MODULI SPACE OF 2|1-DIMENSIONAL COMPLEX
ASSOCIATIVE ALGEBRAS

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Abstract. In this paper, we study the moduli space of 2|1-dimensional complex associative algebras, which is also the moduli space of codifferentials on the tensor coalgebra of a 1|2-dimensional complex space. We construct the moduli space by considering extensions of lower dimensional algebras. We also construct miniversal deformations of these algebras. This gives a complete description of how the moduli space is glued together via jump deformations.

1. Introduction

The classification of associative algebras was instituted by Benjamin Peirce in the 1870’s [12], who gave a partial classification of the complex associative algebras of dimension up to 6, although in some sense, one can deduce the complete classification from his results, with some additional work. The classification method relied on the following remarkable fact:

Theorem 1.1. Every finite dimensional algebra which is not nilpotent contains a nontrivial idempotent element.

A nilpotent algebra $A$ is one which satisfies $A^n = 0$ for some $n$, while an idempotent element $a$ satisfies $a^2 = a$. This observation of Peirce eventually leads to two important theorems in the classification of finite dimensional associative algebras. Recall that an algebra is said to be simple if it has no nontrivial proper ideals, and it is not the 1-dimensional nilpotent algebra over $\mathbb{K}$, given by the trivial product.

Theorem 1.2 (Fundamental Theorem of Finite Dimensional Associative Algebras). Suppose that $A$ is a finite dimensional algebra over a field $\mathbb{K}$. Then $A$ has a maximal nilpotent ideal $N$, called its radical. If $A$ is not nilpotent, then $A/N$ is a semisimple algebra, that is, a direct sum of simple algebras.

Moreover, when $A/N$ satisfies a property called separability over $\mathbb{K}$, then $A$ is a semidirect product of its radical and a semisimple algebra. Over the complex numbers, every semisimple algebra is separable. To apply this theorem to construct algebras by extension, one uses the following characterization of simple algebras.

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Theorem 1.3 (Wedderburn). If $A$ is a finite dimensional algebra over $K$, then $A$ is simple iff $A$ is isomorphic to a tensor product $M \otimes D$, where $M = \mathfrak{gl}(n,K)$ and $D$ is a division algebra over $K$.

In the nongraded case, a division algebra is a unital algebra where every nonzero element has a multiplicative inverse. For $\mathbb{Z}_2$-graded associative algebras, the situation is a bit more complicated. First, we need to consider graded ideals, so that a $\mathbb{Z}_2$-graded algebra is simple when it has no proper nontrivial graded ideals. Secondly, the definition of a division algebra needs to be changed as well, in order to generalize Wedderburn’s theorem to the $\mathbb{Z}_2$-graded case. A $\mathbb{Z}_2$-graded division algebra is a division algebra when every nonzero homogeneous element is invertible.

With these changes, the Fundamental Theorem remains the same, except that the radical is the maximal graded nilpotent ideal, and Wedderburn’s theorem is also true, if we understand that by a matrix algebra, we mean the general linear algebra of a $\mathbb{Z}_2$-graded vector space.

In this paper, we shall be concerned with the moduli space of associative algebras on a $\mathbb{Z}_2$-graded algebra $A$ of dimension $2|1$, so that $A_0$ has dimension 2 and $A_1$ has dimension 1. However, we recall that this space coincides with the equivalence of odd codifferentials of degree 2 on the parity reversion $W = \Pi A$, which has dimension $1|2$, so in this paper, we shall study codifferentials on a space of dimension $1|2$, but the reader should keep in mind that this corresponds to associative algebras on a $2|1$-dimensional space.

The main goal of this paper is to give a complete description of the moduli space of $2|1$-dimensional associative algebras, including a computation of the miniversal deformation of every element.

2. Construction of algebras by extensions

In [3], the theory of extensions of an algebra $W$ by an algebra $M$ is described in the language of codifferentials. Consider the diagram

$$0 \rightarrow M \rightarrow V \rightarrow W \rightarrow 0$$

of associative $K$-algebras, so that $V = M \oplus W$ as a $K$-vector space, $M$ is an ideal in the algebra $V$, and $W = V/M$ is the quotient algebra. Suppose that $\delta \in C^2(W)$ and $\mu \in C^2(M)$ represent the algebra structures on $W$ and $M$ respectively. We can view $\mu$ and $\delta$ as elements of $C^2(V)$. Let $T^{k,l}$ be the subspace of $T^{k+l}(V)$ given recursively by $T^{0,0} = K$,

$$T^{k,l} = M \otimes T^{k-1,l} \oplus V \otimes T^{k,l-1}. $$

Let $C^{k,l} = \text{Hom}(T^{k,l}, M) \subseteq C^{k+l}(V)$. If we denote the algebra structure on $V$ by $d$, we have

$$d = \delta + \mu + \lambda + \psi,$$

where $\lambda \in C^{1,1}$ and $\psi \in C^{0,2}$. Note that in this notation, $\mu \in C^{2,0}$. Then the condition that $d$ is associative: $[d,d] = 0$ gives the following relations:

$$[\delta, \lambda] + \frac{1}{2} [\lambda, \lambda] + [\mu, \psi] = 0, \quad \text{The Maurer-Cartan equation}$$

$$[\mu, \lambda] = 0, \quad \text{The compatibility condition}$$

$$[\delta + \lambda, \psi] = 0, \quad \text{The cocycle condition}$$
Moreover, the group $\lambda$ equation, for the $D\mu$ is an algebra structure, $\psi$, then $D\mu$ is a differential on $C(V)$. Moreover $D\mu : C^{k,l} \to C^{k+1,l}$. Let

$$Z_{\mu}^{k,l} = \ker(D\mu : C^{k,l} \to C^{k+1,l}), \text{ the } (k,l)-\text{cocycles}$$

$$B_{\mu}^{k,l} = \text{Im}(D\mu : C^{k-1,l} \to C^{k,l}), \text{ the } (k,l)-\text{coboundaries}$$

$$H_{\mu}^{k,l} = Z_{\mu}^{k,l}/B_{\mu}^{k,l}, \text{ the } D\mu (k,l)-\text{cohomology}$$

Then the compatibility condition means that $\lambda \in Z^{1,1}$. If we define $D_{\delta+\lambda}(\varphi) = [\delta + \lambda, \varphi]$, then it is not true that $D_{\delta+\lambda}^{2} = 0$, but $D_{\delta+\lambda}D\mu = -D\mu D_{\delta+\lambda}$, so that $D_{\delta+\lambda}$ descends to a map $D_{\delta+\lambda} : H_{\mu}^{k,l} \to H_{\mu}^{k,l+1}$, whose square is zero, giving rise to the $D_{\delta+\lambda}$-cohomology $H_{\mu}^{k,l+1}$. If the pair $(\lambda, \psi)$ give rise to a codifferential $d$, and $(\lambda, \psi')$ give rise to another codifferential $d'$, then if we express $\psi' = \psi + \tau$, it is easy to see that $[\mu, \tau] = 0$, and $[\delta + \lambda, \tau] = 0$, so that the image $\bar{\tau}$ of $\tau$ in $H_{\mu}^{0,2}$ is a $D_{\delta+\lambda}$-cocycle, and thus $\tau$ determines an element $\{\bar{\tau}\}$ in $H_{\mu}^{0,2}$.

If $\beta \in C^{0,1}$, then $g = \exp(\beta) : T(V) \to T(V)$ is given by $g(m, w) = (m + \beta(w), w)$. Furthermore $g^* = \exp(-\text{ad}_{\beta}) : C(V) \to C(V)$ satisfies $g^*(d) = d'$, where $d' = \delta + \mu + \lambda' + \psi'$ with

$$\lambda' = \lambda + [\mu, \beta]$$

$$\psi' = \psi + [\delta + \lambda + \frac{1}{2}[\mu, \beta], \beta]$$

In this case, we say that $d$ and $d'$ are equivalent extensions in the restricted sense. Such equivalent extensions are also equivalent as codifferentials on $T(V)$. Note that $\lambda$ and $\lambda'$ differ by a $D\mu$-coboundary, so $\bar{\lambda} = \lambda'$ in $H_{\mu}^{1,1}$. If $\lambda$ satisfies the MC-equation for some $\psi$, then any any element $\lambda'$ in $\bar{\lambda}$ also gives a solution of the MC equation, for the $\psi'$ given above. The cohomology classes of those $\lambda$ for which a solution of the MC equation exists determine distinct restricted equivalence classes of extensions.

Let $G_{M,W} = \text{GL}(M) \times \text{GL}(W) \subseteq \text{GL}(V)$. If $g \in G_{M,W}$ then $g^* : C^{k,l} \to C^{k,l}$, and $g^* : C^{k}(W) \to C^{k}(W)$, so $d' = g^*(\delta)$ and $\mu' = g^*(\mu)$ are codifferentials on $T(M)$ and $T(W)$ respectively. The group $G_{\delta,\mu}$ is the subgroup of $G_{M,W}$ consisting of those elements $g$ such that $g^*(\delta) = \delta$ and $g^*(\mu) = \mu$. Then $G_{\delta,\mu}$ acts on the restricted equivalence classes of extensions, giving the equivalence classes of general extensions. Also, $G_{\delta,\mu}$ acts on $H_{\mu}^{k,l}$, and induces an action on the classes $\lambda$ of $\lambda$ giving a solution $(\lambda, \psi)$ to the MC equation.

Next, consider the group $G_{\delta,\mu,\lambda}$ consisting of the automorphisms $h$ of $V$ of the form $h = g \exp(\beta)$, where $g \in G_{\delta,\mu}$, $\beta \in C^{0,1}$ and $\lambda = g^*(\lambda) + [\mu, \beta]$. If $d = \delta + \mu + \lambda + \psi + \tau$, then $h^*(d) = \delta + \mu + \lambda + \psi + \tau'$ where

$$\tau' = g^*(\psi) - \psi + [\delta + \lambda + \frac{1}{2}[\mu, \beta], \beta] + g^*(\tau).$$

Moreover, the group $G_{\delta,\mu,\lambda}$ induces an action on $H_{\mu,\delta+\lambda}^{0,2}$ given by $\bar{\tau} \mapsto \{\bar{\tau}'\}$. In fact, $\{g^*(\tau)\}$ is well defined as well, and depends only on $\{\bar{\tau}\}$.

The general group of equivalences of extensions of the algebra structure $\delta$ on $W$ by the algebra structure $\mu$ on $M$ is given by the group of automorphisms of $V$ of the form $h = \exp(\beta)g$, where $\beta \in C^{0,1}$ and $g \in G_{\delta,\mu}$. We have the following classification of such extensions up to equivalence.
Theorem 2.1. The equivalence classes of extensions of $\delta$ on $W$ by $\mu$ on $M$ is classified by the following:

1. Equivalence classes of $\bar{\lambda} \in H^{1,1}_\mu$ which satisfy the MC equation
   \[ [\delta, \lambda] + \frac{1}{2}[\lambda, \lambda] + [\mu, \psi] = 0 \]
   for some $\psi \in C^{0,2}$, under the action of the group $G_{\delta, \mu}$.

2. Equivalence classes of $\{\bar{\tau}\} \in H^{0,2}_{\mu, \delta + \lambda}$ under the action of the group $G_{\delta, \mu, \lambda}$.

Equivalent extensions will give rise to equivalent codifferentials on $V$, but it may happen that two codifferentials arising from nonequivalent extensions are equivalent. This is because the group of equivalences of extensions is the group of invertible block upper triangular matrices on the space $V = M \oplus W$, whereas the the equivalence classes of codifferentials on $V$ are given by the group of all invertible matrices, which is larger.

The fundamental theorem of finite dimensional algebras allows us to restrict our consideration of extensions to two cases. First, we can consider those extensions where $\delta$ is a semisimple algebra structure on $W$, and $\mu$ is a nilpotent algebra structure on $M$. In this case, because we are working over $\mathbb{C}$, we can also assume that $\psi = \tau = 0$. Thus the classification of the extension reduces to considering equivalence classes of $\lambda$.

Secondly, we can consider extensions of the trivial algebra structure $\delta = 0$ on a 1-dimensional space $W$ by a nilpotent algebra $\mu$. This is because a nilpotent algebra has a codimension 1 ideal $M$, and the restriction of the algebra structure to $M$ is nilpotent. However, in this case, we cannot assume that $\psi$ or $\tau$ vanish, so we need to use the classification theorem above to determine the equivalence classes of extensions. In many cases, in solving the MC equation for a particular $\lambda$, if there is any $\phi$ yielding a solution, then $\psi = 0$ also gives a solution, so the action of $G_{\delta, \mu, \lambda}$ on $H^{0,2}_{\mu, \delta + \lambda}$ takes on a simpler form than the general action we described above. In fact, if in addition to $\psi = 0$ providing a solution to the MC equation, any element $h = g \exp(\beta)$ satisfies $[\mu, \beta] = 0$, then the action of $h$ on $H^{0,2}_{\delta, \mu, \lambda}$ is just the action $g^\ast(\{\bar{\tau}\}) = \{g^\ast(\tau)\}$, which is easy to calculate in practice.

3. Associative algebra structures on a 2|1 vector space

Let $A$ be a 2|1-dimensional vector space, and $V = \Pi A$ be the parity reversion of $A$, so that $V$ is 1|2-dimensional. Let $\{v_1, v_2, v_3\}$ be a basis of $V$ with $v_1$ an even element and $v_2, v_3$ odd elements and $d$ be a codifferential on $V$ representing an associative algebra structure on $A$.

By results in [1], there are only two $\mathbb{Z}_2$-graded division algebras, the complex numbers, and a certain 1|1-dimensional algebra. As a consequence, there are no 2|1-dimensional simple algebras, so we can express $V$ as an extension of an algebra structure $\delta$ on $W$ by an algebra structure $\mu$ on $M$, where $V = M \oplus W$, and $M$ is an ideal in $V$.

By the fundamental theorem of finite dimensional associative algebras, we can assume that $\mu$ is a nilpotent algebra structure on $M$. Moreover, $\delta$ is a semisimple algebra structure on $W$, unless $d$ is a nilpotent algebra structure.

Since every nilpotent algebra has a codimension 1 ideal, if $d$ is nilpotent, we can assume that $W$ is 1-dimensional (either even or odd), and that $\delta = 0$. The only semisimple algebras we need to consider are the simple 1|1-dimensional algebra, and
can express
\[ R = M \]
Then we have
de the matrices \( L \) and \( W \) and similarly for the components of the other matrices. Let \( L_k = LE_k + LO_k \) and \( R_k = RE_k + RO_k \). Then
\[
\frac{1}{2}[\lambda, \lambda] = \psi_{m(i)}^w(k)(m(j)w(l))(LE_k)^i_j + \psi_{m(i)}^w(k)(m(j)w(l))(RE_k)^i_j.
\]
where \( LE_k \) and \( RE_k \) are matrices of even maps \( M \to M \), and \( LO_k \) and \( RO_k \) are matrices of odd maps \( M \to M \). For simplicity, we shall denote \( (LE_k)^i_j \) as \( LE_{k,i}^j \) and similarly for the components of the other matrices.

It is important to note that the formula above is given in terms of matrix multiplication. This is significant from the computational view as we shall illustrate below. It is interesting to note that matrices in \( G_{M,W} \), which are block diagonal maps \( \text{diag}(G_1, G_2) \), where \( G_1 \in \text{GL}(M) \) and \( G_2 \in \text{GL}(W) \), act on \( \lambda \) in a manner which can be described in terms of the matrices above. First, \( G_1 \) acts by conjugating all the matrices \( L_k \) and \( R_k \) simultaneously. Secondly, the matrix \( G_2 \) acts on the \( k \) indices. We shall say more about these actions later.

Let us give one concrete application of the remarks above. Suppose that \( W \) is completely odd, and \( M \) is \( r \)-dimensional. Then we can express \( M = \langle v_1, \ldots, v_r \rangle \) and \( W = \langle v_{r+1}, \ldots, v_{r+n} \rangle \). In this case \( L_k = LE_k \) and \( R_k = RE_k \). We can express \( R_k = \text{diag}(T_k, B_k) \), where \( T_k : M_0 \to M_0 \) and \( B_k : M_1 \to M_1 \) and \( M = M_0 \oplus M_1 \) represents the decomposition of \( M \) into its even and odd parts. Then we have
\[
\frac{1}{2}[\lambda, \lambda] = -\psi^k_{i,j}(L_k L_i)^j_i + \psi^k_{i,j}(R_i L_k - L_k R_i)^j_i + \psi^k_{i,j}(R_i R_k)^j_i.
\]
Let \( \delta = \sum_{k=r+n+1}^{r+n} \psi^k_{i,k} \) be the codifferential semisimple algebra \( \mathbb{C}^n \). Then
\[
[\delta, \lambda] = \psi^k_{i,j}(L_k)^j_i + (-1)^j \psi^j_{i,k}(R_k)^j_i.
\]
When considering an extension of the algebra \( \mathbb{C}^n \) by an algebra structure on \( M \), we can assume that the cocycle \( \psi \) vanishes, so the MC equation is just \( [\delta, \lambda] + \frac{1}{2}[\lambda, \lambda] = 0 \), which is equivalent to the following:

\[
L_k = L_k^2, \quad L_k L_l = 0, \quad \text{if } k \neq l
\]
\[
L_k R_l = R_l L_k
\]
\[
T_k = -T_k^2, \quad B_k = B_k^2, \quad R_k R_l = 0, \quad \text{if } k \neq l.
\]
As a consequence, the matrices above give a commuting set of diagonalizable matrices, so they can be simultaneously diagonalized. Moreover, \( L_k \) and \( B_k \) have only 0 and 1 as possible eigenvalues and \( T_k \) has only 0 and \(-1\) as possible eigenvalues. If we consider \( \mu = 0 \) as the algebra structure on \( M \), then the elements \( \text{diag}(G_1, G_2) \in G_{\delta, \mu} \) are given by an arbitrary matrix \( G_1 \) and \( G_2 \) is just a permutation matrix.

Thus we can apply an element \( G \in G_{\delta, \mu} \) to \( \lambda \) to put it in the form where all the matrices are diagonal, and are ordered in such a manner that the nonzero \( L_k \) matrices appear first. Moreover, since the \( L_k \) matrices are orthogonal to each other, there are at most \( m = r + s \) nonzero \( L \) matrices.

Similar considerations apply to the \( R \) matrices, so that in total, there can be no more than \( 2m \) pairs \((L_k, R_k)\), such that at least one matrix does not vanish. Therefore, when \( n \geq 2m \), the number of distinct equivalence classes of extensions of \( \mathbb{C}^n \) by a trivial algebra structure on \( M \) is exactly equal to \( 2m \).

We say that this is the stable situation. The number of extensions is independent of \( n \) as long as it is at least \( 2m \). Moreover, the cohomology and deformation theory also becomes stable, in a natural way. When \( \mu \neq 0 \), the situation is a bit more complicated, but there is also an \( n \) beyond which the situation becomes stable.

We now give a construction of the elements in the moduli space of 1-dimensional simple algebra. The number of extensions is independent of the semisimple algebra structure on \( C \), and is exactly equal to \( 2 \).

Let \( \lambda = \psi^1 \), \( \mu = \psi^2 \), \( \lambda = \psi^3 \), \( \mu = \psi^3 \). Then the of \( \delta \) with the simple 0-dimensional algebra \( \mu = \psi^2 \) gives the unique semisimple 1/2 dimension \( d_1 \). Now consider the extensions of \( \delta \) by the trivial 0/1-dimensional algebra \( \mu = 0 \).

The generic lambda is of the form \( \lambda = \psi^1 \). The only semisimple algebra structure on \( W \) is given by \( \delta = \psi^2 \). Then the of \( \delta \) with the simple \( 0 \)-dimensional algebra \( \mu = \psi^2 \) gives the unique semisimple 1/2 dimension \( d_1 \).

4. Extensions where \( W \) is 1/1-dimensional and \( M \) is 0/1-dimensional

Let \( W = \langle v_1, v_2 \rangle \) and \( M = \langle v_2 \rangle \). The unique 1/1-dimensional simple algebra is given by the codifferential \( \delta = \psi^1 \). Then the of \( \delta \) with the simple 0/1-dimensional algebra \( \mu = \psi^2 \) gives the unique semisimple 1/2 dimension \( d_1 \).

\[
[\delta, \lambda] = \phi^1 \left( LE_{21}^1 + \psi^3 RE_{21}^1 \right), \quad \text{and} \quad \frac{1}{2}[\lambda, \lambda] = -\phi^2 \left( LE_{21}^1 \right)^2 + \psi^3 \left( RE_{21}^1 \right)^2
\]

so the MC equation forces \( \lambda = 0 \). Therefore, the unique extension of \( \delta \) is the direct sum of \( \delta \) and the trivial 1-dimensional algebra, which is the codifferential \( d_2 \).

5. Extensions where \( W \) is 0/2-dimensional and \( M \) is 1/0-dimensional

Let \( W = \langle v_2, v_3 \rangle \) be 0/2-dimensional and \( M = \langle v_1 \rangle \) be 1/0-dimensional. The only semisimple algebra structure on \( W \) is given by \( \delta = \psi^3 \). Then the of \( \delta \) with the simple \( 1 \)-dimensional algebra \( \mu = 0 \). This case is an example of the general case of extending the semisimple algebra structure \( \mathbb{C}^2 \) by the trivial algebra structure on \( M \). In fact, this is the stable case because the dimension of \( W \) is exactly twice the dimension of \( M \).

A generic \( \lambda \) is of the form

\[
\lambda = \psi^2 \left( LE_{11}^1 + \psi^3 R_{11}^1 \right) + \psi^3 \left( LE_{21}^1 + \psi^1 R_{21}^1 \right).
\]

We have exactly five solutions up to equivalence given by

\[
\begin{bmatrix} L_1 & R_1 \\ L_2 & R_2 \end{bmatrix} \in \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\},
\]

corresponding to the codifferentials \( d_3, \ldots, d_7 \).
6. Extensions where $M$ is 1|1-dimensional and $W$ is 0|1-dimensional

Let $M = \langle v_1, v_2 \rangle$ be 1|1-dimensional and $W = \langle v_3 \rangle$ be 0|1-dimensional. The group $G_{M,W}$ consists of diagonal matrices $G = \text{diag}(r,s,t)$, with $rst \neq 0$. A generic element of $C^{1,1}$ is of the form $\lambda = \psi_1^{31} a L_{11}^1 + \psi_1^{33} R_{11}^1 + \psi_2^{23} L_{12}^2 + \psi_2^{23} R_{12}^2$, corresponding to the matrices

$$L_1 = \begin{bmatrix} L_{11} & 0 \\ 0 & L_{12} \end{bmatrix}, \quad R_1 = \begin{bmatrix} R_{11} & 0 \\ 0 & R_{12} \end{bmatrix}. $$

An element of $C^{0,2}$ is of the form $\tau = \psi_2^{33} c_1$ and an element of $C^{0,1}$ is of the form $\beta = \psi_2^{31} b_1$. 

| Codifferential | $H^0$ | $H^1$ | $H^2$ | $H^3$ | $H^4$ |
|-----------------|-------|-------|-------|-------|-------|
| $d_1 = \psi_1^{13} - \psi_1^{31} + \psi_3^{13} + \psi_2^{22} - \psi_3^{33}$ | 2 | 0 | 0 | 0 | 0 |
| $d_2 = \psi_1^{13} - \psi_1^{31} + \psi_3^{13} - \psi_3^{33}$ | 2 | 1 | 1 | 1 | 1 |
| $d_3 = \psi_2^{23} + \psi_3^{33} - \psi_1^{12}$ | 1 | 0 | 0 | 0 | 0 |
| $d_4 = \psi_2^{23} + \psi_3^{33} + \psi_1^{21}$ | 1 | 0 | 0 | 0 | 0 |
| $d_5 = \psi_2^{23} + \psi_3^{33} + \psi_1^{21} - \psi_1^{13}$ | 1 | 0 | 0 | 0 | 0 |
| $d_6 = \psi_2^{23} + \psi_3^{33} + \psi_1^{21} - \psi_1^{12}$ | 3 | 2 | 2 | 2 | 2 |
| $d_7 = \psi_2^{23} + \psi_3^{33}$ | 3 | 1 | 1 | 1 | 1 |
| $d_8 = \psi_1^{33} + \psi_2^{11} + \psi_3^{31} - \psi_1^{13} + \psi_2^{32} + \psi_2^{23}$ | 2 | 1 | 1 | 1 | 1 |
| $d_9 = \psi_3^{33} + \psi_1^{31}$ | 2 | 1 | 2 | 2 | 1 |
| $d_{10} = \psi_2^{33} - \psi_1^{13} + \psi_3^{23}$ | 0 | 3 | 0 | 0 | 0 |
| $d_{11} = \psi_2^{33} + \psi_1^{31} + \psi_3^{23}$ | 0 | 3 | 0 | 0 | 0 |
| $d_{12} = \psi_2^{33} + \psi_1^{31} + \psi_3^{23}$ | 0 | 1 | 0 | 1 | 0 |
| $d_{13} = \psi_2^{33} - \psi_1^{13} + \psi_3^{23}$ | 0 | 1 | 0 | 1 | 0 |
| $d_{14} = \psi_3^{33} + \psi_1^{31} - \psi_1^{13} + \psi_2^{23}$ | 1 | 1 | 1 | 1 | 1 |
| $d_{15} = \psi_3^{33} + \psi_1^{31} - \psi_1^{13} + \psi_2^{23}$ | 1 | 1 | 1 | 1 | 1 |
| $d_{16} = \psi_3^{33} + \psi_2^{23}$ | 1 | 1 | 2 | 2 | 2 |
| $d_{17} = \psi_3^{33} + \psi_3^{32}$ | 1 | 1 | 2 | 2 | 2 |
| $d_{18} = \psi_3^{33} - \psi_1^{13} + \psi_2^{23} + \psi_3^{32}$ | 1 | 1 | 1 | 1 | 1 |
| $d_{19} = \psi_3^{33} + \psi_1^{31} + \psi_3^{23} + \psi_2^{32}$ | 1 | 1 | 1 | 1 | 1 |
| $d_{20} = \psi_3^{33} + \psi_2^{23} + \psi_2^{22}$ | 3 | 2 | 2 | 2 | 2 |
| $d_{21} = \psi_3^{33} - \psi_1^{13}$ | 1 | 1 | 2 | 2 | 2 |
| $d_{22} = \psi_3^{33} + \psi_1^{31}$ | 1 | 1 | 2 | 2 | 2 |
| $d_{23} = \psi_3^{33} - \psi_1^{13} + \psi_1^{31}$ | 3 | 3 | 3 | 3 | 3 |
| $d_{24} = \psi_3^{33} + \psi_1^{31} + \psi_2^{23} + \psi_3^{32}$ | 3 | 4 | 4 | 6 | 24 |
| $d_{25} = \psi_3^{33}$ | 3 | 4 | 8 | 16 | 32 |
| $d_{26} = \psi_1^{11} + \psi_3^{33}$ | 2 | 2 | 3 | 5 | 6 |
| $d_{27} = \psi_1^{11}$ | 3 | 4 | 9 | 18 | 32 |
| $d_{28} = \psi_2^{33}$ | 3 | 5 | 9 | 17 | 33 |

**Table 1.** Cohomology of the 28 families of codifferentials on a 1|2-dimensional space
There is a nontrivial nilpotent 1|1-dimensional algebra. Thus we have to take into account extensions by both the nontrivial and the trivial algebra structure on $M$.

6.0.1. Extensions of the simple 0|1-dimensional algebra. In this case $\delta = \psi^{33}_1$. First, we consider the extensions by the nontrivial algebra $\mu = \psi^{11}_1$. Then $G_{\mu,\delta}$ consists of the diagonal matrices of the form $G = \text{diag}(r, r^2, 1)$, with $r \neq 0$. Now $[\mu, \lambda] = 0$ precisely when $L_{11} = L_{12}^2 = R_{12}^2 = -R_{11}^2$. In other words, $L_1 = \text{diag}(L_{11}, L_{11})$ and $R_1 = \text{diag}(-L_{11}, L_{11})$. Since we already know by the MC equation that the eigenvalues of $L_1$ are either 0 or 1, this gives exactly two codifferentials, corresponding to $d_8$ and $d_9$.

Next suppose that $\mu = 0$. Then $L_1$ is either $I$, $\text{diag}(1, 0)$, $\text{diag}(0, 1)$ or 0, while $R_1$ is either $\text{diag}(-1, 1)$, $\text{diag}(-1, 0)$ $\text{diag}(0, 1)$ or 0. Since these conditions are independent, there are a total of 16 different solutions, giving rise to the codifferentials $d_{10}$ through $d_{25}$.

6.0.2. Extensions of the trivial 0|1-dimensional algebra. In this case, $\delta = 0$. Let us first consider the extensions by the nontrivial nilpotent algebra $\mu = \psi^{11}_1$. Then $G_{\delta,\mu}$ consists of diagonal matrices of the form $G = \text{diag}(r, r^2, t)$, where $rt \neq 0$. The MC equation yields $\lambda = 0$, and it places no condition on $\psi$, so we can take $\psi = 0$.

Then we need to compute $H^{0,2}_{\mu,\delta+\lambda}$, which is spanned by $\psi^{33}_1$. If we let $\tau = \psi^{33}_1 c$, then consider the action of $G_{\delta,\mu,\lambda} = G_{\delta,\mu}$ on $\tau$, and we obtain $g^\ast(\tau) = \tau \ast t$, when $G = \text{diag}(r, r^2, t)$. Therefore, we can reduce to the cases when $t = 1$, which gives $d_{26}$ and $t = 0$, which gives $d_{27}$. These two algebras are nilpotent.

Next, consider the case $\mu = 0$. Then $G_{\delta,\mu} = G_{M, W}$. As in the previous case, we obtain $\lambda = 0$, $\tau = \psi^{33}_1 c$, and the action of $G_{\delta,\mu,\lambda}$ allows us to consider only the cases $c = 1$, which gives $d_{28}$, and $c = 0$, which gives the zero codifferential.

7. Extensions where $M$ is 0|2-dimensional and $W$ is 1|0-dimensional

We have $M = \langle v_2, v_3 \rangle$ and $W = \langle v_1 \rangle$. The group $G_{M, W}$ consists of matrices of the form $G = \begin{bmatrix} g_{11} & 0 \\ 0 & G \end{bmatrix}$, where $G \in \text{GL}(2, \mathbb{C})$. In this case we have $\delta = 0$ and $\lambda = 0$. Moreover $C^{0,2} = \langle \psi^{11}_2, \psi^{11}_3 \rangle$. There are two possibilities for $\mu$, the trivial codifferential and $\mu = \psi^{33}_2$.

Let us consider the nontrivial case for $\mu$ first. Then $G_{\delta,\mu}$ is given by the matrices in $G_{M, W}$ such that $G = \begin{bmatrix} g_{3,3}^2 & g_{3,3} \\ 0 & g_{3,3} \end{bmatrix}$. Let $\psi = \psi^{11}_2 c_1 + \psi^{11}_3 c_2$. Then the MC equation reduces to $[\mu, \psi] = 0$, which forces $c_2 = 0$. Therefore, we can set $\psi = 0$ and $\tau = \psi^{11}_2 c_1$. Now let $g$ be an element of $G_{\delta,\mu}$, given by the matrix $G$ above.

Then $g^\ast(\tau) = \tau \ast \frac{g_{3,3}^2}{g_{3,3}}$, so we need only consider the cases $\tau = \psi^{11}_2$ and $\tau = 0$. The first one duplicates the codifferential $d_{26}$ while the second duplicates $d_{28}$.

Finally, consider the case $\mu = 0$. Then $G_{\delta,\mu} = G_{M, W}$. There is no condition on $\psi$, so we can take $\psi = 0$ and $\tau = \psi^{11}_2 t_1 + \psi^{11}_3 t_2$. If we represent $\tau$ by the column vector $[t_1, t_2]$, then the action of an element $g$ on $\tau$ is given by $T \mapsto g_{1,1} G^{-1} T$. This means we can reduce to the case where $\tau = \psi^{11}_2$ or $\tau = 0$. The first one is equivalent to $d_{27}$, already seen, and the second one is the zero codifferential.

Thus we have completed the classification of the elements in the moduli space of associative algebra structures on a space $V$ of dimension 2|1, or, in other words, the codifferentials on a 1|2-dimensional space.
8. Hochschild Cohomology and Deformations

Hochschild cohomology was introduced in [10], and used to classify infinitesimal deformations of associative algebras. Suppose that 

\[ m_t = m + t \varphi, \]

is an infinitesimal deformation of \( m \). By this we mean that the structure \( m_t \) is associative up to first order. From an algebraic point of view, this means that we assume that \( t^2 = 0 \), and then check whether associativity holds. It is not difficult to show that is equivalent to the following.

\[ t \varphi(b,c) - \varphi(ab,c) + \varphi(a,bc) - \varphi(a,b)c = 0, \]

where, for simplicity, we denote \( m(a,b) = ab \). Moreover, if we let

\[ g_t = I + t \lambda \]

be an infinitesimal automorphism of \( A \), where \( \lambda \in \text{Hom}(A,A) \), then it is easily checked that

\[ g^*_t(m)(a,b) = ab + t(a \lambda(b) - \lambda(ab) + \lambda(a)b). \]

This naturally leads to a definition of the Hochschild coboundary operator \( D \) on \( \text{Hom}(T(A), A) \) by

\[
D(\varphi)(a_0, \ldots, a_n) = a_0 \varphi(a_1, \ldots, a_n) + (-1)^{n+1} \varphi(a_0, \ldots, a_{n-1})a_n \\
+ \sum_{i=0}^{n-1} (-1)^{i+1} \varphi(a_0, \ldots, a_{i-1}, a_ia_{i+1}, a_{i+2}, \ldots, a_n).
\]

If we set \( C^n(A) = \text{Hom}(A^n, A) \), then \( D : C^n(A) \to C^{n+1}(A) \). One obtains the following classification theorem for infinitesimal deformations.

**Theorem 8.1.** The equivalence classes of infinitesimal deformations \( m_t \) of an associative algebra structure \( m \) under the action of the group of infinitesimal automorphisms on the set of infinitesimal deformations are classified by the Hochschild cohomology group

\[ H^2(m) = \ker(D : C^2(A) \to C^3(A))/\text{Im}(D : C^1(A) \to C^2(A)). \]

When \( A \) is \( \mathbb{Z}_2 \)-graded, the only modifications that are necessary are that \( \varphi \) and \( \lambda \) are required to be even maps, so we obtain that the classification is given by \( H^2(A) \), the even part of the Hochschild cohomology.

We wish to transform this classical viewpoint into the more modern viewpoint of associative algebras as being given by codifferentials on a certain coalgebra. To do this, we first introduce the parity reversion \( \Pi A \) of a \( \mathbb{Z}_2 \)-graded vector space \( A \). If \( A = A_e \oplus A_o \) is the decomposition of \( A \) into its even and odd parts, then \( W = \Pi A \) is the \( \mathbb{Z}_2 \)-graded vector space given by \( W_e = A_o \) and \( W_o = A_e \). In other words, \( W \) is just the space \( A \) with the parity of elements reversed.

Denote the tensor (co)-algebra of \( W \) by \( T(W) = \bigoplus_{k=0}^{\infty} W^k \), where \( W^k \) is the \( k \)-th tensor power of \( W \) and \( W^0 = \mathbb{K} \). For brevity, the element in \( W^k \) given by the tensor product of the elements \( v_i \) in \( W \) will be denoted by \( v_1 \cdots v_k \). The coalgebra structure on \( T(W) \) is given by

\[
\Delta(v_1 \cdots v_n) = \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n.
\]
Define $d : W^2 \to W$ by $d = \pi \circ m \circ (\pi^{-1} \otimes \pi^{-1})$, where $\pi : A \to W$ is the identity map, which is odd, because it reverses the parity of elements. Note that $d$ is an odd map. The space $C(W) = \text{Hom}(T(W), W)$ is naturally identifiable with the space of coderivations of $T(W)$. In fact, if $\varphi \in C^k(W) = \text{Hom}(W^k, W)$, then $\varphi$ is extended to a coderivation of $T(W)$ by

$$\varphi(v_1 \cdots v_n) = \sum_{i=0}^{n-k} (-1)^{(v_i + \cdots + v_n)} \varphi v_1 \cdots v_i \varphi(v_{i+1} \cdots v_{i+k}) v_{i+k+1} \cdots v_n.$$  

The space of coderivations of $T(W)$ is equipped with a $\mathbb{Z}_2$-graded Lie algebra structure given by

$$[\varphi, \psi] = \varphi \circ \psi - (-1)^{\varphi \psi} \psi \circ \varphi.$$  

The reason that it is more convenient to work with the structure $d$ on $W$ rather than $m$ on $A$ is that the condition of associativity for $m$ translates into the coderivation property $[d, d] = 0$. Moreover, the Hochschild coboundary operation translates into the coboundary operator $D$ on $C(W)$, given by

$$D(\varphi) = [d, \varphi].$$  

This point of view on Hochschild cohomology first appeared in [16]. The fact that the space of Hochschild cochains is equipped with a graded Lie algebra structure was noticed much earlier [5, 6, 7, 8, 9].

For notational purposes, we introduce a basis of $C^n(W)$ as follows. Suppose that $W = (v_1, \cdots, v_m)$. Then if $I = (i_1, \cdots, i_n)$ is a multi-index, where $1 \leq i_k \leq m$, denote $v_I = v_{i_1} \cdots v_{i_n}$. Define $\varphi_I^j \in C^n(W)$ by

$$\varphi_I^j(v_{j}) = \delta^j_I v_{i_j},$$  

where $\delta^j_I$ is the Kronecker delta symbol. In order to emphasize the parity of the element, we will denote $\varphi_I^j$ by $\psi_I^j$ when it is an odd coderivation.

For a multi-index $I = (i_1, \cdots, i_k)$, denote its length by $\ell(I) = k$. If $K$ and $L$ are multi-indices, then denote $KL = (k_1, \cdots, k_{\ell(K)}, l_1, \cdots, l_{\ell(L)})$. Then

$$\langle \varphi_I^j \circ \varphi_J^j \rangle(v_K) = \sum_{K_1, K_2, K_3 = K} (-1)^{v_{k_1} v_{j}^I} \varphi_I^j \varphi_J^j(v_{K_1}, v_{K_2} v_{K_3})$$  

$$= \sum_{K_1, K_2, K_3 = K} (-1)^{v_{k_1} v_{j}^J} \delta^j_{K_1} \delta_{K_2} \delta^j_{K_3} v_{i_j},$$  

from which it follows that

$$\varphi_I^j \circ \varphi_J^j = \sum_{k=1}^{\ell(I)} (-1)^{(v_{i_1} + \cdots + v_{i_{k-1}}) v_{j}^J} \delta^j_{I} \delta^j_{I} \varphi_J^{(I, k)},$$  

where $(I, J, k)$ is given by inserting $J$ into $I$ in place of the $k$-th element of $I$; i.e., $(I, J, k) = (i_1, \cdots, i_{k-1}, j_1, \cdots, j_{\ell(J)}, i_{k+1}, \cdots, i_{\ell(I)})$.

Let us recast the notion of an infinitesimal deformation in terms of the language of coderivations. We say that

$$d_t = d + t\psi$$  

is a deformation of the codifferential $d$ precisely when $[d_t, d_t] = 0 \mod t^2$. This condition immediately reduces to the cocycle condition $D(\psi) = 0$. Note that we require $d_t$ to be odd, so that $\psi$ must be an odd coderivation. One can introduce a more general idea of parameters, allowing both even and odd parameters, in which
case even coderivations play an equal role, but we will not adopt that point of view in this paper.

For associative algebras, we require that \(d\) and \(\psi\) lie in \(\text{Hom}(W^2, W)\). This notion naturally generalizes to considering \(d\) simply to be an arbitrary odd codifferential, in which case we would obtain an \(A_\infty\) algebra, a natural generalization of an associative algebra.

More generally, we need the notion of a versal deformation, in order to understand how the moduli space is glued together. To explain versal deformations we introduce the notion of a deformation with a local base. A local base \(A\) is a \(\mathbb{Z}_2\)-graded commutative, unital \(K\)-algebra with an augmentation \(\epsilon : A \to \mathbb{K}\), whose kernel \(m\) is the unique maximal ideal in \(A\), so that \(A\) is a local ring. It follows that \(A\) has a unique decomposition \(A = \mathbb{K} \oplus m\) and \(\epsilon\) is just the projection onto the first factor. Let \(W_A = W \otimes A\) equipped with the usual structure of a right \(A\)-module. Let \(T_A(W_A)\) be the tensor algebra of \(W_A\) over \(A\), that is \(T_A(W_A) = \bigoplus_{k=0}^{\infty} T^k_A(W_A)\) where \(T^0_A(W_A) = A\) and \(T^k_A(W_A) = T^k(W_A)_A \otimes_A W_A\). It is a standard fact that \(T^k_A(W_A) = T^k(W) \otimes A\) in a natural manner, and thus \(T_A(W_A) = T(W) \otimes A\).

Any \(A\)-linear map \(f : T_A(W) \to T_A(W)\) is induced by its restriction to \(T(W) \otimes \mathbb{K} = T(W)\) so we can view an \(A\)-linear coderivation \(\delta_A\) on \(T_A(W_A)\) as a map \(\delta_A : T(W) \to T(W) \otimes A\). A morphism \(f : A \to B\) induces a map

\[
f_A : \text{Coder}_A(T_A(W_A)) \to \text{Coder}_B(T_B(W_B))\]

given by \(f_A(\delta_A) = (1 \otimes f)\delta_A\), moreover if \(\delta_A\) is a codifferential then so is \(f_A(A)\). A codifferential \(d_A\) on \(T_A(W_A)\) is said to be a deformation of the codifferential \(d\) on \(T(W)\) if \(\epsilon_*(d_A) = d\).

If \(d_A\) is a deformation of \(d\) with base \(A\) then we can express

\[d_A = d + \varphi\]

where \(\varphi : T(W) \to T(W) \otimes m\). The condition for \(d_A\) to be a codifferential is the Maurer-Cartan equation,

\[D(\varphi) + \frac{1}{2} [\varphi, \varphi] = 0\]

If \(m^2 = 0\) we say that \(A\) is an infinitesimal algebra and a deformation with base \(A\) is called infinitesimal.

A typical example of an infinitesimal base is \(\mathbb{K}[t]/(t^2)\), moreover, the classical notion of an infinitesimal deformation

\[d_t = d + t\varphi\]

is precisely an infinitesimal deformation with base \(\mathbb{K}[t]/(t^2)\).

A local algebra \(A\) is complete if

\[A = \lim_{\to \infty} A/m^k\]

A complete, local augmented \(K\)-algebra will be called formal and a deformation with a formal base is called a formal deformation. An infinitesimal base is automatically formal, so every infinitesimal deformation is a formal deformation.

An example of a formal base is \(A = \mathbb{K}[[t]]\) and a deformation of \(d\) with base \(A\) can be expressed in the form

\[d_t = d + t\psi_1 + t^2\psi_2 + \ldots\]
This is the classical notion of a formal deformation. It is easy to see that the condition for \( d \) to be a formal deformation reduces to

\[
D(\psi_{n+1}) = -\frac{1}{2} \sum_{k=1}^{n} [\psi_k, \psi_{n+1-k}]
\]

An automorphism of \( W_A \) over \( A \) is an \( A \)-linear isomorphism \( g_A : W_A \to W_A \) making the diagram below commute. The map \( g_A \) is induced by its restriction to

\[
\begin{array}{c}
W_A \xrightarrow{g_A} W_A \\
\downarrow \epsilon_w \quad \downarrow \epsilon_w \\
W \xrightarrow{\epsilon} W
\end{array}
\]

\( T(W) \otimes \mathbb{K} \) so we can view \( g_A \) as a map

\[
g_A : T(W) \to T(W) \otimes A
\]

so we can express \( g_A \) in the form

\[
g_A = I + \lambda
\]

where \( \lambda : T(W) \to T(W) \otimes m \). If \( A \) is infinitesimal then \( g_A^{-1} = I - \lambda \).

Two deformations \( d_A \) and \( d'_A \) are said to be equivalent over \( A \) if there is an automorphism \( g_A \) of \( W_A \) over \( A \) such that \( g_A^*(d_A) = d_A' \). In this case we write \( d_A' \sim d_A \).

An infinitesimal deformation \( d_A \) with base \( A \) is called universal if whenever \( d_B \) is an infinitesimal deformation with base \( B \), there is a unique morphism \( f : A \to B \) such that \( f_*(d_A) \sim d_B \).

**Theorem 8.2.** If \( \dim H_{\text{odd}}^2(d) < \infty \) then there is a universal infinitesimal deformation \( d^\infty \) of \( d \). Given by

\[
d^\infty = d + \delta^i t_i
\]

where \( H_{\text{odd}}^2(d) = \langle \bar{\delta}^i \rangle \) and \( A = \mathbb{K}[t_i]/(t_it_j) \) is the base of deformation.

A formal deformation \( d_A \) with base \( A \) is called versal if given any formal deformation of \( d_B \) with base \( B \) there is a morphism \( f : A \to B \) such that \( f_*(d_A) \sim d_B \). Notice that the difference between the versal and the universal property of infinitesimal deformations is that \( f \) need not be unique. A versal deformation is called miniversal if \( f \) is unique whenever \( B \) is infinitesimal. The basic result about versal deformation is:

**Theorem 8.3.** If \( \dim H_{\text{odd}}^2(d) < \infty \) then a miniversal deformation of \( d \) exists.

In this paper we will only need the following result to compute the versal deformations.

**Theorem 8.4.** Suppose \( H_{\text{odd}}^2(d) = \langle \bar{\delta}^i \rangle \) and \( [\delta^i, \delta^j] = 0 \) for all \( i, j \) then the infinitesimal deformation

\[
d^\infty = d + \delta^i t_i
\]

is miniversal, with base \( A = \mathbb{K}[[t_i]] \).
The construction of the moduli space as a geometric object is based on the idea that codifferentials which can be obtained by deformations with small parameters are “close” to each other. From the small deformations, we can construct 1-parameter families or even multi-parameter families, which are defined for small values of the parameters, except possibly when the parameters vanish.

If $d_t$ is a one parameter family of deformations, then two things can occur. First, it may happen that $d_t$ is equivalent to a certain codifferential $d'$ for every small value of $t$ except zero. Then we say that $d_t$ is a jump deformation from $d$ to $d'$. It will never occur that $d'$ is equivalent to $d$, so there are no jump deformations from a codifferential to itself. Otherwise, the codifferentials $d_t$ will all be nonequivalent if $t$ is small enough. In this case, we say that $d_t$ is a smooth deformation.

In [4], it was proved for Lie algebras that given three codifferentials $d$, $d'$ and $d''$, if there are jump deformations from $d$ to $d'$ and from $d'$ to $d''$, then there is a jump deformation from $d$ to $d''$. The proof of the corresponding statement for associative algebras is essentially the same.

Similarly, if there is a jump deformation from $d$ to $d'$, and a family of smooth deformations $d'_t$, then there is a family $d_t$ of smooth deformations of $d$, such that every deformation in the image of $d'_t$ lies in the image of $d_t$, for sufficiently small values of $t$. In this case, we say that the smooth deformation of $d$ factors through the jump deformation to $d'$.

In the examples of complex moduli spaces of Lie and associative algebras which we have studied, it turns out that there is a natural stratification of the moduli space of $n$-dimensional algebras by orbifolds, where the codifferentials on a given strata are connected by smooth deformations, which don’t factor through jump deformations. These smooth deformations determine the local neighborhood structure.

The strata are connected by jump deformations, in the sense that any smooth deformation from a codifferential on one strata to another strata factors through a jump deformation. Moreover, all of the strata are given by projective orbifolds. In fact, in all the complex examples we have studied, the orbifolds either are single points, or $\mathbb{C}P^n$ quotiented out by either $\Sigma_{n+1}$ or a subgroup, acting on $\mathbb{C}P^n$ by permuting the coordinates.

We don’t have a concrete proof at this time, but we conjecture that this pattern holds in general. In other words, we believe the following conjecture.

**Conjecture 8.5** (Fialowski-Penkava). The moduli space of Lie or associative algebras of a fixed finite dimension $n$ are stratified by projective orbifolds, with jump deformations and smooth deformations factoring through jump deformations providing the only deformations between the strata.

9. **Deformations of the Elements in the Moduli Space**

9.1. $d_1 = \psi_1^{13} - \psi_1^{31} + \psi_3^{11} + \psi_2^{22} - \psi_3^{33}$. The matrix of this codifferential is

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix},
$$

This is the only 1|2-dimensional complex semisimple algebra, and is the direct sum of the unique 1|1-dimensional algebra and the unique 0|1-dimensional simple algebra $\mathbb{C}$. Like all semisimple algebras, this algebra is unital. Its center is spanned
by \{v_2, v_3\}. We have \(h^0 = 0|2\) and \(h^n = 0|0\) otherwise, so this algebra is rigid, as is always the case with semisimple algebras. Note that \(h^0\) is always the dimension of the center of the algebra, where by center, we mean the \(\mathbb{Z}_2\)-graded center.

9.2. \(d_2 = \psi_1^{33} - \psi_1^{31} + \psi_3^{11} - \psi_3^{33}\). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix},
\]

This algebra is the direct sum of the 1\(|1\)-dimensional algebra and the trivial 0\(|1\)-dimensional simple algebra \(\mathbb{C}_0\). The algebra is not unital, and its center is spanned by \{\(v_2, v_3\)\}. We have

\[
\begin{align*}
H^0 &= \langle \psi_2, \psi_3 \rangle \\
H^{2n} &= \langle \psi_2^{2n} \rangle \\
H^{2n+1} &= \langle \psi_2^{2n+1} \rangle.
\end{align*}
\]

The versal deformation of this algebra is \(d^\infty = d + \psi_2^2 t\), which means that there is a jump deformation from \(d_2\) to \(d_1\), which is not surprising.

9.3. \(d_3 = \psi_2^{32} + \psi_3^{33} - \psi_1^{12}\). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

This algebra is the first of 5 extensions of the 0\(|2\)-dimensional semisimple algebra \(\mathbb{C}_2\) by the trivial 1\(|0\)-dimensional algebra \(\Pi \mathbb{C}_0\). The algebra is not unital, and its center is spanned by \{\(v_3\)\}, so \(h^0 = 0|1\), but \(h^n = 0|0\) for all \(n > 0\). Thus \(d_3\) is rigid. Its opposite algebra is \(d_4\).

9.4. \(d_4 = \psi_2^{32} + \psi_3^{33} + \psi_1^{21}\). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

This algebra is the second of 5 extensions of the 0\(|2\)-dimensional semisimple algebra \(\mathbb{C}_2\) by the trivial 1\(|0\)-dimensional algebra \(\Pi \mathbb{C}_0\). The algebra is not unital, and its center is spanned by \{\(v_3\)\}, so \(h^0 = 0|1\), but \(h^n = 0|0\) for all \(n > 0\). Thus \(d_4\) is rigid. Its opposite algebra is \(d_3\).

9.5. \(d_5 = \psi_2^{22} + \psi_3^{33} + \psi_1^{21} - \psi_1^{13}\). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

This algebra is the third of 5 extensions of the 0\(|2\)-dimensional semisimple algebra \(\mathbb{C}_2\) by the trivial 1\(|0\)-dimensional algebra \(\Pi \mathbb{C}_0\). The algebra is not unital, and its center is spanned by \{\(v_2 + v_3\)\}, so \(h^0 = 0|1\), but \(h^n = 0|0\) for all \(n > 0\). Thus \(d_5\) is rigid. This algebra is isomorphic to its opposite algebra.
9.6. $d_6 = \psi_2^{22} + \psi_3^{33} + \psi_1^{21} - \psi_1^{12}$. The matrix of this codifferential is
\[
\begin{bmatrix}
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]
This algebra is the fourth of 5 extensions of the 0|2-dimensional semisimple algebra $\mathbb{C}^2$ by the trivial 1|0-dimensional algebra $\Pi_{\mathbb{C}}$. The algebra is unital and commutative. We have
\[
H^0 = \langle \varphi_1 \psi_2, \psi_3 \rangle \\
H^n = \langle \varphi_1^n \psi_2^n \rangle.
\]
Its versal deformation is $d^\infty = d + \psi_1^{11} t$, and this is a jump deformation to $d_1$.

9.7. $d_7 = \psi_2^{22} + \psi_3^{33}$. The matrix of this codifferential is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]
This algebra is the direct of the 0|2-dimensional semisimple algebra $\mathbb{C}^2$ and the trivial 1|0-dimensional algebra $\Pi_{\mathbb{C}}$. The algebra is not unital but is commutative. We have
\[
H^0 = \langle \varphi_1 \psi_2, \psi_3 \rangle \\
H^n = \langle \varphi_1^n \rangle.
\]
Since $h^2 = 1|0$, there are no odd elements in $H^2$, so this algebra is rigid.

9.8. $d_8 = \psi_3^{33} + \psi_1^{11} + \psi_1^{31} - \psi_1^{13} + \psi_2^{32} + \psi_2^{23}$. The matrix of this codifferential is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
This algebra is an extension of the 0|1-dimensional simple algebra $\mathbb{C}$ by the nontrivial nilpotent 1|1-dimensional algebra $\mu = \psi_1^{11}$. The algebra is unital but not commutative. Its center is spanned by $\{v_2, v_3\}$. We have $h^{2n} = 0|1$ and $h^{2n+1} = 1|0$, but a basis for $h^n$ is not obvious. The versal deformation is $d^\infty = d - \psi_3^{11} t + \psi_2^{22} t$, which is a jump deformation to $d_1$.

9.9. $d_9 = \psi_3^{33} + \psi_2^{11}$. The matrix of this codifferential is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
This algebra is the direct sum of the 0|1-dimensional simple algebra $\mathbb{C}$ and the nontrivial nilpotent 1|1-dimensional algebra $\mu = \psi_1^{11}$. The algebra is not unital. Its center is spanned by $\{v_2, v_3\}$. It is difficult to determine its cohomology in general, but $h^2 = 1|1$. The versal deformation is $d^\infty = d - \psi_3^{11} t + \psi_2^{12} t + \psi_2^{22} t$, which is a jump deformation to $d_1$. 
9.10. \( d_{10} = \psi_{33}^{33} - \psi_{13}^{13} + \psi_{23}^{23} \). The matrix of this codifferential is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

This algebra is an extension of the 0\(|\cdot|1\)-dimensional simple algebra \( \mathbb{C} \) by the trivial nilpotent 1\(|\cdot|1\)-dimensional algebra \( \mu = 0 \). The algebra is neither unital nor commutative. Its opposite algebra is \( d_{11} \). We have \( H^1 = \langle \phi_2^{2}, \psi_1^{3}, \psi_1^{1} \rangle \) and \( H^n = 0 \) otherwise, so that this algebra is rigid. In fact, it belongs to a family of rigid extensions of the simple 0\(|\cdot|1\)-dimensional algebra. When \( M \) is 0\(|\cdot|n\)-dimensional, there are always \( n+1 \) elements of the family, but for \( r|s\)-dimensional algebras, there are more elements in the family. In this case, there are 4 such rigid algebras \( d_{10}, \ldots, d_{13} \).

9.11. \( d_{11} = \psi_{33}^{33} + \psi_{31}^{31} + \psi_{23}^{32} \). The matrix of this codifferential is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

This algebra is an extension of the 0\(|\cdot|1\)-dimensional simple algebra \( \mathbb{C} \) by the trivial nilpotent 1\(|\cdot|1\)-dimensional algebra \( \mu = 0 \). The algebra is neither unital nor commutative. Its opposite algebra is \( d_{10} \). We have \( H^1 = \langle \phi_2^{2}, \psi_1^{3}, \psi_1^{1} \rangle \) and \( H^n = 0 \) otherwise, so that this algebra is rigid.

9.12. \( d_{12} = \psi_{33}^{33} + \psi_{31}^{31} + \psi_{23}^{23} \). The matrix of this codifferential is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

This algebra is an extension of the 0\(|\cdot|1\)-dimensional simple algebra \( \mathbb{C} \) by the trivial nilpotent 1\(|\cdot|1\)-dimensional algebra \( \mu = 0 \). The algebra is neither unital nor commutative. Its opposite algebra is \( d_{13} \). We have \( h^2 = 0 \), so this algebra is rigid.

9.13. \( d_{13} = \psi_{33}^{33} - \psi_{13}^{13} + \psi_{23}^{32} \). The matrix of this codifferential is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

This algebra is an extension of the 0\(|\cdot|1\)-dimensional simple algebra \( \mathbb{C} \) by the trivial nilpotent 1\(|\cdot|1\)-dimensional algebra \( \mu = 0 \). The algebra is neither unital nor commutative. Its opposite algebra is \( d_{12} \). We have \( h^2 = 0 \), so this algebra is rigid.

9.14. \( d_{14} = \psi_{33}^{33} + \psi_{31}^{31} - \psi_{13}^{13} + \psi_{23}^{23} \). The matrix of this codifferential is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
This algebra is an extension of the 0|1-dimensional simple algebra \( C \) by the trivial nilpotent 1|1-dimensional algebra \( \mu = 0 \). The algebra is neither unital nor commutative. Its opposite algebra is \( d_{15} \). We have \( H^n = \langle \varphi_1^{1^n} \rangle \), for all \( n \), so this algebra is rigid.

9.15. \( d_{15} = \psi_3^{33} + \psi_3^{31} - \psi_1^{13} + \psi_2^{32} \). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

This algebra is an extension of the 0|1-dimensional simple algebra \( C \) by the trivial nilpotent 1|1-dimensional algebra \( \mu = 0 \). The algebra is neither unital nor commutative. Its opposite algebra is \( d_{14} \). We have \( H^n = \langle \varphi_1^{1^n} \rangle \), for all \( n \), so this algebra is rigid.

9.16. \( d_{16} = \psi_3^{33} + \psi_2^{23} \). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

This algebra is an extension of the 0|1-dimensional simple algebra \( C \) by the trivial nilpotent 1|1-dimensional algebra \( \mu = 0 \). The algebra is not unital and its center is spanned by \( \{ v_1 \} \). Its opposite algebra is \( d_{17} \). Since \( h^2 = 2|0 \), this algebra is rigid.

9.17. \( d_{17} = \psi_3^{33} + \psi_2^{32} \). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

This algebra is an extension of the 0|1-dimensional simple algebra \( C \) by the trivial nilpotent 1|1-dimensional algebra \( \mu = 0 \). The algebra is not unital and its center is spanned by \( \{ v_1 \} \). Its opposite algebra is \( d_{17} \). Since \( h^2 = 2|0 \), this algebra is rigid.

9.18. \( d_{18} = \psi_3^{33} - \psi_1^{13} + \psi_2^{23} + \psi_2^{32} \). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

This algebra is an extension of the 0|1-dimensional simple algebra \( C \) by the trivial nilpotent 1|1-dimensional algebra \( \mu = 0 \). The algebra is neither unital nor commutative. Its opposite algebra is \( d_{19} \). We have

\[
H^{2n} = \langle \psi_2^{2^n} \rangle \\
H^{2n+1} = \langle \varphi_2^{2^{n+1}} \rangle,
\]

so this algebra is rigid.
9.19. $d_{19} = \psi_3^{33} + \psi_3^{31} + \psi_2^{23} + \psi_2^{32}$. The matrix of this codifferential is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
This algebra is an extension of the 0|1-dimensional simple algebra $\mathbb{C}$ by the trivial nilpotent 1|1-dimensional algebra $\mu = 0$. The algebra is neither unital nor commutative. Its opposite algebra is $d_{19}$. We have
\[
H^{2n} = \langle \psi_2^{2n} \rangle, \\
H^{2n+1} = \langle \varphi_2^{2n+1} \rangle,
\]
so this algebra is rigid.

9.20. $d_{20} = \psi_3^{33} + \psi_2^{31} + \psi_2^{32}$. The matrix of this codifferential is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
This algebra is an extension of the 0|1-dimensional simple algebra $\mathbb{C}$ by the trivial nilpotent 1|1-dimensional algebra $\mu = 0$. The algebra is not unital but is commutative.
We have
\[
H^0 = \langle \varphi_1, \psi_2, \psi_3 \rangle, \\
H^{2n} = \langle \varphi_1^{12n}, \psi_3^{2n} \rangle, \\
H^{2n+1} = \langle \varphi_1^{12n+1}, \varphi_2^{2n+1} \rangle.
\]
The versal deformation is $d^\infty = d + \psi_3^{22}t$, which is a jump deformation to $d_7$.

9.21. $d_{21} = \psi_3^{33} - \psi_1^{13}$. The matrix of this codifferential is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
This algebra is an extension of the 0|1-dimensional simple algebra $\mathbb{C}$ by the trivial nilpotent 1|1-dimensional algebra $\mu = 0$. The algebra is neither unital nor commutative. Its opposite algebra is $d_{22}$.
We have
\[
H^0 = \langle \psi_2 \rangle, \\
H^{2n} = \langle \psi_1^{2^{2n-1}}, \psi_2^{2n} \rangle, \\
H^{2n+1} = \langle \varphi_1^{2^{2n+1}}, \varphi_2^{2^{2n+1}} \rangle.
\]
This is the first example in this paper when the versal deformation depends on more than one parameter. The versal deformation is $d^\infty = d + \psi_1^{21}t_1 + \psi_2^{32}t_2$, and there are relations on the base of the versal deformation. Since $h^3 = 2|0$, we should have 2 relations on the base. These relations are $\{0, 2t_1(t_2 - t_1)\}$, so the first relation is trivial, but the second one is not. This means the base of the versal deformation is
\( \mathcal{A} = \mathbb{C}[[t_1, t_2]]/(t_1(t_2 - t_1)) \). It is easy in this case to see that the relations imply that either \( t_1 = 0 \) or \( t_1 = t_2 \). This means that actual deformations lie along two lines, which intersect transversally at the origin. To understand the deformations, we need to consider the two solutions separately.

For the first solution, we have the 1-parameter deformation \( d_4 = d + \psi_2^{22}t \), which is a jump deformation to \( d_5 \). For the second solution, we have the 1-parameter deformation \( d_4 = d + \psi_1^{12}t + \psi_2^{22}t \), which is a jump deformation to \( d_5 \).

9.22. \( d_{22} = \psi_3^{33} + \psi_1^{31} \). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

This algebra is an extension of the 0|1-dimensional simple algebra \( \mathbb{C} \) by the trivial nilpotent 1|1-dimensional algebra \( \mu = 0 \). The algebra is neither unital nor commutative. Its opposite algebra is \( d_{21} \).

We have

\[
H^0 = \langle \psi_2 \rangle \\
H^{2n} = \langle \psi_1^{22n}, \psi_2^{22n} \rangle \\
H^{2n+1} = \langle \phi_1^{22n}, \phi_2^{22n+1} \rangle.
\]

This is the first example in this paper when the versal deformation depends on more than one parameter. The versal deformation is \( d^{\infty} = d + \psi_2^{22}t_1 + \psi_1^{12}t_2 \), and there are relations on the base of the versal deformation. Since \( h^{\infty} = 2|0 \), we should have 2 relations on the base. These relations are \( \{0, 2t_1(t_2 + t_1)\} \), so the first relation is trivial, but the second one is not. This means the base of the versal deformation is \( \mathcal{A} = \mathbb{C}[[t_1, t_2]]/(t_1(t_2 + t_1)) \). It is easy in this case to see that the relations imply that either \( t_1 = 0 \) or \( t_1 = -t_2 \). This means that actual deformations lie along two lines, which intersect transversally at the origin. To understand the deformations, we need to consider the two solutions separately.

For the first solution, we have the 1-parameter deformation \( d_4 = d + \psi_2^{22}t \), which is a jump deformation to \( d_5 \). For the second solution, we have the 1-parameter deformation \( d_4 = d + \psi_1^{12}t - \psi_2^{22}t \), which is a jump deformation to \( d_5 \).

9.23. \( d_{23} = \psi_3^{33} - \psi_1^{13} + \psi_1^{31} \). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

This algebra is an extension of the 0|1-dimensional simple algebra \( \mathbb{C} \) by the trivial nilpotent 1|1-dimensional algebra \( \mu = 0 \). The algebra is not unital but is commutative.

We have

\[
H^{2n} = \langle \phi_1^{22n}, \phi_2^{22n}, \psi_3^{22n} \rangle \\
H^{2n+1} = \langle \phi_1^{22n+1}, \phi_2^{22n+1}, \psi_3^{22n+1} \rangle.
\]
The versal deformation is given by \( d^\infty = d + \psi_3^1 t_1 + \psi_2^2 t_2 \). and there are no relations on the base of the versal deformation. When both \( t_1 \) and \( t_2 \) do not vanish, the deformation is equivalent to \( d_1 \). When \( t_2 = 0 \), the deformation is equivalent to \( d_2 \). When \( t_1 = 0 \), the deformation is equivalent to \( d_7 \). Thus we have jump deformations to \( d_1 \), \( d_2 \) and \( d_7 \).

This is a typical pattern when a deformation depends on more than one parameter. There is a generic case, which holds except on some lower codimension submanifolds. In fact, one can see from this deformation that \( d_2 \) and \( d_7 \) must also have jump deformations to \( d_1 \), a fact which we already encountered.

9.24. \( d_{24} = \psi_3^{33} - \psi_1^{13} + \psi_1^{31} + \psi_2^{23} + \psi_2^{32} \). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

This algebra is an extension of the 0|1-dimensional simple algebra \( \mathbb{C} \) by the trivial nilpotent 1|1-dimensional algebra \( \mu = 0 \). The algebra is both unital and commutative.

We have \( h^2 = 3|3 \). The versal deformation is given by

\[
d^\infty = d + \psi_2^{22} t_1 + \psi_1^{12} t_2 + \psi_2^{11} t_3 - \psi_3^{11} ((t_1 + t_2) t_3) + \psi_3^{22} ((t_1 + t_2) t_2).
\]

and there are 2 nontrivial relations on the base of the versal deformation

\[
t_2(t_1 + t_2) = 0, \quad t_2t_3 = 0,
\]

which have solutions \( t_2 = 0 \) or \( t_3 = 0 \) and \( t_2 = -t_1 \). Thus the base of the versal deformation is a plane and a line which intersects this plane transversally at the origin. Note that this is the first case in this paper where the versal deformation has higher order terms, that is, it is not given by the infinitesimal deformation.

For the first solution, when both \( t_1 \) and \( t_3 \) do not vanish, the deformation is equivalent to \( d_1 \). When \( t_3 = 0 \), the deformation is equivalent to \( d_7 \). When \( t_1 = 0 \), the deformation is equivalent to \( d_8 \). The second solution is a jump deformation to \( d_5 \). Thus the codifferential has jump deformations to \( d_1 \), \( d_5 \), \( d_7 \) and \( d_8 \).

9.25. \( d_{25} = \psi_3^{33} \). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

This algebra is the direct sum of the 0|1-dimensional simple algebra \( \mathbb{C} \) and the trivial nilpotent 1|1-dimensional algebra \( \mu = 0 \). The algebra is not unital but is commutative.

We have \( h^2 = 4|4 \) and \( h^3 = 8|8 \). The versal deformation is given by

\[
d^\infty = d + \psi_2^{22} t_1 + \psi_1^{21} t_2 + \psi_1^{12} t_3 + \psi_2^{11} t_4
\]

and there are 6 nontrivial relations on the base of the versal deformation

\[
t_4(t_2 + t_3) = 0, \quad t_3(t_1 + t_3) = 0, \quad t_4(t_1 + t_3) = 0, \quad t_4(t_1 - t_2) = 0, \quad t_2(t_2 - t_1) = 0,
\]
which have 4 solutions:

\[ t_2 = t_3 = t_4 = 0 \]
\[ t_3 = t_4 = 0, t_1 = t_2 \]
\[ t_2 = t_4 = 0, t_1 = -t_3 \]
\[ t_1 = t_2, t_3 = -t_2 \]

Thus the base of the versal deformation is a plane and three lines which intersects this plane transversally at the origin.

The first solution is a jump deformation to \( d_6 \), the second is a jump deformation to \( d_4 \), and the third is a jump to \( d_3 \). For the fourth solution, when both \( t_2 \) and \( t_4 \) do not vanish, the deformation is equivalent to \( d_1 \). When \( t_4 = 0 \), the deformation is equivalent to \( d_7 \). When \( t_2 = 0 \), the deformation is equivalent to \( d_9 \). Thus the codifferential has jump deformations to \( d_1, d_3, d_4, d_6, d_7 \) and \( d_9 \).

9.26. \( d_{26} = \psi_2^{11} + \psi_2^{33} \). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

This algebra is an extension of the trivial 0|1-dimensional algebra \( \delta = 0 \) by the nontrivial trivial nilpotent 1|1-dimensional algebra \( \mu = \psi_2^{11} \). It is also an extension of the trivial 1|0 dimensional algebra by the nontrivial 0|2-dimensional algebra \( \mu = \psi_2^{33} \). It is nilpotent, and therefore could not be unital, and its center is spanned by \( \{v_2, v_3\} \).

As with all nilpotent algebras, its cohomology is quite complicated. We have \( h^2 = 1|2 \) and \( h^3 = 3|2 \). The versal deformation is given by

\[
d^\infty = d - \psi_1^{12}((t_2 - t_1)t_2) + \psi_1^{21}((t_2 - t_1)t_2) - \psi_1^{31}t_2 + \psi_1^{33}t_2 + \psi_1^{11}t_2 + \psi_1^{22}t_2 + \psi_1^{33}(t_2 - t_1)t_2 + \psi_1^{33}t_2
\]

and there are no relations on the base of the versal deformation.

When \( t_2 \neq 0, t_1 \neq t_2 \) and \( t_1 \neq 2t_2 \), the deformation is equivalent to \( d_1 \). On the line \( t_1 = t_2 \) the deformation jumps to \( d_2 \), on the line \( t_1 = 2t_2 \), it jumps to \( d_8 \), and on the line \( t_2 = 0 \), it jumps to \( d_9 \). Thus, the codifferential has jump deformations to \( d_1, d_2, d_8 \) and \( d_9 \).

9.27. \( d_{27} = \psi_2^{11} \). The matrix of this codifferential is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

This algebra is an extension of the trivial 0|1-dimensional algebra \( \delta = 0 \) by the nontrivial nilpotent 1|1-dimensional algebra \( \mu = \psi_2^{11} \). It is also an extension of the trivial 1|0 dimensional algebra by the trivial 0|2-dimensional algebra \( \mu = 0 \). It is nilpotent, and its center is spanned by \( \{v_2, v_3\} \).

The cohomology of this algebra was computed in [2]. We have \( h^2 = 5|4 \) and \( h^3 = 6|11 \). The versal deformation is given by

\[
d^\infty = d + \psi_3^{33}t_1 - \psi_1^{12}t_2 + \psi_1^{21}t_2 + \psi_2^{22}t_2 + \psi_2^{33}t_3 + \psi_2^{33}t_3 + \psi_1^{31}t_3 - \psi_1^{13}t_3 + \psi_3^{33}t_4.
\]
and there is one nontrivial relation on the base of the versal deformation, given by
\[ t_1 t_2 - t_2^2 + t_3 t_4 = 0, \]
which means the base of the versal deformation is a hypersurface in \( \mathbb{C}^4 \). When \( t_2 \neq 0 \), then if \( t_4 \neq 0 \) the deformation is equivalent to \( d_1 \), and if \( t_4 = 0 \), the deformation is equivalent to \( d_2 \). When \( t_2 = 0 \), then if \( t_3 = 0 \) and \( t_4 \neq 0 \), it jumps to 9, and if \( t_3 = t_4 = 0 \) it jumps to \( d_{20} \). Finally, if \( t_2 = 0 \) and \( t_3 \neq t_4 \), it jumps to \( d_8 \). Thus the codifferential has jump deformations to \( d_1, d_2, d_8, d_9 \) and \( d_{26} \).

9.28. \( d_{28} = \psi_{33}^{33} \). The matrix of this codifferential is
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
This algebra is an extension of the trivial \( 0|\{1\} \)-dimensional algebra \( \delta = 0 \) by the trivial nilpotent \( 1|\{1\} \)-dimensional algebra \( \mu = 0 \). It is also an extension of the trivial \( 1|\{0\} \) dimensional algebra by the nontrivial \( 0|\{2\} \)-dimensional algebra \( \mu = \psi_{33}^{33} \). It is nilpotent and commutative.

The cohomology of this algebra was computed in [21]. We have \( h^2 = 5|4 \) and \( h^3 = 9|8 \). The versal deformation is given by
\[
d^\infty = d + \psi_{2}^{11} t_1 + \psi_{3}^{33} t_1 t_2 t_3 + \psi_{2}^{22} t_3 + \psi_{2}^{33} t_4 + \psi_{1}^{13} t_5
\]
\[
- \psi_{3}^{11} t_1 t_5 - \psi_{1}^{12} ((t_2 + t_5) t_5) + \psi_{1}^{21} ((t_4 - t_2) t_4).
\]
and there are 5 nontrivial relations on the base of the versal deformation, given by
\[
t_4 (t_2 t_4 - t_2^2 + t_3) = 0
\]
\[
t_1 (t_4 + t_5)(t_2 - t_4 + 2t_5) = 0
\]
\[
t_5 (t_3 - t_2^2 - t_2 t_5) = 0
\]
\[
t_1 (t_2 t_5 - t_3 - t_4 t_5) = 0
\]
\[
t_1 (t_4 + t_5) = 0.
\]
Solving, we obtain 6 solutions
\[
t_1 = t_4 = t_5 = 0
\]
\[
t_1 = t_4 = 0, \quad t_3 = t_5 (t_2 + t_5)
\]
\[
t_1 = t_5 = 0, \quad t_3 = t_4 (t_4 - t_2)
\]
\[
t_1 = 0, \quad t_3 = t_5 (t_2 + t_5), \quad t_4 = -t_5
\]
\[
t_1 = 0, \quad t_2 = t_4 - t_5, \quad t_3 = t_4
\]
\[
t_3 = t_4 (t_4 - t_2), \quad t_5 = -t_4,
\]
which means the base of the versal deformation is the union of 5 surfaces and 1 hypersurface through the origin in \( \mathbb{C}^5 \).

On the first surface, except on the curves \( t_3 = 0, t_3 = -t_2^2 / 4 \), the deformation is equivalent to \( d_{6} \), while on the curve \( t_3 = -t_2^2 / 4 \) it jumps to \( d_{20} \), and on the curve \( t_3 = 0 \) it jumps to \( d_{25} \).

On the second surface, except on the curves \( t_5 = 0, t_5 = -t_2 \) and \( t_5 = -t_2^3 / 2 \), the deformation is equivalent to \( d_{3} \), while on the curve \( t_5 = -t_2^3 / 2 \) it jumps to \( d_{18} \), on the curve \( t_5 = -t_2 \) it jumps to \( d_{21} \), and on the curve \( t_5 = 0 \) it jumps to \( d_{25} \).
On the third surface, except on the curves $t_4 = 0$, $t_4 = t_2$ and $t_4 = t_2/2$, the deformation is equivalent to $d_4$, while on the curve $t_4 = t_2/2$ it jumps to $d_{19}$, on the curve $t_4 = t_2$ it jumps to $d_{22}$ and on the curve $t_4 = 0$, it jumps to $d_{25}$.

On the fourth surface, except on the curves $t_5 = 0$, $t_5 = -t_2$ and $t_5 = -t_2/2$, the deformation is equivalent to $d_6$, while on the curve $t_5 = -t_2/2$ it jumps to $d_{21}$, on the curve $t_5 = -t_2$ it jumps to $d_{23}$, and on the curve $t_5 = 0$ it jumps to $d_{25}$.

On the fifth surface, except on the curves $t_4 = 0$, $t_4 = -t_5$ and $t_5 = 0$, the deformation is equivalent to $d_5$, while on the curve $t_4 = 0$ it jumps to $d_{21}$, on the curve $t_5 = 0$ it jumps to $d_{22}$, and on the curve $t_4 = -t_5$ it jumps to $d_{24}$.

On the hypersurface, except on the surfaces $t_4 = t_2/2$, $t_4 = t_2$, $t_1 = 0$ and $t_4 = 0$, the deformation is equivalent to $d_1$. On the surface $t_2 = t_4$, except on the lines $t_2 = 0$ and $t_1 = 0$, the deformation is equivalent to $d_3$. On the surface $t_1 = 0$, except on the lines $t_4 = t_2$, $t_4 = 0$ and $t_4 = t_2/2$, the deformation is equivalent to $d_6$. On the surface $t_4 = t_2/2$, except on the lines $t_1 = 0$ and $t_2 = 0$, the deformation is equivalent to $d_8$. On the surface $t_4 = 0$, except on the lines $t_1 = 0$ and $t_2 = 0$, the deformation is equivalent to $d_9$. On the line $t_4 = t_2$, $t_1 = 0$, the deformation jumps to $d_{23}$. On the line $t_4 = t_2/2$, $t_1 = 0$, the deformation jumps to $d_{21}$. On the line $t_4 = t_1 = 0$, the deformation jumps to $d_{25}$. Finally, on the line $t_2 = t_4 = 0$, the deformation jumps to $d_{26}$.

As a consequence, we see that the $d_{28}$ has jump deformations to $d_1$, $d_2$, $d_3$, $d_4$, $d_5$, $d_6$, $d_7$, $d_8$, $d_9$, $d_{18}$, $d_{19}$, $d_{20}$, $d_{21}$, $d_{22}$, $d_{23}$, $d_{24}$, $d_{25}$ and $d_{26}$.

References

1. L. Brunshidle, M. Penkava, M. Phillipson, and D. Wackwitz, Fundamental theorem of finite dimensional graded associative algebras, preprint, 2009.
2. C. DeCleene, M. Penkava, and M. Phillipson, Classification of two r-s-dimensional nilpotent algebras, Preprint, 2009.
3. A. Fialowski and M. Penkava, Extensions of associative algebras, preprint, 2007.
4. , Formal deformations, contractions and moduli spaces of Lie algebras, International Journal of Theoretical Physics 47 (2007), no. 2, 561–582.
5. M. Gerstenhaber, The cohomology structure of an associative ring, Annals of Mathematics 78 (1963), 267–288.
6. , On the deformations of ringe and algebras I, Annals of Mathematics 79 (1964), 59–103.
7. , On the deformations of ringe and algebras II, Annals of Mathematics 84 (1966), 1–19.
8. , On the deformations of ringe and algebras III, Annals of Mathematics 88 (1968), 1–34.
9. , On the deformations of ringe and algebras IV, Annals of Mathematics 99 (1974), 257–276.
10. G. Hochschild, On the cohomology groups of an associative algebra, Annals of Mathematics 46 (1945), 58–67.
11. M. Mulase and M. Penkava, Ribbon graphs, quadratic differentials on Riemann surfaces, and algebraic curves defined over $\mathbb{Q}$, Asian Journal of Mathematics 2 (1998), 875–920, math-ph/9811024.
12. B. Peirce, Linear associative algebra, American Journal of Mathematics 4 (1881), 97.
13. M. Penkava and A. Schwarz, On some algebraic structures arising in string theory, Perspectives in Mathematical Physics (R Penner and S. Yau, eds.), International Press, 1994, pp. 219–227.
14. , $A_{\infty}$ algebras and the cohomology of moduli spaces, Dynkin Seminar, vol. 169, American Mathematical Society, 1995, pp. 91–107.
15. M. Penkava and P. Vanhaecke, Deformation quantization of polynomial Poisson algebras, Journal of Algebra (2000), no. 227, 365–393.
16. J.D. Stasheff, *The intrinsic bracket on the deformation complex of an associative algebra*, Journal of Pure and Applied Algebra **89** (1993), 231–235.

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