THEORETICAL GRAVITATIONAL LENSING – BEYOND THE WEAK-FIELD SMALL-ANGLE APPROXIMATION

VOLKER PERLICK
Physics Department, Lancaster University, Lancaster LA1 4YB, United Kingdom
v.perlick@lancaster.ac.uk

An overview is given on those theoretical gravitational lensing results that can be formulated in a spacetime setting, without assuming that the gravitational fields are weak and that the bending angles are small. The first part is devoted to analytical methods considering spacetimes in which the equations for light rays (lightlike geodesics) is completely integrable. This includes spherically symmetric static spacetimes, the Kerr spacetime and plane gravitational waves. The second part is devoted to qualitative methods which give some information on lensing properties without actually solving the equation for lightlike geodesics. This includes Morse theory, methods from differential topology and bifurcation theory.

1. Introduction

For many years, theoretical work on gravitational lensing was almost exclusively done in an approximation formalism based on the assumptions that the gravitational fields are weak and that the bending angles are small. In its simplest version this formalism gives rise to the traditional “lens equation”, see e.g. Schneider, Ehlers and Falco,1 Petters, Levine and Wambsganss2 or Wambsganss.3 This lens equation has proven a very powerful tool for investigating gravitational lensing situations.

In recent years, however, some gravitational lensing situations have come into the reach of observability for which the weak-deflection approximation is not applicable. In particular, the discovery that there is a black hole at the center of our galaxy, and probably at the center of most galaxies, is an important motivation for investigating the deflection of light rays that come close to a black hole. For such light rays the bending angle may become, in principle, arbitrarily large because light rays can make arbitrarily many turns around the black hole. There are also some other, more hypothetical, astrophysical objects whose lensing properties may lead to observable effects beyond the traditional lens equation, such as wormholes or monopoles.

If one wants to discuss gravitational lensing without the weak-field small-angle approximation, one has to use the formalism of general relativity. In this formalism, light rays are mathematically described as lightlike geodesics of a Lorentzian manifold. The past-oriented lightlike geodesics issuing from an observation event \( p_O \) make up the past light cone of \( p_O \). All information on what can be seen at the celestial sphere of an observer at \( p_O \) is coded in the geometry of \( p_O \)’s past light cone. In this sense, all lensing features are known if the past light cones are known. Determining the past light cone of \( p_O \) requires solving the equation for lightlike geodesics through \( p_O \). In spacetimes with sufficiently many symmetries this equation is completely integrable, so the light cones can be determined analytically. In spacetimes where
the geodesic equation is not completely integrable one may study some qualitative features of the lightlike geodesics without actually solving the geodesic equation, e.g. with the help of methods from global analysis or differential topology. In this sense the theoretical work on gravitational lensing in a Lorentzian geometry setting can be naturally divided into analytical studies and qualitative studies.

As was already mentioned, the main motivation for investigating gravitational lensing in a spacetime setting without the weak-field small-angle approximation comes from observations: There are observational situations, accessible with present or near-future technology, for which this approximation is simply not adequate. However, a second and almost equally important motivation comes from methodology: One gets a much better understanding of the physics behind gravitational lensing if one develops the relevant formalism as far as possible in the general framework of general relativity and introduces approximations only for those applications for which they are really necessary.

This article gives an overview on methods of how to study gravitational lensing in a Lorentzian geometry setting, without the weak-field small-angle approximation. Chapter 2 reviews analytical work in spacetimes where the geodesic equation is completely integrable. This includes all spherically symmetric static spacetimes such as the Schwarzschild spacetime (Section 2.1), the Kerr spacetime (Section 2.2), and plane gravitational waves (Section 2.3). Chapter 3 reviews some mathematical methods that give qualitative results on lensing features, e.g. on the number of images, without actually solving the geodesic equation. This includes Morse theory (Section 3.1), methods from differential topology (Section 3.2) and bifurcation theory (Section 3.3). Related material can be found in Ref. 4.

2. Analytical Methods

2.1. Spherically Symmetric Static Spacetimes

In a spherically symmetric static spacetime, the geodesic equation is completely integrable, so the lightlike geodesics can be explicitly written in terms of integrals. (In the case of the Schwarzschild solution these are elliptic integrals.) This allows to write an exact lens equation for such spacetimes.

Lensing without weak-field or small-angle approximations in spherically symmetric static spacetimes was pioneered by Darwin\textsuperscript{5,6} and by Atkinson.\textsuperscript{7} Whereas Darwin’s work is restricted to the Schwarzschild spacetime throughout, Atkinson derives all relevant formulas for an unspecified spherically symmetric static spacetime before specializing to the Schwarzschild spacetime in Schwarzschild and in isotropic coordinates. All important features of Schwarzschild lensing are clearly explained in both papers. However, they do not derive anything like a lens equation.

The first version of a lens equation for spherically symmetric static spacetimes that goes beyond the weak-field small-angle approximation was brought forward by Virbhadra, Narasimha and Chitre\textsuperscript{8} and then, in a modified form, by Virbhadra and Ellis.\textsuperscript{9} The Virbhadra-Ellis lens equation was originally applied to the Schwarzschild
spacetime\textsuperscript{9} and later also to other spherically symmetric static spacetimes, e.g. to a boson star by Dąbrowski and Schunck,\textsuperscript{10} to a fermion star by Bilić, Nikolić and Viollier,\textsuperscript{11} and to spacetimes with naked singularities by Virbhadra and Ellis.\textsuperscript{12} The Virbhadra-Ellis lens equation might be called an “almost exact lens equation”. It is approximative insofar as it restricts to asymptotically flat spacetimes, with observer and light source in the asymptotic region and almost aligned (i.e., almost in opposite directions from the center of symmetry). However, the light rays are not restricted to the asymptotic region where the gravitational field is weak. They are rather allowed to enter the central region of the spacetime, on their way from the source to the observer, and to be arbitrarily strongly bent, i.e., to make arbitrarily many turns around the center.

An exact lens equation for spherically symmetric spacetimes, without any restriction on observer or source positions, was suggested by Perlick.\textsuperscript{13} (This can be viewed as a special case of Fritteli and Newman’s exact lens equation in arbitrary spacetimes,\textsuperscript{14} see Section 3.2.) To make use of the symmetry, one places observer and light source on spheres, rather than on planes as in the traditional lens equation, see Fig. 1. The observer is at $r = r_O$ and the light sources are distributed

![Fig. 1. Exact lens equation in spherically symmetric static spacetimes. The observer is at radius coordinate $r = r_O$, the light source at radius coordinate $r = r_S$. The lens equation relates the angle $\Theta$ which gives the image position on the observer’s sky to the angle $\Phi$ which gives the source position in the spacetime, see Ref. 13 for details.](image_url)
at $r = r_S$, where $r$ is the radial coordinate in the spherically symmetric spacetime under consideration. For each past-oriented light ray, leaving the observer at an angle $\Theta$ with respect to the radial outward direction, one can integrate the lightlike geodesic equation to get the azimuthal angle $\Phi$ swept out by the light ray before it reaches the sphere $r = r_S$. In this way one gets a lens equation which gives $\Phi$ as a function of $\Theta$ and, thereby, relates the source position in the spacetime to the image position at the observer’s sky. Of course, one has to take into account that a light ray with $\Phi + 2\pi$ arrives at the same source position as a light ray with $\Phi$. Also, one has to note that light rays may hit the sphere $r = r_S$ several times; if this is the case, the lens equation gives a multi-valued map $\Theta \mapsto \Phi$.

The lensing features are nicely illustrated if one plots $\Phi$ as a function of $\Theta$, as given by the lens equation. This is shown for the Schwarzschild spacetime in the top panel of Fig. 2. (Similar pictures for the spacetimes of an Ellis wormhole and of a Barriola-Vilenkin monopole can be found in Ref. 13.) The multiples of $\pi$ are indicated on the $\Phi$ axis; they correspond to source positions for which the observer sees an Einstein ring. If $\Theta$ approaches a certain value $\delta$, the angle $\Phi$ becomes infinite, corresponding to a light ray that circles around the center forever. This light ray asymptotically spirals towards the photon sphere at $r = 3m$.

A similar behaviour can be observed in any spherically symmetric static spacetime which is asymptotically flat and has an unstable photon sphere. (Here the sphere $r = r_L$ is called a photon sphere if a lightlike geodesic remains on this sphere if it starts tangentially to it. Thus, a photon sphere is filled with circular geodesics. A photon sphere is called stable if all lightlike geodesics with initial values close to those of a circular geodesic stay close to the photon sphere; otherwise it is called unstable. The surface $r = 3m$ in the Schwarzschild spacetime is the best known example of an unstable photon sphere.) For an observer situated between the photon sphere and infinity, the plot of $\Phi$ versus $\Theta$ always looks qualitatively as in Fig. 2. For a certain value of $\Theta$, the angle $\Phi$ diverges, corresponding to a light ray that spirals towards the photon sphere. The divergence is always logarithmic, as was shown by Bozza\textsuperscript{15} using the setting of the Virbhadra-Ellis lens equation. Thus, by logarithmically rescaling the $\Theta$ axis, the graph of $\Phi$ versus $\Theta$ asymptotically becomes a straight line. For the Schwarzschild spacetime this is shown in the middle panel of Fig. 2. Following Bozza, this straight line is called the strong field limit or the strong deflection limit. (The latter name is actually more appropriate; the bending angle goes to infinity but the gravitational field, measured in terms of the tidal force, need not be particularly strong near a photon sphere.) Calculating the strong deflection limit analytically requires some careful handling of integrals with singularities. The relevant formulas were worked out by Bozza.\textsuperscript{15}

The fact that $\Phi$ goes to infinity implies that the observer sees infinitely many images of each light source. This can be read from the bottom panel of Fig. 2: the dotted line, indicating a particular value of $\cos \Phi$, intersects the graph of $\cos \Phi$ over $\Theta$ infinitely many times. For light sources exactly on the axis through the observer position (i.e. $\cos \Phi = \pm 1$), there are infinitely many Einstein rings. For off-axis
Fig. 2. Plot of $\Phi$ versus $\Theta$ for the Schwarzschild spacetime.
light sources (i.e. all other values of Φ), there are two infinite sequences of isolated images. They correspond to light rays that make increasingly many turns around the center, in the positive \(\varphi\)-direction for the first sequence and in the negative \(\varphi\)-direction for the second sequence. Images corresponding to light rays that make at least one full turn around the center are called higher-order images. (The name relativistic images is also in use, but this name is a bit misleading because such images would occur in any theory, relativistic or not, in which photons are acted upon by gravity.) Higher-order images have not been observed until now, but there is some hope that they might be observed sometimes in the future. The most promising candidate for such observations is the black hole at the center of our galaxy, followed by the black hole at the center of M31. The perspectives for observing lensed images (secondary and higher order) of stars near these black holes were discussed at this conference in a talk by Mancini, cf. Refs. 16,17. Of course, even with the most advanced future technology it will never be possible to see all the infinitely many images because there will always be some limits on sensitivity and resolution power.

Note that, if \(\Phi\) is close to a multiple of \(2\pi\), the light returns approximately in the same direction from which it came, after having made one or more turns around the center. This phenomenon, sometimes called retrolensing, was discussed e.g. by Stuckey,\(^{18}\) by Holz and Wheeler\(^ {19}\) and by Eiroa and Torres.\(^ {20}\)

![Past light cone of an event \(p_O\) in the Schwarzschild spacetime. This is a (2+1)-dimensional spacetime diagram, restricting the spatial coordinates to an equatorial plane. Close to the vertex \(p_O\), the light cone looks like the light cone in Minkowski spacetime. Farther away from \(p_O\), the light cone wraps around the horizon and forms a caustic which, in this (2+1)-dimensional picture appears as a transverse self-intersection of the light cone. Actually, with the missing dimension taken into account, at each point of the caustic a circle’s worth of light rays intersect. If one follows the light cone further down into the past, an infinite sequence of caustics is encountered. The second caustic is on the same side of the center as \(p_O\), the third is opposite, and do on. (For a color version of this picture see Fig. 12 in Ref. 4.)](image-url)
If one calculates the strong-deflection limit for observer and light source in the asymptotic region, one always qualitatively gets the same picture as for Schwarzschild, see middle panel of Fig. 2. However, the pitch of the asymptotic straight line and its intersection point with the vertical axis are different for other spacetimes. Thus, the observation of higher-order images would allow to distinguish a Schwarzschild black hole from other black holes, see Bozza for a detailed discussion. For calculations of the strong deflection limit in various black hole spacetimes see e.g. Refs. 21–24

The strong deflection limit has also been investigated for wormhole spacetimes. The qualitative lensing features of wormholes are quite similar to black holes. The reason is that the spacetime of a wormhole, like that of a black hole, is asymptotically flat and has an unstable photon sphere. (The photon sphere is at the neck of the wormhole.) As a consequence, the lens map for a wormhole is qualitatively similar to that for a black hole if observer and light source are in the same asymptotically flat region, compare the Φ-versus-Θ plot for the Schwarzschild spacetime given in Fig. 2 with that for an Ellis wormhole given in Ref. 13.

Iyer and Petters have recently shown that the strong deflection limit in the Schwarzschild spacetime can be viewed as the leading order term in an infinite perturbation series. A similar perturbation series can be set up with the standard weak deflection limit as the leading order term. They are able to show that both series together provide an approximation that deviates by not more than 1% from the exact bending angle for every light ray between the light sphere and infinity.

A completely different approach to analytically evaluating the bending angle formula in spherically symmetric static spacetimes was brought forward by Amore et al. This approach, which makes no assumption on either strong or weak bending, converts the formula for the bending angle into a geometrically convergent series whose terms can be calculated analytically. Example calculations demonstrate that typically only a few terms are needed to get accurate results, even for cases where the bending angle is neither particularly small nor particularly large.

An important motivation for studying gravitational lensing in spherically symmetric static metrics is in the fact that this provides a means for testing alternative theories of gravity. A systematic theoretical framework for investigations of this kind was brought forward by Keeton and Petters on which Keeton reported at this conference, see Ref. 34. The basic idea is to write a spherically symmetric static metric as a Taylor series with respect to the gravitational radius of the central mass. The coefficients of this series are different for different gravitational theories. The linear term corresponds to the weak deflection limit; so far all lensing observations can be satisfactorily explained if only this term is taken into account. However, as outlined in the quoted papers, near-future technology will lead to an accuracy in observation that requires taking second and higher order terms into account. This will give us a new test of Einstein’s theory and of alternative theories of gravity.

Among alternative theories of gravity, braneworld scenarios are particularly fashionable. These scenarios provide spherically symmetric static black hole solu-
tions for which lensing can be studied with the analytical methods discussed above. At this conference Whisker (see Refs. 35,36) and Majumdar (see Refs. 37,38) reported on their recent investigations comparing lensing by braneworld black holes to Schwarzschild lensing.

![Wave fronts in the Schwarzschild spacetime.](image)

Fig. 4. Wave fronts in the Schwarzschild spacetime. If one intersects the light cone from Fig. 3 with a hypersurface $t = \text{constant}$ one gets a wave front. The picture shows such wave fronts, for four different values of $t$, projected to three-space. In contrast to Fig. 3 all three spatial dimensions can be represented in the picture. If one intersects the light cone close to the vertex, the resulting wave front is a slightly deformed sphere. This sphere becomes more and more deformed if the light rays come closer to the horizon. After the caustic has formed, the wave fronts are no longer topological spheres. (For a color version of this picture see Fig. 13 in Ref. 4.)

2.2. Kerr Spacetime

There is strong evidence for a supermassive black hole at the center of our galaxy, and some observations indicate that its spin is non-negligible. A non-vanishing spin modifies the lensing features of a black hole in a characteristic way. Some spin-dependent observable effects were discussed by Zakharov in his talk at this conference, see Ref. 39.

The appropriate mathematical model for describing the spacetime around a spinning black hole is the Kerr metric. Its lightlike (and timelike) geodesics have been studied in great detail, see e.g. Chandrasekhar\cite{Chandrasekhar} or O’Neill.\cite{O’Neill} In the Kerr spacetime the geodesic equation is completely integrable – the solutions can be written in terms of elliptic integrals – but the analysis of the resulting light paths turns out to be rather involved.
The Kerr metric depends on two parameters, the mass $m$ and the specific angular momentum $a$. In standard Boyer-Lindquist coordinates, the metric reads

$$g = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \vartheta \, d\varphi)^2 + \frac{\sin^2 \vartheta}{\rho^2} \left((r^2 + a^2) \, dt - a \, dr\right)^2 + \frac{\rho^2}{\Delta} \, dr^2 + \rho^2 \, d\vartheta^2,$$  

(1)

where $\rho$ and $\Delta$ are defined by

$$\rho^2 = r^2 + a^2 \cos^2 \vartheta \quad \text{and} \quad \Delta = r^2 - 2mr + a^2.$$  

(2)

For $a = 0$ one recovers the Schwarzschild metric. In the following we restrict to the case $a^2 \leq m^2$ which describes a black hole, and we ignore the case $m^2 < a^2$ which describes a naked singularity.

![Fig. 5. The photon region $K$ in the Kerr metric. The picture shows a meridional section, i.e., a half plane $\varphi = \text{constant}, \ t = \text{constant},$ of the Kerr metric for $a = 0.5 \ m$. The outer horizon is at $r_+$, so the domain of outer communication is the region outside of the black ball. The photon region $K$ meets the equatorial plane in two circular lightlike geodesics whose radii $r_{ph}$ and $r_{ph}$ are given by the equation

$$(r_{ph}^2) - 3m r_{ph}^2 + 2a^2 = \pm 2a \sqrt{(r_{ph}^2)^2 - 2 m r_{ph}^2 + a^2},$$

and the axis at radius $r_c$ given by $r_c^3 - 3 r_c^2 m + r_c (a^2 + q^2) + a^2 m = 0.$

We denote the region in the Kerr spacetime where

$$-2 a \ r \sqrt{\Delta} \sin \vartheta \leq 2 r \Delta - (r - m) \rho^2 \leq 2 a \ r \sqrt{\Delta} \sin \vartheta$$  

(3)

by $K$ and we call this the photon region, see Fig. 5. This region is characterized by the fact that it is filled with spherical lightlike geodesics, i.e., lightlike geodesics that...
stay on spheres \( r = \text{constant} \). Along each such geodesic the \( \vartheta \) coordinate oscillates between extremal values \(-\vartheta_0\) and \( \vartheta_0\), whereas the \( \varphi \) and \( t \) coordinates increase or decrease linearly with the affine parameter. In the Schwarzschild limit \( a \to 0 \) the photon region \( K \) shrinks to the photon sphere \( r = 3m \).

In the Schwarzschild spacetime light rays that come close to the photon sphere \( r = 3m \) can make arbitrarily many turns around the center. Similarly, in the Kerr spacetime light rays that come arbitrarily close to the photon region \( K \) can make arbitrarily many turns around the center. Thus, one can define a strong deflection limit in the same sense as for spherically symmetric static spacetimes. Light rays that start tangential to the equatorial plane \( \vartheta = \pi/2 \) remain in that plane. Their analysis is, therefore, not much more difficult than for the spherically symmetric static case. One gets infinitely many images on either side of the center, but in contrast to the spherically symmetric static case the positions of the images are not mirror symmetric. This asymmetry can be interpreted as saying that the light rays are “dragged” along with the rotation of the black hole. The analytic formulas for the strong deflection limit in the equatorial plane of the Kerr metric have been worked out by Bozza.

Away from the equatorial plane, the analysis is much more difficult because then a light ray is not confined to a plane. The strong deflection limit was calculated by Bozza, de Luca, Scarpetta and Sereno for the case that the observer is in the equatorial plane (but the light source need not) and by Bozza, de Luca and Scarpetta for the general case. These results were presented at this conference in a talk by de Luca, cf. Ref. 45. (For another work of Kerr lensing without weak-field assumption, largely based on numerical studies, see Vázquez and Esteban.) The investigations of the strong deflection limit complements ongoing work on the weak deflection limit in the Kerr metric which was presented by Sereno at this conference, cf. Refs. 47,48.

An important aspect of the above-mentioned work on the strong deflection limit in the Kerr metric is in the fact that it analytically corroborates knowledge about the caustics in this spacetime which, in earlier work, has been investigated numerically. If one considers the past light cone of an event \( p_O \) in the Kerr spacetime, one finds that it forms infinitely many caustics. By definition, a caustic is a connected subset of the light cone where neighboring light rays from \( p_O \) intersect each other, at least to first order. At caustic points, the light cone fails to be an immersed submanifold of the spacetime, forming e.g. cuspidal edges. In the Kerr spacetime the caustics, if projected to three-dimensional space, are tubes with astroid cross sections. A picture can be found in a paper by Blandford and Rauch who determined the caustics numerically. In the Schwarzschild case, \( a = 0 \), the caustics are just radial lines in three-space. With increasing \( a \) they open out into tubes with astroid cross-sections, and they start winding around the center. In the limit of an extreme Kerr black hole, \( a = m \), they spiral asymptotically towards the horizon. The analysis of the strong deflection limit gives analytic support to these results which have been found from numerical studies. However, what is still missing is a good picture that
shows how the caustics are situated inside the light cone. Such pictures can easily be produced for the Schwarzschild case in two different ways: First, one can depict the past light cone of an event in the spacetime, restricting oneself to the equatorial plane, see Fig. 3. Second, one can depict in three-space a series of intersections of the light cone with hypersurfaces $t = \text{constant}$ (wavefronts), see Fig. 4. It would be desirable to have similar pictures for the Kerr metric. Some attempts in this direction have been made, beginning with a 1977 paper by Hanni. At this conference Grave presented computer visualizations for lensing, including a movie that shows the propagation of wave fronts in the Kerr spacetime. A picture of wave fronts in the Kerr spacetime, analogous to the Schwarzschild wave fronts shown in Fig. 4, can be found in Ref. 51. This picture nicely shows how the twisting effect, governed by the rotation of the Kerr black hole, produces an asymmetry in the wave fronts. However, the resolution of the picture is too low to show the astroid structure of the caustic. Also, it is hard to read from this picture the structure of the caustic near the horizon. Therefore it seems fair to say that until now the structure of the caustics in the Kerr spacetime has not been clearly brought out in visualizations.

2.3. Plane Gravitational Waves

Standard textbooks on general relativity usually treat gravitational waves as small perturbations of Minkowski spacetime. In this weak-field approximation, it is easy to study the effect of gravitational waves on light rays. However, the resulting picture is rather incomplete. Several interesting qualitative features are brought out only if one goes beyond the weak-field approximation. This can be done by studying plane gravitational waves in terms of exact solutions to Einstein’s field equation. Because of their high symmetry, plane gravitational wave solutions are over-idealized in view of applications; however, they are highly instructive in view of understanding the focusing properties of gravitational waves.

A plane gravitational wave is a spacetime with metric

$$g = -2 \, du \, dv - (f(u) \, (x^2 - y^2) + 2 \, g(u) \, x \, y) \, du^2 + dx^2 + dy^2$$

(4)

where $f(u)^2 + g(u)^2$ is not identically zero. For any choice of $f(u)$ and $g(u)$, the metric (4) has vanishing Ricci tensor, i.e., Einstein’s vacuum field equation is satisfied. With the four coordinates $u, v, x, y$ ranging over all of $\mathbb{R}^4$, the spacetime is geodesically complete. The spacelike coordinates $x$ and $y$ are transverse to the propagation direction of the gravitational wave.

For a metric of the form (4), the geodesic equation is completely integrable, i.e., the light cone structure can be studied explicitly without approximation. It reveals an interesting focusing property of gravitational waves that is not captured if one restricts to weak fields. This focusing property was discussed in great detail by Ehrlich and Ench, cf. Beem, Ehrlich and Easley, Chapter 13. What one finds is the following. Of all the past-oriented lightlike geodesics issuing from an event $p_O$, one goes straight to infinity whereas all the other ones are being refocused in a
Fig. 6. Past light cone of an event \( p_0 \) in the spacetime of a plane gravitational wave. With the exception of one lightlike geodesic \( \lambda \), all the past-oriented lightlike geodesics from \( p_0 \) are being refocused in a spacelike curve \( q \). The picture was produced with profile functions \( f(u) > 0 \) and \( g(u) = 0 \). Then there is focusing in the \( x \)-direction and defocusing in the \( y \)-direction. In the \((2+1)\)-dimensional picture, with the \( y \)-coordinate not shown, the curve \( q \) is represented by a point. – A color version of this picture can be found as Fig. 29 in Ref. 4. For a similar picture, hand-drawn by Roger Penrose, see Ref. 55.

A spacelike curve \( q \) which is completely contained in a hypersurface \( u = \text{constant} \), see Fig. 6. The curve \( q \) is the first caustic of the past light cone of \( p_0 \). This focusing property is universal, i.e., independent of the profile functions \( f(u) \) and \( g(u) \). It is closely related to the fact that a plane gravitational wave spacetime cannot be globally hyperbolic. The latter fact was discovered by Penrose who studied the focusing property for the special case that \( f(u) \) and \( g(u) \) are different from zero only in a sufficiently small interval \( u_1 < u < u_2 \). For this special choice of profile functions there is no second caustic, whereas for other choices of \( f(u) \) and \( g(u) \) the past-oriented lightlike geodesics issuing from \( p_0 \) may be refocused another time after passing through \( q \). It is geometrically evident from Fig. 6 that an inextendible timelike curve \( \gamma_S \) which is sufficiently close to \( q \) intersects the past light cone of \( p_0 \) exactly twice. Thus, an observer at \( p_0 \) sees exactly two images of a light source with worldline \( \gamma_S \). This example demonstrates that, in contrast to the weak-field theory, a transparent gravitational lens need not produce an odd number of images, even in the case of a geodesically complete spacetime with trivial topology.

3. Qualitative Methods

3.1. Morse Theory

Morse theory is a body of mathematical theorems that give some information on the number of solutions to a variational problem. In applications to lensing, the
variational problem is a version of Fermat’s principle and the solutions are the light rays from the source to the observer. Every such light ray corresponds to an image of the source on the observer’s sky. Morse theory can be used to determine whether the number of images is infinite or finite, even or odd. Results of this kind are useful because, if there is an observation which contradicts them, one knows that one has to search for an additional image which has gone unnoticed so far. This may have happened because an image was too faint or too close to another image for being detected.

Morse theory can be applied to gravitational lensing either in the weak-field small-angle approximation, or in a spacetime setting without such approximations. In the following we will be concerned only with the latter, for the former see Petters \cite{56} or the respective chapter in Petters, Levine and Wambsganss. We will concentrate on a version of Morse theory that was brought forward by Uhlenbeck \cite{57} and applied to gravitational lensing first by McKenzie. \cite{58} This version is restricted to globally hyperbolic spacetimes throughout. There have been several attempts to formulate a Morse theory in more general spacetimes, starting out from Kovner’s version \cite{59} of Fermat’s principle, see Fig. 7. Although this variational principle can be formulated in an arbitrary spacetime, a fully-fledged Morse theory requires additional assumptions which come close to global hyperbolicity. For the most general version available, which applies to some subsets-with-boundary of a stably causal spacetime, see Giannoni, Masiello and Piccione. \cite{60}

Fig. 7. Kovner’s version of Fermat’s principle. In an arbitrary general-relativistic spacetime, fix an event $p_O$ and a timelike curve $\gamma_S$. Consider, as the trial curves, all past-oriented lightlike curves from $p_O$ to $\gamma_S$. The solutions to the variational problem are those trial curves for which the arrival time is stationary (i.e., a minimum, a maximum or a saddle). Here “arrival time” refers to a parametrization of $\gamma_S$. This variational problem was brought forward by Kovner. \cite{59} A proof that the solution curves are, indeed, precisely the lightlike geodesics from $p_O$ to $\gamma_S$ was given by Perlick. \cite{61} A local version of this variational principle, restricted to convex normal neighborhoods, can be found already in a 1938 paper by Temple. \cite{62}
Following Uhlenbeck\textsuperscript{57} we consider a 4-dimensional Lorentzian manifold \((M, g)\) that admits a foliation into smooth Cauchy surfaces, i.e., a globally hyperbolic spacetime. (The fact that the original definition of global hyperbolicity is equivalent to the existence of a foliation into \textit{smooth} Cauchy surfaces was completely proven only recently by Bernal and Sánchez.\textsuperscript{63}) Then \(M\) can be written as a product of a three-dimensional manifold \(S\), which serves as the prototype for each Cauchy surface, and a time-axis,

\[
M = S \times \mathbb{R}.
\]  

(5)

In this spacetime we fix an event \(p_O\) and a timelike curve \(\gamma_S\), and we make the following additional assumptions. (We are interested in past-oriented lightlike geodesics from \(p_O\), whereas Uhlenbeck considered future-oriented lightlike geodesics. Therefore, our assumptions are the time-reversals of her’s.)

(a) With respect to the splitting (5), the spacetime has no particle horizons. (An analytical criterion for this to be true is the time-reversed version of Uhlenbeck’s “metric growth condition”.)

(b) The timelike curve \(\gamma_S\) is inextendible and it does not meet the caustic of the past light cone of \(p_O\). (This excludes situations where the observer sees an extended image, such as an Einstein ring.)

(c) There is a sequence \(t_n\) with \(t_n \to -\infty\) for \(n \to \infty\) such that the projection to \(S\) of \(\gamma_S(t_n)\) has an accumulation point. (Roughly speaking, this condition prohibits that \(\gamma_S\) “goes to infinity” in the past.)

Then the Morse inequalities

\[
N_k \geq B_k \quad \text{for all} \quad k \in \mathbb{N}_0
\]  

(6)

and the Morse relation

\[
\sum_{k=0}^{\infty} (-1)^k N_k = \sum_{k=0}^{\infty} (-1)^k B_k
\]  

(7)

hold true, where \(N_k\) denotes the number of past-pointing lightlike geodesics with index \(k\) from \(p_O\) to \(\gamma_S\), and \(B_k\) denotes the \(k\)-th Betti number of the loop space of \(M\). (The index of a geodesic is the number of its conjugate points, counted with multiplicity. The loop space of a connected topological space is the space of all continuous curves connecting two fixed points. For the definition of Betti numbers see, e.g., Frankel.\textsuperscript{64})

The sum on the right-hand side of (7) is, by definition, the Euler characteristic \(\chi\) of the loop space of \(M\). Hence, (7) can also be written in the form

\[
N_+ - N_- = \chi,
\]  

(8)

where \(N_+\) (respectively \(N_-\)) denotes the number of past-pointing lightlike geodesics with even (respectively odd) index from \(p_O\) to \(\gamma_S\).

The Betti numbers of the loop space of \(M = S \times \mathbb{R}\) are, of course, determined by the topology of \(S\). Three cases are to be distinguished.
Case A: $M$ is not simply connected. Then the loop space of $M$ has infinitely many connected components, so $B_0 = \infty$. In this situation (6) says that $N_0 = \infty$, i.e., that there are infinitely many past-pointing lightlike geodesics from $p$ to $\gamma$ that are free of conjugate points.

Case B: $M$ is simply connected but not contractible to a point. Then for all but finitely many $k \in \mathbb{N}_0$ we have $B_k > 0$. This was proven in a classical paper by Serre,\textsuperscript{65} cf. McKenzie.\textsuperscript{58} In this situation (6) implies $N_k > 0$ for all but finitely many $k$. In other words, for almost every positive integer $k$ we can find a past-pointing lightlike geodesic from $p_0$ to $\gamma_S$ with exactly $k$ conjugate points. Hence, there must be infinitely many past-pointing lightlike geodesics from $p_0$ to $\gamma_S$.

Case C: $M$ is contractible to a point. Then the loop space of $M$ is contractible to a point, i.e., $B_0 = 1$ and $B_k = 0$ for $k > 0$. In this case (6) takes the form $N_+ - N_- = 1$ which implies that the total number $N_+ + N_- = 2N_- + 1$ of past-pointing lightlike geodesics from $p_0$ to $\gamma_S$ is (infinite or) odd.

Case C has relevance for asymptotically simple and empty spacetimes. These spacetimes describe the gravitational field around a transparent isolated body, see Hawking and Ellis\textsuperscript{66} for the formal definition. It is well known that asymptotically simple and empty spacetimes are globally hyperbolic, with contractible Cauchy surface, and it is not difficult to verify that they have no particle horizons. Also, it follows from the definition of an asymptotically simple and empty spacetime that $N_+ + N_-$ must be finite. Hence, if $\gamma_S$ satisfies conditions (b) and (c) above, Case C applies and says that the total number of images must be odd. This version of the “odd number theorem” makes no assumption on the weakness of the gravitational field. (It does not apply to the example of a plane gravitational wave, treated in Section 2.3 because the latter is not asymptotically simple.) For further discussions of the odd number theorem and Morse theory see McKenzie.\textsuperscript{58}

Case B has relevance for black hole spacetimes. In particular, it can be used to give a general proof that a Kerr-Newman black hole produces infinitely many images, with only very mild restrictions on the allowed motion of the light source. Here the globally hyperbolic spacetime to be considered is the domain of outer communication of the black hole, i.e., the region outside of the outer horizon. Its Cauchy surfaces have topology $S^2 \times \mathbb{R}$ which is, indeed, simply connected but not contractible. The details are worked out in Ref. 67.

3.2. Differential Topology

The idea of formulating a lens map in an arbitrary general-relativistic spacetime, without weak-field or small-angle approximations, is due to Frittelli and Newman\textsuperscript{14} and was further developed by Ehlers, Frittelli and Newman.\textsuperscript{68,69} In this section we briefly review the construction of their lens map and we discuss how methods from differential topology can be applied to it.

In the standard weak-field small-angle formalism for thin lenses the lens map, resulting from the lens equation, is a map from a deflector plane to a source plane,
To define something analogous in an arbitrary spacetime, without approximations, we fix an observation event $p_O$ and we look for analogues of the deflector plane and the source plane. As to the deflector plane, there is an obvious candidate, namely the *celestial sphere* $S_O$ at $p_O$. This can be defined as the set of all lightlike directions at $p_O$. As to the source plane, however, there is no natural candidate.

Following Frittelli, Newman and Ehlers, we choose a timelike 3-dimensional submanifold $T$ of the spacetime manifold. We assume that $T$ is of the form $T = N \times \mathbb{R}$ where the $\mathbb{R}$-lines are timelike and can be interpreted as the worldlines of light sources. The two-dimensional manifold $N$ is the analogue of the source plane. In this situation the lens map $f : S_O \rightarrow N$ is defined by associating each lightlike direction $Y$ at $p_O$ with the geodesic to which it is tangent, extending this lightlike geodesic into the past until it meets $T$ and then projecting onto $N$, see Fig. 8. In general, this prescription does not give a well-defined map $f$ since neither existence nor uniqueness of the target value is guaranteed. As to existence, there might be some past-pointing lightlike geodesics from $p_O$ that never reach $T$. As to uniqueness, one and the same lightlike geodesic might intersect $T$ several times.

![Fig. 8. Illustration of the lens map $f$](image)

However, there are some situations of physical interest where the lens map is well-defined. An important example is the lens map in asymptotically simple and empty spacetimes. This class of spacetimes was mentioned already in Section 3.1 and it was indicated how Morse theory can be used to prove that, in such a spacetime, the number of images is odd under very general conditions. The same result can also be found with the help of the Frittelli-Newman lens map. The proof, which is worked
out in Ref. 70, proceeds in two steps. First one considers “light sources at infinity”; in this case the lens map is a map from a two-sphere to a two-sphere, and it can be shown that its mapping degree (Brouwer degree) is equal to ±1. In the second step it is proven that this result implies an odd number of images for very general light sources in the asymptotically simple and empty spacetime. Ref. 70 contains a number of other applications of differential topology to the lens map.

Clearly, the “exact lens equation” for spherically symmetric static spacetimes, discussed in Section 2.1, defines a lens map which is a special case of the Frittelli-Newman lens map.

### 3.3. Bifurcation Theory

Bifurcation theory is concerned with variational problems that depend on a real parameter. The goal is to characterize the situation that, in dependence on the parameter, a solution to the variational problem (i.e., a stationary point of the variation functional) bifurcates into two or more solutions. Recently bifurcation theory has been applied to Kovner’s version of Fermat’s principle, recall Fig. 7, by Giambó, Giannoni and Piccione \(^{71}\) and by Javaloyes and Piccione. \(^{72}\) To discuss their approach, we fix a timelike curve \(γ_S\) and a past-oriented lightlike geodesic \(λ\) such that \(λ(0)\) is on \(γ_S\), see Fig. 9. Now for every \(s < 0\) we can consider Fermat’s principle with the given worldline \(γ_S\) and the observation event \(p_O = λ(s)\). This gives us a one-parameter family of variational problems, depending on the parameter \(s\). For any value of \(s\), a section of \(λ\) is a solution curve of the variational problem. With the help of bifurcation theory one can characterize those values \(s_0\) of \(s\) where this solution bifurcates into two or more solutions. It is easy to see that a bifurcation can occur only at points \(λ(s_0)\) which are conjugate to \(λ(0)\). The latter condition means that \(λ(0)\) is in the caustic of the past light cone of the event \(p_O = λ(s_0)\).

Bifurcation theory gives us some information on conjugate points that is difficult to get with other methods. In the following we present one of the results that is proven in the quoted papers.

As a preparation, we recall some facts which can be verified with elementary means. Let \(λ\) be a past-oriented lightlike geodesic and assume that, for some \(s_0 < 0\), the point \(λ(s_0)\) is conjugate to \(λ(0)\), see again Fig. 9. Then it is an elementary exercise to verify that, for every \(ε > 0\), the geodesic from \(λ(s_0 − ε)\) to \(λ(0)\) contains a point conjugate to \(λ(s_0 − ε)\). Hence a known result (Theorem 2 in Ref. 73) implies that there is a timelike curve \(γ_S\) through \(λ(0)\) that can be connected to \(λ(s_0 − ε)\) by a second lightlike geodesic \(λ_ε\).

With the help of bifurcation theory, this statement can be strengthened in the following way, see Proposition 13 in Ref. 72. For \(ε\) sufficiently small,

- the statement is true for all timelike curves \(γ_S\) through \(λ(0)\);
- \(λ_ε\) depends continuously on \(ε\) (i.e., the geodesics \(λ_ε\) can be parametrized such that the map \((s, ε) \mapsto λ_ε(s)\) is differentiable with respect to \(s\) and continuous with respect to \(ε\)).
Thus, an observer at $\lambda(s_0 - \varepsilon)$ sees (at least) two images of the light source $\gamma_S$, one along the lightlike geodesic $\lambda$ and one along the lightlike geodesic $\lambda_\varepsilon$. For $\varepsilon \to 0$ these two images move continuously towards each other until they merge for $\varepsilon = 0$. This property is universal, i.e., it holds for all conjugate points.

![Diagram](image)

Fig. 9. If $\lambda(s_0)$ is conjugate to $\lambda(0)$, events beyond $\lambda(s_0)$ can be connected to $\gamma_S$ by a second lightlike geodesic.

4. Concluding Remarks

With an increase in observational accuracy, theoretical gravitational lensing beyond the weak-field and small-angle approximation becomes more and more relevant in view of applications. At this conference, it was already the majority of contributions to the Theoretical Gravitational Lensing workshop that went beyond these approximations. Of course, studying lensing in weak gravitational fields is of remaining value. At this conference this was illustrated, in addition to the talks already mentioned, by Yoo’s talk on lensing in a clumpy universe, by Miranda’s talk on lensing by galactic halo structures and by Frutos-Alfaro’s talk on computer programs for visualizing and modeling gravitational lenses based on the traditional lens equation. Quite generally, weak-field studies have a wide range of applications and they complement calculations in arbitrarily strong gravitational fields in a useful way. From a methodological point of view, it is desirable to develop the general formalism in a spacetime setting without approximations, as far as possible, and to introduce approximations only afterwards for those applications for which they are necessary.
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