Model for Classical Electron with Finite Mass and Action

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Abstract—It is shown in the tetrad representation that there are Reissner–Nordström solutions with a finite action and total inertial mass equal to the gravitational mass of the considered system. These solutions describe systems of electromagnetic and gravitational fields without any admixture of massive point-charges. The stress tensor for this solutions is shown to be identically zero. This means that there is no need in additional nonelectromagnetic surface-tension, existing in the Lorentz electron model, preventing the system disintegration. The hypothesis that gravitation can play a crucial role in the structure of elementary particles is discussed.

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INTRODUCTION

As is well known the ratio of the electrostatic repulsion between two electrons is larger than their gravitational attraction by a factor of $4.2 \times 10^{42}$. At first sight, this means that gravitational interaction cannot play important role in the elementary particle interaction and in their structure. As will be argued this point of view can probably be wrong.

The main concept of classical electrodynamics (see for instance [1, 2]) assumes the existence of charged massive particles and the electromagnetic field responsible for their electromagnetic interaction. The principle difficulties of this concept are well known: if the fundamental charged particles are point-like, then the energy of the electromagnetic field is infinite even for one particle. If the particle is considered as a drop of charged liquid, as in the Lorentz model of an electron [3], having a finite range in the three-dimensional space, the problem of the nature of non-electromagnetic attractive forces preventing the drop disintegration by the electrostatic forces arises immediately. These problems are not solved up to now.

In quantum electrodynamics (QED), electron-positron field is a bispinor field of point-like particles, which electromagnetic current is a source of the quantized electromagnetic field. The lagrangian of QED predicts an infinite mass and infinite electric charge of the electron and positron (see for instance [4–6]). In order to give physical meaning to divergent integrals and to get the experimentally observed cross sections, the procedure of the mass and charge renormalization was invented. But this infinite renormalization is realized by an introduction into the initial QED lagrangian of contra-terms having infinite coupling constants [6].

In the present paper, another approach will be discussed in which the classical electron is considered as the system of electromagnetic and the gravitational fields described by the Reissner–Nordström (RN) solution of both the Maxwell and Einstein equations. It will be shown that fields are localized in a very small three-dimensional-space region (about $10^{-34}$ cm). To avoid any misunderstanding, we would like to stress that the main results are obtained in the framework of classical physics. Only in Sections 9 and 10, quantum effects will be shortly discussed. It will be demonstrated that in spite of the singularities of the electromagnetic and gravitational fields, the total lagrangian density is integrable function and the action is finite in the tetrad representation for any values of the parameters $e$ and $m$ of the RN solution denoting the electric charge and the gravitational mass of the system, respectively. As will be shown if the parameters $e$ and $m$ obey the relation, considered in the present paper, the solution corresponds to a finite total inertial mass of the system that is equal to its gravitational mass. This means that the solution is in accordance with the equivalence principle. It is this solution with the charge equal to the experimental electron charge which will be quoted as “the classical electron”. Note that the total mass is just the mass of the electromagnetic and gravitational fields. This means that there is no need in any additional fundamental entities such as charged point-like particles (electron-positron field) and the only existing fields are the electromagnetic and gravitational fields.

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1 The article is published in the original.
Let us consider the spherical coordinates \( x^1 = r, \) \( x^2 = \theta, \) \( x^3 = \varphi, \) and \( x^0 = c t, \) where \( r \) denotes a radius, \( \theta, \varphi \) are polar and azimuthal angles, \( c \) is the velocity of light in empty space and \( t \) denotes time. The stationary solution of the Maxwell and Einstein equations, depending only on \( r, \) was found independently by H. Reissner, H. Weyl, G. Nordström, and G.B. Jeffery [7–10], nevertheless it is called usually the Reissner–Nordström solution. From here on, it will be referred to as the RN solution. For this solution, the inertial mass is shown to be finite. It turns out that the stress tensor density is identically zero for the classical electron. It is shown in Section 7 that according to the equivalence principle there is no need in any point-like particles with some bare mass that are usually named electrons. The electrical charge distribution is summarizing in Section 10.

1. REISSNER–NORDSTRÖM SOLUTION

Let us consider the spherical coordinates \( x^1 = r, \) \( x^2 = \theta, x^3 = \varphi, \) and \( x^0 = c t, \) where \( r \) denotes a radius, \( \theta, \varphi \) are polar and azimuthal angles, \( c \) is the velocity of light in empty space and \( t \) denotes time. The stationary solution of the Maxwell and Einstein equations, depending only on \( r, \) was found independently by H. Reissner, H. Weyl, G. Nordström, and G.B. Jeffery [7–10], nevertheless it is called usually the Reissner–Nordström solution. From here on, it will be referred to as the RN solution. For this solution, the metric tensor can be chosen diagonal and its nonzero covariant and contravariant components are

\[
g_{00} = \frac{1}{g^{00}} = -g^{11} = -1/g_{11} = \Lambda, \tag{1}
\]

\[
\Lambda = 1 - \frac{2km}{c^2r} + \frac{ke^2}{c^2r^2}, \tag{2}
\]

\[
g_{22} = \frac{1}{g^{22}} = -r^2, \tag{3}
\]

\[
g_{33} = \frac{1}{g^{33}} = -r^2 \sin^2 \theta. \tag{4}
\]

Here, \( k = 6.67 \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^{-2} \) is the gravitational constant, \( e \) and \( m \) are respectively the electric charge and mass of the considered system. This solution is used in cosmology to describe black holes for which

\[
m > \sqrt{\frac{e^2}{k}}. \tag{5}
\]

In this work, we start our consideration with the cases when the inverse condition

\[
m < \sqrt{\frac{e^2}{k}}, \tag{6}
\]

is fulfilled since it is this condition which is valid for real leptons and quarks. We shall name such a system “the point-charge” which could correspond to the electron if \( e = -4.80 \times 10^{-10} \text{ esu}. \) We shall distinguish “the real electron” with the experimental mass \( m = 9.11 \times 10^{-28} \text{ g} \) (0.511 MeV/c2) and “the classical electron” which properties will be explained later.

It is convenient to rewrite Eq. (2) for \( \Lambda \) in the form

\[
\Lambda = 1 - \frac{r_e^2}{r^2}, \tag{7}
\]

where

\[
r_e = \frac{2km}{c^2}, \tag{8}
\]

\[
r_e^2 = \frac{ke^2}{e^4}, \tag{9}
\]

with \( r_e \) being the Schwarzschild radius [11]. Using values \( c = 2.998 \times 10^{10} \text{ cm/s} \) and \( e, m \) for the real electron one gets \( r_e = 1.35 \times 10^{-55} \text{ cm}, \) \( r_e = 1.38 \times 10^{-34} \text{ cm.} \) As follows from these numerical estimates \( r_e \gg r_e \) for the real electrons.

It is obvious from Eqs. (1) and (7) that the element \( g^{00} \) at \( r \geq 0 \) has no singularity and is positive when \( r_0^2 > 0, \) where \( r_0^2 \) is given by

\[
r_0^2 = r_e^2 - r_e^2/4 \equiv r_e^2 \left( 1 - \frac{km^2}{e^2} \right). \tag{10}
\]

Since according to the numerical estimates presented after Eq. (9) \( r_e/\tau < 1 \) we have from Eq. (10) \( r_e \approx r_e \) with a very high precision. We conclude also from the right-hand side of Eq. (10) that the requirement \( r_0^2 > 0 \) is equivalent to the condition (6). We get from Eq. (6) the numerical estimate \( m < 1.86 \times 10^{-6} \text{ g}. \) This means that condition (6) is fulfilled not only for real electrons but for muons, \( \tau \)-leptons, and quarks.

The tensor of the electromagnetic field \( F^{ik} \) obeys the Maxwell equations in the gravitational field [2, 12]:

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left( \sqrt{-g} g^{ik} F_{kj} \right) = \frac{4\pi}{c} J^i, \tag{11}
\]
with zero electromagnetic current $J = 0$ for all $r$ except $r = 0$. Summing over any pair of identical covariant and contravariant Latin indexes is assumed in Eq. (11) and in all below formulas except formulas in Appendix, where all sums have the symbol $\Sigma$. All Latin indexes can be equal to 0, 1, 2, 3, while the Greek indexes can be equal to 1, 2, 3. Summing over any pair of identical Greek indexes is also assumed everywhere except Appendix. Here, $g$ is the determinant of the matrix $g_{ij}$. The solution of Eq. (11) for the radial component of the electric field density $E_r$ is [7–10]

$$
F^{10} = -F^{01} = -F_{10} = F_{01} = E_r = \frac{e}{r^2}, \quad (12)
$$

all the other components of $F^{ik}$ and $F_{ik}$ are zero.

The formula for the energy-momentum tensor of the electromagnetic field reads [2, 12]

$$
T^i_k = \frac{1}{4\pi} \left[ -F^{ij} F_{kj} + \frac{1}{4} \delta^i_k F_{lm} F^{lm} \right], \quad (13)
$$

where $\delta^i_k$ denotes the Kronecker symbol ($\delta^i_0 = \delta^i_1 = \delta^i_2 = \delta^i_3 = 1$, all the other elements are zero). Using Eqs. (12) and (13) one gets the formulas for the nonzero elements of $T^0_k$:

$$
T_0^0 = T_1^1 = -T_2^2 = -T_3^3 = \frac{e^2}{8\pi r^4}, \quad (14)
$$

The metric tensor $g_{ik}$ and the energy-momentum tensor $T_{ik}$ enter the equations for the gravitational field established by Einstein [13] and Hilbert [14].

2. ISOTROPIC COORDINATES

In order to introduce the isotropic coordinates, we define the new radial variable $\rho$ with the relations

$$
r = \rho \mathcal{D}(\rho), \quad (15)
$$

$$
\mathcal{D}(\rho) = 1 + \frac{r_e}{2\rho} + \frac{r_0^2}{4\rho^2} \equiv \left[ 1 + \frac{r_e}{2\rho} \right]^2 - \frac{r_0^2}{4\rho^2}, \quad (16)
$$

where $r_e$, $r_0^2$, and $r_0^2$ are given by Eqs. (8)-(10). Defining

$$
\mathcal{N}(\rho) \equiv \frac{dr}{d\rho} = 1 + \frac{r_0^2}{4\rho^2}, \quad (17)
$$

we get the formula for $\Lambda$

$$
\Lambda = \frac{\mathcal{N}^2}{\mathcal{D}^2}, \quad (18)
$$

and the following relation for the spacetime interval:

$$
ds^2 = \frac{\mathcal{N}^2}{\mathcal{D}^2} dx_0^2 - \mathcal{D}^2 \left[ (d\rho)^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]. \quad (19)
$$

Let us introduce the pseudo-Euclidean coordinates

$$
\rho^0 = \rho_0 = x^0, \quad \rho_1 = \rho_1, \quad \rho_2 = \rho_2, \quad \rho_3 = \rho_3 \quad (called \ the \ uniform \ coordinates) \ with \ the \ relations
$$

$$
\rho_1 = \rho \sin \theta \cos \varphi, \quad \rho_2 = \rho \sin \theta \sin \varphi, \quad \rho_3 = \rho \cos \theta. \quad (20)
$$

Now, the formula for the spacetime interval looks like

$$
ds^2 = g_{ik} d\rho^i d\rho^k = \frac{\mathcal{N}^2}{\mathcal{D}^2} (d\rho_0^2 - \mathcal{D}^2 (d\rho_1^2 + d\rho_2^2 + d\rho_3^2)). \quad (21)
$$

It defines the metric tensor components $g_{ik}$ for the uniform coordinates $\rho^0, \rho^1, \rho^2, \rho^3$ and also $g^{ij} = 1/g_{ij}$.

According to Eq. (17), $\mathcal{N} > 1$ for $r_0^2 > 0$, hence the dependence of $r$ on $\rho$ is monotonic. For asymptotically large $\rho \rightarrow \infty$, $\mathcal{D} \rightarrow 1$ in accordance with Eq. (16), therefore $r$ and $\rho$ are approximately equal to each other. The minimal value of $r$ equal to zero corresponds to the minimal possible positive value of $\rho = \rho_{min}$. This value is the maximal number obeying the equation $\mathcal{D}(\rho) = 0$ and is equal to

$$
\rho_{min} = r_e/2 - r_0^2/4. \quad (22)
$$

This means that the sphere in the three-dimensional space $(\rho, \rho_1, \rho_2)$ with the radius $\rho = \rho_{min}$ corresponds to the point $r = 0$. There is no contradiction in this respondency since according to Eq. (21) the distance between any two points on the sphere is zero due to the relation $\mathcal{D}(\rho_{min}) = 0$.

3. TETRAD REPRESENTATION

In the tetrad representation proposed in [15], the fundamental variables of the gravitational field are four unit four-vectors $h_{\alpha\beta}$ ($\alpha = 0, 1, 2, 3$ is a counting number of the four-vector) being functions of the coordinates of points in the four-dimensional space-time. Their covariant and contravariant components are related in the usual way by means of the metric tensor:

$$
h_{\alpha\beta} = g_{ik}h^i_{\alpha}, \quad h^i_{\alpha} = g^{ik}h_{\alpha k}. \quad (23)
$$

The four-vector with $\alpha = 0$ is chosen time-like, while all others are space-like, namely

$$
h_{\alpha\beta}h^{\beta}_{(\alpha)} = \eta_{\alpha\beta}, \quad (24)
$$

with the diagonal matrix $\eta_{\alpha\beta} = \text{diag}(1,-1,-1,-1)$. Defining $\eta^{\alpha\beta}$ equal to $\eta_{\alpha\beta}$ the vectors $h^{(\alpha)}$ can be expressed in terms of $h_{\alpha\beta}$ with the equation valid both for the covariant and contravariant components [2, 15]

$$
h^{(\alpha)} = \eta^{\alpha\beta}h_{\beta(\alpha)}, \quad h^{(\alpha)} = \eta^{\alpha\beta}h_{\beta(\alpha)}. \quad (25)
As follows from Eqs. (24) and (25) the orthogonality conditions look like [2, 15]

\[ h_{(a)i} h^{(b)j} = \delta^b_i, \]  
(26)

\[ h_{(a)j} h^{(b)k} = \delta^b_j, \]  
(27)

where \( \delta^b_j \) and \( \delta^b_i \) denote the Kronecker symbols. The simplest consequence of Eq. (27) is the fundamental relation between the tetrads and the metric tensor [2, 12, 15]  

\[ h_{(a)j} h^{(b)k} = g_{ik}, \]  
(28)

\[ h_{(a)j} h^{(b)k} = g^{jk}. \]  
(29)

The partial derivative of \( h_{(a)i} \) over the uniform coordinate \( \rho^j \) is denoted by \( h_{(a)j} \), while \( h_{(a)i} \) denotes the covariant derivative, where by definition,

\[ h_{(a)j} = \frac{\partial h_{(a)i}}{\partial \rho^j} - \Gamma_{ik}^m h_{(a)m} = h_{(a)j} - \Gamma_{ik}^m h_{(a)m}, \]  
(30)

with \( \Gamma_{ik}^m \) being the Christoffel symbols. They are obtained with the help of standard formulas [2, 12]

\[ \Gamma_{ik}^m = g^{mn} \Gamma_{n,jk}, \]  
(31)

\[ \Gamma_{n,jk} = \frac{1}{2} \left[ \frac{\partial g_{nk}}{\partial \rho^j} + \frac{\partial g_{nj}}{\partial \rho^k} - \frac{\partial g_{jk}}{\partial \rho^n} \right]. \]  
(32)

Using Eq. (28) for the metric tensor \( g_{ij} \) in terms of the tetrads and substituting Eqs. (31) and (32) into Eq. (30) one gets the formula of interest [15]:

\[ h_{(a)j} = \frac{1}{2} \left( h_{(a)j} - h_{(a)j} \right) + \frac{h_{(a)n}}{2} \times \left[ h_{(b)n} h_{(b)m} - h_{(b)m} h_{(b)n} \right]. \]  
(33)

In the following, we need the formula for \( h_{(a)jk} \). In order to obtain it, let us calculate the covariant derivative of Eq. (26) that is

\[ h_{(a)j} h^{(b)k} + h_{(a)j} h^{(b)k} = 0. \]  
(34)

Multiplying this equation by \( h^{(b)k} \), summing over \( a \), and making use of Eq. (27) we get

\[ h_{(a)j} h^{(b)k} = -h_{(a)j} h^{(b)k} h_{(a)j} = -h_{(a)j} h^{(b)k} h_{(a)j}. \]  
(35)

Substituting Eq. (33) into Eq. (35), summing over \( a \), and taking into account Eq. (29) we get finally

\[ h_{(a)jk} = \frac{1}{2} h_{(a)m} h^{(b)k} (h_{(a)n} - h_{(a)n}) + \frac{1}{2} g^{ik} (h_{(a)i} - h_{(a)i}) + \frac{1}{2} g^{ik} h_{(a)i} (h_{(a)n} - h_{(a)n}). \]  
(36)

As seen from Eqs. (33) and (36) the covariant derivatives \( h_{(a)j} \) and \( h_{(a)k} \) depend on the partial derivatives \( h_{m,j} \) linearly.

4. LAGRANGIAN FOR THE REISSNER–NORDSTRÖM SOLUTION

The formula for the total lagrangian density reads

\[ \mathcal{L}_{\text{tot}} = \mathcal{L}_g + \mathcal{L}_e, \]  
(37)

where \( \mathcal{L}_g \) is the lagrangian density of gravitational field, while \( \mathcal{L}_e \) denotes the lagrangian density of electromagnetic field. In the tetrad representation, the former is given by the formula [15, 16]

\[ \mathcal{L}_g = \frac{|h|}{2\kappa} (h_{(a)i}h_{(a)j} - h_{(a)i}h_{(a)j}), \]  
(38)

where

\[ \kappa = 8\pi k / c^4, \]  
(39)

and \( |h| \) denotes the determinant of the \( 4 \times 4 \) matrix \( h_{(a)i} \). As is demonstrated by Eq. (33) or (36) the covariant derivatives of the tetrads components depend only on the tetrads components and their first partial derivatives with respect to the coordinates \( \rho^m \). Therefore the lagrangian \( \mathcal{L}_g \) in Eq. (38) depends also on the tetrads and their first partial derivatives.

According to Eq. (21) and relation [15, 16]

\[ g = -|h|^2, \]  
(40)

the formula for \( |h| \) for the uniform coordinates for the RN solution is

\[ |h| = N^{-2}. \]  
(41)

The general formula for the lagrangian density of the electromagnetic field reads [2, 12]

\[ \mathcal{L}_e = \frac{-|h|}{16\pi} F_{ik} F^{ik}. \]  
(42)

Since \( F_{ik} F^{ik} \) is invariant it is possible to use Eq. (12) for the electromagnetic tensor expressing \( r \) with the help of Eq. (15). Using also Eq. (41) the final formula for \( \mathcal{L}_e \) for the RN solution in the uniform coordinates is obtained

\[ \mathcal{L}_e = \frac{-e^2 N}{8\pi \rho G} = \frac{N r_e^2}{\kappa \rho G}. \]  
(43)

In the second representation for \( \mathcal{L}_e \) in Eq. (43) more convenient below, the formula

\[ \frac{e^2}{8\pi} = \frac{r_e^2}{\kappa}, \]  
(44)

is taken into account that follows from definitions given by Eqs. (9) and (39).
For the tetrads with the covariant components
\[ h_{(0)k} = h_{k}^{(0)} = \frac{N}{D_0} x_k, \quad h_{(\mu)k} = -h_{k}^{(\mu)} = D_k \delta^\mu_0 \]  
(45)
for any \( k \) and \( \mu \), relation (28) is fulfilled. Formulas for the contravariant components follow from Eqs. (23), (45), and also from Eq. (21) for the metric tensor
\[ h_{(0)^{\mu}} = g_{(0)^{\mu}} h_{(0)\nu} \delta^\nu_0 = \delta^\mu_0 / N, \]  
(46)
\[ h_{(\mu)^{\nu}} = -h_{(\nu)^{\mu}} = g_{(\nu)^{\mu}} h_{(\nu)\lambda} = -\delta^\mu_\lambda / D. \]  
(47)
These contravariant components of the tetrad obey Eq. (29). Using formulas for \( h_{ij}^{(0k)} \) obtained in Subsection 11.2 of Appendix one gets
\[ \mathcal{L}_e = \frac{N(D)^{\prime}}{\kappa} \left\{ 2 N^{\prime} / N - D \right\}, \]  
(48)
where \( N^{\prime} \) and \( D^{\prime} \) denote derivatives of \( N(p) \) and \( D(p) \) with respect to \( p \).

Substituting \( \mathcal{L}_{em} \) given by Eq. (43) and \( \mathcal{L}_e \) from Eq. (48) into Eq. (37) and using Eqs. (16) and (17) respectively for \( D(p) \) and \( N(p) \) we obtain for the total lagrangian density the very simple formula
\[ \mathcal{L}_{tot} = \frac{\rho^2}{\kappa p^{4}}. \]  
(49)
Since the three-dimensional space \( (\rho_x, \rho_y, \rho_z) \) consists of points for which \( \rho = \sqrt{\rho_x^2 + \rho_y^2 + \rho_z^2} \) obeys the inequality \( \rho \geq \rho_{min} \), the value of the total lagrangian is given by the equation
\[ L_{tot} = \int_{\rho = \rho_{min}} \mathcal{L}_{tot} d\rho_x d\rho_y d\rho_z = \frac{4\pi \rho_{min}^2}{\kappa p^{4}}. \]  
(50)
Making use of Eqs. (8)–(10) and (21) we get the final result:
\[ L_{tot} = c^2 \sqrt{\frac{\rho^2}{k} + mc^2}. \]  
(51)
This formula shows that the lagrangian in the tetrad representation and the action \( S = L_{tot} t \) are finite in spite of the singular behaviour of the electromagnetic and gravitational fields near the point \( r = 0 \) \( (\rho = \rho_{min}) \).

It is not the case when we consider the metric tensor components as the fundamental variables describing the gravitational field. Indeed, it is well known [2, 12, 14, 17] that the scalar curvature \( \mathcal{R} \) and hence the lagrangian density of the gravitational field
\[ \mathcal{L}_g = -\frac{\mathcal{R} \sqrt{-g}}{2k}, \]  
(52)
is zero if the matter is represented by the electromagnetic field only. Since the lagrangian density for the electromagnetic field defined by Eqs. (42) and (12) has nonintegrable singularity at \( r = 0 \) the total lagrangian corresponding to Eqs. (37), (42), and (52) is meaningless, while Eq. (51) provides the finite lagrangian in the tetrad representation. We assume that it is these sixteen functions \( h_{ij}^{(0k)}(p^k) \) of the coordinates \( p^k \) which are the fundamental gravitational variables rather than the components of the metric tensor. The knowledge of a true lagrangian density is of crucial importance since it could be used in calculations of the process amplitudes in quantum theory with the help of the Feynman functional integrals [18, 19].

5. EQUATIONS OF MOTION

Remarkable formula (51) follows from the choice of the tetrad defined by Eqs. (45)–(47) that reproduces the metric tensor corresponding to the RN solution. But the true tetrads are to obey the Lagrange equations
\[ \frac{\partial \mathcal{L}_{tot}}{\partial h_{(c)\rho}} = \frac{\partial}{\partial \rho^\nu} \left[ \frac{\partial \mathcal{L}_{tot}}{\partial h_{(c)\rho q}} \right]. \]  
(53)
The total lagrangian density depends on the variables \( h_{ij}^{(c)} \) describing the gravitational field and their partial derivatives \( h_{ij}^{(c)\rho q} \) with respect to \( \rho^\nu \). The variables for the electromagnetic field are the four-potentials \( A_i \). The electromagnetic field tensor \( F_{ik} \)
\[ F_{ik} = A_{k,i} - A_{i,k} = \frac{\partial A_k}{\partial \rho^i} - \frac{\partial A_i}{\partial \rho^k}, \]  
(54)
is a linear combination of the partial derivatives of \( A_i \) with respect to the spacetime coordinates. Formula (42) for the lagrangian density of the electromagnetic field rewritten in the form
\[ \mathcal{L}_{em} = -\frac{|h|}{16\pi} g^{ik} F_{ik} F_{ij}, \]  
(55)
shows that \( \mathcal{L}_{em} \) depends on \( A_{i,k} \) and \( h_{ij}^{(c)} \) since the determinant \( |h| \) and the metric tensor can be expressed in terms of the tetrad components according to Eq. (29), therefore \( \mathcal{L}_{em} \) does not depend on \( h_{ij}^{(c)} \). The only lagrangian density, which depends on the derivatives \( h_{ij}^{(c)\rho q} \) is that of the gravitational field \( \mathcal{L}_g \) defined by Eq. (38).

Since \( \mathcal{L}_g \) is a sum of bilinear products of \( h_{ij}^{(0k)} \) we need the derivative of \( h_{ij}^{(0k)} \) with respect to \( h_{ij}^{(c)\rho q} \). Making use of Eq. (36) it is easy to get the result
\[ \frac{\partial h_{ij}^{(0k)}}{\partial h_{ij}^{(c)\rho q}} = \frac{1}{2} \left[ h_{ij}^{(c)}(\delta_{ij} h_{(c)pq} - \delta_{ij} h_{(c)pq}) + \frac{1}{2} \delta_{ij} g^{pk} \delta_{ij} - g^{pk} \delta_{ij} \right]. \]  
(56)
Multiplying $h_{i\alpha j}^l$ from Eq. (36) by $\frac{\partial h_{i\alpha k}^l}{\partial h_{p\alpha q}^c}$ from Eq. (56), after some algebra, one gets the simple expression

$$\frac{\partial (h_{i\alpha k}^l h_{\alpha k}^l)}{\partial h_{p\alpha q}^c} = 2h_{c\alpha}^l \gamma_{\alpha i}^p,$$  
(57)

in terms of the tensor $\gamma_{\alpha i}^p$, where its covariant components are defined by [15, 16]

$$\gamma_{\alpha i}^p h^a_{\alpha j} = -h_{a\alpha j} h^a_{\alpha j} = -\gamma_{\alpha i}^p.$$  
(58)

Taking $k = l$ in Eq. (56) and summing over $k$ we get

$$\frac{\partial h_{i\alpha k}^{(b)}}{\partial h_{p\alpha q}^{(c)}} = h^{c}_{\alpha j} h^{(b)cg} - h^{c}_{\alpha j} h^{(b)p}.$$  
(59)

Let us define the four-vector $\Phi^\gamma$ by the relation

$$\Phi^\gamma = \gamma^\gamma.$$  
(60)

Then, taking $k = l$ in Eq. (36), presenting $g^{ik}$ as the bilinear product of the tetrads with the help of Eq. (29), and summing over $l$, we get [15]

$$h_{i\alpha j}^{(b)} = -h^{(b)m}_{\alpha j} \Phi^m.$$  
(61)

The simplest consequence of Eqs. (59) and (61) is the formula

$$\frac{\partial \left( h_{i\alpha k}^{(b)} h_{\alpha k}^{(b)} \right)}{\partial h_{p\alpha q}^{(c)}} = 2 \left( h_{\alpha j}^{(c)} \Phi^\alpha - h_{\alpha j}^{(c)} \Phi^\alpha \right).$$  
(62)

Substituting Eqs. (57) and (62) into Eq. (38) we get finally

$$\frac{\partial}{\partial \Phi^\gamma} \left[ \frac{\partial L_{tot}}{\partial h_{p\alpha q}^{(c)}} \right] = \frac{\partial}{\partial \Phi^\gamma} \left[ \frac{1}{\kappa} \left( h_{\alpha j}^{(c)} \Phi^\alpha + h_{\alpha j}^{(c)} \Phi^\alpha - h_{\alpha j}^{(c)} \Phi^\alpha \right) \right].$$  
(63)

In order to obtain the left-hand side of Eq. (53), one needs the formula for $\frac{\partial L_{tot}}{\partial h_{p\alpha q}^{(c)}}$, where

$$\frac{\partial L_{tot}}{\partial h_{p\alpha q}^{(c)}} = \frac{\partial L_g}{\partial h_{p\alpha q}^{(c)}} + \frac{\partial L_{em}}{\partial h_{p\alpha q}^{(c)}}.$$  
(64)

Using Eq. (38) we get

$$\frac{\partial L_g}{\partial h_{p\alpha q}^{(c)}} = \frac{1}{\kappa} \left[ h_{i\alpha k}^{(b)} \frac{\partial h_{i\alpha k}^{(b)}}{\partial h_{p\alpha q}^{(c)}} - h_{i\alpha k}^{(b)} \frac{\partial h_{i\alpha k}^{(b)}}{\partial h_{p\alpha q}^{(c)}} \right] + \frac{1}{2\kappa} \left[ h_{i\alpha k}^{(b)} h_{i\alpha k}^{(a)} - h_{i\alpha k}^{(a)} h_{i\alpha k}^{(a)} \right] \frac{\partial h_{i\alpha k}^{(b)}}{\partial h_{p\alpha q}^{(c)}}.$$  
(65)

Equation (65) contains $\frac{\partial h_{i\alpha k}^{(b)}}{\partial h_{p\alpha q}^{(c)}}$ that can be obtained from Eq. (36). Differentiating for this aim Eq. (26) with respect to $h_{p\alpha q}^{(c)}$ and using the trivial formulas

$$\frac{\partial h_{m \alpha }^{(b)}}{\partial h_{p\alpha q}^{(c)}} = \delta_{\alpha m}^p \Phi^\alpha, \quad \frac{\partial h_{b\alpha m}^{(c)}}{\partial h_{p\alpha q}^{(c)}} = \eta_{\alpha m}^p \Phi^\alpha,$$  
(66)

we get

$$\frac{\partial h_{i\alpha k}^{(b)}}{\partial h_{p\alpha q}^{(c)}} = -h_{b\alpha m}^{(b)} \Phi^\alpha.$$  
(67)

Differentiating the left-hand side of Eq. (29) and taking into account Eqs. (67), (68) we get the formula for the derivative of $g^{ij}$ with respect to $h_{p\alpha q}^{(c)}$

$$\frac{\partial g^{ij}}{\partial h_{p\alpha q}^{(c)}} = -g^{im} h_{a\alpha j}^{(b)} - g^{mj} h_{a\alpha i}^{(b)},$$  
(69)

Making use of Eqs. (66)–(69) in the calculation of the partial derivative of $h_{i\alpha k}^{(b)}$ over $h_{p\alpha q}^{(c)}$, we get after some algebra

$$h_{i\alpha k}^{(b)} h_{i\alpha k}^{(b)} - h_{i\alpha k}^{(b)} h_{i\alpha k}^{(b)} - \frac{1}{2} g^{ik} h_{i\alpha k}^{(b)} h_{i\alpha k}^{(b)} (\gamma_{\alpha i}^p + \gamma_{\alpha i}^p).$$  
(70)

In order to express the derivative $\frac{\partial h_{i\alpha k}^{(b)}}{\partial h_{p\alpha q}^{(c)}}$ in terms of the tensor $\gamma_{\alpha i}^p$, its definition by Eq. (58) is used and the formula

$$h_{i\alpha k}^{(b)} h_{i\alpha k}^{(b)} = h_{i\alpha k}^{(b)} - h_{i\alpha k}^{(b)} = h_{i\alpha k}^{(b)} (\gamma_{\alpha i}^p - \gamma_{\alpha i}^p).$$  
(71)

is applied. Formula (71) follows from the obvious transformation using Eq. (26) and the definition of $\gamma_{\alpha i}^p$ by Eq. (58)

$$h_{i\alpha k}^{(b)} = \delta_{\alpha m}^p h_{i\alpha k}^{(b)} = h_{i\alpha k}^{(b)} h_{i\alpha k}^{(b)} = h_{i\alpha k}^{(b)} (\gamma_{\alpha i}^p + \gamma_{\alpha i}^p).$$  
(72)

Making use of Eq. (71) the basic formula (36) can easily be rewritten as [15]

$$h_{i\alpha k}^{(b)} = -h_{i\alpha k}^{(b)} \gamma_{\alpha i}^p.$$  
(73)

Combining Eq. (73) with Eq. (70) one gets

$$h_{i\alpha k}^{(b)} h_{i\alpha k}^{(b)} = h_{i\alpha k}^{(b)} \gamma_{\alpha i}^p (\gamma_{\alpha i}^p + \gamma_{\alpha i}^p).$$  
(74)

Taking in Eq. (70) $k = l$ and summing over $k$ we get

$$\frac{\partial h_{i\alpha k}^{(b)}}{\partial h_{p\alpha q}^{(c)}} = h_{i\alpha k}^{(b)} h_{i\alpha k}^{(b)} + h_{i\alpha k}^{(b)} (\gamma_{\alpha i}^p - \gamma_{\alpha i}^p).$$  
(75)
Combining this formula with Eq. (61) we get
\[
\mathcal{L}_{\text{tot}}^{(c)} = \frac{\partial}{\partial h_{0}^{(c)}} \left[ \gamma_{\mu \nu} \Phi_{\mu} - \Phi^{\nu} \Phi^{\mu} \right].
\] (76)

The last formula needed to obtain \( \partial \mathcal{L}_{g} / \partial h_{p}^{(c)} \) is the following [15, 20]:
\[
\frac{\partial \mathcal{L}_{g}}{\partial h_{p}^{(c)}} = \gamma_{\mu \nu} \Phi_{\mu} - \Phi^{\nu} \Phi^{\mu}.
\] (77)

Finally, substituting \( h_{b b}^{(b)} \partial h_{j}^{(b)}/\partial h_{p}^{(c)} \) given by Eq. (74), \( h_{a}^{(b)} \partial h_{k}^{(b)}/\partial h_{p}^{(c)} \) from Eq. (76), and \( \gamma_{\mu \nu} \Phi_{\mu} - \Phi^{\nu} \Phi^{\mu} \) given by Eq. (77) into Eq. (65) we obtain the formula
\[
\frac{\partial \mathcal{L}_{\text{em}}}{\partial h_{p}^{(c)}} = \frac{F_{i} F_{j}}{16\pi} \left[ \gamma_{\mu \nu} \Phi_{\mu} - \Phi^{\nu} \Phi^{\mu} + \gamma_{\mu \nu} \Phi_{\mu} + \frac{h_{p}}{2} \left( \gamma_{\mu \nu} \Phi_{\mu} - \Phi^{\nu} \Phi^{\mu} \right) \right].
\] (78)

In the calculation of \( \mathcal{L}_{\text{em}}^{(c)}/\partial h_{p}^{(c)} \), where
\[
\frac{\partial \mathcal{L}_{\text{em}}}{\partial h_{p}^{(c)}} = \frac{F_{i} F_{j}}{16\pi} \left[ \gamma_{\mu \nu} \Phi_{\mu} - \Phi^{\nu} \Phi^{\mu} + \gamma_{\mu \nu} \Phi_{\mu} + \frac{h_{p}}{2} \left( \gamma_{\mu \nu} \Phi_{\mu} - \Phi^{\nu} \Phi^{\mu} \right) \right].
\] (79)

Eq. (55) for \( \mathcal{L}_{\text{em}} \) is used and the independence of the covariant components of the electromagnetic field tensor \( F_{i} \) on \( h_{p}^{(c)} \) is taken into account. Substituting \( \gamma_{\mu \nu} \Phi_{\mu} - \Phi^{\nu} \Phi^{\mu} \) from Eq. (69) and \( \gamma_{\mu \nu} \Phi_{\mu} - \Phi^{\nu} \Phi^{\mu} \) from Eq. (77) into Eq. (79) we get
\[
\frac{\partial \mathcal{L}_{\text{em}}}{\partial h_{p}^{(c)}} = -\gamma_{\mu \nu} \Phi_{\mu} + \gamma_{\mu \nu} \Phi_{\mu} + \left( \gamma_{\mu \nu} \Phi_{\mu} - \Phi^{\nu} \Phi^{\mu} \right).
\] (80)

where \( T_{i}^{\mu} \) is the energy-momentum tensor of the electromagnetic field defined by Eq. (13).

Substituting \( \partial \mathcal{L}_{g} / \partial h_{p}^{(c)} \) and \( \mathcal{L}_{\text{em}} / \partial h_{p}^{(c)} \) respectively from Eqs. (78) and (80) into Eq. (64) and taking into account Eq. (13) for \( T_{i}^{\mu} \) we get for the left-hand side of motion Eq. (53) the final formula
\[
\frac{\partial \mathcal{L}_{\text{tot}}}{\partial h_{p}^{(c)}} = \frac{\partial}{\partial h_{0}^{(c)}} \left[ \gamma_{\mu \nu} \Phi_{\mu} + \gamma_{\mu \nu} \Phi_{\mu} + \gamma_{\mu \nu} \Phi_{\mu} + \gamma_{\mu \nu} \Phi_{\mu} - \Phi^{\nu} \Phi^{\mu} + \frac{\alpha}{2} \left( \gamma_{\mu \nu} \Phi_{\mu} - \Phi^{\nu} \Phi^{\mu} \right) \right].
\] (81)

Here, \( \mathcal{L}_{\text{tot}} \) is described by very short formula (49) which simplifies any calculations. Note that expression (13) for \( T_{i}^{\mu} \) contains the term proportional to the product of the Kronecker symbol and the lagrangian density \( \mathcal{L}_{\text{em}} \) given by Eq. (42). The lagrangian density \( \mathcal{L}_{g} \) for the gravitational field can be rewritten in the form [15, 16]
\[
\mathcal{L}_{g} = \frac{\partial}{\partial h_{p}^{(c)}} \left[ \gamma_{\mu \nu} \Phi_{\mu} - \Phi^{\nu} \Phi^{\mu} \right].
\] (82)

The sum of the two last terms in the right-hand side of Eq. (78) proportional to \( \mathcal{L}_{g} \) and the term in Eq. (80) proportional to \( \mathcal{L}_{\text{em}} \) gives the term proportional to \( \mathcal{L}_{\text{tot}} \) in Eq. (81).

Thus, the equation of motion for \( h_{p}^{(c)} \) is Lagrange Eq. (53), where its left-hand side is given by Eq. (81), while the right-hand side is given by Eq. (63). It is shown in Subsection 11.4 of Appendix, that the tetrad \( h_{p}^{(c)} \) defined by Eqs. (45) obeys the Lagrange equation.

6. CONSERVATION
OF ENERGY-MOMENTUM FOUR-VECTOR

In the tetrad formalism, the total energy-momentum pseudo-tensor density, \( \mathcal{T}_{i}^{k} \) is related to the superpotential \( U_{i}^{kl} \), proposed by Moller (see [15, 16, 20]), where
\[
U_{i}^{kl} = -U_{i}^{lk} = h_{i}^{(a)} \frac{\partial \mathcal{L}_{\text{tot}}}{\partial h_{i}^{(a)}} = h_{i}^{(a)} \frac{\partial \mathcal{L}_{\text{tot}}}{\partial h_{i}^{(a)}}.
\] (83)

with the formula
\[
\mathcal{T}_{i}^{k} = \frac{\partial U_{i}^{kl}}{\partial l}. \tag{84}
\]

It is shown in [15, 16, 20] that
\[
U_{i}^{kl} = \frac{\partial}{\partial h_{i}^{(a)}} \left[ h_{i}^{(a)} + \Phi_{i}^{(a)} \right] \tag{85}
\]
or in terms of the tensor \( \gamma_{ai} \) and the four-vector \( \Phi^{i} \) defined respectively by Eq. (58) and (60)
\[
U_{i}^{kl} = \frac{\partial}{\partial h_{i}^{(a)}} \left[ h_{i}^{(a)} + \Phi_{i}^{(a)} \right]. \tag{86}
\]

As seen from Eq. (85) Moller’s superpotential is the tensor density under arbitrary coordinate transformations since it is expressed in terms of the tetrad vectors \( h_{i}^{(a)} \), their covariant derivatives \( h_{i}^{(a)} \), and the determinant \( h_{i} \). The superpotential \( U_{i}^{kl} \) is antisymmetric with respect to the indexes \( l, k \), therefore the divergence of the total energy-momentum pseudo-tensor density is zero
\[
\mathcal{T}_{i}^{k} = U_{i}^{kl} \frac{\partial \Phi_{i}^{l}}{\partial l} = 0. \tag{87}
\]

As is well known [2, 12, 15–17, 20] the conservation of the energy-momentum four-vector is a consequence of Eq. (87).

If the total energy-momentum pseudo-tensor is localized in the compact three-dimensional region \( V \), such a system will be called the insular one. Then the
metric tensor $g_{αβ}$ for the insular system goes to its Minkowski limit $ν_{αβ}$ at $ρ → ∞$ as

$$g_{αβ}(ρ) = ν_{αβ}(ρ) + O(ρ).$$

(88)

Here, $O_n$ denotes the quantity which main term at $ρ → ∞$ is proportional to $ρ^{-n}$ for $n = 1, 2, ...$. The standard consideration shows that the energy-momentum four-vector $P_i$ defined by

$$P_i = \frac{1}{c} \int \mathcal{T}_i^α d^3ρ,$$

(89)

is conserved if all fields are zero outside the region $\mathcal{V}$, where $d^3ρ = dρ_1 dρ_2 dρ_3$ is the volume element. Since the formula for $P_i$ can be rewritten in terms of the superpotential components according to Eq. (84) it can be expressed as the surface integral using the Gauss theorem

$$P_i = \frac{1}{c} \int \mathcal{T}_i^{αλ} d^3ρ = \frac{1}{c} \int \mathcal{Λ}_i^{αλ} k_i dσ.$$

(90)

Here, $Σ$ is the closed surface enclosing the region $\mathcal{V}$, while the three-dimensional vector $k_i$ is the unit outer normal to the surface. The element of the surface is defined by

$$k_i dσ = ε_{αμν} dp^μ δσ^ν,$$

(91)

where $ε_{αμν}$ is the totally antisymmetric three-dimensional Levi-Civita symbol, while $dp^μ$ and $δσ^ν$ are infinitesimal three-vectors on the surface $Σ$.

When the matter fields are localized mainly in the region $\mathcal{V}$ and $\mathcal{T}_i^{αλ}$ goes to zero outside $\mathcal{V}$ at $ρ → ∞$ as a quantity of $O_n$ with $n ≥ 4$ the system will be also called the insular system. For this case, the full three-dimensional space ($\mathcal{V}_∞$) is to be considered and for the surface $Σ$, we will choose $Ω_Ω$ with $R → ∞$. Here $Ω_Ω$ denotes the surface of the sphere with the radius $R$. The surface integral in Eq. (90) becomes the limit of the integral on $Ω_Ω$ at $R → ∞$.

Substituting formulas (41) for $|h|$, (46) and (47) for $k_α$, and (141)–(144) for $k_α^{μν}$, obtained in Appendix, into Eq. (85) we get for the nonzero components of the superpotential $U_i^{αλ}$ for the RN solution

$$U_0^{αλ} = -U_0^{λα} = -2N n_Ω dΩ' / kΩ,$$

(92)

where $n_Ω$ is the unit three-vector

$$n_Ω = ρ^λ / ρ$$

(93)

with $ρ^λ$ defined by Eqs. (20). For $μ ≠ λ$, one gets

$$U_μ^{λμ} = -U_μ^{μλ} = \frac{h_k}{k} N.$$

(94)

Note that for the RN solution the full three-dimensional space for the uniform coordinates $ρ_x$, $ρ_y$, $ρ_z$ consists of all points with $ρ = \sqrt{ρ_x^2 + ρ_y^2 + ρ_z^2} ≥ ρ_{min}$, where $ρ_{min}$ is given by Eq. (22) and $Ω(ρ_{min}) = 0$. This means that the surface integral consists of the integrals over the spheres with the radii $ρ = ρ_{min}$ and $ρ = R$ with $R → ∞$. As seen from Eq. (92) all the superpotential components $U_0^{αλ}$ are infinite on the sphere with the radius $ρ_{min}$ since $Ω(ρ_{min}) = 0$ and $U_0^{αλ}$ behave near $ρ_{min}$ as $1/(ρ - ρ_{min})$. This makes the definition of the energy given by Eq. (90) meaningless for any parameter $m$ obeying inequality in Eq. (6) since the energy is infinitely large. Also, there is no solution with $e$ and $m$ obeying Eq. (6) with a finite total inertial mass which can be used as a model for electrons.

But it is not the case when the limit $m → \sqrt{e/k}$ is considered, and

$$f\left(\frac{e}{\sqrt{km}}\right) ≡ \frac{r_e^2}{r_0^2} = 1 - \left(\frac{e}{\sqrt{km}}\right)^2 = 0,$$

(95)

where $r_e$ and $r_0$ are defined by Eqs. (9) and (10), respectively. The solution of this equation with the positive value of the mass $m = m_Ω$, where

$$m_Ω = \sqrt{\frac{e^2}{k}},$$

(96)

gives the mass of the system called “the point-charge”.

In classical electrodynamics, the electric charge can be arbitrary, therefore the typical length of the system $r_c$ given by Eq. (9) can be large for large $|e|$ and the system can be macroscopic. Having in mind to consider elementary particles, we put $|e| = 4.80 × 10^{-10}$ esu for which $r_c = 1.38 × 10^{-34}$ cm. It is this system which will be referred to as “the classical electron”. Its mass is equal to $1.86 × 10^{-6}$ g that is much larger than the experimental value of the real electron mass.

If condition (95) is fulfilled, then according to definitions (8)–(10) and (22)

$$r_0 = 0, \quad r_e = r_c / 2, \quad ρ_{min} = 0.$$

(97)

Note that owing to the singularity at $ρ = 0$ the limit $m → m_Ω$ for fixed $e$ is not trivial. Indeed, at $m = m_Ω$ we have $r_0 = 0$ and the total lagrangian density is zero according to Eq. (49). Nevertheless the total lagrangian being the integral of the lagrangian density over the three-dimensional space is nonzero according to Eq. (51): $L_{tot} = 2m_Ω e^2$.  

PHYSICS OF PARTICLES AND NUCLEI LETTERS  Vol. 16  No. 3  2019
Using Eqs. (97) we get for $\mathcal{D}(\rho)$, $\mathcal{N}(\rho)$, $\mathcal{D}'(\rho)$, $\mathcal{N}'(\rho)$, from Eqs. (16) and (17)

\[
\mathcal{D}(\rho) = 1 + \frac{r_2}{2\rho} = 1 + \frac{r_2}{\rho}, \quad (98)
\]

\[
\mathcal{N}(\rho) = 1, \quad (99)
\]

\[
\mathcal{D}'(\rho) = -\frac{r_2}{2\rho} = -\frac{r_2}{\rho}, \quad (100)
\]

\[
\mathcal{N}'(\rho) = 0. \quad (101)
\]

Due to Eq. (101) formula (94) is simplified:

\[
U^\mu_\mu = -U'^\mu_\mu = 0, \quad (102)
\]

while formula (92) becomes

\[
U^\mu_\mu = -U'^\mu_\mu = \frac{m_\text{e}c^2}{4\pi} \frac{n_\text{e}}{\rho^3 (1 + r_\text{e}/\rho)}, \quad (103)
\]

if Eqs. (98)–(100), (8), (9), and (39) are taken into account.

Formula (90) for the total energy of the system of the electromagnetic and gravitational fields reads now

\[
\mathcal{E} \equiv cP_\text{b} = \int_{\Omega_b} U_0^{\mu\lambda} n_\text{e} d\sigma - \int_{\Omega_b} U_0^{\mu\lambda} n_\text{h} d\sigma, \quad (104)
\]

with $\mathcal{R} \to \infty$ and $\epsilon \to 0$. The first integral over the sphere with the radius $\mathcal{R}$ in the right-hand side of this formula is

\[
\lim_{\mathcal{R} \to \infty} \int_{\Omega_b} \frac{m_\text{e}c^2}{4\pi} \frac{n_\text{e}}{\mathcal{R}^2 (1 + r_\text{e}/\mathcal{R})} \mathcal{R}^2 d\Omega = m_\text{e}c^2, \quad (105)
\]

where $d\sigma = \mathcal{R}^2 d\Omega$ with $d\Omega = \sin \theta d\theta d\varphi$ being the differential of the solid angle. The second integral in Eq. (104) over the sphere with the infinitesimal radius $\epsilon$ is zero. Indeed, we have

\[
\lim_{\epsilon \to 0} \int_{\Omega_b} \frac{m_\text{e}c^2}{4\pi} \frac{n_\text{h}}{\epsilon^3 (1 + r_\text{e}/\epsilon)} \epsilon^2 d\Omega = 0. \quad (106)
\]

This means that in spite of the singular behaviour of the electromagnetic and gravitational fields near $\rho = 0$, the singularity does not contribute to the surface integral, therefore the energy is determined by the field behaviour at large distances ($\mathcal{R} \to \infty$). The net result of Eqs. (104)–(106) is

\[
\mathcal{E} = m_\text{e}c^2 = c^2 \sqrt{\frac{\mathcal{E}}{k}}, \quad (107)
\]

which shows that the total energy of the system of the electromagnetic and gravitational fields is equal to $c^2 \sqrt{\mathcal{E}/k}$ or equivalently to $mc^2$ with $m = m_\text{h}$, according to Eq. (95), where $m$ is the parameter of the RN solution. This parameter is now the inertial mass of the system of the electromagnetic and gravitational fields.

In principle, the point-like particle with a bare mass $m_b$ may contribute to the total inertial mass of the electron to make it equal to the gravitational mass. As will be shown in the next section $m_b = 0$.

The total energy-momentum pseudo-tensor is described with formulas (84), (102), and (103) which give the only nonzero component for the case under consideration

\[
\mathcal{T}_0^0 = \frac{\partial}{\partial \rho^\mu} U_0^\mu \xi, \quad (108)
\]

where $\xi(R)$ denotes the three-dimensional Dirac delta function, and $\mathbf{R} = (\rho_x, \rho_y, \rho_z)$ is the three-vector. Since $\rho_0^0(\mathbf{R}) = 0$ the first term in the square brackets in Eq. (108) does not contribute to $\mathcal{T}_0^0$, hence the final result is

\[
\mathcal{T}_0^0 = \frac{k m_\text{e}^2}{4\pi^2 (1 + r_\text{e}/\mathcal{R})^2} = \frac{e^2}{4\pi^2 (1 + r_\text{e}/\mathcal{R})^2}, \quad (109)
\]

where in the right-hand side of Eq. (109) condition (95) is taken into account. We see that the energy of the electromagnetic and gravitational fields for the classical electron is localized in the space region of the range of about $r_\text{e} = 1.38 \times 10^{-34}$ cm. Strictly speaking it is true only for the convenient coordinate system under consideration since the energy distribution cannot be uniquely defined in field theory [2, 21] even if the time variable is fixed. Nevertheless, $r_\text{e}$ can really characterize the order of magnitude of the system length. This is easily seen from the formula (51) for the total lagrangian which can be rewritten with the help of Eq. (9) for the classical electron in the form

\[
L_{\text{tot}} = 2c^2 \sqrt{\frac{\mathcal{E}}{k}} = 2 \frac{e^2}{r_\text{e}}. \quad (110)
\]

Let us again calculate the total energy of the classical electron with the help of Eqs. (89) and (109) rather than Eqs. (90) and (92). Since now $\rho_{\text{min}} = 0$ we have

\[
\mathcal{E} = \int_0^\infty \mathcal{T}_0^0 (4\pi r^2) d\rho = \int_0^\infty \frac{e^2}{4\pi^2 (1 + r_\text{e}/\rho)^2} d\rho. \quad (111)
\]

As seen from this formula the integrand increases with decreasing of $\rho$ as $\rho^{-2}$ at $\rho \gg r_\text{e}$ but at $\rho \leq r_\text{e}$ the integrand goes to a finite constant. As a result, the integral is convergent and is equal to $m_\text{e}c^2 = c^2 \sqrt{\mathcal{E}/k}$. Indeed, using Eq. (9) we have from Eq. (111)

\[
\mathcal{E} = \int_0^\infty \frac{e^2}{\rho} d\rho = e^2 \int_0^\infty \frac{e^2}{\rho} d\rho = m_\text{e}c^2. \quad (112)
\]
In the absence of gravitation, when \( k = 0 \), the value of \( r_c \) is zero according to Eq. (9) and the integral in Eq. (111) becomes divergent as in classical electrodynamics. Therefore it is the gravitational interaction which makes the classical electron mass finite. Roughly speaking the gravitation contribution reduces the electron mass from infinity (the pure electromagnetic mass) to a finite mass \( \sqrt{e^2 / k} \).

It is interesting to compare formula (109) with the energy-momentum tensor density of the electromagnetic field. Since the tensor component \( T_0^0 \) is the same for coordinate systems \( x^0, r, \theta, \phi \) and \( p_0, p_r, p_\theta, p_\phi \), the expression for the tensor density component \( \bar{T}_0^0 \) of the electromagnetic fields follows from Eqs. (14), (15), (41), (98), and (99)

\[
\bar{T}_0^0 \equiv h|T_0^0| = \frac{e^2 N(p)[D(p)]^2}{8\pi \rho(D(p))} = \frac{e^2}{8\pi \rho^4 \left(1 + \frac{\rho}{m_c} \right)^3}.
\]

A comparison of Eqs. (109) and (113) shows that the total energy-momentum pseudo-tensor density is by a factor of two larger than that of the electromagnetic field. This is the trivial consequence of the Tolman formula [22]

\[
\bar{T}_0^0 = \bar{T}_0^0 - \bar{T}_1^1 - \bar{T}_2^2 - \bar{T}_3^3.
\]

Indeed, the trace of the tensor density \( \bar{T}_i^i \) is zero for the electromagnetic field [1, 2, 12], hence \( -\bar{T}_1^1 - \bar{T}_2^2 - \bar{T}_3^3 = \bar{T}_0^0 \), therefore \( \bar{T}_0^0 = 2\bar{T}_0^0 \) which proves the statement. Since the density of the energy-momentum tensor of the electromagnetic field \( \bar{T}_0^0 \) is one-half of the total pseudo-tensor density \( \bar{T}_0^0 \), the electromagnetic mass of the classical electron is also one-half of the total mass \( m_{cl} \) according to Eq. (89). Note that “the electromagnetic mass” has conditional meaning since its formula contains the gravitational constant \( k \) according to Eqs. \( m_{em} = m_{cl}/2 \) and (96).

The space part of the energy-momentum pseudo-tensor density \( \bar{T}_\mu^\nu \) is zero

\[
\bar{T}_\mu^\nu \equiv 0,
\]

which follows from Eqs. (84) and (102). This assumes the absence of any pressure of any part of the system under discussion (classical electron) on others. Therefore there is no need in the additional surface-tension of the charged liquid (existing, for instance, in the Lorentz model of the electron [3]) which prevents dis-integration of the classical electron. There is an equilibrium between the electrostatic repulsion and gravitational attraction.

Since the components \( U_{\mu}^{\alpha \beta} \) of the superpotential are zero the three-momentum of the classical electron is zero in the rest system of frame. If the four-velocity of the singular point (the point with \( p = 0 \) in the rest system) is \( u_i \) than the four-vector of the electron in any Lorentz system is

\[
P_i = m_c u_i,
\]

while the components of the energy-momentum pseudo-tensor are

\[
\bar{T}_i^k = m_c e^2 \delta(R) u_i u_k,
\]

using the Dirac three-dimensional \( \delta \)-function. This formula is widely used to describe point-like particles.

We would like to stress that Eq. (118) takes into account the electromagnetic field of the classical electron, hence it is not correct to add to the tensor density given by Eq. (118) the energy-momentum tensor density of either the external or the total electromagnetic field as it is often done in classical electrodynamics (see for instance [2, 12]). Let us explain this statement in more details. If the total electromagnetic field tensor \( F_{(tot)}^{lm} \) is equal to the sum of the external field tensor \( F_{(ext)}^{lm} \) and that of the classical electron field \( F_{(cl)}^{lm} \), then the total energy-momentum tensor density of the electromagnetic field is

\[
\bar{T}_k^{(tot)} = |h| \left( T_k^{(cl)} + T_k^{(ext)} + T_k^{(int)} \right).
\]

Here, \( T_k^{(cl)} \) and \( T_k^{(ext)} \) are respectively the energy-momentum tensors of the classical electron and the external field given by Eq. (13) for \( F_{(cl)}^{lm} = F_{(ext)}^{lm} \) and \( F_{(int)}^{lm} = F_{(ext)}^{lm} \) respectively. Here \( F_k^{(cl)} \) and \( F_k^{(ext)} \) are the tensors of the electromagnetic fields of the electron and the external field. The interference term \( T_k^{(int)} \) is

\[
T_k^{(int)} = \frac{1}{4\pi} \times \left( \frac{1}{2} \delta^{kl} F_{(cl)lm} F_{(ext)}^{lm} - F_{(cl)kl} F_{(ext)}^{(kl)} - F_{(cl)kl} F_{(ext)}^{(kl)} \right).
\]

In the zero approximation applied here, we ignore the alteration of the metrics due to the influence of the external electromagnetic field. Since the contribution of \( T_k^{(cl)} \) is taken into account in the term given by Eq. (117) or (118) the additional terms describing the
contribution of the external electromagnetic field is given by the formula
\[ \Delta T^{k}_{(tot)} = \ln \left[ T^{k}_{(ext)} + T^{k}_{(int)} \right], \]  
(121)
rather than \( \ln |T^{k}_{(ext)}| \) or \( \ln |T^{k}_{(tot)}| \). The expression for \( \Delta T^{k}_{(tot)} \) in Eq. (121) is integrable function while \( \ln |T^{k}_{(tot)}| \) is not.

7. EQUVALENCE PRINCIPLE

In order to calculate the total gravitational mass \( m_{gr} \) of any insular system we should consider the \( g_{00} \) component of the metric tensor which asymptotic behaviour at \( \rho \to \infty \) is [2, 12]
\[ g_{00} = 1 + 2 \frac{\phi (\rho)}{c^2} = 1 - \frac{2km_{gr}}{c^2 \rho}, \]
(122)
where \( \phi (\rho) \) denotes the Newtonian gravitational potential. Using formulas (16) and (17) respectively for \( \rho \) and \( N \) we get from Eq. (21) that for any \( e \) and \( m \) at \( \rho \to \infty \)
\[ g_{00} = \frac{N^2}{\rho^2} = 1 - \frac{r_e}{\rho} = 1 - \frac{2km}{c^2 \rho}. \]
(123)
A comparison of Eq. (122) with (123) gives \( m_{gr} = m \) which means that the parameter \( m \) in the RN solution is always the total gravitational mass of the system. For \( m = m_{cl} = \sqrt{e^2/k} \), the parameter \( m \) becomes the inertial mass of the system of the electromagnetic and gravitational fields in accordance with Eqs. (104) and (107). Therefore for this case, the total gravitational mass is equal to the inertial mass of the system of the electromagnetic and gravitational fields. According to the equivalence principle, this means that the bare mass \( m_b \) of the point-like particle is zero.

Therefore the classical electron is a system of the electromagnetic and gravitational fields localized in the space region with the typical length of about 10^{-34} cm. There is no need in the existence of any charged point-like particle which is usually named “electron” in classical electrodynamics.

8. ELECTRICAL CHARGE DISTRIBUTION

In order to study the electrical charge distribution, we should consider the non-trivial properties of the three-dimensional space in the vicinity of the point with \( \rho = 0 \). Let \( \rho \) be equal to infinitesimal \( \epsilon > 0 \). In order to calculate the length of circumference of a circle with the maximal length on the sphere with \( \sqrt{r_x^2 + r_y^2 + r_z^2} = \epsilon \) we should take into account Eq. (19) and put \( \theta \) equal to \( \pi/2 \). We have the formula for the length
\[ l_{e} = \int_{0}^{2\pi} D(\epsilon) \rho^{2} \rho_{e} = 2\pi(\epsilon + r_{c}). \]
(124)
In the transformation of the right-hand side of this equation, expression (98) for \( D(\rho) \) is used. Formula (124) shows that for \( \epsilon = 0 \), one gets a nonzero length \( l_{e} = 2\pi r_{c} \).

In order to understand this paradox, we find the relation between \( r \) and \( \rho \), which follows from Eqs. (15) and (98)
\[ r = \rho + r_{c}. \]
(125)
As seen from Eqs. (125) the point \( \rho = 0 \) in the space \( (\rho, \rho, \rho) \) corresponds to the sphere with the radius \( r_{c} \) in the three-dimensional space \( (x, y, z) \), where
\[ x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta. \]
(126)
The electric charge density on this sphere is
\[ \rho_{ch} = \frac{e}{4\pi r_{c}^{2}} = \frac{e^{4}}{4\pi e^{2}}, \]
(127)
since the metric on this sphere in accordance with Eqs. (3) and (4) is the same as on the sphere in the Euclidean space.

The radial distance between points with \( \rho = \rho_{1} > 0 \) and \( \rho = \rho_{2} > 0 \), the other coordinates being equal, is
\[ l_{12} = \int_{\rho_{1}}^{\rho_{2}} D(\rho) d\rho = \int_{\rho_{1}}^{\rho_{2}} \left[ 1 + \frac{r_{e}}{\rho} \right] d\rho \]
(128)
in accordance with the metrics given by Eq. (19) and expression (98) for \( D(\rho) \). As is seen from the above formula \( l_{12} \to \infty \) if \( \rho_{1} \to 0 \). Therefore the distance between the point with \( \rho_{x} = \rho_{y} = \rho_{z} > 0 \) and any other point is infinite. Since for \( m = m_{cl} \) we have \( r_{g} = 2r_{e} \) and the tensor component \( -g_{11} \) for the coordinates \( r, \theta, \phi \) is equal to \( 1/(1 - r_{g}/r)^{2} \) according to Eqs. (1) and (7). Therefore \( -g_{11} \) goes to infinity when \( r \to r_{e} \). It is obvious from this behavior of \( -g_{11} \) that the distance between any point in the \( (x, y, z) \) space with a finite \( r > r_{e} \) and the sphere with the radius \( r_{e} \), which center has coordinates \( x = y = z = 0 \), is also infinite.

For \( m = m_{cl} \) \( g_{00} = -1/g_{11} = (1 - r_{e}/r)^{2} \) according to Eqs. (1) and (7). The sphere with \( r = r_{e} \) represents the surface of the event horizon \( (g_{00}(r_{e}) = 0) \), therefore no information can be obtained from the internal \( (r < r_{e}) \) region [2, 12]. This means that the coordinates \( x, y, z \)
do not correspond to any physical objects which can be observed by any external observer if \( \sqrt{x^2 + y^2 + z^2} < r_e \). This means also that any observables (energy, electric and magnetic field strengths etc.) can be really measured only in the three-dimensional space with \( r > r_e \). This explains why the integral in Eq. (112) runs only for \( r \geq r_e \) which corresponds to \( \rho \geq 0 \).

Also, the electric charge of the classical electron is uniformly distributed on the sphere with \( r = r_e \). The space-like components of the energy-momentum pseudo-tensor density \( T_{\mu \nu}^0 \) responsible for the forces between parts of the electron are identically zero. Therefore the problem of extra nonelectromagnetic forces preventing the electron disintegration as in the Lorentz model [3], presenting it as a charged liquid drop, is absent. The distance from this charged sphere and any point in the three-dimensional space \((x, y, z)\) is infinite.

9. DISCUSSION OF RESULTS
AND QUANTUM EFFECTS

As is shown above the system of gravitational and electromagnetic fields described by Eqs. (1)–(4) and (12) has a finite total inertial mass if the parameter \( m \) of the RN solution is equal to \( \sqrt{e^2/k} \). For this case \( m \) is both the total inertial and gravitational mass, that is in agreement with the equivalence principle. Such a system is called “the classical electron” when \( e \) is the experimental electron charge. There is no need in any additional charged point-like particle which is usually called the electron. This assumes the absence of the term in the action describing the interaction of the point-charge moving with a three-velocity \( v \) with the electromagnetic field usually considered in classical electrodynamics [1, 2, 4]

\[
S_{pf} = e \int \left[ \frac{v(t)}{c} A(x, y, z, t) - A_0(x, y, z, t) \right] dt. \tag{129}
\]

Note that the term \( S_{pf} \) has no physical meaning since it is infinite. Indeed, we should consider the scalar \( A_0 \) and vector \( A \) potentials of the electromagnetic field. But \( A_0(x, y, z) \) and \( A(x, y, z, t) \) are infinite at any point of a trajectory of the point-charge. Consideration of the external field, usually done in textbooks for the integral in Eq. (129) to make it convergent, is in contradiction with the fundamental concepts of the field theory. Indeed, all the basic quantities should be expressed in terms of variables of the total electromagnetic field as it is the only fundamental notion. The term “external field” for all point-charges assumes the consideration of individual fields of all point-charges but these fields are not fundamental notions.

In the approach under consideration in the present paper, the only existing entities are the electromagnetic and gravitational fields, while the solutions, localized in the space regions of the range of about \( 10^{-34} \) cm, represent the classical electrons. The important thing is the tetrad representation, which makes the action finite. Considering the metric tensor components as the true variables of the gravitational field we conclude that the action for the classical electron is infinite.

It is well known that the quantum effects become important at distances of about the Compton wavelength of the real electron \( \lambda_c = \hbar/(mc) \) where \( \hbar \) denotes the Plank constant and \( m_e \) denotes the experimental electron mass. Since \( \lambda_c = 3.86 \times 10^{-11} \) cm, it is much greater than the typical length \( r_e = 1.38 \times 10^{-10} \) cm for the classical electron. For the first sight, this means that the above consideration has no physical meaning. Nevertheless, highly likely that this argument is not true. Indeed, the divergence of the integral \( \int T_\mu^\nu d^3r \) in classical electrodynamics is very hard, namely the integrand behaves as \( 1/r^2 \) at \( r \to 0 \). Nevertheless the contribution of gravitation makes the energy finite for the parameter \( m \) of the Reissner–Nordström solution equal to \( \sqrt{e^2/k} \). Since the divergence of the integral for the self-energy in QED (the integrand behaves as \( 1/r \) at small \( r \)), the role of gravitation can be much less important. If it changes the behaviour of the integrand, say, to \( 1/r^{1-\delta} \) with a very small positive \( \delta \), the integral becomes convergent. The integral for the self-energy in QED is proportional to \( \alpha_{\text{em}} \ln(\Lambda_c/(mc)) \) where \( \Lambda_c \) is a ultraviolet cutoff parameter which is put equal to infinity and \( \alpha_{\text{em}} = e^2/(\hbar c) \) is the fine-structure constant. As is seen from Eq. (111) the space cutoff parameter arising due to the contribution of gravitation is \( r_e \), hence the natural cutoff parameter in the momentum space is

\[
\Lambda_c = \frac{\hbar c}{M_p}, \quad M_p = \sqrt{\hbar c / k} \tag{130}
\]

where \( M_p = \sqrt{\hbar c / k} \) is the Plank mass. Therefore \( \alpha_{\text{em}} \ln(\Lambda_c/(mc)) \) could be replaced by \( \alpha_{\text{em}} \ln(M_p/m) \) with the same accuracy. In the classical physics, the finite energy of the electron exists for \( m = \sqrt{e^2/k} \) only or, in other words, when the function \( f(e/m\sqrt{k}) \) defined by Eq. (95) is zero. It is not excluded, that for the quantum case the finite energy of the field configuration exists not for all mass parameters \( m \) but for some fixed one for which the quantum analog of the function...
\( f(x) \) is zero with \( x = \alpha_{\text{em}} \ln(M_{\text{Pl}} / m) \). For the experimental mass of the electron \( M_{\text{Pl}} / m_e = 2.4 \times 10^{32} \), while \( \alpha_{\text{em}} = 1/137.04 \), therefore \( x \approx 0.38 \). This means that \( x \) is of the order of unity, hence the argument \( x = \alpha_{\text{em}} \ln(M_{\text{Pl}} / m) \) looks rather natural and equation \( f(x) = 0 \) could give the electron mass close to its experimental value, while, for instance, \( x = h/(r_m c) \) is unnatural (huge).

The most natural question arises: “How to find the quantum analog of the function \( f(x) \) defining the electron mass?” We saw that the total stress tensor \( \mathcal{T}_{\mu}^\nu \) is identically zero in Eq. (115) if \( m = \sqrt{e^2/k} \) and vice versa. It is likely that in the quantum case this condition can be replaced with the demand that some matrix elements of the total stress tensor over the wavefunctions of the electron must be zero to prevent the finite value of the total mass.

In the approach based on the supersymmetry, the negative vacuum energy of the electron-positron field compensates the positive vacuum energy of the photon field. The total vacuum energy of all fundamental fields can be equal to zero if the supersymmetry is not broken. Supersymmetric partners of the existing particles with masses less than about 1 TeV/c^2 are not found up to now (see for instance [23–25]). This can mean that the supersymmetry is not a fundamental symmetry of elementary particles. In the absence of the electron-positron field, the vacuum energy can probably be made finite due to the negative contribution of the gravitational field. Indeed, let us consider only the modes of the quantum oscillations of the electromagnetic field with frequencies \( \omega < \omega_c \). Let us imagine that the vacuum energy density for these modes with the mode energies \( \epsilon_{\text{vac}} = h\omega/2 \) is huge but finite in the absence of the gravitational interaction. If the gravitational interaction is switched on its negative contribution decreases the vacuum energy. If the electromagnetic energy density goes to infinity (when \( \omega \to \infty \)) as in standard QED the modulus of the gravitational interaction contribution increases also to infinity. It is not excluded that the total vacuum energy density is finite. This picture is in an analogy with the classical electron case for which the infinite growth of the electromagnetic energy density at \( \rho \to 0 \) is restricted with the gravitational attraction of small space regions, filled with electromagnetic field, with each other. The cancellation of the contributions to the total mass of the electromagnetic and the gravitational fields leads to the finite value of the total mass.

Electrons take part in weak interaction and this electron property should be taken into account. Therefore we should try to find solutions of equations for the system of the electromagnetic, gravitational, and weak-boson fields. Another property of the electron, that should be taken into account in its realistic description, is the electron spin \( s \) equal to \( h/2 \). But solutions with \( s = h/2 \) are not excluded for the nonlinear boson fields. A well known example provides the Skyrme model [26, 27] in which the solutions with the spin \( h/2 \) (baryons) are constructed though the fundamental field being the nonlinear field of the pseudoscalar pions. Another way to make the spin is to consider a solution (if it exists) which corresponds to an electric charge and a magnetic dipole moment since the electromagnetic field in this case has an angular momentum [28]. The localized states of the quantized electromagnetic, gravitational, and weak-boson fields with the \( h/2 \) spin, the observed values of the electric charge, the weak charge, and the finite masses equal to those of the electron, muon, and \( \tau \)-lepton could exist. In the same way, the localized solutions of the quantized gluon, electromagnetic, weak boson, and gravitational field equations would be quarks and there would not be a need in local bispinor fields corresponding to the point-like massive quarks. This problem cannot probably be solved soon since there is no renormalizable quantum field theory of gravitation though any estimates of the electron mass could be performed in lattice calculations using the continual integrals for electromagnetic and gravitational fields (see review [19]). Nevertheless, we assume that the idea to construct all observed particles as singular or localized (in a very small three-dimensional space region) solutions of fundamental field equations is constructive.

10. CONCLUSIONS

It is shown, that for the Reissner–Nordström solution, the contribution of gravitation makes the total energy-momentum pseudo-tensor density integrable function if the parameter of the solution \( m = \sqrt{e^2/k} \). Nevertheless the singular point exists for this case also. The total inertial mass of the system of the electromagnetic and gravitational fields is finite \( m_{\text{in}} = \sqrt{e^2/k} \) and equal to its total gravitational mass \( m_{\text{gr}} \). According to the equivalence principle \( (m_{\text{in}} = m_{\text{gr}}) \), this means the absence of an additional contribution to the total mass of any charged point-like particle with a nonzero bare mass. In the approach of the present paper, the
classical electron is the system of the electromagnetic and gravitational fields localized in the space region with the typical length of about \(10^{-34}\) cm, \(e = -4.80 \times 10^{-10}\) esu, and \(m = 1.86 \times 10^{-6}\) g.

Since the total stress tensor density \(\mathcal{T}_\nu\) is identically zero there is no need in additional nonelectrostatic forces preventing the disintegration of the classical electron which were introduced in the Lorentz model of the electron (surface tension forces for the charged liquid drop).

As is known the total lagrangian for the Reissner—Nordström solution has nonintegrable singularity if the metric tensor components are considered as fundamental variables of the gravitational field. The total lagrangian for the Reissner—Nordström solution is shown to be finite in the tetrad representation for any values of the parameters \(e\) and \(m\).

We assume that it is not excluded that in quantum field theory, the only existing fundamental entities are gravitational, electromagnetic, weak-boson, and gluon fields. Leptons and quarks are states of these fields having singular points whose positions are usually considered as the positions of corresponding particles.

11. APPENDIX

11.1. Christoffel Symbols in Uniform Coordinates

Using the metric tensor defined by Eq. (21) the Christoffel symbols can be obtained with the help of Eqs. (31) and (32). The nonzero components of \(\Gamma_{i,jk}\) in the uniform coordinates are

\[
\Gamma_{0,0\mu} = \Gamma_{0,\mu 0} = -\Gamma_{\mu,00} = \frac{Nn_\mu}{D'} \{D_N - N'D'\},
\]

\[
\Gamma_{\mu,\mu\mu} = -n_\mu D' D',
\]

where the three-vector \(n_\mu\) is defined in Eq. (93) and

\[
N' = \frac{\partial N}{\partial p} = -\frac{r_0^2}{2p^3},
\]

\[
D' = \frac{\partial D}{\partial p} = -\frac{r_s}{2p} + \frac{r_0^2}{2p^3},
\]

\[
D_N - N'D' = \frac{r_s}{2p^2} \left(1 - \frac{r_0^2}{4p^2}\right) - \frac{r_0^2}{p^4}.
\]

Note that in Appendix, there is no summation over two or three identical indexes, all sums contain the symbol \(\sum\). The functions \(D\) and \(N\) are defined in Eqs. (16) and (17), while \(r_s\) and \(n_\mu\) in Eqs. (8) and (10), respectively. Other nonzero Christoffel symbols for \(\mu \neq \nu\) are

\[
\Gamma_{\mu,\nu\nu} = -\Gamma_{\nu,\mu\nu} = n_\mu D' D'.
\]

The nonzero Christoffel symbols \(\Gamma_{j,k}\) are

\[
\Gamma_{\mu\mu} = \Gamma_{\mu 00} = n_\mu \frac{(D_N' - N'D')}{N'D'},
\]

\[
\Gamma_{0\mu 0} = n_\mu \frac{(D_N' - N'D')}{D'^2},
\]

\[
\Gamma_{\mu\mu\mu} = n_\mu \frac{D'}{D},
\]

while for \(\mu \neq \nu\)

\[
\Gamma_{\nu\nu} = -\Gamma_{\nu\nu\mu} = -n_\nu \frac{D'}{D}.
\]

11.2. Covariant Derivatives of Tetrads

For the tetrads defined by Eqs. (45) and the Christoffel symbols obtained in Subsection 11.1 the calculation of the covariant derivatives of \(h_m^{(a)}\) with using Eq. (30) gives the following nonzero components:

\[
h_{m,0}^{(0)} = -n_\mu \frac{D}{D'} \left\{D_N - N'D'\right\},
\]

\[
h_{m,0}^{(i)} = -n_\mu \frac{N}{D} \left\{D_N - N'D'\right\},
\]

and for \(\mu \neq \nu\)

\[
h_{m,\nu}^{(\mu)} = n_\mu D',
\]

\[
h_{m,\nu}^{(\nu)} = -n_\mu D'.
\]

As a consequence of these formulas, we have

\[
\sum_{0}^{3} h_{m}^{(0)i} = 0,
\]

\[
\sum_{0}^{3} h_{m}^{(i)i} = -n_\mu \left\{\frac{N'}{N} + \frac{D'}{D}\right\},
\]

therefore

\[
\sum_{a,b=0}^{3} \sum_{l,k=0}^{3} h_{m}^{(a,l)} h_{m}^{(b,k)} = -\frac{1}{D} \left\{\frac{N'}{N} + \frac{D'}{D}\right\}^2.
\]

This sum is nothing else than the second sum in the brackets in Eq. (38).

The first sum in the brackets in Eq. (38) is

\[
\sum_{a,b=0}^{3} \sum_{l,k=0}^{3} \left\{h_{m}^{(a,l)} h_{m}^{(b,k)}\right\} = \sum_{a,b=0}^{3} \sum_{l,k=0}^{3} \left\{g_{k}^{kk} g_{l}^{l} h_{m}^{(a,l)} h_{m}^{(b,k)}\right\}
\]

\[
= -\frac{1}{D} \left\{\left(\frac{N'}{N}\right)^2 - 2 \frac{N'}{N} \frac{D'}{D} + 3 \left(\frac{D'}{D}\right)^2\right\},
\]

which follows from Eq. (141)—(144) for \(h_{m}^{(a)}\) and (21) for \(g_{ij} = 1/g_{ji}\). The difference of these two sums multiplied by \(h_{F}/(2\pi)\) provides the formula for the lagrangian density of the gravitational field given by Eq. (48).
if formula (41) for the determinant $|h|$ is taken into account.

11.3. Tensor $\gamma_{\mu\nu}$ and Four-Vector $\Phi^\mu$

Using definition of $\gamma_{\mu\nu}$ given by Eq. (58), Eqs. (141)–(144) for $h^{(0)}_{\mu\nu}$ and Eqs. (45) for $h^{(n)}_{\mu\nu}$ one gets formulas for the nonzero covariant components of the tensor $\gamma_{\mu\nu}$

$$\gamma_{\mu00} = -\gamma_{0\mu0} = n_0 \frac{N}{D} \left\{ D N^* - N^* D \right\},$$

$$\gamma_{\mu0} = -\gamma_{0\mu} = n_0 \frac{D'}{D} \left\{ -N^* \right\},$$

$$\gamma_{\mu\nu} = -\gamma_{\nu\mu} = n_0 \frac{D'}{D} \left\{ N^* \right\},$$

where $\mu \neq \nu$ in Eq. (150).

Applying the metric tensor from Eq. (21) corresponding to the RN solution for the uniform coordinates the nonzero components of $\gamma_{\mu\nu}$ can be obtained

$$\gamma_{\mu0} = -\gamma_{0\mu} = n_0 \frac{N}{D} \left\{ D N^* - N^* D \right\},$$

$$\gamma_{\mu\nu} = -\gamma_{\nu\mu} = n_0 \frac{D'}{D} \left\{ N^* \right\},$$

while for $\mu \neq \nu$ we have

$$\gamma_{\mu\nu} = -\gamma_{\nu\mu} = n_0 \frac{D'}{D} \left\{ N^* \right\},$$

In an analogous way, the formulas

$$\gamma_{\mu0} = -\gamma_{0\mu} = n_0 \frac{N}{D} \left\{ D N^* - N^* D \right\},$$

and the relations for $\mu \neq \nu$

$$\gamma_{\mu\nu} = -\gamma_{\nu\mu} = n_0 \frac{D'}{D} \left\{ N^* \right\},$$

can be obtained.

The contravariant nonzero components of $\gamma^{kl}$ are

$$\gamma^{\mu00} = -\gamma^{0\mu0} = -n_0 \frac{D N^*}{D} \left\{ N^* D \right\},$$

$$\gamma^{\nu\mu} = -\gamma^{\nu\mu0} = -n_0 \frac{D}{D} \left\{ N^* \right\}.$$

Using Eqs. (154) and (154) we get for $\Phi^\mu$ defined by Eq. (60) the result

$$\Phi_0 = 0,$$  

$$\Phi_\mu = -n_0 \left\{ \frac{N^*}{N} + \frac{D'}{D} \right\}.$$

The contravariant components of the vector $\Phi$ follow from the metric tensor $g^{\mu\nu}$ defined by Eq. (21) and from Eqs. (160), (161)

$$\Phi^0 = 0,$$  

$$\Phi^\mu = \frac{n_0}{D} \left\{ \frac{N^*}{N} + \frac{D'}{D} \right\}.$$

11.4. Solution of the Lagrange Equations

According to Eqs. (45) there are two nonzero independent components of the tetrads for the RN solution, namely $h^{(0)}_{\mu0}$ and $h^{(n)}_{\nu\mu} = h^{(1)}_{\nu\mu} = h^{(3)}_{\nu\mu}$, the latter will be denoted from here on $h^{(1)}_{\mu\nu}$. Substituting into Eq. (53) $c = p = 0$ and $q = \nu = 1, 2, 3$ ($q \neq 0$ since all field variables are time independent) we get for the right-hand side of Eq. (53) with the help of Eq. (63)

$$\frac{\partial L}{\partial \Phi^\mu} \left[ \frac{\partial L_{tot}}{\partial h_{(0)}^{0\mu}} \right] = \frac{\partial L}{\partial \Phi^\mu} \left[ \left\{ h_{(0)}^{0\mu} \Phi^0 + h_{(0)}^{0\nu} \Phi^\nu - h_{(0)}^{\nu\Phi^\mu} \right\} \right]$$

$$= \frac{\partial L}{\partial \Phi^\mu} \left[ \left\{ \frac{N^*}{N} \right\} - \frac{n_0}{N} \left( \frac{D}{D} - \frac{D'}{D} \right) \right]$$

$$+ \frac{n_0}{N} \left( \frac{D}{D} + \frac{D'}{D} \right) \right\} = \frac{2}{\kappa} \frac{n_0}{D} \left\{ \frac{D'n_0}{N} \right\}.$$

In the above chain of equations, it is taken into account that $\Phi^0 = 0$ according to Eq. (162) and formula (46) for the $h^{(0)}_{\mu\nu}$ components. Equations (41) for $|h|$, (156) for $\gamma_{0\mu}$, and (163) for $\Phi^\nu$ are also used. Using Eq. (93) for $n_0$ and Eq. (134) for $D'$ one gets

$$\frac{\partial L_{tot}}{\partial h_{(0)}^{0\mu}} = -\frac{n_0^2}{\kappa r^4} \frac{4\pi n_0^2}{\kappa r^4} \delta(R) = \frac{n_0^2}{\kappa r^4}.$$

Since $\rho \geq \rho_{\text{min}} > 0$ for the uniform coordinates the term with the three-dimensional Dirac delta-function $\delta(R)$ in Eq. (165) is zero.

In order to obtain the left-hand side of Eq. (53), we substitute into Eq. (81) $p = c = 0$. This leads to the formula

$$\frac{\partial L_{tot}}{\partial h_{(0)}^{0\mu}} = \sum_{i=0}^4 \gamma_{(0)}^{i0} F_{(0)i} \left( h_{(0)}^{i0} + h_{(0)}^{0i} \right)$$

$$+ \frac{\gamma_{(0)}^{0i}}{\kappa c} \sum_{j=0}^3 \gamma_{(m)}^{ij0} \left( \gamma_{(m)}^{0i0} + \gamma_{(m)}^{0i0j} \right)$$

$$+ \frac{3}{\mu=1} \left( \gamma_{(0)}^{0i0} + \gamma_{(0)}^{0i0j} \right) \Phi_{(i)}.$$
is taken into account also that $\Phi^0 = \Phi_0 = 0$ according to Eqs. (160) and (162), therefore we may replace $\Phi_m$ by $\Phi^0$.

When the variable $\rho$ is used instead of $r$ the nonzero component $F_{01}$ is transformed to $\tilde{F}_{01} = \frac{dr}{d\rho} F_{01} = N F_{01}$ according to Eq. (17). The coordinate transformation from $\rho, \theta, \phi$ to $r^1, r^2, r^3$ causes the electromagnetic tensor transformation $\tilde{F}_{\alpha\beta} = \frac{d\rho}{dr^\lambda} F_{\alpha\beta}$ with $\frac{d\rho}{dr^\lambda} = n_\lambda$.

Therefore the electromagnetic tensor components for the uniform coordinates $r^k$ are $\tilde{F}_{0k} = n_k F_{01}$. Using Eq. (12) for $F_{01}$ and relation (15) we came to the final formula for the nonzero components of $\tilde{F}_{ik}$

$$F_{0k} = -F_{ik} = E_\lambda \frac{e \rho^2}{N} n_\lambda = \frac{e N}{\rho^2 e} n_\lambda. \quad (167)$$

In Eq. (167) and hereafter, more simple notations $F_{ik}$ are used instead of $\tilde{F}_{ik}$ in the uniform coordinate system. Since $F_{0k}$ is zero for $l = 0$ we should write

$$\sum_{l=0}^{3} \sum_{\lambda=1}^{3} F_{l0}^0 F_{0l} = \sum_{l=0}^{3} \sum_{\lambda=1}^{3} g^{00} g^{0l} [F_{0l}]^2 \frac{\partial^2}{\partial \rho^2} \left[ \frac{N}{\rho} e^2 \right]^2 = \frac{\partial^2}{\partial \rho^2} \left[ \frac{N}{\rho} e^2 \right]^2 = \frac{\partial^2}{\partial \rho^2} \left[ e^2 \right]^2. \quad (168)$$

In transformation of Eq. (168), the obvious relation

$$\sum_{\lambda=1}^{3} n^2_\lambda = 1, \quad (169)$$

and Eq. (21) for $g^\lambda = 1$ are taken into account. If Eq. (41) for $|\bar{h}|$, Eq. (46) for $h_{0}^{0}$, Eq. (44) to express $e^2$ through $r_c^2$, and Eq. (168) are used this leads to the formula

$$h_0^0 = \frac{|\bar{h}|}{4\pi} \sum_{l=0}^{3} \sum_{\lambda=1}^{3} F_{l0}^0 F_{0l} = -\frac{2}{3} \frac{r_c^2}{\kappa p^2} \frac{e^2}{N}. \quad (170)$$

According to Eq. (158) the contravariant tensor $\gamma^{0m}$ is nonzero if two indexes are zero. Due to the anti-symmetry with respect to the first and second indexes, $m$ cannot be zero in $\gamma^{0m}$. Hence $l = 0$ in $\gamma^{0m}$. Due to the same reason $\gamma_{0m} = 0$ if $l = 0$. Therefore the sum in Eq. (166) can be simplified

$$\sum_{m=0}^{3} \gamma^{0m} (\gamma_{0m} + \gamma_{00}) = \sum_{m=0}^{3} \gamma^{0m} \gamma_{00}. \quad (171)$$

Finally, using Eq. (170), formula (49) for $L_{\text{tot}}$, Eq. (46) for $h_{0}^{0}$, substituting Eq. (171) into basic Eq. (166), expressing the nonzero components of the tensor $\gamma$ using Eqs. (149), (156), (158) and the four-vector components $\Phi_\mu$ with the help of (161) we get

$$\frac{\partial L_{\text{tot}}}{\partial h_{0}^{0}} = -\frac{r_c^2}{\kappa p^2}. \quad (172)$$

In order to obtain the simple formula (172), we use Eqs. (16), (17), (133), (134) respectively for $\gamma$, $N^\lambda$, $N^0$, $\mathcal{D}$, $\mathcal{D}'$ taking also into account Eq. (10). A comparison of Eqs. (165) and (172) with the master Eq. (53) shows that Lagrange Eq. (53) for $h_0^{0}$ is valid.

Equation (53) for $c = 0$ and $p = v \neq 0$ is satisfied also, though both the right-hand side and left-hand side are zero. Indeed, remembering that $q \neq 0$ in Eq. (63) since both functions are time independent, hence $q = \lambda$ with $\lambda = 1, 2, 3$, we have for this case from Eq. (63)

$$\frac{\partial}{\partial \rho^l} \left[ \frac{\partial L_{\text{tot}}}{\partial h_{0}^{0}} \right] = \frac{\partial}{\partial \rho^l} \left[ |\bar{h}| (h_{0}^{0} \gamma^{\lambda l} + h_{0}^{l} \Phi^\lambda - h_{0}^{0} \Phi^\lambda) \right] = 0. \quad (173)$$

We take into consideration that $h_{0}^{0} = h_{0}^{l} = 0$ in Eq. (173) in the uniform coordinates and also that $h_{0}^{l} \neq 0$ only if $l = 0$. The simplest consequence of the last relation is the following equality: $\gamma^{\lambda l} = \gamma^{0} = 0$ valid according to Eqs. (156) and (157) showing nonzero tensor components of $\gamma^{0l}$. This proves Eq. (173).

For the calculation of the left-hand side of Eq. (53), we substitute into Eq. (81) $c = 0$, $p = v$ that gives

$$\frac{\partial L_{\text{tot}}}{\partial h_{0}^{0}} = \frac{|\bar{h}|}{4\pi} F^{00} F_{00}^{0} + h_{0}^{0} L_{\text{tot}} + \frac{|\bar{h}|}{\kappa} h_{0}^{0} \left[ \sum_{m=0}^{3} \gamma^{m0} (\gamma_{0m} + \gamma_{00}) + \sum_{m=0}^{3} \gamma^{m0} (\gamma_{0m} + \gamma_{00}) \right]. \quad (174)$$

Since $h_{0}^{0} \neq 0$ only for $l = 0$, then the first term in Eq. (174) is zero as $F_{00}^{0} = 0$, while the second term vanishes as $h_{0}^{0} = 0$. The tensor components $\gamma^{0l} = \gamma_{00}^l = 0$ at $l = 0$ since $\gamma^{0l} \neq 0$ when even number of indexes are equal to zero, and also $\Phi_0 = 0$ at $l = 0$ according to Eq. (160). Therefore two last terms in the curl brackets in Eq. (174) are zero. In the sum over m and l in Eq. (174), the tensor $\gamma^{m0}$ is nonzero according to Eqs. (158) and (159) for two cases: $m = l = 0$ or both $m$ and $l$ are nonzero. For the former case and for $l = 0$ $\gamma_{m0} = \gamma_{0m} = 0$, while for the latter case...
\[ \gamma_{m\ell l} = \gamma_{l\ell m} = 0 \text{ since they are absent in Eqs. (149) and (150) for nonzero } \gamma_{ikl}. \text{ As a result we get} \]

\[ \frac{\partial \mathcal{L}_{\text{tot}}}{\partial h_{\mu\nu}^{(0)}} = 0, \quad \text{(175)} \]

that proves the validity of Lagrange Eq. (53) for this case.

For \( p = c = \mu \) and \( q = \lambda \), Eq. (63) looks like

\[ \sum_{\lambda = 1}^{3} \frac{\partial}{\partial \rho_{\lambda}} \left[ \frac{\partial \mathcal{L}_{\text{tot}}}{\partial h_{\mu\lambda}^{(0)}} \right] \sum_{\lambda = 1}^{3} \frac{\partial}{\partial \rho_{\lambda}} \left[ \gamma_{\mu\lambda} \right] \]

\[ \times \left[ \rho_{\lambda} (h_{\lambda\mu}^{(0)} \rho_{\mu\lambda}^{(0)} + h_{\lambda\mu}^{(0)} \rho_{\mu\lambda}^{(0)} - \rho_{\lambda} \rho_{\mu}) \right]. \quad \text{(176)} \]

Expressing \( h_{\mu\lambda}^{(0)} \) with the help of Eq. (47), using Eq. (163) for \( \Phi^{(0)} \) and Eq. (41) for the determinant \(|h|\), Eq. (176) is transformed to the following:

\[ \frac{3}{\lambda = 1} \frac{\partial}{\partial \rho_{\lambda}} \left[ \frac{\partial \mathcal{L}_{\text{tot}}}{\partial h_{\mu\lambda}^{(0)}} \right] \sum_{\lambda = 1}^{3} \frac{\partial}{\partial \rho_{\lambda}} \left[ \rho_{\lambda} \gamma_{\mu\lambda} \right] \]

\[ + \frac{n_{\mu} - n_{\mu} \rho_{\mu}^{(0)}}{\kappa} \frac{\rho_{\lambda} \rho_{\mu}}{N} \left( \frac{\rho_{\lambda} \rho_{\mu}}{D} \right). \quad \text{(177)} \]

Since both \( \gamma_{\mu\mu} = 0 \) and \( n_{\mu} - n_{\mu} \rho_{\mu}^{(0)} = 0 \) at \( \lambda = \mu \), then Eq. (177) can be rewritten in the form

\[ \frac{3}{\lambda = 1} \frac{\partial}{\partial \rho_{\lambda}} \left[ \frac{\partial \mathcal{L}_{\text{tot}}}{\partial h_{\mu\lambda}^{(0)}} \right] \sum_{\lambda = 1}^{3} \frac{\partial}{\partial \rho_{\lambda}} \left[ \frac{\rho_{\lambda} \gamma_{\mu\lambda} \rho_{\mu}}{\kappa} \frac{\rho_{\lambda} \rho_{\mu}}{D} \right] \]

\[ - \frac{n_{\mu} \rho_{\lambda}^{(0)}}{\kappa} \frac{\rho_{\lambda} \rho_{\mu}}{N} \left( \frac{\rho_{\lambda} \rho_{\mu}}{D} \right). \quad \text{(178)} \]

Here, Eq. (157) for \( \gamma_{\mu\mu} \) is taken into account. Finally, using Eq. (16) for \( D \) and Eq. (133) for \( N \) we obtain the formula for the right-hand side of Eq. (53) for \( p = c = \mu \)

\[ \sum_{\lambda = 1}^{3} \frac{\partial}{\partial \rho_{\lambda}} \left[ \frac{\partial \mathcal{L}_{\text{tot}}}{\partial h_{\mu\lambda}^{(0)}} \right] \sum_{\lambda = 1}^{3} \frac{\partial}{\partial \rho_{\lambda}} \left[ \gamma_{\mu\lambda} \right] \]

\[ = - \frac{n_{\mu}^{2}}{\kappa \rho_{\mu}^{(0)} D} \left[ 1 + \frac{r_{\mu}}{4 p} - 2n_{\mu}^{2} \left( 1 + \frac{3r_{\mu}}{8 \rho_{\mu}^{2}} - \frac{n_{\mu}^{2}}{8 \rho_{\mu}^{2}} \right) \right]. \quad \text{(179)} \]

The first sum over \( \nu \) in Eq. (182) is also easily calculated

\[ \sum_{\nu \neq \mu} \gamma^{\nu\nu} \gamma_{\nu\mu} = \sum_{\nu \neq \mu} n_{\nu} \left( \frac{D}{D} \right)^{2} = 2n_{\nu} \left( \frac{D}{D} \right)^{2}, \quad \text{(184)} \]

if Eqs. (150) and (159) for \( \gamma_{\nu\nu} \) and \( \gamma^{\nu\nu} \) are used, respectively. For transformation of the second sum in Eq. (182), one writes

\[ \sum_{\nu \neq \mu} \gamma^{\nu\mu} \gamma_{\nu\mu} = \left( \frac{D}{D} \right)^{2} = 1 - n_{\mu}^{2} \left( \frac{D}{D} \right)^{2}. \quad \text{(185)} \]

Here, the obvious relation for the unit three-vector \( n_{\nu} \)

\[ \sum_{\nu \neq \mu} n_{\nu}^{2} = 1 - n_{\mu}^{2}, \quad \text{(186)} \]

is used in addition to formulas (150) and (159).

Since in the last sum over \( m \) in the curl brackets in Eq. (180) the term with \( m = 0 \) is zero according to
Eq. (160) and \( \gamma_{\mu}^{\mu} = 0 \) due to the antisymmetry property, then using Eqs. (157) and (161) we get

\[
\sum_{l=1}^{3} \partial_{\mu} \nabla_{\lambda}^{\nu} \Phi_{\lambda} = \left( r_{\mu} - 1 \right) \frac{\nabla_{\nu}^{\lambda}}{N_{\lambda}} (N_{\nu}^{\lambda} + \frac{\nabla_{\nu}^{\lambda}}{N_{\lambda}}) \tag{187}
\]

Equation (186) is again used to transform Eq. (187).

Finally, making use of Eq. (41) for the determinant \( \mu_{l} \), Eq. (47) for \( h_{l}^{\mu} \), Eq. (49) for \( L_{\alpha}^{\mu} \), substituting Eqs. (181)–(185) and (187) into formula (180), using also Eqs. (161) and (163) to obtain \( \Phi_{\mu} \), we get

\[
\frac{\partial L_{\text{tot}}}{\partial h_{l}^{\mu}} = \frac{\partial L_{\text{tot}}}{\partial h_{l}^{\mu}} = \frac{-2r_{\mu}^{2}}{4\kappa P^{2}} \left[ 1 + \frac{r_{\mu}}{4} - 2r_{\mu}^{2} \left( 1 + \frac{3r_{\mu}}{8} - \frac{r_{\mu}^{2}}{8} \right) \right] \tag{188}
\]

A comparison of Eqs. (188) with (179) shows that Lagrange Eq. (53) for \( p = \mu = \mu \) is satisfied.

Though the tetrad components \( h_{l}^{\mu} \) for \( \mu \neq \nu \) are zero, the Lagrange equation for this case is nontrivial, and it will be shown that it is satisfied for the tetrad given by Eqs. (45). Substituting in Eq. (63) \( c = \mu \), \( p = \nu \), \( q = \lambda \) and remembering that \( h_{c}^{\mu} \) is nonzero only if \( i = c \) we write

\[
3 \sum_{\lambda=1}^{3} \partial_{\lambda} \left[ \frac{\partial L_{\text{tot}}}{\partial h_{l}^{\mu}} \right] = \sum_{\lambda=1}^{3} \partial_{\lambda} \left[ \frac{\partial L_{\text{tot}}}{\partial h_{l}^{\mu}} \right] = \sum_{\lambda=1}^{3} \partial_{\lambda} \left[ \frac{\partial L_{\text{tot}}}{\partial h_{l}^{\mu}} \right] \tag{189}
\]

Since \( \gamma_{\alpha}^{\mu} = 0 \) if all indexes are different from each other or \( \lambda = \nu \) due to the antisymmetry with respect to these indexes, hence \( \gamma_{\alpha}^{\mu} \neq 0 \) for \( \lambda = \mu \) only. The second term in the brackets in Eq. (189) is zero as \( h_{l}^{\mu} = 0 \) for \( \mu \neq \nu \), while the third term is nonzero only for \( \lambda = \mu \). Therefore Eq. (189) can be transformed to

\[
3 \sum_{\lambda=1}^{3} \partial_{\lambda} \left[ \frac{\partial L_{\text{tot}}}{\partial h_{l}^{\mu}} \right] = \partial_{\lambda} \left[ \frac{\partial L_{\text{tot}}}{\partial h_{l}^{\mu}} \right] (\gamma_{\alpha}^{\mu} - \Phi_{\alpha}) \tag{190}
\]

Using Eqs. (41), (47), (157), and (163) respectively for \( L_{\text{tot}}, h_{l}^{\mu}, \gamma_{\alpha}^{\mu}, \) and \( \Phi_{\alpha} \) we get the final result

\[
3 \sum_{\lambda=1}^{3} \partial_{\lambda} \left[ \frac{\partial L_{\text{tot}}}{\partial h_{l}^{\mu}} \right] = \partial_{\lambda} \left[ \frac{\partial L_{\text{tot}}}{\partial h_{l}^{\mu}} \right] \tag{191}
\]

Here, Eqs. (16) and (133) respectively for \( \nabla_{\mu} \) and \( N_{\mu} \) are taken into account.

For \( \partial L_{\text{tot}}/\partial h_{l}^{\mu} \) we have from Eq. (81)

\[
\frac{\partial L_{\text{tot}}}{\partial h_{l}^{\mu}} = \frac{1}{4\pi} \frac{F_{\alpha}^{(1)}}{g_{\alpha}^{(0)}} \frac{N_{\alpha}^{(0)}}{\kappa} \tag{192}
\]

It is taken into account in Eq. (192) that \( F_{\alpha}^{(1)} \) is nonzero only for \( l = 0 \) only, and \( h_{l}^{(0)} \neq 0 \) only if \( \mu = \nu \). The term \( \frac{F_{\alpha}^{(1)}}{g_{\alpha}^{(0)}} \) is transformed as

\[
\frac{F_{\alpha}^{(1)}}{g_{\alpha}^{(0)}} \frac{N_{\alpha}^{(0)}}{\kappa} = \left[ \frac{1}{4\pi} \frac{F_{\alpha}^{(1)}}{g_{\alpha}^{(0)}} \right] = \left[ \frac{2r_{\mu}^{2}}{4\pi} \right] \tag{193}
\]

with the help of Eq. (44) and Eq. (167) for \( F_{\alpha}^{(1)} \) and \( g_{\alpha}^{(0)} \).

Formula (21) is also used for \( g_{\alpha}^{(0)} \) and \( g_{\alpha}^{(0)} \).

In the sum over \( m \) and \( l \) in the curl brackets of Eq. (192), there is the nonzero term with \( m = l = 0 \) and the term with \( m = l = 0 \). In the sum over \( m \) and \( l \) in the curl brackets of Eq. (192), there is the nonzero term with \( m = l = 0 \) and the term with \( m = 0 \). This gives

\[
3 \sum_{m=0}^{3} (\gamma_{\alpha}^{\mu} + \gamma_{\alpha}^{\mu}) \Phi_{\alpha} = \gamma_{\alpha}^{\mu} \Phi_{\alpha} + \gamma_{\alpha}^{\mu} \Phi_{\alpha} + \gamma_{\alpha}^{\mu} \Phi_{\alpha} \tag{195}
\]

Since \( \gamma_{\alpha}^{\mu} = 0 \). Substituting formulas (193)–(195) into Eq. (192) and using Eqs. (41), (47) respectively for \( L_{\text{tot}}, h_{l}^{\mu}, \gamma_{\alpha}^{\mu}, \) and \( \Phi_{\alpha} \) we get the final result

\[
\frac{\partial L_{\text{tot}}}{\partial h_{l}^{\mu}} = \frac{2r_{\mu}^{2} \kappa}{2\pi} \left[ 1 + \frac{3r_{\mu}}{8} - \frac{r_{\mu}^{2}}{8} \right] \tag{196}
\]
Note that $r_e^2$ from Eq. (193) is combined with $r_0^2/4$ from other terms to give $r_0^2$ according to Eq. (10). A comparison of Eq. (196) with Eq. (191) confirms that Lagrange Eqs. (53) for $\nu, c = \mu,$ and $\nu \neq \mu$ are fulfilled.

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REFERENCES

1. J. D. Jackson, Classical Electrodynamics (Wiley, New York, 1998).
2. L. D. Landau and E. M. Lifshitz, Course of Theoretical Physics, Vol. 2: The Classical Theory of Fields (Nauka, Moscow, 1988; Pergamon, Oxford, 1975).
3. H. A. Lorentz, The Theory of Electrons and its Applications to the Phenomena of Light and Radiant Heat (B. G. Teubner, Leipzig, 1916).
4. R. P. Feynman, Quantum Electrodynamics (Westview, Boulder, 1998).
5. A. I. Akhiezer and V. B. Berestetskii, Quantum Electrodynamics, 3rd ed. (Nauka, Moscow, 1969; Wiley, New York, 1965).
6. N. N. Bogoliubov and D. V. Shirkov, Quantum Fields (Benjamin-Cummings, Reading, MA, 1982).
7. H. Reissner, “Über die Eigengravitation des elektrischen Felders nach der Einsteinschen Theorie,” Ann. Phys. 50, 106–120 (1916).
8. H. Weyl, “Zur Gravitationstheorie,” Ann. Phys. 54, 117–145 (1917).
9. G. Nordström, “On the energy of the gravitational field in Einstein’s theory,” Proc. R. Netherlands Acad. Art Sci. 20, 1238–1245 (1918).
10. G. B. Jeffery, “The field of an electron on Einstein’s theory of gravitation,” Proc. R. Soc. London, Ser. A 99, 123–134 (1921).
11. K. Schwarzschild, “Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie,” Sitz. Preuss. Akad. Wiss., 189–196 (1916); arXiv:physics/9905030v1 [physics.hist-ph].
12. S. Weinberg, Gravitation and Cosmology (Wiley, New York, London, Sydney, Toronto, 1972).
13. A. Einstein, “Die Feldgleichungen der Gravitation,” Sitz. Preuss. Akad. Wiss. Math.-Phys. Kl. 48, 844–847 (1915).
14. D. Hilbert, “Die Grundlagen der Physik,” Nachr. König. Gesell. Wiss. Gött. Math.-Phys. Kl., 395–407 (1915).
15. C. Møller, “Conservation laws and absolute parallelism in general relativity,” Mat. Fys. Skr. Dan. Vid. Selsk. 1 (10), 1–50 (1961).
16. C. Møller, “Momentum and energy in general relativity and gravitational radiation,” Mat. Fys. Medd. Dan. Vid. Selsk. 34 (3), 1–67 (1964).
17. A. Einstein, “Das hamiltonisches Prinzip und allgemeine Relativitätstheorie,” Sitz. Preuss. Akad. Wiss. 2, 1111–1116 (1916).
18. R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill, New York, 1965).
19. L. D. Faddeev and V. N. Popov, “Covariant quantization of the gravitational field,” Sov. Phys. Usp. 16, 777 (1973).
20. C. Møller, “Conservation laws in the tetrad theory of gravitation,” in Proceedings of the Conference on the Theory of Gravitation, Warszawa and Jablonna, 1962 (Gauthier-Villars, PWN, Paris, Warszawa, 1964), pp. 31–42.
21. L. D. Faddeev, “Problem of energy in Einstein’s theory of gravitation,” Sov. Phys. Usp. 25, 130 (1982).
22. R. C. Tolman, “On the use of the energy-momentum principle in general relativity,” Phys. Rev. 35, 875–895 (1930).
23. D. Toback, “Run II searches for supersymmetry,” AIP Conf. Proc. 753, 373–382 (2005); arXiv:0409067 [hep-ex].
24. V. M. Abazov, B. Abbott, M. Abolins, et al., “Search for supersymmetry with gauge-mediated breaking in diphoton events at D0,” Phys. Rev. Lett. 94, 041801–041801-12 (2005); arXiv:0408146 [hep-ex].
25. S. Chatrchyan, V. Khachatryan, A. M. Sirunyan, et al., “Inclusive search for supersymmetry using the razor variables in pp collisions at $\sqrt{s}=7$ TeV,” Phys. Rev. Lett. 111, 081802-1–081802-17 (2013); arXiv:1212.6961 [hep-ex].
26. T. H. R. Skyrme, “A non-linear field theory,” Proc. R. Soc. London, Sect. A 260, 127–138 (1961).
27. T. H. R. Skyrme, “A unified field theory of mesons and baryons,” Nucl. Phys. 31, 556–569 (1962).
28. P. A. M. Dirac, “Quantized singularities in the electromagnetic field,” Proc. R. Soc. London, Ser. A 133, 60–72 (1931).