Comparison of finite-dimensional approximations of the permeability field on solving the inverse problem for the equation of stationary single-phase fluid filtration

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Abstract. The problem of identifying the permeability coefficient for equation of liquid stationary single-phase filtration by the known values of bottomhole pressure on the wells is considered. This problem is reduced to residual function minimization.

1. Introduction
For the solution of tasks of single-phase filtration in oil reservoirs or aquifers, it is necessary to know the their parameters. The inverse coefficient problem of the permeability field identification by measurements of bottomhole pressure in wells is considered in this paper. Methods of solving inverse coefficient problems are classified into direct and indirect [1, 2]. In the direct methods for the solution of the considered problem the values of identified parameters are determined from the solution of a system of algebraic equations, in this case, you must know the pressure at all points of the reservoir. In indirect methods, the identification problem is reduced to the problem of the residual function minimization, and the bottomhole pressure are used only [3,4]. This approach is used in this paper. The model problem is used to compare different types of approximations of the permeability field, and to study the process of refinement of the results with an increase in the dimension of the finite-dimensional approximation with a fixed number of known bottomhole pressure values. The permeability field is determined in classes of piecewise-constant, piecewise-bilinear functions and spline functions.

2. Problem statement
The differential equation of a single-phase stationary filtration of a fluid, obeying the Darcy law, in a two-dimensional reservoir can be written in the form [5]

\[ \nabla \left( \frac{k h}{\mu} \nabla p \right) = \sum_{i=1}^{M} Q_i \delta(x_i, y_i), \]

where \( k = k(x, y) \) is the coefficient of permeability, \( h = h(x, y) \) is the thickness of reservoir, \( \mu \) is the coefficient of fluid viscosity, \( p \) is the pressure, \( Q_i, (x_i, y_i) \) are the flow rate and coordinates of \( i \)-th wells, \( \delta(x_i, y_i) \) is the delta function, \( M \) is the number of wells. For the equation (1) the following initial conditions is assigned:

\[ p_{\Gamma} = p_{\Gamma}, \]
where $\Gamma$ is the boundary of the reservoir $\Omega$.

To solve the system of equations (1)-(2), the coefficients of equation (1) must be known. In this paper $h = h(x, y)$ and $\mu$ are considered known, and the permeability coefficient $k$ is considered unknown. This coefficient is determined by the known measurement of bottomhole pressure in the process of minimizing the residual function, which has the form

$$J(k) = \frac{1}{2} \sum_{i=1}^{M} (p_i - p_i^*)^2,$$

where $p_i$, $p_i^*$ are the bottomhole pressure values obtained as a result of the solution of the system (1)-(2), and the known measurements at the wells. Minimization of residual function is conducted using the Levenberg-Marquardt method [6]. To stop the minimization process, two criteria are used: achieving the specified accuracy by pressure measurements $\max_{j=1,M} |p_j - p_j^*| < 0.01 \text{ m}$

or slow convergence of the minimization process $J^n - J^{n+1} < 0.01J^n$ during 3 iterations, where $n$ is the iteration number.

3. Model task
Reservoir $\Omega$ (8000 m $\times$ 8000 m, thickness of reservoir is 10 m) opens up 39 injection and 250 production wells. The discharge rate of 150 $\text{m}^3$/day is assigned at injection wells, at producing wells flow rate varies from 20 $\text{m}^3$/day up to 95 $\text{m}^3$/day. The radius of wells is 0.1 m. At the boundary of the reservoir is assigned pressure of 20 MPa. The coefficient of liquid viscosity is 10 $\text{mPa}\cdot\text{s}$. The model task is constructed as follows. The permeability values are assigned in the nodes of the square grid. The step of grid equals 2000 m. Further the true permeability field $k^{tr}$ is constructed by the kriging method using given values permeability (Figure 1). Then from the solution of the system (1)-(2) for the field $k^{tr}$ the well pressure values are determined. The system (1)-(2) is solved numerically. Equation (1) is approximated by the method of control volumes. In this case the reservoir $\Omega$ is covered a square grid with a step of 50 m ($L = 25600$ is number of control volumes). The resultant system of linear algebraic equations is solved by the method of conjugate gradients in the form approximate Cholesky factorization [7,8]. The values of bottomhole pressure are calculated by using the Peaceman formula [9]. Further, the permeability field is considered to be unknown and is restored by known bottomhole pressure values using various types of its approximation. For comparison of the calculated permeability fields with the true fields are given values of the deviation $\Delta K_{avr} = \frac{1}{L} \sum_{i=1}^{L} (k^{tr}_i - k_i)^2$, the maximum deviation $\Delta K_{max} = \max_{i=1,L} |k^{tr}_i - k_i|$, the maximum residuals for wells $r_{\text{well max}} = \max_{j=1,M} |p_j - p_j^*|$ and all control volumes $r_{max} = \max_{j=1,L} |p_j - p_j^*|$.

![Figure 1](image-url). The true permeability field $k^{tr}$.
3.1. The piecewise-constant approximation

The reservoir $\Omega$ is divided into $N$ homogeneity zones $\Omega_i$, $\bigcup_{i=1}^{N} \Omega_i = \Omega$. The permeability field of the reservoir is represented by a piecewise-constant function

$$k = k(x, y) = \sum_{i=1}^{N} k_i \varphi_i(x, y),$$

where $k_i$ is the value of permeability in the homogeneity zone $\Omega_i$, $\varphi_i(x, y) = 1$ if $(x, y) \in \Omega_i$ and $\varphi_i(x, y) = 0$ if $(x, y) \not\in \Omega_i$. The number of homogeneity zones is corresponded with number of identified parameters. The results obtained for a different number of homogeneity zones are shown in Figures 2-5 and in Table 1.

Table 1. The piecewise-constant approximation of the permeability field: deviation and maximum deviation of the true permeability values from the calculated values and the maximum residuals.

| $N$ | $\Delta K_{av}$ | $\Delta K_{max}$ | $r_{well}^{max}$ | $r_{max}^{max}$ |
|-----|-----------------|------------------|-----------------|----------------|
| 4   | 0.347           | 1.369            | 13.178          | 13.178         |
| 16  | 0.268           | 1.113            | 7.574           | 7.574          |
| 64  | 0.141           | 0.632            | 3.311           | 3.311          |
| 256 | 0.076           | 0.352            | 0.465           | 0.465          |

3.2. The piecewise-bilinear approximation.

The reservoir $\Omega$ is covered with an additional square grid. The permeability field is represented as

$$k = k(x, y) = \sum_{i=1}^{N} k_i \Phi_i(x, y),$$
where $k_i$ is the value of permeability in node $i$ of grid, $\Phi_i$ is the bilinear basic function, $N$ is the number of grid nodes (it is corresponded to number of identified parameters). The function $\Phi_i$ is equal 0 everywhere in $\Omega$ except for a subdomain composed of grid elements, one of the vertices of which coincides with node $i$. The results obtained for a different number of grid nodes are shown in Figures 6-9 and in table 2.

**Figure 6.** The piecewise-bilinear approximation of the permeability field, $N=9$.

**Figure 7.** The piecewise-bilinear approximation of the permeability field, $N=25$.

**Figure 8.** The piecewise-bilinear approximation of the permeability field, $N=81$.

**Figure 9.** The piecewise-bilinear approximation of the permeability field, $N=289$.

**Table 2.** The piecewise-bilinear approximation of the permeability field: deviation and maximum deviation of the true permeability values from the calculated values and the maximum residuals.

| $N$ | $\Delta K_{av}$ | $\Delta K_{max}$ | $r_{max}^{well}$ | $r_{max}$ |
|-----|-----------------|------------------|------------------|-----------|
| 9   | 0.268           | 1.323            | 8.198            | 8.198     |
| 25  | 0.059           | 0.227            | 2.166            | 2.166     |
| 81  | 0.027           | 0.205            | 0.556            | 0.556     |
| 289 | 0.058           | 0.611            | 0.018            | 0.332     |

3.3. Approximation of spline function

The spline function is constructed on the values of permeability $k_i$ given the interpolation nodes $(x_i, y_i)$ [11]

$$k(x, y) = \sum_{i=1}^{N} c_i r_i^2 \ln r_i^2 + c_{N+1} + c_{N+2}x + c_{N+3}y,$$

where $r_i^2 = (x - x_i)^2 + (y - y_i)^2$, the number of interpolation nodes is corresponded with number of identified parameters. The coefficients $c_i$, $i=1, N+3$, are determined from the solution of equations system
\( k(x_i, y_i) = k_i, \ i = 1, N, \)
\[ \sum_{i=1}^{N} c_i = 0, \sum_{i=1}^{N} x_i c_i = 0, \sum_{i=1}^{N} y_i c_i = 0. \]

The advantage of using the spline function is that do not need to build an additional grid to calculate permeability values. Interpolation nodes can be placed arbitrarily, and a single solution requires that at least three interpolation nodes do not lie on the same line. Two approaches to the choice of interpolation nodes are used in the paper. In the first approach, the interpolation nodes are located in the nodes of the additional square grid. In the second approach, the coordinates of interpolation nodes are coincided with the coordinates of the wells. The results obtained using the first approach for different numbers of interpolation nodes are shown in Figures 10-13 and in table 3. The results obtained using the second approach are shown in Figure 14 and in table 3.

**Figure 10.** The approximation of spline function of the permeability field, \( N=9. \)

**Figure 11.** The approximation of spline function of the permeability field, \( N=25. \)

**Figure 12.** The approximation of spline function of the permeability field, \( N=81. \)

**Figure 13.** The approximation of spline function of the permeability field, \( N=289. \)

**Table 3.** The approximation of spline function of the permeability field: deviation and maximum deviation of the true permeability values from the calculated values and the maximum residuals.

| \( N \) | \( \Delta K_{\text{av}} \) | \( \Delta K_{\text{max}} \) | \( r_{\text{well}} \) | \( r_{\text{max}} \) |
|-----|----------|-----------|----------|----------|
| 9   | 0.649    | 10.735    | 8.674    | 8.674    |
| 25  | 0.138    | 1.339     | 1.675    | 1.675    |
| 81  | 0.037    | 0.493     | 0.570    | 0.570    |
| 289 | 0.158    | 5.550     | 0.013    | 0.432    |
| 289*| 0.009    | 0.149     | 0.004    | 0.202    |

* - the coordinates of interpolation nodes is corresponded to the coordinates of wells.
Figure 14. The approximation of spline function of the permeability field, $N=289$. The coordinates of interpolation nodes is corresponded to the coordinates of wells.

From the above results it is clear that the given pressure accuracy was achieved only in the approximation of permeability field in the form of spline functions with interpolation nodes at the locations of the wells (table 3). In all other cases the minimization process is interrupted according to the criterion of the slow convergence. It should be noted that an increase in the number of identified parameters does not always lead to better identification results. Thus, when using the approximation of the permeability field by piecewise-bilinear function and spline function (interpolation nodes in the nodes of the additional grid) with 289 unknown values, the identification results are worse than using a smaller number of unknown parameters.

4. Conclusions
Model task of the permeability field identification of two-dimensional reservoir under conditions of stationary single-phase filtration by known values of bottomhole pressure on the wells is solved. The permeability field is determined in the process of minimizing a method Levenberg-Marquardt of the residual function, constructed by the values of bottomhole pressure at the wells. Three types of approximation for parameterize of the permeability field were used: piecewise-constant, piecewise-bilinear functions and spline function. The use of the spline function with interpolation nodes in the locations of wells for the identification of the permeability field allowed to obtain the solution closest to the true.

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