CLASSES OF SMOOTH SOLUTIONS TO THE
MULTIDIMENSIONAL BALANCE LAWS OF GAS DYNAMIC
TYPE ON THE RIEMANNIAN MANIFOLDS

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Abstract. The paper is devoted to the special classes of solutions to
multidimensional balance laws of gas dynamic type. In the velocity field
for the solutions of such class the time and space variables are separated.
The simplest case is the solution with the linear profile of velocity in the
Euclidean space.

1. Balance laws on the riemannian manifold

1.1. Preliminaries. Let Σ be a smooth n-dimensional riemannian
 manifold with local curvilinear orthogonal coordinates \( x^i, i = 1, \ldots, n \). Consider the
following system of nonlinear partial differential equations

\[
(R^i(U, x, t))_t + \nabla_j q^{ij}(U, x, t) + Q^i(U, x, t) = 0, \quad i, j = 1, \ldots, n. \tag{1.1}
\]

Here \( U = (u^1, \ldots, u^n) \), \( R = (R^1, \ldots, R^n) \) and \( Q = (Q^1, \ldots, Q^n) \) are the
vector-functions on the tangent bundle of \( \Sigma \) (\( u \) is unknown), \( q^{ij} \) is a smooth
tensor field, \( \nabla_j q^{ij} \) is a divergency of tensor. All differential operations with
respect to space variables are performed in this coordinate system by means
of smooth metric tensor \( g_{ij} \) of the manifold \( \Sigma \), the variable \( t \) denotes the time,
as usual. Recall that the covariant derivative of any contravariant vector
\( X(X^1, \ldots, X^n) \) is the tensor \( \nabla_j X^i = \frac{\partial X^i}{\partial x^j} + \Gamma^i_{jk} X^k, \ i, j, k = 1, \ldots, n \), where
\( \Gamma^i_{jk} \) are the Christoffel symbols.

If the tensor field \( q^{ij} \) is sufficiently smooth, then the system (1.1) can be
re-written in the form

\[
G^i_j(U, x, t)(u^i)_t + F^{ik}_j(U, x, t)\nabla_k u^j + Q^i(U, x, t) = 0, \quad i = 1, \ldots, n, \tag{1.2}
\]

with new tensor fields \( G^i_j \) and \( F^{ik}_j \), or

\[
A^0(U, x, t)\frac{\partial U}{\partial t} + \sum_{k=1}^n A^k(U, x, t)\frac{\partial U}{\partial x^k} + \bar{Q}(U, x, t) = 0, \tag{1.3}
\]

where \( A^0 = G^i_j, A^k = F^{ik}_j, \bar{Q}^i = Q^i + F^{ik}_m \Gamma^m_{kj} u^j \).

Remark that the system (1.2) includes only covariant derivatives, which
coincide with the usual partial ones only in the case of the euclidean space.

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Nevertheless, the representation of the covariant derivatives through the partial ones and the Christoffel symbols \[2\] adds the derivatives free terms only.

We restrict ourself by the systems of gas dynamic type, namely, by the system of the Euler equation with the right hand sides of special form.

The important fact is the possibility to reduce these balance laws to the symmetric hyperbolic form.

Recall that the system of form (1.3) for the unknown function \(U(t, x)\) is called symmetric hyperbolic if the matrices \(A^j(U, x, t)\), \(j = 0, ..., n\), are symmetric and the matrix \(A^0(U, x, t)\) is positively defined in addition. The following classical result is well known for the euclidean space \(\mathbb{R}^n\) (\[3\], \[4\], particular cases in \[5\], \[6\], \[7\], \[8\]): if the matrices \(A^j(p, x, t)\) and the right hand side \(\bar{Q}(p, x, t)\) smoothly depend on their arguments, have the continuous bounded derivatives up to order \(m+1\) with respect to the variables \((p, x)\) under bounded \(p\), and the initial data \(U(0, x)\) and the function \(\bar{Q}(0, x, t)\) belong to the Sobolev class \(H^m(\mathbb{R}^n)\), then the corresponding Cauchy problem has locally in time unique solution in the class \(\bigcap_{j=0}^m C^j([0, T); H^{m-j}(\mathbb{R}^n))\). Moreover,

\[
\lim_{t \to T-0} \sup \|U\|_{L^\infty} + \|\nabla_x U\|_{L^\infty} = +\infty.
\]

(Remark that on the initial data less strict requirements may be imposed, for example, the requirement of belonging uniformly local in space to \(H^m(\mathbb{R}^n)\). \[19\], \[4\].)

As usual in the problem arising from physics the coefficients of system (1.3) depend only on the solution \(U\). The fact simplifies significantly the formulation of the result. Namely, for the local in time existence of the Cauchy problem in the described class it is sufficiently to require the implementation of the condition \(\bar{Q}(0) = 0\) besides of the smoothness of coefficients.

If the manifold \(\Sigma\) is diffeomorphic to \(\mathbb{R}^n\), then the theorem on the existence of smooth local in time solution can be transmitted to the case.

The author don’t know the analogous result for the case of an arbitrary riemannian manifold.

1.2. The system of the Euler equations. Further we shall consider the following system given on the riemannian manifold \(\Sigma\) in \(n\) dimensions with the metric tensor \(g_{ij}\) (below we denote it (E1–E3))

\[
\rho(\partial_t \mathbf{V} + (\mathbf{V}, \nabla) \mathbf{V}) = -\nabla p + \rho \mathbf{F}(\rho, p, \mathbf{V}, x, t), \tag{1.2.1}(E1)
\]

\[
\partial_t \rho + \text{div} (\rho \mathbf{V}) = 0 \tag{1.2.2}(E2)
\]

\[
\partial_t p + (\mathbf{V}, \nabla p) + \gamma p \text{div} \mathbf{V} = 0. \tag{1.2.3}(E3)
\]

Here \(\rho, p, \mathbf{V}\) are the density, the pressure and the velocity vector, respectively. The constant \(\gamma\) is the heat ratio, \(\gamma > 1\), moreover, in the physical case \(\gamma \leq 1 + \frac{2}{n}\).

The right-hand side \(\rho \mathbf{F}\) is the forcing term, it is supposed to be smooth with respect to all arguments.
For example, it may describe the friction if \( F = -\mu V|V| \) with the non-negative function \( \mu(t, x) \) and the constant \( \sigma \geq 0 \), or the Coriolis term. In the last case \( F = [V \times \omega] \), \( \omega \) is a constant vector, corresponding to the vector of angular velocity (for \( n = 3 \)) or \( F(V) = lV_\perp \), \( l = l(x, t) \) is the Coriolis parameter, \( V_\perp = e_{ij}V^j \), \( e_{ij} \) is the skew-symmetric discriminant tensor of the surfaces (for \( n = 2 \)).

2. The Euclidean space. Solutions with the linear profile of velocity and their properties

We consider the classical solutions to (E1–E3) with the density sufficiently quickly decreasing as \( |x| \to \infty \) to guarantee a convergency of the integral \( \int_{\mathbb{R}^n} \rho|x|^2 dx \) (so called solutions with a finite moment).

As well known, solutions to the Cauchy problem for system (E1–E3) may lose the initial smoothness for a finite time. Sometimes there is a possibility to estimate the time of a singularity formation from above (see, e.g., [10] and references therein). Moreover, in the case \( F = 0 \) the singularity appears in any solution with compactly supported initial data (f.e., [11]).

At the same time, there exist some nontrivial classes of globally smooth solutions. In the section our special interest will be the solutions with linear profile of velocity

\[
V = A(t)r + b(t),
\]

where \( A(t) \) and \( b(t) \) are a matrix \( (n \times n) \) and an \( n \)-vector, dependent on time, \( r \) is a radius-vector of the point.

2.1. Construction of solutions with linear profile of velocity and integral functionals.

2.1.1. The system on the plane \((n = 2)\). Now we construct solutions with linear profile of velocity for the important case \( F = LV \), with matrix \( L = \begin{pmatrix} -\mu & -l \\ l & -\mu \end{pmatrix} \), \( \mu = \text{const} \geq 0 \), \( l = \text{const} \). This right-hand side describes in a simplest way the Coriolis force and the Rayleigh friction in the meteorological model neglecting the vertical processes.

We consider only smooth solutions to (E1–E3) with the density (and pressure) vanishing at \( |x| \to \infty \) rather quickly to guarantee the convergency of all integrals involved (whereas the velocity components may even grow). For the solution the total mass

\[
\mathcal{M} = \int_{\mathbb{R}^2} \rho dx
\]

is conserved, the total energy

\[
\mathcal{E}(t) = \int_{\mathbb{R}^2} \left( \frac{\rho|V|^2}{2} + \frac{p}{\gamma - 1} \right) dx = \mathcal{E}_k(t) + \mathcal{E}_p(t)
\]
is nonincreasing function. Let us involve integral functionals

\[ G(t) = \frac{1}{2} \int_{\mathbb{R}^2} \rho |r|^2 \, dx, \]

\[ N_i(t) = \int_{\mathbb{R}^2} X_{1i} \rho \, dx, \quad i = 1, 2, \]

\[ I_i(t) = \int_{\mathbb{R}^2} V_i \rho \, dx, \quad i = 1, 2, \]

\[ F_i(t) = \int_{\mathbb{R}^2} (\mathbf{V}, X_i) \rho \, dx, \quad i = 1, 2, \]

where \( \mathbf{X}_1 = \mathbf{r} = (x, y), \mathbf{X}_2 = \mathbf{r}_\perp = (y, -x) \). Note that \( G(t) > 0 \).

For the classical (and for the piecewisely smooth) solutions to (E1 – E3), the following relations hold [12] (see also [13]):

\[ G'(t) = F_1(t), \quad (2.1.1) \]

\[ N_i'(t) = I_i(t), \quad i = 1, 2, \quad (2.1.2) \]

\[ I_1'(t) = -lI_2(t) - \mu I_2(t), \quad (2.1.3) \]

\[ I_2'(t) = lI_1(t) - \mu I_2(t), \quad (2.1.4) \]

\[ F_1'(t) = 2(\gamma - 1)\mathcal{E}_p(t) + 2\mathcal{E}_k(t) - lF_2(t) - \mu F_1(t), \quad (2.1.5) \]

\[ F_2'(t) = lF_1(t) - \mu F_2(t), \quad (2.1.6) \]

\[ \mathcal{E}'(t) = -2\mu \mathcal{E}_k(t). \quad (2.1.7) \]

The result can be obtained by means of the Green’ formula.

Generally speaking, the functions \( \mathcal{E}_p(t) \) and \( \mathcal{E}_k(t) \) cannot be expressed through \( G(t), F_1(t), F_2(t), I_1(t), I_2(t), N_1(t), N_2(t) \). But if we choose the velocity field with the linear profile \( \mathbf{V} = A(t) \mathbf{r} + \mathbf{b}(t) \), we obtain the closed system of ODE for finding the coefficients of the matrix \( A(t) \) and of the components of vector \( \mathbf{b}(t) \). In general case it is rather complicated and can be solved only numerically.

More simple result can be obtained if

\[ \mathbf{V} = \alpha(t) \mathbf{r} + \beta(t) \mathbf{r}_\perp + \mathbf{b}(t) = \]

\[ = \left( \alpha(t) I + \beta(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \mathbf{r} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}, \quad (2.1.8) \]

with the identity matrix \( I \). It is easy to see that in this case

\[ I_1(t) = \alpha(t) N_1(t) + \beta(t) N_2(t) + b_1(t) \mathcal{M}, \]

\[ I_2(t) = \alpha(t) N_2(t) - \beta(t) N_1(t) + b_2(t) \mathcal{M}, \]

\[ F_1(t) = 2\alpha(t) G(t) + b_1(t) N_1(t) + b_2(t) N_2(t), \]

\[ F_2(t) = 2\beta(t) G(t) + b_2(t) N_1(t) - b_1(t) N_2(t), \]

\[ \mathcal{E}_k(t) = (\alpha^2(t) + \beta^2(t)) G(t) + \alpha(t) b_1(t) N_1(t) + \beta(t) b_1(t) N_2(t) + \]

\[ + \alpha(t) b_2(t) N_2(t) + \beta(t) b_2(t) \mathcal{M}. \]
\[ +\alpha(t)b_2(t)N_2(t) + \beta(t)b_2(t)N_1(t) + \frac{1}{2}(b_1^2(t) + b_2^2(t))M, \]

\[ \mathcal{E}_p(t) = KG^{1-\gamma}(t), K = \mathcal{E}_p(0)G^{\gamma-1}(0). \]

In that way, all functions involved in the system (2.1.1–2.1.7) are expressed through \( G(t), N_1(t), N_2(t), \alpha(t), \beta(t), b_1(t), b_2(t) \).

2.1.2. \( A(t) \) of special form, \( b(t) = 0 \). The case is simplest. Namely, \( F_1(t), F_2(t), E_k(t), E_p(t) \) can be expressed through \( G(t), \alpha(t), \beta(t) \). Therefore, if we denote for the convenience \( G_1(t) = 1/G(t) \), then we obtain the system

\[ G_1'(t) = -2\alpha(t)G_1(t), \]

\[ \beta'(t) = \alpha(t)(l - 2\beta(t)) - \mu\beta(t), \]

\[ \alpha'(t) = -\alpha^2(t) + \beta^2(t) - l\beta(t) - \mu\alpha(t) + (\gamma - 1)KG_1^2(t). \]

For \( \gamma > 1 \), the functions \( \alpha(t) \) and \( \beta(t) \) are bounded, it follows from the expression for the total energy. Really,

\[ \mathcal{E}(t) = (\alpha(t)^2 + \beta(t)^2)G(t) + \mathcal{E}_p(0)G^{\gamma-1}(0) \frac{1}{G^{\gamma-1}(t)} \leq \mathcal{E}(0), \]

it implies

\[ \alpha^2(t) + \beta^2(t) \leq \mathcal{E}(0)G_1(t) - \mathcal{E}_p(0)G_1^{1-\gamma}(0)G_1'(t) < +\infty, \]

\(|\text{div } \mathbf{V}| < \infty\). Consequently, the density and pressure are bounded for all solutions of the class we want to construct.

If \( \mu = 0 \), the system (2.1.9) can be integrated and we can obtain

\[ \alpha(G_1) = \pm \sqrt{C_2G_1^3 - C_1^2G_1^2 + (\mathcal{E}(0) - lC_1)G_1 - l^2/4}, \]

\[ \beta(t) = C_1G_1(t) + 1/2, \quad - \int_{G_1(0)}^{G_1(t)} \frac{dG_1}{2G_1\alpha(G_1)} = t, \]

where \( C_1 = \frac{2\beta(0) - l}{2C_1^2} \), \( C_2 = (\alpha^2(0) + C_1^2G_1^2(0) - (\mathcal{E}(0) - lC_1)G_1(0) + l^2/4)/G_1'(0) \).

In the cases \( \mu > 0 \) and \( \mu = l = 0 \), there is a unique stable equilibrium in the origin. We have the following asymptotics of the solution components as \( t \to \infty \):

If \( \mu > 0, l = 0 \), then \( \alpha(t) \sim \frac{1}{2\gamma}t^{-1}, \beta(t) \sim C_1 \left( \frac{\mu}{2K_1\gamma} \right)^{1/\gamma}t^{-1/\gamma}\exp\{-\mu t\}, \)

\[ G_1(t) \sim \left( \frac{\mu}{2K_1\gamma} \right)^{1/\gamma}t^{-1/\gamma}; \]

If \( \mu > 0, l \neq 0 \), then \( \alpha(t) \sim \frac{1}{2\gamma}t^{-1}, \beta(t) \sim \frac{l}{2\mu\gamma}t^{-1}, G_1(t) \sim \left( \frac{l^2 + \mu^2}{2K_1\mu\gamma} \right)^{1/\gamma}t^{-1/\gamma}; \)

If \( \mu = l = 0 \), then \( \alpha(t) \sim \left( \sqrt{G_1(0)}/\mathcal{E}(0) + t \right)^{-1}, G_1(t) \sim \alpha^2(t), \beta(t) = C_1G_1(t), \)
where $K_1 = (\gamma - 1)E_p(0)G_1^{1-\gamma}(0)$.

Knowing $\alpha(t)$ and $\beta(t)$, in the standard way we can find components of density and entropy from the equations (E1), (E3), linear with respect to $\rho$ and $p$, respectively. We obtain that

$$\rho(t, |r|, \phi) = \exp(-2 \int_0^t \alpha(\tau) d\tau) \rho_0(|r|) \exp(-\int_0^t \alpha(\tau) d\tau),$$

$$p(t, |r|, \phi) = \exp(-2\gamma \int_0^t \alpha(\tau) d\tau) p_0(|r|) \exp(-\int_0^t \alpha(\tau) d\tau).$$

From (E2) and (2.1.9) we get that on the classical solution of the system (E1–E3) the relation

$$\nabla p = -(\gamma - 1)G_1^{1-\gamma}(0)E_p(0)G_1^\gamma(t)\rho r$$

must be satisfied. Hence it follows that the components of the initial data $\rho_0$ and $p_0$ must be asymmetrical and compatible, i.e. connected as follows:

$$\nabla p_0 = -(\gamma - 1)G_1(0)E_p(0)\rho_0 r. \quad (2.1.10)$$

For example, one can choose

$$p_0 = \frac{1}{(1 + |r|^2)^a}, \quad a = const > 3, \quad (2.1.11)$$

$$\rho_0 = \frac{2a}{(\gamma - 1)G_1(0)E_p(0)(1 + |r|^2)^{a+1}}. \quad (2.1.12)$$

2.1.3. $A(t)$ of general form, $b(t) = 0$. To consider the velocity field (2.1) with the matrix

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix},$$

we introduce the functionals

$$G_x(t) = \frac{1}{2} \int_{\mathbb{R}^2} \rho x^2 \text{d}x, \quad G_y(t) = \frac{1}{2} \int_{\mathbb{R}^2} \rho y^2 \text{d}x, \quad G_{xy}(t) = \frac{1}{2} \int_{\mathbb{R}^2} \rho xy \text{d}x.$$

Then involve the auxiliary variables

$$G_1(t) = G_x(t)\Delta^{-(\gamma+1)/2}(t), \quad G_2(t) = G_y(t)\Delta^{-(\gamma+1)/2}(t),$$

$$G_3(t) = G_{xy}\Delta^{-(\gamma+1)/2}(t),$$

where $\Delta(t) = G_x(t)G_y(t) - G_{xy}^2(t)$ is a positive function on solutions to (E1–E3). Note that the behaviour of $\Delta(t)$ is governed by the equation

$$\Delta'(t) = 2(a(t) + d(t))\Delta(t), \quad (2.1.14)$$

and the potential energy $E_p(t)$ is connected with $\Delta(t)$ as follows:

$$E_p(t) = E_p(0)\Delta^{(\gamma-1)/2}(0)\Delta^{(-\gamma+1)/2}(t).$$
To find $G_1$, $G_2$, $G_3$, and the elements of the matrix $A(t)$ we get the system of equations
\[ G'_1(t) = ((1 - \gamma)a(t) - (1 + \gamma)d(t))G_1(t) + 2b(t)G_3(t), \]
\[ G'_2(t) = ((1 - \gamma)d(t) - (1 + \gamma)a(t))G_2(t) + 2c(t)G_3(t), \]
\[ G'_3(t) = c(t)G_1(t) + b(t)G_2(t) - \gamma(a(t) + d(t))G_3(t), \]
\[ a'(t) = -a^2(t) - b(t)c(t) + lc(t) - \mu a(t) + K_2G_2(t), \]
\[ b'(t) = -b(t)(a(t) + d(t)) + \mu b(t) - K_2G_3(t), \]
\[ c'(t) = -c(t)(a(t) + d(t)) - la(t) - \mu c(t) - K_2G_3(t), \]
\[ d'(t) = -d^2(t) - b(t)c(t) - \mu b(t) + K_2G_1(t), \]
with $K_2 = \frac{\gamma - 1}{2}(E_p(0)\Delta^{(\gamma - 1)/2}(0))$.

Now we obtain an analytical result on a behaviour of the coefficients of the matrix $A(t)$ as $t \to \infty$.

**Proposition 2.** 1. Suppose $\mu = l = 0$, that is $\mathbf{F} = 0$. If the system (E1–E3) has the solution with the linear profile of velocity $\mathbf{V} = A(t)\mathbf{r}$, then $A(t) \sim \delta I$ as $t \to \infty$, with the identity matrix $I$ and a constant $\delta > \frac{1}{2}$.

**Proof.** Go to the new variables $a_1(t) = a(t) - d(t), b_1(t) = b(t) + c(t), c_1(t) = b(t) - c(t), d_1(t) = a(t) + d(t), G_1(t) = G_1(t) + G_2(t), G_5(t) = G_1(t) - G_2(t)$.

In variables $a_1, b_1, c_1, d_1, G_3, G_4, G_5$ system (2.1.15) has the form:
\[ a'_1(t) = -a_1(t)(a_1(t) - K_2G_5(t)), \]
\[ b'_1(t) = -b_1(t)d_1(t) - 2K_2G_3(t), \]
\[ c'_1(t) = -c_1(t)d_1(t), \]
\[ d'_1(t) = -\frac{1}{2}(a_1^2(t) + d_1^2(t)) - \frac{1}{2}(b_1^2(t) - c_1^2(t)) + K_2G_4(t), \]
\[ G'_3(t) = -\gamma d_1(t)G_3(t) + \frac{1}{2}b_1(t)G_4(t) - \frac{1}{2}c_1(t)G_5(t), \]
\[ G'_4(t) = -\gamma d_1(t)G_4(t) + a_1(t)G_5(t) + 2b_1(t)G_3(t), \]
\[ G'_5(t) = -\gamma d_1(t)G_5(t) + a_1(t)G_4(t) - 2c_1(t)G_3(t). \]

Let as $t \to \infty$ the asymptotics of functions involved in the system be the following: $a_1(t) \sim L_1t^{l_1}, b_1(t) \sim L_2t^{l_2}, c_1(t) \sim L_3t^{l_3}, d_1(t) \sim L_4t^{l_3}, G_3(t) \sim Nt^q, G_4(t) \sim M_1t^{p_1}, G_5(t) \sim M_2t^{p_2}$, where $L_i$, $i = 1, 2, 3, 4$; $M_j$, $j = 1, 2, 3$; $N$ are certain constants not equal to zero. Note that $p_2 \leq p_1$ and in virtue of $\Delta > 0$ the estimate $q \leq p_1$ holds.

From (2.1.18) we get immediately that
\[ l_3L_3t^{l_3-1} = -L_3L_4t^{l_3+l_4}, \]

hence $l_4 = -1$, $L_4 = -l_3$.

Taking the fact into account, from (2.1.16) we have
\[ l_1L_1t^{l_1-1} = -L_1L_4t^{l_1-1} - 2K_2M_2t^{p_2}. \]

The following variants are possible:
\( p_2 < l_1 - 1 \), hence \( l_1 = -L_4 \), i.e. \( l_1 = l_3 \),

\( p_2 = l_1 - 1 \), hence

\[
l_1 = -L_4 - \frac{K_2 M_2}{L_1} = l_3 - \frac{K_2 M_2}{L_1}.
\]

From (2.1.17) we have analogously

\[
l_2 L_2 t^{l_2 - 1} = -L_2 L_4 t^{l_2 - 1} - 2K_2 N t^q.
\]

There are the variants:

- \( q < l_2 - 1 \), hence \( l_2 = -L_4 \), i.e. \( l_2 = l_3 \),
- \( q = l_2 - 1 \), hence

\[
l_2 = -L_4 - \frac{2K_2 N}{L_2} = l_3 - \frac{2K_2 N}{L_2}.
\]

From (2.1.19) we get

\[
-L_4 t^{-2} = -\frac{1}{2}(L_1^2 t^{l_1} + L_2^2 t^{-2}) - \frac{1}{2}(L_2^2 t^{l_2} - L_3^2 t^{l_3}) + K_2 M_1 t^{p_1}.
\]

If \( l_i < -1 \), \( i = 1, 2, 3 \), then at \( t \to \infty \)

\[
a(t) \sim \frac{1}{2}(L_1 t^{l_1} + L_4 t^{-1}), \quad b(t) \sim \frac{1}{2}(L_2 t^{l_2} + L_3 t^{l_3}),
\]

\[
c(t) \sim \frac{1}{2}(L_2 t^{-2} - L_3 t^{l_3}), \quad d(t) \sim \frac{1}{2}(L_4 t^{-1} - L_1 t^{l_1}),
\]

that is \( A(t) \sim \frac{L_4}{2} I, \) \( L_4 = \text{const} > 0 \).

In the case \( L_4 = -l_3 > 1 \), therefore \( \delta > \frac{1}{2} \).

We shall show below that others situations do not appear.

From (2.1.25) one can deduce the impossibility of the situation \( l_3 = -1, l_1 < -1, l_2 < -1, p_1 \leq -2 \). Really, in the case \( l_3 = -L_4 = -1 - \sqrt{1 + L_2^2} < -1 \) or \( l_3 = -L_4 = -1 - \sqrt{1 + L_3^2} + 2K_2 M_1 < -1 \), it contradicts to the assumption.

Now let \( l_3 > -1 \). Then from (2.1.25) \( l_1 = l_3 \) or (and) \( l_2 = l_3 \). Suppose, for example, that \( l_1 = l_3 \). Then \( p_2 < l_1 - 1 \), from (2.1.22) it follows that

\[
p_2 M_2 t^{p_2 - 1} = -\gamma L_4 M_2 t^{p_2 - 1} + L_1 M_1 t^{p_1 + l_1} + 2L_2 N t^{q + l_2},
\]

and therefore \( p_1 + l_1 = q + l_2 \), \( l_1 \leq l_2 \), i.e. \( l_2 > -1 \). Besides,

\[
L_1 M_1 = -2L_2 N.
\]

From (2.1.20) we have

\[
q N t^{q - 1} = -\gamma L_4 N t^{q - 1} + \frac{1}{2} L_2 M_1 t^{p_1 + l_2} - \frac{1}{2} L_3 M_2 t^{p_2 + l_3}.
\]

As soon as \( q - 1 \leq p_1 - 1 \), and \( p_1 - 1 < p_1 + l_2 \), then \( p_1 + l_2 = p_2 + l_3 \leq p_1 + l_3 \), and therefore \( l_2 \leq l_3 \), and as \( l_1 = l_3 \), then \( l_1 = l_2 = l_3 \) and \( p_1 = q \). Besides,

\[
L_2 M_1 = L_3 M_2.
\]

Further, from (2.1.21) we obtain

\[
p_1 M_1 t^{p_1 - 1} = -\gamma L_4 M_1 t^{p_1 - 1} + L_1 M_2 t^{p_2 + l_1} - 2L_3 N t^{q + l_3}.
\]
As \( q + l_3 > p_1 - 1 \), then \( p_2 + l_1 = q + l_3, p_2 = q \).

\[
L_2 M_1 = 2 L_3 N. \tag{2.1.31}
\]

From (2.1.27) and (2.1.31) we have \( \frac{M_1}{M_2} = -\frac{L_2}{L_3} \), and from (2.1.29) we obtain

\[
\frac{M_1}{M_2} = \frac{L_2}{L_3}, \quad \text{i.e.} \quad L_2 = -L_3, \quad L_2 = L_3 = 0 \text{ in spite of the assumption.}
\]

Suppose now that \( l_2 = l_3 \). Then \( q < l_2 - 1, p_2 \leq l_1 - 1 \). From (2.1.20) we have \( q - 1 < l_2 + p_1 \), since \( l_2 > -1 \), and therefore \( l_2 + p_1 = l_3 + p_2, p_1 = p_2, L_2 M_1 = L_3 M_2 \).

Therefore from (2.1.26) we have \( p_1 + l_1 = q + l_2 \leq p_1 + l_2, l_1 \leq l_2, L_2 M_1 = -2 L_3 N. \)

From (2.1.30) we get \( p_2 + l_1 = q + l_3 \leq p_1 + l_2, p_2 = q, l_1 = l_2, L_1 M_2 = -2 L_3 N. \) In that way, we obtain the contradiction analogous to the previous one.

Now let \( l_3 \leq -1, l_1 > -1 \) and (or) \( l_2 > -1 \). Then from (2.1.25) it follows that \( p_1 = 2 l_1 \) and (or) \( p_1 = 2 l_2 (p_1 > -2) \). For example, if \( p_1 = 2 l_1 \), then from (2.1.27) we get \( p_2 - 1 < p_1 + l_1, p_1 + l_1 = q + l_2 \leq p_1 + l_2, l_1 \leq l_2. \) From (2.1.28)

\[
p_2 + l_3 \leq p_1 - 1, l_2 + p_1 \geq l_1 + p_1 > p_1 - 1, \text{ therefore } q - 1 = l_2 + p_1. \text{ But if } q - 1 \leq p_1 - 1, \text{ therefore } l_2 \leq -1, \text{ and } l_2 \leq -1 \text{ in spite of the assumption.}
\]

If \( p_1 = 2 l_2 \), then from (2.1.28) we get \( l_3 + p_2 \leq p_1 - 1, \text{ therefore } p_1 + l_2 \geq p_2 + l_3, q - 1 = p_1 + l_2, l_1 \leq l_2. \) But \( q - 1 \leq p_1 - 1, \text{ therefore } l_2 \leq -1 \) in spite of the supposition.

So, it remains the unique possibility: \( l_3 \leq -1, l_1 \leq -1 \) and (or) \( l_2 \leq -1 \).

In the case, as follows from (2.1.25)

\[
l_3 = -1 \pm \sqrt{1 - \left( \delta_1 L_1^2 + \delta_2 L_2^2 - \delta_3 L_3^2 - 2 \delta_4 K_2 M_1 \right)}, \tag{2.1.32}
\]

where \( \delta_i = 1, \text{ if } l_i = -1, \text{ and } \delta_i = 0 \text{ otherwise, } i = 1, 2, 3, \delta_4 = 1, \text{ if } p_2 = -2 \) and \( \delta_4 = 0 \) otherwise. Hence

\[
\delta_1 L_1^2 + \delta_2 L_2^2 - \delta_3 L_3^2 - 2 \delta_4 K_2 M_1 \leq 1, \tag{2.1.33}
\]

if inequality (2.1.33) is strict, then \( l_3 < -1 \).

We consider this case, that is \( l_3 < -1, l_1 = -1 \) and (or) \( l_2 = -1 \).

If \( l_1 = -1 \neq l_3, \text{ then } p_2 = l_1 - 1 = -2. \) If \( l_2 = -1 \neq l_3, \text{ then } q = l_2 - 1 = -2. \)

In that way, in any of these cases we have \( p_1 = -2 \).

At \( l_1 = -1, l_2 < -1, l_3 < -1 \), from (2.1.26) we obtain

\[
L_1 M_1 \frac{M_1}{M_2} = -(2 + \gamma l_3), \tag{2.1.34}
\]

from (2.1.30) we get

\[
L_1 M_2 \frac{M_2}{M_1} = -(2 + \gamma l_3), \tag{2.1.35}
\]

and from (2.1.24)

\[
K_2 M_2 \frac{L_2}{L_1} = l_3 + 1. \tag{2.1.36}
\]
From (2.1.34), (2.1.36) after excluding $L_1$ and $M_2$ we obtain

$$K_2 M_1 = -(2 + \gamma l_3)(l_3 + 1).$$  \hspace{1cm} (2.1.37)

As $K_2$ and $M_1$ are positive, and $l_3 \leq -1$, then

$$l_3 > -\frac{2}{\gamma}.$$ \hspace{1cm} (2.1.38)

At $\gamma > 2$ it goes already to the contradiction.

Further, from (2.1.35) and (2.1.36) we have $\frac{L_2^2}{K_2 M_1} = -\frac{2 + \gamma l_3}{l_3 + 1}$, this fact together with (2.1.37) gives

$$L_1^2 = (2 + \gamma l_3)^2.$$ \hspace{1cm} (2.1.39)

From (2.1.32), (2.1.37), (2.1.38) and (2.1.39) we obtain the inequality

$$-\frac{2}{\gamma} < -1 - \sqrt{1 - (2 + \gamma l_3)^2 - 2(2 + \gamma l_3)(l_3 + 1)},$$ \hspace{1cm} (2.1.40)

which cannot be true at $\gamma > 1$.

If $l_1 < -1$, $l_2 = -1$, $l_3 < -1$, then from (2.1.26) we get $p_2 = -\gamma L_4 + \frac{2L_2 N}{M_2}$,

$$\frac{2L_2 N}{M_2} = -2 - \gamma l_3,$$ \hspace{1cm} (2.1.41)

from (2.1.28) $q = -\gamma L_4 + \frac{L_2 M_1}{2N}$,

$$\frac{L_2 M_1}{2N} = -2 - \gamma l_3,$$ \hspace{1cm} (2.1.42)

from (2.1.30)

$$p_2 = -\gamma L_4 = \gamma l_3.$$

In that way, $l_3 = -\frac{2}{\gamma}$, this fact together with (2.1.41) or (2.1.42) contradicts to the fact that $L_2$, $\hat{N}$ and $M_1$ are not equal to zero.

If $l_1 < -1$, $l_2 = -1$, $l_3 = -1$, then from (2.1.26),(2.1.28) and (2.1.30) we get correspondingly

$$\frac{2L_2 N}{M_2} + \frac{L_1 M_1}{M_2} = -2 - \gamma l_3,$$ \hspace{1cm} (2.1.43)

$$\frac{L_2 M_1}{2N} = -2 - \gamma l_3,$$ \hspace{1cm} (2.1.44)

$$\frac{L_1 M_2}{M_1} = -2 - \gamma l_3,$$ \hspace{1cm} (2.1.45)

from (2.1.23), (2.1.24)

$$\frac{K_2 M_2}{L_1} = \frac{2K_2 N}{L_2} = l_3 + 1.$$ \hspace{1cm} (2.1.46)

In that way, multiplying (2.1.43) by (2.1.45), taking into account (2.1.46) we get

$$(2 + \gamma l_3)^2 = L_1^2 + \frac{M_2 L_2^2}{M_1}.$$
Therefore, as follows from (2.1.32)

\[ M_1^2 = M_2^2, \]  
(2.1.45)

that is

\[ (2 + \gamma l_3)^2 = L_1^2 \pm L_2^2. \]  
(2.1.48)

As above, from (2.1.34), (2.1.37), (2.1.38) and (2.1.48) we get the inequality
\[ \lambda > \gamma \]

It remains the unique possibility \( l_1 = l_2 = l_3 = -1 \). In the case \( p_2 < -2, q < -2 \) and, as follows from (2.1.26), \( p_1 = p_2 \) or \( p_1 = q \), that is \( p_1 < -2 \). Therefore, as follows from (2.1.32)

\[ L_1^2 + L_2^2 - L_3^2 = 1. \]  
(2.1.49)

If \( p_1 = p_2 \), then \( |M_2| < M_1 \). If \( p_1 = p_2, q < p_1 \), then from (2.1.26), (2.1.30) we obtain that \( p_2 + \gamma = \frac{L_1 M_1}{M_2}, p_1 + \gamma = \frac{L_1 M_1}{M_2}, \) hence \( M_1^2 = M_2^2 \), it goes to the contradiction.

If \( p_1 = q, p_2 < p_1 \), then from (2.1.26), (2.1.28), (2.1.30) we get

\[ L_1 M_1 = -2L_2 N, \lambda = \frac{L_2 M_1}{2N} = -2L_3 N \frac{M_1}{M_1}, \]  
(2.1.50)

where \( \lambda = q + \gamma = p_1 + \gamma \). It follows, in particular, that \( L_2^2 = -\lambda L_1, L_1 L_3 = -\lambda L_2, \lambda^2 = -L_2 L_3 \). Besides, if \( G_1(t) \sim N_1 t^q, G_2(t) \sim N_2 t^p, t \to \infty \), where \( N_1, N_2 \) are some positive constants, \( q_1 = q_2 = p_1 = q, N_1 = N_2, M_1 = 2N_1 \), and \( N_2^2 \geq N^2 \) in virtue of \( G_1(t) G_2(t) - G_2^2(t) > 0 \). In such way, from (2.1.50) we get that \( L_2^2 \geq \lambda^2, \lambda^2 \geq L_3^2, L_3^2 \geq L_2^2 \), and taking into account (2.1.49), \( L_1^2 \leq 1, L_2^2 \leq L_3^2 \leq 1, \lambda^2 \leq 1 \). That is if \( \gamma > 1, \gamma > 3 \), the conditions mentioned in the paragraph cannot hold together.

At last, consider the case \( p_1 = p_2 = q \). Then from (2.1.26), (2.1.28), (2.1.30) we have \( \lambda = \frac{L_1 M_1}{M_2} + 2\frac{L_2 N}{2N} = \frac{L_2 M_1}{2N} = \frac{L_1 M_2}{M_1} = 2L_3 N \frac{M_3}{M_1} \), that is the system of linear homogeneous equations with respect to the variables \( M_1, M_2, N \)

\[ L_1 M_1 - \lambda M_2 + 2L_2 N = 0, \]
\[ L_2 M_1 - L_3 M_2 - 2\lambda N = 0, \]
\[ \lambda M_1 - L_1 M_2 + 2L_3 N = 0. \]

For the existence of its nontrivial solution the determinant of the system must be equal to zero, i.e.

\[ \lambda L_2^2 - 2\lambda L_2 L_3 + L_1 L_2^2 + L_1 L_3^2 - \lambda^3 = 0. \]

Taking into account (2.1.49), involve the function with respect to the variables \( L_2 \) and \( L_3 \), where \( \lambda \) plays the role of parameter, namely

\[ \Psi_{\lambda}(L_2, L_3) = \lambda(1 - L_2^2 + L_3^2) - 2\lambda L_2 L_3 + (L_2^2 + L_3^2) \sqrt{1 - L_2^2 + L_3^2} - \lambda^3. \]

By the standard methods one can show that the function is not equal to zero at \( \lambda > 1 \), i.e. at \( \gamma > 3 \) (remember, that \( p_1 < -2 \)).

So, it remains to investigate the case \( 1 < \gamma \leq 3 \).
Consider equation (2.1.14), which can be written as
\[ \Delta'(t) = 2d_1(t)\Delta(t). \]
If we suppose that \( \Delta(t) \sim \text{const} \cdot t^m, t \to \infty, \) \( m \) is a constant, then \( m = 2L_1 = 2. \) From (2.1.14) we can get \( E_p(t) \sim \text{const} \cdot t^{1-\gamma} (\to 0), t \to \infty. \) Therefore, if we denote \( E \) the quantity of the total energy of the system, then \( E_k(t) = E - E_p(t) \sim E(1 - C_1 t^{1-\gamma}), \) here and further \( C_i \) are some positive constants. But \( E_k(t) \sim C_2 G_k(t)^{-\Delta' + 3} \sim C_2 G_k t^{\gamma - 1}, \) where \( G_k \) is at least one of functions \( G_1, G_2 \) or \( G_3. \) That is \( G_k \sim C_4 (1 - C t^{1-\gamma}) t^{1-\gamma} \sim C_5 t^{1-\gamma}. \) But then \( 1 - \gamma \leq p_1 < -2, \) \( \gamma > 3, \) and \( \gamma \) does not belong to the interval under consideration.

So the proof of the proposition is over.

**Remark 2.1.1** One can show more shortly, that in the physical case \( 1 < \gamma \leq 2 \) the situation \( l_1 = l_2 = l_3 = -1 \) if impossible. Taking into account (2.1.13) we have \( G_1 G_2 - G_3^2 = \Delta - \gamma \sim \text{const} \cdot t^{-2\gamma}, t \to \infty. \) But the degree of the leading term of the expression \( G_1 G_2 - G_3^2 \) is not greater then \( 2p_1, \) therefore \( p_1 \geq -\gamma, \) as \( p_1 < -2, \) then \( \gamma > 2. \)

**Remark 2.1.2** Actually \( l_3 = -L_4 = -2, G_1 G_2 - G_3^2 \sim \text{const} \cdot t^{-4\gamma}, t \to \infty. \)

2.1.4. \textit{A(t) of special form, b(t) \neq 0.} Suppose that the velocity field is of form (2.1.8).

In the case we need to analyze rather complicated system of 7 equations:
\[ G'(t) = 2\alpha(t)G(t) + b_1(t)N_1(t) + b_2(t)N_2(t), \]
\[ N_1'(t) = \alpha(t)N_1(t) + \beta(t)N_2(t) + b_1(t)M, \]
\[ N_2'(t) = \alpha(t)N_2(t) - \beta(t)N_1(t) + b_2(t)M, \]
\[ (\alpha(t)N_1(t) + \beta(t)N_2(t) + b_1(t)M)' = -\mu(\alpha(t)N_1(t) + \beta(t)N_2(t) + b_1(t)M) - l(\alpha(t)N_2(t) - \beta(t)N_1(t) + b_2(t)M), \]
\[ (\alpha(t)N_2(t) - \beta(t)N_1(t) + b_2(t)M)' = -\mu(\alpha(t)N_2(t) - \beta(t)N_1(t) + b_2(t)M) + l(\alpha(t)N_1(t) + \beta(t)N_2(t) + b_1(t)M), \]
\[ (b_1(t)N_1(t) + b_2(t)N_2(t) + 2\alpha(t)G(t))' = -\mu(2\alpha(t)G(t) + b_1(t)N_1(t) + b_2(t)N_2(t)) + 2(\beta(t)G(t) + b_2(t)N_2(t) - b_1(t)N_1(t)) + (b_1^2(t) + b_2^2(t))M + 2(\gamma - 1)K G_1^{-\gamma}(t) + 2G(t)(\alpha^2(t) + \beta^2(t)) + 2\alpha(t)b_1(t)N_1(t) + 2\beta(t)b_1(t)N_2(t) + 2\alpha(t)b_2(t)N_2(t) - 2\beta(t)b_2(t)N_2(t), \]
\[ (b_2(t)N_1(t) - b_1(t)N_2(t) + 2\beta(t)G(t))' = -l(2\alpha(t)G(t) + b_1(t)N_1(t) + b_2(t)N_2(t)) - \mu(2\beta(t)G(t) + b_2(t)N_1(t) - b_1(t)N_2(t)). \]
We could write the normal form of the system, but it is very long. Mention only that we can always express the derivatives of \( \alpha, \beta, b_1, b_2 \) explicitly, because the determinant of the corresponding algebraic system is equal to
\[
4G^2(t)M^2 - (N_1(t))^2 + (N_2(t))^2.
\]
As the Hölder inequality shows, it is positive for the solutions we consider.

The compatibility condition in the case has the form
\[
\nabla p_0 = \rho_0(c_1 \mathbf{r} + c_2 \mathbf{r}_\perp + \mathbf{c}_0)
\]
with scalar constants \( c_1, c_2 \) and constant vector \( \mathbf{c}_0 \).

2.1.5. \( A(t) \) of general form, \( b(t) \neq 0 \). The corresponding system consists of 11 equations for unknown functions \( a(t), b(t), c(t), d(t), b_1(t), b_2(t), G_x(t), G_y(t), G_{xy}(t), N_1(t), N_2(t) \). It seems that it can be analyzed only numerically.

As soon as the components of velocity field are found, we can always find the density and the pressure. Namely,
\[
\rho(t, \mathbf{x}) = \exp\left(-\int_0^t \text{tr}A(\tau)d\tau\right) \rho_0(M^{-1}(t)\mathbf{x} - \int_0^t M^{-1}(\tau)b(\tau)d\tau),
\]
\[
p(t, \mathbf{x}) = \exp\left(-\gamma\int_0^t \text{tr}A(\tau)d\tau\right) p_0(M^{-1}(t)\mathbf{x} - \int_0^t M^{-1}(\tau)b(\tau)d\tau),
\]
where \( M(t) \) is a solution to the matrix equation \( \frac{dX}{dt} = A(t)X(t) \) \([18]\).

The compatibility condition in the case has the form
\[
\nabla p_0(\mathbf{x}) = \rho_0(\mathbf{x})(C\mathbf{x} + \mathbf{c}_0)
\]
with a constant matrix \( C \) and a constant vector \( \mathbf{c}_0 \).

2.2. The system in the physical space \( (n = 3) \). Now we consider the right-hand side of (E1), corresponding to the Coriolis force and the Rayleigh friction in the real physical space. Namely,
\[
f = -\mu \mathbf{V} + \delta[\mathbf{V} \times \omega] = (-\mu I + \delta \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}) \mathbf{V},
\]
where \( \delta = 0 \) or \( 1 \), \( \omega \) is a constant vector \( (\omega_1, \omega_2, \omega_3) \), \( \mu \geq 0 \) is a constant, \([a \times b]\) is the cross product of the vectors \( \mathbf{a} \) and \( \mathbf{b} \).

Besides of \( G(t) = \frac{1}{2} \int_{\mathbb{R}^3} \rho |\mathbf{r}|^2 d\mathbf{x} \) and \( F_1(t) = \int_{\mathbb{R}^3} (\mathbf{V}, \mathbf{r}) \rho d\mathbf{x} \), the generalization of the functionals, involved in 2.1.1, to the 3D case, we consider
\[
\tilde{F}_2(t) = \int_{\mathbb{R}^3} (\mathbf{V}, [\omega \times \mathbf{r}]) \rho d\mathbf{x},
\]
\[ F_3(t) = \int_{\mathbb{R}^3} ([V \times \omega], [\omega \times r]) \rho \, dx, \]
\[ H(t) = \frac{1}{2} \int_{\mathbb{R}^3} ([\omega \times r])^2 \rho \, dx. \]

For the smooth solutions to the system (E1–E3) with the right-hand side (2.2.1) we have
\[ G' = F_1(t), \]
\[ F_1(t) = 2E_k(t) + 3(\gamma - 1)E_p(t) + \delta \bar{F}_2(t) - \mu F_1(t), \]
\[ \bar{F}_2(t) = \delta F_3(t) - \mu \bar{F}_2(t), \]
\[ H'(t) = 0. \]

We seek the solution with a special linear velocity profile
\[ V = \alpha(t)r + \beta(t)[r \times \omega] = (\alpha(t)E + \beta(t)) \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} r. \]

In this case
\[ F_1(t) = 2\alpha(t)G(t), \]
\[ \bar{F}_2(t) = -2\beta(t)H(0), \]
\[ F_3(t) = -2\alpha H(0), \]
\[ E_k(t) = \alpha^2(t)G(t) + \beta^2(t)H(0), \]
\[ E_p(t) = const \cdot G^{\frac{3(\gamma - 1)}{2}}(t). \]

Changing \( G^{-1}(t) = G_1(t) \), we obtain the closed system of ODE
\[ \alpha'(t) = -\alpha^2(t) - \mu \alpha(t) + \beta^2(t)G_1(t)H(0) - \delta \beta(t)G_1(t)H(0) + \frac{3(\gamma - 1)}{2} K G_1^{\frac{3\gamma - 1}{2}}(t), \]
\[ \beta'(t) = \delta \alpha(t) - \mu \beta(t), \quad G_1'(t) = -2\alpha(t)G_1(t), \]

with \( K = E_p(0)G^{\frac{3\gamma - 1}{2}}(0) \). Since the total energy is nonincreasing, we can draw similarly to the 2D case that \(|\alpha(t)| < \infty, |\beta(t)| < \infty\).

In the situation with \( \mu = \delta = 0 \), with \( \mu > 0, \delta = 0 \) and with \( \mu > 0, \delta = 1 \), there is a stable equilibrium in the origin \( \alpha(t) = \beta(t) = G_1(t) = 0 \).

If \( \mu = 0, \delta = 0 \), we have
\[ \alpha(t) \sim t^{-1}, \beta(t) = \beta(0) = const, G_1(t) \sim const \cdot t^{-2}, t \to \infty. \]

If \( \mu > 0, \delta = 0 \), then
\[ \alpha(t) \sim \frac{1}{3\gamma - 1} t^{-1}, \beta(t) = \beta(0) \exp(-\mu t), G_1(t) \sim const \cdot t^{-\frac{2}{\mu(3\gamma - 1)}}, t \to \infty. \]

In the case \( \mu > 0, \delta = 1 \),
\[ \alpha(t) \sim \frac{1}{3\gamma - 1} t^{-1}, \beta(t) \sim \frac{1}{\mu(3\gamma - 1)} t^{-1}, G_1(t) \sim const \cdot t^{-\frac{2}{\mu(3\gamma - 1)}}, t \to \infty. \]
What about other component of solution, they can be found by the same formulas, as in the 2D case ((2.1.50 – 2.1.51)), and must satisfy the compatibility condition

$$\nabla p_0 = - (\gamma - 1) G_1(0) E_p(0) \rho_0 e,$$

the same as (2.1.10).

**Remark 2.2.1** We can also consider the velocity with linear profile of general form, but the system of integral functionals is complicated and it is difficult to analyze it.

### 2.3. Theorem on the interior solutions.

Further we need to obtain the symmetric form of system (E1-E3). For this purpose we consider the entropy \( S \), connected with the components of solution by the state equation \( p = e^S \rho \gamma \).

Thus, instead of (E1-E3) we obtain other system

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho \mathbf{V}) &= 0, \\
\rho \partial_t \mathbf{V} + (\rho \mathbf{V}, \nabla) \mathbf{V} + \nabla p &= \rho f(x, t, \mathbf{V}, \rho, S), \\
\partial_t S + (\mathbf{V}, \nabla S) &= 0.
\end{align*}
\]

(2.3.1)

(2.3.2)

(2.3.3)

Here \( \rho(t, x), \mathbf{V}(t, x), S(t, x) \) are components of the solution, given in \( \mathbb{R}_+ \times \mathbb{R}^n, n \geq 1 \) (density, velocity and entropy, respectively).

The systems (E1–E3) and (2.3.1–2.3.3) are equivalent for \( \rho > 0 \).

Put the Cauchy problem for (2.3.1–2.3.3):

\[
\begin{align*}
\rho(0, x) &= \rho_0(x) \geq 0, \\
\mathbf{V}(0, x) &= \mathbf{V}_0(x), \\
S(0, x) &= S_0(x). \tag{2.3.4}
\end{align*}
\]

Denote \( p_0(x) = e^{S_0(x)} \rho_0^\gamma(x) \), and remark that the first of the Cauchy conditions can be replaced to \( p(0, x) = p_0(x) \geq 0 \).

**Definition.** We shall call the global-in-time classical solution \((\bar{\rho}(t, x), \bar{\mathbf{V}}(t, x), \bar{S}(t, x))\) to the system (2.3.1–2.3.3) the interior solution, if

\[
(\bar{\rho}^{\frac{2}{\gamma-1}}, \bar{\mathbf{V}}, \bar{S}) \in C^1(\mathbb{R}_+ \times \mathbb{R}^n),
\]

where \( \bar{\rho} = e^\gamma \bar{\rho}^\gamma \), and any solution \((\rho(t, x), \mathbf{V}(t, x), S(t, x))\) to the Cauchy problem (2.3.1–2.3.4) with the sufficiently small norm

\[
\| (p_0^{(\gamma-1)/2\gamma}(x) - \bar{\rho}^{(\gamma-1)/2\gamma}(0, x), \mathbf{V}_0(x) - \bar{\mathbf{V}}(0, x), S_0(x) - \bar{S}(0, x)) \|_{H^m(\mathbb{R}^n)},
\]

\( m > n/2 + 1 \), is also smooth, such that \((\bar{\rho}^{\frac{2}{\gamma-1}}, \bar{\mathbf{V}}, \bar{S}) \in C^1(\mathbb{R}_+ \times \mathbb{R}^n)\).

Note that the trivial solution is not interior at least for \( \mathbf{F} = \mathbf{f} = 0 \). But a set of the interior solutions is not empty. In \cite{15} (a generalization of \cite{16}) for \( \mathbf{f} = 0 \) it was shown that the solution \((0, \bar{V}(t, x), \text{const})\) to (2.3.1–2.3.4) is interior, provided \( \bar{V}(t, x) \) is a globally smooth solution to the equation

\[
\partial_t \mathbf{V} + (\mathbf{V}, \nabla) \mathbf{V} = 0
\]

such that the spectrum of its Jacobian is separated.
initially from the real negative semi-axis, $D^2 \bar{V}(0, x) \in L^\infty(\mathbb{R}^n)$, $D^2 \bar{V}(0, x) \in H^{m-1}(\mathbb{R}^n)$ (we denote by $D^k$ the vector of all spatial derivatives of the order $k$).

The result is clear from a physical point of view: the velocity field with a positive divergency spreads the initially concentrated small mass, that prevents the singularity formation.

As we show below, some of solutions with linear profile of velocity are interior with the density not close to zero.

For the solutions to (2.3.1–2.3.3) with a finite moment we have $\inf \rho = \inf p = 0$, thus we need to use the symmetrization proposed in [1]. After involving the new variable $\Pi = \kappa p^\frac{\gamma + 1}{\gamma - 1}$, where $\kappa = \frac{2\sqrt{\gamma} - 1}{\gamma - 1}$, we obtain the symmetric form of the system (2.3.1–2.3.3) with the solution $U = (\Pi, V, S)$:

$$\begin{align*}
\exp(S) & (\partial_t + (V, \nabla))\Pi + \frac{\gamma - 1}{2} \exp(S) \Pi \text{div} V = 0, \\
(\partial_t + (V, \nabla))V + \frac{\gamma - 1}{2} \exp(S) \Pi & \nabla \Pi = f_1(t, x, \Pi, V, S), \\
(\partial_t + (V, \nabla))S & = 0,
\end{align*}$$

where $f_1 = f(t, x, e^{-\frac{S}{\gamma}}(\Pi)^{\frac{\gamma - 1}{\gamma - 1}}, V, S)$. Let $\Pi_0 := \kappa p_0^\frac{\gamma + 1}{\gamma - 1}$, $U_0 := (\Pi_0, V_0, S_0)$. In this way, the Cauchy problem (2.3.1–2.3.4) is transformed to the problem (2.3.5–2.3.7), with the initial data $U(0, x) = U_0(x)$. Further, suppose that the system (2.3.5–2.3.7) has a classical solution $\bar{U} := (\bar{\Pi}, \bar{V}, \bar{S})$, where $\bar{V} := A(t) r + b(t)$, $\bar{\Pi} := \kappa \bar{p}^\frac{\gamma - 1}{\gamma - 1}$. Denote also $U - \bar{U} := u := (\pi, v, s)$.

Before formulating Theorem 2.1 we perform certain transformations. Firstly we go on to the system with the solution $u$. Then, following [16], we carry out the nondegenerate change of variables such that the infinite semi-axis of time turns to semi-interval.

Assume that there exists a nondegenerate square $(n \times n)$ matrix $A_1(t)$ such that $A_1(t)(A_1^{-1}(t))' = A(t)$, moreover, $A_1(t)A(t) = A(t)A_1(t)$. Choose a positive decreasing function $\lambda(t)$ such that the integral $\int_0^\infty \lambda(\tau) d\tau$ converges to the finite value $\sigma_\infty$, and set $\sigma(t) := \int_0^t \lambda(\tau) d\tau$. Let $(\sigma, y) := (\sigma(t), A_1(t)x)$ be new variables. Then $\nabla_x = A_1^* \nabla_y$, div$_x V = \text{div}_y A_1 V$, $\partial_\sigma = \lambda^{-1}(t)(\partial_t + A_r \nabla_x)$. In that way, the semi-infinite axis of time goes to the semi-interval $[0, \sigma_\infty)$. Note that there exists the inverse function $t = t(\sigma)$. Further, involve the variables $W = \lambda^{-1}(t)A_1(t)v$, $P = \lambda^{-q}(t)\pi$, the constant $q$ will be defined below. Let $U(\sigma, y) := (P, W, s)$, $\bar{P} := \lambda^{-q}\bar{\Pi}$, $\bar{W} := \lambda^{-1}A_1 \bar{V}$, $\bar{U} := (\bar{P}, \bar{W}, \bar{S})$.

So we get a system

$$\begin{align*}
(\partial_\sigma + (W, \nabla_y))P + \frac{\gamma - 1}{2} (P + \bar{P}) \text{div}_y W &= \end{align*}$$
\[-(W, \nabla y)\bar{P} - PQ_1 - \lambda^{-1}(A_1 b, \nabla y P), \tag{2.3.8}\]
\[(\partial_t + (W, \nabla y))W + \frac{\gamma-1}{2}\Psi(S, \sigma)(P + \bar{P})\nabla y P = \]
\[-\frac{\gamma-1}{2}\Psi(S, \sigma)(P + \bar{P})\nabla y P + \frac{\gamma-1}{2}\Psi(S, \sigma)P\nabla y P - \]
\[-Q_2 W + G - \lambda^{-1}(A_1 b, \nabla y) W, \tag{2.3.9}\]
\[(\partial_t + (W, \nabla y))s = -(W, \nabla y)\bar{S} - \lambda^{-1}(A_1 b, \nabla y s). \tag{2.3.10}\]

Here we take into account that \(t = t(\sigma), x = A_1^{-1}(t(\sigma))y, \lambda = \lambda(t(\sigma)), A = A(t(\sigma)), A_1 = A_1(t(\sigma)), b = b(t(\sigma))\) and denote
\[Q_1 = Q_1(t(\sigma)) := \lambda(t)^{-1}\left(\frac{\gamma-1}{2}\operatorname{tr} A(t) + q(\ln \lambda(t))'\right),\]
\[Q_2 = Q_2(t(\sigma)) := \lambda(t)^{-1}((\ln \lambda(t))' E + A(t) + A_1(t)A(t)A_1^{-1}(t)),\]
\[\Psi(S, \sigma) := \exp\left(\frac{S}{\gamma}\right) R A_2,\]
\[A_2 = A_2(t(\sigma)) := A_1(t)A_1^{-1}(t)(\det A_1(t))^{-2/n},\]
\[R = R(t(\sigma)) := \lambda^{2q-2}(t)(\det A_1(t))^{2/n},\]
\[G = G(\sigma, y, U, U') := \lambda^{-2}(t)A_1(t)(f_1(t, x, \lambda(t)(\bar{P} + P),\]
\[\lambda(t)A_1^{-1}(t)(\bar{W} + W), (\bar{S} + s)) - f_1(t, x, \lambda(t)\bar{P}, \lambda(t)A_1^{-1}(t)(\bar{W}, \bar{S})).\]

Note that \(A_2\) is a family of invertible matrices.

The initial data \(U_0\) and the vector-function \(\bar{U}\) induce the Cauchy data \(U_0(y)\) for the system (2.3.8–2.3.10) as follows: \(U_0(y) = \lambda^{-q}(\Pi_0(A_1^{-1} y) - \Pi(0, A_1^{-1} y), \lambda^{-1} A_1(V_0(A_1^{-1} y) - V(0, A_1^{-1} y)), S_0(A_1^{-1} y) - \bar{S}(0, A_1^{-1} y))\) for \(\sigma = 0\).

After multiplying (2.3.9) by \(\Psi^{-1}(S, \sigma)\) the system becomes symmetric hyperbolic. We can apply the theorem on a local existence of the unique solution to the Cauchy problem for symmetric hyperbolic systems [3] to the problem (2.3.8–2.3.10), \(U(0, y) = U_0(y)\), provided \(\Psi^{-1}(S, \sigma)\) is uniformly positive. Sometimes \(\Psi^{-1}(S, \sigma)\) is really uniformly positive (see [10]), and in that cases we can conclude that if \(U_0 \in H^m(\mathbb{R}^n), m > 1 + n/2\), then \(U \in \cap_{j=0}^m C^j([0, \sigma_*); H^{m-j}(\mathbb{R}^n)], \sigma_* > 0\). Since \(\sigma_* \to \infty\), as \(\|U_0\|_{H^m} \to 0\), then choosing the initial data \(U_0\) small in the Sobolev \(H^{m-}\) norm, we can extend the time of existence of the smooth solution \(U\) to the Cauchy problem up to \(\sigma_* \geq \sigma_\infty\). In this case we conclude that \(U \in \cap_{j=0}^m C^j([0, \sigma_*); H^{m-j}(\mathbb{R}^n)]\) and \(u \in \cap_{j=0}^m C^j([0, \infty); H^{m-j}(\mathbb{R}^n)]\).

However, generally speaking, \(\Psi^{-1}(S, \sigma) \to 0\) as \(t \to \infty\), therefore we can extend the time of existence of the smooth solution \(U\) to the Cauchy problem up to \(0, \sigma_*\), for any \(\sigma_* < \sigma_\infty\), choosing the initial data small in the Sobolev \(H^{m-}\) norm, but we cannot guarantee the existence of the solution for \(0, \sigma_\infty\). Thus, we need to act differently.

If \(t_*\) is the supremum of the time of existence of the smooth solution \(U\) for the Cauchy problem (2.3.5–2.3.7), \(U(0, x) = U_0(x)\) (the solution \(u\) preserves the smoothness during the same time), then \(\sigma_* = \sigma(t_*)\). If \(\sigma_* < \sigma_\infty(t_* < \)
linear profile of velocity

Let the system (2.3.1–2.3.3) has a global in time classical solution

Theorem 2.1. Let the function $f_1(t, x, \Pi, V, S)$ have the derivatives with respect to all arguments up to the order $m + 1, n > n/2 + 1$. Suppose that the system (2.3.1–2.3.3) has a global in time classical solution $(\bar{\rho}, \bar{V}, \bar{S})$ with linear profile of velocity $V = A(t)r + b(t)$ such that

a) $p^{m+1} \in C^1([0, \infty); H^{m-j}(\mathbb{R}^n))$;

$$D \bar{S}(t, x) \in \bigcap_{j=0}^{m} C^j([0, \infty); H^{m-j}(\mathbb{R}^n));$$

b) there exists a matrix $A_1(t)$ such that $A(t) = A_1(t)(A_1^{-1}(t))', A(t)A_1(t) = A_1(t)A_1(t)$, det $A_1(t) > 0$, for $t \geq t_0 \geq 0$.

Let us assume that there exist a smooth real-valued decreasing nonnegative function $\lambda(t)$, a constant $q$ and a matrix $U_\phi(t)$ with real coefficients, having the following properties:

$$\int_{t_0}^{+\infty} \lambda(t) d\tau < \infty, \quad (2.3.11)$$

$$\int_{t_0}^{+\infty} \lambda^q(t)(\det A_1(t))^{1/n} d\tau < \infty, \quad (2.3.12)$$

$$R(t)' \geq 0 \quad \text{and} \quad Q_1(t) \geq 0 \quad \text{for} \quad t \geq t_0 \geq 0, \quad (2.3.13)$$

$A_2^{-1}(t)U_\phi(t)$ is a skew-symmetric matrix,

the following functions, vector-functions and matrices are bounded in $[t_0, \infty) \times \mathbb{R}^n$ (under bounded $(\Pi, V, S)$):

$$\lambda^{-q}(t) \exp \left( -\frac{\gamma - 1}{2} \text{tr} A_1(t) \right), \quad \exp \left( -\int_{t_0}^{t} A_1(t) d\tau \right),$$

$$\lambda^{-q}(t)b(t), \quad \lambda^{-1}(t)A_2^{-1}(t)A_2'(t), \quad (2.3.14)$$

$$\lambda^{-1}D^j(\nabla_\Pi f_1), \quad \lambda^{-(q+1)}D^j(\nabla_\Sigma f_1), \quad (2.3.15)$$

$$Q_3 := (Q_2(t) - \lambda^{-1}(t)D^j(A_1(t)\nabla V f_1)A_1^{-1}(t) - U_\phi(t)), \quad |J| = 0, 1, \ldots, m.$$
Remark 2.3.1. If $Q_3$ depends only on time, then we can require only the nonnegativity of this expression for $t \geq t_0$.

Proof. Let $\sigma \in [0, \sigma_*)$. Suppose for the sake of simplicity that $t_0 = 0$, otherwise we can consider the Cauchy problem at $\sigma_0 = \sigma(t_0) < \sigma_*$ with the initial data induced by the Cauchy data $U_0$. Note that choosing $U_0$ sufficiently small in the $H^m$-norm we can prolong the smooth solution $U$ up to $\sigma_0$. Denote $(X, Y, Z) : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ a vector-function from $L_2(\mathbb{R}^{n+2})$. Introduce a norm

$$[X, Y, Z]^2(\sigma) := \int_{\mathbb{R}^n} ((X^2 + Z^2) + Y^* \Psi^{-1}(S, \sigma)Y) dy.$$ 

It is equivalent to the usual $L^2(\mathbb{R}^{n+2})$-norm, but not uniformly in time. For any $p \in \mathbb{N}$ we define

$$E_p(\sigma) = \frac{1}{2} \sum_{|J| = p} [D^J P, D^J W, D^J s]^2(\sigma), \quad F_m(\sigma) = \sum_{p=0}^m E_p(\sigma).$$

Let us compute

$$\frac{dE_p}{d\sigma} = \sum_{|J| = p} \left[ \int_{\mathbb{R}^n} (D^J PD^J \partial_\sigma P) dy + \int_{\mathbb{R}^n} D^J s D^J \partial_\sigma s dy + \int_{\mathbb{R}^n} D^J W \Psi^{-1}(S, \sigma)D^J \partial_\sigma W dy + \int_{\mathbb{R}^n} (D^J W)^* \left( \frac{\partial \Psi^{-1}(S, \sigma)}{\partial \sigma} - \frac{1}{\gamma} \Psi^{-1}(S, \sigma) \partial_\sigma S \right) D^J W dy \right] = I_1 + I_2 + I_3 + I_4,$$

the integrals $I_k$, $k = 1, 2, 3, 4$, are numbered in the consecutive order.

We estimate every of the integrals in the standard way using the Hölder and the Gagliardo-Nirinberg inequalities (see [16], [15]) under assumption (2.3.11 – 2.3.16) of Theorem 1. We denote by $c_i$ certain positive constants, which do not depend on $\sigma$.

Let us begin from $I_1$.

$$I_1 = -\int_{\mathbb{R}^n} D^\alpha P D^\alpha ((W, \nabla_y) P + \frac{\gamma - 1}{2} (P + \bar{P}) \text{div}_y W) + (W, \nabla_y) \bar{P} + PQ_1(t(\sigma))) dy.$$ 

The integral from the terms of higher order $D^\alpha P D^\alpha ((W, \nabla_y) P + \frac{\gamma - 1}{2} P \text{div}_y W)$ can be reduced by integration by parts to

$$-\int_{\mathbb{R}^n} \left\{ \frac{1}{2} D^\alpha P \text{div}_y W + D^\alpha P ((W, \nabla_y (D^\alpha P)) - D^\alpha (W, \nabla_y P)) \right\} dy -$$
\[-\frac{\gamma - 1}{2} \int_{R_n} D^\alpha P \{ (\nabla_y P, D^\alpha W) + (P \text{div}_y D^\alpha W - D^\alpha (P \text{div}_y W)) \} \, dy := I_{11}.
\]

One can show applying the Hölder and the Galiardo-Nirinberg inequalities that

\[ I_{11} \leq \tilde{c}_1 \|D^p P\|_2 (\|D^p P\|_2 \|\nabla_y W\|_\infty + \|D^p W\|_2 \|\nabla_y P\|_\infty) \leq c_1 R^{1/2} F_m^{1/2} E_p \]

(The details in [16, 13].) Further, according to (2.3.5), (2.3.13) and (2.3.14)

\[ \left| \int_{R_n} D^\alpha P D^\alpha (W \nabla_y \tilde{P}) \, dy \right| \leq \tilde{c}_2 R^{1/2} \lambda^{-q} \exp\left(-\frac{\gamma - 1}{2} \text{tr} A\right) F_m \leq c_2 R^{1/2} F_m, \]

\[ \left| \int_{R_n} D^\alpha P D^\alpha (\tilde{P} \text{div}_y W) \, dy \right| \leq c_3 R^{1/2} F_m, \]

\[ \int_{R_n} D^\alpha P D^\alpha (PQ_1) \, dy \geq 0. \]

The integral \( I_2 \) can be estimated analogously:

\[ I_2 \leq c_4 R^{1/2} F_m^{1/2} E_p + c_5 R^{1/2} F_m. \]

Further,

\[ I_3 = \int_{R_n} (D^\alpha W)^* \Psi^{-1}(S, \sigma) \{ (W, \nabla_y) D^\alpha W - D^\alpha ((W, \nabla_y W) \} \, dy + \]

\[ + \frac{1}{2} \int_{R_n} \left\{ (D^\alpha W)^* \Psi^{-1}(S, \sigma) D^\alpha W \text{div}_y W - \frac{(W \nabla_y S)}{\gamma} (D^\alpha W)^* \Psi^{-1}(S, \sigma) D^\alpha W \right\} \, dy + \]

\[ + \frac{\gamma - 1}{2} \int_{R_n} \left\{ (D^\alpha W)^* \Psi^{-1}(S, \sigma) D^\alpha \left( P \Psi(S, \sigma) \nabla_y P + \Psi(S, \sigma) \nabla_y (P \tilde{P}) \right) \right\} \, dy + \]

\[ + \frac{\gamma - 1}{4} \int_{R_n} \left\{ (D^\alpha W)^* \Psi^{-1}(S, \sigma) D^\alpha \left( (\Psi(S, \sigma) - \Psi(S, \sigma)) \nabla_y (\tilde{P}^2) \right) \right\} \, dy + \]

\[ - \int_{R_n} (D^\alpha W)^* \Psi^{-1}(S, \sigma) D^\alpha (Q_2 W - G) \, dy. \]

First two integrals can be estimated from above by the values

\[ c_6 R^{-1} \|D^p W\|_2^2 \|\nabla W\|_\infty + \]

\[ c_7 R^{-1} \|D^p W\|_2^2 \|W\|_2^2 \leq c_8 R^{1/2} F_m^{1/2} E_p, \]

the third, containing the terms of the form

\[ D^p W (\prod_{j=1}^k (D^j S)_{\beta_j}) D^{l-k} P D^{p+1-l} P, \]

\[ D^p W (\prod_{j=1}^k (D^j S)_{\beta_j}) D^{l-k} \tilde{P} D^{p+1-l} P, \]
or

$$D^p W^k \prod_{j=1}^k (D^j S)^{\beta_j} D^{l-k} P D^{p+1-l} \tilde{P}, \; 1 \leq l \leq p, \; 1 \leq k \leq l, \; \sum_j j \beta_j = k,$$

as follows from the Galiardo-Nirenberg inequality and (2.3.14), can be estimated by the sum of form

$$R^{1/2} E_p^{1/2} \sum_{i=9}^{11} c_i F_{m+1} + c_i, \epsilon_i > 0.$$

The fourth integral contains the term of form

$$\exp(\theta s) D^p W^k \prod_{j=1}^k (D^j S)^{\beta_j} D^{l-k} P D^{p+1-l} \tilde{P}, \; 1 \leq l \leq p, \; 1 \leq k \leq l,$$

and can be estimated in a similar way, taking into account that the value of $\|s\|_\infty$ is bounded.

The last integral taking into account condition (2.3.16) can be estimated by the quantity $c_9 F_m$. Remark here that

$$G = \lambda^{-1}(t) A_1(t) \nabla \psi f_1(t, x, \Pi + \theta_1 \pi, \nabla + \theta_2 \psi S + \theta_3 s) A_1^{-1}(t) W +$$

$$\lambda^{\rho-2}(t) \nabla \psi f_1(t, x, \Pi + \theta_1 \pi, \nabla + \theta_2 \psi S + \theta_3 s) P +$$

$$\lambda^{-2}(t) \nabla \psi f_1(t, x, \Pi + \theta_1 \pi, \nabla + \theta_2 \psi S + \theta_3 s),$$

where $\theta_i \in (0, 1), \; i = 1, 2, 3$, and

$$\int_{R^n} (D^\alpha W)^* \Psi^{-1}(S, \sigma) D^\alpha (U_\phi(t) W) dy = 0,$$

since

$$(w, A_2^{-1}(t) U_\phi(t) w) = 0$$

for any real-valued vector $w$.

Further we estimate $I_4$. Remark first of all, that according to (2.3.13) $R'(\sigma)/R(\sigma) \geq 0$. Taking into account (2.3.15) we obtain

$$\frac{\partial \Psi^{-1}(S, \sigma)}{\partial \sigma} = -\frac{\Psi^{-2}(S, \sigma) \Psi(S, \sigma) R'}{R} + O(\Psi^{-1}(S, \sigma)) \leq c_{12} \Psi^{-1}(S, \sigma).$$

Thus,

$$I_4 \leq \tilde{c}_{13} (1 + \|W\nabla_y s\|_\infty + \|W\nabla_y S\|_\infty) E_p \leq c_{13} (1 + R^{1/2} F_m + R^{1/2} F_{m+1/2}) E_p.$$

So we get

$$I_1 \leq c_1 (E_p + R^{1/2} F_m^{1/2} E_p + R^{1/2} F_m),$$

$$I_2 \leq c_2 (R^{1/2} F_m^{1/2} E_p + R^{1/2} F_m),$$
\[ I_3 \leq c_3(F_m + R^{1/2}F_m^{1/2}E_p + R^{1/2}F_m + R^{1/2}E_p^{1/2} \sum_{i=1}^{4} F_m^{1+\gamma_i}), \gamma_i > 0, \]
\[ I_4 \leq c_5(1 + R^{1/2}F_m + R^{1/2}F_m^{1/2})E_p. \]

Using all the estimates we get
\[ F'_m \leq c_6(F_m + R^{1/2}(F_m + \sum_{i=1}^{6} F_m^{3/2+\gamma_i}), \gamma_i \geq 0. \]

Set \( \Lambda_m(\sigma) = e^{-c_6\sigma}F_m \). Then
\[ \Lambda'_m \leq c_7(\Lambda_m + \sum_{i=1}^{6} (\Lambda_m)^{3/2+\gamma_i})R^{1/2}, \gamma_i \geq 0, \quad (2.3.17) \]

where the constant \( c_7 \) depends on \( c_6, \sigma_\infty \). Let
\[ \Theta(\bar{g}) := \int_{\delta}^{\bar{g}} \frac{dg}{g + \sum_{i=1}^{6} g^{3/2+\gamma_i}}. \]
\( \delta > 0 \). The integral diverges in the zero, such that \( \Theta(0) = -\infty \). Integrating inequality (2.3.17) over \( \sigma \) we obtain
\[ \Theta(\Lambda_m(\sigma)) \leq \Theta(\Lambda_m(0)) + c_8 \int_{0}^{\sigma} R^{1/2}(\bar{\sigma})d\bar{\sigma}, \]

moreover, as follows from (2.3.12), the integral in the right-hand side of the last inequality converges as \( \sigma \to \sigma_\infty \) to the constant \( C \), depending only on the initial data. Choosing \( \Lambda_m(0) \) (that is the \( H_m^- \) norm of the initial data) sufficiently small, one can make the value \( \Theta(\Lambda_m(0)) + C \) later then \( \Theta(+\infty) \), it signifies that \( \Lambda_m(\sigma) \) and \( F_m(\sigma) \) are bounded from above for all \( \sigma \in [0, \sigma_\infty) \) and \( \sigma_* = \sigma_\infty \). So, \( u \in \cap_{j=0}^{m} C^3([0, \infty); H^{m-j}(\mathbb{R}^n)) \) and Theorem 2.1 is proved.

**Remark 2.3.2** The solutions with linear profile of velocity, satisfying the conditions of Theorem 2.1, exist. In the case investigated in [16], \( f = 0, \gamma \leq 1 + \frac{2}{n}, A(t) = (E + tA(0))^{-1}A(0), \) \( SpA(0) \cap \mathbb{R}_- = \emptyset, b(t) = 0, A_1(t) = A(t), \) \( \bar{\rho} = 0, S = const. \) Here \( \lambda = (1 + t)^{-2}, q = \frac{n(\gamma-1)}{4}. \)

2.3.1. **Corollary and examples.** The following Corollary helps us to prove that among the solutions to the Euler system constructed in section 2.1 and 2.2 there are interior ones.

**Corollary 2.1.** Let \( f = L(t)V, \) where \( L(t) = -\mu I + U_1(t), \mu \) is a nonnegative constant, \( U_1(t) \) is a one-parameter family of skew-symmetric matrices with the smooth coefficients. Assume that the system (2.3.1–2.3.3) has a global in time classical solution \( (\bar{\rho}, A(t)r, S) \), satisfying the condition a) of Theorem 1 and \( A(t) \sim \alpha(t)I + \beta(t)U_2, \) as \( t \to \infty, \) where \( \alpha(t) \) and \( \beta(t) \) are the real-valued functions, \( U_2 \) is a skew-symmetric matrix with constant coefficients. Moreover,
suppose that $\alpha(t) = \frac{\delta}{t}$, with $\delta = \text{const}$, such that $\delta > 0$ for $\mu > 0$, and $\delta > \frac{1}{2}$ for $\mu = 0$. Then this solution is interior.

Proof. Here $b(t) = 0$. The matrix $A_1(t) = \exp(-\int_{t_0}^{t} A(\tau)d\tau)$ satisfying the condition b) of Theorem 2.1 exists, det $A_1(t) = \exp(-n \int_{t_0}^{t} \alpha(\tau)d\tau)$, as the eigenvalues of a skew-symmetric real-valued matrix have the real part equal to zero. The first and second conditions in (2.3.16) hold, since $f$ is independent on $p$ and $S$, $Q_3$ depends only on $t$(see Remark 2.3.1). We choose as $U_0(t)$ the matrix $A_2(t)A_1(t)(\beta(t)U_2 - U_1(t))A_1^{-1}(t)$ (such that $A_2^{-1}(t)U_0(t)$ is skew-symmetric) to obtain $Q_3 = \lambda^{-1}(t)((\ln \lambda(t))' + 2\alpha(t) + \mu A_2(t))I$. If $\lambda(t) = \exp(-\varepsilon \int_{t_0}^{t} \alpha(\tau)d\tau)$, $\varepsilon = \text{const} > 0$, $\varepsilon \delta > 1$, then both integrals in (2.3.11), (2.3.12) converge. To guarantee the nonnegativity of $(\int_{t_0}^{t} \lambda^{-1}(t)((\ln \lambda(t))' + 2\alpha(t) + \mu A_2(t))I) + (2.3.11)$, $\lambda(t) = \exp(-\varepsilon \int_{t_0}^{t} \alpha(\tau)d\tau)$, $\varepsilon = \text{const} > 0$, $\varepsilon \delta > 1$, then both integrals in (2.3.11), (2.3.12) converge. To guarantee the nonnegativity of $(\int_{t_0}^{t} \lambda^{-1}(t)((\ln \lambda(t))' + 2\alpha(t) + \mu A_2(t))I) + (2.3.11)$, we must satisfy the following inequalities for $\varepsilon$ and $q$: $q < 1$, $\varepsilon \geq \frac{1}{1-q}, \varepsilon \leq \frac{(\gamma-1)n}{2q}$, $\varepsilon \leq 2$ (the last inequality arises only in the case $\mu = 0$), the conditions (2.3.12) hold for $\varepsilon \leq \frac{(\gamma-1)n}{2q}$. For $\mu > 0$, it implies $\frac{1}{1-q} \leq \varepsilon \leq \frac{(\gamma-1)n}{2q}$, $\frac{1}{\delta} < a \leq \frac{(\gamma-1)n}{2q}$, $q < \bar{q} := \min\{\frac{\delta n(\gamma-1)}{2}, \frac{n(\gamma-1)}{2+\gamma n(\gamma-1)}\}$, and we choose $q \in (0, \bar{q})$. If $\mu = 0$, then additionally we have inequalities $\frac{1}{q} < \varepsilon \leq 2$, $\frac{1}{1-q} \leq \varepsilon \leq 2$. Since $\delta < \frac{1}{2}$, then we can choose $\varepsilon$ to satisfy the first inequality, for the second we have $q \leq \frac{1}{2}$. In this case we choose $q \in (0, \min\{\bar{q}, \frac{1}{2}\})$. Thus, all conditions of Theorem 2.1 are satisfied and the Corollary is proved.

Examples. Thus, as follows from Corollary 2.1, in the cases $\mu = l = 0$ and $\mu > 0$ in the Sections 2.1.1 ($n = 2$) and 2.2 ($n = 3$) we have constructed the velocity field for the interior solution. Really, the right-hand side has the form indicated in the Corollary statement, the velocity has linear profile $V = A(t)r$, with $A(t)$ of special form, that is $A(t) = \alpha(t)I + \beta(t)U_2$ with the skew-symmetric $U_2$. For $\mu = l = 0$, we have $\alpha(t) \sim t^{-1}, t \to \infty, \delta = 1 > \frac{1}{2}$.

For $\mu > 0$, we get $\alpha(t) \sim \frac{1}{\delta}t^{-1}, t \to \infty, \delta = \frac{1}{\gamma} > 0$ ($n = 2$), and $\delta = \frac{1}{3\gamma - 1} > 0$ ($n = 3$)(remark that the denotation $\delta$ has here other sense than in the subsection 2.2.)

The solution constructed in the Section 2.1.2 is also interior for $\mu = l = 0$. As follows from Proposition 2.1, in the case $A(t) \sim \alpha(t)I$, where $\alpha(t) = \delta t^{-1}, t \to \infty, \delta > \frac{1}{2}$.

Moreover, as the numerical analysis suggests, all solutions with linear profile of velocity (of general form) are interior if $\mu > 0$, $l = 0$, either for $b(t) = 0$ or $b(t) \neq 0$ (it seems that in the last case $b(t) \sim Bt^{-1}$, with the constant vector $B$). However, the last assertion can be considered only as a hypothesis. The case $\mu > 0$, $l \neq 0$ is more complicated, because it seems that it accepts stable equilibriums except of the origin.

Remark 2.3.3 We can find the entropy function for the solution constructed in Sections 2.1 and 2.2. For example, in the simplest case (Section
2.2.1) 

\[ S(t, |r|, \phi) = S_0(|r| \exp\left(-\int_0^t \alpha(\tau)d\tau\right), \phi + \int_0^t \beta(\tau)d\tau). \]

For example, for the initial data (2.1.12) \( S_0 = const + (a(\gamma - 1) + \gamma) \ln(1 + |r|^2) \).

In spite of the components of the pressure and the density must vanish as \(|x| \to \infty\), the entropy may even increase (remember that the conditions to Theorem 2.1 require only the boundedness of the entropy gradient).

**Remark 2.3.4** In the case \( G(0) \neq 0 \) the density cannot be compactly supported (in contrast with \( G(0) = 0 \)). Really, since \( p = \frac{\pi}{\gamma-1} \), \( \rho = \frac{\pi}{\gamma-1} \cdot e^{-\frac{S}{\gamma}} \), then due to the compatibility conditions \( \pi \nabla \pi \sim const \cdot e^{-\frac{S}{\gamma}} \), \( |x| \to c \), where \( c \) is a point of the support of \( \pi \). Therefore, for \( C^1 \) smooth \( \pi \) it occurs that \( S \to +\infty \), \( |x| \to c \), and we cannot choose any smooth initial data. It is interesting that if one requires only \( C^0 \)– smoothness of \( \pi \) (\( \pi \sim const \cdot (c - |x|)^{1/2} \), \( |x| \to c \), \( \pi = 0 \), \( |x| \geq c > -c \)), the condition may be fulfilled. Moreover, \( \rho \) and \( p \) will be of the \( C^1 \)- class of smoothness, however, neither the theorem on the local in time existence of the smooth solution, no Theorem 2.1 can be applied.

3. **Acceptable velocities: generalization of velocity with linear profile**

3.1. **The auxiliary system of transport equations.** Consider the following system of equations associated with (E1–E3):

\[ \partial_t V + (V, \nabla)V = 0. \quad (3.1.1) \]

Here \( V \) is a vector field from the tangent bundle of \( \Sigma \).

We look for the smooth solution to (3.1.1) of the separated form

\[ V = A(t)\Lambda(x) \quad (3.1.2) \]

with a square \((n \times n)\) matrix \( A(t) \).

The following possibilities can be realized:

**I.**

\[ \nabla_i \Lambda^j = const \cdot \delta_i^j, \quad (3.1.3)(A1) \]

with the Kronecker symbol \( \delta_i^j \), \( A'(t) = -A^2(t) \).

If \( Sp(A(0)) \cap \mathbb{R}_- = \emptyset \), then \( A(t) \) is bounded all over the time \( t \geq \infty \).

It is easy to see that for the euclidean metrics there are the unique possibility \( \Lambda = ar + b \), where \( r \) is the radius-vector of point, \( a, b \) are constants, so we come back to the velocity with the linear profile.

**II.**

\[ A' = A \nabla_j A^i, \quad i = 1, ..., n, \quad (3.1.4)(A2) \]

and \( A(t) = a(t)I \), with the identity matrix \( I \), \( a(t) \) is a function such that \( a'(t) = -a^2(t) \).
In the second case we essentially restrict the set of matrices, but the set of corresponding vector fields $\Lambda$ is more rich then in the case (A1).

**Remark 3.1.1.** The equations (A1) may be incompatible (i.e. for the 2D sphere with natural coordinates).

**Remark 3.1.2.** If $\Lambda$ satisfies (A1), then it satisfies (A2).

### 3.1.1. The vector field satisfying condition (A2) with a constant divergence.

In our further considerations the vector fields $\Lambda$ satisfying (A2) having a constant divergence, play an important role, therefore we study them in detail.

**Theorem 3.1.** If the nontrivial potential vector field satisfying (A2) has a constant divergence $D = D_0$, then $D_0 > 0$.

**Proof.** Really,

$$
(\nabla \Lambda) = \sum_{i,j} \nabla_i (\Lambda^j \nabla_j \Lambda^i) = 2 \sum_{i \neq j} \nabla_i \Lambda^j \nabla_i \Lambda^j + \sum_i (\nabla_i \Lambda^i)^2 + \sum_j \Lambda^j \nabla_j (\nabla \Lambda).
$$

(3.1.5)

(in the formula we do not sum over indices $i$ in $(\nabla_i \Lambda^i)$).

If the field $\Lambda$ is potential (that is there exist a function $\Phi$ such that $\Lambda = \nabla \Phi$), then $\nabla_i \Lambda^j = \nabla_j \Lambda^i$. If we suppose that the divergence is constant, then

$$
D_0 = 2 \sum_{i \neq j} (\nabla_i \Lambda^j)^2 + \sum_i (\nabla_i \Lambda^i)^2.
$$

Thus, $D_0$ is a sum of squares and it is therefore nonnegative. However, if $D, \nabla \Lambda = 0$, then $\nabla_i \Lambda^j = 0$ under all combinations of indices, therefore $\Lambda$ is constant and, as follows from (A2), equal to zero, that is trivial in spite of the assumption.

Theorem 3.1 has the evident corollary.

**Corollary 3.1.** There is no nontrivial potential divergence free vector field satisfying (A2).

**Remark 3.1.3.** As we shall see below, the possibility to choose the divergence free vector field should involve the possibility to construct the solutions to the Euler equations with the properties of localization of mass in a point. But the situation is restricted for the systems, containing pressure.

The case of dimension greater than one is more interesting for us, as for the dimension equal to one the situation is the following: if nontrivial vector field satisfies condition (A2), then its divergence is equal to 1 automatically.

Further we assume that the field $\Lambda$ has at least two times differentiable components.

Let us remark that a smooth vector field with constant nonzero divergence cannot exist on the compact manifold without boundary. Indeed, it contradicts to the fact that the integral from the divergence taken over all manifold has to be zero.
Theorem 3.2. There exists no nontrivial divergency free vector field given on the manifold of dimension \( n \) satisfying condition (A2).

Denote
\[
J_m = \sum_{i_k \in K_m, j_k \neq i_k, k=1,...,m} (\nabla_{i_1} \Lambda^{i_1} \cdots \nabla_{i_m} \Lambda^{i_m} - \nabla_{j_1} \Lambda^{i_1} \cdots \nabla_{j_m} \Lambda^{i_m}),
\]
where the summation is taken over all subsets \( K_m = i_1, ..., i_m, 1 \leq m \leq n \), from the set \( 1, ..., n \) and the set \( j_1, ..., j_m \) is a permutation of elements of \( K_m \).

We need the following lemma

Lemma 3.1. Let the field \( \Lambda \) satisfy (A2) and have a constant divergency \( D \). Then the following representation holds:
\[
D = D^m + \text{const} \cdot J_m + \mathcal{M}(D, J_2, ..., J_{m-1}), \tag{3.1.6}
\]
where \( \mathcal{M}(D, J_2, ..., J_{m-1}) \) is a polynomial of \( k \)-th degree \( 0 < k < m \), \( m = 2, ..., n \).

Proof of Lemma 3.1. From (A2) we get
\[
\Lambda^{i_1} = \Lambda^{i_2} \nabla_{i_2} \Lambda^{i_1} = \Lambda^{i_2} \nabla_{i_3} \Lambda^{i_2} \nabla_{i_2} \Lambda^{i_1} = \Lambda^{i_m} \nabla_{i_m} \Lambda^{i_m-1} \cdots \nabla_{i_2} \Lambda^{i_1},
\]
where \( i_k = 1, ..., n, k = 1, ..., n, m \leq n \). Therefore
\[
D = \sum_{i_1=1}^{n} \nabla_{i_1} \Lambda^{i_1} = \nabla_{i_1} \Lambda^{i_m} \nabla_{i_m} \Lambda^{i_m-1} \cdots \nabla_{i_2} \Lambda^{i_1} + \Lambda^{i_m} \nabla_{i_1} (\nabla_{i_m} \Lambda^{i_m-1} \cdots \nabla_{i_2} \Lambda^{i_1}) + \nabla_{i_1} \Lambda^{i_m} \nabla_{i_m} \Lambda^{i_m-1} \cdots \nabla_{i_2} \Lambda^{i_1} + \Lambda^{i_m} \nabla_{i_1} D.
\]
If \( D = \text{const} \), then \( D = D^m + \text{const} \cdot J_m + \mathcal{M}(D, J_2, ..., J_{m-1}) \), the last polynomial is homogeneous of \( m \)-th degree with respect to \( \nabla \Lambda^j, i, j = 1, ..., n \).

Example. The rather thorough (for \( m > 2 \)) calculation shows that the following formulas are true:
\[
D = D^2 - 2J_2, \tag{3.1.7}
\]
\[
D = D^3 + 3J_3 - 3DJ_2, \tag{3.1.8}
\]
\[
D = D^4 + 4J_4 - 4D^2J_2 + DJ_3 + 2J_2^2. \tag{3.1.9}
\]

Lemma 3.2. For the field \( \Lambda \) satisfying (A2) the following representation takes place
\[
1 - D + J_2 - J_3 + J_4 - ... + J_n = 0, \tag{3.1.10}
\]

Proof of Lemma 3.2. Consider the system, defined by the condition (A2) as a linear homogeneous system with respect to the variables \( \Lambda^i \). The condition (3.1.10) is namely the condition of existence of its nontrivial solution, that is the corresponding determinant of linear system is equal to zero. So the proof is over.
Proof of Theorem 3.2. Suppose that $D = 0$. Then from the representation (3.1.6) we obtain consecutively that $J_m = 0$, $m = 2, ..., n$. This fact contradicts to (3.1.10).

**Theorem 3.3.** Suppose that on the manifold $\Sigma$ of dimension $n$ there exists the vector field $\bar{\Lambda}$ satisfying condition (A1). If a nontrivial vector field, given on $\Sigma$ and satisfying condition (A2) has a constant divergency $D$, then $D$ can take only integer values from 1 to $n$.

**Proof.** Equations (3.1.6) for $m = 2, ..., n$ and (3.1.10) form the system of $n$ algebraic equations for $n$ unknown variables $(D, J_2, ..., J_n)$. We express consecutively $J_2, ..., J_n$ from equations (3.1.6) through the divergency $D$ and substitute the result in (3.1.10). In such way we obtain the equation of $n$-th degree for $D$. Thus, $D$ cannot take more then $n$ different real values. But the field $\bar{\Lambda}$, as well as its projections to the subspaces of dimension 1, ..., $n - 1$, satisfy condition (A2), moreover, their divergency takes exactly $n$ values, namely, 1, 2, ..., $n$. Thus, $D$ cannot take another values and the Theorem 3.3 is proved.

**Remark 3.1.4.** For the euclidean space $\bar{\Lambda} = r$, where $r$ is the radius-vector of point.

**Remark 3.1.5.** To all appearances, the requirement of existence of the field satisfying (A1), is unnecessary, that is in the case where the divergency of the field $\bar{\Lambda}$ with the property (A2) is constant, it is equal to a natural number, later or equal to the dimension of space. At least the fact holds in the spaces of dimensions 2 and 3, as the following proposition shows.

**Proposition 3.1.** If a nontrivial vector-field given on the manifold $\Sigma$ of dimension $n = 2$ or $n = 3$ and satisfying condition (A2) has the constant divergency $D$, then $D$ can take only natural values from 1 to $n$.

**Proof.** a) If $n = 2$, then from (3.1.6), (3.1.10) we have that $D = D^2 - 2J_2$, $1 - D + J_2 = 0$, or $D^2 - 3D + 2 = 0$. The last equation has two roots $D = 1$, $D = 2$.

b) Analogically for $n = 3$ we have the system of equations $D = D^2 - 2J_2$, $D = D^3 + 3J_3 - 3DJ_2$, $1 - D + J_2 - J_3 = 0$. Therefore $D^3 - 6D + 11D - 6 = 0$, and the roots are $D = 1$, $D = 2$, $D = 3$.

3.1.2. A generalization of condition (A2). The solution to the equation (A2) is not unique. Suppose that we succeed in finding of finite or infinite number of the solutions such that $\Lambda_k$, $k \in K$, $K \subseteq N$, having the additional property

$$\Lambda_m \nabla_j \Lambda_l^j + \Lambda_l^j \nabla_j \Lambda_m^j = \sum_k \beta_k \Lambda_k, \ m \neq 1, \ \beta_k = const., k, m, l \in K. \ (3.1.11)$$

Then there exists a solution of the form

$$u = \sum_k a_k(t) \Lambda_k.$$
The functions \( a_k(t) \) satisfy the system of equations
\[
a'_k(t) = -a_k^2(t) - \beta_k a_k(t) \sum_{1 \neq k} a_1(t), \quad k \in \mathbf{K}. \tag{3.1.12}
\]

4. Two-dimensional manifold

The section is devoted to the important for applications case of a two-dimensional manifold.

4.1. Transport equation on the manifold covered by one chart. Let \((x^1, x^2)\) be the coordinates on the manifold, and \((\Lambda^1, \Lambda^2)\) be the components of the solution to equation (3.1.4). Thus, the equations (A2) has the form
\[
\begin{align*}
\Lambda^1 &= \Lambda^1 \frac{\partial \Lambda^1}{\partial x^1} + \Lambda^2 \frac{\partial \Lambda^1}{\partial x^2} + \Gamma_{11}^1 (\Lambda^1)^2 + 2\Gamma_{12} \Lambda^1 \Lambda^2 + \Gamma_{22}^1 (\Lambda^2)^2, \\
\Lambda^2 &= \Lambda^1 \frac{\partial \Lambda^2}{\partial x^1} + \Lambda^2 \frac{\partial \Lambda^2}{\partial x^2} + \Gamma_{11}^2 (\Lambda^1)^2 + 2\Gamma_{12} \Lambda^1 \Lambda^2 + \Gamma_{22}^2 (\Lambda^2)^2
\end{align*}
\tag{4.1.1}
\]

If we denote \(Z = \Lambda^2_x\), then as a corollary of (4.1.1–4.1.2) we obtain the quasilinear equation
\[
\frac{\partial Z}{\partial x^1} + Z \frac{\partial Z}{\partial x^2} = \Gamma_{22}^1 (Z)^3 + (2\Gamma_{12} - \Gamma_{22}^1)(Z)^2 + (\Gamma_{11} - 2\Gamma_{12})Z - \Gamma_{11},
\tag{4.1.3}
\]
which can be solved by the characteristics method:
\[
\begin{align*}
dZ &= \Gamma_{22}^1 (Z)^3 + (2\Gamma_{12} - \Gamma_{22}^1)(Z)^2 + (\Gamma_{11} - 2\Gamma_{12})Z - \Gamma_{11}, \\
\frac{dx^2}{dx^1} &= Z.
\end{align*}
\]
If \(Z(x^1, x^2)\) is found, then from (4.1.1) we obtain the linear equation
\[
\frac{\partial \Lambda^1}{\partial x^1} + Z \frac{\partial \Lambda^1}{\partial x^2} + (\Gamma_{11} + 2\Gamma_{12}Z + \Gamma_{22}^1 (Z)^2)\Lambda^1 = 1,
\tag{4.1.4}
\]
for the function \(\Lambda^1\). As \(\Lambda_2 = Z \Lambda_1\), then the solution components are found.
If \(\lambda_1 = \sqrt{g_{11}} \Lambda^1\) are the physical components of the field \(\Lambda\), then
\[
\begin{align*}
\frac{\partial \lambda_1}{\partial x^1} + \frac{\sqrt{g_{11}} \lambda_2}{\sqrt{g_{22}}} \frac{\partial \lambda_1}{\partial x^2} + \frac{\sqrt{g_{11}}}{\sqrt{g_{22}}} \Gamma_{12} \lambda_2 + \frac{g_{11}}{g_{22}} \Gamma_{22}^1 \frac{(\lambda_2)^2}{\lambda_1} &= \sqrt{g_{11}}, \\
\frac{\partial \lambda_2}{\partial x^1} + \frac{\sqrt{g_{11}} \lambda_2}{\sqrt{g_{22}}} \frac{\partial \lambda_2}{\partial x^2} + \frac{\sqrt{g_{22}}}{\sqrt{g_{11}}} \Gamma_{21} \lambda_1 + \frac{g_{22}}{g_{11}} \Gamma_{21}^2 \lambda_2 &= \sqrt{g_{11}} \lambda_2.
\end{align*}
\]
If \(z = \frac{\lambda_2}{\lambda_1}\), then the function satisfies the equation
\[
\begin{align*}
\frac{\partial z}{\partial x^1} + \frac{\sqrt{g_{11}} z}{\sqrt{g_{22}}} \frac{\partial z}{\partial x^2} + \frac{\sqrt{g_{22}}}{\sqrt{g_{11}}} z^2 + \frac{\sqrt{g_{11}}}{\sqrt{g_{22}}} \Gamma_{12} z - \frac{\sqrt{g_{11}}}{\sqrt{g_{22}}} \Gamma_{21} z^2 - \frac{g_{11}}{g_{22}} \Gamma_{12}^2 z^3 &= 0,
\tag{4.1.5}
\end{align*}
\]

\(\ast\ast\ast\)
the components $\lambda_1$ and $\lambda_2$ can be found from the equation
\[
\frac{\partial \lambda_1}{\partial x^1} + \frac{\sqrt{g_{11}}}{\sqrt{g_{22}}} \frac{\partial \lambda_1}{\partial x^2} + \frac{\sqrt{g_{11}}}{\sqrt{g_{22}}} \Gamma_{12} \lambda_1 z + \frac{g_{11}}{g_{22}} \Gamma_{12}^2 \lambda_1 = \sqrt{g_{11}},
\]
\[
\frac{\partial \lambda_2}{\partial x^1} + \frac{\sqrt{g_{11}}}{\sqrt{g_{22}}} \frac{\partial \lambda_2}{\partial x^2} + \frac{\sqrt{g_{11}}}{\sqrt{g_{22}}} \Gamma_{21} \lambda_2 z + \frac{g_{11}}{g_{22}} \Gamma_{21}^2 \lambda_2 = \sqrt{g_{11}} z.
\] (4.1.6)

Here $\lambda_1 = \frac{\lambda_2}{z}$.

REMARK 4.1.1. As we shall see, the important role plays a possibility to find a potential vector field satisfying (A2). In the terms of the potential $\Phi$ (such that $\Lambda = \nabla \Phi$) condition (A2) takes the form
\[
\nabla \Phi = (\nabla \Phi, \nabla) \nabla \Phi
\]
or
\[
\nabla^1 \Phi = \nabla^1 \Phi \nabla^1 \Phi + \nabla^2 \Phi \nabla^2 \Phi,
\]
\[
\nabla^2 \Phi = \nabla^1 \Phi \nabla^2 \Phi + \nabla^2 \Phi \nabla^2 \Phi.
\]

If we consider the system as linear homogeneous with respect to $\nabla^1 \Phi$ and $\nabla^2 \Phi$ we obtain the necessary condition for the existence of the nontrivial potential $\Phi$:
\[
\text{Hess} \Phi - \Delta \Phi + 1 = 0,
\]
where $\text{Hess} \Phi = \nabla^1 \Phi \nabla^2 \Phi - \nabla^2 \Phi \nabla^1 \Phi$, $\Delta \Phi = \nabla^1 \Phi + \nabla^2 \Phi$.

REMARK 4.1.2. For the equation of the form
\[
\Xi(x^1, x^2, \Lambda) \Lambda^i = \Lambda^j \nabla_j \Lambda^i, \quad i = 1, 2,
\]
whose particular case is (4.1.2), the equation (4.1.3) is true as well. But instead of (4.1.4) we obtain the equation
\[
\frac{\partial \Lambda^1}{\partial x^1} + \frac{\partial \Lambda^1}{\partial x^2} + \left( \Gamma_{11}^1 + 2 \Gamma_{12} Z + \Gamma_{22}^1 (Z)^2 \right) \Lambda^1 = \Xi(x^1, x^2, |\Lambda|). \quad (4.1.7)
\]

In the case of aerodynamic friction, where $F(V) = -\mu_1 V |V|$, $\mu_1 = \text{const} > 0$, we can obtain
\[
\Xi(x^1, x^2, |\Lambda|) = 1 - \mu_1 \text{sign } a(t) |\Lambda^1| \sqrt{1 + (Z)^2}.
\]

REMARK 4.1.3. If the dimension $n$ is greater than 2, by means of introduction of a new variable $Z_{k-1} = \frac{\Lambda^k}{A}$, $k = 2, ..., n$, acting analogously, one can deduce the equation (A2) to the system of $n-1$ equations for $Z_i$, $i = 1, ..., n-1$. In the case of euclidean metrics it will be $n-1$-dimensional system of transport equations as well, therefore the procedure of the dimension reduction we can apply several times up to reduction to one equation.
4.1.1. Nonlinear transport equation with the Coriolis force on a two dimensional surface. We mean the equation
\[ u_t + (u, \nabla)u = l u_\perp, \]
or
\[ u_i^t + u^j \nabla_j u^i = le^i_j u^j, \tag{4.1.8} \]
where \( l = l(x^1, x^2) \) is a Coriolis parameter, \( u_\perp^i = e^i_j u^j \), \( e_{ij} \) is the skew-symmetric Levi-Civita tensor.

We seek for a solution of the equation in the form
\[ u = a(t) \Lambda + \Xi, \tag{4.1.9} \]
where \( \Lambda \) and \( \Xi \) are vectors, which do non depend on time.

In the case \( a'(t) = -a^2(t) \) as before, the vector \( \Lambda \) satisfies condition (A2), the vector \( \Xi \) is a stationary solution to (4.1.8), that is the solution to the equation
\[ le^i_j \Xi^j = \Xi^j \nabla_j \Xi^i, \ i = 1, \ldots, n. \tag{4.1.10} \]

Moreover, the vectors \( \Lambda \) and \( \Xi \) can be compatible in some sense, that is have to satisfy the equation
\[ le^i_j \Lambda^j = \Xi^j \nabla_j \Lambda^i + \Lambda^l \nabla_j \Xi^i, \ i = 1, \ldots, n. \tag{4.1.11} \]

Thus the solution of form (4.1.9) is a sum of a nonstationary solution to the nonrotational equation (4.1.1) and a stationary solution to (4.1.8). Of course, it is not evident before that the condition (4.1.11) takes place.

4.2. The case of the plane. For the euclidean metrics all Christoffel symbols are equal to zero, the geometrical components of the vector \( \Lambda \) are equal to physical ones. Therefore the equation (4.1.3) coincides with the transport equation (here \( (x^1, x^2) = (x^1, x^2) \) are the charthesian coordinates):
\[ \frac{\partial z}{\partial x_1} + z \frac{\partial z}{\partial x_2} = 0. \tag{4.1.12} \]
As well known (f.e., [17]), the solution to equation (4.1.12) can be given implicitly in the form \( z = F(x_2 - zx_1) \), where \( z(0, x_2) = F(x_2) \) is the Cauchy datum on the noncharacteristic curve \( x_1 = 0 \). The functions \( \Lambda_1(x_1, x_2), \Lambda_2(x_1, x_2) \) can be found from equations
\[ \frac{\partial \Lambda_1}{\partial x_1} + z \frac{\partial \Lambda_1}{\partial x_2} = 1, \]
\[ \frac{\partial \Lambda_2}{\partial x_1} + z \frac{\partial \Lambda_2}{\partial x_2} = z. \]

Note that the presence of singularities of a solution to (4.1.12) does not signify the presence of singularities of \( \Lambda_1, \Lambda_2 \).

The solution with linear profile of velocity of the form \( V = a(t)I r + B(t) \)
where \( r \) is a radius-vector of point, corresponds to \( z = \frac{az_1 + bz_2}{ax_1 + by_1} \).
It is not difficult to see that $z = K = \text{const}$ corresponds to the velocity field

$$\Lambda_1 = x_1 + \phi(x_2 - Kx_1), \quad \Lambda_2 = Kx_1 + K\phi(x_2 - Kx_1),$$

(4.1.13)

where $\phi$ is an arbitrary differentiable function. The divergency of the velocity field is equal to 1.

Remark that it can be potential only if it coincides (up to the rotation with respect to the origin) with the linear field, depending only on one coordinate.

Thus, we have found the method of constructing solutions with separated variables of the form $u = a(t)\Lambda$ of the two-dimensional transport equation on the plane.

Stress that the solution cannot be divergency free.

Let us try to find the condition of the existence of a solution of form

$$u = a_1(t)\Lambda_1 + a_2(t)\Lambda_2,$$

where $\Lambda_1$ is given by formula (4.1.13), and $\Lambda_2 = r$ is a radius-vector of point. We can clear up, when condition (4.1.11) is true. The computation shows that it takes place only for $\phi(\xi) = C\xi$, $\beta_1 = 2$, $\beta_2 = 0$, that is if $\Lambda_1^1 = (1 - CK)x + Cy$, $\Lambda_1^2 = K(1 - CK)x + CKy$. The functions $a_i(t)$, $i = 1, 2$ can be found from the system

$$a_1(t) = -a_1^2(t) - 2a_1(t)a_2(t), \quad a_1(t) = -a_1^2(t).$$

However the obtained solution has the form $u = A(t)\Lambda$, $\Lambda = r$, where $A'(t) = -A^2(t)$, that is we have found a solution of form (4.1.2), with the additional condition (A1)). It is the solution with linear profile of velocity described before.

4.2.1. Nonlinear transport equation on the two-dimensional sphere. For the sphere of radius $r$ with the standard orthogonal coordinates $x^1 = \phi \in ]-\pi, \pi]$, $x^2 = \theta \in [0, \pi]$ the metric tensor is the following: $g_{11} = r^2\sin^2\theta$, $g_{22} = r^2$, $\Gamma^2_{11} = -\sin\theta\cos\theta$, $\Gamma^1_{12} = \cot\theta$, the other Christoffel symbols are equal to zero.

Thus, from equation (4.1.5) for the ratio $z = \frac{u}{v}$ of physical components of the velocity $V = (u, v)$ we have

$$\frac{dz}{d\phi} = \cos\theta(1 + z^2), \quad \frac{d\theta}{d\phi} = \sin\theta z.$$

The equation may be solved explicitly. Namely,

$$z = \pm \sqrt{-1 + \sin^2\theta\Psi \left( \phi \pm \int_{\theta_0}^{\theta} \frac{d\tau}{\sin\tau\sqrt{K + \ln\sin^2\tau}} \right)},$$

(4.1.14)

with an arbitrary differential function $\Psi$. Remark that this solution exists only in the strip $|\sin\theta| > \exp(-\frac{K}{2})$, it implies $K > 0$. 

Example. Consider at greater length the simplest case $\Psi = C = \text{const}$. We can see from (4.1.14) that the solution is defined by the formula only if $|\sin \theta| \geq \frac{1}{\sqrt{C}}$, that is in some spherical strip $\Pi$ surrounding the equator, moreover, the solution is zero on the boundary of the strip at $\theta_* = \pm \arcsin \frac{1}{\sqrt{C}}$.

Further, to define the meridional component $v$ we use the equation (4.1.6), so that

$$\frac{\partial v}{\partial \phi} + \sin \theta z \frac{\partial v}{\partial \theta} - \cos \theta \frac{v}{z} = r \sin \theta z.$$ 

Further considerations we carry out for the nonnegative branch $z = \sqrt{C \sin^2 \theta - 1}$. For the branch with the opposite sign we can act in a similar way. Therefore,

$$\frac{\partial v}{\partial \phi} + \sin \theta \sqrt{C \sin^2 \theta - 1} \frac{\partial v}{\partial \theta} - \frac{\cos \theta}{\sqrt{C \sin^2 \theta - 1}} v = r \sin \theta \sqrt{C \sin^2 \theta - 1}.$$ 

Solving the equation we obtain

$$v = -\frac{r}{\sqrt{C}} \arcsin \frac{\sqrt{C \cos \theta \sqrt{C \sin^2 \theta - 1}}}{\sqrt{C - 1}} \frac{1}{\sin \theta} + \frac{\sqrt{C \sin^2 \theta - 1}}{\sin \theta} \Psi_1(\phi - \mathcal{R}(\theta)), \quad (4.1.15)$$

where $\Psi_1$ is an arbitrary differentiable function, and

$$\mathcal{R} = -\ln \tan \frac{\theta}{2} + \frac{\sqrt{C}}{\sqrt{C - 1}} \arctan \frac{\tan^2 \frac{\theta}{2}}{2C} + 1.$$ 

Further, for the parallel directed component we have the formula

$$u = -\frac{r}{\sqrt{C}} \arcsin \frac{\sqrt{C \cos \theta}}{\sqrt{C - 1}} \frac{1}{\sin \theta} + \frac{1}{\sin \theta} \Psi_1(\phi - \mathcal{R}(\theta)). \quad (4.1.16)$$

We see that the meridional component $v$ vanishes on the boundary of the spherical strip $\Pi$, and $u$ remains bounded but, generally speaking, does not vanish. Therefore, the motion takes place inside $\Pi$ independently from zones near poles of the sphere. So the stream flows round the polar zones.

Remark that the vector field from this example is not potential.

5. Constructing solution to the Euler equations

The procedure of constructing solutions to system (E1–E2) consists of the following steps:

1. Find the velocity vector

$$\mathbf{V} = a(t) \mathbf{A}(x), \quad (5.1)$$

with the $C^1$-smooth field $\mathbf{A}$ given on $\Sigma$ satisfying (A2).
2. Supposing that we know \( V \), find \( \rho \) and \( p \) from the equations (E1), (E3), linear with respect to these functions.

3. Choose the initial data \( \rho_0 \) and \( p_0 \) such that the compatibility condition

\[
\nabla p(t,x) = \phi(t)\rho(t,x)\Lambda - \rho(t,x)F(t,x,a(t)\Lambda,\rho,p) \tag{5.2}
\]

holds with some function \( \phi(t) \).

Remark that it may be that for a certain manifold there are no initial conditions \( \rho_0, p_0 \) and the vector field \( \Lambda \) to satisfy (5.2).

However, as we see (section 2), there are examples, where it really holds, at least in the euclidean space.

If we additionally suppose that \( F = F(V) \) and it is homogeneous with respect to \( V \) (the case of dry and aerodynamic friction, and of the Coriolis force as well), then the compatibility condition takes simpler form

\[
\nabla p(t,x) = \phi(t)\rho(t,x)(\Lambda - F(\Lambda)). \tag{5.3}
\]

5.1. The velocity field. Integral functionals for the Euler equations.

5.1.1. The zero right hand side. If \( \Lambda \) is found (for example by the method described in Section 2), then the further problem is to find the function \( a(t) \).

Suppose that in system (E1–E3) the mass \( M = \int_\Sigma \rho d\Sigma \), and the total energy \( E = E_k(t) + E_p(t) \) are conserved. Here the kinetic and potential energies are defined as

\[
E_k(t) = \frac{1}{2} \int_\Sigma \rho |V|^2 d\Sigma, \quad E_p(t) = \frac{1}{\gamma - 1} \int_\Sigma \rho d\Sigma,
\]

This conservation laws always take place for the smooth solutions on the compact manifold. If the manifold is not compact, then it is sufficiently to require a rather quick vanishing of \( \rho \) and \( p \) at infinity with respect to the space variables, to fulfill the convergence of integrals expressing the mass and energy and other integrals, that we mention below, as well.

Recall that we seek for the velocity in the form (5.1).

Introduce the functionals

\[
G_m(t) = \frac{1}{2} \int_\Sigma \rho |\Lambda|^m d\Sigma,
\]

\[
Q(t) = \int_\Sigma \rho D d\Sigma, \quad D = \nabla_4 \Lambda^4, \quad m > 0,
\]

\[
F(t) = \frac{1}{2} \int_\Sigma \rho(\nabla |\Lambda|^2, \nabla) d\Sigma.
\]

Denote \( G_2(t) := G(t) \).
5.1.2. A potential velocity field. Suppose that the field $\mathbf{\Lambda}$ is potential, that is
\[ \nabla_i \Lambda^j = \nabla_j \Lambda^i, \quad i \neq j, \quad R \mathbf{\Lambda} = \frac{1}{2} \nabla |\mathbf{\Lambda}|^2. \]

It is not difficult to see that here for the smooth solutions to the system (E1–E3) the following relations hold
\[
G'(t) = F(t), \quad (5.1.1)
\]
\[
F(t) = 2a(t)G(t), \quad (5.1.2)
\]
\[
G'_m(t) = ma(t)G_m(t), \quad (5.1.3)
\]
\[
F'(t) = 2E_k(t) + Q(t), \quad (5.1.4)
\]
\[
E'_p(t) = -E'_k(t) = - (\gamma - 1) a(t)Q(t), \quad (5.1.5)
\]
\[
E_k(t) = a^2(t)G(t), \quad (5.1.6)
\]
\[
E_p(t) + a^2(t)G(t) = \mathcal{E} = \text{const.} \quad (5.1.7)
\]

It follows that for the solutions with the velocity field of form (5.1) we obtain the system
\[
G'(t) = 2a(t)G(t), \quad (5.1.8)
\]
\[
a'(t) = -a^2(t) + \frac{1}{2} \frac{Q(t)}{G(t)}. \quad (5.1.9)
\]

Moreover, from (5.1.3) we obtain
\[
G_m(t) = K_{ml}(G_l(t))^{\frac{m}{l}}, \quad K_{ml} = G_m(0)G_l(0)^{-\frac{m}{l}} = \text{const.} \quad (5.1.10)
\]

It will be more convenient to involve a new variable $\tilde{G}(t) = \frac{1}{G(t)} > 0$ and instead of (5.1.8–5.1.9) to consider the system
\[
a'(t) = -a^2(t) + \frac{1}{2} \frac{Q(t)}{G(t)} \tilde{G}(t), \quad (5.1.11)
\]
\[
\tilde{G}'(t) = -2a(t)\tilde{G}(t). \quad (5.1.12)
\]

However this system is not closed. Below we consider the cases were there is a possibility to obtain it in a closed form.

Note that the properties of the function $a(t)$ are different from the case of transport equation considered in Section 3.

**Remark 5.1.1.** The functional $Q(t)$ equals to zero identically if $p \equiv 0$ (so called pressure free gas dynamics or in the case of divergence free field $\mathbf{\Lambda}$). As follows from (5.1.9), if $a(0) < 0$, then the velocity $\mathbf{V}$ goes to infinity, and $G(t)$ goes to zero in a finite time. Taking into account the mass conservation we conclude that on the points where $|\mathbf{\Lambda}| = 0$ the mass concentrates on the set of zero measure. The phenomenon really takes place in the case of the pressure free gas dynamics. The second possibility don’t realize as $\mathbf{\Lambda}$ cannot have the zero divergence.
5.1.3. A non-potential velocity field. If the field $\mathbf{A}$ satisfies (A2), but is not potential, then the properties (5.1.1–5.1.5) take place, but the expression (5.1.6) for the kinetic energy $E_k(t)$ becomes more complicated. For example, in the two-dimensional space the following relation takes place:

$$E_k(t) = a^2(t)G(t) + 2 \int_\Sigma \rho J \Lambda^1 \Lambda^2 (1-D) d\Sigma,$$

with $D = \nabla_1 \Lambda^i$, $J = \nabla_2 \Lambda^1 - \nabla_1 \Lambda^2$. It is interesting that if the divergency $D$ is equal to 1, then (5.1.13) coincide with (5.1.6) and we obtain once more the system (5.1.11–5.1.12). Recall that there are two possible values of constant divergency, namely, $D = 1$ and $D = 2$ (see Theorem 3.3), the example of a vector field $\mathbf{A}$ on the plane with the divergency equal to 1 constructed in Subsection 2.1.1.

5.1.4. The field $\mathbf{A}$ with a constant divergency $D$. Suppose once more that $\mathbf{A}$ is potential. It is easy to see that if $D$ is constant, then

$$Q(t) = (\gamma - 1)DE_p(t), \quad (5.1.14)$$

$$E_p(t) = \text{const} \cdot G^{-\frac{(\gamma-1)D}{2}}(t). \quad (5.1.15)$$

Thus, the system (5.1.11–5.1.12) has the closed form

$$\ddot{G}(t) = -2a(t)\dot{G}(t), \quad a'(t) = -a^2(t) + \text{const} \cdot G^{-\frac{(\gamma-1)D}{2}} + 1. \quad (5.1.16)$$

The energy conservation gives

$$\text{const} \cdot G^{-\frac{(\gamma-1)D}{2}}(t) + a^2(t)G(t) = \mathcal{E}.$$ 

So for $D > 0$ the function $G(t)$ cannot go to zero ($a(t)$ to the infinity, respectively), therefore the mass cannot be concentrated in a point.

The only equilibrium point is the origin. The phase portrait is presented on Figure 1.
5.2. The bounded field $\Lambda$ with an arbitrary divergency $D$. If the divergency is not constant, then, generally speaking, we cannot obtain the closed system for finding $a(t)$. However, some properties of $a(t)$ we can get.

It is evident that $\tilde{G}(t) \geq 0$. In the equilibrium points we have $Q(t) = 0$. It may exist several points of such kind on the axis $a = 0$. Anyway, the phase trajectories of (5.1.11–5.1.12) are symmetric with respect to the axis.

Moreover, the function $a(t)$ cannot go to the infinity. Really, from (5.1.12) we have that $\tilde{G}(t) = \tilde{G}(0) \exp \left( -2 \int_{0}^{t} a(\tau) d\tau \right)$, therefore if $a(t) \to -\infty$, $t \to t_*$, then $\tilde{G}(t) \to \infty$ ($G(t) \to 0$) $t \to t_*$.

Further we use the generalization of Lemma from [19].

**Lemma 5.1.** Let $\Sigma$ be a smooth riemannian manifold with the metric tensor $g_{ij}$ such that $|g_{ij}|$ is separated from zero by a positive constant $g_*$. If on the manifold there exists a function $f(x)$ such that $f(x) \geq 0$, $f(x) \in L_\gamma(\Sigma)$, $|x|^2 f(x) \in L_1(\Sigma)$, and $\gamma > 1$, then

$$\int_{\Sigma} f d\Sigma \leq C_{\gamma,g_{ij}} \left( \int_{\Sigma} f^\gamma d\Sigma \right)^{\frac{2\gamma}{(n+2)\gamma-n}} \left( \int_{\Sigma} \left| x \right|^2 f d\Sigma \right)^{\frac{n(\gamma-1)}{(n+2)\gamma-n}},$$

where

$$C_{\gamma,g_{ij}} = g_\ast^{\frac{n(\gamma-1)}{((n+2)\gamma-n)}} \left( \frac{2\gamma}{n(\gamma-1)} - \frac{n(\gamma-1)}{(n+2)\gamma-n} \right).$$

According to (2.3.3) the quantity $S$ remains constant along particle trajectories, so that $S(t,x) \geq s_0$, where $s_0 = \min_{\Sigma} S_0(x)$.

We use Lemma 5.1 to obtain

$$E_p(t) = \frac{1}{\gamma-1} \int \rho^\gamma \exp S d\Sigma \geq \frac{\exp s_0}{\gamma-1} \int \rho^\gamma d\Sigma \geq \frac{K}{G^{\gamma(n-1)/2}(t)},$$

with

$$K = \frac{1}{(\gamma-1)^{2(n+2)(\gamma-n)/2}} \exp s_0 \left( \frac{\mathcal{M}}{C_{\gamma,g_{ij}}} \right)^{(n+2)(\gamma-n)/2},$$

and $\mathcal{M}$ is a conserved total mass.

Therefore, if $G(t) \to 0$, $t \to t_*$, then $E_p(t) \to \infty$, and we obtain the contradiction with the conservation of total energy.

**Remark 5.2.1** From the state equation (see Section 2.3) we have $p \geq \exp s_0 \rho^\gamma$, where $s_0 = \min_{\Sigma} S_0(x)$. The Hölder inequality gives us

$$\mathcal{M}^\gamma = \left( \int_{\Sigma} \rho d\Sigma \right) \leq (\Omega(t))^{\gamma-1} \int_{\Sigma} \rho^\gamma d\Sigma \leq \exp(-s_0) \sup_{t} \Omega(t) \int_{\Sigma} p d\Sigma = \beta_0 E_p(t),$$
with the area of the density support $\Omega(t)$ and the positive constant $\beta_0 = \exp(-s_0) \sup_t \Omega(t)(\gamma - 1)$. Therefore, if the support of the density is bounded, the potential energy cannot vanish and there exists an inaccessible potential energy, which cannot convert to the kinetic one. Namely,

$$E_p(t) \geq M^\gamma \beta_0^{-1}.$$ 

If the support of density is not bounded, then the inaccessible energy does not exist, generally speaking [19] (about the treatment on the inaccessible potential energy see also [20], [12]).

5.2.1. Two-dimensional manifold. This situation is more simple. Let us introduce functionals

$$Q_m(t) = \int_{\Sigma} pD^m d\Sigma, \quad m = 0, 1, \ldots,$$

where $D = \nabla_i \Lambda^i$. It is easy to see that $E_p(t) = (\gamma - 1)Q_0(t)$, the functional $Q(t)$ introduced in Section 5.1.1 is $Q_1(t)$ in the new notation.

Further, from (E3) for the velocity of form (5.1) we have

$$Q'_m(t) = a(t) \int_{\Sigma} p((\mathbf{A}, \nabla D^m) - (\gamma - 1)D^{m+1}) d\Sigma. \quad (5.2.1)$$

From (3.1.5) and (3.1.10) we obtain $(\mathbf{A}, \nabla D) = D - D^2 + 2J$ and $J = D - 1$, therefore (5.2.1) gives

$$Q'_m(t) = -ma(t) \int_{\Sigma} p((1 + \frac{\gamma - 1}{m})D^2 - 3D + 2) d\Sigma, \quad (5.2.2)$$

or

$$Q'_m(t) = -ma(t)(1 + \frac{\gamma - 1}{m})Q_{m+1}(t) + 3Q_m(t) - 2), \quad m \in \mathbb{N}. \quad (5.2.3)$$

For any $m$ the coefficient at $Q_{m+1}$ does not vanish, therefore the system (5.1.8), (5.1.9), (5.2.3) is also unclosed.

However it is interesting that

$$\text{sign } Q'_m(t) = -\text{sign } a(t)$$

for $m = 2k + 1, k = 0, 1, \ldots, m < 8(\gamma - 1)$. It follows from (5.2.2), as the expression

$$(1 + \frac{\gamma - 1}{m})D^2 - 3D + 2$$

is positive and separated from zero for $m < 8(\gamma - 1)$.

In particular,

$$Q_1(t) = -a(t)N(t), \quad (5.2.4)$$
where \( N(t) = \int_{\Sigma} p(\gamma D^2 - 3D + 2)p \, d\Sigma \), moreover

\[
N(t) \geq \alpha_0 E_p(t), \quad \alpha_0 = \frac{(8\gamma - 9)(\gamma - 1)}{4\gamma}, \quad \gamma > \frac{9}{8}.
\]

From (5.1.8) and (5.2.4) we have \( \frac{dQ_1}{dG} = -\frac{N}{2G} < 0 \). Therefore there exists a decreasing function \( Q_1 = Q_1(G) \) and the system (5.1.8), (5.1.9) can be written as

\[
G'(t) = 2a(t)G(t), \quad a'(t) = -a^2(t) + \frac{1}{2} \frac{Q_1(G)}{G}. \tag{5.2.5}
\]

Let us study the equilibriums of (5.2.5). Remark that at the phase plane \((G, a)\) the trajectories are symmetric with respect to the axis \( a = 0 \). The second coordinate of the equilibriums is \( a = 0 \), the first one is \( G \) satisfying the equation \( Q_1(G) = 0 \). Recall that if the root of the equation exists, it is unique. If the root does not exist, the first coordinate of the equilibrium goes to the infinity, in the case if the field \( \Lambda \) is unbounded. Otherwise, if \( G(t) \leq G_+ = \text{const} \), it signifies that there exists no globally in time smooth solution with the properties we try to construct. If we use once more the variable \( \tilde{G} = G^{-1} \), then we get the system

\[
\tilde{G}'(t) = -2a(t)\tilde{G}(t), \quad \tilde{a}'(t) = -a^2(t) + \frac{1}{2} \tilde{Q_1}(\tilde{G})\tilde{G},
\]

and the equilibrium point goes from the infinity to the origin. The phase portrait is similar to the one from Fig.1. If the field \( \Lambda \) is bounded, the restriction for \( \tilde{G} \) has the form \( \tilde{G} > G_+^{-1} \). The situation is presented on Fig.2.

If the manifold is compact, then \( G(t) \leq \frac{1}{2} M \max |\Lambda|^2 \) (if we suppose \( \Lambda \) to be smooth), therefore there is no equilibrium in the infinity. Note that \( Q_1(t) = 0 \) for the constant solution \( \rho = \rho_0, V = 0, p = p_0 \). Here the coordinates of
the only equilibrium are \((G_0, 0)\), with \(G_0 = \frac{1}{2} \int |\Lambda|^2 d\Sigma = \text{const} > 0\), it is a center (see Fig.3).

If the manifold is not compact, it may exist two equilibriums - in the finite point \((G_0, 0)\), where \(G_0\) is a root of the equation \(Q_1(G) = 0\) (a center) and in the infinite point \((G = +\infty, a = 0)\), (the point \((\tilde{G} = 0, a = 0)\) on the plain \((\tilde{G}, a)\) respectively). The situation is presented on Fig.4. The infinite point arises only if \(\Lambda\) is unbounded.

Remark 5.2.2 If we suppose that there exists the limit function \(Q_\infty(t) = \lim_{m \to \infty} \int pD^m d\Sigma\), then from (5.2.3) we obtain that the equation

\[
Q_\infty(t) = -(\gamma - 1)a(t)Q_\infty(t)
\]

holds. Together with (5.1.8) it gives that

\[
Q_\infty(t) = \text{const} \cdot G^{-\frac{2-1}{\gamma}}(t).
\]
Let us stress that it is not clear whether the limit exists in the nontrivial case \(Q_\infty \neq 0\).

5.2.2. Appearance of singularities of smooth solutions. Suppose, for example, that the smooth field \(\Lambda\) is bounded together with its divergency \(D\). Generally speaking, the divergency changes the sign over the manifold, that is \(\inf D < 0\).

From (5.1.4) we obtain

\[
F'(t) = \frac{F^2(t)}{2G(t)} + Q(t) \geq \frac{F^2(t)}{\mathcal{M}_+^2} - (\gamma - 1)E D_-, \tag{5.2.6}
\]

where \(\Lambda_+^2 = \sup |\Lambda|^2\), \(D_- = |\inf D|\).

Moreover,

\[
F(t) \leq 2E_k(t)G(t) \leq \mathcal{M} E \Lambda_+^2. \tag{5.2.7}
\]

It follows from (5.2.6) that if we choose

\[
F(0) > \Lambda_+ \sqrt{(\gamma - 1)D_- \mathcal{M} E}, \tag{5.2.8}
\]

then \(F(t)\) will grow unboundedly, it results in a contradiction with (5.2.7).

As the mass does not concentrate on sets of zero measure, that \(F(t)\) will grow owing to the gradient of density. Really, in the case of a potential field \(\Lambda\) we have

\[
F(t) = \int_\Sigma (\rho \nabla \Lambda) d\Sigma = -\int_\Sigma \nabla (\rho \nabla \Lambda) |\Lambda|^2 d\Sigma = -\int_\Sigma (\nabla \rho, \Lambda) |\Lambda|^2 d\Sigma - \int_\Sigma \rho D |\Lambda|^2 d\Sigma. \tag{5.3.1}
\]

The last integral is bounded by the constant, therefore, the growth of \(F(t)\) corresponds to the growth of \(|\nabla \rho, \Lambda|\) with respect to time.

Remark that one can ask whether the condition (5.2.8) is possible? The comparison of (5.2.7) and (5.2.8) gives the necessary condition for this possibility, namely,

\[
D_- \leq \frac{\mathcal{M} E}{\gamma - 1}.
\]

5.3. The Euler equations with the right hand side of special form. Let \(\Lambda\) satisfy (A2) and the velocity field be of form (5.1). Suppose that for \(F(V)\) the following condition holds

\[
(F(V), \Lambda) = \sum_{k=1}^K \lambda_k(t, a(t)) |\Lambda|^{s_k}, \tag{5.3.1}
\]

\(k \in \mathcal{N}, s_k \geq 0\). Here \(\lambda_k(t, a(t))\) are known functions of \(a(t)\) and \(t\).

Remark 5.3.1 The condition (5.3.1) takes place, for example, in the case of dry and aerodynamic friction.

Recall that for the dry friction

\[
F(V) = -\mu(t, x)V.
\]

Suppose \(\mu(t, x) = \mu(t)\). Then \(K = 1, s_1 = 2, \lambda_1(t, a(t)) = -\mu(t)a(t)\).
For the aerodynamic friction
\[ F(V) = -\mu_1(t, x) V |V|. \]
If \( \mu_1(t, x) = \mu_1(t) \), then \( K = 1, s_1 = 3, \lambda_1(t, a(t)) = -\mu_1(t)a(t) |a(t)|. \)

**Remark 5.3.2** Also (5.3.1) holds for the Navier-Stockes equation in the Euclidean space, where \( F^i = \nabla^j T^j_i \) with the tensor
\[ T^j_i = \alpha(\rho)(\nabla_i V^j - \nabla_j V^i) + \beta(\rho) V^k \delta^j_i, \]
if \( \Lambda \) coincides with the radius-vector \( r \) ((\( F(V) \equiv 0 \)) in the situation).

Thus, from (5.1.1), (5.1.2) we obtain similar to (5.1.3) that
\[ F'(t) = (2a(t)G(t))' = 2E_k(t) + Q(t) + \sum_{k=1}^{K} \lambda_k(t, a(t)) G_{s_k}(t), \]
and
\[ a'(t) = -a^2(t) + \frac{1}{2} \sum_{k=1}^{K} \lambda_k(t, a(t)) \frac{G_{s_k}(t)}{G_2(t)} + \frac{1}{2} \frac{Q(t)}{G_2(t)}. \]

According to (5.1.10) we have \( G_{s_k}(t) = K_{s_k} G_2^{\frac{1}{k}}(t) = K_{s_k} G^{\frac{1}{k}}(t) \), with some constants \( K_{s_k} \), depending on the initial data. Thus,
\[ a'(t) = -a^2(t) + \frac{1}{2} \sum_{k=1}^{K} K_{s_k} 2\lambda_k(t, a(t)) G^{\frac{1}{k}}(t) + \frac{1}{2} \frac{Q(t)}{G(t)}. \]

Introduce for the convenience (similar to Section 5.1.1) the new variable \( \tilde{G}(t) = G^{-1}(t) \), and obtain the system for \( (a(t), \tilde{G}(t), Q), n \in N \):
\[ a'(t) = -a^2(t) + \frac{1}{2} \sum_{k=1}^{K} K_{s_k} 2\lambda_k(t, a(t)) \tilde{G}^{-\frac{1}{k}}(t) + \frac{1}{2} \frac{Q(t)}{\tilde{G}(t)}, \]
\[ \tilde{G}'(t) = -2a(t)\tilde{G}(t). \]

Generally speaking, it is unclosed, but in the case of constant divergency \( \nabla_i \Lambda^i = D \), as in the case of zero right hand side it is possible to express \( Q(t) \) through \( \tilde{G}(t) \) and obtain the closed system for \( (a(t), \tilde{G}(t)) \). Namely,
\[ a'(t) = -a^2(t) + \frac{1}{2} \sum_{k=1}^{K} K_{s_k} 2\lambda_k(t, a(t)) \tilde{G}^{-\frac{1}{k}}(t) + \tilde{K} \tilde{G}^{\frac{(s-1)D+2}{2}}(t), \]
\[ \tilde{G}'(t) = -2a(t)\tilde{G}(t). \]

For the dry friction with constant friction coefficients \( \mu \) the system (5.3.5 – 5.3.4) takes the form
\[ a'(t) = -a^2(t) - \frac{1}{2} \mu K_{s_1} 2a(t) + \tilde{K} \tilde{G}^{\frac{(s-1)D+2}{2}}(t), \]
\[ \tilde{G}'(t) = -2a(t)\tilde{G}(t). \]
For the aerodynamic friction with constant coefficients $\mu_1$ we have

$$a'(t) = -a^2(t) - \frac{1}{2} \mu_1 K_{s1} a(t) |a(t)| \tilde{G}^{-\frac{1}{2}}(t) + \tilde{K} \tilde{G}^{\frac{(\gamma-1)D+2}{2}}(t), \quad (5.3.7)$$

$$\tilde{G}'(t) = -2a(t) \tilde{G}(t).$$

The phase portraits of systems (5.2.6 – 5.2.4) and (5.2.7 – 5.2.4) are presented on Figures 5 and 6, respectively.

In the case of dry friction there are two equilibriums: at the origin (stable) and at the point $(0, -\frac{K_{s1} \mu_1}{2})$ (unstable).

In the case of aerodynamic friction (and for any friction term with $\sigma > 0$ as well (see Section 1.1)) the only equilibrium is at the origin, moreover, it is stable.

6. FURTHER DISCUSSION

There is a lot of interesting problems connecting with this work that would be a subject of further investigations. Let us mention some of them.
1. Euler equations on the two-dimensional sphere
   a) to construct, if it is possible, a potential smooth vector field satisfying (A2).
   b) to find a possibility to close the system (5.1.11 – 5.1.12) and to investigate the behaviour of the function $a(t)$.

2. Interior solutions
   It is rather easy to show that if in the Euclidean space the vector field $\Lambda$ satisfying (A2) differs from the radius-vector up to the function from the Sobolev space, then the corresponding solution is interior.
   Which of solutions with velocity field (5.1) are interior? At least is it true that any solution where $\Lambda$ has a constant divergency is interior?

3. Nonviscous solutions of systems with viscous term
   It is easy to see that in the Euclidean space the solutions with linear profile of velocity ”do not feel” the viscous term (5.3.2) in the Navier-Stokes system. Therefore the solutions solve Navier-Stokes as well. In particular, for the viscous, but pressure free case we can construct the solution with the mass concentrated in the point.
   It is interesting to find the solution of such kind for any manifolds. What is a condition for the metric tensor for the possibility of existence of ”nonviscous solutions”?
   Further, if a solution with linear profile of velocity is interior for the Euler system, can we guarantee that it will be interior for the corresponding Navier-Stokes system?

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