EIGENFUNCTIONS AND A LOWER BOUND ON THE WASSERSTEIN DISTANCE

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Abstract. We prove a conjectured lower bound on the $p$-Wasserstein distance between the positive and negative parts of a Laplace eigenfunction. Our result holds for general RCD($K,\infty$) spaces.

1. Introduction

In recent years, a growing interest has been devoted to the study of Wasserstein distances between positive and negative parts of a function, particularly in relation with uncertainty estimates [24, 26, 25, 6, 7, 21]. Given a closed (i.e. compact, without boundary), smooth, $n$-dimensional Riemannian manifold and denoting by $\mathfrak{m}$ its volume measure, one considers a nice enough function $f$ with zero mean and notices that if it is cheap to transport $f^+\mathfrak{m}$ to $f^-\mathfrak{m}$, then most of the mass of $f^+$ has to be close to most of the mass of $f^-$ and hence the zero set has to be large. The uncertainty principle quantify this relation by providing bounds from below on the quantity

$$W_p(f^+\mathfrak{m}, f^-\mathfrak{m})\mathcal{H}^{n-1}(\{x : f(x) = 0\}).$$

Here $W_p$ denotes the $p$-Wasserstein distance, $\mathcal{H}^{n-1}$ is the ($n-1$)-dimensional Hausdorff measure and $f^+, f^-$ are the positive and negative parts of $f$, respectively.

When $f$ is a Laplace eigenfunction, it is an intriguing problem to understand whether a meaningful upper bound on (1) also holds. Questions related to the geometry of eigenfunctions are of central interest for different areas of mathematics and estimates on the quantity (1) together with Steinerberger’s conjecture [24] allow to get estimates on the measure of nodal sets, in the flavour of Yau’s conjecture [29].

Around 40 years ago, Yau conjectured that there exists a positive constant $C$, depending only on the manifold, such that every eigenfunction $f_\lambda$, of eigenvalue $\lambda$, satisfies

$$C\sqrt{\lambda} \geq \mathcal{H}^{n-1}(\{x : f_\lambda(x) = 0\}) \geq \frac{\sqrt{\lambda}}{C}.$$  

We refer to [17] for a review of results related to Yau’s conjecture, here we limit to mention that the lower bound in (2) was proved by Logunov [16], while the upper bound has been very recently established for open regular subset of $\mathbb{R}^n$ by Logunov, Malinnikova, Nadirashvili and Nazarov [18], but it remains open for compact manifolds (see [15] for a polynomial upper bound).

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Regarding Wasserstein distances, Steinerberger proposed the following conjecture: for any $p \geq 1$ there exists a constant $C$, depending only on $p$ and the manifold, such that for every non-constant eigenfunction $f_\lambda$, of eigenvalue $\lambda$, it holds
\[
\frac{C}{\sqrt{\lambda}} \|f_\lambda\|_{L^p(M)}^{\frac{1}{p}} \geq W_p(f_\lambda^+ m, f_\lambda^- m) \geq \frac{1}{C\sqrt{\lambda}} \|f_\lambda\|_{L^p(M)}^{\frac{1}{p}}.
\] (3)

The upper bound in (3) with the non optimal factor $\sqrt{\log \lambda/\lambda}$ in place of $1/\sqrt{\lambda}$ was established by Steinerberger already in [24]. For $p = 1$, the same non optimal upper bound was then extended to a more general class of spaces, the so called RCD($K, N$) spaces, by Cavalletti and Farinelli [7]. The sharp upper bound is known to hold for closed Riemannian manifold and $p = 1$ thanks to a recent result of Carroll, Massaneda and Ortega-Cerdá [6].

Concerning the lower bound, while this manuscript was in preparation we became aware of [21] where the author obtains the conjectured inequality for 2-dimensional closed Riemannian manifolds and $p = 1$.

Notice that a lower bound on the Wasserstein distance between $f_\lambda^+ m$ and $f_\lambda^- m$ can be used to derive an upper bound on the measure of the nodal set of $f_\lambda$, provided an estimate from above of the quantity (1) is established.

The aim of the present paper is to show that the lower bound in Steinerberger’s conjecture holds for every $p$ and in any dimension.

**Theorem 1.1.** Let $(M, g)$ be a smooth, closed, Riemannian manifold, and $p \geq 1$. Then there exists a constant $C(K, M, p)$ such that for any non-constant eigenfunction of the Laplacian $f_\lambda$, of eigenvalue $\lambda \geq M$, the following inequality is satisfied
\[
W_p(f_\lambda^+ m, f_\lambda^- m) \geq C(K, M, p) \frac{1}{\sqrt{\lambda}} \|f_\lambda\|_{L^p(M)}^{\frac{1}{p}},
\]
with $K$ being a lower bound on the Ricci curvature of the manifold.

We remark that in the estimate there is no dependence on the dimension of the manifold. From Theorem 1.1 and the above mentioned result [6, Theorem 3] it follows exactly the conjecture (3) for $p = 1$ and an equivalent formulation of Yau’s conjecture:

**Corollary 1.2.** Let $(M, g)$ be a smooth, closed, Riemannian manifold. Then there exists a constant $C$, depending only on the manifold, such that for any non-constant eigenfunction $f_\lambda$, of eigenvalue $\lambda$, the following inequality is satisfied
\[
\frac{C}{\sqrt{\lambda}} \|f_\lambda\|_{L^1(M)} \geq W_1(f_\lambda^+ m, f_\lambda^- m) \geq \frac{1}{C\sqrt{\lambda}} \|f_\lambda\|_{L^1(M)}.
\]

As a consequence, Yau’s conjecture holds if and only if there exists a constant $C$, depending only on the manifold, such that for any eigenfunction $f_\lambda$ the following inequality is satisfied
\[
C \|f_\lambda\|_{L^1(M)} \geq W_1(f_\lambda^+ m, f_\lambda^- m) \mathcal{H}^{n-1}(\{x : f_\lambda(x) = 0\}) \geq \frac{\|f_\lambda\|_{L^1(M)}}{C}.
\]

We obtain Theorem 1.1 as an outcome of a more general result, valid for a class of spaces which includes Riemannian manifolds.

**Theorem 1.3.** Let $p \geq 1$, $M > 0$, $K \in \mathbb{R}$ and $(X, d, m)$ be an RCD($K, \infty$) space of finite measure. Then for any non-constant eigenfunction $f_\lambda$ of the Laplacian, of eigenvalue $\lambda \geq M$ and satisfying $\int_X d(\bar{x}, x)^p |f_\lambda| \, dm(x) < +\infty$ for some $\bar{x} \in X$, it holds
\[
W_p(f_\lambda^+ m, f_\lambda^- m) \geq C(p) C(K, M) \frac{1}{\sqrt{\lambda}} \|f_\lambda\|_{L^p(X)}^{\frac{1}{p}},
\]
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where

\[
C(K, M) := \begin{cases} 
  e^{-\frac{1}{2}} & \text{if } K \geq 0, \\
  \left(1 - \frac{K}{M}\right)^{\frac{M}{2} - \frac{1}{2}} & \text{if } K < 0,
\end{cases}
\]

and

\[
C(p) := \begin{cases} 
  2^{\frac{p-1}{p}} & \text{if } 1 \leq p \leq 2, \\
  \frac{2^{p-1}}{\sqrt{p-1}} & \text{if } 2 < p.
\end{cases}
\]

Roughly speaking, \( \text{RCD}(K, \infty) \) spaces are (possibly non-smooth) metric measure spaces having Ricci curvature bounded below by \( K \in \mathbb{R} \) and no upper bound on the dimension, in a synthetic sense. We refer the reader to Section 2 for the precise definition, and here we only mention that the class of \( \text{RCD}(K, \infty) \) spaces was introduced in [3] and includes: weighted Riemannian manifolds with Bakry-Émery Ricci curvature bounded below [27], pmGH-limits of Riemannian manifolds with Ricci curvature bounded below [11], finite dimensional Alexandrov spaces [22]. In particular, every closed Riemannian manifold endowed with the geodesic distance and the volume measure is an \( \text{RCD}(K, \infty) \) space for some \( K \in \mathbb{R} \).

We remark that in Theorem 1.3 we are not requiring any compactness of the space \( (X, d) \), nor are we assuming that the spectrum of the metric measure space is discrete. The assumptions \( m(X) < \infty \) and \( \int_X d(\bar{x}, x)^p|f_\lambda| \, dm(x) < +\infty \), trivially satisfied for compact spaces, are requested here to ensure that the measures \( f_\lambda^+ m, f_\lambda^- m \) have the same total mass and finite \( p \)-moment.

The proof of Theorem 1.3 relies on a crucial inequality which relates the \( p \)-Wasserstein distance between two finite measures with the \( p \)-Hellinger distance of their evolution through the heat flow. This inequality was originally stated and proved by Luise and Savaré in [19, Theorem 5.2] for the case \( 1 \leq p \leq 2 \). In Proposition 3.1 we extended their result to a general \( p \), taking advantage of a functional inequality satisfied by the heat semigroup (see Proposition 3.2).

To conclude the introduction we notice that, up to our knowledge, the upper bound (3) conjectured by Steinerberger is open for any \( p \) in \( \text{RCD}(K, \infty) \) spaces and for \( p > 1 \) even in smooth Riemannian manifolds.

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2. Preliminaries

2.1. Wasserstein and Hellinger distances. Given a complete and separable metric space \((X, d)\), \( M(X) \) is the space of finite, non-negative, Borel measures on \( X \). We write \( \mu \in M_p(X) \) if \( \mu \in M(X) \) and there exists \( \bar{x} \in X \) such that

\[
\int_X d(x, \bar{x})^p \, d\mu(x) < +\infty.
\]
Definition 2.1. Given $\mu_0, \mu_1 \in \mathcal{M}_p(X)$ with the same total mass, $p \in [1, +\infty)$, the $p$-Wasserstein distance is defined as
\[
W_p^p(\mu_0, \mu_1) := \inf \left\{ \int_{X \times X} d(x, y)^p \, d\pi(x, y) \mid \pi \in \mathcal{P}(X \times X), (P_1)_\pi \pi = \mu_0, (P_2)_\pi \pi = \mu_1 \right\},
\]
where $(P_i)_\pi$ is the pushforward through the projection on the $i$-th component.

We recall that $(\mathcal{P}_p(X), W_p)$ is a complete and separable metric space, where $\mathcal{P}_p(X) \subset \mathcal{M}_p(X)$ is the subset of probability measures with finite $p$-moment. The $p$-Wasserstein distance metrizes the weak convergence of measures plus convergence of the $p$-moment (see e.g. [28]).

Definition 2.2. Given $\mu_0, \mu_1 \in \mathcal{M}(X)$ and $p \in [1, +\infty)$, the $p$-Hellinger distance (also called Matusita distance) \cite{He, He2} between $\mu_0$ and $\mu_1$ is defined as
\[
\text{He}_p(\mu_0, \mu_1) := \int_X \left| \rho_0^{1/p} - \rho_1^{1/p} \right|^p \, d\lambda,
\]
where $\lambda$ is any dominating measure of $\mu_0$, $\mu_1$ and $\rho_i$ are the relative densities: $\mu_i \ll \lambda$ and $\mu_i = \rho_i \lambda$ for $i = 0, 1$.

It is easy to show that $\text{He}_p$ is well defined and it is indeed a distance on $\mathcal{M}(X)$.

Proposition 2.3. For any $p \geq 1$ and every $\mu_i = \rho_i \lambda \in \mathcal{M}(X)$, $i = 0, 1$, it holds
\[
\text{He}_p(\mu_0, \mu_1) \leq \text{He}_1(\mu_0, \mu_1) \leq \frac{p}{2} \| \rho_0 + \rho_1 \|_{L^{1/(1-p)}(X, \lambda)} \text{He}_p(\mu_0, \mu_1). \tag{6}
\]

Proof. The left hand side inequality of (6) is a consequence of
\[
\left| a^{1/p} - b^{1/p} \right|^p \leq |a - b|, \quad a, b \geq 0.
\]
For the right hand side, we use the following inequality
\[
|a^p - b^p| \leq \frac{p}{2^{1-1/p}} |a - b| |b^p + a^p|^{1/p}, \quad a, b \geq 0, \tag{7}
\]
so that, by Holder’s inequality,
\[
\int_X |\rho_0 - \rho_1| \, d\lambda \leq \frac{p}{2^{1-1/p}} \int_X |\rho_0^{1/p} - \rho_1^{1/p}| |\rho_0 + \rho_1|^{1-1/p} \, d\lambda \leq \frac{p}{2^{1-1/p}} \text{He}_p(\mu_0, \mu_1) \| \rho_0 + \rho_1 \|_{L^{1/(1-p)}(X, \lambda)}.
\]

To prove (7), one can assume without loss of generality $b \geq a$ and notice that
\[
b^p - a^p = p(b - a) \int_0^1 ((1 - t)a + tb)^{p-1} \, dt \leq p(b - a) \left( \int_0^1 ((1 - t)a + tb)^p \, dt \right)^{1/p}
\]
\[
\leq p(b - a) \left( \frac{a^p + b^p}{2} \right)^{1/p},
\]
where the first inequality follows from Holder’s inequality and the second by the convexity of the function $x \mapsto x^p$ and integration. \qed

Remark 2.4. Proposition 2.3 should be compared with \cite[Theorem 2.5]{19}, which has inspired our result. Inequality (6) allows us to obtain a better estimate of the constant $C(p)$ in Theorem 1.3, for $p \in (1, 2)$.
From the previous proposition it follows immediately that any $p$-Hellinger distance induces the same strong convergence of the 1-Hellinger distance, which is the total variation.

2.2. $\text{RCD}(K,\infty)$ spaces. In this section we recall the definition of $\text{RCD}(K,\infty)$ spaces and some of their basic properties, which will be useful later on. We refer to the survey [1] and the book [12] as general references on the subject.

Our assumption on the space is that $(X,d,m)$ is a metric measure space, briefly m.m.s., in the sense that $(X,d)$ is a complete and separable metric space and $m$ is a non-negative, Borel measure defined on the Borel $\sigma$-algebra given by the metric $d$. We will always assume $m(X) < \infty$ and $\text{supp}(m) = X$.

We define the relative entropy functional with respect to $m$, $\text{Ent}_m : \mathcal{P}_2(X) \to [0, +\infty]$, as

$$\text{Ent}_m(\mu) := \begin{cases} \int_{\{\rho > 0\}} \rho \log(\rho) \, dm & \text{if } \mu = \rho m, \\ +\infty & \text{otherwise.} \end{cases}$$

In order to define the $\text{RCD}(K,\infty)$ condition, we need first to define the $\text{CD}(K,\infty)$ introduced by Lott-Villani in [14] and by Sturm in [27].

**Definition 2.5.** We say that a m.m.s. $(X,d,m)$ satisfies the $\text{CD}(K,\infty)$ condition if for any couple of measures $\mu^0, \mu^1 \in \mathcal{P}_2(X)$ with $\text{Ent}_m(\mu_i) < +\infty$, $i = 0, 1$, there exists a $W_2$-geodesic $\{\mu_t\}_{t \in [0,1]}$ such that $\mu_0 = \mu^0$, $\mu_1 = \mu^1$ and for every $t \in (0,1)$

$$\text{Ent}_m(\mu_t) \leq (1-t)\text{Ent}_m(\mu_0) + t\text{Ent}_m(\mu_1) - \frac{K}{2} t(1-t)W_2^2(\mu_0, \mu_1).$$

For $f \in L^2(X,m)$ we define the Cheeger energy (see [8]) as

$$\text{Ch}(f) := \inf \left\{ \liminf_{n \to \infty} \frac{1}{2} \int |Df_n|^2 \, dm : f_n \in \text{Lip}(X) \cap L^2(X,m), \|f_n - f\|_{L^2} \to 0 \right\},$$

where $|Df_n|(x)$ is the slope of $f_n$ at the point $x$ and

$W^{1,2}(X,d,m) := \{ f \in L^2(X,m) : \text{Ch}(f) < +\infty \}.$

For simplicity, we will often drop the dependence of the metric measure structure and write $W^{1,2}(X)$ in place of $W^{1,2}(X,d,m)$ or $L^p(X)$ in place of $L^p(X,m)$.

For any $f \in W^{1,2}(X)$, the Cheeger energy admits an integral representation

$$\text{Ch}(f) = \frac{1}{2} \int_X |Df|^2 \, dm,$$

where $|Df|_w$ is called minimal weak upper gradient.

**Definition 2.6.** Following [3], we say that a m.m.s. $(X,d,m)$ satisfies the $\text{RCD}(K,\infty)$ condition if it satisfies the $\text{CD}(K,\infty)$ condition and in addition the Cheeger energy $\text{Ch}$ is a quadratic form on $W^{1,2}(X,d,m)$, i.e. for every $f$ and $g \in W^{1,2}(X,d,m)$ the following equality is satisfied

$$\text{Ch}(f + g) + \text{Ch}(f - g) = 2\text{Ch}(f) + 2\text{Ch}(g).$$

We remark that $\text{Ch}$ is a convex and lower semicontinuous functional over $L^2(X,m)$. This implies that $W^{1,2}(X,d,m)$ is a Banach space with the norm

$$\|f\|_{W^{1,2}(X)} := \|Df\|_{L^2(X)} + 2\text{Ch}(f),$$

which turns out to be an Hilbert space if $X$ satisfies the $\text{RCD}(K,\infty)$ condition.

We focus from now on on $(X,d,m)$ satisfying the $\text{RCD}(K,\infty)$ condition.
It is useful to recall the definition of subdifferential for \( \text{Ch} \). Given \( f \in W^{1,2}(X) \), we say that \( g \in \partial^- \text{Ch}(f) \), namely \( g \) is in the subdifferential of \( \text{Ch} \) at \( f \), if
\[
\int_X g(\psi - f) \, dm \leq \text{Ch}(\psi) - \text{Ch}(f) \quad \forall \psi \in L^2(X).
\]

In an \( \text{RCD}(K, \infty) \) space, the subdifferential of \( \text{Ch} \) where non empty is single valued. From the convexity and lower semicontinuity of \( \text{Ch} \) and from the fact that \( W^{1,2}(X) \) is dense in \( L^2(X) \), it follows, using the theory of gradient flows in Hilbert spaces, that for any \( f \in L^2(X) \) there exists a unique locally absolutely continuous curve \( t \mapsto H_t(f), \, t \in (0, +\infty) \), with values in \( L^2(X) \), which satisfies
\[
\begin{cases}
\frac{d}{dt} H_t f = -\partial^- \text{Ch}(H_t f) & \text{a.e. } t > 0, \\
\lim_{t \to 0} H_t f = f & \text{in } L^2(X).
\end{cases}
\]

\( \{H_t\}_{t \geq 0} \) is called the heat semigroup and for any \( t > 0 \), \( f \mapsto H_t f \) is a linear contraction in \( L^2(X) \). By the density of \( L^2(X) \cap L^p(X) \) in \( L^p(X) \), it can be extended to a semigroup of linear contractions in any \( L^p(X), \, p \geq 1 \). It can also be extended to \( L^\infty(X) \) and it is known that \( H_t f, \, f \in L^\infty(X) \), admits an integral representation via the heat kernel.

We remark that in our setting, the heat semigroup satisfies the maximum principle:
\[
H_t f \leq C \quad \text{if } f \leq C \quad \text{m.a.e.,}
\]
from which follows that it is sign preserving. Moreover, \( H_t \) is measure preserving
\[
\int_X H_t f \, dm = \int_X f \, dm, \quad \forall f \in L^1(X), \quad \forall t > 0,
\]
and for any \( f \in L^\infty(X, m) \) we have \( H_t f \in \text{Lip}_b(X) \) with the bound [9, Proposition 3.1]
\[
\| |D H_t f|_w \|_\infty \leq \sqrt{\frac{2K}{\pi(e^{2Kt} - 1)}} \| f \|_\infty \quad \text{if } K \neq 0,
\]
\[
\| |D H_t f|_w \|_\infty \leq \frac{1}{\pi t} \| f \|_\infty \quad \text{if } K = 0,
\]
which is sharp in the case \( K > 0 \). In addition, for \( f \in L^\infty(X) \), being \( H_t f \in \text{Lip}_b(X) \) it is well defined its slope \( |D H_t f| \) and it is known, see [2, Theorem 3.17], that for all \( t > 0 \)
\[
|D H_t f|_w = |D H_t f| \quad \text{m.a.e. in } X.
\]

It is well known that in the setting of \( \text{RCD}(K, \infty) \) spaces the Bakry-Emery inequality holds (see [2]). We recall the following refined version of it, proved in [23, Corollary 3.5]:
\[
|D H_t f|^{2\alpha}_w \leq e^{-2\alpha Kt} H_t(\|D f\|^{2\alpha}_w), \quad \text{m.a.e.}
\]
for any \( f \in W^{1,2}(X) \) and \( \alpha \in [\frac{1}{2}, 1] \).

Finally we add that from (8) and (9), \( H_t \) maps \( C_b(X) \) to \( C_b(X) \) so it is defined its adjoint operator \( H^*_t : \mathcal{P}(X) \to \mathcal{P}(X) \) that satisfies
\[
H^*_t(\rho m) = H_t(\rho)m
\]
for any probability density \( \rho \in L^1_b(X, m) \) (see [2, Proposition 3.2] for details).
Laplacian and Eigenfunctions. From the Cheeger energy arises also the definition of Laplacian:

**Definition 2.7.** Let \((X, d, m)\) satisfying the \(\text{RCD}(K, \infty)\) condition. For \(f \in W^{1,2}(X)\), the Laplacian of \(f\) is defined as \(\Delta f := -\partial^- \text{Ch}(f)\), provided that \(\partial^- \text{Ch}(f)\) is non empty.

We say that a non-zero function \(f_\lambda \in W^{1,2}(X)\) is an eigenfunction of the Laplacian of eigenvalue \(\lambda \in [0, +\infty)\) if \(-\Delta f_\lambda = \lambda f_\lambda\). If one considers the evolution at time \(t\) via the heat flow of an eigenfunction \(f_\lambda\), then

\[
H_t f_\lambda = e^{-\lambda t} f_\lambda.
\]

Every non-zero constant function is an eigenfunction of eigenvalue \(0\) (recall that we are assuming \(m(X) < +\infty\)), and every other eigenfunction has zero mean, meaning that

\[
\int_X f_\lambda^- dm = \int_X f_\lambda^+ dm.
\]

Under our quite general assumptions, the spectrum of the Laplacian may not be discrete. For brevity, we refer the reader to [11, Proposition 6.7] and [10, Theorem 2.17] for some results about the spectrum of the Laplacian on \(\text{RCD}(K, \infty)\) spaces. Here we just mention that the condition \(\text{diam}(X) < \infty\), or \(K > 0\), implies the compactness of the embedding of \(W^{1,2}(X)\) into \(L^2(X)\), and thus the existence of a basis of \(L^2(X)\) formed by eigenfunctions corresponding to a diverging sequence of eigenvalues.

3. **Proof of Theorem 1.3**

Our starting result shows that the regularizing effect of the heat semigroup \(H_t\) allows to control the stronger \(p\)-Hellinger distance in terms of the weaker \(p\)-Wasserstein distance. We recall that the case \(p \in [1, 2]\) has been firstly proved in [19, Theorem 5.2].

First of all, we set

\[
R_K(t) := \begin{cases} 
\frac{e^{2Kt} - 1}{K} & \text{if } K \neq 0, \\
2t & \text{if } K = 0.
\end{cases}
\]

(13)

**Proposition 3.1.** Let \((X, d, m)\) be an \(\text{RCD}(K, \infty)\) metric measure space, \(K \in \mathbb{R}\). For \(p \geq 1\) and \(\mu_0, \mu_1 \in \mathcal{P}_p(X)\) it holds

\[
C_p(R_K(t)) \frac{1}{2} \mathbb{H}_p(H_t^\ast \mu_0, H_t^\ast \mu_1) \leq W_p(\mu_0, \mu_1) \quad \forall t > 0,
\]

where \(R_K(t)\) was defined in (13) and

\[
C_p := \begin{cases} 
1 \leq p \leq 2, \\
\frac{p}{\sqrt{p - 1}} & 2 < p.
\end{cases}
\]

(14)

To prove Proposition 3.1 in the case \(p > 2\) we take advantage of a new (also for manifolds) functional inequality for the heat flow, namely inequality (15). The strategy of the proof of this inequality is based on a classical semigroup interpolation argument which has been extensively used in the last decades, typically by assuming the existence of an algebra of sufficiently regular functions in the domain of the Laplacian (see e.g. [4, 5]); similar results have been obtained also in the context of \(\text{RCD}\) spaces, where some additional technical arguments are needed to perform the computations (see [2, Corollary 2.3] or [9, Proposition 3.1]).
Proposition 3.2. Let $(X,d,m)$ be an RCD$(K,\infty)$ metric measure space, $K \in \mathbb{R}$. Then for every $1 < q < 2$ and $f \in L^\infty(X)$ it holds

$$(q-1)R_K(t)|DH_t(f)|_m^q \leq (H_t(|f|^q))^{\frac{2}{q}} - (H_t(f))^2, \quad \text{m-a.e. in } X, \text{ for any } t > 0. \quad (15)$$

Proof. We approximate the function $t \mapsto |t|^q$, which second derivative is not defined at $t = 0$, with the following function

$$\phi^q_\varepsilon(t) := (t^2 + \varepsilon^2)^{\frac{q}{2}} - \varepsilon^q,$$

which is $C^\infty(\mathbb{R})$, $\phi^q_\varepsilon(0) = 0$. We consider the map $s \mapsto (H_s(\phi^q_\varepsilon(H_{t-s}f)))^{\frac{2}{q}}$ and we have that it is locally Lipschitz from $(0,t)$ to $L^2$. This follows from the very definition of heat flow, that is a locally Lipschitz map from $(0,\infty)$ to $L^2$ and the fact that its extension to $L^p(X)$ preserves this property, together with the fact that $\phi^q_\varepsilon$ is smooth. So in particular using the fundamental theorem of calculus for the Bochner integral in the space $L^2(X)$ and computing the derivative inside the integral, we have that for any fixed and sufficiently small $\delta > 0$ one has

$$\begin{align*}
(H_{t-\delta}(\phi^q_\varepsilon(H_{t}f)))^{\frac{2}{q}} - (H_{\delta}(\phi^q_\varepsilon(H_{t-\delta}f)))^{\frac{2}{q}} &= \int_{\delta}^{t-\delta} \frac{d}{ds}(H_s(\phi^q_\varepsilon(H_{t-s}f)))^{\frac{2}{q}} ds \\
&= \frac{2}{q} \int_{\delta}^{t-\delta} (H_s(\phi^q_\varepsilon(H_{t-s}f)))^{\frac{2}{q} - 1} \left[ \Delta H_s(\phi^q_\varepsilon(H_{t-s}f)) + H_s((\phi^q_\varepsilon)'(H_{t-s}(f)) \frac{d}{ds}H_{t-s}(f)) \right] ds \\
&= \frac{2}{q} \int_{\delta}^{t-\delta} (H_s(\phi^q_\varepsilon(H_{t-s}f)))^{\frac{2}{q} - 1} \left[ H_s(\Delta(\phi^q_\varepsilon(H_{t-s}(f)))) - (\phi^q_\varepsilon)'(H_{t-s}(f)) \Delta(H_{t-s}(f)) \right] ds, \quad (16)
\end{align*}$$

where the last equality follows from the chain rule of the Laplacian thanks to the fact that $(\phi^q_\varepsilon)''$ is locally bounded. Now recalling that $H_t$ is an integral operator, we apply Holder’s inequality

$$H_t(hg) \leq H_t(|h|^p')^{\frac{1}{p'}} H_t(|g|^{q'})^{\frac{1}{q'}} \quad \text{with exponents } p' = \frac{2}{q} \text{ and } q' = \frac{2}{q-2},$$

namely

$$\left[ H_s \left( |D(H_{t-s}f)|^{p''} ((\phi^q_\varepsilon)''(H_{t-s}f))^{\frac{2}{q'}} \right) \right]^{\frac{1}{p'}} \geq H_s \left( |D(H_{t-s}f)|^{q''} ((\phi^q_\varepsilon)''(H_{t-s}f))^{\frac{2}{q'}} \right). \quad (17)$$

We observe that the left hand side of (17) to the power $\frac{q}{2}$ is exactly the integrand of (16), so we get that

$$\begin{align*}
(H_{t-\delta}(\phi^q_\varepsilon(H_{t}f)))^{\frac{2}{q}} - (H_{\delta}(\phi^q_\varepsilon(H_{t-\delta}f)))^{\frac{2}{q}} \\
&\geq \frac{2}{q} \int_{\delta}^{t-\delta} H_s \left[ |D(H_{t-s}f)|^{q''} ((\phi^q_\varepsilon)''(H_{t-s}f))^{\frac{2}{q'}} \right] ds.
\end{align*}$$
Now we take \( \psi \in L^\infty(X) \), \( \psi \geq 0 \). We integrate the previous inequality against \( \psi \) in order to be able to tackle the limit \( \varepsilon \to 0 \):

\[
\int_X (H_{t-\delta}(\phi_\delta^t(H_{\delta}f)))^\frac{2}{q} - (H_{\delta}(\phi_\delta^t(H_{\delta}f)))^\frac{2}{q} \, \psi \, dm
\]

\[
\geq \frac{2}{q} \int_\delta^{t-\delta} \int_X H_s (|D(H_{t-s}f)|_W^2((\phi_s^t)'(H_{t-s}f)))^\frac{2}{q} \, \phi_s^t(H_{t-s}f)^{(1-\frac{2}{q})} \, \psi \, dm \, ds.
\]  \hspace{1cm} (18)

Noticing that \( f \in L^\infty(X), \phi_\delta^t(t) \in C^\infty(\mathbb{R}) \) and using (9), we can infer that the integrands in both sides of (18) are uniformly integrable with respect to \( \varepsilon \).

So we can take the limit for \( \varepsilon \to 0 \) in both sides of the inequality using the dominated convergence theorem, obtaining

\[
\int_X (H_{t-\delta}(|H_{\delta}f|)^{\frac{2}{q}}) - (H_{\delta}(|H_{\delta}f|)^{\frac{2}{q}}) \, \psi \, dm
\]

\[
\geq \frac{2}{q} \frac{q(q-1)}{2} \int_\delta^{t-\delta} \int_X H_s (|D(H_{t-s}f)|_W^2)^\frac{2}{q} \, \psi \, dm \, ds
\]

\[
\geq 2(q-1) \int_\delta^{t-\delta} \int_X |D H_s(H_{t-s}f)|_W^2 e^{2Ks} \, ds \, dm = 2(q-1) \int_\delta^{t-\delta} e^{2Ks} \, ds \int_X |D(H_t f)|_W^2 \, \psi \, dm,
\]

where the last inequality follows from (11) with \( \alpha = \frac{q}{2} \). Now we can send \( \delta \to 0 \) in the previous inequality and get

\[
\int_X (H_t(|f|)^{\frac{2}{q}}) - (H_{t}(f))^2 \psi \, dm \geq (q-1) \frac{e^{2Kt-1}}{K} \int_X |D(H_t f)|_W^2 \, \psi \, dm,
\]

and observing that inequality holds for any non-negative \( \psi \in L^\infty(X) \), the result is proved.

\[ \square \]

Proof of Proposition 3.1, case \( p > 2 \). We closely follow the proof of [19, Theorem 5.2], to which we refer for details. The main difference is that we deduce (21) from Proposition 3.2, while in [19] the equivalent estimate is deduced from the classical Bakry-Émery inequality (which corresponds to (15) with \( q = 2 \)) and Jensen’s inequality.

The proof makes use of the dual dynamic formulations of both the \( p \)-Wasserstein and the \( p \)-Hellinger distances, stated respectively in [19, Theorem 2.10, Proposition 2.8], (the first one precisely in a rescaled version):

\[
\frac{1}{a} \mathbb{W}_p^q(\mu_0, \mu_1) = \sup \left\{ \int_X \xi_1 \, d\mu_1 - \int_X \xi_0 \, d\mu_0 : \xi \in C^1([0,1], \text{Bor}(X)), \partial_s \xi_s + \frac{a^{q-1}}{q^{p-1}} |D\xi_s|^q \leq 0 \right\},
\]

\[
\text{He}_p^q(H_t^* \mu_0, H_t^* \mu_1) = \sup \left\{ \int_X \xi_1 \, d(H_t^* \mu_1) - \int_X \xi_0 \, d(H_t^* \mu_0) : \xi \in C^1([0,1], \text{Bor}(X)), \partial_s \xi_s + (p-1) |\xi_s|^p \leq 0 \right\},
\]

where \( \text{Bor}(X) \) is the space of bounded Borel functions on \( X \).

To get the result it is enough to prove that any function \( \xi \in C^1([0,1], \text{Bor}(X)) \) satisfying

\[
\partial_s \xi_s + (p-1) |\xi_s|^p \leq 0,
\]  \hspace{1cm} (19)
is such that $H_t(\xi_s)$ satisfies
\[ \partial_s H_t(\xi_s) + \frac{a(t)^{q-1}}{q^{p-1}} |DH_t(\xi_s)|^q \leq 0, \quad t > 0, \tag{20} \]
with
\[ a(t) := \left( \frac{1}{p-1} \right)^{\frac{q}{p}} p^p R_K(t)^{\frac{q}{p}}, \]
so that $H_t(\xi_s)$ is an admissible competitor in the definition of $\frac{1}{p} W_p^p(\mu_0, \mu_1)$. The fact that (19) implies (20) follows by applying $H_t$ to the inequality (19), recalling the fact that $H_t$ is sign preserving and then by using the key inequality
\[ (q-1)\frac{q}{p} (R_K(t))^{\frac{q}{p}} |DH_t(\xi_s)|^q \leq H_t(\|\xi_s\|^q), \tag{21} \]
which is a consequence of (15) neglecting the negative term in the right hand side and taking the $\frac{q}{p}$-power. We remark that in (21) we use the slope of $H_t(\xi_s)$ in place of the weak upper gradient thanks to equality (10), since $\xi_s \in L^\infty(X)$.

**Proof of Theorem 1.3.** We use (12) and the inequality (14) to bound from below the cost $W_p$ in terms of the total variation (notice that all the quantities are scale invariant, so we are not restricted to work with *probability* measures):
\[ W_p(f^+_\lambda, f^-_\lambda) \geq C_p(R_K(t))^{\frac{1}{2}} \text{He}_p(H^+_\lambda f^+_\lambda, H^-_\lambda f^-_\lambda) \]
\[ \geq C_p \frac{2^{\frac{p-1}{p}}}{p} (R_K(t))^{\frac{1}{2}} \frac{\|\text{He}_1(H^+_\lambda f^+_\lambda, H^-_\lambda f^-_\lambda)\|_{L^1(X)}}{\|H^+_\lambda(f^+_\lambda) + H^-_\lambda(f^-_\lambda)\|_{L^1(X)}}, \tag{22} \]
where $C_p$ is the one of Proposition 3.1 and the second inequality follows applying (6). We observe that
\[ \text{He}_1(H^+_\lambda f^+_\lambda, H^-_\lambda f^-_\lambda) = \|H^+_\lambda(f^+_\lambda) - H^-_\lambda(f^-_\lambda)\|_{L^1(X)} = e^{-\lambda t} \|\xi_\lambda\|_{L^1(X)}, \]
using the linearity of the heat flow and recalling that $H^+_\lambda(f_\lambda) = e^{-\lambda x} f_\lambda$. In addition, thanks again to the linearity of $H_t$ and using the preservation of mass of the heat flow we have
\[ \|H^+_\lambda(f^+_\lambda) + H^-_\lambda(f^-_\lambda)\|_{L^1(X)} = \|f_\lambda\|_{L^1(X)}. \]
So inequality (22) reads as
\[ W_p(f^+_\lambda, f^-_\lambda) \geq C_p \frac{2^{\frac{p-1}{p}}}{p} (R_K(t))^{\frac{1}{2}} e^{-\lambda t} \|f_\lambda\|_{L^1(X)} \tag{22} \]

In the case $K = 0$ the result follows by choosing $\bar{t} = \frac{1}{2\lambda} \log(\frac{\lambda}{\lambda - K})$ in the previous inequality.

For $K < 0$ we choose instead $\bar{t} = \frac{1}{2\lambda} \log(\frac{\lambda}{\lambda - K})$ in order to obtain
\[ W_p(f^+_\lambda, f^-_\lambda) \geq C_p \frac{2^{\frac{p-1}{p}}}{p} \frac{1}{\sqrt{\lambda}} \sqrt{\frac{-\lambda}{K} \left( e^{(-\frac{\lambda}{K}) \log \frac{\lambda}{\lambda - K}} - e^{(1 - \frac{\lambda}{K}) \log \frac{\lambda}{\lambda - K}} \right) \|f_\lambda\|_{L^1(X)}^{\frac{1}{2}}. \]

The result follows by standard computations, setting $x = -\frac{\lambda}{K} \geq -\frac{M}{K} > 0$ and noticing that the function
\[ x \mapsto \sqrt{\frac{x e^x \log( \frac{1}{x+1}) - e^x (1 + x) \log(1+e^{x})}{x}} = \left( \frac{x}{x+1} \right)^{\frac{1}{2}} \]
is increasing.

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