Carlson’s Theorem for Different Measures

Meredith Sargent

Department of Mathematics, Campus Box 1146, Washington University in St Louis, St Louis, MO 63130

Abstract

We use an observation of Bohr connecting Dirichlet series in the right half plane \( \mathbb{C}_+ \) to power series on the polydisk to interpret Carlson’s theorem about integrals in the mean as a special case of the ergodic theorem by considering any vertical line in the half plane as an ergodic flow on the polytorus. Of particular interest is the imaginary axis because Carlson’s theorem for Lebesgue measure does not hold there. In this note, we construct measures for which Carlson’s theorem does hold on the imaginary axis for functions in the Dirichlet series analog of the disk algebra \( \mathcal{A}(\mathbb{C}_+) \).

Keywords: Dirichlet Series, Ergodic Theorem

1. Introduction

In 1913, Bohr observed that one may connect Dirichlet series converging on the right half plane \( \mathbb{C}_+ = \{ s \in \mathbb{C} : \Re(s) > 0 \} \) to power series on the infinite polydisk using the correspondence

\[
z_1 = 2^{-s}, \ z_2 = 3^{-s}, \ldots, \ z_j = p_j^{-s}, \ldots
\]

where \( p_j \) denotes the \( j \)th prime. For a Dirichlet series \( f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \), we can use the fundamental theorem of arithmetic to factor each integer \( n \) uniquely and then represent \( f \) by a power series \( F \) in the variables \( \{ z_j \} \). As discussed in [2], the Bohr correspondence also allows us to consider any vertical line in \( \mathbb{C} \) as an ergodic flow on the infinite-dimensional polytorus \( \mathbb{T}^\infty \):

\[
(e^{i\theta_1}, e^{i\theta_2}, \ldots) \mapsto (p_1^{-it} e^{i\theta_1}, p_2^{-it} e^{i\theta_2}, \ldots) \in \mathbb{T}^\infty
\]

and in particular, the imaginary axis maps to the boundary of the infinite polydisk (of radius one.) We would like to compare a “space average” of the power series \( F \) on \( \mathbb{T}^\infty \) to a “time average” of the Dirichlet series \( f \) on the ergodic flow described above. For this question, we consider \( \mathcal{H}^\infty \), the Banach space of Dirichlet series of the form

\[
f(s) = \sum_{n=1}^{\infty} a_n n^{-s}
\]

that converge to bounded analytic functions on \( \mathbb{C}_+ \).

A theorem of Carlson [2] tells us about the limit in the mean of a Dirichlet series on an ergodic flow for \( \sigma > 0 \).

**Theorem 1** (Carlson’s Theorem). If a Dirichlet series \( f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \) converges in the right half plane \( \mathbb{C}_+ \) and is bounded in every half plane \( \Re(s) \geq \delta \) for \( \delta > 0 \), then for each \( \sigma > 0 \)

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |f(\sigma + it)|^2 \, dt = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma}
\]
Saksman and Seip showed in [5] that Carlson’s theorem fails to hold on the imaginary axis when we replace \( f(\sigma + it) \) with its non-tangential limit \( f(it) \) (which exists for almost every \( t \)).

**Theorem 2** (Saksman-Seip). The following two statements hold:

(i) There exists a function \( f \) in \( H^\infty \) such that
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |f(it)|^2 dt
\]
does not exist.

(ii) Given \( \epsilon > 0 \), there exists a singular inner function \( g = \sum_{n=1}^{\infty} b_n n^{-s} \) in \( H^\infty \) such that
\[
\sum_{n=1}^{\infty} |b_n|^2 \leq \epsilon.
\]

What this result tells us is that there can be no direct analog of Carlson’s Theorem on the boundary: the limit need not exist and equality need not hold, at least not for Lebesgue measure and for all functions in \( H^\infty \). However, by looking at a smaller space, we can prove an analog of Carlson’s theorem.

The space we consider is \( A(\mathbb{C}_+) \), the set of Dirichlet series which are convergent on \( \mathbb{C}_+ \) and define uniformly continuous functions there. In [1], Aron, Bayart, Gauthier, Maestre, and Nestoridis show that \( A(\mathbb{C}_+) \) is a closed subspace of \( H^\infty \) and prove that it consists exactly of the uniform limits of Dirichlet polynomials:

**Theorem 3.** Given \( f : \mathbb{C}_+ \to \mathbb{C} \) the following are equivalent.

1. \( f \) is the uniform limit on \( \mathbb{C}_+ \) of a sequence of Dirichlet polynomials.
2. \( f \) is represented by a Dirichlet series pointwise on \( \mathbb{C}_+ \) and \( f \) is uniformly continuous on \( \mathbb{C}_+ \).

We now come to the main result:

**Theorem 4.** (i) Let \( \mu \) be a Borel probability measure on the infinite torus \( T^\infty \). There exists a locally finite Borel measure \( \lambda \) on \( \mathbb{R} \), such that, for all \( f \in A(\mathbb{C}_+) \)
\[
\lim_{T \to \infty} \frac{1}{\lambda([0,T])} \int_0^T |f(it)|^2 d\lambda(t) = \int_{T^\infty} |F(z)|^2 d\mu(z).
\] (1.3)

(ii) Let \( \lambda \) be a locally finite Borel measure on \( \mathbb{R} \) such that the limit on the left hand side of (1.3) exists and is finite for all \( f \in A(\mathbb{C}_+) \). Then there exists a unique Borel probability measure \( \mu \) on the infinite torus \( T^\infty \) such that, for all \( f \in A(\mathbb{C}_+) \), (1.3) holds.

To prove Theorem 4, it is helpful to first consider following useful lemma which allows us to first consider linear combinations of point masses and construct corresponding \( \lambda \), and then use that result to construct \( \lambda \) for general Borel measures.

**Lemma 1** ([4]). Let \( X \) be a compact metric space. The set \( V \) of finite linear combinations of point masses is dense in the space of finite Borel measures, \( M(X) \), with the weak-* topology.
2. Construction of \( \lambda \) for when \( \mu \) is a linear combination of point masses

To construct \( \lambda \), it is also helpful to recall Kronecker’s theorem:

**Lemma 2.** [Kronecker’s Theorem] Let \( q_1, \ldots, q_k \in \mathbb{R} \) be linearly independent over \( \mathbb{Q} \) and let \( \gamma_1, \ldots, \gamma_k \in \mathbb{R} \) and \( T, \epsilon > 0 \) be given. Then there exists \( t > T \) and \( q_1, \ldots, q_k \in \mathbb{Z} \) such that

\[
|tq_j - \gamma_j - q_j| < \epsilon, \ 1 \leq j \leq k
\]

**Lemma 3** (Linear combination of point masses). For every measure on \( \mathbb{T}^\infty \) of the form \( d\mu = \sum_{j=1}^{N} c_j \delta_{\omega} \) with \( \sum_{j=1}^{N} c_j = 1 \), there exists an infinite measure on \( \mathbb{R} \), \( \lambda \), such that

\[
\int_{\mathbb{T}^\infty} |F(z)|^2 d\mu = \lim_{T \to \infty} \frac{1}{\lambda([0, T])} \int_0^T |f(it)|^2 d\lambda
\]

for Dirichlet polynomials \( f \).

**Proof.** Let \( F(z) = \sum a_\alpha z^\alpha \) be a polynomial on \( \mathbb{T}^\infty \) and let \( f(it) = \sum a_\alpha (p^\alpha - it)^{-t} \) be the corresponding Dirichlet polynomial. (Note that these have finitely many terms, there is some \( d \in \mathbb{N} \) such that every \( \alpha \) that appears is of the form \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d, 0, \ldots) \).

We will construct \( \lambda \) to be a sum of point masses \( t \in \mathbb{R} \), using Kronecker’s theorem to place them so that their images under the Bohr correspondence \( z \in \mathbb{T}^\infty \) approximate the point masses that make up \( \mu \). In particular, we would like the images \( z \) to fall within \( \delta \)-balls of \( \mathbb{T}^\infty \) where given \( \epsilon > 0 \), \( \delta \) is chosen to be small enough such that if \( |\omega - z| < \delta \), then

\[
|F(w)|^2 - |f(it)|^2 = ||F(w)|^2 - |F(z)|^2| < \epsilon.
\]

The first equality is because of the Bohr correspondence, and then we use the continuity of \( F \).

Let us examine \( |F(w)|^2 \) and \( |f(it)|^2 \):

\[
|F(w)|^2 = \left| \sum a_\alpha \omega^\alpha \right|^2
= \left| \sum |a_\alpha|^2 \omega^\alpha + \sum_{\alpha \neq \beta} a_\alpha \overline{a_\beta} \omega^\alpha \omega^\beta \right|^2
= \sum |a_\alpha|^2 + \sum_{\alpha \neq \beta} a_\alpha \overline{a_\beta} \omega^\alpha \omega^\beta.
\]

Now, expanding \( |f(it)|^2 \),

\[
|f(it)|^2 = \left| \sum a_\alpha [n(\alpha)]^{-it} \right|^2
= \sum |a_\alpha|^2 + \sum_{\alpha \neq \beta} a_\alpha \overline{a_\beta} (p_1^{\alpha_1-\beta_1} \cdots p_d^{\alpha_d-\beta_d})^{-it}
= \sum |a_\alpha|^2 + \sum_{\alpha \neq \beta} a_\alpha \overline{a_\beta} (p_1^{\alpha_1-\beta_1} \cdots p_d^{\alpha_d-\beta_d})^{-it}.
\]

So we want to place point masses \( t \) so that \( (p_1^{\alpha_1-\beta_1} \cdots p_d^{\alpha_d-\beta_d})^{-it} \) is near \( \omega^\alpha \omega^\beta \) for all \( \alpha, \beta \). Examine both sides. Since \( \omega \in \mathbb{T}^\infty \), \( \omega^\alpha = \omega_1^{\alpha_1} \omega_2^{\alpha_2} \cdots \omega_d^{\alpha_d} \) and there are \( \theta_1, \theta_2, \ldots, \theta_d \) so that

\[
\omega_1^{\alpha_1} \omega_2^{\alpha_2} \cdots \omega_d^{\alpha_d} = e^{i\theta_1 \alpha_1} e^{i\theta_2 \alpha_2} \cdots e^{i\theta_d \alpha_d} = e^{i\theta_1 (\alpha_1 - \beta_1)} e^{i\theta_2 (\alpha_2 - \beta_2)} \cdots e^{i\theta_d (\alpha_d - \beta_d)}
\]
On the other side, we have

\[
\left( p_1^{a_1-\beta_1} \cdots p_d^{a_d-\beta_d} \right)^{-it} = e^{-it \log \left( p_1^{a_1-\beta_1} \cdots p_d^{a_d-\beta_d} \right)} = e^{-i(\alpha_1-\beta_1) t \log(p_1) \cdots e^{-i(\alpha_d-\beta_d) t \log(p_d)}
\]

Note that both of these lie on the unit circle, so the problem reduces to finding \( t \) so that \( -t \log p_r \approx \theta_j \mod 2\pi \). We can use Kronecker’s theorem to do this, however, it can only be done for finitely many primes. Because we want the measure \( \lambda \) to be independent of which primes appear in a polynomial, we will construct the measure in steps, so that the point masses farther from zero approximate the \( \omega_j \) more accurately for more primes. This way any prime that appears in a polynomial will appear in our approximation at some level.

We also want \( \lambda \) to have the property that the measures of large intervals far from zero are much larger than the measures of intervals nearer to zero so that error from the poor approximations near zero is divided out when we take the integral in the mean. We will achieve this by placing more point masses for better approximations. Now, let us begin the formal construction.

**Construction of \( \lambda \).** First construct \( \lambda_1 \). By Kronecker’s theorem, we can find \( t_1^{1,1}, \ldots, t_1^{N,1} \) and corresponding integers \( q \) such that

\[
| - t_1^{j,1} \log p_1 - \theta_1^j - 2\pi q | < 2^{-1}; \quad \text{for } j = 1, \ldots, N.
\]

Repeat this to find \( t_1^{j,2} > \max_j t_1^{j,1} \) so that there are two point masses corresponding to each component of \( \mu_l \). (In future steps, we will repeat this so that there are \( 2^k \) point masses for each component.) Now define

\[
\lambda_1 = \sum_{j=1}^N c_j \left( \delta_{t_1^{j,1}} + \delta_{t_1^{j,2}} \right)
\]

and note that \( \| \lambda_1 \| = 2 \).

Inductively construct a sequence of measures \( \{ \lambda_k \}_{k=2}^{\infty} \):

For \( k > 1 \) choose \( T_{k-1} = \max \{ t_k^{j,m} ; j = 1, \ldots, N, m = 1, \ldots, 2^{k-1} \} \). Again using Kronecker’s theorem, find points \( \{ t_k^{1,1}, \ldots, t_k^{N,1} \} > T_{k-1} \) and corresponding integers \( r \) such that

\[
| - t_k^{j,1} \log p_r - \theta_r^j - 2\pi q | < 2^{-k}; \quad \text{for } j = 1, \ldots, N \text{ and } r = 1, \ldots, k. \tag{2.5}
\]

This means that this inequality holds for all \( j \) and for the first \( k \) primes (or, equivalently, the first \( k \) coordinates in \( \mathbb{T}^{\infty} \)). Repeat this to find \( N \) more points \( \{ t_k^{j,1}, \ldots, t_k^{j,2} \}_{j=1}^N \) that satisfy (2.6) and such that \( t_k^{j,2} > \max_j t_k^{j,1} \). Continue until there are \( M_k = 2^k \| \lambda_{k-1} \| \) points for each \( \omega_j \). Define

\[
\gamma_1 = \lambda_1 \\
\gamma_k = \sum_{m=1}^{M_k} \sum_{j=1}^N c_j \delta_{t_k^{j,m}}, \quad \| \gamma_k \| = 2^k \| \lambda_{k-1} \|
\]

and

\[
\lambda_k = \lambda_{k-1} + \gamma_k = \sum_{\ell=1}^k \gamma_\ell, \quad \| \lambda_k \| = \lambda_k ([0, T_k]) = (2^k + 1) \| \lambda_{k-1} \| \tag{2.6}
\]

and then let \( \lambda = \sum_{\ell=1}^\infty \gamma_\ell \). Note that for any \( T \) there is some \( k \) such that \( \lambda ([0, T]) = \lambda_k ([0, T]) \).
\( \lambda \) satisfies \((2.7)\). Now we will verify that this measure gives the correct limit. We will use the continuity of \( |F|^2 \) as in \((2.2)\). Given \( \epsilon > 0 \), there exists \( \delta_j > 0 \) such that \( \|\omega_j - z\|_{\mathbb{T}^d} < \delta_j \Rightarrow \|F(\omega_j)|^2 - |F(z)|^2\| < \epsilon \). Now choose \( \delta = \min_j \delta_j \) so

\[
\|\omega_j - z\|_{\mathbb{T}^d} < \delta \Rightarrow \|F(\omega_j)|^2 - |F(z)|^2\| < \epsilon \quad \forall j.
\]

Using \((2.5)\) and the formula for the length of a chord of a circle, for \( z_{k,r}^{j,m} = e^{-it_k^{j,m} \log p_r} \), and for large \( k \), we have

\[
|z_{k,r}^{j,m} - \omega_{j,r}| < 2 \sin(2^{-k}) \approx 2 \cdot 2^{-k} = 2^{-k+1}
\]

and

\[
\|\omega_j - z_{k,r}^{j,m}\|^2_{\mathbb{T}^d} = |\omega_{j,1} - z_{k,1}^{j,m}|^2 + \cdots + |\omega_{j,d} - z_{k,d}^{j,m}|^2
\]

\[
= 2^{-k+1} \cdot d
\]

For every \( T \) there is some \( k \) such that \( T \in [T_k, T_{k+1}] \), so choose \( T \) large enough that

\[
2^{-k+1} \cdot d < \delta^2.
\]

So for point masses \( t_k^{j,m} \in \text{supp} \lambda \cap [T_k-1, \infty) \), \((2.5)\) holds for all \( j, m \). By the continuity argument \((2.7)\) and the Bohr correspondence,

\[
\left| |F(\omega_j)|^2 - |f(it_k^{j,m})|^2 \right| < \epsilon \quad \forall j, m.
\]

For this \( T \), consider

\[
\left| \frac{1}{\lambda([0,T])} \int_0^T |f(it)|^2d\lambda - \int_{\mathbb{T}^d} |F(z)|^2d\mu \right|
\]

\[
= \left| \frac{1}{\lambda([0,T])} \int_0^{T_{k-1}} |f(it)|^2d\lambda + \frac{1}{\lambda([0,T])} \int_{T_{k-1}}^T |f(it)|^2d\lambda - \int_{\mathbb{T}^d} |F(z)|^2d\mu \right|
\]

\[
\leq \frac{\lambda_k - 1([0,T_{k-1}] \cdot \|f\|_\infty + \frac{1}{\lambda([0,T])} \int_{T_{k-1}}^T |f(it)|^2d\lambda - \int_{\mathbb{T}^d} |F(z)|^2d\mu
\]

\[
= \frac{2^k + 1}{2^k + 1} \||f\|_\infty + \frac{1}{\lambda([0,T])} \int_{T_{k-1}}^T |f(it)|^2d\lambda - \int_{\mathbb{T}^d} |F(z)|^2d\mu
\]

\( \||f\|_\infty \) is bounded, so the first term goes to zero as \( T \) (and therefore \( k \)) goes to infinity, so we only need to consider the second term:

\[
\left| \frac{1}{\lambda([0,T])} \int_{T_{k-1}}^T |f(it)|^2d\lambda - \int_{\mathbb{T}^d} |F(z)|^2d\mu \right|
\]

\[
= \left| \frac{1}{\lambda([0,T])} \left[ \int_{T_{k-1}}^{T_k} |f(it)|^2d\lambda + \int_{T_k}^T |f(it)|^2d\lambda - \int_{\mathbb{T}^d} |F(z)|^2d\mu \right] \right|
\]

5
Evaluate the integrals using the definition of \( \lambda \) and letting \( X_j = \{ t_{k+1}^{i,m} \} \in [T_k, T] \cap \text{supp } \lambda \). (Note that \( \lambda[T_k, T] = \sum_{j=1}^{N} c_j |X_j| \).)

\[
\begin{align*}
&= \left| \frac{1}{\lambda([0, T])} \left[ \sum_{m=1}^{M_k} \sum_{j=1}^{N} c_j |\int f(it_{k}^{i,m})|^2 + \sum_{j=1}^{N} c_j \sum_{t \in X_j} |f(it)|^2 \right] - \int_{-\infty}^{\infty} |F(z)|^2 d\mu \right| \\
&\leq \frac{1}{\lambda([0, T])} \left[ \sum_{m=1}^{M_k} \sum_{j=1}^{N} c_j |\int f(it_{k}^{i,m})|^2 - |F(\omega_j)|^2 \right] + \sum_{j=1}^{N} c_j \sum_{t \in X_j} |f(it)|^2 - |F(\omega_j)|^2 \\
&\quad + \frac{2k\lambda_{k-1}}{\lambda([0, T])} \int |F(z)|^2 d\mu - \int |F(z)|^2 d\mu + \frac{1}{\lambda([0, T])} \sum_{j=1}^{N} c_j \sum_{t \in X_j} |F(\omega_j)|^2 \\
&= \frac{(2k)\lambda_{k-1} + \lambda[T_k, T]}{(2k + 1)\lambda_{k-1} + \lambda[T_k, T]} \epsilon \\
&\quad + \left( \frac{2k\lambda_{k-1}}{\lambda([0, T])} - 1 \right) \int_{-\infty}^{\infty} |F(z)|^2 d\mu + \frac{1}{\lambda([0, T])} \sum_{j=1}^{N} c_j |X_j||F(\omega_j)|^2 \\
&\leq \frac{2k\lambda_{k-1}}{\lambda([0, T])} - 1 \int_{-\infty}^{\infty} |F(z)|^2 d\mu = \frac{2k}{2k + 1} - 1 \int_{-\infty}^{\infty} |F(z)|^2 d\mu \\
\end{align*}
\]

For large \( k \), the first term is small, as needed. For the second term we consider three cases depending on the size of the sets \( X_j \). When we constructed \( \lambda \), we placed the point masses in sets of size \( N \) so that each mass \( \omega_j \) had a representative, and then we repeated this. This means that \( \|X_{j_i} - X_{j_j}\| \leq 1 \) for \( i \neq j \), so we can consider the cases \( |X_j| = 0 \) for all \( j \), \( |X_j| = C \) for all \( j \), and \( |X_j| = C \) for \( j = 1, \ldots, J \) and \( |X_j| = C + 1 \) for \( j = J + 1, \ldots, N \).

**Case 1:** \( |X_j| = 0 \) for all \( j \). In this case, \( \frac{1}{\lambda([0, T])} \sum_{j=1}^{N} c_j |X_j||F(\omega_j)|^2 = 0 \) and \( \lambda([0, T]) = \lambda([0, T_k]) = (2k + 1)\lambda_{k-1} \), so we have

\[
\frac{2k\lambda_{k-1}}{\lambda([0, T])} - 1 \int_{-\infty}^{\infty} |F(z)|^2 d\mu = \frac{2k}{2k + 1} - 1 \int_{-\infty}^{\infty} |F(z)|^2 d\mu
\]

which is small for large \( k \).

**Case 2:** \( |X_j| = C \leq M_{k+1} \) for all \( j \). In this case \( \lambda([0, T]) = \lambda([0, T_k]) + \sum_{j=1}^{N} c_j \cdot C = (2k + 1)\lambda_{k-1} + C \). Also, note that \( \sum_{j=1}^{N} c_j |X_j||F(\omega_j)|^2 = C \int_{-\infty}^{\infty} |F(z)|^2 d\mu \). Then, substituting and
The first term is small as in Case 2, and in the second term

\[ \frac{2^k \| \lambda_{k-1} \|}{\lambda([0, T])} - 1 \int_{T}^\infty |F(z)|^2 d\mu + \frac{1}{\lambda([0, T])} \sum_{j=1}^{N} c_j |X_j| |F(\omega_j)|^2 \]

simplifying gives

\[ \left| \left( \frac{2^k \| \lambda_{k-1} \|}{\lambda([0, T])} - 1 \right) \int_{T}^\infty |F(z)|^2 d\mu + \frac{1}{\lambda([0, T])} \sum_{j=1}^{N} c_j |X_j| |F(\omega_j)|^2 \right| \]

\[ = \left| \left( \frac{2^k \| \lambda_{k-1} \|}{\lambda([0, T])} - 1 \right) \int_{T}^\infty |F(z)|^2 d\mu + \frac{C}{\lambda([0, T])} \sum_{j=1}^{N} c_j |F(\omega_j)|^2 + \frac{1}{\lambda([0, T])} \sum_{j=J+1}^{N} c_j |F(\omega_j)|^2 \right| \]

and this is small for large \( k \).

**Case 3:** \( |X_j| = C \) for \( j = 1, \ldots, J \) and \( |X_j| = C + 1 \) for \( j = J + 1, \ldots, N \). Similarly to the previous case, we get

\[ \left| \left( \frac{2^k \| \lambda_{k-1} \|}{\lambda([0, T])} - 1 \right) \int_{T}^\infty |F(z)|^2 d\mu + \frac{1}{\lambda([0, T])} \sum_{j=1}^{N} c_j |X_j| |F(\omega_j)|^2 \right| \]

\[ = \left| \left( \frac{2^k \| \lambda_{k-1} \|}{\lambda([0, T])} - 1 \right) \int_{T}^\infty |F(z)|^2 d\mu + \frac{C}{\lambda([0, T])} \sum_{j=1}^{N} c_j |F(\omega_j)|^2 + \frac{1}{\lambda([0, T])} \sum_{j=J+1}^{N} c_j |F(\omega_j)|^2 \right| \]

The first term is small as in Case 2, and in the second term \( \frac{C}{\lambda([0, T])} \) is multiplied by a bounded quantity, and so this will be small for large \( T \).

In each case,

\[ \left| \frac{1}{\lambda([0, T])} \int_{0}^{T} |f(it)|^2 d\lambda - \int_{T}^\infty |F(z)|^2 d\mu \right| \]

is small for large \( T \), and so the proof is complete. \( \square \)

This construction also gives us the following lemma

**Lemma 4.** If \( \{ \mu_n \} \) is a finite collection of measures \( d\mu = \sum_{j=1}^{J_n} c_j^\mu \delta_{\omega_j^\mu} \) with \( \sum_{j=1}^{J_n} c_j^\mu = 1 \) and \( \{ F_m \}_{m=1}^{M} \) is a finite set of polynomials. If \( \lambda_n \) is the measure constructed as in Lemma 3 corresponding to \( \mu_n \), then given \( \epsilon > 0 \), there exists \( T_\epsilon \) such that for sufficiently large \( T \gg T_\epsilon \),

\[ \left| \frac{1}{\lambda_n([T_\epsilon, T])} \int_{T_\epsilon}^{T} |f(it)|^2 d\lambda_n - \int_{T}^\infty |F(z)|^2 d\mu_n \right| < \epsilon, \forall m, n. \]

**Proof.** As in the proof of Lemma 3, for each \( (m, n) \) we can find \( T_{(m, n)} \) large enough that for point masses \( t_\epsilon^m \in \text{supp} \lambda_n \cap [T_{(m, n)}, \infty) \) and \( z_\epsilon^m \) corresponding to \( t_\epsilon^m \),

\[ |\omega_j^m - z_\epsilon^m| < \delta \]

and so \[ |F_m(\omega_j^m)|^2 - |f_m(it_\epsilon^m)|^2 < \epsilon \forall j. \]

There are finitely many \( \mu_n \) and \( F_m \), so choose \( T_\epsilon = \max\{T_{(m, n)}\} \). Now we have

\[ \left| |F_m(\omega_j^m)|^2 - |f_m(it_\epsilon^m)|^2 \right| < \epsilon, \forall t_\epsilon^m \in \text{supp} \lambda_n \cap [T_\epsilon, \infty \text{ and for all } j, m, n \quad (2.10) \]
Now, for $T$ large enough that each $\omega^*_j$ has many more than $\left[ \sum_{\mu_n} \sum_{j=1}^d |c_j| \right]$ representatives $t^j_\ell$ in $[T_e, T]$ (i.e. such that $|X_j| = \{|t^j_\ell \in [T_e, T]| \} \gg \left[ \sum_{\mu_n} \sum_{j=1}^d |c_j| \right]$, for each $j$.) Consider

$$\left| \frac{1}{\lambda([T_e, T])} \int_{T_e}^T |f(it)|^2 d\lambda - \int_{T}^{\infty} |F(z)|^2 d\mu \right|$$

$$= \left| \frac{1}{\lambda([T_e, T])} \sum_{j=1}^{J_n} c^j_\ell \sum_{t^j_\ell \in X_j} \left[ |f(it^j_\ell)|^2 - |F(\omega^*_j)|^2 \right] + \frac{1}{\lambda([T_e, T])} \sum_{j=1}^{J_n} c^j_\ell |X_j||F(\omega^*_j)|^2 - \int_{T}^{\infty} |F(z)|^2 d\mu \right|.$$ 

Rearranging and using the triangle inequality yields

$$\leq \frac{1}{\lambda([T_e, T])} \sum_{j=1}^{J_n} c^j_\ell \sum_{t^j_\ell \in X_j} \left[ |f(it^j_\ell)|^2 - |F(\omega^*_j)|^2 \right] + \frac{1}{\lambda([T_e, T])} \sum_{j=1}^{J_n} c^j_\ell |X_j||F(\omega^*_j)|^2 - \int_{T}^{\infty} |F(z)|^2 d\mu \right|.$$ 

The first term is small so we must consider the second term. As in the proof of Lemma 3 we can consider when $|X_j| = C$ for all $j$, and when $|X_j| = C$ for $j = 1, \ldots, J$ and $|X_j| = C + 1$ for $j = J + 1, \ldots, J_n$. Note that $C \geq \left[ \sum_{\mu_n} \sum_{j=1}^d |c_j| \right]$. In the first case, we have

$$\left| \frac{1}{\lambda([T_e, T])} \sum_{j=1}^{J_n} c^j_\ell |X_j||F(\omega^*_j)|^2 - \int_{T}^{\infty} |F(z)|^2 d\mu \right| = \left| \frac{C}{C} \sum_{j=1}^{J_n} c^j_\ell |F(\omega^*_j)|^2 - \int_{T}^{\infty} |F(z)|^2 d\mu \right| = 0.$$ 

In the second case

$$\left| \frac{1}{\lambda([T_e, T])} \sum_{j=1}^{J_n} c^j_\ell |X_j||F(\omega^*_j)|^2 - \int_{T}^{\infty} |F(z)|^2 d\mu \right|$$

$$= \left| \frac{C}{C} \sum_{j=J+1}^{J_n} c^j_\ell |F(\omega^*_j)|^2 - \int_{T}^{\infty} |F(z)|^2 d\mu \right| + \frac{1}{\lambda([T_e, T])} \sum_{j=J+1}^{J_n} c^j_\ell |F(\omega^*_j)|^2$$

$$= \left| \frac{C}{C} \int_{T}^{\infty} |F(z)|^2 d\mu + \frac{1}{\lambda([T_e, T])} \sum_{j=J+1}^{J_n} c^j_\ell |F(\omega^*_j)|^2.$$ 

We chose $T$ large enough that $C \gg \sum_{j=J+1}^{J_n} c_j$, so the first term will be small, and the second term is small because $\| |f|^2 \|_{\infty}$ is bounded. \hfill $\Box$

**Remark 5.** If $T' > T_e$, the upper limit $T$ can be found so that this same estimate holds on $[T', T]$.

### 3. Proof of Theorem 4 part (i)

We may now return to the main result.
Theorem $\text{4(i)}$. Let $\mu$ be a Borel probability measure on the infinite torus $\mathbb{T}^\infty$. There exists a locally finite Borel measure $\lambda$ on $\mathbb{R}$, such that, for all $f \in \mathcal{A}(\mathbb{C}_+)$,

$$\lim_{T \to \infty} \frac{1}{\lambda([0, T])} \int_0^T |f(it)|^2 \, d\lambda(t) = \int_{\mathbb{T}^\infty} |F(z)|^2 \, d\mu(z).$$

Proof. Let $\{F_m\}_{m=1}^\infty$ be a countable set of polynomials which is dense in $\mathcal{A}(\mathbb{D}^N)$. (By definition, the polynomials are dense in $\mathcal{A}(\mathbb{D}^d)$, and there is a countable dense set of polynomials within each $\mathcal{A}(\mathbb{D}^d)$ for finite $d$, so use Cantor diagonalization.) We only need to prove the theorem for $F_m$ (corresponding to a Dirichlet polynomial $f_m$) in this dense set: for $F \in \mathcal{A}(\mathbb{D}^N)$, there is some $F_m$ such that $\sup |f_m|^2 - |F|^2 < \epsilon$ (and similarly $\sup |f_m|^2 - |F|^2 < \epsilon$). So

$$\left| \int_{\mathbb{T}^\infty} |F_m(z)|^2 \, d\mu - \int_{\mathbb{T}^\infty} |F(z)|^2 \, d\mu \right| < \epsilon$$

and

$$\left| \frac{1}{\lambda([0, T])} \int_0^T |f_m(it)|^2 \, d\lambda - \int_{\mathbb{T}^\infty} |F(z)|^2 \, d\mu \right| < \epsilon$$

for any measure $\lambda$ and for all $T$. So now we have

$$\left| \frac{1}{\lambda([0, T])} \int_0^T |f(it)|^2 \, d\lambda - \int_{\mathbb{T}^\infty} |F(z)|^2 \, d\mu \right|$$

$$\leq \left| \frac{1}{\lambda([0, T])} \int_0^T |f_m(it)|^2 \, d\lambda - \frac{1}{\lambda([0, T])} \int_0^T |f(it)|^2 \, d\lambda \right| + \left| \frac{1}{\lambda([0, T])} \int_0^T |f_m(it)|^2 \, d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 \, d\mu \right|$$

$$+ \left| \int_{\mathbb{T}^\infty} |F_m(z)|^2 \, d\mu - \int_{\mathbb{T}^\infty} |F(z)|^2 \, d\mu \right|$$

$$< \epsilon + \left| \frac{1}{\lambda([0, T])} \int_0^T |f_m(it)|^2 \, d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 \, d\mu \right| + \epsilon$$

It remains to show that

$$\left| \frac{1}{\lambda([0, T])} \int_0^T |f_m(it)|^2 \, d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 \, d\mu \right| < \epsilon. \tag{3.1}$$

By Lemma 11 there exists a sequence of linear combinations of point masses $\{\mu_n\}$ that converges weak-* to $\mu$. Now choose a subsequence $\{\mu_{n_j}\}$ such that for $m = 1, \ldots, 2^j$,

$$\left| \int_{\mathbb{T}^\infty} |F_m(z)|^2 \, d\mu_{n_j} - \int_{\mathbb{T}^\infty} |F_m(z)|^2 \, d\mu \right| < 2^{-j}$$
and re-index so that \( \{\mu_j\} = \{\mu_{n_j}\} \). Now, given \( m, \) we have for \( j \geq J = \lfloor \log_2 m \rfloor \), we have control over the convergence:

\[
\left| \int_{\mathbb{R}^n} |F_m(z)|^2 d\mu_j - \int_{\mathbb{R}^n} |F_m(z)|^2 d\mu \right| < 2^{-j}. \tag{3.2}
\]

Also, for each of these \( \mu_j \), there is a corresponding \( \lambda_j \) as constructed above that satisfies \( \|\lambda(1)\| = 2 \) for all Dirichlet polynomials, and in particular, works for all \( F_m \).

Construction of \( \lambda \). This will be by a similar process to in the linear combination case in that we will find an approximation and then repeat it so that better approximations appear more. However, this case, we will not be approximating with point masses, but with the measures \( \lambda_j \) constructed as in the previous case using the Lemma 3.

By Lemma 3 there is some \( T_1 \) such that for \( j, m = 1, 2 \) and for sufficiently large \( T_1^{(1)} \)

\[
\left| \frac{1}{\lambda[T_1,T_1^{(1)}]} \int_{T_1} |f_m(it)|^2 d\lambda_j - \int_{T_1} |F_m(z)|^2 d\mu_j \right| < \frac{1}{2}. \tag{3.3}
\]

Define

\[
\lambda^{(1)} = \gamma_{(1)}^{(1)} = \sum_{j=1}^{2^1} \frac{\lambda_j|\{T_1,T_1^{(1)}\}|}{\lambda_j|\{T_1^{(1)},T_1^{(1)}\}|}
\]

Note that supp \( \lambda^{(1)} = [T_1, T_1^{(1)}] \) and \( \|\lambda^{(1)}\| = 2 \). (This approximation is not repeated, analogously to in the linear combination case where in the first step we only found two point masses.)

Now find \( T_2 \geq T_1^{(1)} \) such that for \( j, m = 1, \ldots, 4 \), and for sufficiently large \( T \),

\[
\left| \frac{1}{\lambda[T_2,T]} \int_{T_2} |f_m(it)|^2 d\lambda_j - \int_{T_2} |F_m(z)|^2 d\mu_j \right| < \frac{1}{2^2}. \tag{3.4}
\]

Let \( T_2^{(1)} \) be large enough that \( (3.4) \) holds. Now we repeat the approximation: by the remark after the proof of Lemma 3 there is some \( T_2^{(2)} \) large enough so that for \( j, m = 1, \ldots, 4 \)

\[
\left| \frac{1}{\lambda[T_2^{(1)},T_2^{(1)}]} \int_{T_2^{(1)}} |f_m(it)|^2 d\lambda_j - \int_{T_2^{(1)}} |F_m(z)|^2 d\mu_j \right| < \frac{1}{2^2}.
\]

Define

\[
\gamma_{(2)}^{(t)} = \sum_{j=1}^{2^2} \frac{\lambda_j|\{T_2^{(t-1)},T_2^{(t)}\}|}{\lambda_j|\{T_2^{(t-1)},T_2^{(t)}\}|}, \quad T_2^{(0)} = T_2
\]

and

\[
\lambda^{(2)} = \lambda^{(1)} + \sum_{t=1}^{\|\lambda^{(1)}\| = 2} \gamma_{(2)}^{(t)}. \tag{\text{2.4)}
\]

Note that \( \|\gamma_{(2)}^{(t)}\| = 2^2 \) and \( \|\lambda^{(2)}\| = \|\lambda^{(1)}\| + 2 \cdot \|\gamma_{(2)}^{(t)}\|. \)

Continue this process, at level \( k \) finding \( T_k \geq T_k^{(\|\lambda^{(k-2)}\|)} \) such that for \( j, m = 1, \ldots, 2^k \), and for sufficiently large \( T \)

\[
\left| \frac{1}{\lambda[T_k,T]} \int_{T_k} |f_m(it)|^2 d\lambda_j - \int_{T_k} |F_m(z)|^2 d\mu_j \right| < \frac{1}{2^k}. \tag{3.5}
\]
Let $T_k^{(1)} = T$ be large enough that (3.5) holds and find $T_k^{(2)}$ such that

$$\left| \frac{1}{\lambda[T_k^{(1)}, T_k^{(2)}]} \int_{T_k^{(1)}}^{T_k^{(2)}} \int |f_m(it)|^2 d\lambda_j - \int |F_m(z)|^2 d\mu_j \right| < \frac{1}{2^k}. $$

Repeat this to get an increasing sequence $\{T_k^{(\ell)}\}_{\ell=0}^{\|\lambda_{k-1}\|}$ (where $T_k^{(0)} = T_k$) such that for $j, m = 1, \ldots, 2^k$ and for $\ell = 1, \ldots, \|\lambda_{k-1}\|,$

$$\left| \frac{1}{\lambda[T_k^{(\ell-1)}, T_k^{(\ell)}]} \int_{T_k^{(\ell-1)}}^{T_k^{(\ell)}} |f_m(it)|^2 d\lambda_j - \int |F_m(z)|^2 d\mu_j \right| < \frac{1}{2^k}. \quad (3.6)$$

Define for $\ell = 1, \ldots, \|\lambda_{k-1}\|$

$$\gamma_k^{(\ell)} = \sum_{j=1}^{2^k} \frac{\lambda_j |T_k^{(\ell-1)}, T_k^{(\ell)}|}{\lambda_j |[T_k^{(\ell-1)}, T_k^{(\ell)}]|}, \quad \|\gamma_k^{(\ell)}\| = 2^k$$

and

$$\lambda^{(k)} = \lambda^{(k-1)} + \sum_{\ell=1}^{\|\lambda^{(k-1)}\|} \gamma_k^{(\ell)}.$$ 

Letting $\lambda = \lim_{k \to \infty} \lambda^{(k)}$, we have that $\lambda[0, T_{k+1}] = \|\lambda^{(k)}\| = (2^k + 1)\|\lambda^{(k-1)}\|$.

$\lambda$ satisfies (3.7). Now, examine

$$\left| \frac{1}{\lambda[0, T]} \int_{0}^{T} |f_m(it)|^2 d\lambda - \int |F_m(z)|^2 d\mu \right|. $$

For any $T$ there is some $k$ such that $T \in [T_{k+1}, T_{k+2}]$. Choose $T$ large enough that $2^k \geq m$. From here, consider three cases: where $T > T_{k+1}^{(\|\lambda^{(k)}\|)},$ where $T < T_{k+1}^{(1)},$ and where $T \in [T_{k+1}^{(q)}, T_{k+2}^{(q+1)}]$ for some $1 \leq q < \|\lambda^{(k)}\|.$

**Case 1:** $T > T_{k+1}^{(\|\lambda^{(k)}\|)}$. This is where $T$ appears after the repetitions:

$$\leftarrow T_{k+1} \quad T_{k+1}^{(1)} \quad T_{k+1}^{(\|\lambda^{(k)}\|-1)} \quad T_{k+1}^{(\|\lambda^{(k)}\|)} \quad T \quad T_{k+2} \rightarrow$$

In this case, $\lambda[0, T] = \lambda[0, T_{k+2}] = (2^{k+1} + 1)\lambda[0, T_{k+1}]$ and

$$\int_{T_{k+1}}^{T} |f_m(it)|^2 d\lambda = \int_{T_{k+1}}^{T_{k+2}} |f_m(it)|^2 d\lambda.$$
So
\[
\left| \frac{1}{\lambda[0,T]} \int_0^T |f_m(it)|^2 |d\lambda| - \int_{T^\infty} |F_m(z)|^2 |d\mu| \right|
\leq \left| \frac{1}{\lambda[0,T_{k+2}]} \int_0^{T_{k+1}} |f_m(it)|^2 |d\lambda| \right| + \left| \frac{1}{\lambda[0,T_{k+2}]} \int_{T_{k+1}}^{T_{k+2}} |f_m(it)|^2 |d\lambda| - \int_{T^\infty} |F_m(z)|^2 |d\mu| \right|
\leq \frac{\lambda[0,T_{k+2}]}{\lambda[0,T_{k+2}]} \|f_m\|^2 + \frac{1}{\lambda[0,T_{k+2}]} \int_{T_{k+1}}^{T_{k+2}} |f_m(it)|^2 |d\lambda| - \int_{T^\infty} |F_m(z)|^2 |d\mu|
\leq \frac{1}{\lambda[0,T_{k+2}]} \|f_m\|^2 + \frac{1}{\lambda[0,T_{k+2}]} \int_{T_{k+1}}^{T_{k+2}} |f_m(it)|^2 |d\lambda| - \int_{T^\infty} |F_m(z)|^2 |d\mu|
\]

The first term here is small for large \( T \), so consider the second term, using the definition of \( \lambda \) and adding and subtracting \( \int |F_m|^2 |d\mu| \) and \( \int |F_m|^2 |d\mu| \) where appropriate

\[
\left| \frac{1}{\lambda[0,T_{k+2}]} \int_{T_{k+1}}^{T_{k+2}} |f_m(it)|^2 |d\lambda| - \int_{T^\infty} |F_m(z)|^2 |d\mu| \right|
\leq \frac{1}{\lambda[0,T_{k+2}]} \sum_{\ell=1}^{k} \sum_{j=1}^{2^{k+1}} \left| \lambda_{j[T^{t_{\ell-1}}],T_{k+1}} \int_{T_{k+1}}^{T_{k+2}} |f_m|^2 |d\lambda| - \int_{T^\infty} |F_m|^2 |d\mu| \right|
\leq \frac{1}{\lambda[0,T_{k+2}]} \|f_m\|^2 + \frac{1}{\lambda[0,T_{k+2}]} \int_{T_{k+1}}^{T_{k+2}} |f_m(it)|^2 |d\lambda| - \int_{T^\infty} |F_m(z)|^2 |d\mu|
\]

Applying (3.6) and using that \( \lambda[0,T_{k+2}] = (2^{k+1} + 1) \lambda[0,T_{k+1}] = (2^{k+1} + 1) \|\lambda^{(k)}\| \) then gives

\[
< \frac{1}{2^{k+1} + 1} \sum_{j=1}^{2^{k+1}} \frac{1}{2^{k+1} + 1} + \frac{1}{2^{k+1} + 1} \sum_{j=1}^{2^{k+1}} \int_{T^\infty} |F_m|^2 |d\mu| - \int_{T^\infty} |F_m|^2 |d\mu| + \frac{1}{2^{k+1} + 1} \int_{T^\infty} |F_m|^2 |d\mu|
= \frac{1}{2^{k+1} + 1} \left[ 1 + \sum_{j=1}^{2^{k+1}} \int_{T^\infty} |F_m|^2 |d\mu| - \int_{T^\infty} |F_m|^2 |d\mu| + \int_{T^\infty} |F_m|^2 |d\mu| \right]
\]

We can get control over the second term by recalling how we chose our sequence as in (3.2) that there is \( J = \lfloor \log_2 m \rfloor \leq k \) such that for \( j \geq J \) we have that \( \int_{T^\infty} |F_m|^2 |d\mu| - \int_{T^\infty} |F_m|^2 |d\mu| < \frac{1}{2^7} \), so considering the worst case, where \( J = k \), we get

\[
< \frac{1}{2^{k+1} + 1} \left[ 1 + \sum_{j=1}^{k-1} \int_{T^\infty} |F_m|^2 |d\mu| - \int_{T^\infty} |F_m|^2 |d\mu| + \sum_{j=k}^{2^{k+1}} 2^{-j} + \int_{T^\infty} |F_m|^2 |d\mu| \right].
\]
Because there are finitely many $j$, \[ \max_{j=1,\ldots,k-1} \left| \int_{\tau} |F_m|^2 d\mu_j - \int_{\tau} |F_m|^2 d\mu \right| \] is bounded, say by $M$, so
\[
< \frac{1}{2^{k+1} + 1} \left[ 1 + [(k - 1)M + 1] + \int_{\tau} |F_m|^2 d\mu \right]
\]
and this is small for large $k$.

**Case 2: $T < T_{k+1}^{(1)}$.** In this case, only $\gamma_{k+1}^{(1)}$ contributes to $\lambda[T_{k+1}, T]$, so $\lambda[T_{k+1}, T] \leq 2^{k+1}$.

The first part of this case is similar to above, giving
\[
\left| \frac{1}{\lambda[0, T]} \int_0^T |f_m(it)|^2 d\lambda - \int_{\tau} |F_m(z)|^2 d\mu \right|
\]
\[
\leq \left| \frac{1}{\lambda[0, T]} \int_0^{T_k} |f_m(it)|^2 d\lambda \right| + \left| \frac{1}{\lambda[0, T]} \int_{T_k}^T |f_m(it)|^2 d\lambda - \int_{\tau} |F_m(z)|^2 d\mu \right|
\]
\[
\leq \frac{1}{2^{k+1} + 1} \|f_m\|_\infty + \left| \frac{1}{\lambda[0, T]} \int_{T_k}^{T_{k+1}} |f_m(it)|^2 d\lambda - \int_{\tau} |F_m(z)|^2 d\mu \right|
\]
\[
+ \left| \frac{1}{\lambda[0, T]} \int_{T_{k+1}}^T |f_m(it)|^2 d\lambda \right|
\]
(3.9)

Here, the first term is small as before. Similarly to Case 1, the second term simplifies down as
\[
\left| \frac{1}{\lambda[0, T]} \int_{T_k}^{T_{k+1}} |f_m(it)|^2 d\lambda - \int_{\tau} |F_m(z)|^2 d\mu \right|
\]
\[
< \frac{1}{2^{k+1} + 1} \left[ 1 + (k - 1)M \right] + \left| \frac{2^k \|\lambda^{(k-1)}\|}{\lambda[0, T]} - 1 \right| \int_{\tau} |F_m(z)|^2 d\mu
\]

By the definition of $\lambda$, we know that $2^k \|\lambda^{(k-1)}\| = \lambda[T_k, T_{k+1}]$, so
\[
= \frac{1}{2^{k+1} + 1} \left[ 1 + (k - 1)M \right] + \left| \frac{\lambda[T_k, T_{k+1}] - \lambda[0, T]}{\lambda[0, T]} \right| \int_{\tau} |F_m(z)|^2 d\mu
\]
\[
= \frac{1}{2^{k+1} + 1} \left[ 1 + (k - 1)M \right] + \left( \frac{\lambda[0, T_k] + \lambda[T_{k+1}, T]}{\lambda[0, T]} \right) \int_{\tau} |F_m(z)|^2 d\mu
\]
\[
\leq \frac{1}{2^{k+1} + 1} \left[ 1 + (k - 1)M \right] + \left( \frac{\|\lambda^{(k-1)}\| + 2^{k+1}}{\lambda[0, T_{k+1}]} \right) \int_{\tau} |F_m(z)|^2 d\mu
\]
Now, noting that \( \lambda[0, T_{k+1}] = (2^k + 1)\|\lambda^{(k-1)}\| = (2^k + 1)(2^{k-1} + 1) \cdots (2^2 + 1) \cdot 2 > 2^{\frac{k(k+1)}{2}} \), we have
\[
< \frac{1}{2^k + 1} [1 + (k - 1)M] + \left( \frac{1}{2^k + 1} + \frac{2^{k+1}}{2^{k+1} + 1} \right) \int_{T} |F_m(z)|^2 d\mu
\]
and for large \( k \), this is small.

We are now left with the last term:
\[
\left| \frac{1}{\lambda[0, T]} \int_{T_{k+1}}^{T} |f_m(it)|^2 d\lambda \right| \leq \frac{\lambda[T_{k+1}, T]}{\lambda[0, T]} \|f_m\|^2 \leq \frac{2^{k+1}}{2^{k+1} + 1} \|f_m\|^2
\]
Again, this is small for large \( k \), so this case is complete.

**Case 3:** \( T_{k+1}^{(q)} \leq T \leq T_{k+1}^{(q+1)} \) for some \( 1 \leq q < \|\lambda^{(k)}\| \).

In this case, \( \lambda[T_k + 1, T] \) consists of the first \( q \) subintervals, and then has a contribution from \([T_{k+1}^{(q)}, T_{k+1}^{(q+1)}]\):
\[
\lambda[T_{k+1}, T] = \sum_{\ell = 1}^{q} \gamma_{k+1}^{(\ell)}[T_{k+1}^{(\ell-1)}, T_{k+1}^{(\ell)}] + \gamma_{k+1}^{(q+1)}[T_{k+1}^{(q)}, T] \quad (3.10)
\]
and
\[
\lambda[0, T] > \|\lambda^{(k)}\| + 2^{k+1}q > (2^{k+1} + 1)q \quad (3.11)
\]
The computation begins similarly to the previous cases:
\[
\left| \frac{1}{\lambda[0, T]} \int_{0}^{T} |f_m(it)|^2 d\lambda - \int_{T} |F_m(z)|^2 d\mu \right|
\leq \frac{1}{2^k + 1} \|f_m\|^2 + \left[ \frac{1}{\lambda[0, T]} \int_{T_{k+1}}^{T} |f_m(it)|^2 d\lambda + \int_{T_{k+1}}^{T} |f_m(it)|^2 d\lambda \right] - \int_{T} |F_m(z)|^2 d\mu
\]
As in Case 2:
\[
\leq \frac{1}{2^k + 1} \|f_m\|^2 + \frac{1}{2^k + 1} [1 + (k - 1)M]
\]
\[
+ \left[ \frac{1}{\lambda[0, T]} \int_{T_{k+1}}^{T} |f_m(it)|^2 d\lambda + \left( \frac{2^{k+1} \|\lambda^{(k-1)}\|}{\lambda[0, T]} - 1 \right) \int_{T} |F_m(z)|^2 d\mu \right].
\]
The first two terms are small for large $k$, so consider the term in absolute values. This is similar to (3.7), where we add and subtract $\int |F_m|^2 d\mu_j$ and $\int |F_m|^2 d\mu$ appropriately, and use the triangle inequality:

\[
\left| \frac{1}{\lambda[0,T]} \int_{T_{k+1}}^{T} |f_m(it)|^2 d\lambda + \left( \frac{2^k\|\lambda^{(k-1)}\|}{\lambda[0,T]} - 1 \right) \int_{T} \int |F_m(z)|^2 d\mu \right|
\leq \frac{1}{\lambda[0,T]} \sum_{j=1}^{q} \int_{T_{k+1}}^{T} \lambda_j(T_{k+1}^{(\ell-1)}, T_{k+1}^{(\ell)}) \int_{j_{k+1}^{(\ell-1)}} \int_{j_{k+1}^{(\ell)}} |f_m(it)|^2 d\lambda_j - \int_{T} \int |F_m|^2 d\mu_j | (I)
\]

\[
+ \frac{1}{\lambda[0,T]} \sum_{j=1}^{q} \int_{T_{k+1}}^{T} \lambda_j(T_{k+1}^{(q)}, T_{k+1}^{(q+1)}) \int_{j_{k+1}^{(q)}} \int_{j_{k+1}^{(q+1)}} |f_m(it)|^2 d\lambda_j | (II)
\]

\[
+ \frac{1}{\lambda[0,T]} \sum_{j=1}^{q} \int_{T_{k+1}}^{T} \int |F_m|^2 d\mu_j - \int_{T} \int |F_m|^2 d\mu | (III)
\]

\[
+ \frac{2^k\|\lambda^{(k-1)}\| + 2^{k+1}q}{\lambda[0,T]} - 1 \int_{T} \int |F_m|^2 d\mu | (IV)
\]

Using (3.11), (II) is bounded by $\frac{2^{k+1}q}{\lambda[0,T] + 2^{k+1}q \parallel \lambda^{(k-1)} \parallel_{\infty}}$. (I) can be simplified using (3.7) and (III) can be simplified similarly to (3.8) yielding:

\[
(I)+(II)+(III)+(IV) < \frac{q}{\lambda[0,T]} \sum_{j=1}^{q} \left( \frac{1}{2^{k+1}} + \frac{q}{\lambda[0,T]} ((k-1)M + 1) + \frac{2^{k+1}}{\lambda[0,T] + 2^{k+1}q \parallel \lambda^{(k-1)} \parallel_{\infty}} \parallel F_m \parallel_{\infty} \right)
\]

\[
+ \frac{2^k\|\lambda^{(k-1)}\| + 2^{k+1}q}{\lambda[0,T]} - 1 \int_{T} \int |F_m|^2 d\mu
\]

As before $\lambda[0,T_{k+1}] > 2^{k+1}q$. Also, because $q < \|\lambda^{(k)}\|$ and $\lambda[0,T] > \|\lambda^{(k)}\| + 2^{k+1}q > (2^{k+1} + 1)q$,

\[
< \frac{1}{2^{k+1} + 1} ((k-1)M + 2) + \frac{2^{k+1}}{2^{k+1} + 2^{k+1}q \parallel \lambda^{(k-1)} \parallel_{\infty}} \parallel F_m \parallel_{\infty}
\]

\[
+ \frac{2^k\|\lambda^{(k-1)}\| + 2^{k+1}q}{\lambda[0,T]} - 1 \int_{T} \int |F_m|^2 d\mu
\]

The first two terms are clearly small for large $k$, so we only need to examine $\left| \frac{2^k\|\lambda^{(k-1)}\| + 2^{k+1}q}{\lambda[0,T]} - 1 \right|$. 

\[
\left| \frac{2^k\|\lambda^{(k-1)}\| + 2^{k+1}q}{\lambda[0,T]} - 1 \right| = \frac{\lambda[T_k, T_{k+1}] + \lambda[T_{k+1}, T_{k+1}^{(q)}] - \lambda[0,T]}{\lambda[0,T]}
\]

\[
= \frac{\lambda[0,T] + \lambda[T_{k+1}^{(q)}, T]}{\lambda[0,T]}
\]

15
Using that $\lambda[T_{k+1}^{(q)}, T] \leq 2^{k+1}$ and $\lambda[0, T] > \|\lambda^{(k)}\| + 2^{k+1} > 2^{\frac{2^{k+1}}{2}} + 2^{k+1}$, 

$$\frac{\lambda[0, T_k] + \lambda[T_{k+1}^{(q)}, T]}{\lambda[0, T]} \leq \frac{\lambda[0, T_k] + 2^{k+1}}{\|\lambda^{(k)}\| + 2^{k+1}} = \frac{\|\lambda^{(k-1)}\|}{(2^k + 1)\|\lambda^{(k-1)}\| + 2^{k+1}} + \frac{2^{k+1}}{2^{\frac{2^{k+1}}{2}} + 2^{k+1}}.$$ 

This is also small for large $k$, completing case three.

Because $k$ depends on $T$, with $T$ large implying that $k$ is large, we’ve shown in each case that for large $T$, 

$$\frac{1}{\lambda(0, T)} \int_0^T |f_m(i t)|^2 d\lambda - \int_{T^\infty} |F_m(z)|^2 d\mu$$ 

is as small as desired, so:

$$\lim_{T \to \infty} \frac{1}{\lambda(0, T)} \int_0^T |f_m(i t)|^2 d\lambda = \int_{T^\infty} |F_m(z)|^2 d\mu.$$ 

\[\square\]

4. Proof of Theorem 4, Part (ii)

Recall the statement of Theorem 4, Part (ii):

**Theorem 4(ii)** Let $\lambda$ be a locally finite Borel measure on $\mathbb{R}$ such that the limit on the left hand side of (1.3) exists and is finite for all $\sum a_n [n(\alpha)]^{-s} \in \mathcal{A}(\mathbb{C}_+)$. Then there exists a unique Borel probability measure $\mu$ on the infinite torus $\mathbb{T}^\infty$ such that, for all $\sum a_n [n(\alpha)]^{-s} \in \mathcal{A}(\mathbb{C}_+)$, (1.3) holds.

$$\lim_{T \to \infty} \frac{1}{\lambda(0, T)} \int_0^T |f(it)|^2 d\lambda(t) = \int_{T^\infty} |F(z)|^2 d\mu(z). \quad (1.3)$$

**Proof.** We will appeal to the Riesz Representation Theorem to show that the left hand side of (1.3) can be used to define a positive bounded linear functional $\phi$ on $C(\mathbb{T}^\infty, \mathbb{R})$ which will give us our unique $\mu$. This $\mu$ can then easily be shown to be a probability measure. We will define this functional $\phi$ on a subalgebra of $C(\mathbb{T}^\infty, \mathbb{R})$ and then apply Stone-Weierstrass to show that the definition extends to the whole of $C(\mathbb{T}^\infty, \mathbb{R})$.

We choose to define $\phi$ on the set

$$\mathcal{U} = \left\{ \sum_{n=1}^N a_n |F_n|^2 : F_n \text{ are polynomials on } \mathbb{T}^\infty \right\}.$$ 

Let

$$\phi : |F|^2 \mapsto \lim_{T \to \infty} \frac{1}{\lambda(0, T)} \int_0^T |f(it)|^2 d\lambda(t). \quad (4.1)$$

Extending this linearly gives the definition of $\phi$ on $\mathcal{U}$, and if we can show that $\mathcal{U}$ is dense in $C(X, \mathbb{R})$, then we can extend $\phi$ to be a linear functional on $C(\mathbb{T}^\infty, \mathbb{R})$. To show that $\mathcal{U}$ is dense, apply Stone-Weierstrass: $\mathcal{U}$ clearly contains the constant functions and is a vector subspace of $C(X, \mathbb{R})$, and it is easy to show that it is closed under multiplication since $|F|^2 |G|^2 = |FG|^2$ holds, and distribution shows that it holds for linear combinations as well. So we have that $\mathcal{U}$ is a subalgebra.

It remains to show that $\mathcal{U}$ separates points on $\mathbb{T}^\infty$: given $z \neq w \in \mathbb{T}^\infty$ we need to find a function in $\mathcal{U}$, $h$ such that $h(z) \neq h(w)$. The points $z$ and $w$ must differ in at least one coordinate, so without
loss of generality, assume they differ in the first one, \( z_1 \neq w_1 \). Consider the linear function in one variable \( P(z) = az_1 + (b + ic) \), where \( a, b, c \neq 0 \). Given two points, the constants can be chosen such that \( h(z) = |P(z)|^2 \) separates those points.

So we have that \( \mathcal{U} \) is dense in \( C(T^\infty, \mathbb{R}) \), and \( \phi \) as defined in (1.1) is clearly linear on \( \mathcal{U} \). Extend \( \phi \) continuously to \( C(T^\infty, \mathbb{R}) \). Because \( C(T^\infty, \mathbb{R}) \) is a Banach space and because of the assumption that the limit exists and is bounded for all elements in \( A(\mathbb{C}_+) \), \( \phi \) is bounded on \( C(T^\infty, \mathbb{R}) \). Then Riesz Representation gives a unique measure \( \mu \) on \( T^\infty \) such that

\[
\phi(h) = \int_{T^\infty} h(z) d\mu.
\]

This \( \mu \) is a probability measure because \( \lim_{T \to \infty} \frac{1}{m([0,T])} \int_0^T |1|^2 \, d\lambda(t) = 1 \), as needed.

\[ \square \]

**Remark: Lebesgue measure**

If \( \mu \) is Lebesgue measure on \( T^\infty \), \( \int_{T^\infty} |\sum a_n z^n|^2 \, d\mu(z) = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma} \). In this case, it is possible to choose \( \lambda \) to be Lebesgue measure, in addition to the measure constructed in the proof. Using the convergence in \( A(\mathbb{C}_+) \) then, we can see that for a sequence of Dirichlet polynomials \( f_m \to f \in A(\mathbb{C}_+) \) and corresponding \( F_m \to F \in A(D^N) \),

\[
\left| \frac{1}{T} \int_0^T |f(it)|^2 \, d\lambda(t) - \int_{T^\infty} |F(z)|^2 \, d\mu(z) \right| \\
\leq \left| \frac{1}{T} \int_0^T |f(it)|^2 \, d\lambda(t) - \frac{1}{T} \int_0^T |f_m(it)|^2 \, d\lambda(t) \right| + \left| \frac{1}{T} \int_0^T |f(it)|^2 \, d\lambda(t) - \int_{T^\infty} |F(z)|^2 \, d\mu(z) \right| \\
+ \left| \int_0^T |F_m(it)|^2 \, d\lambda(t) - \int_0^T |F(it)|^2 \, d\lambda(t) \right| < \epsilon
\]

where the first and third terms are small because of the convergence, and the middle term is small for large \( T \) because Carlson’s theorem holds on the boundary \( \sigma = 0 \) for Dirichlet polynomials. This is not hard to see by simply computing the integrals on either side.

This example lets us see explicitly that given \( \mu \) we do not have uniqueness for \( \lambda \).

**Acknowledgments**

This paper is part of my PhD thesis under the advising of John McCarthy, and I would like to thank him for his suggestions and support.

**References**

References

[1] Aron, R., Bayart, F., Gauthier, P. M., Maestre, M., and Nestoridis, V. Dirichlet approximation and universal Dirichlet series. arXiv:1608.06457 [math.CV] (2016).

[2] Bohr, H. Über die Bedeutung der Potenzreihen unendlich vieler Variablen in der Theorie der Dirichletschen Reihen \( \sum \frac{a_n}{n^s} \). Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. (1913), 441–488.
[3] Carlson, F. Contributions à la théorie des séries de Dirichlet. *Ark. Mat.* 16 (1922), 1–19. Note I.

[4] Folland, G. B. *Real Analysis: Modern Techniques and Their Applications*, 2 ed. John Wiley and Sons, 2013.

[5] Saksman, E., and Seip, K. Integral means and boundary limits of Dirichlet series. *Bull. London Math. Soc.* 41 (2009), 411–422.