Remarks on the nonvanishing of cohomology groups for perverse sheaves on abelian varieties

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Let \( X \) be an abelian variety over an algebraically closed field \( k \) of dimension \( g \) and let \( K \) be an irreducible perverse sheaf in \( D_b^b(X, \Lambda) \) for \( \Lambda = \mathbb{Q}_\ell \). If the base field \( k \) has positive characteristic, we assume that \( K \) is defined over a field that is finitely generated over its prime field with \( \ell \) different from the characteristic. Suppose that not all cohomology groups \( H^\nu(X, K) \) are zero and let denote \( d(K) = \max\{ \nu \mid H^\nu(X, K) \neq 0 \} \). Notice \( d(K) \geq 0 \), by the Hard Lefschetz Theorem.

**Theorem.** For \( d = d(K) > 0 \) we have

\[
\dim_\Lambda(H^{d-1}(X, K)) > 2d/(d + g) \cdot \dim_\Lambda(H^d(X, K)).
\]

If furthermore \( X \) is a simple abelian variety, then \( \dim_\Lambda(H^{d-1}(X, K)) > d \cdot \dim_\Lambda(H^d(X, K)) \).

**Remark.** By the Hard Lefschetz Theorem an immediate consequence of this theorem is the assertion: \( H^\nu(X, K) \neq 0 \) if and only if \( \nu \in [-d(K), d(K)] \). So for character twists \( K_\chi \) [KrW] the sets \( V_i(K) = \{ \chi \mid H^i(X, K_\chi) \neq 0 \} \) satisfy \( V_{i+1}(K) \subseteq V_i(K) \) for all \( i \geq 0 \). For an arbitrary projective smooth variety \( Y \) over \( k \) with Albanese morphism \( f : Y \to X \) and a perverse sheaf \( L \) on \( Y \) the decomposition theorem gives \( R^nf_*(L) \cong \bigoplus i \in \mathbb{Z} H^i(R^nf_*(L))[-i] \) and \( H^\nu(Y, L) \cong \bigoplus j+i=\nu H^j(Y, \mathbb{Q} \cdot R^nf_*(L)) \).

From the relative Hard Lefschetz Theorem and the theorem above applied to the irreducible constituents \( K \) of the semisimple perverse cohomology sheaves \( \mathbb{Q} \cdot R^nf_*(L) \) we therefore obtain

**Corollary 1.** Let \( L \) be an irreducible perverse sheaf \( L \) on a smooth projective variety \( Y \) with \( d = d(K) > 0 \). Suppose the Albanese morphism \( f : Y \to X \) is not trivial and suppose \( H^d(Y, L) \neq H^0(Y, \mathbb{Q} \cdot R^nf_*(L)) \) (e.g. this is the case if the fibers of \( f \) have dimension \( < d \)). Then \( H^\nu(Y, L) \neq 0 \) if and only if \( \nu \in [-d, d] \).

**Proof of the theorem.** First suppose that \( K \) is negligible, i.e. of the form \( K \cong \pi^*(\mathbb{Q})[q] \) for a perverse sheaf \( Q \) on a quotient abelian variety \( \pi : X \to X/A \) defined by an abelian subvariety \( A \subseteq X \) of dimension \( q > 0 \). Then \( d = d(K) = d(Q) + q \) since \( H^\nu(X, K) \cong \bigoplus_{i=0}^{2q} H^\nu(X/A, Q[i + q]) \).

Hence \( H^d(X, K) \cong H^{d(Q)}(X/A, Q) \) and \( H^{d-1}(X, K) \cong 2q \cdot H^{d(Q)}(X/A, Q) \). Since \( 2q > 2d/(d + g) \), our claim follows in this case; similarly \( 2q = 2g > d \) in the case where \( X = A \) is simple. Therefore we now make the

**Assumption.** Suppose \( K \) is irreducible, but not negligible. Furthermore suppose \( d > 0 \).

For the perverse sheaf \( K \) on \( X \) consider the Laurent polynomial \( h_\nu(X, K) = \sum a_\nu t^\nu \) defined by \( a_\nu = \dim_\Lambda(H^\nu(X, K)) \). Then \( d = d(K) \) is the largest integer \( \nu \) such that \( a_\nu \neq 0 \).

Choose an integer \( r \) minimal such that \( r \cdot d > g \). Hence \( r > 1 \) and \( r \cdot d < g + d \). The \( r \)-th convolution power of \( K \) is a direct sum of a perverse sheaf \( K_r \) on \( X \) and a finite direct sum of complexes \( L_\mu[n_\mu] \) with negligible perverse sheaves \( L_\mu \) on \( X \) of the form:
• \( L_\mu = \pi^*_\mu(Q_\mu)[g_\mu] \) for irreducible not negligible perverse sheaves \( Q_\mu \) on \( X/A_\mu \)
• \( \pi_\mu : X \to X/A_\mu \) is the quotient by an abelian subvariety \( A_\mu \) of \( X \) of dimension \( g_\mu > 0 \).

This follows from [KrW], [W], and for this assertion we have to assume that the perverse sheaf \( K \) is defined over a finitely generated field over the prime field in the case of positive characteristic [W].

Then \( h_t(X, L_\mu[n_\mu]) = \sum \dim(H'^\nu(X, L_\mu[n_\mu]) \cdot t'^\nu = \sum_{\nu < d_\mu} b_{\mu\nu} t'^\nu \) for integers \( b_{\mu\nu} \geq 0 \), and we may assume \( b_\mu = b_{\mu d_\mu} \geq 1 \) since we can ignore cohomologically trivial summands in the following. Let \( T \) denote the set of all indices \( \mu \) such that \( d_\mu + g_\mu = r \cdot d \) holds. By well known cohomological bounds [BBD], the cohomology of an irreducible perverse sheaf on \( X \) vanishes in degrees \( g \) unless it is negligible. Since \( r \cdot d \geq g \), the Künneth formula in the form \( H^*(X, K^{r\nu}) \cong H^*(X, K)^{\otimes r} \) and a comparison of coefficients at \( t^{rd} \) implies

\[
(a_d)^r = \sum_{\mu \in T} b_\mu .
\]

Similarly, now using \( r \cdot d \geq g + 1 \), by comparing coefficients at \( t^{rd-1} \) we obtain

\[
r \cdot a_{d-1}(a_d)^{r-1} \geq \sum_{\mu \in T} 2g_\mu b_\mu \geq 2 \cdot \min \{g_\mu\} \cdot (a_d)^r .
\]

Indeed, the second equality follows from the formula \( \sum_{\mu \in T} b_\mu = (a_d)^r \) above. For the first inequality we exploited the fact that all coefficients \( b_{\mu\nu} \) in \( h_t(X, L_\mu[n_\mu]) = (t+2+t^{-1})^{g_\mu} \cdot h_t(X/A_\mu, Q_\mu[n_\mu]) = (t^{g_\nu} + 2g_\mu t^{g_\nu-1} + \cdots)(b_\mu t^{d_\mu} + \cdots) \) are nonnegative. We conclude

\[
a_{d-1} \geq \frac{2 \min \{g_\mu\}}{r} \cdot a_d \geq \frac{2}{r} \cdot a_d \geq \frac{2d}{g + d} \cdot a_d .
\]

where the last inequality follows from \( r \cdot d < g + d \). If \( X \) is simple, then \( \min \{g_\mu\} = g \) and hence \( a_{d-1} \geq \frac{2}{d} a_d \). Now \( r \cdot d < g + d < 2g \) implies \( a_{d-1} > d \cdot a_d \). QED

**Remark.** \( d(K) \) for the intersection cohomology sheaf \( K \) of an irreducible subvariety \( Y \) of \( X \) is the dimension of \( Y \). In this case there exist stronger geometric estimates than those from the theorem above. However, already when \( Y \) is a variety of maximal Albanese dimension and \( K \) is an arbitrary irreducible constituent of the direct image of the intersection cohomology perverse sheaf on \( Y \) under the Albanese morphism \( f: Y \to X = \text{Alb}(Y) \) I am not aware of estimates of the above form in the literature.

Next, consider a finite Galois morphism

\[
\pi : \tilde{Y} \to Y
\]

between smooth complex varieties of dimension \( n \) with Galois group \( \Gamma \), where we view \( \Gamma \) to act on \( \tilde{Y} \) from the right. For every isomorphism class \( \phi \) of irreducible representations \( V_\phi \) of \( \Gamma \) let \( m_\phi(\phi) \) denote the multiplicity of the irreducible representation \( \phi \) of \( \Gamma \) on \( H^{\nu+n}(\tilde{Y}, \mathbb{C}) \).

For simplicity, from now on suppose that \( Y \) is projective and \( f : Y \to \text{Alb}(Y) = X \) is a closed embedding. Then the theorem above implies

**Corollary 2.** If \( d = d(K_\phi) > 0 \), then \( m_{d-1}(\phi) > 2dm_\phi(\phi)/(d + g) > 0 \).
Proof. For every class \( \phi \) there exists an irreducible perverse sheaf \( K_\phi \) on \( Y \) and a \( \Gamma \)-equivariant isomorphism \( H^{*-n}(Y, C) \cong \bigoplus_{\phi} V_{\phi} \otimes_C H^*(Y, K_\phi) \), where \( \Gamma \) acts on \( V_{\phi} \) by \( \phi \) and trivially on \( H^*(Y, K_\phi) \). For unramified \( \pi \), this immediately follows from [KiW], remark 15.3 (d). Applying this remark for the restriction of \( \pi \) to \( \pi^{-1}(U) \), for the open dense subset \( U \subseteq Y \) obtained by removing the ramifications divisor of \( \pi \), by perverse analytic continuation in general it suffices to observe that for \( \delta_Y = C_Y[n] \) the semisimple perverse sheaf \( \pi_*(\delta_Y) \) on \( Y \) has irreducible perverse constituents \( K \) whose restriction to \( U \) are nontrivial. To show this notice that \( Hom(\pi_*(\delta_Y), K) = Hom(\delta_Y, \pi^!(K)) \) vanishes if \( K \) (and hence \( \pi^!(K) \)) is a perverse sheaf with support of dimension \( < \dim(Y) \). Indeed, since \( \delta_Y \) is an irreducible perverse sheaf with support of dimension \( \dim(Y) \), \( Hom(\delta_Y, \pi^!(K)) \) is zero. This being said, we obtain \( m_{\nu, \phi}(\phi) = \dim(H^p(Y, K_\phi)) \). Since \( f \) is a closed immersion, the direct images of \( K_\phi \) under the Albanese morphism again are irreducible perverse sheaves. So we can apply the theorem. QED

Still suppose \( \pi : \tilde{Y} \to Y \) is a Galois covering and \( f : Y \to Alb(Y) \) is a closed embedding. Since \( \chi(Y, K_\phi) = \sum_{\nu} (-1)^{\nu} \dim(H^\nu(Y, K_\phi)) \), for \( \gamma \in \Gamma \) the trace \( tr(\gamma) = \sum_{\nu} (-1)^{\nu} tr(\gamma; H^\nu(Y, \delta_Y)) \) can be written

\[
tr(\gamma) = \sum_{\phi} \chi(Y, K_\phi) \cdot tr(\gamma; V_{\phi}).
\]

\( K_\phi \) has rank \( \dim(V_{\phi}) \) on \( U \), thus generic rank \( \dim(V_{\phi}) \) on \( Y \). Hence the characteristic cycle of the D-module on \( Alb(Y) \) attached to \( f_\phi(K_\phi) \) is a sum of irreducible Lagrangians cycles containing the conormal Lagrangian cycle \( \Lambda_{f(Y)} \subset T^*(X) \) with multiplicity \( \dim(V_{\phi}) \). As shown in [FK], by the theorem of Dubson-Riemann-Roch this implies \( \chi(K_\phi) = \chi(f_\phi(K_\phi)) \geq \dim(V_{\phi}) \cdot deg(\Lambda_{f(Y)}) \).

Furthermore since \( f(Y) \cong Y \) is smooth, the characteristic variety of \( f(Y) \) is \( \Lambda_{f(Y)} \) and hence \( deg(\Lambda_{f(Y)}) \) is the Euler-Poincare characteristic \( \chi_Y \) of the variety \( Y \), again by [FK]. This implies

\[
tr(\gamma) = \sum_{\nu} (\dim(V_{\phi})\chi_Y + a_{\phi}) \cdot \phi(\gamma)
\]

for certain integers \( a_{\phi} \geq 0 \). Hence the virtual representation defined by \( tr(\gamma) \) is \( \chi_Y \) times the regular representation of \( \Gamma \) plus a true representation of \( \Gamma \). Notice that \( \chi_Y \geq 0 \) holds by [FK] and our assumptions on \( f \).

Remark. In the case of surfaces \( Y \), for a nontrivial irreducible representation \( \phi \) of \( \Gamma \) from this Chevalley-Weil type trace formula we obtain the estimate \( m_0(\phi) - 2m_1(\phi) = \dim(V_{\phi})\chi_Y + a_{\phi} \geq 0 \). So, for surfaces and nontrivial \( \phi \) under the assumptions before corollary 2, this improves the previous estimate \( m_0(\phi) \geq 2m_1(\phi)/(g + 1) \) of corollary 2.

References:

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