Four-body problem and BEC-BCS crossover in a quasi-one-dimensional cold fermion gas

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The four-body problem for an interacting two-species Fermi gas is solved analytically in a confined quasi-one-dimensional geometry, where the two-body atom-atom scattering length \( a_{aa} \) displays a confinement-induced resonance. We compute the dimer-dimer scattering length \( a_{dd} \) and show that this quantity completely determines the many-body solution of the associated BEC-BCS crossover phenomenon in terms of bosonic dimers.

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Cold atomic quantum gases continue to attract a lot of attention due to their high degree of control, tunability, and versatility. A main topic of interest has been the exploration of the BEC-BCS crossover in fermionic systems \(^{1,2,3,4,5,6}\). In two or three dimensions, this is still a controversial and not completely settled issue on the theory side \(^{7,8,9,10}\), despite the qualitative agreement between mean-field theories and experimental data. Notably, a similar (but different) crossover phenomenon has been predicted to occur in quasi-one-dimensional (1D) systems \(^{11,12}\), where a cylindrical trap leads to a confinement-induced resonance (CIR) \(^{13,14}\) in the atom-atom interaction strength, analogous to the magnetically tuned Feshbach resonance \(^{8}\). In contrast to what happens in 3D, one always has a two-body bound state (‘dimer’) in 1D, regardless of the sign of the 3D atom-atom scattering length \( a \). We solve the fermionic four-body problem in the confined geometry, and compute the dimer-dimer scattering length \( a_{dd} \) throughout the full BCS-BEC crossover, on each side of the CIR. On the ‘BEC’ side, we establish contact to results for the unconfined case \(^{15}\), while on the ‘BCS’ side, a simple Bethe Ansatz calculation provides exact results. The three-body problem has no trimer solution \(^{16}\), and thus the full many-body crossover solution can be expressed in terms of \( a_{dd} \) alone and is thereby solved completely in this Letter. Since 1D atomic gases can be prepared and probed thanks to recent advances \(^{17,18,19}\), our predictions could be observed in state-of-the-art experiments.

We assume two fermion hyperfine components (denoted by \( \uparrow, \downarrow \)) with identical particle numbers \( N_{\uparrow} = N_{\downarrow} = N/2 \), interacting only via \( s \)-wave interactions. At low energies, the pseudopotential approximation \(^{20}\) for the 3D interaction among unlike fermions applies, \( V(r) = (4\pi\hbar^2a/m_0)\delta(r)\delta_z(r) \) \((m_0 \) is the mass). We consider the transverse confinement potential \( U_c(r) = m_0\epsilon^2(x^2+y^2)/2 \), with lengthscale \( a_{\perp} = (2\hbar/m_0\omega_{\perp})^{1/2} \). The solution of the two-body problem \(^{12,13,14}\) reveals that a single dimer (composite boson) state exists for arbitrary \( a \), where the dimensionless binding energy \( \Omega_B \) and (longitudinal) size \( a_B \),

\[
\Omega_B = -\frac{E_B}{2\hbar^2\omega_{\perp}} = (a_{\perp}/2a_B)^2 > 0,
\]

follow from \( \zeta(1/2,\Omega_B) = -a_{\perp}/a \) with the Hurvitz zeta function. For an experimental verification, see Ref. \(^{18}\). For \( a_{\perp}/a \to -\infty \), the BCS limit with \( \Omega_B \approx (a/a_{\perp})^2 \ll 1 \) and \( a_B \approx a^2/(2|a|) \) is reached, while for \( a_{\perp}/a \to +\infty \), the dimer (or BEC) limit emerges, with \( \Omega_B \approx (a_{\perp}/2a)^2 \gg 1 \) and \( a_{dd} \approx a \). The atom-atom scattering length is

\[
a_{aa} = a_{\perp}(C - a_{\perp}/a)/2, \quad C = -\zeta(1/2) \approx 1.4603.
\]

For low energies, this result is reproduced by the 1D atom-atom interaction \( V_{aa}(z,z') = g_{aa}\delta(z-z') \) with \( g_{aa} = -2\hbar^2/m_0a_{aa} \). The CIR (where \( g_{aa} \to \pm\infty \)) takes place for \( a_{\perp}/a = C \), which is equivalent to \( \Omega_B = 1 \). In this paper, we solve the 1D fermionic four-body \((\uparrow\downarrow\downarrow\downarrow)\) problem and show that this also solves the \( N \)-body problem for arbitrary \( \Omega_B \) in the low-energy regime.

Let us first discuss general symmetries of the four-body problem. We denote the positions of the \( \uparrow \) fermions by \( x_1,4 \) \((x_2,3)\), respectively, and then form distance vectors between unlike fermions, \( r_1 = x_1 - x_3, r_2 = x_4 - x_3 \), and \( r_+ = x_1 - x_3, r_- = x_4 - x_2 \). The distance vector between dimers \(^{12}\) and \(^{34}\) is \( R/\sqrt{2} = (x_1 + x_2 - x_3 - x_4)/2 \). After an orthogonal transformation, the center-of-mass coordinate decouples and the four-body wavefunction \( \Psi \) depends only on \( r_{1,2} \) and \( R \). With respect to dimer interchange, \( \Psi \) is symmetric,

\[
\Psi(r_1, r_2, R) = \Psi(r_2, r_1, -R),
\]

while under the exchange of identical fermions,

\[
\Psi(r_1, r_2, R) = -\Psi(r_\pm, r_\mp, \pm(r_1 - r_2)/\sqrt{2}).
\]

The four-body Schrödinger equation then reads

\[
\frac{\hbar^2}{m_0}\left[\frac{\Delta_r}{\omega_{\perp}} + U_c(r_1) + U_c(r_2) + U_c(R) + V(r_2) - E\right] \Psi = -\sum_{i=1,\pm} V(r_i) \Psi.
\]

\[
\left[\frac{\hbar^2}{m_0}\left(\Delta_{r_+} + \Delta_{r_-} + \Delta_R\right) + U_c(r_1) + U_c(r_2) + U_c(R) + V(r_2) - E\right] \Psi = -\sum_{i=1,\pm} V(r_i) \Psi,
\]
The pseudopotentials on the r.h.s. are incorporated via Bethe-Peierls boundary conditions imposed when a dimer is contracted, e.g.,

$$\Psi(r_1, r_2, R)_{r_1 \to 0} \approx \frac{f(r_2, R)}{4\pi r_1^2} (1 - r_1/a).$$  \hspace{1cm} (6)

All other boundary conditions can also be expressed in terms of $f(r, R)$ using Eqs. 5 and 4, where

$$f(r, R) = f(-r, -R),$$  \hspace{1cm} (7)

expresses (parity) invariance of Eq. 3 under $r_{1,2} \to -r_{1,2}$ and $R \to -R$ in combination with Eq. 3. In order to appreciate the importance of Eq. 4, it is instructive to expand $f(r, R)$ in terms of the single-particle eigenfunctions $\psi_\lambda(r)$ and the two-body scattering states $\Phi_\lambda(r)$ in the presence of the confinement,

$$f(r, R) = \sum_{\mu\nu} f_{\mu\nu} \Phi_\mu(r) \psi_\nu(R).$$  \hspace{1cm} (8)

The quantum numbers $\lambda$ include the 1D momentum $k$, the (integer) angular momentum $m$, and the radial quantum number $n = 0, 1, 2, \ldots$. Explicit expressions for $\psi_\lambda$ and $\Phi_\lambda$ can be found in Refs. 13, 16. While both have the same energy $E_\lambda$, the $\Phi_\lambda$ now include the dimer bound state (denoted by $\lambda = 0$) $\Phi_0(r)$. For relative longitudinal momentum $k$ of the two dimers, the total energy is (excluding zero-point and center-of-mass motion)

$$E = -2\hbar \omega_1 \Omega_B + \frac{\hbar^2k^2}{2m_0}.$$  \hspace{1cm} (9)

We consider the low-energy regime $ka_\perp < 1$, where the relative dimer motion is in the lowest transverse state $(n = m = 0)$ when dimers are far apart. We then have to deal with a 1D dimer-scattering problem in this ‘open’ channel, where the asymptotic 1D scattering state $f_0(Z)$ for $|Z| \gg \max(a_\perp, |a_{\perp\perp}|)$ follows from Eq. 8 as

$$f(r, R) = \Phi_0(r) \psi_\perp 00 \left( e^{iX^2/2 + Y^2} f_0(Z),$$  \hspace{1cm} (10)

where $\psi_\perp 00$ is the transverse part of $\psi_{n=0,m=0}$. The symmetry relation 7 now enforces $f_0(Z) = f_0(-Z)$, reflecting the fact that two (composite) bosons collide, i.e.,

$$f_0(Z) = e^{-ik|Z|} + (1 + 2\tilde{f}(k)) e^{ik|Z|}.$$  \hspace{1cm} (11)

As long as only s-wave scattering is important, symmetry considerations thus rule out odd-parity solutions normally present in 1D scattering problems 13, 16. This crucial observation implies that, assuming analyticity, the 1D scattering amplitude can be expanded in terms of a 1D dimer-dimer scattering length $a_{dd}$ 22,

$$\tilde{f}(k) = -1 + ika_{dd} + \mathcal{O}(k^2).$$  \hspace{1cm} (12)

For $|ka_{dd}| \ll 1$, this also follows from the zero-range 1D dimer-dimer potential

$$V_{dd}(Z, Z') = g_{dd} \delta(Z - Z'), \hspace{1cm} g_{dd} = -\frac{2\hbar^2}{(2\pi\hbar a_{dd})}.$$  \hspace{1cm} (13)

We stress that Eq. 13 holds for arbitrary $a_\perp/a$, and therefore 1D dimer-dimer scattering at low energies is always characterized by a simple $\delta$-interaction.

Let us then analyze the BCS limit, $\Omega_B \ll 1$, where the scattering problem is kinematically 1D on length-scales exceeding $a_\perp$. Projecting Eq. 4 onto the transverse ground state, the 1D Schrödinger equation for four attractively interacting fermions reads with $a_{aa} = a_\perp^2/2|a| \gg a_\perp$, see Eq. 4.

$$\left( \frac{2m_0 E}{\hbar^2} + \sum_{i=1}^{4} \frac{\partial^2}{\partial z_i^2} + \frac{4}{a_{aa}} \sum_{i<j} \delta(z_i - z_j) \right) \Psi = 0,$$  \hspace{1cm} (14)

where the second sum excludes identical fermion pairs, $(i, j)$ corresponding to $\{14\}$ and $\{23\}$. The Bethe Ansatz expresses the wavefunction as a sum of products of plane waves 28. Let us choose the momenta $a_{aa}k_{1,4} = \mp i - u/2$ and $a_{aa}k_{3,2} = \mp i + u/2$ to describe dimer-dimer scattering, and measure lengths in units of $a_{aa}$. The energy of this state is $E = \hbar^2(-2 + u^2/2)/(m_0a_{aa}^2)$ and $u$ the relative momentum of the two dimers. Up to an overall normalization constant, the wavefunction in the domain $D_1 = \{(z_1, z_2) < (z_3, z_2)\}$ must then be given by

$$\Psi_1 = e^{-(z_2 + z_3 - z_4 - z_1)/2} \left( e^{iu(z_2 + z_4 - z_3 - z_1)/2} - e^{iu(z_2 + z_3 - z_4 - z_1)/2} + e^{iu(z_1 + z_2 - z_3 - z_4)/2} - e^{iu(z_1 + z_2 - z_4 - z_3)/2} \right)$$

$$\Psi_2 = 2\epsilon \left( e^{-(z_2 + z_4 - z_3 - z_1)/2} \right)$$

$$+ \left( \frac{2 + iu}{2 + iu} e^{iu(z_2 + z_3 - z_4 - z_1)/2} - 2e^{iu(z_2 + z_4 - z_3 - z_1)/2} \right).$$

The wavefunction in other domains can be found in a similar manner. As a result, for a large dimer-dimer distance $Z$, $\Psi \sim e^{-|z|} e^{-|z|} f_0(Z)$, where $e^{-|z|}$ is the 1D wavefunction of the dimer 13 and 24, respectively. This result shows explicitly that even in the BCS limit, the two dimers are not broken in the collision even for large $k$. There is no coupling to additional fermionic states, and the composite nature of the dimer is not apparent in $\Psi$. The 1D scattering state $f_0(Z)$, see Eq. 11, has the exact scattering amplitude

$$\tilde{f}(k) = -\frac{1}{1 + ika_{dd}}, \hspace{1cm} a_{dd} = \frac{a_{aa}}{2} = \frac{a_\perp^2}{4|a|},$$  \hspace{1cm} (15)

which reproduces the full scattering amplitude derived from Eq. 13 and not just the first order as in Eq. 12.
The bound state at imaginary $k$ predicted by Eq. (15) is however unphysical, since the corresponding Bethe Ansatz solutions are then not normalizable. It would correspond to a non-existent bound four-fermion (tetramer) state, and hence Eq. (15) is restricted to the real axis.

Let us now turn to the many-body problem, starting with the BCS limit. Since dimers are not broken in the collision, the ground state can be described in terms of $N/2$ bosons (bosonization) with the interaction $12$ and $a_{dd} = a_{aa}/2$. The attractively interacting Bose gas is stabilized by the real-$k$ restriction, implying the omission of many-body bosonic bound states. Bosonization is possible for $\rho a_{dd} < 1$, since typical momenta are $k \approx \rho$ for total 1D fermionic density $\rho$. This reasoning immediately leads to the famous Lieb-Liniger (LL) equations $25$:

$$
\frac{E_0}{N} = -\hbar \omega_{\perp} \Omega_B + \frac{1}{\rho} \int_{-K_0}^{K_0} dk \frac{\hbar^2 k^2}{4m_0} f(k),
$$

$$
2\pi f(k) = 1 - \frac{4}{a_{dd}} \int_{-K_0}^{K_0} dp \frac{f(p)}{4/a_{dd}^2 + (p - k)^2},
$$

where $E_0$ is the ground state energy and $K_0$ is fixed by $\rho/2 = \int_{-K_0}^{K_0} dk f(k)$. Notably, since $a_{dd} = a_{aa}/2$, the LL equations coincide with Yang-Gaudin equations for $N$ attractively interacting 1D fermions, thereby explaining a dimer regime is realized for $a_{dd} < a_{dd} \equiv a_{aa}/2$. For $a_{dd} \lesssim a_{dd}$, one leaves the BCS regime and enters the ‘crossover regime’, while (once $a_{dd} < 0$) the dimer regime is realized for $|a_{dd}| \gtrsim a_{dd}$. Within the crossover regime, $|a_{dd}| \lesssim a_{dd}$, we have hard-core bosons that can effectively be fermionized $11,12$, again implying typical momenta $k \approx \rho$. For $\rho a_{\perp} < 1$, the condition $|\kappa a_{dd}| \ll 1$ is imposed by Eq. (12), and hence $|a_{dd}| \lesssim a_{dd}$, and the solution of the 1D four-body problem is sufficient to completely solve the 1D BEC-BCS many-body problem for dilute systems, $\rho a_{\perp} < 1$, in terms of the LL equations $14$.

Next we discuss the 1D four-body problem. Enforcing the boundary condition $15$ or the other equivalent ones,

$$
a_{dd} = -\frac{a_{red,\perp}^2}{2(6a_0)}, \quad a_{red,\perp} = (\hbar^2/m_0 \omega_{\perp})^{1/2}.
$$

The two-body Green’s function $G_{E}(r,0)$ can be found in Ref. $10$. The two degrees of freedom in $f(r, R)$ imply two different types of ‘closed’ channels that may be excited in a dimer-dimer collision: (i) scattering states above the bound state for each dimer (corresponding to $r$ or $\mu$ in Eq. (8)), and (ii) excited states in the transverse direction for the relative motion of two dimers (corresponding to $\mathbf{R}$ or $\nu$ in Eq. (8)). Neglecting both types of closed-channel excitations, Eq. (15) can be solved numerically for arbitrary $a_{1}/a$ as in Ref. $10$. The result is shown in the inset of Fig. 1. Addition, this approximation allows to extract $a_{dd}$ in both limits analytically: in the dimer limit, we find $a_{dd} = -\kappa a_{\perp}^2/(2a_0) + 2\kappa_1 a_0$, with $\kappa_0 = 1/4$ and $\kappa_1 \approx 0.21$, while in the BCS limit, $a_{dd} = \eta_0 a_0^2/a$ with $\eta_0 \approx 0.42$. The exact (numerical) result for arbitrary $a_{1}/a$ agrees to within $\pm 0.05$ in $a_{dd}/a_{\perp}$ with a simple interpolation formula obtained by simply adding these two limiting results. For practical purposes, the interpolation is therefore virtually exact. Let us then turn to the effects of closed-channel excitations. In the BCS limit, excitations of type (ii) are irrelevant $10$, but type-(i) excitations are important. Their inclusion results in the exact value $\eta_0 = 1/4$, see Eq. (13), which also follows from the solution of Eq. (8), including

$$
\text{FIG. 1: Scattering length } a_{dd} \text{ as a function of } \Omega_B. \text{ Dashed curves give exact limiting results, the solid curve interpolates by adding these. Inset: Same but neglecting all closed-channel excitations. Here the solid curve gives the exact result.}
$$
type-(i) excitations \[27\]. In the dimer limit, inclusion of the closed channels leads to the correct value \(\kappa_0 \approx 0.83\), see Eq. \[18\]. Incidentally, the two excitation types can be disentangled \[27\], and we find \(a_{dd}^{1D} \approx 0.66a\) by just neglecting type-(i) excitations, which is already close to the exact value \(a_{dd}^{3D} \approx 0.69\) \[12\]. Type-(ii) excitations are obviously important in the dimer limit, which may be valuable input for diagrammatics \[11, 28\]. The exact limiting results for \(a_{dd}\) are shown in the main part of Fig. 1 as dashed curves. For the full crossover, the additive interpolation formula is again expected to be highly accurate. Notably, this predicts \(a_{dd} = 0\) for \(\Omega_B \approx 0.3\). At this point, a CIR for dimer-dimer scattering occurs, see Eq. \[13\], where the interaction strength \(g_{dd}\) diverges and changes sign. Interestingly, the dimer-dimer CIR takes place at a different value for \(\Omega_B\) (and hence \(a_\perp/a\)) than the atom-atom CIR.

In experiments, quasi-1D regimes can be obtained in arrays of very elongated traps with a shallow confinement in the longitudinal direction. Typical trap frequencies are \(\omega_\perp/2\pi \approx 70\ \text{kHz}\) and \(\omega_z/2\pi \approx 250\ \text{Hz}\), with \(N \approx 100\) atoms per tube to ensure the 1D condition \(N < \omega_\perp/\omega_z\) \[12\]. The BCS-BEC crossover can be investigated using a Feshbach resonance, which leads to changes in the density profile \[11\], excitation gaps \[12\] and ground state energy that can be probed via release energy \[6\] and rf spectroscopy measurements \[4\] \[12\]. A probably more precise approach is to measure collective axial modes. The dipole mode frequency is always \(\omega_z\), irrespective of interactions. Using a sum rule approach \[29\], we calculated the frequency of the lowest compressional (breathing) mode from the mean-square size of the cloud \(\omega^2 = -2 \langle (\ln z^2) / d\omega^2 \rangle^{-1}\), see Fig. 2 by solving Eqs. \[14\] using our results for \(a_{dd}\). Limiting values are \(\omega = \sqrt{3}\omega_z\) in the dimer limit, and \(\omega = 2\omega_z\) both in the BCS limit and close to \(a_{dd} = 0\). We hope that this prediction will soon be tested. - This work was supported by the DFG-SFB TR12.

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