BILINEAR ESTIMATES AND APPLICATIONS TO GLOBAL WELL-POSEDNESS FOR THE DIRAC-KLEIN-GORDON EQUATION ON $\mathbb{R}^{1+1}$

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Abstract. We prove new bilinear estimates for the $X^{s,b}_\pm(\mathbb{R}^2)$ spaces which are optimal up to endpoints. These estimates are often used in the theory of nonlinear Dirac equations on $\mathbb{R}^{1+1}$. The proof of the bilinear estimates follows from a dyadic decomposition in the spirit of Tao [21] and D’Ancona, Foschi, and Selberg [11]. As an application, by using the $I$-method of Colliander, Keel, Staffilani, Takaoka, and Tao, we extend the work of Tesfahun [23] on global existence below the charge class for the Dirac-Klein-Gordon equation on $\mathbb{R}^{1+1}$.

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1. Introduction

We consider the problem of proving bilinear estimates in the Bourgain-Klainerman-Machedon type spaces $X^{s,b}_\pm$ on $\mathbb{R}^2$, where we define the spaces $X^{s,b}_\pm$ via the norm

$$
\|\psi\|_{X^{s,b}_\pm} = \|\langle \tau \pm \xi \rangle^b \langle \xi \rangle^s \psi(\tau,\xi)\|_{L^2_\tau(\mathbb{R}^2)}
$$

with $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. These spaces have been used in the low regularity theory of various nonlinear Dirac equations in one space dimension, [14, 20], as well as the Dirac-Klein-Gordon (DKG) system [17, 19]. Though recently, product Sobolev spaces based on the null coordinates $x \pm t$ have also proved useful [6, 16]. In applications of the $X^{s,b}_\pm$ spaces to low regularity well-posedness, we often require product estimates of the form

$$
\|uv\|_{X^{-s_1,-b_1}_\pm} \lesssim \|u\|_{X^{s_2,b_2}_\pm} \|v\|_{X^{s_3,b_3}_\pm}
$$

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where \( s_j, b_j \in \mathbb{R} \) and \( \pm \) are independent choices of \( \pm \). A number of estimates of this form, for specific values of \( s_j \) and \( b_j \), have appeared previously in the literature \cite{13, 19, 20}. The case where \( \pm_1 = \pm_2 = \pm_3 \) is not particularly interesting, as a simple change of variables reduces \((1)\) to two applications of the 1-dimensional Sobolev product estimate
\[
\|fg\|_{H^{-s_1}(\mathbb{R})} \lesssim \|f\|_{H^{s_2}(\mathbb{R})} \|g\|_{H^{s_3}(\mathbb{R})}.
\]
Thus leading to the conditions
\[
b_j + b_k > 0, \quad b_1 + b_2 + b_3 > \frac{1}{2} \tag{2}
\]
and
\[
s_j + s_k > 0, \quad s_1 + s_2 + s_3 > \frac{1}{2} \tag{3}
\]
where \( j \neq k \). On the other hand, if we have \( \pm_1 = \pm_2 = \mp \) and \( \pm_3 = \mp \), then we can make significant improvements over \((3)\). This observation allows one to exploit the null structure that is often found in nonlinear hyperbolic systems in one dimension, see for instance \cite{20}.

To state our first result we use the following conventions. For a set of real numbers \( \{a_1, a_2, a_3\} \), we let \( a_{\text{max}} = \max_i a_i \), \( a_{\text{min}} = \min_i a_i \), and use \( a_{\text{med}} \) to denote the median. If \( a \in \mathbb{R} \) then we define
\[
a_+ = \begin{cases} 
  a & a > 0 \\
  0 & a \leq 0.
\end{cases}
\]
We state our product estimate in the dual form.

**Theorem 1.** Let \( s_j, b_j \in \mathbb{R}, j = 1, 2, 3 \) satisfy
\[
b_1 + b_2 + b_3 > \frac{1}{2}, \quad b_j + b_k > 0, \quad (j \neq k) \tag{4}
\]
and for \( k \in \{1, 2\} \)
\[
s_1 + s_2 \geq 0, \quad s_k + s_3 \geq -b_{\text{min}}, \quad s_k + s_3 > \frac{1}{2} - b_1 - b_2 - b_3, \tag{5}
\]
\[
s_1 + s_2 + s_3 \geq \left(\frac{1}{2} - b_{\text{max}}\right)_+ + \left(\frac{1}{2} - b_{\text{med}}\right)_+ - b_{\text{min}}.
\]
Then
\[
\left| \int_{\mathbb{R}^2} \Pi_{j=1}^3 \psi_j(t,x) dx dt \right| \lesssim \|\psi_1\|_{X_{+}^{s_1, b_1}} \|\psi_2\|_{X_{+}^{s_2, b_2}} \|\psi_3\|_{X_{+}^{s_3, b_3}}. \tag{6}
\]
Moreover the conditions \((4)\) and \((5)\) are sharp up to equality.

\(^1\)For the sake of exposition, we are ignoring the endpoint cases. The sharp result allows one of the inequalities in \((2)\) to be replaced with an equality, a similar comment applies to the condition \((3)\).
Remark 1. There are cases where we can allow equality in (4) or (5), for instance the case
\[ s_1 = s_2 = s_3 = 0, \quad b_1 = 0, \quad b_2 = b_3 = \frac{1}{2} + \epsilon \]
holds \cite{19} Corollary 1]. We have not attempted to list or prove the endpoint cases here, as this would
significantly complicate the statement of Theorem 1. Additionally, Theorem 1 is sufficient for our intended
application to global well-posedness for the Dirac-Klein-Gordon equation.

Define the Wave-Sobolev spaces \( H^{s,b} \) by using the norm
\[ \| \psi \|_{H^{s,b}} = \| \langle |\tau| - |\xi| \rangle^b \langle |\xi| \rangle^s \tilde{\psi}(\tau, \xi) \|_{L^2_{\tau,\xi}(\mathbb{R}^2)}. \]

Then as a simple corollary to Theorem 1 we can replace one of the \( X^{s,b}_{\pm} \) norms on the righthand side of \( (6) \) with a \( H^{s,b} \) norm.

Corollary 2. Let \( r, s_1, s_2, b_j \in \mathbb{R}, j = 1, 2, 3 \) satisfy
\[ b_1 + b_2 + b_3 > \frac{1}{2}, \quad b_j + b_k > 0, \quad (j \neq k) \]
and for \( k \in \{1, 2\} \)
\[ s_k + r \geq 0, \]
\[ s_k + r > -b_{\text{min}}, \]
\[ s_1 + s_2 > -b_{\text{min}}, \]
\[ s_1 + s_2 > \frac{1}{2} - b_1 - b_2 - b_3, \]
\[ s_1 + s_2 + r > \frac{1}{2} - b_k, \]
\[ s_1 + s_2 + r > \left( \frac{1}{2} - b_{\text{max}} \right)_+ + \left( \frac{1}{2} - b_{\text{med}} \right)_+ - b_{\text{min}}. \]

Then
\[ \left| \int_{\mathbb{R}^2} \Pi_{j=1}^3 \psi_j(t, x) dx dt \right| \lesssim \| \psi_1 \|_{X^{s_1, b_1}} \| \psi_2 \|_{X^{s_2, b_2}} \| \psi_3 \|_{H^{r,b}_3}. \]

Proof. We decompose \( \psi_3 \) into the regions \( \{ (\tau, \xi) \in \mathbb{R}^{1+1} \mid \pm \tau \xi \geq 0 \} \) and observe that on the first region \( \langle |\tau| - |\xi| \rangle = \langle \tau - \xi \rangle \) while in the second region \( \langle |\tau| - |\xi| \rangle = \langle \tau + \xi \rangle \). The corollary now follows from two applications of Theorem 1. \hfill \Box

Remark 2. This result should be compared to the similar estimates contained in \cite{19} and \cite{23}. Also we note that the decomposition used in the proof of Corollary 2 can be used to give bilinear estimates in the Wave-Sobolev spaces \( H^{r,b} \), thus giving an alternative (though closely related) proof of Theorem 7.1 in \cite{10} (up to endpoints).

The second main result contained in this article concerns the global existence problem for the DKG
equation on \( \mathbb{R}^{1+1} \). The DKG equation can be written as
\[ (\gamma_0 \partial_t + \gamma_1 \partial_x) \psi = -iM\psi + i\phi\psi \]
\[ -\Box + m^2 \phi = \langle \gamma_0 \psi, \psi \rangle_{C^2} \]
with initial data
\[ \psi(0) = \psi_0 \in H^r, \quad \phi(0) = \phi_0 \in H^r, \quad \partial_t \phi(0) = \phi_1 \in H^{r-1}. \]
for some values of $s, r \in \mathbb{R}$. The d’Alembertian is defined by $\Box = -\partial_t^2 + \partial_x^2$ and we take the standard representation of the Dirac matrices
\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
The Dirac spinor $\psi \in \mathbb{C}^2$, and the real-valued scalar field $\phi \in \mathbb{R}$, are functions of $(t, x) \in \mathbb{R}^{1+1}$. The notation $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ refers to the standard inner product on $\mathbb{C}^2$, and $m, M \in \mathbb{R}$ are constants.

There are two main features of the DKG equation (7) which we wish to highlight here. The first feature concerns the conservation of charge which can be stated as follows: if $(\psi, \phi)$ is a smooth solution to (7) with sufficient decay at infinity, then for all times $t \in \mathbb{R}$ we have
\[
\|\psi(t)\|_{L^2} = \|\psi(0)\|_{L^2}.
\]
The conservation of charge is crucial in controlling the global behaviour of the solution $(\psi, \phi)$. The second feature we would like to note is that the nonlinearity in the DKG equation has null structure. Roughly speaking, this refers to the fact that the nonlinear terms in (7) behave significantly better than generic products. The null structure is a crucial component in the low regularity existence theory for the DKG equation and has been used by a number of authors [5, 12, 15, 17, 19]. The observation that null structure can be used to improve local existence results for nonlinear wave equations is due to Klainerman and Machedon in [13].

The question of local well-posedness (LWP) for the DKG equation was first considered by Chadam [7]. Subsequently, much progress has been made by numerous authors [3, 4, 7, 12, 17, 19]. The best result to date is due to Machihara, Nakanishi, and Tsugawa [16] where it was shown that (7) with initial data (8) is locally well-posed provided
\[
s > -\frac{1}{2}, \quad |s| \leq r \leq s + 1.
\]
Moreover, this region is essentially sharp, except possibly at the endpoint $s = -\frac{1}{2}$. More precisely, outside this region the solution map is either ill-posed, or fails to be twice differentiable, see [16] for a more precise statement.

In the current article we are interested in the minimum regularity required on the initial data (8) to ensure that the corresponding local in time solution $(\psi, \phi)$ to (7) can be extended globally in time. Global well-posedness (GWP) in the high regularity case $s = r = 1$ was first proven by Chadam [7], this was then progressively lowered to $s \geq 0$ by a number of authors [3, 4, 7, 12, 17] by exploiting the conservation of charge (9) together with the local well-posedness theory. The first result below the charge class was due to Selberg [18] where it was shown that the DKG equation is GWP in the region $-\frac{1}{8} < s < 0$,
\[
-s + \sqrt{s^2 - s} < r \leq s + 1.
\]
Note that when $s < 0$, the conservation of charge cannot be used directly since $\psi \not\in L^2$, thus the problem of global existence is significantly more difficult. Instead Selberg made use of the Fourier truncation method of Bourgain [2], which allows one to take initial data just below a conserved quantity. There is a difficulty in directly applying this method to the DKG equation however, as there is no conservation law

\[2\text{Note that this also gives GWP in the region } s > 0, |s| \leq r \leq s + 1\text{ by persistence of regularity, see for instance } 19.\]
Figure 1. Global well-posedness holds in the shaded region by Theorem 3. Local well-posedness holds inside the the lines $r = |s|$ and $r = s + 1$ for $s > -\frac{1}{2}$ by [16].

for the scalar $\phi$. Instead, one needs to exploit the fact the nonlinearity for $\phi$ depends only on the spinor $\psi$. Thus, as we have control over $\psi$ via the conservation of charge, we should be able to estimate the growth of $\phi$. This strategy was implemented by Selberg via an induction argument involving the cascade of free waves.

Currently, the best result for GWP for the DKG equation is due to Tesfahun [23] where the GWP region of Selberg was extended to

$$\frac{-1}{8} < s < 0, \quad s - \frac{1}{4} + \sqrt{(s - \frac{1}{4})^2 - s} < r \leq s + 1.$$ 

The improvement comes from applying the $I$-method of Colliander, Keel, Staffilani, Takaoka, and Tao, see for instance [8] for an introduction to the $I$-method. In the current article, we prove the following.

**Theorem 3.** The DKG equation (7) is globally well-posed for initial data $\psi_0 \in H^s$, $(\phi_0, \phi_1) \in H^r \times H^{r-1}$ provided

$$\frac{-1}{6} < s < 0, \quad s - \frac{1}{4} + \sqrt{(s - \frac{1}{4})^2 - s} < r \leq s + 1.$$ 

The proof of Theorem 3 follows the argument used in [23] together with the bilinear estimates in Theorem 1. More precisely, we use the $I$-method together with the induction on free waves approach of Selberg. The main idea, following the usual $I$-method, is to define a mild smoothing operator $I$ such that, firstly, for some large constant $N$, we have the estimate

$$\|If\|_{L^2(\mathbb{R})} \lesssim N^{-s}\|f\|_{H^s(\mathbb{R})} \lesssim N^{-s}\|f\|_{L^2}.$$  \hspace{1cm} (10)$$

Secondly, we require $I$ to be the identity on low frequencies. We then try to estimate the growth of $\|I\psi(t)\|_{L^2}$ in terms of $t$. It turns out that despite the fact that $I\psi$ no longer solves the DKG equation, there is sufficient cancelation of frequencies to ensure that the charge $\|I\psi(t)\|_{L^2}$ is almost conserved. This
almost conservation property follows from the usual proof of the conservation of charge, together with a number of applications of Theorem 1. Thus we can estimate the growth of \( \|\psi(t)\|_{H^s} \) from \( (10) \). The induction on free waves approach of Selberg then allows us to control the scalar field \( \phi \) and completes the proof of Theorem 3.

We now give a brief outline of this article. In Section 2 we recall some properties of the \( X^{s,b} \) and \( H^{r,b} \) spaces which we require in the proof of Theorem 3. The proof of Theorem 3 is contained in Section 3. In Section 4 we prove that the conditions in Theorem 1 are sufficient for the estimate (6). Finally, the counter examples showing that Theorem 1 is sharp up to equality are contained in Section 5.

**Notation:** The Fourier transform on \( \mathbb{R} \) of a function \( f \in L^1(\mathbb{R}) \) is denoted by \( \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx \). We use the notation \( \hat{f}(\tau, \xi) \) for the space-time Fourier transform of a function \( f(t,x) \) on \( \mathbb{R}^{1+1} \). We write \( a \lesssim b \) if there is some constant \( C \), independent of the variables under consideration, such that \( a \leq C b \). If we wish to make explicit that the constant \( C \) depends on \( \delta \) we write \( a \lesssim_{\delta} b \). Occasionally we write \( a \ll b \) if \( C < 1 \). We use \( a \approx b \) to denote the inequalities \( a \lesssim b \) and \( b \lesssim a \).

All sums such as \( \sum_N f(N) \) are over dyadic numbers \( N \in 2^\mathbb{N} \). Given dyadic variables \( N_1, N_2, N_3 \in 2^\mathbb{N} \), we use the short hand

\[
\sum_{N_{\max} \leq N_{med}} = \sum_{N_{\max} \in 2^i} \sum_{N_{med} \in 2^i} \sum_{N_{\min} \in 2^i}.
\]

We let \( 1_\Omega \) denote the characteristic function of the set \( \Omega \); we occasionally abuse notation and write \( 1_{\{|x| \leq N\}} \) instead of \( 1_{\{|x| \leq N\}} \). The standard Sobolev space \( H^s \) is defined as the completion of \( C_0^\infty \) using the norm

\[
\|f\|_{H^s} = \|\langle \xi \rangle^s \hat{f}\|_{L^2}.
\]

If \( u \) is a function of \( (t, x) \in \mathbb{R}^{1+1} \) we use the notation

\[
\|u[t]\|_{H^s} = \|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}}.
\]

To handle solutions to the wave equation, we make use of the Banach space \( H^{r,b} \) defined via the norm

\[
\|\varphi\|_{H^{r,b}} = \|\varphi\|_{H^{r,b}} + \|\partial_t \varphi\|_{H^{r-1,b}}.
\]

The proof of Theorem 3 requires the use of the local in time versions of the \( X^{s,b}_\pm \) and \( H^{r,b} \) spaces. Let \( S_{\Delta T} = [0, \Delta T] \times \mathbb{R} \). We define \( X^{s,b}_\pm(S_{\Delta T}) \) by restricting elements of \( X^{s,b}_\pm \) to \( S_{\Delta T} \). More precisely,

\[
X^{s,b}_\pm(S_{\Delta T}) = X^{s,b}_\pm \cap \{ f \in X^{s,b}_\pm \mid |f|_{S_{\Delta T}} = 0 \}.
\]

The local in time space \( X^{s,b}_\pm(S_{\Delta T}) \) is a Banach space with norm

\[
\|\varphi\|_{X^{s,b}_\pm(S_{\Delta T})} = \inf_{u = \varphi \text{ on } S_{\Delta T}} \|u\|_{X^{s,b}_\pm}.
\]

If \( b > \frac 12 \), then we have the continuous embedding \( X^{s,b}_\pm(S_{\Delta T}) \subset C([0, \Delta T], H^s) \). We define the Banach spaces \( H^{r,b}(S_{\Delta T}) \) similarly and note that, if \( b > \frac 12 \), then we have the continuous embedding \( H^{r,b}(S_{\Delta T}) \subset C([0, \Delta T], H^r) \cap C^1([0, \Delta T], H^{r-1}) \).
2. Linear Estimates

Here we briefly recall some of the important properties of the $X^{s,b}$ and $\mathcal{H}^{r,b}$ spaces which we make use of in the proof of Theorem 3 for more details we refer the reader to [9] and [22]. We start by recalling some properties of the localised spaces $X^{s,b}_\pm(S_{\Delta T})$.

**Lemma 4.** Let $s \in \mathbb{R}$, $0 < \Delta T < 1$, and $\nu \in C_0^\infty(\mathbb{R})$. If $-\frac{1}{2} < b_1 \leq b_2 < \frac{1}{2}$ then

$$\left\| \nu \left( \frac{t}{\Delta T} \right) u(t, x) \right\|_{X^{s,b}_\pm} \lesssim \Delta T^{b_2-b_1} \| u \|_{X^{s,b}_\pm}.$$  

Consequently, we have $\| u \|_{X^{s,b}_\pm(S_{\Delta T})} \lesssim \Delta T^{b_2-b_1} \| u \|_{X^{s,b}_\pm(S_{\Delta T})}$. Moreover if $-\frac{1}{2} < b < \frac{1}{2}$ then

$$\| 1_{[0,\Delta T]}(t) u \|_{X^{s,b}_\pm(S_{\Delta T})} \lesssim \| u \|_{X^{s,b}_\pm(S_{\Delta T})}$$  

with constant independent of $\Delta T$.

**Proof.** The first conclusion is well known and can be found in, for instance, [22]. The second conclusion is perhaps not as well known and for the convenience of the reader we include the proof here. The definition of $X^{s,b}(S_{\Delta T})$ together with a change of variables on the frequency side shows that is suffices to prove

$$\| 1_{[0,\Delta T]}(t) f \|_{H^b} \lesssim \| f \|_{H^b}.$$  

By duality we may assume that $0 < b < \frac{1}{2}$. Then by a well-known characterisation of the Sobolev spaces $H^s$, (see for instance [1]) we have

$$\| 1_{[0,\Delta T]} f \|_{H^b}^2 \approx \| 1_{[0,\Delta T]} f \|_{L^2}^2 + \int_{t \in \mathbb{R}} \int_0^{\Delta T} \frac{|1_{[0,\Delta T]}(t) f(t) - 1_{[0,\Delta T]}(t') f(t')|^2}{|t-t'|^{1+2b}} dt'dt$$  

$$\lesssim \| f \|_{L^2}^2 + \int_0^{\Delta T} \int_0^{\Delta T} \frac{|f(t')|^2}{|t-t'|^{1+2b}} dt'dt + 2 \int_0^{\Delta T} \int_{t' \in [0,\Delta T]} \frac{|f(t)|^2}{|t-t'|^{1+2b}} dt'dt$$  

$$\lesssim \| f \|_{H^b}^2 + 2 \int_0^{\Delta T} \int_{t' \in [0,\Delta T]} \frac{|f(t)|^2}{|t-t'|^{1+2b}} dt'dt.$$  

To complete the proof we use Hardy’s inequality (see for instance [22] Lemma A.2]) together with the assumption $0 < b < \frac{1}{2}$ to deduce that

$$\int_0^{\Delta T} \int_{t' \in [0,\Delta T]} \frac{|f(t')|^2}{|t-t'|^{1+2b}} dt'dt \lesssim \int_0^{\Delta T} \frac{|f(t)|^2}{|t-t'|^{1+2b}} dt + \int_0^{\Delta T} \frac{|f(t)|^2}{|t-t'|^{1+2b}} dt$$  

$$\lesssim \| f \|_{L^2}^2 + \| f \|_{H^b}^2.$$  

□

To control the solution to the Dirac equation we make use of the energy estimate for the $X^{s,b}_\pm$ spaces.

**Lemma 5.** Let $s \in \mathbb{R}$, $b > \frac{1}{2}$, and $0 < \Delta T < 1$. Suppose $f \in H^s$, $F \in X^{s,b-1}_\pm(S_{\Delta T})$, and let $u$ be the solution to

$$\partial_t u \pm \partial_x u = F$$  

$$u(0) = f.$$  

Then $u \in X^{s,b}_\pm(S_{\Delta T})$ and we have the estimate

$$\| u \|_{X^{s,b}_\pm(S_{\Delta T})} \lesssim \| f \|_{H^s} + \| F \|_{X^{s,b-1}_\pm(S_{\Delta T})}.$$
We also require the $H^{r,b}$ versions of the above results.

**Lemma 6.** Let $r \in \mathbb{R}$, $0 < \Delta T < 1$, and $\nu \in C_0^\infty(\mathbb{R})$. Then if $-\frac{1}{2} < b_1 \leq b_2 < \frac{1}{2}$ we have

$$\|\nu \left( \frac{t}{\Delta T} \right) u(t,x) \|_{H^{r,b_1}} \lesssim \Delta T^{b_2-b_1} \|u\|_{H^{r,b_2}}.$$ 

Consequently, we have $\|u\|_{H^{r,b_1}(S\Delta T)} \lesssim \Delta T^{b_2-b_1} \|u\|_{H^{r,b_2}(S\Delta T)}$.

**Lemma 7.** Let $r \in \mathbb{R}$, $b > \frac{1}{2}$, $0 < \Delta T < 1$, and $m \in \mathbb{R}$. Suppose $f \in H^r$, $g \in H^{r-1}$, and $F \in H^{r-1,b-1}(S\Delta T)$ and let $u$ be the solution to

$$\Box u = m^2 u + F, \quad u(0) = f, \quad \partial_t u(0) = g.$$ 

Then $u \in H^{r,b}(S\Delta T)$ and we have the estimate

$$\|u\|_{H^{r,b}(S\Delta T)} \lesssim \|f\|_{H^r} + \|g\|_{H^{r-1}} + \|F\|_{H^{r-1,b-1}(S\Delta T)}.$$ 

**Proof.** See [23].

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### 3. Global Well-Posedness for the Dirac-Klein-Gordon Equation

We are now ready to consider the proof of global well-posedness for the DKG equation. To uncover the null structure for the DKG equation, we let $\psi = (\psi_+, \psi_-)^T$. Then the DKG equation (7) can be written as

$$\partial_t \psi_\pm \pm \partial_x \psi_\pm = -iM \psi_\mp + i\phi \psi_\pm$$

with initial data

$$\psi_\pm(0) = f_\pm \in H^s, \quad \phi(0) = \phi_0 \in H^r, \quad \partial_t \phi(0) = \phi_1 \in H^{r-1}. \quad (11)$$

Note that the right hand side of (11) has the bilinear product $\psi_+ \overline{\psi_-}$, which, as we have seen in Theorem 1, behaves significantly better than the corresponding product with $++$. The $+-$ structure can also be seen in the term $\phi \psi_\pm$ via a duality argument [19]. These are the key observations used in the local well-posedness theory for the DKG equation.

To prove the global well-posedness result of Theorem 3, by the local well-posedness result in [19], it suffices to prove that the data norms $\|\psi_\pm(T)\|_{H^r}$, $\|u[T]\|_{H^r}$ remain finite for all large times $0 < T < \infty$. To this end, we make use of the $I$-method together with ideas from [18] and [23]. Let $\rho_0 \in C^\infty$ be even, decreasing, and satisfy

$$\rho_0(\xi) = \begin{cases} 1 & |\xi| < 1 \\ |\xi|^{s} & |\xi| > 2. \end{cases}$$

Let $\rho(\xi) = \rho_0 \left( \frac{\xi}{N} \right)$ and define the $I$ operator by $\hat{I}\psi(\xi) = \rho(\xi)\hat{\psi}(\xi)$. We have the following straightforward estimates. Firstly, since $s < 0$, we have for any $\sigma \in \mathbb{R}$,

$$\|f\|_{H^r} \lesssim \|I f\|_{H^{r-\sigma}} \lesssim N^{-\sigma}\|f\|_{H^r}. \quad (13)$$
In particular, by taking $\sigma = 0$, we observe that to obtain control over $\|\hat{\psi}(t)\|_{L^2}$, it suffices to estimate $\|I\psi(t)\|_{L^2}$. Secondly, if $\text{supp } \hat{g} \subset \{ |\xi| \geq N \}$, $s < 0$, and $s_1 < s_2$, then we can trade regularity for decay in terms of $N$,

$$\|g\|_{H^{s_2}} \lesssim N^{s_1 - s_2}\|g\|_{H^{s_2}} \approx N^{s_1 - s_2 + s}\|Ig\|_{H^{s_2 - \epsilon}}.$$  \hspace{1cm} (14)

Thirdly, we note that the $I$ operator is the identity on low frequencies, so if $\text{supp } \hat{f} \subset \{ |\xi| < N \}$ then $If = f$. Finally, if $f$ is real-valued, then $If$ is also real-valued since $\rho$ was assumed to be even.

The $I$-method proceeds as follows. Assume we have a local solution

$$\psi_\pm \in C([0, \Delta T], H^s), \quad \phi \in C([0, \Delta T], H^s) \cap C^1([0, \Delta T], H^{s-1})$$

to (11), (12). Note that from (13) we have

$$\text{Lemma 8. Let } \phi \in C([0, \Delta T], H^s) \cap C^1([0, \Delta T], H^{s-1})$$

Now as $\phi$ is real-valued, $I^2\phi$ is also real-valued and hence

$$2\Re \left(iI^2\phi(\overline{I\psi_+ I\psi_-} + \overline{T\psi_- I\psi_+}) \right) = 0.$$  \hspace{1cm} (15)

Subtracting this term from (15) and using the fundamental theorem of Calculus then gives

$$\sup_{t' \in [0, \Delta T]} (\|I\psi_+(t')\|_{L^2}^2 + \|I\psi_-(t')\|_{L^2}^2) \leq \|f_+\|_{L^2}^2 + \|f_-\|_{L^2}^2$$

$$+ 2 \sum_{\pm} \sup_{t' \in [0, \Delta T]} \left| \int_0^{t'} \int_{\mathbb{R}} (I(\phi \psi_\pm) - I^2\phi I\psi_\pm) T\psi_\pm dxdt \right|. \hspace{1cm} (16)$$

Thus provided we can show the last term in (16) is small, we can deduce that over a small time $[0, \Delta T]$, $\|I\psi_\pm(t)\|_{L^2}$ does not grow to large. The first step in this direction is the following.

**Lemma 8.** Let $\frac{-1}{4} < s < 0$ and $-s < r \leq 1 + 2s$. Assume $b = \frac{1}{2} + \epsilon$ with $\epsilon > 0$ sufficiently small. Then for any $\Delta T \ll 1$, $N \gg 1$ we have

$$\sup_{t' \in [0, \Delta T]} \left| \int_0^{t'} \int_{\mathbb{R}} (I(\phi u) - I^2\phi Iu) I\psi dxdt \right| \lesssim \Delta T^{\frac{1}{2} - 2\epsilon} N^{2s - r + 2\epsilon} \|I^2\phi\|_{H^{r-2\epsilon, b}(S_{\Delta T})} \|Iu\|_{X_{r}^{b}(S_{\Delta T})} \|I\psi\|_{X_{r}^{b}(S_{\Delta T})} \hspace{1cm} (17)$$

where $S_{\Delta T} = [0, \Delta T] \times \mathbb{R}$.

**Proof.** See Subsection 3.1 below. \hspace{1cm} $\Box$

**Remark 3.** The use of $I^2\phi$ instead of just $\phi$ or $I\phi$ on the right hand side of (17) may require some explanation. Roughly speaking, the larger the negative exponent on $N$ in (17), the better the eventual GWP result will be. Moreover, an examination of the proof of Lemma 8 shows that the exponent on
$N$ depends entirely on the number of derivatives on $\phi$. In other words, we could replace the term $N^{2s-r}\|I^2\phi\|_{H^{r-2s,b}}$ with $N^{ks-r}\|I^k\phi\|_{H^{r-2s,b}}$ for any $k \in \mathbb{N}$ (provided $r - ks \leq 1$). However, the size of $\phi$ with respect to $N$ ends up being of the order $N^{-2s}$. This follows by observing that schematically $\phi$ is a solution to $\Box \phi = \psi^2$, and by \cite{13}, the low frequency component of $\psi^2$ is essentially of size $N^{-2s}$. Thus it is natural to take $I^2\phi$, which via \cite{13}, also has size roughly $N^{-2s}$.

**Remark 4.** The powers of $\Delta T$ and $N$ on the right hand side of (17) are essentially sharp if we are working in the spaces $X^{s,b}_\pm, H^{s,b}$. This follows from the counter examples in Section 5 together with a scaling argument.

Lemma \cite{8} allows us to estimate the growth of $\|I\psi_\pm(t)\|_{L^2}$ on $[0,\Delta T]$, provided that we can control the size of the norms $\|I\psi_\pm\|_{X^{0,b}_\pm(S_{\Delta T})}$ and $\|I^2\phi\|_{H^{r-2s,b}(S_{\Delta T})}$. This control is provided by a modification of the usual local well-posedness theory.

**Lemma 9.** Let $\frac{-1}{6} < s < 0, -s < r \leq \frac{1}{2} + 2s$, and $b = \frac{1}{2} + \epsilon$ with $\epsilon > 0$ sufficiently small. Assume $f_\pm \in H^s$ and $\phi[0] \in H^r \times H^{r-1}$. Choose $\Delta T \ll 1$ and $N \gg 1$ such that

$$\left(\Delta T^{1+r-2s-3s} + N^{r+2s+2}\right)\|I^2\phi[0]\|_{H^{r-2s}} \ll 1$$

and

$$\left(\Delta T^{1-s} + N^{r+2s}\right)\left(\|I_+\|_{L^2} + \|I_-\|_{L^2}\right)^2 \ll 1.$$  

(18)

(19)

Then the Dirac-Klein-Gordon equation (11) with initial data (12) is locally well-posed on the domain $S_{\Delta T} = [0,\Delta T] \times \mathbb{R}$. Moreover, the solution $(\psi, \phi)$ satisfies

$$\|I\psi_+\|_{X^{0,b}_\pm(S_{\Delta T})} + \|I\psi_-\|_{X^{0,b}_\pm(S_{\Delta T})} \lesssim \|I_+\|_{L^2} + \|I_-\|_{L^2}$$

and

$$\|I^2\phi\|_{H^{r-2s,b}(S_{\Delta T})} \lesssim \|I^2\phi[0]\|_{H^{r-2s}} + \left(\|I_+\|_{L^2} + \|I_-\|_{L^2}\right)^2.$$

**Proof.** See Subsection 3.2 below.

**Remark 5.** Note that since $\|I^2\phi[0]\|_{H^{r-2s}} \lesssim N^{-2s}$, by choosing $N$ sufficiently large and $\Delta T$ sufficiently small, we can ensure that the inequality (18) is satisfied. A similar comment applies to (19).

**Remark 6.** The reason that we can extend the work of Tesfahun \cite{23} is due to the conclusions in Lemma \cite{8} and Lemma 9. In more detail, Lemma \cite{8} improves \cite{23} Lemma 8 by adding a power of $\Delta T$ on the right hand side of (17). Since $\Delta T$ will be taken small, this is a significant gain. Similarly, Lemma 9 extends \cite{23} Theorem 8 by having a larger exponent on $\Delta T$ in (13). As a consequence, we can take $\Delta T$ larger, which improves the eventual GWP result. The point here is that the larger $\Delta T$ becomes, the fewer time steps of length $\Delta T$ are required to reach a large time $T$.

We now follow the argument used in \cite{23} and sketch the proof of Theorem 3. The persistence of regularity result in \cite{19} shows that it suffices to prove GWP in the case

$$-\frac{1}{6} < s < 0, \quad s - \frac{1}{4} + \sqrt{\left(s - \frac{1}{4}\right)^2 - s} < \frac{1}{2} + 2s.$$  

(20)

Note that this region is non-empty as the intersection of the curves $s - \frac{1}{4} + \sqrt{\left(s - \frac{1}{4}\right)^2 - s}$ and $\frac{1}{2} + 2s$ occurs at $s = -\frac{1}{6}$.
Choose some large time $T > 0$ and assume $\epsilon > 0$ is small. Let $N$ be some large fixed constant to be chosen later depending on the initial data $\|\psi(0)\|_{H^s}$ and $\|\phi[0]\|_{H^r}$, as well as the various constants appearing in Lemma 8 and Lemma 9. Take $\Delta T = N^{-\frac{4s - 2r}{4s - 2r}}$. If $N$ is sufficiently large then from \(13\)

\[
\left(\Delta T^{\frac{r}{2} - 2s - 3\epsilon} + N^{-r + 2s + 2\epsilon}\right) \|I^2\phi[0]\|_{H^{r - 2s}} \ll 1
\]

\[
\left(\Delta T^{1 - \epsilon} + N^{-\frac{1}{2} + 2s}\right) \left(\|If_+\|_{L^2} + \|If_-\|_{L^2}\right)^2 \ll 1.
\]

Therefore by Lemma 9 we get a solution $(\psi, \phi)$ to (11) on $[0, \Delta T]$. We would now like to repeat this argument $\frac{T}{\Delta T}$ times to advance to the time $T$. The only obstruction is the possible growth of the norms $\|I\psi_{\pm}(t)\|_{L^2}$ and $\|I^2\phi(t)\|_{H^{r - 2s}}$. Our aim is to use Lemma 8 to show that $\|I\psi_{\pm}(t)\|_{L^2}$ is “almost conserved” and consequently obtain large time control over the norm $\|I\psi_{\pm}(t)\|_{L^2}$. This is accomplished by using an induction argument as follows.

Assume $n \leq \frac{T}{\Delta T}$ and suppose we have a solution $(\psi, \phi)$ on $[0, n\Delta T]$ with the bounds

\[
\sup_{t \in [0, n\Delta T]} \left(\|I\psi_+(t)\|_{L^2}^2 + \|I\psi_-(t)\|_{L^2}^2\right) \leq 2\|If_+\|_{L^2}^2 + 2\|If_-\|_{L^2}^2 \quad (21)
\]

and

\[
\sup_{t \in [0, n\Delta T]} \|I^2\phi(t)\|_{H^{r - 2s}} \leq C^* \left(\|I^2\phi[0]\|_{H^{r - 2s}} + \left(\|If_+\|_{L^2}^2 + \|If_-\|_{L^2}^2\right)^2\right) \quad (22)
\]

where the constant $C^*$ is some large constant independent of $N$, $\Delta T$, and $n$. If $N$ is sufficiently large, depending on $C^*$ and the initial data $\|f_0\|_{H^s}$, $\|\phi[0]\|_{H^r}$, then we can apply Lemma 8 with initial data $\psi(n\Delta T)$, $\phi(n\Delta T)$, $\partial_t \phi(n\Delta T)$, and extend the solution to $[0, (n + 1)\Delta T]$. Suppose we could show that the bounds (21) and (22) on $[0, n\Delta T]$ implied that they also hold on the larger interval $[0, (n + 1)\Delta T]$ with the same constant $C^*$. Then by induction we would have (21) and (22) on $[0, T]$. Since $T$ was arbitrary, Theorem 3 would follow. Thus it suffices to verify the estimates (21) and (22) on the interval $[0, (n + 1)\Delta T]$. We break this into two parts, proving the bound on $\|I\psi_{\pm}(t)\|_{L^2}$, and then estimating $\|I^2\phi(t)\|_{H^{r - 2s}}$.

**Bound on the Spinor $\psi_{\pm}$.** Let

\[
\Gamma(z) = \sup_{t \in [0, z]} \left(\|I\psi_+(t)\|_{L^2}^2 + \|I\psi_-(t)\|_{L^2}^2\right).
\]

Note that the bounds (21) and (22) imply that

\[
\Gamma(n\Delta T) \leq AN^{-2s}
\]

\[
\sup_{t \in [0, n\Delta T]} \|I^2\phi(t)\|_{H^{r - 2s}} \leq BN^{-2s} \quad (23)
\]

where $A$ and $B$ depend on the initial data, the constant $C^*$, and $T$, but are independent of $n$, $N$, and $\Delta T$. If we now combine Lemma 8, Lemma 9 together with (16) we obtain the following control on the growth of $\Gamma(t)$.

**Corollary 10** (Almost conservation law). Let $\frac{1}{b} < s < 0$ and $-s < r \leq \frac{1}{2} + 2s$ and $b = \frac{1}{2} + \epsilon$ with $\epsilon > 0$ sufficiently small. Suppose

\[
\Delta T = N^{-\frac{4s - 2r}{4s - 2r}}
\]

and we have the bounds (23). Then provided $N$ is sufficiently large,

\[
\Gamma(\Delta T) \leq \Gamma(0) + C\Delta T^{\frac{r}{2} - 2s} N^{-r + 2s} (A + B) \Gamma(0).
\]
Proof. By Lemma 8, Lemma 9, and (16) it suffices to show that
\[ \Delta T^{\frac{1}{2} + r - 2s - 3\epsilon} N^{-2s} B + N^{-r + 2\epsilon} B \ll 1 \]
equiv and
\[ \Delta T^{1 - r} N^{-2s} A + N^{2r - \frac{1}{2} - 2s} B \ll 1. \]
However these inequalities follow provided \( \Delta T = N^{-\frac{4s+2\epsilon}{4s-4\epsilon}} \) and we choose \( N \) sufficiently large. \( \square \)

We can now iterate the previous corollary to get control over \( \Gamma(t) \) at time \((n+1)\Delta T\)
\[ \Gamma((n+1)\Delta T) \leq \Gamma(0) + Cn\Delta T^{\frac{1}{2} - 2\epsilon} N^{-r + 2\epsilon} (A + B) \Gamma(0). \]
Since the number of steps \( n \leq \frac{T}{\Delta T} \) we get
\[ \Gamma((n+1)\Delta T) \leq \Gamma(0) + CT \Delta T^{\frac{1}{2} - 2\epsilon} N^{-r + 2\epsilon} (A + B) \Gamma(0). \]
We want to make the coefficient of the second term small. Thus we need to ensure that, using the requirement on \( \Delta T \) in Corollary 10,
\[ CT \Delta T^{\frac{1}{2} - 2\epsilon} N^{-r + 2\epsilon} (A + B) \approx N^{-(1 + 2\epsilon)(1 + 2\epsilon)} \Gamma(0) \ll 1. \]
(24)

By choosing \( N \) large, and \( \epsilon > 0 \) sufficiently small, we see that (24) will follow provided \(-2s - r(1 + 2r - 4s) < 0\). Rearranging, we get the quadratic polynomial \( 2r^2 + (1 - 4s)r + 2s > 0 \) and so we need
\[ s - \frac{1}{4} + \sqrt{(s - \frac{1}{4})^2 - s} < r. \]
Therefore, provided we choose \( N \) large enough, depending on \( T \), \( A \), and \( B \), we get
\[ \Gamma((n+1)\Delta T) \leq 2\Gamma(0) \]
as required.

Bound on \( \phi \). Recall that our goal was to show that, if the bounds (21) and (22) hold for \( t \in [0, n\Delta T] \), then in fact they also held on the larger domain \([0, (n+1)\Delta T]\) (with the same constants). The bound for \( \| I\psi \|_{L^2} \) was obtained above. Thus it remains to bound \( \| I^2 \phi[t] \|_{H^{r-2s}} \) on the interval \([0, (n+1)\Delta T]\).
The argument that gives the required bound makes use of an idea due to Selberg in [18] on induction of free waves. The idea is to break \( \phi \) into a sum of homogeneous waves, together with an inhomogeneous term and then use an induction argument to estimate the contribution that each of these homogeneous waves makes to the size of \( \| I^2 \phi[t] \|_{H^{r-2s}} \). We note that this idea was also used in [23].

We begin by observing that the induction assumptions (21) and (22) together with Lemma 9 give for every \( 0 \leq j \leq n \)
\[ \| I\psi_+ \|_{X^{a, b}_{n+1}(S_j)} + \| I\psi_- \|_{X^{a, b}_{n+1}(S_j)} \leq C_1 \left( \| I f_+ \|_{L^2_x} + \| I f_- \|_{L^2_x} \right) \]
(25)
where \( S_j = [j\Delta T, (j+1)\Delta T] \) and the constant \( C_1 \) is independent of \( C^*, j, n, N, \) and \( \Delta T \). Suppose we could show that (25) implies that
\[ \sup_{t \in [n\Delta T, (n+1)\Delta T]} \| I^2 \phi[t] \|_{H^{r-2s}} \leq C_2 \left( \| I^2 \phi[0] \|_{H^{r-2s}} + \left( \| I f_+ \|_{L^2_x} + \| I f_- \|_{L^2_x} \right)^2 \right), \]
(26)
Then by taking $C^* = C_2$ we see that the bound (22) holds for $t \in [0, (n + 1)\Delta T]$. Thus by induction, together with the fact that the constants in (21) and (22) are independent of $n$, we would obtain control over the solution on $[0, T]$ and Theorem 3 would follow.

We now show that (25) implies (26). We make use of the following result which is a variant of a corresponding result in [23].

**Lemma 11.** Let $m \in \mathbb{R}$, $0 < \Delta T < 1$, $\frac{1}{2} < s < 0$, $0 < r < \frac{1}{2} + 2s$, and $b > \frac{1}{2}$. Assume $u \in X_+^{s,b}(S\Delta T)$ and $v \in X_-^{s,b}(S\Delta T)$. Then there exists a unique solution $\Phi \in H^{r,b}(S\Delta T)$ to

\[
\square \Phi = (uv) + m^2 \Phi,
\]

\[
\Phi(0) = \partial_t \Phi(0) = 0.
\]

on $S\Delta T = [0, \Delta T] \times \mathbb{R}$. Moreover we have

\[
\sup_{t \in [0, \Delta T]} \|I^2 \Phi(t)\|_{H^{r-2s}_+} \lesssim (\Delta T + N^{-\frac{1}{2} + 2e})\|Iu\|_{X_+^{0,b}(S\Delta T)}\|Iv\|_{X_-^{0,b}(S\Delta T)}.
\]

**Proof.** The existence/uniqueness claim follows from Lemma 7 together with an application of Theorem 11. To prove (27) we write $\Phi = \Phi_1 + \Phi_2$ where

\[
\square \Phi_1 = (uv) + m^2 \Phi_1,
\]

\[
\Phi_1(0) = 0, \quad \partial_t \Phi_1(0) = 0.
\]

and $\tilde{u}_{\text{low}} = I_{|\xi| < \frac{1}{s+1}} \tilde{u}$, $\tilde{v}_{\text{low}} = I_{|\xi| < \frac{1}{s+1}} \tilde{v}$. The standard representation of solutions to the Klein-Gordon equation, together with the Sobolev product law and the observation that $I^2(uv_{\text{low}}) = u_{\text{low}}v_{\text{low}}$, gives

\[
\sup_{t \in [0, \Delta T]} \|I^2 \Phi_1(t)\|_{H^{r-2s}_+} \lesssim \int_0^{\Delta T} \|u_{\text{low}}(t)v_{\text{low}}(t)\|_{H^{r-2s-1}_+} dt
\]

\[
\lesssim \int_0^{\Delta T} \|u_{\text{low}}(t)\|_{L^2_x} \|v_{\text{low}}(t)\|_{L^2_x} dt
\]

\[
\lesssim \Delta T \|Iu\|_{X_+^{0,b}(S\Delta T)} \|Iv\|_{X_-^{0,b}(S\Delta T)}.
\]

To bound the remaining term, $\Phi_2$, we note that by the energy estimate for $H^{s,b}$ spaces in Lemma 7

\[
\sup_{t \in [0, \Delta T]} \|I^2 \Phi_2(t)\|_{H^{r-2s}_+} \lesssim \|I^2 \Phi_2\|_{H^{r-2s,b}(S\Delta T)}
\]

\[
\lesssim \|I^2(uv - u_{\text{low}}v_{\text{low}})\|_{H^{r-2s-1,b-1}(S\Delta T)}
\]

\[
\lesssim \|u_{\text{low}}v_{\text{hi}}\|_{H^{-\frac{1}{2} + b-1}(S\Delta T)} + \|u_{\text{hi}}v_{\text{low}}\|_{H^{-\frac{1}{2} + b-1}(S\Delta T)} + \|u_{\text{hi}}v_{\text{hi}}\|_{H^{-\frac{1}{2} + b-1}(S\Delta T)}
\]

where $u_{hi} = u - u_{\text{low}}$ is the high frequency component of $u$, $v_{hi}$ is defined similarly, and we used the assumption $r < \frac{1}{2} + 2s$. By Corollary 2 we have the estimate

\[
\|\psi_1 \psi_2\|_{H^{-\frac{1}{2} + b-1}} \lesssim \|\psi_1\|_{X^{-\frac{1}{2} + s_1, 1}} \|\psi_2\|_{X_{-}^{s_1, b}}
\]

for $\frac{1}{2} < s_1 < 0$. To control the first term in (28) we use (29) with $s_1 = -\frac{1}{2} + 2e$ together with (14) to obtain

\[
\|u_{\text{low}}v_{\text{hi}}\|_{H^{-\frac{1}{2} + b-1}(S\Delta T)} \lesssim \|u_{\text{low}}\|_{X_+^{0,b}(S\Delta T)} \|v_{\text{hi}}\|_{X_-^{0,b}(S\Delta T)}
\]

\[
\lesssim N^{-\frac{1}{2} + 2e} \|Iu\|_{X_+^{0,b}(S\Delta T)} \|Iv\|_{X_{-}^{0,b}(S\Delta T)}
\]
A similar application of (29) allows us to estimate the second term in (28). Finally, for the last term in (28) we use (14) and (29) with $s_1 = s$ to deduce that

$$\|u_t v_{th}\|_{H^{-\frac{1}{2}, b}(S_{\Delta T})} \lesssim \|u_{th}\|_{X^{-\frac{1}{2}, a, b}(S_{\Delta T})} \|v_{th}\|_{X_{a, b}^{\frac{1}{2}}(S_{\Delta T})}$$

$$\lesssim N^{-\frac{1}{2} + 2\epsilon} \|Iu\|_{X_{a, b}^{\frac{1}{2}}(S_{\Delta T})} \|Iv\|_{X_{a, b}^{\frac{1}{2}}(S_{\Delta T})}$$

where we needed $-\frac{1}{2} - s + 2\epsilon \leq s$ which holds provided $s > -\frac{1}{4}$ and $\epsilon$ sufficiently small.

We now have the necessary results to control the growth of $\|I^2 \phi[t]\|_{H^{1-2\epsilon}}$. Let $0 \leq j \leq n$ and define $\phi^{(0)}_j$ to be the solution to

$$\square \phi^{(0)}_j = m^2 \phi^{(0)}_j$$

$$\phi^{(0)}_j(j \Delta T) = \phi(j \Delta T), \quad \partial_t \phi^{(0)}_j(j \Delta T) = \partial_t \phi(j \Delta T).$$

Let $\Phi_j = \phi - \phi^{(0)}_j$ be the inhomogeneous component of $\phi$. The inequality (25) together with Lemma [11] and the assumption $\Delta T = N^{\frac{4\epsilon - 2\epsilon}{1 + 2\epsilon - 4\epsilon - 6\epsilon}}$, shows that for every $0 \leq j \leq n$

$$\sup_{t \in [j \Delta T, (j+1) \Delta T]} \|I^2 \Phi_j[t]\|_{H^{1-2\epsilon}} \lesssim \Delta T \left( \|I^f_+\|_{L^2} + \|I^f_-\|_{L^2} \right)^2.$$  

(31)

We now claim that for $1 \leq j \leq n$ we have the estimate

$$\sup_{t \in [0, (n+1) \Delta T]} \|I^2 \phi^{(0)}_j[t]\|_{H^{1-2\epsilon}} \lesssim \sup_{t \in [0, (n+1) \Delta T]} \|I^2 \phi^{(0)}_{j-1}[t]\|_{H^{1-2\epsilon}} + C \Delta T \left( \|I^f_+\|_{L^2} + \|I^f_-\|_{L^2} \right)^2.$$  

(32)

Assume for the moment that (32) holds. Then after $n$ applications of (32), together with the standard energy inequality for the homogeneous wave equation, we obtain

$$\sup_{t \in [0, (n+1) \Delta T]} \|I^2 \phi^{(0)}_n[t]\|_{H^{1-2\epsilon}} \lesssim \sup_{t \in [0, (n+1) \Delta T]} \|I^2 \phi^{(0)}_0[t]\|_{H^{1-2\epsilon}} + C \Delta T \left( \|I^f_+\|_{L^2} + \|I^f_-\|_{L^2} \right)^2$$

$$\lesssim \|I^2 \phi^{(0)}_0\|_{H^{1-2\epsilon}} + C \Delta T \left( \|I^f_+\|_{L^2} + \|I^f_-\|_{L^2} \right)^2.$$  

(33)

If we now combine (31) and (33) we see that since $n \leq T / \Delta T$

$$\sup_{t \in [n \Delta T, (n+1) \Delta T]} \|I^2 \phi[t]\|_{H^{1-2\epsilon}} \lesssim \sup_{t \in [n \Delta T, (n+1) \Delta T]} \|I^2 \phi^{(0)}_n[t]\|_{H^{1-2\epsilon}} + \sup_{t \in [n \Delta T, (n+1) \Delta T]} \|I^2 \Phi_n[t]\|_{H^{1-2\epsilon}}$$

$$\lesssim \|I^2 \phi^{(0)}_0\|_{H^{1-2\epsilon}} + (n + 1) \Delta T \left( \|I^f_+\|_{L^2} + \|I^f_-\|_{L^2} \right)^2$$

$$\lesssim \|I^2 \phi^{(0)}_0\|_{H^{1-2\epsilon}} + \left( \|I^f_+\|_{L^2} + \|I^f_-\|_{L^2} \right)^2$$

where the implied constant is independent of $N$, $C^*$, and $\Delta T$. Thus we obtain (26) as required.

It only remains to prove (32). We begin by observing that

$$(\phi^{(0)}_j - \phi^{(0)}_{j-1})(j \Delta T) = \phi(j \Delta T) - \phi^{(0)}_{j-1}(j \Delta T) = \Phi_{j-1}(j \Delta T).$$

Hence the difference $\phi^{(0)}_j - \phi^{(0)}_{j-1}$ satisfies the equation

$$\square (\phi^{(0)}_j - \phi^{(0)}_{j-1}) = m^2 (\phi^{(0)}_j - \phi^{(0)}_{j-1})$$

$$(\phi^{(0)}_j - \phi^{(0)}_{j-1})(j \Delta T) = \Phi_{j-1}(j \Delta T),$$

$$\partial_t (\phi^{(0)}_j - \phi^{(0)}_{j-1})(j \Delta T) = \partial_t \Phi_{j-1}(j \Delta T).$$
Therefore
\[ \sup_{t \in [0,(n+1)\Delta T]} \| I^2 \phi_j^{(0)} [t] \|_{H^{r-2s}_x} \leq \sup_{t \in [0,(n+1)\Delta T]} \| I^2 \phi_j^{(0)} [t] \|_{H^{r-2s}_x} + \sup_{t \in [0,(n+1)\Delta T]} \| I^2 (\phi_j^{(0)} - \phi_j^{(0)} [t]) \|_{H^{r-2s}_x} \]
\[ \leq \sup_{t \in [0,(n+1)\Delta T]} \| I^2 \phi_j^{(0)} [t] \|_{H^{r-2s}_x} + C\| \Phi_{j-1} [j \Delta T] \|_{H^{r-2s}_x} \]
and so (32) follows from (31). Consequently, we deduce that the induction assumptions (21) and (22) hold on the larger interval $[0, (n+1)\Delta T]$ and hence Theorem 3 follows.

3.1. Proof of Lemma 3. Let $Q(f, g) = I(fg) - I^2 f g$. Note that
\[ Q(f, g)(\xi) = \int_{\mathbb{R}} (\rho(\xi) - \rho(\xi - \eta)^2 \rho(\eta)) \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta. \]
An application of Cauchy-Schwarz together with Lemma 4 gives
\[ \left| \int_0^t \int_{\mathbb{R}} (I(\phi u) - I^2 \phi I u) V dx dt \right| \lesssim \| I \|_{X^{0, -\frac{1}{2} + s}(S_{\Delta T})} \| I \|_{X^{0, \frac{1}{2} + s}(S_{\Delta T})}. \]
Thus, by the definition of $X^{0, b}_\pm(S_{\Delta T})$, it suffices to prove that
\[ \| Q(\phi, u) \|_{X^{0, -\frac{1}{2} + s}(S_{\Delta T})} \lesssim \Delta T \frac{1}{2} - 2s N^{2s - r + 2s} \| I^2 \phi \|_{H^{r-2s, b}} \| I u \|_{X^{0, b}_\pm}. \tag{34} \]
where we may assume that $\phi$ and $u$ are supported in $[-\Delta T, 2\Delta T] \times \mathbb{R}$. Note that since the $I$ operator only acts on the spatial variable $x$, $I^2 \phi$ and $I u$ are also supported in $[-\Delta T, 2\Delta T] \times \mathbb{R}$. Write $\phi = \phi_{\text{low}} + \phi_{\text{hi}}$ and $u = u_{\text{low}} + u_{\text{hi}}$ where, as in the proof of Lemma 11, we define $\phi_{\text{low}} = \mathbbm{1}_{|\xi| < \frac{N}{2}} \hat{\phi}$ and $u_{\text{low}}$ is defined similarly. We consider each of the possible interactions separately.

- **Case 1 (low-low).** In this case we simply note that $Q(\phi, u) = 0$ and hence (34) holds trivially.

- **Case 2 (low-hi).** We need to use the smoothing property of the bilinear form $Q(\phi, u)$ to transfer a derivative from $\phi_{\text{low}}$ to $u_{\text{hi}}$. More precisely, suppose $|\xi - \eta| < \frac{N}{2}$ and $|\eta| > \frac{N}{2}$. Then since $\rho'(z) \lesssim N^{-s}|z|^{s-1}$ for $|z| > \frac{N}{2}$ we have
\[ |\rho(\xi) - \rho(\xi - \eta)^2 \rho(\eta)| = |\rho(\xi) - \rho(\eta)| \lesssim N^{-s}|\eta|^{s-1}|\xi - \eta| \approx \rho(\eta) \frac{|\xi - \eta|}{|\eta|} \lesssim \rho(\eta) \frac{|\xi - \eta|^{r-2s}}{|\eta|^{r-2s}} \]
provided $r - 2s < 1$. Hence
\[ |Q(\hat{\phi}_{\text{low}}, u_{\text{hi}})(\tau, \xi)| \lesssim \int_{\mathbb{R}^2} |\xi - \eta|^{r-2s} |\hat{\phi}_{\text{low}}(\tau - \lambda, \xi - \eta)| |\eta|^{r+2s} \rho(\eta) |\hat{u}_{\text{hi}}(\lambda, \eta)| d\lambda d\eta. \]
Thus we can move the derivative $|\nabla|^{r-2s}$ from $u_{\text{hi}}$ to $\phi_{\text{low}}$, where we let $(|\nabla|^{r-2s})(\xi) = |\xi|^r \hat{f}(\xi)$. This is the essential step which allows us to prove (34) in the low-hi case. We now apply (14) and Theorem 1.
with \(s_1 = s_2 = 0\), \(s_3 = 2\epsilon\), \(b_1 = \frac{1}{2} - \epsilon\), \(b_2 = 0\), and \(b_3 = b\) to obtain
\[
\|Q(\phi_{\text{low}}, u_{\text{hi}})\|_{X^0_{\pm} (S \Delta T)} \lesssim \|\nabla|^{-2s} \phi_{\text{low}} \nabla|^{-r+2s} I u_{\text{hi}}\|_{X^0_{\pm}^{-\frac{1}{2} s}}
\]
\[
\lesssim \|\nabla|^{-2s} \phi_{\text{low}}\|_{L^2_{\xi}} \|\nabla|^{-r+2s} I u_{\text{hi}}\|_{X^2_{\pm}}
\]
\[
\lesssim \Delta T \frac{1}{2} N^{-r+2s+2\epsilon} \|I^2 \phi\|_{L^\infty T^{r-2s}} \|I u\|_{X^0_{\pm}}
\]
where we used the assumption \(\text{supp } \phi \subset \{-\Delta T, 0\} \times \mathbb{R}\).

- **Case 3 (hi-low).** In this case we do not have to transfer any regularity and we simply use the estimate \(\rho(\xi) - \rho(\xi - \eta)^2 \rho(\eta) \lesssim 1\). Then (14) together with an identical application of Theorem 1 to the low-hi case gives
\[
\|Q(\phi_{\text{hi}}, u_{\text{low}})\|_{X^0_{\pm} (\Delta T)} \lesssim \|\phi_{\text{hi}} u_{\text{low}}\|_{X^0_{\pm}^{-\frac{1}{2} s}}
\]
\[
\lesssim \|\phi_{\text{hi}}\|_{L^2_{\xi}} \|u_{\text{low}}\|_{X^2_{\pm}}
\]
\[
\lesssim \Delta T \frac{1}{2} N^{-r+2s+2\epsilon} \|I^2 \phi\|_{L^\infty T^{r-2s}} \|I u\|_{X^0_{\pm}}
\]
where as before, we used the assumption \(\text{supp } \phi \subset \{-\Delta T, 0\} \times \mathbb{R}\).

- **Case 4 (hi-hi).** This is the most difficult case and we need to make full use of the generality of Theorem 1 to obtain the term \(\Delta T^{\frac{1}{2} - \epsilon}\). We decompose \(\phi_{\text{hi}} = \phi_{\text{hi}}^+ + \phi_{\text{hi}}^-\) where
\[
\tilde{\phi}_{\text{hi}}^- = \mathbb{1}_{\{\tau \leq 0\}} \tilde{\phi}_{\text{hi}}
\]
is the restriction of \(\tilde{\phi}_{\text{hi}}\) to the second and fourth quadrants of \(\mathbb{R}^{1+1}\). Note that \(\|\phi^+\|_{X^2_{\pm}} \lesssim \|\phi\|_{H^{r,b}}\).
Assume that we have \(\pm = +, \mp = -\) in (54), it will be clear that the proof will also apply to the \(\pm = -, \mp = +\) case.

- **Case 4a (hi-hi +).** As in hi-low case we start by discarding the smoothing multiplier \(Q\). We now apply Theorem 1 with \(s_1 = -s + 2\epsilon\), \(s_2 = s\), \(s_3 = 0\), \(b_1 = b_2 = \frac{1}{4}\), and \(b_3 = \frac{1}{2} - \epsilon\) to obtain
\[
\|Q(\phi_{\text{hi}}^+, u_{\text{hi}})\|_{X^0_{\pm} (\Delta T)} \lesssim \|\phi_{\text{hi}}^+ u_{\text{hi}}\|_{X^0_{\pm}^{-\frac{1}{2} s}}
\]
\[
\lesssim \|\phi_{\text{hi}}^+\|_{X^{s+2s, \frac{1}{2}}_{\pm}} \|u_{\text{hi}}\|_{X^{s, \frac{1}{2}}_{\pm}}
\]
\[
\lesssim N^{2s-r+2\epsilon} \|I^2 \phi\|_{H^{r-2s, \frac{1}{2}}} \|I u\|_{X^{0, \frac{1}{2}}_{\pm}}
\]
\[
\lesssim \Delta T^{\frac{1}{2} - \epsilon} N^{2s-r+2\epsilon} \|I^2 \phi\|_{H^{r-2s, b}} \|I u\|_{X^{0, b}_{\pm}}
\]
where we needed \(-s < r, \epsilon > 0\) sufficiently small, and in the final line we used the assumption that \(\phi, u\), are compactly supported in the interval \([-\Delta T, 2\Delta T]\) together with Lemma 4 and Lemma 6.
• **Case 4b** (hi-hi - ). Here we first apply Lemma 3 and discard the multiplier \( Q \), and then apply Theorem 1 with \( s_1 = 0, s_2 = -s + \epsilon, s_3 = s, b_1 = b_2 = \frac{1}{2}, \) and \( b_3 = \frac{1}{2} + \epsilon \) to obtain

\[
\|Q(\phi_{hi}, u_{hi})\|_{X_{-\frac{1}{2}+\epsilon}(S_{\Delta T})} \lesssim \Delta T^{\frac{1}{2}-\epsilon}\|\phi_{hi} u_{hi}\|_{X_{-\frac{1}{2}+\epsilon}}
\]

\[
\lesssim \Delta T^{\frac{1}{2}-\epsilon}\|\phi_{hi}\|_{X_{-\frac{1}{2}+\epsilon}}\|u_{hi}\|_{X_{\frac{1}{2}+\epsilon}}
\]

\[
\lesssim \Delta T^{\frac{1}{2}-\epsilon}N^{2s-r+\epsilon}\|I^2\phi\|_{H^{-2s, \frac{1}{2}}}\|I u\|_{X_{2}^{0, b}}
\]

\[
\lesssim \Delta T^{\frac{1}{2}-2\epsilon}N^{2s-r+\epsilon}\|I^2\phi\|_{H^{-2s, \frac{1}{2}}}\|I u\|_{X_{2}^{0, b}}
\]

where, as previously, we used the assumption on the support of \( \phi \) in the last line.

3.2. **Proof of Lemma 9.** Lemma 9 follows by a standard fixed point argument using Lemma 3 and the estimates

\[
\|I(uv)\|_{X_{Z}^{b-1}(S_{\Delta T})} \lesssim \left( \Delta T^{\frac{1}{2}+r-2s-3\epsilon} + N^{-r+2s+2\epsilon} \right) \|I^2 u\|_{H^{-2s, \frac{1}{2}}(S_{\Delta T})} \|I v\|_{X_{Z}^{0, b}(S_{\Delta T})} \tag{35}
\]

and

\[
\|I^2(uv)\|_{H^{-2s, 1-b-1}(S_{\Delta T})} \lesssim \left( \Delta T^{1-\epsilon} + N^{-\frac{3}{2}+2\epsilon} \right) \|I u\|_{X_{Z}^{0, b}(S_{\Delta T})} \|I v\|_{X_{Z}^{0, b}(S_{\Delta T})}. \tag{36}
\]

See for instance [23].

We start by proving (35). As in the proof of Lemma 8 we decompose \( u = u_{low} + u_{hi} \) and \( v = v_{low} + v_{hi} \).

• **Case 1 (low-low).** We split \( u_{low} = u_{low}^+ + u_{low}^- \), where we use the same notation as in Subsection 3.1. Observe that an application of Theorem 1 gives

\[
\int_{\mathbb{R}^3} \Pi_{j=1}^{3} \psi_j \psi \, dt \lesssim \|\psi\|_{X^{0, \epsilon}} \|\psi\|_{X^{r-2s, \frac{1}{2}+r+2s+\epsilon}} \|\psi\|_{X^{0, \frac{1}{2}+\epsilon}} \tag{37}
\]

provided that \( 0 < r - 2s < \frac{1}{2} \) and \( \epsilon > 0 \) is sufficiently small. Hence, using Lemma 3 together with two applications of (37) we see that

\[
\|I(u_{low}v_{low})\|_{X_{Z}^{b-1}(S_{\Delta T})} \lesssim \Delta T^{\frac{1}{2}-2\epsilon}\|u_{low}^+ v_{low}\|_{X_{Z}^{0, \epsilon}(S_{\Delta T})} + \|u_{low}^- v_{low}\|_{X_{Z}^{b-1}(S_{\Delta T})}
\]

\[
\lesssim \Delta T^{\frac{1}{2}-2\epsilon}\|u_{low}^+\|_{X_{Z}^{r-2s, \frac{1}{2}+r+2s+\epsilon}(S_{\Delta T})} \|v_{low}\|_{X_{Z}^{0, \frac{1}{2}+\epsilon}(S_{\Delta T})}
\]

\[
+ \|u_{low}^-\|_{X_{Z}^{r-2s, \frac{1}{2}+r+2s+\epsilon}(S_{\Delta T})} \|v_{low}\|_{X_{Z}^{0, \epsilon}(S_{\Delta T})}
\]

\[
\lesssim \Delta T^{\frac{1}{2}+r-2s-3\epsilon}\|I^2 u\|_{H^{-2s, \frac{1}{2}}(S_{\Delta T})} \|I v\|_{X_{Z}^{0, b}(S_{\Delta T})}.
\]

• **Case 2 (low-hi).** Note that Corollary 2 implies that

\[
\|\psi \phi\|_{X_{Z}^{0, b-1}} \lesssim \|\psi\|_{H^{1, b}} \|\psi\|_{X_{Z}^{2, b}} \tag{38}
\]

provided

\[
s_1 > 0, \quad s_2 > -\frac{1}{2} + \epsilon, \quad s_1 + s_2 > \epsilon.
\]

We now apply (38) with \( s_1 = r - 2s, s_2 = 2s - r + 2\epsilon \) to get

\[
\|I(u_{low}v_{hi})\|_{X_{Z}^{b-1}(S_{\Delta T})} \lesssim \|u_{low}\|_{H^{-2s, \frac{1}{2}}(S_{\Delta T})} \|v_{hi}\|_{X_{Z}^{2s-r+2s, b}(S_{\Delta T})}
\]

\[
\lesssim N^{2s-r+2\epsilon}\|I^2 u\|_{H^{-2s, \frac{1}{2}}(S_{\Delta T})} \|I v\|_{X_{Z}^{0, b}(S_{\Delta T})}.
\]
• **Case 3 (hi-low).** An application of (38) with \(s_1 = 2\epsilon, s_2 = 0\) gives
\[
\|I(u_{hi}v_{low})\|_{X^{0,b-1}_+ (S_{\Delta T})} \lesssim \|u_{hi}\|_{H^{r,b}(S_{\Delta T})} \|v_{low}\|_{X^{0,b}_-(S_{\Delta T})} \\
\lesssim N^{2s-r+2\epsilon} \|u\|_{H^{r-2s,b}(S_{\Delta T})} \|v\|_{X^{0,b}_+(S_{\Delta T})},
\]
where we used the assumption \(r > -s\) together with (14).

We now prove (36). We again break \(u = u_{low} + u_{hi}\) and \(v = v_{low} + v_{hi}\) and consider each of the possible interactions separately.

• **Case 1 (low-low).** Corollary 2 together with the assumption \(r - 2s < \frac{1}{2}\) gives
\[
\|I^2(u_{low}v_{low})\|_{H^{r-2s-1,b-1}(S_{\Delta T})} \lesssim \|u_{low}v_{low}\|_{H^{r,b-1}(S_{\Delta T})} \\
\lesssim \|u_{low}\|_{X^{0,b}_+(S_{\Delta T})} \|v_{low}\|_{X^{0,b}_-(S_{\Delta T})} \\
\lesssim \Delta T^{1-2\epsilon} \|u\|_{X^{0,b}_+(S_{\Delta T})} \|v\|_{X^{0,b}_-(S_{\Delta T})},
\]
which follows from Corollary 2 provided
\[
s_1 > -\frac{1}{2}, \quad s_2 > -\frac{1}{2}, \quad s_1 + s_2 > -\frac{1}{2} + \epsilon.
\]
The low-hi case now follows by taking \(s_1 = 0, s_2 = -\frac{1}{2} + 2\epsilon\) and observing that
\[
\|I^2(u_{low}v_{hi})\|_{H^{r-2s-1,b-1}(S_{\Delta T})} \lesssim \|u_{low}v_{hi}\|_{H^{r,b-1}(S_{\Delta T})} \\
\lesssim \|u_{low}\|_{X^{0,b}_+(S_{\Delta T})} \|v_{hi}\|_{X^{-\frac{1}{2}+2s,b}_-(S_{\Delta T})} \\
\lesssim N^{-\frac{1}{2}+2\epsilon} \|u\|_{X^{0,b}_+(S_{\Delta T})} \|v\|_{X^{0,b}_-(S_{\Delta T})},
\]

• **Case 3 (hi-low).** Follows by taking \(s_1 = -\frac{1}{2} + 2\epsilon, s_2 = 0\) in (39) and using an identical argument to the previous case.

• **Case 4 (hi-hi).** As before, we use (39) with \(s_1 = -\frac{1}{2} + 2\epsilon - s\) and \(s_2 = s\) and apply a similar argument to the above cases.

4. **Bilinear Estimates**

In this section we prove Theorem 11. To help simplify the proof, we start by introducing some notation. Let \(m : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}\) and consider the inequality
\[
\int_{\Gamma} m(\tau, \xi) \Pi^3_{j=1} f_j(\tau_j, \xi_j) d\sigma(\tau, \xi) \lesssim \Pi^3_{j=1} \|f_j\|_{L^2_{\tau,\xi}} \tag{40}
\]
where $\tau, \xi \in \mathbb{R}^3$, $\Gamma = \{\xi_1 + \xi_2 + \xi_3 = 0, \ \tau_1 + \tau_2 + \tau_3 = 0\}$, and $d\sigma$ is the surface measure on the hypersurface $\Gamma$. Without loss of generality, we may assume $f_j \geq 0$ as we are using $L^2$ norms on the right hand side of (10). Note that the $X^{a,b}$ estimate contained in Theorem 1 can be written in the form (10) after applying Plancherel and relabeling.

Following Tao in [21], for a multiplier $m$, we use the notation $\|m\|_{[3,\mathbb{R} \times \mathbb{R}]}$ to denote the optimal constant in (10). This norm $\| \cdot \|_{[3,\mathbb{R} \times \mathbb{R}]}$ was studied in detail in [21]. We recall the following elementary properties. Firstly, if $m_1 \leq m_2$ then it is easy to see that $\|m_1\|_{[3,\mathbb{R} \times \mathbb{R}]} \leq \|m_2\|_{[3,\mathbb{R} \times \mathbb{R}]}$. Secondly, via Cauchy-Schwarz, for $j, k \in \{1, 2, 3\}, j \neq k$, we have the characteristic function estimate

$$\|1_A(\tau_j, \xi_j)1_B(\tau_k, \xi_k)\|_{[3,\mathbb{R} \times \mathbb{R}]} \lesssim \sup_{(\tau, \xi) \in \mathbb{R}^2} \left| \{(\lambda, \eta) \in A : (\tau - \tau_j, \xi - \xi_j) \in B\} \right|^2 \tag{41}$$

where $|\Omega|$ denotes the measure of the set $\Omega \subset \mathbb{R}^2$. We refer the reader to [21] for a proof as well a number of other properties of the norm $\| \cdot \|_{[3,\mathbb{R} \times \mathbb{R}]}$.

Let

$$\lambda_1 = \tau_1 \pm \xi_1, \quad \lambda_2 = \tau_2 \pm \xi_2, \quad \lambda_3 = \tau_3 \mp \xi_3. \tag{42}$$

Note that if $(\tau, \xi) \in \Gamma$, then

$$\lambda_1 + \lambda_2 + \lambda_3 = \pm 2\xi_3.$$  

Let $N_j, L_j \in 2^N$, $j = 1, 2, 3$, be dyadic numbers. Our aim is to decompose the $\xi_j$ and $\lambda_j$ variables dyadically, and reduce the problem of estimating $\|m\|_{[3,\mathbb{R} \times \mathbb{R}]}$ to trying to bound the frequency localised version

$$\left\| m(\tau, \xi)\Pi_j=1 \mathbb{1}_{\{\xi_j \approx N_j, |\lambda_j| \approx L_j\}} \right\|_{[3,\mathbb{R} \times \mathbb{R}]}$$

together with computing a dyadic summation. Note that if we restrict $|\xi_j| = N_j$, then since $\xi_1 + \xi_2 + \xi_3 = 0$ we must have $N_{max} \approx N_{med}$, where, as in the introduction, $N_{max} = \max\{N_1, N_2, N_3\}$, $N_{med}$ and $N_{min}$ are defined similarly. Similarly, if $|\lambda_j| \approx L_j$, then (42) implies that $L_{max} \approx \max\{L_{med}, N_3\}$. Hence

$$1 \approx \sum_{N_{max} \approx N_{med}} \sum_{L_{max} \approx \max\{N_3, L_{med}\}} \Pi_j=1 \mathbb{1}_{\{\xi_j \approx N_j, |\lambda_j| \approx L_j\}}.$$  

Combining these observations with results from [21] leads to the following.

**Lemma 12.**

$$\|m\|_{[3,\mathbb{R} \times \mathbb{R}]} \lesssim \sup_N \sum_{N_{max} \approx N_{med}} \sum_{N_{max} \approx N_{med}} \left\| m(\tau, \xi)\Pi_j=1 \mathbb{1}_{\{\xi_j \approx N_j, |\lambda_j| \approx L_j\}} \right\|_{[3,\mathbb{R} \times \mathbb{R}]}.$$

**Proof.** The inequality follows from the triangle inequality together with [21] Lemma 3.11. Alternatively, we can just compute by hand. For ease of notation, let $a_{N_j} = \|f_1\mathbb{1}_{|\xi_j| \approx N_j}\|_{L^2}$, $b_{N_2} = \|f_2\mathbb{1}_{|\xi_2| \approx N_2}\|_{L^2}$, $c_{N_3} = \|f_3\mathbb{1}_{|\xi_3| \approx N_3}\|_{L^2}$, and $A_{N_1, N_2, N_3} = \|m(\tau, \xi)\Pi_j=1 \mathbb{1}_{|\xi_j| \approx N_j}\|_{[3,\mathbb{R} \times \mathbb{R}]}$. Then since $\xi_j$ lie on the surface $\Gamma$, we have $\xi_1 + \xi_2 + \xi_3 = 0$ and so

$$\int_\Gamma m(\tau, \xi)\Pi_j=1 f_j(\tau_j, \xi_j)d\sigma(\tau, \xi) = \sum_{N_{max} \approx N_{med}} \sum_{N_{min} \approx N_{med}} \int_\Gamma m(\tau, \xi)\Pi_j=1 f_j(\tau_j, \xi_j)\mathbb{1}_{|\xi_j| \approx N_j}d\sigma(\tau, \xi)$$

$$\leq \sum_{N_{max} \approx N_{med}} \sum_{N_{min} \approx N_{med}} a_{N_1}b_{N_2}c_{N_3}A_{N_1, N_2, N_3}.$$
Without loss of generality we may assume that \( N_1 \geq N_2 \geq N_3 \) and so \( N_1 \approx N_2 \). For simplicity we also assume that \( N_1 = N_2 \) as the general case \( N_1 \approx N_2 \) is essentially the same. Then
\[
\int_{\Gamma} m(\tau, \xi) \Pi_{j=1}^{3} f_j(\tau_j, \xi_j) d\sigma(\tau, \xi) \leq \sum_{N_1} a_{N_1} b_{N_1} \sum_{N_3 \leq N_1} c_{N_3} A_{N_1, N_1, N_3} \lesssim \left( \sup_{N_3} c_{N_3} \right) \left( \sup_{N_1} \sum_{N_3 \leq N_1} A_{N_1, N_1, N_3} \right) \sum_{N_1} a_{N_1} b_{N_1} \lesssim \left( \sup_{N_1} \sum_{N_3 \leq N_1} A_{N_1, N_1, N_3} \right) \Pi_{j=1}^{3} f_j \|f\|_{L^2}.
\]
Thus we have
\[
\|m\|_{[3, R \times \mathbb{R}]} \lesssim \sup_{N} \sum_{N_{\text{max}} = N_{\text{med}} \approx N} \sum_{N_{\text{min}} \leq N_{\text{med}}} \|m(\tau, \xi) \Pi_{j=1}^{3} f_j(\tau_j, \xi_j) \|_{[3, R \times \mathbb{R}]} \lesssim \|m\|_{[3, R \times \mathbb{R}]}.
\]
To decompose the \( \lambda_j \) variables follows an similar argument. We omit the details.

We now come to the proof of Theorem 1. To begin with, by taking the Fourier transform and relabeling, the required estimate (3) is equivalent to showing
\[
\left| \int_{\Gamma} m(\tau, \xi) \Pi_{j=1}^{3} f_j(\tau_j, \xi_j) d\sigma(\tau, \xi) \right| \lesssim \Pi_{j=1}^{3} f_j \|f\|_{L^2}(\xi)
\]
where
\[
m(\tau, \xi) = \frac{\langle \xi_1 \rangle^{-s_1} \langle \xi_2 \rangle^{-s_2} \langle \xi_3 \rangle^{-s_3}}{\langle \tau_1 \pm 1 \xi_1 \rangle^{s_1} \langle \tau_2 \pm 2 \xi_2 \rangle^{s_2} \langle \tau_3 \pm 3 \xi_3 \rangle^{s_3}}.
\]
Note that Theorem 1 follows from the estimate \( \|m\|_{[3, R \times \mathbb{R}]} < \infty \). Now since
\[
\|\Pi_{j=1}^{3} 1 \{ |\xi| \approx N_j, |\lambda| \approx L_j \} \|_{[3, R \times \mathbb{R}]} \approx \|\Pi_{j=1}^{3} 1 \{ |\xi| \approx N_j, |\lambda| \approx L_j \} \Pi_{j=1}^{3} N_j^{-s_j} L_j^{-b_j} \|
\]
an application of Lemma 12 shows that suffices to estimate, for every \( N \in \mathbb{N} \),
\[
\sum_{N_{\text{max}} = N_{\text{med}} \approx N} N_1^{-s_1} N_2^{-s_2} N_3^{-s_3} \sum_{L_{\text{max}} = \max(L_{\text{med}}, N_3)} L_1^{-b_1} L_2^{-b_2} L_3^{-b_3} \prod_{j=1}^{3} 1 \{ |\xi_j| \approx N_j, |\lambda_j| \approx L_j \} \|_{[3, R \times \mathbb{R}]}.
\]
The first step to estimate this sum is the following estimate on the size of the frequency localised multiplier.

**Lemma 13.**
\[
\|\Pi_{j=1}^{3} 1 \{ |\xi| \approx N_j, |\lambda| \approx L_j \} \|_{[3, R \times \mathbb{R}]} \lesssim \min \left\{ \frac{N_j^{\frac{1}{2}}}{L_{\min}^{\frac{1}{2}}}, L_1^{\frac{1}{2}}, L_2^{\frac{1}{2}}, L_3^{\frac{1}{2}} \right\}
\]

**Proof.** Let \( I = \|\Pi_{j=1}^{3} 1 \{ |\xi| \approx N_j, |\lambda| \approx L_j \} \|_{[3, R \times \mathbb{R}]} \). If we let \( A = 1_{|\lambda| \approx L_j, |\xi| \approx N_j} \) and \( B = 1_{|\lambda_k| \approx L_k, |\xi_k| \approx N_k} \) in (11), then an application of Fubini gives
\[
I \lesssim \| 1_{|\lambda| \approx L_j} 1_{|\lambda_k| \approx L_k} \|_{[3, R \times \mathbb{R}]} \lesssim \sup_{\lambda, \xi \in \mathbb{R}} \left\{ |\lambda| \approx L_j : |\lambda - \lambda_j| \approx L_k \right\} \|1_{\{ |\xi| \approx N_j \}}| \approx \min \left\{ L_j^\frac{1}{2}, L_k^\frac{1}{2} \right\} \min \left\{ N_j^\frac{1}{2}, N_k^\frac{1}{2} \right\}
\]
and hence \( I \lesssim L_{\min}^\frac{1}{2} N_{\min}^\frac{1}{2} \). On the other hand, another application of (11) together with a change of variables gives
\[
I \lesssim \| 1_{|\lambda_1| \approx L_1} 1_{|\lambda_3| \approx L_3} \|_{[3, R \times \mathbb{R}]} \lesssim \sup_{\tau, \xi \in \mathbb{R}} \left\{ |\tau \pm \xi_1| \approx L_1 : |\tau \mp \xi_j \approx L_3 \right\} \|1_{\{ |\xi_j| \approx N_j \}}| \approx \min \left\{ L_1^\frac{1}{2}, L_3^\frac{1}{2} \right\} \min \left\{ N_j^\frac{1}{2}, N_k^\frac{1}{2} \right\}
\]
and hence \( I \lesssim L_{\min}^\frac{1}{2} N_{\min}^\frac{1}{2} \). On the other hand, another application of (11) together with a change of variables gives
\[
I \lesssim \| 1_{|\lambda_1| \approx L_1} 1_{|\lambda_3| \approx L_3} \|_{[3, R \times \mathbb{R}]} \lesssim \sup_{\tau, \xi \in \mathbb{R}} \left\{ |\tau \pm \xi_1| \approx L_1 : |\tau \mp \xi_j \approx L_3 \right\} \|1_{\{ |\xi_j| \approx N_j \}}| \approx \min \left\{ L_1^\frac{1}{2}, L_3^\frac{1}{2} \right\} \min \left\{ N_j^\frac{1}{2}, N_k^\frac{1}{2} \right\}
\]
and hence \( I \lesssim L_{\min}^\frac{1}{2} N_{\min}^\frac{1}{2} \).
Lemma 14. Let \( \epsilon > 0\). Then for any \( L \)

Now for the first sum in (45) we have

Since the righthand side is symmetric under permutations of \( \{\xi_j, |\xi_j| \approx L_j\} \). We need to decompose further into \( L_{max} = L_3 \) and \( L_{max} \neq L_3 \).

Case 1 (\( L_{med} \leq N_3 \)). Since the righthand side of Lemma 13 does not behave symmetrically with respect to the sizes of the \( L_j \), we need to decompose further into \( L_{max} = L_3 \) and \( L_{max} \neq L_3 \).

Case 1a (\( L_{med} \leq N_3 \) and \( L_{max} \neq L_3 \)). We have by Lemma 13

Since the righthand side is symmetric under permutations of \( \{1, 2, 3\} \), we may assume \( L_1 \geq L_2 \geq L_3 \).

Then for any \( \epsilon > 0\)

Now for the first sum in (45) we have

\[
N_3^{-b_1} \sum_{L_2 \leq N_{min}} L_2^{(\frac{1}{2}-b_3)} + \frac{1}{2}-b_2 \lesssim N_3^{\frac{1}{2}+b_1\frac{1}{2}+b_2} N_{min}^b \log(N_{min})
\]

\[
N_3^{\frac{1}{2}+b_1\frac{1}{2}+b_2} N_{min}^b \log(N_{min}) \lesssim N_3^{\frac{1}{2}+b_1\frac{1}{2}+b_2} N_{min}^b + (\frac{1}{2}-b_{med}) + N_3^{-b_{min}+\frac{1}{2}}.
\]
For the second sum we first consider the case \((\frac{1}{2} - b_3)_+ - b_2 > 0\). Then
\[
N_{\min}^\frac{1}{2} N_3^{-b_1} \sum_{N_{\min} \leq L_2 \leq N_3} L_2^{(\frac{1}{2} - b_3)_+ - b_2} \lesssim N_{\min}^\frac{1}{2} N_3^\frac{1}{2} N_{\min}^{-b_3 + b_1 - b_2}
\]
\[
\lesssim N_{\min}^\frac{1}{2} N_3^\frac{1}{2} (\frac{1}{2} - b_{\max})_+ - b_{\med} - b_{\min}
\]

On the other hand if \((\frac{1}{2} - b_3)_+ - b_2 \leq 0\) we get
\[
N_{\min}^\frac{1}{2} N_3^{-b_1} \sum_{N_{\min} \leq L_2 \leq N_3} L_2^{(\frac{1}{2} - b_3)_+ - b_2} \lesssim N_{\min}^\frac{1}{2} - b_2 + (\frac{1}{2} - b_3)_+ + N_3^{-b_1} \log(N_3)
\]
\[
\lesssim N_{\min}^\frac{1}{2} (\frac{1}{2} - b_{\max})_+ + (\frac{1}{2} - b_{\med})_+ + N_3^{-b_{\min}} + \frac{1}{2}.
\]

Together with (45) this then gives
\[
\sum_{L_{\max} \approx N_3 \geq L_{\med}} L_1^{-b_1} L_2^{-b_2} L_3^{-b_3} \left\| \prod_{j=1}^3 1\{|\xi_j| \approx N_j, |\lambda_j| \approx L_j\} \right\|_{[3,R \times R]} \lesssim N_3^\frac{1}{2} \left( N_3^{\frac{1}{2} - b_{\max}}_+ - b_{\med} - b_{\min} \right) N_{\min}^\frac{1}{2} + N_3^{-b_3} N_{\min}^\frac{1}{2} + (\frac{1}{2} - b_{\med})_+ + N_3^{-b_{\min}}.
\]

where we used the inequality
\[
N_{\min}^\frac{1}{2} N_3^{\frac{1}{2} - b_{\max}}_+ - b_{\med} - b_{\min} \leq N_{\min}^\frac{1}{2} N_3^{\frac{1}{2} - b_1 - b_2 - b_3} + N_{\min}^{\frac{1}{2} - b_{\max}}_+ + (\frac{1}{2} - b_{\med})_+ + N_3^{-b_{\min}}.
\]

which is trivial if \(b_{\max} < \frac{1}{2}\). On the other hand, if \(b_{\max} \geq \frac{1}{2}\), then (46) follows by noting that since \(b_j + b_k > 0\) we have \(b_{\med} > 0\) and so
\[
N_{\min}^\frac{1}{2} N_3^{-b_{\med} - b_{\min}} \lesssim N_{\min}^\frac{1}{2} - b_{\med} N_3^{-b_{\min}} \lesssim N_{\min}^{\frac{1}{2} - b_{\med}} + N_3^{-b_{\min}}
\]
as required.

- **Case 1b** \((L_{\med} \leq N_3 \text{ and } L_{\max} = L_3)\). Lemma (13) together with the assumption \(L_{\max} = L_3\) gives
\[
\left\| \prod_{j=1}^3 1\{ |\xi_j| \approx N_j, |\lambda_j| \approx L_j \} \right\|_{[3,R \times R]} \lesssim N_{\min}^\frac{1}{2}.
\]

Suppose \(L_1 \leq L_2\). Then
\[
\sum_{L_{\max} \approx N_3 \geq L_{\med}} L_1^{-b_1} L_2^{-b_2} L_3^{-b_3} \left\| \prod_{j=1}^3 1\{ |\xi_j| \approx N_j, |\lambda_j| \approx L_j \} \right\|_{[3,R \times R]} \lesssim N_{\min}^\frac{1}{2} N_3^{-b_3} \sum_{L_2 \leq N_3} L_2^{-b_2} \sum_{L_1 \leq L_2} L_1^{-b_1}
\]
\[
\lesssim N_{\min}^\frac{1}{2} N_3^{-b_3} \sum_{L_2 \leq N_3} L_2^{(\frac{1}{2} - b_1)_+ - b_2} \log(L_2)
\]
\[
\lesssim N_{\min}^\frac{1}{2} N_3^{((\frac{1}{2} - b_1)_+ - b_2)_+ - b_3 + \epsilon}
\]

for any \(\epsilon > 0\). If we have
\[
N_{\min}^\frac{1}{2} N_3^{((\frac{1}{2} - b_1)_+ - b_2)_+ - b_3} \leq N_3^{\frac{1}{2} - b_1 - b_2 - b_3} N_{\min}^\frac{1}{2} + N_3^{-b_3} N_{\min}^\frac{1}{2} + N_3^{-b_{\min}} N_{\min}^{\frac{1}{2} - b_{\max}}_+ + (\frac{1}{2} - b_{\med})_+
\]
then we get
\[
N_3^\frac{1}{2} \left( N_3^{\frac{1}{2} - b_1 - b_2 - b_3} N_{\min}^\frac{1}{2} + N_3^{-b_3} N_{\min}^\frac{1}{2} + N_3^{-b_{\min}} N_{\min}^{\frac{1}{2} - b_{\max}}_+ + (\frac{1}{2} - b_{\med})_+ \right)
\]
as required. The case $L_1 \geq L_2$ follows an identical argument and so it remains to show (48). To this end note that if $(\frac{1}{2} - b_1)_+ = b_2 < 0$ then we simply have
\[
N_\min^{\frac{1}{2}} N_3^{((\frac{1}{2} - b_1)_+ - b_2)_+ - b_3} = N_\min^{\frac{1}{2}} N_3^{-b_3}.
\]
On the other hand, if $(\frac{1}{2} - b_1)_+ - b_2 \geq 0$, then by using (46) we have
\[
N_\min^{\frac{1}{2}} N_3^{((\frac{1}{2} - b_1)_+ - b_2)_+ - b_3} \leq N_\min^{\frac{1}{2}} N_3^{(\frac{1}{2} - b_1)_+ - b_2 - b_3} + N_\min^{\frac{1}{2}} N_3^{\frac{1}{2} - b_1 - b_2 - b_3} + N_\min^{\frac{1}{2}} N_3^{(\frac{1}{2} - b_{\max})_+ + (\frac{1}{2} - b_{med})_+ + N_\min^{\frac{1}{2}} N_3^{-b_{\min}}}
\]
and so we obtain (48).

- **Case 2** ($L_{med} \geq N_3$). In this case we have $L_{max} \approx L_{med}$ and by Lemma 13
\[
\|\Pi_{j=1}^3 \{ \| \xi_j \| \approx N_j, \| \lambda_j \| \approx L_j \} \|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim N_\min^{\frac{1}{2}} L_{min}^{\frac{1}{2}}.
\]
Suppose $L_1 \geq L_2 \geq L_3$. Then
\[
\sum_{L_{max} \approx L_{med} \geq N_3} L_1^{-b_1} L_2^{-b_2} L_3^{-b_3} N_\min^{\frac{1}{2}} L_{min}^{\frac{1}{2}} \lesssim N_\min^{\frac{1}{2}} \sum_{L_2 \geq N_3} L_2^{-b_1 - b_2} \sum_{L_3 \leq L_2} L_3^{-b_2} \lesssim N_\min^{\frac{1}{2}} \sum_{L_2 \geq N_3} L_2^{(\frac{1}{2} - b_1)_+ - b_1 - b_2 + \epsilon} \log(L_2) \lesssim N_\min^{\frac{1}{2}} N_3^{(\frac{1}{2} - b_1)_+ - b_1 - b_2 + \epsilon} \lesssim N_\min^{\frac{1}{2}} N_3^{\frac{1}{2} - b_{med} - b_{\min} + \epsilon}
\]
provided $b_1 + b_2 + b_3 > \frac{1}{2}$, $b_j + b_k > 0$, and we choose $\epsilon > 0$ sufficiently small. Since this argument also holds for all other size combinations of the $L_j$, we get from (48)
\[
\sum_{L_{max} \approx L_{med} \geq N_3} L_1^{-b_1} L_2^{-b_2} L_3^{-b_3} \|\Pi_{j=1}^3 \{ \| \xi_j \| \approx N_j, \| \lambda_j \| \approx L_j \} \|_{[3, \mathbb{R} \times \mathbb{R}]} \lesssim N_3^\epsilon \left( N_3^{\frac{1}{2} - b_1 - b_2 - b_3} N_\min^{\frac{1}{2}} + N_3^{-b_{\min}} N_\min^{(\frac{1}{2} - b_{\max})_+ + (\frac{1}{2} - b_{med})_+} \right)
\]
and so lemma follows.

We now come to the proof of Theorem 1.

**Proof of Theorem 1.** By Lemma 12 and Lemma 14 it suffices to estimate the sum
\[
\sup_{N} \sum_{N_{max} \approx N_{med} \approx N} \left( \Pi_{j=1}^3 N_j^{-s_j} \right) N_\min^\alpha N_3^{-\beta} N_\min^{\frac{1}{2}} N_3^{-\frac{1}{2}}
\]
for the pairs
\[
(\alpha, \beta) \in \left\{ \left( \frac{1}{2}, b_1 + b_2 + b_3 - \frac{1}{2} - \epsilon \right), \left( \frac{1}{2}, b_3 - \epsilon \right), \left( \frac{1}{2} - b_{\max} \right)_+ + \left( \frac{1}{2} - b_{med} \right)_+ , b_{\min} - \epsilon \right\}
\]
where $\epsilon > 0$ may be taken arbitrarily small. Let $s'_1 = s_1$, $s'_2 = s_2$, and $s'_3 = s_3 + \beta$. Then we have to show
\[
\sup_{N} \sum_{N_{max} \approx N_{med} \approx N} \left( \Pi_{j=1}^3 N_j^{-s'_j} \right) N_\min^\alpha N_3^{-\beta} \lesssim \sup_{N} \sum_{N_{max} \approx N_{med} \approx N} \left( \Pi_{j=1}^3 N_j^{-s'_j} \right) N_\min^\alpha N_3^{-\beta} < \infty.
\]
Since this summation is symmetric with respect to the \( N_j \), we may assume \( N_1 \leq N_2 \leq N_3 \). Then
\[
\sum_{N_{\text{med}} \leq N_j \leq N_{\text{max}}} \left( \prod_{j=1}^{3} N_j^{s_j'} \right) N_{\text{min}}^{\alpha} \lesssim N^{-s'_3 - s'_4} \sum_{N_1 \leq N} N_1^{-s'_1 + \alpha} < \infty
\]
provided \( s'_j + s'_k \geq 0 \) and \( s'_1 + s'_2 + s'_4 > \alpha \). These conditions hold by the assumptions in Theorem 1 provided we choose \( \epsilon \) sufficiently small.

\( \square \)

5. Counter Examples

Here we prove that the conditions in Theorem 1 are sharp up to equality.

**Proposition 15.** Assume the estimate (43) holds. Then we must have
\[
b_j + b_k \geq 0, \quad b_1 + b_2 + b_3 \geq \frac{1}{2}
\]
and for \( k \in \{1, 2\} \)
\[
s_1 + s_2 \geq 0, \quad s_k + s_3 \geq -b_{\text{min}}, \quad s_k + s_3 \geq \frac{1}{2} - b_1 - b_2 - b_3, \quad s_1 + s_2 + s_3 \geq \frac{1}{2} - b_3, \quad s_1 + s_2 + s_3 \geq \left( \frac{1}{2} - b_{\text{max}} \right)_+ + \left( \frac{1}{2} - b_{\text{med}} \right)_+ - b_{\text{min}}.
\]

**Remark 7.** We note that in some regions the \( \pm \) structure in (1) is redundant and so the counter examples for the Wave-Sobolev spaces used in [11] and [19] would apply. In fact, the counterexamples in [11] already essentially show that we must have [49], [50], and [54]. On the other hand, the conditions [51]-[53] reflect the \( \pm \) structure and thus cannot be deduced from [11].

**Proof.** It suffices to find necessary conditions for the estimate (43). Moreover we may assume \( \pm = + \) since the case \( \pm = - \) follows by a reflection in the \( \tau_j \) variables. Let \( \lambda \gg 1 \) be some large parameter. The main idea is as follows. Assume we have sets \( A, B, C \subset \mathbb{R}^{1+1} \) with
\[
|A| \approx \lambda^{d_1}, \quad |B| \approx \lambda^{d_2}, \quad |C| \approx \lambda^{d_3}.
\]
Moreover, suppose that if \( (\tau_2, \xi_2) \in B \) and \( (\tau_3, \xi_3) \in C \), then
\[
-(\tau_2 + \tau_3, \xi_2 + \xi_3) \in A
\]
and
\[
\langle \xi_2 + \xi_3 \rangle^{-s_1} \langle \xi_2 \rangle^{-s_2} \langle \xi_3 \rangle^{-s_3} \approx (\tau_2 + \tau_3 + \xi_2 + \xi_3)^{b_1} (\tau_2 + \xi_2)^{b_2} (\tau_3 - \xi_3)^{b_3} \approx \lambda^{-\delta}.
\]
Let \( f_1 = 1_A, f_2 = 1_B, f_3 = 1_C \). Then using the conditions (55)-(57) we have
\[
\int_{\Gamma} m(\tau, \xi) \prod_{j=1}^{3} f_j(\tau_j, \xi_j) d\sigma(\tau, \xi) \gtrsim \lambda^{-\delta} \int_{B} \int_{C} dr_3 d\xi_3 dr_2 d\xi_2 \approx \lambda^{d_2 + d_3 - \delta}.
\]
Therefore, assuming that the inequality (43) holds, we must have
\[
\lambda^{d_2 + d_3 - \delta} \lesssim |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} |C|^{\frac{1}{2}} \approx \lambda^{\frac{d_1 + d_2 + d_3}{2}}.
\]
By choosing $\lambda$ large, we then derive the necessary condition
\[
\delta + \frac{d_1 - d_2 - d_3}{2} \geq 0.
\] (58)

Thus it will suffice to find sets $A$, $B$, and $C$ satisfying the conditions (55-57) with particular values of $\delta$, $d_1$, $d_2$, and $d_3$.

- **Necessity of (50).** We first show that $b_j + b_k \geq 0$. Since the estimate (13) is symmetric in $b_1$, $b_2$, it suffices to consider the pairs $(j, k) \in \{(1, 2), (1, 3)\}$. For the first pair, we choose
\[
B = \{ |\tau + \lambda| \leq 1, |\xi| \leq 1 \}, \quad C = \{ |\tau| \leq 1, |\xi| \leq 1 \}, \quad A = \{ |\tau - \lambda| \leq 2, |\xi| \leq 2 \}.
\]
Then the conditions (55-57) hold with $d_1 = d_2 = d_3 = 0$ and $\delta = b_1 + b_2$ and so from (58) we obtain the necessary condition $b_1 + b_2 \geq 0$.

On the other hand, for the pair $(1, 3)$ we choose
\[
B = \{ |\tau| \leq 1, |\xi| \leq 1 \}, \quad C = \{ |\tau + \lambda| \leq 1, |\xi| \leq 1 \}, \quad A = \{ |\tau - \lambda| \leq 2, |\xi| \leq 2 \}.
\]
Then as in the previous case, the conditions (55-57) hold with $d_1 = d_2 = d_3 = 0$ and $\delta = b_1 + b_3$ and so from (58) we obtain the necessary condition $b_1 + b_3 \geq 0$.

To show the second condition in (49) is also necessary, we take
\[
B = \{ |\tau - 2\lambda| \leq \lambda, |\xi| \leq 1 \}, \quad C = \{ |\tau - 2\lambda| \leq \lambda, |\xi| \leq 1 \}, \quad A = \{ |\tau + 4\lambda| \leq 2\lambda, |\xi| \leq 2 \}.
\]
Then (55-57) hold with $d_1 = d_2 = d_3 = 1$ and $\delta = b_1 + b_2 + b_3$ which leads to the condition $b_1 + b_2 + b_3 \geq \frac{1}{2}$.

- **Necessity of (51).** Let
\[
B = \{ |\tau - \lambda| \leq 1, |\xi + \lambda| \leq 1 \}, \quad C = \{ |\tau| \leq 1, |\xi| \leq 1 \}, \quad A = \{ |\tau + \lambda| \leq 2, |\xi - \lambda| \leq 2 \}.
\]
Then (55-57) hold with $d_1 = d_2 = d_3 = 0$ and $\delta = s_1 + s_2$ and so we must have (50).

- **Necessity of (51).** By symmetry we may assume $k = 1$. Suppose $b_{\min} = b_1$ and choose
\[
B = \{ |\tau| \leq 1, |\xi| \leq 1 \}, \quad C = \{ |\tau - \lambda| \leq 1, |\xi - \lambda| \leq 1 \}, \quad A = \{ |\tau + \lambda| \leq 2, |\xi + \lambda| \leq 2 \}.
\]
Then (55-57) hold with $d_1 = d_2 = d_3 = 0$ and $\delta = s_1 + s_3 + b_1$ and so we must have $s_1 + s_3 + b_1 \geq 0$.

On the other hand, if $b_{\min} = b_2$ we let
\[
B = \{ |\tau + 2\lambda| \leq 1, |\xi| \leq 1 \}, \quad C = \{ |\tau - \lambda| \leq 1, |\xi - \lambda| \leq 1 \}, \quad A = \{ |\tau - \lambda| \leq 2, |\xi + \lambda| \leq 2 \}.
\]
Then (55-57) hold with $d_1 = d_2 = d_3 = 0$ and $\delta = s_1 + s_3 + b_2$ and so we obtain the condition $s_1 + s_3 + b_2 \geq 0$.

The final case, $b_{\min} = b_3$, follows by taking
\[
B = \{ |\tau| \leq 1, |\xi| \leq 1 \}, \quad C = \{ |\tau - \lambda| \leq 1, |\xi + \lambda| \leq 1 \}, \quad A = \{ |\tau + \lambda| \leq 2, |\xi - \lambda| \leq 2 \}.
\]
Again the conditions (55-57) hold with $d_1 = d_2 = d_3 = 0$ and $\delta = s_1 + s_3 + b_3$. Hence (51) is necessary.

- **Necessity of (52).** As in the previous case, by symmetry, we may assume $k = 1$. Let
\[
B = \{ |\tau - \lambda| \leq \frac{\lambda}{4}, |\xi| \leq 1 \}, \quad C = \{ |\tau| \leq \frac{\lambda}{4}, |\xi - \lambda| \leq \frac{\lambda}{4} \}, \quad A = \{ |\tau + \lambda| \leq \frac{\lambda}{2}, |\xi + \lambda| \leq \frac{\lambda}{2} \}.
\]
Then \(55 - 57\) hold with \(d_1 = d_3 = 2, d_2 = 1,\) and \(\delta = s_1 + s_3 + b_1 + b_2 + b_3.\) Thus we obtain the necessary condition \(52.\)

\[\bullet \text{ Necessity of } (53).\] In this case we choose
\[B = \left\{ |\tau + \xi| \leq 1, \ |\xi - \lambda| \leq \frac{\lambda}{4} \right\}, \quad C = \left\{ |\tau + \xi| \leq 1, \ |\xi - \lambda| \leq \frac{\lambda}{4} \right\}, \quad A = \left\{ |\tau + \xi| \leq 2, \ |\xi + 2\lambda| \leq \frac{\lambda}{2} \right\}.
\]
Then a simple computation shows that \(55 - 57\) hold with \(d_3 = 2,\) and \(\delta = s_1 + s_2 + s_3 + b_3\). So we see that \(53\) is necessary.

\[\bullet \text{ Necessity of } (54).\] We break this into the 3 conditions
\[s_1 + s_2 + s_3 \geq 1 - b_1 - b_2 - b_3, \quad s_1 + s_2 + s_3 \geq \frac{1}{2} - b_j - b_k, \quad s_1 + s_2 + s_3 \geq -b_{\min}. \quad (59)
\]
For the first inequality, we take
\[B = \left\{ |\tau| \leq \frac{\lambda}{4}, \ |\xi - \lambda| \leq \frac{\lambda}{4} \right\}, \quad C = \left\{ |\tau| \leq \frac{\lambda}{4}, \ |\xi - \lambda| \leq \frac{\lambda}{4} \right\}, \quad A = \left\{ |\tau| \leq \frac{\lambda}{2}, \ |\xi + 2\lambda| \leq \frac{\lambda}{2} \right\}.
\]
Then we have \(55 - 57\) with \(d_1 = d_3 = 2,\) and \(\delta = s_1 + s_2 + s_3 + b_1 + b_2 + b_3.\) Therefore we must have \(s_1 + s_2 + s_3 \geq 1 - b_1 - b_2 - b_3.\)

We now consider the second inequality in \(59.\) By symmetry, it suffices to consider \((j, k) \in \{(1, 2), (1, 3)\}.\) Let
\[B = \left\{ |\tau + (\xi - \lambda)| \leq \frac{\lambda}{4}, \ |\xi - \lambda| \leq \frac{\lambda}{4} \right\}, \quad C = \left\{ |\tau - \xi| \leq 1, \ |\xi - \lambda| \leq \frac{\lambda}{4} \right\}, \quad A = \left\{ |\tau + \xi + 3\lambda| \leq \lambda, \ |\xi + 2\lambda| \leq \frac{\lambda}{2} \right\}.
\]
Then \(55 - 57\) hold with \(d_1 = d_2 = d_3 = 2,\) and \(\delta = s_1 + s_2 + s_3 + b_1 + b_2.\) Therefore we must have \(s_1 + s_2 + s_3 > \frac{1}{2} - b_1 - b_2.\) On the other hand, for the case \((j, k) = (1, 3),\) we take
\[B = \left\{ |\tau + \xi| \leq 1, \ |\xi - \lambda| \leq \frac{\lambda}{4} \right\}, \quad C = \left\{ |\tau| \leq \frac{\lambda}{4}, \ |\xi - \lambda| \leq \frac{\lambda}{4} \right\}, \quad A = \left\{ |\tau + \xi + \lambda| \leq \frac{3\lambda}{4}, \ |\xi + 2\lambda| \leq \frac{\lambda}{2} \right\}.
\]
A simple computation shows that \(55 - 57\) are satisfied with \(d_1 = d_3 = 2, d_2 = 1,\) and \(\delta = s_1 + s_2 + s_3 + b_1 + b_3.\)

Finally, the third condition in \(59\) follows from the conditions \(50\) and \(51.\) \(\square\)

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