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A FLAT PHOTOIONIZATION RATE AT 2 ≤ Z ≤ 4.2: EVIDENCE FOR A STELLAR-DOMINATED UV BACKGROUND AND AGAINST A DECLINE OF COSMIC STAR FORMATION BEYOND Z ∼ 3

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ABSTRACT

We investigate the implications of our measurement of the Lyman-α forest opacity at redshifts 2 ≤ z ≤ 4.2 from a sample of 86 high-resolution quasar spectra for the evolution of the cosmic ultraviolet hydrogen opacity and density and its sources. The derived hydrogen photoionization rate Γ is remarkably flat over this redshift range, implying an increasing comoving ionizing emissivity with redshift. Because the Lyman-α opacity function is strongly peaked near z ≈ 2, star-forming galaxies likely dominate the ionizing emissivity at z ≥ 3. Our measurement argues against a star formation rate density declining beyond z ∼ 3, in contrast with existing state-of-the-art determinations of the cosmic star formation history from direct galaxy counts. Stellar emission from galaxies therefore likely reionized the Universe.

Subject headings: Cosmology: diffuse radiation — methods: data analysis — galaxies: formation, evolution, high-redshift — quasars: absorption lines

1. INTRODUCTION

The opacity of the Lyman-α (Lyα) forest is set by a competition between hydrogen photoionizations and recombinations (Gunn & Peterson 1965) and can thus serve as a direct probe of the photoionization rate (e.g., Rauch et al. 1997). The hydrogen photoionization rate Γ is a particularly valuable quantity as it is an integral over all sources of ultraviolet (UV) radiation in the Universe,

\[ Γ(z) = 4π \int_{\nu_{\text{HI}}}^{\nu_{\text{Lyα}}} \frac{dν}{hν} J_\nu(z) σ(ν), \]

where \( J_\nu \) is the angle-averaged specific intensity of the background, \( σ(ν) \) is the photoionization cross section of hydrogen, and the integral is from the Lyman limit to infinity. As such, it bears a signature of cosmic stellar and quasistellar activity that is not subject to the completeness issues to which direct source counts are prone. Moreover, unlike the redshifted radiation background observed on Earth, the Lyα forest is a local probe of the high-redshift UV radiation, as only sources at approximately the same redshift contribute to Γ at any point in the forest (e.g., Haardt & Madau 1996). In addition to being a powerful probe of galaxy formation and evolution and a fundamental ingredient of cosmological simulations (e.g., Efstathiou 1992), identifying the sources that contribute most to the UV background is key to our understanding of the reionization history of the Universe.

In this Letter, we derive the photoionization rate implied by our measurement of the Lyα forest opacity at 2 ≤ z ≤ 4.2 from a sample of 86 high-resolution quasar spectra (Faucher-Giguère et al. 2008c), for the first time consistently analyzing such a large data set (corrected for both continuum bias and metal absorption) over this redshift interval. We discuss the implications of its flatness over this redshift range for the relative contribution of quasars and star-forming galaxies to the high-redshift cosmic UV background. Throughout, we assume a WMAP5 cosmology (Komatsu et al. 2008). The full details of our analysis, as well as supporting arguments, are presented elsewhere (Faucher-Giguère et al. 2008a).

2. THE PHOTOIONIZATION RATE FROM THE Lyα FOREST

The specific measurement we use is that of the Lyα effective optical depth \( τ_{\text{eff}} \) in \( Δz = 0.2 \) bins corrected for continuum bias and for metal absorption following the results of Schaye et al. (2003) reported by Faucher-Giguère et al. (2008c).

The effective optical depth is defined as

\[ τ_{\text{eff}} = -\ln \langle F(z) \rangle, \]

where \( \langle F \rangle = \langle \exp(-τ) \rangle \) is the mean transmission of the forest at redshift \( z \) and \( τ \) is the local Gunn & Peterson (1965) optical depth. In photoionization equilibrium and for a power-law temperature-density relation for the low-density intergalactic medium (IGM) of the form \( T = T_0(1 + δ)^β \) (Hui & Gnedin 1997),

\[ τ = A(z)(1 + δ)^{-2 - 0.7β}, \]

with

\[ A(z) = \frac{πe^2 f_{\text{Lγα}}}{m_e ν_{\text{Lγα}}} \left( \frac{\rho_{\text{crit}} Ω_\text{h}^2}{m_ν} \right)^2 \frac{1}{H(z)} \times X(1 + 0.5Y R_0 T_0^{-0.7} \frac{Γ}{Γ})^6. \]

Here, \( f_{\text{Lγα}} \) is the oscillator strength of the Lyα transition, \( ν_{\text{Lγα}} \) is its frequency, \( X \) and \( Y \) are the mass fractions of hydrogen and helium (respectively taken to be 0.75 and 0.25; Burles et al. 2001), \( R_0 = 4.2 \times 10^{-13} \text{cm}^2 \text{s}^{-1}/(10^4 \text{K})^{-0.7} \), and \( T_0 \) is the IGM temperature at mean density (δ = 0). This expression is valid when all the intergalactic helium is fully ionized; an error ≤ 8% may arise prior to HeII reionization.
Given a volume-weighted probability density function (PDF) for the gas density \( \Delta = 1 + \delta \),

\[
\langle F \rangle (z) = \int_0^\infty d\Delta P(\Delta; z) \exp (-\tau). \tag{5}
\]

We use the analytical fit to gas-dynamical simulations of Miralda-Escudé et al. (2000) for this PDF. For the IGM temperature, we interpolate between the \( T_0 = 2 \times 10^4 \) K values measured by Zaldarriaga et al. (2001) from the Lyα forest power spectrum, and \( \beta = 0.62 \), appropriate in the limit of early hydrogen reionization (Hui & Gnedin 1997).

With the above, we solve for the unique \( \Gamma \) that reproduces the measured \( \tau_{\text{eff}} \) at each redshift. The results are shown in Figure 1A, where \( \Gamma \approx (0.5 \pm 0.1) \times 10^{-12} \) s\(^{-1}\) is seen to be remarkably flat over the redshift range \( 2 \leq z \leq 4.2 \). Note, however, that the absolute normalization of \( \Gamma \) depends on the cosmology; the thermal history of the IGM, as well as on the gas density distribution (e.g., Bolton et al. 2003), and that a significant scatter between the results of different studies employing the same basic method remains (see Figure 1 of Faucher-Giguère et al. 2008a). These systematic sources of uncertainty are not included in our analysis. On the other hand, the measurement we present consistently samples a large redshift interval with independent statistical errors and should therefore reliably trace the redshift evolution of \( \Gamma \).

3. THE IONIZING SOURCES

The UV background is generally assumed to be produced by quasars and star-forming galaxies, but the relative importance of these two populations remains uncertain. Moreover, it is unclear whether all the sources responsible for the ionization state of the high-redshift IGM are presently accounted for by magnitude-limited surveys.

In Figure 1A, we show the contribution of quasars as calculated using the \( B\)-band realization of the Hopkins et al. (2007) bolometric quasar luminosity function (LF). The curve, subject to overall normalization uncertainties in the mean free path of ionizing photons, the spectral energy distribution of quasars, and the fraction of ionizing photons that they emit that escape into the IGM, has been renormalized to approximately match the total photoionization rate of the Lyα forest at \( z = 2 \). Its shape is however robustly constrained at redshifts \( z \gtrsim 2 \), owing to both an increasing dominance of the brightest quasars to the UV background and obscuration corrections decreasing in importance with redshift. In particular, the quasar contribution to \( \Gamma \) is strongly peaked near \( z = 2 \) and even if these objects produce the entire ionizing background at this redshift, they fall short of accounting for the total \( \Gamma \) measured at \( z = 4 \) by a factor \( \gtrsim 5 \).

To estimate the contribution of star-forming galaxies to the UV background, we consider recent determinations of the galaxy UV LF from Lyman break galaxy (LBG) surveys by Sawicki & Thompson (2006) (Keck Deep Fields), Bouwens et al. (2007) (Hubble Ultra Deep Field [HUDF] and other deep Hubble Space Telescope [HST] deep fields), Steidel et al. (1999) and Reddy et al. (2008) (Keck LBG), and Yoshida et al. (2006) (Subaru Deep Field). These were selected to be the most up-to-date measurements in the fields covered. The LFs have an effective wavelength near 1500 Å, and the specific emissivity at this wavelength is simply obtained by extrapolating and integrating them down to zero luminosity. The error bars we quote are propagated from those on the individual Schechter parameters; because the latter are generally correlated, these will overestimate the true errors on the luminosity densities. Exceptions are the Sawicki & Thompson (2006) points, for which we take the total luminosity densities and errors reported by the authors. In order to compare the UV luminosity den-
sity of LBGs with our measured $\Gamma$, we convert the latter to a comoving emissivity at 1500 Å.

Given the proper mean free path of ionizing photons at the Lyman limit $\lambda_{\text{mfp}}$, we can calculate the total comoving specific emissivity implied by the measured $\Gamma$:

$$\epsilon_{912} \approx \frac{h(\alpha_{\text{HI}} + 3)}{\sigma_{\text{HI}}\lambda_{\text{mfp}}} (1 + z)^{-3}\Gamma$$  (6)

(e.g., Schirber & Bullock 2003) for an ionizing background with a power-law spectrum $J_{\text{HI}} \propto \nu^{-5.0}$. Here, $\sigma_{\text{HI}}$ is the photoionization cross section of hydrogen at the Lyman limit. Letting $\alpha_{\text{UV}}$ be the spectral index between 912 Å and 1500 Å, we can calculate the emissivity at the wavelength probed by the galaxy UV luminosity functions,

$$\epsilon_{1500} = \frac{1}{f_{\text{esc}}} \left(\frac{1500 \, \text{Å}}{912 \, \text{Å}}\right)^{\alpha_{\text{UV}}} \epsilon_{912},$$  (7)

where the escape fraction $f_{\text{esc}}$ accounts for the discontinuity at the Lyman limit owing to Lyman-continuum absorption associated with the host galaxy.

The exact value for $f_{\text{esc}}$ is not well constrained at present, but is likely to be at most a few percent (e.g., Steidel et al. 2001; Shapley et al. 2006; Chen et al. 2007). Here, we simply note that $\epsilon_{1500} \propto \Gamma(1 + z)$ for a mean free path $\lambda_{\text{mfp}} \propto (1 + z)^{4}$, as appropriate if the incidence of Lyman-limit systems increases as $(1 + z)^{1.5}$ (Stengler-Larrea et al. 1995). This is a conservative assumption, as our conclusions regarding the need for an increasing comoving ionizing emissivity with redshift would only be strengthened if the absorbers responsible for limiting the mean free path instead evolve as $(1 + z)^{2}$, as is often assumed on the basis of the better studied lower column density systems (e.g., Madau et al. 1999). We then solve for the normalization that minimizes the $\chi^2$ difference between the emissivities calculated from the UV luminosity functions and our Lyα forest measurement. The result is shown Figure 1B. In Faucher-Giguère et al. (2008a), we show that for fiducial assumptions, only a small $f_{\text{esc}} \sim 0.5\%$ is required for LBGs to solely account for the $z \sim 3$ ionizing background.

Within the large scatter, the redshift evolution of UV emissivity derived from the Lyα forest is reasonably reproduced by the emission from LBGs only. The only hint of a decline of the galaxy UV emissivity near $z = 4$ comes from the highest-redshift point of Sawicki & Thompson (2006). This measurement is inconsistent with the higher points from the Subaru Deep Field (Yoshida et al. 2006) and Steidel et al. (1999). This may owe to cosmic variance and is not uniformly complete. For example, one of the $z \sim 6$ points that drive the Hopkins & Beacom (2006) fit is the estimate of Bunker et al. (2004), which is only complete to 0.1$L^*$ and is based on an extremely small HUDF 11 arcmin$^2$ exposure. We instead consider the analysis of Bouwens et al. (2007), which includes the HUDF data as a subset and yields a higher SFR density, and consistently integrate the LF down to zero luminosity. Alternatively, present surveys may be missing a significant UV luminosity density from very faint galaxies.

The above results have interesting implications for the cosmic star formation history. In particular, several authors have previously found evidence for a peak in the star formation rate (SFR) density near $z \sim 2 - 3$ (e.g., Madau et al. 1998; Hopkins 2004; Hopkins & Beacom 2006). Barring redshift evolution of dust obscuration, the escape fraction, or the initial mass function (IMF) of stars, the SFR density should trace the UV emissivity, which the Lyα forest suggests instead increases continuously from $z = 2$ to $z = 4.2$. As a representative example of state-of-the-art determinations of the high-redshift star formation history, we consider the fit of Hopkins & Beacom (2004) to a large compilation of galaxy surveys.

In Figure 1C, we show the comoving SFR density we derived from the UV emissivities from both the Lyα forest (assuming that it arises solely from galaxies) and from direct measurements of the galaxy UV LF. We convert from specific UV emissivity to SFR density using

$$\rho_{\star} = 1.08 \times 10^{-28} \epsilon_{1500}.$$  (8)

where $\rho_{\star}$ is in units of comoving $M_\odot \text{yr}^{-1} \text{Mpc}^{-3}$ provided $\epsilon_{1500}$ is expressed in comoving erg s$^{-1}$ Hz$^{-1}$ Mpc$^{-3}$. This conversion is appropriate for a “modified Salpeter A” IMF, consistent with the Hopkins & Beacom (2006) fit also shown on the Figure. Other IMFs would result in different conversion factors. However all the data points (and fit) in this plot would be equally renormalized and conclusions with regards to discrepant redshift evolutions would be unaffected. We apply a UV obscuration correction factor of 3.4 over the entire redshift range, corresponding to the “common” obscuration correction applied by Hopkins (2004) and Hopkins & Beacom (2006) at $z \gtrsim 3$. Although this correction is unlikely to be exact, it allows for a consistent comparison with the high-redshift fits to the star formation history by these authors.

We find no compelling evidence for a decline in the comoving SFR density over the redshift range probed by our measurement, either from it or from the directly measured UV LF, in contrast to the best fit of Hopkins & Beacom (2006). Inspection of Figure 1C suggests that the present data are instead roughly consistent with a constant $\rho_{\star} \sim 0.2 M_\odot \text{yr}^{-1} \text{Mpc}^{-3}$ at $2 \lesssim z \lesssim 4.5$. Since our analysis assumes a dust correction consistent with these authors at high redshifts, but is based on more recent data, it thus seems that the SFR density peak suggested by their fit may be an artifact of the scarce high-redshift data in their compilation, which may be affected by cosmic variance and is not uniformly complete. For example, one of the $z \sim 6$ points that drives the Hopkins & Beacom (2006) fit is the estimate of Bunker et al. (2004), which is only complete to 0.1$L^*$ and is based on an extremely small HUDF 11 arcmin$^2$ exposure. We instead consider the analysis of Bouwens et al. (2007), which includes the HUDF data as a subset and yields a higher SFR density, and consistently integrate the LF down to zero luminosity. Alternatively, present surveys may be missing a significant UV luminosity density from very faint galaxies.

It is immediately clear from the scatter in panels B and C of Figure 1 that the total UV luminosity density extrapolated from the measured LF should be interpreted
with caution. In fact, the dispersion between different points at fixed redshift is generally larger than the calculated error bars, indicating that these are unlikely to be uniformly reliable, a situation which is particularly manifest at $z \approx 4$. There are several reasons why this may be the case, including extrapolation to fainter magnitudes than probed by individual surveys, cosmic variance arising from large-scale structure, and parameters (perhaps inaccurately) held fixed in some fits.

A number of previous studies of the LBG UV LF have also found little evidence for a decline of the SFR density beyond $z \approx 3$ (e.g., Steidel et al. 1999; Giavalisco et al. 2003; Yoshida et al. 2006). This finding has in addition been corroborated by measures based on photometric redshifts (e.g., Thompson et al. 2001; Thompson 2003) and is also in qualitative agreement with theoretical models that predict a SFR history peaking at higher redshift (e.g., Springel & Hernquist 2003; Hernquist & Springel 2003).

If only because the SFR density is expected to rise continuously on physical grounds, it must eventually decline toward high redshifts. Bouwens et al. (2007) in fact find evidence for such a decline toward $z = 6$ on the basis of evolving dust obscuration suggested by observed $\beta$-values at this redshift (e.g., Stanway et al. 2003). We simply contend here that neither the present Ly$\alpha$ forest data or the recent UV LF compiled here, especially when considered together with their mutual scatter after extrapolation down to zero luminosity, show convincing evidence for the often-assumed peak in SFR density near $z \sim 2 - 3$. The requirement that the Universe be reionized by $z = 6$ also supports a SFR peaking significantly earlier (Faucher-Giguère et al. 2008a).

5. COMPARISON WITH PREVIOUS WORK

Similar conclusions have been reached in previous studies of the UV background. Bolton et al. (2005), in particular, inferred $\Gamma$ from the Ly$\alpha$ opacity measurement of Schaye et al. (2003) and also found its evolution to be consistent with being constant at $2 \leq z \leq 4$. By comparing with the estimated quasar contribution, they also found evidence for a stellar-dominated UV background at all redshifts. In their analysis using the Miralda-Escudé et al. (2000) PDF, Becker et al. (2007) also derived a flat $\Gamma$ over this redshift range. Our results extend beyond previous analyses in highlighting that common assumptions regarding the star formation history fall short of providing for the ionizing rate of the forest at $z \gtrsim 3$.

Measurements based on the proximity effect (e.g., Scott et al. 2000) have tended to yield $\Gamma$ values higher by a factor of $\sim 3$. However, the overdense regions in which quasars reside are likely to bias these measurements high (Faucher-Giguère et al. 2008a).

6. REIONIZATION

The decline of the quasar LF and the increasing dominance of stellar emission to the high-redshift $z \gtrsim 3$ ionizing background make a compelling case that the Universe was reionized by stars. This gives credibility to analytical and numerical calculations of hydrogen reionization that make this assumption (e.g., Furlanetto et al. 2004; McQuinn et al. 2007a; Zahn et al. 2007). This is encouraging news for upcoming observational probes of the epoch of reionization, such as redshifted 21-cm emission and high-redshift Ly$\alpha$ emitters (e.g., Zaldarriaga et al. 2004; McQuinn et al. 2007a), whose detailed interpretation will rely on our understanding of the morphology of reionization and its origin.

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