Boundedness Condition for Anisotropic Norm of Linear Discrete Time-invariant Systems with Multiplicative Noise

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Abstract. In this paper, the anisotropy-based analysis problem for the linear discrete time-invariant system with state-multiplicative noises is considered. The mean anisotropy of stationary input sequence is assumed to be bounded by given nonnegative value. For this setting, the boundedness condition for anisotropic norm of the system is obtained in terms of Riccati equations.

1. Introduction

Basic concepts of anisotropy-based theory were established in the 1990s [1,2]. I.G. Vladimirov and co-authors applied some ideas of information theory to $H_\infty$ control theory and its methods. Stochastic approach to $H_\infty$ theory gives several advantages: class of external disturbances gets wider; designed controllers and estimators demonstrate less conservatism in contrast to standard $H_\infty$ controllers and estimators. This “modification” of $H_2/H_\infty$ control theory was called anisotropy-based theory. The main definitions of the theory are the anisotropy of a random vector, the mean anisotropy of a stationary ergodic sequence of random vectors, and the anisotropic norm of operator corresponding to some linear system.

For linear dynamic systems, a problem of analysis was studied in [3]. Control and estimation problems were successfully solved there. Both time-invariant and time-varying cases were considered. In these cases matrices of dynamic system were deterministic. Anisotropy-based analysis for systems with random matrices was studied in [4]. After that, the problem of analysis for multiplicative noise systems was solved [5]. Linear discrete time-varying system and finite time horizon was considered in [5]. The first attempt to calculate upper bound of anisotropic norm was made in [6], where majorant estimation to several subsystems was applied.

Multiplicative noise systems were chosen not by accident – this type of systems has an important role in different practical issues: dynamics of mechanical, hybrid, and financial systems; sensor networks are described by a linear system with multiplicative noises. The explicit expression for anisotropic norm was described in [7] for time-invariant systems. Outstanding results for estimation of a time-varying system with dropouts and its correction was obtained in [8].

In this paper, condition of boundedness for anisotropic norm of time-invariant systems with multiplicative noise is derived. Similar results for non-random type of linear discrete time-invariant systems are studied in [9].
2. Preliminaries
A definition of anisotropy was considered in [10]. Initially, it was a measure between two distributions. One of them was supposed to be uniform distribution on a unit sphere (reference distribution). Anisotropy of random distribution shows quantitatively its closeness to the reference one. Later, anisotropy of a random vector was introduced. Anisotropy of $m$-dimensional random vector $W$ with probability density function (pdf) $f$ is defined in [11] as follows:

$$A(w) = \min_{\lambda \geq 0} D(f||p_{m,\lambda}),$$

(1)

where $p_{m,\lambda}$ is a pdf for Gaussian random vector

$$p_{m,\lambda}(x) = (2\pi\lambda)^{-m/2} \exp\left(-\frac{x^T x}{2\lambda}\right), x \in \mathbb{R}^m.$$

The left side of (1) represents Kullback-Leibler divergence or differential entropy of pdf $f$ with respect to $p_{m,\lambda}$.

$$D(f||p_{\lambda}) = E_f\left[\ln \frac{f}{p_{\lambda}}\right] = \int f(x) \ln \left(\frac{f(x)}{p_{\lambda}(x)}\right) dx,$$

where $E_f[\cdot]$ stands to expectation with respect to $f$. If $W$ belongs to Gaussian random vectors with pdf

$$f(x) = ((2\pi)^m \det \Sigma)^{-1/2} \exp\left(-\frac{1}{2} x^T \Sigma^{-1} x\right), x \in \mathbb{R}^m,$$

where $\Sigma$ is a covariance matrix, the anisotropy of $W$ is equal to

$$A(W) = -\frac{1}{2} \ln \det(\frac{m\Sigma}{\text{tr} \Sigma}).$$

Hereinafter, det denotes the determinant of the matrix and tr denotes the trace of the matrix. Alongside anisotropy of a random vector, a mean anisotropy for sequence of random vectors $\{W_k\}$ with fixed pdf is used [1]:

$$\overline{A}(W) = \lim_{N \to \infty} \frac{A(W_{0:N})}{N + 1}.$$

Extended vector $W_{0:N} = (W_0^T, ..., W_N^T)^T \in \mathbb{R}^{m(N+1)}$ is an assembling of $N + 1$ random vector with common pdf. Mean anisotropy can be considered as time averaging of anisotropy for expanding vector.

Performance criterion in anisotropy-based theory is always related to anisotropic norm of linear system. Let us consider the linear discrete time-invariant system $F \in \mathcal{H}_{\infty}^{p\times m}$

$$F: \begin{cases} x_{k+1} = Ax_k + Bw_k, \\ z_k = Cx_k + Dw_k \end{cases}$$

with $n$-dimensional state $x_k$, $m$-dimensional input $w_k$ and $p$-dimensional output $z_k$, $\mathcal{H}_{\infty}^{p\times m}$ denotes the Hardy space of stable transfer functions in open unit circle. Matrices $A, B, C, D$ are known of appropriate dimensions. The $\alpha$-anisotropic norm $|||F|||_{\alpha}$ of system $F$ can be defined as [1]

$$|||F|||_{\alpha} = \sup_{\overline{A}(W_{\alpha})} \frac{||z||_2}{||w||_2},$$

where $w$ denotes input sequence with bounded mean anisotropy level $\alpha$, $z$ denotes output of the system $F$. 

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\begin{align*}
\|x\|_2 &= \left(\sum_{k=0}^{\infty} E[|x_k|^2]\right)^{1/2},
\end{align*}

\(|x_k|\) is Euclidean norm of \(x_k\). Anisotropic norm has important limiting cases: if mean anisotropy tends
to zero, the anisotropic norm tends to scaled \(H_2\)-norm of the system; if mean anisotropy became
infinite, the anisotropic norm is equal to \(H_\infty\)-norm

\[
\frac{\|F\|_2}{\sqrt{m}} = \lim_{a \to 0} \|F\|_a \leq \|F\|_a \leq \lim_{a \to \infty} \|F\|_a = \|F\|_\infty.
\]

Let us remind how \(H_2\) and \(H_\infty\) norms of the systems look like:

\[
\|F\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \text{tr} \left( F^*(\omega)\bar{F}(\omega) \right) d\omega \right)^{1/2}, \quad F(\omega) = F(e^{i\omega}), \quad \|F\|_\infty = \sup_{|z| \leq 1} \|F(z)\|.
\]

How to calculate anisotropic norm in state space (using matrices of the system) will be described
below.

3. Problem Statement

Let us consider the linear discrete time-invariant system of the form

\[
\begin{align*}
F: \begin{cases} 
 x_{k+1} &= Ax_k + Bw_k, \\
 z_k &= Cx_k + Dw_k,
\end{cases}
\end{align*}
\]

where \(x_k \in \mathbb{R}^n\) is a state, \(w_k \in \mathbb{R}^m\) is an input, \(z_k \in \mathbb{R}^p\) is an output. Matrix \(A\) has special form:

\[
A = A_0 + \sum_{i=1}^{M} \xi_{i,k} A_i.
\]

Matrices \(A_i, B, C, D\) are deterministic and known and have appropriate
dimensions. Random variables \(\xi_{i,k}, i = 1, M\), are mutually independent (on both indexes \(k\) and \(i\))
and have zero expectations and unit dispersions: \(E[\xi_{i,k}] = 0, E[\xi_{i,k}\xi_{j,s}] = \delta_{ij} \cdot \delta_{k,s}\), where \(E[\cdot]\) denotes
expectation, \(\delta_{ij}\) denotes the Kronecker symbol. Besides, random variables \(\xi_{i,k}\) are statically
independent with input \(w_k\). Mean anisotropy of stationary ergodic sequence of random vectors \(\{w_k\}\)
is bounded by given non-negative value \(\alpha\).

The problem is to find any condition on matrices of the system (2), such that anisotropic norm of the
system

\[
|||F|||_a = \sup \left\{ \frac{\|FW\|_2}{\|W\|_2} : \text{\(A(W) \leq \alpha\)} \right\}
\]

is bounded by threshold \(\gamma \geq 0\). Here \(FW\) is associated with output and

\[
||W||_2 = \left(\sum_{k=0}^{+\infty} \text{tr} \left( E[w_k^T w_k] \right) \right)^{1/2}.
\]

In contrast to paper [9], where spectral radius of non-random matrix \(A\) of the system (2) is
bounded by unit, in this paper, another constraint is considered: \(\lim_{k \to \infty} \rho \left( (E[A^k])^{1/k} \right) < 1\). It
equals to stability (in some sense) for the system (2) with random matrix \(A\).

4. Main Result

One important property of the dynamic system is considered in anisotropy-based theory. This property
is called innerness. It means that norm for output of the dynamic system is equal to input norm of the
system. Algebraic criterion of innerness for system (2) can be obtained as follows:

\[
D^T D + B^T R B = I_m,
\]
where matrices $P$ and $R$ correspond to the following Lyapunov equations:

$$P = A_cPA_c^T + BB^T,$$

$$R = \sum_{i=0}^{M} A_i^T R A_i + C^T C. \tag{8}$$

**Theorem:** Anisotropic norm $||F||_a$ of the system (2) is bounded by given $\gamma$, if (10)-(13) are solvable and the special type inequality

$$-\frac{1}{2} \ln \det \left((1 - q\gamma^2)S\right) \geq a \tag{9}$$

holds true.

Figure 1 shows an auxiliary system.

![Fig.1. Innerness criterion](image)

Let us describe the system presented in Fig. 1. It helps derive an inner algebraic criterion for system (2). Here $F$ denotes the system (2), $G$ belongs to former filter generating colored sequence $\{w_k\}$, $G^{-1}$ denotes the inverse filter to $G$. Former filter $G$ uses the white noise $\{v_k\}$ as input and generates the sequence $\{w_k\}$ as output. The mean anisotropy $\{v_k\}$ is zero and mean anisotropy of $\{w_k\}$ is bounded by given $a$, $0 \leq A(W) \leq a$. If the following matrix equations are solvable:

$$R_1 = \sum_{i=0}^{M} A_i^T R A_i + qC^T C, \tag{10}$$

$$R_2 = A_0^T R A_0 + L^T S^{-1} L, \tag{11}$$

$$S = (I_m - qD^T D - B^T R_2 B - B^T R_2 B)^{-1}, \tag{12}$$

$$L = S(qD^T C + B^T R_1 A_0 + B^T R_2 A_0), \tag{13}$$

the system $\left[\sqrt{qF^T, G^{-T}}\right]^T, q \in [0, \|F\|^2_\infty)$ is inner. This immediately follows from inner criterion (5)-(6) for multiplicative noise systems. This auxiliary system $\left[\sqrt{qF^T, G^{-T}}\right]^T$ or some similar type of dynamic system is always applied in anisotropy-based theory to examine the innerness property (or outerness property). The inequality (9) provides the condition of mean anisotropy boundedness, see [12] for details.

5. Conclusion

In this paper, calculation of the anisotropic norm for linear discrete time-invariant systems with multiplicative noise is considered. The state-space realization of the systems is used and two matrix equations and one inequality are derived.

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