Statistical mechanics of the multi-constraint continuous knapsack problem

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Abstract. We apply the replica analysis established by Gardner to the multi-constraint continuous knapsack problem, which is one of the linear programming problems and a most fundamental problem in the field of operations research (OR). For a large problem size, we analyse the space of solution and its volume, and estimate the optimal number of items to go into the knapsack as a function of the number of constraints. We study the stability of the replica symmetric (RS) solution and find that the RS calculation cannot estimate the optimal number of items in knapsack correctly if many constraints are required. In order to obtain a consistent solution in the RS region, we try the zero entropy approximation for this continuous solution space and get a stable solution within the RS ansatz. On the other hand, in replica symmetry breaking (RSB) region, the one step RSB solution is found by Parisi’s scheme. It turns out that this problem is closely related to the problem of optimal storage capacity and of generalization by maximum stability rule of a spherical perceptron.

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1. Introduction

Recently, Korutcheva, Opper and López [1] pointed out that statistical mechanical analysis based on the replica calculation [2] can be used to investigate some specific statistical properties of optimal solutions for an optimization problem, the so-called knapsack problem. The knapsack problem they studied is one of the integer programming problems in which the variables are all integer. The problem is stated as follows. Let us suppose that a man climbs a mountain. He puts $N$ items $s_1, s_2, \cdots, s_N$ in his knapsack. Each item has its own weight $a_1, a_2, \cdots, a_N$ and “utility” $c_1, c_2, \cdots, c_N$ which means, for example, its own “price” or “necessity in climbing a mountain”. He cannot bring all the $N$ items in his knapsack and he has to leave some of them. He must decide which combination of items is best for him. We can write down the above situation as

$$\max \left\{ \sum_{j=1}^{N} c_j s_j \mid \sum_{j=1}^{N} a_j s_j \leq b, s_j \in \{0, 1\}, j = 1, \cdots, N \right\}$$

(1)

where the decision variable $s_j = 1$ is defined such that the $j$th item goes into the knapsack, and $s_j = 0$ otherwise. The constant $b$ represents the weight limit. If we have to take account not only of the strength of the man but also of the capacity of the knapsack itself (i.e. the volume of the knapsack is limited), this new constraint must be considered and the problem becomes more complex. Thus it is meaningful for us to generalize our problem to a new form with $K$ constraints as follows.

$$\max \left\{ \sum_{j=1}^{N} c_j s_j \mid \sum_{j=1}^{N} a_{kj} s_j \leq b_k, k = 1, \cdots, K, s_j \in \{0, 1\}, j = 1, \cdots, N \right\}$$

(2)

Following Korutcheva et al [1], we set the utilities $c_j$ all $1/2$, and we set the limits $b_k$ all representing capacities of the man, of his knapsack, etc, to $b$.

In order to treat the multi-constraint ($K \sim N$) knapsack problem in a more familiar way for physicists, we convert the decision variables $s_j = (1, 0)$ into spin variables $\hat{s}_j = (-1, 1)$, by transformation $\hat{s}_j = 2(s_j - 1/2)$. The load $a_{kj}$ is also transformed as $a_{kj} = 1/2 + \xi_{kj}$, where the quenched random variable $\xi_{kj}$ has zero mean and variance $\sigma^2$ and is assumed to obey the distribution

$$P(\xi_{ki}) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left(-\frac{(\xi_{ki})^2}{2\sigma^2}\right)$$

(3)

As loads $a_{ki}$, which means “weights” for example, should be positive, we must choose the above variance $\sigma^2$ small enough and for this small variance, the fraction of items for which $a_{ki} < 0$ for all $k$ vanishes exponentially with increasing $K$. In this paper, we use $\sigma = 1/12$ following Korutcheva et al [1]. Using these transformations, our problem is rewritten as follows.

$$\max \left\{ U = \frac{N}{4} + \frac{1}{2} \sum_{j=1}^{N} \hat{s}_j \mid Y(\{s\}, \{a_{ki}\}, b) \leq 0, k = 1, \cdots, K, \hat{s}_j \in \{1, -1\}, j = 1, \cdots, N \right\}$$

(4)
where
\[ Y(\{s\}, \{a_{ki}\}, b) = \frac{1}{2} \sum_{j=1}^{N} (1 + \hat{s}_j) \xi_{kj} + \frac{1}{4} \sum_{j=1}^{N} \hat{s}_j + \frac{N}{4} - b \]  

(5)

From the constraints appearing in Eqs.(4) and (5), if we treat the case of \(N/4 \ll b\), we can put most of the items into the knapsack. On the other hand, the number of items to be in the knapsack is too small when we set \(N/4 \gg b\). For this reason, we treat the case of \(N/4 = b\) which means that about half of the items go into the knapsack.

The case of finite \(K\) in the limit \(N \to \infty\) was investigated by Meanti et al. \[3\] by the Lagrangian relaxation technique. They gave the explicit results for the special cases of \(K = 1\) and \(K = 2\). Recently, Fontanari \[4\] presented an explicit calculation of the annealed entropy of the configuration \(s^*\) which satisfies \(K\) constraints and gives the total benefit \(E = \sum_j s^*_j/2\). He estimated the upper limit of the total benefit using the zero entropy condition.

He compared it with the exact result from Lagrangian relaxation for the case of \(K = 1, 2\) (exact solutions) and \(K > 2\) to conclude that the annealed approximation becomes good as \(K\) increases.

The original knapsack problem consists of only one constraint, or at most, several constraints. One may feel that only such a case is worth investigating. However, in actual operations research field, we sometimes face multi-constraint (\(K \sim N\)) knapsack problems \[6\], an example of which is investigated in this paper.

If we look at the above problem from an actual operations research (OR) point of view, difficulties lie in the discreteness of knapsack variables. When hardship of this sorts confronts us, it is customary to apply the “linear programming relaxation” technique \[5\]. Linear programming relaxation is an approach to solve the integer programming problem approximately, which brings us to the neighborhood of the exact solution. Actually, for linear programming problems, a lot of useful methods, for example, the simplex method or the interior point method, have been proposed and improved \[5\].

In addition, it is worth while to investigate the objective function for a linear programming problem itself (not as a relaxation of integer programming problem such as the orignal knapsack problem) \[6\]. Linear programming problem is defined as follows

\[ \max \left\{ \frac{1}{2} \sum_{j=1}^{N} \hat{s}_j \left| \sum_{j=1}^{N} a_{kj} \hat{s}_j \leq b, k = 1, \cdots, K, 0 \leq \hat{s}_j \leq \infty, j = 1, \cdots, N \right\} \]  

(6)

Problems of the above style occurs very often in our life. “The diet (nutrition) problem” is one of them. The diet problem ask us to determine the intake \(\{s_j\}\) of each food. Each food \(s_j\) has its own nutrients \(a_{kj}\), i.e., calcium, protein, vitamin, energy, etc., and one has to take each nutrient within the limit, i.e. \(\sum_j a_{kj} \leq b\) (\(k = 1, \cdots, K\)), where \(a_{kj}\) is \(k\)th nutrient of the \(j\)th food. We should notice that valuables in this problem are real numbers, not integers. And linear programming problem has more application than for integer programing problem \[6\]. For two reason, it is worth investigating the multi-constraint (\(K \sim N\)) continuous knapsack problem.

When we look at our knapsack problem from the stand-point of linear programing relaxation, the knapsack valuable \(s_i\), which has a real value and determines whether the \(i\)th item goes into knapsack, takes any value in the real subspace satisfies \(\sum_i s_i^2 = N\).
We treat the next real-variable problem in the present paper.

\[
\max \left\{ U = \frac{N}{4} + \frac{1}{2} \sum_{j=1}^{N} \hat{s}_j \left| Y(\{s\}, \{a_{ki}\}, b) \leq 0, k = 1, \ldots, K, \hat{s}_j \in \{-\infty, +\infty\}, j = 1, \ldots, N \right\} \right. 
\]

(7)

and Eq. (5). Although this relaxation of constraint \( \sum_i \hat{s}_i^2 = N \) and \( \hat{s}_i \in \{-\infty, +\infty\} \) in Eq. (7) somewhat obscures the direct significance of the variable \( s_i \), the global macroscopic behavior of the system is expected not to be affected by this approximation as was mentioned above. And this constraint could be justified as simply the spherical version of the problem investigated by Korutcheva et al [1].

Using this linear programming relaxation technique, we can obtain the approximate configuration \( s^* = (s^*_1, \ldots, s^*_N) \) and we get higher value of the objective function (or the total profit)

\[
E = \frac{1}{2} \sum_{j=1}^{N} s_j
\]

(8)

In this paper, we investigate the knapsack problem by the linear programming relaxation technique and estimate the objective function when the number of constraints is given for a large problem size.

### 2. Replica symmetric theory

For the parameter regions mentioned in the previous section, we consider the case with continuous variables under the normalization constraint \( \sum_{i=1}^{N} (\hat{s}_i)^2 = N \). For a large problem size, the fractional volume of solution \( V(M, \{\xi_{ki}\}) \) to Eq. (7) is written by introducing the “fluctuation of the magnetization” around 0, which is of order \( O(1) \),

\[
M = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \hat{s}_i
\]

(9)

as

\[
V(M, \{\xi_{ki}\}) = \prod_{k=1}^{K} \int d\hat{s}_i \Theta \left( -\sum_{i=1}^{N} (1 + \hat{s}_i) \xi_{ki} \sqrt{\frac{N}{M}} + \frac{M}{2} \right) \delta \left( \sum_{i=1}^{N} (\hat{s}_i)^2 - N \right) \delta \left( \sqrt{NM} - \sum_{i=1}^{N} \hat{s}_i \right)
\]

(10)

This expression is similar to Gardner’s volume of a binary spherical perceptron with local field \( -\sum_i \hat{s}_i \xi_{ki} / \sqrt{N} \) and stability criterion constant \( -M/2 \) appearing in the problem of optimal capacity [3] or of the generalization of maximum stability rule [4]. Thus it is possible to use the same technique. And this V is the value of configuration \( \{\hat{s}_i\} \) which have a fixed \( M \), implying also a fixed utility \( U \). The typical value of \( V \) is given by \( \exp(\ll \log V \gg) \), where \( \ll \cdots \gg \) denotes the averaging over quenched disorder. We perform the average of \( \log V \) with the distribution (3) over the different sets of...
solutions using the replica trick as follows.

\[ \langle \log V \rangle = \lim_{n \to 0} \frac{\langle V^n \rangle - 1}{n} \]  

(11)

The averaging of the power \( \langle V^n \rangle \) is accomplished by introducing an ensemble of \( n \) identical replicas

\[ \langle V^n \rangle = \prod_{\alpha=1}^{n} \left( \int d\hat{s}_i^\alpha \Theta \left( -\left( \sum_{i=1}^{N} \frac{(1 + \hat{s}_i^\alpha) \xi_{ki}}{\sqrt{N}} + \frac{M}{2} \right) \delta \left( \sum_{i=1}^{N} (\hat{s}_i^\alpha)^2 - N \right) \delta \left( \sqrt{N} M - \sum_{i=1}^{N} \hat{s}_i^\alpha \right) \right) \prod_{k=1}^{K} \int d\hat{s}_i^\alpha \delta \left( \sum_{i=1}^{N} (\hat{s}_i^\alpha)^2 - N \right) \right) \]  

(12)

Then we introduce the order parameter which is the overlap between two solutions labeled by two replica indices \( \alpha \) and \( \beta \),

\[ q_{\alpha \beta} = \frac{1}{\sqrt{N}} \sum_{i} \hat{s}_i^\alpha \hat{s}_i^\beta \]  

(13)

By the standard procedure, we obtain within the replica symmetric ansatz

\[ \langle V^n \rangle = \exp \left[ N n \left\{ \text{extr}_{q, \hat{E}, \hat{q}, \hat{M}} G(q, \hat{E}, \hat{q}, \hat{M}) + O\left( \frac{1}{N} \right) \right\} \right] \]  

(14)

where \( \text{extr}_{q, \hat{E}, \hat{q}, \hat{M}} \) denotes the extremization with respect to the parameters \( q, \hat{E}, \hat{q} \) and \( \hat{M} \). Here

\[ G = \alpha G_1 + G_2 - \frac{i}{2} \hat{q} \hat{q} + i \hat{E} \]  

(15)

\[ G_1 = \log \prod_{\alpha} \int_{-M/2}^{M/2} \frac{d\lambda_{\alpha}}{2\pi} \int_{-\infty}^{\infty} dx_{\alpha} \exp \left[ i \sum_{\alpha} \lambda_{\alpha} - \sigma^2 \sum_{\alpha} (x_{\alpha})^2 - \sigma^2 (1 + q) \sum_{\beta} \sum_{\alpha < \beta} x_{\alpha} x_{\beta} \right] \]  

(16)

\[ G_2 = \log \int_{-\infty}^{\infty} \prod_{\alpha} d\hat{s}_\alpha \exp \left[ -i \hat{M} \sum_{\alpha} \hat{s}_\alpha - i \hat{q} \sum_{\beta} \sum_{\alpha < \beta} \hat{s}_\alpha \hat{s}_\beta - i \hat{E} \sum_{\alpha} (\hat{s}_\alpha)^2 \right] \]  

(17)

Using the saddle point equation with respect to \( \hat{M} \), we obtain

\[ \hat{M} = 0 \]  

(18)

Using this result and saddle point equation with respect to \( \hat{E} \), we get

\[ G(M, q) = \alpha \int_{-\infty}^{\infty} Dt \log H \left( \frac{M(2\sigma) - t \sqrt{1 + q}}{\sqrt{1 - q}} \right) + \frac{1}{2} \log(1-q) + \frac{1}{2} \frac{q}{1-q} \]  

(19)

where

\[ Dt \equiv \frac{\exp(-t^2/2)}{\sqrt{2\pi}} \]  

(20)
and

\[ H(x) = \int_x^\infty Dt \]  

Finally we estimate the saddle point of Eq.(19) with respect to \( q \) in the limit \( q \to 1 \). This means that only one optimal solution (which satisfies all \( K = \alpha N \) conditions in Eq.(7)) is selected for a given number of constraints in this limit and then optimal \( M_{\text{opt}} \) can be calculated by \( q \to 1 \) for this optimal \( \hat{s} \) which satisfies all \( K \) constraints (7). Thus we obtain \( M_{\text{opt}} \) as a function of \( \alpha = K/N \). This result is shown in figure 1.

From this result, we see that \( M - \alpha \) lines show the next behavior: As the number of constraints increases, \( M \) decreases and as a result, total utilities decrease (we should remember that total utilities are given as \( U = N/4 + \sum_{j=1}^N \hat{s}_j/2 \) and \( M = \sum_{i=1}^N \hat{s}_i/\sqrt{N} \)). In figure 2, we also plotted the optimal profit \( M_{\text{opt}} \) for the case of Ising variables using the zero entropy condition of solution discrete space taken from [1]. This result shows that we can obtain the larger optimal profit than that of the original knapsack problem by our relaxation of variables.

3. Three relevant lines

3.1. AT line

In order to investigate the local stability of the RS solution, we follow the usual Almeida and Thouless [10] argument and introduce the fluctuations of the order parameters around the RS order parameter as follows.

\[
\begin{align*}
q_{\alpha\beta} &= q + \delta q_{\alpha\beta} \\
\hat{q}_{\alpha\beta} &= \hat{q} + \delta \hat{q}_{\alpha\beta} \\
M_{\alpha} &= M + \delta M_{\alpha} \\
\hat{M}_{\alpha} &= \hat{M} + \delta \hat{M}_{\alpha} \\
\hat{E}_{\alpha} &= \hat{E} + \delta \hat{E}_{\alpha}
\end{align*}
\]  

(22)

It turns out that only fluctuations in \( q_{\alpha\beta} \) and \( \hat{q}_{\alpha\beta} \) lead to instability. The function to be investigated has the form

\[
\alpha G_1(q_{\alpha}, M_{\alpha}) + G_2(q_{\alpha\beta}, \hat{M}_{\alpha}, \hat{E}_{\alpha}) + i \sum_{\beta} \sum_{\alpha < \beta} q_{\alpha\beta} \hat{q}_{\alpha\beta}
\]  

(23)

From this expression, we can apply Gardner’s [6, 7] analysis to our case and obtained AT-line as

\[
\alpha \left[ \int_{-\infty}^{\infty} \! Dt \left\{ 1 - \frac{\int_{I_z} z^2 Dz}{\int_{I_z} Dz} + \left( \frac{\int_{I_z} z Dz}{\int_{I_z} Dz} \right)^2 \right\} \right]^2 < 1
\]  

(24)

where

\[
\int_{I_z} \equiv \int_{(M/(2\sigma)-\sqrt{1+q\ell})/\sqrt{1-q}}^{\infty}
\]  

(25)

and \( q \) is the order parameter obtained by the replica symmetric calculation and is given as

\[
q = \alpha \left( \frac{\sqrt{1-q}}{\sqrt{1+q}} - q \right) \int D t \left( \frac{M}{2\sigma} \sqrt{1+q - 2t} \right)
\]
\[ \times \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(M/(2\sigma) - t\sqrt{1+q})^2}{2(1-q)} \right) / H \left( \frac{M/(2\sigma) - t\sqrt{1+q}}{\sqrt{1-q}} \right) \] (26)

We plot this condition Eq.(24) in figure 1.

From this figure, we see that the replica symmetry is stable for \( M > M_{AT} = 0.100 \) \((\alpha < \alpha_{AT} = 0.846)\). However, when \( M \) becomes smaller than this critical value, the replica symmetry breaking occurs and the replica symmetric saddle point becomes unstable. The AT point \((\alpha_{AT}(M_{AT}))\) is a critical point where the replica symmetric order parameter \( q_{RS} = q \) splits into (one step) replica symmetry breaking saddle point represented by \( q_0 \) and \( q_1 \). If this RSB transition at \( \alpha_{AT} \) is continuous, the AT condition Eq.(24) gives a correct criterion of replica symmetry breaking. On the other hand, if the RSB transition is discontinuous (first order), we should regard this discontinuity point as the symmetry breaking point rather than the AT stability limit. Fortunately, as we see in the next section, this transition is continuous and the AT argument is correct.

In order to understand this situation physically, we can investigate the replica symmetry breaking from the dis-connectivity of the solution space \( \hat{s} \) according to Monasson and O’Kane [12], who treated RSB of perceptron with non-monotonic transfer function.

The solution condition Eq.(7) can be rewritten as

\[ \sum_i \hat{s}_i \frac{\xi_{ki}}{\sqrt{N}} \leq -\frac{M}{2} + \sum_i \frac{\xi_{ki}}{\sqrt{N}} = -\frac{M}{2} + \mathcal{O}(1) \] (27)

Then if \( M \) is positive, for many constraints \( K \), the space of \( \hat{s} \) which satisfies Eq.(7) as well as the normalization constraint \( \sum(\hat{s})^2 = N \) consists of a single domain and the solution space is connected. On the other hand, as \( M \) becomes negative, the space of \( \hat{s} \) splits into a lot of domains and the solution space is disconnected.

Here we can estimate the \( \mathcal{O}(1) \) term roughly appearing on the right hand side of the above inequality as \( \sum_j \xi^\mu_j / \sqrt{N} \sim (1/\sqrt{N}) \times \sqrt{N} \sigma = 1/12. \) We should notice that this term is of \( \mathcal{O}(1) \) and we used the standard deviation \( \sigma = 1/12 \). Therefore if we investigate the dis-connectivity of solution space, replica symmetry must be broken for \( M < 1/6 = 0.1667 \). This value is not so far from the value obtained by the AT argument. We conclude that this dis-connectivity leads to replica symmetry breaking.

For this RSB region, we will later find the one step replica symmetry breaking solution following the scheme of Parisi [13].

### 3.2 Zero entropy line

In the previous subsection, we calculated the volume of solution space \( \exp(\log \ll V \gg) \) for quenched random loads \( \xi_{ki} \). For optimal \( M_{opt} \), this volume shrinks to zero continuously.

Thus the increase in the number of constraints leads to a decrease of the solution space \( \hat{s} \) in a discrete manner, and the entropy of solution space is \( S(M_{opt}) = 0 \) at \( M = M_{opt} \). Following the idea suggested by Krauth and Mézard [11] we expect that the optimal \( M \) for continuous variables may be obtained when the volume of solution contains typically a single point of hypercube. From analogy with the discrete-variable problem, this should occur around

\[ V \sim \frac{1}{2^N} \] (28)
3.2.1. Annealed calculation  We first calculate the condition Eq.(28) by the annealed approximation as
\[
\log \langle V \rangle = G - H = -\log 2 \tag{29}
\]
This condition is easily calculated by
\[
G = \alpha \log H \left( \frac{M}{2\sqrt{2}} \right) + \frac{1}{2} \left( 1 + \log(2\pi) \right) \tag{30}
\]
\[
H = \frac{1}{2} \left( 1 + \log(2\pi) \right) \tag{31}
\]
Finally we get
\[
\alpha \log \left( \frac{M}{2\sqrt{2}} \right) = -\log 2 \tag{32}
\]
This is shown in figure 2.

3.2.2. Quenched calculation  Next we calculate the above zero entropy condition Eq.(28) by the quenched calculation defined as
\[
\langle \log V \rangle = \frac{1}{2^n} \tag{33}
\]
This leads to
\[
\alpha \int \! Dt \, \log H \left( \frac{M/2\sigma - t\sqrt{1+q}}{\sqrt{1-q}} \right) + \frac{1}{2} \frac{q}{1-q} + \frac{1}{2} \log(1-q) = -\log 2 \tag{34}
\]
We plotted this result also in Figure 2.

From figure 2 we find that this line Eq.(34) obtained by the quenched zero entropy condition is stable in the RS region. In figure 4 we plotted the behavior of the order parameter q as a function of α which gives zero entropy and satisfies Eq.(34). From these results, we may conclude that our zero entropy approach gives a meaningful and consistent criterion for the optimal \( M_{\text{opt}} \).

3.3. One step RSB solution
In this subsection, we calculate the one step replica symmetry breaking solution following Parisi [13][14]. We try to find a first candidate for matrices \(q\) and \(\hat{q}\) appearing in the free energy. Using Parisi’s suggestion, we divide the \(n\) replicas into \(m/n\) groups of \(m\) replicas. Next we set \(q_{\alpha\beta} = q_1, \hat{q}_{\alpha\beta} = \hat{q}_1\), if \(\alpha\) and \(\beta\) belong to the same group, and \(q_{\alpha\beta} = q_0, \hat{q}_{\alpha\beta} = \hat{q}_0\), if \(\alpha\) and \(\beta\) belong to different groups and we set \(q_{\alpha\alpha} = 0\). We express \(q\) and \(\hat{q}\) in terms of a tensor product,
\[
q = (q_1 - q_0)_{1/n/m} \otimes e_m e_m^T + q_0 e_n e_n^T \tag{35}
\]
\[
\hat{q} = (\hat{q}_1 - \hat{q}_0)_{1/n/m} \otimes e_m e_m^T + \hat{q}_0 e_n e_n^T \tag{36}
\]
where \( \mathbf{1}_k \) is a \( k \)-dimensional unit matrix and \( \mathbf{e}_k^T = (1, \cdots, 1) \). Using this form of broken symmetry of replica, we get the free energy as follows,

\[
f_{1\text{RSB}}(q_1, q_0, m) = \frac{\alpha}{m} \int D\mu \int Dz \left\{ H \left( \frac{M/(2\sigma)-y\sqrt{1+q_0}-z\sqrt{1-q_0}}{1-q_1} \right) \right\}^m
+ \frac{1}{2m} \log[1-q_1+m(q_1-q_0)] + \frac{1}{2} \left( 1 - \frac{1}{m} \right) \log(1-q_1) + \frac{q_0}{2[1-q_1+m(q_1-q_0)]}
\]

where we used the saddle point equations with respect to \( \mu, Q \) and \( q_0 \). For the set of order parameters \( q \) and \( \alpha \), give the critical values \( \alpha \). From this table, we see that until \( \alpha \sim \alpha_{AT} = 1.20 \), the replica symmetric saddle point (which is \( q_0 = q_1 = q \) and is independent of \( m \)) is stable and the one step RSB solution does not exist. When we exceed the critical value \( \alpha_{AT} \), the replica symmetric saddle point becomes unstable and order parameter \( q \) splits into one step RSB order parameters \( q_0 \) and \( q_1 \) continuously as shown also in figure 3. From this continuous transition, we conclude that the AT argument in the previous section is consistent. Comparing the free energy \( f_{RS} \) and \( f_{1\text{RSB}} \), we see that the global minimum of this system goes from the RS saddle point \( q \) to the one step RSB saddle point \( (q_0, q_1) \) at the AT instability point. Therefore, the one step RSB solution actually exists.

In order to obtain the solution in the limit \( q_1 \to 1 \), we use the scaling \( q_1 = 1-\epsilon, q_0 = Q \) and \( m = \mu \epsilon \) to get

\[
F_{1\text{RSB}}(Q, \mu, \epsilon) = \frac{1}{\epsilon} f_{1\text{RSB}}(Q, \mu) + O(\log \epsilon)
\]

Here

\[
f_{1\text{RSB}}(Q, \mu) = \frac{\alpha}{\mu} \int D\mu\log[H_1 + H_2 + \frac{1}{2\mu} \log[1+\mu(1-Q)] + \frac{Q}{2[1+\mu(1-Q)]}]
\]

where

\[
H_1 = H \left( \frac{M/(2\sigma)-y\sqrt{1+Q}}{\sqrt{1-Q}} \right)
\]

and

\[
H_2 = \exp(\frac{\mu(M/(2\sigma)-y\sqrt{1+Q})^2}{2[1+\mu(1-Q)]}) \cdot H \left( \frac{M/(2\sigma)-y\sqrt{1+Q}}{\sqrt{1-Q}\sqrt{1+\mu(1-Q)}} \right)
\]

The saddle point equations with respect to \( \mu, Q \) and \( \epsilon \)

\[
\left. \frac{\partial F}{\partial \mu} \right|_{\mu, Q, \alpha, \epsilon} = \left. \frac{\partial F}{\partial Q} \right|_{\mu, Q, \alpha, \epsilon} = F \left|_{\mu, Q, \alpha, \epsilon} \right. = 0
\]

give the critical values \( \alpha, \epsilon \) calculated by the one step RSB scheme. We solved Eq.(42) numerically and show the result in figure 2.
From this figure, we see that the replica symmetry breaking decreases the optimal $M_{opt}$ slightly. We also confirm this result from the fact that replica symmetry breaking point $q(\alpha_{AT})$ comes close to 1. In figure 5, we also show the order parameter $q_0 = Q$ as a function of the optimal profit $M_{opt}$. From this figure we see that, as $M_{opt}$ becomes close to 0.100, i.e., as replica symmetry breaking becomes weak, $Q$ becomes 1 gradually ($Q = q_0 = 0.9912$ for $m = 0.100$). This result is reasonable from the meaning of replica symmetry breaking. The reason is that in critical limit of $\alpha$, the one step RSB order parameter $q_0$ and $q_1$ becomes close to $q_{RS} = q$ and $q \rightarrow 1$ if replica symmetry does not break down.

In order to confirm that this one step RSB solution is exact, we must check the stability of the one step RSB saddle point by the same technique as that used for the AT line and investigate if the free energy of one step calculation is lower than that of the second step calculation. Or the distribution of sizes of the “disconnected” domains of solution space $s$ must be computed analytically by the same technique as in Monasson and O’Kane [12]. This is a highly non-trivial problem. However, we can conjecture that the two step replica symmetry breaking can hardly decrease the optimal profit even if two step RSB solution exists, because the optimal profit of replica symmetric calculation and that of one step replica symmetry breaking calculation are very close to each other as we saw above.

4. Summary and Discussion

We have calculated the optimal profit $M_{opt}$ explicitly for the case of continuous knapsack variables and the number of constraints $K$ is of the same order with the number of items $N$ by replica symmetric calculation explicitly. From the AT argument, the replica symmetric solution becomes unstable for $M < 0.100$. We also investigated this symmetry breaking point by minimizing the one step replica symmetry breaking free energy directly and saw that this transition is continuous and the AT line is valid. For the argument of the problem of the optimal storage capacity of the perceptron with non-monotonic transfer function investigated by Monasson and O’Kane [12], the physical meaning of this RSB can be understood as the dis-connectivity of the solution space. Therefore, from the condition of dis-connectivity of solution space, we roughly estimated where replica symmetry breaking begins. For the RSB region, it was necessary to try to find a replica symmetry breaking solution according to Parisi’s procedure [13] [14]. Thus using Parisi’s scheme, we obtained the one step RSB solution and the result shows that replica symmetry breaking makes the $M_{opt}$ decrease slightly. Because it is not easy to confirm that this one step RSB calculation is exact, we can roughly expect that even if we calculate the two step RSB solution or three step RSB solution or further step RSB solution, the optimal profit obtained at these further RSB saddle point should hardly increase. In stead of investigation in this direction, in the replica symmetric region, we can find a consistent solution using the zero entropy approach and this result agrees with the discrete variables case [1] for the most part. This result may also support the validity of their RS calculation. The present statistical mechanical approach is expected to be applicable to the other linear programming problems.
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Figure captions

Figure 1. $M_{opt} - \alpha$ line calculated by the replica symmetric theory, AT line and one step replica symmetry breaking solution. In each line, we set $\sigma = 1/12$. The replica symmetric solution is stable above $M \sim 0.100$. In this figure, the region where the replica symmetry is broken is shown enlarged. The optimal profit $M_{opt}$ decreases slightly by the one step replica symmetry breaking calculation.

Figure 2. The zero entropy lines (quenched and annealed calculations). The quenched zero entropy line is stable in the RS region. The optimal profit for Ising variables is also plotted. This line is very close to the quenched zero entropy line.

Figure 3. Order parameter $q$ as a function of the relative number of constraints $\alpha$ for the case of $M_{opt} = 0$. Replica symmetric order parameter $q$ splits into the one step replica symmetry breaking order parameter $(q_0, q_1)$ continuously at $\alpha_{AT} \sim 1.20$.

Figure 4. Overlap $q$ which satisfies the zero entropy condition.
**Figure 5.** One step RSB order parameter \( q_1 = Q \) as a function of the optimal profit \( M_{\text{opt}} \) in the limit of \( q_1 \to 1 \). As replica symmetry breaking becomes weak ( as \( M_{\text{opt}} \) closes to 0.10, \( \alpha(M_{\text{opt}} = 0.100) = \alpha_{\text{AT}} \), \( Q \) closes to 1.

### Tables and table captions

| \( \alpha \) | \( q_{RS} \) | \( q_0 \) | \( q_1 \) | \( m \) | \( \hat{f}_{RS} \) | \( \hat{f}_{RSB} \) |
|---------------|--------------|--------|--------|-----|--------|--------|
| 0.90          | 0.7129       | ———   | ———   | —   | -1.5336| ———   |
| 1.00          | 0.7773       | ———   | ———   | —   | -1.9346| ———   |
| 1.10          | 0.8408       | ———   | ———   | —   | -2.4859| ———   |
| 1.20          | 0.8901       | ———   | ———   | —   | -3.3794| ———   |
| 1.25          | 0.9073       | 0.8213 | 0.9901 | 0.1688 | -3.7642 | -5.7847 |
| 1.30          | 0.9211       | 0.8155 | 0.9902 | 0.1858 | -4.4110 | -6.0763 |

**Table 1.** The replica symmetric order parameter \( q \) and the 1-step replica symmetry breaking order parameters \((q_0,q_1,m)\) are listed with the free energies for several values of \( \alpha \). Until \( \alpha = \alpha_{\text{AT}} = 1.20 \) the RS saddle point is stable and the 1-step RSB solution does not exist. Above the \( \alpha_{\text{AT}} \), the free energy of 1-step RSB approximation is lower than that of RS approximation.
Figure 1  J. Inoue

\[ M_{\text{opt}} \]

\[ \alpha \]

- 1 step RSB
- RS
- AT line

RS  RSB
Figure 2  J. Inoue

- RS
- AT line
- Zero entropy for Ising variables
- Quenched zero entropy
- Annealed zero entropy

\[ M_{opt} \] vs. \[ \alpha \]
Figure 3  J. Inoue

\[ q \]

\[ q_{RS} \]

\[ q_0 \]

\[ q_1 \]
Figure 4  J. inoue

\[ q \]
\[ \alpha \]
Figure 5  J.inoue

\[ Q \]

\[ M_{opt} \]