Index Iteration Theory for Symplectic Paths with Applications to Nonlinear Hamiltonian Systems

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Abstract

In recent years, we have established the iteration theory of the index for symplectic matrix paths and applied it to periodic solution problems of nonlinear Hamiltonian systems. This paper is a survey on these results.

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Since P. Rabinowitz’s pioneering work of 1978, variational methods have been widely used in the study of existence of solutions of Hamiltonian systems. But how to study the geometric multiplicity and stability of periodic solution orbits obtained by variational methods has kept to be a difficulty problem. For example let $x = x(t)$ be a $\tau$-periodic solution of a Hamiltonian system

$$\dot{x}(t) = JH'(x(t)), \quad \forall t \in \mathbb{R}.$$ (0.1)

The $m$-th iteration $x^m$ of $x$ is defined by induction $m - 1$ times via $x(t + \tau) = x(t)$ for $t > 0$. It runs $m$-times along the orbit of $x$. Geometrically these iterations produce the same solution orbit of (0.1), but they are different as critical points of corresponding functionals. This multiple covering phenomenon causes major difficulties in the study.

A natural way to study solution orbits found by variational methods is to study the Morse-type index sequences of their iterations. But when one studies general Hamiltonian systems, the Morse indices of the critical points of the corresponding functional are always infinite. To overcome this difficulty, in their celebrated paper of 1984, C. Conley and E. Zehnder defined an index theory for any non-degenerate paths in $\text{Sp}(2n)$ with $n \geq 2$, i.e., the so called Conley-Zehnder index.
theory. This index theory was further defined for non-degenerate paths in $\text{Sp}(2)$ by E. Zehnder and the author in [23] of 1990. The index theory for degenerate linear Hamiltonian systems was defined by C. Viterbo in [39] and the author in [20] of 1990 independently. In [24] of 1997, this index was extended to any degenerate symplectic matrix paths.

Motivated by the iteration theories for the Morse type index theories established by R. Bott in 1956 and by I. Ekeland in 1980s, in recent years the author extended the index theory mentioned above, introduced an index function theory for symplectic matrix paths, and established the iteration theory for the index theory of symplectic paths. Applying this index iteration theory to nonlinear Hamiltonian systems, interesting results on periodic solution problems of Hamiltonian systems are obtained. Here a brief survey is given on these subjects. Readers are referred to the author’s recent book [30] for further details.

### 1. Index function theory for symplectic paths

As usual we define the symplectic group by $\text{Sp}(2n) = \{ M \in \text{GL}(\mathbb{R}^{2n}) \mid M^TJM = J \}$, where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, $I$ is the identity matrix on $\mathbb{R}^n$, and $M^T$ denotes the transpose of $M$. For $\omega \in U$, the unit circle in the complex plane $\mathbb{C}$, we define the $\omega$-singular subset in $\text{Sp}(2n)$ by $\text{Sp}(2n)_0^\omega = \{ M \in \text{Sp}(2n) \mid \omega^{-n}\det(\gamma(t) - \omega I) = 0 \}$. Here for any $M \in \text{Sp}(2n)_0^\omega$, we define the orientation of $\text{Sp}(2n)_0^\omega$ at $M$ by the positive direction $\frac{d}{dt}M \exp(tJ)|_{t=0}$. Since the fundamental solution of a general linear Hamiltonian system with continuous symmetric periodic coefficient $2n \times 2n$ matrix function $B(t)$,

$$\dot{x}(t) = JB(t)x(t), \quad \forall t \in \mathbb{R},$$

is a path in $\text{Sp}(2n)$ starting from the identity, for $\tau > 0$ we define the set of symplectic matrix paths by $\mathcal{P}_\tau(2n) = \{ \gamma \in C([0, \tau], \text{Sp}(2n)) \mid \gamma(0) = I \}$. For any two path $\xi$ and $\eta : [0, \tau] \to \text{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, as usual we define $\eta * \xi(t)$ by $\xi(2t)$ if $0 \leq t \leq \tau/2$, and $\eta(2t - \tau)$ if $\tau/2 \leq t \leq \tau$. We define a special path $\zeta : [0, \tau] \to \text{Sp}(2n)$ by

$$\zeta(t) = \text{diag}(2 - \frac{t}{\tau}, \ldots, 2 - \frac{t}{\tau}, \frac{2 - \frac{t}{\tau}}{1}, \ldots, \frac{2 - \frac{t}{\tau}}{1}), \quad \text{for } 0 \leq t \leq \tau.$$

**Definition 1.** (cf. [27]) For any $\tau > 0$, $\omega \in U$, and $\gamma \in \mathcal{P}_\tau(2n)$, we define the $\omega$-nullity of $\gamma$ by

$$\nu_\omega(\gamma) = \dim_{\mathbb{C}} \ker_{\mathbb{C}}(\gamma(t) - \omega I).$$

If $\gamma$ is $\omega$ non-degenerate, i.e., $\nu_\omega(\gamma) = 0$, we define the $\omega$-index of $\gamma$ by the intersection number

$$i_\omega(\gamma) = |\text{Sp}(2n)_0^\omega : \gamma * \zeta|.$$  \hspace{1cm} \text{(1.3)}

If $\gamma$ is $\omega$ degenerate, i.e., $\nu_\omega(\gamma) > 0$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of $\gamma$ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf \{ i_\omega(\beta) \mid \beta \in U, \nu_\omega(\beta) = 0 \}. \hspace{1cm} \text{(1.4)}$$
Then we call \((i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\}\) the index function of \(\gamma\) at \(\omega\).

The relation of this index \((i_1(\gamma), \nu_1(\gamma))\) with the Morse index of \(\tau\)-periodic solutions of the problem (1) was proved by C. Conley, E. Zehnder, and the author in [4, 33], and [20] (cf. Theorem 6.1.1 of [30]).

2. Iteration theory of the index for symplectic paths

Given a path \(\gamma \in \mathcal{P}_\tau(2n)\), its iteration is defined inductively by \(\gamma(t + \tau) = \gamma(t)\gamma(\tau)\) for \(t \geq 0\), i.e.,

\[
\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j, \quad j\tau \leq t \leq (j + 1)\tau, j = 0, 1, \ldots, m - 1,
\]

for any \(m\) in the natural integer set \(\mathbb{N}\). For our applications of this index theory to nonlinear Hamiltonian systems, we are facing two types of problems:

1. knowing the end point \(\gamma(\tau)\) of a path \(\gamma \in \mathcal{P}_\tau(2n)\), the initial index \((i_1(\gamma), \nu_1(\gamma))\), and the iteration time \(m\), want to find the index \(i_1(\gamma^m)\) of the \(m\)-th iterated path \(\gamma^m\);

2. knowing the end point \(\gamma(\tau)\) of a path \(\gamma \in \mathcal{P}_\tau(2n)\), the initial index \((i_1(\gamma), \nu_1(\gamma))\), and the index \((i_1(\gamma^m), \nu_1(\gamma^m))\) of the \(m\)-th iterated path \(\gamma^m\), want to find the iteration time \(m\).

To solve these problems, we first generalize Bott’s formula of the iterated Morse index for closed geodesics to the index theory for general symplectic paths:

**Theorem 2** (cf. [27]). For any \(\tau > 0\), \(\gamma \in \mathcal{P}_\tau(2n)\), \(z \in U\), and \(m \in \mathbb{N}\), there hold:

\[
i_z(\gamma^m) = \sum_{\omega^m = z} i_\omega(\gamma), \quad \nu_z(\gamma^m) = \sum_{\omega^m = z} \nu_\omega(\gamma).
\]

By (2.2) it is easy to see that the mean index \(\bar{i}(\gamma) = \lim_{m \to +\infty} i_1(\gamma^m)/m\) for any \(\gamma \in \mathcal{P}_\tau(2n)\) is always a finite real number.

To further solve the problems (1) and (2), we need to go beyond the Bott-type formula (2.2). For a given path \(\gamma\) we consider to deform it to a new path \(\eta\) in \(\mathcal{P}_\tau(2n)\) so that

\[
i_1(\gamma^m) = i_1(\eta^m), \quad \nu_1(\gamma^m) = \nu_1(\eta^m), \quad \forall m \in \mathbb{N},
\]

and that \((i_1(\eta^m), \nu_1(\eta^m))\) is easy enough to compute. This leads to finding homotopies \(\delta : [0, 1] \times [0, \tau] \to \text{Sp}(2n)\) starting from \(\gamma\) in \(\mathcal{P}_\tau(2n)\) and keeping the end points of the homotopy always stay in a certain suitably chosen maximal subset of \(\text{Sp}(2n)\) so that (2.3) always holds. By (2.2), this set is defined to be the path connected component \(\Omega^0(M)\) containing \(M = \gamma(\tau)\) of the set

\[
\Omega(M) = \{ N \in \text{Sp}(2n) \mid \sigma(N) \cap U = \sigma(M) \cap U, \text{ and } \nu_\lambda(N) = \nu_\lambda(M) \forall \lambda \in \sigma(M) \cap U \}.
\]

Here we call \(\Omega^0(M)\) the homotopy component of \(M\) in \(\text{Sp}(2n)\).

Using normal forms of symplectic matrices (cf. [22], [13]), we then decompose \(\gamma(\tau)\) within \(\Omega^0(\gamma(\tau))\) into product of 10 special \(2 \times 2\) and \(4 \times 4\) symplectic normal
form matrices, which we call basic normal forms. Correspondingly by the homotopy invariance and symplectic additivity of the index theory, the computations in are reduced to iterations of those paths in $\text{Sp}(2)$ or $\text{Sp}(4)$ whose end points are one of the 10 basic normal form matrices. The study of the index for iterations of any symplectic paths is carried out for paths in $\text{Sp}(2)$ via the $\mathbb{R}^3$-cylindrical coordinate representation of $\text{Sp}(2)$, then for hyperbolic and elliptic paths in $\text{Sp}(2n)$. This yields the precise iteration formula obtained in [29] of the index theory for any symplectic path $\gamma \in \mathcal{P}_\tau(2^n)$ in terms of the basic norm form decomposition of $\gamma(\tau)$, $(i(\gamma, 1), \nu(\gamma, 1))$, and the iteration time $m$.

For any $M \in \text{Sp}(2n)$, its splitting numbers at an $\omega \in \mathbb{U}$ is defined in [27] by

$$S^\pm_M(\omega) = \lim_{\epsilon \to 0^+} i\omega \exp(\pm \sqrt{-1} \epsilon)(\gamma) - i\omega(\gamma),$$

(2.5) via any $\gamma \in \mathcal{P}_\tau(2^n)$ satisfying $\gamma(\tau) = M$. Then it is proved that the splitting numbers of $M$ at $\omega$ can be characterized algebraically.

Motivated by the precise iteration formulae of [29], the following second index iteration formula of any symplectic path is established by C. Zhu and the author. Here we denote by $(i(\gamma, m), \nu(\gamma, m)) = (i_1(\gamma^m), \nu_1(\gamma^m))$.

**Theorem 3** (cf. [34]). For any $\tau > 0$, $\gamma \in \mathcal{P}_\tau(2^n)$, and $m \in \mathbb{N}$, there holds:

$$i(\gamma, m) = m(i(\gamma, 1) + S^+_M(1) - C(M)) + 2 \sum_{\theta \in (0, 2\pi)} E\left(\frac{m\theta}{2\pi}\right) S_M(e^{\sqrt{-1}\theta}) - (S^+_M(1) + C(M)),$$

(2.6)

where $M = \gamma(\tau)$, $C(M) = \sum_{0 < \theta < 2\pi} S_M(e^{\sqrt{-1}\theta})$, and $E(a) = \min\{k \in \mathbb{Z} | k \geq a\}$ for any $a \in \mathbb{R}$.

In order to solve problems on nonlinear Hamiltonian systems, various index iteration inequalities for any path $\gamma \in \mathcal{P}_\tau(2^n)$ and $m \in \mathbb{N}$ are proved by D. Dong, C. Liu, C. Zhu and the author in [7], [16], [17], and [34].

**Theorem 4.** For any $\gamma \in \mathcal{P}_\tau(2^n)$ and $m \in \mathbb{N}$, the following iteration inequalities always hold.

**Estimate via mean index** (cf. [10], [17]):

$$m\hat{i}(\gamma) - n \leq i(\gamma, m) \leq \hat{m}i(\gamma) + n - \nu(\gamma, m).$$

(2.7)

**Estimate via initial index** (cf. [18]):

$$m(i(\gamma, 1) + \nu(\gamma, 1) - n) + n - \nu(\gamma, 1) \leq i(\gamma, m) \leq m(i(\gamma, 1) + n) - n - (\nu(\gamma, m) - \nu(\gamma, 1)).$$

(2.8)

**Successive index estimate** (cf. [34]):

$$\nu(\gamma, m) - \frac{e(\gamma(\tau))}{2} \leq i(\gamma, m + 1) - i(\gamma, m) - i(\gamma, 1) \leq \nu(\gamma, 1) - \nu(\gamma, m + 1) + \frac{e(\gamma(\tau))}{2}.$$
Here we define $e(M)$ to be the total multiplicity of eigenvalues of $M$ on $U$ and call it the elliptic height of $M$.

A consequence of the iteration inequality (2.8) together with the necessary and sufficient conditions for any equality in (2.8) to hold for some $m$ yields a new proof of the following theorem of D. Dong and the author on controlling the iteration time $m$ via indices:

**Theorem 5** (cf. [7]). For any $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbb{N}$, suppose $i(\gamma, m) \leq n+1$, $i(\gamma, 1) \geq n$, and $\nu(\gamma, 1) \geq 1$. Then $m = 1$.

Note also that the inequality (2.9) yields a way to estimate the ellipticity of solution orbits of Hamiltonian systems obtained by variational methods via their iterated indices.

In order to study the properties of solution orbits of the system (0.1) on a given energy hypersurface, when the number of orbits is finite, we need to study common properties of any given finite family of symplectic paths $\gamma_j \in \mathcal{P}_\tau_j(2n)$ with $1 \leq j \leq q$. This leads to the following common index jump theorem of C. Zhu and the author proved in [34]. For any $\gamma \in \mathcal{P}_\tau(2n)$, its $m$-th index jump $\mathcal{G}_m(\gamma)$ is defined to be the open interval $\mathcal{G}_m(\gamma) = (i(\gamma, m) + \nu(\gamma, m) - 1, i(\gamma, m + 2))$.

**Theorem 6** (cf. [34]). Let $\gamma_j \in \mathcal{P}_\tau_j(2n)$ with $1 \leq j \leq q$ satisfying

$$\hat{i}(\gamma_j) > 0, \quad i(\gamma_j, 1) \geq n, \quad 1 \leq j \leq q.$$  \hfill (2.10)

Then there exist infinitely many positive integer tuples $(N, m_1, \ldots, m_q) \in \mathbb{N}^{q+1}$ such that

$$\emptyset \neq [2N - \kappa_1, 2N + \kappa_2] \subset \bigcap_{j=1}^{q} \mathcal{G}_{2m_j - 1}(\gamma_j),$$  \hfill (2.11)

where $\kappa_1 = \min_{1 \leq j \leq q}(i(\gamma_j, 1) + 2S_{\gamma_j(\tau)}^{-1}(1) - \nu(\gamma_j, 1))$ and $\kappa_2 = \min_{1 \leq j \leq q}i(\gamma_j, 1) - 1$.

In order to prove this theorem, we need to make each index jump to be as big as possible, and to make their largest sizes happen simultaneously to guarantee the existence of a non-empty largest common intersection interval among them. By the term $E(\frac{m}{\tau})$ in the abstract iteration formula (2.6), such a problem is reduced to a dynamical system problem on a torus, and is solved by properties of closed additive subgroups of tori.

### 3. Applications to nonlinear Hamiltonian systems

So far, we have applied our index iteration theory to three important problems on periodic solutions of nonlinear Hamiltonian systems. Let $T > 0$ and suppose $x$ is a non-constant $T$-periodic solution of the nonlinear Hamiltonian system (0.1). Suppose the minimal period of $x$ is $\tau = \frac{T}{k}$ for some $k \in \mathbb{N}$. We denote by $\gamma_x \in \mathcal{P}_\tau(2n)$ the fundamental solution of the linearized Hamiltonian system (1.1) at $x$ with $B(t) = H''(x(t))$, and the iterated index of $x$ by $(i(x, m), \nu(x, m)) = (i(\gamma_x, m), \nu(\gamma_x, m))$ for all $m \in \mathbb{N}$.
3.1. Prescribed minimal period solution problem

In [35] of 1978, P. Rabinowitz posed a conjecture on whether the Hamiltonian system possesses periodic solutions with prescribed minimal period when the Hamiltonian function satisfies his superquadratic conditions. This conjecture is studied by D. Dong and the author as an application of our index iteration theory. Note that for a non-constant $\tau$-periodic solution $x$ of the autonomous system (0.1), the condition on the nullity in Theorem 5 always holds. Thus Theorem 5 yields:

**Theorem 7** (cf. [7]). For any non-constant $\tau$-periodic solution $x$ of (0.1), denote its minimal period by $\tau/m$ for some $m \in \mathbb{N}$. Suppose $i(x|_{[0,\tau]}, 1) \leq n+1$ and $n \leq i(x|_{[0,\tau/m]}, 1)$. Then $m = 1$, i.e., $\tau$ is the minimal period of $x$.

Here the first estimate on the index holds if $x$ is obtained by minimax or minimization methods, and the second estimate on the index holds if the Hamiltonian function $H$ is convex in a certain weak sense along the orbit of $x$. This result reveals the intrinsic relationship between the minimal period of a periodic solution and its indices, and unifies all the results on Rabinowitz’s conjecture under various convexity conditions. Specially, it recovers the famous theorem of I. Ekeland and H. Hofer in 1985 (cf. [11]) who solved Rabinowitz’s conjecture for convex superquadratic Hamiltonian systems.

3.2. Periodic points of the Poincaré map of Lagrangian systems on tori

In 1984, C. Conley stated a conjecture on whether the Poincaré map of any 1-periodic time dependent Hamiltonian system defined on the standard torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ always possesses infinitely many periodic points which are produced by contractible periodic solutions of the corresponding Hamiltonian system on $T^{2n}$. A celebrated partial answer to this conjecture was given by D. Salamon and E. Zehnder in 1992 (cf. [37]) for a large class of symplectic manifolds on which every contractible integer periodic solution of the Hamiltonian system has at least one Floquet multiplier not equal to 1. So far Conley conjecture is still open and seems far from being completely understood.

In [28], we studied the Lagrangian system version of this conjecture. Consider

$$\frac{d}{dt}L_x(t, x, \dot{x}) - L_x(t, x, \dot{x}) = 0, \quad x \in \mathbb{R}^n, \quad (3.1)$$

where $L_x$ and $L_{\dot{x}}$ denote the gradients of $L$ with respect to $\dot{x}$ and $x$ respectively. The main result is the following:

**Theorem 8** (cf. [28]). Suppose the Lagrangian function $L$ satisfies

(L1) $L(t, x, p) = \frac{1}{2}A(t)p \cdot p + V(t, x)$, where $\frac{1}{2}A(t)p \cdot p \geq \lambda|p|^2$ for all $(t, p) \in \mathbb{R} \times \mathbb{R}^n$ and some fixed constant $\lambda > 0$.

(L2) $A \in C^3(\mathbb{R}, \mathcal{L}_s(\mathbb{R}^n))$, $V \in C^3(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, both $A$ and $V$ are 1-periodic in all of their variables, where $\mathcal{L}_s(\mathbb{R}^n)$ denotes the set of $n \times n$ real symmetric matrices.

Then the Poincaré map $\Psi$ of the system (3.1) possesses infinitely many periodic points on $T^{2n}$ produced by contractible integer periodic solutions of the system (3.1).
In the proof of Theorem 8, the above inequality (2.7) plays a crucial role. By this inequality, at very high iteration level, a global homological injection map can be constructed which maps a generator of a certain non-trivial local critical group to a nontrivial homology class $[\sigma]$ in a global homology group, if the number of contractible integer periodic solution towers of the system (5.1) is finite. But on the other hand, by a technique of V. Bangert and W. Klingenberg in [3], it is shown that this homology class $[\sigma]$ must be trivial globally. This contradiction then yields the conclusion of Theorem 8.

### 3.3. Closed characteristics on convex compact hypersurfaces

Denote the set of all compact strictly convex $C^2$-hypersurfaces in $\mathbb{R}^{2n}$ by $\mathcal{H}(2n)$. For $\Sigma \in \mathcal{H}(2n)$ and $x \in \Sigma$, let $N_\Sigma(x)$ be the outward normal unit vector at $x$ of $\Sigma$. We consider the problem of finding $\tau > 0$ and a curve $x \in C^1([0, \tau], \mathbb{R}^{2n})$ such that

$$
\begin{align*}
\dot{x}(t) &= JN_\Sigma(x(t)), \quad x(t) \in \Sigma, \quad \forall t \in \mathbb{R}, \\
x(\tau) &= x(0).
\end{align*}
$$

(3.1)

A solution $(\tau, x)$ of the problem (3.1) is called a closed characteristic on $\Sigma$. Two closed characteristics $(\tau, x)$ and $(\sigma, y)$ are geometrically distinct, if $x(\mathbb{R}) \neq y(\mathbb{R})$. We denote by $\mathcal{T}(\Sigma)$ the set of all geometrically distinct closed characteristics $(\tau, x)$ on $\Sigma$ with $\tau$ being the minimal period of $x$. Note that the problem (3.1) can be described in a Hamiltonian system version and solved by variational methods. A closed characteristic $(\tau, x)$ is non-degenerate, if 1 is a Floquet multiplier of $x$ of precisely algebraic multiplicity 2, and is elliptic, if all the Floquet multipliers of $x$ are on $U$. Let $\#A$ denote the total number of elements in a set $A$.

This problem has been studied for more than 100 years since at least A. M. Liapunov in 1892. A long standing conjecture on the multiplicity of closed characteristics is whether

$$
\# \tilde{\mathcal{J}}(\Sigma) \geq n, \quad \forall \Sigma \in \mathcal{H}(2n).
$$

(3.2)

The first breakthrough on this problem in the global sense was made by P. Rabinowitz [35] and A. Weinstein [40] in 1978. They proved $\# \mathcal{T}(\Sigma) \geq 1$ for all $\Sigma \in \mathcal{H}(2n)$. Besides many results under pinching conditions, in 1987–1988, I. Ekeland-L. Lassoued, I. Ekeland-H. Hofer, and A. Szulkin proved $\# \mathcal{T}(\Sigma) \geq 2$ for all $\Sigma \in \mathcal{H}(2n)$ and $n \geq 2$. In 1998, H. Hofer, K. Wysocki, and E. Zehnder proved in [14]: $\# \mathcal{T}(\Sigma) = 2$ or $+\infty$ for every $\Sigma \in \mathcal{H}(4)$. In recent years C. Liu, C. Zhu, and the author gave the following answers to the conjecture (3.2):

**Theorem 9** (cf. [34]). There holds

$$
\# \mathcal{T}(\Sigma) \geq \left[\frac{n}{2}\right] + 1, \quad \forall \Sigma \in \mathcal{H}(2n),
$$

(3.3)

where $[a] = \max\{k \in \mathbb{Z} | k \leq a\}$ for any $a \in \mathbb{R}$. Moreover, if all the closed characteristics on $\Sigma$ are non-degenerate, then $\# \mathcal{T}(\Sigma) \geq n$.

**Theorem 10** (cf. [19]). For any $\Sigma \in \mathcal{H}(2n)$, if $\Sigma$ is symmetric with respect to the origin, i.e., $x \in \Sigma$ implies $-x \in \Sigma$, then $\# \mathcal{T}(\Sigma) \geq n$.

Very recently, Y. Dong and the author further proved the following result.
Theorem 11 (cf. [8]). Let $\Sigma \in H(2n)$ be $P$-symmetric with respect to the origin, i.e., $x \in \Sigma$ implies $P x \in \Sigma$, where $P = \text{diag}(-I_{n-k}, I_k, -I_{n-k}, I_k)$ for some fixed integer $k \in [0, n-1]$. Let $\Sigma(k) = \{(x, y) \in (\mathbb{R}^k)^2 \mid (0, x, y) \in \Sigma\}$. Suppose \# $\mathcal{T}(\Sigma(k)) \leq k$ or \# $\mathcal{T}(\Sigma(k)) = +\infty$ holds. Then \# $\mathcal{T}(\Sigma) \geq n - 2k$.

Proof of Theorem 11 depends on a new index iteration theory for symplectic paths iterated by the formula $\gamma(t + \tau) = P\gamma(t)P\gamma(\tau)$ for $t \geq 0$.

The second long standing conjecture on closed characteristics is whether there always exists at least an elliptic closed characteristic on any $\Sigma \in H(2n)$. Up to the author’s knowledge, the existence of one elliptic closed characteristic on $\Sigma \in H(2n)$ was proved by I. Ekeland in 1990 when $\Sigma$ is $\sqrt{2}$-pinched by two spheres, and by G.-F. Dell’Antonio, B. D’Onofrio, and I. Ekeland in 1992 when $\Sigma$ is symmetric with respect to the origin. Recently using an enhanced version of the iteration estimate (2.9) on the elliptic height, based on results in [29] the following result was further proved by C. Zhu and the author.

Theorem 12 (cf. [34]). For $\Sigma \in H(2n)$, suppose \# $\mathcal{T}(\Sigma) < +\infty$. Then there exists at least an elliptic closed characteristic on $\Sigma$. Moreover, suppose $n \geq 2$ and \# $\mathcal{T}(\Sigma) \leq 2[n/2]$. Then there exist at least two elliptic elements in $\mathcal{T}(\Sigma)$.

The main ingredient in the proofs of Theorems 9 to 12 is our index iteration theory mentioned above. To illustrate this method, we briefly describe below the main idea in the proof of (3.3) in Theorem 9. Because each closed characteristic on $\Sigma$ corresponds to infinitely many critical values of the related dual action functional, our way to solve the problem is to study how the index intervals of iterated closed characteristics cover the set of integers $2N - 2 + n$ to count the number of closed characteristics on $\Sigma$. Suppose $q = \# \mathcal{J}(\Sigma) < +\infty$. In the proof of the multiplicity claim (3.3) of Theorem 9, the most important ingredient is the following estimates:

\[
q \geq \# \left( (2N - 2 + n) \cap \bigcap_{j=1}^q \mathcal{G}_{2m_j-1}(\gamma_{x_j}) \right) \\
\geq \# \left( (2N - 2 + n) \cap [2N - \kappa_1, 2N + \kappa_2] \right) \\
\geq \left[ \frac{n}{2} \right] + 1, \tag{3.4}
\]

The first inequality in (3.4) is a new version of the Liusternik-Schnirelman theoretical argument at the iterated index level, which distinguishes solution orbits geometrically instead of critical points only as usual methods do. The second inequality in (3.4) uses the common index jump Theorem 6. The last inequality in (3.4) uses the Morse theoretical approach. Roughly speaking, the common index jump theorem picks up as many as possible points of $2N - 2 + n$ in the interval $[2N - \kappa_1, 2N + \kappa_2] \subset \bigcap_{j=1}^q \mathcal{G}_{2m_j-1}(\gamma_{x_j})$, which yields a lower bound for \# $\mathcal{T}(\Sigma)$.

As usual, a hypersurface $\Sigma \subset \mathbb{R}^{2n}$ is star-shaped if the tangent hyperplane at any $x \in \Sigma$ does not intersect the origin. Closed characteristics on $\Sigma$ can be defined by (3.1) too. In this case, the result \# $\mathcal{J}(\Sigma) \geq 1$ was proved by P. Rabinowitz in [35] of 1978. Then multiplicity results were proved under certain pinching conditions on star-shaped $\Sigma$. Recently, the following result for the free case was proved by X. Hu and the author.

Theorem 13 (cf. [15]). Let $\Sigma$ be a star-shaped compact $C^2$-hypersurface in $\mathbb{R}^{2n}$. Suppose all the closed characteristics on $\Sigma$ and all of their iterates are non-
degenerate. Then \( \# T(\Sigma) \geq 2 \). Moreover, if \( n = 2 \) and \( \# T(\Sigma) < +\infty \) further holds, then there exist at least two elliptic closed characteristics on \( \Sigma \).

Here the crucial point is to prove \( i(x, 1) \geq n \) when \((\tau, x)\) is the only geometrically distinct closed characteristic on \( \Sigma \). This conclusion is proved by using our index iteration theory and an identity of non-degenerate closed characteristics on \( \Sigma \) proved by C. Viterbo in 1989.

Because of Theorem 9 and other indications, we suspect that the following holds:

\[
\{ \# T(\Sigma) \mid \Sigma \in \mathcal{H}(2n) \} = \{ k \in \mathbb{Z} \mid \left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq n \} \cup \{ +\infty \}.
\] (3.5)

We also suspect that closed orbits of the Reeb field on a compact contact hypersurfaces in a symplectic manifold may have similar properties.

Many other problems related to iterations of periodic solution orbits are still open, for example, the Seifert conjecture on the existence of at least \( n \) brake orbits for the given energy problem of classical Hamiltonian systems on \( \mathbb{R}^n \) (cf. [38], [11] and the references there in), and the conjecture on the existence of infinitely many geometrically distinct closed geodesics on every compact Riemannian manifold (cf. [2] and the solution for \( S^2 \) by J. Franks and V. Bangert). We believe that our index iteration theory for symplectic paths and the methods we developed to establish and apply it to nonlinear problems will have the potential to play more roles in the study on these problems and in other mathematical areas.

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