Method for searching higher symmetries for quad-graph equations

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Received 4 April 2011, in final form 27 June 2011
Published 15 July 2011
Online at stacks.iop.org/JPhysA/44/325202

Abstract

A generalized symmetry integrability test for discrete equations on the square lattice is studied. Integrability conditions are discussed. A method for searching higher symmetries (including non-autonomous ones) for quad-graph equations is suggested based on characteristic vector fields.

PACS number: 02.30.Ik

1. Introduction

Consider nonlinear equations on the quad-graph (or double discrete chains) of the form

\[ u_{1,1} = f(u, u_1, \bar{u}_1). \]  

(1)

Here the unknown \( u = u(m, n) \) is a function of two discrete variables \( m, n \). For the sake of convenience, we use the following notations: \( u_k = u(m + k, n) \), \( \bar{u}_k = u(m, n + k) \), \( u_{1,1} = u(m + 1, n + 1) \). The function \( f \) is supposed to be locally smooth. It depends essentially on all three arguments, i.e. there is a domain in the three-dimensional space of the variables \( u, u_1, \bar{u}_1 \) where the derivatives \( \frac{\partial f}{\partial u}, \frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial \bar{u}_1} \) do not vanish identically. This means that equation (1) can be solved for any of the variables \( u, u_1, \bar{u}_1 \). In other words, equation (1) can be rewritten in any of the following forms:

\[ u_{i,j} = f^{i,j}(u, u_i, \bar{u}_j), \quad \text{with} \quad i = \pm 1, \quad j = \pm 1. \]  

(2)

Equations of the form (1) have important applications and the problem of the complete description of their integrable cases is challenging. At present there are various approaches for studying the integrability of discrete models. In [1–5], the property of consistency around a cube is considered as a criterion of integrability. Another classification method called vanishing of the algebraic entropy is developed in [6–9]. The symmetry approach is approved to provide a powerful classification tool for integrable discrete and continuous models [10–17].
It was observed earlier that characteristic vector fields provide an effective classification tool for integrable models either in the case of Darboux integrability [18–21] or S-integrability [22–24]. Recall that in the discrete case, these vector fields are defined as follows: $Y_1 = \tilde{D}^{-1} \frac{\partial}{\partial u_1} \tilde{D}$ and $Y_{-1} = D_{\bar{u}_{-1}} \tilde{D}^{-1}$ and have the following coordinate representation [19]:

$$Y_1 = \frac{\partial}{\partial u} + x \frac{\partial}{\partial u_1} + \frac{1}{x-1} \frac{\partial}{\partial u_{-1}} + x x_1 \frac{\partial}{\partial u_2} + \frac{1}{x-1} \frac{\partial}{\partial u_{-2}} + \cdots,$$

and

$$Y_{-1} = \frac{\partial}{\partial u} + y \frac{\partial}{\partial u_1} + \frac{1}{y-1} \frac{\partial}{\partial u_{-1}} + y y_1 \frac{\partial}{\partial u_2} + \frac{1}{y-1} \frac{\partial}{\partial u_{-2}} + \cdots,$$

where $x = \tilde{D}^{-1} \left( \frac{\partial f(u,u_1,\bar{u}_{-1})}{\partial u} \right) = -\frac{\partial g(u,u_1,\bar{u}_{-1})}{\partial u}$ and $y = \tilde{D} \left( \frac{\partial f(u_1,u,\bar{u}_{-1})}{\partial u_1} \right) = -\frac{\partial g(u,u_1,\bar{u}_{-1})}{\partial u_1}$. Here the shift operators $D$ and $\bar{D}$ act due to the rule $Dh(m, n) = h(m + 1, n)$, $\bar{D}h(m, n) = h(m, n + 1)$. The vector fields $Y_1$, $Y_{-1}$, $X_1 = \frac{\partial}{\partial u_1}$ and $X_{-1} = \frac{\partial}{\partial u_{-1}}$ considered as operators applied to the variables $\bar{u}_1$, $u_{-1}$, $u, u_{\pm 1}, u_{\pm 2}, \ldots$ satisfy the following conjugation relations:

$$DX_1 D^{-1} = p X_1, \quad DY_1 D^{-1} = \frac{1}{x} Y_1, \quad DX_{-1} D^{-1} = q X_{-1}, \quad DY_{-1} D^{-1} = \frac{1}{y} Y_{-1},$$

where $p = D \left( \frac{\partial f^{-1}(u,u_1,\bar{u}_{-1})}{\partial u_1} \right) = \frac{1}{\frac{\partial f^{-1}(u,u_1,\bar{u}_{-1})}{\partial u_1}}$ and $q = D \left( \frac{\partial f^{-1}(u,u_1,\bar{u}_{-1})}{\partial u_{-1}} \right) = \frac{1}{\frac{\partial f^{-1}(u,u_1,\bar{u}_{-1})}{\partial u_{-1}}}$.

To stress the close connection of these operators with the symmetry approach, we note that the restrictions of the characteristic vector fields $Y_1$ and $Y_{-1}$ on the set of functions depending only on $u, u_1, u_{-1}$ coincide with the differential operators, introduced by P E Hydon [10] and successfully applied in [11, 12, 14] to look for conservation laws and symmetries:

$$A = \frac{\partial}{\partial u} - \frac{f_u^{-1}}{f_{u_1}^{-1}} \frac{\partial}{\partial u_1} - \frac{f_{u_{-1}}^{-1}}{f_{u_{-1}}^{-1}} \frac{\partial}{\partial u_{-1}},$$

and

$$B = \frac{\partial}{\partial u} - \frac{f_u}{f_{u_1}} \frac{\partial}{\partial u_1} - \frac{f_{u_{-1}}}{f_{u_{-1}}^{-1}} \frac{\partial}{\partial u_{-1}}.$$

To compare operators (3), (4), (6), (7), use the following formulas:

$$1/x_1 = -f_u^{-1} / f_{u_1}^{-1} \quad \text{and} \quad 1/y_{-1} = -f_u^{-1} / f_{u_{-1}}^{-1}.$$

Our work was stimulated by the results obtained by Levi and Yamilov. In the above-mentioned articles [11, 12], they made a good progress in a formalization of the theory of integrable discrete models and gave an effective tool to look for autonomous symmetries of (1) of the form

$$u_1 = g(u, u_1, u_{-1}, \bar{u}_1, \bar{u}_{-1}).$$

The goal of the present paper is to suggest a method suitable for searching higher symmetries of equation (1) of general form (13) depending also on discrete arguments $m, n$. It is important since there are integrable equations (1) having no any symmetry of the form (9) but having symmetries (13). We know at least two examples of such a kind, one of them is equation (11) considered below and the other is Adler’s discretization of the Tzitzeica equation (AdT) recently suggested in [25].
In [16] such a basic concept of the symmetry approach as a formal recursion operator is extended to the discrete case. Actually recursion operators are very important, they allow one to describe effectively the whole hierarchy of higher symmetries of the chain when the seed symmetry is known. The necessary integrability conditions for chain (1) are derived in [16] under an additional assumption that the chain admits a recursion operator given as a formal series of the form

$$R = r_1 D + r_2 D^{-1} + r_3 D^{-2} + \cdots,$$

with coefficients $r_j$. Note that this assumption is restrictive. For instance, the above-mentioned AdT equation does not admit such a kind of recursion operator.

This paper is organized as follows. In section 2, we discuss on a sequence of integrability conditions implied by the requirement of existence of symmetries (13) of sufficiently high order without any additional assumption. These integrability conditions are derived by consecutive differentiation of the linearized equation with respect to the dynamical variables. They are in fact functional equations. We give the first four of the conditions in an explicit form (see proposition 1 below). Among them, the first two were deduced earlier in [11].

In a particular case when the chain under investigation admits a symmetry of the form (9), integrability conditions formulated in proposition 1 can easily be transformed to the integrability conditions found earlier in [16] (see equations (34), (35), (38), (39) in [16]). In section 3 by using characteristic vector fields we deduce differential consequences of the integrability conditions from section 2. The consequences are given as systems of linear first-order partial differential equations (PDEs). Compatibility conditions of these systems are the effective necessary integrability conditions since they are easily checked for any given quad-graph equation of the form (1). At the same time, the systems provide an algorithm of constructing symmetries. In lemma 2, it is proved that any symmetry (13) below splits down into a sum of two functions like $g(m, n, u_j, \bar{u}_j, \bar{u}_{j-1}, \ldots, \bar{u}_k) = F(m, n, u_j, \bar{u}_{j-1}, \ldots, \bar{u}_k) + G(m, n, u_j, \bar{u}_{j-1}, \ldots, \bar{u}_k)$. Recall that in the particular case (9), this fact has been proved in [15]. Efficiency of the symmetry-finding algorithm is demonstrated in sections 4 and 5. In section 4, a non-autonomous symmetry

$$g = (-1)^{m+n}C\frac{u(u^2-1)(u_1u_{-1}+1)}{(uu_1+u_1-u+1)(uu_{-1}-u_{-1}+u+1)}$$

(10)

is found for the equation

$$u_{1,1}u(u_1-1)(\bar{u}_1+1)+(u_1+1)(\bar{u}_1-1) = 0$$

(11)

suggested in [9]. This equation was supposed to be integrable by the algebraic entropy criterion. Then it was proved in [12] that equation (11) has no five-point autonomous symmetries of the form (9). In the recent paper [26], the connection of this equation with a discrete SG equation is discussed. In section 5, a fifth-order symmetry $u_\xi = g$ is evaluated for the same equation (11), where

$$g = \frac{uu_1(u_1-1)(u^2-1)}{(u_1+1+uu_1-u)^2(1-u_1+u_2+uu_1)} - \frac{uu_1(u(u_1-1))}{(uu_1-u_1)(uu_{-1}-u_1)} - \frac{uu_1(u(u_1-1))}{(u_1+1+uu_1-u)^2(1-u_{-1}+uu_{-1}+u)^2}.$$  

(12)

For certain cases, the systems (27), (30), (37), (44) derived in section 3 are sufficient to find the final form of a fifth-order symmetry if it does exist. It is the case for
equation (11) mentioned above. However, they are not the only differential consequences of the integrability conditions listed in proposition 1. In section 6, we briefly discuss how to derive additional systems of linear first-order PDEs, consistency of which is also necessary for the existence of higher symmetries.

2. Higher symmetries and integrability conditions

The existence of higher symmetries is believed to be an important indication of integrability. Assume that chain (1) admits a higher symmetry (generally non-autonomous) of the form

\[ u_j = g(m, n, u_{j'}, u_{j'-1}, \ldots, u_{j'}, \tilde{u}, \tilde{u}_{k-1}, \ldots, \tilde{u}_k), \]

(13)

where \( j \geq 1, j' \leq -1, k \geq 1, k' \leq -1 \). As it is well known, the function \( g \) should satisfy the linearization of chain (1):

\[ D\bar{D}g = f_{u_k}Dg + f_{\tilde{u}_k}\bar{D}g + f_{u}g. \]

(14)

Lemma 1. Linearized equation (14) can be rewritten in any of the following equivalent forms:

\[ D^{-1}\bar{D}g = f^{-1}_{u_k}D^{-1}g + f^{-1}_{\tilde{u}_k}\bar{D}g + f^{-1}_{u}g, \]

\[ D\bar{D}^{-1}g = f^{-1}_{u_k}D\bar{D}^{-1}g + f^{-1}_{\tilde{u}_k}\bar{D}g + f^{-1}_{u}g, \]

(15)

\[ D^{-1}\bar{D}^{-1}g = f^{-1}_{u_k}D^{-1}\bar{D}^{-1}g + f^{-1}_{\tilde{u}_k}\bar{D}^{-1}g + f^{-1}_{u}g. \]

Proof. Let us prove the first statement of lemma 1. To this end, apply the operator \( D^{-1} \) to equation (14), and divide the result by \( D^{-1}f_{u_k} \):

\[ D^{-1}\bar{D}g = -D^{-1}(f_u/f_{u_k})D^{-1}g + D^{-1}(1/f_{u_k})\bar{D}g - D^{-1}(f_{\tilde{u}_k}/f_{u_k})g. \]

Due to the relations

\[ D^{-1}(f_u/f_{u_k}) = -f^{-1}_{u_k}, \quad D^{-1}(1/f_{u_k}) = f^{-1}_{u_k}, \quad D^{-1}(f_{\tilde{u}_k}/f_{u_k}) = -f^{-1}_{u_k}, \]

the latter is reduced to the first equation in (15). The other formulas are verified in a similar way. \( \square \)

Differentiation of equation (14) with respect to the highest order variable \( u_{j+1} \) yields (see [11])

\[ D\bar{D}g_{u_k}D^{j+1}f_{u_k} = f_{u_k}Dg_{u_k}. \]

(16)

Set \( z := \log g_{u_k} \) and rewrite the last equation in the form of a conservation law

\[ \bar{z}_1 = z + \log D^{-1} - D^{j+1} \log f_{u_k}. \]

(17)

Differentiation of (15) with respect to the lowest order variable \( u_{j-1} \) yields

\[ D^{-1}\tilde{D}g_{u_k}D^{j+1}f_{u_k}^{-1} = f_{u_k}^{-1}Dg_{u_k}. \]

(18)

Obviously, equation (18) implies for \( v := \log g_{u_{j'}} \):

\[ \bar{v}_1 = v + \log (D - D^{j'+1}) \log f^{-1}_{u_k}. \]

(19)

Actually, equations (17) and (19) provide the necessary conditions of the integrability of equation (1): there exist numbers \( j \) and \( j' \) such that the functions \( (D^{-1} - D^{j+1}) \log f_{u_k} \) and \( (D - D^{j'+1}) \log f^{-1}_{u_k} \) belong to the image of the operator \( E = \bar{D} \), where \( E \) is the identity operator. These integrability conditions derived for the first time in [11] are highly nonlocal, since each equation contains the values of unknowns \( z, v \) taken at
two different values of their arguments \( z = z(m, n, u_j, u_{j-1}, \ldots, u_{j'}, \tilde{u}_{k-1}, \ldots, \tilde{u}_k), \)
\( v = v(m, n, u_j, u_{j-1}, \ldots, u_{j'}, \tilde{u}_{k-1}, \ldots, \tilde{u}_k) \) and \( \tilde{z} = z(m, n + 1, u_j, u_{j-1}, \ldots, u_{j'}, \tilde{u}_{k+1}, \tilde{u}_{k-1}, \ldots, \tilde{u}_k) \).

In other words, equations (17) and (19) are functional equations.

To derive the next necessary condition of the existence of a symmetry of the form (13)
with \( j \geq 2 \), differentiate equation (14) with respect to the variable \( u_j' \): 
\[
D_\tilde{D}(g_{a_j,\nu})D^{j-1}(f_{a_j}) + D_\tilde{D}(g_{a_j})[D^j(f_{a_j})D^{j-1}(f_{a_j})] = f_{a_j}D(g_{a_{j+1}})
\]
\[
+ f_{a_{j+1}}\tilde{D}(g_{a_j})D^{j-1}(f_{a_{j+1}}) + f_{a_j}g_{a_{j+1}}.
\]

Put \( z^{(1)} := g_{a_{j+1}} \) and rewrite equation (20) as follows:
\[
z^{(1)} = r(m, n, \tilde{u}_1, u_j, u_{j-1}, \ldots, u_{j'})z^{(1)} + R(m, n, \tilde{u}_1, u_j, u_{j-1}, \ldots, u_{j'}),
\]

where \( r = \frac{D^{j-1}(f_{a_j})}{D^{j-2}(f_{a_j})} \) and
\[
R = \frac{1}{D^{j-2}(f_{a_j})}D^{-1}(f_{a_j})D^{j-1}(f_{a_j}) + f_{a_j}g_{a_{j+1}} - D\tilde{D}(g_{a_j})D^j(f_{a_j}) - D(g_{a_j})D^{j-1}(f_{a_{j+1}}).
\]

Thus a symmetry exists only when the function \( R \) is in the image of the operator \( \tilde{D} - rE \).

Differentiating equation (15) with respect to the variable \( u_j' \) one obtains for any \( j' \leq -2 \)
\[
D^{-1}\tilde{D}(g_{a_j,\nu})D^{j+1}(f_{a_j}) + D\tilde{D}(g_{a_j})[D^{j+1}(f_{a_j}) + D'(f_{a_{j+1}})D^{j+1}(f_{a_{j+1}})]
\]
\[
= f_{a_{j+1}}^{-1}D^{-1}(g_{a_{j+1}}) + f_{a_{j+1}}^{-1}\tilde{D}(g_{a_j})D^{j+1}(f_{a_{j+1}}) + f_{a_{j+1}}^{-1}g_{a_{j+1}}.
\]

Put \( \bar{v}^{(1)} := g_{a_{j+1}} \) and rewrite the last equation as follows:
\[
\bar{v}^{(1)} = \tilde{r}(m, n, \tilde{u}_1, u_j, u, \ldots, u_{j'})v^{(1)} + \tilde{R}(m, n, \tilde{u}_1, u_j, u_{j-1}, \ldots, u_{j'}),
\]

where the functions
\[
\tilde{r} = \frac{D(f_{a_{j+1}}^{-1})}{D^{j+2}(f_{a_{j+1}}^{-1})}
\]
and
\[
\tilde{R} = \frac{1}{D^{j+2}(f_{a_{j+1}}^{-1})}D\left[f_{a_{j+1}}^{-1}\tilde{D}(g_{a_j})D^{j+1}(f_{a_{j+1}}) + f_{a_{j+1}}^{-1}g_{a_{j+1}} - D^{-1}\tilde{D}(g_{a_j})D^{j+1}(f_{a_{j+1}})\right]
\]
\[
- D(g_{a_j})D^{j+1}(f_{a_{j+1}}^{-1})
\]
are expressed through the known quantities \( f_{j+1}^{-1}, v \), their derivatives and shifts. Obviously equation (23) provides one more integrability condition: the function \( \tilde{R} \) should be in the image of the operator \( \tilde{D} - \tilde{r}E \).

Finalizing the reasonings above, we obtain the following statement.

**Proposition 1.** If an equation of the form (1) admits a higher symmetry of sufficiently great order, then the following conditions hold for some entire \( j \) and \( j' \):

1. \( (D^{-1} - D^{j-1}) \log f_{a_j} \in \text{Im}(E - \tilde{D}) \) for \( j \geq 1 \);
2. \( (D - D^{j+1}) \log f_{a_{j+1}}^{-1} \in \text{Im}(E - \tilde{D}) \) for \( j' \leq -1 \);
3. \( R(m, n, \tilde{u}_1, u_j, u_{j-1}, \ldots, u_{j'}) \in \text{Im}(E - \tilde{D}) \) for \( j \geq 2 \);
4. \( \tilde{R}(m, n, \tilde{u}_1, u_j, u_{j+1}, u_{j-1}, \ldots, u_{j'}) \in \text{Im}(E - \tilde{D}) \) for \( j' \leq -2 \).

In such a way, one can derive a sequence of integrability conditions given as functional equations. Obviously, in this form integrability conditions are not very effective. In the next section, we suggest a rule to derive some differential consequences of these equations which have a form of systems of first-order PDEs and can effectively be studied by standard methods of differential algebra.
3. Algorithm of finding symmetries

Concentrate first on equation (17). Note that \( z \) might depend only on the variables \( u_j, u_{j-1}, \ldots, u_f \), it does not depend on \( \bar{u}_j \) with \( j \neq 0 \). Indeed, suppose that \( z \) depends on \( \bar{u}_k, k > 0 \), and does not depend on \( \bar{u}_s, s > k \); then evidently \( \bar{z}_1 \) depends on \( \bar{u}_{k+1} \) that is not possible since it contradicts equation (17). The case \( k < 0 \) is studied analogously. Reduce (17) to a system of first-order PDEs. To this end, applying the operator \( \bar{D}_{-1} \frac{\partial}{\partial \bar{u}_1} \) to both sides of the equation gives

\[
Y_1 z = f^{(1)},
\]

where \( f^{(1)} = \bar{D}_{-1} \frac{\partial}{\partial \bar{u}_1} (\bar{D}^{-1} - D_i^{-1}) \log f_{u_i} \) and \( Y_1 = \bar{D}_{-1} \frac{\partial}{\partial \bar{u}_1} \bar{D} \) (see formula (3) above). Note that the coefficients of equation (24) depend on the variable \( \bar{u}_{-1} \), in spite of the solution \( z \) cannot depend on it. That is why, one has to put an additional condition

\[
X_1 z = 0,
\]

where \( X_1 = \frac{\partial}{\partial \bar{u}_{-1}} \).

In a similar way the functional equation (17) implies two more differential equations. Indeed, the equation can be represented as \( \bar{z}_{-1} = z - \bar{D}^{-1} (\bar{D}^{-1} - D_i^{-1}) \log f_{u_i} \). Applying the operator \( \bar{D} X_1 \) gives

\[
Y_{-1} z = f^{(-1)},
\]

where \( f^{(-1)} = -Y_{-1} (\bar{D}^{-1} - D_i^{-1}) \log f_{u_i} \) and \( Y_{-1} = \bar{D}_{-1} \frac{\partial}{\partial \bar{u}_{-1}} \bar{D}^{-1} \) (see formula (4) above).

Thus, we come up to a system of the first-order non-homogeneous linear equations

\[
\begin{align*}
X_1 z &= 0, \\
Y_1 z &= f^{(1)}, \\
X_{-1} z &= 0, \\
Y_{-1} z &= f^{(-1)}. 
\end{align*}
\]

Note that for \( j = 1, j' = -1 \), the system (27) in essence coincides with that suggested in [11]. Compatibility of the system (27) is necessary for the existence of symmetry (13). Emphasize that generally the system (27) is not closed. To close it, we have to add all its linearly independent differential consequences obtained by taking cross applications of the operators such as \([X_1, Y_1] z = X_1 f^{(1)} \), \([Y_1, Y_{-1}] z = Y_1 f^{(-1)} - Y_{-1} f^{(1)} \) etc. In such a way, we arrive at a system of the form

\[
L_s z := \sum_{i=j}^{j'} a_{s,i} \frac{\partial z}{\partial u_i} = F(s), \quad s = 1, 2, \ldots, N, \\
X_1 z = 0, \\
X_{-1} z = 0,
\]

satisfying the following condition: any further cross application of the operators \( L_s \) gives an equation of the form \([L_s, L_r] z = L_s (F(r)) - L_r (F(s)) \) linearly expressed through the already known equations (28). Then the system is closed.

Introduce the notations \( p^{(s)} = \frac{\partial z}{\partial u_i} \) and rewrite the system (28) as a system of linear algebraic equations:

\[
\sum_{i=j}^{j'} a_{s,i} p^{(i)} = F(s), \quad s = 1, 2, \ldots, N.
\]
The last two equations in (28) are valid automatically since \( z \) does not depend on \( \bar{u}_1, \bar{u}_{-1} \).

Due to the well-known Kronecker–Capelli theorem, the system of linear equations (29) is compatible if and only if the rank of the coefficient matrix \( A = (a_{ij}) \) is equal to that of the augmented matrix \( B \) obtained from \( A \) by adding the column of free terms \( F^{(1)} \). Thus, the condition \( \text{rank}(A) = \text{rank}(B) \) is necessary for the existence of a symmetry (13).

Let us deduce some differential consequences of equation (19). Just applying the reasonings above to this equation, we obtain

\[
\begin{align*}
X_1 v &= 0, \\
Y_1 v &= \tilde{f}^{(1)}, \\
X_{-1} v &= 0, \\
Y_{-1} v &= \tilde{f}^{(-1)},
\end{align*}
\]

where \( \tilde{f}^{(1)} = D^{-1} \frac{\partial}{\partial \bar{u}_1} (D - D^{(1)}) \log f_{u_{-1}}^{(-1)} \) and \( \tilde{f}^{(-1)} = -Y_{-1} \left[(D - D^{(-1)}) \log f_{u_{-1}}^{(-1)}\right]. \)

Similar to the previous case, one can close the system (30) and then reduce it to a system of linear algebraic equations. We omit this part.

Study the third condition of proposition 1, which requires the consistency of the functional equation (21). Note that again unknown \( z^{(1)} \) does not depend on the variable \( \bar{u}_1 \); hence, applying the operator \( D^{-1} \frac{\partial}{\partial \bar{u}_1} \) to equation (21) yields

\[
Y_1 z^{(1)} = D^{-1} (r \bar{u}_1) z^{(1)} + D^{-1} R \bar{u}_1.
\]

Exclude \( z^{(1)} \) by means of the same equation (21) to obtain

\[
Y_1 z^{(1)} = A^{(1)} z^{(1)} + B^{(1)},
\]

where \( A^{(1)} = D^{-1} (\log r) \bar{u}_1 \) and \( B^{(1)} = D^{-1} (R \bar{u}_1 - R (\log r) \bar{u}_1) \). Rewrite equation (21) as follows:

\[
z^{(1)} = \frac{1}{r} (\bar{z}_1^{(1)} - R)
\]

and shift it back by applying the operator \( D^{-1} \)

\[
\bar{z}_{-1}^{(1)} = D^{-1} \left( \frac{1}{r} \right) z^{(1)} - D^{-1} \left( \frac{R}{r} \right).
\]

Apply the operator \( D \frac{\partial}{\partial \bar{u}_{-1}} \) to the last equation to obtain

\[
Y_{-1} z^{(1)} = Y_{-1} \left( \frac{1}{r} \right) \bar{z}_{-1}^{(1)} - Y_{-1} \left( \frac{R}{r} \right),
\]

which obviously is rewritten as follows:

\[
Y_{-1} z^{(1)} = A^{(-1)} z^{(1)} + B^{(-1)},
\]

where \( A^{-1} = r Y_{-1} (1/r) \) and \( B^{-1} = -\frac{1}{r} Y_{-1} (R) \). As was done above, we put two more equations \( X_1 z^{(1)} = 0 \) and \( X_{-1} z^{(1)} = 0 \). Thus, we find a consequence of the necessary condition (3) of proposition 1. The following system of the first-order linear PDEs should be consistent:

\[
\begin{align*}
Y_1 z^{(1)} &= A^{(1)} z^{(1)} + B^{(1)}, \\
X_1 z^{(1)} &= 0, \\
Y_{-1} z^{(1)} &= A^{(-1)} z^{(1)} + B^{(-1)}, \\
X_{-1} z^{(1)} &= 0.
\end{align*}
\]
Deduce differential consequences from the fourth condition in proposition 1; to this end, apply the operator \( \bar{D}^{-1} \frac{\partial}{\partial \bar{z}} \) to equation (23):

\[
Y_{1} v^{(1)} = \bar{D}^{-1}(\bar{g}_{\bar{u}}) v^{(1)} + \bar{D}^{-1} \bar{R}_{\bar{u}}. 
\]

(38)

Exclude \( \bar{v}_{-1}^{(1)} \) by means of equation (23) to obtain

\[
y_{1} \bar{v}^{(1)} = \bar{A}^{(1)} v^{(1)} + \bar{B}^{(1)},
\]

where \( \bar{A}^{(1)} = \bar{D}^{-1}(\log \bar{r})_{\bar{u}} \), and \( \bar{B}^{(1)} = \bar{D}^{-1}(\bar{R}_{\bar{u}} - \bar{R}(\log \bar{r})_{\bar{u}}) \). Rewrite equation (23) as follows:

\[
v^{(1)} = \frac{1}{\bar{r}} (\bar{v}_{1}^{(1)} - \bar{R})
\]

(40)

and shift it back by applying the operator \( \bar{D}^{-1} \):

\[
\bar{v}_{-1}^{(1)} = \bar{D}^{-1} \left( \frac{1}{\bar{r}} \right) (v^{(1)} - \bar{D}^{-1} \left( \frac{\bar{R}}{\bar{r}} \right)).
\]

(41)

Apply the operator \( \bar{D} \frac{\partial}{\partial \bar{z}} \) to the last equation to obtain

\[
Y_{-1} v^{(1)} = Y_{-1} \left( \frac{1}{\bar{r}} \right) \bar{v}_{1}^{(1)} - Y_{-1} \left( \frac{\bar{R}}{\bar{r}} \right).
\]

(42)

which obviously is rewritten as follows:

\[
Y_{-1} v^{(1)} = \bar{A}^{(-1)} v^{(1)} + \bar{B}^{(-1)},
\]

(43)

where \( \bar{A}^{-1} = \bar{Y}_{-1}(1/\bar{r}) \) and \( \bar{B}^{-1} = -\frac{1}{2} \bar{Y}_{-1}(\bar{R}) \). As was done above, we put two more equations \( X_{1} v^{(1)} = 0 \) and \( X_{-1} v^{(1)} = 0 \). Thus, we find a consequence of the necessary condition (4) of proposition 1. The following system of the first-order linear PDEs should be compatible:

\[
\begin{align*}
Y_{1} v^{(1)} &= A^{(1)} z^{(1)} + B^{(1)}, \\
X_{1} v^{(1)} &= 0, \\
Y_{-1} v^{(1)} &= A^{(-1)} v^{(1)} + B^{(-1)}, \\
X_{-1} v^{(1)} &= 0.
\end{align*}
\]

(44)

**Proposition 2.** The compatibility conditions of the systems (27), (30), (37) and (44) provide the effective necessary integrability conditions for chain (1).

Recall that integrability is understood here as the existence of symmetries of sufficiently high order. Continuing this way, one can derive the \( \mathfrak{h} \)th integrability condition which is formulated as a compatibility condition for a system of equations for the function \( z^{(\mathfrak{h})} := g_{\bar{u}} \) with \( j' \leq s \leq \mathfrak{h} \). It is easy to prove by induction that all of the functions \( g_{\bar{u}} \) depend only on the variables \( m, n, u_{j}, u_{j-1}, \ldots, u_{j'} \). This observation allows us to prove the following statement on the decomposition of a symmetry of the lattice (1).

**Lemma 2.** Any symmetry (13) of the lattice (1) splits down into a sum of two functions:

\[
g(m, n, u_{j}, u_{j-1}, \ldots, u_{j'}) = F(m, n, u_{j}, u_{j-1}, \ldots, u_{j'}) + G(m, n, \bar{u}_{k}, \bar{u}_{k-1}, \ldots, \bar{u}_{k'}).
\]

Here it is supposed that \( u = \bar{u} = u \).

**Proof.** We observed above that

\[
g_{\bar{u}} = a_{s}(m, n, u_{j}, u_{j-1}, \ldots, u_{j'}), \quad \text{where} \quad j' \leq s \leq \mathfrak{h}.
\]

These formulas yield

\[
\begin{align*}
g_{\bar{u}, \bar{u}} &= 0, \quad \text{where} \quad j' \leq s \leq \mathfrak{h}, \quad k' \leq r \leq k, \quad r \neq 0, \quad s \neq 0.
\end{align*}
\]

The latter immediately proves the lemma. \( \square \)
4. Evaluation of a non-autonomous symmetry for the quad-graph equation

It is proved in [12] that equation (11) does not admit any autonomous symmetry of the form (9), but it is conjectured there that it might have a non-autonomous symmetry:

\[ u_t = g(m, n, u_{-1}, u, u_1). \]  
(45)

Apply the scheme above to equation (11) to look for its symmetry (45). Due to the results of section 3, the function \( z = \log g_{u_1} \) solves the system of equations (27). Let us study the second and fourth equations in (27) since the other ones hold automatically. Rewrite them in coordinates:

\[
\frac{u^2 - 1}{2u} \frac{\partial z}{\partial u_{-1}} + \frac{\partial z}{\partial u} + \frac{2u_1}{u^2 - 1} \frac{\partial z}{\partial u_1} = \frac{u^2 - 2u - 1}{u(u^2 - 1)}; \\
\frac{2u_{-1}}{u^2 - 1} \frac{\partial z}{\partial u_{-1}} - \frac{\partial z}{\partial u} + \frac{u^2 - 1 - 1}{2u} \frac{\partial z}{\partial u_1} = \frac{-u^2 u_1 + 2u^2 - u_1}{u(u^2 - 1)}. 
\]

Note that the latter is not closed. To close this system, one has to add one more equation

\[ [Y_1, Y_{-1}] z = Y_1 \tilde{f}^{(1)} - Y_{-1} f^{(1)}. \]  

Now solve the obtained system of three equations with respect to the partial derivatives of \( z \):

\[
\frac{\partial z}{\partial u_{-1}} = 0, \quad \frac{\partial z}{\partial u_1} = -\frac{2(u_1 - 1)}{u_1(u + 1) + 1 - u}, \\
\frac{\partial z}{\partial u} = -\frac{2(u_1 - 1)}{u_1(u + 1) + 1 - u} + \frac{1}{u} + \frac{1}{u + 1} + \frac{1}{u - 1} 
\]

This triple of equations is an overdetermined system of equations for \( z \). It is directly checked to be consistent. Hence, its general solution \( z \) is easily found. It contains an arbitrary function \( C_1(m, n) \) depending on both discrete variables:

\[ z = \log \frac{C_1(m, n) u(u^2 - 1)}{(u u_1 + u_1 - u + 1)^2}. \]

Substitution of \( z \) into (17) yields \( \log \frac{-C_1(m, n + 1)}{C_1(m, n)} = 0 \). The last equation is easily solved, \( C_1(m, n) = (-1)^n C_2(m) \), where \( C_2(m) \) is an arbitrary function of one discrete variable.

Therefore, by solving the equation \( z = \log g_{u_1} \), we find

\[ g = -(-1)^n \frac{C_2(m) u(u - 1)}{u u_1 + u_1 - u + 1} + g_2(m, n, u_{-1}, u). \]

For the further specification, consider \( v(u_{-1}, u) = \log g_{u_{-1}} = \log g_{2u_{-1}}, \) which due to the general scheme solves the system of equations (30), the essential part of which is reduced to the form

\[
\frac{u^2 - 1}{2u} \frac{\partial v}{\partial u_{-1}} + \frac{\partial v}{\partial u} = \frac{2u^2 - 2uu_{-1} + u_{-1}}{u(u^2 - 1)}; \\
\frac{2u_{-1}}{u^2 - 1} \frac{\partial v}{\partial u_{-1}} - \frac{\partial v}{\partial u} = \frac{1 - u^2 - 2u}{u(u^2 - 1)}. 
\]

Now solve them with respect to the derivatives

\[
\frac{\partial v}{\partial u_{-1}} = -\frac{2(u - 1)}{u_{-1}(u - 1) + u + 1}, \\
\frac{\partial v}{\partial u} = -\frac{2(u_{-1} + 1)}{u_{-1}(u - 1) + 1 + u} + \frac{1}{u} + \frac{1}{u + 1} + \frac{1}{u - 1} 
\]
and then find \( v \):
\[
v = \log \frac{C_3(m, n)u(u^2 - 1)}{(uu_{-1} - u_{-1} + u + 1)^2}.
\]
Substitute it into the difference equation (19) and obtain the relation
\[
\log \frac{-C_3(m, n + 1)}{C_3(m, n)} = 0,
\]
the solution to which is \( C_3(m, n) = (-1)^n C_4(m) \), where \( C_4(m) \) is an arbitrary function of \( m \).

As a result, the function \( g \) takes the form
\[
g = -(-1)^n C_2(m)u(u - 1) + (-1)^n C_4(m)u(u + 1) + g_3(u).
\]

Thus using two necessary conditions (27) and (30) allowed us to find the symmetry searched up to the unknown function \( g_3(u) \) of one variable. To find \( g_3(u) \), one can use directly the linearized equation (14). By substituting the function obtained into the linearized equation and applying to the result the following operator:
\[
\frac{(u_{-1} - 1)^2}{2} D^{-1} \frac{\partial}{\partial u_1} \frac{1}{u_1} \frac{\partial}{\partial u_1} \frac{1}{u_1} - 1,
\]
we come to the relation
\[
g''(u) - \frac{1}{u} g'(u) + \frac{1}{u^2} g(u) = \frac{u^2 + 1}{(uu_{-1} - u_{-1} + u + 1)^2} (C_2(m - 1) - C_4(m)),
\]
which immediately implies
\[
g''(u) - \frac{1}{u} g'(u) + \frac{1}{u^2} g(u) = 0, \quad C_4(m) = C_2(m - 1).
\]

Therefore \( g_3(u) = C_5(m, n)u + C_6(m, n)u \log u \), where the coefficients do not depend on \( u \) but might depend on \( m, n \).

Now we have
\[
g = -(-1)^n \frac{C_2(m)u(u - 1)}{uu_1 + u_1 - u + 1} - (-1)^n \frac{C_2(m - 1)u(u + 1)}{uu_{-1} - u_{-1} + u + 1} + C_5(m, n)u + C_6(m, n)u \log u.
\]

Substitute it into the linearized equation and apply the operator
\[
\frac{1}{u_1} D^{-1} \frac{\partial}{\partial u_1} \frac{1}{u_1} \frac{1}{u_1^2} - 1,
\]
to obtain
\[
2(u^2 + 1)C_5(m, n) + 2((u^2 + 1) \log u - u^2 + 1)C_6(m, n) + 2(u^2 - 1)C_6(m + 1, n) - C_2(m)(u - 1)^2 + C_2(m - 1)(u + 1)^2 = 0.
\]

Collecting the coefficients before linearly independent functions \( u^2 + 1, u^2 - 1, 2u, u^2 + 1 \), we obtain the system of equations
\[
C_6(m, n) = 0, \quad C_6(m + 1, n) = 0, \quad C_2(m - 1) + C_2(m) = 0, \quad C_2(m - 1) - C_2(m) + 2C_5(m, n) = 0.
\]

Thus,
\[
C_2(m) = -(-1)^n C, \quad C_6(m, n) = 0, \quad C_5(m, n) = (-1)^n (-1)^n C.
\]

Due to the last equalities, we find the final form of the symmetry desired:
\[
g = (-1)^{mn} C \frac{u(u^2 - 1)(uu_1 + 1)}{(uu_1 + u_1 - u + 1)(uu_{-1} - u_{-1} + u + 1)}.
\]
5. Evaluation of a fifth-order symmetry for the quad-graph equation

In this section, we proceed with the testing of our symmetry-finding algorithm. Apply it again to equation (11). Now we will look for a fifth-order symmetry for it:

\[ u_z = g(u_2, u_1, u, u_{-1}, u_{-2}). \]  (46)

Start with equations (24) and (26) for the function \( z = \log g_{u_z} \). These equations being as follows \( Y_1z = f^{(1)} \), \( Y_{-1}z = f^{(-1)} \), in an enlarged form are

\[
\begin{align*}
\frac{(u^2_{-1} - 1)(u^2_{-2} - 1)}{4u_{-1}u} \frac{\partial z}{\partial u_{-2}} + \frac{u^2_{-1} - 1}{u^2} \frac{\partial z}{\partial u_{-1}} + \frac{\partial z}{\partial u} + \frac{2u_1}{u^2 - 1} \frac{\partial z}{\partial u_1} + \frac{4u_2u_1}{(u^2 - 1)u} \frac{\partial z}{\partial u_2} = \frac{1}{u} - \frac{4u_1}{u^2 - 1};
\end{align*}
\]  (47)

\[
\begin{align*}
\frac{4u_{-2}u_{-1}}{(u^2 - 1)(u^2_{-1} - 1)} \frac{\partial z}{\partial u_{-2}} - \frac{2u_{-1}}{u^2 - 1} \frac{\partial z}{\partial u_{-1}} + \frac{\partial z}{\partial u} = \frac{1}{u^2 - 1} - \frac{2u_2(u^2_{-1} - 1)}{2u_1u}.\end{align*}
\]  (48)

Obviously the system (47), (48) is not closed. To close it, we add its differential consequences obtained by taking cross applications of the operators \( Y_1, Y_{-1} \):

\[
\begin{align*}
Y_1z = f^{(1)},
Y_{-1}z = f^{(-1)},
[Y_1, Y_{-1}]z = Y_1f^{(-1)} - Y_{-1}f^{(1)} = : \tilde{f},
[Y_1, [Y_1, Y_{-1}]]z = Y_1\tilde{f} - [Y_1, Y_{-1}]f^{(1)},
[Y_{-1}, [Y_1, Y_{-1}]]z = Y_{-1}\tilde{f} - [Y_1, Y_{-1}]f^{(-1)}.
\end{align*}
\]  (49)

The system (49) is closed and consistent. Solving it with respect to the derivatives yields

\[
\begin{align*}
\frac{\partial z}{\partial u_{-2}} &= 0, \quad \frac{\partial z}{\partial u_{-1}} = 0, \quad \frac{\partial z}{\partial u_2} = -\frac{2(u_1 + 1)}{1 - u_1 + u_2 + u_2u_1},
\end{align*}
\]

\[
\begin{align*}
\frac{\partial z}{\partial u} = \frac{1}{u} + \frac{2(1 - u_1)}{u_1 + 1 + u_1u - u} + \frac{2u}{u^2 - 1},
\end{align*}
\]

\[
\begin{align*}
\frac{\partial z}{\partial u_1} = \frac{1}{u_1} - \frac{2(u + 1)}{u_1 + 1 + u_1u - u} + \frac{2(1 - u_2)}{1 - u_1 + u_2 + u_2u_1} + \frac{2u_1}{u^2_{-1} - 1}.
\end{align*}
\]

Find \( z \) by integrating

\[
\begin{align*}
z &= \log \frac{u_1u(u^2_{-1} - 1)(u^2 - 1)(u + 1)}{(1 - u_1 + u_2 + u_2u_1)^2(u_1 + 1 + u_1u - u)^2},
\end{align*}
\]

which yields immediately an intermediate representation of \( g \):

\[
\begin{align*}
g &= \frac{u_1u(u^2_{-1} - 1)(u^2 - 1)}{(u_1 + 1 + u_1u - u)^2(1 - u_1 + u_2 + u_2u_1)} + g_{(u_1, u, u_{-1}, u_{-2})}.
\end{align*}
\]

For the further specification of \( g \), let us find \( v(u_1, u, u_{-1}, u_{-2}) = \log g_{u_z} = \log g_{1, u_{-2}} \). The function \( v = v(u_1, u, u_{-1}, u_{-2}) \) should satisfy the equations (compare with (30))

\[
\begin{align*}
\frac{(u^2_{-1} - 1)(u^2_{-2} - 1)}{4u_{-1}u} \frac{\partial v}{\partial u_{-2}} + \frac{u^2_{-1} - 1}{u^2} \frac{\partial v}{\partial u_{-1}} + \frac{\partial v}{\partial u} + \frac{2u_1}{u^2 - 1} \frac{\partial v}{\partial u_1} + \frac{4u_2u_1}{(u^2 - 1)u} \frac{\partial v}{\partial u_2} = \frac{2u}{(u + 1)(u - 1)} - \frac{(u_1 + 1)u_2(u_{-1} - 1)}{2u_{-1}u};
\end{align*}
\]  (50)
The system of equations (50), (51) can be closed by taking cross application of the operators \( Y_1 \) and \( Y_{-1} \). After some manipulations of such a kind, which are omitted, one finds the partial derivatives of \( v \):

\[
\frac{\partial v}{\partial a_{-2}} = \frac{2(1 - u_{-1})}{1 + u_{-1} - u_{-2} + u_{-2}u_{-1}}, \quad \frac{\partial v}{\partial u_{1}} = 0,
\]

\[
\frac{\partial v}{\partial u_{-1}} = \frac{1}{u_{-1}} - \frac{2(u_{-2} + 1)}{1 + u_{-1} - u_{-2} + u_{-2}u_{-1}} = \frac{2(u_{-1} + 1)}{u_{-1} + u_{-2} + u_{-2}u_{-1} - 1} + \frac{2u_{-1}}{u_{-1} + 1 + u_{-2} + u_{-2}u_{-1} - 1}.
\]

Thus \( v \) is

\[
v = \log C - u_{-1}u(u_{-1}^2 - 1)(u^2 - 1)
\]

Since \( v = \log g_{u_{-2}} = \log g_{1,u_{-2}} \), by integrating we obtain more detailed representation for \( g \):

\[
g = \frac{uu_{1}(u_{1} - 1)(u^2 - 1)}{(u_{1} + 1 + u_{1}u - u)^2(1 - u_{1} + u_{2} + u_{2}u_{1})} + C \frac{u_{-1}u(u_{-1} + 1)(u^2 - 1)}{(-u_{-1} + u_{-2}u + u_{-2}u_{-1})^2(1 + u_{-1} - u_{-2} + u_{-2}u_{-1})} + g_{2}(u_{-1}, u, u_{1}),
\]

containing an unknown tail \( g_{2}(u_{-1}, u, u_{1}) \). Due to equation (32), the function \( y^{(1)} = \partial g_{2}/\partial u_{1} \) closely connected with \( z^{(1)} = \partial g/\partial u_{1} \) solves the following system of equations:

\[
\frac{u_{-1}^2 - 1}{2u} \frac{\partial y^{(1)}}{\partial u_{-1}} + \frac{2u_{1}}{u^2 - 1} \frac{\partial y^{(1)}}{\partial u_{1}} = \frac{(u_{1}^2 - 1)(u^2 - 1) y^{(1)}}{u(u^2 - 1)},
\]

\[
\frac{2u_{1} - 1}{u^2 - 1} y^{(1)} = \frac{2u_{1}}{u^2 - 1} \frac{\partial y^{(1)}}{\partial u_{1}} + \frac{2u_{1}^2 - 1}{2u} \frac{\partial y^{(1)}}{\partial u} + \frac{2u_{1}^2 + u_{1}^2 u_{1} - u_{1}}{u(u^2 - 1)} y^{(1)} = \frac{u(u_{1}^2 - 1)^2}{u^2 - 1}
\]

\[
\times \left( \frac{2}{(u_{1} + 1 + uu_{1} - u)^3} + \frac{-2u_{1}^2 u_{1} - u + 3uu_{1}^2 + 2u_{1} - 1 - u_{1}^2}{(-u_{1} + 1 + uu_{1} + u)^2(u_{1} + 1 + uu_{1} - u)^2} \right).
\]

These equations imply

\[
\frac{\partial y^{(1)}}{\partial u_{1}} = \frac{4u_{1}^2(u^2 - 1)^2(u_{1}u_{1} - 1)}{(u_{1} + uu_{1} - u)^3};
\]

\[
\frac{\partial y^{(1)}}{\partial u_{-1}} = \frac{2(u_{1} + 1)}{u_{1} + uu_{1} - u} y^{(1)} = \frac{2(u_{1} - 1)(u^2 - 1)^2 u}{(u_{1} + uu_{1} - u)^4}.
\]

We omit the expression for \( \frac{\partial y^{(1)}}{\partial u_{1}} \) since it is huge.
Thus, we obtain the next representation for the searched $g$:

$$g = \frac{uu_1(u_1 - 1)(u^2 - 1)}{(u_1 + 1 + uu_1 - u^2)(1 - u_1 + u_2 + uu_1)} + C \frac{u - 1 + uu_1 + u}{(u_1 + 1 + uu_1 - u)(u_1 + 1 + uu_1 + u)}$$

$$= \frac{uu_1(u_1 - 1)(u^2 - 1)}{(u_1 + 1 + uu_1 - u^2)(1 - u_1 + u_2 + uu_1)} + C \frac{u - 1 + uu_1 + u}{(u_1 + 1 + uu_1 - u)(u_1 + 1 + uu_1 + u)}$$

where $C_2$ is the constant of integration. By substituting this representation of $g$ into the linearized equation (14) and differentiating it with respect to $u_1$, we obtain $C_2 = 0$. In order to look for $g_3$, differentiate the linearized equation with respect to $u_1$ and study according to the scheme above the linear equations for $u_1$, we find the next specification of $g$:

$$u_{-1}^2 - 1 \frac{\partial y^{(2)}}{\partial u_{-1}} + \frac{\partial y^{(2)}}{\partial u} + u_1^2 - 2u_1^2 - u_1 - u_1 \frac{\partial y^{(2)}}{\partial u} = \frac{(3uu_1 + u + uu_1 + 1)(u - 1)}{(u_1 + 1)(1 - uu_1 + uu_1 + uu_1 + u)^3}$$

$$\frac{2u_1}{u_1^2 - 1} \frac{\partial y^{(2)}}{\partial u_{-1}} - \frac{\partial y^{(2)}}{\partial u} + \frac{(u_1^2 - 1 + 2u_1)}{(u_1 - 1)(u_1 + 1)u} = \frac{(u - 1)(1 + uu_1 + uu_1 + uu_1 - uu_1)}{(u_1 + 1)(1 - uu_1 + uu_1 + uu_1 + u)^3},$$

By solving these equations, we find the next specification of $g$:

$$g = \frac{uu_1(u_1 - 1)(u^2 - 1)}{(u_1 + 1 + uu_1 - u^2)(1 - u_1 + u_2 + uu_1)} + C_1 \frac{uu_1(u_1 - 1)(u^2 - 1)}{(u_1 + 1 + uu_1 - u^2)(1 - u_1 + u_2 + uu_1)}$$

$$= \frac{uu_1(u_1 - 1)(u^2 - 1)}{(u_1 + 1 + uu_1 - u^2)(1 - u_1 + u_2 + uu_1)} + C_1 \frac{uu_1(u_1 - 1)(u^2 - 1)}{(u_1 + 1 + uu_1 - u^2)(1 - u_1 + u_2 + uu_1)}$$

By substituting it into the linearized equation and differentiating the result with respect to the variable $u_1$, we prove that $C_3 = 0$ and that $g_4(u)$ solves a linear homogeneous equation which has a unique solution $g_4 = 0$. Thus, the final form of the symmetry searched is (see also (12) above)

$$u_1 = g = \frac{uu_1(u_1 - 1)(u^2 - 1)}{(u_1 + 1 + uu_1 - u^2)(1 - u_1 + u_2 + uu_1)} + C_1 \frac{uu_1(u_1 - 1)(u^2 - 1)}{(u_1 + 1 + uu_1 - u^2)(1 - u_1 + u_2 + uu_1)}$$
6. Higher order characteristic vector fields and additional differential consequences of integrability conditions

For certain cases, the systems (27), (30), (37), (44) derived in section 3 are sufficient to find the final form of the symmetry searched if it does exist. But for the other ones, these systems do not provide enough information to find the symmetry desired or to make the final decision on its existence. It happens, for instance, when \( \text{rank}(A) = \text{rank}(B) < j - j' + 1 \); see (29). Here we briefly discuss how to derive additional systems of linear first-order PDEs, the consistency of which is also necessary for the existence of higher symmetries. To this end, we can make use of the higher characteristic vector fields defined in such a way that

\[
Y_k = \bar{D}^{-k}X_{-1}\bar{D}^k \quad \text{and} \quad Y_{-k} = \bar{D}^kX_1\bar{D}^{-k} \quad \text{for} \quad \forall k > 1.
\]

(52)

It follows from equation (17) that \( \bar{z}_2 = z + (1 + \bar{D})(D^{-1} - D^{-1}) \log f_{u_1} \). By applying the operator \( \bar{D}^{-2}X_{-1} \) to the last equation, we find

\[
Y_2z = f^{(2)},
\]

where \( f^{(2)} = \bar{D}^{-2} \frac{\partial}{\partial u_2} (1 + \bar{D})(D^{-1} - D^{-1}) \log f_{u_1} \). The coordinate representation of the vector field \( Y_2 \) is as follows:

\[
Y_2 = X_1 + \bar{D}^{-1}(Y_1f) \frac{\partial}{\partial u_1} + \bar{D}^{-1}(Y_1f^{-1.1}) \frac{\partial}{\partial u_{-1}} + \bar{D}^{-1}(Y_1f_1) \frac{\partial}{\partial u_2} + \cdots.
\]

(54)

The coefficients of this equation depend on the extra variable \( u_{-2} \); hence, we require

\[
X_2z = 0,
\]

(55)

where \( X_2 = \frac{\partial}{\partial u_2} \).

Represent now equation (17) as \( \bar{z}_2 = z - (\bar{D}^{-1} + \bar{D}^{-1})(D^{-1} - D^{-1}) \log f_{u_1} \). By applying the reasonings above, one obtains a pair of equations

\[
Y_{-2}z = f^{(-2)}, \quad X_{-2}z = 0,
\]

(56)

where \( f^{(-2)} = -Y_{-2}(1 + \bar{D})(D^{-1} - D^{-1}) \log f_{u_1} \), and \( X_{-2} = \frac{\partial}{\partial u_{-2}} \).

Iterating this way, one obtains a system of the form

\[
Y_iz = f^{(i)}, \quad X_iz = 0, \quad -N \leq i \leq N, \quad i \neq 0,
\]

(57)

which should be consistent for any value of the natural \( N \). Here \( X_i = \frac{\partial}{\partial u_i} \), and the functions \( f^{(i)} \) are expressed through the given functions \( f, f^{-1.1}, f^{-1.1}, f^{-1.1} \) and their derivatives and shifts. Recall that the functions \( f^{(1)}, f^{(-1)} \) are defined earlier in (24) and (26).

Obviously such a kind of extended consequences can be deduced from any integrability condition discussed in section 2.

7. Conclusions

Integrability conditions for quad-graph equations are studied based on the symmetry approach. The effective consequences of the existence of higher symmetries are derived by using characteristic vector fields. These conditions can be successfully used for testing certain classes of quad-graph equations, depending on a finite number of undetermined constants or unknown functions of one or two variables. The symmetry approach might be effective in classifying
integrable quadrilinear equations of the form $Q(u, u_1, \bar{u}_1, u_1, 1) = 0$ having a freedom only on constants. At present, this work is in progress. An algorithm for evaluating higher symmetries (generally non-autonomous) for autonomous quad-graph equations is suggested. The efficiency of the algorithm is approved by applying it to an example recently found in [9]. Third- and fifth-order symmetries for this equation are evaluated, and the former turns out to be non-autonomous. Even though the chain itself is autonomous, nevertheless it can have non-autonomous symmetries. The algorithm discussed above in section 3 is effective to look for either autonomous or non-autonomous symmetries. But in the latter case, when solving the corresponding differential equations one should keep in mind that the constants of integration appearing might depend on the discrete arguments $m, n$.

Acknowledgments

This work is partially supported by Russian Foundation for Basic Research (RFBR) grants 10-01-91222-CT-a, 11-01-97005-r-povoljie-a and 10-01-00088-a.

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