UNO’S CONJECTURE ON REPRESENTATION TYPES OF HECKE ALGEBRAS

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Abstract. Based on a recent result of the author and A. Mathas, we prove that Uno’s conjecture on representation types of Hecke algebras is true for all Hecke algebras of classical type.

1. Introduction

Let $K$ be a field of characteristic $l$, $A$ a finite dimensional $K$–algebra. We always assume that $K$ is a splitting field of $A$. We say that $A$ is of finite representation type if there are only finitely many isomorphism classes of indecomposable $A$–modules. If $A$ is a group algebra, then the following theorem answers when $A$ is of finite representation type.

Theorem 1. Let $G$ be a finite group, $A = KG$ its group algebra. Then $A$ is of finite representation type if and only if the Sylow $l$–subgroups of $G$ are cyclic.

We restrict ourselves to the case where $G$ is a finite Weyl group and see consequences of this result.

Theorem 2. Let $W$ be a finite Weyl group. Then $KW$ is of finite representation type if and only if $l^2$ does not divide the order $|W|$.

For the proof see Appendix. Note that Theorem 2 does not hold if we consider finite Coxeter groups. Dihedral groups $W(I_2(m))$ with odd $l$ and $l^2 | m$ are obvious counterexamples.

Uno conjectured a $q$–analogue of Theorem 2. Let $q \in K$ be an invertible element, $(W, S)$ a finite Weyl group. We denote by $\mathcal{H}_q(W)$ the (one parameter) Hecke algebra associated to $W$. The quadratic relation we choose is $(T_s + 1)(T_s - q) = 0$ ($s \in S$). The Poincaré polynomial $P_W(x)$ is defined by

$$P_W(x) = \sum_{w \in W} x^{l(w)}.$$

Let $e$ be the smallest positive integer such that $q^{e-1} + \cdots + 1 = 0$. If $q = 1$ then $e = l$, and if $q \neq 1$ then $e$ is the multiplicative order of $q$. As the condition that $l^2$ does not divide $|W|$ is the same as $(\frac{x^{e-1} - 1}{x-1})^2$ evaluated at $x = 1$ does not divide $P_W(1)$, the following is a reasonable guess. We call this Uno’s conjecture.

Conjecture 3. Assume that $q \neq 1$ and $K$ a splitting field of $\mathcal{H}_q(W)$. Then $\mathcal{H}_q(W)$ is of finite representation type and not semisimple if and only if $q$ is a simple root of $P_W(x) = 0$, that is, if and only if $(x - q)^2$ does not divide $P_W(x)$.

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It is well–known that $H_q(W)$ is semisimple if and only if $P_W(q) \neq 0$. See \cite{[1]} Proposition 2.3 for example. This fact will be used later.

**Remark 4.** In \cite{[1]}, it is proved that the conjecture is true for $H_q(I_2(m))$. So, unlike the case of $q = 1$, we may ask the same question for finite Weyl groups instead of finite Weyl groups.

The following theorem is proved in \cite{[1]}

**Theorem 5.** \cite{[1] Proposition 3.7,Theorem 3.8} Suppose that $q \neq 1$ and denote its multiplicative order by $e$. Then $H_q(A_{n−1})$ is of finite representation type if and only if $n < 2e$.

As $P_W(x) = \prod_{i=1}^n \frac{x^i − 1}{x−1}$ in this case, a primitive $e^{th}$ root of unity is a simple root if and only if $n < 2e$. In particular, the conjecture is true if $W = W(A_{n−1})$. The purpose of this article is to prove.

**Theorem 6.** (Main Theorem) Assume that $W$ is of classical type and that $K$ is a splitting field of $H_q(W)$. Then $H_q(W)$ is of finite representation type and not semisimple if and only if $q$ is a simple root of $P_W(x) = 0$.

We remark that the exceptional cases are settled recently by Miyachi \cite{[M]} under the assumption that the characteristic $l$ of the base field $K$ is not too small.

### 2. Reduction to Hecke algebras associated to irreducible Weyl groups

This is proved by using the complexity of modules. Let $A$ be a self–injective finite dimensional $K$–algebra, $M$ a finite dimensional $A$–module, $P^a \rightarrow M$ be its minimal projective resolution. Then the complexity $c_A(M)$ is the smallest integer $s \geq 0$ such that $\dim_K(P^t)/(t+1)^{s−1} (t = 0,1,\ldots)$ is bounded. The following lemma is fundamental.

**Lemma 7.** Suppose that $A$ is self–injective as above. Then

1. $c_A(M) = 0$ if and only if $M$ is a projective $A$–module.
2. $A$ is semisimple if and only if $c_A(M) = 0$ for all indecomposable $A$–modules $M$.
3. If $A$ is of finite representation type and not semisimple then $c_A(M) \leq 1$ for all indecomposable $A$–modules $M$ and the equality holds for some $M$.

**Proposition 8.** Let $S$ be a set of irreducible finite Weyl groups. If Uno’s conjecture is true for all $H_q(W)$ with $W \in S$ then the conjecture is true for $H_q(W_1 \times \cdots \times W_r)$ with $W_1,\ldots,W_r \in S$.

**Proof.** Write $W = W_1 \times \cdots \times W_r$. Then $H_q(W) = H_q(W_1) \otimes \cdots \otimes H_q(W_r)$ and $P_W(x) = P_{W_1}(x) \cdots P_{W_r}(x)$.

First assume that $q \neq 1$ is a simple root of $P_W(x) = 0$. Then $q$ is a simple root of $P_{W_i}(x) = 0$ and $P_{W_j}(q) \neq 0$ for all $j \neq i$. Then $H_q(W_j)$ for $j \neq i$ are all semisimple and $H_q(W_i)$ is of finite representation type and not semisimple. Thus $H_q(W)$ is of finite representation type and not semisimple.

Next assume that $q$ is a multiple root of $P_W(x) = 0$. If $q$ is a multiple root of $P_{W_i}(x) = 0$, for some $i$, then $H_q(W_i)$ is of infinite representation type by assumption. Thus $H_q(W)$ is of infinite representation type. If $q$ is a simple root of $P_{W_i}(x) = 0$ and $P_{W_j}(x) = 0$, for some $i \neq j$, then $H_q(W_i)$ and $H_q(W_j)$ are of finite
representation type and not semisimple. By Lemma 3, there exist an indecomposable $H_q(W_i)$-module $M_i$ and an indecomposable $H_q(W_j)$-module $M_j$ such that $c_{H_q(W_i)}(M_i) = 1$ and $c_{H_q(W_j)}(M_j) = 1$. Write $M = M_i \otimes M_j$. We shall prove that the complexity of $M$ as an indecomposable $H_q(W_i) \otimes H_q(W_j)$-module is equal to 2; if we use the fact that $c_{H_q(W_i \times W_j)}(M)$ is the growth rate of $\text{Ext}^*(M, M)$ then the Kunneth formula implies the result. In a more concrete manner, the proof of the assertion follows.

Let $P_i^t$ and $P_j^t$ be minimal projective resolutions of $M_i$ and $M_j$ respectively. Then $c_{H_q(W_i)}(M_i) = 1$ and $c_{H_q(W_j)}(M_j) = 1$ imply that there exists a constant $C$ such that $1 \leq \dim_K(P_i^t) \leq C$ and $1 \leq \dim_K(P_j^t) \leq C$ for all $t$. As $P^t = P_i^t \otimes P_j^t$ is a minimal projective resolution of $M$, we have

$$t + 1 \leq \dim_K(P^t) = \sum_{s=0}^{t} \dim_K(P_i^s) \dim_K(P_j^{t-s}) \leq C^2(t + 1).$$

Therefore, the complexity of $M$ is exactly 2. As a result, $H_q(W_i) \otimes H_q(W_j)$ is of infinite representation type by Lemma 2 and (3). Thus $H_q(W)$ is also of infinite representation type.

3. Type $B$ and Type $D$

To prove Theorem 6 it is enough to consider type $B$ and type $D$ by virtue of Theorem 5 and Proposition 8.

Let $q$ and $Q$ be invertible elements of $K$. The (two parameter) Hecke algebra $H_{q,Q}(B_n)$ of type $B_n$ is the unital associative $K$-algebra defined by generators $T_0, T_1, \ldots, T_{n-1}$ and relations

$$(T_0 + 1)(T_0 - Q) = 0, \quad (T_i + 1)(T_i - q) = 0 \quad (1 \leq i \leq n - 1),$$

$$T_0T_1T_0T_1 = T_1T_0T_1T_0, \quad T_{i+1}T_iT_{i+1}T_i = T_iT_{i+1}T_i \quad (1 \leq i \leq n - 2),$$

$$T_iT_j = T_jT_i \quad (0 \leq i < j - 1 \leq n - 2).$$

The following theorem, together with Theorem 5, will allow us to assume that $-Q$ is a power of $q$.

Theorem 9. [DJ Theorem 4.17] Suppose that $Q \neq -q^f$ for any $f \in \mathbb{Z}$. Then $H_{q,Q}(B_n)$ is Morita equivalent to $\bigoplus_{m=0}^n H_q(A_{m-1}) \otimes H_q(A_{n-m-1})$.

Corollary 10. Assume that $q$ is a primitive $e^{th}$ root of unity with $e \geq 2$ as above. If $Q \neq -q^f$ for any $f \in \mathbb{Z}$ then $H_{q,Q}(B_n)$ is of finite representation type if and only if $n < 2e$.

Proof. If $n \geq 2e$ then $H_q(A_{n-1})$ is of infinite representation type by Theorem 5. Thus, $H_{q,Q}(B_n)$ is of infinite representation type by Theorem 9.

If $n < 2e$ then one of $m$ and $n - m$ is smaller than $e$ for each $m$. Thus, one of $H_q(A_{m-1})$ and $H_q(A_{n-m-1})$ is semisimple and the other is of finite representation type for each $m$. Thus, $H_{q,Q}(B_n)$ is of finite representation type by Theorem 9.

Theorem 11. [AM Theorem 1.4] Suppose that $-Q = q^f$ $(0 \leq f < e)$ and $e \geq 3$. Then $H_{q,Q}(B_n)$ is of finite representation type if and only if

$$n < \min\{e, 2\min\{f, e-f\} + 4\}.$$  

It is also easy to prove that Theorem 11 is valid for $e = 2$; see [AM2].

Corollary 12. Uno’s conjecture is true if $W = W(B_n)$.
Assume that $T$ does not contain partitions. Further, $P$ is injective and we identify $H$ of the simple $D$.

Recall from [AM] that simple $H$ is of finite representation type if and only if $n < e$.

Let $D$ be a $H_q(D_n)$–module which affords the sign representation $T_i^D \mapsto -1$, for $0 \leq i \leq e - 1$. We denote its projective cover by $P$.

Recall from [AM] that simple $H_q(B_n)$–modules are indexed by Kleshchev bipartitions. Further, $\lambda = ((0),(1^e))$ is Kleshchev and if $e \geq 3$ is even then the projective cover $P^\lambda$ of the simple $H_q(B_n)$–module $D^\lambda$ has the property that $\text{Rad}P^\lambda/\text{Rad}^2P^\lambda$ contains $D^\lambda \oplus D^\lambda$. See the proof of [AM] Theorem 4.1] in p.12. $D^\lambda$ affords the representation $T_i^B \mapsto -1$ ($0 \leq i \leq n - 1$).

**Lemma 14.** Assume that $e$ is even. Then $P \simeq \text{Res}(P^\lambda)$.

**Proof.** First note that the characteristic $l$ of the base field is odd since $e$ is even. Let $D^\mu$ be the simple $H_q(B_n)$–module which affords the representation $T_0^D \mapsto 1$ and $T_0^B \mapsto -1$ ($1 \leq i \leq e - 1$). Then $\text{Ind}(D) = D^\lambda \oplus D^\mu$ since the left hand side is given by

$$
T_0^B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_i^B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1 \leq i \leq e - 1),
$$
and \( T_0 \) is diagonalizable because \( l \) is odd. Now the surjection \( P \to D \) induces surjective homomorphisms \( \text{Ind}(P) \to D^\lambda \) and \( \text{Ind}(P) \to D^\mu \). Hence these induce surjective homomorphisms \( \text{Ind}(P) \to P^\lambda \) and \( \text{Ind}(P) \to P^\mu \). We have that both \( P^\lambda \) and \( P^\mu \) are direct summands of \( \text{Ind}(P) \). On the other hand, Mackey’s formula implies that

\[
\text{Res}(\text{Ind}(P)) \simeq P \oplus \tau P,
\]

where \( \tau P \) is the indecomposable projective \( H_q(D_e) \)-module which is the twist of \( P \) by \( \tau \). As the twist of \( D \) is \( D \) itself, we have \( \text{Res}(\text{Ind}(P)) \simeq P \oplus P \). As \( \text{Res}(P^\lambda) \) and \( \text{Res}(P^\mu) \) are direct summands of \( \text{Res}(\text{Ind}(P)) \), we conclude that \( \text{Res}(P^\lambda) \) and \( \text{Res}(P^\mu) \) are isomorphic to \( P \).

Next lemma is obvious.

**Lemma 15.** Let \( A \) be a finite dimensional \( K \)-algebra and \( B \) a \( K \)-subalgebra such that \( B \) is a direct summand of \( A \) as a \((B,B)\)-bimodule. If \( A \) is of finite representation type then so is \( B \).

Recall that \( P_W(x) = (x^n - 1)\prod_{i=1}^{n-1} (x^{2^i} - 1) \) in type \( D_n \). Thus \( q \) is a simple root of \( P_W(x) = 0 \) if and only if either \( e \) is odd and \( e < n < 2e \) or \( e \) is even and \( \frac{n}{2} + 1 \leq n < e \).

**Proposition 16.** Uno’s conjecture is true if \( W = W(D_n) \).

**Proof.** First assume that \( e \) is odd. If \( n < 2e \) then Lemma 15 implies that \( H_q(D_n) \) is of finite representation type since \( H_q(B_n) \) is of finite representation type by Lemma 15(1). If \( n \geq 2e \) then it is enough to prove that \( H_q(D_{2e}) \) is of infinite representation type by Lemma 15. Using the same lemma again, we further know that it is enough to prove that \( H_q(A_{2e-1}) \) is of infinite representation type. However, this is nothing but the result of Theorem 8.

Next assume that \( e \) is even. If \( n < e \) then Lemma 15 implies that \( H_q(D_n) \) is of finite representation type since \( H_q(B_n) \) is of finite representation type by Lemma 15(2). If \( n \geq e \) then it is enough to prove that \( H_q(D_e) \) is of infinite representation type by Lemma 15. Note that \( W(D_2) = W(A_1) \times W(A_1) \) and the conjecture is true in this case by Proposition 8. Thus we may assume that \( e \geq 4 \). In particular, we have \( q \neq -1 \), and this implies that \( \text{Ext}^i(D,D) = 0 \); if we write

\[
T_i = \begin{pmatrix} -1 & a_i \\ 0 & -1 \end{pmatrix},
\]

then \( T_i - q \) is invertible and thus \( a_i = 0 \).

Let \( \mathcal{T} = \text{Res}(P^\lambda/\text{Rad}^3P^\lambda) \). Lemma 15(3) and \( \text{Ext}^i(D,D) = 0 \) imply that \( \mathcal{T} \) has Loewy length 3. Since \( \mathcal{T} \) has unique head \( D \) by Lemma 14, there exists a surjective homomorphism \( P \to \mathcal{T} \). Further, as \( \mathcal{T} \) contains \( D \oplus D \) as a \( H_q(D_e) \)-submodule, \( \text{Rad}^2P/\text{Rad}^3P \) contains \( D \oplus D \). On the other hand, \( \text{Ext}^i(D,D) = 0 \) implies that \( \text{Rad}P/\text{Rad}^2P \) does not contain \( D \). Hence we conclude that \( \text{End}_{H_q(D_e)}(P/\text{Rad}^3P) \) is isomorphic to \( K[X,Y]/(X^2, XY, Y^2) \), which is not isomorphic to any of the truncated polynomial rings \( K[X]/(X^N) \) (\( N = 1, 2, \ldots \)). As we assume that the base field is a splitting field of \( H_q(D_e) \), this implies that \( H_q(D_e) \) is of infinite representation type. \( \square \)
4. Appendix

In this section, we prove Theorem[2]. If the reader is familiar with the structure of the Sylow subgroups of exceptional Weyl groups then (s)he would not need this proof to know that Theorem[2] is true. In the proof below, we use standard facts about the structure of exceptional Weyl groups; they can be found in [Bo] or [Hu, 2.12]. First we consider irreducible Weyl groups.

Type $A_{n-1}$;
W has cyclic Sylow $l$–subgroups if and only if $n < 2l$, and this is equivalent to the condition that $l^2$ does not divide $|W| = n!$.

Type $B_n$;
W has cyclic Sylow $l$–subgroups if and only if either $l > 2$ and $n < 2l$ or $l = 2$ and $n < 2$, and this is equivalent to the condition that $l^2$ does not divide $|W| = 2^n n!$.

Type $D_n$;
W has cyclic Sylow $l$–subgroups if and only if either $l > 2$ and $n < 2l$ or $l = 2$ and $n < 2$, and this is equivalent to the condition that $l^2$ does not divide $|W| = 2^{n-1} n!$.

Type $F_4$;
As $|W(F_4)| = 2^7 \cdot 3^2$, we prove that the Sylow $l$–subgroups for $l = 2, 3$ are not cyclic. Let $\Delta(F_4)$ be the root system of type $F_4$. The long roots form a root system which is isomorphic to $\Delta(D_4)$. Let $\Gamma$ be the Dynkin automorphism group of the Dynkin diagram of type $D_4$. Then it is known that $W(F_4)$ is isomorphic to the semi–direct product of $W(D_4)$ and $\Gamma$.

Assume that $l = 2$. Since $W(D_4)$ contains $C_2 \times C_2$, the Sylow 2–subgroup of $W(F_4)$ cannot be cyclic.

Assume that $l = 3$. Since $\Gamma$ is isomorphic to the symmetric group of degree 3, we can choose $\sigma \in \Gamma$ of order 3. Let $P$ be a Sylow 3–subgroup of $W(F_4)$ containing $\sigma$. As the Sylow 3–subgroup of $W(D_4)$ is a cyclic group of order 3, we have $|W(D_4) \cap P| \leq 3$. On the other hand, as $(\sigma)$ is a Sylow 3–subgroup of $\Gamma$ and $|P| = 9$, both $W(D_4) \cap P$ and $P/W(D_4) \cap P$ are isomorphic to the cyclic group of order 3. Let $\tau$ be a generator of $W(D_4) \cap P$. Then we have the following split exact sequence.

\[ 1 \rightarrow \langle \tau \rangle \simeq C_3 \rightarrow P \rightarrow \langle \sigma \rangle \simeq C_3 \rightarrow 1. \]

As $\text{Aut}(C_3) \simeq C_2$, $\sigma$ acts on $\langle \tau \rangle$ trivially and $P \simeq C_3 \times C_3$.

Type $E_n$;
Recall that $|W(E_6)| = 2^7 \cdot 3^4 \cdot 5$, $|W(E_7)| = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$ and $|W(E_8)| = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$.

Assume that $l = 2$ or $l = 3$. Since $W(F_4) \subset W(E_n)$, The Sylow $l$–subgroup of $W(E_n)$ contains $C_l \times C_l$. Thus it cannot be a cyclic group.

Assume that $l = 5$. Let $Q$ be the root lattice of type $E_8$ with scalar product normalized to $(\alpha_i, \alpha_i) = 2$ for simple roots $\alpha_i$ ($1 \leq i \leq 8$). Then $q(x) = \frac{(x, x)}{2}$ (mod 2) defines a quadratic form on $Q/2Q \cong \mathbb{F}_2^8$. Note that if we choose simple roots as a basis, we can write down $q(x)$ explicitly, and the computation of its Witt decomposition shows that its Witt index is 4. Thus, by choosing a different basis, we may assume that $q(x) = \sum_{i=1}^{8} x_{2i-1}x_{2i}$. Let $O_8(2)$ be the orthogonal group associated to this form. Then it is known that there is an exact sequence

\[ 1 \rightarrow \{ \pm 1 \} \rightarrow W(E_8) \rightarrow O_8(2) \rightarrow 1. \]
Let \( q'(x) = x_1x_2 + x_3^2 + x_3x_4 + x_4^2 \) be a quadratic form on \( \mathbb{F}_4^4 \). If we write \( O_4^{-}(2) \) for the orthogonal group associated to this form, we know that its Sylow 5–subgroups are cyclic of order 5. Now explicit computation of Witt decomposition again shows that \( q' \oplus q' \) has Witt index 4. Thus \( O_8(2) \) contains \( C_5 \times C_5 \). As a result, the Sylow 5–subgroup of \( W(E_8) \) is isomorphic to \( C_5 \times C_5 \).

Type \( G_2 \);

This is the dihedral group of order 12 and its Sylow 2–subgroups are not cyclic.

Now let \( W \) be a general finite Weyl group. That is, \( W \) is a product of the groups listed above. Then \( W \) has a cyclic Sylow \( l \)–group if and only if at most one component of the product has a cyclic Sylow \( l \)–group and all the other components have trivial Sylow \( l \)–groups. This is equivalent to the condition that \( l^2 \) does not divide \( |W| \).

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