Application of non-bijective transformations to various potentials

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Paper presented at the XVIth International Colloquium on Group Theoretical Methods in Physics (Varna, Bulgaria, 15-20 June 1987) and published in “Group Theoretical Methods in Physics”, edited by H.-D. Doebner, J.-D. Hennig and T.D. Palev (Springer-Verlag, Berlin, 1988): Lecture Notes in Physics 313, 238 (1988).
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ABSTRACT

Some results about non-bijective quadratic transformations generalizing the Kustaanheimo-Stiefel and the Levi-Civita transformations are reviewed in §1. The three remaining sections are devoted to new results: §2 deals with the Lie algebras under constraints associated to some Hurwitz transformations; §3 and §4 are concerned with several applications of some Hurwitz transformations to wave equations for various potentials in $R^3$ and $R^5$.

1. Non-bijective canonical transformations

We start with a $2m$-dimensional ($2m = 2, 4, 8, \ldots$) Cayley-Dickson algebra $A(c)$ where $c$ stands for a $p$-uple $(c_1, c_2, \ldots, c_p)$ such that $c_i = \pm 1$ for $i = 1, 2, \ldots, p$ and $2^p = 2m$. (Remember that $A(-1)$, $A(-1, -1)$ and $A(-1, -1, -1)$ are nothing but the algebras of complex numbers, usual quaternions and usual octonions, respectively.) Let $u = u_0 + \sum_{i=1}^{2m-1} u_i e_i$ be an element of $A(c)$ where $u_0, u_1, \ldots, u_{2m-1}$ are the components of $u$ and $\{e_1, e_2, \ldots, e_{2m-1}\}$ is a system of generators for $A(c)$. We associate to the hypercomplex number $u$ the element $\hat{u}$ of $A(c)$ defined by $\hat{u} = u_0 + \sum_{i=1}^{2m-1} \epsilon_i u_i e_i$ where $\epsilon_i = \pm 1$ for $i = 1, 2, \ldots, 2m - 1$. Let us consider the right (or left) application $A(c) \rightarrow A(c) : u \mapsto x = uu$ (or $\hat{u}u$).

Three cases can occur according to the form taken by $\hat{u}$.

A. For $\hat{u} = u$: The (right = left) application $u \mapsto x = u^2$ defines a map $R^{2m} \rightarrow R^{2m}$ which constitutes an extension of the Levi-Civita map [1] corresponding to $2m = 2$ and $c_1 = -1$ and of the map introduced in [2] and corresponding to $2m = 4$ and $c_1 = c_2 = -1$. The maps $R^{2m} \rightarrow R^{2m}$ for $2m = 2, 4, 8, \ldots$ correspond to the quasiHurwitz transformations of [3]. From a geometrical viewpoint, the map $R^{2m} \rightarrow R^{2m}$ for fixed $2m$ is associated to a fibration on spheres with discrete fiber in the compact case where $c_i = -1$ ($i = 1, 2, \ldots, p$) and to a fibration on
hyperboloids with discrete fiber in the remaining non-compact cases.

B. For \( \hat{u} = j(u) \): By \( j(u) \) we mean that the coefficients \( \epsilon_i \) \((i = 1, 2, \ldots, 2m-1)\) are such that the application \( j : A(c) \to A(c) : u \mapsto \hat{u} = j(u) \) defines an anti-involution of the algebra \( A(c) \). Then, the right (or left) application \( u \mapsto x = uj(u) \) (or \( j(u)u \)) defines a map \( R^{2m} \to R^{2m-n} \) with \( n = m - 1 + \delta(m, 1) \) or \( 2m - 1 \) being the number of zero components of \( x \). The latter map constitutes an extension of the Kustaanheimo-Stiefel map [4] which corresponds to \( 2m = 4, n = 1, c_1 = c_2 = -1 \) and, for example, \( \epsilon_1 = -\epsilon_2 = -\epsilon_3 = -1 \). The map introduced by Iwai [5] is obtained when \( 2m = 4, n = 1, c_1 = -c_2 = -1 \) and, for example, \( \epsilon_1 = -\epsilon_2 = -\epsilon_3 = -1 \). The maps \( R^{2m} \to R^{2m-n} \) for \( 2m = 2, 4, 8, \ldots \) correspond to the Hurwitz transformations of [3]. From a geometrical viewpoint, the maps \( R^{2m} \to R^{2m-n} \) for \( 2m = 2, 4, 8 \) and 16 are associated to the classical Hopf fibrations on spheres with compact fiber in the compact cases and to fibrations on hyperboloids with either compact or non-compact fiber in the non-compact cases.

C. For \( \hat{u} \neq u \) or \( j(u) \): This case does not lead to new transformations for \( 2m = 2 \) and 4. Some new transformations, referred to as pseudoHurwitz transformations in [3], arise for \( 2m \geq 8 \). In particular for \( 2m = 8 \) and \( \sum_{i=1}^{7} \epsilon_i = -3 \) or 5, the application \( u \mapsto uu \) (or \( uu \)) defines a map \( R^8 \to R^7 \). Such a map is associated to a Hopf fibration on spheres with compact fiber for \( (c_1, c_2, c_3) = (-1, -1, -1) \) and to fibrations on hyperboloids with either compact or non-compact fiber for \( (c_1, c_2, c_3) \neq (-1, -1, -1) \).

The various transformations mentioned above may be presented in matrix form. (A detailed presentation can be found in [3].) Let us define the \( 2m \times 2m \) matrix \( \epsilon = \text{diag}(1, \epsilon_1, \epsilon_2, \ldots, \epsilon_{2m-1}) \) and let \( u \) and \( x \) be the \( 2m \times 1 \) column-vectors whose entries are the components of the hypercomplex numbers \( u \) and \( x \), respectively. Then, the \( R^{2m} \to R^{2m-p} \) transformation defined by \( u \mapsto x = uu \) (where \( p = 0 \) and \( 2m = 2, 4, 8, \ldots \) for quasiHurwitz transformations, \( p = m - 1 + \delta(m, 1) \) or \( 2m - 1 \) and \( 2m = 2, 4, 8, \ldots \) for Hurwitz transformations, and \( p \geq 1 \) and \( 2m \geq 8 \) for pseudoHurwitz transformations) can be described by \( x = A(u) \epsilon u \) where \( A(u) \) is a \( 2m \times 2m \) matrix. For \( 2m = 2, 4 \) or 8, the matrix \( A(u) \) may be written in terms of Clifford matrices and constitutes an extension of the Hurwitz matrices occurring in the Hurwitz factorization theorem (see [3]). Note that there are \( p \) zero entries in the \( 2m \times 1 \) column-vector \( x \) with, in particular, \( p = 2m - 1 \) or \( m - 1 + \delta(m, 1) \) for Hurwitz transformations.

2. Lie algebras under constraints
In this section, we shall restrict ourselves to the Hurwitz transformations for $2m = 2, 4$ and $8$ and, more specifically, to the $R^{2m} \to R^{2m-n}$ transformations with $n = m - 1 + \delta(m, 1)$. These transformations are clearly non-bijective and this fact may be transcribed as follows. Let us consider the $2m \times 1$ column-vector $2A(u) \epsilon du$ where $du$ is the differential of the $2m \times 1$ column-vector $u$. It can be seen that $2m - n$ components of $2A(u) \epsilon du$ may be integrated to give the $2m - n$ non-zero components of $x = A(u) \epsilon u$. Further, the remaining $n$ components of $2A(u) \epsilon du$ are one-forms $\omega_1, \omega_2, \ldots, \omega_n$ which are not total differentials. In view of the non-bijective character of the map $R^{2m} \to R^{2m-n}$, we can assume that $\omega_i = 0$ for $i = 1, 2, \ldots, n$. To each one-form $\omega_i$, we may associate a vector field $X_i$ which is a bilinear form in the $u_\alpha$ and $p_\alpha = \partial/\partial_\alpha$ for $\alpha = 0, 1, \ldots, 2m - 1$. An important property, for what follows, of the latter vector fields is that $X_i \psi = 0$ ($i = 1, 2, \ldots, n$) for any function $\psi$ of class $C^1(R^{2m-n})$ and $C^1(R^{2m})$ in the variables of type $x$ and $u$, respectively. In addition, it can be verified that the $n$ operators $X_i$ ($i = 1, 2, \ldots, n$) span a Lie algebra. We denote $L_0$ this algebra and refer it to as the constraint Lie algebra associated to the $R^{2m} \to R^{2m-n}$ Hurwitz transformation.

Now, it is well known that the $2m(4m + 1)$ bilinear forms $u_\alpha u_\beta$, $u_\alpha p_\beta$ and $p_\alpha p_\beta$ for $\alpha, \beta = 0, 1, \ldots, 2m - 1$ span the real symplectic Lie algebra $sp(4m, R)$ of rank $2m$. We may then ask the question: what remains of the Lie algebra $sp(4m, R)$ when we introduce the $n$ constraint(s) $X_i = 0$ ($i = 1, 2, \ldots, n$) into $sp(4m, R)$. (It should be noted that each constraint $X_i = 0$ may be regarded as a primary constraint in the sense of Dirac, cf. [6,7].) This amounts in last analysis to look for the centralizer of $L_0$ in $L = sp(4m, R)$ [8]. The resulting Lie algebra $L_1 = cent_L L_0 / L_0$ is referred to as the Lie algebra under constraints associated to the $R^{2m} \to R^{2m-n}$ Hurwitz transformation. An important result is the following.

**Result 1.** For fixed $2m$, the constraint Lie algebra $L_0$ and the Lie algebra under constraints $L_1$ are characterized by the (compact or non-compact) nature of the fiber of the fibration associated to the $R^{2m} \to R^{2m-n}$ Hurwitz transformation. The determination of $L_1$ has been achieved, in partial form, for one of the cases $(2m, 2m - n) = (4, 3)$ in [9] and, in complete form, for all the cases $(2m, 2m - n) = (2, 1), (4, 3)$ and $(8, 5)$ in [8].

3. **Generalized Coulomb potentials in $R^3$ and $R^5$**

Let us begin with the “Coulomb” potential ($-Ze^2$ is a coupling constant):

$$V_5 = -Ze^2/(x_0^2 - c_2x_2^2 + c_1c_2x_3^2 - c_3x_4^2 + c_1c_3x_5^2)^{1/2}$$

in $R^5$ equipped with the metric $\eta_5 = diag(1, -c_2, c_1c_2, -c_3, c_1c_3)$ and consider
the Schrödinger equation for this potential and this metric. (The corresponding
generalized Laplace operator is $\tilde{\nabla} \eta \nabla$ in the variables $(x_0, x_2, x_3, x_4, x_5)$.) We
now apply a $R^8 \to R^5$ Hurwitz transformation to the considered problem. The
knowledge of the transformation properties of the generalized Laplace operators
under the $R^8 \to R^5$ Hurwitz transformations leads to the following result.

Result 2. The Schrödinger equation for the potential $V_5$ in $R^5$ with the
metric $\eta_5$ and the energy $E$ is equivalent to a set consisting of (i) one Schrödinger
equation for the harmonic oscillator potential

$$V_8 = -4E(u_0^2 - c_1u_1^2 - c_2u_2^2 + c_1c_2u_3^2 - c_3u_4^2 + c_1c_3u_5^2 + c_2c_3u_6^2 - c_1c_2c_3u_7^2)$$

in $R^8$ with the metric $\eta_8 = diag(1, -c_1, -c_2, c_1c_2, -c_3, c_1c_3, c_2c_3, -c_1c_2c_3)$ and the
energy $4Ze^2$ and (ii) three first-order differential equations associated to the $n = 3$
constraints of the $R^8 \to R^5$ Hurwitz transformations.

A similar result is obtained under the evident replacements: $V_5 \to V_3 =
-Ze^2/(x_0^2 - c_2x_2^2 + c_1c_2x_3^2)^{1/2}$, $\eta_5 \to \eta_3 = diag(1, -c_2, c_1c_2)$, $V_8 \to V_4 = -4E(u_0^2 -
c_1u_1^2 - c_2u_2^2 + c_1c_2u_3^2)$, $\eta_8 \to \eta_4 = diag(1, -c_1, -c_2, c_1c_2)$ and $n = 3 \to n = 1$. The
so-obtained result for the generalized Coulomb potential $V_3$ in $R_3$ equipped with
the metric $\eta_3$ thus corresponds to $c_3 = 0$. Note that the usual Coulomb potential
$-Ze^2/(x_0^2 + x_2^2 + x_3^2)^{1/2}$ corresponds to $c_3 = 0$ and $c_1 = c_2 = -1$. (The case of
$V_3$ with $c_1 = c_2 = -1$ has been investigated in [10,11,12] and the case of $V_5$ with
$c_1 = c_2 = c_3 = -1$ has been recently considered in [3,12,13].)

As a corollary, information on the spectrum of the hydrogen atom in $R^5$ ($R^3$)
with the metric $\eta_5$ ($\eta_3$) can be deduced from the knowledge of the spectrum of the
harmonic oscillator in $R^8$ ($R^4$) with the metric $\eta_8$ ($\eta_4$). By way of illustration, we
shall continue with $c_1 = c_2 = c_3 = -1$ (i.e., the case of a Coulomb potential in $R^5$
with unit metric), on the one hand, and with $c_1 = c_2 = c_3 = -1$ (i.e., the case of
the usual Coulomb potential), on the other hand. The non-invariance dynamical
algebras for the corresponding isotropic harmonic oscillators in $R^8$ and $R^4$ are
clearly $sp(16, R)$ and $sp(8, R)$, respectively. Then, the non-invariance dynamical
algebras for the corresponding hydrogen atoms in $R^5$ and $R^3$ are nothing but
the Lie algebras under constraints $L_1$ associated to the $R^8 \to R^5$ and $R^4 \to R^3$
compact Hurwitz transformations, respectively. The results for $L_1$ of [8,9] yield
the non-invariance dynamical algebras $L_1 = so(6, 2)$ and $so(4, 2)$ for the hydrogen
atoms in $R^5$ and $R^3$, respectively.

To close this section, let us show how to obtain the discrete spectra for
the $R^5$ and $R^3$ hydrogen atoms under consideration. A careful examination of
the hydrogen-oscillator connection shows that energies and coupling constants are
exchanged in such a connection. As a matter of fact, we have

\[(1/2)\mu(2\pi\nu)^2 = -4E \quad h\nu(\sum_{\alpha=0}^{2m-1} n_\alpha + m) = 4Ze^2 \quad n_\alpha \in N \quad 2m = 8 \text{ or } 4\]

where \(\mu(2\pi\nu)^2\) is the coupling constant for the oscillator (whose mass \(\mu\) is the reduced mass of the Coulomb system) and \(n_\alpha\) for \(\alpha = 0, 1, \ldots, 2m - 1\) are the (Cartesian) quantum numbers for the isotropic harmonic oscillator in \(R^{2m}\) (\(2m = 8\) or \(4\)). Furthermore, the \(n\) constraint(s) associated to the \(R^{2m} \rightarrow R^{2m-n}\) Hurwitz transformations for \(2m = 8\) and \(4\) yield \(\sum_{\alpha=0}^{2m-1} n_\alpha + 2 = 2k\) where \(k\) (\(= 1, 2, 3, \ldots\)) plays the role of a principal quantum number (cf. [11]). By eliminating the frequency \(\nu\) from the formulas connecting coupling constants and energies, we end up with

\[E = E_0/(k + m/2 - 1)^2 \quad E_0 = -\mu Z^2 e^4/(2\hbar^2) \quad k = 1, 2, 3, \ldots\]

where \(m/2 = 1\) and \(2\) for the \(R^3\) and \(R^5\) hydrogen atoms, respectively, in agreement with the Bohr-Balmer formula in arbitrary dimension (see, for example, [7,14]).

4. Axial potentials in \(R^3\)

A. Generalized Hartmann potential in \(R^3\). Let us consider the potential

\[W_3 = -\eta\sigma^2/r + (1/2)q\eta^2\sigma^2/\rho^2\]

in \(R^3\) equipped with the metric \(\eta_3\). The variables \(r\) and \(\rho\) are “distances” in \(R^3\) and \(R^2\) given by \(r = (x_0^2 - c_2 x_2^2 + c_1 c_2 x_3^2)^{1/2}\) and \(\rho = (-c_2 x_2^2 + c_1 c_2 x_3^2)^{1/2}\), respectively. The parameters \(\eta\) and \(\sigma\) are positive and the parameter \(q\) is such that \(0 \leq q \leq 1\). The potential \(W_3\) is an extension of the so-called Hartmann potential which is of interest in the quantum chemistry of ring-shaped molecules. Indeed, the Hartmann potential corresponds to \(c_1 = c_2 = -1\) and \(q = 1\) (cf. [15]). The potential \(W_3\) for \(q = 1\) and \((c_1, c_2)\) arbitrary shall be called generalized Hartmann potential. It is to be observed that for \(q = 0\) and \(\eta\sigma^2 = Ze^2\), the potential \(W_3\) identifies to the (generalized) Coulomb potential \(V_3\) in \(R^3\) equipped with the metric \(\eta_3\). (The parameter \(q\) is thus simply a distinguishing parameter which, for \(0 \leq q \leq 1\), may be restricted to take the values \(0\) or \(1\).)

It is possible to find an \(R^4 \rightarrow R^3\) Hurwitz transformation to transform the \(R^3\) Schrödinger equation, with the metric \(\eta_3\), for the generalized Hartmann potential into an \(R^4\) Schrödinger equation, with the metric \(\eta_4\), for a non-harmonic oscillator plus a constraint equation. Each of the two obtained equations can
be separated into two $R^2$ equations. This leads to the following result where $E$ denotes the energy of a particle of (reduced) mass $\mu$ in the potential $W_3$. Such a result generalizes the one derived in [15] for the special case $c_1 = c_2 = -1$.

Result 3. The $R^3$ Schrödinger equation, with the metric $\eta_3$, for the generalized Hartmann potential $W_3$ is equivalent to a set comprising (i) two coupled $R^2$ Schrödinger equations for two two-dimensional oscillators with mass $\mu$, one with the metric $\text{diag}(1, -c_1)$ and the potential $V_{01} = -4E(u_0^2 - c_1 u_1^2) + (1/2)q\eta^2\sigma^2/(u_0^2 - c_1 u_1^2)$, the other with the metric $\text{diag}(-c_2, c_1 c_2)$ and the potential $V_{23} = -4E(-c_2 u_2^2 + c_1 c_2 u_3^2) + (1/2)q\eta^2\sigma^2/(-c_2 u_2^2 + c_1 c_2 u_3^2)$, and (ii) two coupled $R^2$ constraint equations.

B. Generalized Coulomb + Aharonov-Bohm potential in $R^3$. Let us consider

$$X_3 = Ze'e''/r + (2\mu \rho^2)^{-1}[A + iB(x_2 \partial/\partial x_3 + c_1 x_3 \partial/\partial x_2)]$$

in $R^3$ equipped with the metric $\eta_3$. Here again, we have $r = (x_0^2 - c_2 x_2^2 + c_1 c_2 x_3^2)^{1/2}$ and $\rho = (-c_2 x_2^2 + c_1 c_2 x_3^2)^{1/2}$. The generalized Hartmann potential $W_3$ can be obtained as a special case of $X_3$: when $Ze'e'' = -\eta\sigma^2$, $A/\mu = q\eta^2\sigma^2$ and $B = 0$, the potential (energy) $X_3$ identifies to $W_3$. In the (compact) case $c_1 = c_2 = -1$ and for $A = (e'/f)/c^2$ and $B = 2e'\hbar f/c$ with $f = F/(2\pi)$, the (velocity-dependent) operator $X_3$ describes the interaction of a particle of charge $e'$ and (reduced) mass $\mu$ with a potential $(A, V)$, where the scalar potential $V = Ze''/(x_0^2 + x_2^2 + x_3^2)^{1/2}$ is of the Coulomb type and the vector potential $A = (A_{x_2} = -[x_2/(x_2^2 + x_3^2)]f, A_{x_3} = [x_2/(x_2^2 + x_3^2)]f, A_{x_0} = 0)$ is of the Aharonov-Bohm type (cf. [16]). We are now in a position to list the following result which generalizes the one obtained in [16] for the special case $c_1 = c_2 = -1$.

Result 4. It is possible to find a Hurwitz transformation to convert the $R^3$ Schrödinger equation, with the metric $\eta_3$, for $X_3$ into an $R^4$ Schrödinger equation, with the metric $\eta_4$, accompanied by a constraint condition. The separation of variables from $R^4$ to $R^2 \times R^2$ is possible here again and this leads to a result similar to Result 3 with the replacements $V_{01} \rightarrow -4E(u_0^2 - c_1 u_1^2) + (A - mB)/[2\mu(u_0^2 - c_1 u_1^2)]$ and $V_{23} \rightarrow -4E(-c_2 u_2^2 + c_1 c_2 u_3^2) + (A - mB)/[2\mu(-c_2 u_2^2 + c_1 c_2 u_3^2)]$, with $im$ being a separation constant.

C. Coulomb + Sommerfeld + Aharonov-Bohm + Dirac potential in $R^3$. We close this paper with a brief study of the potential (energy):

$$Y_3 = Ze'e''/r + s/r^2 + (2\mu \rho^2)^{-1}[\alpha(x) + i\beta(x)(x_2 \partial/\partial x_3 - x_3 \partial/\partial x_2)]$$

in $R^3$ equipped with the unit metric. The distances $r$ and $\rho$ are given here by $r = (x_0^2 + x_2^2 + x_3^2)^{1/2}$ and $\rho = (x_2^2 + x_3^2)^{1/2}$. The potential $Y_3$ includes a Coulomb
term $Ze'e''/r$ and a Sommerfeld term $s/r^2$. Further, we take $\alpha(x) = [e'/c] f(x)^2$ and $\beta(x) = (2e'h/c)f(x)$ with $f(x) = F/(2\pi) + g(1-x_0/r)$ so that the two other terms in $Y_3$ describe the interaction of a particle of charge $e'$ and (reduced) mass $\mu$ with the vector potential $A = (-[x_3/(x^2_2+x^2_3)]f(x), [x_2/(x^2_2+x^2_3)]f(x), 0)$ where $F$ refers to an Aharonov-Bohm potential and $g$ to a Dirac monopole potential. (The vector potential $A$ corresponds to the magnetic field $B = gr/r^3$). It is to be noted that the potential $Y_3$ with $s = g = 0$ yields the potential $X_3$ with $c_1 = c_2 = -1$.

We can apply a compact Hurwitz transformation to the $R^3$ Schrödinger equation for the potential $Y_3$. This leads to a system comprizing an $R^4$ Schrödinger equation and a constraint equation. The latter system is not always separable from $R^4$ to $R^2 \times R^2$. Separability is obtained for $2\mu s = (e'g/c)^2$. This may be precised with the following preliminary result to be developed elsewhere.

**Result 5.** The $R^3$ Schrödinger equation, with the usual metric $\text{diag}(1,1,1)$, for the potential $Y_3$ with $2\mu s = (e'g/c)^2$ is equivalent to a set comprizing (i) two coupled $R^2$ Schrödinger equations for two two-dimensional isotropic oscillators involving each a centrifugal term and (ii) two coupled $R^2$ constraint equations.

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