ON NSOP\textsubscript{2} THEORIES

SCOTT MUTCHNIK

Abstract. Answering a question of Džamonja and Shelah, we show that every NSOP\textsubscript{2} theory is NSOP\textsubscript{1}.

Contents

1. Introduction \hfill 2
2. Preliminaries \hfill 4
3. Canonical coheirs in any theory \hfill 7
4. Canonical coheirs in NSOP\textsubscript{2} theories \hfill 12
5. Conant-independence in NSOP\textsubscript{2} theories \hfill 14
6. NSOP\textsubscript{2} and NSOP\textsubscript{1} theories \hfill 18
References \hfill 20
1. Introduction

One of the most exciting areas of research in modern model theory is the classification along various dividing lines of non-simple but otherwise tame theories, especially NSOP\(_n\) theories for \(1 \leq n \leq 3\). The first two of these properties, introduced in [12], require the nonexistence of certain trees:

**Definition 1.1.** A theory \(T\) is NSOP\(_1\) if there does not exist a formula \(\varphi(x, y)\) and tuples \(\{b_\eta\}_{\eta \in 2^{<\omega}}\) so that \(\{\varphi(x, b_\sigma^n)\}_{n \in \omega}\) is consistent for any \(\sigma \in 2^\omega\), but for any \(\eta_2 \supseteq \eta_1 \prec (0)\), \(\{\varphi(x, b_{\eta_2}), \varphi(x, b_{\eta_1 \prec \langle 1\rangle})\}\) is inconsistent. Otherwise it is SOP\(_1\).

**Definition 1.2.** A theory \(T\) is NSOP\(_2\) if there does not exist a formula \(\varphi(x, y)\) and tuples \(\{b_\eta\}_{\eta \in 2^{<\omega}}\) so that \(\{\varphi(x, b_\sigma^n)\}_{n \in \omega}\) is consistent for any \(\sigma \in 2^\omega\), but for incomparable \(\eta_1\) and \(\eta_2\), \(\{\varphi(x, b_{\eta_1}), \varphi(x, b_{\eta_2})\}\) is inconsistent. Otherwise it is SOP\(_2\).

The property NSOP\(_3\) is introduced in [29] as part of a family of notions NSOP\(_n\) for \(n \geq 3\):

**Definition 1.3.** A theory \(T\) is NSOP\(_n\) (that is, does not have the \(n\)-strong order property) if there is no definable relation \(R(x_1, x_2)\) with no \(n\)-cycles, but with tuples \(\{a_i\}_{i \in \omega}\) with \(\models R(a_i, a_j)\) for \(i < j\). Otherwise it is SOP\(_n\).

**Fact 1.4.** ([29], [12]) Simple theories are NSOP\(_1\), and NSOP\(_n\) theories are NSOP\(_m\) for \(n \leq m\).

In [30] it is shown that \(T_{\text{feq}}^*\), the model companion of the theory of parametrized equivalence relations, is NSOP\(_1\) but not simple; a limited number of further examples have since been found by various authors. Yet the main problem, posed by Džamonja and Shelah in [12], has remained unsolved:

**Problem 1.5.** Are all NSOP\(_3\) theories NSOP\(_2\)? Are all NSOP\(_2\) theories NSOP\(_1\)?

In this paper we answer the latter question in the positive:

**Theorem 1.6.** All NSOP\(_2\) theories are NSOP\(_1\).

One reason for the significance of this problem comes from Shelah and Usvyatsov’s proposal in [30] to characterize classes of theories both internally in terms of the structure of their sufficiently saturated models, and externally in terms of orders on theories. The NSOP\(_2\) theories have a deep external characterization: under the generalized continuum hypothesis, Džamonja and Shelah [12] show that maximality in the order \(\triangleleft^*\), an order related to the Keisler order, implies a combinatorial property related to SOP\(_2\), which Shelah and Usvyatsov then show in [30] to be the same as SOP\(_2\); later, Malliaris and Shelah in [24] show the equivalence between SOP\(_2\) and \(\triangleleft^*\)-maximality under the generalized continuum hypothesis. On the other hand, NSOP\(_1\) theories can be characterized internally not only in terms of trees, but through the theory of independence, in analogy with stability theory. It is well known that simple theories are characterized as those theories where forking and dividing behave in certain ways as they do in stable theories; for example, symmetry of forking characterizes simple theories. In [16], Kaplan and Ramsey show that Kim-forking, or forking witnessed by invariant Morley sequences, is the correct way of extending the
theory of forking to NSOP$_1$ theories from simple theories. By relaxing the requirement of base monotonicity, they extend the Kim-Pillay characterization of simple theories in terms of the existence of abstract independence relations to NSOP$_1$ theories, and, more concretely, characterize NSOP$_1$ theories by the symmetry of Kim-independence, by the independence theorem for Kim-independence, and by a variant of Kim’s lemma in simple theories, asserting that $\text{Kim-dividing}$ of a formula, rather than dividing, is witnessed by any invariant Morley sequence. Our result that NSOP$_1$ theories coincide with NSOP$_2$ theories therefore shows a surprising agreement between dividing lines related to Keisler’s order and dividing lines related to independence.

We outline the paper and give a word on the strategy for the proof. In section 3, we develop in general theories a version of a construction used by Chernikov and Kaplan in [6] to study forking and dividing in NTP$_2$ theories. In [1], Adler initiated the study of abstract relations between sets in a model, generalizing some of the properties of forking-independence, coheirs, and other concrete relations from model theory, and provided a set of potential axioms for these relations$^1$. We notice that the construction of Chernikov and Kaplan can be relativized to relations between sets satisfying certain axioms, obtaining new relations between sets from old ones, and iterate this construction to obtain a canonical class of coheirs in any theory.

In section 4, we study this canonical class of coheirs in NSOP$_2$ theories. Before the development of Kaplan and Ramsey’s theory of Kim-independence in NSOP$_1$ theories in [16], Chernikov [5] proposed finding a theory of independence for NSOP$_2$ theories, and the proof of our main result comes from our efforts to answer this proposal. Just as in [6], Chernikov and Kaplan’s construction gives maximal classes in the dividing order of Ben Yaacov and Chernikov [33], we show that in NSOP$_2$ theories our variant of this construction gives minimal classes in the restriction of this order to coheir Morley sequences, proving an analogue of Kim’s lemma. As a by-product of this construction, we also initiate the theory of independence in a class related to the NATP theories of Ahn and Kim [2], the study of which was further developed by Ahn, Kim and Lee in [3], showing that under this assumption Kim-forking and Kim-dividing coincide for coheir Morley sequences. (See [20] for the question of finding an analogue for NSOP$_1$ theories of the role that NTP$_2$ theories play relative to simple theories, and developing Kim-independence in that analogue; that

$^1$Other than Adler’s work in [1] and Conant’s work on free amalgamation theories in [8], an additional observation which ultimately led us to the proof of this result is found in [11], where d’Elbée proposes the problem of explaining the apparent ubiquity of additional independence relations with no known concrete model-theoretic independence relations in NSOP$_1$ theories, such as strong independence existing alongside Kim-independence in the theory ACFG (introduced as part of a more general class in [13]) of algebraically closed fields with a generic additive subgroup. He observes that just as in the case of free amalgamation of generic functional structures in [22] or generic incidence structures in [10], these stronger independence relations can be used to prove the equivalence of forking and dividing for complete types in many known NSOP$_1$ theories. Before proving Theorem 1.6, we gave some very weak axioms (including stationarity, a feature of the examples considered by [11]) for abstract relations between sets over a model, which appeared to be very common in NSOP theories including strictly NSOP$_1$ theories and NSOP$_4$ theories, and proved that theories with such a relation could not be NSOP$_2$; instead of considering Morley sequences in canonical coheirs as in the below, we used $\downarrow$-independent sequences for the abstract relation $\downarrow$, in the sense of Definition 7.5 of [8]. Note also that the property quasi-strong finite character considered below is a property of the examples in [11].
Kim-forking coincides with Kim-dividing for coheir Morley sequences in a related class gives us preliminary evidence that NATP completes this analogy.

In section 5, we investigate behavior similar to NSOP$_1$ theories in NSOP$_2$ theories. We introduce the notion of Conant-independence, which will generalize the relation $A \downarrow^a_M B$ defined by $\text{acl}(MA) \cap \text{acl}(MB) = M$ in the free amalgamation theories introduced by Conant [8] (based on concepts used to study the isometry groups of Urysohn spheres in [32]); see the following section. While it will end up coinciding with Kim-independence in our case, we studied a version of Conant-independence in a potentially strictly NSOP$_1$, potentially SOP$_3$ generalization of free amalgamation theories in [26]. Conant-independence in NSOP$_2$ theories can be defined as Kim-independence relative to canonical Morley sequences, just as $\rightarrow^a$ is Kim-independence relative to free amalgamation Morley sequences (as in lemma 7.7 of [8]); it can also be defined by forcing Kim’s lemma on Kim-independence, requiring a formula to divide with respect to every Morley sequence instead of just one, as suggested in tentative remarks of Kim in [17] in his discussion of strong dividing in subtle theories. We show that many of Ramsey and Kaplan’s arguments on Kim-independence in NSOP$_1$ theories in [16] can be generalized to Conant-independence in NSOP$_2$ theories, including a chain condition, symmetry and a weak independence theorem. (But as is apparent in [8] and [26], similar behavior can occur in a SOP$_3$ theory, which is why the following section is essential to the proof of our main result.)

In section 6, we conclude the proof of Theorem 1.6. One consequence of Conant’s free amalgamation axioms (say, the freedom, closure and stationarity axioms, in Definition 2.1 in [8]) is the following:

Let $\downarrow$ denote free amalgamation and $A_1 \downarrow^a_M B$, $A_2 \downarrow^a_M C$, and $B \downarrow_M C$ with $A_1 \equiv_M A_2$. Then there is some $A \downarrow^a_M BC$ with $A \equiv_{MB} A_1$ and $A \equiv_{MC} A_2$.

We will have shown in the prior section that Conant-independence is symmetric, and that a similar fact holds, roughly, when replacing free amalgamation with canonical coheirs and $\downarrow^a$ with Conant-independence. Conant shows in [8] that modular free amalgamation theories must either be simple or SOP$_3$ (see [14] for a related result on countably categorical Hrushovski constructions), starting with a failure of forking-independence to coincide with $\downarrow^a$ (because forking-independence cannot be symmetric unless a theory is simple) and using the above fact to build up a configuration giving SOP$_3$. Starting, analogously, with the assumption that an NSOP$_2$ theory $T$ is SOP$_1$, so Kim-dividing independence is not symmetric and therefore fails to coincide with Conant-independence, we simulate Conant’s construction of an instance of SOP$_3$. In short, we show that a NSOP$_2$ theory is either NSOP$_1$ or SOP$_3$. But a NSOP$_2$ theory is of course not SOP$_3$, so it must be NSOP$_1$.

2. Preliminaries

We let $a, b, c, d, e, A, B, C$ denote sets, potentially with an enumeration depending on context, and $x, y, z, X, Y, Z$ denote tuples of variables. We let $\mathbb{M}$ denote a sufficiently saturated model of a theory $T$ and let $M$ denote an elementary submodel. We write $AB$ to denote the union (or concatenation) of the sets $A$ and $B$, and write $I, J, \text{etc.}$ for infinite sequences (or sometimes trees) of tuples or an infinite linearly ordered set.

Relations between sets
Roughly following the axioms for abstract independence relations in \cite{P}, as well as others that are standard in the literature, we define the following axioms for relations $A \downarrow_{\text{M}} B$ between sets over a model:

Invariance: For all $\sigma \in \text{Aut}(\text{M})$, $A \downarrow_{\text{M}} B$ implies $\sigma(A) \downarrow_{\sigma(\text{M})} \sigma(B)$.

Full existence: For $M \subseteq A, B \subseteq \text{M}$, there is always some $A' \equiv_{\text{M}} A$ with $A \downarrow_{\text{M}} B$.

Left extension: If $A \downarrow_{\text{M}} B$ and $A \subseteq C$, there is some $B' \equiv_{A} B$ with $C \downarrow_{\text{M}} B'$.

Right extension: If $A \downarrow_{\text{M}} B$ and $B \subseteq C$, there is some $A' \equiv_{B} A$ with $A' \downarrow_{\text{M}} C$.

Left monotonicity: If $A \downarrow_{\text{M}} B$ and $M \subseteq A' \subseteq A$, then $A' \downarrow_{\text{M}} B$.

Right monotonicity: If $A \downarrow_{\text{M}} B$ and $M \subseteq B' \subseteq B$, then $A \downarrow_{\text{M}} B'$.

(We will refer to the two previous properties, taken together, as monotonicity.)

Symmetry: If $A \downarrow_{\text{M}} B$ then $B \downarrow_{\text{M}} A$.

**Coheirs and Morley sequences**

A global type $p$ is a complete type over $\text{M}$. For $p \in S(A)$ for $M \subseteq A$, we say $p$ is finitely satisfiable over $M$ or a coheir extension of its restriction to $M$ if every formula in $p$ is satisfiable in $M$. Global types $p$ finitely satisfiable in $M$ are invariant over $M$: whether $\varphi(x, b)$ belongs to $p$ for $\varphi$ a formula without parameters, depends only on the type of the parameter $b$ over $M$. We write $a \downarrow^{u}_{M} b$ to denote that $tp(a/Mb)$ is finitely satisfiable in $M$. We let $a \downarrow^{b}_{M} b$ denote $b \downarrow^{u}_{M} a$. The relation $\downarrow^{u}$ (over models) is well-known to satisfy all of the above properties other than symmetry. We say $\{b_{i}\}_{i \in I}$, for $I$ potentially finite, is a coheir sequence over $M$ if $b_{i} \downarrow^{u}_{M} b_{<i}$ for $i \in I$. We say a coheir sequence $\{b_{i}\}_{i \in I}$, for $I$ infinite, is moreover a coheir Morley sequence over $M$ if there is a fixed global type $p(x)$ finitely satisfiable in $M$ so that $b_{i} \models p(x)|_{Mb_{i}}$ for $i \in I$. The type of a coheir Morley sequence over $M$ (indexed by a given set) is well-known to depend only on $p(x)$, and coheir Morley sequences are known to be indiscernible; the type of a coheir sequence over $M$ depends only on the global coheirs over $M$ extending the $tp(b_{i}/Mb_{<i})$.

**NSOP$_{1}$ theories and Kim-dividing**

In this paper we use nonstandard terminology: Kim-dividing, etc. are defined in terms of Morley sequences in invariant types over $M$ rather than finitely satisfiable types over $M$ in \cite{L}. The reason why we do this is that $\downarrow^{u}$ is known to satisfy left extension. This will do us no harm for our main result, though when we briefly consider Kim-forking in some NATP theories, we will note the nonstandard usage.

**Definition 2.1.** A formula $\varphi(x, b)$ Kim-divides over $M$ if there is a coheir Morley sequence $\{b_{i}\}_{i \in \mathbb{ω}}$ starting with $b$ so that $\{\varphi(x, b_{i})\}_{i \in \mathbb{ω}}$ is inconsistent (equivalently, $k$-inconsistent for some $k$: any subset of size $k$ is inconsistent). A formula $\varphi(x, b)$ Kim-forks over $M$ if it implies a (finite) disjunction of formulas $\varphi(x, b)$ over $M$. We write $a \downarrow^{Kd}_{M} b$, and say that $a$ is Kim-dividing independent from $b$ over $M$ if $tp(a/Mb)$ does not include any formulas Kim-dividing over $M$.

The following follows directly from Proposition 5.2 of \cite{L}; see also Proposition 3.22 of \cite{L} (where the evident argument for the version for invariant types is given) and Theorem 5.16 of \cite{L} for the full symmetry characterization of NSOP$_{1}$.

**Fact 2.2.** Symmetry of $\downarrow^{Kd}$ implies NSOP$_{1}$.
NSOP\(_2\) theories

A characterization of SOP\(_2\) as k-TP\(_1\) was proven by Kim and Kim in \([18]\), where they also introduce the notion of weak k-TP\(_1\), prove that it implies SOP\(_1\), and conjecture that it also implies SOP\(_2\):

**Definition 2.3.** The theory \(T\) has weak k-TP\(_1\) if there exists a formula \(\varphi(x, y)\) and tuples \(\{b_\eta\}_{\eta \in \omega^{< \omega}}\) so that \(\{\varphi(x, b_{\sigma|\eta})\}_{\eta \in \omega}\) is consistent for any \(\sigma \in \omega^\omega\), but for pairwise incomparable \(\eta_1 \ldots, \eta_k \in \omega^{< \omega}\) with common meet, \(\{\varphi(x, b_{\eta_i})\}_{i=1}^k\) is inconsistent.

Later, Chernikov and Ramsey, in Theorem 4.8 of \([7]\), claim to show that weak k-TP\(_1\) implies SOP\(_2\), but their proof is incorrect; the embedded tree \(\{b_\eta\}_{\eta \in \omega^{< \omega}}\) in the proof of that theorem is not actually strongly indiscernible over the parameter set \(C\). In an earlier version of this paper, we used this result. In this section, we will introduce an equivalent form of SOP\(_2\) that will suffice for our argument, and use the same method as \([7]\) to give a proof that will work to show this equivalence despite failing for weak k-TP\(_1\).

**Definition 2.4.** (Proposition 2.51, item IIIa, \([102]\)) A list \(\eta_1 \ldots, \eta_k \in \omega^{< \omega}\) is a descending comb if and only if it is an antichain so that \(\eta_1 <_{\text{lex}} \ldots <_{\text{lex}} \eta_n\), and so that, for \(1 \leq k < n\), \(\eta_1 \land \ldots \land \eta_k < \eta_1 \land \ldots \land \eta_k\).

So for example, all descending combs of length \(n\) have the same quantifier-free type in the language \(\{<_{\text{lex}}, <, \land\}\) as the descending comb \(\langle 0 \rangle^{n-1} \approx \langle 1, \ldots, 1 \rangle\); meanwhile, \(\langle 00 \rangle, \langle 01 \rangle, \langle 10 \rangle, \langle 11 \rangle\) is an example of a lexicographically ordered antichain that is not a descending comb.

**Definition 2.5.** (Definitions 11 and 12, \([31]\)) For tuples \(\overline{\eta}, \overline{\eta}' \in \omega^{< \omega}\) of elements of \(\omega^{< \omega}\), we write \(\overline{\eta} \equiv_0 \overline{\eta}'\) to mean that \(\overline{\eta}\) has the same quantifier-free type in the language \(\{<_{\text{lex}}, <, \land\}\) as \(\overline{\eta}'\). For \(\{b_\eta\}_{\eta \in \omega^{< \omega}}\) a tree-indexed set of tuples and \(\overline{\eta} = \eta_1 \ldots, \eta_n \in \omega^{< \omega}\) an \(n\)-tuple of elements of \(\omega^{< \omega}\), we write \(b_{\overline{\eta}} := b_{\eta_1} \ldots b_{\eta_n}\), and call \((b_\eta)_{\eta \in \omega^{< \omega}}\) strongly indiscernible over a set \(A\) if for all tuples \(\overline{\eta}, \overline{\eta}' \in \omega^{< \omega}\) of elements of \(\omega^{< \omega}\) with \(\overline{\eta} \equiv_0 \overline{\eta}'\), \(b_{\overline{\eta}} \equiv_A b_{\overline{\eta}'}\).

**Fact 2.6.** (Theorem 16, \([31]\); see \([28]\) for an alternate proof) Let \((b_\eta)_{\eta \in \omega^{< \omega}}\) be a tree-indexed set of tuples, and \(A\) a set. Then there is \((c_\eta)_{\eta \in \omega^{< \omega}}\) strongly indiscernible over \(A\) so that for any tuple \(\overline{\eta} \in \omega^{< \omega}\) of elements of \(\omega^{< \omega}\) and \(\varphi(x) \in L(A)\), if \(\models \varphi(b_{\overline{\eta}})\) for all \(\overline{\eta} \equiv_0 \eta\), then \(\models \varphi(c_{\eta})\).

**Definition 2.7.** The theory \(T\) has k-DCTP\(_1\) if there exists a formula \(\varphi(x, y)\) and tuples \(\{b_\eta\}_{\eta \in \omega^{< \omega}}\) so that \(\{\varphi(x, b_{\sigma|\eta})\}_{\eta \in \omega}\) is consistent for any \(\sigma \in \omega^\omega\), but for any descending comb \(\eta_1 \ldots, \eta_k \in \omega^{< \omega}\), \(\{\varphi(x, b_{\eta_i})\}_{i=1}^k\) is inconsistent.

**Lemma 2.8.** For any \(k > 1\), a theory has SOP\(_2\) if and only if it has k-DCTP\(_1\).

**Proof.** (\(\Rightarrow\)) The property 2-DCTP\(_1\) follows directly from Fact 4.2, \([7]\).

(\(\Leftarrow\)) We follow the proof of theorem 4.8 of \([7]\), which is incorrect for the claimed result. Let \(\{b_\eta\}_{\eta \in \omega^{< \omega}}\) witness DCTP\(_1\) with the formula \(\varphi(x, y)\). By fact 2.6 we can assume \(\{b_\eta\}_{\eta \in \omega^{< \omega}}\) is strongly indiscernible (as paths and descending combs are preserved under \(\equiv_0\)-equivalence), and will produce a witness to SOP\(_2\). Let \(\eta_i = (0)^i \approx \langle 1 \rangle\) (so that, say, \(\eta_n, \ldots, \eta_0\) will form a descending comb), and let \(n\) be maximal so that
\{ \varphi(x, b_{\eta_n}) : i < n, \alpha < \omega \}

is consistent; by consistency of the paths, n will be at least 1, and by inconsistency of descending combs of size k, n will be at most k. Let \( C = \{ b_{\eta_n} : i < n - 1, \alpha < \omega \} \). We see that, say, \( \mu = (0)_{n-1} \) sits strictly above the meets of any two or more of the \( \eta_i \) for \( i < n - 1 \) in the order \( < \), and is incomparable to and lexicographically to the left of \( \eta^{n-2} \) when \( n > 1 \), so the appropriately tree-indexed subset \( \{ c_{\eta} \}_{\eta \in \omega^{<\omega}} \) of \( \{ b_{\eta} \}_{\eta \in \omega^{<\omega}} \) consisting of those \( b_{\eta} \) with \( \mu \leq \eta \) (that is, where \( \eta = b_{\mu \cdot \eta} \)) really is strongly indiscernible over \( C \).

By strong indiscernibility of \( \{ b_{\eta} \}_{\eta \in \omega^{<\omega}} \) and the fact that \( \{ \varphi(x, b_{\eta_n}) : i < n, \alpha < \omega \} \) is consistent, \( \{ \varphi(x, c_{\eta_n}) : i < n, \alpha < \omega \} \cup \{ \varphi(x, c) : c \in C \} \) is consistent; let \( d \) realize it, and by Ramsey, compactness and an automorphism over \( C \), we can assume \( \{ c_{\eta_n} : \eta \in \omega^{<\omega} \} \) is indiscernible over \( dC \). On the other hand, for \( p(y, z) = \text{tp}(d, \{ c_{\eta} : \eta \in \omega^{<\omega}/C \}) \), we see that \( p(y, \{ c_{\eta_n} : \eta \in \omega^{<\omega} \} \cup \{ c_{\eta_n} : \eta \in \omega^{<\omega} \}) \) is inconsistent, by strong indiscernibility of \( \{ b_{\eta} \}_{\eta \in \omega^{<\omega}} \) and inconsistency (by maximality of \( n \)) of \( \{ \varphi(x, b_{\eta_n}) : i < n, \alpha < \omega \} \) (noting that, say, \( p(y, \{ c_{\eta_n} : \eta \in \omega^{<\omega} \}) \) contains \( \{ b_{\eta_n} : i < n - 1, \alpha < \omega \} \)). This is exactly what the “path collapse lemma,” Lemma 4.6 of \([7]\), tells us that we need to obtain SOP\(_\omega\). \( \square \)

Though the proof of Theorem 4.8 of \([7]\) is incorrect, that theorem (albeit, not a “local” version) will be a corollary of our main result, Theorem \([16]\) and the result of \([18]\) that weak \( k\)-TP\(_1\) implies SOP\(_1\). (Note that SOP\(_2\) is just weak 2-TP\(_1\)).

**Corollary 2.8.1. (to Theorem \([7,7]\))** For any \( k \), a theory has weak \( k\)-TP\(_1\) if and only if it has SOP\(_2\).

### 3. Canonical coheirs in any theory

The following section will require no assumptions on \( T \). Iterating a similar construction to the one used by Chernikov and Kaplan in \([6]\) to prove the equivalence of forking and dividing for formulas in NTP\(_2\) theories, we will contruct a canonical class of coheir extensions in any theory. This class will end up satisfying a variant of the “Kim’s lemma for Kim-dividing” in NSOP\(_1\) theories (Theorem 3.16 of \([16]\)) when considered in a NSOP\(_2\) theory.

**Proposition 3.1.** Let \( T \) be any theory. Consider relations \( \downarrow \), between sets over a model that are stronger that \( \downarrow^h \), satisfy invariance, monotonicity, full existence and right extension, and satisfy the coheir chain condition: if a \( \downarrow_M^b I = \{ b_i \}_{i \in \omega} \) is a coheir Morley sequence starting with \( b \), then there is some \( I' \equiv_M I \) with a \( \downarrow_M I' \) and each term of \( I' \) satisfying \( \text{tp}(b/Ma) \). There is a weakest such relation \( \downarrow^c_M \).

The “weakest” clause is not necessary for the main result, but we include it anyway to show our construction is canonical.

We start by relativizing the notions of Kim-dividing, Kim-forking, and quasi-dividing (Definition 3.2 of \([6]\)) to an \( M \)-invariant ideal on the definable subsets of \( M \).

**Definition 3.2.** Let \( I \) be an \( M \)-invariant ideal on the definable subsets of \( M \). A formula \( \varphi(x, b) \) \( I \)-Kim-divides over \( M \) if there is a coheir Morley sequence \( \{ b_i \}_{i \in \omega} \) starting with \( b \) so that for some \( k \), the intersection of some (any) \( k \)-element subset of \( \{ \varphi(M, b_i) \}_{i \in \omega} \) belongs...
Lemma 3.3. We say \( \varphi(x,b) \) \( \mathcal{I} \)-Kim-forks over \( M \) if it implies a (finite) disjunction of formulas \( \mathcal{I} \)-Kim-dividing over \( M \). We say \( \varphi(x,b) \) \( \mathcal{I} \)-quasi-divides over \( M \) if there are \( b_1, \ldots, b_n \) with \( b \equiv_M b_i \) so that \( \cap_{i=1}^n \varphi(M, b_i) \in \mathcal{I} \).

We say \( \varphi(x,b) \vdash^\mathcal{I} \psi(x,c) \) if \( \varphi(M,b) \setminus \psi(M,c) \in \mathcal{I} \).

The proof of the following lemma is adapted straightforwardly from the proof of the “broom lemma” of Chernikov and Kaplan (Lemma 3.1 of [6]).

For the convenience of the reader we give a simplified proof of the modified version; note that this version is just a rephrasing in terms of ideals of Lemma 4.19 in [4]:

**Lemma 3.3. ("\( \mathcal{I} \)-broom lemma")** Suppose

\[
\alpha(x,e) \vdash^\mathcal{I} \psi(x,c) \lor \bigvee_{i=1}^N \varphi_i(x,a_i)
\]

with \( \varphi_i(x,a_i) \) \( \mathcal{I} \)-Kim-dividing over \( M \) with respect to \( P(x) \) and \( c \downarrow^n a_1 \ldots a_n \). Then there are some \( e_1, \ldots, e_m \) with \( e_i \equiv_M e \) so that \( \bigwedge_{i=1}^m \alpha(x,e_i) \vdash^\mathcal{I} \psi(x,c) \). In particular, \( \mathcal{I} \)-Kim-forking implies \( \mathcal{I} \)-quasi-dividing over \( M \).

**Proof.** We need the following claim:

**Claim 3.4.** Let \( a^1, \ldots, a^n \) begin a coheir Morley sequence in a global type \( q \) finitely satisfiable over \( M \). Let \( a \equiv_M a^1 \) and let \( b \) be any tuple. Then there are \( b^1, \ldots, b^n \) so that \( b^1a^1, \ldots, b^na^n \) begin a coheir Morley sequence and \( b^1a^1 \equiv_M ba \). (The same is true for Coheir morley sequences themselves, rather than just their initial segments).

**Proof.** Left extension for \( \downarrow^u \) gives a global type \( r \) finitely satisfiable over \( M \) extending both \( q \) and \( \tp(ab/M) \). Now take a coheir Morley sequence in \( r \) and apply an automorphism. The parenthetical is similar. \( \square \)

Now we can prove the lemma by induction on \( N \). Write \( \bigvee_{i=1}^{N-1} \varphi_i(x,a_i) \) as \( \varphi(x,b) \), and let \( a = a_N \). Let \( p \) be a global coheir extension of \( \tp(c/Mba) \). Let \( (a^i)^n_{i=1} \) be such that \( a^i \downarrow^u a^i-1, \ldots, a^1 \) and \( a^i \equiv_M a \) for \( 1 \leq i \leq n \) and \( \bigwedge_{i=1}^n \varphi(N,x,a^i) \vdash^\mathcal{I} \perp \). By the claim, find \( b^1, \ldots, b^n \) so that \( a^i b^i \downarrow^u a^{i-1} b^{i-1} \ldots a^1 b^1 \) and \( a^i b^i \equiv_M ab \) for \( 1 \leq i \leq n \). Then we can assume \( c \models p |\downarrow^u a^1 b^1 \ldots a^n b^n \). From \( c \downarrow^u a^1 b^1 \ldots a^n b^n \), together with \( a^i b^i \downarrow^u a^{i-1} b^{i-1} \ldots a^1 b^1 \) for \( 1 \leq i \leq n \), it is easy to check \( ca^{i+1} b^{i+1} \ldots a^n b^n \downarrow^u M a^i b^i \) for \( 0 \leq i \leq n \), and therefore

\[
\bigvee_{i=0}^{n} c b^{i+1} \ldots b^n \downarrow^u M b^i
\]

for \( 0 \leq i < n \).

Now for \( 1 \leq i \leq n \) we have \( ca^i b^i \equiv_M cab \). Let \( e_i c^i a^i b^i \equiv_M eca b^i \) for \( 1 \leq i \leq n \). Then

\[\text{References:}\]

1. [4]: Chernikov and Kaplan, “The Regularity Lemma for Simple Theories,” Journal of Symbolic Logic, 2012.
2. Alex Kruckman, in a personal communication with the author, discussed an alternative to this proof for showing the properness of the ideal corresponding to the independence result of \( \downarrow^u \), with the broom lemma as a corollary, which works for invariant Morley sequences as well as coheir Morley sequences; it is based on unpublished work of James Hanson on the concept of “fracturing,” a generalization of quasi-forking and quasi-dividing.
\[ \bigwedge \alpha(x, e_i) \vdash^I \psi(x, c) \lor \bigvee_{i=1}^n \varphi(x, b^i) \lor \bigwedge_{i=1}^n \varphi_N(x, a^i) \]

But by choice of the \( a^i \),

\[ \bigwedge \alpha(x, e_i) \vdash^I \psi(x, c) \lor \bigvee_{i=1}^n \varphi(x, b^i) \]

Now for \( 1 \leq i \leq n \), because \( b^i \equiv_M b \), \( \varphi(x, b^i) \) will be of the form \( \bigvee_{j=1}^{n-1} \varphi_j(x, a'_j) \) for \( \varphi_j(x, a'_j) \) \( \mathcal{I} \)-Kim-dividing over \( M \). So, as the first of \( n \) steps, we can apply \( cb^2 \ldots b^n \downarrow^w b^1 \) and the inductive hypothesis on \( N \) to find some conjunction \( \beta(x, \overline{c}) \) of conjugates of \( \bigwedge \alpha(x, e_i) \) so that

\[ \beta(x, \overline{c}) \vdash^I \psi(x, c) \lor \bigvee_{i=2}^n \varphi(x, b^i) \]

Repeating \( n - 1 \) more times, we are done.

We now begin our construction. The following terminology comes from the notion of strong finite character (used in e.g. [7]).

**Definition 3.5.** Let \( \downarrow \) be an invariant relation between sets over a model. We say that \( \downarrow \) satisfies quasi-strong finite character if for \( p, q \) complete types over some model \( M \), \( \{a, b \models p(x) \cup q(y) : a \downarrow_M b\} \) is type-definable.

**Definition 3.6.** Let \( \downarrow \) be an invariant relation between sets over a model satisfying monotonicity, right extension and quasi-strong finite character, and fix a complete type \( P(x) \) over a model \( M \).

1. A set of formulas \( \{\varphi_i(x, b_i)\}_{i \in I} \) is \( h \downarrow \)-inconsistent with respect to \( P(x) \) if there is no \( a \models P(x) \) with \( a \downarrow_M \{b_i\}_{i \in I} \) and \( a \models \varphi_i(a, b_i) \) for all \( i \in I \).
2. A formula \( \varphi(x, b) \) \( h \downarrow \)-Kim-divides with respect to \( P(x) \) if there is a coheir Morley sequence \( \{b_i\}_{i \in \omega} \) starting with \( b \) so that \( \{\varphi(x, b_i)\}_{i \in \omega} \) is \( h \downarrow \)-inconsistent with respect to \( P(x) \).
3. A formula \( h \downarrow \)-Kim-forks with respect to \( P(x) \) if it implies a disjunction of formulas \( h \downarrow \)-Kim-dividing with respect to \( P(x) \).
4. A formula \( \varphi(x, b) \) \( h \downarrow \)-quasi-divides over \( M \) with respect to \( P(x) \) if there are \( b_1, \ldots, b_n \) with \( b_i \equiv_M b \) and \( \{\varphi(x, b_i)\}_{i=1}^n \) \( h \downarrow \)-inconsistent with respect to \( P(x) \).

**Lemma 3.7.** (1) The sets defined by formulas \( \varphi(x, b) \) so that \( \{\varphi(x, b)\} \) is \( h \downarrow \)-inconsistent with respect to \( P(x) \) form an \( M \)-invariant ideal \( \mathcal{I}_{P(x)} \).

(2) A set \( \{\varphi_i(x, b_i)\}_{i \in I} \) is \( h \downarrow \)-inconsistent with respect to \( P(x) \) if and only if some finite subset is (so its conjunction defines a set in the ideal \( \mathcal{I}_{P(x)} \)).

**Proof.** For (1), it suffices to show (a) that if \( \models \forall x (\varphi(x, b) \rightarrow \psi(x, c)) \), and \( \psi(x, c) \) is \( h \downarrow \)-inconsistent with respect to \( P(x) \), then \( \varphi(x, b) \) is \( h \downarrow \)-inconsistent with respect to \( P(x) \),
and (b) that if both $\varphi(x, b)$ and $\psi(x, c)$ are $h \downarrow$-inconsistent with respect to $P(x)$ then so is $\varphi(x, b) \lor \psi(x, c)$. For (a), suppose otherwise; then there is some realization $a$ of $P(x)$ with $\models \varphi(a, b)$ and $a \downarrow_M b$. By right extension, we can assume $a \downarrow_M b c$. But then $\models \psi(a, c)$, and by right monotonicity, $a \downarrow_M c$, contradicting that $\psi(x, c)$ is $h \downarrow$-inconsistent with respect to $P(x)$. For (b), suppose otherwise; then there is some realization $a$ of $P(x)$ with $\models \varphi(a, b) \lor \psi(a, c)$ and $a \downarrow_M b c$; without loss of generality, $\models \varphi(a, b)$, and by right monotonicity, $a \downarrow_M b$, contradicting that $\varphi(x, b)$ is $h \downarrow$-inconsistent with respect to $P(x)$. The proof of (a) also gives us the fact that a set $\{\varphi_i(x, b_i)\}_{i \in I}$ is $h \downarrow$-inconsistent with respect to $P(x)$ if some finite subset is (so its conjunction defines a set in the ideal $\mathcal{I}_{P(x)}$).

To complete (2), we show the “only if” direction. If $\{\varphi_i(x, b_i)\}_{i \in I}$ is $h \downarrow$-inconsistent with respect to $P(x)$ then there is no realization $a$ of $P(x) \cup \{\varphi_i(x, b_i)\}_{i \in I}$ with $a \downarrow_M \{b_i\}_{i \in I}$. But the set of realizations $a$ of $P(x)$ that satisfy $a \downarrow_M \{b_i\}_{i \in I}$ is, by quasi-strong finite character, type-definable. So by compactness, there must be some finite $I_0 \subseteq I$ so there is no realization $a$ of $P(x) \cup \{\varphi_i(x, b_i)\}_{i \in I_0}$ with $a \downarrow_M \{b_i\}_{i \in I_0}$. But if there is a realization $a$ of $P(x) \cup \{\varphi_i(x, b_i)\}_{i \in I_0}$ with $a \downarrow_M \{b_i\}_{i \in I_0}$, then we can even get $a \downarrow_M \{b_i\}_{i \in I}$ by right-extension, so $\{\varphi_i(x, b_i)\}_{i \in I_0}$ will be as desired.

**Corollary 3.7.1.** For all formulas, $h \downarrow$-Kim-forking with respect to $P(x)$ implies $h \downarrow$-quasi-dividing with respect to $P(x)$.

**Proof.** By Lemma 3.7, $h \downarrow$-Kim-dividing with respect to $P(x)$ is just $\mathcal{I}_{P(x)}$-Kim-dividing. Apply Lemma 3.3 to $\mathcal{I}_{P(x)}$-Kim-dividing.

**Lemma 3.8.** If a formula $\varphi(x, b)$ is $h \downarrow$-inconsistent with respect to $P(x)$, then it is $h \downarrow$-inconsistent with respect to any complete type $Q(x, y)$ extending $P(x)$. So the same is true for $h \downarrow$-Kim-dividing and $h \downarrow$-Kim-forking.

**Proof.** Suppose otherwise. Then there is a realization $ac$ of $Q(x, y) \cup \{\varphi(x, b)\}$ with $ac \downarrow_M b$. So by left monotonicity, $a \downarrow_M b$, but $a$ realizes $P(x) \cup \{\varphi(x, b)\}$, a contradiction.

We are now in a position to study derived independence relations:

**Definition 3.9.** Let $\downarrow$ be an invariant relation between sets over a model satisfying monotonicity, right extension and quasi-strong finite character. Then we define $a \downarrow'M b$ to mean that $\text{tp}(a/Mb)$ does not contain any formulas $h \downarrow$-Kim-forking with respect to $\text{tp}(a/M)$.

**Lemma 3.10.** Suppose $\downarrow$ is an invariant relation between sets over a model satisfying monotonicity, right extension, quasi-strong finite character, and full existence. Then so is $\downarrow'$.

**Proof.** Invariance is obviously inherited from $\downarrow$. Quasi-strong finite character is by construction and right extension is also standard from the construction: if $a \downarrow'M b$ but, for some $c \in M$ there is no $a' \equiv_{Mb} a$ with $a \downarrow'M bc$, then $\text{tp}(a/Mb)$ must imply a disjunction of formulas with parameters in $Mbc$ $h \downarrow$-Kim-forking with respect to $P(x)$; some formula
in $\text{tp}(a/Mb)$ must then imply this disjunction, which will then $h \downarrow$-Kim-fork with respect to $P(x)$, contradicting $a \downarrow_M' b$. Right monotonicity is by definition. Left monotonicity is Lemma 3.8. It remains to show full existence; the proof is a straightforward generalization of the proof of Lemma 3.7 of [6]. By right extension, it suffices to show that $b \downarrow_M' M$ for any tuple $b$ (the “existence” property that is implied by full existence). Suppose otherwise; then $\text{tp}(b/M)$ contains a formula $\varphi(x, m)$ for $m \in M$ that $h \downarrow$-Kim-forks over $M$. By Corollary 3.7.1, $\varphi(x, m)$ $h\downarrow$-quasi-divides over $M$. Since $m \in M$, this just means that $\varphi(x, m) \in \mathcal{I}_{\text{tp}(b/M)}$. But since $\varphi(x, m) \in \text{tp}(b/M)$, this contradicts full existence for $\downarrow$. \hfill \Box

The next observation is required to produce a relation with the coheir chain condition:

**Lemma 3.11.** Let $\downarrow$ be as in Lemma 3.10 and suppose $a \downarrow_M' b$. Then for $I = \{b_i\}_{i \in \omega}$ a coheir Morley sequence starting with $b$, there is $I' \equiv_M I$ with $a \downarrow_M' I'$ and each term of $I'$ satisfying $\text{tp}(b/Ma)$. In particular, $\downarrow'$ implies $\downarrow$, so $h \downarrow$-Kim-forking implies $h \downarrow'$-Kim-forking.

**Proof.** Suppose otherwise: then for $q = \text{tp}(a, b/M)$, $\cup_{i \in \omega} q(x, b_i)$ is $h \downarrow$-inconsistent with respect to $\text{tp}(a/M)$, so by part (2) of Lemma 3.7, some finite subset must be $h \downarrow$-inconsistent with respect to $\text{tp}(a/M)$. This gives us a formula in $q(x, b)$ that $h \downarrow$-Kim divides with respect to $\text{tp}(a/M)$, a contradiction. \hfill \Box

Note that $\downarrow^h$ satisfies the assumptions of Lemma 3.10. Now define inductively, $\downarrow^{(0)} = \downarrow^h$, $\downarrow^{(n+1)} = (\downarrow^{(n)})'$. Let $\downarrow^{\text{CK}} = \bigcap_{n=0}^{\infty} \downarrow^{(n)}$. Then because $h \downarrow^{(n)}$-Kim-forking implies $h \downarrow^{(n+1)}$-Kim-forking, and $a \downarrow^{\text{CK}} M b$ means that $\text{tp}(a/Mb)$ does not contain a $h\downarrow^{(n)}$-Kim-forking formula for any $n$, right extension and quasi-strong finite character are standard. Monotonicity and invariance follows from monotonicity and invariance of the $\downarrow^{(n)}$. By right extension for $\downarrow^{\text{CK}}$, full existence for $\downarrow^{\text{CK}}$ would follow from the existence property $b \downarrow^{\text{CK}} M$ for any $b$, but this just follows from full existence for each of the $\downarrow^{(n)}$. Finally, the coheir chain condition follows from Lemma 3.11 together with quasi-strong finite character for the $\downarrow^{(n)}$ and compactness.

It remains to show that $\downarrow^{\text{CK}}$ is the weakest relation implying $\downarrow^h$ and satisfying these properties. Let $\downarrow$ be some other such relation and assume by induction that $\downarrow$ implies $\downarrow^{(n)}$. Assume $a \downarrow_M b$; we show $a \downarrow^{(n+1)} M b$. Suppose otherwise; by right extension for $\downarrow$, we can assume $\text{tp}(a/Mb)$ contains a formula $\varphi(x, b)$ that $h \downarrow^{(n)}$-Kim-divides with respect to $\text{tp}(a/M)$. Let $I = \{b_i\}_{i \in \omega}$ be a coheir Morley sequence starting with $b$ witnessing this. Then by the coheir chain condition for $\downarrow$, there is some $a'$ with $a' \downarrow_M I$, so in particular $a' \downarrow^{(n)} M I$ by induction, and with $a'b_i \equiv_M ab$ for $i \in \omega$, so in particular with $a'$ satisfying $\{\varphi(x, b_i)\}_{i \in \omega}$, a contradiction.

This completes the proof of Proposition 3.11.

**Remark 3.12.** If $M' \succ M$ is a very large (sufficiently saturated) model, then $\downarrow^{\text{CK}}$ as computed in $M'$ restricts to $\downarrow^{\text{CK}}$ as computed in $M$. We can see that $\downarrow^{\text{CK}}$ has this property as it is true for $\downarrow^h$ and is preserved by going from $\downarrow^{(n)}$ to $\downarrow^{(n+1)}$. However,
it is also immediate that invariance, monotonicity, full existence and right extension, and the coheir chain condition are preserved on restriction.

4. Canonical coheirs in \( \text{NSOP}_2 \) theories

The goal of this section is to prove a version of “Kim’s lemma for Kim-dividing” for canonical Morley sequences in \( \text{NSOP}_2 \) theories.

**Lemma 4.1.** Let \( p(x) \) be a type over \( M \). Then it has a global extension \( q(x) \) so that for all tuples \( b \in M \), if \( c \models q|Mb \), then \( b \downarrow_{\text{CK}}^M c \). So in particular, \( q \) is a global coheir of \( p(x) \).

**Proof.** In a very large \( M' \succ M \), full existence and invariance for \( \downarrow_{\text{CK}} \), and an automorphism, gives us a realization \( c' \) of \( p(x) \) with \( M \downarrow_{\text{CK}}^M c' \). Now take \( q(x) \) to be tp(\( c'/M \)), and the lemma follows by monotonicity on the left. \( \square \)

**Definition 4.2.** We call \( q(x) \) as in Lemma 4.1 a canonical coheir, and a coheir Morley sequence in it a canonical Morley sequence.

**Theorem 4.3.** Let \( T \) be \( \text{NSOP}_2 \). Suppose a canonical Morley sequence witnesses Kim-dividing of a formula \( \varphi(x,y) \) over \( M \). Then there is a finite bound (depending only on \( \varphi(x,y) \) and the degree of Kim-dividing witnessed by the canonical Morley sequence) on the length of a coheir sequence \( \{b_i\}_{i=1}^n \) over \( M \) of realizations of \( \text{tp}(b_0|/M) \) so that \( \{\varphi(x,b_i)\}_{i=1}^n \) is consistent. In particular, every coheir Morley sequence starting with \( b \) witnesses Kim-dividing of \( \varphi(x,y) \) over \( M \).

To start, we introduce the notion of a coheir tree in a general theory \( T \).

**Definition 4.4.** Let \( p \) be any type over \( M \). We say that a tree \( \{b_\eta\}_{\eta \in \omega \leq n} \) of realizations of \( p \) is a coheir tree if

1. for each \( \mu \in \omega^{<n} \), \( \{b_\eta\}_{\eta \in \mu \sim (i)} \downarrow_{\text{CK}}^M (the sequence consisting of the subtrees above a fixed node) is a coheir Morley sequence over \( M \).
2. there are global coheir extensions \( q_0, \ldots, q_n \) of \( p \) so that for each \( \mu \in \omega^{n-m} \), \( b_\mu \models q_m|\{b_\eta\}_{\eta \in \mu} \).

The key lemma of this section allows us to construct coheir trees in any theory so that sequences of nodes with common meet are canonical Morley sequences. Abusing the language by nodes, paths, etc. we often refer to the tuples which they index; the term “descending comb” will have a similar meaning in a tree of finite height or a set of subtrees as it does in \( \omega^\omega \).

**Lemma 4.5.** Let \( p(x) \) be any type over \( M \). Let \( q(x) \) be a canonical coheir extension of \( p(x) \). Let \( b_0, \ldots, b_n \) be a coheir sequence over \( M \) of realizations of \( p \). Then there is a coheir tree indexed by \( \omega \leq n \), any path of which, read in the direction of the root, realizes \( \text{tp}(b_0 \ldots b_n|/M) \), and descending comb of which, read in lexicographic order, begin a canonical Morley sequence in \( q(x) \).

**Proof.** We need the following claim:

**Claim 4.6.** If \( a \downarrow_{\text{CK}}^M b \) and \( I \) is a coheir tree in \( \text{tp}(b/M) \), then there is some \( I' \equiv_M I \) with \( a \downarrow_{\text{CK}}^M I' \) (so in particular \( I' \downarrow^u_M a \)) each term of which satisfies \( \text{tp}(b/Ma) \).
ON NSOP$_2$ THEORIES

Proof. Let $I = \{b_\eta\}_{\eta \in \omega \leq n}$; we find $I' = \{b'_\eta\}_{\eta \in \omega \leq n}$ as desired. The proof is by downward induction on $k$: suppose $\{b'_\eta\}_{\eta \in \omega \leq n}$ is already constructed, and we construct $\{b'_\eta\}_{\eta \in \omega \leq n(k+1)}$. First, $\{b'_\eta\}_{\eta \in \omega \leq n-(k+1)}$ comes directly from the chain condition. Second, by left extension for $\bigcup^n$, find some copy $J$ of $\{b'_\eta\}_{\eta \in \omega \leq n-(k+1)}$ over $M$ with $J \downarrow^u_M \{b'_\eta\}_{\eta \in \omega \leq n(k+1)}$ and some arbitrary term of $J$ satisfying the conjugate to $M\{b'_\eta\}_{\eta \in \omega \leq n-(k+1)}$ of $\text{tp}(b_{n(k+1)})/M\{b'_\eta\}_{\eta \in \omega \leq n-(k+1)}$ (that is, $g_{k+1}(x)|_{M\{b'_\eta\}_{\eta \in \omega \leq n-(k+1)}}$ from Definition [4.3]). Then use the chain condition to find some $J' \equiv_{M\{b'_\eta\}_{\eta \in \omega \leq n-(k+1)}} J$ with $J' \equiv_{M\{b'_\eta\}_{\eta \in \omega \leq n-(k+1)}}$ and $\downarrow^\text{CK}_M \{b'_\eta\}_{\eta \in \omega \leq n-(k+1)}$. Finally, using monotonicity on the right, discard all the terms of $J'$ other than the one corresponding to the chosen term of $J$, to obtain $b'_\eta$.

Now by induction, it suffices to show this for a coheir sequence $b_0, \ldots, b_{n+1}$ assuming $I_n = \{b_\eta\}_{\eta \in \omega \leq n}$ is already constructed for $b_0, \ldots, b_n$. First, we find a long coheir sequence $\{I'_n\}_{n=0}^\omega$ of realizations of $\text{tp}(I_n/M)$ so that each node of $I'_n$ satisfies $\phi(x)|_{M\{I'_n\}_{n<\gamma}}$; then having taken it long enough, we can find a coheir Morley sequence $\{I_n\}_{n=0}^\omega$ with the same property, preserving the condition on descending combs. (Any descending comb inside of these copies will either lie inside of one copy of $I_n$, so will of course begin a descending Morley sequence inside of $\phi(x)$ by the induction hypothesis, or will consist of a descending comb inside one copy $I_n$ followed by an additional node of a later copy $I_n$ for $i < j$, which will indeed continue the Morley sequence in $\phi(x)$ begun by the previous nodes.) Suppose $\{I'_n\}_{n<\gamma}$ already constructed; taking $a = \{I'_n\}_{n<\gamma}$ in the above claim and $b \models \phi(x)|_{M\{I'_n\}_{n<\gamma}}$, we can choose $I_n$ to be the $I'$ given by the claim.

Now let $q_{n+1}$ be a global extension, finitely satisfiable in $M$, of $\text{tp}(b_{n+1}/M0 \ldots b_n)$. Then we take $b \models q_{n+1}(x)|_{M\{I_n\}_{n<\gamma}}$ as the new root, guaranteeing the condition on paths. Now reindex accordingly.

We can now prove Theorem [4.3]. Let $\phi(x)$ be a canonical coheir extension of $\text{tp}(b/M)$ and $k$ the degree of Kim-dividing for $\phi(x, b)$ witnessed by a canonical Morley sequence in $\phi(x)$. Let $\{b_i\}_{i=0}^n$ be a coheir sequence over $M$ of realizations of $\text{tp}(b/M)$ so that $\{\phi(x, b_i)\}_{i=0}^n$ is consistent. Then the coheir tree given by the previous lemma gives the first $n + 1$ levels of an instance of $k$-DCTP$_1$: the $k$-dividing witnessed by canonical Morley sequences in $\phi(x)$ gives the inconsistency condition for descending combs of size $k$, and the consistency of $\{\phi(x, b_i)\}_{i=0}^n$ gives the consistency of the paths. So if $n$ is without bound, we must have $k$-DCTP$_1$ for $\phi(x, y)$ by compactness, and thus SOP$_2$ by lemma [2.8]. This concludes the proof of [4.3].

We have some applications of this proof to a notion related to the NATP theories introduced by Ahn and Kim in [2], and studied in greater depth by Ahn, Kim and Lee in [3], assuming the NATP analogue of lemma [2.8]. The result for NATP theories would be interesting because while NSOP$_1$ theories are NATP [2], as Ahn, Kim and Lee have shown in [3], there are examples of NATP SOP$_1$ theories. The following is the original definition from [2]:
Definition 4.7. The theory $T$ has NATP (the negation of the antichain tree property) if there does not exist a formula $\varphi(x,y)$ and tuples $\{b_\eta\}_{\eta \in \omega}$ so that $\{\varphi(x,b_{\sigma[\eta]})\}_{\eta \in \omega}$ is 2-inconsistent for any $\sigma \in 2^\omega$, but for pairwise incomparable $\eta_1, \ldots, \eta_l \in 2^\omega$, $\{\varphi(x,b_\eta)\}_{i=1}^l$ is consistent.

In [3], Ahn, Kim and Lee define a theory to have $k$-ATP if the above fails replacing $2$-inconsistency with $k$-inconsistency, and show that for any $k \geq 2$, a theory fails to be NATP (that is, has 2-ATP) if and only if it has $k$-ATP. That is, they show the analogue for NATP theories of results of Kim and Kim in [18] on NSOP 2 theories, but of not those claimed by Chernikov and Ramsey in [4], nor of the above Lemma 2.8. One might ask whether, for any $k$, the following definition is equivalent to the failure of NATP:

Definition 4.8. The theory $T$ has $k$-DCTP 2 if there exists a formula $\varphi(x,y)$ and tuples $\{b_\eta\}_{\eta \in \omega}$ so that $\{\varphi(x,b_{\sigma[\eta]})\}_{\eta \in \omega}$ is $k$-inconsistent for any $\sigma \in 2^\omega$, but for any descending comb $\eta_1, \ldots, \eta_l \in 2^\omega$, $\{\varphi(x,b_\eta)\}_{i=1}^l$ is consistent.

If so, then the following applies to NATP theories:

Theorem 4.9. Let $T$ be a theory so that, for all $k \geq 2$, $T$ does not have $k$-DCTP 2. Let $M$ be any model and $b$ any tuple. Then there is a global type extending $tp(b/M)$, finitely satisfiable in $M$, so that for any formula $\varphi(x,y)$ with parameters in $M$, if coheir Morley sequences in this type do not witness Kim-dividing of $\varphi(x,b)$, no coheir Morley sequence over $M$ starting with $b$ witnesses Kim-dividing of $\varphi(x,b)$ over $M$.

This follows from the same construction. The following corollary is standard; see Corollary 3.16 of [6] for a similar argument:

Corollary 4.9.1. If, for all $k \geq 2$, $T$ does not have $k$-DCTP 2, then Kim-forking (with respect to coheir Morley sequences) coincides with Kim-dividing (with respect to coheir Morley sequences).

5. Conant-independence in NSOP 2 theories

We introduce a notion of independence which will generalize, in the proof of the main result of this paper, the role played by $\downarrow^a$ in the free amalgamation theories introduced in [8]. The notation $\downarrow^K$ comes from the related notion of Kim-independence from [16], $\downarrow^K$; a similar notion involving dividing with respect to all (invariant) Morley sequences is suggested in tentative remarks of Kim in [17].

Definition 5.1. Let $M$ be a model and $\varphi(x,b)$ a formula. We say $\varphi(x,b)$ Conant-divides over $M$ if for every coheir Morley sequence $\{b_i\}_{i \in \omega}$ over $M$ starting with $b$, $\{\varphi(x,b_i)\}_{i \in \omega}$ is inconsistent. We say $\varphi(x,b)$ Conant-forks over $M$ if and only if it implies a disjunction of formulas Conant-dividing over $M$. We say $a$ is Conant-independent from $b$ over $M$, written $a \downarrow^K b$, if $tp(a/Mb)$ does not contain any formulas Conant-forking over $M$.

Note that this definition differs from the standard definition of Conant-independence given in [26], in that it uses coheir Morley sequences rather than invariant Morley sequences. In [21] Alex Kruckman and the author show how to carry out this proof with the standard Conant-independence. We may also dualize Theorem 3.10 of [19].
Proposition 5.2. In any theory $T$, Conant-forking coincides with Conant-dividing for formulas, and $\downarrow^K_\ast$ has right extension.

Proof. We see first of all that Conant-dividing is preserved under adding and removing unused parameters: it suffices to show that if $\models \forall x \varphi(x, a) \leftrightarrow \varphi'(x, ab)$ then $\varphi(x, a)$ Conant-divides over $M$ if and only if $\varphi'(x, ab)$ Conant-divides over $M$. Let $\{a_i b_i\}_{i \in \omega}$ be a coheir Morley sequence starting with $ab$ witnessing the failure of Conant-dividing of the latter; then $\{a_i\}_{i \in \omega}$ witnesses the failure of Conant-dividing of the former. Conversely, let $\{a_i b_i\}_{i \in \omega}$ be a coheir Morley sequence starting with $a$ witnessing the failure of Conant-dividing of $\varphi(x, a)$; then by Claim 3.3 and an automorphism there are $\{b_i\}_{i \in \omega}$ so that $\{a_i b_i\}_{i \in \omega}$ is a coheir Morley sequence starting with $ab$, and this will witness the failure of Conant-dividing of $\varphi'(x, ab)$. The result is now standard, following, say, the proof in [16] of the analogous fact for Kim-dividing under Kim’s lemma. Suppose $\varphi(x, b)$ Conant-forks over $M$ but does not Conant-divide over $M$; by the above we can assume it implies a disjunction of the form $\bigvee_{i=1}^n \varphi_i(x, b)$ where $\varphi_i(x, b)$ Conant-divides over $M$. Let $\{b_i\}_{i \in \omega}$ be a coheir Morley sequence starting with $b$ witnessing the failure of Conant-dividing, so there is some $\alpha$ realizing $\{\varphi(x, b_i)\}_{i \in \omega}$. Then by the pigeonhole principle, there is some $1 \leq k \leq n$ so that $\alpha$ realizes infinitely many of the $\varphi_k(x, b_i)$. By an automorphism this contradicts Conant-dividing of $\varphi_k(x, b)$.

Right extension is standard and exactly as in Lemma 3.10: if $a \downarrow^K_\ast b$ but there is no $a' \equiv_{Mb} a$ with $a' \downarrow^K_M bc$, then $\text{tp}(a/Mb)$ must imply a disjunction of formulas with parameters in $Mbc$ Conant-forking over $M$; some formula in $\text{tp}(a/Mb)$ must then imply this disjunction, which will then Conant-fork over $M$, contradicting $a \downarrow^K_\ast b$. \hfill \Box

The following is immediate from Theorem 4.3:

Corollary 5.2.1. Let $T$ be $\text{NSOP}_2$. Then a formula Conant-divides (so Conant-forks) over $M$ if and only if it Kim-divides with respect to some (any) canonical Morley sequence.

We develop the theory of Conant-independence in $\text{NSOP}_2$ theories in analogy with the theory of Kim-independence in $\text{NSOP}_1$ theories.

Proposition 5.3. ( Canonical Chain Condition): Let $T$ be $\text{NSOP}_2$ and suppose $a \downarrow^K_\ast b$. Then for any canonical Morley sequence $I$ starting with $b$, we can find some $I' \equiv_{Mb} I$ indiscernible over $a$; any such $I'$ will satisfy $a \downarrow^K_\ast I'$.

Proof. This is similar to the proof of, say, the analogous fact about Kim-independence in $\text{NSOP}_1$ theories (Proposition 3.21 of [16]). The existence of such an $I'$ follows from the previous corollary by Ramsey and compactness. To get $a \downarrow^K_\ast I'$, let $I' = \{b_i\}_{i \in \omega}$; it suffices to show $a \downarrow^K_\ast b_0 \ldots b_{n-1}$ for any $n$. But $\{b_0 b_1 \ldots b_{n+(n-1)}\}_{i \in \omega}$ is a coheir Morley sequence over $M$ starting with $b_0 \ldots b_{n-1}$, each term of which satisfies $\{b_0 \ldots b_{n-1}/Ma\}$, so $a \downarrow^K_\ast b_0 \ldots b_{n-1}$ follows. \hfill \Box

Theorem 5.4. Let $T$ be $\text{NSOP}_2$. Then Conant-independence is symmetric.

Proof. Suppose otherwise, so for some $a, b \in M$, $a \downarrow^K_\ast b$ but $b$ is Conant-dependent on $a$ over $M$. We use $a \downarrow^K_\ast b$ to build trees as in the proof of symmetry of Kim-independence
for NSOP$_1$ theories (the construction is Lemma 5.11 of [16].) Specifically, what we want is, for any $n$, a tree $(I_n, J_n) = ((a_\eta)_{\eta \in \omega < n}, (b_\sigma)_{\sigma \in \omega^n})$, infinitely branching at the first $n + 1$ levels and then with each $a_\sigma$ for $\sigma \in \omega^n$ at level $n + 1$ followed by a single additional leaf $b_\sigma$ at level $n + 2$, satisfying the following properties:

1. For $\eta \leq \sigma$, $a_\eta b_\sigma \equiv_M ab$
2. For $\eta \in \omega < n$, the subtrees above $\eta$ form a canonical coheir sequence indiscernible over $a_\eta$, so by Proposition 5.3 $a_\eta$ is Conant-independent over $M$ from those branches taken together.

Suppose $(I_n, J_n)$ already constructed; we construct $(I_{n+1}, J_{n+1})$. We see that the root $a_0$ of $(I_n, J_n)$ is Conant-independent from the rest of the tree, $(I_n, J_n)^*$: for $n = 0$ this is just the assumption $a \downarrow M^* b$, where we allow $a_0 b_0 = ab$, while for $n > 0$ this is (2). So by extension we find $a_0' \equiv_M (I_n, J_n)^* a_0$ (so guaranteeing (1)), to be the root of $(I_{n+1}, J_{n+1})$, with $a \downarrow M^* I_n J_n$. Then by Proposition 5.3 find some canonical Morley sequence $\{(I_n, J_n)^i\}_{i \in \omega}$ starting with $(I_n, J_n)$ indiscernible over $Ma'_n$, guaranteeing (2), and reindex accordingly.

Now let $\varphi(x, a) \in tp(b/Ma)$ (so $\varphi(x, y)$ is assumed to have parameters in $M$) witness the Conant-dependence of $b$ on $a$ over $M$ and let $k$ be the (strict) bound supplied by Theorem 4.3. We show $I_n$ gives the first $n + 1$ levels of an instance of $k$-DCTP$_1$ for $\varphi(x, y)$, giving a contradiction to NSOP$_2$ by compactness and lemma 2.8. Consistency of the paths comes from (1). As for the inconsistency of a descending comb of size $k$, it follows from (2) (and the same reasoning as in the proof of Lemma 4.5) that a descending comb forms a coheir sequence, so the inconsistency follows by choice of $k$.

Note that by constructing a tree of size $\kappa$ and using an Erdős-Rado version of fact 2.6 (see Lemma 5.10 of [16] for a result of this kind for similar kind of indiscernible tree, itself based on Theorem 1.13 of [15]), we could have assumed the tree we constructed in the above proof to be strongly indiscernible. It follows that we could have only used that if a canonical Morley sequence witnesses Kim-dividing of a formula, then so does any coheir Morley sequence; the statement of Theorem 4.3 is somewhat stronger. (In fact, by using a local version of the chain condition—i.e., if $a \downarrow_M^* b$ and $\models \varphi(a, b)$, then there is some coheir Morley sequence $I = \{b_i\}_{i \in \omega}$ so that $b_i \equiv_M b$, $\models \varphi(a, b_i)$ for $i \in \omega$, and $a \downarrow_M^* I$—we could have avoided Theorem 4.3 altogether up to this point, but we have not yet found a suitable replacement for the below “weak independence theorem” that does not require it. We leave the details to the reader.)

We next aim to prove a version of the “weak independence theorem.” To formulate this, we need the following strengthening of Lemma 4.1.

**Lemma 5.5.** Let $p(x)$ be a type over $M$. Then there is some global extension $q(x)$ of $p(x)$ so that, for all tuples $b \in M$, $c \in M$ if $c \models q(x)|M_b$, then for any $a \in M$ there is $a' \equiv_M a$ with $a' \in M$ so that $tp(a'c/Mb)$ extends to a canonical coheir of $tp(a'c/M) = tp(ac/M)$. So in particular, $q(x)$ is a canonical coheir of $p(x)$.

**Proof.** Working again in a very large $M' \gg M$, find $M_1 \equiv_M M$ with $M \downarrow M^* M_1$ using full existence for $\downarrow M^*$. Find a realization $c''$ of $p(x)$ in $M_1$ and let $q(x)$ be its type over $M$. Now suppose $b \in M$ and $c \in M$ with $c \models q(x)|Mb$, and let $a \in M$. Then there is some $a'' \in M_1$ with $a''c'' \equiv_M ac$. Because $c'' \equiv_M c$, there is some $a' \in M$ with $a''c'' \equiv_M a'c$. Together
with \(a''c'' \equiv_M ac\), it follows that \(a' \equiv_{Mc} a\). And \(tp(a'/c/Mb)\) extends to \(tp(a''c''/M)\), which it remains to show is canonical. But by right monotonicity, \(M \downarrow^{CK}_M a''c''\), so the result follows by left monotonicity (see also the proof of Lemma [4.1]).

**Definition 5.6.** We call \(q(x)\) as in Lemma [5.2] a strong canonical coheir, and a coheir Morley sequence in it a strong canonical Morley sequence.

The proof of the following is as in Proposition 6.10 of [16]:

**Proposition 5.7.** (Weak Independence Theorem) Assume \(T\) is \(\text{NSOP}_2\). Let \(a_1 \downarrow^M K^* b_1\), \(a_2 \downarrow^M K^* b_2\), \(a_1 \equiv_M a_2\), and \(tp(b_2/Mb_1)\) extends to a strong canonical coheir \(q(x)\) of \(tp(b_2/M)\). Then there exists a realization \(a\) of \(tp(a_1/Mb_1) \cup tp(a_2/Mb_2)\) with \(a \downarrow^M K^* b_1 b_2\).

**Proof.** We start with the following claim, proven exactly as in [16] but with canonical rather than invariant Morley sequences:

**Claim 5.8.** There exists some \(b_2'\) with \(a_1 b_2' \equiv_M a_2 b_2\) and \(a_1 \downarrow^M K^* b_1 b_2'\).

**Proof.** It is enough by symmetry of \(\downarrow^M K^*\) to find \(b_2'\) with \(a_1 b_2' \equiv_M a_2 b_2\) and \(b_1 b_2' \downarrow^M K^* a_1\). If \(p(x, a_2) = tp(b_2/Ma_2)\) (leaving implied, throughout the proof of this claim, any parameters in \(M\) in types and formulas), then by \(a_2 \downarrow^M K^* b_2\) and symmetry we have \(b_2 \downarrow^M K^* a_2\), so because \(a_1 \equiv_M a_2\) we know that \(p(x, a_1)\) contains no formulas Conant-forking over \(M\). It suffices to show consistency of

\[
p(x, a_1) \cup \{-\varphi(x, b_1, a_1) : \varphi(x, y, a_1) \text{ Conant-forks over } M\}
\]

Otherwise, by compactness and equivalence of Conant-forking with Conant-dividing, we must have \(p(x, a_1) \vdash \varphi(x, b_1, a_1)\) for some \(\varphi(x, y, z)\) with \(\varphi(x, y, a_1)\) Conant-dividing over \(M\). By symmetry, \(b_1 \downarrow^M K^* a_1\). So Proposition [5.3] yields a canonical Morley sequence \(\{a_i^1\}_{i \in \omega}\) starting with \(a_1\) and indiscernible over \(Mb_1\). So

\[
\bigcup_{i=0}^{\omega} p(x, a_i^1) \vdash \{\varphi(x, b_1, a_i^1)\}_{i \in \omega}
\]

But because \(p(x, a_1)\) contains no formulas Conant-dividing over \(M\) and \(\{a_i^1\}_{i \in \omega}\) is a canonical Morley sequence, \(\bigcup_{i=0}^{\omega} p(x, a_i^1)\) is consistent, so \(\{\varphi(x, b_1, a_i^1)\}_{i \in \omega}\) and therefore \(\{\varphi(x, y, a_i^1)\}_{i \in \omega}\) is consistent. But this contradicts the fact that \(\varphi(x, y, a_1)\) Conant-divides over \(M\). 

We now complete the proof of the proposition. Let \(p_2(x, b_2) = tp(a_2/Mb_2)\) (with parameters in \(M\) left implied); we have to show that \(tp(a_1/Mb_1) \cup p_2(x, b_2)\) has a realization \(a\) with \(a \downarrow^M K^* b_1 b_2\). So for \(b_2'' \equiv_{Mb_1} b_2\) with \(b_2'' \models q(x)|_{Mb_1}\), it suffices to show that \(tp(a_1/Mb_1) \cup p_2(x, b_2'')\) has a realization \(a\) with \(a \downarrow^M K^* b_1 b_2''\). Using \(b_2'' \equiv_{M} b_2 \equiv_{M} b_2'\), we find \(b_1'\) with \(b_1' b_2'' \equiv_{M} b_1 b_2''\); using Lemma [5.5] we can assume \(tp(b_1' b_2'' /Mb_1 b_2)\) extends to a canonical coheir of its restriction to \(M\). So \(b_1' b_2''\) begins a canonical Morley sequence \(I\) over \(M\), and by Proposition [5.3] and an automorphism, there is some \(a \equiv_{Mb_1 b_2' a_1}\) with \(a \downarrow^M K^* I\) and therefore \(a \downarrow^M K^* b_1 b_2''\), and with \(I\) indiscernible over \(Ma\). By \(a \equiv_{Mb_1} a_1\) we
have that \( a \) realizes \( tp(a_1/Mb_1) \), and by \( ab'_2 \equiv_M ab'_2 \equiv_M a_1b'_2 \equiv_M a_2b_2 \) we have that \( a \) realizes \( p_2(x, b'_2) \).

\[ \square \]

6. NSOP\(_2\) AND NSOP\(_1\) THEORIES

We are now ready to prove that if \( T \) is NSOP\(_2\), it is NSOP\(_1\). The proof follows Conant’s proof (Theorem 7.17 of [3]) that certain free amalgamation theories are either simple or SOP\(_3\). As anticipated in Section 5, \( \downarrow^{K^*} \) will play the role of \( \downarrow^a \), while (strong) canonical Morley sequences will play the role of Morley sequences in the free amalgamation relation. This makes sense, as Lemma 7.6 of [3] shows that \( \downarrow^a \) is just Kim-independence with respect to Morley sequences in the free amalgamation relation, while Conant-independence in a NSOP\(_2\) theory is Kim-independence with respect to canonical Morley sequences. Similarly to how Conant uses free amalgamation and \( \downarrow^a \) to show that a (modular) free amalgamation theory is either simple or SOP\(_3\), we will show by strong canonical types and \( \downarrow^{K^*} \) that if \( T \) is NSOP\(_2\), then

\( T \) is either NSOP\(_1\) or SOP\(_3\)

and therefore must be NSOP\(_1\). (In [26], we generalize Conant’s work by studying abstract independence relations in potentially strictly NSOP\(_1\) or SOP\(_3\) theories, finding a more general set of axioms for these relations than Conant’s free amalgamation axioms under which the NSOP\(_1\)-SOP\(_3\) dichotomy holds and showing relationships with Conant-independence for invariant rather than coheir Morley sequences—note that in Conant’s free amalgamation theories, this is just \( \downarrow^a \).)

We begin our proof.

Assume \( T \) is NSOP\(_2\) and suppose \( T \) is SOP\(_1\). Obviously Kim-dividing independence, \( \downarrow^{Kd}_M \), implies \( \downarrow^{K^*}_M \); the reverse implication would imply that \( \downarrow^{Kd}_M \) is symmetric, contradicting SOP\(_1\) by Fact [22]. So there are \( a \downarrow^{K^*}_M b \) with a Kim-dividing dependent on \( b \) over \( M \); let \( r(x, y) = tp(a, b/M) \), and let \( \{ b_i \}_{i \in \mathbb{N}} \) be a coheir Morley sequence over \( M \) starting with \( b \) such that \( \{ r(x, b_i) \}_{i \in \mathbb{N}} \) is \( k \)-inconsistent for some \( k \). The following corresponds to Claim 1 of the proof of Theorem 7.17 in [3], but requires a different argument; see also [23] and footnote 1 of [27], for another argument involving the proof of Proposition 3.14 of [16]:

Claim 6.1. We can assume \( k = 2 \). More precisely, there are \( \tilde{a}, \tilde{b} \in M \) with \( \tilde{a} \downarrow^{K^*}_M \tilde{b} \) and some coheir Morley sequence \( \{ \tilde{b}_i \}_{i \in \mathbb{N}} \) over \( M \) starting with \( \tilde{b} \) such that, for \( \tilde{r}(\tilde{x}, \tilde{y}) := tp(\tilde{a}, \tilde{b}/M) \), \( \{ \tilde{r}(\tilde{x}, \tilde{b}_i) \}_{i \in \mathbb{N}} \) is 2-inconsistent.

Proof. In particular there is no realization \( a' \) of \( \{ r(x, b_i) \}_{i < k} \) with \( a' \downarrow^{K^*}_M b_0 \ldots b_{k-1} \). Let \( k^* \) be the maximal value of \( k \) without this property, and \( \tilde{b} = b_0 \ldots b_{k^*-1} \). Then \( \{ \tilde{b}_i \}_{i \in \omega} = \{ b_{ik^*} \ldots b_{ik^*+k^*-1} \}_{i \in \omega} \) is a coheir Morley sequence starting with \( \tilde{b} \). Let \( a' \downarrow^{K^*}_M \tilde{b} \) realize \( \{ r(x, b_i) \}_{i < k^*} \), and let \( r'(x, y) = tp(a', \tilde{b}/M) \). Then by maximality and symmetry, there is no realization \( a'' \) of \( r'(x, \tilde{b}_0) \cup r'(x, \tilde{b}_1) \) with \( \tilde{b}_0 \tilde{b}_1 \downarrow^{K^*}_M a'' \). So there is no coheir Morley
sequence \( \{a'_i\}_{i \in \mathbb{N}} \) starting with \( a' \), every term of which realizes \( r'(x, \tilde{b}_0) \cup r'(x, \tilde{b}_1) \). But by \( a'_i \upharpoonright_{M \tilde{b}}^{K^*} \), symmetry and Proposition 5.3, there is some \( M\tilde{b} \)-indiscernible canonical Morley sequence \( I \) starting with \( a \) so that \( I \upharpoonright_{M \tilde{b}}^{K^*} \). So let \( \tilde{a} \) be \( I \) and \( \tilde{b} \) with \( \tilde{b} \). Since \( \tilde{r}(\tilde{x}, \tilde{b}) = tp(I/M\tilde{b}) \) contains \( \cup_{i=1}^{n} r'(x_i, \tilde{b}_i) \), \( \tilde{a} \) and \( \tilde{b} \) are as desired. \( \square \)

Now replace \( a \) with \( \tilde{a} \) and \( b \) with \( \tilde{b} \), as in Claim 6.1 let \( \rho(x, y) \in r(x, y) = tp(a, b/M) \) be such that \( \{r(x, b_i)\}_{i \in \omega} \) is 2-inconsistent, by compactness. We have \( b_1 \upharpoonright_{M \tilde{b}_0}^{K^*} \) in analogy with Claim 2 of the proof of Theorem 7.17 of \( [S] \), because \( b_1 \upharpoonright_{M \tilde{b}_0}^{u} \) and clearly \( \upharpoonright_{M \tilde{b}}^{u} \) implies \( \upharpoonright_{M \tilde{b}}^{K^*} \).

Fix a strong canonical coheir extension \( q(x) \) of \( p(x) = tp(b/M) \). We wish to construct, by induction, a configuration \( \{b^n_1b^n_2\}_{i \in \omega} \) with the following properties:

1. For \( J_n \) the sequence beginning with \( b^n_1 \) for \( i < n \) and then continuing with \( b^n_i \) for \( i \geq n \), \( J_n \) is a strong canonical Morley sequence in \( q(x) \).
2. For \( i \leq j \), \( b^n_i b^n_j \equiv_{M} b^n_0 b^n_1 \)
3. \( b^n_0 \ldots b^n_i \upharpoonright_{M \tilde{b}}^{K^*} b^n_0 \ldots b^n_n \) for any \( n \in \omega \).

Then by \( a \upharpoonright_{M \tilde{b}}^{K^*} \) (1) gives consistent sequences of instances of \( r(x, y) \), while (2) gives inconsistent pairs by claim 6.1 so we can get an instance of SOP from this configuration exactly as in the argument at the end of the proof of Theorem 7.17 in \( [S] \), which we will reproduce for the convenience of the reader.

We make repeated use of symmetry for \( \upharpoonright_{M \tilde{b}}^{K^*} \) throughout. Suppose \( \{b^n_1b^n_2\}_{i \leq n} \) already constructed. We start by adding \( b^n_{i+1} \) and then add \( b^n_{i+2} \). If we take \( b^n_{i+1} \models q(x)|_{M\tilde{b}_0 \ldots \tilde{b}_i \tilde{b}_0} \) then (1) and (2) are preserved up to this point, and (3) is preserved by the following claim (which also holds of Kim-independence in NSOP\(_1\) theories):

**Claim 6.2.** If \( a \upharpoonright_{M \tilde{b}}^{K^*} \) and \( tp(c/Mab) \) extends to an \( M \)-invariant type \( q(x) \), then \( ac \upharpoonright_{M \tilde{b}}^{K^*} \).

**Proof.** By Proposition 5.3 let \( I = \{b_i\}_{i \in \omega} \) be an \( Ma \)-indiscernible canonical Morley sequence over \( M \) starting with \( b \). Choose \( c^* \models q|_{M\tilde{a}} \), so for \( i < \omega \), \( b_i a \equiv_{M c^*} b_0 a = ba \). Since \( I = \{b_i\}_{i \in \omega} \) form a coheir Morley sequence with \( b_i \equiv_{Ma^*} b \) for \( i < \omega \), \( ac^* \upharpoonright_{M \tilde{b}}^{K^*} \) by 5.2 so \( ac \upharpoonright_{M \tilde{b}}^{K^*} \) as \( c^* \models tp(c/ab) \). \( \square \)

Now by \( b_1 \upharpoonright_{M \tilde{b}}^{K^*} \) and the fact that \( J_0 \) is still a (strong) canonical Morley sequence up to this point, we can find a realization \( b_* \upharpoonright_{M \tilde{b}}^{K^*} b^1_0 \ldots b^1_{n+1} \) of \( \{t(b^i_1, y)\}_{i=1}^{n+1} \) for \( t(x, y) = tp(b_0b_1/M) \) by Proposition 5.3 and an automorphism. Take \( b^* \models q(x)|_{M\tilde{b}_0 \ldots \tilde{b}_i \tilde{b}_0} \), so \( b^* \equiv_{M} b_* \); then this together with (3) allows us to apply Proposition 5.7 to the conjugate \( q_1 \) of \( tp(b^1_0 \ldots b^1_{n+1}/Mb_*) \) under an automorphism taking \( b_* \) to \( b^* \), and \( q_2 = tp(b^1_0 \ldots b^1_{n+1}/M\tilde{b}_0 \ldots \tilde{b}_n) \). This and an automorphism (over \( b^0_0 \ldots b^0_n \), taking the Conant-independent joint realization of \( q_1 \) and \( q_2 \) to \( b^0_0 \ldots b^1_{n+1} \)) gives us our desired \( b^0_{n+1} \) as the image of \( b^* \) under this automorphism.

Now having constructed the configuration, let \( a_n \) realize the consistent set of instances of \( r(x, y) \) coming from \( J_n \), and let \( d_i = (b^i_1, b^i_2) \), \( z = (y^1, y^2) \), \( \phi(x, y) = \rho(x, y_1) \), \( \psi(x, z) = \rho(x, y_2) \). As in the proof of Theorem 7.17 of \( [S] \), these satisfy the hypotheses of the following fact:
Fact 6.3. (Corrected version of proposition 7.2, [8])
Suppose there are sequences $\{a_i\}_{i<\omega}, \{d_i\}_{i<\omega}$, and $\phi(x,y), \psi(x,y)$ so that
(i) $\models \phi(a_i, d_j)$ for all $i < j$ and $\psi(a_i, d_j)$ for all $i \geq j$
(ii) for all $i < j$, $\varphi(x, b_i) \cup \psi(x, b_j)$ is inconsistent
Then $T$ is SOP$_3$.

So $T$ is SOP$_3$.
This concludes the proof of the main result of this paper.

Acknowledgements The author would like to thank Mark Kamsma and Itay Kaplan, as well as seminar participants at Hebrew University of Jerusalem, Imperial College London, and Université Claude Bernard Lyon 1 for many helpful edits and comments.

References
[1] Hans Adler. A geometric introduction to forking and thorn-forking. Journal of Mathematical Logic, 9, 2009.
[2] JinHoo Ahn and Joonhee Kim. SOP$_1$, SOP$_2$, and antichain tree property, preprint. Available at https://arxiv.org/abs/2003.10030. 2020.
[3] JinHoo Ahn, Joonhee Kim, and Junguk Lee. On the antichain tree property, preprint. Available at https://arxiv.org/abs/2106.03779. 2021.
[4] Artem Chernikov. Theories without the tree property of the second kind. Annals of Pure and Applied Logic, 165(2):695–723, 2014.
[5] Artem Chernikov. NTP$_1$ theories, presentation slides. Paris. Available at https://www.math.ucla.edu/chernikov/slides/ParisSeminar2015.pdf. June 2015.
[6] Artem Chernikov and Itay Kaplan. Forking and dividing in NTP$_2$ theories. The Journal of Symbolic Logic, 77(1):1–20, 2012.
[7] Artem Chernikov and Nicholas Ramsey. On model-theoretic tree properties. Journal of Mathematical Logic, 16(2):1650009, 2016.
[8] Gabriel Conant. An axiomatic approach to free amalgamation. The Journal of Symbolic Logic, 82(2):648–671, 2017.
[9] Gabriel Conant, Personal communication. 2023.
[10] GABRIEL CONANT and ALEX KRUCKMAN. Independence in generic incidence structures. Journal of Symbolic Logic, 84(2):750–780, 2019.
[11] Christian D’Elbée. Forking, imaginaries, and other features of ACFG. The Journal of Symbolic Logic, 86(2):669–700, 2021.
[12] Mirna Džamonja and Saharon Shelah. On $\prec^*$-maximality. Annals of Pure and Applied Logic, 125(1-3):119–158, 2004.
[13] Christian d’Elbée. Generic expansions by a reduct. Journal of Mathematical Logic, 21(03):2150016, 2021.
[14] David E. Evans and Mark Wing Ho Wong. Some remarks on generic structures. Journal of Symbolic Logic, 74(4):1143 – 1154, 2009.
[15] Rami P. Grossberg, José Iovino, and Olivier Lessmann. A primer of simple theories. Archive for Mathematical Logic, 41:541–580, 2002.
[16] Itay Kaplan and Nicholas Ramsey. On kim-independence. Journal of the European Mathematical Society, 22, 02 2017.

Gabriel Conant, in a personal communication with the author ([9]), noted this correction to Proposition 7.2 of [8], and plans to publicize this in a future corrigendum. See also Observation 6.15 of [25] for an earlier version of this fact, which can also be used here.
ON NSOP\(_2\) THEORIES

[17] Byunghan Kim. NTP\(_1\) theories, presentation slides. BIRS Workshop, Seoul. Available at https://www.birs.ca/workshops/2009/09w5113/files/Kim.pdf. February 2009.

[18] Byunghan Kim and Hyeung-Joon Kim. Notions around tree property 1. *Annals of Pure and Applied Logic*, 162(9):698–709, 2011.

[19] Joonhee Kim and Hyoyoon Lee. Some remarks on kim-dividing in NATP theories, Preprint. Available at https://arxiv.org/pdf/2211.04213.pdf. 2022.

[20] Alex Kruckman. Research statement., Available at https://akruckman.faculty.wesleyan.edu/files/2019/07/researchstatement.pdf. 2022.

[21] Alex Kruckman and Scott Mutchnik, Personal communication. 2022.

[22] Alex Kruckman and Nicholas Ramsey. Generic expansion and Skolemization in NSOP\(_1\) theories. *Annals of Pure and Applied Logic*, 169(8):755–774, aug 2018.

[23] Hyoyoon Lee, Personal communication. Feb. 10, 2023.

[24] Maryanthe Malliaris and Saharon Shelah. Model-theoretic applications of cofinality spectrum problems. *Israel Journal of Mathematics*, 220(2):947–1014, 2017.

[25] M.E. Malliaris. Edge distribution and density in the characteristic sequence. *Annals of Pure and Applied Logic*, 162(1):1–19, 2010.

[26] Scott Mutchnik. Conant-independence in generalized free amalgamation theories, Preprint. Available at https://arxiv.org/abs/2210.07527. 2022.

[27] Scott Mutchnik. Properties of independence in NSOP\(_3\) theories, Preprint. Available at https://arxiv.org/abs/2305.09908. 2023.

[28] Lynn Scow. Indiscernibles, em-types, and Ramsey classes of trees. *Notre Dame Journal of Formal Logic*, 56(3):429–447, 2015.

[29] Saharon Shelah. Toward classifying unstable theories. *Annals of Pure and Applied Logic*, 80(3):229–255, 1996.

[30] Saharon Shelah and Alexander Usvyatsov. More on SOP\(_1\) and SOP\(_2\). *Annals of Pure and Applied Logic*, 155(1):16–31, 2008.

[31] Kota Takeuchi and Akito Tsuboi. On the existence of indiscernible trees. *Annals of Pure and Applied Logic*, 163(12):1891–1902, 2012.

[32] Katrin Tent and Martin Ziegler. On the isometry group of the Urysohn space. *Journal of the London Mathematical Society*, 87(1):289–303, nov 2012.

[33] Itaï Ben Yaacov and Artem Chernikov. An independence theorem for NTP\(_2\) theories. *The Journal of Symbolic Logic*, 79(1):135–153, 2014.