Hoeffding decomposition in $H^1$ spaces

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Abstract
The well known result of Bourgain and Kwapień states that the projection $P_{\leq m}$ onto the subspace of the Hilbert space $L^2(\Omega^\infty)$ spanned by functions dependent on at most $m$ variables is bounded in $L^p$ with norm $\leq c_m^p$ for $1 < p < \infty$. We will be concerned with two kinds of endpoint estimates. We prove that $P_{\leq m}$ is bounded on the space $H^1(D^\infty)$ of functions in $L^1(T^\infty)$ analytic in each variable. We also prove that $P_{\leq 2}$ is bounded on the martingale Hardy space associated with a natural double-indexed filtration and, more generally, we exhibit a multiple indexed martingale Hardy space which contains $H^1(D^\infty)$ as a subspace and $P_{\leq m}$ is bounded on it.

Keywords Hardy spaces · Hoeffding decomposition · Martingale inequalities · Decoupling

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1 Introduction

The Rademacher functions \((r_i)_{i \in \mathbb{N}}\) generate a well studied subspace of \(L^p[0, 1]\), which we identify with \(L^p(\mathbb{Z}^N_2)\). In particular by Khintchine inequality
\[
\left\| \sum_i c_i r_i \right\|_{L^p} \simeq_p \left( \sum_i |c_i|^2 \right)^{\frac{1}{2}}
\]
for \(0 < p < \infty\), and \(\text{span} (r_i : i \in \mathbb{N})\) is complemented in \(L^p\) for \(1 < p < \infty\) but not for \(p = 1\). We will index the Walsh system by finite subsets of \(\mathbb{N}\), i.e.
\[
w_A = \prod_{i \in A} r_i.
\]
The number \(|A|\) is called the mutiplicity of \(w_A\). Analogous problems for Walsh functions of finite multiplicity have been resolved independently by Bonami [2] and Kiener [13]. Namely, the inequality
\[
\left\| \sum_{|A| \leq m} c_A w_A \right\|_{L^p} \simeq_{p,m} \left( \sum_{|A| \leq m} |c_A|^2 \right)^{\frac{1}{2}}
\]
holds true, and the orthogonal projection \(P_m\) onto \(\text{span} (w_A : |A| \leq m)\) is bounded if and only if \(1 < p < \infty\). Some lower estimates for \(P_m\) are also known, see [12] or [20] for a detailed discussion on this subject.

In [4], Bourgain generalized these results to a setting in which \((\mathbb{Z}_2, \{\emptyset, \{0\}, \{1\}, \{0, 1\}\})\) is replaced with an arbitrary probability space \((\Omega, \mathcal{F}, \mu)\). To be more precise, let \((\Omega^\infty, \mathcal{F}^\infty, \mu^\infty)\) be the infinite product space. Any \(f \in L^2(\Omega^\infty, \mathcal{F}^\infty, \mu^\infty)\) can be decomposed in a unique way into a series
\[
f(x) = \sum_m \sum_{i_1 < \ldots < i_m} f_{i_1,\ldots,i_m}(x_{i_1}, \ldots, x_{i_m})
\]
where \(f_{i_1,\ldots,i_m} \in L^2(\Omega^m)\) is mean zero in each of its \(m\) arguments. Thus, \(P_A\) and \(P_m\) defined by
\[
P_{\{i_1,\ldots,i_m\}} f(x) = f_{i_1,\ldots,i_m}(x_{i_1}, \ldots, x_{i_m}), \quad P_m = \sum_{|A|=m} P_A
\]
are mutually orthogonal orthogonal projections. In the case of \(\Omega = \mathbb{Z}_2\), the image of \(P_A\) is just the one-dimensional space spanned by \(w_A\), so the above definition of \(P_m\) coincides for with the projection onto Walsh functions of multiplicity \(m\). In [4] Bourgain proved that for \(1 \leq p < \infty\),
\[
\left\| \sum_{|A| \leq m} P_A f \right\|_{L^p} \simeq_p \left\| \left( \sum_{|A| \leq m} |P_A f|^2 \right)^{\frac{1}{2}} \right\|_{L^p},
\]
which is a direct generalization of (1.3). Moreover, he proved that \(P_m\) is bounded on \(L^p\) if and only if \(1 < p < \infty\), with norm smaller than \(c_p^m\) where \(c_p \lesssim \frac{\hat{p}^2}{\log \hat{p}}\) and \(\hat{p} = p \vee \frac{p}{p-1} \).

It turns out that the projections \(P_m\) have a well established probabilistic interpretation. In [14], Kwapien connected them to the notion of Hoeffding decomposition, which originated...
from Hoeffding’s work [11]. More precisely, elements of the image of \( P_m \) are what is called
generalized canonical \( U \)-statistics and the decomposition \( f = \sum_m P_m f \) plays a crucial role
in the proofs of many theorems concerning \( U \)-statistics. For more information, we refer the
reader to [16]. Kwapień provided a shorter proof of Bourgain’s result about boundedness of
\( P_m \), with a better constant \( c_p \lesssim \hat{p} \log \hat{p} \).

Let us decribe the main results of this paper, which give certain endpoint estimates for
\( P_m \). One of them (Theorem 4.5 in the text) is obtained by restricting the domain of
\( P_m \).

**Theorem A**  \( P_m \) is bounded on the subspace \( H^1_{all}(\mathbb{T}^{\infty}) \) of \( L^1(\mathbb{T}^{\infty}) \) consisting of functions
analytic in each variable.

We also find a norm stronger than \( L^1 \) and weaker than \( L^p (p > 1) \), in which
\( P_m \) is bounded. The detailed construction is described in Sect. 5.2.

**Theorem B** For any \( m \in \mathbb{N} \), there is a partition of the family of finite subsets of \( \mathbb{N} \) into
\( \bigcup_{i \in I} A_i \) such that the norm

\[
\| f \| := \mathbb{E} \left( \sum_i \left| \sum_{A_i} P_A f \right|^2 \right)^{1/2}
\]

is between \( L^1 \) and all \( L^p (p > 1) \) and \( P_m \) is bounded in this norm.

It is worth noting that Theorem A translates directly to the space \( \mathcal{H}^1 \) of Dirichlet series, i.e.
the closure of polynomials of the form \( \sum_{n=1}^{N} b_n n^{-s} \) in the norm

\[
\left\| \sum_{n=1}^{N} b_n n^{-s} \right\|_{\mathcal{H}^1} := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^{N} b_n n^{-it} \right| dt.
\]

The Bohr lift, dating back to [1], is the map

\[
H^1_{all}(\mathbb{T}^{\infty}) \ni \sum_{k \in \mathbb{N}^{\mathbb{N}}} a_k e^{i(k,t)} \mapsto \sum_{n \in \mathbb{N}} b_n n^{-s} \in \mathcal{H}^1
\]

where \( a_k = b_n \) for \( n \) having the prime number factorization \( n = \prod_j p_j^{k_j} \). It is an isometry
between \( H^1_{all}(\mathbb{T}^{\infty}) \) and the space \( \mathcal{H}^1 \) of Dirichlet series. Thus, our result is equivalent to the
fact that the projection from \( H^1 \) onto

\[
\text{span} \{ n^{-s} : n \text{ has at most } m \text{ prime factors} \} \subset \mathcal{H}^1
\]

is bounded. For a more detailed exposition of Dirichlet series and their relation to polydisc
Hardy spaces, see [21].

The paper is organized as follows. In Sect. 2 we introduce necessary notation and def-
initions. In Sect. 3, we provide a new simple proof of the historic \( L^p \) boundedness result.
The proof of the estimate \( \| P_m \| \leq (e \| P_1 \|)^m \) is done by means of a combinatorial identity
expressing \( P_m \) in terms of tensor products of \( P_1 \). In Sect. 4, we show that the same argument
carries over with little modification showing boundedness of \( P_m \) on \( H^1_{all}(\mathbb{T}^{\infty}) \). In Sect. 5.1,
we define, purely in terms of square functions and not referring to analyticity, a multiple
indexed martingale Hardy space \( H^1_{m \text{last}}(\mathbb{T}^{\infty}) \) of functions on \( \Omega^\infty \) that admits a bounded action
of \( P_m \). It turns out that if \( \Omega = \mathbb{T} \), there is a subspace \( H^1_{m \text{last}}(\mathbb{N}^{\infty}) \) of \( L^1(\mathbb{T}^{\infty}) \), much bigger.
than \( H_{\text{all}}^1 (\mathbb{T}^\infty) \), on which the \( L^1 \) norm is equivalent to \( H^1 [T_m] \) norm. The arguments rely heavily on \( L^1 \) square function theorem for Hardy martingales and decoupling inequality of Zinn. We present two proofs of the latter in Sect. 1.

2 Preliminaries

Probability spaces and conditional expectations In all of the text, \((\Omega, \mathcal{F}, \mu)\) will be a probability space. We will equip sets of the form \( \Omega^I \), where \( I \) is an at most countable index set, with the product measure \( \mu^\otimes I \) defined on \( \mathcal{F}^\otimes I \). In case we are only concerned with the cardinality of \( I \), we will write \( \Omega^n \), where \( n \) is a natural number or \( \infty \). By the natural filtration on \( \Omega^\mathbb{N} \) we mean the filtration \( (\mathcal{F}_n : n = 0, 1, \ldots) \), where \( \mathcal{F}_k \) is generated by the coordinate projection \( \omega \mapsto (\omega_1, \ldots, \omega_k) \) and denote \( \mathbb{E}_k = \mathbb{E} \cdot | \mathcal{F}_k \). In general, for a subset \( A \) of the index set, \( \mathcal{F}_A \) will be the sigma algebra generated by the coordinate projection \( \omega \mapsto (\omega_i)_{i \in A} \) and \( \mathbb{E}_A = \mathbb{E} \cdot | \mathcal{F}_A \). In more explicit terms, measurability with respect to \( \mathcal{F}_A \) is equivalent to being dependent only on variables with indices belonging to \( A \) and the conditional expectation operator \( \mathbb{E}_A \) integrates away the dependence on all other variables, so that the formulas

\[
\mathbb{E}_k f (x) = \int_{\Omega^{k+1, \infty}} f (x_1, \ldots, x_k, y_{k+1}, y_{k+2}, \ldots) \, d\mu^\otimes(k+1, \infty) (y),
\]

\[
\mathbb{E}_A f (x) = \int_{\Omega^{\mathbb{N}\setminus A}} f (x_A, y_{\mathbb{N}\setminus A}) \, d\mu^\otimes\mathbb{N}\setminus A (y)
\]

are satisfied (with the convention that sequences indexed by \( A \) and \( \mathbb{N}\setminus A \) are merged in a natural way into a sequence indexed by \( \mathbb{N} \)). It will often be convenient to identify a function \( f \) defined on \( \Omega^A \) with an \( \mathcal{F}_A \)-measurable function \( \Omega^I \ni \omega \mapsto f ((\omega_i)_{i \in A}) \). In order to save space, we will often write \( dx \) instead of \( d\mu (x) \) whenever the measure is implied by context.

Tensor products Let \( 1 \leq p < \infty \). For \( f_k \in L^p (\Omega_k) \), we will denote by \( \bigotimes_{k=1}^n f_k \) the function on \( \prod_k \Omega_k \) satisfying

\[
\left( \bigotimes_k f_k \right) (x) = \prod_k f_k (x_k).
\]

Because of separation of variables, we have \( \left\| \bigotimes_k f_k \right\|_{L^P (\prod_k \Omega_k)} = \prod_k \| f_k \|_{L^P (\Omega_k)} \). This way we actually define an injection of the algebraic tensor product \( \bigotimes_k L^P (\Omega_k) \) into \( L^P (\prod_k \Omega_k) \), the image of which is dense.

Let \( X_k \) be subspaces (by a subspace we always mean a closed linear subspace) of \( L^P (\Omega_k) \). By \( \bigotimes_k X_k \) we will denote the subspace of \( L^P (\prod_k \Omega_k) \) spanned by functions of the form \( \bigotimes_k f_k \), where \( f_k \in X_k \), and the norm is inherited from \( L^P (\prod_k \Omega_k) \) (care has to be taken, as \( \bigotimes_k X_k \) is not determined solely by \( X_k \) as Banach spaces, but rather by the particular way they are embedded in \( L^P (\Omega_k) \)). If \( T_k : X_k \to L^P (\Omega_k) \) are bounded operators, then we can define an operator \( \bigotimes_k T_k : \bigotimes_k X_k \to L^P (\prod_k \Omega_k) \) by the formula

\[
\left( \bigotimes_k T_k \right) \left( \bigotimes_k f_k \right) = \bigotimes_k T_k f_k,
\]

and easily check that the property

\[
\left\| \bigotimes_k T_k : \bigotimes_k X_k \to L^P (\prod_k \Omega_k) \right\| \leq \prod_k \| T_k : X_k \to L^P (\Omega_k) \|.
\]
is satisfied. Indeed, $\otimes_k T_k = \prod_k \text{id}_{L^p(\prod_{j \neq k} \Omega_j)} \otimes T_k$, and any operator of the form $\text{id} \otimes T$ has norm bounded by $\|T\|$, because $(\text{id} \otimes T) f (\omega_1, \omega_2) = T (f (\omega_1, \cdot)) (\omega_2)$.

**Fourier transform** Let $\mathbb{T}$ be the interval $[0, 2\pi)$ equipped with addition modulo $2\pi$ and normalized Lebesgue measure $d\mu = \frac{dx}{2\pi}$. We will be exclusively dealing with Fourier transforms of functions on $\mathbb{T}$ or some power of $\mathbb{T}$. Since the group dual to $\mathbb{T}$ is $\mathbb{Z}$, the dual group to the product $\mathbb{T}^N$ is the direct sum $\mathbb{Z}^\otimes \mathbb{N}$ (i.e., integer-valued sequences that are eventually 0), on which we define the Fourier transform by

$$\hat{f}(n) = \int_{\mathbb{T}^N} f(x) e^{-in \cdot x} d\mu \otimes \mathbb{N}(x). \quad (2.6)$$

**Hardy spaces of martingales and analytic functions** By $\mathbb{D}$ we denote the unit disk in the complex plane. We can identify $\mathbb{T}$ with the unit circle by the map $t \mapsto e^{it}$. For $N \in \mathbb{N}$, the space $H^1(\mathbb{D}^N)$ is defined as the space of functions analytic in the polydisc $\mathbb{D}^N$ such that the norm

$$\|F\|_{H^1(\mathbb{D}^N)} = \sup_{0 \leq r_1, \ldots, r_N < 1} \left| \int_{\mathbb{T}^N} F \left( r_1 e^{it_1}, \ldots, r_N e^{it_N} \right) \frac{dt}{(2\pi)^N} \right|$$

is finite. It is well-known [10] that such a function has an a.e. radial limit $f(t_1, \ldots, t_n) = \lim_{r \to 1} F \left( r e^{it_1}, \ldots, r e^{it_n} \right)$ on the distinguished boundary $\mathbb{T}^N$ and $F$ can be recovered from $f$ by convolution with a Poisson kernel. This sets a one-to-one correspondence between $H^1(\mathbb{D}^N)$ and the space

$$H^1_{\text{all}}(\mathbb{T}^N) = \text{span} \left\{ e^{in \cdot t} : n_1, \ldots, n_N \geq 0 \right\} \subset L^1(\mathbb{T}^N). \quad (2.8)$$

We also can define $H^1_{\text{all}}(\mathbb{T}^N)$ in the same manner as in (2.8), but care has to be taken, since these functions are no longer be extended analytically to $\mathbb{D}^N$ in general (hence the shorthand $H^1(\mathbb{D}^N)$, which we will sometimes use, is an abuse of notation). Later we will use two more $H^1$ spaces, namely $H^1_{\text{last}}(\mathbb{T}^N)$ (also called Hardy martingales) and $H^1_{m \text{ last}}(\mathbb{T}^N)$, which we will define as follows.

$$H^1_{\text{last}}(\mathbb{T}^N) = H^1_{\text{last}}(\mathbb{T}^N) = \text{span} \left\{ e^{in \cdot t} : n_{i_0} > 0 \text{ for } i_0 = \max \{ i : n_i \neq 0 \} \right\} \subset L^1(\mathbb{T}^N). \quad (2.9)$$

$$H^1_{m \text{ last}}(\mathbb{T}^N) = \text{span} \left\{ e^{in \cdot t} : m \text{ last of nonzero } n_j \text{ 's are } > 0 \right\} \subset L^1(\mathbb{T}^N). \quad (2.10)$$

In the space $H^1_{m \text{ last}}(\mathbb{T}^N)$ we allow characters of the form $e^{in \cdot t}$, where $|\text{supp } n| < m$ and $n_j \geq 0$ for all $j$.

Now we recall the definition of a martingale Hardy space and some related inequalities. A standard reference in this matter is [9]. Let $(\mathcal{F}_n)_{n=0}^\infty$ be an arbitrary filtration on a probability space $(\Omega, \mathcal{F}, \mu)$, where $\mathcal{F}$ is generated by $\bigcup \mathcal{F}_n$. We denote $\mathbb{E}_k = \mathbb{E}(\cdot | \mathcal{F}_k)$, $\Delta_0 = \mathbb{E}_0$, $\Delta_k = \mathbb{E}_k - \mathbb{E}_{k-1}$ for $k \geq 1$, and define the square function and maximal function of $f$ respectively by

$$Sf = \left( \sum_{n=0}^\infty |\Delta_n f|^2 \right)^{1/2}, \quad f^* = \sup_n |\mathbb{E}_n f|. \quad (2.11)$$

This allows us to define the martingale Hardy space.
Definition 2.1 The space $H^1[\mathcal{F}_n_{n=1}^\infty]$ is a function space on $\Omega$ with the norm
\[
\| f \|_{H^1[(\mathcal{F}_n)_{n=1}^\infty]} = \mathbb{E}Sf.
\] (2.12)
We will make use of three following classical martingale inequalities.

Theorem 2.2 (Burkholder, Gundy [7] for $1 < p < \infty$; Davis [8] for $p = 1$) For $1 \leq p < \infty$,
\[
\| Sf \|_{L^p} \simeq_p \| f \|_{L^p}^*.
\] (2.13)

Theorem 2.3 (Burkholder [6]) For $1 < p < \infty$,
\[
\| f \|_{L^p} \simeq \| Sf \|_{L^p}.
\] (2.14)

Theorem 2.4 (Stein [24]) For $1 < p < \infty$ and an arbitrary sequence $(f_n)_{n=0}^\infty$,
\[
\left( \sum_{n=0}^\infty |\mathbb{E}_n f_n|^2 \right)^{\frac{1}{2}} \lesssim_p \left( \sum_{n=0}^\infty |f_n|^2 \right)^{\frac{1}{2}}. \tag{2.15}
\]

Definition 2.5 A martingale atom is a function of the form
\[
a = u - \mathbb{E}_{j-1}u,
\] (2.16)
where
\[
A \in \mathcal{F}_j, \quad \text{supp } u \subset A, \quad \| u \|_{L^2} \leq |A|^{-\frac{1}{2}}. \tag{2.17}
\]

Theorem 2.6 Let $f \in H^1[(\mathcal{F}_n)_{n=1}^\infty]$ be of mean 0. Then there are atoms $a_1, a_2, \ldots$ and scalars $c_1, c_2, \ldots$ such that
\[
f = \sum_{n=1}^\infty c_n a_n \tag{2.18}
\]
and
\[
\sum_{n=1}^\infty |c_n| \lesssim \| f \|_{H^1[(\mathcal{F}_n)_{n=1}^\infty]}.
\] (2.19)

Theorem 2.7 [Fefferman] The dual space to $H^1[(\mathcal{F}_n)_{n=1}^\infty]$ is $BMO[(\mathcal{F}_n)_{n=1}^\infty]$, where
\[
\| g \|_{BMO[(\mathcal{F}_n)_{n=1}^\infty]} \simeq \sup_k \left( \mathbb{E}_k \sum_{n \geq k} |\Delta_n g|^2 \right)^{\frac{1}{2}}_{L^\infty}, \tag{2.20}
\]
where the duality is given by $\langle f, g \rangle = \lim_{n \to \infty} \mathbb{E} (\mathbb{E}_n f \mathbb{E}_n g)$.

Vector-valued inequalities For a Banach space $B$, by $L^p(S, B)$ we denote the Bochner space of strongly measurable $B$-valued random variables equipped with the norm
\[
\| f \|_{L^p(S, B)} = \left( \int_S \| f(x) \|_B^p d\mu(s) \right)^{\frac{1}{p}} \tag{2.21}
\]
(or, equivalently, the closed span of functions of the form $(f \otimes v)(x) = f(x)v$, where $f \in L^p(S)$ and $v \in B$, in the $L^p(S, B)$ norm). For an operator $T$ between subspaces of
Let $X_i \in \ell^2(I_i)$ for $i = 1, 2$, $B$ be a Hilbert space and $T : X_1 \to X_2$ be bounded. Then $T \otimes \text{id}_B : X_1 \otimes B \to X_2 \otimes B$, where $X_i \otimes B$ is treated as a subspace of $L^1(\Omega_i, \ell^2(I_i, B))$, is bounded with norm $\lesssim \|T\|$.  

**Proof** Without loss of generality, $B$ is finite-dimensional, say $B = \ell^2(J)$ for some finite $J$. Let $X_1 \otimes \ell^2(J) \ni f = (f_j)_{j \in J}$, so that $f_j \in X_1$. Let also $r_j$ for $j \in J$ be Rademacher variables. Then, applying $\ell^2(I_2)$-valued Khintchine inequality,

$$\| (T \otimes \text{id}) f \|_{L^1(S_2, \ell^2(I_2 \times J))}$$

$$\leq \| T \| \| \mathbb{E} \int_{S_1} \left( \sum_j r_j f_j(s) \right) \|_{\ell^2(I_2)} \|_{\ell^2(I_1)} \|_{L^1(S_1, \ell^2(I_1 \times J))}. \tag{2.27}$$

Hoeffding decomposition Now we define the main object of our interest. In order to avoid technicalities with convergence in strong operator topology, we will work in a finite product of $\Omega$ (all the results extend automatically to $\Omega^\infty$ by density). We will see in a moment that any function $f \in L^1(\Omega^n)$ can be decomposed in a unique way as

$$f = \sum_{m=0}^n \sum_{1 \leq i_1 < \ldots < i_m \leq n} P_{i_1 \ldots i_m} f,$$

where $P_{i_1 \ldots i_m} f (x_1, \ldots, x_n)$ depends only on $x_{i_1}, \ldots, x_{i_m}$ and is of mean 0 with respect to each of $x_{i_1}, \ldots, x_{i_m}$ (equivalently, $P_A f$ is $\mathcal{F}_A$-measurable and is orthogonal to all $\mathcal{F}_B$-measurable functions for $B \subsetneq A$). This decomposition has been studied in [4,14].
particular, $P_{1,...,m}$ are pairwise orthogonal projections. Let

$$P_m = \sum_{1 \leq i_1 < ... < i_m \leq n} P_{i_1,...,i_m}$$

and $U_m$ be the range of $P_m$. It is known [4,14] that $P_m$ is bounded on $L^p(\Omega^n)$, $1 < p < \infty$, with norm independent on $n$, but this is not true for $L^1(\Omega^n)$.

One of the possible ways to prove the existence of the above decomposition in $L^2(\Omega^n)$ is as follows. First we define the subspace $U_{\leq m} = \text{span} \bigcup_{|A| \leq m} \{ f \in L^2(\Omega^n) : f \text{ is } \mathcal{F}_A\text{-measurable} \} \subset L^2(\Omega^n)$ (2.28) for each $m \geq 0$. The sequence of subspaces $U_{\leq 0}, U_{\leq 1}, \ldots, U_{\leq n}$ is increasing, so by putting

$$U_0 = U_{\leq 0}, \quad U_m = U_{\leq m} \cap U_{\leq m-1}$$

we obtain a decomposition

$$L^2(\Omega^n) = \bigoplus_{m=0}^n U_m$$

into an orthogonal direct sum of $U_m$. We will denote the orthogonal projection onto $U_m$ by $P_m$.

A more explicit formula for $P_m$ can be obtained. For $A \subset [1,n]$, let

$$P_A = (\text{id} - \mathbb{E})^{\otimes A} \otimes \mathbb{E}^{\otimes [1,n]\setminus A},$$

where id and $\mathbb{E}$ are understood to act on $L^2(\Omega)$, and let $U_A$ be the range of the projection $P_A$. It is easy to see that

$$\mathbb{E}_A = (\text{id} - \mathbb{E} + \mathbb{E})^{\otimes A} \otimes \mathbb{E}^{\otimes [1,n]\setminus A} = \sum_{B \subset A} (\text{id} - \mathbb{E})^{\otimes B} \otimes \mathbb{E}^{\otimes [1,n]\setminus B}$$

and, since the subspaces $U_B$ are mutually orthogonal,

$$L^2(\Omega^n, \mathcal{F}_A) = \bigoplus_{B \subset A} U_B.$$ (2.33)

Moreover

$$U_{\leq m} = \text{span} \bigcup_{|A| \leq m} L^2(\Omega^n, \mathcal{F}_A)$$

$$= \text{span} \bigcup_{|A| \leq m} \bigoplus_{B \subset A} U_B$$

$$= \bigoplus_{|B| \leq m} U_B$$

and consequently

$$U_m = \bigoplus_{|B|=m} U_B, \quad P_m = \sum_{|B|=m} P_B.$$ (2.37)

**Decoupling inequalities** We are going to present a special case of a theorem of J. Zinn [26], which will be one of the most important tools.
Corollary 2.11
For all \(i\) as desired.
\(\Box\)

We will provide two new proofs of the above in Sect. 1. Below, we state two corollaries obtained by iterating Zinn’s inequality.

Corollary 2.10
For \(1 \leq a < b \leq N\), let \(f_{a,b} \in L^1(\Omega^N, \mathcal{F}_{[a,b]})\). Denote \((x_a, \ldots, x_b)\) by \(x_{[a,b]}\). Then

\[
\int_{\Omega^N} \left( \sum_{a<b} |f_{a,b}(x_{[a,b]})|^2 \right)^{\frac{1}{2}} \, dx \sim \int_{\Omega^N} \left( \sum_{a<b} |f_{a,b}(z_a, x_{a+1, b-1}, y_b)|^2 \right)^{\frac{1}{2}} \, dx \, dy \, dz
\]

(2.39)

Proof
Let \(F_b \in L^1(\Omega^N, \mathcal{F}_b, \ell^2)\) be defined by \((F_b)_a = f_{a,b}\) for \(a < b\) and 0 otherwise. Then, by Theorem 2.9 applied for functions \(\|F_b\|_{\ell^2}\),

\[
\int_{\Omega^N} \left( \sum_{a<b} |f_{a,b}(x_{[a,b]})|^2 \right)^{\frac{1}{2}} \, dx = \int_{\Omega^N} \left( \sum_b \|F_b(x_{\leq b})\|_{\ell^2}^2 \right)^{\frac{1}{2}} \, dx
\]

(2.40)

\[
\sim \int_{\Omega^N} \left( \sum_b \|F_b(x_{b-1}, y_b)\|_{\ell^2}^2 \right)^{\frac{1}{2}} \, dx \, dy
\]

(2.41)

\[
= \int_{\Omega^N} \left( \sum_{a<b} |f_{a,b}(x_{a+1, b-1}, y_b)|^2 \right)^{\frac{1}{2}} \, dx \, dy.
\]

(2.42)

Analogously, by setting \(y\) as fixed, and applying Theorem 2.9 with reversed order of variables (which we can do, because we are dealing with finite sums),

\[
\int_{\Omega^N} \left( \sum_{a \leq b} |f_{a,b}(x_{[a,b-1], y_b})|^2 \right)^{\frac{1}{2}} \, dx \, dy \sim \int_{\Omega^N} \left( \sum_{a \leq b} |f_{a,b}(z_a, x_{a+1, b-1}, y_b)|^2 \right)^{\frac{1}{2}} \, dx \, dy \, dz
\]

(2.43)
as desired.

Corollary 2.11
For all \(i = (i_1, \ldots, i_m)\) such that \(i_1 < \ldots < i_m\), let \(f_i\) be an \(\mathcal{F}_{[1,i_1-1]\cup[i_1,\ldots,i_m]}\)-measurable function on \(\Omega^N\). Then, treating each \(f_i\) as a function on \(\Omega^N\),

\[
\int_{\Omega^N} \left( \sum_i |f_i(x_{<i_1}, x_{i_1}, \ldots, x_{i_m})|^2 \right)^{\frac{1}{2}} \, dx
\]
\[ \tilde{z}_m \int_{\Omega} \int_{\Omega^m} \left( \sum_{\nu} |f_i(x_{<i_1}, y_{i_1}^{(1)}, \ldots, y_{i_m}^{(m)})|^2 \right)^{\frac{1}{2}} dy^{(1,\ldots,m)} dx, \tag{2.44} \]

where \( y^{(1)}, \ldots, y^{(m)} \) are variables in \( \Omega^m \).

**Proof** Let us fix \( k \in \{1, \ldots, m\} \) and for each \( j \in \mathbb{N} \) define a function \( \varphi_j \) on \( \Omega^{[1,j]} \times (\Omega^m)^{m-k} \) by the formula

\[ \varphi_j \left( x_{\leq j}, y^{(k+1)}, \ldots, y^{(m)} \right) = \left( \sum_{i_1 < \ldots < i_k < j < i_{k+1} < \ldots < i_m} |f_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m} \left( x_{<i_1}, x_{i_1}, \ldots, x_{i_{k-1}}, x_j, y_{i_{k+1}}^{(k+1)}, \ldots, y_{i_m}^{(m)} \right)|^2 \right)^{\frac{1}{2}}. \tag{2.45} \]

Then, for fixed \( y^{(>k)} = (y^{(k+1)}, \ldots, y^{(m)}) \in (\Omega^m)^{m-k} \),

\[ \int_{\Omega} \left( \sum_{i_1 < \ldots < i_m} |f_i \left( x_{<i_1}, x_{i_1}, \ldots, x_{i_k}, y_{i_{k+1}}^{(k+1)}, \ldots, y_{i_m}^{(m)} \right)|^2 \right)^{\frac{1}{2}} dx = \int_{\Omega} \left( \sum_{j \in \mathbb{N}} |\varphi_j \left( x_{\leq j}, y^{(k+1)}, \ldots, y^{(m)} \right)|^2 \right)^{\frac{1}{2}} dx \tag{2.46} \]

\[ \geq \int_{\Omega^m} \int_{\Omega^m} \left( \sum_{j \in \mathbb{N}} |\varphi_j \left( x_{\leq j}, y^{(k+1)}, \ldots, y^{(m)} \right)|^2 \right)^{\frac{1}{2}} dxdy^{(k)} \tag{2.47} \]

\[ = \int_{\Omega^m} \int_{\Omega^m} \left( \sum_{i_1 < \ldots < i_m} |f_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m} \left( x_{<i_1}, x_{i_1}, \ldots, x_{i_{k-1}}, y_{i_{k+1}}^{(k+1)}, \ldots, y_{i_m}^{(m)} \right)|^2 \right)^{\frac{1}{2}} dxdy^{(k)}. \tag{2.48} \]

Here, \( i_k \) plays the role of \( j \) and (2.48) is an application of Theorem 2.9 to functions \( |\varphi_j|^2 \). Integrating the resulting inequality with respect to \( y^{(>k)} \), we get

\[ \int_{\Omega^{m-k}} \int_{\Omega} \left( \sum_{i_1 < \ldots < i_m} |f_i \left( x_{<i_1}, x_{i_1}, \ldots, x_{i_k}, y_{i_{k+1}}^{(k+1)}, \ldots, y_{i_m}^{(m)} \right)|^2 \right)^{\frac{1}{2}} dxdy^{(>k+1)} \]

\[ \geq \int_{\Omega^{m+1}} \int_{\Omega^{m+1}} \left( \sum_{i_1 < \ldots < i_m} |f_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m} \left( x_{<i_1}, x_{i_1}, \ldots, x_{i_{k-1}}, y_{i_{k+1}}^{(k+1)}, \ldots, y_{i_m}^{(m)} \right)|^2 \right)^{\frac{1}{2}} dxdy^{(>k)}, \tag{2.50} \]

which by induction from \( k = m \) to \( k = 1 \) proves (2.44).

\section{3 Boundedness of \( P_m \) on \( L^p(\Omega^m) \)}

The main motivation for this part is the following theorem, proved by Bourgain with \( c_p \lesssim \hat{p}^2 \frac{5}{ln \hat{p}} \) and by Kwapien with \( c_p \lesssim \hat{p} \frac{p}{ln \hat{p}} \), where \( \hat{p} = \max \left( p, \frac{p}{p-1} \right) \).
Theorem 3.1 \([4,14]\) \(P_m\) is bounded on \(L^p(\Omega^N)\) for \(1 < p < \infty\), with norm \(\lesssim c_p^m\).

We will present a proof that yields \(\|P_1 : L^p \circ\| < \infty\) and \(c_p = e\|P_1 : L^p \circ\|\).

**Proof** Without loss of generality, we may assume that we are working in \(L^p(\Omega^{[1,N]}, \mathcal{F}^\otimes N)\).

Indeed, by (2.33) and (2.37), \(P_m\) preserves \(L^2(\Omega^{[1,N]}, \mathcal{F}_1, N)\), which can be canonically identified with \(L^2(\Omega^{[1,N]}, \mathcal{F}^\otimes N)\). Since the sequence \((L^2(\Omega^N, \mathcal{F}_N) : N \in \mathbb{N})\) is increasing and its sum is dense in \(L_p(\Omega^N, \mathcal{F}^\otimes N)\), all we need to prove is

\[
\lim_{N \to \infty} \left\| P_m : L^p \left(\Omega^{[1,N]}, \mathcal{F}^\otimes N\right) \circ \right\| = \frac{1}{e} c_p^m. \tag{3.1}
\]

\(P_0 = \mathbb{E}\) is bounded. The \(L^p\) boundedness of \(P_1\) is essentially a known result \([5]\), but we provide a proof for the sake of completeness. Let \((\mathcal{F}_k : k \in [0, N])\) be the natural filtration and \((\mathcal{F}_k^\otimes N)_{k=0}^N\) be the natural reversed filtration, i.e. \(\mathcal{F}_k^\otimes = \mathcal{F}_{k,N}\). By (2.32) and (2.37) we see that

\[
P_1 = \sum_{k=1}^N P_{[k]}, \quad \Delta_k = \sum_{\max A = k} P_A, \quad \mathbb{E}_k^A = \sum_{A \subset [k,N]} P_A. \tag{3.2}
\]

By mutual orthogonality of \(P_A\)’s

\[
\Delta_k P_1 = P_{[k]} = \mathbb{E}_k^A \Delta_k. \tag{3.3}
\]

Applying Theorem 2.3, (3.3) and Theorem 2.4, we obtain

\[
\|P_1 f\|_{L^p} \simeq_p \left\| \left( \sum_{k=0}^N |\Delta_k P_1 f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \tag{3.4}
\]

\[
= \left\| \left( \sum_{k=0}^N |\mathbb{E}_k^A \Delta_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \tag{3.5}
\]

\[
\lesssim_p \left\| \left( \sum_{k=0}^N |\Delta_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \tag{3.6}
\]

\[
\simeq_p \|f\|_{L^p}. \tag{3.7}
\]

We will now proceed by induction. Suppose that (3.1) is satisfied with \(m - 1\) in the place of \(m\). Let \(N = mn\) and define an operator \(Q_m\) acting on \(L^p(\Omega^{[1,N]})\) by

\[
Q_m = \frac{1}{N} \sum_{A \subset [1,N] \atop |A| = m} \left( P_1 : L^p(\Omega^A) \circ \otimes (P_{m-1} : L^p(\Omega^{[1,N]\setminus A}) \circ) \right). \tag{3.8}
\]

Utilising (2.37) we get

\[
\left( P_1 : L^p(\Omega^A) \circ \right) = \sum_{B_1 \subset A \atop |B_1| = 1} \left( P_{B_1} : L^p(\Omega^A) \circ \right), \tag{3.9}
\]

\[
\left( P_{m-1} : L^p(\Omega^{[1,N]\setminus A}) \circ \right) = \sum_{B_2 \subset [1,N] \setminus A \atop |B_2| = m-1} \left( P_{B_2} : L^p(\Omega^{[1,N]\setminus A}) \circ \right). \tag{3.10}
\]
By (2.31),

\[
(P_{B_1} : L^p (\Omega^A) \circ) \otimes (P_{B_2} : L^p (\Omega^{[1,N] \setminus A}) \circ) = (P_{B_1 \cup B_2} : L^p (\Omega^{[1,N]}) \circ).
\]

(3.11)

Putting the last four equations together, we get

\[
Q_m = \frac{1}{(N^m)} \sum_{A \subseteq \{1, N\}} \sum_{|A| = n} (P_{B_1 \cup B_2} : L^p (\Omega^{[1,N]}) \circ)
\]

(3.12)

\[
= \frac{1}{(N^m)} \sum_{A \subseteq \{1, N\}} \sum_{|B \cap A| = 1, |B \setminus A| = m - 1} (P_B : L^p (\Omega^{[1,N]}) \circ)
\]

(3.13)

\[
= \frac{1}{(N^m)} \sum_{B \subseteq \{1, N\}} |\{A \subseteq \{1, N\} : |A| = n, |B \cap A| = 1\}| P_B = \frac{1}{(N^m)} \sum_{B \subseteq \{1, N\}} m (N - m) P_B
\]

(3.14)

\[
= m (\frac{N - m}{n - 1}) P_m.
\]

(3.15)

However, by (3.8) and the induction hypothesis,

\[
\|Q_m\| \leq \frac{1}{(N^m)} \sum_{A \subseteq \{1, N\}} \|P_1 : L^p (\Omega^A) \circ\| \cdot \|P_{m-1} : L^p (\Omega^{[1,N] \setminus A}) \circ\|
\]

(3.16)

\[
\leq \|P_1 : L^p (\Omega^{[1,N]}) \circ\| \cdot cm^{-1}
\]

(3.17)

\[
= \frac{c m}{e^2}.
\]

(3.18)

Let \(a_n \approx b_n\) denote \(\lim_{n \to \infty} \frac{a_n}{b_n} = 1\). By the Stirling formula,

\[
\binom{nm}{n} \approx \frac{(2\pi nm)^{\frac{1}{2}} \left(\frac{nm}{e}\right)^{nm}}{(2\pi n)^{\frac{1}{2}} \left(\frac{n}{e}\right)^n (2\pi n(m - 1))^{\frac{1}{2}} \left(\frac{n(m-1)}{e}\right)^{nm}}^{n(m-1)}
\]

(3.19)

\[
= \left(\frac{m}{2\pi n(m - 1)}\right)^{\frac{1}{2}} \frac{n^{nm} m^{nm}}{n^n m^{n(m-1)} (m - 1)^n (m-1)}
\]

(3.20)

\[
= \left(\frac{m}{2\pi n(m - 1)}\right)^{\frac{1}{2}} \left(\frac{m^m}{(m - 1)^{m-1}}\right)^n.
\]

(3.21)
Hoeffding decomposition in $H^1$ spaces

Thus

\[
\frac{\binom{nm}{n}}{m^{(n-1)m}} \approx \frac{1}{m} \left( \frac{n+1}{n} \right)^{\frac{1}{m}} \frac{m^m}{(m-1)(m-1)} \\
\approx \left( \frac{m}{m-1} \right)^{m-1}.
\]  

(3.23)

Finally, by (3.15), (3.18) and (3.23),

\[
\lim_{N \to \infty} \| P_m : L^p \left( \Omega^{[1,N]} \right) \| \leq \lim_{n \to \infty} \left\| P_m : L^p \left( \Omega^{[1,nm]} \right) \right\|
\]

(3.24)

\[
= \lim_{n \to \infty} \frac{\binom{nm}{n}}{m^{(nm-m)}} \| Q_m \|
\]

(3.25)

\[
= \left( 1 + \frac{1}{m-1} \right)^{m-1} \| Q_m \|
\]

(3.26)

\[
\leq \frac{c_p^m}{e}.
\]

(3.27)

We provide a short proof of a fact taken from [5] that Theorem 3.1 can not be, extended to $p = 1$ or $\infty$, which motivates the next section.

**Proposition 3.2** If \( \Omega \) is not a single atom, then \( P_m \) for \( m \geq 1 \) is not bounded on \( L^1 (\Omega^\infty) \) or \( L^\infty (\Omega^\infty) \).

**Proof** It is enough to consider \( L^1 (\Omega^\infty) \), because \( P_m \)'s are self-adjoint. Let \( f \in L^2 (\Omega) \) be such that \( \mathbb{E} f = 1, f \geq 0 \) and \( \mu (\text{supp} f) < 1 \). Then \( \mathbb{E} |f-1|^2 > 0 \). For \( F_n = f \otimes^n \in L^2 (\Omega^n) \) we have

\[
\| P_1 F_n \|_{L^1 (\Omega^n)} = \int_{\Omega^n} \left| \sum_i (f (x_i) - 1) \right| \, dx
\]

(3.28)

\[
\simeq \int_{\Omega^n} \left( \sum_i |f (x_i) - 1|^2 \right)^{\frac{1}{2}} \, dx
\]

(3.29)

\[
\geq (n \mathbb{E} |f - 1|^2)^{\frac{1}{2}}
\]

(3.30)

which is not dominated by \( \| F_n \|_{L^1 (\Omega^n)} = \| f \|_{L^1 (\Omega)}^n = 1 \). To prove the unboundedness of \( P_m \) for \( m > 1 \), we simply notice that

\[
P_m \left( (f - 1)^{\otimes (m-1)} \otimes F_n \right) = (f - 1)^{\otimes (m-1)} \otimes P_1 F_n.
\]

(3.31)

\( \Box \)
4 Boundedness of $P_m$ on $H^1(\mathbb{D}^N)$

The projection $P_m$ can be described even more explicitly in the case $\Omega_1 = T$. Indeed, if $n \in \mathbb{Z}^N$ is supported on the set $A$, then

$$P_A e^{i(n, t)} = \bigotimes_{j \in A} (\text{id} - E) e^{in_j t_j} = \prod_{j \in A} e^{in_j t_j} = e^{i(n, t)}.$$  \hspace{1cm} (4.1)

Thus

$$e^{i(n, t)} \in U_{\text{supp } n}$$  \hspace{1cm} (4.2)

and

$$U_m = \text{span} \left\{ e^{i(n, t)} : |\text{supp } n| = m \right\}.$$  \hspace{1cm} (4.3)

In particular, $P_m$ preserves the space $H^2(\mathbb{D}^N) = L^2(T^N) \cap H^1(\mathbb{D}^N)$.

In order to adapt the proof of Theorem 3.1 to the $H^1(\mathbb{D}^N)$ case, we will need a replacement for the argument proving that $P_1$ is bounded. The role of the combination of Burkholder–Gundy and Doob inequalities will be played by the following theorem, which can be found in [3].

**Theorem 4.1** (Bourgain) For $f \in H^{1}_{\text{last}}(T^N)$, there is an equivalence of norms

$$\|f\|_{L^1(T^N)} \simeq \|f\|_{H^1[(\mathcal{F}_n)_{n=0}^{\infty}]} ,$$  \hspace{1cm} (4.4)

where $(\mathcal{F}_n)_{n=0}^{\infty}$ is the natural filtration on $T^N$.

For later use, we note the Hilbert space valued extension.

**Corollary 4.2** Let $B$ be a Hilbert space. For $f \in H^{1}_{\text{last}}(T^N, B)$, there is an equivalence of norms

$$\|f\|_{L^1(T^N, B)} \simeq \|f\|_{H^1[(\mathcal{F}_n)_{n=0}^{\infty}, B]} = \int_{T^N} \left( \sum_{k=0}^{\infty} \|\Delta_k f(t)\|_B^2 \right)^{\frac{1}{2}} dt ,$$  \hspace{1cm} (4.5)

where $(\mathcal{F}_n)_{n=0}^{\infty}$ is the natural filtration on $T^N$.

**Proof** Theorem 4.1 gives a map

$$T : H^{1}_{\text{last}}(T^N) \to L^1(T^N, \ell^2) ,$$  \hspace{1cm} (4.6)

which is an isomorphism onto the subspace of $L^1(T^N, \ell^2)$ consisting of functions $f$ such that $f_k$ is a $k$-th martingale difference and is analytic in the $k$-th variable, defined by

$$T f = (\Delta_k f)_{k=0}^{\infty} .$$  \hspace{1cm} (4.7)

Thus, applying Lemma 2.8 with $I_1$ being a singleton, $I_2 = \mathbb{N}$, $T$ as above (and then the same for $T^{-1}$) we get

$$\|f\|_{H^{1}_{\text{last}}(T^N, B)} \simeq \|T \otimes \text{id}_B f\|_{L^1(T^N, B)} = \int_{T^N} \left( \sum_{k} \|\Delta_k f(t)\|_B^2 \right)^{\frac{1}{2}} dt .$$  \hspace{1cm} (4.8)
The role of the Stein martingale inequality will be played by the following simple observation.

**Corollary 4.3** For any sequence \((f_n : n \in \mathbb{N})\) adapted to the natural filtration on \(\Omega^N\),

\[
\mathbb{E}\left( \sum_{n=1}^{\infty} |f_n|^2 \right)^{\frac{1}{2}} \geq \mathbb{E}\left( \sum_{n=1}^{\infty} |\mathbb{E}[n] f_n|^2 \right)^{\frac{1}{2}}.
\]  

**(Proof)** Let \(\tilde{f}_n\) be a sequence of functions on \(\Omega^N \times \Omega^N\) defined by

\[
\tilde{f}_n(x, y) = f_n(x_1, \ldots, x_{n-1}, y_n).
\]

Applying Theorem 2.9 and conditional expectation with respect to the second of two sets of variables,

\[
\mathbb{E}\left( \sum_{n=1}^{\infty} |f_n|^2 \right)^{\frac{1}{2}} \geq \mathbb{E}\left( \sum_{n=1}^{\infty} |\tilde{f}_n|^2 \right)^{\frac{1}{2}}
\]

\[
\geq \mathbb{E}\left( \sum_{n=1}^{\infty} |(\mathbb{E} \otimes \text{id}) \tilde{f}_n|^2 \right)^{\frac{1}{2}}
\]

\[
= \mathbb{E}\left( \sum_{n=1}^{\infty} |1 \otimes \mathbb{E}[n] f_n|^2 \right)^{\frac{1}{2}}
\]

\[
= \mathbb{E}\left( \sum_{n=1}^{\infty} |\mathbb{E}[n] f_n|^2 \right)^{\frac{1}{2}}.
\]

\(\square\)

By conditioning with respect to the first set of variables, we obtain the inequality

\[
\mathbb{E}\left( \sum_{n=1}^{\infty} |f_n|^2 \right)^{\frac{1}{2}} \geq \mathbb{E}\left( \sum_{n=1}^{\infty} |\mathbb{E}_{n-1} f_n|^2 \right)^{\frac{1}{2}}
\]

due to Lepingle [17].

**Theorem 4.4** For any \(\Omega\), \(P_1\) is bounded on \(H^1[(F_n)_{n=0}^{\infty}]\).

**(Proof)** We proceed as in the proof of Theorem 3.1. First, we reduce the problem to the \(\Omega^{[1,N]}\) realm. Then we notice that

\[
\Delta_k P_1 = P_{[k]} = \mathbb{E}_{[k]} \Delta_k,
\]

which by Corollary 4.3 yields

\[
\| P_1 f \|_{H^1} = \left\| \left( \sum_{k=0}^{N} |\Delta_k P_1 f|^2 \right)^{\frac{1}{2}} \right\|_{L^1}
\]

\[
= \left\| \left( \sum_{k=0}^{N} |\mathbb{E}_{[k]} \Delta_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^1}
\]
\[ \left\| \sum_{k=0}^{N} |\Delta_k f|^2 \right\|_{L^1} \lesssim \left\| f \right\|_{H^1} . \]  
(4.19)

\[ = \| f \|_{H^1} . \]  
(4.20)

**Theorem 4.5**  
\( P_m \) is bounded on \( H^1(\mathbb{D}^N) \) with norm \( \leq \frac{1}{e} c_1^m \), where
\[ c_1 = e \| P_1 : H^1(\mathbb{D}^N) \| . \]  
(4.21)

**Proof**  
The case \( m = 0 \) is trivial, \( m = 1 \) follows directly from Theorems 4.1 and 4.4. The induction step is identical to the proof of Theorem 3.1, up to changing \( L^p(\Omega^I) \) to \( H^1(\mathbb{D}^I) \).

Alternatively, we can prove the same in a single step. Set
\[ Q_{m,n} = \frac{1}{(nm)^m} \sum_{A_1 \cup \ldots \cup A_m = [1,nm]} m \bigotimes_{i=1}^{m} \left( P_1 : H^1(\mathbb{D}^{A_1}) \right) \]  
(4.22)

It is easily seen that for each set \( B \) of cardinality \( m \), \( P_B \) appears \( m! \binom{n-1}{m} \) times in the sum. Therefore
\[ \| P_1 : H^1(\mathbb{D}^N) \|^m \geq \| Q_{m,n} \| \]  
(4.23)

\[ = \frac{m! \binom{n-1}{m}^m}{(nm)^m} \| P_m : H^1(\mathbb{D}^{nm}) \| \]  
(4.24)

and since
\[ \lim_{n \to \infty} \frac{\binom{nm}{n-1,n-1}}{m! \binom{(n-1)m}{n-1,n-1}} = \lim_{n \to \infty} \frac{\binom{nm}{n-1,m}}{m! \binom{(n-1)m}{n-1,m}} = \lim_{n \to \infty} \frac{nn}{m} m^{-m} = \frac{m^m}{m!} \]  
(4.25)

we get
\[ \| P_m : H^1(\mathbb{D}^N) \| \leq \frac{m^m}{m!} \| P_1 : H^1(\mathbb{D}^N) \|^m \leq \frac{1}{(2\pi m)^{1/2}} \left( e \| P_1 : H^1(\mathbb{D}^\infty) \| \right)^m \]  
(4.28)

It has to be noted that our proofs of Theorems 3.1 and 4.5 extend naturally to a vector valued case, respectively UMD and AUMD valued. Indeed, Bourgain’s proof of Theorem 2.4, as presented in [20], extends to the UMD valued version, while Theorem 4.1 is just the statement that a one-dimensional space has the AUMD property. In both cases, the induction follows without change. There is also a second direction in which we can generalize. Namely, by looking carefully at the proof of Theorem 4.1, one can see that the only place in which
analyticity plays a role is the $H^1 = H^2 \cdot H^2$ theorem, which is true for $H^1$ on any compact and connected group with ordered dual $[22]$, which means that we can replace $\mathbb{T}$ with any such group.

Given that Kwapien’s constant $c_p$ in Theorem 3.1 has the best known asymptotics as a function of $p$ for $m = 1$, one can ask about the dependence of $\|P_m : L^p(\Omega^\infty)\|$ and $\|P_m : H^1(\mathbb{D}^N)\|$ on $m$.

**Proposition 4.6** The inequalities

$$\|P_m : L^p(\Omega^\infty)\| \geq \|P_1 : L^p_0(\Omega^\infty)\|^m$$  \hfill (4.29)

for nontrivial $\Omega$ and

$$\|P_m : H^1(\mathbb{D}^N)\| \geq \|P_1 : H^1_0(\mathbb{D}^N)\|^m,$$  \hfill (4.30)

where $L^p_0$ and $H^1_0$ stand for functions of mean 0, are true. Also,

$$\|P_1 : H^1_0(\mathbb{D}^N)\| > 1.$$  \hfill (4.31)

**Proof** Let $f \in L^p(\Omega^\infty)$ be of mean 0. Then $f^{\otimes m} \in L^p(\Omega^{mn})$ and

$$(P_m : L^p(\Omega^{mn}) \otimes (f^{\otimes m}) = ((P_1 : L^p(\Omega^\cdot) \otimes (f))^{\otimes m}.$$  \hfill (4.32)

Indeed, we have $f = \sum_{|A| \geq 1} P_A f$ because $P_0 f = \mathbb{E} f = 0$, hence

$$f^{\otimes m} = \sum_{A_i \subset [n(i-1)+1, ni] \text{ for } i = 1, \ldots, m} \bigotimes_{|A_i| \geq 1} P_{A_i} f.$$  \hfill (4.33)

The only way to get a summand in $U_m$ is to have $|A_i| = 1$ for all $i$ and the sum of such summands is the right hand side of (4.32). Taking an $f$ which is close to attaining the norm of $P_1$ on a respective space proves (4.29) and (4.30).

In order to see (4.31), assume for the sake of contradiction that $P_1$ is a contraction on $H^1_0(\mathbb{D}^2)$. We will test it on functions of the form $\mathbb{T}^2 \ni (w, z) \mapsto F(z) + w + az w \in \mathbb{C}$, where $F \in H^1_0(\mathbb{D})$ and $a$ is a scalar. It is easy to see that

$$\mathbb{E}|\alpha + \beta w| = \mathbb{E}| |\alpha| + |\beta| w|$$  \hfill (4.34)

for $\alpha, \beta \in \mathbb{C}$. Hence, from the inequality

$$\mathbb{E}|F(z) + w(1 + az)| \geq \mathbb{E}|F(z) + w|$$  \hfill (4.35)

we get

$$\mathbb{E}|F(z)| + w(1 + az)| \geq \mathbb{E}|F(z)| + w|.$$  \hfill (4.36)

Since any nonnegative function can be approximated by the modulus of an $H^1_0(\mathbb{D})$ function, (4.35) is true for any nonnegative $F$. In particular, the left hand side attains a local minimum at $a = 0$, so by $|u + v| = |u| + \Re \frac{u}{|u|} + o(v)$ we infer that

$$\Re \mathbb{E}\frac{(F(z) + w) \overline{w} z}{|F(z) + w|} = 0.$$  \hfill (4.37)

Now let

$$\phi(r) = \mathbb{E}\frac{1 + r \overline{w}}{|1 + r \overline{w}|}$$  \hfill (4.38)
for \( r \geq 0 \). This is a continuous function, whose values lie on some curve \( \gamma \) connecting 0 and 1 (because \( \phi(0) = 1 \) and \( \lim_{r \to \infty} \phi(r) = 0 \)). The condition (4.37) can be rewritten as

\[
\Re \mathbb{E} \bar{z} \phi(F(z)) = 0. \tag{4.39}
\]

Since \( F \) was allowed to be any positive function, \( \phi(F) \) can be any function with values in \( \gamma \), making (4.39) obviously false.

\[\square\]

### 5 Martingale Hardy spaces

#### 5.1 Double indexed martingales

Above we noticed that the boundedness of \( P_1 \) on \( H^1(\mathbb{D}^N) \) follows from the boundedness of \( P_1 \) on a bigger space \( H^1((\mathcal{F}_n)) \). It is tempting to find an abstract martingale inequality responsible for the boundedness of \( P_m \) on \( H^1(\mathbb{D}^N) \). We can do this for \( m = 2 \).

By the natural double-indexed filtration on \( \Omega^N \) we will mean the family \((\mathcal{F}_{[a,b]} : a \leq b)\) (note that the inclusion order in the first index is reversed). Let \( \Delta_n = \mathbb{E}_n - \mathbb{E}_{n-1} \) be the martingale differences with respect to \((\mathcal{F}_n)\) and \( \Delta^*_n = \mathbb{E}_n^* - \mathbb{E}_{n+1}^* \) be the martingale differences with respect to \((\mathcal{F}_n^*)\), where \( \mathcal{F}_n = \mathcal{F}_{[n,\infty)} \). We define the martingale differences with respect to \((\mathcal{F}_{[a,b]})\) by

\[
\Delta_{[a,b]} = \Delta^*_a \Delta_b = \mathbb{E}_{[a+1,b-1]} + \mathbb{E}_{[a,b]} - \mathbb{E}_{[a+1,b]} - \mathbb{E}_{[a,b-1]} \tag{5.1}
\]

and an \( H^1 \) norm for this filtration by

\[
\| f \|_{H^1((\mathcal{F}_{[a,b]}))} = \mathbb{E} \left( |f|^2 + \sum_{1 \leq a \leq b} |\Delta_{[a,b]} f|^2 \right)^{\frac{1}{2}}. \tag{5.2}
\]

The definition of double martingale differences coincides with what is considered in [25].

**Corollary 5.1** For \( f \in H^1(\mathbb{D}^N) \), there is an equivalence of norms

\[
\| f \|_{H^1(\mathbb{D}^N)} \simeq \| f \|_{H^1((\mathcal{F}_{[a,b]}))}, \tag{5.3}
\]

where \((\mathcal{F}_{[a,b]}))_{a \leq b} is the natural double-indexed filtration on \( \mathbb{T}^N \).

**Proof** For any \( \pm 1 \)-valued sequence \((\varepsilon_n : n \in \mathbb{N})\), we define operators \( S_\varepsilon \) and \( S^*_\varepsilon \) by

\[
S_\varepsilon f = \mathbb{E} f + \sum_{n=1}^{\infty} \varepsilon_n \Delta_n f, \quad S^*_\varepsilon f = \mathbb{E} f + \sum_{n=1}^{\infty} \varepsilon_n \Delta^*_n f. \tag{5.4}
\]

By Theorem 4.1, \( S_\varepsilon \) is an isomorphism from \( H^1(\mathbb{D}^N) \) to itself, uniformly in \( \varepsilon \). By reversing the order of variables, the same can be said about \( S^*_\varepsilon \). Thus for any \( \varepsilon, \varepsilon' \),

\[
\| f \|_{H^1(\mathbb{D}^N)} \simeq \| S_\varepsilon S^*_{\varepsilon'} f \|_{H^1(\mathbb{D}^N)} \tag{5.5}
\]

\[
= \left\| \mathbb{E} f + \sum_{a \leq b} \varepsilon_a \varepsilon'_b \Delta^*_a \Delta_b f \right\|_{H^1(\mathbb{D}^N)}. \tag{5.6}
\]
Hoeffding decomposition in $H^1$ spaces

\[ \mathbb{E} \left| \mathbb{E} f + \sum_{a \leq b} \varepsilon_a \varepsilon'_b \Delta_{[a,b]} f \right| . \hspace{1cm} (5.7) \]

By averaging the last quantity over all choices of $\varepsilon, \varepsilon'$ and applying the Khintchine-Kahane inequality twice, we get the desired inequalities. \hfill \Box

**Theorem 5.2**  $P_2$ is bounded on $H^1 \left[ \left( \mathcal{F}_{a,b} \right)_{a \leq b} \right]$, for any $\Omega$.

**Proof** As usual, we reduce the problem to the $\Omega^{[1,N]}$ version. By (3.2),

\[ \Delta_{[a,b]} = \sum_{\min A = a}^{\max A = b} P_A . \hspace{1cm} (5.8) \]

Thus

\[ \Delta_{[a,b]} P_2 = P_{[a,b]} = \mathbb{E}_{[a,b]} \Delta_{[a,b]} \hspace{1cm} (5.9) \]

for $a < b$. We can assume that $P_0 f = P_1 f = 0$ (i.e. $\mathbb{E} f = 0$ and $\Delta_{[a,a]} f = 0$ for all $a$), because $U \leq 1$, being the image of $\mathbb{E} + \sum_{a} \Delta_{[a,a]}$ is trivially complemented in the underlying norm and $P_2$ is 0 on $U \leq 1$. By applying Corollary 2.10,

\[ \| f \|_{H^1} = \mathbb{E} \left( \sum_{a < b} |\Delta_{[a,b]} f|^2 \right)^{\frac{1}{2}} \hspace{1cm} (5.10) \]

\[ = \int_{\Omega^N} \left( \sum_{a < b} |\Delta_{[a,b]} f (x_{[a,b]})|^2 \right)^{\frac{1}{2}} dx \hspace{1cm} (5.11) \]

\[ \simeq \int_{(\Omega^N)^3} \left( \sum_{a < b} |\Delta_{[a,b]} f (z_a, x_{[a+1,b-1]}, y_b)|^2 \right)^{\frac{1}{2}} dx dy dz \hspace{1cm} (5.12) \]

\[ \simeq \int_{(\Omega^N)^2} \left( \sum_{a < b} \int_{\Omega^N} \Delta_{[a,b]} f (z_a, x_{[a+1,b-1]}, y_b) dx \right)^{\frac{1}{2}} dy dz \hspace{1cm} (5.13) \]

\[ \simeq \mathbb{E} \left( \sum_{a < b} |\mathbb{E}_{[a,b]} \Delta_{[a,b]} f|^2 \right)^{\frac{1}{2}} \hspace{1cm} (5.14) \]

\[ = \mathbb{E} \left( \sum_{a < b} |\Delta_{[a,b]} P_2 f|^2 \right)^{\frac{1}{2}} \hspace{1cm} (5.15) \]

\[ = \| P_2 f \|_{H^1} \hspace{1cm} (5.16) \]
as desired. \hfill \Box

### 5.2 Multiple indexed martingales

We will make an attempt at generalizing the above for multiple indexed martingales. Suppose there is a family $(T_i, \partial T_i)_{i \in \mathcal{I}}$ of pairs of finite subsets of some set $X$ (finite or not) indexed by some set $\mathcal{I}$, such that $\partial T_i \subseteq T_i$ ($\partial T_i$ is not a boundary in a topological sense - we use this
notation for resemblance with the case where $T_i$ are intervals and $\partial T_i$ are their endpoints). We would like to define operators $\Delta_i$ on $L^2(\Omega^X)$ by the formula

$$\Delta_i = (\text{id} - E)^{\otimes \partial T_i} \otimes \text{id}^{\otimes T_i \setminus \partial T_i} \otimes E^{\otimes T'_i},$$

(5.17)

where $T'_i$ stands for the complement of $T_i$ in $X$. This is supposed to mimic the standard martingale differences when $X = \mathbb{N}, \mathcal{I} = \mathbb{N}, T_i = [0, i], \partial T_i = \{i\}$ and double martingale differences when $\mathcal{I} = \{(a, b) : a \leq b\}, T_{a, b} = [a, b], \partial T_{a, b} = \{a, b\}$. The natural condition

$$\sum_i \Delta_i f = f$$

(5.18)

is guaranteed by

for any finite $A \subset X$, there exists unique $i \in \mathcal{I}$ such that $\partial T_i \subseteq A \subseteq T_i$. (5.19)

Indeed,

$$\Delta_i = (\text{id} - E)^{\otimes \partial T_i} \otimes \text{id}^{\otimes T_i \setminus \partial T_i} \otimes E^{\otimes T'_i}$$

(5.20)

$$= (\text{id} - E)^{\otimes \partial T_i} \otimes E^{\otimes T'_i} \otimes \sum_{B \subseteq T_i \setminus \partial T_i} \left( P_B : L^2(\Omega^{T_i \setminus \partial T_i}) \right)$$

(5.21)

$$= (\text{id} - E)^{\otimes \partial T_i} \otimes E^{\otimes T'_i} \otimes \sum_{B \subseteq T_i \setminus \partial T_i} (\text{id} - E)^{\otimes B} \otimes E^{\otimes T_i \setminus (\partial T_i \cup B)}$$

(5.22)

$$= \sum_{B \subseteq T_i \setminus \partial T_i} P_{\partial T_i \cup B}.$$  

(5.23)

Hence

$$\sum_{i \in \mathcal{I}} \Delta_i = \sum_{i \in \mathcal{I}} \sum_{B \subseteq T_i \setminus \partial T_i} P_{\partial T_i \cup B}$$

(5.24)

and each $P_A$ appears in the above sum exactly once if and only if the condition (5.19) is satisfied. For a family $(T_i, \partial T_i)_{i \in \mathcal{I}}$ we may define a norm by the formula

$$\|f\|_{H^1[(T_i, \partial T_i)_{i \in \mathcal{I}}]} = \mathbb{E} \left( \sum_{i \in \mathcal{I}} |\Delta_i f|^2 \right)^{1/2}$$

(5.25)

and ask the following:

- Is it true that

$$\|f\|_{H^1[(T_i, \partial T_i)_{i \in \mathcal{I}}]} \simeq \|f\|_{H^1(\mathbb{Z}^N)}$$

(5.26)

for $f \in \|f\|_{H^1(\mathbb{Z}^N)}$?

- If yes, is there any interesting example of a set $\mathbb{Z}^{\oplus \infty} \subset \Gamma \subset \mathbb{Z}^{\oplus \infty}$ such that (5.26) is true for $f \in L^1(T^{\infty})$ with supp $\hat{f} \subset \Gamma$?

- For which, if any, $m$ is $P_m$ bounded on $H^1[(T_i, \partial T_i)_{i \in \mathcal{I}}]$?
We are able to answer them in the case when
\[
\mathcal{I} = \{ A \subset \mathbb{N} : |A| \leq m \} \quad (5.27)
\]
\[
\partial T_A = A, \quad T_A = \begin{cases} A & \text{if } |A| \leq m, \\ (0, \min A) \cup A & \text{if } |A| = m. \end{cases} \quad (5.28)
\]

For a finite set \( B \subset \mathbb{N} \), the unique \( A \in \mathcal{I} \) such that \( \partial T_A \subseteq B \subseteq T_A \), which we will denote by \( \partial B \), is
\[
\partial B = \begin{cases} B & \text{if } |B| < m, \\ \text{last elements of } B & \text{if } |B| \geq m. \end{cases} \quad (5.29)
\]

**Theorem 5.3** Let \( m \geq 1 \) be fixed and \( (T_i, \partial T_i)_{i \in \mathcal{I}} \) be defined by (5.27), (5.28). Then
\[
\| f \|_{H^1_{m,\text{last}}(\mathbb{T}^m)} \simeq \| f \|_{H^1[\{(T_i, \partial T_i)_{i \in \mathcal{I}}\}]} \quad (5.30)
\]
for \( f \in H^1_{m,\text{last}}(\mathbb{T}^m) \), where \( \mathbb{T} \) is used as \( \Omega \). Moreover, for \( m' \in \mathbb{N} \) and nontrivial \( \Omega \), the following are equivalent.

(i) \( m' \leq m \)
(ii) \( P_{m'} \) is bounded on \( H^1[\{(T_i, \partial T_i)_{i \in \mathcal{I}}\}] \)
(iii) \( P_{m'} \) is bounded on \( H^1_{m,\text{last}}(\mathbb{T}^m) \).

**Proof** For \( A, B \subset \mathbb{N} \), we will write \( T_A^{(m)}, \partial T_A^{(m)}, \Delta_A^{(m)}, \partial^{(m)}B \) to indicate the value of \( m \) we are currently using. For brevity we will denote \( (T_A^{(m)}, \partial T_A^{(m)})|_{A| \leq m} \) by \( T_m \). For \( |A| < m \), we have \( \Delta_A^{(m)} = (\text{id} - \mathbb{E})^{\otimes A} \otimes \mathbb{E}^{[0,N]|A} = P_A \). In particular, \( \Delta^{(m)}_{\emptyset} = \mathbb{E} \). Therefore, by definition of the \( H^1 \{T_m\} \) norm and Corollary 2.11,
\[
\| f \|_{H^1[T_m]} \simeq \int_{\Omega^m} \sum_{i_1 < \ldots < i_m} \left| \Delta_A^{(m)} f(\{x_{<i_1}, y_{i_1}^{(1)}, \ldots, y_{i_m}^{(m)}\}) \right|^2 d\gamma(1,\ldots,m) dx
\]
\[
+ \sum_{0<s<m} \int_{(\Omega^m)^s} \sum_{i_1 < \ldots < i_s} \left| P_s f(\{y_{i_1}^{(1)}, \ldots, y_{i_s}^{(s)}\}) \right|^2 d\gamma(1,\ldots,s) + |\mathbb{E} f| .
\]
(5.31)

Here, we identify an increasing sequence with the set of its elements, write \( d\gamma(1,\ldots,m) \) to denote \( d\gamma(1) \ldots d\gamma(m) \) and treat \( \Delta_A^{(m)} f \) as a function on \( \Omega^{[1,i_1-1]} \times \Omega^{[i_1,\ldots,i_m]} \). From this expression, we immediately see the implication \((i) \implies (ii)\). Indeed, for \( m' < m \),
\[
\Delta_A^{(m)} P_{m'} = \begin{cases} \Delta_A^{(m)} & \text{if } |A| = m' \\ 0 & \text{if } |A| \neq m'. \end{cases} \quad (5.32)
\]
which trivializes the inequality \( \| f \|_{H^1[T_m]} \gtrsim \| P_{m'} f \|_{H^1[T_m]} \). For \( m' = m \), we notice that
\[
\Delta_A^{(m)} P_m = \begin{cases} \mathbb{E}^{[1,\min A-1]} \Delta_A^{(m)} & \text{if } |A| = m \\ 0 & \text{if } |A| \neq m. \end{cases} \quad (5.33)
\]
and the desired inequality follows from

\[ \|f\|_{H^1_1[T_m]} \geq \int_{\Omega^N} \left( \sum_{i_1 < \cdots < i_m} \left| \Delta_{i_1}^{(m)} f (x_{<i_1}, y_{i_1}^{(1)}, \ldots, y_{i_m}^{(m)}) \right|^2 \right)^{\frac{1}{2}} dy^{(1, \ldots, m)} dx \]

(5.34)

\[ \geq \int_{\Omega^N} \left( \sum_{i_1 < \cdots < i_m} \left| \int_{\Omega^N} \Delta_{i_1}^{(m)} f (x_{<i_1}, y_{i_1}^{(1)}, \ldots, y_{i_m}^{(m)}) dx \right|^2 \right)^{\frac{1}{2}} dy^{(1, \ldots, m)} \]

(5.35)

\[ = \int_{\Omega^N} \left( \sum_{i_1 < \cdots < i_m} \left| \Delta_{i_1}^{(m)} P_m f (x_{<i_1}, y_{i_1}^{(1)}, \ldots, y_{i_m}^{(m)}) \right|^2 \right)^{\frac{1}{2}} dy^{(1, \ldots, m)} \]

(5.36)

\[ = \| P_m f \|_{H^1_1[T_m]} \]

(5.37)

The implication (iii) \(\implies\) (iii) follows from (5.30), which we will prove by induction with respect to \(m\). For \(m = 1\) this is just Theorem 4.1. Suppose it is true for some \(m\) and let \(f \in H^1_{m+1} (\mathbb{T}^N)\). In particular, \(f \in H^1_m (\mathbb{T}^N)\). By (5.30), which is now the induction hypothesis, and (5.31),

\[ \|f\|_{L^1_1(\mathbb{T}^N)} \simeq \int_{\mathbb{T}^N} \int_{\mathbb{T}^N} \left( \sum_{i_1 < \cdots < i_m} \left| \Delta_{i_1}^{(m)} f (x_{<i_1}, y_{i_1}^{(1)}, \ldots, y_{i_m}^{(m)}) \right|^2 \right)^{\frac{1}{2}} dy^{(1, \ldots, m)} dx \]

\[ + \sum_{0 < s < m} \int_{\mathbb{T}^N} \left( \sum_{i_1 < \cdots < i_s} \left| P_i f (y_{i_1}^{(1)}, \ldots, y_{i_s}^{(s)}) \right|^2 \right)^{\frac{1}{2}} dy^{(1, \ldots, s)} + \| \mathbb{E} f \| \]

(5.38)

The last two summands are as they are in the desired expression for \(m + 1\) instead of \(m\) and we only have to deal with the first. For any \(i_1 < \cdots < i_m\) and \(t \in \mathbb{T}^N\) we have

\[ \Delta_{i_1 \ldots i_m}^{(m)} f (t) = \sum_{\hat{g}^{(m)} \text{supp } n = i} \hat{f}(n) e^{i(n,t)}. \]

(5.39)

Thus, treating \(\Delta_{i_1}^{(m)} f\) as a function on \(\mathbb{T}^{[1,i_i - 1]} \times \mathbb{T}^{[i_i, \ldots, i_m]}\),

\[ \Delta_{i_1 \ldots i_m}^{(m)} f (x_{<i_1}, y_1, \ldots, y_m) = \sum_{\hat{g}^{(m)} \text{supp } n = i} \hat{f}(n) e^{i \sum_{j<i_1} n_j x_j + i \sum_{1 \leq j \leq m} n_j y_j}. \]

(5.40)

Let \(y\) be fixed and \(\Delta_{k}^{(1)}\), where \(k \in \mathbb{N} \cup \{0\}\), act with respect to the variable \(x \in \mathbb{T}^N\) (so, technically, \(\Delta_{k}^{(1)} \) stands for \(\Delta_{k}^{(1)} \otimes \text{id}\)). Then

\[ \Delta_{\emptyset}^{(1)} \Delta_{i_1 \ldots i_m}^{(m)} f (x_{<i_1}, y_1, \ldots, y_m) = P_i f (y_1, \ldots, y_m) \]

(5.41)

and

\[ \Delta_{k}^{(1)} \Delta_{i_1 \ldots i_m}^{(m)} f (x_{<i_1}, y_1, \ldots, y_m) = 0 \quad \text{for } k \geq i_1. \]

(5.42)

For \(k < i_1\),

\[ \Delta_{k}^{(1)} \Delta_{i_1 \ldots i_m}^{(m)} f (x_{<i_1}, y_1, \ldots, y_m) \]

(5.43)
\[ \sum_{\partial^{(m)} \text{supp } n = i} \widehat{f}(n) e^{i \sum_{1 \leq j \leq m} n_j y_j} \Delta_k^{(1)} e^{i \sum_{j=i+1} n_{j-1}} (x) \]  
\[ = \sum_{\partial^{(m)} \text{supp } n = i} \widehat{f}(n) e^{i \sum_{1 \leq j \leq m} n_j y_j + i \sum_{j \leq k} n_j x_j} \]  
\[ = \sum_{\partial^{(m+1)} \text{supp } n = [k, i_1, \ldots, i_m]} \widehat{f}(n) e^{i \sum_{j \leq k} n_j x_j + i n_k x_k + i \sum_{1 \leq j \leq m} n_j y_j} \]  
\[ = \Delta_{k, i_1, \ldots, i_m} f \left( x_{<k}, x_k, y_1, \ldots, y_m \right). \]  
(5.47)

By (5.40), \( \Delta_{i_1, \ldots, i_m} f \left( x_{<i_1}, y_1, \ldots, y_m \right) \) is in \( H^1_{\text{last}} (\mathbb{T}^m) \) with respect to \( x \). Therefore, applying Corollary 4.2 to the vector valued function \( x \mapsto \left( \Delta_{i_1} f \left( x_{<i_1}, y^{(1)}_1, \ldots, y^{(m)}_m \right) \right)_{i_1 < \ldots < i_m} \) with fixed \( y^{(1)}, \ldots, y^{(m)} \), plugging in (5.41), (5.42), (5.47) and using Corollary 2.11, we get

\[ \int_{\mathbb{T}^m} \left( \sum_{i_1 < \ldots < i_m} \left| \Delta_{i_1} f \left( x_{<i_1}, y^{(1)}_1, \ldots, y^{(m)}_m \right) \right| \right)^2 \, dx \]  
\[ = \int_{\mathbb{T}^m} \left\| \left( \Delta_{i_1} f \left( x_{<i_1}, y^{(1)}_1, \ldots, y^{(m)}_m \right) \right)_{i_1 < \ldots < i_m} \right\|_{L^2}^2 \, dx \]  
\[ = \left\| \Delta_{i_1} f \left( x_{<i_1}, y^{(1)}_1, \ldots, y^{(m)}_m \right) \right\|_{L^2}^2 \]  
\[ + \int_{\mathbb{T}^m} \left( \sum_{k} \left| \Delta_{k} \Delta_{i_1} f \left( x_{<i_1}, y^{(1)}_1, \ldots, y^{(m)}_m \right) \right| \right)^2 \, dx \]  
\[ = \left( \sum_{i_1 < \ldots < i_m} \left| P_{i_1, \ldots, i_m} f \left( y^{(1)}_1, \ldots, y^{(m)}_m \right) \right| \right)^2 \]  
\[ + \int_{\mathbb{T}^m} \int_{\mathbb{T}^m} \left( \sum_{k < i_1 < \ldots < i_m} \left| \Delta_{k, i_1, \ldots, i_m} f \left( x_{<k}, y^{(0)}_k, y^{(1)}_1, \ldots, y^{(m)}_m \right) \right| \right)^2 \, dx \, dy^{(0)}. \]  
(5.55)

Integrating the resulting equivalence with respect to \( y^{(1)}, \ldots, y^{(m)} \) and plugging into (5.38), we verify that \( \| f \|_{L^1(\mathbb{T}^m)} \simeq H^1 \left[ T_{m+1} \right] \), which finishes the proof of (5.30).

In order to see that (iii) \( \Rightarrow \) (i), let us take \( m' > m \). For any \( g \in L^1(\mathbb{T}^m) \), the function \( G \in L^1(\mathbb{T}^m) \) defined by

\[ G \left( t \right) = g \left( t_1, \ldots, t_n \right) e^{i \sum_{j=m+1}^{m'} t_j} \]  
(5.56)
is in $H^1_{m \text{ last}} \left( \mathbb{T}^n \right)$. But

$$ (P_{m'} G) (t) = \left( P_{m' - m} g \right) (t_1, \ldots, t_n) e^{\sum_{j=0}^{n+m} I_j}, \quad (5.57) $$

so

$$ \left\| P_{m'} : H^1_{m \text{ last}} \left( \mathbb{T}^n \right) \circ \right\| \geq \frac{\int_{\mathbb{T}^n} \left| (P_{m'} G) (t) \right| dt}{\int_{\mathbb{T}^n} \left| G(t) \right| dt}, \quad (5.58) $$

$$ = \frac{\int_{\mathbb{T}^n} \left| (P_{m' - m} g) (t_1, \ldots, t_n) \right| dt}{\int_{\mathbb{T}^n} \left| g \left( t_1, \ldots, t_n \right) \right| dt}, \quad (5.59) $$

which by Proposition 3.2 can be arbitrarily big. \hfill \square

It is worth noting that by repeating the above proof of the equivalence between $H^1_{m \text{ last}}$ norm and $H^1 \left[ T_m \right]$, one can obtain

$$ \| f \|_{H^p \left[ T_m \right]} \sim_p \| f \|_{L^p}, \quad (5.60) $$

where $H^p \left[ T_m \right]$ is defined in a natural way. Moreover, by iterating the $\| f \|_{H^1} \geq \| f \|_{L^1}$ inequality for linearly ordered martingales,

$$ \| f \|_{H^1 \left[ T_m \right]} \gtrsim \| f \|_{L^1}. \quad (5.61) $$

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Appendix

We present two proofs of Theorem 2.9 different from the original one by Zinn.

Let us recall the non-linear telescoping lemma due to Bourgain and Müller.

Lemma 6.1 [3,19] Let $\lambda_1, \ldots, \lambda_n, \varphi_1, \ldots, \varphi_n$ be nonnegative random variables such that

$$ \mathbb{E} \lambda_k \geq \mathbb{E} \left( \varphi_k^2 + \lambda_{k-1}^2 \right)^{\frac{1}{2}}. \quad (6.1) $$

Then

$$ \mathbb{E} \left( \sum_{k=1}^{n} \varphi_k^2 \right)^{\frac{1}{2}} \leq 2 \left( \mathbb{E} \lambda_n \mathbb{E} \max_{1 \leq k \leq n} \lambda_k \right)^{\frac{1}{2}}. \quad (6.2) $$

Corollary 6.2 Let $X_1, \ldots, X_n$ be independent and set

$$ \lambda_0 = 0, \quad \lambda_k = \mathbb{E} \left( X_k^2 + \lambda_{k-1}^2 \right)^{\frac{1}{2}}. \quad (6.3) $$
Then
\[ \lambda_n \leq \mathbb{E} \left( \sum_{k=1}^{n} X_k^2 \right)^{\frac{1}{2}} \leq 2\lambda_n. \] (6.4)

**Proof** The right inequality of (6.4) follows directly from Lemma 6.1, since \( \lambda_k \) is an increasing sequence of constants. To prove the other inequality, we see that conditioning with respect to \( \sigma (X_{k+1}, \ldots, X_n) \) gives
\[
\mathbb{E} \left( \sum_{j=k}^{n} X_j^2 + \lambda_{k-1}^2 \right)^{\frac{1}{2}} = \mathbb{E} \left( \sum_{j=k+1}^{n} X_j^2 + \left( X_k^2 + \lambda_{k-1}^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \geq \mathbb{E} \left( \sum_{j=k+1}^{n} X_j^2 + \lambda_k^2 \right)^{\frac{1}{2}},
\] (6.5)
thus by induction
\[
\mathbb{E} \left( \sum_{k=1}^{n} X_k^2 \right)^{\frac{1}{2}} \geq \mathbb{E} \left( \sum_{j=k+1}^{n} X_j^2 + \lambda_k^2 \right)^{\frac{1}{2}},
\] (6.7)
which for \( k = n \) is the desired inequality. \( \square \)

**Proof of Theorem 2.9** In order to prove the \( \gtrsim \) inequality in (2.38), we merely perform a slight modification of the proof of Lepingle inequality presented in [3]. Let us denote \( N \times \Omega^N \ni (x, y) \mapsto f_n (x_1, \ldots, x_{n-1}, y) \) by \( f_n \). By tensoring \( (f_n) \) against the Rademacher sequence, we may assume that it is a martingale difference sequence. Then the left hand side equals \( \| F \|_{H^1} \), where \( F = \sum_{n=1}^{\infty} f_n \) and \( f_n = \Delta_n F \). By Theorem 2.6 it is enough to check the boundedness of the right hand side in the case when \( F \) is an atom, because we have an a priori bound for finite sums. Let \( F = u - \mathbb{E}_{j-1} u \), where \( u \) satisfies (2.17). Then
\[
f_k = \begin{cases} 0 & \text{if } k < j \\ \Delta_k u & \text{if } k \geq j. \end{cases}
\] (6.8)
By \( A \in \mathcal{F}_j \), the support of \( \mathbb{E}_k u \) for \( k \geq j \) is contained in \( A \) as well, because
\[
\mathbb{E} (|\mathbb{E}_k u| \cdot 1_{\Omega^\infty \setminus A}) \leq \mathbb{E} (|\mathbb{E}_k u| \cdot 1_{\Omega^\infty \setminus A})
\leq \mathbb{E} (|u| \cdot 1_{\mathbb{E}_k \Omega^\infty \setminus A})
\leq 0.
\] (6.9)
Thus for \( k > j \) we have \( \operatorname{supp} f_k \subset \operatorname{supp} \mathbb{E}_k u \cup \operatorname{supp} \mathbb{E}_{k-1} u \subset A \). Consequently
\[
\operatorname{supp} \tilde{f}_k \subset A \times \Omega^\infty \text{ for } k > j,
\] (6.12)
because if \( (x, y) \in \operatorname{supp} \tilde{f}_k \), then \( \{ (x_1, \ldots, x_{k-1}, y) \} \times \Omega^{k+1 \ldots} \subset \operatorname{supp} f_k \subset A \), which by \( A \in \mathcal{F}_j \) implies \( x \in A \). By and (6.8) we have
\[
\mathbb{E} \left| \tilde{f}_j \right| = \mathbb{E} |f_j| = \mathbb{E} |\Delta_j u| \leq 2|u|.
\] (6.13)
Combining (2.17), (6.8), (6.12), (6.13) with the inequality
\[
\| X \|_{L^1} \leq |\operatorname{supp} X|^{\frac{1}{2}} \| X \|_{L^2}
\] (6.14)
and the fact that the projections $\Delta_k$ are mutually orthogonal, we obtain

$$\mathbb{E} \left( \sum_{k=1}^{\infty} f_k^2 \right)^{\frac{1}{2}} \leq \mathbb{E} \left| \tilde{f}_1 \right| + \mathbb{E} \left( \sum_{k=j+1}^{\infty} f_k^2 \right)^{\frac{1}{2}} \quad (6.15)$$

$$\leq 2 \mathbb{E} |u| + \mathbb{E} \left( \sum_{k=j+1}^{\infty} f_k^2 \right)^{\frac{1}{2}} \quad (6.16)$$

$$\leq 2 |A|^{\frac{1}{2}} \left( \mathbb{E} u^2 \right)^{\frac{1}{2}} + |A|^{\frac{1}{2}} \left( \sum_{k=j+1}^{\infty} \mathbb{E} f_k^2 \right)^{\frac{1}{2}} \quad (6.17)$$

$$= 2 |A|^{\frac{1}{2}} \left( \mathbb{E} u^2 \right)^{\frac{1}{2}} + \mathbb{E} \left( \Delta_k u \right)^2 \quad (6.18)$$

$$\leq 3 |A|^{\frac{1}{2}} \left( \mathbb{E} u^2 \right)^{\frac{1}{2}} \quad (6.19)$$

$$\leq 3. \quad (6.20)$$

We will prove the $\leq$ inequality in (2.38) now. It is clear that it is enough to prove it with only finitely many of $f_k$ nonzero. We define the sequence $(\lambda_k(x))_{k=1}^{n}$ of functions in $L^1(\Omega_{\infty})$ inductively by

$$\lambda_0(x) = 0, \quad \lambda_k(x) = \int_{\Omega} \left( f_k^2 \left( x_1, \ldots, x_{k-1}, y_k \right) + \lambda_{k-1}^2 \right) dy_k. \quad (6.21)$$

For any fixed $x \in \Omega_{\infty}$, this sequence coincides with the sequence defined by (6.3) applied to the independent random variables $f_k \left( x_1, \ldots, x_{k-1}, \cdot \right) \in L^1(\Omega)$, so the left part of (6.4) yields the pointwise inequality

$$\int_{\Omega_{\infty}} \left( \sum_{k=1}^{n} f_k^2 \left( x_1, \ldots, x_{k-1}, y_k \right) \right)^{\frac{1}{2}} dy \geq \lambda_n(x). \quad (6.22)$$

By induction it is obvious that $\lambda_k$ is $\mathcal{F}_{k-1}$-measurable. Thus

$$\lambda_k(x) = \int_{\Omega} \left( f_k^2 \left( x_1, \ldots, x_{k-1}, y_k \right) + \lambda_{k-1}^2 \left( x_1, \ldots, x_{k-2} \right) \right) dy_k \quad (6.23)$$

$$= \left( \mathbb{E}_{k-1} \left( f_k^2 + \lambda_{k-1}^2 \right)^{\frac{1}{2}} \right) (x_1, \ldots, x_{k-1}). \quad (6.24)$$

In particular, $\lambda_k$ verify the condition (6.1) with respect to $f_k$ and are pointwise increasing, so Lemma 6.1 gives

$$\mathbb{E} \left( \sum_{k=1}^{n} f_k^2 \right)^{\frac{1}{2}} \leq 2 \mathbb{E} \lambda_n \mathbb{E} \max_{k \leq n} \lambda_k \quad (6.25)$$

Integrating (6.22) with respect to $x$ and applying (6.25) we obtain

$$\mathbb{E} \left( \sum_{k=1}^{n} f_k^2 \right)^{\frac{1}{2}} = \int_{\Omega_{\infty}} \int_{\Omega_{\infty}} \left( \sum_{k=1}^{n} f_k^2 \left( x_1, \ldots, x_{k-1}, y_k \right) \right)^{\frac{1}{2}} dydx \quad (6.26)$$
\[ \geq \int_{\Omega^\infty} \lambda_n(x) \, dx \quad (6.27) \]
\[ \geq \frac{1}{2} \mathbb{E} \left( \sum_{k=1}^{n} f_k^2 \right)^{1/2}. \quad (6.28) \]

**Proof** (Yet another proof of Theorem 2.9) Without loss of generality, we may assume that \( \Omega \) has a structure of a compact abelian group with Haar measure, e.g. by embedding \( \Omega \) in \( \mathbb{T} \). Just like previously, we also may assume that \( f_k \) is a \( k \)-th martingale difference and notice that the left hand side is just \( \| \sum_k f_k \|_{H^1} \). For \( \xi \in \Omega^N \) we define an operator \( T_\xi \) by
\[ T_\xi f(x) = \sum_k \Delta_k f \left( x, \ldots, x_{k-1}, x_k + \xi_k \right). \quad (6.29) \]

Since \( \Delta_k T_\xi f \) is just a translation of \( \Delta_k f \),
\[ \| \Delta_k T_\xi f \|_{L^\infty} = \| \Delta_k f \|_{L^\infty}. \quad (6.30) \]

For \( k > n \), by translation in the variable \( y_k \),
\[ \mathbb{E}_n \left| \Delta_k T_\xi f \right|^2(x) = \int_{\Omega_{>n}} |\Delta_k f \left( x, \ldots, y_n, \ldots, y_k, y_k + \xi_k \right)|^2 \, dy \quad (6.31) \]
\[ = \int_{\Omega_{>n}} \left| \Delta_k f \left( x, \ldots, y_n, \ldots, y_{k-1}, y_k + \xi_k \right) \right|^2 \, dy \quad (6.32) \]
\[ = \mathbb{E}_n \left| \Delta_k f \right|^2(x). \quad (6.33) \]

Therefore
\[ \| T_\xi f \|_{BMO} \lesssim \| f \|_{BMO}. \quad (6.34) \]

We have \( T_\xi^* = T_{-\xi} \), because
\[ \langle T_\xi f, g \rangle = \mathbb{E} \sum_k \Delta_k T_\xi f \Delta_k g \quad (6.35) \]
\[ = \sum_k \int_{\Omega^k} \Delta_k f \left( x, \ldots, x_{k-1}, x_k + \xi_k \right) \Delta_k g \left( x, \ldots, x_k \right) \, dx \quad (6.36) \]
\[ = \sum_k \int_{\Omega^k} \Delta_k f \left( x, \ldots, x_{k-1}, x_k \right) \Delta_k g \left( x, \ldots, x_{k-1} - \xi_k \right) \, dx \quad (6.37) \]
\[ = \langle f, T_{-\xi} g \rangle. \quad (6.38) \]

By Theorem 2.7, \( T_\xi \) are uniformly bounded on martingale \( H^1 \), so
\[ \| T_\xi f \|_{H^1} \lesssim \| f \|_{H^1} = \| T_{-\xi} T_\xi f \|_{H^1} \lesssim \| T_\xi f \|_{H^1} \quad (6.39) \]
and thus
\[ \| f \|_{H^1} \simeq \int_{\Omega^N} \| T_\xi f \|_{H^1} \, d\xi. \quad (6.40) \]

Ultimately, by translating \( \xi \) for fixed \( x \),
\[ \int_{\Omega^N} \left( \sum_k |f_k \left( x, \ldots, x_k \right)|^2 \right)^{1/2} \, dx \]
\[ = \left\| \sum_k f_k \right\|_{H^1} \quad (6.41) \]
\[ \approx \int_{\Omega^N} \left\| T_{\xi} \sum_k f_k \right\|_{H^1} d\xi \quad (6.42) \]
\[ = \int_{\Omega^N} \int_{\Omega^N} \left( \sum_k \left| f_k(x_1, \ldots, x_k - 1, x_k + \xi_k) \right|^2 \right) \frac{1}{2} d\xi dx \quad (6.43) \]
\[ = \int_{\Omega^N} \int_{\Omega^N} \left( \sum_k \left| f_k(x_1, \ldots, x_k - 1, \xi_k) \right|^2 \right) \frac{1}{2} d\xi dx. \quad (6.44) \]

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