CONSTANT MEAN CURVATURE SURFACES IN
SUB-RIEMANNIAN GEOMETRY

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Abstract. We investigate the minimal and isoperimetric surface problems in a large class of sub-Riemannian manifolds, the so-called Vertically Rigid spaces. We construct an adapted connection for such spaces and, using the variational tools of Bryant, Griffiths and Grossman, derive succinct forms of the Euler-Lagrange equations for critical points for the associated variational problems. Using the Euler-Lagrange equations, we show that minimal and isoperimetric surfaces satisfy a constant horizontal mean curvature conditions away from characteristic points. Moreover, we use the formalism to construct a horizontal second fundamental form, $II_0$, for vertically rigid spaces and, as a first application, use $II_0$ to show that minimal surfaces cannot have points of horizontal positive curvature and, that minimal surfaces in Carnot groups cannot be locally horizontally geometrically convex. We note that the convexity condition is distinct from others currently in the literature.

1. Introduction

Motivated by the classical problems of finding surfaces of least area among those that share a fixed boundary (the minimal surface problem) and surfaces of least area enclosing a fixed volume (the isoperimetric problem), several authors have recently formulated and investigated similar problems in the setting of sub-Riemannian or Carnot-Carathéodory spaces. In particular, N. Garofalo and D.M. Nhieu in [11] laid the foundations of the theory of minimal surfaces in Carnot-Carathéodory spaces and provided many of the variational tools necessary to make sense of such a problem. Building on this foundation, Danielli, Garofalo and Nhieu, [7], investigated aspects of minimal and constant mean curvature surfaces in Carnot groups. Among many other results, these authors showed the existence of isoperimetric sets and that, when considering the isoperimetric problem in the Heisenberg group, if one restricts to a set of surfaces with a generalized cylindrical symmetry, then the minimizers satisfy an analogue of the constant mean curvature equation. In this setting, the authors identify the absolute minimizer bounding a fixed volume and show that it is precisely the surface that Pansu conjectured to be the solution to the isoperimetric problem [17]. Further in this direction of the isoperimetric problem, Leonardi and Rigot, [15], independently showed the existence of isoperimetric sets in any Carnot group and investigated some of their properties. Leonardi and Masnou, [13], investigated the geometry of the isoperimetric minimizers in the Heisenberg group and also showed a version of the result in [7] showing the among sets with a cylindrical symmetry, Pansu’s set is the isoperimetric minimizer.

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In addition to this more general work in Carnot groups and Carnot-Carathéodory spaces, a great deal of work has been done on the minimal surface problem in more specialized settings. For example, the second author, in [15], showed a connection between Riemannian minimal graphs in the Heisenberg group and those in the Carnot Heisenberg group and used this connection to prove $W^{1,p}$ estimates for solutions to the minimal surface equation. In addition, he found a number of initial examples of minimal surfaces in the Heisenberg group and used them to demonstrate non-uniqueness of the solution to the Dirichlet problem in the Heisenberg group. Recently, both Garofalo and the second author, [12], and Cheng, Huang, Malchiodi and Yang, [5], independently investigated minimal surfaces in some limited settings. Garofalo and the second author restricted their view to the Heisenberg group and provided, among other results, a representation theorem for smooth minimal surfaces, a horizontal regularity theorem and proved an analogue of the Bernstein theorem, showing that a minimal surface in the Heisenberg group that is a graph over some plane satisfies a type of constant curvature condition. We note that Cheng and Huang, [4], independently showed a more general version of this Bernstein-style theorem by classifying all properly embedded minimal surfaces in $\mathbb{H}$. In [5], the authors investigate $C^2$ minimal surfaces in three dimensional pseudohermitian geometries (including, of course, the Heisenberg group) and, using the techniques of CR geometry, investigate the structure of minimal surfaces in this setting and, among many results, prove, under suitable conditions, a uniqueness result for the Dirichlet problem for minimal surfaces in the Heisenberg group. In [10], the second author extends the representation theorem of [12] to $C^1$ minimal surfaces in $\mathbb{H}^1$, provides examples of continuous (but not smooth) minimal surfaces and shows a geometric obstruction to the existence of smooth minimal solutions to the Plateau Problem in the Heisenberg group. In [6], Cole examines minimal surfaces in spaces of Martinet-type. While this collection of spaces includes the Heisenberg group and many of those considered in [5], Cole’s thesis also treats three dimensional spaces that do not have equiregular horizontal subbundles. In [6], Cole derives the minimal surface equation and examines the geometry and existence of smooth solutions to the Plateau Problem.

While there has been great progress in the understanding of minimal and constant mean curvature surfaces in the setting of Carnot-Carathéodory spaces, there are still many fundamental open questions left to address. Most notably, much of the focus has been on the minimal surface equation and the majority of the work has focused on more limited settings such as the Heisenberg group, groups of Heisenberg type or three dimensional pseudohermitian manifolds. In this paper, we will address more general problems using a new tool to help discriminate between the various types of constant mean curvature surfaces that abound in different Carnot-Carathéodory spaces.

**Question:** In a Carnot-Carathéodory manifold $M$, do the surfaces of least perimeter or the surfaces of least perimeter enclosing a fixed volume satisfy any partial differential equations? Can the solutions be characterized geometrically?
In Euclidean space, there is a beautiful connection between the geometry of surfaces and the solutions to these variational problems: minimal surfaces are characterized as zero mean curvature surfaces while isoperimetric surfaces have constant mean curvature.

In this paper, we restrict ourselves to a large class of sub-Riemannian manifolds which we call vertically rigid sub-Riemannian (VR) spaces. These spaces are defined in Section 2 and include basically all examples already studied (including Carnot groups, Martinet-type spaces and pseudohermitian manifolds) but is a much larger class. On such spaces, we define a new connection, motivated by the Webster-Tanaka connection of strictly pseudoconvex pseudohermitian manifolds, that is adapted to the sub-Riemannian structure. Using this connection and the variational framework of Bryant, Griffiths and Grossman ([2]), we investigate minimal and isoperimetric surface problems. The framework of [2] provides a particularly nice form of the Euler-Lagrange equations for these problems and leads us to define a horizontal second fundamental form, $II_0$, and the horizontal mean curvature, $\text{Trace}(II_0)$, associated to a hypersurface in a Carnot group. Given a noncharacteristic submanifold $\Sigma$ of a VR space $M$, let $\{e_1, \ldots, e_k\}$ be an orthonormal basis for the horizontal portion of the tangent space to $\Sigma$ and let $e_0$ be the unit horizontal normal to $\Sigma$ (see the next section for precise definitions). Then, we define the horizontal second fundamental form as

$$II_0 = \begin{pmatrix}
    \langle \nabla e_1 e_0, e_1 \rangle & \cdots & \langle \nabla e_1 e_0, e_k \rangle \\
    \vdots & \ddots & \vdots \\
    \langle \nabla e_k e_0, e_1 \rangle & \cdots & \langle \nabla e_k e_0, e_k \rangle
\end{pmatrix}$$

and define the horizontal mean curvature as the trace of $II_0$. We note that the notion of horizontal mean curvature has appeared in several contexts (see [1], [5], [7], [8], [12], [18]), and that this notion coincides with the others, possibly up to a constant multiple. However, we emphasize that the version of the mean curvature above applies to all VR spaces (before this work, only [7] deals with mean curvature in any generality, but again is limited to Carnot groups) and has the advantage of being written in an invariant way with respect to the fixed surface. With this notion in place, we have a characterization of $C^2$ solutions to the two variational problems discussed above:

**Theorem 1.1.** Let $M$ be a vertically rigid sub-Riemannian manifold and $\Sigma$ a noncharacteristic $C^2$ hypersurface. $\Sigma$ is a critical point of the first variation of perimeter if and only if the horizontal mean curvature of $\Sigma$ vanishes.

Similarly,

**Theorem 1.2.** Let $M$ be a vertically rigid sub-Riemannian space and $\Sigma$ a $C^2$ hypersurface. If $\Sigma$ is a solution to the isoperimetric problem, then the horizontal mean curvature of $\Sigma$ is locally constant.

Thus, we recover an analogue of the classical situation: the solutions to these two problems are found among the critical points of the associated variational problems. Moreover, these critical points are characterized by having the trace of the second fundamental form be constant.

We note that the characterization of minimal surfaces in terms of both a PDE and in terms of mean curvature was achieved first by Danielli, Garofalo and Nhieu, [7], in Carnot groups and by the second author, [18], for minimal graphs in the
Heisenberg group. The technique described above provides a broad extension of this characterization and describes mean curvature in a geometrically motivated manner. From this point of view, this is most similar to the treatment of mean curvature by Cheng, Huang, Malchiodi and Yang, [5], who use a similar formalism but used the Webster-Tanaka connection. In contrast, some of the earlier definitions of mean curvature relied on the minimal surfaces equation for the definition (as in [18]) or via a different geometric analogue such as a symmetrized horizontal Hessian (as in [7], [8]).

On the other hand, for isoperimetric surfaces, the only known links between isoperimetric sets and constant mean curvature were under the restriction of cylindrical symmetry in the first Heisenberg group ([7], [14]), for $C^2$ surfaces in 3-dimensional pseudohermitian manifolds ([5]) and for $C^2$ surfaces in the first Heisenberg group using mean curvature flow methods due to Bonk and Capogna ([1]) and, recently, for $C^2$ surfaces in all Heisenberg groups due to Rigó and Rosales ([20]). Thus, our treatment of these problems unifies these results and extends them to a much larger class of sub-Riemannian manifolds. Moreover, we provide a number of new, general techniques and tools for investigating these problems in a very general setting.

As mentioned above, the geometric structure of minimal surfaces has only been studied in cases such as the Heisenberg group, pseudohermitian manifolds, and Martinet-type spaces. In general, even in the higher Heisenberg groups, nothing is known about the structure and geometry of minimal surfaces. As an illustration of the power of this framework, we use the horizontal second fundamental form to provide some geometric information about minimal surfaces in any VR space. To better describe minimal surfaces, we introduce some new notions of curvature in VR spaces:

**Definition 1.3.** Let $H_0$ be the horizontal second fundamental form for a $C^2$ non-characteristic surface, $\Sigma$, in a vertically rigid sub-Riemannian manifold $M$. Let $\{\mu_0, \ldots, \mu_k\}$ be the eigenvalues (perhaps complex and with multiplicity) of $H_0$. Then, the horizontal principle curvatures are given by

$$\kappa_i = \Re(\mu_i)$$

for $0 \leq i \leq k$.

Moreover, given $x \in \Sigma$, we say that $\Sigma$ is horizontally positively (non-negatively) curved at $x$ if $H_0$ is either positive (semi-)definite or negative (semi-) definite at $x$ and is horizontally negatively curved at $x$ if there is at least one positive and one negative $\kappa_i$. $\Sigma$ is horizontally flat at $x$ if $\kappa_i = 0$ for $0 \leq i \leq k$.

Let $\Sigma$ be a $C^2$ hypersurface in $M$, a vertically rigid sub-Riemannian manifold. Then, the horizontal exponential surface at $x \in \Sigma$, $\Sigma_0(x)$, is defined to be the union of all the horizontal curves in $\Sigma$ passing through $x$. The notion of horizontal principle curvatures described above gives rise to a new definition of convexity:

**Definition 1.4.** A subset $U$ of a Carnot group $M$ with $C^2$ boundary $\Sigma$ is horizontally geometrically convex (or hg-convex) if, at each noncharacteristic point $x \in \Sigma$, $\Sigma_0(x)$ lies to one side of $T^h_x \Sigma$, the horizontal tangent plane to $\Sigma$ at $x$. We say that $\Sigma$ is locally hg-convex at $x$ if there exists an $\epsilon > 0$ so that $\Sigma_0(x) \cap B(x, \epsilon)$ lies to one side of $T^h_x \Sigma$. 

We note that this notion of convexity is distinct from those described in [8] or [16]. In Section 6, we give explicit examples showing the nonequivalence of the various notions.

With these definitions in place, we prove an analogue to the classical statement that a minimal surfaces in $\mathbb{R}^3$ must be nonpositively curved.

**Theorem 1.5.** Let $\Sigma$ be a $C^2$ noncharacteristic minimal hypersurface in a vertically rigid sub-Riemannian space $M$. Then, $\Sigma$ cannot contain a point of horizontal positive curvature. If we further assume that $M$ is a Carnot group, then $\Sigma$ cannot be locally horizontally geometrically convex.

We emphasize that this is the first description of the geometry of minimal surfaces in a relatively general class of spaces.

The rest of the paper is divided into five sections. In Section 2, we define vertically rigid sub-Riemannian spaces, the adapted connection we mentioned above, and an adapted frame bundle for such objects. In Section 3, we briefly review the relevant machinery from [2]. In Section 4, we address the minimal surface problem using the machinery of Bryant, Griffiths and Grossman. Section 5 addresses the isoperimetric problem in this setting and finally, in Section 6, we define the horizontal second fundamental form and prove the geometric properties of minimal surfaces described above.

## 2. Vertically rigid sub-Riemannian manifolds

We begin with our basic definitions:

**Definition 2.1.** A sub-Riemannian (or Carnot-Carathéodory) manifold is a triple $(M, V_0, \langle \cdot, \cdot \rangle)$ consisting of a smooth manifold $M^{n+1}$, a smooth $k+1$-dimensional distribution $V_0 \subset TM$ and a smooth inner product on $V_0$. This structure is endowed with a metric structure given by

$$d_{cc}(x,y) = \inf \left\{ \int (\dot{\gamma}, \dot{\gamma})^{1/2} \gamma(0) = x, \gamma(1) = y, \gamma \in \mathcal{A} \right\}$$

where $\mathcal{A}$ is the space of all absolutely continuous paths whose derivatives, when they are defined, lie in $V_0$.

**Definition 2.2.** A sub-Riemannian manifold has a vertically rigid complement if there exist

- a smooth complement $V$ to $V_0$ in $TM$
- a smooth frame $T_1, \ldots, T_{n-k}$ for $V$
- a Riemannian metric $g$ such that $V$ and $V_0$ are orthogonal, $g$ agrees with $\langle \cdot, \cdot \rangle$ on $V_0$ and $T_1, \ldots, T_{n-k}$ are orthonormal.
- a partition of $\{1, \ldots, n-k\}$ into equivalence classes such that for all sections $X \in \Gamma(V_0)$, $g([X, T_j], T_i) = 0$ if $j \sim i$.

A sub-Riemannian space with a vertically rigid complement is called a vertically rigid (VR) space.

For a VR space, we shall denote the number of equivalence classes of the partition by $v$ and the size of the partitions by $l_1, \ldots, l_v$. In particular, we then have $l_1 + \cdots + l_v = n-k$. After choosing an order for the partitions, for $j > 0$ we set

$$V_j = \text{span}\{T_i : i \text{ is in the } j\text{th partition.}\}$$
Then
\[ TM = \bigoplus_{j=0}^{r} V_j. \]

After reordering we can always assume that the vector fields \( T_1, \ldots, T_l \) span \( V_1 \), the next \( l_2 \) span \( V_2 \) etc.

There are 3 motivating examples for this definition:

**Example 2.3.** Let \((M, \theta, J)\) be a strictly pseudoconvex pseudohermitian structure (see [21]). Then \( V_0 = \ker \theta \) has codimension 1 and a vertically rigid structure can be defined by letting \( T_1 \) be the characteristic (Reeb) vector field of \( \theta \) and defining \( g \) to be the Levi metric
\[ g(X, Y) = d\theta(X, JY) + \theta(X)\theta(Y). \]

Since \( T_1 \) is dual to \( \theta \) and \( T_1 \cdot d\theta = 0 \) the required commutation property clearly holds.

**Example 2.4.** Let \((M, v_0)\) be a graded Carnot group with step size \( r \). Then the Lie algebra of left-invariant vector fields of \( M \) decomposes as
\[ m = \bigoplus_{j=0}^{r} v_j \]
where \( v_{j+1} = [v_0, v_j] \) for \( j < r \) and \( [v_0, v_r] = 0 \). We then set \( V_j = \text{span}(v_j) \) and construct a (global) frame of left invariant vector fields for each \( v_j, j > 0 \). This clearly yields a vertically rigid structure.

**Example 2.5.** Let \( M = \mathbb{R}^3 \). We define a Martinet-type sub-Riemannian structure on \( M \) by defining \( V_0 \) to be the span of
\[ X = \partial_z + f(x,y)\partial_x, \quad Y = \partial_y + g(x,y)\partial_x \]
where \( f \) and \( g \) are smooth functions. The metric is defined by declaring \( X, Y \) an orthonormal frame for \( V_0 \). (In particular if we take \( f = 0 \) and \( g = x^2 \), we see that the step size is 1 on \( x \neq 0 \) and 2 at \( x = 0 \).) Now define \( T_1 = \partial_z \) and extend the metric so that \( X, Y, T_1 \) are orthonormal. Again the commutation condition clearly holds.

**Example 2.6.** Let \( \{X_1, \ldots, X_k\} \) be a collection of smooth vector fields on \( \mathbb{R}^n \) that satisfy Hörmander’s condition. We will construct the \( \{T_i\} \) as follows. As the \( X_i \) bracket generate, let \( \{T_i\} \) be a basis for the complement of the span of the \( \{X_i\} \) formed by differences of the brackets of the \( X_i \) and linear combinations of the \( X_i \) themselves. These \( T_i \) are naturally graded by counting the number of brackets of \( X_i \)'s it takes to include \( T_i \) in the span. Define a Riemannian inner product that...
makes the \( \{ X_1, \ldots, X_k, T_1, \ldots, T_{n-k} \} \) an orthonormal basis. This structure satisfies all the conditions for a vertically rigid structure except possibly the last.

We note that the majority of the examples in the literature, either from subelliptic PDE, control theory and/or robotic path planning, satisfy the last condition. □

The advantage of vertically rigid structures is that they admit connections which are adapted to analysis in the purely horizontal directions.

**Definition 2.7.** A connection \( \nabla \) on \( TM \) is adapted to a vertically rigid structure if

- \( \nabla \) is compatible with \( g \), i.e. \( \nabla g = 0 \).
- \( \nabla T_j = 0 \) for all \( j \).
- Tor\(_p\)(\( X, Y \)) \( \in V_p \) for all sections \( X, Y \) of \( V_0 \) and \( p \in M \).

The motivating example for this definition is the Webster-Tanaka connection for a strictly pseudoconvex pseudohermitian manifold [21].

**Lemma 2.8.** Every vertically rigid structure admits an adapted connection.

**Proof:** Let \( \nabla \) denote the Levi-Cevita connection for \( g \). Define \( \nabla \) as follows: set \( \nabla T_j = 0 \) for all \( j \). Then for a section \( X \) of \( V_0 \) and any vector field \( Z \) define

\[
\nabla_Z X = (\nabla_Z X)_0
\]

where \((·)_0\) denotes the orthogonal projection onto \( V_0 \). This essentially defines all the Christoffel symbols for the connection. It is easy to see that it satisfies all the required conditions. For example, to show \( \nabla g = 0 \), take vector fields \( X, Y, Z \) and write

\[
X = X_0 + \sum x_i T_i,
\]

\[
Y = Y_0 + \sum y_i T_i.
\]

where \( X_0, Y_0 \in V_0 \). Using the fact that \( \nabla T_i = 0 \), we have

\[
\nabla g(X,Y,Z) = Zg(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)
\]

\[
= Zg(X_0, Y_0) + \sum Z(x_i y_i) - g((\nabla_Z X)_0, Y_0) - g(\sum Z y_i, X)
\]

\[
= g(\nabla_Z X_0, Y_0) - g((\nabla_Z X)_0, Y_0) + g(X_0, \nabla_Z Y_0) - g(X_0, (\nabla_Z Y)_0)
\]

\[
+ \sum Z(x_i y_i) - \sum Z(x_i) y_i - \sum Z(y_i) x_i
\]

\[= 0
\]

The last equality follows since \( X_0 \) and \( Y_0 \) are horizontal vector fields and using the product rule. The statement about torsion follows directly from the definition. ■

**Lemma 2.9.** If \( X, Y, Z \) are horizontal vector fields then

\[
\langle \nabla_X Y, Z \rangle = \frac{1}{2} (X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle + \langle [X, Y]_0, Z \rangle + \langle [Z, X]_0, Y \rangle + \langle [Z, Y]_0, X \rangle).
\]

In particular, this depends solely on the choice of orthogonal complement \( V \).
Proof: We note that since $\nabla g = 0$ and $V_0$ is parallel, 
$$\langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle - \langle Y, \nabla_X Z \rangle.$$ 
Since the torsion of two horizontal vector fields is purely vertical, we also obtain 
$$\langle \nabla_X Y, Z \rangle = \langle \nabla_Y X, Z \rangle + g([X, Y], Z) = \langle \nabla_Y X, Z \rangle + ([X, Y]_0, Z).$$ 
The remainder of the proof is identical to the standard treatment of the Levi-Civita connection on a Riemannian manifold given in [3].

Remark 2.10. We note that the definition of adapted connection leaves some flexibility in its definition. In particular, we have some freedom in defining Christoffel symbols related to $\nabla T_i X$ when $X$ is a section of $V_0$. While we could make choices that would fix a unique adapted connection, we will not do so in order to preserve the maximum flexibility for applications.

To study these connections and sub-Riemannian geometry it is useful to introduce the idea of the graded frame bundle.

Definition 2.11. An orthonormal frame $(e, t) = e_0, \ldots, e_k, t_1, \ldots, t_{n-k}$ is graded if $e_0, \ldots, e_k$ span $V_0$, $t_1, \ldots, t_{n-k}$ span $V$ and each $t_j$ is in the span of $\{T_i : i \sim j\}$.

The bundle of graded orthonormal frames $GF(M) \overset{\pi}{\longrightarrow} M$ is then an $O(k + 1) \times \Pi_{j=1}^{n-k} O(l_j)$-principle bundle.

On $GF(M)$ we can introduce the canonical 1-forms $\omega^j, \eta^j$ defined at a point $f = (p, e, t)$ by
$$\omega^j(X)_f = g_p(\pi_*(X), e_j), \quad \eta^j(X)_f = g_p(\pi_*(X), t_j).$$
An adapted connection can be viewed as an affine connection on $GF(M)$. The structure equations are then determined by the following lemma.

Lemma 2.12. On $GF(M)$ there exist connection 1-forms $\omega^i$, $0 \leq i, j \leq k$ and $\eta^i$, $1 \leq i, j \leq n-k$ together with torsion 2-forms $\tau^i$, $0 \leq i \leq k$ and $\tilde{\tau}^i$, $1 \leq i \leq n-k$ such that
$$d\omega^i = \sum_{0 \leq j \leq k} \omega^j \wedge \omega^i_j + \tau^i,$$
$$d\eta^i = \sum_{i \sim j} \eta^i_j \wedge \eta^i_j + \tilde{\tau}^i.$$ 

Proof: The content of the lemma is in the terms that do not show up from the standard structure equations of an affine connection. However since $V_0$ is parallel we can immediately deduce that there exist forms $\omega^i_k$ such that $\nabla e_k = \omega^i_k \otimes e_j$. Furthermore since each $t_j$ is in the span of $\{T_i : i \sim j\}$ and all the $T_i$ are also parallel we must have $\nabla t_j = \sum_{i \sim j} \eta^i_j \otimes t_i$ for some collections of forms $\eta^i_j$.

Lemma 2.13. The torsion forms for an adapted connection have the following properties:

- $\tau^i(e_a, e_b) = 0$
- $\tilde{\tau}^i(t_i, e_b) = 0$ if $j \sim i$

for any lifts of the vector fields.
Proof: The first of these is a direct rewrite of the defining torsion condition for an adapted connection. For the second we observe that
\[ \text{Tor}(T_i, e_b) = \nabla_{T_i} e_b - \nabla_{e_b} T_i - [T_i, e_b] \]
is orthogonal to \( T_j \) if \( i \sim j \) by the bracket conditions of a vertically rigid structure. The result then follows from noting that torsion is tensorial.

3. Exterior differential systems and variational problems

In this section, we briefly review the basic elements of the formalism of Bryant, Griffiths and Grossman which can be found in more detail in chapter one of [2].

Their formalism requires the following data:

(a) A contact manifold \((M, \theta)\) of dimension \(2n + 1\)
(b) An \(n\)-form, called the Lagrangian, \(\Lambda\) and the associated area functional

\[ \mathcal{F}_\Lambda(N) = \int_N \Lambda \]

where \(N\) is a smooth compact Legendre submanifold of \(M\), possibly with boundary. In this setting, a Legendre manifold is a manifold \(i : N \to M\) so that \(i^*\theta = 0\).

From this data, we compute the Poincaré-Cartan form \(\Pi\) from the form \(d\Lambda\). They show that \(d\Lambda\) can be locally expressed as

\[ d\Lambda = \theta \wedge (\alpha + d\beta) + d(\theta \wedge \beta) \]

for appropriate forms \(\alpha, \beta\). Then,

\[ \Pi = \theta \wedge (\alpha + d\beta) \]

and we often denote \(\alpha + d\beta\) by \(\Psi\). With this setup, Bryant, Griffiths and Gross prove the following characterization of Euler-Lagrange systems ([2] section 1.2):

**Theorem 3.1.** Let \(N\) be a Legendre surface with boundary \(\partial N\) in \(M\) given by \(i : N \to M\) as above. Then \(N\) is a stationary point under all fixed boundary variations, measured with respect to \(\mathcal{F}_\Lambda\), if and only if \(i^*\Psi = 0\).

In the next two sections, we will use this formalism and the previous theorem to investigate the minimal and isoperimetric surface problems.

4. Minimal Surfaces

For a \(C^2\) hypersurface \(\Sigma\) of a vertically rigid sub-Riemannian manifold we define the horizontal perimeter of \(\Sigma\) to be

\[ P(\Sigma) = \int_{\Sigma} |(\nu_g)_0| \nu_{g \perp} dV_g \]

where \(\nu_g\) is the unit normal to \(\Sigma\) with respect to the Riemannian metric \(g\). At noncharacteristic points of \(\Sigma\), i.e. where \(T\Sigma \not\subset V_0\), this can be re-written as

\[ P(\Sigma) = \int_{\Sigma} \nu_{g \perp} dV_g \]

where \(\nu\) is the horizontal unit normal vector, i.e. the projection of the Riemannian normal to \(V_0\). We note that, when restricting to the class of \(C^2\) submanifolds, this
definition is equivalent to the perimeter measure of De Giorgi introduced in [9].

Our primary goal for this section is to answer the following question.

**Question 1.** In a vertically rigid sub-Riemannian manifold, given a fixed boundary can the hypersurfaces spanning the boundary with least perimeter measure be geometrically characterized?

To answer this question, we shall employ the formal techniques of Bryant, Griffiths and Grossman [2] by exhibiting the minimizing hypersurfaces as integrable Legendre submanifolds of a contact covering manifold \( \widetilde{M} \). More specifically, we define \( \widetilde{M} \) to be the bundle of horizontally normalized contact elements,

\[
\widetilde{M} = \{ (p, \nu, T) \in M \times (V_0)_p \times V_p : \| \nu \| = 1 \}.
\]

Thus \( \widetilde{M} \stackrel{\pi_1}{\longrightarrow} M \) has the structure of an \( \mathbb{S}^{k+1} \times \mathbb{R}^{n-k} \)-bundle over \( M \). Next we define a contact form \( \theta \) on \( \widetilde{M} \) by

\[
\theta_p(X) = g_p((\pi_1)_*X, \nu + T).
\]

To compute with \( \theta \) it is useful to work on the graded frame bundle. However as there is no normalization on the \( T \) component of \( \widetilde{M} \), we shall need to augment \( \mathcal{GF}(M) \) to the fiber bundle \( \mathcal{GF}^0(M) \) defined as follows: over each point the fibre is \( O(k+1) \times \bigoplus_{j=1}^{v} O(l_j) \times \mathbb{R}^{l_1+\cdots+l_v} \). The left group action is extend as follows. If \( h = (h_1, h_2) \in O(k+1) \times \bigoplus_{j=1}^{v} O(l_j) \),

\[
h \cdot (p, e, t, a) = (p, (h_1 \cdot e, h_2 \cdot t), ah_2^{-1}).
\]

The natural projection \( \pi \) from \( \mathcal{GF}^0(M) \) to \( M \) now filters through \( \widetilde{M} \) as

\[
\mathcal{GF}^0(M) \stackrel{\pi_2}{\longrightarrow} \widetilde{M} \stackrel{\pi_1}{\longrightarrow} M
\]

where under \( \pi_2 \), \((p, e, t, a) \mapsto (p, e_0, \sum a_j t_j) \). In particular this means \( \pi_2 \circ (id, h_2) = \pi_2 \). This formulation now allows us to pull \( \theta \) back to \( \mathcal{GF}^0(M) \) by

\[
\pi_2^* \theta = \omega^0 + a_j \eta^j.
\]

We shall denote this pullback by \( \theta \) also.

**Remark 4.1.** The augmented frame bundle \( \mathcal{GF}^0(M) \) is not a principle bundle and so we cannot impose an affine connection on it in the usual sense. However since it has the smooth structure of \( \mathcal{GF}(M) \times \mathbb{R}^l \), we can naturally include the canonical forms and the connection structure equations of \( \mathcal{GF}(M) \) into the augmented bundle. Thus the results of Lemma 2.12 and Lemma 2.13 hold on \( \mathcal{GF}^0(M) \) also.

**Lemma 4.2.** The contact manifold \( (\widetilde{M}, \theta) \) is maximally non-degenerate, i.e. \( \theta \wedge d\theta^n \).

**Proof:** We shall work on the augmented graded frame bundle where

\[
d\theta = \omega^j \wedge \omega^j + \tau^0 + da_j \wedge \eta^j + a_j (\eta^j \wedge \eta^j + \tilde{\tau}^j).
\]

We pick out one particular term of the expansion of \( \theta \wedge d\theta^n \), namely

\[
\mu = \omega^0 \wedge \omega^1 \wedge \cdots \wedge \omega^k \wedge \omega^0_1 \wedge \cdots \omega^k_1 \wedge \cdots \eta^{n-k} \wedge da_1 \wedge \cdots da_{n-k}.
\]

The connection forms are vertical (in the principle bundle sense) and the canonical forms are horizontal (again in the bundle sense). Thus \( \mu \) is the wedge of \( n-k \) da terms, \( n+k+1 \) horizontal forms and \( k \) vertical forms. Since each torsion form is purely (bundle) horizontal, \( \mu \) is clearly the only term of this form in \( \theta \wedge d\theta^n \). All
the forms are independent so \( \mu \) does not vanish. Thus we deduce that \( \theta \wedge d\theta^n \neq 0 \) on \( GF^0(M) \) and so cannot vanish on \( \tilde{M} \).

The transverse Legendre submanifolds of \((\tilde{M}, \theta)\) are the immersion \( \iota: \Sigma \rightarrow \tilde{M} \) such that \( \iota^* \theta = 0 \) and \( \pi_2 \circ \iota \) is also an immersion. These are noncharacteristic oriented hypersurface patches in \( M \) with normal directions defined by the contact element in \( \tilde{M} \).

Define
\[
\Lambda = \omega^1 \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \eta^{n-k}
\]
on \( GF^0(M) \). Then \( \Lambda = \pi_2^*(\nu \pi_1^* dV) \) and so \( \Lambda \) is basic over \( \tilde{M} \). Furthermore due to \( (3) \) we see that
\[
P(\Sigma) = \int_{\Sigma} \iota^* \Lambda.
\]

Now on \( GF^0(M) \), Lemma 2.13 implies that \( \tau^j \) has no component of the form \( \omega^0 \wedge \omega^j \) and \( \tilde{\tau}^j \) none of form \( \omega^0 \wedge \eta^j \). Thus we see from \( (1) \) that
\[
d\Lambda = \sum_j (-1)^{j-1} \omega^1 \wedge \cdots \wedge (\omega^j \wedge \omega^0_0) \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^{n-k} + \sum_j (-1)^{k+j-1} \omega^1 \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge (\eta^j \wedge \eta^j_{j}) \wedge \cdots \wedge \eta^{n-k}.
\]
The connection is metric compatible so \( \omega^j_j = 0 \) and \( \eta^j_j = 0 \). Thus the second and third sums vanish identically. This implies \( d\Lambda = \theta \wedge \Psi \) where
\[
\Psi = \sum_j \omega^1 \wedge \cdots \wedge (\omega^j_0) \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^{n-k}.
\]

If \( \Sigma \subset \tilde{M} \) is a transverse Legendre submanifold, then we can construct a graded frame adapted to \( \Sigma \), i.e. with \( e_0 = \nu \). Choosing any section immersing \( \Sigma \) into \( GF^0(M) \) we can then pull \( \Psi \) back to \( \Sigma \). Switching the \( \omega \)'s and \( \eta \)'s to represent the coframe and connection form for this fixed frame, we get
\[
\Psi|_{\Sigma} = \left( \sum_{j=1}^k \omega^j_0(e_j) \right) \omega^1 \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \eta^{n-k}.
\]

**Theorem 4.3.** Suppose \( \Sigma \) is a \( C^2 \) hypersurface in the vertically rigid manifold \( M \). Then \( \Sigma \) is a critical point for perimeter measure in a noncharacteristic neighborhood \( U \subset \Sigma \) if and only if the unit horizontal normal \( \nu \) satisfies the minimal surface equation
\[
H = 0
\]
in \( U \). Equivalently, the horizontal normal must satisfy
\[
div_g \nu = 0
\]
everywhere on \( U \), where the divergence is taken with respect to the Riemannian metric \( g \).
Proof: From the Bryant-Griffiths-Grossman formalism \[2\] we see that \( \nu : \Sigma \hookrightarrow \tilde{M} \) is a stationary Legendre submanifold for \( \Lambda \) in a small neighborhood if and only if \( \nu^* \Psi = 0 \). This condition is just \( \sum_{j=1}^{k} \omega_j^0(\nu) = 0 \) for any local orthonormal frame \((\nu, e_1, \ldots, e_k)\) for \( V_0 \). This can be re-written as

\[
H = \sum \langle \nabla_{e_j} \nu, e_j \rangle = 0.
\]

A standard formula in Riemannian geometry (see for example \[13\]) states that for any connection for which the volume form is parallel, the divergence of a vector field can be computed by

\[
\text{div}_g X = \text{trace}(\nabla X + \text{Tor}(X, \cdot)).
\]

The adapted connection is symmetric for \( g \) and so we can apply this result while noting that by the defining conditions \( \text{trace}(\text{Tor}(\nu, \cdot)) = 0 \). Thus

\[
\text{div}_g \nu = \sum \langle \nabla_{e_j} \nu, e_j \rangle + \sum g(\nabla_{e_j} \nu, t_j) = \sum \langle \nabla_{e_j} \nu, e_j \rangle.
\]

Referring back to (10) then completes the proof.

\[\square\]

Corollary 4.4. The minimal surface equation (9) may depend on the choice of orthogonal complement \( V \), but not on the remainder of the vertically rigid structure or choice of adapted connection.

Proof: After we write \( \text{div}_g \nu = \sum \langle \nabla_{e_j} \nu, e_j \rangle \), the result follows immediately from Lemma \[2.9\].

\[\square\]

Corollary 4.5. Any minimal noncharacteristic patch of a vertically rigid sub-Riemannian manifold \((M, V_0, \langle \cdot, \cdot \rangle)\) with

\[
\dim V_0 = 2
\]

is ruled by horizontal \( \nabla \)-geodesics.

Proof: Extend \( \nu \) off \( \Sigma \) to any unit horizontal vector field. Define \( \nu^\perp \) to be any horizontal unit vector field that is orthogonal to \( \nu \). By the torsion properties of the connection and the arguments of Theorem \[4.3\], the minimal surface equation (9) can be written

\[
0 = \langle \nabla_{\nu^\perp} \nu, \nu^\perp \rangle = -\langle \nu, \nabla_{\nu^\perp} \nu^\perp \rangle.
\]

Since \( \nu^\perp \) has no covariant derivatives in vertical directions, this implies that

\[
\nabla_{\nu^\perp} \nu^\perp = 0.
\]

In other words the integral curves of \( \nu^\perp \) are \( \nabla \)-geodesics. However, these integral curves clearly foliate the noncharacteristic surface patch.

\[\square\]

Remark 4.6. We note that the last corollary is a generalization of the results of Garofalo and the second author \[12\] in the Heisenberg group, those of Cheng, Huang, Malchiodi and Yang \[5\] in three dimensional pseudohermitian manifolds and those of Cole \[6\] in Martinet-type spaces. In those cases, the authors proved the minimal surfaces in those settings were ruled by appropriate families of horizontal curves.
5. CMC SURFACES AND THE ISOPERIMETRIC PROBLEM

We now investigate the following question

**Question 2.** Given a fixed volume, what are the closed surfaces bounding this volume of minimal perimeter.

Using the results of the previous section, we can now define a hypersurface of locally constant mean curvature (CMC) by requiring that $H = \text{constant}$ on each connected component of $\Sigma' = \Sigma - \text{char}(\Sigma)$, where $\Sigma$ is a CMC($\rho$) surface. By comparing to the Riemannian case, these are our prime candidates for solutions to Question 2. Throughout this section we shall make the standing assumption that the volume form $dV_g$ is globally exact, i.e., there exists a form $\mu$ such that $d\mu = dV_g$ on $M$. Since this is always locally true, the results of this section will hold for sufficiently small domains.

For a closed codimension 2 surface $\gamma$ in $M$ we define

$$\text{Span}(\gamma, a) = \left\{ \text{C}^2 \text{ noncharacteristic surface } \Sigma : \partial \Sigma = \gamma, \int_{\Sigma} \mu = a \right\}.$$  

**Lemma 5.1.** If $\text{Span}(\gamma, a)$ is non-empty then any element of minimal perimeter $\Sigma_0$ must have constant mean curvature.

**Proof:** As the boundary of the surfaces are fixed, we can again employ the formalism of [2]. We permit variations that alter the integral $\int_{\Sigma} \mu$ and apply a Lagrange multiplier method to establish a condition for critical points of perimeter. Indeed the minimiser $\Sigma_0$ must be a critical point of the functional

$$\Sigma \mapsto \int_{\Sigma} \Lambda - c \int_{\Sigma} \mu$$

for some constant $c$. Now pulled-back to the contact manifold

$$d(\Lambda - c\mu) = \theta \wedge \Psi - c\pi_1^*dV_g$$

$$= \theta \wedge ((H - c)\omega^1 \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^{n-k})$$

$$:= \bar{\theta} \wedge \bar{\Psi}.$$  

The same methods as Theorem 4.3 then imply that $\bar{\Psi}|_{\Sigma_0} = 0$ and hence $\Sigma_0$ has constant mean curvature.

**Theorem 5.2.** If a $C^2$ domain $\Omega$ minimises surface perimeter over all domains with the same volume, then $\Sigma = \partial \Omega$ is locally CMC.

**Proof:** Let $p \in \partial \Sigma$ be a noncharacteristic point. As $\Sigma$ is $C^2$ there exists a noncharacteristic neighbourhood $U$ of $p$ in $\Sigma$ with at least $C^2$ boundary. Now by Stokes' theorem $\text{Vol}(\Omega) = \int_{\Sigma} \mu$ and so $U$ must minimise perimeter over all noncharacteristic surfaces in $\text{Span}(\partial U, \int_U \mu)$. Therefore $U$ has constant mean curvature by Lemma 5.1.

Under certain geometric conditions we can provide a more intuitive description of $\mu$.

**Definition 5.3.** A dilating flow for a vertically rigid structure is a global flow $F: M \times \mathbb{R} \rightarrow M$. 

\[ (F_{\lambda})_* E_j = e^{\lambda} E_j \text{ for some fixed horizontal orthonormal frame } E_1, \ldots, E_{k+1}. \]

\[ (F_{\lambda})_* T_j = e^{\gamma_j} T_j \text{ for some constant } \gamma_j. \]

Associated to a dilating flow are the dilation operators defined by
\[ \delta_{\lambda} = F_{\log \lambda} \]
and the generating vector field \( X \) defined by
\[ X_p = \frac{d}{d\lambda} |_{\lambda=0} F(\lambda, p). \]

The homogeneous dimension of \( M \) is given by
\[ Q = k + 1 + \sum_{j=1}^{n-k} \gamma_j. \]

The dilating flow is said to have an origin \( O \) if for all \( p \), \( \delta_{\lambda}(p) \to O \) as \( \lambda \to 0 \).

In the sequel, a vertically rigid sub-Riemannian manifold that admits a dilating flow with origin will be referred to as a VRD-manifold.

**Example 5.4.** All Carnot groups admit a dilation with origin. On the Lie algebra level, the dilation is defined merely by defining a linear map with eigenspaces the various levels of the grading, i.e. \( \delta_{\lambda} X = \lambda^{j+1} X \) for \( X \in V_j \). The group dilations are then constructed by exponentiating.

**Example 5.5.** The jointly homogeneous Martinet spaces, i.e. those of Example 2.5 with the functions \( f \) and \( g \) bihomogeneous of degree \( m \). Then the dilations are defined by
\[ \delta_{\lambda}(x, y, z) = (\lambda x, \lambda y, \lambda^{m+1} z). \]

Then clearly
\[ (\delta_{\lambda})_* (\partial_x + f(x, y) \partial_z) = \lambda \partial_x + \lambda^{m+1} f(x, y) \partial_z = \lambda (\partial_x + f(\lambda x, \lambda y) \partial_z) \]
and \( (\delta_{\lambda})_* \partial_z = \lambda^{m+1} \partial_z \).

**Lemma 5.6.** In a VRD-manifold the form \( \mu = Q^{-1} X \lrcorner dV_g \) satisfies
\[ d\mu = dV_g. \]

**Proof:** Let \( \omega^j, \eta^j \) be the dual basis to \( E_j, T_i \). Then
\[ dV_g = \omega^1 \wedge \cdots \wedge \omega^{k+1} \wedge \eta^1 \wedge \cdots \wedge \eta^{n-k}. \]

Now \( F^*_\lambda \omega^j(Y) = \omega^j((F_{\lambda})_* Y) \) so \( F^*_\lambda \omega^j = \lambda \omega^j \). Thus
\[ \mathcal{L}_X \omega^j = \frac{d}{d\lambda} |_{\lambda=0} F^*_\lambda \omega^j = \lambda \omega^j. \]

By a virtually identical argument we see that \( \mathcal{L}_X \eta^j = \gamma_j \eta^j \). Therefore
\[ d\mu = Q^{-1} d(X \lrcorner dV_g) = Q^{-1} \mathcal{L}_X dV_g = dV_g. \]

Every point \( p \) in a VRD-manifold can be connected to the origin by a curve of type \( t \mapsto \delta_t(p) \). For any surface \( \Sigma \) we can then construct the dilation cone over \( \Sigma \) as
\[ (12) \quad \text{cone}(\Sigma) = \{ \delta_t(p) : 0 \leq t \leq 1, p \in \Sigma \}. \]
Lemma 5.7. Suppose $\Sigma$ is $C^2$ surface patch in a VRD-manifold such that any dilation curve intersects $\Sigma$ at most once. If $\Sigma$ is oriented so that the normal points away from the origin then

$$\text{Vol}(\text{cone}(\Sigma)) = \int_{\Sigma} \mu.$$ 

Proof: This is just Stokes’ theorem for a manifold with corners, for

$$\int_{\text{cone}(\Sigma)} dV_g = \int_{\partial \text{cone}(\Sigma)} \mu = \int_{\Sigma} \mu$$

as $\mu$ vanishes when restricted to any surface foliated by dilation curves. 

We can now interpret Lemma 5.1 as minimizing surface perimeter under the constraint of fixed dilation cone volume. Since the volume of a domain is equal to the signed volume of its boundary dilation cone, this yields some geometric intuition for the arguments of Theorem 5.2.

Remark 5.8. In [7] and [14], the authors characterize cylindrically symmetric minimizers in the Heisenberg group as constant mean curvature surfaces in that setting. Our treatment allows for such a characterization in all VR spaces without the assumption of cylindrical symmetry. We note, however, that this method requires some regularity (the surfaces must be at least $C^2$ to ensure the computations work) while the work in [7] and [14] is more general in this respect, allowing for piecewise $C^1$ defining functions.

Remark 5.9. We point out that Theorem 5.2 provides an approach to understanding the isoperimetric problem in VR or VRD spaces via a better understanding of their constant mean curvature surfaces. For a specific example, the reader is referred to Section 6 below.

6. The horizontal second fundamental form

We now present a more classical interpretation of these results by defining an analogue to the second fundamental form.

Definition 6.1. Consider a noncharacteristic point of a hypersurface $\Sigma$ of a VR space $M$ and fix a horizontal orthonormal frame $e_0, \ldots, e_k$ as before with $\nu = e_0$. Then the horizontal second fundamental form as the $k \times k$ matrix,

$$H_0 = \begin{pmatrix} \langle \nabla_{e_1} \nu, e_1 \rangle & \cdots & \langle \nabla_{e_1} \nu, e_k \rangle \\ \vdots & \ddots & \vdots \\ \langle \nabla_{e_k} \nu, e_1 \rangle & \cdots & \langle \nabla_{e_k} \nu, e_k \rangle \end{pmatrix} = \begin{pmatrix} \omega_0^1(e_1) & \cdots & \omega_0^k(e_1) \\ \vdots & \ddots & \vdots \\ \omega_0^1(e_k) & \cdots & \omega_0^k(e_k) \end{pmatrix}.$$ 

Further, we define the horizontal mean curvature, $H$, to be the trace of $H_0$.

By arguments already given we see $\text{div}_g \nu = H$. Thus we have shown that a $C^2$ surface that minimizes perimeter must satisfy $H = 0$ at any noncharacteristic point.

We note that several authors have proposed other candidates for certain types of analogues of the second fundamental form and horizontal mean curvature. For example, Danielli, Garofalo and Nhieu ([7],[8]) use a symmetrized horizontal Hessian.
to analyze minimal and CMC surfaces (in [7]) and convex sets (in [8]). We emphasize that many of the other candidates are symmetrized versions of the second fundamental form while the definition above is explicitly \textit{a priori} non-symmetric.

This definition of mean curvature coincides (up to a constant multiple) with the various definitions of mean curvature in the Carnot group setting (see, for example, [7], [12]). Moreover, our version of the minimal surface equation as $\text{Trace } II_0 = 0$ matches with others in the literature once it is suitably interpreted.

In particular, our formulation, when restricted to the appropriate setting is equivalent to that of [7], [5], [18], [6], and [20].

In addition to the horizontal mean curvature described above, we would also like to define other aspects of horizontal curvature.

**Definition 6.2.** Let $II_0$ be the horizontal second fundamental form for a $C^2$ non-characteristic surface, $\Sigma$, in a vertically rigid sub-Riemannian manifold $M$. Let $\{\mu_0, \ldots, \mu_k\}$ be the eigenvalues (perhaps complex and with multiplicity) of $II_0$. Then, the horizontal principle curvatures are given by

$$\kappa_i = \operatorname{Re}(\mu_i)$$

for $0 \leq i \leq k$.

Moreover, given $x \in \Sigma$, we say that $\Sigma$ is \textbf{horizontally positively (non-negatively) curved} at $x$ if $II_0$ is either positive (semi-)definite or negative (semi-) definite at $x$ and is \textbf{horizontally negatively curved} at $x$ if there is at least one positive and one negative $\kappa_i$. $\Sigma$ is \textbf{horizontally flat} at $x$ if $\kappa_i = 0$ for $0 \leq i \leq k$.

This definition coupled with the observation that $\text{Trace } II_0 = \kappa_0 + \cdots + \kappa_k$ yields the following immediate corollary of Theorem [10]:

**Corollary 6.3.** If $\Sigma$ is a $C^2$ minimal surface in a vertically rigid sub-Riemannian manifold, then $\Sigma$ has no noncharacteristic points of horizontal positive curvature.

This is reflective of the Euclidean and Riemannian cases where minimal surfaces cannot have points of positive curvature.

**Remark 6.4.** We note that having, for example, only positive horizontal principle curvatures at a point is necessary but not sufficient to conclude that the surface is horizontally positively curved.

**Definition 6.5.** Let $c$ be a horizontal curve on $\Sigma$ a $C^2$ hypersurface in a vertically rigid sub-Riemannian manifold. Then at a noncharacteristic point of $\Sigma$, the horizontal curvature of $c$ is given by

$$k_c = \langle \nabla_c \dot{c}, e_0 \rangle$$

We note that, analogous to the Euclidean and Riemannian cases, there is a connection between the horizontal curvature of curves passing through a point on a hypersurface and the horizontal second fundamental form at that point:

**Lemma 6.6.** Let $c$ be a horizontal curve on $\Sigma$ a $C^2$ hypersurface in a vertically rigid sub-Riemannian manifold. Then at noncharacteristic points,

$$k_c = \langle II_0(\dot{c}), \dot{c} \rangle$$

**Proof:** Since $c$ is horizontal, we have that $\dot{c} = c_1 e_1 + \cdots + c_k e_k$ for appropriate functions $c_i$. Differentiating $\langle \dot{c}, e_0 \rangle = 0$, we have:

$$\langle \nabla_c \dot{c}, e_0 \rangle = - \langle \dot{c}, \nabla_c e_0 \rangle$$

$$k_c = - \langle II_0(\dot{c}), \dot{c} \rangle$$
This gives, as an immediate corollary, an analogue of Meusnier’s theorem:

**Corollary 6.7.** All horizontal curves lying on a surface \( \Sigma \) in \( M \), a VR space, that, at a noncharacteristic point \( x \in \Sigma \), have the same tangent vector also have the same horizontal curvature at \( x \).

As in the classical case, Corollary 6.7 allows us to speak of the horizontal curvature associated with a direction rather than with a curve, showing that \( II_0 \) contains all of the horizontal curvature information at a point.

**Lemma 6.8.** Given \( \Sigma \) and \( x \) as above, let \( l \) be the number of distinct principle curvatures at \( x \). Then, there exist curves \( \{ c_1, \ldots, c_l \} \subset \Sigma \) so that

\[
k_{c_i} = \kappa_i
\]

**Proof:** Let \( \{ \lambda_1, \ldots, \lambda_j, \lambda_{j+1} \pm i\beta_{j+1}, \ldots, \lambda_l \pm i\beta_l \} \) be the eigenvalues of \( II_0 \) at \( x \) associated with the distinct principle curvatures. Further, let \( \{ u_1, \ldots, u_j, u_{j+1} \pm iv_{j+1}, \ldots, u_k \pm iv_k \} \) be the associated eigenvectors. Without loss of generality, we have ordered the eigenvalues so that the real eigenvalues appear first and the complex eigenvalues are last. We note that, for each complex conjugate pair of eigenvalues, \( \lambda_j \pm \beta_j \), the associated principle curvatures, \( \kappa_j \) and \( \kappa_{j+1} \), are equal.

Using the eigenvectors, we may replace \( \{ e_1, \ldots, e_k \} \) by a new basis given by

\[
\{ u_1, \ldots, u_j, u_{j+1}, v_{j+1}, \ldots, u_i, v_i, \tilde{e}_{2l-j}, v_k \}
\]

where \( \{ \tilde{e}_i \} \) form a basis for the orthogonal complement of the eigenvectors. Rewriting \( II_0 \) with respect to this new basis, there is a submatrix of \( II_0 \) which is a block matrix where the first block is of the form

\[
A = \begin{pmatrix}
\lambda_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_j
\end{pmatrix}
\]

and there are \( k-j \) remaining blocks of the form

\[
B_i = \begin{pmatrix}
\lambda_i & \beta_i \\
-\beta_i & \lambda_i
\end{pmatrix}
\]

Now, let \( c_i \) be the integral curve of the \( i \)th new basis vector for \( 1 < i < 2l-j \). Then,

\[
k_{c_i} = -< II_0(\tilde{e}_i), \tilde{e}_i > =< II_0(e_i), e_i > = \kappa_i
\]

**Remark 6.9.** We note that since \( II_0 \) is often nonsymmetric, there is often not a full basis of eigenvectors. For example, in the Carnot group \( \mathbb{H} \times \mathbb{R} \) with coordinates \( (x, y, t, s) \) and Lie algebra spanned by \( \{ X_1, X_2, X_3, T \} \) where

\[
\begin{align*}
X_1 &= \partial_x - \frac{y}{2} \partial_t \\
X_2 &= \partial_y + \frac{x}{2} \partial_t \\
X_3 &= \partial_s \\
X_4 &= \partial_t
\end{align*}
\]
Taking $V_0 = \text{span}\{X_1, X_2, X_3\}$ and $V_1 = \text{span}\{X_4\}$ yields a vertically rigid structure. The surface defined by $\frac{2}{\rho^2} - t - s = 0$ has unit horizontal normal given by:

$$\nu = \frac{y}{\sqrt{1 + y^2}} X_1 - \frac{1}{\sqrt{1 + y^2}} X_3$$

and

$$H_0 = \begin{pmatrix} 0 & \frac{y^2}{(y^2+1)^{3/2}} \\ 0 & 0 \end{pmatrix}$$

Thus, this is a minimal surface and $H_0$ has a double eigenvalue of 0 and a single eigenvector $(1, 0)$ in this basis. The presents an entirely different phenomena than the analogous Riemannian or Euclidean situation.

We note that this phenomena and the existence of complex eigenvalues both indicate the existence of a nontrivial bracket structure among the elements of the tangent space to $\Sigma$. Indeed, both of these indicate that there are vector fields $e_i, e_j \in \{e_1, \ldots, e_k\}$ with the property that $[e_i, e_j]$ has a component in the $e_0$ direction. In particular, this is an indication that the distribution $\{e_1, \ldots, e_k\}$ is not integrable and hence, $\Sigma$ cannot be realized as a surface ruled by codimension one horizontal submanifolds.

We pause to note that we can now state an analogue of Corollary 4.5

**Corollary 6.10.** Any CMC($\rho$) noncharacteristic patch of a vertically rigid sub-Riemannian manifold $(M, H, \langle \cdot, \cdot \rangle)$ with

$$\dim V_0 = 2$$

is ruled by horizontal curves with horizontal constant curvature $\rho$.

**Proof:** We follow precisely the same proof as that of Corollary 4.5 to get that

$$<\nabla_\nu \nu^\perp, \nu> = \rho$$

Thus, the integral curves of $\nu^\perp$ have constant horizontal curvature $\rho$. ■

**Remark 6.11.** We note that in specific cases, these rulings can be computed exactly. Basically this amounts to explicitly computing the torsion terms and solving ODEs. For example, it is straightforward to verify that in the Heisenberg group, such curves are geodesics with respect to the Carnot-Carathéodory metric and are horizontal lifts of planar circles of curvature $\rho$. This particular observation is also contained in [5].

**Definition 6.12.** Let $\Sigma$ be a $C^2$ hypersurface in a vertically rigid sub-Riemannian manifold $M$. Then, the horizontal exponential surface at $x \in \Sigma$ is defined to be the union of all the horizontal curves in $\Sigma$ passing through $x$. We denote this subset of $\Sigma$ by $\Sigma_0(x)$.

**Definition 6.13.** Let $\Sigma$ be a $C^2$ hypersurface in a vertically rigid sub-Riemannian manifold $M$. Then, the horizontal tangent plane at a noncharacteristic point $x \in \Sigma$, is defined as

$$T^h_x \Sigma = \{exp_x(v)|g(v, e_0(x)), v \in T_x M\}$$

where $exp$ is the Riemannian exponential map.
This horizontal tangent plane in a Carnot group can also be defined by blowing up the metric at a given point (see [10]).

This gives us a geometric interpretation of these curvature conditions analogous to the Riemannian setting:

**Theorem 6.14.** Let $\Sigma$ be a $C^2$ hypersurface in $M$, a Carnot group, and let $\{\kappa_i\}$ be the set of horizontal principle curvatures of $\Sigma$ at a noncharacteristic point $x$. Then, $\Sigma_0(x)$ locally lies to one side of $T^h_x\Sigma$ if and only if the surface is horizontally positively curved at $x$. Similarly, if $\Sigma$ is horizontally negatively curved at $x$, then any neighborhood of $x$ in $\Sigma_0(x)$ intersects $T^h_x\Sigma$ at points other than $x$.

**Proof:** First assume that $\Sigma_0(x)$ locally lies to one side of the horizontal tangent plane at $x$. Then, as any curves in $\Sigma_0(x)$ must also lie to one side of the tangent plane, we have that $\langle \nabla c_1 e_1, e_0 \rangle$ and $\langle \nabla c_2 e_2, e_0 \rangle$ are either both non-positive or both non-negative at $x$ for any $c_1, c_2 \in \Sigma_0(x)$. Thus, $\Sigma$ is horizontally non-negatively curved at $x$. Conversely, if every curve $c \in \Sigma_0(x)$ has positive horizontal curvature, then $\langle \nabla c(t), e_0 \rangle$ is either non-positive or non-negative. Without loss of generality, we will assume it to be non-negative. But, geometrically, this says that with respect to the connection $\nabla$, each curve $c$ locally changes in the direction of $e_0$ and cannot move towards $-e_0$. To finish the proof, consider a curve $c \in \Sigma_0(x)$ in a small neighborhood of $x$. If we let $v_i$ be the left invariant unit vector field on $M$ so that $v_i(x) = e_i(x)$, we have that $T^h_x\Sigma$ is the integral submanifold of the distribution perpendicular to $v_0$ (with respect to the Riemannian metric). With respect to these vector fields, we write

$$\dot{c}(t) = c_0(t) v_0 + \cdots + c_N(t) v_N$$

where $c_0(0) = 0$. Then, computing with respect to the Riemannian metric, we have

$$\langle \nabla \dot{c}, v_0 \rangle = \ddot{c}_0(t) + \sum_{i,j} \dot{c}_i(t) \dot{c}_j(t) \langle \nabla v_i, v_j, v_0 \rangle$$

$$= \ddot{c}_0(t)$$

The last equation follows because, as a Carnot group is graded, for left invariant horizontal vector fields $v_1, v_2, v_3$, $\langle [v_1, v_2], v_3 \rangle = 0$. Noticing that at $t = 0$, we have

$$\langle \nabla c(0), e_0 \rangle = \langle \nabla \dot{c}(0), v_0(0) \rangle = \ddot{c}_0(0)$$

The hypothesis that $\Pi_0$ is positive semi-definite shows that $\ddot{c}_0(0) \geq 0$ and the result follows. A similar argument shows the last statement.

This leads us to define a notion of convexity in sub-Riemannian manifolds.

**Definition 6.15.** A subset $U$ of a Carnot group $M$ with $C^2$ boundary $\Sigma$ is horizontally geometrically convex (or hg-convex) if, at each noncharacteristic point $x \in \Sigma$, $\Sigma_0(x)$ lies to one side of $T^h_x\Sigma$. We say that $\Sigma$ is locally hg-convex at a non-characteristic point $x$ if there exists an $\epsilon > 0$ so that $\Sigma_0(x) \cap B(x, \epsilon)$ lies to one side of $T^h_x\Sigma$.

With this definition, we have yet another analogue of Euclidean minimal surface theory:

**Corollary 6.16.** If $\Sigma$ is a $C^2$ minimal hypersurface in a Carnot group then $\Sigma$ cannot bound a locally hg-convex set.
Proof: As the distribution $V_0$ is non-integrable, every $C^2$ surface must have at least one noncharacteristic point. The result then follows from Theorem 6.14 and Corollary 6.3.

Theorem 1.5 in the introduction is the combination of this corollary and Corollary 6.3.

We note that this notion of convexity is distinct from the notion of set convexity in Carnot groups introduced by Danielli, Garofalo and Nhieu in $\mathbb{S}$ which is based on weakly convex defining functions (the reader should also see the work of Lu, Manfredi and Straffolini [10] which independently presented a notion of convex functions at the same time). In particular, we point out that there are minimal surfaces in the first Heisenberg group which are the boundary of a convex region in the sense of $\mathbb{S}$ (for example, the plane $t = 0$ and the surface $t = \frac{x^2}{2}$) and others that bound nonconvex sets (for example, $t = \frac{x^2}{2} - \frac{x^3}{3}$). Moreover, we further point out that as $\phi(x,y,t) = \frac{x^2}{2} + g(x) - t$ has minimal level sets in $\mathbb{H}$ for every choice of $C^2$ function $g$, we can easily produce many functions that violate the various convexity conditions presented in $\mathbb{S}$ and [10]. As the level sets of $\phi = x$ are minimal in $\mathbb{H}$ and satisfy all of the convexity properties defined in $\mathbb{S}$, we see that none of these notions can be made equivalent to the notion of $h\sigma$-convexity.

Remark 6.17. We remark that the definitions and proofs in this section are direct generalizations or adaptations of the Euclidean and/or Riemannian machinery. While the proofs are quite straightforward, we point out that this is due to a correct choice of geometric structure, in this case the adapted connection, that allows for the ease of the proofs. Without such machinery, the statements about the relation between horizontal curvature and minimality/isoperimetry above were known only in certain three dimensional sub-Riemannian manifolds where they reduce to much simpler statements.

References

[1] Mario Bonk and Luca Capogna. Mean curvature flow in the Heisenberg group. 2005. Preprint.
[2] Robert Bryant, Phillip Griffiths, and Daniel Grossman. Exterior differential systems and Euler-Lagrange partial differential equations. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2003.
[3] Isaac Chavel. Riemannian geometry—a modern introduction, volume 108 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1993.
[4] Jih-Hsin Cheng and Jenn-Fang Hwang. Properly embedded and immersed minimal surfaces in the Heisenberg group. 2004. Preprint: arxiv math.DG/0407094.
[5] Jih-Hsin Cheng, Jenn-Fang Hwang, Andrea Malchiodi, and Paul Yang. Minimal surfaces in pseudohermitian geometry. 2003. Preprint.
[6] Daniel Cole. On minimal surfaces in Martinet-type spaces. PhD thesis, Dartmouth College, 2005.
[7] D. Danielli, N. Garofalo, and D.-M. Nhieu. Minimal surfaces, surfaces of constant mean curvature and isoperimetry in Carnot groups. August, 2001. Preprint.
[8] Donatella Danielli, Nicola Garofalo, and Duy-Minh Nhieu. Notions of convexity in Carnot groups. Comm. Anal. Geom., 11(2):263–341, 2003.
[9] Ennio De Giorgi. Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad $r$ dimensioni. Ann. Mat. Pura Appl. (4), 36:191–213, 1954.
[10] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. Rectifiability and perimeter in the Heisenberg group. Math. Ann., 321(3):479–531, 2001.
[11] Nicola Garofalo and Duy-Minh Nhieu. Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. *Comm. Pure Appl. Math.*, 49(10):1081–1144, 1996.

[12] Nicola Garofalo and Scott D. Pauls. The Bernstein problem in the Heisenberg group. 2003. Submitted.

[13] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*. John Wiley & Sons, Inc., 1963.

[14] G. P. Leonardi and S. Masnou. On the isoperimetric problem in the Heisenberg group $\mathbb{H}^n$. Preprint, 2002.

[15] G. P. Leonardi and S. Rigot. Isoperimetric sets on Carnot groups. *Houston J. Math.*, 29(3):609–637 (electronic), 2003.

[16] Guozhen Lu, Juan J. Manfredi, and Bianca Stroffolini. Convex functions on the Heisenberg group. *Calc. Var. Partial Differential Equations*, 19(1):1–22, 2004.

[17] Pierre Pansu. Une inégalité isopérimétrique sur le groupe de Heisenberg. *C. R. Acad. Sci. Paris Sér. I Math.*, 295(2):127–130, 1982.

[18] Scott D. Pauls. Minimal surfaces in the Heisenberg group. *Geom. Ded.*, 104:201–231, 2004.

[19] Scott D. Pauls. Obstructions to the existence of smooth solutions to the Plateau problem in the Heisenberg group. 2004. Submitted.

[20] Manuel Ritoré and César Rosales. Rotationally invariant hypersurfaces with constant mean curvature in the Heisenberg group $\mathbb{H}^n$. 2005. Preprint.

[21] N. Tanaka. *A differential geometric study on strongly pseudoconvex manifolds*. Kinokuniya Book-Store Co., Ltd., 1975.