CONLEY CONJECTURE AND LOCAL FLOER HOMOLOGY

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Abstract. In this paper we connect algebraic properties of the pair-of-pants product in local Floer homology and Hamiltonian dynamics. We show that for an isolated periodic orbit the product is non-uniformly nilpotent and use this fact to give a simple proof of the Conley conjecture for closed manifolds with aspherical symplectic form. More precisely, we prove that on a closed symplectic manifold the mean action spectrum of a Hamiltonian diffeomorphism with isolated periodic orbits is infinite.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. In this paper we study the pair-of-pants product in local Floer homology and use its properties to give a simple proof of the Conley conjecture for closed manifolds with aspherical symplectic form. We prove that the product in the local Floer homology is non-uniformly nilpotent unless the periodic orbit is a symplectically degenerate maximum. Then we utilize this fact to prove that on a closed symplectic manifold the mean action spectrum of a Hamiltonian diffeomorphism with isolated periodic orbits is infinite.

To state the results in more detail; recall that for a broad class of symplectic manifolds, every Hamiltonian diffeomorphism has infinitely many simple periodic orbits. Such existence results are usually referred as the Conley conjecture. The example of an irrational rotation of $M = S^2$ shows that the conjecture does not hold unconditionally, or the condition $\omega|_{\pi_2(M)} = 0$ cannot be dropped completely. The
following fact, proven in [GG16], encompasses all known cases of the conjecture: When a closed symplectic manifold \((M, \omega)\) admits a Hamiltonian diffeomorphism with finitely many periodic orbits, there is a class \(A \in \pi_2(M)\) with \(\omega(A) > 0\) and \(\langle c_1(TM), A \rangle > 0\). The proof is a formal consequence of previously known cases combined with the aspherical case established in [GG16]. Ginzburg and Gürel prove the aspherical case by constructing a strictly decreasing sequence of mean action values. In this paper, following Remark 4.4 in [GG16], we show that the mean action spectrum is infinite by using a vanishing property of the pair-of-pants product in local Floer homology.

Here is a brief outline of the argument. Let \(\varphi_H\) be a Hamiltonian diffeomorphism of a closed symplectic manifold generated by a one-periodic Hamiltonian \(H\). We first show that the local Floer algebra \(FA(H, x) = \bigoplus_k HF_\ast(H^\natural k, x^k)\) of a one-periodic orbit \(x\) is non-uniformly nilpotent, if \(x^k\) is isolated and not a symplectically degenerate maximum for all \(k \in \mathbb{N}\) (cf. Prop 5.3 in [GG10]). Then arguing by contradiction, assuming in addition that the mean action spectrum of \(\varphi_H\) is finite, we show that the total Floer algebra \(FA(H) = \bigoplus_k HF_\ast(H^\natural k)\) is nilpotent, which is impossible.

1.2. Main results. Let us now state the main theorems. The conventions and basic definitions are reviewed in Section 2. In what follows a “periodic orbit of a Hamiltonian diffeomorphism” means a “contractible periodic orbit of the time-dependent flow generated by a Hamiltonian”.

**Theorem 1.1.** Let \(\varphi_H\) be a Hamiltonian diffeomorphism of a closed symplectic manifold \((M, \omega)\) generated by a one-periodic Hamiltonian \(H\). Assume that periodic orbits of \(\varphi_H\) are isolated. Then the mean action spectrum of \(H\) is infinite.

**Remark 1.2.** In fact a slightly stronger result holds. Namely, a minor modification of the proof shows that the mean action spectrum must have an accumulation point.

It follows from Theorem 1.1 that when \(\omega\) is aspherical, \(\varphi_H\) has infinitely many simple periodic orbits. In other words, Conley conjecture holds for closed symplectic manifolds \((M, \omega)\) with aspherical \(\omega\).

**Corollary 1.3.** A Hamiltonian diffeomorphism of a closed symplectic manifold \((M, \omega)\) with aspherical \(\omega\) has infinitely many simple periodic orbits.

This is a Lusternik-Schnirelmann type result in the sense that the lower bound for critical points is established by bounding critical values. Theorem 1.1 is proved in Section 3.2. The proof relies on the following vanishing property of the pair-of-pants product.

Let \(M\) be a symplectic manifold and \(x\) be a one-periodic orbit of a Hamiltonian \(H: S^1 \times M \to \mathbb{R}\). Denote by \(x^k\) the \(k\)th iteration of \(x\). The local Floer homology \(HF_\ast(H^\natural k, x^k)\) is defined whenever \(x^k\) is isolated. The pair-of-pants product turns the direct sum \(\bigoplus_k HF_\ast(H^\natural k, x^k)\) into a graded algebra, which we denote by \(FA(H, x)\). We say that the local Floer algebra \(FA(H, x)\) is non-uniformly nilpotent, if for all \(N \in \mathbb{N}\), the product is nilpotent when restricted to \(\bigoplus_{k=1}^N HF_\ast(H^\natural k, x^k)\). The next result (cf. Prop. 5.3 in [GG10]) is proved in Section 3.1. See Section 2.2.1 for the definition of symplectically degenerate maximum.

**Theorem 1.4.** Let \(M\) be a symplectic manifold and let \(x\) be a one-periodic orbit of a Hamiltonian \(H: S^1 \times M \to \mathbb{R}\). Assume that \(x^k\) is isolated and not a symplectically
implies the existence of $S\Z \times \Lo \Z \Z \Lo$ and $\hat{\mu}$ mean index trivial capping, of an autonomous Hamiltonian with small Hessian. The homogeneous with respect to iteration: In this paper, $\mu$ degenerate maximum for all $k \in \N$. Then the local Floer algebra $FA(H, x)$ is non-uniformly nilpotent.

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2. Preliminaries

2.1. Conventions and basic definitions. Let $(M, \omega)$ be a closed symplectic manifold. Throughout the paper there will be no restrictions on $c_1(TM)$, but we will usually assume that $\omega$ is aspherical, i.e., $\omega|_{\pi_2(M)} = 0$.

Let $H$ be a one-periodic in time Hamiltonian on $(M, \omega)$, i.e., $H: S^1 \times M \to \R$, where $S^1 = \R/\Z$. The Hamiltonian vector field $X_H$ of $H$ is defined by $i_{X_H} \omega = -dH$. The time-one map of the time-dependent flow of $X_H$ is denoted by $\varphi_H$. Such time-one maps are referred as Hamiltonian diffeomorphisms.

A capping of a contractible loop $x: S^1 \to M$ is a map $A: D^2 \to M$ such that $A|_{S^1} = x$. The action of a Hamiltonian $H$ on a capped closed curve $\bar{x} = (x, A)$ is

$$A_H(\bar{x}) = -\int_A \omega + \int_{S^1} H(t, x(t)) \, dt.$$ 

The critical points of $A_H$ on the space of capped closed curves are exactly the capped one-periodic orbits of $X_H$. The set of critical values of $A_H$ is called the action spectrum $S(H)$ of $H$ (or of $\varphi_H$). These definitions extend to Hamiltonians of any period in an obvious way.

For $k \in \N$, the $k$th iteration of $H$, by which we simply mean $H$ treated as $k$-periodic, is denoted by $H^{sk}$. With this notation, the mean action spectrum $\bar{S}(H)$ of $H$ (or of $\varphi_H$) is defined as the union of the normalized spectra $S(H^{sk})/k$. Note that the action functional is homogeneous with respect to iteration:

$$A_{H^{sk}}(\bar{x}) = kA_H(\bar{x})$$

where $\bar{x}^k$ is the $k$th iteration of the capped orbit $\bar{x}$. Furthermore, attaching a sphere $A \in \pi_2(M)$ to $\bar{x}^k$ changes $A_{H^{sk}}(\bar{x})$ by $\omega(A)$. So $\bar{S}(H)$ is always infinite when $\omega$ is not aspherical, and when $\omega$ is aspherical Theorem 1.1 implies the existence of infinitely many simple (i.e., not iterated) periodic orbits.

A periodic orbit $x$ of $H$ is called non-degenerate if the linearized return map $d\varphi_H: T_x(0) \to T_x(0)$ has no eigenvalues equal to one. The Conley-Zehnder index $\mu_{cz}(\bar{x}) \in \Z$ of a non-degenerate capped orbit $\bar{x}$ is defined, up to a sign, as in [Sn, SZ]. In this paper, $\mu_{cz}$ is normalized so that $\mu_{cz}(\bar{x}) = n$ when $x$ is a maximum, with trivial capping, of an autonomous Hamiltonian with small Hessian. The mean index $\hat{\mu}_{cz}(\bar{x}) \in \R$ is defined even when $\bar{x}$ is degenerate and depends continuously on $H$ and $\bar{x}$, see [Lo, SZ]. Furthermore, it satisfies $|\hat{\mu}_{cz}(\bar{x}) - \mu_{cz}(\bar{x})| \leq n$, and it is homogeneous with respect to iteration:

$$\hat{\mu}_{cz}(\bar{x}^k) = k \hat{\mu}_{cz}(\bar{x}).$$
2.2. Floer homology. In this section, we recall some properties of local Floer homology and pair-of-pants product, then define Floer algebra. See [FO, GG09, HS, MS, Sa, SZ] for a detailed account on (filtered) Floer homology, and [AS, MS, PSS] for pair-of-pants product.

2.2.1. Local Floer homology. Let $\bar{x}$ be an isolated capped periodic orbit of a Hamiltonian $H$. Local Floer homology $HF_*(H, \bar{x})$ of $\bar{x}$ is defined as in [Gi10, GG09, GG10]. The capping is only used to fix a trivialization of $TM|_x$ and hence to give a $\mathbb{Z}$-grading to $HF_*(H, \bar{x})$. Recapping a capped orbit shifts Conley-Zehnder index by an even integer. So $\mathbb{Z}_2$-graded homology $HF_*(H, x)$ is defined without fixing a trivialization.

The support of $HF_*(H, \bar{x})$ is the collection of integers $m$ such that $HF_m(H, \bar{x}) \neq 0$. A capped periodic orbit $\bar{x}$ is called a symplectically degenerate maximum (SDM) if $HF_{\mu_{CZ}(\bar{x})+n}(H, \bar{x}) \neq 0$, this property is independent of the capping. The mean action spectrum $\tilde{S}(H)$ of a Hamiltonian diffeomorphism $\varphi_H$ with an SDM orbit is infinite, see [Gi10] for details. If $\bar{x}$ is not an SDM, then the support of $HF_*(H, \bar{x})$ is contained in the half-open interval $[\mu_{CZ}(\bar{x})-n, \mu_{CZ}(\bar{x})+n]$.

We will use the properties of $\mathbb{Z}$-grading by Conley-Zehnder index in the proof of Theorem 1.4. When proving Theorem 1.1, for the sake of simplicity, we work with $\mathbb{Z}_2$-graded homology.

2.2.2. Pair-of-pants product. On a closed symplectic manifold $(M, \omega)$ the filtered Floer homology carries the so-called pair-of-pants product; see, e.g., [AS]. On the level of complexes, this product is a map

$$\text{CF}^{(a,b)}_m(H) \otimes \text{CF}^{(c,d)}_l(K) \rightarrow \text{CF}^{(a+c, b+d)}_{m+l-n}(H \natural K)$$

giving rise on the level of homology to an associative, graded-commutative product

$$HF^{(a,b)}_m(H) \otimes HF^{(c,d)}_l(K) \rightarrow HF^{(a+c, b+d)}_{m+l-n}(H \natural K).$$

The product turns the direct sum of the total filtered Floer homology

$$\text{FA}^{(a,b)}(H) := \bigoplus_{k \geq 1} HF_k^{(ka, kb)}(H^\natural)$$

into an associative and graded-commutative non-unital algebra, which we call the filtered Floer algebra. The local Floer algebra $\text{FA}(H, x)$ and total Floer algebra $\text{FA}(H)$ are defined in the same way. In all three, we work with $\mathbb{Z}_2$-graded homology when $c_1(TM) \neq 0$.

By the energy estimates for the product, see [GG16] for details, the filtered Floer algebra splits

$$\text{FA}^{(c-c, c+c)}(H) = \bigoplus_{\mathcal{A}_H(x_i) = c} \text{FA}(H, x_i)$$

as an algebra when the orbits $x_i$ with $\mathcal{A}_H(x_i) = c$ are isolated. As a result, the local algebra can be seen as a building block for the filtered, and hence for the total algebra.

The product structure in total Floer algebra, under the isomorphism with $(\mathbb{Z}_2$-graded if $c_1(TM) \neq 0$) singular homology at each summand, agrees with the dual of cup-product. So there is an infinite sequence of elements in $\text{FA}(H)$, namely the fundamental class for each summand, with non-zero product. We prove Theorem 1.1 by contradiction: We show that $\text{FA}(H)$ would be nilpotent if the mean action spectrum were finite.
3. Proofs

3.1. Proof of Theorem 1.4. Let $x$ be a one-periodic orbit of a Hamiltonian $H: S^1 \times M \to \mathbb{R}$. Assume that $x$ and all of its iterations are isolated, and non-SDM. Choose a capping $\bar{x} = (x, A)$ and iterate it $\bar{x}^k = (x^k, A^k)$. For a fixed $N \in \mathbb{N}$, consider products of the form $w_1, \ldots, w_r \in \text{HF}_l(H^k, \bar{x}^k)$ with $w_i \in \text{HF}_l(H^k, \bar{x}^{ki})$; where $k_i \leq N$, $l = \sum l_i - (r - 1)n$ and $k = \sum k_i$.

Since $\bar{x}^k$ is not an SDM, support of $\text{HF}_s(H^k, \bar{x}^k)$ is contained in the half open interval $[k_i \Delta(\bar{x}) - n, k_i \Delta(\bar{x}) + n)$. So there exists $\delta_{k_i} > 0$ depending on $k_i$ but on the capping, such that $l_i - k_i \Delta(\bar{x}) - n \leq -\delta_i$. Summing over $i$ gives

$$\sum_{i=1}^r l_i - k \Delta(\bar{x}) - rn = l - k \Delta(\bar{x}) - n \leq -\sum_{i=1}^r \delta_{k_i} \leq -r \delta,$$

where $\delta = \min\{\delta_{k_i} | k_i \leq N\}$. So when $r > 2n/\delta$, $l$ goes out of the support of $\text{HF}_s(H^k, \bar{x}^k)$.

**Remark 3.1.** When the mean index is an integer we can choose $\delta = 1$ independent of $N$. Thus, in that case the algebra is uniformly nilpotent. But this is not true in general. For an example, consider the autonomous Hamiltonian $H(x, y) = \lambda(x^2 + y^2)$ where $\lambda > 0$ is a small irrational number. Let $k \in \mathbb{N}$ such that $k\lambda - \lfloor k\lambda \rfloor = \theta > 0$ is small, and $G(x, y) = -\lfloor k\lambda \rfloor(x^2 + y^2)$. Denote by $K$ the composition $G \circ H^{\theta}$; and by $x, y$ the constant orbits of $H$ and $K$.

Using the isomorphisms induced by composing $\varphi_{H^k}$ and $\varphi_{H^{k+1}}$ with the loop diffeomorphism $\varphi_G$, we form a commutative diagram in homology

$$\begin{array}{ccc}
\text{HF}_s(H, x) \otimes \text{HF}_s(H^k, x^k) & \longrightarrow & \text{HF}_s(H^{k+1}, x^{k+1}) \\
\downarrow & & \downarrow \\
\text{HF}_s(H, x) \otimes \text{HF}_s(K, y) & \longrightarrow & \text{HF}_s(H^{\circ K}, x^2 y)
\end{array}$$

where the horizontal arrows are pair-of-pants product. For $C^2$-small autonomous Hamiltonians, e.g. $H$ and $K$, pair-of-pants product in the local Floer homology of an isolated critical point agrees with the dual of cup-product in the local Morse homology of the same point. The latter is non-zero for a local maximum. Hence, by commutativity, the second row in the diagram is also non-zero. Now by taking an infinite sequence of iterations as above, we may construct an infinite non-zero product.

**Remark 3.2.** Roughly speaking, the only restriction on the local Morse homology of an isolated critical point is that with all the algebraic structures it is isomorphic to the homology of a suspension, see [CLOT, Pe]. In particular, the cup-product and the Massey products vanish in the local Morse (co)homology, [Vi], but other cohomology operations need not be trivial (e.g. Steenrod squares). Non-uniform nilpotency of the pair-of-pants product is a Floer theoretic counterpart of this vanishing phenomena. However, similar to the local Morse homology, the local Floer homology can carry many non-nilpotent cohomology operations.

3.2. Proof of Theorem 1.1. As discussed in Section 2.1, the mean action spectrum $\bar{S}(H)$ of $H: S^1 \times M \to \mathbb{R}$ is infinite when $\omega$ is not aspherical; or in the existence of an SDM orbit, see [Gi10] for details. In this section, we assume that $\omega$ is aspherical and none of the orbits of $\varphi_H$ is an SDM.
Arguing by contradiction, assuming in addition that $\hat{S}(H)$ is finite, let $\hat{S}(H) = \{a_1, \ldots, a_m\}$ be the mean action spectrum (ordered) of $H$. Choose $c_i \in \mathbb{R}$ such that $a_i < c_i < a_{i+1}$ and set $c_0 = -\infty$, $c_m = \infty$. By taking an appropriate iteration, if necessary, we may assume that all simple orbits are 1-periodic.

The first step is to argue that the filtered Floer algebra $\mathcal{F}A(c_i, c_{i+1})(H)$ is non-uniformly nilpotent. Then using these as building blocks, we will show that the total algebra $\mathcal{F}A(H)$ is non-uniformly nilpotent (or just nilpotent, since summands are isomorphic), which is impossible (see Section 2.2.2). Recall that the filtered algebra $\mathcal{F}A(c_i, c_{i+1})(H)$ splits:

$$\mathcal{F}A(c_i, c_{i+1})(H) = \bigoplus_{A_H(x_j) = a_i} \mathcal{F}A(C, x_j)$$

as an algebra. Each summand is non-uniformly nilpotent by Theorem 1.4, so the sum is non-uniformly nilpotent.

Now fix $N \in \mathbb{N}$ and let $k, l$ be the degrees of nilpotency of $\mathcal{F}A(c_i, c_{i+1})(H)$, $\mathcal{F}A(c_i, c_{i+1}, c_{i+2})(H)$ when restricted to first $N l$ and $N$ iterations respectively. We will show that $\mathcal{F}A(c_i, c_{i+2})(H)$ is nilpotent with degree at most $k l$ when restricted to first $N$ iterations, and hence non-uniformly nilpotent.

Let $A, B, C$ be the Floer chain complexes which give rise to Floer algebras above, i.e., $\mathcal{H}F_*(A) = \mathcal{F}A(c_i, c_{i+1})(H)$, $\mathcal{H}F_*(B) = \mathcal{F}A(c_i, c_{i+1}, c_{i+2})(H)$, $\mathcal{H}F_*(C) = \mathcal{F}A(c_i, c_{i+2})(H)$. Let $D$ be the quotient chain complex coming from the inclusion $A \otimes A \to C \otimes C$. Using the auxiliary complex $D$ we form a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & A \otimes A & \longrightarrow & C \otimes C & \longrightarrow & D & \longrightarrow & 0 \\
0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & B & \longrightarrow & 0
\end{array}
$$

with exact rows, where the vertical arrows are pair-of-pants product. It induces a commutative diagram

$$
\begin{array}{cccccc}
\mathcal{H}F_*(A) \otimes \mathcal{H}F_*(A) & \longrightarrow & \mathcal{H}F_*(C) \otimes \mathcal{H}F_*(C) & \longrightarrow & \mathcal{H}F_*(D) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{H}F_*(A) & \longrightarrow & \mathcal{H}F_*(C) & \longrightarrow & \mathcal{H}F_*(B)
\end{array}
$$

in homology. Note that the product map $D \to B$ factors through $B \otimes B$; so $\mathcal{H}F_*(D)$ has the same product structure with $\mathcal{H}F_*(B) \otimes \mathcal{H}F_*(B)$.

Now take $k l$ classes $w_i \in \mathcal{H}F_*(C)$ that belongs to first $N$ iterations. If at least $k$ of them are in the image of $P$, using the associativity of the product and the commutativity of the first block in the diagram, we conclude that $w_1 \cdots w_{k l} = 0$.

If not, take $l$ of the classes $w_{i_j}$ that are not in the image of $P$. This time using commutativity of the second block, we conclude that the product $w_{i_1} \cdots w_{i_l}$ is in the image of $P$. We do the same until we obtain $k$ classes (from first $N l$ iterations) in the image of $P$ and go back to first case.

References

[AS] A. Abbondandolo, M. Schwarz, Floer homology of cotangent bundles and the loop product, Geom. Topol., 14 (2010), 1569–1722.

[CLOT] O. Cornea, G. Lupton, J. Oprea and D. Tanre, Lusternik-Schnirelmann Category, Mathematical Surveys and Monographs, vol. 52, AMS, Providene, RI, 2013.
[FO] K. Fukaya, K. Ono, Arnold conjecture and Gromov–Witten invariant, Topology, 38 (1999), 933–1048.

[Gi10] V.L. Ginzburg, The Conley conjecture, Ann. of Math. (2), 172 (2010), 1127–1180.

[GG09] V.L. Ginzburg, B.Z. Gürel, Action and index spectra and periodic orbits in Hamiltonian dynamics, Geom. Topol., 13 (2009), 2745–2805.

[GG10] V.L. Ginzburg, B.Z. Gürel, Local Floer homology and the action gap, J. Sympl. Geom., 8 (2010), 323–357.

[GG16] V.L. Ginzburg, B.Z. Gürel, Conley conjecture revisited, Preprint arXiv:1609.05592.

[HS] H. Hofer, D. Salamon, Floer homology and Novikov rings, in The Floer Memorial Volume, 483–524, Progr. Math., 133, Birkhäuser, Basel, 1995.

[Lo] Y. Long, Index Theory for Symplectic Paths with Applications, Birkhäuser Verlag, Basel, 2002.

[MS] D. McDuff, D. Salamon, J-holomorphic Curves and Symplectic Topology, Colloquium publications, vol. 52, AMS, Providence, RI, 2012.

[Pe] J. Pears, Degenerate critical points and the Conley index, Thesis, University of Edinburgh, 1995.

[PSS] S. Piunikhin, D. Salamon, M. Schwarz, Symplectic Floer–Donaldson theory and quantum cohomology, in Contact and Symplectic Geometry (Cambridge, 1994), Publ. Newton Inst., vol. 8, Cambridge University Press, Cambridge, 1996, 171–200.

[Sa] D.A. Salamon, Lectures on Floer homology, in Symplectic Geometry and Topology, IAS/Park City Math. Ser., vol. 7, Amer. Math. Soc., Providence, RI, 1999, 143–229.

[SZ] D. Salamon, E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, Comm. Pure Appl. Math., 45 (1992), 1303–1360.

[Vi] C. Viterbo, Some remarks on Massey products, tied cohomology classes, and the Lusternik–Shnirelman category, Duke Math. J., 86 (1997), 547–564.

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