Physics of nonlinear oscillations with nonlocal variables

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Abstract. In cases where physical processes cannot be described by linear equations, and nonlinear equations are difficult to solve mathematically, we have to use approximate solutions to such problems. One such example is the description of the Kapitsa pendulum, which is a pendulum with a vibrating suspension point. In contrast to the previously known methods of describing such a problem, in this paper we propose to use additional variables in the form of higher derivatives, which allows us to obtain corrections that give a more detailed contribution to the description of this problem.

1. Introduction

We will consider the physics of nonlinear oscillations on the example of the Kapitsa pendulum. As nonlocal variables, we will use them in the form of higher derivatives. A detailed experimental study of a pendulum with dynamic stability in an inverted position is described in the works of P. L. Kapitsa \cite{1}.

The Kapitsa pendulum is a system consisting of a body attached to a light, inextensible spoke that is attached to a vibrating suspension point. The pendulum named after P. L. Kapitsa, who published a description of such an oscillatory system in 1951 \cite{1, 2}.

When studying this type of pendulum, all attention was focused on the type of movement when the period of oscillation of the suspension T differed little from the period of oscillation of the pendulum itself $\tau$. These studies were reduced to studying the properties of solutions to the Mathieu equation, which describes this motion at small oscillation amplitudes. The nature of the movement of the pendulum in this position and the degree of its stability at high frequencies of suspension vibrations remained completely unexplored.

For the problem of stabilization of a mathematical pendulum with a vibrating suspension point in an inclined position, the conditions for the existence of equilibrium are obtained \cite{1, 2}. It is shown that for any limited movement of the suspension, the exact equilibrium states of the pendulum can only be the vertical (upper and lower) positions. This result is generalized to the case of the laws of motion of the suspension, which have discontinuities in velocity, which mechanically means pulsed effects at the corresponding time points. For inclined positions, control actions are constructed that provide periodic vibration of the suspension point in time and harmonic oscillations of the pendulum relative to a fixed inclined position.

Therefore, a visual description of the movement of such pendulums was made with the help of differential equations of higher derivatives, which led to a formal solution. Ostrogradsky's formalism
using the Lagrange function in such an oscillatory system leads to equations of motion that coincide with Newton’s equations, which is the basis for the expression for the Lagrange function [1,2].

Let the axis of the pendulum suspension oscillates in the vertical direction, the angle between the spoke and the y-axis is denoted by \( \varphi \), then the dependence of the coordinates of the body on time will have the form:

\[
\begin{align*}
x &= l \sin \varphi \\
y &= -(l \cos \varphi + A \cos \omega t)
\end{align*}
\]

where \( l \) – rod length, \( \varphi \) – the angle between the rod and the axis y, \( A \) - amplitude of the oscillations in the vertical direction, \( \omega \) is the cyclic frequency of the oscillations, при чем \( A << l \).

A special case is also considered for the possibility of a theoretical study of the motion of the Kapitsa pendulum, which can be determined using the Euler-Lagrange equation of motion for the phase of the pendulum, limiting ourselves to the first derivative

\[
\frac{\partial L}{\partial \varphi} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}}
\]

The sought differential equation describing the phase evolution of the pendulum is nonlinear due to the presence of the \( \sin \varphi \) factor in it.

\[
\ddot{\varphi} = -\frac{A \dot{\varphi}^2 \cos \theta t \sin \varphi - g}{l} \sin \varphi.
\]

Thus, we derive the equation of motion of the pendulum in time, which was solved numerically by the explicit Runge-Kutta method, and show that the Kapitsa pendulum has two stable positions, which is in agreement with the theory. Such an analytical method allows us to renounce the considered particular, narrower cases.

2. How to find the upward force for the stability of the Kapitsa pendulum?

The Ostrogradsky formalism [3] using the Lagrange function

\[
L = L(q, \dot{q}, \ddot{q}, \ldots q^{(n)}),
\]

let’s consider a vibrating reference frame, then

\[
x = A \sin \omega t.
\]

The behavior of macroscopic mechanical systems in non-inertial reference frames can be described by higher-order differential equations. Here, we consider the case when the contribution of higher derivatives is small compared to lower ones. Therefore, at this stage, we restricted ourselves to only the third derivatives of the coordinates with respect to time. There are many examples of the description of mechanical systems in non-inertial reference frames due to the influence of the backgrounds of random fields and waves. Theoretical descriptions of such cases do not always fully describe the physical reality of processes occurring in this process. Such cases include Kapitsa’s pendulum, the movement of bulk materials upwards, against the action of gravity. For describing vibrating mechanical systems, the principle of least action is traditionally used to obtain critical states of mechanical systems. All such cases are described by second-order differential equations. In this case, the direction of the resultant force remains uncertain. This is the main disadvantage of this method of description.

Here, we use a fourth-order differential equation. This allows to first obtain the correct direction of the resultant force. The differential equation of the Kapitsa pendulum is limited to the fourth-order of the time derivative of the coordinate [4]:

\[
F - ma + \frac{mj}{\omega^2} - \frac{ms}{2\omega^2} = 0
\]

where \( a = \frac{d^2q}{dt^2} \) is the acceleration of the suspension coordinate, \( j = \dot{a} = \frac{d^3q}{dt^3} \), \( s = \ddot{a} = \frac{d^4q}{dt^4} \) is the rate of change in the acceleration of the coordinate \( q \) over time, called the jerk and impact \( s, \text{где } \tau = 1/\omega \) this is the averaging time during the transition from the micro-to the macrocosm, opposite to the average cyclic frequency \( \omega \).
Then
\[ F + m \omega^2 A (\cos \omega t - \sin \omega t) = ma. \]

Ostrogradsky's formalism using Lagrange with high-order derivatives has a great advantage over the study. Namely, this approach makes it possible to consider the properties of the system in a large range of additional variables in the form of higher derivatives.

Let's designate
\[ v_x = \frac{dx}{dt} = \frac{d}{dt} (l \sin \varphi) = l \dot{\varphi} \cos \varphi, \]
\[ v_y = \frac{dy}{dt} = -\frac{d}{dt} (l \cos \varphi + A \cos \omega t) = -l \dot{\varphi} \sin \varphi + A \omega \sin \omega t. \]
\[ a_x = \frac{d^2 x}{dt^2} = \frac{d}{dt} (l \dot{\varphi} \cos \varphi) = l \ddot{\varphi} \cos \varphi - l \dot{\varphi}^2 \sin \varphi, \]
\[ a_y = \frac{d^2 y}{dt^2} = \frac{d}{dt} (l \dot{\varphi} \sin \varphi + A \omega \cos \omega t) = -l \dot{\varphi} \sin \varphi + l \dot{\varphi}^2 \cos \varphi + A \omega^2 \cos \omega t. \]
\[ \ddot{a}_x = \frac{d^3 x}{dt^3} = \frac{d}{dt} (l \ddot{\varphi} \cos \varphi - l \dot{\varphi}^2 \sin \varphi) = -l \dddot{\varphi} \cos \varphi - 3l \dot{\varphi} \ddot{\varphi} \sin \varphi - l \dot{\varphi}^3 \cos \varphi, \]
\[ \ddot{a}_y = \frac{d^3 y}{dt^3} = \frac{d}{dt} (l \ddot{\varphi} \sin \varphi + l \dot{\varphi}^2 \cos \varphi + A \theta^2 \cos \omega t) = -l \dddot{\varphi} \sin \varphi + 3l \dot{\varphi} \ddot{\varphi} \cos \varphi - l \dot{\varphi}^3 \sin \varphi - A \omega^3 \sin \omega t. \]

The energy of the system is determined by the Lagrange function, namely, the difference between the kinetic and potential energy. Because of the symmetry of the Poincare group, the derivatives of the coordinates must be quadratic functions
\[ L = -U + \frac{mv^2}{2} + \frac{ma^2}{2} \]
or
\[ L = -U + \frac{mv^2}{2} + \frac{ma^2}{2 \omega^2} + \frac{mj^2}{4 \omega^4}. \]

Substituting the values, we get
\[ L = -U + \frac{m}{2} ((l \dot{\varphi} \cos \varphi)^2 + (l \dot{\varphi} \sin \varphi + A \omega \sin \omega t)^2) + \frac{m}{2 \omega^2} ((l \dot{\varphi} \cos \varphi - l \dot{\varphi}^2 \sin \varphi)^2 + (l \dot{\varphi} \sin \varphi + l \dot{\varphi}^2 \cos \varphi + A \omega^2 \cos \omega t)^2) + \frac{m}{4 \omega^4} ((l \ddot{\varphi} \cos \varphi - 3l \dot{\varphi} \ddot{\varphi} \sin \varphi - l \dot{\varphi}^3 \cos \varphi)^2 + (l \ddot{\varphi} \sin \varphi + 3l \dot{\varphi} \ddot{\varphi} \cos \varphi - l \dot{\varphi}^3 \sin \varphi - A \omega^3 \sin \omega t)^2) \]

After performing the transformations, the Lagrange function has the form:
\[ L = -U + \frac{m}{2} l^2 \ddot{\varphi}^2 + \frac{3m}{4} A^2 l^2 \omega^2 \sin^2 \omega t + ml \dot{\varphi} \sin \varphi A \omega \sin \omega t + \frac{m}{2 \omega^2} l^2 \dot{\varphi}^4 + \frac{m}{2 \omega^2} l^2 \ddot{\varphi}^2 + \frac{m}{2} A^2 \omega^2 \cos^2 \omega t + ml \dot{\varphi} \sin \varphi A \cos \omega t + ml \dot{\varphi} \cos \varphi A \cos \omega t + \frac{m}{4 \omega^4} l^2 \ddot{\varphi}^2 + \frac{9m}{3m} l^2 \ddot{\varphi}^2 \dddot{\varphi}^2 + \frac{m}{4 \omega^4} l^2 \ddot{\varphi}^2 \dddot{\varphi}^2 + \frac{m}{2 \omega^2} l^2 \ddot{\varphi}^2 \dddot{\varphi}^2 - \frac{m}{2 \omega^4} l^2 \ddot{\varphi}^2 \dddot{\varphi}^2 - \frac{m}{2 \omega} l \dddot{\varphi} \sin \varphi A \sin \omega t \]

The equations of motion of such a system satisfy the Euler-Lagrange equations. From here, we find the equation for the dependence of the pendulum phase \( \varphi \) on time and thus determine the position of the weight by the formula (1). The Euler-Lagrange equation for the pendulum phase is as follows:
\[ \frac{\partial L}{\partial \dot{\varphi}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{\varphi}} - \frac{d^2}{dt^2} \frac{\partial L}{\partial \dot{\varphi}^2} - \frac{d^3}{dt^3} \frac{\partial L}{\partial \dddot{\varphi}} = 0 \]

Using the Lagrange formula, we calculate the derivatives that are used in this expression:
\[
\frac{\partial L}{\partial \varphi} = -\frac{\partial U}{\partial \varphi} + ml\dot{\varphi}\cos\varphi A\omega \sin\omega t + ml\dot{\varphi}\cos\varphi A \cos\omega t - ml\dot{\varphi}^2 \sin\varphi A \cos\omega t \\
- \frac{m}{2\omega} l\dot{\varphi} \cos\varphi A \sin\omega t + \frac{3m}{2\omega} l\ddot{\varphi} \sin\varphi A \sin\omega t + \frac{m}{2\omega} l\dot{\varphi}^3 \cos\varphi A \sin\omega t,
\]
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} = ml^2 \ddot{\varphi} - ml\dot{\varphi} \cos\varphi A \sin\omega t + ml\sin\varphi A^2 \cos\omega t + \frac{6m}{\omega^2} l^2 \ddot{\varphi}^2 \dot{\varphi} + \frac{m}{2} l\dot{\varphi} \cos\varphi A \cos\omega t \\
- \frac{m}{2} l\dot{\varphi}^2 \sin\varphi A \cos\omega t + \frac{9m}{\omega^4} l^2 \ddot{\varphi}^3 + \frac{27m}{2\omega^4} l^2 \ddot{\varphi}^2 \dot{\varphi} \ddot{\varphi} + \frac{15m}{2\omega^4} l^2 \ddot{\varphi}^4 + \frac{3m}{\omega^4} l^2 \dddot{\varphi}^3 \dot{\varphi} \\
- \frac{3m}{2\omega} l^2 \dddot{\varphi}^2 \dot{\varphi}^{(4)} - \frac{3m}{2\omega} l \ddot{\varphi} \cos\varphi A \sin\omega t + \frac{3m}{2\omega} l\dot{\varphi} \sin\varphi A \sin\omega t \\
+ \frac{3m}{2\omega} l\dot{\varphi}^3 \cos\varphi A \sin\omega t,
\]
\[
\frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{\varphi}} = \frac{m}{\omega^2} l^2 \dddot{\varphi}^{(4)} - 2l\dot{\varphi} \cos\varphi A \cos\omega t + 2ml\ddot{\varphi} \sin\varphi A \cos\omega t + 2ml\dot{\varphi} \sin\varphi A \cos\omega t \\
- \frac{m}{2} l\dot{\varphi} \cos\varphi A \omega \sin\omega t - ml \sin\varphi A^2 \cos\omega t,
\]
\[
\frac{d^3}{dt^3} \frac{\partial L}{\partial \dddot{\varphi}} = \frac{m}{2\omega^2} l^2 \dddot{\varphi}^{(6)} - \frac{3m}{\omega^4} l^2 \dddot{\varphi}^3 - \frac{9m}{\omega^4} l^2 \dddot{\varphi}^2 \dot{\varphi} \dddot{\varphi} - \frac{3m}{2\omega^4} l^2 \dddot{\varphi}^2 \dot{\varphi}^{(4)} - \frac{m}{2\omega} l\dddot{\varphi} \cos\varphi A \sin\omega t \\
+ \frac{3m}{2\omega} l\ddot{\varphi} \sin\varphi A \sin\omega t - \frac{m}{2\omega} l\dot{\varphi} \cos\varphi A \cos\omega t + \frac{m}{2\omega} l\dddot{\varphi} \sin\varphi A \sin\omega t \\
+ \frac{3m}{2\omega} l\dot{\varphi}^2 \sin\varphi A \cos\omega t - ml\ddot{\varphi} \cos\varphi A \cos\omega t + \frac{3m}{2} l\ddot{\varphi} \cos\varphi A \omega \sin\omega t \\
+ \frac{m}{2} l\dot{\varphi} \sin\varphi A^2 \cos\omega t.
\]

So
\[
- \frac{\partial U}{\partial \varphi} + 2ml\ddot{\varphi} \sin\varphi A \cos\omega t - \frac{3m}{2\omega} l\dddot{\varphi} \cos\varphi A \sin\omega t - ml\dot{\varphi}^2 - \frac{5m}{2} l \sin\varphi A \omega^2 \cos\omega t \\
- \frac{6m}{\omega^2} l^2 \dddot{\varphi} \dot{\varphi} - \frac{m}{2} l\dot{\varphi} \cos\varphi A \cos\omega t - \frac{9m}{\omega^4} l^2 \dddot{\varphi}^3 - \frac{27m}{2\omega^4} l^2 \dddot{\varphi}^2 \dot{\varphi} \dddot{\varphi} - \frac{15m}{2\omega^4} l^2 \dddot{\varphi}^4 \\
+ \frac{3m}{\omega} l^2 \dddot{\varphi} \dddot{\varphi} + \frac{3m}{2\omega} l^2 \dddot{\varphi}^2 \dot{\varphi}^{(4)} + \frac{3m}{2\omega} l \ddot{\varphi} \cos\varphi A \sin\omega t - \frac{9m}{\omega^2} l\ddot{\varphi} \sin\varphi A \sin\omega t \\
+ \frac{m}{\omega^2} l^2 \dddot{\varphi}^{(4)} - \frac{m}{2\omega^2} l^2 \dddot{\varphi}^{(6)} + \frac{3m}{\omega^4} l^2 \dddot{\varphi}^3 + \frac{9m}{\omega^4} l^2 \dddot{\varphi} \dddot{\varphi}^{(4)} + \frac{3m}{2\omega^4} l^2 \dddot{\varphi}^2 \dot{\varphi}^{(4)} \\
+ \frac{m}{2\omega} l\dddot{\varphi} \cos\varphi A \cos\omega t = 0.
\]

Getting rid of the terms that make a small contribution, we write
\[
- \frac{\partial U}{\partial \varphi} - ml\ddot{\varphi} - \frac{m}{2} l\dot{\varphi} \cos\varphi A \cos\omega t - \frac{5m}{2} l \sin\varphi A\omega^2 \cos\omega t + \frac{3m}{2\omega} l^2 \ddot{\varphi} \cos\varphi A \sin\omega t \\
+ \frac{m}{2\omega} l\dot{\varphi} \cos\varphi A \cos\omega t = 0.
\]
or
\[
ml^2 \ddot{\varphi} + k\dddot{\varphi} + ml(g + A\omega^2 \sin\omega t) \sin\varphi = 0.
\]
The upper position of the pendulum is stable when
\[
\frac{mv^2}{2} = \frac{mA^2 \omega^2}{2} > mgl
\]
or
\[
A \omega > \sqrt{2gl}.
\]
3. Conclusion
The dynamics of mechanical systems in non-inertial reference frames can be described by differential equations above the second. Physics require the concept of inertial reference systems, the description of motion in nonlocal hidden variables in a more general description, and can be named non-inertial mechanics, complementing physics. In this particular case, for example, when using a fourth-order differential equation by the method of variable replacement, it can be represented by two second-order equations. In the general case, non-inertial dynamics can be described by high order differential equations.

The classical problem of the behavior of a pendulum with a vertically oscillating suspension axis is a special case of the physics of nonlinear oscillations. Nonlinear oscillations, which were considered on the example of the Kapitsa pendulum model, cover general patterns that can be described using high-order derivatives of all mechanical systems [5-8] and are more complete than the description with second-order derivatives.

References
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