Hitchin system on singular curves I

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Abstract.

In this paper we study Hitchin system on singular curves. Some examples of such system were first considered by N. Nekrasov (hep-th/9503157), but our methods are different. We consider the curves which can be obtained from the projective line by gluing several points together or by taking cusp singularities. (More general cases of gluing subschemas will be considered in the next paper). It appears that on such curves all ingredients of Hitchin integrable system (moduli space of vector bundles, dualizing sheaf, Higgs field etc.) can be explicitly described, which may deserve independent interest. As a main result we find explicit formulas for the Hitchin hamiltonians. We also show how to obtain the Hitchin integrable system on such a curve as a hamiltonian reduction from a more simple system on some finite-dimensional space. In this paper we also work out the case of a degenerate curve of genus two and find the analogue of the Narasimhan-Ramanan parameterization of SL(2)-bundles. We describe the Hitchin system in such coordinates. As a demonstration of the efficiency of our approach we also rederive the rational and trigonometric Calogero systems from the Hitchin system on cusp and node with a marked point.

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1 Introduction

Hitchin system was introduced in [1] as an integrable system on the cotangent bundle of the moduli space $\mathcal{T}^* \mathcal{M}$ of stable holomorphic bundles on an algebraic curve $\Sigma$. This phase space can be obtained by the Hamiltonian reduction by the gauge group action from the space of pairs $d''_A, \Phi$, where $d''_A$ is the operator defining the holomorphic structure on the bundle $V$ and $\Phi$ is an endomorphism of this bundle, more precisely $\Phi \in \Omega^{0,1}(\Sigma, \text{End}(V))$ where the gauge group is the group of $GL_N$-valued functions on $\Sigma$. The invariant symplectic structure on the “big” space can be written as:

$$\omega = \int_{\Sigma} \text{Tr} \delta \Phi \wedge \delta d''_A.$$  \hspace{1cm} (1)

The zero level of the moment map is described by the condition $d''_A \Phi = 0$ which means that $\Phi$ is holomorphic with respect to the induced holomorphic structure on the bundle $\text{End}(V)$. It turns out that the system of quantities $\text{Tr} \Phi^k$, treated as vector functions on the phase space, Poisson-commute and their number is exactly half the dimension of the phase space.

The importance of Hitchin system and its generalizations [2, 3, 4] in modern mathematical physics cannot be overestimated. Many well-known systems can be obtained as particular cases. Automatically they inherit the universal construction of a family of commuting Hamiltonians as well as the geometric description of the Hamiltonian flows, the Lax representation, and the “action-angle” variables.

This domain is also connected with important questions in mathematical physics like the geometric Langlands correspondence [5, 6, 7], conformal field theory (in a sense Hitchin system is a Knizhnik-Zamolodchikov-Bernard equation on the critical level) [6, 8], non-linear partial differential equations such as KP [9], Davey-Stewartson equation [10], Nahm’s equations describing monopoles [11], and other problems (see for example [12, 13]).

Despite its importance Hitchin system is far from being fully investigated. One of the reasons for such a situation is that the moduli space of vector bundles is a complicated manifold and it is difficult to choose “good” coordinates on it to write down the Hamiltonians explicitly. Several attempts have been done in [2] and in [14]. Nevertheless such descriptions appear to be complicated and do not answer many questions (at least yet). So it is important to work out some examples of Hitchin system which on the one hand are sufficiently simple and on the other hand are rich enough to find out general methods for solving Hitchin system and to understand such phenomena as the separation of variables and the geometric Langlands correspondence.

The approach elaborated in this paper can be applied in rather specific cases, namely when the base algebraic curve is singular and its normalization is a rational curve. Its richness is proved by the number of nontrivial examples. For such curves all ingredients of Hitchin systems (vector bundles, their endomorphisms, the moduli space of vector bundles, the dualizing sheaf, Higgs fields) can be described very explicitly and in a quite simple way. So we hope that the understanding of such systems will shed light on the general case.

We proceed by formulating the main results of this paper.
1.1 Constructing Hitchin system

Consider the curve $\Sigma^{proj}$ which results from gluing $N$ distinct points $P_i$ on $\mathbb{CP}^1$ to one point (i.e. the curve which is obtained by adding the smooth point $\infty$ to the curve $\Sigma^{aff} = \text{Spec}\{f \in \mathbb{C}[z] : \forall i, j f(P_i) = f(P_j)\}$.

- A rank $r$ vector bundle on such a curve corresponds to a rank $r$ module $M_\Lambda$ over the affine part given by the subset of vector-valued functions $s(z)$ on $\mathbb{C}$ i.e. $s(z) \in \mathbb{C}[z]^r$ which satisfy the conditions: $s(P_1) = \Lambda_i s(P_i)$. The moduli space of vector bundles on $\Sigma^{proj}$ is the factor by $GL_r$ of the set of invertible matrices $\Lambda_i, \forall i = 2, ..., N$ where $GL_r$ acts by conjugation. (See section 2.3.2, theorem 1).

- The basis of global sections of the dualizing sheaf on $\Sigma^{proj}$ can be described as meromorphic differentials on $\mathbb{C}$ given by $dz_{\infty} - dz_{P_i}, \forall i = 2, ..., N$ (see section 2.2.1, example 3).

- The endomorphisms of the module $M_\Lambda$ are matrix valued polynomials $\Phi(z)$ such that $\Phi(P_1) = \Lambda_i \Phi(P_i)\Lambda_i^{-1}, \forall i = 2, ..., N$ (see section 2.4.1, proposition 6). The action of $\Phi(z)$ on $s(z)$ is: $s(z) \mapsto \Phi(z)s(z)$. The space $H^1(\text{End}(M_\Lambda))$ can be described as the space of matrix valued polynomials factorized by the subspaces: $\text{End}_{\text{out}} = \{\chi(z) \in \mathfrak{gl}[z]|\chi(z) = \text{const}\}$ and $\text{End}_{\text{in}} = \{\chi(z) \in \mathfrak{gl}[z]|\chi(P_1) = \Lambda_i\chi(P_i)\Lambda_i^{-1}, \forall i = 2, ..., N\}$. The elements of $H^1(\text{End}(M_\Lambda))$ are the tangent vectors to $\Lambda_i$, the element $\chi(z)$ gives the following tangent vector to $\Lambda_i$:

$$\delta_{\chi(z)}\Lambda_i = \chi(P_1)\Lambda_i - \Lambda_i\chi(P_i)$$

(2)

- The global sections of $H^0(\text{End}(M_\Lambda) \otimes \mathcal{K})$ ("Higgs fields") are described as

$$\Phi(z) = \sum_{i=2,\ldots,N} \frac{\Lambda_i \Phi_i \Lambda_i^{-1}}{z - P_i} \, dz + \sum_{i=2,\ldots,N} \frac{\Phi_i}{z - P_i} \, dz,$$

(3)

where $\sum_{i=2,\ldots,N} \Lambda_i \Phi_i \Lambda_i^{-1} + \Phi_i = 0$ (see section 2.5.2, proposition 9). Let us mention that precisely this condition arises as the zero moment level condition (see section 2.7, formula 23). The symplectic form on the cotangent bundle to the moduli space can be described as the reduction of the form on the space $\Lambda_i, \Phi_i, \forall i = 2, ..., N$ given by

$$- \sum_{i=2,\ldots,N} \text{Trd}(\Phi_i \Lambda_i^{-1}) \wedge d\Lambda_i$$

(4)

(see section 2.7, proposition 15).

**Result 1:** The Hitchin system on the curve $\Sigma^{proj}$ can be described as the system with a phase space which is the hamiltonian reduction of the space $\Lambda_i, \Phi_i$ with the symplectic form 4; the reduction is taken by the group $GL(r)$, which acts by conjugation, the Lax operator is given by 3.
Remarks: For the case of gluing two points the same Lax operator has been proposed by N. Nekrasov ([2]), though his methods are different from ours, and the explicit description of bundles, dualizing sheaf, endomorphisms etc is absent in his approach. When one glues several groups of points: \( P_i = P_j, Q_i = Q_j \) ... it is obvious how to modify all propositions above, for example the Lax operator becomes:

\[
\sum_{i=2, \ldots, N} -\Lambda_i \Phi_i \Lambda_i^{-1} \frac{dz}{z-P_i} + \sum_{i=2, \ldots, N} \frac{\Phi_i}{z-P_i} dz + \sum_{i=2, \ldots, N} -\tilde{\Lambda}_i \tilde{\Phi}_i \tilde{\Lambda}_i^{-1} \frac{dz}{z-Q_i} + \sum_{i=2, \ldots, \tilde{N}} \frac{\tilde{\Phi}_i}{z-Q_i} dz.
\]

Actually one can easily guess the Lax operator above from the case of gluing two points: one must first consider the gluing of \( N - 1 \) pairs of points together \( P_2 = R_2, P_3 = R_3, \ldots \), then take \( R_k = P_1 \).

Analogously we obtain all propositions for the case of a curve with several cusps at points \( P_i \) on \( \mathbb{C}P^1 \).

- The curve: \( \Sigma^{aff} = \text{Spec}\{ f \in \mathbb{C}[z] : \forall i, f'(P_i) = 0 \} \).
- The modules: \( s(z) \in \mathbb{C}[z]^r : s'(P_i) = \Lambda_i s(P_i) \).
- The basis of global sections of the dualizing sheaf: \( \frac{dz}{(z-P_i)^2} \). The endomorphisms of the module \( M_{\Lambda} \): \( \Phi(z) \) satisfying the condition \( \Phi'(P_i) = [\Lambda_i, \Phi(P_i)] \).
- The global sections of \( H^0(\text{End}(M_{\Lambda}) \otimes \mathcal{K}) \) ("Higgs fields"):

\[
\Phi(z) = \sum_i \left( \frac{\Phi_i dz}{(z-P_i)^2} + \frac{[\Lambda_i, \Phi_i] dz}{z-P_i} \right),
\]

where \( \sum_{i=1, \ldots, N} [\Lambda_i, \Phi_i] = 0 \).
- The symplectic form:

\[
- \sum_{i=1, \ldots, N} Tr d\Phi_i \wedge d\Lambda_i.
\]

1.2 Narasimhan-Ramanan parameterization

It is known since [15] that the moduli space of \( SL(2) \) vector bundles on a curve of genus 2 is \( \mathbb{C}P^3 \). In this paper we introduce some analogs of the Narasimhan-Ramanan parameters for the \( SL(2) \) vector bundles on a singular curve of genus 2 - the curve with two cusps at points \( P_1, P_2 \). On such curves the vector bundles can be described as bundles corresponding to the modules \( M_{\Lambda} \) defined as \( s(z) \in \mathbb{C}[z]^2 : s'(P_i) = \Lambda_i s(P_i) \), where \( \Lambda_i \in sl(2) \). Our goal is to express analogs of the Narasimhan-Ramanan parameters via the \( \Lambda_1, \Lambda_2 \).

Result 2: The Narasimhan-Ramanan coordinates over a singular curve are:

\[
\tau_1 = Tr \Lambda_1^2, \quad \tau_3 = Tr \Lambda_3^2, \quad \tau_2 = Tr (\Lambda_1 \Lambda_2) + \tau_1 \tau_3 \frac{(P_1 - P_2)^2}{4}.
\]
The Hitchin Hamiltonians in this case are

\[ H_1 = Tr \Phi_1^2 = 4p_1^2 t_1 + p_2^2 t_3 + 4p_1 p_2 t_2; \]

\[ H_2 = 2Tr \Phi_1 \Phi_2 + (z_1 - z_2)^2 Tr[\Lambda_1, \Phi_1]^2 \]
\[ = 4p_1 p_2 t_1 + 4p_2 p_3 t_3 + (8p_1 p_3 + 2p_2^2) t_2 - 2(z_1 - z_2)^2 p_2^2 (t_1 t_3 - t_2^2); \]

\[ H_3 = Tr \Phi_2^2 = 4p_3^2 t_3 + p_2^2 t_1 + 4p_2 p_3 t_2; \]

where

\[ t_1 = Tr \Lambda_1^2, \quad t_2 = Tr \Lambda_1 \Lambda_2, \quad t_3 = Tr \Lambda_2^2 \]

and \( p_i \) are the corresponding conjugated variables.

This paper is organized as follows: the first section contains all algebraic-geometric preliminaries. In the second section we work out the case of a rational curve with double point and cusp and show that the arising systems are the trigonometric and rational Calogero-Moser system with spin. In the third section we treat the case of a rational curve with two cusps, which is a curve of algebraic genus 2. We consider the moduli space of holomorphic \( SL_2 \)-bundles on it and construct the analog of the Narasimhan-Ramanan parameterization in the singular case. In conclusion we state some open problems for future work.

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2 Algebraic-geometric background

2.1 Curves defined by gluing points with multiplicities.

Let us consider a curve \( \Sigma \) and some effective divisor \( D = \sum_i n_i P_i \) \( (n_i > 0) \) such that \( \text{deg} D > 1 \). One defines a new curve \( \Sigma_D \) by, roughly speaking, gluing all points \( P_i \) with multiplicities \( n_i \) to one point \( P \); formally speaking we define the structure sheaf \( O(\Sigma_D) \) to be a subsheaf of \( O(\Sigma) \) with the properties: \( f(P_i) = f(P_j); f^k(P_i) = 0, k = 1, ..., n_i - 1 \). In Serre’s terminology this is “the curve defined by the module \( D \)” (see [16] ch. 4 sect. 4). The new curve \( \Sigma_D \) obviously has one more singular point \( P \), the normalization of \( \Sigma_D \) is \( \Sigma \) (of course, if \( \Sigma \) is a smooth curve).

Example 1 Main example to keep in mind. If we consider \( \Sigma = \mathbb{C}^1 \) and \( D = P_1 + P_2 \) we obtain the curve \( \text{Spec}\{f \in \mathbb{C}[z] : f(P_1) = f(P_2)\} \) which is called node (or double point in another terminology), it is an affine curve which can be defined by the equation \( y^2 = x^2(x + a), z = \frac{y}{x}, P_1 = \sqrt{a}, P_2 = -\sqrt{a} \).
Example 2 If we consider $\Sigma = \mathbb{C}^1$ and $D = 2P$ we obtain the curve $\text{Spec}\{f \in \mathbb{C}[z] : f'(P) = 0\}$ which is called cusp, it is an affine curve which can be defined by the equation $y^2 = (x-a)^3$, $z = \frac{y}{x-a}$, $P = a$.

Proposition 1 Consider $\mathbb{C}P^1$ and the effective divisor $D = \sum n_i P_i$ on it. Consider the curve $\Sigma_D$ which is obtained by gluing points $P_i$ with multiplicities $n_i$ to one point as was explained above. Then the genus (i.e. $\dim H^1(\mathcal{O}_{\Sigma_D})$) of such a curve equals $(\sum n_i) - 1$.

The proposition above is obvious: it is enough to cover the curve by two charts: the first contains the singular point and does not contain infinity, the other does not contain the singular point, and to calculate the Čech cohomology of $\mathcal{O}$. (On the curve such a covering is acyclic because the chosen charts are affine manifolds).

Remark 1 One sees that the genus of the node and cusp curves is equal to 1, the same as for an elliptic curve. It is not surprising due to the fact that these curves are degenerations of elliptic curves and the genus does not change under deformation.

2.2 Canonical (dualizing) sheaf on curves defined by gluing points.

2.2.1 Description of the dualizing sheaf.

Recall that the dualizing sheaf on a curve $\Sigma$ is, roughly speaking, a sheaf $\mathcal{K}$ such that for an arbitrary coherent sheaf $F$ there is a canonical isomorphism $\text{Hom}(F, \mathcal{K}) \cong H^1(F^*)$. On a nonsingular curve the dualizing sheaf is the sheaf of 1-forms, but for a singular curve the notion of “1-form” must be clarified and the naive definition of the Kähler differentials ([17] ch. 2 sect. 8) is not the right object. The general receipt (see [16, 18]) for the description of the dualizing sheaf on a singular curve is the following:

Proposition 2 Let $\Sigma_{\text{norm}}$ be the normalization of $\Sigma$ and $\pi : \Sigma_{\text{norm}} \to \Sigma$ the corresponding projection. The dualizing sheaf $\mathcal{K}$ on the singular curve $\Sigma$ can be described as the sheaf of meromorphic 1-forms $\alpha$ on $\Sigma_{\text{norm}}$ such that $\forall f \in \mathcal{O}(\Sigma)$ and $\forall P \in \Sigma$ it is true that:

$$\sum_{P_i \in \Sigma_{\text{norm}} : \pi(P_i) = P} \text{Res}_{P_i} \tilde{f} \alpha = 0,$$

where $\tilde{f} \in \mathcal{O}(\Sigma_{\text{norm}})$ is the pullback of the function $f$ on the singular curve to its normalization, $P_i$ are points on the normalization such that they map to the point $P$ on the singular curve by the normalization map $\pi$.

Let us describe explicitly the canonical (dualizing) sheaf on a singular curve defined by gluing the points $P_i$ (of the smooth curve $\Sigma$) with multiplicities $n_i > 0$ to one point. Denote by $D$ the divisor of the points with multiplicities $D = \sum_i n_i P_i$.

Corollary 1 The dualizing sheaf $\mathcal{K}_{\Sigma_D}$ is defined as the subsheaf of differential forms $w$ on $\Sigma$ with possible poles in $P_i$ with orders $\text{ord}_{P_i} w \geq -n_i$ (i.e. the subsheaf of $\mathcal{K}_\Sigma(D)$) with the condition $\sum_i \text{Res}_{P_i} w = 0$.

Our convention is $\text{ord}_0 z^n = n$. Obviously $\mathcal{K}_{\Sigma_D}$ is a coherent sheaf on $\Sigma_D$. Moreover one can see directly (or look at [16] ch. 4 sect. 11) that it is a locally free sheaf (i.e. a
line bundle on a singular curve). The dualizing sheaf is not always locally free. It is true for complete intersections and arbitrary plane curves (see [16] for the discussion).

**Example 3** Consider the node curve Σ, i.e., $\mathbb{C}P^1$ with two points $P_1, P_2$ glued together, so the affine part of this curve is $\text{Spec}\{ f \in \mathbb{C}[z] : f(P_1) = f(P_2) \}$. The sections of the sheaf $\mathcal{K}_{\text{node}}$ on the chart without infinity are described as $\frac{czdz}{z-P_1} - \frac{cdz}{z-P_2} + f(z)dz$, where $f(z)$ is holomorphic. On the other charts one obtains the sections of $\mathcal{K}_{\text{node}}$ by the usual localization procedure: on the charts, which do not contain the singular point, the sections of $\mathcal{K}_{\text{node}}$ are the usual holomorphic 1-forms. So the only global section of $\mathcal{K}_{\text{node}}$ is $\frac{cdz}{z-P_1} - \frac{cdz}{z-P_2}$.

One can easily guess what is going on for the case when we glue $n$ points $P_1, ..., P_n$ on $\mathbb{C}P^1$ together: for example the basis of global holomorphic differentials can be given by $\frac{dz}{z-P_i}$, for $i = 2, ..., n$.

**Example 4** Consider the cusp curve Σ, i.e., $\mathbb{C}P^1$ with the point $P$ glued with multiplicity 2, so the affine part of this curve is $\text{Spec}\{ f \in \mathbb{C}[z] : f'(P) = 0 \}$. The sections of the sheaf $\mathcal{K}_{\text{cusp}}$ on the chart without infinity are described by $\frac{cdz}{(z-P)^2} + f(z)dz$, where $f(z)$ is holomorphic. So obviously $\frac{cdz}{(z-P)^2}$ is the only global section of $\mathcal{K}_{\text{cusp}}$.

The description of the canonical class on an $n$-cusp curve when we glue one point $P$ on $\mathbb{C}P^1$ with multiplicity $n$ is analogous. For example the basis of global holomorphic differentials can be given by $\frac{dz}{(z-P)^i}$, for $i = 2, ..., n$.

**Remark 2** One can see that the Serre’s description of a dualizing sheaf is quite consistent with the naive arguments for the node and cusp curve. Consider the node curves: $y^2 = x^2(x + a)$ and $z = \sqrt{\frac{y}{a}}$ - is the coordinate on the normalization of this curve. The holomorphic differential on an elliptic curve is given by the formula:

$$\frac{dx}{y} = \frac{dz^2}{z(z^2 - a)} = \frac{2dz}{z - \sqrt{a}} - \frac{2dz}{z + \sqrt{a}},$$

so we obtain a differential which satisfies Serre’s conditions: the orders of the poles are 1 and the sum of its residues is equal to zero. If one puts $a = 0$ which corresponds to the cusp curve we obtain as a limit the differential $\frac{dz}{z^2}$. It has a pole of order 2 and residue zero. This is also in accordance with Serre’s rule.

### 2.2.2 Serre’s pairing in the Čech description of $H^1(F)$ on a singular curve.

Let us describe the pairing (Serre’s duality) between $H^1(F)$ and $H^0(F^* \otimes \mathcal{K}_{\Sigma_D})$, where $F$ is a flat coherent sheaf on the curve $\Sigma_D$. Recall that $\Sigma$ is a smooth curve and $\Sigma_D$ is a singular curve obtained by gluing the points $P_i$ with multiplicities $n_i$ together, $D = \sum_i n_iP_i$. In [16], the duality is presented in the most general case by using the language of distributions (adels). So for the sake on convenience we write it down explicitly in our simple case.

For smooth curves, the elements of $H^1(F)$ in Dolbeault’s representation are “$d\bar{z}$ forms with values in $F$”. The elements of $\mathcal{K}$ can be represented as holomorphic 1-forms, so the pairing can be given by $\int_{\Sigma} < f, f^* > dzd\bar{z}$. For singular curves Dolbeault’s approach
does not work, at least naively, so we prefer the Čech description of $H^1(\mathcal{F})$, which works perfectly even for singular curves.

Let us cover the curve $\Sigma_D$ by the two charts $U_P = \Sigma_D \setminus \infty$ and $U_\infty = \Sigma_D \setminus P$, where we denote by $\infty$ an arbitrary point in $\Sigma_D$, distinct from $P$ (recall that $P$ is the only singular point obtained by gluing the points $P_i$ together). This choice of covering is the most convenient for our calculation and will be used throughout the paper. One knows that a curve minus any point is an affine curve, so this covering is sufficient to calculate the cohomology of the coherent sheaves: $H^1(\mathcal{F}) = \mathcal{F}(U_\infty \cap U_P)/(\mathcal{F}(U_P) \oplus \mathcal{F}(U_\infty))$.

**Proposition 3** The Serre’s pairing between $f \in H^1(\mathcal{F})$ and $h \otimes w \in H^0(\mathcal{F}^* \otimes K_{\Sigma_D})$ can be described as follows: consider $\tilde{f} \in \mathcal{F}(U_\infty \cap U_P)$ the representative of the element $f$, then the pairing is given by:

$$< f, h \otimes w > = \sum_i \text{Res}_{P_i} < \tilde{f}, h > w$$

This pairing is well-defined (i.e. it does not depend on the choice of the representative $\tilde{f}$) and non-degenerate.

**Corollary 2** So one obviously obtains

$$\dim H^1(\mathcal{O}_{\Sigma_D}) = \dim H^0(K_{\Sigma_D}) = H^1(\mathcal{O}_\Sigma) + \deg D - 1.$$

Let us sketch why the pairing is well-defined and nondegenerate. In order to see that the pairing is well-defined one needs to check that it is zero for $\tilde{f} \in \mathcal{F}(U_P)$ and for $\tilde{f} \in \mathcal{F}(U_\infty)$. Indeed, if $\tilde{f} \in \mathcal{F}(U_P)$ then $g = < \tilde{f}, h >$ belongs to $\mathcal{O}_{\Sigma_D}(U_P)$ and hence its pullback $\tilde{g}$ has the same values at the preimages of the point $P \tilde{g}(P_i) = \tilde{g}(P_j) =: g(P)$ and satisfies the conditions $\tilde{g}^{(k)}(P_i) = 0, k = 1, \ldots, n_i - 1, \forall i$. Hence:

$$< \tilde{f}, h \otimes w > = \sum_i \text{Res}_{P_i} (\tilde{g} \otimes w) - g(P) \sum_i \text{Res}_{P_i} w = 0,$$

where we used the fact that the sum of the residues of a meromorphic differential is zero.

For an element $\tilde{f} \in \mathcal{F}(U_\infty)$ one has

$$< \tilde{f}, h \otimes w > = \sum_i \text{Res}_{P_i} (\tilde{g} \otimes w) = -\text{Res}_{\infty} (\tilde{g} \otimes w) = 0,$$

because both $w$ and $\tilde{g}$ are regular at $\infty$.

To show that this paring is nondegenerate it is sufficient to prove that there is no meromorphic differential $w$ of the prescribed type such that

$$< f, w > = 0 \quad \forall f \in \mathcal{O}(U_\infty \cap U_P).$$

We can present $n - 1$ functions $f_i$ on the normalization with their only pole of sufficiently high order at $\infty$ and with nondegenerate matrix of their derivatives up to $n_i - 1$ order at the points $P_i$. The condition of orthogonality

$$< f_i, \omega > = 0, \quad i = 1, \ldots, n - 1$$

is a system of linear homogeneous equations on the negative coefficients of $\omega$ at the points $P_i$. The only solution of this system is zero vector and one obtains that the pairing is nondegenerate due to the absence of holomorphic differentials on $\mathbb{C}P^1$. 

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2.3 Holomorphic bundles on singular curves

2.3.1 Projective modules over an affine part

Holomorphic bundles on a non-singular manifold can be described by sheaves of its sections. Such sheaves are locally free or equivalently (by a general theory) they are sheaves of projective modules over the structure sheaf. The geometric description of a holomorphic bundle on a singular manifold is problematic in contrast with the algebraic side which is unambiguous.

Definition 1 The holomorphic bundle on a singular curve $\Sigma$ is the sheaf of projective modules over $\mathcal{O}(\Sigma)$.

(It is known that a projective module is locally free (in Zariski’s topology) also for singular manifolds, so it is equivalent to speak about projective or locally free modules). First let us describe the projective modules over the affine curve $\Sigma^\text{aff}$ which is obtained by gluing the points $P_i$ on $\mathbb{C}$ with multiplicities $n_i$ to one point. As usual we denote by $D$ the effective divisor $\sum n_i P_i$. We will describe such modules as submodules of the trivial module on the normalization.

Proposition 4 Consider the curve $\Sigma^\text{aff}$ given by $\text{Spec}\{f \in \mathbb{C}[z] : \forall i, j f(P_i) = f(P_j) = \ldots = f^{n_i-1}(P_i) = 0\}$. Consider the following set of matrices: invertible matrices $\Lambda_2, \ldots, \Lambda_N$ and arbitrary matrices $\Lambda_i^l \in \text{Mat}(r)$, where $i = 1, \ldots, N; l = 1, \ldots, n_i - 1$. The subset of vector-valued polynomials $s(z)$ on $\mathbb{C}$ i.e. $s(z) \in \mathbb{C}[z]^r$ such that they satisfy the conditions: $s(P_i) = \Lambda_i s(P_i); s^{(l)}(P_i) = \Lambda_i^l s(P_i)$ is a projective module of rank $r$ over the algebra $\{f \in \mathbb{C}[z] : \forall i, j f(P_i) = f(P_j); f'(P_i) = \ldots = f^{n_i-1}(P_i) = 0\}$. All projective modules can be obtained in this way.

Notation Let us denote the module and the bundle described above by $M_\Lambda$. (We will use the same notation $M_\Lambda$ for the vector bundles on the projectivization $\Sigma^\text{proj}$ of the curve $\Sigma^\text{aff}$, we hope that it will not be confusing).

Remark 3 One can easily show (calculating the divisor for example) that in the case of rank $r = 1$ all these modules are non isomorphic for different $\Lambda$’s. For the $r > 1$ it is certainly not true, but we will see below that the vector bundles on the projectivization $\Sigma^\text{proj}$ of the curve $\Sigma^\text{aff}$ corresponding to these modules are isomorphic iff all $\Lambda$’s are conjugated by the same constant matrix $C$.

Remark 4 Let us mention that even in the case $r = 1$ if one considers the analytic topology then all bundles $M_\Lambda$ become isomorphic, because from the exponential sequence one can easily see that $H^1(\mathcal{O}^*) = H^1(\mathcal{O})$ and $H^1(\mathcal{O}) = 0$ on any affine curve. But, for projective curves the GAGA principle guarantees the same results for both the algebraic and analytic setups. We will be interested in projective curves so one must not pay too much attention to the remarks above.

Sketch of Proof. This proposition is quite simple so let us only sketch out its proof. Let $\pi : \mathbb{C} \to \Sigma^\text{aff}$ be the normalization map. Consider any torsion free module $\mathcal{F}$ of rank $r$. So $\pi^*\mathcal{F}$ is a torsion free rank $r$ module, but all such modules over $\mathbb{C}[z]$ are trivial, so $\pi^*\mathcal{F} = \mathbb{C}[z]^r$. Consider $\pi_*\pi^*\mathcal{F}$. It is isomorphic to $\mathbb{C}[z]^r$ considered as a module over $\mathcal{O}(\Sigma_D) = \{f \in \mathbb{C}[z] : \forall i, j f(P_i) = f(P_j); f'(P_i) = \ldots = f^{n_i-1}(P_i) = 0\}$, it is a torsion
free, but not a projective module (the fiber at the singular point \( P \) jumps). We have the exact sequence \( \mathcal{F} \to \pi_+ \pi^* \mathcal{F} \to \mathbb{C}^r \), where \( \mathbb{C}^r \) is a skyscraper sheaf at the point \( P \), \( r \) is the rank, and \( g = \text{deg}D - 1 \).

So we see that any torsion free module \( \mathcal{F} \) can be described as the kernel of the map \( \phi : \mathbb{C}[z]^r \to \text{skyscraper at } P \). Such maps \( \phi \) bijectively correspond to the maps \( \phi : \text{fiber at } P \text{ of module } \mathbb{C}[z]^r \to \mathbb{C}^r \). The finite dimensional linear space \( \text{fiber at } P \text{ of module } \mathbb{C}[z]^r \) in our case is the space \( \oplus_P \mathbb{C}^r(n_i+1) \). So in general the kernel of such a map can be described by the maps: \( \Lambda_i : \mathbb{C}_{r_i} \to \mathbb{C}_{r_{i+1}} \) and \( \Lambda_i : \mathbb{C}_{0,r_i} \to \mathbb{C}_{r_i,r_{i-1}} \). One can easily see that if we are not in the general case or if \( \Lambda_i \)'s are not invertible the modules will not be projective. ■

Let us recall that the fiber at a point \( P \) of a module \( M \) over a ring \( R \) is defined as \( M^{loc}/I^{loc}M^{loc} \), where \( I \) is the maximal ideal of the point \( P \) and \( loc \) means “localization at point \( P \)”.

**Example 5**

Consider the node (or double point) curve: \( \Sigma = \text{Spec}\{ f \in \mathbb{C}[z] : f(P_1) = f(P_2) \} \). Then the rank 1 modules (line bundles or rank one torsion free sheaf) are parameterized by one complex number \( \lambda \in \mathbb{C} \). They are given by the condition \( \{ s(z) \in \mathbb{C}[z] : s(P_1) = \lambda s(P_2) \} \). Obviously \( M_\lambda \) is a torsion free module. For \( \lambda = 0 \) one can see that it is not a projective module, because the fiber at the point zero jumps and becomes two-dimensional, which is impossible for locally free modules. It is a nice exercise to calculate the divisor of the line bundle \( M_\lambda \). For \( \lambda \neq 0 \) one can see that this module is locally free (hence projective). (A rank 1 projective module becomes free on any open set which does not contain any representative of its divisor). This example illustrates also that the moduli space of line bundles (the so called generalized Jacobian) on a singular curve is non compact. In this case \( \text{Jac} \cong \mathbb{C}^* \) and it is also an isomorphism of groups, where as usually one considers the tensor product as a group operation on line bundles. \( \text{Jac} \) can be compactified by the torsion free modules. In this case one should add one module corresponding to \( \lambda = 0 \), (it coincides with the module \( \lambda = \infty \), i.e. the module \( \{ s \in \mathbb{C}[z] : 0 = s(P_2) \} \) ). It can be shown that if one constructs properly the algebraic structure on the set of torsion free sheaves of rank 1 as a manifold it coincides with the curve \( \Sigma^{proj} \) itself. This result is related to the fact that the Jacobian of an elliptic curve is isomorphic to this curve. This can be done by constructing the Poincaré line bundle on the product of the curve with itself.

Given the divisor \( D = P_1 + P_2 + \ldots + P_N \) one obtains the curve \( \Sigma_D = \text{Spec}\{ f \in \mathbb{C}[z] : f(P_1) = f(P_2) = \ldots = f(P_N) \} \) by gluing the points \( P_i \in \mathbb{C} \) together. The rank \( r \) modules can be described as subsets of vector-valued polynomials \( s(z) \) on \( \mathbb{C} \) i.e. \( s(z) \in \mathbb{C}[z]^r \) such that they satisfy the conditions: \( s(P_1) = \Lambda_i s(P_i), i = 2, \ldots, N \), where \( \Lambda_i \) are arbitrary invertible matrices.

**Example 6**

Consider the cusp curve: \( \Sigma = \text{Spec}\{ f \in \mathbb{C}[z] : f'(P) = 0 \} \), recall that it means that we glued the point \( P \) with itself with multiplicity 2. The modules can be described by \( \{ s(z) \in \mathbb{C}[z] : s'(P) = \lambda_1 s(P) \} \). In this example for all \( \lambda_1 \in \mathbb{C} \) these modules are projective. So \( \mathbb{C} \) is the moduli space of line bundles. It can be compactified by adding one point \( \lambda_1 = \infty \) (i.e. the module \( \{ s \in \mathbb{C}[z] : 0 = s(0) \} \), which is the same as the maximal ideal of the singular point \( z = 0 \), and the same as the direct image of
we define the sheaf which is a trivial rank $r$ lots of bundles $F$ constant matrices.

and not containing the singular point and which is the module an invertible map of modules $K$. The degree of such bundles equals zero. The vector bundles are equivalent if there exists a module over the affine chart and the corresponding vector bundle on the projectivization).
The degree of such bundles equals zero. The vector bundles are equivalent if there exists an invertible map of modules $K(z) : M_\Lambda \to M_{\tilde{\Lambda}}$.

Recall that we denoted by $\Sigma_D^{proj}$ the projective curve which we obtain from the affine curve $Spec\{ f \in \mathbb{C}[z] : \forall i, j : f(P_i) = f(P_j), f^k(P_i) = 0, k = 1, ..., n_i \}$ by adding one smooth point at infinity. The modules $M_\Lambda$ give a vector bundle over $\Sigma^{proj}$ in an obvious way: we define the sheaf which is a trivial rank $r$ module over the chart containing infinity and not containing the singular point and which is the module $M_\Lambda$ (or more precisely its localization) over the chart which contains the singular point. Let us denote these bundles by $\tilde{M}_\Lambda$ (we hope that it will not be too confusing to denote by the same symbol). The modules $\tilde{M}_\Lambda$ are isomorphic if there exist a constant $\Lambda$. But nevertheless the general stable and possibly semistable bundles satisfy the property that $\tilde{M}_\Lambda$ is the trivial bundle on $\mathbb{C}P^1$, where $\pi : \mathbb{C}P^1 \to \Sigma^{proj}$ is the normalization map. Obviously there are lots of bundles $F$ of degree zero on $\Sigma^{proj}$ such that $\pi^*F$ are not trivial bundles but some bundles of the type $\oplus_{k=1,...,r} O(t_k)$ such that $\sum t_k = 0$. So by no means we obtain all bundles on $\Sigma^{proj}$ as bundles $\tilde{M}_\Lambda$ for some $\Lambda$. But nevertheless the general stable and possibly semistable bundles satisfy the property that $\pi^*F$ is a trivial bundle on $\mathbb{C}P^1$, and so it is easy to see from our previous description of projective modules that the general semistable bundles can be obtained as the bundles $\tilde{M}_\Lambda$ for some $\Lambda$.

2.3.2 Vector bundles over the projectivization

The modules $M_\Lambda$ and $M_{\tilde{\Lambda}}$ are equivalent, if there exists an invertible map of modules $K(z) : M_\Lambda \to M_{\tilde{\Lambda}}$.

Recall that we denoted by $\Sigma_D^{proj}$ the projective curve which we obtain from the affine curve $Spec\{ f \in \mathbb{C}[z] : \forall i, j : f(P_i) = f(P_j), f^k(P_i) = 0, k = 1, ..., n_i \}$ by adding one smooth point at infinity. The modules $M_\Lambda$ give a vector bundle over $\Sigma^{proj}$ in an obvious way: we define the sheaf which is a trivial rank $r$ module over the chart containing infinity and not containing the singular point and which is the module $M_\Lambda$ (or more precisely its localization) over the chart which contains the singular point. Let us denote these bundles by $\tilde{M}_\Lambda$ (we hope that it will not be too confusing to denote by the same symbol). The modules $\tilde{M}_\Lambda$ are isomorphic if there exist a constant $\Lambda$. But nevertheless the general stable and possibly semistable bundles satisfy the property that $\tilde{M}_\Lambda$ is the trivial bundle on $\mathbb{C}P^1$, where $\pi : \mathbb{C}P^1 \to \Sigma^{proj}$ is the normalization map. Obviously there are lots of bundles $F$ of degree zero on $\Sigma^{proj}$ such that $\pi^*F$ are not trivial bundles but some bundles of the type $\oplus_{k=1,...,r} O(t_k)$ such that $\sum t_k = 0$. So by no means we obtain all bundles on $\Sigma^{proj}$ as bundles $\tilde{M}_\Lambda$ for some $\Lambda$. But nevertheless the general stable and possibly semistable bundles satisfy the property that $\pi^*F$ is a trivial bundle on $\mathbb{C}P^1$, and so it is easy to see from our previous description of projective modules that the general semistable bundles can be obtained as the bundles $\tilde{M}_\Lambda$ for some $\Lambda$.

We obtain the following corollary:

**Proposition 5** The vector bundles $M_\Lambda$ over $\Sigma^{proj}$ are isomorphic if there exist a constant matrix $K$ such that $\forall i, j : \Lambda_i = K\Lambda_i K^{-1}; \Lambda^j_i = K\tilde{\Lambda}_i K^{-1}$.

We obtain the following corollary:

**Theorem 1** The open subset in the space of semistable vector bundles of degree zero and rank $r$ over the curve $\Sigma^{proj}$ can be described as

$$\mathcal{M} = \Lambda/GL_r$$

where $\Lambda$ is the set of matrices $\{\Lambda_i, \Lambda^j_k\}; i = 2, ..., N; k = 1, ..., N; j = 1, ..., n_i$, with $\Lambda_i$ invertible and $\Lambda^j_k$ arbitrary, and the $GL_r$-action is defined by the common conjugation by constant matrices.

**Remark 5** Let us also note that for the bundle $M_\Lambda$ the pullback $\pi^*M_\Lambda$ is the trivial bundle on $\mathbb{C}P^1$, where $\pi : \mathbb{C}P^1 \to \Sigma^{proj}$ is the normalization map. Obviously there are lots of bundles $F$ of degree zero on $\Sigma^{proj}$ such that $\pi^*F$ are not trivial bundles but some bundles of the type $\oplus_{k=1,...,r} O(t_k)$ such that $\sum t_k = 0$. So by no means we obtain all bundles on $\Sigma^{proj}$ as bundles $\tilde{M}_\Lambda$ for some $\Lambda$. But nevertheless the general stable and possibly semistable bundles satisfy the property that $\pi^*F$ is a trivial bundle on $\mathbb{C}P^1$, and so it is easy to see from our previous description of projective modules that the general semistable bundles can be obtained as the bundles $\tilde{M}_\Lambda$ for some $\Lambda$. 


2.4 Endomorphisms of $M_A$

2.4.1 Endomorphisms of the module $M_A$ over an affine chart

In this section we will describe endomorphisms of the bundles over the curves obtained by gluing distinct points $P_i$ together and for the cusp curve; the case of gluing points with multiplicities is more complicated and will be treated in \[20\].

Recall that the module $M_A$ over the algebra \{ $f \in \mathbb{C}[z] : \forall i, j \ f(P_i) = f(P_j)$ \}, is defined as the subset of the vector-valued polynomials $s(z)$ on $\mathbb{C}$, i.e. $s(z) \in \mathbb{C}[z]^r$, which satisfy the conditions: $s(P_i) = \Lambda_i s(P_i)$, $i = 2, ..., N$. It is natural to look for endomorphisms of $M_A$ as endomorphisms of $\mathbb{C}[z]^r$ which preserve the submodule $M_A$.

**Proposition 6** An endomorphism of the module $M_A$ can be described as a matrix polynomial $\Phi(z) : s(z) \mapsto \Phi(z)s(z)$, which satisfy the condition

$$\Phi(P_1) = \Lambda_1 \Phi(P_1) \Lambda_1^{-1}$$

(11)

The condition above implies that $\Phi(z)s(z)$ satisfies: $\Phi(P_1)s(P_1) = \Lambda_i \Phi(P_i)s(P_i)$, so $\Phi(z)s(z)$ is again an element of $M_A$ and $\Phi(z) : s(z) \mapsto \Phi(z)s(z)$ is an endomorphism of $M_A$.

**Example 7** In the abelian case (i.e. rank 1 modules over any manifold) the condition above is empty and any element $\Phi(z)$ defines an endomorphism i.e. the sheaf of endomorphisms of any rank 1 coherent sheaf is just $\mathcal{O}$ as in the regular case.

**Example 8** Consider the node (or double point) curve $\text{Spec}\{ f \in \mathbb{C}[z], f(1) = f(0) \}$. An endomorphisms of the module $M_A$ (which is defined as \{ $s \in \mathbb{C}[z]^r, s(1) = \Lambda s(0)$ \}), for some matrix $\Lambda$) is given by a matrix-valued polynomial $\Phi(z) = \Phi_0 + \Phi_1 z + \Phi_2 z^2 + ...$ such that $\Phi(1) = \Lambda \Phi(0) \Lambda^{-1}$. Hence $\Phi(z) = \Phi_0 + (\Lambda \Phi_0 \Lambda^{-1} - \Phi_0) z + z(z - 1) \hat{\Phi}(z)$, where $\hat{\Phi}(z)$ is arbitrary. When one considers the projectivization of our curve and the bundle corresponding to $M_A$ on it, we see that in order to be regular at infinity one must only consider constant endomorphisms $\Phi(z) = \Phi_0$. So in order to satisfy the condition $\Phi(1) = \Lambda \Phi(0) \Lambda^{-1}$ one must request that the matrix $\Phi_0$ commutes with $\Lambda$. As a corollary we see that there is only $r$-dimensional space of global endomorphisms for a general module $M_A$.

**Example 9** Consider the node (or double point) curve $\text{Spec}\{ f \in \mathbb{C}[z], f(A) = f(B) \}$. An endomorphism of the module $M_A$ is given by a matrix-valued polynomial $\Phi(z) = \Phi_0 + \Phi_1 z + \Phi_2 z^2 + ...$ such that $\Phi(A) = \Lambda \Phi(B) \Lambda^{-1}$. Hence

$$\Phi(z) = \Phi_0 + \Phi_1 z + (z - A)(z - B)(\hat{\Phi}(z)),$$

where $\Phi_0, \Phi_1$ must satisfy $\Phi_0 + A \Phi_1 = \Lambda (\Phi_0 + B \Phi_1) \Lambda^{-1}$ and $\hat{\Phi}(z)$ is arbitrary. It is more convenient to rewrite this expression as follows:

$$\Phi(z) = \Theta(z - A) - \Lambda \Theta \Lambda^{-1} (z - B) + (z - A)(z - B) \Theta(z),$$

where $\Theta$ is an arbitrary constant matrix. So global endomorphisms are given by $\Phi(z) = \Theta(B - A)$, with $\Theta$ commuting with $\Lambda$. 

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Example 10 Consider the triple point curve \( \text{Spec}\{ f \in \mathbb{C}[z], f(P_1) = f(P_2) = f(P_3) \} \).

So an endomorphism of module \( M_\Lambda \) is given by a matrix-valued polynomial \( \Phi(z) = \Phi_0 + \Phi_1 z + \Phi_2 z^2 + \ldots \) such that \( \Phi(P_1) = \Lambda_2 \Phi(P_2) \Lambda_2^{-1}, \Phi(P_1) = \Lambda_3 \Phi(P_3) \Lambda_3^{-1} \).

Hence

\[
\Phi(z) = \Phi(P_1) \frac{(z-P_2)(z-P_3)}{(P_1-P_2)(P_1-P_3)} + \Lambda_2^{-1} \Phi(P_1) \frac{(z-P_1)(z-P_3)}{(P_2-P_1)(P_2-P_3)} + \\
+ \Lambda_3^{-1} \Phi(P_1) \Lambda_3 \frac{(z-P_1)(z-P_2)}{(P_3-P_1)(P_3-P_2)} + (z-P_1)(z-P_2)(z-P_3) \Phi(z).
\]

Let us consider the cusp curve \( \text{Spec}\{ f \in \mathbb{C}[z] : f'(P) = 0; \} \), recall that the module \( M_\Lambda \) is defined as the subset of vector-valued polynomials \( s(z) \) on \( \mathbb{C} \) i.e. \( s(z) \in \mathbb{C}[z]^r \) which satisfy the conditions: \( s'(P) = \Lambda s(P) \).

Proposition 7 An endomorphism of the module \( M_\Lambda \) on a cusp can be described as a matrix polynomial \( \Phi(z) : s(z) \mapsto \Phi(z)s(z) \), which satisfy the condition

\[
\Phi'(P) = [\Lambda, \Phi(P)] \tag{12}
\]

The condition above is obviously equivalent to \( (\Phi(P)s(P))' = \Lambda \Phi(P) \) which means that \( \Phi \) is really an endomorphism.

2.4.2 Endomorphisms of the bundle \( M_\Lambda \) over the projectivization

Consider the projective curve \( \Sigma_D^{\text{proj}} \) which is obtained by adding one smooth point \( \infty \) to the curve \( \text{Spec}\{ f \in \mathbb{C}[z] : \forall i, j \, f(P_i) = f(P_j); f'(P_i) = \ldots = f^{n_i-1}(P_i) = 0 \} \). An endomorphism of the bundle \( M_\Lambda \) is given by endomorphisms of the corresponding modules over each chart. So an endomorphism of the bundle \( M_\Lambda \) is an endomorphisms of the module \( M_\Lambda \) over the affine chart which is regular at infinity. In order to be regular at infinity an endomorphism \( \Phi(z) \) must be constant \( \Phi(z) = \Phi_0 \), on the other hand an endomorphism must satisfy the conditions 11, 12, so we see that:

Proposition 8 A global endomorphism of the bundle \( M_\Lambda \) over \( \Sigma_D^{\text{proj}} \) is given by a constant matrix \( \Phi_0 \) which commute with all \( \Lambda_i, \Lambda_i' \).

Remark 6 We see that, if the genus of the curve is greater than 1, for the general bundle the endomorphisms are only scalar matrices; this fact reflects the stability of general bundles. In the case when the genus equals one (node and cusp curves), the general bundle has an \( r \)-dimensional space of endomorphisms, which corresponds to the fact that on genus one curves the general bundle is a sum of linear bundles.
2.5 Description of $\text{End}(M_\Lambda) \otimes K$

2.5.1 Examples of node and cusp curves

Example 11 Consider the node curve $\text{Spec}\{ f \in \mathbb{C}[z], f(A) = f(B) \}$. An endomorphism of the module $M_\Lambda$ is given (see example 9) by

$$\Theta(z) = \Theta(z - A) - \Lambda \Theta^{-1}(z - B) + (z - A)(z - B)\tilde{\Theta}(z),$$

where $\Theta, \tilde{\Theta}(z)$ are arbitrary. The sections of the dualizing module are given by

$$\omega = \frac{c dz}{z - A} + \frac{c dz}{z - B} + \text{holomorphic in } z.$$ 

So the sections of $\text{End}(M_\Lambda) \otimes K$ can be described as

$$\Phi(z) = \left( B - A \right) \left( \frac{\Theta dz}{z - A} - \frac{\Theta dz}{z - B} \right) + \text{holomorphic in } z. \quad (13)$$

Hence the global sections $H^0(\text{End}(M_\Lambda) \otimes K)$ over the projectivization are $\Phi(z)$’s which are regular at infinity. The condition that $\Phi(z)$ has no pole of order greater than 2 gives that there is no holomorphic term in the expression (13). The condition that the residue at infinity is zero is equivalent to $\Lambda \Theta^{-1} - \Theta = 0$. Hence the global sections are:

$$\Phi(z) = \left( B - A \right) \left( \frac{\Theta dz}{z - A} - \frac{\Theta dz}{z - B} \right)$$

where $\Lambda \Theta = \Theta \Lambda$. We can see in this case that

$$H^0(\text{End}(M_\Lambda) \otimes K) = H^0(\text{End}(M_\Lambda)) \otimes H^0(K).$$

Example 12 Consider the cusp curve $\text{Spec}\{ f \in \mathbb{C}[z], f'(P) = 0 \}$. An endomorphisms of the module $M_\Lambda$ is given by

$$\Theta(z) = \Theta + [\Lambda, \Theta](z - P) + (z - P)^2\tilde{\Theta}(z)$$

where $\Theta, \tilde{\Theta}(z)$ are arbitrary. The sections of the canonical module are given by

$$\frac{c_1 dz}{(z - P)^2} + c(z)dz$$

where $c(z)$ is holomorphic. So the sections of $\text{End}(M_\Lambda) \otimes K$ can be described as

$$\Phi(z) = \left( \Theta + [\Lambda, \Theta](z - P) \right) \left( \frac{dz}{(z - P)^2} + \text{holomorphic in } z \right)$$

The global sections $H^0(\text{End}(M_\Lambda) \otimes K)$ are $\Phi(z)$’s which are regular at infinity. Hence $[\Theta, \Lambda] = 0$, and the global sections are:

$$\Phi(z) = \left( B - A \right) \frac{\Theta dz}{(z - P)^2}$$

where $\Lambda \Theta = \Theta \Lambda$. Here we have the same observation as in the node case

$$H^0(\text{End}(M_\Lambda) \otimes K) = H^0(\text{End}(M_\Lambda)) \otimes H^0(K).$$
2.5.2 Curves obtained by gluing points without multiplicities

Consider the curve $\Sigma_{\text{proj}}$ which is the result of gluing $N$ distinct points on $\mathbb{C}P^1$, (we consider the case of gluing without multiplicities). Recall that the affine part of $\Sigma_{\text{proj}}$ is given by $\text{Spec}\{f(z) \in \mathbb{C}[z] : \forall i, j \ f(P_i) = f(P_j)\}$. The bundle $M_\Lambda$ corresponds to the module $\{s(z) \in \mathbb{C}[z]^r: \forall i = 2, \ldots, N s(P_i) = \Lambda_i s(P_i)\}$ over the affine part.

**Proposition 9** A sections of $\text{End}(M_\Lambda) \otimes K$ over the chart without infinity can be described as a matrix polynomial:

$$\Phi(z) = \sum_{i=2}^{N} -\Lambda_i \Phi_i \Lambda_i^{-1} \frac{dz}{z - P_i} + \sum_{i=2}^{N} \frac{\Phi_i}{z - P_i} dz + \text{holomorphic in } z \text{ terms}, \quad (14)$$

where $\Phi_i$ are arbitrary matrices.

The global sections $H^0(\Sigma_{\text{proj}}, \text{End}(M_\Lambda) \otimes K)$ are described by the formula

$$\Phi(z) = \sum_{i=1}^{N} \frac{\Phi_i}{z - P_i} dz \quad (15)$$

imposing the conditions

$$\sum_{i=1}^{N} \Phi_i = 0; \quad \sum_{i=2}^{N} -\Lambda_i \Phi_i \Lambda_i^{-1} + \Phi_i = 0. \quad (16)$$

**Proof.** The claim about the section over the affine chart is trivial. Indeed, over the affine part $\Gamma(\text{End}(M_\Lambda) \otimes K) = \Gamma(\text{End}(M_\Lambda)) \otimes \Gamma(K)$. It is the straightforward consequence of the triviality of $K$ over the affine chart even in the algebraic setup. So we can represent the section as

$$\Phi(z) = \sum_{i=2}^{N} \Phi_i(z) \otimes \omega_i, \quad (17)$$

where

$$\omega_i = \frac{dz}{(z - P_i)(z - P_i)}$$

provide the basis of holomorphic differentials on $\Sigma_{\text{proj}}$ and $\Phi_i(z)$ are the sections of $\text{End}(M_\Lambda)$ over the affine part, which means that $\Phi_i(P_1) = \Lambda_i \Phi_i(P_k) \Lambda_k^{-1}$. Calculating the residues of the expression for $\Phi(z)$ one obtains formula (14) where $\Phi_i = \Phi_i(P_i)$.

To prove the second part of the proposition we must realize that global sections correspond to sections over the affine chart which can be continued to regular functions at infinity. The expression (14) is regular at infinity if it has no term holomorphic in $z$ and if its residue at infinity is zero. This imposes the additional condition

$$\sum_{i=1}^{N} \Phi_i = 0.$$

Conversely, for every $\{\Phi_i; i = 1, \ldots, N\}$ subject to the conditions (16) there exist a module $M_\Lambda$ endomorphism $\{\Phi_i(z); i = 2, \ldots, N\}$ such that

$$\Phi(z) = \sum_{i=1}^{N} \frac{\Phi_i}{z - P_i} \otimes dz = \sum_{i=2}^{N} \Phi_i(z) \otimes \omega_i.$$
To prove it let us take local endomorphisms $\Phi'_i(z)$ such that $\Phi'_i(P_i) = (P_i - P_1)\Phi_i$, $i = 2, \ldots, N$. The residue at $P_i$ of

$$\Phi'(z) = \sum_{i=2}^{N} \Phi'_i(z) \otimes \omega_i$$

is $Res_{P_i} \Phi' = - \sum_{i=2}^{N} \Lambda_i \Phi_i \Lambda_i^{-1} = \Phi_i$. The difference $\Phi(z) - \Phi(z)'$ is regular at the affine chart so it has no residue at $\infty$. Consider now the expression

$$\tilde{\Phi}(z) \otimes dz = \prod_{i=1}^{N} (z - P_i) \tilde{\Phi}(z) \otimes \frac{dz}{\prod_{i=1}^{N} (z - P_i)}$$

which is also the section of $\text{End}(M_\Lambda \otimes K)$ over the affine chart for an arbitrary matrix function $\tilde{\Phi}(z)$. It has arbitrary poles at $\infty$ of order greater then 2 and has no residue. One can find a function $\tilde{\Phi}(z)$ such that $\tilde{\Phi}(z) = \Phi'(z) + \tilde{\Phi}(z) \otimes dz$ but the latter is also an expression of the type (17), which ends the proof.

\section{2.6 Description of $H^1(\text{End}(M_\Lambda))$}

\subsection{2.6.1 Curves obtained by gluing points without multiplicities}

Consider the curve $\Sigma^{\text{proj}}$ which is the result of gluing $N$ distinct points on $\mathbb{C}P^1$, (we consider the case of gluing without multiplicities). Recall that the affine part of $\Sigma^{\text{proj}}$ is given by $\text{Spec}\{f(z) \in \mathbb{C}[z] : \forall i, j f(P_i) = f(P_j)\}$, the bundle $M_\Lambda$ corresponds to the module which, over the affine part, is described as $\{s(z) \in \mathbb{C}[z]^r : \forall i = 2, ..., N s(P_i) = \Lambda_i s(P_i)\}$.

**Proposition 10** The space of matrix polynomials $\chi(z) = \sum_{i=0}^{N-1} \chi_i z^i$ maps surjectively to $H^1(\text{End}(M_\Lambda))$. The kernel of this map is the sum of two linear subspaces in the space of matrix polynomials $\chi(z)$: the first space is the space of constant polynomials $\chi(z) = \chi_0$ and the second space is made of matrix polynomials which satisfy the condition: $\chi(P_i) = \Lambda_i \chi(P_i) \Lambda_i^{-1}$, for $i = 2, ..., N$. (Let us mention that the intersection of these two subspaces is precisely $H^0(\text{End}(M_\Lambda))$ and that the second subspace consists of $\chi(z)$ which gives endomorphism of the module $M_\Lambda$ over the affine chart without infinity.)

**Proof.** The proposition is quite obvious from the point of view of the Čech’s description of $H^1(\text{End}(M_\Lambda))$. Let us cover our singular curve by the charts $U_P = \Sigma \setminus \infty$, $U_\infty = \Sigma \setminus P$, where $P$ is the singular point. Then $H^1(\text{End}(M_\Lambda)) = \text{End}(M_\Lambda)(U_P \cap U_\infty)/\text{End}(M_\Lambda)(U_P) \oplus \text{End}(M_\Lambda)(U_\infty)$. As we know from proposition 8 $\text{End}(M_\Lambda)(U_P)$ are precisely polynomials $\chi(z)$ which satisfy $\chi(P_i) = \Lambda_i \chi(P_i) \Lambda_i^{-1}$. So we obviously come to the desired conclusion.

**Remark 7** From the proposition above we see that the Riemann-Roch theorem for the bundle $M_\Lambda$ becomes obvious. The description of $H^1(\text{End}(M_\Lambda))$ given above shares similarities with the description done in Krichever’s and adelic approaches (see e.g. [21]). These descriptions hypothetically can be used for the proof of the general Riemann-Roch theorem.
Example 13 In the abelian case (i.e. when $M_\Lambda$ is a rank 1 module) for any $\Lambda$ it is known that $M_\Lambda$ is just $O$. So in the abelian case the proposition claims that $H^1(O)$ is the factor space of the space of all polynomials $\sum_{i=0}^{N-1} \chi_i z^i$ by the space of polynomials $\chi(z)$ which satisfy the conditions $\chi(P_i) = \chi(P_i)$. For example this can be seen from the exact sequence: $O \rightarrow O^{\text{norm}} \rightarrow \mathbb{C}_P$ which gives: $H^1(O) = H^0(\mathbb{C}_P)$.

It is well-known that the vector space $H^1(\text{End}(M_\Lambda))$ is the tangent space to deformations of $M_\Lambda$ as an algebraic vector bundle, on the other hand we know that all vector bundles are given by $\Lambda$. Our goal is to determine $\Delta_\Lambda$ corresponding to the element $\chi(z) = \chi_i z^i$.

Proposition 11 The matrix polynomial $\chi(z) = \sum_{i=0}^{N-1} \chi_i z^i$, (which is considered as an element of $H^1(\text{End}(M_\Lambda))$ due to the proposition above), gives the following deformation of $\Lambda_i$:

$$\delta\chi(z) \Lambda_i = \chi(P_i)\Lambda_i - \Lambda_i\chi(P_i).$$

Remark 8 One knows that $H^2(\text{Coherent sheaves}) = 0$ for the case of curves and so by the general theory the map from $H^1(\text{End}(M_\Lambda))$ to the tangent space of deformations of the bundle $M_\Lambda$ is a bijection. So the formula above can be taken as a definition of the map from the space of matrix polynomials $\chi(z) = \sum_i \chi_i z^i$ to the space $H^1(\text{End}(M_\Lambda))$. It means that we can (by definition) associate with the matrix polynomial $\chi(z) = \sum_i \chi_i z^i$ an element of $H^1(\text{End}(M_\Lambda))$ which deforms the bundle $M_\Lambda$ by the formula $\Lambda_i \mapsto \Lambda_i + \chi(P_i)\Lambda_i - \Lambda_i\chi(P_i)$. What must be proved after such a definition is how to describe the Serre’s pairing between $H^0(\text{End}(M_\Lambda) \otimes K)$ and $H^1(\text{End}(M_\Lambda))$. We describe Serre’s pairing in proposition 12.

Corollary 3 The matrix polynomial $\chi(z) = \sum_j \chi_j z^j$, which for $i = 2, ..., N$ satisfies the condition $\chi(P_i) = \Lambda_i\chi(P_i)\Lambda_i^{-1}$ does not change $\Lambda_i$. This fact is in full agreement with proposition 10 which says that such a polynomial gives a zero element in $H^1(\text{End}(M_\Lambda))$.

Corollary 4 The matrix polynomial $\chi(z) = \chi_0$, changes $\Lambda$ to its conjugated by a constant matrix, so it gives the same vector bundle. This fact is in full agreement with proposition 10 which says that such a polynomial gives a zero element in $H^1(\text{End}(M_\Lambda))$.

Proof. This proposition can be demonstrated as follows: consider an element $1 + \delta\chi(z)$, where $\delta^2 = 0$, it is an infinitesimal automorphism of the module $\mathbb{C}[z]^r$ corresponding to the endomorphism $\chi(z)$. Having such an automorphism it is clear how to deform the module $M_\Lambda$: the new module is the set of elements $(1 + \delta\chi(z))s(z)$, where $s(z)$ is an element of $M_\Lambda$. The elements of the type $\tilde{s}(z) = (1 + \delta\chi(z))s(z)$ satisfy the condition:

$$\tilde{s}(P_i) = (1 + \delta\chi(P_i))\Lambda_i(1 + \delta\chi(P_i))^{-1}\tilde{s}(P_i)$$

for $i = 2, ..., N$. Hence

$$\tilde{s}(P_i) = (\Lambda_i + \delta\chi(P_i)\Lambda_i - \Lambda_i\delta\chi(P_i))\tilde{s}(P_i).$$

We see that the new module is the module $M_{\Lambda_i + \chi(P_i)\Lambda_i - \Lambda_i\chi(P_i)}$. ■
Our reasoning has been the following: the module $M_A$ is embedded in the module $\mathbb{C}[z]^r$, this module cannot be deformed, so the deformations of $M_A$ are governed only by the deformations of the embedding $M_A \to \mathbb{C}[z]^r$. The elements of $H^1(\text{End}(M_A))$ have been identified with some elements of $\text{End}(\mathbb{C}[z]^r)$ due to the fact that $\mathbb{C}[z]^r \to \mathbb{C}^p$ is a resolution of $M_A$. The elements of $\text{End}(\mathbb{C}[z]^r)$ act on the embeddings of $M_A \to \mathbb{C}[z]^r$. But the formulas in the paragraph above are more convincing than any words.

There exists another way to demonstrate the proposition above. It is more transparent at the level of ideas but much longer at the level of formulas. Let us use the Čech’s description of cohomologies of sheaves. We consider the covering of our projective curve consisting of two charts: the first is everything except the singular point, the second is not really an open set but a limit of the open sets - infinitesimal neighborhood of the singular point i.e. $\text{Spec}\{f \in \mathbb{C}(z), f$ is regular at $P_i$ and $f(P_i) = f(P_j)\}$. It is convenient to consider such an infinitesimal neighborhood. Only in such a neighborhood of the singular point all modules become trivial because it is the spectrum of a local ring. So any module on a singular curve can be given by gluing two trivial modules by the gluing function on the intersection of two charts. In this case the intersection is the “general point” i.e. $\text{Spec}(\mathbb{C}(z))$. So the first task is to describe the module $M_A$ by the gluing function. After that it is obvious how to calculate which deformation corresponds to an element of $H^1(\text{End}(M_A))$. We represent an element of $H^1(\text{End}(M_A))$ as an element $\chi \in \text{End}(M_A)$ on the intersection of the two charts and one must simply multiply the gluing function by the element $1 + \delta \chi$. So we obtain a new gluing function. The new bundle can be again represented in the form $M_A$, so we obtain the deformation of $M_A$ and this construction gives the same results as above.

Let us give examples illustrating propositions 10 and 11.

**Example 14** Consider the node curve with the affine part $\text{Spec}\{f(A) = f(B)\}$, where $A, B \in \mathbb{C}$. Consider the matrix polynomial

$$\chi(z) = \chi_0 + \chi_1 z + (z - A)(z - B)\bar{\chi}(z).$$

According to proposition 11 it acts on $\Lambda$ by the formula $\delta \chi \Lambda = \chi(A)\Lambda - \Lambda \chi(B)$. The part $(z - A)(z - B)\bar{\chi}(z)$ does not act on $\Lambda$. So we can consider only the linear part $\chi(z) = \chi_0 + \chi_1 z$. According to proposition 10 the $H^1(\text{End}(M_A))$ is the factor of the space $\chi_0 + \chi_1 z$ by the sum of the spaces $\chi(z) = \chi_0$ and $\chi(z) = \Theta(z - A) - \Lambda \Theta \Lambda^{-1}(z - B)$, where $\chi_0, \Theta$ are arbitrary matrices. (It would be nice to have an explicit parameterization of the orthogonal complement with respect to the Killing form to the sum of these two subspaces and to generalize it to the case of schematic points). The intersection of these two subspaces is the subspace of $\chi(z) = \chi_0$ such that $\chi_0$ commutes with $\Lambda$. This intersection is $H^0(\text{End}(M_A))$. So we see that $\dim H^1(\text{End}(M_A)) = \dim H^0(\text{End}(M_A))$.

This observation coincides with the calculation done from the Riemann-Roch theorem:

$$\dim H^0(\text{End}(M_A)) - \dim H^1(\text{End}(M_A)) = \deg(\text{End}(M_A)) - n^2(1 - \dim H^1(\mathcal{O})) = 0.$$
Corollary 5 Consider the matrix polynomial \( \chi(z) = \sum_k \chi_k z^k \) which satisfies the conditions \( \chi(P_i) = \Lambda_i \chi(P_i) \Lambda_i^{-1} \) for \( i = 2, \ldots, N \). The Serre's pairing (given by formula [19]) between \( \chi(z) \) and an arbitrary \( \Phi(z) \in H^0(\text{End}(M_\Lambda) \otimes K) \) is identically zero. This fact is in full agreement with proposition [18] which says that such a polynomial gives a zero element in \( H^1(\text{End}(M_\Lambda)) \).

Corollary 6 Consider the matrix polynomial \( \chi(z) = \chi_0 \). The Serre's pairing given by formula [19] between \( \chi(z) \) and an arbitrary \( \Phi(z) \in H^0(\text{End}(M_\Lambda) \otimes K) \) is identically zero. This fact is in full agreement with proposition [18] which says that such a polynomial gives a zero element in \( H^1(\text{End}(M_\Lambda)) \).

This proposition follows immediately from the general description of Serre's pairing given in proposition [3]. To prove the corollaries we use proposition [9] in order to represent \( \tilde{\Phi}(z) \) as a matrix polynomial:

\[
\sum_{i=2,\ldots,N} -\Lambda_i \Phi_i \Lambda_i^{-1} dz + \sum_{i=2,\ldots,N} \frac{\Phi_i}{z - P_i} dz
\]

for some \( \Phi_i \).

Remark 9 To prove the second corollary we must also use the condition

\[
\sum_{i=2,\ldots,N} -\Lambda_i \Phi_i \Lambda_i^{-1} + \Phi_i = 0.
\]

The first corollary does not use this condition for \( \Phi_i \) and is true for all \( \Phi(z) \) represented in the form

\[
\Phi(z) = \sum_{i=2,\ldots,N} -\Lambda_i \Phi_i \Lambda_i^{-1} dz + \sum_{i=2,\ldots,N} \frac{\Phi_i}{z - P_i} dz
\]

with arbitrary \( \Phi_i \).

2.6.2 The cusp curve

Consider the cusp curve \( \Sigma_{\text{proj}} \). Recall that the affine part of \( \Sigma_{\text{proj}} \) is given by \( \text{Spec}\{ f(z) \in \mathbb{C}[z] : f'(P) = 0 \} \), the bundle \( M_\Lambda \) corresponds to the module which is described as \( \{ s(z) \in \mathbb{C}[z]^r : s'(P) = \Lambda s(P) \} \) over the affine part.

Proposition 13 The space of matrix polynomials \( \chi(z) = \chi_0 + \chi_1(z - P) \) maps surjectively to \( H^1(\text{End}(M_\Lambda)) \). The kernel of this map consists of the sum of two linear subspaces in the space of matrix polynomials \( \chi(z) \): the first space is the space of constant polynomials \( \chi(z) = \chi_0 \) and the second space consists of matrix polynomials which satisfy the condition: \( \chi_1 = [\Lambda, \chi_0] \) for \( i = 2, \ldots, N \). (Let us mention that the intersection of the two subspaces is precisely \( H^0(\text{End}(M_\Lambda)) \) and the second subspace consists of \( \chi(z) \) which gives endomorphisms of the module \( M_\Lambda \) over the affine chart without infinity.)

Proof. This proposition is quite obvious and can be demonstrated in the Čech approach in the same way as proposition [19].
Proposition 14  The matrix polynomial \( \chi(z) = \chi_0 + \chi_1(z - P) \) which is considered as an element of \( H^1(\text{End}(M_\Lambda)) \) due to the proposition above gives the following deformation of \( \Lambda \):

\[
\delta \chi(z) \Lambda = \chi_1 + [\chi_0, \Lambda].
\]

Proof.  This proposition can be demonstrated in the same way as proposition 11. Let us only comment on the key step. Consider an element \( 1 + \delta \chi(z) \), where \( \delta^2 = 0 \), it is an infinitesimal automorphism of the module \( \mathbb{C}[z]^r \) corresponding to the endomorphism \( \chi(z) \). Having such an automorphism it is clear how to deform the module \( M_\Lambda \): the new module is the set of elements \( (1 + \delta \chi(z))s(z) \) where \( s(z) \) is an element of \( M_\Lambda \). The elements of the type \( \tilde{s}(z) = (1 + \delta \chi(z))s(z) \) satisfy the condition:

\[
\tilde{s}'(P) = (1 + \delta \chi(P))'s(P) + (1 + \delta \chi(P))s'(P)
= \delta \chi(P)'(1 - \delta \chi(P))\tilde{s}(P) + (1 + \delta \chi(P))\Lambda(1 - \delta \chi(P))\tilde{s}(P)
= (\Lambda + \delta \chi(P)' + [\delta \chi(P), \Lambda])\tilde{s}(P).
\]

Hence one can see that the new module is the module

\[
M_{\Lambda + \delta \chi(P)' + [\delta \chi(P), \Lambda]}.
\]

The Serre’s pairing can be described exactly in the same way as in subsection above.

2.7 Canonical 1-form on the cotangent bundle to the moduli space of vector bundles in terms of \( \Phi, \Lambda \)

2.7.1 Curves obtained by gluing points without multiplicities

Consider the curve \( \Sigma^{\text{proj}} \) which is result of gluing \( N \) distinct points on \( \mathbb{C}P^1 \) without multiplicities. Recall that the affine part of \( \Sigma^{\text{proj}} \) is given by \( \text{Spec}\{ f(z) \in \mathbb{C}[z] : \forall i, j \ f(P_i) = f(P_j) \} \), the bundle \( M_\Lambda \) over the affine part corresponds to the module

\[
\{ s(z) \in \mathbb{C}[z]^r : \forall i = 2, \ldots, N \ s(P_i) = \Lambda_i s(P_i) \}.
\]

It is well-known that \( H^1(\text{End}(M_\Lambda)) \) is the tangent space to the moduli space of vector bundles at the point \( M_\Lambda \) and \( H^0(\text{End}(M_\Lambda) \otimes \mathcal{K}) \) is the dual space to \( H^1(\text{End}(M_\Lambda)) \) so it is the cotangent space to the moduli space of vector bundles at the same point. According to proposition 9 the sections of \( H^0(\text{End}(M_\Lambda) \otimes \mathcal{K}) \) can be described as:

\[
\Phi(z) = \sum_{i=2,\ldots,N} \Lambda_i \Phi_i \Lambda_i^{-1} \frac{dz}{z - P_i} - \sum_{i=2,\ldots,N} \frac{\Phi_i}{z - P_i} \frac{dz}{z - P_i}
\]

where the matrices \( \Phi_i \) satisfy \( \sum_{i=2,\ldots,N} -\Lambda_i \Phi_i \Lambda_i^{-1} + \Phi_i = 0 \).

This section is devoted to proving the claim which, in expert language, can be formulated as follows:
Claim: The canonical 1-form on the cotangent bundle to the moduli space of vector bundles on the curve $\Sigma^\text{proj}$ in terms of $\Lambda_i, \Phi_i$ can be written in the form:

$$- \sum_{i=2,\ldots,N} \text{Tr} \Phi_i \Lambda_i^{-1} d\Lambda_i$$

(22)

Remark 10 In the abelian case (the case of moduli space of line bundles) the claim above is an exact proposition - $\Lambda_i, \Phi_i$ are “honest” coordinates on the cotangent to the moduli space of line bundles, and the expression above has a clear meaning. In the non-abelian case there is a subtlety due to the fact that the moduli space of vector bundles is the factor of the space of matrices $\Lambda_i$ by conjugation, but the 1-form above is written on the space of $\Lambda_i, \Phi_i$ without factorization. So one must explain what the expression $\Phi_i \Lambda_i^{-1} d\Lambda_i$ means, if $\Lambda_i$ are defined only up to conjugation. It is obvious for the expert, nevertheless, despite it is a bit long. Let us give a correct formulation of the proposition, do not claiming that everybody can translate from the expert's slang to the precise formulation.

Consider the space of matrices $\Lambda_i, \Phi_i$ with the 1-form $\sum_i \Phi_i \Lambda_i^{-1} d\Lambda_i$. Consider the subspace defined by the equation $\sum_{i=2}^N (\Lambda_i \Phi_i \Lambda_i^{-1} - \Phi_i) = 0$. Consider the map $p$ of this subspace to the cotangent to the moduli space of bundles given by $p : (\Lambda_i, \Phi_i) \mapsto (M_\Lambda, \Phi(z))$, where $\Phi(z)$ is defined by formula (21). Let us define the 1-form on the cotangent to the moduli space as follows: restrict the 1-form $\sum_i \Phi_i \Lambda_i^{-1} d\Lambda_i$ to the submanifold $\sum_{i=2}^N (\Lambda_i \Phi_i \Lambda_i^{-1} - \Phi_i) = 0$; check that the 1-form equals zero on the tangent vectors to the fiber of the map $p$; hence the 1-form $\sum_i \Phi_i \Lambda_i^{-1} d\Lambda_i$ can be pushed down to the image of $p$ i.e. to the cotangent to the moduli space of vector bundles. We claim that the result of the push-down of $\sum_i \Phi_i \Lambda_i^{-1} d\Lambda_i$ under the map $p$ coincides with the canonical 1-form on the cotangent bundle to the moduli space. The procedure above is actually a hamiltonian reduction. Now we proceed with formulating the exact result.

Let us consider the cotangent space to the space of matrices $\Lambda_i$ which is

$$\mathcal{T}^*(GL_r^{\times (N-1)}) = \times_{i=2}^N \mathcal{T}^*(GL_r)$$

with the canonical invariant symplectic form on it

$$\omega_1 = \sum_{i=2}^N \text{Tr} d(\Phi_i \Lambda_i^{-1}) \wedge d(\Lambda_i),$$

where $\Phi_i$ are coordinates on $\mathcal{T}^*(GL_r^{\times (N-1)})$ that $\Phi_i \Lambda_i^{-1}$ are coordinates on $\mathcal{T}^*(GL_r^{\times (N-1)})$ canonically conjugated to the coordinates $\Lambda_i$ on $(GL_r^{\times (N-1)})$. (When we say that the matrix $M$ is a “coordinate” we mean that each of its matrix elements is a coordinate). This symplectic form is invariant by the natural action of $GL_r$:

$$g : \Lambda_i \mapsto g^{-1} \Lambda_i g; \quad \Phi_i \mapsto g^{-1} \Phi_i g.$$ The moment map of this action $\mu : \mathcal{T}^*(GL_r^{\times (N-1)}) \to gl_r^*$ can be calculated by using the 1-form $\lambda_1 = \sum_{i=2}^N \text{Tr} \Phi_i \Lambda_i^{-1} d(\Lambda_i)$ due to its invariance as follows:

$$(i(\xi) \lambda_1)[P] = \text{Tr} [\mu[P] \ast \xi)$$
where $P \in T^*(GL_r^{\times(N-1)})$ and we use the identification of the vector spaces $gl^*_r$ and $gl_r$ by means of $< A, B > = TrAB$. Here $\xi$ is the vector field corresponding to the infinitesimal action of $gl_r$ which can be written as: $i(\xi)d\Lambda_i = \xi \Lambda_i - \Lambda_i \xi$. Finally one obtains

$$i(\xi)\lambda_1 = \sum_{i=2}^{N} Tr\Phi_i\Lambda_i^{-1}(\xi \Lambda_i - \Lambda_i \xi)$$

such that the moment map as an element of $gl^*_r$ coincides with

$$\mu[\Lambda_i, \Phi_i] = \sum_{i=2}^{N} (\Lambda_i \Phi_i \Lambda_i^{-1} - \Phi_i).$$

(23)

**Remark 11** We see that the holomorphity condition for $\Phi(z)$ (see proposition 14) coincides with the condition that the moment map equals zero.

Let us consider the hamiltonian reduction for the action above corresponding to the zero moment level

$$T^*(GL_r^{\times(N-1)})//GL_r = \mu^{-1}(0)/GL_r.$$

This space is endowed with the canonical symplectic structure which coincides with the symplectic structure on the cotangent bundle

$$T^*(GL_r^{\times(N-1)}/GL_r).$$

For pedagogical reasons we recall here the proof of the following well known result:

**Lemma 1** Let the group $G$ act on the manifold $M$; then there is an induced action of $G$ on $T^*(M)$ which is the pullback of differential forms. Then the canonical 1-form $\lambda$ on $T^*(M)$ is invariant for this action and the symplectic manifold obtained by the hamiltonian reduction $T^*(M)//G$ with zero moment level is canonically isomorphic to the symplectic space $T^*(M//G)$ with the canonical symplectic form.

**Proof.** Let us describe the cotangent space to $M//G$. We denote by $\tau$ the factorization map $\tau : M \rightarrow M//G$. The tangent space at the point $m \in M//G$ is by definition

$$T_m(M//G) = T_{\tau^{-1}(m)}/\{Im(G)\},$$

where $Im(G)$ is the subspace in $T_{\tau^{-1}(m)}$ which corresponds to the generators of the action of $G$. Hence cotangent vectors to $M//G$ are described by 1-forms such that they vanishes on vector fields coming from the action of the group $G$, but these are precisely 1-forms $\Theta$ such that the pair $(m, \Theta)$ lies in the zero moment level because $< \Theta(x)|\xi_g(x) >= < \lambda(x, \Theta)|\xi_g(x, \Theta) >= < \mu[(x, \Theta)]|g >$, where $\xi_g$ is the canonical lifting of $\xi_g$ from $M$ to $T^*M$, $\xi_g$ is the vector field on $M$ generating the infinitesimal action of the group $G$.

The fact that the canonical symplectic form and the canonical 1-form on $T^*M$ reduce to the canonical symplectic form and the canonical 1-form on $T^*(M//G)$ is quite tautological. It is enough to note that the 1-form $\lambda$ reduced to the zero moment level is correctly defined on the factor space $T^*(M//G)$. Indeed, its value on the vector field $\xi$ at the point $m, \theta$ is defined by $< \lambda, d\pi(\xi) >$ and does not depend on the choice of the representative of an element of $T_{\tau^{-1}(m)}/\{Im(G)\}$. 

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Returning to our specific case we denote the corresponding symplectic form on $T^*(GL_r^{(N-1)})/GL_r$ by $\omega_2$. Due to proposition \ref{prop} we have the identification of spaces

$$T^*(GL_r^{(N-1)})/GL_r \overset{p}{\longrightarrow} T^*\mathcal{M}$$

constructed as follows: for the set of matrices $\{\Lambda_i\}$ one constructs the bundle $M_\Lambda$ and the cotangent vector $\Phi(z)$, which can be expressed from the matrices $\{\Phi_i\}$ by the usual formula (21). This identification is correct by virtue of formula (23) for the moment map. Different choices for the representative $\Lambda_i$ correspond to conjugated matrices $\tilde{\Lambda}_i = g^{-1}\Lambda_ig$, $\tilde{\Phi}_i = g^{-1}\Phi_ig$.

Remark 12 In fact the map $p$ is not only the identification of spaces, it is also a symplectomorphism considering the canonical symplectic structures on these spaces. Moreover, we will show below that this map coincides with the composition of the canonical identification of the spaces $T^*(GL_r^{(N-1)})/GL_r$ with $T^*(GL_r^{(N-1)}/GL_r)$, described in the lemma above and of the identification $p$ of $T^*(GL_r^{(N-1)}/GL_r)$ with $T^*\mathcal{M}$. The significance of this claim is the following: we reduce the problem of describing the symplectic form on $T^*\mathcal{M}$ to the much more simple question - the symplectic form on the cotangent bundle to the group (in our case the group is $GL_r$). This non trivial claim in our description is the definition of the symplectic structure in Beauville’s construction \cite{19}, here we deduce it from the canonical symplectic form on the cotangent bundle.

**Proposition 15** The map $p$ is a symplectomorphism, i.e.

$$p^*(\omega) = \omega_2$$

where $\omega$ is the canonical symplectic form on $T^*\mathcal{M}$.

As a corollary we see the way to calculate the value of the canonical 1-form on the given tangent vector in terms of $\Lambda_i, \Phi_i$:

**Corollary 7** Let $\alpha$ be the canonical 1-form on $T^*\mathcal{M}$. Its value on the vector $\xi \in T_{(\Lambda, \Phi)}^*(T^*\mathcal{M})$ equals to :

$$- \sum_{i=2,\ldots,N} Tr\Phi_i\Lambda_i^{-1}\tilde{\xi}_i,$$

where the point $\Lambda$ in the moduli space of vector bundles is given by the vector bundle $M_\Lambda$, corresponding to the set of matrices $\Lambda_i$; the covector $\Phi$ corresponds to $\Phi(z)$ given by formula (21). The matrices $\tilde{\xi}_i$, considered as a tangent vector to the $(\text{Mat}_r^{(N-1)})$ at the point $\Lambda_i, i = 2,\ldots,N$ are defined by the condition that the projection of such a vector onto the space of vector bundles corresponding to the map $\Lambda_i \mapsto M_\Lambda$ coincides with $\pi_*(\xi)$, where $\pi$ is the canonical projection from the cotangent bundle to the moduli space to the moduli space itself.

**Proof.** The most simple way to prove this proposition consists in calculating the pullback of $\omega$ on $T^*\mathcal{M}$ to the space $\mu^{-1}(0) \subset T^*(\times_{i=2}^N GL_r)$. Let us denote by $\pi_1$ the map $\pi_1 : \mu^{-1}(0) \to T^*\mathcal{M}$ which is the composition of the factorization map and the identification $p$. 

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Due to the invariance of the 1-form $\lambda$ it is sufficient to consider its pullback. By definition the value of the “pullbacked” 1-form on a vector $\xi \in T(\mu^{-1}(0))$ is $\pi_1^*(\lambda)(\xi) = \lambda(d\pi_1 \xi)$. Let us take the vector field $\xi$ and the deformation $\delta_\xi \Lambda_i = i(\xi)d\Lambda_i$. Further we find $\chi_i$ satisfying the equations:

$$\delta_\xi \Lambda_i = \chi_i \Lambda_i - \Lambda_i \chi_i \quad (25)$$

and chose some $\chi(z)$ such that $\chi(P_i) = \chi_i$, $i = 1, \ldots, N$. There is an ambiguity in equation (25), for every $\chi_1$ one can find $\chi_i$. In fact the choice of $\chi_1$ does not matter: $\chi(z)$ constructed with different $\chi_1$ lie in the same class in $H^1(End(E))$ (recall that describing $H^1(End(E))$ in proposition 10 we have factorized the space of matrix polynomials $\chi(z)$ by constant matrices). This map is really the differential of $\pi_1$ because the tangent vector $\chi(z)$ constructed as said defines the same deformation of $\Lambda_i$ due to proposition 10 as the expression (25). Now

$$\pi_1^*(\lambda)(\xi) = \lambda(d\pi_1 \xi) = <\Phi(z), \chi(z)>$$

where $\Phi(z)$ is constructed from $\{\Phi_i\}$ due to the identification $\rho$. In virtue of Serre’s pairing

$$<\Phi(z), \chi(z)> = -\sum_{i=2}^{N} Tr(\Phi_i \chi(P_i)) + Tr(\sum_{i=2}^{N} \Lambda_i \Phi_i \Lambda_i^{-1} \chi(P_i)).$$

On the other hand the canonical 1-form on $\mu^{-1}(0)$ can be calculated on the vector $\xi$:

$$\lambda_1(\xi) = \sum_{i=2}^{N} Tr(\Phi_i \Lambda_i^{-1} \delta_\xi \Lambda_i) = \sum_{i=2}^{N} Tr(\Phi_i \Lambda_i^{-1} (\chi_1 \Lambda_i - \Lambda_i \chi_i)) = <\Phi(z), \chi(z)>$$

2.7.2 Curves with many cusps

According to section 2.5 consider the curve $\Sigma^{proj}$ which is a $\mathbb{C}P^1$ curve with $N$ cusps. Recall that the affine part of $\Sigma^{proj}$ is given by $\text{Spec}(f(z) \in \mathbb{C}[z]: \forall i = 1, ..., N f'(P_i) = 0)$, the bundle $M_\Lambda$ corresponds to the module which over the affine part is described as $\{s(z) \in \mathbb{C}[z]^r : \forall i = 1, ..., N s'(P_i) = \Lambda_i s(P_i)\}$. In section 2.5 we have considered the case of one cusp, but everything can be obviously generalized to the case of many cusps. So the sections of $H^0(End(M_\Lambda) \otimes \mathcal{K})$ can be described as:

$$\Phi(z) = \sum_i \left( \frac{\Phi_i}{(z - P_i)^2} + \frac{[\Lambda_i, \Phi_i]}{z - P_i} \right) \quad (26)$$

where the matrices $\Phi_i$ satisfy $\sum_i [\Lambda_i, \Phi_i] = 0$.

**Claim:** The canonical 1-form on the cotangent bundle to the moduli space of vector bundles on the curve $\Sigma^{proj}$ in the coordinates $\Lambda_i, \Phi_i$ can be written as follows:

$$\sum_{i=1,\ldots,N} \text{Tr} \Phi_i d\Lambda_i \quad (27)$$

One should slightly correct this formulation as was done in the previous subsection, which can be done along the same lines; so, we omit all details.
2.8 Hitchin system

Let us summarize the results obtained up to now.

**Theorem 2** The Hitchin system on a curve $\Sigma^{proj}$ which is the result of gluing $N$ distinct points $P_i$ on $\mathbb{C}P^1$ is the system with phase space obtained by Hamiltonian reduction from the space of matrices $\Lambda_i, \Phi_i, i = 2, ..., N$ where $\Lambda_i$ are invertible with the symplectic form

$$\sum_{i=2,\ldots,N} - Tr d(\Phi_i \Lambda_i^{-1}) \, d\Lambda_i,$$

where $GL(r)$ acts by conjugation. The Lax operator is given by:

$$\Phi(z) = \sum_{i=2,\ldots,N} \frac{-\Lambda_i \Phi_i \Lambda_i^{-1}}{z - P_i} \, dz + \sum_{i=2,\ldots,N} \frac{\Phi_i}{z - P_i} \, dz.$$

Its Hamiltonians can be obtained by expanding $Tr \Phi(z)^k$ under the basis of holomorphic differentials. We will give the direct proof of their commutativity on the nonreduced phase space in the more general case in our next paper [20], so here we omit the details.

Analogously, the Hitchin system on the curve with many cusps can be described according to:

**Theorem 3** The Hitchin system on a curve $\Sigma^{proj}$ which is the curve $\mathbb{C}P^1$ with $N$ cusps at distinct points $P_i$ is the system with phase space obtained by Hamiltonian reduction from the space of matrices $\Lambda_i, \Phi_i, i = 2, ..., N$, with the symplectic form

$$\sum_{i=2,\ldots,N} - Tr d\Phi_i \, d\Lambda_i,$$

where $GL(r)$ acts by conjugation. The Lax operator is given by:

$$\Phi(z) = \sum_i \left( \frac{\Phi_i}{(z - P_i)^2} + \frac{[\Lambda_i, \Phi_i]}{z - P_i} \right).$$

3 Trigonometric and rational Calogero-Moser systems

3.1 Node

As was shown in section 2.2 the dualizing sheaf in this case has one global section $dz(\frac{1}{z - z_1} - \frac{1}{z - z_2})$. Consider the moduli space $\mathcal{M}$ of holomorphic bundles $V$ of rank $n$ on $\Sigma_{node}$ with a fixed trivialization at the point $p, z = z_3$. It means that

$$\mathcal{T}_V \mathcal{M} = H^1(\Sigma, \text{End}(V) \otimes \mathcal{O}(-p)).$$
We restrict to the principal cell of this moduli space which corresponds to the space of equivalence classes of matrices $\Lambda$ with different eigenvalues. The cotangent space is the space of holomorphic sections of $\text{End}^*(V) \otimes \mathcal{K} \otimes \mathcal{O}(p)$. Such sections are matrix-valued functions of $z$ of the form

$$\Phi(z) = \frac{\Phi_1}{z-z_1} - \frac{\Phi_2}{z-z_2} + \frac{\Phi_3}{z-z_3}$$

such that

$$\Phi_1 \Lambda = \Lambda \Phi_2 \quad \text{and} \quad \Phi_1 - \Phi_2 + \Phi_3 = 0. \quad (28)$$

This function is parameterized by $(\Phi_3)_{ij} = f_{ij}, i \neq j$, the eigenvalues $e^{2\pi i}$ of the matrix $\Lambda$ (all calculations are in the diagonal base for this matrix) and the diagonal elements of the matrix $(\Phi_1)_{ii} = p_i$. All other quantities can be expressed in these terms.

We now investigate the symplectic structure. The moduli space is parameterized by the matrix $\Lambda$ and the matrix $U$ which fixes the trivialization at the point $p$. The variables $\Phi_i$ define the cotangent vector.

**Lemma 1** The canonical symplectic form on the cotangent bundle $T^*\mathcal{M}$ can be represented as

$$\omega = \text{Tr}(d(\Lambda^{-1} \Phi_1) \wedge d\Lambda) + \text{Tr}(d(U^{-1} \Phi_3) \wedge dU). \quad (29)$$

**Remark 13** It is a slightly incorrect formulation: we mean that the canonical symplectic form on the cotangent bundle $T^*\mathcal{M}$ can be obtained from that form after the reduction by conjugation (see section 2.7 for the precise formulation).

**Proof.** The canonical symplectic form $\omega$ on the cotangent bundle $T^*X$ to the manifold $X$ is defined as follows: the point of the cotangent bundle is the pair $(x, p)$ where $x \in X$ and $p : T_xX \to \mathbb{C}$. We start by defining the 1-form $\lambda$ on the cotangent bundle by the formula

$$\lambda(x, p)(\xi) = p(\pi_* \xi), \quad (30)$$

where $\xi \in T(T^*X)$, $\pi$ is the projection $T^*X \to X$ and $\pi_*$ is its differential. On the cotangent bundle to $\mathbb{C}^N$ with coordinates $x_1, \ldots, x_N$ and with the corresponding coordinates $p_1, \ldots, p_N$ on the cotangent space this form reads $\lambda = \sum p_i dx_i$. The canonical symplectic form on the cotangent bundle is $\omega = d\lambda$. We have to prove that the form $\lambda = \text{Tr}(\Lambda^{-1} \Phi_1 d\Lambda) + \text{Tr}(U^{-1} \Phi_3 dU)$ is the canonical 1-form.

Let us describe explicitly the deformation of the vector bundle $V$ defined by an element $\chi \in H^1(\Sigma, \text{End}(V) \otimes \mathcal{O}(-p))$. The finite form of this element acts on the vector bundle data as follows:

$$\Lambda \mapsto (1 + \epsilon \chi(z_1))\Lambda(1 - \epsilon \chi(z_2)), \quad U \mapsto (1 + \epsilon \chi(z_3))U.$$

The infinitesimal form of these deformations $\delta_\chi \Lambda = \chi(z_1)\Lambda - \Lambda \chi(z_2)$ and $\delta_\chi U = \chi(z_3)U$ define vector fields $\xi_\chi$ on the moduli space and they can be lifted canonically to the vector fields on the total space of the cotangent bundle. We call this lifting $l : T\mathcal{M} \to T(T^*\mathcal{M})$, and evidently $\pi_* \circ l = id$. Another type of canonical vector fields on the total space of the cotangent bundle is the vertical vector fields $\xi_{\text{vert}}$. They act only on the coordinates on the space of sections $\Phi$ and $\xi_{\text{vert}} \in \ker \pi_*$. So we have to verify that $\lambda'(\text{vert}) = 0$ and
\( \lambda'(l(\xi)) = \langle \Phi, \chi \rangle \). The first is trivial because \( d\Lambda(\xi_{\text{vert}}) = 0 \) and \( dU(\xi_{\text{vert}}) = 0 \). As for the second condition, one has:

\[
\lambda'(l(\xi)) = Tr(\Lambda^{-1}\Phi_1\delta_\chi \Lambda) + Tr(U^{-1}\Phi_3\delta_\chi U)
\]

\[
= Tr(\Lambda^{-1}\Phi_1(\chi(z_1)\Lambda - \Lambda\chi(z_2)) + Tr(U^{-1}\Phi_3\chi(z_3)U)
\]

\[
= Tr(\Phi_1\chi(z_1) - \Phi_2\chi(z_2) + \Phi_3\chi(z_3)) = \langle \Phi, \chi \rangle,
\]

where we have used the paring \( \Phi \) and the relation \( \Phi_1 \Lambda = \Lambda \Phi_2 \).

To obtain the Calogero-Moser system one needs to perform an additional hamiltonian reduction. The form (29) is invariant by the \( GL_n \)-group action

\[
U \mapsto Ug.
\]

The moment map of this action is \( g^{-1}Ug \). We fix it to be diagonal and factorize the level manifold by the stabilizer which is \( n \)-dimensional. This procedure subtracts \( n \times n \) degrees of freedom, corresponding to the auxiliary variable \( U \). Finally the reduced nonsingular manifold is \( C^{2n} \) with a canonical nonsingular structure and the coadjoint orbits of \( GL_n \) generically of rank \( n \). This can be expressed in terms of Poisson brackets as follows:

\[
\{x_i, p_j\} = \delta_{ij}, \quad \{f_{ij}, f_{kl}\} = \delta_{jk}f_{il} - \delta_{il}f_{kj}
\]

(31)

**Remark 14** This expression for the symplectic form can be interpreted within a general approach in terms of coordinates on the nonreduced phase space of connections. For the constant connection \( A \) it is proportional to \( \ln \Lambda \) and (11) gives us

\[
\oint Tr(\delta \Phi \wedge \delta (\ln \Lambda))dz
\]

which is equal to the sum of the residues with the appropriate signs, i.e.

\[
\sum_{i=1}^{3} Tr(d\Phi_i \wedge \Lambda^{-1}d\Lambda).
\]

Let us proceed with the construction of integrals. The general scheme gives the quantities \( Tr\Phi^k(z) \), or more precisely all non trivial negative coefficients of the Laurent expansion of this expression at the poles. They are in involution by construction and by the fact that our symplectic form is obtained from the canonical one on the cotangent bundle by hamiltonian reduction. The trace of \( \Phi^2(z) \) is a rational function of \( z \) the coefficients of which are linear functions of \( Tr(\Phi_1^2) \) and \( Tr(\Phi_1\Phi_2) \). A straightforward calculation shows that:

\[
H_1 = Tr\Phi_1^2 = \sum_{i=1}^{n} p_i^2 - 4 \sum_{i\neq j} \frac{f_{ij}f_{ji}}{\sinh^2(x_i - x_j)};
\]

(32)

\[
H_2 = Tr\Phi_1\Phi_2 = \sum_{i=1}^{n} p_i^2 - 4 \sum_{i\neq j} \frac{f_{ij}f_{ji}}{(e^{2(x_j - 2x_i)} - 1)^2};
\]

(33)

The difference \( H_1 - H_2 \) is equal to the second Casimir element of \( \Phi_3 \), i.e. \( \Sigma_{i\neq j} f_{ij}f_{ji} \). Notice that \( H_1 \) is the Hamiltonian of the trigonometric Calogero-Moser system with spin.
3.2 Cusp

As in the nodal case we consider the principal cell of the moduli space which is parameterized by diagonal matrices Λ and introduce the moduli space with fixed trivialization U at the point \( P, z = z_2 \). The section of \( \text{End}^+(V) \otimes K \otimes \mathcal{O}(p) \) in this case can be represented by

\[
\Phi(z) = \frac{\Phi_1}{(z - z_1)^2} + \frac{\Phi_2}{z - z_1} + \frac{\Phi_3}{z - z_2}
\]

subject to the conditions

\[
\Phi_2 = [\Lambda, \Phi_1] \quad \text{and} \quad \Phi_3 = -\Phi_2. \quad (34)
\]

The coordinates on the phase space are \( x_i \)-the diagonal elements of the matrix Λ, the variables \( p_i \) which are the diagonal elements of the matrix \( \Phi_1 \), the nondiagonal elements of the matrix \( \Phi_2 \) which we denote by \( f_{ij}, i \neq j \) and the matrix elements of \( U \). Then solving the conditions (34) one obtains

\[
\Phi_{1ij} = \frac{f_{ij}}{x_i - x_j}.
\]

The infinitesimal action of the element \( \chi \in H^1(\Sigma, \text{End}(V) \otimes \mathcal{O}(-p)) \) is the following

\[
\delta_\chi \Lambda = [\chi(z_1), \Lambda] + \chi'(z_1), \quad \delta_\chi U = U\chi(z_2).
\]

The paring (9) realizing Serre’s duality in this case specializes to the form

\[
< \Phi, \chi > = \text{Tr}(\Phi_1 \chi'(z_1) + \Phi_2 \chi(z_1) + \Phi_3 \chi(z_2)) = \text{Tr}(\Phi_1 (\chi'(z_1) + [\chi(z_1), \Lambda]) - \Phi_2 \chi(z_2)),
\]

where we used (34). Substituting the explicit form of the action and using the arguments from the previous paragraph we obtain:

**Lemma 2** *The canonical symplectic form on the cotangent bundle \( T^*\mathcal{M} \) can be represented by*

\[
\omega = \text{Tr}(d\Phi_1 \wedge d\Lambda) + \text{Tr}(d(U^{-1}\Phi_2) \wedge dU). \quad (35)
\]

**Remark 15** see remark 13.

As in the previous paragraph we continue by the Hamiltonian reduction on the variable \( U \) and obtain the symplectic variety corresponding to the brackets (34). The coefficients of the function \( \text{Tr} \Phi_2^2(z) \) are linear functions of the expressions \( \text{Tr} \Phi_1^2, \text{Tr} \Phi_2 \Phi_1, \text{Tr} \Phi_2 \Phi_2 \). The first is the Hamiltonian of the rational Calogero-Moser system with spin

\[
H_3 = \text{Tr} \Phi_1^2 = \sum_{i=1}^n p_i^2 + \sum_{i \neq j} \frac{f_{ij} f_{ji}}{(x_i - x_j)^2}; \quad (36)
\]

\[
H_4 = \text{Tr} \Phi_2^2 = \sum_{i \neq j} f_{ij} f_{ji}. \quad (37)
\]

Let us note that \( H_4 \) is the second Casimir element for the matrix \( \Phi_2 \) and \( \text{Tr} \Phi_1 \Phi_2 = 0 \).
4 Curves with two cusps

For smooth curves of genus 2 the moduli space of $SL(2)$-bundles has been identified in \cite{15} with $\mathbb{C}P^3$. In the papers \cite{23, 24} the Hitchin Hamiltonians have been written explicitly as functions on $T^*\mathbb{C}P^3$. In our paper we consider singular curves of genus 2 (we consider the example of a rational curve with only two cusps). We will describe below the analog of the Narasimhan-Ramanan parameterization of $SL(2)$ bundles on such a curve and we will write down the Hitchin hamiltonians.

4.1 Construction

In this section we examine the moduli space of $SL_2$ holomorphic bundles on the curve $\Sigma_2$, defined by the equation

$$y^2 = (z - z_1)^3(z - z_2)^3,$$

of algebraic genus 2 with two singularities which can be realized by contracting all cycles on a genus two Riemann surface. On the normalization which is the rational curve $\mathbb{C}P^1$ obtained by the following blowup: $t = y/(z - z_1)(z - z_2)$ the inverse image of the structure sheaf can be realized as the subsheaf

$$\mathcal{O}_{\Sigma_2} = \{f \in \mathcal{O}_{\mathbb{C}P^1} : \partial_z f|_{z_1} = \partial_z f|_{z_2} = 0\}. \quad (38)$$

The holomorphic bundles $E$ on $\Sigma_2$ are parameterized by the pairs of matrices $(\Lambda_1, \Lambda_2)$ with zero trace up to common conjugation and can be described in terms of their section sheaf as follows:

$$\Gamma_U(E) = \{S \in \mathcal{O}_U(\mathbb{C}P^1) \otimes \mathbb{C}^2 : \partial_z S = \Lambda_1 S|_{z_1} ; \partial_z S = \Lambda_2 S|_{z_2}\}. \quad (39)$$

For constructing the cotangent bundle $T^*\mathcal{M}$ to the moduli space of holomorphic bundles $E$ we again exploit the Kodaira-Spenser correspondence.

Using the same arguments as in previous paragraphs we choose the realization of the canonical bundle on our singular curve $\Sigma_2$ to be $\mathcal{K}(\Sigma_2^{\text{norm}}) \otimes \mathcal{O}(2z_1 + 2z_2)$ on the normalization. The typical section of $\text{End}(V) \otimes \mathcal{K}$ in this case is the matrix-valued function

$$\Phi(z) = \frac{\Phi_1}{(z - z_1)^2} + \frac{\Phi_3}{z - z_1} + \frac{\Phi_2}{(z - z_2)^2} + \frac{\Phi_4}{z - z_2};$$

subject to the relations:

$$\Phi_3 = [\Lambda_1, \Phi_1], \quad \Phi_4 = [\Lambda_2, \Phi_2], \quad \Phi_3 + \Phi_4 = 0. \quad (40)$$

By analogy with the previous section we obtain the following:

**Lemma 3** The canonical symplectic form on the cotangent bundle of $rk = 2$ bundles on $\Sigma_2$ can be represented in the form:

$$\omega = Tr(d\Phi_1 \wedge d\Lambda_1 + d\Phi_2 \wedge d\Lambda_2). \quad (41)$$
Remark 16  see remark 13

As mentioned previously the open subset in the space of holomorphic rk = 2 bundles in consideration is the space of equivalence classes of pairs of 2 × 2-matrices Λ1, Λ2 up to common conjugation. The natural coordinates on this space are the invariant functions

\[ t_1 = TrΛ^2_1, \quad t_2 = TrΛ_1Λ_2, \quad t_3 = TrΛ^2_2. \]

Some technical preliminaries are convenient at this point. Due to the conditions (40) we have

\[ 0 = -Tr[Λ_2, Φ_2]Λ_2 = Tr[Λ_1, Φ_1]Λ_2 = Tr[Λ_2, Λ_1]Φ_1; \]

\[ 0 = -Tr[Λ_1, Φ_1]Λ_1 = Tr[Λ_2, Φ_2]Λ_1 = Tr[Λ_1, Λ_2]Φ_2. \]

Using the fact that the Killing form on \( sl_2 \) is not degenerate we conclude that for common \( Λ_1, Λ_2 \) the matrices \( Φ_1, Φ_2 \) are linear combinations of them:

\[ Φ_1 = p_{i1}Λ_1 + p_{i2}Λ_2 \quad i = 1, 2. \]  \( (42) \)

Also notice that \( p_{12} = p_{21} \). It comes from

\[ p_{12}[Λ_1, Λ_2] = [Λ_1, Φ_1] = -[Λ_2, Φ_2] = -p_{21}[Λ_2, Λ_1]. \]

Finally we introduce new coordinates:

\[ p_1 = p_{11}/2; p_2 = p_{12} = p_{21}; p_3 = p_{22}/2. \]

In these coordinates the linear condition (42) takes a more convenient form

\[ Φ_1 = 2p_1Λ_1 + p_2Λ_2; \]
\[ Φ_2 = p_2Λ_1 + 2p_3Λ_2. \]  \( (43) \)

Lemma 4  The conjugated variables to \( t_1, t_2, t_3 \) subject to the symplectic form (44) are \( p_1, p_2, p_3 \) such that

\[ ω = \sum_{i=1}^{3} dp_i \wedge dt_i. \]

**Proof.** We proceed by comparing the corresponding one-forms \( λ = Tr(Φ_1dΛ_1 + Φ_2dΛ_2) \) and \( λ' = ∑_{i=1}^3 p_idt_i \). The infinitesimal deformation is defined as follows:

\[ δ_χΛ_1 = [χ(z_1), Λ_1] + Χ'z_1); \quad δ_χΛ_2 = [χ(z_2), Λ_2] + Χ'z_2). \]

We have to verify that \( λ(l(χ_χ)) = λ'(l(χ_χ)) \) where \( l \) is some lifting of the vector field on \( M \) to the vector field on \( T^*M \) such that \( π_*l = id. \)

\[ λ(l(χ_χ)) = Tr(Φ_1[χ(z_1), Λ_1] + φ_1χ'(z_1) + φ_2[χ(z_2), Λ_2] + Φ_2χ'(z_2)) \]
\[ = Tr([Λ_1, φ_1](χ(z_1) − χ(z_2)) + φ_1χ'(z_1) + φ_2χ'(z_2)); \]

where we have used (40).

\[ λ'(l(χ_χ)) = Tr(2p_1Λ_1([χ(z_1), Λ_1] + Χ'z_1)) + p_2Λ_1([χ(z_2), Λ_2] + Χ'z_2)) \]
\[ + p_2Λ_2([χ(z_1), Λ_1] + Χ'z_1) + 2p_3Λ_2([χ(z_2), Λ_2] + Χ'z_2)) \]
\[ = Tr((2p_1Λ_1 + p_2Λ_2)χ'(z_1) + (2p_3Λ_2 + p_3Λ_1)χ'(z_2) + p_2(χ(z_1) − χ(z_2))[Λ_1, Λ_2]). \]

Due to the relations (43) one obtains \([Λ_1, Φ_1] = p_2[Λ_1, Λ_2] \) which shows that \( λ(l(χ_χ)) = λ'(l(χ_χ)) \). The fact that these 1-forms coincide on the vertical vector fields is trivial and this ends the demonstration of the lemma. □
4.2 Hitchin Hamiltonians

To obtain the Hamiltonians we use the natural mapping

\[ \text{End}(V) \otimes \mathcal{K}^\otimes_m \to \text{End}(V) \otimes \mathcal{K}^\otimes_m \to \text{End}(V) \otimes \mathcal{K}^\otimes_m \to \mathcal{K}^\otimes_m, \]

where the first arrow is the diagonal, the second is given by the multiplication in \( \text{End}(V) \).

The composition of these mappings induces the mapping on cohomologies

\[ i : H^0(\text{End}(E) \otimes \mathcal{K}) \to H^0(\mathcal{K}^\otimes_m). \] (44)

Any choice of global sections in \( \mathcal{K}^\otimes_m \) gives us a set of functions on our phase space.

We focus our attention on the quadratic Hamiltonians which are defined for \( m = 2 \).

Due to the Riemann-Roch theorem \( H^0(\mathcal{K}^2) = 3 \) and \( H^0(\mathcal{K}) = 2 \). We can take as a basis of global sections of \( \mathcal{K} \) the following

\[ s_1 = \frac{dz}{(z - z_1)^2}; \quad s_2 = \frac{dz}{(z - z_2)^2}. \]

Their tensor quadratic monomials

\[ s_1 \otimes s_1, \quad s_1 \otimes s_2 \quad \text{and} \quad s_2 \otimes s_2 \] (45)

form a basis in the space of global sections for \( \mathcal{K}^2 \). Now consider the Higgs field

\[ \Phi(z) = \left\{ \frac{\Phi_1}{(z - z_1)^2} + \frac{\Phi_2}{(z - z_2)^2} + \frac{(z_1 - z_2)[\Lambda_1, \Phi_1]}{(z - z_1)(z - z_2)} \right\} dz. \]

We calculate the image of (44)

\[ \text{Tr}\Phi^2(z) = \left\{ \frac{\text{Tr}\Phi_1^2}{(z - z_1)^4} + \frac{\text{Tr}\Phi_2^2}{(z - z_2)^4} + \frac{2\text{Tr}\Phi_1\Phi_2 + (z_1 - z_2)^2\text{Tr}[\Lambda_1, \Phi_1]^2}{(z - z_1)^2(z - z_2)^2} \right\} dz \otimes dz. \]

Decomposing \( \text{Tr}\Phi(z)^2 \) on the basis (45) we obtain the coefficients:

\[ H_1 = \text{Tr}\Phi_1^2 = 4p_1^2t_1 + p_2^2t_3 + 4p_1p_2t_2; \]
\[ H_2 = 2\text{Tr}\Phi_1\Phi_2 + (z_1 - z_2)^2\text{Tr}[\Lambda_1, \Phi_1]^2 \]
\[ = 4p_1p_2t_1 + 4p_2p_3t_3 + (8p_1p_3 + 2p_2^2)t_2 - 2(z_1 - z_2)^2p_2^2(t_1t_3 - t_2^2); \]
\[ H_3 = \text{Tr}\Phi_2^2 = 4p_3^2t_3 + p_2^2t_1 + 4p_2p_3t_2; \]

which are the functions on the original phase space.

**Proposition 16** The quantities \( H_i \) are in involution, so one obtains an integrable system. The proof is straightforward.

**Remark 17** Here we have a one-parameter family of a priori non-equivalent integrable systems on \( T^*\mathbb{C}^3 \). However the identification of this family with the Neumann model remains unclear.
4.3 Degenerated Narasimhan-Ramanan parameterization

Here we recall the classical construction from \[15\] \[22\] which identifies the moduli space of the stable bundle on a regular curve $\Sigma$ of genus 2 with $\mathbb{C}P^3$

$$\mathcal{M} \xrightarrow{\Delta} |2\Theta| \cong \mathbb{C}P^3.$$ 

With the bundle $E$ one associates $D_E \subset Pic_1(\Sigma)$ such that for any line bundles $\mathcal{L} \in D_E$ the dimension of $H^0(E \otimes \mathcal{L})$ equals 1. Due to \[15\] \[22\] the divisor $D_E$ lies in the linear system $|2\Theta|$ and this correspondence is an isomorphism. $D_E$ is given by the equation

$$\sum_{i,j=1,2} p_{ij} \theta_{\frac{1}{2}(i-1),\frac{1}{2}(j-1)} = 0.$$ 

The coefficients $p_{ij}$ are projective coordinates on the moduli space.

The degenerate case shares similar considerations. Let us take the $SL_2$-bundle $E$ given by the pair of matrices $\Lambda_1, \Lambda_2$ on $\Sigma_2$ and the linear bundle $\mathcal{L}$ of degree 1 given by the pair of complex numbers $\lambda, \mu$. Their tensor product $E' = E \otimes \mathcal{L}$ is the $rk = 2$ and $deg = 2$ holomorphic bundle. At the singular points this bundle is characterized by the pair of matrices $\Lambda_1 + \lambda 1, \Lambda_2 + \mu 1$. We have to calculate the dimension of $H^0(E \otimes \mathcal{L})$. To describe the space of global sections we use the following covering $U_1 = \Sigma_2 \setminus \infty$ and $U_2 = U_{\infty, z}$-small neighborhood of $\infty$. The fact that the bundle $\mathcal{L}$ is of degree 1 means that the transition function associated to this covering can be chosen in the scalar form $z$ which is invertible in $U_1 \cap U_2$. So, the global sections of $E'$ are linear vector functions $S$ on $U_1$ such that $\partial_z S = \Lambda_1 S|_{z_1}; \partial_\bar{z} S = \Lambda_2 S|_{z_2}$. Notice that these holomorphic vector functions can be continued to the chart $U_1$ because they are linear and $S/z$ is regular at $\infty$.

We rewrite linear defining conditions \[43\] using that the section $S$ is linear, i.e. $S(z) = S_0 + zS_1$ as follows:

$$S_1 = (\Lambda_1 + \lambda 1)(S_0 + z_1 S_1);$$

$$S_1 = (\Lambda_2 + \mu 1)(S_0 + z_2 S_1).$$

The consistency condition for this linear system is

$$Det \begin{pmatrix} \Lambda_1 + \lambda 1 & z_1(\Lambda_1 + \lambda 1) + 1 \\ \Lambda_2 + \mu 1 & z_2(\Lambda_2 + \mu 1) + 1 \end{pmatrix} = 0.$$ 

A straightforward calculation gives the following expression for the determinant:

$$Det(\lambda, \mu) = \lambda \mu(\lambda(z_1 - z_2) - 2)(\mu(z_1 - z_2) + 2) + (\lambda + \mu)^2 + \lambda^2 det(\Lambda_2) + \mu^2 det(\Lambda_1) + 2(\lambda det(\Lambda_2) - \mu det(\Lambda_1))(z_1 - z_2) + det(\Lambda_1) det(\Lambda_2)(z_1 - z_2)^2 + det(\Lambda_1) + det(\Lambda_2) + Tr(\Lambda_1 \Lambda_2). \quad (46)$$

In terms of the variables $t_1, t_2, t_3$ we rewrite \[46\] as:

$$Det(\lambda, \mu) = \frac{t_1 t_3}{4}(z_1 - z_2)^2 + \frac{t_1}{2}(\mu(z_1 - z_2) - 1)^2 - \frac{t_3}{2}(\lambda(z_1 - z_2) + 1)^2 + (\lambda \mu(z_1 - z_2) - \lambda + \mu)^2. \quad (47)$$
Now we interpret the functions
\[ 1, \quad -(\mu(z_1 - z_2) - 1)^2/2, \quad -(\lambda(z_1 - z_2) + 1)^2/2, \quad (\lambda\mu(z_1 - z_2) + \lambda - \mu)^2 \]
as the basis of \(\theta\)-functions of second order for our singular curve and the expressions
\[ \tau_1 = t_1, \quad \tau_2 = t_2 + \frac{t_1 t_3}{4}(z_1 - z_2)^2, \quad \tau_3 = t_3 \]
as affine analogs of Narasimhan-Ramanan parameters.

5 Conclusion

We have shown that the invariant description of Hitchin system on singular curves is consistent and provides an explicit parameterization in all considered cases. However, the problem of constructing nonreduced coordinates remains tricky. We outline below several principal directions in which to continue the study of Hitchin system:

- **Explicit parameterization.** Even in nonsingular cases there are some manners to explicitly parameterize the moduli space of (semi-)stable holomorphic bundles. The Hecke-Turin parameterization is one of the most universal and the problem here is to reexpress the explicit formulas obtained above for the Hamiltonians in terms of the Hecke-Turin parameters (see [14] for the case of nonsingular curves). For singular curves it will be done in [24]. A related question is the construction of separated variables and the understanding of its algebraic nature.

- **General description of Hitchin system on singular curves.** The considered examples show the homogeneity of the analysis in such cases as cusp and node singularities. The subject of the subsequent paper [20] is the universal treatment of a wide class of singular curves.

- **Compactification of moduli space of vector bundles on singular curves.** For the curves considered in this paper (curves with cusps and nodes) the vector bundles can be described very explicitly - so one hopes that one can explicitly describe the compactification of the moduli space of vector bundles. The moduli space of linear bundles on a singular curve must be compactified by torsion free sheaves. So for the case of vector bundles it is most likely that it must be compactified by some "semistable torsion-free sheaves".

- **Lax Pair Representation.** All integrable systems have a Lax pair representation. It is not unique and for almost all known integrable systems the Lax representation arises naturally, as, for example, the auxiliary linear problem for nonlinear PDE’s. It is important to obtain a Lax pair representation for our system and to understand its intrinsic meaning.

(The Lax representation of Calogero-Moser system was discussed in the framework of Hitchin systems in [20]).
• **Classical Solutions.** There is a general prescription to obtain “action-angle” variables for Hitchin system - they are related to the Jacobians of spectral curves. Another way to solve the classical system is the so-called projection method which can be specially effective in our description. It would be interesting to compare these methods and to obtain explicit solutions of the classical equations of motion.

• **Relation to hierarchies of isomonodromic deformations and to the Knizhnik-Zamolodchikov-Bernard equations.** It is known (see for example [8]) that by changing the complex structure on a curve one goes from a Hitchin system to hierarchies of isomonodromic deformations and from a quantum Hitchin system to the Knizhnik-Zamolodchikov-Bernard equations. It would be tempting to consider the movement of double points and cusps and to obtain such equations explicitly in our case.

• **Quantization.** Another interesting goal is to find explicitly the quantum commuting hamiltonians related to the classical ones presented here. One hopes to fully explore the Hitchin system: to find the wave-functions, the spectrum, the statistical sum, the correlators, to explore the so-called duality [27, 28] for Hitchin system. Also it would be interesting to explore the properties of wave-functions: their integral representations, asymptotics, different relations among them and so on. The quantization of Hitchin system goes back to [3] and was discussed later for some particular cases in [6, 7, 14]. The rather general approach of [29] provides the quantization of an analog of the Hitchin system, namely a system over the space of rational matrices (the Beauville system), in terms of separated variables. The quantization of the Hitchin and related Beauville-Mukai systems was discussed recently in [30, 31]. Another approach to quantization is built on the notion of quantum R-matrix and its generalization [32] (let us note that the classical r-matrices for the Hitchin system were found in [3, 33]). It would be very interesting to understand the analogs of all these constructions for Hitchin systems on singular curves.

• **Separation of variables and geometric Langlands correspondence.** One of the most interesting goals is to work out explicitly the separation of variables (31, 29) for the integrable systems considered in our paper and to understand its relation to the geometric Langlands correspondence (35, 27). It seems that the integrable systems considered here are quite simple and explicit so one can try to explicitly understand some complicated constructions of [5] in these cases and to shed some light on the miracle of the Langlands correspondence. One must also note that the curves considered in this paper have been considered in [16] for the construction of ramified geometric class field theories; one can hope that these singular curves will play an analogous role for the ramified version of the geometric Langlands correspondence.
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