NON-MAXIMAL INTEGRAL ELEMENTS IN JET SPACES AND PARTIAL PROLONGATIONS

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Abstract. The space of \( l \)-dimensional horizontal integral elements at a point in a jet space as well as the space of \((l, n)\)-dimensional flags of horizontal integral elements are supplied with a natural Pfaffian system. It is established that these exterior differential systems are the prolongations of a natural system of first order PDEs on sections of a bundle we dubbed partial jet prolongations. Explicit bases and their commutation relations are given for all these distributions.

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Introduction

Let \( J \) be a manifold supplied with an exterior differential system. Consider a point \( \theta \in J \) and denote with \( I^{l}_{\theta} \) the space of \( l \)-dimensional regular integral elements of the exterior differential system at \( \theta \). There is a natural distribution (in the sense of field of tangent planes) on \( I^{l}_{\theta} \): a tangent vector to an integral element \( L \) belongs to this distribution if, considered as an infinitesimal first order motion of \( L \), it leaves \( L \) inside of its polar space. In [Bächtold, Moreno (2014)] we called this the polar distribution. Its existence was pointed out by A. M. Vinogradov [Vinogradov (2008)].
In the case when $J = J^k(2,1)$ ($k^{th}$ order jets of function with two independent and one dependent variables) and the exterior differential system is the contact structure (a.k.a. Cartan distribution) it was proved in [Bächtold (2009)], that the polar distribution on the space of one-dimensional horizontal integral is locally isomorphic to a Cartan distribution on the jet space of a new bundle with one independent and $k + 1$ dependent variables. This was later extended in [Bächtold, Moreno (2014)] to the case of one-dimensional integral elements of contact manifolds, i.e. the case $J = J^l(n,1)$. Both proofs were in local coordinates and gave no hint on the geometrical origin of this new bundle, nor on ways to extend the result to other dimensions.

Here we remedy this by:

(a) generalising the result to horizontal integral elements of arbitrary dimension $l < n$ in the Cartan distribution at a jet $\theta_k \in J^k(n,m)$ in arbitrary numbers of independent and dependent variables $n$ resp. $m$.

(b) giving a coordinate free description of this new bundle and clarifying its geometrical meaning.

I call the bundle mentioned in b) the space of partial jet prolongations of $\theta_k$ and denote its total space with $\vec{J}_l^{\theta_k}$. An element of it can be thought of as a way of extending the $k^{th}$ order jet $\theta_k$ with $k + 1^{st}$ order information in direction of an $l$-dimensional subspace. In terms of local coordinates this means that an element of $\vec{J}_l^{\theta_k}$ adds partial derivatives of order $k + 1$ in $l$ chosen directions to the partial derivatives of order $\leq k$ specified by $\theta_k$ in all directions. The base of the bundle of partial jet prolongations will be the Grassmannian of “directions” along which we prolong. So a section of this bundle specifies a prolongation of $\theta_k$ along each $l$-dimensional direction. There is a natural condition for such a section to be “holonomic”: when all prolongations agree on intersecting directions. These pasting conditions can be formulated as a system of first order linear PDEs on sections of $\vec{J}_l^{\theta_k}$. We then have

**Theorem 1** (Main result, first part). The polar distribution on $I_{\theta_k}^l$ is the $k - 1^{st}$ prolongation of the system of pasting PDEs on $\vec{J}_l^{\theta_k}$.

In the case of 1-dimensional integral elements the pasting conditions are empty as there are no non-trivial intersections of one dimensional directions. So in that case $I_{\theta_k}^l$ is the $k^{th}$ jet space of the bundle $\vec{J}_l^{\theta_k}$, in agreement with the previous results [Bächtold (2009), Bächtold, Moreno (2014)].

We further show

**Theorem 2** (Main result, continued). Prolonging the polar distribution once more leads to the space of $(l,n)$-dimensional partial integral flags with its canonical distribution induced from its double fibration structure. Finally, when $l > 1$, prolonging once more stabilizes the process leading to an involutive distribution whose integral leaves correspond to “full” prolongations of $\theta_k$, i.e. jets of order $k + 1$.

The proof of these theorems proceeds from the top of the prolongation tower down: we consider the space of partial flags of integral elements and exhibit certain natural structures on it. From these we construct the tower of prolongations by reduction. Along the way we construct non holonomic frames adapted to the distributions and compute their commutation relations.

The article is divided into two sections. The first one gives more detailed definitions and a statement of the main result. The second section contains the proofs.

**Motivations and relations to other work.** Spaces of lower dimensional integral elements in the Cartan distribution appear at several places in the theory of PDEs.
They are central to characteristics, Monge cones, geometric singularities of PDEs
[Vinogradov (1987), Lychagin (1988)] and boundary conditions [Moreno (2013)].
They have been used to find differential contact invariants of certain classes of
PDEs [Alekseevsky et al. (2012), Marvan et al. (1987)] . To the authors knowledge,
the associated polar distribution made its first appearance in the literature
in [Bächtold (2009), Bächtold, Moreno (2014)]. Flags of integral elements appear
in the context of the Cartan-Kähler theorem [Bryant et al. (1991)].

1. Definitions and main results

1.1. Conventions on jets. We work in the setting of jets of \( n \)-dimensional
submanifolds in a fixed \( n + m \)-dimensional ambient manifold \( E \). The space of \( k \)th order
such jets is denoted with \( J^k = J^k(E, n) \). The reader not familiar with jets of sub-
manifolds might as well think of the locally isomorphic space of jets of sections of
a bundle with \( m \)-dimensional fibers and \( n \)-dimensional base. We fix throughout a
jet \( \theta_k \in J^k \) of order \( k \geq 1 \) and denote with \( C_{\theta_k} \) the plane of the Cartan
distribution at \( \theta_k \). The terminology Cartan distribution and higher contact structure are used
synonymously. Manifolds are real although all arguments remain valid over any
field of characteristic 0.

We shall make use of several facts on jets and the Cartan distribution which we
recall here. The initiated reader may want to skip ahead to the next subsection
1.2.

For \( k > r \) there are natural projections \( \pi_{k,r} : J^k \to J^r \) “forgetting” higher order
information of jets. We say that \( \theta_k \) restricts to the \( r \)th order jet \( \theta_r \in J^r \) (or that \( \theta_k \) prolongs or extends \( \theta_r \)) when \( \theta_r = \pi_{k,r}(\theta_k) \). In particular, the restriction of \( \theta_k \)
to 0th order is a point in \( E = J^0 \) denoted with \( \theta_0 \).

In agreement with our convention of indexing the fiber of a bundle with its base
point, we denote the manifold of all \( k + 1 \)st order jets prolonging \( \theta_k \) with \( J^{k+1} \).

There is a natural bijection between \( n \)-dimensional horizontal integral planes
\( R \subset C_{\theta_k} \) and jets of order \( k + 1 \) extending \( \theta_k \). Such integral planes are called
R-planes in [Bocharov et al. (1999)]. The R-plane corresponding to \( \theta_{k+1} \in J^{k+1}_{\theta_k} \)
is denoted with \( R_{\theta_{k+1}} \). The R-plane in \( J^0 = E \) corresponding to the 1st order
restriction of \( \theta_k \) will be denoted with \( R \subset T_{\theta_0}E \).

The fiber \( J^{k+1}_{\theta_k} \) is affine with underlying vector space \( S^{k+1}R^* \otimes N \) [Lychagin (1988)],
where \( N \) is the normal tangent space \( T_{\theta_k}E \). We will interpret tensors in \( S^{k+1}R^* \otimes N \)
as homogeneous polynomial maps from \( R^* \) to \( N \) of degree \( k + 1 \).

For a distribution \( \mathcal{E} \) on a manifold \( M \) the curvature form is the skew-symmetric
tensor

\[
\Omega : \mathcal{E} \wedge \mathcal{E} \to [\mathcal{E}, \mathcal{E}] / \mathcal{E}
\]  
(1.1)

induced by the Lie bracket of sections of \( \mathcal{E} \). Here \( [\mathcal{E}, \mathcal{E}] \) denotes the derived distribution of \( \mathcal{E} \), which is the distribution spanned by \( \mathcal{E} \) and Lie-brackets of fields in \( \mathcal{E} \).
The curvature form of the Cartan distribution \( \mathcal{C} \) is called the metasymplectic form.
One can show that \( [\mathcal{C}, \mathcal{C}] / \mathcal{C} |_{\theta_k} \cong S^{k-1}R^* \otimes N \) so

\[
\Omega : C_{\theta_k} \wedge C_{\theta_k} \to S^{k-1}R^* \otimes N.
\]  
(1.2)

In standard local coordinates \( x_i, u^j, u^j_\alpha \) on jet spaces \( J^k \), where \( x_i \) are the independent variables, \( u^j \) are the dependent variables and \( u^j_\alpha \) are jet coordinates with
\( \sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{N}^m \) a multiindex of length \( |\sigma| \leq k \), the metasymplectic structure
acts as
\[ \Omega(D_i, D_j) = 0 \] (1.3)
\[ \Omega\left(\partial_{\omega^k_i}, \partial_{\omega^k_j}\right) = 0 \] (1.4)
\[ \Omega\left(\partial_{\omega^k_i}, D_i\right) = \partial_{\omega^{k-1}_i}. \] (1.5)
Here \( D_i = \partial_{\sigma^i} + \sum_{\sigma | \sigma < k} \omega_{\sigma+i}, \partial_{\omega^k_i} \) are total derivatives and the vertical fields \( \partial_{\omega^k_i} \) correspond to the homogeneous polynomials
\[ \frac{1}{\sigma!}(dx_1)^{\sigma_1} \cdots (dx_n)^{\sigma_n} \otimes \frac{\partial}{\partial w^j} \in \mathcal{S}^{k-1} \mathcal{R}^* \otimes \mathcal{N}. \] (1.6)

We refer the reader to section 2.5.3 for further notational conventions.

1.2. Integral elements and the polar distribution. Recall that a vector subspace \( L \subset \mathcal{C}_{\theta_k} \) is called an integral element [Bryant et al. (1991)] (or involutive subspace in [Bocharov et al. (1999)]) of the Cartan distribution, if all differential forms in the differential ideal generated by the Cartan distribution vanish when restricted to \( L \). Equivalently \( L \) is integral if the metasymplectic form \( \Omega \) vanishes when restricted to \( L \). Such a plane is horizontal if it is transversal to fibers of the projection \( J^k \to J^{k-1} \), which turns out to imply transversality with respect to \( J^k \to J^0 \).

**Definition 1.** The space of horizontal \( l \)-dimensional integral elements of \( \mathcal{C}_{\theta_k} \)

\[ I^l_{\theta_k} := \{ L \subset \mathcal{C}_{\theta_k} | \dim L = l, \Omega|_L = 0 \text{, } L \text{ horizontal} \}. \] (1.7)

Horizontal integral elements of maximal dimension are precisely \( R \)-planes.

To define the polar distribution on \( I^l_{\theta_k} \), recall that the polar space \( L^\perp \) [Bryant et al. (1991)] of an integral element \( L \) is defined as the \( \Omega \)-orthogonal of \( L \):
\[ L^\perp := \{ v \in \mathcal{C}_{\theta_k} | \Omega(v, w) = 0 \text{ for all } w \in L \}. \] (1.8)

Since
\[ T_L \text{Gr(}\mathcal{C}_{\theta_k}, l) \cong L^* \otimes \mathcal{C}_{\theta_k}/L \] [Harris (1995)] and \( I^l_{\theta_k} \subset \text{Gr}(\mathcal{C}_{\theta_k}, l) \), a tangent vector \( \dot{L} \) to \( I^l_{\theta_k} \) at \( L \) may be considered to be a linear map
\[ L \overset{\dot{L}}{\longrightarrow} \mathcal{C}_{\theta_k}/L. \] (1.10)

The image of \( \dot{L} : L \to \mathcal{C}_{\theta_k}/L \), understood in the obvious way as a subspace of \( \mathcal{C}_{\theta_k}/L \), is called the osculator \( \text{osc } \dot{L} \) of the tangent vector. It may be thought of as the span of \( L \) and all infinitesimally near \( L \) \( \in \text{Gr}(\mathcal{C}_{\theta_k}, l) \) reached by the infinitesimal motion \( \dot{L} \).

Using these notions we can give the

**Definition 2.** The plane of the polar distribution \( \mathcal{P} \) on \( I^l_{\theta_k} \) at \( L \in I^l_{\theta_k} \) is
\[ \mathcal{P}_L := \left\{ \dot{L} \in T_L(I^l_{\theta_k}) \mid \text{osc } \dot{L} \subseteq L^\perp \right\}. \] (1.11)

Alternatively we have the simpler but equivalent description
\[ \mathcal{P}_L = \left\{ \dot{L} \in L^* \otimes \mathcal{C}_{\theta_k}/L \mid \Omega(l_1, \dot{L}(l_2)) = 0 \text{ for all } l_1, l_2 \in L \right\}, \] (1.12)
which follows from

**Lemma 1.** A vector \( \dot{L} \in L^* \otimes \mathcal{C}_{\theta_k}/L \) is tangent to \( I^l_{\theta_k} \subset \text{Gr}(\mathcal{C}_{\theta_k}, l) \) at \( L \in I^l_{\theta_k} \) iff
\[ \Omega(l_1, \dot{L}(l_2)) = \Omega(l_2, \dot{L}(l_1)) \text{ for all } l_1, l_2 \in L. \] (1.13)
Proof. Consider a smooth one parameter family $L_t \in \mathcal{I}_{\theta_k}^1$ with $L_0 = L_t \big|_{t=0} = \hat{L} \in L^* \otimes C_0 / L$ and families $l_i(t) \in L_t$ with $l_i(0) = l_i$ for $i = 1, 2$. Since $L_t$ is integral we have

$$\Omega \left( (l_1(t), l_2(t)) \right) = 0. \quad (\text{1.14})$$

Taking the derivative with respect to $t$ at $t = 0$ on both sides of equation 1.14 using the product rule we get

$$\Omega \left( l_1, \hat{L}(l_2) \right) + \Omega \left( \hat{L}(l_1), l_2 \right) = 0, \quad (\text{1.15})$$

which by skew-symmetry of $\Omega$ leads to

$$\Omega(l_1, \hat{L}(l_2)) = \Omega(l_2, \hat{L}(l_1)). \quad (\text{1.16})$$

\[ \square \]

1.3. Bundle of partial jet prolongations. Suppose $\theta_{k+1}, \theta'_{k+1} \in J^k_{\theta_k}$ are two $k+1$st order jets prolonging $\theta_k$. One may think of these as $k+1$st order infinitesimal germs of submanifolds in $E$ having contact of order $k$. Furthermore let $L \in \text{Gr} (\mathbb{R}, l)$ be an $l$–dimensional “direction” in these germs.

**Definition 3.** We say that $\theta_{k+1}$ and $\theta'_{k+1}$ are tangent of order $k + 1$ in direction $L$ if all $k + 1$st order partial derivatives of $\theta_{k+1}$ and $\theta'_{k+1}$ in direction of $L$ coincide.

More precisely, choose splitting coordinates $x_1, \ldots, x_n, u_1, \ldots, u_m$ on $E$ centered at $\theta_0$ such that $L$ is spanned by $\partial_{x_1}, \ldots, \partial_{x_l}$. Let

$$u_j = F_j(x_1, \ldots, x_n) \quad (\text{1.17})$$

and

$$u_j = G_j(x_1, \ldots, x_n) \quad (\text{1.18})$$

be two sets of locally defined functions with $j = 1, \ldots, m$, such that $\theta_{k+1}$ (resp. $\theta'_{k+1}$) is the $k + 1$st jet of 1.17 (resp. 1.18). Hence the jets $\theta_{k+1}$ and $\theta'_{k+1}$ are determined by all partial derivatives of $F$ and $G$ at 0 of order $k + 1$ or less. Since $\theta_{k+1}$ and $\theta'_{k+1}$ are tangent of order $k$ all partial derivatives of $F$ and $G$ at 0 of order $k$ or less agree. That $\theta$ and $\theta'$ are tangent of order $k + 1$ in direction $L$ by definition means that all partial derivatives at 0 of $F$ and $G$ of order $k + 1$ involving only $\partial_{x_1}, \ldots, \partial_{x_l}$ coincide.

An equivalent coordinate independent description is given by

**Lemma 2.** Two jets $\theta_{k+1}$ and $\theta'_{k+1}$ prolonging $\theta_k$ are tangent of order $k + 1$ iff the polynomial $\theta_{k+1} - \theta'_{k+1} \in S^{k+1} \mathbb{R}^r \otimes N$ vanishes when restricted to $L$.

**Proof.** From the properties of the affine $S^{k+1} \mathbb{R}^r \otimes N$ structure on $J^k_{\theta_k}$. See for instance [Lychagin (1988)]. \[ \square \]

**Definition 4.** Tangency of order $k + 1$ along $L$ defines an equivalence relation on the jets of order $k + 1$ prolonging $\theta_k$. The quotient set is denoted by $J^k_{\theta_k} L$ and called the space of partial prolongations of $\theta_k$ along $L$.

An element of $J^k_{\theta_k} L$ can be thought of as the jet $\theta_k$ with the additional $k + 1$st order information in direction of $L$.

Varying $L$ in $\text{Gr} (\mathbb{R}, l)$, the spaces $J^k_{\theta_k} L$ make up the fibers of a bundle we denote with

$$\text{dir} : J^k_{\theta_k} \rightarrow \text{Gr} (\mathbb{R}, l) \quad (\text{1.19})$$

$$\phi \mapsto L = \text{dir} (\phi) \quad (\text{1.20})$$
where the projection $\text{dir}$ maps a partial prolongation $\phi$ to its direction of prolongation $L$. By definition, a section of $\text{dir}$ assigns to any $L \in \text{Gr}(R, l)$ a partial jet prolongation of $\theta_k$ along $L$.

There is an obvious "holonomicity" condition for such a section: a section of partial jet prolongations is holonomic if for any two directions $L, L' \in \text{Gr}(R, l)$ the partial prolongations agree on the intersection $L \cap L'$. We call these the pasting conditions since they express when the partial prolongations can be "glued together" to a whole $k + 1$–order prolongation of $\theta_k$ (this is actually a result we shall prove).

1.4. Infinitesimal pasting conditions. The pasting conditions can be reformulated as a system of 1st order PDEs on sections of $\text{dir}$, which we shall call the infinitesimal pasting conditions. To write down this system of PDEs we introduce local coordinates.

On the base space $\text{Gr}(R, l)$ we choose standard affine coordinates on Grassmannians: fix an element $L_0 \in \text{Gr}(R, l)$, choose a basis $y_1, \ldots, y_d \in L_0^\circ$ (1.21) of the annihilator $L_0^\circ$ and complement it to a basis of $R^*$ with covectors $x_1, \ldots, x_l \in R^*$ (1.22). Then for any plane $L \in \text{Gr}(R, l)$ transversal to the complement $L_{\text{compl}} := \bigcap_{1 \leq j \leq l} \ker x_j$ (1.23) there are unique coefficients $A_{i,j}$ such that the covectors

$$y_i - \sum_{j=1}^l A_{i,j} x_j \quad \text{with} \quad i = 1, \ldots, d$$

(1.24)

form a basis of the annihilator of $L$ and conversely any choice of such coefficients determines such a plane. Hence the $A_{i,j}$ serve as local coordinates on $\text{Gr}(R, l)$.

Coordinates on fibers: each fiber $\rightharpoonup J^l_{\theta_k} \theta_k$ of $\text{dir}$ is an affine quotient of $J^{k+1}_{\theta_k} \theta_k$. Since $J^{k+1}_{\theta_k}$ is affine over $S^{k+1} R^* \otimes N$ we fix a “base jet”

$$\theta_{k+1,0} \in J^{k+1}_{\theta_k}$$

(1.25)

and identify $J^{k+1}_{\theta_k}$ with the vector space $S^{k+1} R^* \otimes N$. Then by lemma 2 each fiber $\tilde{J}_{\theta_k}^l L$ can be identified with $S^{k+1} L^* \otimes N$. Since each $L$ transversal to $L_{\text{compl}}^0$ is further identified with $L_0$, by the splitting $R = L_0 \oplus L_{\text{compl}}^0$, we get an identification of $S^{k+1} L^* \otimes N$ with $S^{k+1} L_0^* \otimes N$. So choosing a basis

$$e_1, \ldots, e_m$$

(1.26)

of $N$, each point in the total space $\tilde{J}_{\theta_k}^l$ is specified by its base coordinates $A_{i,j}$ plus the coefficients $v^h_\lambda$ of a homogeneous polynomial

$$\sum_{\lambda,h} v^h_\lambda x^\lambda \otimes e_h$$

(1.27)

where $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{N}^l$ denotes a multiindex of length $|\lambda| = k + 1$ and $x^\lambda = x_1^{\lambda_1} \cdots x_l^{\lambda_l}$.

In these local coordinates

$$A_{i,j}, v^h_\lambda$$

(1.28)

a section of $\text{dir}$ is given by functions

$$v^h_\lambda (A)$$

(1.29)
where $A$ is short for all the variables $A_{i,j}$. Such a section satisfies the non-
infinitesimal pasting conditions iff for any two planes $L, L' \in \text{Gr} (R, l)$ with 
coordinates $A, A'$ we have
\[ \sum_{\lambda,h} v^h_\lambda (A) x^\lambda \otimes e_h = \sum_{\lambda,h} v^h_\lambda (A') x^\lambda \otimes e_h \]  
whenever $x = (x_1, \ldots, x_l)$ satisfies
\[ \sum_j A_{i,j} x_j = \sum_j A'_{i,j} x_j \]  
for all $i = 1, \ldots, d$.

We turn to derive the infinitesimal pasting conditions from 1.30, 1.31. For this 
fix $L \in \text{Gr} (R, l)$ with coordinates $A$ and consider two continuous perturbations of $L$: 
one perturbation changing entry $A_{i,j}$ of matrix $A$ to $A_{i,j} + t$ with $t$ a perturbation 
parameter and leaving the other entries fixed. The other perturbation changing 
entry $A_{i,j'}$ to $A_{i,j'} + s$ with parameter $s$ and leaving all other entries unperturbed. 
Here $i, j, j'$ are fixed indices. We write the perturbed matrices as
\[ A + tl_{i,j}, \quad A + sl_{i,j'} . \]  
If a section of $\text{dir}$ satisfies the pasting conditions 1.30, 1.31 we get for these two 
perturbations
\[ \sum_{\lambda} v^h_\lambda (A + tl_{i,j}) x^\lambda \otimes e_h = \sum_{\lambda} v^h_\lambda (A + sl_{i,j'}) x^\lambda \otimes e_h \]  
whenever $x = (x_1, \ldots, x_l)$ satisfies
\[ tx_j = sx_{j'} . \]  
Taking the total differential of both sides of equations 1.34, 1.35 (where the variables 
are $s, t, x$ while $A$ is assumed fixed, i.e. $dA = 0$) we get
\[ \sum_{\lambda} \partial_{A_{i,j}} v^h_\lambda (A + tl_{i,j}) x^\lambda dt \otimes e_h + \sum_{\lambda,i} v^h_\lambda (A + tl_{i,j}) \lambda_i x^{\lambda - 1} dx_i \otimes e_h = \]  
\[ \sum_{\lambda} \partial_{A_{i,j}} v^h_\lambda (A + s1_{i,j'}) x^\lambda ds \otimes e_h + \sum_{\lambda,i} v^h_\lambda (A + s1_{i,j'}) \lambda_i x^{\lambda - 1} dx_i \otimes e_h \]  
from 1.34, while from 1.37 we get
\[ x_j dt + tdx_j = x_j ds + sx_{j'} . \]  
Now set $t = s = 0$, so 1.34 1.35 are trivially satisfied while 1.36 becomes
\[ \sum_{\lambda} \partial_{A_{i,j}} v^h_\lambda (A) x^\lambda dt \otimes e_h = \sum_{\lambda} \partial_{A_{i,j'}} v^h_\lambda (A) x^\lambda ds \otimes e_h \]  
after canceling equal terms. Equation 1.37 becomes
\[ x_j dt = x_j ds . \]  
We may multiply both sides of equation 1.38 with $x_j$ and substitute $x_j dt$ for $x_j ds$ 
since $x_j dt = x_j ds$ to find
\[ \sum_{\lambda} \partial_{A_{i,j}} v^h_\lambda (A) x^{\lambda + 1} \otimes e_h = \sum_{\lambda} \partial_{A_{i,j'}} v^h_\lambda (A) x^{\lambda + 1} \otimes e_h \]  
where we have canceled $ds$. Since equations 1.40 hold for arbitrary values $x$ we 
can equate coefficients on both sides to find that in local coordinates a section
v^h_\lambda (A) that satisfies pasting conditions 1.30, 1.31 satisfies the infinitesimal pasting equations
\[ \partial_{A_{i,j}} v^h_\lambda = \partial_{A_{i,j}} v^h_{\lambda'}, \quad \text{whenever} \quad \lambda - 1_j = \lambda' - 1_{j'}, \tag{1.41} \]
\[ \partial_{A_{i,j}} v^h_\lambda = 0 \quad \text{whenever} \quad \lambda_j = 0. \tag{1.42} \]
Observe that when \( l = 1 \) these conditions are trivially satisfied, so the equations are “empty”.

**Remark 1.** If one considers perturbations \( A + t_1 i, j, A + s t_{i', j'} \) with different indices \( i \neq i' \) one finds again equations 1.42. In fact, prolongation theorems 1, 2 establish that all possible differential consequences of the non-infinitesimal pasting conditions 1.30, 1.31 are differential consequences of the infinitesimal pasting conditions 1.41, 1.42.

### 1.5. Partial integral flags and the double fibration structure.

**Definition 5.** A pair \((L, R)\) of subspaces \( L \subset R \subset C_\theta \) with \( R \) an \( R \)-plane and \( L \) of dimension \( l \) is called a *partial flag of \((l, n)\)-dimensional horizontal integral elements*. The space of all such flags is denoted with
\[ I^{l,n}_{\theta_k} := \{ (L, R) \mid L \in I^l_{\theta_k}, \ R \in I^n_{\theta_k}, \ L \subset R \}. \tag{1.43} \]

The space of partial flags is naturally fibered in two ways: one projection forgets the smaller integral element \( L \) and remembers only \( R \). Since \( R \) is an \( R \)-plane corresponding to some jet of order \( k + 1 \) we write this projection as:
\[ \text{pr}_n : I^{l,n}_{\theta_k} \to J^{k+1}_{\theta_k}, \]
\[ (L, R) \mapsto L. \tag{1.44} \]
\[ \text{pr}_l : I^{l,n}_{\theta_k} \to I^l_{\theta_k}, \]
\[ (L, R) \mapsto (R). \tag{1.45} \]

We picture both of these as a double fibration
\[ \begin{array}{ccc}
I^{l,n}_{\theta_k} & \xrightarrow{\text{pr}_n} & J^{k+1}_{\theta_k} \\
\downarrow \nearrow & & \downarrow \nearrow \\
I^l_{\theta_k} & \xrightarrow{\text{pr}_l} & I^l_{\theta_k}
\end{array} \]

**Definition 6.** The sum of the two vertical distributions associated to the fibrations \( \text{pr}_n \) and \( \text{pr}_l \) defines a distribution
\[ \mathcal{D} := V \text{pr}_n + V \text{pr}_l \tag{1.49} \]
on \( I^{l,n}_{\theta_k} \) which we call the *double fibration distribution* on partial flags.

### 1.6. Statement of the main results.

Before we state the main result we recall the notion of prolongation of an exterior differential system with independence conditions. We shall only need the case where the independence conditions are given by transversality conditions with respect to a bundle projection \( \pi : M \to N \), and where the exterior differential system on \( M \) is a distribution \( \mathcal{E} \) (i.e. a Pfaffian system). We refer to [Bryant et al. (1991)] for the general definition.

One defines the first prolongation of \((M, \mathcal{E})\) as the manifold \( M^{(1)} \) of \((\dim N)\)-dimensional \( \pi \)-horizontal integral elements of \( \mathcal{E} \). The prolonged distribution \( \mathcal{E}^{(1)} \) on \( M^{(1)} \) is defined as the lift of the tautological relative distribution along the projection \( \pi^{(1)} : M^{(1)} \to M \) which to each \( S \in M^{(1)} \) associates \( S \) as a subspace
of \( T_{\pi^{(1)}} S M \). Since \( M^{(1)} \) is still a bundle over \( N \) via \( \pi \circ \pi^{(1)} \) we can iterate this construction and define the second prolongation etc.

**Theorem 3** (Main theorem). The \( k - 1 \)st prolongation of the system of pasting equations is the polar distribution on \( I_{\theta_k}^1 \) and the \( k \)th prolongation is the space of partial flags with its double fibration distribution. Moreover, when \( l > 1 \) the \( k + 1 \)st prolongation is an involutive distribution whose maximal integral submanifolds are in one to one correspondence with jets of order \( k + 1 \) prolonging \( \theta_k \). When \( l = 1 \) the pasting conditions are empty and so \( I_{\theta_k}^1 = J^k(\text{dir}) \) and \( I_{\theta_k}^{l,n} = J^{k+1}(\text{dir}) \) and the polar and double fibration distributions are the Cartan distributions on \( J^k(\text{dir}) \) resp. \( J^{k+1}(\text{dir}) \).

2. **Proof of the main result**

Our proof proceeds by exhibiting certain natural distributions on the space of partial integral flags \( I_{\theta_k}^{l,n} \) and constructing the tower of prolongations from these, “top down”.

2.1. **Natural structures on \( I_{\theta_k}^{l,n} \).** Since any integral element \( L \in I_{\theta_k}^l \) is transversal to \( \pi_{k-1,0} \) we may project it down to \( R \subset T_{\theta_k} E \) to obtain a subspace we denote with \( L \in \text{Gr}(R, l) \) (This projection also establishes a canonical isomorphism \( L \cong \mathcal{L} \).

Hence \( I_{\theta_k}^l \) is naturally fibered over \( \text{Gr}(R, l) \):

\[
I_{\theta_k}^l \rightarrow \text{Gr}(R, l)
\]

\[
L \hookrightarrow \mathcal{L}
\]

Using this projection we note the following useful decomposition of the space of partial flags.

**Lemma 3.** The map

\[
I_{\theta_k}^{l,n} \rightarrow \text{Gr}(R, l) \times J^{k+1}_{\theta_k} \quad (2.3)
\]

\[
(L, R_{\theta_k+1}) \mapsto (\mathcal{L}, \theta_{k+1})
\]

is an isomorphism of manifolds.

**Proof.** The inverse can be described by

\[
(\mathcal{L}, \theta_{k+1}) \mapsto \left((R_{\theta_k+1} \cap (T_{\theta_k} \pi_{k,0})^{-1}(\mathcal{L})), R_{\theta_k+1}\right). \quad (2.5)
\]

\[\Box\]

We henceforth use the identification \( I_{\theta_k}^{l,n} = \text{Gr}(R, l) \times J^{k+1}_{\theta_k} \) without explicit mention. From this product decomposition follows that the tangent spaces of \( I_{\theta_k}^{l,n} \) carry nontrivial structure.

**Corollary 1.** The tangent space at \( (L, R) \in I_{\theta_k}^{l,n} \) is naturally isomorphic to

\[
(\mathcal{L}^* \otimes R/\mathcal{L}) \oplus (S^{k+1}R^* \otimes N). \quad (2.6)
\]

**Proof.** The first summand \( \mathcal{L}^* \otimes R/\mathcal{L} \) is just the tangent space to \( \text{Gr}(R, l) \) at \( L \), while the second summand \( S^{k+1}R^* \otimes N \) is the tangent space to \( J^{k+1}_{\theta_k} \).

\[\Box\]

A vector in \( (\mathcal{L}^* \otimes R/\mathcal{L}) \oplus (S^{k+1}R^* \otimes N) \) will usually be denoted with \( h \oplus f \), where \( h \in \mathcal{L}^* \otimes R/\mathcal{L} \) and \( f \in S^{k+1}R^* \otimes N \).

We also record the following

**Lemma 4.** The composition of the natural projections \( I_{\theta_k}^{l,n} \rightarrow I_{\theta_k}^l \) and \( I_{\theta_k}^l \rightarrow \text{Gr}(R, l) \) is the projection onto the first factor in the decomposition \( I_{\theta_k}^{l,n} = \text{Gr}(R, l) \times J^{k+1}_{\theta_k} \).

**Proof.** Immediate.
2.2. A filtration on homogeneous polynomials. The subspace \( L \subset R \) associated to \((L, R)\) gives rise to a filtration on the second component \( S^{k+1} \otimes N \) of the tangent space of \( I^{\theta_k}_{hL} \).

**Definition 7.** For \( p = 0, 1, \ldots, k + 2 \) define \( U^p_L \) to be the vector subspace of \( S^{k+1} \otimes N \) consisting of all homogeneous polynomials that vanish after taking \( p \) derivatives in direction of \( L \). Equivalently, \( U^p_L \) consists of all symmetric \( k + 1 \)-multilinear forms on \( R \) that vanish when inserting \( p \) elements of \( L \).

These subspaces form a natural filtration in \( S^{k+1} \otimes N \) depending on \( L \in \text{Gr}(R, l) \):

\[
U^p_L \subset U^1_L \subset \ldots \subset U^{k+1}_L \subset U^{k+2}_L = S^{k+1} \otimes N.
\]  

(2.7)

A basis of \( U^p_L \) may be constructed as follows: fix a basis \( y_1, \ldots, y_d \) of \( L^* \) and complement it with forms \( x_1, \ldots, x_l \) to a basis of \( R^* \). Denote symmetric monomials of these basic forms with:

\[
y^\delta x^\lambda := y_{\delta_1} \cdots y_{\delta_d} x_{\lambda_1} \cdots x_{\lambda_l}
\]

(2.8)

where \( \delta \) and \( \lambda \) are multi indices: \( \delta = (\delta_1, \ldots, \delta_d) \in \mathbb{N}^d \) and \( \lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{N}^l \). Choose a basis \( e_1, \ldots, e_m \) of \( N \).

**Lemma 5.** \( U^p_L \) is generated by all tensors \( y^\delta x^\lambda \otimes e_h \) which are of degree less than \( p \) in the \( x \)'s. More formally

\[
U^p_L = \{ y^\delta x^\lambda \otimes e_h \mid |\delta| + |\lambda| = k + 1, |\lambda| < p, h = 1, \ldots, m \}.
\]  

(2.9)

We adopt the convention that the zero polynomial is of any degree.

**Proof.** This follows straightforwardly from interpretation of such tensors as polynomial maps. \( \square \)

Denoting with

\[
q := k + 1 - p
\]  

(2.10)

the complementary degree to \( p \), we can also describe \( U^p_L \) as all tensor of degree \( q \) or more in the \( y \)'s.

**Corollary 2.**

\[
U^1_L = S^k \otimes N
\]  

(2.11)

\[
U^{k+1}_L = \text{polynomials vanishing on } L
\]  

(2.12)

The filtration \( U^p_L \) induces two natural filtrations on the tangent spaces of \( I^{\theta_k}_{hL} \) which we call the higher vertical and higher flag distributions. From these we shall obtain the tower of prolongations.

2.3. Higher vertical distributions on \( I^{\theta_k}_{hL} \).

**Definition 8.** For \( p = 0, \ldots, k + 2 \) define the \( p \)th higher vertical distribution \( \mathcal{V}^p \) on \( I^{\theta_k}_{hL} \) at a point \( (L, R) \in I^{\theta_k}_{hL} \) as

\[
\mathcal{V}^p_{(L, R)} := \left\{ 0 \oplus f \in (L^* \otimes R/L) \oplus (S^{k+1} \otimes N) = T_{(L, R)}I^{\theta_k}_{hL} \mid f \in U^p_L \right\}.
\]  

(2.13)

It is clear that

\[
\mathcal{V}^0 = \mathcal{V}^1 \subset \ldots \subset \mathcal{V}^{k+2}
\]  

(2.14)

and the biggest vertical distribution \( \mathcal{V}^{k+2} \) is just the vertical distribution with respect to the projection \( I^{\theta_k}_{hL} \rightarrow \text{Gr}(R, l) \).
Lemma 6. All higher vertical distributions are involutive, their leaves are affine spaces and their spaces of leaves are manifolds.

Proof. The claim is clear for $V^{k+2}$. To check involutivity of the other vertical distributions note that since each $V^p \subset V^{k+2}$ it suffices to check involutivity on each fiber of $I_{\theta_k}^{l,n} \to \text{Gr}(R,l)$. But each fiber $I_{\theta_k}^{l,n} \to \text{Gr}(R,l)$ is the affine space $J_{\theta_k}^{k+1}$ and the distribution $V_{\theta_k}^{l,n} F_{\theta_k}^{k+1}$ is “constantly” equal to $U^p_{\theta_k}$ there. Hence the leaves are affine and the quotient exists for each $L \in \text{Gr}(R,l)$. □

Definition 9. We denote the space of leaves of the distribution $V^p$ with $M^q$ where $q$ is the complementary degree $k + 1 - p$.

This way we get a tower of bundles

\[ \underbrace{M^{k+1}}_{=I_{\theta_k}^{l,n}} \to \underbrace{M^k}_{=I_{\theta_k}^{l,n}} \to \ldots \to \underbrace{M^0}_{=I_{\theta_k}^{l,n}} \to \underbrace{M^{-1}}_{=\text{Gr}(R,l)} \quad (2.15) \]

where each $M^q$ is a bundle over $M^{q-1}$ with affine fibers. This tower will turn out to be the tower of prolongations of the pasting equations.

We start by observing

Lemma 7. The distribution $V^1$ is the vertical distribution of the fibration $I_{\theta_k}^{l,n} \to I_{\theta_k}^{l,n}$, so $M^k = I_{\theta_k}^{l,n}$.

Proof. If $(L, R_{\theta_{k+1}})$ and $(L, R_{\theta_{k+1}'}$) are in the same fiber of $I_{\theta_k}^{l,n} \to I_{\theta_k}^{l,n}$ then $\theta_{k+1} - \theta_{k+1}' \in S^{k+1} R^* \otimes N$ is a polynomial vanishing when taking one derivative in direction of $L$ since $L \subset (R_{\theta_{k+1} \cap R_{\theta_{k+1}'})$, so $\theta_{k+1} - \theta_{k+1}' \in U_{L}^1$. □

Next we have

Lemma 8. $M^0 = J_{\theta_k}^l$.

Proof. Two flags $(L, R_{\theta_{k+1}})$ and $(L', R_{\theta_{k+1}'})$ are in the same leaf of the distribution $V^{k+1}$ if and only if $L = L'$ and $\theta_{k+1} - \theta_{k+1}' \in U_{L}^{k+1}$, but this last condition is precisely the condition that all $k + 1$-st derivatives of $\theta_{k+1}$ and $\theta_{k+1}'$ in direction of $L$ agree, hence they define the same partial jet prolongation of $\theta_k$. □

So we have identified the following components:

\[ \underbrace{M^{k+1}}_{=I_{\theta_k}^{l,n}} \to \underbrace{M^k}_{=I_{\theta_k}^{l,n}} \to \underbrace{M^{k-1}}_{=J_{\theta_k}^l} \to \ldots \to \underbrace{M^0}_{=I_{\theta_k}^{l,n}} \to \underbrace{M^{-1}}_{=\text{Gr}(R,l)} \quad (2.16) \]

Our next aim to show that each space $M^q$ is naturally equipped with a distribution. We start by defining a second chain of distributions on $I_{\theta_k}^{l,n}$.

2.4. Higher flag distributions on $I_{\theta_k}^{l,n}$.

Definition 10. For $p = -1, 0, 1, \ldots, k + 1$ define the $p$th flag distribution $\mathcal{F}_p$ on $I_{\theta_k}^{l,n}$ as the sum of $V_p^{k+1}$ with the distribution vertical to the projection $\text{pr}_n : I_{\theta_k}^{l,n} \to J_{\theta_k}^{k+1}$.

So the plane of the $p$th flag distribution at a point $(L, R)$ is

\[ \mathcal{F}_{(L,N)} = \left\{ h \otimes f \in (L^* \otimes R/L) \otimes (S^{k+1} R^* \otimes N) = T_{(L,R)} I_{\theta_k}^{l,n} \big| f \in U_{L}^{p+1} \right\} \quad (2.17) \]
It is clear that
\[
F^{-1} \subset F^0 \subset \ldots \subset F^{k+1}.
\] (2.18)

Concerning the second smallest distribution \(F^0\) we have

**Lemma 9.** \(F^0\) is the double fibration distribution \(\mathcal{D}\) on \(I^{l,n}_{\theta_k}\).

**Proof.** This is a direct consequence of the definitions and lemma 7. \(\square\)

Moreover for the third smallest \(F^1\) we have

**Proposition 1.** \(F^1\) is the lift of the polar distribution \(\mathcal{P}\) from \(I^l_{\theta_k}\) to \(I^{l,n}_{\theta_k}\) via \(I^{l,n}_{\theta_k} \rightarrow I^l_{\theta_k}\).

**Proof.** Fix \((L, R) \in I^{l,n}_{\theta_k}\). It suffices to show that for any tangent vector at \((L, R) \in I^{l,n}_{\theta_k}\) of the form \(0 \oplus f \in (L^* \otimes R/L) \oplus (S^{k+1}R^* \otimes N)\) the following conditions are equivalent:

1) \(f \in U_2^L\)
2) \(T_{(L,R)}pr_t(0 \oplus f) \in \mathcal{P}_L\)

where \(T_{(L,R)}pr_t\) is the tangent map of \(pr_t : I^{l,n}_{\theta_k} \rightarrow I^l_{\theta_k}\) at \((L, R) \in I^{l,n}_{\theta_k}\). Let \(df : R \rightarrow S^kR^* \otimes N\) denote the total differential of the polynomial \(f \in S^{k+1}R^* \otimes N\) and let \(df|_L : L \rightarrow C_{\theta_k}/L\) denote its restriction to \(L\). In 2.20 we have implicitly used the canonical isomorphism \(L \cong L^*\) and the inclusion \(S^kR^* \otimes N \subset C_{\theta_k}\) as vertical tangent space to the projection \(J^k \rightarrow J^{k-1}\). Then for any \(0 \oplus f \in T_{(L,R)}I^{l,n}_{\theta_k}\)
\[
T_{(L,R)}pr_t(0 \oplus f) = df|_L.
\] (2.21)

We compute with \(l_1, l_2 \in L\)
\[
\Omega(l_1, T_{(L,R)}pr_t(0 \oplus f)(l_2)) = \Omega(l_1, df|_L(l_2))
\] (2.22)
\[
= \partial_{l_1} \partial_{l_2} f
\] (2.23)

where 2.23 follows from the structural properties of the metasymplectic form \(\Omega\). But 2.23 is zero for all \(l_1, l_2 \in L \cong L\) iff \(f \in U^L_2\) so the claim follows from description 1.12. \(\square\)

Recall that a vector field \(X\) is called a characteristic symmetry of a distribution \(\mathcal{E}\), iff it is contained in \(\mathcal{E}\) and a symmetry of \(\mathcal{E}\) (so Lie brackets with \(X\) preserve the distribution). Characteristic symmetries form an involutive sub-distribution of \(\mathcal{E}\) and one may always reduce \(\mathcal{E}\) to a distribution on the space of leaves of the characteristic distribution. Conversely, a distribution lifted from the base of a fiber bundle to the total space contains the vertical vector fields as characteristic symmetries.

Proposition 1 together with lemma 7 implies that \(\mathcal{Y}^1\) consists of characteristic symmetries of \(\mathcal{F}^1\). We have the following generalization which allows us to reduce the higher flag distributions \(\mathcal{F}^p\) to the quotients \(M^q\)

**Theorem 4.** For all \(p = 0, \ldots, k + 1\) the characteristic distribution of \(\mathcal{F}^p\) is \(\mathcal{Y}^p\).

To prove theorem 4 we construct a basis of the higher flag distribution in local coordinates and compute its commutation relations.
2.5. Non-holonomic frames and commutation relations. We start by introducing local coordinates on each component of \( I^{l,n}_{\theta_k} = \text{Gr} (R^1_l) \times J^{k+1}_{\theta_k} \). Since we later introduce a second set of local coordinates on \( I^{l,n}_{\theta_k} \) which descend to the space of leaves \( M^q \), we call this first set \textit{trivial} and the second \textit{adapted}.

2.5.1. Trivial local coordinates. As in subsection 1.4 we use affine coordinates \( A_{i,j} \) on \( \text{Gr} (R^1_l) \) and identify the second component \( J^{k+1}_{\theta_k} \) with the vector space \( S^{k+1} \mathbb{R}^* \otimes \mathbb{N} \) using the chosen "base jet" 1.25.

Using the bases 1.21, 1.22 and 1.26 of subsection 1.4, a basis of \( S^{k+1} \mathbb{R}^* \otimes \mathbb{N} \) is given by divided powers

\[
\frac{1}{\delta! \lambda!} y^\delta x^\lambda \otimes e_h \tag{2.24}
\]

with \(|\delta| + |\lambda| = k + 1\). Here the factorial \( \delta! \) of a multiindex is \( \delta_1! \cdots \delta_d! \).

Definition 11. The dual basis to the divided powers 2.24 will be denoted with \( u^h_{\delta,\lambda} \) and serves as local coordinates on \( J^{k+1}_{\theta_k} \).

2.5.2. A non-holonomic frame adapted to higher vertical distributions. Recall that a tangent vector at a point \((L, R) \in I^{l,n}_{\theta_k}\) can be identified with an element \( h \otimes f \in \left( L^* \otimes \mathbb{R}^* \right) \oplus \left( S^{k+1} \mathbb{R}^* \otimes \mathbb{N} \right) \). Such an \( h \otimes f \) is in the distribution \( \mathcal{V}^p \) at the point \((L, R) \) iff \( f \in U^p_L \) and \( h = 0 \). Hence, according to lemma 5 and the definition of the coordinates \( A_{i,j} \), the “partially” divided powers

\[
\left( \frac{1}{\delta!} (y - \sum A x)^\delta x^\lambda \right) \otimes e_h, \text{ with } |\lambda| < p \tag{2.25}
\]

form a basis of \( \mathcal{V}^p \) at each point of \( I^{l,n}_{\theta_k} \) (we have suppressed the component \( h = 0 \)). In the previous equation the notation \( (y - \sum A x)^\delta \) stands for \( (y_1 - \sum_j A_{1,j} x_j)^{\delta_1} \cdots (y_d - \sum_j A_{d,j} x_j)^{\delta_d} \).

Definition 12. Local vector fields on \( I^{l,n}_{\theta_k} \) corresponding to the partially divided powers 2.25 will be denoted with \( V^h_{\delta,\lambda} \) and called \textit{vertical fields}.

These vertical fields \( V^h_{\delta,\lambda} \) together with the coordinate fields \( \partial_{A_{i,j}} \) clearly form a (non-holonomic) local frame on \( I^{l,n}_{\theta_k} \).

For readers familiar with charts on jet spaces: the \( V^h_{\delta,\lambda} \) will play an analogous role to the horizontal coordinate fields \( \partial_{A_{i,j}} \) on jets [Bocharov et al. (1999)] while the \( \partial_{A_{i,j}} \) will play an analogous role to the total derivatives \( D_i \). For this reason and since we later introduce a second set of coordinates we adopt the following

Definition 13. The fields \( \partial_{A_{i,j}} \) from the current chart will be denoted with \( D_{i,j} \) and called \textit{homogeneous total derivatives}.

It is evident that the frame \( D_{i,j}, V^h_{\delta,\lambda} \) is adapted to the higher vertical and flag distributions in the sense that

\[
\mathcal{V}^p = \left\{ V^h_{\delta,\lambda} \mid |\lambda| < p \right\} \tag{2.26}
\]

\[
\mathcal{F}^p = \left\{ D_{i,j}, V^h_{\delta,\lambda} \mid |\lambda| \leq p \right\} \tag{2.27}
\]

Theorem 5. All commutators of the frame \( D_{i,j}, V^h_{\delta,\lambda} \) are zero except for the commutators \( \left[ V^h_{\delta,\lambda}, D_{i,j} \right] \) when \( \delta_i > 0 \). In that case we have:

\[
\left[ V^h_{\delta,\lambda}, D_{i,j} \right] = V^h_{\delta - 1, \lambda + 1,j}. \tag{2.28}
\]
Proof. That \([D_{i,j}, D_{i',j'}] = 0\) is clear since in the chosen coordinates these fields are just partial derivatives. That \([V^h_{\delta,\lambda}, V^{h'}_{\delta',\lambda'}] = 0\) is also easily seen, since by equation 2.25, the \(V^h_{\delta,\lambda}\) are linear combinations of the coordinate fields \(\partial_{a^h_{\delta,\lambda}}\) with coefficients depending only on the coordinates \(A_{i,j}\).

We are left to consider the Lie brackets \([V^h_{\delta,\lambda}, D_{i,j}]\). We compute how these act on coordinate functions. First note that \([V^h_{\delta,\lambda}, D_{i,j}] (A_{i',j'}) = 0\) since

\[
[V^h_{\delta,\lambda}, D_{i,j}] (A_{i',j'}) = V^h_{\delta,\lambda} (D_{i,j} (A_{i',j'})) - D_{i,j} (V^h_{\delta,\lambda} (A_{i',j'})) = 0. \tag{2.29}
\]

Now consider the action of \([V^h_{\delta,\lambda}, D_{i,j}]\) on a coordinate function \(u^H_{\Delta,\Lambda}\) where \(\Delta \in \mathbb{N}^d\) and \(\Lambda \in \mathbb{N}^n\) are multi-indices and \(H = 1, \ldots, m\):

\[
V^h_{\delta,\lambda} (D_{i,j} (u^H_{\Delta,\Lambda})) - D_{i,j} (V^h_{\delta,\lambda} (u^H_{\Delta,\Lambda})) = -D_{i,j} (V^h_{\delta,\lambda} (u^H_{\Delta,\Lambda})). \tag{2.30}
\]

To continue the computation consider the inner term \(V^h_{\delta,\lambda} (u^H_{\Delta,\Lambda})\) on the r.h.s. When \(h \neq H\) this is obviously 0. In the case \(h = H\) note that \(V^h_{\delta,\lambda} (u^H_{\Delta,\Lambda})\) is the coefficient in front of \(\partial_{a^h_{\delta,\lambda}}\) in the expansion of \(V^h_{\delta,\lambda}\) in the coordinate frame. But this is the same as the coefficient in the expansion of \(\frac{1}{\delta!} (y - \sum Ax)^\delta x^\lambda\). This coefficient may be computed by applying the operator

\[
\partial_y^\alpha \partial_x^\Lambda := \partial_{y_1}^{\Lambda_1} \cdots \partial_{y_d}^{\Lambda_d} \partial_{x_1}^{\Lambda_1} \cdots \partial_{x_j}^{\Lambda_j} \tag{2.31}
\]

to \(\frac{1}{\delta!} (y - \sum Ax)^\delta x^\lambda\) since all polynomials involved are homogenous. So we have

\[
V^h_{\delta,\lambda} (u^H_{\Delta,\Lambda}) = \partial_y^\alpha \partial_x^\Lambda \left( \frac{1}{\delta!} (y - \sum Ax)^\delta x^\lambda \otimes e_h \right). \tag{2.32}
\]

Plugging this in in the r.h.s. of equation 2.30 we obtain

\[
[V^h_{\delta,\lambda}, D_{i,j}] (u^H_{\Delta,\Lambda}) = -\partial_{A_{i,j}} \partial_y^\alpha \partial_x^\Lambda \left( \frac{1}{\delta!} (y - \sum Ax)^\delta x^\lambda \otimes e_h \right). \tag{2.33}
\]

Now we can exchange the order of derivatives on the r.h.s. and derive first w.r.t. \(\partial_{A_{i,j}}\). Using the chain rule we compute:

\[
\partial_{A_{i,j}} \left( \frac{1}{\delta!} (y - \sum Ax)^\delta x^\lambda \right) = -\delta_i \cdot x_j \frac{1}{\delta!} (y - \sum Ax)^{\delta-1_i, x^\lambda} \tag{2.34}
\]

\[
= \begin{cases} 0 & \text{if } \delta_i = 0 \\ -\frac{1}{(\delta-1)_i} (y + \sum Ax)^{\delta-1_i, x^{\lambda+1_j}} & \text{if } \delta_i > 0. \end{cases} \tag{2.35}
\]

So we arrive at:

\[
[V^h_{\delta,\lambda}, D_{i,j}] (u^H_{\Delta,\Lambda}) = \begin{cases} 0 & \text{if } \delta_i = 0 \\ \partial_y^\alpha \partial_x^\Lambda \left( \frac{1}{\delta!} (y + \sum Ax)^{\delta-1_i, x^{\lambda+1_j}} \otimes e_h \right) & \text{if } \delta_i > 0. \end{cases} \tag{2.36}
\]

From this we conclude that \([V^h_{\delta,\lambda}, D_{i,j}] = 0\) if \(\delta_i = 0\) while in the case when \(\delta_i > 0\) the r.h.s. of the last equation is precisely \(V^h_{\delta-1_i, \lambda+1_j} (u^H_{\Delta,\Lambda})\) by equation 2.32. \(\square\)

Commutation relations 2.28 immediately imply theorem 4, hence the flag distribution \(F^p\) reduces to a distribution on \(M^q\) for \(q = 0, \ldots, k + 1\).

**Definition 14.** The reduction of the flag distribution \(F^p\) to \(M^q\) is denoted with \(\underline{F}^q\), where \(p\) is the complementary degree to \(q\).
Since $F^{k+1} = TM^{l,n}$ we have $F^0 = TM^1$. So the tower 2.16 is now enhanced with distributions:

\[
(\mathcal{M}^{k+1}, F^{k+1}) \rightarrow (\mathcal{M}^k, F^k) \rightarrow (\mathcal{M}^{k-1}, F^{k-1}) \rightarrow \ldots \rightarrow (\mathcal{M}^0, F^0) \rightarrow \text{Gr}(R_1, l)
\]

(2.37)

Another direct consequence of the commutation relations 2.28 which we shall not use explicitly is

**Corollary 3.** All flag distributions $F^p$ with $p \geq 1$ are derived distributions of the double fibration distribution $D = F^2$. More precisely

\[
F^{p+1} = [F^p, F^p]
\]

(2.38)

for all $p = 0, \ldots, k$.

Our next aim is to prove that the distributions $(M^q, F^q)$ are the prolongations of $(M^1, F^1)$. To make this precise consider one consecutive fibration

\[
\Pi_{q, q-1} : M^q \rightarrow M^{q-1}
\]

(2.39)

in tower 2.37 and let $q$ be a point in the fiber $M^q_{\phi_{q-1}}$ over $\phi_{q-1} \in M^{q-1}$. Attached to $\phi_q$ is the plane $\mathcal{F}^q_{\phi_q}$ of the distribution $F^q$ which we may project down to $M^q_{\phi_q}$. We denote the projected plane with

\[
Q_{\phi_q} := T_{\phi_{q+1}, \Pi_{q+1, q}}(\mathcal{F}^q_{\phi_{q+1}}).
\]

(2.40)

These “Q-planes” are analogous to R-planes by the following results which establish most of the main claims.

**Proposition 2.** For each $\phi_q \in M^q$ with $q = 1, \ldots, k + 1$, the plane $Q_{\phi_q}$ is a horizontal maximal integral element in $(M^{q-1}, F^{q-1})$ of dimension $\dim(\text{Gr}(R_1, l))$. Horizontal here means transversal to $M^{q-1} \rightarrow M^{q-2}$, which turns out to be equivalent to being transversal to $M^{q-1} \rightarrow M^q$.

**Proposition 3.** For all $q = 1, \ldots, k + 1$, the map

\[
\phi_q \mapsto Q_{\phi_q}
\]

(2.41)

is an injection from the fiber $M^q_{\phi_{q-1}}$ into the space of horizontal maximal integral elements of $\mathcal{F}^{q-1}_{\phi_{q-1}}$.

So we may identify $M^q$ with a subset of maximal horizontal integral elements of $(M^{q-1}, F^{q-1})$. In fact, for $q \geq 2$, any maximal integral elements of $(M^{q-1}, F^{q-1})$ is of the form $Q_{\phi_q}$.

**Proposition 4.** For all $q = 2, \ldots, k + 1$, the map

\[
\phi_q \mapsto Q_{\phi_q}
\]

(2.42)

is a surjection from the fiber $M^q_{\phi_{q-1}}$ to horizontal maximal integral elements of $\mathcal{F}^{q-1}_{\phi_{q-1}}$.

For $q = 1$ we have

**Proposition 5.** The image of the map

\[
\phi_1 \mapsto Q_{\phi_1}
\]

(2.43)

is described precisely by infinitesimal pasting equations 1.41 and 1.42.

To prove propositions 2, 3, 4 and 5 we introduce a second set of coordinates on $I_{\phi_{q-1}}$ which descend to the quotients $M^q$ and compute in these coordinates.
2.5.3. Adapted local coordinates. Since we fixed a jet \( \theta_{k+1,0} \in J_{\theta_k}^{k+1} \) in 1.25 to identify \( J_{\theta_k}^{k+1} \) with the vector space \( S^{k+1}\mathbb{R}^* \otimes N \), we may consider

\[
\text{Gr} (\mathbb{R}, l) \times J_{\theta_k}^{k+1} \to \text{Gr} (\mathbb{R}, l)
\]

(2.44)
to be a vector bundle. The partially divided powers \( \frac{1}{\delta!} (y - \sum A x)^{\delta} x^\lambda \otimes e_h \) then form a basis in each fiber. This frame is “moving” from fiber to fiber as it depends on the base coordinates \( A_{i,j} \). Here \( A_{i,j} \) and \( x, y \) have the same meaning as in subsection 2.5.1.

**Definition 15.** The fiber wise dual one-forms to the frame

\[
\frac{1}{\delta!} (y - \sum A x)^{\delta} x^\lambda \otimes e_h
\]

(2.45)
will be denoted with \( v_{\delta,\lambda}^h \) and provide new coordinates on the fibers of \( \text{Gr} (\mathbb{R}, l) \times J_{\theta_k}^{k+1} \to \text{Gr} (\mathbb{R}, l) \). Together with the coordinates \( A_{i,j} \) on the base \( \text{Gr} (\mathbb{R}, l) \) they constitute another set of local coordinates on \( I_{l,n}^{k+1} \theta_k \) which we call adapted.

Observe that in these adapted coordinates the vector fields \( V_{\delta,\lambda}^h \) are just the partial derivatives

\[
\frac{\partial v_{\delta,\lambda}^h}{\partial v_{\delta,\lambda}^h}
\]

(2.46)
while the fields \( D_{i,j} \) are no longer the coordinate fields \( \partial A_{i,j} \), as in the trivial coordinates.

It is clear from 2.46 that the coordinates \( A_{i,j}, v_{\delta,\lambda}^h \) with |\( \delta \)| \( \leq q \) descend to coordinates on \( M^q \).

Our next aim is to express the fields \( D_{i,j} \) in coordinates \( A_{i,j}, v_{\delta,\lambda}^h \).

**Proposition 6.** We have

\[
D_{i,j}(A_{i',j'}) = \begin{cases} 1 & \text{if } i = i' \text{ and } j = j' \\ 0 & \text{else} \end{cases}
\]

(2.47)

\[
D_{i,j}(v_{\delta,\lambda}^h) = \begin{cases} v_{\delta+1,\lambda-1}^h & \text{if } \lambda_j > 0 \\ 0 & \text{else} \end{cases}
\]

(2.48)
from which the coordinate expansion

\[
D_{i,j} = \partial A_{i,j} + \sum_{\lambda_j > 0} v_{\delta+1,\lambda-1}^h \partial v_{\delta,\lambda}^h
\]

(2.49)
follows. The sum on the r.h.s. of 2.49 runs over all repeated indices \( h, \delta, \lambda \).

**Proof.** Equation 2.47 is obvious if we recall that in the previous trivial coordinates the derivations \( D_{i,j} \) were just the partial derivative with respect to \( A_{i,j} \).

To prove the second equation 2.48 we first express the \( u_{\delta,\lambda}^h \) and \( v_{\delta,\lambda}^h \) as sections of the dual \( S^{k+1}\mathbb{R} \otimes N^* \) using the dual basis to \( y_1, \ldots, y_d, x_1, \ldots, x_l \in \mathbb{R}^* \) and \( e_1^*, \ldots, e_m^* \in N^* \) and the natural isomorphism

\[
S^{k+1}(\mathbb{R}^*) \cong (S^{k+1}\mathbb{R})^*
\]

(2.50)
induced from the non-degenerate pairing

\[
S^{k+1}\mathbb{R} \otimes S^{k+1}(\mathbb{R}^*) \to \mathbb{R}
\]

(2.51)
given by

\[
w_1 \cdot \ldots \cdot w_{k+1} \otimes \alpha_1 \cdot \ldots \cdot \alpha_{k+1} \mapsto \sum_{\varsigma} \prod_{i=1}^{k+1} (w_{\varsigma(i)}, \alpha_i)
\]

(2.52)
where \( \varsigma \) runs through all permutations of the set \( \{1, \ldots, k + 1\} \). If \( r_1, \ldots, r_n \) is a basis of \( R \) and the associated dual basis of \( R^* \) is denoted with \( r_1^*, \ldots, r_n^* \), then under identification \( 2.50 \) the dual basis of \( r^* \in S^{k+1} R \) is mapped to \( \frac{1}{\varsigma} (r^*)^\varsigma \in S^{k+1} R^* \).

So letting \( y_1^*, \ldots, y_d^*, x_1^*, \ldots, x_l^* \in R^* \) denote the basis dual to \( y_1, \ldots, y_d, x_1, \ldots, x_l \in R \) and \( e_1^*, \ldots, e_m^* \in N^* \) the one dual to \( e_1, \ldots, e_m \in N \) we have

\[
u_{\delta,\lambda}^h = (y^*)^\delta (x^*)^\lambda \otimes e_h^.* \tag{2.53}\]

Further, since the basis of \( R \) dual to the basis \( \frac{1}{\varsigma} \) runs through all permutations of the set \( \{1, \ldots, k + 1\} \), \( A_{1,j} x_j \), \( \ldots \), \( A_{d,j} x_j \), \( x_1, \ldots, x_l \) of \( R^* \) is given by

\[
y_1^*, \ldots, y_d^*, (x_1^* + \sum A_{i,1} y_i^*), \ldots, (x_l^* + \sum A_{i,l} y_i^*) \tag{2.54}\]

we have

\[
u_{\delta,\lambda}^h = \frac{1}{\varsigma} (y^*)^\delta (x^* + Ay^*)^\lambda \otimes e_h^.* \tag{2.56}\]

again by \( 2.50 \) and since the \( \nu_{\delta,\lambda}^h \) are by definition dual to the basis \( \frac{1}{\varsigma} y - \sum Ax \). By expanding the powers on the r.h.s. of \( 2.56 \) we could express the coordinates \( \nu_{\delta,\lambda}^h \) as linear combinations of the \( \nu_{\delta,\lambda}^h \) with coefficients depending on the variables \( A_{i,j} \). We shall not do this, instead we recall again that in the coordinates \( \nu_{\delta,\lambda}^h, A_{i,j} \) the derivations \( D_{i,j} \) act as partial derivative with respect to \( A_{i,j} \). Hence applying the chain rule we can compute

\[
D_{i,j}(\nu_{\delta,\lambda}^h) = \frac{\partial}{\partial A_{i,j}} \left( \frac{1}{\varsigma} (y^*)^\delta (x^* + Ay^*)^\lambda \otimes e_h^.* \right) \tag{2.57}
\]

\[
= \lambda_j y_i^* \frac{1}{\varsigma} (y^*)^\delta (x^* + Ay^*)^{\lambda-1}_j \otimes e_h^.* \tag{2.58}
\]

\[
= \begin{cases} 
\frac{1}{\varsigma^{+1}} (y^*)^{\delta+1}_j (x^* + Ay^*)^{\lambda-1}_j \otimes e_h^.* & \text{if } \lambda_j > 0 \\
0 & \text{else}
\end{cases} \tag{2.59}
\]

\[
= \begin{cases} 
\nu_{\delta+1,j-1}^h & \text{if } \lambda_j > 0 \\
0 & \text{else}
\end{cases} \tag{2.60}
\]

\[\square\]

**Definition 16.** The \(q\)-truncated homogeneous total derivatives are the vector fields on \( M^q\) (where \( q \geq 0 \)) defined in local adapted coordinates by

\[
\begin{aligned}
D_{i,j}^{[q]} := \partial_{A_{i,j}} + \sum_{|\delta|<q, \lambda_j > 0} \nu_{\delta+1,j-1}^h \partial_{A_{i,j}}.
\end{aligned} \tag{2.61}
\]

It is clear that \( D_{i,j}^{[k+1]} = D_{i,j}. \)

**Lemma 10.**

a) The fields \( \{\partial_{A_{i,j}}, D_{i,j}^{[q]}\} \) with \( |\delta| \leq q \) make up a frame on \( M^q \).

b) Commutators of this frame are all zero except for the commutators

\[
\left[ \partial_{A_{i,j}}, D_{i,j}^{[q]} \right] = \partial_{A_{i-1,j+1,j}} \tag{2.62}
\]

when \( |\delta| > 0 \).

c) The vertical distribution of \( M^q \to M^{q-1} \) is spanned by \( \partial_{A_{i,j}} \) with \( |\delta| = q \).

d) The fields \( \{\partial_{A_{i,j}}, D_{i,j}^{[q]}\} \) with \( |\delta| = q \) form a local basis of \( \mathcal{E}^q \) and split it into vertical and horizontal part.
Proof. Straightforward from the definitions.

**Corollary 4.** For \( q = 0, \ldots, k + 1 \) any plane \( Q \subset \mathcal{F}^q \) of maximal dimension and horizontal to \( M \to M^{q-1} \) has a basis of the form

\[
C_{i,j} := D_{i,j} + \sum_{|\delta|=q} C_{i,j,h}^{\delta,\lambda} \partial_{\delta,\lambda}^h
\]

with unique coefficients \( C_{i,j,h}^{\delta,\lambda} \). It is hence of dimension \( \dim \text{Gr}(R, l) \) and horizontal to the projection \( M \to \text{Gr}(R, l) \).

**Definition 17.** We denote the curvature form of \( \mathcal{F}^q \) with \( \Omega[0] \). We may compute it directly by using commutators 2.62.

**Lemma 11.** For \( q = 1, \ldots, k + 1 \) a horizontal plane \( Q \subset \mathcal{F}^q \) of dimension \( \text{Gr}(R, l) \) is an integral element of \( \mathcal{F}^q \) if and only if the coefficients \( C_{i,j,h}^{\delta,\lambda} \) of its basis 2.63 satisfy

\[
C_{i,j,h}^{\delta,\lambda} = C_{i',j',h}^{\delta',\lambda'}
\]

whenever the indices satisfy

\[
\begin{align*}
\delta' &> 0, \quad \delta' > 0 \\
\delta - 1_s' &\equiv \delta' - 1_i \\
\lambda + 1_{s'} &\equiv \lambda' + 1_j
\end{align*}
\]

and condition

\[
C_{i,j,h}^{\delta,\lambda} = 0 \quad \text{whenever} \quad \lambda_j = 0 \quad \text{and} \quad l > 1.
\]

**Proof.** The plane \( Q \) is integral if and only if

\[
\Omega[0](C_{i,j}, C_{i',j'}) = 0
\]

for all \( i, j, i', j' \). Expanding the left hand side of 2.69 leads to

\[
\sum_{|\delta|=q, \delta'>0} C_{i,j,h}^{\delta,\lambda} \partial_{\delta,\lambda}^h - \sum_{|\delta|=q, \delta'>0} C_{i',j',h}^{\delta',\lambda'} \partial_{\delta',\lambda'}^h = 0.
\]

Changing indices in the first sum to \( \Delta = \delta - 1_s, \quad \Lambda = \lambda + 1_{s'} \) and in the second to \( \Delta = \delta - 1_i, \quad \Lambda = \lambda + 1_j \) transforms equation 2.70 to

\[
\sum_{|\Delta|=q-1, \Lambda_j>0} C_{i,j,h}^{\Delta+1_s,\Lambda-1_i} \partial_{\Delta,\Lambda}^h - \sum_{|\Delta|=q-1, \Lambda_{s'}>0} C_{i',j',h}^{\Delta+1_i,\Lambda-1_s} \partial_{\Delta,\Lambda}^h = 0.
\]

Collecting bases we find

\[
\sum_{|\Delta|=q-1, \Lambda_{s'}>0} \left( C_{i,j,h}^{\Delta+1_s,\Lambda-1_i} - C_{i',j',h}^{\Delta+1_i,\Lambda-1_s} \right) \partial_{\Delta,\Lambda}^h +
\]

\[
+ \sum_{|\Delta|=q-1, \Lambda_{s'}>0} C_{i,j,h}^{\Delta+1_s,\Lambda-1_i} \partial_{\Delta,\Lambda}^h + \sum_{|\Delta|=q-1, \Lambda_{s'}=0} C_{i',j',h}^{\Delta+1_i,\Lambda-1_s} \partial_{\Delta,\Lambda}^h = 0.
\]

Equating coefficients to zero and returning to the previous indices we find conditions 2.64 from the first summand of 2.72 while from the second and third summands (which are only present when \( l > 1 \)) we find condition 2.68.

□
Lemma 12. For $q = 1, \ldots, k + 1$ a horizontal plane $Q \subset \mathcal{F}^{q-1}_{q-1}$ is of the form $Q_{\phi_q}$ for some $\phi_q \in M^q_{q-1}$ if and only if the coefficients $C_{i,j,h}^{\delta,\lambda}$ of its basis $2.63$ satisfy

$$C_{i,j,h}^{\delta,\lambda} = C_{i',j',h}^{\delta',\lambda'}$$

whenever the indices satisfy

$$\lambda_j > 0, \lambda'_j > 0$$

$$\delta + 1_i = \delta' + 1_i$$

$$\lambda - 1_j = \lambda' - 1_j$$

and condition

$$C_{i,j,h}^{\delta,\lambda} = 0 \quad \text{whenever} \quad \lambda_j = 0.$$  \hfill (2.77)

Proof. We start by showing that the basis of a plane $Q_{\phi_q}$ satisfies 2.73 and 2.77. By lemma 10 the plane $\mathcal{F}^{q}_{q-1}$ is spanned by the fields $D_{i,j}^{[q]}$ and vertical fields $\partial_{\phi_q}$ with $|\delta| = q$. The vertical ones are annihilated when projecting to $M^{q-1}$ while the $D^{[q]}_{i,j}$ are mapped to

$$C_{i,j} := D_{i,j}^{[q-1]} + \sum_{|\delta|=q-1 \atop \lambda_j > 0} v_{\delta+1,\lambda-1,j}^h \partial_{\phi_q}$$

where now the numbers $v_{\delta+1,\lambda-1,j}^h$ on the r.h.s of 2.78 are to be understood as the coordinates of the point $\phi_q$ in the fiber over $\phi_q$. Vectors 2.78 are a basis of $Q_{\phi_q}$ of the form 2.63 with $C_{i,j,h}^{\delta,\lambda} = v_{\delta+1,\lambda-1,j}^h$. It is straightforward to see that these coefficients satisfy 2.73 and 2.77.

Conversely suppose the basis $C_{i,j}$ of a plane $Q \subset \mathcal{F}^{q-1}_{q-1}$ satisfies conditions 2.73 and 2.77. We need to find a point $\phi_q \in M^q_{q-1}$ such that $Q = Q_{\phi_q}$. For any multiindex $(\Delta, \Lambda) \in \mathbb{N}^d \times \mathbb{N}^f$ with $|\Delta| = q$, $|\Delta| + |\Lambda| = k + 1$ and any $h \in 1, \ldots, m$ define the numbers

$$v_{\Delta,\Lambda}^h := C_{i,j,h}^{\Delta-1,\Lambda+1,j}$$

where we choose $i$ in such a way that $\Delta_i > 0$, which is always possible since $|\Delta| \geq 1$. By 2.73 this definition is independent of the choices of $i, j$. By further taking into consideration condition 2.77 we see that

$$C_{i,j} = D_{i,j}^{[q-1]} + \sum_{|\delta|=q-1 \atop \lambda_j > 0} v_{\delta+1,\lambda-1,j}^h \partial_{\phi_q}$$

which by 2.78 proves that $Q$ is of the form $Q_{\phi_q}$ with the point $\phi_q \in M^q_{q-1}$ determined by the fiber coordinates 2.79. \hfill $\square$

Proof of proposition 2. By lemma 12 the basis $C_{i,j}$ of $Q_{\phi_q}$ satisfies conditions 2.73 and 2.77, which for $q > 1$ are the same as conditions 2.64 and 2.68 of lemma 11, hence $Q_{\phi_q}$ is integral. When $q = 1$, $Q_{\phi_1}$ is integral since $\mathcal{F}^0 = TM^0$. \hfill $\square$

Proof of proposition 3. If $\phi_q \neq \tilde{\phi}_q$ are two distinct points over $\phi_{q-1}$ there must be indices $\delta, \lambda, h$ such that the corresponding fiber coordinates of the points differ $v_{\delta,\lambda}^h \neq v_{\delta,\lambda}^h$. Since $q > 0$ there is an $i$ such that $\delta_i \neq 0$. Then the coefficients in front of $\partial_{\phi_q}$ in the bases 2.78 of $Q_{\phi_q}$ and $Q_{\tilde{\phi}_q}$ differ, hence $Q_{\phi_q} \neq Q_{\tilde{\phi}_q}$ by uniqueness of the bases $C_{i,j}$. \hfill $\square$
Proof of proposition 4. For the range of indices $q$ under consideration conditions 2.64 and 2.68 of lemma 11 coincide with conditions 2.73 and 2.77 of lemma 12 hence an integral $Q$ is of the form $Q_{\phi_q}$. □

Proof of proposition 5. Observe first that coordinates $A_{i,j}, v^h_{\lambda}$ used in the description of the infinitesimal pasting conditions 1.41 and 1.41 are precisely the adapted coordinates $A_{i,j}, v^h_{0,\lambda}$ on $M^0$ (where now $\delta = 0$). Fix a point $\phi_0 \in M^0$. Any $\dim \text{Gr}(R^l) \cdot$-dimensional horizontal plane $Q \subset T_{\phi_0}M^0$ is now of the form

$$C_{i,j} = \partial A_{i,j} + \sum C_{0,\lambda}^{0,\lambda} \partial v^h_{0,\lambda}$$

(2.81)

with unique coefficients $C_{0,\lambda}^{0,\lambda}$ which may be thought of as fiber coordinates $v^h_{0,\lambda}$ in the first jet bundle of $\text{dir}$ corresponding to partial derivatives $\partial A_{i,j}$.

By lemma 12, $Q$ is of the form $Q_{\phi_1}$ iff the coefficients $C_{0,\lambda}^{0,\lambda}$ satisfy

$$C_{i,j}^{0,\lambda} = C_{i,j'}^{0,\lambda'}$$

(2.82)

whenever the indices satisfy

$$\lambda_j > 0, \lambda'_{j'} > 0$$

(2.83)

$$\lambda - 1_j = \lambda' - 1_{j'}$$

(2.84)

and condition

$$C_{i,j}^{0,\lambda} = 0 \text{ whenever } \lambda_j = 0.$$  

(2.85)

These are precisely the pasting conditions 1.41 and 1.41. □

We finish with

Lemma 13. When $l > 1$ The only maximal integral elements of $(I_{l,n}^{n,\theta_k}, D)$ transversal to $I_{l,n}^{n,\theta_k} \to I_{l,n}^{n,\theta_k}$ are the vertical tangent spaces of the projection

$$\text{pr}_n: \text{Gr}(R^l) \times J_{\theta_k}^{k+1} \to J_{\theta_k}^{k+1}$$

(2.86)

i.e. planes of the distribution $F^{-1}$. So the maximal integral submanifolds of $F^0$ are the fibers of $J_{\theta_k}^{l,n} \to J_{\theta_k}^{k+1}$ and hence correspond bijectively to jets of order $k + 1$ prolonging $\theta_k$.

Proof. Follows directly from lemma 11 equation 2.68 since in this case $\lambda = 0$. □

This finishes the proof of main theorem 3.

Notational conventions

For a finite dimensional vector space $W$ over a field $\mathbb{K}$, and $V \subset W$ a subspace

(1) $\text{Gr}(W, l)$ denotes the Grassmannian of all $l$ dimensional subspaces of $W$.

(2) $S^k W$ denotes the $k^{th}$ symmetric tensor product of $W$.

(3) $W^*$ denotes the dual $\text{hom}(W, \mathbb{K})$.

(4) $V^\circ \subset W^*$ denotes the annihilator of $V$.

(5) $W/V$ denotes the quotient.

(6) $\langle S \rangle$ denotes the span of the subset $S \subset W$. For manifolds $M, N$ and a map $f: M \to N$

(1) $Tf: TM \to TN$ denotes the tangent map.

(2) $f^{-1}(S)$ denotes the preimage of subset $S \subset N$ under $f: M \to N$.

(3) $M_q := f^{-1}(\{q\})$ denotes the fiber of the over $q \in N$ when $f: M \to N$ is a bundle.

(4) An $f$–horizontal plane is a tangent subspace of $M$ transversal to the fibers of $f$.

(5) $Vf$ denotes the vertical distribution of $f$ when it is a bundle.
A relative distribution along \( f \) is a sub-bundle of the pullback of the tangent bundle \( TN \) to \( M \).

The lift of a relative distribution \( D \) along \( f \) is the distribution \( f^{-1}D \) on \( M \) defined by \( f^{-1}D|_p = T_p f^{-1}(D_p) \).

For a chart \( x_1, \ldots, x_n \) on \( N \) the associated coordinate fields are denoted with \( \partial_{x_i} \).

For a multiindex \( \delta = (\delta_1, \ldots, \delta_n) \in \mathbb{N}^n \) and variables \( x_1, \ldots, x_n \):

1. \( x^\delta = x_1^{\delta_1} \cdots x_n^{\delta_n} \).
2. \( \delta! = \delta_1! \cdots \delta_n! \) is the factorial of the multi-index.
3. \( |\delta| = \delta_1 + \ldots + \delta_n \) denotes the length of the multi-index.
4. \( 1_j \) denotes the multi-index with all zero entries except for the entry at position \( j \) equaling 1.

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