Group theoretical study of nonstrange and strange mixed symmetric baryon states \([N_c - 1, 1]\) in the \(1/N_c\) expansion

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Abstract

Using group theory arguments we extend and complete our previous work by deriving all SU(6) exact wave functions associated to the spectrum of mixed symmetric baryon states \([N_c - 1, 1]\) in the \(1/N_c\) expansion. The extension to SU(6) enables us to study the mass spectra of both strange and nonstrange baryons, while previous work was restricted to nonstrange baryons described by SU(4). The wave functions are specially written in a form to allow a comparison with the approximate, customarily used wave functions, where the system is separated into a ground state core and an excited quark. We show that the matrix elements of the flavor operator calculated with the exact wave functions acquire the same asymptotic form at large \(N_c\), irrespective of the spin-flavor multiplet contained in \([N_c - 1, 1]\), while with the approximate wave function one cannot obtain a similar behaviour. The isoscalar factors of the permutation group of \(N_c\) particles derived here can be used in any problem where a given fermion system is described by the partition \([N_c - 1, 1]\), and one fermion has to be separated from the rest.

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I. INTRODUCTION

The $1/N_c$ expansion method [1–4], where $N_c$ is the number of colors, has led to a better understanding of the spin-flavor structure of baryons in the context of QCD. Much work has been devoted to the ground state of light [5–10] and heavy baryons [11, 12]. For $N_c \to \infty$ the ground state is governed by an exact contracted SU($2N_f$) symmetry [3, 4], where $N_f$ is the number of flavors. Accordingly, baryon masses are degenerate at $N_c \to \infty$. For finite $N_c$ the mass splitting starts at order $1/N_c$. It is customary to drop higher order corrections in the mass formula. The results on spectra and mass relations prove that the large $N_c$ world is sufficiently close to $N_c = 3$.

For ground state baryons the study is systematic and straightforward because the orbital wave function is symmetric and also irrelevant in the calculations. The spin-flavor wave function is also symmetric which makes quite easy to deal with it.

For excited states the symmetry is enlarged to SO(3) × SU($2N_f$) to include orbital excitations. Then the system acquires a new degree of freedom, described by a specific orbital wave function, the symmetry of which must match that of the spin-flavor part, in order to lead to a totally symmetric state in the orbital-spin-flavor space. The excited states described by symmetric wave functions in both the orbital and spin-flavor degrees of freedom are nearly as simple as the ground state. Results are available for the Roper resonance $[56', 0^+]$ [13], and for states belonging to the $[56, 2^+]$ [14] and to the $[56, 4^+]$ [15] multiplets respectively. For finite $N_c$ the mass splitting starts at order $1/N_c$, like for the ground state.

More complicated are the mixed symmetric states in both orbital and spin-flavor space. They belong to the $[70, \ell^P]$ multiplet with parity $P = (-)^\ell$. Starting from group theoretical arguments, here we study the SU(6) [Nc − 1, 1] multiplet for arbitrary $N_c$ and thus complete our previous work [16], restricted to SU(4) (for a review see [17] and [18]). We recall that the SU(6) generators are $S^i$, $T^a$ and $G^{ia}$ acting on spin, flavor and spin-flavor respectively.

So far the most extensively studied is the $[70, 1^-]$ multiplets. This is the simplest and the best known experimentally group of states. Historically, the first approach, presently a standard procedure, was based on the decoupling of the system into a ground state core described by a symmetric spin-flavor state of $N_c - 1$ quarks and an orbitally excited quark
Accordingly the SU(2N_f) generators were written as

\[ S_i = S^i_c + s^i, \quad T^a = T^a_c + t^a, \quad G^{ia} = G^{ia} + g^{ia}, \]  

(1)

where the operators carrying the lower index \( c \) act on the core and the lower case operators act on the excited quark. This method is in the spirit of the Hartree picture and the system is described by an approximate wave function, where the orbital part has a configuration of type \( s^{N_c-1}p \) (no antisymmetrization) which is combined with an approximate (truncated) spin-flavor part. The splitting (1) of the generators amounts to an excessively large number of independent operators to be included in the mass formula, difficult to handle when the data is restricted, as it usually happens.

As being the first proposal in large \( N_c \) baryon spectroscopy, we have also applied it to the \([70, \ell^+]\) multiplet (\( \ell = 0, 2 \)) \[25, 26\]. Consistenly with previous studies, we found that the splitting starts at order \( \mathcal{O}(N_0^0) \). There are many interesting papers in the field to be cited. However here we have to restrict the list to our specific goal.

From the studies we performed on the \([70, \ell^+]\) multiplet, we understood that a simpler procedure can as correctly be used, where no quark is decoupled from the system, all identical quarks being treated on the same footing, with an exact wave function in the orbital-flavor-spin space \[27\]. We found out that the key to the problem was the knowledge of the matrix elements of all SU(2N_f) generators, \( S^i, T^a \) and \( G^{ia} \) for mixed symmetric states \([N_c-1, 1]\). For SU(4) they were derived in Ref. \[27\] and for SU(6) in Ref. \[28\]. In the standard procedure the problem was simplified, by truncating the wave function to a part where a quark is decoupled from the whole system, the rest remaining in a ground state symmetric core. In this way the problem was reduced to the knowledge of the matrix elements of SU(2N_f) generators for the core, in the symmetric representation \([N_c-1]\).

To better clarify our purpose, let us give an example in the SU(4) standard procedure, in connection to the isospin operator. The SU(2)-isospin Casimir operator was written as \( T^2 = T^2_c + 2t \cdot T_c + 3/4 \), i.e. formed of three independent pieces, corresponding to the terms in this decomposition. In SU(4) \( T^2_c \) and \( S^2_c \) have identical matrix elements because the spin and isospin states of a symmetric core are identical, so that \( T^2_c \) can be neglected. But \( t \cdot T_c \) has different matrix elements from \( s \cdot S_c \) as one can clearly see from Table II of Ref. \[21\]. Then, in the decoupling scheme the isospin can be introduced only through \( t \cdot T_c \). In Ref. \[16\] Table VI we have shown that the introduction of the operators \( \frac{1}{N_c} t \cdot T_c \) together with
\[ \frac{1}{N_c} S_c^2 \text{ and } \frac{1}{N_c} s \cdot S_c \text{ separately deteriorates the fit. This may explain why } \frac{1}{N_c} t \cdot T_c \text{ has been avoided in previous numerical fits in SU}(4) \]

Physical consequences of the neglect of the isospin operator were discussed in Ref. [27] for SU(4), where it was shown that the isospin term, neglected in the standard procedure, becomes as dominant in \( \Delta \) as the spin term in \( N \) resonances. As a first physical application of this work, where we extend our procedure to SU(6) we ask again the question why the operator \( \frac{1}{N_c} t \cdot T_c \), as well as \( T_c^2 \) were avoided in previous numerical fits in SU(6) [24].

Before presenting our work we wish to point out that the calculation of the matrix elements of the operators appearing in the mass formula, with the approximate wave function of the standard procedure is not however an approximation. In the framework of a large \( N_c \) quark model, by using properties of the permutation group, Pirjol and Schat [29] have shown that one can pass from the exact wave function to that of Ref. [21] without making any approximation. This implies that an approximate wave function can be used in an effective theory, provided the constraints imposed by a given quark model are satisfied. These constraints represent fixed ratios between specific coefficients in the mass formula in terms of well defined radial integrals.

By analogy to our previous work [16], here we analyze the spin and flavor terms both for the exact and the truncated, or else, approximate, wave function of the [70, \( t^P \)] multiplet, without any prejudice. In other words we extend our previous work from SU(4) (\( N_f = 2 \)) to SU(6) (\( N_f = 3 \)). We briefly recall the standard procedure based on the core+excited quark separation. The relation between the approximate wave function [21] and the exact one, has been presented already in Ref. [16]. To test the validity of the approximate wave function we compare the analytical expressions of the matrix elements of the spin and flavor operators entering the mass formula. For the first time we explicitly show that at large \( N_c \) the approximate wave function provides matrix elements of the flavor operator where the flavor singlet \( 2^1 \) behaves asymptotically different from the octets \( 2^8, 4^8 \) and the decuplet \( 2^{10} \) (the notation corresponding to \( N_c = 3 \)), which means that the large \( N_c \) counting rule is broken (see Sec. III). By contrast, the exact wave functions lead to identical analytic forms at large \( N_c \) for all spin-flavor multiplets belonging to the representation 70 of SU(6) allowing a consistent definition of the flavor operator.

The paper is organized as follows. In the next section we recall the relation between the
exact and approximate wave function and derive the isoscalar factors needed for the flavor singlet $^21$. In Sec. III we derive analytic expressions of some dominant operators entering the mass formula and discuss their behavior at large $N_c$ for the multiplets which become $^28$, $^48$, $^210$ and $^21$ when $N_c = 3$, both for the approximate and the exact wave function. The last section presents our conclusions. In Appendix A we recall some isoscalar factors obtained previously, but needed in this work. In Appendix B we describe the procedure to obtain general analytic expression of new $S_N$ isoscalar factors associated with the wave function of the $^21$ multiplet. Appendix C is devoted to the derivation of the matrix elements of the generator $G^{ja}$ of SU(6) and exhibits the SU(6) isoscalars factors calculated in this work.

II. THE WAVE FUNCTION

We deal with a system of $N_c$ quarks where one quark carries $\ell$ units of orbital excitation. Therefore the orbital ($O$) wave function must have a mixed symmetry $[N_c - 1, 1]$, which describes the lowest excitations in a baryon. The $N_c - 1$ independent basis states of the $[N_c - 1, 1]$ irreducible representation (irrep) corresponding to $N_c - 1$ Young tableaux, as presented below, is equivalent to a basis written in terms of $N_c - 1$ internal Jacobi coordinates, thus the center of mass motion is automatically removed. The center of mass motion is then described by the symmetric state $[N_c]$ with one excited quark.

The color wave function being antisymmetric, the orbital-spin-flavor wave part must be symmetric. Then the spin-flavor ($FS$) part must have the same symmetry as the orbital part in order to obtain a totally symmetric state in the orbital-spin-flavor space. We recall that the general form of such a wave function is \[ |[N_c]\rangle = \frac{1}{\sqrt{d_{[N_c-1,1]}}} \sum_Y |[N_c - 1, 1]Y\rangle_O |[N_c - 1, 1]Y\rangle_{FS}, \] (2)

where $d_{[N_c-1,1]} = N_c - 1$ is the dimension of the representation $[N_c - 1, 1]$ of the permutation group $S_{N_c}$ and $Y$ is a symbol for a Young tableau (Yamanouchi symbol). The sum is performed over all possible standard Young tableaux. By convention, in each term the first basis vector represents the orbital space and the second the spin-flavor space. In this sum there is only one $Y$ (the normal Young tableau) where the last particle is in the second row and $N_c - 2$ terms where the last particle is in the first row. The explicit form of the orbital part is not needed in this work.
More precisely, we write $Y = (pqy)$ where $p$ is the row of the $N_c$-th particle, $q$ the row of the $(N_c - 1)$-th particle and $y$ is the Young tableau of the remaining particles. Let us denote by $p, p'$ and $p''$ the position of the last particle in the spin-flavor, spin and flavor Young tableaux respectively. They are indicated by crosses in the example given by Eqs. (7)-(8) below. Similarly for the $(N_c - 1)$-th particle we have $q, q'$ and $q''$ and for the rest $y, y'$ and $y''$.

We need now to decompose the spin-flavor wave function into its spin and flavor parts. For this purpose we use the Clebsch-Gordan (CG) coefficients of $S_{N_c}$, denoted by $S([f']p'q'y'[f'']p''q''y'')[f]pqy$ and their factorization property \[30\]. Denoting by $K([f']p'[f'']p'')$ the isoscalar factors of $S_{N_c}$ we have \[16\]

$$S([f']p'q'y'[f'']p''q''y'')[f]pqy) = K([f']p'[f'']p'')S([f']q'y'[f'']p'')q''y'')[f]pqy),$$

(3)

where the second factor in the right-hand side is a CG coefficient of $S_{N_c-1}$ containing the partitions $[f''], [f'p'']$ and $[f_p]$ obtained after the removal of the $N_c$-th quark. Keeping in mind that, for a given $p$, the quantum numbers of the SU(6) wave function are the same and by using the above property we can write the spin-flavor part of the wave function as

$$\langle [N_c - 1, 1]p; (\lambda\mu)YII_3; SS_3 \rangle = \sum_{p'p''} K([f']p'[f'']p'')\langle [N_c - 1, 1]p|SS_3; p'\rangle (\lambda\mu)YII_3; p''),$$

(4)

where $\langle SS_3; p'\rangle (\lambda\mu)YII_3; p''$ contains the CG coefficients $S([f'']q'y'[f'p']q'y'')[f]pqy$ and includes a sum over $q'y'$ and $q''y''$. These CG coefficients sum up to 1 by normalization. Then in the matrix elements of every SU(6) opearator we shall have one term with $p = 2$ and $N_c - 2$ terms with $p = 1$ (see example in the next section).

In the wave function \[4\] the spin part $|SS_3; p'\rangle$ is defined by the SU(2) coupling

$$|SS_3; p'\rangle = \sum_{m_{1},m_{2}} \left( \begin{array}{c} S \cr m_{1} \cr \frac{1}{2} \cr m_{2} \cr S_{3} \end{array} \right) |S_{c}m_{1}\rangle \frac{1}{\sqrt{2m_{2}}},$$

(5)

with $S_{c} = S - 1/2$ for $p' = 1$ and $S_{c} = S + 1/2$ for $p' = 2$ and the flavor part by the SU(3) coupling

$$|(\lambda\mu)YII_3, p'\rangle =$$

$$\sum_{y_{c},j_{c},i_{c}} \left( \begin{array}{c} (\lambda_{c}y_{c}) \cr (\lambda\mu) \cr YI \cr I_{c} \cr i_{c} \cr I_{3} \end{array} \right) |Y_{c}I_{c}I_{c}\rangle |(10)yii_{3}\rangle,$$

(6)
with \((\lambda_c, \mu_c) = (\lambda - 1, \mu)\) for \(p'' = 1\), \((\lambda_c, \mu_c) = (\lambda + 1, \mu - 1)\) for \(p'' = 2\) and \((\lambda_c, \mu_c) = (\lambda, \mu + 1)\) for \(p'' = 3\). Here \(\lambda\) and \(\mu\) are consistent with the partition \([f'']\) from Tables I, IV, V and VI respectively. As usually, in Eq. (6), the SU(3) CG coefficient has been factorized into an isoscalar factor and an SU(2)-isospin factor [31].

By taking \(N_c = 7\) let us first illustrate Eq. (4) in terms of Young tableaux for the case presently under study, namely the flavor singlet of spin \(S = 1/2\). We have two linearly independent spin-flavor states

\[
\begin{align*}
\begin{array}{c}
\times \end{array} & = K([43]1[331]3|[61]2) \times \begin{array}{c}
\times \end{array}, \\
\begin{array}{c}
\times \end{array} & = K([43]1[331]2|[61]1) \times \begin{array}{c}
\times \end{array} + K([43]2[331]2|[61]1) \times \begin{array}{c}
\times \end{array} + K([43]2[331]3|[61]1) \times \begin{array}{c}
\times \end{array},
\end{align*}
\]

where the cross in the left-hand side indicates that the states (7) and (8) correspond to \(p = 2\) and to \(p = 1\) respectively. We remind that in the right-hand side, if one removes the crossed box, the first and second Young tableaux describe the spin and flavor states respectively. In fact each such product represents a spin-flavor state of \(S_6\) of partition \([6]\) and \([51]\) for \(p = 2\) and \(p = 1\) respectively, coupled to the 7th quark in a given spin-flavor state. When \(N_c = 3\) we recover the \(2^1\) flavor singlet. This case is new and completes our work on the \([N_c - 1, 1]\) states by allowing to incorporate the \(\Lambda\) baryons.

We recall that the approximate wave function [21] contains only terms with \(p = 2\) as discussed in Ref. [27].

The isoscalar factors \(K([f']p'[f'']p''|[f]'p)\) of \(S_{N_c}\) for the spin-flavor states corresponding \(2^8\) and \(4^8\) and \(2^{10}\) multiplets, when \(N_c = 3\), have been obtained in Ref. [16] and as we need them again, for self-consistency they are reproduced in Appendix A.

The analytic forms obtained here for the isoscalar factors needed for the states corresponding to the flavor singlet \(2^1\) when \(N_c = 3\) are reproduced in Table I. Details of the calculations are given in Appendix B. Note that the analytic expressions of Table I hold for \(N_c\) odd only, because the partitions must contain integer numbers.
TABLE I: Isoscalar factors $K([f']p'[f'']p''|[f]p)$ for $S = 1/2$, corresponding to $^21$ when $N_c = 3.$

| $[f']p'[f'']p''$ | $[N_c - 1, 1]1$ | $[N_c - 1, 1]2$ |
|------------------|----------------|-----------------|
| $[N_c + 1, N_c - 1]/2, [N_c - 1, N_c - 1]/2]$ | 1 $[N_c - 1, N_c - 1]/2, 1]2$ | $1 \sqrt{(N_c - 3)(N_c + 1)/N_c(N_c - 2)}$ 0 |
| $[N_c + 1, N_c - 1]/2, [N_c - 1, N_c - 1]/2]$ | 1 $[N_c - 1, N_c - 1]/2, 1]3$ | 0 1 |
| $[N_c + 1, N_c - 1]/2, [N_c - 1, N_c - 1]/2$ | 2 $[N_c - 1, N_c - 1]/2, 1]2$ | $1 \sqrt{3(N_c - 3)(N_c + 1)/N_c(N_c - 2)}$ 0 |
| $[N_c + 1, N_c - 1]/2, [N_c - 1, N_c - 1]/2$ | 2 $[N_c - 1, N_c - 1]/2, 1]3$ | $-\sqrt{3/N_c(N_c - 2)}$ 0 |

In Table I and those of Appendix A, the isoscalar factors from the columns with $p = 1$ and $p = 2$ obey the orthogonality property defined generally as

$$\sum_{p',p''} K([f']p'[f'']p''|[f]p)K([f']p'[f'']p''|[f_1]p_1) = \delta_{f_1} \delta_{pp_1}. \quad (9)$$

The expressions exhibited in Table I have been checked for $N_c = 3, 5, 7$ and 9, by using the recurrence relation described in Ref. [32] which allows to obtain isoscalar factors of $S_{N-1}$ from those of $S_N$. For consistency the same phase convention must be constantly applied. Tables I, IV, V and VI prove that this is the case, one has the same phase irrespective of $N_c$. Thus they offer a convenient test to check the phase convention rule. The results of Table I and those reproduced in Appendix A can be used for any fermion system described by the partition $[N_c - 1, 1]$ where one fermion must be separated from the rest.

III. MATRIX ELEMENTS

It is very important to apply the $1/N_c$ expansion method to both nonstrange and strange baryons together. First, we have at our disposal a larger number of experimental data than for nonstrange baryons alone and second, we can get an unified picture of all light baryons.

In the following we consider $N_f = 3$. When the SU(3)-flavor symmetry is exact, the $1/N_c$ expansion mass operator describing an excited state can be written as the linear combination

$$M^{(1)} = \sum_i c_i O_i, \quad (10)$$

8
where \( c_i \) are unknown coefficients which parametrize the QCD dynamics and the operators \( O_i \) are combinations of SU(6) and SO(3) generators \( L^i \). The presence of \( L^i \) is necessary in describing excited states.

For the purpose of our analysis and as an extension of the previous work \[16\], here it is enough to consider some of the most dominant operators, namely the spin and flavor operators. Previous experience indicates that the most dominant operators to order \( \mathcal{O}(1/N_c) \) included, are those constructed from SU(2\(N_f\)) exclusively \[27\], the operators containing \( L^i \) bringing usually smaller contributions.

We recall that in the standard procedure, based on core+quark separation, these operators are \( s \cdot S_c, \, S_c^2, \, t \cdot T_c\) and \( T_c^2 \). The analytic expressions of the expectation values, calculated both with the approximate and the exact wave functions, as defined in the previous section, are presented in Tables III and [III].

Regarding the spin operators the only change with respect to SU(4) \[16\] is the addition of the last row where the result is naturally identical to that of first and third ones when the exact wave function is used, because \( ^28, ^210 \) and \( ^21 \) have the same spin. The approximate wave function leads to different results for \( ^28, ^210 \) and \( ^21 \) because the wave function is truncated. Note that for the approximate wave function we agree with Ref. \[24\]. As a matter of fact, for the approximate wave function, the matrix elements of \( s \cdot S_c \) and \( S_c^2 \) are independent of \( N_c \) for \( ^48, ^210 \) and \( ^21 \) the reason being again the truncation of the wave function.

The expressions and the order in \( N_c \) of the expectation values of \( t \cdot T_c \) and \( T_c^2 \) with SU(6) wave functions are naturally different from those of SU(4) \[16\]. Using the wave function described by Eqs. \[\text{(4)-(6)}\] we have first obtained the general form of the expectation value of \( T_c^2 \) at fixed \( p \). This is

\[
\langle T_c^2 \rangle_p = \frac{1}{3} \sum_{p'p''} K([f']p'[f''p''|p])^2 g_{\lambda_c\mu_c}
\]

where

\[
g_{\lambda_c\mu_c} = (\lambda_c^2 + \mu_c^2 + \lambda_c\mu_c + 3\lambda_c + 3\mu_c),
\]

where \( \lambda_c \) and \( \mu_c \) depend on \( p'' \) as defined below Eq. \[\text{(10)}\].

Taking \( p = 2 \) we recover the expressions of \( \langle T_c^2 \rangle \) with the approximate wave function as exhibited in Table III, column 3.
TABLE II: Matrix elements of the spin operators calculated with the approximate and the exact wave functions.

|       | \langle s \cdot S_c \rangle | \langle S^2_c \rangle |
|-------|-----------------------------|---------------------|
|       | approx. w.f. | exact w.f. | approx. w.f. | exact w.f. |
| $^{28}$ | \frac{N_c + 3}{4N_c} | \frac{3(N_c - 1)}{4N_c} | \frac{N_c + 3}{2N_c} | \frac{3(N_c - 1)}{2N_c} |
| $^{48}$ | \frac{1}{2} | \frac{3(N_c - 5)}{4N_c} | 2 | \frac{3(3N_c - 5)}{2N_c} |
| $^{210}$ | -1 | \frac{3(N_c - 1)}{4N_c} | 2 | \frac{3(N_c - 1)}{2N_c} |
| $^{21}$ | 0 | \frac{3(N_c - 1)}{4N_c} | 0 | \frac{3(N_c - 1)}{2N_c} |

TABLE III: Matrix elements of the flavor operators calculated with the approximate and the exact wave functions.

|       | \langle t \cdot T_c \rangle | \langle T^2_c \rangle |
|-------|-----------------------------|---------------------|
|       | approx. w.f. | exact w.f. | approx. w.f. | exact w.f. |
| $^{28}$ | \frac{N_c(N_c - 4) - 9}{12N_c} | \frac{(N_c - 9)(N_c - 1)}{12N_c} | 18 + N_c + 4N^2_c + N^3_c | \frac{(N_c - 1)[18 + N_c(N_c + 5)]}{12N_c} |
| $^{48}$ | \frac{N_c - 13}{12} | \frac{(N_c - 9)(N_c - 1)}{12N_c} | \frac{19 + N_c(N_c + 4)}{12} | \frac{(N_c - 1)[18 + N_c(N_c + 5)]}{12N_c} |
| $^{210}$ | \frac{N_c + 5}{12} | \frac{45 + N_c(N_c - 10)}{12N_c} | \frac{(N_c + 2)^2 + 15}{12} | \frac{-90 + N_c[49 + N_c(N_c + 4)]}{12N_c} |
| $^{21}$ | \frac{-N_c + 5}{6} | \frac{N_c(N_c - 16) - 9}{12N_c} | \frac{(N_c - 1)(N_c + 5)}{12} | \frac{18 + N_c[7 + N_c(N_c - 2)]}{12N_c} |

For the exact wave function both $p = 1$ and $p = 2$ contribute. According to the discussion following Eq. (2) the expectation value of $T^2_c$ becomes

$$\langle T^2_c \rangle = \frac{1}{N_c - 1} \left[ \langle T^2 \rangle_{p=2} + (N_c - 2)\langle T^2 \rangle_{p=1} \right]. \quad (13)$$

Note that such a combination of $p = 1$ and $p = 2$ terms is required for any operator in the mass formula (10) when the matrix elements are calculated with the exact wave function. For $\langle T^2_c \rangle$ the results are presented in the last column of Table III.

Knowing that $\langle T^2 \rangle = g_{\lambda \mu}/3$ with $g_{\lambda \mu}$ defined as in Eq. (12), but with $\lambda_c \mu_c \rightarrow \lambda \mu$, one
can then derive the matrix element of $t \cdot T_c$ as
\[ \langle t \cdot T_c \rangle = \frac{1}{2} \left[ \langle T^2 \rangle - \langle T_c^2 \rangle - \frac{4}{3} \right], \tag{14} \]
both for the exact and the approximate wave functions. At fixed $p$ this is in agreement with Eq. (A15) of Ref. [26].

Actually we are interested in the operators $\frac{1}{N_c} t \cdot T_c$ and $\frac{1}{N_c} T_c^2$, entering the mass formula (10). One can see that with the exact wave functions the matrix elements of the operator $\frac{1}{N_c} t \cdot T_c$ are of order $O(N_c^0)$ and those of the operator $\frac{1}{N_c} T_c^2$ of order $O(N_c)$ for all spin-flavor multiplets of mixed symmetric states. To fulfill the large $N_c$ counting a solution would be to make the replacement
\[ \frac{1}{N_c} t \cdot T_c \rightarrow \frac{1}{N_c} \left( t \cdot T_c - \frac{1}{12} \mathbb{1} \right), \tag{15} \]
where the shift is due to the subtraction of the dominant operator $O_1 = \mathbb{1}$ of order $O(N_c)$, and similarly, $\frac{1}{N_c} T_c^2$ can be replaced by
\[ \frac{1}{N_c} T_c^2 \rightarrow \frac{1}{N_c} \left( T_c^2 - \frac{N_c}{12} \mathbb{1} \right), \tag{16} \]
because for both operators the extracted terms are identical for all spin-flavor multiplets of the mixed representation $[N_c - 1, 1]$. By compensation, in the mass operator, these terms can provide an additional contribution to the leading orders $N_c^0$ and $N_c$ respectively.

A similar procedure is impossible for the approximate wave function because the $^21$ multiplet has a different large $N_c$ analytic form than the other spin-flavor multiplets of $[N_c - 1, 1]$, as one can see from Table III. Thus there is no unique term to be subtracted.

Based on the standard procedure with the approximate wave function, the authors of Ref. [33] observed that the matrix elements of the shifted operator (15), denoted in their work by $O_5$ vanish for “all states in multiplets with $Y_{max} = \frac{N_c}{3}$ (which includes all nonstrange states in the “$70$”). For $Y_{max} = \frac{N_c}{3} - 1$ multiplets the matrix elements of $O_5$ are found to be $-1/4$”. The latter value is consistent with the expression in the last row of our Table III for the approximate wave function because $\langle t \cdot T_c \rangle \rightarrow -N_c/6$ in the large $N_c$ limit. Note that the vanishing of the expectation values of the operator (15) takes place in fact only at large $N_c$ for the $^28$, $^48$ and $^210$ multiplets. This makes us to believe that, to some extent, it was known that the $^21$ multiplet had a different large $N_c$ behavior than the other multiplets, but from the statement of Ref. [33] it is not quite clear that the cancellation takes place at large $N_c$ only and that it does not hold for the $^21$ multiplet of “$70$”.

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Therefore from the analysis of Table III we conclude that the approximate wave function does not lead to the same large $N_c$ limit for $\langle t \cdot T_c \rangle$, irrespective of the spin-flavor multiplet contained in $[N_c - 1, 1]$. In addition, note that for the $21$ multiplet the sign of the matrix element of $\langle t \cdot T_c \rangle$ is negative, consistent with Ref. [33]. However, let us note that in the symmetric core approach there is no problem in obtaining five towers of states because $t \cdot T_c/N_c$ appears among the four $\mathcal{O}(N_c^0)$ needed operators.

We have already encountered the problem with the large $N_c$ behavior of $T^2$ in Ref. [28]. Based on the correspondence between a Young diagram and an irrep of SU(3), we have shown that using SU(6) generators acting on the whole system, here $T^a$ of Eq. (1), a general elegant solution is to redefine the operator $T^a T^a$ as

$$\frac{1}{N_c} \left[ T^a T^a - \frac{1}{12} N_c (N_c + 6) \right]$$

(17)

where the subtracted term, $\frac{1}{12} N_c (N_c + 6)$, appears in the Casimir expectation value of all irreducible representations of SU(3) contained in $70$. The operator (17) is of order $N_c^0$ for $N_f = 3$. Interestingly, this definition recovers the expectation values of the isospin operator $O_4 = \frac{1}{N_c} T_i T^i$ ($i = 1, 2, 3$) for $N_f = 2$ [27]. Indeed, from the expectation value of the first term with $f = 0$ in Eq. (30) of Ref. [28] we obtain

$$\langle O_4 \rangle = \frac{1}{4 N_c} \lambda (\lambda + 2).$$

(18)

Taking $\lambda = 2I$ we recover the SU(4) results of Table 3, column 5 of Ref. [27] in a single formula, which shows that the matrix elements of the isospin operator are order $1/N_c$, as expected.

But, on the other hand, for the flavor singlet $21$ the eigenvalues of the operator (17) are of order $\mathcal{O}(N_c^0)$. This can be seen by writing the eigenvalue of the Casimir operator not in terms of $\lambda$ and $\mu$ but in terms of $\lambda$ and $f$, where $f$ is the number of columns filled with 3 boxes and $\mu = (N_c - \lambda - 3f)/2$ [28]. In that case we have [28]

$$\frac{1}{N_c} T^a T^a = \frac{1}{12 N_c} \left\{ N_c (N_c + 6) + 3\lambda (\lambda + 2) - 3f [2(N_c + 3) - 3f] \right\}.$$  

(19)

From this expression the first term cancels out with the last term in Eq. (17) and the remaining quantity is of order $\mathcal{O}(N_c^0)$. Therefore the flavor operator introduces a shift between the flavor singlet $21$ and the other multiplets, namely $28$, $48$ and $210$. This is entirely consistent with the results of Cohen and Lebed [33] of five irreps labelled by the
grand spin $K$. Then the 5 independent mass eigenvalues are split by $\mathcal{O}(N_c^0)$. By analogy, in our approach the five towers will be due to the operator $O_1 = N_c \mathbb{1}$ of order $\mathcal{O}(N_c)$ and to 4 other operators of order $\mathcal{O}(N_c^0)$ which are $L \cdot S$, $\frac{1}{N_c} \left[ T \cdot T - \frac{1}{12} N_c (N_c + 6) \right]$, $\frac{1}{N_c} L \cdot T \cdot G$ and $\frac{1}{N_c} L^{(2)} \cdot G \cdot G$. Their matrix elements will be presented elsewhere.

Moreover, from Table III one can see that the matrix elements of $t \cdot T_c$ and $T_c^2$ with the approximate wave function associated with the octets $^2S^2$ and $^4S^2$ are different from each other, which is not natural, because the flavor operators are independent of spin. By contrast, the exact wave function leads to identical expressions, which is correct. In this situation, a quantitative discussion between results obtained, on the one hand, with the exact, and on the other hand, with the approximate wave functions, cannot be made, as it was done for SU(4).

IV. CONCLUSIONS

Let us briefly present our conclusions. In the scheme based on the decoupling of the system into a symmetric core of $N_c - 1$ quarks and an excited quark, the flavor operator was decomposed in three independent parts, namely $T_c^2$, $t \cdot T_c$ and a constant both in SU(4) [21] and in SU(6) [24].

In SU(4), describing nonstrange baryons, $S_c^2$ and $T_c^2$ have identical matrix elements because the spin and isospin states of a symmetric core are identical, so that $T_c^2$ can be neglected. In SU(6) the situation is different. One must include $T_c^2$ as well in the mass formula.

Thus in SU(4), in the decoupling scheme, the isospin can be introduced only through the operator $t \cdot T_c$. However $t \cdot T_c$ has been ignored in all studies of the spectrum based on this scheme, although it has entirely different matrix elements from those of $s \cdot S_c$ [21]. In Ref. [16], Table VI, we have shown that the introduction of $\frac{1}{N_c} t \cdot T_c$, together with $\frac{1}{N_c} S_c^2$ and $\frac{1}{N_c} s \cdot S_c$ as independent operators, deteriorates the fit. This may explain why $\frac{1}{N_c} t \cdot T_c$ has been avoided in previous numerical fits in SU(4) [21].

In SU(6) both $t \cdot T_c$ and $T_c^2$ are necessary and both have been ignored [24]. Actually it would be simpler and more physically to consider the full Casimir operator $T^2$ of SU(3) instead of independent $t \cdot T_c$ and $T_c^2$ operators. Such a procedure was used in SU(4) [27] for
TABLE IV: Isoscalar factors $K([f']p'[f'']p' || [f]p)$ for $S = I = 1/2$, corresponding to $^28$ when $N_c = 3$.

| $[f']p'[f'']p''$ | $[N_c - 1, 1]_1$ | $[N_c - 1, 1]_2$ |
|-----------------|-----------------|-----------------|
| $[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]_1 [\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]_1$ | 1 | 0 |
| $[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]_2 [\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]_2$ | $\sqrt{\frac{N_c - 3}{2(N_c - 2)}}$ | $\frac{N_c + 3}{4N_c}$ |
| $[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]_2 [\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]_2$ | $-\frac{1}{2}\sqrt{\frac{N_c - 1}{N_c}}$ | 0 |
| $[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]_2 [\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]_2$ | $-\frac{1}{2}\sqrt{\frac{N_c - 1}{N_c - 2}}$ | 0 |

Due to the failure of the approximate wave functions [21] to lead to the same large $N_c$ counting of $\frac{1}{N_c}t \cdot T_c$ for all spin-flavor multiplets of $70$, a quantitative estimate between the two approaches, one based on the exact, the other on the approximate wave function, could not presently be made for SU(6) as it was previously made for SU(4) [16].

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Appendix A: Isoscalar factors needed for $^28$, $^48$ and $^210$

Here we find it useful to recall the isoscalar factors of $S_N$ obtained in Ref. [16], which are also needed in this study. They are exhibited in Tables IV, V and VI for $S = I = 1/2$, $S = 3/2$, $I = 1/2$ and $S = 1/2$, $I = 3/2$ states respectively, for arbitrary $N_c$.

Appendix B: Equations for isoscalar factors needed for $^21$

Here we shortly describe the method used to obtain general analytic forms for the isoscalar factors of $S_{N_c}$ presented in Table II.
TABLE V: Isoscalar factors $K([f']p'[f'']p''|[f]p)$ for $S = 3/2$, $I = 1/2$, corresponding to $^8_4$ when $N_c = 3$.

| $[f']p'[f'']p''$ | $[N_c - 1, 1]1$ | $[N_c - 1, 1]2$ |
|------------------|----------------|----------------|
| $[N_c + 3, N_c - 3]_2$ | $[N_c + 1, N_c - 1]_2$ | $1$ | $1/2 \sqrt{(N_c - 1)(N_c + 3)/N_c(N_c - 2)}$ | $0$ |
| $[N_c + 3, N_c - 3]_2$ | $[N_c + 1, N_c - 1]_2$ | $2$ | $1/2 \sqrt{5(N_c - 1)(N_c - 3)/2N_c(N_c - 2)}$ | $0$ |
| $[N_c + 3, N_c - 3]_2$ | $[N_c + 1, N_c - 1]_2$ | $1$ | $1/2 \sqrt{(N_c - 3)(N_c + 3)/2N_c(N_c - 2)}$ | $1$ |
| $[N_c + 3, N_c - 3]_2$ | $[N_c + 1, N_c - 1]_2$ | $2$ | $0$ | $0$ |

TABLE VI: Isoscalar factors $K([f']p'[f'']p''|[f]p)$ for $S = 1/2$, $I = 3/2$, corresponding to $^{10}_2$ when $N_c = 3$.

| $[f']p'[f'']p''$ | $[N_c - 1, 1]1$ | $[N_c - 1, 1]2$ |
|------------------|----------------|----------------|
| $[N_c + 1, N_c - 1]_2$ | $[N_c + 3, N_c - 3]_2$ | $1$ | $1/2 \sqrt{(N_c - 1)(N_c + 3)/N_c(N_c - 2)}$ | $0$ |
| $[N_c + 1, N_c - 1]_2$ | $[N_c + 3, N_c - 3]_2$ | $2$ | $1/2 \sqrt{5(N_c - 1)(N_c - 3)/2N_c(N_c - 2)}$ | $0$ |
| $[N_c + 1, N_c - 1]_2$ | $[N_c + 3, N_c - 3]_2$ | $1$ | $1/2 \sqrt{(N_c - 3)(N_c + 3)/2N_c(N_c - 2)}$ | $1$ |
| $[N_c + 1, N_c - 1]_2$ | $[N_c + 3, N_c - 3]_2$ | $2$ | $0$ | $0$ |

Recall that we deal with the partitions

$$[f'] = \left[ \frac{N_c + 1}{2}, \frac{N_c - 1}{2} \right], \quad [f''] = \left[ \frac{N_c - 1}{2}, \frac{N_c - 1}{2}, 1 \right], \quad [f] = [N_c - 1, 1].$$  \hspace{1cm} (B1)

The case $p = 2$ is trivial for $^2_1$, $^8_4$ and $^{10}_2$, see Tables I, V and VI. Using Young diagrams it becomes obvious that there is only one non-vanishing isoscalar factor, which according to the orthogonality relation (9) reads

$$K([f']1[f'']3|f]2) = 1.$$  \hspace{1cm} (B2)
For example, Eq. (7) contains a single term in the right-hand side and therefore the corresponding isoscalar factor has the value 1, consistent with the normalization properties and the phase convention [30, 32].

For \( p = 1 \) there are three non-vanishing isoscalar factors, so we need three equations to derive them. Our first idea was to use the same equations as in Ref. [16]. They turn out to be quadratic.

The first equation is the orthogonality relation (9) which for \( p = 1 \) becomes

\[
(K([f']1[f'']2|[N_c - 1, 1]1))^2 + (K([f']2[f'']2|[N_c - 1, 1]1))^2 + (K([f']2[f'']3|[N_c - 1, 1]1))^2 = 1.
\]

(B3)

A second equation is provided by the diagonal matrix elements of the SU(6) generator \( G^{ia} \), written as a sum of two generators: \( G^{ia}_c \) which acts on a subsystem of \( N_c - 1 \) quarks, where \( c \) stands for "core", and \( g^{ia} \) which acts on the last quark separated from the rest. One has

\[
\langle G^{ia} \rangle = \langle G^{ia}_c \rangle + \langle g^{ia} \rangle
\]

(B4)

Note that for \( p = 1 \) the core has the partition \([N_c - 2, 1]\). This equation gives the following relation between the isoscalar factors

\[
\frac{N_c + 3}{4}(K([f']1[f'']2|[N_c - 1, 1]1))^2 - \frac{N_c(N_c - 6) - 11}{4(N_c - 1)}(K([f']2[f'']2|[N_c - 1, 1]1))^2
+ \frac{(N_c - 3)(N_c + 3)}{2(N_c - 1)}(K([f']2[f'']3|[N_c - 1, 1]1))^2
+ \frac{\sqrt{3}(N_c - 1)}{2}K([f']1[f'']2|[N_c - 1, 1]1)K([f']2[f'']2|[N_c - 1, 1]1)
+ \frac{4\sqrt{(N_c - 3)(N_c + 1)}}{N_c - 1}K([f']2[f'']2|[N_c - 1, 1]1)K([f']2[f'']3|[N_c - 1, 1]1)
+ \frac{N_c - 3}{2} = 0
\]

(B5)

From the Casimir identity, one can derive a third equation

\[
\frac{\langle s \cdot S_c \rangle}{3} + \frac{\langle t \cdot T_c \rangle}{2} + 2\langle g \cdot G_c \rangle = \frac{5N_c - 11}{12},
\]

(B6)

which gives another quadratic equation

\[
\frac{(N_c - 1)^2}{48}(K([f']1[f'']2|[N_c - 1, 1]1))^2
+ \frac{83 + N_c(N_c - 15) + 7}{48(N_c - 1)}(K([f']2[f'']2|[N_c - 1, 1]1))^2
\]
\[
\begin{align*}
&- \frac{91 - N_c(N_c(N_c + 3) - 9)}{48(N_c - 1)} (K([f'][2][f''][3][N_c - 1, 1, 1])^2 \\
&+ \frac{\sqrt{3}(N_c - 1)}{4} K([f'][1][f''][2][N_c - 1, 1, 1])K([f'][2][f''][2][N_c - 1, 1, 1]) \\
&+ \frac{2\sqrt{(N_c - 3)(N_c + 1)}}{N_c - 1} K([f'][2][f''][2][N_c - 1, 1, 1])K([f'][2][f''][3][N_c - 1, 1, 1]) \\
&- \frac{(N_c - 19)(N_c - 1)}{48} = 0. \quad \text{(B7)}
\end{align*}
\]

However, one can show that the system of equations \[(B3), (B5)\] and \[(B6)\] is linearly dependent. By using a recurrence relation described in Refs. \([30, 32]\), we have derived the values of the isoscalar factors for \(N_c = 3, 5, 7\) and \(9\). From them we could quite easily make a generalization to an arbitrary \(N_c\). These expressions are presented in Table I. They verify the three equations presented above.

**Appendix C: Matrix elements of \(\langle g \cdot G_c \rangle\)**

Equation \[(B7)\] requires the knowledge of the matrix element of \(\langle g \cdot G_c \rangle\) for which we first need the matrix elements of \(G_c^{ja}\). Using the generalized Wigner-Eckart theorem, the matrix elements of \(G_c^{ja}\) have been obtained in Ref. \([28]\) as

\[
\begin{align*}
&\langle [N_c - 1, 1]p; (\lambda \mu') Y'I'I_3'; S'm_s'|G_c^{ja}|[N_c - 1, 1]p; (\lambda \mu) Y'I_3; Sm_s \rangle = \\
&(-1)^{1/2} \sqrt{2S + 1} \frac{C^{[f]}(SU(6))}{2} \begin{pmatrix} S & 1 \\ m_s & j \end{pmatrix} \begin{pmatrix} I & I_a \\ I_3 & I_3' \end{pmatrix} \\
&\times \sum_{p',p'',p',p''} (-1)^{S_c}(-1)^{\lambda - \lambda_c + \lambda' - \lambda_c'}(-1)^{\mu - \mu_c + \mu' - \mu_c'} \sqrt{(2S_c' + 1)}K([f']p'[f''][p''][[N_c - 1, 1]p) \\
&\times K([f']p'[f''][p''][[N_c - 1, 1]p) \\
&\times U((10)(\lambda_c \mu_c)(\lambda' \mu')(11); (\lambda \mu)_\rho(\lambda' \mu'_{\rho_c})_{\rho_c}) \begin{pmatrix} [f_c] & [214] \\ [f_c] & [11] \end{pmatrix} \begin{pmatrix} (\lambda_c \mu_c)S_c (11) \\ (\lambda' \mu'_{\rho_c})S_c' \end{pmatrix} \end{align*}
\]

where \(C^{[f]}(SU(6))\) is the SU(6) Casimir operator associated to the irreducible representation described by the partition \([f_c]\). For \(p = 1, 2\) one has \([f_c] = [N_c - 2, 1]\) and \([f_c] = [N_c - 1]\) respectively, in agreement with the discussion in Sec. II. Note that in Ref. \([28]\) there is a misprint which has been corrected here: the factor \(\sqrt{2S_c' + 1}\) has been included in the sum.
over \( \rho' \) in agreement with the definition of \( S'_c \) following Eq. (4). Also the upper index of the SU(6) Casimir operator has been corrected by replacing \([f]\) by \([f_c]\).

For a single quark the matrix element of the generator \( g^{j a} \) takes a simpler form

\[
\langle [N_c - 1, 1] \rho; (\lambda' \mu') Y'I'O_3; S'm'_s | g^{j a} | [N_c - 1, 1] \rho; (\lambda \mu) YI3; Sm_s \rangle = \\
(-1)^{S'_c - 1/2} \sqrt{2(2S + 1)} \left( \begin{array}{c}
S' \hfill \hfill S' \hfill \hfill 1 \\
1 \hfill I \hfill 1 \\
m_s \hfill j \hfill m'_s \\
I \hfill I \hfill 1 \\
I_3 \hfill I_3 \hfill I'_3
\end{array} \right) \\
\times \sum_{p', p''} (1 - 1)^{S_c} K([f']p'[f']p''|[N_c - 1, 1] \rho) K([f]p_1[f]p_1|[N_c - 1, 1] \rho) \left( \begin{array}{c}
S \hfill 1 \\
1/2 \hfill S_c \hfill 1/2 \\
\end{array} \right)
\]

Note that in Ref. [28] there is a misprint in the Racah coefficient \( U((\lambda_c \mu_c)(10)(\lambda' \mu')(11); (\lambda \mu)_\rho(10)) \) where the lower index \( \rho \) has been inadvertently shifted.

The above matrix elements lead to

\[
\langle [N_c - 1, 1] \rho; (\lambda' \mu') Y'I'O_3; S'm'_s | g \cdot G_c | [N_c - 1, 1] \rho; (\lambda \mu) YI3; Sm_s \rangle = \\
\delta_{SS'} \delta_{m_m'} \delta_{II'} \delta_{I_3 I'_3} (-1)^{S - 1/2} \sqrt{\frac{C[f_c]}{2}} \left( \begin{array}{c}
C[f_c](SU(6)) \\
2
\end{array} \right) \times \\
\times \sum_{p', p''} (1 - 1)^{S_c} K([f']p'[f']p''|[N_c - 1, 1] \rho) K([f]p_1[f]p_1|[N_c - 1, 1] \rho) \left( \begin{array}{c}
1/2 \hfill S' \hfill S \hfill 1/2 \hfill 1 \\
S_c \hfill 1/2 \hfill 1
\end{array} \right)
\]

\[
\times \sum_{\rho_c} U((\lambda_c \mu_c)(11)(\lambda \mu)(10); (\lambda' \mu'_c)_{\rho_c}(10)) \left( \begin{array}{c}
[f_c] \hfill [21^4] \\
(\lambda_c \mu_c)_{S_c} (11)1 \hfill (\lambda' \mu'_c)_{S'_c} \rho_c
\end{array} \right)
\]

Eqs. (C1) or (C3) contain new isoscalar factors of SU(6) which we derived in this study using the same method as in Ref. [28]. They are presented in Table VII for \( \rho = 1 \) and 2. The last row corresponds to \( \rho = 1 \), the only non-vanishing case. In a similar way we have also obtained

\[
\left( \begin{array}{c|c}
[N_c - 1, 1] \hfill [21^4] \hfill [N_c - 1, 1] \\
(\lambda - 1, \mu + 1) S \hfill (11)1 \hfill (\lambda - 1, \mu - 1) S
\end{array} \right) = \frac{1}{5} \sqrt{\frac{6(2S - 1)(S + 1)(N_c - 2)(S - 2)}{(N_c - 2)(S - 1)N_c(5N_c + 18)}},
\]

(C4)
TABLE VII: Isoscalar factors of the SU(6) generator $G^{ja}$ needed for the $^21$ multiplet.

| $(\lambda_1\mu_1)S_1$ | $(\lambda_2\mu_2)S_2$ | $\rho$ | $\frac{[N_c - 1, 1]}{[21^4]}$ | $[N_c - 1, 1]$ | $(\lambda_1\mu_1)S_1$ $(\lambda_2\mu_2)S_2$ | $(\lambda + 1, \mu - 2)S$ |
|----------------------|----------------------|-------|-----------------------------|----------------|-----------------------------|-------------------------|
| $(\lambda + 1, \mu - 2)S$ | (11)1 | 1 | $[N_c(4S + 7) + 6(S + 1)]$ | $\frac{2S}{(S + 1)(N_c^2 + 12S(S + 2))N_c(5N_c + 18)}$ |
| $(\lambda + 1, \mu - 2)S$ | (11)1 | 2 | $\frac{4(S(S + 3) - 1) - N_c(N_c + 6)}{2S + 1}$ | $\frac{3S(2S + 1)(2S + 3)(N_c + 2S)(N_c - 2(S + 1))}{2(S + 1)(N_c - 2S)(N_c + 2(S + 2))(N_c^2 + 12S(S + 2))N_c(5N_c + 18)}$ |
| $(\lambda + 1, \mu - 2)S + 1$ | (11)1 | 1 | $3N_c$ | $\frac{2(2S + 3)}{(S + 1)(N_c^2 + 12S(S + 2))N_c(5N_c + 18)}$ |
| $(\lambda + 1, \mu - 2)S + 1$ | (11)1 | 2 | $[12S(S + 2) - N_c(N_c + 6)]$ | $\frac{3(N_c + 2S)(N_c - 2(S + 2))}{2(S + 1)(2S + 1)(N_c + 2(S + 2))(N_c^2 + 12S(S + 2))(N_c - 2S)N_c(5N_c + 18)}$ |
| $(\lambda_\mu)S + 1$ | (11)1 | / | $\frac{-S}{S + 1}$ | $\frac{6(N_c - 2(S + 1))(2S + 3)}{(2S + 1)(N_c - 2S)N_c(5N_c + 18)}$ |
where $\rho$, when unspecified, by convention corresponds to $\rho = 1$. When applying the above formulas one has to be rather careful with the meaning of $\lambda$ and $\mu$ when they are expressed in terms of $S$ as above. For example for the flavor singlet $^21$ one has $S = 1/2$, which implies that the label $(\lambda - 1, \mu - 1)$ corresponds to the flavor singlet, inasmuch as, by definition, we take $\lambda = 2S$ and $\mu = (N_c - 2S)/2$. The isoscalar factors of Table VII can be used in other studies based on SU(6).

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