Completeness Property of One-Dimensional Perturbations of Normal and Spectral Operators Generated by First Order Systems

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Abstract. The paper is concerned with the completeness property of rank one perturbations of the unperturbed operators generated by special boundary value problems (BVP) for the following $2 \times 2$ system

$$Ly = -iB^{-1}y' + Q(x)y = \lambda y, \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad (0.1)$$

on a finite interval assuming that the potential matrix $Q$ is summable, and $b_1b_2^{-1} \notin \mathbb{R}$ (essentially non-Dirac type case). We assume that the unperturbed operator generated by a BVP belongs to one of the following three subclasses of the class of spectral operators: (a) normal operators; (b) operators similar either to a normal or almost normal; (c) operators that meet Riesz basis property with parentheses; We show that in each of the three cases there exists (non-unique) operator generated by a quasi-periodic BVP and its certain rank-one perturbations (in the resolvent sense) generated by special BVPs which are complete while their adjoint are not. In connection with the case (b) we investigate Riesz basis property of quasi-periodic BVP under certain assumptions on the potential matrix $Q$. We also find a simple formula for the rank of the resolvent difference for operators corresponding to two BVPs for $n \times n$ system in terms of the coefficients of linear boundary forms.

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1. Introduction

1.1. During the last two decades there appeared numerous papers devoted to completeness and Riesz basis properties in $L^2([0,1]; \mathbb{C}^n)$ of boundary value problems (BVP) for general first order system of ODE

\[ Ly := L(B,Q)y := -iB^{-1}y' + Q(x)y = \lambda y, \quad y = \text{col}(y_1, \ldots, y_n). \quad (1.1) \]

Here $B$ is a nonsingular diagonal $n \times n$ matrix with complex entries, $B = \text{diag}(b_1, b_2, \ldots, b_n) \in \mathbb{C}^{n \times n}$, and $Q =: (Q_{jk})_{j,k=1}^n \in L^1([0,1]; \mathbb{C}^{n \times n})$ is the potential matrix.

To obtain a BVP, equation (1.1) is subject to the following boundary conditions (BC)

\[ Cy(0) + Dy(1) = 0, \quad C = (c_{jk}), \quad D = (d_{jk}) \in \mathbb{C}^{n \times n}. \quad (1.2) \]

We always impose the maximality condition $\text{rank}(CD) = n$.

With system (1.1) one associates, in a natural way, the maximal operator $L_{\text{max}} := L_{\text{max}}(B,Q)$ acting in $L^2([0,1]; \mathbb{C}^n)$ on the domain

\[ \text{dom}(L_{\text{max}}) = \{ y \in AC([0,1]; \mathbb{C}^n) : Ly \in L^2([0,1]; \mathbb{C}^n) \}. \]

Clearly, $\text{dom}(L_{\text{max}}) = W^{1,2}([0,1]; \mathbb{C}^n)$ whenever $Q \in L^2([0,1]; \mathbb{C}^{n \times n})$. In this case the minimal operator $L_{\text{min}} := L_{\text{min}}(B,Q)$ is the restriction of $L_{\text{max}}$ to

\[ \text{dom}(L_{\text{min}}) = W^{1,2}_0([0,1]; \mathbb{C}^n) := \{ y \in W^{1,2}([0,1]; \mathbb{C}^n) : y(0) = y(1) = 0 \}. \]

Denote by $L_{C,D} := L_{C,D}(B,Q)$ the operator associated in $L^2([0,1]; \mathbb{C}^n)$ with the BVP (1.1)–(1.2). It is defined as the restriction of $L_{\text{max}}(B,Q)$ to the set of functions satisfying (1.2).

Apparently, the spectral problems (1.1)–(1.2) have first been investigated by Birkhoff and Langer [8]. Namely, they have extended certain previous results due to Birkhoff and Tamarkin on non-selfadjoint BVP for ODE to the case of BVP (1.1)–(1.2). More precisely, they introduced the concepts of regular and strictly regular boundary conditions and investigated the asymptotic behavior of eigenvalues and eigenfunctions of the corresponding
operator $L_{C,D}(B,Q)$ assuming that the potential matrix $Q$ is continuous. Moreover, they proved a pointwise convergence result on spectral decompositions of the operator $L_{C,D}(B,Q)$ corresponding to the BVP (1.1)–(1.2) with regular boundary conditions.

The completeness property of the root vectors system of the general BVP for the equation (1.1) has first been investigated in the recent paper [36]. In this paper the concept of weakly regular boundary conditions (1.2) for the system (1.1) was introduced and the completeness of the root vectors for such type of BVPs was proved (see Theorem 5.3 in Appendix).

In the recent paper [26] the Riesz basis property with parentheses for system (1.1) subject to various classes of boundary conditions with the potential $Q \in L^\infty([0,1];\mathbb{C}^{n \times n})$ was established.

1.2. Turning the attention to the case $n = 2$ we consider the system

$$\mathcal{L}y = -iB^{-1}y' + Q(x)y = \lambda y, \quad y = \text{col}(y_1, y_2), \quad x \in [0,1],$$

with nonsingular matrix $B$ and complex valued potential matrix $Q$,

$$B = \text{diag}(b_1, b_2), \quad \text{and} \quad Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix} \in L^1([0,1];\mathbb{C}^{2 \times 2}).$$

In this case it is more convenient to rewrite conditions (1.2) as

$$U_j(y) := a_{j1}y_1(0) + a_{j2}y_2(0) + a_{j3}y_1(1) + a_{j4}y_2(1) = 0, \quad j \in \{1,2\},$$

where the linear forms $\{U_j\}_{j=1}^2$ are assumed to be linearly independent. We also write $L_U$ instead of $L_{C,D}$.

As opposed to the general problem (1.1)–(1.2), BVP (1.3)–(1.5) with $B = \text{diag}(-1,1)$ (Dirac system) has been investigated in great detail. First we mention that the completeness property of irregular and even degenerate BVP (1.1)–(1.2) was investigated in [36], [23]. Besides, P. Djakov and B. Mityagin [10] imposing certain smoothness condition on $Q$ proved equiconvergence of the spectral decompositions for $2 \times 2$ Dirac equations subject to general regular boundary conditions. Moreover, the Riesz basis property for $2 \times 2$ Dirac operators $L_U$ has been investigated in numerous papers (see [7,9,11,12,19,24,27,32,38–40] and references therein, and discussion in Remark 3.14).

1.3. In this paper considering the case of $n = 2$ we always assume that

$$B = \text{diag}(b_1, b_2) \quad \text{and} \quad b_1b_2^{-1} \notin \mathbb{R}.$$  \hspace{1cm} (1.6)

Here we exploit system (1.3) to indicate explicitly realizations of expression (1.3) that are peculiarly complete in the sense of the following definition.

**Definition 1.1.** (i) An operator $S$ with a discrete spectrum in a Hilbert space $\mathcal{H}$ is called complete if the system of its root vectors is complete in $\mathcal{H}$.

(ii) We call an operator $S$ peculiarly complete if $S$ is complete while the adjoint operator $S^*$ is not and the span of its root vectors has infinite codimension in $\mathcal{H}$. 
We emphasize that our interest in peculiarly complete operators has been influenced by a recent remarkable result by A. Baranov and D. Yakubovich [5, 6], which we reformulate for unbounded operators.

**Theorem 1.2.** [4–6] For any normal operator \( T \in C(\mathcal{H}) \) with simple point spectrum there exists a peculiarly complete operator \( S \) such that the resolvent difference \( (S - \lambda)^{-1} - (T - \lambda)^{-1} \) is one-dimensional.

In fact, this result was proved in [5, 6] only for \( T = T^* \) and was extended to the case of normal operators \( T \) in the recent preprint by Baranov [4].

Note in this connection that the first (highly nontrivial) example of a peculiarly complete operator \( S \) was constructed by Hamburger [20]. Later on Deckard et al. [13] found a simpler construction. However, in these examples the resolvent \( (S - \lambda)^{-1} \) of the corresponding operator \( S \) is an infinite dimensional perturbation of a compact resolvent \( (T - \lambda)^{-1}, T = T^* \). Surprisingly, in accordance with Theorem 1.2 one can find such examples among rank one perturbations.

Inspired by Theorem 1.2 we were seeking for such pairs \( \{T, S\} \) among BVP for systems (1.3). However, we have not found peculiarly complete operators among realizations \( S := L_V(B, Q) \) of formally symmetric Dirac type differential expressions (see discussion in Example 4.19). At the same time it happens that such operators do exist among realizations \( S \) of the expression (1.3) provided that the matrix \( B \) satisfies (1.6). Namely, we show (see Proposition 4.8) that under certain restrictions on a potential matrix \( Q \) the realization \( S \) with boundary conditions

\[
V_1(y) = y_1(0) - h_1y_2(0) = 0, \quad V_2(y) = y_1(1) - h_2y_2(0) = 0, \quad h_1h_2 \neq 0,
\]

(1.7)
is peculiarly complete. Moreover, in the case \( Q = 0 \) for each peculiarly complete realization \( S \) corresponding boundary conditions are equivalent to (1.7) with some \( h_1, h_2 \in \mathbb{C}\setminus\{0\} \).

The second aim of the paper is to investigate existence of “good” unperturbed realizations \( T := L_U(B, Q) \) of the expression (1.3) that satisfy

\[
\text{rank}\left((S - \lambda)^{-1} - (T - \lambda)^{-1}\right) = 1,
\]

(1.8)

for some peculiarly complete realization \( S \). The realization \( T \) is called “good” whenever it belongs to one of the three subclasses of spectral operators: (i) normal operators; (ii) operators similar to almost normal operators, in particular to normal ones; (iii) operators having Riesz basis property with parentheses. It is shown that for the peculiarly complete realization \( S = L_V(B, Q) \) with \( V_1, V_2 \) of the form (1.7) there exists a one parameter family of “good” realizations \( T = L_U(B, Q) \) with

\[
U_1(y) := y_1(0) - d_1y_1(1) = 0, \quad U_2(y) := y_2(0) - d_2y_2(1) = 0, \quad d_1d_2 \neq 0,
\]

(1.9)

\( d_1 = h_1^{-1}h_2 \), and certain \( d_2 \in \mathbb{C}\setminus\{0\} \) and such that condition (1.8) holds. Moreover, we indicate certain conditions on \( Q \) ensuring that \( T \) belongs to one of the above mentioned classes (see Theorems 4.11, 4.14, 4.15).
Theorem 1.2 substantially complements the classical results by Keldysh, Macaev, and others on completeness of weak perturbations of a selfadjoint compact operator (cf. [17, 21, 41]). It follows from the Macaev theorem (see discussion in Remark 4.12) that in Theorem 1.2 the operator \( S \) is always a singular (=non-additive) perturbation of \( T = T^* \), i.e. \( \text{dom} S \neq \text{dom} T \). This observation naturally leads to investigation of BVPs in order to achieve the effect described in Theorem 1.2 by means of changing the boundary conditions.

To describe the area of applicability of Theorem 1.2 to BVPs let us consider the following simple example.

Example 1.3. Let \( T \) be the Dirichlet realization of \(-d^2/dx^2\) in \( L^2[0,1] \), i.e. \[
\text{dom} T = \text{dom} D^2_0 = \{ f \in W^{2,2}[0,1] : f(0) = f(1) = 0 \},
\]
(1.10)
One could not reach the effect described in Theorem 1.2 by means of changing boundary conditions. Indeed, the BVP for \(-d^2/dx^2\) is complete in \( L^2[0,1] \) if and only if the BC are non-degenerate (see [29, Theorem 1.3.1]). Therefore, the corresponding operators \( S \) and \( S^* \) are complete only simultaneously.

A similar effect for Dirac operator with \( Q \in L^1 \) is discussed in Example 4.19.

To treat these examples in the general framework of BVPs we first recall the definition of a dual pair of operators and its proper extensions.

Definition 1.4. (i) A pair \( \{ A_1, A_2 \} \) of closed densely defined operators in \( H \) is called a dual pair of operators if \( A_1 \subset A_2^* \) \( (\Leftrightarrow A_2 \subset A_1^* \)).

(ii) An operator \( T \) is called a proper extension of the dual pair \( \{ A_1, A_2 \} \) and is put in the class \( \text{Ext}\{A_1, A_2\} \) if \( A_1 \subset T \subset A_2^* \).

In connection with Theorem 1.2 the following problem naturally arises.

Problem 1. Given a dual pair of operators \( \{ A_1, A_2 \} \) find all pairs of proper extensions \( T, S \in \text{Ext}\{A_1, A_2\} \) such that \( T \) is normal, \( S \) is peculiarly complete, and the resolvent difference \((S - \lambda)^{-1} - (T - \lambda)^{-1}\) is one-dimensional.

Note that in comparison with the assumptions of Theorem 1.2 we restrict the class of perturbations \( S \) by the class \( \text{Ext}\{A_1, A_2\} \) assuming that it contains a normal extension \( T \). Example 1.3 demonstrates significance of this restriction. Namely, Problem 1 has no solution for the dual pair \( \{ A, A \} \), where \( A = D^2_{\text{min}}, \text{dom} D^2_{\text{min}} = W^{2,2}_0[0,1] \), is the minimal symmetric operator generated in \( L^2[0,1] \) by the expression \(-d^2/dx^2\). At the same time, in accordance with Theorem 1.2 the Dirichlet extension \( T = D^2_0 = T^* \) of \( A \), where \( D^2_0 \) is given by (1.10), has rank one peculiarly complete perturbation \( S \), which necessarily is not a proper extension of \( A \). On the other hand, one of our main results, Theorem 4.11, shows that Problem 1 has an affirmative solution for the dual pair \( \{ L_{\text{min}}(B,0), L_{\text{min}}(B^*,0) \} \) with \( B \) satisfying (1.6).

We also consider two weaker versions of Problem 1. Namely, in Problem 2 the normality of the operator \( T \) is replaced with the similarity to a normal or almost normal operator, and in Problem 3 it is replaced with the Riesz basis property with parentheses. Note in this connection that by Wermer...
theorem (see [14, Theorem XV.6.4]) a spectral operator is of scalar type if and only if it is similar to a normal operator.

The paper is organized as follows. In Sect. 2 we find explicit formula for the rank of the resolvent difference of arbitrary operators $L_{C, D}(B, Q)$ and $L_{\tilde{C}, \tilde{D}}(B, Q)$ in general $n \times n$ case. Namely, we show that it is equal to

$$\text{rank} \left( \begin{array}{cc} C & D \\ \tilde{C} & \tilde{D} \end{array} \right) - n.$$ 

We also refine this formula in the case of $n = 2$ and boundary conditions (1.7) for one of the operators.

In Sect. 3 we establish the Riesz basis property of realizations $L_{U}(B, Q)$ with $B$ satisfying (1.6), boundary conditions (1.9), and imposing certain assumptions on $Q$. In particular, we indicate conditions on $Q$ ensuring similarity of the realization $L_{U}(B, Q)$ either to a normal or to an almost normal operator. In Proposition 3.8 we also prove the asymptotic behavior of the eigenvalues of the operator $L_{U}(B, Q)$.

In Sect. 4 we prove our main results answering Problems 1–3. In particular, we prove Theorem 4.11 that gives a complete solution to Problem 1. Namely, we describe all pairs $\{T, S\}$ of proper extensions of the dual pair $\{L_{\text{min}}(B, Q), L_{\text{min}}(B^*, Q^*)\}$ with normal $T = L_{U}(B, Q)$ and realization $S := L_{V}(B, Q)$ being peculiarly complete. The proof is rather involved and relies on two ingredients: description of all normal realizations of (1.3), and description of all peculiarly complete realizations. In particular, it is shown that normal realizations of $L$ exist only for special constant $Q$, hence an affirmative solution to Problem 1 is exceptional in character.

In Theorems 4.14 and 4.15 we show that in contrast with Problem 1, both Problems 2 and 3 have an affirmative solution for a wide class of the potential matrices $Q$. To this end we use the main results of Sect. 3 as well as our results from [25] on Riesz basis property of realizations $T = L_{U}(B, Q)$. Moreover, we present simple explicit examples (see Example 4.16) of pairs $\{A_0, A\}$ of compact operators with normal $A_0$ and $A$ being one dimensional peculiarly complete perturbation of $A_0$. Namely, setting $2J_R := J + J^*$ and $A_0 = b_1J_R \oplus b_2J_R$, where $J : f \to i \int_0^x f(t)dt$ is the integration operator, i.e.

$$A_0 f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} ib_1 \int_0^x f_1(t)dt \\ ib_2 \int_0^x f_2(t)dt \end{pmatrix} = \frac{1}{2} \begin{pmatrix} ib_1 \int_0^1 f_1(t)dt \\ ib_2 \int_0^1 f_2(t)dt \end{pmatrix},$$

(1.11)

and setting

$$A = A_0 + K, \quad \text{with} \quad K f = \frac{1}{2} \begin{pmatrix} 0 \\ ib_1 \int_0^1 f_1(t)dt + ib_2 \int_0^1 f_2(t)dt \end{pmatrix},$$

(1.12)

we obtain a desired pair of operators $\{A_0, A\}$ acting on $L^2[0, 1] \oplus L^2[0, 1]$. It is worth mentioning that the peculiarly complete operator $A$ is one dimensional perturbation of the Volterra operator $b_1J \oplus b_2J$. These examples complement the above mentioned examples from [20] and [13] and are much simpler.

A preliminary version of the paper was published as a preprint [2].
Notation. Let \( \mathcal{C}(T) \) be the set of closed densely defined operators in a Hilbert space \( \mathcal{H} \); for any \( T \in \mathcal{C}(T) \) denote by \( \sigma(T) \) and \( \rho(T) = \mathbb{C} \setminus \sigma(T) \) its spectrum and resolvent set, respectively.

2. Resolvent Difference Properties of the Operators \( L_{C,D}(B,Q) \)

2.1. Formula for the Rank of the Resolvent Difference

In this subsection we consider operators \( L_{C,D} := L_{C,D}(B,Q) \) associated with BPV (1.1)–(1.2) in the general \( n \times n \) case. We will find an explicit formula for the rank of the resolvent difference of any two such operators. Recall that for a bounded operator \( A \) acting in a Hilbert space \( \mathcal{H} \) its rank is the dimension of its range, \( \text{rank} A := \dim(\text{ran} A) \).

Let \( \lambda \in \mathbb{C} \) and \( \Phi(\cdot, \lambda) \in AC([0,1]; \mathbb{C}^{n \times n}) \) be the fundamental matrix of the system (1.1), i.e.

\[
-ib^{-1}\Phi'(x, \lambda) + Q(x)\Phi(x, \lambda) = \lambda \Phi(x, \lambda), \quad \text{for a.e. } x \in [0,1], \quad \Phi(0, \lambda) = I_n.
\]

(2.1)

It is well-known that \( \Phi(x, \lambda) \) is nonsingular for all \( x \in [0,1] \) and thus, \( \Phi^{-1}(\cdot, \lambda) \in AC([0,1]; \mathbb{C}^{n \times n}) \).

In what follows we denote by \( R_{C,D}(\lambda) := (L_{C,D} - \lambda)^{-1} \) the resolvent of the operator \( L_{C,D} \) associated to the BVP (1.1)–(1.2). First we recall the simple lemma from [25].

**Lemma 2.1.** [25, Corollary 4.2] Let \( \lambda \in \rho(L_{C,D}) \). Then

\[
(R_{C,D}(\lambda)f)(x) = (K_{C,D}(\lambda)f)(x) - \Phi(x, \lambda)M_{C,D}(\lambda)(K_{C,D}(\lambda)f)(1),
\]

(2.2)

where

\[
M_{C,D}(\lambda) := (C + D\Phi(1, \lambda))^{-1}D,
\]

(2.3)

\[
(K_{C,D}(\lambda)f)(x) := \Phi(x, \lambda)\int_{0}^{x} \Phi^{-1}(t, \lambda)ibf(t)dt.
\]

(2.4)

Alongside the operator \( L_{C,D} \) we consider the operator \( L_{\tilde{C},\tilde{D}} := L_{\tilde{C},\tilde{D}}(B,Q) \) associated to equation (1.1) subject to the boundary conditions

\[
\tilde{C}y(0) + \tilde{D}y(1) = 0, \quad \tilde{C}, \tilde{D} \in \mathbb{C}^{n \times n}, \quad \text{rank}(\tilde{C} \quad \tilde{D}) = n.
\]

(2.5)

The following formula for the rank of the resolvent difference is immediately implied by Lemma 2.1.

**Lemma 2.2.** Let \( \lambda \in \rho(L_{C,D}) \cap \rho(L_{\tilde{C},\tilde{D}}) \). Then

\[
\text{rank}(R_{\tilde{C},\tilde{D}}(\lambda) - R_{C,D}(\lambda)) = \text{rank} \hat{M}(\lambda),
\]

(2.6)

where

\[
\hat{M}(\lambda) := M_{C,D}(\lambda) - M_{\tilde{C},\tilde{D}}(\lambda).
\]

(2.7)
Moreover, if the common rank in (2.6) is equal to 1 then \( \hat{\mathcal{M}}(\lambda) \) admits representation
\[
\hat{\mathcal{M}}(\lambda) = \alpha(\lambda) \cdot \beta(\lambda)^* = \left( \alpha_j(\lambda) \bar{\beta}_k(\lambda) \right)_{j,k=1}^n,
\]
(2.8)
\[
\alpha(\lambda) = \text{col}(\alpha_1(\lambda), \ldots, \alpha_n(\lambda)), \quad \beta(\lambda) = \text{col}(\beta_1(\lambda), \ldots, \beta_n(\lambda)),
\]
(2.9)
and for any \( f \in L^2([0,1]; \mathbb{C}^n) \) we have
\[
(R_{\tilde{C},\tilde{D}}(\lambda) - R_{C,D}(\lambda))f = (f, \Psi^*(\cdot, \lambda)\beta(\lambda))_{L^2([0,1]; \mathbb{C}^n)} \cdot \Phi(\cdot, \lambda)\alpha(\lambda),
\]
(2.10)
where
\[
\Psi(\cdot, \lambda) := i\Phi(1, \lambda)\Phi^{-1}(\cdot, \lambda)B.
\]
(2.11)
Here dot between multipliers denotes the standard matrix multiplication, e.g. columns \( \alpha(\lambda) \) and \( \beta(\lambda) \) are considered as matrices of size \( n \times 1 \).

Proof. (i) It follows from Lemma 2.1 (formula 2.2) that
\[
(R_{\tilde{C},\tilde{D}}(\lambda) - R_{C,D}(\lambda)f)(x) = \Phi(x, \lambda)(M_{C,D}(\lambda) - M_{\tilde{C},\tilde{D}}(\lambda))[(K_\lambda f)(1)],
\]
(2.12)
for any \( f \in \mathcal{H} := L^2([0,1]; \mathbb{C}^n) \). It easily follows from definition of \( K_\lambda \) (formula (2.4)) that
\[
\{(K_\lambda f)(1) : f \in \mathcal{H}\} = \mathbb{C}^n.
\]
(2.13)
Namely, for \( u \in \mathbb{C}^n, (K_\lambda f)(1) = u \), if we set \( f(x) = \Psi^{-1}(x, \lambda)u \). Since \( \Phi(\cdot, \lambda), \Phi^{-1}(\cdot, \lambda) \in AC([0,1]; \mathbb{C}^{n \times n}) \), formula (2.6) immediately follows from (2.12), (2.13) and (2.7).

(ii) If the common rank in (2.6) is equal to 1 then \( \hat{\mathcal{M}}(\lambda) \) has rank 1 and thus admits the representation (2.8). It follows now from (2.12) and definition of \( K_\lambda \) and \( \Psi(\cdot, \lambda) \) (formulas (2.4) and (2.11)) that for any \( f \in \mathcal{H} \)
\[
(R_{\tilde{C},\tilde{D}}(\lambda) - R_{C,D}(\lambda))f = \Phi(\cdot, \lambda)\alpha(\lambda) \cdot \beta(\lambda)^* \cdot \int_0^1 \Psi(t, \lambda)f(t)dt
\]
\[
= \Phi(\cdot, \lambda)\alpha(\lambda) \cdot \int_0^1 \langle f(t), \Psi^*(t, \lambda)\beta(\lambda) \rangle_{\mathbb{C}^n} dt
\]
\[
= (f, \Psi^*(\cdot, \lambda)\beta(\lambda))_{\mathcal{H}} \cdot \Phi(\cdot, \lambda)\alpha(\lambda),
\]
which finishes the proof. \( \square \)

The following result gives an explicit formula for the rank of the resolvent difference of operators \( L_{\tilde{C},\tilde{D}} \) and \( L_{C,D} \) in terms of matrices \( C, D, \tilde{C}, \tilde{D} \).

Proposition 2.3. Let \( \lambda \in \rho(L_{C,D}) \cap \rho(L_{\tilde{C},\tilde{D}}) \). Then
\[
\text{rank}(R_{\tilde{C},\tilde{D}}(\lambda) - R_{C,D}(\lambda)) = \text{rank}\left( \begin{array}{cc} C & D \\ \tilde{C} & \tilde{D} \end{array} \right) - n.
\]
(2.14)
Proof. Let us set \( A := A(\lambda) := C + D\Phi(1, \lambda) \) and \( \tilde{A} := \tilde{A}(\lambda) := \tilde{C} + \tilde{D}\Phi(1, \lambda) \). Note that matrices \( A \) and \( \tilde{A} \) are nonsingular since \( \lambda \in \rho(L_{C,D}) \cap \rho(L_{\tilde{C},\tilde{D}}) \). Taking this into account we get
\[
\begin{align*}
\text{rank} \left( \begin{pmatrix} C & D \\ \tilde{C} & \tilde{D} \end{pmatrix} \right) &= \text{rank} \left( \begin{pmatrix} C + D\Phi(1, \lambda) & D \\ \tilde{C} + \tilde{D}\Phi(1, \lambda) & \tilde{D} \end{pmatrix} \right) = \text{rank} \left( \begin{pmatrix} A & D \\ \tilde{A} & \tilde{D} \end{pmatrix} \right) \\
&= \text{rank} \left( \begin{pmatrix} A & 0 \\ 0 & \tilde{A} \end{pmatrix} \right) \begin{pmatrix} I_n & A^{-1}D \\ I_n & \tilde{A}^{-1}\tilde{D} \end{pmatrix} \right) = \text{rank} \left( I_n \begin{pmatrix} A^{-1}D \\ \tilde{A}^{-1}\tilde{D} \end{pmatrix} \right) \\
&= \text{rank} \left( 0 \begin{pmatrix} A^{-1}D - \tilde{A}^{-1}\tilde{D} \end{pmatrix} \right) = n + \text{rank} \left( A^{-1}D - \tilde{A}^{-1}\tilde{D} \right) \\
&= n + \text{rank} \left( (C + D\Phi(1, \lambda))^{-1}D - (\tilde{C} + \tilde{D}\Phi(1, \lambda))^{-1}\tilde{D} \right) \\
&= n + \text{rank} \left( M_{C,D}(\lambda) - M_{\tilde{C},\tilde{D}}(\lambda) \right) = n + \text{rank} \tilde{M}(\lambda). \tag{2.15}
\end{align*}
\]

Formula (2.14) now follows from (2.6) and (2.15). \(\square\)

**Remark 2.4.** Here we shortly mention that formula (2.2) can be treated as a special case of Krein’s type formula for resolvents for dual pairs. The latter is proved in [35] in general framework of extension theory of dual pairs and reads as follows

\[
(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma_\Pi(\lambda)(\Theta - M(\lambda))^{-1}\gamma_\Pi^* (\lambda), \quad \lambda \in \rho(A_\Theta) \cap \rho(A_0). \tag{2.16}
\]

Here \(A_0\) is a reference extension of a dual pair, \(A_\Theta\) is arbitrary proper extension with non-empty resolvent set parameterized by a linear relation \(\Theta\), and \(M(\cdot)\) and \(\gamma_\Pi(\cdot)\) are the corresponding Weyl function and \(\gamma_\Pi\)-field, respectively.

Considering the dual pair \(\{L_{\min}(B,Q), L_{\min}(B^*,Q^*)\}\) and reference extension \(L_{I_{n,0}}\) given by (2.5) (with \(C = I_n\) and \(D = 0\)), and applying formula (2.16) to extensions \(L_{C,D}\) and \(L_{I_{n,0}}\) and computing the corresponding Weyl function and \(\gamma_\Pi\)-field we arrive at (2.2). Note however, that the direct computations are simpler.

Note also that formula (2.14) with the same proof remains valid for two proper extensions of a dual pair with equal finite “deficiency indices”.

### 2.2. Resolvent Difference Properties for 2 × 2 System

Let \(\Phi(x, \lambda)\) be the fundamental matrix of the system (1.3) defined in the previous subsection and

\[
\Phi(x, \lambda) := \begin{pmatrix} \Phi_1(x, \lambda) & \Phi_2(x, \lambda) \end{pmatrix}, \quad \Phi_j(x, \lambda) := \begin{pmatrix} \varphi_{1j}(x, \lambda) \\ \varphi_{2j}(x, \lambda) \end{pmatrix}, \quad j \in \{1, 2\}. \tag{2.17}
\]

The eigenvalues of the problem (1.3)–(1.5) are the roots of the characteristic equation \(\Delta(\lambda) := \Delta_U(\lambda) := \det \Phi_U(\lambda) = 0\), where

\[
\Phi_U(\lambda) := \begin{pmatrix}
U_1(\Phi_1(x, \lambda)) & U_1(\Phi_2(x, \lambda)) \\
U_2(\Phi_1(x, \lambda)) & U_2(\Phi_2(x, \lambda))
\end{pmatrix}. \tag{2.18}
\]

Further, let us set for \(j, k \in \{1, \ldots, 4\}\),

\[
A_{jk} := A_{jk}^U := \begin{pmatrix} a_{1j} & a_{1k} \\ a_{2j} & a_{2k} \end{pmatrix} \quad \text{and} \quad J_{jk} := J_{jk}^U = \det A_{jk}^U. \tag{2.19}
\]
Note, that boundary conditions (1.5) takes the form (1.2) if we set $C := A_{12}$ and $D := A_{34}$. In particular, $\Phi_U(\lambda) = C + D\Phi(1, \lambda)$.

Taking into account notations (2.19) we arrive at the following expression for the characteristic determinant

$$\Delta(\lambda) = J_{12} + J_{34} e^{i(b_1 + b_2)\lambda} + J_{32}\varphi_{11}(\lambda) + J_{13}\varphi_{12}(\lambda) + J_{42}\varphi_{21}(\lambda) + J_{14}\varphi_{22}(\lambda),$$

(2.20)

where $\varphi_{jk}(\lambda) := \varphi_{jk}(1, \lambda)$ (see [26, 36]). If $Q = 0$ then $\varphi_{12}(x, \lambda) = \varphi_{21}(x, \lambda) = 0$, and the characteristic determinant $\Delta_0(\cdot) := \Delta_{0,U}(\cdot)$ has the form

$$\Delta_0(\lambda) = J_{12} + J_{34} e^{i(b_1 + b_2)\lambda} + J_{32} e^{ib_1\lambda} + J_{14} e^{ib_2\lambda}.$$  

(2.21)

In what follows we denote by $R_U(\lambda) := (L_U - \lambda)^{-1}$ the resolvent of the operator $L_U$ associated to the BVP (1.3)–(1.5). Straightforward calculations lead to explicit formula for the matrix function $M_U(\lambda) := M_{C,D}(\lambda)$ given by (2.3), via determinants $J_{jk}$ from (2.19).

**Lemma 2.5.** Let $\lambda \in \rho(L_U)$. Then $\Delta(\lambda) \neq 0$, $M_U(\lambda)$ is well defined and admits the following representation

$$M_U(\lambda) = \frac{1}{\Delta(\lambda)} \begin{pmatrix} J_{32} + J_{34}\varphi_{22}(\lambda) & J_{42} - J_{34}\varphi_{12}(\lambda) \\ J_{13} - J_{34}\varphi_{21}(\lambda) & J_{14} + J_{34}\varphi_{11}(\lambda) \end{pmatrix}.$$  

(2.22)

Moreover,

$$\det M_U(\lambda) = \frac{\det D}{\det(C + D\Phi(1, \lambda))} = \frac{J_{34}}{\Delta(\lambda)},$$  

(2.23)

where $C = A_{12}$ and $D = A_{34}$.

**Proof.** According to definition (2.18) $\Delta(\lambda) := \det \Phi_U(\lambda) = \det(C + D\Phi(1, \lambda))$ and

$$C + D\Phi(1, \lambda) = \begin{pmatrix} a_{11} + a_{13}\varphi_{11}(\lambda) + a_{14}\varphi_{21}(\lambda) & a_{12} + a_{13}\varphi_{12}(\lambda) + a_{14}\varphi_{22}(\lambda) \\ a_{21} + a_{23}\varphi_{11}(\lambda) + a_{24}\varphi_{21}(\lambda) & a_{22} + a_{23}\varphi_{12}(\lambda) + a_{24}\varphi_{22}(\lambda) \end{pmatrix}.$$  

(2.24)

Hence the following formula for the inverse matrix holds

$$(C + D\Phi(1, \lambda))^{-1} = \frac{1}{\Delta(\lambda)} \begin{pmatrix} a_{22} + a_{23}\varphi_{12}(\lambda) + a_{24}\varphi_{22}(\lambda) & -(a_{12} + a_{13}\varphi_{12}(\lambda) + a_{14}\varphi_{22}(\lambda)) \\ -(a_{21} + a_{23}\varphi_{11}(\lambda) + a_{24}\varphi_{21}(\lambda)) & a_{11} + a_{13}\varphi_{11}(\lambda) + a_{14}\varphi_{21}(\lambda) \end{pmatrix}.$$  

(2.25)

Multiplying (2.25) by $D$ from the right we arrive at formula (2.22). E.g. for the first entry we have

$$\Delta(\lambda)[M_U(\lambda)]_{11} = (a_{22} + a_{23}\varphi_{12}(\lambda) + a_{24}\varphi_{22}(\lambda))a_{13} - (a_{12} + a_{13}\varphi_{12}(\lambda) + a_{14}\varphi_{22}(\lambda))a_{23}$$

$$= (a_{22}a_{13} - a_{12}a_{23}) + (a_{23}a_{13} - a_{13}a_{23})\varphi_{12}(\lambda) + (a_{24}a_{13} - a_{14}a_{23})\varphi_{22}(\lambda) = J_{32} + J_{34}\varphi_{22}(\lambda).$$  

(2.26)
The remaining equalities in (2.22) are verified similarly. \( \square \)

Alongside the operator \( L_U \) we consider the operator \( L_V := L_V(B, Q) \) associated to equation (1.3) subject to the boundary conditions

\[
V_j(y) := a_{j1}^V y_1(0) + a_{j2}^V y_2(0) + a_{j3}^V y_1(1) + a_{j4}^V y_2(1) = 0, \quad j \in \{1, 2\}. \quad (2.27)
\]

Similarly to (2.19) we define matrices \( A_{jk}^V \) and the corresponding determinants \( J_{jk} \), \( j, k \in \{1, \ldots, 4\} \), with \( a_{jk}^V \) in place of \( a_{jk} \). This leads to the formula for the characteristic determinant \( \Delta_V(\cdot) \) of the operator \( L_V \) similar to (2.20).

Note that \( L_V = L_{\tilde{C}, \tilde{D}} \) with \( \tilde{C} := A_{12}^V \) and \( \tilde{D} := A_{34}^V \). The following result immediately follows from Proposition 2.3.

**Corollary 2.6.** Let \( L_U \neq L_V \) and \( \lambda \in \rho(L_U) \cap \rho(L_V) \). Then the resolvent difference \( R_V(\lambda) - R_U(\lambda) \) is one-dimensional if and only if

\[
\det \begin{pmatrix} A_{12}^U & A_{34}^U \\ A_{12}^V & A_{34}^V \end{pmatrix} = 0, \quad (2.28)
\]

which in turn is equivalent to

\[
J_{12}^V J_{34}^V + J_{13}^V J_{42}^V + J_{14}^V J_{23}^V + J_{12}^U J_{34}^U + J_{13}^U J_{42}^U + J_{14}^U J_{23}^U = 0. \quad (2.29)
\]

**Proof.** Since \( \operatorname{rank}(A_{12}^U \ A_{34}^U) = \operatorname{rank}(A_{12}^V \ A_{34}^V) = 2 \) and \( L_U \neq L_V \) it follows that

\[
r := \operatorname{rank} \begin{pmatrix} A_{12}^U & A_{34}^U \\ A_{12}^V & A_{34}^V \end{pmatrix} \in \{3, 4\}. \quad (2.30)
\]

Hence, \( r = 3 \) if and only if condition (2.28) holds. In turn, \( r = 3 \) is equivalent to the fact that the resolvent difference \( R_V(\lambda) - R_U(\lambda) \) is one-dimensional due to Proposition 2.3.

Finally, applying Laplace expansion by the first 2 rows to the determinant in (2.28) and taking into account definition of \( J_{jk}^U \) and \( J_{jk}^V \) we get equivalence of (2.28) and (2.29). \( \square \)

### 2.3. Special Boundary Conditions

Next we consider system (1.3),

\[
\mathcal{L} y = -iB^{-1}y' + Q(x)y = \lambda y, \quad y = \col(y_1, y_2), \quad x \in [0, 1]. \quad (2.31)
\]

with \( Q \) given by (1.4), subject to the special boundary conditions

\[
V_1(y) := y_1(0) - h_1 y_2(0) = 0, \quad V_2(y) = y_1(1) - h_2 y_2(0) = 0, \quad h_1 h_2 \neq 0. \quad (2.32)
\]

Denote by \( L_V = L_V(B, Q) \) the operator associated to the problem (2.31)–(2.32) in \( \mathfrak{H} = L^2([0, 1]; \mathbb{C}^2) \).

In the following proposition we indicate simple algebraic condition on coefficients of general problem (1.3)–(1.5) ensuring that the resolvent difference of operators \( L_U \) and \( L_V \) is one-dimensional. Moreover, we give explicit form of this resolvent difference.

**Proposition 2.7.** Let \( L_V \neq L_U \) and \( \lambda \in \rho(L_U) \cap \rho(L_V) \).
(i) Then the resolvent difference $R_V(\lambda) - R_U(\lambda)$ is one-dimensional if and only if

$$J_{34}^U h_2 + J_{14}^U h_1 = J_{42}^U.$$  \tag{2.33}

(ii) Let condition (2.33) be fulfilled and in addition

$$\gamma(\lambda) := J_{14}^U + J_{34}^U \varphi_{11}(\lambda) \neq 0,$$  \tag{2.34}

then the resolvent difference $R_V(\lambda) - R_U(\lambda)$ admits representation (2.10) with the vector-functions $\alpha =: \text{col}(\alpha_1, \alpha_2)$ and $\beta =: \text{col}(\beta_1, \beta_2)$ given by

$$\alpha_1(\lambda) = h_1 - \frac{J_{34}^U \Delta_V(\lambda)}{\gamma(\lambda)}, \quad \alpha_2(\lambda) = 1, \quad \beta_1(\lambda) = \frac{J_{13}^U - J_{34}^U \varphi_{21}(\lambda)}{\Delta_U(\lambda)} - \frac{1}{\Delta_V(\lambda)}, \quad \beta_2(\lambda) = \frac{\gamma(\lambda)}{\Delta_U(\lambda)}. \tag{2.35}$$

\textbf{Proof.} (i) It follows from (2.32) that $A_{12}^V = \begin{pmatrix} 1 & -h_1 \\ 0 & -h_2 \end{pmatrix}$ and $A_{34}^V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Hence by definition of $J_{jk}^V$ we have

$$J_{12}^V = -h_2, \quad J_{13}^V = 1, \quad J_{14}^V = h_1, \quad J_{42}^V = J_{34}^V = 0. \tag{2.37}$$

Thus condition (2.29) transforms into (2.33). Corollary 2.6 now finishes the proof of part (i).

(ii) Due to (i) and Lemma 2.2 condition (2.33) yields that rank $\hat{M}(\lambda) = 1$. Hence $\hat{M}(\lambda)$ admits representation (2.8) which for $n = 2$ turns into

$$\hat{M}(\lambda) = \begin{pmatrix} \alpha_1(\lambda) \beta_1(\lambda) \\ \alpha_2(\lambda) \beta_1(\lambda) \end{pmatrix} = \frac{\alpha_1(\lambda) \beta_2(\lambda)}{\alpha_2(\lambda) \beta_2(\lambda)}. \tag{2.38}$$

Let us verify formulas (2.35)–(2.36) for $\alpha_1(\lambda), \alpha_2(\lambda), \beta_1(\lambda), \beta_2(\lambda)$. It follows from (2.20), (2.22) and (2.37) that

$$\Delta_V(\lambda) = -h_2 + h_1 \varphi_{11}(\lambda) + \varphi_{12}(\lambda), \quad M_V(\lambda) = \frac{1}{\Delta_V(\lambda)} \begin{pmatrix} h_1 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.39}$$

Put

$$M_U(\lambda) =: \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \tag{2.40}$$

where for convenience we omitted dependency on $\lambda$. It follows from (2.39) and (2.40) that

$$\hat{M}(\lambda) = \begin{pmatrix} m_{11} - \frac{h_1}{\Delta_V(\lambda)} & m_{12} \\ m_{21} - \frac{1}{\Delta_V(\lambda)} & m_{22} \end{pmatrix} =: \begin{pmatrix} \hat{m}_{11} & \hat{m}_{12} \\ \hat{m}_{21} & \hat{m}_{22} \end{pmatrix}. \tag{2.41}$$

Since $\gamma(\lambda) \neq 0$, it follows from (2.22) that $\hat{m}_{22} = \gamma(\lambda) \neq 0$. The equality $\det \hat{M}(\lambda) = 0$ now yields that the representation (2.38) takes place with

$$\alpha_1(\lambda) = \frac{\hat{m}_{12}}{\hat{m}_{22}}, \quad \alpha_2(\lambda) = 1, \quad \beta_1(\lambda) = \hat{m}_{21}, \quad \beta_2(\lambda) = \hat{m}_{22}. \tag{2.42}$$
It follows from (2.41), (2.22) and definition (2.34) of $\gamma(\lambda)$ that
\[
\hat{m}_{12} = \frac{J_{12}^U - J_{34}^U \varphi_{12}(\lambda)}{\Delta_U(\lambda)}, \quad \hat{m}_{21} = \frac{J_{13}^U - J_{34}^U \varphi_{21}(\lambda)}{\Delta_U(\lambda)} - \frac{1}{\Delta_V(\lambda)}, \quad \hat{m}_{22} = \frac{\gamma(\lambda)}{\Delta_U(\lambda)}.
\]
Formula (2.36) now immediately follows from (2.42) and (2.43). For $\alpha_1(\lambda)$ we derive from (2.43), (2.33), (2.39) and (2.34)
\[
\alpha_1(\lambda) = \frac{\hat{m}_{12}}{\hat{m}_{22}} = \frac{J_{12}^U - J_{34}^U \varphi_{12}(\lambda)}{\gamma(\lambda)} = \frac{J_{14}^U h_1 + J_{34}^U (h_2 - \varphi_{12}(\lambda))}{\gamma(\lambda)} = \frac{J_{14}^U h_1 + J_{34}^U (h_1 \varphi_{11}(\lambda) - \Delta_V(\lambda))}{\gamma(\lambda)} = h_1 - \frac{J_{34}^U \Delta_V(\lambda)}{\gamma(\lambda)}.
\]
This completes the proof. \(\Box\)

Next we show that for almost each BVP (1.3)–(1.5), there exist BVP (2.31)–(2.32) such that the corresponding resolvent difference is one-dimensional.

**Corollary 2.8.** Let $L_U$ be an operator associated to the problem (1.3)–(1.5) and let $J_{jk}^U$ be given by (2.19). Assume that among numbers $\{J_{14}^U, J_{42}^U, J_{34}^U\}$ either at least two are non-zero or all zero. Then there exists a pair $\{h_1, h_2\}$ with $h_1 h_2 \neq 0$ such that the resolvent difference $R_V(\lambda) - R_U(\lambda)$ is one-dimensional.

In particular, the latter holds for any regular boundary conditions $U_1, U_2$ (see Definition 5.1).

**Proof.** By Proposition 2.7 it suffices to choose $h_1, h_2 \neq 0$ satisfying (2.33). If all $J_{14}^U, J_{42}^U, J_{34}^U$ are zero any pair $\{h_1, h_2\}$ with $h_1 h_2 \neq 0$ is suitable. If at least two of these numbers are non-zero, then existence of required numbers $\{h_1, h_2\}$ is immediate from (2.33).

Now assume boundary conditions to be regular. In both cases $b_1/b_2 \in \mathbb{R}$ and $b_1/b_2 \notin \mathbb{R}$ it implies that $J_{14}^U J_{32}^U \neq 0$ (see Appendix). In this case $J_{34}^U$ and $J_{42}^U$ cannot equal zero simultaneously, since otherwise $J_{32}^U$ would be zero. And thus, among numbers $J_{14}^U, J_{42}^U, J_{34}^U$ at least two are non-zero. \(\Box\)

3. Riesz Basis Property for $2 \times 2$ System

In this section we study the Riesz basis property for the system of root vectors of the operator $L_U(B, Q)$ associated with the BVP (1.3)–(1.5) under the assumption (1.6) and additional smoothness assumption that $Q_{12}$ and $Q_{21}$ admit an analytic continuation to the disk $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$ for some sufficiently large $R$. We denote it as $Q \in A(\mathbb{D}_R; C^{2 \times 2})$.

3.1. Transformation Operators and Characteristic Determinant

First, we recall a special case of Theorem 3.2 from [33] on existence of a triangular transformation operator for a general system (1.1) with the analytical potential matrix $Q$. We set
\[
\|Q\| := \|Q\|_{C[0,1]} := \max\{\|Q_{12}\|_{C[0,1]}, \|Q_{21}\|_{C[0,1]}\}.
\]
Proposition 3.1. [33] Let $Q \in A(\mathbb{D}_R; \mathbb{C}^{2 \times 2})$. Assume that $e_{\pm}(\cdot, \lambda)$ are the solutions of the Cauchy problem for system (1.3)–(1.5) satisfying the initial conditions $e_{\pm}(0, \lambda) = \left(\begin{array}{c} 1 \\ \pm 1 \end{array}\right)$. Then they admit the following triangular representations

$$e_{\pm}(x, \lambda) = e_{\pm}^0(x, \lambda) + \int_0^x K_{\pm}(x, t)e_{\pm}^0(t, \lambda)dt,$$

where

$$e_{\pm}^0(x, \lambda) := \begin{pmatrix} e^{ib_1 \lambda x} \\ \pm e^{ib_2 \lambda x} \end{pmatrix}, \quad K_{\pm} = (K_{jk}^\pm)_{j,k=1}^2 \in C^\infty(\Omega; \mathbb{C}^{2 \times 2}),$$

and $\Omega := \{(x, t) : 0 \leq t \leq x \leq 1\}$. Moreover, the following estimates hold

$$\|K_{jk}^\pm\|_{C(\Omega)} \leq C_0 \|Q\| \cdot \exp(C_1 \|Q\|), \quad j, k \in \{1, 2\},$$

with some constants $C_0, C_1 > 0$.

Note that estimates (3.4) are easily extracted from the proof of [33, Theorem 3.2].

Let $\Phi(\cdot, \lambda)$ be the fundamental matrix solution of the system (1.3) given by (2.1) and let $\Phi_j(\cdot, \lambda), \varphi_{jk}(\cdot, \lambda), j, k \in \{1, 2\}$, be given by (2.17).

In the sequel we follow the scheme proposed in [27] for investigating the Riesz basis property of BVP for Dirac type system $(B = B^*)$ with a summable potential matrix $Q$. The following result is similar to that of Proposition 3.1 from [27].

Lemma 3.2. Let $Q \in A(\mathbb{D}_R; \mathbb{C}^{2 \times 2})$. Then the functions $\varphi_{jk}(\cdot, \lambda), j, k \in \{1, 2\}$, admit the following representations

$$\varphi_{jk}(x, \lambda) = \delta_{jk}e^{ib_k \lambda x} + \int_0^x R_{jk1}(x, t)e^{ib_1 \lambda t}dt + \int_0^x R_{jk2}(x, t)e^{ib_2 \lambda t}dt,$$

where $R_{jkh} \in C^\infty(\Omega)$ and there exists constants $C_0, C_1 > 0$ such that

$$\|R_{jkh}\|_{C(\Omega)} \leq C_0 \|Q\| \cdot \exp(C_1 \|Q\|), \quad j, k, h \in \{1, 2\}.$$  

Proof. Comparing initial conditions and applying the Cauchy uniqueness theorem one easily gets $2\Phi_1(\cdot, \lambda) = 2\begin{pmatrix} \varphi_{11}(\cdot, \lambda) \\ \varphi_{21}(\cdot, \lambda) \end{pmatrix} = e_+(\cdot, \lambda) + e_-(\cdot, \lambda)$. Inserting in place of $e_+(\cdot, \lambda)$ and $e_-(\cdot, \lambda)$ their expressions from (3.2) one arrives at (3.5)–(3.6) for $k = 1$. Relations (3.5)–(3.6) for $k = 2$ are proved similarly. \hfill \Box

Next we obtain a formula for the characteristic determinant $\Delta(\cdot)$ of the BVP (1.3)–(1.5) similar to that used in [27, Lemma 4.1].

Lemma 3.3. Let $Q \in A(\mathbb{D}_R; \mathbb{C}^{2 \times 2})$. The characteristic determinant $\Delta(\cdot)$ of the BVP (1.3)–(1.5) is an entire function admitting the following representation

$$\Delta(\lambda) = \Delta_0(\lambda) + \int_0^1 g_1(t)e^{ib_1 \lambda t}dt + \int_0^1 g_2(t)e^{ib_2 \lambda t}dt.$$  

(3.7)
Here \( g_j \in C^\infty[0,1], \ j \in \{1,2\} \), and
\[
\|g_j\|_{C[0,1]} \leq C_0\|Q\| \cdot \exp(C_1\|Q\|), \quad j \in \{1,2\}, \tag{3.8}
\]
with some constants \( C_0 > 0 \) and \( C_1 > 0 \).

**Proof.** Inserting formulas (3.5) with \( x = 1 \) into (2.20) and taking expression (2.21) for \( \Delta_0(\cdot) \) into account, we arrive at the expression (3.7) for \( \Delta(\cdot) \).

Since \( g_j(\cdot), j \in \{1,2\}, \) are linear combinations of the functions \( R_{jkh}(1,\cdot), \ j,k,h \in \{1,2\}, \) estimates (3.8) easily follow from estimates (3.6). \( \square \)

### 3.2. Riesz Basis Property for Quasi-Periodic BVP

In the rest of this section we study BVP generated by equation (1.3)–(1.4),
\[
Ly = -iB^{-1}y' + Q(x)y = \lambda y, \quad y = \text{col}(y_1, y_2), \quad x \in [0,1]. \tag{3.9}
\]
subject to the following “quasi-periodic” boundary conditions
\[
U_1(y) = y_1(0) - d_1y_1(1) = 0, \quad U_2(y) = y_2(0) - d_2y_2(1) = 0, \quad d_1, d_2 \in \mathbb{C}\setminus\{0\}. \tag{3.10}
\]

First we study characteristic determinant \( \Delta_0(\lambda) \) of the operator \( L_U(B,Q) \).

**Definition 3.4.** (i) A sequence \( \Lambda := \{\lambda_n\}_{n \in \mathbb{Z}} \) of complex numbers is said to be **separated** if for some positive \( \delta > 0 \),
\[
|\lambda_j - \lambda_k| > 2\delta \quad \text{whenever} \quad j \neq k. \tag{3.11}
\]
In particular, all entries of a separated sequence are distinct.

(ii) The sequence \( \Lambda \) is said to be **asymptotically separated** if for some \( n_0 \in \mathbb{N} \) the subsequence \( \Lambda_{n_0} := \{\lambda_n\}_{|n| > n_0} \) is separated.

**Lemma 3.5.** Let \( \Delta_0(\cdot) \) be the characteristic determinant of the problem (3.9)–(3.10) with \( Q = 0 \), and let \( b_1b_2^{-1} \notin \mathbb{R} \). Then the following statements hold:

(i) The sequence of zeros \( \Lambda_0 \) of \( \Delta_0(\cdot) \) counting multiplicity is of the form
\[
\Lambda_0 = \{\lambda_{nj}^0\}_{n \in \mathbb{Z}, j \in \{1,2\}}, \quad \lambda_{nj}^0 = 2\pi b_j^{-1}(\gamma_j + n), \quad n \in \mathbb{Z}, \ j \in \{1,2\}, \tag{3.12}
\]
where
\[
\gamma_j := \alpha_j + i\beta_j := \frac{i}{2\pi} \log d_j = \frac{i}{2\pi} (\log |d_j| + i\arg d_j). \tag{3.13}
\]
In particular, the sequence \( \Lambda_0 \) is asymptotically separated and all its entries but possibly one are simple.

(ii) Let \( b_1b_2^{-1} = c_1 + ic_2, \ c_1, c_2 \in \mathbb{R} \) (note that \( c_2 \neq 0 \)). The sequence \( \Lambda_0 \) is separated if and only if
\[
\alpha_1 \neq \left\{ \frac{c_1\beta_1 - (c_1^2 + c_2^2)\beta_2}{c_2} \right\} \quad \text{or} \quad \alpha_2 \neq \left\{ \frac{\beta_1 - c_1\beta_2}{c_2} \right\}, \tag{3.14}
\]
where \( \{x\} \) denotes the fractional part of \( x \in \mathbb{R} \).

(iii) For any \( \varepsilon > 0 \) there exists \( \widetilde{C}_\varepsilon > 0 \) such that
\[
|\Delta_0(\lambda)| > \widetilde{C}_\varepsilon (e^{-1\text{Im}(b_1\lambda)} + 1) (e^{-1\text{Im}(b_2\lambda)} + 1), \quad \lambda \in \mathbb{C}\setminus\Omega_\varepsilon, \tag{3.15}
\]
where \( \Omega_\varepsilon := \bigcup_{\lambda_0 \in \Lambda_0} \mathbb{D}_\varepsilon(\lambda_0) \) and \( \mathbb{D}_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\} \).
Proof. (i) It follows from (2.21) and (3.13) that the characteristic determinant \( \Delta_0(\cdot) \) of the problem (3.9)–(3.10) (with \( Q = 0 \)) is given by
\[
\Delta_0(\lambda) = 1 + d_1 d_2 e^{i(b_1+b_2)\lambda} - d_1 e^{ib_1\lambda} - d_2 e^{ib_2\lambda}
= (d_1 e^{ib_1\lambda} - 1)(d_2 e^{ib_2\lambda} - 1) = \Delta_{01}(\lambda) \cdot \Delta_{02}(\lambda),
\]
(3.16)
\[
\Delta_{0j}(\lambda) := e^{i(b_j\lambda - 2\pi\gamma_j)} - 1, \quad j \in \{1, 2\}.
\]
(3.17)
It is clear that the sequence \( \Lambda_{0j} := \{\lambda_{nj}^0\}_{n \in \mathbb{Z}} \) is the sequence of the zeros of \( \Delta_{0j}(\cdot), j \in \{1, 2\} \). Hence formulas (3.12) for zeros of \( \Delta_0(\cdot) \) are immediate from factorization (3.16).

(ii) The sequences \( \Lambda_{0j} := \{\lambda_{nj}^0\}_{n \in \mathbb{Z}}, j \in \{1, 2\} \), form two arithmetic progressions. Since \( b_1 b_2^{-1} \notin \mathbb{R} \), the sequence \( \Lambda_0 \) is separated if and only if the sequences \( \Lambda_{01} \) and \( \Lambda_{02} \) have no common entries. This in turn is reduced to solving the following Diophantine equation
\[
\lambda_{n1}^0 = 2\pi b_1^{-1}(\gamma_1 + n) = 2\pi b_2^{-1}(\gamma_2 + m) = \lambda_{m2}^0, \quad n, m \in \mathbb{Z}.
\]
(3.18)
Inserting formula \( b_1 b_2^{-1} = c_1 + ic_2 \) and representations (3.13) for \( \gamma_1, \gamma_2 \), in (3.18), one reduces this equation to
\[
\alpha_1 + i\beta_1 + n = (c_1 + ic_2)(\alpha_2 + i\beta_2 + m).
\]
(3.19)
Separating real and imaginary parts in (3.19) we arrive at the system
\[
\begin{align*}
\alpha_1 + n &= c_1 \alpha_2 - c_2 \beta_2 + c_1 m, \\
\beta_1 &= c_2 \alpha_2 + c_1 \beta_2 + c_2 m.
\end{align*}
\]
(3.20)
Since \( c_2 \neq 0 \), the solution to this system is given by
\[
\begin{align*}
m &= \frac{\beta_1 - c_1 \beta_2}{c_2} - \alpha_2, \\
n &= \frac{c_1 \beta_1 - (c_1^2 + c_2^2) \beta_2}{c_2} - \alpha_1.
\end{align*}
\]
(3.21)
Since \( \alpha_1, \alpha_2 \in [0, 1) \), both \( m \) and \( n \) in (3.21) are integers if and only if both conditions (3.14) are violated. Hence Diophantine equation (3.18) does not have integer solutions if and only if condition (3.14) holds. This completes the proof.

(iii) Let \( \Omega_0 := \bigcup_{n \in \mathbb{Z}} \mathbb{D}_\varepsilon(\pi n) \). We need the following well known estimate from below (see e.g. [22, Theorem I.5.7])
\[
|\sin z| > A_\varepsilon \cdot e^{\Re z} \geq 2^{-1} A_\varepsilon \cdot (e^{1 \Re z} + e^{-1 \Re z}), \quad z \in \mathbb{C} \backslash \Omega_0^\varepsilon.
\]
(3.22)
for some \( A_\varepsilon > 0 \). Since \( 2|\sin z| = |e^{2iz} - 1| \cdot e^{\Re z} \), it follows from (3.22) that
\[
|e^{2iz} - 1| > A_\varepsilon \cdot e^{-2 \Re z} + 1, \quad z \in \mathbb{C} \backslash \Omega_0^\varepsilon.
\]
(3.23)
Since the sequence \( \Lambda_{0j} := \{\lambda_{nj}^0\}_{n \in \mathbb{Z}} \) is the sequence of zeros of \( \Delta_{0j}(\cdot) \), estimate (3.23) yields the following estimate from below on \( |\Delta_{0j}(\cdot)| \):
\[
|\Delta_{0j}(\lambda)| = |e^{i(b_j\lambda - 2\pi\gamma_j)} - 1| > A_\varepsilon \cdot (e^{\Re (2\pi\gamma_j)} - e^{-\Re (b_j\lambda)}) + 1
\geq A_\varepsilon \cdot (e^{-\Re (b_j\lambda)} + 1), \quad \lambda \in \mathbb{C} \backslash \bigcup_{n \in \mathbb{Z}} \mathbb{D}_\varepsilon(\lambda_{nj}^0), \quad j \in \{1, 2\},
\]
(3.24)
for some $A_{\varepsilon j} > 0, j \in \{1, 2\}$. Since the sequence $\Lambda_{0j}$ is the subsequence of $\Lambda_0$ for $j \in \{1, 2\}$, estimate (3.24) holds automatically for $\lambda \in \mathbb{C}\setminus \Omega_{\varepsilon}$. Inserting inequalities (3.24) into factorization identity (3.16) yields the desired estimate (3.15) with $\tilde{C}_\varepsilon = A_{\varepsilon 1}A_{\varepsilon 2}$.

The following estimate will be of importance below.

**Corollary 3.6.** Let $\Delta_0(\cdot)$ and $\Omega_{\varepsilon}$ be as in Lemma 3.5. Then for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left| \Delta_0(\lambda) \right| > C_\varepsilon (e^{-\operatorname{Im}(b_1\lambda)} + e^{-\operatorname{Im}(b_2\lambda)} + 2), \quad \lambda \in \mathbb{C}\setminus \Omega_{\varepsilon}. \quad (3.25)$$

**Proof.** This estimate is immediate from (3.15) with $C_\varepsilon = \tilde{C}_\varepsilon/2$. □

In what follows we need the following version of the Riemann–Lebesgue lemma (see e.g. [27, Lemma 3.5]).

**Lemma 3.7.** Let $g \in L^1[0, 1]$ and $c \in \mathbb{C}\setminus \{0\}$. Then for any $\delta > 0$ there exists $M_\delta > 0$ such that

$$\left| \int_0^1 g(t)e^{ict\lambda}dt \right| < \delta (e^{-\operatorname{Im}(c\lambda)} + 1), \quad |\lambda| > M_\delta. \quad (3.26)$$

The following asymptotic formula for the eigenvalues of the problem (3.9)–(3.10) plays a crucial role in the study of Riesz basis property.

**Proposition 3.8.** Let $\Delta(\cdot)$ be the characteristic determinant of the problem (3.9)–(3.10) with $Q$ of the form (1.4) and let $\Lambda_0 = \{\lambda_{nj}^0\}_{n \in \mathbb{Z}, j \in \{1, 2\}}$ be a sequence given by (3.12). Then the following statements hold:

(i) The sequence $\Lambda$ of its zeros can be ordered as $\Lambda = \{\lambda_{nj}\}_{n \in \mathbb{Z}, j \in \{1, 2\}}$ in such a way that the following asymptotic formula holds

$$\lambda_{nj} = \lambda_{nj}^0 + o(1), \quad \text{as} \quad n \to \infty, \quad n \in \mathbb{Z}, \quad j \in \{1, 2\}. \quad (3.27)$$

In particular, the sequence $\Lambda$ is asymptotically separated.

(ii) Let in addition condition (3.14) be satisfied. Then there exists a constant $C > 0$ such that the sequence $\Lambda$ is separated whenever $\|Q\|_{C[0,1]} < C$.

**Proof.** (i) Let $\Delta_0(\cdot)$ be the characteristic determinant of the problem (3.9)–(3.10) with $Q = 0$ and let $\varepsilon > 0$. Combining Lemma 3.3 with Lemma 3.7 yields the following estimate for any $\delta > 0$,

$$|\Delta(\lambda) - \Delta_0(\lambda)| < \delta (e^{-\operatorname{Im}(b_1\lambda)} + e^{-\operatorname{Im}(b_2\lambda)} + 2), \quad |\lambda| > M_\delta, \quad (3.28)$$

with certain constant $M_\delta > 0$. By Lemma 3.5, (iii), there exists a constant $C_\varepsilon > 0$ such that the estimate (3.25) holds. Combining estimates (3.25) and (3.28) with $\delta = C_\varepsilon$ we arrive at the following important estimate

$$|\Delta(\lambda) - \Delta_0(\lambda)| < |\Delta_0(\lambda)|, \quad \lambda \in \mathbb{C}\setminus \tilde{\Omega}_\varepsilon, \quad \tilde{\Omega}_\varepsilon := \mathbb{D}_R \cup \Omega_{\varepsilon}, \quad R := M_{C_\varepsilon}. \quad (3.29)$$

Since $\Omega_{\varepsilon} = \bigcup_{\lambda_0 \in \Lambda_0} \mathbb{D}_\varepsilon(\lambda_0)$, where $\Lambda_0$ is a sequence of zeros of the determinant $\Delta_0(\cdot)$, estimate (3.29) makes it possible to apply the classical Rouche theorem. Its applicability ensures that all zeros of the determinant $\Delta(\cdot)$ lie in the domain $\tilde{\Omega}_\varepsilon$. Moreover, each connected component of $\tilde{\Omega}_\varepsilon$ contains the
same number of zeros of determinants $\Delta(\cdot)$ and $\Delta_0(\cdot)$ counting multiplicity. Since in accordance with Lemma 3.5, (i), the sequence of zeros $\Lambda_0$ is asymptotically separated, it follows that for $\varepsilon$ sufficiently small, discs $D_\varepsilon(\lambda_0)$, $\lambda_0 \in \Lambda_0$, $|\lambda_0| > N_0$, do not intersect each other, where $N_0$ does not depend on $\varepsilon$. Hence $\tilde{\Omega}_\varepsilon \setminus D_{N_\varepsilon}$, $N_\varepsilon := \max\{R_\varepsilon, N_0\}$, is a union of disjoint disc parts $D_\varepsilon(\lambda_0) \setminus D_{N_\varepsilon}$, $\lambda_0 \in \Lambda_0$, $|\lambda_0| > N_\varepsilon - \varepsilon$. Thus, each of these disc parts contains exactly one (simple) zero of $\Delta(\cdot)$. Since $\varepsilon > 0$ is arbitrary small, the latter implies the desired asymptotic formula (3.27) and also that the sequence $\Lambda$ is asymptotically separated.

(ii) By Lemma 3.5, (ii), condition (3.14) ensures that the sequence of zeros $\Lambda_0$ is disjoint. By Corollary 3.6, there exists $C_\varepsilon > 0$ such that estimate from below (3.25) holds. Choose $C > 0$ so small that $C_0 C \cdot \exp(C_1 C) < C_\varepsilon$, where $C_0, C_1$ are the constants from inequality (3.8). Assuming that $\|Q\| = \|Q\|_{C[0,1]} < C$ one easily gets from (3.8) that

$$|g_j(t)| < C_0 C \cdot \exp(C_1 C) < C_\varepsilon, \quad t \in [0,1], \ j \in \{1, 2\}. \quad (3.30)$$

Further, it is clear that

$$|e^{ib_j \lambda t}| < e^{-\Im(b_j \lambda)} + 1, \quad t \in [0,1], \ \lambda \in \mathbb{C}, \ j \in \{1, 2\}. \quad (3.31)$$

Combining formula (3.7) with estimates (3.30), (3.31) and (3.25) we arrive at

$$|\Delta(\lambda) - \Delta_0(\lambda)| \leq \int_0^1 |g_1(t)| \cdot |e^{ib_1 \lambda t}| dt + \int_0^1 |g_2(t)| \cdot |e^{ib_2 \lambda t}| dt$$

$$< C_\varepsilon (e^{-\Im(b_1 \lambda)} + e^{-\Im(b_2 \lambda)} + 2)$$

$$< |\Delta_0(\lambda)|, \quad \lambda \in \mathbb{C} \setminus \tilde{\Omega}_\varepsilon^0. \quad (3.32)$$

Since the discs $D_\varepsilon(\lambda_0)$, $\lambda_0 \in \Lambda_0$, are disjoint, the Rouche theorem now implies that each of these discs contains exactly one (simple) zero of $\Delta(\cdot)$, which implies that the sequence $\Lambda$ is separated. \hfill \Box

Next we recall classical definitions of Riesz basisness and Riesz basisness with parentheses (see e.g. [17, 37]).

**Definition 3.9.** (i) A sequence $\{f_k\}_{k=1}^\infty$ of vectors in $\mathcal{H}$ is called a **Riesz basis** if it admits a representation $f_k = T e_k$, $k \in \mathbb{N}$, where $\{e_k\}_{k=1}^\infty$ is an orthonormal basis in $\mathcal{H}$ and $T : \mathcal{H} \to \mathcal{H}$ is a bounded operator with bounded inverse.

(ii) A sequence of subspaces $\{\mathcal{H}_k\}_{k=1}^\infty$ is called a **Riesz basis of subspaces** in $\mathcal{H}$ if there exists a complete sequence of mutually orthogonal subspaces $\{\mathcal{H}_k\}_{k=1}^\infty$ and a bounded operator $T$ in $\mathcal{H}$ with bounded inverse such that $\mathcal{H}_k = T \mathcal{H}_k'$, $k \in \mathbb{N}$.

(iii) A sequence $\{f_k\}_{k=1}^\infty$ of vectors in $\mathcal{H}$ is called a **Riesz basis with parentheses** if each of its finite subsequences is linearly independent, and there exists an increasing sequence $\{n_k\}_{k=0}^\infty \subset \mathbb{N}$ such that $n_0 = 1$ and the sequence $\mathcal{H}_k := \text{span}\{f_j\}_{j=n_k-1}^{n_k-1}$, forms a Riesz basis of subspaces in $\mathcal{H}$. Subspaces $\mathcal{H}_k$ are called blocks.
Note that if $A$ is an operator in $H$ with discrete spectrum, then the property of its root vectors (eigenvectors) to form a Riesz basis with parentheses (Riesz basis) in $H$ can be retranslated in terms of $A$ to be close to a certain “good” operator.

To retranslate this property we recall that an eigenvalue $\lambda_0$ of an operator $A$ is called algebraically simple if $\ker(A - \lambda_0) = \mathcal{R}(\lambda_0, A)$, where $\mathcal{R}(\lambda_0, A)$ is the root subspace of $A$. It is equivalent to the fact that $\lambda_0$ is a simple pole (i.e. a pole of order one) of the resolvent $(A - \lambda)^{-1}$. This means that all Jordan cells corresponding to $\lambda_0$ are of size one.

In applications to BVP it is convenient to reformulate Riesz basis property (with and without parentheses) of systems of root functions of operators with discrete spectrum in terms of their similarity to certain subclasses of the class of spectral operators. Recall that an operator is called a spectral operator if it admits a countably additive (generally non-orthogonal) resolution of identity defined on Borel subsets of the complex plane.

Next we collect several definitions in a form most suitable for the purposes of our paper.

**Definition 3.10.** (i) A bounded operator $N$ in $H$ is called **quasi-nilpotent** if $\sigma(N) = \{0\}$.

(ii) A closed operator $S$ in $H$ is called an **operator of scalar type** if it is similar to a normal operator.

(iii) A closed operator $T$ in $H$ will be called **almost normal** if it admits an orthogonal decomposition $T = T_1 \oplus T_2$ where $T_1$ is finite dimensional and $T_2$ is normal.

Note that according to [14, Theorem XVIII.2.28] the operator $T = S + N$, where $S$ is an operator of scalar type and $N$ is a quasi-nilpotent operator that commutes with $S$, is a spectral operator. Such a representation characterizes bounded spectral operators, while becomes false, in general, for unbounded operators (see [14, Section XVIII.2]).

Note also that the definition of a scalar type operator is given in accordance with the Wermer theorem, see [14, Theorem XV.6.4].

**Lemma 3.11.** Let $A$ be a closed densely defined operator in $H$ with discrete spectrum. Then

(i) **For the operator $A$ to be similar to a normal operator in $H$** it is necessary and sufficient that its eigenvalues are algebraically simple and its system of eigenvectors $\{f_k\}_{k \in \mathbb{N}}$ forms a Riesz basis in $H$.

(ii) **For the operator $A$ to be similar to an almost normal operator in $H$** it is necessary and sufficient that all its eigenvalues but finitely many are algebraically simple and its system of eigen- and associated vectors $\{f_k\}_{k \in \mathbb{N}}$ forms a Riesz basis in $H$.

(iii) **The operator $A$ is similar to an orthogonal direct sum of finite dimensional operators if and only if the system of its root vectors forms a Riesz basis with parentheses. In particular, in this case $A$ is a spectral operator.**
Proof. (i) The necessity is obvious because of the corresponding properties of normal operator and Definition 3.9(i).

To prove sufficiency let \( \{\lambda_k\}_{k\in\mathbb{N}} \) be a sequence of eigenvalues of \( A \), counting (geometric) multiplicities, \( Af_k = \lambda_k f_k, \ k \in \mathbb{N} \). Since \( \{f_k\}_{k=1}^{\infty} \) forms a Riesz basis in \( \mathfrak{H} \), there exists a bounded operator \( T : \mathfrak{H} \to \mathfrak{H} \) with bounded inverse and an orthonormal basis \( \{e_k\}_{k=1}^{\infty} \) in \( \mathfrak{H} \) such that \( f_k = Te_k, \ k \in \mathbb{N} \).

Define a diagonal operator \( S \) in \( \mathfrak{H} \) by setting \( Se_k = \lambda_k e_k, \ k \in \mathbb{N} \), and extending it by linearity to the natural (maximal) domain of definition. Clearly, \( S \) is normal and \( T^{-1}ATe_k = Se_k, \ k \in \mathbb{N} \). With any \( \lambda_0 \in \rho(A) \) these relations can be rewritten as \( T^{-1}(A - \lambda_0)^{-1}Te_k = (S - \lambda_0)^{-1}e_k, \ k \in \mathbb{N} \). By linearity and continuity these relations lead to the similarity of the resolvents: \( T^{-1}(A - \lambda_0)^{-1}T = (S - \lambda_0)^{-1} \), i.e. to the similarity of \( A \) and \( S \).

(ii) and (iii) are proved similarly if one defines an operator \( S \) accordingly in an orthogonal Jordan chain chosen in each (necessarily finite-dimensional) root subspace \( \mathcal{R}(\lambda_0, A) \) corresponding to each eigenvalue \( \lambda_0 \) of \( A \).

Clearly, an orthogonal sum of finite dimensional operators admits the representation \( S + N \), where \( S \) is normal operator and \( N \) is a quasi-nilpotent operator that commutes with \( S \). Hence the operator \( A \) in part (iii) is similar to a spectral operator according to [14, Theorem XVIII.2.28] and thus is a spectral operator itself.

To prove the main result of the section let us recall a result from [26] on Riesz basis property with parentheses for the operator \( L_U(B,Q) \).

**Proposition 3.12.** [26, Proposition 5.9] Let \( Q \in L^\infty([0,1]; \mathbb{C}^{2\times2}), \ b_1b_2^{-1} \notin \mathbb{R} \), and let \( L := L_U(B,Q) \) be an operator associated with the BVP (3.9)–(3.10). Then the system of root functions of \( L \) forms a Riesz basis with parentheses in \( L^2([0,1]; \mathbb{C}^2) \).

Finally, we are ready to state the main result of the section on Riesz basis property of the operator \( L_U(B,Q) \).

**Theorem 3.13.** Let \( Q \in A(\mathbb{D}; \mathbb{C}^{2\times2}), \ b_1b_2^{-1} \notin \mathbb{R} \) and let \( L := L_U(B,Q) \) be the operator associated with the BVP (3.9)–(3.10). Then the following statements hold:

(i) The operator \( L \) is similar to an almost normal operator. In particular, all eigenvalues of \( L \) but finitely many are algebraically simple and the system of root functions of \( L \) forms a Riesz basis in \( L^2([0,1]; \mathbb{C}^2) \).

(ii) Let in addition condition (3.14) hold. Then there exists \( C > 0 \) such that the operator \( L \) is similar to a normal operator whenever \( \|Q\|_{C[0,1]} < C \).

**Proof.** (i) Due to Proposition 3.12 the system of root functions of the operator \( L \) forms a Riesz basis with parentheses in \( L^2([0,1]; \mathbb{C}^2) \), where each block is constituted by the root subspaces corresponding to the eigenvalues of \( L \) that are mutually \( \varepsilon \)-close with respect to the sequence \( \Psi := \{-\varphi_1, -\varphi_2, \pi - \varphi_1, \pi - \varphi_2\} \). Here \( \varphi_j = \arg b_j, j \in \{1, 2\} \), and \( \varepsilon > 0 \) is sufficiently small. Recall that numbers \( \lambda, \mu \in \mathbb{C} \) are called \( \varepsilon \)-close with respect to the sequence \( \{\psi_k\}_{k=1}^{m} \) if for some \( k \in \{1, \ldots, m\} \) they belong to a small angle of size \( 2\varepsilon \) with the bisectrix \( l_+(\psi_k) := \{\lambda \in \mathbb{C} : \arg \lambda = \psi_k\} \) and their projections on this ray differ no more than by \( \varepsilon \) (see [26, Definition 5.4]).
Let us prove that for \( \varepsilon(>0) \) sufficiently small the above blocks are asymptotically of size one, i.e. \( n_{k+1} = n_k + 1 \) for sufficiently large \( k \) (see Definition 3.9, (iii)). Due to Proposition 3.8, (i), eigenvalues of \( L \) are asymptotically simple and separated and, due to asymptotic formula (3.27) and the form (3.12) of the sequence \( \Lambda_0 = \{\lambda_n^0\}_{n \in \mathbb{Z}, j \in \{1,2\}} \), they are located along two different non-parallel lines of \( \mathbb{C} \) that are parallel to the rays \( l_+(-\varphi_1) \), \( l_+(-\varphi_2) \). It is clear now that for sufficiently small \( \varepsilon > 0 \) and sufficiently large \( n, m \in \mathbb{Z} \) different numbers \( \lambda_n \) and \( \lambda_m \) are not \( \varepsilon \)-close with respect to the sequence \( \Psi \), for \( j, k \in \{1,2\} \). Indeed, if \( j \neq k \) then they do not belong to a small angle with the bisectrix \( l_+ (\psi) \) for any \( \psi \in \Psi \), because they are close to two different non-parallel lines of \( \mathbb{C} \). If \( j = k \) then \( n \neq m \) and numbers \( \lambda_n \) and \( \lambda_m \) belong to a small angle with the bisectrix \( l_+ (\psi) \) for some \( \psi \in \Psi \). From the form (3.12) of the sequence \( \Lambda_0 \) and asymptotic formula (3.27) it is clear that projections of \( \lambda_n \) and \( \lambda_m \) on \( l_+ (\psi) \) are separated.

Thus, \( n_{k+1} = n_k + 1 \) for sufficiently large \( k \), hence the system of root functions of \( L \) forms a Riesz basis (without parentheses) in \( L^2([0,1]; \mathbb{C}^2) \).

(ii) By (i) the system of root functions of the operator \( L \) forms a Riesz basis in \( L^2([0,1]; \mathbb{C}^2) \). On the other hand, in accordance with Proposition 3.8, (ii), eigenvalues of \( L \) are (algebraically and geometrically) simple and separated provided that \( \|Q\| < C \), for certain \( C > 0 \). To get a similarity of \( L \) to a normal operator it remains to apply Lemma 3.11, (i).

\( \square \)

Remark 3.14. (i) As distinguished from the case \( b_1 b_2^{-1} \notin \mathbb{R} \), the Riesz basis property for \( 2 \times 2 \) Dirac operators \( L_U \) has been investigated in numerous papers (see [7,9,11,12,18,19,24,27,32,34,38–40] and references therein). The most complete result was recently obtained independently and by different methods in [24,27] and [38–40]. Namely, assuming that \( B = B^* \) and \( Q \in L^1([0,1]; \mathbb{C}^{2 \times 2}) \) it is proved in [24,27] (the general case of \( b_1 b_2 < 0 \)) and [38–40] (the Dirac case, \( b_1 = -b_2 \)) that the system of root vectors of equation (1.3) subject to regular boundary conditions constitutes a Riesz basis with parentheses in \( L^2([0,1]; \mathbb{C}^2) \) and ordinary Riesz basis provided that BC are strictly regular. Recently A.S. Makin [32] completed these results by investigating existence of the Riesz basis property (without parentheses) for non-periodic type regular but not strictly regular BC. We also mention the criterion for the system of root vectors of periodic BVP for Dirac operator to form a Riesz basis (without parentheses) obtained by P. Djakov and B.S. Mityagin [12].

(ii) Note also that an important role in proving Riesz basis property in [24,27] and [40], [38,39], is playing the following asymptotic formula

\[
\lambda_n = \lambda_n^0 + o(1), \quad n \to \infty, \quad n \in \mathbb{Z},
\]

for the eigenvalues \( \{\lambda_n\}_{n \in \mathbb{Z}} \) of the operator \( L_{C,D}(B, Q) \) with regular BC (and summable potential matrix \( Q \)), where \( \{\lambda_n^0\}_{n \in \mathbb{Z}} \) is the sequence of eigenvalues of the unperturbed operator \( L_{C,D}(B, 0) \). Note also that formula (3.33) has recently been applied to investigate the spectral properties of Dirac systems on star graphs [1].

(iii) Note that the periodic problem for system (1.3)–(1.6) substantially differs from that for Dirac operators. Namely, periodic BVP for system (1.3)–(1.6) is always strictly regular, while for Dirac system it is only regular.
Another proof of Theorem 3.13(i) can also be obtained in just the same way as the proof of Riesz basis property for Dirac operators in [24,27]. The proof ignores Proposition 3.12 and is completely relied on transformation operators.

Remark 3.15. Numerous papers are devoted to the completeness and Riesz basis property for the Sturm–Liouville operator (see the recent survey [31] by Makin and the papers cited therein). In connection with Theorem 3.13 we especially mention the recent achievements for periodic (anti-periodic) Sturm–Liouville operator \(-d^2/dx^2 + q(x)\) on \([0,\pi]\). Namely, Gesztesy and Tkachenko [15,16] for \(q \in L^2[0,\pi]\) and Djakov and Mityagin [11] for \(q \in W^{-1/2}[0,\pi]\) established by different methods a criterion for the system of root functions to contain a Riesz basis.

Remark 3.16. Here we have restricted ourself to the case of analytical \(Q\) and “quasi-periodic” BC (3.10). In fact, the main results of the section, Proposition 3.8 and Theorem 3.13, remain valid for \(Q \in L^1([0,1];\mathbb{C}^{2\times2})\) and strictly regular boundary conditions instead of “quasi-periodic” ones. These results will be treated by another method elsewhere.

4. Completeness Property Under Rank One Perturbations

4.1. Normal Realizations

Let us start with some examples of dual pairs (see Definition 1.4).

Example 4.1. One obtains a typical example of a dual pair by choosing \(\{A_1, A_2\}\) to be the minimal operators associated in \(H = L^2[0,1]\) with the Sturm–Liouville differential expressions \(L(q) = -d^2/dx^2 + q\) and \(L(\overline{q}) = -d^2/dx^2 + \overline{q}\), respectively. Assuming that \(q \in L^2[0,1]\) one gets that the minimal operators \(L_{\text{min}}(q)\) and \(L_{\text{min}}(\overline{q})\) are given by the differential expressions \(L(q)\) and \(L(\overline{q})\) on the domain
\[
\text{dom}(L_{\text{min}}(q)) = \text{dom}(L_{\text{min}}(\overline{q})) = W^{2,2}_0[0,1].
\]

Example 4.2. Another example can be obtained by choosing \(A_1\) and \(A_2\) to be the minimal operators associated in \(L^2([0,1];\mathbb{C}^n)\) with expression (1.1) and its formal adjoint \(L(B^*,Q^*) = -i(B^*)^{-1}d/dx + Q^*(x)\):
\[
A_1 := L_{\text{min}}(B,Q) \quad \text{and} \quad A_2 := L_{\text{min}}(B^*,Q^*).
\]

If \(Q \in L^2([0,1];\mathbb{C}^{n\times n})\), then
\[
\text{dom}(A_1) = \text{dom}(A_2) = W^{1,2}_0([0,1];\mathbb{C}^n).
\]

Here we describe normal realizations \(L_U(B,Q)\) generated by the BVP (1.3)–(1.6). To this end and for the reader’s convenience we recall the definition of the operator \(L_U(B,Q)\). Let matrices \(B\) and \(Q\) be given by
\[
B = \text{diag}(b_1, b_2), \quad b_1b_2^{-1} \notin \mathbb{R}, \quad \text{and} \quad Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix} \in L^1([0,1];\mathbb{C}^{2\times2}).
\]
(4.2)
Denote by \( L_U(B, Q) \) the operator generated by the equation
\[
Ly := -iB^{-1}y' + Q(x)y = \lambda y, \quad y = \text{col}(y_1, y_2), \quad x \in [0, 1],
\] subject to the boundary conditions
\[
U_j(y) := a_{j1}y_1(0) + a_{j2}y_2(0) + a_{j3}y_1(1) + a_{j4}y_2(1) = 0, \quad j \in \{1, 2\}.
\] Setting \( C := A_{12} \) and \( D := A_{34} \) we see that boundary conditions (4.4) become
\[
Cy(0) + Dy(1) = 0, \quad \text{rank}(C,D) = 2.
\] (4.5)

First, we recall following [28] a description of normal extensions of the dual pair \( \{A_1, A_2\} \) of the form (4.1), i.e. all normal operators \( L_U(B, Q) \) generated by the BVP (4.3)–(4.4).

To this end, we recall the result from [26] describing normal operators \( L_{C,D}(B, 0) \) with zero potential in general \( n \times n \) case.

**Lemma 4.3.** [26, Lemma 5.1] The operator \( L_{C,D}(B, 0) \) given by (1.1)–(1.2) is normal if and only if
\[
CBC^* = DBD^*. \tag{4.6}
\] In this case boundary conditions (1.2) are regular, i.e. \( \det T_{izB}(C,D) \neq 0 \) for each admissible \( z \) (see Definition 5.1 in Appendix).

Going back to the case \( n = 2 \) we can obtain the following explicit form of normal boundary conditions.

**Lemma 4.4.** [28] Let \( n = 2 \) and \( b_1b_2^{-1} \notin \mathbb{R} \). The realization \( L_{C,D}(B, 0) \) is a normal operator if and only if
\[
D = C \cdot \text{diag}(d_1, d_2), \quad |d_1| = |d_2| = 1, \quad \det C \neq 0. \tag{4.7}
\]

**Proof.** (i) **Necessity.** Let the operator \( L_{C,D}(B, 0) \) be normal. Then Lemma 4.3 implies condition (4.6) and that the boundary conditions (4.5) are regular. This yields that \( \det C = J_{12} \neq 0 \) (see Definition 5.1 in Appendix). Multiplying boundary conditions (4.4) by \( C^{-1} \) from the left we can assume without loss of generality that \( C = I_2 \) which transforms condition (4.6) into \( DBD^* = B \).

The equality \([DBD^*]_{11} = b_1 \) takes the form
\[
|d_{11}|^2 + b_2b_1^{-1}|d_{12}|^2 = 1. \tag{4.8}
\]
Since \( b_2b_1^{-1} \notin \mathbb{R} \) it follows from (4.8) that \( d_{12} = 0 \) and \( |d_{11}| = 1 \). Similarly, equality \([DBD^*]_{22} = b_2 \) turns into \( b_1b_2^{-1}|d_{21}|^2 + |d_{22}|^2 = 1 \) and yields \( d_{21} = 0, |d_{22}| = 1 \). Summing up we arrive at representation (4.7).

(ii) **Sufficiency.** It follows from (4.7) that
\[
DBD^* = C \cdot \text{diag}(d_1, d_2) \cdot \text{diag}(b_1, b_2) \cdot \text{diag}(d_1, d_2) \cdot C^* = C \cdot \text{diag}(b_1|d_1|^2, b_2|d_2|^2) \cdot C^* = CBC^*. \tag{4.9}
\]
By Lemma 4.3, equality (4.9) yields normality of the operator \( L_{C,D}(B, 0) \).

Rewriting Lemma 4.4 explicitly in terms of linear forms \( U_1 \) and \( U_2 \) we get the following result.
Lemma 4.5. Let $b_1b_2^{-1} \notin \mathbb{R}$. Then the operator $L_U(B,0)$ is normal if and only if the boundary conditions $\{U_1,U_2\}$ are of the form
\[
U_1(y) = y_1(0) - d_1 y_1(1) = 0, \quad U_2(y) = y_2(0) - d_2 y_2(1) = 0, \quad |d_1| = |d_2| = 1. \tag{4.10}
\]

Finally, we present an explicit description of the class of normal operators $L_U(B,Q)$ with $b_1b_2^{-1} \notin \mathbb{R}$ and non-zero potential matrix $Q$. A first glance this class should contain all differential expressions (4.3) with normal matrices $Q$. In fact, the class of such normal operators is disappointingly small.

Proposition 4.6. \[28\] Let $B$ and $Q$ be given by (4.2), $Q \in L^1([0,1]; \mathbb{C}^{2 \times 2})$, $Q \neq 0$. Then the operator $L_U(B,Q)$ is normal if and only if:

(i) the potential matrix $Q$ is a constant matrix of the form
\[
Q(x) = \begin{pmatrix} b_1^{-1} - b_2^{-1} & 0 \\ 0 & q \end{pmatrix} \neq Q^*(x), \quad x \in [0,1], \quad q \in \mathbb{C}\setminus\{0\}, \tag{4.11}
\]

(ii) the boundary conditions (4.4) are of the form
\[
y(1) = e^{i\varphi} y(0), \quad \varphi \in [-\pi,\pi). \tag{4.12}
\]

Proposition 4.8. Let $B$ and $Q$ be given by (4.2). Assume also that $Q_{12}$ vanishes at the neighborhood of the endpoint $1$, i.e. for some $a \in (0,1)$
\[
Q_{12}(x) = 0, \quad \text{for a.e.} \quad x \in [a,1]. \tag{4.14}
\]
Then the operator $L_V(B,Q)$ corresponding to the BVP (4.3), (4.13), is peculiarly complete (see Definition 1.1).
Hence the system of eigenfunctions \( \{a,\} \) is orthogonal to the system \( \{b,\} \), on the interval \( [0,1] \). In the first of the equations in (4.17) we conclude that a solution of this system on the interval \([0,1]\) is
\[
L^* y := -i (B^*)^{-1} y' + Q^* (x) y = \lambda y, \tag{4.15}
\]
and boundary conditions
\[
V_{*,1} y = \overline{h}_1 y_1 (0) + \overline{b}_1 \overline{b}_2 y_2 (0) = 0, \quad V_{*,2} y = y_2 (1) = 0, \tag{4.16}
\]
i.e. \( L_V^*:=(L_V)^* = L_{V_*} (B^*, Q^*) \).

Let \( \lambda \in \sigma (L_V^*) = \{\lambda_j\}_{1}^{\infty} \) and let \( f = (f_1, f_2) \in \ker (L^*_V - \lambda) \) be the corresponding eigenvector. Then the equation
\[
(L^*_V - \lambda) f = 0
\]
splits into the following system
\[
\begin{cases}
-ib_1^{-1} f'_1 (x) + Q_{21} (x) f_2 (x) = \lambda f_1 (x), \\
-ib_2^{-1} f'_2 (x) + Q_{12} (x) f_1 (x) = \lambda f_2 (x).
\end{cases}
\tag{4.17}
\]
Since \( Q_{12} (x) = 0 \) for \( x \in [a,1] \), the solution of the second equation in (4.17) on the interval \([a,1]\) is \( f_2 (x) = C_2 e^{i b_2 \lambda x} \). The second boundary condition in (4.16) implies that \( C_2 = 0 \), hence \( f_2 (x) = 0 \) for \( x \in [a,1] \). Inserting this relation in the first of the equations in (4.17) we conclude that a solution of this system on the interval \([a,1]\) is proportional to a vector
\[
f(x) = \left( e^{i b_1 \lambda x}, 0 \right), \quad x \in [a,1].
\tag{4.18}
\]
Hence the system of eigenfunctions \( \{u_j (\cdot)\}_{j=1}^{\infty} \) of the problem (4.15)–(4.16) on the interval \([a,1]\) reads as follows
\[
u_j (x) := \left( u_{1j} (x), u_{2j} (x) \right) = \left( e^{i b_1 \lambda_j x}, 0 \right), \quad \lambda_j \in \sigma (L^*_V).
\]
Therefore each vector \( \left( 0 \right) g \in L^2 ([0,1] ; \mathbb{C}^2) \) with \( g \) satisfying \( \text{supp} \, g \subset [a,1] \) is orthogonal to the system \( \{u_j\}_{1}^{\infty} \). Thus, the system \( \{u_j\}_{1}^{\infty} \) is not complete in \( L^2 ([0,1] ; \mathbb{C}^2) \) and its orthogonal complement in \( L^2 ([0,1] ; \mathbb{C}^2) \) is infinite dimensional. \( \square \)

Remark 4.9. Here we show that incompleteness property of the adjoint operator \((L_V (B,Q))^*\) in Proposition 4.8 can also be extracted from [26, Corollary 4.7]. Let us recall that in the case of \( n = 2 \) it states that if one of the boundary conditions is \( y_1 (0) = 0 \) and \( Q_{12} \) vanishes at a neighborhood of 0, then the system of root vectors of the operator \( L_V (B,Q) \) is incomplete and its span is of infinite codimension in \( L^2 ([0,1] ; \mathbb{C}^2) \).

Applying trivial linear transformations \( i_1 : (y_1, y_2) \mapsto (y_2, y_1) \) and \( i_2 : y (x) \mapsto y (1-x) \) to the operator \( L_{V_*} (B^*, Q^*) \) we reduce it to the operator \( L_{\hat{U}_*} (B^*, \hat{Q}) \) where the new boundary condition \( \hat{U}_{*,2} \) takes the form \( y_1 (0) = 0 \) and \( \hat{Q}_{12} (x) = Q_{12} (1-x) \). Hence \( \hat{Q}_{12} \) vanishes at the neighborhood of 0. Now incompleteness property of \( L_{\hat{U}_*} (B^*, \hat{Q}) \), and hence of the operator \((L_V (B^*, Q^*))^*\) is implied by [26, Corollary 4.7].
In what follows we need the following definition.

**Definition 4.10.** We call a pair of BC $U_1(y) = U_2(y) = 0$ equivalent to a pair of BC $V_1(y) = V_2(y) = 0$, if they can be transformed to each other by means of the simplest linear transforms $i_1 : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}$ and $i_2 : y(x) \mapsto y(1 - x)$.

Now we are ready to state and prove the main result of this subsection. It describes all pairs of operators $\{T, S\}$ giving an affirmative solution to the **Problem 1** for the dual pair $\{L_{\min}(B, Q), L_{\min}(B^*, Q^*)\}$ of minimal first order differential operators admitting normal extensions $L_U(B, Q)$. It happen, in particular, that for such pair of operators to exist the potential matrix $Q$ is necessarily zero.

**Theorem 4.11.** Let $n = 2$, $b_1 b_2^{-1} \notin \mathbb{R}$, and let $T := L_U(B, Q)$ and $S := L_V(B, Q)$ be any two proper extensions of the dual pair $\{L_{\min}(B, Q), L_{\min}(B^*, Q^*)\}$. Then the pair $\{T, S\}$ consists of a normal operator $T$ and peculiarly complete $S$, if and only if the following conditions hold: (i) $Q \equiv 0$; (ii) boundary conditions $\{U_1, U_2\}$ are equivalent to (4.10); (iii) boundary conditions $\{V_1, V_2\}$ are equivalent to (4.13).

Moreover, for such a pair of operators $\{T, S\}$ the resolvent difference $(S - \lambda)^{-1} - (T - \lambda)^{-1}$ is one-dimensional if and only if $h_1 = d_1 h_2$.

**Proof.** (i) **Necessity.** Let $T = L_U(B, Q)$ be a normal operator and $S = L_V(B, Q)$ be peculiarly complete. Assume that $Q \neq 0$. Then by Proposition 4.6 matrix-function $Q$ is constant of the form (4.11). Thus, $Q$ is an entire matrix function and $Q_{12}(0) Q_{12}(1) Q_{21}(0) Q_{21}(1) \neq 0$. Therefore in accordance with [3, Corollary 1.7] this implies completeness of the system of root vectors of any operator $L_{\widetilde{U}} := L_U(B, Q)$ and its adjoint $L_{\widetilde{U}}^*$, unless boundary conditions $\widetilde{U}_1(y) = \widetilde{U}_2(y) = 0$ represent one of the Cauchy problems, $y(0) = 0$ or $y(1) = 0$. Clearly in the case of the Cauchy problem both operators $L_{\widetilde{U}}$ and $L_{\widetilde{U}}^*$ have no spectrum. Since $S$ is a complete operator, it follows that boundary conditions $\{V_1, V_2\}$ does not represent initial value problem. Hence $S^*$ is also complete which contradicts our assumption. Thus, $Q \equiv 0$. Equivalence of boundary conditions $\{U_1, U_2\}$ to conditions (4.10) is now implied by Lemma 4.5.

Next we investigate boundary conditions $\{V_1, V_2\}$ generated the operator $S$. If they are weakly $B$-regular (see Definition 5.2) then, by Theorem 5.3, both operators $S$ and $S^*$ are complete. Thus, boundary conditions $\{V_1, V_2\}$ are not weakly $B$-regular. By [3, Lemma 2.7] it means that they are either equivalent to BC (4.13) with $h_1 h_2 \neq 0$ or to the boundary conditions

$$y_2(1) = 0, \quad a_{21} y_1(0) + a_{22} y_2(0) + a_{23} y_1(1) = 0.$$ 

In the latter case Proposition 4.8 implies that the system of root vectors of the corresponding BVP is not complete. Thus, boundary conditions of the operator $S = L_V(B, Q)$ are necessarily equivalent to boundary conditions (4.13).

(ii) **Sufficiency.** Let $Q \equiv 0$ and BC of the operators $T = L_U(B, Q)$ and $S = L_V(B, Q)$ are equivalent to (4.10) and (4.13), respectively. Since $Q$ vanishes at 0 and 1, Proposition 4.8 yields that $S$ is peculiarly complete. Normality of the operator $T = L_{C,D}(B, 0)$ follows from Corollary 4.5.
Finally, we investigate the rank of the resolvent difference \((S - \lambda)^{-1} - (T - \lambda)^{-1}\) for the operators \(S\) and \(T\) generated by boundary conditions (4.13) and (4.10), respectively. Let \(J_{jk}\) be determinants given by (2.19) for linear forms (4.10). It follows from (4.10) that \(J_{42} = 0\), \(J_{14} = -d_2\) and \(J_{34} = d_1d_2\). Hence, condition (2.33) of Proposition 2.7 transforms into \(h_1 = d_1h_2\). Thus, by Proposition 2.7 the corresponding resolvent difference is one-dimensional if and only if \(h_1 = d_1h_2\).

Remark 4.12. Theorem 1.2 substantially complements the classical results by Keldysh, Macaev, and others on completeness of weak perturbations of a selfadjoint compact operator (cf. [17,21,41]). To discuss their connection we recall the “unbounded” counterpart of the Macaev theorem.

Theorem 4.13. [17, Theorem 5.10.2] Let \(T = T^*\) be a selfadjoint operator in \(\mathcal{H}\) with discrete spectrum and let \(KT^{-1} \in \mathcal{S}_\omega\). Then the operators \(S = T + K\) and \(S^*\) have discrete spectrum and are complete.

This result complements the classical Keldysh result [17, Theorem 5.10.1] where instead of the condition \(KT^{-1} \in \mathcal{S}_\omega\) a stronger inclusion \(T^{-1}KT^{-1} \in \mathcal{S}_p, p < \infty\), is assumed. Here \(\mathcal{S}_\omega\) and \(\mathcal{S}_p\) denote, respectively, the Macaev ideal and Neumann-Schatten ideal in \(\mathcal{B}(\mathcal{H})\) (see [17, Chapter 3]).

Note, that under the assumptions of the Keldysh and Macaev theorems one has the relation \(\text{dom } S = \text{dom } T\), meaning that \(S\) is an additive perturbation of \(T\). It follows that under the conditions of Theorem 1.2 the operator \(S\) is always a singular (=non-additive) perturbation of \(T = T^*\), i.e. \(\text{dom } S \neq \text{dom } T\). Indeed, if \(\text{dom } S = \text{dom } T\) and \(K := S - T\), then \(\text{rank } K = \text{rank } ((S - \lambda)^{-1}K(T - \lambda)^{-1}) = \text{rank } ((S - \lambda)^{-1} - (T - \lambda)^{-1}) = 1\).

Since \(\text{dom } K = \text{dom } T\), the operator \(KT^{-1}\) is well defined and one-dimensional. Thus, Theorem 4.13 ensures that both \(S\) and \(S^*\) are complete.

In applications to BVPs representation \(S = T + K\) of the differential operator \(S\) means that a \((T\text{-compact})\) perturbation \(K\) can change coefficients of low order terms of the differential expression \(T\) while boundary conditions remain unchanged. On the other hand, Theorem 4.11 shows that the effect described in Theorem 1.2 can be achieved by means of changing the boundary conditions, i.e. by means of singular perturbations.

4.3. Peculiarly Complete Perturbations of Realizations with Riesz Basis Property

Proposition 4.6 shows that the class of normal operators generated by BVP for equation (4.3) is disappointingly small. This result together with Theorem 4.11 makes it reasonable to pose more general version of Problem 1. Namely we consider Problem 2 just replacing in formulation of Problem 1 a normal operator \(T\) by an operator similar either to a normal operator or to an almost normal operator.

Theorem 4.14. Let \(b_1b_2^{-1} \notin \mathbb{R}\), \(n = 2\), and let \(T := L_U(B,Q)\) and \(S := L_V(B,Q)\). Assume also that \(Q_{21}(\cdot)\) admits a holomorphic continuation to an
entire function and $Q_{12} \equiv 0$, i.e. condition (4.14) is satisfied with $a = 0$. Let, finally, boundary conditions $\{U_1,U_2\}$, $\{V_1,V_2\}$, be given by

\begin{align*}
U_1(y) &= y_1(0) - d_1y_1(1) = 0, & U_2(y) &= y_2(0) - d_2y_2(1) = 0, \ (4.19) \\
V_1(y) &= y_1(0) - h_1y_2(0) = 0, & V_2(y) &= y_1(1) - h_2y_2(0) = 0, \ (4.20)
\end{align*}

where $d_1d_2h_1h_2 \neq 0$. Then:

(i) The operator $T$ is similar to an almost normal operator. In particular, all eigenvalues of $T$ but finitely many are algebraically simple, and the system of root vectors of $T$ forms a Riesz basis.

(ii) Assume in addition that $\|Q_{21}\|_{C[0,1]}$ is sufficiently small and the algebraic condition (3.14) holds. Then the operator $T$ is similar to a normal operator.

(iii) The operator $S$ is peculiarly complete in $L^2([0,1];\mathbb{C}^2)$.

(iv) Resolvent difference $(S - \lambda)^{-1} - (T - \lambda)^{-1}$ is one-dimensional if and only if $h_1 = d_1h_2$.

Proof. (i) This statement is immediate from Theorem 3.13(i).

(ii) By Theorem 3.13, (ii), there exists $C > 0$ such that the operator $L$ is similar to a normal operator whenever $\|Q\|_{C[0,1]} = \|Q_{21}\|_{C[0,1]}$ is sufficiently small and condition (3.14) holds.

(iii)–(iv) These statements are proved in just the same way as in Theorem 4.11. \hfill \Box

Finally, we consider Problem 3, the most general version of Problem 1. Namely, Problem 3 is obtained from Problem 1 by replacing the normality of $T$ by the property of its roots vectors to constitute a Riesz basis with parentheses in $S$. It happen that for the dual pair $\{L_{\text{min}}(B,Q),L_{\text{min}}(B^*,Q^*)\}$ Problem 3 has an affirmative solution for much wider class of potential matrices $Q$ than in both previous cases.

**Theorem 4.15.** Let $b_1b_2^{-1} \notin \mathbb{R}$, $n = 2$, and let $T := L_U(B,Q)$ and $S := L_V(B,Q)$. Let in addition, $Q \in L^\infty([0,1];\mathbb{C}^{2\times 2})$, $Q_{12}(\cdot)$ vanishes at a neighborhood of the endpoint 1, i.e. condition (4.14) holds, and let boundary forms $\{U_1,U_2\}$, $\{V_1,V_2\}$ be given by (4.19)–(4.20) with $d_1d_2h_1h_2 \neq 0$. Then:

(i) The system of root vectors of the operator $T$ forms a Riesz basis with parentheses in $L^2([0,1];\mathbb{C}^2)$.

(ii) The operator $S$ is peculiarly complete in $L^2([0,1];\mathbb{C}^2)$.

(iii) The resolvent difference $(S - \lambda)^{-1} - (T - \lambda)^{-1}$ is one-dimensional if and only if $h_1 = d_1h_2$.

Proof. Statement (i) is immediate from Proposition 3.12. Other statements are proved in the same way as in Theorem 4.11. \hfill \Box

Finally, we illustrate main results by considering the following example which was our first initial observation while studying this problem (see in this connection also [6]).

**Example 4.16.** Consider equation (4.3) with $Q = 0$. Setting $d_1 = d_2 = -1$ in (4.10) we arrive at anti-periodic boundary conditions

\begin{align*}
U_1(y) &= y_1(0) + y_1(1) = 0, & U_2(y) &= y_2(0) + y_2(1) = 0. \ (4.21)
\end{align*}
Denote by $L_{ap}$ the operator generated in $L^2([0, 1]; \mathbb{C}^2)$ by the boundary value problem (4.3), (4.2), (4.21). Recall that $L_V$ is an operator generated by the boundary value problem (4.3), (4.13). Assuming that $h_2 \neq h_1$ one easily finds inverse operators $L^{-1}_V(0)$ and $L^{-1}_{ap}(0)$:

$$L^{-1}_V(0)f = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} ib_1 f_1(x) \\ ib_2 f_2(x) \end{pmatrix} + \frac{ib_1}{h_2 - h_1} \int_0^1 f_1(t) dt \quad (4.22)$$

and

$$L^{-1}_{ap}(0)f = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} ib_1 f_1(x) \\ ib_2 f_2(x) \end{pmatrix} - \frac{i}{2} \left( b_1 \int_0^1 f_1(t) dt \right), \quad (4.23)$$

Further, let

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in L^2([0, 1]; \mathbb{C}^2). \quad (4.24)$$

Combining relation (4.22) with (4.23) one gets

$$(L^{-1}_V(0) - L^{-1}_{ap}(0))f = \frac{ib_1}{h_2 - h_1} (f, e_1) \left( 2^{-1}(h_1 + h_2) \right) + \frac{ib_2}{2} (f, e_2) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.25)$$

Thus, $\text{rank}(L^{-1}_V(0) - L^{-1}_{ap}(0)) \leq 2$ and, if $h_2 + h_1 = 0$, then the resolvent difference $L^{-1}_V(0) - L^{-1}_{ap}(0)$ is one-dimensional. Moreover, the operator $L_V(0)$ is complete while its adjoint $L_V(0)^*$ is not and the codimension of the span of its root functions is infinite.

Note that for $d_1 \neq 1$ and $d_2 \neq 1$ one obtains a similar formula for the difference of the inverse operators,

$$(L^{-1}_V(0) - L^{-1}_{ap}(0))f = \frac{ib_1}{h_2 - h_1} (f, e_1) \left( \frac{d_1 h_2 - h_1}{d_1 - 1} \right) + \frac{ib_2 d_2}{d_2 - 1} (f, e_2) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.26)$$

Summing up we get a one parameter family of normal operators and their one dimensional peculiarly complete perturbations. The latter happens precisely when $d_1 h_2 = h_1$. These examples complement results from [20] and [13].

**Remark 4.17.** Note that the authors of [6] (see also [5]) investigated in great detail the completeness property of one-dimensional non-weak perturbations of a compact self-adjoint operator $A$ in $\mathcal{H}$. They obtain new criteria for completeness of rank one (non-dissipative) perturbations, for joint completeness of the operator and its adjoint, as well as for the spectral synthesis.

**Remark 4.18.** Note, that non-degenerate separated boundary conditions are always strictly regular, hence the root vectors of the corresponding BVP constitute a Riesz basis [9]. Earlier these results were proved for Dirac operator with $Q \in L^2([0, 1]; \mathbb{C}^{2 \times 2})$ by P. Djakov and B. Mityagin [9] (see also [7]).

**Example 4.19.** Let us briefly discuss BVPs for Dirac type systems.

(i) First we consider BVPs (1.1)–(1.2) with a nonsingular $n \times n$ diagonal matrix $B = \text{diag}(b_1, b_2, \ldots, b_n) = B^*$ and $Q = 0$. It is shown in [36] that in this case the operator $L_{C,D}(B, 0)$ is complete if and only if BC (1,2) are regular ($\iff$ weakly regular), i.e. the conditions (5.2) are satisfied. Therefore,
by Theorem 5.3, the operators $L_{C,D}(B,0)$ and $L_{C,D}(B,0)^*$ are complete only simultaneously.

This example, as well Example 1.3, gives a negative solution to Problem 1 for simplest Dirac-type and Sturm–Liouville operators, respectively. Both examples are opposite to the one given by the BVP (1.3)–(1.5) with $Q = 0$ and the diagonal matrix $B$ satisfying $b_1b_2^{-1} \not\in \mathbb{R}$.

(ii) Next we consider $2 \times 2$ Dirac type system (4.3) with

$$B = \text{diag}(b_1, b_2) = B^*, \quad \text{and} \quad Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix} \in L^1([0, 1]; \mathbb{C}^{2 \times 2}).$$

assuming that $b_1b_2 < 0$. In the case $Q \neq 0$ several sufficient conditions of completeness of non-regular (and even degenerate) BVPs (1.3)–(1.5) (operators $L_U(B, Q)$) were obtained in [36], [23]. An interesting feature of these results is that they ensure completeness of both operators $L_U(B, Q)$ and $(L_U(B, Q))^*$.

In connection with Problem 1 it is interesting to examine the BVP (4.3), (4.13). Since $J_{14} = J_{24} = 0$ and $J_{32}J_{13} = h_1 \cdot 1 \neq 0$, boundary conditions (4.13) meet all the assumptions of Theorem 5(i) from [23] but one. Namely, Theorem 5(i) from [23] ensures completeness of the problem (4.3), (4.13) and its adjoint provided that $Q \in W^{k,2}([0, 1]; \mathbb{C}^{2 \times 2})$ and $Q_{12}^{(j)}(1) \neq 0$ for some $j \in \{0, \ldots, k - 1\}$. It happens, in particular, if either $Q$ is analytic at the endpoint 1 and $Q \neq 0$, or $Q \in C^{\infty}([1 - \varepsilon, 1]; \mathbb{C}^{2 \times 2})$ and $Q_{12}^{(j-1)}(1) \neq 0$ for some $j \in \mathbb{N}$. In all these cases the Problem 1 with $Q = Q^*$ has no solution.

However, Problem 1 remains open for the operator $L$ generated by the problem (4.3), (4.13) with non-analytic $Q = Q^* \in C^{\infty}([1 - \varepsilon, 1]; \mathbb{C}^{2 \times 2})$ and satisfying $Q_{12}^{(j-1)}(1) = 0$ for all $j \in \mathbb{N}$.

Example 4.20. Consider Sturm–Liouville equation

$$L(q)y := -y'' + q(x)y = \lambda y, \quad q \in L^1[0, 1],$$

subject to general linear boundary conditions

$$U_j(y) := a_{j1}y(0) + a_{j2}y'(0) + a_{j3}y(1) + a_{j4}y'(1) = 0, \quad j \in \{1, 2\},$$

where the linear forms $\{U_j\}_{j=1}^2$ are assumed to be linearly independent. Denote by $L_U(q)$ the operator associated in $L^2[0, 1]$ with the BVP (4.28)–(4.29).

Recall that BC for Sturm–Liouville equation are called nondegenerate if the characteristic determinant $\Delta(\cdot)$ of the BVP is not reduced to a constant, $\Delta \neq \text{const}$. It is well known (see e.g. [29, Theorem 1.3.1]) that the BVP (4.28)–(4.29) (the operator $L_U(q)$) is complete whenever BC are non-degenerate. In this case both operators $L_U(q)$ and $(L_U(q))^*$ are complete simultaneously.

Passing to degenerate BC we note they are equivalent (see e.g. [34]) to a pair of conditions of the following one-parameter family

$$U_{\alpha,1}(y) := y(0) - \alpha y(1), \quad U_{\alpha,2}(y) := y'(0) + \alpha y'(1), \quad \alpha \in \mathbb{C} \setminus \{0\}. \quad (4.30)$$

We put $L_{\alpha}(q) := L_{U_{\alpha}}(q)$ and note that the adjoint operator $L_{\alpha}(q)^* := (L_{\alpha}(q))^* = L_{\beta}(q)$, i.e. $L_{\alpha}(q)^*$ is given by expression (4.28) with $\bar{q}$ instead of $q$ and the BC $U_{\beta,1}$ and $U_{\beta,2}$ of the form (4.30) with $\beta = -1/\alpha$. 

Completeness property for BVP (4.28)–(4.29) with degenerate BC was investigated in [34] and [30]. All known sufficient conditions ensure completeness of operators $L_\alpha(q)$ for all $\alpha \in \mathbb{C}\backslash\{0\}$ and, in particular, completeness of $L_\alpha(q)^* = L_\beta(\overline{q})$. For instance, a result from [34] guaranties completeness of these operators whenever $q_{\text{odd}}(x) := q(x) - q(1 - x)$ is smooth and $q_{\text{odd}}^{(k-1)}(0) \neq 0$ for some $k \in \mathbb{N}$. However, the following problem remains open:

Does there exist a non-analytic potential $q = \overline{q} \in C^\infty[0, 1]$ satisfying $q_{\text{odd}}^{(j-1)}(0) = 0$ for all $j \in \mathbb{N}$ and such that the operator $L_{q, \alpha}$ is peculiarly complete?

The existence of such a real potential $q$ would lead to a positive solution to Problem 1 for Sturm–Liouville operator (4.28) with such $q$. However, we conjecture that the answer is negative.

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5. Appendix. Regular and Weakly Regular Boundary Value Problems

Let us recall the definition of regular boundary conditions from [8, p.89]. We use the following construction. Let $A = \text{diag}(a_1, \ldots, a_n)$ be a diagonal matrix with entries $a_k$ (not necessarily distinct) that are not lying on the imaginary axis, $\text{Re } a_k \neq 0$. Starting from arbitrary matrices $C, D \in \mathbb{C}^{n \times n}$, we define the auxiliary $n \times n$ matrix $T_A(C, D)$ as follows:

- if $\text{Re } a_k > 0$, then the $k$th column in the matrix $T_A(C, D)$ coincides with the $k$th column of the matrix $C$,
- if $\text{Re } a_k < 0$, then the $k$th column in the matrix $T_A(C, D)$ coincides with the $k$th column of the matrix $D$.

It is clear that $T_A(C, D) = T_{-A}(D, C)$.

Definition 5.1. The boundary conditions (1.2) are called regular whenever $\det T_{izB}(C, D) \neq 0$ for every admissible $z \in \mathbb{C}$, i.e. for such $z$ that $\text{Re}(izB)$ is nonsingular.

To understand this definition better consider the lines $\{\lambda \in \mathbb{C} : \text{Re}(ib_j\lambda) = 0\}$, $j \in \{1, 2, \ldots, n\}$, of the complex plane. They divide the complex plane in $m = 2m' \leq 2n$ sectors. Denote these sectors by $\sigma_1, \sigma_2, \ldots, \sigma_m$. Let $z_1, z_2, \ldots, z_m$ be complex numbers such that $z_j$ lies in the interior of $\sigma_j, j \in \{1, \ldots, m\}$. The boundary conditions (1.2) are regular whenever

$$\det T_{iz_jB}(C, D) \neq 0, \quad j \in \{1, \ldots, m\}.$$  \hspace{1cm} (5.1)

Let us recall the concept of weakly regular boundary conditions from [36] and completeness results for BVP with such conditions.
Definition 5.2. ([36]) The boundary conditions (1.2) are called weakly $B$-regular (or, simply, weakly regular) if there exist three complex numbers $\{z_j\}_{j=1}^3$ satisfying the following conditions:

(a) the origin is an interior point of the triangle $\triangle z_1 z_2 z_3$;

(b) $\det T_{iz_j}(C, D) \neq 0$ for $j \in \{1, 2, 3\}$.

The following result is contained in [36, Theorem 1.2 and Corollary 3.3].

Theorem 5.3. Let $Q \in L^1([0, 1]; \mathbb{C}^{n \times n})$ and let boundary conditions (1.2) be weakly $B$-regular. Then the system of root functions of the BVP (1.1)–(1.2) (of the operator $L_{C,D}(Q)$) is complete and minimal in $L^2([0, 1]; \mathbb{C}^n)$.

Moreover, the system of root functions of the adjoint operator $L_{C,D}(Q)^*$ is also complete and minimal in $L^2([0, 1]; \mathbb{C}^n)$.

Corollary 5.4. Let $Q \in L^1([0, 1]; \mathbb{C}^{n \times n})$ and let the matrices $T_{iz}(C, D)$ and $T_{-iz}(C, D) = T_{iz}(D, C)$ be nonsingular for some $z \in \mathbb{C}$. Then

(i) The boundary conditions (1.3) are weakly $B$-regular.

(ii) The system of EAF of the operator $L_{C,D}(Q)$ is complete and minimal in $L^2([0, 1]; \mathbb{C}^n)$.

For $n \times n$ Dirac type system ($B = B^*$) the concept of weakly regular BC (1.2) coincides with that of regular ones and reads as follows

$$\det(CP_+ + DP_-) \neq 0 \quad \text{and} \quad \det(CP_- + DP_+) \neq 0. \quad (5.2)$$

Here $P_+$ and $P_-$ are the spectral projections onto “positive” and “negative” parts of the spectrum of $B = B^*$, respectively. Hence, by Theorem 5.3, under conditions (5.2) both operators $L_{C,D}(Q)$ and $L_{C,D}(Q)^*$ are complete and minimal. In the case $n = 2$ condition (5.2) turns into $J_{14}J_{32} \neq 0$.

Consider system (1.3) with the matrix $B = \text{diag}(b_1^{-1}, b_2^{-1}) \neq B^*$ assuming that $b_1/b_2 \notin \mathbb{R}$. In this case the lines $\{ \lambda \in \mathbb{C} : \text{Re}(ib_j\lambda) = 0 \}$, $j \in \{1, 2\}$, divide the complex plane in two pairs of vertical sectors and Corollary 5.4 guarantees the completeness and the minimality of the root system of problem (1.3)–(1.5) in the following cases:

(i) $J_{14}J_{32} \neq 0$ or (ii) $J_{12}J_{34} \neq 0$. \quad (5.3)

Note that the regularity of boundary conditions (1.2) implies in particular that

$$J_{14}J_{32}J_{12}J_{34} \neq 0.$$

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