WEAK COMPLETIONS OF PARATOPOLOGICAL GROUPS

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This article is respectfully dedicated to Professor Jerzy Mioduszewski on the occasion of his 93rd anniversary

Abstract. Given a $T_0$ paratopological group $G$ and a class $C$ of continuous homomorphisms of paratopological groups, we define the $C$-semicompletion $C[G]$ and $C$-completion $C[G]$ of the group $G$ that contain $G$ as a dense subgroup, satisfy the $T_0$-separation axiom and have certain universality properties. For special classes $C$, we present some necessary and sufficient conditions on $G$ in order that the (semi)completions $C[G]$ and $C[G]$ be Hausdorff. Also, we give an example of a Hausdorff paratopological abelian group $G$ whose $C$-semicompletion $C[G]$ fails to be a $T_1$-space, where $C$ is the class of continuous homomorphisms of sequentially compact topological groups to paratopological groups. In particular, the group $G$ contains an $\omega$-bounded sequentially compact subgroup $H$ such that $H$ is a topological group but its closure in $G$ fails to be a subgroup.

1. Introduction

It is well known [1, §3.6] that each topological group $G$ has the Raïkov completion, $\varrho G$, which coincides with the completion of $G$ with respect to the two-sided group uniformity of the group. The Raïkov completion has a nice categorial characterization. It turns out that $\varrho G$ is a unique topological group, up to topological isomorphism, that contains $G$ as a dense subgroup and has the following extension property: Every continuous homomorphism $h: X \to G$ defined on a dense subgroup $X$ of a topological group $\tilde{X}$ admits a unique extension to a continuous homomorphism $\tilde{h}: \tilde{X} \to \varrho G$ (see [1, Proposition 3.6.12]). A topological group $G$ is Raïkov complete if and only if it is complete in its two-sided uniformity if and only if $G = \varrho G$.

In [2] and [3], it is shown that a kind of the Raïkov completion can also be defined for paratopological groups. By a paratopological group we understand a group $G$ endowed with a topology $\tau$ making the group multiplication $G \times G \to G$, $(x, y) \mapsto xy$ continuous. If, in addition, the inversion $G \to G$, $x \mapsto x^{-1}$, is continuous, then $(G, \tau)$ is a topological group. A topology $\tau$ on a group $G$ is called a topological group topology on $G$ if $(G, \tau)$ is a topological group.

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Let us recall that a topology \( \tau \) on a set \( X \) satisfies the \( T_0 \) separation axiom or, equivalently, \( \tau \) is a \( T_0 \)-\textit{topology} if for any distinct points \( x, y \in G \) there exists an open set \( U \in \tau \) such that \( U \cap \{x, y\} \) is a singleton. It is well known, on the one hand, that each topological group satisfying the \( T_0 \) separation axiom is automatically Tychonoff. On the other hand, the topology \( \tau = \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\} \) turns the additive group of the reals into a \( T_0 \) paratopological group which does not satisfy the \( T_1 \) separation axiom. So \( T_0 \) does not imply \( T_1 \) in paratopological groups. From now on we assume that all paratopological groups considered here satisfy the \( T_0 \) separation axiom.

Each paratopological group \( \langle G, \tau \rangle \) admits a stronger topological group topology \( \tau^\sharp \), whose neighborhood base at the identity \( e \) consists of the set \( U \cap U^{-1} \) where \( e \in U \in \tau \). So, \( \langle G, \tau^\sharp \rangle \) is topologically isomorphic to the subgroup \( \{\langle x, x^{-1} \rangle : x \in G\} \) of the product group \( G \times G \). The topological group \( G^\sharp = \langle G, \tau^\sharp \rangle \) is called the \textit{group coreflection} of the paratopological group \( G = \langle G, \tau \rangle \). If the topology \( \tau \) satisfies the \( T_0 \) separation axiom, then the topological group topology \( \tau^\sharp \) is Hausdorff and hence Tychonoff. A paratopological group \( \langle G, \tau \rangle \) is a topological group if and only if \( \tau = \tau^\sharp \). A paratopological group \( G \) is called \( \sharp \)-\textit{complete} if the topological group \( G^\sharp \) is Raïkov complete. The Sorgenfrey line \( \mathbb{S} \) is an easy example of a \( \sharp \)-complete paratopological group since \( \mathbb{S}^\sharp \) is the discrete group of the reals.

A subset \( D \) of a paratopological group \( \langle G, \tau \rangle \) is called \( \sharp \)-\textit{dense} in \( G \) if \( D \) is dense in \( \langle G, \tau^\sharp \rangle \). Since \( \tau \subseteq \tau^\sharp \), each \( \sharp \)-dense subset is dense (but not vice versa).

According to [4] or [3], each paratopological group \( G \) is a \( \sharp \)-dense subgroup of a \( \sharp \)-complete paratopological group \( \tilde{G} \) that has the following extension property: \textit{any continuous homomorphism} \( h : X \to G \) defined on a \( \sharp \)-dense subgroup \( X \) of a paratopological group \( \tilde{X} \) admits a unique extension to a continuous homomorphism \( h : \tilde{X} \to \tilde{G} \). This extension property of \( \tilde{G} \) implies that the \( \sharp \)-complete paratopological group \( \tilde{G} \) containing \( G \) as a \( \sharp \)-dense subgroup is unique up to a topological isomorphism. In [3] this unique paratopological group \( \tilde{G} \) is called the \textit{Raïkov completion} of \( G \) and in [4] it is called the \textit{bicompletion} of \( G \). By [3], a neighborhood base at the identity \( e \) of the paratopological group \( \tilde{G} \) consists of the sets \( \overline{U^\sharp} \), where \( U \) is a neighborhood of \( e \) in \( G \) and \( \overline{U^\sharp} \) is the closure of \( U \) in the Raïkov completion \( \emptyset G^\sharp \) of the topological group \( G^\sharp \). So as a group, \( \tilde{G} \) coincides with the Raïkov completion of \( G^\sharp \). If \( G \) is a topological group, then the Raïkov completion \( \tilde{G} \) of \( G \) as a paratopological group coincides with the usual Raïkov completion \( \emptyset G \) of \( G \).

By [3], for any regular paratopological group \( G \), its Raïkov completion \( \tilde{G} \) is regular and hence Tychonoff (for the latter implication, see [2]). However, the Raïkov completion of a Hausdorff paratopological group \( G \) is not necessarily Hausdorff (see [3] Example 2.9]). To bypass this pathology of \( \tilde{G} \), in this article we consider some weaker notions of completion of paratopological groups which preserve Hausdorffness in some special cases.

Our weaker notion of completion depends on a class \( \mathcal{C} \) of continuous homomorphisms between paratopological groups. For example, \( \mathcal{C} \) can be the class of all continuous homomorphisms of topological groups (possessing some property like precompactness, pseudocompactness, countable compactness, sequential compactness, etc.) to paratopological groups. So, we fix a class \( \mathcal{C} \) of continuous homomorphisms between paratopological groups.
A paratopological group $\tilde{G}$ is called a $\mathcal{C}$-semicompletion of a paratopological group $G$ if $G \subseteq \tilde{G}$ and any homomorphism $h: X \to G$ in the class $\mathcal{C}$ has a unique extension $\tilde{h}: \tilde{X} \to \tilde{G}$ to the Raïkov completion $\tilde{X}$ of the paratopological group $X$. The extension property of the Raïkov completion $\tilde{G}$ ensures that $\tilde{G}$ is a $\mathcal{C}$-semicompletion of $G$. So every paratopological group has at least one $\mathcal{C}$-semicompletion.

A $\mathcal{C}$-semicompletion $\tilde{G}$ of a paratopological group $G$ is called minimal if $\tilde{G} = H$ for any subgroup $H \subseteq G$ which is a $\mathcal{C}$-semicompletion of $G$. Every $\mathcal{C}$-semicompletion $\tilde{G}$ of a paratopological group $G$ contains a unique minimal $\mathcal{C}$-semicompletion of $G$, which is equal to the intersection $\bigcap S$ of the family $S$ of subgroups $H \subseteq \tilde{G}$ such that $H$ is a $\mathcal{C}$-semicompletion of $G$. The Hausdorffness of the group coreflections implies that $\bigcap S$ is indeed a $\mathcal{C}$-semicompletion of $G$.

The $\mathcal{C}$-semicompletion $\mathcal{C}(G)$ of a paratopological group $G$ is defined to be the smallest $\mathcal{C}$-semicompletion of $G$ in its Raïkov completion $\tilde{G}$. It is equal to the intersection $\bigcap S$ of the family $S$ of all groups $H$ with $G \subseteq H \subseteq \tilde{G}$ such that $H$ is a $\mathcal{C}$-semicompletion of $G$. Let us note that the $\mathcal{C}$-semicompletion $\mathcal{C}(G)$ of $G$ is topologically isomorphic to any minimal $\mathcal{C}$-semicompletion of $G$. Indeed, given a minimal $\mathcal{C}$-semicompletion $H$ of $G$, consider the Raïkov completion $\tilde{H}$ of the paratopological group $H$. Since $G \subseteq H \subseteq \tilde{H}$, the Raïkov completion $\tilde{G}$ of $G$ can be identified with the $\tilde{\pi}$-closure of $G$ in $\tilde{H}$. Now the minimality of the $\mathcal{C}$-semicompletion $H$ implies that $H = H \cap \tilde{G} \subseteq \tilde{G}$ and, finally, $H = \mathcal{C}(G)$.

Having defined the $\mathcal{C}$-semicompletions of paratopological groups, we can pose the following two general problems.

**Problem 1.1.** Explore the categorial properties of $\mathcal{C}$-semicompletions.

**Problem 1.2.** Characterize the paratopological groups $G$ for which the $\mathcal{C}$-semicompletion $\mathcal{C}(G)$ of $G$ is Hausdorff.

Some partial answers to these problems will be given in Sections 2 and 3.

Again, let $\mathcal{C}$ be a class of continuous homomorphisms of paratopological groups. Now we introduce the notion of a $\mathcal{C}$-completion of a paratopological group. A paratopological group $G$ is $\mathcal{C}$-complete if $G$ is a $\mathcal{C}$-semicompletion of $G$. Equivalently, a paratopological group $G$ is $\mathcal{C}$-complete if each homomorphism $h: X \to G$ in the class $\mathcal{C}$ extends to a continuous homomorphism $\tilde{h}: \tilde{X} \to G$ defined on the Raïkov completion of $X$. By the extension property of Raïkov completions, each $\tilde{\pi}$-complete paratopological group $G$ (hence, $\tilde{G}$) is $\mathcal{C}$-complete. In particular, the Sorgenfrey line $S$ is $\mathcal{C}$-complete.

By the $\mathcal{C}$-completion, $\mathcal{C}(G)$, of a paratopological group $G$ we understand the intersection $\bigcap S$ of the family $S$ of all $\mathcal{C}$-complete subgroups $H \subseteq \tilde{G}$ that contain $G$. The Hausdorffness of the groups coreflections of paratopological groups implies that the $\mathcal{C}$-completion $\mathcal{C}(G)$ of any paratopological group $G$ is $\mathcal{C}$-complete and, hence, $\mathcal{C}(G) \subseteq \mathcal{C}(G)$.

It is clear from the above definitions that if $\mathcal{C}$ is the empty class (or, more generally, every homomorphism $h: X \to G$ with $h \in \mathcal{C}$, if exists, is trivial), then $G$ is $\mathcal{C}$-complete and, therefore, $G = \mathcal{C}(G) = \mathcal{C}(G)$. 


Problem 1.3. Explore the categorial properties of the operation of $\mathcal{C}$-completion in the category of paratopological groups and their continuous homomorphisms.

Problem 1.4. Characterize the paratopological groups $G$ such that the $\mathcal{C}$-completion $\mathcal{C}[G]$ of $G$ is Hausdorff.

Problem 1.5. For which paratopological groups (and classes $\mathcal{C}$ of homomorphisms) do their $\mathcal{C}$-semicompletions coincide with their $\mathcal{C}$-completions?

2. Categorial properties of $\mathcal{C}$-completions and $\mathcal{C}$-semicompletions

A class $\mathcal{C}$ of continuous homomorphisms between paratopological groups is called composable if for any homomorphism $f : X \to Y$ in the class $\mathcal{C}$ and any continuous homomorphism $g : Y \to Z$ of paratopological groups, the composition $g \circ f$ is in $\mathcal{C}$. For example, for any class $\mathcal{P}$ of paratopological groups, the class $\mathcal{C}$ of continuous homomorphisms $h : X \to Y$ between paratopological groups with $X \in \mathcal{P}$ is composable.

The following proposition shows that for a composable class $\mathcal{C}$, the constructions of $\mathcal{C}$-semicompletion and $\mathcal{C}$-completion are functorial in the category of paratopological groups and their continuous homomorphisms.

Proposition 2.1. Let $\mathcal{C}$ be a composable class of continuous homomorphisms of paratopological groups. For any continuous homomorphism $h : X \to Y$ of paratopological groups, its continuous extension $\tilde{h} : \tilde{X} \to \tilde{Y}$ satisfies $\tilde{h}[\mathcal{C}[X]] \subseteq \mathcal{C}[Y]$ and $\tilde{h}[\mathcal{C}[X]] \subseteq \mathcal{C}[Y]$.

Proof. To see that $\tilde{h}[\mathcal{C}[X]] \subseteq \mathcal{C}[Y]$, it suffices to check that the preimage $\tilde{h}^{-1}[\mathcal{C}[Y]]$ is a $\mathcal{C}$-semicompletion of $X$. Given any homomorphism $f : Z \to X$ in the class $\mathcal{C}$, consider its continuous homomorphic extension $\tilde{f} : \tilde{Z} \to \tilde{X}$. Then $\tilde{h} \circ \tilde{f} : \tilde{Z} \to \tilde{Y}$ is a continuous extension of the homomorphism $h \circ f : Z \to Y$. Taking into account that $\mathcal{C}[Y]$ is a $\mathcal{C}$-semicompletion of $Y$ and the topology of the group reflection of $\tilde{Y}$ is Hausdorff, we conclude that $(\tilde{h} \circ \tilde{f})[\tilde{Z}] \subseteq \mathcal{C}[Y]$ and hence $\tilde{f}[\tilde{Z}] \subseteq \tilde{h}^{-1}[\mathcal{C}[Y]]$. This implies that $\tilde{h}^{-1}[\mathcal{C}[Y]]$ is a $\mathcal{C}$-semicompletion of $X$ and $\mathcal{C}[X] \subseteq \tilde{h}^{-1}[\mathcal{C}[Y]]$, by the minimality of $\mathcal{C}[X]$.

Next, we show that the preimage $\tilde{h}^{-1}[\mathcal{C}[Y]]$ is a $\mathcal{C}$-complete paratopological group. Given any homomorphism $f : Z \to \tilde{h}^{-1}[\mathcal{C}[Y]]$ in the class $\mathcal{C}$, consider its unique continuous extension $\tilde{f} : \tilde{Z} \to \tilde{X}$. Then $\tilde{h} \circ \tilde{f} : \tilde{Z} \to \tilde{Y}$ is a unique continuous extension of the homomorphism $h \circ f : Z \to \mathcal{C}[Y]$. Taking into account that the paratopological group $\mathcal{C}[Y]$ is $\mathcal{C}$-complete and the topology of the group reflection of $\tilde{Y}$ is Hausdorff, we conclude that $(\tilde{h} \circ \tilde{f})[\tilde{Z}] \subseteq \mathcal{C}[Y]$, which implies the inclusion $\tilde{f}[\tilde{Z}] \subseteq \tilde{h}^{-1}[\mathcal{C}[Y]]$. Therefore, the paratopological group $\tilde{h}^{-1}[\mathcal{C}[Y]]$ is $\mathcal{C}$-complete and $\mathcal{C}[X] \subseteq \tilde{h}^{-1}[\mathcal{C}[Y]]$, by the minimality of $\mathcal{C}[X]$. □

3. The Hausdorff property of $\mathcal{C}$-completions of paratopological groups

In this section, we present some necessary and some sufficient conditions on a paratopological group $G$ in order that the $\mathcal{C}$-semicompletion $\mathcal{C}[G]$ or $\mathcal{C}$-completion $\mathcal{C}[G]$ of $G$ be Hausdorff. Let us recall that for a regular paratopological group $G$, its Raïkov completion $\tilde{G}$ is regular and so are the paratopological groups $\mathcal{C}[G]$ and $\mathcal{C}[G]$. 

Proposition 3.1. The $C$-completion $C[G]$ of a paratopological group $G$ is Hausdorff if and only if $G$ is a subgroup of a Hausdorff $C$-complete paratopological group.

Proof. The “only if” part is trivial. To prove the “if” part, assume that $G$ is a subgroup of a Hausdorff $C$-complete paratopological group $H$. Taking into account that $G \subseteq H$ and $G^t \subseteq H^t$, we conclude that $\hat{G} \subseteq \hat{H}$. The $C$-completeness of the paratopological groups $\hat{G}$ and $H$ implies that the paratopological group $\hat{G} \cap H$ is $C$-complete and hence $C[G] \subseteq \hat{G} \cap H$ is Hausdorff, being a subgroup of the Hausdorff paratopological group $H$. \(\square\)

Let $\mathcal{P}$ be a property. A paratopological group $H$ is said to be projectively $\mathcal{P}$ if every neighborhood of the identity element in $H$ contains a closed invariant (equivalently, “normal” in the algebraic sense) subgroup $N$ such that the quotient paratopological group $H/N$ has $\mathcal{P}$. Similarly, a paratopological group $X$ is said to be projectively $\Psi$ if for every neighborhood $U$ of the identity in $X$, there exists a continuous homomorphism $h: X \to Y$ to a Hausdorff paratopological group of countable pseudocharacter such that $h^{-1}(e_Y) \subseteq U$; here $e_Y$ denotes the identity of the group $Y$.

It is easy to see that the projectively $\Psi$ paratopological groups are exactly the projectively $\mathcal{P}$ groups, where $\mathcal{P}$ is the property of being a Hausdorff space of countable pseudocharacter. Indeed, let $G$ be a projectively $\Psi$ paratopological group and $U$ be a neighborhood of the identity in $G$. By our assumption, there exists a continuous homomorphism $h: G \to H$ onto a Hausdorff paratopological group $H_\alpha$ of countable pseudocharacter such that $h^{-1}(e_H) \subseteq U$, where $e_H$ is the identity of $H$. Let $N = h^{-1}(e_H)$ be the kernel of $h$ and $p: G \to G/N$ be the quotient homomorphism. Clearly there exists a continuous one-to-one homomorphism $j: G/N \to H$ satisfying $h = j \circ p$. Thus $j$ is a continuous bijection of $G/N$ onto $H$ and, hence, the quotient group $G/N$ is Hausdorff and has countable pseudocharacter. Therefore, $G$ is projectively $\mathcal{P}$. The inverse implication is evident.

According to [6, Proposition 2] every projectively Hausdorff paratopological group is Hausdorff. In particular, all projectively $\Psi$ paratopological groups are Hausdorff. This fact also follows from the characterization of projectively $\Psi$ groups presented in the next lemma.

Lemma 3.2. A paratopological group $G$ is projectively $\Psi$ if and only if $G$ is topologically isomorphic to a subgroup of a product of Hausdorff paratopological groups of countable pseudocharacter.

Proof. The sufficiency is evident, so we verify only the necessity. Assume that $G$ is a projectively $\Psi$ paratopological group. Applying [6, Proposition 2] we conclude that $G$ is Hausdorff. Let $\{U_\alpha : \alpha \in A\}$ be a neighborhood base at the identity $e$ of $G$. For every $\alpha \in A$, take an open neighborhood $V_\alpha$ of $e$ in $G$ such that $V_\alpha^2 \subseteq U_\alpha$. By our assumption, there exists a continuous homomorphism $f_\alpha$ of $G$ onto a Hausdorff paratopological group $H_\alpha$ of countable pseudocharacter such that $f_\alpha^{-1}(e_\alpha) \subseteq V_\alpha$, where $e_\alpha$ is the identity of $H_\alpha$. Let $N_\alpha$ be the kernel of $f_\alpha$ and $p_\alpha: G \to G/N_\alpha$ be the quotient homomorphism. Clearly there exists a continuous one-to-one homomorphism $j_\alpha: G/N_\alpha \to H_\alpha$ satisfying $f_\alpha = j_\alpha \circ p_\alpha$. Thus $j_\alpha$ is a continuous bijection of $G/N_\alpha$ onto $H_\alpha$ and, hence, the quotient group $G/N_\alpha$ is Hausdorff and has countable pseudocharacter.
Denote by \( p \) the diagonal product of the family \( \{ p_\alpha : \alpha \in A \} \). Then \( p \) is a continuous homomorphism of \( G \) to \( P = \prod_{\alpha \in A} G/N_\alpha \). We claim that \( p \) is an isomorphic topological embedding. First, the kernel of \( p \) is trivial. Indeed, if \( x \in G \) and \( x \neq e_G \), take a neighborhood \( U \) of \( e_G \) such that \( x \notin U \). There exists \( \alpha \in A \) such that \( p^{-1}_\alpha(e_\alpha) \subseteq U_\alpha \subseteq U \), whence it follows that \( p_\alpha(x) \neq e_\alpha \). Hence \( p(x) \neq e_H \) and \( p \) is injective.

Further, let \( U \) be an arbitrary neighborhood of \( e \) in \( G \). Take \( \beta \in A \) with \( U_\beta \subseteq U \). Then \( f_\beta \{ f_\beta(V_\beta) = V_\beta N_\beta \subseteq V_\beta^2 \subseteq U_\beta \subseteq U \). Since \( j_\beta \) is one-to-one, it follows from \( f_\beta = j_\beta \circ p_\beta \) that \( p_\beta^{-1} p_\beta(V_\beta) = f_\beta^{-1} f_\beta(V_\beta) \subseteq U \). Thus the open neighborhood \( p_\beta(V_\beta) \) of the identity in \( H_\beta \) satisfies \( p_\beta^{-1} p_\beta(V_\beta) \subseteq U \). Denote by \( \pi_\beta \) the projection of \( \prod_{\alpha \in A} G/N_\alpha \) to the factor \( G/N_\beta \). Applying the equality \( p_\beta = \pi_\beta \circ p \) we deduce that the open neighborhood \( W = p(G) \cap \pi_\beta^{-1} p_\beta(V_\beta) \) of the identity in \( p(G) \) satisfies \( p^{-1}(W) = p_\beta^{-1}(p_\beta(V_\beta)) \subseteq U \). We have thus proved that for every neighborhood \( U \) of the identity in \( G \), there exists a neighborhood \( W \) of the identity in \( p(G) \) such that \( p^{-1}(W) \subseteq U \). This property of the continuous monomorphism \( p \) implies that \( p \) is an isomorphic topological embedding of \( G \) into \( \prod_{\alpha \in A} G/N_\alpha \) (see also [1, Section 3.4]).

A paratopological group \( G \) is called \( \omega \)-balanced if for any neighborhood \( U \) of the identity in \( G \), there exists a countable family \( \mathcal{V} \) of neighborhoods of the identity in \( G \) such that for each \( x \in G \), one can find \( V \in \mathcal{V} \) satisfying \( xVx^{-1} \subseteq U \).

The Hausdorff number \( Hs(G) \) of a Hausdorff paratopological group \( G \) is the smallest cardinal \( \kappa \geq 1 \) such that for every neighborhood \( U \subseteq G \) of the identity in \( G \), there exists a family \( \{ V_\alpha \}_{\alpha \in \kappa} \) of neighborhoods of the identity in \( G \) such that \( \bigcap_{\alpha \in \kappa} V_\alpha V_\alpha^{-1} \subseteq U \). The Hausdorff number was introduced and studied in [7].

According to [7, Theorem 2.7], every \( \omega \)-balanced paratopological group \( G \) with \( Hs(G) \leq \omega \) is a subgroup of a Tychonoff product of first-countable Hausdorff paratopological groups. By Lemma 3.2, the latter implies that the \( \omega \)-balanced paratopological groups with countable Hausdorff number are projectively \( \Psi \).

**Theorem 3.3.** Let \( \mathcal{C} \) be a subclass of the class of continuous homomorphisms from pseudocompact topological groups to paratopological groups. If a paratopological group \( G \) is projectively \( \Psi \), then its \( \mathcal{C} \)-completion \( \mathcal{C}[G] \) is Hausdorff. If \( G \) is \( \omega \)-balanced and satisfies \( Hs(G) \leq \omega \), then the paratopological group \( \mathcal{C}[G] \) is \( \omega \)-balanced, Hausdorff, and satisfies \( Hs(\mathcal{C}[G]) \leq \omega \).

**Proof.** Let \( G \) be a projectively \( \Psi \) paratopological group. By Lemma 3.2, \( G \) is topologically isomorphic to a subgroup of a Tychonoff product \( Y = \prod_{\alpha \in A} Y_\alpha \) of Hausdorff paratopological groups of countable pseudocharacter. We claim that the paratopological group \( Y \) is \( \mathcal{C} \)-complete. Consider any homomorphism \( f : Z \to Y \) in the class \( \mathcal{C} \). By the choice of \( \mathcal{C} \), \( Z \) is a pseudocompact topological group. For every \( \alpha \in A \), denote by \( \pi_\alpha : Y \to Y_\alpha \) the natural projection and consider the continuous homomorphism \( f_\alpha = \pi_\alpha \circ f : Z \to Y_\alpha \). Let \( e_\alpha \) be the identity of the group \( Y_\alpha \). Clearly \( Z_\alpha = f_\alpha^{-1}(e_\alpha) \) is a closed invariant subgroup of \( Z \) and the quotient topological group \( Z/Z_\alpha \) admits a continuous injective homomorphism to the group \( Y_\alpha \) of countable pseudocharacter. Hence \( Z/Z_\alpha \) has countable pseudocharacter as well and we can apply [8, Proposition 2.3.12] to conclude that \( Z/Z_\alpha \) is a compact
metrizable group. Let \( q_\alpha : Z \to Z/Z_\alpha \) be the quotient homomorphism and \( g_\alpha : Z/Z_\alpha \to Y_\alpha \) be a unique continuous injective homomorphism such that \( f_\alpha = g_\alpha \circ q_\alpha \). Since the compact topological group \( Z/Z_\alpha \) is Raïkov complete, the quotient homomorphism \( q_\alpha : Z \to Z/Z_\alpha \) admits a continuous homomorphic extension \( \tilde{q}_\alpha : \tilde{Z} \to Z/Z_\alpha \). Then \( \tilde{f}_\alpha = g_\alpha \circ \tilde{q}_\alpha : \tilde{Z} \to Y_\alpha \) is a continuous extension of the homomorphism \( f_\alpha \).

The diagonal product of the homomorphisms \( \tilde{f}_\alpha : \tilde{Z} \to Y_\alpha \) with \( \alpha \in A \) is a continuous homomorphism \( \tilde{f} : \tilde{Z} \to Y = \prod_{\alpha \in A} Y_\alpha \) extending the homomorphism \( f \) and witnessing that the paratopological group \( Y \) is \( C \)-complete. By Proposition 3.1, the \( C \)-completion \( C[X] \) of \( X \) is Hausdorff.

Now assume that the paratopological group \( G \) is \( \omega \)-balanced and satisfies \( Hs(G) \leq \omega \). By [7, Theorem 2.7], \( G \) is a subgroup of a Tychonoff product \( Y = \prod_{\alpha \in A} Y_\alpha \) of first-countable Hausdorff paratopological groups. Hence \( G \) is projectively \( \Psi \), by Lemma 3.2. By the above argument, the paratopological group \( Y \) is \( C \)-complete and, hence, \( C[G] \) can be identified with a subgroup of \( Y \). By Propositions 2.1–2.3 in [7], the subgroup \( C[G] \) of the Tychonoff product \( Y \) of first-countable Hausdorff paratopological groups is \( \omega \)-balanced and satisfies \( Hs(C[G]) \leq \omega \). \( \square \)

A topological group \( G \) is called precompact if its Raïkov completion \( \rho G \) is compact. This happens if and only if for any neighborhood \( U \) of the identity in \( G \), there exists a finite set \( F \subseteq G \) such that \( G = UF = FU \).

**Proposition 3.4.** Let \( \mathcal{C} \) be a subclass of the class of continuous homomorphisms from precompact topological groups to paratopological groups. A Hausdorff paratopological group \( G \) is \( \mathcal{C} \)-complete if and only if for any homomorphism \( h : X \to G \) in the class \( \mathcal{C} \), the image \( h[X] \) has compact closure in \( G \).

**Proof.** Sufficiency. Take any homomorphism \( h : X \to G \) in the class \( \mathcal{C} \) and assume that the image \( h[X] \) has compact closure \( \overline{h[X]} \) in \( G \). It follows from [1, Proposition 1.4.10] that \( H = \overline{h[X]} \) is a Hausdorff compact topological semigroup and \( H \), being a subsemigroup of a group, is cancellative. By Numakura’s theorem (see [5] or [1, Theorem 2.5.2]), \( H \) is a compact topological group. Since \( H \) is Raïkov complete, the homomorphism \( h : X \to H \) has a continuous homomorphic extension \( \tilde{h} : \tilde{X} \to H \subseteq G \), witnessing that the paratopological group \( G \) is \( \mathcal{C} \)-complete.

Necessity. Assume that the paratopological group \( G \) is \( \mathcal{C} \)-complete. Then every homomorphism \( h : X \to G \) in the class \( \mathcal{C} \) has a continuous homomorphic extension \( \tilde{h} : \tilde{X} \to G \) to the Raïkov completion \( \tilde{X} = \rho X \) of the topological group \( X \). The precompactness of \( X \) guarantees that the topological group \( \tilde{X} \) is compact and so is its image \( \tilde{h}[X] \) in \( G \). Since the space \( G \) is Hausdorff, the compact subspace \( \tilde{h}[\tilde{X}] \) is closed in \( G \) and the closure \( \overline{\tilde{h}[\tilde{X}]} \) of \( \tilde{h}[X] \) in \( G \), being a closed subset of the compact space \( \tilde{h}[\tilde{X}] \), is compact. \( \square \)

A subset \( F \) of a topological space \( X \) is called functionally closed (or else a zero-set) if \( F = f^{-1}(0) \) for some continuous function \( f : X \to \mathbb{R} \). A paratopological group \( G \) is said to be simply sm-factorizable if for every functionally closed set \( A \) in \( G \), there exists a
continuous homomorphism \( h : G \to H \) onto a separable metrizable paratopological group \( H \) such that \( A = h^{-1}[B] \), for some closed set \( B \subseteq H \) (see [11, Definition 5.6]).

A subspace \( X \) of a topological space \( Y \) is called \( C\)-embedded in \( Y \) if each continuous real-valued function on \( X \) has a continuous extension over \( Y \).

**Proposition 3.5.** Every regular simply \( \mathrm{sm} \)-factorizable paratopological group \( G \) is \( C\)-embedded in its \( C\)-completion \( C[G] \) provided \( C \) is a subclass of the class of continuous homomorphisms of pseudocompact topological groups to paratopological groups.

**Proof.** The space \( G \) is Tychonoff, by [2, Corollary 5]. We apply [9, Theorem 4.3] according to which the realcompactification \( \upsilon G \) of the space \( G \) admits the structure of paratopological group containing \( G \) as a dense paratopological subgroup. Since the closure of every pseudocompact subspace in \( \upsilon G \) is compact and every pseudocompact topological group is precompact, the paratopological group \( \upsilon G \) is \( C\)-complete according to Proposition 3.4. Then \( G \subseteq C[G] \subseteq \upsilon G \), which implies that \( G \) is \( C\)-embedded in \( C[G] \), being \( C\)-embedded in \( \upsilon G \). \( \square \)

Now we present a necessary condition on a paratopological group \( G \) for the Hausdorffness of its \( C\)-semicompletion \( C[G] \). In view of the inclusion \( C[G] \subseteq C[G] \), the same condition is necessary for the Hausdorffness of \( C[G] \).

**Proposition 3.6.** Let \( C \) be a subclass of the class of continuous homomorphisms from precompact topological groups to paratopological groups. If the \( C\)-semicompletion \( C[G] \) of \( G \) is Hausdorff, then for any homomorphism \( h : X \to G \) in \( C \), the closure of \( h[X] \) in \( G \) is a precompact topological group.

**Proof.** Let \( h : X \to G \) be any homomorphism in the class \( C \). By the definition of a \( C\)-semicompletion, the homomorphism \( h \) has a continuous extension \( \hat{h} : \hat{X} \to C[G] \). The precompactness of \( X \) ensures that \( \hat{X} = \rho X \) is a compact topological group. Assume that the paratopological group \( C[G] \) is Hausdorff. Then the compact subspace \( \hat{h}[\hat{X}] \) of \( C[G] \) is closed. It follows that \( \hat{h}[\hat{X}] \) is a compact topological group that is topologically isomorphic to a quotient group of the compact topological group \( \hat{X} \). Then the closure of \( h[X] \) in \( G \) is a precompact topological group, which is equal to the intersection \( G \cap \hat{h}[\hat{X}] \) of the group \( G \) with the compact topological group \( \hat{h}[\hat{X}] \subseteq C[G] \).

Finally we present an example of a Hausdorff paratopological group \( G \) with a non-Hausdorff \( C\)-semicompletion, where \( C \) is the class of all continuous homomorphisms from sequentially compact topological groups to paratopological groups. As usual, a space \( X \) is sequentially compact if each sequence in \( X \) contains a convergent subsequence. Also, \( X \) is said to be \( \omega\)-bounded if the closure in \( X \) of every countable set is compact.

**Example 3.7.** Let \( C \) be the class of continuous homomorphisms from sequentially compact topological groups to paratopological groups. There exists a Hausdorff paratopological abelian group \( G \) whose \( C\)-semicompletion \( C[G] \) fails to be a \( T_1 \)-space. In addition, \( G \) contains a subgroup \( H \) such that \( H \) is a sequentially compact \( \omega\)-bounded topological group but its closure in \( G \) is not a group.
Proof. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the torus group with its usual topology and multiplication inherited from the complex plane. So the identity of $\mathbb{T}$ is 1. Denote by $\Sigma$ the subgroup of the Tychonoff product $\mathbb{T}^{\omega_1}$ defined as follows:

$$
\Sigma = \{x \in \mathbb{T}^{\omega_1} : |\{\alpha \in \omega_1 : x_\alpha \neq 1\}| \leq \omega\}.
$$

According to Corollaries 1.6.33 and 1.6.34 of [1], the space $\Sigma$ is Fréchet-Urysohn and $\omega$-bounded. Hence $\Sigma$ is sequentially compact.

Take an element $c \in \mathbb{T}^{\omega_1}$ of infinite order such that $\langle c \rangle \cap \Sigma = \{c\}$, where $e$ is the identity element of $\mathbb{T}^{\omega_1}$ and $\langle c \rangle$ is the cyclic group generated by $c$. For every open neighborhood $U$ of $e$ in $\mathbb{T}^{\omega_1}$, we define a subset $O_U$ of $G = \Sigma \times \mathbb{Z}$ by letting

$$
O_U = \bigcup_{n \in \omega} (c^n U \cap \Sigma) \times \{n\}.
$$

Here $\omega$ is identified with the subset $\{0, 1, 2, \ldots\}$ of $\mathbb{Z}$. A routine verification shows that the sets $O_U$ with $U$ as above, constitute a base at the identity $(e, 0) \in G$ for a Hausdorff paratopological group topology on $G$.

It turns out that the subgroup $H = \Sigma \times \{0\}$ of $G$ has the required properties. Let us show that the closure of $H$ of $G$ is the set $\Sigma \times (-\omega)$, where $-\omega = \{0, -1, -2, \ldots\}$, so this closure is not a subgroup of $G$. First, we claim that any element $g = (x, k) \in G$ with $k > 0$ is not in the closure of $H$. Indeed, let $U$ be an arbitrary open neighborhood of $e$ in $\mathbb{T}^{\omega_1}$. Then the set $gO_U$ is an open neighborhood of $g$ in $G$ disjoint from $H$.

Further, consider an element $g = (x, -k) \in G$, where $x \in \Sigma$ and $k \in \omega$. If $k = 0$, then $g \in \Sigma \times \{0\} = H$. Assume that $k > 0$ and take a basic open neighborhood $O_U$ of the identity in $G$, where $U$ is an open neighborhood of $e$ in $\mathbb{T}^{\omega_1}$. Then

$$
g \cdot O_U = g \cdot \bigcup_{n \in \omega} ((c^n U \cap \Sigma) \times \{n\}) = \bigcup_{n \in \omega} (xe^n U \cap \Sigma) \times \{n - k\}.
$$

Therefore, $g \cdot O_U \cap H = g \cdot O_U \cap (\Sigma \times \{0\}) = (xe^k U \cap \Sigma) \times \{k\} \neq \emptyset$. Hence $g$ is in the closure of $H$. We have thus shown that the closure of $H$ in $G$ is the asymmetric subset $\Sigma \times (-\omega)$ of $G$, which implies the second claim of the example.

The definition of the topology of the paratopological group $G$ implies that the topology of the group coreflection $G^\sharp$ coincides with the product topology on $\Sigma \times \mathbb{Z}$ and then the Raïkov completion $\rho G^\sharp$ can be identified with the product $\mathbb{T}^{\omega_1} \times \mathbb{Z}$, where $\mathbb{Z}$ is endowed with the discrete topology. By [3] §2.2], a neighborhood base for the topology of the Raïkov completion $\hat{G}$ at its identity consists of the closures $\overline{O_U}$ of the basic neighborhoods $O_U$ in the product space $\mathbb{T}^{\omega_1} \times \mathbb{Z}$. It is easy to see that for any neighborhood $U$ of $e$ in $\mathbb{T}^{\omega_1}$, the closure $\overline{O_U}$ of $O_U = \bigcup_{n \in \omega} (c^n U \cap \Sigma) \times \{n\}$ coincides with the set $\bigcup_{n \in \omega} (c^n U) \times \{n\}$. We see, therefore, that each neighborhood of the identity in $\hat{G}$ contains the point $(c, 1)$, which means that the Raïkov completion $\hat{G}$ of $G$ does not satisfy the $T_1$ separation axiom.

It remains to prove that $\hat{G}$ is the $C$-semicompletion of $G$. Since the subgroup $\Sigma$ of $\mathbb{T}^{\omega_1}$ is sequentially compact, the continuous homomorphism

$$
h : \Sigma \to \Sigma \times \{0\} \subseteq G, \quad h : x \mapsto \langle x, 0 \rangle,
$$

...
extends to a continuous homomorphism $\bar{h}: T^{ω_1} \to \mathcal{C}(G) \subseteq \mathcal{G}$. Since $T^{ω_1}$ is a topological group, the homomorphism $\bar{h}$ remains continuous with respect to the topology of the Raǐkov completion $\rho G^♯ = T^{ω_1} \times \mathbb{Z}$ of the topological group $G^♯$. Now the compactness of $T^{ω_1}$ implies that $T^{ω_1} \times \{0\} = \bar{h}|T^{ω_1}] \subseteq \mathcal{C}(G)$ and, hence, $\mathcal{C}(G) = \mathcal{G}$. □

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