EQUIVALENCE OF FAMILIES OF SINGULAR SCHEMES ON THREEFOLDS AND ON RULED FOURFOLDS

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ABSTRACT. The main purpose of this paper is twofold. We first want to analyze in details the meaningful geometric aspect of the method introduced in the previous paper [12], concerning regularity of families of irreducible, nodal "curves" on a smooth, projective threefold $X$. This analysis highlights several fascinating connections with families of other singular geometric "objects" related to $X$ and to other varieties.

Then, we generalize this method to study similar problems for families of singular divisors on ruled fourfolds suitably related to $X$.

INTRODUCTION

The theory of families of singular curves with fixed invariants (e.g. geometric genus, singularity type, number of irreducible components, etc.) and which are contained in a projective variety $X$ has been extensively studied from the beginning of Algebraic Geometry and it actually receives a lot of attention, partially due to its connections with several fields in Geometry and Physics.

Nodal curves play a central role in the subject of singular curves. Families of irreducible and $\delta$-nodal curves on a given projective variety $X$ are usually called Severi varieties of irreducible, $\delta$-nodal curves in $X$. The terminology "Severi variety" is due to the classical case of families of nodal curves on $X = \mathbb{P}^2$, which was first studied by Severi (see [23]).

The case in which $X$ is a smooth projective surface has recently given rise to a huge amount of literature (see, for example, [4], [5], [6], [7], [11], [14], [15], [21], [22] just to mention a few. For a chronological overview, the reader is referred for example to Section 2.3 in [10] and to its bibliography). This depends not only on the great interest in the subject, but also because for a Severi variety $V$ on an arbitrary projective variety $X$ there are several problems concerning $V$ like non-emptyness, smoothness, irreducibility, dimensional computation as well as enumerative and moduli properties of the family of curves it parametrizes.

On the contrary, in higher dimension only few results are known. Therefore, in [12] we focused on what is the next relevant case, from the point of view of Algebraic Geometry: families of nodal curves on smooth, projective threefolds.

The aim of this paper is twofold: first, we want to study in details the meaningful geometric aspect of the method introduced in [12]. As a result of this analysis, we discover several intriguing and fascinating connections with families of other singular geometric "objects" related to $X$. Then, we generalize this method to study similar problems for families of singular divisors on ruled fourfolds suitably related to $X$.

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To be more precise, let \( X \) be a smooth projective threefold and let \( \mathcal{F} \) be a rank-two vector bundle on \( X \), which is assumed to be globally generated with general global section \( s \) having its zero-locus \( V(s) \) a smooth, irreducible curve \( D = D_s \) in \( X \). The geometric genus of \( D \) is given by
\[
2g(D) - 2 = 2p_a(D) - 2 = \text{deg}(\mathcal{L} \otimes \omega_X \otimes \mathcal{O}_D),
\]
where \( \mathcal{L} := c_1(\mathcal{F}) \in \text{Pic}(X) \) and \( \omega_X \) is the canonical sheaf of \( X \).

Take now \( \mathbb{P}(H^0(X, \mathcal{F})) \); from our assumptions on \( \mathcal{F} \), its general point parametrizes a global section whose zero-locus is a smooth, irreducible curve. This projective space somehow gives a scheme dominating a subvariety in which the curves move.

Given a positive integer \( \delta \leq p_a(D) \), it makes sense to consider the locally closed subscheme:
\[
\mathcal{V}_\delta(\mathcal{F}) := \{ [s] \in \mathbb{P}(H^0(X, \mathcal{F})) \mid C_s := V(s) \subset X \text{ is irreducible with only } \delta \text{ nodes as singularities} \};
\]
(cfr. \( \mathbb{P}^3 \)). These are usually called \textit{Severi varieties} of global sections of \( \mathcal{F} \) whose zero-loci are irreducible, \( \delta \)-nodal curves in \( X \), of arithmetic genus \( p_a(D) \) and geometric genus \( g = p_a(D) - \delta \) (cfr. \( \mathbb{P}^3 \), for \( X = \mathbb{P}^3 \), and \( \mathbb{P}^4 \) in general). This is because such schemes are the natural generalization of the (classical) Severi varieties on smooth, projective surfaces recalled before.

When \( \mathcal{V}_\delta(\mathcal{F}) \) is not empty then its expected codimension in \( \mathbb{P}(H^0(X, \mathcal{F})) \) is \( \delta \) (see Proposition \( \mathcal{P}^5 \)). Thus, one says that a point \( [s] \in \mathcal{V}_\delta(\mathcal{F}) \) is a regular point if it is smooth and such that \( \dim_{[s]}(\mathcal{V}_\delta(\mathcal{F})) \) equals the expected one (cfr. Definition \( \mathcal{P}^6 \)).

It is clear from the definition of regularity that it is fundamental to determine the tangent space to a Severi variety at a given point. In \( \mathbb{P}^4 \) we introduced the following cohomological description of the tangent space \( T_{[s]}(\mathcal{V}_\delta(\mathcal{F})) \).

\section*{Theorem 1 (cfr. Theorem \( \mathcal{P}^7 \))} Let \( X \) be a smooth projective threefold. Let \( \mathcal{F} \) be a globally generated rank-two vector bundle on \( X \) and let \( \delta \) be a positive integer. Fix \([s] \in \mathcal{V}_\delta(\mathcal{F}) \) and let \( C = V(s) \subset X \). Denote by \( \Sigma \) the set of nodes of \( C \).

Let
\[
\mathcal{P} := \mathbb{P}_X(\mathcal{F}) \xrightarrow{\pi} X
\]
be the projective space bundle together with its natural projection \( \pi \) on \( X \) and denote by \( \mathcal{O}_\mathcal{P}(1) \) its tautological line bundle.

Then, there exists a zero-dimensional subscheme \( \Sigma^1 \subset \mathcal{P} \) of length \( \delta \), which is a set of \( \delta \) rational double points for the divisor \( G_s \in |\mathcal{O}_\mathcal{P}(1)| \) corresponding to the given section \( s \in H^0(X, \mathcal{F}) \).

In particular, each element \([s] \in \mathcal{V}_\delta(\mathcal{F}) \) corresponds to a divisor \( G_s \in |\mathcal{O}_\mathcal{P}(1)| \), which contains the \( \delta \) fibres \( L_{p_i} = \pi^{-1}(p_i) \subset \mathcal{P} \), for \( p_i \in \Sigma, 1 \leq i \leq \delta \), and which has \( \delta \) rational double points each of which are on exactly one of the \( \delta \) fibres \( L_{p_i} \).

By using the above result, one can translate the regularity property of \([s] \in \mathcal{V}_\delta(\mathcal{F}) \) into the surjectivity of some maps among spaces of sections of suitable sheaves on \( X \) (cfr. Proposition \( \mathcal{P}^8 \) and Remark \( \mathcal{P}^{14} \)). This allows us to find several equivalent and sufficient conditions for the regularity of Severi varieties \( \mathcal{V}_\delta(\mathcal{F}) \) on \( X \) (cfr. \( \mathbb{P}^4 \) and also Corollaries \( \mathcal{P}^{11} \), \( \mathcal{P}^{12} \), \( \mathcal{P}^{14} \), \( \mathcal{P}^{16} \), \( \mathcal{P}^{18} \) and Theorem \( \mathcal{P}^{21} \) in this paper).

On the other hand, Theorem 1 also introduces a meaningful and fascinating connection between elements in \( \mathcal{V}_\delta(\mathcal{F}) \) and other singular schemes in \( X \) and in \( \mathcal{P} \). The aim of
this paper is to investigate in details the deep geometric meaning of \( T_{[s]}(V_\delta(\mathcal{F})) \) and its several connections with these families of singular geometric objects related to \( X \) and to \( \mathcal{P} \).

More precisely, we show that there exists a natural connection among:

- a nodal section \([s] \in V_\delta(\mathcal{F})\) on \( X \),
- the corresponding singular divisor \( G_s \) in \( |\mathcal{O}_\mathcal{P}(1)| \) on \( \mathcal{P} \) having \( \delta \) rational double points over the nodes of \( C = V(s) \),
- for any \( s + \epsilon s' \in T_{[s]}(V_\delta(\mathcal{F})) \), the surface \( V(s \wedge s') \subset X \) which is singular along \( \Sigma \) and which belongs to the linear system \( |\mathcal{O}_X \otimes \mathcal{L}| \) on \( X \),
- given \( G_s \) and \( G_{s'} \) in \( |\mathcal{O}_\mathcal{P}(1)| \) corresponding to \( s \) and \( s' \) respectively, the "complete intersection" surface in \( \mathcal{P} \), \( S_{s,s'} := G_s \cap G_{s'} \), which is singular along \( \Sigma^1 \) and which dominates \( V(s \wedge s') \),

(cf. Propositions 3.4, 4.2 and Remarks 4.10, 5.9).

In particular, we prove:

**Theorem. 2 (cf. Theorem 5.7 and Corollary 5.11)** The following conditions are equivalent:

(i) \( s + \epsilon s' \in T_{[s]}(V_\delta(\mathcal{F})) \), where \( \epsilon^2 = 0 \);
(ii) \( V(s \wedge s') \subset X \) is a surface which contains \( C \) and which is singular along \( \Sigma \);
(iii) the divisor \( G_{s'} \) passes through \( \Sigma^1 \)
(iv) the surface \( S_{s,s'} := G_s \cap G_{s'} \subset \mathcal{P} \) is singular along \( \Sigma^1 \);

By using Theorem 2 we can also give several further equivalent conditions for the regularity of \([s] \in V_\delta(\mathcal{F})\) (cf. Remark 5.12).

Furthermore, thanks to the correspondence introduced in Theorems 1 and 2, we also consider a generalization of Severi varieties of nodal curves. Indeed, we denote by

\[ \mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1)) := \{ G_s \in |\mathcal{O}_\mathcal{P}(1)| \text{ s.t. } [s] \in V_\delta(\mathcal{F}) \} \]

the schemes parametrizing families of expected codimension \( \delta \) in \( |\mathcal{O}_\mathcal{P}(1)| \), whose elements correspond to divisors which are irreducible and with only \( \delta \) rational double points as singularities. For brevity sake, these are called \( \mathcal{P}_\delta \)-Severi varieties (cf. Definition 6.1 and 6.3).

One can obviously give a similar definition of regularity for \( \mathcal{P}_\delta \)-Severi varieties (cf. Definition 6.4). We prove:

**Theorem. 3 (cf. Theorem 6.5 and Corollary 6.12)** Let \([G_s] \in \mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1))\) on \( \mathcal{P} \) and let \( \Sigma^1 \) be the zero-dimensional scheme of the \( \delta \)-rational double points of \( G_s \subset \mathcal{P} \). Then

\[ T_{[G_s]}(\mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1))) \cong H^0(\mathcal{O}_{\mathcal{P}}(1) \otimes \mathcal{O}_\mathcal{P}(1)) / G_s \]

In particular,

\[ [G_s] \in \mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1)) \text{ is a regular point } \iff [s] \in V_\delta(\mathcal{F}) \text{ is a regular point.} \]

Finally, we first improve some regularity results of [12] for Severi varieties \( V_\delta(\mathcal{F}) \) of irreducible, \( \delta \)-nodal sections on \( X \); then, we use Theorem 3 to deduce regularity results also for \( \mathcal{P}_\delta \)-Severi varieties \( \mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1)) \) on \( \mathcal{P} \). Precisely, we have:

**Theorem. 4 (cf. Theorems 7.1 and 7.3)** Let \( X \) be a smooth projective threefold, \( \mathcal{E} \) be a globally generated rank-two vector bundle on \( X \), \( M \) be a very ample line bundle on \( X \)
and \( k \geq 0 \) and \( \delta > 0 \) be integers. Let \( \mathcal{P} := \mathbb{P}_X(\mathcal{E} \otimes M^{\otimes k}) \) and \( \mathcal{O}_\mathcal{P}(1) \) be its tautological line bundle. If

\[
(\ast) \quad \delta \leq k + 1,
\]

then both \( \mathcal{V}_\delta(\mathcal{E} \otimes M^{\otimes k}) \) on \( X \) and \( \mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1)) \) on \( \mathcal{P} \) are regular at each point.

The upper-bounds in \((\ast)\) are also shown to be almost-sharp (cf. Remark 7.4).

What we want to stress is the following fact: the regularity condition for the schemes \( \mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1)) \) on \( \mathcal{P} \) is equivalent to the separation of suitable zero-dimensional schemes by the linear system \( |\mathcal{O}_\mathcal{P}(1)| \) on the fourfold \( \mathcal{P} \) (cf. Corollary 6.12). In general, it is well-known how difficult is to establish separation of points in projective varieties of dimension greater than or equal to three (cf. e.g. [1], [9] and [18]). In some cases, some separation results can be found by using technical tools like multiplier ideals as well as the Nadel and the Kawamata-Viehweg vanishing theorems (see, e.g. [8], for an overview). In our situation, thanks to the correspondence between \( \mathcal{V}_\delta(\mathcal{F}) \) on \( X \) and \( \mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1)) \) on \( \mathcal{P} \), we deduce regularity conditions for \( \mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1)) \) from those already obtained for \( \mathcal{V}_\delta(\mathcal{F}) \).

The paper consists of seven sections. Section 1 contains some terminology and notation. In Section 2 we briefly recall some fundamental definitions in [12], which are frequently used in the whole paper. Section 3 briefly recall one of the main result in [12] concerning the correspondence between elements in \( \mathcal{V}_\delta(\mathcal{F}) \) on \( X \) and those in \( \mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1)) \) on \( \mathcal{P} \) (cf. Theorem 3.1).

In Section 4 we describe how to associate elements of \( T_{|\delta|} \mathcal{V}_\delta(\mathcal{F}) \) to singular divisors in \( X \). Section 5 contains one of the main result of the paper (cf. Theorem 5.1) which proves the equivalence of several singular geometric "objects" related to \( X \) and to \( \mathcal{P} \). In Section 6 we focus on \( \mathcal{P} \)-Severi varieties on \( \mathcal{P} \); we give a description of tangent spaces at points of such schemes as well as we find conditions for their regularity (cf. Theorem 6.5 and Corollary 6.12). Section 7 is devoted to the determination of almost-sharp upper-bounds on \( \delta \) implying the regularity of \( \mathcal{V}_\delta(\mathcal{F}) \) on \( X \) as well as of \( \mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1)) \) on \( \mathcal{P} \).

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1. Notation and Preliminaries

We work in the category of algebraic \( \mathbb{C} \)-schemes. \( Y \) is a \( m \)-fold if it is a reduced, irreducible and non-singular scheme of finite type and of dimension \( m \). If \( m = 1 \),
then $Y$ is a (smooth) curve; $m = 2$ and $3$ are the cases of a (non-singular) surface and
threefold, respectively. If $Z$ is a closed subscheme of a scheme $Y$, $\mathcal{I}_Z/Y$ denotes the
ideal sheaf of $Z$ in $Y$, $\mathcal{N}_Z/Y$ the normal sheaf of $Z$ in $Y$ whereas $\mathcal{N}_Z/Y \cong \mathcal{I}_Z/Y^2$
is the conormal sheaf of $Z$ in $Y$. As usual, $h^i(Y, -) := \dim H^i(Y, -)$.

Given $Y$ a projective scheme, $\omega_Y$ denotes its dualizing sheaf. When $Y$ is a smooth
variety, then $\omega_Y$ coincides with its canonical bundle and $K_Y$ denotes a canonical divisor
s.t. $\omega_Y \cong \omega_Y(K_Y)$. Furthermore, $\mathcal{T}_Y$ denotes its tangent bundle whereas $\Omega_1^1_Y$
denotes its cotangent bundle.

If $D$ is a reduced curve, $p_a(D) = h^1(\mathcal{O}_D)$ denotes its arithmetic genus, whereas
g($D$) = $p_y(D)$ denotes its geometric genus, the arithmetic genus of its normalization.

Let $Y$ be a projective $m$-fold and $\mathcal{E}$ be a rank-$r$ vector bundle on $Y$; $c_i(\mathcal{E})$ denotes the
$i$-th Chern class of $\mathcal{E}$, $1 \leq i \leq r$. As in [17] - Sect. II.7 - $\mathbb{P}_Y(\mathcal{E})$ denotes the projective
space bundle on $Y$, defined as $\text{Proj} (\text{Sym}(\mathcal{E}))$.

There is a surjection $\pi^*(\mathcal{E}) \rightarrow \mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1)$, where $\mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1)$ is the tautological line
bundle on $\mathbb{P}_Y(\mathcal{E})$ and where $\pi: \mathbb{P}_Y(\mathcal{E}) \rightarrow Y$ is the natural projection morphism. Recall
that $\mathcal{E}$ is said to be an ample (resp. nef) vector bundle on $Y$ if $\mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1)$ is an ample
(resp. nef) line bundle on $\mathbb{P}_Y(\mathcal{E})$ (see, e.g. [16]).

For non reminded terminology, the reader is referred to [3], [13] and [17].

2. Families of nodal "curves" on smooth, projective threefolds

In this section we briefly recall some definitions and results from [12] which will be
frequently used in the sequel.

Let $X$ be a smooth projective threefold and let $\mathcal{F}$ be a rank-two vector bundle on $X$.
If $\mathcal{F}$ is globally generated on $X$, it is not restrictive if from now on we assume that the
zero-locus $V(s)$ of its general global section $s$ is a smooth, irreducible curve $D = D_s$ in
$X$ (for details, see [12]; for general motivations and backgrounds, the reader is referred to
e.g. [19] and to [24], Chapter IV).

From now on, denote by $\mathcal{L} \in \text{Pic}(X)$ the line bundle on $X$ given by $c_1(\mathcal{F})$. Thus, by the
Koszul sequence of $(\mathcal{F}, s)$:

\begin{equation}
0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{F} \rightarrow \mathcal{I}_{V(s)} \otimes \mathcal{L} \rightarrow 0,
\end{equation}

we compute the geometric genus of $D$ in terms of the invariants of $\mathcal{F}$ and of $X$. Precisely

\begin{equation}
2g(D) - 2 = 2p_a(D) - 2 = \text{deg}(\mathcal{L} \otimes \omega_X \otimes \mathcal{O}_D).
\end{equation}

This integer is easily computable when, for example, $X$ is a general complete inter-
section threefold. In particular, when $\text{Pic}(X) \cong \mathbb{Z}$ (e.g $X = \mathbb{P}^3$ or $X$ either a prime
Fano or a complete intersection Calabi-Yau threefold) one can use this isomorphism
to identify line bundles on $X$ with integers. Therefore, if $A$ denotes the ample
generator class of $\text{Pic}(X)$ over $\mathbb{Z}$ and if $\mathcal{F}$ is a rank-two vector bundle on $X$ such that
c_1(\mathcal{F}) = nA$, we can also write $c_1(\mathcal{F}) = n$ with no ambiguity.

Thus, if e.g. $X = \mathbb{P}^3$ and if we put $c_i = c_i(\mathcal{F}) \in \mathbb{Z}$, we have

\begin{equation}
\text{deg}(D) = c_2 \text{ and } g(D) = p_a(D) = \frac{1}{2}(c_2(c_1 - 4)) + 1,
\end{equation}

i.e. $D$ is subcanonical of level $(c_1 - 4)$.
Take now \( \mathbb{P}(H^0(X, \mathcal{F})) \); from our assumptions on \( \mathcal{F} \), the general point of this projective space parametrizes a global section whose zero-locus is a smooth, irreducible curve in \( X \). Given a positive integer \( \delta \leq p_a(D) \), it makes sense to consider the subset

\[
\mathcal{V}_\delta(\mathcal{F}) := \{ [s] \in \mathbb{P}(H^0(X, \mathcal{F})) | C_s := V(s) \subset X \text{ is irreducible with only } \delta \text{ nodes as singularities} \};
\]

therefore, any element of \( \mathcal{V}_\delta(\mathcal{F}) \) determines a curve in \( X \) whose arithmetic genus \( p_a(C_s) \) is given by (2.2) and whose geometric genus is \( g = p_a(C_s) - \delta \). We recall that \( \mathcal{V}_\delta(\mathcal{F}) \) is a locally closed subscheme of the projective space \( \mathbb{P}(H^0(X, \mathcal{F})) \); it is usually called the Severi variety of global sections of \( \mathcal{F} \) whose zero-loci are irreducible, \( \delta \)-nodal curves in \( X \) (cf. \[2\], for \( X = \mathbb{P}^3 \), and \[12\] in general). This is because such schemes are the natural generalization of the (classical) Severi varieties of irreducible and \( \delta \)-nodal curves in linear systems on smooth, projective surfaces (see \[5\], \[4\], \[7\], \[11\], \[14\], \[15\], \[21\], \[22\] and \[23\], just to mention a few).

For brevity sake, we shall usually refer to \( \mathcal{V}_\delta(\mathcal{F}) \) as the Severi variety of irreducible, \( \delta \)-nodal sections of \( \mathcal{F} \) on \( X \).

First possible questions on such Severi varieties are about their dimensions as well as their smoothness properties.

A preliminary estimate is given by the following standard result:

**Proposition 2.5.** Let \( X \) be a smooth projective threefold, \( \mathcal{F} \) be a globally generated rank-two vector bundle on \( X \) and \( \delta \) be a positive integer. Then

\[
\text{expdim}(\mathcal{V}_\delta(\mathcal{F})) = \begin{cases} 
h^0(X, \mathcal{F}) - 1 - \delta, & \text{if } \delta \leq h^0(X, \mathcal{F}) - 1 = \dim(\mathbb{P}(H^0(\mathcal{F}))), \\
-1, & \text{if } \delta \geq h^0(X, \mathcal{F}). \end{cases}
\]

**Proof.** See Proposition 2.10 in \[12\]. \(\square\)

**Assumption 1.** From now on, given \( X \) and \( \mathcal{F} \) as in Proposition 2.5, we shall always assume \( \mathcal{V}_\delta(\mathcal{F}) \neq \emptyset \). We write \([s] \in \mathcal{V}_\delta(\mathcal{F})\) to intend that the global section \( s \in H^0(X, \mathcal{F}) \) determines the corresponding point \([s]\) of the scheme \( \mathcal{V}_\delta(\mathcal{F}) \). We simply denote by \( C \) - instead of \( C_s \) - the zero-locus of the given section \( s \), when it is clear from the context that \( s \) is fixed. We finally consider \( \delta \leq \min\{h^0(X, \mathcal{F}) - 1, \ p_a(C)\} \) - the latter is because we want \( C = V(s) \) to be irreducible, for any \([s] \in \mathcal{V}_\delta(\mathcal{F})\).

By Proposition 2.5 it is natural to state the following:

**Definition 2.6.** Let \([s] \in \mathcal{V}_\delta(\mathcal{F})\), with \( \delta \leq \min\{h^0(X, \mathcal{F}) - 1, \ p_a(C)\} \). Then \([s]\) is said to be a regular point of \( \mathcal{V}_\delta(\mathcal{F}) \) if:

(i) \([s] \in \mathcal{V}_\delta(\mathcal{F})\) is a smooth point, and

(ii) \(\dim|_s(\mathcal{V}_\delta(\mathcal{F})) = \expdim(\mathcal{V}_\delta(\mathcal{F})) = \dim(\mathbb{P}(H^0(X, \mathcal{F}))) - \delta\).

\(\mathcal{V}_\delta(\mathcal{F})\) is said to be regular if it is regular at each point.

One of the main result in \[12\] has been to present a cohomological description of the tangent space \( T_{[s]}(\mathcal{V}_\delta(\mathcal{F})) \) which translates the regularity property of a given point \([s] \in \mathcal{V}_\delta(\mathcal{F})\) into the surjectivity of some maps among spaces of sections of suitable sheaves on the threefold \( X \). This description allowed us to find also several sufficient conditions for the regularity of Severi varieties \( \mathcal{V}_\delta(\mathcal{F}) \) on \( X \).
One of the aim of this paper is to study in more details the deep geometric meaning of the cohomological description of the tangent space $T_{[s]}(\mathcal{V}_{\delta}(\mathcal{F}))$ and its several connections with families of other singular geometric objects related to $X$ and to $\mathcal{F}$.

To do this, we have first to recall some results contained in [12], since these are the starting point of our analysis.

3. Association of elements of $\mathcal{V}_{\delta}(\mathcal{F})$ on $X$ to singular divisors in $\mathbb{P}_{X}(\mathcal{F})$

In this section we want to briefly recall the correspondence given in [12] between elements of $\mathcal{V}_{\delta}(\mathcal{F})$ on $X$ and suitable singular divisors in the tautological linear system $|0_{\mathcal{P}}(1)|$ on the projective space bundle $\mathcal{P} := \mathbb{P}_{X}(\mathcal{F})$, which is a fourfolds ruled over $X$. Instead of referring the reader to [12], we prefer to briefly recall here the proofs of some results contained in there, not only because we give here more precise statements and proofs, but mainly because the strategy of the proofs as well as their technical details will be fundamental for the analysis in the whole paper.

From now on, with conditions as in Assumption [1] let $[s] \in \mathcal{V}_{\delta}(\mathcal{F})$. Then:

**Theorem 3.1.** (cf. Theorem 3.4 (i) in [12]) Let $X$ be a smooth projective threefold. Let $\mathcal{F}$ be a globally generated rank-two vector bundle on $X$ and let $\delta$ be a positive integer. Fix $[s] \in \mathcal{V}_{\delta}(\mathcal{F})$ and let $C = V(s) \subset X$. Denote by $\Sigma$ the set of nodes of $C$.

Let

$$\mathcal{P} := \mathbb{P}_{X}(\mathcal{F}) \xrightarrow{\pi} X$$

be the projective space bundle together with its natural projection $\pi$ on $X$ and denote by $0_{\mathcal{P}}(1)$ its tautological line bundle.

Then, there exists a zero-dimensional subscheme $\Sigma^{\delta} \subset \mathcal{P}$ of length $\delta$, which is a set of $\delta$ rational double points for the divisor $G_{s} \in |0_{\mathcal{P}}(1)|$ corresponding to the given section $s \in H^{0}(X, \mathcal{F})$.

In particular, each element $[s] \in \mathcal{V}_{\delta}(\mathcal{F})$ corresponds to a divisor $G_{s} \in |0_{\mathcal{P}}(1)|$, which contains the $\delta$ fibres $L_{p_{i}} = \pi^{-1}(p_{i}) \subset \mathcal{P}$, for $p_{i} \in \Sigma$, $1 \leq i \leq \delta$, and which has $\delta$ rational double points each of which are on exactly one of the $\delta$ fibres $L_{p_{i}}$.

**Proof.** Consider the smooth, projective, ruled fourfold

$$\mathcal{P} := \mathbb{P}_{X}(\mathcal{F}) \xrightarrow{\pi} X,$$

together with its tautological line bundle $0_{\mathcal{P}}(1)$ such that $\pi_{*}(0_{\mathcal{P}}(1)) \cong \mathcal{F}$. Since $[s] \in \mathcal{V}_{\delta}(\mathcal{F})$, in particular $s \in H^{0}(X, \mathcal{F})$; then, one also has

$$0 \to 0_{\mathcal{P}} \xrightarrow{s} 0_{\mathcal{P}}(1).$$

Therefore, the nodal curve $C \subset X$ corresponds to a divisor - say $G_{s}$ - on the fourfold $\mathcal{P}$ which belongs to the tautological linear system $|0_{\mathcal{P}}(1)|$.

It is clear that $G_{s}$ contains all the $\pi$-fibres over the zero-locus $C = V(s)$; precisely, it contains the surface $\mathbb{F} := \mathbb{P}^{3}_{C} = \text{Proj}(\text{Sym}(\mathcal{F}|_{C}))$ which is ruled over $C$. Therefore, in particular $G_{s}$ contains the locus $\Lambda := \bigcup_{i=1}^{\delta} L_{p_{i}}$, where $\Sigma = \{p_{1}, \ldots, p_{\delta}\}$.

The geometry of $G_{s}$ is strictly related to the one of $C$. Indeed, if $p \in \Sigma = \text{Sing}(C)$, take $U_{p} \subset X$ an affine open set containing $p$, where the vector bundle $\mathcal{F}$ trivializes. Since $p$ is a planar singularity, one can choose local coordinates $\underline{x} = (x_{1}, x_{2}, x_{3})$ on $U_{p} \cong \mathbb{A}^{3}$ such that $\underline{x}(p) = (0, 0, 0)$ and such that the global section $s$ is

$$s|_{U_{p}} = (x_{1}x_{2}, x_{3}).$$
For what concerns the divisor $G_s \in |O_P(1)|$ corresponding to $s \in H^0(X, \mathcal{F})$, since $U_p$ trivializes $\mathcal{F}$, then $P|_{U_p} \cong U_p \times \mathbb{P}^1$. Taking homogeneous coordinates $[u, v] \in \mathbb{P}^1$, we have $O_P(\pi^{-1}(U_p)) \cong \mathbb{C}[x_1, x_2, x_3, u, v]$. Thus,

$$O_{G_s}(\pi^{-1}(U_p)) \cong \mathbb{C}[x_1, x_2, x_3, u, v]/(ux_1x_2 + vx_3).$$

This implies that the local equation of $G_s$ in $\pi^{-1}(U_p)$ is given by

$$ux_1x_2 + vx_3 = 0. \quad (3.2)$$

In the open chart where $v \neq 0$, $G_s$ is smooth, whereas where $u \neq 0$, we see that (3.2) is the equation of a quadric cone in $\mathbb{A}^4$ having vertex at the origin. This means that $G_s$ has a rational double point along the $\pi$-fibre $L_p := \pi^{-1}(p) \subset P$.

Globally speaking, one can state that there exist $\delta$ distinguished points on $P$. Such distinguished points determine a $0$-dimensional subscheme

$$\Sigma^1 \subset P$$

along which the divisor $G_s \subset P$ is singular. Thus $\Sigma^1$ is a set of $\delta$ rational double points for $G_s$, each line of $\Lambda = \pi^{-1}(\Sigma) = \bigcup_{i=1}^\delta L_{p_i}$ containing only one of such $\delta$ points; more precisely, each point $p_i^1 \in \Sigma^1$ is a rational double point for $G_s$ and it belongs to the fibre $L_{p_i} \subset P$, where $p_i \in \Sigma$ is a node of $C = V(s)$. Therefore, the isomorphism

$$\Sigma^1 \cong \Sigma$$

is directly given by the natural projection $\pi : P \to X$.

The above result introduces a correspondence between elements of $\mathcal{V}_\delta(\mathcal{F})$ and suitable divisors in $|O_P(1)|$; this correspondence has been used in [12] to give a cohomological description of the tangent space $T_{[s]}(\mathcal{V}_\delta(\mathcal{F}))$, which has been a fundamental point in order to determine several sufficient conditions for the regularity of Severi varieties $\mathcal{V}_\delta(\mathcal{F})$ on $X$ (cf. Theorems 4.5, 5.9, 5.25, 5.28 and 5.36 in [12]).

The main ideas are as follows: $C$ is local complete intersection in $X$, whose normal sheaf is the rank-two vector-bundle $\mathcal{N}_{C/X} \cong \mathcal{F}|_C$. Let $T_C^1$ be the first cotangent sheaf of $C$, i.e. $T_C^1 \cong \mathcal{E}xt^1(\Omega_C^1, O_C)$, where $\Omega_C^1$ is the sheaf of Kähler differentials of the nodal curve $C$ (for details, see [20]). Since $C$ is nodal, $T_C^1$ is a sky-scrapersheaf supported on $\Sigma$, such that $T_C^1 \cong \bigoplus_{i=1}^\delta C(i)$. Furthermore, one has the exact sequence:

$$0 \to \mathcal{N}_C^\prime \to \mathcal{N}_{C/X} \xrightarrow{\gamma} T_C^1 \to 0, \quad (3.3)$$

where $\mathcal{N}_C^\prime$ is defined as the kernel of the natural surjection $\gamma$ (see, for example, [22]).

**Proposition 3.4.** (cf. Theorem 3.4 (ii) in [12]) With assumptions and notation as in Theorem 3.2, denote by $3\Sigma^1/P$ the ideal sheaf of $\Sigma^1$ in $P$. Then the subsheaf of $\mathcal{F}$, defined by

$$\mathcal{F}_{\Sigma} := \pi_*(3\Sigma^1/P \otimes O_P(1)), \quad (3.5)$$

is such that its global sections (modulo the one dimensional subspace $<s>$) parametrize first-order deformations of $s \in H^0(X, \mathcal{F})$ which are equisingular.

Precisely, we have

$$H^0(X, \mathcal{F}_{\Sigma}, <s>) \cong T_{[s]}(\mathcal{V}_\delta(\mathcal{F})) \subset T_{[s]}(\mathbb{P}(H^0(\mathcal{F}))) \cong H^0(X, \mathcal{F}, <s>). \quad (3.6)$$
Proof. By the correspondence given in Theorem 3.1, we can consider the closed immersion $\Sigma^1 \subset \mathcal{P}$ and so the natural exact sequence

\[(3.7) \quad 0 \to \mathcal{O}_{\Sigma^1/P} \otimes \mathcal{O}_P(1) \to \mathcal{O}_P(1) \to \mathcal{O}_{\Sigma^1} \to 0,\]

which is defined by restricting $\mathcal{O}_P(1)$ to $\Sigma^1$.

Since $\pi_*(\mathcal{O}_P(1)) \cong \mathcal{F}$, $\pi_*(\mathcal{O}_{\Sigma^1}) = \pi_*(\mathcal{O}_{\Sigma^1}) = \pi_*(\mathcal{O}_{\Sigma}) \cong \mathcal{O}_\Sigma$ and since we have $\mathcal{F} \to \mathcal{O}_\Sigma$, by applying $\pi_*$ to the exact sequence \((3.7)\), we get $\mathcal{R}_1\pi_* (\mathcal{O}_{\Sigma^1/P} \otimes \mathcal{O}_P(1)) = 0$. Thus, we define $\mathcal{F}^{\Sigma}$ as in \((3.5)\), so that

\[(3.8) \quad 0 \to \mathcal{F}^{\Sigma} \to \mathcal{F} \to \mathcal{O}_\Sigma \to 0,\]

holds.

Observe that $\mathcal{F}^{\Sigma}$ fits in the following exact diagram:

\[
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & \downarrow & & \\
0 & \mathcal{O}_{\mathcal{C}/\mathcal{X}} \otimes \mathcal{F} & \cong & \mathcal{O}_{\mathcal{C}/\mathcal{X}} \otimes \mathcal{F} \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{F}^{\Sigma} & \to & \mathcal{F} \to \mathcal{O}_\Sigma \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{N}'_C & \to & \mathcal{F}|_C \to \mathcal{T}^1_C \to 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0.
\end{array}
\]

From the commutativity of diagram \((3.9)\), the vector space

\[
\bar{H}^0(X, \mathcal{F}^{\Sigma}) < s >
\]

parametrizes the first-order deformations of $[s]$ in $\mathbb{P}(\bar{H}^0(X, \mathcal{F}))$ which are equisingular; indeed, these are exactly the global sections of $\mathcal{F}$ which go to zero at $\Sigma$ in the composition

\[(3.10) \quad \mathcal{F} \to \mathcal{F}|_C \to \mathcal{T}^1_C \cong \mathcal{O}_\Sigma. \]

Notice that \((3.6)\) gives a completely general characterization of the tangent space $T_{[s]}(\mathcal{V}_\delta(\mathcal{F}))$ on $X$. In particular, with assumptions and notation as in Theorem 3.1 and in Proposition 3.4, we get the following results:

Corollary 3.11. Each global section $s' \in \bar{H}^0(X, \mathcal{F}^{\Sigma})$ corresponds to a divisor $G_{s'} \in |\mathcal{O}_{\Sigma^1/P} \otimes \mathcal{O}_P(1)|$ on $\mathcal{P}$. In particular, for $\epsilon \in \mathbb{C}[T]/(T^2)$, we have:

$s + \epsilon s' \in T_{[s]}(\mathcal{V}_\delta(\mathcal{F}))$ on $X$ if and only if $s' \in \bar{H}^0(X, \mathcal{F}^{\Sigma}) \Leftrightarrow G_{s'} \in |\mathcal{O}_P(1)|$ and $\Sigma^1 \subset G_{s'}$.

Proof. It directly follows from the definition of $\mathcal{F}^{\Sigma}$ and from the correspondence in Theorem 3.1 \qed

Corollary 3.12. (cf. Corollary 3.9 in [12]) From \((3.8)\), it follows that

\[(3.13) \quad [s] \in \mathcal{V}_\delta(\mathcal{F}) \text{ is regular} \quad \Leftrightarrow \quad \bar{H}^0(X, \mathcal{F}) \xrightarrow{\mu_X} \bar{H}^0(X, \mathcal{O}_\Sigma) \quad \Leftrightarrow \quad \bar{H}^0(\mathcal{P}, \mathcal{O}_P(1)) \xrightarrow{\mu^\mathcal{P}} \bar{H}^0(\mathcal{P}, \mathcal{O}_{\Sigma^1}). \]

Proof. It follows from Proposition 2.5 from Theorem 3.1 and from Proposition 3.4 \qed
Remark 3.14. Note that, on the one hand, the map $\mu_X$ in (3.13) is not defined by restricting the global sections of $\mathcal{F}$ to $\Sigma$ because (3.8) - i.e. the second row of diagram (3.9) - does not coincide with the restriction sequence

$$0 \to \mathcal{F}\big|_\Sigma \otimes \mathcal{F} \to \mathcal{F} \to \mathcal{F}\big|_\Sigma \to 0;$$

indeed $\mathcal{F}\big|_\Sigma$ has rank two at each node, whereas $\mathcal{F}\Sigma$ has rank one at each node.

On the other hand, by the Leray isomorphism the exact sequence (3.7) on the fourfold $\mathcal{P}$ is equivalent in cohomology to the one in (3.8) but it is more naturally defined by restricting the line bundle $\mathcal{O}_\mathcal{P}(1)$ to $\Sigma^1$. Therefore, the map $\rho_\mathcal{P}$ in (3.13) is a standard restriction map.

To better understand the geometric meaning of the map $\mu_X$, we briefly recall the local description of (3.8). Therefore, it suffices to consider $\delta = 1$. Assume $\{p\} = Sing(C) = \Sigma$ and take, as before, $U_p \subset X$ an affine open set containing $p$, where the vector bundle $\mathcal{F}$ is trivial. Take local coordinates $x = (x_1, x_2, x_3)$ on $U_p \cong \mathbb{A}^3 = \mathbb{C}^3$ such that $\mathfrak{g}(p) = (0, 0, 0)$ and such that the global section $s$, whose zero-locus is $C$, is given by $s|_{U_p} = (x_1, x_2, x_3)$. Since $C = V(x_1, x_2, x_3) \subset U_p$, around the node $x(p) = 0$ the Jacobian map:

$$J(s) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{pmatrix} = \begin{pmatrix} x_2 & x_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Put $\text{Im}(J(s)) = \langle x_2 e_1', x_1 e_1', e_2' \rangle$, where $\{e_1', e_2'\}$ a local basis for $\mathcal{N}_{C/\mathbb{C}^3}$. Thus, $e_2'$ goes to zero in $\mathfrak{T}_C$, so, by this local description, it follows that the map $\mu_X$ is exactly the composition of the evaluation at $p$ of global sections together with the projection

$$\mathbb{C}^2_{(p)} \xrightarrow{\pi_1} \mathbb{C}_{(p)},$$

where $\mathbb{C}^2_{(p)} \cong \mathcal{F} \otimes \mathcal{O}_p$, $\mathbb{C}_{(p)} \cong \mathfrak{T}_{C,p}$ and $\pi_1((x, y)) = x$ (for more details, cf. §3 in [12]).

4. ASSOCIATION OF ELEMENTS IN $T_{[s]}\mathcal{V}_\delta(\mathcal{F})$ TO SINGULAR SURFACES IN $|\mathcal{J}_{C/X} \otimes c_1(\mathcal{F})|$ ON $X$

The connection between singular schemes defined in Sections 2 and 3 can be further analyzed in order to determine interesting geometric interpretations of first-order deformations given by sections in $H^0(X, \mathcal{F}\Sigma)$.

With notation as in §2 and in Assumption 1, let $[s] \in \mathcal{V}_\delta(\mathcal{F})$, $C = V(s)$, $\Sigma = Sing(C)$. Denote by $\mathcal{L} := c_1(\mathcal{F}) \in \text{Pic}(X)$.

By (2.1), one has:

$$T_{[s]}(\mathcal{V}_\delta(\mathcal{F})) \subset T_{[s]}(\mathbb{P}(H^0(X, \mathcal{F}))) \cong \frac{H^0(X, \mathcal{F})}{H^0(X, \mathcal{O}_X)} \hookrightarrow H^0(X, \mathcal{J}_{C/X} \otimes \mathcal{L}).$$

Therefore, first-order deformations of $[s]$ in $\mathbb{P}(H^0(X, \mathcal{F}))$, as well as in the Severi variety $\mathcal{V}_\delta(\mathcal{F})$, can be related to suitable divisors moving in the linear system $|\mathcal{L}|$ on $X$ and containing the nodal curve $C$.

Indeed, we have:
Proposition 4.2. Let $X$ be a smooth projective threefold. Let $\mathcal{F}$ be a globally generated rank-two vector bundle on $X$ and let $\mathcal{L} = c_1(\mathcal{F})$. Let $\delta$ be a positive integer, $[s] \in \mathcal{V}_\delta(\mathcal{F})$ and $C = V(s)$ be the corresponding irreducible, nodal curve in $X$. Denote by $\Sigma$ the set of nodes of $C$. Let $\mathcal{F}^\Sigma$ be the sheaf on $X$ defined in Proposition 3.4.

Then:

(i) Each global section $s' \in H^0(X, \mathcal{F}^\Sigma) \setminus \{s\}$ determines a divisor $V(s \wedge s') \subseteq |\mathcal{F} \otimes \mathcal{L}|$ in $X$ which is singular along $\Sigma$. Precisely,

\[ s' \in H^0(X, \mathcal{F}^\Sigma) \iff V(s \wedge s') \text{ contains } C \text{ and it is singular along } \Sigma. \tag{4.3} \]

(ii) The singularities of $V(s \wedge s')$ are along $\Sigma$ and along the (possibly empty) intersection scheme $C \cap V(s')$.

Proof. (i) Consider $s' \in H^0(X, \mathcal{F}^\Sigma)$. By Proposition 3.4 and Remark 3.14, this corresponds to a global section of $\mathcal{F}$ which is in the kernel of the map $\mu_X$ - i.e. a global section which goes to zero in the composition $\mathcal{F} \rightarrow \mathcal{F}|_C \rightarrow T^1_C$ in diagram (3.9).

Since the situation is local, we may work locally around each node, in some open subset where $\mathcal{F}$ trivializes. Thus, fix a node $p \in \Sigma$ and a suitable neighborhood $U = U_p$ of $p$, whose local coordinates are denoted by $(x_1, x_2, x_3)$. We assume that $C$ is defined in $U$ by two equations $f_1 = f_2 = 0$, where $s|_U = (f_1, f_2), f_1, f_2 \in \mathcal{O}_X(U)$.

Thus, the kernel of $\mu_X$ is given by those sections which are, at each node $p$, in the image of the Jacobian map

\[ \mathcal{T}_{\mathbb{C}^3}|_C \xrightarrow{J(s)} \mathcal{N}_{\mathbb{C}^3} \rightarrow T^1_C \]

given by

\[ J(s) := \left( \begin{array}{ccc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{array} \right) \tag{4.5} \]

in $U$.

Assume $s' \in H^0(X, \mathcal{F}^\Sigma)$ and suppose $s'|_U = (g_1, g_2)$. Therefore, for each $p \in \Sigma$, $s' \in H^0(X, \mathcal{F}^\Sigma)$ if and only if $s'(p) = (g_1(p), g_2(p)) \in \mathcal{F}_p$ is linear dependent on each pair $(\frac{\partial f_1}{\partial x_i}(p), \frac{\partial f_2}{\partial x_i}(p)), 1 \leq i \leq 3$. This is equivalent to the following conditions:

\[ \det \left( \begin{array}{cc} g_1(p) & g_2(p) \\ \frac{\partial f_1}{\partial x_1}(p) & \frac{\partial f_2}{\partial x_1}(p) \end{array} \right) = 0, \text{ for each } 1 \leq i \leq 3, \]

i.e.

\[ g_1(p) \frac{\partial f_2}{\partial x_i}(p) - g_2(p) \frac{\partial f_1}{\partial x_i}(p) = 0, \text{ for each } 1 \leq i \leq 3. \tag{4.6} \]

On the other hand observe that, since in particular $s' \in H^0(X, \mathcal{F})$, then it defines a divisor in $|\mathcal{L}|$ on $X$ containing $C$. Indeed, consider

\[ \tau := (s, s') : \mathcal{O}_X \oplus \mathcal{O}_X \rightarrow \mathcal{F}, \]

where

\[ \tau := \left( \begin{array}{cc} f_1 & g_1 \\ f_2 & g_2 \end{array} \right) \]

in the given open subset $U$. The degeneration locus of the map $\tau$ is given by

\[ V(\det(\tau)) = V(s \wedge s'), \]
whose local equation in $U$ is given by
\begin{equation}
(4.7) 
  f_1g_2 - f_2g_1 = 0.
\end{equation}

Thus, since $\mathcal{L} = c_1(\mathcal{F})$, $V(s \wedge s')$ corresponds to a divisor in $|\mathcal{L}|$ containing $C = V(s)$, i.e. it belongs to the linear system $|\mathcal{I}_{C/X} \otimes \mathcal{L}|$ on $X$.

By the local equation of $V(s \wedge s')$ in $U$ and by the fact that $p \in C = V(s)$, we have that
\begin{equation}
(4.8) 
  \frac{\partial}{\partial x_i}(f_1g_2 - f_2g_1)(p) = g_1(p)\frac{\partial f_2}{\partial x_i}(p) - g_2(p)\frac{\partial f_1}{\partial x_i}(p), \quad \text{for each } 1 \leq i \leq 3.
\end{equation}

Therefore, $s' \in H^0(X, \mathcal{F}_\Sigma^c)$ if and only if the associated divisor in $|\mathcal{I}_{C/X} \otimes \mathcal{L}|$ is singular at the nodes of $C$ (cf. [2], for the case $X = \mathbb{P}^3$).

(ii) Let $q \in C$ be a point and let $s = (f_1, f_2)$, $s' = (g_1, g_2)$ be the local expressions of $s$ and $s'$ around $q$. Fix $(x_1, x_2, x_3)$ local coordinates of $X$ around $q$. Since $C = V(s)$ and $q \in C$, we have that (4.3) holds, for each $1 \leq i \leq 3$.

Assume that $q \in C \setminus \Sigma$ and that $s'(q) \neq (0, 0)$; in this case, if (4.8) is equal to 0 at $q$, for each $1 \leq i \leq 3$, we would have that $(g_1(q), g_2(q))$ is linear dependent on each pair
\begin{equation}
(4.9) 
  (\frac{\partial f_1}{\partial x_1}(q), \frac{\partial f_2}{\partial x_1}(q)), \quad 1 \leq i \leq 3.
\end{equation}

In particular, the three pairs in (4.9) would be linearly dependent. This is a contradiction; indeed, since $q \in C \setminus \Sigma$, the Jacobian map in (4.3) and (4.4) is surjective at such a point $q$, i.e. $T_{c,q}^1 = 0$. On the other hand, since $\mathcal{N}_{C/X,q} \cong \mathcal{O}_{C,q}^3$, then we must have that two of the three pairs in (4.9) are linearly independent.

This implies that $V(s \wedge s')$ cannot be singular outside $\Sigma \cup (C \cap V(s'))$. Indeed, in the other cases - i.e. either $q \in \Sigma$ or $q \in (C \cap V(s'))$ - or both - it is easy to observe that (4.8) always vanishes at $q$, so that $V(s \wedge s')$ is singular at each such a point.

By using the "divisorial" approach introduced in Theorem 3.1, we shall give in Theorem 5.1 other interpretations of the equivalence in Proposition 4.2, which highlights the deep connection between these apparently distinct approaches.

**Remark 4.10.** From (4.8), we see that among the global sections in $H^0(X, \mathcal{F}_\Sigma^c)$, there are those global sections $s^*$ such that $s^*(p_i) = (0, 0)$, for $p_i \in \Sigma$, $1 \leq i \leq \delta$.

By the very definition of $\mathcal{F}_\Sigma^c$, we find that the inclusion
\[
\mathcal{I}_{\Sigma/X} \otimes \mathcal{F} \subseteq \mathcal{F}_\Sigma^c
\]
is not an isomorphism of sheaves on $X$. Indeed, the global sections of $H^0(X, \mathcal{F}_\Sigma^c)$ are those satisfying a condition like (4.6) at each node in $\Sigma$; by (4.8), condition (4.6) in particular holds if we consider global sections in the vector space $H^0(X, \mathcal{I}_{\Sigma/X} \otimes \mathcal{F})$. Such a vector space has an expected codimension equal to $2\delta$ in $H^0(X, \mathcal{F})$ as it follows by the exact sequence:
\begin{equation}
(4.11) 
  0 \to \mathcal{I}_{\Sigma/X} \otimes \mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_\Sigma \cong \mathcal{O}_{\Sigma}^3 \to 0,
\end{equation}

therefore, $\mathcal{I}_{\Sigma/X} \otimes \mathcal{F}$ is a proper subsheaf of $\mathcal{F}_\Sigma^c$.

To sum up, surfaces in $X$ given by $V(s \wedge s')$, where $[s] \in \mathcal{V}_b(\mathcal{F})$ and $s' \in H^0(X, \mathcal{F}_\Sigma^c)$, are certainly singular along $\Sigma$ if the zero-locus $V(s')$ passes there; however, they can be also singular along $\Sigma$ even if $V(s')$ does not pass there. This happens since $C = V(s)$ is singular along $\Sigma$ and when (4.6) holds.
There is also a "divisorial" interpretation of the fact that $H^0(X, j_{\Sigma/X} \otimes F)$ does not span the whole $H^0(X, \mathcal{F}^\Sigma)$. From the correspondence of Theorem 3.11 and from what stated in Corollary 3.11 we know that the general section $s' \in H^0(X, \mathcal{F}^\Sigma)$ corresponds to a divisor $G_{s'}$ in the tautological linear system $|0_\mathcal{P}(1)|$ on $\mathcal{P}$, which simply passes through the scheme $\Sigma^1$ of $\delta$ rational double points of the divisor $G_s \in |0_\mathcal{P}(1)|$, where $G_s$ corresponds to the given section $[s] \in \mathcal{V}_b(\mathcal{F})$ we started with. Recall that $G_s$ instead contains the whole $\pi$-fibres over $\Sigma$ and that it is singular along $\Sigma^1$.

Thus, if one considers $s^* \in H^0(X, j_{\Sigma/X} \otimes \mathcal{F})$, such an element corresponds to a divisor $G_{s^*} \in |j_{\Lambda/\mathcal{P}} \otimes 0_\mathcal{P}(1)|$, where $\Lambda = \bigcup_{i \in \Sigma} L_{p_i}$. In this case, we have

\[
0 \to j_{\Lambda/\mathcal{P}} \otimes 0_\mathcal{P}(1) \to 0_\mathcal{P}(1) \to 0_\mathcal{P}(1) \otimes 0_\Lambda \cong \bigoplus_{i=1}^\delta 0_{L_{p_i}}(1) \to 0,
\]

where $\Sigma = \{p_1, \ldots, p_\delta\}$. Observe that

\[
0_{L_{p_i}}(1) \cong 0_{p_1}(1), \quad \forall 1 \leq i \leq \delta;
\]

obviously, $|j_{\Lambda/\mathcal{P}} \otimes 0_\mathcal{P}(1)|$ is properly contained in $|j_{\Sigma^1/\mathcal{P}} \otimes 0_\mathcal{P}(1)|$, since the general element of the latter linear system simply passes through $\Sigma^1$ but does not contain the whole scheme $\Lambda$ as each element of $|j_{\Lambda/\mathcal{P}} \otimes 0_\mathcal{P}(1)|$ does. Therefore, $|j_{\Sigma^1/\mathcal{P}} \otimes 0_\mathcal{P}(1)|$ has expected codimension equal to $2\delta$ in $|0_\mathcal{P}(1)|$ as we found by using (4.11).

For completeness sake, we conclude by observing that, in this correspondence, the subsheaf

\[
j_{\mathcal{C}/X} \otimes \mathcal{F} \subset \mathcal{F}^\Sigma
\]
gives global sections which are related to divisors on $\mathcal{P}$ belonging to $|j_{\mathcal{F}/\mathcal{P}} \otimes 0_\mathcal{P}(1)|$, where $\mathbb{P}^1_C$ is the ruled surface contained in $\mathcal{P}$ with $\pi$-fibres over the base curve $C$.

5. Connection among various singular subschemes of $X$ and of $\mathcal{P}$

The aim of this section is to study in details the deep connection among:

- a nodal section $[s] \in \mathcal{V}_b(\mathcal{F})$ on $X$,
- the corresponding singular divisor $G_s$ in $|0_\mathcal{P}(1)|$ on $\mathcal{P}$ having $\delta$ rational double points over the nodes of $C = V(s)$,
- for any $s' \in H^0(X, \mathcal{F}^\Sigma) \setminus [s]$, the singular surface $V(s \cap s') \subset X$ belonging to the linear system $|j_{\mathcal{C}/X} \otimes \mathcal{L}|$ on $X$,
- the singular "complete intersection" surface in $\mathcal{P}$, $S_{s,s'} := G_s \cap G_{s'}$, dominating $V(s \cap s')$ (cf. Remark 5.9).

As a consequence of the analysis given in Sections 3.4 we have the following:

**Theorem 5.1.** Let $X$ be a smooth projective threefold. Let $\mathcal{F}$ be a globally generated rank-two vector bundle on $X$ and let $\delta$ be a positive integer. Fix $[s] \in \mathcal{V}_b(\mathcal{F})$ and let $C = V(s)$ be the corresponding nodal curve on $X$. Denote by $\Sigma$ the set of nodes of $C$.

Let $\mathcal{P} := \mathbb{P}_{X} (\mathcal{F})$ be the projective space bundle, $0_\mathcal{P}(1)$ be its tautological line bundle and $\Sigma^1 \subset \mathcal{P}$ be the zero-dimensional scheme of $\delta$-rational double points of the divisor $G_s \in |0_\mathcal{P}(1)|$ which corresponds to $s \in H^0(X, \mathcal{F})$ (cf. Theorem 3.7). Let $\mathcal{F}^\Sigma$ be the subsheaf of $\mathcal{F}$ defined as in Proposition 3.7.

Then, the following conditions are equivalent:

(i) $s' \in H^0(X, \mathcal{F}^\Sigma) \setminus [s]$;

(ii) $V(s \cap s') \subset X$ is a surface which contains $C$ and which is singular along $\Sigma$;
(iii) the divisor \( G_{s'} \) passes through \( \Sigma^1 \).
(iv) the surface \( S_{s,s'} := G_s \cap G_{s'} \subset \mathcal{P} \) is singular along \( \Sigma^1 \);

Proof. Some of the implications are already proved in the previous results. Indeed:

(i) \( \iff \) (ii): this has already been proved in Proposition 4.2.

(ii) \( \iff \) (iii): By the very definition of \( \mathcal{F}_{\Sigma} \), it is obvious that (iii) is equivalent to

(i); therefore, from the step above we would have finished. Anyhow, we give a direct proof of the equivalence of this two conditions, since it highlights the strict connection between the two apparently different approaches and it allows to give another interpretation of the equivalence in (4.3).

To this aim, let \( [s] \in \mathcal{V}_\delta(\mathcal{F}) \) and let \( G_s \) be the corresponding divisor in \( \mathcal{P} \) which is singular along \( \Sigma^1 \). As usual, take \( p \in \Sigma \), \( U = U_p \) an open subset such that \( U \cap (\Sigma \setminus \{p\}) = \emptyset \), where \( \mathcal{F} \) trivializes and whose local coordinates are \( x = (x_1, x_2, x_3) \), such that \( \mathcal{F}(p) = 0 \in U \cong \mathbb{A}^3 \). If \( s|_U = (f_1, f_2) \), we recall that the local equation of \( G_s \) in \( \pi^{-1}(U) \) is

\[
F(x_1, x_2, x_3, u, v) := uf_1 + vf_2
\]

Therefore, we have a singular point on \( G_s \) in \( \pi^{-1}(U) \) if, and only if, there exists a solution of

\[
F = \frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial x_3} = \frac{\partial F}{\partial u} = \frac{\partial F}{\partial v} = 0. \tag{5.2}
\]

Observe that (5.2) is equivalent to

\[
uf_1 + vf_2 = \frac{\partial f_1}{\partial x_1}u + \frac{\partial f_2}{\partial x_1}v = \frac{\partial f_1}{\partial x_2}u + \frac{\partial f_2}{\partial x_2}v = \frac{\partial f_1}{\partial x_3}u + \frac{\partial f_2}{\partial x_3}v = f_1 = f_2 = 0. \tag{5.3}
\]

By the last two equations, we find as in Theorem 3.1 that the singular point of \( G_s \) must be on the \( \pi \)-fibre over a point of \( C \). Let \( q \in U \cap C \) be such a point and let \( L = L_q \) be this fibre.

We can restrict the system (5.3) to \( L \). We thus get:

\[
\frac{\partial f_1}{\partial x_1}(q)u + \frac{\partial f_2}{\partial x_1}(q)v = \frac{\partial f_1}{\partial x_2}(q)u + \frac{\partial f_2}{\partial x_2}(q)v = \frac{\partial f_1}{\partial x_3}(q)u + \frac{\partial f_2}{\partial x_3}(q)v = 0. \tag{5.4}
\]

Therefore, there exists a solution \( [u, v] \in L \cong \mathbb{P}^1 \) if, and only if, (5.4) has rank less than or equal to one. This is equivalent to saying that

\[
\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(q) \\
\frac{\partial f_1}{\partial x_2}(q) \\
\frac{\partial f_1}{\partial x_3}(q) \end{pmatrix} = \lambda \begin{pmatrix} \frac{\partial f_2}{\partial x_1}(q) \\
\frac{\partial f_2}{\partial x_2}(q) \\
\frac{\partial f_2}{\partial x_3}(q) \end{pmatrix}, \tag{5.5}
\]

for some \( \lambda \in \mathbb{C}^* \); this is equivalent to:

\[
\frac{\partial f_1}{\partial x_1}(q) \frac{\partial f_2}{\partial x_2}(q) - \frac{\partial f_2}{\partial x_1}(q) \frac{\partial f_1}{\partial x_2}(q) = \frac{\partial f_1}{\partial x_2}(q) \frac{\partial f_2}{\partial x_3}(q) - \frac{\partial f_2}{\partial x_2}(q) \frac{\partial f_1}{\partial x_3}(q) = \frac{\partial f_1}{\partial x_3}(q) \frac{\partial f_2}{\partial x_1}(q) - \frac{\partial f_2}{\partial x_3}(q) \frac{\partial f_1}{\partial x_1}(q) = 0. \tag{5.6}
\]

In such a case, we have:

\[
[u, v] = [-\frac{\partial f_2}{\partial x_1}(q), \frac{\partial f_2}{\partial x_2}(q)] = [-\frac{\partial f_2}{\partial x_2}(q), \frac{\partial f_1}{\partial x_2}(q)]
= [-\frac{\partial f_2}{\partial x_3}(q), \frac{\partial f_1}{\partial x_3}(q)] = [-\lambda, 1]. \tag{5.7}
\]
Therefore, \( G_s \) is singular at \([-\lambda, 1] \in L \) if, and only if, \( C = V(s) \) has a node at \( q \in U \). This implies that \( q = p \), since \( p \) was the only node in \( U \) by assumption; thus \( L = L_p \) and once again - the singularities of \( G_s \) are on the \( \pi \)-fibres of the nodes of \( C \).

Let now \( s' \in H^0(X, F^\Sigma) \). Assume that \( s'|_U = (g_1, g_2) \), for some \( g_1, g_2 \in \mathcal{O}_X(U) \). Then, \( G_{s'} \) passes through the singular point of \( G_s \) along \( L_p \), i.e. \([-\lambda, 1]\) if, and only if,

\[
[-g_2(p), g_1(p)] = [-\lambda, 1].
\]

This means that \([-g_2(p), g_1(p)] \) is a solution of (5.4) (where \( q = p \), as proved above). This is exactly equivalent to (4.6) and so to the fact that the surface \( V(s \wedge s') \) is singular at \( p \).

\((iii) \Leftrightarrow (iv)\): trivial consequence of the fact that \( G_s \) is always singular at \( \Sigma^1 \).

\[\square\]

**Remark 5.9.** The proof of the equivalence of conditions \((ii)\) and \((iii)\) in Theorem 5.1 shows that the local computations on \( X \) in Proposition 4.2 are exactly equivalent to the local computations on \( P \) via the correspondence introduced in Theorem 3.1 and Proposition 3.4.

In particular, from (5.7) and (5.8), it follows that

\[
[-g_2(p), g_1(p)] = [-\partial f_2/\partial x_i(q), \partial f_2/\partial x_i(q)], \text{ for each } 1 \leq i \leq 3.
\]

This means that \((g_1(p), g_2(p))\) linearly depends on each pair \((\partial f_2/\partial x_i(q), \partial f_2/\partial x_i(q))\), \(1 \leq i \leq 3\), as already obtained in (4.9).

It is interesting to give also a direct proof of the equivalence of conditions \((ii)\) and \((iv)\), in order to relate it with what observed in Remark 4.10. Thus:

\((ii) \Rightarrow (iv)\): Since the computations are local, we can fix one node \( p \in \Sigma \). Since \( V(s \wedge s') \) is singular along \( \Sigma \) by assumption, from (4.8) it follows that either \( s' \) passes through \( p \) or \( s'(p) \) is proportional to each pair as in (4.9) evaluated at \( p \). In the former case, we have that \( G_{s'} \in |\mathcal{I}_{L_p/P} \otimes \mathcal{O}_P(1)| \), where \( L_p \) is the \( \pi \)-fibre over \( p \in \Sigma \); in the latter case, by the very definition of \( F^\Sigma \) and by (4.4) and (4.5), we have that \( G_{s'} \in |\mathcal{I}_{p} \otimes \mathcal{O}_P(1)| \), where \( p' \in \Sigma^1 \) is the corresponding point to \( p \in \Sigma \). In any case, \( G_{s'} \) passes through \( p^1 \).

If we globalize this approach, in any case, \( G_{s'} \) passes through \( \Sigma^1 \). Now, since \( \Sigma^1 \subseteq \text{Sing}(G_s) \), then it obviously follows that \( S_{s,s'} := G_s \cap G_{s'} \) is singular along \( \Sigma^1 \).

\((iv) \Rightarrow (ii)\): Once again, we can focus on one of the nodes in \( \Sigma \), say \( p \). Take \( U = U_p \) an open neighborhood of \( p \) in \( X \) where \( F \) trivializes. Then, we can assume that the local expression of \( s \) in \( U \) is \( s|_U = (f_1, f_2) \), where \( f_1, f_2 \in \mathcal{O}_X(U) \). Denote by \( m_p \) the maximal ideal of the point \( p \) in the stalk \( \mathcal{O}_{X,p} \). Since by assumption \( [s] \in \mathcal{V}_0'(\mathcal{F}) \) and \( p \in \Sigma \), we can assume that the reduction of \( s \) in \( \mathcal{F} \otimes (\mathcal{O}_{X,p}/m_p^2) \) is \((1, 0)\). This means that if we consider homogeneous coordinates \([u, v]\) on the \( \pi \)-fibre \( L_p \cong \mathbb{P}^1 \) over \( p \), the corresponding rational double point \( p^1 \) for \( G_s \) on \( L_p \) has coordinates \([0, 1]\) on such a line.

Similarly, take \( s'|_U = (g_1, g_2) \), where \( g_1, g_2 \in \mathcal{O}_X(U) \). Since by assumption \( S_{s,s'} := G_s \cap G_{s'} \) is singular at \( p^1 = [0, 1] \), in particular \( G_{s'} \) passes through \( p^1 \). Therefore, we can assume that the reduction of \( s' \) in \( \mathcal{F} \otimes (\mathcal{O}_{X,p}/m_p) \) is \((a, 0)\). If \( a = 0 \), this means that \( G_{s'} \) contains \( L_p \); otherwise, as in (2.2), the local equation of \( G_{s'} \) is given by \( \{au = 0\} \), so that the intersection point between \( G_{s'} \) and \( L_p \) is indeed \( p^1 = [0, 1] \).
In any case, we have that \( g_2 \in \mathbb{m}_p \) and \( g_1 = a + j_1 \), where \( j_1 \in \mathbb{m}_p \). Analogously, we have that \( f_1 \in \mathbb{m}_p \) and \( f_2 \in \mathbb{m}_p^2 \). Therefore,

\[
\det \begin{pmatrix} f_1 & f_2 \\ a + j_1 & g_2 \end{pmatrix} = f_1 g_2 - f_2(a + j_1) \in \mathbb{m}_p^2.
\]

This implies that \( V(s \wedge s') \) is singular at \( p \in \Sigma \).

Recall that, in Remark 3.10 we observed that surfaces in \( X \) given by \( V(s \wedge s') \), with \( s' \in H^0(X, F^\Sigma) \) are certainly singular along \( \Sigma \) if the zero-locus \( V(s') \) passes there; however, they can be also singular along \( \Sigma \) even if \( V(s') \) does not pass there, precisely when \( (4.6) \) holds at each point of \( \Sigma \). From the correspondence between \( V(s \wedge s') \) and \( S_{s,s'} \) we see that, in the former case, the surface \( S_{s,s'} \) has to contain \( \Lambda = \pi^{-1}(\Sigma) = \bigcup_{i=1}^j L_{p_i} \), whereas in the latter, \( S_{s,s'} \) has to pass through the point \( p_i \in L_{p_i} \), for each \( p_i \in \Sigma \), which is singular for \( G_s \) so - a fortiori - for \( S_{s,s'} \).

In any case, differently from \( V(s \wedge s') \), the surface \( S_{s,s'} \) always contains \( \Sigma^1 \) if \( s' \in H^0(X, F^\Sigma) \). Notice also that \( S_{s,s'} \) is a "complete intersection" in \( P \) which dominates \( V(s \wedge s') \). Indeed, if \( s' \in H^0(X, F^\Sigma) \) is such that either \( V(s') \cap \Sigma \neq \emptyset \) or \( V(s') \cap C \neq \emptyset \) then, as above, the \( \pi \)-fibres over these intersection schemes are properly contained in \( S_{s,s'} \). In particular, if \( s' \) is the general section in \( H^0(X, F^\Sigma) \) such that \( V(s') \cap C = \emptyset \) then, by the Zariski Main Theorem, \( S_{s,s'} \) is isomorphic via \( \pi \) to the surface \( V(s \wedge s') \).

Observe that by Theorem 5.1 we can improve the statement of Corollary 3.11.

**Corollary 5.11.** \( s + cz \in \mathcal{T}[s](\mathcal{V}_\delta(F)) \), s.t. \( c^2 = 0 \) \( \iff V(s \wedge s') \) is singular along \( \Sigma \) \( \iff S_{s,s'} \in |j_1| \cap \mathcal{O}_P(1) \) \( \iff S_{s,s'} := G_s \cap G_{s'} \subset P \) is singular along \( \Sigma^1 \).

**Remark 5.12.** By using Theorem 5.1 we can also give another interpretation of the equivalence of regularity conditions in Corollary 3.12. Recall that the map \( \rho_\mathcal{P} \) in \( (3.13) \) is a standard restriction map.

Therefore, \( |\mathcal{O}_\mathcal{P}(1)| \) does not separate \( \Sigma^1 \) if, and only if, each divisor in \( |\mathcal{O}_\mathcal{P}(1)| \) passing through all but one point \( p_j \) of \( \Sigma^1 \) passes also through the point \( p_j \), for some \( 1 \leq j \leq \delta \). By Theorems 3.1 and 5.1 and by Proposition 3.4 this happens if, and only if, for each \( [s] \in \mathcal{V}_\delta(F) \) and for each \( s' \in H^0(X, F^\Sigma) \), the surface \( S_{s,s'} = G_s \cap G_{s'} \) which is singular along all but one point \( p_j \) of \( \Sigma^1 \) is singular also at the remaining point \( p_j \), for some \( 1 \leq j \leq \delta \). This happens if, and only if, for each \( [s] \in \mathcal{V}_\delta(F) \) and for each \( s' \in H^0(X, F^\Sigma) \), the surface \( V(s \wedge s') \subset X \) which is singular along all but one point \( p_j \) of \( \Sigma \) is singular also at the remaining point \( p_j \), for some \( 1 \leq j \leq \delta \); this is equivalent to the non-surjectivity of the map \( \mu_X \), since the section \( s' \in H^0(X, F) \) which vanishes in the composition \( F \to F|_C \to \mathcal{O}_{\Sigma \setminus \{p_j\}} \) also vanishes in the composition \( F \to F|_C \to \mathcal{O}_{\{p_j\}} \).

To sum up, the regularity condition for points of the scheme \( \mathcal{V}_\delta(F) \) on \( X \) not only translates into the independence of conditions imposed by a 0-dimensional scheme on the tautological linear system \( |\mathcal{O}_\mathcal{P}(1)| \) on \( \mathcal{P} \), but also to the independence of singularity conditions imposed on suitable families of singular surfaces in \( X \) as well as in \( \mathcal{P} \).

6. \( \mathcal{P}_3 \)-SEVERI VARIETIES OF SINGULAR DIVISORS ON \( \mathcal{P} \)

The aim of this section is to study families of singular divisors in \( \mathcal{P} \), which are related to Severi varieties \( \mathcal{V}_\delta(F) \) of nodal sections of \( F \) on \( X \).
Definition 6.1. Given \( \mathcal{P} \) and \( \delta \) as in Assumption \[ \text{Definition 6.1.} \] Theorem 3.1 and Proposition 3.4 consider the scheme

\[
R_\delta(\mathcal{O}_\mathcal{P}(1)) := \{ G_s \in |\mathcal{O}_\mathcal{P}(1)| \text{ s.t. } [s] \in \mathcal{V}_\delta(\mathcal{F}) \}.
\]

These schemes parametrize families of divisors in the tautological linear system of \( \mathcal{P} \), \( |\mathcal{O}_\mathcal{P}(1)| \), which are irreducible and have only \( \delta \) rational double points as the only singularities. For brevity sake, these will be called \( \mathcal{P}_\delta \)-Severi varieties.

By the above definition and by Assumption \[ \text{Assumption 5.1.} \] from now on we consider \( \delta \leq \min\{h^0(\mathcal{F})-1, \frac{1}{2}\deg(\mathcal{L} \otimes \omega_X \otimes \mathcal{O}_D) + 1\} \) and \( R_\delta(\mathcal{O}_\mathcal{P}(1)) \neq \emptyset \).

It is clear that:

\[
exdim(R_\delta(\mathcal{O}_\mathcal{P}(1))) = \dim(|\mathcal{O}_\mathcal{P}(1)|) - \delta;
\]

indeed, imposing a rational double point gives at most 5 conditions on \( |\mathcal{O}_\mathcal{P}(1)| \); each such point varies on any of the \( \pi \)-fibre over \( X \).

As in Definition 2.6 from \( (6.3) \) it is natural to give the following:

Definition 6.4. Let \([G_s] \in R_\delta(\mathcal{O}_\mathcal{P}(1))\). Then \([G_s]\) is said to be a regular point of \( R_\delta(\mathcal{O}_\mathcal{P}(1)) \) if:

(i) \([G_s] \in R_\delta(\mathcal{O}_\mathcal{P}(1))\) is a smooth point, and

(ii) \(\dim_{[G_s]}(R_\delta(\mathcal{O}_\mathcal{P}(1))) = \exdim(R_\delta(\mathcal{O}_\mathcal{P}(1))) = \dim(|\mathcal{O}_\mathcal{P}(1)|) - \delta.\)

The \( \mathcal{P}_\delta \)-Severi variety \( R_\delta(\mathcal{O}_\mathcal{P}(1)) \) is said to be regular if it is regular at each point.

As in Theorem 3.1 in order to find regularity conditions for a given point \([G_s] \in R_\delta(\mathcal{O}_\mathcal{P}(1))\), it is crucial to give a description of the tangent space at \([G_s]\) to the given \( \mathcal{P}_\delta \)-Severi variety.

Theorem 6.5. Let \([G_s] \in R_\delta(\mathcal{O}_\mathcal{P}(1))\) on \( \mathcal{P} \) and let \( \Sigma^1 \) be the zero-dimensional scheme of the \( \delta \)-rational double points of \( G_s \subset \mathcal{P} \). Then, we have:

\[
T_{[G_s]}(R_\delta(\mathcal{O}_\mathcal{P}(1))) \cong \frac{H^0(\mathcal{O}_{\Sigma^1/P} \otimes \mathcal{O}_\mathcal{P}(1))}{\langle G_s \rangle}.
\]

In particular, if \( \epsilon \in \mathbb{C}[T]/(T^2) \), then:

\[
G_s + \epsilon G_r \in T_{[G_s]}(R_\delta(\mathcal{O}_\mathcal{P}(1))) \Leftrightarrow G_r \in |\mathcal{O}_\mathcal{P}(1)| \text{ and } \Sigma^1 \subset G_r.
\]

Proof. The divisor \( G_s \subset \mathcal{P} \), related to the point \([G_s] \in R_\delta(\mathcal{O}_\mathcal{P}(1))\), corresponds to a section \([s] \in \mathcal{V}_\delta(\mathcal{F})\) on \( X \).

Therefore, as in the proof of Theorem 3.1 we may locally work around a node of \( C = V(s) \subset X \). Let \( p \in \Sigma = \text{Sing}(C) \) be such a node and let \( U = U_p \subset X \) be an affine open set containing \( p \), where the vector bundle \( \mathcal{F} \) trivializes. We thus can choose local coordinates \( x = (x_1, x_2, x_3) \) on \( U \cong \mathbb{A}^3 \) and homogeneous coordinates \([u,v] \in \mathbb{P}^1\), such that \( \varphi(p) = (0,0,0) \), \( s|_U = (x_1 x_2, x_3) \) and the local equation of \( G_s \) in \( \pi^{-1}(U) \cong U_p \times \mathbb{P}^1 \) is given by \( u x_1 x_2 + v x_3 = 0 \) (cf. \( (3.2) \)).

Recall that in the open chart where \( v \neq 0 \), \( G_s \) is smooth, whereas, in the open chart where \( u \neq 0 \), we see that the local equation of \( G_s \) in \( \mathbb{A}^3 \times \mathbb{A}^1 \cong \mathbb{A}^4 \) is

\[
G_s = V(x_1 x_2 + x_3 t), \text{ where } t = \frac{v}{u}.
\]

This is the equation of a quadric cone in \( \mathbb{A}^4 \) having vertex at the origin of the \( \mathbb{A}^4 \) having coordinates \((x_1, x_2, x_3, t)\).
We can consider the Jacobian map of $G_s$ in this $\mathbb{A}^4$. This is given by:
\[
\begin{align*}
\mathcal{T}_{\mathbb{A}^4|G_s} & \xrightarrow{J_{G_s}} \mathcal{N}_{G_s/\mathbb{A}^4} \\
\partial/\partial x_1 & \mapsto x_2 \\
\partial/\partial x_2 & \mapsto x_1 \\
\partial/\partial x_3 & \mapsto t \\
\partial/\partial t & \mapsto x_3,
\end{align*}
\]
where $\mathcal{N}_{G_s/\mathbb{A}^4}$ is locally free of rank one on $G_s$. It is then clear that $J_{G_s}$ is surjective except at the origin $0 = (0, 0, 0, 0)$.

By the local computations we analytically get:
\[
coker(J_{G_s}) \cong \mathbb{C}[[x_1, x_2, x_3, t]]/(x_1x_2 + x_3t) \cong \mathbb{C};
\]
Globally speaking, given $G_s \subset \mathcal{P}$, whose singular scheme is $\Sigma^1$, we have the exact sequence of sheaves on $G_s$:
\[
(6.8) \quad \mathcal{T}_{\mathcal{P}|G_s} \xrightarrow{J_{G_s}} \mathcal{N}_{G_s/\mathcal{P}} \to T^1_{G_s} \to 0,
\]
where $T^1_{G_s}$ is a sky-scraper sheaf supported on $\Sigma^1$ and of rank one at each point.

As in (3.3) for nodal curves, denote by $\mathcal{N}'_{G_s}$ the kernel of $J_{G_s}$ in (6.8). This is the so-called equisingular sheaf, whose global sections give equisingular first-order deformations of $G_s$ in $\mathcal{P}$.

By standard exact sequences, one sees that there is an injection
\[
H^0(\mathcal{P}, J_{\Sigma^1/\mathcal{P}} \otimes \mathcal{O}_\mathcal{P}(1)) \hookrightarrow H^0(G_s, \mathcal{N}'_{G_s}),
\]
which is an isomorphism when $\mathcal{P}$ - equivalently $X$ - is regular, i.e. $h^1(\mathcal{P}, \mathcal{O}_\mathcal{P}) = 0$. Therefore, the vector space on the left-hand-side of the injection actually parametrizes equisingular first-order deformations of $G_s$ in $|\mathcal{O}_\mathcal{P}(1)|$. □

**Remark 6.9.** Recall that when one studies classical Severi varieties of irreducible, $\delta$-nodal curves on a smooth projective surface, there is also a parametric approach for equisingular first-order deformations (cf., e.g [4] and [22]).

Precisely, let $S$ be an arbitrary smooth, projective surface, $|D|$ be a complete linear system on $S$, whose general element is assumed to be a smooth and irreducible curve, which is a divisor on $S$; one considers the Severi variety $V_{|D|,\delta} \subset |D|$, for any $0 \leq \delta \leq p_a(D)$, which parametrizes reduced, irreducible curves in $|D|$ having $\delta$-nodes as the only singularities. If $[C] \in V_{|D|,\delta}$, this point corresponds to a curve $C \sim D$ on $S$, such that $N := Sing(C) \subset S$ is the 0-dimensional scheme of its $\delta$ nodes and one can consider:
\[
(6.10) \quad \tilde{C} \subset \tilde{S} \quad \downarrow \varphi_N \quad \downarrow \mu_N \quad C \subset S,
\]
where
- $\mu_N$ is the blow-up of $S$ along $N$,
- $\varphi_N$ is the normalization of $X$,
- $\tilde{C}$ is a smooth curve of (geometric) genus $g = g(\tilde{C}) = p_a(D) - \delta$.  

It is a standard result that \( T_{[X]}(V[\delta]) \cong H^0(S, \mathcal{N}_{\varphi_N}/\mathcal{S}) \) is isomorphic to a (proper) subspace of \( H^0(\mathcal{N}_{\varphi_N}) \), where \( \mathcal{N}_{\varphi_N} \) is the normal bundle to map \( \varphi_N \), which is the line bundle on the smooth curve \( \tilde{C} \) defined by:
\[
0 \to T_{\tilde{C}} \to \varphi^*(\mathcal{S}) \to \mathcal{N}_{\varphi_N} \to 0.
\]
It is well-known that \( H^0(\tilde{C}, \mathcal{N}_{\varphi_N}) \) parametrizes equisingular first-order deformations of \( C \) in \( S \) and that the subspace mentioned above coincides with the whole space when \( S \) is a regular surface.

Therefore, for irreducible nodal curves on surfaces, the parametric approach coincides with the Cartesian approach, which makes use of the equisingular sheaf \( \mathcal{N}'_{\widetilde{S}} \) defined by
\[
0 \to \mathcal{N}'_{\widetilde{S}} \to \mathcal{N}_{X/S} \to T_{X} \to 0.
\]
Indeed, in this case (and only in this case) one has \( \mathcal{N}'_{\widetilde{S}} \cong \varphi^*(\mathcal{N}_{\varphi_N}) \).

Even if the elements of \( R_{\delta}(\mathcal{O}_P(1)) \) are divisors in \( P \) - as curves on surfaces - the same does not occur for these families. For simplicity, assume that \( X \) - and so \( P \) - is a regular threefold, i.e. \( h^1(X, \mathcal{O}_X) = 0 \). Let
\[
\mu_{\Sigma^1} : \tilde{P} \to P
\]
be the blow-up of \( P \) along \( \Sigma^1 \) and
\[
\varphi_{\Sigma^1} : \tilde{G}_s \to G_s
\]
the desingularization of \( G_s \), which is induced by \( \mu_{\Sigma^1} \), by a diagram similar to the one in (6.10), and by the fact that \( \Sigma^1 \) is a scheme of ordinary double points for \( G_s \). Let \( B := \Sigma_{i=1}^\delta E_i \) be the \( \mu_{\Sigma^1} \)-exceptional divisor. Thus,
\[
\mu^*_{\Sigma^1}(G_s) = \tilde{G}_s + 2B, \quad \mu^*_{\Sigma^1}(K_{\tilde{P}}) = K_{\tilde{P}} - 3B.
\]
By the exact sequence:
\[
0 \to T_{\tilde{G}_s} \to \varphi^*_{\Sigma^1}(T_{\tilde{P}}) \to \mathcal{N}_{\varphi_{\Sigma^1}} \to 0
\]
and by the adjunction formula on \( \tilde{P} \), we get that:
\[
(6.11) \quad \mathcal{N}_{\varphi_{\Sigma^1}} \cong \mathcal{O}_{\tilde{G}_s}(\mu^*_{\Sigma^1}(G_s) + B).
\]
Tensoring by \( \mathcal{O}_{\tilde{G}_s}(\mu^*_{\Sigma^1}(G_s) + B) \) the exact sequence
\[
0 \to \mathcal{O}_{\tilde{P}}(-\mu^*_{\Sigma^1}(G_s) + 2B) \to \mathcal{O}_{\tilde{P}} \to \mathcal{O}_{\tilde{G}_s} \to 0,
\]
by the regularity of \( P \) and by Fujita’s Lemma, we see that \( H^0(\mathcal{N}_{\varphi_{\Sigma^1}}) \) is not isomorphic to \( H^0(\mathcal{N}'_{\tilde{G}_s}) \), i.e. the first-order deformations given by general vectors in \( H^0(\mathcal{N}_{\varphi_{\Sigma^1}}) \) are not equisingular.

To conclude the section, as in (3.13), by
\[
\rho_P : H^0(P, \mathcal{O}_P(1)) \to H^0(P, \mathcal{O}_{\Sigma^1})
\]
the natural restriction map. Then, from Theorem 6.5, it immediately follows:

**Corollary 6.12.** With assumptions and notation as in Theorem 6.5, we have:

\[
[G_s] \in R_\delta(\mathcal{O}_P(1)) \text{ a regular point} \iff \rho_P \text{ is surjective}
\]
\[
\iff [s] \in V_\delta(\mathcal{F}) \text{ is a regular point (in the sense of Definition 2.6).}
\]
Proof. The first equivalence is a direct consequence of (6.3) and Theorem 6.5. The other follows from Corollary 6.12. □

7. SOME UNIFORM REGULARITY RESULTS FOR $V_\delta(F)$ AND $R_\delta(\mathcal{O}_P(1))$

In this section we first improve some regularity results of [12] for Severi varieties $V_\delta(F)$ of irreducible, $\delta$-nodal sections of $X$, then we use Corollary 6.12 to deduce regularity results also for $P_\delta$-Severi varieties $R_\delta(\mathcal{O}_P(1))$ of irreducible divisors in $|\mathcal{O}_P(1)|$ on $P$. We find upper-bounds on the number $\delta$ of singular points which ensure the regularity of $V_\delta(F)$ as well as of $R_\delta(\mathcal{O}_P(1))$; these upper-bounds are shown to be almost sharp (cf. Remark 7.4).

What we want to stress is the following fact: even if the regularity of the schemes $R_\delta(\mathcal{O}_P(1))$ on $P$ is defined by means of separation of suitable zero-dimensional schemes by the linear system $|\mathcal{O}_P(1)|$ on the fourfold $P$, one can avoid to consider this intricate situation. Indeed, it is well-known how difficult is to enstablish separation of points in projective varieties of dimension greater than or equal to three (cf. e.g. [1], [9] and [18]). In general some results can be found by using the technical tools of multiplier ideals together the Nadel and the Kawamata-Viehweg vanishing theorems (see, e.g. [8], for an overview). Anyhow, sometimes no answers are given by using these techniques.

In our situation, thanks to the correspondence between $V_\delta(F)$ on $X$ and $R_\delta(\mathcal{O}_P(1))$ on $P$, we deduce regularity conditions for the scheme $R_\delta(\mathcal{O}_P(1))$ from those of the scheme $V_\delta(F)$.

From now on, let $X$ be a smooth projective threefold, $E$ be a globally generated rank-two vector bundle on $X$, $M$ be a very ample line bundle on $X$ and $k \geq 0$, $\delta > 0$ be integers. With notation and assumptions as in Section 2, we shall always take

$$F = E \otimes M^{\otimes k}$$

and consider the scheme $V_\delta(E \otimes M^{\otimes k})$ on $X$.

By using Theorem 3.1 and Corollary 3.12 here we determine conditions on the vector bundle $E$ and on the integer $k$ and uniform upper-bounds on the number of nodes $\delta$ implying that each point of $V_\delta(E \otimes M^{\otimes k})$ is regular. Indeed, the next result is a generalization of Theorem 4.5 in [12].

**Theorem 7.1.** Let $X$ be a smooth projective threefold, $E$ be a globally generated rank-two vector bundle on $X$, $M$ be a very ample line bundle on $X$ and $k \geq 0$ and $\delta > 0$ be integers. If

$$\delta \leq k + 1,$$

then $V_\delta(E \otimes M^{\otimes k})$ is regular.

**Proof.** If $k = 0$, then we consider $\delta = 1$; therefore, by the hypothesis on $E$, it follows that

$$H^0(E) \to H^0(\mathcal{O}_p^{\otimes 2})$$

is surjective, for each $p \in X$. By the description of the map $\mu_X$ in Remark 3.14, this implies that $V_1(E)$ is regular at each point.

When $k > 0$, first of all observe that, since $M$ is very ample, then $M^{\otimes k}$ separates any set $\Sigma$ of $\delta$ distinct point of $X$ with $\delta \leq k + 1$. This is equivalent to saying that the restriction map

$$\rho_k : H^0(X, M^{\otimes k}) \to H^0(\mathcal{O}_\Sigma)$$

is surjective, for each such $\Sigma \subset X$. Thus, (7.2) implies there exist global sections $\sigma_1, \ldots, \sigma_\delta \in H^0(X, M^{\otimes k})$ s. t.

$$\sigma_i(p_j) = 0 \in \mathbb{C}^\delta, \text{ if } i \neq j, \text{ and } \sigma_i(p_i) = (0, \ldots, 1, \ldots, 0), 1 \leq i \leq \delta.$$ 

On the other hand, since $\mathcal{E}$ is globally generated on $X$, the evaluation morphism

$$H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \overset{ev}{\rightarrow} \mathcal{E}$$

is surjective. This means that, for each $p \in X$, there exist global sections $s^{(p)}_1, s^{(p)}_2 \in H^0(X, \mathcal{E})$ such that

$$s_1^{(p)}(p) = (1, 0), \quad s_2^{(p)}(p) = (0, 1) \in \mathcal{O}_{X,p}^{\oplus 2}.$$ 

Therefore, it immediately follows that

$$H^0(X, \mathcal{E} \otimes M^{\otimes k}) \rightarrow H^0(\mathcal{O}_{\Sigma}^{\oplus 2}) \cong \mathbb{C}^{2\delta}. $$

If we compose with diagram (3.9), we get:

$$
\begin{array}{ccc}
H^0(\mathcal{E} \otimes M^{\otimes k}) & \rightarrow & H^0(\mathcal{O}_{\Sigma}^{\oplus 2}) \cong \mathbb{C}^{2\delta} \\
\downarrow \mu_X & & \downarrow \\
H^0(\mathcal{O}_{\Sigma}) & \rightarrow & H^0(\mathcal{O}_{\Sigma}) \cong \mathbb{C}^{\delta}. \\
\end{array}
$$

Thus $\mu_X$ is surjective (cf. Remark [3.14]). By (5.13), one can conclude.

\begin{remark}
Observe that the bound (7.2) is uniform, i.e. it does not depend on the postulation of nodes of the curves which are zero-loci of sections parametrized by $V_\delta(\mathcal{E} \otimes M^{\otimes k})$. We remark that Theorem 7.1 improves our Theorem 4.5 in [12]. Both these results generalize what proved by Ballico and Chiantini in [2] mainly because our approach more generally holds for families of nodal curves on smooth projective threefolds but also because, even in the case of $X = \mathbb{P}^3$, main subject of [2], our regularity results are effective and not asymptotic as Proposition 3.1 in [2]. Furthermore, in [12] we observed that the bound $\delta \leq k + 1$ is almost sharp. Indeed, one can easily construct examples of non-regular points $[s] \in V_{k+4}(\mathcal{O}_{\mathbb{P}^3}(k + 1) \oplus \mathcal{O}_{\mathbb{P}^3}(k + 4))$, for any $k \geq 3$, whose corresponding curve $C$ has its $(k + 4)$ nodes lying on a line $L \subset \mathbb{P}^3$; anyhow, one can also show that $V_{k+4}(\mathcal{O}_{\mathbb{P}^3}(k + 1) \oplus \mathcal{O}_{\mathbb{P}^3}(k + 4))$ is generically regular.

For what concerns $\mathcal{P}_\delta$-Severi varieties on $\mathcal{P}$, we get:

\begin{theorem}
Let $X$ be a smooth projective threefold, $\mathcal{E}$ be a globally generated rank-two vector bundle on $X$, $M$ be a very ample line bundle on $X$ and $k \geq 0$ and $\delta > 0$ be integers.

Let $\mathcal{P} := \mathbb{P}_X(\mathcal{E} \otimes M^{\otimes k})$ and let $\mathcal{O}_\mathcal{P}(1)$ be its tautological line bundle. Let $\mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1))$ be the $\mathcal{P}_\delta$-Severi variety of irreducible divisors on $\mathcal{P}$ having $\delta$-rational double points on $\mathcal{P}$. Then, if:

$$\delta \leq k + 1,$$

$\mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1))$ is regular.

\begin{proof}
From Theorem 7.1 we know that (7.6) is a sufficient condition for the regularity of $V_\delta(\mathcal{E} \otimes M^{\otimes k})$ on $X$. One can conclude by using Corollary 6.12.
\end{proof}
\end{theorem}
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