The dc Josephson current in a long multi-channel quantum wire

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The dc Josephson current flowing at zero voltage bias across an SNS-junction at temperature $T$ as a consequence of an applied phase difference $\chi$ between the superconducting leads, is generally obtained by taking the derivative of the system free energy $F$ with respect to $\chi$, that is $I[\chi; T] = 2e \frac{dF}{d\chi}$. Using Bardeen-Cooper-Schrieffer (BCS) approximation for the leads and ignoring interactions within the normal region, assuming spin rotational symmetry for the whole system, the free energy $F$ is simply obtained by summing over all the individual single-quasiparticle energies $E_n$, so that the current is given by\textsuperscript{2}

$$I[\chi; T] = 2e \sum_n f(E_n) \frac{dE_n}{d\chi}, \quad (1.1)$$

with $f(E) = 1/[e^{E/T} + 1]$ being the Fermi distribution function which, at $T = 0$, gives the zero-temperature current $I[\chi; T = 0] = 2e \sum_{E_n \leq 0} \frac{dE_n}{d\chi}$. The factor of 2 in Eq. (1.1) accounts for the spin degeneracy of each level, due to the spin rotational symmetry of the system.\textsuperscript{2}

In general, to compute $I[\chi; T]$ one has to pertinently sum over contributions from both sub-gap Andreev bound states (ABS’s) localized in the central normal region, with wavefunctions exponentially decaying within the superconducting leads, as well as from propagating scattering states (SS’s), with energy $|E| > \Delta$, $\Delta$ being the superconducting gap.\textsuperscript{6,7} Accurately summing over all types of states is, in general, quite hard, due to the delicate cancellation between various contributions, yielding a small final result from differences of very large terms.\textsuperscript{8}

In Ref. \textsuperscript{9}, based on an adapted version of the formalism developed in Refs.\textsuperscript{10,11}, we rewrite Eq. (1.1) as a contour integral in the complex energy plane, explicitly involving the determinant of the analytically continued $S$-matrix, from which we show that, at low temperatures, the formula for the dc Josephson current is greatly simplified in the long junction limit. In particular, we prove that, for a long junction and at low temperatures, $I[\chi; T]$ depends only on data at the Fermi level, namely, on the single-particle normal- and Andreev-reflection amplitudes at the SN interfaces. Specifically, in \textsuperscript{9} we consider a one-dimensional model for the SNS-junction, with just one active channel, within both the central region, and the superconducting leads, both in the continuum formulation ("Blonder-Tinkham-Klapwijk (BTK) model\textsuperscript{12}") as well as in a tight-binding version of the model, such as the one discussed in Ref. \textsuperscript{13}. Nevertheless, ballistic SNS junctions realized with point contacts between superconducting leads ("superconducting quantum contacts")\textsuperscript{10,14} as well as by connecting, for instance, a carbon nanotube to two superconductors,\textsuperscript{15} are typically characterized by several open one-dimensional channels, both within the leads and in the central region. To keep in touch with such realistic models of ballistic SNS junctions, in this paper we discuss the generalization of the main results of \textsuperscript{9} to a long SNS junction at low temperatures, with an arbitrary number of open channels $N_L$ and $N_R$ within the left-hand (L) and the right-hand superconducting lead (R), respectively, and a generic number $K$ of (noninteracting) open electronic channels within the central region C. In particular, on providing explicit applications

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I. INTRODUCTION

The dc Josephson current\textsuperscript{1} flowing at zero voltage bias across an SNS-junction at temperature $T$ as a consequence of an applied phase difference $\chi$ between the superconducting leads, is generally obtained\textsuperscript{2} by summing contributions from sub-gap Andreev bound states, as well as from continuum states propagating within the superconducting leads. We show that, in a long multi-channel SNS-junction, at low temperatures, all these contributions add up, so that the current can be entirely expressed in terms of single-particle normal- and Andreev reflection amplitudes at the Fermi level at both SN interfaces. Our derivation applies to a generic number of channels in the normal region and/or in the superconducting leads, without assumptions about scattering processes at the SN interfaces: if the channels within the central region have the same dispersion relation, it allows for expressing the current in terms of a simple integral involving only scattering amplitudes at the Fermi level. Our result motivates using a low energy effective boundary Hamiltonian formalism for computing the current, which is crucial for treating Luttinger liquid interaction effects.

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of our approach to specific model-calculations of \( I[\chi; T] \), we show that, the simple closed-form formula for \( I[\chi; T] \) \((I[\chi; T = 0])\) given in Eq. (3) of \([9]\) takes a nice generalization to the multi-channel SNS junction in the case of equivalent channels within \( C \), that is, in the case in which all the channels within \( C \) are characterized by the same dispersion relation (but not necessarily by the same tunneling amplitudes with the leads). Even when this last condition is not met, it is possible to write compact formulas for \( I[\chi; T] \) at low temperatures, which eventually allow for a straightforward calculation of the current.

In treating ballistic multi-channel SNS junctions, relevant results have been obtained by using a quasiclassical approach based on the Eilenberger equations written for the slowly varying (on atomic distances) part of the Matsubara-Green functions \([16]\) in which the scattering at SN interfaces is accounted for by means of simple linear conditions, rather than the “standard” Zaitsev boundary conditions \([17]\). Such an approach has revealed itself to be quite effective in computing the dc Josephson current in a variety of physically relevant situations, such as Josephson junctions with a series of insulating barriers, SHS junctions, where \( H \) is a “half-metal”, that is, fully spin polarized materials acting as an insulator for electrons with one of the two spin directions, junctions realized with spin-active SN-interfaces, or junctions realized with single- or multi-layer graphene contacted with two superconducting electrodes \([18]\).

As we outline in Appendix E in the long-junction limit, the results of Ref. \([18]\) can be recovered from our simple formulas, in the limit in which one neglects scattering between different channels at the SN-interfaces. Thus, while being consistent with the well-grounded method based on Eilenberger equations, our approach constitutes a remarkable simplification of the technique of Ref. \([18]\), as it provides an explicit formula for the current at low temperatures in a generic long ballistic multi-channel SNS junction, without going though a limiting procedure of complex formulas. Moreover, the fact that we explicitly prove the cancellation between contributions to \( I[\chi; T] \) from finite-energy states, motivates resorting to a simplified model calculation, in which the superconducting leads are integrated out and traded for a pertinent boundary interaction Hamiltonian, only involving the single-electron field operators at the endpoints of the central region. This result is important, as it provides an effective method for including interaction effects in the normal region, based on boundary conformal field theory techniques \([13,22]\) and can be readily applied to study, for instance, the signature on the dc Josephson current of the emergence of nontrivial fixed points in junctions involving topological superconductors \([26]\) quantum Josephson junction networks \([27]\) etc.

It is worth remarking that, while, in order to present our technique, throughout this paper we work with a model Hamiltonian in which the leads are pictured as one-dimensional s-wave superconductors and the whole SNS-Hamiltonian is \( SU(2) \)-invariant, so that spin is conserved in a single-quasiparticle scattering at the SN-interfaces, with pertinent modifications to the model Hamiltonian used for the calculations, our derivation is expected to be effective in providing a reliable long-junction limit of the dc Josephson current in the system studied, for instance, in \([19,20]\).

The paper is organized as follows:

- In Section II we implement a pertinent version of the \( S \)-matrix approach, to derive the general formula for the dc Josephson current across a multi-channel SNS junction.
- In Section III we compute the dc Josephson current at low temperatures across a long multi-channel SNS junction. We show that, to leading order in \( \ell^{-1} \), the current is fully determined only by scattering amplitudes at the Fermi level. In particular, in the case of \( K \) equivalent channels within \( C \), the current can be presented in a simple closed-form formula, in terms of the roots of an algebraic equation of the form \( \mathcal{P}(u; \chi) = 0 \), with \( \mathcal{P}(u; \chi) \) being, at fixed \( \chi \), a \( 2K \)-degree polynomial of the (complex) unknown \( u \). In the case of \( K \) inequivalent channels within \( C \), at low temperatures the leading contribution to the current in \( \ell^{-1} \) can be recast in an integral formula that can be easily computed numerically.
- Section IV contains our conclusions.
- In the appendices, we provide mathematical details of our derivation.

II. THE DC JOSEPHSON CURRENT FOR A MULTI-CHANNEL SNS JUNCTION

In this section, based on a minimal set of reasonable assumptions, we provide a general formula for \( I[\chi; T = 0] \) and eventually discuss the extension of the result to \( I[\chi; T] \).

To simplify the derivation of the general formula for \( I[\chi; T = 0] \), we assume that, while the dispersion relation within the superconducting lead can be different for different channels, the number of channels in the two leads is the same, that is, \( N_L = N_R = N \). In addition, we assume that the superconducting order parameter is the same for each channel (see Appendix A for a detailed review of the simplifying assumptions.) As discussed above, while, as a model calculation, we consider the case of s-wave superconducting leads, pictured as one-dimensional superconductors described by the model Hamiltonian introduced in \([12]\), but our derivation is expected to apply...
equally well, for instance, to lattice models, to SHS-junctions, to junctions with spin-active interfaces, or to the case in which the leads are realized with superconductors with unconventional pairing, which has recently become of great relevance to engineering SN-interfaces hosting localized Majorana fermions.

In Fig. 1a, we provide a sketch of a generic multi-channel SNS junction. The corresponding Hamiltonian is given in Eqs. (A1), (A13) and (A14). Note that it has s-wave pairing and SU(2) spin symmetry. Any superconducting region \( r \) of the junction is described by a model Hamiltonian of the form

\[
H_r - \mu N_r = \int_{x \in r} dx \sum_{\lambda=1}^{N_r} \left\{ \sum_{\sigma} \Psi^\dagger_{r,\lambda,\sigma}(x) h_{0,\lambda}(x) \Psi_{r,\lambda,\sigma}(x) + \Delta e^{i \chi_r} \Psi_{r,\lambda,\uparrow}(x) \Psi_{r,\lambda,\downarrow}(x) + \Delta e^{-i \chi_r} \Psi^\dagger_{r,\lambda,\uparrow}(x) \Psi^\dagger_{r,\lambda,\downarrow}(x) \right\} ,
\]

with \( N_r \) being the number of active channels and \( N_r \) being the total particle number within \( r \), \( h_{0,\lambda}(x) \) being the normal Hamiltonian of \( r \), \( \Delta \) being the superconducting order parameter and \( \chi_r \) being the corresponding phase. \( \Psi_{r,\lambda,\sigma}(x) \) is the single-electron field operator for a particle in channel \( \lambda \) with spin \( \sigma \). The key quantity required to compute the dc Josephson current across a junction as such is the \( S \)-matrix for single quasiparticle states. Indeed, in general it can be shown that all the contribution to the dc Josephson current add up to an integral formula which only depends on the determinant of \( S \)-matrix. In Appendix A1 we discuss in detail the derivation of the single-particle wavefunctions in the leads from the BDG equations and the corresponding definition of the \( S \)-matrix. Since \( S \) is defined in terms of the “asymptotic” (that is, far enough from the central region) behavior of the wave functions, the formulas we derive in this section do not rely on any specific assumptions concerning \( C \) and the SN-interfaces, such as the ones we will introduce to discuss the long-junction limit, and hold independently of the specific behavior of the superconducting gap at the SN-interfaces and/or of the particular form of the Hamiltonian within \( C \).

In Ref. [9], the key step to systematically work out the formula for \( I[\chi;T] \) in the long-junction limit in the single-channel case was the possibility of expressing the determinant of the \( S \)-matrix at fixed \( E \) and \( \chi \), \( \det[S(E;\chi)] \) as

\[
\det[S(E;\chi)] = \frac{\mathcal{F}[E;\chi]}{\mathcal{G}[E;\chi]} ,
\]

with \( \mathcal{F}[E;\chi] \) and \( \mathcal{G}[E;\chi] \) in Eq. (2.2) being functions of \( E \) in the complex \( E \)-plane which we choose to obey the following properties (generally met in physically relevant models)\[\]

i) They are always finite for finite \( E \). This can be easily achieved by shifting poles of \( \mathcal{G} \) into zeroes of \( \mathcal{F} \) and vice versa;
ii) They have no common zeroes. [Possible common zeroes (e.g. \( E_0 \)], could always be cancelled by a redefinition: \( \mathcal{F}(E;\chi) \rightarrow \mathcal{F}[E;\chi]/(E - E_0) \), \( \mathcal{G}[E;\chi] \rightarrow \mathcal{G}[E;\chi]/(E - E_0) \), without changing Eq. (2.2)];
iii) \( \mathcal{F}[E;\chi] = \mathcal{G}^\ast[E;\chi] \). Here this equation refers to complex conjugating the function without complex conjugating its argument, \( E \). This condition is consistent with the requirement that \( |\det[S]| = 1 \) for scattering states.
iv) \( \mathcal{G}[E;\chi] \) can be defined to have branch cuts along the real \( E \)-axis, corresponding to the nonzero density of scattering states in the leads. This is due to the fact that \( \mathcal{G}[E;\chi] \) depends on \( E \) via the particle and hole momenta \( \beta_p \) and \( \beta_h \) and that they become double-valued functions of \( E \), for \( |E| > \Delta \).
v) \( \partial_n \ln \mathcal{G}[E;\chi] \) vanishes rapidly at \( |E| \rightarrow \infty \) along any ray not parallel to the real axis.
vii) \( \mathcal{G}[E;\chi] \) is real in the bound state region: the real axis with \( -\Delta \leq E \leq \Delta \).

Once the above conditions are met, from Eq. (2.2), by deforming the integration path as displayed in Fig. 2a, one can first of all show\[\]
that \( I[\chi;T = 0] \) can be written in terms of just one integral over the imaginary axis as

\[
I[\chi; T = 0] = \frac{2e}{2\pi} \int_{-\infty}^{\infty} d\omega \ \partial_\chi \ln \mathcal{G}[i\omega;\chi] .
\]

At variance, as sketched in Fig. 2b, at finite-\( T \) the deformation of the integration path in the energy plane yields a sum over the fermionic Matsubara frequencies \( \omega_\nu = 2\pi T (\nu + \frac{1}{2}) \), with \( \nu \) being a relative integer, so that one obtains\[\]

\[
I[\chi; T] = 2eT \sum_\nu \partial_\chi \ln \mathcal{G}[i\omega_\nu;\chi] .
\]
For a single-channel junction, the function $G$ left-hand lead to the central region $C$, and $R$ right-hand lead. The $S$-matrix is fully determined by the scattering processes that happen in the region corresponding to the central box of the figure. $M$ are exact formulas, independently of the details of the SNS junction. To generalize them to the multi-channel $(A10)$ of Appendix A. Eqs.(2.4,2.3) are formally equivalent to Eq. (1.1) and to its zero-temperature limit respectively, so, they are exact formulas, independently of the details of the SNS junction. To generalize them to the multi-channel case, we first of all introduce a pertinent labeling of the transmission matrix element, namely, we label each matrix elements $[M(E;\chi)]_{j\lambda,j'\lambda'}$ with the pair of indices $(j,\lambda)$ and $(j',\lambda')$. $\lambda,\lambda'(=1,\ldots,N)$ are the channel indices, while $j,j'=1,\ldots,4$ label the forward/backward-propagating particle/hole-solutions, exactly as in the single-channel case. In order to generalize Eq. (2.2) to the multi-channel case, we have to derive the generalized $\mathcal{F}[E;\chi],\mathcal{G}[E;\chi]$ functions. This is done in Appendix B where we prove that one gets

$$
\mathcal{F}[E;\chi] = \det[M^A(E;\chi)],
\mathcal{G}[E;\chi] = \det[M^B(E;\chi)],
$$

with $M^A, M^B$ in Eq. (2.6) being $4N \times 4N$-matrices that are given by

$$
[M^A(E;\chi)]_{j\lambda,j'\lambda'} = (\delta_{j,1} + \delta_{j,3})[M(E;\chi)]_{j\lambda,j'\lambda'} + (\delta_{j,2} + \delta_{j,4})\delta_{j,j'}\delta_{\lambda,\lambda'},
$$

and

$$
[M^B(E;\chi)]_{j\lambda,j'\lambda'}(\delta_{j,2} + \delta_{j,4})[M(E;\chi)]_{j\lambda,j'\lambda'} + (\delta_{j,1} + \delta_{j,3})\delta_{j,j'}\delta_{\lambda,\lambda'}.
$$
Eqs. (2.2-2.6,2.8) encode the key result of this section. Based on these equations, in the following, we discuss the “symmetric” behavior of the function $G$. For the sake of the presentation, we will separately discuss the “symmetric” detail the simplifications that occur to Eqs.(2.3,2.4) in the long junction limit, also providing a few explicit model Eqs.(2.2,2.6,2.7,2.8) encode the key result of this section. Based on these equations, in the following, we discuss in

I. JOSEPHSON CURRENT ACROSS A LONG MULTI-CHANNEL SNS JUNCTION

In order to discuss the long-junction limit, we assume that the system exhibits sharp interfaces between the leads and the central region as we sketch in Fig. 1. In particular, we assume that the central region $C$ runs from $x = 0$ to $x = \ell$ (as from now on $\ell$ will be the key variable of the expansion we perform, we will explicitly display it among the argument of the various functions.) Thus, as we are considering a ballistic SNS junction, the long-junction limit is defined by $E_{\text{Th}} \ll \Delta$ with the Thouless energy $E_{\text{Th}} \sim v/\ell$, $v$ being of the order of the Fermi velocity within $C$. As we discuss in detail in Appendix A.2, the transmission matrix $M(E; \chi; \ell)$ can then be written in a factorized form as

$$M(E; \chi; \ell) = R(E; \chi) \cdot M^C(E; \ell) \cdot L(E; \chi)$$

(3.1)

with $L(E; \chi)$ ($R(E; \chi)$) being the $4N \times 4N$ ($4K \times 4N$) transmission matrix at the left- (right-) hand interface, and $M^C(E; \ell)$ being the transmission matrix of the central region. For a ballistic junction, we then obtain $[M^C(E; \chi)]_{(j', \rho')\langle(\ell; \rho)} = e^{i\alpha_{j', \rho'}} \delta_{j', \rho'} \delta_{\lambda, \lambda'}$ and $\alpha_{1(2); \rho} = \pm \alpha_{p, \rho}$, $\alpha_{3(4); \rho} = \mp \alpha_{h, \rho}$ ($\rho, \rho' = 1, \ldots, K$), with the energy-$E$ particle- and hole-momenta within channel $\rho$, $\alpha_{p, \rho}$, $\alpha_{h, \rho}$, defined in Eq. (A19) of Appendix A. From Eq. (3.1), one obtains that the transmission matrix elements are given by

$$[M(E; \chi; \ell)]_{j, \lambda; (j', \lambda')} = \sum_{j=1}^{K} \sum_{\rho=1}^{K} [R(E; \chi)]_{j, \lambda; (j', \rho)} [L(E; \chi)]_{(j', \rho); j, \lambda'} e^{i\alpha_{j', \rho} \ell}$$

(3.2)

To compute $G(E; \chi; \ell)$, we use the formula for the $M^B$-matrix in Eq. (2.8) which implies that $G(E; \chi; \ell)$ is given by the determinant of a $2N \times 2N$-matrix, whose entries are given by the matrix elements $[M(E; \chi; \ell)]_{(2k, \lambda); (2k', \lambda')}$ with $k, k' = 1, 2$. From Eq. (3.2) we obtain

$$[M(E; \chi; \ell)]_{(2k, \lambda); (2k', \lambda')} = \sum_{\rho=1}^{K} \left\{ e^{i\alpha_{p, \rho} \ell} R_{(2k, \lambda); (1, \rho)} L_{(1, \rho); (2k', \lambda')} + e^{-i\alpha_{p, \rho} \ell} R_{(2k, \lambda); (2, \rho)} L_{(2, \rho); (2k', \lambda')} \right\}$$
\[ + e^{-i\alpha_p,\rho} R_{\ell(2k,\lambda),(3,\rho)} L_{(3,\rho),(2k',\lambda')} + e^{i\alpha_p,\rho} R_{\ell(2k,\lambda),(4,\rho)} L_{(4,\rho),(2k',\lambda')} \right) . \] (3.3)

When computing the determinant of the matrix in Eq. (3.3), one readily sees that it cannot contain a term proportional, for instance, to \( e^{2i\alpha_p,\rho} \), for any \( \rho \). Indeed, in the determinant, a term of this form should arise from a sum of the form \( c_{a_1,\ldots,a_N}^{a_1',\ldots,a_N'} e^{2i\alpha_p,\rho} R_{a_1,(1,\rho)} L_{(1,\rho),a_1'} R_{a_2,(1,\rho)} L_{(1,\rho),a_2'} \cdots \), with \( a_j \) and \( a_j' \) corresponding to a pair of indices such as \((j, \lambda)\) and \((j', \lambda')\) and \( c_{a_1,\ldots,a_N}^{a_1',\ldots,a_N'} \) being the fully antisymmetric tensor. Clearly, a term such as the one shown before is equal to 0. Therefore, one obtains

\[ G[E; \chi; \ell] = \prod_{\rho=1}^{K} \sum_{\{a_{\rho}, b_{\rho}\} \in \{-1,0,1\}} \left[ \delta_{a_{\rho}0} \delta_{b_{\rho}0} + \delta_{[a_{\rho},1]} \delta_{[b_{\rho},1]} \right] \times e^{i[a_{\rho}\alpha_{\rho}(\epsilon) + b_{\rho}\alpha_{\rho}(\epsilon')]} G_{\{a_{\rho}, b_{\rho}; a_1, b_1; a_2, b_2; \ldots; a_K, b_K\}}(E; \chi) , \] (3.4)

with the coefficients \( G_{\{a_{\rho}, b_{\rho}\}}(E; \chi) \) being fully determined by the \( L_\tau \) and \( R_\tau \)-matrix elements. Note that, in writing Eq. (3.3), we have evidenced that the nonzero contributions are either characterized by \( a_\rho = b_\rho = 0 \), or by \( \{a_{\rho}, b_{\rho}\} \in \{-1,1\} \). From Eq. (3.3) we see that, in the specific case of \( K \) equivalent (that is, with the same dispersion relation) channels within \( C \), i.e., assuming that \( \alpha_{j,\rho} \) is independent of \( \rho \), the contributions to Eq. (3.4) can be grouped together, so that one obtains

\[ G[E; \chi; \ell] = \sum_{a, b = -K}^{K} G_{a,b}(E; \chi) e^{i[a\alpha_{\rho} + b\alpha_{\rho}^{\prime}]} , \] (3.5)

where \( \sum \) means that the sum is taken over \( |a - b| = 0 \) (mod 2) and the coefficients \( G_{a,b}(E; \chi) \) being defined by comparing Eq. (3.3) to Eq. (3.4). Eqs. (3.4, 3.5) are exact and provide the multi-channel generalization of the analogous formulas of Ref. [9]. In the following, we will use them to derive the Josephson current in the long-junction limit. As the formal manipulations required to recover the formulas for the Josephson current are, in general, different whether the \( K \) channels are equivalent, or not, in the following we separately consider the case of equivalent and non-equivalent channels within \( C \).

### A. The Josephson current in the case of equivalent channels within the central region

In the case of \( K \) equivalent channels, \( G[E; \chi; \ell] \) is given in Eq. (3.5). In using Eq. (2.3) to compute \( I[\chi; T = 0] \) in the long-junction limit, we employ the same approximation used in Ref. [9] in the single-channel case, that is, we set \( \alpha_{\rho}/\hbar \approx \alpha_F \pm i\omega/\nu \), with the Fermi momentum \( \alpha_F = \sqrt{2mE} \) and the Fermi velocity \( \nu = \alpha_F/m \), and \( m \) being the effective mass, \( \mu \) the chemical potential. At the same time, we approximate \( G_{a,b}(E; \chi) \approx G_{a,b}(E = 0; \chi) \equiv G_{a,b}(\chi) \), which is correct up to subleading contributions in \( \ell^{-1} \) to the current. Eq. (2.3) eventually yields

\[ I[\chi; T = 0] = \frac{2e}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{\partial \chi}{\partial \omega} \ln \left[ \sum_{j = -K}^{K} P_{j + K}(\chi) e^{-2i\omega j} \right] + \ldots , \] (3.6)

with the coefficients \( P_j(\chi) \) fully determined by the matrix elements of \( L(E = 0; \chi) \) and of \( R(E = 0; \chi) \) and the ellipses corresponding to terms going to zero faster than \( \ell^{-1} \) in the large-\( \ell \) limit, which we will neglect henceforth. It can be shown that \( P_0 = P_{2K} = 1 \). (See Appendix D.) Switching to the integration variable \( u = e^{-2i\omega j} \), we obtain

\[ I[\chi; T = 0] = \frac{2e \nu}{2\pi} \int_{0}^{\infty} \frac{du}{u} \left\{ \frac{\sum_{j=1}^{2K-1} \partial \chi P_j(\chi) u^j}{u^{2K} + \sum_{j=1}^{2K-1} P_j(\chi) u^j + 1} \right\} \] . (3.7)

As discussed in detail in Appendix C, pertinently computing the integral in Eq. (3.7) and observing that, denoting with \( u_j(\chi) \) \((j = 1, \ldots, 2K)\) the roots of the equation \( \sum_{j=1}^{2K-1} P_j(\chi) u^j + 1 = 0 \), one obtains \( \Pi_{j=1}^{2K} u_j(\chi) = 1 \), one eventually gets

\[ I[\chi; T = 0] = \frac{e \nu}{4\pi \ell} \sum_{j=1}^{2K} \partial \chi \ln[u_j(\chi)] \] . (3.8)
While, in general, the coefficients $G_{a,b}(E;\chi)$ are complicated functions of the $L(E;\chi)$- and of the $R(E;\chi)$-matrix elements, Eq. (3.8) only involves quantities evaluated at the Fermi level. This allows for building a simplified algorithm for constructing the polynomial $P(u;\chi) = u^{2K} + \sum_{j=1}^{2K-1} P_j(\chi) u^j + 1$, which we discuss in detail in Appendix D.

The generalization of Eq. (3.8) to $T$ finite, but still much lower than the superconducting gap, can be again worked out by deforming the integration path in the complex energy plane, so that the final integral over the imaginary axis is traded for a sum of integrals over small circles surrounding the points $i\omega_n$ over the imaginary axis (see Fig. 2), with $\omega_n$ being the $\nu^{th}$ fermionic Matsubara frequency, $\omega_n = \pi T (2\nu + 1)$. To work out the modification of Eq. (3.6) at finite-$T$, we also consider that, due to the particle-hole symmetry of the Bogoliubov-de Gennes equations near the Fermi level, one obtains that $P_{K+j}(\chi) = P_{K-j}(\chi)$. Thus, performing the integrals over each circle and adding up the results, one obtains

$$I[\chi;T] = 2 e T \sum_{\nu=-\infty}^{\infty} \partial_{\chi} \ln \left[ 2 \cosh \left( \frac{2K\omega_n\ell}{v} \right) + 2 \sum_{j=0}^{K-1} P_j(\chi) \cosh \left( \frac{2j\omega_n\ell}{v} \right) \right] + \ldots ,$$

(3.9)

with, again, the ellipses corresponding to terms going to zero faster than $\ell^{-1}$ in the large-$\ell$ limit. As $\ell T/v \to 0$, the sum in Eq. (3.9) can be traded for an integral over a continuous variable $\omega$, thus leading back to Eq. (3.6). For $N = K = 1$, it is easy to check that one obtains Eq. (53) of [9] for the finite-temperature dc Josephson current in this case. Finally, in the regime $\ell T/v \gg 1$, $I[\chi;T]$ exhibits an exponential decay in $T$, again consistent with the result of [9].

From Eqs. (3.8, 3.9), we see that the key quantity needed to compute the current, both at $T = 0$ and at $T > 0$, is the polynomial $P(u;\chi)$. In Appendix D we discuss in detail the algorithm for constructing $P(u;\chi)$ in general and carry out the whole calculation in the specific case $N = 1$. In particular, we show that, provided $K \geq 2$, the calculation can be always reduced to a model with $K = 2$ channels within $C$ coupled to the $L$- and to the $R$-channel with strengths of the form $\langle t_{L1}, t_{L2} \rangle = t_{L} (\cos(\theta), \sin(\theta))$ and $\langle t_{R1}, t_{R2} \rangle = t_{R}(1,0)$, respectively. As a specific model calculation, we explicitly compute Eq. (3.8) in the case in which the two channels within $C$ effectively coupled to the leads both exhibit perfect Andreev reflection. In this case, one obtains (see Appendix D for details)

$$P_\theta(u;\chi) = u^4 + 1 - 2\cos^2(\theta)[\cos(2\alpha F \ell) + \cos(\chi)](u^3 + u) + 2 \{ \cos(2\theta) + 2 \cos^2(\theta) \cos(2\alpha F \ell) \cos(\chi) \} u^2 ,$$

(3.10)

with the suffix $\theta$ added to evidence the dependence of $P$ on this parameter, as well. The equation $P_\theta(u;\chi) = 0$ may be straightforwardly solved by means of elementary algebraic techniques. Its roots are given by

$$u_1(\theta;\chi) = z_1(\theta;\chi) + i \sqrt{1 - z_1^2(\theta;\chi)}$$

$$u_2(\theta;\chi) = z_1(\theta;\chi) - i \sqrt{1 - z_1^2(\theta;\chi)}$$

$$u_3(\theta;\chi) = z_2(\theta;\chi) + i \sqrt{1 - z_2^2(\theta;\chi)}$$

$$u_4(\theta;\chi) = z_2(\theta;\chi) - i \sqrt{1 - z_2^2(\theta;\chi)} ,$$

(3.11)

with

$$z_1(\theta;\chi) = \frac{1}{2} \left\{ \cos^2(\theta)[\cos(2\alpha F \ell) + \cos(\chi)] + \sqrt{4\sin^2(\theta) - 4 \cos^2(\theta) \cos(2\alpha F \ell) \cos(\chi) + \cos^4(\theta)[\cos(2\alpha F \ell) + \cos(\chi)]^2} \right\}$$

$$z_2(\theta;\chi) = \frac{1}{2} \left\{ \cos^2(\theta)[\cos(2\alpha F \ell) + \cos(\chi)] - \sqrt{4\sin^2(\theta) - 4 \cos^2(\theta) \cos(2\alpha F \ell) \cos(\chi) + \cos^4(\theta)[\cos(2\alpha F \ell) + \cos(\chi)]^2} \right\}$$

(3.12)

From Eqs. (3.11) we see that it is possible to write

$$u_{1,2}(\theta;\chi) = e^{\pm i\theta_1[\theta;\chi]} , \quad u_{3,4}(\theta;\chi) = e^{\pm i\theta_2[\theta;\chi]} ,$$

(3.13)

with $\theta_1[\theta;\chi] = \arccos\{ z_1(\theta;\chi) \}$. Thus, we eventually obtain that the dc Josephson current is given by (making explicit the dependence on the parameter $\theta$, as well)
that 

\[
I[\chi; T = 0] = \frac{e\nu}{\pi \ell} \left\{ \vartheta_1^2(\theta; \chi) + \vartheta_2^2(\theta; \chi) \right\}.
\]

To check the consistency of Eq. (3.14), we notice that, for \( \theta = 0 \), \( I[\chi; \theta = 0; T = 0] \) reduces back to Ishii’s sawtooth behavior corresponding to perfect Andreev reflection at both boundaries. At variance, for \( \theta = \frac{\pi}{2} \) one obtains \( I[\chi; \theta = \frac{\pi}{2}; T = 0] = 0 \), as it is appropriate to a situation where only channel-1 within C is coupled to the left-hand lead and only channel-2 is coupled to the right-hand lead. To evidence the effect of a finite value of \( \theta \) such that \( 0 < \theta < \frac{\pi}{2} \), in Fig. 3 we plot \( I[\chi; \theta; T = 0] \) vs. \( \chi \) for three values of \( \theta \), including \( \theta = 0 \) (see caption for details). It is interesting to note that a finite discontinuity takes place at \( \chi = \pi \) (mod 2\( \pi \)) for any value of \( \theta \) and that \( I[\chi = \pi^+; \theta; T = 0] - I[\chi = \pi^-; \theta; T = 0] \propto \cos^2(\theta) \). This is a typical feature of junctions exhibiting perfect Andreev reflection at the SN interfaces; formally, it is a consequence of the fact that, as it can be readily seen from Eqs. (3.12), \( \vartheta_1(\theta; \chi) \) always reaches the value \(-1\) as \( \chi \to \pi \) (mod 2\( \pi \)), irrespectively of the values of \( \alpha_F \ell \) and \( \theta \). Thus, though \( \vartheta_2(\theta; \chi) \) is continuous at \( \chi = \pi \), it exhibits a cusp, with a corresponding finite discontinuity in its derivative. This is what determines the discontinuity in the plots of \( I[\chi; T = 0] \) vs. \( \chi \) in Fig. 3. An important remark about Eq. (3.14) is that, though, at a first glance, it looks similar to what one would get by only summing the contributions to \( I[\chi; \theta; T = 0] \) arising from ABS’s near the Fermi energy, in fact, as a result of the cancellations between large contributions to the current from states far from the Fermi energy, the result is exact, to leading order in \( \ell^{-1} \), as we proved before.

While in “conventional” multi-channel junctions the various channels do not exhibit equivalence, as they typically have different Fermi velocities, an SNS junction with two equivalent channels can be realized for instance by connecting a non-chiral metallic carbon nanotube to two spinful s-wave superconductors. Electrons around the two non-equivalent Dirac points in the single-electron spectrum of the carbon nanotube act as two spinful independent channels, thus realizing the system we discuss in detail in appendix D and in which, in this section, we explicitly solve for \( I[\chi; T = 0] \) in the special case of pure Andreev reflection in each channel coupled to the superconducting leads.

When there is no equivalence between the channels within C, from the discussion we make in Section III one expects that the current in the long-junction limit is still determined by reflection coefficients at the Fermi level, even in the case of inequivalent channels. However, as we are going to outline in the following section, it is in general not possible to resort to a simple and compact analytical expression, such as the one in Eqs. (3.17,3.19) and, therefore, one has to numerically evaluate the resulting integral which is expected to depend on a number of parameters, including the asymmetries between the channels.

### B. The Josephson current in the case of inequivalent channels within the central region

In the case of inequivalent channels within C, \( G[E; \chi; \ell] \) is given in Eq. (3.14). In the large \( \ell \) limit one may again perform the approximation used in Subsection IIIA and discussed in [6]. As a result, Eq. (2.3) for \( I[\chi; T = 0] \) generalizes to

\[
I[\chi; T = 0] = \frac{2e}{2\pi} \frac{U}{\ell} \int dz \vartheta_\chi \left\{ \ln \prod_{\rho=1}^{K} \sum_{\{a_\rho,b_\rho\} \in \{-1,0,1\}} \left[ \delta_{a_\rho,0} \delta_{b_\rho,0} + \delta_{[a_\rho,1]} \delta_{[b_\rho,1]} \right] e^{i(a_\rho - b_\rho)\alpha_F^p \ell} e^{-w_\rho(a_\rho + b_\rho)z} \right\}.
\]
\[ \times \tilde{G}_{\{a_1, b_1, \ldots, a_K, b_K\}}(\chi) + \ldots, \quad (3.15) \]

with \( \tilde{G}_{\{a_1, b_1, \ldots, a_K, b_K\}}(\chi) = G_{\{a_1, b_1, \ldots, a_K, b_K\}}(E = 0; \chi), \alpha_F^{(\rho)}, v^{(\rho)} \) being respectively the Fermi momentum and the Fermi velocity for channel-\( \rho \), \( U^K = \prod_{\rho=1}^{K} v^{(\rho)} \), and \( w_{\rho} = U/v^{(\rho)} \). Note that, in Eq. (3.15), we have introduced the rescaled integration variable \( z = \omega U/U \) and that, as in the similar equations above, the ellipses correspond to subleading contributions going to zero faster than \( \ell^{-1} \) in the large-\( \ell \) limit. Similarly, Eq. (3.9) for \( I[\chi; T] \) now generalizes to

\[ I[\chi; T] = 2eT \sum_{\nu = -\infty}^{\infty} \partial_{\chi} \{ \ln \prod_{\rho=1}^{K} \sum_{(a_{\rho}, b_{\rho}) \in \{-1, 0, 1\}} \left[ [\delta_{a_{\rho}, 0} \delta_{b_{\rho}, 0} + \delta_{|a_{\rho}|, 1} \delta_{|b_{\rho}|, 1}] e^{i(a_{\rho} - b_{\rho})\alpha_F^{(\rho)} \ell} e^{-w_{\rho}(a_{\rho} + b_{\rho})\omega} \right] \times \tilde{G}_{\{a_1, b_1, \ldots, a_K, b_K\}}(\chi) \} . \quad (3.16) \]

As it clearly appears from Eqs. (3.15, 3.10), the general result that in the long-junction limit the current only depends on backscattering amplitudes at the Fermi level holds in the case of inequivalent channels, as well. The key function one has to derive, in order to compute \( I[\chi; T] \) in the long-junction limit, is the function \( \Phi[\omega; \chi] \), defined as

\[ \Phi[\omega; \chi] = \prod_{\rho=1}^{K} \sum_{(a_{\rho}, b_{\rho}) \in \{-1, 0, 1\}} \left[ [\delta_{a_{\rho}, 0} \delta_{b_{\rho}, 0} + \delta_{|a_{\rho}|, 1} \delta_{|b_{\rho}|, 1}] e^{i(a_{\rho} - b_{\rho})\alpha_F^{(\rho)} \ell} e^{-w_{\rho}(a_{\rho} + b_{\rho})\omega} \right] \times \tilde{G}_{\{a_1, b_1, \ldots, a_K, b_K\}}(\chi) \} . \quad (3.17) \]

with the coefficients \( \tilde{G}_{\{a_1, b_1, \ldots, a_K, b_K\}}(\chi) \) defined as in Eq. (3.15). In Appendix E we discuss the systematic procedure to construct \( \Phi[\omega; \chi] \); clearly, the final result will apply in general, including the case of equivalent channels within \( C \). In this latter case, however, as we discuss in subSection III B, once expressed in terms of the variable \( u = e^{-2\omega} \), \( \Phi[\omega; \chi] \) reduces to the \( 2K \)-degree polynomial \( P(u; \chi) \) in the variable \( u \).

As an example of the effectiveness of our procedure, we compute \( I[\chi; T = 0] \) for \( N = 1, K = 2 \) in the case in which the two channels within \( C \) are characterized by Fermi momenta \( \alpha_{F}^{(1)}, \alpha_{F}^{(2)} \) and by Fermi velocities \( v^{(1)}, v^{(2)} \), respectively, and the couplings at the SN interfaces are \( ([t_{L}], [t_{R}] = t_{L}\cos(\varphi), \sin(\varphi)) \) and \( ([t_{R}], [t_{R}] = t_{R}\cos(\varphi), \sin(\varphi)) \). As we are going to show in the following, the absence of the symmetry between the two channels makes even this simple case quite interesting to consider. To derive \( \Phi[\omega; \chi] \), we use the formula in Eq. (E3) of Appendix E which, in the specific case we are dealing with, yields

\[ \Phi[\omega; \chi] = 4 \cos^4(\varphi) \cos(2\alpha_{F}^{(1)} \ell) [\cos(\chi) - \cosh(2\omega w)] + 4 \sin^4(\varphi) \cos(2\alpha_{F}^{(2)} \ell) \left[ \cos(\chi) - \cosh \left( \frac{2\omega}{w} \right) \right] \\
- 4 \left[ \cos^4(\varphi) \cos \left( \frac{2\omega}{w} \right) + \sin^4(\varphi) \cosh(2\omega w) \right] \left[ \cos(\chi) + 2 \cosh \left( 2\omega w + \frac{2\omega}{w} \right) + [1 + \cos(4\varphi)] \cosh \left( 2\omega w - \frac{2\omega}{w} \right) \right] \\
+ 8 \sin^2(\varphi) \left[ \cosh \left( \frac{\omega}{w} \right) \cosh(\omega w) \sin(\alpha_{F}^{(1)} \ell) \sin(\alpha_{F}^{(2)} \ell) \sin \left( \frac{\chi}{2} \right) - \sin \left( \frac{\omega}{w} \right) \sinh(\omega w) \cos(\alpha_{F}^{(1)} \ell) \cos(\alpha_{F}^{(2)} \ell) \cos^2 \left( \frac{\chi}{2} \right) \right] \right), \quad (3.18) \]

with \( w = \sqrt{\frac{v^{(1)} v^{(2)}}{v^{(1)} v^{(2)}}} \). Clearly, the “relative contribution” of the two channels within \( C \) to the total current depends on the angle \( \varphi \). For instance, if \( \varphi \) is closer to 0 than to \( \pi/2 \), channel-1 is expected to provide a contribution higher than the one provided by channel-2. Thus, on tuning the asymmetry between the two channels, in this case we expect the current to increase (decrease), if the asymmetry “weights” more the contribution from channel-1 (channel-2). To check this point, in Fig. 4 we plot the current at fixed \( \varphi \), \( I[\chi; T = 0] \) vs. \( \chi \), numerically computed using the formula for \( \Phi[\omega; \chi] \) in Eq. (4.15), with \( \varphi = \pi/10 \) and the other parameters fixed as detailed in the caption of the figure. As expected, at fixed \( \chi \), we see that the smaller is \( w \) (that is, the higher is the Fermi velocity in channel-1 with respect to the one in channel-2), the higher is the current.

As we showed, once \( \Phi[\omega; \chi] \) is computed as discussed in Appendix E, Eqs. (3.15, 3.10) provide a simple and effective tool to compute \( I[\chi; T = 0] \) and \( I[\chi; T] \) for generic values of the parameters. The important information they encode is that, in the long junction limit, the current is fully expressed only in terms of reflection amplitudes computed at the Fermi energy. Once the reflection amplitudes are known, the integral and/or the sum can be computed numerically, which (especially for a long junction) is enormously simpler than performing a sum over contributions from all kind of states at any energy. Our result holds in general, independently of the symmetry between the channels within \( C \) and, in the symmetric case, it is possible to work out simple closed-form formulas for the current, as the ones we provide in Eqs. (3.15, 3.10).
FIG. 4. Plot of $I[\chi; T=0]$ vs. $\chi$, numerically computed as from Eq. (3.15) with $\Phi[\omega; \chi]$ given in Eq.(3.18). The parameters are chosen so that $\alpha_0^{(1)} = 0.48\pi$, $\alpha_0^{(2)} = 0.51\pi$, $\varphi = \pi/10$, while $w = .5$ for the blue curve, $w = 1$ for the black curve, $w = 1.5$ for the green curve.

IV. CONCLUSIONS

In this paper, we go through a systematic application of the analytic properties of the S-matrix for a long multi-channel SNS junction, to show that the dc Josephson current across the junction at low temperatures can be fully expressed in terms of scattering amplitudes at the Fermi level only. When the dispersion relations for the channels within the central region are equal to each other, the current can be expressed in terms of a simple, closed-form formula, given in Eq. (3.8) in the zero-temperature limit, in Eq. (3.9) at finite temperature. In general, the current can still be simply computed, by evaluating integrals involving only scattering amplitudes at the Fermi level. Besides providing a simple and effective algorithm for computing the current, our results justify resorting to a low energy Hamiltonian approach,13 which is crucial for treating Luttinger liquid interaction effects. While we choose a model Hamiltonian in which the leads are pictured as one-dimensional s-wave superconductors, our results are expected to readily generalize to situations in which the leads are realized, for instance, as topological p-wave superconductors, where the dc Josephson current is, in general, strongly affected by the possible presence of emerging Majorana fermions at the SN-interfaces.

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Appendix A: Scattering solutions of the Bogoliubov - de Gennes equations

In this appendix we review the derivation of the single-quasiparticle and of the single-quasihole scattering solutions to the Bogoliubov - de Gennes equations for a multi-channel SNS junction. In particular, we first derive the asymptotic form of the scattering solutions within the superconducting leads, which is expected to apply to a generic junction, without specializing to the long-junction limit. Thereafter, we will discuss in detail the case of a long SNS junction.

1. Asymptotic solutions within the superconducting leads

We perform our derivation within a straightforward multi-channel generalization of the continuum one-dimensional model for a spinful superconductor discussed in [12]. Besides the simplifying assumption $N_R = N_L = N$, since by a phase redefinition we can always choose the phases of the order parameter to be equal and opposite in the left/right leads, we also require that the phase difference $\chi$ between the leads is uniformly distributed between the two sides, namely, that the phase of the superconducting order parameter $\to \chi$ for $x \to -\infty$ and $\to -\chi$ for $x \to \infty$, and that the superconducting gap $\Delta$ is the same for all the channels. Thus, the second-quantized Hamiltonians for leads L and R, $H_L$ and $H_R$, are respectively given by
\[ H_L - \mu N_L = \int_{x \in L} dx \sum_{\lambda=1}^{N} \left\{ \sum_{\sigma} \Psi_{L,\lambda,\sigma}^\dagger(x) h_{0,\lambda}(x) \Psi_{L,\lambda,\sigma}(x) + \Delta e^{i\chi} \Psi_{L,\lambda,\uparrow}(x) \Psi_{L,\lambda,\downarrow}(x) + \Delta e^{-i\chi} \Psi_{L,\lambda,\downarrow}(x) \Psi_{L,\lambda,\uparrow}(x) \right\} \]

\[ H_R - \mu N_R = \int_{x \in R} dx \sum_{\lambda=1}^{N} \left\{ \sum_{\sigma} \Psi_{R,\lambda,\sigma}^\dagger(x) h_{0,\lambda}(x) \Psi_{R,\lambda,\sigma}(x) + \Delta e^{i\chi} \Psi_{R,\lambda,\uparrow}(x) \Psi_{R,\lambda,\downarrow}(x) + \Delta e^{-i\chi} \Psi_{R,\lambda,\downarrow}(x) \Psi_{R,\lambda,\uparrow}(x) \right\} \] (A1)

with \( \Psi_{L,\lambda,\sigma}(x), \Psi_{R,\lambda,\sigma}(x) \) being the fermion annihilation operators for an electron in channel-\( \lambda \) with spin \( \sigma \) in lead L and R, respectively. \( h_{0,\lambda}(x) = -\frac{1}{2m_{S,\lambda}} \frac{d^2}{dx^2} + V_{S,\lambda} \) is the normal lead Hamiltonian in channel-\( \lambda \), \( m_{S,\lambda} \) and \( V_{S,\lambda} \) are the corresponding effective electron mass and potential, respectively. The BDG equations within L and R are derived starting from the Bogoliubov-Valatin transformations, which enable us to express an energy eigenmode operator of \( H_L, \gamma_{L,\lambda,\sigma}(E) \), as

\[
\gamma_{L,\lambda,\sigma}(E) = \int_{x \in L} dx \{ u_{L,\lambda,E}(x) \Psi_{L,\lambda,\sigma}(x) + \sigma v_{L,\lambda,E}(x) \Psi_{L,\lambda,\sigma}^\dagger(x) \} \quad . \tag{A2}
\]

Requiring that \( [\gamma_{L,\lambda,\sigma}(E), H_L] = E\gamma_{L,\lambda,\sigma}(E) \) yields the BDG equations for the wavefunctions within L, \( u_{L,\lambda,E}(x), v_{L,\lambda,E}(x) \), given by

\[
h_{0,\lambda}(x)u_{L,\lambda,E}(x) + \Delta e^{i\chi} v_{L,\lambda,E}(x) = Eu_{L,\lambda,E}(x) \\
\Delta e^{-i\chi} u_{L,\lambda,E}(x) - h_{0,\lambda}(x)v_{L,\lambda,E}(x) = Ev_{L,\lambda,E}(x) \quad , \tag{A3}
\]

with \( x \in \mathbb{L} \). Similarly, writing an energy eigemode operator of \( H_R, \gamma_{R,\lambda,\sigma}(E) \), as

\[
\gamma_{R,\lambda,\sigma}(E) = \int_{x \in R} dx \{ u_{R,\lambda,E}(x) \Psi_{R,\lambda,\sigma}(x) + \sigma v_{R,\lambda,E}(x) \Psi_{R,\lambda,\sigma}^\dagger(x) \} \quad , \tag{A4}
\]

and requiring that \( [\gamma_{R,\lambda,\sigma}(E), H_R] = E\gamma_{R,\lambda,\sigma}(E) \) yields the BDG equations for the wavefunctions within R, \( u_{R,\lambda,E}(x), v_{R,\lambda,E}(x) \), given by

\[
h_{0,\lambda}(x)u_{R,\lambda,E}(x) + \Delta e^{-i\chi} v_{R,\lambda,E}(x) = Eu_{R,\lambda,E}(x) \\
\Delta e^{i\chi} u_{R,\lambda,E}(x) - h_{0,\lambda}(x)v_{R,\lambda,E}(x) = Ev_{R,\lambda,E}(x) \quad , \tag{A5}
\]

with \( x \in \mathbb{R} \). Thus, one sees that a scattering solutions at energy \( E \) asymptotically obeys Eqs.(A3) within L and Eqs.(A5), within R. As a consequence, for each channel \( \lambda \) one finds four independent solutions to Eqs.(A3): a forward/backward particle-like- \((1,2)\) and a forward/backward hole-like \((3,4)\) solution, respectively given by

\[
\left[ \begin{array}{c} u_{L,\lambda,E}(x) \\ v_{L,\lambda,E}(x) \end{array} \right]_{1,2} = \left[ \begin{array}{c} \cos \left( \frac{\chi}{2} \right) \\ -e^{i\phi} \sin \left( \frac{\chi}{2} \right) \end{array} \right] e^{\pm i\beta_{p,\lambda} x} \right] \right]_{3,4} = \left[ \begin{array}{c} -e^{-i\phi} \sin \left( \frac{\chi}{2} \right) \\ \cos \left( \frac{\chi}{2} \right) \end{array} \right] e^{-i\beta_{h,\lambda} x} \quad , \tag{A6}
\]

with \( \beta_{p/h,\lambda} = 2m_{S,\lambda} \{ \mu_{S,\lambda} \pm (E^2 - \Delta^2)^{1/2} \} \) and \( \Psi \equiv -\arcsin(\Delta/E) \). Similarly, for each channel one finds four analogous independent solutions to Eqs.(A5), given by

\[
\left[ \begin{array}{c} u_{R,\lambda,E}(x) \\ v_{R,\lambda,E}(x) \end{array} \right]_{1,2} = \left[ \begin{array}{c} \cos \left( \frac{\chi}{2} \right) \\ -e^{-i\phi} \sin \left( \frac{\chi}{2} \right) \end{array} \right] e^{\pm i\beta_{p,\lambda}(x-\ell)} \right] \right]_{3,4} = \left[ \begin{array}{c} -e^{i\phi} \sin \left( \frac{\chi}{2} \right) \\ \cos \left( \frac{\chi}{2} \right) \end{array} \right] e^{i\beta_{h,\lambda}(x-\ell)} \quad , \tag{A7}
\]

where the right lead is at \( x > \ell \). Thus, in each channel \( \lambda \), a generic wavefunction within \( L \) (R), \( \left[ \begin{array}{c} u_{L(R),\lambda,E}(x) \\ v_{L(R),\lambda,E}(x) \end{array} \right] \), can be written as a linear superpositions of the four kinds of plane wave quasiparticle and quasihole solutions in Eqs.(A6,A7) so that, in general, one obtains
The transmission matrix $M(E; \chi)$ relates the $A_{j,\lambda}^+(E; \chi)$-amplitudes to the $A_{j,\lambda}^-(E; \chi)$-ones. Thus, it appears natural to label the $M$-matrix elements with two pairs of indices, $(j, \lambda), (j', \lambda')$, referring to the quasiparticle character and to the channel, respectively, so that the matrix elements $[M(E; \chi)]_{(j,\lambda),(j',\lambda')}$ satisfy

$$A_{j,\lambda}^+(E; \chi) = \sum_{j'=1}^N \sum_{\lambda'=1}^N [M(E; \chi)]_{(j,\lambda),(j',\lambda')} A_{j',\lambda'}^-(E; \chi) .$$

(A10)

At variance, the $S$-matrix relates to each other incoming (in) and outgoing (out) quasiparticle amplitudes. These are related to the $A_{j,\lambda}^\pm(E; \chi)$-amplitudes as

$$
\begin{bmatrix}
A_{1,\lambda}^{in}(E; \chi) \\
A_{2,\lambda}^{in}(E; \chi) \\
A_{3,\lambda}^{in}(E; \chi) \\
A_{4,\lambda}^{in}(E; \chi)
\end{bmatrix} = \begin{bmatrix}
A_{1,\lambda}^{-}(E; \chi) \\
A_{2,\lambda}^{-}(E; \chi) \\
A_{3,\lambda}^{-}(E; \chi) \\
A_{4,\lambda}^{-}(E; \chi)
\end{bmatrix} \begin{bmatrix}
A_{1,\lambda}^{out}(E; \chi) \\
A_{2,\lambda}^{out}(E; \chi) \\
A_{3,\lambda}^{out}(E; \chi) \\
A_{4,\lambda}^{out}(E; \chi)
\end{bmatrix} = \begin{bmatrix}
A_{1,\lambda}^{+}(E; \chi) \\
A_{2,\lambda}^{+}(E; \chi) \\
A_{3,\lambda}^{+}(E; \chi) \\
A_{4,\lambda}^{+}(E; \chi)
\end{bmatrix} .
$$

(A11)

Thus, the $S$-matrix elements satisfy

$$\sqrt{v_{j,\lambda}} A_{j,\lambda}^{out}(E; \chi) = \sum_{j'=1}^N \sum_{\lambda'=1}^N [S(E; \chi)]_{(j,\lambda),(j',\lambda')} \sqrt{v_{j',\lambda'}} A_{j',\lambda'}^{in}(E; \chi) ,$$

(A12)

with the velocities $v_{j,\lambda} = \left| \frac{dE}{d\xi_{j,\lambda}} \right|$ for $j = 1, 2$, and $v_{j,\lambda} = \left| \frac{dE}{d\xi_{j,\lambda}} \right|$ for $j = 3, 4$.

2. **Bogoliubov-de Gennes equations for a long SNS junction**

   We now consider the BDG equations within a long SNS junction, such as the one sketched in Fig. 1b). We assume that the central region $C$ runs from $x = 0$ to $x = \ell$ and, consistently, that the lead $L$ extends from $x = -\infty$ to $x = 0$, while the lead $R$ extends from $x = \ell$ to $x = \infty$. Letting $K$ be the number of open electronic channels within $C$, one finds that the second-quantized Hamiltonian for the system is given by $H = H_L + H_R + H_C + H_T$, with $H_L, H_R$ given in Eqs. (A11), with the integrals respectively computed from $-\infty$ to 0 and from $\ell$ to $+\infty$, and $H_C$ given by

$$H_C - \mu N_C = \int_0^\ell dx \sum_{\rho=1}^K \left\{ \sum_{\sigma} \Psi_{C,\rho,\sigma}^\dagger(x) h_{\rho}(x) \Psi_{C,\rho,\sigma}(x) \right\} ,$$

(A13)

with $\Psi_{C,\rho,\sigma}(x)$ being the annihilation operator for an electron in channel-$\rho$ with spin $\sigma$ within $C$, $N_C$ being the total particle number within $C$, and $h_{\rho}(x) = -\frac{2m_{\rho}^e}{\hbar^2} \frac{d^2}{dx^2} + V_{\rho}$ is the corresponding single-fermion Hamiltonian, with $m_{\rho}$ being the effective electron mass and $V_{\rho}$ being the potential within channel $\rho$. In addition, in order for $H_L, H_R$ in Eqs. (A11) and $H_C$ in Eq. (A13) to be well-defined, we impose boundary conditions on $\Psi_{L,\lambda,\sigma}(x)$ and $\Psi_{C,\rho,\sigma}(x)$ at $x = 0$, as well as on $\Psi_{R,\lambda,\sigma}(x)$ and $\Psi_{C,\rho,\sigma}(x)$ at $x = \ell$, by requiring that all the derivatives with respect to $x$ vanish, so that the fields themselves are non-zero at the interfaces. The tunneling Hamiltonian $H_T$ encodes the coupling between $C$ and the leads. We assume it to take the generic form.
\[ H_T = \sum_{\lambda=1}^{N} \sum_{\rho=1}^{K} \sum_{\sigma=\uparrow,\downarrow} \{ t_{L,\lambda,\rho} \Psi_{L,\lambda,\sigma}(0) \Psi_{C,\rho,\sigma}(0) + \text{h.c.} \} + \sum_{\lambda=1}^{N} \sum_{\rho=1}^{K} \sum_{\sigma=\uparrow,\downarrow} \{ t_{R,\lambda,\rho} \Psi_{R,\lambda,\sigma}(\ell) \Psi_{C,\rho,\sigma}(\ell) + \text{h.c.} \} , \]  

where \( t_{L,\lambda,\rho}, t_{R,\lambda,\rho} \) are tunneling amplitude matrices independent of \( \sigma \). Writing an energy eigemode operator of \( H_C \), \( \gamma_{C,\rho,\sigma}(E) \), as

\[ \gamma_{C,\rho,\sigma}(E) = \int_0^{\ell} dx \{ u_{C,\rho,E}(x) \Psi_{C,\rho,\sigma}(x) + \sigma v_{C,\rho,E}(x) \Psi_{C,\rho,\sigma}^\dagger(x) \} , \]

and requiring that \( [\gamma_{C,\rho,\sigma}(E), H_C] = E \gamma_{C,\rho,\sigma}(E) \) yields the BDG equations for the wavefunctions within \( C \), \( u_{C,\rho,E}(x), v_{C,\rho,E}(x) \), given by

\[ \begin{bmatrix} -\frac{1}{2m_\rho} \frac{d^2}{dx^2} + V_\rho \end{bmatrix} u_{C,\rho,E}(x) = E u_{C,\rho,E}(x) \]
\[ \begin{bmatrix} \frac{1}{2m_\rho} \frac{d^2}{dx^2} - V_\rho \end{bmatrix} v_{C,\rho,E}(x) = E v_{C,\rho,E}(x) , \]  

with \( 0 < x < \ell \). A generic solution to the Eqs. (A16) can then be written as

\[ \begin{bmatrix} u_{C,\rho,E}(x) \\ v_{C,\rho,E}(x) \end{bmatrix}^{1(2)} = \sum_{j=1}^{4} C_{j,\rho}(E; \chi) \begin{bmatrix} u_{C,\rho,E}(x) \\ v_{C,\rho,E}(x) \end{bmatrix}_j , \]

with

\[ \begin{bmatrix} u_{C,\rho,E}(x) \\ v_{C,\rho,E}(x) \end{bmatrix}^{1(2)} = \begin{bmatrix} e^{\pm i\alpha_{\rho,E}x} \\ 0 \end{bmatrix} , \]
\[ \begin{bmatrix} u_{C,\rho,E}(x) \\ v_{C,\rho,E}(x) \end{bmatrix}^{3(4)} = \begin{bmatrix} 0 \\ e^{\mp i\alpha_{\rho,E}x} \end{bmatrix} , \]

and

\[ \alpha_{\rho,E} = \sqrt{2m_\rho(V_\rho + E)} \]
\[ \alpha_{\rho,E} = \sqrt{2m_\rho(V_\rho - E)} . \]

The scattering processes at the interfaces are determined by the specific form of \( H_T \) in Eq. (A14) and are encoded in the transmission matrix from \( L \) to \( C \), \( L(E) \), and in the transmission matrix from \( C \) to \( R \), \( R(E) \). In general, \( L(E) \) and \( R(E) \) are a \( 4K \times 4N \) and a \( 4N \times 4K \)-rectangular matrix, respectively, defined so that

\[ C_{j,\rho}(E; \chi) = \sum_{j'=1}^{N} \sum_{\lambda=1}^{N} [L(E; \chi)]_{(j',\rho),(j,\lambda)} A_{j',\lambda}^+(E; \chi) \]
\[ A_{j,\rho}^+(E; \chi) = \sum_{j'=1}^{K} \sum_{\rho'=1}^{K} [R(E; \chi)]_{(j,\rho),(j',\rho')} C_{j',\rho'}(E; \chi) . \]  

Once the transmission matrices at the interfaces are defined as in Eq. (A20), the factorizability of the total transmission matrix readily yields Eq. (3.1) of the main text.
Appendix B: Derivation of Eqs. (2.2, 2.6) for a multi-channel junction

In this appendix, we derive Eq. (2.2) in the multi-channel case, together with the relation between \( F[E; \chi], G[E; \chi] \) and the \( M \)-matrix elements (Eq. (2.6)) (for notational simplicity, we will drop throughout all the appendix the dependence of the amplitudes and of the matrix elements on \( E \) and \( \chi \)). To do so, we consider a solution of the BDG equations for a multi-channel system discussed in Appendix A with boundary conditions corresponding to putting the system in a large box, ranging from \( x = -L/2 \) to \( x = L/2 + \ell \), that is, we require that the wavefunctions are equal to 0 both at \( x = -L/2 \) and at \( x = L/2 + \ell \). This constrains the form of the solutions, leading to consistency relations between the momenta, which can be either expressed in terms of the \( M \), or of the \( S \)-matrix elements. Equating corresponding quantities expressed in formally different ways, we eventually derive Eqs. (2.2, 2.6).

Imposing vanishing boundary conditions as described above implies, at the left-hand boundary of the box (\( x = -L/2 \)) \( u_{L,\lambda,E}(x = -L/2) = v_{L,\lambda,E}(x = -L/2) = 0 \). Thus, from Eqs. (A8) we obtain

\[
\begin{align*}
\cos \left( \frac{\Psi}{2} \right) \{ e^{-\frac{i}{2} \beta_{p,\lambda} L} A_{1,\lambda}^- + e^\frac{i}{2} \beta_{p,\lambda} L A_{2,\lambda}^- \} - \sin \left( \frac{\Psi}{2} \right) e^\frac{i}{2} \beta_{h,\lambda} L A_{3,\lambda}^- + e^{-\frac{i}{2} \beta_{h,\lambda} L} A_{4,\lambda}^- \} = 0 \\
- \sin \left( \frac{\Psi}{2} \right) \{ e^{-\frac{i}{2} \beta_{p,\lambda} L} A_{1,\lambda}^- + e^\frac{i}{2} \beta_{p,\lambda} L A_{2,\lambda}^- \} + \cos \left( \frac{\Psi}{2} \right) e^\frac{i}{2} \beta_{h,\lambda} L A_{3,\lambda}^- + e^{-\frac{i}{2} \beta_{h,\lambda} L} A_{4,\lambda}^- \} = 0 \ ,
\end{align*}
\]

with \( \lambda = 1, 2, \ldots, N \). Similarly, at the right-hand boundary of the box, we impose \( u_{R,\lambda,E}(x = L/2 + \ell) = v_{L,\lambda,E}(x = L/2 + \ell) = 0 \). As a result, from Eqs. (A9) we obtain

\[
\begin{align*}
\cos \left( \frac{\Psi}{2} \right) \{ \sum_{\lambda'=1}^{N} \sum_{j'=1}^{4} M_{(1,\lambda'),(j',\lambda')} A_{j',\lambda'}^- + \sum_{\lambda'=1}^{N} \sum_{j'=1}^{4} M_{(2,\lambda'),(j',\lambda')} A_{j',\lambda'}^- \} \\
- \sin \left( \frac{\Psi}{2} \right) \{ \sum_{\lambda'=1}^{N} \sum_{j'=1}^{4} M_{(3,\lambda'),(j',\lambda')} A_{j',\lambda'}^- + \sum_{\lambda'=1}^{N} \sum_{j'=1}^{4} M_{(4,\lambda'),(j',\lambda')} A_{j',\lambda'}^- \} = 0 \\
- \sin \left( \frac{\Psi}{2} \right) \{ \sum_{\lambda'=1}^{N} \sum_{j'=1}^{4} M_{(1,\lambda'),(j',\lambda')} A_{j',\lambda'}^- + \sum_{\lambda'=1}^{N} \sum_{j'=1}^{4} M_{(2,\lambda'),(j',\lambda')} A_{j',\lambda'}^- \} \\
+ \cos \left( \frac{\Psi}{2} \right) \{ \sum_{\lambda'=1}^{N} \sum_{j'=1}^{4} M_{(3,\lambda'),(j',\lambda')} A_{j',\lambda'}^- + \sum_{\lambda'=1}^{N} \sum_{j'=1}^{4} M_{(4,\lambda'),(j',\lambda')} A_{j',\lambda'}^- \} = 0 \ ,
\end{align*}
\]

with \( \lambda = 1, 2, \ldots, N \). Eqs. (B1, B2) can be regarded as a homogenous system in the \( 4N \) unknowns \( A_{j',\lambda'}^- \), which can be rewritten as

\[
\sum_{j'=1}^{4} \sum_{\lambda'=1}^{N} A_{(j,\lambda),(j',\lambda')} A_{j',\lambda'}^- = 0 \ ,
\]

with the matrix elements \( A_{(j,\lambda),(j',\lambda')} \) given by

\[
A_{(j,\lambda),(j',\lambda')} = \sum_{j''=1}^{4} \sum_{\lambda''=1}^{N} [\delta_{\lambda,\lambda''} M_{j,j''} \delta_{(j',\lambda''),(j',\lambda')} ] \quad ,
\]

with \( \mathcal{M} \) being a \( 4 \times 4 \) matrix defined as

\[
\mathcal{M} = \begin{bmatrix}
0 & \cos \left( \frac{\Psi}{2} \right) & 0 & -e^{-\frac{i}{2} \beta_h} \sin \left( \frac{\Psi}{2} \right) \\
0 & -e^{-\frac{i}{2} \beta_h} \sin \left( \frac{\Psi}{2} \right) & 0 & \cos \left( \frac{\Psi}{2} \right) \\
\cos \left( \frac{\Psi}{2} \right) & 0 & -e^{-\frac{i}{2} \beta_h} \sin \left( \frac{\Psi}{2} \right) & 0 \\
-e^{-\frac{i}{2} \beta_h} \sin \left( \frac{\Psi}{2} \right) & 0 & \cos \left( \frac{\Psi}{2} \right) & 0
\end{bmatrix} \quad ,
\]

and
\[ \alpha_{(j',\lambda')(j',\lambda')} = \delta_{j',1}e^{\mp \beta_{,\lambda'}L}M_{(1,\lambda')(j',\lambda')} + e^{-\mp \beta_{,\lambda'}L}M_{(2,\lambda')(j',\lambda')} \]

\[ + \delta_{j',2}\{e^{\mp \beta_{,\lambda'}L}\delta_{j',1} + e^{\mp \beta_{,\lambda'}L}\delta_{j',2}\}\delta_{\lambda',\lambda'} \]

\[ + \delta_{j',3}\{e^{\mp \beta_{,\lambda'}L}M_{(3,\lambda')(j',\lambda')} + e^{\mp \beta_{,\lambda'}L}M_{(4,\lambda')(j',\lambda')} \}
\]

\[ + \delta_{j',4}\{e^{\mp \beta_{,\lambda'}L}\delta_{j',3} + e^{\mp \beta_{,\lambda'}L}\delta_{j',4}\}\delta_{\lambda',\lambda'} . \]

(B6)

The consistency condition for having nonzero solutions for the amplitudes \( A_{j',\lambda} \) then reads \( \text{det} \| A_{(j',\lambda), (j',\lambda')} \| = 0 \), that is, \( \cos^{2N}(\Psi)\text{det} \| \alpha_{(j,\lambda), (j',\lambda')} \| = 0 \Rightarrow \text{det} \| \alpha_{(j,\lambda), (j',\lambda')} \| = 0 \). By pertinently grouping powers of \( e^{\pm \beta_{,\lambda}L} \) and of \( e^{\mp \beta_{,\lambda'}L} \), this latter condition gives rise to the equation

\[ \sum_{\lambda=1}^{N} c_A e^{i[\beta_{,\lambda'} - \beta_{,\lambda}]L} + \ldots + c_B \sum_{\lambda=1}^{N} e^{-i[\beta_{,\lambda'} - \beta_{,\lambda}]L} = 0 \],

(B7)

where we have introduced the ellipses to represent terms \( \propto \sum_{\lambda=1}^{N} e^{i[\alpha_{,\lambda} \beta_{,\lambda} - \beta_{,\lambda} \beta_{,\lambda'}]L} \), with \( \alpha_{,\lambda} \), \( b_{,\lambda} \) = \pm 1 \) and at least one of the \( \alpha_{,\lambda} \) and/or \( b_{,\lambda} \) different from the others. It is, now, clear that \( c_A \sum_{\lambda=1}^{N} e^{i[\beta_{,\lambda'} - \beta_{,\lambda}]L} \) is given by the determinant of the matrix obtained from \( \| \alpha_{(j,\lambda), (j',\lambda')} \| \) by setting to 0 all the contributions not proportional to either \( e^{\mp \beta_{,\lambda}L} \), or to \( e^{\pm \beta_{,\lambda'}L} \), that is, one obtains

\[ \sum_{\lambda=1}^{N} c_A e^{i[\beta_{,\lambda'} - \beta_{,\lambda}]L} = \text{det} \| \alpha_A^{(j,\lambda), (j',\lambda')} \| , \]

(B8)

with

\[ \alpha_A^{(j,\lambda), (j',\lambda')} = e^{\mp \beta_{,\lambda}L}\delta_{j,1}M_{(1,\lambda), (j',\lambda')} + \delta_{j,2}\delta_{j',2}\delta_{\lambda',\lambda'} \]

\[ + e^{\mp \beta_{,\lambda}L}\delta_{j,3}M_{(3,\lambda), (j',\lambda')} + \delta_{j,4}\delta_{j',4}\delta_{\lambda',\lambda'} \].

(B9)

This can be rewritten as the matrix product of a diagonal matrix containing all the \( \beta \)-dependence and the matrix \( M^A \) defined in Eq. (2.7):

\[ \alpha_{(j,\lambda), (j',\lambda')} = 4 \sum_{j''=1}^{N} \sum_{\lambda''=1}^{N} \{e^{\mp \beta_{,\lambda}L}\delta_{j',1}\delta_{j'',1} + \delta_{j',2}\delta_{j'',2}\}\delta_{\lambda',\lambda''} \]

\[ + e^{\mp \beta_{,\lambda}L}\delta_{j',3}\delta_{j'',3}\delta_{\lambda',\lambda''} + \delta_{j',4}\delta_{j'',4}\delta_{\lambda',\lambda''} \] .

(B10)

This shows that

\[ c_A = \text{det}[M^A] . \]

(B11)

Going through similar arguments, one readily proves that

\[ \sum_{\lambda=1}^{N} c_B e^{i[\beta_{,\lambda'} - \beta_{,\lambda}]L} = \text{det} \| \alpha_B^{(j,\lambda), (j',\lambda')} \| , \]

(B12)

with

\[ \alpha_B^{(j,\lambda), (j',\lambda')} = e^{-\mp \beta_{,\lambda}L}\delta_{j,1}M_{(2,\lambda), (j',\lambda')} + \delta_{j,2}\delta_{j',2}\delta_{\lambda',\lambda'} \]

\[ + e^{\mp \beta_{,\lambda}L}\delta_{j,3}M_{(4,\lambda), (j',\lambda')} + \delta_{j,4}\delta_{j',4}\delta_{\lambda',\lambda''} \].

(B13)

A factorization similar to the one in Eq. (B10) takes place in this case, as well, in the form

\[ \alpha_B^{(j,\lambda), (j',\lambda')} = 4 \sum_{j''=1}^{N} \sum_{\lambda''=1}^{N} \{e^{\mp \beta_{,\lambda}L}\delta_{j',1}\delta_{j'',1} + \delta_{j',2}\delta_{j'',2}\}\delta_{\lambda',\lambda''} \]

\[ + e^{\mp \beta_{,\lambda}L}\delta_{j',3}\delta_{j'',3}\delta_{\lambda',\lambda''} + \delta_{j',4}\delta_{j'',4}\delta_{\lambda',\lambda''} \] .

(B14)
with \(M^B\) given in Eq. \((2.8)\), which implies

\[
c_B = \det[M^B] .
\]  

(B15)

As a result, we then see that Eq. \((B7)\) can be recast in the form

\[
\begin{bmatrix}
\det[M^A] \\
\det[M^B]
\end{bmatrix}
\prod_{\lambda=1}^{N} e^{2i[\beta_{p,\lambda} - \beta_{h,\lambda}]L} + \ldots + 1 = 0 .
\]

(B16)

To relate \(\frac{\det[M^A]}{\det[M^B]}\) to the determinant of the \(S\)-matrix, we use Eq. \((A12)\) to trade Eqs. \((B1)-(B2)\) for an algebraic system of \(4N\)-equations in the unknowns \(A_{j,\lambda}^{in}\). The resulting system is

\[
\begin{align*}
\cos \left( \frac{\Psi}{2} \right) & \left\{ e^{-\frac{\pi}{2} \beta_{p,\lambda} L} A_{1,\lambda}^{1} + e^{\frac\pi2 \beta_{p,\lambda} L} \sum_{j'=1,\lambda'=1}^{N} \sqrt{v_{j',\lambda}} S_{(3,\lambda),(j',\lambda')} A_{j',\lambda'}^{in} \right\} \\
& - e^{\frac{\pi}{2} \gamma L} \sin \left( \frac{\Psi}{2} \right) \left\{ e^{\frac\pi2 \beta_{p,\lambda} L} A_{2,\lambda}^{2} + e^{-\frac{\pi}{2} \beta_{p,\lambda} L} \sum_{j'=1,\lambda'=1}^{N} \sqrt{v_{j',\lambda}} S_{(4,\lambda),(j',\lambda')} A_{j',\lambda'}^{in} \right\} = 0
\end{align*}
\]

(B17)

As \(\lambda = 1, \ldots, N\), Eqs. \((B17)\) define a \(4N\)-equation system in the unknowns \(A_{j,\lambda}^{in}\), which can be rewritten as

\[
\sum_{j'=1,\lambda'=1}^{N} B_{(j,\lambda),(j',\lambda')} A_{j',\lambda'}^{in} = 0 \quad ,
\]

(B18)

with the matrix elements \(B_{(j,\lambda),(j',\lambda')}\) given by

\[
B_{(j,\lambda),(j',\lambda')} = \sum_{j''=1,\lambda''=1}^{N} [\delta_{\lambda,\lambda''} M_{j,j''} \beta_{(j'',\lambda''),(j',\lambda')} \]

(B19)

and
\[
\beta_{(j,\lambda),(j',\lambda')} = \delta_{j,1} \left\{ e^{i\beta_p,\lambda L} \frac{v_j,\lambda'}{v_1,\lambda} S_{(1,\lambda),(j',\lambda')} + e^{-i\beta_p,\lambda L} \delta_{j',3} \delta_{\lambda,\lambda'} \right\} \\
+ \delta_{j,2} \left\{ e^{-i\beta_p,\lambda L} \delta_{j',1} \delta_{\lambda,\lambda'} + e^{i\beta_p,\lambda L} \frac{v_j,\lambda'}{v_3,\lambda} S_{(3,\lambda),(j',\lambda')} \right\} \\
+ \delta_{j,3} \left\{ e^{-i\beta_h,\lambda L} \frac{v_j,\lambda'}{v_2,\lambda} S_{(2,\lambda),(j',\lambda')} + e^{i\beta_h,\lambda L} \delta_{j',4} \delta_{\lambda,\lambda'} \right\} \\
+ \delta_{j,4} \left\{ e^{i\beta_h,\lambda L} \delta_{j',2} \delta_{\lambda,\lambda'} + e^{-i\beta_h,\lambda L} \frac{v_j,\lambda'}{v_4,\lambda} S_{(4,\lambda),(j',\lambda')} \right\} . 
\] (B20)

The consistency condition for having nonzero solutions therefore reads \( \det \parallel B_{(j,\lambda),(j',\lambda')} \parallel = \cos^{2N}(\Psi) \det \parallel \beta_{(j,\lambda),(j',\lambda')} \parallel = 0 \), which implies \( \beta_{(j,\lambda),(j',\lambda')} \parallel = 0 \). As we have done before, by pertinently grouping powers of \( e^{\pm i\beta_p,\lambda L} \) and of \( e^{\pm i\beta_h,\lambda L} \), we trade the condition on the determinant for the equivalent equation

\[
\delta_A \prod_{\lambda=1}^{N} e^{i[\beta_p,\lambda - \beta_h,\lambda]L} + \ldots + \delta_B \prod_{\lambda=1}^{N} e^{-i[\beta_p,\lambda - \beta_h,\lambda]L} = 0 . 
\] (B21)

As we have done before, we therefore compute \( \delta_A \prod_{\lambda=1}^{N} e^{i[\beta_p,\lambda - \beta_h,\lambda]L} \) as \( \delta_A \prod_{\lambda=1}^{N} e^{i[\beta_p,\lambda - \beta_h,\lambda]L} = \det \parallel \beta_A^{(j,\lambda),(j',\lambda')} \parallel \), with

\[
\beta_A^{(j,\lambda),(j',\lambda')} = e^{i\beta_p,\lambda L} \left\{ \delta_{j,1} \frac{v_j,\lambda'}{v_1,\lambda} S_{(1,\lambda),(j',\lambda')} + \delta_{j,2} \frac{v_j,\lambda'}{v_3,\lambda} S_{(3,\lambda),(j',\lambda')} \right\} \\
+ e^{-i\beta_h,\lambda L} \left\{ \delta_{j,3} \frac{v_j,\lambda'}{v_2,\lambda} S_{(2,\lambda),(j',\lambda')} + \delta_{j,4} \frac{v_j,\lambda'}{v_4,\lambda} S_{(4,\lambda),(j',\lambda')} \right\} . 
\] (B22)

At variance, we obtain \( \delta_B \prod_{\lambda=1}^{N} e^{-i[\beta_p,\lambda - \beta_h,\lambda]L} = \det \parallel \beta_B^{(j,\lambda),(j',\lambda')} \parallel \), with

\[
\beta_B^{(j,\lambda),(j',\lambda')} = e^{-i\beta_p,\lambda L} \left\{ \delta_{j,1} \delta_{j',3} \delta_{\lambda,\lambda'} + \delta_{j,2} \delta_{j',1} \delta_{\lambda,\lambda'} \right\} \\
+ e^{i\beta_h,\lambda L} \left\{ \delta_{j,3} \delta_{j',4} \delta_{\lambda,\lambda'} + \delta_{j,4} \delta_{j',2} \delta_{\lambda,\lambda'} \right\} . 
\] (B23)

Thus, we obtain

\[
\delta_A = (-1)^N \det[S] \\\n\delta_B = (-1)^N . 
\] (B24)

As a consequence of Eqs. (B24), we see that Eq. (B21) can be recast in the form

\[
\det[S] \prod_{\lambda=1}^{N} e^{2i[\beta_p,\lambda - \beta_h,\lambda]L} + \ldots + 1 = 0 . 
\] (B25)

Though Eqs. (B21) have been obtained following two alternative routes, they must clearly coincide with each other, once the coefficients are consistently normalized, as we did. As a result, the coefficients of \( \prod_{\lambda=1}^{N} e^{2i[\beta_p,\lambda - \beta_h,\lambda]L} \) must be equal to each other, which implies Eqs. (22) of the main text.

Appendix C: Derivation of Eq. (3.8)

Eq. (3.8) is one of the key results of this paper, as it provides us with a closed-form formula to exactly expressing \( I(\chi; T = 0) \) in the case of equivalent channels. To derive Eq. (3.8), we start from the result in Eq. (3.6) and from the observation that, based on general properties of the transmission matrix elements, as well as on the explicit calculation
of $\mathcal{P}(u; \chi)$, one obtains that $P_0(\chi) = P_{2K}(\chi) = 1$. As a first intermediate step, let us define $z = -\frac{2\pi\ell}{u}$, so that Eq. (3.6) becomes

$$I[\chi; T = 0] = \frac{2e\nu}{4\pi\ell} \int_{-\infty}^{\infty} dz \left[ \frac{\sum_{j=-K+1}^{K-1} \partial_\chi P_j(\chi)(e^z)^j}{\sum_{j=-K}^{K} P_j(\chi)(e^z)^j} \right].$$  \hspace{1cm} (C1)

Next, let us multiply the numerator and the denominator of Eq. (C1) by $\ln\chi$. We then obtain

$$I[\chi; T = 0] = \frac{2e\nu}{4\pi\ell} \int_{-\infty}^{\infty} dz \left[ \frac{\sum_{j=-K+1}^{K-1} \partial_\chi P_j(\chi)(e^z)^j + K}{\sum_{j=-K}^{K} P_j(\chi)(e^z)^j + K} \right].$$  \hspace{1cm} (C2)

Finally, let us define $u = e^z$ and use $u$ as integration variable. This implies

$$I[\chi; T = 0] = \frac{2e\nu}{4\pi\ell} \int_{0}^{\infty} du \left[ \frac{\sum_{j=-K+1}^{K-1} \partial_\chi P_j(\chi)u^j}{\sum_{j=-K}^{K} P_j(\chi)u^j + K} \right].$$  \hspace{1cm} (C3)

On introducing the polynomial $\mathcal{P}(u; \chi) = \sum_{j=0}^{2K} P_j(\chi)u^j = 1 + u^{2K} + \sum_{j=1}^{2K-1} P_j(\chi)u^j$, with, in general, $P_j(\chi) \neq 1$, for $j = 1, \ldots, 2K - 1$, Eq. (C3) can be rewritten as

$$I[\chi; T = 0] = \frac{2e\nu}{4\pi\ell} \int_{0}^{\infty} du \left[ \frac{\partial_\chi \mathcal{P}(u)}{\mathcal{P}(u)} \right] = \frac{2e\nu}{4\pi\ell} \int_{0}^{\infty} du \partial_\chi \ln\left[ \prod_{j=1}^{2K}(u_j(\chi)) \right],$$ \hspace{1cm} (C4)

with $u_j(\chi)$ being the roots of $\mathcal{P}(u; \chi) = 0$. Eq. (C4) can then be rewritten as

$$I[\chi; T = 0] = \frac{2e\nu}{4\pi\ell} \sum_{j=1}^{2K} \int_{0}^{\infty} du \frac{\partial_\chi u_j(\chi)}{u (u_j(\chi) - u)}.$$ \hspace{1cm} (C5)

The argument of the integral in Eq. (C5) looks like it diverges as $u^{-1}$ as $u \to 0$. However, the integral is convergent, due to the condition $\prod_{j=1}^{2K} u_j(\chi) = 1$, which implies $\sum_{j=1}^{2K} \ln u_j(\chi) = 0$. To evidence this, we introduce a scale $\epsilon$ to control the small-$u$ divergence and (though it is not strictly necessary), a cutoff $\Lambda$ to keep under control the behavior of the integral in the large-$u$ region. This means that we rewrite Eq. (C5) as

$$I[\chi; T = 0] = \frac{2e\nu}{4\pi\ell} \lim_{\Lambda \to \infty} \lim_{\epsilon \to 0} \sum_{j=1}^{2K} \int_{\epsilon}^{\Lambda} du \frac{\partial_\chi u_j(\chi)}{u (u_j(\chi) - u)}.$$ \hspace{1cm} (C6)

Computing the integrals at finite cutoffs and eventually getting rid of the cutoffs by sending $\epsilon \to 0$ and $\Lambda \to \infty$, by using the relations between the roots listed above, one obtains

$$I[\chi; T = 0] = \frac{2e\nu}{4\pi\ell} \sum_{j=1}^{2K} \{\ln u_j(\chi)\partial_\chi \ln u_j(\chi)\} = \frac{e\nu}{4\pi\ell} \sum_{j=1}^{2K} \partial_\chi \ln^2[u_j(\chi)].$$ \hspace{1cm} (C7)

In the specific case $K = 1$, which was considered in Ref. [3], we obtain (using $u_2(\chi) = 1/u_1(\chi) \Rightarrow \partial_\chi \{\ln u_1(\chi) + \ln[u_2(\chi)]\} = 0$)

$$\partial_\chi \ln^2[u_1(\chi)] + \partial_\chi \ln^2[u_2(\chi)] = \frac{1}{2} \partial_\chi \{\ln[u_1(\chi)] + \ln[u_2(\chi)]\}^2 + \frac{1}{2} \partial_\chi \{\ln[u_1(\chi)] - \ln[u_2(\chi)]\}^2 = \frac{1}{2} \ln^2 \left( \frac{u_1(\chi)}{u_2(\chi)} \right).$$ \hspace{1cm} (C8)
From Eq. (C8) we eventually get, for $K = 1$,

$$I[\chi; T = 0] = \frac{e^u}{4\pi \ell} \partial_\chi \ln^2 \left( \frac{u_1(\chi)}{u_2(\chi)} \right) .$$  \hspace{1cm} (C9)$$

From Eq. (C9), setting

$$u_1(\chi) = e^{i\delta(\chi)},$$
$$u_2(\chi) = e^{-i\delta(\chi)},$$  \hspace{1cm} (C10)

which implies

$$\frac{u_1(\chi)}{u_2(\chi)} = e^{2i\delta(\chi)},$$  \hspace{1cm} (C11)

one obtains Eq. (3) of Ref. \cite{3}. 

**Appendix D: Construction of the polynomial $P(u; \chi)$**

In this appendix, we work out the algorithm to explicitly construct the polynomial $P(u; \chi)$ we introduce in Section \[III\] to fully characterize the formula for the dc Josephson current in the symmetric case. In particular, we first construct $P(u; \chi)$ in full generality, that is, for generic $N$ and $K$, by expressing it as a function of the reflection matrices at the interfaces evaluated at the Fermi level. As a specific example, we then provide the explicit formula for $N = 1$ and $K$ generic, by showing that, for $N = 1$, any system with $K(\geq 2)$ equivalent channels within $C$ can be reduced to the one with $K = 2$.

The starting point is that, as $|E| < \Delta$, there are no transmitted waves outside of $C$. This means that, within the left-hand lead, there will be no $\left[ \begin{array}{c} u_{L,\lambda,E}(x) \\ v_{L,\lambda,E}(x) \end{array} \right]_{1,3}$-solutions, while $\left[ \begin{array}{c} u_{L,\lambda,E}(x) \\ v_{L,\lambda,E}(x) \end{array} \right]_{2,4}$, will behave as evanescent waves, as $x \to -\infty$. As a result, we obtain $2K$ linear relations between the coefficients of the solution to Eq. \[A17\]. To formally express them, we introduce the $2K \times 2K$ reflection matrix at the left-hand interface, $\| [R_L(E; \chi)](a,\rho)(a',\rho') \|$, with $a, a' = 1, 2$ and $\rho, \rho' = 1, \ldots, K$, such that

$$C_{2a-1,\rho} = \sum_{a'=1,2} \sum_{\rho'=1}^K \sqrt{\frac{\partial R_L(E; \chi)_{(a,\rho)}(a',\rho')}{\partial a_{a',\rho'}}} R_L(E; \chi)_{(a',\rho')} C_{2a'-1,\rho'},$$  \hspace{1cm} (D1)

with $\psi_{1,\rho}^C = \left[ \frac{dE}{da_{a',\rho'}} \right]$ and $\psi_{2,\rho}^E = \left[ \frac{dE}{d\chi} \right]$. Similarly, within the right-hand lead, there will be no $\left[ \begin{array}{c} u_{L,\lambda,E}(x) \\ v_{L,\lambda,E}(x) \end{array} \right]_{2,4}$-solutions, while $\left[ \begin{array}{c} u_{L,\lambda,E}(x) \\ v_{L,\lambda,E}(x) \end{array} \right]_{1,3}$, will behave as evanescent waves, as $x \to \infty$. This allows for deriving $2K$ additional linear relations between the coefficients of the solution to Eq. \[A17\], in terms of the $2K \times 2K$ reflection matrix at the right-hand interface, $\| [R_R(E; \chi)](a,\rho)(a',\rho') \|$, such that

$$C_{2a,\rho} = e^{i\alpha\rho^{(2)}} \sum_{a'=1,2} \sum_{\rho'=1}^K \sqrt{\frac{\partial R_R(E; \chi)_{(a',\rho')} e^{i\alpha\rho^{(2)}}}{\partial a_{a',\rho'}}} R_R(E; \chi)_{(a',\rho')} C_{2a'-1,\rho'},$$  \hspace{1cm} (D2)

with $\alpha^{(2)}_{\rho} = \alpha_{\rho,\rho}$ and $\alpha^{(2)}_{\rho'} = -\alpha_{\rho',\rho'}$. Putting together Eqs. \[D1\] and \[D2\], one obtains a homogeneous equation for $C_{2,\rho}, C_{4,\rho}$, given by

$$\sum_{a'=1,2} \sum_{\rho'=1}^K \left[ \delta_{a,a'} \delta_{\rho,\rho'} - e^{i\alpha^{(2)}_{\rho'}} \sqrt{\frac{\partial R_R(E; \chi)_{(a',\rho')} e^{i\alpha^{(2)}_{\rho'}}}{\partial a_{a',\rho'}}} \sum_{a'=1,2} \sum_{\rho'=1}^K [R_R(E; \chi)](a,\rho)(a',\rho') e^{i\alpha^{(2)}_{\rho'}} [R_L(E; \chi)](a',\rho')(a',\rho') \right] C_{2a',\rho'} = 0 . \hspace{1cm} (D3)$$
In order to obtain nontrivial solutions to the system of equations reported in Eq. (D3), the consistency condition

$$\det \left[ \delta_{a,a'} \delta_{\rho,\rho'} - e^{i\alpha_{\rho} \tau^3} \sum_{a'=1,2} e^{i\alpha_{\rho'} \tau^3} \sum_{a'=1,2} K \left[ R_{H}(E; \chi) \right]_{a,a'(a',\rho')} e^{i\alpha_{\rho} \tau^3} \left[ R_{L}(E; \chi) \right]_{a,(a',\rho')} \right] = 0 \quad ,$$  \hspace{1cm} \text{(D4)}

must be imposed. Eq. (D4) is the secular equation for the energies of the Andreev states localized within C. Restricting ourselves to the symmetric case, we therefore assume that $\alpha_a^{\rho} = \ldots = \alpha_a^{K} = \alpha_a$. To recover the long junction limit, we then substitute $e^{i\alpha_{\rho} \tau^3} \approx \delta_{a,a'} e^{i\alpha_{\rho} \tau^3} |a,a'e^{-i\delta_{\rho,\rho'}}$, where $\delta_{\rho,\rho'} = \delta_{\rho,\rho}$ form the third Pauli matrix. In addition (which amounts to neglecting to subleading powers of $\ell^{-1}$), we set $E = 0$ in the matrices $R_{L}(E; \chi)$ and $R_{R}(E; \chi)$ and in the quasiparticle velocities $v_{a,\rho}^{\sigma}$. In particular, this latter approximation, together with the fact that we are assuming that the Fermi velocities are independent of $\rho$, implies $\frac{v_{a,\rho}^{\sigma}}{v_{a',\rho'}^{\sigma}} = 1$, independently of $a, a'$. Once the approximations described above have been performed, Eq. (D4) must coincide with $P(u; \chi) = 0$, provided the normalization of the coefficients in the two of them has been properly chosen. As a result, multiplying Eq. (D4) by $u^K$, we eventually get

$$P(u; \chi) = \| u \delta_{a,a'} \delta_{\rho,\rho'} - e^{i\alpha_{\rho} \tau^3} \sum_{a'=1,2} e^{i\alpha_{\rho'} \tau^3} \sum_{a'=1,2} K \left[ R_{H}(0; \chi) \right]_{a,a' \rho,a' \rho} e^{i\alpha_{\rho} \tau^3} \left[ R_{L}(0; \chi) \right]_{a,a' \rho,a' \rho} \| \quad . \hspace{1cm} \text{(D5)}$$

An important remark is that Eq. (D5) implies $P_{\rho}(\chi) = P_{2\rho}(\chi) = 1$ since, as a general property of the solutions of the Bogoliubov - de Gennes equations, one has that $\det[R_{H}(0; \chi)] = \det[R_{R}(0; \chi)] = 1$.

As a specific example of application of Eq. (D5), we now consider the case $N = 1$. As we are going to argue next, $N = 1$ is special, in that any system with $K \geq 2$ can be traced out to a unitary equivalent one with $K = 2$. To work out the formula for the current in this case, we consider the tunneling Hamiltonian in Eq. (A14) in the specific case $N = 1$. Defining $t_{L(R)} = \sqrt{\sum_{a=1}^{K} \left[ t_{L(R)} \right]_{a,a}}$ and $\hat{t}_{L(R)} = \left( t_{L(R)} \right)_{1,1}, \ldots, \left( t_{L(R)} \right)_{K,1}$, we now rotate, at fixed spin polarization $\sigma$, the fields $\Psi_{C;\rho,\sigma}(x)$ by means of an unitary transformation $U \in U(K)$:

$$\begin{bmatrix} \tilde{\Psi}_{C,1,\sigma}(x) \\ \tilde{\Psi}_{C,2,\sigma}(x) \\ \vdots \\ \tilde{\Psi}_{C,K,\sigma}(x) \end{bmatrix} = U^\dagger \begin{bmatrix} \Psi_{C,1,\sigma}(x) \\ \Psi_{C,2,\sigma}(x) \\ \vdots \\ \Psi_{C,K,\sigma}(x) \end{bmatrix} , \hspace{1cm} \text{(D6)}$$

with $U$ defined so that

$$U \hat{t}_{R} = t_{R} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} , \quad U \hat{t}_{L} = t_{L} e^{i\delta} \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \\ 0 \end{bmatrix} . \hspace{1cm} \text{(D7)}$$

The phases, $\delta$ and $\theta$ are determined by the scalar product of the two vectors:

$$\hat{t}_{R} \cdot \hat{t}_{L} = t_{L} t_{R} e^{i\delta} \cos \theta . \hspace{1cm} \text{(D8)}$$

The first phase $\delta$ can be adsorbed into $\chi$, the phase difference of the order parameters in the two leads. Thus, for a generic number of equivalent channels $K$, we can simply work with a $K = 2$-model with real tunneling matrix elements given by the right hand sides of Eq. (D7), with $\delta = 0$. Therefore, with no loss of generality, from now on we will assume $K = 2$. For $K = 2$, the general form of $R_{I}(E; \chi)$ and $R_{R}(E; \chi)$ may be inferred by noting that the allowed physical processes at each interface are the ones corresponding to a particle (hole) incoming with spin $\sigma$ from channel 1 (2) and emerging as a particle (hole) with spin $\sigma$ in channel 1 (2) after a normal reflection process, or as a hole (particle) with spin $-\sigma$ in channel 1 (2) after an Andreev reflection process. In addition, there will be inter-channel reflection processes, in which a particle (hole) incoming with spin $\sigma$ from channel 1 (2) can emerge as a particle (hole) with spin $\sigma$ in channel 2 (1) after a normal reflection process, or as a hole (particle) with spin $-\sigma$ in
channel 2 (1) after an Andreev reflection process. For notational simplicity, when dealing with the $K = 2$-problem, in the remainder of this appendix and in next one, we will order the $[R_{L(R)}(E; \chi)]_{(\rho, \rho')}(\sigma, \sigma')$-matrix elements in square matrices $R_{L(R)}(E; \chi)$, so that, denoting with $N_{L(R)}^{p(h)}(\rho, \rho')(E; \chi)$ and with $A_{L(R)}^{p(h)}(\rho, \rho')(E; \chi)$ the single-particle(hole) normal and Andreev scattering amplitude at the left-(right-)hand interface from channel-$\rho'$ to channel-$\rho$ respectively, the matrices $R_{L(R)}(E; \chi)$ are given by (dropping for simplicity the arguments $E$ and $\chi$ from the matrix elements)

$$R_{L(R)}(E; \chi) = \begin{bmatrix} N_{L(R)}^{p}(1,1) & A_{L(R)}^{p}(1,1) & N_{L(R)}^{p}(1,2) & A_{L(R)}^{p}(1,2) \\ A_{L(R)}^{h}(1,1) & N_{L(R)}^{h}(1,1) & A_{L(R)}^{h}(1,2) & N_{L(R)}^{h}(1,2) \\ N_{L(R)}^{p}(2,1) & A_{L(R)}^{p}(2,1) & N_{L(R)}^{p}(2,2) & A_{L(R)}^{p}(2,2) \\ A_{L(R)}^{h}(2,1) & N_{L(R)}^{h}(2,1) & A_{L(R)}^{h}(2,2) & N_{L(R)}^{h}(2,2) \end{bmatrix}.$$  \hspace{1cm} (D9)

At the Fermi level, Eq. (D9) yields the matrices $\bar{R}_{L(R)}(\chi)$, defined as

$$\bar{R}_{L(R)}(\chi) = R_{L(R)}(E = 0; \chi) = \begin{bmatrix} \bar{N}_{L(R)}^{p}(1,1) & \bar{A}_{L(R)}^{p}(1,1) & \bar{N}_{L(R)}^{p}(1,2) & \bar{A}_{L(R)}^{p}(1,2) \\ \bar{A}_{L(R)}^{h}(1,1) & \bar{N}_{L(R)}^{h}(1,1) & \bar{A}_{L(R)}^{h}(1,2) & \bar{N}_{L(R)}^{h}(1,2) \\ \bar{N}_{L(R)}^{p}(2,1) & \bar{A}_{L(R)}^{p}(2,1) & \bar{N}_{L(R)}^{p}(2,2) & \bar{A}_{L(R)}^{p}(2,2) \\ \bar{A}_{L(R)}^{h}(2,1) & \bar{N}_{L(R)}^{h}(2,1) & \bar{A}_{L(R)}^{h}(2,2) & \bar{N}_{L(R)}^{h}(2,2) \end{bmatrix},$$  \hspace{1cm} (D10)

with the bar generically used to denote quantities evaluated at the Fermi level. By virtue of the charge-conjugation symmetry of the Bogoliubov - de Gennes equations, one finds that the following relations hold for the reflection amplitudes at the Fermi level:

$$\bar{N}_{L(R)}^{p}(\rho, \rho') = [\bar{N}_{L(R)}^{h}(\rho, \rho')]^* , \quad \bar{A}_{L(R)}^{p}(\rho, \rho') = [\bar{A}_{L(R)}^{h}(\rho, \rho')]^*.$$  \hspace{1cm} (D11)

As a result, after dropping the indices $p$ and $h$ and setting $\bar{N}_{L(R)}(\rho, \rho') \equiv \bar{N}_{L(R)}^{p}(\rho, \rho')$ and $\bar{A}_{L(R)}(\rho, \rho') \equiv \bar{A}_{L(R)}^{p}(\rho, \rho')$, Eq. (D10) can be rewritten as

$$\bar{R}_{L(R)}(\chi) = \begin{bmatrix} \bar{N}_{L(R)}(1,1) & \bar{A}_{L(R)}(1,1) & \bar{N}_{L(R)}(1,2) & \bar{A}_{L(R)}(1,2) \\ [\bar{A}_{L(R)}(1,1)]^* & [\bar{N}_{L(R)}(1,1)]^* & [\bar{A}_{L(R)}(1,2)]^* & [\bar{N}_{L(R)}(1,2)]^* \\ \bar{N}_{L(R)}(2,1) & \bar{A}_{L(R)}(2,1) & \bar{N}_{L(R)}(2,2) & \bar{A}_{L(R)}(2,2) \\ [\bar{A}_{L(R)}(2,1)]^* & [\bar{N}_{L(R)}(2,1)]^* & [\bar{A}_{L(R)}(2,2)]^* & [\bar{N}_{L(R)}(2,2)]^* \end{bmatrix}.$$  \hspace{1cm} (D12)

Let us, now, compute $P(\psi; \chi)$. Consistently with Eq. (D8), we assume

$$[t_L]_1 = t_L \cos(\theta) , \quad [t_L]_2 = t_L \sin(\theta) \quad , \quad [t_R]_1 = t_R , \quad [t_R]_2 = 0.$$  \hspace{1cm} (D13)

For the sake of computing $P(\psi; \chi)$, it is useful to use the equivalence between the electronic channels within C to rotate $\Psi_{C,1,\sigma}(x), \Psi_{C,2,\sigma}(x)$ to $\Psi_{C,1,\sigma}(x), \Psi_{C,2,\sigma}(x)$, defined as

$$\begin{bmatrix} \Psi_{C,1,\sigma}(x) \\ \Psi_{C,2,\sigma}(x) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \Psi_{C,1,\sigma}(x) \\ \Psi_{C,2,\sigma}(x) \end{bmatrix}.$$  \hspace{1cm} (D14)

Clearly, at the left(right)-hand SN interface, $\tilde{\Psi}_{C,2,\sigma}(x) (\tilde{\Psi}_{C,2,\sigma}(x))$ is fully decoupled from the superconducting lead and can only exhibit normal reflection at the Fermi level. As a result, in the basis of the operators $\Psi_{C,1,\sigma}(x), \Psi_{C,2,\sigma}(x)$, one finds

$$\bar{R}_L = \begin{bmatrix} \bar{R}_L^{(1)} & 0 \\ 0 & -I \end{bmatrix},$$  \hspace{1cm} (D15)

with $\bar{R}_L^{(1)}$ being the $(2 \times 2)$ backscattering matrix for channel 1 at the left-hand interface, evaluated at the Fermi level. Similarly, in the basis of the operators $\Psi_{C,1,\sigma}(x), \Psi_{C,2,\sigma}(x)$, one finds
with \( \bar{R}_R^{(1)} \) being the \((2 \times 2)\) backscattering matrix for channel 1 at the right-hand side, evaluated at the Fermi level. Taking into account the need for rotating back and forth from the original basis \((\Psi_{C,1,\sigma}(x), \Psi_{C,2,\sigma}(x))\) to the basis \((\bar{\Psi}_{C,1,\sigma}(x), \bar{\Psi}_{C,2,\sigma}(x))\), in which the matrices \(\bar{R}_R\) and \(\bar{R}_L\) are respectively block-diagonal, one finds that Eq. (D15) yields

\[
\mathcal{P}(u; \chi) = \det \left\{ I_4 u - \begin{bmatrix} \bar{R}_L^{(1)} & 0 \\ 0 & -I \end{bmatrix} \cdot \begin{bmatrix} e^{i\sigma_x \alpha_F \ell} & 0 \\ 0 & e^{i\sigma_y \alpha_F \ell} \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta) I & -\sin(\theta) I \\ \sin(\theta) I & \cos(\theta) I \end{bmatrix} \cdot \begin{bmatrix} \bar{R}_R^{(1)} & 0 \\ 0 & -I \end{bmatrix} \cdot \begin{bmatrix} e^{i\sigma_x \alpha_F \ell} & 0 \\ 0 & e^{i\sigma_y \alpha_F \ell} \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta) I & \sin(\theta) I \\ -\sin(\theta) I & \cos(\theta) I \end{bmatrix} \right\}
\]

with the suffix \( \theta \) added to \(\mathcal{P}(u; \chi)\) to explicitly evidence its dependence on \( \theta \). As a consistency check of Eq. (D17), we notice that, as \( \theta \to 0 \), we obtain

\[
\mathcal{P}_{\theta=0}(u; \chi) = (u - e^{2i\alpha_F \ell})(u - e^{-2i\alpha_F \ell})P_2(u; \chi)
\]

with

\[
P_2(u; \chi) = u^2 - 2u \text{ Re} \left\{ e^{2i\alpha_F \ell} \tilde{N}_{L,(1,1)} \tilde{N}_{R,(1,1)} + \tilde{A}_h^{L,(1,1)} \tilde{A}_h^{R,(1,1)} \right\} + 1
\]

Clearly, the only roots of \( \mathcal{P}_{\theta=0}(u; \chi) \) that depend on \( \chi \), \( u_\pm(\chi) \), are the solutions of \( P_2(u; \chi) = 0 \). Setting \( u_\pm(\chi) = e^{\pm i\vartheta(\chi)} \) and using Eqs. (33), one then finds the main result of the derivation of [9], that is

\[
I[\chi] = -\frac{e\mu}{\pi \epsilon} \partial_\chi \vartheta^2(\chi)
\]

This is definitely consistent with Eq. (D18) being the extension of the result of Eqs. (3,4) of Ref. [3] to the case of a generic angle \( \theta \) between the couplings at the two SN interfaces. In the case we discuss at the end of Section III that is, two interfaces exhibiting perfect Andreev reflection, but with non-symmetric couplings between the interfaces, that is, with \( \theta \neq 0 \), one gets

\[
\bar{R}_L^{(1)} = \begin{bmatrix} 0 & e^{-\frac{i}{2} \chi} \\ e^{\frac{i}{2} \chi} & 0 \end{bmatrix}, \quad \bar{R}_R^{(1)} = \begin{bmatrix} 0 & e^{\frac{i}{2} \chi} \\ e^{-\frac{i}{2} \chi} & 0 \end{bmatrix}
\]

Inserring the matrices \( \bar{R}_L^{(1)}, \bar{R}_R^{(1)} \) into Eq. (D17), one obtains the polynomial \( \mathcal{P}_\theta(u; \chi) \) in Eq. (3.10) of the main text.

**Appendix E: Construction of the function \( \Phi[\omega; \chi] \)**

In this appendix we develop a technique to derive the function \( \Phi[\omega; \chi] \) defined in Section III B similar to the one we use in Appendix D to construct the polynomial \( \mathcal{P}(u) \). Moreover, we show how the main formula of Ref. [18] for the zero-temperature dc Josephson current across a SINIS junction with \( K \) channels within C can be recovered as a particular limit of our results. The starting point is Eq. (D4) of Subsection D which we now develop without eventually
imposing the symmetry constraint. On expanding the momenta analytically continued to imaginary energies we have now to take into account the explicit dependence of the Fermi velocities on \(\rho\), which yields

\[
\begin{align*}
\alpha^2_{\rho} & \approx \alpha_F - i \frac{\omega}{v(\rho)} \\
\alpha^2_{\rho} & \approx -\alpha_F - i \frac{\omega}{v(\rho)}
\end{align*}
\]  

(E1)

with \(v(\rho)\) being the Fermi velocity in channel-\(\rho\), as defined after Eq. (3.15). Therefore, in the large-\(\ell\) limit, Eqs. (E1) motivate substituting in Eq. (D1) \(e^{i\alpha^2_{\rho} \ell}\) with \([e^{i\alpha^2_{\rho} \ell}]_{a,a} e^{-i\omega \ell}\). Moreover, just as we have done in the derivation in the symmetric case outlined in Appendix D, we set \(E = 0\) in the matrices \(R_L(E; \chi)\) and \(R_R(E; \chi)\) and in the quasiparticle velocities \(v_{\chi}^{\alpha,\rho}\), which implies \(\sum_{\alpha''} \frac{v_{\chi}^{\alpha',\rho'}}{v_{\chi}^{\alpha,\rho}} \approx \sum_{\alpha''} \frac{v_{\chi}^{\alpha',\rho'}}{v(\rho)}\). As a result, one sees that, in the large-\(\ell\) limit, Eq. (D1) can be approximated as

\[
\det || \delta_{a,a'} \delta_{\rho,\rho'} - \frac{v(\rho')}{v(\rho)} (e^{i\sigma^z \alpha_F \ell})_{a,a} e^{-i\omega \ell} \sum_{\rho''=1,2} \sum_{\rho'''=1} [R_L(0; \chi)]_{(a,\rho),(a'',\rho''')} (e^{i\sigma^z \alpha_F \ell})_{a'',a'} e^{-i\omega \ell} [R_L(0; \chi)]_{(a',\rho'''),(a',\rho')} || = 0 
\]

(E2)

From the definition of the function \(\Phi[\omega; \chi]\) we give in Eq. (3.17), we see that, once regarded as an equation in \(\omega\) at fixed \(\chi\), Eq. (E2) must have the same solutions as the equation \(\Phi[\omega; \chi] = 0\). Therefore, apart from an over-all multiplicative non-zero coefficient, we obtain that \(\Phi[\omega; \chi]\) must coincide with the left-hand side of Eq. (D2). By direct investigation, one finds that the appropriate multiplicative factor is given by \(\prod_{\rho=1}^{K} e^{-i\omega \ell}\). Thus, one eventually obtains

\[
\Phi[\omega; \chi] = \prod_{\rho=1}^{K} \sum_{\{a, b\}_{\rho} \in \{-1, 0, 1\}} \left[ \delta_{a,a'} \delta_{b,b'} + \delta_{a,a'} \delta_{b,b'} \right] e^{i(\alpha_{a,a'} - \alpha_{b,b'}) \ell} e^{-i\omega (a + b) \ell} \\
\times \bar{G}_{\{a_1, b_1, \ldots, a_K, b_K\}}(\chi) \\
= \det || \delta_{a,a'} \delta_{\rho,\rho'} e^{i\omega \ell} \\
- \frac{v(\rho')}{v(\rho)} (e^{i\sigma^z \alpha_F \ell})_{a,a} \sum_{\rho''=1,2} \sum_{\rho'''=1} [R_L(0; \chi)]_{(a,\rho),(a'',\rho''')} (e^{i\sigma^z \alpha_F \ell})_{a'',a'} e^{-i\omega \ell} [R_L(0; \chi)]_{(a',\rho'''),(a',\rho')} || 
\]

(E3)

As a simple model calculation, let us now compute \(\Phi[\omega; \chi]\) for \(N = 1\) and \(K = 2\) inequivalent channels within \(C\). In particular, to simplify the derivation, we choose \(H_F\) as in Eq. (A13) with \(K = 2\), but setting \(\theta_L = \theta_R = \varphi\) in Eq. (D3). As it happens in the example of Appendix D, also here only a linear combination of the operators for the two channels within \(C\) couples to the leads. Let \(\tilde{R}_L^{(1)}, \tilde{R}_R^{(1)}\) be the corresponding \(2 \times 2\) reflection amplitude matrix at the left-hand side and at the right-hand side interface for the coupled channel, respectively. One then obtains (defining the square matrices \(R(E; \chi)\) just as we did in Appendix D)

\[
R_{L(R)}(0; \chi) = \begin{bmatrix}
\cos^2(\varphi)\tilde{R}_L^{(1)} + \sin^2(\varphi)I & \cos(\varphi)\sin(\varphi)(\tilde{R}_L^{(1)} - I)w \\
\cos(\varphi)\sin(\varphi)(\tilde{R}_L^{(1)} - I)\frac{1}{w} & \sin^2(\varphi)\tilde{R}_L^{(1)} + \cos^2(\varphi)I
\end{bmatrix},
\]

(E4)

with \(w = \sqrt{(\frac{v(2)}{v(1)})}^2\). It is, now, simple to check that, for \(\varphi = 0\) or \(\varphi = \frac{\pi}{2}\), respectively setting \(u = e^{-\omega w}\) and \(u = e^{-\varphi}\), Eq. (E3) gives back the (second-order) polynomial \(P(u; \chi)\) for a single-channel, with Fermi velocity and Fermi momentum equal to \(v(1), \alpha_F,1\) and to \(v(2), \alpha_F,2\), respectively. The same result is clearly obtained for a generic value of \(\varphi\), on setting \(\alpha_{F,1} = \alpha_{F,2} = \alpha_F\) and \(v^{(1)} = v^{(2)} = v\), which implies \(w = 1\). In general, once \(\Phi[\omega; \chi]\) computed with Eqs. (E3) (E4) is put into Eqs. (3.13) (3.16), one recovers a simple and effective tool to compute \(I[\chi; T = 0]\) and \(I[\chi; T]\) for generic values of the parameters by means of pertinent numerical techniques, as we do at the end of Section III B by assuming perfect Andreev reflection at both interfaces, that is, by assuming that \(\tilde{R}_L^{(1)}\) and \(\tilde{R}_R^{(1)}\) are the matrices given in Eq. (D2).
As mentioned in the introduction, from Eqs. (3.10, 13, 15) it is possible to recover the main result of Ref. [18] for the dc Josephson current in a multi-channel SINIS-junction. To do so, one has to assume that there are no scattering processes at the interfaces between different channels within C. Formally, this means that both the \( R_L(E; \chi) \) and the \( R_R(E; \chi) \) matrices (and, consequently, the \( R_L, R_R \) matrices) have to be diagonal in the channel index \( \rho \), that is

\[
[R_L(R; \chi)|_{(a, \rho), (a', \rho')} = [R_L(R; \chi)_{a, a'}(E; \chi)] \delta_{\rho, \rho'} .
\]  

(E5)

Accordingly, Eq. (E3) for \( \Phi[\omega; \chi] \) simplifies to

\[
\Phi[\omega; \chi] = \prod_{\rho=1}^{K} \Phi_{\rho}[\omega; \chi],
\]  

(E6)

with

\[
\Phi_{\rho}[\omega; \chi] = e^{w_{\rho} \omega} \det \begin{pmatrix}
I_2 - R^\rho_L(0; \chi) & e^{i \alpha^{(\rho)}(\omega)} e^{-w_{\rho} \omega} & 0 \\
0 & e^{-i \alpha^{(\rho)}(\omega)} e^{-w_{\rho} \omega} & 0 \\
0 & 0 & e^{-i \alpha^{(\rho)}(\omega)} e^{-w_{\rho} \omega}
\end{pmatrix} \cdot R^\rho_R(0; \chi) \cdot \begin{pmatrix}
e^{i \alpha^{(\rho)}(\omega)} e^{-w_{\rho} \omega} & 0 \\
0 & e^{-i \alpha^{(\rho)}(\omega)} e^{-w_{\rho} \omega}
\end{pmatrix}. \]  

(E7)

As a result, at finite \( T \), \( I[\chi; T] \) can be written as

\[
I[\chi; T] = \sum_{\rho=1}^{K} \left\{ 2eT \sum_{\nu = -\infty}^{\infty} \partial_\chi \Phi_{\rho}[\omega_\nu; \chi] \right\}, \quad \sum_{\rho=1}^{K} I_{\rho}[\chi; T].
\]  

(E8)

Similarly, at \( T = 0 \) one obtains

\[
I_{\rho}[\chi; T = 0] = \sum_{\rho=1}^{K} \left\{ \frac{2e}{2 \pi} U \int_{-\infty}^{\infty} d\omega \partial_\chi \Phi_{\rho}[\omega; \chi] \right\} = \sum_{\rho=1}^{K} I_{\rho}[\chi; T = 0]. \]  

(E9)

\( I_{\rho}[\chi; T] \) and \( I_{\rho}[\chi; T = 0] \) are the current for a single-channel SNS junction at finite \( T \) and at \( T = 0 \), respectively. They can be readily computed following the derivation of Ref. [9]. To compare with the result of Ref. [18], we then compute \( I_{\rho}[\chi; T = 0] \), which is given by

\[
I_{\rho}[\chi; T = 0] = -\frac{e^{u^{(\rho)}}}{\pi \ell} \partial_\chi \vartheta^2_{\rho}(\chi),
\]  

(E10)

with

\[
\vartheta_{\rho}(\chi) = \arccos \left\{ \operatorname{Re}[N_{R,\rho}^p \bar{N}_{L,\rho}^p e^{2i \alpha^{(\rho)}(\omega)}] + \bar{A}_{R,\rho}^p \bar{A}_{L,\rho}^h \right\},
\]  

(E11)

and \( \bar{N}_{R,\rho}^p, \bar{A}_{R,\rho}^p, \bar{A}_{L,\rho}^h \) respectively being the normal and the Andreev single-particle/hole reflection amplitudes within channel-\( \rho \) at the right/left-hand S-N interface evaluated at the Fermi level only. It is now straightforward to check that Eqs. (E10, E11) give back the result of Ref. [18] for a \( K \)-channel SINIS junction provided that, for a generic channel \( \rho \), one first of all relates the reflection and the transmission coefficients at the left/right-hand SIN-interface, respectively given by \( B_{L,\rho}, D_{L,\rho} (B_{R,\rho}, D_{R,\rho}) \) to the modulus of the normal and Andreev reflection coefficients, according to the equations

\[
|\bar{A}_{R,\rho}^h| = \frac{D_{R,\rho}}{1 + B_{R,\rho}},
\]

\[
|\bar{A}_{L,\rho}^h| = \frac{2\sqrt{B_{R,\rho} D_{R,\rho}}}{1 + B_{R,\rho}},
\]  

(E12)
and indentifies the phase $\phi_{\rho}$ in Eq. (23) of Ref. \cite{18} with $\arg[\tilde{N}_{R,\rho}^p,\tilde{N}_{L,\rho}^p]$. It is therefore likely that, where the range of applicability of our approach overlaps with the one of the approach based on Eilenberger equations, equivalent results are obtained. It would be interesting to check this point by repeating, for instance, the calculations of Refs. \cite{19,20} with our technique, but this goes beyond the scope of this work, which is mainly a presentation of our approach. It is important to recall that, as already remarked before, our derivation is amenable for trading complicated model Hamiltonians describing the whole SNS junctions for simple boundary models, which is the key steps for treating Luttinger liquid interaction effects in the central region.

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