ON 3-UNIFORM HYPERGRAPHS
WITHOUT A CYCLE OF A GIVEN LENGTH

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Abstract. We study the maximum number of hyperedges in a 3-uniform hypergraph on
n vertices that does not contain a Berge cycle of a given length ℓ. In particular we prove
that the upper bound for $C_{2k+1}$-free hypergraphs is of the order $O(k^2n^{1+1/k})$, improving
the upper bound of Győri and Lemons [10] by a factor of $Θ(k^2)$. Similar bounds are shown
for linear hypergraphs.

1. A generalization of the Turán problem

Counting substructures is a central topic of extremal combinatorics. Given two (hy-
per)graphs $G$ and $H$ let $N(G; H)$ denote the number of subgraphs of $G$ isomorphic to $H$.
(Usually we consider a labelled host graph $G$). Note that $N(G; K_2) = e(G)$, the number of
edges of $G$. More generally, $N(G; H)$ is the maximum of $N(G; H)$ where $G \in \mathcal{G}$, a class of
graphs. In most cases, in Turán type problems, $\mathcal{G}$ is a set of $n$-vertex $\mathcal{F}$-free graphs, where
$\mathcal{F}$ is a collection of forbidden subgraphs. This maximum is denoted by $N(n, \mathcal{F}; H)$. So
$N(n, \mathcal{F}; H)$ is the maximum number of copies of $H$ in an $\mathcal{F}$-free graph on $n$ vertices. The
Turán number $\text{ex}(n, H)$ is defined as $N(n, \mathcal{F}; K_2)$. Let $\text{ex}(m, n, \mathcal{F})$ be the maximum number
edges in a bipartite graph with parts of order $m$ and $n$ vertices that do not contain any
member of $\mathcal{F}$. $C_\ell$ is the family of all cycles of length at most $\ell$. For any graph $G$ and any
vertex $x$, we let $t(G)$ and $t(x)$ denote the number of triangles in $G$ and the number of triangles
containing $x$, respectively. Let $t_\ell(n) := N(n, C_\ell; K_3)$.

Our starting point is the Bondy-Simonovits [3] theorem, $\text{ex}(n, C_{2k}) \leq 100kn^{1+1/k}$. Recall
two contemporary versions due to Pikhurko [15], Bukh and Z. Jiang [4], respectively, and a
classical result by Kővári, T. Sós, and Turán [11]. For all $k \geq 2$ and $n \geq 1$, we have
\[
\begin{align*}
ex(n, C_{2k}) &\leq (k-1)n^{1+1/k} + 16(k-1)n, \quad (1) \\
ex(n, C_{2k}) &\leq 80\sqrt{k\log kn^{1+1/k}} + 10k^2n, \quad (2) \\
ex(n, n, C_4) &\leq n^{3/2} + 2n. \quad (3)
\end{align*}
\]

Erdős [6] conjectured that a triangle-free graph on $n$ vertices can have at most $(n/5)^5$ five cycles and that equality holds for the blown-up $C_5$ if $5|n$. Győri [9] showed that a triangle-free graph on $n$ vertices contains at most $c(n/5)^5$ copies of $C_5$, where $c < 1.03$. Grzesik [8], and independently, Hatami et al. [13] confirmed that Erdős’ conjecture is true by using Razborov’s method of flag algebras, i.e., $N(n, C_3; C_5) \leq (n/5)^5$.

Bollobás and Győri [2] asked a related question: how many triangles can a graph have if it does not contain a $C_5$. They obtained the upper bound $t_5(n) \leq (1 + o(1))(5/4)n^{3/2}$ which yields the correct order of magnitude.

Later, Győri and Li [12] provided bounds on $t_{2k+1}(n)$.
\[
\binom{k}{2} \ex \left( \frac{n}{k+1}, \frac{n}{k+1}, C_{2k} \right) \leq t_{2k+1}(n) \leq \frac{(2k-1)(16k-2)}{3} \ex(n, C_{2k}). \quad (4)
\]

In Section 3 we improve the upper bound by a factor of $\Omega(k)$.

**Theorem 1.** For $k \geq 2$,
\[
\begin{align*}
t_{2k+1}(n) &\coloneqq N(n, C_{2k+1}; K_3) \leq 9(k-1) \ex \left( \left\lfloor \frac{n}{3} \right\rfloor, \left\lfloor \frac{n}{3} \right\rfloor, C_{2k} \right), \quad (5) \\
t_{2k}(n) &\leq \frac{2k-3}{3} \ex(n, C_{2k}). \quad (6)
\end{align*}
\]

The inequalities (1), (3) and (5) give $t_{2k+1}(n) \leq 9(k-1)^2 ((2/3)n)^{1+1/k} + O(n)$ for $k \geq 3$ and $t_5(n) \leq \sqrt{3n^{3/2}} + O(n)$. This latter one is not better than the Bollobás-Győri bound. However, our constant factor in Theorem 1 is the best possible in the following sense. It is widely believed that that the Turán numbers in the above statements are ‘smooth’, i.e., there are constants $a_k, b_k$ depending only on $k$ such that $\ex(n, n, C_{2k}) = (a_k + o(1))n^{1+1/k}$ and $\ex(n, n, C_{2k}) = (b_k + o(1))n^{1+1/k}$. If these are indeed true then the ratio of the upper bound in (5) and the lower bound in (4) is bounded by a constant factor of $O(a_k/b_k)$. It is also believed that the sequence $a_k/b_k$ is bounded (as $k \to \infty$), so further essential improvement is probably not possible.

Since the first version of this manuscript (2011) Alon and Shikhelman [1] improved the upper bound in Theorem 1 by a constant factor to $(16/3)(k-1) \ex([n/2], C_{2k})$ and showed that $t_5(n) \leq (1 + o(1))(\sqrt{3}/2)n^{3/2}$. Nevertheless, we include our proof in Section 3 for completeness, and because we use Theorem 1 in our main result in the next section.
2. Berge cycles

A Berge cycle of length \( k \) is a family of distinct hyperedges \( H_0, \ldots, H_{k-1} \) such that there are distinct vertices \( v_0, \ldots, v_{k-1} \) satisfying

\[ v_i v_{i+1} \subset H_i \text{ for } 0 \leq i \leq k-1 \pmod{k}. \]

A hypergraph is linear, also called nearly disjoint, if every two edges meet in at most one vertex. Let \( C^{(3)}_\ell \) be the collection of 3-uniform Berge cycles of length \( \ell \).

We write \( \text{ex}_r(n, F) \) (\( \text{ex}^{\text{lin}}_r(n, F) \), resp.) to denote the maximum number of hyperedges in a \( r \)-uniform (and linear, resp.) hypergraph on \( n \) vertices that does not contain any member of \( F \). Győri and Lemons [10] showed that

\[
\text{ex} \left( \left\lfloor \frac{n}{3} \right\rfloor, \left\lfloor \frac{n}{3} \right\rfloor, C_{2k} \right) \leq \text{ex}_3(n, C_{2k+1}^{(3)}) < 4k^4n^{1+\frac{1}{k}} + 15k^4n + 10k^2n. \tag{7}
\]

The order of magnitude of the upper bound probably cannot be improved (as \( k \) is fixed and \( n \to \infty \)).

Győri and Lemons [11] extended their result to \( C_{2k}^{(3)} \)-free 3-uniform hypergraphs (and also to \( m \)-uniform hypergraphs) by showing that the same lower bound as in (7) holds for \( \text{ex}_3(n, C_{2k}^{(3)}) \) and that \( \text{ex}_3(n, C_{2k}^{(3)}) \leq c(k)n^{1+\frac{1}{k}} \). The construction showing the lower bound in (7) is defined by considering a balanced bipartite graph \( G \) on \( n/3 + n/3 \) vertices which is extremal not containing any members of \( C_{2k} \). A 3-uniform \( C_{2k}^{(3)} \)-free hypergraph \( \mathcal{H} \) is formed by doubling each vertex in one of the parts of \( G \), thus turning each edge of \( G \) to a hyperedge of \( \mathcal{H} \). The number of hyperedges in \( \mathcal{H} \) is \( e(G) = \text{ex}(n/3, n/3, C_{2k}) \).

In this paper, we make improvements on the bounds on \( \text{ex}_3(n, C_{2k+1}^{(3)}) \) and \( \text{ex}_3(n, C_{2k}^{(3)}) \). First, observe that trivially

\[
\text{ex}_3(n, C_{2k+1}^{(3)}) \leq \text{ex}_3(n, C_{2k}^{(3)}). \tag{8}
\]

(Consider the triple system defined by the triangles of a \( C_{2k+1} \)-free graph). So (1) gives a lower bound which (probably) improves the lower bound in (7) by a factor of \( \Omega(k) \).

The aim of this paper is to improve the upper bound in (7) by a factor of (at least) \( \Omega(k^2) \) and also to simplify the original proof. In Section 4 we reduce the upper bound into three subproblems as follows.

**Theorem 2.** For \( k \geq 2 \) we have

\[
\text{ex}_3(n, C_{2k+1}^{(3)}) \leq t_{2k+1}(n) + 4\text{ex}(n, C_{2k}) + 12\text{ex}^{\text{lin}}_3(n, C_{2k+1}^{(3)}), \tag{9}
\]

\[
\text{ex}_3(n, C_{2k}^{(3)}) \leq t_{2k}(n) + \text{ex}(n, C_{2k}). \tag{10}
\]

The first and the third terms in (9) are both lower bounds, and probably the middle term is the smallest one. In Section 5 we estimate the third term.
Theorem 3. For $k \geq 2$ we have

$$\text{ex}^\text{lin}_3(n, C_{2k+1}^{(3)}) \leq 2kn^{1+1/k} + 9kn. \quad (11)$$

We were not able to relate the left hand side directly to $\text{ex}(n, C_{2k})$. In fact, just like in Győri and Lemons’ proof \[10\], we reiterate a version of the original proof of Bondy and Simonovits \[3\] (as everybody else did in \[16\], \[15\], \[5\], and in \[4\]). Our rendering is much simpler than \[10\]. For the even case $\text{ex}^\text{lin}_3(n, C_{2k}^{(3)}) \leq \text{ex}(n, C_{2k})$ is obvious by selecting a pair from each hyperedge in a linear $C_{2k}$-free triple system. We have no matching lower bound for $\text{ex}^\text{lin}_3(n, C_{\ell}^{(3)})$ other than what follows from the random method. Collier, Graber and Jiang \[5\] proved that $\text{ex}^\text{lin}_r(n, C_{2k+1}^{(r)}) \leq \alpha_{k,r} n^{1+1/k}$, but their $\alpha_{k,r}$ is greater than $r(2k)^r$. They find not only a Berge cycle but a linear cycle, i.e., a cyclic list of triples such that consecutive sets intersect in exactly one element and nonconsecutive sets are disjoint.

Theorems 1, 2 and 3 together with (1) imply

$$\text{ex}_3(n, C_{2k+1}^{(3)}) \leq (9k^2 + 10k + 5)n^{1+1/k} + O(k^2n)$$

and $\text{ex}_3(n, C_{2k}^{(3)}) \leq \frac{1}{2}(2k + 9)(k - 1)n^{1+1/k} + O(k^2n)$. Using (2) one can lower the main coefficient to $O(k^{3/2} \sqrt{\log k})$. If the smoothness conjectures concerning $\text{ex}(n, C_{2k})$ and $\text{ex}(n, n, C_{2k})$ hold, then the ratio of the upper bound \[\text{9}\] and lower bound \[\text{8}\] is of $O(a_k/b_k)$.

3. Counting Triangles in $C_{2k}$-Free and $C_{2k+1}$-Free Graphs

We need the following classical result of Erdős and Gallai \[7\] on paths.

$$\text{ex}(n, P_k) \leq \frac{k - 2}{2}n. \quad (12)$$

Lemma 4. If $G$ is a $C_\ell$-free graph, then $t(G) \leq \frac{1}{3}(\ell - 3)e(G)$.

Proof. For any vertex $x$, $t(x)$ equals to the number of edges induced by $N(x)$. Therefore,

$$t(G) = \frac{1}{3} \sum_{x \in V(G)} t(x) = \frac{1}{3} \sum_{x \in V(G)} e(G[N(x)]).$$

The subgraph induced by $N(x)$ does not contain $P_{k-1}$, because $G$ is $C_\ell$-free. Therefore, by \[12\], we have

$$e(G[N(x)]) \leq \frac{1}{2}(\ell - 3)\deg(x).$$

We obtain

$$t(G) \leq \frac{1}{3} \sum_{x \in V(G)} \frac{1}{2}(\ell - 3)\deg(x) = \frac{1}{3}(\ell - 3)e(G). \square$$

Note that Lemma \[4\] implies the upper bound \[6\] for $t_{2k}(n)$.
Proof of Theorem 1. Let $G$ be a $C_{2k+1}$-free graph, $k \geq 2$, with the $n$ element vertex set $V$. Let $\mathcal{H}$ be the family of triangles in $G$. Given any 3-partition (or 3-coloring) $\{V_1, V_2, V_3\}$ of $V$ let $\mathcal{H}(V_1, V_2, V_3)$ be the 3-partite induced subhypergraph of $\mathcal{H}$ with these parts, i.e., $\mathcal{H}(V_1, V_2, V_3) := \{T \in \mathcal{H} : |T \cap V_i| = 1 \text{ for all } 1 \leq i \leq 3\}$. Standard averaging argument shows that there is a partition such that each color class $V_i$ with color $i$ has size $\lceil (n+i-1)/3 \rceil$, $1 \leq i \leq 3$, and the number of triples in $\mathcal{H}' := \mathcal{H}(V_1, V_2, V_3)$ is at least $2/9$‘th of the number of triples in $\mathcal{H}$. So we have $|\mathcal{H}| \leq (9/2)|\mathcal{H}'|$. Let $G'$ be the edges of $G$ contained in any triple from $\mathcal{H}'$. Since $t(G) = |\mathcal{H}|$ and $t(G') = |\mathcal{H}'|$, we have $t(G) \leq (9/2)t(G')$. From now on, our aim is to give an upper estimate for $t(G')$. Since $t(G') \leq \frac{1}{3}(2k-2)e(G')$ by Lemma 4, we have that

$$t(G) \leq \frac{9}{2}t(G') \leq 3(k-1)e(G').$$

To complete the proof of Theorem 1 we only need an appropriate upper bound on $e(G')$.

Let $G_{ij}$ be the bipartite subgraph of $G'$ induced by the vertex set $V_i \cup V_j$, $1 \leq i < j \leq 3$. Assume that there exists a copy $L$ of $C_{2k}$ in $G_{ij}$ for some $i$ and $j$. Let $x$ and $y$ be two adjacent vertices in $L$. Since there exists a triangle in $G'$ with vertices $x, y, z$ for some $z \in V_k$ ($k \neq i, j$), there exists a copy of $C_{2k+1}$ in $G$ with the edge set $(E(L) - \{xy\}) \cup \{xz, yz\}$, a contradiction. Therefore, $G_{ij}$ is $C_{2k}$-free. We obtain

$$e(G') = \sum_{1 \leq i < j \leq 3} e(G_{i,j}) \leq 3 \text{ex}([n/3], [n/3], C_{2k}).$$

4. $C_{\ell}^{(3)}$-free 3-uniform hypergraphs

Proof of Theorem 2

For a pair of vertices $u$ and $v$, $\deg_{\mathcal{H}}(u, v)$ (or just $\deg(u, v)$) denotes the number of hyperedges of $\mathcal{H}$ containing both $u$ and $v$.

Proposition 5. Let $\mathcal{H}$ be a $C_{\ell}^{(3)}$-free hypergraph, $\ell \geq 3$. Let $G_2 := G_2(\mathcal{H})$ be the graph on the vertex set of $\mathcal{H}$ such that $E(G_2) := \{uv : \deg(u, v) \geq 2\}$. Then, $G_2$ is $C_{\ell}$-free.

Proof. Suppose, on the contrary, that $L$ is a cycle of length $\ell$ in $G_2$. Let $\mathcal{H}(e)$ be the set of triples from $\mathcal{H}$ containing the pair $e$. Suppose that $\ell \geq 4$, the case $\ell = 3$ is trivial. Then every triple $E \in \mathcal{H}$ contains at most two edges from $E(L)$, but every $e \in E(L)$ is contained in at least two triples, Hall condition holds. I.e., every $i$ edges of $E(L)$ (for $1 \leq i \leq \ell$) are contained in at least $i$ triples. So by Hall’s theorem one can choose a distinct hyperedge from $\mathcal{H}(e)$ for each edge $e$ of $L$. These are forming a Berge cycle of length $\ell$, a contradiction. 

The upper bound on $\text{ex}_3(n, C_{2k+1}^{(3)})$. Let $\mathcal{H}$ be a 3-uniform hypergraph that does not contain $C_{2k+1}^{(3)}$ as a subgraph. Let $G_2$ be defined as in Proposition 5. Then $G_2$ is $C_{2k+1}$-free. Let $\mathcal{H}_2$ be the collection of triples from
\( \mathcal{H} \) having all the three pairs covered at least twice. The edges of \( \mathcal{H}_2 \) induce triangles in \( G_2 \), hence we have

\[
|\mathcal{H}_2| \leq N(G_2; C_3) \leq t_{2k+1}(n). \tag{13}
\]

Let \( \mathcal{H}_1 \) be the set of triples \( E \) from \( \mathcal{H} \) having a pair \( P(E) \) such that \( P(E) \) is contained only in \( E \). Note that \( |\mathcal{H}| = |\mathcal{H}_1| + |\mathcal{H}_2| \). In the following, we find an upper bound for \( |\mathcal{H}_1| \) by defining further subfamilies \( \mathcal{H}_3, \ldots, \mathcal{H}_6 \).

Color the vertices of \( \mathcal{H}_1 \) randomly with two colors. The probability that for an edge \( E \in \mathcal{H}_1 \) the pair \( P(E) \) gets the same color and the vertex \( E \setminus P(E) \) has the opposite color is 1/4. This implies that there is a partition \( V_1 \cup V_2 \) of \( V(\mathcal{H}) \) and a subfamily \( \mathcal{H}_3 \subset \mathcal{H}_1 \) such that \( |\mathcal{H}_3| \geq (1/4)|\mathcal{H}_1| \) and every edge \( E \) of \( \mathcal{H}_3 \) has two vertices in \( V_i \) and one vertex in \( V_{3-i} \) for some \( i \in \{1, 2\} \) such that \( V_i \cap E = P(E) \). Split \( \mathcal{H}_3 \) into two subfamilies as follows.

\[
\mathcal{H}_4 := \{\{u, v, w\} \in \mathcal{H}_3 : P(E) = \{u, v\} \subset V_i, w \in V_{3-i}, \max(\deg(w, u), \deg(w, v)) \geq 3, i \in \{1, 2\}\}
\]

and let \( \mathcal{H}_5 := \mathcal{H}_3 \setminus \mathcal{H}_4 \).

We claim that the graph \( G_4 \) consisting of the pairs \( P(E), E \in \mathcal{H}_4 \), is \( C_{2k} \)-free. Indeed, suppose, on the contrary, that \( L = (v_1, \ldots, v_{2k}) \) is a cycle of \( G_4 \). Since \( G_4 \) has no edge joining \( V_1 \) and \( V_2 \) we may suppose that \( L \subset V_1 \). Consider the triples of \( \mathcal{H}_4 \) containing the edges of \( L, E_i := \{v_i, v_{i+1}, w_i\}, (1 \leq i \leq 2k - 1) \), and \( E_{2k} := \{v_{2k}, v_1, w_{2k}\} \). The vertices \( w_1, \ldots, w_{2k} \) are in \( V_2 \), so they are not on \( L \). Assume that \( \deg(v_1, w_1) \geq 3 \). Then, there is a hyperedge \( E_0 = \{v_1, w_1, u\} \in \mathcal{H} \) different from \( E_1, \ldots, E_{2k} \). The hyperedges \( \{E_0, E_1, E_2, \ldots, E_{2k}\} \) are containing the consecutive pairs \( \{v_1, w_1, v_2, \ldots, v_{2k}\} \) in this cyclic order, so form a Berge cycle of length \( 2k + 1 \). Thus,

\[
|\mathcal{H}_4| = e(G_4) \leq \text{ex}(|V_1|, C_{2k}) + \text{ex}(|V_2|, C_{2k}) \leq \text{ex}(n, C_{2k}). \tag{14}
\]

Because the multiplicity of the pairs in any edge \( E \) in \( \mathcal{H}_5 \) is at most 2, one can use a greedy algorithm to find a subfamily \( \mathcal{H}_6 \subset \mathcal{H}_5 \) such that \( |\mathcal{H}_6| \geq (1/3)|\mathcal{H}_5| \), where \( \mathcal{H}_6 \) is linear, that is each vertex-pair is covered at most once by an edge of \( \mathcal{H}_6 \).

Finally,

\[
|\mathcal{H}| = |\mathcal{H}_1| + |\mathcal{H}_2| \leq 4|\mathcal{H}_3| + |\mathcal{H}_2| = |\mathcal{H}_2| + 4|\mathcal{H}_4| + 4|\mathcal{H}_5| \leq |\mathcal{H}_2| + 4|\mathcal{H}_4| + 12|\mathcal{H}_6|.
\]

This with (13), (14), and the linearity of \( \mathcal{H}_6 \) completes the proof of (9).

The upper bound on \( \text{ex}_3(n, C_{2k}^{(3)}) \).
Let \( \mathcal{H} \) be a 3-uniform hypergraph that does not contain \( C_{2k}^{(3)} \) as a subgraph. Let \( G_2, \mathcal{H}_1, \mathcal{H}_2 \) be defined for \( \mathcal{H} \) as before. By Proposition 5 \( G_2 \) is \( C_{2k} \)-free. Hence, \( |\mathcal{H}_2| \leq N(G_2; C_3) \leq t_{2k}(n) \).
Recall that for each hyperedge $E$ in $H_1$, there exists a vertex-pair, $P(E)$, such that $P(E)$ is contained only in $E$ in $H$. Let $G_1$ be the graph defined by its edge set as $E(G_1) := \{P(E) : E \in H_1\}$. We have that $|H_1| = e(G_1)$. Since $G_1$ is obviously $C_{2k}$-free we get

$$|H| = |H_1| + |H_2| \leq t_{2k}(n) + \text{ex}(n, C_{2k}).$$

\[\square\]

5. $C^{(3)}_{\ell}$-free 3-uniform Linear Hypergraphs

A theta graph of order $\ell$, denoted by $\Theta_{\ell}$, is a cycle $C_{\ell}$ with a chord, where $\ell \geq 4$. The following result was used implicitly in [3] and is stated as a separate lemma in [16] Lemma 2 and also used in [1] and [15]. Let $F$ be a $\Theta$-graph of order $\ell$ and $\ell > t \geq 2$. Let $A \cup B$ be a partition of $V(F)$ with $A, B \neq \emptyset$ such that every path of length $t$ in $F$ that starts in $A$ necessarily ends in $A$. Then $F$ is bipartite with parts $A$ and $B$. We need the following corollary, whose proof is left to the reader.

**Corollary 6.** Let $F$ be a $\Theta$-graph of order $\ell$, where $\ell > t \geq 1$ and $t$ is an odd integer. Let $A \cup B$ be a partition of $V(F)$, $A \neq \emptyset$ such that every path of length $t$ in $F$ that starts in $A$ necessarily ends in $A$. Then $A = V(F)$. \[\square\]

We also use the following easy fact, which is used in [3], [4] and [15], too. If the $n$-vertex graph $G$ contains no $\Theta$-graph of order at least $\ell \geq 4$, then $e(G) \leq (\ell - 2)n$. In other words

$$\text{ex}(n, \Theta_{\ell}) \leq (\ell - 2)n.$$ (15)

**Proof of the upper bound on $\text{ex}^{\text{lin}}_3(n, C^{(3)}_{2k+1})$ in Theorem 3.**

Let $H$ be a 3-uniform hypergraph on $n$ vertices such that no two hyperedges meet in two vertices. Suppose that $H$ contains no $C^{(3)}_{2k+1}$ and let $\delta$ be the third of the the average degree. We have $\sum_{v \in V(H)} \deg(v) = 3|H| = 3\delta n$. Then, there exists a subhypergraph $H'$ on $n'$ vertices such that the degree of each vertex of $H'$ is at least $\delta$. Therefore, we may suppose that every degree of $H$ is at least $\delta$, and also that $\delta \geq 11k$.

The mapping $\pi : H \to \binom{[n]}{2} \cup \emptyset$ is called a *choice function* if $\pi(E) \subset E$ for each $E \in H$. There are $4^{|H|}$ such choice functions. Let $\partial H$ be the set of vertex-pairs contained in the members of $H$ and consider a coloring of $\partial H$, where the color of each pair is given by the single hyperedge of $H$ containing it. We call a subgraph $G$ of $\partial H$ *multicolored*, if all edges of $G$ have different colors under this coloring. For a choice function $\pi$ on $H$, define the graph $G_{\pi}$ as the graph induced by the edge set $\{\pi(E) : \pi(E) \neq \emptyset, E \in H\}$. Because $H$ is a linear hypergraph, for two different hyperedges $E$ and $E'$ in $H$ we have $\pi(E) \neq \pi(E')$. First, we consider the properties of arbitrary multicolored $G_{\pi}$, later we will define a special $\pi$. Clearly, $G_{\pi}$ has no cycle $C_{2k+1}$.
Lemma 7. Let $T$ be a subtree (not necessarily spanning) in $G_{\pi}$, let $x \in V(T)$ be an arbitrary vertex, and let $V_i := N_i(x)$ in $T$, the set of vertices of distance $i$ from $x$ in the tree $T$. Consider $G_i := G_{\pi}[V_i]$, the subgraph of $G_{\pi}$ restricted to $V_i$. Then $G_i$ has no $\Theta$-graph of order $2k$ or larger.

Corollary 8. $e(G_i) \leq (2k - 2)|V_i|$ for $1 \leq i \leq k$.

Proof of Lemma 7. We use induction on $i$. Since $V_0 = x$, and $V_1$ (more exactly $G_1$) contains no path of $2k$ vertices, it does not contain a $\Theta_{\geq 2k}$ either. From now on, we may suppose that $i \geq 2$.

Suppose, on the contrary, that $F$ is a $\Theta$ subgraph of $G_i$ of order $\ell \geq 2k$, $i \geq 2$. For arbitrary $y \in V_i$, let $V_i(y)$ be the subset of descendants of $y$ in $V_i$ in the tree $T$. Consider the partition of $V_i$ defined as $\{V_i(y) : y \in V_i\}$. There exists a $y_1 \in V_i$ such that $A := V(y_1) \cap V(F) \neq \emptyset$.

We claim that $F$ is contained in $V(y_1)$. Note that there is no path $P(a, b)$ of $F$ (neither of $G_i$) of length $2k + 1 - 2i$ that starts in some vertex $a \in A \subset V_i(y_1)$ and ends in another vertex $b \in V_i \setminus V(y_1)$. Otherwise, the $xy_1a$ and $xb$ paths on $T$ have only a single common vertex (namely $x$), have lengths $i$ so together with $P(a, b)$ they form a $C_{2k+1}$ in $G_{\pi}$, a contradiction. Therefore, every path of length $2k + 1 - 2i$ in $F$, that starts in $A$ ends in $A$. Corollary 6 implies that $A = V(F)$, i.e., $V(F) \subset V(y_1)$.

To finish the proof of Lemma 7 simply use induction to the subtree $T_1$ of $T$ consisting of all descendants of $y_1$ and Corollary 6 implies that $A = V(F)$, i.e., $V(F) \subset V(y_1)$.

We say for two sets of sequences of integers $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_k)$ that $\alpha > \beta$, if there is an $i$ such that $a_i > b_i$ and $a_j = b_j$ for all $j < i$. This is called the lexicographical ordering, and it is indeed a linear order.

We are ready to define a concrete $T$ and a choice function $\pi$. Fix a vertex $x \in V(\mathcal{H})$ arbitrarily, let $V_0 := \{x\}$. Consider all choice functions $\pi$ and all multicolored trees of $G_{\pi}$ with root and center $x$ and radius at most $k$. Let $T$ be such a tree for which the sequence of the neighborhood sizes $(|N_1(x)|, \ldots, |N_k(x)|)$ takes its maximum in the lexicographic order. Since $\mathcal{H}$ is linear we have $|N_1(x)| = \deg_\mathcal{H}(x)$. Recall that $N_i(x)$ is denoted by $V_i$, $0 \leq i \leq k$. Our aim is to prove that the sizes of the $|V_i|$’s increase rapidly as follows.

Lemma 9. For $1 \leq i \leq k - 1$ we have $|V_{i+1}| \geq \frac{\delta - 7k}{2k}|V_i|$.

This lemma completes the proof, because we obtain $n \geq |V_k| \geq (\delta - 7k)^{k-1} (2k)^{-k+1} |V_1|$. This and $|V_1| = \deg_\mathcal{H}(x) \geq \delta$ give $2kn^{1/k} + 7k \geq \delta$.

Proof of Lemma 9. Let $\mathcal{H}_i$ be the hyperedges of $\mathcal{H}$ containing the edges of $T$ joining $V_i$ to $V_{i+1}$, $0 \leq i \leq k - 1$, we have $|\mathcal{H}_i| = |V_{i+1}|$. If $uvw = E \in \mathcal{H}_i$ with $u \in V_i$, $v \in V_{i+1}$, then
w \notin V_j$ with $j < i$. Otherwise, leaving out the edge $uv$ from $T$ and joining $uv$ results in a multicolored tree preceding $T$ in the lexicographic order.

Let $B_i$ be the set of hyperedges from $H \setminus (H_0 \cup H_1 \cup \cdots \cup H_i)$ meeting $V_i$, but not meeting $\cup_{j<i} V_j$, $0 \leq i \leq k - 1$. We have $B_0 = \emptyset$. If $E \in B_i$, then $E \subseteq V_i \cup V_{i+1}$. Otherwise, if $v \in E \cap V_i$ and $v \in E \setminus (V_i \cup V_{i+1})$ then truncating our tree at $V_0 \cup V_1 \cup \cdots \cup V_{i+1}$ and joining the edge $uv$ result in another tree lexicographically larger than $T$.

Let $B^\alpha_i$, $0 \leq i \leq k - 1$, be the set of those hyperedges from $B_i$, that meet $V_i$ exactly in $\alpha$ vertices, $\alpha = 1, 2$ or 3. The graph $G_i$, for $1 \leq i \leq k - 1$, is defined on the vertex set $V_i$ as follows. It contains exactly one vertex-pair from each member of $B^3_i$ and the pairs $E \cap V_i$ for $E \in B^2_i \cup B^1_i$. For $i = k$, the edge set of $G_k$ consists only of the sets $\{E \cap V_k : E \in B^1_{k-1}\}$, since $B_k$ is undefined. The graph $G_{\pi}$ consisting of the edges of $T$ and the $G_i$'s, $1 \leq i \leq k$, is a multicolored subgraph. So Corollary 8 implies that

$$e(G_i) \leq (2k - 2)|V_i|. \quad \text{(16)}$$

Consider the $H$-degrees of the elements of $V_i$, $(1 \leq i \leq k - 1)$. Their total sum is at least $\delta|V_i|$. Obviously,

$$\sum_{v \in V_i} \deg_B(v) = \sum_{E \in E} |E \cap V_i|.$$ 

The edges of $H$ meeting $V_i$ belong to some $H_j$, $j < i$, or to $B_{i-1} \cup B_i$. An edge $E \in H_j$ can meet $V_i$ in at least two elements, only if $j$ is equal to $i - 1$ or $i$. We obtain for $1 \leq i \leq k - 1$

$$\delta|V_i| \leq \sum_{v \in V_i} \deg_B(v) = \sum_{E \in E} |E \cap V_i|$$

$$\leq \left( \sum_{0 \leq j \leq i-2} |H_j| \right) + 2|H_{i-1}| + 2|H_i| + |B^2_{i-1}| + 2|B^1_{i-1}| + 3|B^3_i| + 2|B^2_i| + |B^1_i|.$$ 

Inequality (16) implies that

$$|B^2_{i-1}| \leq e(G_{i-1}) \leq (2k - 2)|V_{i-1}|,$$

$$2|B^1_{i-1}| + 3|B^3_i| + 2|B^2_i| \leq 3(|B^3_{i-1}| + |B^1_i| + |B^2_i|) = 3e(G_i) \leq (6k - 6)|V_i|,$$

$$|B^1_i| \leq e(G_{i+1}) \leq (2k - 2)|V_{i+1}|.$$ 

Using these inequalities and the fact that $|H_j| = |V_{j+1}|$ we obtain that

$$\delta|V_i| \leq \left( \sum_{1 \leq j \leq i-1} |V_j| \right) + 2|V_i| + 2|V_{i+1}| + (2k - 2)|V_{i-1}| + (6k - 6)|V_i| + (2k - 2)|V_{i+1}|.$$ 

By rearranging we have

$$(\delta - (6k - 4))|V_i| \leq \left( \sum_{1 \leq j \leq i-1} |V_j| \right) + (2k - 2)|V_{i-1}| + 2k|V_{i+1}|. \quad \text{(17)}$$
For $i = 1$ the fact that $\mathcal{B}_0 = \emptyset$ implies the slightly stronger $(\delta - (6k - 4))|V_1| \leq 2k|V_2|$. So Lemma 9 holds for $i = 1$. For larger $i$ we use induction and (17) to prove first that $2|V_i| \leq |V_{i+1}|$ for all $i < k$ and then the sharper inequality of Lemma 9. □

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