_a}{dt}$. For a Hamiltonian $H(t)$ that depends on parameters $\lambda^1, \ldots, \lambda^4$, when one traces out a path in the parameter space the time derivative of $c_b$ becomes $\frac{dc_b}{dt} = -(A_{\mu})_{ba} c_a \frac{d\lambda^\mu}{dt}$, where $(A_{\mu})_{ba} = i\langle \psi_b(\partial_\mu \psi_a) \rangle$. Zee and Wilczek’s approach, $|\psi_a\rangle$ plays the role of $\vec{e}_a$

In the early 2000’s Serna and Cahill used the Narasimhan and Ramanan theorem to visualize the geometry of simple electromagnetic fields with an $SO(2)$ gauge group. To gain some visual intuition they found $SO(2)$ gauge basis vectors $\vec{e}_i$ embedded in an $\mathbb{R}^3$ trivial fiber for certain vector potentials \cite{20}. For a matter vector on the gauge fiber, as represented visually in their work, a clockwise rotation in the momentum direction corresponded to a positive charge, while a counterclockwise rotation corresponded to a negative charge. In addition to this they observed an indication for a geometry-based explanation of charge quantization. This is similar to the representation of charge in Kaluza-Klein. Although all free fundamental particles have $\pm e$ charge, quarks have fractional charge. The fractional charge would follow from a GUT gauge theory, such as that of $U(1) \times SU(2) \times SU(3) \subset SU(5) \subset SU(10)$. These GUTs always enable one to absorb $e \cdot A = A'$. We can only absorb $e$ into $A_\mu$ if every field couples with the same coefficient as in most GUTs.

2.3 The Hidden-Spatial-Geometry Camp

The next camp maps a hidden spatial geometry onto the gauge fields. The gauge potential transforms inhomogeneously and makes unclear the physical nature of the theory. In 1978, Goldstone and Jackiw \cite{32} made the electric field in an $SU(2)$ gauge theory diagonal, which made easier separating the gauge-invariant parts of gauge angles. They wed these ideas to a 4-space ‘spinning top’ analogy.

In 1994 Lunev \cite{33} formulated a tetrad-based mapping from $A^a_\mu$ to a tetrad variable. In 1995 Freedman, Haugensen, Johnson, and Latorre also introduced tetrad variables $u^a_\mu$, as a replacement to $A^a_\mu$ \cite{33,34,35,36}, where the index $a$ denotes the color index and $\hat{j}$ denotes the spatial index. In our work we follow the notation of Haugensen and Johnson. The $u^a_\mu$ variables serve as a mapping from the basis vectors that span an internal color space at a space-time point to the coordinate tangent vectors of a hidden spatial metric at that space-time point. For this approach to work, the color-space dimension of the tetrad must be equal to the space-time dimension of a chosen slice.

Haugensen and Johnson used an $SU(2)$ gauge group, with structure constants $f^{abc} = \varepsilon^{abc}$ for the color index and $GL(3, \mathbb{R})$ for the spatial component. They worked in the temporal gauge $A^0_\mu = 0$ so they could map the three vector potentials $A^0_\mu$ to the three dimensions of the space slice. The constraint imposed on $u^a_\mu$ was that the color index had to transform as a covariant vector under $SU(2)$ and the spatial index had to transform as $GL(3, \mathbb{R})$. The condition that $u^a_\mu$ transform as a vector leads to the gluon ‘spin’ operator
where $\Gamma$ is a tetrad in the first 4 dimensions the tangent vectors used in eq 10 are given by

$$\Gamma_{ij} = \frac{\varepsilon_{imn} \partial_m u^a_k(u^b_j u^c_k - \frac{1}{2} u^b_j u^c_k)}{\det u},$$

which is similar to the spinning top analogy given in Jackiw and Goldstone. The end result is that, for a given set of tetrads $u^a_j$, a unique vector potential $A^a_j$ can be found; however, the other direction is not unique. For a given $A^a_j$ several $u^a_j$ exist. Given a set of tetrad fields $u^a_j$, the $SO(3)$ vector potential is given by

$$A^a_j = (\varepsilon^{nmk} \partial_m u^a_k)(u^b_j u^c_k - \frac{1}{2} u^b_j u^c_k).$$

In using the constraint of eq 5, a hidden spatial metric was implicitly introduced. The anti-symmetric tensor in eq 5 implies

$$\partial_j u^a_k + \varepsilon^{abc} A^b_j u^c_k = \Gamma^a_{jk} u^a_k,$$

where $\Gamma^a_{jk}$ is a quantity symmetric in the indices $j,k$. With further manipulation this turns into

$$\Gamma^a_{jk} = \frac{1}{2} g^{lm}(\partial_j g_{mk} - \partial_k g_{jm} - \partial_m g_{jk}),$$

where $g_{ij} = u^a_i u^a_j$. These are the familiar Christoffel symbols for a Riemannian metric associated with a tetrad $u$. Thus, imposing eq 5 implicitly introduced a Riemannian geometry with a tetrad $u$ and a metric.

Towards the end of the nineties Schiappa adapted these local gauge-invariant variables for supersymmetric gauge theory [38]. An independent variation of this camp was pursued by Slizovskiy and Niemi [39].

In summary, the variables $u^a_j$ map basis vectors that span the internal color space to coordinate tangent vectors of a hidden spatial metric.

### 3 Connecting the Kaluza-Klein and Grassmannian Approaches

We now wish to connect the Kaluza-Klein camp of section 2.1 to the Grassmannian camp of section 2.2. The Kaluza-Klein metric places the gauge field $A_\mu$ in the $5$ off-diagonal element of the metric

$$g_{5\mu} = t^\mu \cdot \bar{t}_5 \propto A_\mu.$$  

To connect to the Grassmannian camp we construct an explicit isometric emersion. We insert the space-time and gauge fiber into an $SO(1,3+N)$ embedding with a Lorentzian signature $\eta = \text{diag}(-1,1,1,\ldots,1)$. The explicit embedding being considered is

$$\vec{X} = (t, x, y, z, \vec{R} \cos \frac{x_5}{R} + \vec{R} \sin \frac{x_5}{R}),$$

where $\vec{e}_n$ is an $N$-dimensional vector from section 2.2 which depends only on the first four coordinates $x^\mu$. This inserts a ring in the embedding space. The tangent vectors used in eq 10 are given by $\vec{t}_A = \partial_A \vec{X}$. For the first 4 dimensions the tangent vectors $t^\mu$ are given by

$$t^\mu = \partial_\mu X^k = \delta^\mu_k + \Theta(k-5) R \left( \cos \frac{x^5}{R} \partial_\mu \varepsilon_1^{k-4} + \sin \frac{x^5}{R} \partial_\mu \varepsilon_2^{k-4} \right),$$

where the discrete Heaviside function $\Theta(x-a)$ is defined to be 1 if $x \geq a$ and 0 when $x < a$ and $k$ indexes the $4+N$ coordinates of the embedding space. The tangent vector $t_5$ is given by

$$t_5 = \partial_5 \left( \frac{\vec{R} \varepsilon_1^{k-4} \cos \frac{x_5}{R} + \vec{R} \varepsilon_2^{k-4} \sin \frac{x_5}{R}}{R} \right).$$

The resulting five-dimensional space-time metric $g_{AB}$ for this embedding to first order in $R$ is $g_{\mu\nu} = t^k_{\mu} \delta^k_\mu \eta_{kl} = \eta_{\mu\nu} + O(R^2)$, $g_{5\mu} = t_5^k \delta^k_\mu \eta_{kl} = RA_\mu$, and $g_{55} = t_5^k t_5^l \eta_{kl} = 1$ where we have used eq 4 from the Grassmannian.
camp applied to $SO(2)$ to relate $A_\mu = \vec{e}^2 \cdot \partial_\mu \vec{e}_1 = -\vec{e}^1 \cdot \partial_\mu \vec{e}_2$ and $\vec{e}_a = \vec{e}^a$. The dilaton field $\Phi(x)$ follows if we allow the size to vary $R \to R\Phi(x)$. The geometry of the embedding, which reproduced the Kaluza-Klein metric, has a compactified ring at every point in space-time on the same plane spanned by the embedding-camp basis vectors. Visual examples will be shown in section 5.

Although we have shown a map between the Kaluza-Klein theory and the Grassmannian camp, the Kaluza Klein camp has a different representation of the wave function. In the Grassmannian camp there is one wave function at a space-time point and it is a vector on the tangent space spanned by the $\vec{e}_a$. The dilaton field $\Phi(x)$ follows if we allow the size to vary $R \to R\Phi(x)$. The geometry of the embedding, which reproduced the Kaluza-Klein metric, has a compactified ring at every point in space-time on the same plane spanned by the embedding-camp basis vectors. Visual examples will be shown in section 5.

4 Connecting the Grassmannian Camp and the Hidden-Spatial-Metric Camp

Next we show how the Grassmannian camp of section 2.2 and the hidden-spatial-metric camp of section 2.3 are related. We show an emersion of the hidden spatial metric into the larger-dimensional space of the Grassmannian camp using the a trivial vector bundle. This causes the planes spanned by the projection operator $P^j_k = \epsilon^j_1 \epsilon^k_1$ to become the tangent planes associated with a surface. Again the color-space dimension of the tetrad must be equal to the dimension of the space-time slice under consideration. For $SO(3)$ we need a three-dimensional slice of space-time to identify with the three color dimensions of the $SO(3)$ gauge fiber. For the $SO(2)$ representation used in section 5 we need a two-dimensional slice of space-time to identify with the two real dimensions of $SO(2)$.

The metric in the hidden-spatial-metric camp is expressed with the familiar tetrad equations (but now in a new context)

$$g_{jk} = u^a_j u^a_k = \vec{t}_j \cdot \vec{t}_k,$$

where $\vec{t}_j$ and $\vec{t}_k$ are tangent vectors in the trivial vector bundle for some shape parameterized by space-time coordinates. The variables $u^a_j$ from the hidden-spatial-metric camp are then seen as tetrads that map the tangent vectors to the orthonormal frame of the Grassmannian camp:

$$\vec{e}_a = u_a^i \vec{t}_i,$$

or its dual,

$$\vec{e}^a = u^a_i \vec{t}^i.$$

The tetrads $u^a_i$ may be obtained by using the Gram-Schmidt orthogonalization process on the coordinate tangent vectors of the hidden spatial metric.

Now to show how the tetrads lead to the gauge field $A_\mu$. If we express eq 4 not in terms of the gauge basis vectors but the coordinate tangent vectors $\vec{t}$ associated with the hidden spatial metric of the Grassmannian camp via eq 15 then eq 4 becomes

$$(u^a_i \vec{t}^i) \cdot \partial_\mu (u^b_k \vec{t}^k) = -i A_j^a_b,$$

$$(u^a_i \delta^i_j \partial_\mu u^b_k + i A_j^a_b + u^a_k u^b_i \Gamma^j_{ik} = 0. $$

Multiplying by $-u^b_k$ and using the identity $u^b_k \partial_\mu u^a_k + u^a_k \partial_\mu u^b_k = 0$ yields

$$\partial_\mu u^a_k - i A_j^a_b u^b_k \delta^j_i - u^a_i \Gamma^j_{ik} = 0.$$

Now we specialize to $SU(2)$, where the form of the generators are

$$\left(T^c\right)_b^a = -i \epsilon^{abc}. $$

\footnote{The distinction between lower and upper indices are dropped in the epsilon term for convenience (see Weinberg [48], chapter 15 appendix A).}
Thus eq (17) leads to
\[ \partial_j u^a_k + \varepsilon^{abc} A^b_j u^c_k = u^a_i \Gamma^i_{jk}, \] (21)
which is eq (7) of the hidden-spatial-metric camp, but this time derived by the Grassmannian camp’s methods. A similar relationship was independently observed by Refs. [49, 30] but without noting the generality of the relationships to the long research records of the two camps.

As for the wave function in the Grassmannian camp, it is a vector on the gauge fiber in the internal space. In the hidden-spatial-metric camp, it is a vector on the tangent space. As these are the same spaces the wave functions are the same in these two camps. This is in contrast to the Kaluza-Klein camp, where the wave function is a scalar function of each point in the five-dimensional space-time.

5 Examples in Electromagnetism

We now apply the results above to an abelian U(1) gauge theory, namely ordinary electromagnetism. We work with SO(2) (which is isomorphic to U(1)) so that everything is real. In SO(2), there is only one generator; the gauge potential is
\[ (A_j)^a_b = (T)^a_b A_j, \] (22)
where \( T^a_b = -i \varepsilon^{ab}. \)

Each electromagnetic field configuration has at least one (sometimes many) geometric representations in each camp. In order to demonstrate the hidden spatial geometry, we need to select a spacetime slice of equal dimension to the dimension of the gauge fiber. For SO(2) we will need to select two-dimensional slices. We will analyze two-dimensional slices of space-time denoted by \( x^\mu(\sigma, \tau) \) and show each camp’s representation in this slice. We use the pullback to map the four-dimensional field-strength tensor \( F_{\mu\nu} \) and vector potential \( A_\mu \) on the 2-D slice of space-time using
\[ F_{ij} = \frac{\partial x^\mu}{\partial x^i} \frac{\partial x^\nu}{\partial x^j} F_{\mu\nu}, \quad A_j = \frac{\partial x^\mu}{\partial x^j} A_\mu. \] (23)

The SO(2) analog of eq (21) is
\[ \partial_j u^a_k + \varepsilon^{ab} A_j u^b_k - u^a_i \Gamma^i_{jk} = 0. \] (24)
The Grassmannian and Kaluza-Klein camp’s equations are unaltered in specializing to SO(2) examples.

We now proceed to illustrate the above connections for three elementary electromagnetic field configurations: a y-polarized plane wave, an electrically charged ring, and a spherical charge. These examples were chosen for their familiarity and because their hidden spatial metrics correspond to a sphere, a paraboloid, and a funnel-shaped object respectively.

5.1 The Y-Polarized Plane Wave

Consider the four-potential for a y-polarized plane wave traveling in the x-direction
\[ A_y = A_0 \cos(k(x - t)), \] (25)
where \( A_0 = \frac{E_0}{k} = \frac{B_0}{k}. \) We take a yt slice of space-time. This is parameterized by \( t(\sigma, \tau) = \tau, x(\sigma, \tau) = x_0, y(\sigma, \tau) = \sigma, \) and \( z(\sigma, \tau) = z_0, \) where \( x_0 \) and \( z_0 \) are fixed coordinates.

Using the pullback the SO(2) vector potential for the plane wave is
\[ A_\sigma = A_0 \cos(k(x_0 - \tau)). \] (26)

The question now is what two-dimensional shape from the hidden-spatial-metric camp has this specific vector potential? Using trial and error, we considered shapes until we found the ones whose tangent vectors led to eq (26). The plane wave follows from a sphere parameterized as
\[ \vec{X} = \begin{pmatrix} \frac{\sigma \sin(k(x_0 - \tau)) \cos(A_0\sigma)}{\varrho \cos(k(x_0 - \tau))} \\ \frac{\sigma \sin(k(x_0 - \tau)) \sin(A_0\sigma)}{\varrho \cos(k(x_0 - \tau))} \end{pmatrix}, \] (27)
where \( \sigma = y \) and \( \tau = t \), and \( \varrho \) is a positive real value on which \( A_i \) and \( F_{ij} \) do not depend.

Figure 3a shows the domain of the variables \( \sigma \) and \( \tau \), which parameterize the \( yt \) slice of space-time. There are two lines, one in the direction of increasing \( \sigma \) on the outer ring and one in the direction of increasing \( \tau \) on the inner ring. Figure 3b shows the diagram as it appears parameterized on the surface of the hidden spatial geometry where we let \( \varrho = 1 \, \text{m}, \, k = 1 \, \text{m}^{-1} \), and \( A_0 = 1 \, \text{m}^{-1} \). This corresponds to an average intensity beam of about \( 5 \times 10^{-17} \) Watt/\( \text{m}^2 \). Here we see that increasing \( \sigma \) corresponds to a line of longitude on the sphere, whereas increasing \( \tau \) corresponds to a line of latitude.

Let us verify that the embedding from eq 27 produces eq 26. The coordinate tangent vectors, \( \mathbf{t}_\sigma \) and \( \mathbf{t}_\tau \), are found by differentiating eq 27 by the respective coordinates \( \mathbf{t}_\sigma = \partial_\tau \mathbf{X} \) and \( \mathbf{t}_\tau = \partial_\sigma \mathbf{X} \). The hidden-spatial-metric \( g_{ij} \) is found by taking the dot products between each of the tangent vectors \( g_{ij} = \mathbf{t}_i \cdot \mathbf{t}_j \). The resulting line element is

\[
ds^2 = k^2 g^{ij} \sigma^i \sigma^j + A_0^2 g^{ij} \sin^2(k(x_0 - \tau)) d\tau^2.
\]

(28)

The lack of off-diagonal terms in the line-element means that the tangent vectors are orthogonal (such is the geometry where we let \( g_{ij} = 0 \)). The tetrads \( u^a_\sigma \), which map \( \mathbf{t}_j \) to \( \mathbf{e}_a \), are \( u^a_\sigma = \frac{1}{|\mathbf{t}_j|} = \frac{1}{k \varrho} \), \( u^0_\sigma = 0 \), \( u^1_\sigma = 0 \), and \( u^2_\sigma = \frac{1}{k \varrho \sin(k \tau)} \).

Normalizing the tangent vectors gives the Grassmannian’s basis vectors

\[
\mathbf{e}_1 = \left( -\sin(A_0 \sigma), \cos(A_0 \sigma) \right), \quad \mathbf{e}_2 = \left( -\cos(k(x_0 - \tau)) \cos(A_0 \sigma), \sin(k(x_0 - \tau)) \cos(A_0 \sigma) \right).
\]

(29)

The only nonzero value of \( A_j \) is

\[
(A_\sigma)^a_b = i\mathbf{e}^a \partial_\sigma \mathbf{e}_b = -i A_0 \cos(k(x_0 - \tau)) \mathbf{e}^{ab}
\]

(30)

which shows that this parameterization of the sphere leads to the geometry of the \( yt \) slice of the \( y \)-linearly-polarized plane wave.

These vectors \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) are visualized in figure 4a as the red basis vectors which span the disks. The figure is to be interpreted as in figure 2 but without the bubbles. The reference pattern from figure 3a is again shown to help visualize the directions of \( \sigma \) and \( \tau \) in both spaces. A rotation of the red basis vectors on the disks corresponds to a gauge transformation. Next figure 4b shows the disks from the Grassmannian-camp representation rearranged into the shape associated with the hidden-spatial-metric camp.

From the Kaluza-Klein picture, we have at each point in space-time a curled up fifth dimension. This is represented by a ring in the Grassmannian’s space on the same tangent plane. The ring’s embedding is parameterized by \( x^5 \) as

\[
\mathbf{r}(x^5) = R \mathbf{e}_1 \cos \left( \frac{x^5}{R} \right) + R \mathbf{e}_2 \sin \left( \frac{x^5}{R} \right),
\]

(31)

where \( \mathbf{r} \) is a three-dimensional vector in a trivial Grassmannian’s embedding space over space-time. Figure 5 shows the representation of the Kaluza-Klein camp, where the rings are shown at several space-time points for our given slice. We have changed the scale of \( R \) to better see the ring. Thus, figures 3, 4, and 5 are the three camp’s geometrical representations of the plane wave; all share a common set of tangent planes as represented in the embedding.

Alternatively for the \( A_\mu \) of eq 25, we want to understand the magnetic field. We consider the \( xy \) slice of space-time: \( t(\sigma, \tau) = t_0, x(\sigma, \tau) = \sigma, y(\sigma, \tau) = \tau \), and \( z(\sigma, \tau) = z_0 \) where \( t_0 \) and \( z_0 \) are fixed values. The pullback of the vector potential is

\[
A_\sigma = \frac{B_0 \cos(k(\sigma - t_0))}{k}.
\]

(32)

The shape corresponding to this slice is also a sphere:

\[
\mathbf{X} = \begin{pmatrix}
\varrho \sin(k(\sigma - t_0)) \cos(A_0 \tau) \\
\varrho \sin(k(\sigma - t_0)) \sin(A_0 \tau) \\
\varrho \cos(k(\sigma - t_0))
\end{pmatrix}
\]

(33)

To the best of our understanding, the similarity to eq 27 is not general.
(a) Reference pattern which will be shown parameterized on the geometries of the three camps.

Figure 3: A $y$-polarized plane wave for a $yt$-slice of space-time.

(b) A hidden-spatial-geometry representation of the $y$-polarized plane wave, $yt$ slice.

(a) A Grassmannian representation of the $y$-polarized plane wave, $yt$ slice.

(b) A hidden-spatial-geometry representation with the disks of the Grassmannian representation mapped to their corresponding location of the shape.

Figure 4: The geometry of a $y$-polarized plane wave for a $yt$ slice, with $k = A_0 = 1 \text{ m}^{-1}, \varrho = 1 \text{ m}$. 
5.2 The Electrically Charged Ring

Next consider the electric field due to a ring of charge \( -Q \) and radius \( b \) centered at the origin on the \( xy \) plane:

\[
E_z = \frac{-Qz}{4\pi(b^2 + z^2)^{3/2}}. \tag{34}
\]

The associated vector potential pulled-back onto a \( zt \) slice (\( z = \sigma, t = \tau, x = 0, y = 0 \)) is given by

\[
A_{\tau} = \frac{Q}{4\pi} \frac{1}{\sqrt{b^2 + \sigma^2}}. \tag{35}
\]

We found the hidden-spatial-geometry shape to be a paraboloid parameterized as

\[
\vec{X} = \left( \begin{array}{c}
\frac{\sigma}{2b} \cos\left(\frac{Q}{4\pi} \tau \right) \\
\frac{\sigma}{2b} \sin\left(\frac{Q}{4\pi} \tau \right) \\
\frac{\sigma^2}{2b} \end{array} \right). \tag{36}
\]

After calculating the tangent vectors \( \vec{t}_j \) and the tetrads \( u^a_j \) that create the basis vectors \( \vec{e}_a \), we calculate the associated vector potential \( A_{\tau} \) to verify that it gives the same \( z \)-directed electric field of the negatively charged ring.

Figure 6a shows the charged ring from the Grassmannian camp with the choice of \( b = \frac{1}{2} \) m and \( Q = 2\pi \). The associated electric-field pictured corresponds to a 1.24 \( \mu \)C ring when converted to SI units. Figure 6b shows the gauge subspace from the Kaluza-Klein camp, and figure 6c shows the hidden-spatial-metric picture. Figure 6d shows the hidden-spatial-metric picture built from the disks of the Grassmanian camp. From the Kaluza-Klein picture, we see a repetition in the disk arrangements in the \( \tau \) direction. This corresponds to the periodicity of the cosine function with respect to \( \tau \). The direction of increasing \( \sigma \) is along the vertical dimension of the paraboloid. As \( \sigma \) increases the shape becomes flatter which corresponds to distances farther from the ring (with a weaker electric field along the axis). The \( \tau \) direction is along the circular dimension of the paraboloid. The tangent planes oscillate with time in all three representations.

Note that the charged ring has an explicit time dependence in all three gauge-invariant geometric representations, as shown by \( t = \tau \) in eq 36. The time dependence disappears when we project to the scalar potential and the electric field. Although the charged ring gives a static electric field, the geometrical representations makes clear there is a hidden gauge-invariant time dependence.

5.3 The Spherically Charged Shell

For a bounded scalar potential \( A_t = \Phi(\vec{x}) \) of a static electric field, a funnel-shaped surface can be found for a given two-dimensional slice of space-time. In this case, \( A_{\mu} = ( \Phi, \ 0, \ 0, \ 0 ) \), and \( A_{\tau} = \frac{\partial t}{\partial \tau} A_t \) is a
Figure 6: The geometry of an electrically charged ring, along the $z$ axis only, for $Q = 2\pi$ and $b = \frac{1}{2}$ m. In SI units this is a charge of 1.24 $\mu$C.

function of $\sigma$ only. The first derivative of $A_0$ must be strictly negative, and $0 < A_\tau \leq \omega$. If we use the shape

$$
\vec{X} = \begin{pmatrix}
\frac{A_\tau (\sigma)}{\omega} \sin(\omega \tau) \\
\frac{A_\tau (\sigma)}{\omega} \cos(\omega \tau) \\
\sqrt{1 - \left(\frac{A_\tau (\sigma)}{\omega}\right)^2} + \ln \left(\frac{A_\tau (\sigma)}{1 + \sqrt{1 - \left(\frac{A_\tau (\sigma)}{\omega}\right)^2}}\right)
\end{pmatrix}
$$

with $t(\tau) = \tau, x^i = x^i(\sigma)$ slice of space. The variable $\omega$ represents a continuous class of geometries which give rise to a single $A_\tau$. Notice that $\omega$ must be nonzero and larger than the maximum value of $A_\mu$ in the given domain.

Consider a spherical shell of charge $q$ with radius smaller than $q/4\pi\omega$, and let $x(\sigma) = \sigma, y = 0, z = 0$. The only nonzero component of the field tensor, when looking strictly along the $x$-axis, is $F_{tx} = E_x = \frac{q}{4\pi \sigma^2}$. The pullback gives $F_{\tau\tau} = \frac{q}{4\pi \sigma^2}$ and $A_\tau = \frac{q}{4\pi \sigma^2}$. Using eq 37 we find that the surface that is associated with
Figure 7: The geometry of a spherical charge for $q = 4\pi$ and $\omega = 1\text{nm}^{-1}$.

The point charge, looking along the $x$-axis, is

$$X^i = \left( \begin{array}{c} \frac{q}{4\omega\pi\sigma} \sin(\omega\tau) \\ \frac{q}{4\omega\pi\sigma} \cos(\omega\tau) \\ \sqrt{1 - \left( \frac{q}{4\omega\pi\sigma} \right)^2} + \ln\left(1 + \sqrt{1 - \left( \frac{q}{4\omega\pi\sigma} \right)^2}\right) \end{array} \right).$$ (38)

Letting $q = 4\pi$ and $\omega = 1\text{nm}^{-1}$, figure 7a shows the spherical charge from the Grassmannian camp, figure 7b is the hidden-spatial-metric picture, figure 7c shows the hidden-spatial-metric picture built from the tangent planes of the Grassmannian camp, and figure 7d shows it from the Kaluza-Klein camp picture. In SI units this corresponds to the field of a $2.2 \times 10^{-17}$ C charge where $\sigma > 1$ nm. From the reference figure 7a, we see
that increasing $\sigma$ is down toward the tip of the funnel and increasing $\tau$ is on the circular dimension. This makes geometrical sense, as $\sigma$ increases, the radius of the funnel shape gets larger, and the shape gets more cylinder-like. Given that we have potential $A_\tau = \frac{q}{4\pi \sigma}$, as we move farther from the sphere, the weaker the field becomes, leading to a less curved surface.

Finally, note that the value of $\omega [> \text{max}(A_\tau)]$ is arbitrary in this case. The time dependence of $\omega$, which is clearly present in all three gauge-invariant geometrical representations, vanishes in the scalar potential and electric field. Again, the geometric relationships make it clear that there is a hidden gauge-invariant time dependence in the electric field of the charged sphere.

6 Discussion and Conclusion

We have shown how three geometric representations of gauge theory used by physicists in the last century are related geometrically. For the Kaluza-Klein camp every point in space-time has a bundled-up fifth dimension. With a proper emersion we can recover the Kaluza-Klein metric and visualize this 5th dimension. The Grassmannian camp uses vector bundles to describe gauge fields. We can visualize the subspace represented by these disks at each point in space-time using an embedding space inside at each space-time point. Finally, by combining the embedding with a shape unearthed in the hidden-spatial-metric camp, we can associate a spatial geometry with a gauge field configuration.

The similarities are deep. All three camps share a common gauge-invariant tangent plane. The wave function and projection operator are invariants of the gauge transformation corresponding to a rotation of the basis vectors in the Grassmannian camp and hidden-spatial-metric camps, as well as an invariant of $x^5$ coordinate transforms in Kaluza-Klein theory. The tetrads change the coordinate basis vectors of the hidden spatial geometry to orthonormal basis vectors on the gauge fiber, and the Kaluza-Klein ring lies along the gauge fiber. There are also similarities related to charge: in the Grassmannian and hidden-spatial-metric camps a positive or negative charge corresponds to a clockwise or counterclockwise rotation of a matter field vector on the gauge fiber, and in the Kaluza-Klein camp a positive or negative charge corresponds to opposite movement along the ring (which lies along the gauge fiber). The opposite charge corresponds to reverse rotation in all three camps.

Each of the three camps considered in this paper use geometry to isolate some gauge-invariant structure. For example in the charged ring and the charged sphere, we show that even static electric fields have a hidden, gauge-invariant time dependence. Is this time dependence physical or an artifact of the geometric representation? The mappings we provide between three independently derived geometric-representation camps show that this gauge-invariant time dependence exists for all three. This agreement suggests it should be taken seriously. It is as if the constant electric field is like a vector field describing the steady flow of a substance like a fluid. A static electric field is like a constant flow-rate of the fluid, but there is real motion of a substance not described by the vector field. For example the fluid could be a slow moving dense substance or a fast moving light substance, but the vector fields would the same. For the static electric fields we describe here, the gauge-invariant parameter $\omega$ could parameterize a similar ambiguity. If we take the gauge-invariant features uncovered here seriously, there are several questions to pursue in future research. What is the ‘fluid’ moving in the case of gauge theories? Is the hidden time dependence connected to the source field through a deeper form of Gauss’s law? Are there any new observable consequences? This unified geometric representation could lead to a deeper form of Gauss’s law and may help address the mysteries of the mass gap in Yang-Mills theory.

Another area of development lies within the study of instanton and other less-well-known semi-classical solutions. Many of the papers previously published in each camp were geared towards identifying and studying instanton solutions. By using the mappings we identify, checking and categorizing instanton solutions may become easier, and it would be another tool for finding other semi-classical solutions that would need to be included in the path integral quantization.\[4\]

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