ALGEBRAIC MORSE THEORY AND HOMOLOGICAL PERTURBATION THEORY

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Abstract. We show that the main result of algebraic Morse theory can be obtained as a consequence of the perturbation lemma of Brown and Gugenheim.

1. Introduction

Robin Forman introduced discrete Morse theory in [For98] as a combinatorial adaptation of the classical Morse theory suited for studying the topology of CW-complexes. Its fundamental idea is also applicable in purely algebraic situations (see e.g. [Jon03], [Koz05], [JW09], [Skö06]).

Homological perturbation theory on the other hand builds on the perturbation lemma [Bro65], [Gug72]. In addition to its applications in algebraic topology, it has also found uses in e.g. the study of group cohomology [Lam92], [Hue89], resolutions in commutative algebra [JLS02] as well as in operadic settings, [Ber09].

In this note we show how to derive the main result of algebraic Morse theory from the perturbation lemma. In related work, Berglund [Ber], has also treated connections between algebraic Morse theory and homological perturbation theory.

2. Definitions

We will briefly review the definitions of the main objects of study.

A contraction is a diagram of chain complexes of (left or right) modules over a ring $R$

$$
\begin{array}{c}
D \\
\xrightarrow{f} \\
\xrightarrow{g} \\
\xleftarrow{h}
\end{array}
\quad C
$$

where $f$ and $g$ are chain maps and $h$ is a degree 1 map satisfying the identities

$$
fg = 1, \quad gf = 1 + dh + hd
$$

and

$$
fh = 0, \quad hg = 0, \quad h^2 = 0.
$$

A contraction is filtered if there is a bounded below exhaustive filtration on the complexes which is preserved by the maps $f$, $g$ and $h$. A perturbation of a chain complex $C$ is a map $t : C \to C$ of degree $-1$ such that $(d + t)^2 = 0$. Given a perturbation $t$ on $C$, we let $C^t$ be the complex obtained by equipping $C$ with the new differential $d + t$.

We can now state the perturbation lemma.

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Theorem 1 (Brown, Gugenheim). Given a filtered contraction

\[ \begin{array}{ccc} D & \xrightarrow{g} & C \\ f & & \circlearrowleft h \end{array} \]

and a filtration lowering perturbation \( t \) of \( C \), the diagram

\[ \begin{array}{ccc} D' & \xrightarrow{g'} & C' \\ f' & & \circlearrowleft h' \end{array} \]

where

\[ f' = f + fSh, \quad g' = g + hSg, \quad h' = h + hSh, \quad t' = fSg \]

and

\[ S = \sum_{n=0}^{\infty} t(ht)^n \]

defines a contraction.

Let us next review some terminology of algebraic Morse theory. By a based complex of \( R \)-modules we mean a chain complex \( C \) of \( R \)-modules together with direct sum decompositions \( C_n = \bigoplus_{\alpha \in I_n} C_{\alpha} \) where \( \{I_n\} \) is a family of mutually disjoint index sets. For \( f : \bigoplus_n C_n \to \bigoplus_n C_n \) a graded map, we write \( f_{\beta,\alpha} \) for the component of \( f \) going from \( C_{\alpha} \) to \( C_{\beta} \), and given a based complex \( C \) we construct a digraph \( G(C) \) with vertex set \( V = \bigcup_n I_n \) and with a directed edge \( \alpha \to \beta \) whenever the component \( d_{\beta,\alpha} \) is non-zero.

A subset \( M \) of the edges of \( G(C) \) such that no vertex is incident to more than one edge of \( M \) is called a Morse matching if, for each edge \( \alpha \to \beta \) in \( M \), the corresponding component \( d_{\beta,\alpha} \) is an isomorphism, and furthermore there is a well founded partial order \( \prec \) on each \( I_n \) such that \( \gamma \prec \alpha \) whenever there is a path \( \alpha^{(n)} \to \beta \to \gamma^{(n)} \) in the graph \( G(C)^M \), which is the graph obtained from \( G(C) \) by reversing the edges from \( M \).

Given the matching \( M \), we define the set \( M^0 \) to be the vertices that are not incident to an arrow from \( M \).

For \( \alpha \) and \( \beta \) vertices in \( G(C)^M \) we can now consider all directed paths from \( \alpha \) to \( \beta \). For each such path \( \gamma \), we get a map from \( C_{\alpha} \) to \( C_{\beta} \) by, for each edge \( \sigma \to \tau \) in \( \gamma \) which is not in \( M \) take the map \( d_{\tau,\sigma} \), and for each edge \( \sigma \to \tau \) in \( \gamma \) which is the reverse of an edge in \( M \) take the map \( -d_{\tau,\sigma}^{-1} \) and composing them. Summing these maps over all paths from \( \alpha \) to \( \beta \) defines the map \( \Gamma_{\beta,\alpha} : C_{\alpha} \to C_{\beta} \).

3. The main result

From the based complex \( C \) with \( C_n = \bigoplus_{\alpha \in I_n} C_{\alpha} \) furnished with a Morse matching \( M \), we define another based complex \( \tilde{C} \) by letting it be isomorphic to \( C \) as a graded module, and defining the differential \( \tilde{d} \) in \( \tilde{C} \) as

\[ \tilde{d}(x) = \begin{cases} d_{\beta,\alpha}(x), & \text{if } \alpha \to \beta \in M, \\ 0, & \text{otherwise;} \end{cases} \]

for \( x \in C_{\alpha} \).
We also need a based complex coming from the vertices in $M^0$, so we define $\tilde{C}^M$ by

$$\tilde{C}^M_n = \bigoplus_{\alpha \in I_n \cap M^0} C_\alpha, \quad d_{\tilde{C}^M} = 0,$$

and maps $\tilde{f} : \tilde{C} \to \tilde{C}^M$, $\tilde{g} : \tilde{C}^M \to \tilde{C}$ and $\tilde{h} : \tilde{C} \to \tilde{C}[1]$ given by

\[
\tilde{f}(x) = \begin{cases} x, & \text{if } \alpha \in M^0, \\ 0, & \text{otherwise}, \end{cases} \quad x \in C_\alpha.
\]

\[
\tilde{g}(x) = x, \quad x \in C_\alpha.
\]

\[
\tilde{h}(x) = \begin{cases} -d_{\alpha,\beta}^{-1}(x), & \text{if } \beta \to \alpha \in M, \\ 0, & \text{otherwise}; \end{cases}
\]

With this notation we can now formulate the following lemma.

**Lemma 1.** The diagram

$$\tilde{C}^M \xrightarrow{\tilde{f}} \tilde{C} \xrightarrow{\tilde{g}} \tilde{C} \xrightarrow{\tilde{h}}$$

is a contraction.

**Proof.** We first need to verify that $\tilde{f}$ and $\tilde{g}$ are chain maps, which is readily seen. Next we check the identities

$$\tilde{f}\tilde{g} = 1, \quad \tilde{g}\tilde{f} = 1 + \tilde{d}\tilde{h} + \tilde{h}\tilde{d}.$$  

The first one is obvious, and the second follows from the fact that for a basis element $x \in C_\alpha$, $\tilde{d}\tilde{h}(x) = -x$ if there is an edge $\beta \to \alpha$ in $M$, and 0 otherwise; and similarly $\tilde{h}\tilde{d}(x) = -x$ if there is an edge $\alpha \to \beta$ in $M$, and 0 otherwise. The identities

$$\tilde{h}\tilde{g} = 0, \quad \tilde{f}\tilde{h} = 0, \quad \tilde{h}^2 = 0$$

follow from that vertices in $M^0$ are not incident to any edge in $M$ (the first two) and that no vertex is incident to more than one edge in $M$ (the third). \qed

Let us now define the perturbation $t$ on $\tilde{C}$ as $t = d - \hat{d}$, where $d$ is the differential on $C$, so

$$t(x) = \sum_{\alpha \to \beta \notin M} d_{\beta,\alpha}(x)$$

for $x \in C_\alpha$. This makes $\tilde{C}^t$ and $C$ isomorphic as based complexes.

**Lemma 2.** The diagram

$$C^M \xrightarrow{f} C \xrightarrow{g} h$$

where, for $x \in C_\alpha$ with $\alpha \in I_n$,

$$d_{C^M}(x) = \sum_{\beta \in M^0 \cap I_{n-1}} \Gamma_{\beta,\alpha}(x) \quad f(x) = \sum_{\beta \in M^0 \cap I_n} \Gamma_{\beta,\alpha}(x)$$

$$g(x) = \sum_{\beta \in I_n} \Gamma_{\beta,\alpha}(x) \quad h(x) = \sum_{\beta \in I_{n+1}} \Gamma_{\beta,\alpha}(x)$$

is a filtered contraction.
Proof. From Lemma 1 together with the fact that there are no infinite paths in $G(C)^M$, the Morse graph of $C$, we can deduce that $ht$ is locally nilpotent, and we can thus invoke the perturbation lemma. It is not so hard to see that the perturbed differential on $\tilde{C}^M$ is given by

$$d(x) = \sum_{i=0}^{\infty} t(ht)^i(x) = \sum_{\beta \in M^0 \cap I_{n-1}} \Gamma_{\beta,\alpha}(x)$$

and the maps $f$, $g$ and $h$ by

$$f(x) = \sum_{i=0}^{\infty} f(ht)^i(x) = \sum_{\beta \in M^0 \cap I_n} \Gamma_{\beta,\alpha}(x)$$

$$g(x) = \sum_{i=0}^{\infty} g(ht)^i(x) = \sum_{\beta \in I_n} \Gamma_{\beta,\alpha}(x)$$

$$h(x) = \sum_{i=0}^{\infty} (ht)^i h(x) = \sum_{\beta \in I_{n+1}} \Gamma_{\beta,\alpha}(x)$$

where $x \in C_{\alpha}$. □

The above result is also shown (without the use of the perturbation lemma) in [Ber] using a result from [JW09].

From the preceding lemma, the main result of algebraic Morse theory now follows.

**Theorem 2.** Let $C$ be a based complex with a Morse matching $M$, then there is a differential on the graded module $\bigoplus_{\alpha \in M^0} C_{\alpha}$ such that the resulting complex is homotopy equivalent to $C$.

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