Basic optics, aerosol optics, and the role of scattering for sky radiance

Helmut Horvath *

University of Vienna, Faculty of Physics, Aerosol and Environmental Physics, Boltzmanngasse 5, 1090 Vienna, Austria

Abstract

The radiance of the night sky is determined by the available light and the scattering properties of the atmosphere (particles and gases). The scattering phase function of the aerosol has a strong dependence on the scattering angle, and depending on the viewing direction different parts of the atmosphere and the ground reflectivity give the most important contribution. The atmospheric radiance cannot be altered by optical instruments. On the other hand the light flux of a distant star increases with the size of the telescope, thus fainter stars become visible. Light extinction, scattering function, atmospheric radiance, ground reflectivity, color effects and others are discussed in detail and a simple theoretical treatment is given.

1. Introduction

In 1666, a remarkable letter written by Adrian Auzout, (French astronomer, 1622–1691) to Monsieur l’Abbé Charles was published as extract in Le Journal des Scavans [3]. The immediate reason was Abbé Charles’ review of a book entitled Ravaglio di nuove osservazioni by Giuseppe Campiani which reports new observations with new telescopes, published in the first issue of the journal. Auzout, being an expert, designer, and user of telescopes has some doubts and puts down some points, which will be investigated in the following.

(1) In relation to their size one cannot see further with bigger telescopes, since bigger telescopes increase the size of the images, but not the light. The aperture of bigger telescopes does not increase in relation to their increase in magnification.

(2) The bigger the telescopes, the more they magnify the vapors, the dust and other small things the air is always full of (les vapeurs, la poussière & les autres petits corps dont l’air est toujours plein), therefore the object looks like seen through a veil.

(3) It will not be possible to see animals on the Moon.

Using today’s nomenclature his theses (1) and (2) would read

(1) Bigger telescopes cannot increase the radiance; too small aperture even reduces the observed radiance.
(2) Light scattering by aerosol particles causes an additional radiance, which reduces contrast (the definition of the aerosol as “the vapors, the dust and other small things the air is always full of” is remarkable, and to the present author’s opinion the first one).

We will discuss these findings in the following, especially with the aspect of light pollution.

2. Animals on the Moon

Auzout’s argumentation about animals on the Moon was done with the knowledge of his time and the knowledge of...
an experienced astronomer. The idea was in the air, to produce telescopes with a length of 1000 or 10,000 ft, the longer the telescope the higher the magnification. Auzout, using his experience, concludes that the maximum length possible would be 400 ft, and then the Moon looks like seen from a distance of 60 miles, and this makes recognition of animals impossible. Since the Moon is at a distance of 38,400 km, the 400 ft. telescope would have magnification of 4000 x. Today's argument would be different, but in 1666 wave optics was unknown, Huyghens wave theory was published in 1678, Newton's Opticks appeared 1705. Refractive index variations in the atmosphere also were not discussed. Now we know, that wave optics requires a minimum diameter of the first aperture such that the angle of the diffraction limit (1.22(l/d)) equals the minimum angular resolution desired. For recognizing the shape of an elephant on the Moon it would be needed to recognize 10 cm at a distance of 384,000 km, thus the minimal angular resolution would be 2.7 x 10^{-10} rad, requiring an aperture (lens/mirror) diameter of 2500 m. Even if a telescope of this size was used, the refractive index variations would make it impossible to see a detail of 10 cm on the Moon.

3. No radiance increase by telescopes

For a human observer, a film, or a CCD, the number of photons per unit time entering the photoreceptor of the eye, interacting with the AgCl crystal of a film, or hitting the charged capacitor of a CCD pixel, is the important magnitude for the signal. The light flux, being the energy per time passing through or emitted by a (real or fictitious) surface solid angle, dω = (dA/A^2), (sr). Solid angle under which a surface dA is seen at distance x, dA is perpendicular to direction x, otherwise the projected surface single scattering albedo, σ = (σ_s/σ_a) = 1 − (σ_a/σ_s), (Dimensionless), Ratio of light scattering coefficient to extinction coefficient.
of the lens. Let us further consider one photoreceptor on the retina having an area of dA and at distance x₁ of a surface emitting light. This can be part of a light source, or an illuminated and reflecting solid, or a fictitious surface. A small part, dA₁, of the surface is imaged on the photoreceptor, see Fig. 1. The aperture of the eye is seen by surface dA₁ at the solid angle dω₁ = (R²π/ω₁²) All the light flux dΦ₁ emitted by surface element dA₁ into this solid angle dω₁ is focused by the eye into the photoreceptor. It is obvious that the light flux is proportional to the solid angle dω₁ and we write dΦ₁ = L₁ dA₁ dω₁; the proportionality constant L₁ is called the radiance of the image within the solid angle dω₁ leaving the image. Obviously then less light reaches the photoreceptor, thus the sensed radiance of the image is less compared to the case considered before.

Let us now consider an observer looking at this image: the light rays leaving the image are contained within the solid angle dω₁; if these rays completely fill the entrance pupil of the eye, all the light flux enters the eye, and the eye senses a surface with radiance L (Fig. 2, left part). But it is also possible that the cone of light fills only part of the eye’s pupil (Fig. 2, right), e.g. caused by a lens with a small diameter and thus a small opening of the cone of light rays leaving the image. Obviously then less light reaches the photoreceptor, thus the sensed radiance of the image is less compared to the case considered before.

The maximum angular subtense of rays leaving an optical instrument can be best characterized by the exit pupil. We can see this by simple geometric optics consideration. A telescope consists basically of a first lens (objective, focal length f₁) and a second lens (eyepiece, focal length f₂). The real image of the object (in infinity) is located in the focal plane of the eyepiece, and viewed through the latter, accommodated to infinity, thus the distance of the two lenses is f₁ + f₂. It is desirable, that all rays entering the objective lens find their way into the pupil of the eye. We consider the first lens simply as a circular stop. This circular hole (the lens mount) is imaged by the second lens at a location behind the lens: its distance from the eyepiece is obtained from the lens equation as f₂(f₁ + f₂)/f₁ and for a long telescope this is close to the focus on the right side (see Fig. 3); we call this the exit pupil and it is the image of the mounting of the first lens. The diameter of the exit pupil is obtained by similarity considerations using the limiting rays through the center of the eyepiece. If D is the diameter of the objective, the diameter d of the exit pupil is d = (D/M) with M the magnifying power of the telescope, M = (f₁/f₂). Obviously all rays entering the objective and refracted into the eyepiece will pass through the exit pupil. Therefore it is desirable to...
position the eye’s pupil at this location, as usually is done in optical instruments. To get optimum light in the eye, the exit pupil should have the same diameter as the eye’s pupil. This is the case for good telescopes. E.g., a binocular with 30 mm lens diameter and 6 × magnification has \( d = 5 \text{ mm} \), which is the size of the eye’s pupil. The 400 in. telescope envisaged by Auzout could have been a Hevelius telescope \[8\]; for a lens diameter of \( 40 \text{ cm} \) and a 4000 \( \times \) magnification, the size of the exit pupil would be 0.1 mm. Thus compared to a good binocular only the fraction \( (0.1/5)^2 = 0.0004 = 4 \times 10^{-4} \) of the radiance of an optimal telescope could be seen, as criticized by Auzout \[3\].

5. Optical properties of aerosols

The atmosphere is illuminated by radiation coming from various sources: sunlight, moonlight, starlight and artificial lights. This light is scattered by molecules and particles in the atmosphere, causing the blue sky, white clouds, colored or colorless haze and smoke, skylow, the colorful sunset, haloes and other optical phenomena.

Light scattering can best be understood by considering the interaction of electromagnetic light waves with molecules/particles. Each molecule can be regarded as a dipole, the electric vector of the light wave causes periodic changes of the dipole moment, making the dipole radiate light in various directions. The radiation has a distinct angular dependence, which is simple for single molecules and complex for particles due to interferences of the scattering by the various dipoles. Furthermore damping of the oscillations takes place, eventually leading to conversion of part of the light’s energy into heat by absorption. Details about the processes of scattering and absorption and calculation methods are subject of textbooks (e.g. \[5\]) and will not be pursued further. Here only the outcome, i.e. optical characteristics of the aerosol/atmosphere will be used in the following.

5.1. Scattering function

An infinitesimal volume element \( dV \) containing gas and particles be illuminated by a parallel beam of light having a flux density, \( S \) (i.e. flux per unit surface). Due to interaction of light with the molecular dipoles, light is scattered in various directions. For the moment we consider the direction given by the scattering angle \( \theta \), which is the angle between the direction of the propagation of light and the direction of interest (see Fig. 4). The flux \( d^2\Phi \) scattered in this direction obviously is proportional to illuminating flux density, \( S \), the size of the volume element, \( dV \), and the solid angle, \( d\omega \), the pencil of scattered rays subtends. Thus we can write \( d^2\Phi = S \chi(\theta) d\omega dV \). The proportionality constant, \( \chi(\theta) \), is called the volume scattering function, its unit is \( \text{m}^{-1} \text{sr}^{-1} \) and is a property of the aerosol/gas. For irregular shaped particles the value of the scattering function, \( \chi(\theta) \), also depends on the orientation of the particle. We keep this in mind but will not state it explicitly in the following. For rotationally symmetric particles or a collective of irregular particles performing Brownian rotation, \( \chi(\theta) \) only depends on the scattering angle.

5.2. Scattering coefficient

So far, only the flux scattered into a cone with the opening \( d\omega \) has been considered. The total flux, scattered by the volume \( dV \) of particles/gas is obtained by integration over all directions, i.e. \( d\Phi = S dV \int_4 \chi(\theta) d\omega \). The expression \( \int_4 \chi(\theta) d\omega = \sigma_s \) is called the scattering coefficient, the unit of \( \sigma_s \) is \( \text{m}^{-1} \). Its meaning is evident immediately: assume the volume element to be a cuboid, extending \( dx \) in the direction of the propagation of light and having a front surface \( dA \) (see Fig. 4, insert). Then \( dV = dA dx \). The light flux entering the volume element from the left is \( \Phi = S dA \) and the total light flux scattered by the volume element is \( d\Phi = S dV \int_4 \chi(\theta) d\omega = S dA dx \sigma_s = \Phi \sigma_s, dx \). The light flux continuing in the same direction and leaving the volume element to the right is reduced by this amount, thus we can write for the flux leaving the volume element: \( \Phi + d\Phi \), with \( d\Phi = -\Phi \sigma_s, dx \). A simple interpretation of the meaning of \( \sigma_s \) is possible by this equation: the scattering coefficient is the relative reduction of light flux \( d\Phi/\Phi \) per unit length. The solution of this differential equation is the well known exponential attenuation law, for parallel light passing a distance \( x \) through an aerosol, \( \Phi_{\text{out}} = \Phi_{\text{in}} \exp(-\sigma_s x) \), see Fig. 4.

5.3. Phase function

The volume scattering is an extensive property, as well as its integral, the scattering coefficient, i.e. it is additive for independent, non-interacting systems, and the property is proportional to the amount of material (particles, molecules) present in the considered volume. An aerosol containing twice as many particles has twice the volume scattering function. Dividing the volume scattering function by its integral (the scattering coefficient) makes it independent of particle number. Usually a factor of \( 4\pi \) is added, and the intensive physical property \( P(\theta) = \sigma_s (\chi(\theta)/\sigma_s) \) is called the phase function. Its unit is \( \text{sr}^{-1} \). The addition of the factor of \( 4\pi \) is a matter of beauty, for an isotropic scattering medium the phase function is \( 1 \text{ sr}^{-1} \). Some authors use the definition without the factor of \( 4\pi \). Examples for phase functions for aerosols consisting of different particle sizes are shown in Fig. 5.
not absorb light, the extinction coefficient equals the scattering coefficient.

5.5. Single scattering albedo, absorption number

Scattering coefficient, absorption coefficient and extinction coefficient are extensive properties. In order to be independent of the number/mass of particles, the intensive property single scattering albedo, \( \sigma_s = (\sigma_a / \sigma_e) = 1 - (\sigma_a / \sigma_e) \), is used. It is dimensionless and has a value of 1 for a purely scattering medium and 0 for a purely absorbing medium. For a slightly absorbing aerosol \( \sigma_s = 0.95 - 0.85 \), black smoke can have \( \sigma_s = 0.6 - 0.5 \), for an aerosol consisting of graphite nanoparticles with 20 nm diameter, it is \( \sigma_s = 0.00163 \) at a wavelength of 550 nm. For completeness the absorption number \( a \) is also mentioned: It is the difference of \( \sigma_s \) to 1, i.e. \( a = 1 - \sigma_s \).

5.6. Relations between the optical properties of the aerosol

The following relations can easily derived from the above definitions \( \sigma_a = \sigma_s \sigma_e \), \( \sigma_a = a \sigma_e \), \( \gamma(\theta) = \sigma_s \sigma_e (1/4 \pi) P(\theta) \).

5.7. Wavelength dependence

It has not been mentioned explicitly, but all optical properties depend on wavelength. The wavelength dependence is a function shape, refractive index, material, mixture and size distribution of the particles. Very frequently the extinction coefficient or scattering function of the particles is larger for blue than for red light, causing red sunset, but exceptions exist, like observations of a blue sun [6], green or purple colors (e.g. during volcanic eruptions, savannah fires, sand storms [9]). A wavelength dependence of the extinction coefficient observing a power law, \( \sigma_e \propto (\lambda / \lambda_0)^{-\alpha} \), has been first reported by Ångström [12]. The reference wavelength, \( \lambda_0 \), usually is 1 \( \mu \)m. The Ångström exponent \( \alpha \) is \(+4\) for the Rayleigh scattering coefficient and the scattering function of gases and particles smaller than 20 nm in diameter, it has values between 1 and 2 for the usual atmospheric aerosol and has no wavelength dependence for particles larger than a few micrometers (e.g. fog or cloud droplets). Junge [7] has found a theoretically funded relation between \( \alpha \) and characteristics of the size distribution; a simple rule of thumb is the smaller the particles, the larger the Ångström exponent \( \alpha \).

Since the single scattering albedo and the phase function are obtained by division of two extensive properties, their wavelength dependence is less distinct.

6. Radiance due to aerosol light scattering

An aerosol being illuminated by light will scatter part of this light and an observer will note some radiance. This radiance can be obtained easily using the illuminating flux density and the volume scattering function, see Fig. 6. We will consider a volume element \( dV \) of aerosol, being a cube of thickness \( dx \) and a front surface of \( dA \), thus \( dV = dA \ dx \). The direction of sight of an observer has an angle of \( \psi \) with the perpendicular of the front surface. The angle between the direction of illumination and observation is the scattering angle \( \theta \). Using the definition of the
volume scattering function, the flux scattered into the solid angle $d\omega$ is $d^2\Phi = S(\theta) d\omega$. Using the definition of radiance, and considering that the surface $dA$ has angle of $\psi$ with the direction of observation, the radiance $dL$ of the volume element is obtained as $dL = (d^2\Phi / dA 	imes \cos \psi) = S(\theta) d\omega / \cos \psi$. This radiance is independent of the size of $dA$ of the front surface and only depends on illuminative flux density, volume scattering function, thickness of the lamina and the angle between the direction of observation and the considered direction of the front surface. Using the above relations between volume scattering function, phase function, scattering coefficient extinction coefficient and single scattering albedo, the above formula can also be written as $dL = S(1/4\pi)\varphi(\theta)(\sigma_{se} \times dx / \cos \psi)$. Note that no assumptions on the orientation of the cuboid volume element have to be made. The expression $(dx / \cos \psi)$ can interpreted as the extent of the lamina into the direction of observation. If this volume element is observed from a distance $x$, the attenuation by the aerosol by $\exp(-\sigma_{se}x)$ has to be taken into account.

For many considerations the layers are assumed to be parallel to the Earth's surface. The angle $\psi$ then is the zenith distance, $z$, of the direction of observation, see Fig. 6, right side. Let us write $h$ for the vertical height and for abbreviation $\cos z = \mu$. Then $\sigma_{se} \times dx = (\sigma_{se} \times dh / \cos z) = (\sigma_{se} \times dh / \mu) = (\delta \times \mu)$, therefore $dL = S(1/4\pi)\varphi(\theta)(\sigma_{se} \times \delta / \mu)$.

7. Flux density caused by radiance

A surface having the radiance $L$ is assumed to illuminate another surface. The flux density through the latter can be calculated easily, using the definition of the radiance. In atmospheric optics many layers contribute to the radiance e. g. of the sky and then it is difficult to imagine the location of the surfaces from which the radiation comes. In the following a relation between radiance and flux density will be given which is independent of the locations of the emitting surfaces.

First consider two surfaces $dA$ and $dA'$, see Fig. 7, being the distance $R$ apart. The straight line between $dA$ and $dA'$ has the angles with the respective perpendiculars to the surfaces of $\eta$ and $\eta'$. Surface $dA'$ emits radiation (radiance $L$) towards $dA$ at an angle of $\eta$. When looking from $dA'$, the surface $dA$ extends the solid angle $d\omega' = (dA \cos \eta / R^2)$. The flux $d^2\Phi$ reaching $dA$ is obtained by $d^2\Phi = L 	imes dA' \times \cos \eta' \times d\omega'$ and inserting $d\omega'$ yields $d^2\Phi = (L 	imes dA' \times \cos \eta' \times dA \cos \eta / R^2)$. The flux density through $dA$ is obtained by dividing through the projected area $dA \times \cos \eta$, thus $dS = (L \times dA' \times \cos \eta / R^2)$. This result can be interpreted in the following way. The emitting surface $dA'$ is seen from $dA$ at a solid angle $d\omega = (dA' \times \cos \eta / R^2)$, therefore $dS = L \times d\omega$. This result does neither contain the distance to the surface $dA'$, nor its size. The radiance $L$ within a solid angle $d\omega$ thus produces a flux density of $dS = L \times d\omega$. 

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**Fig. 6.** Radiance of a lamina of aerosol. Left: the volume element is illuminated by a flux density $S$. The light is scattered at an angle $\psi$ with the perpendicular of the front surface. The scattering angle $\theta$ is between the incident illumination and the considered direction of the scattered light. Right: usual orientation of laminae and zenith distance/angle.

**Fig. 7.** Flux density caused by radiance: Left: a surface $dA'$ sends a radiance $L$ at an angle $\eta'$ perpendicular towards surface $dA$. The surface $dA$ is seen from $dA'$ at a solid angle $d\omega'$. From $dA$ the surface $dA'$ is seen at a solid angle $d\omega$. **
8. Reflectance of surfaces

A surface illuminated by light can reflect light both specular and diffuse light. Except for extremely glossy surfaces, the diffuse reflection is responsible for the majority of the reflected light. A complete description of the reflectivity is characterized by the bidirectional reflectance [11,12]. A surface is illuminated by light of a flux density $S$, coming from a direction characterized by azimuth and zenith angle. When observed in direction also given by its (different) azimuth and zenith angle this surface has a radiance of $L$. The ratio of $L$ and $S$ is called the bidirectional reflectance and is a function of the four angles. This completely describes the reflective properties of the surface, any reflectance data can be obtained from the hemispheric distribution of incident radiation and knowledge of bidirectional reflectance. Integral properties such as albedo can be derived by simple mathematic operations. Values for a few surfaces (sand, red soil, turf) can be found in Overington [11].

A very simple and idealized reflector is a perfectly diffusing white surface (Lambertian reflector). This can be experimentally easily produced by coating a surface with freshly prepared magnesium oxide. It is characterized by two facts: (1) all light energy is conserved and (2) the radiance of the surface is independent of the direction of observation. This surface produces a radiance $L_0$, independent of the angle $z$ (see Fig. 8). Using the definition of radiance, $L_0 = (d^2 \phi_0)/(dA dz \cos \psi)$, the flux emitted into the solid angle $d\omega$ is $d^2 \phi = L_0 dA d\omega \cos \psi$. For reasons of rotational symmetry the solid angle $d\omega$ for all directions in the angular range $[z, z+dz]$ is $d\omega = 2\pi \sin z \, dz$. Integrating over all possible angles $z$ yields the total emitted flux $d\Phi = L_0 dA 2\pi \int_0^{12} \sin z \cos z \sin z \, dz = L_0 dA \pi$, or $L_0 = (1/\pi) (d\phi/dA)$. The expression $(d\phi/dA)$ is the total emitted flux per area of the diffusing surface, which equals the incoming flux density, $S$, due to conservation of light energy, thus $L_0 = (1/\pi)S$.

9. Example for radiance of an illuminated layer

With the knowledge of the previous chapters let us solve the following problem (see Fig. 9):

**Problem:** Determine the radiance of a 1 m thick horizontal aerosol layer, located at an altitude of 1 km, which is irradiated by the light of a 1 m$^2$ horizontal surface, being a Lambertian diffuse reflector with an albedo of 0.5. This surface is illuminated by light with flux density of 1 W m$^{-2}$ at 45$^\circ$. Light from the diffuse reflector radiates to the layer under consideration at 30$^\circ$ zenith distance. The observation is at a zenith distance of 45$^\circ$. The atmospheric extinction coefficient is $\sigma_e = 0.2 \text{ km}^{-1} = 2 \times 10^{-4} \text{ m}^{-1}$, (20 km visibility), the phase function of the aerosol is the one shown in Fig. 5 for particles with a diameter of 600 nm. The particles are slightly absorbing having a single scattering albedo of $\omega = 0.95$.

**Solution:** Since the illumination is at 45$^\circ$, the flux density onto the surface is $S = 1 \text{ W m}^{-2} \cos 45^\circ = 0.707 \text{ W m}^{-2}$. The radiance of the ideal diffuse surface with albedo $A$ is $L_0 = (A/\pi)S$, thus $L_0 = 0.113 \text{ Wm}^{-2} \text{ sr}^{-1}$. The considered lamina is at a distance of $x_1 = 1 \text{ km} / \cos 30^\circ = 1155 \text{ m}$, and attenuation takes place by a factor $\exp(-\sigma_e x_1) = 0.794$, thus the radiance at the location of the layer is $L_1 = 0.0893 \text{ W m}^{-2} \text{ sr}^{-1}$. The diffuse surface is seen from the 1 m layer at 1000 m height under a solid angle of $d\omega = ((1 \text{ m}^2 \cos 30^\circ) / (1155 \text{ m})^2) = 6.49 \times 10^{-7} \text{ sr}$, therefore the flux density $S$ at this location is $S = L_1 \frac{d\omega}{d\psi} = 5.80 \times 10^{-5} \text{ W m}^{-2}$. The scattering angle $\psi$ is the angle between the transmitted beam from the diffuse surface and the direction from the observer is $\theta = 105^\circ$. From Fig. 5 we obtain the phase function for particles with 600 nm diameter at 105$^\circ$ as $P(105^\circ) = 0.146 \text{ sr}^{-1}$. The radiance of the 1 m layer at 1000 m thus is

$$dl_2 = \frac{S}{4\pi P(\theta) \omega_e dx} \frac{d\omega}{d\psi} \cos z \times 5.80 \times 10^{-8} \text{ W m}^{-2} \frac{1}{4\pi} \frac{0.146 \text{ sr}^{-1}}{\cos 45^\circ} \times 0.95 \times 2 \times 10^{-4} \text{ m}^{-1} \times 1 \text{ m} \cos 45^\circ = 1.81 \times 10^{-13} \text{ W m}^{-2} \text{ sr}^{-1}$$

This radiance is attenuated on its way to the observer by $\exp(-\omega x_2)$ with $x_2 = (1000 \text{ m} / \cos 45^\circ) = 1414 \text{ m}$, thus...
the radiance, \( dl \), seen by the observer is \( dl = 1.364 \times 10^{-13} \text{ W m}^{-2} \text{ sr}^{-2} \).

The total radiance, the observer sees, is the sum of the contributions of the radiances of all layers. Furthermore the flux densities by all illuminated surface elements on the ground have to be considered. This means three integrations, one with respect to the height and two with respect to the surface. For this the characteristics of the aerosol at all locations and the bidirectional reflectance of the ground and the hemispheric distribution of radiation to the ground has to be known. All this is difficult to assess, so simplified assumption for models need to be used. Furthermore the scattered light within the atmosphere causes additional radiance, so radiative transfer methods are applied to determine the multiple scattered light.

10. Influence of the phase function and the extinction coefficient

The light scattered by a volume element is determined by the scattering function of the aerosol, the illuminating light, and the attenuation in the atmosphere. The radiance \( L_0 \) be the radiance on the ground caused by light sources and reflections from the ground. Let us consider two extreme cases and neglecting multiple scattering:

1. **Radiance by the aerosol directly above the light sources**. This would be the case when looking overhead at the sky in the center of an illuminated town. A schematic is shown in Fig. 10, left side. The radiance from the ground is attenuated by \( \exp(-\sigma_x x) \) as well as the scattered radiation. The observed radiance at the ground of a layer of aerosol at a distance \( x \) is

\[
\frac{dl}{dx} = L_0 \int_{4\pi} P(\theta) d\omega \frac{\sigma_x}{\cos z} dx \exp(-2\sigma_x x)
\]

The scattering angle is close to 180°. Looking at Fig. 5, one can see that the phase function has a low value for typical atmospheric aerosols (e.g. particles with diameters between 400 and 600 nm) thus the contribution by the aerosol is low. This is also the case for micrometer sized irregular shaped particles. In contrary the phase function for pure air is higher by a factor of 10, and even for the lower extinction coefficient of the gas molecules, the contribution to the sky radiance is considerable.

The extinction coefficient of the aerosol is wavelength dependent and we assume the Ångström relation \( \sigma_x = 0.2 \text{ km}^{-1} (\lambda/550 \text{ nm})^{-1.5} \) for the aerosol. The concentration of aerosol particles decreases with altitude. The scale height, being the altitude, at which the concentration decreases by a factor of \( e^{-1} \), is a simple characterization of this decrease. A usual value for the scale height is 1 km, so we assume the layer at an altitude of 1 km. Since we look at the aerosol directly above, \( \cos z = 1.0 \). As first approximation \( P(\theta) \) is independent of wavelength. The wavelength dependence of \( dl \) is determined by the wavelength dependence of the extinction coefficient \( \sigma_x(\lambda) \) and the dependence of \( \exp(-2\sigma_x x) \). For \( x=1 \text{ km} \) this exponential function is 0.525 and 0.754 at 400 and 700 nm, i.e. the transmission for blue light is a factor of 0.693 lower for red light. The extinction coefficients at the two wavelengths are \( 3.22 \times 10^{-4} \text{ m}^{-1} \) and \( 1.39 \times 10^{-4} \text{ m}^{-1} \). The product of the exponential function and the extinction coefficient yields \( 1.69 \times 10^{-4} \text{ m}^{-1} \) and \( 1.05 \times 10^{-4} \text{ m}^{-1} \) at 400 nm and 700 nm respectively. This means that for the light scattered by the aerosol the blue light is 60% larger than red light in comparison to the light sources. Thus there is a slight blue shift in the color of the scattered skylight.

2. **Radiance by the aerosol of a distant light source**. A schematic is shown in Fig. 10, right side. Let us assume the light from ground and the observer are 20 km apart and again a layer at a height of 1 km is considered. Then \( x=10.05 \text{ km} \) and \( \exp(-2\sigma_x x) \) has values of 0.0015 and 0.061 at 400 and 700 nm, thus the transmission for blue light is only 2.52% of the one for red light. The higher scattering coefficient in the blue cannot compensate this. The product of the exponential and the extinction coefficient yields \( 4.94 \times 10^{-7} \text{ m}^{-1} \) and \( 8.47 \times 10^{-8} \text{ m}^{-1} \) at 400 nm and 700 nm respectively, i.e. the radiance of the red light is 17 times more than for the blue light, thus a considerable red shift occurs, which is a well known fact for sky glow from distant sources. Due to the strong attenuation of the light one might have the impression, that this effect is small, but it must be considered, that for forward scattering at an angle of 11.4° the phase...
function is 60 times larger than for backscattering, thus the radiance is even higher compared to looking to the zenith in the town.

The pattern of the phase function (Fig. 5) strongly depends on particle size and shape. For micrometer sized spherical particles there is a considerable increase in backscattering which is not the case for particles with irregular shape, which are more likely. Thus except for micrometer sized drops the backscattering is small. The extinction coefficient of air at standard conditions is 0.0116 km\(^{-1}\) at 550 nm. Because of the symmetry, the phase function for backscattering is 7.5 times higher than for the 600 nm particles, thus despite of the much smaller extinction coefficient backscattering can be comparable to the aerosol particles in clean air conditions.

11. Atmospheric radiance and visibility of stars

The scattering of light in the atmosphere causes a radiance which is determined by many factors as discussed above, and this radiance is omnipresent in the atmosphere and cannot be altered by any telescope. When looking at a star we see this atmospheric radiance as background and the light coming from the star. If the light coming from the star causes a higher stimulus in the photoreceptor than the background, the star is visible. Physiological investigations have revealed, that a difference in stimuli is recognized, if the contrast between an “object” and its background exceeds the threshold value \(\varepsilon\). In Fig. 11 the definition of the contrast is shown schematically. The threshold value \(\varepsilon\) amounts to 0.005 under optimum daylight viewing conditions and increases for the dark adapted eye [4].

When looking at a star, an image is formed on the retina. Applying geometric optics, the image of a-centauri, having a diameter of 1.707 \(\times\) 10\(^{16}\) m at a distance of 4.016 \(\times\) 10\(^{16}\) m, would be 0.85 nm for the 20 mm focus of the eye’s lens. Such a small image is not possible, since the eye’s image is diffraction limited; the Airy disk is about 3 \(\mu\)m, in agreement with the size of the receptors. Even if the telescope of Mt. Palomar with \(f = 16.8\) m was used, the image would be 714 nm, which is also smaller than its diffraction limit being 2.2 \(\mu\)m for the mirror of 5.1 m diameter. Thus we can conclude that under optimum conditions the image of a star falls on one photoreceptor.

When looking at the sky having a radiance \(L_{\text{sky}}\) and the photoreceptors receive a flux proportional to this value, let us call it \(\phi_{\text{sky}}\). The photoreceptor on which the image of the star is formed receives an additional flux thus in total \(\phi_{\text{sky}} + \phi_{\text{star}}\). Therefore the contrast between the photoreceptor seeing the sky and the one seeing the star is \(C = (\phi_{\text{star}} \div \phi_{\text{sky}})\). If this contrast is larger than the threshold the star is visible. Let us assume a star for which \(C = \varepsilon\), i.e. the star is just visible. If the sky radiance increases, due to light pollution, the denominator of \(C = (\phi_{\text{star}} \div \phi_{\text{sky}})\) gets bigger and the contrast falls below the threshold, the star is invisible. When a telescope with a larger aperture is used, more light from the star, is imaged on the photosensor. The surrounding sensors still receive the same flux, since no optical instrument can increase the radiance, thus smaller stars are visible.

Let us call the \(S_{\text{star}}\) the flux density of the star, then \(\phi_{\text{star}} = S_{\text{star}} R^2 \pi\) with \(R\) the radius of the aperture of the telescope. Let us compare two telescopes with aperture radii \(R_1\) and \(R_2\), being able to just observe stars with flux densities \(S_{\text{star1}}\) and \(S_{\text{star2}}\). This means that the contrast to the sky radiance for both telescopes is the threshold, \((S_{\text{star1}} R^2 \div \phi_{\text{sky}}) = \varepsilon\) or \((S_{\text{star1}} \div S_{\text{star2}}) = (R_2^2 \div R_1^2)\). The difference in magnitude of stars with flux densities \(S_{\text{star1}}\) and \(S_{\text{star2}}\) is \(\Delta \text{Mag} = -2.5 \log (S_{\text{star2}} \div S_{\text{star1}})\), therefore the difference in magnitude of the faintest observable stars for the two telescopes is \(\Delta \text{Mag} = -5 \log (R_1 \div R_2)\). For the unaided eye having a pupil diameter of 5 mm, the smallest observable magnitude is \(\text{Mag} = 6\); a telescope with a diameter of the mirror of 5.1 m should be able to observe stars down to a magnitude of \(6 - \log(2.5 \times 10^{-3} \text{ m}^2 \div 2.55 \text{ m}) = 21.04\), assuming the same observation conditions.

12. Conclusions

The radiance of the sky is determined by (1) the illumination which is caused by the various light sources and their reflections from surfaces, (2) the scattering properties of the particles and molecules in the atmosphere and (3) the attenuation of light in the atmosphere. Determining the sky radiance in principle is straightforward, but requires the knowledge of many parameters: the emissive properties of the light sources, the reflective properties of objects, the scattering and extinction properties of the atmosphere, all of them vary in space and time. This information is almost impossible to assess completely, therefore we have to rely on models, which are continuously improved.

As already known in 1666 the radiance cannot be increased by telescopes, which also holds for the atmospheric radiance. Using telescopes with larger apertures gets more light from the stars, but still the same radiance form the atmosphere, thus fainter stars can be seen.

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