MATHEMATICS IN AND OUT OF STRING THEORY

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Dedicated to the memory of my father and mother
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The upsurge of excitement amongst theoretical physicists, over the subject of string theory, has filtered through to an appreciable extent into the mathematical community. Whereas the basic reason for the excitement amongst the physicists has been the vision of unification for the fundamental forces of nature via string theory, in mathematics its interest has been the wide range of deep ideas that have been involved, and the fascinating interconnections that have emerged.

“Stringy” ideas in mathematics include:

(a) The investigation of path integrals over the spaces of Riemann surfaces, leading to a natural modular-invariant measure (“the Polyakov volume form”) on the Teichmüller spaces. The description of this measure by complex geometry of the Teichmüller spaces, involving the Mumford isomorphisms.

(b) A search for infinite-dimensional “Universal Teichmüller spaces” of Riemann surfaces that parametrize simultaneously complex structures on surfaces of all topologies, – and the canonical relationships between various natural candidates for such a moduli space. This is important for a non-perturbative formulation of string theory.

(c) The study of the unitary (and projective unitary) representations of the diffeomorphism group of the circle (the closed string!); at the infinitesimal level, this is the representation theory of the Virasoro algebra. Indeed, there is an intimate relationship ([NV]) between the group $\text{Diff}(S^1)$ and the Teichmüller spaces – demonstrating that (c) is deeply related to (b).

It goes without saying that the above topics by no means exhaust the mathematical challenges raised by string theory. Owing to restrictions of space and time, and, more importantly, of the author’s knowledge, we shall deal in these notes only with some matters pertaining to items (a) and (b) above. For more directions, see the references cited. We will provide here an exposition of the Polyakov-Mumford construction on

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1Expanded version of the opening lecture in the 37th International Taniguchi Symposium: “Topology and Teichmüller spaces”, July 1995. [Preprint no.:imsc/95/32]
the Teichmüller space $T_h$ of Riemann surfaces of fixed genus $h \geq 2$; we will then explain our recent work (with Indranil Biswas and Dennis Sullivan) coherently fitting together this construction over the Universal Commensurability Teichmüller space, $T_\infty$. In fact, $T_\infty$ qualifies as a parameter space of the type desired in (b), because it comprises compact Riemann surfaces of all genus.

The connecting thread intertwining all the mathematics we discuss is the natural appearance of the Teichmüller/moduli spaces of Riemann surfaces. The most fundamental point is that, mathematically speaking, the quantum theory of the closed bosonic string is the theory of a sum over random surfaces ("world-sheets") that are swept out by strings propagating in spacetime. Owing to conformal invariance properties of the "Polyakov action", that sum finally reduces to an integral over the parameter space of Riemann surfaces. Namely, the quantum string theory, as a sum over random surfaces, precipitates a natural measure – the Polyakov measure – on each moduli space $M_h$.

We adopt the attitude that we are addressing mathematicians with no prior exposure to (the pulling of) strings. We have taken particular pains (Sections I and II) to explain to a mathematical audience the reduction of the Polyakov functional integral from an ill-defined and infinite dimensional situation to a finite dimensional one.

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The paper is organized as follows:

Section I: Polyakov action and the string path integral.
Section II: Polyakov volume form on the moduli space.
Section III: Geometry of Teichmüller space and Polyakov volume.
Section IV: The Universal Teichmüller space of compact surfaces.
Section V: Universal Polyakov-Mumford on $T_\infty$.

I: POLYAKOV ACTION AND THE STRING PATH INTEGRAL

I.A. Mechanics: The uninitiated mathematician may not object to being reminded of how path integrals arise in the first place. Mechanics, whether classical or quantum can be formulated as arising from a Lagrangian that allows one to assign a weight, called the "action", to any choice of an admissible path in configuration space connecting given initial and final boundary conditions.
For instance, suppose a particle (or a mechanical system) is moving in configuration space $\mathbb{R}^d$, with position at time $t$ being $x(t) = (x_1(t), \ldots, x_d(t))$ from $x(t_0) = x_0$ to $x(t_1) = x_1$. Then the Lagrangian, $L(x(t), \dot{x}(t), t)$, is a functional of the path and the action for that choice of path is defined by:

$$S(x(t)) = \int_{t_0}^{t_1} L(x(t), \dot{x}(t), t) \, dt \quad (1.1)$$

**Example:** Particle moving in a potential $V : \mathbb{R}^d \to \mathbb{R}$. $L$ may be taken as “kinetic energy minus potential energy”, namely, $L = \frac{m}{2} \| \dot{x}(t) \|^2 - V(x(t))$.

Amongst all admissible paths interpolating between $x(t_0) = x_0$ and $x(t_1) = x_1$, the actual classical path followed by the particle is the “best” path - namely the value of the action should be extremal (minimal). Thus the classical equations of motion are the Euler-Lagrange equations for the variational problem of minimising the action (1.1). Applied to the case of the example above, the reader can easily check that the Euler-Lagrange equations are simply Newton’s equations of motion.

In quantum mechanics [Feynman’s path integral formulation] there is no determined “preferred” path from $(x_0, t_0)$ to $(x_1, t_1)$. Rather “all” joining paths are possible histories of transition, and the proper question therefore is not which path the particle follows, but what is the probability amplitude that a particle at $x_0$ (at time $t_1$) will be at $x_1$ (at time $t_1$). That probability is taken to be a certain weighted average over all interpolating paths, where a path is weighted by $\exp(- \text{Action}(\text{path}))$. Thus notice that the classical path (with the minimum action) is given the highest weight, and paths near to the classical one would get relatively high weighting. The path integral answering the basic quantum query is

$$Z = \int_{\{x(t)\}} e^{-S(x(t))} \, Dx \quad (1.2)$$

where $\{x(t)\}$ is the family of all admissible (say continuous) paths joining the given initial and final conditions, and $Dx$ represents some Wiener-type measure on this path-family.

The analog of this infinite-dimensional path integral is what we will now describe for Polyakov’s bosonic string. The crucial discovery is that in a particularly happy situation (namely when the spacetime dimension $d = 26$), the integral reduces from the infinite-dimensional space of possible paths (=world-sheets) to the finite-dimensional moduli spaces of Riemann surfaces.

**I.B. String Theory:**

*String theory is the theory of fundamental particles considered as being one dimensional, “strings”, rather than as zero dimensional (“point-like”) objects. Thus a*
closed string, (therefore a circle – there being only one closed 1-manifold), propagating in a spacetime $\mathbb{R}^d$ sweeps out a 2-dimensional surface called its world-sheet. Several strings can interact, and more may be created or annihilated, without the world-sheet (which is a 2-manifold) becoming singular.

$\Sigma$

**Figure I.1: String world-sheet**

The figure illustrates that a configuration of two strings at an initial time can become (say) three at a later time while sweeping out a non-singular history $\Sigma$. Compare this with the case for point-like particles!

In Polyakov’s string theory [Pol1] the action assigned to any particular world-sheet $\Sigma$ depends on its location (embedding) in $\mathbb{R}^d$ as well as on an arbitrarily chosen smooth Riemannian metric on $\Sigma$. Both these freedoms, in the choice of embedding and metric, have to be integrated out (i.e., averaged over) in setting up the path integral.

The fundamental path integral that one needs to evaluate is the “vacuum to vacuum” amplitude – meaning that both the initial and final configurations are taken to be empty. Consequently, the world-sheets are without boundary – namely closed surfaces embedded in Euclidean space of $d$ dimension, (which is taken to be the background spacetime). We therefore assume henceforth that the world-sheets are closed and orientable surfaces, and attempt to work out the contribution to the path integral for each fixed genus.

Fix for reference a closed oriented smooth surface $\Sigma$ of genus $h$ (number of handles), and consider an arbitrary smooth embedding of $\Sigma$ in $\mathbb{R}^d$:

$$X \equiv (X^1, \ldots, X^d) : \Sigma \longrightarrow \mathbb{R}^d$$

(1.3)

and simultaneously consider an arbitrary Riemannian metric $g$ on $\Sigma$:

$$ds^2 = g_{ij}d\sigma^id\sigma^j, \quad i, j = 1, 2$$

(1.4)
Here \((\sigma^1, \sigma^2)\) are local smooth coordinates on \(\Sigma\). The image \(X(\Sigma)\) is to be considered as a typical vacuum-to-vacuum world-sheet (path) and the random metric \(((g_{ij}))\) (having nothing to do with the metric induced on \(\Sigma\) via the embedding into Euclidean space) is an extra dynamical variable which also has to be summed over in the Polyakov string theory.

The fundamental definition is the **Polyakov action** for that world-sheet and that assigned metric:

\[
S(X, g) = \int \int_{\Sigma} \left[ g^{ab} \frac{\partial X^{\mu}}{\partial \sigma^a} \frac{\partial X^{\mu}}{\partial \sigma^b} \right] \sqrt{g} d\sigma^1 d\sigma^2 \tag{1.5}
\]

Clearly, the part of the integrand (i.e., Lagrangian) in square brackets is a real-valued function on \(\Sigma\), and it is being integrated with respect to the area-element, \(d_g(vol) = \sqrt{g} d\sigma^1 d\sigma^2\), induced on \(\Sigma\) by the metric \(g\).

**Notation**: Summation conventions over all repeated indices are in use in (1.4), (1.5), and subsequently. In (1.5), indices \(a\) and \(b\) are summed over \(1 \leq a, b \leq 2\), and \(\mu\) is summed over \(1 \leq \mu \leq d\). Furthermore, \(\sqrt{g}\) signifies the density \(\sqrt{\det(g_{ij})}\), and \(((g^{ab}))\) denotes the inverse matrix to \(((g_{ij}))\), as usual.

The basic problem therefore is to analyse the functional integral:

\[
Z = \int_{\{g_{ij}\}} \int_{\{X\}} e^{-S(X, g)} DX.Dg \tag{1.6}
\]

That will represent the basic “partition function”, or vacuum-to-vacuum amplitude, for string propagations over world-sheets with \(h\) handles.

The remarkable thing is that, after taking care of certain symmetries in the Polyakov action, the above integral does have a sensible reduction to a finite dimensional integration over the moduli space \(M_h\) of complex structures on a genus \(h\) surface, provided the spacetime dimension \(d\) equals 26. Our first purpose, therefore, is to explain concisely to a mathematical audience how the action prescription (1.5) leads to a canonical measure – called naturally the Polyakov measure – on each moduli space \(M_h\).

**Remark on the classical theory for (1.5)**: For a fixed choice of embedding \(X\), one may enquire as to what is the “best” (extremal) metric for the action (1.5). It is easily derived that the action is extremised precisely for the metric \(g\) on \(\Sigma\) induced from the (Euclidean) target space \(R^d\) via the embedding \(X\). Namely, \(g_{ij} = \sum_{\mu=1}^{d} \frac{\partial X^{\mu}}{\partial \sigma^i} \frac{\partial X^{\mu}}{\partial \sigma^j}\), is the “classical” metric. So Polyakov action tells us to give this induced metric the highest weight but average over all metrics in the path-integral (1.6). Note that this fact, about the classical metric being the one induced from the target \(R^d\), remains true even if we choose an arbitrary Riemannian metric \(((G_{\mu\nu}))\) in the background spacetime \(R^d\); here, of course, we replace the Lagrangian integrand in (1.5) by the more general:

\[
\left[ g_{ab} \frac{\partial X^a}{\partial \sigma^a} \frac{\partial X^b}{\partial \sigma^b} G_{\mu\nu} \right].
\]

As we will see, however, for the corresponding partition function
(1.6), the integral over the embedding variables (for fixed \((g_{ij})\)) is not any more an (infinite dimensional) “Gaussian” if \((G_{\mu\nu})\) is non-flat. For our purposes, therefore, we restrict to the case \((G_{\mu\nu}) = \text{Euclidean}, \) just as in (1.5).

I.C. Symmetries of the Polyakov action:

To analyze (1.6), we first need to note that Polyakov’s action has certain symmetries:

\[
S(X^\mu + c^\mu, g) = S(X^\mu, g), \text{ any } c^\mu \in \mathbb{R}^d. \quad (1.7)
\]

\[
S(f^*X^\mu, f^*g) = S(X^\mu, g), \text{ any } f \in \text{Diff}^+(\Sigma). \quad (1.8)
\]

\[
S(X^\mu, e^\phi g) = S(X^\mu, g), \text{ any } \phi \in C^\infty(\Sigma, \mathbb{R}). \quad (1.9)
\]

(1.7) corresponds to the fact that the action remains unchanged if the embedded surface is simply translated in \(\mathbb{R}^d\). (1.8) says that, since (1.5) is invariantly defined, independent of choice of coordinates on \(\Sigma\), if we use any (orientation preserving) diffeomorphism of \(\Sigma\) to pullback both the metric and the embedding, the action remains unperturbed. [Explicitly, \(f^*X^\mu = X^\mu \circ f\), and \(f^*g\) is the metric on \(\Sigma\) which assigns to any curve the length that \(g\) assigns to the \(f\)-image of the curve.] Finally, (1.9) is the truly non-trivial symmetry, and says that, for a fixed embedding, the value of the action depends only on the conformal class of the metric \(g\). [Conformal scaling, \(g \mapsto \text{(scaling function)} g\) is called a “Weyl-scaling” by physicists.] Verification of (1.9) is immediate since the Weyl factor cancels off between the square-bracketed Lagrangian and the area-element term.

Clearly then, if the path integral (1.6) were actually computed over all embeddings \(X\) and all Riemannian metrics \(g\), then one would be getting infinite answers simply because one is overcounting by (a) the “volume of \(\mathbb{R}^d\)”, corresponding to the arbitrary \(c^\mu\) of (1.7); (b) the “volume of Diff(\(\Sigma\))” because of (1.8); (c) the “volume of positive functions (conformal factors) on \(\Sigma\)” because of (1.9). In other words, we can only expect to make sense of (1.6) by quotienting out these symmetries – namely by integrating on the quotient space:

\[
\{\text{Embeddings}\} \times \{\text{Metrics}\} / \{\mathbb{R}^d \times \text{Diff}^+(\Sigma) \times \text{Conf}(\Sigma)\} \quad (1.10)
\]

Notation: Write \(\text{Emb}(\Sigma) = \{X^\mu\}\) for the space of all (smooth) embeddings of \(\Sigma\) in \(\mathbb{R}^d\), and \(\text{Met}(\Sigma)\) for the space of all (smooth) Riemannian metrics on \(\Sigma\).

I.D. The integral over \(\{X^\mu\}\):

For every fixed \(g \in \text{Met}(\Sigma)\) on \(\Sigma\) we will show below that, in analogy with Gaussian (multivariate normal distribution) integrals, the integral over \(\{\text{Embeddings}\} / \mathbb{R}^d\) can be carried out to produce a reasonable answer. One should consider this section as providing heuristic motivation for the following:
Proposition/Definition 1.1.: For every fixed $g$ in $\text{Met}(\Sigma)$, the inner integral in (1.6) is assigned the value:

$$
\int_{\text{Emb}(\Sigma)} e^{-S(X,g)} DX = \left[ \frac{\text{det}'(-\Delta_g)}{\int_\Sigma d_g(\text{vol})} \right]^{d/2} 
$$

where $\Delta_g$ is the Laplace-Beltrami operator on functions on $\Sigma$, and $\text{det}'(A)$ denotes the determinant of an operator $A$ after discarding any zero eigenvalues.

The Polyakov integral (1.6) therefore becomes:

$$
Z = \int_{\text{Met}(\Sigma)} \left[ \frac{\text{det}'(-\Delta_g)}{\int_\Sigma d_g(\text{vol})} \right]^{d/2} Dg 
$$

with respect to a suitable volume element $[Dg]$ on the space of Riemannian metrics on $\Sigma$.

Note: Determinants will be computed by heat-kernel (zeta-function) regularization.

Since $(\Sigma, g)$ is a Riemannian manifold, we introduce the standard $L^2$ inner-product on functions on $\Sigma$ by

$$
<f_1, f_2> = \int \int_\Sigma (f_1 f_2) d_g \text{vol}
$$

and obtain the Hilbert space $L^2(\Sigma)$ of square-integrable real-valued functions with respect to this scalar product. The Laplace-Beltrami operator, $\Delta_g$, is the formally self-adjoint operator defined on sufficiently smooth functions on $\Sigma$ by:

$$
\Delta_g(f) = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial \sigma^a} \left( \sqrt{g} \frac{\partial f}{\partial \sigma^b} \right) \right]
$$

Lemma 1.2: The Polyakov action (1.5) can be rewritten as

$$
S(X, g) = \sum_{\mu=1}^d <X^\mu, (-\Delta_g)X^\mu>
$$

Notice that $(-\Delta_g)$ is a positive operator.

Proof: Triangulate $\Sigma$ so that each triangle falls in a typical $(\sigma^1, \sigma^2)$ coordinate patch. Then integrating by parts in (1.5) with respect to $\sigma^a$, on any triangle, gives an integral over all boundary of the triangle plus the term

$$
-\int \int_{\text{triangle}} X^\mu \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial \sigma^a} \left( \sqrt{g} g^{ab} \frac{\partial X^\mu}{\partial \sigma^b} \right) \right] \sqrt{g} d\sigma^1 d\sigma^2
$$

Summing up over all the triangles, the boundary terms cancel off and we are left with (1.14).
Now we are ready to give some heuristic arguments for (1.11). Since all the \(d\) target coordinates are on an equal footing, the equation (1.14) shows that \(\int e^{-S(X,g)} DX\) will be the product of \(d\) identical integrals:

\[
\int e^{<X^\mu, \Delta_g X^\mu>} DX^\mu, \mu = 1, 2, \ldots, d.
\]

Thus we are reduced to motivating the equation:

\[
\int_{\{Y : \Sigma \rightarrow \mathbb{R}\}} e^{-<Y, -\Delta_g Y>} dY = \left[\frac{\text{det}'(-\Delta_g)}{\text{Area}(\Sigma, g)}\right]^{-1/2} \tag{1.11'}
\]

This is clear. Indeed, suppose \(\{e_o, e_1, e_2, \ldots\}\) is an orthonormal basis for \(L^2(\Sigma)\) consisting of eigenfunctions of \(-\Delta_g\). Since the \(\Delta_g\)-harmonic functions are the constants, we have a 1-dimensional space of eigenfunctions with eigenvalue \(\lambda_o = 0\) generated by the constant function \(e_o\). We set \((-\Delta_g)e_k = \lambda_k e_k\), with \(\lambda_k > 0\) for all \(k = 1, 2, \ldots\)

Expanding in Fourier series an arbitrary \(Y : \Sigma \rightarrow \mathbb{R}\) in terms of the basis, we set

\[
Y = \sum_{k=0}^{\infty} y_k e_k
\]

with \(y_k = <Y, e_k>\).

Then one has

\[
-<Y, -\Delta_g Y> = -\sum_{k=0}^{\infty} |y_k|^2 \lambda_k \tag{1.15}
\]

If we take the functional measure “\(DY\)” in (1.11’) to mean \(dy_0 dy_1 dy_2 dy_3 \ldots\) with Lebesgue measure in each factor, we see that we have on our hands a Gaussian integral. Ignoring the \(dy_o\) integral for the time being, this should produce \((\lambda_1 \lambda_2 \lambda_3 \ldots)^{-1/2}\) by comparison with the standard multivariate normal integrals. Thus we make the reasonable definition that:

\[
\int e^{-\sum_{k=1}^{\infty} y_k^2 \lambda_k} [\prod_{k=1}^{\infty} dy_k] = [\text{det}'(-\Delta_g)]^{-1/2} \tag{1.16}
\]

Now, the integral over \(y_o\), which should simply be “the volume of \(y_o\)-space”, clearly contributes an infinity which we wish to understand and cancel against the infinity produced by the translation symmetry of embeddings expressed in equation (1.7). We will explain this.

Since the eigenvector \(e_o\) (corresponding to \(\lambda_o = 0\)) is a constant, the normalization \(\|e_o\|^2 = 1\) shows that the value of \(e_o\) is \((\text{Area}(\Sigma, g))^{-1/2}\). In order to perform the \(y_o\)-integral uniformly over all choices of the underlying metric \(g\), we stipulate the normalization \(\int e^{-\|y_o\|^2} dy_o = 1\). \(|| \cdot ||\) always denotes the \(L^2\) norm from (1.12). The “1” inside the norm means the (unnormalized) eigenvector given by the constant
(harmonic) function $1$. Note also that we ignore the $\sqrt{2\pi}$ factors that appear in finite
dimensional Gaussian integrals.]

We were interested in analyzing

$$\int e^{-\|y_o - e_o\|^2/2} \lambda_o dy_o,$$

where we think of $\lambda_o$ as a small (but positive) eigenvalue that we will ultimately send to zero. By the normalized
integral above, and the value of $e_o$, we are forced to the conclusion that the integral above must behave like

$$(\lambda_o e_o)^{-1/2} = \lambda_o^{-1/2} (\text{Area}(\Sigma, g))^{1/2}.$$  

So putting all the integrals over $(y_o, y_1, y_2 \ldots)$ together produces:

$$\lambda_o^{-1/2} (\text{Area}(\Sigma, g))^{1/2} (\text{det}'(-\Delta_g))^{-1/2}$$  

(1.17)

Now remember that the symmetry exhibited in (1.7); in fact, $Y \mapsto Y + c$, affects
only the $y_o$-variable and produces the overcounting infinity mentioned before. We
think of this infinity as “cancelling off” the $\lambda_o^{-1/2}$ infinity appearing in (1.17) (since $\lambda_o \to 0$). Thus (1.17) becomes (1.11'), which in turn motivates (1.11) itself.

In any event, we are only claiming that (1.11) is a well-motivated definition, and
a little thought about the above arguments shows that it is the definition that is mathematically natural.

I.E. Met$(\Sigma)$ and $\mathcal{M}_h$ :

Taking care of the symmetries enjoyed by the action law, the Polyakov integral
(1.11*) has now metamorphosed to the following (still rather enigmatic) shape :

$$Z = \frac{1}{\text{vol}(\text{Diff}^+(\Sigma)) \times \text{vol}(\text{Conf}(\Sigma))} \int_{\text{Met}(\Sigma)} \left[ \frac{\text{det}'(-\Delta_g)}{\text{Area}(\Sigma, g)} \right]^{-d/2} Dg$$

$$= \int_{\text{Met}(\Sigma)/\text{Diff}^+(\Sigma) \times \text{Conf}(\Sigma)} \left[ \frac{\text{det}'(-\Delta_g)}{\text{Area}(\Sigma, g)} \right]^{-d/2} Dg$$  

(1.18)

The final space over which one is now supposed to be integrating is nothing other
than the moduli space $\mathcal{M}_h$ parametrizing all complex analytic structures on $\Sigma$. In
fact, two metrics $g_1$ and $g_2$ in $\text{Met}(\Sigma)$ are equivalent in $\mathcal{M}_h$ provided $g_2 = f^*(e^\phi g_1)$,
for some $e^\phi \in \text{Conf}(\Sigma)$ and some $f \in \text{Diff}^+(\Sigma)$. Thus the complex structures, $\tau_1$ and $\tau_2$, obtained via isothermal parameters from the metrics $g_1$ and $g_2$, respectively,
are then biholomorphically equivalent via the biholomorphism $f : \Sigma_{\tau_2} \longrightarrow \Sigma_{\tau_1}$. So we define:

$$\mathcal{M}_h = \text{Met}(\Sigma)/\{\text{Diff}^+(\Sigma) \times \text{Conf}(\Sigma)\}$$  

(1.19)

The fundamental question, therefore, is whether the integrand in (1.18) produces
a well-defined measure on $\mathcal{M}_h$ for some choice of the free parameter $d$ (the spacetime
dimension). We will sketch in the next sections how that pleasant state of affairs transpires exactly when $d = 26$.  

9
II: POLYAKOV VOLUME FORM ON THE MODULI SPACE

II.A. The Riemannian structure of $\text{Met}(\Sigma)$

Consider the infinite dimensional manifold of Riemannian metrics, $\text{Met}(\Sigma)$, on the fixed smooth surface $\Sigma$. At the metric $g \in \text{Met}(\Sigma)$ we can assign the following natural inner product to the tangent space of $\text{Met}(\Sigma)$ thereat:

$$<\delta g^1_{ab}, \delta g^2_{cd}> = \int_\Sigma (g^{ac} g^{bd} \delta g^1_{ab} \delta g^2_{cd}) d_g (\text{vol})$$  \hspace{1cm} (2.1)

where $\delta g^1_{ab}, \delta g^2_{cd}$ – two symmetric 2nd-order covariant tensors on $\Sigma$ – are small perturbations of $g$, representing two tangent vectors to $\text{Met}(\Sigma)$. Thus $\text{Met}(\Sigma)$ itself qualifies as a Riemannian manifold (of infinite dimension). The Polyakov path integral, (1.11*) or (1.18), is to be interpreted as an integration over $\text{Met}(\Sigma)$ with respect to the volume form induced on $\text{Met}(\Sigma)$ by this Riemannian structure (2.1).

Remark: It is possible to make the mathematical considerations regarding (2.1) completely rigorous by working with the Hilbert manifold $\text{Met}(\Sigma)$ that one obtains by considering all metrics belonging to a suitable Sobolev class $H^s$. Since the finite-dimensional mathematics on the Teichmüller space that we finally arrive upon is independent of these technical subtleties, we will not say more about this matter in this exposition. In what follows it is possible to restrict oneself to metrics, conformal scalings and diffeomorphisms that are all of class $C^\infty$ on $\Sigma$.

As we know, the infinite dimensional manifold, $\text{Met}(\Sigma)$, is acted on by the infinite dimensional group $\mathcal{G} = \text{Diff}^+(\Sigma) \times \text{Conf}(\Sigma)$, (this is actually a semi-direct product), producing the quotient space $\mathcal{M}_h$ – an orbifold of finite dimension $(6h - 6)$.

Remark: If one assigns $\infty$ as the ‘dimension’ of the space of (local) functions on $\Sigma$, then the choice of a Riemannian metric tensor involves essentially three arbitrary functions – so $\text{Met}(\Sigma)$ is $3\infty$ dimensional. In that sense $\text{Conf}(\Sigma)$ is $\infty$ dimensional, and $\text{Diff}^+(\Sigma)$ is $2\infty$ dimensional. Thus, the trading off of the parameters in $\text{Met}(\Sigma)$ for those in the gauge group leaves a residual finite number of “moduli parameters” – leading to the interesting equation “$3\infty - 3\infty = 6h - 6$”!

We shall work with the universal covering space of $\mathcal{M}_h$ in the orbifold covering sense; that is the Teichmüller space, $\mathcal{T}(\Sigma) = \mathcal{T}_h$. To define it, replace the group $\text{Diff}^+(\Sigma)$ in the above action by its identity component $\text{Diff}_0(\Sigma)$ (comprising diffeomorphisms homotopic to the identity). We obtain:

$$\mathcal{T}_h = \text{Met}(\Sigma)/\{\text{Diff}_0(\Sigma) \times \text{Conf}(\Sigma)\}$$  \hspace{1cm} (2.2)

Concomitantly, we shall denote the quotient projection from $\text{Met}(\Sigma)$ to $\mathcal{T}_h$ by:

$$\mathcal{P} : \text{Met}(\Sigma) \to \mathcal{T}_h$$  \hspace{1cm} (2.3)
Notice that the quotient \( Met(\Sigma)/Conf(\Sigma) \) is precisely the space of (smooth) Beltrami coefficients on \( \Sigma \) (see [N1]).

**Representing \( \mathcal{T}_h \) as a slice within \( Met(\Sigma) \):** We henceforth assume that the genus \( h \) is at least two; (the situation for spheres is trivial, and for tori the case is special and easily treated.) Then \( \mathcal{T}_h \) is a smooth manifold of real dimension \((6h - 6)\) (with a natural complex manifold structure); the discrete “mapping class group” (or “modular group”), \( MCG(\Sigma) = MC_h = Diff^+(\Sigma)/Diff_0(\Sigma), \) acts biholomorphically and proper discontinuously on \( \mathcal{T}_h \) producing the quotient \( \mathcal{M}_h \) as the (complex analytic) orbifold.

The space \( \mathcal{T}_h \) can be concretely pictured as the space of conjugacy classes of “marked” Fuchsian groups \( \{\Gamma\} \) which are cocompact and which produce quotient surfaces of the given genus. A Fuchsian group \( \Gamma \) is marked by the choice of an isomorphism of the fundamental group of \( \Sigma \) onto it. The moduli space \( \mathcal{M}_h \) is just the set of conjugacy classes of these Fuchsian groups (without markings). See [N1] for this basic material.

Any smooth section (=right-inverse) of the quotient map \( P \) will be called a slice in \( Met(\Sigma) \).

*Juliet:* “What’s in a name? that which we call a rose,
By any other name would smell as sweet;”

Thus a slice is an embedded copy of Teichmüller space, \( \mathcal{T}_h \), in \( Met(\Sigma) \). Slices are \((6h - 6)\) dimensional submanifolds of \( Met(\Sigma) \), transverse to the orbits of the “gauge group” \( \mathcal{G} \). Mathematically speaking, a slice represents the variation of the conformal moduli as variation of Riemannian metric; in physics one says that choosing a slice “fixes the gauge freedom”.

**The Poincare slices and Weil-Petersson:** The uniformization theorem guarantees that every Riemann surface structure on \( \Sigma \) arises from a Poincare (hyperbolic) metric of constant negative curvature \((-1)\), that metric being uniquely determined up to an arbitrary diffeomorphism. Define therefore the following subset (infinite dimensional submanifold) of \( Met(\Sigma) \):

\[
Hyp(\Sigma) = \{ g \in Met(\Sigma) : \text{curvature}(g) \equiv -1 \} \tag{2.4}
\]

The quotient \( Met(\Sigma)/Conf(\Sigma) \) is therefore in natural bijection with the above submanifold of hyperbolic metrics.

Using the uniformization theorem with parameters, (see [N1]), we can choose a smoothly varying family of hyperbolic metrics \( \gamma(t) \) in \( Hyp(\Sigma) \) representing the Teichmüller space. Any such slice we call a “Poincare slice”. Thus \( Hyp(\Sigma)/Diff_0(\Sigma) \) is the Teichmüller space, and we can choose special Poincare slices that are orthogonal...
with respect to (2.1) to the orbits of the gauge group $Diff_0(\Sigma)$. A convenient name we will adopt for such a slice is “horizontal Poincare slice”. The fundamental metric (2.1), restricted to any horizontal Poincare slice, gives a Riemannian structure to $T_h$ that is known classically as the Weil-Petersson metric on the Teichmüller space. See Section III.A for more in this direction.

Remark: Having fixed a Poincare slice $\gamma(t)$, any other Poincare slice is then given by $f_t^*(\gamma(t))$, where the $f_t \in Diff_0(\Sigma)$ are an arbitrary family of diffeomorphisms that are chosen to depend smoothly on $t \in T_h$. It is therefore clear that the metric (2.1) cannot induce the same metric on $T_h$ via an arbitrarily chosen Poincare slice; however, (2.1) does induce Weil-Petersson on the horizontal Poincare slices defined above.

A natural way to select a Poincare slice is to utilize the unique harmonic diffeomorphism (Eells-Sampson), that exists in the homotopy class of the identity, between any two hyperbolic metrics on $\Sigma$. One may pullback the target metric via this diffeomorphism to obtain a specific choice of hyperbolic metric on $\Sigma$ representing the Teichmüller class of the target metric. This Poincare slice is horizontal (because harmonic Beltrami coefficients are orthogonal to the $Diff_0$ directions). See [J],[N3],[W] and the references therein.

Facts about (2.1): (a) The surface diffeomorphisms, $Diff^+(\Sigma)$, act on $Met(\Sigma)$ as isometries of the Riemannian structure (2.1).

(b) However, the action of the Weyl-rescalings on $Met(\Sigma)$ do not enjoy this compatibility with the metric (2.1).

Remarks: These are easily established. Fact(b) above, namely that conformal rescalings fail to preserve the metric on $Met(\Sigma)$, is at the root of the “conformal anomaly” that we will have to grapple with in the material below.

II.B. Change of coordinates in $Met(\Sigma)$

In order to analyse the integral (1.18) restricted to any slice, we must clearly understand the nature of the metric (2.1) on $Met(\Sigma)$ in coordinates that are along the gauge orbits, and complementary coordinates in the Teichmüller directions along the slice. Indeed, we want to factor out of the Polyakov integral the integrals over the orbits of $Diff^+(\Sigma)$ and $Conf(\Sigma)$, as explained in (1.18). Therefore it is natural to want to express the integral (1.18) as an iterated integral over these gauge orbits and over the slice.

So fix once and for all some real analytic coordinates $((t)) = ((t_i))$, $i$ running from 1 to $N = 6h - 6$, on $T_h$; (the $((t_i))$ can be chosen as the real and imaginary parts of holomorphic coordinates, and this can even be done globally over Teichmüller space). We shall identify $T_h$ as this domain in $((t))-space$, whenever convenient.
Let us fix a slice $K$, by choosing a right-inverse of $P$:

$$\gamma : \mathcal{T}_h \rightarrow \text{Met}(\Sigma), \quad \gamma(\mathcal{T}_h) = K$$

and note that an arbitrary metric $\rho \in \text{Met}(\Sigma)$ has a unique expression:

$$\rho = f^*[e^\phi \gamma(\{(t_i)\})]$$

(2.6)

Here $\{(t_i)\}$ represents the Teichmüller point to which $\rho$ projects by $P$, $e^\phi$ is a conformal rescaling, and $f \in Diff_0(\Sigma)$. Thus the new coordinates for $\rho$ are $(f, \phi, \{(t_i)\}) \in Diff_0(\Sigma) \times Conf(\Sigma) \times \mathcal{T}_h$. Rescale the entire slice of metrics by the fixed $e^\phi$, and set:

$$g = g(\{(t)\}) = e^\phi \gamma(\{(t)\})$$

$g(\{(t)\})$ is an associated gauge-fixing slice, and note that $f$ becomes an isometry from the metric $\rho$ to the the metric $g$ because $\rho = f^*(g)$. We remark that since $Diff_0(\Sigma)$ is the arc-component of the identity in $Diff^+(\Sigma)$, it is possible to express $f$ as $\exp(\xi)$ where $\xi$ is a smooth vector-field on $\Sigma$.

We need to compute the Riemannian structure (2.1) of $\text{Met}(\Sigma)$ in these new coordinates. At any given point $\rho \in \text{Met}(\Sigma)$, we must understand small metrical variations $\delta \rho_{ab}$ in terms of changes in these new $Gauge \times \mathcal{T}_h$ coordinates. Instead of working with a small change of $\rho$ we will work with a corresponding small change of $g$; remembering (Fact (a) above), that the pullback action by the fixed diffeomorphism $f$ is an isometric automorphism of $\text{Met}(\Sigma)$, we lose nothing by this.

To this end we first write the arbitrary small change in the metric in the form:

$$\delta \rho = f^*(\delta g)$$

(2.7)

utilizing the same diffeomorphism $f$ as in (2.6). The definition of $\delta g$ implies that it involves: a diffeomorphism close to the identity (which we write as $\exp(\xi)$), a small change in the Weyl rescaling, $\delta \phi$, and small changes $\{(\delta t_i)\}$ in the Teichmüller coordinates; $\delta g = (\exp(\xi))^*[e^{\phi+\delta \phi} \gamma(\{(t + \delta t)\})] - g$

We will work in the tangent space to $\text{Met}(\Sigma)$ at the point $g$. Recall that the trace (with respect to $g_{ab}$) of a symmetric tensor $\sigma_{ab}$ is by definition the contraction $g^{ab}\sigma_{ab}$. Observe that the traceless symmetric second-order covariant tensors, $dh_{ij}$, and the pure-trace tensors of the form $\delta \phi g_{ab}$, constitute orthogonal spaces with respect to the fundamental inner product (2.1). It is therefore convenient to express the general metrical deformation $\delta g$ as a sum of traceless and pure-trace parts. That will be done below.

Notation: We shall sum over repeated indices in the formulae of this article. The roman letters $a, b$ etc will usually vary over the surface coordinates (i.e., $1 \leq a, b \leq 2$),
whereas the indices $i, j, m, n$ will usually run through Teichmüller coordinates and thus range from 1 to $N = (6h - 6)$. Indices will be raised and lowered with respect to $g$. $D(\xi)$ will denote covariant derivative of the vector field $\xi$, with respect to $g$.

**Lemma II.1:** The infinitesimal change $\delta g$ defined above decomposes as:

$$\delta g_{ab} = \delta \phi g_{ab} + (P\xi)_{ab} + \delta t T^i_{ab}$$

with the last two terms constituting the tracefree part. Here

$$P : \{\text{Vector fields on } \Sigma\} \to \{\text{Symmetric traceless (2,0) tensors on } \Sigma\}$$

is the 1st order elliptic operator (depending on $g$), given by

$$(P\xi)_{ab} = D_a(\xi_b) + D_b(\xi_a) - g_{ab}D_c(\xi^c)$$

and,

$$T^i_{ab} = \frac{\partial}{\partial t^i}[g_{ab}(t)] - \frac{1}{2}g^{cd}\frac{\partial}{\partial t^i}[g_{cd}(t)]$$

The $T^i((t))$ are the (trace free parts of) the tangent vectors in the Teichmüller directions to the gauge slice $g((t))$.

**Proof:** Differential geometry on the surface gives: $(\exp(\xi))_* g_{ab} = g_{ab} + D_a(\xi_b) + D_b(\xi_a)$; and by Taylor expansion one sees: $e^{\delta \phi} g_{ab} = g_{ab} + \delta \phi g_{ab} + o(\delta \phi)$. Computing with the help of these gives (2.8). In (2.8), by suitably redefining $\delta \phi$, all the pure-trace component of the $\delta g$ has been absorbed in the first term on the right hand side.  

Now, the various spaces of tensor-fields on the surface $(\Sigma, g)$ carry natural inner products induced by the metric $g$. For the space of vector-fields the pairing is explicitly:

$$(\xi, \eta)_g = \int \int_{\Sigma} [\xi^a \eta^b g_{ab}] g_{vol}$$

and for the covariant 2nd-order traceless tensors the pairing is:

$$(R, S)_g = \int \int_{\Sigma} [R_{ab} S_{cd} g^{ac} g^{bd}] g_{vol}$$

(Whenever the metric $g$ is clear from the context we will take the liberty of suppressing that subscript.)

We can think of the operator $P$ as an unbounded closed operator defined on the appropriate dense domain of the Hilbert space of vector fields. Then, by basic functional analysis, the target Hilbert space of symmetric 2nd order tracefree tensors decomposes into the orthogonal direct sum: \{2nd order symmetric tracefree tensors\} = \{Range $P$\} $\oplus$ \{Ker $P^*$\}. 

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The range of $P$ constitutes the piece of the tangent space to $\text{Met}(\Sigma)$ arising from pullbacks of the metric $g$ by diffeomorphisms close to the identity. The Weyl (conformal) scalings of $g$ are already absorbed in the pure-trace part of the infinitesimal deformation $\delta g$. Consequently, the residual piece given by $\text{Ker} P^*$ is the finite-dimensional part comprising tangent vectors in the Teichmüller (slice) directions.

The final upshot is that the tangent space at $g$ to the infinite dimensional space $\text{Met}(\Sigma)$ decomposes as an orthogonal direct sum:

$$T_g(\text{Met}(\Sigma)) = \{\text{pure trace}\} \oplus \{\text{Range } P\} \oplus \{\text{Ker } P^*\} \quad (2.10)$$

In order to understand the Teichmüller deformations piece, which is of central interest to us, we need therefore to analyse the adjoint of $P$ (computed with respect to the inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_b$). One easily verifies that $P^*$ is nothing other than the d-bar operator on the space of (smooth) quadratic differentials on $(\Sigma, g)$ (thought of as a Riemann surface). Consequently, the kernel, $\text{Ker} P^*$, is the $6h - 6$ dimensional (real) subspace of the tracefree 2nd order symmetric tensors consisting of the (real and imaginary parts of) holomorphic quadratic differentials.

**Teichmüller’s Lemma:** $T^*_t(\mathcal{T}_h) \equiv H^0(X_t, K(X_t)^2)$: The upshot of the discussion above will be recognized by those familiar with Teichmüller theory as a variant of “Teichmüller’s lemma”; that lemma describes the complex cotangent space (at the conformal structure $g$) to the Teichmüller space $\mathcal{T}_h$, as precisely (namely, canonically isomorphic to) the vector space of holomorphic quadratic differentials on $(\Sigma, g)$. See [N1]. Indeed, the tangent space at $t \in \mathcal{T}_h$ to Teichmüller space, is, by Kodaira-Spencer, canonically isomorphic to the first cohomology $H^1(X_t, K^{-1})$, which, by Serre duality, is canonically isomorphic to the dual space of global holomorphic sections of $K^2$. By the Riemann-Roch theorem, this last space is a complex vector space of complex dimension $(3h - 3)$ (for each $t \in \mathcal{T}_h$). (Note: $K = K(X_t)$ denotes the holomorphic cotangent bundle of the Riemann surface $X_t$.)

**Bases for $H^0(X_t, K^2)$ as $X_t$ varies in $\mathcal{T}_h$:** To facilitate our work, we therefore introduce an arbitrary auxiliary choice of basis for the holomorphic quadratic differentials on each of the Riemann surfaces $X_t = (\Sigma, g((t)))$ – i.e., all over the Teichmüller space $\mathcal{T}_h$. Thus let $Q^m((t))$, $1 \leq m \leq (6h - 6)$, be a basis of the $6h - 6$ dimensional (over reals) vector space of holomorphic quadratic differentials on the Riemann surface $(\Sigma, g((t)))$, where the $Q^m$ are assumed to vary smoothly with $((t)) \in \mathcal{T}_h$. (That is easily accomplished by, for instance, utilizing Poincare theta-series to define the $Q^m$ with respect to real-analytically varying Fuchsian groups $\Gamma_{((t))}$.) Notice the important fact that once we have chosen bases as above on each surface along a particular slice, the job is accomplished for all slices, – because the $Q^m$ depend only on the conformal class of the metric under consideration.
Clearly, Teichmüller’s lemma, asserts that each $Q^m((t))$ can be considered as a nowhere vanishing 1-form over $T_h$. We will need this interpretation in Lemma II.5 below.

Substituting (2.8) into the Riemannian structure formula (2.1) we obtain immediately the desired re-expression for the Riemannian norm on the tangent space to $\text{Met}(\Sigma)$ at $g$:

$$||\delta g||^2 = ||\delta \phi||^2 + (P\xi, P\xi) + (T^i, Q^m)((Q^m, Q^n))^{-1}(Q^n, T^j)\delta t_i\delta t_j$$  \hspace{1cm} (2.11)

We explain the notations: The $L^2$ norm for functions on the surface $(\Sigma, g)$ was already shown in (1.12), – and that defines the first term $||\delta \phi||^2$. The $T^i((t))$ are, as shown above, the (trace free parts of) the tangent vectors in the Teichmüller directions.

Each $T^i$ is, of course, a second-order symmetric traceless covariant 2nd-order tensor on $(\Sigma, g)$. All the pairings appearing in the second and third terms on the right of (2.11) are the inner products exhibited in (2.9b). Thus, setting $R = S = P\xi$ in (2.9b) gives the second term, and similarly for the pieces in the third term. (Note that the inversion involved in the last term is matrix inversion.)

**Proposition II.2:** The volume measure induced by the Riemannian structure (2.1) on the infinite dimensional manifold $\text{Met}(\Sigma)$ expressed in the coordinates $(\phi, \xi, ((t)_i))$ is:

$$[Dg] = [(\det P^* P)^{1/2}][\det((Q^m, Q^n))]^{-1/2}[\det(T^i, Q^m)][D\phi][D\xi]dt_1 \wedge \cdots \wedge dt_{6h-6}$$  \hspace{1cm} (2.12)

By $[D\phi]$ we mean the volume measure on $\text{Conf}(\Sigma)$ induced by the $L^2$-norm (1.12) on the space of rescaling functions $\phi$. Similarly, $[D\xi]$ is the volume element on $\text{Diff}_0(\Sigma)$ arising from the inner product (2.9a) on vector fields.

**Proof:** The expression of the Riemannian structure (2.1) in the form (2.11) is adapted precisely to the slice $K$. Thus the usual formula $dg_{vol} = \sqrt{\det(g_{ij})}dx_1 \wedge \cdots \wedge dx_M$ for the Riemannian volume element on a $M$-dimensional manifold, when applied formally to this infinite dimensional $\text{Met}(\Sigma)$, gives (2.12). Indeed, in the infinite dimensional pieces corresponding to the first two terms on the right of (2.11), one needs to compute the determinants of the operators $I = \text{Identity}$ and $P$, respectively. For later convenience we have substituted $|\det P| = |(\det P^* P)^{1/2}|$.

Recall that the integrations over the gauge variables, $(\phi, \xi)$, is expected to produce the volumes of the gauge groups that we have been desiring to factor out (see (1.18)). Therefore, ignoring the $[D\phi][D\xi]$ in the above volume element of $\text{Met}(\Sigma)$ is clearly a reasonable method of accomplishing that aim, – and it is the physicist’s way of reducing the Polyakov integral to a finite dimensional integral over the given slice.

Dropping $[D\phi][D\xi]$ from (2.12) we obtain therefore a volume element on the slice $K$:

$$d\mu_K = \left[\frac{(\det P^*_g P_g)}{\det((Q^m, Q^n)_g)}\right]^{1/2}[\det((T^i, Q^m)_g)]dt_1 \wedge \cdots \wedge dt_{6h-6}$$  \hspace{1cm} (2.13)

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We emphasize that as the metric \( g = g((t)) \) varies over \( K \), the operator \( P = P_g \) varies, and thus the infinite determinant \( ([det P^* P]^{1/2}) \) above is a non-trivial function of \( ((t)) \).

Now we have reached the goal of expressing the Polyakov integral as a well-defined finite dimensional integration over any chosen slice:

**II.3 The Polyakov prescription:** In the light of Proposition II.2 and the discussion above, we interpret the Polyakov integral (1.18) as the following integral over the (arbitrarily chosen) slice:

\[
\int_{\text{Slice } K} \left[ \frac{det'(-\Delta_g)}{\text{Area}(\Sigma, g)} \right]^{-d/2} d\mu_K
\]  

(2.14)

The integrand above is (for any choice of \( d \)) a well-defined volume form on the slice \( K \), and since \( K \) is an embedded copy of \( \mathcal{T}_h \) the volume form can be considered as living on \( \mathcal{T}_h \).

**Remark:** We have been working over the Teichmüller space, \( \mathcal{T}_h \), rather than over the moduli space, \( \mathcal{M}_h \) – as was prescribed by (1.18). So, strictly speaking, the Polyakov integral (2.14) should be divided by the “cardinality of the mapping class group \( \text{MCG}_h \)”.

Alternatively, we should think of the integral in (2.14) only over a fundamental domain for the action of \( \text{MCG} \). In any event, we are only interested in the volume form = integrand on \( \mathcal{T}_h \) or \( \mathcal{M}_h \), rather than in the integral itself. The basic fact is that the Polyakov measure on \( \mathcal{T}_h \) is \( \text{MCG} \)-invariant, but its total integral over \( \mathcal{M}_h \) (i.e., over any fundamental domain) is infinite (see (3.3) below).

**II.C. Finally! The Polyakov measure on \( \mathcal{T}_h \):**

Now that we have reduced ourselves from the original Polyakov path integral (1.6) to the integral (2.14), the fundamental question is whether (2.14) is independent of the choice of the slice. The main result is:

**Theorem II.4:** When \( d/2 = 13 \), the integrand in (2.14), as a volume form on the Teichmüller space \( \mathcal{T}_h \), is independent of the choice of the slice \( K \). This is the Polyakov volume form, \( d(\text{Poly}) \), on \( \mathcal{T}_h \). It is mapping class group invariant, is absolutely continuous with respect to the Lebesgue measure class on \( \mathcal{T}_h \).

**Addendum to Theorem:** \( d(\text{Poly}) \) has the following expression in terms of the Weil-Petersson volume form on \( \mathcal{T}_h \):

\[
d(\text{Poly}) = [(Z_G'(1))^{-13} Z_G(2)] d(\text{Weil - Pet})
\]  

(2.15)

Here \( Z_G(s) \) is the Selberg zeta function (an entire function) associated to the (variable) Fuchsian group \( G \).

**Remark:** Throughout the above considerations, we are ignoring overall constant factors in expressions for the Polyakov form. It is our attitude that the Polyakov volume
form on each genus moduli is an interesting mathematical object defined only up to an arbitrary scaling constant. See (3.3) below also.

The notations in (2.15) will be explained as we go along. Combining (2.13) and (2.14) we realize that we have to deal with the following volume form on the slice $K$:

$$dP_K = \left[ \frac{\det(-\Delta_g)}{\text{Area}(\Sigma, g)} \right]^{\frac{d}{2}} \left[ \frac{\det(P^* g)}{\det((Q^m, Q^n)_g)} \right]^{1/2} \left[ \det((T^i, Q^m)_g) \right] dt_1 \wedge \cdots \wedge dt_{6h-6} \quad (2.16)$$

(the metric $g$ varies over $K$.)

The aim is to understand how far the above volume density depends on the choice of the Teichmüller slice. Let us break up the work into natural pieces:

**Lemma II.5:** The volume form on $\mathcal{T}_h$ given by:

$$d\lambda = \left[ \det((T^i, Q^m)_g) \right] dt_1 \wedge \cdots \wedge dt_{6h-6} \quad (2.17)$$

is independent of the choice of the slice. (It depends only on the choice of the bases of quadratic differentials $Q^m((t))$, which we have fixed, once and for all, over the entire Teichmüller space $\mathcal{T}_h$.)

Indeed, recalling that the $Q^m((t))$ can be considered as 1-forms over $\mathcal{T}_h$, this volume form is the following top dimensional form:

$$d\lambda = Q^1 \wedge \cdots \wedge Q^{6h-6} \quad (2.18)$$

**Proof:** A new slice $K^\sharp$ is obtained from the slice $K$ by scaling the metrics comprising $K$ in a $(t)$-dependent fashion: $g^\sharp((t)) = e^{\phi((t))} g((t))$. The trace-free parts of the tangent vectors to the slices are related by: $T_i^\sharp = e^\phi T_i$. Computing the relevant pairings now, by the law (2.9b) (with respect to the metrics $g$ and $g^\sharp$, respectively), we obtain

$$\langle T_i^\sharp, Q^m \rangle_{g^\sharp} = \langle T_i, Q^m \rangle_g \quad (2.19)$$

Note that the above equality simply represents the fact that the $T_i$ are tangent vectors to the Teichmüller space, whereas the $Q^m$ are co-tangent vectors to the same space, and that the pairing is just the duality pairing of tangents with cotangents. Hence, it is not surprising that (2.19) holds independent of scalings in the metric.

Consequently, (2.17) is independent of the choice of the slice, as claimed. The rest is straightforward. □

Consider therefore the fundamental function $F$ defined below (for each value of the unspecified “space-time” dimension $d$). Each $F = F_d$ is a real-valued function on the entire space of metrics, $\text{Met}(\Sigma)$:

$$F_d(g) = \left[ \frac{\det(-\Delta_g)}{\text{Area}(\Sigma, g)} \right]^{-\frac{d}{2}} \left[ \frac{\det(P^* g)}{\det((Q^m, Q^n)_g)} \right]^{1/2} \quad (2.20)$$
We shall see that $F_{26}$ is singled out!:

**Proposition II.6:** The function $F_d$ is invariant along the orbits of the gauge group $\{\text{Diff}^+(\Sigma) \times \text{Conf}(\Sigma)\}$ if and only if $d = 26$; $F_{26}$ thus descends as a function to the moduli space $\mathcal{M}_h$. Namely one has the remarkable invariance:

$$ F_{26}(f^*(e^\phi g)) = F_{26}(g) \quad (2.21) $$

for all $g \in \text{Met}(\Sigma)$, under all conformal rescalings and pullbacks by diffeomorphisms.

**Proof:** Let us first note the significance of the two infinite dimensional determinant factors that constitute the function $F_d$. Define:

$$ D_0(g) = \left[ \frac{\text{det}'(-\Delta_g)}{\text{Area}(\Sigma, g)} \right] \quad (2.22) $$

$$ D_1(g) = \left[ \frac{(\text{det}P^*P_g)}{\text{det}((Q^m, Q^n)_g)} \right] \quad (2.23) $$

Remember that the operator $\Delta_g$ is the Laplacian acting on smooth functions on $\Sigma$, and that $P^*P_g$ is the Laplacian acting on smooth vector-fields.

The problem is to understand the variation of these two determinantal functions on $\text{Met}(\Sigma)$ under a conformal rescaling of the metric. This is a standard type of problem for the application of heat kernel techniques – the results are clearly exposed in Alvarez[Alv]. See also [AN]. Recall the heat-kernel (i.e., zeta-function) regularized determinant for operators in infinite dimensions:

$$ \log(\text{det}'D) = -\lim_{\epsilon \to 0} \int_\epsilon^{\infty} [\text{Trace}'(e^{-tD})] \frac{dt}{t} \quad (2.24) $$

The notation $\text{det}'$ and $\text{Trace}'$ mean that the determinant or trace is being calculated after discarding the zero eigenvalues – namely on the orthogonal complement of the kernel of $D$.

Consider an arbitrary infinitesimal conformal scaling of the metric $g$ to $\gamma = e^{\delta \phi} g$. The corresponding perturbation in the non-zero eigenvalues of the relevant operators appearing in (2.22) and (2.23) can be calculated. The first variations turn out as below, ($R_g$ denotes the scalar curvature for metric $g$):

$$ \delta(\log D_0) = \frac{1}{12\pi} \int_\Sigma R_g \delta \phi d_g(\text{vol}) \quad (2.25) $$

$$ \delta(\log D_1) = \frac{13}{12\pi} \int_\Sigma R_g \delta \phi d_g(\text{vol}) \quad (2.26) $$
Therefore, $\delta F_d$ is \(\frac{1}{12\pi}(13 - \frac{d}{2})\) times the integral appearing on the right of the formulae (2.25) and (2.26). The invariance property (2.21) for $F_{26}$ follows immediately.

**Proof of Theorem II.4:** Combining the gauge-invariant function $F_{26}$ on $Met(\Sigma)$ with the slice-independent volume form $d\lambda$ of Lemma II.5, we see that we have proved that the Polyakov volume form on $T_h$ has the expression:

$$d(Poly) = F_{26}d\lambda$$  \hspace{1cm} (2.27)

Since the bases for quadratic differentials were introduced simply as an artifice to help simplify our formulae, it is obvious that the volume form $d(Poly)$ above is also independent of the choice of these bases $\{Q^m((t))\}$. [Note: Glancing at (2.16), we may see directly that the combination of $Q$-dependent pieces: $[\det((T^2, Q^m)_g)][\det((Q^m, Q^p)_g)]^{-1/2}$, evidently remains invariant when the $Q^m$ are subjected to an arbitrary (in general $t$-dependent) change of basis.]

We have therefore established the existence of the Polyakov volume form on $T_h$, independent of all arbitrary choices, as stated in Theorem II.4. \qed

As general references for the matter we presented see [Alv], [AN], [Nel]. The expression for $d(Poly)$ in terms of $d(Weil - Pet)$ will be dealt with in the next section.

**III: GEOMETRY OF TEICHMÜLLER SPACE AND POLYAKOV VOLUME**

**III.A. Weil-Petersson and Polyakov:**

**Weil-Petersson Riemannian structure on $T_h$:** Since the cotangent space to $T_h$ at any point $t$ is canonically isomorphic to the vector space of holomorphic quadratic differentials on the Riemann surface $X_t = (\Sigma, \gamma(t))$, a hermitian structure is induced on $T_h$ by utilizing the Petersson pairing of holomorphic quadratic differentials on $X_t$. In fact, let $\gamma(t)$ be a hyperbolic metric representing the conformal structure $t \in T_h$; if the metric is expressed as $\gamma = \lambda(z)|dz|$ in terms of local holomorphic (=isothermal) coordinates, then set

$$(Q^1, Q^2)_{WP} = \int_{\Sigma} Q^1(z)\overline{Q^2(z)}\lambda(z)^{-2}dz \wedge d\bar{z}$$  \hspace{1cm} (3.1)

This is Weil-Petersson hermitian metric of $T_h$. See [IT], [J], [N1]. The corresponding volume form on $T_h$ we shall denote by $d(Weil - Pet)$.

**Lemma III.1:** (a) The Riemannian structure (2.1) of $Met(\Sigma)$, restricted to a horizontal Poincare slice, is Weil-Petersson on $T_h$.

(b) Choose the bases $\{Q^m((t))\}$ for the spaces of holomorphic quadratic differentials to be orthonormal (pairing (2.9b)) with respect to the corresponding hyperbolic metrics
Then the (slice-independent) volume form \( d\lambda = Q^1 \wedge \cdots \wedge Q^{6h-6} \) described in Lemma II.5 is \( d(\text{Weil} - \text{Pet}) \).

**Proof (a):** A holomorphic quadratic differential \( Q \) on \((\Sigma, \gamma)\) is a cotangent vector to \( T_h \) there; the Weil-Petersson pairing sets up an isomorphism of the cotangent space with the tangent space – thus \( Q \) corresponds to a certain tangent vector \( v_Q \) to \( T_h \). Identifying \( T_h \) with the slice, the tangent \( v_Q \) is given by an infinitesimal change in the hyperbolic metric \( \gamma \). But that perturbation of metric, say \( \delta \gamma_Q \), is, in its turn, represented by a certain Beltrami differential on \((\Sigma, \gamma)\). Indeed, one sees that the co-vector \( Q \) corresponds to the harmonic (Bers') Beltrami differential \( \mu_Q = Q(z)y^2 \), \((z = x + iy)\).

Above, we are utilizing uniformization to represent the Riemann surface \((\Sigma, \gamma)\) as \( H/\Gamma \), \( H \) being the Poincare upper half-plane, and \( \Gamma \) a Fuchsian group operating thereon. The hyperbolic metric \( \gamma = y^{-1}|dz| \) therefore gets deformed to \( \gamma + \delta \gamma = y^{-1}|dz + \varepsilon \mu(z)d\bar{z}| \) with \( \mu = \mu_Q \) as shown. See \([N1]\) for details.

Expanding out the right side we may compute the \( \delta \gamma = \delta \gamma_Q \); then substituting in the formula (2.1) we get \((\delta \gamma_{Q_1}, \delta \gamma_{Q_2}) = (Q^1, Q^2)_{WP} \), as desired.

(b): Clearly, at every point \((t) \in T_h \), the co-vectors \( \{Q^m((t))\} \) now constitute an orthonormal basis for \( T^*_h((t))(T_h) \) with respect to the Weil-Petersson inner-product. Therefore, \( Q^1 \wedge \cdots \wedge Q^{6h-6} = d(\text{Weil} - \text{Pet}) \)

**Remark:** The bases of quadratic differentials above can be constructed using Poincare theta series with respect to varying Fuchsian group \( \Gamma((t)) \) and then applying the Gram-Schmidt orthonormalization procedure. Since the bases are arbitrary up to any smooth choice \( T_h \to GL(6h - 6, \mathbb{R}) \), we note that the corresponding measure \( Q^1 \wedge \cdots \wedge Q^{6h-6} = d\lambda \) can be, a priori, chosen quite arbitrarily.

**Selberg zeta and determinants for Laplacians:** Associated to the hyperbolic Riemann surface \((\Sigma, \gamma) = H/\Gamma \), \( \Gamma \) the uniformizing (purely hyperbolic, cocompact) Fuchsian group, Selberg defines

\[
Z_\Gamma(s) = \prod_\alpha \prod_{n \geq 0} \left[ 1 - \exp \left\{-l(g_\alpha)(s + n)\right\} \right]
\]

where \( \{g_\alpha\} \) are a complete set of representatives for the conjugacy classes of the primitive elements of \( \Gamma \) (i.e., those allowing no non-trivial \( n \)-th roots in \( \Gamma \)); \( l(g) \) denotes the length of the geodesic in the free homotopy class determined by \( g \). This function is defined by the above product for \( \text{Re}(s) > 1 \), and can be shown to have an analytic continuation as an entire function in the whole \( s \)-plane.

The Selberg trace formula, that goes hand in hand with the above zeta function, allows one to express the heat-kernel regularized determinants of the Laplace operators on \((\Sigma, \gamma)\) – on functions, vector-fields and higher order tensor fields – as certain...
values of the above holomorphic function. As explained, for example, in [DP], one gets $\text{det}'(-\Delta) = Z'(1)$ (up to a constant independent of the Fuchsian group), and similarly, $\text{det}(P^*P) = Z(2)$. (All the numbers $Z(k)$, integral $k$, have interpretations as determinants of Laplacians acting on suitable spaces of tensors on $(\Sigma, \gamma)$.)

**Proof of Addendum to Theorem II.4:** Glancing now at (2.22), (2.23), and armed with (b) of Lemma III.1, we have clearly obtained the expression for $d(\text{Poly})$ claimed in the Addendum to Theorem II.4. (Note: Over hyperbolic metrics the Gauss-Bonnet shows that $\text{Area}(\Sigma, \gamma) = 4\pi(h-1)$; moreover, the determinant appearing in the denominator of (2.23) we had already normalized to unity by our choice of orthonormal bases for quadratic differentials.)

**Remark:** The asymptotics of the Selberg zeta functions, when the Fuchsian groups approach the boundary (Deligne-Mumford boundary) of $\mathcal{M}_h$, can be used to show that the Polyakov volume form blows up near the boundary so fast that the fundamental amplitude integral is divergent:

$$\int_{\mathcal{M}_h} d(\text{Poly}) = \infty$$  \hspace{1cm} (3.3)

(Recall that the Weil-Petersson volume of $\mathcal{M}_h$, on the other hand, is finite.) (3.3) demonstrates that no finite answer is forthcoming via the Polyakov prescription for the “vacuum-to-vacuum” amplitude of the bosonic string.

**III.B. Complex geometry and Polyakov:**

We are now at the point where we can explain our deepest reason for excitement about the Polyakov measure. **In fact, $d(\text{Poly})$ has a natural and simple construction (over each $\mathcal{T}_h$) by simply the complex analytic geometry of $\mathcal{T}_h$ – with no inputs whatsoever from physics.**

**The universal family and allied bundles over $\mathcal{T}_h$:**

The Teichmüller spaces are fine moduli spaces. Namely, the total space $\Sigma \times \mathcal{T}_h$ admits a natural complex structure such that the projection to the second factor

$$\psi_h : \mathcal{C}_h := \Sigma \times \mathcal{T}_h \to \mathcal{T}_h$$  \hspace{1cm} (3.4)

gives the universal Riemann surface over $\mathcal{T}_h$. This means that for any $\eta \in \mathcal{T}_h$, the submanifold $\Sigma \times \eta$ is a complex submanifold of $\mathcal{C}_h$, and the complex structure on $\Sigma$ induced by this embedding is represented by $\eta$. As is well-known, (Chapter 5 in [N1]), the family $\mathcal{C}_h \to \mathcal{T}_h$ is the *universal* object in the category of holomorphic families of genus $h$ marked Riemann surfaces.

We recall now the *determinant of cohomology* construction for obtaining line bundles over any parameter space of Riemann surfaces. Let $X$ be a compact Riemann
surface and $L$ be a holomorphic line bundle on $X$. The determinant of cohomology of $L$ is defined to be the 1-dimensional complex vector space $\text{det}H^*(L) = \text{det}(L)$:

$$\text{det}(L) = (\wedge^{\text{top}} H^0(X,L)) \bigotimes (\wedge^{\text{top}} H^1(X,L)^*)$$  \hspace{1cm} (3.5)

We are ready to pass to families. Let $\pi : \mathcal{X} \rightarrow S$ be a holomorphic family of compact Riemann surfaces parametrized by a base $S$ which is complex analytic space. The prototypical instance is the universal family described over $\mathcal{T}_h$. The definition demands that $\pi$ be a holomorphic submersion with compact and connected fibers of complex dimension one – each fiber being of genus $h$.

For any point $s \in S$, the construction (3.5) gives a complex line $\text{det}(L_s)$. The basic fact is that these lines fit together to give a holomorphic line bundle on $S$ (see [BGS]), which is called the determinant of cohomology bundle of $L_S$, and is denoted by $\text{det}(L_S)$. (Note: This is not to be confused with the top exterior power of $L_S$ – which is a line bundle over the total space $\mathcal{X}$.) The general definition arises from the direct image functors $R_*$ of algebraic geometry. See [KM], [D].

Over each genus Teichm"uller space we thus have a sequence of natural determinants of cohomology bundles arising from the powers of the relative tangent bundles along the fibers of the universal curve. Indeed, let $\omega_h \rightarrow \mathcal{C}_h$ be the relative cotangent bundle for the projection $\psi_h$ in (3.4). The determinant line bundle over $\mathcal{T}_h$ arising from its $n$-th tensor power is fundamental for us, and we shall denote it by:

$$\text{DET}_{n,h} := \text{det}(\omega_h^n) \rightarrow \mathcal{T}_h, \quad n \in \mathbb{Z}$$  \hspace{1cm} (3.6)

Applying Serre duality shows that there is a canonical isomorphism $\text{DET}_{n,h} = \text{DET}_{1-n,h}$, for all $n$. $\text{DET}_{0,h} = \text{DET}_{1,h}$ is called the Hodge line bundle over $\mathcal{T}_g$.

Remarks: (i) The determinant of cohomology bundle, $\text{det}(L_S)$, is functorial with respect to base change. Given any morphism $\gamma : S' \rightarrow S$ this means, in particular, that there is a canonical isomorphism of the determinant of cohomology associated to the pulled back family over $S'$ with the pullback by $\gamma$ of the original determinant bundle over $S$. See [BNS] for details.

(ii) The determinant of cohomology construction, $\text{det}(L_S)$, produces a bundle over the parameter space $S$ induced by the bundle over the total space $\mathcal{X}$; now, the Grothendieck-Riemann-Roch (GRR) theorem (see [BGS], [D]) gives a canonical isomorphism of $\text{det}(L_S)$ with a combination of certain bundles (on $S$) obtained from the direct images of the bundle $L_S$ and the relative tangent bundle $T_{X/S}$. The GRR theorem is important for our work below.

(iii) Whenever the bundles $L_S$ and the relative tangent bundle above are assigned smooth hermitian structures, the determinant of cohomology inherits a smooth hermitian structure, due to Quillen [Q], in a functorial way. See [BGS], [D], [BNS].
The canonical isomorphisms arising from base change (see (i)) then become isometric isomorphisms.

The vertical (=relative) tangent bundle for the universal family, namely the bundle \( T_{\mathcal{C}_h}/T_h \rightarrow \mathcal{C} \) comprising the tangent bundles of the fibering Riemann surfaces, carries a canonical hermitian structure by assigning the Poincaré hyperbolic metric on each surface (utilizing the uniformization theorem with moduli parameters). As a consequence, applying remark (iii) above, the holomorphic line bundles \( DET_{n,h} \) carry natural Quillen hermitian metric arising from the Poincaré metrics on the fibers of \( \mathcal{C}_h \).

Observe that by the naturality of the above constructions it follows that the action of \( MC_h \) on \( T_h \) has a natural lifting as automorphisms of these \( DET \) bundles. These automorphisms are unitary with respect to the Quillen structure.

**The fibers of \( DET_{n,h} \):** Notice that the fiber of Hodge, over \( X \in T_h \), is the top exterior power of the vector space of holomorphic Abelian differentials on \( X \). Similarly, the fiber of \( DET_{2,h} \) is the \((3h - 3)\) exterior power of the vector space of holomorphic quadratic differentials on \( X \), and so on. By some complex geometry (e.g., Poincaré theta series with respect to holomorphically varying quasi-Fuchsian groups), one can create the natural \( MCG \) equivariant holomorphic vector bundle, \( V_{n,h} \) over \( T_h \) by attaching over \( X \) the fiber \( H^0(X, K_X^n) \) \((n = 1, 2, ..)\). Therefore, (3.6) shows that the top exterior powers of these vector bundles over \( T_h \) are nothing other than the determinant of cohomology bundles \( DET_{n,h} \) we are describing.

In particular, by Teichmüller’s lemma we know \( V_{2,h} \) is canonically isomorphic to the holomorphic cotangent bundle of Teichmüller space. Thus, there are canonical isomorphisms:

\[
(3h - 3) \wedge V_{2,h} = \wedge^\text{top} T^*T_h = DET_{2,h}
\]

(3.7)

each of the above being a description of the canonical line bundle over \( T_h \).

We also recall that holomorphic 1-forms can be paired naturally, in the \( L^2 \) sense, on any Riemann surface \( X \) via the “Hodge pairing”

\[
(\alpha, \beta) = -i \int_X \alpha \wedge \bar{\beta}
\]

(3.8)

This gives a \( MCG \) invariant hermitian metric on \( V_{1,h} \), and hence induces a \( MCG \) invariant hermitian structure (called the Hodge metric) on the Hodge bundle. (The Hodge metric has to be modified suitably by a factor given by the determinant of the Laplacian to obtain the Quillen metric on that bundle.)

**Mumford isomorphisms and Polyakov volume:** These natural line bundles over \( T_h \) will be considered as \( MCG \)-equivariant line bundles, and the isomorphisms we talk about will be \( MCG \)-equivariant isomorphisms. By applying the Grothendieck-
Riemann-Roch theorem indicated above, Mumford [Mum] had shown that $\text{DET}_{n,h}$ is canonically isomorphic to a certain fixed (genus-independent!) tensor power of the Hodge bundle.

**Proposition III.2:**

$$\text{DET}_{n,h} \cong \text{DET}_{1,h} \otimes (6n^2 - 6n + 1)^{(3.9)}$$

The above isomorphism is isometric with respect to the Quillen metrics. An isomorphism (3.9) is ambiguous only up to a non-zero scalar.

**Remarks:** (i) It can be shown that the Picard group of (isomorphism classes of) all $MC_h$-equivariant holomorphic line bundles over $\mathcal{T}_h$, (which can be identified as the group of holomorphic line bundles over the moduli space $\mathcal{M}_h$), is a cyclic group generated by the Hodge bundle. This can be shown from work of Harer, Mumford and Arbarello-Cornalba. That cyclic group is of order 10 for $h = 2$, and is infinite cyclic for $h > 2$. The above Proposition shows that the natural sequence of determinant of cohomology line bundles pick out a certain explicit subsequence from the Picard group.

(ii) The $MCG(\equiv MC_h)$ invariant holomorphic functions on $\mathcal{T}_h$ – namely those that descend to $\mathcal{M}_h$ – are it constant, for all $h > 2$. This follows from the Satake compactification theory for $\mathcal{M}_h$. Thus, although $\mathcal{M}_h$ is non-compact, in certain aspects it behaves like a compact analytic space. Consequently, (at least when $h > 2$), any two isomorphisms between bundles over $\mathcal{M}_h$ can only be ambiguous up to a scale factor, as asserted.

There is a remarkable connection, discovered by Belavin and Knizhnik [BK], between the Mumford isomorphism above for the case $n = 2$, [i.e., that $\text{DET}_2$ is the 13-th tensor power of Hodge], and the existence of the Polyakov string measure on the moduli space $\mathcal{M}_g$. In fact this 13 is the same lucky number as the value of $d/2$ in Proposition II.6. See [BK],[Bos],[Nel].

First, it is elementary to see that assigning a hermitian metric on the canonical bundle of any complex space gives a measure on that space. Indeed, fixing a volume density on a space simply amounts to fixing a fiber metric on the canonical line bundle, $K$, – because then we know the unit vectors in $K$ – and absolute square gives volume. Now, $K$ for the Teichmüller space is nothing other than $\text{DET}_{2,h}$ – recall (3.7) above. Thus, any natural hermitian structure on $\text{DET}_{2,h}$ becomes a choice of a natural volume form on $\mathcal{T}_h$.

**Theorem III.3:** The Hodge bundle, $\text{Hodge} \equiv \text{DET}_{1,h}$, has its natural Hodge metric, arising from (3.8). We may transport the corresponding metric on $\text{Hodge}^{13}$ to $\text{DET}_{2,h}$ by Mumford’s isomorphism, (the choice of isomorphism being unique up to scalar) – thereby obtaining a $MCG$ invariant volume form, say $d(Mum)$, on $\mathcal{T}_h$ (also unambiguous up to the choice of a scalar). This $d(Mum)$ is none other than the Polyakov
volume form \(d(\text{Poly})\) of Theorem II.4.

**Sketch of Proof:** Choose a local holomorphic frame, \((\omega_1(t), \cdots, \omega_h(t))\) for the rank \(h\) vector bundle \(V_{1,h}\) over a local holomorphic \(t\)-patch in \(T_h\). Suppose that \((\psi_1(t), \cdots, \psi_{3h-3}(t))\) is a local holomorphic frame for \(V_{2,h}\) over the same neighbourhood such that under the Mumford isomorphism:

\[
[\omega_1 \wedge \cdots \wedge \omega_h]^{13} \mapsto (\psi_1 \wedge \cdots \wedge \psi_{3h-3}) \quad (3.10)
\]

Then the Mumford volume form on the \(t\)-coordinate patch is immediately seen to have the expression:

\[
d(Mum) = [\det((\omega_i(t), \omega_j(t)))]^{-13}d\lambda \quad (3.11)
\]

with the matrix of pairings being the Hodge pairings of (3.8), and where the measure \(d\lambda\) is given by Lemma II.5, equation (2.18): \(d\lambda = \psi_1 \wedge \cdots \wedge \psi_{3h-3} \wedge \overline{\psi_1} \wedge \cdots \wedge \overline{\psi_{3h-3}}\).

Utilizing therefore the bases \(Q^m\) for quadratic differentials given by precisely these \(\psi_k\) and their conjugates, we may compare \(d(Mum)\) of (3.11) with \(d(\text{Poly})\) as shown in equation (2.20); we derive that the ratio of these two volume forms on \(T_h\) is the following function:

\[
\frac{d(\text{Poly})}{d(Mum)} = F_{26}(g(t))[\det((\omega_i(t), \omega_j(t)))]^{13}
\]

\[
= \left[\frac{\det(-\Delta_g)}{\text{Area}(\Sigma, g)}\right]^{-13} \left[\frac{\det P_g^* P_g}{\det((\psi_m, \psi_n)_g)}\right]^{1/2} [\det((\omega_i(t), \omega_j(t)))]^{13} \quad (3.12)
\]

Above, \(g = g(t)\) is any Riemannian metric representing the conformal structure \(t \in T_h\).

Since both volume forms under consideration are, by construction, modular invariant, we note that the above ratio, say \(G(t)\), is a real valued MCG invariant function on the Teichmüller space.

**Claim:** \(\frac{d(\text{Poly})}{d(Mum)} = G(t)\) is the absolute value square of a global holomorphic function \(f\) on \(T_h\).

Basically one computes \(\delta \delta \log G(t)\) and shows that this is identically zero. This local computation is explained in [BK],[Nel] and other references. Thus holomorphic \(f_U\) exists on any local \((t)\)-neighbourhood \(U\) satisfying \(G(t) = |f(t)|^2\). But \(T_h\) is simply connected, and it is clear that these local \(f_U\) can be patched up to analytically continue along all paths in \(T_h\). It follows directly that there is a holomorphic function \(f\) defined on the entire \(T_h\) such that \(G(t) \equiv |f(t)|^2\).

Finally, we show that this global \(f\) is necessarily modular invariant. In fact, under any modular transformation \(f\) can only be multiplied by factor of absolute value one – thus \(f\) gives rise to a character for the modular group. But, since that group, MCG,
is generated by elements of finite order (!), which all have fixed points in their action on $\mathcal{T}_h$, it follows that the character must be trivial. That is as desired.

But such $f$, and hence $G$, must be constant (recall the Satake compactification remark) – and we are through. $\square$

Remarks: (i) The expression for $d(Mum)$ in (3.11) also allows one to study the blow up of the volume form as one approaches the Deligne-Mumford boundary of the moduli space. This can be used to reprove the divergence of the string amplitude stated in (3.3).

(ii) As we said above Theorem III.3, any natural hermitian structure on $DET_{2,h}$ is the assignment of a natural volume form on $\mathcal{T}_h$. Therefore the Quillen hermitian structure on this $DET$ bundle assigns also a volume element to Teichmüller/moduli space. This is being studied in relation to the Polyakov volume. For example, is the Quillen volume of $\mathcal{M}_h$ finite?

IV: THE UNIVERSAL TEICHMÜLLER SPACE OF COMPACT SURFACES

The moral of the story above is that the presence of the Mumford isomorphisms over the moduli space of genus $g$ Riemann surfaces describes the Polyakov measure structure thereon! It is consequently a natural problem to try to create genus-independent versions of these constructions by working over some universal parameter space that parametrizes Riemann surfaces of varying genus. Indeed, the string may sweep over world-sheets of arbitrary genus – so that the problem raised is fundamental to non-perturbative string theory.

In this and the following section we give a concise report of our work [BNS] where we succeed in finding a genus-independent description of the Mumford isomorphisms over a certain universal parameter space $\mathcal{T}_\infty$. We would like to emphasize one point: all our constructions are equivariant under the action on $\mathcal{T}_\infty$ of an intriguing new mapping class group $MC_\infty$.

IV.A. The direct limit, $\mathcal{T}_\infty$, of classical Teichmüller spaces:

We start with a fundamental topological situation. Let

$$\pi : \tilde{X} \longrightarrow X$$

be an unramified covering map, orientation preserving, between two compact connected oriented two manifolds $\tilde{X}$ and $X$ of genera $\tilde{g}$ and $g$, respectively. Assume $g \geq 2$. The degree of the covering $\pi$, which will play an important role, is the ratio of the respective Euler characteristics; namely, $\text{deg}(\pi) = (\tilde{g} - 1)/(g - 1)$.

Given any complex structure on $X$, we may pull back this structure via $\pi$ to a complex structure on $\tilde{X}$. Now, the homotopy lifting property guarantees that there
is a unique diffeomorphism \( \tilde{f} \in \text{Diff}_0(\tilde{X}) \) which is a lift of any given \( f \in \text{Diff}_0(X) \). Mapping \( f \) to \( \tilde{f} \) defines an injective homomorphism of \( \text{Diff}_0^+(X) \) into \( \text{Diff}_0(\tilde{X}) \). Consequently, \( \pi \) induces an injection of the smaller Teichmüller space into the larger one:

\[
\mathcal{T}(\pi) : \mathcal{T}_g \longrightarrow \mathcal{T}_{\tilde{g}}
\]  

(4.2)

It is known that this map \( \mathcal{T}(\pi) \) is a proper holomorphic embedding between these finite dimensional complex manifolds; \( \mathcal{T}(\pi) \) respects the quasiconformal-distortion (=Teichmüller) metrics. From the definitions it is evident that this embedding between the Teichmüller spaces depends only on the (unbased) isotopy class of the covering \( \pi \).

At the level of Fuchsian groups, one should note that any covering space \( \pi \) corresponds to the choice of a subgroup \( H \) of finite index (=\( \text{deg}(\pi) \)) in the uniformizing group \( G \) for \( X \), and the embedding (4.2) is then the standard inclusion mapping \( \mathcal{T}(G) \rightarrow \mathcal{T}(H) \); (see Chapter 2, [N1]).

Remark: One notices that \( \mathcal{T} \) is a contravariant functor from the category of closed oriented topological surfaces, morphisms being covering maps, to the category of finite dimensional complex manifolds and holomorphic embeddings. We shall have more to say along these lines below.

We construct a category \( \mathcal{A} \) of certain topological objects and morphisms: the objects, \( \text{Ob}(\mathcal{A}) \), are a set of compact oriented topological surfaces each equipped with a base point \((\star)\), there being exactly one surface of each genus \( g \geq 0 \); let the object of genus \( g \) be denoted by \( X_g \). The morphisms are based isotopy classes of pointed covering mappings

\[
\pi : (X_{\tilde{g}}, \star) \rightarrow (X_g, \star)
\]

there being one arrow for each such isotopy class. Note that the monomorphism of fundamental groups induced by (any representative of) the based isotopy class \( \pi \), is unambiguously defined.

The direct system of classical Teichmüller spaces: Fix a genus \( g \) and let \( X = X_g \). Observe that all the morphisms with the fixed target \( X_g \):

\[
M_g = \{ \alpha \in \text{Mor}(\mathcal{A}) : \text{Range}(\alpha) = X_g \}
\]  

(4.3)

constitute a directed set under the partial ordering given by factorisation of covering maps. Thus if \( \alpha \) and \( \beta \) are two morphisms from the above set, then \( \beta \succ \alpha \) if and only if the image of the monomorphism \( \pi_1(\beta) \) is contained within the image of \( \pi_1(\alpha) \); that happens if and only if there is a commuting triangle of morphisms: \( \beta = \alpha \circ \theta \). It is important to note that the factoring morphism \( \theta \) is uniquely determined because we are working with base points. [Remark: Notice that the object of genus 1 in \( \mathcal{A} \) only has morphisms to itself – so that this object together with all its morphisms (to and from) form a subcategory.]
By (4.2), each morphism of $A$ induces a proper, holomorphic, Teichmüller-metric preserving embedding between the corresponding finite-dimensional Teichmüller spaces. We can thus create the natural direct system of Teichmüller spaces over the above directed set $M_g$, by associating to each $\alpha \in M_g$ the Teichmüller space $T(X_g(\alpha))$, where $X_g(\alpha) \in \text{Ob}(A)$ denotes the domain surface for the covering $\alpha$. To each $\beta \succ \alpha$ one associates the corresponding holomorphic embedding $T(\theta)$ (with $\theta$ as in the paragraph above). From this direct system we form the direct limit Teichmüller space over $X = X_g$:

$$T_\infty(X_g) = T_\infty(X) := \text{ind.lim.} T(X_g(\alpha))$$ (4.4)

$T_\infty(X)$ is a metric space with the Teichmüller metric, and it also has a natural Weil-Petersson Riemannian structure obtained from scaling the Weil-Petersson pairing on each finite dimensional stratum, $T_g$, by the factor $(g - 1)^{-1}$. See [NS].

$T_\infty$ is our commensurability Teichmüller space – which will serve as the base space for universal Mumford isomorphisms.

**The Teichmüller space, $T(H_\infty)$, of the hyperbolic solenoid:** Over the same directed set $M_g$ we may also define a natural inverse system of surfaces, by associating to $\alpha \in M_g$ a certain copy, $S_\alpha$ of the pointed surface $X_g(\alpha)$. [Note: Fix a universal covering over $X = X_g$. $S_\alpha$ can be taken to be the universal covering quotiented by the action of the subgroup $\text{Im}(\pi_1(\alpha)) \subset \pi_1(X, \star)$.] If $g \geq 2$, then the inverse limit of this system is the universal solenoidal surface $H_\infty(X) = \text{proj.lim.} X_g(\alpha)$ that was studied in [S],[NS].

In fact, $H_\infty$ is a compact topological space that fibers over $X$ with the fibers being Cantor sets. The path components of $H_\infty$, (with leaf-topology), are simply connected two-manifolds restricted to each of which the projection $\pi_\infty : H_\infty \rightarrow X$ becomes a universal covering. There are uncountably many of these path components ("leaves") in $H_\infty$, and each is a dense subset in $H_\infty$. Each leaf is thus, morally speaking, a hyperbolic plane. That is why we call $H_\infty$ the universal hyperbolic solenoid. The facts above follow from a careful study of this inverse system of surfaces, the main tool being the lifting of paths in $X$ to its coverings.

As explained in [S],[NS], the solenoid $H_\infty$ has a natural Teichmüller space comprising equivalence classes of complex structures on the leaves – the leaf complex structures being required to vary continuously in the fiber (Cantor) directions. In particular, any complex structure assigned to any of the surfaces $X_g(\alpha)$ appearing in the inverse tower can be pulled back to all the surfaces above it – and therefore assigns a complex structure of the sort demanded on $H_\infty$ itself. These complex structures that arise from some finite stage can be characterized as the "transversely locally constant" (TLC) ones (see [NS]), and they comprise precisely the dense subset $T_\infty(X)$ sitting within the separable Banach manifold $T(H_\infty(X))$. We collect some of these
thoughts in the:

**Proposition IV.1:** The “ind-space” $\mathcal{T}_\infty(X)$, (see [Sha]), is the inductive limit of finite dimensional complex manifolds, and hence carries a complex structure defined strata-wise. The completion of $\mathcal{T}_\infty(X)$ with respect to the Teichmüller metric is the separable complex Banach manifold $\mathcal{T}(H_\infty(X))$.

Alternatively, $\mathcal{T}_\infty(X)$ can be embedded in Bers’ universal Teichmüller space, $\mathcal{T}(\Delta)$, as a directed union of the Teichmüller spaces of Fuchsian groups. (The Fuchsian groups vary over the finite index subgroups of a fixed cocompact Fuchsian group $G$, $X = \Delta/G$.) The closure in $\mathcal{T}(\Delta)$ of $\mathcal{T}_\infty(X)$ is a Bers-embedded copy of $\mathcal{T}(H_\infty(X))$.

**Remark:** It is evident, but important to note, that the spaces $\mathcal{T}_\infty(X)$ and $\mathcal{T}(H_\infty(X))$ we are dealing with do not really depend on the choice of $X$. If we were to start with a surface $X'$ of different genus (also greater than one), then we could pass to a common covering surface (always available!), and hence the limit spaces we construct would be isomorphic.

**IV.B. The commensurability mapping class group** $MC_\infty = Aut(\mathcal{T}_\infty)$:

A remarkable but obvious fact about the above construction is that every morphism $\pi : Y \to X$ of $\mathcal{A}$ induces a natural Teichmüller metric preserving homeomorphism

$$\mathcal{T}_\infty(\pi) : \mathcal{T}_\infty(Y) \to \mathcal{T}_\infty(X) \quad (4.5)$$

$\mathcal{T}_\infty(\pi)$ is invertible simply because the morphisms of $\mathcal{A}$ with target $Y$ are cofinal with those having target $X$ (thus all finite ambiguities are forgotten in passing to the inductive limits!). It is also clear that $\mathcal{T}_\infty(\pi)$ is a biholomorphic identification (with respect to the strata-wise complex structures). [Note that $\mathcal{T}_\infty$ is covariant – whereas the Teichmüller functor $\mathcal{T}$ itself was contravariant.]

It follows that each $\mathcal{T}_\infty(X)$, and so also its metric completion $\mathcal{T}(H_\infty(X))$, is equipped with a large automorphism group – one from each undirected cycle of morphisms of $\mathcal{A}$ starting from $X$ and returning to $X$. By repeatedly using pull-back diagrams (i.e., by choosing appropriate connected component of the fiber product of covering maps), it is fairly easy to see that the automorphism of $\mathcal{T}_\infty(X)$ arising from any (many arrows) cycle can be obtained simply from a two-arrow cycle $\hat{X} \to X$.

Namely, whenever we have (the isotopy class of) a “self-correspondence” of $X$ given by any two non-isotopic coverings, say $\alpha$ and $\beta$,

$$\hat{X} \to X \quad (4.6)$$

we can create a corresponding automorphism $R \in Aut(\mathcal{T}_\infty(X))$ defined as the composition: $R = \mathcal{T}_\infty(\beta) \circ (\mathcal{T}_\infty(\alpha))^{-1}$. 

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These automorphisms constitute a group that we shall call the *commensurability modular group*:

\[ MC_{\infty}(X) = Aut(T_{\infty}(X)) \]  

acting on \( T_{\infty}(X) \) and on \( T(H_{\infty}(X)) \).

To clarify matters further, consider the abstract graph (1-complex), \( \Gamma(\mathcal{A}) \), obtained from the topological category \( \mathcal{A} \) by looking at the objects as vertices and the (undirected) arrows as edges. It is clear from the definition above that the fundamental group of this graph, viz. \( \pi_1(\Gamma(\mathcal{A}), X) \), is acting on \( T_{\infty}(X) \) as these automorphisms. (We may fill in triangular 2-cells in this abstract graph whenever two morphisms (edges) compose to give a third edge; the thereby-reduced fundamental group of this 2-complex also produces on \( T_{\infty}(X) \) the action of \( MC_{\infty}(X) \).)

**Making explicit the genus one situation:** For the genus one object \( X_1 \) in \( \mathcal{A} \), we know that the Teichmüller space for any unramified covering is but a copy of the upper half-plane \( H \). The maps \( T(\pi) \) are Möbius identifications of copies of the half-plane with itself, and we easily see that the pair \( (T_{\infty}(X_1), CM_{\infty}(X_1)) \) is identifiable as \( (H, PGL(2, \mathbb{Q})) \). Notice that the action has dense orbits in this case. Anticipating for a moment the definition of \( Vaut \) given below, we remark that \( GL(2, \mathbb{Q}) \cong Vaut(\mathbb{Z} \oplus \mathbb{Z}) \), and \( Vaut^+ \) is precisely the subgroup of index 2 therein, as expected.

On the other hand, if \( X \in Ob(\mathcal{A}) \) is of any genus \( g \geq 2 \), then we get an infinite dimensional “ind-space” as \( T_{\infty}(X) \) with the action of \( G(X) \) on it as described. Since the tower of coverings over \( X \) and \( Y \) (both of genus higher than 1) eventually become cofinal, it is clear that for any choice of genus higher than one we get one isomorphism class of pairs \( (T_{\infty}, MC_{\infty}) \).

**Virtual automorphism group of \( \pi_1(X) \) and \( MC_{\infty} \):** In the classical situation, the action of the mapping class group \( MCG(X) \) on \( T(X) \) was induced by the action of (isotopy classes of) self-homeomorphisms of \( X \); in the direct limit set up we now have the more general (isotopy classes of) self-correspondences of \( X \) inducing the new mapping class automorphisms on \( T_{\infty}(X) \). In fact, we will see that our group \( MC_{\infty} \) corresponds to “virtual automorphisms” of the fundamental group \( \pi_1(X) \), – generalizing exactly the classical situation where the usual \( Aut(\pi_1(X)) \) appears as the action via modular automorphisms on \( T(X) \).

Given any group \( G \), one may look at its “partial” or “virtual” automorphisms; as opposed to usual automorphisms that are defined on all of \( G \), for virtual automorphisms we demand only that they be defined on some finite index subgroup. To be precise, consider isomorphisms \( \rho : H \to K \) where \( H \) and \( K \) are subgroups of finite index in \( G \). Two such isomorphisms (say \( \rho_1 \) and \( \rho_2 \)) are considered equivalent if there is a finite index subgroup (sitting in the intersection of the two domain groups) on which they coincide. The equivalence class \([\rho]\) – which is like the *germ* of the isomor-
phism $\rho$ - is called a virtual automorphism of $G$; clearly the virtual automorphisms of $G$ constitute a group, christened $\text{Vaut}(G)$, under the obvious law of composition, (i.e., compose after passing to deeper finite index subgroups, if necessary).

Clearly $\text{Vaut}(G)$ is trivial unless $G$ is infinite (though there do exist infinite groups - see [MT] - such that $\text{Vaut}$ is trivial). Also evident is the fact that $\text{Vaut}(\text{group}) = \text{Vaut}(\text{any finite index subgroup})$. Since we shall apply this concept to the fundamental group of a surface of genus $g$, ($g > 1$), the last remark shows that our $\text{Vaut}(\pi_1(X_g))$ is genus independent!

In fact, $\text{Vaut}$ presents us a neat way of formalizing the “two-arrow cycles” (4.6) which we introduced to represent elements of $\text{MC}_\infty$. Letting $G = \pi_1(X)$, (recall that $X$ is already equipped with a base point), we see that the diagram (4.6) corresponds exactly to the following virtual automorphism of $G$:

$$\rho = [\beta \circ \alpha^{-1} : \alpha_*(\pi_1(\tilde{X})) \to \beta_*(\pi_1(\tilde{X}))] \quad (4.8)$$

Here $\alpha_*$ denotes the monomorphism of the fundamental group $\pi_1(\tilde{X})$ into $\pi_1(X) = G$, and similarly $\beta_*$ etc.. We let $\text{Vaut}^+(\pi_1(X))$ denote the subgroup of $\text{Vaut}$ arising from pairs of orientation preserving coverings.

As we said, all the automorphisms arising from arbitrarily complicated cycles of coverings (i.e., any finite sequence of morphisms starting and ending at $X$), are each obtained from these simple two-arrow pictures (4.6). The reduction of any many-arrow cycle in $\Gamma(A)$ to a two-arrow cycle utilizes successive fiber product diagrams; there is some amount of choice in this reduction process, and one may obtain different two-arrow cycles starting from the same cycle - but the virtual automorphism that is defined is unambiguous. The final upshot is:

**Proposition IV.2:** One has natural surjective group homomorphisms:

$$\pi_1(\Gamma(A),X) \to \text{Vaut}^+(\pi_1(X)) \to \text{Aut}(\mathcal{T}_\infty(X)) \equiv \text{MC}_\infty(X)$$

**Remarks:** (i) The concept of $\text{Vaut}$ has already arisen in group theory papers - for example [Ma],[MT]. I am grateful to Chris Odden for pointing out these references to me, and for interesting discussions.

(ii) There is a natural representation of $\text{Vaut}(\pi_1(X))$ in the homeomorphism group of the unit circle $S^1$, by the standard theory of boundary homeomorphisms (see, for example, [N1]). This leads to many obvious questions whose answers are not obvious. In recent work, Robert Penner and the present author have proved an isomorphism between the group $\text{MC}_\infty$ above and a direct limit of Penner’s Ptolemy groups that operate on his “Tessellations” version of universal Teichmüller space. That isomorphism is connected with the following conjecture, and we hope to report on these matters in forthcoming publications.
Topological transitivity of $MC_\infty$ on $T_\infty$ and allied issues: Does $MC_\infty$ act with dense orbits in $T_\infty$? That is a basic query. This question is directly seen to be equivalent to the following old conjecture which, we understand, is due to L.Ehrenpreis and C.L.Siegel:

**Conjecture IV.3:** Given any two compact Riemann surfaces, $X_1$ (of genus $g_1 \geq 2$) and $X_2$ (of genus $g_2 \geq 2$), and given any $\epsilon > 0$, can one find finite unbranched coverings $\pi_1$ and $\pi_2$ (respectively) of the two surfaces such that the corresponding covering Riemann surfaces $\tilde{X}_1$ and $\tilde{X}_2$ are of the same genus and there exists a $(1 + \epsilon)$ quasiconformal homeomorphism between them. (Namely, $\tilde{X}_1$ and $\tilde{X}_2$ come $\epsilon$-close in the Teichmüller metric.)

Remark: Since the uniformization theorem guarantees that the universal coverings of $X_1$ and $X_2$ are exactly conformally equivalent, the conjecture asks whether we can obtain high finite coverings that are approximately conformally equivalent.

V: UNIVERSAL POLYAKOV-MUMFORD ON $T_\infty$:

The goal is to construct natural geometrical fiber bundles over the commensurability Teichmüller space, which when restricted to the finite dimensional strata, $T_g$, become the Hodge (or higher $n$) $DET$ bundles thereon. The bundles over $T_\infty$ should be related by universal Mumford isomorphisms, which restrict to the strata as the finite dimensional Mumford isomorphisms explained in (3.9). We are able to carry out this entire program in a $MC_\infty$ equivariant fashion ([BNS]).

V.A. The Main Lemma: We invoke into play the arbitrary unramified finite covering $\pi : \tilde{X} \to X$, and recall that $T(\pi)$ (equation (4.2)) is the associated holomorphic embedding of Teichmüller spaces.

The fundamental question is whether there exists any natural relationship between the line bundle $DET_{n,g}$ over the Teichmüller space $T_g = T(X)$ and the bundle $T(\pi)^* DET_{n,\tilde{g}}$ obtained as the pullback of the corresponding determinant of cohomology bundle over the larger Teichmüller space $T_{\tilde{g}} = T(\tilde{X})$. For example, we are asking, is there any natural relationship between the two Hodge bundles?

We have an elegant answer to this question that forms the foundation for our genus-independent description of Mumford isomorphisms. In effect, $DET_{n,g}$ raised to the tensor power $deg(\pi)$, simply extends naturally over the larger Teichmüller space $T_{\tilde{g}}$ as the $DET_{n,\tilde{g}}$ bundle thereon! We prove this by utilizing the Grothendieck-Riemann-Roch (GRR) theorem (of [D]) in crucial ways.

**Lemma V.1:** The two holomorphic line bundles (with Quillen hermitian structures), $(DET_{n,g})^{deg(\pi)}$ and $T(\pi)^* DET_{n,\tilde{g}}$, on $T_g$ are canonically isometrically isomorphic for every integer $n$. (The isomorphism is canonical up to the choice of a 12th root of
unity.) In other words, there is a canonical isometrical line bundle morphism $\Gamma(\pi)$ lifting $T(\pi)$ and making the following diagram commute:

$$
\begin{array}{ccc}
DET_{n,g} & \xrightarrow{\deg(\pi)} & DET_{n,\tilde{g}} \\
\downarrow & & \downarrow \\
\mathcal{T}_g & \xrightarrow{T(\pi)} & \mathcal{T}_{\tilde{g}}
\end{array}
$$

The maps $\Gamma(\pi)$ are functorial, so that for any commuting triangle of morphisms in the category $\mathcal{A}$, the corresponding $\Gamma$-lifts also commute.

**Curvature forms of $DET$ bundles and Lemma V.1**: The existence of the canonical relating morphism between the above determinant bundles (fixed $n$) in the fixed covering situation was first conjectured and deduced by us (see [BN]) utilizing the differential geometry of the Quillen metrics. Recall that the Teichmüller spaces $\mathcal{T}_g$ and $\mathcal{T}_{\tilde{g}}$ carry natural symplectic forms – the Weil-Petersson Kähler forms – which are in fact the **curvature forms of the Quillen metrics** of these $DET$ bundles ([Wol], [ZT], [BGS]). If the covering $\pi$ is unbranched of degree $d$, a direct calculation shows that this natural WP form on $\mathcal{T}_{\tilde{g}}$ (appropriately renormalized by the degree $d$) pulls back to the WP form of $\mathcal{T}_g$ by $T(\pi)$. Equality of the curvature forms leads one to expect the isomorphism of $DET_{n,g}^d$ with $T(\pi)^*DET_{n,\tilde{g}}$. This intuition is what is behind the more sophisticated GRR proof of the Lemma above.

**V.B. Power-law principal bundle morphisms over Teichmüller spaces**

We desire to obtain certain canonical geometric objects over the inductive limit of the finite dimensional Teichmüller spaces by coherently fitting together the determinant line bundles $DET_{n,g}$ thereon. To this end it is necessary to find a canonical mapping relating $DET_{n,g}$ itself to $DET_{n,\tilde{g}}$ utilizing the Lemma above.

Now, given any complex line bundle $\lambda \to T$ over any base $T$, there is a certain **canonical** mapping of $\lambda$ to any positive integral ($d$-th) tensor power of itself, given by:

$$
\omega_d : \lambda \to \lambda^\otimes d
$$

where $\omega_d$ on any fiber of $\lambda$ is the map $z \mapsto z^d$. Observe that $\omega_d$ maps $\lambda$ minus its zero section to $\lambda^\otimes d$ minus its zero section by a map which is of degree $d$ on the $\mathbb{C}^*$ fibers. $\omega_d$ is a **homomorphism of the associated principal $\mathbb{C}^*$ bundles**. When $T$ is a complex manifold, and $\lambda$ is a line bundle in that category, then the map $\omega_d$ is a holomorphic morphism between the total spaces of the source and target bundles.

In the situation of Lemma V.1, therefore, we may define a canonical principal bundle morphism relating the relevant bundles:

$$
\Omega(\pi) := \Gamma(\pi) \circ \omega_{\deg(\pi)} : DET_{n,g} \to DET_{n,\tilde{g}}
$$
where $\Gamma(\pi)$ is the canonical (GRR) line bundle morphism provided by the Lemma.

The canonical and functorial choice of these connecting maps, $\Omega(\pi)$, provides us with a direct system of line/principal bundles over the direct systems of Teichmüller spaces.

Given a direct system $T_\alpha$ of complex manifolds, and line bundles $\xi_\alpha$ over these, whenever there are connecting maps as the $\Omega(\pi)$ above, we may pass to the direct limit of the bundles themselves, simultaneously with passing to the limit of the base spaces. That precipitates our main “non-perturbative” result:

**Theorem V.2:** Fix integer $n$. Starting from any “base” surface $X \in \text{Ob}(\mathcal{A})$, we obtain a direct system of principal $\mathbb{C}^*$ bundles $\mathcal{L}_n(\tilde{X}) := \text{DET}_{n,g(\tilde{X})}$ over the Teichmüller spaces $T(\tilde{X})$ with connecting holomorphic homomorphisms $\Omega(\pi)$ between their total spaces.

Passing to the direct limit, one therefore obtains over the universal commensurability Teichmüller space, $T_\infty(X)$, a principal $\mathbb{C}^* \otimes \mathbb{Q}$ bundle:

$$ \mathcal{L}_{n,\infty}(X) = \text{ind.lim.} \mathcal{L}_n(Y) $$

Since the maps $\Omega(\pi)$ preserved the Quillen unit circles, the limit object also inherits such a Quillen “hermitian” structure. The commensurability modular group action $\text{CM}_\infty(X)$ on $T_\infty(X)$ has a natural lifting to $\mathcal{L}_{n,\infty}(X)$ – acting by isometrical automorphisms.

Finally, the Mumford isomorphisms persist as isometrical $\text{MC}_\infty$ equivariant isomorphisms:

$$ \mathcal{L}_{n,\infty}(X) = (6n^2 - 6n + 1)\mathcal{L}_{1,\infty}(X) $$

**Remarks:**

(i) To be explicit, the Mumford isomorphism in the above theorem means that $\mathcal{L}_{n,\infty}$ and $\mathcal{L}_{1,\infty}$ are equivariantly isomorphic relative to the automorphism of $\mathbb{C}^* \otimes \mathbb{Q}$ induced by the homomorphism of $\mathbb{C}^*$ that raises to the power exhibited.

(ii) We could have used the Quillen hermitian structure to reduce the structure group from $\mathbb{C}^*$ to $U(1)$, and thus obtain direct systems of $U(1)$ bundles over the Teichmüller spaces. Passing to the direct limit would then produce $U(1) \otimes \mathbb{Q} := \text{“tiny circle” bundles}$ over $T_\infty$.

(iii) Another interpretation of this construction over $T_\infty$ is to produce “rational line bundles” over it, with relating Mumford isomorphisms, utilizing Lemma V.1 but not involving the power law mappings. See [BNS] for details.

(iv) We have also glued together the universal bundles over direct limits of the moduli spaces $\mathcal{M}_g$, by considering the tower of characteristic coverings over $X$. See [BN].
Above we have succeeded in fitting together the Hodge and higher $DET_n$ bundles over the ind-space $T_\infty$, together with the relating Mumford isomorphisms – our entire construction being $MC_\infty$ equivariant. We thus have a structure on $T_\infty$ that suggests a genus-independent and universal version of the finite dimensional Polyakov structure that we delineated in Sections I to III of this paper.

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