The Young-Laplace’s equation for solid

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Abstract

The Young-Laplace’s equation is established based on liquid membrane without shearing resistance. It is not valid for solid. By taking into account the in-plane shearing and transverse shearing within the surface layer, we reconstruct the Young-Laplace’s equation so as to characterize the surface of solid. A new version of the Young-Laplace’s equation is proposed. It shows that the surface equilibrium of solid is determined by the bulk stress, surface membrane stress and surface transverse stress together. The transverse shear stress depends on the gradient of the Gaussian curvature of surface and strain. The intrinsic membrane stress and surface transverse shear stress cause the residual stresses to appear in the interior of solid. The intrinsic surface transverse shear stress only occurs in the non-spherical body.

Key words: Young-Laplace’s equation, generalized Young-Laplace’s equation, surface stress, surface transverse stress, Tolman’ formula

1 Introduction

Surface is a thin layer with finite thickness rather than a film of zero-thickness. For liquid, its surface can be perfectly represented by a film only subjected to tension, because a liquid cannot support shearing stress indefinitely. Characterization to equilibrium of the liquid film leads to the Young-Laplace’s equation [1]. On the other hand, a solid can be in equilibrium under a shear stress. By introducing the in-plane shear deformation, Gurtin and Murdoch extended the Young-Laplace’s equation into the generalized Young-Laplace’s equation so as to characterize the surface of elastic solid [2]. Steigmann and Ogden further proposed a reinforced generalized Young-Laplace equation by taking into account the bending stiffness of the surface film [3]. Recently, Javili and Mosler et al revisited and carefully examined the surface/interface elasticity theory [4].

So far, various models based on the Young-Laplace equation and the generalized Young-Laplace equation have been presented in several contexts. For example, Wang and Feng [5] investigated the influence of surface elasticity and residual surface tension on the natural frequency of micro beams. It is not the purpose of this short letter to list and review these abundant works. The reader can refer to the reviews by Wang et al [6], Muller and Saul [7], Sun [8] and Duan et al [9] on the relevant literature.

However, all works mentioned above are developed based on the model of film with zero-thickness. As a result, in the existing theories and models it is impossible to take into account transverse shearing effects within the surface layer of solid. In fact, since non-uniformity of the excess energy profile across the surface layer causes the energy gradient to appear, the transverse shear stress inevitably exists on the cross-section of surface layer. Meanwhile, if there is shearing on the internal boundary surface of the surface layer, it will also cause the transverse shearing effects to occur within the surface layer. To the best of our knowledge, hardly anyone realizes existence of the transverse shear stress and its influence on the equilibrium of a solid surface, while this influence cannot be neglected. Therefore, this problem will be investigated in this paper, the emphasis will be placed on reconstructing the Young-Laplace’s equation for solid from the angle of theory.

The paper is outlined as follows. In Section 2, we propose a Lagrangian to describe the effects caused by the excess energy within the surface layer and the Lagrangian equation and curvature-dependent natural boundary condition. This boundary condition is simplified into a new generalized Young-Laplace’s equation involved with the transverse shear stress in Section 3. In Section 4, we analyze the characters of intrinsic membrane stress, surface transverse stress and bulk residual stress and their mutual relation. Finally, the summary and comment on the results in this paper are given.
Let $\mathbf{x} = \{x^i\}$ be a 3-dimensional position vector in $\Omega$ and $t \in [t_0, t_1]$ be time. A vector field defined on $[t_0, t_1] \cup \Omega$ is denoted by $\phi_k = \phi_k(t, \mathbf{x})$. The Lagrangian of the field $\phi_k$ is written as $L = L(\phi_k, \phi_k, \partial_t \phi_k)$.

By Eq. (2), the action of field can be represented as \[ L(\phi_k, \phi_k, \partial_t \phi_k) \] where $\mathbf{g}_4$ is the unit base vector defined on the tangent plane of $\partial \Omega$ and $\mathbf{n}$ the unit normal vector. By the identity $\nabla \cdot \mathbf{n} = -2H$ \[ [10, 11], \text{Eq.}(1) \] is rewritten as $\Gamma = \gamma + \partial_t \mathbf{n}^2 = -2H$, where $H$ is the mean curvature of $\partial \Omega$. Not losing generality, we introduce a scale parameter $\chi(x)$ which is defined as the ratio of $2\Gamma(\mathbf{r}, \phi_k, \partial_t \phi_k)$ to $\gamma(\mathbf{r}, \phi_k, \partial_t \phi_k)$. As thus, we have

\[ \Gamma = (1 - \chi H)\gamma + \partial_t \mathbf{n}^2, \] (2)

By Eq. (2), the action of field can be represented as \[ (1 - \chi H)\gamma + \partial_t \mathbf{n}^2 \]

Taking the variation of $A[\phi_k]$ leads to

\[
\delta A = \int_{t_0}^{t_1} \left[ \frac{\partial L}{\partial \phi_k} - \frac{d}{dt} \frac{\partial L}{\partial (\partial_t \phi_k)} - \partial_t \left[ \frac{\partial L}{\partial (\partial_t \phi_k)} \right] \right] \delta \phi_k \, dt + \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial (\partial_t \phi_k)} n_j + (1 - \chi H) \left( \frac{\partial \gamma}{\partial \phi_k} \frac{d}{dt} \frac{\partial \gamma}{\partial (\partial_t \phi_k)} - \partial_t \left[ \frac{\partial \gamma}{\partial (\partial_t \phi_k)} \right] \right) + \frac{\partial \gamma}{\partial (\partial_t \phi_k)} \partial_t (\chi H) \right\} \delta \phi_k \, dt, \]

where $n_j$ denotes the unit normal vector on $\partial \Omega$. The Hamilton's principle asserts that $\delta A[\phi_k] = 0$. Therefore, according to the fundamental lemma of variation, we have

\[
\frac{\partial L}{\partial \phi_k} - \frac{d}{dt} \frac{\partial L}{\partial (\partial_t \phi_k)} - \partial_t \left[ \frac{\partial L}{\partial (\partial_t \phi_k)} \right] = 0, \quad x^k \in \Omega. \] (5)

Natural boundary condition:

\[
\frac{\partial L}{\partial (\partial_t \phi_k)} n_j = (1 - \chi H) \left[ \frac{\partial \gamma}{\partial \phi_k} \frac{d}{dt} \frac{\partial L}{\partial (\partial_t \phi_k)} + \partial_t \left[ \frac{\partial \gamma}{\partial (\partial_t \phi_k)} \right] - \frac{\partial \gamma}{\partial \phi_k} \partial_t (\chi H) \right], \quad x^k \in \partial \Omega. \] (6)

Eq. (5) and (6) show that the surface Lagrangian has no influence on the Euler-Lagrange equation, but it contributes to the natural boundary condition and causes the natural boundary condition to be correlated with the mean curvature and its gradient of boundary surface. As a boundary condition, Eq. (6) is universal but complicated. Next, we turn to simplification and discussion to Eq. (6).
3 The generalized Young-Laplace’s equation and Tolman’s length

In the following, we stipulate that $\phi_k$ is a displacement field. Only concerned with a quasi-static system, Eq.(6) can be simplified into

$$\frac{\partial L}{\partial (\partial_j \phi_k)} n_j = (1 - \chi H)(\partial_l [\frac{\partial \gamma}{\partial (\partial_k \phi_l)}] - \frac{\partial \gamma}{\partial \phi_k}) - \frac{\partial \gamma}{\partial (\partial_k \phi_k)} \partial_a (\chi H), \quad x^k \in \partial \Omega. \quad (7)$$

The surface Lagrangian $\gamma$ must be invariant under the translational transformation of $\phi_k$. So $\gamma$ is necessarily independent of $\phi_k$ itself, and Eq.(7) reduces to

$$\frac{\partial L}{\partial (\partial_j \phi_k)} n_j = \partial_a [(1 - \chi H)\frac{\partial \gamma}{\partial (\partial_k \phi_k)}], \quad x^k \in \partial \Omega. \quad (8)$$

Introduce two signs as follows

$$\sigma^{kj} = \frac{\partial L}{\partial (\partial_j \phi_k)}, \quad \tilde{\sigma}^{k} = \frac{\partial \gamma}{\partial (\partial_k \phi_k)} \quad (9)$$

In physics, $\sigma^{kj}$ and $\tilde{\sigma}^{k}$ can be interpreted as bulk stress and surface stress. By Eq.(9), Eq.(8) is represented as

$$\sigma^{kj} n_j = \partial_a [(1 - \chi H)\tilde{\sigma}^k], \quad x^k \in \partial \Omega. \quad (10)$$

Set a local coordinate system with the base vectors $(g_1, g_2, g_3) = (g_A, n)$ on the surface $\partial \Omega$, where $g_A (A = 1, 2)$ is the the covariant base vectors corresponding to the curvilinear coordinate on $\partial \Omega$ and $n$ the unit normal vector. In such a coordinate system, we have

$$\sigma = \sigma^{kj} g_k \otimes g_j, \quad \tilde{\sigma} = \tilde{\sigma}^k g_k = \tilde{\sigma}^{AB} g_A \otimes g_B + \tilde{\sigma}^{33} g_3 \otimes n. \quad (11)$$

Clearly, $\sigma^{AB}$ is the membrane stress component of surface and $\sigma^{33}$ is the transverse stress component on the cross-section of surface layer. Let

$$\sigma_\gamma = (1 - \chi H)\tilde{\sigma}, \quad x^k \in \partial \Omega. \quad (12)$$

It is easy to see that Eq.(12) is just the Tolman’s formula [13] in which $\tilde{\sigma}$ represents the surface stress of a flat surface, while $\sigma_\gamma$ is the curvature-dependent surface stress. So the scale parameter $\chi$ is also referred as to the Tolman’s length. The Tolman’s formula has been extensively applied to analyze the surface size effects of micro/nano-scale liquid droplet and solid particle [14, 15]. By Eq.(11) and (12), Eq.(10) can be equivalently written as

$$\sigma \cdot n = \nabla_s \cdot \sigma_\gamma, \quad x^k \in \partial \Omega. \quad (13)$$

Eq.(13) is the so-called generalized Young-Laplace’s equation, but it is a new version taking into account the curvature effect and transverse shearing effect of surface layer. To clarify this point, firstly let us to calculate $\nabla_s \cdot \sigma_\gamma$ as follows

$$\nabla_s \cdot \sigma_\gamma = \{ \partial_a [(1 - \chi H)\tilde{\sigma}^{AB}] - (1 - \chi H)(\tilde{\sigma}^{AB} b_A^B) g_B + \{ \partial_a [(1 - \chi H)\tilde{\sigma}^{33}] + (1 - \chi H)\tilde{\sigma}^{AB} b_{AB} \} n, \quad x^k \in \partial \Omega, \quad (14)$$

where

$$\partial_a [(1 - \chi H)\tilde{\sigma}^{AB}] = [(1 - \chi H)\tilde{\sigma}^{AB} g_A + (1 - \chi H)(\tilde{\sigma}^{CB} \Gamma^{A}_{CA} + \tilde{\sigma}^{AC} \Gamma^{B}_{AC}), \quad x^k \in \partial \Omega, \quad (15)$$

$$\partial_a [(1 - \chi H)\tilde{\sigma}^{33}] = [(1 - \chi H)\tilde{\sigma}^{33} g_A + (1 - \chi H)\tilde{\sigma}^{AB} \Gamma^{3}_{AC}], \quad x^k \in \partial \Omega, \quad (16)$$

where $\Gamma^{3}_{AC}$ and $b_{AB}$ (or $b_A^B$) are the connection coefficients and curvature tensor of the surface $\partial \Omega$, respectively. Substituting Eq.(14) into (13), and then projecting it onto the tangential plane and normal direction of the surface $\partial \Omega$, we have

$$\mathbf{P} \cdot \sigma \cdot n = \{ \partial_a [(1 - \chi H)\tilde{\sigma}^{AB}] - (1 - \chi H)(\tilde{\sigma}^{AB} b_A^B) g_B, \quad x^k \in \partial \Omega, \quad (17)$$

$$n \cdot \sigma \cdot n = \partial_a [(1 - \chi H)\tilde{\sigma}^{33}] + (1 - \chi H)\tilde{\sigma}^{AB} b_{AB}, \quad x^k \in \partial \Omega, \quad (18)$$

where $\mathbf{P}$ is the projection operator, which reads $\mathbf{P} = g^{ij} g_i \otimes g_j - n \otimes n$. Eq.(17) and (18) are another form of the generalized Young-Laplace’s equation. It is obvious that they contain both the surface transverse shearing effect and curvature-dependent effect of membrane stress.
4 Intrinsic membrane stress and surface transverse stress

In a general case, although no external traction is prescribed, the membrane stress $\bar{\sigma}^{AB}$ and surface transverse stress $\sigma^A$ also exist due to the excess energy within the surface layer. We refer to $\bar{\sigma}^{AB}$ and $\sigma^A$ as the intrinsic membrane stress and intrinsic surface transverse stress if they are only caused by the excess energy within the surface layer. For convenience, the intrinsic membrane stress and intrinsic surface transverse stress are denoted by $\bar{\sigma}_{0}^{AB}$ and $\tau^A$, respectively.

In terms of the Shuttleworth-Herring equation $\gamma = \bar{\gamma} s^{AB} + \partial \bar{\gamma} / \partial \phi_{AB}$, the intrinsic surface transverse stress reads

$$\bar{\sigma}_{0}^{AB} = \bar{\sigma}^{AB} |_{\phi_{A}\phi_{B}=0} = \bar{\gamma} s^{AB},$$

where $\bar{\gamma}$ is the surface tension. Eq.(19) shows that the intrinsic membrane stress always exists, irrelevant to the curvature of surface. However, the intrinsic surface transverse stress is different from the intrinsic membrane stress. Under some special cases, the intrinsic surface transverse stress does not occur. For example, no intrinsic surface transverse stress appears on the surface of a spherical grain, due to the spherical symmetry.

Differential geometry tells us: a closed surface is a spherical surface if and only if its Gaussian curvature is a constant $[16]$. It follows immediately that $\nabla_{x} \kappa = 0$, where $\kappa$ is the Gaussian curvature. As thus, the physical fact that the intrinsic surface transverse stress does not on a spherical surface but it occurs on a non-spherical surface shows that the intrinsic surface transverse stress $\tau^A$ is related with $\bar{\sigma}^{A} \kappa$. Meanwhile, $\tau^A$ is also dependent on the shear modulus $\mu$. So under a general case, we have $\tau^A = \bar{\sigma}^{A} f(\mu, \kappa)$. In terms of $\pi$ theorem of the dimensional analysis $[17]$, $f(\mu, \kappa)$ can be concretely represented as $f(\mu, \kappa) = \mu \varepsilon \kappa^{-1}$, where $\varepsilon$ is a dimensionless constant. Let $\tau = \mu \varepsilon$. Noticing that both $\mu$ and $\varepsilon$ are constants, we have

$$\tau^A = \bar{\sigma}^{A} |_{\phi_{A}\phi_{B}=0} = \tau \bar{\sigma}^{A} \kappa^{-1}$$

(20)

In physics, the constant $\varepsilon$ can be interpreted as a transverse shear strain caused by the excess energy within the surface layer. Thus, $\tau$ is a residual shear stress on the cross section of the surface layer. Substituting Eq.(19) and (20) into (17) and (18) lead to

$$P \cdot \sigma = \{ \bar{\sigma}^{B}(1 - \chi H) - \tau (1 - \chi H) b_{B}^{R} \bar{\sigma}^{A} \kappa^{-1} \} g_{B}, \quad x^k \in \partial \Omega, \quad (21)$$

$$n \cdot \sigma = \partial_{A} [\tau (1 - \chi H) \bar{\sigma}^{A} \kappa^{-1}] + 2 \bar{\gamma} (1 - \chi H) H, \quad x^k \in \partial \Omega. \quad (22)$$

Eq.(21) and (22) show that the excess energy within surface layer can give rise to the residual stresses in the interior of solid. It should be emphasized that the surface tension $\bar{\gamma}$ differs from the surface Lagrangian $\gamma$. The correlation between them can be represented as.

$$\gamma = \int_{0}^{\partial_{A} \phi_{B}} (\bar{\gamma} s^{AB} + \frac{\partial \bar{\gamma}}{\partial (\partial_{A} \phi_{B})}) | \partial_{A} \phi_{B} | d(\partial_{A} \phi_{B}) + \int_{0}^{\partial_{A} \phi_{1}} (\tau \bar{\sigma}^{A} \kappa^{-1} + \frac{\partial \bar{\gamma}}{\partial (\partial_{A} \phi_{1})}) | \partial_{A} \phi_{1} | d(\partial_{A} \phi_{1}),$$

(23)

Inserting Eq.(23) in $(9)_2$, we have

$$\sigma^{AB} = \frac{\partial \gamma}{\partial (\partial_{A} \phi_{B})} = \bar{\gamma} s^{AB} + \frac{\partial \gamma}{\partial (\partial_{A} \phi_{B})}, \quad (24)$$

$$\sigma^{A} = \frac{\partial \gamma}{\partial (\partial_{A} \phi_{1})} = \tau \bar{\sigma}^{A} \kappa^{-1} + \frac{\partial \bar{\gamma}}{\partial (\partial_{A} \phi_{1})}. \quad (25)$$

which are the constitutive equations characterizing the mechanical behaviors of the solid surface.

For a liquid droplet in the static equilibrium, $\tau$ is identical to zero and $\bar{\gamma}$ is a constant. Therefore, if $\chi = 0$, Eq.(21) and (22) reduce to $n \cdot \sigma \cdot n = 2 \bar{\gamma} H$. This is just the original version of the Young-Laplace’s equation for liquid.

5 Conclusion

In the framework of the Lagrangian field theory, we propose a surface Lagrangian to characterize the surface effects of field, and reconstruct the generalized Young-Laplace’s equation for solid. Based on this equation, the conclusions are summarized as follows.

1. On the surface of solid, there exists the transverse shear stress induced by the excess energy within the surface layer. The transverse shear stress depends on the gradient of the Gaussian curvature of surface and deformation.
2. For the surface of a solid, its equilibrium is determined by the bulk stress, surface membrane stress and surface transverse stress together.
3. The intrinsic membrane stress and surface transverse shear stress cause the residual stresses to appear in the interior of solid. The intrinsic surface transverse shear stress only occurs in the non-spherical body.

Finally, it should be pointed out that the influence of surface on a bulk solid becomes obvious only at micro/nano scale. Meanwhile, it also requires that the characteristic dimension of the solid must be much larger than the thickness of surface layer so that the surface layer can be treated as a surface of vanishing thickness. Otherwise, atomistic or quantum models are necessary.

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References

[1] P.R. Pujado, C. Huh, L.E. Scriven, On the contribution of an equation of capillary to Young and Laplace, Journal of Colloid and Interface Science 38 (1972) 662-663.
[2] M.E. Gurtin, A.I. Murdoch, A continuum theory of elastic material surfaces, Archive for Rational Mechanics and Analysis 57 (1975) 291-323.
[3] D.J. Steigmann, R.W. Ogden, Elastic surface-substrate interactions, Proceedings of the Royal Society of London A 455 (1999) 437-474.
[4] A. Javili, N.S. Ottosen, M. Ristinmaa, J. Aspects of interface elasticity theory, Mathematics and Mechanics of Solids DOI: 10.1177/1081286517699041 (2017) 1-21.
[5] Wang, G.F., Feng, X.Q.: Effects of surface elasticity and residual surface tension on the natural frequency of micro beams. Appl. Phys. Lett. 90, 231904 (2007).
[6] J. Wang, Z. Huang, H. Duan, S. Yu, X. Feng, G. Wang, et al, Surface stress effect in mechanics of nanostructured materials, Acta Mechanica Solida Sinica 24 (2011) 52-82.
[7] Pierre Muller and Andres Saul, Elastic effects on surface physics, Surface Science Reports 54 (2004) 157-258.
[8] C.Q. Sun, Size dependence of nanostructures: Impact of bond order deficiency, Prog. Solid State Chem. 35 (2007) 1-159.
[9] Duan, H.L., Wang, J.X., Karihaloo, B.L.: Theory of elasticity at the nano-scale. Adv. Appl. Mech. 42, 1C68 (2009).
[10] J.G. Simmonds, A Brief on Tensor Analysis, Springer, New York, 1994.
[11] Y. Yin, J. Wu, J. Yin, Symmetrical fundamental tensor, differential operators and integral theorems in differential geometry, Tsinghua Science & Technology 13 (2008) 121-126.
[12] Zaixing Huang, Torsional wave and vibration subjected to constraint of surface elasticity, Acta Mechanica, Vol.229, 2018 (3), pp1171-1182
[13] R.C. Tolman, The effect of droplet size on surface tension, Journal of Chemical Physics 17 (1949) 333-337.
[14] A. Malijevsky, G. Jackson, A perspective on the interfacial properties of nanoscopic drops, Journal of Physics: Condensed Matter 24 (2012) 464121.
[15] Z. Huang, P. Thomson, S.L. Di, Lattice contractions of nanoparticle due to the surface tension: A model of elasticity, J. Phys. Chem. Solids 68 (2007) 30-535.
[16] J. Oprea, Differential Geometry and Its Applications, Prentice Hall, New York, 2004.
[17] H.E. Huntley, Dimensional Analysis, Dover, New York, 1967.