Selfdual spin 2 theory in a 2+1 dimensional (A)dS space–time

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The lagrangian constraint analysis of the selfdual massive spin 2 theory in a 2+1 dimensional flat space–time and its extension to a curved one were performed. Demanding consistency of the degrees of freedom and causality in the model with gravitational interaction, gives rise to physical restrictions on non minimal coupling terms and background. Finally, a constant curvature scenario was explored, showing the existence of classes of forbidden mass values. Notorious aspects related with the construction of the reduced action and the one-particle exchange amplitude were mentioned.

Keywords higher spin theory, Lagrangian formulation, lower dimensional gravity

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1 Introduction

In the context of ordinary field theory, there has been great interest about the lagrangian study of higher spin fields with external interaction [1–17]. These theories are only known in certain backgrounds (i.e., constant curvature, non Einstein’s spaces) because generally, a consistent higher spin field theory with interaction does not exist as a result of the non conservation of the degrees of freedom and causality violation.

It is well known [18, 19] that the introduction of auxiliary fields, that vanish on shell, is needed in order to obtain a Lagrangian formulation, in a flat space–time, for a free massive field with spin $s$ described by a symmetric, transverse, and traceless, rank $s$ tensor (i.e., $\partial^\mu h_{\mu...} = 0$, $h^{\mu...} = 0$). However, when an arbitrary interaction is turned on, auxiliary fields become dynamic. Hence, they could modify the number of local degrees of freedom.

Causality violation has also been noted [17, 21–23]. For this, let us use the following notation [17]: The equations of motion for an integer spin field, $h_{\alpha_1...\alpha_2}$, which come from some lagrangian formulation can be written as $\left(\mathcal{M}_{\beta_3\beta_4...}^{\alpha_1...\alpha_2...}\right)_{\alpha_3...\alpha_4...} = 0$, with the help of the lagrangian constraints [24]. Let the vector $n_\mu$ used to define the characteristic matrix $\mathcal{M}_{AB}(n) \equiv \mathcal{M}_{AB}^{\mu\nu} n_\mu n_\nu$, where $A, B$ are composed indexes. The solutions of the characteristic equation $\det \mathcal{M}_{AB}(n) = 0$ define characteristic surfaces that might describe some propagation process. If the solution of the characteristic equation gives rise to a real $n_0$, the system is called hyperbolic. A hyperbolic system is called causal if there is no time like vectors among the solutions of the characteristic equation. On the contrary, if there exists time like vectors, the corresponding characteristic surfaces are space like and violate causality. When an arbitrary external interaction is considered, the characteristic matrix $\mathcal{M}_{AB}(n)$ does not necessarily define a hyperbolic-causal equation of motion system.

In this work, we are interested in the study of some distinctive features related to the aforementioned problems in the lagrangian formulation of the selfdual massive spin 2 field in 2+1 dimensions [25, 26] coupled with gravity. This letter is organized as follows. We will start with a brief review of the lagrangian constraints analysis of selfdual massive spin 2 theory in a flat space–time without external interaction. Next, we introduce the coupling between selfdual massive spin 2 field with an arbitrary gravitational background through a suitable set of non-minimal terms in the lagrangian formulation, and we will discuss the physical restrictions that arise in order to preserve a consistent interaction. As expected, one can find a constant curvature space solution in which the degrees of freedom must be consistently preserved and causality must take place. There, we obtain forbidden mass values.
for the selfdual massive spin 2 field. The construction of the reduced action in constant curvature space–time is noted. In the end, we discuss the one-particle exchange amplitude. Finally, some remarks will be stated.

2 Lagrangian analysis of the selfdual theory

The action of selfdual massive spin 2 field [25], in flat space–time is

\[ S_{sd} = \left( \frac{m}{2} \epsilon^{\mu \nu \lambda \rho} \partial_{\mu} h_{\nu \lambda} - \frac{m^2}{2} (h_{\mu \nu} h^{\mu \nu} - h^2) \right) \]  

where \( h = h^\mu \), \( \epsilon^{012} = \epsilon^{12} = +1 \), \( \epsilon \) \( \equiv \int d^3x \) and Minkowski’s metric is \( \text{diag}(+ + +) \). The equation of motion coming from \( S_{sd} \) provides nine primary constraints

\[ \phi^{(1)\mu \rho} = m \epsilon^{\mu \nu \lambda \rho} \partial_{\nu} h_{\lambda \rho} + m^2 (\partial^\mu h - h_{\mu \nu} \partial^\nu h) \approx 0 \]  

Preservation of (2) takes us to the secondary constraints

\[ \phi^{(2)\rho} \equiv \phi^{(1)\mu \rho} \partial_{\mu} h_{\lambda \rho} \approx 0 \]  

We observe that (3) can be replaced with the combination \( \phi^{(2)\rho} \equiv \partial_{\mu} \phi^{(1)\mu \rho} - m \epsilon^{\mu \nu \lambda \rho} \partial_{\nu} h_{\lambda \rho} \approx 0 \), which enforces \( h_{\mu \nu} \) to be symmetric. Relations \( \phi^{(1)\rho} = 0 \) allow us to find the following accelerations

\[ \ddot{h}_{\rho \mu} = \partial_{\nu} h_{\rho \mu} + m \epsilon_{\rho \beta} (\delta_{\beta} h_{\mu} - h_{\mu \rho}) \]  

and the \( \ddot{h}_{\rho \rho} \) remains unknown.

The procedure continues with the preservation of \( \phi^{(2)\rho} \approx 0 \), which gives rise to three additional constraints

\[ \phi^{(3)\rho} \equiv \phi^{(2)\rho} \approx -m^3 \epsilon^{\rho \mu \nu} h_{\mu \nu} \approx 0 \]  

saying that the symmetry property is consistent with time evolution. If we look at (5), the \( \rho = 0 \) component can be rewritten, on shell, as \( \phi^{(3)0} \approx \partial_{\nu} \phi^{(2)0} \approx -m^3 \epsilon^{\rho \mu \nu} \partial_{\nu} h_{\rho \mu} \approx 2m^4 h \approx 0 \), which shows the tracelessness property of the tensor field. Then, preserving \( \phi^{(3)\rho} \approx 0 \), we obtain the last constraint

\[ \phi^{(4)} \equiv 2m^4 \dddot{h} \approx 0 \]  

and two relations for the remaining accelerations, which means that \( -m^3 \epsilon^{\rho \mu \nu} \ddot{h}_{\mu \nu} \approx 0 \). These allow us to obtain

\[ \dddot{h}_{\rho \mu} = \dddot{h}_{\mu \rho} = \partial_{\nu} h_{\rho \mu} + m \epsilon_{\rho \beta} (\delta_{\beta} h_{\mu} - h_{\mu \rho}) \]  

The analysis of the lagrangian constraints ends with the preservation of (6). This provides one more relation for the accelerations \( m^4 \dddot{h} \approx 0 \) from which we obtain

\[ \dddot{h}_{\rho \mu} = -\dddot{h}_{\mu \rho} = -\partial_{\nu} h_{\rho \mu} - m \epsilon_{\rho \beta} (\delta_{\beta} h_{\mu} - h_{\mu \rho}) \]  

Therefore, it can be shown that the 16 lagrangian constraints indicate the existence of one propagated excitation, and it is described by a symmetric, transverse, and traceless tensor field. In other words,

\[ h^{(s)\mu \nu} h_{\nu \rho} = h^{(s)\rho \nu} \partial_{\nu} h_{\mu \rho} = 0, \quad h^{(s)\nu \rho} = 0 \]  

respectively, satisfying the field equation:

\[ \epsilon^{\mu \nu \lambda \rho} \partial_{\nu} h^{(s)\lambda \rho} - m h^{(s)\nu \rho} = 0 \]  

A hyperbolic-causal equation, \((\Box - m^2) h^{(s)\nu \rho} = 0\) is obtained from (10) using (9).

It can be observed that restrictions (9) leave just two free components of the nine in \( h_{\mu \nu} \) but relying on dynamic restrictions (10). It can be seen that only one degree of freedom is locally propagated. From the action point of view, one can also expose this unique excitation through the construction of the reduced action (\( S_{sd} \)), which starts performing a \( 2 + 1 \) splitting [27] for \( h_{\mu \nu} \), this means, \( n = h_{00}, N_i = h_{i0}, M_i = h_{0i}, h^{(s)}_{ij} = \frac{1}{2} (h_{ij} + h_{ji}), V = \frac{1}{2} \epsilon_{ij} h_{ij}, \) in action (1). Then, a transverse-longitudinal decomposition is realized introducing new variables defined by: \( \gamma_i \equiv \epsilon_{jk} \partial_j h_{NT}, N_i \equiv \epsilon_{ik} \partial_k h_{LT} + \partial_i h_{NT}, M_i \equiv \epsilon_{ik} \partial_k h_{NT} + \partial_i h_{LT} + 2 \epsilon_{ik} \partial_k h_{NL} (h^{(s)}_{ij}) + \epsilon_{ik} \partial_k h_{NL} \). This decomposition establishes an easy way to obtain the reduced action using the corresponding field equations, \( S_{sd} = \int d^3x \left\{ P \cdot Q - \frac{1}{2} \delta_0^2 \int d^3x \frac{1}{2} Q(\Delta - m^2)Q \right\} \), where \( Q = \sqrt{\Box} \Delta h_{TT} \) and \( P = \sqrt{2} \int d^3x h_{TT} \), which describes a single massive mode.

3 Coupling with gravity

Now, we outline the model of selfdual massive spin 2 field non-minimally coupled with gravity in a torsionless space–time as follows:

\[ S_{sdg} = \left( \frac{m}{2} \epsilon^{\mu \nu \lambda \rho} h_{\mu \nu} \partial_{\rho} h_{\lambda \rho} + \frac{\alpha \beta \gamma \delta}{2} h_{\alpha \beta} h_{\gamma \delta} \right) \]  

where \( \nabla_{\mu} \) is the covariant derivative, \( \epsilon^{\nu \lambda \rho} \equiv \frac{\epsilon^{\nu \lambda \rho}}{\sqrt{-g}} \), and now, \( \langle \rangle \equiv \int d^3x \sqrt{-g} \). Due to the fact that in 2 + 1 dimensions, the Riemann curvature tensor can be written in terms of the Ricci tensor (i.e., \( R_{\mu \nu \rho \sigma} = g_{\mu \rho} R_{\nu \sigma} - g_{\lambda \rho} R_{\mu \lambda \nu} + g_{\mu \nu} R_{\rho \lambda \sigma} - \frac{R}{2} (g_{\nu \lambda} g_{\rho \sigma} - g_{\nu \rho} g_{\lambda \sigma}) \)), the non-minimal coupling in (11) is characterized by a tensor \( \Omega^{\alpha \beta \gamma \delta} \), whose general form is

\[ \Omega^{\alpha \beta \gamma \delta} \equiv m^2 (g^{\alpha \lambda} g^{\beta \gamma} - g^{\alpha \gamma} g^{\beta \lambda}) + a_1 (R^{\alpha \beta \gamma \delta} g^{\rho \sigma} + R^{\alpha \beta \gamma \delta} g^{\rho \sigma}) + a_2 (R^{\alpha \beta \gamma \delta} g^{\rho \sigma} + R^{\alpha \beta \gamma \delta} g^{\rho \sigma}) + a_3 R^{\alpha \beta \gamma \delta} g^{\rho \sigma} + a_4 R^{\alpha \beta \gamma \delta} g^{\rho \sigma} \]
pling parameters in arbitrary background is equivalent to six relations for the accelerations (\(\Phi^{(1)\alpha} \approx 0\) is preserved)

\[
\Phi^{(2)\alpha} \approx \nabla_{\mu} \Phi^{(1)\alpha} \equiv \Omega^{\mu\alpha\sigma} \nabla_{\mu} h_{\sigma}\lambda
\]

where

\[
B^{\alpha\sigma\lambda} \equiv \frac{m}{2} \varepsilon^{\mu\nu\rho} (R^{\alpha\lambda}_{\mu
u\rho} \delta^\sigma_{\rho} - R^{\sigma}_{\rho\nu\mu} g^{\alpha\lambda})
\]

(15)

On the other hand, preservation of \(\Phi^{(1)\alpha} \approx 0\) leads to six relations for the accelerations \((m \neq 0, \text{ as in the flat case})\)

\[
\nabla_{\sigma} h_{\alpha}^2 \equiv -\frac{\varepsilon_{\alphalj}}{m} \nabla_{\sigma} \alpha \lambda \nabla_{\sigma} h_{\alpha} \lambda
\]

\[
-\left(\frac{\varepsilon_{\alphalj}}{m} \nabla_{\sigma} \alpha \lambda \nabla_{\sigma} h_{\alpha} \lambda + R_{\sigma0j0} g^{\alpha\lambda}\right) h_{\sigma} \lambda
\]

\[
+\nabla_{j} \nabla_{\sigma} h_{0} \alpha + R_{\sigma j0} h_{0\alpha}
\]

(16)

remaining the unknown accelerations, \(\nabla_{\sigma} h_{0\alpha}\).

At this point, the lagrangian analysis with free coupling parameters in arbitrary background is equivalent to the flat space–time case. The next step is the preservation of the constraint \(\Phi^{(2)\alpha} \approx 0\). In other words,

\[
\Phi^{(3)\alpha} \approx \Omega^{\alpha0\lambda} \nabla_{0} h_{\alpha \lambda} + (\Omega^{\alpha0\lambda} + \Omega^{j00\lambda}) \nabla_{j} \nabla_{0} h_{\alpha \lambda}
\]

\[
+ \left(\frac{\varepsilon_{\alphalj}}{m} \nabla_{\sigma} \alpha \lambda \nabla_{\sigma} h_{\alpha} \lambda + \nabla_{\sigma} \nabla_{\sigma} h_{\alpha\lambda}\right)
\]

\[
+ \left[\frac{\varepsilon_{\alphalj}}{m} \nabla_{\sigma} \alpha \lambda \nabla_{\sigma} h_{\alpha} \lambda + \nabla_{\sigma} \nabla_{\sigma} h_{\alpha\lambda}\right] \approx 0.
\]

(17)

and we expect that this expression represents three additional constraints, as in the flat case. However, from (17), it would be impossible to obtain any relation for the accelerations \(\nabla_{\sigma} h_{\alpha \lambda}\), because (17) constitutes a complete system for the aforementioned accelerations. We demand that all matrices \(3 \times 3, 2 \times 2, \text{ and } 1 \times 1\) built with \(\Omega^{\alpha0\lambda}\) have null determinants (i.e., \(\Omega^{\alpha0\lambda}\) totally degenerated), this means

\[
\Omega^{\alpha0\lambda} = 0
\]

(18)

This condition gives rise to restrictions on coupling parameters. Using (12),

\[
a_1 = -a_2 = a
\]

\[
a_4 = -a_5 = b
\]

\[
a_3 = a_4 = a_7 = 0
\]

(19)

and just two free parameters remain. Then,

\[
\Omega^{\alpha\beta\lambda} = a \, R^{\alpha\beta\lambda} + \left[m^2 + \left(\frac{a}{2} - b\right)R\right]
\]

(20)

and now,

\[
\Omega^{\alpha\beta\lambda} = \Omega^{\alpha\beta\lambda} - \Omega_{\alpha\lambda\beta}
\]

(21)

The object \(B^{\alpha\sigma\lambda}\), defined by (15), can be rewritten in terms of the Einstein’s tensor, \(G_{\mu\nu} = R_{\mu\nu} - \frac{g_{\mu\nu}}{2} R\) as follows:

\[
B^{\alpha\sigma\lambda} \equiv m \varepsilon^{\alpha\lambda\beta} G_{\beta} \lambda + \nabla_{\mu} \Omega^{\alpha\mu\sigma\lambda} = -B^{\lambda\sigma\alpha}
\]

(22)

with an antisymmetric property in virtue of (21).

With the help of (20), we can write the three constraints, \(\Phi^{(3)\rho} \equiv \nabla_{0} \Phi^{(2)\rho} \approx 0\) as follows:

\[
\Phi^{(3)\alpha} = N^{\alpha\lambda} \nabla_{0} \nabla_{0} h_{\alpha \lambda} + \nabla_{0} \Omega^{\alpha0\lambda} \nabla_{0} h_{\alpha \lambda} + A^{\lambda \sigma} h_{\alpha \lambda}
\]

\[
+ \Omega^{\alpha\lambda\alpha} \nabla_{i} \nabla_{j} h_{\lambda \alpha} + D^{\alpha\lambda\alpha} h_{\lambda \alpha} \approx 0.
\]

(23)

where

\[
N^{\alpha\lambda} \equiv \frac{1}{m} \varepsilon_{0kl} \Omega^{0\alpha kl} \Omega^{0\lambda k} + B^{0\alpha \lambda} = -N^{\lambda\alpha}
\]

\[
A^{\alpha\lambda} \equiv -\frac{1}{m} \Omega^{0\alpha kl} \Omega^{0\lambda k} + m R_{\alpha\lambda}^{0} + m R_{\alpha\lambda}^{0} \rho_{0} ^{9} + \nabla_{9} B^{0\alpha\lambda} - \Omega^{0\alpha\lambda} R_{\rho\rho 0}^{9}
\]

\[
= -\Omega^{0\alpha\lambda} R_{\rho\rho 0}^{9}
\]

\[
= -\Omega^{0\alpha\lambda} R_{\rho\rho 0}^{9} + \nabla_{0} B^{0\alpha\lambda} + \nabla_{0} B^{0\alpha\lambda}
\]

(25)

\[
D^{\alpha\lambda} \equiv -\frac{1}{m} \varepsilon_{0kl} \Omega^{0\alpha kl} \Omega^{0\lambda k} + \nabla_{0} B^{0\alpha\lambda}
\]

\[
= -\Omega^{\alpha\lambda} R_{\rho\rho 0}^{9} + \nabla_{0} B^{\alpha\lambda} - \Omega^{0\alpha\lambda} R_{\rho\rho 0}^{9}
\]

(26)

(27)

Going on the lagrangian procedure, preservation of \(\Phi^{(3)\rho} \approx 0\) must represent, as in the flat case, two expressions for accelerations \(\nabla_{0} h_{\alpha \lambda}\) and one for the last constraint (whose preservation allow us to get the re-
maining accelerations, and the procedure ends. Let us consider $3 \times 3$ and $2 \times 2$ arrays built with objects $N^{\alpha \lambda}$, the last request means that
\[ \det(N^{\alpha \lambda}) = 0 \] (28)
\[ \det(N^{ij}) \neq 0 \] (29)

Because the antisymmetry property of the odd rank matrix $(N^{\alpha \lambda})$, relation (28) is identically satisfied. Expression (29) gives a physical restriction on the gravitational field, and it conduces to
\[ \varepsilon_{0ij} N^{ij} \neq 0 \] (30)

It can be shown that this restriction will include non-Einsteinian solutions. Although these type of solutions exist, (30) will impose conditions on them. For illustration, let us consider $R_{\lambda \mu \rho \nu} = \frac{f(x)}{6} (g_{\lambda \nu} g_{\mu \rho} - g_{\lambda \rho} g_{\mu \nu})$.

Then, restriction (30) enforces a constraint for $f(x)$ (i.e., $6M^4 - m^2 f(x) + m \sigma b \partial_\nu f(x) \neq 0$, with $\sigma \equiv \frac{2}{3} a - b$). Our interest is focused on the particular solution $\partial_\nu f(x) = 0$ (hence, (30) relates the mass with cosmological constant), which is a dS/AdS type.

Considering a constant curvature space–time, with cosmological constant $\lambda$, has been related to a dS space $(\lambda > 0)$ or AdS space $(\lambda < 0)$ via Einstein’s equation, $R_{\mu \nu} - \frac{g_{\mu \nu}}{2} R - \lambda g_{\mu \nu} = 0$, where Riemann and Ricci tensors are
\[ R_{\lambda \mu \rho \nu} = \frac{R}{6} (g_{\lambda \nu} g_{\mu \rho} - g_{\lambda \rho} g_{\mu \nu}) , \quad R_{\mu \nu} = \frac{R}{3} g_{\mu \nu} \] (31)

respectively, and
\[ R = -6\lambda \] (32)

Tensor (20) is
\[ \Omega^{\alpha \beta \sigma \lambda} = M^2 (g^{\alpha \beta} g^{\sigma \lambda} - g^{\alpha \sigma} g^{\beta \lambda}) \] (33)

where
\[ M^2 = m^2 + \sigma R \] (34)

with $\sigma \equiv \frac{2}{3} a - b$. Using (33), the action (11) takes the form
\[ S_{sd\lambda} = \frac{m}{2} \varepsilon^{\mu \lambda \alpha \beta} \nabla_\mu h_{\alpha \beta} - \frac{M^2}{2} (h_{\mu \nu} h^{\mu \nu} - h^2) \] (35)

and with the help of (22) and (33), the object $N^{ij}$ becomes
\[ N^{ij} \equiv \left( \frac{6M^4 - Rm^2}{m} \right) \varepsilon^{ij} \] (36)

Consistence relation (30) is now
\[ 6M^4 - Rm^2 \neq 0 \] (37)

Considering (34), we can think about this relation as a restriction on $m^2$ in terms of scalar curvature and $\sigma$. This means that
\[ m^2 \neq m^2 \pm \frac{R}{12} (1 - 12\sigma \pm \sqrt{1 - 24\sigma}) \] (38)

showing the existence of some forbidden mass values in order to have consistency, which represents a well known fact in context of higher spin theories [3–5].

Now, lagrangian constraints are revisited, this time in dS/AdS space. The primary nine, (13) are
\[ \phi^{(1)\mu \alpha} \equiv m \varepsilon^{\mu \alpha \lambda} \nabla_\nu h_{\lambda \sigma} + M^2 (g^{\mu \alpha} h - h^{\alpha \mu}) \approx 0 \] (39)

Secondary constraints (14)
\[ \phi^{(2)\alpha} \approx M^2 (\nabla_\alpha h - \nabla_\mu h^{\mu \alpha}) + \frac{mR}{6} \varepsilon^{\alpha \sigma \lambda} h_{\sigma \lambda} \approx 0 \] (40)

can be written as
\[ \phi^{(2)\alpha} \approx \nabla_\mu \phi^{(1)\mu \alpha} - \frac{M^2}{m} \varepsilon^{\alpha \sigma \lambda} \phi^{(1) \mu \alpha} = \left( \frac{m^2 R - 6M^4}{6m} \right) \varepsilon^{\alpha \sigma \lambda} h_{\sigma \lambda} \approx 0 \]

Therefore, the symmetry property for the $h_{\mu \nu}$ field, in virtue of (37), is gained. Preservation of $\phi^{(2)\alpha}$ provides three more constraints
\[ \phi^{(3)\alpha} \approx \left( \frac{m^2 R - 6M^4}{6m} \right) \varepsilon^{\alpha \sigma \lambda} \nabla_\delta h_{\sigma \lambda} \approx 0 \] (41)

Its temporal component, $\phi^{(3)0} \approx 0$ is expressed as:
\[ \phi^{(3)0} \approx \nabla_\mu \phi^{(2)\mu} + \left( \frac{6M^4 - m^2 R}{6m^2} \right) \phi^{(1)\mu} \]

\[ = \frac{M^2}{3m^2} (6M^4 - m^2 R) h \approx 0 \] (42)

It says that the spin 2 field is traceless (obviously, if $M^2 \neq 0$). The last constraint arise from the preservation of $\phi^{(3)0} \approx 0$,
\[ \phi^{(4)} \equiv \nabla_\delta \phi^{(3)0} \approx \frac{M^2}{3m^2} (6M^4 - m^2 R) \nabla_\delta h \approx 0 \] (43)

We observe that traceless and transverse properties for a consistent description of the selfdual field demands the additional condition
\[ M^2 \neq 0 \] (44)

as a consequence of equations (40), (42), and (43). If one relaxes this restriction on $M^2$ (i.e., $M^2 = 0$), the lagrangian system will not furnish the expected number of degrees of freedom. Imposing (44), we can construct an hyperbolic-causal field equation for $h^{(\alpha) T \mu \nu}$, as follows:
\[ \left( \square - \frac{M^4}{m^2} + \frac{R}{2} \right) h^{(\alpha) T \mu \nu} = 0 \] (45)
with $\Box \equiv \nabla_\alpha \nabla^\alpha$. Therefore, the selfdual massive theory studied holds forbidden mass values in dS/AdS spaces because of (37) and (44). These facts can be resumed as follows:

| $R$   | $\sigma$ | forbidden $m$ |
|-------|----------|---------------|
| $> 0$ (AdS) | $0 \leq \sigma \leq \frac{1}{24}$ | $m_{\pm}$ |
| $> 0$ (AdS) | $< 0$ | $\sqrt{-\sigma R}$ |
| $< 0$ (dS) | $> 0$ | $\sqrt{-\sigma R}$ |

We note that, in the study of the selfdual massive spin 2 field theory coupled with gravity, a well-known fact is verified: inside a possible set of solutions, those of constant curvature spaces respect the number of degrees of freedom and causality. However, in contrast to other type of spin 2 theories [17], the selfdual massive one does not have massless limit, and $M^2 \neq 0$ is demanded in order to guarantee equivalence between constraint system and symmetry, traceless, and transverse properties of selfdual massive field with a hyperbolic and causal equation provided.

There are other issues related with the condition $M^2 \neq 0$. On one hand, this condition, in a dS/AdS background, maintains the selfdual lagrangian, (35), conformally variant due to the non null trace of the energy-momentum associated with the selfdual field, $T^\mu_\mu = -\frac{M^2}{2} h^{(s)Tt_\mu}_\mu h^{(s)Tt_\mu}$.

Moreover, the critical value $M^2 = 0$ reveals the existence of an expected discontinuity in the degrees of freedom count, because it gives rise to a non consistent set of lagrangian constraints, which is associated with the nature of the quadratical terms in the action and not to the features of the gravitational interaction. This kind of discontinuity can be illustrated in a flat space–time as follows. Let us consider the two parameter action

$$S_{m_1, m_2} = \frac{1}{2} \epsilon^{\mu\nu\lambda} h_\mu^\alpha \partial_\nu h_\lambda^\alpha - \frac{m_2^2}{2} (h_{\mu\nu} h^{\mu\nu} - h^2)$$

which reproduces the selfdual massive model when we choose $m_1 = m_2 = m$. Particularly, when $m_2 = 0$, the action (46) describes a model with no degrees of freedom (in fact, the reduced action becomes null identically). However, if $m_2 \neq 0$ is considered during the procedure that leads us to the reduced action, one can arrive to the expected relation: $S_{m_1, m_2} = \int d^2x \left\{ \tau \dot{Q} - \frac{1}{2} p^2 + \frac{1}{2} Q (\Delta - M^2) \dot{Q} \right\}$, with $M \equiv \frac{m_2^2}{m_1}$, $P \equiv \sqrt{2} m_2 \Delta h^{(s)TL}$ and $Q \equiv \sqrt{2} m_2^3 \Delta h^{(s)TT}$, which is a singular function at $m_2 = 0$, saying that the model (46) does not have a well defined limit at $m_2 = 0$.

In an analogous way, a discontinuity does appear in the selfdual massive model when we consider a dS/AdS background, (35) at the critical value $M^2 = 0$. In fact, this behavior is manifested if we observe that equation (39) is now gauge invariant under

$$\delta h^{(s)Tt_\mu}_\mu = \left( \nabla_\mu \nabla_\nu - \frac{R}{6} g_{\mu\nu} \right) \omega (x)$$

which says that the only degree of freedom due to $h^{(s)Tt_\mu}_\mu$ can be gauged away, and the theory with $M^2 = 0$ does not propagate degrees of freedom as in the flat case.

### 4 The reduced action

If in the context of a curved space–time, we want to realize a procedure in order to obtain a reduced action for selfdual massive spin 2 theory, one could try to perform a description of the only propagated degree of freedom through a field like $h^{(s)Tt_\mu}_\mu$. In the flat space–time it can be seen that the symbol “±” is associated with a propagation of spin ±2 [28]. However, in a non-flat space–time this “flat” procedure will find serious obstacles. Essentially, this is related with the problem of the Fourier transform in curved spaces [29] and the definition of arbitrary powers of D’Alembertian and as a consequence of the obscure business to obtain projectors.

However, we can say something following a covariant procedure in order to obtain the reduced action, starting with the a symmetric-antisymmetric decomposition:

$$h_{\mu\nu} \equiv h^{(s)\mu\nu}_\mu + \epsilon_{\mu\nu\lambda\rho} V^\lambda$$

Using this in (35) conduce us to

$$S_{sd\lambda} = \frac{m^2}{2} \epsilon^{\mu\sigma\rho} g^{\beta\alpha} h^{(s)}_{\mu\nu} \nabla_\sigma h^{(s)}_{\sigma\alpha}$$

$$\quad - \frac{M^2}{2} (h^{(s)}_{\mu\nu} h^{(s)\mu\nu}_\mu - h^{(s)2})$$

$$\quad + m V^\mu (\nabla_\mu h^{(s)} - \nabla_\nu h^{(s)\nu})$$

$$\quad - \frac{m^2}{2} \epsilon^{\mu\nu\sigma} V^\mu \nabla_\nu V_\sigma - M^2 V^\mu V^\mu$$

(49)

with field equations

$$m \epsilon^{\mu\nu\lambda\rho} \nabla_\nu h^{(s)}_{\alpha} + m \epsilon^{\mu\alpha\beta} \nabla_\beta h^{(s)}_{\nu} - 2 M^2 h^{(s)2}_{\lambda\alpha}$$

$$+ 2 m^2 \epsilon^{\lambda\alpha} (h^{(s)}_{\mu\nu} - 2 m g^{\lambda\alpha} \nabla_\mu V^\nu$$

$$+ m (\nabla^\lambda V^\alpha + \nabla^\alpha V^\lambda) = 0$$

(50)

and

$$m \epsilon^{\mu\nu\lambda\rho} \nabla_\nu V_\rho + 2 M^2 V^\lambda - m \nabla^\lambda h^{(s)} + m \nabla_\mu h^{(s)\mu\lambda} = 0$$

(51)

The trace and divergence of (50) give

$$M^2 h^{(s)} - m \nabla^\mu V_\mu = 0$$

(52)

$$m \epsilon^{\mu\nu\lambda\rho} \nabla_\nu \mathcal{H}_\lambda - 2 M^2 \mathcal{H}^\mu + 2 M^2 \nabla^\mu h^{(s)} + m \Box V^\mu$$

(53)
\[-m \nabla^\mu \nabla_\mu V^\alpha - \frac{mR}{3} V^\mu = 0\] (53)

with the notation \(H_\lambda \equiv \nabla_\alpha h^{(s)}_\lambda^\alpha\). Using (53) in (51), we get

\[(Rm^2 - 6M^4)V_\sigma = 0\] (54)

Taking into account the restriction (37), we get \(V_\sigma = 0\). This last relation with (52) gives the supplementary \(h^{(s)} = 0\), and equation (51) assures \(H_\lambda \equiv \nabla_\alpha h^{(s)}_\lambda^\alpha = 0\). Then, it is confirmed that in constant curvature spaces the self-dual massive spin 2 theory is described by a symmetric-transverse-traceless field, \(h^{(s)}T^{\mu\nu}\), and the reduced action will take the form

\[S_{sd\xi}^{(2)*} = \frac{m}{2} \varepsilon^{\mu\sigma\rho}\varepsilon^{\sigma\tau\nu} h^{(s)}T^{\mu\rho} \nabla_{\nu} h^{(s)}T^{\tau\sigma}\]

\[-\frac{M^2}{2} h^{(s)}T^{\mu\nu} h^{(s)}T^{\mu\nu}\] (55)

which the equations of motion

\[m \varepsilon^{\mu\nu\sigma}\nabla_{\nu} h^{(s)}T_{\mu}^{\beta} - M^2 h^{(s)}T^{\sigma\beta} = 0\] (56)

From this, the causal propagation (45) is obtained.

At a point \(p\) of the manifold \(M\), it can be attached a tangent space \(T_p(M)\) with locally coordinates \(\xi^a\) provided. Therefore, in this reference, the hyperbolic-causal equation is

\[\left(\nabla_\xi - \frac{M^4}{m^2} + \frac{R}{2}\right) h^{(s)}T^{\mu\nu}(\xi) = 0\] (57)

with \(\nabla_\xi \equiv \partial_\lambda\partial_\lambda\). Next, we define the locally “+” and “−” parts of \(h^{(s)}T^{\mu\nu}(\xi)\) in the way

\[h^{(s)}T^{\mu\nu}_{+\pm} = \frac{1}{2} \left(\frac{\delta^d_a \delta^c_b}{q} \pm \delta^d_a \delta^c_b \frac{\partial_\xi}{\nabla_\xi}\right) h^{(s)}T^{\mu\nu}_{dc}\] (58)

where the parameter \(q \equiv \sqrt{1 - \frac{2Rm^2}{2M^4}}\). Then, with the local on-shell relation (57), it can be obtained that

\[\left(\nabla_\xi - \frac{M^4}{m^2} + \frac{R}{2}\right) h^{(s)}T^{\mu\nu}_{+\pm}(\xi) = 0\] (59)

\[h^{(s)}T^{\mu\nu}_{+\mp}(\xi) = 0\] (60)

saying that the only degree of freedom locally propagated is described through \(h^{(s)}T^{\mu\nu}_{+\pm}(h^{(s)}T^{\mu\nu}_{+\mp})\), if the spin is +2(−2). It can be observed that expression (58) can be rewritten as \(h^{(s)}T^{\mu\nu}_{+\pm} \equiv P_{+\pm}^{dc} h^{(s)}T^{\mu\nu}_{dc}\), where

\[P_{+\pm}^{dc} \equiv \frac{1}{4} \left(1 \pm \frac{1}{q} \left(\delta^d_a \delta^c_b + \delta^c_a \delta^d_b\right) \pm \left(\delta^d_a \delta^c_b + \delta^c_a \delta^d_b\right) \frac{\partial_\xi}{\nabla_\xi}\right)\] (61)

is not a projector (i.e., \(P_{+\pm}^{dc} P_{+\pm}^{ab} \neq P_{+\pm}^{dc} e_{df}\)).

5 The one-particle exchange amplitude

Finally, we examine the one-particle exchange amplitude that describes the interaction between sources. This starts with the selfdual massive spin 2 action in the form (49) but now minimally coupled with an external, symmetric and conserved source \(T^{(s)}_{\mu\nu}(x)\) (i.e., \(\nabla_\mu T^{(s)}_{\nu\rho}(x) = 0\)) as follows:

\[S_{sd\xi} = \left(\frac{m}{2} \varepsilon^{\mu\nu\sigma}\varepsilon^{\sigma\tau\nu} h^{(s)}_{\mu\rho} \nabla_{\nu} h^{(s)}_{\tau\rho}\right) - \frac{M^2}{2} \left(h^{(s)}_{\mu\rho} h^{(s)}_{\nu\rho} - h^{(s)}_{\mu\nu}\right)\]

\[+ m \varepsilon^{\mu\nu\sigma}\nabla_{\nu} h^{(s)}_{\lambda\rho} - m^2 h^{(s)}_{\mu\rho} V_\rho + m (\nabla^\lambda V^\rho + \nabla^\rho V^\lambda)\]

\[+ \kappa h^{(s)}_{\mu\rho} T^{(s)}_{\nu\rho}\] (62)

where \(\kappa\) is a coupling parameter.

The field equations are emerging from (62) are

\[m \varepsilon^{\mu\nu\lambda}\nabla_\nu h^{(s)}_{\lambda\rho} + m \varepsilon^{\mu\nu\alpha}\nabla_\mu h^{(s)}_{\lambda\rho} - 2M^2 h^{(s)}_{\mu\rho} = 0\]

\[+ 2M^2 \varepsilon^{\rho\sigma\lambda\mu} h^{(s)}_{\nu\rho} - 2m \nabla^\lambda V^\rho + m (\nabla^\lambda V^\rho + \nabla^\rho V^\lambda) = -2 \kappa T^{(s)}_{\lambda\rho}\] (63)

and

\[m \varepsilon^{\mu\nu\lambda}\nabla_\nu \varepsilon^{\lambda\rho \alpha} - m \nabla^\lambda V^\rho + m \nabla_\lambda V^\rho + m \nabla_\lambda V^\rho = 0\] (64)

Divergence and trace of (63) give

\[\varepsilon^{\lambda\rho \sigma\nu} \nabla_\nu h^{(s)}_{\lambda\rho} = - \frac{2M^2}{m} \nabla_\rho h^{(s)}_{\mu\rho} + \frac{2M^2}{m} \nabla^\mu h^{(s)}_{\mu\rho}\]

\[-2m \nabla^\lambda V^\rho + \nabla_\lambda \nabla^\rho V^\lambda + \nabla V^\rho = 0\] (65)

and

\[2M^2 h^{(s)}_{\mu\rho} - 2m \nabla_\rho V^\mu = - \kappa T^{(s)}_{\mu\rho}\] (66)

The curl of (64) is

\[-\varepsilon^{\mu\rho \sigma\nu} \nabla_\nu h^{(s)}_{\lambda\rho} - \nabla_\nu V_\sigma + \nabla_\lambda \nabla_\sigma V^\lambda\]

\[- \frac{2M^2}{m} \varepsilon^{\mu\rho \sigma\nu} \nabla_\nu V_\mu = 0\] (67)

and with the help of (65) conduces us to

\[(m^2 R - 6M^4)V_\sigma = 0\] (68)

which again says that \(V_\sigma = 0\). Then, we can rewrite (63), (64), and (66) as follows:
\[ m\varepsilon^{\mu\nu\lambda} \nabla_\mu h^{(s)}_{\nu} - m\varepsilon^{\mu\rho\sigma} \nabla_\mu h^{(s)}_{\nu} - 2M^2 h^{(s)\lambda\alpha} + 2M^2 g^{\alpha\lambda} h^{(s)}_{\nu} = -2\kappa T^{(s)\lambda\alpha} \]  
(69)

\[ \nabla_\lambda h^{(s)}_{\nu} - \nabla_\mu h^{(s)\mu\lambda} = 0 \]  
(70)

\[ 2M^2 h^{(s)}_{\nu} = -\kappa T^{(s)} \]  
(71)

For the computation of the exchange amplitude, we need the decomposition

\[ h^{(s)}_{\mu\nu} = h^{(s)T}_{\mu\nu} + \nabla_\mu a^T_{\nu} + \nabla_\nu a^T_{\mu} + \nabla_\mu \nabla_\nu \phi + g_{\mu\nu} \psi \]  
(72)

where \( \nabla_\mu a^T_{\nu} = 0 \). The following relations arise from (72)

\[ h^{(s)} = \Box \phi + 3\psi \]  
(73)

\[ \left(-\Box + \frac{R}{3}\right) a^T_{\mu} + \frac{2}{m} \nabla_\mu \phi + 2\nabla_\mu \psi = 0 \]  
(74)

where the last one is obtained with the help of (70). Divergence of (74) provides \( R\Box \phi + 6\Box \psi = 0 \), and using this in (73) with (71), we get

\[ \left(\Box - \frac{R}{2}\right) \psi = \frac{R\kappa}{12M^2} T^{(s)} \]  
(75)

Now, we need to write down \( h^{(s)T}_{\mu\nu} \) in terms of the source. The \( T^t \) part of (76) is

\[ m\varepsilon^{\mu\nu\lambda} \nabla_\mu h^{(s)T\lambda}_{\nu} - M^2 h^{(s)\mu\alpha} = -\kappa T^{(s)\mu\alpha} \]  
(76)

from which we obtain the hyperbolic-causal equation for \( h^{(s)T}_{\mu\nu} \)

\[ \left(\Delta^{(2)} + \frac{M^4}{m^2} + \frac{R}{2}\right) h^{(s)T\mu\nu} = -\frac{\kappa M^2}{m^2} T^{(s)T\mu\nu} + \frac{\kappa}{m} \varepsilon^{\mu\rho\sigma} \nabla_\rho T^{(s)T\sigma\alpha} \]  
(77)

where \( \Delta^{(2)} \) is the Lichnerowicz operator that obeys the following properties [30]:

\[ \Delta^{(0)} \phi = -\Box \phi \]  
(78)

\[ \nabla_\mu \Delta^{(1)} V_\mu = \Delta^{(0)} \nabla_\mu V_\mu \]  
(79)

\[ \Delta^{(2)} (\nabla_\mu V_\nu) = \nabla_\mu \Delta^{(1)} V_\nu \]  
(80)

\[ \nabla_\mu \Delta^{(2)} h^{(s)}_{\mu\nu} = \Delta^{(1)} \nabla_\mu h^{(s)}_{\mu\nu} \]  
(81)

\[ \Delta^{(2)} (g_{\mu\nu} \phi) = g_{\mu\nu} \Delta^{(0)} \phi \]  
(82)

and \( T^{(s)T\mu\nu} \) is given by

\[ T^{(s)T\mu\nu} = T^{(s)\mu\nu} - \frac{g_{\mu\nu}}{2} T^{(s)} \]  
(83)

The exchange amplitude between two covariant conserved sources is

\[ A = \int d^3x \sqrt{-g}A, \]  
where \( A \equiv T^{(s)\mu\nu} h^{(s)\mu\nu} \). Up to boundary terms, we can write \( A \) using

\[ A = T^{(s)\mu\nu} h^{(s)T\mu\nu} + T^{(s)} \psi \]  
(84)

Considering (75), (77) and (83), we finally obtain

\[ \frac{A}{\kappa} = -\frac{M^2}{m^2} T^{(s)\alpha\beta} (\Delta^{(2)} + \mu^2)^{-1} T^{(s)\alpha\beta} \]  

\[ -\frac{M^2}{2m^2} T^{(s)\alpha\beta} (\Box - \mu^2)^{-1} T^{(s)} \]  

\[ + 2 T^{(s)\alpha\beta} (\Delta^{(2)} + \mu^2)^{-1} \varepsilon^{\alpha\mu\sigma} \nabla_\mu T^{(s)T\beta\sigma} \]  

\[ - R \frac{\kappa}{12M^2} T^{(s)} \left(\Box - \frac{R}{2}\right)^{-1} T^{(s)} \]  

\[ + M^2 \frac{R}{12M^2} T^{(s)} \left(\Box - \mu^2\right)^{-1} \left(\Box - \frac{R}{2}\right)^{-1} T^{(s)} \]  
(85)

where \( \mu^2 \equiv \frac{M^4}{m^2} + \frac{R}{2} \).

If one takes a look at the first three terms in (85), two of them are proportional to \( \frac{M^2}{m^2} \) and the other to \( \frac{2}{m^2} \). Then, at the flat limit, they correspond to the amplitude of a massive selfdual massive spin 2 in a 2+1 dimensional Minkowski space–time.

The two remaining terms in (85) give a cosmological contribution, which disappears in the flat limit. In the non-flat case, it can be observed that these last terms have an unphysical pole at \( \Box = \frac{R}{2} \), which do not propagate whatever the sign of the cosmological constant (i.e., the residue in the amplitude is \( \frac{M^2 R}{12m^2} \left(-\frac{R}{2} + \mu^2\right)^{-1} - \frac{R}{12M^2} = 0 \)). On the other hand, the physical pole \( \Box = \mu^2 \) has the residue

\[ \mathcal{R}(\Box = \mu^2) = -\frac{M^2}{2m^2} \left(1 - \frac{\lambda M^2}{M^4}\right) \]  
(86)

which is clearly non null only in an AdS space–time.

### 6 Conclusion

It is confirmed that the well known fact that a consistent construction of the theory of a higher spin coupled with gravity only can occur in constant curvature space–times or non-Einstein ones. The adjustment of free coupling parameters is not sufficient in order to respect causality and the number of degrees of freedom. Therefore, severe restrictions must arise on the background.

However, the selfdual massive spin 2 model has a
distinctive property that makes this model remarkable. This means, there is non null mass limit. Then, in a constant curvature space–time, the selfdual massive spin 2 model exhibits forbidden mass values, obeying restrictions, which are given by $6M^4 - Rm^2 \neq 0$ and $M^2 \neq 0$ with $M^2 = m^2 + \sigma R$. Both contain information about the background, and they match in the flat space–time limit with the known consistence condition of the model: $m \neq 0$. Parameter $M^2$ appears as a quadratical power of a “mass” in the action and this can be thought as the non-flat version of the two parameters flat action, which contains the selfdual massive spin 2 model as a special case. Moreover, condition $M^2 \neq 0$ guarantees a conformally variant selfdual massive model in dS/AdS, matching with a similar situation in flat theory. However, one can distinguish between the models defined in a dS or AdS because the residue $R (\Box = \mu^2)$ in the one-particle exchange amplitude is not sign defined when $\lambda > 0$ (i.e., dS space).

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