Bogoliubov wave turbulence in Bose-Einstein condensates

Kazuya Fujimoto¹ and Makoto Tsubota¹,²

¹Department of Physics, Osaka City University, Sumiyoshi-ku, Osaka 558-8585, Japan
²The OCU Advanced Research Institute for Natural Science and Technology (OCARINA), Osaka City University, Sumiyoshi-ku, Osaka 558-8585, Japan

(Dated: March 6, 2015)

We theoretically and numerically study Bogoliubov wave turbulence in three-dimensional atomic Bose-Einstein condensates with the Gross-Pitaevskii equation, investigating three correlation functions for the macroscopic wave function, the density distribution, and the Bogoliubov wave distribution. In this turbulence, Boboliubov waves play an important role in the behavior of the correlation functions, so that we call it Bogoliubov wave turbulence. In a previous study [D. Proment et al., Phys. Rev. A 80, 051603(R) (2009)], a −3/2 power law in the correlation function for the macroscopic wave function was suggested by using weak wave turbulence theory, but we find that another −7/2 power law appears in both theoretical and numerical calculations. Furthermore, we focus on the density correlation function, which can be observed in experiments, discussing the possibility of experimental observation. Through these analytical and numerical calculations, we also demonstrate that the previously neglected condensate dynamics induced by the Bogoliubov waves is remarkably important.

PACS numbers: 03.75.Kk,05.45.-a

I. INTRODUCTION

In nature, complex and chaotic flow universally appears in various systems from small to large scale, for instance, in watercourses, rivers, oceans, and the atmosphere. Such a flow, called turbulence, has attracted considerable attention in various fields because understanding it leads to significant development of fundamental and applied sciences such as mathematics, physics, and technology. Though turbulence has been studied for a long time, however, it is considered to be one of the unresolved problems and becomes an important issue in modern physics.

Originally, turbulence has been studied in classical fluids, where it is important to investigate statistical quantities such as the probability density function of the velocity field [1,2]. One of the most significant quantities is the kinetic energy spectrum corresponding to a two-point correlation function of the velocity field, which is known to exhibit the famous Kolmogorov −5/3 power law. This Kolmogorov law appears as a result of the constant energy transfer in wave number space, and the observation of this power law is very important in the study of turbulence. In this paper, we call turbulence with constant transfer for focusing quantities Kolmogorov turbulence.

In quantum fluids, turbulence also appears and has been investigated for many years [3–7]. This turbulence is called quantum turbulence (QT), in contrast to classical turbulence (CT), and was originally developed in the field of superfluid helium. QT offers rich phenomena such as thermal counterflow QT [8,9], Kolmogorov cascades [10,12], Kelvin wave cascades [13,14], the bottleneck effect [15], velocity statistics [16], and the reconnection of quantized vortices [17], leading to a deep understanding of turbulence, and it has been actively studied until today.

Recently, atomic Bose-Einstein condensates (BECs) have been providing a novel stage for turbulence studies in quantum fluids [3–7]. This system has three advantages for investigating turbulence in quantum fluids. The first advantage is the experimental visualization of the density and vortex distribution [18], which gives us a large amount of information about turbulence. Such an observation is difficult in superfluid helium, though the vortices have been visualized recently by use of tracer particles [10,17]. The second advantage is the high controllability of the system by optical techniques [19], which enable us to change the shape of the quantum fluid and the interaction between particles. This can access new problems; the geometrical dependence of turbulence such as two-dimensional QT. This kind of study is being actively investigated. The third advantage is the realization of multicomponent BECs such as binary [20] and spinor BECs [21,22]. The hydrodynamics in this kind of quantum fluid will give us novel physics, and investigations in this area have recently begun [23,29]. Thus, turbulence in the atomic BECs can address exotic phenomena not found in superfluid helium.

At present, there is some research on turbulence in atomic BECs, and most theoretical and numerical studies consider turbulence with many quantized vortices. In three- and two-dimensional systems, when many quantized vortices are nucleated, the Kolmogorov −5/3 power law is numerically confirmed in incompressible kinetic energy [30,34]. Recently, two-dimensional QT has been actively investigated, with discussion of features characteristic of two-dimensional systems such as inverse energy cascades [32,34], vortex clustering [34,36], negative temperature [38], and decay of vortex number [37].
In experiments, some groups have succeeded in generating turbulence in atomic BECs [38–41], where it has been possible to obtain turbulence with many quantized vortices. All these experimental studies focus on quantized vortices, investigating the anomalous expansion [38], vortex dynamics [39], annihilation of vortices [41], and so on. However, the energy spectrum corresponding to a two-point correlation function of the velocity field has not yet been observed. This means that Kolmogorov turbulence has not been observed. It would be very important to observe Kolmogorov turbulence in atomic BECs; however, observing the superfluid velocity directly in turbulence is difficult, so we cannot confirm whether Kolmogorov turbulence appears as of now.

To shed new light on the above problem, we focus on weak wave turbulence (WT) with a strong condensate in atomic BECs because the density profile of the BEC is observable and thus plays an important role in observing Kolmogorov turbulence. In this turbulence, Bogoliubov waves are significant, so we call it Bogoliubov wave turbulence in this paper.

Originally, in classical fluids, WT, which is turbulent flow dominated by waves, has been studied [42, 43]. It is much different from the usual hydrodynamic turbulence (HT) because vortex structures are regarded as important in HT [1, 2]. There are weak and strong kinds of WT, depending on whether the nonlinearity is weak or strong. In weak WT, a constant transfer of energy or wave action generates power law behaviors in correlation functions, and Kolmogorov turbulence is known to occur in various wave systems such as those involving fluid surfaces (gravity and capillary waves) [44], magnetic substances (spin waves) [45], and elastic media (acoustic waves) [46].

We expect that it is possible to confirm Kolmogorov turbulence in atomic BECs by the observation of the density profile in Bogoliubov wave turbulence. In this paper, by applying weak WT theory [42, 43] to the Gross-Pitaevskii (GP) equation, we theoretically and numerically study the power law behaviors for three correlation functions of a macroscopic wave function, density distribution, and Bogoliubov wave distribution, discussing the experimental possibility for observing Kolmogorov turbulence, and an inconsistency of our analytical calculation. Finally, we summarize our study in Sec. VI.

II. FORMULATION

We address a one-component BEC in a uniform system at zero temperature. This system is well described by the macroscopic wave function $\psi$ obeying the GP equation [19] given by

$$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + g|\psi|^2 \psi$$

(1)

with particle mass $m$ and interaction coefficient $g$.

In this paper, we focus on spectra for the macroscopic wave function $\psi$, the density distribution $\rho = |\psi|^2$, and the Bogoliubov wave distribution $b$, which correspond to the two-point correlation functions of each variable. The details of the Bogoliubov wave distribution are defined in Sec. III B. The spectrum for the wave function is defined by

$$C_{\psi}(k) = \frac{1}{\Delta k} \sum_{k-\Delta k/2 < |k_1| < k + \Delta k/2} \langle |\tilde{\psi}(k_1)|^2 \rangle$$

(2)

with a resolution of $\Delta k = 2\pi/L$ in wave number space and a system size $L$. The brackets indicate an ensemble average. The function $\tilde{\psi}(k)$ is the Fourier component of the macroscopic wave function calculated by $\mathcal{F}[\psi(r)]$ with $\langle \rangle = \int e^{-i k r} dV/L^d$ and spatial dimension $d$. In the same way, we can define the spectra for the density and Bogoliubov wave distributions as

$$C_{\rho}(k) = \frac{1}{\Delta k} \sum_{k-\Delta k/2 < |k_1| < k + \Delta k/2} \langle |\tilde{\rho}(k_1)|^2 \rangle,$$

(3)

$$C_{b}(k) = \frac{1}{\Delta k} \sum_{k-\Delta k/2 < |k_1| < k + \Delta k/2} \langle |\tilde{b}(k_1)|^2 \rangle.$$  

(4)

with the Fourier components of the density distribution $\tilde{\rho}(k) = \mathcal{F}[\rho(r)]$ and the Bogoliubov wave distribution $b(k)$. 

lytically deriving the $-7/2$ power law in the correlation function for the macroscopic wave function and numerically confirm this power law. We find that the condensate dynamics induced by the fluctuation, which is neglected in the previous studies, is important for Bogoliubov wave turbulence. Furthermore, we focus on the density correlation function, obtaining the $-3/2$ power law, and discuss the experimental possibility of this power law.

The article is organized as follows. Section II describes the GP equation and the correlation functions for some quantities. In Sec. III, we apply weak WT theory to the GP equation, deriving the power laws for the correlation functions. In Sec. IV, we show our numerical result for this Bogoliubov wave turbulence. Section V discusses comparison between the previous results and ours, the observation of Kolmogorov turbulence, and an inconsistency of our analytical calculation. Finally, we summarize our study in Sec. VI.
III. APPLICATION OF WEAK WAVE TURBULENCE THEORY

We study weak WT in a uniform BEC with the strong condensate, deriving the power exponents of $C_w(k)$, $C_d(k)$, and $C_b(k)$. For this purpose, we apply weak WT theory to the GP equation (1). In this application, there is an important points, which is the nonlinear dynamics for the condensate. We take the second-order of the fluctuation for the macroscopic wave function, treating this nonlinear dynamics. In this case, we can also use weak WT theory, obtaining these power exponents.

A. Equations of the condensate and the fluctuation

We now derive the equations of the condensate and the fluctuation. First, we define the condensate $\psi$ and the fluctuation $\phi$ as

$$\psi = \psi_0 (1 + \phi),$$

where $\psi_0$ is defined by

$$\psi_0 = \frac{1}{L^d} \int \psi \, dV.$$  

From Eqs. (5) and (6), we can derive

$$\int \phi \, dV = 0.$$  

Equations (5) and (7) mean that the condensate and the fluctuation are the $k = 0$ and $k \neq 0$ Fourier components of $\psi(k)$, respectively. In this paper, we consider a weak fluctuation from a strong condensate, assuming the weak nonlinear condition $|\phi| \ll 1$.

We substitute Eq. (5) into the GP equation (1), deriving equations for $\psi_0$ and $\phi(k) = \mathcal{F}[\phi]$ within second order of the fluctuation:

$$i\hbar \frac{\partial}{\partial t} \psi_0 = g\rho_0 |\phi|^2 \psi_0 \left[ 1 + \sum_{k_1} (2|\phi(k_1)|^2 + |\phi(k_1)|^2 |\phi(-k_1)|^2) \right],$$

which is discussed in Sec. IV B.

B. Diagonalization of the Hamitonian and Bogoliubov wave distribution equation

To diagonalize the one-body Hamiltonian $H_2$, we use the Bogoliubov transformation defined by

$$\tilde{\phi}(k) = u(k) b(k) + v(k) b^*(-k),$$

where $u(k)$ is given by

$$u(k) = \sqrt{\frac{1}{2} \left( \frac{\epsilon_0(k) + g\rho_0}{\epsilon_b(k)} + 1 \right)},$$

where $\rho_0$ is the condensate density $|\psi_0|^2$ and $\delta(\cdot)$ is the Kronecker delta. Substituting Eq. (8) into Eq. (9), we obtain the following equation for the fluctuation:

$$i\hbar \frac{\partial}{\partial t} \tilde{\phi}(k) = \frac{\hbar^2 k^2}{2m} \tilde{\phi}(k) + g\rho_0 \left[ \tilde{\phi}(k) + \tilde{\phi}^*(-k) \right] + 2 \sum_{k_1, k_2} \tilde{\phi}^*(k_1) \tilde{\phi}(k_2) \delta(k + k_1 - k_2) + \sum_{k_2, k_3} \tilde{\phi}(k_2) \tilde{\phi}(k_3) \delta(k - k_2 - k_3).$$
where $b(k)$ is the canonical variable for Bogoliubov waves (Bogoliubov wave distribution) and $\epsilon_b(k)$ is the dispersion relation for this wave defined by

$$
\epsilon_b(k) = \sqrt{\epsilon_0(k)\epsilon_0(k) + 2\rho_0},
$$

with $\epsilon_0(k) = \hbar^2k^2/2m$. Applying this transformation to the Hamiltonian of Eqs. (13) and (14) leads to

$$
H_2 = \sum_{k_1} \epsilon_b(k_1)|b(k_1)|^2,
$$

$$
H_3 = \sum_{k_1,k_2,k_3} \delta(k_1 - k_2 - k_3) V(k_1,k_2,k_3) \times \left( b^*(k_1)b(k_2)b(k_3) + b(k_1)b^*(k_2)b^*(k_3) \right)
+ \sum_{k_1,k_2,k_3} \delta(k_1 + k_2 + k_3) W(k_1,k_2,k_3) \times \left( b^*(k_1)b^*(k_2)b^*(k_3) + b(k_1)b(k_2)b(k_3) \right),
$$

where $V$ and $W$ are the interaction functions for Bogoliubov waves. These functions are given by

$$
V(k_1,k_2,k_3) = \rho_0 \left( u_1u_2u_3 + v_1u_2v_3 + v_1v_2u_3 + u_1v_2u_3 + u_1u_2v_3 \right),
$$

$$
W(k_1,k_2,k_3) = \rho_0 \left( u_1v_2u_3 + u_1u_2v_3 \right),
$$

where we use simple expressions such as $u_1$ and $v_1$ for $u(k_1)$ and $v(k_1)$.

We must notice that the Bogoliubov coefficients $u(k)$ and $v(k)$ have time dependence through the condensate density $\rho_0$. Thus, the transformation of Eq. (15) is not the canonical transformation. However, as shown in the following, the time dependence of these coefficients are found to give only small corrections in the weak nonlinear condition, so that the transformation of Eq. (15) approximately becomes the canonical transformation. To indicate this, we derive the time development equation for $\rho_0$ from Eq. (18), which is given by

$$
\frac{\partial}{\partial t} \rho_0 = \frac{g_0^2}{i\hbar} \sum_{k_1} \left[ b(k_1)b(-k_1) - b^*(k_1)b^*(-k_1) \right].
$$

Then, the time derivative of the Bogoliubov coefficient $u(k)$ becomes

$$
\frac{\partial}{\partial t} u(k) = \frac{\partial u(k)}{\partial \rho_0} \frac{\partial \rho_0}{\partial t},
$$

which means that the value of this derivative is the second-order of the fluctuation. The same thing is true

in the time derivative of $v(k)$. As a result, in the weak nonlinear condition, the time derivative of Eq. (14) is

$$
\frac{\partial}{\partial t} \tilde{\phi}(k) \approx u(k) \frac{\partial}{\partial \rho_0} \rho_0 + v(k) \frac{\partial}{\partial \rho_0} \rho_0^*,
$$

since the third-order of the fluctuation is neglected in Eq. (10). Thus, the Bogoliubov transformation of Eq. (15) approximately becomes the canonical transformation.

Based on the above calculation, the time development equation for $b(k)$ becomes

$$
\frac{i\hbar}{\partial t} b(k) = \frac{\delta H}{\delta b^*(k)}.
$$

This equation can describe weak nonlinear dynamics for Bogoliubov waves. In the following subsection, we apply weak WT theory to Eq. (26), discussing the behavior of the spectra for the wave function, density distribution, and Bogoliubov wave distribution.

C. Kinetic equation for Bogoliubov waves

In weak WT theory, the kinetic equation for the correlation function of the canonical variable $b(k)$ can be derived (12, 13). The correlation function is defined by

$$
n(k) = \left( \frac{L}{2\pi} \right)^d \langle |b(k)|^2 \rangle.
$$

Applying weak WT theory to Eq. (26), we obtain the following kinetic equation:

$$
\frac{\partial}{\partial t} n(k) = \int \left( \frac{\partial}{\partial t} R(k,k_1,k_2) - R(k_1,k_2,k) - R(k_2,k,k_1) \right) dk_1 dk_2,
$$

$$
R(k,k_1,k_2) = 2\pi |V(k,k_1,k_2)|^2 \times \delta_4(\epsilon_b(k) - \epsilon_b(k_1) - \epsilon_b(k_2)) \delta_4(k - k_1 - k_2)
\times (n(k_1)n(k_2) - n(k_2)n(k_1) - n(k)n(k_1))
$$

with the Dirac delta function $\delta_4(\cdot)$. In this application, the contributions from $b_1b_2b_3$ and $b_1^*b_2^*b_3^*$ terms in Eq. (20) are found to be smaller than those of other terms (43, 48, 51) because these terms accompany $\delta_4(\epsilon_b(k) + \epsilon_b(k_1) + \epsilon_b(k_2))$. In Bogoliubov waves, the argument in this Dirac delta function is never satisfied, so that the interaction $W(k_1,k_2,k_3)$ does not appear in Eqs. (28) and (29). The details of this calculation are described in (43), where a nonlinear canonical transformation causes the interaction $W(k_1,k_2,k_3)$ to vanish. The transformation does not change the expression for $V(k_1,k_2,k_3)$, but it does change the expression for $b(k)$.

However, this change gives only a small contribution to $n(k)$. 
Note that the application of weak WT theory to Eq. (26) is slightly different from usual cases where the dispersion relation $\epsilon_b$ and the interaction $V$ do not have time dependence. In our case, the condensate density $\rho_0$ depends on time, which makes $\epsilon_b$ and $V$ dependent on time. Thus, we cannot use the conventional weak WT theory in [43], but we can derive the kinetic equation (28). The detail of this derivation is described in Appendix A.

We are interested in the large-scale dynamics of Bogoliubov waves because, in experiments, it may be difficult to observe small-scale dynamics. Here, to elucidate the meaning of large scale, let us note the dispersion relation of Eq. (21), obtaining the following expression:

$$\epsilon_b(k) \approx \hbar c_s k \left[ 1 + \frac{1}{2} \left( \frac{k}{k_0} \right)^2 \right].$$

As shown by the two delta functions of Eq. (29), Bogoliubov waves must satisfy both momentum and energy conservation laws, which are expressed by $k_1 = k_2 + k_3$ and $\epsilon_b(k_1) = \epsilon_b(k_2) + \epsilon_b(k_3)$. These conservations lead to rather simple form of Eq. (32). In the low-wave-number region, the dispersion relation becomes

$$\epsilon_b(k) \approx \hbar c_s k \left[ 1 + \frac{1}{2} \left( \frac{k}{k_0} \right)^2 \right].$$

We substitute Eqs. (30) and (31) into the interaction $V$ of Eq. (24), obtaining the following expression:

$$V(k_1, k_2, k_3) \approx \frac{g\rho_0}{8} \sqrt{\frac{k_b^3}{k_1 k_2 k_3}} \left[ -2\left( k_1 - k_2 - k_3 \right) + \frac{1}{2k_b^2} \left( k_1^3 - k_2^3 - k_3^3 + k_1^2 k_2 - k_1 k_2^2 + k_2^2 k_3 + k_1 k_2 k_3 - k_1 k_3^2 + 12 k_1 k_2 k_3 \right) \right].$$

As shown above, the effective interaction $V_b$ for Bogoliubov waves becomes

$$V_b(k_1, k_2, k_3) = \frac{3g\rho_0}{2k_b^{3/2}} \sqrt{k_1 k_2 k_3}.$$  (34)

The derivation of this interaction is described in Appendix B. This interaction satisfies the scaling law $V_b(\lambda k_1, \lambda k_2, \lambda k_3) = \lambda^{5/2} V_b(k_1, k_2, k_3)$, which is directly related to the power exponent of the spectra. In previous studies different canonical variables were used to derive the interaction function, which is different from Eq. (34), but the scaling law is the same [47, 48].

### D. Interaction of Bogoliubov waves in the low-wave-number region

We now derive the expression for $V(k_1, k_2, k_3)$ in the low-wave-number region. First, we assume that this interaction is local in the turbulent state, which means that the dominant contribution to the interaction comes from wave numbers of the same order [48]. This locality is also discussed in Appendix D. Thus, the wave number region $k_j = k_h$ ($j = 1, 2, 3$) is considered in the following.

In the low-wave-number region, the Bogoliubov coefficients $u(k)$ and $v(k)$ are approximated as

$$u(k) \approx \frac{1}{2} \sqrt{\frac{k_b}{k}} \left[ 1 + \frac{k}{k_b} + \frac{1}{4} \left( \frac{k}{k_b} \right)^2 - \frac{1}{4} \left( \frac{k}{k_b} \right)^3 \right],$$

$$v(k) \approx \frac{1}{2} \sqrt{\frac{k_b}{k}} \left[ -1 + \frac{k}{k_b} - \frac{1}{4} \left( \frac{k}{k_b} \right)^2 - \frac{1}{4} \left( \frac{k}{k_b} \right)^3 \right].$$

We substitute Eqs. (30) and (31) into the interaction $V$ of Eq. (24), obtaining the following expression:

$$V(k_1, k_2, k_3) \approx \frac{g\rho_0}{8} \sqrt{\frac{k_b^3}{k_1 k_2 k_3}} \left[ -2\left( k_1 - k_2 - k_3 \right) + \frac{1}{2k_b^2} \left( k_1^3 - k_2^3 - k_3^3 + k_1^2 k_2 - k_1 k_2^2 + k_2^2 k_3 + k_1 k_2 k_3 - k_1 k_3^2 + 12 k_1 k_2 k_3 \right) \right].$$

### E. Derivation of power exponents of the correlation functions

In a stationary state, we can derive the power exponents of the correlation functions by using the kinetic equation (28) with the interaction of Eq. (34). Our derivation makes use of the fact that a Bogoliubov wave with sound velocity $c_s$ has linear dispersion. In this region, Bogoliubov wave turbulence is similar to acoustic turbulence, for which, in a two-dimensional system, the validity of the kinetic equation is controversial [42, 50, 51]. Thus, in the following, we address three-dimensional systems.

In the next section, to investigate the behavior of the spectra in the low-wave-number region, we calculate an expression for the interaction $V$ for the low-wave-number limit.

$$V(k_1, k_2, k_3) \approx \frac{g\rho_0}{8} \sqrt{\frac{k_b^3}{k_1 k_2 k_3}} \left[ -2\left( k_1 - k_2 - k_3 \right) + \frac{1}{2k_b^2} \left( k_1^3 - k_2^3 - k_3^3 + k_1^2 k_2 - k_1 k_2^2 + k_2^2 k_3 + k_1 k_2 k_3 - k_1 k_3^2 + 12 k_1 k_2 k_3 \right) \right].$$
\[ f_1(k, k_1) = \theta_t(k - k_1) \left( n(k) n(k - k_1) - n(k) n(k_1) - n(k) n(k - k_1) \right), \quad (37) \]

\[ f_2(k, k_1) = n(k_1) n(k) - n(k + k_1) n(k) - n(k_1) n(k + k_1) \quad (38) \]

with the Heaviside step function \( \theta_t(\cdot) \). Here we assume the isotropy of \( n(k) = n(k) \), which is reasonable in isotropic turbulence. In Appendix C, we show the derivation of Eqs. (35) - (38).

From Eq. (33), we can derive the continuity equation of the Bogoliubov wave energy spectrum given by

\[ \frac{\partial}{\partial t} E_b(k) + \frac{\partial}{\partial k} \Pi(k) = 0, \quad (39) \]

where \( E_b(k) \) and \( \Pi(k) \) are the Bogoliubov wave energy spectrum and the energy flux defined by

\[ E_b(k) = \int \epsilon_b(k) n(k) d\Omega_k = 4\pi k^2 \epsilon_b(k) n(k), \quad (40) \]

\[ \Pi(k) = -4\pi \int_0^k k_1^2 \left[ \epsilon_b(k_1) I(k_1) + n(k_1) \frac{\partial}{\partial t} \epsilon_b(k_1) \right] dk_1, \quad (41) \]

In the statistically stationary state, the time derivatives of \( E_b(k) \) in Eq. (39) is zero, which means that the energy flux \( \Pi(k) \) is independent of the wave number. Also, the time derivative of \( \epsilon_b(k) \) is zero because the condensate density \( \rho_0 \) is stationary in this state. Assuming the power law \( n(k) \propto k^{-m} \) and apply the transformation \( k_1 = k k_1 \) to Eq. (41), we obtain

\[ \Pi(k) \propto k^{9 - 2m}, \quad (42) \]

which exhibits that the energy flux is constant if the power exponent \( m \) is 9/2. Thus, we derive the power law in the Bogoliubov wave turbulence:

\[ n(k) \propto k^{-9/2}. \quad (43) \]

This power law is found to be the local, so that the locality assumption in Sec. IV D is justified, which is described in Appendix D. Therefore, the correlation function integrated over the solid angle yields

\[ C_b(k) \propto k^{-5/2}. \quad (44) \]

We can derive the power exponent for \( C_w \) by using Eq. (44). In the low-wave-number region, the relation between \( \tilde{\phi}(k) \) and \( b(k) \) becomes

\[ \tilde{\phi}(k) \propto \frac{b(k)}{\sqrt{k}} - \frac{b^*(-k)}{\sqrt{k}}, \quad (45) \]

which is obtained from Eq. (15). Then, we assume isotropy for \( b(k) \) and use the approximation to derive Eq. (28), obtaining

\[ \langle |\tilde{\phi}(k)|^2 \rangle \propto \frac{1}{k} \langle |b(k)|^2 \rangle. \quad (46) \]

As a result, in the low-wave-number region, the spectrum for the macroscopic wave function exhibits the power law behavior given by

\[ C_w(k) \propto k^{-7/2}. \quad (47) \]

In this derivation, we use the Fourier component \( F[\psi] = \psi_0(\delta(k) + \tilde{\phi}(k)) \).

For the density correlation function, the following expression for \( \psi \) is useful to derive the power exponent of \( C_d \):

\[ \psi(r) = \sqrt{\rho_0 + \delta \rho(r)} e^{i(\theta_0 + \delta \theta(r))} \]

\[ \simeq \psi_0 \left( 1 + \frac{\delta \rho(r)}{2\rho_0} + i \delta \theta(r) \right), \quad (48) \]

where \( \psi_0 \) is written as \( \sqrt{\rho_0} \exp(i\theta_0) \), and \( \delta \rho \) and \( \delta \theta \) are the density and phase fluctuations, respectively, around it. Then, the fluctuations of the wave function and the density are obtained from

\[ \phi(r) = \frac{\delta \rho(r)}{2\rho_0} + i \delta \theta(r), \quad (49) \]

\[ \delta \rho(r) = \rho_0 \left( \phi(r) + \phi^*(r) \right), \quad (50) \]

from which we can express the Fourier component of \( \delta \rho \) with the canonical variable \( b \). Performing a similar calculation for Eqs. (49) - (51), we can derive the power law in the spectrum for the density distribution:

\[ C_d(k) \propto k C_b(k) \propto k^{-3/2}. \quad (51) \]

We consider that this power law is a candidate for an experimentally observable quantity to confirm the presence of Kolmogorov turbulence. In the next section, by numerically calculating the GP equation, we discuss these power laws.

**IV. NUMERICAL RESULTS**

We now present our numerical results for Bogoliubov wave turbulence with the GP equation. One of the main results is in good agreement with the power exponents of Eq. (11), (17), and (51) derived by using weak WT theory.
A. Numerical method

Our numerical calculation treats a three-dimensional BEC without a trapping potential. Time and length are normalized by $\tau = \hbar / g \rho_0$ and $\xi = \hbar / \sqrt{2mg\rho_0}$, and the numerical system size $L \times L \times L$ is $256\xi \times 256\xi \times 256\xi$ with spatial resolution $dx/\xi = 1$. In this situation, we numerically solve the GP equation by using the pseudo-spectral method. The time propagation is performed by using a fourth-order Runge-Kutta method with time resolution $dt/\tau = 5 \times 10^{-3}$.

We use an initial state in which energy is injected in the large scale to confirm the direct energy cascade solutions corresponding to Eq. (44), (47), and (51). The initial state is prepared by the random numbers as follows:

$$\psi(r) = \sqrt{\rho_0} \left(1 + \phi(r)\right),$$  \hspace{1cm} (52)

$$\mathcal{F}[\phi](k) = 3000 (R_1 + iR_2) k^4 \xi^4 \exp(-k^2 \xi^2 / 0.0016),$$  \hspace{1cm} (53)

where $R_1$ and $R_2$ are the random numbers in the range [0.5, -0.5]. Figure 1(a) shows the density distribution in the initial state and demonstrates that the energy is injected at the large scale.

In our calculation, dissipation is phenomenologically included by replacing $i\hbar \partial / \partial t$ with $(i - \gamma \theta_\text{H}(k - k_\text{b})) \hbar \partial / \partial t$ in the Fourier transformed $|k|$ of Eq. (11) [11, 31]. Here, $k_\text{b}$ is $2\sqrt{mg\rho_0}/\hbar$, and $\gamma$ is the strength of the dissipation. In our calculation, we use $\gamma = 0.03$. We comment on the dissipation region. In this paper, we focus on the low-wave-number region, adopting the dissipation working in the region larger than $k_\text{b}$. There is arbitrariness in the choice of the expression for the dissipation. However, in a previous paper [52], the dissipation was found to work in the high-wave-number region at low temperature, so that we use this expression for the dissipation.

B. Time development of the condensate and the fluctuation

We numerically confirm that our turbulence satisfies the weak nonlinear condition $|\phi| \ll 1$. Figure 2 shows the time dependence of the spatially averaged absolute values $\phi_0$ and $\phi_1$ for real and imaginary parts of the fluctuation $\phi$, from which the order of the fluctuation $|\phi|$ is found to be about 0.13.

Figure 3 shows the time development of the amplitude $\rho_0$ and phase $\theta_0$ of the condensate $\psi_0$; we find that the condensate does not obey $\sqrt{\rho_0} \exp(-i\mu t/\hbar)$. This means...
C. Time development of the spectra

We next numerically calculate the spectra $C_b$, $C_w$, and $C_d$, finding that the numerical results show good agreement with our analytical results of Eqs. 44, 47, and 51. Figure 4 shows the time development of the three spectra, demonstrating the direct energy cascade from small to large wave numbers. This cascade is also confirmed by the density distribution as shown in Figs. 1(a)–1(d), in which small scale structure is seen to grow. In each spectrum, as time passes, the power law behaviors corresponding to Eq. 44, 47, and 51 gradually grow, and the scaling range finally becomes $0.05 \lesssim k\xi \lesssim 0.6$. The fluctuation in $k\xi \lesssim 0.2$ is large, which may be the finite size effect. Particularly, the time development of $C_b$ is large.

We note the calculation of the Bogoliubov wave distribution $b(k)$. As pointed in Sec. IV B, in previous works 44, 49, the condensate function is assumed to be $\sqrt{\rho_0} \exp(-i\mu t/\hbar)$. When we use this condensate function and calculate the Bogoliubov wave distribution, the spectrum $C_b$ does not show a $-5/2$ power law and the deviation from this law is much larger. When we use the condensate function defined by Eq. 9, however, the spectrum $C_b$ shows good agreement with Eq. 44. Therefore, it is important to take account of the dynamics of the condensate coupled with the fluctuation. Such a condensate has been called a quasi-condensate 52, but the condensate dynamics of Eq. 8 has not been treated in previous studies.

V. DISCUSSION

In this paper, we derive the three power exponents for $C_b$, $C_w$, and $C_d$. In Sec. V A, we detail the difference between the power exponents in previous results and ours. In Sec. V B, we propose some conditions to observe Kolmogorov turbulence in atomic BECs and comment on the turbulence theory beyond the weak WT one. We consider that our result for $C_d$ may be the key to achieve this observation. In Sec. V C, we discuss an inconsistency of our analytical calculation in Sec. III.

A. Comparison between previous results and our results

In a previous study 49, the spectrum of the macroscopic wave function was suggested to obey a $-3/2$ power law in a three-dimensional system. However, our result has another $-7/2$ power law.

To discuss the reason for this difference, we review the derivation of the $-3/2$ power law in this previous paper 49. In 48, Zakharov and Nazarenko studied Bogoliubov wave turbulence using the two canonical variables
\( \varphi = \sqrt{\tau} \) and \( \Pi = 2\hbar \rho \theta \) with \( \psi = \sqrt{\tau} \exp(i \theta) \). To study the fluctuation around the strong condensate, they took expressions \( \varphi = \varphi_0 + \varphi_1 \) and \( \Pi = \Pi_1 \), where the quantity \( \varphi_0 \) corresponds to the condensate and \( \varphi_1 \) and \( \Pi_1 \) correspond to the fluctuations. By using the following canonical transformation for these canonical variables, they introduced new canonical variables \( B(k) \), diagonalizing a one-body Hamiltonian as

\[
H' = \frac{1}{\hbar} \sum_{k_1} \epsilon_0(k) |B(k_1)|^2.
\] (54)

The canonical variables \( B(k) \) were defined by

\[
\tilde{\varphi}_1(k) = \frac{1}{2} \sqrt{\frac{\epsilon_0(k)}{\hbar \epsilon_1(k)}} \left( B(k) + B^*(-k) \right),
\] (55)

\[
\tilde{\Pi}_1(k) = -i \sqrt{\frac{\hbar \epsilon_1(k)}{\epsilon_0(k)}} \left( B(k) - B^*(-k) \right),
\] (56)

with the Fourier components \( \tilde{\varphi}_1(k) = \mathcal{F}[\varphi_1] \) and \( \tilde{\Pi}_1(k) = \mathcal{F}[\Pi_1] \). Here, note that \( H'_2 \) is different from \( H_2 \) of Eq. (59). Then, they derived the \(-3/2\) power law in the Bogoliubov energy spectrum defined by

\[
\mathcal{E}_B(k) = \frac{1}{\hbar} \Delta k \sum_{k_1 < \Delta k/2} \epsilon_0(k) |B(k)|^2.
\] (57)

The previous paper [49] makes use of this power law to derive the power exponent of the spectrum for the macroscopic wave function. In this previous paper \( \mathcal{E}_B(k) \) is estimates as

\[
\frac{1}{\Delta k} \sum_{k_1 < \Delta k/2} \epsilon_0(k) |\tilde{\psi}(k)|^2
\] (58)

with \( \epsilon_0(k) = g \rho_0 + \epsilon_1(k) \). Then, in the low-wave-number region, \( \epsilon_0(k) \approx g \rho_0 \) is approximately satisfied, so that they derived the \(-3/2\) power law in the spectrum \( C_w \) for the macroscopic wave function.

However, owing to the weak fluctuation, the macroscopic wave function \( \psi \) can be expanded as

\[
\psi \simeq \varphi_0 + \varphi_1 + i \frac{\Pi_1}{2\hbar \varphi_0}.
\] (59)

By applying the Fourier transformation to Eq. (59), we obtain

\[
\tilde{\psi}(k) \simeq \varphi_0 \delta(k) + \tilde{\varphi}_1(k) + i \frac{\tilde{\Pi}_1(k)}{2\hbar \varphi_0}.
\] (60)

As a result, in the low-wave-number region, the approximation used in weak WT theory gives

\[
|\tilde{\psi}(k)|^2 \simeq \frac{mc}{\hbar^2 \varphi_0^2 k} (|B(k)|^2),
\] (61)

which obviously leads to the \(-7/2\) power law in \( C_w \) because \( \mathcal{E}_B(k) \propto k (|B(k)|^2) k^2 \propto k^{-3/2} \). Thus, the derivation of the \(-3/2\) power law for \( C_w \) in the previous work [49] seems not to be correct.

### B. Experimental possibility of Kolmogorov turbulence in atomic BECs

In atomic BEC experiments, observations can be made of the density distribution, which accesses the spectrum \( C_d \). This can be the key to confirming the presence of Kolmogorov turbulence; if we can observe the \(-3/2\) power law in this spectrum, then Kolmogorov turbulence is present.

We discuss three points related to observing the \(-3/2\) power law in \( C_d \). The first point is that one must take a wide scaling region; that is, one must prepare an atomic BEC that is much larger than the coherence length. The \(-3/2\) power law appears in the low-wave-number region \( k < k_b \), but, in the trapped system, there is a limit on the system size of the order the Thomas-Fermi radius \( R_{TF} \). This scale can be the lower limit of the scaling region for the \(-3/2\) power law. Thus, the radius \( R_{TF} \) should be at least a factor of 10 larger than \( 2\pi/k_b \).

The second point concerns how to excite the system weakly in the low-wave-number region. The \(-3/2\) power law is valid under the weak nonlinearity condition \( |\phi| \ll 1 \), so that fluctuations of both density and phase should be very weak. In our calculation, the density fluctuation is about 5%, as shown in Fig. 1. However, this may be too weak to observe. We must investigate the spectrum dependence on the strength of the fluctuation, for instance, the condition at which the prediction of weak WT theory is broken; this, however, is a subject future work. Also, it is necessary to excite a large-scale modulation because the \(-3/2\) power law results from the direct energy cascade. Thus, candidates for the excitation are considered to be repulsive or attractive Gaussian potentials moving slower than the sound velocity or a shaking trapping potential with fixed frequency. However, the excitation mode induced by these methods should depend on size, strength, and the velocity of the Gaussian potential or frequency and displacement of the shaking, etc. Understanding the excitation methods is essential for observing Kolmogorov turbulence in atomic BECs and should be numerically investigated hereafter.

The third point is that one must take surface effects into consideration. In a three-dimensional system, we can obtain the density distribution integrated along the incident direction of the probe light, so that this distribution contains information of the surface profile as well as the bulk. Since our result should be applicable to the bulk of the trapped BEC, the surface effect may alter the power law of Eq. (51). If the system size is much larger than the damping length of the surface wave into the bulk [49], this power law is expected to be relevant.

Finally, we comment on turbulence theory beyond weak WT theory. Recently, Yoshida and Arimitsu have studied strong turbulence with the spectral closure approximation, deriving the some power exponents [54].
These power exponents have not been confirmed numerically. These power laws may be another key to confirmation of Kolomogorov turbulence in atomic BECs because, in this theory, the fluctuation is large and it is easy to observe.

C. An inconsistency of our analytical calculation

In Sec. III and Appendix A, we apply weak WT theory to Eqs. (23) and (20), obtaining Eq. (28). However, this derivation has an inconsistency for the perturbative expansion in Eqs. (5) and (10). In this expansion, we keep the second-order of the fluctuation $\phi$ in both equations, which is not consistent because the right-hand sides of Eqs. (5) and (10) are the first- and second-order of smallness, respectively. To do a reliable expansion, we must take the third-order of $\phi$ in Eq. (10), but this calculation is much difficult.

Our numerical calculation exhibits that the condensate dynamics is different from $\sqrt{n_0}\exp(-i\mu t/h)$, so that we must take the nonlinear dynamics for $\psi$. In this paper, although our calculation contains the inconsistency for the perturbative expansion, we treat the coupled nonlinear dynamics for the condensate and the fluctuation. Thus, our theory for this coupled dynamics is insufficient. The construction of the improved theory is the future work. In spite of this inconsistency, the analytical results exhibit good agreement with the numerical results.

As another possibility, our numerical calculation seems not to be substantially satisfied with the weak nonlinear condition. However, if we use the weaker fluctuation, the computational time for the formation of the power law is much long, which cannot be confirmed at present.

VI. CONCLUSION

We have theoretically and numerically studied Bogoliubov wave turbulence in a uniform BEC at zero temperature by using the GP equation. Our application of weak WT theory to the GP equation leads to the three power exponents $-5/2$, $-7/2$, and $-3/2$, corresponding to the correlation functions for the Bogoliubov distribution $C_h$, the macroscopic wave function $C_w$, and the density distribution $C_d$. By numerically calculating the GP equation, we have confirmed the good agreement with these power exponents. In our analytical and numerical calculation, we found that taking account of the condensate dynamics coupled with the fluctuation, an aspect not treated in previous works [47–49], was important.

Acknowledgments

The authors thank S. Nazarenko for pointing out the importance of the dimension in the acoustic wave turbulence. KF was supported by a Grant-in-Aid for JSPS Fellows (Grant No. 262524). MT was supported by JSPS KAKENHI Grant No. 26400366 and MEXT KAKENHI Fluctuation & Structure Grant No. 26103526.

Appendix A: Application of weak WT theory

We apply weak WT theory in [43] to Eqs. (23) and (20), deriving Eq. (28).

In this theory, the separation of time scale is important. There are two time scales, which are a linear time scale $T_L \sim 2\pi \hbar/\epsilon_0$ and a nonlinear one $T_{NL}$. In weak WT, the nonlinear term is very weak, which leads to the long nonlinear time scale $T_{NL} \gg T_L$. In weak WT theory, we make use of this time scale separation, filtering out the fast linear oscillation. For this purpose, let us introduce auxiliary intermediate time $T$ satisfying $T_{NL} \gg T \gg T_L$. In the following, we solve Eqs. (23) and (20) with a formal perturbative expansion, finding the canonical variables $b(k)$ at $t = T$.

Before the perturbative expansion, we rewrite Eqs. (23) and (20) with interaction variables $c(k)$ defined by

$$b(k) = \epsilon c(k) e^{-i\alpha(k)}, \quad (A1)$$

$$\alpha(k) = \frac{1}{\hbar} \int_0^t \epsilon_0(k, t) dt_1, \quad (A2)$$

where we introduce the small parameter $\epsilon$ in order to elucidate the power of smallness of each term in the perturbative expansion. Then, Eqs. (23) and (20) become

$$i\hbar \frac{\partial}{\partial t} c(k) = \epsilon \sum_{k_1, k_2} \left[ \delta(k - k_1 - k_2) V(k, k_1, k_2) \times \Delta_1(k, k_1, k_2) c(k_1)c(k_2) + 2\delta(k_1 - k_2 - k)V(k_1, k_2, k) \times \Delta_2(k_1, k_2, k)c(k_1)c^*(k_2) \right], \quad (A3)$$

$$\frac{\partial}{\partial t} \rho_0 = \epsilon^2 \frac{\partial^2}{\partial t^2} A \quad (A4)$$

with $\Delta_1(k, k_1, k_2) = \exp[i(\alpha(k) - \alpha(k_1) - \alpha(k_2))]$ and $\Delta_2(k_1, k_2, k) = \exp[-i(\alpha(k_1) - \alpha(k_2) - \alpha(k))]$. We do not show the expression of $A$ because it is not important. In this calculation, as noted in Sec. III C, the terms with $W$ are neglected.

We calculate the canonical variable $c(k)$ and the condensate density $\rho_0$ at $t = T$ up to the second-order of $\epsilon$ by using the perturbative expansion:

$$c(k) = c^{(0)}(k) + \epsilon c^{(1)}(k) + \epsilon^2 c^{(2)}(k) + \cdots, \quad (A5)$$
\[ \rho_0 = \rho_0^{(0)} + \epsilon \rho_0^{(1)} + \epsilon^2 \rho_0^{(2)} + \cdots. \quad \text{(A6)} \]

From Eqs. (A1) and (A4), the expansion of \( \rho_0 \) is given by

\[ \rho_0(T) = \rho_0(0) + \epsilon^2 \rho_0^{(2)}(T) + \cdots. \quad \text{(A7)} \]

The function \( F(\rho_0) \) such as \( V, \Delta_1, \) and \( \Delta_2 \) has \( \rho_0 \) as a variable, being expanded in the following:

\[ F(\rho_0(T)) \approx F(\rho_0(0)) + \frac{\partial F(\rho_0(0))}{\partial \rho_0(0)} \epsilon^2 \rho_0^{(2)}(T). \quad \text{(A8)} \]

Thus, the second-order of the condensate density \( \rho_0^{(2)} \) can be neglected in Eq. (A3) because such terms lead to the third-order of \( \epsilon \) in Eq. (A3). Therefore, the time dependence of \( \rho_0 \) vanishes in the application of weak WT theory, so that we can use the conventional derivation of the kinetic equation in [42], obtaining the Eq. (28).

**Appendix B: Derivation of Eq. (34)**

The derivation of Eq. (34) is described. In the low wave number region, the momentum and energy resonant conditions of the Bogoliubov waves give the particular class of the wave number, where the wave vectors are nearly parallel to each other because of the linear dispersion [42], [50] [51]. Then, the resonant condition \( \epsilon_0(k_1) = \epsilon_0(k_2) + \epsilon_0(|k_1 - k_2|) \)

\[ k_1 - k_2 + \frac{1}{2k_b^2} (k_1^2 - k_2^2) \simeq |k_1 - k_2| + \frac{1}{2k_b^2} |k_1 - k_2|^2 \] \text{(B1)}

\[ I(k) = \frac{4 \pi^2}{\hbar c_s} \int_0^\infty dk_1 \int_0^\pi \sin \theta_1 d\theta_1 \left[ \frac{2(k_1 - k)}{k} |V_b(k_1, k_1, |k - k_1|)|^2 \theta_1 (k_1 - k_1) \delta (\theta_1^2 - \theta_2^2) f(k, k_1, k - k_1) \right. \]

\[ - \left. \frac{2(1 + k)}{k} |V_b(|k_1 + k|, k_1, k_1)|^2 \theta_1 (k_1 - k_1) \delta (\theta_1^2 - \theta_2^2) f(k_1, k_1, k - k_1) \right] \]

\[ - \left. \frac{2(1 + k)}{k} |V_b(|k_1 + k|, k_1, k_1)|^2 \delta (\theta_1^2 - \theta_2^2) f(k_1, k_1, k - k_1) \right], \quad \text{(C2)} \]

with \( f(k, k_1, k_2) = n(k_1)n(k_2) - n(k_2)n(k) - n(k)n(k_1), \)

\[ \Theta_1 = \sqrt{3}|k_1 - k_1|/\hbar_b, \] \text{ and } \[ \Theta_2 = \sqrt{3}|k + k_1|/\hbar_b. \] \] Finally, we can perform the integration by substitution with \( y = \theta_1^2 \)

\[ \text{and } \sin(\Theta_3) \simeq \Theta_3 \quad (j = 1, 2), \] obtaining Eq. (35).

**Appendix C: Locality of the power law**

The locality condition of the power law is derived. In this derivation, we consider the convergence condition of collisional integral in Eq. (35).

The power law is assumed to be \( n(k) \propto k^{-m} \) with the positive power exponent \( m \). Then, from Eq. (35), we find that the collisional integral may diverge at three points \( k_1 = 0, k, \infty \). The point \( k_1 = k \) means the \( k_2 = 0 \), so that the convergence condition at \( k_1 = k \) is same as that at \( k_1 = 0 \) because of the exchange symmetry \( (k_1 \leftrightarrow k_2) \)

\[ |k_1 - k_2| \simeq \sqrt{(k_1 - k_2)^2 + k_1 k_2 \theta^2}, \quad \text{(B2)} \]

where the angle \( \theta \) formed by the two vectors \( k_1 \) and \( k_2 \) is small. Taking the second order of the small quantities, we obtain the angle \( \theta \simeq \sqrt{3(k_1 - k_2)/k_b} \) [42], which gives

\[ |k_1 - k_2| \simeq (k_1 - k_2) \left( 1 + \frac{3k_1 k_2}{k_b^2} \right). \quad \text{(B3)} \]

Here we use the condition \( k_1 > k_2 \) because of the resonant condition. Thus, the effective interaction between the Bogoliubov waves can be calculated by \( V(k_1, k_2, k_1 - k_2) \) of Eq. (32). As a result, we obtain

\[ V_b(k_1, k_2, k_1 - k_2) = \frac{3g \rho_0}{2k_b^{3/2}} \sqrt{k_1 k_2 (k_1 - k_2)}. \quad \text{(B4)} \]

**Appendix D: Locality of the power law**

The locality condition of the power law is derived. In this derivation, we consider the convergence condition of collisional integral in Eq. (35),
which is approximately expressed by
\[
2 \int_0^\infty dk_1 \left[ k_1^4 \left( n(k_1 + k) - n(k_1 - k) \right) \right] \\
\approx 4 \int_0^\infty dk_1 k_1^4 n(k) \frac{\partial n(k_1)}{\partial k_1}
\]
\[\propto \int_0^\infty dk_1 k_1^{4-m}. \tag{D1}\]

Thus, the convergence condition at \( k_1 = \infty \) is \( 4 < m \).

On the other hand, in the small \( k_1 \) region, the dangerous terms appear in the first and third ones of Eq. \( \text{(55)} \). This term becomes
\[
\int_0^\infty dk_1 \left[ k_1^4 n(k_1) \left( n(k_1) + n(k_1 - 2n(k_1)) \right) \right] \\
\approx \int_0^\infty dk_1 k_1^4 n(k_1) \frac{\partial^2 n(k_1)}{\partial k_1^2}
\]
\[\propto \int_0^\infty dk_1 k_1^{4-m}, \tag{D2}\]
which shows that the convergence condition at \( k_1 = 0 \) is \( m < 5 \).

In summary, the convergence condition of the collisional integral is \( 4 < m < 5 \). In the Bogoliubov wave turbulence, the power exponent is \( 9/2 \), so that this power law is local.

\[\text{[1]}\] P. A. Davidson, Turbulence: An Introduction for Scientists and Engineers (Oxford University Press, Oxford, 2004).
\[\text{[2]}\] U. Frisch, Turbulence (Cambridge University Press, Cambridge, 1995).
\[\text{[3]}\] Progress in Low Temperature Physics, edited by W. P. Halperin and M. Tsubota (Elsevier, Amsterdam, 2008), Vol. XVI.
\[\text{[4]}\] W. F. Vinen, J. Low Temp. Phys. 161, 419 (2010).
\[\text{[5]}\] L. Skrbek and K. R. Sreenivasan, Phys. Fluids 24, 011301 (2012).
\[\text{[6]}\] A. C. White, B. P. Anderson, and V. S. Bagnato, Proc. Natl. Acad. Sci. U.S.A. 111, 4719 (2013).
\[\text{[7]}\] C. F. Barenghi, L. Skrbek, and K. R. Sreenivasan, Proc. Natl. Acad. Sci. U.S.A. 111, 4647 (2014).
\[\text{[8]}\] W. F. Vinen, Proc. R. Soc. A 240, 114 (1957); 240, 128 (1957); 242, 493 (1957); 243, 400 (1958).
\[\text{[9]}\] J. T. Toug, in Progress in Low Temperature Physics, edited by D. F. Brewer (North-Holland, Amsterdam, 1982), Vol. VIII.
\[\text{[10]}\] C. Nore, M. Abid, and M. E. Brachet, Phys. Rev. Lett. 78, 3896 (1997); Phys. Fluids 9, 2644 (1997).
\[\text{[11]}\] M. Kobayashi and M. Tsubota, Phys. Rev. Lett. 94, 065302 (2005); J. Phys. Soc. Jpn. 74, 3248 (2005).
\[\text{[12]}\] J. Mauer and P. Tabeling, Europhys. Lett. 43, 29 (1998).
\[\text{[13]}\] E. Kozik and B. Svistunov, Phys. Rev. Lett. 92, 035301 (2004).
\[\text{[14]}\] V. S. L’vov and S. Nazarenko, J. Low. Temp. Phys. Lett. 91, 428 (2008).
\[\text{[15]}\] V. S. L’vov, S. V. Nazarenko, and O. Rudenko, Phys. Rev. B 76, 024520 (2007).
\[\text{[16]}\] M. S. Paolelitti, R. B. Fiorito, K. R. Sreenivasan, and D. P. Lathrop, J. Phys. Soc. Jpn. 77, 111007 (2008).
\[\text{[17]}\] M. S. Paolelitti, M. E. Fisher, K. R. Sreenivasan, and D. P. Lathrop, Phys. Rev. Lett. 101, 154501 (2008).
\[\text{[18]}\] T. W. Neely, D. F. Brewer, and K. R. Sreenivasan, Proc. Natl. Acad. Sci. U.S.A. 80, 63611 (2012).
\[\text{[19]}\] H. Takeuchi, N. Suzuki, K. Kasamatsu, H. Saito, and M. Tsubota, Phys. Rev. B 81, 094517 (2010).
\[\text{[20]}\] M. Karl, B. Nowak, and T. Gasenzer, Phys. Rev. A 88, 063615 (2013).
\[\text{[21]}\] D. Kobayakov, A. Bezett, E. Lundh, M. Marklund, and V. Bychkov, Phys. Rev. A 89, 013631 (2014).
\[\text{[22]}\] B. Villaseñor, R. Zamora-Zamora, D. Berna, and V. Romero-Rochín, Phys. Rev. A 89, 033611 (2014).
\[\text{[23]}\] K. Fujimoto and M. Tsubota, Phys. Rev. A 90, 013629 (2014).
\[\text{[24]}\] N. G. Parker and C. S. Adams, Phys. Rev. Lett. 95, 145301 (2005).
\[\text{[25]}\] M. Kobayashi and M. Tsubota, Phys. Rev. A 76, 045603 (2007).
\[\text{[26]}\] T.-L. Horng, C.-H. Hsueh, S.-W. Su, Y.-M. Kao, and T.-S. Chen, Phys. Rev. A 80, 023618 (2009).
\[\text{[27]}\] A. S. Bradley and B. P. Anderson, Phys. Rev. X 2, 041001 (2012).
\[\text{[28]}\] M. T. Reeves, T. P. Billam, B. P. Anderson, and A. S. Bradley, Phys. Rev. Lett. 110, 104501 (2013).
\[\text{[29]}\] A. C. White, C. F. Barenghi, and N. P. Proukakis, Phys. Rev. A 86, 013635 (2012).
\[\text{[30]}\] T. Simula, M. J. Davis, and K. Helmerson, Phys. Rev. Lett. 113, 165302 (2014).
\[\text{[31]}\] G. W. Stagg, A. J. Allen, N. G. Parker, and C. F. Barenghi, Phys. Rev. A 91, 013612 (2015).
\[\text{[32]}\] E. A. L. Henn, J. A. Seman, G. Roati, K. M. F. Magalhaes, and V. S. Bagnato, Phys. Rev. Lett. 103, 045301 (2009).
\[\text{[33]}\] M. T. Reeves, B. P. Anderson, and A. S. Bradley, Phys. Rev. A 86, 053621 (2012).
\[\text{[34]}\] T. W. Neely, A. S. Bradley, E. C. Samson, S. J. Rooney, E. M. Wright, K. J. H. Law, R. Carretero-González, P. G. Kevrekidis, M. J. Davis, and B. P. Anderson, Phys. Rev. Lett. 111, 235301 (2013).
\[\text{[35]}\] W. J. Kwon, G. Moon, J. Y. Choi, S. W. Seo, and Y. I. Shin, Phys. Rev. A 90, 063627 (2014).
\[\text{[36]}\] V. E. Zakharov, V. S. L’vov, and G. Falkovich, Kolmogorov Spectra of Turbulence I: Wave Turbulence (Springer, Berlin, 1992).
\[\text{[37]}\] S. Nazarenko, Wave Turbulence, Lecture Notes in Physics Vol. 825 (Springer, Heidelberg, 2011).
\[\text{[38]}\] E. Falcon, C. Larroche, and S. Fauve, Phys. Rev. Lett.
98, 094503 (2007).
[45] V. S. Lutovinov and V. R. Chechetkin, Zh. Eksp. Teor. Fiz. 76, 223 (1979).
[46] G. Diöring, C. Josserand, and S. Rica, Phys. Rev. Lett. 97, 025503 (2006).
[47] S. Dyachenko, A. C. Newell, A. Pushkarev, and V. E. Zakharov, Physica D 57, 96 (1992).
[48] V. Zakharov and S. Nazarenko, Physica D 201, 203 (2005).
[49] D. Proment, S. Nazarenko, and M. Onorato, Phys. Rev. A 80, 051603(R) (2009).
[50] G. E. Falkovich, Zh. Eksp. Teor. Fiz. 93, 117 (1987).
[51] V. S. L'vov, Yu. L'vov, A. C. Newell, and V. Zakharov, Phys. Rev. E, 56, 390 (1997).
[52] M. Kobayashi and M. Tsubota, Phys. Rev. Lett. 97, 145301 (2006).
[53] S. Nazarenko, M. Onorato, and D. Proment, Phys. Rev. A 90, 013624 (2014).
[54] K. Yoshida and T. Arimitsu, J. Phys. A: Math. Theor. 46, 335501 (2013).