Relativistic fluid dynamics is an important model to understand various collective phenomena in astrophysics and heavy-ion collisions. However, the relativistically covariant extension of the Navier-Stokes equations is acausal and unstable \[1\]. The reason is that the irreversible currents (the shear stress tensor $\sigma^{\mu\nu}$, the bulk viscous pressure $\Pi$ etc.) are linearly proportional to the thermodynamic forces (the shear tensor $\sigma^{\mu\nu}$, the expansion scalar $\theta$ etc.), with the constant of proportionality being the shear viscosity coefficient $\eta$, the bulk viscosity coefficient $\zeta$ etc.. Thus, the forces have an instantaneous influence on the currents, which obviously violates causality and leads to instabilities. These problems are solved by introducing retardation into the definitions of the irreversible currents, leading to equations of motion for these currents which thus become independent dynamical variables. Theories of this type are called causally relativistic dissipative fluid dynamics (CRDF).

With the retardation, in general the irreversible currents and the thermodynamic forces are no longer linearly proportional to each other. As a consequence, the transport coefficients for CRDF cannot be computed applying the methods commonly used for Navier-Stokes fluids, such as the Green-Kubo-Nakano (GKN) formula. So far, there are several approaches to derive the transport coefficients of CRDF \[4–7\]. This formula is the analogue of the GKN formula of CRDF from time-correlation functions was proposed \[4–7\]. This formula is the analogue of the GKN formula in Navier-Stokes fluids. Since this formula is derived from quantum field theory, it will be applicable even to dense fluids.

However, in a leading-order perturbative calculation which should apply in the dilute limit, i.e., the regime of applicability of the kinetic approach, the field-theoretical formula gives results which are different from those of the IS calculation. Is this inconsistency due to a problem with the field-theoretical formula or with the kinetic calculation? Or is there simply no correspondence between the field-theoretical and the kinetic derivation of the transport coefficients of CRDF? This would come as a surprise, as this correspondence does exist in the case of Navier-Stokes fluids.

In this letter, we show that the field-theoretical and kinetic calculations are indeed consistent, even for CRDF. The key point is that the 14-moment approximation employed by Israel and Stewart is not unique. We suggest a new method to obtain equations of motion for the irreversible currents, which leads to expressions for the transport coefficients which are different from the IS results. The new transport coefficients turn out to be consistent with those obtained from the field-theoretical formula.

In the original IS calculation, the evolution equations of the shear stress tensor and the bulk viscous pressure are obtained from the second moment of the Boltzmann equation,

$$
\partial_\mu \int \frac{d^3k}{(2\pi)^3 E_k} K^\mu K^\nu f = \int \frac{d^3k}{(2\pi)^3 E_k} K^\nu K^\rho C[f],
$$

(1)

where $f$ is the single-particle distribution function, $K^\mu = (E_k, \mathbf{k})$ with $E_k = \sqrt{k^2 + m^2}$, and $C[f]$ is the collision term. In order to obtain a closed set of equations, one assumes a specific form for $f$,

$$
f = f_0 + f_0 (1 - a f_0) (e + e_\mu K^\mu + e_\mu_\nu K^\mu K^\nu),
$$

(2)

where $e$, $e^\mu$, and $e^{\mu\nu}$ constitute a set of 14 independent parameters related to the irreversible currents by matching conditions and $f_0 = (e^{\beta \omega_0 K^\mu} + a)^{-1}$ is the single-particle distribution function in local equilibrium, with $a = \pm 1$ for fermions/bosons; $\beta \equiv 1/T$ is the inverse temperature. Then Eq. \[1\] is decomposed into scalar, vector, and tensor parts which are interpreted as the evolution equations of the bulk viscous pressure, the particle diffusion (heat conduction) current, and the shear stress tensor, respectively.

The idea of the new kinetic calculation is as follows \[11\]. In the kinetic approach, the shear stress tensor and
the bulk viscous pressure are always defined by

$$
\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \frac{d^3k}{(2\pi)^3 E_k} \kappa^\alpha K^\beta (f - f_0),
$$

(3)

$$
\Pi = -\frac{m^2}{3} \frac{d^3k}{(2\pi)^3 E_k} (f - f_0).
$$

(4)

The tensor $\Delta^{\mu\nu\alpha\beta} = (\Delta^{\mu\alpha} \Delta^{\nu\beta} + \Delta^{\mu\beta} \Delta^{\nu\alpha} - (2/3)\Delta^{\mu\nu} \Delta^{\alpha\beta})/2$, with $\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}$. The evolution equations for $\pi^{\mu\nu}$ and $\Pi$ are now obtained directly by applying the comoving time derivative and substituting the Boltzmann equation together with Eq. (2). Details will be presented elsewhere. 

The final result for the evolution equations is

$$
\Delta^{\mu\nu}_{\alpha\beta} u^{\nu} \partial_\mu \pi^{\alpha\beta} = -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2(\beta_\eta + \eta_{\Pi} \Pi) \sigma^{\mu\nu}
-2\eta_{\tau_\pi} \Delta^{\nu}_{\alpha\beta} \pi^{\alpha\beta} \eta_{\nu} + 2\Delta^{\mu\nu}_{\alpha\beta} \sigma^{\alpha\beta} \lambda + 2\eta_{\Pi} \delta^{\mu\nu} \lambda \eta + \Pi_{\eta} \pi^{\mu\nu} \sigma_{\mu\nu},
$$

(5)

$$
u^{\nu} \partial_\mu \Pi = -\frac{\Pi}{\tau_\Pi} - (\beta_\zeta + \zeta_{\Pi} \Pi) \lambda + \Pi_{\Pi} \pi^{\mu\nu} \sigma_{\mu\nu},
$$

(6)

where $\nabla^{\mu} = \Delta^{\mu\nu} \partial_\nu$, $\theta = \partial_\nu u^{\nu}$ is the expansion scalar, $\sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \nabla^{\alpha} w^{\beta}$ is the shear tensor, and $\omega^{\mu\nu} = (\nabla^{\mu} u^{\nu} - \nabla^{\nu} u^{\mu})/2$ is the vorticity. The relaxation times for shear and bulk viscous pressure are $\tau_\pi$ and $\tau_\Pi$, respectively, and the transport coefficients $\beta_\eta \equiv \eta/\tau_\pi$ and $\beta_\zeta \equiv \zeta/\tau_\Pi$. The other coefficients in Eqs. (5, 6) play no role in the following discussion.

In general, the values of the transport coefficients depend on the collision term. However, in the ratios $\beta_\eta$, $\beta_\zeta$ of viscosity coefficients to relaxation times the collision term drops out; these ratios are simply thermodynamic functions

$$
\beta_\eta = \frac{15}{1} \left[ 9P + \varepsilon - \frac{n}{3} \frac{d^3k}{(2\pi)^3 E_k} f_0(k) \right],
$$

(7)

$$
\beta_\zeta = \left( \frac{1}{3} - c_s^2 \right) (\varepsilon + P) - \frac{2}{9} (\varepsilon - 3P)
- \frac{m^2}{9} \frac{d^3k}{(2\pi)^3 E_k^3} f_0(k).
$$

(8)

Here, $\varepsilon$, $P$, and $c_s^2 = dP/d\varepsilon$ are the energy density, the pressure, and the velocity of sound squared, respectively. In the following, we shall prove that the values [7, 8] obtained in the new kinetic calculation are consistent with the field-theoretical approach.

The field-theoretical approach uses the projection operator method [9, 10]. First, we discuss the shear viscosity. The expression for $\beta_\eta$ is

$$
\beta_{\eta, ft} = (\varepsilon + P) \frac{d^3x \left( T^{\mu\nu} (x), T^{\mu\nu} (0) \right)}{d^3x \left( T^{\mu\nu} (x), T^{\mu\nu} (0) \right)},
$$

(9)

where $T^{\mu\nu}$ is the energy-momentum tensor and the inner product is defined by Kubo’s canonical correlation, $(X, Y) = \int_0^3 d\lambda \beta^{-1} (e^{\lambda H} X e^{-\lambda H} Y)_{eq}$. Here, $H$ is the Hamiltonian and $(\cdots)_{eq}$ indicates the thermal expectation value.

In order to compare with kinetic theory, it is sufficient to calculate this correlation functions appearing in Eq. (9) in the free gas approximation. Using $\int d^3x \left( T^{\mu\nu} (x), T^{\mu\nu} (0) \right) = (\varepsilon + P)/\beta$, we obtain for bosons

$$
\beta_{\eta, ft} = -\beta \frac{\partial}{\partial \beta} \left[ \frac{d^3k}{(2\pi)^3} \frac{(k^2)^2}{15 E_k} f_0(k) \right.
+ \frac{d^3k}{(2\pi)^3} \frac{(k^2)^2}{30 E_k} \left( 1 + 2 f_0(k) \right) \bigg],
$$

(10)

The first term in Eq. (10) contains the derivative with respect to $\beta$, which is re-expressed using

$$
\frac{\partial f_0(k)}{\partial \beta} = \frac{E_k}{\beta}
\times \lim_{P \to 0} \frac{f_0(k + p)f_0(k) - f_0(k)f_0(k + p)}{E_{k + p} - E_k}.
$$

(11)

In many-body physics [12], this term can be interpreted as the contribution from the collisions of bosons. The second term in Eq. (11) contains an ultraviolet divergent term due to the vacuum self-energy and is re-expressed as

$$
1 + 2 f_0(k) = \lim_{p \to 0} \left[ f_0(k + p)f_0(k) - f_0(k)f_0(k + p) \right],
$$

(12)

This term corresponds to the pair annihilation-creation (PAC) part.

Usually, pair annihilation and creation processes are not considered in the Boltzmann equation. Thus, for the sake of a consistent comparison, we compare only the collision part in Eq. (10) with the kinetic result. A straightforward integration by parts gives the result [7], i.e., it is identical with the result of the new kinetic calculation. The full result, i.e., including the PAC part, is

$$
\beta_{\eta, ft} = P,
$$

(13)

which has already been quoted in Ref. [4]. Here, the vacuum contribution has been neglected.

The temperature dependence of $\beta_\eta/(\varepsilon + P)$ is shown in Fig. 1. One observes that the values obtained from the field-theoretical formula are larger than those for the new kinetic and the IS calculation. The reason is that the PAC part makes a non-negligible contribution to $\beta_{\eta, ft}$. But even without this part, the new kinetic calculation gives values which are above the original IS result. Since $\beta_\eta/(\varepsilon + P)$ is related to the signal propagation speed in CRDF, we expect effects on the collective behavior of relativistic fluids [1]. The same transport coefficients of CRDF were calculated from the Boltzmann equation in Ref. [8], but the results depend on the coupling strength and are different from our results.
The detailed derivation will be given in Ref.\cite{7}.

For the sake of comparison, the result from an AdS/CFT calculation, \( \beta_n/(\varepsilon + P) = 1/[2(2 - \ln 2)] \), is shown by the dash-dotted line.

Next, we consider \( \beta_\zeta \). In the field-theoretical approach, this is given by

\[
\beta_{\zeta,ft} = (\varepsilon + P) \frac{\int d^3x \langle \Pi(x), \Pi(0) \rangle}{\int d^3x \langle T^{0x}(x), T^{0x}(0) \rangle},
\]

where \( \Pi = [(1 - 3 c_s^2) T^{00} - T^{\mu\nu}] \) \cite{8}. For bosons in the free gas approximation, we obtain

\[
\beta_{\zeta,ft} = -\beta \int \frac{d^3k}{(2\pi)^3} \frac{1}{9 E_k^{3/2}} \left( k^2 - 3 c_s^2 E_k^3 \right)^2 \frac{\partial}{\partial \beta} f_0(k)
+ \frac{m^4}{18} \int \frac{d^3k}{(2\pi)^3} E_k \left[ 1 + 2 f_0(k) \right].
\]

Similarly to the case of \( \beta_n,ft \), the first and second terms are interpreted as the collision part and the PAC part, respectively. The collision part can be re-expressed using integration by parts, thermodynamic relationships, and the definition of \( c_s^2 \). It is then found to be exactly the same as the new kinetic result \cite{9}.

By taking the PAC part into account, the field-theoretical approach yields

\[
\beta_{\zeta,ft} = \left( \frac{1}{3} - c_s^2 \right) (\varepsilon + P) - \frac{2}{9} (\varepsilon - 3P).
\]

The detailed derivation will be given in Ref.\cite{10}.

The temperature dependence of \( \beta_\zeta/(\varepsilon + P) \) is shown in Fig.\text{2}. Similarly to the case of the shear viscosity, the new kinetic calculation and the field-theoretical formula predict larger values than the IS calculation. However, the difference is now more than an order of magnitude.

For \( \beta_\zeta \) the PAC part qualitatively changes the behavior, in particular, at low temperatures. If we neglect the PAC part, \( \beta_\zeta \) increases rapidly in the low-temperature region, and starts to decrease at high temperature. This behavior is the same in all kinetic calculations. However, when we consider the PAC part, \( \beta_\zeta \) becomes a monotonously decreasing function of temperature. This is the same as the behavior predicted by a string theoretical calculation \cite{10}.

In the case of the kinetic calculations, \( \beta_n \) and \( \beta_\zeta \) have the same forms for fermions as for bosons. This is true even for the field-theoretical approach, if the PAC part is neglected. Then, \( \beta_n \) and \( \beta_\zeta \) for fermions are given by Eqs.\text{7} and \text{8}, respectively, i.e., the field-theoretical approach and the new kinetic calculation yield the same results. However, the contributions from the PAC terms depend on the statistics. For fermions, the full results of the field-theoretical calculation are

\[
\beta_{n,ft} = 0,
\]
 \[
\beta_{\zeta,ft} = \left( \frac{1}{3} - c_s^2 \right) (\varepsilon + P) - \frac{1}{3} (\varepsilon - 3P).
\]

The results of the field-theoretical calculation may be expressed in a unified way as

\[
\beta_{n,ft} = |3 - \alpha| P,
\]
 \[
\beta_{\zeta,ft} = \left( \frac{s}{3} \frac{d}{ds} - \frac{\alpha}{9} \right) (\varepsilon - 3P),
\]
where $\alpha = 2$ for boson and $\alpha = 3$ for fermion, and $s$ is the entropy density. Note that, for a mixed system of bosons and fermions, the correlation functions which appear in the numerators and denominators of Eqs. (14) and (19) are the sum of both contributions, respectively.

Note that these calculations are leading-order results and will be modified by the effect of interactions. For example, the exact expression for $\tau_p/\beta$ is given by the ratio of the real and imaginary parts of the retarded Green’s function of $T^{\mu\nu}$. However, in the leading-order calculation, the real part is approximated by the result for the free-gas approximation, while the imaginary part is not. The divergent $\tau_p$ for fermions, leading to a vanishing $\beta_{s,\tau}$ in Eq. (19), will be rendered finite by a more complete calculation.

As shown in Ref. [13], there is a sum rule for the bulk viscous pressure. There, the correlation function for bulk viscous pressure was calculated for interacting gauge bosons. In the weak-coupling limit, the result is reproduced by setting $\alpha = 4$ in Eq. (20). Similar correlation functions were studied in Lattice QCD [14].

Now we can answer the question posed in the introduction. The inconsistency between the field-theoretical and kinetic calculations is due to a problem with the IS calculation. We have suggested an alternative kinetic calculation, the results of which are consistent with the field-theoretical approach. Thus, even in CRDF, there is a relation between the field-theoretical and kinetic calculations, just as for Navier-Stokes fluids.

At the same time, we found that, in the field-theoretical formula, there is a contribution from the PAC part which is not included in the kinetic calculation. In a relativistic setting, particle annihilation and creation processes may occur, which re-distribute momenta as well as influence chemical equilibrium, thus affecting both the shear and the bulk viscosities. Therefore, one must not neglect the PAC part. When taking the PAC part into account, we have seen that $\beta_{s,\tau}$ and $\beta_{\zeta,\tau}$ can be expressed solely by thermodynamic quantities such as $\varepsilon$, $P$, and $\alpha_s$ as shown in Eqs. (19) and (20).

A natural question that arises from our work is whether it is possible to introduce the PAC term in the kinetic calculation. Remember that $\beta_\eta$ and $\beta_\zeta$ are independent of the collision term. Thus, even if we improve the collision term by introducing PAC processes, these coefficients will not change. Therefore we have to extend the Boltzmann equation itself to include the PAC effect discussed here.

In this letter, we simply dropped the temperature-independent, divergent vacuum contribution term in the calculation of the coefficients. Strictly speaking, however, we do not know the precise renormalization scheme for these quantities. The problem of renormalization is very important in order to reliably determine the values of the transport coefficients and should be studied more carefully in the future.

We have discussed the consistency of the field-theoretical and kinetic calculations. We have also computed the 1+1-dimensional scaling flow using our new kinetic coefficients and compared with a numerical simulation of the Boltzmann equation. We find better agreement with the new coefficients than with those of the IS calculation [11]. This may serve as another justification for the validity of our results.

In order to discuss particle diffusion (heat conduction), the kinetic approach should be generalized to a multi-component fluid, considering the flows of particles and anti-particles on an equal footing. For the field-theoretical approach, it is not yet known what is the appropriate projection operator to derive the corresponding formula for the particle diffusion. This is a challenge for the future.

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