Tracking through equality

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Abstract

We give a tracker solution for the quintessence scalar field for Ratra–Peebles or SUGRA potentials, holding before, during and after the equality epoch ($\rho_m = \rho_r$) and nicely fitting the numerical behavior.

\textit{Key words:} methods: analytical, numerical – cosmology: theory – dark energy

1 Introduction

Dark Energy (DE) is one of the main puzzles of cosmology. It could be a scalar field $\phi$, self–interacting through a potential $V(\phi)$ (Wetterich 1988, Ratra \& Peebles 1988), so that

$$\rho_{DE} = \rho_{k,DE} + \rho_{p,DE} \equiv \ddot{\phi}^2/2a^2 + V(\phi), \quad p_{DE} = \rho_{k,DE} - \rho_{p,DE} = w \rho_{DE},$$

provided that $\rho_{k,DE}/V \ll 1/2$, so that $-1/3 > w > -1$. Here

$$ds^2 = a^2(\tau)(-d\tau^2 + dx_i dx^i), \quad (i = 1, \ldots, 3)$$

is the metrics and dots indicate differentiation with respect to $\tau$ (conformal time). This kind of DE is dubbed \textit{dynamical} DE (dDE) or \textit{quintessence}.

The most significant potentials are those allowing tracker solutions, so limiting the impact of initial conditions. Among them, we consider here the SUGRA

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(Brax & Martin 1999, 2001; Brax, Martin & Riazuelo 2000) and Ratra & Peebles (1988, 2003) potentials

\[ SUGRA : \quad V(\phi) = (\Lambda^{4+\alpha}/\phi^\alpha) \exp\left(4\pi \phi^2 / m_p^2\right) \] (3)

\[ RP : \quad V(\phi) = \Lambda^{4+\alpha}/\phi^\alpha \] (4)

\( (m_p = G^{-1/2}: \) Planck mass); in the recent times, they yield \(|w|\) close to unity and fastly variable or \(|w|\) farther from unity and slowly variable, respectively. In a flat model, once the DE density parameters \(\Omega_{DE}\) is assigned, either \(\alpha\) or the energy \(\Lambda\) can still be freely chosen.

SUGRA and RP behave differently when \(\phi\) approaches \(m_p\). Here, however, we shall not deal with late epochs and our expressions, worked out for RP, suit SUGRA as well.

Tracker solutions are then usually found by considering the equation of motion

\[ \ddot{\phi} + 2(\dot{a}/a)\dot{\phi} = \alpha a^2 \Lambda^{4+\alpha}/\phi^{1+\alpha} \] (5)

and seeking solutions of the form

\[ \phi = \phi_i (\tau/\tau_i)^\beta \] (6)

(\(\tau_i\) is a reference time and \(\phi_i\) is the field value at that time). They are fixed once the time dependence of the scale factor \(a(\tau)\) is set. In turn, we can however write

\[ \dot{a}/a = u(\tau)/\tau, \] (7)

so that \(u = 1 (u = 2)\) in the radiation (matter) dominated eras and is actually \(\tau\) dependent around matter–radiation equality.

In this note we give a single expression for the tracker solution holding before, during and after equality, occurring at \(\tau_e\). This expression improves the tracker solution expression, known for \(\tau \ll \tau_e\), when \(\tau\) approaches \(\tau_e\). It also neatly improves the tracker solution known for \(\tau \gg \tau_e\), for a large range of redshifts. Furthermore, it fits the behavior of the \(\phi\) field across equality, with quite a small discrepancy from the numerical solution, keeping mostly well below 1\%.
2 Tracker solutions through equality

Let be $R = a/a_e$ and $\theta = \tau/\tau_e$ ($a_e, \tau_e$: equality scale factor, time); it is easy to verify that the expression
\[
R = c^2 \theta^2 + 2c \theta \quad \text{with} \quad c = \sqrt{2} - 1 ,
\] (8)
gives the scale factor behavior before, during and after equality. This follows the integration of the Friemann equation
\[
\dot{R} = H_e a_e (R + 1)^{1/2} \quad (9)
\]
\[
(3H_e = 8\pi G \rho_{e,m} = 8\pi G \rho_{e,r}; \rho_{e,m} = \rho_{e,r} \text{ are matter and radiation energy densities at equality}) \quad \text{and noticing that} \quad H_e a_e = 2c/\tau_e . \quad \text{It is then also easy to see that}
\]
\[
u(\theta) = 1 + c \theta / (1 + c \theta/2) \quad (10)
\]
so that, clearly, $u \to 1$ ($u \to 2$) for $\theta \to 0$ ($\theta \to \infty$).

Let us now rewrite eq. (5) in the form
\[
Y'' + 2\frac{u}{\theta}Y' = \alpha R^2 Y^{-(1+\alpha)} . \quad (11)
\]
Here $Y = \phi/\sigma$ with $\sigma^{2+\alpha} = (a_e \tau_e)^2 \Lambda^{4+\alpha}$ and $'$ indicates differentiation in respect to $\theta$. Let us then seek solutions of eq. (11) of the form
\[
Y = Y_e (R \theta)^b \quad \text{i.e.} \quad \phi = \sigma Y_e (a \tau/a_e \tau_e)^b . \quad (12)
\]
Clearly
\[
Y' = \frac{b}{\theta} (1 + u)Y , \quad Y'' = \frac{1}{\theta^2} \left[ b^2 (u+1)^2 + bu' \theta - b(u+1) \right] Y \quad (13)
\]
so that eq. (11) yields
\[
Y_e^{2+\alpha} (R \theta)^{b(2+\alpha)} = \alpha (R \theta)^2 G(\theta) \quad (14)
\]
Here
\[
G^{-1}(\theta) = b^2 (u+1)^2 + b(u+1)(2u-1) + bu' \theta \quad (15)
\]
Fig. 1. Comparison between the tracker solution $\phi(\tau)$ obtained in this work (GT) and the numerical integral $\phi_n(\tau)$ of the $\phi$–field equation. Dotted lines show RT and MT solutions vs. the numerical solution. The GT is consistent with the numerical integral, apart a narrow interval, where its discrepancy from the numerical integral marginally approaches 1%. RT is a fair approximation until $\tau_e$. MT performs much worse.

depends on $\theta$, however not so strongly as is meant by a power law. Let then be

$$b = \frac{2}{2 + \alpha}, \quad Y_e = \alpha G,$$

although this means that also $Y_e$ depends on $\theta$ and, therefore, the expressions (13) should contain further terms. The $\theta$ dependence of $G$ arises from the $\theta$ dependence of $u$; should we neglect the former one, the last term in the expression (15) should be consistently omitted.

In the next section we shall compare the behavior of $\phi$, as obtained from the expressions (12), (15), (16), denominated GT, with a numerical solution of eq. (5).

### 3 High and low $\theta$ limits

Let us first consider the solution (12) in the limit $\theta \ll 1$. Then $u \simeq 1$, $G \simeq 1/[(2b)^2 + (2b)]$ and, according to eq. (8), $R \simeq 2e\theta$. Accordingly, we obtain

$$Y_r = \left[ \frac{4e^2\alpha}{\beta_r^2 + \beta_r} \right]^{\frac{1}{2 + \alpha}} \theta^{\beta_r}, \quad \text{with} \quad \beta_r = \frac{4}{2 + \alpha}$$

so that, as expected, we recover the tracker solution for the radiation dominated regime (herebelow RT).
In quite the same way, for \( \theta \gg 1 \), we have \( u \simeq 2, G \simeq 1/[(3b)^2 + 3(3b)] \) and, according to eq. (8), \( R \simeq c^2\theta^2 \). Accordingly, we obtain

\[
Y_m = \left[ \frac{c^4\alpha}{\beta_m^2 + 3\beta_m} \right]^{\frac{1}{2\pi\alpha}} \theta^{3\beta_m}, \quad \text{with} \quad \beta_m = 6/(2 + \alpha)
\]

(18)

and, again, we recover the tracker solution for a matter dominated regime (herebelow MT).

4 Comparison with numerical integrals

The tracker solution (17) can be used to set the initial conditions to the eq. (11), performing then its numerical integration. In doing so we can use the expression (8) and (10) so that the system to integrate amounts to 2 first order equations. We shall indicate the numerical integral by \( Y_n(\theta) \). In Figure 1 we plot the ratios \( Y/Y_n, Y_r/Y_n, Y_m/Y_n \).

The symmetry between RT and MT solution is only apparent. Dotted curves are symmetric in respect to \( \tau \approx 10\tau_e \), and this means that at such redshift, inside the matter dominated era, the RT solution still performs as well as the MT solution. Using the expressions (17) and (18) it is easy to see that the two solutions intersect for

\[
\theta^2 = \frac{18\beta_r^2/2 + \beta_r}{c^2 \beta_r^2 + \beta_r}
\]

(19)

i.e., for \( \theta = \tau/\tau_e \sim \sqrt{18}/(\sqrt{2} - 1) \sim 10 \) and this confirms what is shown by the plot.
Altogether, this means that a fair tracker solution, since \( \sim \tau_e \), can only be given by using the GT expression, holding until DE itself begins to play a significant role as a source of the cosmic expansion. The maximum discrepancy \( \Delta \phi / \phi \) between GT and numerical solution is \( \sim 1\% \), slightly after \( \tau_e \), for any reasonable \( \alpha \) value. Its \( \alpha \) dependence is plotted in Figure 2.

5 Conclusions

Tracker solutions play an important role in the analysis of dDE cosmologies. A good tracker solution for the radiation dominated regime can be easily given. A tracker solution for the matter dominated regime also exists, but its usual analytical expression is rather far from a fair numerical behavior.

Here we gave a fair solution performing as well as the usual one in the radiation dominated era, fitting the numerical solution also about the equality (the residual discrepancy approaches \( \sim 1\% \) just in a narrow interval) and fitting the numerical solution all through the matter dominated era. It reads

\[
\phi = \phi_e \left( \frac{\tau}{\tau_e a} \right)^b
\]

with

\[
b = \frac{2}{2 + \alpha}, \quad \phi_e = \frac{\alpha \Lambda (a_e \tau_e \Lambda)^b}{b^2 (u + 1)^2 - b(u + 1)(2u - 1)}.
\]

It was found by using an exact analytical expression for \( a(t) \) through the equality period, also to work out the \( \tau \) dependence of \( a = \tau \dot{a}/a \).

When initial conditions to any numerical problem are to be set, one usually needs to go back to the radiation dominated era, so to rely on a fair tracking. As an example of problems where initial conditions in the matter dominated era are useful, let us remind the study of the evolution of a spherical fluctuation, the prediction of cluster mass function and its redshift evolution (Mainini, Macciò & Bonometto 2003, Mainini et al. 2003, Mainini 2005) using a Press & Schechter (1974) or similar (Sheth & Tormen 1999, 2002, Jenkins et al. 2001) approach, and performing N–body simulations (see, e.g., Klypin et al. 2003, Macciò et al. 2004, Solevi et al. 2006).

Using the GT expression, given here, this can be safely avoided, and initial conditions can be given at any time, until DE itself begins to be important for the overall cosmic expansion.
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