The Complexity of Temporal Logic over the Reals

M. Reynolds
Murdoch University, Australia
February 1, 2008

Abstract

It is shown that the decision problem for the temporal logic with until and since connectives over real-numbers time is PSPACE-complete.

1 Introduction

There are a variety of temporal logics appropriate for a variety of reasoning tasks. Propositional reasoning on a natural numbers model of time has been well studied via the logic now commonly called PLTL which was introduced in [Pnu77]. However, it has long been acknowledged that dense or specifically real-numbers time models may be better for many applications, ranging from philosophical, natural language and AI modelling of human reasoning to computing and engineering applications of concurrency, refinement, open systems, analogue devices and metric information. See for example [KMP94] or [BG85].

The most natural and useful such temporal logic is propositional temporal logic over real-numbers time using the Until and Since connectives introduced in [Kam68]. We will call this logic RTL in this paper. We know from [Kam68] that this logic is sufficiently expressive for many applications: technically it is expressively complete and so at least as expressive as any other usual temporal logic which could be defined over real-numbers time and as expressive as the first-order monadic logic of the real numbers. We have, from [GH90] and [Rey92], complete axiom systems to allow derivation of the validities of RTL. We know from [BG85] that RTL is decidable, ie that an algorithm exists for deciding whether a given RTL formula is a validity or not. Unfortunately, it has seemed difficult to develop the reasoning procedures any further. It is not even clear from the decision procedure in [BG85] (via Rabin’s non-elementarily complex decision procedure for the second-order monadic logic of two successors) how computationally complex it might be to decide validities in RTL.

This is in marked contrast to the situation with PLTL which has been shown to have a PSPACE-complete decision problem in [SC85]. A variety of practical reasoning methods for PLTL have been developed.
Here we show that as far as determining validity is concerned, RTL is just as easy to reason with as PLTL. In particular, the complexity of the decision problem is PSPACE-complete.

This opens the way for the development of efficient reasoning procedures for RTL and for many practical applications. For example, it is commonly required to determine consequence relations between finite sets of formulas, e.g., a detailed description of the running of a system and a desirable overall property. Such a question is equivalent to a validity question.

Note that there has been some work on restricted versions of temporal logic over the reals. In [Rab98] and [KMP94] the assumption of finite variability is made, i.e., it is supposed that atoms do not change their truth values densely in time. Under such an assumption, standard discrete time techniques can be used to develop decision procedures. We do not make any such assumptions.

The proof here uses new techniques in temporal logic. In particular, we further develop the idea of linear time mosaics as seen in [Reyed]. Mosaics were used to prove decidability of certain theories of relation algebras in [Nem95] and have been used since quite generally in algebraic logic and modal logic. These mosaics are small pieces of a model, in our case, a small piece of a real-flowed structure. We decide whether a finite set of small pieces is sufficient to be used to build a real-numbers model of a given formula. This is also equivalent to the existence of a winning strategy for one player in a two-player game played with mosaics. The search for a winning strategy can be arranged into a search through a tree of mosaics which we can proceed through in a depth-first manner. By establishing limits on the depth of the tree (a polynomial in terms of the length of the formula) and on the branching factor (exponential) we can ensure that we have a PSPACE algorithm as we only need to remember a small fixed amount of information about all the previous siblings of a given node.

In the case of the real numbers in this paper we do not emphasize the game aspect of this search but instead study certain structures which correspond to tactics in the game. By ensuring that mosaics get simpler as we get deeper in the tree we can respect the depth bound and also capture the Dedekind completeness of the underlying flow. By ensuring that certain thorough mutually recursive relationships (called shuffles) between mosaics in the trees include at least one very simple pair of mosaics, we can also capture the separability property of the reals.

The proof also vaguely suggests a tableau based method for determining validity but developing such a method will need some more work.

2 The logic

Fix a countable set \( \mathcal{L} \) of atoms. Here, frames \((T, <)\), or flows of time, will be irreflexive linear orders. Structures \( \mathcal{T} = (T, <, h) \) will have a frame \((T, <)\) and a valuation \( h \) for the atoms i.e., for each atom \( p \in \mathcal{L} \), \( h(p) \subseteq T \). Of particular importance will be real structures \( \mathcal{T} = (\mathbb{R}, <, h) \) which have the real numbers flow (with their usual irreflexive
linear ordering).

The language $L(U,S)$ is generated by the 2-place connectives $U$ and $S$ along with classical $\neg$ and $\wedge$. That is, we define the set of formulas recursively to contain the atoms and for formulas $\alpha$ and $\beta$ we include $\neg\alpha$, $\alpha \wedge \beta$, $U(\alpha,\beta)$ and $S(\alpha,\beta)$.

Formulas are evaluated at points in structures $T = (T, <, h)$. We write $T, x \models \alpha$ when $\alpha$ is true at the point $x \in T$. This is defined recursively as follows. Suppose that we have defined the truth of formulas $\alpha$ and $\beta$ at all points of $T$. Then for all points $x$:

- $T, x \models p$ iff $x \in h(p)$, for $p$ atomic;
- $T, x \models \neg \alpha$ iff $T, x \not\models \alpha$;
- $T, x \models \alpha \wedge \beta$ iff both $T, x \models \alpha$ and $T, x \models \beta$;
- $T, x \models U(\alpha,\beta)$ iff there is $y > x$ in $T$ such that $T, y \models \alpha$ and for all $z \in T$ such that $x < z < y$ we have $T, z \models \beta$; and
- $T, x \models S(\alpha,\beta)$ iff there is $y < x$ in $T$ such that $T, y \models \alpha$ and for all $z \in T$ such that $y < z < x$ we have $T, z \models \beta$.

Often, definitions, results or proofs will have a mirror image in which $U$ and $S$ are exchanged and $<$ and $>$ swapped.

A formula $\phi$ is $\mathbb{R}$-satisfiable if it has a real model: i.e. there is a real structure $S = (\mathbb{R}, <, h)$ and $x \in \mathbb{R}$ such that $S, x \models \phi$. A formula is $\mathbb{R}$-valid iff it is true at all points of all real structures. Of course, a formula is $\mathbb{R}$-valid iff its negation is not $\mathbb{R}$-satisfiable.

Let RTL-SAT be the problem of deciding whether a given formula of $L(U,S)$ is $\mathbb{R}$-satisfiable or not. The main result of this paper, proved in lemma 26 and lemma 27 below, is:

**THEOREM 1** RTL-SAT is PSPACE-complete.

### 3 Mosaics for $U$ and $S$

We will decide the satisfiability of formulas by considering sets of small pieces of real structures. The idea is based on the mosaics seen in [Nem95] and used in many other subsequent proofs.

Each mosaic is a small piece of a model, i.e. a small set of objects (points), relations between them and a set of formulas for each point indicating which formulas are true there in the whole model. There will be coherence conditions on the mosaic which are necessary for it to be part of a larger model.

We want to show the equivalence of the existence of a model to the existence of a certain set of mosaics: enough mosaics to build a whole model. So the whole set of mosaics also has to obey some conditions. These are called saturation conditions. For example, a particular small piece of a model might require a certain other piece to exist somewhere else in the model. We talk of the first mosaic having a defect which is cured by the latter mosaic.

Our mosaics will only be concerned with a finite set of formulas:
DEFINITION 1 For each formula $\phi$, define the closure of $\phi$ to be $\text{Cl}\phi = \{ \psi, \neg\psi \mid \chi \leq \phi \}$ where $\chi \leq \psi$ means that $\chi$ is a subformula of $\psi$.

We can sometimes think of $\text{Cl}\phi$ as being closed under negation: we could treat $\neg\neg\alpha$ as if it was $\alpha$. To be more rigorous, we introduce the following notation.

DEFINITION 2 For each $\alpha \in L(U,S)$, define $\sim\alpha$ to mean $\beta$ if $\alpha = \neg\beta$ and $\neg\alpha$ otherwise.

Note that if $\alpha \in \text{Cl}\phi$ then $\sim\alpha \in \text{Cl}\phi$. Note also that in many places in the proof we explicitly use $\neg\alpha$ when we can be sure it is in $\text{Cl}\phi$, for example when $U(\alpha,\beta) \in \text{Cl}(\phi)$.

DEFINITION 3 Suppose $\phi \in L(U,S)$ and $S \subseteq \text{Cl}\phi$. Say $S$ is propositionally consistent (PC) iff there is no substitution instance of a tautology of classical propositional logic of the form $\neg(\alpha_1 \land \ldots \land \alpha_n)$ with each $\alpha_i \in S$. Say $S$ is maximally propositionally consistent (MPC) iff $S$ is maximal in being a subset of $\text{Cl}\phi$ which is PC.

We will define a mosaic to be a triple $(A,B,C)$ of sets of formulas. The intuition is that this corresponds to two points from a structure: $A$ is the set of formulas (from $\text{Cl}\phi$) true at the earlier point, $C$ is the set true at the later point and $B$ is the set of formulas which hold at all points strictly in between.

DEFINITION 4 Suppose $\phi$ is from $L(U,S)$. A $\phi$-mosaic is a triple $(A,B,C)$ of subsets of $\text{Cl}\phi$ such that:

0.1 $A$ and $C$ are maximally propositionally consistent, and

0.2 for all $\beta \in \text{Cl}(\phi)$ with $\neg\beta \in \text{Cl}(\phi)$ we have $\neg\beta \in B$ iff $\sim\beta \in B$

and the following four coherency conditions hold:

C1. if $\neg U(\alpha,\beta) \in A$ and $\beta \in B$ then we have both:

C1.1. $\neg\alpha \in C$ and either $\neg\beta \in C$ or $\neg U(\alpha,\beta) \in C$; and

C1.2. $\neg\alpha \in B$ and $\neg U(\alpha,\beta) \in B$.

C2. if $U(\alpha,\beta) \in A$ and $\neg\alpha \in B$ then we have both:

C2.1 either $\alpha \in C$ or both $\beta \in C$ and $U(\alpha,\beta) \in C$; and

C2.2. $\beta \in B$ and $U(\alpha,\beta) \in B$.

C3-4 mirror images of C1-C2.

DEFINITION 5 If $m = (A,B,C)$ is a mosaic then $\text{start}(m) = A$ is its start, $\text{cover}(m) = B$ is its cover and $\text{end}(m) = C$ is its end.

If we start to build a model using mosaics then, as we have noted, we may realise that the inclusion of one mosaic necessitates the inclusion of others: defects need curing.

DEFINITION 6 A defect in a mosaic $(A,B,C)$ is either
1. a formula $U(\alpha, \beta) \in A$ with either
   1.1 $\beta \notin B$,
   1.2 $(\alpha \notin C$ and $\beta \notin C)$, or
   1.3 $(\alpha \notin C$ and $U(\alpha, \beta) \notin C)$;
2. a formula $S(\alpha, \beta) \in C$ with either
   2.1 $\beta \notin B$,
   2.2 $(\alpha \notin A$ and $\beta \notin A)$, or
   2.3 $(\alpha \notin A$ and $S(\alpha, \beta) \notin A)$; or
3. a formula $\beta \in \text{Cl}_\phi$ with $\sim \beta \notin B$.

We refer to defects of type 1 to 3 (as listed here). Note that the same formula may be both a type 1 or 2 defect and a type 3 defect in the same mosaic. In that case we count it as two separate defects.

We will need to string mosaics together to build linear orders. This can only be done under certain conditions. Here we introduce the idea of composition of mosaics.

**DEFINITION 7** We say that $\phi$-mosaics $(A', B', C')$ and $(A'', B'', C'')$ compose iff $C' = A''$. In that case, their composition is $(A', B' \cap C' \cap B'', C'')$.

It is straightforward to prove that this is a mosaic and that composition of mosaics is associative.

**LEMMA 1** If mosaics $m$ and $m'$ compose then their composition is a mosaic.

**LEMMA 2** Composition of mosaics is associative.

Thus we can talk of sequences of mosaics composing and then find their composition. We define the composition of a sequence of length one to be just the mosaic itself. We leave the composition of an empty sequence undefined.

**DEFINITION 8** A decomposition for a mosaic $(A, B, C)$ is any finite sequence of mosaics $(A_1, B_1, C_1), (A_2, B_2, C_2), ..., (A_n, B_n, C_n)$ which composes to $(A, B, C)$.

It will be useful to introduce an idea of fullness of decompositions. This is intended to be a decomposition which provides witnesses to the cure of every defect in the decomposed mosaic.

**DEFINITION 9** The decomposition above is full iff the following three conditions all hold:

1. for all $U(\alpha, \beta) \in A$ we have
   1.1 $\beta \in B$ and either $(\beta \in C$ and $U(\alpha, \beta) \in C)$ or $\alpha \in C$,
   1.2 or there is some $i$ such that $1 \leq i < n$, $\alpha \in C_i$, for all $j \leq i$, $\beta \in B_j$
   and for all $j < i$, $\beta \in C_j$;
2. the mirror image of 1.; and
3. for each $\beta \in \text{Cl}_\phi$ such that $\sim \beta \notin B$ there is some $i$ such that $1 \leq i < n$
   and $\beta \in C_i$. 

5
If 1.2 above holds in the case that \( U(\alpha, \beta) \in A \) is a type 1 defect in \((A, B, C)\) then we say that a cure for the defect is witnessed (in the decomposition) by the end of \((A_i, B_i, C_i)\) (or equivalently by the start of \((A_{i+1}, B_{i+1}, C_{i+1})\)). Similarly for the mirror image for \( S(\alpha, \beta) \in C \). If \( \beta \in C_i \) is a type 3 defect in \((A, B, C)\) then we also say that a cure for this defect is witnessed (in the decomposition) by the end of \((A_i, B_i, C_i)\). If a cure for any defect is witnessed then we say that the defect is cured.

**LEMMA 3** If \( m_1, \ldots, m_n \) is a full decomposition of \( m \) then every defect in \( m \) is cured in the decomposition.

## 4 Satisfiability and relativization

Because mosaics represent linear orders with end points, it is inconvenient for us to continue to work directly with \( \mathbb{R} \). Because we want to make use of some simple tricks with the metric at several places in the proof, we will move to work in the unit interval \([0, 1]\) instead.

If \( x < y \) from \( \mathbb{R} \) then let \([x, y]\) denote the open interval \( \{ z \in \mathbb{R} | x < z < y \} \) and \([x, y]\) denote the closed interval \( \{ z \in \mathbb{R} | x \leq z \leq y \} \). Similarly for half open intervals.

One can get a mosaic from any two points in a structure.

**DEFINITION 10** If \( T = (T, <, h) \) is a structure and \( \phi \) a formula then for each \( x < y \) from \( T \) we define \( \text{mos}^\phi_T(x, y) = (A, B, C) \) where:

- \( A = \{ \alpha \in \text{Cl}_T | T, x \models \alpha \} \),
- \( B = \{ \beta \in \text{Cl}_T | \text{for all } z \in T, \text{ if } x < z < y \text{ then } T, z \models \beta \} \), and
- \( C = \{ \gamma \in \text{Cl}_T | T, y \models \gamma \} \).

It is straightforward to show that this is a mosaic.

**LEMMA 4** \( \text{mos}^\phi_T(x, y) \) is a mosaic.

If \( T \) and \( \phi \) are clear from context then we just write \( \text{mos}(x, y) \) for \( \text{mos}^\phi_T(x, y) \).

**DEFINITION 11** Suppose \( T \subseteq \mathbb{R} \). Let \( < \) also denote the restriction of \( < \) to any such \( T \). We say that a \( \phi \)-mosaic is \( T \)-satisfiable iff it is \( \text{mos}^\phi_T(x, y) \) for some \( x < y \) from \( T \) and some structure \( T = (T, <, h) \).

**DEFINITION 12** We say that a \( \phi \)-mosaic is fully \([0, 1]\)-satisfiable iff it is \( \text{mos}^\phi_{[0, 1]}(0, 1) \) from some structure \( T = ([0, 1], <, h) \).

We will now relate the satisfiability of a formula \( \phi \) to that of certain mosaics.
DEFINITION 13 Given \( \phi \) and an atom \( q \) which does not appear in \( \phi \), we define a map \( * = *_q^\phi \) on formulas in \( \text{Cl}(\phi) \) recursively:

1. \( *p = p \land q \),
2. \( *\neg\alpha = \neg(\alpha) \land q \),
3. \( *(\alpha \land \beta) = \alpha \land \beta \land q \),
4. \( *U(\alpha, \beta) = U(\alpha, \beta) \land q \), and
5. \( *S(\alpha, \beta) = S(\alpha, \beta) \land q \).

So \( *_q^\phi(\phi) \) will be a formula using only \( q \) and atoms from \( \phi \).

LEMMA 5 \( *_q^\phi(\phi) \) is at most 3 times as long as \( \phi \).

LEMMA 6 If \( \alpha \leq \phi \) then \( *\alpha \leq *\phi \).

DEFINITION 14 We say that a \( *_q^\phi(\phi) \)-mosaic \((A, B, C)\) is \((\phi, q)\)-relativized iff

1. \( \neg q \) is in \( A \) and no \( S(\alpha, \beta) \) is in \( A \);
2. \( q \in B \) and \( \neg *_q^\phi(\phi) \notin B \); and
3. \( \neg q \in C \) and no \( U(\alpha, \beta) \) is in \( C \).

LEMMA 7 Suppose that \( \phi \) is a formula of \( L(U, S) \) and \( q \) is an atom not appearing in \( \phi \). Then \( \phi \) is \( \mathbb{R} \)-satisfiable iff there is some fully \([0, 1]\)-satisfiable \((\phi, q)\)-relativized \( *_q^\phi(\phi) \)-mosaic.

PROOF: Let \( * = *_q^\phi \) and let \( \zeta : ]0, 1[ \to \mathbb{R} \) be any order preserving bijection. Suppose that \( \phi \) is \( \mathbb{R} \)-satisfiable. Say that \( \mathcal{S} = (\mathbb{R}, <, g) \), \( s_0 \in \mathbb{R} \) and \( \mathcal{S}, s_0 \models \phi \). Let \( \mathcal{T} = ([0, 1], <, h) \) where:

1. for atom \( p \neq q \), \( h(p) = \{ t \in ]0, 1[ | \zeta(t) \in g(p) \} \); and
2. \( h(q) = ]0, 1[ \).

An easy induction on the construction of formulas in \( \text{Cl}(\phi) \) shows that \( \mathcal{T}, \zeta^{-1}(s_0) \models *\phi \) and so \( \text{mos}_\mathcal{T}(0, 1) \) is the right mosaic.

Suppose mosaic \((A, B, C) = \text{mos}(0, 1)\) from structure \( \mathcal{T} = ([0, 1], <, h) \) is a \((\phi, q)\)-relativized \(*(\phi)\)-mosaic. Thus \( q \in B \) and \( \neg q \in A \cap C \). Define \( \mathcal{S} = (\mathbb{R}, <, g) \) via \( s \in g(p) \) iff \( \zeta^{-1}(s) \in h(p) \) for any atom \( p \) (including \( p = q \)). As \( \neg *\phi \notin B \) there is some \( z \) such that \( 0 < z < 1 \) and \( \mathcal{T}, z \models *\phi \). It is easy to show that \( \mathcal{S}, \zeta(z) \models \phi \). \( \Box \)

Our satisfiability procedure will be to guess a relativized mosaic \((A, B, C)\) and then check that \((A, B, C)\) is fully \([0, 1]\)-satisfiable. Thus we now turn to the question of deciding whether a relativized mosaic is satisfiable.
5 Shuffles

A game can be played by two players with mosaics: one player provides full decompositions for the mosaics chosen by the other. We will not develop this idea here but we will examine some structures which correspond to tactics in this game. In this section we will consider the most complex such structure: the shuffle.

We shall write \( \langle p_1, \ldots, p_n \rangle \) for the sequence of mosaics containing \( p_1, \ldots, p_n \) in that order. We shall write \( \pi \uparrow \rho \) for the sequence resulting from the concatenation of sequences \( \pi \) and \( \rho \) in that order. Sequences will always be finite.

**DEFINITION 15** Suppose \( 0 \leq r \), each \( \lambda_i (1 \leq i \leq r) \) is a non-empty composing sequence of \( \phi \)-mosaics, and \( P_0, \ldots, P_s (0 \leq s) \) are maximally propositionally consistent subsets of \( \text{Cl}\phi \).

Suppose \( \phi \)-mosaic \( o = (A, B, C) \) and:

\[
m' = (A, B, P_0); \quad y_i = (P_i, B, P_{i+1}) \ (0 \leq i \leq s - 1); \quad y_s = (P_s, B, P_0);
m'' = (P_0, B, C); \quad \text{and} \quad \mu = \langle y_0, \ldots, y_s \rangle.
\]

If \( r = 0 \) suppose \( \lambda = \langle \rangle \), the empty sequence, but otherwise, if \( r > 0 \), suppose:

- \( A_i \) is the start of the first mosaic in \( \lambda_i (1 \leq i \leq r) \);
- \( C_i \) is the end of the last mosaic in \( \lambda_i (1 \leq i \leq r) \);
- \( x_0 = (P_0, B, A_1) \);
- \( x_i = (C_i, B, A_{i+1}) \), \( 1 \leq i \leq r - 1 \);
- \( x_r = (C_r, B, P_0) \);
- \( \lambda = \langle x_0 \rangle \uparrow \lambda_1 \uparrow \langle x_1 \rangle \uparrow \ldots \uparrow \lambda_r \uparrow \langle x_r \rangle \).

Further suppose that \( m' \), \( m'' \), and each \( y_i \) and \( x_i \) are mosaics.

Then we say that \( o \) is fully decomposed by the tactic shuffle \( \langle P_0, \ldots, P_s \rangle, \langle \lambda_1, \ldots, \lambda_r \rangle \) iff the following conditions all hold:

\[
\text{F1. } o \text{ is fully decomposed by } \langle m' \rangle \uparrow \lambda \uparrow \mu \uparrow \langle m'' \rangle;
\]

\[
\text{F2. } \text{if } r > 0, x_0 \text{ is fully decomposed by } \lambda \uparrow \mu \uparrow \langle x_0 \rangle;
\]

\[
\text{F3. } \text{if } 0 < i < r, x_i \text{ is fully decomposed by } \\
\langle x_i \rangle \uparrow \lambda_{i+1} \uparrow \langle x_{i+1} \rangle \uparrow \ldots \uparrow \lambda_r \uparrow \langle x_r \rangle \uparrow \mu \uparrow \langle x_0 \rangle \uparrow \lambda_1 \uparrow \langle x_1 \rangle \uparrow \ldots \uparrow \lambda_i \uparrow \langle x_i \rangle;
\]

\[
\text{F4. } \text{if } r > 0, x_r \text{ is fully decomposed by } \langle x_r \rangle \uparrow \mu \uparrow \lambda;
\]

\[
\text{F5. } \text{if } 0 \leq i < s, y_i \text{ is fully decomposed by } \langle y_i, y_{i+1}, \ldots, y_s \rangle \uparrow \lambda \uparrow \langle y_0, \ldots, y_i \rangle;
\]

\[
\text{F6. } y_s \text{ is fully decomposed by } \langle y_s \rangle \uparrow \lambda \uparrow \mu.
\]

The term shuffle has been used in the literature (see, for example, [LL60] or [BG83]) to refer to a certain method of constructing a monadic linear structure from a thorough mixture of smaller linear structures. The intention here is similar.

Note that as \( s \geq 0 \) there is at least one \( P_i \) involved in the shuffle. In a general linear order setting we could define a shuffle with no \( P_i \)s (provided that then \( r > 0 \)) but over the reals it turns out to be crucial to require at least one \( P_i \). This ensures that the mosaic is satisfiable in a structure on a separable linear frame.
For the purposes of algorithmic checking of shuffles we find it convenient to have a different characterization of shuffles. First a couple of helpful properties.

**DEFINITION 16** Suppose \( \phi \in L(U,S) \) and \( m \) is a \( \phi \)-mosaic. We say that an MPC set \( Q \subseteq Cl(\phi) \) satisfies the forward \( K(m) \) property iff for any \( U(\alpha, \beta) \in Cl(\phi) \) we have \( U(\alpha, \beta) \in Q \) iff both \( \beta \in \text{cover}(m) \) and (at least) one of the following holds:

1. \( K1 \) \( \sim \alpha \notin \text{cover}(m) \);
2. \( K2 \) \( \alpha \in \text{end}(m) \); or
3. \( K3 \) \( \beta \in \text{end}(m) \) and \( U(\alpha, \beta) \in \text{end}(m) \).

The mirror image is the backwards \( K(m) \) property.

**LEMMA 8** Suppose \( \phi \in L(U,S) \), \( m = (A, B, C) \) is a \( \phi \)-mosaic, and each \( P_i \subseteq Cl(\phi) \) (0 \( \leq i \leq s \)) and each \( \lambda_i \) (1 \( \leq i \leq r \)) is a sequence of \( \phi \)-mosaics.

Then \( m \) is fully decomposed by the tactic shuffle \( \langle P_0, ..., P_s \rangle, \langle \lambda_1, ..., \lambda_r \rangle \) iff the following seven conditions hold:

1. \( S0 \) \( B \) is a subset of each \( P_i \) and of the start, end and cover of each mosaic in each \( \lambda_i \);
2. \( S1 \) each \( P_i \) satisfies both the forward and backwards \( K(m) \) property;
3. \( S2 \) the start of the first mosaic in each \( \lambda_i \) satisfies the backwards \( K(m) \) property;
4. \( S3 \) the end of the last mosaic in each \( \lambda_i \) satisfies the forwards \( K(m) \) property;
5. \( S4 \) \( A \) satisfies the forward \( K(m) \) property;
6. \( S5 \) \( C \) satisfies the backwards \( K(m) \) property;
7. \( S6 \) if \( \beta \in Cl(\phi) \) but \( \sim \beta \notin B \) then either \( \beta \) is contained in some \( P_i \) or \( \beta \) is contained in the start or end of some mosaic in some \( \lambda_i \).

**PROOF:**

Consider the forward direction of the proof. Suppose \( m = (A, B, C) \) is fully decomposed by the tactic shuffle \( \langle P_0, ..., P_s \rangle, \langle \lambda_1, ..., \lambda_r \rangle \).

By \( F0 \), we have a decomposition for \( m \) including each \( \lambda_i \) and mosaics with each \( P_i \) in their starts or ends. \( S0 \) follows.

We now establish condition \( S1 \). Each \( P_i \) is an MPC by the definition of a shuffle. To show the forward \( K(m) \) property for \( P_i \), suppose that \( U(\alpha, \beta) \in P_i \). We consider the case when \( i < s \): the case with \( i = s \) is similar. We know \( F4-F5 \) that \( y_i = (P_i, B, P_{i+1}) \) is fully decomposed by \( \langle y_i, ..., y_s \rangle \quad \lambda \quad \langle y_0, ..., y_i \rangle \). If \( U(\alpha, \beta) \) is a type 1 defect in \( y_i \) then it is cured in this decomposition and we can conclude that \( \beta \) is in the cover \( B \) of the first mosaic \( y_i \). If \( U(\alpha, \beta) \) is not a type 1 defect in \( y_i \) then \( \beta \in \text{cover}(y_i) = B \) as well. Thus in any case \( \beta \in B \).

I claim that \( U(\alpha, \beta) \in A \). If not then \( \sim U(\alpha, \beta) \in \text{start}(m) \) and coherency \( C1.2 \) of \( m \) implies that \( \sim U(\alpha, \beta) \in B \subseteq P_i \). This is a contradiction to the consistency of \( P_i \).

So \( U(\alpha, \beta) \in A \) is either a type 1 defect in \( m \) or not. In the former case it is cured in the full decomposition \( F1 \) for \( m \) and so \( \alpha \) appears in the start or end of a mosaic in some \( \lambda_j \) or in some \( P_j \). Thus \( \sim \alpha \notin \text{cover}(m) = B \). This is \( K1 \).
If \(U(\alpha, \beta) \in A\) is not a type 1 defect in \(m\) then K2 or K3 holds by definition.

We now show the converse part of the forward \(K(m)\) property for \(P_i\). Suppose that both \(\beta \in \text{cover}(m)\) and K1 holds: the cases of K2 or K3 holding are straightforward. Thus \(\sim \alpha \notin B\), \(\alpha\) is a type 3 defect in \(m\) and so a cure is witnessed in some \(P_j\) or in the start or end of some mosaic in some \(\lambda_j\). Now look in the decomposition F5 (or F6) for \(y_i\) in which we have \(\beta\) holding in all starts, ends and covers and \(\alpha\) appearing somewhere. A simple induction shows that we must have \(U(\alpha, \beta)\) in the very start \(P_i\) as required.

To show the backwards \(K(m)\) property is the mirror image.

Very similar arguments establish conditions S2 – S6. To show condition S2 we just use the full decomposition (F2-F3) for \(x_{i-1}\) and reason about type 2 defects. To show condition S3, use the full decomposition (F2-F4) for \(x_i\) and reason about type 1 defects. Conditions S4, S5 and S6 follow from using the full decomposition (F1) for \(m\) and reasoning about type 1, 2 and 3 defects respectively.

Now consider the converse: suppose that the seven conditions S0–S6 hold for mosaic \(m = (A, B, C)\).

First we must show that each of \(m', m'', \) each \(y_i\) and any \(x_i\) (as defined from \(m\), the \(P_i\) and the \(\lambda_i\) in the definition of a shuffle) are mosaics. This follows from

**CLAIM 1** If the MPC \(D \subseteq \text{Cl}(\phi)\) satisfies the forward \(K(m)\) property, the MPC \(E \subseteq \text{Cl}(\phi)\) satisfies the backwards \(K(m)\) property and \(B \subseteq D \cap E\) then \((D, B, E)\) is a mosaic.

**PROOF:** We must check the first two coherency conditions. The mirror images are mirror images.

(C1). Suppose \(\neg U(\alpha, \beta) \in D\) and \(\beta \in B\).

First we establish that we must have \(U(\alpha, \beta) \notin A\). Suppose not for contradiction. Since \(U(\alpha, \beta) \notin D\), K1 does not hold and so \(\neg \alpha \in B\), K2 does not hold and so \(\neg \alpha \in C\) and K3 does not hold and so either \(\beta \notin C\) or \(U(\alpha, \beta) \notin C\). We have a contradiction to the coherency (C2.1) of \(m\).

(C1.1). First, we show \(\neg \alpha \in E\). Otherwise, \(\neg \alpha \notin B \subseteq E\). Thus, by K1, \(U(\alpha, \beta) \in D\) and we have our contradiction.

Next we show that either \(\neg \beta \in E\) or \(\neg U(\alpha, \beta) \in E\). Suppose instead that \(\beta \in E\) and \(U(\alpha, \beta) \in E \supseteq B\). Thus \(\neg U(\alpha, \beta) \notin B\). By coherency C1.2 of \((A, B, C)\), we must have \(U(\alpha, \beta) \in A\) which is a contradiction.

(C1.2). We show that \(\neg \alpha \in B\) and \(\neg U(\alpha, \beta) \in B\). We can not have \(\neg \alpha \notin B\), as then K1 implies \(U(\alpha, \beta) \in D\). We can not have \(\neg U(\alpha, \beta) \notin B\) as then coherency (C1.2) of \(m\) implies \(U(\alpha, \beta) \in A\), a contradiction.
(C2). Assume \( U(\alpha, \beta) \in D \) and \( \neg \alpha \in B \). By the forward \( K(m) \) property for \( D, \beta \in B \subseteq E \) and, since \( \neg \alpha \in B \), either K2 or K3 holds (for C). By the coherency C1.1 of \( m \) we can conclude that we can not have \( \neg U(\alpha, \beta) \in A \). Thus \( U(\alpha, \beta) \in A \) and C2.2 of \( m \) implies that \( U(\alpha, \beta) \in B \subseteq E \) as required. \( \square \)

Next we must check the fullness of the decompositions. This follows by

**CLAIM 2** Suppose \( D \) and \( E \) are as in the previous claim.

Furthermore, suppose the sequence \( \sigma \) of mosaics composes to \( (D, B, E) \) such that for each \( \beta \in \mathsf{Cl}(\phi) \) with \( \sim \beta \notin B \), there is a mosaic in \( \sigma \) other than the very first which includes \( \beta \) in its start.

Then \( (D, B, E) \) is fully decomposed by \( \sigma \).

**PROOF:** **Type 1 defects:** Suppose \( U(\alpha, \beta) \in D \) is a type 1 defect of \( (D, B, E) \). By the forward \( K(m) \) property for \( D, \beta \in B \).

As \( U(\alpha, \beta) \) is a type 1 defect \( \alpha \notin E \) and either \( \beta \notin E \) or \( U(\alpha, \beta) \notin E \). By coherency C2 of \( (D, B, E) \), \( \neg \alpha \notin B \). So \( \sim \alpha \notin B \) and \( \alpha \) must appear in a non-first mosaic in \( \sigma \) and we have our cure.

**Type 2 defects:** mirror image.

**Type 3 defects:** Suppose \( \beta \in \mathsf{Cl}(\phi) \) but \( \sim \beta \notin B \). Thus \( \beta \) appears in the start of a non-first mosaic in \( \sigma \). We have our witness.

\( \square \)

Thus \( m \) is fully decomposed by the tactic shuffle \( (P_0, ..., P_s), (\lambda_1, ..., \lambda_r) \) as required. \( \square \)

### 6 Real Mosaic Systems

In this section we define a concept of a collection or system of mosaics in which each member is decomposable in terms of simpler members. First another tactic for decomposition.

**DEFINITION 17** Suppose \( \phi \in L(U, S) \), \( m \) is a \( \phi \)-mosaic and \( \sigma \) is a non-empty sequence of \( \phi \)-mosaics. Then, we say that \( m \) is fully decomposed by the tactic \( \mathsf{lead}(\sigma) \) iff \( \langle m \rangle \uparrow \sigma \) is a full decomposition of \( m \). We say that \( m \) is fully decomposed by the tactic \( \mathsf{trail}(\sigma) \) iff \( \sigma \uparrow \langle m \rangle \) is a full decomposition of \( m \).

**DEFINITION 18** For \( \phi \in L(U, S) \), suppose \( S \) is a set of \( \phi \)-mosaics and \( n \geq 0 \).

A \( \phi \)-mosaic \( m \) is a level \( n^+ \) member of \( S \) iff \( m \) is the composition of a sequence of mosaics, each of them being either a level \( n \) member of \( S \) or fully decomposed by the tactics \( \mathsf{lead}(\sigma) \) or \( \mathsf{trail}(\sigma) \) with each mosaic in \( \sigma \) being a level \( n \) member of \( S \).

A \( \phi \)-mosaic \( m \) is a level \( (n+1)^- \) member of \( S \) iff \( m \) is the composition of a sequence of mosaics, each of them being either a level \( n^+ \) member of \( S \) or fully decomposed by the tactics \( \mathsf{lead}(\sigma) \) or \( \mathsf{trail}(\sigma) \) with each mosaic in \( \sigma \) being a level \( n^+ \) member of \( S \).
A $\phi$-mosaic $m \in S$ is a level $n$ member of $S$ iff $m$ is the composition of a sequence of mosaics with each of them being either a level $n^-$ member of $S$ or a mosaic which is fully decomposed by the tactic shuffle($\langle P_0, ..., P_s \rangle$, $\langle \sigma_1, ..., \sigma_r \rangle$) with each mosaic in each $\sigma_i$ being a level $n^-$ member of $S$.

Note that it is generally possible for mosaics to be level 0 members of some $S$ provided that they are compositions of mosaics which can be fully decomposed by shuffles in which there are no sequences (ie, $r = 0$).

**DEFINITION 19** For $\phi \in L(U, S)$, a real mosaic system of $\phi$-mosaics is a set $S$ of $\phi$-mosaics such that for every $m \in S$ there exists some $n$ such that $m$ is a level $n$ member of $S$. For any $n$ we say that $S$ is a real mosaic system of depth $n$ iff every $m \in S$ is a level $n$ member of $S$.

7 Realizing Mosaics

In this section we show that relativized mosaics which appear in real mosaic systems are satisfiable. To do so we define a concept of realization intended to capture the idea of a mosaic being satisfiable as far as internal information is concerned: ie we ignore formulas of the form $U(\alpha, \beta)$ in the end or $S(\alpha, \beta)$ in the start.

**DEFINITION 20** Suppose that $x < y$ from $[0, 1]$. We say that $\phi$-mosaic $m$ is realised by the map $\mu$ on the closed interval $[x, y]$ iff the following conditions all hold:

R1. for each $z \in [x, y]$, $\mu(z)$ is a maximally propositionally consistent subset of $\text{Cl}_\phi$;

R2. Suppose $z \in [x, y]$. Then $U(\alpha, \beta) \in \mu(z)$ iff either

R2.1, there is $u$ such that $z < u \leq y$ and $\alpha \in \mu(u)$ and for all $v$, if $z < v < u$ then $\beta \in \mu(v)$ or

R2.2, $\beta \in \mu(y)$, $U(\alpha, \beta) \in \mu(y)$ and for all $v$, if $z < v < y$ then $\beta \in \mu(v)$;

R3. the mirror image of R2 for $S(\alpha, \beta)$;

R4. $\mu(x)$ is the start of $m$;

R5. $\mu(y)$ is the end of $m$; and

R6. for each $\beta \in \text{Cl}_\phi$, $\beta$ is in the cover of $m$ iff for all $u$, if $x < u < y$, $\beta \in \mu(u)$.

**LEMMA 9** If $m$ is the composition of $m'$ and $m''$ with each of $m'$ and $m''$ having a realization on any closed interval of $[0, 1]$ then for any $x < y$ from $[0, 1]$, there is $\mu$ which realises $m$ on $[x, y]$.

**PROOF:** Given $x < y$ from $[0, 1]$ choose any $w$ with $x < w < y$. Let $\mu'$ realize $m'$ on $[x, w]$ and $\mu''$ realize $m''$ on $[w, y]$. Define $\mu : [x, y] \rightarrow \phi(\text{Cl}_\phi)$ via:

$$
\mu(u) = \begin{cases} 
\mu'(u), & x \leq u \leq w \\
\mu''(u), & w < u \leq y
\end{cases}
$$
If $m$ is fully decomposed by the tactic $\text{lead}(\sigma)$ with each mosaic in $\sigma$ having a realization on any closed interval of $[0, 1]$ then for any $x < y$ from $[0, 1]$, there is $\mu$ which realizes $m$ on $[x, y]$. There is a mirror image result for $\text{trail}(\sigma)$.

PROOF: Say $\sigma = \langle b_1, ..., b_k \rangle$. Choose a sequence $x < ... < y_2 < y_1 < y_0 = y$ converging to $x$. For each $i = 1, 2, ...,$ and each $j \in J = \{1, ..., k\}$, let $\mu_{i,j}$ realize $b_j$ on $[y_{ik-j+1}, y_{ik-j}]$.

Define $\mu : [x, y] \rightarrow \emptyset(\text{Cl}\emptyset)$ via $\mu(x) = \text{start}(m), \mu(y) = \text{end}(m) = \text{start}(b_k)$ and if $z \in [y_{ik-j+1}, y_{ik-j}]$, then put $\mu(z) = \mu_{i,j}(z)$.

I claim that $\mu$ realizes $m$ on $[x, y]$. Consider the six realization conditions. The harder cases are conditions R2 and R6. There are several subcases and their converses and they all involve similar sorts of reasoning so we will just present a few for illustration purposes.

To show the forward direction of R2 assume that $z \in [x, y]$ and $U(\alpha, \beta) \in \mu(z)$. The subcases concern whether $z = x$, $z$ equals some $y_{ik-j} < y_0$ or $z$ is in some $[y_{ik-j+1}, y_{ik-j}]$. We must show that R2.1 or R2.2 holds.

Suppose $x = z$ and $U(\alpha, \beta) \in \mu(x)$ is a type 1 defect in $m$. As $\langle m, b_1, ..., b_k \rangle$ is a full decomposition of $m$, a cure to this defect is witnessed in this sequence. We can conclude that $\beta$ is in the cover of $m$ and so in the starts, covers and ends of each of the $b_i$. We can also conclude that $\alpha$ is in the start of some $b_i$ and in the end of the preceding one $b_j$ (with $j = k$ if $i = 1$). R2.1 follows easily with $u = y_{ik-j}$.

Suppose $x = z$ and $U(\alpha, \beta) \in \mu(x)$ is not a type 1 defect in $m$. So $\beta \in \text{cover}(m)$. If $\alpha \in \text{end}(m)$ then R2.1 holds. Otherwise R2.2 holds.

Suppose $U(\alpha, \beta) \in \mu(z)$ and $z \in [y_{ik-j+1}, y_{ik-j}]$. So $U(\alpha, \beta) \in \mu_{i,j}(z)$. Now $b_j$ is realized by $\mu_{i,j}$ on $[y_{ik-j+1}, y_{ik-j}]$ and so by R2 (for $\mu_{i,j}$) either R2.1 holds and we are almost immediately done or R2.2 holds. In this latter case $\beta \in \text{end}(b_j)$, $U(\alpha, \beta) \in \text{end}(b_j)$, we may suppose $-\alpha \in \text{end}(b_j)$ and for all $v$, if $z < v < y_{ik-j}$ then $\beta \in \mu_{i,j}(v) = \mu(v)$.

Possibly there are some $i' > 0$ and $j' \in J$ such that $0 \leq i'k - j' < ik - j$ (so $y_{ik-j} \leq y_{i'k-j'+1} < y_{i'k-j'} \leq y_0$) and either one of the following five holds: $\beta \notin
cover(b_{j'}) \land \neg \alpha \not\in \text{cover}(b_{j'}) \land \beta \not\in \text{end}(b_{j'}) \land \alpha \in \text{end}(b_{j'}) \land U(\alpha, \beta) \not\in \text{end}(b_{j'})

If there is no such \( i', j' \) then it is straightforward to show that R2.2 holds and we are done. If there are such \( i', j' \) then we can suppose that they are chosen so that \( i'k - j' \) is greatest possible. It follows that \( U(\alpha, \beta) \in \text{start}(b_{j'}) \). If R2.1 holds of \( \mu_{i', j'} \) then it is easy to finish. So suppose not. Thus R2.2 holds of \( \mu_{i', j'} \) and we can conclude via R6 that \( \beta \in \text{cover}(b_{j'}) \land \beta \in \text{end}(b_{j'}) \land U(\alpha, \beta) \in \text{end}(b_{j'}) \). Because R2.1 does not hold of \( \mu_{i', j'} \) we can also conclude via R6 that \( \neg \alpha \in \text{cover}(b_{j'}) \land \neg \alpha \in \text{end}(b_{j'}) \). This contradicts our choice of \( i' \) and \( j' \) and we are done.

For the converse direction of R2, we assume that \( z \in [x, y] \) and either R2.1 or R2.2 holds. The subcases concern whether R2.1 or R2.2 holds and whether \( z = x \), \( z \) equals some \( y_{ik - j} < y_0 \) or \( z \) is in some \( y_{ik - j}, y_{ik - j} \]. We must show that \( U(\alpha, \beta) \in \mu(z) \).

Suppose R2.2 holds with \( z \) in some \( y_{ik - j}, y_{ik - j} \]. So \( \beta \in \mu(y) \land U(\alpha, \beta) \in \mu(y) \land \beta \in \mu(v) \). A straightforward induction on \( i'k - j' \), using R2.2 for each \( \mu_{i', j'} \) shows that for all such numbers with \( 0 \leq i'k - j' \leq ik - j \), we have \( U(\alpha, \beta) \in \mu(y_{ik - j'}) \). That \( U(\alpha, \beta) \in \mu_{i,j}(z) \) follows immediately by using R2.2 on \( \mu_{i,j} \).

For the forward direction of condition R6, suppose \( \beta \) is in the cover of \( m \). Thus \( \beta \) is in the start, end and cover of each \( b_i \) as they compose (with \( m \) itself) to \( m \). Also note that the end of \( b_k \) is the same as the start of \( b_1 \). By conditions R4, R5 and R6 for each \( \mu_{i,j} \), \( \beta \in \mu(z) \) for each \( z \in [y_{ik-j+1}, y_{ik-j}] \) as required.

For the converse direction of condition R6, suppose, for all \( u \in [x, y] \), \( \beta \in \mu(u) \). It is clear that \( \beta \) is in the cover, start and end of each \( b_i \). If \( \beta \) was not in the cover of \( m \) then the fact that \( \langle m, b_1, ..., b_k \rangle \) is a full decomposition of \( m \) would imply that \( \sim \beta \) would be in the start of some \( b_i \). Hence, by contradiction, \( \beta \) is in the cover of \( m \) as required. \( \square \)

Recall that a linear order \( (T, <) \) is separable iff there is countable set \( Q \subseteq T \) such that if \( s < t \) are from \( T \) then there is \( q \in Q \) such that \( s < q < t \). Clearly \( \mathbb{R} \) is separable with \( \mathbb{Q} \) being a dense countable suborder.

**LEMMA 11** Suppose \( x < y \) are from \([0, 1]\), \( 0 \leq r \) and \( 0 \leq s \).

Then there are sets \( K_1, ..., K_r \) of closed intervals of \( ]x, y[ \) and sets \( R_0, ..., R_s \) of elements of \( ]x, y[ \) such that:

- if \( [a, b] \in K_i \) and \( [c, d] \in K_j \) and \( [a, b] \) and \( [c, d] \) are not disjoint then \( i = j \), \( a = c \) and \( b = d \);
- if \( [a, b] \in K_i \) then \( [a, b] \) is disjoint from \( R_j \);
- if \( i \neq j \) then \( R_i \) and \( R_j \) are disjoint;
- if \( u < v \) are from \( ]x, y[ \) and are not both in the same interval in some \( K_i \) then for each \( j = 1, ..., r \) there is an interval in \( K_j \) which begins strictly after \( u \) and ends strictly before \( v \) and for each \( j = 0, ..., s \) there is some \( z \in R_j \) such that \( u < z < v \);
• every $z \in ]x, y[\, appears\ in\ some\ R_j\ or\ in\ some\ interval\ in\ some\ K_i$.

PROOF: We can proceed in a two stage construction as follows. Stage one is the construction of the $K_i$. If $r = 0$ skip this stage.

Stage one proceeds in $\omega$ rounds starting with round 0. Start with all the $K_i$ empty. Before each round $K_i = \bigcup_{1 \leq i \leq r} K_i$ will contain finitely many closed intervals within $]x, y[\,$. So there will be finitely many open maximal intervals partitioning the complement of $\bigcup_{u,v} K_i u, v]$ within $]x, y[\,$. Call these the spaces left before that round.

In round 0 put $\left(\frac{2x+y}{3}, \frac{x+2y}{3}\right)$ in $K_1$. In general, for each space $]u,v[\, left\ before\ round\ pr + q + 1$ (for integers $p \geq 0$ and $q$ with $0 \leq q < r$), put $\left(\frac{2u+v}{3}, \frac{u+2v}{3}\right)$ in $K_{q+1}$. Notice that we leave spaces on each side of the new intervals and these spaces are one third as wide as the original space.

After $\omega$ rounds we have our final $K_i$’s.

We will now use the separability property of $\mathbb{R}$ to show that there are still plenty of points of $]x, y[\, not\ in\ any\ interval\ in\ any\ K_i$. Let $R$ be the set of these points.

I claim that between every pair of intervals from $\bigcup K_i$ there are some elements of $R$. To show this by contradiction suppose that no element of $R$ lies between $[a, b]$ and $[c, d]$ (where $b < c$). So, for every $w \in ]b, c[\, there is some interval $I_w \in \bigcup K_i$ with $w \in I_w$.

Let $(S, \prec)$ be the ordering of intervals from $\bigcup K_i$ which lie within $]b, c[\,$ inherited from their elements. This is isomorphic to the rationals order as it is countable, dense and without endpoints.

Thus the order $(S, \prec)$ has an uncountable order $(G, \prec)$ of gaps. Define a map $f : G \to \mathbb{R}$ as follows: given a gap $\gamma$ in $S$, let $X = \{x \in \mathbb{R} \mid x \text{ lies in some } [u, v] \in S \text{ with } [u, v] \prec \gamma\}$. Let $f(\gamma) = \sup(X)$ which exists as $X \subseteq \mathbb{R}$. Clearly $f$ is order preserving and one-to-one. Furthermore, if $\gamma < \delta$ are gaps of $S$ then there is $[u, v] \in S$ between them. Thus $b < f(\gamma) < u < v < f(\delta) < c$ and $u$ must also be strictly between the interval $I_{f(\gamma)}$ from $S$ containing $f(\gamma)$ and the interval $I_{f(\delta)}$ from $S$ containing $f(\delta)$. These two intervals must be disjoint.

Thus $\{I_{f(\gamma)} \mid \gamma \in G\}$ is an uncountable set of pairwise disjoint non-singleton intervals of $\mathbb{R}$. This clearly contradicts separability.

Thus $R$ is a set of points densely located between the intervals in the $K_i$ or, in the case that $r = 0$, $R = ]x, y[\,\.$

It is straightforward to partition $R$ densely into the pairwise disjoint $R_0, ..., R_s$ as required.

□

**LEMMA 12** If $m$ is fully decomposed by the tactic shuffle($\langle P_0, ..., P_s\rangle, \langle \lambda_1, ..., \lambda_r\rangle$) with each mosaic in each $\lambda_i$ having a realization on any closed interval of $[0, 1]$ then for any $x < y$ from $[0, 1]$, there is $\nu$ which realises $m$ on $[x, y]$.
PROOF: Let $K_1, ..., K_r$ and $R_0, ..., R_s$ be as constructed for $[x, y]$ in lemma 10. Suppose $1 \leq i \leq r$ and $\lambda_i = (s_1, ..., s_{e(i)})$. For each interval $[u, v] \subseteq K_i$, choose a sequence $u = w_0 < w_1 < w_2 < ... < w_{e(i)} = v$ and for each $j = 1, ..., e(i)$, let $I(u, v, j)$ be the interval $[w_{j-1}, w_j]$. Let $\nu_{u, v, j}$ realize $s_j$ on $I(u, v, j)$.

Define $\nu$ via $\nu(x) = \text{start}(m)$, $\nu(y) = \text{end}(m)$, for each $z$ in $I(u, v, j)$ within an interval $[u, v]$ from $K_i$, $\nu(z) = \nu_{u, v, j}(z)$ and for each $z \in R_i$, $\nu(z) = P_i$. Note that $z$ may lie at the end of some $I(u, v, j)$ and the beginning of $I(u, v, j + 1)$. In that case, the fact that the mosaics in each $\lambda_i$ compose will guarantee that $\mu(z)$ is well-defined.

I claim that $\nu$ realizes $m = (A, B, C)$ on $[x, y]$. Consider the six conditions. The harder cases are conditions R2 and R6. It is useful to consider condition R6 first.

For the forward direction of condition R6, suppose $\beta$ is in the cover of $m$. By lemma 8, $\beta$ is in each $P_i$ and in the start, cover and end of each mosaic in each $\lambda_i$. So $\beta$ is in $\nu(z)$ for each $z$ in each $R_i$ and in each $I(u, v, j)$ in each $[u, v]$ in each $K_i$. Thus $\beta \in \nu(z)$ for each $z \in [x, y]$ as required.

For the converse direction of condition R6, suppose, for all $u \in [x, y]$, $\beta \in \nu(u)$. For contradiction suppose that $\beta$ is not in the cover of $m$. Thus $\sim \beta$ is a type 3 defect in $m$ and this is cured in the full decomposition $F_1$. Thus $\sim \beta$ appears in the start of a mosaic in $\lambda$ or in $\mu$ or in the start of $m''$. Thus $\sim \beta$ appears in the start of a mosaic in one of the $\lambda_i$ or appears in one of the $A_i$, $C_i$ or $P_i$. Thus $\sim \beta \in \nu(w)$ for $w$ being the start of some $I(u, v, j)$ for some $[u, v]$ in some $K_i$ or for $w$ where some $[u, w]$ is in some $K_i$ or for $w$ in some $R_i$. Thus $\beta$ cannot be in $\nu(w)$ and we have our contradiction. Hence, $\beta$ is in the cover of $m$ as required.

To show the forward direction of R2 assume that $z \in [x, y]$ and $U(\alpha, \beta) \in \nu(z)$. The subcases concern whether $z = x$, $z$ is in some $R_i$ or $z$ is in some $I(u, v, j)$ for some $[u, v]$ in some $K_i$. We must show that R2.1 or R2.2 holds. Again there are several subcases and their converses using similar sorts of arguments. We give a selection for illustration purposes.

First consider $z = x$. So $U(\alpha, \beta) \in \nu(x) = A$. Now $m$ is fully decomposed by $\langle m' \rangle ^{\lambda} \langle \mu \rangle ^{\langle m'' \rangle}$ so, by definition of a full decomposition, either (1) $U(\alpha, \beta)$ is a type 1 defect cured in the decomposition or (2) $\beta \in B$ and either (i) $\beta \in C$ and $U(\alpha, \beta) \in C$ or (ii) $\alpha \in C$. These latter conditions (2) give us the desired result immediately.

If $U(\alpha, \beta) \in A$ is cured in the full decomposition of $m$ then it is clear that $\beta$ is in the cover of the first mosaic, $m'$. But this cover is $B$ itself so $\beta \in B$ and $\beta \in \nu(v)$ for all $v \in [x, y]$. Now $\alpha$ appears in the end of a mosaic in the full decomposition and so in $\nu(u)$ for some $u \in [x, y]$. Thus we are done.

Now consider the case of $z \in R_i$ with $U(\alpha, \beta) \in \nu(z) = P_i$. The case of $i = s$ is a slightly special case of what follows and can be proved with slight modifications so we will omit that case. Assume $0 \leq i < s$.

Thus $y_i = (P_i, B, P_{i+1})$ is fully decomposed by $\langle y_i, ..., y_s \rangle ^{\lambda} \langle y_0, ..., y_i \rangle$ and
we must have $\beta \in \text{cover}(y_i) = B$. This is whether or not $U(\alpha, \beta) \in P_i$ is a type 1 defect in $y_i$ or not. By the argument above for the R6 case, $\beta \in \nu(v)$ for all $v \in [x, y]$.

If $\sim \alpha \not\in B$ then $\alpha$ is a type 3 defect in $m$ and thus is cured in the full decomposition. Thus $\alpha$ appears in the start of a mosaic in $\lambda$ or in $\mu$ or in the start of $m''$. Thus $\alpha$ appears in the start of a mosaic in one of the $\lambda_i$, or appears in one of the $A_i, C_i$ or $P_i$. Thus $\alpha \in \nu(w)$ for $w > z$ being the start of some $I(u, v, j)$ for some $[u, v]$ in some $K_i$ or for $w$ where some $[u, w]$ is in some $K_i$ or for $w$ in some $R_i$. Combined with the observation about $\beta$ this gives us R2.1.

Otherwise, $\sim \alpha \in B$ and so coherency C1.2 along with the fact that $\sim U(\alpha, \beta) \not\in B$ gives us $U(\alpha, \beta) \in A$. By the fullness of the decomposition of $m$, either $\alpha$ appears in the start of a mosaic in $\lambda$ or in $\mu$ or in the start of $m''$ (and we proceed as above), $\alpha \in C$ (and R2.1 holds) or $\beta \in C$ and $U(\alpha, \beta) \in C$ (and R2.2 holds). We are done.

The case of $z$ in some $I(u, v, j)$ for $[u, v] \in K_i$ is similar but a little more complex.

For the converse direction of R2, we assume that $z \in [x, y]$ and either R2.1 or R2.2 holds. The subcases concern whether $z = x$, $z$ is in some $R_i$ or $z$ is in some $I(u, v, j)$ for some $[u, v]$ in some $K_i$. We must show that $U(\alpha, \beta) \in \nu(z)$.

Consider just the case of R2.1 holding for $z$ in some $I(u', v', j)$ for some $[u', v']$ in some $K_i$. Let $u' = w_0 < w_1 < ... < w_{e(i)} = v'$ be such that each $I(u', v', j) = [w_{j-1}, w_j]$. Thus $w_{j-1} \leq z \leq w_j$. We have $z < u \leq y$ and $\alpha \in \nu(u)$ and for all $v, \beta \in \nu(v)$. There are three possibilities for $u$.

Suppose $z < u \leq w_j$. Thus R2.1 holds for $\nu_{w', v, j}$ and so $U(\alpha, \beta) \in \nu(z) = \nu_{w', v, j}(z)$.

Suppose $z \leq w_j \leq w_{j'} < u \leq w_{j'+1} \leq w_{e(i)}$. By R2.1 or R2.2, $U(\alpha, \beta) \in \nu(w_{j'})$. An easy induction using R2.2 establishes that $U(\alpha, \beta) \in \nu(w_j)$. Then R2.2 gives us $U(\alpha, \beta) \in \nu(z)$ as required.

Suppose $w_{e(i)} < u \leq y$. For each $i = 0, ..., s$, choose $u \in [w_{e(i)}, u]$ with $w \in R_i$. So $\beta \in \nu(w) = P_i$. For each $i = 1, ..., r$, choose $[u'', v''] \in K_i$ with $w_{e(i)} < u'' < v'' < u$. Say $\lambda_{v'} = \langle s_1, ..., s_{e(v')} \rangle$ and for each $j' = 1, ..., e(v')$, $I(u'', v'', j') = [w_{j'-1}, w_{j'}]$. Now $\beta \in \nu(w_{j'-1}) = \text{start}(s_j), \beta \in \bigcap_{w'' \in [w_{j'-1}, w_{j'}]} \nu(w'') = \text{cover}(s_{j'})$ and $\beta \in \nu(w_{j'}) = \text{end}(s_{j'})$. We can conclude $\beta \in \nu(v')$ for all $v' \in [x, y]$ and so by R6 that $\beta \in B = \text{cover}(B)$ and so in the cover of all mosaics in each $\lambda_{v'}$ and each $x_{v'}$ and each $y_{v'}$.

There are two cases now: either $\sim \alpha \in B$ or not. Suppose $\sim \alpha \in B$ so that $\sim \alpha \in \nu(w)$ for all $w \in [x, y]$. So $\alpha \not\in \nu(w)$ for any such $w$. We know that $\alpha \in \nu(u)$ so it follows that $u = y$. Thus $\alpha \in \text{end}(m)$. Coherency C1.1 implies that $U(\alpha, \beta) \in \text{start}(m)$ and C2.2 gives us $U(\alpha, \beta) \in \text{cover}(m) = B$. By R6, $U(\alpha, \beta) \in \nu(z)$ as required.

The other case is that $\sim \alpha \not\in B$ so that $\alpha$ is a type 3 defect in $m$ and so appears in the start of a mosaic in some $\lambda_{v'}$, at the end of some $\lambda_{v'}$ or in some $P_{v'}$. If
LEMMA 13 Suppose that $\phi \in L(U, S)$, $q$ is an atom not appearing in $\phi$ and $m$ is a $(\phi, q)$-relativized $*_q^\phi(\phi)$-mosaic which appears in a real mosaic system. Then $m$ is fully $[0, 1]$-satisfiable.

PROOF: let $* = *_q^\phi$. Given the real mosaic system $S$ of $*_q^\phi(\phi)$-mosaics, we can easily proceed by induction on $k$ to show that, for any $x < y$ from $[0, 1]$, for any level $k$ member $m \in S$ there is $\mu$ which realises $m$ on $[x, y]$. Each step of the induction is just a use of one or two of the preceding lemmas \[\square\], its mirror image and \[\square\].

So we have $\mu$ which realizes $m$ on $[0, 1]$. Define $h$ by $t \in h(p)$ iff $p \in \mu(t)$ and let $T = ([0, 1], <, h)$.

I claim, for all $\alpha \in \Cl(*_q^\phi(\phi))$, for all $t \in [0, 1]$, $T, t \models \alpha$ iff $\alpha \in \mu(t)$.

This is a straightforward proof by induction on the construction of $\alpha$. The case of $U(\alpha, \beta)$ is as follows.

Note that if $U(\alpha, \beta) \in \Cl(*_q^\phi(\phi))$ then $U(\alpha, \beta) \leq *_q^\phi(\phi)$ and so both $\alpha$ and $\beta$ are also in $\Cl(*_q^\phi(\phi))$ by lemma \[\square\].

First suppose $T, t \models U(\alpha, \beta)$. Thus there is $s \in [0, 1]$ with $0 \leq t < s \leq 1$, $T, s \models \alpha$ and for all $u$, if $t < u < s$ then $T, u \models \beta$. By the inductive hypothesis, $\alpha \in \mu(s)$ and for all $u$, if $t < u < s$ then $\beta \in \mu(u)$. By R2, $U(\alpha, \beta) \in \mu(t)$ as required.

Now suppose $U(\alpha, \beta) \not\in \mu(t)$. Note that $t < 1$ as no $U(\gamma, \delta)$ is in $\mu(1) = \text{end}(m)$ as $m$ is relativized. By R2, either R2.1 or R2.2 holds and it can not be the latter as that entails $U(\alpha, \beta) \in \mu(1)$ amongst other things. So R2.1 holds and there is $s$ with $t < s \leq 1$, $\alpha \in \mu(s)$ and for all $u$, if $t < u < s$ then $\beta \in \mu(u)$. It follows via the inductive hypothesis that $T, t \models U(\alpha, \beta)$ as required.

From the claim and conditions R4, R5 and R6 on realization, it follows that start$(m) = \{ \alpha \in \Cl(*_q^\phi(\phi)) | T, 0 \models \alpha \}$, end$(m) = \{ \alpha \in \Cl(*_q^\phi(\phi)) | T, 1 \models \alpha \}$ and the cover of $m$ contains exactly those $\alpha \in \Cl(*_q^\phi(\phi))$ which hold at all points in between 0 and 1. Thus $m = \text{mos}_T^\phi(0, 1)$ as required. \[\square\]
8 Decomposition trees

In this section we begin to show the converse of the last lemma, to show that satisfiable mosaics appear in real mosaic systems. Here we show how to arrange decompositions for satisfiable mosaics into a tree structure.

**LEMMA 14** Suppose \( T = ([0, 1], <, h) \) is a structure, \( \phi \in L(U, S) \), and \( 0 \leq x_0 < x_1 < \ldots < x_n \leq 1 \).

Then the composition of \( \langle \text{mos}(x_0, x_1), \text{mos}(x_1, x_2), \ldots, \text{mos}(x_{n-1}, x_n) \rangle \) is \( \text{mos}(x_0, x_n) \).

**LEMMA 15** Suppose \( \phi \in L(U, S) \) and \( T = ([0, 1], <, h) \). If \( m = \text{mos}(x, y) \) for some \( x < y \) from \([0, 1]\) then there is some sequence \( x = x_0 < x_1 < \ldots < x_n = y \) such that \( \langle \text{mos}(x_0, x_1), \ldots, \text{mos}(x_{n-1}, x_n) \rangle \) is a full decomposition of \( m \). Furthermore, the \( x_i \) can be chosen so that no \( x_{j+1} - x_j \) is greater than half of \( y - x \).

**PROOF:** We will choose a finite set of points from \([x, y]\) at which we will decompose \( \text{mos}(x, y) \). For each defect \( \delta \) in \( \text{mos}(x, y) = (A, B, C) \) choose some \( u_\delta \) or \( z_\delta \) witnessing its cure between \( x \) and \( y \) as follows.

If \( \delta = U(\alpha, \beta) \in A \) is a type 1 defect then it is clear that there must be \( u_\delta \in [x, y] \) with \( T, u_\delta \models \alpha \) and for all \( v \in [x, u_\delta] \), \( T, v \models \beta \). Similarly find \( u_\delta \in [x, y] \) witnessing a cure for type 2 defects.

If \( \delta \in \text{Cl}_\phi \) is a type 3 defect in \( \text{mos}(x, y) \) then it is clear that there is \( z_\delta \in [x, y] \) with \( T, z_\delta \models \delta \).

Collect all the \( u_\delta \)s and \( z_\delta \)s so defined into a finite set and add the midpoint \((x + y)/2\) of \( x \) and \( y \). Order these points between \( x \) and \( y \) as \( x = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} = y \). Note that some points might be in this list for two or more reasons.

It is clear that because of our choice of witnesses, the sequence of \( \text{mos}(x_{j-1}, x_j) \) is a full decomposition. \( \Box \)

**DEFINITION 21**

1. A tree here is just a set (of nodes), partially-ordered by a binary irreflexive ancestor relation such that the set of ancestors of any node is finite and well-ordered (by the ancestor relation) and there is a (unique) root (ie, ancestor of every other node).

2. The depth of a node with \( n \) ancestors is \( n + 1 \). So the root has depth 1.

3. An ordered tree is a tree with finite numbers of children for each node and an earlier-later relation which totally orders siblings.

4. A decomposition tree is an ordered tree with each node labelled by a pair \((x, y)\) of elements of \([0, 1]\) such that \( x < y \) and such that if node \( g \) is labelled by \((x, y)\) and has children labelled by \((x_0, y_0), \ldots, (x_n, y_n)\) in order then \( x = x_0 < y_0 = x_1 < y_1 = x_2 < \ldots < y_{n-1} = x_n < y_n = y \).
5. A decomposition tree is tapering iff for all nodes $g$, and children $h$ of $g$, if $g$ is labelled by $(x, y)$ and $h$ is labelled by $(u, v)$ then $v - u$ is at most half of $y - x$.

6. In the context of a structure $([0, 1], <, h)$ and a formula $\phi$, we say that the $\phi$-mosaic $m$ is the mosaic label of a node $g$ of a decomposition tree iff $g$ is labelled by $(x, y)$ and $m = \text{mos}(x, y)$.

7. If $T = ([0, 1], <, h)$ is a structure and $\phi \in L(U, S)$, then we say that a decomposition tree is $(T, \phi)$-full iff for each node $g$, if $g$ has children then the mosaic labels on the children in order form a full decomposition of the mosaic label of $g$.

8. A $(T, \phi)$-full decomposition tree is complete iff every node has children.

In diagrams, we will represent earlier-later by left-to-right ordering and ancestors above descendents.

An ordered tree has a depth-first earlier-later total ordering of its nodes. We will call this the lexical ordering and sometimes restrict it to leaf nodes.

**Lemma 16** Suppose $T = ([0, 1], <, h)$ is a structure, $\phi \in L(U, S)$, and $0 \leq x < y \leq 1$. Then $\text{mos}(x, y)$ is the mosaic label of the root of a complete and tapering $(T, \phi)$-full decomposition tree.

**Proof:** Say $m = \text{mos}(t^-, t^+)$ for $t^- < t^+$ from $[0, 1]$. Construct a decomposition tree with root labelled by $(t^-, t^+)$ by repeated use of lemma 15.

□

Consider the sequence of labels $(u, v)$ along any infinite branch $\eta$ of a tapering $(T, \phi)$-full decomposition tree $D$. Because we have included the mid-points of each $(u, v)$ in the labels of the children of that node, this sequence of pairs will converge to some $r \in [0, 1]$, i.e. if the labels of nodes in order along $\eta$ are $(x_0, y_0), (x_1, y_1), \ldots$ then both sequences $x_0 \leq x_1 \leq \ldots$ and $y_0 \geq y_1 \geq \ldots$ converge to $r$. To see this, just note that the spread $(v - u)$ of a node’s label is at most half that of its parents. Call $r = \text{limit}(\eta)$ the limit of $\eta$.

Note that if node $f$ is a child of node $g$ then the cover of the mosaic label of $f$ includes (as a subset) the cover of the mosaic label of $g$. This is a simple property of compositions of mosaics. The cover of the mosaic label of the child may be strictly bigger. However, it may be equal. We are interested in infinite branches in such a $D$ along which the cover of the mosaic labels remains the same forever.

**Definition 22** Suppose $H \subseteq \text{Cl}\phi$. Say that the infinite branch $\eta \subset D$ is an infinite $H$-branch iff there is a node $e \in \eta$ labelled with a mosaic of cover $H$ such that every node $f \in \eta$ which is a descendant of $e$ is labelled with a mosaic with cover $H$.

**Definition 23** We say that the infinite branch $\eta$ lies after the node $k$ in a decomposition tree iff $k$ does not lie on $\eta$ and there are some nodes of $\eta$ which lie lexically after $k$.

We say that the infinite branch $\eta$ lies before the node $k$ in a decomposition tree iff all the nodes of $\eta$ lie before $k$.

We say that the branch $\eta$ lies before the branch $\theta$ iff some node of $\theta$ lies after $\eta$.
Note the slight asymmetry in the definitions here reflecting the choice that descendents of a node will be lexically ordered after the node, rather than before.

**LEMMA 17** Suppose a node lies on an infinite \( H \)-branch. Then there is a lexically first such branch and a lexically last one.

**PROOF:** To find the first branch, start at the node and recursively move to the first child of the current node which lies on an infinite \( H \)-branch. \( \square \)

**LEMMA 18** Suppose \( k \) is a node labelled by \((k_-, k_+)\) in a decomposition tree with root labelled by \((g_-, g_+)\).

Then there is a sequence \( k_+ = x_0 < x_1 < ... < x_n \) (possibly with \( n = 0 \) and \( k_+ = x_0 = x_n \)) with each \((x_i, x_{i+1})\) the label of a leaf node of the tree and either:

- \( x_n = g_+ \) and there are no infinite branches after \( k \), or
- there is a node \( e \) labelled by \((e_-, e_+)\) lying on an infinite branch with \( x_n = e_- \).

There is a mirror image.

**LEMMA 19** Suppose that \( E \) is a tapering \((T, \phi)\)-full decomposition tree such that every sibling of a leaf is itself a leaf.

Suppose the root node \( g \in E \) is labelled by \((x, y)\) such that \( \text{mos}(x, y) \) has cover \( H \). Also suppose that \( g \) lies on an infinite \( H \)-branch. Suppose \( \eta \) is the lexically first infinite \( H \)-branch on which \( g \) lies and \( \text{limit}(\eta) = r \).

Then \( x \leq r \).

If \( x < r \) then there is a sequence \( x = x_0 < x_1 < x_2 < ... < x_n \leq r \) such that \( n > 0 \) and each \((x_i, x_{i+1})\) is the label of the parent of leaf node in \( E \).

Furthermore, the sequence can be chosen such that if \( x_n < r \) then there is a sequence \( x_n = y_0 < y_1 < ... < y_m < r \) such that \( m > 0 \) and:

- each \((y_i, y_{i+1})\) is the label of a leaf node in \( E \);
- \( \text{mos}(y_0, r) \) is fully decomposed by \( \langle \text{mos}(y_0, y_1), ..., \text{mos}(y_{m-1}, y_m), \text{mos}(y_m, r) \rangle \);
- and \( \text{mos}(y_m, r) = \text{mos}(y_0, r) \).

There is a mirror image result with \( \eta \) being the lexical last branch on which \( g \) lies.

**PROOF:** Suppose that the mosaic \( b \) appears infinitely often as a mosaic label along \( \eta \) in the tree \( E \). Thus the cover of \( b \) is \( H \).

For each \( \delta = S(\alpha, \beta) \in \text{Cl} \phi \) such that \( T, r \models \delta \) choose \( u_\delta < r \) such that \( T, u_\delta \models \alpha \) and for all \( w \in ]u_\delta, r[ \), \( T, u_\delta \models \beta \). We say that \( \beta \in \text{Cl}(\phi) \) is constantly true for a while before \( r \) iff there is some \( x' < r \) such that if \( x' < w < r \) then
$\mathcal{T}, x_\beta \models \beta$. For each $\beta \in \text{Cl}\phi$ such that $\beta$ is true for a while before $r$ choose some $x_\beta < r$ such that if $x_\beta < w < r$ then $\mathcal{T}, x_\beta \models \beta$. Now choose any node $k_0 \in \eta$ strictly below $g$ and labelled by $(u', v')$ such that $u'$ is strictly greater than each $u_k$ and each $x_\beta$. This can be done as $v' - u'$ halves in each generation but always $u' \leq r \leq v'$.

Now say that $b$ appears as $\text{mos}(k_-, k_+)$ for some node $k \in \eta$ below $k_0$ and labelled by $(k_-, k_+)$. 

Let $f_1, ..., f_n$ be the sequence of parents of leaf nodes of $E$ before $k$ in order. By the mirror image of lemma 18 there is a sequence $x = x_0 < x_1 < ... < x_n = k_-$ such that the label of each $f_i$ is $(x_{i-1}, x_i)$. Note that we apply lemma 18 to the subtree of $E$ without the leaf nodes of $E$. In the statement of the lemma we required that all siblings of leaves are themselves leaves in order to guarantee that this subtree is a decomposition tree.

If $k_- = r$ then we are done. Assume $k_- < r$.

We say that a formula $\gamma$ is true (in $\mathcal{T}$) arbitrarily recently before a point $z \in T$ iff for every $z' < z$ there is some $z'' \in T$ with $z' < z'' < z$ and $\mathcal{T}, z'' \models \gamma$. For each $\beta \in \text{Cl}\phi$ which is true arbitrarily soon before $r$, choose some $s_\beta$ such that $k_- < s_\beta < r$ and $\mathcal{T}, s_\beta \models \beta$. Now find a node $k' \in \eta$ below $k$ and labelled by $(k_-, k_+)$ such that $\text{mos}(k'_-, k'_+) = b$ and $k'_+$ is strictly greater than each $s_\beta$.

Let $g_1, ..., g_p$ be the sequence of parents of leaf nodes of $E$ below $k$ and before $k'$ in order. Let $h_1, ..., h_{m-1}$ be the sequence of children in $E$ of the $g_i$ in order. By the mirror image of lemma 18 (used in the subtree rooted at $k$), there is a sequence $k_- = y_0 < y_1 < ... < y_{m-1} = k'_-$ such that the label of each $h_i$ is $(y_{i-1}, y_i)$.

If $k'_- = r$ then let $f_{n+1}, ..., f_{n'}$ be the parents of leaf nodes of $E$ below $k$ and before $k'$ in order. By lemma 18 (applied to the subtree of $E$ rooted at $k$ and not including the leaf nodes from $E$), there is a sequence $k_- = x_n < x_{n+1} < ... < x_{n'} = k'_- = r$ such that each $(x_i, x_{i+1})$ is the label of $f_i$. The long sequence $x = x_0 < x_1 < ... < x_n < ... < x_{n'} = r$ is as required and we are done.

Now assume $k'_- < r$. Let $b' = \text{mos}(k_-, r)$. I claim that $\text{mos}(k'_-, r) = b'$ too. The start of both is the start of $b$. The end of both is just $\{\alpha \in \text{Cl}\phi | \mathcal{T}, r \models \alpha\}$. Finally the cover of both is just the set of $\beta \in \text{Cl}\phi$ such that $\beta$ holds constantly for a while before $r$.

We will now show that $\sigma = \langle \text{mos}(y_0, y_1), ..., \text{mos}(y_{m-1}, y_m), \text{mos}(y_m, r) \rangle$ fully decomposes $b'$. The composition is $b' = \text{mos}(y_0, r)$ by lemma 14.

Before we show that the decomposition is full consider the nodes below $k$. Say that the children of node $k$ are $k_1, ..., k_q$ labelled by $(u_0, u_1), ..., (u_{q-1}, u_q)$ respectively. Thus $\langle \text{mos}(u_0, u_1), ..., \text{mos}(u_{q-1}, u_q) \rangle$ is a full decomposition of $\text{mos}(k_-, k_+) = b$. Also $k_- = u_0 < u_1 < ... < u_{q-1} < u_q = k_+$ and there is some $j$ such that $u_{j-1} < r < u_j$ and $k_j$ lies on $\eta$. Because $\eta$ is the lexically first infinite branch on which $k$ lies there are only a finite number of nodes (in $E$) below any $k_i$ with $i = 1, ..., j - 1$. Also, we can start with the sequence $k_1, ..., k_{j-1}$ and repeatedly replace a node from $E$ by the sequence of its children in order.
and end up with a prefix sequence of $h_1, ..., h_{m-1}$ and a corresponding sequence $k_-=u_0=y_0<y_1<y_2<...<y_M=u_{j-1}$. Note that $u_0<u_1<...<u_{j-1}$ will be a subsequence of this.

Now let us return to consider defects in $b'$.

**Type 1 defects:** Suppose $U(\alpha, \beta) \in \text{start}(b') = \text{start}(b)$ is a type 1 defect of $b'$.

If it happens that $\beta \notin H = \text{cover}(b)$ then $U(\alpha, \beta)$ is also a type 1 defect in $b$ and thus cured in the full decomposition $\langle \text{mos}(u_0, u_1), ..., \text{mos}(u_{q-1}, u_q) \rangle$. Thus the cure of $U(\alpha, \beta)$ is witnessed in this sequence. Say $\alpha \in \text{end}(\text{mos}(u_{i-1}, u_i))$ and $\beta \in \bigcap_{l=1}^{i-1} (\text{cover}(\text{mos}(u_{l-1}, u_l)) \cap \text{end}(\text{mos}(u_{l-1}, u_l))) \cap \text{cover}(\text{mos}(u_{i-1}, u_i))$. As $k_j$ labelled by $(u_{j-1}, u_j)$ lies on $\eta$, cover(\text{mos}(u_{j-1}, u_j)) = H does not contain $\beta$. Thus $i < j$. However, we have seen that then $u_i$ appears as one of the $y_r$ and thus we can find a witness to the cure of the type 1 defect $U(\alpha, \beta)$ of $b'$ in the decomposition $\sigma$ as required.

Now assume $\beta \in H = \text{cover}(b) \subseteq \text{cover}(b')$. For $U(\alpha, \beta)$ to be a type 1 defect in $b'$ we thus must have $\alpha \notin \text{end}(b')$ and either $\beta \notin \text{end}(b')$ or $U(\alpha, \beta) \notin \text{end}(b')$. Since $U(\alpha, \beta)$ holds at $k_-$ we must have $\alpha$ true somewhere between $k_-$ and $r$. So $\alpha$ is a type 3 defect of $\text{mos}(k_-, k_+)$ and so cured in $\langle \text{mos}(u_0, u_1), ..., \text{mos}(u_{q-1}, u_q) \rangle$. Thus $\alpha$ must appear in the end of $\text{mos}(u_{i-1}, u_i)$ say. It can not appear after $r$ as $U(\alpha, \beta)$ is not true at $r$ so $i < j$. As above this implies $\alpha$ appears in the end of a mosaic in $\text{mos}(y_r, y_{r+1})$ in $\sigma$.

**Type 2 defects:** No type 2 defects are possible in $b'$. Suppose $S(\alpha, \beta) \in \text{end}(b')$. It is not possible that $\beta \notin \text{cover}(b')$ as $u_{S(\alpha, \beta)} < k_-$. It is not possible that $\alpha \notin \text{start}(b')$ and $\beta \notin \text{start}(b')$ as $u_{S(\alpha, \beta)} < k_-$. It is not possible that $\alpha \notin \text{start}(b')$ and $S(\alpha, \beta) \notin \text{start}(b')$ as $u_{S(\alpha, \beta)} < k_-.$

**Type 3 defects:** Suppose $\beta \in \text{Cl} \phi$ but $\sim \beta \notin \text{cover}(b')$. So $\sim \beta$ does not hold constantly for a while before $r$ and so is true arbitrarily recently before $r$. So $y_0 = k_- < s_\beta < y_{m-1} = k'_r < r$ and $T, s_\beta \models \beta$. Say that $y_{r-1} \leq s_\beta < y_r$.

So $s_\beta$ is within the label of $h_{j'}$ whose parent is $g_q$ say. Maybe $\beta$ is in the start or end of $\text{mos}(a_-, a_+)$ where $(a_-, a_+)$ is the label of $g_q$. Otherwise $\sim \beta \notin \text{cover}(\text{mos}(a_-, a_+))$ and so $\beta$ appears in the end of a mosaic in the full decomposition of $g_q$. So $\beta$ is witnessed in $\langle \text{mos}(y_0, y_1), ..., \text{mos}(y_{m-2}, y_{m-1}) \rangle$ as required. \[\Box\]

**DEFINITION 24** Say that the infinite $B$-branch $\eta$ in a tapering $(T, \phi)$-full decomposition tree is a $B$-stick iff there is a node $e \in \eta$ which lies on only one infinite $B$-branch.

9 Satisfiability implies existence

In this section we do the main work of the paper and show that satisfiable mosaics appear in real mosaic systems with a certain bound on the depth.
LEMMA 20 Suppose $\psi \in L(U, S)$ has length $L$ and that $\psi$-mosaic $m_0$ is $[0, 1]$-satisfiable. Then there is a real mosaic system of depth $2L$ containing $m_0$.

PROOF: Say $T = ([0, 1], <, h), 0 \leq t_0^- < t_0^+ \leq 1$ and $m_0 = \text{mos}_T(t_0^-, t_0^+)$. Let $S$ be the set of all $\text{mos}(x, y)$ for $x < y$ from $[0, 1]$. Clearly $m_0 \in S$. I claim that $S$ is a real mosaic system of depth $2L$.

In fact, I show that for all $m \in S$, for all $c = 0, \ldots, 2L$, if $m$ has cover containing at least $2L - c$ formulas then $m$ is a level $c$ member of $S$. We proceed by induction on $c$. Suppose that we have shown this for every $c' \leq c$ and mosaic $m = (A, B, C) \in S$ has cover containing at least $2L - (c + 1)$ formulas. All full trees will be $(T, \psi)$-full trees.

CLAIM 3 There is a tapering full decomposition tree $E$ with root with mosaic label $m = (A, B, C)$ such that:
1. all siblings of leaves are leaves,
2. each leaf and each parent of a leaf is labelled by a level $c^+$ member of $S$,
3. if a node of $E$ lies on an infinite branch then its mosaic label has cover $B$, and
4. $E$ has no $B$-sticks in it.

PROOF: Choose any $g_- < g_+$ such that $m = \text{mos}(g_-, g_)$. Use lemma [10] to find a complete and tapering full decomposition tree $D$ with root with label $(g_-, g_+)$. Let $E_0$ be the sub-tree of $D$ containing only the nodes with mosaic label with cover $B$ and all their children and grandchildren. Thus, any leaf node in $E_0$ and any parent of a leaf node in $E_0$ will have cover strictly including $B$ and, by the inductive hypothesis will be a level $c$ member of $S$. Also, any sibling of a leaf node of $E_0$ will also be a leaf node of $E_0$.

Enumerate the $B$-sticks in $E_0$. This can be done as for each stick $\xi$ we can choose some node $e_\xi$ which lies on it and on no other infinite $B$-branches.

We can use a step by step process of gradually constructing $E$ from $E_0$. Each step removes one stick $\xi$ by only changing the subtree of $E_0$ rooted at $e_\xi$. The step introduces no other infinite $B$-branches. So it suffices to just show how to so remove one $B$-stick $\xi$ from $E_0$ to make a tree $E'$.

Choose any node $f \in \xi$ below $e_\xi$ (so $f$ lies on no other infinite $B$-branches apart from $\xi$). Say that $f_1, \ldots, f_a$ are the children of $f$ in order and $f_d$ lies on $\xi$. Say that $f_d$ is labelled by $(k_-, k_+)$ and that the limit of $\xi$ is $r$ (so $k_- \leq r \leq k_+$).

To make $E'$ we will just replace $f_d$ from $E_0$ and all its descendents by a sequence of new children of $f$ who will be parents of leaf nodes
in \( E' \). The new children of \( f \) will lie later than \( f_1, \ldots, f_d-1 \) and earlier than \( f_d+1, \ldots, f_n \). In fact, we may replace \( f_d \) by one sequence of children with labels partitioning the interval \([k_-, r] \) and another later sequence with labels partitioning the interval \([r, k_+] \). Such a change can be seen to be as required in effecting a removal of \( \xi \) without any other infinite branches being introduced or even affected. Note that as the start of the mosaic label of the first new child of \( f \) will just be the same as the end of \( f_d-1 \) (or the start of \( f \) in case that \( d = 1 \)), and similarly for the end of the last new mosaic label, the children of \( f \) in \( E' \) will still carry a full decomposition of the mosaic label of \( f \).

If \( k_- = r \) or \( r = k_+ \) then we do not add any new children in the first or second sequence respectively. Note that there will be some new children to add in one or other or both sequences as we can not have \( k_- = r = k_+ \). Here we just show how to construct the first sequence of new children with labels partitioning \([k_-, r] \) in the case that \( k_- < r \). Constructing the later second sequence is via a mirror image argument.

So suppose \( k_- < r \). Lemma 19 applied to the subtree \( E_0' \) of \( E_0 \) consisting of \( f_d \) and all its descendents in \( E_0 \) tells us we have two cases.

Possibly there is a sequence \( k_- = x_0 < x_1 < \ldots < x_n = r \) such that each \((x_i, x_{i+1})\) is the label of a parent, \( g_i \), say, of a leaf node in \( E_0' \). In this case the earlier sequence of new children of \( f \) in \( E' \) will be new nodes \( e_1, \ldots, e_{n+1} \) with each \( e_i \) labelled by \((x_{i-1}, x_i)\). We also give each \( e_i \) leaf node children with exactly the same labels as the leaf node children of \( g_i \). We are done.

In the other case there is a sequence \( k_- = x_0 < x_1 < \ldots < x_n < x_{n+1} = r \) such that each \((x_i, x_{i+1})\) with \( i < n \) is the label of a parent \( g_i \) of a leaf node in \( E_0' \) and a sequence \( x_n = y_0 < y_1 < \ldots < y_m < y_{m+1} = r \) such that each \((y_i, y_{i+1})\) with \( i < m \) is the label of a leaf node in \( E_0' \) and \( mos(y_m, r) = mos(y_0, r) \) is fully decomposed by \( mos(y_0, y_1), \ldots, mos(y_m, r) \). In this case the earlier sequence of new children of \( f \) in \( E' \) will be new nodes \( e_0, \ldots, e_n \) with each \( e_i \) labelled by \((x_i, x_{i+1})\). For \( i < n \), we give each \( e_i \) leaf node children with exactly the same labels as the leaf node children of \( g_i \). Thus for \( i < n \) each \( mos(x_i, x_{i+1}) \) is a level \( c \) member of \( S \). For the node \( e_n \) labelled by \((x_n, r)\) we give it \( m+1 \) children, \( e_0', \ldots, e_m' \) in that order.

We label each \( e_j' \) by \((y_j, y_{j+1})\). Now \( mos(y_0, r) = mos(y_m, r) \) is fully decomposed by tactic trail\((mos(y_0, y_1), \ldots, mos(y_{m-1}, y_m))\) and each of these mosaics are mosaic labels of leaf nodes in \( E_0' \) and so are level \( c \) members of \( S \). Thus \( mos(y_0, r) = mos(y_m, r) \), the mosaic label of both \( e_n \) and \( e_m' \) is a level \( c^+ \) member of \( S \). Again we are done. \( \square \)
Construct such an $E$ with root $g$ labelled by $(g_-, g_+)$. If $E$ has no infinite branches then it is clear that the mosaic labels on the leaf nodes taken in lexical order form a decomposition of $m$. They are all in $S$ and so we have our required decomposition. Thus $m \in S$ is a level $(c+1)^-$ member of $S$ and hence trivially a level $(c+1)$ member of $S$ and we are done.

We can thus assume that $E$ has two or more infinite branches: a lone one would be a stick. Say that $\eta_\infty$ is the lexically first one and $\theta_\infty$ is the lexically last one.

Possibly $g_- = \lim(\eta_\infty)$. If not, ie if $g_- < \lim(\eta_\infty)$, then we can use lemma 13 to find either a sequence of level $c^+$ members of $S$ which compose to $\text{mos}(g_-, \lim(\eta_\infty))$ or sequences $\sigma_0$ and $\rho_0$ of level $c^+$ members of $S$ and a mosaic $b_1$, such that $b_1$ is fully decomposed by $\text{trail}(\rho_0)$ and $\sigma_0 \land (b_1)$ composes to $\text{mos}(g_-, \lim(\eta))$.

It follows that $\text{mos}(g_-, \lim(\eta_\infty))$ is the composition of level $(c+1)^-$ members of $S$. Similarly, with $\theta_\infty$, and $g_+$. We are done when we show that $o = \text{mos}(\lim(\eta_\infty), \lim(\theta_\infty))$ is fully decomposed by a shuffle of level $(c+1)^-$ members of $S$. Then it follows that $\text{mos}(g_-, g_+)$ is a level $(c+1)$ member of $S$ as required. Note that if $\lim(\eta_\infty) = \lim(\theta_\infty)$ then we would already be done, so we are assuming $\lim(\eta_\infty) < \lim(\theta_\infty)$.

Let $K = \{ \beta \in \text{Cl}_{\psi} \mid \beta \notin B \}$, the set of type 3 defects in $m$.

Let $h_1$ be the deepest node on both $\eta_\infty$ and $\theta_\infty$. Let $h_2$ be any descendent of $h_1$ on $\eta_\infty$ which has two children which each lie on an infinite $B$-branch. Such a node exists as $\eta_\infty$ is not a stick. Let $h_3$ be a child of $h_2$ which does not lie on $\eta_\infty$ but does lie on another infinite $B$-branch. Clearly $h_3$ has mosaic label with cover $B$ and $h_3$ is lexically after every node on $\eta_\infty$.

Say that the children of $h_3$ are $g_0, ... , g_N$ in order. Because the children are labelled with a full decomposition, each $\beta \in K$ appears in the start of some $g_i$ for $i > 0$.

**CLAIM 4** For each $i = 1, ..., N$, if $g_i$ is labelled by $(x, y)$, there are two infinite $B$-branches $\theta'$ and $\eta'$ such that $\lim(\theta') \leq x \leq \lim(\eta')$ and, if $\lim(\theta') < \lim(\eta')$ then, $\text{mos}(\lim(\theta'), \lim(\eta'))$ can be decomposed as a non-empty sequence of level $(c+1)^-$ member of $S$ which includes some mosaic with start or end equal to start($\text{mos}(x, y)$).

**PROOF:** We find $\eta'$ and, if $x < \lim(\eta')$, a sequence $\mu$ of level $c^+$ members of $S$ which composes to $\text{mos}(x, \lim(\eta'))$. Finding $\theta'$ and a similar sequence $\nu$ which composes to $\text{mos}(\lim(\theta'), x)$ is (almost) a mirror image. The sequence $\nu \land \mu$ will be as required.

Note that as $i \geq 1$, $g_i$ will have a next earlier sibling $g_{i-1}$ which will be labelled with $(w, x)$ for some $w$.

Use lemma 13 applied to $g_{i-1}$ to find a node $e$ labelled by $(e_-, e_+)$ lying on an infinite branch of $E$ and a sequence $x = x_0 < ... < x_n =$
$e_-$ with each $(x_i, x_{i+1})$ the label of a leaf node of $E$. Note that there is an infinite branch of $E$ which lies after $g_{i-1}$ as $\theta_\infty$ does.

Possible $x = e_-$ in which case let $\sigma'$ be the empty sequence of mosaics. Otherwise, if $x < e_-$ then let $\sigma'$ be the sequence of $\text{mos}(x_i, x_{i+1})$s in order. These are each level $c^+$ members of $S$ and the composition of the sequence is $\text{mos}(x, e_-)$.

Let $\eta'$ be the lexically first infinite branch on which $e$ lies. If $e_\infty = \text{limit}(\eta')$ then we are done as $\mu = \sigma'$ will do. So suppose that $e_\infty < \text{limit}(\eta')$. We will find a sequence $\mu'$ of level $(c + 1)^-$ members of $S$ which compose to $\text{mos}(e_\infty, \text{limit}(\eta'))$ and then we can put $\mu = \sigma' \land \mu'$ and we will be done. By lemma 19 we have two cases.

Possibly there is a sequence of mosaic labels of parents of leaf nodes in $E$ which compose to $\text{mos}(e_\infty, \text{limit}(\eta'))$ and we can use that as our $\mu'$.

The other possibility is that we have a sequence $\tau$ of mosaic labels of parents of leaf nodes in $E$ followed by one final mosaic $b$ such that $\mu' = \tau \land \langle b \rangle$ composes to $\text{mos}(e_\infty, \text{limit}(\eta'))$ and $b$ is fully decomposed by the tactic trail$(\rho_\infty)$ where $\rho$ is a sequence of mosaic labels of leaf nodes of $E$. Again the mosaics in $\tau$ are level $c^+$ members of $S$ and $b$ is a level $(c + 1)^-$ member of $S$ and so we are done. \(\square\)
Let \{((\theta_{-1}, \eta_{-1}), ..., (\theta_{-s}, \eta_{-s}))\} and \{(\theta_1, \eta_1), ..., (\theta_r, \eta_r)\} be the sets of all the pairs of infinite \(B\)-branches got using claim 4. For each \(i = 1, ..., N\), let \(\theta_{-j}\) and \(\eta_{-j}\) have equal limits but each \(\theta_j\) and \(\eta_j\) do not. For each \(j = 1, ..., s\), let \(P_j = \{\beta \in \text{Cl}\psi|\mathcal{T}, \lim(\eta_{-j}) \models \beta\}\). For each \(j = 1, ..., r\) let \(\lambda_j\) be a sequence of level \((c + 1)^{-}\) member of \(S\) which compose to \(\text{mos}(\lim(\theta_j), \lim(\eta_j))\). By the claim and our original choice of \(g_0, ..., g_N\), we can do this and ensure that for each \(\beta \in K\), either there is a \(P_j\) with \(\beta \in P_j\) or a \(\lambda_j\) with \(\beta\) in the start or end of some mosaic in \(\lambda_j\).

Now choose any infinite branch \(\theta_0\) of \(E\) as follows. Start at \(g\) and proceed recursively. Choose a child of the current node which lies on an infinite branch. When there is a choice of such children (as there will be infinitely often) infinitely often choose an earlier child, infinitely often a later child. Let \(s\) be the limit of \(\theta_0\). Let \(P_0 = \{\alpha \in \text{Cl}\psi|\mathcal{T}, s \models \alpha\}\). Also let \(\eta_0 = \theta_0\).

Say that an infinite \(B\)-branch \(\kappa\) is left dense if for all nodes \(e \in \kappa\) there is a descendent \(f\) of \(e\) in \(\kappa\) such that \(f\) has at least two children \(f'\) and \(f''\) on infinite \(B\)-branches such that \(f'\) is earlier than \(f''\) but \(f''\) lies on \(\kappa\).

Define right dense as the mirror image.

Note that due to the absence of sticks each infinite \(B\)-branch is either left dense or right dense or both.

**CLAIM 5** Each \(\theta_j(-s \leq j \leq r)\) and \(\theta_\infty\) is left dense and each \(\eta_j(-s \leq j \leq r)\) and \(\eta_\infty\) is right dense.

**PROOF:** Each \(\theta_i(i \neq 0)\) and \(\theta_\infty\) is found as the lexically last infinite \(B\)-branch on which some node lies. Thus it can not be right dense. As it is not a stick it must be left dense. Similarly with \(\eta_\infty\) and each \(\eta_i(i \neq 0)\).

\(\theta_0\) was chosen to be left dense and right dense by construction.

\(\square\)

**CLAIM 6** If \(U(\alpha, \beta)\) is true at the limit of a right dense infinite branch of \(E\) then \(\beta\) is in the cover of \(m\).

**PROOF:** Suppose \(\eta'\) is a right dense infinite branch with limit \(s\) and \(P = \{\gamma \in \text{Cl}\psi|\mathcal{T}, s \models \gamma\}\). If \(\mathcal{T}, s \models U(\alpha, \beta)\) then there is some \(t > s\) such that \(\mathcal{T}, t \models \alpha\) and for all \(u\), if \(s < u < t\) then \(\mathcal{T}, u \models \beta\).

Choose some node \(h \in \eta'\) labelled by \((h_-, h_+)\) such that \(h_+ - h_- < t - s\) which we can do as the width of labels halves with each generation.

Since \(\eta'\) is right dense we can choose some descendent \(h'\) of \(h\) with a child \(f\) on an infinite \(B\)-branch and an earlier child \(f'\) lying on \(\eta'\). Say that \(f\) is labelled with \((u, v)\).

Clearly \(h_- \leq s \leq u < v \leq h_+ < t\). Thus \(\beta\) is in the cover of \(\text{mos}(u, v)\) which is just \(B\). \(\square\)
CLAIM 7 If $\alpha$ is a type 3 defect in $m$ and $\theta$ is a left dense infinite branch then $\alpha$ is true arbitrarily recently before $\operatorname{limit}(\theta)$. There is a mirror image using right dense infinite branches and arbitrarily soon afterwards.

PROOF: Let $\operatorname{limit}(\theta) = s$ and $t < s$. Choose a node $n_1$ on $\theta$ labelled with $(n_1^-, n_1^+)$ such that $t < n_1^- \leq s$. Choose a node $n_2$ on $\theta$ below $n_1$ with two children $n_3$ before $n_4$ on $\theta$ and $n_3$ on another infinite branch. So $t < n_1^- \leq n_3^- < n_3^+ \leq n_4^- \leq s$. Now the cover of $\operatorname{mos}(n_3^-, n_4^-)$ is $B$ and $\alpha$ is a type 3 defect in $m$ and so in $\operatorname{mos}(n_3^-, n_4^-)$. Consider the full decomposition exhibited by the children of $n_3$. Thus there is some non-first child $n_5$ of $n_3$ with $\alpha \in \operatorname{start}(\operatorname{mos}(n_5^-, n_5^+))$. Thus $t < n_1^- \leq n_3^- < n_3^+ \leq n_5^- \leq s$ and $T, n_5^- \models \alpha$ as required. $\square$

CLAIM 8 The cover of $o$ is $B$.

PROOF: As $g_- \leq \operatorname{limit}(\eta_\infty) < \operatorname{limit}(\theta_\infty) \leq g_+$, the cover is contained in $B$. For each $\beta \in \operatorname{Cl}(\psi) \setminus B$, $\sim \beta$ is a type 3 defect in $m$ and so by claim [3], $\beta$ is true arbitrarily soon before $\operatorname{limit}(\eta_\infty)$. Thus $\sim \beta$ is also not in the cover of $o$. $\square$

CLAIM 9 If $\eta$ is a right dense infinite branch of $E$ then $\operatorname{limit}(\eta) < \operatorname{limit}(\theta_\infty)$.
(And mirror image).

PROOF: It is clear that $\operatorname{limit}(\eta) \leq \operatorname{limit}(\theta_\infty)$. We must rule out the case of $\operatorname{limit}(\eta) = \operatorname{limit}(\theta_\infty)$. Choose any node $n$ labelled by $(n_-, n_+)$ say on $\eta$ which has a later sibling $n'$ labelled by $(n'_-, n'_+)$ on another infinite branch $\kappa$ say. Thus $\operatorname{limit}(\eta) \leq n_+ \leq n'_\leq \operatorname{limit}(\kappa') \leq \operatorname{limit}(\theta_\infty)$. Now choose a descendent $p$ of $n$ labelled by $(p_-, p_+)$ which lies on $\eta$ and has a later sibling $p'$ lying on another infinite branch. Thus $\operatorname{limit}(\eta) \leq p_+ \leq p'_\leq p'_+ \leq n_+ \leq \operatorname{limit}(\theta_\infty)$ as required. $\square$

CLAIM 10 If $\eta$ is a right dense infinite branch of $E$ then $Q = \{ \gamma \in \operatorname{Cl}(\eta) | T, \operatorname{limit}(\eta) \models \gamma \}$ satisfies the forward $K(o)$ property. There is a mirror image.

PROOF: Suppose $\eta$ is a right dense infinite branch. First suppose $U(\alpha, \beta) \in Q$. By claim [4], $\beta \in \operatorname{cover}(m) = B = \operatorname{cover}(o)$.

Now either $\alpha$ is a type 3 defect in $m$ so $\sim \alpha \notin \operatorname{cover}(m)$ (so K1) or $\alpha$ is not a type 3 defect of $m$ so $\sim \alpha \in B$. However, $U(\alpha, \beta)$ is true at $\operatorname{limit}(\eta)$ so there is $s > \operatorname{limit}(\eta)$ with $\alpha$ true at $s$ and $\beta$ true everywhere in $[\operatorname{limit}(\eta), s[$. Thus $s \geq g_+$ and $\beta$ is true everywhere in
It follows that $\beta$ and $U(\alpha, \beta)$ hold everywhere in $[g_-, s]$. Now $g_- \leq \lim(\theta_\infty) \leq g_+ \leq s$ so $\alpha$ holds at $\lim(\theta_\infty)$ if $\lim(\theta_\infty) = g_+ = s$ or $\beta$ and $U(\alpha, \beta)$ hold at $\lim(\theta_\infty)$ if $\lim(\theta_\infty) < s$. Thus K2 or K3 holds as required.

To show the converse suppose that $\beta \in B$ and K1, K2 or K3 holds with respect to $U(\alpha, \beta)$ and the forward $K(o)$ property. Thus $\beta$ holds everywhere between $\lim(\eta)$ and $\lim(\theta_\infty)$. If K2 or K3 holds then it is clear that $U(\alpha, \beta)$ is true at $\lim(\eta)$. If K1 holds then claims 4 and 5 tell us that $\alpha$ is true somewhere in between $\lim(\eta)$ and $\lim(\theta_\infty)$. Again it follows that $U(\alpha, \beta)$ is true at $\lim(\eta)$ as required. □

CLAIM 11 The mosaic $o = \text{mos}(\lim(\eta_\infty), \lim(\theta_\infty))$ is fully decomposed by the tactic shuffle $(\langle P_0, \ldots, P_s \rangle, \langle \lambda_1, \ldots, \lambda_r \rangle)$.

PROOF: Let $A_i$ and $C_i$ be as in the definition of a shuffle. We use lemma 8. All the necessary forward and backward $K(o)$ properties (S1–S5) follow from claim 10 above. S0 holds by virtue of claim 4. S6 holds by choice of the $P_i$ and $\lambda_i$. □

This gives us our result as $\text{mos}(\lim(\eta_\infty), \lim(\theta_\infty))$ is fully decomposed by a shuffle in which each mosaic in each $\lambda_i$ is a level $(c+1)^{-}$ member of $S$. □

10 Summary so far

Let us summarize.

DEFINITION 25 Suppose $\psi \in L(U, S)$. Let $\text{RMS}(\psi)$ be the set of all $\psi$-mosaics which appear in any real mosaic system.

LEMMA 21 Suppose $\psi \in L(U, S)$. Then $\text{RMS}(\psi)$ is a real mosaic system and the following are equivalent for any $\psi$-mosaic $m$ and any $n \geq 0$:

1. $m$ is a level $n$ member of $\text{RMS}(\psi)$.
2. $m$ is a level $n$ member of some real mosaic system.

PROOF: To show 2 implies 1 is straightforward and it follows that $\text{RMS}(\psi)$ is a real mosaic system. It is then clear that 1 implies 2. □

THEOREM 2 Suppose $\phi$ is a formula of $L(U, S)$ and $q$ is an atom not appearing in $\phi$.

Suppose $\psi = s^q_q(\phi)$ has length $N$.

Then the following are equivalent:

1. $\phi$ is $\mathbb{R}$-satisfiable;
2. there is a $(\phi, q)$-relativized $\psi$-mosaic which appears in some real mosaic system;
3. there is a $(\phi, q)$-relativized $\psi$-mosaic which is a level $2N$ member of $\text{RMS}(\psi)$. 30
PROOF: (1 ⇒ 3) If \( \phi \) is satisfiable then lemma 7 implies there exists a \((\phi, q)\)-relativized \( \psi \)-mosaic \( m \) which is fully \([0, 1] \)-satisfiable, and so is \([0, 1] \)-satisfiable. Lemma 20 implies \( m \) appears in a real mosaic system of depth \( 2N \).

(3 ⇒ 2) follows from lemma 21.

(2 ⇒ 1). If \((\phi, q)\)-relativized \( \psi \)-mosaic \( m \) appears in a real mosaic system then lemma 13 implies that \( m \) is fully \([0, 1] \)-satisfiable. Thus lemma 7 tells us that \( \phi \) is \( \mathbb{R} \)-satisfiable. □

11 The width of the decompositions

In this section we place bounds on the number of mosaics needed in various decompositions. This is to allow us to determine termination conditions during nondeterministic algorithms.

Suppose \( \phi \in L(U, S) \) has length \( L \). There are at most \( 2^L \) formulas in \( Cl(\phi) \) and so there are at most \( 2^L \cdot 2^L = 2^{2L} \) different \( \phi \)-mosaics.

**Lemma 22** Suppose \( \phi \in L(U, S) \) has length \( L \). If the sequence \( \sigma \) of \( \phi \)-mosaics composes to \( m \) then there is a subsequence \( \sigma' \) of \( \sigma \) of length at most \( 2^{7L+1} \) which also composes to \( m \).

PROOF: For each \( \beta \in K = \{ \beta \in Cl(\phi) | \sim \beta \notin \text{cover}(m) \} \) choose a mosaic from \( \sigma \) to witness \( \beta \). We can choose either a non-first mosaic which has \( \beta \) in its start, a non-last mosaic which has \( \beta \) in its end or any mosaic from \( \sigma \) which does not have \( \sim \beta \) in its cover. Call these \( \leq 2L \) mosaics the important ones in \( \sigma \). Construct \( \sigma' \) by including the important mosaics and a composing subsequence of the mosaics between each consecutive pair of important mosaics which contains no repeated mosaics. A simple iterative procedure allows us to successively remove one copy of each repeat and the mosaics in between. Thus there will be at most \( 2^{6L} \) mosaics in \( \sigma' \) in between important mosaics. The maximum length of \( \sigma' \) will be \( 2L \cdot 2^{6L} = 2^{7L+1} \). It is straightforward to check that the composition of \( \sigma' \) is \( m \): the cover is right because of the inclusion of the important mosaics. □

**Lemma 23** If a \( \phi \)-mosaic \( m \) is fully decomposed by the tactic \( \text{lead}(\sigma) \) (or \( \text{trail}(\sigma) \)) then there is a subsequence \( \sigma' \) of \( \sigma \) of length at most \( 2^{7N+1} \) such that \( m \) is fully decomposed by the tactic \( \text{lead}(\sigma') \) (or \( \text{trail}(\sigma') \) respectively).

PROOF: Use the idea of important mosaics as in the proof of the previous lemma but include, as important, a witness for the cure of each defect in \( m \). □
LEMMA 24 If a $\phi$-mosaic $m$ is fully decomposed by the tactic shuffle $(\langle P_0, ..., P_s \rangle, \langle \lambda_1, ..., \lambda_r \rangle)$ then $m$ is fully decomposed by a tactic shuffle $(\langle P'_0, ..., P'_{s'} \rangle, \langle \lambda'_1, ..., \lambda'_{r'} \rangle)$ where $r' + s' \leq 2L$, each $P'_i$ is one of the $P_j$, each $\lambda'_i$ is a subsequence of one of the $\lambda_j$ and each $\lambda'_i$ has length at most $2^{7L+1}$.

PROOF: By lemma [3] we need only enough $P_i$ and $\lambda_i$ such that each element of $K = \{ \beta \in \text{Cl}(\phi) \mid \sim \beta \notin \text{cover}(m) \}$ appears in some $P_i$ or in the start or end of a mosaic in some $\lambda_i$. Thus $r'$ and $s'$ can be chosen so that $r' + s' \leq 2L$.

As in the proofs of the previous lemmas we can reduce each of the chosen $\lambda_i$ to be of length $\leq 2^{7L+1}$ by removing repeats in between important mosaics: in this case just the witnesses of elements of $K$. □

12 RTL-SAT in PSPACE

Recall that we have defined RTL-SAT to be the problem of deciding satisfiability of formulas in the language $L(U, S)$ over real flows of time. So, the idea is that we enter a formula as input into a machine and we get a yes or no answer as output corresponding to satisfiability or unsatisfiability respectively. Here we show that RTL-SAT is in PSPACE.

We need to specify how formulas of $L(U, S)$ are fed into a Turing machine. There is a particular question about the symbolic representation of atomic propositions since we allow them to be chosen from an infinite set of atoms. A careful approach (seen in a similar example in [HU79]) is to suppose (by renaming) that the propositions actually used in a particular formula are $x_1, ..., x_n$ and to code $x_i$ as the symbol $x$ followed by $i$ written in binary. Of course this means that the input to the machine might be a little longer than the length of the formula. In fact a formula of length $n$ may correspond to an input of length about $n \log_2 n$. However, for a PSPACE algorithm the difference is not enough for us to need to carefully distinguish between the length of the formula and the length of the input.

In the proof we shall make use of non-deterministic Turing machines. We use the definition of NPSPACE (as in [vEB90]) which requires all possible computations of such a machine to terminate on any input after using space polynomial in the size of the input.

DEFINITION 26 We consider several algorithms, each of which is given a formula $\phi$ of $L(U, S)$, a natural number $n$, and a $\phi$-mosaic $m$ (or more correctly a triple $(A, B, C)$ where $A, B$ and $C$ are subsets of Cl($\phi$)). We say that a possibly nondeterministic algorithm is a $\phi$-NPSPACE one iff there is some polynomial $p(L)$ such that on any input with $\phi$ of length $\leq L$ and $n \leq 2L$ the algorithm returns a yes or no answer after using at most $p(L)$ tape spaces.

LEMMA 25 There are $\phi$-NPSPACE algorithms which do the following for each $\phi$, $n$ and $m$:
• $SH(\phi, n, m)$ decides whether or not there exists $P_i$s and $\lambda_j$s such that $m$ can be fully decomposed by the tactic shuffle $((P_0, ..., P_s), (\lambda_1, ..., \lambda_r))$ where each mosaic in each $\lambda_i$ is a level $n^-$ member of $RMS(\phi)$.

• $LV(\phi, n, m)$ decides whether or not $m$ is a level $n$ member of $RMS(\phi)$;

• $LD(\phi, n, m)$ decides whether or not there is some $\sigma$ such that $m$ can be fully decomposed by tactic $lead(\sigma)$ with each mosaic in $\sigma$ being a level $n$ member of $RMS(\phi)$;

• $TR(\phi, n, m)$ decides whether or not there is some $\sigma$ such that $m$ can be fully decomposed by tactic $trail(\sigma)$ with each mosaic in $\sigma$ being a level $n$ member of $RMS(\phi)$;

• $CP(\phi, n, m)$ decides whether or not $m$ is a level $n^+$ member of $RMS(\phi)$;

• $LD'(\phi, n, m)$ decides whether or not there is some $\sigma$ such that $m$ can be fully decomposed by tactic $lead(\sigma)$ with each mosaic in $\sigma$ being a level $n^+$ member of $RMS(\phi)$;

• $TR'(\phi, n, m)$ decides whether or not there is some $\sigma$ such that $m$ can be fully decomposed by tactic $trail(\sigma)$ with each mosaic in $\sigma$ being a level $n^+$ member of $RMS(\phi)$;

• $CM(\phi, n, m)$ decides whether or not $m$ is a level $(n + 1)^-$ member of $RMS(\phi)$;

PROOF: 1. Description of algorithms. The algorithms are defined in terms of each other. First consider $SH$. Given $\phi$ of length $N$, $n$ and $m$, first check whether $m$ is a mosaic and check that its start satisfies the forward $K(m)$ property and its end satisfies the backwards $K(m)$ property. Return the answer “no” if any of these checks or subsequent checks fail. Also collect the set $DEF$ of type 3 defects in $m$ and guess $s \leq 2N$.

For each $i = 0$ to $s$, guess $P_i$ and check that it is an MPC containing the cover of $m$ and satisfying the forwards and backwards $K(m)$ property and remove any $\beta \in P_i$ from the set $DEF$.

Guess $r \in \{0, 1, ..., 2N - s\}$. For each $i = 1, ..., r$ (if any), guess the start of the first mosaic in $\lambda_i$ and check that it satisfies the backwards $K(m)$ property and guess the end of the last mosaic in $\lambda_i$ and check that it satisfies the forwards $K(m)$ property. Also guess and check “on the fly” a composing sequence $\lambda_i$ of up to $2^{6N + 1}$ mosaics (with appropriate start and ends). Check (via $CM(\phi, n - 1, m')$) that each $m'$ of these is a level $n^-$ member of $RMS(\phi)$ and remove from $DEF$ any formula which appears in the start or end of $m'$. Check that the start, cover and end of each $m'$ contains the cover of $m$.

Return “yes” if $DEF$ ends up empty. Otherwise return “no”.

Now consider $LV$. To decide whether or not $m$ is a level $n$ member of $RMS(\phi)$ we need to guess a sequence of mosaics which compose to $m$ and check that each of these, $m'$ say, is either a level $n^-$ member of $RMS(\phi)$ (so use $LV(\phi, n - 1, m')$) or is fully decomposed by a shuffle with each mosaic
in each sequence in the shuffle being a level \( n \) member of \( RMS(\phi) \) (so use \( SH(\phi, n, m') \)).

LD is as follows. To decide whether or not there is some \( \sigma \) such that \( m \) can be fully decomposed by tactic lead(\( \sigma \)) with each mosaic in \( \sigma \) being a level \( n \) member of \( RMS(\phi) \), we need to guess and check a sequence \( \sigma \wedge \langle m \rangle \) which is a full decomposition of \( m \) and check that each mosaic in \( \sigma \) returns yes from \( LV(\phi, n, m) \).

TR is similar to LD.

CP is easy: we already know how to guess and check decompositions. LD' and TR' are very similar to LD and TR. CM uses LD' and TR' in the same way CP uses LD and TR. We already know how to guess and check decompositions.

2. **The algorithms use polynomial space and are correct.** Fix \( \phi \) of length \( N \). We proceed by induction on the number \( n \) used. Assume \( n \geq 0 \) and that we have shown that the algorithms work for any \( n' < n \) and any \( m \).

By lemmas 24 and 8 and the inductive hypothesis, \( SH \) gives the correct result. By lemmas 22 and 23, the other algorithms are correct.

The space bounds follow as each algorithm needs only a small constant amount of information about each mosaic and the composition so far in a possibly long composing sequence of mosaics. They may also need about \( 7N \) bits to represent, in binary, the value of a counter as we check that the sequence is not too long. Each call that they make to another algorithm also requires a polynomial amount of space but we know that the depth of nesting of such calls is just linear in \( N \). \( \square \)

We conclude

**LEMMA 26** \( RTL-SAT \) is in \( PSPACE \).

**PROOF:** An NPSPACE algorithm is as follows. Given \( \phi \) of length \( L \), choose some atom \( q \) not appearing in \( \phi \). Guess a \((\phi, q)\)-relativized \( \ast^q_\phi(\phi) \)-mosaic \( m = (A, B, C) \) (checking that it is is straightforward and uses polynomial space). Use \( LV \) from lemma 23 to check whether there is a real mosaic system of depth 6L including \( m \). By theorem 2, this approach gives “yes” answers to satisfiable input and the approach does not give incorrect “yes” answers.

By a theorem in [Sav70] the problem is also in \( PSPACE \). \( \square \)

13 **RTL-SAT is PSPACE-hard**

This part of the result is relatively straightforward.

**LEMMA 27** \( RTL-SAT \) is \( PSPACE \)-hard.
PROOF: The proof of lemma 15 in [Reye7] (as one possible example amongst many in the literature) contains a formula which we can easily modify. The idea is to simulate the running of any polynomial space bounded Turing machine in a formula.

Let $M = (Q, \Sigma, \zeta, V_A, V_R, q_0)$ be a one-tape deterministic Turing Machine where $Q$ is the set of states, $\Sigma$ is the alphabet including blank $\#$, $\zeta : (Q \times \Sigma) \to (Q \times \Sigma \times \{L, R\})$, $V_A \subseteq Q$ is the set of accepting states, $V_R \subseteq Q$ is the set of rejecting states and $q_0 \in Q$ is the initial state. Suppose that $M$ is $S(n)$-space bounded, where $S(n)$ is bounded by a polynomial in $n$. Without loss of generality, we may assume that $M$ is $2^{B(n)}$-time bounded where $B(n)$ is also bounded by a polynomial in $n$. We also assume that once $M$ enters a state in $V_A$ (or $V_R$) then it stays in states in $V_A$ (or $V_R$ resp.). Let $a = a_1...a_n$ be an input to $M$.

We can represent runs of $M$ via tape configurations in the usual way. These may be supposed to be sequences of $S(n)$ symbols each from $\Sigma \cup (Q \times \Sigma)$.

We are going to effectively construct an $L(U)$ formula $\phi$ which is of polynomial size in $n$ such that the satisfiability of $\phi$ is equivalent to the acceptance of $a$ by $M$.

The atoms we use for $\phi$ are from $Q \cup (Q \times \Sigma) \cup \{\text{tick, } \ast\}$ along with $B(n)$ new atoms $r_1, ..., r_{B(n)}$.

The idea of the proof will be that $\phi$ is $\mathbb{R}$-satisfiable iff it is satisfiable in a certain structure $T$ in this language. $T$ will represent an accepting run of $M$ on $a$ in a straightforward way. $T$ has an initial tick point $0$ say. From then on, every tick point has a successor tick point so we can name the points $0, 1, 2, ...$ etc but there may be more tick points after those. At every $(S(n) + 1)$th tick point, starting at $0$, the atom $\ast$ will hold. The $S(n)$ tick points in between $* \ast$ points will represent the contents of $M$'s tape configuration at a particular instant. The points $1, ..., S(n)$ represent the tape configuration at the initial instant of $M$'s run with input $a$. For $1 \leq i \leq S(n)$, the atom $a_i$ from $\Sigma$ will be true at the $i$th point. The atom $(q_0, a_1) \in Q \times \Sigma$ will hold at point $1$. The $S(n)$ points in between the $* \ast$ at point $S(n) + 1$ and the $* \ast$ at point $2S(n) + 2$ will similarly contain the tape configuration at the second instant of $M$'s run. And so on.

We will use the $r_i$s to count up to $2^{B(n)}$ in binary at $* \ast$ points because we are only interested in the first $2^{B(n)}$ steps in $M$’s computation.

The formula $\phi$ will be the conjunction of $\phi_1, ..., \phi_{15}$ as defined below. It should be clear that $\phi$ is satisfiable iff it is satisfiable in a model like $T$ which represents a run of $M$ (on input $a$) which is accepting, iff $M$ accepts $a$. That will complete our proof.

We use abbreviations $\bot = \neg \top$, $X\alpha = U(\text{tick} \land \alpha, \neg \text{tick})$, $F\alpha = U(\alpha, \top)$ and $G\alpha = \neg F(\neg \alpha)$. We also write $X^{m+1}\alpha$ for $XX^m\alpha$ and $X^1\alpha = X\alpha$. Note that $F$ and $G$ are thus strict.

The discreteness of ticks is given by $\phi_1 = \text{tick} \land \neg S(\text{tick}, \top) \land G(\text{tick} \to \text{X} \top)$. 

35
The distribution of *s is given by $\phi_2 = * \land X^{S(n)+1} * \land G(* \rightarrow X^{S(n)+1} *)$.

$\phi_3$, which we will not write out in detail just prevents any two different configuration symbol atoms from $Q \cup (Q \times \Sigma) \cup \{*\}$ from holding at any one point and prevents any of these symbols holding at non-tick points.

The initial configuration is given by $\phi_4 = \beta_0$ defined as follows. Let $a_k = \#$ for each $k > n$. Now define each $\beta_k$ by recursion down from $\beta_{S(n)}$ to $\beta_0$. $\beta_{S(n)} = \top$, each $\beta_{k-1} = X(a_k \land \beta_k)$ ($k > 0$), and $\beta_0 = X((q_0, a_1) \land \beta_1)$. This is a formula of length $< 5S(n)$.

The start of the second configuration is determined by $\phi_5 = X^{S(n)+2}(a' \land X(q', a_2))$ where $q'$ and $a'$ are such that $\zeta(q, a_1) = (q', a', R)$. (M must move right at first).

The relationship between a consecutive sequence of three symbols in any configuration and the corresponding symbols at the next step is given in cases by $\phi_6, \ldots, \phi_{12}$.

$\phi_6$ is the conjunction of all $G((* \land X((q, a) \land Xb)) \rightarrow X^{S(n)+1}(* \land X(a' \land X(q', b))))$ for each $q, q' \in Q, a, b, a' \in \Sigma$ such that $\zeta(q, a) = (q', a', R)$.

$\phi_7$ is the conjunction of all $G((a \land X((q, b) \land Xc)) \rightarrow X^{S(n)+1}((q', a) \land X(b' \land Xc)))$ for each $q, q' \in Q, a, b, c, b' \in \Sigma$ such that $\zeta(q, b) = (q', b', L)$.

$\phi_8$ is the conjunction of all $G((a \land X((q, b) \land Xc)) \rightarrow X^{S(n)+1}(a \land X(b' \land X(q', c))))$ for each $q, q' \in Q, a, b, c, b' \in \Sigma$ such that $\zeta(q, b) = (q', b', R)$.

$\phi_9$ is the conjunction of all $G((a \land X((q, b) \land X*)) \rightarrow X^{S(n)+1}((q', a) \land X(b' \land X*)))$ for each $q, q' \in Q, a, b, c, b' \in \Sigma$ such that $\zeta(q, b) = (q', b', L)$.

$\phi_{10}$ is the conjunction of all $G((a \land X(b \land Xc)) \rightarrow X^{S(n)+1}(Xb))$ for each $a, b, c \in \Sigma$.

$\phi_{11}$ is the conjunction of all $G((* \land X(b \land Xc)) \rightarrow X^{S(n)+1}(Xb))$ for each $b, c \in \Sigma$.

$\phi_{12}$ is the conjunction of all $G((a \land X(b \land X*)) \rightarrow X^{S(n)+1}(Xb))$ for each $a, b \in \Sigma$.

It is straightforward to show that $\phi_5, \ldots, \phi_{12}$ along with $\phi_3$ ensures the progress of configurations represented in any model of $\phi$ matches those of a run of $M$.

$\phi_{13}$ says that of the $r_i$s only $r_1$ holds at time point $S(n) + 1$.

$\phi_{14}$ forces the $r_i$s to count * points. This large conjunct of $\phi$ is still of size
polynomial in \( n \). It is
\[
G \bigwedge_{i=1}^{B(n)} \left[ \left( \ast \land \bigwedge_{j<i} r_j \land \neg r_i \right) \rightarrow \left( X^{S(n)+1} \left( \bigwedge_{j<i} \neg r_j \land r_i \right) \land \bigwedge_{j>i} (r_j \leftrightarrow X^{S(n)+1} r_i) \right) \right].
\]

\( \phi_{15} \) says that when the \( r_i \)'s are next all false at an \( \ast \) point then from then on the only \((q,a)\) atoms holding are those with \( q \in V_A \). This forces the structure to be representing an accepting run of \( M \) as described above.

\( \square \)

14 Conclusion

We have shown that the decision problem for the temporal logic with until and since connectives over real-numbers time is PSPACE-complete.

There is a simple corollary using the expressive completeness ([GHR94]) of RTL over the reals. Consider a usual temporal logic, ie one with connectives defined by first-order truth tables as defined in [GHR94]. It follows that deciding any usual temporal logic over the reals is a PSPACE problem (but not necessarily PSPACE-hard).

In the introduction I suggested that the mosaic proof here suggests a tableau style theorem-proving procedure for the logic. The idea would be to generate all \( \phi \)-mosaics for a given \( \phi \) and then systematically remove those which can be decomposed into simpler mosaics (in the sense of a real mosaic system). This would give an exponential time procedure along the lines of that seen in [Pra79]. We leave further development of this idea as future work.

References

[BG85] J. P. Burgess and Y. Gurevich. The decision problem for linear temporal logic. *Notre Dame J. Formal Logic*, 26(2):115–128, 1985.

[GH90] D. M. Gabbay and I. M. Hodkinson. An axiomatisation of the temporal logic with until and since over the real numbers. *Journal of Logic and Computation*, 1(2):229 – 260, 1990.

[GHR94] D. Gabbay, I. Hodkinson, and M. Reynolds. *Temporal Logic: Mathematical Foundations and Computational Aspects, Volume 1*. Oxford University Press, 1994.

[HU79] J. Hopcroft and J. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley, 1979.

[Kam68] H. Kamp. *Tense logic and the theory of linear order*. PhD thesis, University of California, Los Angeles, 1968.
[KMP94] Y. Kesten, Z. Manna, and A. Pnueli. Temporal verification of simulation and refinement. In *A decade of concurrency: reflections and perspectives: REX school/symposium, Noordwijkerhout, the Netherlands, June 1–4, 1993*, pages 273–346. Springer–Verlag, 1994.

[LL66] H. Läuchli and J. Leonard. On the elementary theory of linear order. *Fundamenta Mathematicae*, 59:109–116, 1966.

[Nem95] I. Németi. Decidable versions of first order logic and cylindric-relativized set algebras. In L. Csirmaz, D. Gabbay, and M. de Rijke, editors, *Logic Colloquium ’92*, pages 171–241. CSLI Publications, 1995.

[Pnu77] A. Pnueli. The temporal logic of programs. In *Proceedings of the Eighteenth Symposium on Foundations of Computer Science*, pages 46–57, 1977. Providence, RI.

[Pra79] V. R. Pratt. Models of program logics. In *Proc. 20th IEEE. Symposium on Foundations of Computer Science, San Juan*, pages 115–122, 1979.

[Rab98] A. Rabinovich. On the decidability of continuous time specification formalisms. *Journal of Logic and Computation*, 8:669–678, 1998.

[Rey92] M. Reynolds. An axiomatization for Until and Since over the reals without the IRR rule. *Studia Logica*, 51:165–193, May 1992.

[Reyed] M. Reynolds. The complexity of the temporal logic with until over general linear time, submitted.

[Sav70] W. J. Savitch. Relationships between non-deterministic and deterministic tape complexities. *J. Comput. Syst. Sci.*, 4:177–192, 1970.

[SC85] A. Sistla and E. Clarke. Complexity of propositional linear temporal logics. *J. ACM*, 32:733–749, 1985.

[vEB90] P. van Emde Boas. Machine models and simulations. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume A. Elsevier, Amsterdam, 1990.