HYDRODYNAMICS OF A RELATIVISTIC FIREBALL: THE COMPLETE EVOLUTION

SHIHO KOBAYASHI,1 TSVI PIRAN, AND RE‘EM SARI
Racah Institute of Physics, The Hebrew University, Jerusalem, 91904 Israel; shiho@alf.hji.ac.il, tsvi@shemesh.fiz.huji.ac.il, sari@shemesh.fiz.huji.ac.il

Received 1998 March 19; accepted 1998 June 26

ABSTRACT

We study numerically the evolution of an adiabatic relativistic fireball expanding into a cold uniform medium. We follow the stages of initial free expansion and acceleration, coasting, and then deceleration and slowing down to a nonrelativistic velocity. We compare the numerical results with simplified analytical estimates. We show that the relativistic self-similar Blandford-McKee solution describes well the relativistic deceleration epoch. It is an excellent approximation throughout the relativistic deceleration stage, down to $\gamma \sim 5$, and a reasonable approximation even down to $\gamma \sim 2$, though the solution is rigorous only for $\gamma \gg 1$. We examine the transition into the Blandford-McKee solution and the transition from the solution to the nonrelativistic self-similar Sedov-Taylor solution. These simulations demonstrate the attractive nature of the Blandford-McKee solution and its stability to radial perturbations.

Subject headings: gamma rays: bursts — hydrodynamics — relativity — shock waves

INTRODUCTION

Sedov (1946), Taylor (1950), and von Neumann (1947) discovered, in the 1940s, a self-similar solution of the strong explosion problem, in which a large amount of energy is released on a short timescale in a small volume. This solution is known today as the Sedov-Taylor self-similar solution. It describes a shock wave propagating into a uniform-density surrounding. The shock wave and the matter behind it decelerate as more and more mass is collected. This solution describes well the adiabatic stage of a supernova remnant evolution.

Blandford & McKee (1977) later established a self-similar solution describing the extreme relativistic version of the strong explosion problem. In this solution the Lorentz factor of the shock and the fluid behind it is much larger than unity. Such high Lorentz factors arise if the rest mass contained within the region where the energy $E$ is released is much smaller than $E$, in other words, when the region containing the energy is “radiation-dominated” rather than matter-dominated. Such a region was later termed a “fireball”.

Cavallo & Rees (1978) have considered the physical processes relevant in a radiation-dominated fireball as a model for gamma-ray bursts (GRBs). Goodman (1986) and Paczynski (1986) have considered the evolution of such a fireball. They have shown that initially the radiation-pair plasma in a purely radiative fireball behaves like a fluid, and it expands and accelerates under its own pressure until the local temperature drops to $\sim 20$ keV, when the last pairs annihilate and the fireball becomes optically thin. Later Shemi & Piran (1990) considered matter-contaminated fireballs. They showed that, quite generally, all the initial energy will be transferred to the baryons in such fireballs, whose final outcome is a shell of relativistic, freely expanding baryons. Piran, Shemi, & Narayan (1993) and Mészáros, Laguna, & Rees (1993) later carried out these calculations in greater detail. These works show that an initially homogeneous fireball will first accelerate while expanding and then coast freely as all its internal energy is transformed to kinetic energy.

The surrounding matter will eventually influence the fireball, after enough external matter has been collected and most of the energy has been transferred from the shell to the ISM. If the surrounding matter is diluted enough, this will take place only after the initial free acceleration phase. This influence was considered by Mészáros & Rees (1992) and by Katz (1994), who suggested that the GRB is produced during this stage. The detailed shock evolution was later studied by Sari & Piran (1995). It is only after these stages, when the fireball has lost most of its energy to the ISM, that the self-similar deceleration solution of Blandford-McKee applies. When the shock decelerates enough so that it is no longer relativistic, it is described by the Sedov-Taylor solution.

Today it is widely accepted that GRBs involve relativistic expanding matter of this kind. While the GRB itself is most likely produced via internal shocks (Narayan, Paczynski, & Piran 1992; Rees & Mészáros 1994; Sari & Piran 1997), the observed GRB afterglow corresponds to the slowing down of this relativistic flow. This has led to an increasing interest in the fireball solution and its various regimes. In this paper we study the whole evolution of a homogeneous fireball, focusing on its interaction with the surrounding matter. We do not consider here internal shocks, which arise owing to interaction within the relativistic flow and require nonuniform velocity.

We have developed a spherically symmetric relativistic Lagrangian code based on a second-order Gudnov method with an exact Riemann solver to solve the ultrarelativistic hydrodynamics problem. With this code, it is possible to track the full hydrodynamical evolution of the fireball within a single computation, from its initial “radiation-dominated” stage at rest through its acceleration, coasting, and shock formation, up to the relativistic deceleration, and finally to the Newtonian deceleration. With typical parameters, this computation spans more than 8 orders of magnitude in the size of the fireball and more than 20 orders of magnitude in its density. We describe these computations here. We show that, quite generically, the solution converges during the relativistic deceleration phase to the Blandford-McKee solution, and then it transforms to the...
Sedov-Taylor solution. Even though the attractive nature of the Blandford-McKee solution suggests that it is stable, we explicitly explore the stability of this solution and show that it is stable to radial perturbations.

In § 2 we review the current analytic understanding of the fireball evolution through the following stages: (1) free acceleration and coasting, (2) energy transfer, (3) the relativistic self-similar solution, and (4) the Newtonian Sedov-Taylor solution. We discuss the numerical results in § 3. In § 4 we examine the evolution of perturbations to the Blandford-McKee solution. We discuss the implications of these results in § 5.

2. ANALYTIC ESTIMATES

The evolution of a fireball is characterized by several phases. The transitions between these phases are determined by several critical radii, which are summarized in Table 1.

2.1. Free Acceleration and Coasting

We consider a homogeneous fireball of energy $E$ and a baryonic load of total mass $M_\odot$ confined initially in a sphere of radius $R_0$. We define the dimensionless entropy (or the initial random Lorentz factor) $\eta \equiv E/M_\odot$. This fireball expands into a surrounding low-density medium (with a density $\rho_1$), which we will refer to as the ISM. This can be considered to be a free expansion in its initial stage. After a short acceleration phase, the motion becomes highly relativistic. Conservations of baryon number, energy, and momentum yield the following conservation laws along a null flow line of each fluid element in the shell (Piran et al. 1993):

$$r^2 \rho \gamma, \quad r^3 \rho^{3/4} \gamma, \quad \text{and} \quad r^4 (4p + \rho) \eta^2 = \text{constant},$$

where $r(t)$, $\gamma(t)$, $p(t)$, and $\rho(t)$ are the radius, Lorentz factor, pressure, and rest-mass density of the fluid element, respectively. In this paper, distance, time, velocity, and the corresponding Lorentz factors are measured in the observer frame. Thermodynamic quantities ($p$ and $\rho$) are measured in the local fluid frame. We use units in which the speed of light $c = 1$. The above equations assume an adiabatic gas index of $\frac{4}{3}$.

Initially the fireball is extremely hot ($p \gg \rho$), so that equation (1) yields

$$\gamma \propto r, \quad \rho \propto r^{-3} \quad \text{and} \quad p \propto r^{-4},$$

(Goodman 1986; Paczyński 1986; Shemi & Piran 1990). The fireball is approximately homogeneous in the local frame, but owing to relativistic effects, it appears to an observer to be at rest as a narrow shell with a radial width $\Delta \sim r/\gamma \sim R_0$ (Shemi & Piran 1990; Piran et al. 1993).

As the fireball expands, the internal energy is converted to kinetic energy of the baryons. At the radius $R_L \equiv \eta R_0$, the fireball uses up all the internal energy, and the approximation $p \gg \rho$ breaks down. This is the end of the acceleration phase.

Now the internal energy of the fireball becomes negligible compared to the rest-mass energy ($p \ll \rho$), and equation (1) yields

$$\gamma = \text{constant}, \quad \rho \propto r^{-2}, \quad \text{and} \quad p \propto r^{-8/3},$$

(Piran et al. 1993). The fireball behaves like a pulse of energy with a frozen radial profile propagating at almost the speed of light.

2.2. Spreading, the Reverse Shock, and Energy Transfer

This frozen pulse approximation on which equation (1) is based breaks down ultimately at the radius $R_\ast \equiv R_0 \eta^2$. Each fluid shell moves with a slightly different velocity, and the fireball begins to spread at $R_\ast$. Internal shocks will take place around $R_\ast$ if the fireball is inhomogeneous and the velocity is not a monotonic function of the radius. As mentioned earlier, these shocks most likely produce the observed GRB. However, even under optimal conditions they cannot convert more than about a quarter of the kinetic energy to radiation. Hence, an inhomogeneous shell will continue to carry ample kinetic energy beyond this stage. We consider here only homogeneous fireballs. The possible effect of internal shocks on the fireball evolution has been discussed extensively in other papers (Kobayashi, Piran, & Sari 1997; Daigle & Mochkovitch 1998).

The coating can also end if the surrounding matter begins to influence the shell. The interaction between the shell and the ISM can be described by two shocks: a forward shock propagating into the ISM and a reverse shock propagating into the shell. Sari & Piran (1995) have defined three critical radii in this respect: $R_N \equiv l^{1/2}/\Delta^{1/4} \eta^2$, where the energy density produced by the shocks becomes high enough so that the reverse shock is relativistic and begins to change the Lorentz factor of the shell considerably; $R_A \equiv l^{3/4}/\Delta^{3/4} \eta$, where the reverse shock crosses the shell; and $R_\ast \equiv l/\eta^{2/3}$, where the mass of the shocked ISM is $M_\odot/\eta$. Here $l \equiv (E/\rho_1)^{1/3}$ is the Sedov length. Fortunately, a simple relation between these four radii can be given in terms of the dimensionless variable $\xi \equiv (l/\Delta)^{1/2} \eta^{-4/3}$:

$$\xi^2 R_\ast = \sqrt{\xi} R_A = R_\gamma = R_N/\xi.$$

If initially $\xi > 1$, then $R_\gamma$ is the smallest radius. The shell begins to spread at $R_\ast$. After that the width $\Delta$ satisfies $\Delta \sim r/\gamma^2 \propto r$, and the scaling of the shell parameters becomes

$$\gamma = \text{constant}, \quad \rho \propto r^{-3} \quad \text{and} \quad p \propto r^{-4}.$$

During the spreading phase, the value of $\xi > 1$ decreases. However, as long as $\xi > 1$, the relation $R_\ast < R_A < R_\gamma < R_N$ is valid. When $\xi \sim 1$ these different radii become comparable: the reverse shock crosses the shell, it becomes mildly relativistic, and an ISM mass of $M_\odot/\eta$ can be collected. Since the reverse shock is only mildly relativistic at this stage, the shell’s Lorentz factor has changed only by a factor of order unity. We call this the Newtonian reverse shock (NRS) case, since the reverse shock is Newtonian relative to the unshocked shell. This is not to be confused with the fact that $\gamma \gg 1$ in this case, and the forward shock is ultrarelativistic.

| TABLE 1 | CRITICAL RADII |
|---------|----------------|
| $R_0$   | Initial fireball size |
| $R_L$   | Coasting |
| $R_s$   | Spreading (and internal shocks) |
| $R_A$   | External shocks (the RRS case) |
| $R_c$   | External shocks (the NRS case) |
| $R_g$   | Relativistic reverse shock |
| $l$      | Sedov length |

$R_0 \eta^2$ |

$R_0 \eta^2$ |

$R_0 \eta^2$ |

$R_0 \eta^2$ |

$R_0 \eta^2$ |

$R_0 \eta^2$ |

$R_0 \eta^2$ |

$R_0 \eta^2$ |

$R_0 \eta^2$ |

$R_0 \eta^2$ |

$R_0 \eta^2$ |
If initially $\xi < 1$, then $R_N$ is the smallest radius. The reverse shock becomes relativistic before it crosses the shell. At $r > R_N$ the reverse shock begins to reduce considerably the Lorentz factor of the shell's matter, which it crosses. The Lorentz factor of the shocked material (Sari 1997) is

$$\gamma(r) = 1^{3/4} \Delta^{-1/4} r^{-1/2}. \quad (6)$$

The shell has decelerated significantly by the time that the reverse shock has crossed the shell at $R_N$. At this stage, the Lorentz factor is reduced from its initial value of $\eta$ to $\eta \xi^{3/4} = (l/\Delta)^{3/8} < \eta$. We call this the relativistic reverse shock (RRS) case, as most of the deceleration is done by a strong relativistic reverse shock.

The deceleration radius, when the shell begins to coast freely, $R_L$, is related to the other radii by a simple relation, $\eta \xi^2 R_L = \xi^2 R_\ast = \xi^{1/2} R_\ast = R_\ast / \xi$. In the RRS case, $\xi > 1$ and $R_L$ is the smallest radius. All the initial thermal energy is converted to kinetic energy before the shell begins to decelerate. In the RRS case ($\xi < 1$), it is possible that the deceleration, owing to external matter, begins before all the energy of the fireball is converted to kinetic energy. The conditions $R_L < R_N$ and $\xi < 1$ yield $(l/\Delta)^{3/8} < \eta < (l/\Delta)^{1/2}$ (see Fig. 1): $\xi$ should be larger than $(\Delta/l)^{1/6}$ in order that $R_L < R_N$. We limit the discussion here to the case of $R_L < R_N$, i.e., one in which the fireball transforms all its internal energy to kinetic energy before the ISM begins to influence its evolution.

The RRS case is relevant if the initial fireball is small and contains relatively many baryons (or, equivalently, when $\eta$ is relatively small), while the RRS case is relevant if the fireball is large and is less polluted by baryon. The shell has given the ISM most of its energy either at $R_s = R_N = R_\ast$, in the RRS case, or at $R_A$, in the RRS case.

2.3 Relativistic Self-similar Deceleration

For $r > R_s$ in the RNS case, or for $r > R_A$ in the RRS case, the shocked ISM contains most of the energy, $E$. After this stage the shell plays a negligible part in the consequent evolution. The profile of the shocked ISM is determined now by only two parameters, $E$ and $\rho_1$.

When the forward shock reaches a radius $R$, the fireball has a mass $4\pi \rho_1 R^3/3$. The energy in the shocked fluid is therefore $\propto \rho_1 R^3$. Since this equals the total energy of the system $E$, we obtain a scaling law, $\gamma \propto (E/\rho_1)^{1/3} R^{-2/3}$. The exact proportionality constant depends on the profiles behind the shock. Blandford & McKee (1976) describe an analytic self-similar solution, in which the Lorentz factor, the density, and the pressure are given by

$$\gamma(r, t) = \frac{1}{\sqrt{2}} \Gamma^{-1/2} \rho(t, r) = 2 \sqrt{2} \rho_1 \Gamma^{-5/4},$$

$$p(t, r) = \frac{2}{3} \rho_1 \Gamma^{-2} \chi^{-1/2},$$

where $\Gamma(t) \equiv (17/8 \pi) R l^{-3/2}$ is the Lorentz factor of the shock itself, and the similarity variable $\chi$ is defined by $\gamma(t, r) \equiv 1 + 8 \Gamma^2 (1 - r/R)$. The shock radius is given by $R = R(t) \sim (1 - 1/8 \Gamma^2)^{1/4}$. In the previous section $r$ denoted the radius of a fluid element in the shell, but here $r$ and $t$ are independent coordinates.

2.4 The Sedov-Taylor Solution

The Blandford-McKee self-similar solution is derived with the assumption $\gamma, \Gamma \gg 1$. This assumption breaks down when the shock sweeps out a volume $\sim l^3$ of ISM and its motion becomes nonrelativistic. At radius $l$ the Sedov-Taylor solution (Sedov 1946; von Neumann 1947; Taylor 1950) becomes a good approximation. A characteristic length scale at time $t$, which we can form from the two parameters $E$ and $\rho_1$, gives the shock radius $R(t) \equiv A(E \rho_1^{-2})^{1/5}$ within a numerical constant factor $A$, depending on the adiabatic constant $\gamma$ ($x = 0.99$ for $\gamma = 4/3$). The velocity of the shock is $u \equiv dR/dt = 2R/5t$. The velocity $\beta_2$, the density $\rho_2$, and the pressure $p_2$ just behind the shock can be expressed in terms of $u$: $\beta_2 = 6u/7, \rho_2 = 7\rho_1, p_2 = 6\rho_1 u^2/7$ for the adiabatic index $\gamma = 4/3$. The profile throughout the region behind the shock is given by the velocity $\beta = 2u/5t$, the density $\rho = \rho_1 G$, and the pressure $p = 3\rho_1 G/Z25t^2$ (Landau & Lifshitz 1987). The dimensionless variables $V, G, Z$ are functions only of the similarity variable $\zeta \equiv r/R$ and are given in an implicit analytical form by the following equations:

$$\zeta^5 = \frac{1}{5} V^{-2} \frac{1}{2^7} (5 - 3V)^{-232/99} \frac{1}{7} (4V - 3)^{5/11}, \quad (8)$$

$$Z = \frac{7}{5} V^2 (1 - V) (4V - 3)^{-1}, \quad (9)$$

$$G = \frac{4}{35} (1 - V)^{-3} \frac{1}{2^7} (4V - 3)^{9/11} \frac{1}{7} (5 - 3V)^{116/33}. \quad (10)$$

The pressure ratio $p/p_2$ tends to a constant as $\zeta \to 0$, while $\beta/\beta_2 \propto \zeta, \rho/\rho_2 \propto \zeta^6$ in this limit.

3. NUMERICAL RESULTS

3.1 Initial Conditions

We consider an initial uniform spherical fireball surrounded by a uniform cold ISM. The initial conditions are determined by four parameters: the total energy $E$, the baryonic mass $M_0 = E/\eta$, the radius $R_b$ of the fireball, and the ISM density $\rho_1$. A “cold” ISM means that its pressure is negligible compared to the pressure behind the shocks throughout the whole evolution. For convenience, we set the initial time as $R_0$ rather than zero. We have chosen two sets of initial conditions to represent the RRS and the RNS cases: $E = 10^{52}$ ergs, $\rho_1 = 1$ proton cm$^{-3}$, $\eta = 50$, and $R_0 = 3 \times 10^{10}$ cm for the RNS case ($\zeta = 43$); and $E = 10^{52}$ ergs, $\rho_1 = 1$ proton cm$^{-3}$, $\eta = 10^3$, and $R_0 = 4.3 \times 10^9$ cm for the RRS case ($\zeta = 0.1$).
We assume that the fluid is described by a constant adiabatic index $\gamma = 4/3$, although in reality this is not always true. The shell’s matter may cool during the coasting phase. However at the end of the coasting stage, the reverse shocks heat the fireball shell and the ISM shocked by the relativistic forward shock carries most of the energy $E$. The main part of the system is subject to relativistic particles again. The forward shock is decelerated as $\Gamma \propto R^{-5/2}$ and it becomes Newtonian at $l$. After that the scaling laws in the Newtonian regime depend on the adiabatic constant. But even then the shocked electrons remains relativistic for a long time after this transition, and the adiabatic index will remain around $\frac{4}{3}$.

### 3.2. Numerical Results

Figures 2 and 3 depict the evolution of the Lorentz factors for the NRS and the RRS cases, respectively. One clearly sees the initial acceleration phase in which the Lorentz factor increases linearly with time. This is followed by a coasting phase in which the Lorentz factor is a constant, $\eta$. This stage ends when the effect of ISM becomes significant, and most of the kinetic energy is dissipated at $R_L$ or $R_A$. Then a self-similar phase begins in which the Lorentz factor of the forward shock decreases like $\Gamma \propto R^{-3/2}$. At $l$ the solution becomes nonrelativistic, and it turns into the Sedov-Taylor solution. A difference appears between the NRS and the RRS fireballs only in the energy-transfer phase. As expected, a sharp decrease in $\gamma$ is seen in the RRS case. A more gradual transition is seen in the NRS case.

In the following sections we compare the numerical results with the analytic estimates. We examine the validity of the estimates of $R_L$, $R_s$, $R_A$, $\gamma$, and $l$ as indicators of transition scales.

#### 3.2.1. Free-Acceleration Stage

A highly relativistic shell is formed after a short acceleration phase (see Fig. 4). The width of the shell is constant in this acceleration stage (Fig. 5). The density and the pressure peak at the same position in the local fluid frame. In the observer frame, the outer part of the shell has a higher Lorentz factor and the density peak is wider than the pressure peak. The density peak (dotted line) moves ahead of the pressure peak (solid line) in the observer frame (Fig. 5).

The average Lorentz factor of the each fluid element in the shell increases as the radius of the shell increases, $\gamma \sim r/l$. However, the Lorentz factor of the outermost layers is well above the average. This results from the initial sharp edges of the fireball. The initial acceleration of the outermost layers depends on the steepness of the initial pressure distribution at the edge of the fireball. The initial step function distribution that we have chosen leads to large acceleration of the outermost layers. We can regard the thin region at the boundary of the fireball as expanding in the free-expansion velocity, independent of the fireball thick-
ness $\Delta$. However, this fast layer is thin, and its mass is negligible. Except of the evolution of the maximal Lorentz factor (at the outermost layers) the evolution of the bulk of the fireball is not affected by the choice of the initial steepness of the boundary. The initial conditions are washed out later, when the interaction with the ISM is significant, and even the maximal Lorentz factor is then independent of the initial conditions. Therefore, we do not discuss this maximal value but instead consider the average Lorentz factor over the "uniform shell". We define the average value as
\[
\langle f \rangle \equiv \int f m r^2 dr / \int m r^2 dr ,
\]
where $m \equiv \gamma \rho + (3 + \beta^2)p$ is the effective mass density in the observer frame, and the integrals are defined from the origin to a radius at which the Lorentz factor takes the maximal value. The evolution of the average Lorentz factor $\langle \gamma \rangle$ for the NRS and the RRS cases is shown in Figures 2 and 3 (thick lines), respectively. The Lorentz factors increase linearly with time. The average internal energy and the mass density in the observer frame are shown in Figure 6. $R_L$ is a good indicator to the transition from the acceleration stage to the coasting stage. The mass density equals the internal energy density at $\sim R_L$ (see Fig. 6).

3.2.2. Coasting Stage

After the fireball uses up all the internal energy at $R_L$, it coasts with a Lorentz factor $\eta$ (see Figs 2 and 3). In the RRS case the shell has a frozen radial profile through this stage, while in the NRS case the frozen pulse approximation breaks down at $R_L$ and the shell begins to expand before the ISM has most of the system energy at $R_s$ (see Fig. 5). We can see the transition of the scaling laws of the density and the pressure at $R_s$ in Figure 6.

3.2.3. Energy Transfer Stage: The RRS Case

In the RRS case the reverse shock becomes relativistic at $R_N$ before it crosses the shell at $R_s$. From this moment onward the Lorentz factor of the coasting shell is reduced considerably after the passage of the reverse shock. Figure 7 shows the deceleration by a relativistic reverse shock. There are four regions in the figure: the ISM, the shocked ISM, the shocked shell, and the unshocked shell, which are
separated by the forward shock (FS), the contact discontinuity (CD), and the reverse shock (RS).

Using the shocks’ jump conditions and the equality of the pressure and the velocity at the contact discontinuity, we can estimate the Lorentz factor of the shocked region, \( \gamma_{CD} \), and the Lorentz factor of the reverse shock, \( \Gamma_{RS} \), for a planner geometry (Sari & Piran 1995):

\[
\gamma_{CD} = f^{1/4} \sqrt{\gamma_4/2}, \quad \Gamma_{RS} = \gamma_{CD}/\sqrt{2},
\]

where \( f \equiv \rho_4/p_4 \). The quantities just ahead of (or just behind) the reverse shock are denoted by the subscript 4 (or 3). For spherical geometry, the pressure is not constant in the shocked region bounded by the two shocks. The ratio \( f \) in the above equations should be replaced with \((p_2/p_3) f\). However, in our simulations, \( p_2/p_3 \) is a factor of a few at most. Then equation (12) can roughly explain the numerical result. The time it takes for the reverse shock to cross a distance \( dx \) in the shell material is \( dt = dx \sqrt{f/2} \). As the reverse shock compresses the shell material, the width \( dx \) becomes \( dx/2 \) after the shock passes through it. Then the distance between the contact discontinuity and the reverse shock is \( t/2 \sqrt{f} (\propto t^2) \). Figure 8 shows the numerical result.

The relativistic reverse shock passes through the shell as \( t^2 \), and it decelerates the coasting shell drastically. After this drastic deceleration, the shocked shell slows down as \( t^{-1/2} \) (Fig. 3) owing to the pressure difference between \( p_2 \) and \( p_3 \). At \( R_{A} \), when the reverse shock crosses the shell, \( \gamma_{CD} \) reaches effectively the value of \( \gamma_4 \) expected from the relativistic self-similar solution, and the profile of the shocked ISM region begins to approach the self-similar one. The transition into the self-similar solution is shown in Figure 9.

At the beginning of the self-similar deceleration stage, the density of the shocked shell is much larger than that of the shocked ISM. There is a large gap in the density at the contact discontinuity. However, the density perturbation, as we discuss in § 4, does not effect \( \gamma \) and \( p \), and it does not propagate in the local fluid frame. As the blast wave expands, it leaves the gap, and the ratio between \( \rho_2 \) and the density of the shocked shell damps.

\[
\gamma_{CD} = f^{1/4} \sqrt{\gamma_4/2}, \quad \Gamma_{RS} = \gamma_{CD}/\sqrt{2},
\]

where \( f \equiv \rho_4/p_4 \). The quantities just ahead of (or just behind) the reverse shock are denoted by the subscript 4 (or 3). For spherical geometry, the pressure is not constant in the shocked region bounded by the two shocks. The ratio \( f \) in the above equations should be replaced with \((p_2/p_3) f\). However, in our simulations, \( p_2/p_3 \) is a factor of a few at most. Then equation (12) can roughly explain the numerical result. The time it takes for the reverse shock to cross a distance \( dx \) in the shell material is \( dt = dx \sqrt{f/2} \). As the reverse shock compresses the shell material, the width \( dx \) becomes \( dx/2 \) after the shock passes through it. Then the distance between the contact discontinuity and the reverse shock is \( t/2 \sqrt{f} (\propto t^2) \). Figure 8 shows the numerical result.

The relativistic reverse shock passes through the shell as \( t^2 \), and it decelerates the coasting shell drastically. After this drastic deceleration, the shocked shell slows down as \( t^{-1/2} \) (Fig. 3) owing to the pressure difference between \( p_2 \) and \( p_3 \). At \( R_{A} \), when the reverse shock crosses the shell, \( \gamma_{CD} \) reaches effectively the value of \( \gamma_4 \) expected from the relativistic self-similar solution, and the profile of the shocked ISM region begins to approach the self-similar one. The transition into the self-similar solution is shown in Figure 9.

At the beginning of the self-similar deceleration stage, the density of the shocked shell is much larger than that of the shocked ISM. There is a large gap in the density at the contact discontinuity. However, the density perturbation, as we discuss in § 4, does not effect \( \gamma \) and \( p \), and it does not propagate in the local fluid frame. As the blast wave expands, it leaves the gap, and the ratio between \( \rho_2 \) and the density of the shocked shell damps.

\[
\gamma_{CD} = f^{1/4} \sqrt{\gamma_4/2}, \quad \Gamma_{RS} = \gamma_{CD}/\sqrt{2},
\]

where \( f \equiv \rho_4/p_4 \). The quantities just ahead of (or just behind) the reverse shock are denoted by the subscript 4 (or 3). For spherical geometry, the pressure is not constant in the shocked region bounded by the two shocks. The ratio \( f \) in the above equations should be replaced with \((p_2/p_3) f\). However, in our simulations, \( p_2/p_3 \) is a factor of a few at most. Then equation (12) can roughly explain the numerical result. The time it takes for the reverse shock to cross a distance \( dx \) in the shell material is \( dt = dx \sqrt{f/2} \). As the reverse shock compresses the shell material, the width \( dx \) becomes \( dx/2 \) after the shock passes through it. Then the distance between the contact discontinuity and the reverse shock is \( t/2 \sqrt{f} (\propto t^2) \). Figure 8 shows the numerical result.

The relativistic reverse shock passes through the shell as \( t^2 \), and it decelerates the coasting shell drastically. After this drastic deceleration, the shocked shell slows down as \( t^{-1/2} \) (Fig. 3) owing to the pressure difference between \( p_2 \) and \( p_3 \). At \( R_{A} \), when the reverse shock crosses the shell, \( \gamma_{CD} \) reaches effectively the value of \( \gamma_4 \) expected from the relativistic self-similar solution, and the profile of the shocked ISM region begins to approach the self-similar one. The transition into the self-similar solution is shown in Figure 9.

At the beginning of the self-similar deceleration stage, the density of the shocked shell is much larger than that of the shocked ISM. There is a large gap in the density at the contact discontinuity. However, the density perturbation, as we discuss in § 4, does not effect \( \gamma \) and \( p \), and it does not propagate in the local fluid frame. As the blast wave expands, it leaves the gap, and the ratio between \( \rho_2 \) and the density of the shocked shell damps.
value of the Lorentz factor just behind the forward shock is \( \sim 15 \), and the value drops to \( \sim 5 \) in the inner region. The numerical results agree well with the self-similar solution. In the bottom panel of the figure, the value of the Lorentz factor just behind the forward shock is \( \sim 8 \), and the value drops to \( \sim 2 \) at the inner region, where the numerical results deviate from the self-similar solution. The relativistic self-similar solution is derived with an assumption that each fluid element is highly relativistic, but it is still a good approximation at \( \gamma \sim 5 \).

### 3.2.6. Transition to the Sedov-Taylor Solution

The Lorentz factor of the forward shock decreases and becomes nonrelativistic around \( l (l \sim 1.9 \times 10^{18} \text{ in this case}) \). The scaling laws of the velocity \( \beta_2 \), the density \( \rho_2 \), and the pressure \( p_2 \) also gradually shift from the Blandford-McKee solution to the Sedov-Taylor solution around \( l \) (see Fig. 11). Circles 2 \((R_{c2} \equiv 0.61l)\) and 3 \((R_{c3} \equiv 0.50l)\) in Figure 11b give a rough estimate of the radii where the relativistic self-similar solution becomes invalid and the Newtonian self-similar solution becomes valid \((\gamma_2 \sim 1.4, \beta_2 \sim 0.70 \text{ at circle } 2; \gamma_2 \sim 1.7, \beta_2 \sim 0.81 \text{ at circle } 3)\). There is an overlap region in which the both self-similar solutions are valid. The relation between \( \beta_2 \) and \( R \) in the Newtonian self-similar solution is already valid at circle 3, but the shock radius \( R \) is still smaller than the radius expected by the Newtonian self-similar solution. The shock radius reaches within a 15% error line at circle 4 \((R_{c4} \equiv 0.85l)\) in Figure 11c, where \( \gamma_2 \sim 1.1, \beta \sim 0.47 \). The relation between \( \beta_2 \) and \( t \) in the Newtonian self-similar solution becomes valid at circle 5 \((R_{c5} \equiv 1.3l)\) in Figure 11c, where \( \gamma_2 \sim 1.02 \) and \( \beta_2 \sim 0.22 \).

The transition of the profiles in the shocked ISM region from the relativistic stage to the Newtonian stage is shown in Figure 13. The jump condition for a strong shock gives a simple relation between the ISM density \( \rho_1 \) and the shocked density \( \rho_2 (= \rho_2/\gamma_2) \). The ISM is then swept up at a radius \( R \), the thickness of the blast wave is approximately \( R/\gamma_2 (4\gamma_2 + 3) \propto R^5 \) for the relativistic stage. The thickness is \( \propto R \) for the Newtonian stage. As the blast wave expands, it becomes broader (Fig. 13a). The density profile approaches the one expected by the Newtonian self-similar solution at \( R_{o3} \) (see Fig. 13b). The inner part of the velocity distribution begins to approach the Newtonian self-similar.

---

**Fig. 10.**—Transition into the relativistic self-similar solution for a NRS solution. Profiles of \( \gamma, \rho, \) and \( p \) at different times \((t = 0.70R, 0.75R, 0.82R, 0.87R, 0.92R, 0.98R, 1.1R, \) and \( 1.1R \)) The x-axis is the distance from the contact discontinuity (dashed line).

**Fig. 11.**—Deceleration stage. (a) \( \beta_2, \gamma_2 \) (solid lines), density (dashed lines), and pressure (dash-dotted lines) just behind the shock vs. the shock radius. The numerical results are thick lines and the relativistic and the Newtonian self-similar solutions are thin lines. We plot \( \gamma_2 (\beta_2) \) rather than \( \beta_2, \gamma_2 \) for the relativistic self-similar solution (the Newtonian self-similar solution). The initial parameters are the same as in Fig. 3. (b) The ratio between the numerical results and the analytic estimates plotted against the shock radius \( \gamma_2 (\beta_2) \) and \( \beta_2, \gamma_2 \) (solid line), density (dashed line), and pressure (dash-dotted line) just behind the shock. The numerical results are compared with the relativistic (thick line) and the Newtonian (thin line) self-similar solutions. \( \gamma_2 (\beta_2) \) is compared with the analytic estimate in the relativistic regime (the Newtonian regime). Circles 1, 2, and 3, indicating the cross points, are at \( R = 1.4R_{c2}, 0.61l, \) and \( 0.50l \), respectively. (c) The ratio between the numerical results and the analytic estimates plotted against the time. The dotted line indicates the ratio between the numerical shock radius and the analytic estimates. Circles 4 and 5, indicating the cross points, are at \( t = 1.0l (R = 0.85l) \) and \( t = 2.6l (R = 1.4l) \), respectively.
when the Lorentz factor behind the shock is uniform ISM of the density 1 proton cm

The latter is an entropy perturbation with no change of

The other is an entropy-vortex wave moving with the fluid. These waves propagate

perturbations to the self-similar profile. One type is a sound wave

stable. In this section we consider the evolution of spherical perturbations to the


Then the energy is transferred to the ISM. This takes place

its initial radiation energy to kinetic energy and then coasts.

After the outgoing pulse boosts the forward shock, it departs from

the forward shock much faster than a density perturbation. After the outgoing pulse boosts the forward shock, the flow profile is almost self-similar.

A pressure perturbation (Fig. 14c) also induces perturbations in \( \gamma \) and \( \rho \). The perturbations consist of three components: two propagating components and a standing density perturbation. After an outgoing compression pulse and an ingoing shock propagate, a density perturbation remains at the position where the initial pressure perturbation was (in the local fluid frame). The ingoing shock leaves the forward shock quickly. The density perturbation departs from the forward shock as discussed earlier. The outgoing pulse is basically the same as the one that appears in the case of an initial velocity perturbation. The pulse boosts the forward shock, and at this stage the profile is almost self-similar.

5. CONCLUSIONS

We have explored numerically the evolution of a relativistic fireball interacting with a uniform ISM through the stages of initial acceleration, coasting, energy transfer to the ISM, and deceleration. These calculations begin when the fireball is at rest. They follow the acceleration to a relativistic velocity and the subsequent slowing down to a velocity far below the speed of light. These calculations span more than eight orders of magnitude in the size of the fireball. The current analytic understanding of fireball evolution explains our numerical results well. Initially, the Lorentz factor increases linearly with the radius during the initial free-acceleration stage. At \( R_s \), the fireball transfers all its initial radiation energy to kinetic energy and then coasts. Then the energy is transferred to the ISM. This takes place at \( R_s \) for the NRS case and at \( R_s \) for the RRS case. After that, the shocked ISM carries most of the initial energy of the fireball. The profile of the shocked ISM is then described well by the relativistic self-similar Blandford-McKee solution. The forward shock decelerates as
Fig. 13.—Transition of the flow profile of the shocked ISM from the relativistic Blandford-McKee solution to the Newtonian Sedov-Taylor solution: (a) $\beta$ vs. $r$; (b) $\rho/\rho_2$ vs. $r/R$; (c) $\beta/\beta_2$ vs. $r/R$; and (d) $p/p_2$ vs. $r/R$. The dotted lines shows the Newtonian self-similar solution. The numbers denote the shock radii $R = (1) 0.21 l, (2) 0.33 l, (3) 0.52 l, (4) 0.76 l, (5) 1.0 l, (6) 1.2 l, (7) 1.3 l, (8) 1.5 l, (9) 1.6 l, (10) 1.8 l, (11) 2.0 l, (12) 2.2 l, and (13) 2.4 l. The initial parameters are the same as in Fig. 3.
the transition into the relativistic self-similar solution and the transition from this solution to the nonrelativistic self-similar solution.

With the spherical code, we have shown that the hydrodynamical profiles of the shocked ISM approaches the relativistic self-similar solution at the end of the coasting stage, even though the initial conditions of the simulation do not have self-similar profiles. This implies that the relativistic self-similar solution is attractive and stable for a spherical perturbation. We have tracked the evolution of the spherical density, velocity, and pressure perturbations to the relativistic self-similar solution in order to verify the stability of this solution.

The Blandford-McKee solution is the basis of much of the GRB afterglow theory. It is used to obtain an explicit expression for the radial profile during the deceleration stage. We have shown that this model is reasonable even down to $\gamma \sim 2$. It is also valid even in the case of a radial inhomogeneity in the ISM.

Previous numerical studies (Panaitescu et al. 1997) calculated the radial evolution only around the radius where most of the energy is transferred from the fireball to the ISM. Their parameters correspond to the Newtonian reverse shock case. They constructed the light curves of the GRB and the afterglow (Panaitescu & Mészáros 1998a, 1998b). Though the results presented in this paper are intended more as a general explanation of the strong explosion problem than as detailed fits of the GRB afterglow, comparisons of the numerically predicted light curves based on these computations in detailed realistic models with recent and upcoming observations will enable us to determine free parameters in the GRB model that cannot be calculated from first principles, such as the total energy of the system, the surrounding density, and the fraction of the energy that is given by the shocks to electrons or to the magnetic field.

S. K. gratefully acknowledges the support of a Golda Meir postdoctoral fellowship. R. S. thanks the Clore Foundations for support. This work was supported in part by US-Israel BSF grant 95-328 and by a NASA grant NAG 5-3516.

REFERENCES

Blandford, R. D., & McKee, C. F., 1976, Phys. Fluids, 19, 1130
Cavali, & Rees. 1978, MNRAS, 183, 359
Daighe, F., & Mochkovitch, R. 1997, MNRAS, 296, 275
Goodman, J. 1986, ApJ, 308, L47
Landau, L. D., & Lifshitz, E. M., 1987, Fluid Mechanics (2d ed.; London: Pergamon)
Katz, J., 1994, ApJ, 422, 248
Kobayashi, S., Piran, T., & Sari, R. 1997, ApJ, 490, 92
Mészáros, P., Laguna, P., & Rees, M. J. 1993, ApJ, 415, 181
Mészáros, P., & Rees, M. J. 1992, ApJ, 397, 570
Narayan, R., Paczyński, B., & Piran, T. 1992, ApJ, 395, L83
Paczyński, B. 1986, ApJ, 308, L51
Panaitescu, A., & Mészáros, P. 1998a, ApJ, 492, 683
—. 1998b, ApJ, 501, 722
Panaitescu, A., Wen, L., Laguna, P., & Mészáros, P. 1997, ApJ, 482, 942
Piran, T., Shemi, A., & Narayan, R. 1993, MNRAS, 263, 861
Rees, M. J., & Mészáros, P. 1994, ApJ, 430, L93
Sari, R. 1997, ApJ, 489, L37
Sari, R., & Piran, T. 1995, ApJ, 455, L143
—. 1997, ApJ, 485, 270
Sedov, L. I. 1946, Prikl. Mat. i Mekh., 10, 241
Shemi, A., & Piran, T. 1990, ApJ, 365, L55
Taylor, G. I. 1950, Proc. R. Soc. London A 201, 159
von Neumann, J. 1947, Los Alamos Sci. Lab. Tech. Ser. 7