Symmetries in the projection evolution model

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Abstract.
Some introductory results concerning symmetries and conservation laws within the Projection Evolution (PEv) formalism are presented. It is shown that the PEv formalism conserves the quantum probability distributions after unitary transformations of the evolution operators represented by the orthogonal resolution of unity. Some conditions for transformations selecting the states with equal transition probabilities are found. A relation between a given symmetry represented by a Lie group and PEv conservation of the eigenvalues of the Casimir operator is derived.

1. Introduction
Until the end of the XIX century, in classical physics, time played the role of a parameter which orders events. In the next century, the relativistic theory demonstrated the necessity of treating time on the same footing as the space variables. The development of similar ideas in quantum mechanics was stopped for many years by the Pauli theorem [1, 2, 3]. This theorem states that it is not possible to form a Hermitian operator representing the position in time. The conclusion from this principle is that time cannot be a dynamical variable but only a parameter, similarly as it was in the classical mechanics. A detailed look at the theorem shows, however, that some of its assumptions can be weakened [4, 5]. Thanks to this, the construction of the time operator is possible. This paves the way to work on problem of time as a variable which plays equivalent role to the space variables.

The main goal of the projection evolution model is to rebuild the idea of the evolution and to change the role of time. The basic assumption in the projection evolution model is that the state of the Universe, which is described by the four-dimensional spacetime, can spontaneously change, according to some probability distribution. The probability depends on the state at the previous
step of the evolution and the resulting state. Thus the evolution means the transformation of
the state from one step of the evolution to another driven by operators which fulfill appropriate
conditions. This excludes the unitary evolution where time is a parameter, as it is in the standard
quantum mechanics. This type of evolution should be regarded as an approximation of the PEv
only.

This approach to the problem of time in quantum physics gives rise to the projection evolution
model [6]–[14]. This model allows to describe many problems and physical phenomena in a
natural way. It allows to discuss the position in time operator [6], the interference in time [7, 9],
particle oscillations [10], delayed-choice experiments [11] and others.

The idea of symmetries is a very strong, well developed and widely used tool in physics. In
the projection evolution model, due to its specificity it is still an open problem. On the one hand,
because in PEv the time is a quantum observable, any symmetry analysis has to be performed
in the full spacetime. On the other hand, in this model the evolution is driven not only by the
unitary transformations, which requires to discover new rules for symmetries and their relations
to some conservation laws.

This leads to two separate types of problems in the symmetry analysis. One is to consider
symmetry as an invariance in a single step of the evolution. In other words, we have to analyze
the invariance of the states and required operators in the four dimensional spacetime. This
analysis should give results analogical to the symmetry analysis in the standard quantum
mechanics. The second problem is to consider symmetry along the evolution path, i.e., the
chain of states which is formed by the subsequent steps in the evolution process. This is a new
kind of problem because it requires to find invariance properties of every state in the evolution
path. Such an analysis requires new theoretical methods which have to be developed. Because
of the current lack of such methods, the symmetry analysis which is introduced in this paper
does not form a systematical review. It is rather a piecemeal set of ideas how to approach this
problem.

In the first section of the paper the main idea of the projection evolution model is introduced.
It contains the description of the main theoretical assumptions of the model and introduces the
mathematical tools used later. In the second section some elements of the symmetry analysis
along the evolution path of the system are introduced. The last section contains a few remarks
and open questions which appeared during the work on this problem and which can be the
starting point for further research.

2. The projection evolution
The main postulate in the projection evolution (PEv) is the evolution principle. The evolution
principle states that the evolution of a quantum system is a random process caused by the
changes of the Universe. Changes of the Universe are the primary process. Every state of the
Universe and any of its subsystems is, in this model, a function in the spacetime, which contains
the full history including the past, the present, and the future. This happens, however, only
until the next step of the evolution. The next step of the evolution modifies the state in the
full spacetime, creating new history of the system. The change of the state can even affect the
past, probably on very short time intervals only. It divests the time of the role of a parameter
for the ordering of events. The problem of the broken causality can be observed even in the
case of a semiclassical treatment of time [8]. In this model, the causality ordering relation is
not necessarily determined by the direction of time. This property is a completely new idea and
carry many new consequences.

The random process of spontaneous changes of the Universe is parametrized by the parameter
$\tau$. The parameter $\tau$ should be understood as an ordering parameter, thus in the set $\tau$ belongs
to, a linear order relation must be defined. The ordering in the set of $\tau$ has a physical meaning
of the quantum causality. $\tau$ is not understood as an additional dimension of the spacetime.
As the results of an evolution process one obtains the chain of states numbered by $\tau$. At every evolution step $\tau_n$, for the system under consideration there is a set of available states $\rho(\tau_n; \nu_k^{(n)})$. The operator $\rho(\tau_n; \nu_k^{(n)})$ denotes the density operator which represents the state of the system. The first parameter $\tau_n$ denotes the evolution step and the second parameter $\nu_k^{(n)}$ distinguishes between different states available at the step $\tau_n$. The realization of such a step-by-step evolution is called the evolution path,

$$\rho(\tau_1; \nu_k^{(1)}) \rightarrow \rho(\tau_2; \nu_k^{(2)}) \rightarrow \ldots \rightarrow \rho(\tau_n; \nu_k^{(n)}) \rightarrow \ldots$$

(1)

The labels $\nu_k^{(n)}$ can be understood as quantum numbers which are attached to quantum observables. We write $\nu^{(n)} \in Q(\tau_n)$, where $Q(\tau_n)$ is a family of sets of quantum numbers defining available states for the evolution step $\tau_n$. The set of available states for a given $\tau_n$ is denoted by:

$$S(\tau_n) = \{ \rho(\tau_n; \nu_k^{(n)}); \nu_k^{(n)} \in Q(\tau_n) \}.$$  

(2)

The transitions among the states are described by the evolution operator $\mathcal{F}(\tau; \nu, \rho)$:

$$\mathcal{F}(\tau; \nu; \cdot) : T_1^+(\mathcal{K}) \rightarrow T^+(\mathcal{K}),$$

(3)

where $\mathcal{K}$ represents the state space of our physical system, $T_1^+(\mathcal{K})$ is the space of operators with trace equal to one and $T^+(\mathcal{K})$ is the space of operators with positive and finite trace.

The operator $\mathcal{F}(\tau; \nu, \rho)$ is Hermitian and positively defined:

$$\mathcal{F}(\tau; \nu, \rho) = \mathcal{F}(\tau; \nu, \rho)^\dagger, \quad \mathcal{F}(\tau; \nu, \rho) \geq 0.$$  

(4)

For every fixed step of the evolution, the sum over available states of traces of $\mathcal{F}(\tau; \nu, \rho)$ is finite:

$$\sum_\nu \text{Tr}[\mathcal{F}(\tau; \nu, \rho)] < \infty.$$  

(5)

During the evolution, the next state is obtained from the previous one by using the operator $\mathcal{F}(\tau; \nu, \rho)$:

$$\rho(\tau_n; \nu_k^{(n)}) = \frac{\mathcal{F}(\tau_n; \nu_k^{(n)}, \rho(\tau_{n-1}; \nu_l^{(n-1)}))}{\text{Tr}[\mathcal{F}(\tau_n; \nu_k^{(n)}, \rho(\tau_{n-1}; \nu_l^{(n-1)}))]}.$$  

(6)

It is chosen randomly from the set of available states with the probability distribution for the chooser given by

$$\text{pev}(\tau_n; \nu_l^{(n-1)} \rightarrow \nu_k^{(n)}) \equiv \text{Prob} \left( \rho(\tau_{n-1}; \nu_l^{(n-1)}) \rightarrow \rho(\tau_n; \nu_k^{(n)}) \right).$$  

(7)

The function (7) denotes the probability of the transition from the previous state $\rho(\tau_{n-1}; \nu_l^{(n-1)})$, defined by the quantum numbers $\nu_l^{(n-1)} = (\nu_1^{(n-1)}, \nu_2^{(n-1)}, \ldots, \nu_l^{(n-1)})$, to the next state $\rho(\tau_n; \nu_k^{(n)})$ defined by the quantum numbers $\nu_k^{(n)} = (\nu_k^{(n)}, \nu_k^{(n)}, \ldots, \nu_k^{(n)})$.

The most important realization of the evolution operators $\mathcal{F}(\tau; \nu, \rho)$ can be constructed from the Krauss-like operators [15]:

$$\mathcal{F}(\tau; \nu, \rho) = \sum_a \mathcal{E}(\tau; \nu, a) \rho \mathcal{E}(\tau; \nu, a)^\dagger,$$  

(8)

$$= \sum_a \mathcal{E}(\tau; \nu, a) \rho \mathcal{E}(\tau; \nu, a)^\dagger.$$  

(8)
where the summation over $a$ dependent on the quantum numbers $\nu$. This form of the operators $\mathcal{F}(\tau; \nu, \rho)$ is important for a few reasons. Firstly, if for a given step of the evolution the $\mathcal{E}(\tau; \nu)$ operators form an orthogonal resolution of unity,

$$
\mathcal{E}(\tau_n; \nu_k^{(n)}) \equiv \mathcal{E}(\tau_n; \nu_k^{(n)}, a = 1) = \text{projection operators}, \quad (9)
$$

$$
\mathcal{E}(\tau_n; \nu_k^{(n)}) \mathcal{E}(\tau; \nu_k^{(n)}) = \delta_{kk'} \mathcal{E}(\tau; \nu_k^{(n)}), \quad (10)
$$

$$
\sum_k \mathcal{E}(\tau_n; \nu_k^{(n)}) = 1, \quad (11)
$$

then the transition probability (7) is given in the following form:

$$
\text{pev}(\tau_n; \nu_l^{(n-1)} \rightarrow \nu_k^{(n)}) = \text{Tr} \left[ \mathcal{E}(\tau_n; \nu_k^{(n)}) \rho(\tau_{n-1}; \nu_l^{(n-1)}) \right]. \quad (12)
$$

In general, the method of finding the transition probability distribution from one mixed state, described by the density operator $\rho(\tau_n; \nu_k^{(n)})$, to another mixed state at the next step of the evolution is still an open problem. The above formula is valid for a special case in which the transition probability distribution is known. In the remainder of the paper, unless noted, the operator $\mathcal{F}(\tau; \nu, \rho)$ will be implemented by operators $\mathcal{E}(\tau; \nu)$ in the form (8).

The second important property of the decomposition (8) is, that this kind of the evolution operators can be defined by means of the so-called evolution generators. The projection evolution generators $\hat{\mathcal{W}}$ for a given $\tau$ are defined as a family of Hermitian operators whose spectral decompositions (plus possible constraints) give the orthogonal resolution of unity. They serve as the evolution operators:

$$
\hat{\mathcal{W}}(\tau_n) = \sum_k w_k \mathcal{E}(\tau_n; \nu_k^{(n)}). \quad (13)
$$

By using the appropriate generator one can reproduce the standard quantum mechanical evolution as a limiting case. For example, the generators:

$$
\hat{\mathcal{W}}(\tau) = \hat{p}_0 - \hat{\mathcal{H}}(\tau) \quad \text{(Schrödinger),} \quad (14)
$$

$$
\hat{\mathcal{W}}(\tau) = \hat{p}_\mu \hat{p}_\nu \quad \text{(free Klein-Gordon),} \quad (15)
$$

$$
\hat{\mathcal{W}}(\tau) = \gamma^\mu \hat{p}_\mu \quad \text{(free Dirac),} \quad (16)
$$

result in the Schrödinger, the free Klein-Gordon, and the free Dirac evolution, respectively. This fact shows that the projection evolution model reproduces the results from the standard quantum mechanics.

3. Symmetries

The symmetry of an object is a property which relates the invariance of this object with respect to an operation. In this paper we use a very general definition of symmetry. Let $X$ be a set of objects and relation $\sim$ be an equivalence relation defined on $X$. The operation $S : X \rightarrow X$ is a symmetry of $x \in X$ if and only if it does not change the relation $\sim$,

$$
x \sim S(x). \quad (17)
$$

The set of operations $S$ forms a group.

Depending on the physical requirements, the relation $\sim$ can be defined differently. For example, it can be defined as an invariance of a special functional, as it is done in the case of the action functional in the classical mechanics, the relativity relations among equivalent observers.
Because of the generality of this definition, it contains a wide spectrum of applications to physical problems.

In this paper we discuss the problem of symmetry analysis along the evolution path only. In the following we consider the evolution operators $\mathcal{E}(\tau; \nu)$ which are either a combination of the unitary operators or they form an orthogonal resolution of unity.

The problem is to find either the observable, or the expectation value of an observable, or the property of the physical state, or a similar object which is invariant at every step of the evolution along the evolution path. In other words, transformation from one evolution step to another does not change this object. The main motivation for studying of this problem is the unknown relation between the symmetries and the conservation laws.

We start from the definition of transformations of the operator $\mathcal{E}(\tau; \nu, \rho)$. Because $\mathcal{E}(\tau; \nu, \rho)$ transforms the quantum state $\rho$ into another quantum state (not normalized), the resulting image of $\mathcal{E}(\tau; \nu, \rho)$ has to be compatible with the transformation of arguments.

To explain this, let us consider the problem of the rotation of some vector function. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. The values of the rotated function $f'$ on the rotated argument $x'$ should be equal to the rotation of the value of the original function on the original argument, $f'(x') = R f(x)$.

In an analogical way the transformation of $\mathcal{E}(\tau; \nu, \rho)$ is defined.

Let $\mathcal{E}(\tau_n; \nu, \rho)$ be the evolution operator defined in (3). Let $G$ be a group with two realizations $S_1(g) : \mathcal{T}_1^+(\mathcal{K}) \rightarrow \mathcal{T}_1^+(\mathcal{K})$ and $S(g) : \mathcal{T}^+(\mathcal{K}) \rightarrow \mathcal{T}^+(\mathcal{K})$. The transformation of the evolution operator $\mathcal{E}$ is defined as:

$$
\mathcal{E}'(\tau_n; \nu, \rho') = S(g) \mathcal{E}(\tau_n; \nu, \rho) S(g^{-1}),
$$

i.e., the resultant evolution operator for the transformed state is equal to the appropriately chosen transformation of the evolution operator for the original state. This idea can be expressed in a more convenient form:

$$
\mathcal{E}'(\tau_n; \nu, \rho) = S(g) \mathcal{E}(\tau_n; \nu, S_1(g^{-1}) \rho S_1(g)) S(g^{-1}).
$$

(19)

The group $G$ provides the same physical interpretation for the operators $S(g)$ and $S_1(g)$ in the above definition.

In the same way one can define the transformation (19) for the evolution generated by the operators $\mathcal{E}(\tau_n; \nu_k^{(n)})$. For shortness we use the notation $S(g) = S_1(g) = g$. For the $n$-th step of the evolution, equation (19) has the following form:

$$
\left[\mathcal{E}(\tau_n; \nu_k^{(n)}) \rho(\tau_{n-1}; \nu_k^{(n-1)}) \mathcal{E}(\tau_n; \nu_k^{(n)})\right]' = g \mathcal{E}(\tau_n; \nu_k^{(n)}) g^{-1} \rho(\tau_{n-1}; \nu_k^{(n-1)}) g \mathcal{E}(\tau_n; \nu_k^{(n)}) g^{-1}.
$$

(20)

The above equation shows that the transformation (19) for the operator $\mathcal{E}(\tau_n; \nu_k^{(n)})$ is equivalent to the transformation by the group element $g$,

$$
\left[\mathcal{E}(\tau_n; \nu_k^{(n)})\right]' = g \mathcal{E}(\tau_n; \nu_k^{(n)}) g^{-1}.
$$

(21)

Definition (18) allows to conserve the probability distributions for unitary equivalent images of quantum mechanics. Let us assume that the transition probability from the evolution step $\tau_{n-1}$ to $\tau_n$ is given by

$$
\text{Prob} [\rho(\tau_{n-1}; \nu_k^{(n-1)}) \rightarrow \rho(\tau_n; \nu_k^{(n)})] = \text{Tr} [\mathcal{E}(\tau_n; \nu_k^{(n)}, \rho(\tau_{n-1}; \nu_k^{(n-1)})]].
$$

(22)

Similarly, the transition probability of the transformed by the group $G$ state $\rho'(\tau_{n-1}; \nu_k^{(n-1)})$ and, at the same time, the transformed evolution operator $\mathcal{E}'$, is given by

$$
\text{Prob} [\rho'(\tau_{n-1}; \nu_k^{(n-1)}) \rightarrow \rho'(\tau_n; \nu_k^{(n)})] = \text{Tr} [\mathcal{E}'(\tau_n; \nu_k^{(n)}, \rho'(\tau_{n-1}; \nu_k^{(n-1)})]].
$$

(23)
It follows from equation (18) that both probabilities are equal:

$$\text{Prob} \left[ \rho(\tau_{n-1}; \nu_l^{(n-1)}) \rightarrow \rho(\tau_{n}; \nu_k^{(n)}) \right] = \text{Prob} \left[ \rho'(\tau_{n-1}; \nu_l^{(n-1)}) \rightarrow \rho'(\tau_{n}; \nu_k^{(n)}) \right]. \quad (24)$$

The consequence of such a symmetry for every step of the evolution is the fact that $g$ does not change the structure of the possible evolution paths, which means that the probability distribution of the potential evolution paths is conserved. In this case, the operations $g \in G$ can also be interpreted as operations which transform among equivalent descriptions of a given model (covariance), or as the transformation between equivalent observers (a kind of “relativity”).

As the second kind of symmetry let us consider invariance of the transition probability when the final states are transformed by the group $G$. In this case, the symmetry group $G$ is responsible for creating the sets of states which are chosen by the evolution in a completely random way. This specific conservation of transition probability can be defined as:

$$\text{Prob} \left[ \rho(\tau_{n-1}; \nu_l^{(n-1)}) \rightarrow \rho(\tau_{n}; \nu_k^{(n)}) \right] = \text{Prob} \left[ \rho(\tau_{n-1}; \nu_l^{(n-1)}) \rightarrow \rho'(\tau_{n}; \nu_k^{(n)}) \right]. \quad (25)$$

The transition probabilities among the transformed states are:

$$\text{Prob} \left[ \rho(\tau_{n-1}; \nu_l^{(n-1)}) \rightarrow \rho'(\tau_{n}; \nu_k^{(n)}) \right] = \text{Tr} \left[ g \mathcal{E}(\tau_{n}; \nu_k^{(n)}) g^{-1} \rho(\tau_{n-1}; \nu_l^{(n-1)}) g \mathcal{E}(\tau_{n}; \nu_k^{(n)}) \right]. \quad (26)$$

It is easy to see that in two special cases,

$$[g, \mathcal{E}(\tau_{n}, \nu^{(n)})] = 0 \quad \text{or} \quad [g, \rho(\tau_{n-1}, \nu^{(n-1)})] = 0, \quad (27)$$

this probability is the same as for the original states.

The next important problem is the relation between symmetries and conservation laws. The correct formulation of this problem in the projection evolution model is problematic. In the most intuitive way one can say that we are looking for conditions under which the expectation value of given observable $A$ is conserved within the PEv approach:

$$\langle A \rangle_{\rho(\tau_{n-1}; \nu_k^{(n)})} = \langle A \rangle_{\rho(\tau_{n-1}; \nu_k^{(n)})} = \ldots = \langle A \rangle_{\rho(\tau_{n-1}; \nu_k^{(n)})} = \ldots \quad (28)$$

The required conditions may involve special relations between the evolution operators, density operators and the observable operator. In the case when the evolution is described by the operators $\mathcal{E}(\tau_{n}; \nu_k^{(n)})$, the conservation of the expectation value $\langle A \rangle$ has the form:

$$\text{Tr} \left[ A \rho(\tau_{n-1}; \nu_l^{(n-1)}) \right] = \frac{\text{Tr} \left[ A \mathcal{E}(\tau_{n}; \nu_k^{(n)}) \rho(\tau_{n-1}; \nu_l^{(n-1)}) \mathcal{E}(\tau_{n}; \nu_k^{(n)}) \right]}{\text{Tr} \left[ \mathcal{E}(\tau_{n}; \nu_k^{(n)}) \rho(\tau_{n-1}; \nu_l^{(n-1)}) \mathcal{E}(\tau_{n}; \nu_k^{(n)}) \right]} \quad (29)$$

for a series of subsequent $n$.

Assume that the operator $\mathcal{E}(\tau_{n}; \nu_k^{(n)})$ is unitary. Under this assumption the condition (29) takes the following form:

$$\text{Tr} \left[ A \rho(\tau_{n}; \nu_k^{(n)}) \right] = \frac{\text{Tr} \left[ A \mathcal{E}(\tau_{n}; \nu_k^{(n)}) \rho(\tau_{n-1}; \nu_l^{(n-1)}) \mathcal{E}(\tau_{n}; \nu_k^{(n)}) \right]}{\text{Tr} \left[ \mathcal{E}(\tau_{n}; \nu_k^{(n)}) \rho(\tau_{n-1}; \nu_l^{(n-1)}) \mathcal{E}(\tau_{n}; \nu_k^{(n)}) \right]} \quad (30)$$

$$= \text{Tr} \left[ \mathcal{E}(\tau_{n}; \nu_k^{(n)}) \mathcal{E}(\tau_{n}; \nu_k^{(n)}) \right] A \mathcal{E}(\tau_{n}; \nu_k^{(n)}) \rho(\tau_{n-1}; \nu_l^{(n-1)})]. \quad (31)$$

In this case the expectation value of the observable $A$ is conserved if the operator $A$ commutes with the evolution operator, i.e., $[A, \mathcal{E}(\tau_{n}; \nu_k^{(n)})] = 0$. This fact has its counterpart in the standard
quantum mechanics – if the Hamiltonian commutes with the operator \( A \), its expectation value is conserved during the unitary evolution generated by this Hamiltonian.

The requirement of the commutation of the observable and the evolution operator is a rather strong condition. In some cases, it is possible to weaken this condition considering an approximate invariance of \( \langle A \rangle \) for the unitary evolution. Assume that the observable \( A \) and the evolution operator do not commute, \( [A, \hat{E}(\tau_n; \nu_k^{(n)})] \neq 0 \), and let the evolution operator be unitary in the form:

\[
\hat{E}(\tau_n; \nu_k^{(n)}) = \exp(-i\sigma_n \hat{S}) = I + (e^{-i\sigma_n} - 1)\hat{S},
\]

where \( \hat{S} \) is a projection operator and \( \sigma_n \) is a number which depends on the evolution step \( \tau_n \).

The expectation value of \( A \) in the step \( n \) is in this case given by

\[
\text{Tr}[A\rho(\tau_n; \nu_k^{(n)})] = \text{Tr}[A\rho(\tau_{n-1}; \nu_l^{(n-1)})] + (e^{i\sigma_n} - 1)\text{Tr}[\hat{S}A\rho(\tau_{n-1}; \nu_l^{(n-1)})] + (e^{-i\sigma_n} - 1)\text{Tr}[\hat{A}\hat{S}\rho(\tau_{n-1}; \nu_l^{(n-1)})] + 2(1 - \cos \sigma_n)\text{Tr}[\hat{A}\hat{S}\rho(\tau_{n-1}; \nu_l^{(n-1)})\hat{S}].
\]

The expectation value of \( A \) for the step \( n \) is equal to the sum of the expectation value of \( A \) at the previous evolution step and the part which depend on \( \sigma_n \). Approximate invariance can be achieved by taking \( \sigma_n \) to minimize the part which depends on this parameter.

The idea of the unitary evolution can be generalized in different ways. For example, it is possible to consider the state at the evolution step \( n \) as a result of a unitary evolution of the previous state through a few different channels. In this case, the state for the evolution step \( n \) is a linear combination of the products of different unitary evolutions of the previous state,

\[
\rho(\tau_n; \nu_k^{(n)}) = \frac{\sum_{m=1}^{N} U_m(\tau_n)\rho(\tau_{n-1}; \nu_l^{(n-1)})U_m(\tau_n)\dagger}{\text{Tr}[\sum_{m=1}^{N} U_m(\tau_n)\rho(\tau_{n-1}; \nu_l^{(n-1)})U_m(\tau_n)\dagger]},
\]

where \( U_m = U_m^{-1} \) for \( m = 1, \ldots, N \).

The second example, which leads to a unitary evolution, is when the evolution generator \( \hat{W}(\tau_n) \) evolves unitarily \( \hat{W}(\tau_n) = U(\tau_n)\hat{W}_0U(\tau_n)\dagger \). Both examples open a new problem in the analysis of symmetries in the PEv model and they require further investigation.

The next interesting case of relations between the conservation laws and symmetries is the invariance of both the evolution operator and the quantum observable.

Let \( G \) be a group, \( \mathcal{C} \) be a Casimir operator of the group \( G \), and \( |\kappa \Gamma \alpha \rangle \) be a vector of an irreducible representation. Assume that the operators \( \hat{E}(\tau_n) \) commute with the group \( [\hat{G}, \hat{E}(\tau_n)] = 0 \). Then the Casimir operator and the density operator can be described in a basis of irreducible representation in the following way:

\[
\mathcal{C} = \sum_{\Gamma} C_{\Gamma} \sum_{\kappa,\alpha} |\kappa \Gamma \alpha \rangle \langle \kappa \Gamma \alpha | \quad \text{and} \quad \hat{E}(\tau_n, \nu_k^{(n)}) = \sum_{\kappa \alpha} |\kappa \Gamma \alpha \rangle \langle \kappa \Gamma \alpha |,
\]

where \( \Gamma \) depends on \( \tau_n \). Then

\[
\text{Tr}[\mathcal{C}\rho(\tau_n; \nu_k^{(n)})] = \frac{\text{Tr}[\mathcal{C}\hat{E}(\tau_n; \nu_k^{(n)})\rho(\tau_{n-1}; \nu_l^{(n-1)})\hat{E}(\tau_n; \nu_k^{(n)})]}{\text{Tr}[\hat{E}(\tau_n; \nu_k^{(n)})\rho(\tau_{n-1}; \nu_l^{(n-1)})\hat{E}(\tau_n; \nu_k^{(n)})]} = C_{\Gamma}.
\]

It follows that if the evolution operators are invariant with respect to the group \( G \), the expectation value of the Casimir operator \( \mathcal{C} \) of the group \( G \) is conserved during the evolution.

This fact has its analogy in the standard quantum mechanics. Let us assume that the Hamiltonian \( \hat{H} \) is invariant with respect to a group \( G \). The eigenvectors of \( \hat{H} \) belong to the invariant subspaces spanned by the bases of the irreducible representations of the group \( G \). In this case the expectation value of the Casimir operator is conserved during the unitary evolution generated by this Hamiltonian.
4. Summary
In this preliminary report a few special cases of symmetries and possible conservation laws have been considered. Because the projection evolution allows for different types of evolution, we have focused on the most typical cases.

The first one is the unitary evolution, where the evolution parameter $\tau$ enumerates subsequent evolution steps. The unitary evolution operators $E(\tau; \nu) = U(\tau)$ act in the full spacetime, contrary to the standard unitary evolution operator $\exp(-iHt/\hbar)$ which acts in the three dimensional space only, with time $t$ playing the role of the events ordering parameter.

The second, very probably the most interesting case is the projection evolution generated by the evolution operators $E(\tau; \nu)$ which furnish the orthogonal resolution of unity. The standard, Schrödinger type unitary evolution is a special case of the projection evolution generated by the appropriate orthogonal resolution of unity with the wave functions sharply localized in time.

As the first result in this paper we have found the basic conditions which have to be fulfilled to satisfy the covariance properties of the projection evolution with respect to a given group of transformations.

When the evolution operator is a four-dimensional unitary evolution, the results are similar to those which can be obtained in the standard Schrödinger type evolution. In this case the commutation of an observable $A$ and the evolution operators is a sufficient condition for the conservation of the expectation value $\langle A \rangle_\rho$ between the steps of the projection evolution.

The last case shows a simple relation between the given symmetry group and the conservation of appropriate Casimir operators while the system evolves according to a specific form of the evolution operators.

It is obvious that the symmetries and conservation laws are the most basic features of any quantum formalism. The PEv approach also requires the full analysis of these problems. They will be the subject of further investigation.

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