Darboux transformations for twisted so(p,q) system and local isometric immersion of space forms

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Abstract

For the n-dimensional integrable system with a twisted so(p, q) reduction, Darboux transformations given by Darboux matrices of degree 2 are constructed explicitly. These Darboux transformations are applied to the local isometric immersion of space forms with flat normal bundle and linearly independent curvature normals to give the explicit expression of the position vector. Some examples are given from the trivial solutions and standard imbedding $T^n \rightarrow \mathbb{R}^{2n}$.

1 Introduction

The theory of integrable system has been widely used to study some differential geometric problems such as minimal submanifolds, submanifolds with constant mean curvature, harmonic maps etc. Especially, the isometric immersion of space form $M_1(K_1)$ of curvature $K_1$ into space form $M_2(K_2)$ of curvature $K_2$ was studied in various papers. If $K_1 \neq K_2$, the nonlinear wave equation and the nonlinear sine-Gordon equation are considered to describe some special problems. When $K_1 = K_2$, the local isometric immersion with flat normal bundle and linearly independent curvature normals was proposed in [17]. That problem was also dealt with by a purely geometric way.

In this paper, we use the Darboux transformation to get the explicit expressions of the local isometric immersions of the space forms of the same curvature with flat normal bundle and linearly independent curvature normals.

In this problem, the Lax pair has a twisted so(n) reduction. When the Lie algebra is $gl(n, \mathbb{C})$ or $sl(n, \mathbb{C})$, there is a systematic construction of Darboux transformations (for the problems discussed later, see [8, 9, 13, 15]), which is now a useful method to get explicit solutions of nonlinear integrable partial differential equations. If the Lie algebra is $su(p, q)$, there is also a general algorithm to choose the spectral parameters for the Darboux transformation of degree 1. In this algorithm, the spectral parameters
can take only two mutually conjugate values. As a subalgebra of $su(p,q)$ ($p + q$ even), $so(p,q)$ ($p + q$ even) problem can be dealt with in a similar way, provided that the real condition can be realized. However, for $so(p,q)$ ($p + q$ odd) problem, this method is not applicable directly\cite{4,11} since the spectral parameters should be conjugate and cannot be real (or purely imaginary if written in another way) so that the Darboux transformation is not trivial. It is known that the pure $so(p,q)$ problem can be dealt with by a Darboux transformation of degree 2.\cite{7,18} Here, for the twisted $so(p,q)$ problem ($so(p,q)$ problem with an additional involution condition), we construct such a Darboux transformation using a limit process so that the Darboux matrix has only two (not four) eigenvalues. For the twisted $so(p,q)$ system, the method in \cite{18} can give the same Darboux matrix under some assumptions, but the method here is more direct and has no assumptions.

§2 describes the linear system and the properties of its solutions. In §3, we get explicit expressions of the Darboux transformations of degree 2 for twisted $so(p,q)$ reduction. Owing to the isomorphism between $su(1,1)$ and $so(2,1)$, we can see that the Darboux transformation for MKdV and sine-Gordon equations given by the Darboux matrix here is actually the well-known standard Darboux transformation.

Using the above conclusions, §4 gives the Darboux transformation for the local isometric immersion from $M_n(K)$ to $M_{2n}(K)$ with flat normal bundle and linearly independent curvature normals, the Lax set of which was proposed by \cite{17}. We present the general expression of the transformation for the position vector of $M_n(K) \rightarrow M_{2n}(K)$. §4 also gives some interesting examples, including the submanifolds derived from trivial solutions for all $K = 0, 1, -1$ cases and the submanifold derived from the standard torus $T^n$ in $\mathbb{R}^{2n}$.

2 Linear system

Let

$$\mathfrak{g} = so_{ex}(p,q,r) = \{ X \in gl(p + q + r, \mathbb{R}) \mid X^T C + CX = 0 \}$$

(1)

where $C = I_{p,q,r} = \text{diag}(1, \cdots, 1, -1, \cdots, -1, 0, \cdots, 0)$. Clearly, $so_{ex}(p,q,0) = so(p,q)$.

Here we consider $so_{ex}(p,q,r)$ instead of $so(p,q)$ for the unified treatment in §4.

Let $\sigma$ be a diagonal matrix such that $\sigma^2 = 1$ ($\sigma \neq 1$). Then the transformation $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$, $X \mapsto \sigma X \sigma$ is an involution on $\mathfrak{g}$. Hence there is a decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are the $+1$ and $-1$ eigenspaces of $\theta$ respectively. Moreover, we suppose that there is a maximum commutative subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{g}_1$.

Let

$$\mathfrak{L}(\mathfrak{g}) = \{ \sum_{j=0}^{n} X_j \lambda^j \mid X_j \in \mathfrak{g}, j = 0, 1, \cdots, n \}$$

(2)

be a subalgebra of the loop algebra of $\mathfrak{g}$,

$$\mathfrak{L}(\mathfrak{g})^\sigma = \{ v \in \mathfrak{L}(\mathfrak{g}) \mid \sigma v(\lambda) \sigma = v(-\lambda) \}$$

(3)
be a subalgebra of $L(g)$.

First we consider the linear system

$$\Phi_x = U(x, \lambda)\Phi$$  \hspace{1cm} (4)

where $U \in L^\sigma(g)$. Denote

$$U(x, \lambda) = \sum_{j=0}^{n} U_j(x)\lambda^j, \quad U_j(x) \in \mathfrak{g}[j], \quad [j] = \begin{cases} 0 & j \text{ even}, \\ 1 & j \text{ odd}, \end{cases}$$ \hspace{1cm} (5)

then $U \in L^\sigma(g)$ if and only if $\sigma U(x, \lambda)\sigma = U(x, -\lambda), (U(x, \lambda))^*C + CU(x, \bar{\lambda}) = 0$. In this paper, we always suppose $\Phi(x, \lambda)$, a solution of (4), is smooth with respect to both $x$ and $\lambda$. To discuss the Darboux transformation for (4), we need the following lemmas.

**Lemma 1** If $\Phi$ is a solution of (4) with $\lambda = \lambda_0$, then $\sigma\Phi$ is a solution of (4) with $\lambda = -\lambda_0$.

**Lemma 2** If $\Phi$ is a solution of (4) with $\lambda = \lambda_0$, $\Psi$ is a solution of (4) with $\lambda = \bar{\lambda}_0$, then $(\Psi^*C\Phi)_x = 0$.

**Proof.** The conclusion follows from

$$\Phi_x = U(x, \lambda_0)\Phi,
\Psi_x^*C = -\Psi^*CU(x, \lambda_0).$$ \hspace{1cm} (6)

**Lemma 3** Suppose $\mu \in \mathbb{R}$, $\Phi$ is a solution of (4) such that $\Phi|_{\lambda = \mu}$ is real, then $\Phi^TC\Phi|_{\lambda = \mu}$ is independent of $x$.

**Proof.**

$$(\Phi_\lambda)_x = U(x, \lambda)\Phi_\lambda + U_\lambda(x, \lambda)\Phi,
\Phi_x^*C = -\Phi^*CU(x, \lambda),$$ \hspace{1cm} (7)

hold at $\lambda = \mu$, hence

$$(\Phi^*C\Phi_\lambda)|_{\lambda = \mu} = \Phi^*CU(\lambda, \lambda)\Phi|_{\lambda = \mu} = 0$$ \hspace{1cm} (8)

since $CX$ is antisymmetric for any $X \in \mathfrak{g}$.

### 3 Darboux transformations

We say a real symmetric matrix $M$ is semidefinite if for any nonzero vector $\xi$, $\xi^TM\xi \geq 0$ or $\xi^TM\xi \leq 0$. The Darboux transformation for (4) can be constructed in the following two ways. The first one is applicable when $C$ is not semidefinite and the second one is applicable when $C\sigma$ is not semidefinite. Since $\sigma$ is not $\pm I$, these two cases cover all the possible situations.
3.1 Construction of Darboux transformation when $C$ is not semidefinite

Let $\mu \in \mathbb{R}$. Let $H$ be a real vector solution of (4) with $\lambda = \mu$ and satisfies $H^*CH = 0$. (This is possible due to Lemma 2.)

Take $\lambda_1^{(e)} = \mu + \sqrt{-1}\varepsilon$, $\lambda_2^{(e)} = \mu - \sqrt{-1}\varepsilon$. Let $h_1^{(e)}$ be a vector solution of (4) with $\lambda = \lambda_1^{(e)}$ satisfying $h_1^{(e)}(0) = H$ and $h_1^{(e)}|_{x=x_0} = H(x_0)$ for some fixed $x_0$. According to Lemma 1, $h_1^{(e)}$ is a solution of (4) with $\lambda = \lambda_2^{(e)}$.

Let $\Gamma^{(e)}_{jk} = \frac{h_j^{(e)*}Ch_k^{(e)}}{\lambda_k^{(e)} - \lambda_j^{(e)}}$ (9)

then

$$G^{(e)}(\lambda) = 1 - 2\sum_{j,k=1}^{2} \frac{1}{\lambda - \lambda_k^{(e)}} h_j^{(e)}(\Gamma^{(e)-1})_{jk} h_k^{(e)*}C$$

is a Darboux matrix for (4) without considering the $L^\sigma(g)$ reduction, that is, for any solution $\Phi$ of (4), $\tilde{\Phi} = G^{(e)}\Phi$ satisfies

$$\tilde{\Phi}_x = \tilde{U}^{(e)}(x, \lambda)\tilde{\Phi}$$

where $\tilde{U}^{(e)} \in \mathfrak{gl}(p + q + r, \mathbb{C})$. Now we calculate $\Gamma^{(e)}$, $G^{(e)}$ and their limits as $\varepsilon \to 0$. First,

$$\Gamma^{(e)} = \begin{pmatrix} h_1^{(e)*}Ch_1^{(e)} & h_1^{(e)*}C\sigma h_1^{(e)} \\ 2\sqrt{-1} \varepsilon & 2\mu h_1^{(e)*}C\sigma h_1^{(e)} \\ -2\varepsilon & -2\sqrt{-1} \varepsilon \end{pmatrix}.$$ (12)

By Lemma 3 and the assumptions $H^*CH = 0$, $\left.\frac{\partial h_1^{(e)}}{\partial \varepsilon}\right|_{x=x_0} = 0$,

$$\lim_{\varepsilon \to 0} \frac{h_1^{(e)*}Ch_1^{(e)}}{2\sqrt{-1} \varepsilon} = -\sqrt{-1} \lim_{\varepsilon \to 0} \operatorname{Re}\left(h_1^{(e)*} C \frac{\partial h_1^{(e)}}{\partial \varepsilon}\right)$$

$$= -\sqrt{-1} \lim_{\varepsilon \to 0} \operatorname{Re}\left(h_1^{(e)*} C \frac{\partial h_1^{(e)}}{\partial \varepsilon}\right)|_{x=x_0} = 0.$$ (13)

Hence

$$\Gamma = \lim_{\varepsilon \to 0} \Gamma^{(e)} = \frac{1}{2\mu} \begin{pmatrix} 0 & -H^T C \sigma H \\ H^T C \sigma H & 0 \end{pmatrix} = \frac{H^T C \sigma H}{2\mu} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ (14)

$$\Gamma^{-1} = \frac{2\mu}{H^T C \sigma H} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ (15)
\[ G(\lambda) = \lim_{\varepsilon \to 0} G^{(\varepsilon)}(\lambda) = 1 - \frac{2\mu}{H^T C \sigma H} \left( \frac{HH^T \sigma}{\lambda + \mu} - \frac{\sigma HH^T}{\lambda - \mu} \right) C \]
\[ = \frac{1}{\lambda^2 - \mu^2} \left( \lambda^2 + \frac{2\lambda \mu}{H^T C \sigma H} [\sigma, HH^T] C + \frac{2\mu^2(\sigma HH^T + HH^T \sigma) C}{H^T C \sigma H} - \mu^2 \right). \tag{16} \]

It is easy to prove that
\[ (G(\tilde{\lambda}))^* CG(\lambda) = C, \]
\[ \sigma G(\lambda) \sigma = G(-\lambda). \tag{17} \]

Using these facts, we know that
\[ \tilde{U} = GUG^{-1} + G_2 G^{-1} \tag{18} \]
satisfies \( \tilde{U} \in \mathcal{L}^\sigma(g) \).

Therefore, the following theorem holds.

**Theorem 1** Suppose \( C \) is not semidefinite. Let \( \mu \in \mathbb{R} \). Let \( H \) be a real vector solution of (4) with \( \lambda = \mu \) such that \( H^T C H = 0 \). Let
\[ G(\lambda) = \frac{1}{\lambda^2 - \mu^2} \left( \lambda^2 + \frac{2\lambda \mu}{H^T C \sigma H} [\sigma, HH^T] C + \frac{2\mu^2(\sigma HH^T + HH^T \sigma) C}{H^T C \sigma H} - \mu^2 \right), \tag{19} \]
then \( \tilde{U} \in \mathcal{L}^\sigma(g) \). Moreover, for any solution \( \Phi \) of (4), \( \tilde{\Phi} = G\Phi \) satisfies \( \tilde{\Phi} x = \tilde{U} \tilde{\Phi} \).

### 3.2 Construction of Darboux transformation when \( C\sigma \) is not semidefinite

When \( C \) is semidefinite, Theorem 1 is no longer valid, since the required \( H \) does not exist in general. Here we discuss the problem when \( C\sigma \) is not semidefinite. We use the following transformation to change the problem to a problem dealt with in Part 3.1.

Consider the linear system
\[ \Phi_x = U(\sqrt{-1}\lambda)\Phi. \tag{20} \]

Let \( \tau \) be a complex diagonal matrix such that \( \tau^2 = \sigma \) and \( \tau \) has only two eigenvalues 1, \( \sqrt{-1} \). Clearly \( \tau^* \tau = 1 \). Let \( V(\lambda) = \tau U(\sqrt{-1}\lambda)\tau^* \), then when \( \lambda \) is real,
\[ \begin{align*}
1 & \quad V(\lambda) = \tau^* U(-\sqrt{-1}\lambda) \tau = \tau^* \sigma U(-\sqrt{-1}\lambda) \sigma \tau = \tau U(\sqrt{-1}\lambda) \tau^* = V(\lambda), \\
2 & \quad (V(\lambda))^* C \sigma = \tau (U(\sqrt{-1}\lambda))^* \tau^* C \sigma = -\tau C U(-\sqrt{-1}\lambda) \tau^* = -\tau C \sigma U(\sqrt{-1}\lambda) \tau^* = -\sigma C V(\lambda), \\
3 & \quad V(-\lambda) = \tau U(-\sqrt{-1}\lambda) \tau^* = \tau \sigma U(\sqrt{-1}\lambda) \sigma \tau^* = \sigma V(\lambda). \tag{21} \end{align*} \]

Hence, \( V(\lambda) \in L^\sigma(g') \) where
\[ g' = \{ X \in gl(p + q + r, \mathbb{R}) \mid X^T C \sigma + C \sigma X = 0 \} \cong so_c(p', q', r) \tag{22} \]
with \( p' + q' = p + q \).

Let \( \Psi = \tau \Phi \), then the linear system (20) is changed to
\[
\Psi_x = V(\lambda)\Psi.
\] (23)

When \( C\sigma \) is not semidefinite, we can use Theorem 1 to this problem.

Let \( G'(\lambda) = \tau G(\sqrt{-1}\lambda)^* \), then \( \tilde{\Psi} = G'(\lambda)\Psi \) satisfies
\[
\tilde{\Psi}_x = \tilde{V}(\lambda)\Psi
\]

where \( \tilde{V}(\lambda) = \tau \tilde{U}(\sqrt{-1}\lambda)^* \).

Let \( \mu \in \mathbb{R} \). Let \( H' \) be a real solution of (23) with \( \lambda = -\mu \). Then Theorem 1 implies that \( G'(\lambda) \) can be chosen as
\[
G'(\lambda') = \frac{1}{\lambda^2 - \mu^2} \left( \lambda^2 - \frac{2\lambda' \mu}{H^T C\sigma H} \sigma C + \frac{2\mu^2(\sigma H' H'^T + H' H'^T \sigma) \sigma C}{H'^T C\sigma H} - \mu^2 \right)
\]

since \( C \) should be replaced by \( \sigma C \) here. However, \( \tau^{-1}H' \) is a solution of (20) with \( \lambda = -\mu \), i.e. it is a solution of (23) with \( \lambda = \sqrt{-1}\mu \). Let \( H = \tau^{-1}H' \), then, for \( \lambda = -\sqrt{-1}\lambda' \),
\[
G(\lambda) = \tau^{-1}G(\lambda')\tau
\]
\[
= \frac{1}{\lambda^2 + \mu^2} \left( \lambda^2 + \frac{2\sqrt{-1}\lambda \mu}{H^T C\sigma H} \sigma C - \frac{2\mu^2(\sigma H H^* + H H^* \sigma) \sigma C}{H^* C H} + \mu^2 \right).
\]

Therefore, we have

**Theorem 2** Suppose \( C\sigma \) is not semidefinite. Let \( \mu \in \mathbb{R} \). Let \( H \) be a complex vector solution of (23) with \( \lambda = \sqrt{-1}\mu \) such that \( \tau H \) is real and \( H^* C\sigma H = 0 \). Let
\[
G(\lambda) = \frac{1}{\lambda^2 + \mu^2} \left( \lambda^2 + \frac{2\sqrt{-1}\lambda \mu}{H^T C\sigma H} \sigma C - \frac{2\mu^2(\sigma H H^* + H H^* \sigma) \sigma C}{H^* C H} + \mu^2 \right),
\]
\[
\tilde{U} = GUG^{-1} + G_x G^{-1},
\]
then \( \tilde{U} \in \mathcal{L}^{\sigma}(g) \). Moreover, for any solution \( \Phi \) of (23), \( \tilde{\Phi} = G\Phi \) satisfies \( \tilde{\Phi}_x = \tilde{U}\tilde{\Phi} \).

### 3.3 Example: MKdV equation and sine-Gordon equation

The MKdV equation
\[
u_t = u_{xxx} + 6u^2u_x
\] (25)

has a well-known Lax pair
\[
\Psi_x = \frac{\lambda}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi + \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \Psi,
\]
\[
\Psi_t = \frac{\lambda^3}{2} + \lambda u^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi + (\lambda^2 u + u_{xx} + 2u^2) \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \Psi + \lambda u_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi.
\] (26)
The real Lie algebra $\mathfrak{su}(1,1)$ generated by
\[ e_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \] (27)
is isomorphic to $\mathfrak{so}(2,1)$, given by the correspondence
\[ e_1 \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_2 \rightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_3 \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (28)

Now take $g = \mathfrak{so}(2,1), C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \sigma = C$. Then,
\[ g_0 = \text{Span} \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \] (29)
\[ g_1 = \text{Span} \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\}. \]

Take the Cartan subalgebra
\[ \mathfrak{h} = \text{Span} \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \] (30)
which corresponds to $e_1$.

Using the correspondence (28), we have the new Lax pair
\[ \Phi_x = \begin{pmatrix} 0 & 2u & 0 \\ -2u & 0 & \lambda \\ 0 & \lambda & 0 \end{pmatrix} \Phi, \]
\[ \Phi_t = \begin{pmatrix} 0 & 0 & 2\lambda^2u + 2u_{xx} + 4u^3 & -2\lambda u_x \\ -2\lambda^2u - 2u_{xx} - 4u^3 & 0 & \lambda^3 + 2\lambda^2u^2 & 0 \\ \lambda^3 + 2\lambda^2u^2 & 0 & 0 \end{pmatrix} \Phi. \] (31)

Let $\mu \in \mathbb{R}$. Let $H = (\alpha, \beta, \gamma)^T$ be a real solution of (31) with $\lambda = \mu$ and satisfies $\alpha^2 + \beta^2 - \gamma^2 = 0$, then Theorem 1 gives
\[ (\lambda^2 - \mu^2)G(\lambda) = \lambda^2 - 2\frac{\lambda \mu}{\gamma^2} \begin{pmatrix} 0 & 0 & \alpha \gamma \\ \alpha \gamma & 0 & \beta \gamma \\ 0 & \beta \gamma & 0 \end{pmatrix} + 2\frac{\mu^2}{\gamma^2} \begin{pmatrix} \alpha^2 & \alpha \beta & 0 \\ \alpha \beta & \beta^2 & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix} - \mu^2, \] (32)
\[ \tilde{u} = u - \frac{\mu \alpha}{\gamma}. \] (33)
Let $\theta = \sin^{-1}(\alpha/\gamma)$, then (31) gives $\theta_x = 2u - \mu \sin \theta$. Therefore, (33) is just the same as the standard Darboux transformation for MKdV equation given by a $2 \times 2$ Darboux matrix, which is linear in the spectral parameter, although the $3 \times 3$ Darboux matrix here is quadratic in $\lambda$.

For the sine-Gordon equation, the $x$-part of the Lax pair is the same as that of the MKdV equation. Hence the Darboux transformation (33) is also the standard Darboux transformation for sine-Gordon equation.

4 Local isometric immersion of space forms with flat normal bundle and linearly independent curvature normals

[17] gives the Lax sets for the equations describing local isometric immersions with flat normal bundles and linearly independent curvature normals. Here we apply the above theorems to this problem and give the Darboux transformations of the position vectors.

First we list some useful conclusions in [17], and rewrite the structure equation to a form to which the Darboux transformation can be easily applied.

Let $M_n(K)$ be an $n$-dimensional space form of curvature $K$ where $K = 0, 1, -1$. For the local isometric immersion $M_n(K) \to M_{2n}(K)$ with flat normal bundle and linearly independent curvature normals, there always exist local coordinates $x = (x_1, \cdots, x_n)$ on $M_n(K)$ and the parallel orthonormal normal vector fields $(e_{n+1}, \cdots, e_{2n})$ such that the first and second fundamental forms are

\begin{align}
I &= \sum_{i,j=1}^{n} g_{ij} \, dx_i \, dx_j = \sum_{i=1}^{n} \rho_i^2(x) \, dx_i^2, \\
II &= \sum_{i,j,\alpha=1}^{n} \Omega_{ij}^\alpha \, dx_i \, dx_j \, e_{n+\alpha} = \sum_{i,j=1}^{n} \rho_i(x) \omega_{i\alpha}(x) \, dx_i^2 \, e_{n+\alpha}
\end{align}

where $\omega(x) = (\omega_{ij}(x)) \in O(n)$.

Denote

\[ R_n(K) = \begin{cases} 
\mathbb{R}^n, & \text{if } K = 0, 1, \\
\mathbb{R}^{n-1,1}, & \text{if } K = -1,
\end{cases} \]

(35)

where $\mathbb{R}^{n-1,1}$ has the metric $x_1^2 + \cdots + x_{n-1}^2 - x_n^2$.

Let $i_n(K) : M_n(K) \to R_{n+1}(K)$ be the standard imbedding given by

\begin{align}
M_n(0) &= \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\}, \\
M_n(1) &= \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 + x_{n+1}^2 = 1\}, \\
M_n(-1) &= \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1\}.
\end{align}

(36)

Then we can consider the immersion

\[ \bar{V} : M_n(K) \to M_{2n}(K) \xrightarrow{i_{2n}(K)} R_{2n+1}(K). \]

(37)
Clearly, $\vec{V} \in \mathbb{R}^{2n}$ for $K = 0$, $\vec{V} \cdot \vec{V} = K$ for $K = \pm 1$. Here “$\cdot$” refers to the inner product in $R_{2n+1}(K)$.

Remark 1 We imbed $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n+1}$ only for the unification of the three cases.

The structure equations for such immersions are

$$\partial_j \vec{V}_i = \Gamma_{ij}^k \vec{V}_k + \Omega_{ij}^\alpha \vec{n}_\alpha - Kg_{ij} \vec{V},$$
$$\partial_i \vec{n}_\alpha = -g^{lk} \Omega_{il}^\alpha \vec{V}_k,$$ \hspace{1cm} (38)

where $\vec{V}_i = \partial_i \vec{V}$, $\Gamma_{ij}^k$’s are the Christofel symbols corresponding to the metric $(g_{ij})$, $\partial_i = \partial/\partial x_i$.

From (34),

$$g_{ij} = \rho_i^2 \delta_{ij}, \quad \Omega_{ij}^\alpha = \rho_i \omega_i \delta_{ij}.$$ \hspace{1cm} (39)

Then (38) becomes

$$\partial_j \vec{V}_i = \frac{\partial_j \rho_i}{\rho_i} \vec{V}_i + \frac{\partial_i \rho_j}{\rho_j} \vec{V}_j, \quad (i \neq j),$$
$$\partial_i \vec{V}_i = -\sum_{k \neq i} \rho_i \partial_k \rho_i \vec{V}_k + \frac{\partial_i \rho_i}{\rho_i} \vec{V}_i + \rho_i \omega_i \vec{n}_\alpha - K \rho_i^2 \vec{V},$$
$$\partial_i \vec{n}_\alpha = -\omega_{i\alpha} \vec{V}_i.$$ \hspace{1cm} (40)

The Gauss-Codazzi equations are the integrability conditions for (40), which are

$$\partial_i \gamma_{ij} + \partial_j \gamma_{ji} = \sum_{l \neq i, l \neq j} \gamma_{il} \gamma_{lj}, \quad (i \neq j),$$
$$\partial_k \gamma_{ij} + \gamma_{ik} \gamma_{kj} = 0, \quad (i \neq j, i \neq k, j \neq k),$$
$$\partial_i \gamma_{ji} + \partial_j \gamma_{ij} - \sum_{l \neq i, l \neq j} \gamma_{il} \gamma_{lj} - K \rho_i \rho_j = 0, \quad (i \neq j),$$
$$\partial_j \rho_i + \gamma_{ij} \rho_j = 0, \quad (i \neq j),$$
$$\partial_j \omega_{i\alpha} + \gamma_{ij} \omega_{j\alpha} = 0, \quad (i \neq j),$$
$$\partial_i \omega_{i\alpha} = \sum_{k \neq i} \gamma_{ki} \omega_{k\alpha},$$ \hspace{1cm} (41)

where

$$\gamma_{ij} = \begin{cases} -\partial_j \rho_i / \rho_j, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$ \hspace{1cm} (43)

Let

$$\bar{p}_i = \sum_\alpha \omega_{i\alpha} \vec{n}_\alpha, \quad \bar{q}_i = \rho_i^{-1} \vec{V}_i, \quad \bar{r} = \bar{V},$$ \hspace{1cm} (44)
then (40) becomes
\[ \begin{align*}
\partial_i \vec{p}_i &= \sum_{k \neq i} \gamma_{ki} \vec{p}_k - \vec{q}_i, \\
\partial_j \vec{p}_i &= -\gamma_{ij} \vec{p}_j \quad (i \neq j), \\
\partial_i \vec{q}_i &= \sum_{k \neq i} \gamma_{ik} \vec{q}_k + \vec{p}_i - K \rho_i \vec{r}, \\
\partial_j \vec{q}_i &= -\gamma_{ji} \vec{q}_j \quad (i \neq j), \\
\partial_i \vec{r} &= \rho_i \vec{q}_i.
\end{align*} \]  

(45)

For simplicity, we write
\[ \vec{p} = \begin{pmatrix} \vec{p}_1 \\ \vdots \\ \vec{p}_n \end{pmatrix}, \quad \vec{q} = \begin{pmatrix} \vec{q}_1 \\ \vdots \\ \vec{q}_n \end{pmatrix}. \]

(46)

In [17], a spectral parameter was proposed so that (41) has a Lax set (generalized Lax pair)
\[ \begin{align*}
\partial_i u_i &= \sum_{k \neq i} \gamma_{ki} u_k - \lambda v_i, \\
\partial_j u_i &= -\gamma_{ij} u_j \quad (i \neq j), \\
\partial_i v_i &= \sum_{k \neq i} \gamma_{ik} v_k + \lambda u_i - K \rho_i w, \\
\partial_j v_i &= -\gamma_{ji} v_j \quad (i \neq j), \\
\partial_i w &= \rho_i v_i.
\end{align*} \]  

(47)

Note that (45) is a special case of (47) for \( \lambda = 1 \).

Let \( \Phi(\lambda) = (u_1, \ldots, u_n, v_1, \ldots, v_n, w)^T \), the above system of equations can be written in matrix form as
\[ \partial_t \Phi = (\lambda J_i + [J_i, P]) \Phi, \]

(48)

where
\[ J_i = \begin{pmatrix} 0 & -E_{ii} & 0 \\ E_{ii} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & \sum_{i \neq j} \gamma_{ij} E_{ij} & \sum_i \rho_i e_i^T \\ -\sum_{i \neq j} \gamma_{ij} E_{ij} & 0 & -K \sum_i \rho_i e_i \\ \sum_i \rho_i e_i^T & 0 & 0 \end{pmatrix}, \]

(49)

\[ [J_i, P] = \begin{pmatrix} \sum_{j \neq i} \gamma_{ijj} (E_{ij} - E_{jj}) & 0 & 0 \\ 0 & \sum_{j \neq i} \gamma_{ijj} (E_{ij} - E_{jj}) & -K \rho_i e_i \\ 0 & 0 & \rho_i e_i^T \end{pmatrix}. \]

Here \( E_{ij} \) is a constant matrix whose \((i, j)\)th entry is one and the rest entries are zero, \( e_i \) is a column matrix whose \(i\)th entry is one and the rest entries are zero.

If we write a \((2n + 1) \times (2n + 1)\) matrix \( M \) as a block matrix
\[ \begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{pmatrix} = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_1 \end{pmatrix} \]

(50)

then
\[(\rho_i) = P_{[3,1]}, \quad (\gamma_{ij}) = P_{[1,2]}, \quad (\omega_{ij}) = \Phi_{[1,1]}(0)L_\omega, \quad \begin{pmatrix} \vec{p} \\ \vec{q} \\ \vec{r} \end{pmatrix} = \Phi(1) \vec{L} \]  

(51)

where \( L_\omega \) is an \( n \times n \) constant matrix, \( \vec{L} \) is a \((2n + 1) \times 1\) constant matrix.

For the unification of three cases \( K = 0, \pm 1 \), we apply Theorem 2 to (48). Notice that (45) for \((\vec{p}_i, \vec{q}_i, \vec{r}_i)\) is a special case of (47) for \((u_i, v_i, w)\) when \( \lambda = 1 \). Moreover, \( \omega_{i\alpha}'s \) satisfy the same equations (in (42)) as \( u_i's \) do in (47) with \( \lambda = 0 \).

For \( K = 0, 1, -1 \), \( \lambda J_i + [J_i, P] \in \Sigma^o(\mathfrak{g}) \) which corresponds to \( C = \text{diag}(1, \ldots, 1, 1, -1, \ldots, -1) \), \( \sigma = \text{diag}(1, \ldots, 1, -1, \ldots, -1) \).

Hence

\[ C\sigma = \text{diag}(1, \ldots, 1, -1, \ldots, -1, -K) \]  

(53)

is always not semidefinite. Therefore, we can use Theorem 2.

Let \( \mu \) be a real number. Let \( H = (\xi_1, \ldots, \xi_n, \sqrt{-1} \eta_1, \ldots, \sqrt{-1} \eta_n, \sqrt{-1} \zeta) \) be a solution of (47) with \( \lambda = \sqrt{-1} \mu \) such that \( \sum_i \xi_i^2 = \sum_i \eta_i^2 + K \zeta^2 \). Then \( \tau H \) is real and \( H^*C\sigma H = 0 \). Written explicitly, the components of \( H \) satisfy

\[
\begin{align*}
\partial_i \xi_i &= \sum_{k \neq i} \gamma_{ki} \xi_k + \mu \eta_i, \\
\partial_j \xi_i &= -\gamma_{ij} \xi_j \quad (i \neq j), \\
\partial_i \eta_i &= \sum_{k \neq i} \gamma_{ik} \eta_k + \mu \xi_i - K \rho_i \xi, \\
\partial_j \eta_i &= -\gamma_{ji} \eta_j \quad (i \neq j), \\
\partial_i \zeta &= \rho_i \eta_i.
\end{align*}
\]

The Darboux matrix in Theorem 2 is

\[
G(\lambda) = \frac{1}{\lambda^2 + \mu^2} \begin{pmatrix}
\lambda^2 + \mu^2 & \frac{2\lambda \mu}{\Delta} & 0 & -\xi \eta^T & -K \zeta \xi \\
0 & \eta \xi^T & 0 & -K \zeta \xi & 0 \\
\frac{2\mu}{\Delta} & 0 & K \eta \zeta & 0 & 0 \\
0 & 0 & K \eta \zeta & K \zeta^2 & 0
\end{pmatrix}
\]

(55)

where \( \xi = (\xi_1, \ldots, \xi_n)^T, \eta = (\eta_1, \ldots, \eta_n)^T, \Delta = \sum_{l=1}^n \xi_l^2 \).

The corresponding geometric quantities are given by

\[
\tilde{P} = P - \frac{2\sqrt{-1} \mu}{H^*CH}[\sigma, HH^*]\sigma C, \quad \tilde{\omega} = G(0)_{[1,1]}\omega, \quad \omega = G(0)_{[1,1]}\omega,
\]

(56)
\[
(p_1, \ldots, p_n, \tilde{q}_1, \ldots, \tilde{q}_n, \tilde{r})^T = G(1)(p_1, \ldots, p_n, q_1, \ldots, q_n, r)^T.
\]

Written explicitly, they are

\[
\begin{align*}
\tilde{\rho}_i &= \rho_i - \frac{2\mu\zeta_i}{\Delta}, \\
\tilde{\gamma}_{ij} &= \gamma_{ij} + \frac{2\mu\xi_j}{\Delta} - \frac{Kb\zeta_i}{\Delta} r,
\end{align*}
\]

\[
\begin{align*}
\tilde{\omega}_{i\alpha} &= \omega_{i\alpha} - \sum_j 2\xi_i \xi_j \omega_{j\alpha}, \\
\tilde{\vec{p}}_i &= \vec{p}_i - \sum_j b\mu\xi_j \vec{p}_j - \sum_j b\xi_j \eta_j \vec{q}_j - \frac{Kb\zeta_i}{\Delta} \vec{r},
\end{align*}
\]

\[
\begin{align*}
\tilde{\vec{q}}_i &= \vec{q}_i + \sum_j b\eta_i \xi_j \vec{p}_j - \sum_j b\mu \eta_j \xi_j \vec{q}_j - \frac{Kb\mu\zeta_j}{\Delta} \vec{r},
\end{align*}
\]

\[
\tilde{\vec{r}} = \vec{r} + \sum_j b\zeta_j \vec{p}_j - \sum_j b\mu \zeta_j \vec{q}_j - \frac{Kb\mu\zeta_j^2}{\Delta} \vec{r},
\]

where \( b = \frac{2\mu}{1 + \mu^2} \).

Using (44) and (40), we have

\[
\tilde{\vec{V}} = \sum_j b\zeta_j \frac{\partial V}{\partial p_j} + \sum_j b\xi_j \frac{\partial V}{\partial \rho_j} - \xi_j \partial_j \rho_j + \sum_{k \neq j} \zeta_k \partial_j \rho_k - \mu \rho_j \eta_j \frac{\partial V}{\partial x_j} + \left( 1 - \frac{Kb\mu\zeta_j}{\Delta} \right) \vec{V}.
\]

Therefore, we have the following general procedure to get the local isometric immersion \( M_n(K) \to M_{2n}(K) \) with flat normal bundle and linearly independent curvature normals. Since we consider the linear system (47), the condition \( \rho_i \neq 0 \) can be removed temporarily, provided that \( \tilde{\rho}_i \neq 0 \) for derived submanifold.

1) Suppose we know a solution of (41) and can solve the linear system (47) to get the fundamental solution \( \Phi(x_1, \ldots, x_n, \lambda) \).

2) Let \( \lambda = \sqrt{-1} \mu \) and get a solution \( (\xi, \eta, \zeta) \) of (54) such that

\[
(\xi, \sqrt{-1} \eta, \sqrt{-1} \zeta)^T = \Phi(x_1, \ldots, x_n, \sqrt{-1} \mu) C
\]

where \( C \) is a constant matrix.

3) Using (58), one gets the position vector \( \tilde{\vec{r}} \) and the corresponding quantities. When \( \rho_i \neq 0 \), (59) gives a more direct answer. Moreover, the corresponding solution of (47) is \( \tilde{\Phi}(x_1, \ldots, x_n, \lambda) = G(x_1, \ldots, x_n, \lambda) \Phi(x_1, \ldots, x_n, \lambda) \).

These three steps give the new solution of (41), (45). When \( \tilde{\rho}_i \neq 0 \), the solution represents a real geometric immersion. For \( \tilde{\vec{r}} \), the corresponding solution of (47) is known. Hence, we can repeat step 2) and 3) to get another solution \( \tilde{\vec{r}} \). Continuing this process, a series of immersions are
obtained by an algebraic algorithm and one should solve a system of linear ODEs only once in step 1).

Now we give some examples. The simplest examples are the totally geodesic submanifolds. However, since their second fundamental forms are zero, the Darboux transformation here is not applicable (also, their curvature normals are linearly dependent). Therefore, we seek other solutions as seed solutions of Darboux transformation.

(1) Solutions derived from the trivial solutions

We take the seed solution of (41) as \( \rho_i = 0, \gamma_{ij} = 0, \omega_i = \delta_i \). Although this does not correspond to a real geometric object, and (59) is not valid, we can still get \((\tilde{\rho}, \tilde{q}, \tilde{r})\) from (58) and then get the nontrivial local immersion \( \tilde{r} \).

To get a solution, we first solve (45) to get \((\vec{p}, \vec{q}, \vec{r})\) and solve (54) to get \((\xi, \eta, \zeta)\). Then normalize the first fundamental form by changing coordinates \(\{x_i\}\). After that, the last equation of (58) gives the position vector of the immersion.

First, we solve (45) to get
\[
\vec{p}_i = \vec{E}_i \cos x_i + \vec{F}_i \sin x_i,
\vec{q}_i = \vec{E}_i \sin x_i - \vec{F}_i \cos x_i,
\vec{r} = \vec{R},
\]

where \(\vec{E}_i, \vec{F}_i, \vec{R} (i = 1, \cdots, n)\) are constant vectors in \(\mathbb{R}_{2n+1}(K)\) such that
\[
\vec{E}_i \cdot \vec{E}_j = \delta_{ij}, \quad \vec{E}_i \cdot \vec{F}_j = 0, \quad \vec{F}_i \cdot \vec{F}_j = \delta_{ij}, \quad \vec{E}_i \cdot \vec{R} = \vec{F}_i \cdot \vec{R} = 0,
\vec{R} \cdot \vec{R} = K \text{ (if } K = \pm 1).}

A) \(K = 0\).

The solution of (54) is
\[
\xi_i = A_i e^{\mu x_i}, \quad \eta_i = A_i e^{\mu x_i}, \quad \zeta = C
\]
where \(A_i, C\) are nonzero real constants.

From (58),
\[
\tilde{\rho}_i = \frac{2CA_i e^{\mu x_i}}{\sum_k A_k^2 e^{2\mu x_k}}.
\]

Let
\[
\tilde{\mu}_i = \frac{2CA_i e^{\mu x_i}}{\sum_k A_k^2 e^{2\mu x_k}},
\]

then
\[
\tilde{l} = \sum_i \tilde{\rho}_i^2 dx_i^2 = \sum_i \tilde{\mu}_i^2 dz_i^2.
\]
Choose $\vec{R} = 0$, then
\[
\vec{r} = \sum_i \frac{\mu}{1 + \mu^2} z_i (\vec{E}_i \cos x_i + \vec{F}_i \sin x_i) - \sum_i \frac{\mu^2}{1 + \mu^2} z_i (\vec{E}_i \sin x_i - \vec{F}_i \cos x_i)
\]
\[
= \sum_i \frac{\mu z_i}{\sqrt{1 + \mu^2}} (\vec{E}_i \cos \bar{x}_i + \vec{F}_i \sin \bar{x}_i)
\]
(67)

where
\[
\bar{x}_i = x_i + \tan^{-1} \frac{1}{\mu} \ln \left( \frac{2C z_i}{A_i \sum_k z_k} \right) + \tan^{-1} \mu.
\]
(68)

This local immersion is only defined for $(z_1, \cdots, z_n)$ with $A_i C z_i > 0$.

For $n = 2$ and $A_i = 1, C > 0$, let $z_1 = r \cos \phi, z_2 = r \sin \phi$,
\[
\tilde{k}_1 = \frac{\vec{E}_1 + \mu \vec{F}_1}{\sqrt{1 + \mu^2}}, \quad \tilde{k}_2 = \frac{\vec{F}_1 - \mu \vec{E}_1}{\sqrt{1 + \mu^2}}.
\]
\[
\tilde{k}_3 = \frac{\vec{E}_2 + \mu \vec{F}_2}{\sqrt{1 + \mu^2}}, \quad \tilde{k}_4 = \frac{\vec{F}_2 - \mu \vec{E}_2}{\sqrt{1 + \mu^2}}.
\]

then $\{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4\}$ is also an orthonormal frame of $\mathbb{R}^4$, and
\[
\vec{r} = \frac{\mu}{\sqrt{1 + \mu^2}} \left( r \cos \phi \cos \left( \frac{1}{\mu} \ln \left( \frac{2C}{r} \cos \phi \right) \right) \tilde{k}_1 + r \cos \phi \sin \left( \frac{1}{\mu} \ln \left( \frac{2C}{r} \cos \phi \right) \right) \tilde{k}_2
\]
\[
+ r \sin \phi \cos \left( \frac{1}{\mu} \ln \left( \frac{2C}{r} \sin \phi \right) \right) \tilde{k}_3 + r \sin \phi \sin \left( \frac{1}{\mu} \ln \left( \frac{2C}{r} \sin \phi \right) \right) \tilde{k}_4 \right) \quad (69)
\]

$r > 0, 0 < \phi < \frac{\pi}{2}$

Written in terms of the coordinates $(x_1, x_2)$,
\[
\vec{r} = \frac{2\mu C}{\sqrt{1 + \mu^2}} \left( 1 + e^{2\mu x_1} \cos x_1 \tilde{k}_1 + e^{2\mu x_1} \sin x_1 \tilde{k}_2
\]
\[
+ e^{2\mu x_2} \cos x_2 \tilde{k}_3 + e^{2\mu x_2} \sin x_2 \tilde{k}_4 \right).
\]
(70)

Remark 2 When $\mu C$ is finite and $\mu \to 0$, except for $(0, 0, 0, 0)$, each point on the submanifold (70) tends to the standard torus in $\mathbb{R}^4$.

\[
\vec{r} = \mu C (\cos x_1 \tilde{k}_1 + \sin x_1 \tilde{k}_2 + \cos x_2 \tilde{k}_3 + \sin x_2 \tilde{k}_4).
\]
(71)

The coefficient is $\mu C$ because the metric on the surface is
\[
I = \mu^2 C^2 \sum_{i=1}^2 dx_i^2.
\]

When $C$ is finite and $\mu \to \infty$, the pointwise limit of the submanifold given by (69) is
\[
\vec{r} = r \cos \phi \tilde{k}_1 + r \sin \phi \tilde{k}_3,
\]
(72)

which is a plain in $\mathbb{R}^4$.
Remark 3 In the coordinate $\{z_i\}$, (45) or (40) is no longer satisfied, because those equations depend on the special coordinate $\{x_i\}$.

The following figures show the manifold with $\mu = -0.2$, $C = 1$. In Fig. 1, the three axes are $(r_1, r_2, r_3)$ if $\vec{r}$ is written as $\vec{r} = \sum_{j=1}^{4} r_j \vec{k}_j$. Fig. 2 is the corresponding contour plot of Fig. 1 on $(r_1, r_2)$ plane. In Fig. 3, the axes are $(r_1, r_2, \pm \sqrt{r_3^2 + r_4^2})$. The dark spiral in Fig. 2 represents the boundary of the manifold.
B) $K = 1$.

The solution of (54) is

$$\xi_i = A_i e^{\mu x_i} + B_i e^{-\mu x_i}, \quad \eta_i = A_i e^{\mu x_i} - B_i e^{-\mu x_i}, \quad \zeta = C$$

(73)

where $A_i, B_i, C$ are nonzero real constants with $C^2 = 4 \sum_i A_i B_i$.

The corresponding

$$\bar{\rho}_i = -\frac{2\mu C(A_i e^{\mu x_i} + B_i e^{-\mu x_i})}{\sum_k (A_k e^{\mu x_k} + B_k e^{-\mu x_k})^2},$$

(74)

Let

$$z_i = \frac{A_i e^{\mu x_i} - B_i e^{-\mu x_i}}{C},$$

(75)

then

$$\bar{I} = \sum_i \bar{\rho}_i^2 dx_i^2 = \frac{4 \sum_i dz_i^2}{(\sum_k z_k^2 + 1)^2},$$

(76)

$$\bar{r} = \left(1 - \frac{b\mu}{\sum_k z_k^2 + 1}\right) \bar{R} + \frac{b\sqrt{z_i^2 + C_i}}{\sum_k z_k^2 + 1} (\bar{E}_i \cos x_i + \bar{F}_i \sin x_i)$$

$$- \frac{b\mu z_i}{\sum_k z_k^2 + 1} (\bar{E}_i \sin x_i - \bar{F}_i \cos x_i)$$

(77)

$$= \left(1 - \frac{2\mu^2}{1 + \mu^2 \sum_k z_k^2 + 1}\right) \bar{R} + \frac{2\mu \sqrt{(1 + \mu^2)z_i^2 + C_i}}{(1 + \mu^2)(\sum_k z_k^2 + 1)} (\bar{E}_i \cos \bar{x}_i + \bar{F}_i \sin \bar{x}_i)$$
with $C_i = 4A_iB_i/C^2$ (therefore $\sum_i C_i = 1$),

$$\bar{x}_i = \frac{1}{\mu} \ln \left( \frac{C}{2A_i} (z_i + \sqrt{z_i^2 + C_i}) \right) + \tan^{-1} \frac{\mu z_i}{\sqrt{z_i^2 + C_i}}. \quad (78)$$

It is clear that this immersion is smooth for $|z| < \infty$. It can be verified that at $|z| = \infty$ (by changing coordinates) it is not smooth.

C) $K = -1$.

The solution of (54) is

$$\xi_i = A_i e^{\mu x_i} - B_i e^{-\mu x_i}, \quad \eta_i = A_i e^{\mu x_i} + B_i e^{-\mu x_i}, \quad \zeta = C \quad (79)$$

where $A_i, B_i, C$ are nonzero real constants with $C^2 = 4 \sum_i A_iB_i$. Then

$$\tilde{\rho}_i = -2 \mu C(A_i e^{\mu x_i} - B_i e^{-\mu x_i}) \sum_k (A_k e^{\mu x_k} - B_k e^{-\mu x_k})^2. \quad (80)$$

Let

$$z_i = \frac{C(A_i e^{\mu x_i} + B_i e^{-\mu x_i})}{\sum_k (A_k e^{\mu x_k} + B_k e^{-\mu x_k})^2}, \quad (81)$$

then

$$\tilde{I} = \sum_i \rho_i^2 dx_i^2 = \frac{4 \sum_i dz_i^2}{(1 - \sum_k z_k^2)^2}, \quad (82)$$

$$\tilde{r} = \left( 1 + \frac{2\mu^2}{1 + \mu^2 \frac{\sum_i z_i^2}{1 - \sum_k z_k^2}} \right) \tilde{R} + \frac{2\mu \sqrt{(1 + \mu^2) z_i^2 - C_i (\sum_k z_k^2)^2}}{(1 + \mu^2)(1 - \sum_k z_k^2)} (\tilde{E}_i \cos \tilde{x}_i + \tilde{F}_i \sin \tilde{x}_i) \quad (83)$$

with $C_i = 4A_iB_i/C^2$ (also $\sum_i C_i = 1$),

$$\bar{x}_i = \frac{1}{\mu} \ln \left( \frac{C}{2A_i} (z_i + \sqrt{z_i^2 + C_i}) \right) + \tan^{-1} \frac{\mu z_i}{\sqrt{z_i^2 + C_i}}. \quad (84)$$

Now $M_n(-1)$ is the disk $\sum_k z_k^2 < 1$, but the immersion (83) can only be defined in a region of $M_n(-1)$. For example, when all $A_i > 0, B_i > 0$, it is defined in $C_i (\sum_k z_k^2)^2 < z_i^2$, which contains the center $(0,0,\cdots,0)$ of the disk.

(2) Solutions derived from the torus $T^n \to \mathbb{R}^{2n}$

Let

$$\bar{r} = \sum_{i=1}^n \left( \cos x_i \bar{E}_i + \sin x_i \bar{F}_i \right), \quad (85)$$

where $\bar{E}_i, \bar{F}_i$ are constant vectors satisfying

$$\bar{E}_i \cdot \bar{E}_j = \delta_{ij}, \quad \bar{E}_i \cdot \bar{F}_j = 0, \quad \bar{F}_i \cdot \bar{F}_j = \delta_{ij}. \quad (86)$$
This is the standard torus $T^n$ in $\mathbb{R}^{2n}$, whose $g_{ij} = \delta_{ij}$, $\omega_{ij\alpha} = \delta_{i\alpha}$, and the corresponding

$$\vec{p}_i = -\cos x_i \vec{E}_i - \sin x_i \vec{F}_i, \quad \vec{q}_i = -\sin x_i \vec{E}_i + \cos x_i \vec{F}_i.$$  \hfill \(87\)

Solving (54) with $\sum_i \xi_i^2 = \sum_i \eta_i^2$, we have

$$\xi_i = A_i e^{\mu x_i}, \quad \eta_i = A_i e^{\mu x_i}, \quad \zeta = \sum_{k=1}^{n} \frac{A_k}{\mu} e^{\mu x_k} + C,$$  \hfill \(88\)

where $A_i, C$ are real constants. Then (58) gives

$$\tilde{r} = \sum_{i=1}^{n} (\cos x_i \vec{E}_i + \sin x_i \vec{F}_i)$$

$$- \frac{b}{\Delta} \left( \sum_{k=1}^{n} \frac{A_k}{\mu} e^{\mu x_k} + C \right) \sum_{i=1}^{n} A_i e^{\mu x_i} \left( (\cos x_i - \mu \sin x_i) \vec{E}_i + (\sin x_i + \mu \cos x_i) \vec{F}_i \right),$$  \hfill \(89\)

$$\tilde{\rho}_i = 1 - \frac{2A_i e^{\mu x_i}}{\Delta} \left( \sum_{k=1}^{n} A_k e^{\mu x_k} + \mu C \right),$$  \hfill \(90\)

where

$$\Delta = \sum_{k} A_k^2 e^{2\mu x_k}, \quad b = \frac{2\mu}{1 + \mu^2}.$$  \hfill \(91\)

This submanifold is defined in the region where $\tilde{\rho}_i \neq 0$ for all $1 \leq i \leq n$.

By the local change of coordinates

$$z_i = x_i - \frac{2A_i e^{\mu x_i}}{\mu \Delta} \left( \sum_{k=1}^{n} A_k e^{\mu x_k} + \mu C \right),$$  \hfill \(92\)

the metric is changed to

$$\tilde{I} = \sum_{i=1}^{n} dz_i^2.$$  

Fig. 4 shows this submanifold (surface) for $n = 2, A_i = 1, C = 1, \mu = -0.2$. Write $\vec{r} = r_1 \vec{E}_1 + r_2 \vec{F}_1 + r_3 \vec{E}_2 + r_4 \vec{F}_2$, then the coordinates are taken as $(2+r_1)r_3, (2+r_1)r_4, r_2)$.

Fig. 5 is the corresponding figure for the standard torus.
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References

[1] M. J. Ablowitz, R. Beals and K. Tenenblat, *On the solution of the generalized wave and generalized sine-Gordon equation*, Stud. in Appl. Math. 74 (1986), 177.

[2] J. L. Barbosa, W. Ferreira and K. Tenenblat, *Submanifolds of constant sectional curvature in pseudo-Riemannian manifolds*, Ann. Global Anal. and Geo. 14 (1996), 381–401.

[3] R. Beals and K. Tenenblat, *An intrinsic generalization for the wave and sine-Gordon equation*, Differential Geometry, Pitman Monograph 52, Longman, Harlow (1991), 25–46.

[4] Jan Cieśliński, *An algebraic method to construct the Darboux matrix*, Jour. Math. Phys. 36 (1995), 5670–5706.

[5] M. Dajczer and R. Tojeiro, *Submanifolds with nonparallel first normal bundle*, Canad. Math. Bull. 37 (1994), 330–337.

[6] M. Dajczer and R. Tojeiro, *Isometric immersions and the generalized Laplace and elliptic sinh-Gordon equations*, J. Reine Angew. Math. 467 (1995), 109–147.

[7] V. S. Gerdjikov, *The Zakharov-Shabat dressing method and the representation theory of the semisimple Lie algebras*, Phys. Lett. A126 (1987), 184–188.

[8] C. H. Gu, *On the Darboux form of Bäcklund transformations*, in Integrable System, Nankai Lectures on Math. Phys., World Scientific Publishing Company, Singapore (1989), 162–168.

[9] C. H. Gu, *On the interaction of solitons for a class of integrable systems in the space-time $\mathbb{R}^{n+1}$*, Lett. Math. Phys. 26 (1992), 199–209.

[10] C. H. Gu and H. S. Hu, *Explicit solutions to the intrinsic generalization for the wave and sine-Gordon equations*, Lett. Math. Phys 29 (1993), 1–11.

[11] C. H. Gu and Z. X. Zhou, *Explicit form of Bäcklund transformations for $GL(N), U(N)$ and $O(2N)$ principal chiral fields*, Nonlinear evolution equations: integrability and spectral methods, Manchester University Press, Manchester (1990), 115–123.

[12] C. H. Gu and Z. X. Zhou, *On Darboux transformations for soliton equations in high dimensional space-time*, Lett. Math. Phys. 32 (1994), 1–10.

[13] V. B. Matveev & M. A. Salle, Darboux transformations and solitons, Springer-Verlag, Heidelberg (1991).

[14] S. Novikov, S. V. Manakov, L. P. Pitaevskii, V. E. Zakharov, Theory of Solitons, Consultants Bureau, New York (1984).
[15] D. H. Sattinger and V. D. Zurkowski, *Gauge theory of Bäcklund transformations II*, Physica 26D (1987), 225-250.

[16] K. Tenenblat, *Bäcklund’s theorem for submanifolds of space forms and a generalized wave equation*, Boll. Soc. Brasil. Mat. 16 (1985), 67–92.

[17] C. L. Terng, *Soliton equations, Kac-Moody algebras and differential geometry*, Jour. Diff. Geo. 45 (1996), 407–445.

[18] V. E. Zakharov and A. V. Mikhailov, *On the integrability of classical spinor models in two-dimensional space-time*, Comm. Math. Phys. 74 (1980), 21–40.

[19] Z. X. Zhou, *Soliton solutions for some equations in 1+2 dimensional hyperbolic su(N) AKNS system*, Inverse Problems 12 (1996), 89–109.