POSITIVE SOLUTIONS OF DOUBLY COUPLED MULTICOMPONENT NONLINEAR SCHRÖDINGER SYSTEMS

JIABAO SU AND RUSHUN TIAN*
School of Mathematical Science, Capital Normal University
Beijing 100048, China

ZHIXIANG WANG*
Center for Applied Mathematics, Tianjin University
Tianjin 300072, China
and
Department of Mathematics and Statistics, Utah State University
Logan, UT 84322, USA

Abstract. In this paper, we study the following doubly coupled multicomponent system

\[
\begin{aligned}
-\Delta u_j + \lambda_j u_j + \sum_{k \neq j} \gamma_{jk} u_k &= \mu_j u_j^3 + u_j \sum_{k \neq j} \beta_{jk} u_k^2, \\
u_j(x) &\geq 0 \text{ and } u_j \in H_0^1(\Omega),
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N \) and \( N = 2, 3; \lambda_j, \gamma_{jk} = \gamma_{kj}, \mu_j, \beta_{jk} = \beta_{kj} \) are constants, \( j, k = 1, 2, \ldots, n \geq 2 \). We prove some existence and nonexistence results for positive solutions of this system. If the system is fully symmetric, i.e. \( \lambda_j \equiv \lambda, \gamma_{jk} \equiv \gamma, \mu_j \equiv \mu, \beta_{jk} \equiv \beta \), we study the multiplicity and bifurcation phenomena of positive solution.

1. Introduction. Consider the following doubly coupled multicomponent nonlinear Schrödinger system

\[
\begin{aligned}
-\Delta u_j + \lambda_j u_j + \sum_{k \neq j} \gamma_{jk} u_k &= \mu_j u_j^3 + u_j \sum_{k \neq j} \beta_{jk} u_k^2, \\
u_j &\in H_0^1(\Omega),
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N \) is a smooth domain, \( N = 2, 3; \lambda_j > 0, \gamma_{jk} = \gamma_{kj}, \mu_j > 0, \beta_{jk} = \beta_{kj} \) are constants, \( j, k = 1, 2, \ldots, n \) and \( n \geq 2 \).

System (1) describes the standing wave solutions of doubly coupled Schrödinger equations, where linear and nonlinear coupling terms are both in presence. Due to the important applications in physics, particularly in the study of nonlinear optics and Bose-Einstein condensates, Schrödinger systems have been studied extensively in recent years. We refer to [12, 21, 22] for more background in physics, and [1]-[10], [14]-[19], [23]-[27], [29] and references therein for mathematical studies. Most of these papers dealt with nonlinear coupling terms only. But the linearly couplings also carry important physical information, see [11] for more interpretation in this case.

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* Corresponding author: Rushun Tian and Zhi-Qiang Wang.
respect. Roughly speaking, the linearly coupling coefficients affect the eigenvalues of the linear operator corresponding to the linearized system of \((1)\), and bring a different kind of influence on the existence, multiplicity and bifurcations of positive solutions to system \((1)\).

We define some notations first, then present our main results.

1.1. Notations. In this paper, we consider attractive self-interactions, i.e., all \(\mu_j\) are positive. \(\Omega \subset \mathbb{R}^N\) is a smooth bounded domain or \(\mathbb{R}^N\). Denote by \(\mathcal{H}\) the product space \([H^1(\Omega)]^n\) or \([H^1(\mathbb{R}^N)]^n\), and \(u = (u_1, ..., u_n) \in \mathcal{H}\). We classify solution as follows:

- **nontrivial solution**: if it has at least one nonzero component;
- **nonnegative solution**: if all components are nonnegative and \(u \neq 0\);
- **positive solution**: if all components are positive.

Denote by \(J : \mathcal{H} \to \mathbb{R}\) the energy functional corresponding to \((1)\),
\[
J(u) = \int \sum_{j=1}^n \left[ \frac{1}{2} \langle \nabla u_j, \nabla u_j \rangle + \frac{1}{4} \mu_j u_j^4 \right] + \sum_{j < k} \gamma_{jk} u_j u_k - \frac{1}{2} \beta_{jk} u_j^2 u_k^2 \quad (2)
\]

By Sobolev embeddings, \(J\) is a \(C^2\) functional, whose critical points are weak solutions of \((1)\). Two Nehari type manifolds associated to \(J\) are considered in this paper: the one-constraint Nehari manifold
\[
\mathcal{N}_1 = \{ u \neq 0 : I(u) := \langle J'(u), u \rangle = 0 \}
\]
and the \(n\)-constraint Nehari manifold
\[
\mathcal{N}_n = \{ u \neq 0 : \langle J'(u), u_j \rangle = 0, u_j \neq 0, j = 1, 2, ..., n \}
\]
where \(u_j\) has only one nonzero component located at the \(j\)-th position. Clearly, \(\mathcal{N}_1\) contains all nontrivial solutions of \((1)\). We call a nontrivial solution a ground state solution, if it achieves the minimum value of \(J\) on \(\mathcal{N}_1\).

Let \(\lambda_* = \min \{ \lambda_j \}, \lambda^* = \max \{ \lambda_j \}\) and \(\gamma^* = \max \{ |\gamma_{jk}| \}\). If \(0 \leq \gamma^* < \frac{\lambda_0}{n-1}\), we can define a norm on \(\mathcal{H}\)
\[
\| u \|_{\mathcal{H}} := \left( \sum_{j=1}^n \| \nabla u_j \|^2 + \frac{1}{2} \sum_{1 \leq j < k \leq n} \gamma_{jk} \int u_j u_k \right)^{\frac{1}{2}}.
\]

Direct calculation shows
\[
\| u \|_{\mathcal{H}}^2 \geq \int \left( \sum_{j=1}^n \| \nabla u_j \|^2 + [\lambda_j - (n-1)\gamma^*] u_j^2 \right) + \gamma^* \int \sum_{1 \leq j < k \leq n} (|u_j| - |u_k|)^2 \geq \min\{1, \lambda_* - (n-1)\gamma^*\} \sum_{j=1}^n \| u_j \|_{H^1(\mathbb{R}^N)}^2.
\]

On the other hand,
\[
\| u \|_{\mathcal{H}}^2 \leq \max\{1, \lambda_* + (n-1)\gamma^*\} \sum_{j=1}^n \| u_j \|_{H^1(\mathbb{R}^N)}^2,
\]
thus \(\| \cdot \|_{\mathcal{H}}\) is equivalent to the standard product norm on \(\mathcal{H}\). Since
\[
\langle I'(u), u \rangle = 2\| u \|_{\mathcal{H}}^2 - 4\| u \|_{\mathcal{H}}^2 = -2\| u \|_{\mathcal{H}}^2 < 0, \text{ for any } u \in \mathcal{N}_1,
\]
\(\mathcal{N}_1\) is smooth and is of co-dimension one.
1.2. Main results. First, we have some nonexistence results of nonnegative solutions to (1).

**Theorem 1.1.** Assume $\beta_{jk} = \beta$ for $1 \leq i, j \leq n$. System (1) has no nontrivial nonnegative solutions in the following cases:

(i) $\Omega = \mathbb{R}^N$ and $\beta \geq 0$. If $\gamma_{jk} < 0$ for any $1 \leq j \neq k \leq n$, and 
$$\lambda^* + \sum_{j \neq k}^n \gamma_{jk} \leq 0 \text{ for } k = 1, \ldots, n.$$ 
In particular, it is the case if $\gamma_{jk} \leq \frac{\lambda^*}{n-1}$ for $1 \leq j \neq k \leq n$.

(ii) $\Omega = \mathbb{R}_+^N$ and $\beta \geq 0$. If $\gamma_{jk} < 0$ for any $1 \leq j \neq k \leq n$, and 
$$\lambda^* + \sum_{j \neq k}^n \gamma_{jk} < 0 \text{ for } k = 1, \ldots, n.$$ 
In particular, it is the case if $\gamma_{jk} < -\frac{\lambda^*}{n-1}$ for $1 \leq j \neq k \leq n$.

**Remark 1.** Note that semi-trivial nonnegative solutions are also ruled out in Theorem 1.1, where the assumption $\gamma_{jk} < 0$ plays an important role. If we replace this condition by $\gamma_{jk} \leq 0$, then there might be semi-trivial nonnegative solution. As a simple example, it is easy to see that a three-component system with $-\lambda^* < \gamma_{12} < 0$ and $\gamma_{13} = \gamma_{23} = 0$ has semi-positive solution in the form $(u, v, 0)$ and $(0, 0, w)$.

**Remark 2.** System (1) possesses a coefficient-component symmetry, which means if $u = (u_1, u_2, \ldots, u_n)$ is solution corresponding to a set of linear coupling coefficients $\{\gamma_{jk}\}_{1 \leq j < k \leq n}$, then $\tilde{u} \in \mathcal{H}$, which only differs from $u$ by changing the sign of the $j$-th component ($1 \leq j \leq n$), is also a solution of (1) with all related coefficients $\gamma_{jk}$ replaced by $-\gamma_{jk}$, $k \neq j$. For instance, in the case $n = 2$, if $(\gamma, u, v)$ is a solution, then $(-\gamma, -u, v)$ and $(-\gamma, u, -v)$ are also solutions. Therefore, Theorem 1.1 has $2^{n-1}$ equivalent versions.

The next theorem concerns the existence of positive solutions of (1). Similar problem has been obtained in [14, 27] for $n = 2$. Here we shall extend such results to multicomponent cases. If $\Omega = \mathbb{R}^N$ and the following symmetric conditions hold

$$\lambda_j \equiv \lambda, \gamma_{jk} \equiv \gamma, \mu_j \equiv \mu, \beta_{jk} \equiv \beta, \quad j, k = 1, \ldots, n,$$  
(3)

the existence of positive solutions can be verified for a larger range of coupling parameters. Indeed, there exists a synchronized solution branch whose components are positive constant multiples of the non-degenerate positive radial solution of

$$-\Delta w + w = w^3, \quad w > 0 \text{ in } \mathbb{R}^N.$$  
(4)

Precisely, we have

**Theorem 1.2.** (i) Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain or $\mathbb{R}^N$. If $\gamma_{jk} \in (-\frac{\lambda^*}{n-1}, 0)$ for all $1 \leq j \neq k \leq n$, then system (1) has a positive ground state solution for any $\beta_{jk} = \beta_{kj} \in \mathbb{R}$.

(ii) Let $\Omega = \mathbb{R}^N$. If (3) holds, then system (1) has a synchronized solution surface parametrized by $(\beta, \gamma)$,

$$T_\omega = \left\{ (\beta, \gamma, Aw, \ldots, Aw) : \beta > \frac{\mu}{n-1}, \gamma > -\frac{\lambda}{n-1} \right\} \subset \mathbb{R}^2 \times \mathcal{H}$$  
(5)

where

$$A = \sqrt{\frac{\lambda + (n-1)\gamma}{\mu + (n-1)\beta}}, \quad w_\gamma(x) = w(\sqrt{\lambda + (n-1)\gamma}x),$$

and $w$ is the unique positive radial solution of (4).
Thirdly, we consider the multiplicity of positive solutions when condition (3) holds. In this case, system (1) is invariant under the action of \( \mathbb{Z}_n \), the cyclic group generated by \( \sigma : \mathcal{H} \rightarrow \mathcal{H}, \)

\[
\sigma(u_1, u_2, \ldots, u_{n-1}, u_n) = (u_n, u_1, \ldots, u_{n-1}),
\]

i.e. \( \mathbb{Z}_n = \{ id, \sigma, \sigma^2, \ldots, \sigma^{n-1} \} \). The \( \mathbb{Z}_n \)-index was first introduced by Wang in [28] and then refined by Tian-Wang in [24] for studying the multiplicity of positive solutions to system (1) with \( \gamma = 0 \). When \( \gamma \neq 0 \) and \( n = 2 \), the multiplicity of (1) has been studied by Li-Zhang in [14] using \( \mathbb{Z}_2 \) symmetry, where the arguments are adopted from [10] by Dancer-Wei-Weth. Our next result generalizes [24] by bringing linear coupling terms into consideration, and also extends the multiplicity results in [14] for \( n > 2 \).

Let \( u_1, u_2 \in \mathcal{H} \). If there exists an integer \( j \) such that \( u_2 = \sigma^j u_1 \), then we say \( u_1 \) and \( u_2 \) belong to the same \( \mathbb{Z}_n \)-orbit.

**Theorem 1.3.** Let \( \Omega \subset \mathbb{R}^N \) be a smooth bounded domain or \( \mathbb{R}^N \). Assume (3) hold. For any \( \gamma \in (-\frac{1}{n-1}, 0) \),

(a) If \( \beta \leq -\frac{\mu}{n-1} \), then system (1) has an infinite sequence of \( \mathbb{Z}_n \)-orbits of solutions.

(b) For any positive integer \( m \), there exists a \( \beta_m \in (-\frac{\mu}{n-1}, 0) \), such that for \( \beta \in (-\frac{\mu}{n-1}, \beta_m) \), system (1) has at least \( m \) \( \mathbb{Z}_n \)-orbits of solutions.

At last, assuming symmetric condition (3), we consider the bifurcations of positive solutions of system (1). Some related work can be found in [4] and [6], where the nonlinearly coupled case, i.e. \( \gamma = 0 \), were studied. In both papers, sequences of \( \beta_j \)'s were found and proved to be bifurcation parameters, where the attractive self-interaction was studied in [4], and the repulsive and mixed self-interactions were considered in [6]. More recently, when both linear and nonlinear couplings are in presence, [27, 9] studied the bifurcation phenomena of (1) for \( n = 2 \).

Without loss of generality, assume \( \lambda = \mu = 1 \). Let \( \omega \) be the unique positive radial solution of (4) and \( T_\omega \) be given by (5). Since there are two coupling parameters, we consider two types of bifurcations:

**\( \beta \)-bifurcation:** bifurcation of positive solutions with respect to \( T_\omega|_\gamma \subset \mathbb{R} \times \mathcal{H} \) for \( \gamma \in (-\frac{1}{n-1}, 0) \) fixed. If \( \beta_\ast \) is a bifurcation parameter, then the corresponding bifurcation point is denoted by \( (\beta_\ast, u_{\beta_\ast}) \).

**\( \gamma \)-bifurcation:** bifurcation of positive solutions with respect to \( T_\omega|_\beta \subset \mathbb{R} \times \mathcal{H} \) for \( \beta \in (-\frac{1}{n-1}, \frac{3}{n-1}) \) fixed. If \( \gamma_\ast \) is a bifurcation parameter, then the corresponding bifurcation point is denoted by \( (\gamma_\ast, u_{\gamma_\ast}) \).

To discuss \( \gamma \)-bifurcations, a dilation transformation will be used, as it is suggested by the expression of \( T_\omega \). Therefore we only consider this type of bifurcation on \( \Omega = \mathbb{R}^N \). The \( \gamma \)-bifurcation on bounded domains is also an interesting problem, but new technique seems to be required to overcome the difficulty of losing scaling invariance. In contrast, \( \beta \)-bifurcation have been studied on both bounded and unbounded (radial) domains. We refer to [4, 5, 25, 26] for more details.

Firstly, we have the following results about \( \beta \)-bifurcations:

**Theorem 1.4.** Assume (3) holds and \( \lambda = \mu = 1 \).

(i) For \( \gamma \in (-\frac{1}{n-1}, 0) \) fixed, there is a sequence of \( \beta \)-bifurcation parameters \( \{ \beta_k \} \) with respect to \( T_\omega|_\gamma \). Moreover, \( \beta_k \downarrow -\frac{1}{n-1} \) as \( k \rightarrow \infty \).
(ii) For each integer $k \geq 2$ there exists a connected set $S_k \subset \mathbb{R} \times \mathcal{H}$ of positive solution of (1), such that $S_k \cap \mathcal{T}_\omega|_\gamma = \{ (\beta_k, \gamma, u_k) \}$.

Secondly, about $\gamma$-bifurcations, we have

**Theorem 1.5.** Assume (3) holds and $\lambda = \mu = 1$. For each fixed $\beta > -\frac{1}{n-1}$ satisfying $\frac{3+(n-3)\beta}{1+(n-1)\beta} \geq \lambda^1$, there are finitely many bifurcations with respect to $\mathcal{T}_\omega|_\beta$, where $\lambda^1$ is the principal eigenvalue of

$$-\Delta \phi + \phi = \lambda \nu^2 \phi, \quad \phi \in H^1(\mathbb{R}^N).$$

At last, we discuss the two-parameter bifurcations with respect to $\mathcal{T}_\omega$. A few notations are needed. Let $\Gamma_{\gamma_0} = \mathcal{T}_\omega|_{\gamma_0} \cap \mathcal{O}$, where $\gamma_0 \in (-\frac{1}{n-1}, 0)$ is arbitrary.

**Theorem 1.6.** Assume (3) holds and $\lambda = \mu = 1$. Let $\mathcal{O} = [(-\frac{1}{n-1}, 0)]^2 \times \mathcal{H}_r$. Then there exists a sequence of bifurcation surfaces $C_k \subset \mathcal{O}$ and $C_k \cap \Gamma_{\gamma_0} \neq \emptyset$, whose dimension at each point is at least 2. Moreover, each $C_k$ satisfies one of the following properties:

(i) $C_k$ is unbounded in $\mathcal{O}$,

(ii) $C_k \cap \partial \mathcal{O} \neq \emptyset$,

(iii) $C_k$ and $\Gamma_{\gamma_0}$ have more than one interactions.

**Remark 3.** Let $\mathcal{P} = \{ P_1, \ldots, P_m \}$ be a partition of $\{ 1, \ldots, n \}$, i.e.

$$\bigcup_{i=1}^m P_m = \{ 1, \ldots, n \} \text{ and } P_i \cap P_j = \emptyset \text{ if } i \neq j.$$

We call $u$ a partially synchronized solution subjected to $\mathcal{P}$, if $u_j(x)/u_k(x)$ is a constant as long as $j, k \in P_k$ for some $1 \leq k \leq m$. One can also obtain bifurcation results for partially synchronized solutions using the arguments in [4].

The paper is organized as follows. In Section 2, we establish the existence and nonexistence results using variational methods. In Section 3, we use $Z_n$-index theory to find multiple positive solutions and give proof of Theorem 1.3. In Section 4, we find and verify the $\gamma$-bifurcations of positive solutions to (1) in linear coupling parameter $\gamma$. Some calculations are contained in the Appendix.

2. **Nonexistence and existence of nonnegative solution.** In this section, we apply a Liouville type theorem [10, Theorem 2.3] to prove Theorem 1.1. Then we use variational methods and some elementary calculations to establish Theorem 1.2. Similar problem has been studied in [27, 14] for $n = 2$.

**Proof of Theorem 1.1.** Let $u = (u_1, \ldots, u_n)$ be a solution of (1) whose components have the same sign. Without loss of generality, assume $u$ is nonnegative. Moreover, by standard regularity argument, $u \in [C^2(\mathbb{R}^N)]^n$.

For (i), add the equations of (1) together and let $U = \sum_{k=1}^n u_k$, then

$$-\Delta U \geq -\Delta U + \left( \lambda^* + \sum_{j \neq k}^n \gamma_{jk} \right) U \geq \sum_{j=1}^n \mu_j u_j^3 \geq C U^3 \quad (6)$$

where $C$ is independent of $U$. Now applying part (i) of [10, Theorem 2.3], we can see $U \equiv 0$.

For (ii), since $\lambda^* + \sum_{j \neq k}^n \gamma_{jk} < 0$, $1 \leq k \leq n$, there exists a positive constant $\delta > 0$ such that $-\Delta U \geq \delta U$. By part (ii) of [10, Theorem 2.3], we obtain $U \equiv 0$. □
In order to prove Theorem 1.2, we need a few estimates on functional $J$. For any $u \in \mathcal{N}_1$, \[\|u\|_{\mathcal{H}}^2 = \sum_{j=1}^n \int \mu_j u_j^4 + \sum_{j \neq k} \int \beta_{jk} u_j^2 u_k^2 \leq C(\lambda_j, \gamma_{jk}, \mu_j, \beta_{jk}) \|u\|_{\mathcal{H}}^4\] therefore there exists a constant $C_0 > 0$, which only depends on $\lambda_j, \gamma_{jk}, \mu_j, \beta_{jk}$ such that \[\|u\| \geq C_0, \quad \text{for all } u \in \mathcal{N}_1.\] On the other hand, \[J|_{\mathcal{N}_1}(u) = \frac{1}{4} \|u\|_{\mathcal{H}}^2,\] thus $J$ is coercive on $\mathcal{N}_1$ and \[\inf_{u \in \mathcal{N}_1} J \geq \frac{C_0}{4} > 0.\] The following lemma indicates an important influence exerted by the linear coupling terms, which helps us to rule out nonnegative semi-trivial solutions.

**Lemma 2.1.** Assume $\gamma_{jk} \in (\frac{\lambda_j}{\alpha}, 0)$. Let $\{u_m\} \subset \mathcal{N}_1$ be a positive PS sequence of $J$, i.e., \[J'(u_m) \to 0, \quad J(u_m) \to c, \quad u_m \in \mathcal{N}_1, \quad (u_m)_j \geq 0, \quad j = 1, 2, \ldots, n.\] Then a subsequence of $\{u_m\}$ converges to a positive solution $u$ of (1).

**Proof.** Clearly, $c \geq C_0$. Since $J$ is coercive on $\mathcal{N}_1$, $\{u_m\}$ is bounded in $\mathcal{H}$. Therefore, there exists $u \in \mathcal{H}$ and a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$ for simplicity) such that \[u_m \rightharpoonup u \quad \text{in } \mathcal{H}, \quad \text{as } m \to \infty.\] If $\Omega \subset \mathbb{R}^N$ is bounded, by compact Sobolev embedding $u$ is a nonnegative solution of (1).

If $\Omega$ is unbounded, by applying the Concentration Compactness Principle as in [14], one can find a new PS sequence with positive components on $\mathcal{N}_1$ which strongly converges to $u$. Precisely, we claim: there exists at least one component $(u_m)_j$ such that \[\lim_{m \to \infty} \sup_{x \in \Omega} \int_{B_1(x)} (u_m)_j^2(y)dy = \alpha > 0\] Otherwise, we have $u_m \to 0$ as $m \to \infty$ in $[L^4(\Omega)]^n$. Then \[\|u_m\|_{\mathcal{H}}^2 = \sum_{j=1}^n \int (u_m)_j^4 + \sum_{j \neq k} \int_{\Omega} (u_m)_j^2(u_m)_k^2 \to 0 \quad \text{as } m \to \infty,\] which contradicts with $\inf_{\mathcal{N}_1} J > 0$. Assume the claim holds for the first component, then up to a subsequence, there exist $\{x_m\} \subset \Omega$ such that \[\int_{B_1(x_m)} (u_m)_1^2(y)dy > \frac{\alpha}{2} \quad \text{for any } m \text{ large.} \tag{7}\] Let $v_m(\cdot) = u_m(\cdot - x_m)$. It is easy to check that $\{v_m\}$ is also a positive PS sequence of $J$. Let $u$ be the weak limit of $\{v_m\}$ in $\mathcal{H}$, then $v_m \to u$ in $[L^2_{\text{loc}}(\Omega)]^n$. Note that (7) implies $\int_{B_1(0)} (v_m)_1^2(y)dy \geq \frac{\alpha}{2}$, therefore $u \neq 0$. On the other hand, \[\|J'(v_m)|_{\mathcal{H}} = \|J'(u)|_{\mathcal{H}}, \] thus $J'(u) = 0$, i.e. $u_0$ is a nontrivial nonnegative critical point of $J$. 


If $u_0$ has zero component, for example $u_n = 0$, then the last equation of (1) gives
$$\gamma_{1n}u_1 + \gamma_{2n}u_2 + \ldots + \gamma_{n-1,n}u_{n-1} = 0$$
therefore $u_j = 0, j = 1, \ldots, n$. But, on the other hand, the lower semi-continuity of norm implies
$$\|u\|_\mathcal{H}^2 \geq 4C_0 > 0$$
A contradiction. Thus $u$ is a positive solution of (1). Then $u \in \mathcal{N}$ and has at least one nontrivial component. By the same argument as above, we deduce that $u$ is a positive solution of (1).

Lemma 2.2. If $\{u_m\}$ is a PS sequence of $J|_{\mathcal{N}}$, then it is also a PS sequence of $J$.

Proof. By assumption, $J|_{\mathcal{N}}(u_m) \to J, J|_{\mathcal{N}}(u_m) \to c$ for some $c \geq C_0$ as $m \to \infty$. By
$$J(u_m) \geq \inf_{u \in \mathcal{N}} J(u) > 0$$
$\{u_m\}$ is bounded away from zero. There exists a Lagrange multiplier $l_m$ for each $m$ such that
$$J'(u_m) = J'(u_m) - l_m I'(u_m)$$
Apply both sides to $u_m$ and use the definition of $\mathcal{N}$, we find
$$o(1) = \langle J'(u_m), u_m \rangle = -l_m \langle I'(u_m), u_m \rangle = 2l_m \|u_m\|_{\mathcal{H}}^2$$
Since $\|u_m\|_{\mathcal{H}}$ has a positive lower bound, $l_m \to 0$ as $m \to \infty$. The coerciveness of $J|_{\mathcal{N}}$ implies the boundedness of $\|u_m\|_{\mathcal{H}}$, which further implies the boundedness of $I'(u_m)$. Thus $\{u_m\}$ is also a PS sequence of $J$.

Proof of Theorem 1.2. For part (i), we find a positive ground state solution to (1). There exists a minimizing sequence $\{u_m\}$ for
$$\inf_{u \in \mathcal{N}} J(u) > 0.$$ 
By Ekeland’s variational principle, $\{u_m\}$ is a PS sequence of $J|_{\mathcal{N}_1}$. Since $\gamma < 0$, there holds
$$\gamma_{jk} \int_{\mathbb{R}^N} u_j u_k \geq \gamma_{jk} \int_{\mathbb{R}^N} |u_j| |u_k| \quad \text{and} \quad \|u\|_{\mathcal{H}} \leq \|u\|_{\mathcal{H}}.$$ 
For any fixed $u \in \mathcal{N}_1$, let $t_0 > 0$ be a constant such that $t_0 |u| \in \mathcal{N}_1$. Then
$$0 = \langle J'(t_0 |u|), t_0 |u| \rangle = t_0^2 \|u\|_{\mathcal{H}}^2 - t_0^4 \|u\|_{\mathcal{H}}^2 = t_0^2 (\|u\|_{\mathcal{H}}^2 - t_0^2 \|u\|_{\mathcal{H}}^2),$$
which implies $t_0 = \|u\|_{\mathcal{H}} / \|u\|_{\mathcal{H}} \leq 1$ and
$$J(t_0 |u|) = t_0^2 \|u\|_{\mathcal{H}}^2 - \frac{t_0^4}{4} \|u\|_{\mathcal{H}}^2 = \frac{t_0^2}{4} \|u\|_{\mathcal{H}}^2 \leq \frac{1}{4} \|u\|_{\mathcal{H}}^2 = J(u).$$
Therefore we may assume that all components of $u_m$ are positive, i.e. $J|_{\mathcal{N}_1}$ has a positive PS sequence. By Lemma 2.2, $\{u_m\}$ is also a positive sequence of $J$. At last, apply Lemma 2.1, $\{u_m\}$ converges to a positive solution $u$ of system (1).

(ii) Assume $\Omega = \mathbb{R}^N$. Let $w$ be the unique least energy positive solution of (4). In order to obtain positive solutions, we set $u_j(x) = A_j w(tx)$, where $t > 0, A_1 > 0, \ldots, A_n > 0$ are unknown parameters to be determined, then (1) becomes
$$t^2 A_j (-\Delta w) + \left( \lambda A_j + \gamma \sum_{k \neq j} A_k \right) w = \left( \mu A_j^3 + \beta A_j \sum_{k \neq j} A_k^2 \right) w^3.$$
or equivalently,
\[-\Delta w + \frac{1}{t^2} \left( \lambda + \frac{\gamma}{A_j} \sum_{k \neq j} A_k \right) w = \frac{1}{t^2} \left( \mu A_j^2 + \beta \sum_{k \neq j} A_k^2 \right) w^3.\]

for \(j = 1, 2, ..., n\). According to the uniqueness of \(w\) that solves scalar equation (4), a system of equations about \(A_j, t, \beta\) and \(\gamma\) can be derived. From the coefficient of \(w\), we get
\[
\begin{cases}
\gamma = 0 \text{ then } t = 1, \\
\gamma > -\frac{\lambda}{n-1} \text{ and } \sum_{k=1}^{n} A_k = 0, \\
A_k = A_l, \forall 1 \leq j < k \leq n
\end{cases}
\] (8)

From the coefficient of \(w^3\), we get
\[
\begin{cases}
\beta = \mu \text{ then } t^2 = \sum_{k=1}^{n} A_k^2, \\
\beta > -\frac{\mu}{n-1} \text{ and } A_k^2 = A_l^2, \forall 1 \leq j < k \leq n
\end{cases}
\] (9)

Set \(A_j \equiv A\) for \(j = 1, ..., n\), then \(A\) and \(t\) satisfy
\[
\lambda + (n-1)\gamma = t^2, \quad [\mu + (n-1)\beta]A^2 = t^2.
\]

for any fixed \(\gamma > -\frac{\lambda}{n-1}\) and \(\beta > -\frac{\mu}{n-1}\). Solve the above equations for \(A\) and \(t\),
\[
A = \sqrt[4]{\lambda + (n-1)\gamma}, \quad t = \sqrt[4]{\mu + (n-1)\beta},
\] (10)

then we obtain the synchronized solution branch \(T_\omega\) as defined in (5).

\[\square\]

Remark 4. If \(\Omega = \mathbb{R}^N\), then the existence of positive solution claimed by part (i) of Theorem 1.2 is easier to be obtained by applying compact embedding between radial spaces.

Remark 5. It is easy to see that other synchronized solutions besides \(T_\omega\) can be derived from equations (8) and (9). Two of the special cases are given below for example:

(a) Set \(\gamma = 0\) and \(\beta = \mu\) in (8), (9) respectively. In this case, system (1) has infinitely many solution of the form \((A_1 w, A_2 w, ..., A_n w)\), as long as the constant coefficients satisfying \(\sum_{k=1}^{n} A_k^2 = 1\). See also [7] for the case \(n = 2\).

(b) For any \(\gamma > -\frac{\lambda}{n-1}\) and \(\beta > -\frac{\mu}{n-1}\), if \(\sum_{k=1}^{n} A_k = 0\) and \(A_k^2 = A_l^2, \forall 1 \leq j < k \leq n\), then we must have \(n = 2m\) for some \(m \in \mathbb{N}\), and the deduced synchronized solution must have \(m\) positive components and \(m\) negative components.

In this paper, we are interested in positive solutions, thus other types of synchronized solution branches will not be studied further. But it is worth to point out that the bifurcation results can also be extended to these types of solutions.
3. Multiplicity of positive solutions. In this section, we study the multiplicity of positive solutions to (1) under symmetric assumption (3). The proof of Theorem 1.3 is similar to [24] and we shall be sketchy here. First, in order to get positive solution of (1), we consider the nontrivial solution of the following modified system

\[
\begin{align*}
-\Delta u_j + \lambda u_j + \gamma \sum_{k \neq j} u_k &= \mu(u_j^+)^3 + \beta u_j \sum_{k \neq j} u_k^2, \\
&\quad u_j \in H_0^1(\Omega), \quad j = 1, 2, \ldots, n.
\end{align*}
\]

Lemma 3.1. Let \( \gamma \in (-\frac{1}{n-1}, 0) \) and \( \beta < 0 \). Any nontrivial solution of (11) is a positive solution of (1).

Proof. Multiply the \( j \)-th equation of system (11) by \( u_j^- \) and integrate over \( \Omega \), then add the equations together,

\[
\| (u_1^-, \ldots, u_n^-) \|^2_{H, \gamma} \leq \sum_{j=1}^n \left( |\nabla u_j^-|^2 + \lambda |u_j^-|^2 + \gamma \int_{\Omega} u_j^- \sum_{k \neq j} u_k \right) = \beta \int_{\Omega} \sum_{j=1}^n (u_j^-)^2 \sum_{k \neq j} u_k^2 \leq 0
\]

Thus \( (u_1^-, \ldots, u_n^-) = (0, \ldots, 0) \), i.e. \( u \) is a positive solution of (1).

Let \( N = p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s} \) be the prime factorization of \( N \), where \( 1 < p_1 < \cdots < p_s \) are prime numbers and \( t_1, \cdots, t_s \) are positive integers. Let \( 1 = q_0 < q_1 < q_2 < \cdots < q_a < n \), for some integer \( a \geq 0 \)

be all the distinct factors of \( n \). Correspondingly, define \( n = n_0 > n_1 > n_2 > \cdots > n_a > 1 \) by \( n_k = n/q_k \) for \( 0 \leq b \leq a \). In this section, we use the codimension \( n \) manifold \( N_n \). Define the least energy of \( J \) on the sets of fixed points of \( \sigma^{q_b} \),

\[
c^{q_b}(\beta, \gamma) := \inf \{ J(u) \mid u \in N_n, \sigma^{q_b}(u) = u \} \quad b = 0, \cdots, a
\]

and \( c^{q_b}(\beta, \gamma) = \infty \) if \( \sigma^{q_b} \) has no fixed point on \( N_n \).

Lemma 3.2. Assume that \( b \) is an integer satisfying \( 0 \leq b \leq a \), then \( c^{q_b}(\beta, \gamma) = \infty \) if \( \beta \leq -\frac{\mu}{n_b-1} \), and \( \lim_{\beta \to -\frac{\mu}{n_b-1}} c^{q_b}(\beta, \gamma) = \infty \).

Proof. If \( \beta \leq -\frac{\mu}{n_b-1} \) and \( \sigma^{q_b}(u) = u \), i.e. \( u = (u_1, \cdots, u_{q_b}, \cdots, u_1, \cdots, u_{q_b}) \), system (11) reduces to \( q_b \) equations. Multiply the \( j \)-th equation by \( u_j \) and integrate over \( \Omega \),

\[
|\nabla u_j|^2 + \lambda |u_j|^2 + \gamma \int (n_b - 1)u_j^2 + n_b u_j \sum_{k \neq j} u_k
\]

\[
= \mu|u_j^+|_4^4 + \beta(n_b - 1)|u_j|^4 + \beta n_b \int \sum_{k \neq j} u_k^2 u_j^2
\]
Then add these \( q_b \) equalities together, we find
\[
\text{L.H.S} = \int \sum_{j=1}^{q_b} \left[ (\nabla u_j)^2 + \left[ \lambda + \gamma (n_b - 1) \right] u_j^2 \right] + 2n_b \gamma \sum_{1 \leq j < k \leq q_b} u_j u_k = \int \sum_{j=1}^{q_b} \left[ (\nabla u_j)^2 + \left[ \lambda + \gamma (n_b - 1) \right] u_j^2 \right] + n_b \gamma \sum_{1 \leq j < k \leq q_b} (u_j^2 + u_k^2) - n_b \gamma \sum_{1 \leq j < k \leq q_b} (u_j - u_k)^2
\]
\[
\geq \int \sum_{j=1}^{q_b} \left[ (\nabla u_j)^2 + \left[ \lambda + \gamma (n - 1) \right] u_j^2 \right] = \frac{1}{n_b} \|u\|^2_{H^s_{n,\lambda}}
\]
and
\[
\text{R.H.S} \leq [\mu + (n_b - 1)\beta] \sum_{j=1}^{q_b} |u_j^+|^4_{L^4} \leq 0.
\]
Thus, \( \|u\|_{H^s_n} = 0 \), i.e. \( u = 0 \). On the other hand, \( 0 \notin \mathcal{N}_n \), so \( \sigma^{n_b} \) has no fixed point on \( \mathcal{N}_n \). By definition, \( e^{n_b}(\beta) = \infty \).

If \( -\frac{\mu}{n_b - 1} < \beta < 0 \) and \( \sigma^{n_b}(u) = u \), then similar calculation as above yields
\[
\|u\|^2_{H^s_n} \leq n_b [\mu + (n_b - 1)\beta] \sum_{j=1}^{q_b} |u_j^+|^4_{L^4} \leq [\mu + (n_b - 1)\beta] \sum_{j=1}^{n} |u_j|^4_{L^4} \leq C [\mu + (n_b - 1)\beta] \|u\|^4_{H^s_{n,\lambda}}
\]
where \( C > 0 \) only depends on \( \gamma \) and Sobolev embedding constant. Thus
\[
J|_{\mathcal{N}_n}(u) = \frac{1}{4} \|u\|^2_{H^s_{n,\lambda}} \geq \frac{1}{4C[\mu + (n_b - 1)\beta]}
\]
and \( J|_{\mathcal{N}_n}(u) \to \infty \) as \( \beta \to -\frac{\mu}{n_b - 1} \).

Recall the \( Z_n \)-index defined in [24].

**Definition 3.3.** For any closed \( \sigma \)-invariant subset \( A \subset \mathcal{N}_n \), define index \( \kappa(A) \) as the smallest \( m \in \mathbb{N} \cup \{0\} \) such that there exists a continuous map \( h : A \to \mathbb{C}^m \setminus \{0\} \) satisfying
\[
h(\sigma u) = e^{i\frac{\pi}{2\pi}} h(u).
\]
If there is no such a map, set \( \kappa(A) = \infty \). Define \( \kappa(\emptyset) = 0 \). In particular, if \( A \) contains a fixed point of \( \sigma^{n_b} \) for \( 0 \leq b \leq a \), then \( \kappa(A) = \infty \).

The properties of \( Z_n \)-index can be found in [24, Lemma 3.2]. Now, we construct a sequence of Liusternik-Schnirelman type levels on \( \mathcal{N}_n \) using this \( Z_n \)-index.
\[
c_k := \inf \{ c \in \mathbb{R} | \kappa(\mathcal{N}_n^c) \geq k \}, \quad k = 1, 2, \ldots
\]
(12)
Denote by \( K \) the set of critical points of \( J \). We give an estimate of the index \( \gamma \) near the critical levels.
Lemma 3.4. For any \( c < \min_{0 \leq b \leq a} \{ c^b(\beta) \} \), the \( Z_N \)-index of \( K_c \) is finite, i.e. \( \gamma(K_c) < \infty \). And there exists \( \epsilon > 0 \) such that

\[
\gamma(M^{c+\epsilon}) \leq \gamma(M^{c-\epsilon}) + \gamma(K_c).
\]

Comparing with the case \( n = 2 \) considered in [10, 14], we need more technical arguments to estimate \( c_k \). Recall the continuous map \( \psi : S^{2m-1} \to N^m \) constructed in [24],

\[
\psi(e^{i\frac{2\pi}{2m} U}) = \sigma \psi(U), \quad \text{for all } U \in S^{2m-1}.
\]

Let \( S^{2m-1} \) be the unit sphere in \( \mathbb{C}^m \). The constructions given in [24, Proposition 4.1] and [24, Proposition 4.2] both work for the doubly coupled system (11).

Lemma 3.5. Let \( \{ c_k \} \) be given by (12).

(i) For every \( m \), \( c_m < \infty \) is bounded independent of \( \beta < 0 \).
(ii) \( c_m \to c^* \) as \( m \to \infty \), where \( \min_{0 \leq b \leq a} \{ c^b(\beta) \} \leq c^* \leq \infty \).
(iii) If \( c := c_m = c_{m+1} = \cdots = c_l < \min_{0 \leq b \leq a} \{ c^b(\beta) \} \) for some \( l \geq m \), then \( \gamma(K_c) \geq l - m + 1 \).
(iv) If \( c_m < \min_{0 \leq b \leq a} \{ c^b(\beta) \} \), then \( K_{c_m} \neq \emptyset \), and \( N_{c_m}^m \) contains at least \( m \) \( Z_n \) orbits of critical points of \( J \).
(v) If \( \beta \leq -\frac{1}{n_{b-1}} \), then there is no fixed point of \( \sigma^{q_\gamma} \) on \( N \) for \( 0 \leq d \leq b \).

The proof in [24, Lemma 4.5] is still valid, so we omit the details here.

Proof of Theorem 1.3. If \( n \) is a prime number, then by similar arguments as the proof of [24, Theorem 1.1], Theorem 1.3 holds.

Now assume that \( n \) is not a prime number. If \( u \) is a solution satisfying \( \sigma^{q_\beta}(u) = u \), then \( u \) is also a solution of system

\[
-\Delta u_j + [\lambda + \gamma(n_b - 1)]u_j + \gamma n_b \sum_{k \neq j} u_k = [\mu + \beta(n_b - 1)]u_j^3 + \beta n_b u_j \sum_{k \neq j} u_k^2
\]

Denote \( \bar{\lambda} = \lambda + \gamma(n_b - 1) \), \( \bar{\gamma} = \gamma n_b \), \( \bar{\mu} = [\mu + \beta(n_b - 1)] \), \( \bar{\beta} = \beta n_b \). Then we see that the reduction does not change the structure of the system in the sense:

- the coefficient \( \beta \) only depends on the number of equations, i.e. \( \bar{\beta} \) depends on \( q_b \) in the same way as \( \beta \) depends on \( n \).

- \( \gamma \in (-\frac{\lambda}{n-1}, 0) \) is equivalent to \( \bar{\gamma} \in (-\frac{\bar{\lambda}}{n_{b-1}}, 0) \)

Therefore the arguments in the proof of [24, Theorem 1.1] can be applied here with simple changes of the linear coupling coefficients.

\( \square \)

4. Linearized system and bifurcation parameters. In this section, symmetric condition (3) is always assumed. Without loss of generality, assume \( \lambda = \mu = 1 \).

In [4], the \( \beta \)-bifurcations for \( \gamma = 0 \) and \( n \geq 3 \) were studied. As an important difference with the doubly coupled system, when \( \gamma = 0 \), the existence of \( T_\omega \) does not require \( \mu_j \equiv \mu \). With similar arguments, we can generalize the \( \beta \)-bifurcations to all \( \gamma \in (-\frac{1}{n-1}, 0) \).

Proof of Theorem 1.4 The proof is similar to [5] and we omit the details here. \( \square \)

From now on, we shall focus on \( \gamma \)-bifurcations. We first investigate the linearized system of (1) along \( T_\omega \).

\[
-\Delta \phi + D(\gamma) \phi = A^2 w \zeta C(\beta) \phi
\]
where \( \phi = (\phi_1, \phi_2, \ldots, \phi_n)^T \),

\[
D(\gamma) = \begin{pmatrix}
1 & \gamma & \cdots & \gamma \\
\gamma & 1 & \cdots & \gamma \\
\vdots & \vdots & \ddots & \vdots \\
\gamma & \gamma & \cdots & 1
\end{pmatrix}_{n \times n},
\]

\[
C(\beta) = \begin{pmatrix}
3 + (n-1)\beta & 2\beta & \cdots & 2\beta \\
2\beta & 3 + (n-1)\beta & \cdots & 2\beta \\
\vdots & \vdots & \ddots & \vdots \\
2\beta & 2\beta & \cdots & 3 + (n-1)\beta
\end{pmatrix}_{n \times n}.
\]

Clearly, \( D(\gamma) \) and \( C(\beta) \) are both symmetric and diagonally dominated if

\[
\gamma > -\frac{1}{n-1} \quad \text{and} \quad \frac{3}{n-1} > \beta > -\frac{1}{n-1}.
\]  

**Proposition 1.** Assume (14), then the dimension of kernel space of (1), i.e. the dimension of solution space of system (13) equals \( l(n-1) \) for some positive integer \( l \).

**Proof.** By (14), \( D(\gamma) \) and \( C(\beta) \) can be diagonalized with the same matrix \( P \) and its inverse \( P^{-1} \).

\[
P = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & -1
\end{pmatrix}, \quad P^{-1} = \frac{1}{n} \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 1-n & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1-n
\end{pmatrix}
\]

In fact, the linearized system (13) can be written as

\[
-\Delta P^{-1} \phi + P^{-1} P P^{-1} D(\gamma) PP^{-1} \phi = A^2 w_\gamma^2 P^{-1} P P^{-1} C(\beta) PP^{-1} \phi
\]  

where \( A \) is defined in (10). Then (15) is equivalent to

\[
-\Delta P^{-1} \phi + \begin{pmatrix}
1 + (n-1)\gamma & 0 & \cdots & 0 \\
0 & 1-\gamma & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1-\gamma
\end{pmatrix} P^{-1} \phi
= A^2 w_\gamma^2 \begin{pmatrix}
3[1 + (n-1)\beta] & 0 & \cdots & 0 \\
0 & 3 + (n-3)\beta & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 3 + (n-3)\beta
\end{pmatrix} P^{-1} \phi
\]

Also note

\[
P^{-1} \phi = \frac{1}{n} \left( \sum_{k=1}^{n} \phi_k, \sum_{k=1}^{n} \phi_k - n\phi_2, \ldots, \sum_{k=1}^{n} \phi_k - n\phi_n \right)^T
\]
so the first equation of (15) is

\[-\Delta (\sum_{k=1}^{n} \phi_k) + [1 + (n-1)\gamma] \sum_{k=1}^{n} \phi_k = 3A^2 w^2_\gamma [1 + (n-1)\beta] \sum_{k=1}^{n} \phi_k = 3w^2_\gamma [1 + (n-1)\gamma] \sum_{k=1}^{n} \phi_k\]

If let \(\xi(tx) = \sum_{k=1}^{n} \phi_k(x)\) where \(t\) is defined in (10), then \(\xi\) satisfies \(-\Delta \xi + \xi = 3\omega^2 \xi\). The non-degeneracy of \(w\) in radial space \(H_n(\mathbb{R})\) implies \(\xi = 0\), i.e. \(\sum_{k=1}^{n} \phi_k = 0\).

On the other hand, the remaining \(n-1\) equations of (15) are

\[-\Delta \phi_j + (1 - \gamma)\phi_j = [3 + (n-3)\beta] A^2 w^2_\gamma \phi_j = \frac{[3 + (n-3)\beta][1 + (n-1)\gamma]}{1 + (n-1)\beta} w^2_\gamma \phi_j\]

for \(j = 2, ..., n\). Clearly, all \(\phi_j\)'s satisfy the same equation

\[-\Delta \psi + \frac{1 - \gamma}{1 + (n-1)\gamma} \psi = \frac{3 + (n-3)\beta}{1 + (n-1)\beta} w^2 \psi.\quad (16)\]

Therefore, we conclude that:

1. Every solution \(\phi\) of system (13), satisfies

   \[\sum_{k=1}^{n} \phi_k = 0.\]

2. Every component \(\phi_k\) of \(\phi\) can be found by solving the scalar equation (16).

Denote \(e_i\), the \(n\)-vector whose \(i\)-th component equals to 1 and all the other components equals to 0. Assume that \(\psi\) is a solution of (16), then \(\phi_j = \psi(e_j + e_n)\), \(j = 1, ..., n-1\), give the base of the solution set to (13).

Therefore, if equation (16) has \(l\) nontrivial solutions, for a fixed set of \(\gamma, \beta\) and \(n\), then the kernel space of (13) has dimension \(l(n-1)\).

4.1. **Local \(\gamma\)-bifurcations.** In this subsection, we find the possible \(\gamma\)-bifurcations for fixed values of \(\beta\) satisfying (14).

Denote

\[g(\gamma) = \frac{1 - \gamma}{1 + (n-1)\gamma}\quad \text{and} \quad f(\beta) = \frac{3 + (n-3)\beta}{1 + (n-1)\beta}.\]

It is easy to check that \(f\) is a decreasing function of \(\beta\) and \(f((-\frac{1}{n-1}, \frac{3}{2(n-1)})) = (\frac{3(n-2)}{2(n-1)}, \infty)\). The eigenvalue problem

\[-\Delta \psi + g(\gamma) \psi = \lambda(\gamma) w^2 \psi, \quad \psi = 0 \text{ on } \partial \Omega.\quad (17)\]

has a sequence of eigenvalues \(\lambda_1(\gamma) < \lambda_2(\gamma) \leq \cdots \leq \lambda_k(\gamma) \leq \cdots\), which are all continuous functions of \(\gamma\), see Lemma 5.1 for the proof.

Fix \(\beta^* > -\frac{1}{n-1}\), if there exists \(\gamma^*\) such that \(f(\beta^*) = \lambda_j(\gamma^*)\), then

\[\left(\beta^*, \gamma^*, \left\langle \frac{1 + (n-1)\gamma^*}{1 + (n-1)\beta^*} w_{\gamma^*}, ..., \frac{1 + (n-1)\gamma^*}{1 + (n-1)\beta^*} w_{\gamma_{j-1}} \right\rangle \right) \in T_\omega\]

may be a bifurcation point. For fixed \(\beta\) and \(j\), denote the \(\gamma\)-bifurcation parameter by \(\gamma_{j}(\beta)\). Then we have
Lemma 4.1. For each fixed \( \beta > -\frac{1}{n-1} \) such that \( f(\beta) \geq \lambda_1(0) \), there are finitely many bifurcation points with respect to \( T_{\omega, \beta} \). Moreover, denote by

\[
K(\beta) = \{ \gamma_j \mid (\gamma_j, u_{\gamma_j}, v_{\gamma_j}) \text{ is a bifurcation point with respect to } T_{\omega, \beta}, j \geq 1 \},
\]

and \( \sharp K(\beta) \) the number of elements of \( K(\beta) \), then \( \sharp K(\beta) \to \infty \) as \( \beta \to -\frac{1}{n-1} \).

Proof. This proof is similar to the proof [27, Lemma 3.3]. We omit the details. \( \square \)

We now apply [20, Theorem 8.9] to verify bifurcations indeed happen at each \( \gamma_j(\beta) \). Then we need the Hessian of \( J \) at \( u(\beta, \gamma) \)

\[
H(v) := \langle J''(u), v \rangle = \sum_{k=1}^{n} \int \left| \nabla v_k \right|^2 + \int \langle D(\gamma)v, v \rangle - \int A^2 w^2 \langle C(\beta)v, v \rangle
\] (18)

Denote by \( b_k = e_1 - e_k, k = 2, 3, \ldots, n \) the eigenvectors of \( 1 - \gamma \) for matrix \( D(\gamma) \), and \( 3 + (n-3)\beta \) for matrix \( C(\beta) \) respectively.

Proposition 2. Fix \( \beta \in (-\frac{1}{n-1}, \frac{3}{n-1}) \), then the Morse index of \( J \) changes as \( \gamma \) crosses \( \gamma_j(\beta) \).

In [27, Lemma 3.3], Proposition 2 is proved for \( n = 2 \), but the calculation of \( \frac{\partial H}{\partial \gamma} \) there contains a gap. We shall provide a modified proof in the Appendix.

Proof of Theorem 1.5. By Lemma 4.1 and Proposition 2, the assumption of [20, Theorem 8.9] is satisfied. Thus the Theorem 1.5 holds. \( \square \)

4.2. Two dimensional bifurcation branches. In this subsection we discuss the global bifurcations of \( (1) \). Define operator \( F : \mathbb{R} \times \mathcal{H} \to \mathcal{H} \)

\[
F(\beta, \gamma, u) = u - (-\Delta + D(\gamma))^{-1}T(\beta, u)
\] (19)

where

\[
\Delta_{n \times n} = \begin{pmatrix}
\Delta & 0 & \cdots & 0 \\
0 & \Delta & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Delta
\end{pmatrix}, \quad T(\beta, u) = \begin{pmatrix}
u_1^2 + \beta u_1 \sum_{j \neq 1} u_j^2 \\
\vdots \\
u_n^2 + \beta u_n \sum_{j \neq n} u_j^2
\end{pmatrix}.
\]

By the compact embedding \( H^1_t(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), 2 < p < 2^*, \) the map \( \mathbb{A} : I^2 \times \mathcal{H} \to \mathcal{H} \)

\[
\mathbb{A}(\beta, \gamma, u) = (-\Delta_{n \times n} + D(\gamma))^{-1}T(\beta, u)
\] (20)

is completely continuous, where \( I = (-\frac{1}{n-1}, \infty) \). To get positive solutions, we confine the problem in the nonnegative cone

\[
\mathbb{P} = \{(u_1, u_2) \in \mathcal{H}_r \mid u_1 \geq 0, u_2 \geq 0\},
\]

and let \( r : \mathcal{H}_r \to \mathbb{P} \) be a retraction. According to the positive definiteness of the operator, it is easy to see that \( \mathbb{A}(\beta, \gamma, \mathbb{P}) \subset \mathbb{P} \).

Proof of Theorem 1.6. To prove the 2-dimension bifurcations, we apply the multiparameter bifurcation theorem [13, Theorem 2.4].

Fix \( \gamma_0 \in (-\frac{1}{n-1}, 0) \). Let \( \beta(\gamma_0) \) be the local \( \beta \)-bifurcation parameter corresponding to \( \lambda_1 \) and \( \mathbb{b}_{\gamma_0} = (\beta(\gamma_0), \gamma_0) \in \Gamma_{\gamma_0} \). To apply [13, Theorem 2.4], we rewrite the system as follows. Denote \( \omega_A = (A|_{\mathbb{b}_{\gamma_0}})\omega_1, \ldots, A|_{\mathbb{b}_{\gamma_0}}\omega) \in \mathcal{T}_\omega \) and \( \mathbb{v} = u - \omega_A \). Define \( G : \mathcal{O} \to \mathcal{H}_r \) by

\[
G(b, v) = A(b, r(\omega_A + v)) - \omega_A.
\] (21)
Clearly, \( G \) is a completely continuous mapping satisfying \( G(b, 0) = 0 \) whenever \((b, 0) \in \mathcal{O}\). We refer to \( \{I^2 \times \{0\}\} \cap \mathcal{O} \) as the trivial solutions. Moreover, \( G(b, v) = v \) is equivalent to \( u = \omega_A + v \in \mathbb{P} \) solving \( \Lambda(b, r(u)) = u \).

Notice that \( h \) is continuously differentiable and has 0 as a regular value. To simplify notations, we identify \( \Gamma \) with \( \Gamma \times \{0\} \subset \mathcal{O} \). For any constant

\[
0 < \varepsilon < \frac{\min\{|\beta_k(\gamma_0) - \beta_{k+1}(\gamma_0)|, |\beta_k(\gamma_0) - \beta_{k-1}(\gamma_0)|\}}{2}
\]

taking

\[
b = (\beta_k(\gamma_0) - \varepsilon, \gamma_0) \quad \text{and} \quad \overline{b} = (\beta_k(\gamma_0) + \varepsilon, \gamma_0).
\]

It is easy to see that \( b \) and \( \overline{b} \) lie in the same component of \( \Gamma_{\gamma_0,} \) neither \( b \) nor \( \overline{b} \) are \( \Gamma \) bifurcation points of \( G := I - G \). It is easy to get that

\[
\text{ind}(\mathcal{G}_b, 0) = (-1)\sum_{i=1}^{k-1} \dim \lambda_i(\gamma_0) \neq (-1)\sum_{i=1}^{k} \dim \lambda_i(\gamma_0) = \text{ind}(\mathcal{G}_\overline{b}, 0).
\]

Now, all the assumptions of [13, Theorem 2.4] are satisfied. Then there exists a connected 2-dimensional bifurcation branch \( C_k \) emanating from \( T_\omega \), and one of the three possibilities listed in Theorem 1.6 holds. \( \square \)

5. **Appendix.** In this section, we first prove a lemma regarding the monotonicity and continuity of \( \lambda_j(\gamma) \), \( j = 1, 2, \ldots \) which are defined in (17). Note that the continuity has been observed in [27], but the original proof contained a gap, which is then fixed in [9]. For completeness, we provide a revised proof below. Then we prove Proposition 2.

Let \( Q(\gamma) = \frac{1-\gamma}{1+\gamma} \), then \( Q \) is strictly decreasing on \((-\frac{1}{n-1}, 0)\).

Define

\[
\Lambda(\phi, \gamma) = \frac{\int_{\mathbb{R}^N} |\nabla \phi|^2 + Q(\gamma)\phi^2}{\int_{\mathbb{R}^N} w^2 \phi^2}, \quad \phi \neq 0
\]

Recall the variational characterization of \( \lambda_j(\gamma) \),

\[
\lambda_j(\gamma) = \sup_{E_{j-1}} \inf_{\phi \in E_{j-1}^+ \setminus \{0\}} \Lambda(\phi, \gamma),
\]

where \( E_j \) denotes a \( j \)-dimensional subspace of \( H_j^1(\mathbb{R}^N) \), \( E_j^+ \) denotes the orthogonal space of \( E_j \). It is easy to see that \( \Lambda(\phi, \cdot) \) is a continuous and decreasing function of \( \gamma \) for fixed \( \phi \).

**Lemma 5.1.** The eigenvalue \( \lambda_j(\gamma) \), \( j \geq 1 \), is a continuous and strictly decreasing function of \( \gamma \in (-\frac{1}{n-1}, \frac{1}{n-1}) \). Moreover, for any \( j \geq 1 \), one has that \( \lambda_j(\gamma) \to +\infty \) as \( \gamma \to (-\frac{1}{n-1})^+ \).

**Proof.** **Monotonicity.** Let \( -\frac{1}{n-1} < \gamma_1 < \gamma_2 < \frac{1}{n-1} \) and \( E_{j-1}^+ \) be the \( j-1 \)-dimensional subspace of \( H_j^1(\mathbb{R}^N) \) associated to \( \lambda_j(\gamma_2) \), i.e.

\[
\lambda_j(\gamma_2) = \inf_{\phi \in (E_{j-1}^+)^\perp} \Lambda(\phi, \gamma_2).
\]

In the case \( j = 1 \), \( (E_0^+)^\perp = H_1^1(\mathbb{R}^N) \). By the characterization of \( \lambda_j \),

\[
\lambda_j(\gamma_1) - \lambda_j(\gamma_2) \geq \inf_{\phi \in (E_{j-1}^+)^\perp} \Lambda(\phi, \gamma_1) - \inf_{\phi \in (E_{j-1}^+)^\perp} \Lambda(\phi, \gamma_2)
\]
Then the monotonicity of \( Q(\gamma) \) implies that \( \lambda_j(\gamma) \) is decreasing, for any \( j \geq 1 \).

**Continuity in \((-\frac{1}{n-1}, \frac{1}{n-1})\).** For any \( j \geq 1 \) and \(-\frac{1}{n-1} < \gamma_1 < \gamma_2 < \frac{1}{n-1}\),

\[
\lambda_j(\gamma_1) - \lambda_j(\gamma_2) = \sup_{E_{j-1}} \inf_{\phi \in E_{j-1}^+} \Lambda(\phi, \gamma_1) - \sup_{E_{j-1}} \inf_{\phi \in E_{j-1}^+} \Lambda(\phi, \gamma_2)
\leq \sup \{ \inf_{\phi \in E_{j-1}^+} \Lambda(\phi, \gamma_1) - \inf_{\phi \in E_{j-1}^+} \Lambda(\phi, \gamma_2) \}
\leq (Q(\gamma_1) - Q(\gamma_2)) \frac{\int_{\mathbb{R}^2} (\phi_j^2)^2 \, dx}{\int_{\mathbb{R}^2} \phi_j^2 \, dx}
\leq \left[ \frac{Q(\gamma_1)}{Q(\gamma_2)} - 1 \right] \lambda_j(\gamma_2),
\]

where \( \phi_j^2 \) is a minimizer of \( \Lambda(\phi, \gamma_2) \) in \( E_{j-1}^+ \).

Now for any \( \gamma_0 \in \left(-\frac{1}{n-1}, \frac{1}{n-1}\right) \), \( j \geq 1 \), we see from (22) and (23) that

\[
0 \leq \lim_{\gamma \to \gamma_0} (\lambda_j(\gamma) - \lambda_j(\gamma_0)) \leq \lim_{\gamma \to \gamma_0} \left( \frac{Q(\gamma)}{Q(\gamma_0)} - 1 \right) \lambda_j(\gamma_0) = 0
\]

and also

\[
0 \leq \lim_{\gamma \to \gamma_0} (\lambda_j(\gamma_0) - \lambda_j(\gamma)) \leq \lim_{\gamma \to \gamma_0} \left( \frac{Q(\gamma_0)}{Q(\gamma)} - 1 \right) \lambda_j(\gamma) \leq \lim_{\gamma \to \gamma_0} \left( \frac{Q(\gamma_0)}{Q(\gamma)} - 1 \right) \lambda_j(\gamma_0) = 0
\]

Combine (24) and (25),

\[
\lim_{\gamma \to \gamma_0} \lambda_j(\gamma) = \lambda_j(\gamma_0).
\]

Hence \( \lambda_j(\gamma) \) is a continuous and strictly decreasing function of \( \gamma \).

For the right limit of \( \lambda_j \) at \(-\frac{1}{n-1}\), we fix any \( E_{j-1} \), then

\[
\lambda_j(\gamma) = \sup_{E_{j-1}} \inf_{\phi \in E_{j-1}^+} \Lambda(\phi, \gamma)
\geq \inf_{\phi \in E_{j-1}^+} \Lambda(\phi, \gamma) = \inf_{\phi \in E_{j-1}^+} \frac{\int_{\mathbb{R}^2} |\nabla \phi|^2 \, dx + Q(\gamma) \int_{\mathbb{R}^2} \phi^2 \, dx}{\int_{\mathbb{R}^2} \phi^2 \, dx}
\geq \frac{Q(\gamma)}{||w||_{L^\infty}^2},
\]

Therefore \( \lambda_j(\gamma) \to +\infty \) as \( \gamma \to \left(-\frac{1}{n-1}\right)^+ \). The proof is completed. \(\square\)

**Proof of Proposition 2.** We first calculate the derivative of \( H \), which is defined by (18), with respect to \( \gamma \), then evaluate at \( \gamma_j \)

\[
\frac{\partial H}{\partial \gamma}
\bigg|_{\gamma = \gamma_j, \psi_k = b_k}
= \frac{\partial}{\partial \gamma} \left( \int \langle D(\gamma)\phi_j b_k, \phi_j b_k \rangle - \frac{1 + (n-1)\gamma}{1 + (n-1)\beta} \int w_\gamma^2 (C(\beta)\phi_j b_k, \phi_j b_k) \right)_{\gamma = \gamma_j}
\]
\[
\frac{\partial}{\partial \gamma} \left( 2(1 - \gamma) \int \phi_j^2 - \frac{2[1 + (n - 1)\gamma][3 + (n - 3)\beta]}{1 + (n - 1)\beta} \int \psi_j^2 \cdot \nabla \psi_j \right)_{\gamma = \gamma_j} = -2 \int \phi_j^2 - 2f(\beta) \left[ (n - 1) \int w_\gamma \phi_j^2 + \int \nabla w_\gamma \cdot \sqrt{1 + (n - 1)\gamma} \right]_{\gamma = \gamma_j}
\]

where \( y = \sqrt{1 + (n - 1)\gamma}x \), \( \psi_j(y) = \phi_j(x) \). It is easy to see that

\[
-\Delta \psi_j + g(\gamma_j)\psi_j = f(\beta)w^2\psi_j
\]

and in radial coordinates

\[
-\psi_j'' - \frac{N - 1}{r} \psi_j' + g(\gamma_j)\psi_j = f(\beta)w^2\psi_j
\]

Calculate the last term of \( \frac{\partial H}{\partial \gamma} \mid_{\gamma = \gamma_j} \)

\[
f(\beta) \int_{\mathbb{R}^N} w\psi_j^2 \nabla w \cdot ydy
\]

\[
= f(\beta)\sigma(S^{N-1}) \int_0^\infty \psi_j^2 r^N \psi_j' dr
\]

\[
= \frac{f(\beta)\sigma(S^{N-1})}{2} \int_0^\infty \psi_j^2 r^2 dw^2
\]

\[
= \frac{f(\beta)\sigma(S^{N-1})}{2} \left[ w^2 \psi_j^2 r^N \right]_0^\infty - \int_0^\infty w^2 d(\psi_j^2 r^N)
\]

\[
= - \frac{f(\beta)\sigma(S^{N-1})}{2} \left[ \int_0^\infty 2w^2 \psi_j r^N \psi_j' dr + N \int_0^\infty \psi_j^2 r^{N-1} dr \right]
\]

\[
= -\sigma(S^{N-1}) \int_0^\infty f(\beta)w^2 \psi_j^2 + N \int_{\mathbb{R}^N} w^2 \psi_j^2
\]

Since \( -\psi_j'' - \frac{N - 1}{r} \psi_j' + g(\gamma_j)\psi_j = f(\beta)w^2\psi_j \),

\[
f(\beta) \int_{\mathbb{R}^N} w\psi_j^2 \nabla w \cdot y dy
\]

\[
= \sigma(S^{N-1}) \int_0^\infty \left( \psi_j'' + \frac{N - 1}{r} \psi_j' - g(\gamma_j)\psi_j \right) \psi_j^2 r^N dr - \frac{N f(\beta)}{2} \int_{\mathbb{R}^N} w^2 \psi_j^2
\]

\[
= \sigma(S^{N-1}) \int_0^\infty \left[ \frac{1}{2} (\psi_j')^2 r^N + (N - 1)(\psi_j')^2 r^{N-1} - \frac{g(\gamma_j)}{2} (\psi_j')^2 r^N \right]
\]

\[
- \frac{N f(\beta)}{2} \int_{\mathbb{R}^N} w^2 \psi_j^2
\]

\[
= \sigma(S^{N-1}) \frac{1}{2} r^N (\psi_j')^2 \bigg|_0^\infty - \frac{N f(\beta)}{2} \int_{\mathbb{R}^N} w^2 \psi_j^2
\]

\[
= -\sigma(S^{N-1}) \frac{g(\gamma_j)}{2} \psi_j^2 r^N \bigg|_0^\infty + \frac{N g(\gamma_j)}{2} \int_{\mathbb{R}^N} \psi_j^2 r^{N-1} dr - \frac{N f(\beta)}{2} \int_{\mathbb{R}^N} w^2 \psi_j^2
\]

\[
= -\frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla \psi_j|^2 + \frac{N g(\gamma_j)}{2} \int_{\mathbb{R}^N} \psi_j^2 - \frac{N f(\beta)}{2} \int_{\mathbb{R}^N} w^2 \psi_j^2
\]
Therefore
\[
\frac{\partial H}{\partial \gamma} \bigg|_{\gamma = \gamma_j, \psi = \phi, b_k} = -\frac{2}{(1 + (n-1)\gamma_j)^n} \left[ \int_{\mathbb{R}^N} \psi_j^2(y) dy + (n-1)f(\beta) \int_{\mathbb{R}^N} w^2 \psi_j^2 + f(\beta) \int_{\mathbb{R}^N} w \psi_j^2 \nabla y \cdot dy \right]
\]
\[= -\frac{2}{(1 + (n-1)\gamma_j)^n} \left[ (n-2) \int_{\mathbb{R}^N} |\nabla \psi_j|^2 + [1 + (n-1)g(\gamma_j)] \int_{\mathbb{R}^N} \psi_j^2 \right] < 0
\]
i.e. the Hessian of \( J \) is strictly increasing at \( \gamma_j \). Hence the Morse index changes at each \( \gamma_j \). By [20, Theorem 8.9], \( \gamma_j \) is a \( \gamma \)-bifurcation parameter. The proof is completed. \( \square \)

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E-mail address: sujb@cnu.edu.cn
E-mail address: rushun.tian@cnu.edu.cn
E-mail address: zhi-qiang.wang@usu.edu