Abstract. - This is a set of introductory lecture notes on black holes in string theory. After reviewing some aspects of string theory such as dualities, brane solutions, supersymmetric and non-extremal intersection rules, we analyze in detail extremal and non-extremal 5d black holes. We first present the D-brane counting for extremal black holes. Then we show that 4d and 5d non-extremal black holes can be mapped to the BTZ black hole (times a compact manifold) by means of dualities. The validity of these dualities is analyzed in detail. We present an analysis of the same system in the spirit of the adS/CFT correspondence. In the “near-horizon” limit (which is actually a near inner-horizon limit for non-extremal black holes) the black hole reduces again to the BTZ black hole. A state counting is presented in terms of the BTZ black hole.
1 Introduction

The physics of 20th century is founded on two pillars: quantum theory and general theory of relativity. Quantum theory has been extremely successful in describing the physics at microscopic scales while general relativity has been equally successful with physics at cosmological scales. However, attempts to construct a quantum theory of gravity stubble upon the problem of the non-renormalizability of the theory. Is it really necessary to have a quantum theory of gravity? Why not having gravity classical and matter quantized? Is it just an aesthetic question or is there an internal inconsistency if some of the physical laws are classical and some quantum? If some of the interactions are classical then one could use only these interactions in order to arbitrarily obtain the position and the velocity of particles, thus violating Heiseberg’s uncertainty principle. Therefore, at the fundamental level, if some of the physical laws are quantum, all of them have to be quantum.

It is amusing to see what happens if we insist on both classical general relativity and the uncertainty principle. Suppose we want to measure a spacetime coordinate with accuracy $\delta x$, then by the uncertainty principle there will be energy of order $1/\delta x$ localized in this region. But if $\delta x$ is very small then the energy will be so large that a black hole will be formed, and the spacetime point will be hidden behind a horizon! One can estimate[1] that the scale that leads to a black hole formation (through the uncertainty principle) is of order of the Planck length $l_p$. Therefore, classical general relativity and quantum mechanics become incompatible at scales of order $l_p$.

One of the most fascinating objects that general relativity predicts is black holes. Classically, black holes are completely black. Objects inside their event horizon are eternally trapped. Even light rays are confined by the gravitational force. In addition, there is a singularity hidden behind the horizon. In the early seventies, a number of laws that govern the physics of black holes were established[2–4]. In particular, it was found that there is a very close analogy between these laws and the four laws of thermodynamics[3]. The black hole laws become that of thermodynamics if one replaces the surface gravity $\kappa$ of the black hole by the temperature $T$ of a body in thermal equilibrium, the area of the black hole $A$ by the entropy $S$[4], the mass of the black hole $M$ by the energy of the system $E$ etc. It is natural to wonder whether this formal similarity is more than just an analogy. At the classical level one immediately runs into a problem if one tries to take this analogy seriously: classically black holes only absorb so their temperature is strictly zero. In a seminal paper[5], however, Hawking showed that quantum mechanically black holes emit particles with thermal spectrum. The temperature was found to be $T = \kappa/2\pi$!
Then from the first law follows the “Bekenstein-Hawking entropy formula”,

\[ S = \frac{A}{4G_N} \]

where \( G_N \) is Newton’s constant. Having established that black hole laws are thermodynamic in nature one would like to understand what is the underlying microscopic theory. What are the microscopic degrees of freedom that make up the black hole?

Since black holes radiate, they lose mass and they may eventually evaporate. Observing such a phenomenon is rather unlikely since one can estimate the lifetime of a black hole of stellar mass to have lifetime\(^1\) longer than the age of the universe. The fact, however, that black holes Hawking radiate and may eventually evaporate leads to an important paradox. The matter that falls into black hole has structure. The outgoing radiation, however, is structure-less since it is thermal. What happens to the information stored in the black hole if the black hole completely evaporates? If it gets lost then the evolution is not unitary. Hawking argued that these considerations imply that quantum mechanics has to be modified. There is great controversy over the question of the final state of black holes, and there is no completely satisfactory scenario. We will not enter into this question in this lectures. Let us note, however, that the resolution of this problem may be related to the question of understanding the microscopic description of black holes. Radiation from stars also has a thermal spectrum. However, we do not claim that information is lost in stars. The thermal spectrum is due to averaging over microscopic states.

We have seen that semi-classical considerations yield a number of important issues. Any consistent quantum theory of gravity should provide answers to the questions raised in the previous paragraphs. The leading candidate for a quantum theory of gravity is string theory. Therefore, string theory ought to resolve these issues. Issues involving black holes are non-perturbative in nature. Up until recently, however, we only had a perturbative formulation of string theory. The situation changed dramatically over the last few years. Dualities have led to a unified picture of all string theories [6, 7]. Moreover, new non-perturbative objects, the D-branes, were discovered[8]. These new ingredients made possible to tackle some of the problems mentioned above.

In this lectures we review recent progress in understanding black holes using string theory. We start by briefly reviewing perturbative strings, D-branes and dualities in section 2. In particular, we review in some detail T-duality in backgrounds with isometries. In section 3 we present the brane solutions of type II and eleven dimensional supergravity, their connections through dualities, and a set of intersection rules that yields new

\(^1\)For black holes of mass \( M \) the Hawking temperature is of order \( T \sim 10^{-6}(M_\odot/M) \) \( K \) and their lifetime of order \( 10^{71}(M_\odot/M)^{-3} \) s.
solutions describing configurations of intersecting branes. We use these results in section 4 in order to study extremal and non-extremal black holes. In section 4.1 we analyze extremal $5d$ black holes. We show that one can derive the Bekenstein-Hawking entropy formula by counting D-brane states. In section 4.2 we show that $4d$ and $5d$ non-extremal black holes can be mapped to the BTZ black hole\cite{9, 10} (times a compact manifold) by means of dualities. We show that a general U-duality transformation preserves the thermodynamic characteristics of black holes. Then we critically examine the so-called shift transformation that removes the constant part from harmonic functions. We show that this transformation also preserves the thermodynamic characteristics of the original black hole. In general, however, it is not a symmetry of the theory. Section 4.3 contains a short introduction to adS/CFT duality, and its application to black holes. The low-energy decoupling limit employed in the adS/CFT correspondence (which is a near inner-horizon limit for non-extremal black holes) also yields a connection with the BTZ black hole. We use the connection to the BTZ black hole to infer a state counting for the higher dimensional black holes.

Previous reviews for black holes in string theory include \cite{11–13}. 
2 String theory and dualities.

In this section we present some aspects of string theory. The main purpose is to set our conventions and to review certain material that they will be of use in later sections.

2.1 Bosonic string and D-branes

The worldsheet action for the bosonic string is given by

\[ S = \frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \sqrt{\hbar h^{ab}\partial_a X^\mu \partial_b X_\mu}. \]  

(2)

where \( h \) is the worldsheet metric. The tension of the string is given by \( T = 1/(2\pi\alpha') \) (\( \alpha' \) is the square of the string length \( l_s \)). Varying the action we obtain

\[ \delta S = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{\hbar} \delta X^\mu \square X^\mu + \frac{1}{2\pi\alpha'} \int d\tau [\sqrt{\hbar} \partial_\sigma X_\mu \delta X^\mu]^{\sigma=\pi}_{\sigma=0} \]  

(3)

In order to have a well-defined variational problem the last term should vanish. This implies three different types of boundary conditions

\[ X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \pi) \quad \text{closed string} \]

\[ \partial_\sigma X^\mu(\sigma = 0) = \partial_\sigma X^\mu(\sigma = \pi) = 0 \quad \text{open string with Neumann BC} \]

\[ X^\mu(\sigma = 0) = \text{const}, \quad X^\mu(\sigma = \pi) = \text{const}, \quad \text{open string with Dirichlet BC} \]

The Neumann boundary conditions for the open string imply that there is no momentum flow at the end of the string. With Dirichlet boundary conditions, however, there is momentum flowing from the string to the hypersurface where the string ends. Therefore, this hypersurface, the D-brane, is a dynamical object.

One may (first) quantize the string using standard methods. The closed string consist of left and right movers. We denote the left and right level by \( N \) and \( \tilde{N} \), respectively. For open strings we have only one kind of oscillators. The perturbative spectrum for the three kind of boundary conditions listed above is given by

\[ M^2_{\text{closed}} = \frac{2}{\alpha'}(N + \tilde{N} - 2) \]

\[ M^2_{\text{open},N} = \frac{1}{\alpha'}(N - 1) \]

\[ M^2_{\text{open},D} = \left( \frac{l}{2\pi\alpha'} \right)^2 + \frac{1}{\alpha'}(N - 1) \]  

(4)

The term \( l/2\pi\alpha' = lT \) is the energy of a string of length \( l \) stretched between two D-branes.
From (4) follows that the massless spectrum of closed strings consist of a graviton $G_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$ and a dilaton $\phi$. The massless spectrum of open strings with Neumann boundary conditions consists of a photon $A_\mu$. Finally, for a string that ends on a Dp-brane, i.e. the open string endpoints are confined to the $p + 1$-dimensional worldvolume of the D-brane, we get a vector field $A_m, m = 0, \ldots, p$, that lives on the worldvolume of the D-brane, and $(25 - p)$ scalars. The latter encode the fluctuations of the position of the D-brane.

The string coupling constant is not a new parameter but the expectation value of the dilaton field, $\langle e^\phi \rangle = g_s$. String theory perturbation theory is weighted by $g_s^\chi$, where $\chi$ is the Euler number of the string worldsheet. A compact surface can be built by adding $g$ handles, $c$ cross-caps and $b$ boundaries to the sphere. Its Euler number is given by $\chi = 2 - 2g - b - c$. Hence, the closed string coupling constant is proportional to the square of the open string coupling constant.

One may calculate the tension of D-branes [8, 14]

$$T_p \sim \frac{1}{g_s l_p^{p+1}}.$$  

(5)

Since the tension of the D-brane depends on the inverse of the string coupling constant, D-branes are non-perturbative objects. Notice that this behavior is different from the behavior of field theory solitons whose mass goes as $1/g^2$, where $g$ is the field theory coupling constant. The existence of such non-perturbative objects is required by string duality [7].

2.2 Superstrings

There are five consistent string theories; type IIA and IIB, type I, heterotic $SO(32)$ and heterotic $E_8 \times E_8$. All of them are related through dualities. In this review we shall concentrate on type II theories, so we briefly present some aspects of them.

The bosonic massless sector of type II theories consist of the following fields

| Type IIA | $g_{\mu\nu}$ | $B_{\mu\nu}$ | $\phi$ | $C_{\mu}^{(1)}$ | $C_{\mu\nu\lambda}^{(3)}$ |
|----------|---------------|---------------|--------|-----------------|------------------|
| Type IIB | $g_{\mu\nu}$ | $B_{\mu\nu}$ | $\phi$ | $C^{(0)}_{\mu\nu}$ | $C^{(2)}_{\mu\nu\lambda} + C^{(4)}_{\mu\nu\lambda\rho}$ |

where $C^{(p)}$ are $p$-index antisymmetric gauge fields. The + in $C^{(4)+}$ indicates that the field strength is self-dual. The graviton $g_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$ and the dilaton $\phi$ make up the NSNS sector. These fields couple to perturbative strings. The RR sector (i.e. the antisymmetric tensors $C^{(p+1)}$), however, does not couple to perturbative strings but rather to Dp-branes.
Extended objects naturally couple to antisymmetric tensors. The prototype example is the coupling of the point particle to electromagnetic field, \( \int A_\mu dx^\mu \). Similarly, fundamental string naturally couple to \( B_{\mu\nu} \), and Dp-branes to \( C^{(p+1)} \)

\[
\int_\Sigma B_{\mu\nu} dx^\mu \wedge dx^\nu \\
\int_{\mathcal{M}_{p+1}} C^{(p+1)}_{\mu_1 \cdots \mu_{p+1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}}
\]

where \( \Sigma \) and \( \mathcal{M}_{p+1} \) is the string worldsheet and Dp-brane worldvolume, respectively. To each “electric” \( p \)-brane there is also a dual “magnetic” \( (6-p) \)-brane. (To see this notice that \( *dC^{(p+1)} = d\tilde{C}^{(7-p)} \)). In particular, there is a solitonic 5-brane (NS5) that is the magnetic dual of a fundamental string F1. In addition, strings can carry momentum. This corresponds in low energy to gravitational waves (W). The (Hodge) dual to waves are Kaluza-Klein monopoles (KK) (see section 3).

In summary, we have the following objects in type II theory (D(-1) are D-instanton and D9 are spacetime-filling branes)

| Type IIA | W | F1 | NS5 | KK | D0 | D2 | D4 | D6 | D8 |
|----------|---|----|-----|----|----|----|----|----|----|
| Type IIB | W | F1 | NS5 | KK | D(-1) | D1 | D3 | D5 | D7 | D9 |

We have deduced the existence of dynamical extended objects by considering perturbative string theory. These states, however, preserve half of maximal supersymmetry and therefore continue to exit at all values of the string coupling constant.

### 2.3 Dualities

A central element in the recent developments are the duality symmetries of string theory. The duality symmetries are believed to be exact discrete gauge symmetries spontaneously broken by scalar vev’s.

The best-understood duality symmetry is T-duality. This symmetry is visible in string perturbation theory but it is non-perturbative on the worldsheet. T-duality relates compactifications on a manifold of (large) volume \( v \) to compactifications on a manifold of (small) volume \( 1/v \). The simplest case is compactification on a circle. Upon such compactification the two type II theories, and heterotic \( E_8 \times E_8 \) and heterotic \( SO(32) \) theories are equivalent,

\[
[IIA]_R \stackrel{T}{\longleftrightarrow} [IIB]_{1/R} \\
[Het \ E_8 \times E_8]_R \stackrel{T}{\longleftrightarrow} [Het \ SO(32)]_{1/R}
\]
where the subscript indicates that the theory is compactified on a circle of radius $R (1/R)$.

The action of T-duality on the various objects present in II theories is given in Table 1. The T-duality may be performed along one of the worldvolume directions or along a transverse direction (for the KK monopole the transverse direction is taken to be the nut direction (see section 3)). More generally, T-duality asserts that different spacetimes with isometries may be equivalent in string theory. We shall present the argument in some detail in the next section since we will make use of these results.

A (conjectured) non-perturbative symmetry is S-duality. This is non-perturbative because it acts on the dilaton as $g_s \to 1/g_s$. Thus, S-duality relates the strong coupling regime of one theory to the weak coupling regime of another. In particular we have

$$
IIB \quad \xleftrightarrow{S} \quad IIB
$$

$$
Het \; SO(32) \quad \xleftrightarrow{S} \quad Type \; I
$$

Actually, IIB string theory is believed to have an exact non-perturbative $SL(2, \mathbb{Z})$ symmetry. In the following we shall only make use of the $Z_2$ subgroup that sends $\tau = C^{(0)} + ie^{-\phi}$ to $-1/\tau$, interchanges $B_{\mu\nu}$ with $C^{(2)}_{\mu\nu}$, and leaves invariant $C^{(4)}$ (so, in terms of branes, S-duality interchanges F1 with D1, NS5 with D5, and leaves invariant the D3 brane).

S-duality allows one to get a handle to the strong coupling limit of three of the five string theories. In turns out that the strong coupling limit of IIA and heterotic $E_8 \times E_8$ theories is of a more “exotic” nature. One gets instead an 11 dimensional theory, the M-theory[7,15]. M-theory on a small circle of radius $R_{11} = g_s l_s$ yields IIA theory with string coupling constant $g_s[7]$. Since perturbative string theory is an expansion around $g_s = 0$, the eleventh dimensions is not visible perturbatively. Likewise, M-theory on an interval gives $E_8 \times E_8$ string theory[16]. Actually, all string theories can be obtained in suitable limits from eleven dimensions.

| Parallel | transverse |
|----------|------------|
| $D_p$    | $D(p-1)$   |
| $F_1$    | $W$        |
| $W$      | $F_1$      |
| NS5      | NS5        |
| KK       | NS5        |

Table 1: T-duality along parallel and transverse directions
Although we do not have a fundamental understanding of what M-theory is, we know that in low-energies M-theory reduces to 11 dimensional supergravity\[17]\. Eleven-dimensional supergravity compactified on a torus yields a lower dimensional Poincaré supergravity with a certain duality group. The discretized version of this duality group is conjectured\[6]\ (and widely believed) to be an exact symmetry of M-theory. T and S duality combine to yield this bigger group, the U-duality group.

2.3.1 Buscher’s duality

Consider the sigma model

\[
S = \frac{1}{4\pi\alpha'} \int d^2 \sigma \sqrt{h} \left( (h^{ab} g_{\mu\nu} + i \frac{\epsilon^{ab}}{\sqrt{h}} B_{\mu\nu}) \partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)}(\phi) \right),
\]

where \( h \) and \( R^{(2)} \) is the worldsheet metric and curvature, \( g \) is the target space metric and \( B \) is a potential for the torsion 3-form \( H = dB \). This action is invariant under the transformation

\[
\delta X^\mu = \epsilon k^\mu
\]

when the vector field \( k^\mu \) is a Killing vector, the Lie derivative of \( B \) is a total derivative and the dilaton is invariant,

\[
\mathcal{L}_k g_{ij} = k_{ij} + k_{ji} = 0,
\]

\[
\mathcal{L}_k B = \iota_k dB + d \iota_k B = d(v + \iota_k B)
\]

\[
\mathcal{L}_k \phi = k^\mu \partial_\mu \phi = 0
\]

One can now choose adapted coordinates \( \{X^\mu\} = \{x, x^i\} \) such that the isometry acts by translation of \( x \), and all fields \( g, B \) and \( \phi \) are independent of \( x \). In adapted coordinates, the killing vector is equal to \( k^\mu \partial / \partial X^\mu = \partial / \partial x \).

To obtain the dual theory we first gauge the symmetry and add a Lagrange multiplier \( \chi \) that imposes that the gauge connection is flat \[18\]. The result (in the conformal gauge and omitting the dilaton term) is (see \[19\], \[20\] for details)

\[
S_1 = \frac{1}{2\pi\alpha'} \int d^2 z [(g_{\mu\nu} + B_{\mu\nu}) \partial X^\mu \partial X^\nu + (J_k - \partial \chi) \bar{A} + (\bar{J}_k + \partial \bar{\chi}) A + k^2 A \bar{A}]
\]

where \( J_k = (k + v)_\mu \partial X^\mu \), \( \bar{J}_k = (k - v)_\mu \bar{\partial} X^\mu \) are the components of the Noether current associated with the symmetry. If one integrates out the Lagrange multiplier field \( \chi \), on a topologically trivial worldsheet the gauge fields are pure gauge, \( A = \partial \theta, \bar{A} = \bar{\partial} \bar{\theta} \), and one recovers the original model (8).
If one integrates out the gauge fields $A, \bar{A}$ one finds the dual model. One obtains (8) but with dual background fields $\tilde{g}, \tilde{B}, \tilde{\phi}$. In adapted coordinates $\{X^\mu\} = \{x, x^i\}$,

\[
\begin{align*}
\tilde{g}_{xx} &= \frac{1}{g_{xx}}, \quad \tilde{g}_{xi} = \frac{B_{xi}}{g_{xx}} \quad \tilde{g}_{ij} = g_{ij} - \frac{g_{xi}g_{xj} - B_{xi}B_{xj}}{g_{xx}} \\
\tilde{B}_{xi} &= \frac{g_{xi}}{g_{xx}} \quad \tilde{B}_{ij} = B_{ij} + \frac{g_{xi}B_{xj} - B_{xi}g_{xj}}{g_{xx}} \\
\tilde{\phi} &= \phi - \frac{1}{2} \ln g_{xx}
\end{align*}
\] (12)

The dilaton shift is a quantum mechanical effect [21] (see [22] for recent careful discussion).

Another useful way to write these transformation rules is to re-write the metric as

\[
ds^2 = g_{xx}(dx + A_i dx^i)^2 + \bar{g}_{ij} dx^i dx^j
\] (13)

where $A_i = g_{xi}/g_{xx}$. Then the duality transformations take the form[23]

\[
\tilde{g}_{xx} = \frac{1}{g_{xx}}, \quad \tilde{A}_i = B_{xi}, \quad \tilde{\bar{B}}_{xi} = A_i, \quad \tilde{\bar{B}}_{ij} = B_{ij} - 2A_{[i}B_{j]}x
\]

\[
\tilde{\phi} = \phi - \frac{1}{2} \ln g_{xx} \quad \tilde{g}_{ij} \text{ invariant}
\] (14)

This form of the transformation rules exhibits most clearly the spacetime interpretation of the duality transformations. The form of the metric in (13) is the standard KK ansatz for reduction over $x$. Dimensional reduction over $x$ leads to a $(d-1)$-dimensional theory which is invariant under the transformations in (14). These transformations act only on the matter fields and not on the pure gravitational sector.

Let us now discuss under which conditions the dual models are truly equivalent as conformal field theories.

- **Compact vs non-compact isometries**

In our discussion above we assumed that the worldsheet is trivial. Let us relax this condition. Suppose also that we deal with a compact isometry. The constraint on $A, \bar{A}$ that comes from integrating out the Lagrange multiplier $\chi$ implies $A, \bar{A}$ are flat, but in principle they still may have nontrivial holonomies around non-contractible loops. These holonomies can be constrained to vanish if $\chi$ has appropriate period[19, 20]. In summary, dualizing along a compact isometry one obtains a dual geometry which also has a compact isometry. The periods of the original and dual coordinate are reciprocal to each other. If this condition does not hold, the two models are not fully equivalent but related via an orbifold construction.

Non-compact isometries can be considered as a limiting case. Since in this case $x$ takes any real value, the dual coordinate $\chi$ must have period zero. The dual manifold is an orbifold obtained by modding out the translations in $\chi$. 

10
**Isometries with fixed points**

In our analysis we also assumed that the isometry is spacelike. If the isometry is timelike then it follows from (11) that the integration over the gauge field yields a divergent factor. If the isometry is null then the quadratic in the gauge field term in (11) vanishes. Therefore these cases require special attention. We refer to [24–26] for work concerning dualization (or the closely related issue of dimensional reduction) along timelike or null isometries.

A spacelike isometry may act freely or have fixed points. A typical example of an isometry without fixed points are the translational symmetries on tori. On the other hand, rotational isometries have fixed points. At the fixed point \( k^2 = 0 \). It follows from (12) (using \( k^2 = g_{xx} \)) that the dual geometry appears to have a singularity at the fixed point.

Taking the curvature of the spacetime to be small in string units (which is required for consistency for strings propagating in a background that only solves the lowest order beta functions) we see that we may approximate the vicinity of the fixed point by flat space. In adapted coordinates, which are just polar coordinates, the isometry direction being the angular coordinate, we have

\[
ds^2 = dr^2 + r^2 d\theta^2. \tag{15}
\]

Dualizing along \( \theta \) we obtain

\[
ds^2 = dr^2 + \frac{1}{r^2} d\theta^2, \quad \phi = -\frac{1}{2} \ln r^2. \tag{16}
\]

So indeed the fixed point of the isometry, i.e. \( r = 0 \), becomes a singular point after the duality transformation. Since the curvature now diverges at \( r = 0 \) we cannot trust the (first order in \( \alpha' \)) sigma model analysis. A more careful conformal field theory analysis[27] shows that the duality yields an exact equivalence but the operator mapping includes all orders in \( \alpha' \). We can read this results as follows: All order \( \alpha' \) corrections resolve the singularity present in the spacetime described by (16) yielding an exact non-singular conformal field theory.

Studies of T-duality along a rotational isometry can be found in [28–30].
3 Brane solutions

String theory has a mass gap of order $1/l_s$. At low enough energies only the massless fields are relevant. We can decouple the massive modes by sending $\alpha' \to 0$ (so the mass of the massive modes goes to infinity). The interactions of the massless fields are described by an effective action. For IIA and IIB superstring theories the low energy theory is IIA and IIB supergravity, respectively. We have seen that in type II string theories there exist dynamical objects other than strings, namely D-branes, and solitonic branes. For each of these objects there is a corresponding solution of the low energy supergravity. The purpose of this section is to describe these solutions. For reviews see [31–33].

The relevant part of the supergravity action, in the string frame, is

$$S = \frac{1}{128\pi^7 g_s^2 \alpha'^4} \int d^{10}x \sqrt{-g}[e^{-2\phi}(R + 4(\partial\phi)^2) - \frac{1}{12}|H_3|^2 - \frac{1}{2(p + 2)!}|F_{p+2}|^2] \quad (17)$$

We use the convention to keep the asymptotic value of $\phi$ in Newton’s constant ($G_N^{(10)} = 8\pi^6 g_s^2 \alpha'^4$), so the asymptotic value of $e^\phi$ below is equal to 1.  

The equations of motion of the above action have solutions that have the interpretation of describing the long range field of fundamental strings (F1), D$p$-branes and solitonic fivebranes (NS5). These solutions are given by[34]

$$ds_{st}^2 = H_{i} \left[ H_{i}^{-1} ds^2(E^{(p,1)}) + ds^2(E^{(9-p)}) \right]$$

$$e^\phi = H_{i}^3$$

$$A_{0_{(p+1)}} = H_{i}^{-1} - 1, \quad \text{“electric”, or } F_{8-p} = * dH_{i}, \quad \text{“magnetic”} \quad (18)$$

where $A^{(p+1)}$ is either the RR potential $C^{(p+1)}$, or the NSNS 2-form $B$, depending on the solution. $*$ is the Hodge dual of $E^{(9-p)}$. The subscript $i = \{p,F1,NS5\}$ denotes which solution (D$p$-brane, fundamental string or solitonic fivebrane, respectively) we are describing. In order (18) to be a solution $H_i$ must be a harmonic function on $E^{(9-p)}$,

$$\nabla^2 H_i = 0 \quad (19)$$

Let $r$ be the distance from the origin of $E^{(9-p)}$. The choice

$$H_i = 1 + \frac{Q_i}{r^{(7-p)}} \quad p < 7 \quad (20)$$
yields the long-range fields of $N$ infinite parallel planar $p$-branes near the origin. The constant part was chosen equal to one in order the solution to be asymptotically flat. The values of the parameters $\alpha$ and $\beta$ for each solution are given in Table 2. In the same table we also give the values of the charges $Q_i$. The constant $d_p$ is equal to

$$d_p = (2\sqrt{\pi})^{5-p} \Gamma\left(\frac{7-p}{2}\right).$$

| D$p$-branes | $\alpha$ | $\beta$ | $Q_p = d_p Ng_s l_s^{7-p}$ |
|-------------|---------|---------|-----------------------------|
| F1          | $\alpha = 0$ | $\beta = -1/2$ | $Q_{F1} = d_1 Ng_s^2 l_s^6$ |
| NS5         | $\alpha = 1$ | $\beta = 1/2$ | $Q_{NS5} = N l_s^2$ |

Table 2: $p$-brane solutions of Type II theories.

Apart from these solutions, there are also purely gravitational ones. There is a solution describing the long range field produced by momentum modes carried by a string. This is the gravitational wave solution,

$$ds^2 = -K^{-1}dt^2 + K(dx_1 - (K^{-1} - 1)dt)^2 + dx_2^2 + \cdots + dx_9^2$$ (21)

where $K = 1 + Q_K/r^6$ is again a harmonic function and $Q_K = d_1 g_s^2 N\alpha'/R^2$. $R$ is the radius of $x_1$.

Finally, there is a solution describing a Kaluza-Klein (KK) monopole (the name originates from the fact that upon dimensional reduction over $\psi$ the KK gauge field that one gets is the monopole connection):

$$ds^2 = ds^2(\mathbb{R}^{6,1}) + ds_{TN}^2$$

$$ds_{TN}^2 = H^{-1}(d\psi + Q_M \cos \theta d\varphi)^2 + H dx^i dx^i, \quad i = 1, 2, 3$$

$$H = 1 + \frac{Q_M}{r}, \quad r^2 = x_1^2 + x_2^2 + x_3^2$$ (22)

where $TN$ stands for Taub-NUT, $\theta$ and $\psi$ are the angular coordinates of $x_1, x_2, x_3$, $Q_M = NR/2$, $N$ is the number of coincident monopoles and $R$ is the radius of $\psi$.

S-duality leaves invariant the action in the Einstein frame. To reach the Einstein frame we need to do the Weyl rescaling $g_E = e^{-\phi/2}g_{st}$. Using the fact that under S-duality $\phi \rightarrow -\phi$ (and $g_s \rightarrow 1/g_s$) we get $g_{\mu\nu} \rightarrow e^{-\phi}g_{\mu\nu}$. The compactification radii are measured using the string metric. So, they change under S-duality. One can take care of this by changing the string scale, $\alpha' \rightarrow \alpha' g_s$. We therefore get the following S-duality transformation rules

$$\phi \rightarrow -\phi \quad (g_s \rightarrow 1/g_s), \quad \alpha' \rightarrow \alpha' g_s$$

$$g_{\mu\nu} \rightarrow e^{-\phi}g_{\mu\nu}, \quad B_{\mu\nu} \leftrightarrow C^{(2)}_{\mu\nu}$$ (23)
With these conventions Newton’s constant, $G_N^{(10)} = 8\pi^6 g_s^2 \alpha'^4$, is invariant under S-duality.

T-duality acts as in (12) in the NSNS sector. In particular, dualization along a coordinate of radius $R$ yields

$$ R \rightarrow \frac{\alpha'}{R}, \quad g_s \rightarrow g_s \frac{l_s}{R} $$

(24)

For the RR fields we get[35]

$$ C_{\mu_1 \cdots \mu_{p+1}} \rightarrow C_{\mu_1 \cdots \mu_{p+1} x}, \quad x \notin \{x_{\mu_1}, \ldots, x_{\mu_{p+1}}\} $$

(25)

$$ C_{x_{\mu_1} \cdots x_{\mu_{p+1}}} \rightarrow C_{\mu_1 \cdots \mu_{p+1}} $$

depending on whether we dualize along a coordinate transverse or parallel to the brane.

It is easy to see that the values of the charges $Q_i$ are consistent with dualities. For instance, under S-duality: $Q_{NS5} = N\alpha' \leftrightarrow Ng_s\alpha' = Q_5$. Actually, dualities determine both the value of Newton’s constant and the charges (including the numerical coefficients) [12]: The mass $M$ of an object can be calculated from the deviation of the Einstein metric from the flat metric at infinity. In particular[36],

$$ g_{E,00} = \frac{16\pi G_N^{(d)} M}{(d-2)\omega_{d-2} r^{d-3}} $$

(26)

where $\omega_d = 2\pi^{(d+1)/2} \Gamma(\frac{d+1}{2})^2$ is the volume of the unit sphere $S^d$. Completely wrapping a given brane on torus and dimensionally reducing we get a spacetime metric in $d = 10 - p$ dimensions,

$$ ds_{E,d}^2 = -H^{-\frac{d-3}{4}} dt^2 + H^{\frac{1}{4}} ds^2 (E^{(d-1)}) $$

(27)

This result is obtained by using the dimensional reduction rules[37]

$$ ds_{E,d}^2 = e^{-\frac{d-3}{4} \phi_d} ds_{st}^2, \quad e^{-2\phi_d} = e^{-2\phi} \sqrt{det g_{int}} $$

(28)

where $g_{int}$ is the component of the metric in the directions we dimensionally reduce. If $H = 1 + c^{(d)}/r^{d-3}$ then,

$$ c^{(d)} = \frac{16\pi G_N^{(d)} M}{(d-3)\omega_{d-2}} $$

(29)

The mass $M$ appearing in this formula is the same as the mass measured in the string frame since we used the convention to leave a factor of $g_s^2$ in Newton’s constant. These masses can be easily obtained by U-duality. Knowledge of one of the coefficients in (29) is sufficient to determine $G_N$ and therefore all other coefficients as well. In [12] the value of $c_{NS5}$ was determined from the Dirac quantization condition. Perhaps the simplest way to proceed is to observe that the coefficient in the harmonic function of the KK monopole is fixed by requiring that the solution is non-singular.
All these solutions are BPS solution and preserve half of maximal supersymmetry. This implies that certain quantities do not renormalize. Let us sketch the argument. The supersymmetry algebra has the form

$$\{Q_\alpha, Q_\beta\} \sim (C \Gamma^\mu)_{\alpha\beta} P_\mu + (C \Gamma^{\mu_1 \cdots \mu_p})_{\alpha\beta} Z^{(p)}_{\mu_1 \cdots \mu_p}$$

(30)

where $C$ the charge conjugation matrix, $Q_\alpha$ are the supercharges, $P_\mu$ is the momentum generator, and $Z^{(p)}$ are central charges. These are the charges carried by $p$ branes.

Taking the expectation value of (30) between a physical state $|A\rangle$ and going to the rest frame we get

$$\langle A|\{Q_\alpha, Q_\beta\}|A\rangle = (M_A - c|Z|)_{\alpha\beta} \geq 0$$

(31)

where $M^A_{\alpha\beta}$ is the mass matrix, $c$ is a constant, and we used the fact that $\{Q_\alpha, Q_\beta\}$ is a positive definite matrix.

If the matrix in the right hand side has no zero eigenvalues, then one can take suitable linear combinations of the supercharges so that the superalgebra takes the form of fermionic oscillator algebra. Then half of oscillators can be regarded as creation and half as annihilation operators. This means that a supermultiplet contains $2^{16}$ states.

If the matrix in the right hand side of (31) has a zero eigenvalue (so the mass is proportional to the charge, $M = c|Z|$, i.e. we have a BPS state) then some of the generators annihilate the state. The remaining supercharges can again be divided into half creation and half annihilation operators. Thus, the BPS supermultiplet is a short multiplet. For 1/2 supersymmetric states, such as the branes we have been discussing, this means that we have $2^8$ states instead of $2^{16}$.

If we vary adiabatically the parameters of theory (i.e. no phase transition) the number of states cannot change abruptly, so the number of BPS states remains invariant and the mass/charge relation does not renormalize[38]. (Here we also assume that we do not cross curves of marginal stability).

3.1 M-branes

We briefly describe the connection of the brane solutions described in the previous section to M-theory. M-theory at low-energies is described by eleven dimensional supergravity. The bosonic field content of eleven dimensional supergravity consists of a metric, $G_{MN}$, and a three-form antisymmetric tensor, $A_{MNP}$. We therefore expect that this theory has solutions describing extended objects coupled “electrically” and “magnetically” to $A_{MNP}$. Indeed, one finds a 2-brane solution, M2, and a fivebrane solution, M5[39, 40].
The explicit form of the solution is as in (18), with \( \alpha_{M2} = 1/3 \) for the M2 and \( \alpha_{M5} = 2/3 \) for the M5 (there is no dilaton field in 11d supergravity, so \( \beta = 0 \)). In addition, we have the purely gravitational solutions describing traveling waves and KK monopoles.

From the solutions of eleven dimensional supergravity one can obtain the solution of IIA supergravity upon dimensional reduction. The Kaluza-Klein ansatz for the bosonic fields leading to the string frame 10d metric is

\[
 ds_{11}^2 = e^{-\frac{2}{3}\phi(x)} g_{\mu\nu} dx^\mu dx^\nu + e^{\frac{4}{3}\phi(x)} (dx_{11} + C^{(1)}_{\mu} dx^\mu)^2 \\
 A = C^{(3)} + B \wedge dx_{11}
\]

(32)

where \( B \) is the NSNS antisymmetric tensor and \( C^{(1)} \) and \( C^{(3)} \) are the RR antisymmetric tensors of IIA theory. Dimensionally reducing the M-branes along a worldvolume or a transverse direction one obtains all solution of IIA as follows:

| 11d SUGRA | W | M2 | M5 | KK |
|-----------|---|----|----|----|
| IIA SUGRA | D0 | W  | F1 | D2 | D4 | NS5 | D6 | KK |

3.2 Intersection rules

In the previous section we described brane solutions of supergravity theories. These solutions can be used as building blocks in order to construct new solutions [41–51] (for a review see [52]). The new solutions can be interpreted as intersecting (or in some cases overlapping) branes. In order to obtain a supersymmetric solution only certain intersections are allowed.

The intersection rules are as follows:

One superimposes the single brane solutions using the rule that all pairwise intersections should belong to a set of allowed intersections. If all harmonic functions are taken to depend on the overall transverse directions (i.e. the directions transverse to all branes) we are dealing with a “standard” intersection. Otherwise the intersection will be called “non-standard”. In \( D = 11 \) there are three standard intersections, \( (0\parallel 2 \perp 2) \), \( (1\parallel 2 \perp 5) \) and \( (3\parallel 5 \perp 5) \) [42, 43, 47], and one non-standard \( (1\parallel 5 \perp 5) \) [53, 43]. In the latter intersection the harmonic functions depend on the relative transverse directions (i.e. the directions which are worldvolume coordinates of the one but transverse coordinates of the other fivebrane).

In addition, one can add a wave solution along a common string. The intersection rules in ten dimensions can be derived from these by dimensional reduction plus T and S-duality. We collect the standard and non-standard intersection rules in the table below.

\(^4\)The notation \((q|p_1 \perp p_2)\) denotes a \(p_1\)-brane intersecting with a \(p_2\)-brane over a \(q\)-brane.
(For intersections rules involving KK monopoles see [49]). When both standard and non-standard intersection rules are used (as for instance in the solutions of [54]), one has to specify which coordinates each harmonic function depends on. This is usually clear by inspection of the intersection, but it can also be further verified by looking at the field equation(s) for the antisymmetric tensor field(s).

|          | standard                  | non-standard              |
|----------|----------------------------|---------------------------|
| $D = 11$ | $(0|M2 \perp M2)$         |                           |
|          | $(1|M2 \perp M5)$          |                           |
|          | $(3|M5 \perp M5)$          | $(1|M5 \perp M5)$         |
| $D = 10$ | $(\frac{1}{2}(p + q - 4)|Dp \perp Dq)$ | $(\frac{1}{2}(p + q - 8)|Dp \perp Dq)$ |
|          | $(1|F1 \perp NS5)$         |                           |
|          | $(3|NS5 \perp NS5)$        | $(1|NS5 \perp NS5)$       |
|          | $(0|F1 \perp Dp)$          |                           |
|          | $(p - 1)|NS5 \perp Dp)$     | $(p - 3)|NS5 \perp Dp)$   |

Table 3: Standard and non-standard intersections in ten and eleven dimensions.

There is a simple algorithm which leads to non-extreme version of a given supersymmetric solution (constructed according to standard intersection rules)[44]. We will give these rules for M-brane intersections. This is sufficient as dimensional reduction and duality produce all standard intersections of type II branes. It consists of the following steps:

1. Make the following replacements in the $d$-dimensional transverse spacetime part of the metric:
   \[ dt^2 \rightarrow f(r)dt^2, \quad dx_1^2 + \cdots + dx_{d-1}^2 \rightarrow f^{-1}(r)dr^2 + r^2d\Omega_{d-2}^2, \quad f(r) = 1 - \frac{\mu^{d-3}}{r^{d-3}}, \]  
   (33)

and use the following harmonic functions,
   \[ H_T = 1 + \frac{Q_T}{r^{d-3}}, \quad Q_T = \mu^{d-3}\sinh^2 \alpha_T, \]
   \[ H_F = 1 + \frac{Q_F}{r^{d-3}}, \quad Q_F = \mu^{d-3}\sinh^2 \alpha_F, \]  
   (34)

for the constituent two-branes and five-branes, respectively.

2. In the expression for the field strength $\mathcal{F}_4$ of the three-form field make the following replacements:
   \[ H_T'^{-1} \rightarrow H_T'^{-1} = 1 - \frac{Q_T}{r^{d-3}}H_T^{-1}, \quad Q_T = \mu^{d-3}\sinh \alpha_T \cosh \alpha_T, \]
\[ H_F \to H'_F = 1 + \frac{Q_F}{\sqrt{d-3}} \quad \quad Q_F = \mu^{d-3} \sinh \alpha_F \cosh \alpha_F , \quad (35) \]
in the “electric” (two-brane) part, and in the “magnetic” (five-brane) part, respectively. In the extreme limit \( \mu \to 0, \alpha_F \to \infty, \) and \( \alpha_T \to \infty, \) while the charges \( Q_F \) and \( Q_T \) are kept fixed. In this case \( Q_F = Q_F \) and \( Q_T = Q_T, \) so that \( H'_T = H_T. \) The form of \( F_4 \) and the actual value of its “magnetic” part does not change compared to the extreme limit.

(3) In the case there is a common string along some direction \( x, \) one can add momentum along \( x. \) Then

\[ -f(r)dt^2 + dx^2 \to -K^{-1}(r)f(r)dt^2 + K(r) \left( dx - [K'^{-1}(r) - 1]dt \right)^2 \quad (36) \]

where

\[ K = 1 + \frac{Q_K}{\sqrt{d-3}}, \quad Q_K = \mu^{d-3} \sinh^2 \alpha_K, \]

\[ K'^{-1} = 1 - \frac{Q_K}{\sqrt{d-3}} K^{-1}, \quad Q_K = \mu^{d-3} \sinh \alpha_K \cosh \alpha_K . \quad (37) \]

In the extreme limit \( \mu \to 0, \alpha_K \to \infty, \) the charge \( Q_K \) is held fixed, \( K = K' \) and thus the metric (36) becomes \( du dv + (K - 1)du^2, \) where \( u, v = x \pm t. \)
4 Black holes in string theory

Black holes arise in string theory as solutions of the corresponding low-energy supergravity theory. String theory lives in 10 dimensions (or 11 from the M-theory perspective). Suppose the theory is compactified on a compact manifold down to $d$ spacetime dimensions. Branes wrapped in the compact dimensions will look like pointlike objects in the $d$-dimensional spacetime. So, the idea is to construct a configuration of intersecting wrapped branes which upon dimensional reduction yields a black hole spacetime. If the brane intersection is supersymmetric then the black hole will be extremal supersymmetric black hole. On the other hand, non-extremal intersections yield non-extremal black holes.

In general, the regime of the parameter space in which supergravity is valid is different from the regime in which weakly coupled string theory is valid. Thus, although we know that a given brane configuration becomes a black hole when we go from weak to strong coupling, it would seem difficult to extract information about the black hole from this fact.

For supersymmetric black holes, however, the BPS property of the states allows one to learn certain things about black holes from the weakly coupled D-brane system. For example, one can count the number of states at weak coupling and extrapolate the result to the black hole phase. In this way, one derives the Bekenstein-Hawking entropy formula (including the precise numerical coefficient) for this class of black holes\cite{55, 56}. We will review this calculation in section 4.1.

In the absence of supersymmetry, we cannot in general follow the states from weak to strong coupling. However, one could still obtain some qualitative understanding of the black hole entropy. On general grounds, one might expect that the transition from weakly coupled strings to black holes happens when the string scale becomes approximately equal to the Schwarzschild radius (or more generally to the curvature radius at the horizon). This point is called the correspondence point. Demanding that the mass and the all other charges of the two different configurations match, one obtains that the entropies also match \cite{57}. These considerations correctly provide the dependence of the entropy on the mass and the other charges, but the numerical coefficient in the Bekenstein-Hawking entropy formula remains undetermined.

In \cite{58} a different approach was initiated. Instead of trying to determine the physics of black holes using the fact that at weak coupling they become a set of D-branes, the symmetries of M-theory are used in order to map the black hole configuration to another black hole configuration. Since the U-duality group involves strong/weak transitions one
does not, in general, have control over the microscopic states that make up a generic configuration. We will see, however, that the situation is better when it comes to black holes! U-duality maps black holes to black holes with the same thermodynamic characteristics, i.e. the entropy and the temperature remain invariant. This implies that the number of microstates that make up the black hole configuration remains the same. Notice that to reach this conclusion we did not use supersymmetry, but the fact that the area of the horizon of a black hole (divided by Newton’s constant) tell us how many degree of freedom the black hole contains. We discuss this approach in section 4.2.

The effect of the U-duality transformations described in section 4.2 is to remove the constant part from certain harmonic functions (and also change the values of some moduli). One can achieve a similar result by taking the low-energy limit $\alpha' \to 0$ while keeping fixed the masses of strings stretched between different D-branes. Considerations involving this limit lead to the adS/CFT correspondence[59]. This will be discussed in section 4.3.

4.1 Extremal black holes and the D-brane counting

We will analyze five dimensional black holes. Four dimensional ones [60] can be analyzed in a completely analogous manner[61, 62]. Rotating black holes have been discussed in [63–65].

4.1.1 5d Extremal Black Holes

To study extremal charged five dimensional black holes we build a configuration of intersecting branes using the supersymmetric intersection rules. In particular, we consider the configuration of $N_5$ D5-brane wrapped in $x_1, \ldots, x_5$, $N_1$ D1-brane wrapped in $x_1$, with $N_K$ momentum modes along $x_1$. The coordinates $x_i, i = 1, \ldots, 5$ are taken periodic with periods $R_i$. Explicitly, the spacetime fields are

$$ds^2 = H_1^{1/2}H_5^{1/2} \left[ H_1^{-1}H_5^{-1} \left( -K^{-1}dt^2 + K(dx_1 - (K^{-1} - 1)dt)^2 \right) + H_5^{-1}(dx_2^2 + \cdots + dx_5^2) + dx_6^2 + \cdots + dx_9^2 \right]$$

and

$$e^{-2\phi} = H_1^{-1}H_5, \quad C_{01}^{(2)} = H_1^{-1} - 1$$

$$H_{ijk} = \frac{1}{2} \epsilon_{ijkl} \partial_l H_5, \quad i, j, k, l = 6, \ldots, 9$$

$$r^2 = x_6^2 + \cdots + x_9^2$$
The harmonic functions are equal to

\[
H_1 = 1 + \frac{Q_1}{r^2}, \quad Q_1 = \frac{N_1 g_s \alpha'^3}{V} \\
H_5 = 1 + \frac{Q_5}{r^2}, \quad Q_5 = N_5 g_s \alpha' \\
K = 1 + \frac{Q_K}{r^2}, \quad Q_K = \frac{N_K g_s^2 \alpha'^4}{R_1^2 V}
\]

where \( V = R_2 R_3 R_4 R_5 \) and the charges have been calculated using (29).

Upon dimensional reduction over the periodic coordinates \( x_1, \ldots, x_5 \), using (28), we obtain

\[
ds^2_{E,5} = \lambda^{2/3} dt^2 + \lambda^{1/3} (dr^2 + r^2 \delta \Omega_3^2)
\]

where

\[
\lambda = H_1 H_5 K = (1 + \frac{Q_1}{r^2})(1 + \frac{Q_5}{r^2})(1 + \frac{Q_K}{r^2})
\]

This is an extremal charged black hole. The horizon is located at \( r = 0 \). The area of the horizon and the five dimensional Newton’s constant are equal to

\[
A_5 = (r^2 \lambda^{1/3})^{3/2}|_{r=0} = \sqrt{Q_1 Q_2 Q_K (2\pi^2)} \\
G_N^{(5)} = \frac{G_N^{(10)}}{(2\pi)^5 R_1 V}
\]

Therefore, the entropy is equal to

\[
S = \frac{A_5}{4G_5} = 2\pi \sqrt{N_1 N_5 N_K}
\]

For the supergravity to be valid we need to suppress string loops and \( \alpha' \) corrections. We suppress string loops by sending \( g_s \to 0 \), while keeping the charges \( Q_i \) fixed. These charges are the characteristic scales of the system. In order to suppress \( \alpha' \) corrections they should be much larger than the string scale,

\[
Q_1, \ Q_5, \ Q_K \gg \alpha'
\]

Taking the compactification radii to be of order \( l_s \) we obtain

\[
g_s N_1, \ g_s N_5, \ g_s^2 N_K \gg 1
\]

This means that \( N_K \gg N_1 \sim N_5 \gg 1 \).
4.1.2 D-brane counting

We now turn to the weak-coupling D-brane configuration in order to compute the D-brane entropy. Counting the degeneracy of D-brane states translates into the question of counting BPS states in the D-brane worldvolume theory[66–70]. For the system we are interested in, and taking the torus $T^4$ in the relative transverse directions to be small, $R_2, R_3, R_4, R_5 \ll R_1$, the relevant worldvolume theory is 1+1 dimensional. This theory is the infrared limit of the Higgs branch of the 1+1 gauge theory, and it has been argued to be a deformation of the supersymmetric $\mathcal{N} = (4,4)$ sigma model with target space $(T^4)^{N_1 N_5}/S^{N_1 N_5}$[68]. Since $(T^4)^{N_1 N_5}/S^{N_1 N_5}$ is a hyperkaehler manifold of dimension $4N_1 N_5$, the sigma model has central charge equal to $6N_1 N_5$. This is the central charge of $4N_1 N_5$ bosonic and fermionic degrees of freedom (since scalars contribute 1 and fermions 1/2 to the central charge). Roughly, these degrees of freedom are the ones describing the motion of the D1 brane inside the D5 brane. For details we refer to [12].

In the worldvolume theory we get that the right movers are in their ground state and the left movers carry $N_K$ momentum modes. Thus, the degeneracy of the D-brane system is given by the degeneracy of the conformal field theory of central charge $c = 6N_1 N_5$ at level $N_K$. For a unitary conformal field theory the degeneracy is given by Cardy’s formula[71]

$$d(c, N_K) \sim \exp(2\pi \sqrt{\frac{c}{6} N_K})$$ (48)

Therefore, the entropy is equal to

$$S = \log d(c, N_K) = 2\pi \sqrt{N_1 N_5 N_K}$$ (49)

This is in exact agreement with (45).

Let us now inspect the regime of validity of the D-brane picture. Open string diagrams pick up a factor $g_s N_{1,5}$ because the open string coupling constant is $g_s$ and there are $N_{1,5}$ branes where the string can end (or equivalently one should sum over the Chan-Paton factors). Processes involving momenta involve a factor $g_s^2 N_K$[72]. Therefore, conventional D-brane perturbation theory is good when

$$g_s N_1, g_s N_5, g_s^2 N_K \ll 1 \Rightarrow Q_1, Q_5, Q_K \ll \alpha',$$ (50)

which is precisely the opposite regime to (47) where the classical supergravity solution is good. The D-brane/string perturbation theory and black hole regimes are thus complementary. This feature is related to open-closed string duality. Due to supersymmetry, however, one can extrapolate results obtained in the D-brane phase to the black hole phase.
4.2 Non-extremal black holes and the BTZ black hole

In this section we review the approach of [58]. The idea is to use U-dualities in order to connect higher dimensional black holes to lower dimensional ones. Such ideas also appeared in [73]. The U-duality transformation essentially maps the initial black hole to its near-horizon region (but Schwarzschild black holes are also included as a limiting case). In particular, four and five dimensional black holes are mapped to the three dimensional BTZ black hole. The U-duality group of string (or M) theory on a torus does not change the number of non-compact dimensions. However, black hole spacetimes always contain an extra timelike isometry. This extra isometry allows for a duality transformation, the shift transformation[74], that yields trans-dimensional transformations. A thorough discussion (that includes global issues) of the shift transformation is given in section 4.2.2.

4.2.1 U-duality and entropy

Let us discuss whether one can use U-duality in order to infer a state counting for a given black hole from the counting of a U-dual configuration. The U-duality group is conjectured (and widely believed) to be an exact symmetry of M-theory. This symmetry, however, is spontaneously broken by the vacuum. The vacua of M-theory (compactified on some manifold) are parametrized by a set constants. These constants are expectation values of scalar fields arising from the compactification. U-duality acts on these scalars, so it transforms one vacuum to another. Therefore, from a state on a given vacuum one can deduce by U-duality the existence of another state in a new vacuum. Since the U-duality group contains S-duality which is strong/weak coupling duality, one cannot in general continue the new state back to the original vacuum, unless this state is protected from quantum corrections. States with this property are BPS states. Therefore, the spectrum of BPS states is invariant under U-duality transformations. This implies in particular that if we want to count the number of states that make up an extremal supersymmetric black hole, we may use any U-duality configuration. Indeed, the entropy formula for extremal black holes is U-duality invariant[75–78].

The question is whether it is justified to use U-duality in more general context. A remarkable fact about S and T duality transformations is that they leave invariant both the entropy and the temperature of black holes connected by S and T transformations. For S-duality this follows from the fact that S-duality leaves invariant the Einstein metric. For T-duality, this has been shown in [23]. We review this argument here.

Consider a black hole solution with a timelike isometry \( \partial/\partial t \), a compact spacelike
isometry $\partial/\partial x$, and a NSNS 2-form $B$ turned on. Smoothness near the horizon requires[23] that the $B_{tx}$ vanishes at the horizon. In order the T-dual geometry to also be smooth (i.e. the dual 2-form to vanish at the horizon) we require in addition that $A_x = 0$ at the horizon (see (13)-(14)). (This can always be achieved by a coordinate transformation.) RR potentials that can be turned into $B_{xt}$ by dualities are also required to vanish at the horizon.

Let us first discuss the entropy. In $d$ dimensions the Einstein metric is given by (see (28)),

$$ds^2_E = e^{-4\phi/(d-2)}[g_{xx}(dx + A_idx)^2 + \bar{g}_{ij}dx^idx^j]$$

The metric induced on the horizon is of the same form but with $i, j$ taking values only over the $d-3$ angular variables. Therefore, the area is equal to

$$A_d = \int \sqrt{(e^{-4\phi/(d-2)})^{d-2}g_{xx} \det \bar{g}} = \int e^{-2\phi}\sqrt{g_{xx}}\sqrt{\det \bar{g}}$$

One may check that $e^{-2\phi}\sqrt{g_{xx}}$ is a T-duality invariant combination (and $\bar{g}$ was invariant to start with). Therefore, the entropy of black holes is T-duality invariant.

Let us also note that the entropy formula is invariant under dimensional reduction

$$S = \frac{A_{10}}{4G_N^{(10)}} = \frac{A_d}{4G_N^{(d)}}$$

since $A_d = A_{10}/V_{10-d}$ and $G_N^{(d)} = G_N^{(10)}/V_{10-d}$, where $V_{10-d}$ is the volume of the compactification space.

We now turn to the discussion of the behavior of the Hawking temperature under duality transformations. Perhaps the simplest way to compute the Hawking temperature is to analytically continue to Euclidean space by taking $t \rightarrow \tau = -it$. The black hole spacetime becomes then a non-singular Riemannian manifold provided that the Euclidean time is periodically identified with period the inverse Hawking temperature. Suppose that the horizon is at $r = \mu$. One can calculate the temperature to be equal to (we assume that the event horizon is non-degenerate)

$$T_H = \frac{\partial_{r}g_{\tau\tau}}{4\pi \sqrt{g_{\tau\tau}g_{rr}}} \bigg|_{r=\mu}$$

It follows by inspection that the Hawking temperature is invariant under non-singular Weyl rescaling. Hence, it does not make any difference whether we consider the Einstein or the string frame. We choose to work with the string frame. From (13) we get

$$g_{\tau\tau} = \bar{g}_{\tau\tau} + g_{xx}A_\tau A_\tau, \quad g_{rr} = \bar{g}_{rr} + g_{xx}A_r A_r$$
Assuming that $A_r$ is finite at the horizon (in all case we will consider $A_r = 0$), and using $g_{xx}|_{r=\mu} = A_\tau |_{r=\mu} = 0$ we obtain

$$T_H = \frac{\partial_\tau g_{\tau\tau} |_{r=\mu}}{4\pi \sqrt{g_{\tau\tau} g_{\tau\tau}} |_{r=\mu}}$$

which is manifestly T-duality invariant.

Therefore, an arbitrary combination of S and T transformations will lead to a black hole solution with the same entropy and temperature as the original one. This implies that black holes connected by U-duality transformations have the same number of microstates. This is somewhat surprising since for non-supersymmetric black holes we cannot follow the states during U-duality transformations. As we move from one configuration to a U-dual one, some states may disappear. However, an equal number of states has to appear, since the final configuration has the same entropy. We do not have a microscopic derivation of this fact. We believe that such derivation will be an important step towards further understanding of black holes.

A general U-duality transformation may involve strong/weak transitions. The U-duality transformations, however, that we will use below do not involve such strong/weak transitions. Actually we shall exclusively be in the black hole phase. We will only consider transformations, call them $U_T$, that are connected to T-dualities by a similarity transformation

$$U_T = U^{-1}TU$$

where $U$ denotes a generic U-duality transformation and $T$ a sequence of two T-duality transformations (so $U_T$ acts within the same theory).

### 4.2.2 The shift transformation

As we have discussed, we construct black holes configurations using appropriate non-extremal intersections of extremal branes. These configurations are solutions of the field equations provided the various harmonic functions $H_i$ appearing in the solution satisfy Laplace’s equations,

$$\nabla^2 H_i = 0,$$  \hspace{1cm} (58)

where $\nabla$ is the Laplacian in the overall transverse space. The constant part of the harmonic function is usually set to one in order the solution to be asymptotically flat. Clearly, up to normalization, the only other choice is to set this constant to zero. This choice has the dramatic effect of changing the asymptotics of the solution. We will see, however, that there is a duality transformation, the shift transformation, that removes the one from the
harmonic function. This duality transformation has been appeared in the past in various contexts [79, 20, 28, 73, 74, 58, 80, 25].

Consider the solution describing a non-extremal fundamental string in \( d+1 \) dimensions

\[
\begin{align*}
    ds^2 &= H^{-1}(r)(-f(r)dt^2 + dx_1^2) + f^{-1}(r)dr^2 + r^2d\Omega^2_{d-2} \\
    B_{tx_1} &= H'^{-1} - 1 + \tanh \alpha \\
    e^{-2\phi} &= H
\end{align*}
\]

(59)

The coordinate \( x_1 \) is periodic with period \( R_1 \). The various harmonic functions are equal to

\[
H = 1 + \frac{\mu^{d-3}\sinh^2 \alpha}{r^{d-3}}, \quad H'^{-1} = 1 - \frac{\mu^{d-3}\sinh \alpha \cosh \alpha}{r^{d-3}}H^{-1}, \quad f = 1 - \frac{\mu^{d-3}}{r^{d-3}}
\]

(60)

The constant part of the antisymmetric tensor \( B_{tx_1} \) is fixed by the requirement that \( B_{tx_1} \) vanishes at the horizon. This is required by regularity[23], as described in the previous section. The entropy and the temperature are given by

\[
S = \frac{1}{4G_N^{(d+1)}}2\pi R_1 \cosh \alpha \mu^{d-2} \omega_{d-2}, \quad T_H = \frac{(d-3)}{4\pi \mu \cosh \alpha}
\]

(61)

Notice that in order to calculate the area one first has to reach the Einstein frame.

We now perform the following sequence of T-dualities that we call the shift transformation:

\[
\text{shift} = T_{\partial/\partial t'}(\partial/\partial x_1') \circ T_{\partial/\partial x_1}(\partial/\partial x_1)
\]

(62)

where

\[
\frac{\partial}{\partial x_1'} = -e^{-\alpha} \frac{\partial}{\partial t} + \frac{1}{\cosh \alpha} \frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial t'} = \cosh \alpha \frac{\partial}{\partial t}
\]

(63)

The notation \( T_{k_1}(k_2) \) indicates a T-duality transformation along the killing vector \( k_2 \) keeping \( k_1 \) fixed.

Let us give the details. After the first T-duality, \( T_{\partial/\partial t}(\partial/\partial x_1) \), we get a non-extremal wave solution,

\[
\begin{align*}
    ds^2 &= -H^{-1}(r)f(r)dt^2 + H(r) \left( dx_1 - (H'^{-1}(r) - 1 + \tanh \alpha)dt \right)^2 \\
    &\quad + f^{-1}(r)dr^2 + r^2d\Omega^2_{d-2}
\end{align*}
\]

(64)

The radius of \( x_1 \) is now \( \alpha'/R_1 \). In addition, \( g_s \rightarrow l_s/R_1 \), so \( G_N^{(d+1)} \rightarrow G_N^{(d+1)} \alpha'/R_1^2 \). One can check that this solution has the same entropy and temperature as the solution in (59).
We would like now to dualize along (63). To do this we first reach adapted coordinates

\[
\begin{pmatrix}
  t \\
x_1
\end{pmatrix} = \begin{pmatrix}
  \cosh \alpha & -e^{-\alpha} \\
  0 & \frac{1}{\cosh \alpha}
\end{pmatrix} \begin{pmatrix}
  t' \\
x'_1
\end{pmatrix}.
\] (65)

The metric in the new coordinates takes the form (we have dropped the primes)

\[
ds^2 = -\tilde{H}^{-1}(r)f^2 dt^2 + \tilde{H}(r) \left(dx_1 - (\tilde{H}^{-1}(r) - 1)dt\right)^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{d-2}^2.
\] (66)

where now

\[
\tilde{H}(r) = \frac{\mu^{d-3}}{r^{d-3}}.
\] (67)

The radius of \(x_1\) also changes to \(\cosh \alpha / R_1\). In addition, there is a again a change in Newton’s constant. One can calculate the temperature and entropy of this solution. The result for the entropy is the same in (61). The temperature is equal to \(T_H = (d - 3)/4\pi \mu\). This differs by a factor of \(\cosh \alpha\) from (61). This is due to the fact that the timelike killing vectors \(\partial/\partial t\) and \(\partial/\partial t'\) differ by a factor of \(\cosh \alpha\) (see (63)).

To summarize, the effect of the shift transformation (62) is to change the solution by removing the constant part of the harmonic functions. All the dependence of the metric and the antisymmetric tensor on the non-extremality angle \(\alpha\) resides in the radius of the compact direction which after the shift transformation is equal to \(R_1 / \cosh \alpha\). In addition, \(g_s \rightarrow g_s / \cosh \alpha\), so \(G_N^{(d+1)} \rightarrow G_N^{(d+1)} / \cosh^2 \alpha\).

The orbits of the killing vector \(\partial/\partial x'_1\) are non-compact since the time coordinate is non-compact. This means that (59) and (68) are not equivalent. To make the duality transformation a symmetry we need to compactify the orbits of the killing vector \(\partial/\partial x'_1\).\(^5\)

The fact, however, that the entropy and temperature of the one black hole can be deduced

\(^5\)One way to make the orbits compact is to compactify time with appropriately chosen radius. It has been argued in [81] that a spatially wrapped brane should also be wrapped in time in order to avoid conical singularities at the horizon. The two issues may be related. The time coordinate is naturally compact in Euclidean black holes, the radius of the time coordinate being the inverse of the Hawking temperature. One may try to formulate the analysis in the Euclidean framework. The problem is then that the coordinate transformation (65) is complex.
from the entropy and temperature of the other indicates that the two solutions are in the same universality class (in a loose sense).

The norm of the killing vector (63) is

$$\left| \frac{\partial}{\partial x_1'} \right|^2 = \frac{\mu^{d-3}}{\gamma}$$

therefore the isometry is spacelike everywhere but it becomes null at spatial infinity. Let us examine the \((r, x_1)\) part of the metric close to spatial infinity. From (66) we get

$$ds^2_{(r, x_1)} = dr^2 + \mu^{d-3} r^3 dx_1^2$$

For \(d = 5\), which will be the case in the next section where we discuss five dimensional black holes, this is exactly the same metric as in (16). This suggest to consider \(r, x_1\) as polar coordinates and the isometry in \(x_1\) as a rotational isometry with a fixed point at infinity.

### 4.2.3 Connection of 5d and 4d black holes to the BTZ black hole

We are now ready to use our results to study non-extremal 5d and 4d black holes. We will explicitly work out the case of 5d black holes. The analysis of 4d black holes is completely analogous [58]. Four and five dimensional black holes can also be mapped by similar operations [73, 58, 82] to two dimensional black holes[83]. Let us also note that the manipulations we describe here cannot connect the BTZ black hole to higher than five dimensional black holes [58]. The relation between the near-horizon limit of higher-dimensional black holes and the BTZ black hole has also been investigated in [84].

The solution we will study is the non-extremal version of (38). Explicitly, the metric, the dilaton and the antisymmetric tensor are given by

$$ds^2_{10} = H_1^{1/2} H_5^{1/2} \left[ H_1^{-1} H_5^{-1} \left( -K^{-1} f dt^2 + K \left( dx_1 - (K' - 1) dt \right)^2 \right) + H_5^{-1} (dx_2^2 + \cdots + dx_5^2) + (f^{-1} dr^2 + r^2 d\Omega_3^2) \right] ,$$

and

$$e^{-2\phi} = H_1^{-1} H_5 , \quad C^{(2)}_{01} = H_1^{r-1} - 1 + \tanh \alpha_1 ,$$

$$H_{ijk} = \frac{1}{2} e_{ijkl} \partial_l H_5^i , \quad i, j, k, l = 6, \ldots, 9 ,$$

$$f = 1 - \frac{\mu^2}{r^2} , \quad r^2 = x_6^2 + \cdots + x_9^2 ,$$
The coordinates \( x_i, i = 1, \ldots, 5 \), are assumed to be periodic, each with radius \( R_i \).

The various harmonic function are given by
\[
K = 1 + \frac{Q_K}{r^2}, \quad K' = 1 - \frac{Q_K}{r^2} K^{-1}, \quad Q_K = \mu^2 \sinh^2 \alpha_K, \quad Q_K = \mu^2 \sinh \alpha_K \cosh \alpha_K
\]
\[
H_1 = 1 + \frac{Q_1}{r^2}, \quad H_1' = 1 - \frac{Q_1}{r^2} H_1^{-1}, \quad Q_1 = \mu^2 \sinh^2 \alpha_1, \quad Q_1 = \mu^2 \sinh \alpha_1 \cosh \alpha_1
\]
\[
H_5 = 1 + \frac{Q_5}{r^2}, \quad H_5' = 1 + \frac{Q_5}{r^2}, \quad Q_5 = \mu^2 \sinh^2 \alpha_5, \quad Q_5 = \mu^2 \sinh \alpha_5 \cosh \alpha_5 , \quad (73)
\]

Dimensionally reducing in \( x_1, x_2, x_3, x_4, x_5 \), one gets a 5d non-extremal black hole, whose metric in the Einstein frame is given by
\[
ds_{E,5}^2 = -\lambda^{-2/3} f dt^2 + \lambda^{1/3} (f^{-1} dr^2 + r^2 d\Omega_3^2) , \quad (74)
\]
where
\[
\lambda = H_5 H_1 K = \left(1 + \frac{Q_5}{r^2}\right) \left(1 + \frac{Q_1}{r^2}\right) \left(1 + \frac{Q_K}{r^2}\right) . \quad (75)
\]
This black hole is charged with respect to the Kaluza-Klein gauge fields originating from the antisymmetric tensor fields and the metric. When all charges are set equal to zero one obtains the 5d Schwarzschild black hole. The metric (74) has an outer horizon at \( r = \mu \) and an inner horizon at \( r = 0 \).

The Bekenstein–Hawking entropy may easily be calculated to be
\[
S = \frac{A_5}{4 G_N^{(5)}} = \frac{1}{4} \frac{(2\pi)^5 V R_1}{G_N^{(10)}} \mu^3 \omega_3 \cosh \alpha_5 \cosh \alpha_1 \cosh \alpha_K , \quad (76)
\]
where \( V = R_2 R_3 R_4 R_5 \) is the compactification volume in the relative transverse directions, \( \omega_3 \) is the volume of the unit 3-sphere and \( G_N^{(5)} \) and \( G_N^{(10)} \) are Newton’s constant in five and ten dimensions, respectively. The temperature is given by
\[
T_H = \frac{1}{2\pi \mu \cosh \alpha_1 \cosh \alpha_5 \cosh \alpha_K} \quad (77)
\]

We will now show that one can connect this black hole to the BTZ black hole times a 3-sphere using transformations of the form (57). A U-transformation is used to map a given brane to a fundamental string. The T transformation is the shift transformation (62).

For the case at hand we need to perform the shift transformation to the D1 and the D5 brane. The final result is given by the metric in (71), but with
\[
H_1 = \frac{\mu^2}{r^2} , \quad H_5 = \frac{\mu^2}{r^2} , \quad (78)
\]
and, in addition,

\[ e^{-2\phi} = 1 , \quad C_{01}^{(2)} = H_1^{-1} - 1 , \]
\[ H_{ijk} = \frac{1}{2} \epsilon_{ijk} \partial_l (H_5 - 1) , \quad i, j, k, l = 6, \ldots, 9 . \]  

(79)

In addition the compactification volume becomes \( V \to V/(\cosh \alpha_1 \cosh \alpha_5) \) (here, for convenience in the presentation, we assume that the U-duality transformation mapped the D1 and D5 into a fundamental string wrapped in one of the relative transverse directions). Furthermore, \( G_N^{(10)} \to G_N^{(10)}/(\cosh^2 \alpha_1 \cosh^2 \alpha_5) \). Notice that the parameters \( \alpha_1 \) and \( \alpha_5 \) associated to the charges of the original D1 and D5 brane do not appear in the background fields anymore.

Dimensionally reducing along \( x_2, x_3, x_4, x_5 \) we find

\[ ds^2_{E,6} = ds^2_{BTZ} + l^2 d\Omega_3^2 , \]

(80)

where

\[ ds^2_{BTZ} = - \frac{(\rho_+^2 - \rho_-^2)(\rho^2 - \rho_-^2)}{l^2 \rho^2} dt^2 + \rho^2 (d\varphi + \frac{\rho_+ \rho_-}{l \rho^2} dt)^2 + \frac{l^2 \rho^2}{(\rho_+^2 - \rho_-^2)(\rho^2 - \rho_-^2)} d\rho^2 \]  

(81)

is the metric of the non-extremal BTZ black hole in a space with cosmological constant \( \Lambda = -1/l^2 \), with inner horizon at \( \rho = \rho_- \) and outer horizon at \( \rho = \rho_+ \). The mass and the angular momentum of the BTZ black hole are equal to

\[ M = \frac{\rho_+^2 + \rho_-^2}{8G_N^{(3)} l^2} , \quad J = \frac{\rho_+ \rho_-}{4G_N^{(3)} l} . \]  

(82)

In terms of the original variables:

\[ l = \mu , \quad \varphi = \frac{x_1}{l} , \quad \rho^2 = r^2 + l^2 \sinh^2 \alpha_K , \]
\[ \rho_+^2 = l^2 \cosh^2 \alpha_K , \quad \rho_-^2 = l^2 \sinh^2 \alpha_K . \]  

(83)

In addition,

\[ \phi = 0 , \quad C_{t\varphi}^{(0)} = (\rho^2 - \rho_+^2)/l , \quad H = l^2 \epsilon_3 , \]

(84)

where \( \epsilon_3 \) is the volume form element of the unit 3-sphere. Therefore, the metric (80) describes a space that is a product of a 3-sphere of radius \( l \) and of a non-extremal BTZ black hole. Notice that the BTZ and the sphere part are completely decoupled.

We can now calculate the entropy of the resulting black hole. The area of the horizon is equal to

\[ A_3 = 2\pi \frac{R_1}{\mu} \mu \cosh \alpha_K , \]

(85)

30
whereas Newton’s constant is given by

\[ G_N^{(3)} = \frac{G_N^{(10)}}{(2\pi)^4 V(cosh \alpha_1 cosh \alpha_5)(\mu^2 \omega_3)} . \]  \hspace{1cm} (86)

It follows that \( S = A_3/(4G_N^{(3)}) \) equals (76), i.e. the Bekenstein–Hawking entropy of the final configuration is equal to the one of the original 5d black hole. Notice that the Newton constant in (86) contains the parameter \( \alpha_1, \alpha_5 \), i.e. carries information on the charge of the original D1 and D5 brane. The temperature of the BTZ black hole is equal to

\[ T_{BTZ} = \frac{\rho_+^2 - \rho_-^2}{2\pi \rho_+ l^2} . \]  \hspace{1cm} (87)

Transforming to the original variables we get

\[ T_{BTZ} = \frac{1}{2\pi \mu \cosh \alpha_K} = \cosh \alpha_1 \cosh \alpha_5 T_H \]  \hspace{1cm} (88)

precisely as predicted by the duality transformations.

We finish this section by pointing out a remarkable fact: We have started with the solution (71) of the low-energy supergravity. This solution is expected to get \( \alpha' \) corrections. Then we used the T-duality rules (12) which are also valid only to first order in \( \alpha' \). The final result, however, is valid to all orders in \( \alpha' \)!

The fields in (81), (84) have their canonical value, so that both the BTZ and the sphere part are separately exact classical solutions of string theory,\(^6\) i.e. there is an exact CFT associated to each of them. For the BTZ black hole the CFT corresponds to an orbifold of the WZW model based on \( SL(2, \mathbb{R}) \) \cite{79,85,86}, whereas for \( S^3 \) and the associated antisymmetric tensor with field strength \( H \), given in (84), the appropriate CFT description is in terms of the \( SO(3) \) WZW model. The same result also holds in the case of 4d black holes\cite{58}. This time the black hole is mapped to \( BTZ \times S^2 \). Again all fields are such that there is an exact CFT associated to each factor. The one associated with \( S^2 \) is the monopole CFT of \cite{87}.

The situation seems quite similar to the case described at the end of section 2.3.1: There we had the singular solution (16) of the lowest order in \( \alpha' \) beta function equation which becomes an exact CFT after dualization with respect to a killing vector whose norm vanishes at spatial infinity. However, to establish equivalence one needs all order in \( \alpha' \).

In the case of black holes we have:

\(^6\)For the D1-D5 system that we discuss we obtain a CFT describing a D-string. One gets a fundamental string from the S-dual system of F1-NS5.
becomes, after dualization with respect to a killing vector whose norm vanishes at spatial infinity (plus other dualities), the BTZ black hole which contains no curvature singularity and is an exact CFT. (So, one could argue that the original singularity is resolved by $\alpha'$ corrections).

We find these similarities quite suggestive. However, it is difficult to see how one could overcome the problem of the non-compactness of the orbits of the killing vector in (63).

4.3 Low-energy limit and the near-horizon geometry

4.3.1 Near-horizon limit of branes

We have argued that the physical system describing a black hole in strong coupling becomes a set of intersecting branes in weak coupling. We emphasize that there is only one physical system. Its description, however, in terms of some weakly coupled theory changes as we change the parameters of the theory, and furthermore, at any given regime of the parameter space, there is only one weakly coupled description.

One may view the different descriptions as effective theories that are adequate to describe the system at specific range of the parameter space. As we go outside this range new degrees of freedom become important and a new description takes over. In some cases, however, a given theory may still be well-defined for any value of the coupling constant. In this case we get a dual description of this theory.

Let us consider $N$ coincident D$^p$-branes. At weak coupling they have a description as hypersurfaces where string can end. There is worldvolume theory describing the collective coordinates of the brane. The worldvolume fields interact among themselves and with the bulk fields. We would like to consider a limit which decouples the bulk gravity but still leaves non-trivial dynamics on the worldvolume. In low energies gravity decouples. So, we consider the limit $\alpha' \to 0$, which implies that the gravitation coupling constant, i.e. Newton’s constant, $G_N \sim \alpha'^4$, also goes to zero. We want to keep the worldvolume degrees of freedom and their interactions. Since the worldvolume dynamics are governed by open string ending on the D-branes, we keep fixed the masses of strings stretched between D-branes as we take the limit $\alpha' \to 0$. In addition, we keep fixed the coupling constant of the worldvolume theory, so all the worldvolume interactions remain present. For $N$ coincident D-branes, the worldvolume theory is an $SU(N)$ super Yang-Mills theory (we ignore the center of mass part). The YM coupling constant is equal (up to numerical
constants) to $g^2_M \sim g_s(\alpha')(p-3)/2$. Thus we get that the following limit,

$$\alpha' \to 0, \quad U = \frac{r}{\alpha'} = \text{fixed}, \quad g^2_M = \text{fixed} \quad (89)$$

yields a decoupled theory on the worldvolume.

At strong coupling the D$p$ branes are described by the black $p$-brane spacetimes (18). Let us consider the limit (89) for this spacetime. One gets that the harmonic function becomes,

$$H_i \to g^2_Y M(\alpha')^{-2}U^{p-7} \quad (90)$$

The limit (89) is a near-horizon limit since $r = U\alpha' \to 0$ and there is a horizon at $r = 0$. We see that the effect of the limit (89) is similar to the effect of the shift transformation, namely the one is removed from the harmonic function. Inserting (90) back in the metric one finds that the spacetime becomes conformal to $adS_{p+2} \times S^{8-p}$ [54, 88] (for M-branes, one gets $adS_4 \times S^7$ for the M2 brane and $adS_7 \times S^4$ for the M5 brane [89]).

Let us now consider the particular case of $N$ coincident D3-branes. The worldvolume theory is $d = 4, N = 4$ SU($N$) SYM theory. This is a finite unitary theory for any value of the its coupling constant. On the other hand, this system has a description as a black 3-brane at strong coupling. In the limit (89) we get that the spacetime becomes $adS_5 \times S^5$. In order to suppress string loops we need to take $N$ large. For the supergravity description to be valid 't Hooft’s coupling constant[90], $g^2_M N$, must be large. We therefore get that the strong ('t Hooft) coupling limit of large $d = 4, N = 4$ SU($N$) SYM is described by $adS$ supergravity[59]!

$N = 4 d = 4$ SYM theory is a well-defined unitary finite theory, whereas supergravity is a non-renormalizable theory. It is best to think about it as the low energy effective theory of strings. Therefore, one should really consider strings on $adS_5 \times S^5$. In this way we reach the celebrated $adS$/CFT duality[59]7:

_Four dimensional $\mathcal{N} = 4$ SU($N$) SYM is dual to string theory on $adS_5 \times S^5$._

This conjecture was made precise in [92, 93], where a prescription for evaluation of correlation functions was proposed. Subsequently a large number of papers appeared, all of them supporting the $adS$/CFT duality.

Let us examine again our result. We obtained that five dimensional $adS$ gravity is equivalent to $d = 4, N = 4$ SYM theory. In other words, a gravity theory in $d + 1(=5)$ dimensions is described in terms of a field theory without gravity in $d(=4)$ dimensions.

---

7Many of the elements leading to this conjecture appeared in [91]. In [58], the worldvolume theory of the D3 brane was argued to be mapped to the singleton of $adS_5$ by the shift transformation.
This is just holography[94,95]! One can further show that the boundary theory indeed has one degree of freedom per Planck area[96].

Similar results hold for other brane configurations as one can always consider the low energy limit. In the case of conformal worldvolume theories there is an \( adS \) factor on the gravity side. In these cases the worldvolume theory is valid at all energy scales, and these considerations provide a weakly coupled gravity description of a strongly coupled theory. In the non-conformal cases the worldvolume SYM theory is a theory with a cut-off. As we change the cut-off new degrees of freedom become relevant and the description in terms of a SYM theory may not be valid. In these cases one finds that as we change the parameters of the theory there is always some perturbative description[97, 98]. The black \( p \)-brane solution becomes conformal to anti-de Sitter spacetime and the gravity description is in terms of gauged supergravities which have domain-wall vacua[88].

4.3.2 Low-energy limit of black hole spacetimes

Let us discuss the low energy limit for black hole configurations. We will discuss in detail the \( 5d \) case. The \( 4d \) case is very similar [99]. Rotating black holes have been considered in [100].

Consider the black hole configuration in (38). We go to low energies keeping fixed the masses of stretched strings, the radius of coordinate which the string is wrapped in and the radii of the relative transverse directions in string units,

\[
\alpha' \to 0, \quad U = \frac{r}{\alpha'} \text{ fixed}, \quad R_1, r_i = \frac{R_i}{\sqrt{\alpha'}} \text{ fixed } \quad i = 2, \ldots, 5 \quad (91)
\]

Notice that \( R_1 \gg R_i, i = 2, \ldots, 5 \), as in section 4.1.2. Since the horizon is at \( r = 0 \) and \( r = U\alpha' \to 0 \) this is at the same time a near-horizon limit. Therefore, the resulting configuration has the same number of degrees of freedom as the original one (since the area of the horizon is a measure of the degrees of freedom).

In the limit (91) the harmonic functions (41) become

\[
H_1 \to \frac{1}{\alpha'} \tilde{Q}_1, \quad \tilde{Q}_1 = \frac{g_s N_1}{v}, \\
H_5 \to \frac{1}{\alpha'} \tilde{Q}_5, \quad \tilde{Q}_5 = g_s N_5 \\
K \to 1 + \frac{\tilde{Q}_K}{U^2}, \quad \tilde{Q}_K = \frac{g^2_s N_K}{R_1^2 v} \quad (92)
\]

where \( v = r_2 r_3 r_4 r_5 \). Notice that the low-energy limit removes the one from the harmonic
function of the D1 and D5 brane exactly as in section 4.2. Let us define new variables

\[ \rho^2 = U^2 + \rho_0^2, \quad \phi = x_1/R_1, \quad t_{BTZ} = t \frac{\tilde{Q}_1 \tilde{Q}_5}{R_1^2} \]

\[ \rho_0^2 = \tilde{Q}_K, \quad l^2 = \frac{\tilde{Q}_1 \tilde{Q}_5}{R_1^2} \]

(93)

The metric (38) becomes

\[ ds^2 = \alpha' \frac{R_1^2}{\sqrt{\tilde{Q}_1 \tilde{Q}_5}} [ds^2_{BTZ} + \frac{\tilde{Q}_1}{R_1^2} (dx_2^2 + \cdots + dx_5^2) + \frac{\tilde{Q}_1 \tilde{Q}_5}{R_1^2} d\Omega_3^2] \]

\[ e^{-2\phi} = \frac{\tilde{Q}_5}{\tilde{Q}_1} \]

(94)

where \( ds^2_{BTZ} \) is the metric (81) with \( \rho_+ = \rho_- = \rho_0 \), i.e. the metric of the extremal BTZ black hole. The overall factor in (94) originates from the fact that we want to have the angle \( \phi \) with unit radius. We move this overall factor to Newton’s constant by a Weyl rescaling. The three dimensional Newton’s constant is then equal to (taking into account the dilaton, and arranging such that the 3d metric is the standard BTZ metric (81))

\[ G_N^{(3)} = \frac{g_s^2}{4 R_1 v \sqrt{\tilde{Q}_1 \tilde{Q}_5}} \]

(95)

Notice that all the factors of \( \alpha' \) have canceled out. The mass, the angular momentum and the area of the horizon of the BTZ black hole are equal to

\[ M = Jl, \quad J = \frac{\rho_0^2}{4 G_N^{(3)} l} = N_K, \quad A = 2\pi \rho_0 = 2\pi \sqrt{\tilde{Q}_K} \]

(96)

Therefore,

\[ S = 2\pi \frac{R_1 v}{g_s^2} \sqrt{\tilde{Q}_1 \tilde{Q}_5 \tilde{Q}_K} = 2\pi \sqrt{N_1 N_5 N_K} \]

(97)

as in (45) (as it should since we just took the near-horizon limit). Therefore, at low energies the physics of extremal black holes is governed by the BTZ black hole.

Let us now move to non-extremal black holes. In this case, the low energy limit is supplemented by the condition[101],

\[ \mu_0 = \frac{\mu}{\alpha'} \text{ fixed} \]

(98)

The non-extremal black hole (71) has an outer horizon at \( r = \mu \) and an inner horizon at \( r = 0 \). Therefore, the low-energy limit (91), (98) is a near inner-horizon rather than near
outer-horizon limit. As a result the entropies do not agree in general. To see this observe
that the effect of the low energy limit (91), (98) is to remove the one from the harmonic
functions $H_1$ and $H_5$ but leave $K$ unchanged[101]. Since before we take the low energy
limit, $H_i(r = \mu) = \cosh^2 \alpha_i$, $i = 1, 5$ and after the low energy limit $H_i(r = \mu) = \sinh^2 \alpha_i$,
the entropies of the two configurations differ by a factor of $\tanh \alpha_1 \tanh \alpha_5$. Unless this
factor is equal to one, the low energy configuration will contain different number of degrees
of freedom. This factor is equal to one in the dilute gas approximation [102]

$$\alpha_1, \alpha_5 \gg 1,$$ (99)

and therefore the entropies agree in this approximation. Far from extremality the number
of degrees of freedom changes as we go to low energies. In all cases the low energy regime
is governed by the BTZ black hole. This result should be contrasted with the result in
the previous section. There we also found that 4$d$ and 5$d$ black holes are connected to
the BTZ black hole. All our transformations, however, were isoentropic, and there was
no limit involved. We only needed that the supergravity approximation is valid.

Let us finish by presenting a microscopic derivation of the Bekenstein-Hawking entropy
formula for extremal black hole (38) using the results of this section. It has been shown
by Brown and Henneaux [103] that the asymptotic symmetry group of $adS_3$ is generated
by two copies of the Virasoro algebra with central charge

$$c = \frac{3l}{2G_N^{(3)}}$$ (100)

This central charge was also derived through the adS/CFT correspondence in [104]. Therefore, any consistent theory of gravity on $adS_3$ is conformal field theory with central charge equal to (100).

The generators of the asymptotic Virasoro are related to the mass and angular mo-
momentum as

$$M = \frac{1}{l}(L_0 + \bar{L}_0),$$
$$J = L_0 - \bar{L}_0$$ (101)

where we have normalized $L_0, \bar{L}_0$ such that they vanish for the massless black hole.

In the case of the 5$d$ extremal black hole, and after the low-energy limit is taken, we
obtain a geometry of the form $BTZ \times S^3 \times T^4$. One may dimensionally reduce over the
compact spaces to obtain the BTZ black hole and a set of matter fields. The BTZ black
hole is asymptotically $adS_3$ so quantum theory in this space is described by a CFT. We
can calculate the central charge using (93), (95). The result is

\[ c = 6N_1N_5 \]  \hspace{1cm} (102)

This is the same value as the one we found in section 4.1.2! In addition, from (96) we obtain \( L_0 = J = N_K \), \( \bar{L}_0 = 0 \). Thus, we get the same description as in the D-brane side. This is the same unitary CFT but we are now at strong coupling. Therefore, Cardy’s formula apply and, (for large black holes, so \( N_K \gg 1 \)) we get correctly (97).

This counting of states generalizes immediately to non-extremal BTZ black holes [105, 106]. (From (101) we get \( L_0, \bar{L}_0 \) in terms of \( M \) and \( J \). We also know \( c \) from (100). Applying Cardy’s formula we get the Bekenstein-Hawking entropy formula). A crucial point is that in order Cardy’s formula to apply we need the CFT to be unitary. The BTZ black hole, however, induces a Liouville theory at spatial infinity[108, 109]. This means that the effective central charge is equal to one[114] instead of \( c = 2l/3G_N^{(3)} \), and one does not get correctly the Bekenstein-Hawking entropy formula (see [113] for further discussion). We argued that for the case we are discussing we have a unitary CFT because of the connection to D-branes. We find likely that the CFT corresponding to the BTZ is unitary only when the latter is embedded in string theory.

Acknowledgments

I would like to thank Jan de Boer for reading the manuscript and for discussions and comments. Research supported by the Netherlands Organization for Scientific Research (NWO).

References

[1] S. Doplicher, K. Fredenhagen and J. Roberts, The Quantum Structure of Spacetime at the Planck Scale and Quantum Fields, Commun. Math. Phys. 172 (1995) 187-220.

[2] D. Christodoulou, Phys. Rev. Lett. 25 (1970) 1596-1597;
   D. Christodoulou and R. Ruffini, Phys. Rev. D4 (1971) 3552;

\footnote{A different counting of the BTZ microstates was presented in [107]. There it was used the fact that three dimensional gravity is topological. The Einstein action can be rewritten as a Chern-Simons action[110,111]. A Chern-Simons theory on a manifold with a boundary induces a WZW model in the boundary[112]. The degrees of freedom in the boundary are would-be gauge degrees of freedom that cannot be gauged away because of the boundary. Assuming that the horizon is a boundary and imposing certain boundary condition one gets that the boundary degrees of freedom can account for the black hole entropy[107]. A problem with this derivation is that some of the states counted have negative norms.}
R. Penrose and R. Floyd, Nature 229 (1971) 77;  
S. Hawking, Phys. Rev. Lett. 26 (1971) 1344;  
B. Carter, Nature 238 (1972) 71.

[3] J.M. Bardeen, B. Carter and S.W. Hawking, *The four laws of black hole mechanics*, Commun. Math. Phys. 31 (1973) 161-170.

[4] J.D. Bekenstein, *Black Holes and the Second Law*, Lett. Nuovo. Cimento 4 (1972) 737; *Black Holes and Entropy*, Phys. Rev. D7 (1973) 2333; *Generalized second law of thermodynamics in black-hole physics*, Phys. Rev. D9 (1974) 3292.

[5] S.W. Hawking, *Black hole explosions?*, Nature 248 (1974) 30; *Particle Creation by Black Holes* Commun. Math. Phys. 43 (1975) 199.

[6] C.M. Hull and P.K. Townsend, *Unity of Superstring Dualities* Nucl. Phys. B438 (1995) 109, hep-th/9410167.

[7] E. Witten, *String Theory Dynamics In Various Dimensions* Nucl. Phys. B443 (1995) 85, hep-th/9503124.

[8] J. Polchinski, *D-branes and RR-charges* Phys. Rev. Lett. 75 (1995) 4724, hep-th/9510017.

[9] M. Bañados, C. Teitelboim and J. Zanelli, *The Black Hole in Three Dimensional Space Time*, Phys. Rev. Lett. 69 (1992) 1849, hep-th/9204099.

[10] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, Phys. Rev. D48 (1993) 1506.

[11] G. Horowitz, *The Dark Side of String Theory: Black Holes and Black Strings*, hep-th/9210119.

[12] J. Maldacena, *Black holes in string theory*, hep-th/9607235.

[13] A. Peet, *The Bekenstein Formula and String Theory (N-brane Theory)*, Class. Quant. Grav. 15 (1998) 3291-3338, hep-th/9712253.

[14] J. Polchinski, *TASI Lectures on D-Branes*, hep-th/9611050.

[15] P.K. Townsend, *The eleven-dimensional supermembrane revisited*, Phys. Lett. B350 (1995) 184-187, hep-th/9501068.

[16] P. Horava, E. Witten, *Heterotic and Type I string dynamics from eleven dimensions* Nucl. Phys. B460 (1996) 506, hep-th/9510209, and *Eleven-Dimensional Supergravity on a Manifold with Boundary*, Nucl. Phys. B475 (1996) 94-114, hep-th/9603142.

[17] E. Cremmer, B. Julia and J. Scherk, *Supergravity in eleven dimensions*, Phys. Lett. B76 (1978) 409.

[18] T. Buscher, *A symmetry of the string background field equations*, Phys. Lett. B194 (1987) 59.

[19] M. Roček and E. Verlinde, *Duality, quotients and currents*, Nucl. Phys. B373 1992 630-646, hep-th/9110053.

[20] E. Alvarez, L. Alvarez-Gaume, J.L.F. Barbon and Y. Lozano, *Some global aspects of duality in string theory*, Nucl.Phys. B415 (1994) 71-100, hep-th/9309039.

[21] T. Buscher, *Path integral derivation of quantum duality in non-linear sigma models*, Phys. Lett. B201 (1988) 466.
22] J. De Jaegher, J. Raymaekers, A. Sevrin and W. Troost, *Dilaton transformation under abelian and non-abelian T-duality in the path-integral approach*, hep-th/9812207.

23] G. Horowitz and D. Welch, *Duality Invariance of the Hawking Temperature and Entropy*, Phys. Rev. **D49** (1994) 590, hep-th/9308077.

24] G. Moore, *Finite in All Directions*, hep-th/9305139; C.M. Hull and B. Julia, *Duality and Moduli Spaces for Time-Like Reductions*, Nucl. Phys. **B534** (1998) 250-260, hep-th/9803239; C.M. Hull, *Timelike T-Duality, de Sitter Space, Large N Gauge Theories and Topological Field Theory*, J.High Energy Phys. 9807 (1998) 021, hep-th/9806146.

25] E. Cremmer, I.V. Lavrinenko, H. Lu, C.N. Pope, K.S. Stelle and T.A. Tran, *Euclidean-signature Supergravities, Dualities and Instantons*, Nucl. Phys. **B534** (1998) 40-82, hep-th/9803259.

26] B. Julia and H. Nicolai, *Null Killing Vector Dimensional Reduction and Galilean Geometrodynamics*, Nucl. Phys. **B439** (1995) 291, hep-th/9412002.

27] E. Alvarez, L. Alvarez-Gaume and I. Bakas, *T-duality and Space-time Supersymmetry*, Nucl. Phys. **B457** (1995) 3, hep-th/9507112.

28] I. Bakas, *Spacetime interpretation of S-duality and supersymmetry violations of T-duality*, Phys. Lett. **343B** (1995) 103, hep-th/9410104.

29] I. Bakas and K. Sfetsos, *T-duality and world-sheet supersymmetry*, Phys. Lett. **B349** (1995) 448-457, hep-th/9502065.

30] R. Gregory, J.A. Harvey and G. Moore, *Unwinding strings and T-duality of Kaluza-Klein and H-Monopoles*, hep-th/9708086.

31] M.J. Duff, R.R. Khuri, J.X. Lu, *String solitons*, Phys. Rept. 259 (1995) 213-326, hep-th/9412184.

32] K. S. Stelle, *Lectures on Supergravity p-branes*, hep-th/9701088.

33] D. Youm, *Black Holes and Solitons in String Theory*, hep-th/9710046.

34] G.T. Horowitz and A. Strominger, *Black strings and p-branes*, Nucl. Phys. **B360** (1991) 197.

35] E. Bergshoeff, C.M. Hull and T. Ortin, *Duality in the Type–II Superstring Effective Action*, Nucl. Phys. **B451** (1995) 547, hep-th/9504081.

36] R. Myers and M. Perry, *Black holes in higher dimensional spacetimes*, Annals Phys. **172** (1986) 304.

37] J. Maharana and J. Schwarz, *Noncompact Symmetries in String Theory*, Nucl. Phys. **B390** (1993) 3, hep-th/9207016; A. Sen, *Electric magnetic duality in string theory*, Nucl. Phys. **B404** (1993) 109, hep-th/9207053.

38] E. Witten and D. Olive, *Supersymmetry algebras that include topological charges*, Phys. Lett. **78B** (1978) 97.

39] M.J. Duff and K.S. Stelle, *Multimembrane solutions of D = 11 supergravity*, Phys. Lett. **B253** (1991) 113-118.

40] R. Güven, *Black p-brane solutions of D = 11 supergravity theory*, Phys. Lett. **B276** (1991) 49-55.

41] G. Papadopoulos and P.K. Townsend, *Intersecting M-branes*, Phys. Lett. **380B** (1996) 273, hep-th/9603087.
[42] A. Tseytlin, *Harmonic superpositions of M-branes*, Nucl. Phys. **B475** (1996) 149, hep-th/9604035 and *No-force condition and BPS combinations of p-branes in 11 and 10 dimensions*, Nucl. Phys. **B487** (1997) 141, hep-th/9609212.

[43] J.P. Gauntlett, D.A. Kastor and J. Traschen, *Overlapping Branes in M-Theory*, Nucl. Phys. **B478** (1996) 544, hep-th/9604179.

[44] M. Cvetič and A.A. Tseytlin, *Non-extreme black holes from non-extreme intersecting M-branes*, Nucl. Phys. **B478** (1996) 431, hep-th/9606033.

[45] E. Bergshoeff, M. de Roo, E. Eyras, B. Janssen and J. P. van der Schaar, *Multiple Intersections of D-branes and M-branes*, Nucl. Phys. **B494** (1997) 119-143, hep-th/9612095.

[46] I.Ya. Aref’eva and O.A. Rytchkov, *Incidence Matrix Description of Intersecting p-brane Solutions*, hep-th/9612236; I.Ya. Aref’eva, K.S. Viswanathan, A.I. Volovich and I.V. Volovich, *Composite p-branes in various dimensions*, Nucl. Phys. Proc. Suppl. **56B** (1997) 52-60, hep-th/9701092; I.Ya. Aref’eva, M.G. Ivanov and I.V. Volovich, *Non-extremal Intersecting p-branes in Various Dimensions*, Phys. Lett. **B406** (1997) 44-48, hep-th/9702079.

[47] R. Argurio, F. Englert and L. Houart, *Intersection Rules for p-Branes*, Phys. Lett. **398B** (1997) 61, hep-th/9701042.

[48] N. Ohta, *Intersection Rules for Non-Extreme p-Branes*, Phys. Lett. **B403** (1997) 218-224, hep-th/9702095.

[49] E. Bergshoeff, M. de Roo, E. Eyras, B. Janssen and J.P. van der Schaar, *Intersections involving waves and monopoles in eleven dimensions*, Class. Quant. Grav. **14** (1997) 2757, hep-th/9704120.

[50] N. Ohta and J.-G. Zhou, *Towards the Classification of Non-Marginal Bound States of M-Branes and Their Construction Rules*, Int. J. Mod. Phys. **A13** (1998) 2013-2046, hep-th/9706153.

[51] J.D. Edelstein, L. Tataru and R. Tatar, *Rules for localized overlappings and intersections of p-branes*, High Energy Phys. 9806 (1998) 003, hep-th/9801049.

[52] J.P. Gauntlett, *Intersecting branes*, hep-th/9705011.

[53] R.R. Khuri, *A Comment on String Solitons*, Phys. Rev. **D48** (1993) 2947, hep-th/9305143.

[54] H.J. Boonstra, B. Peeters and K. Skenderis, *Branes intersections, anti-de Sitter spacetimes and dual superconformal theories*, Nucl. Phys. **B533** (1998) 127-162, hep-th/9803231.

[55] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, Phys. Lett. **B379** (1996) 99, hep-th/9601029.

[56] C.G. Callan and J.M. Maldacena, *D-brane approach to black hole quantum mechanics*, Nucl. Phys. **B472** (1996) 591, hep-th/9602043.

[57] G.T. Horowitz and J. Polchinski, *A Correspondence Principle for Black Holes and Strings*, Phys. Rev. **D55** (1997) 6189, hep-th/9612146.

[58] K. Sfetsos and K. Skenderis, *Microscopic derivation of the Bekenstein-Hawking entropy formula for non-extremal black holes*, Nucl. Phys. **B517** (1998) 179-204, hep-th/9711138.

[59] J.M. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200.
[60] R. Kallosh, A. Linde, T. Ortin, A. Peet and A. Van Proeyen, *Supersymmetry as a Cosmic Censor*, Phys. Rev. D**46** (1992) 5278-5302, hep-th/9205027; M. Cvetic and D. Youm, *Dyonic BPS Saturated Black Holes of Heterotic String on a Six-Torus*, Phys. Rev. D**53** (1996) 584-588, hep-th/9507090; M. Cvetic and A.A. Tseytlin, *General class of BPS saturated dyonic black holes as exact superstring solutions*, Phys. Lett. B**366** (1996) 95, hep-th/9510097, and *Solitonic Strings and BPS Saturated Dyonic Black Holes*, Phys. Rev. D**53** (1996) 5619-5633; Erratum-ibid. D**55** (1997) 3907, hep-th/9512031.

[61] J. Maldacena and A. Strominger, *Statistical Entropy of four-dimensional extremal black holes*, Phys. Rev. Lett. 77 (1996) 428-429, hep-th/9603060.

[62] C. Johnson, R. Khuri and R. Myers, *Entropy of 4D extremal black holes*, Phys. Lett. B**378** (1996) 78-86, hep-th/9603061.

[63] J.C. Breckenridge, R.C. Myers, A.W. Peet, C. Vafa, *D–branes and Spinning Black Holes*, Phys. Lett. B**391** (1997) 93-98, hep-th/9602065.

[64] J. C. Brekenridge, D. A. Lowe, R. C. Myers, A. W. Peet, A. Strominger and C. Vafa, *Macroscopic and Microscopic Entropy of Near-Extremal Spinning Black Holes*, Phys. Lett. B**381** (1996) 423-426, hep-th/9603078.

[65] M. Cvetic and D. Youm, *Entropy of Non-Extreme Charged Rotating Black Holes in String Theory*, Phys. Rev. D**54** (1996) 2612-2620, hep-th/9603147.

[66] E. Witten, *Bound States Of Strings And p-Branes*, Nucl. Phys. B**460** (1996) 335, hep-th/9510135.

[67] A. Sen, *A Note on Marginally Stable Bound States in Type II String Theory*, Phys. Rev. D**54** (1996) 2964-2967, hep-th/9510229; and *U-duality and Intersecting D-branes*, Phys. Rev. D**53** (1996) 2874-2894, hep-th/9511026.

[68] C. Vafa, *Gas of D-Branes and Hagedorn Density of BPS States*, Nucl. Phys. B**463** (1996) 415-419, hep-th/9511088.

[69] M. Bershadsky, V. Sadov and C. Vafa, *D-Branes and Topological Field Theories*, Nucl. Phys. B**463** (1996) 420-434, hep-th/9512222.

[70] C. Vafa, *Instantons on D-branes*, Nucl. Phys. B**463** (1996) 435-442, hep-th/9512078.

[71] J.L. Cardy, *Operator content of two-dimensional conformally invariant theories*, Nucl. Phys. B**270** (1986) 186.

[72] M. Douglas, J. Polchinski and A. Strominger, *Probing Five-Dimensional Black Holes with D-Branes*, J.High Energy Phys. 9712 (1997) 003, hep-th/9703031.

[73] S. Hyum, *U-duality between Three and Higher Dimensional Black Holes*, hep-th/9704005.

[74] H.J. Boonstra, B. Peeters and K. Skenderis, *Duality and asymptotic geometries*, Phys. Lett. 411B (1997) 59, hep-th/9706192, and *Branes and anti-de Sitter spacetimes*, hep-th/9801076.

[75] S. Ferrara and R. Kallosh, *Supersymmetry and Attractors*, Phys. Rev. D**54** (1996) 1514-1524, hep-th/9602136.

[76] G. Horowitz, J. Maldacena and A. Strominger, *Nonextremal black hole microstates and U-duality*, Phys. Lett. B**383** (1996) 151-159, hep-th/9603109.
[77] M. Cvetic and C.M. Hull, *Black Holes and U-Duality*, Nucl. Phys. **B480** (1996) 296-316, hep-th/9606193.

[78] S. Ferrara and J.M. Maldacena, *Branes, central charges and U-duality invariant BPS conditions*, Class. Quant. Grav. **15** (1998) 749-758, hep-th/9706097.

[79] G. Horowitz and D. Welch, *Exact Three Dimensional Black Holes in String Theory*, Phys. Rev. Lett. **71** (1993) 328, hep-th/9302126.

[80] E. Bergshoeff and K. Behrndt, *D-Instantons and asymptotic geometries*, Class. Quant. Grav. **15** (1998) 1801-1813, hep-th/9803090.

[81] G.W. Gibbons, *Wrapping Branes in Space and Time*, hep-th/9803206.

[82] E. Teo, *Statistical entropy of charged two-dimensional black holes*, Phys. Lett. **B430** (1998) 57-62, hep-th/9803064.

[83] M.D. McGuigan, C.R. Nappi and S.A. Yost, *Charged Black Holes in Two-Dimensional String Theory*, Nucl. Phys. **B375** (1992) 421-452, hep-th/9111038; G.W. Gibbons and M. Perry, *The Physics of 2-d Stringy Spacetimes*, Int.J.Mod.Phys. **D1** (1992) 335-354, hep-th/9204090; C.R. Nappi and A. Pasquinucci, *Thermodynamics of Two-Dimensional Black-Holes*, Mod. Phys. Lett. **A7** (1992) 3337-3346, gr-qc/9208002.

[84] Y. Satoh, *BTZ black holes and the near-horizon geometry of higher-dimensional black holes*, hep-th/9810135.

[85] N. Kaloper, *Miens Of The Three Dimensional Black Hole* Phys. Rev. **D48** (1993) 2598, hep-th/9303007.

[86] A. Ali and A. Kumar, *O(\tilde{d}, \tilde{d}) Transformations and 3D Black Hole* Mod. Phys. Lett. **A8** (1993) 2045-2052.

[87] I. Antoniadis, C. Bachas and A. Sagnotti, *Gauged supergravity vacua in string theory*, Phys. Lett. **B235** (1990) 255; S.B. Giddings, J. Polchinski and A. Strominger, *Four-dimensional black holes in string theory*, Phys. Rev. **D48** (1993) 5784, hep-th/9305083.

[88] H.J. Boonstra, K. Skenderis and P.K. Townsend, *The domain-wall/QFT correspondence*, J.High Energy Phys. 9901 (1999) 003, hep-th/9807137.

[89] G.W. Gibbons and P.K. Townsend, *Vacuum interpolation in supergravity via super p-branes*, Phys. Rev. Lett. **71** (1993) 3754, hep-th/9307049.

[90] G. ’t Hooft, *A planar diagram theory for strong interactions*, Nucl. Phys. **B72** (1974) 461.

[91] I.R. Klebanov, *World Volume Approach to Absorption by Non-dilatonic Branes*, Nucl. Phys. **B496** (1997) 231, hep-th/9702076; S.S. Gubser, I.R. Klebanov, A.A. Tseytlin, *String Theory and Classical Absorption by Threebranes*, Nucl. Phys. **B499** (1997) 217, hep-th/9703040; S.S. Gubser, I.R. Klebanov, *Absorption by Branes and Schwinger Terms in the World Volume Theory*, Phys. Lett. **413B** (1997) 41, hep-th/9708005.

[92] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge Theory Correlators from Non-Critical String Theory*, Phys. Lett. **B428** (1998) 105, hep-th/9802109.

[93] E. Witten, *Anti-de Sitter space and Holography*, Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150.
[94] G. ’t Hooft, *Dimensional reduction in Quantum gravity*, Class. Quant. Grav. **11** (1994) 621, gr-qc/9310006.

[95] L. Susskind, *The World as a Hologram*, J. Math. Phys. **36** (1995) 6377, hep-th/9409089.

[96] L. Susskind and E. Witten, *The Holographic Bound in Anti-de Sitter Space*, hep-th/9805114.

[97] N. Itzhaki, J. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and The Large N Limit of Theories With Sixteen Supercharges*, Phys. Rev. **D58** (1998) 046004, hep-th/9802042.

[98] A.W. Peet, J. Polchinski, *UV/IR Relations in AdS Dynamics*, hep-th/9809022.

[99] V. Balasubramanian and F. Larsen, *Near Horizon Geometry and Black Holes in Four Dimensions*, Nucl. Phys. **B528** (1998) 229, hep-th/9802198.

[100] M. Cvetic and F. Larsen, *Near Horizon Geometry of Rotating Black Holes in Five Dimensions*, Nucl. Phys. **B531** (1998) 239-255 and *Microstates of Four-Dimensional Rotating Black Holes from Near-Horizon Geometry*, hep-th/9805146.

[101] J. Maldacena and A. Strominger, *AdS$_3$ black holes and a stringy exclusion principle*, hep-th/9804085.

[102] J. Maldacena, A. Strominger, *Black Hole Greybody Factors and D-Brane Spectroscopy*, Phys. Rev. **D55** (1997) 861-870, hep-th/9609026.

[103] J.D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: An example from three-dimentional gravity*, Commun. Math. Phys. **104** (1986) 207.

[104] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, J.High Energy Phys. 9807 (1998) 023, hep-th/9806087; *Holography and the Weyl anomaly*, hep-th/9812032.

[105] A. Strominger, *Black Hole Entropy from Near-Horizon Microstates*, J.High Energy Phys. 9802 (1998) 009, hep-th/9712251.

[106] D. Birmingham, I. Sachs and S. Sen, *Entropy of Three-Dimensional Black Holes in String Theory*, Phys. Lett. **B424** (1998) 275-280, hep-th/9801019.

[107] S. Carlip, *The Statistical Mechanics of the $(2+1)$-Dimensional Black Hole*, Phys. Rev. **D51** (1995) 632, gr-qc/9409052 and *The Statistical Mechanics of the Three-Dimensional Euclidean Black Hole*, **D55** (1997) 878, gr-qc/9606043.

[108] S. Carlip, *Inducing Liouville theory from topologically massive gravity*, Nucl. Phys. **B362** (1991) 111-124.

[109] O. Coussaert, M. Henneaux and P. van Driel, *The asymptotic dynamics of three-dimensional Einstein gravity with negative cosmological constant*, Class. Quant. Grav. **12** (1995), 2961-2966.

[110] A. Achúcarro and P. K. Townsend, *A Chern-Simons action for three-dimensional anti-de Sitter supergravity theories*, Phys. Lett. **B180** (1986) 89.

[111] E. Witten, *(2 + 1)-dimensional gravity as an exactly soluble system*, Nucl. Phys. **B311** (1988) 46.

[112] G. Moore and N. Seiberg, *Taming the conformal zoo*, Phys. Lett. **B220** (1989) 422; S. Elitzur, G. Moore, A. Schwimmer and Nathan Seiberg, *Remarks on the canonical quantization of the Chern-Simons-Witten theory*, Nucl. Phys. **B326** (1989) 108.
[113] S. Carlip, What We Don't Know about BTZ Black Hole Entropy, Class. Quant. Grav. 15 (1998) 3609-3625, hep-th/9806026.

[114] D. Kutasov and N. Seiberg, Number of degrees of freedom, density of states and tachyons in string theory and CFT, Nucl. Phys. B358 (1991) 600-618.