SEMISTABLE MINIMAL MODELS OF THREEFOLDS
IN POSITIVE OR MIXED CHARACTERISTIC

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ABSTRACT. We extend the minimal model theorem to the 3-dimensional schemes which are projective and have semistable reduction over the spectrum of a Dedekind ring.

0. Introduction

The classification theory of algebraic surfaces is characteristic free (cf. [Mu]), and it also holds for log surfaces, i.e., pairs consisting of surfaces and divisors (cf. [K1], [F], [TM]). But the minimal model theory of algebraic 3-folds is only completed over the field of characteristic 0, because the vanishing theorem of Kodaira type is necessary in the course of the proof (cf. [KMM], see also [Ko]). The purpose of this paper is to extend the minimal model theory to the positive or mixed characteristic case for 3-dimensional schemes which have semistable reduction over the spectrum of a Dedekind ring. We refer to [KMM] for the terminology of the minimal model theory.

Because our 3-fold is fibered by a family of surfaces, we can prove the Cone Theorem (Theorem 1.3) easily by using the results on log surfaces. It says that there exists an extremal ray if the given 3-fold is not minimal relatively over the base. Since the vanishing theorem of Kodaira type is false in positive characteristic, we cannot prove the Base Point Free Theorem in general. But using certain weaker vanishing theorems (Lemmas 2.1 and 2.2), we prove the Contraction Theorem (Theorem 2.3): there exists a contraction morphism associated to an extremal ray.

There are several things about which we should be careful in the positive or mixed characteristic case. Because we do not have the Grauert-Riemenschneider vanishing theorem, we do not know whether terminal singularities are always Cohen Macauley. So we put a technical assumption (6) at (1.1) instead. This condition is automatically satisfied in the case of characteristic 0, but we do not know whether it is true or not in the general case. We shall prove that this assumption is preserved under the birational transformations in the process toward the minimal models, i.e., the divisorial contractions and flips (Theorems 5.3 and 5.5).

In order to prove the Flip Theorem in §5, we classify the terminal singularities which appear inevitably in the course of the minimal model program in §§3-4. Here we may assume that the base ring is a complete discrete valuation ring whose residue...
field is algebraically closed. The canonical cover is an inseparable morphism if its degree is divisible by the characteristic, and the global canonical covering space may have non-normal points, e.g., for a quasi-hyperelliptic surface (cf. [BM]). But we can prove similar statements for a local canonical cover as in the case of characteristic 0 by using a general position argument (§3). Then the singularities of the canonical covers are classified by using the assumption (1.1) (§4). Here we can use the method of the toric geometry (cf. [KKMS]), because it concerns the set of monomials, and independent of the characteristic.

The termination of the sequence of flips (Theorem 5.2) is proved by looking at the minimal discrepancy coefficients of exceptional divisors (cf. [K4]). Then by induction on the maximal index of the relative canonical divisor, we prove the Flip Theorem (Theorem 5.5): the flip exists for a small contraction. This proof is new and simple compared with the previously known proofs when it is applied to the case of characteristic 0 (cf. [T], [K2], [S], [Mo2]). But the additional condition (6) of (1.1) made our proof longer.

As in [KMM], the combination of the Cone, the Contraction and the Flip Theorems yields the main theorem of this paper, the Minimal Model Theorem (Theorem 5.7): the given morphism has a birational model which is minimal or has a Mori fiber space structure.

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1. Cone theorem

Let $\Delta = \text{Spec } A$ be the spectrum of a Dedekind ring, and let $f : X \to \Delta$ be a flat quasi-projective morphism from a normal scheme $X$ of dimension 3, or a morphism obtained by a completion from such a morphism. In this paper, $f$ is said to be semistable if the following conditions are satisfied:

1. the generic fiber $X_\eta$ is smooth,
2. $X$ is regular, and the fibers $X_s$ for the closed points $s \in \Delta$ are geometrically reduced normal crossing divisors,
3. the irreducible components of any closed fiber $X_s$ are geometrically irreducible and smooth.

The purpose of this paper is to construct a minimal model for the given semistable and projective morphism $f : X \to \Delta$.

The relative canonical sheaf $\omega_{X/\Delta} = H^{-2}(\omega^\bullet_{X/\Delta})$ is a coherent reflexive sheaf of rank 1. The corresponding Weil divisor $K_{X/\Delta}$ is called the relative canonical divisor.

The construction of a minimal model is inductive, and we shall deal with the category of morphisms given below. Note that a semistable morphism is in this category as well as its minimal model, but the latter is not necessarily semistable.

1.1. Assumption. Throughout this paper, we assume the following conditions:

1. the generic fiber $X_\eta$ is smooth,
2. the fibers $X_s$ for the closed points $s \in \Delta$ are geometrically reduced and satisfy the condition $(S_2)$,
3. the irreducible components $S_i$ of any closed fiber $X_s$ are geometrically irreducible and geometrically normal $\mathcal{O}$-Cartier divisors on $X$. 

(4) there exists a finite set \( \Sigma \) of closed points on \( X \) such that \( X \setminus \Sigma \) is regular, and \( X \) has only terminal singularities at \( \Sigma \), i.e., (i) the relative canonical divisor \( K_{X/\Delta} \) is \( \mathbb{Q} \)-Cartier, and (ii) for any birational quasi-projective morphism \( \mu : X' \to X \) from a normal scheme \( X' \) and any prime divisor \( E \) on \( X' \) such that \( \mu(E) \subset \Sigma \), we have
\[
v_E(K_{X'/\Delta} - \mu^*K_{X/\Delta}) > 0
\]
where \( v_E \) denotes the discrete valuation at the generic point of \( E \),

(5) any closed fiber \( X_s \) is a normal crossing divisor on \( X \setminus \Sigma \), and the pair \((X, X_s)\) has only log terminal singularities at \( \Sigma \), i.e., for any \( \mu : X' \to X \) and \( E \) as in (4), we have
\[
v_E(K_{X'/\Delta} - \mu^*(K_{X/\Delta} + X_s)) > -1.
\]

From §3 on, we shall assume the additional condition:

(6) the \( \omega_{X/\Delta}^{[m]} = O_X(mK_{X/\Delta}) \) satisfy the condition \((S_3)\) for all \( m \in \mathbb{Z} \).

\( f : X \to \Delta \) is said to be minimal if \( f \) is projective and \( K_{X/\Delta} \) is \( f \)-nef. The condition (4) actually follows from (5), since \( X_s \) is a Cartier divisor. By the adjunction of the log canonical divisors, we deduce from (3) and (5) that the pairs \((S_i, D_i)\) for
\[
D_i = \sum_{j \neq i} S_j |_{S_i}
\]
have only log terminal singularities at \( \Sigma \). We note that we excluded the pinch point and its quotients from the singularities of \( X_s \) by the condition (3). The minimal resolutions of the log terminal singularities of surfaces are classified by [TM]. In characteristic 0, the condition (6) follows from the condition (4), since the terminal singularities are Cohen-Macauley in this case.

1.2. Definition. In this paper, a curve is an irreducible and reduced closed subscheme \( C_j \) of a fiber \( X_s \) for \( s \in \Delta \) which has dimension 1 over the residue field \( \kappa(s) = A_s/m_s \). As in [KMM], when \( f \) is projective, we define
\[
N^1(X/\Delta) = \left\{ \text{Cartier divisor } L \text{ on } X \right\} / \sim_{\text{num}/\Delta} \otimes \mathbb{R}
\]
\[
N_1(X/\Delta) = \left\{ \text{1-cycle } \sum d_j C_j \text{ in the fibers of } f \right\} / \sim_{\text{num}/\Delta} \otimes \mathbb{R}
\]
where \( \sim_{\text{num}/\Delta} \) denotes the numerical equivalence over \( \Delta \). A \( \mathbb{Q} \)-Cartier divisor \( L \) gives a linear functional \( h_L \) on \( N_1(X/\Delta) \) defined by \( h_L(z) = (L \cdot z) \). We also define the Hironaka-Kleiman-Mori cone (or the cone of curves) \( \overline{NE}(X/\Delta) \) to be the closed convex cone in \( N_1(X/\Delta) \) generated by the numerical classes of curves.

1.3. Cone Theorem. Under the assumption (1.1), assume moreover that \( f \) is projective. Then \( \overline{NE}(X/\Delta) \) is locally polyhedral in the half space defined by \( h_{K_{X/\Delta}}(z) < 0 \). Moreover, any extremal ray in this half space is generated by a class of a curve on a fiber of \( f \).

Proof. We have \( K_{X/\Delta}|_{S_i} = K_{S_i} + D_i \) for any irreducible component \( S_i \) of any fiber \( f \). By [Mo1] and [TM], the Hironaka-Kleiman-Mori cone \( \overline{NE}(S_i) \subset N_1(S_i) \).
is locally polyhedral in the half spaces $h_{K_{S_i} + D_i}(z) < 0$, and any extremal ray is generated by a curve on $S_i$.

The images of the $\text{NE}(S_i)$ by the surjective homomorphism

$$\bigoplus_i N_1(S_i) \to N_1(X/\Delta)$$

generate $\text{NE}(X/\Delta)$ as a closed convex cone.

Let $X_s$ be a smooth closed fiber and $C$ a curve on $X_s$ which generates an extremal ray of $\text{NE}(X_s)$. We shall prove that the class of $C$ in $N_1(X/\Delta)$ is contained in the image of $\text{NE}(X_\eta)$. Let $\mathcal{H}$ be an irreducible component of the relative Hilbert scheme of $f : X \to \Delta$ which contains the point $[C]$ corresponding to $C$. Let $\bar{k}$ be an algebraic closure of the residue field $k = \kappa(s)$, and $\bar{A}$ a complete discrete valuation ring which dominates $A_s$ and whose residue field is $\bar{k}$. The base change to $\bar{k}$ gives a closed subscheme $\bar{C}$ of $X_\bar{s}$.

By [Mo1, 2.7], (i) $\bar{C}$ is a disjoint union of $(-1)$-curves, (ii) $\bar{C}$ is a union of fibers of a conic bundle, or (iii) $\rho(X_s) = 1$. By an infinitesimal calculus, the geometric point $[\bar{C}] \in \mathcal{H}(\bar{k})$ corresponding to $\bar{C}$ can be extended to a geometric point $\bar{C} \in \mathcal{H}(\bar{A})$ in the cases (i) and (ii). In the case (iii), we have the same assertion if we replace $C$ by a hyperplane section of $X_s$ belonging to an $f$-very ample linear system. By the universality of $\mathcal{H}$, we conclude that there exists a closed subscheme $C_\eta$ of $X_\eta$ which has the same image as $C$ in $N_1(X/\Delta)$.

Since there are only a finite number of non-smooth fibers of $f$, we have the theorem. Q.E.D.

We can prove a relative version of the cone theorem, where $N_1(X/B)$, etc. are defined similarly as in Definition 1.2 (cf. [KMM]).

1.3’. Theorem. Let $B$ be a scheme with a flat and quasi-projective morphism $g : B \to \Delta$, or one which is obtained by a completion from such a scheme, and $f' : X \to B$ a projective morphism such that $f = g \circ f'$ satisfies (1.1). Then the closed convex cone $\text{NE}(X/B) \subset N_1(X/B)$ is locally polyhedral in the half space defined by $h_{K_{X/B}}(z) < 0$. Moreover, any extremal ray in this half space is generated by a class of a curve on a fiber of $f'$.

2. Contraction theorem

2.1. Lemma. Let $(S, D)$ be a pair of a normal surface over an algebraically closed field $k$ and a $\mathbb{Q}$-divisor which has only log terminal singularities (thus $\lfloor D \rfloor = 0$), $L$ a Cartier divisor on $S$, and let $f : S \to B$ be a projective surjective morphism whose fibers are at most 1 dimensional. Assume that $L - (K_S + D)$ is nef and $f$-big (cf. [KMM]). Then $R^1 f_* \mathcal{O}_S(L) = 0$.

Proof. Let $g : S' \to S$ be the minimal resolution. We define $K_{S'} + D' = g^*(K_S + D)$ and $L' = g^*L$. Then the pair $(S', D')$ and $L'$ satisfy the conditions of the lemma with respect to $g$ and $f \circ g$. Therefore, we may assume that $S$ is nonsingular. We may also assume that $L - (K_S + D)$ is $f$-ample.

Let $E = \sum e_i E_i$ be a nonzero effective divisor on $S$ whose irreducible components $E_i$ are contained in a fiber of $f$. Suppose that $H^1(E, \mathcal{O}_E(L)) \neq 0$. Then by Serre duality, there is a nonzero element $s \in H^0(E, \mathcal{O}_E(K_S + E - L))$. If we decrease the $e_i$, we may assume that $s$ generates $\mathcal{O}_E(K_S + E - L)$ at the generic points of the $E_i$. Then $((K_S + E - L) \cdot E_i) \geq 0$, hence $((E - D) \cdot E_i) > 0$, a contradiction. Q.E.D.
2.2. Lemma. Let $(S, D)$ be a pair of a normal projective surface over an algebraically closed field $k$ and a $\mathbb{Q}$-divisor which has only log terminal singularities. Assume that $-(K_S + D)$ is ample. Then $H^1(S, \mathcal{O}_S) = 0$.

Proof. We shall prove that $S$ is a rational surface. Let $g : S' \to S$ be a minimal resolution. Since $(S, D)$ has only log terminal singularities, there exists a $\mathbb{Q}$-divisor $D'$ on $S'$ such that

1. $K_{S'} + D' - g^*(K_S + D)$ is an effective divisor supported on the exceptional locus of $g$,
2. $(S', D')$ has only log terminal singularities,
3. $-(K_{S'} + D')$ is ample.

Since $|mK_{S'}| = \emptyset$ for all $m > 0$, $S'$ is birationally equivalent to a ruled surface. By (3), $\text{NE}(S')$ is spanned by extremal rays, which are generated by rational curves. Hence the genus of the base curve of the ruling is 0. Q.E.D.

2.3. Contraction Theorem. Under the assumption of Theorem 1.3, let $R$ be an extremal ray of $\text{NE}(X/\Delta)$ in the half space $h_{K_X/\Delta}(z) < 0$. Then there exists a projective surjective morphism $\varphi : X \to Z$ to a normal scheme $Z$ which is projective over $\Delta$ such that $\varphi_* \mathcal{O}_X = \mathcal{O}_Z$ and that any curve $C$ on any fiber of $f$ is mapped to a point by $\varphi$ if and only if $\text{cl}(C) \in R$.

Proof. We shall use the contraction theorem of log surfaces (cf. [TM]). By definition of an extremal ray, there exists a line bundle $L$ on $X$ such that $(L \cdot C) \geq 0$ for any curve $C$ on any fiber of $f$, and $(L \cdot C) = 0$ if and only if $\text{cl}(C) \in R$. We shall prove that the linear system $|mL|$ is relatively free over $\Delta$ for a positive integer $m$, i.e., the canonical homomorphism $f^* f_* \mathcal{O}_X (mL) \to \mathcal{O}_X (mL)$ is surjective.

Let us fix a closed point $s$ of $\Delta$. Let $\bar{k}$ be an algebraic closure of the residue field $k = \kappa(s)$. $X_{\bar{s}}$ the corresponding geometric fiber, $\bar{S}_i$ the irreducible components of $X_{\bar{s}}$, and $\bar{D}_i$ the divisors on the $\bar{S}_i$ obtained from the $D_i$ by the base change. As in the proof of Theorem 1.3, let $\hat{A}$ be a complete discrete valuation ring which dominates $\bar{A}_s$ and whose residue field $\hat{A}/\mathfrak{m}_{\bar{s}} = \bar{k}$. Moreover, we assume that $\mathfrak{m}_{\bar{s}} = \mathfrak{m}_s \hat{A}$. Let $\hat{f} : \hat{X} \to \hat{A}$ be the morphism obtained by the base change $\bar{A}_s \to \hat{A}$. By the flat base change theorem, it is sufficient to prove that $\hat{f}^* \hat{f}_* \mathcal{O}_{\hat{X}} (m\hat{L}) \to \mathcal{O}_{\hat{X}} (m\hat{L})$ is surjective.

Let $C_{\bar{s}}$ be set of the curves on $X_{\bar{s}}$ which are irreducible components of the 1-dimensional closed subschemes obtained by the base change from the curves $C$ on $X_s$ such that $\text{cl}(C) \in R$. Then the curves in $C_{\bar{s}}$ generates an extremal face $R_{\bar{s}}$ of $\text{NE}(X_{\bar{s}})$. Let $E_{\bar{s}}$ be the union of these curves.

There are four cases:

1. there are only a finite number of curves in $C_{\bar{s}}$,
2. $E_{\bar{s}}$ contains an irreducible component $\bar{S}_i$ of $X_{\bar{s}}$ but $E_{\bar{s}} \neq X_{\bar{s}}$,
3. $E_{\bar{s}} = X_{\bar{s}}$ but $\text{cl}(\bar{C}) \notin R_{\bar{s}}$ for some curve $\bar{C}$ on $X_{\bar{s}}$,
4. $\text{cl}(\bar{C}) \in R_{\bar{s}}$ for all $\bar{C}$ on $X_{\bar{s}}$.

Case (1). Let $\bar{C}$ be a curve $C_{\bar{s}}$. If $\bar{C}$ is contained in two irreducible components $\bar{S}_i$ and $\bar{S}_j$ of $X_{\bar{s}}$, then $(\bar{S}_i \cdot C) < 0$ and $(\bar{S}_j \cdot C) < 0$, because $\bar{C}$ is contractible in the normal surfaces $\bar{S}_i$ and $\bar{S}_j$. Hence $\bar{C}' \subset \bar{S}_i \cap \bar{S}_j$ for any $\bar{C}' \in C_{\bar{s}}$. Here we note that we used the condition (3) of (1.1). If $\bar{C} \subset \bar{S}_i$, $\bar{C} \not\subset \bar{D}_i$ and $\bar{C} \cap \bar{D}_i \neq \emptyset$, then $(\bar{S}_i \cdot C) < 0$, hence $\bar{C}' \subset \bar{S}_i$ for any $\bar{C}' \in C_{\bar{s}}$. Finally, if $\bar{C} \subset \bar{S}_i$ and $\bar{C} \cap \bar{D}_i = \emptyset$, then $\bar{C}' \subset \bar{S}_i$, in any of the above cases, one
can prove that there exist a projective scheme $Z_s$ of dimension 2 which satisfies the condition $(S_2)$ and a birational morphism $\varphi_s : X_s \to Z_s$ whose exceptional locus coincides with $E_s$.

By Lemma 2.1, one can easily check that $R^1\varphi_s^*O_{X_s} = 0$, and there exists an ample line bundle $H_s$ on $Z_s$ such that $L_s = \varphi_s^*H_s$ (cf. [A1]). Therefore, the linear system $|mL_s|$ is free and $H^1(X_s, O_{X_s}(mL_s)) = 0$ for a positive integer $m$ by the Serre vanishing theorem, hence our assertion.

If we choose $m$ large enough, then the image $Z$ satisfies conditions $(S_3)$ and $(R_2)$, and in particular, it is normal.

Case (2). The extremal face $R_s$ determines an extremal face of $\overline{NE}(\tilde{S}_i)$ with respect to $K_{\tilde{S}_i} + D_i$, hence a contraction morphism $\varphi_{\tilde{S}_i} : \tilde{S}_i \to \varphi_{\tilde{S}_i}(\tilde{S}_i) = \tilde{B}$. We have two subcases:

(2a) $\dim \tilde{B} = 1$,
(2b) $\dim \tilde{B} = 0$.

In the subcase (2a), suppose that a general fiber $\tilde{C}$ of $\varphi_{\tilde{S}_i}$ does not intersect $\tilde{D}_i$. Then we have $(\tilde{S}_j \cdot \tilde{C}) = 0$ for any $j$ and any $\tilde{C} \in C_s$. If $\tilde{C}''$ is an irreducible component of $\tilde{D}_i$, then $\tilde{C}''$ is contained in a fiber of $\varphi_{\tilde{S}_i}$, hence $(\tilde{S}_i \cdot \tilde{C}'') = 0$. So if $\tilde{C}'' \subset \tilde{S}_j$, then the contraction morphism from $\tilde{S}_j$ determined by the face $R_s$ is of fiber type again, i.e., $\tilde{S}_j \subset E_s$. Hence $X_s \subset E_s$, a contradiction.

Therefore, there is an irreducible component $\tilde{S}_j$ such that $\tilde{S}_i \cap \tilde{S}_j$ is a section of $\varphi_{\tilde{S}_i}$. Then $(\tilde{S}_i \cdot \tilde{C}) < 0$ for a general fiber $\tilde{C}$ of $\varphi_{\tilde{S}_i}$, and $(\tilde{S}_i \cdot \tilde{C}') < 0$ for any $\tilde{C}' \in C_s$. Hence $E_s = \tilde{S}_i$. Let $\tilde{C}''$ be an irreducible component of $\tilde{D}_i$ in a fiber of $\varphi_{\tilde{S}_i}$. Since $(\tilde{S}_i \cdot \tilde{C}'') > 0$, $\tilde{C}''$ intersects $\tilde{S}_j$ at a nonsingular point of $\tilde{S}_i$. In particular, $\tilde{C}''$ is the only component of $\tilde{D}_i$ in the same fiber as $\tilde{C}''$.

Then one can construct a projective scheme $Z_s$ of dimension 2 which satisfies the condition $(S_2)$ and a morphism $\varphi_s : X_s \to Z_s$ which coincides with $\varphi_{\tilde{S}_i}$ on $\tilde{S}_i$ and is isomorphic outside $\tilde{S}_i$. As in the case (1), we have only to prove that $R^1\varphi_s^*O_{X_s} = 0$.

Let $X'_s$ be the union of the irreducible components of $X_s$ other than $\tilde{S}_i$. Thus $\tilde{D}_i = X'_s \cap \tilde{S}_i$. We have exact sequences

\[0 \to O_{X'_s}(-\tilde{D}_i) \to O_{X_s} \to O_{\tilde{S}_i} \to 0\]
\[0 \to O_{X'_s}(-\tilde{D}_i) \to O_{X'_s} \to O_{\tilde{D}_i} \to 0.\]

We have $R^1\varphi_{\tilde{S}_i}^*O_{\tilde{S}_i} = 0$ by Lemma 2.1. Since $R^1\varphi_s^*O_{X_s} = 0$, and $\tilde{D}_i$ is connected, we have also $R^1\varphi_s^*O_{X'_s}(-\tilde{D}_i) = 0$. Hence $R^1\varphi_{\tilde{S}_i}^*O_{X'_s} = 0$.

In the subcase (2b), since $(\tilde{S}_i \cdot \tilde{C}) < 0$ for any $\tilde{C} \in C_s$, we have again $E_s = \tilde{S}_i$. Moreover, since any irreducible component of $\tilde{D}_i$ is an ample $\mathbb{Q}$-Cartier divisor on $\tilde{S}_i$, it intersects any curve on $\tilde{S}_i$. But since $K_{\tilde{S}_i} + \tilde{D}_i$ is negative, $\tilde{D}_i$ consists of 1 or 2 irreducible components. Then Lemma 2.2 combined with the argument in the subcase (2a) yields the result.

Case (3). Let $\tilde{S}_i$ be an irreducible component of $X_s$ such that the image of the associated contraction morphism $\varphi_{\tilde{S}_i}$ is a curve. If $\tilde{D}_i$ contains a section of $\varphi_{\tilde{S}_i}$, then we have $(\tilde{S}_i \cdot \tilde{C}) < 0$ for $\tilde{C} \in C_s$, and $E_s = \tilde{S}_i$ as in the case (2), a contradiction. Hence $(\tilde{S}_i \cdot \tilde{C}) = 0$ for any $\tilde{C} \in C_s$, and no irreducible component of $X_s$ is contracted to a point by the associated contraction morphism. Any irreducible component of any intersection $\tilde{S}_i \cap \tilde{S}_j$ is at the same time a fiber of $\varphi_{\tilde{S}_i}$ and $\varphi_{\tilde{S}_j}$. Thus we have a
1-dimensional reduced scheme $Z_{\bar{s}}$ with only ordinary double points as singularities and a contraction morphism $\varphi_{\bar{s}} : X_{\bar{s}} \to Z_{\bar{s}}$ with $R^1\varphi_{\bar{s}}^*O_{X_{\bar{s}}} = 0$, and we are done.

Case (4). We have just $\varphi = f$. Q.E.D.

We have again a relative version of the contraction theorem.

2.3'. Theorem. Under the assumption of Theorem 1.3', if $R$ is an extremal ray of $NE(X/B)$ in the half space $h_{K_{X/\Delta}}(z) < 0$, then there exists a projective surjective morphism $\varphi : X \to Z$ which is projective over $B$ such that $\varphi_*O_X = O_Z$ and that any curve $C$ on any fiber of $f'$ is mapped to a point by $\varphi$ if and only if $cl(C) \in R$.

3. Canonical cover

We shall extend the construction of the canonical cover (cf. [R]). We assume that the morphism $f : X \to \Delta$ satisfies (1.1) including the condition (6). Moreover, we assume that $A$ is a complete discrete valuation ring whose residue field $k$ is algebraically closed. For a closed point $P \in \Sigma$, the positive integer

$$r = \min\{m \in \mathbb{N} : mK_{X/\Delta} \text{ is Cartier at } P\}$$

is called the index of $K_{X/\Delta}$ at $P$. We note that $K_{X/\Delta}$ is Cartier on $X \setminus \Sigma$.

Let us fix a point $P \in \Sigma$. By replacing $X$ by a small neighborhood of $P$ if necessary, we assume that $K_{X/\Delta}$ is Cartier on $U = X \setminus \{P\}$ and there exists a nowhere vanishing section $\theta$ of $O_X(rK_{X/\Delta})$. Then an $O_X$-algebra structure on

$$\mathcal{F} = \bigoplus_{m=0}^{r-1} O_X(-mK_{X/\Delta})t^m$$

is defined by an equation $t^r = \theta$. Let $Y = \text{Spec } \mathcal{F}$, $\pi : Y \to X$ the natural projection, and $Q = \pi^{-1}(P)$. By the condition (6) of (1.1), $Y$ satisfies the condition $(S_3)$.

The group scheme $\mu_r = \text{Spec } \mathbb{Z}[\zeta]/(\zeta^r - 1)$ with co-multiplication $\zeta \mapsto \zeta \otimes \zeta$ acts on $Y$ by the co-action

$$x \mapsto \zeta^m \otimes x \text{ if } x \in O_X(-mK_{X/\Delta})t^m.$$ 

The ring of $\mu_r$-invariants of $O_Y$ coincides with $O_X$.

Since the choice of $\theta$ is large, the canonical cover $\pi : Y \to X$ is not unique even locally in the etale topology.

3.1. Theorem. If $X$ is a small enough neighborhood of $P$ and $\theta$ is chosen sufficiently general, then $f \circ \pi : Y \to \Delta$ satisfies the conditions of (1.1). Moreover, $Y$ is Gorenstein.

Proof. By [H],

$$\pi_*\omega_Y/\Delta \simeq \text{Hom}_{O_X}(\pi_*O_Y, \omega_{X/\Delta}) = \bigoplus_{m=0}^{r-1} O_X((m+1)K_{X/\Delta})t^{-m} \simeq \pi_*O_Y \cdot t$$

as $\pi_*O_Y$-modules. Hence $Y$ is Gorenstein.
One can decompose $\pi$ as follows. Let $p$ be the characteristic of $k$, and let $r = \tilde{r}p^{e}$ with $(\tilde{r}, p) = 1$. We set $\tilde{t} = t^{\tilde{r}^e}$, and $\tilde{Y} = \text{Spec} \left( \bigoplus_{m=0}^{\tilde{r}-1} \mathcal{O}_{X}(-mp^{e}K_{X/\Delta})^{\tilde{r}^m} \right)$ with $\tilde{r}^e = \theta$. Let $\tilde{\pi} : \tilde{Y} \to X$ and $\sigma : Y \to \tilde{Y}$ be the natural morphisms. Then $\tilde{\pi}$ is etale over $X \setminus \Sigma$ and $\sigma$ is purely inseparable.

Let $T_i$ be the inverse image by $\pi$ of an irreducible component $S_i$ of $X_s$. If $T_i$ is not irreducible, then its irreducible components meet only at $Q$. But there is a positive integer $n$ such that $nT_i$ is a Cartier divisor on $Y$ and satisfies the condition (S2), a contradiction. Hence $T_i$ is an irreducible $\mathbb{Q}$-Cartier divisor.

We shall prove that $Y$ is normal. If $r = \tilde{r}$, then it is clear that $Y \setminus \{Q\}$ is regular, hence $Y$ is normal. We assume that $p|r$ in the following. For a point $P' \neq P$ of $X$, let $\theta_0$ be a germ of a generating section of $\mathcal{O}_{X}(K_{X/\Delta})$ at $P'$, and write $\theta = h\theta_0$ for a germ of an invertible function $h$ at $P'$. Let $s$ be a nonzero global section of $\mathcal{O}_{X}(K_{X/\Delta})$. If we write $s = c\theta_0$ for a germ of a regular function $c$ at $P'$, then $s^r = c^r\theta_0^r = g\theta$ for a global regular function $g$. Thus a local section $dh$ of $\Omega_{X/\Delta}^1$ vanishes if and only if $dg/g$ vanishes. If $U'$ is the open subset of $X$ on which $f$ is smooth and $dg/g$ does not vanish, then $f \circ \pi$ is smooth over $\pi^{-1}(U')$. If we replace $\theta$ by $u\theta$ for an invertible function $u$, then $dg/g$ is replaced by $dg/g - du/u$. By Bertini’s theorem, $du/u$ has no zero on $U$ for general $u$. The 1-form $dg/g$ gives a rational section of the projection $\mathbb{P}_x(\Omega_{U/\Delta}) \to U$, and its intersection with the section given by $du/u$ is at most 2-dimensional for general $u$, since the latter is free. Then one can arrange $u$ so that the zero locus of $dg/g - du/u$ is at most 1-dimensional. Therefore, $X \setminus U'$ is at most 1-dimensional, and $Y$ is normal. The above argument also showed that the condition (2) of (1.1) holds for $Y$.

We shall prove that $Y$ has only terminal singularities. Then it follows the conditions (1) and (4) of (1.1) for $Y$. Let $\mu : X' \to X$ be a quasi-projective birational morphism from a normal scheme $X'$, and $E$ a prime divisor on $X'$ such that codim $\mu(E) \geq 2$. We have $K_{X'/\Delta} = \mu^*K_{X/\Delta} + \Gamma$ for some effective $\mathbb{Q}$-divisor $\Gamma$ whose support contains $E$; we have $\Gamma = \frac{a}{r}E + ...$ with $a \in \mathbb{N}$. We define an $\mathcal{O}_{X'}$-algebra structure on

$$\mathcal{F}' = \bigoplus_{m=0}^{r-1} \mathcal{O}_{X'}(-mK_{X'/\Delta} + m\Gamma_{\Delta})t^{m}$$

by $t^r = \mu^*\theta \in \mathcal{O}_{X'}(rK_{X'/\Delta} - r\Gamma)$. Let $Y' = \text{Spec} \mathcal{F}'$, and $\pi' : Y' \to X'$ the projection. Since there are natural homomorphisms

$$\mu^*\mathcal{O}_{X}(-mK_{X/\Delta}) \to \mathcal{O}_{X'}(-mK_{X'/\Delta} + m\Gamma_{\Delta})$$

we have a birational morphism $\mu' : Y' \to Y$ which covers $\mu$.

We have to prove that $Y'$ is normal. By construction, $Y'$ satisfies the condition (S2). Since $\mu'$ is an isomorphism outside the support of $\Gamma$, it is enough to prove that $Y'$ is regular above the generic point $\eta_E$ of $E$. Let $v$ and $\theta_0^r$ be generators of the maximal ideal $\mathfrak{m}_{E,X'}$ and $\omega_{X'/\Delta}$ at $\eta_E$, respectively. We can write $\mu^*\theta = hv_0\theta_0^{tr}$ for an invertible function $h$ at $\eta_E$. Let $c = (r,a)$, $r = cr'$ and $a = ca'$. Then $am_0 - (\frac{r}{n}am_0/r'n - 1)r' = 1$ for an integer $m_0$ such that $0 < m_0 \leq r'$. We set

$$u_j = \theta_0^{-jr'}v^{-ja'}t^{jr'} \in \mathcal{O}_{X',\eta_E}(-jr'K_{X'/\Delta} + jr'\Gamma_{\Delta})t^{jr'}$$

for $0 \leq j < c$

$$\sigma' = \theta_0^{-cm_0} - \frac{c}{h_0}c_0^r + 1_{c_0^r} \in \mathcal{O}_{X,\eta_E}(-cm_0K_{X/\Delta} + c_0^r\Gamma_{\Delta})t^{cm_0}.$$
Then $u_j^r = h^j \in \mathcal{O}_{X', \eta_E}$ is invertible, and $u'^r = h^{m_0}$. Hence $u'$ generates the maximal ideals at the points of $Y'$ above $\eta_E$, and $Y'$ is regular there. Therefore, $Y'$ is isomorphic to the normalization of the fiber product $Y \times_X X'$.

We have

$$\pi'_* \omega_{Y'/\Delta} \simeq \bigoplus_{m=0}^{r-1} \mathcal{O}_{X'}((m+1)K_{X'/\Delta} - \sum m\Gamma_\nu)t^{-m},$$

and there are natural homomorphisms

$$\mu^* \mathcal{O}_X((m+1)K_{X/\Delta}) \to \mathcal{O}_{X'}((m+1)K_{X'/\Delta} - \sum m\Gamma_\nu)$$

which is not isomorphic at $\eta_E$ for $m = r - 1$. Hence the discrepancy $K_{Y'/\Delta} - \mu^* K_{Y/\Delta}$ has positive coefficients at prime divisors of $Y'$ which lie above $E$. Thus $Y$ has only terminal singularities.

Next, we shall prove that the pair $(Y, Y_s)$ has only a log terminal singularity at $Q$. Supposing that $\mu(E) = P$, let us write $\mu^* X_s = bE + \ldots$ for $b \in \mathbb{N}$. Since $(X, X_s)$ is log terminal at $P$, we have $a/r - b > -1$. The coefficient of the discrepancy $K_{Y'/\Delta} - \mu^* (K_{Y/\Delta} + Y_s)$ at the points above $\eta_E$ is $r'(a - \sum a(r-1)/r + b) = r'(a/r - b)^{-1} \geq 0$. Hence $(Y, Y_s)$ is log terminal at $Q$.

Let $T'_i$ be the normalizations of the $T_i$, and $F_i = \sum_{j \neq i} T_j | T'_i$. Since the pair $(Y, Y_s)$ is log terminal at $Q$, so are the pairs $(T'_i, F_i)$ above $Q$ by the adjunction. Then it follows easily that the $F_i$ are reduced, hence $Y_s$ is a normal crossing divisor on $Y \setminus \{Q\}$. The restriction of a local section of $\mathcal{O}_{T_i}$ on $T_i \setminus \{Q\}$ to $F_i \setminus \{Q\}$ extends to sections of the $\mathcal{O}_{T_j}$ on the $T_j \setminus \{Q\}$ for $j \neq i$, hence a section of $\mathcal{O}_{Y_s}$ over $Y_s \setminus \{Q\}$. Since $Y_s$ satisfies the condition (S2), it extends over to $Y_s$, hence the $T_i$ are normal. We note that this is the point where we needed the condition (6) of (1.1). This completes the proof of the theorem. Q.E.D.

4. Classification of singularities

Let $f : X \to \Delta$ be as in §3. We shall classify the singular points of $X$ (cf. [Mo2]).

4.1. Theorem. Let $P \in X$ be a point of index $r > 1$, and $\pi : Y \to X$, etc. as in §3. Then the completion of $\mathcal{O}_{Y,Q}$ at $Q = \pi^{-1}(P)$ with the $\mu_r$-action is isomorphic to the completion of $A[x, y, z]/(F)$ for some semi-invariant coordinates $x, y$ and $z$, where

$$F = xy + G(z^r)$$

for some polynomial $G \in A[z^r]$, and the action of $\mu_r$ is given by

$$x \mapsto \zeta^a \otimes x, \quad y \mapsto \zeta^{-a} \otimes y, \quad z \mapsto \zeta \otimes z$$

for some positive integer $a$ such that $(r, a) = 1$. Moreover, if $\mathcal{O}_{Y_s,Q}$ is not integral, then

$$F = xy + \tau$$

for a generator $\tau$ of the maximal ideal of $A$.

Proof. By the classification of log terminal singularities of surfaces of index 1, the completion of $\mathcal{O}_{Y_s,Q}$ with the $\mu_r$-action is isomorphic to a hypersurface singularity $k[[x, y, z]]/(F_\ast)$ for some semi-invariant coordinates $x, y$ and $z$, where one of the following holds (cf. [A2]):

1. $F_\ast = xy$;
2. $F_\ast$ has at most an isolated singularity and $\text{ord}(F_\ast) \leq 2$.

Therefore, we can use the classification theorem for curves to find a semi-invariant coordinate $z$ such that $\zeta \otimes z$ is an isolated singularity, and $\zeta^a \otimes x$ for some $a$.

Next, we shall prove that the pair $(Y, Y_s)$ has only a log terminal singularity at $Q$. Supposing that $\mu(E) = P$, let us write $\mu^* X_s = bE + \ldots$ for $b \in \mathbb{N}$. Since $(X, X_s)$ is log terminal at $P$, we have $a/r - b > -1$. The coefficient of the discrepancy $K_{Y'/\Delta} - \mu^* (K_{Y/\Delta} + Y_s)$ at the points above $\eta_E$ is $r'(a - \sum a(r-1)/r + b) = r'(a/r - b)^{-1} \geq 0$. Hence $(Y, Y_s)$ is log terminal at $Q$.

Let $T'_i$ be the normalizations of the $T_i$, and $F_i = \sum_{j \neq i} T_j | T'_i$. Since the pair $(Y, Y_s)$ is log terminal at $Q$, so are the pairs $(T'_i, F_i)$ above $Q$ by the adjunction. Then it follows easily that the $F_i$ are reduced, hence $Y_s$ is a normal crossing divisor on $Y \setminus \{Q\}$. The restriction of a local section of $\mathcal{O}_{T_i}$ on $T_i \setminus \{Q\}$ to $F_i \setminus \{Q\}$ extends to sections of the $\mathcal{O}_{T_j}$ on the $T_j \setminus \{Q\}$ for $j \neq i$, hence a section of $\mathcal{O}_{Y_s}$ over $Y_s \setminus \{Q\}$. Since $Y_s$ satisfies the condition (S2), it extends over to $Y_s$, hence the $T_i$ are normal. We note that this is the point where we needed the condition (6) of (1.1). This completes the proof of the theorem. Q.E.D.
Since $Y_s = X_s \times_X Y$, the completion of $O_{Y,Q}$ with the $\mu_r$-action is isomorphic to the completion of $A[x,y,z]/(F)$ such that $F \equiv F_s (\mod \tau A[x,y,z])$. We note that the elements in $A$ are $\mu_r$-invariants. Since the action of $\mu_r$ is nontrivial and $Y$ has only isolated singularity, the subspace of Spec $A[x,y,z]$ defined by $F = 0$ intersects the fixed locus of $\mu_r$ only at the origin. In fact, the fixed locus of codimension two or more has toric singularities, while if there is only one coordinate which is not invariant, then $\omega_{X/\Delta,P}$ is generated by one element over $O_{X,P}$, a contradiction. Therefore, none of $x$, $y$ nor $z$ is $\mu_r$-invariant for an integer $r'$ with $r'|r$ and $r' > 1$. Moreover, $F$ contains a constant term on $x$, $y$ and $z$, hence is a $\mu_r$-invariant. Then so is $F_s$.

Thus we have the first assertion of the theorem in the case (1). We write $F = xyz + \tau G'(z^r)$. Since $Y$ has only isolated singularities, $G'$ is not divisible by $\tau$. Since the $S_i$ are $\mathbb{Q}$-Cartier, the prime divisor on $Y$ defined by $x = \tau = 0$ should be $\mathbb{Q}$-Cartier. Then $G'$ is invertible, hence the last assertion.

In the case (2), if $\text{ord}(F_s) = 1$, then there is an invariant coordinate, a contradiction. Hence $\text{ord}(F_s) = 2$. If $F_s$ contains a degree 2 term of the form $xy$, then we are done. If it contains $x^2$ and there are no other terms of degree 2, then $r = 2$ and there is a degree 3 term, a contradiction to the fact that none of the coordinates $x$, $y$ and $z$ are $\mu_r$-invariants. Q.E.D.

4.2. Corollary. There exists a projective birational morphism $\mu : X' \rightarrow X$ from a normal scheme $X'$ such that the discrepancy coefficient of one of the exceptional divisors with respect to $K_{X/\Delta}$ is equal to $1/r$.

Proof. We take a weighted blowing up with weights $\frac{1}{r}(a, r - a, 1, r)$ for the semi-invariants $(x, y, z, \tau)$ (cf. [KKMS] and [K4]). Q.E.D.

In the above theorem, the axial multiplicity of the singular point $P \in X$ is defined to be largest integer $n$ such that $G(\mod z^r A[z^r]) \in \tau^n A$. If $G_s \equiv G(\mod \tau A[z^r]) \in z^r k[z^r]$ is equal to 0 or contains a linear term in $z^r$, then the singular point $P \in X$ is said to be of simple type. Then we call the above $\mu : X' \rightarrow X$ the standard blowing up of $X$. In this case, the exceptional locus $E$ of $\mu$ is an irreducible divisor with $\rho(E) = 1$, and $X'$ satisfies the conditions in (1.1) and has at most 3 terminal singular points which are of simple type again; it has singular points of indices $a$ and $r - a$ if $a > 1$ and $r - a > 1$, respectively, and a singular point of index $r$ and axial multiplicity $n - 1$ if $n > 1$.

4.3. Corollary. Let $S_i$ be an irreducible component of the closed fiber $X_s$ of $f$, and let $D_i$ be as before. Then the singularities of the pair $(S_i, D_i)$ are toric; a point on $S_i \setminus D_i$ is of type $\frac{1}{hr}(bhr - 1, 1)$, while $S_i$ is of type $\frac{1}{r}(b, 1)$ at a point on $D_i$, for some positive integers $r$, $b$ and $h$ such that $0 < b < r$ and $(r, b) = 1$. Moreover, the former is of simple type if and only if $h = 1$, and the latter is always of simple type.

Proof. In the former case, we have $F_s = xy + z^{hr}$ for some positive integer $h$. Then we set $ab = cr + 1$ for a positive integer $c$. Q.E.D.

4.4. Theorem. Let $P \in X$ be a point of index 1. If $f$ is not smooth at $P$, then the completion of $O_{X,P}$ is isomorphic to the completion of $A[x,y,z]/(F)$ for some coordinates $x$, $y$ and $z$, where one of the following holds:

1. $F = xyz + \tau$,
2. $F = xy + \tau$.

Proof. We take a weighted blowing up with weights $\frac{1}{r}(a, r - a, 1, r)$ for the semi-invariants $(x, y, z, \tau)$ (cf. [KKMS] and [K4]). Q.E.D.
(3) \( F \equiv F_s(\mod \tau A[x, y, z]) \) for a polynomial \( F_s \in k[x, y, z] \) which defines a rational double point.

**Proof.** By the classification of log terminal singularities of surfaces of index 1, we have either \( F_s = xyz \), \( F_s = xy \), or the case (3). Since the \( S_i \) are \( \mathbb{Q} \)-Cartier, we have (1) or (2) in the former cases as in the proof of Theorem 4.1. Q.E.D.

5. **Flip theorem**

5.1. **Definition.** A flip in this paper is a diagram

\[ X \xleftarrow{\varphi} Z \xrightarrow{\varphi^+} X^+ \]

which satisfies the following conditions:

1. \( \varphi : X \to Z \) and \( \varphi^+ : X^+ \to Z \) are projective birational morphisms,
2. \( Z \) is a normal scheme with a flat quasi-projective morphism \( g : Z \to \Delta \), or one which is obtained by a completion from such a scheme, and the induced morphisms \( f : X \to \Delta \) and \( f^+ : X^+ \to \Delta \) satisfy the conditions in (1.1),
3. \( \varphi_\eta : X_\eta \to Z_\eta \) and \( \varphi_\eta^+ : X_\eta^+ \to Z_\eta \) are isomorphisms,
4. \( \varphi \) and \( \varphi^+ \) contract only finitely many curves on finite numbers of closed fibers of \( f \) and \( f^+ \), respectively,
5. \(-K_{X/\Delta}\) is \( \varphi \)-ample and \( K_{X^+/\Delta}\) is \( \varphi^+ \)-ample.

5.2. **Theorem.** Let \( X \xleftarrow{\varphi} Z \xrightarrow{\varphi^+} X^+ \) be a flip, and let \( r \) (resp. \( r^+ \)) be the maximum of the indices of \( K_{X/\Delta} \) (resp. \( K_{X^+/\Delta} \)) at the points on the exceptional locus \( \text{Exc}(\varphi) \) (resp. \( \text{Exc}(\varphi^+) \)). Then \( r > r^+ \geq 1 \). In particular, the termination of flips holds, i.e., there exists no infinite sequence of flips over \( \Delta \) as follows:

\[ X^{(0)} \xrightarrow{\varphi^{(0)}} Z^{(0)} \xleftarrow{\varphi^{(0)+}} X^{(1)} \xrightarrow{\varphi^{(1)}} Z^{(1)} \xleftarrow{\varphi^{(1)+}} X^{(2)} \xrightarrow{\varphi^{(2)}} \ldots \]

**Proof.** We may assume that \( A \) is a complete discrete valuation ring whose residue field is algebraically closed. First, we shall prove that \( r > 1 \). Suppose that \( r = 1 \).

Let \( C \) be an irreducible component of \( \text{Exc}(\varphi) \). Then there is only one irreducible component \( S_i \) of the closed fiber \( X_s \) which contains \( C \). In fact, if \( C \) is contained in another component \( S_j \), then we have \( (S_i \cdot C) \leq -1 \) and \( (S_j \cdot C) \leq -1 \) by Theorem 4.4. Since \( (X_s \cdot C) = 0 \), there are at least 2 other components of \( X_s \) which intersect \( C \) transversally. But then we have \( (K_{X/\Delta} \cdot C) = ((K_{S_i} + D_i) \cdot C) \geq 0 \), a contradiction. Thus \( C \notin D_i \).

Since the strict transform of \( C \) in the minimal resolution of \( S_i \) is a \((-1)\)-curve, if \( C \cap D_i \neq \emptyset \), then \( (K_{X/\Delta} \cdot C) \geq 0 \). Thus \( C \cap D_i = \emptyset \). The contractibility of \( C \) in \( S_i \) implies that there is at most one singular point of \( S_i \) on \( C \), and it is of type A. Then the surface \( \varphi(S_i) \) is smooth at \( Q = \varphi(C) \), and so is \( Z \) at \( Q \). But \( K_Z \) must not be a \( \mathbb{Q} \)-Cartier divisor at \( Q \), a contradiction, hence \( r > 1 \).

If \( r^+ > 1 \), then there exists an exceptional divisor \( E \) over \( X^+ \) whose discrepancy coefficient with respect to \( K_{X^+/\Delta} \) is equal to \( 1/r^+ \). Then the discrepancy coefficient of \( E \) with respect to \( K_{X/\Delta} \) is less than \( 1/r^+ \), hence \( 1/r < 1/r^+ \). Q.E.D.
5.3. Theorem. If the exceptional locus $E$ of the contraction morphism $\varphi : X \to Z$ in Theorems 2.3 or 2.3’ coincides with an irreducible component $S_i$ of a closed fiber $X_s$, then the induced morphism $g : Z \to \Delta$ satisfies the conditions of (1.1). If $X$ is $\mathbb{Q}$-factorial, and if $E$ contains a prime divisor of $X$ but is not equal to the whole $X$, then $E$ coincides with the prime divisor, $g : Z \to \Delta$ satisfies the conditions of (1.1), and $Z$ is again $\mathbb{Q}$-factorial.

Proof. All statements are clear by construction except the condition (6) of (1.1) (cf. [KMM]). We may assume that $A$ is a complete discrete valuation ring whose residue field $k$ is algebraically closed, because the depth is preserved by the base change to such $A$, but we lose the condition that $\rho(X/Z) = 1$. We fix a positive integer $m$.

First, we consider the case in which the closed fiber $E_i$ of $E$ is 1-dimensional in the latter half of the theorem. We write $K_{X/\Delta} = \varphi^*K_{Z/\Delta} + dE$ for a positive rational number $d$. Then we have $R^1\varphi_*\mathcal{O}_X(mK_{X/\Delta} - \gamma mdE) = 0$ by Lemma 2.1 as in the proof of Theorem 2.3. Since $\varphi_*\mathcal{O}_{X,s}(mK_{X,s} - \gamma mdE) = \mathcal{O}_{Z,s}(mK_{Z,s})$ satisfies the condition $(S_2)$, $\varphi_*\mathcal{O}_X(mK_{X/\Delta} - \gamma mdE) = \mathcal{O}_Z(mK_{Z/\Delta})$ satisfies $(S_3)$.

In the rest of the proof, we assume that $E$ is an irreducible component of $X_s$. We write $S = S_i = E$ and $D = D_i$. For any Weil divisor $W$ on $X$, we shall write $\mathcal{O}_D(W) = (\mathcal{O}_X(W) \otimes \mathcal{O}_D)/\text{torsion}$ in this proof. Let $d$ be the smallest positive integer such that $\mathcal{O}_D(mK_{X/\Delta} - dS)$ has a nonzero global section. We set $L = mK_{X/\Delta} - dS$. Let $X'_s$ be the union of the irreducible components of $X_s$ except $S$. We have an exact sequence

$$0 \to \mathcal{O}_S(L + S) \to \mathcal{O}_{X,s}(L) \to \mathcal{O}_{X'_s}(L) \to 0.$$ 

We shall prove

(1) $R^1\varphi_*\mathcal{O}_S(L + S) = 0$.

(2) $R^1\varphi_*\mathcal{O}_{X'_s}(L) = 0$.

Then (1) implies that the natural homomorphism $\varphi_*\mathcal{O}_{X,s}(L) \to \varphi_*\mathcal{O}_{X'_s}(L)$ is surjective, hence $\varphi_*\mathcal{O}_{X_s}(L)$ satisfies the condition $(S_2)$. By the results in §4, $\mathcal{O}_X(L)$ satisfies $(S_3)$, and the natural homomorphism $\mathcal{O}_X(L) \to \mathcal{O}_{X,s}(L)$ is surjective. Then by (2), we have $R^1\varphi_*\mathcal{O}_{X,s}(L) = 0$, and the natural homomorphism $\mathcal{O}_Z(mK_{Z/\Delta}) = \varphi_*\mathcal{O}_X(L) \to \varphi_*\mathcal{O}_{X,s}(L)$ is surjective, hence $\mathcal{O}_Z(\gamma mdE) = \mathcal{O}_Z(mK_{Z/\Delta})$ satisfies $(S_3)$.

Our assertions follow immediately from Lemma 2.1 except (1) when $\varphi(S)$ is a point. There are 3 cases:

(i) $D$ consists of 2 rational curves,

(ii) $D$ is an irreducible rational curve and there are at most 2 singular points of $S$ on $D$,

(iii) $D$ is an irreducible rational curve and there are 3 singular points of $S$ on $D$.

We consider the cases (i) and (ii). Then $\mathcal{O}_D(-K_{X/\Delta})$ has a nonzero global section, hence so has the $\mathcal{O}_D(L - nK_{X/\Delta})$ for any positive integer $n$. If $n$ is sufficiently large, then we have $H^1(S, \mathcal{O}_S(L - D - nK_{X/\Delta})) = 0$ by the Serre vanishing theorem. By the exact sequence

$$0 \to \mathcal{O}_S(L - D - nK_{X/\Delta}) \to \mathcal{O}_S(L - nK_{X/\Delta}) \to \mathcal{O}_D(L - nK_{X/\Delta}) \to 0,$$

we have $H^0(S, \mathcal{O}_S(L - nK_{X/\Delta})) \neq 0$ if $H^1(S, \mathcal{O}_S(L - D - nK_{X/\Delta})) = 0$. Let $M_n$ be an effective Weil divisor on $S$ which corresponds to a nonzero global section of $\mathcal{O}_S(L - nK_{X/\Delta})$. If $\varphi(S)$ is a point, then $\varphi_*\mathcal{O}_{X,s}(L)$ satisfies $(S_3)$, and $\varphi_*\mathcal{O}_{X_s}(L)$ is surjective. Hence $\mathcal{O}_Z(mK_{Z/\Delta})$ satisfies $(S_3)$.
of \( \mathcal{O}_S(L - nK_{X/\Delta}) \). Then it is an ample \( \mathbb{Q} \)-Cartier divisor. By [LM, Lemmas 1 and 2], where the same arguments can be extended to the \( \mathbb{Q} \)-divisors, we have \( H^0(M_n, \mathcal{O}_{M_n}) = k \). Since \( H^1(S, \mathcal{O}_S) = 0 \) by Lemma 2.2, we have \( H^1(S, \mathcal{O}_S(-M_n)) = 0 \), hence \( H^1(S, \mathcal{O}_S(L - D - (n - 1)K_{X/\Delta})) = 0 \). By the descending induction on \( n \), we obtain \( H^1(S, \mathcal{O}_S(L - D)) = 0 \), i.e., the assertion (1).

In the case (iii), if \( \rho(S) = 1 \), then we have our theorem by the following lemma. We shall finish the proof in the case \( \rho(S) > 1 \) by using Theorem 5.5. Here we note that the proof of Theorem 5.5 uses Theorem 5.3 in the case in which \( \rho(S_1/\mathcal{O}(S_1)) = 1 \).

We may replace \( Z \) by its completion at \( \varphi(S) \). Then \( \rho(X/Z) = \rho(S) \). In fact, we have \( R^2\varphi_*\mathcal{O}_X = 0 \) by Lemma 2.2, hence \( R^2\varphi_*\mathcal{O}_X = 0 \). By an exact sequence
\[
0 \to \mathcal{O}_X(-X_s) \to \mathcal{O}_X^* \to \mathcal{O}_X^* \to 0
\]
the natural homomorphism \( H^1(X, \mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) \) is surjective. Since \( H^1(X, \mathcal{O}_X^*) \to H^1(S, \mathcal{O}_S^*) \) is also surjective, we have our claim.

We apply the minimal model program to the morphism \( \varphi : X \to Z \). By Theorems 5.5 and 5.2, there exists a sequence of flips \( X = X_0 \to \cdots \to X_i \to \cdots \to X_i \) and a divisorial contraction \( X_i \to Z \). By the previous arguments, \( Z' \to \Delta \) satisfies the conditions (1.1). Since the induced morphism \( \psi : Z' \to Z \) contracts only a finite number of curves, we have \( K_{Z'/\Delta} = \psi^*K_{Z/\Delta} \). By Lemma 2.1, we have \( R^1\psi_*\mathcal{O}_{Z'}(mK_{Z'/\Delta}) = 0 \) for any integer \( m \), hence \( \psi_*\mathcal{O}_{Z'}(mK_{Z'/\Delta}) = \mathcal{O}_Z(mK_{Z/\Delta}) \) satisfies (1.1). Q.E.D.

5.4. Lemma. In the above theorem, assume that \( A \) is a complete discrete valuation ring with an algebraically closed residue field, \( \mathcal{O}(S_i) \) is a point, \( D_i \) is irreducible, \( \rho(S_i) = \dim N_1(S_i) = 1 \), and that there are 3 singular points of \( S_i \) on \( D_i \). Then the conclusion of the theorem holds.

Proof. We shall use the arguments of [MT]. We explain which part we should pick up from [MT]. We note that the characteristic is assumed to be zero in [MT], but the arguments which we pick up are characteristic free.

We use the following notation of [MT]; \( \bar{V} = S_i, \bar{E} = D_i, f : V \to \bar{V} \) is the minimal resolution, \( E \) is the strict transform of \( \bar{E} \) by \( f \), \( D \) is the sum of \( E \) and the exceptional divisors of \( f \), \( D = \sum_i D_i \) is the decomposition into irreducible components, \( K_V + D^\# = f^*(K_{\bar{V}} + \bar{E}) \), \( T_i \) for \( i = 1, 2, 3 \) are exceptional divisors of \( f \) above the 3 singular points of \( \bar{V} \) on \( \bar{E} \), \( C_i \) is the irreducible component of \( T_i \) which meets \( E \), \( C_1 = T_1, (C_1^2) = -2 \), and \( F = E + T_1 + T_2 + T_3 \).

A permissible linear pencil of rational curves \( \Lambda \) on \( V \) is defined on [MT, p.274]. \( L_i \) for \( i = 1, 2, 3 \) are singular members of \( \Lambda \), and \( L_1 = C_1 + 2\ell + D_1 \). \( E \) becomes a cross section of \( \Lambda \) ([MT, Lemma 2.2]). Since \( (f_*(\ell)^2) > 0 \), \( D_1 \) intersects some irreducible components \( D_2, \ldots, D_{e+1} \) of \( D - D_1 \). They are also cross sections of \( \Lambda \). By [MT, 2.3.1], the \( L_i \) are the only singular fibers of \( \Lambda \), and they consist of \((-1\)-curves and irreducible components of \( D \). If \( e = 1 \), then there are exactly one \((-1\)-curve on each \( L_i \) by [MT, 2.3.2], and they do not intersect the section \( D_2 \). Thus the connected component of \( D \) containing \( D_2 \) cannot be contracted to a singular point described in Corollary 4.3, because its dual graph is a fork, a contradiction. If \( e > 2 \), then we have a fork with the central component \( D_1 \). If \( e = 2 \), then there exists a singular fiber \( L_2 \) which has only one \((-1\)-curve by [MT, 2.3.2]. Thus \( D_2 \) and \( D_1 \) should intersect an irreducible component \( L_i \) which is contained in \( D \).
Hence we have a cycle in $D$, a contradiction. Therefore, it is sufficient to prove the existence of a permissible linear pencil of rational curves in the following.

In [MT, §3], by contracting the curves in $D$ except $E$ and $C_1$ on $V$, we obtain a morphism $g$ to factorize $f$ as $V \xrightarrow{g} W \xrightarrow{g'} V$. Let $\bar{C}_1 = g_*(C_1)$ and $\bar{E} = g_*(E)$. Since $\rho(W) = 2$ and $(\bar{C}_1^2) = -2$, $\overline{NE}(W)$ is generated by $C_1$ and an extremal ray $\bar{\ell}$ with respect to $K_W$. We may assume that the strict transform $\ell$ of $\bar{\ell}$ on $V$ is a $(-1)$-curve, because $(K_V \cdot \ell) \leq (K_W \cdot \bar{\ell}) < 0$ and we may take an $\ell$ which passes through a singular point of $W$ if $(\bar{\ell}^2) = 0$. We set $\bar{\ell} = f_*(\ell)$.

There are 2 possibilities: $\ell \neq E$ or $\ell = E$. We assume first that $\ell \neq E$. Since $(\bar{E}^2) > 0$ and $(\bar{\ell}^2) \leq 0$, we have $(C_1 \cdot \ell) > 0$. But $((K_V + D^\#) \cdot \ell) < 0$, hence $(C_1 \cdot \ell) = 1$ and $(E \cdot \ell) = 0$. In [MT, Lemma 3.4] it is proved that $\ell$ does not intersect $F$ except $C_1$. Then by [MT, 4.1], $\ell$ intersects an irreducible component $D_1$ of $D - F$. If $(D_1^2) = -2$, then $(D_1 \cdot \ell) = 1$, because $(\bar{\ell}^2) \leq 0$, hence $|C_1 + 2\ell + D_1|$ gives a permissible pencil of rational curves. If $(D_1^2) = -3$, then we derive a contradiction with $(\bar{\ell}^2) > 0$ by the argument in [MT, p.295, up to line 17].

We shall assume that $\ell = E$ in the following. Then we have $(\bar{E}^2) \leq 0$ and $(E^2) = -1$. The calculation in [MT, Lemma 3.1] shows that the combination of singularities of $\bar{V}$ along $\bar{E}$ are as in one of the cases (1), (2), (3), (5), (7), (9), (10), (11) or (13). In the cases (1) with $r = 1$, (2), (5) or (9), if $D_i = S_i \cap S_j$ in the notation of Theorem 5.3, then $D_i$ is contracted in $S_j$ to a rational double point. Then $Z_*$ is Gorenstein, and so is $Z$, and the claim of the theorem is clear.

In the case (1) with $r > 1$, we consider a linear pencil $\Lambda = |C_1 + 2E + C_2|$. Then $A_r = C_3$ becomes a double section, and there is only one $(-1)$-curve on each singular fiber by [MT, 5.3.1.(i)]. Let $L$ be a singular fiber containing $A_{r-1}$, the irreducible component of $F_3$ next to $C_3$. By the argument in [MT, p.309, lines 5-11], since $(A_r \cdot A_{r-1}) = 1$, we have $(A_r \cdot \ell) = 0$, and the multiplicity of $A_{r-1}$ in $L$ is 2, hence the graph of $L$ is either the bottom one on [MT, p.299] or that on [MT, p.300]. In the former case, $D - F$ has a fork except when $s = 1$. But if $s = 1$, then $(B_1^2) = -(r + 1)$ in the notation there, and the three curves at the right end cannot be contracted to a singular point of the type described in Corollary 4.3. In the latter case, similar argument holds for the $s + 1$ curves at the right end.

In the remaining cases, we shall first prove that there exists a $(-1)$-curve $h$ which does not meet $F - C_1$ and such that $(h \cdot C_1) = 1$. This is done in the cases (3), (7) and (13) at [MT, 5.5 and the latter half of 5.3.2], and in the cases (10) and (11) at [MT, 5.6 and 5.7 using 5.2 similarly as 5.4].

We shall find an irreducible component $D_1$ of $D$ such that $(D_1^2) = -2$ and $(h \cdot D_1) = 1$. Then the pencil $|C_1 + 2h + D_1|$ is permissible. Let $\tilde{h} = g_*(h)$. In [MT, Lemma 6.1] one calculates that $(\tilde{h}^2) < 1$ in our cases. By the first argument of [MT, 6.2], there exists an irreducible component $D_1$ of $D$ which meets $h$ and such that $(D_1^2) = -2$ or $-3$. Let $\mu = (h \cdot D_1)$. In the latter case, we have $\mu = 1$. In the former case, if we write $g^*\tilde{h} = h + \alpha_1 D_1 + \ldots$ (instead of the equation at [MT, p.306, line 10]), then $0 = (g^*\tilde{h} \cdot D_1) \geq \mu - 2\alpha_1$, and $(\tilde{h}^2) \geq -1 + \mu^2/2$, hence $\mu = 1$. Then the rest is the same as in the last part of the proof in the case in which $\ell \neq E$.

Q.E.D.

5.5. **Flip Theorem.** Let $\varphi : X \to Z$ and $g : Z \to \Delta$ be morphisms which satisfy the conditions in (5.1). Then there exists a morphism $\varphi^+ : X^+ \to Z$ which satisfies the remaining conditions of (5.1), i.e., the flip of $\varphi$. Moreover, if $\text{Exc}(\varphi)$
is a geometrically irreducible curve, then so is $\text{Exc}(\varphi^+)$

**Proof.** We may assume that $A$ is a complete discrete valuation ring whose residue field is algebraically closed. In fact, the existence of the flip is equivalent to the surjectivity of the homomorphisms

$$\varphi^* \mathcal{O}_X(m_0 K_{X/\Delta}) \otimes \varphi^* \mathcal{O}_X(n m_0 K_{X/\Delta}) \to \varphi^* \mathcal{O}_X((n + 1)m_0 K_{X/\Delta})$$

for a positive integer $m_0$ and all positive integers $n$, and is stable by the flat base change.

Let $r(\varphi)$ be the maximum of the indices of $K_{X/\Delta}$ at the points on $\text{Exc}(\varphi)$. We shall prove the theorem by induction on $r(\varphi)$. Let $r_0$ be a positive integer, and assume that the theorem is true for any $\varphi$ such that $r(\varphi) < r_0$. Let us fix a $\varphi$ such that $r(\varphi) = r_0$.

Let $Q \in Z$ be a point in the image of $\text{Exc}(\varphi)$. Since the existence of the flip is a local question, we replace $Z$ by its completion at $Q$, and then $X$, etc., by its fiber product over $Z$. We note that the properties (1.1) are preserved under this replacement.

Let $e$ be the number of irreducible components of $\text{Exc}(\varphi)$. Then we have $\rho(X/Z) = e$ as in the proof of Theorem 5.3. We may assume that $e = 1$ in the following. In fact, if we have our theorem for $e = 1$, then applying the minimal model program to $\varphi: X \to Z$, we obtain a sequence of flips whose exceptional loci are irreducible curves, which terminates by Theorem 5.2. So we obtain $\varphi^+: X^+ \to Z$ such that $K_{X^+/\Delta}$ is $\varphi^+$-nef. Let $\psi_s: X^+_s \to X^+_s$ be the contraction morphism of all the curves whose intersection numbers with $K_{X^+/\Delta}$ are 0. Then we have $R^1 \psi_* \mathcal{O}_{X^+_s} = 0$, and $\psi_s$ can be extended to a morphism $\psi: X^+ \to X^+$ as in the proof of the contraction theorem. Since we can prove that $R^1 \psi_* \mathcal{O}_{X^+_s}(mK_{X^+/\Delta}) = 0$ for any integer $m$, $X^+$ also satisfies the conditions in (1.1). Therefore, the induced morphism $\varphi^+: X^+ \to Z$ gives the flip of $\varphi$.

Let $C = \text{Exc}(\varphi)$, and $\Lambda(\varphi) = \{P \in C: \text{ the index of } K_{X/\Delta} \text{ at } P \text{ is } r_0\}$. Let $n(P)$ be the axial multiplicity at $P \in \Lambda(\varphi)$, and let $n(\varphi) = \sum_{P \in \Lambda(\varphi)} n(P)$ be the total axial multiplicity.

First, we shall prove our theorem under the additional assumption that all the points $P \in \Lambda(\varphi)$ are of simple type, and we proceed by induction on $n(\varphi)$. We note that any non-normal point of $X_s$ is of simple type by Theorem 4.1.

**5.6. Lemma.** Let $P$ be a point of index $r_0$ on $C$, $\mu: X^+ \to X$ the standard blowing up at $P$, and $C'$ the strict transform of $C$. Then $(K_{X^+/\Delta} \cdot C') \leq 0$.

**Proof.** There are three cases:

1. $C \subset S_1 \cap S_2$ for two irreducible components $S_1$ and $S_2$ of $X_s$,
2. $C \subset S_1$ for the only one irreducible component $S_1$ of $X_s$ and $C \cap D_1 \neq \emptyset$,
3. $C \subset S_1$ for the only one irreducible component $S_1$ of $X_s$ and $C \cap D_1 = \emptyset$.

Let $E$ be the exceptional divisor of $\mu$, and $S'_i$ the strict transforms of the $S_i$ by $\mu$.

Case (1). Since $C$ is contractible, we have $(S_i \cdot C) < 0$ for $i = 1, 2$. Since $(X_s \cdot C) = 0$, there exists another irreducible component $S_3$ such that $(S_3 \cdot C) > 0$. Since the index of $K_{X/\Delta}$ at the point $S_1 \cap S_2 \cap S_3$ is 1, $P$ is not contained in $S_2$. Since the contraction of $C$ in $S_1$ yields a log terminal singularity, $P$ is the only singular point of $S_2$ on $C$. Then $C'$ intersects two irreducible components $E$ and $S'_1$. Since $E \cap S'_1$ is a normal crossing, and $K_{X^+/\Delta} + C$ is $\mu$-ample, we have $(K_{X^+/\Delta} \cdot C') = 0$. Since $\mu^*$ is an isomorphism on $X^+_s$ and $\mu^* K_{X^+/\Delta} = (K_{X/\Delta} + C)|_{X^+_s}$, we have $(K_{X^+/\Delta} \cdot C') = 0$. Therefore, $(K_{X^+/\Delta} \cdot C') \leq 0$.

Case (2). The proof is similar to Case (1).

Case (3). Since $C$ is contractible, we have $(S_i \cdot C) < 0$ for $i = 1, 2$. Since $(X_s \cdot C) = 0$, there exists another irreducible component $S_3$ such that $(S_3 \cdot C) > 0$. Since the index of $K_{X/\Delta}$ at the point $S_1 \cap S_2 \cap S_3$ is 1, $P$ is not contained in $S_2$. Since the contraction of $C$ in $S_1$ yields a log terminal singularity, $P$ is the only singular point of $S_2$ on $C$. Then $C'$ intersects two irreducible components $E$ and $S'_1$. Since $E \cap S'_1$ is a normal crossing, and $K_{X^+/\Delta} + C$ is $\mu$-ample, we have $(K_{X^+/\Delta} \cdot C') = 0$. Since $\mu^*$ is an isomorphism on $X^+_s$ and $\mu^* K_{X^+/\Delta} = (K_{X/\Delta} + C)|_{X^+_s}$, we have $(K_{X^+/\Delta} \cdot C') = 0$. Therefore, $(K_{X^+/\Delta} \cdot C') \leq 0$. The proof is complete.
and $S_3'$ of $X'$ besides $S_1'$ and $S_2'$. Hence we have $(S_i' \cdot C') = -1$ for $i = 1, 2$, and $(K_{X'/\Delta} \cdot C') = 0$, i.e., $C'$ is a $(-1, -1)$-curve.

Case (2). Since $C$ contracts to a log terminal singularity inside $S_1$, it follows that $C \cap D_1$ consists of a single point $P_1$, and there are at most another singular point $P_2$ of $S_1$ on $C$; if $S_1$ is nonsingular along $C \setminus \{P_1\}$, we let $P_2$ be an arbitrary point of $C \setminus \{P_1\}$. Let $r_i \geq 1$ be the indices of $K_{X/\Delta}$ at the points $P_i$ for $i = 1, 2$. Let $S_2$ be the other irreducible component of $X$ which intersects $C$, and $D$ the irreducible component of $D_1$ which contains $P_1$. We set $\frac{1}{r_1}(1, a_1)$ the type of the singularity of $S_1$ at $P_1$. Let $\alpha : S_1' \to S_1$ be the minimal resolution, and let $D''$ and $C''$ be the strict transforms of $D$ and $C$ by $\alpha$, respectively. Then $C''$ is a $(-1)$-curve, and the exceptional divisors $F_{i,j}$ ($i = 1, 2$ and $1 \leq j \leq d_i$ for some $d_i \geq 0$) of $\alpha$ over $P_i$ satisfy

$$
(D'' \cdot F_{1,1}) = (F_{1,1} \cdot F_{1,2}) = \ldots = (F_{1,d_1-1} \cdot F_{1,d_1}) = 1
$$

$$
(C'' \cdot F_{2,1}) = (F_{2,1} \cdot F_{2,2}) = \ldots = (F_{2,d_2-1} \cdot F_{2,d_2}) = 1
$$

under the suitable numbering. There are two subcases:

(2a) $P_2$ is a rational double point of type $A$ and $(C'' \cdot F_{1,j_0}) = 1$ for some $j_0$, and all other intersections are 0,

(2b) $(C'' \cdot F_{1,d_1}) = 1$, and all other intersections are 0.

In the subcase (2a), since $r_2 = 1$, we have $r_0 = r_1$ and $P = P_1$. Then $E_1 = E \cap S_1'$ is the image of $F_{1,1}$, and it is clear that $(K_{X'/\Delta} \cdot C') \leq 0$, where the equality holds if and only if $j_0 = 0$.

In the subcase (2b), let $\frac{1}{h_2r_2^2}(b_2h_2r_2 - 1, 1)$ be the type of $P_2 \in S_1$ for some positive integers $b_2$ and $h_2$ such that $b_2 < r_2$ (cf. Corollary 4.3). We also set $a_1b_1 = c_1r_1 + 1$ for some integers $b_1$ and $c_1$ such that $0 < b_1 \leq r_1$. The singularity of $S_1$ at $P_1$ (resp. $P_2$) can be described as a toroidal embedding whose lattice of 1-parameter subgroups is

$$
N = \mathbb{Z}^2 + \frac{1}{r_1}(1, a_1)\mathbb{Z} \quad \text{(resp. } \mathbb{Z}^2 + \frac{1}{h_2r_2^2}(b_2h_2r_2 - 1, 1)\mathbb{Z}),
$$

and $D''$, $F_{1,d_1}$, $C''$ (resp. $C''$, $F_{2,1}$) correspond to $(0, 1)$, $(b_1/r_1, 1/r_1)$, $(1, 0)$ (resp. $(1, 0)$, $((b_2h_2r_2 - 1)/h_2r_2^2, 1/h_2r_2^2)$) in $N_{\mathbb{Q}}$. Then the total transform of $C$ on $S_1''$ has coefficient $b_1/r_1$ (resp. $(b_2h_2r_2 - 1)/h_2r_2^2$) at $F_{1,d_1}$ (resp. $F_{2,1}$) and

$$(C^2) = -1 + b_1/r_1 + (b_2h_2r_2 - 1)/h_2r_2^2 < 0,$$

because $C$ is contractible. The discrepancy coefficient of $F_{1,d_1}$ (resp. $F_{2,1}$) with respect to $K_{S_1} + D$ is given by

$$(b_1 + 1)/r_1 - 1 - 1/r_1 = b_1/r_1 - 1 \quad \text{(resp. } b_2/r_2 - 1),$$

hence

$$
((K_{S_1} + D) \cdot C) = -1 - (b_1/r_1 - 1) - (b_2/r_2 - 1) = 1 - b_1/r_1 - b_2/r_2 < 0.
$$

Thus

$$
b_1/r_1 + b_2/r_2 - 1/b_2r_2^2 \leq 1 < b_1/r_1 + b_2/r_2 - 1 - c_1/r_1 < 0.
$$
for a positive integer e. Hence \(1/h_2r_2^2 > 1/r_1r_2\), i.e., \(r_1 > h_2r_2\). Therefore, \(r_0 = r_1\) and \(P = P_1\).

The exceptional divisor \(E_1 = E \cap S'_1\) is the image of \(F_{1,1}\), which corresponds to \((1/r_1, a_1/r_1)\). The new singularity at the intersection \(E_1 \cap C'\) is of type \(1/a_1(1,-r_1) = \frac{1}{a_1}(c_1,1)\), and

\[
(K_{X'/\Delta} \cdot C') = ((K_{S'_1} + D' + E_1) \cdot C') = -1 - (c_1/a_1 - 1) - (b_2/r_2 - 1) = 1/a_1r_1 - e/r_1r_2.
\]

Since \(b_1r_2 = (r_2 - b_2)r_1 + e \equiv e \pmod{r_1}\), we have \(r_2 \equiv a_1e \pmod{r_1}\), hence \(r_2 \leq a_1e\).

Therefore, \((K_{X'/\Delta} \cdot C') \leq 0\).

Case (3). If there is only one singular point of index greater than one on \(C\), then it is clear that \((K_{X'/\Delta} \cdot C') \leq 0\). We assume that there are more than one such singularities in the following. Then there are exactly 2 singular points, say \(P_1\) and \(P_2\), because \(C \subset S_1\) contracts to a rational singularity. Let \(\alpha : S''_1 \rightarrow S_1, C''\), and \(F_{i,j} (i = 1,2\) and \(1 \leq j \leq d_i\) for some \(d_i \geq 0\) be as in the case (2) except the intersection number with \(D''\). As before, let \(\frac{1}{h_1r_i^2}(b_ih_ir_i - 1,1)\) be the types of the \(P_i \in S_1\) for some positive integers \(r_i, b_i\) and \(h_i\) such that \(b_i < r_i\) for \(i = 1,2\). We set also \(a_ib_i = c_ir_i + 1\) for some integers \(a_i\) and \(c_i\) such that \(0 < a_i < r_i\). There are 2 subcases:

1. \((C'') \cdot F_{1,d_1}) = 1\), and all other intersections are 0,
2. \((C'') \cdot F_{1,j_0}) = 1\) for some \(j_0\) with \(1 < j_0 < d_1\), and all other intersections are 0.

In the subcase (3a), \(C\) is contracted to a log terminal singularity of type A. We may assume \(r_0 = r_1 \geq r_2, h_1 = 1\) and \(P = P_1\). Then

\[
(C^2) = -1 + (b_1r_1 - 1)/r_1^2 + (b_2h_2r_2 - 1)/h_2r_2^2 < 0
\]

\[
(K_{X/\Delta} \cdot C) = -1 - (b_1/r_1 - 1) - (b_2/r_2 - 1) < 0,
\]

thus

\[
b_1/r_1 + b_2/r_2 - 1/r_1^2 - 1/h_2r_2^2 < 1 < b_1/r_1 + b_2/r_2 = 1 + e/r_1r_2
\]

for a positive integer e. The exceptional divisor \(E_1 = E \cap S'_1\) corresponds to \((r_1 - a_1)/r_1^2, a_1/r_1^2) \in \mathbb{N}_0\) as before. We have

\[
((K_{S'_1} + E_1) \cdot C') = -1 - (c_1/a_1 - 1) - (b_2/r_2 - 1) = 1/a_1r_1 - e/r_1r_2.
\]

Since \(b_1r_2 = (r_2 - b_2)r_1 + e \equiv e \pmod{r_1}\), we have \(r_2 \equiv a_1e \pmod{r_1}\), hence \(r_2 \leq a_1e\). Therefore, \((K_{X'/\Delta} \cdot C') \leq 0\).

We shall prove that the subcase (3b) does not occur. If \(C\) is contracted to a log terminal singularity of type D, then we have one of the following:

1. \(d_1 = 3, j_0 = 2, (F_{1,1}^2) = (F_{1,3}^2) = -2,\) and \((F_{1,2}^2) = \ell\) for an integer \(\ell \geq 3,\)
2. \(d_2 = 1, j_0 = 2, (F_{1,1}^2) = -2,\) and \((F_{2,1}^2) = -3.\)

But then, \(P_1\) (resp. \(P_2\)) in the case (3b1) (resp. (3b2)) is of type \(\frac{1}{4\ell - 1}(2\ell - 1,1)\) (resp. \(\frac{1}{3}(1,1)\)), and cannot be of type \(\frac{1}{hr^r}(bhr - 1,1)\) for some positive integers \(r, b\) and \(h\) with \(0 < b < r\).

Next, assume that a type E singularity appears after the contraction. As for \(P_2\), a similar consideration as above shows that \(d_1 = 3, (F_{1,3}^2) = -4,\) or \(d_1 = 2,\) and...
\((F_{2,1}^2) = (F_{2,2}^2) = -3\). Then for \(P_1\), we have \(j_0 = 2\), and one of the following holds for an integer \(\ell \geq 2\):

- (3b3) \(d_1 = 3\), \((F_{1,1}^2) = -2\), \((F_{1,2}^2) = -\ell\), and \((F_{1,3}^2) = -3\),
- (3b4) \(d_1 = 3\), \((F_{1,1}^2) = -2\), \((F_{1,2}^2) = -\ell\), and \((F_{1,3}^2) = -4\),
- (3b5) \(d_1 = 3\), \((F_{1,1}^2) = -2\), \((F_{1,2}^2) = -\ell\), and \((F_{1,3}^2) = -5\),
- (3b6) \(d_1 = 4\), \((F_{1,2}^2) = -\ell\), and \((F_{1,3}^2) = (F_{1,4}^2) = -2\),
- (3b7) \(d_1 = 4\), \((F_{1,1}^2) = (F_{1,3}^2) = -2\), \((F_{1,2}^2) = -\ell\), and \((F_{1,4}^2) = -3\),
- (3b8) \(d_1 = 4\), \((F_{1,1}^2) = (F_{1,4}^2) = -2\), \((F_{1,2}^2) = -\ell\), and \((F_{1,3}^2) = -3\),
- (3b9) \(d_1 = 5\), \((F_{1,2}^2) = -\ell\), and \((F_{1,3}^2) = (F_{1,4}^2) = (F_{1,5}^2) = -2\),
- (3b10) \(d_1 = 6\), \((F_{1,2}^2) = -\ell\), and \((F_{1,3}^2) = (F_{1,4}^2) = (F_{1,5}^2) = (F_{1,6}^2) = -2\).

Then the only possible cases are (3b3) with \(\ell = 5\), (3b4) with \(\ell = 3\), and (3b7) with \(\ell = 6\). But in these cases, we have

\[
(K_{X/\Delta} \cdot C) = -1 + 1/2 + 4/5 > 0
\]

\[
(K_{X/\Delta} \cdot C) = -1 + 1/2 + 2/3 > 0
\]

\[
(K_{X/\Delta} \cdot C) = -1 + 1/2 + 6/7 > 0,
\]

a contradiction. Q.E.D.

**Proof of Theorem 5.5 continued.** If \((K_{X'/\Delta} \cdot C') < 0\), we put \(X^{(0)} = X'\). Otherwise, we have \((K_{X'/\Delta} \cdot C') = 0\). Then there exists a projective birational morphism \(\psi : X' \to Z'\) over \(\Delta\) which contracts only \(C'\) as in the contraction theorem. We shall construct a flop of \(\psi\):

\[
X' \xrightarrow{\psi} Z' \xleftarrow{\psi^+} X^{(0)}.
\]

In the case (1) of Lemma 5.6, the blowing up of \(C'\) and the contraction in the other direction give us the flop. In other cases, since \(R^1\psi_* \mathcal{O}_{X'}(mK_{X'/\Delta}) = 0\) for any integer \(m\), \(Z'\) satisfies the conditions in (1.1) except that the irreducible components of \(Z'_s\) are not necessarily \(\mathbb{Q}\)-Cartier. Let \(Q' = \psi(C')\) and \(r'\) the index of \(K_{Z'/\Delta}\) at \(Q'\). As in Theorems 4.1 or 4.4, we can prove that the completion of the canonical cover of \(\mathcal{O}_{Z',Q'}\) with the \(\mu_{r'}\)-action is isomorphic to that of \(A[x,y,z]/(F')\) with \(F' = xy + \tau G'(r')z\) for some semi-invariant coordinates \(x, y, z\) and a generator \(\tau\) of the maximal ideal of \(A\), where the action is given by \(x \mapsto \zeta^{a'} \otimes x, y \mapsto \zeta^{-a'} \otimes y,\) and \(z \mapsto \zeta \otimes z\) for some positive integer \(a'\) such that \((r', a') = 1\). Since the strict transforms of \(S_1\) and \(E\) on \(Z'\) are not \(\mathbb{Q}\)-Cartier at \(Q'\), \(G'\) is not invertible. So if \(\psi\) is obtained from the blowing up at the ideal \((x, \tau)\), then the blowing up at \((y, \tau)\) gives us the flop \(\psi^+ : X^{(0)} \to Z'\).

Now we apply the minimal model program to the induced morphism \(f^{(0)} : X^{(0)} \to Z\). We note that \(\rho(X^{(0)}/Z) = 2\). Since \(K_{X^{(0)}/\Delta}\) is not relatively nef over \(Z\), there exists a contraction morphism \(\varphi^{(0)} : X^{(0)} \to Z^{(0)}\) with respect to an extremal ray of \(\overline{\text{NE}}(X^{(0)}/Z)\). In the case \((K_{X'/\Delta} \cdot C') < 0\), we take \(\varphi^{(0)}\) to be the contraction of \(C'\).

If \(\varphi^{(0)}\) is a small contraction, then the flip \(\varphi^{(0)+} : X^{(1)} \to Z^{(0)}\) exists, since \(r(\varphi^{(0)}) \leq r_0\), and if the equality holds, then \(n(\varphi^{(0)}) < n(\varphi)\). Let \(\varphi^{(1)} : X^{(1)} \to Z^{(1)}\) be the contraction morphism with respect to an extremal ray of \(\overline{\text{NE}}(X^{(1)}/Z)\). If it is a small contraction again, then the flip \(\varphi^{(1)+} : X^{(2)} \to Z^{(1)}\) exists by the same reason. We continue this process. By Theorem 5.2, we obtain a divisorial contraction \(\varphi^{(k)} : X^{(k)} \to Z^{(k)} = X^+\) for a nonnegative integer \(k\), where \(k = 0\) may
occur only if \((K_{X'/\Delta} \cdot C') = 0\). By Theorem 5.3, \(X^+\) satisfies the conditions in (1.1).

We shall prove that the composite birational map \(X \dashrightarrow X^+\) is not the identity. Then the induced morphism \(\varphi^+ : X^+ \to Z\) gives us the flip of \(\varphi\), because \(\rho(X^+/Z) = 1\) and \(K_{X^+/\Delta}\) must be \(\varphi^+\)-ample. Let \(v\) be a discrete valuation of the fraction field of \(X\) which has the center \(C'\) on \(X'\). Let \(d\) and \(d^+\) be the discrepancy coefficients of \(v\) over \(X\) and \(X^+\), respectively. If \(s = 0\), then \(d < d^+\), because \(C'\) is not contained in \(E\), while the flopped curve is contained in the strict transform of \(E\). If \(s > 1\), then \(d < d^+\), because a flip or a divisorial contraction increases the discrepancy. Therefore, the theorem is proved under the additional assumption that all the singular points of index \(r_0\) are of simple type.

Next, we consider the general case. By the proof of Lemma 5.6, we have to consider only the case (3) there. Let \(P \in \Lambda(\varphi)\). By Theorem 4.1, the completion of \(X\) at \(P\) has an equation \(xy + G_P(z^{r_0}) = 0\) in the \(\mu_{r_0}\)-quotient of the completion of \(\text{Spec} A[x,y,z]\). Since \(X_0\) is smooth, the generic fiber of the 1-dimensional scheme defined by \(G_P(w) = 0\) in the completion of \(\text{Spec} A[w]\) is geometrically reduced.

Let \(\bar{A}\) be complete discrete valuation ring which is finite over \(A\) and such that \(\bar{\Delta} = \text{Spec} \bar{A}\) dominates all the irreducible components of these 1-dimensional schemes for all \(P \in \Lambda(\varphi)\). Let \(\bar{X} = X \times_{\Delta} \bar{\Delta}\), \(\bar{Z} = Z \times_{\Delta} \bar{\Delta}\), \(\sigma : \bar{X} \to X\) and \(\bar{\varphi} : \bar{X} \to \bar{Z}\) the natural morphisms, \(\bar{P} = \sigma^{-1}(P)\) for \(P \in \Lambda(\varphi)\), and \(\bar{\tau}\) be a generator of the maximal ideal of \(\bar{\Delta}\). Since \(X_0\) is normal, the completion of \(\bar{X}\) at \(\bar{P}\) is described by an equation \(xy + \prod_j G_{P,j}(z^{r_0}) = 0\) where the \(G_{P,j}(\mod \bar{A}[z^{r_0}]) \in z^{r_0}k[z^{r_0}]\) contain linear terms in \(z^{r_0}\). By the argument in [K3, p.116], there exists a projective birational morphism \(\lambda : Y \to \bar{X}\) which contracts only a finite number of curves and such that \(K_{Y/\bar{\Delta}} = \lambda^*K_{\bar{X}/\bar{\Delta}}\); the induced morphism \(g : Y \to \bar{\Delta}\) satisfies the conditions in (1.1), and that \(Y\) has only singularities of simple type. If we apply the minimal model program to the small morphism \(\bar{\varphi} \circ \lambda : Y \to \bar{Z}\), then there exists a projective birational morphism \(\bar{\varphi}^+ : Y^+ \to \bar{Z}\) such that \(K_{Y^+/\bar{\Delta}}\) is \(\bar{\varphi}^+\)-nef by the first part of the proof. Then there exists a positive integer \(m_0\) such that the natural homomorphisms

\[
\bar{\varphi}_*O_{\bar{X}}(m_0K_{\bar{X}/\bar{\Delta}}) \otimes \bar{\varphi}_*O_{\bar{X}}(nm_0K_{\bar{X}/\bar{\Delta}}) \to \bar{\varphi}_*O_{\bar{X}}((n+1)m_0K_{\bar{X}/\bar{\Delta}})
\]

are surjective for any positive integer \(n\). By the flat base change theorem, we deduce the existence of the flip of \(\varphi\). Q.E.D.

As explained in [KMM], the cone, contraction, and flip theorems combined yield the following:

5.7. Minimal Model Theorem. Let \(f^{(0)} : X^{(0)} \to \Delta\) be a semistable projective morphism of relative dimension 2. Then there exist a projective flat morphism \(f : X \to \Delta\) from a \(Q\)-factorial scheme \(X\) satisfying the conditions in (1.1) and a birational map \(\alpha : X^{(0)} \dashrightarrow X\) over \(\Delta\) such that one of the following holds:

(a) \(K_{X/\Delta}\) is \(f\)-nef,
(b) \(\rho(X/\Delta) = 2\), and there exist a projective flat morphism \(g : Z \to \Delta\) of relative dimension 1 and a projective surjective morphism \(\varphi : X \to Z\) over \(\Delta\) such that \(-K_{X/\Delta}\) is \(\varphi\)-ample,
(c) \(\rho(X/\Delta) = 1\), and \(-K_{X/\Delta}\) is \(f\)-ample.

In the case (a), if \(\kappa(X^{(0)}_{\eta}) = 0\), then \(O_X(12K_{X/\Delta}) \in f^*\text{Pic}(\Delta)\) by the upper semicontinuity theorem.
5.8. Theorem. In the case (a) of Theorem 5.7, if \( \kappa(X^{(0)}_q) = 2 \), then there exists a positive integer \( m_0 \) such that the natural homomorphism

\[
f^*_s \mathcal{O}_X(m_0K_{X/\Delta}) \to \mathcal{O}_X(m_0K_{X/\Delta})
\]

is surjective. In particular, the relative canonical ring

\[
\bigoplus_{m=0}^{\infty} f^*_s \mathcal{O}_X^{(0)}(mK_{X^{(0)}_s/\Delta})
\]

is finitely generated over \( A \).

**Proof.** We may assume that \( A \) is a complete discrete valuation ring whose residue field is algebraically closed. First, we shall prove that \( |m_1K_{X_s}| \) is free for a positive integer \( m_1 \). The proof is similar to that in [K5, middle of p.356 to 357]. But we do not use the log canonical cover since it is not necessarily normal. Let \( \kappa = \kappa(S_i, K_{S_i} + D_i) \) for the irreducible components \( S_i \) of \( X_s \). We may assume that \( D_i \neq 0 \) for all \( i \) when we glue sections of pluricanonical sheaves. Under this assumption, if \( \nu_i = 0 \), then \( S_i \) is birationally equivalent to a ruled surface over a curve \( C_i \) of genus \( g_i \). If \( g_i > 0 \), then \( D_i \) contains either 2 sections or a double section which may be an inseparable section. Hence \( g_i \leq 1 \).

We claim that \( D_i \) is connected if \( \nu_i = 0 \) and \( S_i \) is rational. If we apply the log minimal model program to the pair \((S_i, 0)\), we obtain a projective birational morphism \( \mu_i : S_i \to S'_i \) to a surface \( S'_i \) with only log terminal singularities such that \( \rho(S'_i) = 1 \), or that \( \rho(S'_i) = 2 \) and there is a surjective morphism \( \varphi_i : S'_i \to C_i \) whose fibers are irreducible. Then \( D'_i = \mu_i(D_i) \) is connected. Tracing back the morphism \( \mu_i \), we conclude that \( D_i \) is also connected.

If \( \nu_i = 0 \), then \( ((K_{S_i} + D_i) \cdot \Gamma) = 0 \) for any \( \Gamma \subset D_i \), and there are the following possibilities:

1. \( S_i \) is a rational surface, and \( D_i \) is a nonsingular elliptic curve,
2. \( S_i \) is a rational surface, and \( D_i \) is a rational curve with a node or a cycle of nonsingular rational curves,
3. \( S_i \) is a rational surface, and \( D_i \) is a nonsingular rational curve on which there are 4 ordinary double points of \( S_i \), or a rod of nonsingular rational curves each of whose 2 end components carries 2 ordinary double points of \( S_i \),
4. \( S_i \) is a rational surface, and \( D_i \) is a nonsingular rational curve on which there are 3 singular points of \( S_i \),
5. \( S_i \) is birationally equivalent to an elliptic ruled surface, and \( D_i \) consists of 2 disjoint nonsingular elliptic curves which are sections of the ruling,
6. \( S_i \) is birationally equivalent to an elliptic ruled surface, and \( D_i \) is a nonsingular elliptic curve which is a double section of the ruling.

In the case (4), the log indices of \( K_{S_i} + D_i \) at the singular points are \( (2, 3, 6), (2, 4, 4) \), or \( (3, 3, 3) \). Therefore as in [K5], we obtain canonical sections in \( H^0(D_i, 12(K_{S_i} + D_i)) \), hence a section in \( H^0(B(0), 12m_2K_{X_s}) \) for a positive integer \( m_2 \), where \( B(0) \) is the union of the \( S_i \) with \( \nu_i = 0 \). Then the rest is the same as in [K5] and \( \mathcal{O}_{X_s}(m_1K_{X_s}) \) is generated by global sections for a positive integer \( m_1 \).

Next, we shall prove that \( H^1(X_s, m_0K_{X_s}) = 0 \) for a positive multiple \( m_0 \) of \( m_1 \). Then the sections in \( H^0(X_s, m_1K_{X_s}) \) are extended to \( X_s \) and the proof of finiteness follows.
the theorem is completed. The free linear system \([m_1K_{X_s}]\) gives a morphism \(\psi : X_s \to Y_s\) with connected fibers to a 2-dimensional projective scheme \(Y_s\) such that \(m_1K_{X_s} = \psi^*H\) for an ample divisor \(H\) on \(Y_s\). Thus it is enough to prove that 
\[ R^1\psi_*\mathcal{O}_{X_s} = 0. \]
The proof is similar to that of Theorem 2.3. If \(\nu_i = 2\), then \(\psi|_{S_i}\) contracts only a finite number of curves, and we have 
\[ R^1\psi_*\mathcal{O}_{S_i}(-D_i) = 0. \]
For \(\nu_i = 1\), we can use Lemma 2.1. So we have to prove that 
\[ H^1(B(0), \mathcal{O}_{B(0)}) = 0. \]

Let \(K^0 = \bigoplus_i \mathcal{O}_{S_i}, K^1 = \bigoplus_{i>j} \mathcal{O}_{D_{ij}},\) and \(K^2 = \bigoplus_{i>j>k} \mathcal{O}_{P_{ijk}},\) where \(D_{ij} = S_i \cap S_j,\) 
\(P_{ijk} = S_i \cap S_j \cap S_k,\) and the summations are taken for all \(i, j, k\) such that \(\nu_i = \nu_j = \nu_k = 0.\) Then we have an exact sequence

\[ 0 \to \mathcal{O}_{B(0)} \to K^0 \to K^1 \to K^2 \to 0. \]

Since there are components \(S_i\) such that \(\nu_i = 2,\) we can check that the homomorphism \(\bigoplus_i H^1(S_i, \mathcal{O}_{S_i}) \to \bigoplus_{i>j} H^1(D_{ij}, \mathcal{O}_{D_{ij}})\) is injective, and that the dual graph of intersections of irreducible components of \(B(0)\) has no cycle. Then by the spectral sequence \(E_1^{p,q} = H^q(K^p) \Rightarrow H^{p+q}(B(0), \mathcal{O}_{B(0)}),\) we obtain \(H^1(B(0), \mathcal{O}_{B(0)}) = 0.\)

Q.E.D.

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