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- putting up results and their proofs for further research
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Part 1. Randomness via Kolmogorov complexity

1. Yu: A trivial observation on mutual information

Definition 1.1 (Levin). For oracles $x, y$,

$$I(x : y) = \log \sum_{n, m} 2^{-K(\langle m, n \rangle) - K^x(m) - K^y(n) + K(m) + K(n)}.$$ 

Definition 1.2 (Levin).

$$I'(x : y) = \log \sum_{n} 2^{-K^x(n) - K^y(n) + K(n)}.$$ 

Note that $I'(x, y) = \infty$ implies $I(x, y) = \infty$. In [26], Hirschfeldt and Weber asked whether there is a non-trivial $x$ so that for any $y$ with $I(x : y) = \infty$, $y$ must compute $x \oplus \emptyset'$. Actually by their own proof, the answer is negative.
Proposition 1.3. If $x$ is not $K$-trivial, then $\{ y \mid I'(x : y) = \infty \}$ is comeager.

Proof. Essentially due to Hirschfeldt and Weber. For any $n$, there is some $k_n$ so that $K^y(x \upharpoonright k_n) \leq K(x \upharpoonright k_n) - n$. If $y$ is sufficiently generic, then for any $n$, there is some $i_n > n$ so that $K^y(x \upharpoonright i_n) \leq i_n$. Then

$$I'(x : y) \geq \log \sum_n 2^{-K^y(x \upharpoonright k_n) - K^y(x \upharpoonright i_n) + K(x \upharpoonright k_n)} \geq \log \sum_n 2^{i_n - i_n} = \infty.$$

$\square$

2. Nies: An analog of the coincidence of $\leq_{LR}$ and $\leq_{LK}$

Let $Y, B$ be sets. Recall that $Y \leq_{LR} B$ if $\text{MLR}^B \subseteq \text{MLR}^Y$. Kjos-Hanssen, Miller and Solomon [34] showed that this LR-reducibility coincides with LK-reducibility, where $Y \leq_{LK} B$ if $\forall x K^B(x) \leq^+ K^Y(x)$. We will weaken both relationships by replacing the objects c.e. in $Y$ by objects computable in $Y$. This is applied in a recent manuscript by Greenberg, Miller and Nies on subclasses of the $K$-trivials.

A computable measure machine (c.m.m.) is a prefix free machine $M$ such that $\lambda[\text{dom } M] \prec \mu$ is a computable real [57, 3.5.14].

Definition 2.1.
(i) We write that $Y \leq_{wLR} B$ if $\text{MLR}^B \subseteq \text{SR}^Y$.
(ii) $Y \leq_{wLK} B$ if $\forall x K^B(x) \leq^+ K^M(x)$ for each computable measure machine $M$ relative to $Y$.

Note that these relations are not transitive. Barmpalias, Miller and Nies [5] have shown that $Y \leq_{wLR} B$ iff $Y$ is c.e. traceable by $B$: there is a computable bound $h$ such that each function $f \leq_T Y$ has an $h$-bounded trace c.e. in $B$. $\emptyset' \leq_{wLR} B$ means that $B$ is “weakly LR-hard”. For c.e. sets, array recursive is the same as c.e. traceable, and it is known that such sets can be properly low$_2$. Hence, by jump inversion for ML-random sets, some weakly LR-hard ML-random $\Delta^0_2$ set $B$ is properly high$_2$, and in particular not LR-hard. Every random set above a smart $K$-trivial in the sense of [8] is not OW-random, hence LR-hard, and in particular high. So the diamond class of weak LR-hardness is properly contained in the $K$-trivials. In fact by the result in [5] it is contained in the diamond class of JT-hardness, which was previously known to be properly contained in the $K$-trivials (see [57, 8.5]).

We adapt the proof of the Kjos-Miller-Solomon result given as Theorem 5.6.5 in Section 5.6 of [57]. Item numbers below refer to [57]. We only give proofs when they are not straightforward adaptations. The following is our new version of 5.6.5.

Theorem 2.2. Let $Y, B$ be sets. We have $Y \leq_{wLK} B \iff Y \leq_{wLR} B$.

Proof. $\Rightarrow$: This follows from the characterisation of ML-randomness via $K$ (Levin-Schnorr), and the characterisation of Schnorr-randomness via $K^M$ for c.m.m. $M$, all relativized appropriately.

$\Leftarrow$: That implication depended on a number of foregoing results, some of them in Section 5.1.

New version of 5.1.10, “partially” relativized to $B$. 


Proposition 2.3. The following are equivalent for a set $A$.

- $Y \leq_{wLR} B$
- For each computable measure machine $M$ relative to $Y$, there is a $\Sigma^0_1(B)$ set $S$ such that
  \[ \lambda S < 1 \land \forall z [K_M(z) \leq |z|-1 \rightarrow [z] \subseteq S]. \]

New version of 5.6.3 (which extends 5.1.10).

Lemma 2.4. $Y \leq_{wLR} B$ if and only if each $\Sigma^0_1(Y)$ class $G$ such that $\lambda G < 1$ and $\lambda G$ is computable in $Y$ is contained in a $\Sigma^0_1(B)$ class $S$ such that $\lambda S < 1$.

For a function $f : \mathbb{N} \rightarrow \mathbb{N}$, let $\mu_f$ be the measure on $\mathcal{P} (\mathbb{N})$ given by $\mu_f ([n]) = 2^{-f(n)}$. A set $I \subseteq \mathbb{N}$ is called $f$-small if $\mu_f (I)$ is finite.

New version of 5.6.4.

Lemma 2.5. $Y \leq_{wLR} B$ if and only if for each computable function $f$, each $f$-small $Y$-c.e. set $I$ such that $\mu_f (I)$ is computable in $Y$, $I$ is contained in an $f$-small $B$-c.e. set $R$.

Proof. As before, we use the fact that for a sequence of real numbers $(a_n)_{n \in \mathbb{N}}$ such that $0 \leq a_n < 1$ for each $n$, we have
\begin{equation}
\sum_{n=0}^{\infty} a_n < \infty \Leftrightarrow \prod_{n=0}^{\infty} (1 - a_n) > 0.
\end{equation}

To see this one works with $g(x) = -\ln(1 - x)$ for $x \in [0, 1]^\mathbb{R}$. Since $e^y \geq 1 + y$ for each $y \in \mathbb{R}$, we have $x \leq g(x)$ for each $x$. On the other hand $g'(x) = 1/(1 - x)$, so $g'(0) = 1 = \lim_{x \to 0} (g(x) - g(0))/x$. Hence there is $\varepsilon > 0$ such that $g(x) \leq 2x$ for each $x \in [0, \varepsilon]$.

In the present setting, we also need that if the sum is a (finite) computable real, then so is the product. We may assume that each tail sum is less than $\varepsilon$. It now suffices to verify that $\prod_{n=k}^{\infty} (1 - a_k) \leq g( \sum_{n=k}^{\infty} a_k )$ for each $n$.

As before we use (2) to infer Lemma 2.5 from the implication “$\Rightarrow$” of Lemma 2.4. It suffices to observe that the class $P$ defined in the original version now has $Y$-computable positive measure because the product is $Y$-computable. Now let $G = 2^N - P$ and apply $\Rightarrow$ of 2.4. \qed

We can now complete the proof of $\Leftarrow$: of Thm. 2.2. Let $f$ be the computable function given by $f(\langle r, y \rangle) = r$ (the book has the typo 2$^r$ there). Let $M$ be a c.m.m. relative to $Y$. The set $I = \{ \langle |\sigma|, y \rangle : M(\sigma) = y \}$ is a bounded request set relative to $A$ and hence $f$-small, and $\mu_f (I)$ is computable in $Y$. So by Lemma 2.5, $I$ is contained in an $f$-small $B$-c.e. set $\tilde{R}$. Let $R \subseteq \tilde{R}$ be a bounded request set relative to $B$ such that $\tilde{R} - R$ is finite. Then, applying to $R$ the Machine Existence Theorem [57, 2.2.17] relative to $B$, we may conclude that $\forall y K^B(y) \leq^+ K_M (y)$. \qed

Part 2. Randomness via algorithmic tests

3. Downey, Nandakumar and Nies:
MULTIPLE RECURRENCE AND RANDOMNESS VIA ALGORITHMIC TESTS

We determine the level of randomness needed for a point so that the multiple recurrence theorem of Furstenberg holds for iterations starting at the point.

3.1. Background in ergodic theory. Let \((X, \mathcal{B}, \mu)\) be a probability space. A measurable operator \(T: X \to X\) is called measure preserving if \(\mu T^{-1}(A) = \mu A\) for each \(A \in \mathcal{B}\).

**Theorem 3.1** (Furstenberg strong multiple recurrence theorem; see [21] Thm. 7.15).

Let \((X, \mathcal{B}, \mu)\) be a probability space. Let \(T_1, \ldots, T_k\) be commuting measure preserving operators on \(X\). Let \(A \in \mathcal{B}\) with \(\mu A > 0\). We have

\[
0 < \liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu \left( \bigcap_{1 \leq i \leq k} T_i^{-n}(A) \right)
\]

As a consequence, there is a positive measure set of points so that the same number of iterations of each of the operators \(T_i\), starting from each of the points, ends in \(A\).

**Corollary 3.2** (Furstenberg multiple recurrence theorem).

With the hypotheses of Thm. 3.1, there is \(n > 0\) such that

\[
0 < \mu \bigcap_{1 \leq i \leq k} T_i^{-n}(A).
\]

For this paper the following variant of Cor. 3.2 will matter.

**Corollary 3.3.** With the hypotheses of Thm. 3.1, for \(\mu\text{-a.e. } x \in A\), there is an \(n > 0\) such that \(x \in \bigcap_{1 \leq i \leq k} T_i^{-n}(A)\).

This statement clearly yields Cor. 3.2 because it implies that \(\bigcap_{1 \leq i \leq k} T_i^{-n}(A)\) has positive measure for some \(n\). Conversely, let us show that Cor. 3.2 yields Cor. 3.3. Let \(R_n = \bigcap_{1 \leq i \leq k} T_i^{-n}(A)\). We recursively define a sequence \(\langle n_p \rangle_{p < N}\) of numbers and a descending sequence \(\langle A_p \rangle_{p < N}\) of sets, where \(0 < N \leq \omega\).

Let \(n_0 = 0\), and \(A_0 = A\). Suppose \(n_p\) and \(A_p\) have been defined. If \(\mu A_p = 0\) let \(N = p + 1\) and finish. Otherwise, let \(n_{p+1}\) be the least \(n > n_p\) such that \(\mu(A_p \cap R_n) > 0\), and let \(A_{p+1} = A_p - R_n\).

Let \(A_N = \bigcap_{p < N} A_p\). Then \(\mu A_N = 0\). This is clear if \(N\) is finite. If \(N = \omega\) and \(\mu A_N > 0\), by Cor. 3.2 there is \(n\) such that \(\mu(R_n \cap A_N) > 0\). This contradicts the definition of \(A_{p+1}\) where \(n_p < n \leq n_{p+1}\).

Since \(\mu A_N = 0\), Cor. 3.3 follows.

We will be mainly interested in the special case where \(T_i = V^i\) for a measure-preserving operator \(V\).

**Corollary 3.4.** Let \((X, \mathcal{B}, \mu)\) be a probability space. Let \(V\) be a measure preserving operator. Let \(A \in \mathcal{B}\) and \(\mu A > 0\). For each \(k\), for \(\mu\text{-a.e. } x \in A\) there is \(n\) such that \(\forall i.1 \leq i \leq k\ [x \in V^{-ni}(A)]\).

In fact we will mostly assume that \((X, \mathcal{B}, \mu)\) is Cantor space \(2^\mathbb{N}\) with the product measure \(\lambda\). In the following \(X, Y, Z\) will denote elements of Cantor space. We will work with the shift \(T\) as the measure preserving operator. Thus, \(T(Z)\) is obtained by deleting the first entry of the bit sequence \(Z\). We note that this operator is (strongly) mixing, and hence strongly ergodic,
namely, all of its powers are ergodic. We will write $Z_n$ for $T^n(Z)$, the tail of $Z$ starting at bit position $n$. Thus, for any $C \subseteq 2^N$, $Z \in 2^N$ and $k \in \mathbb{N}$, $Z \in T^{-k}(C) \iff Z_k \in C$.

For a set of strings $S \subseteq 2^{\omega}$, by $[S]$ we denote the open set $\{ Y \in 2^N : \exists \sigma \in S[\sigma \prec Y] \}$. We write $\lambda[S]$ for the measure of this set, namely $\lambda([S])$.

### 3.2. The connection with algorithmic randomness

For the remainder of the paper, we consider multiple recurrence for closed sets. Note that for (multiple) recurrence in the sense of Cor. 3.2, this is not an essential restriction, because any set of positive measure contains a closed subset of positive measure. The following is our central definition. Let $T : 2^N \to 2^N$ denote the shift operator.

**Definition 3.5.** Let $P \subseteq 2^N$ be closed, and let $Z \in 2^N$. We say that $Z$ is $k$-recurrent in $P$ if there is $n \geq 1$ such that

$$Z \in \bigcap_{1 \leq i \leq k} T^{-ni}(P).$$

We say that $Z$ is multiply recurrent in $P$ if $Z$ is $k$-recurrent in $P$ for each $k \geq 1$.

In other words, $Z$ is $k$-recurrent in $P$ if there is $n$ such that taking $n, 2n, \ldots, kn$ bits off $Z$ takes us into $P$.

We analyse how weaker and weaker effectiveness conditions on $P$ ensure multiple recurrence when starting from a sequence $Z$ that satisfies a stronger and stronger randomness property for an algorithmic test notion. We begin with the strongest effectiveness condition, being clopen; in this case it is easily seen that weak (or Kurtz) randomness of $Z$ suffices. As most general effectiveness condition we will consider being effectively closed (i.e. $\Pi^0_1$); Martin-Löf-randomness turns out to be the appropriate notion. The proof will import some method from the case of a clopen $P$. Note that $\Pi^0_1$ subsets of Cantor space are often called $\Pi^0_1$ classes. For background on randomness notions see [57, Ch. 3] or [13, Ch ?].

### 3.3. Multiple recurrence for weakly random sequences

Recall that $Z$ is weakly (or Kurtz) random if $Z$ is in no null $\Pi^0_1$ class.

**Proposition 3.6.** Let $P \subseteq 2^N$ be a non-empty clopen set. Each weakly random bit sequence $Z$ is multiply recurrent in $P$.

**Proof.** Suppose $Z$ is not $k$-recurrent in $P$ for some $k \geq 1$. We define a null $\Pi^0_1$ class $Q$ containing $Z$. Let $n_0$ be least such that $P = [F]$ for some set of strings of length $n_0$. Let $n_t = n_0(k + 1)^t$ for $t \geq 1$. Let

$$Q = \bigcap_{t \in \mathbb{N}} \left\{ Y : \bigvee_{1 \leq i \leq k} Y_{in_t} \not\in P \right\}.$$ 

By definition of $n_0$ the conditions in the same disjunction are independent, so we have

$$\lambda\left( \bigvee_{1 \leq i \leq k} Y_{in_t} \not\in P \right) = 1 - (\lambda P)^k < 1.$$
By definition of the $n_t$ for $t > 0$, the class $\mathcal{Q}$ is the independent intersection of such classes indexed by $t$. Therefore $\mathcal{Q}$ is null. Clearly $\mathcal{Q}$ is $\Pi^0_1$.

By hypothesis $Z \in \mathcal{Q}$. So $Z$ is not weakly random. \hfill \Box

3.4. Multiple recurrence for Schnorr random sequences.

**Theorem 3.7.** Let $\mathcal{P} \subseteq 2^\mathbb{N}$ be a $\Pi^0_1$ class such that $0 < p = \lambda\mathcal{P}$ and $p$ is a computable real. Each Schnorr random $Z$ is multiply recurrent in $\mathcal{P}$.

We note that this also follows from a particular effective version of Furstenberg multiple recurrence (Cor. 3.3), as explained in Remark 3.14 below. However, we prefer to give a direct proof avoiding Cor. 3.3.

**Proof.** We extend the previous proof, working with an effective approximation $\mathcal{B} = 2^\mathbb{N} - \mathcal{P} = \bigcup_s \mathcal{B}_s$ where the $\mathcal{B}_s$ are clopen. We may assume that $2^\mathbb{N} - \mathcal{B}_s = [B_s]^s$ for some effectively given set $B_s$ of strings of length $s$.

We fix an arbitrary $k \geq 1$ and show that $Z$ is $k$-recurrent in $\mathcal{P}$. Given $v \in \mathbb{N}$ we will define a null $\Pi^0_1$ class $\mathcal{Q}_v \subseteq 2^\mathbb{N}$ which plays a role similar to the class $\mathcal{Q}$ before. We also define an “error class” $\mathcal{G}_v \subseteq 2^\mathbb{N}$ that is $\Sigma^0_1$ uniformly in $v$. Further, $\lambda\mathcal{G}_v$ is computable uniformly in $v$ and $\lambda\mathcal{G}_v \leq 2^{-v}$, so that $(\mathcal{G}_v)_{v \in \mathbb{N}}$ is a Schnorr test. If $Z$ passes this Schnorr test then $Z$ behaves essentially like a weakly random in the proof of Proposition 3.6, which shows that $Z$ is $k$-recurrent for $\mathcal{P}$.

For the details, given $v \in \mathbb{N}$, we define a computable sequence $\langle n_t \rangle$. Let $n_0 = 1$. Let $n = n_t \geq (k + 1)n_{t-1}$ be so large that

$$\lambda (\mathcal{B} - \mathcal{B}_n) \leq 2^{-t-v-k}.$$

As in the proof of Proposition 3.6, the class $\mathcal{Q}_v = \{ Y: \forall t \quad \bigvee_{1 \leq i \leq k} Y_{i n_t} \in \mathcal{B}_{n_t} \}$ is $\Pi^0_1$ and null. The “error class” for $v$ at stage $t$ is $\mathcal{G}_v^t = \{ Y: \quad \bigvee_{1 \leq i \leq k} Y_{i n_t} \in \mathcal{B} - \mathcal{B}_{n_t} \}$.

Notice that $\lambda\mathcal{G}_v^t \leq k2^{-t-v-k}$, and this measure is computable uniformly in $v, t$. Let $\mathcal{G}_v = \bigcup_t \mathcal{G}_v^t$. Then $\lambda\mathcal{G}_v$ is also uniformly computable in $v$, and bounded above by $2^{-v}$, as required.

If $Z$ is Schnorr random, there is $v$ such that $Z \notin \mathcal{G}_v$. Also, $Z \notin \mathcal{Q}_v$, so that for some $t$ we have $Z_{i n_t} \in \mathcal{P}$ for each $i$ with $1 \leq i \leq k$, as required. \hfill \Box

3.5. Multiple recurrence for ML-random sequences. For general $\Pi^0_1$ classes, the right level of randomness to obtain multiple recurrence is ML-randomness. We first remind the reader that even the case of 1-recurrence characterizes ML-randomness. This is a well-known result of Kučera [35].

**Proposition 3.8.** $Z$ is ML-random $\iff$ $Z$ is 1-recurrent in each $\Pi^0_1$ class $\mathcal{P}$ with $0 < p = \lambda\mathcal{P}$.

**Proof.** $\Rightarrow$: see e.g. [57, 3.2.24] or [13, where?].

$\Leftarrow$: ML-randomness of a sequence $Z$ is preserved by adding bits at the beginning. By the Levin-Schnorr Theorem, the $\Pi^0_1$ class $\mathcal{P} = \{ Y: \forall nK(Y|n)_n$
\( ) \geq n - 1 \) consists entirely of ML-randoms. So, if \( Z \) is not ML-random, then no tail of \( Z \) is in the \( \Pi^0_1 \) class \( \mathcal{P} \). Further, \( \lambda \mathcal{P} \geq 1/2 \).

**Theorem 3.9.** Let \( \mathcal{P} \subseteq 2^\omega \) be a \( \Pi^0_1 \) class with \( 0 < p = \lambda \mathcal{P} \). Each Martin-Löf random \( Z \) is multiply recurrent in \( \mathcal{P} \).

**Proof.** As before we fix an arbitrary \( k \geq 1 \) in order to show that \( Z \) is \( k \)-recurrent in \( \mathcal{P} \). First we prove the assertion under the additional assumption that \( 1 - 1/k < p \). This generalises Kučera’s argument in \( \Rightarrow \)' of the proposition above, where \( k = 1 \) and the additional assumption \( 0 < p \) is already satisfied.

Let \( B \subseteq 2^\omega \) be a prefix-free c.e. set such that \( [B]^\omega = 2^\omega - \mathcal{P} \). We may assume that \( B_0 = \emptyset \) and for each \( t > 0 \), if \( \sigma \in B_t - B_{t-1} \) then \( |\sigma| = t \).

We define a uniformly c.e. sequence \( (C^r) \) of prefix-free sets with the same property that at stage \( t \) only strings of length \( t \) are enumerated.

For a string \( \eta \) and \( u \leq |\eta| \), we write \( (\eta)_u \) for the string \( \eta \) with the first \( u \) bits removed. Let \( C^0 \) only contain the empty string, which is enumerated at stage 0. Suppose \( r > 0 \) and \( C^{r-1} \) has been defined. Suppose \( \sigma \) is enumerated in \( C^{r-1} \) at stage \( s \) (so \( |\sigma| = s \)). For strings \( \eta > \sigma \) we search for the failure of \( k \)-recurrence in \( \mathcal{P} \) that would be obtained by taking \( s \) bits off \( \eta \) for \( k \) times.

At stage \( t > (k+1)s \), for each string \( \eta \) of length \( t \) such that \( \eta > \sigma \) and

\[
\bigvee_{1 \leq i \leq k} (\eta)_{si} \in B_{t-si},
\]

and no prefix of \( \eta \) is in \( C^r_{t-1} \), put \( \eta \) into \( C^r \) at stage \( t \).

**Claim 3.10.** \( C^r \) is prefix-free for each \( r \).

This holds for \( r = 0 \). For \( r > 0 \) suppose that \( \eta \leq \eta' \) and both strings are in \( C^r \). Let \( t = |\eta| \). By inductive hypothesis the string \( \eta \) was enumerated into \( C^r \) via a unique \( \sigma < \eta \), where \( \sigma \in C^{r-1} \). Then \( \eta = \eta' \) because we chose the string in \( C^r \) minimal under the prefix relation. This establishes the claim.

By hypothesis \( 1 > q = k\lambda|B|^{<} \).

**Claim 3.11.** For each \( r \geq 0 \) we have \( \lambda|C^r|^{<} \leq q^r \).

This holds for \( r = 0 \). Suppose now that \( r > 0 \). Let \( \sigma \in C^{r-1} \). The local measure above \( \sigma \) of strings \( \eta \), of a length \( t \), such that \( \bigvee_{1 \leq i \leq k} \eta_{si} \in B_{t-si} \) is at most \( q \). The estimate follows by the prefix-freeness of \( C^r \).

If \( Z \) is not \( k \)-recurrent in \( \mathcal{P} \), then \( Z \in [C^r]^{<} \) for each \( r \), so \( Z \) is not ML-random.

We now remove the additional assumption that \( 1 - 1/k < p \). We define the sets \( C^r \) as before. Note that any string in \( C^r \) has length at least \( r \). Everything will work except for Claim 3.11: if \( \lambda|B|^{<} \geq 1/k \) then \( \lambda|C^r|^{<} \) could be 1. To remedy this, we choose a finite set \( D \subseteq B \) such that the set \( \tilde{B} = B - D \) satisfies \( \lambda|\tilde{B}|^{<} < 1/k \). Let \( N = \max\{|\sigma|: \sigma \in D\} \). We modify the argument of Prop. 3.6, where the clopen set \( \mathcal{P} \) there now becomes \( 2^\omega - [D]^{<} \).

Let \( C = \bigcup_s C^s \). Let \( G_m \) be the set of prefix-minimal strings \( \eta \) such that \( \eta \in C \), and there exist \( m \) many \( s > N \) as follows.

- \( \eta |_s \in C \), and
- for some \( i \) with \( 1 \leq i \leq k \), \( \eta |_{si,s(i+1)} \) extends a string in \( D \).
we have \( \sigma |_{[s_i, s_{i+1})} \) may extend a string in \( D \), rather than one in \( B \).

The sets \( G_m \) are uniformly \( \Sigma^0_1 \). By choice of \( N \) and independence, as in the proof of Prop. 3.6 we have \( \lambda(G_{m+1}) \approx (1-\eta^k)\lambda G_m \), where \( \eta = \lambda(2^N - [D] \approx) \).

If \( Z \) is ML-random we can choose a least \( m^* \) such that \( Z \notin [G_{m*}] \approx\).

Note that \( m^* > 0 \) since \( G_0 = \{ \emptyset \} \). So choose \( \rho < Z \) such that \( \rho \in G_{m^*-1} \). Then \( \rho \in C^r \) for some \( r \), and no \( \tau \) with \( \rho \approx \tau \approx Z \) is in \( G_{m*} \).

We define a ML-test that succeeds on \( Z \). Let \( C^r = C^r \). Suppose \( u > r \) and \( C^u \) has been defined. For each \( \sigma \in C^u-1 \), put into \( C^u \) all the strings \( \eta \approx \sigma \) in \( C^u \) so that \( \eta \not\approx \sigma \) in \( C^u \) and \( \rho \approx \eta \not\approx \sigma \approx Z \) can be strengthened to \( \bigvee_{1 \leq i \leq k} (\eta_i \approx \bar{B}_{r-i}s) \), where \( s = |\sigma| \).

Let \( q = k\lambda[\bar{B}] \approx \). Note that \( \lambda(\bar{C}^u) \approx \leq q^u \) as before. By the choice of \( m^* \) we have \( Z \in \bigcap_{u \geq r} \bar{C}^u \approx \), so since \( q < 1 \), an appropriate refinement of the sequence of open sets \( \left( \bar{C}^u \right)_{u \in \mathbb{N}} \) shows \( Z \) is not ML-random.

\[ \square \]

3.6. Towards the general case.

3.6.1. Recurrence for \( k \) shift operators. The probability space under consideration is now \( \mathcal{X} = \{0, 1\}^{\mathbb{N}} \) with the product measure. For \( 1 \leq i \leq k \), the operator \( T_i: \mathcal{X} \rightarrow \mathcal{X} \) takes one “face” of bits off in direction \( i \). That is, for \( Z \in \mathcal{X} \),

\[ T_i(Z)(u_1, \ldots, u_k) = Z(u_1, \ldots, u_i + 1, \ldots, u_k) \]

\( Z \) is \( \text{recurrent} \) in a class \( \mathcal{P} \subseteq \mathcal{X} \) if \( Z \in \bigcap_{i \leq k} T_{i}^{-n}(\mathcal{P}) \) for some \( n \).

Algorithmic randomness notions for points in \( \mathcal{X} \) can be defined via the effective measure preserving isomorphism \( \mathcal{X} \rightarrow 2^\mathbb{N} \) given by a computable bijection \( \mathbb{N}^k \rightarrow \mathbb{N} \). Modifying the methods above, we show the following.

**Theorem 3.12.** Let \( \mathcal{P} \subseteq \mathcal{X} \) be a \( \Pi^0_1 \) class with \( 0 < p = \lambda \mathcal{P} \). Let \( Z \in \mathcal{X} \).

If \( Z \) is (a) Kurtz (b) Schnorr (c) ML-random, then \( Z \) is \( \text{recurrent} \) in \( \mathcal{P} \) in case (a) \( \mathcal{P} \) is copen (b) \( \lambda \mathcal{P} \) is computable (c) for any \( \mathcal{P} \).

**Proof.** For the duration of this proof, by an array we mean a map \( \sigma: \{0, \ldots, n-1\}^k \rightarrow \{0, 1\} \). We call \( n \) the size of \( \sigma \) and write \( n = |\sigma| \). The letters \( \sigma, \tau, \rho, \eta \) now denote arrays. For \( s \leq n \) and \( i \leq k \) let \( (\sigma)_{i,s} \) be the array \( \tau \) of size \( n - s \) such that

\[ \tau(u_1, \ldots, u_k) = \sigma(u_1, \ldots, u_i + s, \ldots, u_k) \]

for \( u_1, \ldots, u_k \leq n - s \). This operation removes \( s \) faces in direction \( i \), and then cuts the opposite faces in the remaining directions in order to obtain an array. For a set \( S \) of arrays we define \( [S] \approx = \{ Y \in \mathcal{X}: \exists \sigma \in S | \sigma \approx Y \} \) where the “prefix” relation \( \prec \) is defined as expected.

Suppose that \( Z \) is not \( k \)-recurrent in \( \mathcal{P} \) for some \( k \geq 1 \).

(a) As in Prop. 3.6 we define a null \( \Pi^0_1 \) class \( \mathcal{Q} \subseteq \mathcal{X} \) containing \( Z \). Let \( n_1 \) be least such that \( \mathcal{P} = [F] \approx \) for some set of arrays of that all have size \( n_1 \). Let

\[ Q = \bigcap_{r \geq 1} \{ Y \in \mathcal{Y}: \bigvee_{1 \leq i \leq k} T_{i}^{n}(Y) \notin \mathcal{P} \}. \]
By the choice of $n_1$ the conditions in the same disjunction are independent, so we have
\[
\lambda(\bigvee_{1 \leq i \leq k} T_i^{p^n}(Y) \notin \mathcal{P}) = 1 - p^k < 1.
\]

The $\Pi^0_1$ class $\mathcal{Q}$ is the independent intersection of such classes indexed by $r$. Therefore $\mathcal{Q}$ is null. By hypothesis $Z \in \mathcal{Q}$. So $Z$ is not weakly random.

(b). We could modify the previous argument. However, this also follows by the general fact in Remark 3.14 below.

(c). The argument is similar to the proof of Theorem 3.9 above. The definition of the c.e. set $B$ and its enumeration are as before, except that each string of length $n$ is now an array of size $n$. In particular, an array enumerated at a stage $s$ has size $s$.

Let $C^0$ only contain the empty array, which is enumerated at stage 0. Suppose $r > 0$ and $C^{r-1}$ has been defined. Suppose $\sigma$ is enumerated in $C^{r-1}$ at stage $s$ (so $|\sigma| = s$).

At a stage $t > 2s$, for each array $\eta$ of size $t$ such that $\eta > \sigma$ and
\[
\bigvee_{1 \leq i \leq k} (\eta)_{i,s} \in B_{t-s},
\]
and no array that is a prefix of $\eta$ is in $C^r_{t-1}$, put $\eta$ into $C^r$ at stage $t$. As before one checks that $C^r$ is prefix-free for each $r$.

Choose a finite set of arrays $D \subseteq B$ such that the set $\tilde{B} = B - D$ satisfies $\lambda[\tilde{B}] < 1/k$. Let $N = \max\{|\sigma| : \sigma \in D\}$. Let $C = \bigcup_r C^r$. Let $G_m$ be the set of prefix-minimal arrays $\eta$ such that $\eta \in C$, and there exist $m$ many $s > N$ as follows.

- $\eta |_{\{0, \ldots, s-1\}} \in C$, and
- for some $i$ with $1 \leq i \leq k$, $(\eta)_{i,s}$ extends an array in $D$.

The sets $G_m$ are uniformly $\Sigma^0_1$. By choice of $N$ and independence $\lambda[G_{m+1}]^c \leq (1 - t^k)\lambda[G_m]$, where $t = \lambda[2^N - |D|^c]$. If $Z$ is ML-random we can choose a least $m^*$ be such that $Z \notin [G_{m^*}]^c$, and $m^* > 0$ since $G_0 = \emptyset$. So choose $\eta < Z$ such that $\eta \in G_{m^*-1}$. Then $\eta \in C^r$ for some $r$, and no $\tau$ with $\eta \preceq \tau < Z$ is in $G_{m^*}$.

Let $\tilde{C}^r = C^r$. Suppose $u > r$ and $\tilde{C}^{u-1}$ has been defined. For each $\sigma \in \tilde{C}^{u-1}$, put into $\tilde{C}^u$ all the arrays $\eta > \sigma$ in $C^u$ so that (*) can be strengthened to $\bigvee_{1 \leq i \leq k} (\eta)_{i,s} \in \tilde{B}_{t-s}$, where $s = |\sigma|$ and $t = |\eta|$.

Let $q = k\lambda[\tilde{B}]$. Then $\lambda[\tilde{C}^u]^c \leq q^u$ as before. By the choice of $m^*$ we have $Z \in \bigcap_{u \geq r} [\tilde{C}^u]^c$, so since $q < 1$, $Z$ is not ML-random. \hfill \Box

3.6.2. The putative full result. It is likely that a multiple recurrence theorem holds in greater generality. For background on computable probability spaces and how to define randomness notions for points in them, see e.g. [22].

**Conjecture 3.13.** Let $(X, \mu)$ be a computable probability space. Let $T_1, \ldots, T_k$ be computable measure preserving transformations that commute pairwise. Let $\mathcal{P}$ be a $\Pi^0_1$ class with $\mu\mathcal{P} > 0$.

If $z \in \mathcal{P}$ is ML-random then $\exists n[z \in \bigcap_{i \leq k} T_i^{-n}(\mathcal{P})]$. 
Remark 3.14. Let \( U_n \) be the open set \( \{ x : x \notin \bigcap_{i<k} T_i^{-n}(P) \} \). Then \( \mu(P \cap \bigcap_{n} U_n) = 0 \) by the classic multiple recurrence theorem in the version of Cor. 3.3. Since \( P \cap \bigcap_{n} U_n \) is \( \Pi^0_2 \), weak 2-randomness of \( z \) suffices for the \( k \)-recurrence.

Jason Rute has pointed out that if \( X \) is Cantor space and \( \mu P \) is computable, then \( \exists z \in \bigcap_{i<k} T_i^{-n}(P) \) for every Schnorr random \( z \in P \). For in this case \( \hat{\mu} U_n \) is uniformly computable where \( \hat{\mu} U_n = \bigcap_{i<n} U_i \). Let \( P = \bigcap_{n} P_n \) where the \( P_n \) are clopen sets computed uniformly in \( n \). Let \( G_n = P_n \cap \hat{\mu} U_n \). Then \( G_n \) is uniformly \( \Sigma^0_1 \) and \( \mu(G_n) \) is uniformly computable. Refining the sequence \( \langle G_n \rangle \) we obtain a Schnorr test capturing \( z \).

For general \( P \)'s, an interesting first case would be when \( T = S^i \) where \( S \) is the rotation of the unit circle of the form \( z \to z e^{2\pi i} \) for irrational computable \( \alpha \). Such an \( S \) is ergodic, but not weakly mixing.

4. Greenberg, Turetsky and Westrick:
   Degrees of halves of left-c.e. randoms

For \( \alpha \) a real, let \( \pi_0(\alpha) \) denote the real made from the even bits of \( \alpha \)'s binary expansion, and \( \pi_1(\alpha) \) the real made from the odd bits. Thus \( \alpha = \pi_0(\alpha) \oplus \pi_1(\alpha) \). A nonempty subset of Greenberg, Miller, Nies and Turetsky wondered the following: If \( \alpha \) and \( \beta \) are left-c.e. random reals, must \( \{ \deg(\pi_0(\alpha)), \deg(\pi_1(\alpha)) \} = \{ \deg(\pi_0(\beta)), \deg(\pi_1(\beta)) \} \). Note that these are unordered pairs, so the question is whether \( \pi_0(\beta) \) must have the same degree as one of \( \pi_0(\alpha) \) or \( \pi_1(\alpha) \), and \( \pi_1(\beta) \) the same degree as the other.

Greenberg, Turetsky and Westrick have answered the question in the negative, via the following:

**Lemma 4.1.** If \( \alpha \) is random, and \( d = 2k + 1 \) is an odd integer with \( d \geq 3 \), then \( \pi_i(\alpha/d) \oplus \pi_j(\alpha/d) \) can derandomize \( \pi_{1-i}(\alpha) \) for \( i, j < 2 \).

Presumably a stronger result is possible: relative to \( \pi_j(\alpha/d) \), \( \pi_i(\alpha) \) should have effective Hausdorff dimension at most \( 1/2 \), with no assumptions on \( \alpha \).

**Proof.** The proof is by long division. For \( \sigma \in 2^{< \omega} \), let \( \text{int}(\sigma) \) be the integer denoted by \( \sigma \) as a big-endian binary representation. That is, \( \text{int}(\sigma) = \sum_{\ell < |\sigma|} \sigma(\ell) \cdot 2^{|\sigma| - \ell - 1} \). We will be making reference to \( \text{int}(\alpha|_n) \). We assume \( \alpha \) is a real between 0 and 1, so we identify it with the infinite sequence of 0s and 1s in its binary expansion to the right of the radix symbol. When passing from \( \alpha \) to \( \alpha|_n \) to \( \text{int}(\alpha|_n) \), we drop the radix symbol to obtain an integer.

Recall the long division algorithm. When performing the division \( \text{int}(\alpha|_n) \div d \), the quotient is \( \text{int}((\alpha/d)|_n) \), and there is some remainder \( r < d \). Further, if \( b_0 \) and \( b_1 \) are the next two bits of \( \alpha \) and \( c_0 \) and \( c_1 \) are the next two bits of \( \alpha/d \), so that \( (\alpha|_{n+2} = (\alpha|_n) \ast b_0 b_1 \) and \( (\alpha/d)|_{n+2} = ((\alpha/d)|_n) \ast c_0 c_1 \), then the quotient of \( (4r + 2b_0 + b_1) \div d \) is \( \text{int}(c_0 c_1) = 2c_0 + c_1 \), again with some remainder. This is simply the “carry” procedure of long division, performed over two bits at a time rather than a single bit.

Recall that \( d = 2k + 1 \). Since \( \alpha \) is random, there are infinitely many \( n \) such that \( \text{int}(\alpha|_n) \div d \) has remainder \( k \). In fact, there are infinitely many such \( n \) which are even and infinitely many which are odd. For such an \( n \), let \( b_0, b_1, c_0 \)
and $c_1$ be as above. Observe that $4k+2b_0+b_1 \geq 2(2k+1)$ iff $b_0 = 1$. So $c_0 = 1$ iff $b_0 = 1$. Also, if $b_0 = 1$, then $4k+2b_0+b_1 < 2(2k+1)+(2k+1)$, since $d \geq 3$ and thus $k \geq 1$. On the other hand, if $b_0 = 0$, then $4k + 2b_0 + b_1 > 2k + 1$, again since $k \geq 1$. Thus $c_1 = 1$ iff $b_0 = 0$.

We now describe a martingale computable from $\pi_i(\alpha) \oplus \pi_j(\alpha/d)$ that succeeds on $\pi_{1-i}(\alpha)$. By reading bits of $\pi_{1-i}(\alpha)$ and combining this with $\pi_d(\alpha)$ from its oracle, our martingale can obtain initial segments of $\alpha$. For each $\alpha|_n$ it computes the remainder of $\text{int}(\alpha|_n) \div d$. When it sees that the remainder is $k$, and $n$ is such that the next bit of $\alpha$ is from $\pi_{1-i}(\alpha)$ ($n$ is even or odd, as appropriate), it is ready to bet on the next bit of $\pi_{1-i}(\alpha)$. By the calculations above, both the $n$th bit $c_0$ and the $n+1$st bit $c_1$ of $\alpha/d$ determine the $n$th bit of $\alpha$. From the $\pi_j(\alpha/d)$ in the oracle, our martingale knows one of these bits, and so it knows the next bit of $\pi_{1-i}(\alpha)$. So it bets all its money on this bit. Our arguments above show that this martingale succeeds on $\pi_{1-i}(\alpha)$. $\square$

Now, suppose $\alpha$ is left-c.e. and random. Then $\beta = \alpha/3$ is also left-c.e. and random. However, neither of $\pi_0(\beta)$ or $\pi_1(\beta)$ can be computable from $\pi_0(\alpha)$, as the lemma would then say that $\pi_0(\alpha)$ could derandomize $\pi_1(\alpha)$, contrary to van Lambalgen’s theorem.

Part 3. Randomness and analysis

5. **Rute: Research directions and open problems for ARA 2014 Japan**

(By Jason Rute, Pennsylvania State University.) This is a revised version on a list of research directions and open problems for Analysis, Randomness, and Applications (ARA) 2014 in Japan\(^1\). (The author was not in attendance and sent these notes in absentia.)

5.1. **Which randomness notions are natural?**

5.1.1. **Determine the natural randomness notions.** As already pointed out by Schnorr [71] and even, perhaps, Martin-Löf [43], there is not just one natural randomness notion. There are at least two—ML-randomness and Schnorr-randomness—and probably more—computable randomness, $n$-randomness, weak $n$-randomness, and higher randomness notions. Here are some problems attempting to systemically understand the collection of randomness notions.

**Problem 1.** Axiomatize randomness.

Axioms will probably include versions of preservation of randomness, no randomness from nothing, and van Lambalgen’s theorem—or possibly an entirely different approach. Van Lambalgen [75] attempted an axiomatization. Recently, both Rute and Simpson\(^2\) have also been working separately on axiomatizations.

**Problem 2.** Characterize the nice randomness notions.

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1 [http://kenshi.miyabe.name/ara2014/](http://kenshi.miyabe.name/ara2014/)
2 [See http://homepages.inf.ed.ac.uk/als/Talks/leeds-istr15.pdf](http://homepages.inf.ed.ac.uk/als/Talks/leeds-istr15.pdf).
This is just another way of stating the last problem. I suspect that the “nice randomness” notions will come out to be those equal to Schnorr randomness relative to a class of oracles. (For example, it is already known $x$ is ML random is and only if it is equal to computable randomness relative to a PA degree. This follows from the paper of Brattka, Miller, and Nies.)

Problem 3. Show that the “weird but natural” randomness notions are more natural randomness notions in disguise.

This has already been done for strong $s$-randomness and $s$-energy randomness. They are equivalent to randomness for capacities (or equivalently randomness for effectively compact classes of measures). I conjecture UD-randomness is really Schnorr randomness for a class of measures, and the differentiability points of all Lipschitz functions of type $\mathbb{R}^n \to \mathbb{R}^m$ are the computable randoms for a certain class of measures.

Problem 4. Let $m \geq 1$ and $n \geq 1$. Characterize the points of differentiability of all computable Lipschitz functions of type $\mathbb{R}^n \to \mathbb{R}^m$.

More specifically, is the following conjecture true? If $x \in \mathbb{R}^n$ is a point of differentiability of all computable Lipschitz functions of type $\mathbb{R}^n \to \mathbb{R}^m$ iff

1. $(m \geq n)$ $x$ is computably random w.r.t. $\mathbb{R}^n$ with the Lebesgue measure.

2. $(m = 1 < n)$ $x$ is computably random w.r.t. some measure $\mu$ (possibly non-computable) with $n$ independent Alberti’s representations (equivalently, $(\mathbb{R}^n, \mu)$ is a Lipschitz differentiability space). (Possibly some smaller subclass of measures $\mu$ is sufficient.)

3. $(m < n)$ $x$ is computably random w.r.t. some measure $\mu$ in a nice class of measures.

On a related note, we need to be more careful what we call randomness notions. Randomness notions are associated to a measure (or class of measures). Many things we call randomness notions are not randomness notions on the Lebesgue measure. Here I am defining a randomness notion as a class of “computable tests” $T$, each of which is identified with a null set, and this class of tests can also be relativized to oracles. A “randomness notion” should only be considered a randomness notion for Lebesgue measure if for each Lebesgue null set $N$, there is a test $T$ relative to an oracle such that $N \subseteq T$. Therefore Kurtz randomness and UD randomness are not randomness notions on the Lebesgue measure!

Problem 5. Understand Martin-Łöf randomness for lower-semicomputable semi-measures better.

There are a lot of definitions out there, both of lower-semicomputable semi-measure, and of randomness for such an object. This needs to be

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3This problem is a major part of his Alex Galicki’s ongoing PhD thesis. I communicated my conjectures to him in Spring 2014 suggesting that he work this out.

4Compare with Preiss and Speight [64, Paragraph above Thm 1.1].

5Compare with Alberti, Csörnyei, and Preiss [1] (also the slide “Differentiability and singular measures” in https://www.ljll.math.upmc.fr/~lemenant/GMT/preiss.pdf) and with Bates [6] which generalizes Alberti, Csörnyei, and Preiss.
worked out. We actually implicitly use randomness for semi-measures already in many of our proofs. (For example, the proof of Miller and Yu’s theorem that if $X$ is MLR, $X \leq_T Y$, and $Y$ is MLR relative to $Z$, then $X$ is MLR relative to $Z$.)

5.2. **Studying the structure of the “random degrees”**. The LR degrees, while a good start, are too coarse for this purpose. The Turing degrees have very little connection to randomness. The truth-table degrees are better; the set $\text{MLR}_{\text{comp}} = \{x \mid x \in \text{MLR}, \mu \text{ computable}\}$ is downward closed in the truth-table degrees (and the same for $\text{SR}_{\text{comp}}$). However, we can do better. I have a number of ideas for this project (see my slides from the 2013 Nancy ARA meeting at [http://www.personal.psu.edu/jmr71/talks/rute_2013_ARA.pdf](http://www.personal.psu.edu/jmr71/talks/rute_2013_ARA.pdf)). Here are some related problems.

**Problem 6.** (Bienvenu and Porter) If $T$ is a computable measure-preserving map and $y$ is Schnorr random, is there some Schnorr random $x$ such that $T(x) = y$?

(Update: Problem 6 has been since answered negatively by Rute.)

For this next problem, if $\mu$ is a computable measure on $2^N \times 2^N$, then there is a kernel measure $\mu(\cdot \mid x)$ defined by $\mu(\sigma \mid x) = \lim_n \frac{\mu([x_n] \times \{\sigma\})}{\mu([x_n] \times 2^N)}$. If $x$ is computably random on the marginal measure $\mu_1$ (where $\mu_1(\sigma) = \mu(\{\sigma\} \times 2^N)$) then $\mu(\cdot \mid x)$ is a measure, although it may not be computable from $x$.

**Problem 7.** (Shen, Takahashi, Bauwens) Suppose $\mu$ is computable measure on $2^N \times 2^N$ and $x$ is Martin-Lof random on $\mu_1$. Characterize the set $\{y \mid (x,y) \in \text{MLR}_\mu\}$ in terms of $x$ and $\mu(\cdot \mid x)$. Is this possible without knowing $\mu$?

Takahashi conjectured that $(x,y) \in \text{MLR}_\mu$ if and only if $x \in \text{MLR}_{\mu_1}$ and $y$ is blind random for $\mu(\cdot \mid x)$ relative to $x$. But this is wrong. Bauwens gave a counterexample (communicated to Shen and me).

**Problem 8.** If $x$ is computably random, and $y$ is computably random relative to $x$, then is $(x,y)$ necessarily computably random?

5.3. **Extending results about MLR to SR**. Most of the results about MLR and analysis extend to SR. This includes almost all the work on basic measure theory, almost all the work on Brownian motion, probably much of the work on effective dimension, and a good deal of the results about ergodic theory. (Miyabe has also done a great job extending the results on LR degrees.) The tools to do this are almost mature. They include

1. A good understanding of the connection between measurable functions and Schnorr randomness, including Schnorr layerwise computability and preservation of Schnorr randomness. (Miyabe [48]; Pathak, Rojas, Simpson [63]; Rute [69])

2. A good understanding of the connection between effective rates of convergence and Schnorr randomness. (Gács, Hoyrup, Rojas [22]; Galatolo, Hoyrup, Rojas [24]; Rute [69])

3. A good-enough version of no-randomness-from-nothing. (Rute⁶)

⁶In preparation. See slides at [http://www.personal.psu.edu/jmr71/talks/rute_2013_ARA.pdf](http://www.personal.psu.edu/jmr71/talks/rute_2013_ARA.pdf).
(4) Van Lambalgen’s theorem holds for Schnorr randomness under uniform reducibility. (Miyabe [47]; Miyabe, Rute [49])
(5) Schnorr randomness can be defined on non-computable measures. (Rute?)

Now the problem is to actually do the work. (Update: Rute has slowly been compiling a bunch of facts about Schnorr randomness and Brownian motion.)

5.4. Gathering the information we have on randomness and analysis. There has been a lot of work done in the last decade on randomness and analysis, but it is spread throughout a bunch of papers. There are not any books yet on the subject. (The closest approximations are Gács’s lecture notes [23]; the article by Bienvenu, Gács, Hoyrup, Rojas, and Shen [9]; and the second part of Rute’s thesis [69].) It would be nice to gather this material better. (Update: Rute is currently writing a survey article on randomness and analysis.)

Part 4. Computability theory and its connections to other areas

6. Patey: effectively bi-immune sets and computably bounded DNC functions

The following section has been written by Ludovic Patey in March 2015.

Definition 6.1. A function $f$ is $h$-bounded for some computable $h$ if $f(e) \leq h(e)$ for all $e$. A set $A = \{x_0 < x_1 < \ldots\}$ is $h$-bounded if its principal function $(p_A : n \mapsto x_n)$ is $h$-bounded. A function $f$ is fixed-point free if $W_{f(e)} \neq W_e$ for all $e$. A function $f$ is diagonally non-computable (DNC) if $f(e) \neq \Phi_e(e)$ for all $e$. A function $f(\cdot, \cdot)$ is escaping if $|W_e| \leq n \rightarrow f(e,n) \notin W_e$ for all $e$.

The degrees of DNC functions are known to be equivalent to the degrees of effectively immune sets. Jockusch and Lewis [31] proved that one can compute a bi-immune set from a DNC function, and asked whether every DNC function computes an effectively bi-immune set. Beros [7] answered negatively with an elaborate construction. We prove that every effectively bi-immune set computes a computably bounded DNC function. This fact is sufficient to answer Jockusch and Lewis question, since it is known that there exists a DNC function computing no computably bounded DNC function [2].

Lemma 6.2. Every effectively co-immune set is computably bounded.

Proof. Let $X$ be an $h$-co-immune set for some computable function $h$. We first build a computable function $f$ such that $f(n)$ bounds the $n$th element of $X$. Let $j$ be a computable function which on input $n$ returns the value $n + h(e)$ for some $e$ such that $W_e = |n, n + h(e)|$. Such a function is computable by uniformity of Kleene’s recursion theorem. Consider the following $f$ defined by

$$f(0) = 0 \text{ and } f(n + 1) = j(f(n))$$

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In preparation. See slides at [http://www.personal.psu.edu/jmr71/talks/rute_2014_jmm.pdf](http://www.personal.psu.edu/jmr71/talks/rute_2014_jmm.pdf).
We prove by induction over $n > 0$ that the $n$th element of $X$ is smaller than $f(n)$. Suppose that the first $n$ elements of $X$ are smaller than $f(n)$. $f(n + 1) = j(f(n)) = f(n) + h(e)$ for some $e$ such that $W_e = [f(n), f(n) + h(e))$. $|W_e| \geq h(e)$ so by $h$-co-immunity of $X$, $W_e \not\subseteq X$ and so there must be an element of $X$ in the interval $[f(n), f(n) + h(e))$ and therefore the $(n+1)$th element of $X$ is smaller than $f(n) + h(e) = f(n+1)$.

**Theorem 6.3.** Fix a set $X$. The following are equivalent:

(i) $X$ computes a computably bounded effectively immune set.

(ii) $X$ computes a computably bounded fixed-point free function.

(iii) $X$ computes a computably bounded DNC function.

(iv) $X$ computes a computably bounded escaping function.

*Proof.* This is exactly the standard proof of equivalent between effectively immune sets, fixed-point free functions, DNC functions and escaping function, noticing that we can transmit the computable bound.

**Corollary 6.4.** Every effectively bi-immune set computes a computably bounded DNC function.

**Corollary 6.5 (Beros).** There exists a DNC function computing no effectively bi-immune set.

7. Up to $2 \cdot 2 \cdot 2^{\aleph_0}$ Cardinal Invariants, and Their Counterparts in Computability Theory

Jörg Brendle and André Nies met in Kobe, Japan. They discussed $2 \cdot 2$ families of cardinal invariants parameterised by reals. There are two families, each with a duals. The first family is defined in terms of a bound $h$ on functions in $\omega^\omega$. The second is defined by a real parameter as a bound on asymptotic density. All characteristics have analogs in computability theory.

Separation is unknown in many interesting cases, in both areas. One particular separation in computability would solve the Gamma question posed in [3].

We don’t claim originality for all the notions. A lot of them are at least implicit in previous work. Set theory: Goldstern and Shelah; Kihara; Computability theory: Andrews et al.; Monin and Nies.

7.1. Background. We follow Brendle et al. [11], some of which in turn relies on work of Rupprecht [67] and his thesis [68].

Let $R \subseteq X \times Y$ be a relation between spaces $X,Y$ (such as Baire space) satisfying $\forall x \exists y (xRy)$ and $\forall y \exists x \neg (xRy)$. Let $S = \{(y, x) \in Y \times X: \neg xRy\}$.

**Definition 7.1.** We write

$$d(R) = \min \{|G| : G \subseteq Y \land \forall x \in X \exists y \in G xRy\}.$$ 

$$b(R) = d(S) = \min \{|F| : F \subseteq X \land \forall y \in Y \exists x \in F \neg xRy\}.$$
7.2. The parameterised families of relations. We will study $d(R)$ and $b(R)$ for two types of relations $R$.

1. Let $h : \omega \to \omega$ (usually unbounded). Define for $x \in \omega$ and $y \in \Pi_n \{0, \ldots, h(n) - 1\}$,

$$x \neq^*_h y \iff \forall \infty n \{x(n) \neq y(n)\}.$$

2. Let $0 \leq p \leq 1/2$. Define, for $x, y \in \omega^2$

$$x \sim^*_p y \iff \rho(x \leftrightarrow y) > p,$$

where $x \leftrightarrow y$ is the set of $n$ such that $x(n) = y(n)$, and $\rho$ denotes the lower density: $\rho(z) = \lim \inf_n |z \cap n|/n$.

7.3. The cardinal characteristics. It will be helpful to express Definition 7.1 for these relations in words.

$\mathfrak{d}(\neq^*_h)$ is the least size of a set $G$ of $h$-bounded functions so that for each function $x$ there is a function $y$ in $G$ such that $\forall \infty n \{x(n) \neq y(n)\}$. (Of course it suffices to require this for $h$-bounded $x$.)

$\mathfrak{b}(\neq^*_h)$ is the least size of a set $F$ of functions such that for each $h$-bounded function $y$, there is a function $x$ in $F$ such that $\exists \infty n x(n) = y(n)$. (Of course we can require that each function in $F$ is $h$-bounded.)

The characteristics $\mathfrak{b}(\neq^*_h)$ have been studied in [62] within a more general framework; the notation there is $c^3_{h,1}$. See Thm. 7.5 below.

$\mathfrak{d}(\sim^*_p)$ is the least size of a set $G$ of bit sequences so that for each sequence $x$ there is a sequence $y$ in $G$ so that $\rho(x \leftrightarrow y) > p$.

$\mathfrak{b}(\sim^*_p)$ is the least size of a set $F$ of bit sequences such that for each bit sequence $y$, there is a sequence $x$ in $F$ such that $\rho(x \leftrightarrow y) \leq p$.

7.4. Basic facts about the cardinal characteristics. We first record the obvious monotonicity properties.

Fact 7.2.

(i) $h \leq^* g$ implies $\mathfrak{d}(h) \geq \mathfrak{d}(g)$ and $\mathfrak{b}(h) \leq \mathfrak{b}(g)$.

(ii) $p \leq q$ implies $\mathfrak{d}(\sim^*_p) \leq \mathfrak{d}(\sim^*_q)$ and $\mathfrak{b}(\sim^*_p) \geq \mathfrak{b}(\sim^*_q)$.

The following facts are somewhat less obvious.

Fact 7.3.

(i) Let $h$ be bounded. Then (a) $\mathfrak{d}(\neq^*_h) = 2^{\aleph_0}$ and (b) $\mathfrak{b}(\neq^*_h) = 2$.

(ii) (a) $\mathfrak{d}(\sim^*_0.5) = 2^{\aleph_0}$ and (b) $\mathfrak{b}(\sim^*_0.5) = 2$.

Proof. (i.a) missing

(i.b) missing

(ii.a) Suppose $G$ is as in the definition above so that $|G| = \mathfrak{d}(\sim^*_0.5)$. Define a map $\Theta : G \to \mathcal{P}(\omega)$ by $\Theta(y)(n) = 1$ iff at least half the bits of $y$ in the
interval $I_n = [n!, (n+1)!]$ are 1. Given $x$ that is constant on each $I_n$, pick $y$ such that $\rho(x \leftrightarrow y) > 0.5$. Then $\Theta(y) = x$. This shows that $|G| = 2^{\aleph_0}$.

(ii.b) In the definition of $b(\sim_{0.5})$, let $F = \{0^\infty, 1^\infty\}$. □

7.5. Placement within the Cichoń Diagram. A version of Cichoń’s Diagram is in Figure 1.

![Cichoń’s diagram](image)

Figure 1. Cichoń’s diagram.

We conjecture that one can insert the new characteristics in the right lower, and left upper regions of the diagram.

**Proposition 7.4.** Let $h$ be a function. Let $0 < p < 1/2$.

(i) $\text{cover}(M) \leq d(\neq^*_h) \leq \text{non}(N)$ and $\text{cover}(M) \leq d(\sim_p) \leq \text{non}(N)$.

Dually,

(ii) $\text{cover}(N) \leq b(\neq^*_h) \leq \text{non}(M)$ and $\text{cover}(N) \leq b(\sim_p) \leq \text{non}(M)$.

**Partial proof.** First we settle the case of characteristics involving $\neq^*_h$. It is trivial that $d(\neq^*_h) \leq d(\neq^*_h)$ and $b(\neq^*_h) \leq b(\neq^*_h)$. Using the equalities in the diagram, this yields two inequalities involving $\neq^*_h$.

Next we consider the case of characteristics involving $\sim_p$.

$b(\sim_p)$: For each bit sequence $x$, for almost every $y$ we have $\rho(x \leftrightarrow y) = 1/2$ by law of large numbers, so the set $\{y: \rho(x \leftrightarrow y) \leq p\}$ is null. Hence any set $F$ as in the definition of $b(\sim_p)$ yields a collection of null sets of size at most $|F|$ with union $2^{\aleph_0}$. Hence $\text{cover}(N) \leq b(\sim_p)$.

$d(\sim_p)$: Let $V$ be a non-null set. For each $x$, as said the set $\{y: \rho(x \leftrightarrow y) > p\}$ is co-null and hence contains an element $y \in V$. Therefore $d(\sim_p) \leq \text{non}(N)$. □

7.6. Consistency of separation of uncountably many $b(\neq^*_h)$.

**Theorem 7.5** (Kamo and Osuga [62], Thm. 1). Let $\delta$ be an ordinal and let $\langle \lambda_\alpha \rangle_{\alpha < \delta}$ be a strictly increasing sequence of regular cardinals. Let $\kappa \geq \delta$ be a cardinal such that $\kappa = \kappa^{<\lambda_\alpha}$ for each $\alpha < \delta$. 

There is a forcing notion $P$ with the c.c.c. that forces: there is a sequence of functions $\langle h_\alpha \rangle_{\alpha < \delta}$ such that $b(\neq^*_h) = \lambda_\alpha$ for each $\alpha$, and $\kappa = 2^{\aleph_0}$.

Moreover, if $\delta \leq b$ then the sequence $\langle h_\alpha \rangle_{\alpha < \delta}$ can be chosen in the ground model.

7.7. The corresponding highness properties in computability theory. We will re-obtain some properties that are at least close to some well known classes. Others are new.

As before, we follow [11]. Again, let $R \subseteq X \times Y$ be a relation between spaces $X, Y$, and let $S = \{ (y, x) \in Y \times X : \neg xRy \}$. Suppose we have specified what it means for objects $x$ in $X$, $y$ in $Y$ to be computable in a Turing oracle $A$. We denote this by for example $x \equiv^T_A$. In particular, for $A = \emptyset$ we have a notion of computable objects.

Let the variable $x$ range over $X$, and let $y$ range over $Y$. We define the highness properties

$$\mathcal{B}(R) = \{ A : \exists y \leq_T A \forall x \text{ computable } [xRy] \}$$

$$\mathcal{D}(R) = \mathcal{B}(S) = \{ A : \exists x \leq_T A \forall y \text{ computable } [\neg xRy] \}$$

Let $h$ be computable, and let $p \in [0, 1]$. Recall the relations $\neq^*_h$ and $\sim_p$ from Subsection 7.2. Expressing the definitions above in words,

$\mathcal{D}(\neq^*_h)$ is the class of oracles $A$ that compute a function $x$ such that for each computable function $y \leq h$, we have $\exists n [x(n) = y(n)]$. This is called “$h$-infinitely often equal” in [50].

$\mathcal{B}(\neq^*_h)$ is the class of oracles $A$ that compute a function $y \leq h$ such that for each computable function $x$, we have $\forall n x(n) \neq y(n)$.

$\mathcal{D}(\sim_p)$ is the class of oracles $A$ that compute a set $x$ such that for each computable set $y$, we have $\varrho(x \leftrightarrow y) \leq p$. We note that $\Gamma(A) < p \Rightarrow A \in \mathcal{D}(\sim_p) \Rightarrow \Gamma(A) \leq p$.

The right arrow cannot obviously be reversed. It could be that $\Gamma(A) \leq p$ because $\gamma(x)$ gets arbitrarily close to $p$ from above, for sets $x \leq_T A$. For $p = 1/2$, the reverse arrow holds by Fact 7.7 below. (There is a related open question at the end of the paper Andrews et al. [4].)

$\mathcal{B}(\sim_p)$ is the class of oracles $A$ that compute a set $y$ such that for each computable set $x$, we have $\varrho(x \leftrightarrow y) > p$. This is some kind of dual $\Gamma$ class.

7.8. Basic facts about the highness properties. Again we note obvious monotonicity properties.

Fact 7.6.

(i) $h \leq^* g$ implies $\mathcal{D}(h) \supseteq \mathcal{D}(g)$ and $\mathcal{B}(h) \subseteq \mathcal{B}(g)$.

(ii) $p \leq q$ implies $\mathcal{D}(\sim_p) \subseteq \mathcal{D}(\sim_q)$ and $\mathcal{B}(\sim_p) \supseteq \mathcal{B}(\sim_q)$.

We proceed to the analog of Fact 7.3.

Fact 7.7.
(i) Let \( h \) be bounded. Then
\[(a) \ D(\neq^*_h) = \text{non-computable} \quad \text{and} \quad (b) \ B(\neq^*_h) = \emptyset \]
(ii) \( (a) \ D(\sim 0.5) \Leftrightarrow \text{non-computable} \quad \text{and} \quad (b) \ B(\sim 0.5) = \emptyset \).

Proof. (i.a) This is nontrivial: see Monin and Nies [50, Thm. IV.1].
(i.b) Trivial: take constant functions \( x \) with value up to the bound on \( h \).

(ii.a) \( A \in D(\sim 0.5) \) implies \( \Gamma(A) \leq 0.5 \), so \( A \) is non-computable.
Now suppose \( A \) is non-computable, and let \( x(k) = A(n) \) for each \( k \in I_n \)
defined as in the corresponding fact above. Then for each computable \( y \) we
have \( \rho(x \leftrightarrow y) \leq 0.5 \), else we could decide \( A(n) \) for almost all \( n \) by looking
at the majority of values of \( y \) in \( I_n \).
(ii.b) Trivial again: take a computable \( x \) and its complement. □

### 7.9. Placement within the Cichoń Diagram.

The computability theoretic Cichoń Diagram is given in Figure 2.

![Cichoń Diagram](image)

Figure 2. The analog of Cichoń’s diagram in computability.

(ia) and (ib) below is known; see [50]. We conjecture the analogs (iia) and (iib).

**Proposition 7.8.** Let \( h \) be an order function. Let \( 0 < p < 1/2 \) be computable. Let \( A \subseteq \omega \).

- (ia) \( A \) is of h.i. degree \( \Rightarrow \) \( A \in D(\neq^*_h) \Rightarrow \) \( A \) is weakly Schnorr engulfing.
- (ib) \( A \) is of h.i. degree \( \Rightarrow \) \( A \in D(\sim p) \Rightarrow \) \( A \) is weakly Schnorr engulfing.
- (iia) \( A \) computes a Schnorr random \( \Rightarrow \) \( A \in B(\neq^*_h) \Rightarrow \) \( A \) is high or d.n.c.
- (iib) \( A \) computes a Schnorr random \( \Rightarrow \) \( A \in B(\sim p) \Rightarrow \) \( A \) is high or d.n.c.
7.10. Relating the two types of highness properties.

**Theorem 7.9** (Monin and Nies [50], Thm III.4 restated). Let $h$ be sufficiently fast growing in that $\forall n h(n) \geq 2^{(dn)}$, for some $d > 1$. Then,

$$A \in \mathcal{D}(\neq) \Rightarrow A \in \mathcal{D}(\sim p) \text{ for each } p > 0; \text{ equivalently, } \Gamma(A) = 0.$$ 

**Part 5. Reverse mathematics**

The following two sections have been written by Ludovic Patey in March 2015.

8. Patey: Pseudo Ramsey’s theorem for pairs

Pseudo Ramsey’s theorem for pairs has been introduced by Murakami, Yamazaki and Yokoyama in [52]. They proved that it is between the chain antichain principle and the ascending descending sequence principle, and asked whether it was equivalent to one of them. We answer positively.

**Definition 8.1** (Ascending descending sequence). ADS is the statement “Every linear order has an infinite ascending or descending sequence”.

**Definition 8.2** (Pseudo Ramsey’s theorem). A coloring $f : [\mathbb{N}]^2 \to 2$ is *semi-transitive* if whenever $f(x,y) = 1$ and $f(y,z) = 1$, then $f(x,z) = 1$ for $x < y < z$. A set $H = \{ x_0 < x_1 < \ldots \}$ is *pseudo-homogeneous* for a coloring $f : [\mathbb{N}]^n \to k$ if $f(x_i, \ldots, x_{i+n-1}) = f(y_j, \ldots, y_{j+n-1})$ for every $i, j \in \mathbb{N}$. psRT$_2^k$ is the statement “Every coloring $f : [\mathbb{N}]^n \to k$ has an infinite pseudo-homogeneous set”.

**Theorem 8.3.** RCA$_0 \vdash$ psRT$_2^2 \leftrightarrow$ ADS

*Proof.* The direction psRT$_2^2 \rightarrow$ ADS is Theorem 24 in [52]. We prove that ADS $\rightarrow$ psRT$_2^2$. Let $f : [\mathbb{N}]^2 \to 2$ be a coloring. The reduction is in two steps. We first define a $\Delta^0_1$ semi-transitive coloring $g : [\mathbb{N}]^2 \to 2$ such that every infinite set pseudo-homogeneous for $g$ computes an infinite set pseudo-homogeneous for $f$. Then, we define a $\Delta^0_1$ linear order $h : [\mathbb{N}]^2 \to 2$ such that every infinite set pseudo-homogeneous for $h$ computes an infinite set pseudo-homogeneous for $g$. We conclude by applying ADS over $h$.

*Step 1:* Define the coloring $g : [\mathbb{N}]^2 \to 2$ for every $x < y$ by $g(x, y) = 1$ if there exists a sequence $x = x_0 < \cdots < x_l = y$ such that $f(x_i, x_{i+1}) = 1$ for every $i < l$, and $g(x, y) = 0$ otherwise. The function $g$ is a semi-transitive coloring. Indeed, suppose that $g(x, y) = 1$ and $g(y, z) = 1$, witnessed respectively by the sequences $x = x_0 < \cdots < x_m = y$ and $y = y_0 < \cdots < y_n = z$. The sequence $x = x_0 < \cdots < x_m = y_0 < \cdots < y_n = z$ witnesses $g(x, z) = 1$. We claim that every infinite set $H = \{ x_0 < x_1 < \ldots \}$ pseudo-homogeneous for $g$ computes an infinite set pseudo-homogeneous for $f$. If $H$ is pseudo-homogeneous with color 0, then $f(x_i, x_{i+1}) = 0$ for each $i$, otherwise the sequence $x_i < x_{i+1}$ would witness $g(x_i, x_{i+1}) = 1$. Thus $H$ is pseudo-homogeneous for $f$ with color 0. If $H$ is pseudo-homogeneous with color 1, then define the set $H_1 \supseteq H$ to be the set of integers in the sequences witnessing $g(x_i, x_{i+1}) = 1$ for each $i$. The set $H_1$ is $\Delta^0_1(\neq H)$ and pseudo-homogeneous for $f$ with color 1.
Step 2: Define the coloring $h : [N]^2 \to 2$ for every $x < y$ by $h(x, y) = 0$ if there exists a sequence $x = x_0 < \cdots < x_l = y$ such that $g(x_i, x_{i+1}) = 0$ for every $i < l$, and $h(x, y) = 1$ otherwise. For the same reasons as for $g$, $h(x, z) = 0$ whenever $h(x, y) = 0$ and $h(y, z) = 0$ for $x < y < z$. We need to prove that if $h(x, z) = 0$ then either $h(x, y) = 0$ or $h(y, z) = 0$ for $x < y < z$. Let $x = x_0 < \cdots < x_l = z$ be a sequence witnessing $h(x, z) = 0$. If $y = x_i$ for some $i < l$ then the sequence $x = x_0 < \cdots < x_i = y$ witnesses $h(x, y) = 0$. If $y \neq x_i$ for every $i < l$, then there exists some $i < l$ such that $x_i < y < x_{i+1}$. By semi-transitivity of $g$, either $g(x_i, y) = 0$ or $g(y, x_{i+1}) = 0$. In this case $h(x, y) = 0$ or $y < x_{i+1} < \cdots < x_l = z$ witnesses $h(y, z) = 0$. Therefore $h$ is a linear order. For the same reasons as for $g$, every infinite set pseudo-homogeneous for $h$ computes an infinite set pseudo-homogeneous for $g$. This last step finishes the proof. \hfill \Box

9. Patey: Increasing polarized Ramsey’s theorem for pairs

The Ramsey-type weak König’s lemma has been introduced by Flood in [18] under the name RKL, and later renamed RWKL by Bienvenu, Patey and Shafer. Independently, the increasing polarized Ramsey’s theorem has been introduced by Dzhafarov and Hirst [14] to find new principles between stable Ramsey’s theorem for pairs and Ramsey’s theorem for pairs. We prove that the two principles are equivalent over RCA₀.

Definition 9.1 (Ramsey-type weak König’s lemma). Given an infinite set of strings $S \subseteq 2^{<\mathbb{N}}$, let $T_S$ denote the downward closure of $S$, that is, $T_S = \{ \tau \in 2^{<\mathbb{N}} : (\exists \sigma \in S)[\sigma \leq \tau] \}$. A set $H \subseteq \mathbb{N}$ is homogeneous for a $\sigma \in 2^{<\mathbb{N}}$ if $(\exists c < 2)(\forall i \in H)(i < |\sigma| \to \sigma(i) = c)$, and a set $H \subseteq \mathbb{N}$ is homogeneous for an infinite tree $T \subseteq 2^{<\mathbb{N}}$ if the tree $\{ \sigma \in T : H \text{ is homogeneous for } \sigma \}$ is infinite. 2-RWKL is the statement “For every set of strings $S$, there is an infinite set which is homogeneous for $T_S$”.

Definition 9.2 (Increasing polarized Ramsey’s theorem). A set increasing $p$-homogeneous for $f : [N]^n \to k$ is a sequence $\langle H_1, \ldots, H_n \rangle$ of infinite sets such that for some color $c < k$, $f(x_1, \ldots, x_n) = c$ for every increasing tuple $\langle x_1, \ldots, x_n \rangle \in H_1 \times \cdots \times H_n$. IPTₖ is the statement “Every coloring $f : [N]^n \to k$ has an infinite increasing $p$-homogeneous set”.

Theorem 9.3. RCA₀ $\vdash$ IPT₃ ↔ 2-RWKL

Proof. IPT₃ $\rightarrow$ 2-RWKL: Let $S = \{ \sigma_0, \sigma_1, \ldots \}$ be an infinite set of strings such that $|\sigma_i| = i$ for each $i$. Define the coloring $f : [N]^2 \to 2$ for each $x < y$ by $f(x, y) = \sigma_y(x)$. By IPT₃, let $\langle H_1, H_2 \rangle$ be an infinite set increasing $p$-homogeneous for $f$ with some color $c$. We claim that $H_1$ is homogeneous for $T_S$ with color $c$. We will prove that the set $I = \{ \sigma \in T_S : H_1 \text{ is homogeneous for } \sigma \}$ is infinite. For each $y \in \mathbb{N}$, let $\tau_y$ be the string of length $y$ defined by $\tau_y(x) = f(x, y)$ for each $x < y$. By definition of $f$, $\tau_y \in S$ for each $y \in \mathbb{N}$. By definition of $\langle H_1, H_2 \rangle$, $\tau_y(x) = c$ for each $x \in H_1$ and $y \in H_2$. Therefore, $H_1$ is homogeneous for $\tau_y$ with color $c$ for each $y \in H_2$. As $\{ \tau_y : y \in H_2 \} \subseteq I$, the set $I$ is infinite and therefore $H_1$ is homogeneous for $T_S$ with color $c$.

2-RWKL $\rightarrow$ IPT₃: Let $f : [N]^2 \to 2$ be a coloring. For each $y$, let $\sigma_y$ be the string of length $y$ such that $\sigma_y(x) = f(x, y)$ for each $x < y$, and
let $S = \{ \sigma_i : i \in \mathbb{N} \}$. By 2-RWKL, let $H$ be an infinite set homogeneous for $T_S$ with some color $c$. Define $\langle H_1, H_2 \rangle$ by stages as follows. At stage 0, $H_{1,0} = H_{2,0} = \emptyset$. Suppose that at stage $s$, $|H_{1,s}| = |H_{2,s}| = s$, $H_{1,s} \subseteq H$ and $\langle H_{1,s}, H_{2,s} \rangle$ is a finite set increasing $p$-homogeneous for $f$ with color $c$. Take some $x \in H$ such that $x > \max(H_{1,s}, H_{2,s})$ and set $H_{1,s+1} = H_{1,s} \cup \{ x \}$. By definition of $H$, there exists a string $\tau < \sigma_y$ for some $y > x$, such that $|\tau| > x$ and $H$ is homogeneous for $\tau$ with color $c$. Set $H_{2,s+1} = H_{2,s} \cup \{ y \}$. We now check that the finite set $\langle H_{1,s+1}, H_{2,s+1} \rangle$ is an increasing $p$-homogeneous for $f$ with color $c$. By induction hypothesis, we need only to check that $f(z, y) = c$ for every $z \in H_{1,s+1}$. By definition of homogeneity and as $H_{1,s+1} \subset H$, $\sigma_y(z) = c$ for every $y \in H_{1,s+1}$. By definition of $\sigma_y$, $f(z, y) = c$ for every $z \in H_{1,s+1}$. This finishes the proof. \hfill \Box

Part 6. Higher Randomness

10. Yu and Zhu: On $NCR_L$

This is a joint work of Liang Yu (Heidelberg) and Yizheng Zhu (Münster). In Logic Blog 2014, Prop. 9.4, it was proved that $NCR_L$ is a $\Pi^1_3$-countable set.

We assume $PD$ throughout this section.

**Proposition 10.1.** If $A$ is a $\Pi^1_2$-countable set, then $A \subseteq NCR_L$.

**Proof.** Suppose that $\varphi$ is a $\Pi^1_2$-formula and $x$ is a so that $\varphi(x)$ and the set $\{ y \mid \varphi(y) \}$ is countable. Suppose that there is a continuous measure $\rho$ so that $x$ is $L$-random respect to $\rho$. Note that $L[\rho, x] \models \varphi(x)$ by the Shoenfield absoluteness. Then $p \models \varphi(\dot{x})$ for some condition $p$. Then for any $L$-random real $y \in p$, $L[\rho, y] \models \varphi(y)$. By Shoenfield absoluteness again, $\varphi(y)$ is true which contradicts to the assumption. \hfill \Box

Note that there is a contractible real which does not belong to any $\Pi^1_2$-countable set.

**Definition 10.2.** $Q_3 = \{ x \mid \exists \alpha < \omega_1 \forall \dot{z} (|z| = \alpha \rightarrow x \leq \Delta^1_2 z) \}$.

$Q$-theory was introduced and studied by Harrington, Kechris, Martin, Solovay and Woodin.

**Proposition 10.3.** For any real $x$, there is a real $y \geq_T x$ so that there is a continuous measure $\rho \leq_T y$ so that $y$ is $L$-random respect to $\rho$.

**Proof.** For any $x$, let $r$ be $L[x]$-random. Then $y = x \oplus r$ is $L$-random respect to $\rho$ for the follow continuous measure $\rho \leq_T x \oplus r$.

$$
\rho(\sigma^i) = \begin{cases} 
\frac{\rho(\sigma)}{2}, & |\sigma| \text{ is odd}, \\
\rho(\sigma), & |\sigma| \text{ is even} \land i = x(\frac{|\sigma|}{2}), \\
0, & \text{Otherwise}.
\end{cases}
$$

\hfill \Box

Let

$$B = \{ y \mid \exists \rho \leq_T y (\rho \text{ is a continuous measure and } y \text{ is } L\text{-random respect to } \rho) \}.$$ 

Then $B$ has cofinally many $L$-degrees.
Proposition 10.4. \(2^\omega \setminus B\) is uncountable.

Actually \(B\) is \(\Pi^1_2\).

Let \(D = \{y_0 \mid \forall y (y \geq T y_0 \rightarrow y \in B)\}\). Then \(D\) is a nonempty \(\Pi^1_2\)-set and so contains a real \(z_0 < T y_{0,3}\) but \(y_{0,3} \notin Q_3\) which is a base for \(D\), where \(y_{0,3}\) is the \(Q_3\)-complete real.

**Lemma 10.5.** For any \(y \notin Q_3\) \(z_0\), \(y \oplus z_0\) is \(L\)-random respect to some continuous measure \(\rho \leq T y \oplus z_0\). Further more, \(y\) must be \(L\)-random respect to some continuous measure.

**Proof.** Suppose that \(y \neq Q_3 z_0\), then by Posner-Robinson Theorem relative to \(z_0\) (due to Woodin), there is a real \(z_1 \geq T z_0\) so that \(y \oplus z_1 \geq T y_{0,3}\). Then by the discussion above relative to \(z_1\), \(y\) is \(L\)-random respect to some measure \(\rho \leq T y \oplus z_1\). \(\square\)

So \(NCR_L \subseteq \{r \mid r \leq_L z_0\}\).

Since \(NCR_L \subseteq C_3\), we have that \(NCR_L \subseteq Q_3\).

So we have the following result.

**Theorem 10.6.** \(NCR_L\) is a proper subset of \(Q_3\).

Let \(P_2 = \{x \mid \forall y (\kappa^x \leq \kappa^y \implies x \leq_L y)\}\), where \(\kappa^x = ((\aleph_1^+)_{L[x]}\). The following lemma is obvious.

**Lemma 10.7.** \(P_2 \subseteq NCR_L\).

Now let \(M_1\) be the minimal model with a Woodin cardinal. By Steel’s result, \(2^\omega \cap M_1 = Q_3\).

**Theorem 10.8.** \(NCR_L\) is cofinal (in the \(L\)-degree sense) in \(Q_3\).

An immediate conclusion of Theorem 10.8 is

**Corollary 10.9.** \(NCR_L\) is not \(\Sigma^1_3\).

Note that, by 6E14 in Msochvakis book, \(NCR_L \cap \Delta^1_3\) is cofinal in \(\Delta^1_3\).

**Part 7. Group theory and its connections to logic**

11. Tent: Low-tech notes on group extensions

(By Katrin Tent) We explain in a “low-tech” way how to describe and understand group extensions.

11.1. Background. It is well-known that group extensions of a group \(N\) by a group \(G\) can be classified via the second cohomology groups of certain associated modules. See e.g. [66, Ch. 11]. Since this theory is quite involved, we give here an easy description of the class of possible group extensions \(E\) of \(N\) by \(G\), i.e. groups \(E\) containing \(N\) (or an isomorphic copy of \(N\)) as a normal subgroup such that \(E/N \cong G\). One writes this as

\[ 1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1. \]

Suppose that a group \(G\) has the presentation

\[ G = \langle s_1, \ldots, s_k \mid r_1, \ldots, r_m \rangle \cong F_k/R \]
where $F_k$ is the free group of rank $k$ on generators $s_1, \ldots, s_k$ and $R$ is the normal subgroup of $F_k$ generated (as a normal subgroup) by $r_1, \ldots, r_m$.

Now suppose we have an extension $E$ such that

$$1 \longrightarrow N \longrightarrow E \longrightarrow G = \langle s_1, \ldots, s_k \rangle \longrightarrow 1.$$ 

Let $\hat{s}_1, \ldots, \hat{s}_k \in E$ be lifts of $Rs_1, \ldots, Rs_k \in G$, i.e. the canonical projection of $\hat{s}_i$ is $Rs_i$ for $i = 1, \ldots, k$. Clearly the $\hat{s}_i$ act on $N$ by conjugation and hence any word $w = w(s_1, \ldots, s_k)$ in the free group $F_k$ with generators $s_1, \ldots, s_k$ acts as an automorphism of $N$ via the natural conjugation action of $w(\hat{s}_1, \ldots, \hat{s}_k) \in E$. Hence any group extension $E$ of $N$ by a $k$-generated group $G = \langle s_1, \ldots, s_k \rangle$ comes with an action of $F_k = F(s_1, \ldots, s_k)$ on $N$.

11.2. Describing $E$. Towards describing $E$ we will have to express this $F_k$ action on $N$. But this is not sufficient. We are missing the natural maps from $R$ to $N$ that transfer from the “view” $F_k/R$ of $G$ to the view $E/N$. Define $\varphi_E : R \rightarrow N$ by $w(s_1, \ldots, s_k) \mapsto w(\hat{s}_1, \ldots, \hat{s}_k)$ for $w(s_1, \ldots, s_k) \in R$. By $\text{Hom}_{F_k}(R, N)$ we denote the set of homomorphisms from $R$ to $N$ that preserve the $F_k$ action. The following is easy to verify.

Lemma 11.1. Using the previous notation we have $\varphi_E \in \text{Hom}_{F_k}(R, N)$.

The next lemma states that the group $E$ is determined – up to isomorphism over $N$ – by the action of $F_k$ on $N$ and the homomorphism $\varphi_E$:

Lemma 11.2. Using the previous notation suppose that $E^1, E^2$ are groups with a common normal subgroup $N$ such that $E^i/N \cong G, i = 1, 2$. Let $\hat{s}_i^j \in E^i, j = 1, 2, i = 1, \ldots, k$ be lifts of $Rs_1, \ldots, Rs_k$, respectively.

Suppose that the induced $F_k$-actions agree, i.e. for all $a \in N$ we have

$$a\hat{s}_i^1 = a\hat{s}_i^2,$$

Then $\varphi_{E^1} = \varphi_{E^2} \iff E^1$ and $E^2$ are isomorphic over $N$ via an isomorphism $g$ such that $g(\hat{s}_i^1) = \hat{s}_i^2, i = 1, \ldots, k$.

Proof. First suppose that $\varphi_{E^1} = \varphi_{E^2}$. Define

$$g : E_1 \longrightarrow E_2, w(\hat{s}_1^1, \ldots, \hat{s}_k^1)n \mapsto w(\hat{s}_1^2, \ldots, \hat{s}_k^2)n.$$

Note that

$$w(\hat{s}_1^1, \ldots, \hat{s}_k^1)n = w(\hat{s}_1^1, \ldots, \hat{s}_k^1)n'$$

if and only if

$$w(\hat{s}_1^1, \ldots, \hat{s}_k^1)(w(\hat{s}_1^1, \ldots, \hat{s}_k^1))^{-1} \in N$$

if and only if

$$w(s_1, \ldots, s_k)(w(s_1, \ldots, s_k))^{-1} \in R.$$

Since $\varphi_{E^1} = \varphi_{E^2}$, we see that indeed $g$ is well-defined and injective.

Note that $f$ is an $F_2$-homomorphism because the $F_2$-actions on $N$ agree. Since $E^2$ is generated by $N$ and $\hat{s}_j^1, \hat{s}_j^2, j = 1, 2$, this now implies that $g$ is surjective and hence an isomorphism.

For the other direction, suppose that $g : E^1 \longrightarrow E^2$ is an isomorphism over $N$ with $g(\hat{s}_i^1) = \hat{s}_i^2, i = 1, \ldots, k$ and $g \mid N = \text{id}$. For any $w(s_1, \ldots, s_k) \in R$ we thus have $\varphi_{E^1}(w(s_1, \ldots, s_k)) = w(\hat{s}_1^1, \ldots, \hat{s}_k^1) = g(w(\hat{s}_1^1, \ldots, \hat{s}_k^1)) = w(\hat{s}_1^2, \ldots, \hat{s}_k^2) = \varphi_{E^2}(w(s_1, \ldots, s_k))$, proving the claim. □
11.3. **Action of** $\text{Hom}_F(R, Z(N))$. In the following to simplify notation we let $k = 2$. We consider the role of the $F$-homomorphisms from $R$ to the centre of $N$. Recall that a group action is called regular if it is transitive and point stabilizers are trivial.

**Lemma 11.3.** Let $C$ be the center of $N$. The group $\text{Hom}_F(R, C)$ acts regularly on the set

$$X = \{ \varphi_E : E \text{ is extension of } N \text{ by } G \text{ with prescribed } F_2\text{-action on } N \}$$

via $\varphi^n(w(s, t)) = \varphi(w(s, t)) \alpha(w(s, t))$ for $\alpha \in \text{Hom}_F(R, C)$ and $\varphi \in X$.

**Proof.** It is clear that point stabilizers are trivial. To see that the action is transitive, notice that for extensions $E_1, E_2$ of $N$ by $G$ with the prescribed $F_2$-action on $N$, and lifts $\hat{s}_i, t_i, i = 1, 2$ as before we have for all $n \in N$

$$n \varphi_{E_1}(w(s, t)) = n w(\hat{s}_1, \hat{t}_1) = n w(s, t) = n w(\hat{s}_2, \hat{t}_2) = n \varphi_{E_2}(w(s, t))$$

and hence $\varphi_{E_1}(w(s, t))(\varphi_{E_2}(w(s, t)))^{-1} \in C$ and so $\varphi_{E_1}$ and $\varphi_{E_2}$ differ by an element in $\text{Hom}_F(R, C)$.

We next verify that $\varphi^n_{E_1} = \varphi^n_{E_2}$ for some extension $E'$ with the same prescribed $F_2$-action on $N$. Define $E'$ by choosing a transversal $T$ for $F_2/R$ so that any element $w(s, t) \in F_2$ can be written uniquely as $w(s, t) = v(s, t)r(s, t)$ where $v(s, t) \in T, r(s, t) \in R$.

We now define the elements of $E'$ as $nw(s, t) = nv(s, t)\varphi_E(r(s, t))\alpha(r(s, t))$ with the induced multiplication. Then $E'$ is an extension with the prescribed $F(s, t)$ action and $\varphi^n_{E'} = \varphi^n_{E}$. □

12. **Doucha and Nies: groups with bi-invariant metric**

Michal Doucha and Nies worked in Auckland and at the Research Centre Coromandel during Michal’s visit to New Zealand December 2014-January 2015. One topic of their discussions was groups that are equipped with a bi-invariant metric. Michal has already submitted or published several papers on this.

**Definition 12.1.** A metric $d$ on a group $G$ is called bi-invariant if the left and the right translations are isometries, that is, $d(gx, gy) = d(xg, yg) = d(x, y)$ for each $x, y, g \in G$.

Every group $G$ has a trivial such metric, namely $d(x, y) = 0$ if $x = y$, and 1 otherwise. For a a topological group $G$, a natural question is whether $G$ admits a compatible bi-invariant metric, that is, one which induces the given topology on $G$. Recall that a topological group $G$ is metrizable iff $1_G$ has a countable base of neighbourhoods. A well known useful fact is the following. Completely metrizable means that the group admits some complete compatible metric; this includes Polish group in particular.

**Proposition 12.2.** Any compatible bi-invariant metric $d$ on a completely metrizable group $G$ is complete.

To see this, let $\hat{G}$ be the metric completion of $G$ with respect to $d$. Using bi-invariance, one can easily check that for every $x, y, u, v \in G$ we have $d(x, y) = d(x^{-1}, y^{-1})$ and $d(xy, uv) \leq d(x, u) + d(y, v)$, thus the inverse operation is isometric and the multiplication is Lipchitz. It follows that the
group operations extend to the completion $\bar{G}$ (note that this is not true in general for left-invariant metrics, where the inverse does not always extend to the completion). Next, it is a well-known fact from general topology that a completely metrizable subset of a metrizable space is $G_δ$. Thus $G$ is a dense $G_δ$ subset of $\bar{G}$. However, if $G \neq \bar{G}$, then there are left-cosets of $G$ in $\bar{G}$ which are disjoint and still dense $G_δ$. That contradicts the Baire category theorem.

The following is another well known fact.

**Proposition 12.3.** A topological group $G$ has a compatible bi-invariant metric iff $1_G$ has a countable base of neighbourhoods so that each member is closed under conjugation.

For instance, Abelian Polish groups and compact Polish groups admit a compatible bi-invariant metric. For another set of examples, let $M$ be a bounded metric space. Let $G$ be the group of isometries of $M$ with the supremum distance $d(\sigma, \tau) = \sup_{g \in G} d(g\sigma, g\tau)$. Then $d$ is bi-invariant. Note that this group is in general not separable (it is separable if $M$ is compact).

$SL_2(\mathbb{R})$ is an example of a Polish group that does not admit a bi-invariant metric \[25, Exercise 2.1.9\]. Given that abelian Polish groups admit a compatible bi-invariant metric, it is natural to ask what happens for groups that in some sense close to abelian, such as a group that is nilpotent of class 2. For instance, the Heisenberg group $UT_3(\mathbb{R})$, which consists of the upper triangular $3 \times 3$ matrices with 1’s on the main diagonal, is nilpotent of class 2. We equip this group with the usual Euclidean topology of $\mathbb{R}^3$.

**Proposition 12.4.** $UT_3^3(\mathbb{R})$ does not admit a compatible bi-invariant metric.

**Proof.** Suppose that $d$ is a compatible metric on $UT_3^3(\mathbb{R})$. Given matrices

$$A' = \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$A^{-1}A'A = \begin{pmatrix} 1 & a' & c' - ab' + a'b' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Since $d$ is compatible there exists $r > 0$ such that $B_r^d(I)$, the open ball of radius 1 centred at $I$ with respect to $d$, is contained in the open set of matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with $|x|, |y|, |z| < r$. Again since $d$ is compatible, we can choose $A'$ with $a' \neq 0$ and $d(A', I) < 1$. Choosing $b$ sufficiently large we have $|c' - ab' + a'b| > r$. It follows that $d(A^{-1}A'A, I) \geq 1$. So $d(A', I) \neq d(A^{-1}A'A, I)$, whence $d$ is not bi-invariant. □

There however are non-abelian groups with finite-dimensional Euclidean topology equipped with a compatible bi-invariant metric. Consider for example the group $U(n)$ of unitary $n \times n$ matrices. This group is compact and
hence has a bi-invariant metric. To be more concrete, it can for instance be equipped with the Hilbert-Schmidt norm \( \| \cdot \|_{HS} \), i.e. for \( u \in U(n) \) we have \( \| u \|_{HS} = \sqrt{\sum_{i,j} |u_{i,j}|^2} \). Then the corresponding distance is bi-invariant. Indeed, notice that the definition of the norm can be written using the trace, i.e. \( \| u \| = \sqrt{\text{tr}(u^*u)} \). Then direct calculation shows, using the property that \( \text{tr}(t^*ut) = \text{tr}(u) \), that for any \( u, v, t, t' \in U(n) \) we have
\[
\text{tr}((tut' - tvt')(tut' - tvt')) = \text{tr}((u - v)^*(u - v)).
\]

Can the two cases when a Polish group has a bi-invariant metric be combined?

**Question 12.5.** Suppose the Polish group \( G \) has a closed abelian normal subgroup \( A \) such that the Polish group \( G/A \) is compact. Does \( G \) admit a compatible bi-invariant metric?

For finite groups, there is an interesting question:

**Question 12.6.** Does the class \( C \) of finite groups with bi-invariant metric form a Fraisse class?

In effect we are asking whether \( C \) has the amalgamation property (AP). The AP is known for finite groups by an old result of Hall (or possibly Neumann?). It seems to be unknown as well for finite groups equipped with a left (say) invariant metric.

13. Nies: descriptive set theory for profinite groups

A compact topological group \( G \) is called profinite if the clopen sets form a basis for the topology. Equivalently, the open normal subgroups form a base of neighbourhoods of the identity. Since open subgroups of a compact group have finite index, this means that \( G \) is the inverse limit of its system of finite quotients with the natural projection maps.

In a group that is finitely generated as a profinite group, all subgroups of finite index are open [61]. This deep theorem implies that the topological structures is determined by the group theoretic structure. In particular, all abstract homomorphisms between such groups are continuous.

Let \( \hat{F}_k \) be the free profinite group in \( k \) generators (\( k \leq \omega \)). This is the inverse limit of the inverse system \( \langle F_k/N \rangle \) with the canonical maps, where \( N \) ranges over the normal subgroups of finite index. (More generally, every countable residually finite group \( G \) is embedded into a profinite group \( \hat{G} \) in this way.)

A presentation of a profinite group has the form
\[
\hat{F}_k/N
\]
where \( N \) is a closed normal subgroup of \( \hat{F}_k \). One can also think of a presentation as an “expression”
\[
\langle x_1, x_2, \ldots | r_1, r_2, \ldots \rangle
\]
where the list of generators \( x_1, \ldots \) has length \( k \), and the list of relators \( r_i \in \hat{F}_k \) has length at most \( \omega \), and of course the list of relators is finite for the case of f.p. profinite groups. We will see below that the two views are equivalent.
in a “Borel” way. This means that the equivalence is carried out by functions that are Borel between suitable Polish spaces of presentations on the one hand, and closed normal subgroups of \( \hat{F}_k \) on the other hand; isomorphism is preserved in both directions. Generally, if \( X,Y \) are Polish spaces and \( E,F \) equivalence relations on \( X,Y \) respectively, one writes \( X,E \leq_B Y,F \) (or simply \( E \leq_B F \)) if there is a Borel function \( g : X \to Y \) such that \( Euv \leftrightarrow Fg(u)g(v) \) for each \( u,v \in X \).

**Proposition 13.1.** The isomorphism relation \( E_{f.g.} \) between finitely generated profinite groups is Borel equivalent to \( \text{id}_\mathbb{R} \), the identity equivalence relation on \( \mathbb{R} \). The same holds for the isomorphism relation \( E_{f.p.} \) of finitely presented profinite groups.

An equivalence relation that is Borel-below \( \text{id}_\mathbb{R} \) is called smooth. We thank Alex Lubotzky for pointing out the crucial fact in [39, Prop. 2.2] used below to show this smoothness of the isomorphism relation.

**Proof.** To show that \( \text{id}_\mathbb{R} \leq_B E_{f.p.} \), we use the argument of Lubotzky [40, Prop 6.1] that there are continuum many non-isomorphic profinite groups that are f.p. as profinite groups. For a set \( P \) of primes let

\[
G_P = \prod_{p \in P} \text{SL}_2(\mathbb{Z}_p) = \text{SL}_2(\hat{\mathbb{Z}})/\prod_{q \notin P} \text{SL}_2(\mathbb{Z}_q).
\]

Here \( \mathbb{Z}_p \) is the profinite ring of \( p \)-adic integers, and \( \hat{\mathbb{Z}} \) is the completion of \( \mathbb{Z} \), which is isomorphic to \( \prod_{p \text{ prime}} \mathbb{Z}_p \). The second equality shows that \( G_P \) is finitely presented as a profinite group. Clearly the map \( P \to G_P \) is Borel, and \( P = Q \leftrightarrow G_P \cong G_Q \).

We now show that \( E_{f.g.} \leq_B \text{id}_\mathbb{R} \). We note that Silver’s dichotomy theorem, e.g. [25, 5.3.5], implies that any equivalence relation strictly Borel below \( \text{id}_\mathbb{R} \) has countably many classes. So the plain result of Lubotzky [40, Prop 6.1] now already yields the Borel equivalence \( E_{f.p.} \equiv_B E_{f.g.} \equiv_B \text{id}_\mathbb{R} \).

However, by the proof of the result explained above, we in fact don’t need Silver’s result.

The idea is as follows. At first let us only consider presentations in a fixed number \( k \) of generators. Let \( \mathcal{N}(\hat{F}_k) \) be the Polish space of normal closed subgroups of \( \hat{F}_k \) (detail below). The Polish group \( G = \text{Aut}(\hat{F}_k) \) acts continuously on \( \mathcal{N}(\hat{F}_k) \). For \( S,T \in \mathcal{N}(\hat{F}_k) \), we have

\[
\hat{F}_k/S \cong \hat{F}_k/T \leftrightarrow \exists \theta \in G \mid \theta(S) = T
\]

by [39, Prop. 2.2] (which uses a profinite version of Gaschütz’s Lemma on lifting generating sets of finite groups). Note that \( G \) is compact, and in fact, profinite [77, Ex. 6 on page 52] (but not f.g.) So its orbit equivalence relation on \( \mathcal{N}(\hat{F}_k) \) is closed, and hence smooth; see e.g. [25, 5.4.7].

We give some detail on the descriptive set theory. To see that \( \mathcal{N}(\hat{F}_k) \) is a Polish space, note that the compact subsets of a compact metric space \( M \) form a complete metric space \( \mathcal{K}(M) \) with the Hausdorff distance. This space is compact as well, and is the completion of the space of finite subsets of \( M \).

In the case of a compact Polish group \( G \), which we equip with a bi-invariant metric \( d \) as explained in Section 12, the normal subgroups form a closed
subset of \( \mathcal{K}(G) \). Firstly, if \( (U_n)_{n \in \mathbb{N}} \) is a sequence of subgroups converging to \( U \), then \( U \) is a subgroup: if \( a, b \in U \), for each \( \epsilon \), for large \( n \) we can choose \( a_n, b_n \in U_n \) with \( d(a_n, a) < \epsilon \) and \( d(b_n, b) < \epsilon \). Then \( a_nb_n \in U_n \) and \( d(a_nb_nb_nb_n, ab) < 3\epsilon \). Secondly, since the metric is bi-invariant, if all the \( U_n \) are normal in \( G \), then so is \( U \).

(One can avoid compactness of \( G \), as long as there is a bi-invariant metric compatible with the topology: the closed normal subgroups of a Polish group \( G \) form a Polish space, being a closed subset of the Effros space \( \mathcal{F}(G) \) of non-empty closed sets in \( G \). While this space is usually seen as a Borel space, it can be topologized using the Wijsman topology, the weakest topology that makes all the maps \( C \to d(g, C), C \in \mathcal{F}(G), \) continuous.)

The presentations in the second sense above form a Polish space \( \hat{F} \). Let us check that one can pass between the two views of presentations in a Borel way. For one direction, given \( \hat{F}/N \), by the Selection Theorem of Kuratowski/Ryll-Nardzewski (e.g. [25, 14.1.4], we can pick a countable dense subset \( r_1, r_2, \ldots \) of \( N \) in a Borel way.

For the converse direction, suppose we are given a presentation in the sense of (3). We have to find in a Borel way the least closed normal subgroup \( N \) of \( \hat{F}_k \) containing all the relators \( r_i \). Since \( \mathcal{K}(\hat{F}_k) \) is the completion of the metric space of finite subsets of \( \hat{F}_k \) under the Hausdorff distance, it suffices to find for each \( n \) a finite subset \( V_n \) of \( \hat{F}_k \) such that \( d(V_n, N) \leq 1/n \): then \( \langle V_n \rangle_{n \in \mathbb{N}} \) is a Cauchy sequence converging to \( N \).

Let \( X \) be the abstract group generated by the \( x_i \), and let \( R \) be the countable subgroup of \( \hat{F}_k \) that is generated by all the conjugates of the relators by elements of \( X \). Then \( \overline{R} = N \). By compactness of \( \hat{F}_k \), for each \( n \) we can find in a Borel way a finite \( V_n \subseteq R \) such that \( d(V_n, R) \leq 1/n \), and hence \( d(V_n, N) \leq 1/n \) as required.

### 13.1. Complexity of isomorphism for separable profinite groups.

We now consider profinite groups that aren’t necessarily finitely generated. As before, we think of such a group as being given by a presentation \( \hat{F}_k/N \), \( N \) closed, but now always \( k = \omega \). Note that now one has to explicitly require that isomorphisms are continuous (while this was automatic for f.g. groups).

The \( N \in \mathcal{N}(\hat{F}_\omega) \) with \( (\hat{F}_\omega)' \subseteq N \) (commutator subgroup) form a closed subset of \( \mathcal{N}(\hat{F}_\omega) \). To see this, note that it suffices to require that the (dense) countable group \( (\hat{F}_\omega)' \) is contained in \( N \). This is the space \( N_{ab}(\hat{F}_\omega) \) of presentations of separable abelian profinite groups.

As pointed out by A. Melnikov, even isomorphism of these groups is quite complex. Pontryagin duality (see e.g. [28]) is a functor on the category of abelian locally compact groups that associates to each \( G \) the group \( G^* \) of continuous homomorphisms from \( G \) into the circle \( \mathbb{T} \), with the compact-open topology (which coincides with the topology inherited from the product topology if \( G \) is discrete). For a morphism \( \alpha : G \to H \) let \( \alpha^* : H^* \to G^* \) be the morphism defined by \( \alpha^*(\psi) = \alpha \circ \psi \).

The Pontryagin duality theorem says that \( G \cong (G^*)^* \) via the application map, for each locally compact abelian group \( G \). A special case of this states that (discrete) abelian torsion groups \( A \) correspond to abelian profinite groups (see [65, Thm. 2.9.6] for a self-contained proof of this special
case). Then, as $A$ ranges through the abelian countable torsion groups, $A^*$ ranges through the separable abelian profinite groups, with $A \cong B$ iff $B^* \cong A^*$. So isomorphism for abelian countable torsion groups is Borel equivalent to continuous isomorphism of separable abelian profinite groups.

The former isomorphism relation is closely related to the equivalence relation $\mathbf{id}(2^{<\omega_1})$ discussed in [25, Section 9.2], which is modelled on the classification of countable abelian torsion groups via Ulm invariants. Also see the diagram [25, p. 351] which shows that $\mathbf{id}(2^{<\omega_1})$ is strictly between $\mathbf{id}_\mathbb{R}$ and graph isomorphism.

Nies has shown that isomorphism for general separable profinite groups is $S_\infty$-complete, which means it is of the same Borel complexity as isomorphism of countable graphs. The hardness part uses a construction of Alan Mekler [44] coding “nice” countable graphs $A$ into countable nilpotent groups $G(A)$ of class 2 and exponent an odd prime $p$. The idea is now to replace $G(A)$ by a certain profinite completion, and show that the graph can still be recovered by an interpretation.

A computable Polish group is given by a computable metric space together with computable group operations on it. The groups $\mathbb{Z}_p$ and $\hat{F}_k$ discussed above are computable.

**Question 13.2. Determine the complexity of isomorphism for f.g. computable profinite groups.**

14. **Khelif: a free metabelian group of rank at least 2 is bi-interpretable with the ring of integers**

(Translated from French and slightly expanded by Nies.)

Let $G$ be a free metabelian group of rank at least 2. Anatoly Khelif (ca. 2006) has shown that $G$ is bi-interpretable (see e.g. [27, Ch. 5] or [56]) with $(\mathbb{N},+\times)$. This implies that $G$ and $\mathbb{N}$ have the same model theoretic properties. For one thing, $G$ is a prime model of its theory. It also implies that $G$ is quasi-finitely axiomatizable in the sense of Nies [60]: there is a first-order sentence $\varphi$ of which $G$ is, up to isomorphism, the only finitely generated model. Another example of a quasi-finitely axiomatizable group is the Heisenberg group $UT_3^1(\mathbb{Z})$ over the integers. However, Khelif has shown that $UT_3^1(\mathbb{Z})$ is not bi-interpretable with $\mathbb{Z}$ even with parameters (see Thm. 7.16 in [56]). $G$ is also interesting because it is a quasi-finitely axiomatizable group that is not finitely presented. In [60] the first example of such a group was obtained: the restricted wreath product $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$. See the 2007 survey [56] for background.

**Theorem 14.1 (A. Khelif).** Let $G$ be a free metabelian group of rank $m \geq 2$. The group $G$ is bi-interpretable (in parameters) with the ring of integers.

*Proof. The commutator subgroup $G'$ is definable in $G$ by the formula $\varphi(x) \equiv \forall y [x, x^y] = 1$ by a result of Mal’cev [42]. (It is easy to see that each commutator $[p, q]$ satisfies the formula, because $[p, q]^9$ is also a commutator.)

Now let $H = [G, G']$ be the third term of the descending central series of $G$. The subgroup $H$ is also definable as the products of at most $m$ commutators of the form $[x, y]$ where $x \in G, y \in G'$.**
Let $S = G/H$. Let $CC(x)$ denote the bi-commutator of $x$, namely centralizer of the centralizer of $x$ (which contains all the powers of $x$). We say that $u \in S$ is primitive if it satisfies the formula

$$\forall v \in CC(u) \forall w \exists t [u, t] = [v, w].$$

It is clear that a minimal set of generators of the group $S$ only contains primitive elements.

14.1. The copy of $\mathbb{Z}$ defined in $S$. Unless otherwise noted quantifiers range over $S$. We say that a pair of elements $(u_0, u_1)$ in $S$ is admissible if $u_0$ is primitive and $u_1 \in CC(u_0)$. On the set $A$ of admissible pairs we define an equivalence relation by

$$(u_0, u_1) \sim (v_0, v_1) \iff \forall w, t[u_0, w] = [v_0, t] \to [u_1, w = v_1, w]$$

Claim 14.2. $(u_0, u_1) \sim (v_0, v_1)$ iff $\exists k \in \mathbb{Z} (u_1 = u_0^k \land v_1 = v_0^k)$.

So we can view $A/\sim$ as the domain of the copy of $\mathbb{Z}$. For the operations, note that for every pair $(u', v')$ and $u$ there is $v$ such that $(u'v') \sim (u, v)$. Given that, it suffices to note that the following first-order definitions are compatible with $\sim$:

$$(u, v_0) \oplus (u, v_1) = (u, v_0v_1)$$
$$(u, v_0) \odot (u, v_1) = (u, w)$$

where $\forall r, s([u, r] = [v_0, r] \to [u, v_1] = [u, w])$.

Claim 14.3. $(A, \oplus, \odot)/\sim$ is isomorphic to $(\mathbb{Z}, +, \times)$.

14.2. Defining exponentiation inside $G$. We can define exponentiation on $S$ internally: given $u \in S$ and $k \in \mathbb{Z}$, represented by $(v, v^k) \in A$, $u^k$ is the unique element $u' \in S$ such that

$$\forall t [u, w] = [v, t] \to [u', w = v^k, t].$$

To do this in $G$ rather than $S = G/H$, let $p: G \to S$ be the canonical projection. If $u \notin H$, $u^k$ is the unique element $u'$ of $CC(u)$ such that $p(u') = p(u)^k$. If $u \in H$, then $u^k$ is the unique element $u' \in H$ such that for each $v \in G - H$, there is $w \in H$ with $(uv)^k = u'[v, w]$.

14.3. Defining in $G$ the module structure of $G'$. The conjugation action of $G$ on $G'$ introduces automorphisms of the abelian group $G'$. They commute pairwise because $x|g.h| = x$ for each $g, h \in G, x \in G'$. Let $u_1, \ldots, u_m$ be a minimal set of generators for $G$, and let $f_i$ be the automorphism of $G'$ induced by $f_i$. Then $G'$ is a $\mathbb{Z} \{X_1, \ldots X_m, X_1^{-1}, \ldots X_m^{-1}\}$ module in a canonical way. This module is free with generators all the $[u_i, u_j]$ for $i \neq j$.

We can now define the multiplication of an element of the ring and an element of $G'$. Let $P$ be a polynome in $\mathbb{Z} \{X_1, \ldots X_m\}$ and $u \in G'$. Then $P \cdot u$ is the element $v \in G'$ such that for each $m$-tuple of integers $\lambda_1, \ldots, \lambda_m$, $v - P(\lambda_1, \ldots, \lambda_m)u$ is in the submodule of $G'$ generated by the images of the $f_i^{-\lambda_i}$. (...) We can therefore define a bijection between $G'$ and $\mathbb{N}$. Since every element of $G$ is expressed uniquely as a product of elements in $CC(u_1), \ldots, CC(u_m)$ and $G'$, one gets a definable bijection (in the sense of $G$ with parameters $u_1, \ldots, u_m$) between $\mathbb{N}$ and $G$. $\square$
Part 8. General topics

Principles common throughout mathematics. On the occasion of talks to mathematics and general logic audiences in Münster and Paris, André Nies thought about a unifying approach to mathematics. The goal was to isolate ideas and principles that occur in lots of areas perceived to be disconnected.

15. Objects of greatest complexity in their class

The following situation arises in many areas of mathematics and theoretical computer science. Given a class of objects, together with a method to compare their complexity, is there a most complicated object in the class? Is such an object uniquely determined?

- The halting problem is a most complicated object in the class of computably enumerable sets under many-one reducibility \( \leq_m \). It is unique up to computable permutations of \( \mathbb{N} \) by Myhill [53].
- The satisfiability problem SAT is a most complicated object in NP (Cook/Levin, 1971/1973) under polynomial time many-one reducibility \( \leq_p \).
- Chaitin’s \( \Omega \) is complete for left-c.e. reals under Solovay reducibility \( \leq_S \). It can be described up to Solovay equivalence \( \equiv_S \) as the unique left-c.e. ML-random [36].
- There is a most complicated \( K \)-trivial set with respect to ML-reducibility by [8].
- There is a complete object in the class of countable abelian groups with embedding of structures.
- Conjugacy of ergodic transformations is \( \Sigma^1_1 \)-complete [19].
- Isomorphism of separable \( C^* \)-algebras is Borel is complete for orbit equivalence relations ([16] together with [70]). The same is true for homeomorphism of compact metric spaces [78].

Suppose we are given a preordering \( \leq \) to compare the complexity of objects in the class \( C \). Why are we interested in complete objects \( S \) for a class \( C \)? The answer depends on whether we start with \( C \), or with \( S \).

1. Starting with \( C \). The preordering is often very simple, and should definitely be simpler that the objects it is supposed to compare. (In fact it is sometimes not even mentioned explicitly.) Usually \( C \) is also downward closed under \( \leq \). So the single object \( S \), together with the simple preordering \( \leq \), describes the whole class \( C \).

2. Starting with \( S \). A complete object \( S \) for \( C \) is often interesting on its own right. Reflecting Tao and others, one can ask the question: is \( S \) random, or structured? For instance, Chaitin’s \( \Omega \) and the Rado graph are random. The halting problem and SAT are structured. The complexity of \( S \) is completely determined by proving it is complete for the natural class \( C \) of objects it belongs to. It is less clear how its randomness content is related to \( C \).

We now give some detail for each example. Then we return to the general metamathematical goal, by discussing analogs of Post’s problem which stems from computability theory.
15.1. C.e. sets and the halting problem. Sets of natural number can be compared via many-one (m, for short) reducibility: \( B \leq_m A \) if \( B = \emptyset \) or \( B = f^{-1}(A) \) for some computable function \( f \). Let \( W_e \) be the \( e \)-th c.e. set. One version of the halting problem is the effective join of all the \( W_e \), which is clearly 1-complete. By the Myhill isomorphism theorem [53], an \( m \)-complete set is unique up to a computable permutation of \( \mathbb{N} \).

Being \( m \)-complete is equivalent to being creative. Interestingly, creativity is first-order definable in the lattice of c.e. sets by a result of Harrington (see [72] or [57, 1.7.20]).

15.2. Languages in NP and the satisfiability problem SAT. Languages in complexity theory can be compared via polynomial time \( m \)-reducibility: \( B \leq_{m^p} A \) if \( B = \emptyset \) or \( B = f^{-1}(A) \) for some polynomial time computable function \( f \). The Cook-Levin theorem from the early 1970s says that SAT is NP-complete.

The 1976 Berman-Hartmanis Conjecture asks whether all NP-complete sets are polynomial time isomorphic. Hundreds of NP-complete problems have been studied. They are all “paddable”. Any two paddable polytime \( m \)-equivalent sets are polytime isomorphic. However, these problems are all “natural”- it is not clear if this can be considered as evidence for the conjecture. Mahaney [41] proved that sparse sets (i.e. with polynomial upper density) cannot be NP-complete unless \( P = NP \). Is this also known for sets that merely have sub-exponential upper density?

15.3. Left-c.e. reals and Chaitin’s \( \Omega \). A left-c.e. real \( \beta \) is given by \( \beta = \sup_s \beta_s \), where \( \langle \beta_s \rangle_{s \in \mathbb{N}} \) is a nondecreasing, computable sequence of rationals. Solovay reducibility is defined by \( \beta \leq_S \alpha \) if for given computable approximations \( \langle \beta_s \rangle_{s \in \mathbb{N}} \) of \( \beta \) and \( \langle \alpha_s \rangle_{s \in \mathbb{N}} \) of \( \alpha \), there is a computable increasing function \( g \) such that \( \beta - \beta_g(s) = O(\alpha - \alpha_s) \). Equivalently, there is a left-c.e. real \( \gamma \) such that \( 2^{-d} \beta + \gamma = \alpha \). See [57, 3.2.28] or the monumental monograph [13, Section 9.1].

An example of an \( \leq_S \) complete c.e. real is \( \sum_e 2^{-e} \beta_e \), where \( \langle \beta_e \rangle \) is an effective listing of the left-c.e. reals in \([0,1]\). Unlike the halting problem, we don’t get any uniqueness other than being Solovay complete. However, Kučera and Slaman [36] gave a description of the class of \( S \)-complete left-c.e. reals as the ones that are Martin-Löf random.

15.4. \( K \)-trivials. There is no largest \( K \)-trivial with respect to \( \leq_T \), because each \( K \)-trivial is low: [57, Theorem 5.3.22] can be used to build a c.e. \( K \)-trivial not below a given low c.e. set.

In a sense, Turing reducibility \( \leq_T \) is too fine for a meaningful complexity analysis of the \( K \)-trivials. We define ML-reducibility by \( B \leq_{ML} A \) if for each ML-random \( Y \), \( Y \geq_T A \) implies that \( Y \geq_T B \).

Bienvenu et al. [8] have shown that some set \( A \), which they call a “smart \( K \)-trivial”, is complete for \( \leq_{ML} \) within the \( K \)-trivials. They define Oberg-wolfach randomness and show that the ML-randoms failing this stronger randomness property are precisely the ones computing all \( K \)-trivials. Then they build a c.e. \( K \)-trivial such that no ML-random Turing above it is Oberg-wolfach random. No direct characterisation of the class of smart \( K \)-trivials is known at present.
15.5. **Structures under embedding.** We look at a class of countable structures under embedding \( \preceq \). Complete structures in this setting are often called *universal*.

- For (symmetric) graphs there is a complete structure, the Rado (or random) graph.
- For linear orders, \((\mathbb{Q},<)\) is complete.
- For abelian groups, there is a complete countable group \(A\). This is because the f.g. abelian groups have the amalgamation property. So one can build \(A\) as a Fraisse limit.
- There is no countable group that is \( \preceq \)-complete for countable groups. For, there are continuum many non-isomorphic 2-generated groups (Higman, Neumann, and Neumann). Only countably many can be isomorphic to a subgroup of a given single countable group.

**Question 15.1.** *Does every variety (in the sense of universal algebra) have the amalgamation property for finitely generated structures?*

In that case, if there are only countably many f.g. structures in the variety, there is a \( \preceq \)-complete countable structure, namely the Fraisse limit of the f.g. structures.

The substructure relation \( B \preceq A \) doesn’t always say that \( B \) is less complex than \( A \). The larger structure can “erase” information from \( B \), for instance when a linear order \( B \) is embedded into the dense linear order \( B \times \mathbb{Q} \) (with the lex ordering). All the complexity now lies in the embedding. If \( B \) is complicated, then the range of the embedding into \( B \times \mathbb{Q} \cong \mathbb{Q} \) is also complicated.

If we require the embeddings to be in some sense effective, this is no longer possible, and a meaningful theory of relative complexity emerges. We will study this in the simple case that the structure is a set with an equivalence relation (ER). Firstly we consider the case of domain \( \mathbb{N} \), with computable embeddings. Thereafter we proceed to the case that the domain is an uncountable Polish space.

15.6. **Complexity of equivalence relations on \( \mathbb{N} \), and presentations of groups.** We define \( m \)-reducibility between ER by \( F \leq_m E \) if there is a computable function \( g: \mathbb{N} \to \mathbb{N} \) such that \( Fuv \leftrightarrow Eg(u)g(v) \).

**\( \Pi^0_n \) completeness.** Ianovski et al. [30] showed that there is an \( m \)-complete \( \Pi^0_1 \) equivalence relation, and no complete \( \Pi^0_n \) equivalence relation for \( n \geq 2 \). For a natural example given by structures, isomorphism of certain polynomial time computable trees is a complete \( \Pi^0_1 \) equivalence relation by [29]. It is unknown whether the \( \Pi^0_1 \) equivalence relation of isomorphism of automatic ER is a complete; for background on this question see [38].

It is easy to build a \( \Sigma^0_n \) complete ER for each \( n \): they can be listed effectively, so it is sufficient to take the disjoint sum. How about \( \Sigma^0_n \) completeness for naturally occurring equivalence relations?

**\( \Sigma^0_n \) completeness.** In computability theory, Ianovski et al. [30] showed completeness at the relevant level for a number of degree equivalences on the c.e. sets. For instance, \( \equiv_T \) among c.e. sets is \( \Sigma^0_3 \) complete.
Consider a finitely axiomatised variety $\mathcal{V}$ of groups, such as all groups, or the metabelian groups. It is easy to see that isomorphism of $\mathcal{V}$-finitely presentable groups is $\Sigma^0_1$.

C.F. Miller [46, p. 80] has proved that isomorphism of finitely presented groups is $m$-complete for $\Sigma^0_1$ equivalence relations. We don’t know of similar results for more restricted varieties, such as the groups that are solvable at a fixed level.

We can describe a f.g. nilpotent group by $F_c(n)/N$ where $F_c(n)$ is the free nilpotent group of class $c$ and rank $n$, and $N$ is finitely generated as a normal subgroup.

Nilpotency is a $\Sigma^0_1$ property of a finite presentations of a group. This can be seen as follows: for having nilpotency class $c$, it is sufficient that the generators $g_1,\ldots,g_k$ satisfy the finitely many relations $[x_1,\ldots,x_{c+1}] = 1$ (for in that case, all the $[x_1,\ldots,x_c]$ are in the centre, and so, if $c > 1$, inductively $G/Z(G)$ is nilpotent of class $c - 1$). This is a $\Sigma^0_1$ event.

So isomorphism is $\Sigma^0_1$ as well. We can effectively list all the presentations of nilpotent groups as $P_0, P_1,\ldots$ and see isomorphism as a relation among the $P_k$.

Of course, we can also take a finite presentation of $F_c(n)$, and add its relators to a finite presentation of a group in $n$ variables. The versions of the isomorphism problem for class $c$-nilpotent groups we obtain by describing nilpotent groups in two different ways are $m$-equivalent.

Isomorphism of abelian f.g. groups is decidable.

**Question 15.2.** For $c > 1$ is isomorphism of f.g. class $c$-nilpotent groups decidable?

15.7. **Completeness for preorders.** $m$-reductions between c.e. preorders and the corresponding completeness notions have been studied beginning with [51], and later e.g. in [30]. Implication of sentences under PA is $\Sigma^0_1$ complete [51], and weak truth table reducibility on c.e. sets is $\Sigma^0_3$ complete [30].

**Question 15.3.** The substructure relation $G \preceq H$ among f.p. groups is merely $\Sigma^0_2$ by definition. Is it properly $\Sigma^0_2$?

The relation that $G$ is a retract of $H$ is $\Sigma^0_1$. Is it $\Sigma^0_1$-complete as a preorder?

For examples of $\Sigma^1_1$-complete ER on $\mathbb{N}$, see Part IV of the 2013 Logic Blog [15].

15.8. **Preliminaries: Choquet theory.** We now move on to examples of completeness in descriptive set theory. First some preliminaries. Choquet\textsuperscript{8} theory starts out with a locally convex topological vector space $V$ over $\mathbb{R}$. Such a vector space has a basis of the topology consisting of the translations of convex sets $C$ that are balanced (if $x \in C$ then $\lambda x \in C$ for each $|\lambda| \leq 1$), and absorbent ($V = \bigcup_n nC$). This generalises the situation of balls in $\mathbb{R}^n$.

For instance, a normed space with the weak topology is locally convex. More generally, given any vector space $V$ and a collection $\mathcal{F}$ of linear functionals on it, $V$ can be turned into a locally convex topological vector space

\textsuperscript{8}Gustave Choquet was a student of the analyst Arnaud Denjoy at ENS Paris in the 1930s.
by giving it the weakest topology that makes all the linear functionals in $F$ continuous.

Locally convex topological vector spaces, more general than the normed spaces, allow us to study interesting compact sets, such as the closed unit ball in $W^*$ with the weak * topology, for a Banach space $W$. (Banach himself in the year 1932 showed that the unit ball is compact in this topology for separable Banach space $W$ via a diagonalization argument not relying on the axiom of choice. Alaoglu proved it in full generality in his 1938 thesis at the Univ. of Chicago, using Tychonoff’s theorem, which needs the axiom of choice.)

Consider a compact convex set $C \subseteq V$. The set of extreme points $E$ is the set of points $x$ in $S$ such that $2x = y_0 + y_1$ implies that $x = y_0 = y_1$. Given a vector space $W$, a function $f : C \to W$ is affine if $f(\frac{1}{2}(x_0 + x_1)) = \frac{1}{2}(f(x_0) + f(x_1))$.

Let $V$ be the dual space of a Banach space $W$. As mentioned, this is locally convex with the weak * topology, and the closed unit ball of $V$ is convex and compact. For $W = C[0, 1]$ this unit ball can be seen as the set of measures $\mu$ on $[0, 1]$ with a mass of at most 1.

For a related example, consider a dynamical system $\langle X, T \rangle$ where $X$ is a topological space and $T : X \to X$. The probability measures on $X$ for which $T$ is invariant form a compact convex set in the space of (signed) Borel measures on $X$, which is the dual space of $C(X)$ with the weak topology. The extreme points are the ergodic measures.

**Definition 15.4.** A Choquet simplex is a compact convex set $S \subseteq V$ with the averaging condition that every $x \in S$ is the barycentre of a unique probability measure $\mu$ on the set $E$ of extreme points: $f(x) = \int f \, d\mu$ for each continuous affine function $f : S \to \mathbb{R}$.

**15.9 Equivalence relations and Polish group actions.** We consider equivalence relations on Polish spaces. Let $X, Y$ denote Polish spaces and $E, F$ equivalence relations. We define Borel reducibility by $(Y, F) \leq_B (X, E)$ if there is a Borel function $g : X \to Y$ such that $Euv \iff Fg(u)g(v)$.

Consider a Polish group action $G \curvearrowright X$. The corresponding orbit equivalence relation (OER) is $E^{G\times G}_X = \{ (u, v) : \exists g \in G \text{ such that } g \cdot u = v \}$.

An equivalence relation is *orbit complete* if it is Borel equivalent to an orbit equivalence relation, and every orbit equivalence relation is Borel reducible to it.

Separable structures can be encoded in various ways as points in a Polish space. Polish spaces themselves are given via completion by a distance matrix on a chosen dense sequence. The space $\mathcal{M}$ of all Polish metric spaces is then a $G_δ$ subset of $\mathbb{R}^{\mathbb{N} \times \mathbb{N}}$, namely the functions satisfying the axioms for metric spaces. The compact metric spaces form a $\Pi^0_3$ subset of $\mathcal{M}$ using that for metric spaces, compact $\iff$ (complete & totally bounded).

Let $B(H)$ denote the algebra of continuous operators on separable Hilbert space with the topology given by the operator norm. A separable $C^*$-algebra is a closed self-adjoint subalgebra of $B(H)$. Elliott et al. [16] proved that isomorphism of separable $C^*$-algebras (with a suitable encoding as Polish metric structures) is Borel below an orbit equivalence relation. They left open the question of orbit completeness.
Farah, Toms and Tornquist [17, Cor. 5.2] Borel-reduced the affine homeomorphism relation on Choquet simplices to isomorphism of separable $C^*$-algebras (in fact, of a subclass, the unital simple AI-algebras). Sabok [70] then obtained this orbit completeness by showing that isometry of Polish metric spaces is Borel reducible to affine homeomorphism of Choquet simplices.

Using Sabok’s result, Zielinski [78] proved that the homeomorphism relation $\sim_h$ of compact metric spaces is orbit complete. Similar to [17], he Borel-reduced affine homeomorphism of Choquet simplices to $\sim_h$.

Let $C(X)$ be the space of complex valued continuous functions on $X$. Since for compact spaces we have $X \sim_h Y \iff C(X) \sim_h C(Y)$ as $C^*$-algebras, this shows that even isomorphism of commutative $C^*$-algebras is orbit complete.

An example of an orbit complete OER is the OER obtained from the Borel action of $\text{Iso}(U)$ on $F(U)$, the Effros algebra of Urysohn space. As a byproduct of Zielinsky’s result and its proof, one obtains another example of an orbit complete OER, possibly more natural than the previously known ones.

- Let $Q$ be Hilbert cube $[0,1]^\mathbb{N}$ with the standard metric
  
  $d((x,y)) = \sum_n 2^{-n-1}|x_n - y_n|$.  

- Let $G$ be the group of autohomeomorphisms of $Q$, which is a Polish group with the metric $d(f,g) = d_\infty(f,g) + d_\infty(f^{-1},g^{-1})$, where $d_\infty(f,g) = \sup_x d(f(x),g(x))$.

- Let $X = \mathcal{K}(Q)$ be the Polish space of compact (i.e. closed) subsets of $Q$ with the Hausdorff distance.

The natural action $G \curvearrowright X$ has an orbit complete OER $E$. Intuitively, to any compact metric space $M$ one can in a Borel fashion assign “small” compact set $C_M \subseteq Q$ that is homeomorphic to $M$, e.g. by using that $Q \times Q$ is homeomorphic to $Q$. The smallness of these sets implies that any homeomorphism between two of them extends to an autohomeomorphism of $Q$. Thus $M \sim_h N$ iff $C_M EC_N$.

15.10. Ergodic theory. A Borel probability space is given by a probability measure on the Borel sets of a standard Polish space. Foreman, Rudolph and Weiss [19] showed:

**Theorem 15.5.** Conjugacy of ergodic transformations on a non-atomic Borel probability space is analytic complete.

By a result of von Neumann, all non-atomic Borel probability spaces are measure theoretically isomorphic to the unit interval $[0,1]$ with Lebesgue measure $\mu$. So we can restrict ourselves to conjugacy of ergodic transformations in MPT, the group of measure preserving transformations of $[0,1]$, with transformations $T_0,T_1$ identified if they agree outside a null set.

MPT as a Polish space. To make sense of Theorem 15.5 we need a Polish topology on MPT. Here is some background.

An element $T$ of MPT gives rise to the unitary operator $U_T$ on the separable Hilbert space $\mathcal{H} = L^2([0,1],\mu)$ such that $U_T(f) = f \circ T$. Note that the equivalence classes of bounded measurable functions are dense in $\mathcal{H}$.

A unitary operator $U$ is of the form $U_T$ iff
both $U$ and $U^{-1}$ preserve $L^\infty(X)$, i.e. the boundedness of (equivalence classes of) functions, and

- $U(fg) = U(f)U(g)$ for bounded $f, g$.

See [76, Thm. 2.4].

The strong operator topology on $\mathcal{B}(\mathcal{H})$ coincides with the weak operator topology on the set of unitary transformations $\mathcal{U}(\mathcal{H})$. The space $\mathcal{U}(\mathcal{H})$ with this topology is separable; a compatible complete metric is for instance

$$d(S, T) = \sum_n 2^{-n-1}||S(x_n) - T(x_n)|| + ||S^{-1}(x_n) - T^{-1}(x_n)||,$$

where $(x_n)$ is a dense sequence in the unit ball of $\mathcal{H}$ [32, I.9B]. The conditions above make MPT a $G_\delta$ subset of $\mathcal{U}(\mathcal{H})$, so it forms a Polish space. One can also directly induce this topology on MPT using the Halmos metric, which is analogous to the metric above:

$$d(S, T) = \sum_n 2^{-n-1}[\mu(S(E_n)\Delta T(E_n)) + \mu(S^{-1}(E_n)\Delta T^{-1}(E_n))],$$

where $(E_n)_{n \in \mathbb{N}}$ is a list of sets generating the $\sigma$-algebra, such as the rational closed intervals. This directly turns MPT into a Polish metric space.

An operator $T \in \text{MPT}$ is called ergodic if each $T$-invariant set is null or conull. Ergodicity is known to be a $G_\delta$ property on MPT. To see this, one uses that $T$ is ergodic iff the Lebesgue measure $\mu$ is an extreme point of the convex set of probability measures on $[0, 1]$ for which $T$ is invariant.

*On the proof of Theorem 15.5 due to [19].* Given a subtree $B$ of $2^{<\omega}$, Foreman, Rudolph and Weiss build an ergodic operator $T_B$ such that $B$ has an infinite branch iff $T_B$ is conjugate to its inverse in MPT.

They list the strings in $B$ as $(\sigma_n)$ so that $\sigma_n \prec \sigma_k$ implies that $n < k$. Next, they define sets $W_n(B)$ of words over $\{0, 1\}$. If $\sigma_n \prec \sigma_k$ then all the words in $W_k(B)$ are concatenations of words in $W_n(B)$. Let $W(B) = \bigcup_n W_n(B)$.

Let $\mathbb{K}(B)$ be the set of $f \in \{0, 1\}^\mathbb{Z}$ such that each block $f\mid_{[u,v]}$ is in $W(B)$. In symbolic dynamics, such a set is called a sub-shift, namely it is closed and shift-invariant. Let $T_B$ be the shift on $\mathbb{K}(B)$. Since the base space $\mathbb{K}(B)$ is compact, it carries a shift-invariant (non-atomic?) probability measure $\mu$; by choosing the $W_n(B)$ in the right way, they show that it is unique, which makes the system $(\mathbb{K}(B), \mu, T_B)$ ergodic. They then verify that $[B] \neq \emptyset$ iff $T_B$ is conjugate to $T_B^{-1}$.

It is not clear whether the construction is effective, because they use some probabilistic argument near the end of the 58-page paper.

By the von Neumann result mentioned above, and the fact that it is a Borel translation, we can assume that $T$ is in MPT.

The big open question is:

**Question 15.6.** *Is the relation $E$ of conjugacy of ergodic measure-preserving transformations $\leq_B$-complete for orbit equivalence relations?*

Foreman had announced at some point that $E$ is $\leq_B$-hard for OER given by $S_\infty$-actions; no paper on this has appeared so far.

### 15.11. Post’s problem

Post’s problem. Frequently, objects turn out to be the most complicated in their class. This fact is familiar from computability theory (a remote branch of mathematical logic formerly known as recursion theory). Post’s problem asked whether there is a c.e. set intermediate between the
computable sets and the halting problem in the sense of Turing reducibility. The answer was yes. However, natural c.e. sets that aren’t outright computable usually end up having the same complexity as the halting problem.

In some areas mentioned above, this is different.

Subsection 15.9 on orbit equivalence relations (OER): $S_\infty$ is the Polish group of permutations of $\mathbb{N}$. Graph isomorphism is $\leq_B$-complete for $S_\infty$-OER. This has been coded into lots of other ER, even isomorphism of countable Boolean algebras by Camerlo and Gao [12]. On the other hand, as they pointed out, isomorphism of countable torsion abelian groups is not complete. This uses Ulm invariants, which are certain countable sequences of countable ordinals. The result was proved by Friedman and Stanley [20].

Subsection 15.10 on ergodic theory: Instead of conjugacy of ergodic transformations $S, T$ in MPT, one can also consider the weaker relation of conjugacy of $U_S, U_T$ in the unitary group $U(H)$ (i.e., one allows conjugating by elements that are not necessarily of the form $U_R$ for any $R$ in MPT). Via spectral theory, one can show that this relation is Borel.

16. Describing a structure within a class

We want to describe a structure in a class up to isomorphism, using an appropriate formal language. Containment in the class is given as an external condition.

For finite structures in a fixed finite signature, there is always a description in first-order logic of length comparable to the size of the structure. An interesting question is how short such a description can be. Nies and Katrin Tent [59] answered this question for finite groups, compressing the group $G$ via a first-order description of length $O(\log^3 |G|)$. The Higman-Sims formula states that the number of non-isomorphic groups of order $p^n$ is $p^{O(n^{5/2})+2n^3/27}$. By a counting argument, this shows that the bound obtained is close to optimal.

The Kolmogorov complexity of a finite mathematical object is the length of a shortest description within an appropriate universal system of descriptions, such as a universal Turing machine. If we encode a finite structure by a string, we can apply this measure of complexity; however, it is not invariant under isomorphism. It would be worthwhile to study the invariant Kolmogorov complexity $K_{inv}(G)$ of a finite group $G$, which is defined as the least Kolmogorov complexity of any $H \cong G$. It is not hard to see that $K_{inv}(G)$ is bounded above by the length of a shortest first-order description of $G$ (plus a fixed additive constant). By the same counting argument, the Higman Sims formula implies that for a $p$-group $G$, $K_{inv}(G)$ and the length of a shortest first-order description are in fact quite close: both are of the order $\log^3 |G|$. What happens if we restrict to other classes of finite groups?

Within the class of finitely generated groups, an interesting question is whether a group can be described at all by a single first-order sentence. If so we call the group quasi-finitely axiomatizable (QFA), a notion introduced in [55]. For instance, this is the case for the Heisenberg group over $\mathbb{Z}$, and for the restricted wreath product of a finite cyclic group with $\mathbb{Z}$ (the latter example is interesting here because it is not finitely presented).
Within the class of countable structures over a countable signature $S$, there is always a description in $L_{\omega_1^\omega}(S)$, the extension of first-order language that allows countable disjunctions over a set of formulas with a shared finite reservoir of free variables (Scott). For the class of separable complete metric spaces, a similar result holds. The most natural logic here is an extension of Lipschitz logic for $S$ that allows countably infinite disjunctions.

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