Noncommutative Gravity and the $\star$-Lie algebra of diffeomorphisms

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Abstract

We construct functions and tensors on noncommutative spacetime by systematically twisting the corresponding commutative structures. The study of the deformed diffeomorphisms (and Poincaré) Lie algebra allows to construct a noncomutative theory of gravity.

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This article is based on common work with Christian Blohmann, Marija Dimitrijević, Frank Meyer, Peter Schupp, Julius Wess [1, 2] and on [3].

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1 Introduction

We present a differential geometry theory on noncommutative spacetime. The simplest and most discussed example is canonical noncommutative spacetime. There we have $x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$. After introducing noncommutative functions and tensors we study their infinitesimal transformations laws and the corresponding deformed Lie algebra of infinitesimal diffeomorphisms (and as a particular case Poincaré transformations). Starting from these basic notions we can then define covariant derivatives (so that we can implement the principle of general covariance on noncommutative spacetime), and torsion and curvature tensors. With these geometric tools we formulate noncommutative Einstein gravity. We here for simplicity consider noncommutative spacetime to be $\mathbb{R}^4$ with canonical commutation relations $[1, 3]$. Gravity on more general noncommutative manifolds, and its coordinate independent formulation is presented in [2].

Among the motivations for the study of noncommutative field theories and gravity I would like to recall that in the passage from classical mechanics to quantum mechanics classical observables become noncommutative. Similarly we expect that in the passage from classical gravity to quantum gravity, gravity observables, i.e. spacetime itself, with its coordinates and metric structure, will become noncommutative. Thus by formulating Einstein gravity on noncommutative spacetime we may learn about quantum gravity.

Planck scale noncommutativity is further supported by Gedanken experiments that aim at probing spacetime structure at very small distances. They show that due to gravitational backreaction one cannot test spacetime at those distances. For example, in relativistic quantum mechanics the position of a particle can be detected with a precision at most of the order of its Compton wave length $\lambda_C = h/mc$. Probing spacetime at infinitesimal distances implies an extremely heavy particle that in turn curves spacetime itself. When $\lambda_C$ is of the order of the Planck length, the spacetime curvature radius due to the particle has the same order of magnitude and the attempt to measure spacetime structure beyond Planck scale fails.

This Gedanken experiment supports finite reductionism. It shows that the description of spacetime as a continuum of points (a smooth manifold) is an assumption no more justified at Planck scale. It is then natural to relax this assumption and conceive a noncommutative spacetime, where uncertainty relations and discretization naturally arise. In this way the dynamical feature of spacetime that prevents from testing sub-Planckian scales is explained by incorporating it at a deeper kinematical level. A similar mechanism happens for example in the passage from Galilean to special relativity. Contraction of distances and time dilatation can be explained in Galilean relativity: they are a consequence of the interaction between ether and the body in motion. In special relativity they have become a kinematical feature.
2 $\star$-Products and Twists

The star product that implements the $x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$ noncommutativity is given by

\[
(h \star g)(x) = e^{i\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}} h(x) g(y)|_{x=y}
\]

(2.1)

where $h$ and $g$ are arbitrary functions. This star-product between functions can be obtained from the usual pointwise product $(h g)(x) = h(x) g(x)$ via the action of a twist operator $\mathcal{F}$ [4, 5]

\[
f \star g := \mu \circ \mathcal{F}^{-1}(f \otimes g),
\]

(2.2)

where $\mu$ is the usual pointwise product between functions, $\mu(f \otimes g) = fg$, and the twist operator and its inverse are

\[
\mathcal{F} = e^{-\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}}, \quad \mathcal{F}^{-1} = e^{\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}}.
\]

(2.3)

Despite the indices $^{\mu\nu}$ notation, we will consistently consider the entries $\theta^{\mu\nu}$ of the antisymmetric matrix $\theta$ as fundamental dimensionful constants, like $c$ or $\hbar$. In particular the deformed spacetime symmetries we consider will leave invariant the $\theta$ matrix. The point is that the exponent of $\mathcal{F}$,

\[
\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}
\]

is not the Poisson tensor associated to the $\star$-product. The difference lies in the tensor-product $\otimes$. The Poisson tensor is

\[
\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes_A \frac{\partial}{\partial x^\nu}
\]

(2.4)

where we have explicitly written that the tensorproduct is over the algebra $A = Fun(\mathbb{R}^4)$ of functions on spacetime. On the other hand the tensorproduct in $\mathcal{F}$ is over the complex numbers, we should write

\[
\mathcal{F} = e^{-\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}}.
\]

That is why $\theta^{\mu\nu}$ in $\mathcal{F}$ is not a tensor but a set of constants. In this respect, a better notation for $\mathcal{F}$ is

\[
\mathcal{F} = e^{-\frac{i}{2} \theta^{ab} X_a \otimes X_b}.
\]

(2.5)

where $a, b = 1, \ldots, 4$ and $X_1 = \frac{\partial}{\partial x^1}, \ldots, X_4 = \frac{\partial}{\partial x^4}$, are globally defined vectorfields on spacetime.

We shall frequently write (sum over $\alpha$ understood)

\[
\mathcal{F} = f^\alpha \otimes f_\alpha, \quad \mathcal{F}^{-1} = \overline{f}^\alpha \otimes \overline{f}_\alpha,
\]

(2.6)
so that
\[ f \star g := \overline{\Gamma}^\alpha(f) \overline{\Gamma}_\alpha(g) . \] (2.7)

Explicitly we have
\[
\mathcal{F}^{-1} = e^{i \theta^{\alpha \nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial \nu}} = \sum_n \frac{1}{n!} \left( \frac{i}{2} \right)^n \theta^{\mu_1 \nu_1} \ldots \theta^{\mu_n \nu_n} \partial_{\mu_1} \ldots \partial_{\mu_n} \otimes \partial_{\nu_1} \ldots \partial_{\nu_n} = \overline{\Gamma}^\alpha \otimes \overline{\Gamma}_\alpha ,
\] (2.8)

so that \( \alpha \) is a multi-index. We also introduce the universal \( R \)-matrix
\[
\mathcal{R} := \mathcal{F}_{21} \mathcal{F}^{-1}
\] (2.9)

where by definition \( \mathcal{F}_{21} = f_\alpha \otimes f^\alpha \). In the sequel we use the notation
\[
\mathcal{R} = R^\alpha \otimes R_\alpha , \quad \mathcal{R}^{-1} = \overline{R}^\alpha \otimes \overline{R}_\alpha .
\] (2.10)

In the present case we simply have \( \mathcal{R} = \mathcal{F}^{-2} \) but for more general twists this is no more the case. The \( R \)-matrix measures the noncommutativity of the \( \star \)-product. Indeed it is easy to see that
\[
h \star g = \overline{R}^\alpha(g) \star \overline{R}_\alpha(h) .
\] (2.11)

The permutation group in noncommutative space is naturally represented by \( \mathcal{R} \). Formula (2.11) says that the \( \star \)-product is \( \mathcal{R} \)-commutative in the sense that if we permute (exchange) two functions using the \( R \)-matrix action then the result does not change.

Note: The class of \( \star \)-products that can be obtained from a twist \( \mathcal{F} \) is quite rich, (for example we can obtain star products that give the commutation relations \( x \star y = qy \star x \) in two or more dimensions). Moreover we can consider twists and \( \star \)-products on arbitrary manifolds not just on \( \mathbb{R}^4 \). For example, given a set of mutually commuting vectorfields \( \{ X_a \} (a = 1, 2, \ldots n) \) on a \( d \)-dimensional manifold \( M \), we can consider the twist
\[
\mathcal{F} = e^{\frac{i}{2} \theta^{ab} X_a \otimes X_b} .
\] (2.12)

Another example is \( \mathcal{F} = e^{\frac{i}{2} H \otimes \ln(1+\lambda E)} \) where the vectorfields \( H \) and \( E \) satisfy \([H, E] = 2E\). In these cases too the \( \star \)-product defined via (2.2) is associative and properly normalized in the sense that for any function \( h \) we have \( h \star 1 = 1 \star h = h \). In general an element \( \mathcal{F} \) is a twist if it is invertible, if it satisfies a cocycle condition and if it is properly normalized [6]. The cocycle and the normalization conditions imply associativity of the \( \star \)-product and the normalization \( h \star 1 = 1 \star h = h \). A deformed gravity theory based on this class of \( \star \)-products can also be constructed [2].

3
3 Vectorfields and Tensorfields

We now use the twist to deform the commutative geometry on spacetime into the twisted noncommutative one. The guiding principle is the one used to deform the product of functions into the $\star$-product of functions. Every time we have a bilinear map

$$\mu : X \times Y \to Z$$

where $X, Y, Z$ are vectorspaces, and where there is an action of $\mathcal{F}^{-1}$ on $X$ and $Y$ we can combine this map with the action of the twist. In this way we obtain a deformed version $\mu_\star$ of the initial bilinear map $\mu$:

$$\mu_\star := \mu \circ \mathcal{F}^{-1}, \quad (3.1)$$

$$\mu_\star : X \times Y \to Z \quad (x, y) \mapsto \mu_\star(x, y) = \mu(\Gamma^\alpha(x), \Gamma_\alpha(y)) .$$

The $\star$-product on the space of functions is recovered setting $X = Y = A = \text{Fun}(M)$. We now study the case of vectorfields and tensorfields.

**Vectorfields $\Xi_\star$.** We deform the product $\mu : A \otimes \Xi \to \Xi$ between the space $A = \text{Fun}(M)$ of functions on spacetime $M$ and vectorfields. A generic vectorfield is $v = v^\nu \partial_\nu$. Partial derivatives acts on vectorfields via the Lie derivative action

$$\partial_\mu(v) = [\partial_\mu, v] = \partial_\mu(v^\nu)\partial_\nu . \quad (3.2)$$

According to (3.1) the product $\mu : A \otimes \Xi \to \Xi$ is deformed into the product

$$h \star v = \Gamma^\alpha(h)\Gamma_\alpha(v) . \quad (3.3)$$

Since $\mathcal{F}^{-1} = e^{\frac{i}{\hbar} \theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$, iterated use of (3.2) gives

$$h \star v = \Gamma^\alpha(h)\Gamma_\alpha(v) = \Gamma^\alpha(h)\Gamma_\alpha(v^\nu)\partial_\nu = (h \star v^\nu)\partial_\nu . \quad (3.4)$$

It is then easy to see that $h \star (g \star v) = (h \star g) \star v$, i.e. that the $\star$-multiplication between functions and vectorfields is consistent with the $\star$-product of functions. We denote the space of vectorfields with this $\star$-multiplication by $\Xi_\star$. As vectorspaces $\Xi = \Xi_\star$, but $\Xi$ is an $A$-module while $\Xi_\star$ is an $A_\star$-module.

**Tensorfields $\mathcal{T}_\star$.** Tensorfields form an algebra with the tensorproduct $\otimes$ (over the algebra of functions). We define $\mathcal{T}_\star$ to be the noncommutative algebra of tensorfields. As vectorspaces $\mathcal{T} = \mathcal{T}_\star$; the noncommutative and associative tensorproduct is obtained by applying (3.1):

$$\tau \otimes_\star \tau' := \Gamma^\alpha(\tau) \otimes \Gamma_\alpha(\tau') . \quad (3.5)$$
There is a natural action of the permutation group on undeformed tensorfields:

\[ \tau \otimes \tau' \xrightarrow{\sigma} \tau' \otimes \tau. \]

In the deformed case it is the \( R \)-matrix that provides a representation of the permutation group on \(*\)-tensorfields:

\[ \tau \otimes * \tau' \xrightarrow{\sigma_R} \overline{R}_\alpha(\tau') \otimes * \overline{R}_\alpha(\tau). \]

It is easy to check that, consistently with \( \sigma_R \) being a representation of the permutation group, we have \((\sigma_R)^2 = id\).

If we consider the local coordinate expression of two tensor fields, for example of the type

\[ \tau = \tau_{\mu_1 \ldots \mu_m} \partial_{\mu_1} \otimes \ldots \otimes \partial_{\mu_m} \]
\[ \tau' = \tau'_{\nu_1 \ldots \nu_n} \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_n} \]

then their \(*\)-tensor product is

\[ \tau \otimes * \tau' = \tau_{\mu_1 \ldots \mu_m} * \tau'_{\nu_1 \ldots \nu_n} \partial_{\mu_1} \otimes * \ldots \otimes * \partial_{\mu_m} \otimes \partial_{\nu_1} \otimes \ldots \otimes \partial_{\nu_n}. \]  (3.6)

\section{\(*\)-Lie algebra of Diffeomorphisms}

In the commutative case we have two derivations that map tensors to tensors (of the same type): the Lie derivative \( \mathcal{L}_v \) and the covariant derivative \( \nabla_v \) along the vectorfield \( v \). If we act on functions (tensors of type 0) Lie derivative and covariant derivative coincide and are the usual action of the vectorfield \( v \) on a function \( h \). The result of this action is the variation of the function \( h \) under the infinitesimal transformation generated by \( v \),

\[ \delta_v h = \mathcal{L}_v h = v(h). \]

In this section we study the \(*\)-action of a vectorfield on a function, i.e. we deform the notion of Lie derivative (infinitesimal diffeomorphism), and study the deformed Lie algebra of these infinitesimal diffeomorphisms. Once this deformed derivation is understood then the notion of deformed covariant derivative will naturally emerge, and the construction of the Riemann curvature tensor, the Ricci tensor and the curvature scalar will follow. These tensors will transform covariantly under infinitesimal diffeomorphisms.

The \(*\)-Lie derivative on the space of functions \( A_* \) is obtained following the general rule \[3.1\]. We combine the usual Lie derivative on functions \( \mathcal{L}_u h = u(h) \) with the twist \( \mathcal{F} \)

\[ \mathcal{L}^*_u (h) := \overline{F}^{\alpha}(u)(\overline{F}_\alpha(h)). \]  (4.1)
By recalling that every vectorfield can be written as \( u = u^\mu \partial_\mu = u^\mu \partial_\mu \) we have

\[
\mathcal{L}_u^*(h) = \mathring{\Gamma}^\alpha (u^\mu \partial_\mu (\mathring{f}(h))) = \mathring{\Gamma}^\alpha (u^\mu) \partial_\mu (\mathring{f}(h)) \\
= u^\mu \star \partial_\mu (h) ,
\]

(4.2)

where in the second equality we have considered the explicit expression (2.8) of \( \mathring{\Gamma}^\alpha \) in terms of partial derivatives, and we have iteratively used the property \([\partial_\nu, u^\mu \partial_\mu] = \partial_\nu (u^\mu) \partial_\mu\). In the last equality we have used that the partial derivatives contained in \( \mathring{f}(\alpha) \) commute with the partial derivative \( \partial_\mu \).

The differential operator \( \mathcal{L}_u^* \) satisfies the deformed Leibniz rule

\[
\mathcal{L}_u^*(h \star g) = \mathcal{L}_u^*(h) \star g + R^\alpha (h) \star \mathcal{L}_{\mathring{R}_\alpha (u)}^*(g) .
\]

(4.3)

This deformed Leibnitz rule is intuitive: in the second addendum we have exchanged the order of \( u \) and \( h \), and this is achieved by the action of the \( R \)-matrix that as observed in Section 2 and 3 provides a representation of the permutation group. A proof of (4.3) is not difficult

\[
\mathcal{L}_u^*(h \star g) = u^\mu \star \partial_\mu (h \star g) = u^\mu \star \partial_\mu (h) \star g + u^\mu \star h \star \partial_\mu (g)
= \mathcal{L}_u^*(h) \star g + \mathring{R}^\alpha (h) \star \mathring{R}_\alpha (u^\mu) \star \partial_\mu (g)
= \mathcal{L}_u^*(h) \star g + \mathring{R}^\alpha (h) \star \mathcal{L}_{\mathring{R}_\alpha (u)}^*(g) .
\]

(4.4)

From (4.2) it is also immediate to check the compatibility condition

\[
\mathcal{L}_{f \star u}^*(h) = f \star \mathcal{L}_u^*(h) ,
\]

(4.5)

that shows that the action \( \mathcal{L}^* \) on functions is the one compatible with the \( A_* \) module structure of vectorfields.

In the commutative case the commutator of two vectorfields is again a vectorfield, we have the Lie algebra of vectorfields. In this \( \star \)-deformed case we have a similar situation. We first calculate

\[
\mathcal{L}_u^* \mathcal{L}_v^*(h) = \mathcal{L}_u^*(\mathcal{L}_v^*(h)) = u^\mu \star \partial_\mu (v^\nu) \star \partial_\nu (h) + u^\mu \star v^\nu \star \partial_\nu \partial_\mu (h)
\]

Then instead of considering the composition \( \mathcal{L}_u^* \mathcal{L}_v^* \) we consider \( \mathcal{L}_{\mathring{R}^\alpha (v)}^* \mathcal{L}_{\mathring{R}_\alpha (u)}^* \), indeed the usual commutator is constructed permuting (transposing) the two vectorfields, and we have just remarked that the action of the permutation group in the noncommutative case is obtained using the \( R \)-matrix. We have

\[
\mathcal{L}_{\mathring{R}^\alpha (v)}^* \mathcal{L}_{\mathring{R}_\alpha (u)}^*(h) = \mathcal{L}_{\mathring{R}^\alpha (v)}^*(\mathring{R}_\alpha (u^\mu) \star \partial_\mu h) = \mathring{R}^\alpha (v^\nu) \star \partial_\nu (\mathring{R}_\alpha (u^\mu) \star \partial_\mu h)
= \mathring{R}^\alpha (v^\nu) \star \mathring{R}_\alpha (\partial_\nu u^\mu) \star \partial_\mu h + \mathring{R}^\alpha (v^\nu) \star \mathring{R}_\alpha (u^\mu) \star \partial_\nu \partial_\mu h
\]


In conclusion
\begin{equation}
\mathcal{L}_u^* \mathcal{L}_v^* - \mathcal{L}_{\overline{R}_\alpha^*(v)}^* \mathcal{L}_{\overline{R}_\alpha(u)}^* = \mathcal{L}_{[u,v]}^* \tag{4.6}
\end{equation}
where we have defined the new vectorfield
\begin{equation}
[u, v]_* := (u^\mu * \partial_\mu v^\nu) \partial_\nu - (\partial_\nu u^\mu * v^\nu) \partial_\mu . \tag{4.7}
\end{equation}
The bracket \([ , ]_*\) is a bilinear map
\begin{equation}
[ , ]_* : \Xi_* \times \Xi_* \rightarrow \Xi_*
(u, v) \mapsto [u, v]_* \tag{4.8}
\end{equation}
and the space of vectorfields equipped with this bracket becomes the \(*\)-Lie algebra of vectorfields \(\Xi_*\). It is easy to see that the bracket \([ , ]_*\) has the \(*\)-antisymmetry property
\begin{equation}
[u, v]_* = -[\overline{R}_\alpha^*(v), \overline{R}_\alpha(u)]_* . \tag{4.9}
\end{equation}
\(*\)-Jacoby identities can be proven as well
\begin{equation}
[u, [v, z]_*]_* = [[u, v]_, z]_* + [\overline{R}_\alpha^*(v), [\overline{R}_\alpha(u), z]_*] . \tag{4.10}
\end{equation}

In conclusion we have constructed the deformed Lie algebra of vectorfields \(\Xi_*\). As vector spaces \(\Xi = \Xi_*\), but \(\Xi_*\) is a \(*\)-Lie algebra. We previously constructed \(\Xi_*\) as an \(A_*\)-module. The compatibility between these two structures reads
\begin{equation}
[u, h * v]_* = \mathcal{L}_u^*(h) * v + \overline{R}_\alpha(h) * [\overline{R}_\alpha(u), v]_* .
\end{equation}

We stress that a \(*\)-Lie algebra is not a generic name for a deformation of a Lie algebra. Rather it is a quantum Lie algebra in the sense of [10], see also [11, 12], [3] and [13]. In this respect the deformed Leibnitz rule (1.3), that states that only vector fields (or the identity) can act on the second argument \(g\) in \(h * g\) (no higher order differential operators are allowed on \(g\)) is of fundamental importance, and later on it will be a key ingredient for the definition of a covariant derivative along a generic vectorfield. Another main property of quantum Lie algebras, and of the quantum Lie algebra of infinitesimal diffeomorphisms we have presented, is that it can be shown that the bracket \([u, v]_*\) is the \(*\)-adjoint action of \(u\) on \(v\). Had we chosen an exactly antisymmetric bracket of vectorfields, we would have lost this geometric property.

A general comment on the approach adopted is now in order. Given the deformation \(Fun_*(M)\) of the algebra of functions \(Fun(M)\), we can

- consider the derivations of \(Fun_*(M)\), i.e. the infinitesimal transformations of \(Fun_*(M)\) that satisfy the usual Leibnitz rule, \(v(h * g) = v(h) * g + h * v(g)\). As is
easily seen expanding in power series of $\theta^{\mu\nu}$, these maps are only the vectorfields that leave invariant the Poisson tensor (2.4). Thus while in the commutative case any vectorfield is a derivation, in the deformed case the space of derivations is smaller. This viewpoint for our purposes is too restrictive, for example infinitesimal Poincaré transformations are not derivations. In this approach we have that Poincaré invariance is spontaneously broken by the presence of $\theta^{\mu\nu}$.

- consistently deform the notion of derivation so that to any infinitesimal transformation of $\text{Fun}(M)$ there correspond one and only one deformed infinitesimal derivation. This is what we have achieved with the map $L_v \rightarrow L_v^*$, where $L_v^*$ satisfies the deformed Leibnitz rule (4.3). This is the quantum groups and quantum spaces approach [10–12, 14]. The bonus of this approach is that instead of dealing with a spontaneously broken diffeomorphisms (or Poincaré) symmetry we have an unbroken quantum diffeomorphisms (or Poincaré) symmetry. In this way we retain a symmetry property that is as strong as the one of commutative spacetime.

**Deformed Poincaré Lie algebra**

The quantum or twisted approach in the case of Poincaré symmetry of canonical noncommutative spacetime was considered in [4, 5, 7], where the twisted Poincaré Hopf algebra is presented (see also [8, 9]). The description of this twisted symmetry in terms of $\star$-Poincaré Lie algebra, i.e. the quantum Lie algebra of the Poincaré Hopf algebra, is in [3]. Using the bracket (4.7), we have that the infinitesimal generators

$$P_\mu = i\partial_\mu , \ M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) ,$$  

(4.11)

close the $\star$-Poincaré Lie algebra

$$[P_\mu , P_\nu]_* = 0 ,$$

$$[P_\rho , M_{\mu\nu}]_* = i(\eta_\rho\mu P_\nu - \eta_\rho\nu P_\mu) ,$$

$$[M_{\mu\nu} , M_{\rho\sigma}]_* = -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}) .$$  

(4.12)

In this particularly simple case of canonical noncommutativity it happens that the structure constants are the same as in the undeformed case, however the $\star$-bracket differs from the commutator, $\mathcal{L}_{[M_{\mu\nu} , M_{\rho\sigma}]} \neq \mathcal{L}_{M_{\mu\nu}}^* \mathcal{L}_{M_{\rho\sigma}}^* - \mathcal{L}_{M_{\rho\sigma}}^* \mathcal{L}_{M_{\mu\nu}}^*$. The deformed Leibnitz rule of these derivations, according to (1.3), reads

$$\mathcal{L}_{M_{\mu\nu}}^*(h \star g) = \mathcal{L}_{M_{\mu\nu}}^*(h) \star g + h \star \mathcal{L}_{M_{\mu\nu}}^*(g) - i\theta^{\alpha\beta} \eta_{\beta\mu} \partial_\alpha(h) \star \partial_\nu(g) + i\theta^{\alpha\beta} \eta_{\beta\nu} \partial_\alpha(h) \star \partial_\mu(g) .$$

From (4.10) or immediately from (4.12) we obtain the $\star$-Jacoby identities:

$$[t , [t' , t'']]_* + [t' , [t'' , t]]_* + [t'' , [t , t']]_* = 0 ,$$  

(4.13)

for all elements $t , t' , t''$ of the $\star$-Poincaré Lie algebra.
5 Covariant Derivative, Torsion and Curvature

The noncommutative differential geometry set up in the previous section allows to develop the formalism of covariant derivative, torsion, curvature and Ricci tensors just by following the usual classical formalism.

We define a $\ast$-covariant derivative $\nabla^\ast_u$ along the vector field $u \in \mathfrak{X}$ to be a linear map $\nabla^\ast_u : \mathfrak{X} \to \mathfrak{X}$ such that for all $u, v, z \in \mathfrak{X}$, $h \in A^\ast$:

$$\nabla^\ast_u (u + v) z = \nabla^\ast_u u z + \nabla^\ast_v z,$$

$$\nabla^\ast_{h \ast u} v = h \ast \nabla^\ast_u v,$$

$$\nabla^\ast_u (h \ast v) = \mathcal{L}^\ast_u (h) \ast v + \overline{R}^\ast (h) \ast \nabla^\ast_{\overline{R}_\alpha (u)} v$$

(5.1)

(5.2)

(5.3)

where in the last line we have used the same deformed Leibnitz rule that appears in (4.3). Expression (5.3) is well defined because $\overline{R}_\alpha (u)$ is again a vectorfield.

The covariant derivative is extended to tensorfields using again the deformed Leibniz rule

$$\nabla^\ast_u (v \otimes \ast z) = \nabla^\ast_u (v) \otimes \ast z + \overline{R}^\ast (v) \otimes \ast \nabla^\ast_{\overline{R}_\alpha (u)} (z).$$

The (noncommutative) connection coefficients $\Gamma^\ast_{\mu \nu} \sigma$ are given by

$$\nabla^\ast_\mu \partial_\nu = \Gamma^\ast_{\mu \nu} \sigma \ast \partial_\sigma = \Gamma^\ast_{\mu \nu} \sigma \partial_\sigma,$$

(5.4)

where $\nabla^\ast_\mu = \nabla^\ast_\partial_\mu$. Let $z = z^\mu \ast \partial_\mu$, $u = u^\nu \ast \partial_\nu$, then (5.2) and (5.3) imply

$$\nabla^\ast_\mu (u) = z^\mu \ast \partial_\mu (u^\nu) \partial_\nu + z^\mu \ast u^\nu \ast \Gamma^\ast_{\mu \nu} \sigma \partial_\sigma.$$

(5.5)

Similarly the covariant derivative on 1-forms is given by

$$\nabla^\ast_\mu (\omega_\rho d x^\rho) = \partial_\mu (\omega_\rho) d x^\rho - \Gamma^\ast_{\mu \nu} \ast \partial_\nu (\omega_\rho) d x^\rho$$

(5.6)

and $\nabla^\ast_\mu = z^\mu \ast \partial_\mu$.

The torsion $T$ and the curvature $R$ associated to a connection $\nabla^\ast$ are the linear maps $T : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$, and $R^\ast : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ defined by

$$T(u, v) := \nabla^\ast_u v - \nabla^\ast_v u - [u, v],$$

$$R(u, v, z) := \nabla^\ast_u \nabla^\ast_v z - \nabla^\ast_v \nabla^\ast_u z - \nabla^\ast_{[u, v]} z,$$

(5.7)

(5.8)

for all $u, v, z \in \mathfrak{X}$. From the $\ast$-antisymmetry property of the bracket $[\ , \ ]_\ast$, see (4.9), it easily follows that the torsion $T$ and the curvature $R$ have the following $\ast$-antisymmetry property

$$T(u, v) = - T(\overline{R}^\ast (v), \overline{R}_\alpha (u)),$$

$$R(u, v, z) = - R(\overline{R}^\ast (v), \overline{R}_\alpha (u), z).$$
The presence of the $R$-matrix in the definition of torsion and curvature insures that $T$ and $R$ are left $A_*$-linear maps [2], i.e.

$$T(f \star u, v) = f \star T(u, v) \quad , \quad T(\partial_\mu, f \star v) = f \star T(\partial_\mu, v)$$

(for any index $\mu$), and similarly for the curvature. The $A_*$-linearity of $T$ and $R$ insures that we have a well defined torsion tensor and curvature tensor.

One can also prove (twisted) first and second Bianchi identities [2].

The coefficients $T_{\mu\nu\rho}^\sigma$ and $R_{\mu\nu\rho\sigma}^\sigma$ with respect to the partial derivatives basis $\{\partial_\mu\}$ are defined by

$$T(\partial_\mu, \partial_\nu) = T_{\mu\nu\rho} \partial_\rho \quad , \quad R(\partial_\mu, \partial_\nu, \partial_\rho) = R_{\mu\nu\rho\sigma} \partial_\sigma$$

and they explicitly read

$$T_{\mu\nu\rho} = \Gamma_{\mu\nu\rho} - \Gamma_{\nu\mu\rho} \quad , \quad R_{\mu\nu\rho\sigma} = \partial_\mu \Gamma_{\nu\rho\sigma} - \partial_\nu \Gamma_{\mu\rho\sigma} + \Gamma_{\nu\beta}^\sigma \Gamma_{\mu\beta\rho} - \Gamma_{\mu\rho}^\beta \Gamma_{\nu\beta\sigma} .$$

As in the commutative case the Ricci tensor is a contraction of the curvature tensor,

$$\text{Ric}_{\mu\nu} = R_{\rho\mu\nu}^\rho .$$

A definition of the Ricci tensor that is independent from the $\{\partial_\mu\}$ basis is also possible.

### 6 Metric and Einstein Equations

In order to define a $\star$-metric we need to define $\star$-symmetric elements in $\Omega_\star \otimes \Omega_\star$ where $\Omega_\star$ is the space of 1-forms. Recalling that permutations are implemented with the $R$-matrix we see that $\star$-symmetric elements are of the form

$$\omega \otimes_\star \omega' + R^\star(\omega') \otimes_\star R_\alpha(\omega) .$$

In particular any symmetric tensor in $\Omega \otimes \Omega$,

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu ,$$

$g_{\mu\nu} = g_{\nu\mu}$, is also a $\star$-symmetric tensor in $\Omega_\star \otimes \star \Omega_\star$ because

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = g_{\mu\nu} \star dx^\mu \otimes_{\star} dx^\nu$$

and the action of the $R$-matrix is the trivial one on $dx^\nu$. We denote by $g^{\mu\nu}$ the star inverse of $g_{\mu\nu}$,

$$g^{\mu\rho} \star g_{\rho\nu} = g_{\nu\rho} \star g^{\rho\mu} = \delta^\mu_\nu .$$
The metric $g_{\mu\nu}$ can be expanded order by order in the noncommutative parameter $\theta^{\rho\sigma}$. Any commutative metric is also a noncommutative metric, indeed the $\star$-inverse metric can be constructed order by order in the noncommutativity parameter. Contrary to [15], we see that in our approach there are infinitely many metrics compatible with a given noncommutative differential geometry, noncommutativity does not single out a preferred metric.

A connection that is metric compatible is a connection that for any vectorfield $u$ satisfies, $\nabla^*_ug = 0$, this is equivalent to the equation

$$\nabla^*_\mu g_{\rho\sigma} - \Gamma_{\mu\rho}^{\nu} \star g_{\nu\sigma} - \Gamma_{\mu\sigma}^{\nu} \star g_{\rho\nu} = 0.$$  \quad (6.5)

Proceeding as in the commutative case we obtain that there is a unique torsion free metric compatible connection [1]. It is given by

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2} (\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu}) \star g^{*\sigma\rho} \quad (6.6)$$

We now construct the curvature tensor and the Ricci tensor using this uniquely defined connection. Finally the noncommutative Einstein equations (in vacuum) are

$$\text{Ric}_{\mu\nu} = 0$$  \quad (6.7)

where the dynamical field is the metric $g$.

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