\( \sigma \) - models on the quantum group manifolds \( SL_q(2, R) \), \( SL_q(2, R)/U_h(1) \), \( C_q(2[0]) \) and infinitesimal transformations.

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The differential and variational calculus on the \( SL_q(2, R) \) group is constructed. The spontaneous breaking symmetry in the WZNW model with \( SL_q(2, R) \) quantum group symmetry and in the \( \sigma \)-models with \( SL_q(2, R)/U_h(1) \) \( C_q(2[0]) \) quantum group symmetry is considered. The Lagrangian formalism over the quantum group manifolds is discussed. The classical solution of \( C_q(2[0]) \) \( \sigma \)-model is obtained.

1. Differential calculus on the \( SL_q(2, R) \) group.

The matrix quantum group [1] \( G = SL_q(2, R) \) is defined by the q-commutation relations (C.R.) of its group parameters. Let

\[
g = \begin{pmatrix} a^1 & a^2 \\ a^3 & a^4 \end{pmatrix}, \quad \begin{pmatrix} a^1a^2 = qa^2a^1, \quad a^1a^4 = qa^4a^1, \quad a^2a^3 = a^3a^2, \quad a^3a^4 = qa^4a^3, \quad a^1a^4 = a^4a^1 + (q - q^{-1})a^2a^3 \end{pmatrix} \tag{1}
\]

\( a^k \) - hermitian, \( |q| = 1, \text{Det}_q g = a^1a^4 - qa^2a^3 = 1 \).

For any elements \( g, g' \in SL_q(2, R) \) element \( g'' = g'g \) will belong \( SL_q(2, R) \) if \( a^k a^l = a^la^k \).

In the Gauss decomposition [2]

\[
g = \begin{pmatrix} 1 & \varphi_- \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varphi_+ & 1 \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} = \begin{pmatrix} \rho + \varphi_- \varphi_+ \rho & \varphi_- \rho^{-1} \\ \varphi_+ \rho & \rho^{-1} \end{pmatrix}
\] \tag{2}

the C.R. are:

\[
\rho \varphi_\pm = q\varphi_\pm \rho, \quad \varphi_- \varphi_+ = q^2 \varphi_+ \varphi_-
\] \tag{3}

Let the quantum group is a manifolds of any possible transformations \( g' = gg_0 \). There are two kinds of the variation: the variation in the neighborhood of the arbitrary point of the group space \( g' = g + dg \) and variation in the neighborhood of the unit of the group \( g = 1 + \delta g \). First variation defines the group invariants: element of the distance between two neighboring points, element of the volume around the point. Second variation defines the group symmetry of this invariants. The C.R. between variation \( dg \) and \( g \) define the type of the differential calculus. The left-invariant differential calculus [3] on the \( GL_q(2, C) \) group, matched with the differential calculi on the \( SL_q(2, C) \) subgroup and on the Borel subgroups \( B_L(C), B_U(C) \), was constructed in [4,5]. Let \( \omega = g^{-1}dg \) is the left differential Kartan 1-form

\[
\omega = \begin{pmatrix} \omega^1 & \omega^2 \\ \omega^3 & \omega^4 \end{pmatrix}, \quad Tr_q \omega = q^2 \omega^1 + \omega^4 = 0
\] \tag{4}

The differential calculus on the \( SL_q(2, R) \) group is defined by the C.R.

\[
\omega^1 \rho = \frac{1}{q^2} \rho \omega^1, \quad \omega^2 \rho = \frac{1}{q^2} \rho \omega^2, \quad \omega^3 \rho = \frac{1}{q^2} \rho \omega^3
\]

\[
\omega^1 \varphi_\pm = \varphi_\pm \omega^1, \quad \omega^2 \varphi_\pm = \varphi_\pm \omega^2, \quad \omega^3 \varphi_\pm = \varphi_\pm \omega^3
\] \tag{5}

The C.R. between the group parameters and their differentials are more complicated:

\[
d\rho \rho = \frac{1}{q^2} \rho d\rho, \quad d\varphi_+ \varphi_+ = \frac{1}{q^2} \varphi_+ d\varphi_+ + (q^4 - 1) \varphi_+^3 d\varphi_-
\]

\[
d\varphi_- \varphi_- = q^2 \varphi_- d\varphi_-, \quad d\rho \varphi_- = q \varphi_- d\rho
\]
The Cartan 1-forms are:

\[ \omega^1 = \rho^{-1}d\rho + \varphi_+d\varphi_-, \quad \omega^3 = \frac{1}{q^2}\rho^2d\varphi_+ - q^3\varphi_+^2\rho^2d\varphi_- \]
\[ \omega^2 = q\rho^{-2}d\varphi_- \]
\[ (\omega^1)^2 = (\omega^2)^2 = (\omega^3)^2 = 0 \]
\[ \omega^1\omega^2 + q^4\omega^2\omega^1 = 0 \]
\[ \omega^1\omega^3 + q^{-4}\omega^3\omega^1 = 0 \]

The left vector fields \( \nabla_k \) can be obtained from the applying the left differential to an arbitrary function on the quantum group \( df = (f_{\partial a^i})da^k = (f\nabla_k)\omega^k \).

\[ \nabla = \left( \begin{array}{c} \nabla_1 \\ \nabla_2 \\ \nabla_3 \\ \nabla_4 \end{array} \right) \quad \nabla_1 = \nabla_1 - q^2\nabla_4 \quad \nabla_4 = \nabla_1 + \nabla_4 \]

The C.R. for vector fields have following form

\[ \rho \nabla_1 = \frac{1}{q}\nabla_1\rho + \rho \]
\[ \rho \nabla_2 = \frac{1}{q}\nabla_2\rho - \varphi_+\rho^3 \]
\[ \rho \nabla_3 = \frac{q}{\rho}\nabla_3\rho \]
\[ \rho \nabla_4 = \frac{1}{q}\nabla_4\rho + \varphi_+\rho^3 \]

\[ q^2\nabla_1\nabla_3 - \frac{1}{q^2}\nabla_3\nabla_1 = (q^2 + 1)\nabla_3, \quad q^2\nabla_2\nabla_1 - \frac{1}{q^2}\nabla_1\nabla_2 = (q^2 + 1)\nabla_2, \quad \nabla_3\nabla_2 - \frac{1}{q^2}\nabla_2\nabla_3 = \nabla_1 \]

and \( \nabla_k \) have the form

\[ \nabla_1 = \frac{\partial}{\partial \rho}, \quad \nabla_2 = \frac{1}{q}\frac{\partial}{\partial \varphi_-}\rho^2 - \frac{\partial}{\partial \rho}\varphi_+\rho^3 + q\frac{\partial}{\partial \varphi_+}\varphi_+^2\rho^2, \quad \nabla_3 = \frac{q}{\rho}\frac{\partial}{\partial \varphi_+}\rho^2 \]

The left vector fields and the left derivatives act on the any function of the group parameters from the right side.

**2. WZNW model on the \( SL_q(2, R) \) group.**

The existing of the quantum group structure in the WZNW model was shown in [6,7]. The \( \sigma \)-models with a quantum group symmetry was considered in [2,8,9,10]. To construct the WZNW model with \( SL_q(2, R) \) group symmetry, we consider the space \( M^{1,1} \oplus SL_q(2, R) \), where \( M^{1,1} \) is the commutative (undeformed) space. The element of the volume in \( M^{1,1} \) space, which is the invariant of \( SL_q(2, R) \), is

\[ Tr_q[\omega(d) \wedge dz^\mu][\omega(d) \wedge dz^\mu] = Tr_q(\omega_\mu\omega^\mu)d^2z, \]

where \( \omega(d) = \omega_\mu dz^\mu, z^\mu \epsilon M^{1,1}, \mu = 1,2. \) For any \( 2 \times 2 \) matrix \( A, Tr_q A = q^2 A^1 + A^4. \) As a result we have

\[ Tr_q(\omega_\mu\omega^\mu) = q^5[2]q^\rho\partial_\mu\partial^\mu\rho + q^5[2]q^\rho^{-1}\varphi_+\partial_\mu\varphi_-\partial^\mu\rho + \frac{1}{q}\partial_\mu\rho\partial^\mu\varphi_- + \]


\[(\partial_{\mu}\varphi_{-}\partial^{\mu}\varphi_{+} + q^{2}\partial_{\mu}\varphi_{+}\partial^{\mu}\varphi_{-}) - q^{2}(q^{4} - 1)\varphi_{+}^{2}\partial_{\mu}\varphi_{-}\partial^{\mu}\varphi_{-}\]  

(12)

The C.R. are now in the same space-time point \(d\rho = \partial_{\mu}\rho dz^{\mu}, d\varphi_{\pm} = \partial_{\mu}\varphi_{\pm} dz^{\mu}\) and \([n]_{q} = \frac{q^{n} - q^{-n}}{q - q^{-1}}\). The Wess-Zumino term

\[Tr_{q}(\omega(d) \wedge \omega(d) \wedge \omega(d)) = \frac{q[2]_{q}[3]_{q}2\varepsilon_{\mu\nu\lambda}\partial_{\lambda}(\rho^{-1}\partial_{\mu}\partial_{\nu}\varphi_{-}\varphi_{+})d^{3}z\]  

(13)

is the total derivative. Finally, the WZNW-action with the \(SL_{q}(2,R)\) quantum group symmetry describes the 2-dimensional relativistic string in the background gravity and antisymmetric fields

\[S[\rho, \varphi_{-}, \varphi_{+}] = \frac{k}{4\pi} \int d^{2}z(G_{AB}\partial_{\mu}X^{A}\partial^{\mu}X^{B} + B_{AB}\varepsilon_{\mu\nu}\partial^{\mu}X^{A}\partial^{\nu}X^{B}),\]  

(14)

where \(X^{A} = (\rho, \varphi_{-}, \varphi_{+})\) and the background gravity and antisymmetric fields have the following form:

\[G_{AB} = \begin{pmatrix}
q^{5}[2]_{q}\rho^{-2} & q^{4}[2]_{q}\rho^{-1}\varphi_{+} & 0 \\
q^{5}[2]_{q}\rho^{-1}\varphi_{+} & -q^{2}(q^{4} - 1)\varphi_{+}^{2} & 1 \\
0 & 0 & 0
\end{pmatrix},\]

\[B_{AB} = \frac{q^{3}[2]_{q}[3]_{q}}{6}\varphi_{+}\rho^{-1}\begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}\]

The group symmetry of this model is \(SL_{q}(2,R) \otimes SL_{q}(2,R)\), because under the left multiplication on the group \(g' = g_{0}g\) the differential forms of Kartan are invariant, \(\omega' = \omega\), and under the right multiplication \(g' = gg_{0}\) the differential forms are covariant, \(\omega' = g^{-1}_{0}\omega g_{0}\). But \(Tr_{q}A\) is invariant of the transformation \(A' = g^{-1}_{0}Ag_{0}\), because the elements of matrix \(A\) commute with the elements of matrix \(g_{0}\), by definition of the quantum group. Therefore, this model describes the spontaneous breaking of the \(SL_{q}(2,R) \otimes SL_{q}(2,R)\) symmetry to the \(SL_{q}(2,R)\) one.

**3. \(\sigma\)-model on the \(SL_{q}(2,R)/U_{h}(1)\) group.**

Let us consider the spontaneous breaking symmetry in the \(\sigma\)-model with the \(SL_{q}(2,R)/U_{h}(1)\) group symmetry. Let \(G = KH, K\)-coset, \(H\)-subgroup.

The Kartan 1-forms

\[k^{-1}dk = \begin{pmatrix}
q^{2}\varphi_{+}d\varphi_{-} & d\varphi_{-} & -d\varphi_{+}
\end{pmatrix} = \omega + \theta,\]  

(15)

where \(\omega \in K, \theta \in H\) and the coset elements \(\varphi_{\pm}\) commute with the subgroup parameter \(\rho\) and satisfy to C.R. of \(SL_{q}(2,R)\) group among themselves. There is a question: how do coset and subgroup separate from \(k^{-1}dk\)? In opposite to the classical case, there is the 3-parametric family of the \(U(1)\) subgroups. The Lagrangian has the following form:

\[L_{n} = \frac{1}{2}Tr_{q}(\omega_{\mu}\omega^{\mu}) = \frac{(q^{4} + 1)}{4q^{4}}(\partial_{\mu}\varphi_{-}\partial^{\mu}\varphi_{+} + q^{2}\partial_{\mu}\varphi_{+}\partial^{\mu}\varphi_{-}) - c_{n}(q)\varphi_{+}\partial_{\mu}\varphi_{-}\partial^{\mu}\varphi_{-},\]  

(16)

where \(c_{n}(q)\) depends on the choice of a subgroup. There are three most interesting examples.

**1) Undeformed \(U(1)\) subgroup: \(c_{1} = \frac{2q^{4} - q^{2} + 1}{2}\)**

\[\omega = \begin{pmatrix}
(q^{2} - 1)\varphi_{+}d\varphi_{-} & d\varphi_{-} \\
d\varphi_{+} - q^{2}\varphi_{+}d\varphi_{-} & 0
\end{pmatrix}, \quad \theta = \varphi_{+}d\varphi_{-}\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}\]  

(17)
The algebra symmetry of this Lagrangian is defined by the Maurer-Kartan equations:

\[
d\theta = -\begin{pmatrix} q^{-2} & 0 \\ 0 & 1 \end{pmatrix} \omega \omega + (q^2 - 1) \omega \theta, \quad d\omega = -\begin{pmatrix} q^{-2} & 0 \\ 0 & 0 \end{pmatrix} \omega \omega - q^2[2] \omega \theta, \quad \theta \omega = q^4 \omega \theta
\]

The C.R. between the coset and the subgroup forms are common for all of the examples

\[
\begin{align*}
\omega^1 \omega^3 + q^4 \omega^3 \omega^1 &= 0, \\
\omega^2 \omega^3 + q^2 \omega^3 \omega^2 &= 0 \\
\omega^1 \omega^3 + q^4 \omega^3 \omega^1 &= 0, \\
\omega^4 \omega^3 + q^4 \omega^3 \omega^4 &= 0
\end{align*}
\]

(18)

2) Classical coset structure: \( c_2 = \frac{q^2+1}{4} \)

\[
\omega = \begin{pmatrix} 0 \\ d\varphi_- + q^2 \varphi^2_- d\varphi_- \end{pmatrix}, \quad \theta = \varphi_+ d\varphi_- \begin{pmatrix} q^2 & 0 \\ 0 & -1 \end{pmatrix}
\]

(19)

\[
d\theta = -\omega \omega, \quad d\omega = -\omega \theta - \theta \omega, \quad \theta \omega = q^2 \omega \theta
\]

3) There is one of the examples of the 2-parametric family \( U_q(1) \) subgroups:

\[
c_3 = \frac{2q^4 - 2q^4 + 1}{2q^4}, \quad \omega = \begin{pmatrix} (q^{-1})^2 \varphi_+ d\varphi_- & d\varphi_- \\ d\varphi_+ - q^2 \varphi^2_+ d\varphi_- & 0 \end{pmatrix}, \quad \theta = \frac{1}{q^4} \varphi_+ d\varphi_- \begin{pmatrix} 1 & 0 \\ 0 & -q^2 \end{pmatrix}
\]

(20)

\[
d\theta = -\begin{pmatrix} q^{-4} & 0 \\ 0 & 1 \end{pmatrix} \omega \omega + (q^4 - 1) \omega \theta, \quad d\omega = -\begin{pmatrix} q^{-4} & 0 \\ 0 & 0 \end{pmatrix} \omega \omega - q^4[2] \omega \theta, \quad \theta \omega = q^6 \omega \theta
\]

Why we have obtained different algebras of a symmetry for the same subgroup? That is possible because we can use the different map from the algebra to the group, for example:

\[
g = \exp(\varphi_+ \tau_+) \exp(\varphi_+ \tau_-) \exp(\ln \rho \tau_3),
\]

(21)

where \( \tau \) are the Pauli matrices – the fundamental representation of the \( U_q(SL(2, R)) \) algebra. The group stability of the vacuum is \( U(1) \). In the another parametrization

\[
g = \exp(\varphi_- \tau_+) \exp(\varphi_+ \tau_-) (1 - \frac{(q^2 - 1)}{q^2} \nabla^3) \nabla_3, \quad \nabla^3 = \begin{pmatrix} 1 & 0 \\ 0 & -q^2 \end{pmatrix}
\]

(22)

the group stability of the vacuum is \( U_q(1) \). The group symmetry of this Lagrangians is \( SL_q(2, R) \) spontaneously breaked to \( U_h(1) \), \( h = q^{\pm 2n}, n = 0, 1, ... \). Under the left multiplication on the group \( G' = G_0 G \), the differential form \( G^{-1} dG = H^{-1} (\omega + \theta) H = G'^{-1} dG' \). Therefore, \( \omega' + \theta' = H' H^{-1} (\omega + \theta) H H^{-1} \). Again, the decomposition on the coset and the subgroup forms is not unique after transformation. The group transformation can transform the Lagrangian with the \( U_h(1) \) subgroup of the vacuum stability to the Lagrangian with the \( U_{h_2}(1) \) subgroup.

4. Variational calculus on the \( SL_q(2, R) \) group.

It is possible to obtain the variational calculus on the group by two ways: from the C.R. between the left vector fields and group parameters and from the infinitesimal transformations on the group. Let us multiply the C.R. (8) between \( \nabla_n \) and group parameters on the parameters of transformation \( R^n \). The form of the infinitesimal transformations of the group parameters is obtained under the requirement

\[
[X_A, \nabla_n R^n] = X_A \delta R^n, \quad X_A = (\rho, \varphi_-, \varphi_+), \quad [A, B] = AB - BA
\]

(23)
By imposing the C.R. between the parameters of infinitesimal transformations and group parameters

\[
\begin{align*}
\rho R^1 &= q^2 R^1 \rho \\
\varphi_- R^1 &= R^1 \varphi_- \\
\varphi_+ R^1 &= R^1 \varphi_+ \\
\rho R^2 &= q R^2 \rho \\
\varphi_- R^2 &= R^2 \varphi_- \\
\varphi_+ R^2 &= R^2 \varphi_+ \\
\rho R^3 &= q R^3 \rho \\
\varphi_- R^3 &= R^3 \varphi_- \\
\varphi_+ R^3 &= R^3 \varphi_+
\end{align*}
\]

we obtain the infinitesimal transformation of the group parameters

\[
\rho \delta = \rho R^1 - \varphi_+ \rho^3 R^2 \\
\varphi_- \delta = \frac{1}{q} \rho^2 R^2 \\
\varphi_+ \delta = q \varphi_+^2 \rho^2 R^2 + q \rho^{-2} R^3
\]

The same result we can obtain from the right infinitesimal multiplication on the group \( g' = gg_0 \), where \( g_0 = 1 + \delta g_0 \). For

\[
\delta g_0 = \begin{pmatrix} R^1 & R^2 \\
R^3 & -q^2 R^1 \end{pmatrix}
\]

we see, that \( dg = g \delta g_0 \) and C.R. for \( \delta g_0 \) are the same as for left forms \( \omega \) simultaneously with condition \( R^4 = -q^2 R^1 \). The C.R. of the variational calculus

\[
\begin{align*}
(\rho \delta) \rho &= \frac{1}{q} \rho (\rho \delta) \\
(\varphi_- \delta) \rho &= \frac{1}{q} \rho (\varphi_- \delta) \\
(\varphi_+ \delta) \rho &= \frac{1}{q} \rho (\varphi_+ \delta) - \frac{q^2 - 1}{q} \varphi_+ (\rho \delta) \\
(\rho \delta) \varphi_- &= \varphi_- (\rho \delta) \\
(\varphi_- \delta) \varphi_- &= \varphi_- (\varphi_- \delta) \\
(\varphi_+ \delta) \varphi_- &= \varphi_- (\varphi_+ \delta) \\
(\rho \delta) \varphi_+ &= \varphi_+ (\rho \delta) \\
(\varphi_- \delta) \varphi_+ &= \varphi_+ (\varphi_- \delta) \\
(\varphi_+ \delta) \varphi_+ &= \varphi_+ (\varphi_+ \delta)
\end{align*}
\]

are consistent with the C.R. (3) and are simpler than the C.R. of the differential calculus (6). The \( U_q(SL(2, R)) \) algebra is the condition of the compatibility of the relations (25)

\[
\begin{align*}
X^A (q^2 R^1 \delta R^3 - q^{-2} \delta R^3 R^1) &= (q^2 + 1) \delta R^3 \\
X^A (q^2 \delta R^2 \delta R^1 - q^{-2} \delta R^1 \delta R^2) &= (q^2 + 1) \delta R^2 \\
X^A (\delta R^3 \delta R^2 - q^2 \delta R^2 \delta R^3) &= \delta R^1
\end{align*}
\]

5. Equations of motion

We use the extremum principle of the action to obtain the equations of motion and we must to commute the variations of fields and their derivatives on the right or on the left side. We can use both variation \( dX^A \) and \( \delta X^A \) to do this. The C.R. of the differential calculus on the \( SL_q(2, R) \) group are insufficient to do this. Therefore, we need in the differential calculus on the Lagrangian manifolds \( (\rho, \varphi_\pm, \dot{\rho}, \varphi_\pm, \rho, \phi) \). This is not the quantum group manifold and we can not use the formalism of 1-forms. We can require, that the Lagrangian equation of motion be coincident with the conservation law \( \partial_\mu \omega^\mu = 0 \) for Lagrangian with \( SL_q(2, R) \) group symmetry. At last, we can investigate the 1- dimensional \( \sigma \) models. The variational calculus is more suitable to obtain the equations of motion. The C.R. between the \( X^A, \dot{X}^A, \ddot{X}^A \) and \( R^a, \dot{R}^a, \ddot{R}^a \) can obtain by differentiating the relations (24).

\[
\begin{align*}
\dot{\rho} R^1 &= q^2 R^1 \dot{\rho} \\
\dot{\rho} R^2 &= q R^2 \dot{\rho} \\
\dot{\rho} R^3 &= q R^3 \dot{\rho} \\
\dot{\rho} \dot{R}^1 &= q^2 \dot{R}^1 \dot{\rho} \\
\dot{\rho} \dot{R}^2 &= q \dot{R}^2 \dot{\rho} \\
\dot{\rho} \dot{R}^3 &= q \dot{R}^3 \dot{\rho}
\end{align*}
\]

The derivatives of \( \varphi_\pm \) commute with the derivatives of \( R^a \).

5. One dimensional \( \sigma \) model on the quantum plane \( (C_q(2|0)) \).

The differential calculus on the \( C_q(2|0) \) is coincide with the differential calculus on the Borel
subgroup of $SL_q(2, \mathbb{C})$ and can be obtained from the differential calculus on the $SL_q(2, \mathbb{C})$ by surjection: $\pi: SL_q(2, \mathbb{C}) \rightarrow B_L$ such that $\pi(b) = 0$.

$$g = \begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix}, \quad xy = qyx, \quad yy = q^{-2}y\dot{y}, \quad \dot{x}x = q^{-2}x\dot{x}, \quad \dot{y}y = q^{-1}y\dot{x}$$

$$\omega = \begin{pmatrix} x^{-1}dx & 0 \\ xdy - qydx & -q^2x^{-1}dx \end{pmatrix}, \quad \dot{y}x = q^{-1}x\dot{y} - \frac{(q^2 - 1)}{q^2}y\dot{x}$$

In term of the variables $\rho, \varphi_\pm$

$$g = \begin{pmatrix} \rho & 0 \\ \varphi_+ \rho & \rho^{-1} \end{pmatrix}; \quad \omega = \begin{pmatrix} \rho^{-1}d\rho & 0 \\ \frac{1}{q^2}\rho^2d\varphi_+ & -q^2\rho^{-1}d\rho \end{pmatrix}$$ (30)

$$L = \frac{1}{2}Tr_q(\omega_{\mu}\omega^\mu) = \frac{q^4(2^1 + 1)}{2}\rho^{-2}\rho^2$$

The equation of motion $\dot{\omega}^1 = \rho^{-1}\dot{\rho} - \rho^2\rho^{-1}\rho^2 = 0$ will coincide with Lagrangian equation, if we impose the C.R. $\delta\dot{\rho}\delta\dot{\rho} = \frac{1}{q^2}\dot{\rho}\delta\dot{\rho}$. The classical solution of this equation is

$$\rho = \alpha \exp(\beta t), \quad \alpha\beta = q^2\beta\alpha$$ (32)

and C.R.

$$\rho(t)\rho(t') = \rho(q^2t')\rho(\frac{1}{q^2}t), \quad \rho(t)\rho(t') = \exp[q^2(q^2 - 1)\beta(t - t')]\rho(t')\rho(t)$$ (33)

There are $4 \times 4$ matrix representations of $\alpha, \beta$ such, that $det_q\alpha = 0$ or $det_q\beta = 0$. Therefore, we can rewrite this Lagrangian as a $4 \times 4$ matrix model for the commuting fields. In conclusion, note that 2-dimensional $\sigma$-model on the quantum plane

$$L = \frac{q^4(2^1 + 1)}{2}\rho^{-2}\partial_\mu\rho\partial^\mu\rho$$ (34)

leads to the C.R. $\delta\dot{\rho}\delta\dot{\rho} = \frac{1}{q^2}\dot{\rho}\delta\dot{\rho}$ and the equation of motion $\partial_\mu\partial^\mu\rho - q^2\rho^{-1}\partial_\mu\rho\partial^\mu\rho = 0, \quad \mu = 1, 2$.

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