Initial and Boundary Value Problems for the Caputo Fractional Self-Adjoint Difference Equations

Kevin Ahrendt
University of Nebraska-Lincoln
Department of Mathematics
Lincoln, NE 68588-0130 USA
kahrendt@huskers.unl.edu

Lydia DeWolf
Department of Mathematics
Union College
Jackson, TN, 38305 USA
lydia.dewolf@my.uu.edu

Liam Mazurowski
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213 USA
lmazurow@andrew.cmu.edu

Kelsey Mitchell
Department of Mathematics
Buena Vista University
Storm Lake, IA 50588 USA
mitckel@bv.eu.edu

Tim Rolling
Department of Mathematics
University of Nebraska-Lincoln
Lincoln, NE 68588-0130 USA
trolling2@unl.edu

Dominic Veconi
Department of Mathematics
Hamilton College
Clinton, NY 13323 USA
dveconi@hamilton.edu

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Abstract

In this paper we develop the theory of initial and boundary value problems for the self-adjoint nabla fractional difference equation containing a Caputo fractional nabla difference that is given by

$$\nabla[p(t+1)f_a^\nu x(t+1)] + q(t)x(t) = h(t),$$

where $0 < \nu \leq 1$. We give an introduction to the nabla fractional calculus with Caputo fractional differences. We investigate properties of the specific self-adjoint nabla fractional difference equation given above. We prove existence and uniqueness theorems for both initial and boundary value problems under appropriate conditions. We introduce the definition of a Cauchy function which allows us to give a variation of constants formula for solving initial value problems. We then show that this Cauchy function is important in finding a Green’s function for a boundary value problem with Sturm-Liouville type boundary conditions. Several inequalities concerning a certain Green’s function are derived. These results are important in using fixed point theorems for proving the existence of solutions to boundary value problems for nonlinear fractional equations related to our linear self-adjoint equation.

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1 Preliminary definitions and theorems

In this section we will develop the basic definitions and theorems needed for our results. For an introduction to whole order difference calculus, see Kelley and Peterson [6]. For more information in the fractional results using the backwards difference operator, see Hein et al. [5] and Ahrendt et al. [1].

Definition 1. [1] Let $a \in \mathbb{R}$. Then the set $\mathbb{N}_a$ is given by $\{a, a + 1, a + 2, \ldots\}$. Furthermore, if $b \in \mathbb{N}_a$, then $\mathbb{N}_b$ is given by $\{a, a + 1, \ldots, b - 1, b\}$.

Definition 2. [1] Let $f : \mathbb{N}_a \to \mathbb{R}$. Then the nabla difference of $f$ is defined by

$$(\nabla f)(t) := f(t) - f(t - 1),$$

for $t \in \mathbb{N}_{a+1}$. For convenience, we will use the notation $\nabla f(t) := (\nabla f)(t)$. For $N \in \mathbb{N}$, we have that the $N$th order fractional difference is recursively defined as

$$\nabla^N f(t) := \nabla(\nabla^{N-1} f(t)),$$

for $t \in \mathbb{N}_{a+N}$.

Definition 3. [1] Let $f : \mathbb{N}_a \to \mathbb{R}$ and let $c, d \in \mathbb{N}_a$. Then the definite nabla integral of $f$ from $c$ to $d$ is defined by

$$\int_c^d f(s) \nabla s := \begin{cases} \sum_{s=c+1}^d f(s), & c < d, \\ 0, & d \leq c. \end{cases}$$
Definition 4. Let $t \in \mathbb{R}$ and let $n \in \mathbb{Z}^+$. Then the rising function is defined by
\[ t^n := (t)(t+1) \cdots (t+n-1) = \frac{\Gamma(t+n)}{\Gamma(t)}, \]
where $\Gamma$ is the gamma function. For $\nu \in \mathbb{R}$, the generalized rising function is then defined by
\[ t^\nu := \frac{\Gamma(t+\nu)}{\Gamma(t)} \]
for $t$ and $\nu$ such that $t + \nu \notin \{\ldots, -2, -1, 0\}$. If $t$ is a non-positive integer and $t + \nu$ is not a non-positive integer then we take by convention $t^\nu = 0$.

Theorem 5 (Fundamental Theorem of Nabla Calculus). Assume the function $f : \mathbb{N}_a \to \mathbb{R}$ and let $F$ be a nabla antidifference of $f$ on $\mathbb{N}_a$, then
\[ \int_a^b f(t) \nabla t = F(t) \bigg|_a^b := F(b) - F(a). \]

The following definitions extend the nabla difference and nabla integral to fractional value orders.

Definition 6. Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$, $\nu \in \mathbb{R}$. The $\nu$th order nabla fractional sum of $f$ is defined as
\[ \nabla_{-\nu}^a f(t) := \sum_{s=a+1}^t \frac{(t-s+1)^{-\nu}}{\Gamma(\nu)} f(s), \]
for $t \in \mathbb{N}_a$.

Definition 7. Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu \in \mathbb{R}$, $\nu > 0$, and $N = \lceil \nu \rceil$. Then $\nu$th order nabla fractional difference of $f$ is defined as
\[ \nabla_{\nu}^a f(t) := \nabla^N \nabla_{-N-\nu}^a f(t), \]
for $t \in \mathbb{N}_{a+N}$.

The next theorem gives results for composing nabla fractional sums and differences in certain cases.

Theorem 8 (Composition Rules). Let $\mu, \nu > 0$ and let $f : \mathbb{N}_a \to \mathbb{R}$. Set $N = \lceil \nu \rceil$. Then
\[ \nabla_{-\nu}^a \nabla_{-\mu}^a f(t) = \nabla_{-\nu-\mu}^a f(t), \quad t \in \mathbb{N}_a, \]
and
\[ \nabla_{\nu}^a \nabla_{\mu}^a f(t) = \nabla_{\nu-\mu}^a f(t), \quad t \in \mathbb{N}_{a+N}. \]

While we gave the traditional definition of a nabla fractional difference above, we will focus on the Caputo nabla fractional difference for the rest of the paper and only appeal to the previous definition when needed. The following definition has been adapted from Anastassiou in [2].
**Definition 9.** Let $f : \mathbb{N}_a - N + 1 \to \mathbb{R}, \nu > 0, \nu \in \mathbb{R}, N = [\nu]$. The $\nu$th order Caputo nabla fractional difference is defined as

$$\nabla^\nu_a f(t) := \nabla_a^{(N-\nu)}(\nabla^N f(t)),$$

for $t \in \mathbb{N}_a$. Note that the Caputo difference operator is a linear operator.

**Theorem 10** (Discrete Whole-Order Taylor’s Formula). [3] Fix $N \in \mathbb{N}_1$ and let $f : \mathbb{N}_a - N + 1 \to \mathbb{R}$. Then

$$f(t) = \sum_{k=0}^{N-1} \frac{(t-a)^k}{k!} \nabla^k f(a) + \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^t \frac{(t-s+1)^{N-1}}{(N-1)!} \nabla^N f(s),$$

for $t \in \mathbb{N}_a$.

The following theorem is adapted from Anastassiou in [2] where we use our definition of the Caputo nabla fractional difference.

**Theorem 11** (Caputo Discrete Taylor’s Theorem). [2] Let $\nu \in \mathbb{R}, \nu > 0, N = [\nu], \text{ and } f : \mathbb{N}_a - N + 1 \to \mathbb{R}$. Then for all $t \in \mathbb{N}_a$, the representation holds

$$f(t) = \sum_{k=0}^{N-1} \frac{(t-a)^k}{k!} \nabla^k f(a) + \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^t \frac{(t-s+1)^{N-1}}{(N-1)!} \nabla^N f(s),$$

for $t \in \mathbb{N}_a$. By Definition 6,

$$\nabla^N a \nabla^N f(t) = \frac{1}{(N-1)!} \int_a^t (t-s+1)^{N-1} \nabla^N f(s) \nabla s,$$

Similarly,

$$\nabla^\nu_a \nabla^\nu_a f(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-s+1)^{\nu-1} \nabla^\nu_a f(s) \nabla s.$$

So from Theorem 11,

$$f(t) = \sum_{k=0}^{N-1} \frac{(t-a)^k}{k!} \nabla^k f(a) + \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^t \frac{(t-s+1)^{\nu-1}}{(N-1)!} \nabla^\nu_a f(s),$$

for $t \in \mathbb{N}_a$, proving the result.
2 Nabla fractional initial value problems

We are interested in solutions to the nabla fractional initial value problem (IVP)

\[
\begin{aligned}
\nabla^\nu_a f(t) &= h(t), \quad t \in \mathbb{N}_{a+1}, \\
\nabla^k f(a) &= c_k, \quad 0 \leq k \leq N - 1,
\end{aligned}
\]  

(1)

where \(a, \nu \in \mathbb{R}, \nu > 0, \ N = \lceil \nu \rceil, \ c_k \in \mathbb{R} \) for \(0 \leq k \leq N-1\), and \(f : \mathbb{N}_{a-N+1} \to \mathbb{R}\).

**Theorem 12.** The solution to the IVP (1) is uniquely determined by

\[
f(t) = \sum_{k=0}^{N-1} \frac{(t-a)^k}{k!} c_k + \frac{1}{\Gamma(\nu)} \sum_{\tau=a+1}^{t} (t-\tau+1)^{\nu-1} h(\tau),
\]

for \(t \in \mathbb{N}_{a+1}\).

**Proof.** Let \(f : \mathbb{N}_{a-N+1} \to \mathbb{R}\) satisfy

\[
\nabla^k f(a) = c_k,
\]

for \(0 \leq k \leq N - 1\). Note that this uniquely determines the value of \(f(t)\) for \(a-N+1 \leq t \leq a\). For \(t \in \mathbb{N}_{a+1}\), let \(f(t)\) satisfy

\[
\nabla^N f(t) = h(t) - \sum_{s=a+1}^{t-1} \frac{(t-s+1)^{N-\nu-1}}{\Gamma(N-\nu)} \nabla^N f(s).
\]

This recursive definition uniquely determines \(f(t+1)\) from the values of \(f(a-N+1), ..., f(t)\), so the function is uniquely defined for all \(t \in \mathbb{N}_{a-N+1}\). So for any \(t \in \mathbb{N}_{a+1}\),

\[
\nabla^N f(t) + \sum_{s=a+1}^{t-1} \frac{(t-s+1)^{N-\nu-1}}{\Gamma(N-\nu)} \nabla^N f(s) = h(t).
\]

Equivalently,

\[
\frac{\Gamma(\nu)}{\Gamma(1)\Gamma(\nu)} \nabla^N f(t) + \sum_{s=a+1}^{t-1} \frac{(t-s+1)^{N-\nu-1}}{\Gamma(N-\nu)} \nabla^N f(s) = \frac{(1)^{\nu-1}}{\Gamma(\nu)} \nabla^N f(t) + \sum_{s=a+1}^{t-1} \frac{(t-s+1)^{N-\nu-1}}{\Gamma(N-\nu)} \nabla^N f(s)
\]

\[
= \sum_{s=a+1}^{t} \frac{(t-s+1)^{N-\nu-1}}{\Gamma(N-\nu)} \nabla^N f(s)
\]

\[
= \nabla^{\nu}(N-\nu) \nabla^N f(t)
\]

\[
= \nabla^\nu_a f(t)
\]

\[
= h(t).
\]
Therefore, \( f(t) \) solves the IVP \( \text{[1]} \). Conversely, if we suppose that there is a function \( f : \mathbb{N}_{a-N+1} \rightarrow \mathbb{R} \) that satisfies the IVP, reversing the above algebraic steps would lead to the same recursive definition. Therefore the solution to the IVP is uniquely defined. By the Caputo Discrete Taylor’s Theorem, \( f(t) \) must satisfy the representation

\[
 f(t) = \sum_{k=0}^{N-1} \frac{(t-a)^k}{k!} \nabla^k f(a) + \frac{1}{\Gamma(\nu)} \sum_{\tau=a+1}^{t} (t-\tau+1)^{-\nu-1} \nabla^\nu_{a+\tau} f(\tau)
\]

for \( t \in \mathbb{N}_{a+1} \).

**Example 13.** Solve the IVP

\[
\begin{align*}
\nabla_{0,7}^a f(t) &= t, \quad t \in \mathbb{N}_1, \\
f(0) &= 2.
\end{align*}
\]

Applying the variation of constants formula yields the following expression for \( f(t) \).

\[
f(t) = \sum_{k=0}^{0} \frac{t^k}{k!} 2 + \nabla_{0}^{-0.7} h(t)
\]

\[
= \sum_{s=1}^{t} \frac{(t-s+1)^{-0.3}}{\Gamma(0.7)} s^2 + 2
\]

After summing by parts, we have

\[
f(t) = \frac{t^{1.7}}{\Gamma(2.7)} + 2.
\]

### 3 General properties of the fractional self-adjoint nabla difference equation

For development of these properties in the continuous setting, see Kelley and Peterson \([7]\).

Let \( D_a := \{ x : \mathbb{N}_a \rightarrow \mathbb{R} \} \) and let the self-adjoint fractional operator \( L_a \) be defined by

\[
(L_a x)(t) := \nabla[p(t+1)\nabla^\nu_{a+\tau} x(t+1)] + q(t) x(t), \quad t \in \mathbb{N}_{a+1},
\]

where \( x \in D_a, 0 < \nu < 1, p : \mathbb{N}_{a+1} \rightarrow (0, \infty) \) and \( q : \mathbb{N}_{a+1} \rightarrow \mathbb{R} \). Note that while the operator is for values of \( t \in \mathbb{N}_{a+1} \), the function \( x \) is defined on \( \mathbb{N}_a \). Note that \( L_a \) is a linear operator.
Theorem 14 (Existence and Uniqueness for Self-Adjoint IVPs). Let \( A, B \in \mathbb{R} \), and let \( h : \mathbb{N}_{a+1} \to \mathbb{R} \). The IVP
\[
\begin{align*}
L_a x(t) &= h(t), \quad t \in \mathbb{N}_{a+1}, \\
x(a) &= A, \\
\nabla x(a+1) &= B,
\end{align*}
\]
has a unique solution \( x : \mathbb{N}_a \to \mathbb{R} \).

Proof. Let \( x : \mathbb{N}_a \to \mathbb{R} \) satisfy the initial conditions
\[
\begin{align*}
x(a) &= A, \\
x(a+1) &= A+B.
\end{align*}
\]
Furthermore, for \( t \in \mathbb{N}_{a+1} \), let \( x(t+1) \) satisfy the recursive equation
\[
x(t+1) = x(t) - \sum_{s=a+1}^{t} \frac{(t-s+2)^{-\nu}}{\Gamma(1-\nu)} \nabla x(s) \\
+ \frac{1}{p(t+1)} \left[ h(t) - q(t)x(t) + p(t) \sum_{\tau=a+1}^{t} \frac{(t-\tau+1)^{-\nu}}{\Gamma(1-\nu)} \nabla x(\tau) \right].
\]
Note that as defined, \( x(t+1) \) is uniquely determined from the values of \( x(a) \), \( x(a+1) \), \ldots, \( f(t-1) \), \( f(t) \), for \( t \in \mathbb{N}_{a+1} \). Furthermore,
\[
\begin{align*}
\nabla x(t+1) + \sum_{s=a+1}^{t} \frac{(t-s+2)^{-\nu}}{\Gamma(1-\nu)} \nabla x(s) \\
= \frac{1^{-\nu}}{\Gamma(1-\nu)} \nabla x(t+1) + \sum_{s=a+1}^{t} \frac{(t-s+2)^{-\nu}}{\Gamma(1-\nu)} \nabla x(s) \\
= \sum_{s=a+1}^{t+1} \frac{(t-s+2)^{-\nu}}{\Gamma(1-\nu)} \nabla x(s) \\
= \nabla_a^{(1-\nu)} \nabla x(t+1) \\
= \nabla_a^{\nu} x(t+1).
\end{align*}
\]
So we have that
\[
\nabla_a^{\nu} x(t+1) = \frac{1}{p(t+1)} \left[ h(t) - q(t)x(t) + p(t) \sum_{\tau=a+1}^{t} \frac{(t-\tau+1)^{-\nu}}{\Gamma(1-\nu)} \nabla x(\tau) \right].
\]
Then
\[ p(t + 1)\nabla^\nu_{a+1} x(t + 1) \]
\[ = h(t) - q(t)x(t) + p(t) \sum_{\tau=a+1}^{t} \frac{(t - \tau + 1)^{-\nu}}{\Gamma(1 - \nu)} \nabla x(\tau) \]
\[ = h(t) - q(t)x(t) + p(t) \nabla_{a+1}^{-1+\nu} x(t) \]
\[ = h(t) - q(t)x(t) + p(t)\nabla^\nu_{a+1} x(t), \]
which implies
\[ \nabla[p(t + 1)\nabla^\nu_{a+1} x(t + 1)] = h(t) - q(t)x(t). \]

Thus, by rearranging, we have that
\[ \nabla[p(t + 1)\nabla^\nu_{a+1} x(t + 1)] + q(t)x(t) = h(t). \]

Therefore, for any value of \( t \in \mathbb{N}_{a+1}, \) \( x(t) \) satisfies the IVP, So a solution exists. Reversing the preceding algebraic steps shows that if some function \( y(t) \) is a solution to the IVP, it must be the same solution as our original \( x(t) \). Therefore a unique solution exists. \( \square \)

The following lemma shows that initial conditions behave nicely when dealing with the Caputna nabla fractional difference.

**Lemma 15.** Let \( 0 < \nu < 1 \) and let \( x : \mathbb{N}_a \to \mathbb{R} \). Then
\[ \nabla^\nu_{a+1} x(a + 1) = \nabla x(a + 1) \]
for \( t \in \mathbb{N}_{a+1} \).

**Proof.** Let \( 0 < \nu < 1 \). Then by definition of the Caputna difference,
\[ \nabla^\nu_{a+1} x(a + 1) = \nabla_{a+1}^{-\nu} \nabla x(a + 1) \]
\[ = \sum_{\tau=a+1}^{a+1} \frac{(a + 1 - \tau + 1)^{-\nu}}{\Gamma(1 - \nu)} \nabla x(\tau) \]
\[ = \nabla x(a + 1), \]
for \( t \in \mathbb{N}_{a+1} \). \( \square \)

The next theorem and corollary show that the self-adjoint fractional nabla difference equation behaves very similar to a second order difference equation.

**Theorem 16 (General Solution of the Homogeneous Equation).** Suppose \( x_1, x_2 : \mathbb{N}_a \to \mathbb{R} \) are linearly independent solutions to \( L_a x(t) = 0 \). Then the general solution to \( L_a x(t) = 0 \) is given by
\[ x(t) = c_1 x_1(t) + c_2 x_2(t), \]
for \( t \in \mathbb{N}_{a+1}, \) where \( c_1, c_2 \in \mathbb{R} \) are arbitrary constants.
Proof. Let \( x_1, x_2 : \mathbb{N}_a \to \mathbb{R} \) be linearly independent solutions of \( L_a x(t) = 0 \). Then there exist constants \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \) for which \( x_1, x_2 \) are the unique solutions to the IVPs

\[
\begin{aligned}
L x_1 &= 0, \quad t \in \mathbb{N}_{a+1}, \\
x_1(a) &= \alpha, \\
\nabla x_1(a + 1) &= \beta,
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
L x_2 &= 0, \quad t \in \mathbb{N}_{a+1}, \\
x_2(a) &= \gamma, \\
\nabla x_2(a + 1) &= \delta.
\end{aligned}
\]

Since \( L_a \) is a linear operator, we have for any \( c_1, c_2 \in \mathbb{R} \)
\[
L_a[c_1 x_1(t) + c_2 x_2(t)] = c_1 L_a x_1(t) + c_2 L_a x_2(t) = 0,
\]
so \( x(t) = c_1 x_1(t) + c_2 x_2(t) \) solves \( L_a x(t) = 0 \). Conversely, suppose \( x : \mathbb{N}_a \to \mathbb{R} \) solves \( L_a x(t) = 0 \). Note that \( x(t) \) solves the IVP

\[
\begin{aligned}
L x &= 0, \quad t \in \mathbb{N}_{a+1}, \\
x(a) &= A, \\
\nabla x(a + 1) &= B,
\end{aligned}
\]

for some \( A, B \in \mathbb{R} \). We show that the matrix equation

\[
\begin{bmatrix}
x_1(a) & x_2(a) \\
\nabla x_1(a + 1) & \nabla x_2(a + 1)
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
A \\
B
\end{bmatrix} \tag{2}
\]

has a unique solution for \( c_1, c_2 \). The above matrix equation can be equivalently expressed as

\[
\begin{bmatrix}
\alpha & \gamma \\
\beta & \delta
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
A \\
B
\end{bmatrix}.
\]

Suppose by way of contradiction that

\[
\begin{bmatrix}
\alpha & \gamma \\
\beta & \delta
\end{bmatrix}
= 0.
\]

Then, without loss of generality, there exists a constant \( k \in \mathbb{R} \) for which \( \alpha = k \gamma \) and \( \beta = k \delta \). Then \( x_1(a) = \alpha = k \gamma = k x_2(a) \), and \( \nabla x_1(a + 1) = \beta = k \delta = k \nabla x_2(a + 1) \). Since \( k x_2(t) \) solves \( L_a x(t) = 0 \), we have that \( x_1(t) \) and \( k x_2(t) \) solve the same IVP. By uniqueness, \( x_1(t) = k x_2(t) \). But then \( x_1(t) \) and \( x_2(t) \) are linearly dependent, which is a contradiction. Therefore, the matrix equation \([2]\) must have a unique solution, so \( x(t) \) and \( c_1 x_1(t) + c_2 x_2(t) \) solve the same IVP, and so by uniqueness in Theorem 14 every solution to \( L_a x(t) = 0 \) can be uniquely expressed as a linear combination of \( x_1(t) \) and \( x_2(t) \). \( \square \)

**Corollary 17** (General Solution of the Nonhomogeneous Equation). Suppose \( x_1, x_2 : \mathbb{N}_a \to \mathbb{R} \) are linearly independent solutions of \( L_a x(t) = 0 \) and \( y_0 : \mathbb{N}_a \to \mathbb{R} \) is a particular solution to \( L_a x(t) = h(t) \) for some \( h : \mathbb{N}_{a+1} \to \mathbb{R} \). Then the general solution of \( L_a x(t) = h(t) \) is given by

\[
x(t) = c_1 x_1(t) + c_2 x_2(t) + y_0(t),
\]

for \( t \in \mathbb{N}_{a+1} \) and where \( c_1, c_2 \in \mathbb{R} \) are arbitrary constants.
Proof. Since $L_a$ is a linear operator, one can show that $x(t) = c_1 x_1(t) + c_2 x_2(t) + y_0(t)$ solves $L_a x(t) = h(t)$ for any $c_1, c_2 \in \mathbb{R}$ in a similar way as in Theorem 16. Conversely, suppose $x : \mathbb{N}_a \to \mathbb{R}$ solves $L_a x(t) = h(t)$. Again note that $x(t)$ solves the IVP

$$\begin{cases}
L x = h(t), & t \in \mathbb{N}_{a+1}, \\
x(a) = A, \\
x(a + 1) = B,
\end{cases}$$

for some $A, B \in \mathbb{R}$. Since $y_0(t)$ is a particular solution of $L_a x(t) = h(t)$, there exist unique constants $c_1, c_2 \in \mathbb{R}$ for which $x_h(t) = c_1 x_1(t) + c_2 x_2(t)$ solves the IVP

$$\begin{cases}
L x_h = 0, \\
x_h(a) = A - y_0(a), \\
x_h(a + 1) = B - y_0(a + 1),
\end{cases}$$

for $t \in \mathbb{N}_{a+1}$. Observe that $x_h(t) + y_0(t)$ satisfies $L_a x(t) = h(t)$. Further,

$$x_h(a) + y_0(a) = A - y_0(a) + y_0(a) = A,$$

and

$$x_h(a + 1) + y_0(a + 1) = B - y_0(a + 1) + y_0(a + 1) = B.$$

Therefore $x(t)$ and $x_h(t) + y_0(t)$ solve the same IVP. Then by uniqueness of IVPs, $x(t) = c_1 x_1(t) + c_2 x_2(t) + y_0(t)$, and thus any solution to $L_a x(t) = h(t)$ may be written in this form. $lacksquare$

## 4 Initial value problems for the fractional self-adjoint equation

In this section we develop techniques to solve initial value problems for the fractional self-adjoint operator involving the Caputo difference. See Brackins [4] for a similar development using the Riemann-Liouville definition of a fractional difference.

**Definition 18.** The Cauchy function for $L_a x(t)$ is the function $x : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ that satisfies the IVP

$$\begin{cases}
L_s x(t, s) = 0, & t \in \mathbb{N}_{s+1}, \\
x(s, s) = 0, \\
\nabla x(s + 1, s) = \frac{1}{p(s + 1)},
\end{cases}$$

for any fixed $s \in \mathbb{N}_a$.

**Remark 19.** Note by Lemma 15, the IVP (3) is equivalent to the IVP

$$\begin{cases}
L_s x(t, s) = 0, & t \in \mathbb{N}_{s+1} \\
x(s, s) = 0, \\
\nabla' x(s + 1, s) = \frac{1}{p(s + 1)}
\end{cases}$$
for any fixed \( s \in \mathbb{N}_a \).

**Theorem 20** (Variation of Constants). Let \( h : \mathbb{N}_{a+1} \to \mathbb{R} \). Then solution to the IVP

\[
\begin{aligned}
L_a y(t) &= h(t), \quad t \in \mathbb{N}_{a+1}, \\
y(a) &= 0, \\
\nabla y(a + 1) &= 0,
\end{aligned}
\]

is given by

\[
y(t) = \int_a^t x(t, s) h(s) \nabla s,
\]

where \( x(t, s) \) is the Cauchy function for the homogenous equation and where \( y : \mathbb{N}_a \to \mathbb{R} \).

**Proof.** Note that

\[
y(a) = \sum_{s=a+1}^a x(a, s) h(s) = 0,
\]

so the first initial condition holds. By the definition of the Caputo difference,

\[
\nabla^{\nu}_{s+1} x(t + 1, s) = \nabla^{-(1-\nu)}_{s} \nabla x(t + 1, s) = \sum_{\tau=s+1}^{t+1} \frac{(t+1-\tau+1)^{-\nu}}{\Gamma(1-\nu)} \nabla x(\tau, s).
\]

Then

\[
\nabla[p(t+1)\nabla^{\nu}_{s+1} x(t + 1, s)]
\]

\[
= p(t+1) \sum_{\tau=s+1}^{t+1} \frac{(t-\tau+2)^{-\nu}}{\Gamma(1-\nu)} \nabla x(\tau, s) - p(t) \sum_{\tau=s+1}^{t} \frac{(t-\tau+1)^{-\nu}}{\Gamma(1-\nu)} \nabla x(\tau, s)
\]

\[
= p(t+1) \sum_{\tau=s+1}^{t+1} \frac{(t-\tau+2)^{-\nu}}{\Gamma(1-\nu)} \nabla x(\tau, s) - p(t) \sum_{\tau=s+1}^{t+1} \frac{(t-\tau+1)^{-\nu}}{\Gamma(1-\nu)} \nabla x(\tau, s)
\]

\[
+ p(t) \frac{(t-t+1+1)^{-\nu}}{\Gamma(1-\nu)} \nabla x(t + 1, s)
\]

\[
= \sum_{\tau=s+1}^{t+1} \frac{(t-\tau+2)^{-\nu}}{\Gamma(1-\nu)} p(t+1) - (t-\tau+1)^{-\nu} p(t) \nabla x(\tau, s).
\]

So we have that

\[
\nabla[p(t+1)\nabla^{\nu}_{s+1} x(t + 1, s)]
\]

\[
= \sum_{\tau=s+1}^{t+1} \frac{(t-\tau+2)^{-\nu}}{\Gamma(1-\nu)} p(t+1) - (t-\tau+1)^{-\nu} p(t) \nabla x(\tau, s). \quad (4)
\]
Now we consider $y(t)$. Note that

$$\nabla y(t) = \sum_{s=a+1}^{t} x(t, s)h(s) - \sum_{s=a+1}^{t-1} x(t - 1, s)h(s)$$

$$= \sum_{s=a+1}^{t-1} x(t, s)h(s) + x(t, t)h(t) - \sum_{s=a+1}^{t-1} x(t - 1, s)h(s)$$

$$= \sum_{s=a+1}^{t-1} \nabla x(t, s)h(s).$$

From here, observe that

$$\nabla y(a + 1) = \sum_{s=a+1}^{a+1} x(t, s)h(s) = 0,$$

so the second initial condition holds. Then by the definition of the Caputo difference,

$$\nabla_{\alpha}^\nu y(t) = \nabla_{\alpha}^{-(1-\nu)} \nabla y(t)$$

$$= \sum_{\tau=a+1}^{t} \frac{(t - \tau + 1)^{-\nu}}{\Gamma(1 - \nu)} \nabla y(\tau)$$

$$= \sum_{\tau=a+1}^{t} \frac{(t - \tau + 1)^{-\nu}}{\Gamma(1 - \nu)} \sum_{s=a+1}^{\tau-1} \nabla x(\tau, s)h(s)$$

$$= \sum_{\tau=a+1}^{t} \sum_{s=a+1}^{\tau-1} \frac{(t - \tau + 1)^{-\nu}}{\Gamma(1 - \nu)} \nabla x(\tau, s)h(s).$$
Theorem 21 (Variation of Constants with Non-Zero Initial Conditions). The

Then from (4)

\[ \nabla[p(t+1) \nabla_{\alpha}^\nu y(t+1)] = p(t+1) \sum_{\tau=a+1}^{t+1} \sum_{s=a+1}^{\tau-1} \frac{(t-\tau+2-\nu)}{\Gamma(1-\nu)} \nabla x(\tau, s) h(s) \]

\[ - p(t) \sum_{\tau=a+1}^{t} \sum_{s=a+1}^{\tau-1} \frac{(t-\tau+1-\nu)}{\Gamma(1-\nu)} \nabla x(\tau, s) h(s) \]

\[ = \sum_{\tau=a+1}^{t+1} \sum_{s=a+1}^{\tau-1} \frac{(t-\tau+2-\nu)p(t+1) - (t-\tau+1-\nu)p(t)}{\Gamma(1-\nu)} \nabla x(\tau, s) h(s) \]

\[ = p(t+1) \nabla x(t+1, t) h(t) + \sum_{s=a+1}^{t-1} \nabla[p(t+1) \nabla_{\alpha}^\nu x(t+1, s)] h(s) \]

\[ = h(t) + \sum_{s=a+1}^{t-1} \nabla[p(t+1) \nabla_{\alpha}^\nu x(t+1, s)] h(s). \]

Therefore

\[ L_{\alpha}y(t) = \nabla[p(t+1) \nabla_{\alpha}^\nu y(t+1)] + q(t) y(t) \]

\[ = h(t) + \sum_{s=a+1}^{t-1} \nabla[p(t+1) \nabla_{\alpha}^\nu x(t+1, s)] h(s) \]

\[ + \sum_{s=a+1}^{t-1} q(t) x(t, s) h(s) + q(t) x(t, t) h(t) \]

\[ = h(t) + \sum_{s=a+1}^{t-1} \left[ \nabla[p(t+1) \nabla_{\alpha}^\nu x(t+1, s)] + q(t) x(t, s) \right] h(s) \]

\[ = h(t) + \sum_{s=a+1}^{t-1} L_{\alpha} x(t, s) h(s) \]

\[ = h(t). \]

Thus \( y : N_\alpha \rightarrow \mathbb{R} \) solves the IVP for \( t \in N_{a+1} \). ■

**Theorem 21** (Variation of Constants with Non-Zero Initial Conditions). The
solution to the IVP

\[
\begin{align*}
L_a y(t) &= h(t), \quad t \in \mathbb{N}_{a+1}, \\
y(a) &= A, \\
\nabla y(a + 1) &= B,
\end{align*}
\]

where \(A, B \in \mathbb{R}\) are arbitrary constants, is given by

\[
y(t) = y_0(t) + \int_a^t x(t, s) h(s) \nabla s,
\]

where \(y_0(t)\) solves the IVP

\[
\begin{align*}
L_a y_0(t) &= 0, \quad t \in \mathbb{N}_{a+1}, \\
y_0(a) &= A, \\
\nabla y_0(a + 1) &= B.
\end{align*}
\]

Proof. The proof follows from Theorem 20 by linearity. ■

Example 22. Find the Cauchy function for

\[
\nabla [p(t + 1) \nabla^\nu_{a+1} y(t + 1)] = 0, \quad t \in \mathbb{N}_{a+1}.
\]

Consider \(\nabla [p(t + 1) \nabla^\nu_{a+1} x(t + 1, s)] = 0\). Integrating both sides from \(s\) to \(t\) and applying the Fundamental Theorem of Nabla Calculus along with the second initial condition from Remark 19 yields

\[
p(t + 1) \nabla^\nu_{a+1} x(t + 1, s) - p(s + 1) \nabla^\nu_{a+1} x(s + 1, s) = 0
\]

\[
p(t + 1) \nabla^\nu_{a+1} x(t + 1, s) - 1 = 0
\]

\[
\nabla^\nu_{a+1} x(t + 1, s) = \frac{1}{p(t + 1)}.
\]

By the definition of the Caputo difference, this is equivalent to

\[
\nabla^{1-\nu}_{s} \nabla x(t + 1, s) = \frac{1}{p(t + 1)}
\]

\[
\nabla^{1-\nu}_{s} \nabla^{1-\nu}_{s} \nabla x(t + 1, s) = \nabla^{1-\nu}_{s} \frac{1}{p(t + 1)}
\]

\[
\nabla x(t + 1, s) = \nabla^{1-\nu}_{s} \frac{1}{p(t + 1)}.
\]

After applying a composition rule from Theorem 8, replacing \(t + 1\) with \(t\) yields

\[
\nabla x(t, s) = \nabla^{1-\nu}_{s} \frac{1}{p(t)}
\]
Applying the Fundamental Theorem after integrating both sides from \( s \) to \( t \) and applying the first initial condition from Remark 19 yields

\[
x(t, s) - x(s, s) = \int_s^t \nabla_s^{1-\nu} \frac{1}{p(\tau)} \nabla_\tau \\
x(t, s) = \int_s^t \nabla \nabla_s^{-\nu} \frac{1}{p(\tau)} \nabla_\tau \\
= \left[ \nabla_s^{-\nu} \frac{1}{p(\tau)} \right]_{\tau=s}^{\tau=t} \\
= \nabla_s^{-\nu} \frac{1}{p(t)} - \nabla_s^{-\nu} \frac{1}{p(s)} \\
= \nabla_s^{-\nu} \frac{1}{p(t)}.
\]

Therefore the Cauchy function is

\[
x(t, s) = \nabla_s^{-\nu} \frac{1}{p(t)} = \sum_{\tau=s+1}^t \frac{(t - \tau + 1)^{\nu - 1}}{\Gamma(\nu) \Gamma(\nu + 1)} \left( \frac{1}{p(\tau)} \right).
\]

**Example 23.** Find the Cauchy function for

\[
\nabla \nabla_s^{\nu} y(t + 1) = 0, \quad t \in \mathbb{N}_{a+1}.
\]

Notice that this is a particular case of the previous example, where \( p(t) \equiv 1 \).

Then the Cauchy function is

\[
x(t, s) = (\nabla_s^{-\nu} 1)(t) \\
= \int_s^t \frac{(t - \tau + 1)^{\nu - 1}}{\Gamma(\nu) \Gamma(\nu + 1)} \nabla_\tau \\
= \frac{(t - \tau + 1)^{\nu - 1}}{\Gamma(\nu + 1)} \bigg|_{\tau=s}^{\tau=t} \\
= \frac{(t - s)^{\nu}}{\Gamma(\nu + 1)}.
\]

Also note that if you take \( \nu = 1 \) as in the whole order self-adjoint case, the Cauchy function simplifies to \( x(t, s) = t - s \).

**Example 24.** Find the solution to the IVP

\[
\begin{aligned}
\nabla[p(t + 1) \nabla_s^{\nu} y(t + 1)] &= h(t), \quad t \in \mathbb{N}_{a+1}, \\
y(a) &= 0, \\
\nabla y(a + 1) &= 0.
\end{aligned}
\]

From Theorem 27, we know that the solution is given by

\[
y(t) = \int_a^t x(t, s) h(s) \nabla s.
\]
By Example 22, we know the Cauchy function for the above difference equation is \( x(t, s) = \nabla_s^{-\nu} \frac{1}{p(t)} \). Then the solution is given by

\[
y(t) = \int_{a}^{t} \nabla_s^{-\nu} \frac{1}{p(t)} h(s) \nabla s.
\]

**Example 25.** Solve the IVP

\[
\begin{aligned}
\nabla\nabla^0_{a^\frac{1}{6}} x(t + 1) &= t, \quad t \in \mathbb{N}_1, \\
x(0) &= 0, \\
\nabla x(1) &= 0.
\end{aligned}
\]

This is a particular case of Example 24 where \( h(t) = t, \ a = 0, \) and \( \nu = 0.6. \) Then

\[
y(t) = \int_{0}^{t} x(t, s)s \nabla s = \sum_{s=1}^{t} \frac{1}{\Gamma(1.6)} (t - s)^{0.6} s,
\]

and after summing by parts and applying Theorem 4 we get that the solution is

\[
y(t) = \frac{(t - 1)^{2.6}}{\Gamma(3.6)}
\]

5 **Boundary value problems of the fractional self-adjoint equation**

In this section we develop techniques to solve boundary value problems for the fractional self-adjoint operator involving the Caputo difference. See Brackins [4] for a similar development using the Riemann-Liouville definition of a fractional difference.

We are interested in the boundary value problems (BVPs)

\[
\begin{aligned}
L_a x(t) &= 0, \quad t \in \mathbb{N}_{a+1}^b, \\
ax(a) - \beta \nabla x(a + 1) &= 0, \\
\gamma x(b) + \delta \nabla x(b) &= 0,
\end{aligned}
\]

and

\[
\begin{aligned}
L_a x(t) &= h(t), \quad t \in \mathbb{N}_{a+1}^b, \\
ax(a) - \beta \nabla x(a + 1) &= A, \\
\gamma x(b) + \delta \nabla x(b) &= B,
\end{aligned}
\]
where \( h : \mathbb{N}^{\mathbb{R}}_{n+1} \to \mathbb{R} \) and \( \alpha, \beta, \gamma, \delta, A, B \in \mathbb{R} \) for which \( \alpha^2 + \beta^2 > 0 \) and \( \gamma^2 + \delta^2 > 0 \). Note that despite the fact that the difference equations above hold for \( t \in \mathbb{N}^{\mathbb{R}}_{n+1} \), the solution \( x(t) \) for each BVP is defined on the domain of \( \mathbb{N}^{\mathbb{R}}_{n+1} \). We are primarily interested in cases where the BVP \( (5) \) has only the trivial solution.

**Theorem 26.** Assume \( (5) \) has only the trivial solution. Then \( (6) \) has a unique solution.

**Proof.** Let \( x_1, x_2 : \mathbb{N}_a \to \mathbb{R} \) be linearly independent solutions to \( L_a x(t) = 0 \). By Theorem 13, a general solution to \( L_a x(t) = 0 \) is given by

\[
x(t) = c_1 x_1(t) + c_2 x_2(t),
\]

where \( c_1, c_2 \in \mathbb{R} \) are arbitrary constants. If \( x(t) \) solves the boundary conditions in \( (5) \), then \( x(t) \) is the trivial solution, which is true if and only if \( c_1 = c_2 = 0 \). This is true if and only if the system of equations

\[
\begin{aligned}
\alpha [c_1 x_1(a) + c_2 x_2(a)] - \beta \nabla_a^\nu [c_1 x_1(a + 1) + c_2 x_2(a + 1)] &= 0, \\
\gamma [c_1 x_1(b) + c_2 x_2(b)] + \delta \nabla_a^\nu [c_1 x_1(b) + c_2 x_2(b)] &= 0,
\end{aligned}
\]

or equivalently,

\[
\begin{aligned}
c_1 [\alpha x_1(a) - \beta \nabla_a^\nu x_1(a + 1)] + c_2 [\alpha x_2(a) - \beta \nabla_a^\nu x_2(a + 1)] &= 0, \\
c_1 [\gamma x_1(b) + \delta \nabla_a^\nu x_1(b)] + c_2 [\gamma x_2(b) + \delta \nabla_a^\nu x_2(b)] &= 0,
\end{aligned}
\]

has only the trivial solution where \( c_1 = c_2 = 0 \). In other words, \( x(t) \) solves \( (5) \) if and only if

\[
D := \begin{vmatrix}
\alpha x_1(a) - \beta \nabla_a^\nu x_1(a + 1) & \alpha x_2(a) - \beta \nabla_a^\nu x_2(a + 1) \\
\gamma x_1(b) + \delta \nabla_a^\nu x_1(b) & \gamma x_2(b) + \delta \nabla_a^\nu x_2(b)
\end{vmatrix} = 0.
\]

Now consider \( (6) \). By Corollary 17, a general solution to \( L_a y(t) = h(t) \) is

\[
y(t) = a_1 x_1(t) + a_2 x_2(t) + y_0(t),
\]

where \( a_1, a_2 \in \mathbb{R} \) are arbitrary constants and \( y_0 : \mathbb{N}_a \to \mathbb{R} \) is a particular solution of \( L_a y(t) = h(t) \). Consider the system of equations

\[
\begin{aligned}
a_1 [\alpha x_1(a) - \beta \nabla_a^\nu x_1(a + 1)] + a_2 [\alpha x_2(a) - \beta \nabla_a^\nu x_2(a + 1)] &= A - \alpha y_0(a) + \beta \nabla_a^\nu y_0(a + 1), \\
a_1 [\gamma x_1(b) + \delta \nabla_a^\nu x_1(b)] + a_2 [\gamma x_2(b) + \delta \nabla_a^\nu x_2(b)] &= B - \gamma y_0(b) - \delta \nabla_a^\nu y_0(b),
\end{aligned}
\]

for arbitrary \( A, B \in \mathbb{R} \) as in \( (6) \). Since \( D \neq 0 \), this system has a unique solution for \( a_1, a_2 \). It may be shown algebraically that this system is equivalent to

\[
\begin{aligned}
\alpha [a_1 x_1(a) + a_2 x_2(a) + y_0(a)] - \beta \nabla_a^\nu [a_1 x_1(a + 1) + a_2 x_2(a + 1) + y_0(a + 1)] &= A, \\
\gamma [a_1 x_1(b) + a_2 x_2(b) + y_0(b)] + \delta \nabla_a^\nu [a_1 x_1(b) + a_2 x_2(b) + y_0(b)] &= B,
\end{aligned}
\]

so \( y(t) \) satisfies the boundary conditions for \( (6) \). Therefore for any \( A, B \in \mathbb{R} \), \( (6) \) has a unique solution. \( \blacksquare \)
Theorem 27. Let
\[ \rho := \alpha \gamma \nabla_{a}^{-\nu} \frac{1}{p(b)} + \frac{\alpha \delta}{p(b)} + \frac{\beta \gamma}{p(a + 1)} \]

Then the BVP
\[ \begin{align*}
\nabla[p(t + 1)\nabla_{a}^{\nu} x(t + 1)] &= 0, \quad t \in \mathbb{N}_{a+1}^{b-1}, \\
\alpha x(a) - \beta \nabla_{a}^{\nu} x(a + 1) &= 0, \\
\gamma x(b) + \delta \nabla_{a}^{\nu} x(b) &= 0,
\end{align*} \]

has only the trivial solution if and only if \( \rho \neq 0 \).

Proof. Note that \( x_1(t) = 1, x_2(t) = \nabla_{a}^{-\nu} \frac{1}{p(t)} \) are linearly independent solutions to
\[ \nabla[p(t + 1)\nabla_{a}^{\nu} x(t + 1)] = 0. \]

Then a general solution of the difference equation is given by
\[ x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 + c_2 \nabla_{a}^{-\nu} \frac{1}{p(t)}. \]

Consider the boundary conditions \( \alpha x(a) - \beta \nabla_{a}^{\nu} x(a + 1) = 0, \) and \( \gamma x(b) + \delta \nabla_{a}^{\nu} x(b) = 0. \) These boundaries give us
\[ c_1 \alpha + c_2 \left( -\gamma \frac{1}{p(a + 1)} \right) = 0, \]
\[ c_1 \gamma + c_2 \left( \frac{\delta}{p(b)} + \frac{\gamma}{p(b)} \frac{1}{p(b)} \right) = 0. \]

Converting this into a linear system yields
\[ \begin{bmatrix} \alpha & -\frac{\beta}{p(a + 1)} \\ \gamma & \frac{\delta}{p(b)} + \frac{\gamma}{p(b)} \frac{1}{p(b)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]

Consider the determinant of the coefficient matrix,
\[ \begin{vmatrix} \alpha & -\frac{\beta}{p(a + 1)} \\ \gamma & \frac{\delta}{p(b)} + \frac{\gamma}{p(b)} \frac{1}{p(b)} \end{vmatrix} = \alpha \gamma \nabla_{a}^{-\nu} \frac{1}{p(b)} + \frac{\alpha \delta}{p(b)} + \frac{\beta \gamma}{p(a + 1)} = \rho. \]

By properties of invertible matrices, the BVP has only the trivial solution if and only if \( \rho \neq 0. \)

Definition 28. Assume that (5) has only the trivial solution. Then we define the Green’s function for the homogeneous BVP (4), \( G(t, s), \) by
\[ G(t, s) := \begin{cases} 
    u(t, s), & a \leq t \leq s \leq b, \\
    v(t, s), & a \leq s \leq t \leq b,
\end{cases} \]
where \( u(t, s) \) solves the BVP

\[
\begin{align*}
L_a u(t) &= 0, \quad t \in \mathbb{N}^{b-1}_{a+1}, \\
\alpha u(a, s) - \beta \nabla u(a + 1, s) &= 0, \\
\gamma u(b, s) + \delta \nabla u(b, s) &= -[\gamma x(b, s) + \delta \nabla x(b, s)],
\end{align*}
\]

for each fixed \( s \in \mathbb{N}^b_a \) and where \( x(t, s) \) is the Cauchy function for \( L_a x(t) \). Then we define

\[
v(t, s) := u(t, s) + x(t, s).
\]

**Theorem 29 (Green’s Function Theorem).** If (5) has only the trivial solution, then the solution to (6) where \( A = B = 0 \) is given by

\[
y(t) = \int_a^b G(t, s) h(s) \nabla s,
\]

where the \( G(t, s) \) is the Green’s function for the homogeneous BVP (7).

**Proof.** First note that by Theorem 26, \( u(t, s) \) for each fixed \( s \in \mathbb{N}^b_a \) is well-defined. Let

\[
y(t) = \int_a^b G(t, s) h(s) \nabla s = \int_a^t G(t, s) h(s) \nabla s + \int_t^b G(t, s) h(s) \nabla s
\]

\[
= \int_a^t v(t, s) h(s) \nabla s + \int_t^b u(t, s) h(s) \nabla s
\]

\[
= \int_a^t [u(t, s) + x(t, s)] h(s) \nabla s + \int_t^b u(t, s) h(s) \nabla s
\]

\[
= \int_a^b u(t, s) h(s) \nabla s + \int_a^t x(t, s) h(s) \nabla s
\]

\[
= \int_a^b u(t, s) h(s) \nabla s + z(t),
\]

where \( z(t) := \int_a^t x(t, s) h(s) \nabla s \). Since \( x(t, s) \) is the Cauchy function for \( L_a x(t) = 0 \), by Theorem 26, \( z(t) \) solves the IVP

\[
\begin{align*}
L_a z(t) &= h(t), \quad t \in \mathbb{N}^{b-1}_{a+1}, \\
z(a) &= 0, \\
\nabla z(a + 1) &= 0.
\end{align*}
\]

Then,

\[
L_a y(t) = \int_a^b L_a u(t, s) h(s) \nabla s + L_a z(t)
\]

\[
= 0 + h(t) = h(t),
\]
for \( t \in \mathbb{N}_a^{b-1} \), so the difference equation is satisfied. Now we check the boundary conditions. At \( t = a \), we have
\[
\alpha y(a) - \beta \nabla y(a + 1) = \int_a^b [\alpha u(a, s) - \beta \nabla u(a + 1, s)] h(s) \nabla s + [\alpha z(a) - \beta \nabla z(a + 1)] = 0,
\]
and at \( t = b \), we have
\[
\gamma y(b) + \delta \nabla y(b) = \gamma z(b) + \int_a^b \gamma u(b, s) h(s) \nabla s + \delta \nabla z(b)
= \gamma \int_a^b x(b, s) h(s) \nabla s + \delta \nabla \int_a^b x(b, s) h(s) \nabla s + \int_a^b [\gamma u(b, s) + \delta \nabla u(b, s)] h(s) \nabla s
= - \int_a^b [\gamma x(b, s) + \delta \nabla x(b, s)] h(s) \nabla s + \int_a^b [\gamma x(b, s) + \delta \nabla x(b, s)] h(s) \nabla s
= 0.
\]

**Corollary 30.** If (5) has only the trivial solution, then the solution to (6) with \( A, B \in \mathbb{R} \) is given by
\[
y(t) = z(t) + \int_a^b G(t, s) h(s) \nabla s,
\]
where \( z : \mathbb{N}_a^b \rightarrow \mathbb{R} \) is the unique solution to
\[
\begin{align*}
L_{a, \alpha} z(t) &= 0, \\
\alpha z(a) - \beta \nabla z(a + 1) &= A, \\
\gamma z(b) + \delta \nabla z(b) &= B.
\end{align*}
\]

**Proof.** This corollary follows directly from Theorem 29 by linearity. ■

**Example 31.** Find the Green’s function for the boundary value problem
\[
\begin{align*}
\nabla [\nabla_{a, \alpha} u(t + 1)] &= 0, & t \in \mathbb{N}_a^{b-1}, \\
y(a) &= 0, \\
y(b) &= 0.
\end{align*}
\]

The Green’s function is given by
\[
G(t, s) = \begin{cases} 
 u(t, s), & a \leq t \leq s \leq b, \\
 v(t, s), & a \leq s \leq t \leq b,
\end{cases}
\]
where \( u(t, s) \), for each fixed \( s \in \mathbb{N}_a^b \), solves the BVP
\[
\begin{align*}
\nabla [\nabla_{a, \alpha} u(t + 1)] &= 0, & t \in \mathbb{N}_a^{b-1}, \\
u(a, s) &= 0, \\
u(b, s) &= -x(b, s),
\end{align*}
\]
and \( v(t, s) = u(t, s) + x(t, s) \). By inspection, we find that \( x_1(t) = 1 \) is a solution of
\[
\nabla[\nabla_a^\nu y(t+1)] = 0,
\]
for \( t \in \mathbb{N}_{a+1} \). Let \( x_2(t) = (\nabla_{a+1}^-)^{(1)}(t) \). Consider
\[
\nabla[\nabla_a^\nu x_2(t+1)] = \nabla[\nabla_a^\nu \nabla_a^{-(\nu)}(1)] = \nabla[1] = 0,
\]
using Theorem 8. So we have that \( x_2(t) \) solves \( \nabla[\nabla_a^\nu y(t+1)] = 0 \). Since \( x_1(t) \) and \( x_2(t) \) are linearly independent, by Theorem 16, the general solution is given by
\[
y(t) = c_1 + c_2(\nabla_a^{-(\nu)}(t) = c_1 + c_2 \frac{(t-a)^\nu}{\Gamma(1+\nu)},
\]
and it follows that
\[
u(t, s) = c_1(s) + c_2(s) \frac{(t-a)^\nu}{\Gamma(1+\nu)}.
\]
The boundary condition \( u(a, s) = 0 \) implies that \( c_1(s) = 0 \). The boundary condition \( u(b, s) = -x(b, s) \) then yields
\[
-x(b, s) = u(b, s) = c_2(s) \frac{(b-a)^\nu}{\Gamma(1+\nu)}.
\]
From Example 23, we know that
\[
x(b, s) = (\nabla_{a+1}^-)^{(1)}(b) = \frac{(b-s)^\nu}{\Gamma(1+\nu)},
\]
and thus
\[
c_2(s) = -\frac{(b-s)^\nu}{(b-a)^\nu}.
\]
Hence the Green’s function is given by
\[
G(t, s) = \begin{cases} 
-\frac{(b-s)^\nu(t-a)^\nu}{\Gamma(1+\nu)(b-a)^\nu}, & a \leq t \leq s \leq b, \\
-\frac{(b-s)^\nu(b-a)^\nu}{\Gamma(1+\nu)} + \frac{(t-s)^\nu}{\Gamma(1+\nu)}, & a \leq s \leq t \leq b.
\end{cases}
\]
Remark 32. Note that in the continuous and whole-order discrete cases, the Green’s function is symmetric for the equivalent BVP in Example 37.
is not necessarily true in the fractional case. By way of counterexample, take $a = 0$, $b = 5$, and $\nu = 0.5$. Then computing we find that

$$G(2, 3) = u(2, 3) = -\frac{(2)^{0.5}(2)^{0.5}}{\Gamma(1.5)(5)^{0.5}} = -\frac{32}{35},$$

but

$$G(3, 2) = v(3, 2) = -\frac{(3)^{0.5}(3)^{0.5}}{\Gamma(1.5)(5)^{0.5}} + \frac{1^{0.5}}{\Gamma(1.5)} = -\frac{3}{7}.$$

Thus for this particular BVP, unlike in the continuous and whole-order discrete cases, is not symmetric.

**Theorem 33.** The Green’s function for the BVP

$$\begin{cases}
\nabla_{a, s}^{\nu}x(t + 1) = 0, \\
x(a) = x(b) = 0,
\end{cases}$$

for $t \in \mathbb{N}_{a+1}^b$, given by

$$G(t, s) = \begin{cases}
\frac{(b - s)^{\nu}(t - a)^{\nu}}{\Gamma(1 + \nu)(b - a)^{\nu}}, & a \leq t \leq s \leq b, \\
\frac{(b - s)^{\nu}(t - a)^{\nu}}{\Gamma(1 + \nu)(b - a)^{\nu}} + \frac{(t - s)^{\nu}}{\Gamma(1 + \nu)}, & a \leq s \leq t \leq b,
\end{cases}$$

satisfies the inequalities

1. $G(t, s) \leq 0$,
2. $G(t, s) \geq -\left(\frac{b - a}{4}\right)\left(\frac{\Gamma(b - a + 1)}{\Gamma(\nu + 1)\Gamma(b - a + \nu)}\right)$,
3. $\int_a^b |G(t, s)| \nabla s \leq \frac{(b - a)^2}{4\Gamma(\nu + 2)}$,

for $t \in \mathbb{N}_a^b$, and

4. $\int_a^b |\nabla G(t, s)| \nabla s \leq \frac{b - a}{\nu + 1}$

for $t \in \mathbb{N}_{a+1}^b$.

**Proof.** (1) Let $a \leq t \leq s \leq b$. Then

$$G(t, s) = u(t, s) = -\frac{(t - a)^{\nu}(b - s)^{\nu}}{\Gamma(\nu + 1)(b - a)^{\nu}} \leq 0,$$

for each fixed $s \in \mathbb{N}_a^b$. Now let $a \leq s < t \leq b$. Then $G(t, s) = v(t, s)$, so we wish to show that $v(t, s)$ is non-positive. First, we show that $v(t, s)$ is increasing. Taking the nabla difference with respect to $t$ yields

$$\nabla_t \left[-\frac{(t - a)^{\nu}(b - s)^{\nu}}{\Gamma(\nu + 1)(b - a)^{\nu}} + \frac{(t - s)^{\nu}}{\Gamma(1 + \nu)}\right] = -\frac{(t - a)^{\nu-1}(b - s)^{\nu}}{\Gamma(\nu)(b - a)^{\nu}} + \frac{(t - s)^{\nu-1}}{\Gamma(\nu)}.$$
This expression is nonnegative if and only if
\[
\frac{(t-a)^{\nu-1}(b-s)^{\nu}}{\Gamma(\nu)(b-a)^{\nu}} \leq \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}.
\]
Since \( t-s \) is positive, this happens if
\[
\frac{(t-a)^{\nu-1}(b-s)^{\nu}}{(b-a)^{\nu}(t-s)^{\nu-1}} \leq 1.
\]
Now, by definition of the rising function,
\[
\frac{(t-a)^{\nu-1}(b-s)^{\nu}}{(b-a)^{\nu}(t-s)^{\nu-1}} = \frac{\Gamma(t-a+\nu-1)\Gamma(b-s+\nu)}{\Gamma(t-a)\Gamma(b-s+\nu-1)} \frac{\Gamma(b-a)\Gamma(t-s)}{\Gamma(b-a+\nu)\Gamma(t-s+\nu-1)}
\]
\[
= \frac{(t-s+\nu-1)(t-s+\nu)\cdots(t-a+\nu-2)}{(t-s)(t-s+1)\cdots(t-a-1)} \frac{(b-s+\nu)(b-s+\nu+1)\cdots(b-a+\nu-1)}{(b-s)(b-s+1)\cdots(b-a-1)} \frac{(t-s+\nu-1)(t-s+\nu)\cdots(t-a+\nu-2)}{(t-s)(t-s+1)\cdots(t-a-1)} \frac{(b-s+\nu)(b-s+\nu+1)\cdots(b-a+\nu-1)}{(b-s)(b-a-1)}
\]
\[
\leq (1)(1)(1)(1)(1) = 1.
\]

Next, we check \( v(t,s) \) at the right endpoint, \( t=b \),
\[
v(b,s) = -\frac{(b-a)^{\nu}(b-s)^{\nu}}{\Gamma(\nu+1)(b-a)^{\nu}} + \frac{(b-s)^{\nu}}{\Gamma(\nu+1)} = 0.
\]
Thus, \( v(t,s) \) is nonpositive for \( a \leq s < t \leq b \). Also note that \( v(t,s) = u(t,s) \) for \( t = s \). Therefore, for \( t \in \mathbb{N}_a \), \( G(t,s) \) is nonpositive.

(2) Since we know that \( v(t,s) \) is always increasing for \( a \leq s < t \leq b \) and that for \( s = t \), \( v(t,s) = u(t,s) \), it suffices to show that
\[
u(t,s) \geq -\frac{\Gamma(b-a+1)}{\Gamma(\nu+1)\Gamma(b-a+\nu)} \left( \frac{b-a}{4} \right).
\]
Let \( a \leq t \leq s \leq b \). Then
\[
G(t,s) = u(t,s) = -\frac{(t-a)^{\nu}(b-s)^{\nu}}{\Gamma(\nu+1)(b-a)^{\nu}} = -\frac{(s-a)^{\nu}(b-s)^{\nu}}{\Gamma(\nu+1)(b-a)^{\nu}} \geq -\frac{(s-a)^{\nu}(b-s)^{\nu}}{\Gamma(\nu+1)(b-a)^{\nu}}
\]
Note that for \( \alpha \in \mathbb{N}_1 \) and \( 0 < \nu < 1 \),
\[
\alpha^{\nu} = \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)} \leq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha^\nu.
\]
So

\[
\frac{-(s-a)^\nu(b-s)^\nu}{\Gamma(\nu+1)(b-a)^\nu} \geq \frac{-(s-a)^\nu(b-s)^\nu}{\Gamma(\nu+1)(b-a)^\nu}
\]

\[
\geq \frac{(a+b-a)(b-a^2)}{\Gamma(\nu+1)(b-a)^\nu}
\]

\[
= \frac{(b-a)(b-a)}{4\Gamma(\nu+1)\Gamma(b-a+\nu)}
\]

\[
= \frac{(b-a)(b-a+1)}{4\Gamma(\nu+1)\Gamma(b-a+\nu)}
\]

Therefore,

\[
G(t, s) \geq -\left(\frac{b-a}{4}\right)\left(\frac{\Gamma(b-a+1)}{\Gamma(\nu+1)\Gamma(b-a+\nu)}\right).
\]
Since all the factors in the above equation are individually positive, we get that

IVPs and BVPs for a Self-Adjoint Caputo Nabla Fractional Difference Equation

(3) Consider

\[
\begin{align*}
\int_a^b |G(t, s)|\nabla s &= \int_a^t |v(t, s)|\nabla s + \int_t^b |u(t, s)|\nabla s \\
&= \int_a^t \left| -\frac{(t-a)^\nu (b-s)^\nu}{\Gamma(\nu + 1)(b-a)^\nu} + \frac{(t-s)^\nu}{\Gamma(1+\nu)} \right| \nabla s \\
&\quad + \int_t^b \frac{(t-a)^\nu (b-s)^\nu}{\Gamma(\nu + 1)(b-a)^\nu} \nabla s \\
&= \int_a^t \left[ -\frac{(t-a)^\nu (b-s)^\nu}{\Gamma(\nu + 1)(b-a)^\nu} + \frac{(t-s)^\nu}{\Gamma(1+\nu)} \right] \nabla s \\
&\quad + \int_t^b \frac{(t-a)^\nu (b-s)^\nu}{\Gamma(\nu + 1)(b-a)^\nu} \nabla s \\
&= \int_a^b \frac{(t-a)^\nu (b-s)^\nu}{\Gamma(\nu + 1)(b-a)^\nu} \nabla s - \int_a^t \frac{(t-s)^\nu}{\Gamma(1+\nu)} \nabla s \\
&= -\frac{(t-a)^\nu (b-s-1)^{\nu+1}}{\Gamma(\nu + 2)(b-a)^\nu} \bigg|_{s=b}^{s=t} + \frac{(t-s-1)^{\nu+1}}{\Gamma(\nu + 2)} \bigg|_{s=a}^{s=t} \\
&= \frac{(t-a)^\nu (b-a-1)^{\nu+1}}{\Gamma(\nu + 2)(b-a)^\nu} - \frac{(t-a-1)^{\nu+1}}{\Gamma(\nu + 2)} \\
&= \frac{(t-a)^\nu (b-a-1)}{\Gamma(\nu + 2)(b-a)^\nu} \\
&= \frac{(t-a)^\nu (b-a-1)}{\Gamma(\nu + 2)} \\
&= \frac{(t-a)^\nu (b-1)}{\Gamma(\nu + 2)} \\
&\leq \frac{(t-a)(b-t)}{\Gamma(\nu + 2)} \\
&\leq \frac{\frac{(a+b)}{2} - a}{\Gamma(\nu + 2)} \frac{(b-a)}{\Gamma(\nu + 2)} \\
&= \frac{(a+b-a)(b-a)}{4\Gamma(\nu + 2)} \\
&= \frac{(b-a)^2}{4\Gamma(\nu + 2)}.
\end{align*}
\]

(4) We can assume that \( b - a > 1 \) as if \( b - a = 1 \), it would be an initial value problem and not a boundary value problem. Taking the difference of \( u(t, s) \) with respect to \( t \), we have

\[
\nabla_t u(t, s) = \nabla_t \frac{-(t-a)^\nu (b-s)^\nu}{\Gamma(\nu + 1)(b-a)^\nu} = -\nu(t-a)^{\nu-1}(b-s)^\nu /\Gamma(\nu + 1)(b-a)^\nu.
\]

Since all the factors in the above equation are individually positive, we get that
\[ \nabla u(t, s) \text{ is nonpositive. Let } t \in \mathbb{N}_a. \text{ We therefore know} \]

\[
\begin{align*}
\int_a^b |\nabla_t G(t, s)| \nabla s &= \int_a^1 |\nabla_t G(t, s)| \nabla s + \int_{t-1}^b |\nabla_t G(t, s)| \nabla s \\
&= \int_a^1 |\nabla_t \nu(t, s)| \nabla s + \int_{t-1}^b |\nabla_t u(t, s)| \nabla s \\
&= \int_a^b \left[ \nabla_t \frac{-(t-a)^\nu(b-s)^\nu}{\Gamma(\nu+1)(b-a)^\nu} + \nabla_t \frac{(t-s)^\nu}{\Gamma(\nu+1)} \right] \nabla s \\
&= \int_a^b \nabla_t \frac{(t-a)^\nu(b-s)^\nu}{\Gamma(\nu+1)(b-a)^\nu} \nabla s + \int_a^{t-1} \nabla_t \frac{(t-s)^\nu}{\Gamma(\nu+1)} \nabla s \\
&= \int_a^b \frac{-(t-a)^\nu(b-s)^\nu}{\Gamma(\nu+1)(b-a)^\nu} \nabla s + \int_a^{t-1} \frac{\nu(t-s)^{\nu-1}}{\Gamma(\nu+1)} \nabla s \\
&= \frac{-\nu(t-a)^{\nu-1}}{\Gamma(\nu+1)(b-a)^\nu} \left[ \frac{-1}{\nu+1} (b-s-1)^{\nu+1} \right]^{s=t-1}_{s=a} \\
&\quad + \frac{\nu}{\Gamma(\nu+1)} \left[ \frac{-1}{\nu} (t-s-1)^{\nu} \right]^{s=t-1}_{s=a} \\
&\quad + \frac{\nu(t-a)^{\nu-1}}{\Gamma(\nu+1)(b-a)^\nu} \left[ \frac{-1}{\nu+1} (b-s-1)^{\nu+1} \right]^{s=t-1}_{s=t-1} \\
&= \frac{\nu(t-a)^{\nu-1}}{\Gamma(\nu+2)(b-a)^\nu} \left[ (b-t)^{\nu+1} - (b-a-1)^{\nu+1} \right] \\
&\quad - \frac{1}{\Gamma(\nu+1)} \left[ (t-t+1-1)^{\nu} - (t-a-1)^{\nu} \right] \\
&\quad + \frac{-\nu(t-a)^{\nu-1}}{\Gamma(\nu+2)(b-a)^\nu} \left[ (b-b-1)^{\nu+1} - (b-t)^{\nu+1} \right] \\
&= \frac{2\nu(t-a)^{\nu-1}(b-t)^{\nu+1}}{\Gamma(\nu+2)(b-a)^\nu} + \frac{(t-a-1)^{\nu}}{\Gamma(\nu+1)} - \frac{\nu(t-a)^{\nu-1}(b-a-1)}{\Gamma(\nu+2)}.
\end{align*}
\]
Suppose $t = b$. This would imply that

$$
\int_a^b |\nabla G(t, s)| \nabla s = \frac{2\nu(b-a)^{\nu-1}(0)^{\nu+1}}{\Gamma(\nu+2)(b-a)^{\nu+1}} + \frac{(b-a-1)^{\nu}}{\Gamma(\nu+1)}
- \frac{\nu(b-a)^{\nu-1}(b-a-1)}{\Gamma(\nu+2)}
= \frac{(\nu+1)(b-a-1)^{\nu}}{\Gamma(\nu+2)} - \frac{\nu(b-a)^{\nu-1}(b-a-1)}{\Gamma(\nu+2)}.
$$

For $t = b$ and $b-a = 2$, this becomes

$$
\int_a^b |\nabla G(t, s)| \nabla s = \frac{(\nu+1)(1)^{\nu}}{\Gamma(\nu+2)} - \frac{\nu(2)^{\nu-1}(1)}{\Gamma(\nu+2)}
= \frac{(\nu+1)\Gamma(\nu+1)}{\Gamma(\nu+2)} - \frac{\nu\Gamma(\nu+1)}{\Gamma(\nu+2)}
= 1 - \frac{\nu}{\nu+1}
= \frac{1}{\nu+1}
\leq \frac{2}{\nu+1}
= \frac{b-a}{\nu+1}.
$$

For $t = b$ and $b-a = 3$, we have

$$
\int_a^b |\nabla G(t, s)| \nabla s = \frac{(\nu+1)(2)^{\nu}}{\Gamma(\nu+2)} - \frac{\nu(3)^{\nu-1}(2)}{\Gamma(\nu+2)}
= \frac{(\nu+1)\Gamma(\nu+2)}{\Gamma(\nu+2)} - \frac{2\nu\Gamma(\nu+2)}{\Gamma(\nu+2)\Gamma(3)}
= \nu + 1 - \nu
= 1
= \frac{3}{3}
\leq \frac{b-a}{\nu+1}.
$$
For $t = b$ and $b - a \geq 4$, the result holds since

$$
\int_a^b |\nabla_t G(t, s)| \, ds = \frac{(\nu + 1)(b - a - 1)^\nu}{\Gamma(\nu + 2)} - \frac{\nu(b - a)^{\nu - 2}(b - a - 1)}{\Gamma(\nu + 2)}
$$

$$
= \frac{(\nu + 1)(b - a - 2 + \nu) \cdots (2 + \nu)}{\Gamma(b - a - 1)}
$$

$$
- \frac{\nu\Gamma(b - a - 1 + \nu)(b - a - 1)}{\Gamma(2 + \nu)\Gamma(b - a)}
$$

$$
= \frac{(\nu + 1)(b - a - 2 + \nu) \cdots (2 + \nu)}{\Gamma(b - a - 1)}
$$

$$
- \frac{\nu\Gamma(b - a - 1 + \nu)}{\Gamma(2 + \nu)\Gamma(b - a - 1)}
$$

$$
= \frac{(\nu + 1)(b - a - 2 + \nu) \cdots (2 + \nu)}{(b - a - 2)!}
$$

$$
- \frac{\nu(b - a - 2 + \nu) \cdots (2 + \nu)}{(b - a - 2)!}
$$

$$
= \frac{(b - a - 2 + \nu) \cdots (2 + \nu)}{(b - a - 2)!}
$$

$$
= \frac{(b - a - 1)(b - a - 2 + \nu) \cdots (2 + \nu)}{(b - a - 1)!}
$$

$$
\leq \frac{1}{2}(b - a - 1)^{(b - a - 1) \cdots (3)}
$$

$$
\leq \frac{1}{2}(b - a - 1)!(b - a - 1)
$$

$$
\leq \frac{b - a - 1}{\nu + 1}
$$

So the result holds if $t = b$ generally. Now, assume $t < b$. If $t = a + 1$, then we
have

\[
\int_a^b |\nabla_t G(t, s)| \, \nabla s = \frac{2\nu(1^{\nu-1})(b - a - 1)^{\nu+1} + (\nu + 1)(0^\nu)(b - a)^\nu}{\Gamma(\nu + 2)(b - a)^\nu} \\
- \frac{\nu(1^{\nu-1})(b - a - 1)^{\nu+1}}{\Gamma(\nu + 2)(b - a)^\nu} \\
= \frac{2\nu\Gamma(\nu)(b - a - 1)\nu - \nu\Gamma(\nu)(b - a - 1)\nu}{\Gamma(\nu + 2)(b - a)^\nu} \\
= \frac{2\nu\Gamma(\nu)(b - a - 1) - \nu\Gamma(\nu)(b - a - 1)}{\Gamma(\nu + 2)} \\
= \frac{\Gamma(\nu + 1)(b - a - 1)}{\Gamma(\nu + 2)} \\
= \frac{\Gamma(\nu + 1)(b - a - 1)}{(\nu + 1)\Gamma(\nu + 1)} \\
= \frac{b - a - 1}{\nu + 1} \\
\leq \frac{b - a}{\nu + 1}.
\]

If \( t = a + 2 \), then

\[
\int_a^b |\nabla_t G(t, s)| \, \nabla s \\
= \frac{2\nu(2^{\nu-1})(b - a - 2)^{\nu+1} + (\nu + 1)(1^\nu)(b - a)^\nu - \nu(2^{\nu-1})(b - a - 1)^{\nu+1}}{\Gamma(\nu + 2)(b - a)^\nu} \\
= \frac{2\nu\Gamma(\nu + 1)(b - a - 2)^{\nu+1} + (\nu + 1)(\nu)\Gamma(\nu)(b - a)^\nu}{\Gamma(\nu + 2)(b - a)^\nu} \\
- \frac{\nu\Gamma(\nu + 1)(b - a - 1)^{\nu+1}}{\Gamma(\nu + 2)(b - a)^\nu} \\
= \frac{2\nu\Gamma(\nu + 1)(b - a - 1)(b - a - 2) + 1 - \nu(b - a - 1)}{\nu + 1} \\
\leq \frac{2\nu(b - a - 2) + 1 - \nu(b - a - 1)}{\nu + 1} \\
= \frac{2\nu(b - a - 2) + \nu + 1 - \nu(b - a - 1)}{\nu + 1} \\
= \frac{\nu(b - a - 2) + 1}{\nu + 1} \\
\leq \frac{b - a - 1}{\nu + 1} \\
\leq \frac{b - a}{\nu + 1}.
\]
If \( t = a + 3 \), then

\[
\int_a^b |\nabla \varphi(t)|| \nabla s \, ds = \frac{2\nu(3^{\nu-1})(b - a - 3)^{\nu+1} + (\nu + 1)(2^\nu)(b - a)^{\nu} - \nu(3^{\nu-1})(b - a - 1)^{\nu+1}}{\Gamma(\nu + 2)(b - a)^{\nu}}
\]

\[
= \frac{2\nu \Gamma(2 + \nu)\Gamma(b - a - 2 + \nu)\Gamma(b - a)}{\Gamma(3)\Gamma(b - a + \nu)\Gamma(\nu + 2)\Gamma(b - a - 3)} + (\nu + 1)
\]

\[
- \frac{\nu\Gamma(\nu + 2)(b - a - 1)}{\Gamma(\nu + 2)\Gamma(3)}
\]

\[
= \frac{\nu(b - a - 1)(b - a - 2)(b - a - 3)}{(b - a - 1 + \nu)(b - a - 2 + \nu)} + (\nu + 1) - \frac{\nu(b - a - 1)}{2}
\]

\[
\leq \nu(b - a - 3) + \nu + 1 - \frac{\nu(b - a - 1)}{2}
\]

\[
= \frac{2\nu b - 2\nu a - 6\nu + 2\nu}{2} - \nu b + \nu a + \nu
\]

\[
= \frac{\nu(b - a - 3) + 2}{2}
\]

\[
\leq \frac{b - a - 3 + 2}{2}
\]

\[
= \frac{b - a - 1}{2}
\]

\[
\leq \frac{b - a}{\nu + 1}.
\]
Now suppose that \( t = a + k \), where \( k \in \mathbb{N}^{b-a-1} \). Then

\[
\int_a^b |\nabla_t G(t, s)| \nabla s \\
= \frac{2\nu(k)^{\nu-1}(b - a - k)^{\nu-1}}{(b - a)^{\nu+1}} + \frac{(\nu + 1)(k - 1)^{\nu+1}}{\Gamma(\nu + 2)} - \frac{\nu(k)^{\nu-1}(b - a - 1)}{\Gamma(\nu + 2)} \\
= 2\nu(\nu + 2)\ldots(\nu + k - 2)(b - a - 1)\ldots(b - a - k) \\
\frac{(k - 1)!}{(b - a - 1 + \nu)\ldots(b - a - (k - 1) \ldots \nu)} + \frac{(\nu + 1)\ldots(\nu + k - 2)}{(k - 2)!} \\
\frac{\nu(\nu + 2)\ldots(\nu + k - 2)(b - a - 1)}{(k - 1)!} \\
= \nu(\nu + 2)\ldots(\nu + k - 2)(b - a - 1\ldots(b - a - k) \\
\frac{(k - 1)!}{(k - 1)!} + \frac{(\nu + 1)\ldots(\nu + k - 2)}{(k - 1)!} \\
\frac{\nu(\nu + 2)\ldots(\nu + k - 2)(b - a - 1\ldots(b - a - k) \\
\frac{(k - 1)!}{(k - 1)!} + \frac{(\nu + 1)\ldots(\nu + k - 2)}{(k - 1)!} \\
\frac{(1)(3)(4)\ldots(k - 1)(b - a + 1 - 2k) + (k - 1)(\nu + 1)\ldots(\nu + k - 2)}{(k - 1)!} \\
= \frac{1}{2}(k - 1)! \frac{(b - a + 1 - 2k) + (k - 1)(\nu + 1)\ldots(\nu + k - 2)}{(k - 1)!} \\
\frac{(b - a + 1 - 2k) + 2(k - 1)}{(k - 1)!} \\
= \frac{b - a - 1}{2} \\
\leq \frac{b - a}{\nu + 1}.
\]

These properties of the Green’s function are important in proving the existence and uniqueness of solutions of BVPs for nonlinear difference equations.
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