Robust adaptive backstepping control for a class of constrained non-affine nonlinear systems via self-organizing Hermite-polynomial-based neural network disturbance observer

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Abstract
The article proposes a robust control approach based on self-organizing Hermite-polynomial-based neural network disturbance observer for a class of non-affine nonlinear systems with input saturation, state constraint, and unknown compound disturbance. Using Taylor series expansion, a hyperbolic tangent function, the non-affine nonlinear system with input saturation is transformed into time-varying affine system without input saturation, which can reduce step of the backstepping technique compared with conventional method. Next, a self-organizing Hermite-polynomial-based neural network disturbance observer is proposed to estimate the compound disturbance online. Then, the auxiliary systems are designed to solve state constraint for subsystems, and hyperbolic tangent function is used to approximate the saturated control input. Simulation results proved the effectiveness of the proposed control scheme.

Keywords
Non-affine nonlinear system, input saturation, state constraint, self-organizing Hermite-polynomial-based neural network disturbance observer, robust control, fast-convergence nonlinear differentiator, backstepping

Introduction
Since several decades ago, besides traditional proportional–integral/proportional–integral–derivative (PI/PID) control, many control approaches of nonlinear system have been proposed, including feedback linearization method, backstepping control and the method based on neural network. In Bidram et al., an input–output feedback linearization control method is proposed for multi-agent systems with nonlinear and heterogeneous dynamics via which nonlinear dynamics of agents is transformed to linear dynamics. An adaptive backstepping scheme is designed to control lateral/directional dynamics of unmanned aerial vehicles in Swarnkar and Kothari. The researches above aim at affine nonlinear system rather than non-affine system.

For the reason that the control input of non-affine system plays a role in the nonlinear implicit way, the control schemes of affine system can’t be applied to non-affine system, and they should be developed for the non-affine system.

Efficient control schemes have been studied widely owing to the fact that most practical engineering projects are non-affine systems. The adaptive fuzzy control scheme is proposed on the basis of the affine-like
equivalent mode.\textsuperscript{12,13} In Wu et al.,\textsuperscript{14} an adaptive fuzzy tracking controller with the corresponding parameter updating laws is designed to estimate the influence of the compound disturbance with unknown upper bound. The control schemes based on neural network are proposed in previous studies.\textsuperscript{15–20} In He et al.\textsuperscript{15} and Karimi and Menhaj,\textsuperscript{16} radial basis function (RBF) neural network is applied to controlling the non-affine systems. In Meng et al.,\textsuperscript{17} the non-affine system is transformed into affine system by the combination of a low-pass filter and state transformation. And a neural network observer is designed to estimate the unavailable state. Also, input saturation is a common phenomenon in the practical project. The hyperbolic tangent function and Nussbaum function are employed to handle the saturation in Chen et al.\textsuperscript{21} and Askari et al.\textsuperscript{22} In Chen and Yu\textsuperscript{23} and Li et al.,\textsuperscript{24} the auxiliary systems are constructed to compensate the influence of input saturation on tracking error. In Rehan et al.,\textsuperscript{25} the anti-windup compensator (AWC) is designed for Lipschitz bound. The control schemes based on neural network updating laws is designed to estimate the influence of tracking controller with the corresponding parameter systems. In Meng et al.,\textsuperscript{17} the non-affine system is neural network is applied to controlling the non-affine system. Also, input saturation is a common phenomenon in the practical project. The hyperbolic tangent function and Nussbaum function are employed to handle the saturation in Chen et al.\textsuperscript{21} and Askari et al.\textsuperscript{22} In Chen and Yu\textsuperscript{23} and Li et al.,\textsuperscript{24} the auxiliary systems are constructed to compensate the influence of input saturation on tracking error. In Rehan et al.,\textsuperscript{25} the anti-windup compensator (AWC) is designed for Lipschitz nonlinear systems with input saturation, and its necessary condition is the stabilizing controller. In Shojaei,\textsuperscript{26} a saturated tracking controller is designed to reduce the risk of actuator saturation using generalized saturation functions. Furthermore, the control scheme should be further developed for the non-affine system with input saturation.

To estimate the unknown compound disturbance, the article proposes a self-organizing Hermite-polynomial-based neural network disturbance observer (SHNND). The number of hidden neurons, the weight matrix, the center vector, and width vector will change as the adaptive laws, which guarantees the estimate accuracy. Compared with SHNND, the accuracy of radial basis function disturbance observer (RBFDO) will be influenced by the weight matrix and the set-value of the center vector and width vector of the basis function and fuzzy disturbance observer (FDO) will be influenced by the choice of adaptive laws. As a result, it has the advantages of less calculation and higher accuracy compared with RBFDO\textsuperscript{27} and FDO.\textsuperscript{28,29} Sliding mode disturbance observer (SMDO)\textsuperscript{30–32} is of simple design and reliable control performance, but the chattering will affect the control performance of sliding mode in the practical projects. Hence, we get the conclusion that the SHNND has the obvious advantages compared with the disturbance observer discussed above.

In this article, the non-affine nonlinear system with input saturation and state constraint is transformed into affine nonlinear system by Taylor series expansion and hyperbolic tangent function, which simplifies the calculation of auxiliary systems and backstepping scheme compared with Chen et al.\textsuperscript{21} Furthermore, SHNND is proposed to estimate the unknown compound disturbance, which is of minor calculation and better approximation performance. Then, the fast-convergence nonlinear differiator is designed to handle the derivatives of virtual control laws. Finally, backstepping controllers are designed and the stability of the close-loop system is proved by Lyapunov method.

The rest of the article is organized as follows. The problem statement is described in section “Problem statement and preparation” and section “Self-organizing Hermite-polynomial-based neural network disturbance observer design” designs SHNND. The control scheme based on SHNND for non-affine nonlinear system with input saturation and unknown disturbance is presented in section “Based on SHNND robust controller design.” The simulation results are given to show effectiveness of the proposed control scheme in section “Simulation analysis” and conclusion is described in section “Conclusion.”

Problem statement and preparation

Consider a class of multi-input and multi-output (MIMO) non-affine nonlinear system with input saturation and state constraint as follows

\[
\begin{align*}
\dot{x}_i &= F_i(x_{i+1}) + D_i, \quad i = 1, 2, \ldots, n - 1 \\
\dot{x}_n &= F_n(x, u(v)) + D_n \\
y &= x_1
\end{align*}
\]

where \(x_i = [x_1^T, x_2^T, \ldots, x_i^T]^T\) and \(x = [x_1^T, x_2^T, \ldots, x_n^T]^T \in \mathbb{R}^n\) are system vectors, \(u = [u_1(v_1), u_2(v_2), \ldots, u_n(v_n)]^T \in \mathbb{R}^n\) is the actual control input, and \(y \in \mathbb{R}^n\) is the output of system. \(F_i(x_{i+1})\) and \(F_n(x, u(v))\) are smooth nonlinear functions, and \(v = [v_1, v_2, \ldots, v_n]^T\) is saturator input.

The saturation function \(u_i(v_i)\) is defined as

\[
u_i(v_i) = \text{sat}(v_i) = \begin{cases} 
-u_M, & v_i \leq -u_M \\
u_i - u_M < v_i < u_M \\
u_M, & v_i \geq u_M
\end{cases}
\]

where \(\text{sat}(\cdot)\) is saturated function, \(v_i\) is the \(i\)th actuator input, \(u_i\) is the \(i\)th actuator output, and \(u_M\) is the known bound of \(u_i(v_i)\).

To proceed the design of the robust tracking control of the non-affine nonlinear systems, these assumptions are required.

Assumption 1. Output signal \(y(t)\) and reference signal \(y^d(t)\) are continuous, differentiable, and bounded about time.

Assumption 2. \(\frac{\partial F_i(x_{i+1})}{\partial x_{i+1}}\) and \(\frac{\partial F_n(x, u(v))}{\partial u(v)}\) are bounded.
Assumption 3. The unknown compound disturbance \(D_i\) \((i = 1, 2, \ldots, n)\) is continuous and bounded.

To get the time-varying affine nonlinear system with input saturation, Taylor series expansion is applied to \(F_i(x, u(v))\) at \(x = 11\) and \(F_i(x, u(v))\) at \(u(v) = u_i(v_i)\) of equation (1). Therefore, the affine form of system is obtained.

Then, to solve the control input constraint, we define hyperbolic tangent function to approximate the saturated function.

\[
h(v_i) = u_M \tanh \left( \frac{v_i}{u_M} \right)
\]

\[= u_M \left( e^{v_i/u_M} - e^{-v_i/u_M} \right) \quad (3)
\]

And the saturation function can be redefined as

\[
sat(v_i) = h(v_i) + d(v_i)
\]

where \(d(v_i)\) is the error of saturated function and hyperbolic tangent function and satisfies

\[
|d(v_i)| = |sat(v_i) - h(v_i)| \leq u_M (1 - \tanh(1)) \quad (5)
\]

The saturation function and hyperbolic tangent function is given as Figure 1.

Similarly, according to Taylor series expansion, we obtain

\[
h(v_i) = h(v_{i,0}) + \frac{\partial h(v_i)}{\partial v_i} |v_i - v_{i,0}| + O(\cdot) \quad (6)
\]

where \(v_{i,0}\) is the filter value of \(v_i\) and is obtained by robust sliding mode filter as equation (10). \(O(\cdot)\) is the high-order terms of Taylor series, and it is handled later. Let \(v_{i,0} = 0\), then \(h(v_{i,0}) = h(0) = 0\). So equation (6) can be simplified to

\[
h(v_i) = \frac{\partial h(v_i)}{\partial v_i} |v_i - v_{i,0}| + O(\cdot) \quad (7)
\]

Let

\[
H_n = \begin{bmatrix}
\frac{\partial h(v_i)}{v_i} & v_i - v_{i,0} \\
\frac{\partial h(v_i)}{v_i} & v_i - v_{i,0} \\
\vdots & \vdots \\
\frac{\partial h(v_i)}{v_i} & v_i - v_{i,0}
\end{bmatrix}
\]

Therefore, the non-affine nonlinear system with input saturation and unknown compound disturbance can be restated as follows

\[
\begin{cases}
\dot{x}_i = f_i(x_i) + g_i x_{i+1} + d_i, & i = 1, 2, \ldots, n - 1 \\
x_n = f_n(x) + g_n y + d_n \\
y = x_1
\end{cases} \quad (9)
\]

where \( f_i(x_i) = F_i(x_i, x_{i+1}) - \frac{dF_i}{dx} x_{i+1} \), \( f_n(x) = F_n(x, u_n(v)) - \frac{dF_n}{dx} u_n(v) \), \( g_i = \frac{dF_i}{dx} \), \( d_i = D_i + \Delta(\cdot) \), \( i = 1, 2, \ldots, n - 1 \), \( d_n = D_n + \Delta(\cdot) + \frac{dF_n}{du} H_n d(v) + \frac{dF_n}{du} H_n O(\cdot) \) is the redefined compound disturbance, \( \Delta(\cdot) \) is the high-order terms of Taylor series, and \( u_n(v) \) is robust sliding mode filter state and is given by

\[
\dot{u}_n(v) = - \frac{u_n(v) - u(v)}{\rho} - \xi_1 (u_n(v) - u(v)) \left\| u_n(v) - u(v) \right\| + \xi_1 \quad (10)
\]

where \( \rho \) is filter time constant, \( \xi_1 > 0 \) and \( \xi_1 > 0 \) are designed switching gain and the switching rate to adjust sliding mode, respectively, \( u_n(v) \) is the filter value of \( u(v) \).

From equation (9), we see that nonlinear terms are considered as unknown disturbance and it is really true that there is ambiguity. In this case, we design robust sliding mode filter as equation (10). While \( \lim \| u_n(v) - u(v) \| = 0 \), there exists that \( \lim \| \Delta(\cdot) \| = 0 \). As a result, if \( \rho \) in equation (10) is small enough, the influence of higher-order terms on approximate method can be ignored.

Assumption 4. The control gain matrix \( g_i \) and \( g_n \) are reversible.

Remark 1. It can be seen that the non-affine nonlinear system with input saturation is transformed into affine nonlinear system without input saturation via the combination of Taylor series and hyperbolic tangent function. Based on it, the computation of backstepping is reduced and we only need to design \( n - 1 \) auxiliary systems to compensate the influence of state constraint, and the effect of input saturation is solved by hyperbolic tangent function.

\[\text{Figure 1. Saturation function and hyperbolic tangent function.}\]
Figure 2 depicts the control scheme based on SHNNDO for non-affine nonlinear system with input saturation and state constraint in this article. To ensure that the output $y$ tracks the desired signal $y^d$ in the non-affine system with input saturation, we first proposed SHNNDO to estimate the unknown disturbance and then design auxiliary systems to handle the influence of input saturation. Finally, the controller based on backstepping is presented.

**SHNNDO design**

From section “Problem statement and preparation,” it can be noted that the redefined disturbance $d_i (i = 1, 2, ..., n)$ is still unknown and then we design a SHNNDO to estimate it. The number of hidden neurons can schedule with the disturbance, which guarantees the accuracy of the estimation. Specifically, while the disturbance gets larger, SHNNDO generates new hidden neurons. While the disturbance is small, the redundant hidden neurons can be canceled according to the competition index.

**Description of self-organizing Hermite-polynomial-based neural network**

There are five layers in self-organizing Hermite-polynomial-based neural network (SHNN) as shown in Figure 3, and the operation function of each layer is described as follows:

*Layer 1 (input layer).* In this layer, the node only transmits input signals to next layer and no function is operated.

*Layer 2 (Hermite layer).* In this layer, the operation functions are given as

$$
\phi_i = \frac{1}{\sqrt{2^i i! \sqrt{\pi}}} e^{-\frac{s^2}{2}} H_i(s), \quad i = 0, 1, 2, \ldots, m
$$

where the orthogonal polynomials are as follows

$$
\begin{align*}
H_0(s) &= 1 \\
H_1(s) &= 2s \\
& \vdots \\
H_i(s) &= 2sH_{i-1}(s) - 2(i-1)H_{i-2}(s), \quad i \geq 2
\end{align*}
$$

*Layer 3 (reception layer).* In this layer, the operation function is as presented

$$
\mathbf{s} = \sum_{i=0}^{m} \phi_i
$$

Generally, the higher-order orthogonal Hermite polynomial basis functions are selected, and the better approximated performance will be obtained.

*Layer 4 (hidden layer).* In this layer, the node output signals are as follows

$$
\theta_j = \exp \left( -\frac{(s - c_j)^2}{2v_j} \right), \quad j = 1, 2, \ldots, n
$$

where $c_j$ and $v_j$ are parameters of center vector and width vector of the basis function, respectively.

*Layer 5(output layer).* The operation function of this layer is the output of the SHNN, that is

$$
\hat{d} = \sum_{j=1}^{n} w_j \theta_j = \mathbf{W}^T \mathbf{\theta}(\mathbf{c}, \mathbf{v})
$$

where $\mathbf{W} = [W_1^T, W_2^T, \ldots, W_n^T]^T$ is weight matrix, $\mathbf{\theta} = [\theta_1^T, \theta_2^T, \ldots, \theta_n^T]^T$ is basis function, $\mathbf{c} = [c_1^T, c_2^T, \ldots, c_n^T]^T$ and $\mathbf{v} = [v_1^T, v_2^T, \ldots, v_n^T]^T$ are center vector and width vector of basis function, respectively, and $\hat{d} = [\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_{n_b}]^T$, where $n_b$ is the number of the output of SHNNDO.
where \( l_{\text{min}} \) follows.

Ron, the distance between the input neuron includes (1) generating a new hidden neuron and (2) canceling the existing inappropriate neurons.

In this section, the learning algorithm of hidden layer neuron includes (1) generating a new hidden neuron and (2) canceling the existing inappropriate neurons. The flowchart of learning algorithm is given in Figure 4.

To determine whether to generate a new hidden neuron, the distance between the input \( s \) of hidden layer and the center of existing hidden neuron is denoted as follows

\[
l_j = |s - c_j|, \quad j = 1, 2, \ldots, n
\]

If \( l_{\text{min}} \geq I_{h} \), a new hidden neuron should be added, where \( l_{\text{min}} = \min_{1 \leq j \leq n} l_j \) is the minimum distance, \( I_{h} \) is a pre-given threshold, and \( n(k) \) is the number of existing hidden neurons at the \( k \)th sample time.

The parameters of added neuron are as follows

\[
\begin{align*}
    c_{\text{new}} &= s \\
    v_{\text{new}} &= \tilde{v} \\
    W_{\text{new}} &= 0_{1 \times n_{h}}
\end{align*}
\]

where \( \tilde{v} \) is the constant being designed.

The hidden layer is updated as following

\[
I_j(k+1) = \begin{cases} 
    I_j(k)e^{-\tau}, & l_j > \rho \\
    I_j(k), & l_j \leq \rho 
\end{cases} 
\]

where \( I_j(k) \) is the competition index of the \( j \)th hidden neuron during the \( k \)th sample time and its initial value is 1, and \( \rho \) is the competition threshold value and \( \tau \) is the reduction speed constant.

If \( I_j(k) \leq I_{h} \), the \( j \)th hidden neuron will be pruned and \( I_{h} \) is the pre-given threshold.

The parameters are updated as follows

\[
\begin{align*}
    c(k + 1) &= Ac(k) \\
    v(k + 1) &= Av(k) \\
    \theta(k + 1) &= A\theta(k) \\
    W(k + 1) &= AW(k)
\end{align*}
\]

where \( A = \begin{bmatrix} I_{j-1} & 0_{(j-1) \times 1} & 0_{(j-1) \times (n-j)} \end{bmatrix} \).

**Design of SHNND**

The Taylor expansion linearization technique is applied to \( \theta \) and we obtain

\[
\tilde{\theta} = A^T\tilde{c} + B^T\tilde{v} + \Delta
\]

where \( \tilde{\theta} = \theta^* - \hat{\theta} \), \( \tilde{c} = c^* - \hat{c} \), and \( \tilde{v} = v^* - \hat{v} \) are estimated values of \( \theta \), \( c \), and \( v \), respectively; \( \Delta \) is the higher-order terms, \( A = (\partial \theta / \partial \hat{c})^T \), and \( B = (\partial \theta / \partial \hat{v})^T \).

Defining \( \tilde{d} = d^* - \hat{d} \), \( \tilde{W} = W^* - W \) as corresponding estimated values, unknown disturbance \( d \) can be described by SHNN, that is

\[
d = \tilde{W}^T\hat{\theta} + \epsilon = \tilde{W}^T\theta* + \epsilon, \quad ||\epsilon|| < \tilde{\epsilon}
\]
where \( \epsilon \) is the approximate error of SHNDO, \( \bar{\epsilon} \) is the unknown upper bound of \( \epsilon \).

**Theorem 1.** Considering the non-affine nonlinear system (equation (1)), Assumption 2, and the approximate method (equations (3)–(8)), the SHNDO is designed as follows

\[
\begin{align*}
\dot{e}_i &= x_i - s_i \\
\dot{s}_i &= K_i e_i + \psi_i + \hat{d}_i + \lambda e_i \\
\hat{W}_i &= h_1(\theta e_i - T_1 \hat{W}_i) \\
\hat{c}_i &= h_2(A \hat{W}_i e_i - T_2 \hat{c}_i) \\
\hat{v}_i &= h_3(B \hat{W}_i e_i - T_3 \hat{v}_i)
\end{align*}
\]  
(23)

where \( e_i \) is the error of \( i \)th disturbance observer; \( s_i \) is the \( i \)th state of SHNDO; \( K_i = \text{diag}\{k_1, k_2, \ldots, k_n\} \) is the positive definite matrix to be designed; \( h_1, h_2, \) and \( h_3 \) are designed positive constants; \( T_1, T_2, \) and \( T_3 \) are designed positive definite matrices; \( \lambda \) is a positive constant, \( \psi_i = f_i + g_i x_i, i = 1, 2, \ldots, n - 1 \); and \( \psi_n = f_n + g_n y \). If \( \hat{W}_i, \hat{c}_i, \) and \( \hat{v}_i \) are chosen as in equation (24), the \( e, \hat{W}, \hat{c}, \) and \( \hat{v} \) will be proved to be uniformly and ultimately bounded.

**Proof.** Substituting equations (9), (21), and (22) into equation (23), we have

\[
\dot{e} = \dot{x} - \dot{s} = -K e + \hat{W}^T \hat{\theta} + \hat{c}^T A \hat{W} + \hat{v}^T B \hat{W} + l - \lambda e
\]  
(25)

where the approximate error becomes \( l = \hat{c}^T A \hat{W} + \hat{v}^T B \hat{W} + \hat{W}^T \Delta + \hat{W}^T \Delta + e \), and it satisfies \( 0 \leq |l| \leq L \), in which \( L \) is a positive constant.

Considering the Lyapunov function candidate

\[
V = \frac{1}{2} e^T e + \frac{1}{2h_1} \hat{W}^T \hat{\theta} - \frac{1}{2h_2} \hat{c}^T \hat{c} + \frac{1}{2h_3} \hat{v}^T \hat{v}
\]  
(26)

Differentiating \( V \) and invoking equations (24) and (25), we have

\[
\dot{V} = e^T (-K e + \hat{W}^T \hat{\theta} + \hat{c}^T A \hat{W} + \hat{v}^T B \hat{W} + l - \lambda e)
\]
\[
+ \frac{1}{h_1} \hat{w}^T \hat{\theta} + \frac{1}{h_2} \hat{c}^T \hat{c} + \frac{1}{h_3} \hat{v}^T \hat{v}
\]
\[
\leq -K\|e\|^2 - \frac{T_1}{2} \| \hat{W} \|^2 - T_2 \| \hat{c} \|^2 - T_3 \| \hat{v} \|^2 + \frac{T_2}{2} \| e \|^2
\]
\[
+ \frac{T_3}{2} \| \hat{v} \|^2 + \frac{T_1}{2} \| \hat{W} \|^2 + \frac{T_2}{2} \| e \|^2 + \frac{T_3}{2} \| \hat{v} \|^2 + L^2 4\alpha
\]
\[
\leq -K\|e\|^2 - \frac{T_1}{2} \| \hat{W} \|^2 - T_2 \| \hat{c} \|^2 - T_3 \| \hat{v} \|^2 + C_0
\]
\[
\leq -\kappa V + C_0
\]  
(27)

where \( \kappa = \min\{2K, T_1, T_2, T_3\} > 0 \), where \( K, T_1, T_2, \) and \( T_3 \) are the minimum eigenvalues of \( K, T_1, T_2, \) and \( T_3 \), respectively; \( C_0 = \frac{T_2}{2} \| e \|^2 + \frac{T_3}{2} \| \hat{v} \|^2 + \frac{L^2}{4\alpha} \).

Integrating equation (27), we obtain

\[
0 \leq V \leq C_0 + \left( V(0) - C_0 \right) e^{-\kappa t}
\]  
(28)

where \( V(0) = \frac{1}{2} \| e(0) \|^2 + \frac{1}{2h_1} \| \hat{W}(0) \|^2 + \frac{1}{2h_2} \| \hat{c}(0) \|^2 + \frac{1}{2h_3} \| \hat{v}(0) \|^2 \).

From equation (25), \( \hat{d} \to d \) when \( e \to 0 \). Furthermore, when \( t \to \infty \), \( V \) satisfies that \( 0 \leq V \leq V(0) \). Choosing the reasonable parameters \( h_1, h_2, h_3, T_1, T_2, \) and \( T_3 \), the estimated errors \( e, \hat{W}, \hat{c}, \) and \( \hat{v} \) are uniformly and ultimately bounded. Thus, it implies that the designed SHNDO will estimate the unknown disturbance exactly.

**Remark 2.** Compared with RBF neural network disturbance observer, SHNDO can adjust the number of hidden neurons according to learning algorithm. Meanwhile, \( e, \hat{W}, \hat{c}, \) and \( \hat{v} \) are the same as equations (18) and (20) and adaptive laws are changed as equation (24).

**Based on SHNDO robust controller design**

In this section, a controller based on SHNDO is designed through backstepping scheme, and the stability is proved. To tackle the input saturation, we use hyperbolic tangent function to approximate the saturation function and then use Taylor series expansion. The auxiliary systems are designed to compensate the influence of state constraints. Finally, the output \( y \) will track the pre-given reference signal \( y^d \).

**Design of fast-convergence nonlinear differentiator**

The fast-convergence nonlinear differentiator is designed to estimate the derivatives of virtual control laws, which can avoid the growing of differential orders effectively.

**Lemma 1.** Consider system

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= f(z_1, z_2)
\end{align*}
\]  
(29)

where \( z_1, z_2 \in R \) are outputs of the system. If there exists \( z_1 \to 0 \) and \( z_2 \to 0 \), when \( t \to \infty \) for any solution, then \( x_1 \) in system
where $a_0, a_1, b_0, b_1 > 0$ and $0 < \theta < 1$ are the constants to be designed. $\text{sign}(x) = |x| \text{sign}(x)$, $k > 0$, and $x \in \mathbb{R}$.

In equation (31), if $a_0 = b_0 = 0$ and $a_1, b_1 > 0$, the fast-convergence nonlinear differentiator corresponds to linear differentiator. If $a_0, b_0 > 0$ and $a_1 = b_1 = 0$, the fast-convergence nonlinear differentiator corresponds to nonlinear differentiator. If and only if $a_0, a_1, b_0, b_1 > 0$, equation (31) is fast-convergence nonlinear differentiator. To illustrate the above clearly, the fast-convergence nonlinear differentiator is given as Figure 5.

According to Lemma 1, to prove the stability of the fast-convergence nonlinear differentiator (31), the Lemma 2 should be proved.

**Lemma 2.** Consider the system (32)

\[
\begin{align*}
\dot{z}_{d1} &= z_{d2} \\
\dot{z}_{d2} &= f(z_{d1}, z_{d2}) \\
&= -a_0 z_{d1} - a_1 \text{sign}(z_{d1})^{\theta} - b_0(z_{d2}) - b_1 \text{sign}(z_{d2})^{\theta}
\end{align*}
\]  

(32)

There exists that $(z_{d1}, z_{d2}) = (0, 0)$ is the equilibrium of the system and $\lim_{t \to \infty} z_{d1} = 0$ and $\lim_{t \to \infty} z_{d2} = 0$.

\[
\textbf{Figure 5.} \text{ Convergent velocity in rapid-convergent form.}
\]

**Proof:** Chose the Lyapunov function as follows

\[
V_d = a_1 \frac{2 - \theta}{\theta} |z_{d1}|^{\theta} + \frac{1}{2} a_0 z_{d1}^2 + \frac{1}{2} z_{d2}^2
\]  

(33)

Differentiating equation (33), we obtain

\[
\dot{V}_d = a_1 |z_{d1}|^{\theta} \text{sign}(z_{d1}) \dot{z}_{d2} + a_0 z_{d1} \dot{z}_{d1} + z_{d2} \dot{z}_{d2}
\]

\[
= a_1 \text{sign}(z_{d1})^{\theta} z_{d2} + a_0 z_{d1} \dot{z}_{d2} + z_{d2} \dot{z}_{d2} - a_0 z_{d1} - a_1 \text{sign}(z_{d1})^{\theta} \dot{z}_{d1} - b_0(z_{d2}) - b_1 \text{sign}(z_{d2})^{\theta}
\]

\[
\leq -b_0 z_{d2}^2 - b_1 |z_{d2}|^{\theta + 1}, \quad (z_{d1}, z_{d2}) \neq (0, 0)
\]  

(34)

This means that the system equilibrium of $\dot{V}_d = 0$ is $(0, 0)$, that is, system (32) is asymptotically stable at origin. As a result, the conditions that $\lim z_{d1} = 0$ and $\lim z_{d2} = 0$ hold.

Consequently, the conclusions of equation (31) are obtained

\[
\lim_{t \to \infty} \int_0^T |x_1 - v| = 0
\]  

(35)

that is

\[
x_{d1} = v, \quad x_{d2} = v
\]  

(36)

In this article, fast-convergence nonlinear differentiator is applied to estimating the derivatives of the virtual control laws. From the proof above, we obtain that fast-convergence nonlinear differentiator is of good tracking performance and the growing of differential orders of virtual laws is solved.

**Backstepping scheme**

Aiming at system (equation (9)), we propose the backstepping scheme based on the SHNDO.

From section “Problem statement and preparation,” we conclude that the input saturation is tackled by hyperbolic tangent function, so we only design the auxiliary systems to handle the state constraint. The auxiliary systems are proposed as follows

\[
\begin{align*}
\gamma_i &= A_{d(i)} \gamma_i + g_i(\Delta x_{i+1} + \gamma_{i+1}), \quad i = 1, 2, \ldots, n - 2 \\
\gamma_{n-1} &= A_{d(n-1)} \gamma_{n-1} + g_{n-1} \Delta x_n
\end{align*}
\]  

(37)

where $\gamma_i(1 = 1, 2, \ldots, n - 1)$ is state of the $i$th auxiliary system, $A_{d(i)}(1 = 1, 2, \ldots, n - 1)$ is the positive matrix to be designed, and $\Delta x_{i+1} = x_{i+1}^d - x_{i+1}^*$.

**Step 1.** Define the compensated error as $z_1 = y - \gamma_1 - y^d$. From system equation (9), the dynamic equation for $z_1$ is
\[
\dot{z}_1 = \dot{y} - \dot{\gamma}_1 - \dot{y}^d
\]
where the compound disturbance \(d_1\) is unknown, but it can be estimated in section “Self-organizing Hermite-polynomial-based neural network disturbance observer design.” Consider the following virtual control laws and design the robust term with exponential convergence properties as follows

\[
\begin{align*}
\mathbf{x}_2^* &= -g_1^{-1}(\Lambda_1 z_1 + f_1 + \hat{d}_1 - A_{a_1} \gamma_1 + \mathbf{r}_1 - \dot{y}^d) \\
\mathbf{r}_1 &= \frac{E_1}{\eta_1 + (1 - \eta_1)e^{-\eta_1|z_1|^p}}|z_1|
\end{align*}
\]

where \(E_1 > |d_1 + \hat{e}_1|, 0 < \eta_1 < 1, v_1, \) and \(p_1\) are strictly positive integers.

Consider the following Lyapunov function candidate for the tracking error

\[
V_1 = \frac{1}{2} z_1^T z_1
\]

Using equations (38) and (39), we have

\[
\dot{V}_1 = z_1^T \dot{z}_1
\]

\[
= z_1^T (f_1 + d_1 - A_{a_1} \gamma_1 + g_1(z_2 + x_2^*) - \dot{y}^d)
\]

\[
= z_1^T (-\Lambda_1 z_1 - r_1 + g_1 z_2 + \hat{d}_1 + \epsilon_1)
\]

\[
\leq -\Delta_1 \|z_1\|^2 + C_1 + z_1^T \mathbf{r}_1 g_1 z_2
\]

Because \(E_1 > |d_1 + \hat{e}_1|, \) we get that \(C_1 = \|z_1\|:\n\]

\[
\|d_1 + \hat{e}_1\| - \frac{E_1}{\eta_1 + (1 - \eta_1)e^{-\eta_1|z_1|^p}}|z_1| < 0,
\]

\[
\dot{V}_1 \leq -\Delta_1 \|z_1\|^2 + z_1^T \mathbf{r}_1 g_1 z_2
\]

Step 1 \((2 \leq i < n-1)\). Let \(z_i = x_i - \gamma_i - x_i^d\), the derivative of \(z_i\) is calculated as follows

\[
z_i = \dot{x}_i - \dot{\gamma}_i - \dot{x}_i^d
\]

\[
= f_i + d_i - A_{a_i} \gamma_i + g_i(z_{i+1} + x_{i+1}^*) - \dot{x}_i^d
\]

To cancel the coupling terms of the previous step, the virtual control law is described as follows

\[
\begin{align*}
x_i^* &= -g^{-1}_i(\Lambda_iz_i + f_i + \hat{d}_i - A_{a_i} \gamma_i + r_i - \dot{x}_i^d + g_i^T z_{i+1}) \\
r_i &= \frac{E_i}{\eta_i + (1 - \eta_i)e^{-\eta_i|z_i|^p}}|z_i|
\end{align*}
\]

where \(E_i > |d_i + \hat{e}_i|\).

Let the Lyapunov function be \(V_i = V_{i-1} + \frac{1}{2} z_i^T z_i\), and the time derivative of \(V_i\) is

\[
\dot{V}_i = \sum_{j=1}^{i-1} (-\Lambda_j V_j + C_j) + z_i^T g_{i-1} z_{i-1} + z_i^T
\]

\[
(f_i + d_i - A_{a_i} \gamma_i + g_i(z_{i+1} + x_{i+1}^*) - \dot{x}_i^d)
\]

\[
= \sum_{j=1}^{i-1} (-\Lambda_j V_j + C_j) + z_i^T g_{i-1} z_{i-1} + z_i^T
\]

\[
(-\Lambda_i z_i - r_i + g_i z_{i+1} + \hat{d}_i + \epsilon_i)
\]

\[
\leq \sum_{j=1}^{i} (-\Delta_i \|z_i\|^2 + C_j) + z_i^T g_i z_{i+1}
\]

Similar to step 1, \(C_j < 0\), we have

\[
\dot{V}_i \leq \sum_{j=1}^{i} (-\Delta_i \|z_i\|^2 + z_i^T g_i z_{i+1})
\]

Step n In order to handle the control input saturation, the hyperbolic tangent function is used to approximate it, then Taylor series expansion is applied to it.

Let \(z_n = x_n - x_n^d\), and the dynamic equation of \(z_n\) is as follows

\[
z_n = \dot{x}_n - \dot{x}_n^d
\]

\[
= f_n + g_n z_n + d_n - \dot{x}_n^d
\]

Define the control input term as follows

\[
\begin{align*}
y &= -g_n^{-1}(\Lambda_n z_n + f_n + d_n - \dot{x}_n^d + r_n + g_n^T z_{n-1}) \\
r_n &= \frac{E_n}{\eta_n + (1 - \eta_n)e^{-\eta_n|z_n|^p}}|z_n|
\end{align*}
\]

To prove that tracking errors of the subsystems converge to zero, the Lyapunov function is chosen as \(V_n = V_{n-1} + (1/2)z_n^T z_n\), and the time derivative is

\[
\dot{V}_n = \sum_{i=1}^{n} (-\Delta V_i + C_i) + z_n^T g_{n-1} z_{n-1} + z_n^T (f_n + g_n y + d_n - x_n^d)
\]

\[
= \sum_{i=1}^{n} (-\Delta V_i + C_i) + z_n^T g_{n-1} z_n - \Delta_i z_n - r_n + z_n^T (d_n + e_n)
\]

\[
\leq \sum_{i=1}^{n} (-\Delta V_i + C_i)
\]

where \(\Delta_i\) is the minimum eigenvalue of \(A_i, E_n > |d_n + \hat{e}_n|; \) therefore, \(C_i = \|z_i\| \cdot |d_i + \hat{e}_i| - \frac{E_n}{\eta_n + (1 - \eta_n)e^{-\eta_n|z_i|^p}}|z_i| < 0\) in the equation (49), we have

\[
\dot{V}_n \leq \sum_{i=1}^{n} -\Delta V_i
\]
Considering the design of SHNNDO and controllers, we get the conclusion as follows.

**Theorem 2.** Consider the non-affine nonlinear system (1), approximation methods (9), SHNNDO (23), and the parameter adaptive law (24), virtual law (39) and (44), control input (48), under Assumptions 1–4. All variables $z_i$ asymptotically converge to zero about the disturbance estimated errors $e_i$ and adaptive parameters $\mathbf{W}_i, \mathbf{c}_i, \mathbf{\dot{v}}_i$.

**Proof.** According to equation (50), we have

$$
\dot{V}_n \leq \sum_{i=1}^{n} -\Delta_i V_i \leq -KV_n
$$

where $K = \min \{2\Delta_i \}$. Let $V_n(0) = \sum_{i=1}^{n} \|z_i(0)\|^2$, and integration of equation (51) yields

$$
V_n(t) \leq e^{-kt} V_n(0)
$$

Then, the stability of the system is proved.

**Remark 3.** The robust terms with exponential convergence properties designed above is of great convergence. $E_i/(\eta + (1-\eta)e^{-\nu_k|x|})$ varies between $E_i$ and $|E_i/\eta|$ as the variation of $|z_i|$. If $|z_i|$ reduces, $(E_i/(\eta + (1-\eta)e^{-\nu_k|x|}))$ converges to $E_i$. Therefore, the robust terms with exponential convergence properties can converge rapidly and is of good robustness and tracking performance.

**Simulation analysis**

In this section, the proposed control scheme is tested via simulation of near-space vehicle (NSV) with unknown compound disturbance. Consider the system

$$
\begin{align*}
\mathbf{\dot{\Omega}} &= \mathbf{f}_\Omega + \mathbf{g}_\Omega \mathbf{w} + \mathbf{D}_\Omega \\
\mathbf{\dot{w}} &= \mathbf{f}_w + \mathbf{g}_w \mathbf{u}(\mathbf{v}) + \mathbf{D}_w \\
\mathbf{y} &= \mathbf{\Omega}
\end{align*}
$$

where $\mathbf{\Omega} = [\alpha \ \beta \ \mu]^T$ is the vector of attitude angles, with $\alpha$ is the angle of attack, $\beta$ is the sideslip angle, and $\mu$ is the roll angle; $\mathbf{w} = [p \ q \ r]^T$ is the vector of angular velocity and is constrained, $p$ is roll angle rate, $q$ is the pitch angular velocity, and $r$ is the yaw angular velocity; $\mathbf{u}(\mathbf{v}) = [I_{ctrl} \ m_{ctrl} \ n_{ctrl}]^T$ is the designed actual control input vector, with $l_{ctrl}, m_{ctrl},$ and $n_{ctrl}$ are moments of ailerons, elevator, and rudder, respectively; $\mathbf{d}_d$ and $\mathbf{d}_f$ are disturbance of slow loop and fast loop, respectively; $\mathbf{d}_f$ is the external disturbance exerted in the style of moment on fast loop; $\mathbf{f}_i = [f_{a} \ f_{b} \ f_{\mu}]^T$.

The parameters are selected as $n_R = 11$, $s$, $\Lambda_x = diag\{4,3,2\}$, $\Lambda_y = diag\{2,2,1\}$, $E_x = E_f = 15$, $\eta_s = \eta_f = 0.5$, $\nu_s = \nu_f = 3$, and $p_s = p_f = 3$. The saturation of rate velocity is set as $\omega_{\text{max}} = [2.8, 2.8, 2.8]^T$ deg/s and $\omega_{\text{min}} = [-2.8, -2.8, -2.8]^T$ deg/s.

The initial values are set as follows:

$$
\begin{align*}
M &= 136615 \text{ kg}, \quad T(0) = 400 \text{ kN} \\
H(0) &= 28 \text{ km}, \quad V(0) = 2.3 \text{ km/s} \\
\mathbf{Q}(0) &= [0.2, 0, 1]^T, \quad \omega(0) = [0, 0, 0]^T
\end{align*}
$$
Suppose that there are 30% uncertainties on aerodynamic coefficients, the disturbance moments chosen as follows are exerted on fast loop when $t(0) = 0$

$$d_f(t) = \begin{bmatrix} 1.2 \times 10^2 \cos(6t + 0.3) \\ 2.4 \times 10^2 \sin(5t + 0.1) \\ 2.0 \times 10^5 \sin 8t \end{bmatrix}$$

The tracking desired guidance commands are given as follows

$$y^d = \begin{bmatrix} \alpha^d \\ \beta^d \\ \mu^d \end{bmatrix} = \begin{cases} 1.6 & 0 \leq t < 4 \\ 2.6 & 4 \leq t < 8 \\ 1 & 0 \leq t < 4 \\ 2 & 4 \leq t < 8 \\ 3 & 0 \leq t < 4 \\ 4 & 4 \leq t < 8 \end{cases}$$

And the filter $4/(s + 4)$ is chosen to guarantee the flight quality.

Figure 6 demonstrates the tracking performance of the proposed controller based on SHNND0. The red dashed line shows the actual attitude angle, and the black solid line corresponds to the desired guidance commands. Furthermore, Figures 6 and 7 show that the tracking errors of attitude angles which tend to zero in the flight.

In Figure 8, the behavior of the actual rate velocity $\omega$, the desired rate velocity $\omega^*$, and the saturated rate velocity $\omega^d$ follows the expectations as attitude angles converges to their desired guidance commands.

Figure 9 shows that the behavior of deflection angles is constrained and tend to zero to guarantee the stability of flight. From above analysis, we obtain that the system has good robustness and the control scheme proposed in this article is effective.

Figure 10 demonstrates the behavior of saturated angle velocity, as we can see, even though there exists
state constraint, the influence can be eliminated to guarantee the robustness of system.

According to the analysis above, we obtain that the system is of satisfactory tracking performance based on the proposed control scheme.

Conclusion

This article applies the backstepping scheme based on SHNDDO to non-affine nonlinear system with state constraint and input saturation to improve the robustness to unknown compound disturbance. We use approximation methods to get the affine nonlinear system. Furthermore, in the backstepping controller, we design a fast-convergence nonlinear differentiator to estimate the derivative of virtual control laws, however, the SHNDDO is proposed to approximate the external disturbance, model errors, and parameter uncertainties. And the stability is proved by Lyapunov function. Finally, the system is simulated on Simulink. Simulation results illustrate the effectiveness of the proposed control scheme.

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