ALGEBRAIC APPROXIMATION OF COHEN-MACaulAY ALGEBRAS

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Abstract. This paper shows that Cohen-Macaulay algebras can be algebraically approximated in such a way that their Cohen-Macaulayness and minimal Betti numbers are preserved. This is achieved by showing that finitely generated modules over power series rings can be algebraically approximated in a manner that preserves their diagrams of initial exponents and their minimal Betti numbers. These results are also applied to obtain an approximation result for flat homomorphisms from rings of power series to Cohen-Macaulay algebras.

1. Introduction

Throughout what follows $K$ will denote an arbitrary field, unless specified otherwise. Also for a fixed integer $n$, $x$ will denote the $n$-tuple of variables $(x_1, \ldots, x_n)$. The ring of formal power series in variables $x$ with coefficients in $K$ will be denoted by $K[[x]]$. A formal power series $F \in K[[x]]$ is called algebraic if it satisfies a non-trivial polynomial relation. The set of all such power series forms a ring that is called the ring of algebraic power series and is denoted by $K\langle x \rangle$. In the case when $K$ is a complete real valued field, the notation $K\{x\}$ is used for the ring of convergent power series in variables $x$. Further, throughout this paper, for a $K[[x]]$-module $M$, the notation $\dim M$ will be used for Krull dimension of $M$.

Let $M \subseteq K[[x]]^p$ be a module generated by $F_1, \ldots, F_s \in K[[x]]^p$. A module $M_\mu$ is called an algebraic approximation of $M$ of order $\mu$ if $M_\mu$ is generated by power series vectors whose entries are algebraic power series, and that agree with the generators $F_1, \ldots, F_s$ up to order $\mu$. The main result proved in this paper, Theorem 4.1, establishes the existence of algebraic approximations of arbitrarily high order to modules $M \subseteq K[[x]]^p$, that share specified algebraic properties with $M$. These algebraic properties are Hironaka’s Diagram of Initial exponents (see Section 2.2 for a precise definition), and the minimal Betti numbers of $M$. (Recall that the minimal Betti numbers are the ranks of the free modules appearing in a minimal free resolution of $M$.) Aside from standard facts in commutative algebra, the two main tools used to prove this theorem are Artin’s Approximation Theorem [2, Theorem 1.10], and the theory of standard bases of modules $M \subseteq K[[x]]$, which is analogous to the theory of Gröbner bases of modules over polynomial rings and was developed by T. Becker in [3].

The motivation for the choice of the particular algebraic properties above is the generalization to arbitrary fields of the result [1, Theorem 8.1], proved by J. Adamus and the author, on the existence of arbitrarily high order algebraic approximations

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of Cohen-Macaulay local analytic algebras over \( \mathbb{R} \) or \( \mathbb{C} \) that preserve the Hilbert-Samuel function. The diagram of initial exponents of an ideal completely determines its Hilbert-Samuel function (see Theorem 2.1). Recall that for an ideal \( I \subseteq \mathbb{K}[[x]] \) the Hilbert-Samuel function of it is defined as

\[
H_I(\eta) = \dim \mathbb{K}[[x]]/I + m^{\eta+1}, \quad \text{for } \eta \in \mathbb{N},
\]

where \( m \) is the maximal ideal of \( \mathbb{K}[[x]] \). Also, recall that for \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \) a local analytic algebra is a quotient of the form \( \mathbb{K}\{x\}/I \) where \( I \subseteq \mathbb{K}\{x\} \) is an ideal.

The property of Cohen-Macaulayness is related to the minimal Betti numbers of \( \mathbb{K}\{x\}/I \); specifically, it can be determined by the number of non-zero Betti numbers (see Theorem 3.2). The result proved in this paper, that achieves the generalization of [1, Theorem 8.1], is Theorem 6.1 and follows directly from Theorem 4.1. In fact, Theorem 6.1 achieves more than just the generalization of [1, Theorem 8.1] to arbitrarily fields, as the minimal Betti numbers are a finer grained property than Cohen-Macaulayness. As a consequence of this, an approximation result analogous to [1, Theorem 8.1] for Gorenstein local algebras is obtained as a corollary to Theorem 6.1. It should be noted here that generalization of the finite determinacy result proved by J. Adamus and the author in [1] for algebras \( \mathbb{K}\{x\}/I \) that are complete intersections, [1, Theorem 7.3], to arbitrary fields, already exists and was proved by Srinivas and Trivedi in [11] using techniques different from those used in [1]. In fact, an even stronger result exists in the special case of isolated complete intersection singularities, due to Greuel and Pham [9, Theorem 1.5].

A generalization of the algebraic counterpart of the result on flat maps from [10, Theorem 1.2] is also obtained as a by-product of the proofs of Theorem 4.1 and Theorem 6.1 (Theorem 6.4). Briefly, [10, Theorem 1.2] states that given a flat analytic map from a real or complex analytic germ whose local ring is Cohen-Macaulay into a germ of euclidean space, one can find arbitrarily high order algebraic approximations to the domain that are Cohen-Macaulay, and to the map that are flat, which preserve the Hilbert-Samuel function of the special fibre. In this paper a homomorphism of rings \( \phi : A \rightarrow B \) will be called flat if it makes \( B \) into a flat \( A \)-module, and in the case when \( A \) and \( B \) are local rings with maximal ideals \( m_A \) and \( m_B \) respectively, the ring \( B/\phi(m_A)B \) will be called the special fibre of \( \phi \). With these definitions Theorem 6.4 can be stated as follows: Let \( y \) denote the \( m \)-tuple of variables \( (y_1, \ldots, y_m) \). If one has a flat homomorphism from \( \mathbb{K}\{y\} \) to \( \mathbb{K}[[x]]/I \), and if \( \mathbb{K}[[x]]/I \) is Cohen-Macaulay, then there exist arbitrarily high order algebraic approximations to the homomorphism and \( \mathbb{K}[[x]]/I \) such that the Hilbert-Samuel function and minimal Betti numbers of the special fibre of the approximants is the same as those of the special fibre of the original homomorphism and such that the approximating homomorphisms are flat. The precise notion of algebraic approximation in the context of homomorphisms of rings is defined in a manner analogous to that for modules (see the statement of Theorem 6.4). The result [10, Theorem 1.2] is a direct consequence of Theorem 6.4 and remarkably, its proof via Theorem 4.1 and Theorem 6.1 has fewer dependencies and is conceptually simpler than that of [10, Theorem 1.2].

The structure of this paper is as follows: Section 2 presents relevant definitions and theorems used in the proofs of the main theorems of the paper. Section 3 presents the proof of Theorem 4.1. Section 5 provides justification for the validity of the results of this paper for rings of convergent power series in the case when the field \( \mathbb{K} \) is a complete real valued field. Section 6 presents the applications of
Theorem 4.1 specifically the proof of Theorem 6.1 in Subsection 6.1 and the proof of Theorem 6.4 in Subsection 6.2.

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2. Background

2.1. Power series vectors. For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), the monomial \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) will be denoted by \( x^\alpha \). For an integer \( p \in \mathbb{N} \), let \( e_1, \ldots, e_p \in \mathbb{K}[x]^p \) denote column vectors with 1 in the \( i \)-th entry and zeros everywhere else, these are called the standard basis vectors of \( \mathbb{K}[x]^p \). An element of \( \mathbb{K}[x]^p \) of the form \( x^\alpha e_i \) for \( (\alpha, i) \in \mathbb{N}^n \times \{1, \ldots, p\} \) is called a monomial term and \( (\alpha, i) \) is called its exponent. Let \( F \in \mathbb{K}[x]^p \) be a power series vector. With the above notation \( F \) can be expressed as a sum of monomial terms as follows:

\[
F = \sum_{(\alpha, i) \in \mathbb{N}^n \times \{1, \ldots, p\}} f_{\alpha, i} x^\alpha e_i.
\]

The support of \( F \) is \( \text{supp}(F) = \{ (\alpha, i) \in \mathbb{N}^n \times \{1, \ldots, p\} : f_{\alpha, i} \neq 0 \} \). If \( \mu \in \mathbb{N} \) then

\[
j^\mu F = \sum_{(\alpha, i) \in \mathbb{N}^n \times \{1, \ldots, p\}, |\alpha| \leq \mu} f_{\alpha, i} x^\alpha e_i.
\]

where \( |\alpha| = \alpha_1 + \cdots + \alpha_n \), is called the \( \mu \)-jet of \( F \).

2.2. The diagram of initial exponents. For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), let \( |\alpha| = \sum_{i=1}^n \alpha_i \). Then, the lexicographic ordering of the \( n + 2 \)-tuples \( (|\alpha|, j, \alpha_1, \ldots, \alpha_n) \), where \( 1 \leq j \leq p \), defines a total ordering on \( \mathbb{N}^n \times \{1, \ldots, p\} \). The initial exponent of \( F \in \mathbb{K}[x]^p \) is \( \exp(F) = \min\{ (\alpha, i) : (\alpha, i) \in \text{supp}(F) \} \), where the minimum is taken with respect to the total ordering just defined. If \( M \subseteq \mathbb{K}[x]^p \) is a finitely generated module then the diagram of initial exponents of \( M \) is \( \text{\textit{H}}(M) = \{ \exp(F) : F \in M \} \). There exists a unique smallest (finite) set \( \mathcal{V}(M) \subseteq \text{\textit{H}}(M) \) such that \( \mathcal{V}(M) = \mathcal{V}(M) + \mathbb{N}^n = \{ (\alpha + \beta, i) : (\alpha, i) \in \mathcal{V}(M), \beta \in \mathbb{N}^n \} \). The elements of \( \mathcal{V}(M) \) are called the vertices of the diagram \( \text{\textit{H}}(M) \).

In the case when the module under consideration is an ideal \( I \subseteq \mathbb{K}[x] \), the diagram of initial exponents is related to the Hilbert-Samuel function of the ideal \( I \) as follows:

**Lemma 2.1** ([11] Lemma 6.2). Let \( I \in \mathbb{K}[x] \) be an ideal. Then,

\[
H_I(\eta) = \# \{ \beta \in \mathbb{N}^n \setminus \text{\textit{H}}(I) : |\beta| \leq \eta \}, \quad \text{for all } \eta \geq 1.
\]

**Remark 2.2.**

(i) For large values of \( \eta \in \mathbb{N} \), the Hilbert-Samuel function \( H_I(\eta) \) coincides with a polynomial called the Hilbert-Samuel polynomial.

(ii) The degree of the Hilbert-Samuel polynomial of an ideal \( I \subseteq \mathbb{K}[x] \) is equal to \( \dim \mathbb{K}[x]/I \).
2.3. **Standard bases and standard representations.** In what follows a module in \(\mathbb{K}[x]^p\) generated by \(F_1, \ldots, F_s \in \mathbb{K}[x]^p\), will be denoted by \((F_1, \ldots, F_s)\). Let \(M = (F_1, \ldots, F_s) \subseteq \mathbb{K}[x]^p\) be a finitely generated module. A set \(\{G_1, \ldots, G_t\} \subseteq M\) is called a **standard basis for** \(M\) if \(\forall (M) \subseteq \{\exp(G_1), \ldots, \exp(G_t)\}\). It is a consequence of Hironaka’s Division Theorem \([5,\text{Theorems 3.1, 3.4}]\) and \([4]\) that a standard basis for \(M\) generates it.

If \(F, H_1, \ldots, H_r \in \mathbb{K}[x]^p\), then \(F\) has a **standard representation** in terms of \(H_1, \ldots, H_r\) if there exist \(Q_1, \ldots, Q_r \in \mathbb{K}[x]\) such that,

\[
F = \sum_{i=1}^{r} Q_i H_i \text{ and } \exp(F) = \min\{\exp(Q_i H_i) : i = 1, \ldots, r\}.
\]

In the above, by convention, it is assumed that \(\exp(F) < \exp(0)\) for all \(F \neq 0\).

Suppose now that \(F = \sum_{(\alpha,i) \in \mathbb{N}_0^{n} \times \{1, \ldots, p\}} f_{\alpha,i} x^{\alpha e_i}\), and that \(G = \sum_{(\alpha,i) \in \mathbb{N}_0^{n} \times \{1, \ldots, p\}} g_{\alpha,i} x^{\alpha e_i}\). Let \((\alpha_F, i_F) = \exp(F), (\alpha_G, i_G) = \exp(G)\), and \(x^\gamma = \text{lcm}(x^{\alpha_F}, x^{\alpha_G})\). Define,

\[
P_{F,G} = \begin{cases} f_{\alpha_F,i_F} x^{\gamma - \alpha G} & \text{if } i_F = i_G, \\ 0, & \text{otherwise,} \end{cases}
\]

\[
P_{G,F} = \begin{cases} g_{\alpha_G,i_G} x^{\gamma - \alpha F} & \text{if } i_F = i_G, \\ 0, & \text{otherwise} \end{cases}
\]

With the above, the **s-series vector** of \(F\) and \(G\) is \(S(F,G) = P_{F,G}F - P_{G,F}G\). The following theorem, which follows directly from the corresponding result for ideals in power series rings \([3,\text{Theorem 4.1}]\), gives a criterion for determining when a collection of power series vectors in \(\mathbb{K}[x]^p\) forms a standard basis for the module generated by it.

**Theorem 2.3** (cf. \([3,\text{Theorem 4.1}]\)). If \(S\) is a finite subset of \(\mathbb{K}[x]^p\) then the elements of \(S\) form a standard basis for the module generated by them if and only if for each pair \(G_1, G_2 \in S\), the s-series vector \(S(G_1, G_2)\) has a standard representation in terms of elements of \(S\).

**Remark 2.4.**

(i) This is a power series analogue of the corresponding result for submodules of free modules over \(\mathbb{K}[x]\) called Buchberger’s criterion \([7]\).

(ii) The result \([3,\text{Theorem 4.1}]\), is for the case of ideals generated by collections of power series in \(\mathbb{K}[x]\), however, the proof of the corresponding result, Theorem 2.3 above, for modules follows by arguments that are almost identical to those used in the proof of \([3,\text{Theorem 4.1}]\).

(iii) The theory of standard basis and the diagram of initial exponents in Sections 2.3 and 2.4 can be developed for orderings other than the one used here, however, this will not be used in this paper.

(iv) Suppose that \(F_1, \ldots, F_s\) is a standard basis for a module \(M \subseteq \mathbb{K}[x]^p\) and that \(G_1, \ldots, G_s\) is a standard basis for a module \(N \subseteq \mathbb{K}[x]^p\), then \(\{\exp(G_1), \ldots, \exp(G_s)\} = \{\exp(F_1), \ldots, \exp(F_s)\}\) implies that \(\mathcal{N}(M) = \mathcal{N}(N)\). This follows directly from Hironaka’s Division Theorem \([5,\text{Theorem 3.1}]\), \([4]\).
2.4. Facts on homomorphisms between free modules over \( \mathbb{K}[x] \). By choosing a suitable basis of \( \mathbb{K}[x]^m, \mathbb{K}[x]^n \) for \( m, n \in \mathbb{N} \), a homomorphism \( \phi : \mathbb{K}[x]^m \to \mathbb{K}[x]^n \) can be represented by a matrix of dimension \( n \times m \) with entries in \( \mathbb{K}[x] \).

A homomorphism \( \phi : \mathbb{K}[x]^m \to \mathbb{K}[x]^n \) is called \textit{algebraic} if the matrix of \( \phi \) has entries in \( \mathbb{K}(x) \). The following condition for the injectivity of \( \phi \) follows immediately from [6, Proposition I.2.9].

**Lemma 2.5.** A homomorphism \( \phi : \mathbb{K}[x]^m \to \mathbb{K}[x]^n \) is injective if and only if none of the the \( m \times m \) minors of a matrix representation of \( \phi \) is 0.

If \( \phi : \mathbb{K}[x]^m \to \mathbb{K}[x]^n \) is given by the following matrix,

\[
\phi = \begin{pmatrix}
S_1 & \cdots & S_m
\end{pmatrix}
\]

where \( S_1, \ldots, S_m \in \mathbb{K}[x]^n \) then the \( \mu \)-jet of \( \phi \) is,

\[
j^\mu \phi = \begin{pmatrix}
j^\mu S_1 & \cdots & j^\mu S_m
\end{pmatrix}
\]

3. Free resolutions

Throughout this section let \( M \subseteq \mathbb{K}[x]^p \) be a finitely generated module. For some integer \( c \), and integers \( n_0, \ldots, n_c \) an exact sequence of the form

\[
\mathcal{F}_M : 0 \to \mathbb{K}[x]^{n_0} \xrightarrow{\phi_0} \mathbb{K}[x]^{n_1} \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{c-1}} \mathbb{K}[x]^{n_c} \xrightarrow{\phi_c} M \to 0
\]

is called a \textit{finite free resolution of} \( M \). By [7, Corollary 19.6], all finitely generated \( \mathbb{K}[x] \)-modules have a finite free resolution. The number \( c \) in the above is called the \textit{length} of the free resolution \( \mathcal{F}_M \). If the homomorphisms \( \phi_i \) for \( 1 \leq i \leq c \) in a finite free resolution \( \mathcal{F}_M \) have the property that \( \text{im}(\phi_i) \subseteq m \mathbb{K}[x]^{n_{i-1}} \), then \( \mathcal{F}_M \) is called a \textit{minimal free resolution of} \( M \). By [7, Lemma 19.4] the minimality of \( \mathcal{F}_M \) is equivalent to the condition that for each \( 1 \leq i \leq c \), a basis for \( \mathbb{K}[x]^{n_{i-1}} \) maps onto a minimal set of generators (i.e., a set of generators with minimal cardinality) of \( \text{coker}(\phi_i) \). Further, by [7, Theorem 20.2], all minimal free resolutions of \( M \) have the same length, which is called the \textit{projective dimension of} \( M \), and denoted by \( \text{pd}_{\mathbb{K}[x]}(M) \). The numbers \( n_0, n_1, \ldots, n_c \) in the minimal free resolution \( \mathcal{F}_M \) of a finitely generated module \( M \subseteq \mathbb{K}[x]^p \) are called the \textit{minimal Betti numbers of} \( M \) and are denoted by \( \beta^M_0, \ldots, \beta^M_c \).

**Remark 3.1.** Suppose that \( I \subseteq \mathbb{K}[x] \) is an ideal. Then there exists an exact sequence \( 0 \to I \xrightarrow{\iota} \mathbb{K}[x] \xrightarrow{\pi} \mathbb{K}[x]/I \to 0 \), where \( \iota \) is the canonical inclusion of \( I \) in \( \mathbb{K}[x] \) and \( \pi \) is the canonical homomorphism from \( \mathbb{K}[x]/I \) to \( \text{coker}(\iota) = \mathbb{K}[x]/I \). Consequently, if

\[
\mathcal{F}_I : 0 \to \mathbb{K}[x]^{n_0} \xrightarrow{\phi_0} \mathbb{K}[x]^{n_1} \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{c-1}} \mathbb{K}[x]^{n_c} \xrightarrow{\phi_c} I \to 0
\]

is a minimal free resolution of \( I \), then

\[
\mathcal{F}_{\mathbb{K}[x]/I} : 0 \to \mathbb{K}[x]^{n_0} \xrightarrow{\phi_0} \cdots \xrightarrow{\phi_{c-1}} \mathbb{K}[x]^{n_c} \xrightarrow{\phi_c} \mathbb{K}[x]/I \xrightarrow{\pi} \mathbb{K}[x]/I \to 0
\]

is a minimal free resolution of \( \mathbb{K}[x]/I \). Note that in the above, by abuse of notation, \( \phi_0 \) is used to denote both the homomorphism \( \phi_0 : \mathbb{K}[x]^{n_0} \to I \) and the composition \( \iota \circ \phi_0 : \mathbb{K}[x]^{n_0} \to \mathbb{K}[x] \).
The following characterization of Cohen-Macaulay rings of the form \( K[[x]]/I \) follows directly from [7, Corollary 19.15] and the fact that \( K[[x]] \) is a regular local ring.

**Theorem 3.2** (cf. [7 Corollary 19.15]). If \( I \subseteq K[[x]] \) is an ideal, then \( K[[x]]/I \) is Cohen-Macaulay if and only if

\[
\text{pd}_{K[[x]]}(K[[x]]/I) = \dim K[[x]] - \dim K[[x]]/I.
\]

If \( K[[x]]/I \) is Cohen-Macaulay, and has a minimal free resolution, such as \( F_\bullet \) above, then the last minimal Betti number \( \beta^{K[[x]]/I}_{c+1} = n_c \) of \( K[[x]]/I \) is called the Cohen-Macaulay type of \( K[[x]]/I \).

Given \( F_1, \ldots, F_s \in K[[x]]^p \), the module generated by all \( H = (H_1, \ldots, H_s)^T \in K[[x]]^s \) such that \( \sum_{i=1}^{s} H_i F_i = 0 \) is called the module of syzygies on \( F_1, \ldots, F_s \) and is denoted by \( \text{Syz}(F_1, \ldots, F_s) \). The specification of a minimal free resolution such as (3.1) for a module \( M \subseteq K[[x]]^p \) is equivalent to the specification of certain syzygy modules as follows: If the homomorphism \( \phi_0 \) is given by,

\[
\phi_0 = \begin{pmatrix} S_{1,0} & \cdots & S_{n_0,0} \\ \vdots & \ddots & \vdots \\ S_{1,k} & \cdots & S_{n_k,k} \end{pmatrix},
\]

then \( S_{1,0}, \ldots, S_{n_0,0} \in K[[x]]^p \) are a minimal basis of generators of the module \( M \). For \( 1 \leq k \leq c \), if the homomorphism \( \phi_k \) is given by the matrix,

\[
\phi_k = \begin{pmatrix} S_{1,k} & \cdots & S_{n_k,k} \\ \vdots & \ddots & \vdots \\ S_{1,k} & \cdots & S_{n_k,k} \end{pmatrix},
\]

then \( S_{1,k}, \ldots, S_{n_k,k} \in K[[x]]^{n_k-1} \) are a minimal basis of generators of the module \( \text{Syz}(S_{1,k-1}, \ldots, S_{n_k-1,k-1}) \). If \( M = (F_1, \ldots, F_s) \) and \( F_1, \ldots, F_s \) form a standard basis, the following theorem gives us a basis of generators for \( \text{Syz}(F_1, \ldots, F_s) \).

**Theorem 3.3** (cf. [7 Theorem 15.10], [5 Theorem 6.2]). Suppose that \( F_1, \ldots, F_s \in K[[x]]^p \) form a standard basis of the module they generate. Further, suppose that the s-series vectors of pairs \( F_i, F_j \) for \( 1 \leq i < j \leq s \) are \( S(F_i, F_j) = P_{i,j} F_i - P_{j,i} F_j \) and that \( Q_{m}^{i,j} \in K[[x]] \), for \( 1 \leq m \leq s \) and \( 1 \leq i < j \leq s \), are such that the following are standard representations of the s-series vectors of pairs in \( F_1, \ldots, F_s \):

\[
P_{i,j} F_i - P_{j,i} F_j = \sum_{m=1}^{s} Q_{m}^{i,j} F_m \quad \text{for} \quad 1 \leq i < j \leq s
\]

Then,

\[
\Xi_{i,j} = P_{i,j} e_i - P_{j,i} e_j - \sum_{m=1}^{s} Q_{m}^{i,j} e_m \quad \text{for} \quad 1 \leq i < j \leq s
\]

form a basis of generators of the module \( \text{Syz}(F_1, \ldots, F_s) \), where the \( e_i \) for \( 1 \leq i \leq s \) are the standard basis vectors of \( K[[x]]^s \).

**Remark 3.4.** (i) In fact, [7 Theorem 15.10] and [5 Theorem 6.2] also provide an ordering with respect to which the \( \Xi_{i,j} \) form a standard basis for \( \text{Syz}(F_1, \ldots, F_s) \), however this fact is not used in this paper.
4. Approximation of Modules

Theorem 4.1. Let \( M = (F_1, \ldots, F_s) \subseteq \mathbb{K}[x]^p \) be a finitely generated module. Further, let

\[
\mathcal{F}_M : 0 \to \mathbb{K}[x]^{n_0} \xrightarrow{\phi_0} \mathbb{K}[x]^{n_{c-1}} \xrightarrow{\phi_{c-1}} \cdots \xrightarrow{\phi_1} \mathbb{K}[x]^{n_0} \xrightarrow{\phi_0} M \to 0
\]

be a minimal free resolution of \( M \). Then there is an integer \( \mu_0 \in \mathbb{N} \) such that for each \( \mu \geq \mu_0 \) there exist \( F_1^{(\mu)}, \ldots, F_s^{(\mu)} \in \mathbb{K}(x)^p \), which generate a module \( M_\mu \subseteq \mathbb{K}[x]^p \), and algebraic homomorphisms \( \phi_j^{(\mu)} : \mathbb{K}(x)^n \to \mathbb{K}(x)^{n_{j-1}} \) for \( 1 \leq j \leq c \), \( \phi_0^{(\mu)} : \mathbb{K}(x)^{n_0} \to M_\mu \) such that,

(i) \( j^\mu F_i = j^\mu F_i^{(\mu)} \) for \( 1 \leq i \leq s \).

(ii) \( j^\mu \phi_i = j^\mu \phi_i^{(\mu)} \) for \( 0 \leq i \leq c \).

(iii) \( \mathfrak{N}(M) = \mathfrak{N}(M_\mu) \).

(iv) The following is a minimal free resolution of \( M_\mu \)

\[
\mathcal{F}_{M_\mu} : 0 \to \mathbb{K}[x]^{n_0} \xrightarrow{\phi_0^{(\mu)}} \mathbb{K}[x]^{n_{c-1}} \xrightarrow{\phi_{c-1}^{(\mu)}} \cdots \xrightarrow{\phi_1^{(\mu)}} \mathbb{K}[x]^{n_0} \xrightarrow{\phi_0^{(\mu)}} M_\mu \to 0
\]

Proof. Suppose that \( \phi_0 \) is given by the following \( p \times n_0 \) matrix,

\[
\phi_0 = \begin{pmatrix}
S_{1,0} & \cdots & S_{n_0,0}
\end{pmatrix}
\]

then \( S_{1,0}, \ldots, S_{n_0,0} \in \mathbb{K}[x]^p \) form a minimal basis for \( M \). Similarly for \( 1 \leq i \leq c \), if \( \phi_i : \mathbb{K}[x]^{n_i} \to \mathbb{K}[x]^{n_{i-1}} \) is given by the \( n_{i-1} \times n_i \) matrix,

\[
\phi_i = \begin{pmatrix}
S_{1,i} & \cdots & S_{n_i,i}
\end{pmatrix},
\]

then \( S_{1,i}, \ldots, S_{n_i,i} \in \mathbb{K}[x]^{n_{i-1}} \) form a minimal basis for the module \( \text{Syz}(S_{1,i-1}, \ldots, S_{n_{i-1},i-1}) \). Further, let \( G_{1,0}, \ldots, G_{r_0,0} \in \mathbb{K}[x]^p \) be a standard basis for \( M \), and for \( 1 \leq i \leq c-1 \), let \( G_{1,i}, \ldots, G_{r_i,i} \in \mathbb{K}[x]^{n_{i-1}} \) be a standard basis for \( \text{Syz}(S_{1,i-1}, \ldots, S_{n_{i-1},i-1}) \). Then, there exist \( H_{ik,j} \in \mathbb{K}[x] \) for \( 0 \leq j \leq c-1, 1 \leq k \leq n_j, 1 \leq i \leq r_j \) such that,

\[
G_{i,j} = \sum_{k=1}^{n_i} H_{ik,j} S_{k,j} \quad \text{for } 0 \leq j \leq c-1, 1 \leq i \leq r_j
\]

Further, there exist \( L_{ik,0} \in \mathbb{K}[x] \) for \( 1 \leq i \leq n_0 \) and \( 1 \leq k \leq s \), such that

\[
S_{1,0} = \sum_{k=1}^{s} L_{ik,0} F_k.
\]

For \( 0 \leq k \leq c-1 \), Theorem 2.3 implies that there exist monomials \( P_{ij,k}(x), P_{ji,k}(x) \in \mathbb{K}[x] \) for \( 1 \leq i < j \leq n_k \) such that the following is a standard representation,

\[
P_{ij,k}(x)G_{i,k} - P_{ji,k}(x)G_{j,k} = \sum_{l=1}^{r_k} Q_{ijl,k} G_{l,k} \quad \text{for } 0 \leq k \leq c-1, 1 \leq i < j \leq n_k.
\]

Explicitly, the condition on the initial exponents for the above to be a standard basis is,

\[
\exp(P_{ij,k}(x)G_{i,k} - P_{ji,k}(x)G_{j,k}) = \min\{\exp(Q_{ijl,k} G_{l,k}) : 1 \leq l \leq r_k\}
\]
Let $e_{1,k}, \ldots, e_{n_k,k}$ be the standard basis vectors of $K[[x]]^{n_k}$ for $0 \leq k \leq c-1$. By Theorem 3.3 and 4.1, the following are a basis of generators of $Syz(S_{1,k}, \ldots, S_{n_k,k})$

$$
\Xi_{ij,k} = P_{ij,k}(x)\left(\sum_{l=1}^{n_k} H_{il,k}e_l,k\right) - P_{ji,k}(x)\left(\sum_{l=1}^{n_k} H_{jl,k}e_l,k\right)
$$

(4.5)

$$
- \sum_{l=1}^{n_k} Q_{ijl,k}\left(\sum_{q=1}^{r_k} H_{qj,k}c_{q,k}\right) \text{ for } 0 \leq k \leq c-1, 1 \leq i < j \leq r_k.
$$

Since, $S_{1,k+1}, \ldots, S_{n_k+1,k}$ are a minimal basis of generators of $Syz(S_{1,k}, \ldots, S_{n_k,k})$, there exist $R_{ijl,k} \in K[[x]]$ such that the following relation holds,

$$
S_{i,k+1} = \sum_{1 \leq i < j \leq r_k} \Xi_{ij,k} R_{ijl,k} \text{ for } 0 \leq k \leq c-1, 1 \leq i \leq n_{k+1}
$$

(4.6)

Now treating the equations (4.1), (4.2), (4.3), (4.5), and (4.6) as a system of polynomial equations with polynomial coefficients in variables $S_{i,k}, G_{i,k}, H_{ij,k}, Q_{ijl,k},$ and $L_{ij,0}$, $\Xi_{ij,k}$ and $R_{ijl,k}$ with coefficients in $K[x]$ (for allowable values of the indices), the above argument establishes the existence of a formal power series solution. In what follows let $n_{-1} = p$ for simplicity. Therefore, for any integer $\mu \in \mathbb{N}$, by Artin’s Approximation Theorem 2 (Theorem 1.10) there exist

1. $F_{i}^{(\mu)} \in K(x)^p$ such that $j^{\mu}F_{i}^{(\mu)} = j^{\mu}F_{i}$ for $1 \leq i \leq s$,
2. $S_{i,k}^{(\mu)} \in K(x)^{n_{k-1}}$ such that $j^{\mu}S_{i,k}^{(\mu)} = j^{\mu}S_{i,k}i$ for $0 \leq k \leq c$ and $1 \leq i \leq n_k$,
3. $G_{i,k}^{(\mu)} \in K(x)^{n_{k-1}}$ such that $j^{\mu}G_{i,k}^{(\mu)} = j^{\mu}G_{i,k}i$ for $0 \leq k \leq c-1$ and $1 \leq i \leq r_k$,
4. $H_{ij,k}^{(\mu)} \in K(x)$ such that $j^{\mu}H_{ij,k}^{(\mu)} = j^{\mu}H_{ij,k}i$ for $0 \leq k \leq c-1$, $1 \leq i \leq r_k$, and $1 \leq j \leq n_k$,
5. $Q_{ijl,k}^{(\mu)} \in K(x)$ such that $j^{\mu}Q_{ijl,k}^{(\mu)} = j^{\mu}Q_{ijl,k}i$ for $0 \leq k \leq c-1$, $1 \leq l \leq r_k$ and $1 \leq i < j \leq r_k$,
6. $L_{ij,0}^{(\mu)} \in K(x)$ such that $j^{\mu}L_{ij,0}^{(\mu)} = j^{\mu}L_{ij,0}i$ for $1 \leq i \leq n_0$ and $1 \leq j \leq s$,
7. $\Xi_{ij,k}^{(\mu)} \in K(x)^{n_k}$ such that $j^{\mu}\Xi_{ij,k}^{(\mu)} = j^{\mu}\Xi_{ij,k}i$ for $0 \leq k \leq c-1$ and $1 \leq i < j \leq r_k$, and $1 \leq l \leq n_{k+1}$,
8. $R_{ijl,k}^{(\mu)} \in K(x)$ such that $j^{\mu}R_{ijl,k}^{(\mu)} = j^{\mu}R_{ijl,k}i$ for $0 \leq k \leq c-1$, $1 \leq i < j \leq r_k$, and $1 \leq l \leq n_{k+1}$.

that also solve the system of equations given by (4.1), (4.2), (4.3), (4.5), and (4.6). Let $(\tilde{a}_k, i_k) = \max\{\exp(G_{1,k}), \ldots, \exp(G_{r_k,k})\}$ for $0 \leq k \leq c$, and let $\mu_0 = \max\{|\tilde{a}_0|, \ldots, |\tilde{a}_c|\}$. Taking $\mu > \mu_0$, will ensure that the approximants in item (3) above will satisfy the criterion (4.3), which will, in turn, ensure that the corresponding approximants $\Xi_{ij,k}^{(\mu)}$ for $1 \leq i < j \leq r_k$ in item (7) from a basis of generators of $Syz(S_{1,k}^{(\mu)}, \ldots, S_{n_k,k}^{(\mu)})$, for $0 \leq k \leq c-1$. Let $M_{\mu} = (F_{1}^{(\mu)}, \ldots, F_{s}^{(\mu)})$. Now, for $0 \leq k \leq c$, define $\phi_k^{(\mu)} : K[[x]]^{n_k} \to K[[x]]^{n_{k-1}}$ to be the algebraic homomorphism given by the matrix

$$
\phi_k^{(\mu)} = \begin{pmatrix}
S_{1,k}^{(\mu)} & \cdots & S_{n_k,k}^{(\mu)}
\end{pmatrix}
$$

Then, for $\mu > \mu_0$ one has the following
Let the following be a minimal free resolution of

\[ \begin{array}{c}
\mathbb{K}[[x]] \to \mathbb{K}[[x]]^{n_0} \to \cdots \to \mathbb{K}[[x]]^{n_{c-1}} \to \mathbb{K}[[x]]^{n_c} \to \cdots \to \mathbb{K}[[x]] \to M_\mu \to 0.
\end{array} \]

□

5. Regarding convergence

In the case when \( K \) is a complete valued field, it is possible to define the ring of convergent power series \( \mathbb{K}\{x\} \). Theorem 4.1 remains valid if \( \mathbb{K}[[x]] \) is replaced by \( \mathbb{K}\{x\} \). Specifically,

**Remark 5.1.**

(i) Theorem 4.1 depends on Hironaka’s Division Theorem, which is valid for \( \mathbb{K}\{x\} \) by [3] Theorem 3.4.

(ii) Theorem 4.1 is also only dependent on [5] Theorem 3.4, and is proved in this context as [3] Theorem 6.2.

(iii) All the theory in Section 3 is valid for \( \mathbb{K}[[x]] \) replaced by any regular local ring, in particular, \( \mathbb{K}\{x\} \).

(iv) Artin’s Approximation Theorem [2] Theorem 1.10 used in the proof of Theorem 4.1 is also valid for approximations of solutions in \( \mathbb{K}\{x\} \) of polynomial systems of equations.

6. Applications

6.1. Approximation of Cohen-Macaulay algebras.

**Theorem 6.1.** Let \( I = (F_1, \ldots, F_s) \) be an ideal in \( \mathbb{K}[[x]] \) such that \( \mathbb{K}[[x]]/I \) is Cohen-Macaulay, has dimension \( \dim \mathbb{K}[[x]]/I = n - k \) and has minimal Betti numbers \( \beta_0^\mathbb{K}[[x]]/I, \ldots, \beta_k^\mathbb{K}[[x]]/I \). Then, there exists an integer \( \mu_0 \) such that for each \( \mu \geq \mu_0 \) there exist, \( F_1^{(\mu)}, \ldots, F_s^{(\mu)} \in \mathbb{K}\{x\} \) which generate an ideal \( I_\mu \subseteq \mathbb{K}[[x]] \) such that,

(i) \( j^\mu F_i^{(\mu)} = j^\mu F_i \) for \( 1 \leq i \leq s \).

(ii) The ring \( \mathbb{K}[[x]]/I_\mu \) is Cohen-Macaulay with \( \dim \mathbb{K}[[x]]/I_\mu = \dim \mathbb{K}[[x]]/I \).

(iii) The minimal Betti numbers of \( \mathbb{K}[[x]]/I_\mu \) satisfy \( \beta_i^{\mathbb{K}[[x]]/I_\mu} = \beta_i^\mathbb{K}[[x]]/I \) for \( 0 \leq i \leq k \).

(iv) \( H_{I_\mu} = H_I \).

**Proof.** Let the following be a minimal free resolution of \( I \)

\[ \begin{array}{c}
\mathbb{K}[[x]] \to \mathbb{K}[[x]]^{n_0} \to \cdots \to \mathbb{K}[[x]]^{n_{c-1}} \to \mathbb{K}[[x]]^{n_c} \to \cdots \to \mathbb{K}[[x]] \to I \to 0.
\end{array} \]
Then by Remark 5.1 and Theorem 6.2 \( c = k - 1 \) and the following is a minimal free resolution of \( \mathbb{K}[x]/I \)

\[
F_{\mathbb{K}[x]/I} : 0 \rightarrow \mathbb{K}[x]^{n_k-1} \xrightarrow{\phi_{k-1}} \cdots \xrightarrow{\phi_1} \mathbb{K}[x]^{n_0} \xrightarrow{\phi_0} \mathbb{K}[x] \xrightarrow{\pi} \mathbb{K}[x]/I \rightarrow 0.
\]

Now, by Theorem 4.1 there exists \( \mu \in \mathbb{N} \), such that for each \( \mu \geq \mu_0 \) there exist, \( F_1^{(\mu)}, \ldots, F_s^{(\mu)} \in \mathbb{K}(x) \), generating \( I_\mu \subseteq \mathbb{K}[x] \) such that \( j_\mu F_1^{(\mu)} = j_\mu F_1, \mathbb{N}(I) = \mathbb{N}(I_\mu) \), and the following is a minimal free resolution of \( I_\mu \)

\[
F_{I_\mu} : 0 \rightarrow \mathbb{K}[x]^{\mu_0(n_k)} \xrightarrow{\phi_{k-1}^{(\mu)}} \cdots \xrightarrow{\phi_1^{(\mu)}} \mathbb{K}[x]^{\mu_0(n_0)} \xrightarrow{\phi_0^{(\mu)}} \mathbb{K}[x] \xrightarrow{\pi} \mathbb{K}[x]/I_\mu \rightarrow 0,
\]

where \( \phi_i^{(\mu)} \) are algebraic homomorphisms for \( 0 \leq i \leq k - 1 \). This implies that

1. The following is a minimal free resolution of \( \mathbb{K}[x]/I_\mu \)

\[
F_{\mathbb{K}[x]/I_\mu} : 0 \rightarrow \mathbb{K}[x]^{\mu_0(n_k)} \xrightarrow{\phi_{k-1}^{(\mu)}} \cdots \xrightarrow{\phi_1^{(\mu)}} \mathbb{K}[x]^{\mu_0(n_0)} \xrightarrow{\phi_0^{(\mu)}} \mathbb{K}[x] \xrightarrow{\pi} \mathbb{K}[x]/I_\mu \rightarrow 0.
\]

2. \( H_{I_\mu} = H_I \), by Lemma 2.1

Item (2) above implies that the Hilbert-Samuel polynomials of \( I \) and \( I_\mu \) and the same which by Remark 2.2 implies that \( \dim \mathbb{K}[x]/I = \dim \mathbb{K}[x]/I_\mu = n - k \). Now, because of item (1) above, Theorem 3.2 implies that \( \mathbb{K}[x]/I_\mu \) is minimal. Item (1) also implies that \( \beta_i^{\mathbb{K}[x]/I_\mu} = \beta_i^{\mathbb{K}[x]/I} \), for all \( 0 \leq i \leq k \).

The above implies in particular that the Cohen-Macaulay type of \( \mathbb{K}[x]/I \) and \( \mathbb{K}[x]/I_\mu \) are the same for all \( \mu \geq \mu_0 \). By [7, Corollary 21.16], \( \mathbb{K}[x]/I \) is Gorenstein if and only if the Cohen-Macaulay type of \( \mathbb{K}[x]/I \) is one. Therefore, an immediate corollary of the above is

**Corollary 6.2.** Let \( I = (F_1, \ldots, F_s) \) be an ideal in \( \mathbb{K}[x] \) such that \( \mathbb{K}[x]/I \) is Gorenstein, and has dimension \( \dim \mathbb{K}[x]/I = n - k \). Then, there exists an integer \( \mu_0 \) such that for each \( \mu \geq \mu_0 \) there exist, \( F_1^{(\mu)}, \ldots, F_s^{(\mu)} \in \mathbb{K}(x) \) which generate an ideal \( I_\mu \subseteq \mathbb{K}[x] \) such that

(i) \( j_\mu F_i^{(\mu)} = j_\mu F_i \) for \( 1 \leq i \leq s \).

(ii) The ring \( \mathbb{K}[x]/I_\mu \) is Gorenstein with \( \dim \mathbb{K}[x]/I_\mu = \dim \mathbb{K}[x]/I \).

(iii) \( H_{I_\mu} = H_I \).

**Remark 6.3.**

(i) In the case where \( \mathbb{K} \) is a complete real-valued field, by the remarks in Section 5, Theorem 6.1 and Theorem 6.2 remain true when \( \mathbb{K}[x] \) is replaced by \( \mathbb{K}[x] \).

(ii) The result [11, Theorem 8.1] follows immediately from Theorem 6.1

### 6.2. Approximation of flat maps

Throughout this section \( m \) will be a fixed integer and \( y \) will denote the \( m \)-tuple of variables \( (y_1, \ldots, y_m) \). Consider a homomorphism rings \( \phi : \mathbb{K}[y] \rightarrow \mathbb{K}[x]/I \), for some ideal \( I \subseteq \mathbb{K}[x] \). Such a map can be completely specified by specifying the images of \( y_1, \ldots, y_m \). The images of \( y_1, \ldots, y_m \) under \( \phi \) can belong to \( \mathbb{K}[x]/I \). Let \( \pi : \mathbb{K}[x] \rightarrow \mathbb{K}[x]/I \) be the canonical projection. A homomorphism \( \phi : \mathbb{K}[y] \rightarrow \mathbb{K}[x]/I \) can be completely specified by giving \( \phi(y_i) = \phi_i(x) \in \mathbb{K}[x] \), for \( 1 \leq i \leq m \), for some \( \phi : \mathbb{K}[y] \rightarrow \mathbb{K}[x] \) such that \( \phi = \pi \circ \phi \). In such a case the homomorphism \( \phi \) is said to be defined by \( \phi_i(x) \in \mathbb{K}[x] \) for \( 1 \leq i \leq m \). The homomorphism \( \phi \) is called algebraic if it is
defined by \( \phi_i(x) \in \mathbb{K}(x) \) for \( 1 \leq i \leq m \). Note that the special fibre of a homomorphism \( \phi : \mathbb{K}[y] \to \mathbb{K}[x]/I \) is isomorphic to \( \mathbb{K}[x]/(I + J) \) where, \( J \) is the ideal generated by any set of power series \( \phi_i(x) \in \mathbb{K}[x] \) defining \( \phi \).

**Theorem 6.4.** Suppose that \( I = (F_1, \ldots, F_s) \subseteq \mathbb{K}[x] \) is an ideal, \( \mathbb{K}[x]/I \) is Cohen-Macaulay and that \( \phi : \mathbb{K}[y] \to \mathbb{K}[x]/I \) is flat homomorphism of rings, defined by \( \phi_i(x) \in \mathbb{K}[x] \) for \( 1 \leq i \leq m \). Then there exists a \( \mu_0 \in \mathbb{N} \) such that for each \( \mu \geq \mu_0 \) there exist \( F_1^\mu, \ldots, F_s^\mu \in \mathbb{K}(x) \) which generate an ideal \( I_\mu \subseteq \mathbb{K}[x] \) and an algebraic homomorphism \( \phi_\mu : \mathbb{K}[y] \to \mathbb{K}[x] \) defined by \( \phi_i(\mu) \in \mathbb{K}(x) \), \( 1 \leq i \leq m \), such that,

1. \( j^\mu F_i(\mu) = j^\mu F_i \) for \( 1 \leq i \leq s \).
2. \( j^\mu \phi_i(\mu) = j^\mu \phi_i \) for \( 1 \leq i \leq m \).
3. \( \phi_\mu \) is a flat homomorphism.
4. If \( J = (F_1, \ldots, F_s, \phi_1, \ldots, \phi_m) \) and \( J_\mu = (F_1^\mu, \ldots, F_s^\mu, \phi_1^\mu, \ldots, \phi_m^\mu) \), then \( H_I = H_{J_\mu} \) and the minimal Betti numbers of \( \mathbb{K}[x]/J \) and \( \mathbb{K}[x]/J_\mu \) are the same.
5. \( H_I = H_{J_\mu} \) and the minimal Betti numbers of \( \mathbb{K}[x]/I \) and \( \mathbb{K}[x]/I_\mu \) are the same.

**Proof.** By the flatness criterion, \cite[Theorem B.8.11]{8}, the flatness of \( \phi \), and the Cohen-Macaulayness of \( \mathbb{K}[x]/I \) imply that

\[
\dim \mathbb{K}[x]/I = \dim \mathbb{K}[y] + \dim \mathbb{K}[x]/J,
\]

where \( J \) is defined as in point (iv) in the statement of the theorem. Further, by \cite[Theorem B.8.15]{8}, the flatness of \( \phi \), and the Cohen-Macaulayness of \( \mathbb{K}[x]/I \) imply that \( \mathbb{K}[x]/J \) is also Cohen-Macaulay. Now, in the proofs of Theorem 4.1 and Theorem 5.1 the systems of equations corresponding to the approximations of \( I = (F_1, \ldots, F_s) \) and \( J = (F_1, \ldots, F_s, \phi_1, \ldots, \phi_m) \) can be solved simultaneously, and thus, there exists \( \mu_0 \in \mathbb{N} \), such that for each \( \mu \geq \mu_0 \) there exist \( F_1^\mu, \ldots, F_s^\mu, \phi_1^\mu, \ldots, \phi_m^\mu \in \mathbb{K}(x) \) such that if \( I_\mu = (F_1^\mu, \ldots, F_s^\mu) \) and \( J_\mu = (F_1^\mu, \ldots, F_s^\mu, \phi_1^\mu, \ldots, \phi_m^\mu) \), then \( H_I = H_{I_\mu} \) and \( H_J = H_{J_\mu} \), \( \dim \mathbb{K}[x]/I = \dim \mathbb{K}[x]/I_\mu \), \( \dim \mathbb{K}[x]/J = \dim \mathbb{K}[x]/J_\mu \), and \( \mathbb{K}[x]/I_\mu \) and \( \mathbb{K}[x]/J_\mu \) are both Cohen-Macaulay. Also, the minimal Betti numbers of \( \mathbb{K}[x]/I \) and \( \mathbb{K}[x]/J \) are the same as those of \( \mathbb{K}[x]/I_\mu \) and \( \mathbb{K}[x]/J_\mu \) respectively. By \cite[Theorem B.8.11]{8} again, the Cohen-Macaulayness of \( \mathbb{K}[x]/I_\mu \) and the relationship between the dimensions above, imply that the homomorphism \( \phi_\mu : \mathbb{K}[y] \to \mathbb{K}[x]/I_\mu \) defined by \( \phi_i(\mu) \) for \( 1 \leq i \leq m \) is flat.

**Remark 6.5.**

1. (i) When \( \mathbb{K} \) is a complete real valued field, by the comments in Section \cite[Theorem 6.4]{5} remains valid if \( \mathbb{K}[y] \) and \( \mathbb{K}[x] \) are replaced by \( \mathbb{K}\{y\} \) and \( \mathbb{K}\{x\} \) respectively.
2. (ii) The result \cite[Theorem 1.2]{10} follows from Theorem 6.4 above immediately.

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