Bi-Hamiltonian representation of Stäckel systems

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Abstract

It is shown that the separation relations are fundamental objects for integration by quadratures of separable Liouville integrable systems (the so-called Stäckel systems). These relations are further employed for the classification of separable systems. Moreover, we prove that any separable Liouville integrable system can be lifted to a bi-Hamiltonian system of Gel’fand-Zakharevich type. In conjunction with other known result this implies that the existence of bi-Hamiltonian representation of Liouville integrable systems is a necessary condition for separability.

1 Introduction

The Hamilton-Jacobi (HJ) theory is one of the most powerful methods of integration by quadratures a wide class of systems described by nonlinear ordinary differential equations, with a long history as a part of analytical mechanics. The theory in question is closely related to the Liouville integrable Hamiltonian systems. The milestones of this theory include the works of Stäckel, Levi-Civita, Eisenhart, Woodhouse,Kalnins, Miller, Benenti and others. The majority of results was obtained for a very special class of integrable systems, important from the physical point of view, namely for the systems with quadratic in momenta first integrals.

The first efficient construction of the separation variables for dynamical systems was discovered by Sklyanin [1]. He adapted the methods of soliton theory, i.e., the Lax representation and r-matrix theory for systematic derivation of separation coordinates. In this approach the integrals of motion in involution appear as coefficients of characteristic equation (spectral curve) of the Lax matrix. This method was successfully applied for separating variables in many integrable systems [1]-[8].

Recently, a modern geometric theory of separability on bi-Poisson manifolds was developed [9]-[15]. The theory in question includes Liouville integrable systems with integrals of motion being functions quadratic in momenta as a very special case. In this approach the constants of motion are closely related to the so-called separation curve which is intimately related to the Stäckel separation relations. The separation curve arising in the geometric approach is closely related to its counterpart in the r-matrix approach. In fact, these curves are identical for linear r-matrix and related by exponentiation of momenta in the spectral curve for dynamical (quadratic) r-matrix [6], [14].

In the present paper we develop in a systematic fashion a separability theory of the Liouville integrable systems which are of the GZ type, including as a special case the class of systems with quadratic in momenta first integrals. First of all, we treat Stäckel separable systems according to the form of separation relations and make some observations related to their classification. Then we construct a quasi-bi-Hamiltonian representation of Stäckel systems on $2n$ dimensional phase space and lift them to the related GZ bi-Hamiltonian systems on the extended $2n + k$ dimensional phase space. This result proves that bi-Hamiltonian property is common to all classes of the Stäckel systems considered. In other words, we prove that the existence of bi-Hamiltonian representation for Liouville integrable systems is a necessary condition for separability. Finally let us mention that up to now such a proof was available only for a distinguished class of the so-called Benenti systems [22], where $k = 1$ and all constants of motion are quadratic in momenta.
Consider a Liouville integrable system on a $2n$-dimensional phase space $M$. Thus, we have $M \ni u = (q_1, \ldots, q_n, p_1, \ldots, p_n)^T$ and there are $n$ functions $H_i(q,p)$ in involution with respect to the canonical Poisson tensor $\pi$

\[
\{H_i, H_j\}_\pi = \pi(dH_i, dH_j) = \langle dH_i, \pi dH_j \rangle = 0, \quad i, j = 1, \ldots, n,
\]

where $\langle \cdot, \cdot \rangle$ is the standard pairing of $TM$ and $T^*M$. Canonicity of $\pi$ means that the only nonzero Poisson brackets among the coordinates are $\{q_i, p_j\}_\pi = \delta_{ij}$. The functions $H_i$ generate $n$ Hamiltonian dynamic systems

\[
u_{t_i} = \pi dH_i = X_{H_i}, \quad i = 1, \ldots, n,
\]

where $X_{H_i}$ are called the Hamiltonian vector fields.

The Hamilton-Jacobi (HJ) method for solving (1) essentially amounts to the linearization of the latter via a canonical transformation

\[(q,p) \rightarrow (b,a), \quad a_i = H_i, \quad i = 1, \ldots, n.\]

In order to find the conjugate coordinates $b_i$ it is necessary to construct a generating function $W(q,a)$ of the transformation (2) such that

\[
b_j = \frac{\partial W}{\partial q_j}, \quad p_j = \frac{\partial W}{\partial a_j}.
\]

The function $W(q,a)$ is a complete integral of the associated Hamilton-Jacobi equations

\[H_i \left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = a_i, \quad i = 1, \ldots, n.\]

In the $(b,a)$ representation the $t_i$-dynamics is trivial:

\[\frac{d}{dt} (a_j)_{t_i} = 0, \quad \frac{d}{dt} (b_j)_{t_i} = \delta_{ij},\]

whence

\[
b_j(q,a) = \frac{\partial W}{\partial q_j} = t_j + c_j, \quad j = 1, \ldots, n,
\]

where $c_j$ are arbitrary constants.

Equations (4) provide implicit solutions for (1). Solving them for $q_j$ is known as the inverse Jacobi problem. The reconstruction in explicit form of trajectories $q_j = q_j(t_i)$ is in itself a highly nontrivial problem from algebraic geometry which is beyond the scope of this paper.

The main difficulty in applying the above method to a given Liouville integrable system in given canonical coordinates $(q,p)$ consists in solving the system (3) for $W$. In general this is a hopeless task, as (3) is a system of nonlinear coupled partial differential equations. In essence, the only hitherto known way of overcoming this difficulty is to find distinguished canonical coordinates, denoted here by $(\lambda, \mu)$, for which there exist $n$ relations

\[
\varphi_i(\lambda_1, \mu; a_1, \ldots, a_n) = 0, \quad i = 1, \ldots, n, \quad a_i \in \mathbb{R}, \quad \det \left[ \frac{\partial \varphi_i}{\partial a_j} \right] \neq 0,
\]

such that each of these relations involves only a single pair of canonical coordinates (1). The determinant condition in (5) means that we can solve the equations (4) for $a_i$ and express $a_i$ in the form $a_i = H_i(\lambda, \mu)$, $i = 1, \ldots, n$.

If the functions $W_i(\lambda_i, a)$ are solutions of a system of $n$ decoupled ODEs obtained from (5) by substituting $\mu_i = \frac{dW_i(\lambda_i, a)}{d\lambda_i}$

\[
\varphi_i \left(\lambda_1, \mu_i, a_1, \ldots, a_n\right) = 0, \quad i = 1, \ldots, n,
\]

then the function

\[W(\lambda, a) = \sum_{i=1}^n W_i(\lambda_i, a)\]
is an additively separable solution of all the equations \eqref{Ham-Jac}, and simultaneously it is a solution of all Hamilton-Jacobi equations \eqref{Ham-Jac-2} because solving \eqref{Ham-Jac} to the form $a_i = H_i(\lambda, \mu)$ is a purely algebraic operation. The Hamiltonian functions $H_i$ Poisson commute since the constructed function $W(\lambda, a)$ is a generating function for the canonical transformation $(\lambda, \mu) \rightarrow (b, a)$. The distinguished coordinates $(\lambda, \mu)$ for which the original Hamilton-Jacobi equations \eqref{Ham-Jac-2} are equivalent to a set of separation relations \eqref{gen-Stackel} are called the separation coordinates.

Of course, the original Jacobi formulation of the method was a bit different from the one presented above, and was made for a particular class of Hamiltonians, nevertheless it contained all important ideas of the method. Jacobi himself doubted whether there exists a systematic method for construction of separation coordinates. Indeed, for many decades of development of separability theory, the method did not exist. Only recently, at the end of the 20th century, after more than one hundred years of efforts, two different constructive methods were suggested, the first related to the Lax representation and the second related to the bi-Hamiltonian representation for a given integrable system.

We would like to stress that all results of the present paper are derived directly from the separation relations \eqref{gen-Stackel}, thus confirming their fundamental role in the modern separability theory.

In what follows we restrict ourselves to considering a special case of \eqref{gen-Stackel} when all separation relations are affine in $H_i$:

$$
\sum_{k=1}^{n} S^k_i(\lambda_i, \mu_i) H_k = \psi_i(\lambda_i, \mu_i), \quad i = 1, \ldots, n,
$$

where $S^k_i$ and $\psi_i$ are arbitrary smooth functions of their arguments. The relations \eqref{gen-Stackel} are called the generalized Stäckel separation relations and the related dynamical systems are called the Stäckel separable ones. The matrix $S = (S^k_i)$ will be called a generalized Stäckel matrix. The reason behind this name is the fact that the conditions \eqref{gen-Stackel} with $S^k_i$ being $\mu$-independent and $\psi_i$ being quadratic in momenta $\mu$ equivalent to the original Stäckel conditions for separability of Hamiltonians $H_i$. To recover the explicit Stäckel form of the Hamiltonians it suffices to solve the linear system \eqref{gen-Stackel} with respect to $H_i$.

Although the restriction of linearity appears to be very strong, for some reasons (certainly deserving further investigation), for all separable systems known to the author, the general separation conditions can be reduced to the form \eqref{gen-Stackel} upon suitable choice of integrals of motion $H_i$.

If in \eqref{gen-Stackel} we further have $S^k_i(\lambda_i, \mu_i) = S^k(\lambda_i, \mu_i)$ and $\psi_i(\lambda_i, \mu_i) = \psi(\lambda_i, \mu_i)$ then the separation conditions can be represented by $n$ copies of the curve

$$
\sum_{k=1}^{n} S^k(\lambda, \mu) H_k = \psi(\lambda, \mu)
$$

in $(\lambda, \mu)$ plane, called a separation curve. The copies in question are obtained by setting $\lambda = \lambda_i$ and $\mu = \mu_i$ for $i = 1, \ldots, n$.

**Remark.** There is an important special case when \eqref{gen-Stackel} is an arbitrary nonsingular compact Riemann surface $\Gamma$, i.e., when $S^k(\lambda, \mu)$ and $\psi(\lambda, \mu)$ are polynomials of $\lambda$ and $\mu$ of certain specific form. Then one can find the genus of this curve and basic holomorphic differentials in a standard fashion and the Jacobi inversion problem \eqref{Jacobi-inversion} can be equivalently expressed by the Abel map of the Riemann surface $\Gamma$ into its Jacobi variety and solved in the language of Riemann theta functions (see \cite{18} and references therein).

From now on we will consider Stäckel separable systems with separation relations of the most general form \eqref{gen-Stackel}. For reasons to be explained in the next section, we collect the terms from the l.h.s. of \eqref{gen-Stackel} as follows:

$$
\sum_{k=1}^{m} \varphi^k_i(\lambda_i, \mu_i) H^{(k)}_i(\lambda_i) = \psi_i(\lambda_i, \mu_i), \quad i = 1, \ldots, n,
$$

where

$$
H^{(k)}_i(\lambda) = \sum_{i_1}^{n_k} \lambda^{n_k-i} H_i^{(k)}, \quad n_1 + \cdots + n_m = n
$$

and impose the normalization $\varphi^m_i(\lambda_i, \mu_i) = 1$.

As the separation relations \eqref{gen-Stackel} play the fundamental role in the Hamilton-Jacobi theory, it is natural to employ them for classification of Stäckel systems. The form of separation relations \eqref{gen-Stackel} allows us
to classify the associated Stäckel systems. Actually, any given class of Stäckel separable systems can be represented by a fixed Stäckel matrix $S$ and the functions $\psi$. The matrix $S$ is uniquely defined by $m$ vectors $\varphi^k = (\varphi^k_1, \ldots, \varphi^k_n)^T$, $k = 1, \ldots, m$, and the partition $(n_1, \ldots, n_m)$ of $n$. Note that in our normalization we have $\varphi^m = (1, \ldots, 1)^T$.

For example, the most intensively studied systems in the 20th century, those related to one-particle separable dynamics on Riemannian manifolds with flat or constant curvature metrics, belong to the simplest class with $m = 1$ and the functions $\psi_i$ being quadratic in the momenta $\mu_i$:

$$\sum_{j=1}^n H_j \lambda_i^{n-j} = \frac{1}{2} f_i(\lambda_i) \mu_i^2 + \gamma_i(\lambda_i), \quad i = 1, \ldots, n. \tag{10}$$

This class, which will hereinafter be referred to as the Benenti class, includes systems generated by conformal Killing tensors [19–22], as well as bi-cofactor systems, generated by a pair of conformal Killing tensors [23–27]. Here the functions $f_i$ define the Stäckel metric while the functions $\gamma_i$ define a separable potential. When $f_i = f(\lambda_i)$ and $f$ is a polynomial of order not higher then $n + 1$, then the associated Stäckel metric is of constant curvature.

Another class of separable systems also has $m = 1$ but the functions $\psi_i$ are now exponential in the momenta

$$\sum_{j=1}^n H_j \lambda_i^{n-j} = f_i(\lambda_i) \exp(a \mu_i) + f_2(\lambda_i) \exp(-b \mu_i) + \gamma_i(\lambda_i), \quad i = 1, \ldots, n,$$

where $\gamma_i$ define a separable potential. This class includes such systems as the periodic Toda lattice [13], the KdV dressing chain [14], the Ruijsenaars-Schneider system [11] and others.

We also know some particular examples from the classes with $m > 1$. For instance, stationary flows of the Boussinesq hierarchy belong to the class with $m = 2$, $n_1 = 1$, $n_2 = n - 1$, $\varphi^1_1 = \mu_i$ [12], [11]. Dynamical system on loop algebra $\mathfrak{sl}(3)$ belongs to the class with $m = 2$, $n_1 = 2$, $n_2 = 4$, $\varphi^1_1 = \mu_i$ [15].

In both cases the functions $\psi_i$ are cubic in the momenta, so these separation relations belong to the following class:

$$\mu_i \sum_{j=1}^{n_1} H_j^{(1)} \lambda_i^{n_1-j} + \sum_{j=1}^{n_2} H_j^{(2)} \lambda_i^{n_1-j} = \frac{1}{2} f_i(\lambda_i) \mu_i^2 + \mu_i \gamma_1(\lambda_i) + \gamma_2(\lambda_i), \quad i = 1, \ldots, n. \tag{11}$$

where $\mu_\gamma_1$ and $\gamma_2$ give rise to the separable potentials.

Finally, systems from the classes with $1 < m \leq n$, $\varphi^k_i = \lambda_i^{(\alpha_k)}$, $\alpha_k \in \mathbb{N}$ and with $\psi_i$ quadratic in the momenta, i.e.

$$\sum_{k=1}^m \lambda_i^{(\alpha_k)} H^{(k)}(\lambda_i) = \frac{1}{2} f_i(\lambda_i) \mu_i^2 + \gamma_i(\lambda_i), \quad i = 1, \ldots, n. \tag{12}$$

were constructed in [31] and [32].

### 3 Bi-Hamiltonian property of Stäckel systems

We start this section with a few definitions important for further considerations. As the Hamiltonian formalism is of tensorial type, there is no need to restrict ourselves to nondegenerate canonical representation of Hamiltonian vector fields. Given a manifold $\mathcal{M}$, a Poisson operator $\pi$ on $\mathcal{M}$ is a bivector (second order contravariant tensor field) with vanishing Schouten bracket

$$[\pi, \pi]_S = 0.$$

Then the bracket

$$\{f_1, f_2\}_\pi := \langle df_1, \pi(df_2) \rangle, \quad f_1, f_2 \in C^\infty(\mathcal{M}),$$

is the Lie bracket, i.e., it is skew-symmetric and satisfies the Jacobi identity. A function $c : \mathcal{M} \to \mathbb{R}$ is called the Casimir function of the Poisson operator $\pi$ if for an arbitrary function $f : \mathcal{M} \to \mathbb{R}$ we have $\{f, c\}_\pi = 0$ (or, equivalently, if $\pi dc = 0$). A linear combination $\pi_\lambda = \pi_1 - \lambda \pi_0$ ($\lambda \in \mathbb{R}$) of two Poisson
operators \( \pi_0 \) and \( \pi_1 \) is called a \textit{Poisson pencil} if the operator \( \pi_\lambda \) is Poisson for any value of the parameter \( \lambda \), i.e., when \( [\pi_0, \pi_1]_\lambda = 0 \). In this case we say that \( \pi_0 \) and \( \pi_1 \) are \textit{compatible}. Given a Poisson pencil \( \pi_\lambda = \pi_1 - \lambda \pi_0 \) we can often construct a sequence of vector fields \( X_i \) on \( \mathcal{M} \) that have two Hamiltonian representations (the so-called \textit{bi-Hamiltonian chain})

\[
X_i = \pi_1 d\lambda_i = \pi_0 d\lambda_{i+1},
\]

where \( h_i \in C^\infty(\mathcal{M}) \) are called the Hamiltonians of the chain (13) and where \( i \) is a discrete index. This sequence of vector fields may or may not terminate in zero depending on the existence of the Casimir functions for the pencil.

Consider a bi-Poisson manifold \( (\mathcal{M}, \pi_0, \pi_1) \) of dim \( \mathcal{M} = 2n + m \), where \( \pi_0, \pi_1 \) is a pair of compatible Poisson tensors of rank \( 2n \). We further assume that the Poisson pencil \( \pi_\lambda \) admits \( m \) Casimir functions which are polynomial in the pencil parameter \( \lambda \) and have the form

\[
h^{(j)}(\lambda) = \sum_{i=0}^{n_j} \lambda^{n_j-i} h_i^{(j)}, \quad j = 1, \ldots, m,
\]

so that \( n_1 + \cdots + n_m = n \) and \( h_i^{(j)} \) are functionally independent. The collection of \( n \) bi-Hamiltonian vector fields

\[
\pi_\lambda d h^{(j)}(\lambda) = 0 \iff X_i^{(j)} = \pi_1 d h_i^{(j)} = \pi_0 d h_{i+1}^{(j)}, \quad i = 1, \ldots, n_j, \quad j = 1, \ldots, m,
\]

is called the Gel’fand-Zakharevich (GZ) system of the bi-Poisson manifold \( \mathcal{M} \). Notice that each chain starts from a Casimir of \( \pi_0 \) and terminates with a Casimir of \( \pi_1 \). Moreover, all \( h_i^{(j)} \) pairwise commute with respect to both Poisson structures

\[
X_i^{(j)}(h_i^{(k)}) = \langle d h_i^{(k)}, \pi_0 d h_{i+1}^{(j)} \rangle = \langle d h_i^{(k)}, \pi_1 d h_i^{(j)} \rangle = \{h_i^{(k)}, h_i^{(j)}\}_{\pi_\lambda} = \{h_i^{(k)}, h_i^{(j)}\}_{\pi_0} = 0.
\]

In the following section we prove that an arbitrary Stäckel system on the phase space \( \mathcal{M} \) with separation conditions given by (4) can be lifted to a GZ bi-Hamiltonian system on the extended phase space \( \mathcal{M} \).

As recently proved in [15], the Stäckel Hamiltonians from separation relations (7) admit the following quasi-bi-Hamiltonian representation

\[
\Pi_1 dH_i = \sum_{j=1}^{n} F_{ij} \Pi_0 dH_j, \quad i = 1, \ldots, n,
\]

where \( \Pi_0 \) is a canonical Poisson tensor

\[
\Pi_0 = \begin{pmatrix}
0 & I_n \\
-\Lambda_n & 0
\end{pmatrix},
\]

\( I_n \) is an \( n \times n \) unit matrix, \( \Pi_1 \) is a noncanonical Poisson tensor of the form

\[
\Pi_1 = \begin{pmatrix}
0 & \Lambda_n \\
-\Lambda_n & 0
\end{pmatrix}, \quad \Lambda_n = \text{diag}(\lambda_1, \ldots, \lambda_n),
\]

compatible with \( \Pi_0 \), and the \textit{control matrix} \( F \) has the form

\[
F = (S^{-1} \Lambda_n S),
\]

where \( S \) is the associated Stäckel matrix.

To have a better insight into the functions \( F_{ij} \), we will find another representation for the entries \( F_{ij} = (S^{-1} \Lambda_n S)_{ij} \) of \( F \). To this end consider a system of \( n \) linear equations for \( V_k, k = 1, \ldots, n; \)

\[
\sum_{k=1}^{n} S^k_i (\lambda_i, \mu_i) V_k = \sum_{j=1}^{n} \lambda_i S^j_i (\lambda_i, \mu_i) a_j, \quad i = 1, \ldots, n,
\]

(18)
where \( a_i, i = 1, \ldots, n \) are some parameters. The solution of this system has the form

\[
V_r = \sum_{p=1}^{n} a_r p a_p, \quad \alpha_r = \frac{\det(S^{(rp)})}{\det(S)},
\]

where \( S^{(rp)} \) is the matrix \( S \) with the \( r \)-th column replaced by \((\lambda_1 S_1^{r}, \lambda_1 \mu_1), \ldots, \lambda_n S_n^{r} (\lambda_n \mu_n))^{T} \), the string of coefficients at the parameter \( a_p \). On the other hand, as \( V = (V_1, \ldots, V_n)^T \) and \((a_1, \ldots, a_n)^T \), system (18) can be written in the matrix form as

\[
JV = \Lambda_n J a \implies V = J^{-1} \Lambda_n J a = \alpha a,
\]

where \( \alpha_{ij} = (J^{-1} \Lambda_n J)_{ij} \). Comparing this result with (17) and (19) we find

\[
F_{ij} = (J^{-1} \Lambda_n J)_{ij} = \frac{\det(S^{(ij)})}{\det(S)}.
\]

Now, the important question is which entries \( F_{ij} \) are nonzero when the separation relations take the form (9). In other words, we want to know for which linearly dependent columns.

To answer this question, we first rewrite the quasi-bi-Hamiltonian chain (16) in the equivalent form

\[
\Pi_1 dH_i^{(k)} = \sum_{l=1}^{n} \sum_{j=1}^{m} F_{i,j}^{k,l} \Pi_0 dH_j^{(l)}, \quad k = 1, \ldots, m, \quad i = 1, \ldots, n_k
\]

adapted to the separation relations written in the form (9). Then a simple inspection shows that

\[
F_{i,i+1}^{k,k} = 1, \quad F_{i,1}^{k,l} = F_{i}^{k,l} \neq 0.
\]

Hence, the quasi-bi-Hamiltonian representation (21) takes the form

\[
\Pi_1 dH_i^{(k)} = \Pi_0 dH_{i+1}^{(k)} + \sum_{l=1}^{m} F_{i}^{k,l} \Pi_0 dH_1^{(l)}, \quad H_{n_k+1}^{(k)} = 0,
\]

where

\[
F_{i}^{k,l} = \frac{\det(S_i^{(k,l)})}{\det(S)},
\]

and \( S_i^{(k,l)} \) is the St"ackel matrix \( S \) with \((n_1 + \cdots + n_{k-1} + i)\)-th column replaced by \((\varphi_1^{(i)}, \ldots, \varphi_n^{(i)} n_i)\).

In the rest of this section we show that the representation (22) can be lifted to a GZ bi-Hamiltonian form. First, we extend the \( 2n \) dimensional phase space \( M \) to \( \tilde{M} = M \times \mathbb{R}^m \) with additional coordinates \( \psi_1, \ldots, \psi_{n_k+1} \) on \( \mathbb{R}^m \). Then, we extend the Hamiltonians as follows:

\[
H_i^{(k)}(q, p) \rightarrow \tilde{H}_i^{(k)}(q, p, c) = H_i^{(k)}(q, p) - \sum_{l=1}^{m} F_{i}^{k,l}(q, p) c_l.
\]

From (18)-(20) we infer that the separation relations for \( \tilde{h}_i^{(k)} \) read

\[
\sum_{k=1}^{m} \varphi_i^k(\lambda_i, \mu_i) \tilde{h}_i^{(k)}(\lambda_i) = \psi_i(\lambda_i, \mu_i), \quad i = 1, \ldots, n,
\]

where

\[
\tilde{h}_i^{(k)}(\lambda) = \sum_{i=0}^{n_k} \lambda^{n_k-i} h_i^{(k)}(\lambda), \quad \tilde{h}_0^{(k)} = c_k, \quad n_1 + \cdots + n_m = n.
\]

Moreover, for the functions \( F_{i}^{k,l} \) we have the same quasi-bi-Hamiltonian representation as for \( H_i^{(k)} \):

\[
\Pi_1 dF_{i}^{k,l} = \Pi_0 dF_{i+1}^{k,l} + \sum_{r=1}^{m} F_{i}^{k,r} \Pi_0 dF_{1}^{r,l},
\]
This was proved for arbitrary Stäckel systems in [15].

Denote the push-forwards of the Poisson tensors $\Pi_0$ and $\Pi_1$ to $\mathcal{M}$ by $\pi_0$ and $\pi_{1D}$. Both $\pi_0$ and $\pi_{1D}$ are degenerate and possess common Casimirs $c_i$, $i = 1, \ldots, m$. We have

$$\pi_0 = \left( \begin{array}{cc} \Pi_0 & 0 \\ 0 & 0 \end{array} \right), \quad \pi_{1D} = \left( \begin{array}{cc} \Pi_1 & 0 \\ 0 & 0 \end{array} \right).$$

(26)

Relations (22)-(26) imply that on $\mathcal{M}$ we have a quasi-bi-Hamiltonian representation with respect to the Poisson tensors $\pi_0$ and $\pi_{1D}$ of the form

$$\pi_{1D} \, dh_i^{(k)} = \pi_0 \, dh_{i+1}^{(k)} + \sum_{l=1}^{m} F_{i,l}^{k,l} \, \pi_0 \, dh_{l}^{(l)} , \quad F_{0}^{k,l} = -\delta_{kl}, \quad h_{nk+1}^{(k)} = 0.$$

(27)

Now introduce the bivector

$$\pi_1 := \pi_{1D} + \sum_{k=1}^{m} X_1^{(k)} \wedge Z_k,$$

where

$$X_1^{(k)} = \pi_0 \, dh_{1}^{(k)}, \quad Z_k = \frac{\partial}{\partial c_k}.$$

First we show that the bivector $\pi_1$ is Poisson. Using the properties of the Schouten bracket we have

$$[\pi_1, \pi_1]_S = 2 \sum_i Z_i \wedge L_{X_1^{(i)}} \pi_{1D} + 2 \sum_{i,j} [X_1^{(i)}, Z_j] \wedge Z_i \wedge X_1^{(j)},$$

(28)

where $L_X$ means the Lie derivative in the direction of $X$, and $[\cdot, \cdot]$ is the commutator of vector fields.

Now, let us prove that

$$L_{X_1^{(r)}} \Pi_{1D} = \sum_l \pi_0 \, dF_{1}^{r,l} \wedge X_1^{(l)}.$$

(29)

From (27) we have

$$Y_k := \pi_{1D} \, dh_{nk}^{(k)} = \sum_l F_{nk}^{k,l} X_1^{(l)},$$

$$\pi_{1D} \, dF_{nk}^{k,l} = \sum_r F_{nk}^{k,r} \pi_0 \, dF_{1}^{r,l}.$$ 

From the Poisson property of $\pi_{1D}$ it follows that

$$0 = L_{Y_k} \pi_{1D}$$

$$= \sum_l (F_{nk}^{k,l} L_{X_1^{(i)}} \pi_{1D} - \pi_{1D} dF_{nk}^{k,l} \wedge X_1^{(l)})$$

$$= \sum_l \left( F_{nk}^{k,l} L_{X_1^{(i)}} \pi_{1D} - \left( \sum_r F_{nk}^{k,r} \pi_0 \, dF_{1}^{r,l} \wedge X_1^{(l)} \right) \right)$$

$$= \sum_r F_{nk}^{k,r} L_{X_1^{(r)}} \pi_{1D} - \sum_r F_{nk}^{k,r} \sum_l \pi_0 \, dF_{1}^{r,l} \wedge X_1^{(l)}$$

$$= \sum_r F_{nk}^{k,r} \left( L_{X_1^{(r)}} \pi_{1D} - \sum_l \pi_0 \, dF_{1}^{r,l} \wedge X_1^{(l)} \right),$$

hence (29) is satisfied. On the other hand,

$$[X_1^{(i)}, Z_j] = \pi_0 \, dF_{1}^{i,j},$$

so (28) becomes

$$[\pi_1, \pi_1]_S = 2 \sum_{i,j} Z_i \wedge \pi_0 \, dF_{1}^{i,j} \wedge X_1^{(j)} + 2 \sum_{i,j} \pi_0 \, dF_{1}^{i,j} \wedge Z_i \wedge X_1^{(j)} = 0.$$
and thus π₁ is Poisson.

Moreover, the Poisson bivectors π₀ and π₁ are compatible as

\[ [\pi_0, \pi_1]_S = \sum_i (Z_i \wedge L_{X_i} \Pi_0 - X_i^{(i)} \wedge L_{Z_i} \pi_0) = 0. \]

Finally, the vector fields \( X_i^{(k)} \) form bi-Hamiltonian chains with respect to \( \pi_0, \pi_1 \). Indeed, we have

\[
\pi_1 dh_i^{(k)} = \pi_0 dh_i^{(k+1)} + \sum_{l=1}^{m} F_{i}^{k,l} \pi_0 dh_i^{(l)} + \sum_{l=1}^{m} (X_1^{(l)} \wedge Z_i)dh_i^{(k)} \\
= \pi_0 dh_i^{(k+1)} = X_i^{(k+1)},
\]

as

\[
\sum_{l=1}^{m} (X_1^{(l)} \wedge Z_i)dh_i^{(k)} = -\sum_{l=1}^{m} F_{i}^{k,l} X_1^{(l)} = -\sum_{l=1}^{m} F_{i}^{k,l} \Pi_0 dh_i^{(l)}.
\]

Quite obviously,

\[ \pi_1 dh_{n_k}^{(k)} = 0, \quad X_1^{(l)} = \pi_1 dc_l = \Pi_1 dh_{l_0}^{(l)}, \]

so \( h^{(k)}(\lambda) \) are polynomial in \( \lambda \) Casimir functions of the Poisson pencil \( \pi_\lambda = \pi_1 - \lambda \pi_0 \).

4 Examples

Here we illustrate the above results with three examples of separable systems with three degrees of freedom. Two of them are classical Stäckel systems with separation relations quadratic in the momenta while the third example has separation relations cubic in the momenta.

Example 1.

Consider the separation relations on a six-dimensional phase space given by the following bare (potential-free) separation curve

\[
H_1 \lambda^2 + H_2 \lambda + H_3 = \frac{1}{2} \mu^2
\]

from the class \([10]\). This curve corresponds to geodesic motion for a classical Stäckel system (of Benenti type). The transformation \((\lambda, \mu) \to (q, p)\) to the flat coordinates of associated metric follows from the point transformation

\[
\sigma_1(q) = q_1 = \lambda^1 + \lambda^2 + \lambda^3, \quad \sigma_2(q) = \frac{1}{4} q_1^2 + \frac{1}{4} q_2^2 = \lambda^1 \lambda^2 + \lambda^3 \lambda^3, \quad \sigma_3(q) = \frac{1}{4} q_1 q_2 + \frac{1}{2} q_3 = \lambda^1 \lambda^2 \lambda^3.
\]

In the flat coordinates the Hamiltonians take the form

\[
H_1 = p_1 p_3 + \frac{1}{2} p_2^2, \quad H_2 = \frac{1}{2} q_1 q_3^2 - \frac{1}{2} q_1 q_2^2 + \frac{1}{2} q_2 p_2 p_3 - \frac{1}{2} p_1 p_2 - \frac{1}{2} q_1 p_1 p_3, \quad H_3 = \frac{1}{2} q_1 q_3^2 + \frac{1}{2} q_1 q_2^2 + \frac{1}{2} q_2 p_1 p_3 + \frac{1}{2} q_2 p_2 p_3 - \frac{1}{2} q_1 q_2 p_2 p_3 - \frac{1}{2} q_3 p_2 p_3
\]

and admit a quasi-bi-Hamiltonian representation \([10]\) with the operators \( \Pi_0 \) and \( \Pi_1 \) of the form

\[
\Pi_0 = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}, \quad \Pi_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & q_1 & -1 & 0 \\ 0 & 0 & 0 & q_2 & 0 & -1 \\ 0 & 0 & 0 & 2q_3 & q_2 & q_1 \\ -q_1 & -q_2 & -2q_3 & 0 & p_2 & p_3 \\ 1 & 0 & -q_2 & -p_2 & 0 & 0 \\ 0 & 1 & -q_1 & -p_3 & 0 & 0 \end{pmatrix}. \quad (31)
\]
and the control matrix

\[ F = \begin{pmatrix} q_1 & 1 & 0 \\ -\frac{1}{2}q_1^2 - \frac{1}{2}q_2^2 & 0 & 1 \\ \frac{1}{2}q_1q_2 + \frac{1}{2}q_3 & 0 & 0 \end{pmatrix}. \]

On the extended phase space of dimension seven, with an additional coordinate \( c \), we have the extended Hamiltonians \( H_{\bar{h}} \)

\[ h_0 = c, \]
\[ h_1 = p_1p_3 + \frac{1}{2}p_2^2 - cq_1, \]
\[ h_2 = \frac{1}{2}q_3p_3 - \frac{1}{2}q_1p_2^2 + \frac{1}{2}q_2p_2p_3 - \frac{1}{2}q_1p_2 - \frac{1}{2}q_1p_1p_3 + \left( \frac{1}{2}q_1^2 + \frac{1}{2}q_2^2 \right)c, \]
\[ h_3 = \frac{1}{2}q_3^2p_3^2 + \frac{1}{2}q_1^2p_2^2 + \frac{1}{2}p_1^2 + \frac{1}{2}q_1p_1p_2 + \frac{1}{2}q_2p_1p_3 - \frac{1}{2}q_1q_2p_2p_3 - \frac{1}{2}q_3p_2p_3 - \left( \frac{1}{2}q_1q_2 + \frac{1}{2}q_3 \right)c. \]

They form a bi-Hamiltonian chain

\[ \pi_0 dh_0 = 0, \]
\[ \pi_0 dh_1 = X_1 = \pi_1 dh_0, \]
\[ \pi_0 dh_2 = X_2 = \pi_1 dh_1, \]
\[ \pi_0 dh_3 = X_3 = \pi_1 dh_2, \]
\[ 0 = \pi_1 dh_3, \]

with the Poisson operators \( \pi_0 \) and \( \pi_1 \), where

\[ \pi_0 = \begin{pmatrix} \Pi_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_1 = \begin{pmatrix} \Pi_1 & X_1 \\ -X_3 & 0 \end{pmatrix}. \]

The separation curve for the extended system takes the form

\[ c\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 = \frac{1}{8}\mu^2. \]

**Example 2.**

Consider now separation relations on a six-dimensional phase space given by the following bare separation curve

\[ \bar{H}_1\lambda^3 + \bar{H}_2\lambda^2 + \bar{H}_3 = \frac{1}{8}\mu^2 \]

from the class \( (12) \). When written using the notation \( (2) \), this curve takes the form

\[ \lambda^2(H_1^{(1)}\lambda + H_2^{(1)}) + H_1^{(2)} = \frac{1}{8}\mu^2 \]

and again represents geodesic motion for a classical Stäckel system (this time of non-Benenti type). Using the coordinates, the Hamiltonians, and the functions \( \sigma_i \) from the previous example we find that

\[ \bar{H}_1 = H_1^{(1)} = -\frac{1}{\sigma_2}H_2, \]
\[ \bar{H}_2 = H_2^{(1)} = \bar{H}_1 - \frac{\sigma_1}{\sigma_2}H_2, \]
\[ \bar{H}_3 = H_1^{(2)} = H_3 - \frac{\sigma_1}{\sigma_2}H_2 \]

and thus we see that the Hamiltonians \( \bar{H}_i \) are related to \( H_i \) through the so-called generalized Stäckel transform (see [32] for further details on the latter). One can show that the metric tensor associated to the Hamiltonian \( \bar{H}_1 \) is not flat anymore.

The Hamiltonians \( \bar{H}_i \) form a quasi-bi-Hamiltonian chain \( (10) \) with the Poisson tensors \( (30),(31) \) and the control matrix

\[ F = \begin{pmatrix} \sigma_1 - \frac{\sigma_2}{\sigma_1} & \sigma_1 & -\frac{1}{\sigma_2} \\ -\sigma_2 + \frac{\sigma_1}{\sigma_2} & 0 & \frac{\sigma_1}{\sigma_2} \\ \sigma_2^2 & \sigma_2 & 0 \end{pmatrix}. \]
On the extended phase space of dimension eight, with additional coordinates \( c_1, c_2 \), we have the extended Hamiltonians (23):

\[
\begin{align*}
h^{(1)}_0 &= c_1, \\
h^{(1)}_1 &= H^{(1)}_1 - (\sigma_1 - \frac{c_1}{\sigma_2})c_1 + \frac{1}{\sigma_2}c_2, \\
h^{(1)}_2 &= H^{(1)}_2 + (\sigma_2 - \frac{c_2}{\sigma_1})c_1 - \frac{c_1}{\sigma_2}c_2, \\
h^{(2)}_0 &= c_2, \\
h^{(2)}_1 &= H^{(2)}_1 - \frac{\sigma_3^2}{\sigma_2}c_1 - \frac{\sigma_1}{\sigma_2}c_2.
\end{align*}
\]

They form two bi-Hamiltonian chains

\[
\begin{align*}
\pi_0 dh_0 &= 0 \\
\pi_0 dh^{(1)}_0 &= X^{(1)}_1 = \pi_1 dh^{(1)}_1 \\
\pi_0 dh^{(1)}_2 &= X^{(1)}_2 = \pi_1 dh^{(1)}_2 \\
0 &= \pi_1 dh^{(1)}_2 \\
\pi_0 dh^{(2)}_0 &= 0 \\
\pi_0 dh^{(2)}_1 &= X^{(2)}_1 = \pi_1 dh^{(2)}_1 \\
\pi_0 dh^{(2)}_1 &= 0 = \pi_1 dh^{(2)}_1,
\end{align*}
\]

with the Poisson operators \( \pi_0 \) and \( \pi_1 \) of the form

\[
\begin{align*}
\pi_0 &= \begin{pmatrix} \Pi_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\pi_1 &= \begin{pmatrix} \Pi_1 & X^{(1)}_1 & X^{(2)}_1 \\ -(X^{(1)}_1)^T & 0 \end{pmatrix}.
\end{align*}
\]

The separation curve for the extended system takes the form

\[
\lambda^2 (c_1\lambda^2 + h^{(1)}_1\lambda + h^{(1)}_2) + c_2\lambda + h^{(2)}_1 = \frac{1}{8}\mu^2.
\]

**Example 3.**

Consider separation relations on a six-dimensional phase space given by the following bare separation curve cubic in the momenta

\[
\mu H^{(1)}_1 + H^{(2)}_1\lambda + H^{(2)}_2 = \mu^3
\]

from the class (11). The transformation \((\lambda, \mu) \to (q, p)\) to new canonical coordinates in which all Hamiltonians are of a polynomial form is obtained from the following two transformations:

\[
\begin{align*}
u_1 &= 3q_3 - 3q_3, \\
u_2 &= -q_1p_3 - q_1p_3 + 3q_3^3 + 5q_1^3 - 6q_2q_3, \\
u_3 &= -q_3^3 - 9q_1q_3 + q_1q_3p_2 + q_1q_3p_3 - \frac{1}{27}q_1^3q_2 + q_2^3p_1 + 3q_2q_3^2, \\
v_1 &= \frac{-1}{q_1}, \\
v_2 &= \frac{3q_2 - 2q_3}{q_1}, \\
v_3 &= p_3 + \frac{2}{7}p_2 - \frac{q_2^3}{q_1} + \frac{3q_2q_3}{q_1} - 4q_1^2,
\end{align*}
\]

and

\[
\begin{align*}
u_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\
u_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\
u_3 &= \lambda_1\lambda_2\lambda_3, \\
m_i &= v_1\lambda_i^2 + v_2\lambda_i + v_3, \quad i = 1, 2, 3.
\end{align*}
\]
In the \((q, p)\)-coordinates the Hamiltonians take the form

\[
\begin{align*}
H_1^{(1)} &= p_2 p_3 + \frac{1}{3} p_2^3 + p_3^3 - 7 q_1^2 p_3 - 4 q_1^2 p_2 - 3 q_2 p_1 + 18 q_1 q_2^2 + 13 q_1^4 + 12 q_3 q_2 q_1, \\
H_1^{(2)} &= 12 q_1^2 q_2 + 8 q_1^3 q_3 - 2 q_1^2 p_1 + ( -6 q_1 q_2 - 4 q_1 q_1) p_3 + p_1 p_3, \\
H_2^{(2)} &= \frac{1}{3} p_2 p_3^3 + \frac{1}{3} p_2^3 p_3^2 + \frac{2}{3} p_3^3 - q_1^2 p_3^2 - q_2 p_1 p_2 - q_1 p_1^2 - \frac{10}{3} q_1^2 p_3 p_2 \\
&\quad + (q_3 - 3 q_2) p_1 p_3 + (21 q_1^2 q_2 + 6 q_3 q_1^2) p_1 + (4 q_3 q_1 q_2 + 6 q_1 q_2^2 + \frac{2}{3} q_4^3) p_2 \\
&\quad + (7 q_1^4 + 18 q_2^2 - 6 q_3 q_1 q_2 - 4 q_1^3 q_3) p_3 - 8 q_1^2 q_2^2 - 72 q_3 q_2 q_1^2 - 90 q_1^2 q_2^2 - 12 q_4^6.
\end{align*}
\]

They form a quasi-bi-Hamiltonian chain \([10]\) with the non-canonical Poisson operator

\[
\Pi_1 = \begin{pmatrix}
0 & 0 & 0 & -q_3 & 3 q_1 & 2 q_2 \\
0 & 0 & -\frac{1}{3} q_1 & A & 3 q_2 - q_3 & -q_2 \\
0 & \frac{1}{2} q_1 & 0 & 2 q_1^2 & 0 & -q_3 \\
- q_3 & -A & -2 q_1^2 & 0 & C & \\
1 - 3 q_1 & -3 q_2 + q_3 & 0 & -C & 0 & -24 q_1^4 \\
0 & q_2 & q_3 & -B & 24 q_1^4 & 0
\end{pmatrix},
\]

where \(A = -\frac{1}{3} p_2 + \frac{1}{3} p_3 - 3 q_1^2\), \(B = 5 q_1 q_2 + 24 q_1 q_3 - 3 p_1\), \(C = -24 q_1 q_2 - 12 q_1 q_3 + p_1\) and the control matrix

\[
F = \begin{pmatrix}
-q_3 & -q_1 & 0 \\
-\frac{1}{3} p_2 + q_1^2 & -2 q_3 + 3 q_2 & 1 \\
5 q_3 q_1^2 + q_3 q_2 - q_1 p_1 & 4 q_3 q_1^3 - q_3^2 + 3 q_2 q_3 + \frac{2}{3} q_1 p_2 + q_1 p_3 & 0
\end{pmatrix}.
\]

On the extended phase space of dimension eight, with additional coordinates \(c_1, c_2\), the extended Hamiltonians \([23]\) are

\[
\begin{align*}
h_0^{(1)} &= c_1, \\
h_1^{(1)} &= H_1^{(1)} + q_3 c_1 + q_1 c_2, \\
h_0^{(2)} &= c_2, \\
h_1^{(2)} &= H_1^{(2)} + (\frac{1}{3} p_2 - q_1^2) c_1 + (2 q_3 - 3 q_2) c_2, \\
h_2^{(2)} &= H_2^{(2)} - (5 q_3 q_1^2 + 6 q_1^2 q_2 - q_1 p_1 - \frac{4}{3} q_3 p_2) c_1 - (-4 q_1^3 - q_3^2 + 3 q_2 q_3 + \frac{2}{3} q_1 p_2 + q_1 p_3) c_2.
\end{align*}
\]

They form two bi-Hamiltonian chains

\[
\begin{align*}
\pi_0 dh_0^{(1)} &= 0, \\
\pi_0 dh_1^{(1)} &= X_1^{(1)} = \pi_1 dh_0^{(1)}, \\
0 &= \pi_1 dh_1^{(1)}, \\
\pi_0 dh_0^{(2)} &= 0, \\
\pi_0 dh_1^{(2)} &= X_1^{(2)} = \pi_1 dh_0^{(2)}, \\
\pi_0 dh_2^{(2)} &= X_2^{(2)} = \pi_1 dh_1^{(2)}, \\
0 &= \pi_1 dh_2^{(2)}
\end{align*}
\]

with the corresponding Poisson operators \(\pi_0\) and \(\pi_1\)

\[
\pi_0 = \begin{pmatrix}
\Pi_0 \\
0 \\
0
\end{pmatrix}, \quad \pi_1 = \begin{pmatrix}
\Pi_1 \\
-(X_1^{(1)})^T \\
-(X_2^{(2)})^T
\end{pmatrix}.
\]

The separation curve for the extended system takes the form

\[
\mu (c_1 \lambda + h_1^{(1)}) + c_2 \lambda^2 + h_2^{(2)} \lambda + h_2^{(2)} = \mu^3.
\]
5 Summary

We have considered the Stäckel systems classified using their separation relations. The most general form of the separation relations considered in the present paper is

$$\sum_{k=1}^{m} S^k_i(\mu_i, \lambda_i) H^{(k)}(\lambda_i) = \psi_i(\lambda_i, \mu_i), \quad m \leq n, \quad i = 1, \ldots, n,$$

where

$$H^{(k)}(\lambda_i) = \sum_{j=1}^{n_k} \lambda_i^{n_k-j} H_j^{(k)}, \quad n_1 + \cdots + n_m = n$$

and $S^k_i, \psi_i$ are smooth functions of their arguments. Moreover, we have proved that all systems whose separation relations are of the above form admit (after the lift to an extended phase space) Gel’fand-Zakharevich bi-Hamiltonian representation. This confirms universality of the latter property for the Stäckel systems. As a consequence, a geometric separability theory, based on the existence of GZ bi-Hamiltonian representation of a given system, is applicable for all Liouville integrable systems from the classes we considered.

Quite obviously, the knowledge of quasi-bi-Hamiltonian representation is sufficient for separability of a given Liouville integrable system. Unfortunately, there is no systematic method available for the construction of such representation. On the other hand, there are some systematic methods for finding the bi-Hamiltonian representation. From this point of view the result presented in this paper is of interest, as it shows that the existence of bi-Hamiltonian representation is an inherent property of Stäckel systems.

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