1. Introduction

A dynamical system is said to be uniquely ergodic when it admits a unique, hence necessarily ergodic, invariant measure. In most classical ergodic-theoretic settings, the dynamical system consists of a measure space along with a single map, or at most a countable semigroup of transformations; unique ergodicity
has been of longstanding interest for such systems. In contrast, unique ergodicity for systems consisting of a “larger” space of transformations (such as the automorphism group of a structure) has been a focus of more recent research, notably that of Glasner and Weiss [GW02], and of Angel, Kechris, and Lyons [AKL14].

When studying ergodicity for a class of dynamical systems, it is natural to consider the minimal such systems. For continuous dynamical systems, one often studies minimal flows, i.e., continuous actions on compact Hausdorff spaces such that each orbit is dense; [AKL14] considers unique ergodicity in this setting. We are interested in unique ergodicity of actions where the underlying space need not be compact and there is just one orbit. In particular, we consider the logic action of the group $S_\infty$ on an orbit, and we characterize all such actions that are uniquely ergodic.

Any transitive $S_\infty$-space is isomorphic to the action of $S_\infty$ on the isomorphism class of a countable structure, restricted to a fixed underlying set. Ackerman, Freer, and Patel [AFP12] characterized those countable structures in a countable language whose isomorphism classes (of structures having a fixed underlying set) admit $S_\infty$-invariant measures. Here we characterize those countable structures whose isomorphism classes admit exactly one such measure, and show via a result of Peter J. Cameron that the five reducts of $(\mathbb{Q},<)$ are essentially the only ones. Furthermore, if the isomorphism class of a countable structure admits more than one $S_\infty$-invariant measure, it must admit continuum-many ergodic such measures.

1.1. Motivation and main results. In this paper we consider, for a given countable language $L$, the collection of countable $L$-structures having the set of natural numbers $\mathbb{N}$ as underlying set. This collection can be made into a measurable space, denoted $\text{Str}_L$, in a standard way, as we describe in Section 2.

The group $S_\infty$ of permutations of $\mathbb{N}$ acts naturally on $\text{Str}_L$ by permuting the underlying set of elements. This action is known as the logic action of $S_\infty$ on $\text{Str}_L$, and has been studied extensively in descriptive set theory. For details, see [BK96, §2.5] or [Gao09, §11.3]. Observe that the $S_\infty$-orbits of $\text{Str}_L$ are precisely the isomorphism classes of structures in $\text{Str}_L$.

By an invariant measure on $\text{Str}_L$, we will always mean a Borel probability measure on $\text{Str}_L$ that is invariant under the logic action of $S_\infty$. We are specifically interested in those invariant measures on $\text{Str}_L$ that assign probability 1 to a single orbit, i.e., the isomorphism class in $\text{Str}_L$ of some countable $L$-structure $\mathcal{M}$. In this case we say that the orbit of $\mathcal{M}$ admits an invariant measure, or simply that $\mathcal{M}$ admits an invariant measure.

When a countable structure $\mathcal{M}$ admits an invariant measure, this measure can be thought of as providing a symmetric random construction of $\mathcal{M}$. The main result of [AFP12] describes precisely when such a construction is possible: a structure $\mathcal{M} \in \text{Str}_L$ admits an invariant measure if and only if definable
closure in $\mathcal{M}$ is trivial, i.e., the pointwise stabilizer in $\text{Aut}(\mathcal{M})$ of any finite tuple fixes no additional elements. But even when there are invariant measures concentrated on the orbit of $\mathcal{M}$, it is not obvious how many different ones there are.

Many natural examples admit more than one invariant measure. For instance, consider the Erdős–Rényi \cite{ER59} construction $G(\mathbb{N}, p)$ of the Rado graph, a countably infinite random graph in which edges have independent probability $p$, where $0 < p < 1$. This yields continuum-many ergodic invariant measures concentrated on the orbit of the Rado graph, as each value of $p$ leads to a different ergodic invariant measure.

On the other hand, some countable structures admit just one invariant measure. One such example is well-known: The rational linear order $(\mathbb{Q}, <)$ admits a unique invariant measure, which can be described as follows. For every $n \in \mathbb{N}$ and $n$-tuple of distinct elements of $\mathbb{Q}$, by $S_\infty$-invariance, each of the $n!$-many orderings of the $n$-tuple must be assigned the same probability. This collection of finite-dimensional marginal distributions determines a (necessarily unique) invariant measure on $\text{Str}_L$, by the Kolmogorov extension theorem (see, e.g., \cite[Theorems 6.14 and 6.16]{Kal02}). This probability measure can be shown to be concentrated on the orbit of $(\mathbb{Q}, <)$ and is sometimes known as the Glasner–Weiss measure; for details, see \cite[Theorems 8.1 and 8.2]{GW02}. We will discuss this measure and its construction further in Section 4.

In fact, these examples illustrate the only possibilities: Either a countable structure admits no invariant measure, or a unique invariant measure, or continuum-many ergodic invariant measures. (Convex combinations of invariant measures are also invariant, but our main result implies that a countable structure admitting more than one invariant measure in fact admits continuum-many ergodic ones.) Furthermore, a countable structure admits a unique invariant measure precisely when it has the property known as high homogeneity. The main result of this paper is the following.

**Theorem 1.1.** Let $\mathcal{M}$ be a countable structure in a countable language $L$. Then exactly one of the following holds:

1. The structure $\mathcal{M}$ has nontrivial definable closure, in which case there is no $S_\infty$-invariant Borel probability measure on $\text{Str}_L$ that is concentrated on the orbit of $\mathcal{M}$.
2. The structure $\mathcal{M}$ is highly homogeneous, in which case there is a unique $S_\infty$-invariant Borel probability measure on $\text{Str}_L$ that is concentrated on the orbit of $\mathcal{M}$.
3. There are continuum-many ergodic $S_\infty$-invariant Borel probability measures on $\text{Str}_L$ that are concentrated on the orbit of $\mathcal{M}$. 


Moreover, by a result of Peter J. Cameron, the case where $\mathcal{M}$ is highly homogeneous is equivalent to $\mathcal{M}$ being interdefinable with a definable reduct (henceforth reduct) of $(\mathbb{Q}, <)$, of which there are five. In particular, this shows the known invariant measures on these five to be canonical.

1.2. Additional motivation. The present work has been motivated by further considerations, which we now describe.

Fouché and Nies [Nie13, §15] provide a certain definition of an algorithmically random presentation of a given countable structure; see also [Fou13], [Fou12], and [FP98]. In the case of the rational linear order $(\mathbb{Q}, <)$, they note that their notion of randomness is in a sense canonical by virtue of the unique ergodicity of the orbit of $(\mathbb{Q}, <)$. Hence one may ask which other orbits of countable structures are uniquely ergodic. Theorem 1.1, along with the result of Cameron, shows that the orbits of $(\mathbb{Q}, <)$ and of its reducts are essentially the only instances.

We also note a connection with “Kolmogorov’s example” of a transitive but non-ergodic action of $S_\infty$, described in [Ver03]. Many settings in classical ergodic theory permit at most one invariant measure. For example, when a separable locally compact group $G$ acts continuously and transitively on a Polish space $X$, there is at most one $G$-invariant probability measure on $X$ [Ver03, Theorem 2]. The action of $S_\infty$, however, allows for continuum-many ergodic invariant measures on the same orbit, as noted above in the case of the Erdős–Rényi constructions; for more details, see [Ver03, §3]. Indeed, many specific orbits with this property are known, but the present work strengthens the sense in which this is typical for $S_\infty$ and uniqueness is rare: There are essentially merely five exceptions to the rule of having either continuum-many ergodic invariant measures or none.

2. Preliminaries

In this paper, $L$ will always be a countable language. We consider the space $\text{Str}_L$ of countable $L$-structures having underlying set $\mathbb{N}$, equipped with the $\sigma$-algebra of Borel sets generated by the topology described in Definition 2.1.

Recall that $\mathcal{L}_{\omega_1,\omega}(L)$ denotes the infinitary language based on $L$ consisting of formulas that can have countably infinite conjunctions and disjunctions, but only finitely many quantifiers and free variables; for details, see [Kec95, §16.C].

**Definition 2.1.** Given a formula $\varphi \in \mathcal{L}_{\omega_1,\omega}(L)$ and $n_0, \ldots, n_{j-1} \in \mathbb{N}$, where $j$ is the number of free variables of $\varphi$, define

$$[\varphi(n_0, \ldots, n_{j-1})] := \{ \mathcal{M} \in \text{Str}_L : \mathcal{M} \models \varphi(n_0, \ldots, n_{j-1}) \}.$$ 

Sets of this form are closed under finite intersection, and form a basis for the topology of $\text{Str}_L$. 
Consider $S_\infty$, the permutation group of the natural numbers $\mathbb{N}$. This group acts on $\text{Str}_L$ via the logic action: for $g \in S_\infty$ and $\mathcal{M}, \mathcal{N} \in \text{Str}_L$, we define $g \cdot \mathcal{M} = \mathcal{N}$ to hold whenever

$$R^\mathcal{N}(s_0, \ldots, s_{k-1}) \quad \text{if and only if} \quad R^\mathcal{M}(g^{-1}(s_0), \ldots, g^{-1}(s_{k-1}))$$

for all relation symbols $R \in L$ and $s_0, \ldots, s_{k-1} \in \mathbb{N}$, where $k$ is the arity of $R$, and similarly with constant and function symbols. Observe that the orbit of a structure under the logic action is its isomorphism class in $\text{Str}_L$; every such orbit is Borel by Scott's isomorphism theorem. For more details on the logic action, see [Kec95, §16.C].

We define an invariant measure on $\text{Str}_L$ to be a Borel probability measure on $\text{Str}_L$ that is invariant under the logic action of $S_\infty$ on $\text{Str}_L$. When an invariant measure on $\text{Str}_L$ is concentrated on the orbit of some structure in $\text{Str}_L$, we may restrict attention to this orbit, and speak equivalently of an invariant measure on the orbit itself.

A structure $\mathcal{M} \in \text{Str}_L$ has trivial definable closure when the pointwise stabilizer in $\text{Aut}(\mathcal{M})$ of an arbitrary finite tuple of $\mathcal{M}$ fixes no additional points:

**Definition 2.2.** Let $\mathcal{M} \in \text{Str}_L$. For a tuple $a \in \mathcal{M}$, the definable closure of $a$ in $\mathcal{M}$, written $\text{dcl}_\mathcal{M}(a)$, is the set of elements of $\mathcal{M}$ that are fixed by every automorphism of $\mathcal{M}$ that fixes $a$ pointwise. The structure $\mathcal{M}$ has trivial definable closure when $\text{dcl}_\mathcal{M}(a) = a$ for every (finite) tuple $a \in \mathcal{M}$.

We will often use the notation $\mathbf{x}$ to denote the tuple of variables $x_0 \cdots x_{n-1}$, where $n = |\mathbf{x}|$.

**2.1. Canonical structures.** In the proof of our main theorem, we will work in the setting of ultrahomogeneous structures. The notions of canonical languages and canonical structures provide such a setting. We provide a brief description of these notions here; for more details, see [AFP12, §2.5].

**Definition 2.3.** Let $G$ be a closed subgroup of $S_\infty$, and consider the action of $G$ on $\mathbb{N}$. Define the canonical language for $G$ to be the (countable) relational language $L_G$ that consists of, for each $k \in \mathbb{N}$ and $G$-orbit $E \subseteq \mathbb{N}^k$, a $k$-ary relation symbol $R_E$. Then define the canonical structure for $G$ to be the structure $C_G \in \text{Str}_{L_G}$ in which, for each $G$-orbit $E$, the interpretation of $R_E$ is the set $E$.

**Definition 2.4.** Given a structure $\mathcal{M} \in \text{Str}_L$, define the canonical language for $\mathcal{M}$, written $L_{\overline{\mathcal{M}}}$, to be the countable relational language $L_G$ where $G := \text{Aut}(\mathcal{M})$. Similarly, define the canonical structure for $\mathcal{M}$, written $\overline{\mathcal{M}}$, to be the countable $L_{\overline{\mathcal{M}}}$-structure $C_G$.

We now define when two structures are interdefinable; any such structures will be regarded as interchangeable for purposes of our classification.
Definition 2.5. Let $\mathcal{M}$ and $\mathcal{N}$ be structures in (possibly different) countable languages, both having underlying set $\mathbb{N}$. Then $\mathcal{M}$ and $\mathcal{N}$ are said to be interdefinable when they have the same canonical language and same canonical structure.

Equivalently, by [AFP12, Lemma 2.14], two structures are interdefinable if and only if there is an $L_{\omega_1, \omega}$-interdefinition between them, in the terminology of [AFP12, Definition 2.11].

By Definitions 2.4 and 2.5, it is immediate that a structure $\mathcal{M} \in \text{Str}_L$ and its canonical structure $\overline{\mathcal{M}}$ are interdefinable.

Proposition 2.6. Let $\mathcal{M} \in \text{Str}_L$. There is a Borel bijection, respecting the action of $S_\infty$, between the orbit of $\mathcal{M}$ in $\text{Str}_L$ and the orbit of its canonical structure $\overline{\mathcal{M}}$ in $\text{Str}_{L_{\overline{\mathcal{M}}}}$. In particular, this map induces a bijection between the sets of ergodic invariant measures on the orbit of $\mathcal{M}$ and those on the orbit of $\overline{\mathcal{M}}$.

Proof. First observe that the orbit of $\mathcal{M}$ and the orbit of $\overline{\mathcal{M}}$ are each Borel spaces that inherit the logic action. The structures $\mathcal{M}$ and $\overline{\mathcal{M}}$ are interdefinable, and so [AFP12, Lemma 2.15] applies. The proof of this lemma provides explicit maps between $\text{Str}_L$ and $\text{Str}_{L_{\overline{\mathcal{M}}}}$ which, when restricted respectively to the orbit of $\mathcal{M}$ and of $\overline{\mathcal{M}}$, have the desired property. \qed

We immediately obtain the following corollary.

Corollary 2.7. Let $\mathcal{M} \in \text{Str}_L$. Then $\mathcal{M}$ and its canonical structure $\overline{\mathcal{M}}$ admit the same number of ergodic invariant measures.

The following result is straightforward (see, e.g., [AFP12, Lemma 2.16]).

Lemma 2.8. Let $\mathcal{M}$ and $\mathcal{N}$ be interdefinable structures in (possibly different) countable languages, both having underlying set $\mathbb{N}$. Then $\mathcal{M}$ has trivial definable closure if and only if $\mathcal{N}$ does.

We therefore have the following.

Corollary 2.9. Let $\mathcal{M} \in \text{Str}_L$. Then $\mathcal{M}$ has trivial definable closure if and only if its canonical structure $\overline{\mathcal{M}}$ has trivial definable closure.

2.2. Ultrahomogeneous structures. Countable ultrahomogeneous structures play an important role throughout this paper: For any $\mathcal{M} \in \text{Str}_L$, we will see in Lemma 2.15 that $\mathcal{M}$ is highly homogeneous if and only if $\overline{\mathcal{M}}$ is; combining this fact with Corollaries 2.7 and 2.9, when proving Theorem 1.1 it suffices to consider instead the number of invariant measures admitted by the canonical structure $\overline{\mathcal{M}}$. In fact, every canonical structure is ultrahomogeneous and has a relational language; these properties will allow us to simplify several arguments and constructions, by restricting attention to ultrahomogeneous structures in a relational language.
Definition 2.10. A countable structure $\mathcal{M}$ is ultrahomogeneous if every isomorphism between finitely generated substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$.

The following fact is standard.

Proposition 2.11 ([AFP12, Proposition 2.17]). Let $\mathcal{M} \in \text{Str}_L$. The canonical structure $\mathcal{M}$ is ultrahomogeneous.

The age of an $L$-structure $\mathcal{M}$, denoted $\text{Age}(\mathcal{M})$, is defined to be the class of finitely generated $L$-structures that can be embedded in $\mathcal{M}$. Observe that any age is closed under isomorphism. (For details, see [Hod93, §7.1].) When $\mathcal{M}$ is ultrahomogeneous, $\text{Age}(\mathcal{M})$ satisfies the following three properties: the hereditary property, stating that the age is closed under substructures; the joint embedding property, stating that for any two elements of the age, there is a third element into which they each embed; and the amalgamation property, stating that any two elements of the age can be amalgamated over their intersection to obtain some other element of the age.

When the language $L$ is relational and $\mathcal{M}$ is ultrahomogeneous, $\mathcal{M}$ having trivial definable closure is equivalent to saying that $\text{Age}(\mathcal{M})$ has the strong amalgamation property, which requires the amalgamation of a pair of elements in the age to be injective on the union of their underlying sets, not just on their intersection; for more details see [Hod93, Theorem 7.1.8] or [Cam90, §2.7].

Ultrahomogeneous structures can be given particularly convenient $L_{\omega_1,\omega}(L)$ axiomatizations via pithy $\Pi_2$ sentences, which can be thought of as “one-point extension axioms”.

Definition 2.12 ([AFP12, Definitions 2.3 and 2.4]). A sentence in $L_{\omega_1,\omega}(L)$ is $\Pi_2$ when it is of the form $(\forall x)(\exists y)\psi(x,y)$, where the (possibly empty) tuple $xy$ consists of distinct variables, and $\psi(x,y)$ is quantifier-free. A countable theory $T$ of $L_{\omega_1,\omega}(L)$ is $\Pi_2$ when every sentence in $T$ is $\Pi_2$.

A $\Pi_2$ sentence $(\forall x)(\exists y)\psi(x,y) \in L_{\omega_1,\omega}(L)$, where $\psi(x,y)$ is quantifier-free, is said to be pithy when the tuple $y$ consists of precisely one variable. A countable $\Pi_2$ theory $T$ of $L_{\omega_1,\omega}(L)$ is said to be pithy when every sentence in $T$ is pithy. Note that we allow the degenerate case where $x$ is the empty tuple and $\psi$ is of the form $(\exists y)\psi(y)$.

The following result follows from essentially the same proof as [AFP12, Proposition 2.17].

Proposition 2.13. Let $L$ be relational and let $\mathcal{M} \in \text{Str}_L$ be ultrahomogeneous. There is a countable $L_{\omega_1,\omega}(L)$-theory, every sentence of which is pithy $\Pi_2$, and all of whose countable models are isomorphic to $\mathcal{M}$.

We will call this theory the Fraïssé theory of $\mathcal{M}$.
2.3. **Highly homogeneous structures.** *High homogeneity* is the key notion in our characterization of structures admitting a unique invariant measure.

**Definition 2.14 ([Cam90, §2.1]).** A structure \( M \in \text{Str}_L \) is **highly homogeneous** when for each \( k \in \mathbb{N} \) and for every pair of \( k \)-element sets \( X, Y \subseteq M \) there is some \( f \in \text{Aut}(M) \) such that \( Y = \{ f(x) : x \in X \} \).

The following lemma is immediate, and allows us to generalize the notion of high homogeneity to permutation groups.

**Lemma 2.15.** A structure \( M \in \text{Str}_L \) is highly homogeneous if and only if its canonical structure is.

**Definition 2.16 ([Cam90, §2.1]).** A closed permutation group \( G \) on \( \mathbb{N} \) is called **highly homogeneous** when its canonical structure \( C_G \) is highly homogeneous.

The crucial fact about highly homogeneous structures is the following.

**Lemma 2.17.** Let \( L \) be relational and let \( M \in \text{Str}_L \) be ultrahomogeneous. Then \( \text{Aut}(M) \) is highly homogeneous if and only if for any \( k \in \mathbb{N} \), all \( k \)-element structures in \( \text{Age}(M) \) are isomorphic.

*Proof.* Let \( M \in \text{Str}_L \) be ultrahomogeneous, and suppose \( X, Y \in \text{Age}(M) \) are of size \( k \). Without loss of generality, we may assume that \( X \) and \( Y \) are substructures of \( M \). If \( \text{Aut}(M) \) is highly homogeneous, then \( X \) and \( Y \) are isomorphic via the restriction of any \( f \in \text{Aut}(M) \) such that \( Y = \{ f(x) : x \in X \} \).

Conversely, suppose \( X, Y \subseteq M \) are \( k \)-element sets (necessarily substructures, as \( L \) is relational). If there is some isomorphism of structures \( g : X \to Y \), then by the ultrahomogeneity of \( M \), there is some \( f \in \text{Aut}(M) \) extending \( g \) to all of \( M \). \( \square \)

Highly homogeneous structures have been classified explicitly by Cameron [Cam76], and characterized (up to interdefinability) as the five reducts of \((\mathbb{Q},<)\), as we now describe.

Let \((\mathbb{Q},<)\) be the set of rational numbers equipped with the usual order. The following three relations are definable within \((\mathbb{Q},<)\):

1. The ternary linear *betweenness* relation \( B \), given by
   \[
   B(a,b,c) \iff (a < b < c) \lor (c < b < a).
   \]
2. The ternary *circular order* relation \( K \), given by
   \[
   K(a,b,c) \iff (a < b < c) \lor (b < c < a) \lor (c < a < b).
   \]
3. The quaternary *separation* relation \( S \), given by
   \[
   S(a,b,c,d) \iff (K(a,b,c) \land K(b,c,d) \land K(c,d,a))
   \lor (K(d,c,b) \land K(c,b,a) \land K(b,a,d)).
   \]
The structure \((\mathbb{Q}, B)\) can be thought of as forgetting the direction of the order, \((\mathbb{Q}, K)\) as gluing the rational line into a circle, and \((\mathbb{Q}, S)\) as forgetting which way is clockwise on this circle.

The following is a consequence of Theorem 6.1 in Cameron [Cam76]; see also (3.10) of [Cam90, §3.4]. For further details, see [Mac11, Theorem 6.2.1].

**Theorem 2.18** (Cameron). Let \(G\) be a highly homogeneous structure. Then \(G\) is interdefinable with one of the set \(\mathbb{Q}\) (in the empty language), \((\mathbb{Q}, <)\), \((\mathbb{Q}, B)\), \((\mathbb{Q}, K)\), or \((\mathbb{Q}, S)\).

Notice that these five structures all have trivial definable closure; this will imply, in Lemma 4.1, that the orbit of each highly homogeneous structure admits a unique invariant measure.

### 2.4. Ergodic invariant measures and Borel \(L\)-structures.

Recall that our main result, Theorem 1.1, characterizes the number of ergodic invariant measures on an orbit. In the case where an orbit admits at least two invariant measures, there are always continuum-many such measures, because a convex combination of any two gives an invariant measure on that orbit, and these combinations yield distinct measures. By a standard result [Kal05, Theorem A1.3], every invariant measure is a mixture of extreme invariant measures, i.e., ones that cannot be written as a nontrivial convex combination of invariant measures. By another standard result [Kal05, Lemma A1.2], the extreme invariant measures coincide with the ergodic ones, which we define below; for more details, see [Kal05, Appendix A1]. Thus when counting invariant measures on an orbit, the interesting quantity to consider is the number of ergodic invariant measures.

Given an action of a group \(G\) on a set \(S\), and an element \(g \in G\), we write \(gx\) to denote the image of \(x \in S\) under \(g\), and for \(X \subseteq S\) we write \(gX := \{gx : x \in X\}\).

**Definition 2.19.** Consider a Borel action of a Polish group \(G\) on a standard Borel space \(S\). A measure \(\mu\) on \(S\) is **ergodic** when for every Borel \(B \subseteq S\) satisfying \(\mu(B \triangle g^{-1}B) = 0\) for each \(g \in G\), either \(\mu(B) = 0\) or \(\mu(B) = 1\).

In our setting, \(S\) will be one of \(\mathbb{R}^\omega\) or \(\text{Str}_L\), and we will consider the group action of \(S_\infty\) on \(\mathbb{R}^\omega\) that permutes coordinates, and the logic action of \(S_\infty\) on \(\text{Str}_L\).

Aldous, Hoover, and Kallenberg have characterized ergodic invariant measures on \(\text{Str}_L\) in terms of a certain sampling procedure involving continuum-sized objects; for more details, see [Aus08] and [Kal05]. We will obtain ergodic invariant measures via a special case of this procedure, by sampling from a particular kind of continuum-sized structure, called a Borel \(L\)-structure.

**Definition 2.20** ([AFP12, Definition 3.1]). Let \(L\) be relational, and let \(\mathcal{P}\) be an \(L\)-structure whose underlying set is the set \(\mathbb{R}\) of real numbers. We say that
$\mathcal{P}$ is a **Borel $L$-structure** if for all relation symbols $R \in L$, the set
\[
\{ a \in \mathcal{P}^j : \mathcal{P} \models R(a) \}
\]
is a Borel subset of $\mathbb{R}^j$, where $j$ is the arity of $R$.

For more on the connection between Borel $L$-structures and the Aldous–
Hoover–Kallenberg representation, see [AFP12, §6.1].

Now consider the following map that takes each sequence of elements of $\mathcal{P}$ to the corresponding structure with underlying set $\mathbb{N}$.

**Definition 2.21** ([AFP12, Definition 3.2]). Let $L$ be relational and let $\mathcal{P}$ be a Borel $L$-structure. The map $\mathcal{F}_\mathcal{P} : \mathbb{R}^\omega \to \text{Str}_L$ is defined as follows. For $t = (t_0, t_1, \ldots) \in \mathbb{R}^\omega$, let $\mathcal{F}_\mathcal{P}(t)$ be the $L$-structure with underlying set $\mathbb{N}$ satisfying
\[
\mathcal{F}_\mathcal{P}(t) \models R(n_1, \ldots, n_j) \iff \mathcal{P} \models R(t_{n_1}, \ldots, t_{n_j})
\]
for all $n_1, \ldots, n_j \in \mathbb{N}$ and for every relation symbol $R \in L$, and for which equality is inherited from $\mathbb{N}$, i.e.,
\[
\mathcal{F}_\mathcal{P}(t) \models (m \neq n)
\]
if and only if $m$ and $n$ are distinct natural numbers.

The map $\mathcal{F}_\mathcal{P}$ is Borel measurable [AFP12, Lemma 3.3]. Furthermore, $\mathcal{F}_\mathcal{P}$ is an $S_{\infty}$-map, i.e., $\sigma \mathcal{F}_\mathcal{P}(t) = \mathcal{F}_\mathcal{P}(\sigma t)$ for every $\sigma \in S_{\infty}$ and $t \in \mathbb{R}^\omega$.

The pushforward of $\mathcal{F}_\mathcal{P}$ gives rise to the invariant measures we will use.

**Definition 2.22** ([AFP12, Definition 3.4]). Let $L$ be relational, let $\mathcal{P}$ be a Borel $L$-structure, and let $m$ be a probability measure on $\mathbb{R}$. Define the measure $\mu(\mathcal{P}, m)$ on $\text{Str}_L$ to be
\[
\mu(\mathcal{P}, m) := m_\infty \circ \mathcal{F}_\mathcal{P}^{-1}.
\]

Note that $m_\infty(\mathcal{F}_\mathcal{P}^{-1}(\text{Str}_L)) = 1$, and so $\mu(\mathcal{P}, m)$ is a probability measure, namely the distribution of a random element in $\text{Str}_L$ induced via $\mathcal{F}_\mathcal{P}$ by an $m$-i.i.d. sequence on $\mathbb{R}$.

By Lemma 3.5 of [AFP12], $\mu(\mathcal{P}, m)$ is an invariant measure on $\text{Str}_L$. In fact, $\mu(\mathcal{P}, m)$ is ergodic: Aldous showed that the sampling procedure we have described is ergodic for finite relational languages [Kal05, Lemma 7.35]. For our setting of a countable relational language, we require the extension by Kallenberg to languages of unbounded arity [Kal05, Lemmas 7.22 and 7.28 (iii)]. For completeness, we include a proof of this result here.

**Proposition 2.23.** Let $L$ be relational, let $\mathcal{P}$ be a Borel $L$-structure, and let $m$ be a probability measure on $\mathbb{R}$. Then the measure $\mu(\mathcal{P}, m)$ is ergodic.

**Proof.** First note that the measure $m_\infty$ on $\mathbb{R}^\omega$ is ergodic by the Hewitt–Savage 0-1 law; for details, see [Kal05, Corollary 1.6] and [Kal02, Theorem 3.15].

Write $\mu := \mu(\mathcal{P}, m)$. Let $B \subseteq \text{Str}_L$ be Borel and suppose that $\mu(B \vartriangle \sigma^{-1}B) = 0$ for every $\sigma \in S_{\infty}$. We will show that either $\mu(B) = 0$ or $\mu(B) = 1$. 
Let $t \in \mathbb{R}^\omega$ and $\sigma \in S_\infty$. We have
\[ t \in \sigma^{-1}F_P^{-1}(B) \iff F_P(\sigma t) \in B, \]
where $\sigma$ and $\sigma^{-1}$ act on $\mathbb{R}^\omega$, and
\[ t \in F_P^{-1}(\sigma^{-1}B) \iff \sigma F_P(t) \in B, \]
where $\sigma$ and $\sigma^{-1}$ act on $\text{Str}_L$ via the logic action.

Now, $\sigma F_P(t) = F_P(\sigma t)$, and so
\[ F_P^{-1}(\sigma^{-1}B) = \sigma^{-1}F_P^{-1}(B). \]
Using this fact, we have
\[ 0 = \mu(B \Delta \sigma^{-1}B) = m^\infty(F_P^{-1}(B \Delta \sigma^{-1}B)) = m^\infty(F_P^{-1}(B) \Delta F_P^{-1}(\sigma^{-1}B)) = m^\infty(F_P^{-1}(B) \Delta \sigma^{-1}F_P^{-1}(B)) = m^\infty(A \Delta \sigma^{-1}A), \]
where $A := F_P^{-1}(B)$.

Because $m^\infty$ is ergodic and $m^\infty(A \Delta \sigma^{-1}A) = 0$ for every $\sigma \in S_\infty$, either $m^\infty(A) = 0$ or $m^\infty(A) = 1$ must hold. But then as $\mu(B) = m^\infty(A)$, either $\mu(B) = 0$ or $\mu(B) = 1$, as desired. \qed

Not all ergodic invariant measures are of the form $\mu(P,m)$: For example, it can be shown that the distribution of an Erdős–Rényi graph $G(N,\frac{1}{2})$, which is concentrated on the orbit of the Rado graph, is not of this form. However, Petrov and Vershik [PV10] show that the orbit of the Rado graph admits an invariant measure of the form $\mu(P,m)$ (in our terminology). More generally, the proof of [AFP12, Corollary 6.1] shows that whenever an orbit admits an invariant measure, it admits one of the form $\mu(P,m)$. Note that this class of invariant measures also occurs elsewhere; see Kallenberg’s notion of simple arrays [Kal99] and, in the case of graphs, the notions of random-free graphons [Jan13, §10] or $0–1$ valued graphons [LS10].

We now consider how to obtain ergodic invariant measures concentrated on a particular orbit. We will do so by obtaining ergodic invariant measures concentrated on the class of models in $\text{Str}_L$ of some Fraïssé theory $T$, and this class will be precisely the desired orbit.

A measure $m$ on $\mathbb{R}$ is said to be nondegenerate when every nonempty open set has positive measure, and continuous when it assigns measure zero to every singleton.

**Definition 2.24** ([AFP12, Definition 3.8]). Let $\mathcal{P}$ be a Borel $L$-structure and let $m$ be a probability measure on $\mathbb{R}$. Suppose $T$ is a countable pithy $\Pi_2$ theory of $\mathcal{L}_{\omega_1,\omega}(L)$. We say that the pair $(\mathcal{P},m)$ witnesses $T$ if for every sentence
\[(\forall x)(\exists y)\psi(x, y) \in T, \text{ and for every tuple } a \in P \text{ such that } |a| = |x|, \text{ we have either} \]

(i) \[ P \models \psi(a, b) \text{ for some } b \in a, \]

(ii) \[ m(\{b \in P : P \models \psi(a, b)\}) > 0. \]

We say that \( P \) **strongly witnesses** \( T \) when, for every nondegenerate continuous probability measure \( m \) on \( \mathbb{R} \), the pair \( (P, m) \) witnesses \( T \).

**Theorem 2.25** ([AFP12, Theorem 3.10]). *Let \( L \) be relational, let \( T \) be a countable pithy \( \Pi_2 \) theory of \( \mathcal{L}_{\omega_1, \omega}(L) \), and let \( P \) be a Borel \( L \)-structure. Suppose \( m \) is a continuous probability measure on \( \mathbb{R} \) such that \( (P, m) \) witnesses \( T \). Then \( \mu(P, m) \) is concentrated on the set of structures in \( \text{Str}_L \) that are models of \( T \).*

Later in this paper we will build a Borel \( L \)-structure \( P \) that strongly witnesses a Fraïssé theory \( T \), so that for nondegenerate continuous \( m \) on \( \mathbb{R} \), the corresponding invariant measure \( \mu(P, m) \) is concentrated on the set of models of \( T \) in \( \text{Str}_L \).

Finally, we state a lemma that we will use in the next section.

**Lemma 2.26** ([AFP12, Lemma 3.9]). *Let \( L \) be relational, let \( P \) be a Borel \( L \)-structure, and let \( T \) be a countable pithy \( \Pi_2 \) theory of \( \mathcal{L}_{\omega_1, \omega}(L) \). If \( P \) strongly witnesses \( T \), then \( P \models T \).*

### 3. Multiple invariant measures

We now show that for any ultrahomogeneous \( M \in \text{Str}_L \), if \( M \) is not highly homogeneous and admits an invariant measure, then it admits infinitely many invariant measures. We will review the construction from [AFP12], using Borel \( L \)-structures, of an invariant measure concentrated on a given orbit, and then show how this construction can be modified to produce multiple invariant measures.

In Section 5, we will show (by a different method) that such an \( M \) in fact admits continuum-many ergodic invariant measures. On the other hand, in Section 4, we will show that any highly homogeneous structure admits precisely one invariant measure.

#### 3.1. Trivial definable closure if and only if at least one invariant measure

In [AFP12] it is shown that a countably infinite structure in a countable language admits at least one invariant measure if and only if it has trivial definable closure. For completeness, we include a proof.

**Theorem 3.1** ([AFP12, Theorem 4.1]). *Let \( M \in \text{Str}_L \). If \( M \) does not have trivial definable closure, then \( M \) does not admit an invariant measure.*

**Proof.** Suppose that \( \mu \) is an invariant measure on the orbit of \( M \), and that \( M \) does not have trivial definable closure. Let \( a \in M \) be such that \( \text{dcl}_M(a) \neq a \), and let \( b \in \text{dcl}_M(a) - a \).
Let $p(xy) \in \mathcal{L}_{\omega_1, \omega}(L)$ be a formula such that $\mathcal{M} \models p(ab)$ and such that whenever $\mathcal{M} \models p(cd)$ there is an automorphism of $\mathcal{M}$ taking $ab$ to $cd$ pointwise. Note that $\mathcal{M} \models (\exists xy)p(xy)$.

Because $\mu$ is concentrated on the isomorphism class of $\mathcal{M}$ in $\text{Str}_L$, and hence on $L$-structures satisfying $(\exists xy)p(xy)$, we have

$$\mu\left(\llbracket (\exists xy)p(xy) \rrbracket\right) = 1.$$

But then by the countable additivity of $\mu$, there is some $mn \in \mathbb{N}$ such that $\mu\left(\llbracket (\exists y)p(my) \rrbracket\right) \geq \mu\left(\llbracket p(mn) \rrbracket\right) > 0$. Define $\alpha := \mu\left(\llbracket p(mn) \rrbracket\right)$. By the $S_\infty$-invariance of $\mu$, for all $j \in \mathbb{N} - m$ we have

$$\mu\left(\llbracket p(mj) \rrbracket\right) = \alpha.$$

Now if $\mathcal{M} \models p(ac)$ for some $c \in \mathbb{N}$, then there is an automorphism of $\mathcal{M}$ that fixes $a$ pointwise and sends $b$ to $c$. Therefore, as $b \in \text{dcl}(a)$, such an element $c$ is equal to $b$. Hence

$$\mathcal{M} \models (\forall xy_1y_2)((p(xy_1) \land p(xy_2)) \rightarrow (y_1 = y_2)).$$

Therefore

$$\llbracket p(mj) \rrbracket \cap \llbracket p(mk) \rrbracket = \emptyset$$

whenever $j, k \in \mathbb{N}$ are distinct, and so

$$\mu\left(\llbracket (\exists y)p(my) \rrbracket\right) = \sum_{j \in \mathbb{N}} \mu\left(\llbracket p(mj) \rrbracket\right).$$

Hence

$$\mu\left(\llbracket (\exists y)p(my) \rrbracket\right) = \sum_{j \in \mathbb{N}} \mu\left(\llbracket p(mj) \rrbracket\right) = \sum_{j \in \mathbb{N} - m} \alpha,$$

a contradiction as $\mu\left(\llbracket (\exists y)p(my) \rrbracket\right) > 0$ and $\mu$ is a probability measure. $\square$

We now, in Theorem 3.3, construct an invariant measure on the orbit of a structure $\mathcal{M} \in \text{Str}_L$ when $\mathcal{M}$ has trivial definable closure. We first construct, in Proposition 3.2, a Borel $L$-structure that strongly witnesses the Fraïssé theory of $\mathcal{M}$. The proof of Proposition 3.2 that we present here is somewhat different from that in [AFP12]. As described in the beginning of §2.2, here we restrict attention to the case where $\mathcal{M}$ is ultrahomogeneous and $L$ is relational.

We will write $qf$-type to mean quantifier-free type.

**Proposition 3.2 ([AFP12, Theorem 3.19]).** Let $L$ be relational and let $\mathcal{M} \in \text{Str}_L$ be ultrahomogeneous. If $\mathcal{M}$ has trivial definable closure, then there is a Borel $L$-structure $\mathcal{P}$ that strongly witnesses the Fraïssé theory of $\mathcal{M}$.
Proof. Assume that $\mathcal{M}$ has trivial definable closure. Recall from Section 2.2 that $\mathcal{M}$ therefore has the strong amalgamation property.

Let $\mathcal{F} = \text{Age}(\mathcal{M})$, let $\{A_i\}_{i \in \mathbb{N}}$ be some enumeration of any countable subset $Z$ of $\mathcal{F}$ that includes at least one element of each isomorphism type, and let $\{q_i\}_{i \subseteq \mathbb{N}}$ be an enumeration of $\mathbb{Q}$. Consider triples of the form $\langle A, B, \alpha \rangle$, where $A, B \subseteq Z$ are such that $|B| = |A| + 1$ and $\alpha : A \rightarrow B$ is an embedding. Let $\{H_i\}_{i \subseteq \mathbb{N}}$ be an enumeration of all such triples, in which each triple occurs infinitely many times.

Suppose that $[s_0, s_1), [s_2, s_3), \ldots, [s_{2h}, s_{2h+1})$ are pairwise disjoint half-open intervals with rational endpoints. We say that the tuple $\langle t_0, \ldots, t_h \rangle$ falls into such a sequence of intervals when each $t_i \in [s_{2i}, s_{2i+1})$. When the intervals are contiguous, i.e., $s_{2i+1} = s_{2i+2}$ for $0 \leq i < h$, we say that $\langle t_0, \ldots, t_h \rangle$ is separated by $\langle s_0, s_2, \ldots, s_{2h-2}, s_{2h}, s_{2h+1} \rangle$. (A similar notion is described in [AFP12, Definition 3.15].)

We construct a Borel $L$-structure $\mathcal{P}$ over the course of a countably infinite sequence of steps, in which we determine the qf-type of every finite tuple of reals.

At the beginning of each step (other than Step 0) we let $r = \langle r_0, r_1, \ldots, r_\ell \rangle$ denote the increasing tuple of all rationals mentioned by the end of the previous step. That is, we redefine each term $r_i$, and use $\ell$ to denote one less than the length of the tuple of all rationals mentioned by the end of the previous step. The inductive hypothesis is that by the end of the previous step we will have determined the qf-type $p$ of every tuple separated by $r$, and furthermore, that the qf-type of every such tuple will correspond to some $D \subseteq \mathcal{F}$, by which we mean that it will be equal to the qf-type of some ordering of the elements of $D$.

In the present step, we in turn extend this information so as to determine the qf-type $p'$ of every tuple separated by the tuple of rationals $r'$ mentioned during the present step, and ensure that $p'$ corresponds to some element of $\mathcal{F}$.

**Step 0:** Let $r' = (0, 1)$, and let $p'$ be some qf-type of some structure in $\mathcal{F}$ of size 1. Declare the qf-type of any tuple separated by $r'$ to be $p'$.

We now define, for $k \in \mathbb{N}$, Steps $3k + 1$, $3k + 2$, and $3k + 3$.

**Step $3k + 1$:** Let $r = \langle r_0, r_1, \ldots, r_\ell \rangle$ be the increasing tuple of rationals that have already been mentioned. By hypothesis, we have determined $p$, the qf-type of every tuple separated by $r$. Consider $q_k$. If $q_k$ is an element of $r$, we do nothing, i.e., set $r' = r$ and $p' = p$. Otherwise, extend $r$ by $q_k$ to form the increasing tuple $r' = \langle r_0', r_1', \ldots, r_{\ell+1}' \rangle$, and determine the qf-type of every tuple separated by $r'$ as follows.

If $q_k$ is one of $r_0'$ or $r_{\ell+1}'$ and $A \subseteq \mathcal{F}$ is such that $p$ corresponds to $A$, take any $B \subseteq \mathcal{F}$ such that $A$ embeds into $B$ and $|B| = |A| + 1$. Let $p'$ be a qf-type extending $p$ that corresponds to $B$, and declare it to be the qf-type of every tuple separated by $r'$.
Otherwise, let $i$ be such that $q_k \in [r_i, r_{i+1})$. By the strong amalgamation property, there is a qf-type $p'$ such that $p'(x_0, \ldots, x_i, x_i', x_{i+1}, \ldots, x_{\ell})$ implies both $p(x_0, \ldots, x_i, x_{i+1}, \ldots, x_{\ell})$ and $p(x_0, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_{\ell})$. Declare the qf-type of every tuple separated by $r'$ to be $p'$.

**Step 3k + 2:** Again let $r = \langle r_0, r_1, \ldots, r_\ell \rangle$ be the increasing tuple of rationals already mentioned; by hypothesis, the qf-type $p$ of every tuple separated by $r$ has been determined. Consider $A_k$. Pick some increasing sequence of rationals $r_{\ell+1}, \ldots, r_{\ell+h}$ all greater than $r_\ell$, where $h = |A_k|$. Let $p^*$ be a qf-type that corresponds to $A_k$. Declare that the qf-type of every tuple separated by $\langle r_\ell, \ldots, r_{\ell+h} \rangle$ is $p^*$. By the strong amalgamation property, there is a qf-type $p'$ extending $p$ and $p^*$. Declare the qf-type of every tuple separated by the tuple $r' = \langle r_0, \ldots, r_{\ell+h} \rangle$ to be $p'$.

**Step 3k + 3:** Once again let $r = \langle r_0, r_1, \ldots, r_\ell \rangle$ be the increasing tuple of rationals already mentioned, and $p$ the qf-type of every tuple separated by $r$. Let $C \in \mathcal{F}$ be some structure to which $p$ corresponds. Consider $H_k$, and let $A, B, \alpha$ be such that $H_k = \langle A, B, \alpha \rangle$. Let $n$ be the number of embeddings of $A$ into $C$, and let $\xi_0, \ldots, \xi_{n-1}$ be an enumeration of these embeddings. Pick some increasing sequence of rationals $r_{\ell+1}, \ldots, r_{\ell+n}$ all greater than $r_\ell$.

We now determine by induction the qf-type $p'_i$ of every tuple separated by $r' = \langle r_0, \ldots, r_{\ell+n} \rangle$. Suppose that we have already determined the qf-type $p_i$ of every tuple separated by $\langle r_0, \ldots, r_{\ell+i} \rangle$, where $0 \leq i < n$. Now we determine the qf-type $p_{i+1}$ of every tuple separated by $\langle r_0, \ldots, r_{\ell+i+1} \rangle$. Consider $\xi_i: A \to C$. Now $p$ corresponds to $C$, and is the qf-type of every tuple separated by $r$. Hence some qf-type $p^i$ implied by $p$ corresponds to the image of $A$ in $C$ under $\xi_i$. Further, $p^i$ is the qf-type of every tuple that falls into some sequence of pairwise disjoint intervals $[r_{j_0}, r_{j_0+1}), \ldots, [r_{j_{n-1}}, r_{j_{n-1}+1})$, where $0 \leq j_0, \ldots, j_{n-1} < \ell$ and $h = |A|$. We may extend $p^i$ to the qf-type $p^*$ of every tuple that falls into

$$[r_{j_0}, r_{j_0+1}), \ldots, [r_{j_{n-1}}, r_{j_{n-1}+1}), [r_{\ell+i}, r_{\ell+i+1})$$

so that $p^*$ is the qf-type of $B$. Using the strong amalgamation property, define the qf-type $p_{i+1}$ of every tuple separated by $\langle r_0, \ldots, r_{\ell+i+1} \rangle$ to be an extension of $p^*$ and $p_i$.

Finally, let $p'$ be the qf-type $p_n$.

**Claim.** At the end of the above construction, the qf-type of every tuple of distinct reals is determined.

**Proof of Claim.** Let $t$ be an increasing tuple of reals. By the above construction, there is some step at which we have defined an increasing tuple of rationals $\langle r_0, \ldots, r_\ell \rangle$ for which there are $j_0, \ldots, j_{n-1}$ such that $0 \leq j_0, \ldots, j_{n-1} < \ell$ and $t$ falls into the sequence of intervals

$$[r_{j_0}, r_{j_0+1}), \ldots, [r_{j_{n-1}}, r_{j_{n-1}+1}).$$
By the end of that step, the qf-type of every tuple separated by \( \langle r_0, \ldots, r_\ell \rangle \) has been determined. This implies that the qf-type of \( t \) has been determined. □

**Claim.** The Borel \( L \)-structure \( \mathcal{P} \) strongly witnesses the Fraïssé theory of \( M \).

**Proof of Claim.** Suppose \( m \) is a nondegenerate continuous probability measure on \( \mathbb{R} \). Note that an \( m \)-i.i.d. sequence of reals almost surely consists of distinct elements and is dense in \( \mathbb{R} \). Conditioned on this event, we have the following.

- The qf-type of any tuple of reals that consists of distinct elements corresponds to some structure in \( \mathcal{F} \).
- For any \( A \in \mathcal{F} \) there are non-overlapping intervals \( I_0, \ldots, I_{h-1} \) such that the qf-types of all tuples that fall into \( I_0, \ldots, I_{h-1} \) are the same and correspond to \( A \).
- For any \( A, B \in \mathcal{F} \) such that \( |B| = |A| + 1 \) and embeddings \( \alpha: A \to B \) and \( \beta: A \to \mathcal{P} \), there is an interval \( I \) such that \( \beta(A) \cap I \neq \emptyset \) and for every \( t \in I \) there is an embedding \( \gamma: B \to \mathcal{P} \) such that \( \beta = \gamma \circ \alpha \) and \( t \in \text{rng}(\gamma) \).

Therefore \( (\mathcal{P}, m) \) witnesses the Fraïssé theory of \( M \), as claimed. □

Hence \( \mathcal{P} \) is as desired, completing the proof of Proposition 3.2. □

**Theorem 3.3 ([AFP12, Theorem 3.21]).** Let \( L \) be relational and let \( \mathcal{M} \in \text{Str}_L \) be ultrahomogeneous. If \( \mathcal{M} \) has trivial definable closure, then \( \mathcal{M} \) admits an invariant measure.

**Proof.** There is a Borel \( L \)-structure \( \mathcal{P} \) that strongly witnesses the Fraïssé theory of \( \mathcal{M} \), by Proposition 3.2. Let \( m \) be any nondegenerate continuous probability measure on \( \mathbb{R} \) (e.g., a Gaussian). Then by Theorem 2.25, the measure \( \mu_{(\mathcal{P}, m)} \) is an invariant probability measure on \( \text{Str}_L \) that is concentrated on the set of models of the Fraïssé theory of \( \mathcal{M} \) in \( \text{Str}_L \). In particular, \( \mu_{(\mathcal{P}, m)} \) is concentrated on the orbit of \( \mathcal{M} \). □

### 3.2. Trivial definable closure and non-high homogeneity implies infinitely many invariant measures.

In this subsection, \( L \) will again be relational and \( \mathcal{M} \) ultrahomogeneous. Now suppose that \( \mathcal{M} \) is not highly homogeneous and that its orbit admits an invariant measure. We prove that the orbit of \( \mathcal{M} \) then admits infinitely many invariant measures. In Section 5, we will show that such an orbit must in fact admit continuum-many ergodic invariant measures.

We first establish a lemma about distributions of the form \( \mu_{(\mathcal{P}, m)} \). Recall the notation \( \llbracket \varphi \rrbracket \) from Definition 2.1.

**Lemma 3.4.** Let \( L \) be relational, let \( \mathcal{M} \in \text{Str}_L \) be ultrahomogeneous, and let \( T \) be the Fraïssé theory of \( \mathcal{M} \). Suppose that \( \mathcal{P} \) is a Borel \( L \)-structure that strongly
witnesses $T$. Let $m$ be a nondegenerate continuous probability measure on $\mathbb{R}$. Then for every $n \in \mathbb{N}$ and every $\mathcal{L}_{\omega_1, \omega}(L)$-formula $\varphi$ having $n$ free variables,

$$\mu_{(\mathcal{P}, m)}(\{\varphi(0, \ldots, n-1)\}) = m^n(\{a \in \mathbb{R}^n : \mathcal{P} \models \varphi(a)\}).$$

**Proof.** By Lemma 3.6 of [AFP12], because $m$ is continuous, $\mu_{(\mathcal{P}, m)}$ is concentrated on the isomorphism classes in $\text{Str}_L$ of countably infinite substructures of $\mathcal{P}$.

Because $\mathcal{M}$ is ultrahomogeneous, for every $\mathcal{L}_{\omega_1, \omega}(L)$-formula $\varphi(x)$ there is some quantifier-free $\psi(x)$ such that $\mathcal{M} \models (\forall x)(\varphi(x) \leftrightarrow \psi(x))$.

Since $\mathcal{P} \models T$ by Lemma 2.26, we also have $\mathcal{P} \models (\forall x)(\varphi(x) \leftrightarrow \psi(x))$.

In particular, if a sequence of reals determines a substructure of $\mathcal{P}$ that is isomorphic to $\mathcal{M}$, then this substructure is in fact $\mathcal{L}_{\omega_1, \omega}(L)$-elementary.

Therefore, as $\mathcal{P}$ strongly witnesses $T$, by Theorem 2.25 and Definition 2.22 the probability measure $m^\infty$ concentrates on sequences of reals that determine elementary substructures of $\mathcal{P}$. Hence the probability that a structure sampled according to $\mu_{(\mathcal{P}, m)}$ satisfies $\varphi(0, \ldots, n-1)$ is equal to $m^n(\{a \in \mathbb{R}^n : \mathcal{P} \models \varphi(a)\})$, as desired. \hfill \Box

For an age $\mathfrak{F}$ and $k \in \mathbb{N}$, define $\mathfrak{F}_k$ to be the set of structures in $\mathfrak{F}$ of size $k$ with underlying set $\{0, \ldots, k-1\}$. For every $A \in \mathfrak{F}$ and $X \in \mathfrak{F}_k$, let $t^A(X)$ denote the fraction of those functions from the underlying set of $X$ to the underlying set of $A$ that are embeddings of $X$ into $A$, and for $I \subseteq \mathfrak{F}_k$ define

$$s^I_A := \sum_{X \in I} t^A(X).$$

In other words, $s^I_A$ is the probability that uniformly independently choosing $k$ elements of $A$ with replacement yields an induced structure that is isomorphic to an element of $I$.

We now prove a technical lemma, which along with a Ramsey-theoretic argument will yield the main result of this section.

**Lemma 3.5.** Let $L$ be relational, let $\mathcal{M} \in \text{Str}_L$ be ultrahomogeneous, and suppose that the orbit of $\mathcal{M}$ admits an invariant measure. Let $\mathfrak{F} = \text{Age}(\mathcal{M})$ and suppose that there are $k, q \in \mathbb{N}$ with $k \leq q$, some $Q \in \mathfrak{F}_q$, and some $I \subseteq \mathfrak{F}_k$ such that $s^I_Q = 0$ and such that for every $\epsilon > 0$ there is some $U \in \mathfrak{F}$ such that $s^I_U > 1 - \epsilon$. Then the orbit of $\mathcal{M}$ admits infinitely many invariant measures.
Proof. By Theorem 3.1, because the orbit of \( \mathcal{M} \) admits an invariant measure, \( \mathcal{M} \) has trivial definable closure. Let \( \mathcal{P} \) be a Borel \( I \)-structure strongly witnessing the Fraïssé theory of \( \mathcal{M} \), constructed as in the proof of Proposition 3.2. Let \( \xi_0, \ldots, \xi_{\ell-1} \) be arbitrary nondegenerate continuous probability measures on \( \mathbb{R} \).

We will find another nondegenerate continuous probability measure \( \gamma \) on \( \mathbb{R} \) such that the invariant measure \( \mu(\mathcal{P}, \gamma) \) is distinct from each of \( \mu(\mathcal{P}, \xi_0), \ldots, \mu(\mathcal{P}, \xi_{\ell-1}) \). There will hence be infinitely many invariant measures of this form, by induction. This will complete the proof of the lemma, as by Theorem 2.25, they are all concentrated on the orbit of \( \mathcal{M} \).

Denote by \( \mathfrak{A} \subseteq \text{Str}_L \) the set of those structures whose restriction to \( \{0, \ldots, k - 1\} \) is isomorphic to some element of \( I \). Because \( s_{1}^I = 0 \) by hypothesis, \( \mathfrak{S}_k - I \) is non-empty. Hence \( \mu(\mathcal{P}, \xi_j)(\mathfrak{A}) < 1 \) for \( 0 \leq j \leq \ell - 1 \), as each \( \mu(\mathcal{P}, \xi_j) \) is concentrated on the orbit of \( \mathcal{M} \). Therefore we may choose some \( \epsilon > 0 \) such that

\[
(1 - \epsilon)^{k+1} > \max_{0 \leq j \leq \ell - 1} \mu(\mathcal{P}, \xi_j)(\mathfrak{A}).
\]

By hypothesis, there is some \( U \in \mathfrak{S} \) such that \( s^I_U > 1 - \epsilon \). Let \( n = |U| \), and let \( p \) be the qf-type of (some ordering of) \( U \). From the construction of \( \mathcal{P} \), there are disjoint intervals \( J_0, \ldots, J_{n-1} \) such that for any \( n \)-tuple \( r = \langle r_0, \ldots, r_{n-1} \rangle \) with each \( r_i \in J_i \), the tuple \( r \) satisfies \( p \). Let \( \gamma \) be any nondegenerate continuous probability measure on \( \mathbb{R} \) such that

\[
\gamma(J_i) = (1 - \epsilon) \frac{1}{n}
\]

for \( 0 < i \leq n - 1 \). In particular,

\[
\gamma^\infty((\bigcup_{i=0}^{n-1} J_i)^k \times \mathbb{R}^\omega) = (1 - \epsilon)^k.
\]

Let \( \mathfrak{R} \subseteq \mathbb{R}^k \) be the collection of those tuples that determine \( k \)-element substructures of \( \mathcal{P} \) that are isomorphic to some element of \( I \). Note that

\[
\frac{\gamma^\infty((\mathfrak{R} \times \mathbb{R}^\omega) \cap ((\bigcup_{i=0}^{n-1} J_i)^k \times \mathbb{R}^\omega))}{\gamma^\infty((\bigcup_{i=0}^{n-1} J_i)^k \times \mathbb{R}^\omega)} = s^I_U.
\]

By Lemma 3.4, we have

\[
\mu(\mathcal{P}, \gamma)(\mathfrak{A}) = \gamma^k(\mathfrak{R}) = \gamma^\infty(\mathfrak{R} \times \mathbb{R}^\omega) \geq \gamma^\infty((\mathfrak{R} \times \mathbb{R}^\omega) \cap ((\bigcup_{i=0}^{n-1} J_i)^k \times \mathbb{R}^\omega)).
\]

Now,

\[
\gamma^\infty((\mathfrak{R} \times \mathbb{R}^\omega) \cap ((\bigcup_{i=0}^{n-1} J_i)^k \times \mathbb{R}^\omega)) = (1 - \epsilon)^k s^I_U > (1 - \epsilon)^{k+1} > \mu(\mathcal{P}, \xi_j)(\mathfrak{A}),
\]

where \( 0 \leq j \leq \ell - 1 \). Therefore \( \mu(\mathcal{P}, \gamma)(\mathfrak{A}) \neq \mu(\mathcal{P}, \xi_j)(\mathfrak{A}) \) for each \( j \), and so \( \mu(\mathcal{P}, \gamma) \) is distinct from each \( \mu(\mathcal{P}, \xi_j) \), as desired. \( \square \)

**Proposition 3.6.** Let \( L \) be relational and let \( \mathcal{M} \in \text{Str}_L \) be ultrahomogeneous. Suppose that \( \mathcal{M} \) is not highly homogeneous and admits an invariant measure. Then \( \mathcal{M} \) admits infinitely many invariant measures.
Proof. Let $\mathcal{F} = \text{Age}(\mathcal{M})$. Because $\mathcal{M}$ is not highly homogeneous, there is some $k \in \mathbb{N}$ such that there are non-isomorphic structures $A_0, A_1 \in \mathcal{F}_k$. Using Ramsey’s theorem, for each $r \geq k$ let $n_r \in \mathbb{N}$ be such that whenever each $k$-element subset of $n_r$ is assigned one of two colors, there is an $r$-element subset of $n_r$ all of whose $k$-element subsets are assigned the same color.

For each $r \geq k$, let $C_r \in \mathcal{F}_{n_r}$, and color each $k$-element subset of $n_r$ one of two colors according to whether it determines a structure isomorphic to $A_0$ or not. Then by the choice of $n_r$, there is a substructure $B_r$ of $C_r$ of cardinality $r$ such that all $k$-element substructures of $B_r$ are isomorphic to $A_0$, or (ii) for every $r \geq k$ there is some substructure $B_r$ of $C_r$ of cardinality $r$ such that all $k$-element substructures of $B_r$ are not isomorphic to $A_0$.

Set $I_0 = \{A_0\}$ and $I_1 = \mathcal{F}_k - I_0$. If (i) holds, observe that $s_{I_0}^{A_1} = 0$, where $s_{A_1}$ is as defined before Lemma 3.5. As every $k$-element substructure of $B_r$ is isomorphic to $A_0$, the quantity $s_{B_r}^{I_0}$ is bounded below by the proportion of functions from $k$ to $r$ that are injections, i.e.,

$$s_{B_r}^{I_0} \geq \frac{r(r-1) \cdots (r-k+1)}{r^k}.$$ 

Because $\lim_{r \to \infty} \frac{r(r-1) \cdots (r-k+1)}{r^k} = 1$, for any $\epsilon > 0$ there is an $r$ such that $s_{B_r}^{I_0} > 1 - \epsilon$. Similarly, if (ii) holds, $s_{B_r}^{I_1} = 0$ and for any $\epsilon > 0$ there is an $r$ such that $s_{B_r}^{I_1} > 1 - \epsilon$. Because we have assumed that $\mathcal{M}$ admits an invariant measure, in either case we may apply Lemma 3.5 to obtain infinitely many invariant measures concentrated on the orbit of $\mathcal{M}$.

4. Unique invariant measures

We now show that every highly homogeneous structure admits a unique invariant measure.

4.1. High homogeneity implies exactly one invariant measure. Recall Cameron’s result, Theorem 2.18, that the highly homogeneous structures are (up to interdefinability) precisely the five reducts of the rational linear order $(\mathbb{Q}, <)$.

Lemma 4.1. Let $L$ be relational and let $\mathcal{M} \in \text{Str}_L$ be ultrahomogeneous. If $\mathcal{M}$ is highly homogeneous, then there is an invariant measure on the isomorphism class of $\mathcal{M}$ in $\text{Str}_L$.

Proof. We can check directly that each reduct of $(\mathbb{Q}, <)$ has trivial definable closure. By Theorem 2.18 and the hypothesis that $\mathcal{M}$ is highly homogeneous, $\mathcal{M}$ is interdefinable with one of these five. Hence $\mathcal{M}$ also has trivial definable closure by Lemma 2.8. Therefore by Theorem 3.3, there is an invariant measure on the isomorphism class of $\mathcal{M}$ in $\text{Str}_L$. □
Alternatively, instead of applying Theorem 3.3, there are several more direct ways of constructing a measure on each of the five reducts of \((\mathbb{Q}, <)\). We sketched the construction of the Glasner–Weiss measure on \((\mathbb{Q}, <)\) in §1.1, as the weak limit of the uniform measures on \(n\)-element linear orders; each of the other four also arises as the weak limit of uniform measures.

Another way to construct the Glasner–Weiss measure is as the ordering on the set \(\mathbb{N}\) of indices induced by an \(m\)-i.i.d. sequence of reals, where \(m\) is any nondegenerate continuous probability measure on \(\mathbb{R}\). The invariant measures on the remaining four reducts may be obtained in a similar way from an i.i.d. sequence on the respective reduct of \(\mathbb{R}\).

For example, for the countable dense circular order, the unique invariant measure can be obtained as either the weak limit of the uniform measure on circular orders of size \(n\) with the (ternary) clockwise-order relation, or from the ternary relation induced on the set \(\mathbb{N}\) of indices by the clockwise-ordering of an \(m\)-i.i.d. sequence, where \(m\) is a nondegenerate continuous probability measure on the unit circle.

Note that the existence of an invariant measure on the orbit of each highly homogeneous structure \(\mathcal{M}\) is a consequence of Exercise 5 of [Cam90, §4.10]; this exercise implies that the weak limit of uniform measures on \(n\)-element substructures of \(\mathcal{M}\) is invariant and concentrated on the orbit of \(\mathcal{M}\).

We are now in a position to prove that every highly homogeneous structure admits a unique invariant measure. Write \(S_n\) to denote the group of permutations of \(\{0, 1, \ldots, n - 1\}\).

**Lemma 4.2.** Let \(\mathcal{M} \in \text{Str}_L\). If \(\mathcal{M}\) is highly homogeneous, then there is at most one invariant measure on the isomorphism class of \(\mathcal{M}\) in \(\text{Str}_L\).

**Proof.** Let \(n \in \mathbb{N}\) and let \(p\) be a qf-type of \(\mathcal{L}_{\omega_1, \omega}(L)\) in \(n\) variables that is realized in \(\mathcal{M}\). Because \(\mathcal{M}\) is highly homogeneous, for any qf-type \(q\) of \(\mathcal{L}_{\omega_1, \omega}(L)\) in \(n\) variables realized in \(\mathcal{M}\), there is some \(\tau \in S_n\) such that

\[
\mathcal{M} \models (\forall x_0 \cdots x_{n-1}) \ (p(x_0, \ldots, x_{n-1}) \leftrightarrow q(x_{\tau(0)}, \ldots, x_{\tau(n-1)})).
\]

Suppose \(\mu\) is an invariant measure on \(\text{Str}_L\) concentrated on the orbit of \(\mathcal{M}\). Then for any \(k_0, \ldots, k_{n-1} \in \mathbb{N}\), we have

\[
\mu([p(k_0, \ldots, k_{n-1})]) = \mu([q(k_{\tau(0)}, \ldots, k_{\tau(n-1)})]).
\]

By the \(S_\infty\)-invariance of \(\mu\), we have

\[
\mu([q(k_{\tau(0)}, \ldots, k_{\tau(n-1)})]) = \mu([q(k_0, \ldots, k_{n-1})]).
\]

By the high homogeneity of \(\mathcal{M}\), there are at most \(n!\)-many distinct qf-types of \(\mathcal{L}_{\omega_1, \omega}(L)\) in \(n\)-many variables realized in \(\mathcal{M}\); let \(\alpha_n\) be this number. Then

\[
\mu([p(k_0, \ldots, k_{n-1})]) = \frac{1}{\alpha_n}.
\]
Sets of the form $[p(k_0, \ldots, k_{n-1})]$ generate the $\sigma$-algebra of Borel subsets of the isomorphism class of $\mathcal{M}$ in $\text{Str}_L$, and so $\mu$ must be the unique measure determined in this way. \hfill \Box

Putting the previous two results together, we obtain the following.

**Proposition 4.3.** Let $L$ be relational and let $\mathcal{M} \in \text{Str}_L$ be ultrahomogeneous. If $\mathcal{M}$ is highly homogeneous, then there is a unique invariant measure on the isomorphism class of $\mathcal{M}$ in $\text{Str}_L$.

**Proof.** By Lemma 4.1, there is an invariant measure on the isomorphism class of $\mathcal{M}$ in $\text{Str}_L$. On the other hand, by Lemma 4.2, this is the only invariant measure on the isomorphism class of $\mathcal{M}$ in $\text{Str}_L$. \hfill \Box

### 5. Continuum-many ergodic invariant measures

In order to complete the proof of our main result, Theorem 1.1, we need to show that there are continuum-many ergodic invariant measures on an orbit whenever there are at least two. When a countable ultrahomogeneous structure in a relational language admits an invariant measure but is not highly homogeneous, we will construct a continuum-sized class of reweighted measures $m^W$ that give rise to distinct invariant measures $\mu(\mathcal{P}, m^W)$ on its orbit, for some appropriate $\mathcal{P}$. We start with some definitions.

#### 5.1. More than one invariant measure implies continuum-many ergodic invariant measures.

**Definition 5.1.** A partition of $\mathbb{R}$ is a collection of subsets of $\mathbb{R}$ that are non-overlapping and whose union is $\mathbb{R}$. By half-open interval, we mean a non-empty, left-closed, right-open interval of $\mathbb{R}$, including the cases $\mathbb{R}$, $(-\infty, c)$, and $[c, \infty)$ for $c \in \mathbb{R}$. A weight $\mathcal{W}$ consists of a partition of $\mathbb{R}$ into a finite set $\mathcal{I}_W$ of finite unions of half-open intervals, along with a map $u_W: \mathcal{I}_W \to \mathbb{R}^+$ that assigns a positive real number to each element of $\mathcal{I}_W$ and satisfies

$$\sum_{I \in \mathcal{I}_W} u_W(I) = 1.$$  

Given a measure $m$ on $\mathbb{R}$, the **reweight**ing $m^W$ of $m$ by a weight $\mathcal{W}$ is the measure on $\mathbb{R}$ defined by

$$m^W(B) = \sum_{I \in \mathcal{I}_W} u_W(I) \frac{m(B \cap I)}{m(I)}$$

for all Borel sets $B \subseteq \mathbb{R}$.

The following is immediate from the definition of a weight.
Lemma 5.2. Let \( m \) be a nondegenerate continuous probability measure on \( \mathbb{R} \), and let \( W \) be a weight. Then \( m^W \), the reweighting of \( m \) by \( W \), is also a nondegenerate continuous probability measure on \( \mathbb{R} \).

We then obtain the following corollary.

Corollary 5.3. Let \( L \) be relational, let \( \mathcal{P} \) be a Borel \( L \)-structure that strongly witnesses a pithy \( \Pi_2 \) theory \( T \), and let \( m \) be a nondegenerate continuous probability measure on \( \mathbb{R} \). Let \( W \) be a weight. Then \( \mu(\mathcal{P}, m^W) \) is concentrated on the set of structures in \( \text{Str}_L \) that are models of \( T \), just as \( \mu(\mathcal{P}, m) \) is.

Proof. Let \( L, \mathcal{P}, T, m, \) and \( W \) be as stated. By Lemma 5.2, \( m^W \) is also a nondegenerate continuous probability measure. Therefore, because \( \mathcal{P} \) strongly witnesses \( T \), both \( (\mathcal{P}, m) \) and \( (\mathcal{P}, m^W) \) witness \( T \). Hence both \( \mu(\mathcal{P}, m) \) and \( \mu(\mathcal{P}, m^W) \) are concentrated on the set of models of \( T \) in \( \text{Str}_L \) by Theorem 2.25. \( \square \)

We now show that continuum-many invariant measures on the isomorphism class of \( M \) in \( \text{Str}_L \) may be obtained by reweighting, when \( M \) is not highly homogeneous but admits an invariant measure. Specifically, suppose \( L \) is relational, \( M \in \text{Str}_L \) is ultrahomogeneous, and \( T \) is the Fraïssé theory of \( M \). Then as \( W \) ranges over weights, we will see that there are continuum-many measures \( \mu(\mathcal{P}, m^W) \), where \( \mathcal{P} \) is a Borel \( L \)-structure that strongly witnesses \( T \) and \( m \) is a nondegenerate continuous probability measure on \( \mathbb{R} \).

We start with two technical results. Recall that \( S_n \) is the group of permutations of \( \{0, 1, \ldots, n-1\} \).

Lemma 5.4. Fix \( n, \ell \in \mathbb{N} \). Suppose \( \{a_s\}_{s \in \{0,1,\ldots,\ell\}^n} \) is a collection of non-negative reals with the following properties:

(a) For each \( \sigma \in S_n \) and \( s, t \in \{0, 1, \ldots, \ell\}^n \), if \( s \circ \sigma = t \) then \( a_s = a_t \).
(b) For some \( s, t \in \{0, 1, \ldots, \ell\}^n \), we have \( a_s \neq a_t \).

Then as the variables \( \lambda_0, \ldots, \lambda_\ell \) range over positive reals such that \( \lambda_0 + \cdots + \lambda_\ell = 1 \),

the polynomial

\[
\sum_{s \in \{0,1,\ldots,\ell\}^n} a_s \lambda_{s(0)} \cdots \lambda_{s(n-1)}
\]

(\(\spadesuit\))

assumes continuum-many values.

Proof. In (\(\spadesuit\)) substitute \( 1 - \sum_{i=0}^{\ell-1} \lambda_i \) for \( \lambda_\ell \) to obtain a polynomial \( P \) in \( \ell \)-many variables \( \lambda_0, \ldots, \lambda_{\ell-1} \). We will show that \( P \) is a non-constant polynomial, and therefore assumes continuum-many values as \( \lambda_0, \ldots, \lambda_{\ell-1} \) range over positive reals such that

\[
\lambda_0 + \cdots + \lambda_{\ell-1} < 1.
\]

Suppose towards a contradiction that \( P \) is a constant polynomial. Let \( a^* := a_u \), where \( u \in \{0, 1, \ldots, \ell\}^n \) is the constant function taking the value \( \ell \). Consider, for \( 0 \leq k \leq n \), the following claim (\(\diamondsuit_k\)).
(▷k) For every j such that 0 ≤ j ≤ k, whenever s ∈ \{0,1, ..., ℓ\}^n is such that exactly j-many of s(0), ..., s(n − 1) are different from ℓ, then a_s = a^∗.

The statement (▷n) implies that for every s ∈ \{0,1, ..., ℓ\}^n, we have a_s = a^∗, thereby contradicting (b). Hence it suffices to prove (▷n), which we now do by induction on k.

The statement (▷0) is clear. Now let k be such that 1 ≤ k ≤ n, and suppose that (▷k−1) holds. We will show that (▷k) holds. Let s ∈ \{0,1, ..., ℓ\}^n be such that exactly k-many of s(0), ..., s(n − 1) are different from ℓ; we must prove that a_s = a^∗.

Since (▷k−1) holds, by (a) we may assume without loss of generality that none of s(0), s(1), ..., s(k − 1) equals ℓ and that
\[ s(k) = s(k + 1) = \cdots = s(n − 1) = ℓ. \]

For 0 ≤ r ≤ ℓ − 1 let k_r denote the number of times that r appears in the sequence s(0), ..., s(k − 1). In particular,
\[ \lambda s(0) \lambda s(1) \cdots \lambda s(k − 1) = \lambda^k_0 \lambda^k_1 \cdots \lambda^k_{ℓ − 1}, \]
and k = k_0 + k_1 + \cdots + k_{ℓ − 1}.

Let \( \beta \) be the coefficient of \( \lambda^k_0 \lambda^k_1 \cdots \lambda^k_{ℓ − 1} \) in P. For \( t_0, \ldots, t_{ℓ − 1} \in \mathbb{N} \) such that \( t_0 + \cdots + t_{ℓ − 1} ≤ n \), let \( \Gamma(t_0, t_1, \ldots, t_{ℓ − 1}) \in \{0,1, \ldots, ℓ\}^n \) be the non-decreasing sequence of length n consisting of \( t_0 \)-many 0’s, \( t_1 \)-many 1’s, ..., \( t_{ℓ − 1} \)-many \( t_{ℓ − 1} \)-’s, and \((n − \sum_{i=0}^{ℓ−1} t_i)\)-many ℓ’s. Define \( C := \frac{n!}{k_0!k_1! \cdots k_{ℓ − 1}!(n − k)!} \). Then
\[
\beta = C \sum_{t_0 \leq k_0, \ldots, t_{ℓ − 1} \leq k_{ℓ − 1}} a_{Γ(t_0,t_1,\ldots,t_{ℓ − 1})} \binom{k_0}{t_0} \cdots \binom{k_{ℓ − 1}}{t_{ℓ − 1}} (-1)^{k − \sum_{i=0}^{ℓ−1} t_i}. \quad (♥)
\]

Note that \( a_s = a_{Γ(k_0,k_1,\ldots,k_{ℓ − 1})} \). By (▷k−1), we also have \( a^∗ = a_{Γ(t_0,t_1,\ldots,t_{ℓ − 1})} \) if \( \sum_{i=0}^{ℓ−1} t_i < k \), in particular whenever each \( t_i \leq k_i \) for 0 ≤ i ≤ ℓ − 1 and \( (t_0,t_1,\ldots,t_{ℓ − 1}) \neq (k_0,k_1,\ldots,k_{ℓ − 1}) \). In other words, all subexpressions \( a_{Γ(t_0,t_1,\ldots,t_{ℓ − 1})} \) appearing in (♥) other than (possibly) \( a_{Γ(k_0,k_1,\ldots,k_{ℓ − 1})} \) are equal to \( a^∗ \).

By the multinomial and binomial theorems,
\[
\sum_{t_0 \leq k_0, \ldots, t_{ℓ − 1} \leq k_{ℓ − 1}} \binom{k_0}{t_0} \cdots \binom{k_{ℓ − 1}}{t_{ℓ − 1}} (-1)^{k − \sum_{i=0}^{ℓ−1} t_i} = \sum_{t_0 \leq k} \sum_{t_1 \leq k_1, \ldots, t_{ℓ − 1} \leq k_{ℓ − 1}} \binom{k_0}{t_0} \cdots \binom{k_{ℓ − 1}}{t_{ℓ − 1}} (-1)^{k − \sum_{i=0}^{ℓ−1} t_i}
\]
\[ = \sum_{t \leq k} \binom{k}{t} (-1)^{k − t} = (1 − 1)^k = 0. \]

Therefore
\[
\beta = 0 + (a_s − a^∗) \binom{k_0}{k_0} \binom{k_{ℓ − 1}}{k_{ℓ − 1}} (-1)^{k} = a_s − a^∗.
\]
But by the assumption that $P$ is a constant polynomial, $\beta = 0$, and so $a_s = a^*$, as desired. \qed

Using this lemma, we can prove the following.

**Proposition 5.5.** Let $m$ be a nondegenerate continuous probability measure, and let $n$ be a positive integer. Suppose $A \subseteq \mathbb{R}^n$ is an $S_n$-invariant Borel set such that $0 < m^n(A) < 1$. Then the family of reals $\{(m^W)^n(A) : W$ is a weight $\}$ has cardinality equal to the continuum.

**Proof.** Because $m^n(A) > 0$, we may define $\tilde{m}$, the conditional distribution of $m^n$ given $A$, by

$$\tilde{m}(B) := \frac{m^n(A \cap B)}{m^n(A)}$$

for every Borel set $B \subseteq \mathbb{R}^n$.

Because $m^n(A) < 1$, we have $m^n(\mathbb{R}^n - A) = 1 - m^n(A) > 0$. Furthermore $\tilde{m}(\mathbb{R}^n - A) = 0$, and so $m^n \neq \tilde{m}$. Therefore there are some half-open intervals $X_0, X_1, \ldots, X_{n-1}$ such that

$$m^n(\prod_{i=0}^{n-1} X_i) \neq \tilde{m}(\prod_{i=0}^{n-1} X_i).$$

Because $m$ is nondegenerate, $m(X_i) > 0$ for each $i \leq n - 1$.

Define the partition $\mathcal{J}$ of $\mathbb{R}$ to be the family of non-empty sets of the form $X_0^{e_0} \cap \ldots \cap X_{n-1}^{e_{n-1}}$ for some $e_0, \ldots, e_{n-1} \in \{+, -\}$, where $X_j^+ := X_j$ and $X_j^- := \mathbb{R} - X_j$. Let $\ell := |\mathcal{J}| - 1$, and let $Y_0, \ldots, Y_{\ell}$ be some enumeration of $\mathcal{J}$.

For $s \in \{0, 1, \ldots, \ell\}^n$, set

$$a_s := \frac{m^n(A \cap \prod_{i=0}^{n-1} Y_{s(i)})}{m^n(\prod_{i=0}^{n-1} Y_{s(i)})}.$$ 

Note that there exists some $v \in \{0, 1, \ldots, \ell\}^n$ such that

$$m^n(\prod_{i=0}^{n-1} Y_{v(i)}) \neq \tilde{m}(\prod_{i=0}^{n-1} Y_{v(i)}),$$

i.e.,

$$m^n(\prod_{i=0}^{n-1} Y_{v(i)}) \neq \frac{m^n(A \cap \prod_{i=0}^{n-1} Y_{v(i)})}{m^n(A)},$$

and hence

$$m^n(A) \neq \frac{m^n(A \cap \prod_{i=0}^{n-1} Y_{v(i)})}{m^n(\prod_{i=0}^{n-1} Y_{v(i)})} = a_v.$$

Observe that if for $s \in \{0, 1, \ldots, \ell\}^n$, the values $a_s$ are all equal, then this value is $m^n(A)$. But we have just shown that $a_v \neq m^n(A)$, and so $a_s \neq a_t$ for some $s, t$. Further, since $A$ is $S_n$-invariant, from the definition of $a_s$ we have that for every $\sigma \in S_n$, and every $s$ and $t$, if $s \circ \sigma = t$ then $a_s = a_t$. 


Hence the assumptions of Lemma 5.4 are satisfied, and so the expression
\[ \sum_{s \in \{0, 1, \ldots, \ell\}^n} a_s \lambda_s(0) \cdots \lambda_s(n-1) \]
takes continuum-many values as \( \lambda_0, \ldots, \lambda_\ell \) range over positive reals satisfying \( \lambda_0 + \cdots + \lambda_\ell = 1 \). Each such \( \lambda_0, \ldots, \lambda_\ell \) together with the partition \( J \) yields a weight \( W \) via \( u_W(y_i) := \lambda_i \) for \( i \leq \ell \). Then the corresponding reweightings \( m^W \) satisfy
\[
(m^W)^n(A) = \sum_{s \in \{0, 1, \ldots, \ell\}^n} \frac{m^n(A \cap \prod_{i=0}^{n-1} Y_{s(i)})}{m^n(\prod_{i=0}^{n-1} Y_{s(i)})} \lambda_s(0) \cdots \lambda_s(n-1)
= \sum_{s \in \{0, 1, \ldots, \ell\}^n} a_s \lambda_s(0) \cdots \lambda_s(n-1).
\]

We conclude that the family \( \{(m^W)^n(A) : W \text{ is a weight}\} \) has cardinality equal to the continuum. \(\square\)

Now we may prove our main result about weights.

**Proposition 5.6.** Let \( L \) be relational, let \( M \in \text{Str}_L \) be ultrahomogeneous, and let \( T \) be the Fraïssé theory of \( M \). Suppose that \( M \) is not highly homogeneous. Further suppose that \( \mathcal{P} \) is a Borel \( L \)-structure that strongly witnesses \( T \), and that \( m \) is a nondegenerate continuous probability measure on \( \mathbb{R} \). Then there are continuum-many measures \( \mu(\mathcal{P}, m^W) \), as \( W \) ranges over weights.

**Proof.** By Theorem 2.25, \( \mu(\mathcal{P}, m) \) is concentrated on the orbit of \( M \). Because \( M \) is not highly homogeneous, by Lemma 2.17 there are non-isomorphic \( n \)-element substructures \( A_0, A_1 \) of \( M \) for some \( n \in \mathbb{N} \). Fix an enumeration of each of \( A_0, A_1 \), and for \( i = 0, 1 \) let \( \varphi_i \) be a quantifier-free \( \mathcal{L}_{\omega_1, \omega}(L) \)-formula in \( n \)-many free variables that is satisfied by \( A_i \) and not by \( A_{1-i} \), (in their respective enumerations). Note that
\[
0 < \mu(\mathcal{P}, m)\left( \bigsqcup_{\sigma \in S_n} \varphi_i(\sigma(0), \ldots, \sigma(n-1)) \right)
\]
for \( i = 0, 1 \), as \( \varphi_i \) is realized in \( M \). Furthermore, as
\[
\mu(\mathcal{P}, m)\left( \bigsqcup_{\sigma \in S_n} \varphi_0(\sigma(0), \ldots, \sigma(n-1)) \right) + \mu(\mathcal{P}, m)\left( \bigsqcup_{\sigma \in S_n} \varphi_1(\sigma(0), \ldots, \sigma(n-1)) \right) \leq 1,
\]
we have
\[
\mu(\mathcal{P}, m)\left( \bigsqcup_{\sigma \in S_n} \varphi_i(\sigma(0), \ldots, \sigma(n-1)) \right) < 1
\]
for \( i = 0, 1 \).
Then, by Lemma 3.4, we have
\[ 0 < m^n \left( \{(a_0, \ldots, a_{n-1}) \in \mathbb{R}^n : P \models \bigvee_{\sigma \in S_n} \varphi_0(a_{\sigma(0)}, \ldots, a_{\sigma(n-1)}) \} \right) < 1. \]

Hence by Proposition 5.5, as \( W \) ranges over weights,
\[ (m^W)^n \left( \{(a_0, \ldots, a_{n-1}) \in \mathbb{R}^n : P \models \bigvee_{\sigma \in S_n} \varphi_0(a_{\sigma(0)}, \ldots, a_{\sigma(n-1)}) \} \right) \]
takes on continuum-many values. Again by Lemma 3.4,
\[ \mu(P, m^W) \left( \left[ \bigvee_{\sigma \in S_n} \varphi_i(\sigma(0), \ldots, \sigma(n-1)) \right] \right) \]
takes on continuum-many values as \( W \) ranges over weights; in particular, the \( \mu(P, m^W) \) constitute continuum-many different measures. \( \square \)

We are now able to complete the proof of our main theorem.

**Proof of Theorem 1.1.** Given a countable structure \( \mathcal{N} \) in a countable language, by Corollaries 2.7 and 2.9 and Lemma 2.15, its canonical structure \( \overline{\mathcal{N}} \) admits the same number of ergodic invariant measures as \( \mathcal{N} \), is highly homogeneous if and only if \( \mathcal{N} \) is, and has trivial definable closure if and only if \( \mathcal{N} \) does. Hence it suffices to prove the theorem in the case where \( \mathcal{M} \in \text{Str}_L \) is the canonical structure of some countable structure in a countable language; in particular, by Proposition 2.11, where \( \mathcal{M} \) is ultrahomogeneous and \( L \) is relational.

By Theorem 3.1, if \( \mathcal{M} \) has nontrivial definable closure then its orbit does not admit an invariant measure, as claimed in (0).

By Proposition 4.3, if \( \mathcal{M} \) is highly homogeneous then its orbit admits a unique invariant measure, as claimed in (1).

Clearly, the orbit of \( \mathcal{M} \) admitting 0, 1, or continuum-many invariant measures are mutually exclusive possibilities. Hence it remains to show that if \( \mathcal{M} \) is not highly homogeneous and its orbit admits an invariant measure, then this orbit admits continuum-many ergodic invariant measures.

Again by Theorem 3.1, because the orbit of \( \mathcal{M} \) admits an invariant measure, \( \mathcal{M} \) must have trivial definable closure. Since \( \mathcal{M} \) is ultrahomogeneous and \( L \) is relational, by Proposition 3.2 there is a Borel \( L \)-structure \( \mathcal{P} \) that strongly witnesses the Fraïssé theory of \( \mathcal{M} \).

Let \( m \) be a nondegenerate continuous probability measure on \( \mathbb{R} \). By Proposition 5.6, as \( W \) ranges over weights, there are continuum-many different measures \( \mu(P, m^W) \). By Corollary 5.3, they are all invariant probability measures concentrated on the orbit of \( \mathcal{M} \). Finally, by Proposition 2.23, each such invariant measure is ergodic. \( \square \)
Acknowledgments

The authors would like to thank Willem Fouché, Yonatan Gutman, Alexander Kechris, André Nies, Arno Pauly, Jan Reimann, and Carol Wood for helpful conversations.

This research was facilitated by the Dagstuhl Seminar on Computability, Complexity, and Randomness (January 2012), the conference on Graphs and Analysis at the Institute for Advanced Study (June 2012), the Buenos Aires Semester in Computability, Complexity, and Randomness (March 2013), the Arbeitsgemeinschaft on Limits of Structures at the Mathematisches Forschungsinstitut Oberwolfach (March–April 2013), the Trimester Program on Universality and Homogeneity of the Hausdorff Research Institute for Mathematics at the University of Bonn (September–December 2013), and the workshop on Analysis, Randomness, and Applications at the University of South Africa (February 2014).

Work on this publication by C. F. was made possible through the support of ARO grant W911NF-13-1-0212 and grants from the John Templeton Foundation and Google. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

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