An Upbound of Hausdorff’s Dimension of the Divergence Set of the fractional Schrödinger Operator on $H^s(\mathbb{R}^n)$

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Abstract This paper shows

\[
\sup_{f \in H^s(\mathbb{R}^n)} \dim_H \left\{ x \in \mathbb{R}^n : \lim_{t \to 0} e^{i(t-\Delta)^s}f(x) \neq f(x) \right\} \leq n+1-\frac{2(n+1)s}{n} \quad \text{under} \quad \begin{cases} n \geq 2; \\ \alpha > \frac{1}{2}; \\ \frac{n}{2(n+1)} < s \leq \frac{n}{2}. \end{cases}
\]

1 Introduction

1.1 Statement of Theorem 1.1

From now on, suppose that $S(\mathbb{R}^n)$ is the Schwartz space of all functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

\[
f \in C^\infty(\mathbb{R}^n) \quad \& \quad \lim_{|x| \to \infty} x^\beta \partial^\gamma f(x) = 0 \quad \forall \text{ multi-indices } \beta, \gamma.
\]

Also, let $H^s(\mathbb{R}^n)$ be the $\mathbb{R} \ni s$-Sobolev space of all tempered distributions $f \in S'(\mathbb{R}^n)$ whose Fourier transforms $\hat{f}$ obey

\[
\|f\|_{H^s(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \left( 1 + |\xi|^2 \right)^s |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty.
\]

If $(-\Delta)^s f$ stands for the $(0, \infty) \ni \alpha$-pseudo-differential operator defined by the Fourier transformation acting on $f \in S'(\mathbb{R}^n)$:

\[
((-\Delta)^s f)^{\wedge}(x) = |x|^{2\alpha} \hat{f}(x) \quad \forall \ x \in \mathbb{R}^n,
\]

then

\[
(1.1) \quad u(x, t) = e^{i(t-\Delta)^s}f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} e^{i|\xi|^{2\alpha}} \hat{f}(\xi)d\xi
\]

exists as a distributional solution to the $\alpha$-Schrödinger equation:

\[
(1.2) \quad \left\{ \begin{align*}
(i\partial_t + (-\Delta)^s)u(x, t) &= 0 \quad \forall \ (x, t) \in \mathbb{R}^n \times \mathbb{R}; \\
u(\cdot, 0) &= f(\cdot) \in H^s(\mathbb{R}^n).
\end{align*} \right.
\]

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While understanding the Carleson problem of deciding such a critical regularity number \( s_c \) that
\[
\lim_{t \to 0} e^{it(-\Delta)^s} f(x) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n \quad \text{holds for all } f \in H^s(\mathbb{R}^n) \quad \& \quad s > s_c,
\]
we are suggested to determine the Hausdorff dimension of the divergence set of the \( \alpha \)-Schrödinger operator \( e^{it(-\Delta)^s} f(x) \):
\[
d(s, n, \alpha) = \sup_{f \in H^s(\mathbb{R}^n)} \dim_H \left\{ x \in \mathbb{R}^n : \lim_{t \to 0} e^{it(-\Delta)^s} f(x) \neq f(x) \right\},
\]
thereby discovering the case \( \alpha > \frac{1}{2} \).

**Theorem 1.1.**
\[
d(s, n, \alpha) \leq n + 1 - \frac{2(n + 1)s}{n} \quad \text{under } n \geq 2 \quad \& \quad \alpha > \frac{1}{2} \quad \& \quad \frac{n}{2(n + 1)} < s \leq \frac{n}{2}.
\]

**1.2 Relevance of Theorem 1.1**
Here, it is appropriate to say more words on evaluating \( d(s, n, \alpha) \).

> In general, we have the following development.

- Theorem 1.1 actually recovers Cho-Ko’s [7] a.e.-convergence result:
\[
f \in H^s(\mathbb{R}^n) \quad \& \quad s > \frac{n}{2(n + 1)} \Rightarrow \lim_{t \to 0} e^{it(-\Delta)^s} f(x) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n.
\]

- A trivial part of Theorem 1.1 reveals:
\[
\|f\|_{L^{\infty}(\mathbb{R}^n)} \leq \|f\|_{H^s(\mathbb{R}^n)} \Rightarrow d(s, n < 2s, \alpha) = 0.
\]

Moreover, Theorem 1.1 improves (1.8) under
\[
\frac{n}{2(n + 1)} < s \leq \frac{n + 1}{4},
\]
as stated below:

* In [22] Sjögren-Sjölin showed
\[
d(s, n, \alpha) < n + 1 - 2s \quad \text{as } \frac{1}{2} < s \leq \frac{n}{2} \quad \& \quad \alpha > \frac{1}{2}.
\]

* In [1] and [29] it was proved by Barceló-Bennett-Carbery-Rogers and Žubrinič that
\[
d(s, n, \alpha) = n - 2s \quad \text{as } \frac{n}{4} \leq s \leq \frac{n}{2}.
\]
An Upper Bound of Hausdorff’s Dimension of the Divergence Set of \( \exp(\imath t(-\Delta)^{\alpha}) f(\cdot) \) for \( f \in H^s(\mathbb{R}^n) \)

* In [11] Barceló-Bennett-Carbery-Rogers gave

\[
\mathcal{d}(s, n, \alpha) \leq \begin{cases} 
  n + 1 - 2s & \text{as } \frac{1}{2} < s \leq \frac{n}{4}; \\
  \frac{3n}{4} + 1 - 4s & \text{as } \frac{n}{4} < s \leq \frac{n+1}{4}; \\
  n - 2s & \text{as } \frac{n+1}{4} < s \leq \frac{n}{2}.
\end{cases}
\]

In particular, we have the following case-by-case treatment.

1. **Case** \( \alpha = 1 \). Under this setting, Theorem 1.1 coincides with Du-Zhang’s [14, Theorem 2.4] since (1.1) turns out to be the classical Schrödinger operator \( e^{-\imath t\Delta} f(\cdot) \). (1.3) was first proposed in [6] by Carleson for this special case, and then intensively studied in e.g. [2, 3, 4, 15, 20, 21, 24, 26, 27, 28]. Upon combining the results in [6, 10, 4, 12, 14], we conclude \( s_c = n^2(n+1) \). Furthermore, in [22] Sjögren-Sjölin considered \( \mathcal{d}(s, n, 1) \). Note that the Sobolev embedding ensures \( \mathcal{d}(s, n < 2s, 1) = 0 \). So it is enough to calculate \( \mathcal{d}(s, n \geq 2s, 1) \).

2. Bourgain’s counterexample in [4] and Lucà-Rogers’ result in [19] showed

\[
\mathcal{d}(s, n, 1) = n \quad \text{as } \frac{n}{2} \leq s \leq \frac{n}{2}.
\]

3. The results in Žubrinić [29] and Barceló-Bennett-Carbery-Rogers [1] found

\[
\mathcal{d}(s, n, 1) = n - 2s \quad \text{as } \frac{n}{4} \leq s \leq \frac{n}{2}.
\]

Accordingly,

\[
\frac{n}{2(n+1)} = \frac{n}{4} = \frac{1}{4} \Rightarrow \mathcal{d}(s, 1, 1) = 1 - 2s.
\]

4. On the one hand, in [14] Du-Zhang proved

\[
\mathcal{d}(s, n, 1) \leq n + 1 - \frac{2(n+1)s}{n} \quad \text{as } \frac{n}{2(n+1)} < s < \frac{n}{4} \quad \text{& } n \geq 2.
\]

On the other hand, in [19, 18] Lucà-Rogers obtained

\[
\mathcal{d}(s, n, 1) \geq \begin{cases} 
  n + \frac{n}{n-1} - \frac{2(n+1)s}{n-1} & \text{as } \frac{n}{2(n+1)} \leq s < \frac{n+1}{8}; \\
  n + 1 - \frac{2(n+2)s}{n} & \text{as } \frac{n+1}{8} \leq s < \frac{n}{4}.
\end{cases}
\]

Thus there is still a gap to determine the exact value of \( \mathcal{d}(s, n, 1) \); see also [13, 14, 17, 18, 19] for more information.

5. **Case** \( \alpha \in (2^{-1}, \infty) \). Sjölin [23] proved \( s_c = 2^{-2} \) for \( n = 1 \). By the iterative argument developed in [3], Miao-Yang-Zheng [20] proved that (1.3) holds for

\[
s > \frac{3}{8} \quad \text{& } n = 2.
\]
Very recently, Cho-Ko [7] proved that (1.3) holds for
\[ s > \frac{n}{2(n + 1)} \quad \text{&} \quad n \geq 2. \]

It seems that the case \( \alpha > 2^{-1} \) shares the same critical index with the case \( \alpha = 1 \). So far there has been no counterexample to verify this problem.

- Case \( \alpha \in (0, 2^{-1}] \). It is uncertain that Theorem 1.1 can be extended to the fractional Schrödinger operator \( e^{it(-\Delta)^{\alpha}} f(x) \) & \( 0 < \alpha \leq 2^{-1} \). So, an investigation of this extension coupled with the foregoing counterexample will be the subject of future articles.

In the sequel of this paper, we always assume \( \alpha > \frac{1}{2} \).

In §2 we verify Theorem 1.1 via Proposition 2.1 & Theorem 2.2 - a global \( L^1 \) & a local \( L^2 \) estimates for the maximal operator living on a compactly-supported Borel measure and \( e^{it(-\Delta)^{\alpha}} f(x) \). However, the proof of Theorem 2.2 is given in §3 via Theorem 3.1 - an \( L^{\frac{2(n+1)}{n+1}} \)-estimate for \( e^{it(-\Delta)^{\alpha}} f(x) \) and its Corollary 3.2 - an \( L^2 \)-estimate for \( e^{it(-\Delta)^{\alpha}} f(x) \). Thanks to a highly nontrivial analysis, §4 is devoted to presenting a proof of Theorem 3.1 which essentially relies on Theorems 4.1 & 4.4 - the broad \( 1 \leq k \leq n+1 \) linear refined Strichartz estimates in dimension \( n+1 \) and Lemma 4.5 - the narrow \( L^{\frac{2(n+1)}{n+1}} \)-estimate for \( e^{it(-\Delta)^{\alpha}} f(x) \).

Notation. In what follows, \( A \leq B \) stands for \( A \leq CB \) for a constant \( C > 0 \) and \( A \sim B \) means \( A \leq B \leq A \). Further more, for given large number \( R \) and small enough \( 0 < \epsilon < 1 \), \( A \leq B \) stands for \( A \leq CR^\epsilon B \) for a constant \( C > 0 \) and \( A \approx B \) means \( A \ll B \ll A \).

2 Theorem 2.2⇒ Theorem 1.1

2.1 Proposition 2.1 & its proof

In order to determine the Hausdorff dimension of the divergence set of \( e^{it(-\Delta)^{\alpha}} f(x) \), we need a law for \( H^s(\mathbb{R}^n) \) to be embedded into \( L^1(\mu) \) with a lower dimensional Borel measure \( \mu \) on \( \mathbb{R}^n \).

Proposition 2.1. For a nonnegative Borel measure \( \mu \) on \( \mathbb{R}^n \) and \( 0 \leq \kappa \leq n \), let
\[
C_\kappa(\mu) = \sup_{(x,r) \in \mathbb{R}^n \times (0, \infty)} r^{-\kappa} \mu(B^n(x, r)) \quad \text{with} \quad B^n(x, r) = \{ y \in \mathbb{R}^n : |y - x| < r \}
\]

and \( M^\kappa(\mathbb{R}^n) \) be the class of all probability measures \( \mu \) with \( C_\kappa(\mu) < \infty \) and being supported in the unit ball \( \mathbb{B}^n = B^n(0, 1) \). Suppose
\[
\begin{align*}
0 < s &\leq \frac{n}{2}; \\
\kappa &> \kappa_0 \geq n - 2s; \\
(N, f, \mu) &\in [1, \infty) \times H^s(\mathbb{R}^n) \times M^\kappa(\mathbb{R}^n); \\
\psi(r) &\equiv \exp(-r^2); \\
e^{it(-\Delta)^{\alpha}} f(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \psi\left(\frac{|\xi|}{R}\right) e^{it(x\cdot\xi + \frac{x^2\xi^2}{2})} \mathring{f}(\xi) d\xi.
\end{align*}
\]
(i) If $t \in \mathbb{R}$, then

\[
\sup_{1 \leq N < \infty} \left\| \sup_{0 < t < 1} \left| e^{it(-\Delta)^\alpha} f \right| \right\|_{L^1(\mu)} \leq \sqrt{C_\epsilon(\mu)\|f\|_{H^s(\mathbb{R}^n)}}.
\]

(ii) If

\[
\sup_{0 < t < 1} \left| e^{it(-\Delta)^\alpha} f \right| \leq \sqrt{C_\epsilon(\mu)\|f\|_{H^s(\mathbb{R}^n)}},
\]

then $d(s, n, \alpha) \leq \kappa_0$.

**Proof.** (i) This (2.1) is the elementary stopping-time-maximal inequality [11] (4).

(ii) The argument is split into two steps.

**Step 1.** We show the following inequality:

\[
\sup_{1 \leq N < \infty} \left\| \sup_{0 < t < 1} \left| e^{it(-\Delta)^\alpha} f \right| \right\|_{L^1(\mu)} \leq \sqrt{C_\epsilon(\mu)\|f\|_{H^s(\mathbb{R}^n)}}.
\]

In a similar way to verify [11, Proposition 3.2], we achieve

\[
\sup_{N \geq 1} e^{it(-\Delta)^\alpha} f(x) \leq \left| e^{it(-\Delta)^\alpha} f(x) \right| + \int_1^\infty \left| \frac{d}{dN} e^{it(-\Delta)^\alpha} f(x) \right| dN.
\]

It is not hard to obtain (2.3) if we have the following two inequalities:

\[
\sup_{0 < t < 1} \left| e^{it(-\Delta)^\alpha} f \right| \leq \sqrt{C_\epsilon(\mu)\|f\|_{H^s(\mathbb{R}^n)}}
\]

and

\[
\int_1^\infty \left\| \sup_{0 < t < 1} e^{it(-\Delta)^\alpha} \left( \frac{1}{N^2} \right)^{\frac{(\cdot)}{N}} \hat{f}(\cdot) \right\|_{L^1(\mu)} dN \leq \sqrt{C_\epsilon(\mu)\|f\|_{H^s(\mathbb{R}^n)}}.
\]

(2.4) follows from the fact that (2.2) implies

\[
\left\| \sup_{0 < t < 1} \left| e^{it(-\Delta)^\alpha} f \right| \right\|_{L^1(\mu)} = \left\| \sup_{0 < t < 1} \int_{\mathbb{R}^n} e^{i(x \xi + t|\xi|^2)} \psi(\xi) \hat{f}(\xi) d\xi \right\|_{L^1(\mu)} \leq \sqrt{C_\epsilon(\mu)\|f\|_{H^s(\mathbb{R}^n)}}.
\]
To prove (2.5), we utilize
\[ \psi^\prime \left( \frac{|\xi|}{N} \right) \leq \sum_{k \geq 0} 2^{-2nk} \chi_{B^r(0, 2^k N)} (\xi) \]
to calculate
\[ (2.6) \quad \left\| \psi^\prime \left( \frac{\xi}{N^2} \right) \hat{f} (\cdot) \right\|_{H^s(\mathbb{R}^n)} \leq \left\| (1 + |\cdot|^2)^{\frac{1}{2}} \sum_{k \geq 0} 2^{-2nk} \chi_{B^r(0, 2^k N)} (\cdot) \hat{f} (\cdot) \right\|_{L^2(\mathbb{R}^n)} \]
\[ \leq \sum_{k \geq 0} 2 N^{1+\epsilon} \left\| (1 + |\cdot|^2)^{\frac{1}{2}} \chi_{B^r(0, 2^k N)} (\cdot) \hat{f} (\cdot) \right\|_{L^2(\mathbb{R}^n)} \]
\[ \leq \frac{1}{N^1+\epsilon} \| f \|_{H^s(\mathbb{R}^n)}. \]

By (2.2) and (2.6), we obtain
\[ \int_1^\infty \left\| \sup_{0<\tau<1} \left\| e^{i(t-\Delta)^\alpha} \left( \frac{\psi^\prime \left( \frac{\xi}{N^2} \right)}{N^2} \right) \right\|_{L^1(\mu)} \right\|_t \leq \int_1^\infty \left\| \frac{\psi^\prime \left( \frac{\xi}{N^2} \right) \hat{f} (\cdot)}{N^2} \right\|_{H^s(\mathbb{R}^n)} \leq \int_1^\infty \frac{C_s(\mu)}{N^1+\epsilon} \| f \|_{H^s(\mathbb{R}^n)} dN \]
\[ \leq \sqrt{C_s(\mu)} \| f \|_{H^s(\mathbb{R}^n)}, \]
thereby reaching (2.5).

Step 2. We are about to show:
\[ d(s, n, \alpha) \leq \kappa_0 \quad \forall \quad \kappa_0 \in [n - 2s, \kappa). \]

By the definition, we have
\[ (2.7) \quad \mu \left\{ x \in \mathbb{B}^n : \lim_{t \to 0} e^{i(t-\Delta)^\alpha} f(x) \neq f(x) \right\} = \mu \left\{ x \in \mathbb{B}^n : \lim_{t \to 0} \lim_{N \to \infty} e^{i(t-\Delta)^\alpha} f(x) \neq \lim_{N \to \infty} e^{i(t-\Delta)^\alpha} f(x) \right\}. \]

For any
\[ f \in H^s(\mathbb{R}^n) \quad \& \quad 0 < \epsilon \ll 1, \]
there exists
\[ g \in S(\mathbb{R}^n) \quad \text{such that} \quad \| f - g \|_{H^s(\mathbb{R}^n)} < \epsilon. \]
Accordingly, if
\[ \mu \in M^s(\mathbb{B}^n) \quad \& \quad \kappa > \kappa_0 \geq n - 2s, \]
then a combination of (2.3) and (2.1) gives
\[ (2.8) \quad \mu \left\{ x \in \mathbb{B}^n : \lim_{t \to 0} \lim_{N \to \infty} \left| e^{i(t-\Delta)^\alpha} f(x) - e^{i(t-\Delta)^\alpha} f(x) \right| > \lambda \right\} \]
Proof of Theorem 1.1

We begin with a statement of the following key result whose proof will be presented in §3 due to its high nontriviality.

**Theorem 2.2**. If

\[
\begin{aligned}
&n \geq 2; \\
&0 < \kappa \leq n; \\
&C_\kappa(\mu) < \infty; \\
&R \geq 1; \\
&d\mu_R(x) = R^\kappa d\mu\left(\frac{x}{R}\right); \\
&f \in H^s(B^n); \\
&\text{supp } \hat{f} \subset A(1) = \{\xi \in \mathbb{R}^n : |\xi| \sim 1\},
\end{aligned}
\]

then

\[
\left\| \sup_{0 < r < R} e^{it(-\Delta)^{\mu}} f \right\|_{L^2(B^n(0,R); d\mu_R)} \lesssim R^{\frac{s-1}{2} \frac{\kappa}{\kappa+1}} |f|_{L^2(B^n)}.
\]

Upon letting \( \varepsilon \to 0 \) firstly and \( \lambda \to 0 \) secondly, we have

\[
\mu\left\{ x \in \mathbb{B}^n : \lim_{t \to 0} e^{it(-\Delta)^{\mu}} f(x) \neq f(x) \right\} = 0.
\]

If \( \mathbb{H}^\kappa \) denotes the \( \kappa \)-dimensional Hausdorff measure which is of translation invariance and countable additivity, then Frostman’s lemma is used to derive

\[
\mathbb{H}^\kappa\left\{ x \in \mathbb{B}^n : \lim_{t \to 0} e^{it(-\Delta)^{\mu}} f(x) \neq f(x) \right\} = 0,
\]

and hence

\[
d(s,n,\alpha) = \sup_{f \in H^s(\mathbb{R}^n)} \dim_H \left\{ x \in \mathbb{R}^n : \lim_{t \to 0} e^{it(-\Delta)^{\mu}} f(x) \neq f(x) \right\} \leq \kappa_0.
\]
Consequently, we have the following assertion.

**Corollary 2.3.** If

\[
\begin{align*}
\begin{cases}
 n \geq 2; \\
 0 < \kappa \leq n; \\
 s > \frac{\kappa}{2(n + 1)} + \frac{n - \kappa}{2}; \\
 C_\kappa(\mu) < \infty; \\
 f \in H^s(\mathbb{R}^n),
\end{cases}
\end{align*}
\]

then

\begin{equation}
\left\| \sup_{0 < t \leq 1} \left| e^{i(t-\Delta)^s} f \right| \right\|_{L^2(\mathbb{R}^n)} \leq \sqrt{C_\kappa(\mu)} \| f \|_{H^s(\mathbb{R}^n)}.
\end{equation}

*Proof.* Upon using Theorem 2.2 and its notations as well as \[7\] (cf. \[8, 15, 16, 20\]), we get

\begin{equation}
\left\| \sup_{0 < t \leq T} \left| e^{i(t-\Delta)^s} f \right| \right\|_{L^2(B^0(0,R) \mu(\xi))} \leq R^{\frac{\alpha}{1 - \alpha}} \| f \|_{L^2(\mathbb{R}^n)}.
\end{equation}

Next, we use parabolic rescaling. More precisely, if

\[
\begin{align*}
\xi &= R^{-1} \eta; \\
x &= RX; \\
t &= R^{2a} T; \\
f_R(x) &= f(Rx); \\
\text{supp} \cap R \subset A(R) &= \{ \xi \in \mathbb{R}^n : |\xi| \sim R \},
\end{align*}
\]

then

\[
e^{i(t-\Delta)^s} f(x) = \int_{\mathbb{R}^n} e^{i(x-\xi + |\xi|^2 s)} \hat{f}(\xi) d\xi
\]

\[
= \int_{\mathbb{R}^n} e^{i(X-\eta + |\eta|^2 s)} \hat{f}(\eta) d\eta
\]

\[
= \int_{\mathbb{R}^n} e^{i(X-\eta + |\eta|^2 s)} \hat{f}_R(\eta) d\eta
\]

\[
e^{i(T-\Delta)^s} f_R(X),
\]

and hence

\[
\left\| \sup_{0 < t \leq R^{2a}} \left| e^{i(t-\Delta)^s} f \right| \right\|_{L^2(B^0(0,R) \mu(\xi))} = R^{\frac{\alpha}{2}} \| \sup_{0 < T \leq 1} \left| e^{iT-\Delta)^s} f_R \right| \right\|_{L^2(\mathbb{R}^n)};
\]

\[
\left\| f_R \right\|_{L^2(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f_R(x)|^2 dx \right)^{\frac{1}{2}} \leq R^{-\frac{\alpha}{2}} \| f \|_{L^2(\mathbb{R}^n)};
\]

\[
R^{\frac{\alpha}{2}} \| \sup_{0 < T \leq 1} \left| e^{iT-\Delta)^s} f_R \right| \right\|_{L^2(\mathbb{R}^n)} \leq R^{\frac{1}{2} - \alpha} R^{\frac{\alpha}{2}} \| f_R \|_{L^2(\mathbb{R}^n)}.
\]
Consequently, if \( T = t \) and \( X = x \), then

\[
\left\| \sup_{0 < r < 1} |e^{it(-\Delta)\rho} f_k| \right\|_{L^2(\mathbb{R}^n, dx)} \leq R^\frac{n+\alpha}{n+1} \|f_k\|_{L^2(\mathbb{R}^n)},
\]

and hence Littlewood-Paley’s decomposition yields

\[
\begin{cases}
    f = f_0 + \sum_{k \geq 1} f_k; \\
    \text{supp}\, \hat{f}_0 \subset A(1); \\
    \text{supp}\, \hat{f}_k \subset A(2^k) = \{ \xi \in \mathbb{R}^n : |\xi| \sim 2^k \}.
\end{cases}
\]

Finally, by Minkowski’s inequality and (2.12) as well as

\[
\frac{\kappa}{2(n+1)} + \frac{n - \kappa}{2},
\]

we arrive at

\[
\left\| \sup_{0 < r < 1} |e^{it(-\Delta)\rho} f| \right\|_{L^2(\mathbb{R}^n, dx)} \leq \left\| \sup_{0 < r < 1} |e^{it(-\Delta)\rho} f_0| \right\|_{L^2(\mathbb{R}^n, dx)} + \sum_{k \geq 1} \left\| \sup_{0 < r < 1} |e^{it(-\Delta)\rho} f_k| \right\|_{L^2(\mathbb{R}^n, dx)} \leq \left\| f_0 \right\|_{L^2(\mathbb{R}^n)} + \sum_{k \geq 1} 2^{k\left(\frac{n+\alpha}{n+1} + \frac{n}{2}\right)} \left\| f_k \right\|_{L^2(\mathbb{R}^n)} \leq \left\| f \right\|_{L^2(\mathbb{R}^n)} + \sum_{k \geq 1} 2^{k\left(\frac{n+\alpha}{n+1} + \frac{n}{2}\right)} \left\| f \right\|_{L^2(\mathbb{R}^n)} \leq \left\| f \right\|_{L^2(\mathbb{R}^n)}.
\]

Proof of (Corollary 2.3)⇒Theorem 1.1. An application of the Hölder inequality and (2.10) in Corollary 2.3 derives

\[
\left\| \sup_{0 < r < 1} |e^{it(-\Delta)\rho} f| \right\|_{L^1(\mathbb{R}^n, dx)} \leq \sqrt{C_n} \|f\|_{H^\nu(\mathbb{R}^n)},
\]

whence (2.12) follows up. So, Proposition 2.1 yields

\[
d(s, n, \alpha) \leq \kappa_0 \in [n - 2s, \kappa).
\]

Also, since

\[
s > \frac{\kappa}{2(n+1)} + \frac{n - \kappa}{2},
\]

we have

\[
n \geq \kappa > n + 1 - \frac{2(n + 1)s}{n}.
\]

Upon choosing

\[
\kappa_0 = n + 1 - \frac{2(n + 1)s}{n},
\]

we make a two-fold analysis below:
On the one hand, we ask for
\[
n + 1 - \frac{2(n + 1)s}{n} \geq n - 2s \iff s \leq \frac{n}{2}.
\]

On the other hand, it is nature to request
\[
n + 1 - \frac{2(n + 1)s}{n} < n \iff s > \frac{n}{2(n + 1)}.
\]

Accordingly,
\[
\frac{n}{2(n + 1)} < s \leq \frac{n}{2}
\]
is required in the hypothesis of Theorem 1.1.

\[\square\]

3 Theorem 3.1 \(\Rightarrow\) Theorem 2.2

3.1 Theorem 3.1 \(\Rightarrow\) Corollary 3.2

We say that a collection of quantities are dyadically constant if all the quantities are in the same interval of the form \((2^j, 2^{j+1}]\), where \(j\) is an integer. The key ingredient of the proof of Theorem 2.2 is the following Theorem 3.1 which will be proved in §4.

**Theorem 3.1.** Let
\[
\begin{aligned}
(n, R) &\in \mathbb{N} \times [1, \infty); \\
supp \hat{f} &\subset \mathbb{B}^n; \\
p &= \frac{2(n + 1)}{n - 1}.
\end{aligned}
\]

Then for any \(0 < \epsilon < \frac{1}{100}\), there exist constants
\[
C_\epsilon > 0 \quad \& \quad 0 < \delta = \delta(\epsilon) \ll \epsilon
\]
such that if:

(i) \(Y = \bigcup_{k=1}^M B_k\) is a union of lattice \(K^2\)-cubes in \(B^{n+1}(0, R)\) and each lattice \(R\mathbb{Z}^2\)-cube intersecting \(Y\) contains \(\sim \lambda\) many \(K^2\)-cubes in \(Y\), where \(K = R^6\);

(ii) \(\|e^{it(-\Delta)^\alpha} f\|_{L^p(B_k)}\) is dyadically a constant in \(k = 1, 2, \cdots, M\);

(iii) \(1 \leq k \leq n + 1\) and \(\gamma\) is given by
\[
\gamma = \max_{B^{n+1}(x', r) \subset B^{n+1}(0, R)} \max_{\ell' \in \mathbb{R}^{n+1}, j' \geq K^2} \frac{\# \{B_k : B_k \subset B^{n+1}(x', r)\}}{r^\epsilon},
\]

then
\[
\|e^{it(-\Delta)^\alpha} f\|_{L^p(Y)} \leq C_\epsilon M^{-\frac{1}{n+1}} \gamma^{\frac{2}{n+2}} \lambda^{\frac{n}{n+2}} R^{\frac{8}{n+2} + \epsilon} \|f\|_{L^2(\mathbb{R}^n)}.
\]
From Theorem 3.1, we can get the following $L^2$-restriction estimate.

**Corollary 3.2.** Let 
\[(n, R) \in \mathbb{N} \times [1, \infty) \quad \& \quad \text{supp} \hat{f} \subset \mathbb{B}^n.\]

Then for any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that if:

(i) $X = \bigcup_k B_k$ is a union of lattice unit cubes in $B_{n+1}(0, R)$;

(ii) $1 \leq k \leq n + 1$ and $\gamma$ is given by
\[
(3.3) \quad \gamma = \max_{B_{n+1}(x', r) \subset B_{n+1}(0, R)} \frac{\# \{ B_k : B_k \subset B_{n+1}(x', r) \}}{r^k},
\]

then
\[
(3.4) \quad \| e^{it(-\Delta)^\gamma} f \|_{L^2(X)} \leq C_\epsilon \lambda \frac{1}{\mu} R^\frac{n+1}{n\epsilon}\| f \|_{L^2(\mathbb{R}^n)}.
\]

**Proof.** For any $1 \leq \lambda \leq R^\frac{1}{n\epsilon}(1)$, we introduce the notation $Z_\lambda = \{ B_k : B_k \subset X \text{ such that any } R^\frac{1}{n\epsilon} \text{ cube contains } \sim \lambda \text{ unit cubes } B_k \text{ in it.} \}$ By pigeonholing, we fix $\lambda$ such that
\[
\| e^{it(-\Delta)^\gamma} f \|_{L^2(X)} \leq \| e^{it(-\Delta)^\gamma} f \|_{L^2(\bigcup_{B_k \in Z_\lambda} B_k)}.
\]

It is easy to see that
\[
\lambda \leq \gamma R^\frac{1}{n\epsilon}
\]
by taking $r = R^\frac{1}{n\epsilon}$ in (3.3).

Next we assume the following inequality holds and we will prove this inequality later.
\[
(3.5) \quad \| e^{it(-\Delta)^\gamma} f \|_{L^2(\bigcup_{B_k \in Z_\lambda} B_k)} \leq \gamma \lambda \frac{1}{\mu} R^\frac{n+1}{n\epsilon}\| f \|_{L^2(\mathbb{R}^n)},
\]

thereby reaching
\[
\| e^{it(-\Delta)^\gamma} f \|_{L^2(X)} \leq C_\epsilon \gamma \lambda \frac{1}{\mu} R^\frac{n+1}{n\epsilon}\| f \|_{L^2(\mathbb{R}^n)}.
\]

Hence it remains to prove (3.5). Denote $Z = \bigcup_{B_k \in Z_\lambda} B_k$.

We can sort them into at most $O(\log R)$ many subsets of $Z$ according to the value of $\| e^{it(-\Delta)^\gamma} f \|_{L^p(B_k)}$. In each subset the value of $\| e^{it(-\Delta)^\gamma} f \|_{L^p(B_k)}$ is dyadically a constant. Among the subsets we can find a set $Z' \subset Z$ such that
\[
\{ \| e^{it(-\Delta)^\gamma} f \|_{L^p(B_k)} : B_k \subset Z' \} \text{ is dyadically a constant}
\]
and
\[
\| e^{it(-\Delta)^\gamma} f \|_{L^2(Z')} \leq \| e^{it(-\Delta)^\gamma} f \|_{L^2(Z)}.
\]

Upon writing
\[
M = \# \{ B : B \text{ is unit cube and } B \subset Z' \},
\]
and using Hölder’s inequality, we have
\[
\| e^{it(-\Delta)^\gamma} f \|_{L^2(Z)} \leq \| e^{it(-\Delta)^\gamma} f \|_{L^2(Z')} \leq \| e^{it(-\Delta)^\gamma} f \|_{L^p(Z')} \leq M \frac{1}{\mu} \| e^{it(-\Delta)^\gamma} f \|_{L^p(Z')}.
\]
So, in order to prove (3.5), it suffices to prove

\[
\left\| e^{it(-\Delta)^s} f \right\|_{L^p(Z')} \lesssim M^{-\frac{1}{m+1}} \gamma^{\frac{2}{m+1}} \lambda^{\frac{m}{m+1}} R^{\frac{m+2}{m+1}} \|f\|_{L^2(\mathbb{R}^n)}.
\]

In order to use the result of Theorem 3.1, we need to extend the size of the unit cube to $K^2$-cube according to the following two steps.

Step 1. Let $\beta$ be a dyadic number and $\mathcal{B}_\beta := \{ B : B \subset Z' \}$ and for any the lattice $K^2$-cube $\tilde{B} \supset B$ such that $\|e^{it(-\Delta)^s} f\|_{L^p(\tilde{B})} \sim \beta$, and set $\tilde{\mathcal{B}}_\beta = \{ \tilde{B} : \text{the relevant } K^2 - \text{cubes} \}$.

Step 2. Next, fixing $\beta$, letting $\lambda'$ be a dyadic number, and denoting

\[
\begin{align*}
\mathcal{B}_{\beta, \lambda'} &= \{ B \in \mathcal{B}_\beta : \text{cube } Q \text{ contains } \lambda' \text{ many } K^2 - \text{cubes from } \tilde{\mathcal{B}}_\beta \}, \\
\tilde{\mathcal{B}}_{\beta, \lambda'} &= \{ \tilde{B} : \text{the relevant } K^2 - \text{cubes} \},
\end{align*}
\]

we find that the pair $\{ \beta, \lambda' \}$ satisfies

\[
M' = \# \tilde{\mathcal{B}}_{\beta, \lambda'} \gtrsim M.
\]

From the definition of $\lambda$ and $\gamma$, we have

\[
\lambda' \leq \lambda,
\]

\[
\gamma' = \max_{B^m(x', r) \subset B^m(0, R)} \frac{\# \{ B : B \in \mathcal{B}_{\beta, \lambda'}, B \subset B^m(x', r) \}}{r^m} \leq \gamma.
\]

If $Y = \bigcup_{\tilde{B} \in \tilde{\mathcal{B}}_{\beta, \lambda'}}$, then Theorem 3.1 yields

\[
\left\| e^{it(-\Delta)^s} f \right\|_{L^p(Z')} \lesssim \left\| e^{it(-\Delta)^s} f \right\|_{L^p(Y)} \lesssim M^{-\frac{1}{m+1}} \gamma'^{\frac{2}{m+1}} \lambda^{\frac{m}{m+1}} R^{\frac{m+2}{m+1}} \|f\|_{L^2(\mathbb{R}^n)} \lesssim M^{-\frac{1}{m+1}} \gamma^{\frac{2}{m+1}} \lambda^{\frac{m}{m+1}} R^{\frac{m+2}{m+1}} \|f\|_{L^2(\mathbb{R}^n)},
\]

which is the desired (3.6).
3.2 Proof of Theorem 2.2

In this section, we use Corollary 3.2 to prove Theorem 2.2.

Proof of (Corollary 3.2 ⇒ Theorem 2.2). This proceeds below.

- We have
  \[ \text{supp} \hat{f} \subset \mathbb{R}^n \Rightarrow \text{supp} (e^{i\lambda \Delta^n f} \wedge \subset \mathbb{R}^{n+1}. \]

  Thus,
  \[ \exists \psi \in \mathcal{S}(\mathbb{R}^{n+1}) \land \hat{\psi} \equiv 1 \text{ on } B^{n+1}(0, 2) \text{ such that } (e^{i\lambda \Delta^n f})^2 = (e^{i\lambda \Delta^n f})^2 \ast \psi. \]

- If
  \[ \sup_{|y-(x,t)| \leq \epsilon} |\psi(y)| = \psi_1(x,t) \]
  which decays rapidly, then for any \((x, t) \in \mathbb{R}^{n+1}, \)
  \[ \hat{m}(x, t) = (m, m_{n+1}) = (m_1, \ldots, m_n, m_{n+1}) \]
  denotes the center of the unit lattice cube containing \((x, t), \) and hence
  \[ (|e^{i\lambda \Delta^n f}|^2 \ast |\psi|)(x, t) \leq (|e^{i\lambda \Delta^n f}|^2 \ast \psi_1)(\hat{m}(x, t)). \]

Accordingly,

(3.7) \[ \left\| \sup_{0 < \lambda < R} \left| e^{i\lambda \Delta^n f} \right| \right\|_{L^2(B^n(0,R)\setminus{\mathbb{R}})}^2 \]

\[ = \int_{B^n(0,R)} \sup_{0 < \lambda < R} \left| e^{i\lambda \Delta^n f} \right|^2 d\mu_R(x) \]

\[ \leq \int_{B^n(0,R)} \sup_{0 < \lambda < R} \left| e^{i\lambda \Delta^n f} \right|^2 \ast |\psi| (x, t) d\mu_R(x) \]

\[ \leq \int_{B^n(0,R)} \sup_{0 < \lambda < R} \left| e^{i\lambda \Delta^n f} \right|^2 \ast \psi_1 (\hat{m}(x, t)) d\mu_R(x) \]

\[ \leq \sum_{m=(m_1, \ldots, m_n) \in \mathbb{Z}^n} \left( \int_{|x-m| \leq 10} d\mu_R(x) \right) \cdot \sup_{m \in \mathbb{Z}^n} \left( \left| e^{i\lambda \Delta^n f} \right|^2 \ast \psi_1 (m, m_{n+1}) \right). \]

- For each \( m \in \mathbb{Z}^n, \) let \( b(m) \) be an integer in \([0, R] \) such that
  \[ \sup_{m_{n+1} \in \mathbb{Z}, 0 \leq m_{n+1} \leq R} \left( \left| e^{i\lambda \Delta^n f} \right|^2 \ast \psi_1 \right)(m, m_{n+1}) = \left( \left| e^{i\lambda \Delta^n f} \right|^2 \ast \psi_1 \right)(m, b(m)). \]

Next, via defining

\[ \nu_m = \int_{|x-m| \leq 10} d\mu_R(x) \leq 1, \]
and using (3.7), we have

\begin{equation}
\left\| \sup_{0<r<R} e^{it(-\Delta)^\mu} f \right\|_{L^2(B^n(0,R);\mu_R)}^2 \lesssim \sum_{v \text{ dyadic}} \sum_{v_m \sim v} v \cdot \left( \left| e^{it(-\Delta)^\mu} f \right|^2 \psi_1 \right)(m, b(m)) + R^{-9n}.
\end{equation}

(3.8)

By pigeonholing, we get that for any small \( \epsilon > 0 \),

\begin{equation}
\left\| \sup_{0<r<R} e^{it(-\Delta)^\mu} f \right\|_{L^2(B^n(0,R);\mu_R)}^2 \lesssim \sum_{v \text{ dyadic}} \sum_{v_m \sim v} v \cdot \left( \int_{B^{n+1}(m, b(m), R^\epsilon)} \left| e^{it(-\Delta)^\mu} f \right|^2 \right) + R^{-8n}
\end{equation}

(3.9)

\begin{equation}
\lesssim v \cdot \int_{\bigcup_{\alpha \in \Lambda_1} B^{n+1}(m, b(m), R^\epsilon)} \left| e^{it(-\Delta)^\mu} f \right|^2 + R^{-8n}.
\end{equation}

\( \triangleright \) Note that

\[ X_v = \bigcup_{m \in \mathbb{Z}^n : |m| \leq R} \text{ and } v_m \sim v \] is not only a union of some distinct \( R^\epsilon \)-balls but also a union of some unit balls. So, these balls’ projections onto the \((x_1, \cdots, x_n)\)-plane are essentially disjoint (a point can be covered \( \leq R^\epsilon \) times). For every \( r > R^2 \epsilon \), the definition of \( \{|m| \leq R \text{ and } v_m \sim v\} \) ensures that the intersection of \( X_v \) and any \( r \)-ball can be contained in \( \leq R^{10n} \epsilon^{-1} r^\kappa \) disjoint \( R^\epsilon \)-balls. Hence we can apply Corollary 3.2 to \( X_v \) with

\[ \gamma \lesssim R^{100n} \epsilon^{-1} \quad \text{and} \quad 1 \leq \kappa \leq n+1. \]

By (3.9), we reach (2.9) via

\[ \left\| \sup_{0<r<R} e^{it(-\Delta)^\mu} f \right\|_{L^2(B^n(0,R);\mu_R)}^2 \lesssim v \left( \gamma^{\frac{n+1}{n+2}} R^{\frac{n+1}{n+2}} \|f\|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{2}{n+2}} \lesssim v^{\frac{n+1}{n+2}} R^{\frac{n+1}{n+2}} \|f\|_{L^2(\mathbb{R}^n)}^2 \lesssim R^{\frac{n}{n+1}} \|f\|_{L^2(\mathbb{R}^n)}^2. \]

\( \square \)

4 Conclusion

4.1 Proof of Theorem 3.1 - \( R \leq 1 \)

In what follows, we always assume

\[ p = \frac{2(n+1)}{n+1}; \]

\[ q = \frac{2(n+2)}{n}; \]

\[ \text{supp} \hat{f} \subset \mathbb{B}^n. \]
But nevertheless, the estimate (3.2) under $R \leq 1$ is trivial. In fact, from the assumptions of Theorem 3.1, we see

$$M \sim \lambda \sim \gamma \sim R \sim 1.$$ 

Furthermore, by the short-time Strichartz estimate (see [9,11]), we get

$$\|e^{it(-\Delta)^\nu}f\|_{L^p(Y)} \leq \|e^{it(-\Delta)^\nu}f\|_{L^p([0,1] \times \mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)},$$

thereby verifying Theorem 3.1 for $R \leq 1$.

4.2 Proof of Theorem 3.1 - $R \gg 1$

This goes below.

1stly, we decompose the unit ball in the frequency space into disjoint $K^{-1}$-cubes $\tau$. Write

$$S = \{\tau : K^{-1} - \text{cubes } \tau \subset \mathbb{B}^n\};$$

$$f = \sum \tau f_\tau;$$

$$\hat{f}_\tau = \hat{f}_{\chi_\tau};$$

$$S(B) = \{\tau \in S : \|e^{it(-\Delta)^\nu}f_\tau\|_{L^p(B)} \geq \frac{1}{100nK}\|e^{it(-\Delta)^\nu}f\|_{L^p(B)}\} \text{ for a } K^2 - \text{cube } B.$$ 

Then

$$\left\|\sum_{\tau \in S(B)} e^{it(-\Delta)^\nu}f_\tau\right\|_{L^p(B)} \sim \|e^{it(-\Delta)^\nu}f\|_{L^p(B)},$$

2ndly, we recall the definitions of narrow cube and broad cube.

- We say that a $K^2$-cube $B$ is narrow if there is an $n$-dimensional subspace $V$ such that for all $\tau \in S(B)$

$$\xi(G(\tau), V) \leq \frac{1}{100nK},$$

where $G(\tau) \subset \mathbb{S}^n$ is a spherical cap of radius $\sim K^{-1}$ given by

$$G(\tau) = \left\{\frac{(-2\xi, 1)}{|(-2\xi, 1)|} \in \mathbb{S}^n : \xi \in \tau\right\},$$

and $\xi(G(\tau), V)$ denotes the smallest angle between any non-zero vector $v \in V$ and $v' \in G(\tau)$.

- Otherwise we say that the $K^2$-cube $B$ is broad. In other words, a cube being broad means that the tiles $\tau \in S(B)$ are so separated such that the norm vectors of the corresponding spherical caps can not be in an $n$-dimensional subspace - more precisely - for any broad $B$,

$$(4.2) \quad \exists \tau_1, \cdots, \tau_{n+1} \in S(B) \text{ such that } |v_1 \wedge v_2 \wedge \cdots \wedge v_{n+1}| \gtrsim K^{-n} \forall v_j \in G(\tau_j).$$
3rdly, with the setting:
\[
\begin{align*}
Y_{\text{broad}} &= \bigcup_{B_k} \text{broad } B_k; \\
Y_{\text{narrow}} &= \bigcup_{B_k} \text{narrow } B_k,
\end{align*}
\]
we will handle \( Y \) according to the sizes of \( Y_{\text{broad}} \) and \( Y_{\text{narrow}} \).

(1) We call it the broad case if \( Y_{\text{broad}} \) contains \( \geq \frac{M}{2} \) many \( K^2 \)-cubes and we will deal with the broad case using the multilinear refined Strichartz estimates.

(2) We call it the narrow case if \( Y_{\text{narrow}} \) contains \( \geq \frac{M}{2} \) many \( K^2 \)-cubes and we will handle the narrow case by \( L^2 \)-decoupling, parabolic rescaling and induction on scales.

### 4.2.1 The broad case.

In this case, we consider the same generalized Schrödinger operators as Cho-Ko [7]. The idea here is to take it as a close perturbation of the typical curve \( |\xi|^2 \) in very small scale and keep this perturbation under parabolic scaling. This can not be true for \( |\xi|^{2\alpha} \) with \( \alpha > \frac{1}{2} \). But it is true for its quadratic term. This is the reason to introduce the following set \( NPF(L, c_0) \) and apply induction in this set. Let us recall the two definitions in [7].

- Let \( \Phi(D) \) be a multiplier operator defined on \( \mathbb{R}^n \) which satisfies:
\[
\begin{align*}
\Phi(\xi) &\text{ is smooth at } \xi \neq 0; \\
|D^\beta \Phi(\xi)| &\leq |\xi|^{2\alpha-|\beta|} \quad \forall \text{ multi-index } \beta; \\
The Hessian matrix of } \Phi \text{ is positive definite.}
\end{align*}
\]

- Let \( 0 < c_0 \ll 1 \) and \( L \in \mathbb{N} \) be sufficiently large. We consider a collection of the normalized phase functions:
\[
NPF(L, c_0) = \left\{ \Phi \in C^\infty_0(B^n(0,2)) : \right\| \Phi(\xi) - \frac{|\xi|^2}{2} \right\|_{C^l(\mathbb{R}^n)} \leq c_0 \}.
\]

**Theorem 4.1.** (*Linear refined Strichartz estimate in dimension \( n + 1 \)). Suppose that

(i) \( \Phi \) is in \( NPF(L, c_0) \) for sufficiently small \( c_0 > 0 \);

(ii) \( \{Q_j\} \) is a sequence of the lattice \( \mathbb{R}^+ \)-cubes in \( B^{n+1}(0,R) \) with \( \| e^{i\Phi} f \|_{L^q(Q_j)} \) being essentially constant in \( j \);

(iii) \( \{Q_j\} \) is arranged in horizontal slabs of the form \( \mathbb{R} \times \cdots \times \mathbb{R} \times \{t_0, t_0 + \mathbb{R}^+\} \) which contains \( \sim \sigma \) cubes \( Q_j \).

Then
\[
\| e^{i\Phi} f \|_{L^q(\bigcup_j Q_j)} \leq C_{\sigma} R^\epsilon \sigma^{-\frac{\epsilon}{2}} \| f \|_{L^2(\mathbb{R}^n)} \quad \forall \epsilon > 0.
\]
Remark 4.2. On the one hand, by taking $\Phi(\xi) = |\xi|^2$, we can rediscover the results for the Schrödinger operator by Du-Guth-Li \[12\] in $\mathbb{R}^{2+1}$ and \[13\] in higher dimensional cases. Similar results can also be found in \[7\] with an extra restriction condition on the support of $f$.

On the other hand, for $\Phi(\xi) = |\xi|^2$ with $\alpha > \frac{1}{2}$ we can reduce $\Phi$ satisfying \[4.3\] to a function in $NP\overline{F}(L, c_0)$. Denote by $H\Phi(\xi_0)$ the Hessian matrix of $\Phi(\xi)$ at point $\xi_0$. Since the Hessian matrix of $\Phi$ is positive definite, we can write it as $H\Phi(\xi_0) = P^{-1}DP$ with $P$ a symmetric matrix $D = (\lambda_1 e_1, \ldots, \lambda_n e_n)$ and $\lambda_1 > 0, \ldots, \lambda_n > 0$. We introduce a new function around point $\xi_0$:

\begin{equation}
\Phi_{\rho, \xi_0}(\xi) = \rho^{-2} \left( \Phi(\rho H^{-1} \xi + \xi_0) - \Phi(\xi_0) - \rho \nabla \Phi(\xi_0) \cdot H^{-1} \xi \right),
\end{equation}

From Cho-Ko \[7\], we have $\Phi_{\rho, \xi_0} \in NP\overline{F}(L, c_0)$ for a sufficiently small $\rho = \rho(\Phi, L, c_0) > 0$. Moreover

$$
|e^{i\rho f}(x)| = (2\pi)^{-n} \left| \int_{\mathbb{R}^n} e^{i(x, r)-(\xi, \Phi(\xi))} \hat{f}(\xi) d\xi \right|
$$

$$
= (2\pi)^{-n} \left| \int_{\mathbb{R}^n} e^{i(x, r)-(\rho H^{-1} \eta + \xi_0)} \hat{f}(\rho H^{-1} \eta + \xi_0) d\eta \right|
$$

$$
= \rho^n |H|^{-1} (2\pi)^{-n} \left| \int_{\mathbb{R}^n} e^{i(\rho H^{-1} \eta + \rho H^{-1} \nabla \Phi(\xi_0), \rho^2 t') - (\xi_0, \Phi_{\rho, \xi_0}(\eta))} \hat{f}(\rho H^{-1} \eta + \xi_0) d\eta \right|
$$

Next, we use

$$
\begin{aligned}
&\begin{cases}
  x' = \rho H^{-1}(x + t\nabla \Phi(\xi_0)); \\
  t' = \rho^2 t;
\end{cases} \\
&\int_{\mathbb{S}} f_{\rho, \xi_0}(\eta) = \rho^2 |H|^{-1} \int_{\mathbb{S}} \hat{f}(\rho H^{-1} \eta + \xi_0);
\end{aligned}
$$

$$
\|f\|_{L^2(\mathbb{R}^n)} = \|f_{\rho, \xi_0}\|_{L^2(\mathbb{S})},
$$

to get

$$
\|e^{i\rho f}\|_{L^q(S)}^q = \int_S |e^{i\rho f}(x)|^q dx dt
$$

$$
= \int_S \rho^n |H|^{-1} (2\pi)^{-n} \left| \int_{\mathbb{R}^n} e^{i(\rho H^{-1} \eta + \rho H^{-1} \nabla \Phi(\xi_0), \rho^2 t') - (\xi_0, \Phi_{\rho, \xi_0}(\eta))} \hat{f}(\rho H^{-1} \eta + \xi_0) d\eta \right|^q dx dt
$$

$$
= \rho^{nq} |H|^{-q} \int_{\mathbb{S}} (2\pi)^{-n} \left| \int_{\mathbb{R}^n} e^{i(\rho H^{-1} \eta + \rho H^{-1} \nabla \Phi(\xi_0), \rho^2 t') - (\xi_0, \Phi_{\rho, \xi_0}(\eta))} \hat{f}(\rho H^{-1} \eta + \xi_0) d\eta \right|^q \left| \rho^{-n} |H| d\eta \rho^{-2} dt' \right|
$$

$$
= \rho^{nq - n - \frac{nq}{2}} |H|^{-q + n + 1 + \frac{nq}{2}} \int_{\mathbb{S}} (2\pi)^{-n} \left| \int_{\mathbb{R}^n} e^{i(\rho H^{-1} \eta + \rho H^{-1} \nabla \Phi(\xi_0), \rho^2 t') - (\xi_0, \Phi_{\rho, \xi_0}(\eta))} \hat{f}(\rho H^{-1} \eta + \xi_0) d\eta \right|^q dx' dt'
$$

$$
= \rho^{nq - n - 2 - \frac{nq}{2}} |H|^{-q + \frac{n}{2} + 1} \int_{\mathbb{S}} (2\pi)^{-n} \left| \int_{\mathbb{R}^n} e^{i(\rho H^{-1} \eta + \rho H^{-1} \nabla \Phi(\xi_0), \rho^2 t') - (\xi_0, \Phi_{\rho, \xi_0}(\eta))} \hat{f}_{\rho, \xi_0}(\eta) d\eta \right|^q dx' dt'
$$

$$
= \rho^{nq - n - 2 - 1} |H|^{-q + \frac{n}{2} + 1} \left\| e^{i\rho f}_{\rho, \xi_0} \right\|_{L^q(S)}^q.
$$

In short, we have

\begin{equation}
\|e^{i\rho f}\|_{L^q(S)} = \rho^{\frac{n}{2} - \frac{n+2}{q}} |H|^{-\frac{q}{2} - \frac{1}{2}} \left\| e^{i\rho f}_{\rho, \xi_0} \right\|_{L^q(S)}.
\end{equation}

Note that

$$
\frac{n}{2} - \frac{n+2}{q} = 0 \quad \text{and} \quad |H| \sim 1 \quad \text{(since supp} \hat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \sim 1\})
$$
and the change of variables does not change the value of \(\sigma\). So (4.4) is also true for the generalized phase functions \(\Phi\) satisfying (4.3) which contains \(\Phi(\xi) = |\xi|^{2\alpha}\) with \(\alpha > \frac{1}{2}\).

**Lemma 4.3. (Bourgain-Demeter’s \(L^2\)-decoupling inequality [5])**. Suppose that \(\hat{g}\) is supported in a \(\sigma\)-neighborhood of an elliptic surface \(S\) in \(\mathbb{R}^n\). If \(\tau\) is a rectangle of size \(\sigma^{\frac{1}{2}} \times \cdots \times \sigma^{\frac{1}{2}} \times \sigma\) inside \(\sigma\)-neighborhood of \(S\), \(\hat{g}_\tau = \hat{g} \chi_\tau\) and \(\epsilon > 0\), then

\[
\|g\|_{L^p(\mathbb{R}^n)} \leq C_{\epsilon} \sigma^{-\epsilon} \left( \sum_{\tau} \|g_{\tau}\|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}}.
\]

**Proof of Theorem 4.1**

Now we prove linear refined Strichartz estimate in dimension \(n + 1\) by four steps.

1. Firstly, we consider the wave packet decomposition of \(f\). For any smooth function \(f : \mathbb{R}^n \rightarrow \mathbb{R}\), we decompose it into wave packets and each wave packet supported in a ball \(\theta\) of radius \(R^{\frac{1}{2}}\). Then we divide the physical space \(B^0(0, R)\) into balls \(D\) of radius \(R^{\frac{1}{2}}\). From [25], we have

\[
f = \sum_{\theta, D} f_{\theta, D} \quad \text{and} \quad f_{\theta, D} = (\hat{f}_{\theta})^\vee \chi_D.
\]

And we have the functions \(f_{\theta, D}\) are approximately orthogonal, thereby getting

\[
\|f\|_{L^2(\mathbb{R}^n)}^2 \sim \sum_{\theta, D} \|f_{\theta, D}\|_{L^2(\mathbb{R}^n)}^2.
\]

By computation, we have the restriction of \(e^{i\partial_x} f_{\theta, D}(x)\) to \(B^{n+1}(0, R)\) is essentially supported on a tube \(T_{\theta, D}\) which is defined as follows:

\[
T_{\theta, D} = \{(x, t) : (x, t) \in B^{n+1}(0, R) \quad \text{and} \quad |x - c(D) - t\nabla \Phi(c(\theta))| \leq R^{\frac{1}{2} + \delta} \quad \text{and} \quad 0 < t < R\}.
\]

Here \(c(\theta)\) and \(c(D)\) denote the centers of \(\theta\) and \(D\) respectively. Therefore, by decoupling theorem, we have

\[
\|e^{i\partial_x} f\|_{L^2(Q)} \leq \left( \sum_T \|e^{i\partial_x} f_T\|_{L^2(Q)}^2 \right)^{\frac{1}{2}},
\]

where \(T_{\theta, D} = T\). In fact, we take \(\eta_Q \in S(\mathbb{R}^{n+1})\) such that \(\text{supp} \eta_Q \subset Q^*\) and \(Q^*\) is \(R^{-\frac{1}{2}}\)-cube. And we have \(|\eta_Q| \sim 1\) on \(Q\). By Lemma 4.3, we obtain

\[
\|e^{i\partial_x} f\|_{L^2(Q)} \leq \|e^{i\partial_x} f \eta_Q\|_{L^2(\mathbb{R}^{n+1})} \leq \left( \sum_T \|e^{i\partial_x} f_T \eta_Q\|_{L^2(\mathbb{R}^{n+1})}^2 \right)^{\frac{1}{2}} \leq \left( \sum_T \|e^{i\partial_x} f_T\|_{L^2(Q)}^2 \right)^{\frac{1}{2}}.
\]

2. Secondly, we use parabolic rescaling and induction on radius \(R^{\frac{1}{2}}\). It goes as follows:

Suppose that:

- \(\{S_j\}\) are \(R^{\frac{1}{2}} \times \cdots \times R^{\frac{1}{2}} \times R^{\frac{1}{2}}\)-tubes in \(T\) which is parallel to the long axes of \(T\);
An upper bound of Hausdorff’s dimension of the divergence set of \( \exp(it(-\Delta)^\alpha)f(x) \forall f \in H^r(\mathbb{R}^n) \)

- \( \|e^{i\Phi} f_T\|_{L^q(S_j)} \) is essentially dyadically constant in \( j \);
- these tubes are arranged into \( R^\frac{2}{3} \)-slabs running parallel to the short axes of \( T \) which contains \( \sim \sigma_T \) tubes \( S_j \);
- \( Y_T = \bigcup_j S_j \).

Then

\[
\|e^{i\Phi} f_T\|_{L^q(Y_T)} \leq C_k R^\frac{2}{3} \sigma_T^{-\frac{2}{3} m} \|f_T\|_{L^2(\mathbb{R}^n)}.
\]

In fact, as in Remark 4.2, we get

\[
\|e^{i\Phi} f\|_{L^q(S)} = \rho^\frac{2}{3} \frac{m}{3} |H|^{\frac{1}{2} - \frac{2}{3}} \|e^{i\Phi} f_{\rho,\xi_0}\|_{L^q(S')} ;
\]

\[
\|e^{i\Phi} f\|_{L^q(Y_T)} = \|f_{\rho,\xi_0}\|_{L^2(\mathbb{R}^n)}.
\]

If

\[
\rho = R^\frac{2}{3} \quad \text{and} \quad \xi_0 = c(D) \quad \text{and} \quad S = Y_T \quad \text{and} \quad S' = \overline{Y},
\]

then \( \overline{Y} \), as the image of \( Y_T \) under the new coordinate, is a union of \( R^\frac{2}{3} \)-cubes inside an \( R^\frac{2}{3} \)-cube. These \( R^\frac{2}{3} \)-cubes are arranged in \( R^\frac{2}{3} \)-horizontal slabs, and

\[
\#(R^\frac{2}{3} \text{- cubes} : \text{cubes are arranged in } R^\frac{2}{3} \text{- horizontal slabs}) \sim \sigma_T,
\]

and hence

\[
\|e^{i\Phi} f\|_{L^q(Y_T)} = |H|^{-\frac{1}{3}} \|e^{i\Phi} f_{\rho,\xi_0}\|_{L^q(\overline{Y})}.
\]

From induction we have

\[
\|e^{i\Phi} f_{\rho,\xi_0}\|_{L^q(\overline{Y})} \leq C_k R^\frac{2}{3} \sigma_T^{-\frac{2}{3} m} \|f_{\rho,\xi_0}\|_{L^2(\mathbb{R}^n)},
\]

thereby getting that if \( f = f_T \) then

\[
\|e^{i\Phi} f_T\|_{L^q(Y_T)} \leq C_k |H|^{-\frac{1}{3}} R^\frac{2}{3} \sigma_T^{-\frac{2}{3} m} \|f_T\|_{L^2(\mathbb{R}^n)} \leq R^\frac{2}{3} \sigma_T^{-\frac{2}{3} m} \|f_T\|_{L^2(\mathbb{R}^n)}, \quad (\text{thanks to } |H| \sim 1)
\]

namely, (4.7) holds.

3rdly, we shall choose an appropriate \( Y_T \). For each \( T \), we classify tubes in \( T \) in the following ways.

- For each dyadic number \( \lambda \), we define \( \mathbb{S}_\lambda = \left\{ S_j : S_j \subset T \quad \& \quad \|e^{i\Phi} f_T\|_{L^q(S_j)} \sim \lambda \right\} \).
- For any dyadic number \( \eta \), we define \( \mathbb{S}_{\lambda,\eta} = \left\{ S_j : S_j \in \mathbb{S}_\lambda \quad \& \quad (S_j, S_j \subset R^\frac{2}{3} \quad \text{slab}) \sim \eta \right\} \).

We denote

\[
Y_{T,\lambda,\eta} = \bigcup_{S_j \in \mathbb{S}_{\lambda,\eta}} S_j.
\]
thereby getting

\[ e^{i\Phi f} = \sum_{\lambda, \eta} \left( \sum_T e^{i\Phi f} f_T \cdot \chi_{Y_T, \lambda, \eta} \right). \]

For each \( \lambda, \eta \), there are \( O(\log R) \) choices. By pigeonholing, we can choose \( \lambda, \eta \) so that

\[ \| e^{i\Phi f} \|_{L^q(Y_j)} \lesssim (\log R)^2 \left\| \sum_T e^{i\Phi f} f_T \cdot \chi_{Y_T, \lambda, \eta} \right\|_{L^q(Q_j)}, \]

holds for \( \approx 1 \) of all cubes \( Q_j \subset Y \), where \( Y = \bigcup_j Q_j \). In fact, we have \#(Q) \leq R^{\frac{2}{n+1}} \& \#(\lambda, \eta) \leq \log R \). Since \( \log R \ll R^{\frac{2}{n+1}} \), this inequality holds for \( \approx 1 \) of all cubes \( Q_j \subset Y \). Here \((\lambda, \eta)\) is independent of \( Q_j \).

– First of all, we fix \( \lambda, \eta \) in the sequel of the proof of refined Strichartz estimate in dimension \( n + 1 \). Let \( Y_T, \lambda, \eta = Y_T \) for convenience. Note that \( Y_T \) satisfies the hypotheses for our inductive estimate, where \( \sigma_T = \eta \). By the definition of \( Y_T \) \& \( \sigma_T \) and the direction of \( T \), we have \( Y_T \) contains \( \lesssim \sigma_T \) cubes \( Q_j \) in any \( R^{\frac{2}{n+1}} \)-horizontal slab. Therefore,

\[(4.9) \quad |Y_T \cap Y| \lesssim \frac{\sigma_T}{\sigma} |Y|.\]

– Next, we choose the tubes \( Y \) according to the dyadic size of \( \| f_T \|_{L^2(\mathbb{R}^n)} \). We can restrict matters to \( O(\log R) \) choices of this dyadic size, and so we can choose a set of \( T \)'s, \( T \) such that

\[ \| f_T \|_{L^2(\mathbb{R}^n)} \text{ is essentially constant} \]

and

\[(4.10) \quad \| e^{i\Phi f} \|_{L^q(Q_j)} \lesssim \left\| \sum_{T \in \mathcal{T}} e^{i\Phi f} f_T \cdot \chi_{Y_T} \right\|_{L^q(Q_j)} \text{ holds for } \approx 1 \text{ of all cubes } Q_j \subset Y.\]

– Last of all, we choose the cubes \( Q_j \subset Y \) according to the number of \( Y_T \) that contain them. Denote by

\[ Y' = \{ Q_j : Q_j \subset Y \text{ which obey } (4.10) \text{ and each } Q_j \text{ lie in } \sim \nu \text{ of the sets } \{ Y_T \}_{T \in \mathcal{T}} \}. \]

Because \( (4.10) \) holds for \( \approx 1 \) cubes \( \nu \) are dyadic numbers, we can use \( (4.9) \) to get

\[ |Y'| \approx |Y| \text{ \& } |Y_T \cap Y'| \leq |Y_T \cap Y| \lesssim \frac{\sigma_T}{\sigma} |Y| \approx \frac{\sigma_T}{\sigma} |Y'|, \]

thereby finding

\[ (4.11) \quad \nu \lesssim \frac{\sigma_T}{\sigma} |\mathbb{T}|. \]

– 4thly, we combine all our ingredients and finish our proof of Theorem 4.1.
- By (4.10) and the decoupling as well as Hölder’s inequality, we have that if \( Q_j \subset Y' \) then
\[
\left\| e^{i\varphi f} \right\|_{L^q(Q_j)} \lesssim \sqrt{2} \left( \sum_{T \in \mathbb{T}, Q_j \subset Y_T} \left\| e^{i\varphi f_T} \right\|_{L^q(Y_T)}^q \right)^{\frac{1}{q}}.
\]

- Via making a sum over \( Q_j \subset Y' \) and using our inductive hypothesis at scale \( R^{\frac{1}{4}} \), we obtain
\[
\left\| e^{i\varphi f} \right\|_{L^q(Y')} \lesssim \sqrt{2} \sum_{T \in \mathbb{T}} \left\| e^{i\varphi f_T} \right\|_{L^q(Y_T)}^q \lesssim \sqrt{2} \sum_{T \in \mathbb{T}} \left( \sigma_T^{-\frac{1}{2}} \left\| f_T \right\|_{L^2(\mathbb{R}^n)}^q \right) \]
\[
\approx \sqrt{2} \sum_{T \in \mathbb{T}} \sigma_T^{-\frac{2}{q}} \left\| f_T \right\|_{L^2(\mathbb{R}^n)}^q.
\]

- For each \( Q_j \subset Y \), since
\[
\left\| e^{i\varphi f} \right\|_{L^q(Q_j)} \text{ is essentially constant in } j \text{ and } |Y'| \approx |Y|,
\]
we get
\[
\left\| e^{i\varphi f} \right\|_{L^q(Y)} \approx \left\| e^{i\varphi f} \right\|_{L^q(Y')},
\]
thereby utilizing (4.11) and the fact that \( \left\| f_T \right\|_{L^2(\mathbb{R}^n)} \) is essentially constant among all \( T \in \mathbb{T} \) to derive
\[
\left\| e^{i\varphi f} \right\|_{L^q(Y)}^q \approx \left\| e^{i\varphi f} \right\|_{L^q(Y')}^q
\]
\[
\lesssim \sqrt{2} \sum_{T \in \mathbb{T}} \sigma_T^{-\frac{2}{q}} \left\| f_T \right\|_{L^2(\mathbb{R}^n)}^q
\]
\[
\approx \sqrt{2} \left( \sum_{T \in \mathbb{T}} \left\| f_T \right\|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{q+2}{q}}
\]
\[
\lesssim \sigma_T^{-\frac{2}{q}} \left\| f \right\|_{L^2(\mathbb{R}^n)}^q.
\]

Taking the \( q \)-th root in the last estimation produces
\[
\left\| e^{i\varphi f} \right\|_{L^q(Y)} \lesssim \sigma_T^{-\frac{1}{2}} \left\| f \right\|_{L^2(\mathbb{R}^n)} \quad & \quad Y = \bigcup_j Q_j.
\]

Moreover, Theorem 4.1 can be extended to the following form which can be verified via 13 and Theorem 4.1

**Theorem 4.4.** (Multilinear refined Strichartz estimate in dimension \( n+1 \)). For \( 2 \leq k \leq n+1 \) \& \( 1 \leq i \leq k \), let \( f_i : \mathbb{R}^n \to \mathbb{C} \) have frequencies \( k \)-transversely supported in \( \mathbb{B}^n \) - i.e. -

\[
1 \leq \bigwedge_{i=1}^k G(\xi_i) \quad & \quad G(\xi_i) = \frac{(-2\xi_i, 1)}{|(-2\xi_i, 1)|} \in \mathbb{S}^n \quad \forall \xi_i \in \text{supp} f_i.
\]
Suppose that $Q_1, Q_2, \cdots, Q_N$ are lattice $\mathbb{R}^2$-cubes in $B^{n+1}(0,R)$ so that each $\|e^{it(-\Delta)^\mu} f_i\|_{L^p(Q_i)}$ is essentially dyadically constant in $j$. If $Y = \bigcup_{j=1}^N Q_j$ and $\epsilon > 0$, then

$$
\| \prod_{j=1}^k e^{it(-\Delta)^\mu} f_j \|_{L^p(Y)} \leq C_r R^e N^{-\frac{k-1}{n+\epsilon}} \sum_{j=1}^k \| f_j \|_{L^2(\mathbb{R}^2)}^{\frac{1}{p}}.
$$

Proof of Theorem 3.1 - the broad case. In the broad case, there are $\geq M$ many broad $K^2$-cubes $B$. Denote the collection of $(n + 1)$-tuple of transverse caps by $\Gamma$:

$$
\Gamma = \{ \tilde{\tau} = (\tau_1, \cdots, \tau_{n+1}) : \tau_j \in S \text{ & (4.2) holds for any } v_j \in G(\tau_j) \}.
$$

Then for each $B \in Y_{\text{broad}}$,

$$
\| e^{it(-\Delta)^\mu} f \|_{L^p(B)}^p \leq K^{O(1)} \prod_{j=1}^{n+1} \left( \int_B \| e^{it(-\Delta)^\mu} f_{\tau_j} \|^p \right)^{\frac{1}{p}}
$$

for some $\tilde{\tau} = (\tau_1, \cdots, \tau_{n+1}) \in \Gamma$.

In order to exploit the transversality and make good use of the locally constant property, we break $B$ into small balls as follows.

- We cover $B = B^{n+1}(c(B), K^2)$ by cubes $B = B^{n+1}(c(B) + v, 2)$, where $v \in B^{n+1}(0, K^2) \cap \mathbb{Z}^{n+1}$. By the locally constant property, we can choose $v_j \in B^{n+1}(0, K^2) \cap \mathbb{Z}^{n+1}$ such that $\| e^{it(-\Delta)^\mu} f_{\tau_j} \|_{L^p(B)}$ is attained in $B^{n+1}(c(B) + v_j, 2)$, and writing

$$
v_j = (x_j, t_j) \text{ & } f_{\tau_j,v_j}(\xi) = f_{\tau_j}(\xi)e^{i(x_j,\xi)\xi^j\tau_j},
$$

we deduce that

$$
e^{it(-\Delta)^\mu} f_{\tau_j,v_j}(x) = e^{it(x_j,t_j)(-\Delta)^\mu} f_{\tau_j}(x + x_j)
$$

and $|e^{it(-\Delta)^\mu} f_{\tau_j,v_j}(x)|$ reaches $\| e^{it(-\Delta)^\mu} f_{\tau_j} \|_{L^p(B)}$ in $B^{n+1}(c(B), 2)$. Therefore

$$
\int_B \| e^{it(-\Delta)^\mu} f_{\tau_j,v_j} \|^p \leq K^{O(1)} \int_{B^{n+1}(c(B), 2)} \| e^{it(-\Delta)^\mu} f_{\tau_j,v_j} \|^p.
$$

- Now for each broad $B$, we find some

$$
\tilde{\tau} = (\tau_1, \cdots, \tau_{n+1}) \in \Gamma \text{ & } \tilde{v} = (v_1, \cdots, v_{n+1})
$$

such that

(4.12)

$$
\| e^{it(-\Delta)^\mu} f \|_{L^p(B)}^p \leq K^{O(1)} \prod_{j=1}^{n+1} \left( \int_{B^{n+1}(c(B), 2)} \| e^{it(-\Delta)^\mu} f_{\tau_j,v_j} \|^p \right)^{\frac{1}{p}} \leq K^{O(1)} \int_{B^{n+1}(c(B), 2)} \prod_{j=1}^{n+1} \| e^{it(-\Delta)^\mu} f_{\tau_j,v_j} \|^p.
$$

Since $\#(\tilde{\tau}) \leq K^{O(1)}$ & $\#(\tilde{v}) \leq K^{O(1)}$, we can choose some $\tilde{\tau}$ and $\tilde{v}$ such that (4.12) holds for $\geq K^{-C} M$ broad balls $B$. Next we fix $\tilde{\tau}$ and $\tilde{v}$, and let $f_{\tau_j,v_j} = f_j$. After that we further sort the collection $B$ of remaining broad balls as follows:
For a dyadic number $A$, let

$$B_A = \left\{ B : B \in \mathcal{B} \text{ and for each } B \text{ we have } \left\| \prod_{j=1}^{n+1} e^{ij(-\Delta)^\rho f}_j \right\|_{L^p(B^{n+1}(c(B),2))} \sim A \right\}. $$

Fix $A$, for dyadic numbers $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n+1}$, let $B_{A, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n+1}}$ consist of all $B \in B_A$ for which $R^+_\lambda$-cube $Q \ni B$ contains $\sim \tilde{\lambda}$ cubes from $B_A$ and obeys $\|e^{it(-\Delta)^{\rho} f}\|_{L^p(Q)} \sim I_j$ for $j = 1, 2, \ldots, n+1$.

Without loss of generality, we may assume $\|f\|_{L^2(\mathbb{R}^n)} = 1$ and we can also assume all the above dyadic numbers are between $R^{-C}$ and $R^C$, where $C$ is a large constant. Therefore, there exist some dyadic numbers $A, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n+1}$ such that $\#B_{A, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n+1}} \geq K^{-C}M$. Fix $A, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n+1}$ and set $B_{A, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n+1}} = \mathcal{B}$. Then, by (4.12) and the definition of $B_A$, we have

$$\|e^{it(-\Delta)^{\rho} f}\|_{L^p(Y)} \leq K^{O(1)} \left( \frac{M}{\lambda} \right)^{-\frac{n}{p+1} + \frac{1}{n+1}} \|f\|_{L^2(\mathbb{R}^n)},$$

where $Q = \{ Q : \text{ the relevant } R^+_\lambda - \text{cubes } Q \text{ defining } \mathcal{B} \}$. Note that

$$\begin{cases} (\#Q)\lambda \geq (\#Q)\sim \#B \geq K^{-C}M; \\ \tilde{N} = \#Q \geq K^{-C}M/\lambda. \end{cases}$$

So, by Theorem 4.4 we get

$$\|e^{it(-\Delta)^{\rho} f}\|_{L^p(Y)} \leq K^{O(1)} M^{-\frac{n}{p+1} + \frac{1}{n+1}} \|f\|_{L^2(\mathbb{R}^n)},$$

thereby getting via (4.13),

$$\|e^{it(-\Delta)^{\rho} f}\|_{L^p(Y)} \leq K^{O(1)} M^{-\frac{n}{p+1} + \frac{1}{n+1}} \|f\|_{L^2(\mathbb{R}^n)} \leq K^{O(1)} M^{-\frac{n}{p+1}} A^{\frac{n}{p+1} + 1} \|f\|_{L^2(\mathbb{R}^n)}. $$

Our goal is to prove

$$\|e^{it(-\Delta)^{\rho} f}\|_{L^p(Y)} \leq C \varepsilon M^{-\frac{1}{p+1}} \frac{n}{p+1} \lambda^{\frac{n}{p+1} + \frac{1}{n+1}} R^{\frac{n}{p+1} + \frac{1}{n+1}} + \varepsilon \|f\|_{L^2(\mathbb{R}^n)},$$

So it remains to verify

$$\|e^{it(-\Delta)^{\rho} f}\|_{L^p(Y)} \leq K^{O(1)} M^{-\frac{n}{p+1}} \frac{n}{p+1} \lambda^{\frac{n}{p+1} + \frac{1}{n+1}} R^{\frac{n}{p+1} + \frac{1}{n+1}} + \varepsilon \text{ i.e. } - M \leq K^{O(1)} \gamma^2 R^K.$$
4.2.2 The narrow case.

In order to prove the narrow case of Theorem 3.1, we have the following lemma which is essentially contained in Bourgain-Demeter [5].

Lemma 4.5. Suppose that:

(i) $B$ is a narrow $K^2$-cube in $\mathbb{R}^{n+1}$ and takes $c(B)$ as its center;

(ii) $S$ denotes the set of $K^{-1}$-cubes which tile $\mathbb{R}^n$;

(iii) $\omega_B$ is a weight function which is essentially a characteristic function on $B$ - more precisely -

$$
\text{supp} \hat{\omega}_B \subset B(0, K^{-2}) \& \chi_B(\tilde{x}) \leq \omega_B(\tilde{x}) \leq \left(1 + \frac{|\tilde{x} - c(B)|}{K^2}\right)^{-1000n}.
$$

Then

$$
\left\| e^{it(-\Delta)^{\alpha}} f \right\|_{L^p(B)} \leq C_\epsilon K^{\epsilon} \left( \sum_{\tau \in S} \left\| e^{it(-\Delta)^{\alpha}} f_{\tau} \right\|_{L^p(\omega_B)}^2 \right)^{1/2}, \quad \forall \, \epsilon > 0.
$$

Proof of Theorem 3.1 - the narrow case. The main method we used is the parabolic rescaling and induction on radius. Next we prove the narrow case step by step.

- 1stly, we consider the wave packet decomposition which is similar to Theorem 4.1 but with different scale. We break the physical ball $B^\alpha(0, R)$ into $\frac{R}{K}$-cubes $D$. From [25], we have

$$
f = \sum_{\tau \in T} f_{\tau, D} \& f_{\tau, D} = (\hat{f}_\tau)^\vee \chi_D.
$$

By computation, we have $e^{it(-\Delta)^{\alpha}} f_{\tau, D}$ (whenever restricted to $B^{\alpha+1}(0, R)$) is essentially supported on an $\frac{R}{K} \times \cdots \times \frac{R}{K} \times R$-box, denoted by

$$
T_{\tau, D} = \left\{ (x, t) : (x, t) \in B^{\alpha+1}(0, R) \& |x - c(D) - 2t\alpha|c(\tau)|^{2\alpha-2}c(\tau)| \leq \frac{R}{K} \& 0 < t < R \right\}.
$$

Here $c(\tau) \& c(D)$ denote the centers of $\tau \& D$ respectively. For a fixed $\tau$, the different tubes $T_{\tau, D}$ tile $B^{\alpha+1}(0, R)$. Next we write $f = \sum_T f_T$ for convenience. Therefore, by decoupling theorem, for each narrow $K^2$-cube $B$, we have

$$
\left(4.15\right) \quad \left\| e^{it(-\Delta)^{\alpha}} f \right\|_{L^p(B)} \leq K^{\epsilon_d} \left( \sum_T \left\| e^{it(-\Delta)^{\alpha}} f_T \right\|_{L^p(\omega_B)}^2 \right)^{1/2}.
$$

The reason to take $K^{\epsilon_d}$ in (4.15) is that there is a $\frac{1}{K^{\epsilon_d}}$ satisfying $\frac{K^{\epsilon_d}}{K^{\epsilon}} \ll 1$ at the end of the proof.
2ndly, we perform a dyadic pigeonholing to get our inductive hypothesis for each \( f_T \). Note that 

\[
\begin{align*}
K &= R^\delta = R^{100}; \\
R_1 &= \frac{R}{R^2} = R^{1-2\delta}; \\
K_1 &= R^\delta = R^{1-2\delta^2}.
\end{align*}
\]

So, not only tiling the box \( T \) by \( KK_1 \times \cdots \times KK_1 \times K^2K_1^2 \)-tubes \( S \), but also tiling the box \( T \) by \( R^+ \times \cdots \times R^+ \times KR^+ \)-tubes \( S' \) which are running parallel to the long axis of box \( T \), we utilize the parabolic rescaling to reveal that the box \( T \) becomes an \( R_1 \)-cube as well as the tubes \( S' \) and \( S \) become lattice \( R^+ \)-cubes and \( K_1^2 \)-cubes respectively. See 7thly for more details.

3rdly, we classify the tubes \( S \) and \( S' \) inside each \( T \) as follows.

- For dyadic numbers \( \eta, \beta_1 \), let \( S_{T, \eta, \beta_1} = \{ S : S \subset T \text{ each of which contains } \sim \eta \text{ narrow } K^2 - \text{cubes in } Y_{\text{narrow}} \text{ and } \| e^{it(-\Delta)^{\mu}} f_T \|_{L^p(S)} \sim \beta_1 \} \).

- Fix \( \eta, \beta_1 \), and for dyadic number \( \lambda_1 \), let \( S_{T, \eta, \beta_1, \lambda_1} = \{ S : S \in S_{T, \eta, \beta_1} \text{ and the tube } S' \supset S \text{ contains } \sim \lambda_1 \text{ tubes from } S_{T, \eta, \beta_1} \} \).

- For the fixed \( \eta, \beta_1, \lambda_1 \), we sort the boxes \( T \). For dyadic numbers \( \beta_2, M_1, \gamma_1 \), let \( B_{\eta, \beta_1, \lambda_1, \beta_2, M_1, \gamma_1} \) denote the collection of boxes \( T \) each of which satisfies 

\[
\| f_T \|_{L^2(\mathbb{R}^n)} \sim \beta_2 \quad \# S_{T, \eta, \beta_1, \lambda_1} \sim M_1
\]

and

\[
\max_{T_r \subset T : r \geq k_1} \frac{\# \{ S : S \in S_{T_{\eta, \beta_1, \lambda_1}} \text{ and } S \subset T_r \}}{r^n} \sim \gamma_1,
\]

where \( T_r \) are \( K r \times \cdots \times K r \times K^2 r \)-tubes in \( T \) which are parallel to the long axis of \( T \).

4thly, let

\[
Y_{T, \eta, \beta_1, \lambda_1} = \bigcup_{S \in S_{T, \eta, \beta_1, \lambda_1}} S.
\]

Then, for \( Y_{\text{narrow}} \) we can write

\[
e^{it(-\Delta)^{\mu}} f = \sum_{\eta, \beta_1, \lambda_1, \beta_2, M_1, \gamma_1} \int_{T \in B_{\eta, \beta_1, \lambda_1, \beta_2, M_1, \gamma_1}} e^{it(-\Delta)^{\mu}} f_T \cdot \chi_{Y_{T, \eta, \beta_1, \lambda_1}} + O(R^{-1000n}) \| f \|_{L^2(\mathbb{R}^n)}.
\]

The error term \( O(R^{-1000n}) \| f \|_{L^2(\mathbb{R}^n)} \) can be neglected.

- In particular, on each narrow \( B \) we have

\[
e^{it(-\Delta)^{\mu}} f = \sum_{\eta, \beta_1, \lambda_1, \beta_2, M_1, \gamma_1} \int_{T \in B_{\eta, \beta_1, \lambda_1, \beta_2, M_1, \gamma_1}} e^{it(-\Delta)^{\mu}} f_T.
\]
– Without loss of generality, we assume
\[
\begin{aligned}
\|f\|_{L^2(\mathbb{R}^n)} &= 1; \\
1 \leq \eta \leq K^{O(1)}, R^{-10n} \leq \beta_1 \leq K^{O(1)}, 1 \leq \lambda_1 \leq R^{O(1)}; \\
R^{-10n} \leq \beta_2 \leq 1, 1 \leq M_1 \leq R^{O(1)}, K^{-2n} \leq \gamma_1 \leq R^{O(1)}. 
\end{aligned}
\]

Therefore, there are only \(O(\log R)\) significant choices for each dyadic number.

– By (4.17), the pigeonholing and (4.15), we can choose \(\eta, \beta, \lambda, \beta, M_1, \gamma_1\) such that

\[
\begin{aligned}
\|e^{i(t-\Delta)^{\nu}} f\|_{L^p(B)} \leq (\log R)^6 K^{\epsilon_4} \left( \sum_{T \in \bigcup_{\beta_1, \lambda_1} B \subset Y_{T, \beta_1, \lambda_1}} \|e^{i(t-\Delta)^{\nu}} f_T\|_{L^p(\omega_B)}^2 \right)^{\frac{1}{2}}.
\end{aligned}
\]

holds for \(\geq (\log R)^{-6}\) narrow \(K^2\)-cubes \(B\).

5thly, we fix \(\eta, \beta_1, \lambda_1, \beta_2, M_1, \gamma_1\) for the rest of the proof. Let

\[
Y_{T, \beta_1, \lambda_1} = Y_T \cap \mathbb{B}_{\eta, \beta_1, \lambda_1} = \mathbb{B}.
\]

Let \(Y' \subset Y_{\text{narrow}}\) be a union of narrow \(K^2\)-cubes \(B\) each of which obeys (4.18) and

\[
\begin{aligned}
\# \{ T : T \in \mathbb{B} \text{ and } B \subset Y_T \} &\sim \nu \text{ for some dyadic number } 1 \leq \nu \leq K^{O(1)}; \\
\# \{ B : B \subset Y' \text{ and } B \text{ is } K^2 \text{ cubes} \} &\geq (\log R)^{-7} M.
\end{aligned}
\]

By our assumption that \(\|e^{i(t-\Delta)^{\nu}} f\|_{L^p(B)}\) is essentially constant in \(k = 1, 2, \ldots, M\), in the narrow case we have

\[
\|e^{i(t-\Delta)^{\nu}} f\|_{L^p(Y')} \leq (\log R)^7 \sum_{B \subset Y'} \|e^{i(t-\Delta)^{\nu}} f\|_{L^p(B)}.
\]

For each \(B \subset Y'\), it follows from (4.18), Hölder’s inequality and (4.19) that

\[
\begin{aligned}
\|e^{i(t-\Delta)^{\nu}} f\|_{L^p(B)}^p &\leq (\log R)^6 K^{\epsilon_4} \left( \sum_{T \in \mathbb{B} : B \subset Y_T} \|e^{i(t-\Delta)^{\nu}} f_T\|_{L^p(\omega_B)}^2 \right)^{\frac{p}{2}} \\
&\leq (\log R)^6 K^{\epsilon_4} \nu^{\frac{p}{2} - 1} \sum_{T \in \mathbb{B} : B \subset Y_T} \|e^{i(t-\Delta)^{\nu}} f_T\|_{L^p(\omega_B)}^p.
\end{aligned}
\]

Via (4.20) and (4.21), we have

\[
\begin{aligned}
\|e^{i(t-\Delta)^{\nu}} f\|_{L^p(Y')} &\leq (\log R)^7 \left( \sum_{B \subset Y'} \|e^{i(t-\Delta)^{\nu}} f\|_{L^p(B)}^p \right)^{\frac{1}{2}}
\end{aligned}
\]
Finally, regarding each $\|e^{i(-\Delta)^\alpha}f_T\|_{L^p(Y_T)}$, we apply the parabolic rescaling and induction on radius. For each $K^{-1}$-cube $\tau = \tau_T$ in $\mathbb{B}^n$, we write $\xi = \xi_0 + K^{-1}\eta \in \tau$, where $\xi_0 = c(\tau)$. Similarly to the argument of (4.6), we also consider a collection of the normalized phase functions

$$\mathcal{NP}F(L, c_0) = \left\{ \Phi \in C_0^\infty(B'(0, 2)) : \|\Phi(\xi) - \frac{|\xi|^2}{2}\|_{C^1_0(\mathbb{R}^n)} \leq c_0 \right\}.$$ 

Via the similar parabolic rescaling,

$$\begin{cases}
\tilde{x} = K^{-1}H^{-1}(x + i\nabla \Phi(\xi_0)); \\
\tilde{t} = K^{-2}t,
\end{cases}$$

we reach

$$(4.23) \quad \|e^{i\Phi}f_T(x)\|_{L^p(Y_T)} = K^{-\frac{1}{n+1}t} \|H|^{-\frac{1}{n+1}t}|e^{i\Phi_{K^{-1}\xi_0}g(\tilde{x})}\|_{L^p(\tilde{Y})} \sim K^{-\frac{1}{n+1}t} \|e^{i\Phi_{K^{-1}\xi_0}g(\tilde{x})}\|_{L^p(\tilde{Y})},$$

where

$$\begin{cases}
|H| \sim 1 \text{ (since } |\xi| \sim 1); \\
\text{supp} \tilde{g} \subset \mathbb{B}^n; \\
\|g\|_{L^2(\mathbb{R}^n)} = \|f_T\|_{L^2(\mathbb{R}^n)},
\end{cases}$$

as well as $\tilde{Y}$ is the image of $Y_T$ under the new coordinates and $\Phi_{K^{-1}\xi_0}$ is similar to (4.5).

Finally, we apply inductive hypothesis (3.2) (replacing $(-\Delta)^\alpha$ with $\Phi$) at scale $R_1 = \frac{R}{K\tau}$ to $\|e^{i(-\Delta)^\alpha}g(\tilde{x})\|_{L^p(\tilde{Y})}$ with $M_1, \gamma_1, A_1, R_1$. Under parabolic rescaling, the relation between preimage and image is as follows:

$$\begin{cases}
T \left( \frac{R}{K} \times \cdots \times \frac{R}{K} \times R - \text{tube} \right) \longrightarrow \tilde{T} \text{ (tube)}; \\
S' \left( R^\frac{1}{2} \times \cdots \times R^\frac{1}{2} \times KR^\frac{1}{2} - \text{tube} \right) \longrightarrow \tilde{S}' \left( R_1^\frac{1}{2} - \text{tube} \right); \\
S \left( KK_1^2 \times \cdots \times K K_1^2 \times K^2 K_1 - \text{tube} \right) \longrightarrow \tilde{S} \left( K^2 - \text{tube} \right).
\end{cases}$$

More precisely, we have

$$\# \{ \tilde{S} : \tilde{S} \subset \tilde{T} \& \tilde{S} \subset \tilde{Y} \} \sim M_1$$
and the $K_1^2$-cubes $\tilde{S}$ are organized into $R_1^{-1}$-cubes $\tilde{S}'$ such that

$$\#(\tilde{S} : \tilde{S} \subset \tilde{S}') \sim \lambda_1.$$  

Moreover, $\|e^{i(\Delta)^{\alpha}}g(\tilde{x})\|_{L^p(S)}$ is dyadically a constant in $S \subset Y_T$. By our choice of $\gamma_1$, we have

$$\max_{B^{\alpha+1}(x', r) \subset \tilde{S}} \frac{\#(\tilde{S} : \tilde{S} \subset B^{\alpha+1}(x', r))}{r^\gamma} \sim \gamma_1.$$  

Hence, by the inductive hypothesis (3.2) (replacing $(-\Delta)^\alpha$ with $\Phi$) at scale $R_1$, we have

$$\|e^{i\Phi_{K^{-1}, \delta_0}g(\tilde{x})}\|_{L^p(Y')} \lesssim M_1^{-\frac{1}{\gamma_1} \left( \frac{2}{n+1+\gamma_2} R / K \right)^{1/n+1+\gamma_2}} \|g\|_{L^2(\mathbb{R}^n)}.$$  

By (4.23) and $\|g\|_{L^2(\mathbb{R}^n)} = \|f_T\|_{L^2(\mathbb{R}^n)}$, we get

$$\|e^{i\Phi_{K^{-1}, \delta_0}g(\tilde{x})}\|_{L^p(Y')} \lesssim K^{-\frac{1}{\gamma_1}} M_1^{-\frac{1}{\gamma_1} \left( \frac{2}{n+1+\gamma_2} R / K \right)^{1/n+1+\gamma_2}} \|g\|_{L^2(\mathbb{R}^n)}.$$  

Since (4.24) also holds whenever replacing $\Phi$ with $(-\Delta)^\alpha$, we get

$$\|e^{i(-\Delta)^{\alpha}}f_T(x)\|_{L^p(Y')} \lesssim K^{-\frac{1}{\gamma_1}} M_1^{-\frac{1}{\gamma_1} \left( \frac{2}{n+1+\gamma_2} R / K \right)^{1/n+1+\gamma_2}} \|f_T\|_{L^2(\mathbb{R}^n)}.$$  

By (4.22) and (4.25), we obtain

$$\|e^{i(-\Delta)^{\alpha}}f\|_{L^p(Y)} \lesssim (\log R)^{1/3} K^{d_{\mathbb{B}}/3} \left( \sum_{T \in \mathbb{B}} \left( \frac{1}{\lambda_1} \langle \frac{R}{K} \rangle^{1/n+1+\gamma_2} \sum_{f_T} \|f_T\|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2} \right)$$

$$\lesssim K^{d_{\mathbb{B}}/3} \left( \frac{V}{\lambda_2} \right)^{1/n+1+\gamma_2} \sum_{T \in \mathbb{B}} \left( \frac{1}{\lambda_1} \langle \frac{R}{K} \rangle^{1/n+1+\gamma_2} \sum_{f_T} \|f_T\|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2}$$

$$\lesssim K^{d_{\mathbb{B}}/3} \left( \frac{V}{\lambda_2} \right)^{1/n+1+\gamma_2} \langle \frac{R}{K} \rangle^{1/n+1+\gamma_2} \|f\|_{L^2(\mathbb{R}^n)},$$

where the third inequality follows from the assumption that $\|f_T\|_{L^2(\mathbb{R}^n)}$ is essentially constant in $T \in \mathbb{B}$ and then implies

$$\left( \sum_{T \in \mathbb{B}} \|f_T\|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2} \leq \left( \frac{V}{\lambda_2} \right)^{1/n+1+\gamma_2} \left( \sum_{T} \|f_T\|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2} \lesssim \left( \frac{V}{\lambda_2} \right)^{1/n+1+\gamma_2} \|f\|_{L^2(\mathbb{R}^n)}.$$  

\(\triangleright\) 8thly, we consider the lower bound and the upper bound of $\#(T, B) : T \in \mathbb{B} \& B \subset Y_T \cap Y'.$

- On the one hand, by the definition of $\nu$ as in (4.19), there is a lower bound

$$\#(T, B) : T \in \mathbb{B} \& B \subset Y_T \cap Y' \gtrsim (\log R)^{-7} M_{\nu}.$$
– On the other hand, by our choices of \( M_1 \) and \( \eta \), for each \( T \in \mathbb{B} \),

\[
\begin{aligned}
\# \{ S : S \subset Y_T \} &\sim M_1; \\
\# \{ B : B \subset S \ \& \ B \subset Y_{\text{narrow}} \} &\sim \eta.
\end{aligned}
\]

so

\[
\# \{ (T, B) : T \in \mathbb{B} \ \& \ B \subset Y_T \cap Y' \} \lesssim (\# \mathbb{B}) M_1 \eta.
\]

Therefore, we get

\[
\nu \frac{\# \mathbb{B}}{M} \lesssim \frac{(\log R)^7 M_1 \eta}{M}.
\]

(4.27)

\[\triangleright\] 9thly, we want to obtain the relation between \( \gamma \) and \( \gamma_1 \). By our choices of \( \gamma_1 \) as in (4.16) and \( \eta \),

\[
\gamma_1 \cdot \eta \sim \max_{T_j \subset T : r \geq K^2} \frac{\# \{ S : S \subset Y_T \cap T_j \}}{r^k} \cdot \# \{ B : B \subset Y_{\text{narrow}} \text{ for any fixed } S \subset Y_T \}
\]

\[
\leq \max_{T_j \subset T : r \geq K^2} \frac{\# \{ B : B \subset Y \ \& \ B \subset T_j \}}{r^k}
\]

\[
\leq \frac{K \gamma (Kr)^k}{r^k}
\]

\[
= \gamma K^{k+1}.
\]

Hence,

\[
\eta \lesssim \frac{\gamma K^{k+1}}{\gamma_1}.
\]

(4.28)

\[\triangleright\] 10thly, we complete the proof of Theorem 3.1

– On the one hand,

\[
\begin{aligned}
\# \{ S : S \subset S' \ \& \ S \subset Y_T \} &\sim \lambda_1; \\
\# \{ B : B \subset S \ \& \ B \subset Y_{\text{narrow}} \} &\sim \eta.
\end{aligned}
\]

– On the other hand, we can cover \( S' \) by \( \sim K \) finitely overlapping \( R^2 \)-balls and each \( R^2 \)-ball contains \( \lesssim \lambda \) many \( K^2 \)-cubes in \( Y \).

Thus it follows that

\[
\lambda_1 \lesssim \frac{K \lambda}{\eta}.
\]

(4.29)

Inserting (4.27), (4.29) and (4.28) into (4.26) gives

\[
\| e^{it(-\Delta)^s} f \|_{L^p(Y)} \lesssim K^{2^d} \left( \frac{(\log R)^7 M_1 \eta}{M} \right)^{\frac{1}{n+1}} K^{-\frac{n}{n+1}} M_1^{-\frac{n}{n+1}} \gamma_1^{\frac{n}{n+1}(n+1)+1} \left( \frac{K \lambda}{\eta} \right)^{\frac{n}{n+1}(n+1)+1} \left( \frac{R}{K^2} \right)^{\frac{n}{n+1}(n+1)+1} \epsilon \| f \|_{L^2(\mathbb{R}^n)}
\]
\[
\begin{aligned}
&\leq \frac{K^{3\epsilon} e^4}{K^{2\epsilon}} \left( \frac{\|f\|_{L^2(\mathbb{R}^n)}}{K^{n+1}} \right)^{\frac{2}{n(n+1)}} M^{\frac{1}{2\pi}} A_{\frac{n(n+1)}{2}} R^{\frac{\epsilon}{n(n+1)}} \|f\|_{L^2(\mathbb{R}^n)} \\
&\leq \frac{K^{3\epsilon} e^4}{K^{2\epsilon}} M^{\frac{1}{2\pi}} A_{\frac{n(n+1)}{2}} R^{\frac{\epsilon}{n(n+1)}} \|f\|_{L^2(\mathbb{R}^n)},
\end{aligned}
\]

where the last inequality follows from (4.28). It is not hard to see that \( \frac{K^{3\epsilon} e^4}{K^{2\epsilon}} \ll 1 \) and the induction concludes the argument for the narrow case.

□

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