DYNAMICS OF SPATIALLY HETEROGENEOUS VIRAL MODEL WITH TIME DELAY

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Abstract. A delayed reaction-diffusion virus model with a general incidence function and spatially dependent parameters is investigated. The basic reproduction number for the model is derived, and the uniform persistence of solutions and global attractivity of the equilibria are proved. We also show the global attractivity of the positive equilibria via constructing Lyapunov functional, in case that all the parameters are spatially independent. Numerical simulations are finally conducted to illustrate these analytical results.

1. Introduction. In recent years, mathematical modeling of in vivo dynamics of viral infections has been employed to understand the mechanisms of disease transmission. For instance, the dynamics of a general in-host model with intracellular delay and spatial homogeneity were studied in [5]; Shu et al. in [15] considered a viral model with nonlinear incidence functions, state dependent removal functions, infinitely distributed intracellular delays, and the cytotoxic T lymphocyte response (CTL). In the framework of mathematical modeling by ordinary differential equations, Perelson and Ribeiro [12] reviewed some developments in HIV modeling, emphasizing quantitative findings about HIV biology uncovered by studying acute infection, the response to drug therapy and the rate of generation of HIV variants that escape immune responses.

When spatial heterogeneity is taken into account in the modelling, many partial differential equations for describing virus dynamics are also proposed. In [4], Lewis et al. studied a spatially-independent model for West Nile virus, proving the
existence of traveling waves and calculating the spatial spreading speed of infection. Zhao and Lou in [6] put forward a nonlocal, time-delayed reaction-diffusion malaria model, where some parameters in the model is assumed to be spatially heterogeneous for designing the spatial allocation of resources. In [17], the author considered the dynamics of steady states of an in-host virus dynamics model with spatial heterogeneity on the general bounded domain. Xu and Ma in [22] investigated a hepatitis B virus (HBV) model with spatial diffusion and saturation response of the infection rate, where the intracellular incubation period was also considered by adding a discrete time delay to the model. Another diffusive HBV model with delayed Beddington-DeAngelis response is studied in [27]. The authors proved the global stability of the two steady states for the model, by constructing proper Lyapunov functionals. Other than that, they also showed the existence of traveling wave solutions connecting the these two steady states when $R_0 > 1$, while there is no traveling wave solution connecting the uninfected steady state to itself when $R_0 < 1$. For more references, we refer the readers to [1, 9, 18, 23, 24, 25].

A basic in-host model of viral dynamics was initially proposed in [11]:

$$
\begin{align*}
S'(t) &= \lambda - dS(t) - \beta S(t)v(t), \\
I'(t) &= \beta S(t)v(t) - aI(t), \\
v'(t) &= kI(t) - uv(t).
\end{align*}
\tag{1}
$$

The model includes three compartments: uninfected target cells ($S$), infected target cells that produce virus ($I$), and free virus particles ($v$). Uninfected target cells are assumed to be produced at a constant rate $\lambda$, and die at the rate of $dS(t)$; infection of target cells by free virus is assumed to occur at the rate of $\beta S(t)v(t)$; the removal rates of infected cells and virus are $a$ and $u$, respectively, new viruses are produced from infected cells at the rate of $kI(t)$. The basic reproductive number of (1) is $R_0 = \frac{\lambda k \beta}{d a u}$. It is argued in [11] that, (1) may be adopted to describe in vivo dynamics of HIV-1, HBV, and other viruses. By additionally considering spatial movement of virus, Wang and Wang in [18] introduced the following the viral infection model with spatial heterogeneity:

$$
\begin{align*}
\frac{\partial S(x,t)}{\partial t} &= \lambda - dS(x,t) - \beta S(x,t)v(x,t), \\
\frac{\partial I(x,t)}{\partial t} &= \beta S(x,t)v(x,t) - aI(x,t), \\
\frac{\partial v(x,t)}{\partial t} &= D \Delta v(x,t) + kI(x,t) - uv(x,t),
\end{align*}
\tag{2}
$$

where $\Delta$ is the Laplacian operator and $D$ the diffusion coefficient. There are no diffusion terms in the first two compartments, by assuming that the uninfected and infected cells move extremely slow in the host environment. In [18], the authors proved the existence of traveling waves via the geometric singular perturbation approach. For (2) with periodic boundary conditions on a bounded square domain, when the recruitment parameter $\lambda$ is spatially-dependent, Brauner et al. in [1] studied the dynamics of the system by analyzing the principle eigenvalue. Later, by assuming that all parameters are spatially dependent except for the diffusion coefficient $D$, Wang et al. in [17] further extended the results in [1] to (2) on a general bounded domain in a finite (not necessarily two) dimensional space, with zero-flux boundary condition.
Motivated by the work of [17], in the paper we consider
\[
\begin{cases}
\frac{\partial S}{\partial t} = \lambda(x) - dS(x, t) - f(x, S(x, t))v(x, t), \\
\frac{\partial I}{\partial t} = e^{-n\tau}f(x, S(x, t-\tau))v(x, t-\tau) - aI(x, t), \\
\frac{\partial v}{\partial t} = D\Delta v(x, t) + k(x)I(x, t) - uv(x, t).
\end{cases}
\]
(3)

Compared with (2), a time delay $\tau$ is involved in the system, for describing the period between the entry of a virus into a target cell and the production of a new virus. Due to this time delay, the term $e^{-n\tau}$ should be included, representing the probability of surviving rate for infected cells from time $t-\tau$ to $t$. Here, the parameter $n$ is the constant death rate for infected cells that can produce virus. In addition, the parameters $\lambda$ and $k$ are assumed to be spatially dependent. Again, both uninfected and infected cells are assume to be spatially immobile. For (3), we assign the following initial and boundary conditions
\[
\begin{align*}
S^0(x) &= \varphi_1(x, \theta) > 0, \quad I^0(x) = \varphi_2(x, 0) \geq 0, \\
v^0(x) &= \varphi_3(x, \theta) \geq 0, \quad x \in \Omega, \quad -\tau \leq \theta \leq 0,
\end{align*}
\]
and
\[
\frac{\partial v}{\partial \nu} = 0, \quad t > 0, \quad x \in \partial \Omega,
\]
(5)
where $\Omega \subset \mathbb{R}^N$ is bounded with smooth boundary $\partial \Omega$. Throughout this paper, we assume that $k : \Omega \to (0, \infty)$ and $\lambda : \Omega \to [0, \infty)$ are both continuous, and the rest parameters $a$, $d$, $D$, $n$, $\tau$ and $u$ are positive constants. The initial functions $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ are also assumed to be uniformly continuous, and $f(\cdot, S) \in C([0, \infty)) \cap C^1((0, \infty))$ satisfies the following properties:
\[
\begin{cases}
f(\cdot, 0) = 0, \\
f'(\cdot, S) > 0, \quad f''(\cdot, S) \leq 0 \text{ for all } S \in (0, \infty).
\end{cases}
\]

The rest of this paper is organized as follows. In section 2, we first prove some basic results of spatially heterogeneous systems, including existence, uniqueness, positivity, as well as boundedness of solutions. Then, we discuss the compactness and persistence properties of the solution semiflow generated by (3). The basic reproduction number $R_0$ of (3) is also derived. Section 3 devotes to the global attractivity of the equilibria for (3) without spatial homogeneity (on $k$ and $\lambda$). In section 4, some numerical simulations are carried out to illustrate the analytic results.

2. Spatial heterogeneity. In this section, we shall investigate the global attractivity of the disease-free equilibrium and the uniform persistence of solutions for (3).

2.1. Basic properties of solutions. Let $X := C(\bar{\Omega}, \mathbb{R}^3)$. Denote
\[
X^+ := C(\bar{\Omega}, \mathbb{R}^3_+), \quad \mathbb{R}^3_+ = \{(S, I, v)|S \geq 0, \quad I \geq 0, \quad v \geq 0\},
\]
\[
Y = C([-\tau, 0], X) \quad \text{and} \quad Y^+ = C([-\tau, 0], X^+).
\]
Then, $(X, X^+)$ and $(Y, Y^+)$ are ordered Banach spaces. For any function $z \in C([-\tau, a], X)$ with some $a > 0$ and any $t \in [0, a)$, we define $z_t \in Y$ by $z_t(\theta) = z(t + \theta)$, $\theta \in [-\tau, 0].$
Lemma 2.1. For any \( \varphi \in \mathbb{Y}^+ \), there exists a unique solution of system (3)-(5) defined on \([0, +\infty)\), which is nonnegative and ultimately bounded in \( \mathbb{Y}^+ \). Furthermore, the set
\[
\mathcal{D} = \left\{ (\varphi_S, \varphi_I, \varphi_v) \in \mathbb{Y}^+ : 0 \leq \|\varphi_S + \varphi_I\| \leq \frac{\bar{\lambda}}{\zeta}, \ 0 \leq \|\varphi_v\| \leq \frac{k\bar{\lambda}}{\zeta_c} \right\},
\]
is positively invariant, and any solution of system (3) falls into \( \mathcal{D} \) after finite time, where
\[
\bar{\lambda} = \max_{x \in \Omega} \lambda(x), \ \zeta = \min\{a, d\} \quad \text{and} \quad \bar{k} = \max_{x \in \Omega} k(x).
\]

Proof. For given \( \varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in \mathbb{Y}^+ \), define
\[
T_1(t)\varphi_1 = e^{-dt}\varphi_1, \ T_2(t)\varphi_2 = e^{-at}\varphi_2.
\]
Let \( \Gamma \) be the Green function associated with \( D\Delta - u \) and the Neumann boundary condition, and
\[
(T_3(t)\varphi_3)(x) = \int_\Omega \Gamma(x, y, t, D)\varphi_3(y)dy, \ t \geq 0.
\]
From the results in [13], it follows that \( T_3(t) : C(\bar{\Omega}, \mathbb{R}) \rightarrow C(\bar{\Omega}, \mathbb{R}) \) is compact and strongly positive for all \( t > 0 \). Define \( \tilde{F} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3) : \mathbb{Y}^+ \rightarrow \mathbb{X} \) by
\[
\tilde{F}_1(\varphi)(x) = \lambda(x) - f(x, \varphi_1(x, 0))\varphi_3(x, 0), \\
\tilde{F}_2(\varphi)(x) = e^{-\eta \tau} f(x, \varphi_1(x, -\tau))\varphi_3(x, -\tau), \\
\tilde{F}_3(\varphi)(x) = k(x)\varphi_2(x, 0).
\]
Then, system (3)-(5) can be rewritten as the integral equation:
\[
z(t) = T(t)\varphi(0) + \int_0^t T(t - \theta)\tilde{F}(z_\theta)d\theta, \ t \geq 0, \\
z_0 = \varphi \in \mathbb{Y}^+,
\]
where
\[
z(t) = \begin{pmatrix} S(t) \\ I(t) \\ v(t) \end{pmatrix}, \ T(t) = \begin{pmatrix} T_1(t) & 0 & 0 \\ 0 & T_2(t) & 0 \\ 0 & 0 & T_3(t) \end{pmatrix}.
\]
It is easy to obtain that
\[
\lim_{h \rightarrow 0^+} \frac{1}{h} \text{dist}(\varphi(0) + h\tilde{F}(\varphi), \mathbb{X}^+) = 0, \ \forall \varphi \in \mathbb{Y}^+.
\]
By [8], we can obtain that there exists a unique noncontinuable solution \( z(x, t; \varphi) \) of system (3)-(5) on \([0, t_\varphi)\) with \( z(\cdot, \theta; \varphi) = \varphi(\cdot, \theta) \) for all \( (x, \theta) \in \Omega \times [-\tau, 0] \) and \( z_t \in \mathbb{Y}^+ \) for \( t \geq 0 \). Denote \( G = (G_1, G_2, G_3) : \mathbb{Y}^+ \rightarrow \mathbb{X} \) by
\[
G_1(x, t) = \lambda(x) - dS(x, t) - f(x, S(x, t)v(x, t), \\
G_2(x, t) = e^{-\eta \tau} f(x, S(x, t - \tau))v(x, t - \tau) - aI(x, t), \\
G_3(x, t) = k(x)I(x, t) - uv(x, t).
\]
System (3) becomes
\[
\begin{align*}
\frac{\partial S}{\partial t} &= G_1(x,t), \\
\frac{\partial I}{\partial t} &= G_2(x,t), \\
\frac{\partial v}{\partial t} &= D\Delta v(x,t) + G_3(x,t).
\end{align*}
\] (10)

Since \( G \) is quasipositive, the solutions of system (3) remain nonnegative for all \( t \geq 0 \) (see [8, Remark 1.1]).

It is easy to see that \( D \) is a positively invariant set with respect to model (3).

Now, we prove the boundedness of solutions. Let \( U(x,t) := S(x,t) + I(x,t + \tau) \). From the first two equations of system (3), \( U(x,t) \) satisfies
\[
\frac{\partial U}{\partial t} \leq \bar{\lambda} - \xi U(x,t), \quad x \in \Omega, \quad t > 0,
\] (11)
where \( \bar{\lambda} \) and \( \xi \) are defined in (6). Consider
\[
\frac{\partial U_1(x,t)}{\partial t} := \bar{\lambda} - \xi U_1(x,t), \quad x \in \Omega, \quad t > 0.
\]

For \( U_1 \), it is straightforward that
\[
\lim_{t \to \infty} \sup U_1(x,t) \leq \frac{\bar{\lambda}}{\xi} \quad \text{for all} \quad x \in \bar{\Omega}.
\]

By the comparison principle, it follows that
\[
U(\cdot, t) \leq U_1(\cdot, t) \quad \text{for all} \quad t \geq 0.
\] (12)

From (12) we know \( v \) satisfies the following system
\[
\begin{cases}
\frac{\partial v}{\partial t} - D\Delta v \leq \frac{\bar{k}\bar{\lambda}}{\xi} - uv, & x \in \Omega, \quad t \geq t_0, \\
\frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0,
\end{cases}
\] (13)

where \( \bar{k} = \max_{x \in \Omega} k(x) \). Let \( v_1(t) \) be the solution to the following equation
\[
\frac{\partial v_1(x,t)}{\partial t} = \frac{\bar{k}\bar{\lambda}}{\xi} - u v_1(x,t), \quad x \in \Omega, \quad t > t_0.
\] (14)

Then, we obtain
\[
\lim_{t \to \infty} \sup v_1(x,t) \leq \frac{\bar{k}\bar{\lambda}}{uw} \quad \text{for all} \quad x \in \bar{\Omega},
\]
and therefore, \( S(t), I(t) \) and \( v(t) \) are ultimately bounded in \( \mathbb{Y}^+ \).

\[\square\]

2.2. Compactness. In this part, we shall discuss the compactness property of the semiflow generated by system (3)-(5). Define the \( \Psi_t : \mathbb{Y}^+ \to \mathbb{Y}^+ \) according to system (3)-(5) by
\[
\Psi_t(\varphi) = z_t(\cdot; \theta; \varphi), \quad t \geq 0,
\] (15)
where \( z_t(\cdot; \theta; \varphi) \) is the solution of system (3)-(5) with \( z_0(\cdot; \theta; \varphi) = \varphi \in \mathbb{Y}^+ \). Since the first two equations in system (3) have no diffusion terms, the semiflow associated with system (3) is not compact. However, we can show the asymptotic compactness of forward orbits, with the aid of the Kuratowski measure \( \kappa \) of noncompactness (see [2]). By the similar arguments as in [26, Lemma 2.1], we have
Lemma 2.2. For every $\varphi \in \mathbb{Y}^+$, the forward orbit $\gamma^+(\varphi) := z_t(\cdot, \theta; \varphi), t \geq 0$ for system (3) is asymptotically compact in the sense that for any sequences $t_n \to \infty$, there exists a subsequence $t_{n_k}$ such that $z_{t_{n_k}}(\cdot, \theta; \varphi)$ converges in $\mathbb{Y}^+$ as $k \to \infty$.

Proof. For a given $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{Y}^+$, there exists $\theta > 0$ such that

$$|S(x, t; \varphi)| \leq \theta, \quad |I(x, t; \varphi)| \leq \theta, \quad |v(x, t; \varphi)| \leq \theta \quad \text{for all } x \in \bar{\Omega}, \ t \geq 0.$$

In view of the Arezla-Ascoli theorem, it suffices to prove that $\{z_{t_n}(x, \theta; \varphi_n)\}_{n \geq 1}$ is equicontinuous in $x \in \Omega$ for all $n \geq 1$. We first show that this is the case for $\{S_{t_n}(x, \theta; \varphi_n)\}_{n \geq 1}$. It is easy to see that

$$g(x, t) = \lambda(x) - f(x, S(x, t))v(x, t)$$

is uniformly continuous in $x \in \bar{\Omega}$, uniformly for $t \geq 0$, that is, for any given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|g(x_1, t) - g(x_2, t)| < \epsilon^2$$

for any $t \geq 0$, and $x_1, x_2 \in \bar{\Omega}$ satisfying $|x_1 - x_2| < \delta$. Set $S_{t_n}^n(x, \theta) = S_{t_n}(x, \theta; \varphi_n)$, for $t_n \geq 0, x \in \Omega$, and $\tilde{S}_{t_n}^n(x, \theta) = S_{t+tn}^n(x, \theta)$, for $t+t_n \geq 0, x \in \Omega$. Obviously,

$$\frac{\partial}{\partial t} [\tilde{S}_{t_n}^n(x_1, \theta) - \tilde{S}_{t_n}^n(x_2, \theta)]^2$$

$$= 2(\tilde{S}_{t_n}^n(x_1, \theta) - \tilde{S}_{t_n}^n(x_2, \theta)) [g(x_1, t+t_n) - g(x_2, t+t_n) - d(\tilde{S}_{t_n}^n(x_1, \theta) - \tilde{S}_{t_n}^n(x_2, \theta))]
$$

$$\leq 4\theta |g(x_1, t+t_n) - g(x_2, t+t_n)| - 2d(\tilde{S}_{t_n}^n(x_1, \theta) - \tilde{S}_{t_n}^n(x_2, \theta))^2$$

$$\leq 4\theta \epsilon^2 - 2d(\tilde{S}_{t_n}^n(x_1, \theta) - \tilde{S}_{t_n}^n(x_2, \theta))^2$$

for any $t \geq -t_n$, and $x_1, x_2 \in \bar{\Omega}, |x_1 - x_2| < \delta$. By the variation of constants formula and the comparison argument, we have

$$|\tilde{S}_{t_n}^n(x_1, \theta) - \tilde{S}_{t_n}^n(x_2, \theta)|^2 \leq e^{-2d(t-s)} |\tilde{S}_{s_1}^n(x_1, \theta) - \tilde{S}_{s_1}^n(x_2, \theta)|^2 + 4\theta \epsilon^2 \int_s^t e^{-2d(t-\theta)} d\theta.$$

Letting $t = 0$ and $s = -t_n$ in the above inequality, we further obtain

$$|\tilde{S}_{0}^n(x_1, \theta) - \tilde{S}_{0}^n(x_2, \theta)|^2 \leq e^{-2dn} |\tilde{S}_{-t_n}^n(x_1, \theta) - \tilde{S}_{-t_n}^n(x_2, \theta)|^2 + 2\theta \epsilon^2 d,$$

that is,

$$|S_{t_n}(x_1, \theta; \varphi_n) - S_{t_n}(x_2, \theta; \varphi_n)|^2 \leq |\varphi_1(x_1, 0) - \varphi(x_2, 0)|^2 + 2\theta \epsilon^2 d,$$

for all $n \geq 1, |x_1 - x_2| < \delta, x_1, x_2 \in \bar{\Omega}$. Since $\varphi_1(x, 0)$ is uniformly continuous for $x \in \Omega$, there exists $\delta_1 > 0$, such that $|\varphi_1(x_1, 0) - \varphi_1(x_2, 0)| < \epsilon$, whenever $|x_1 - x_2| < \delta_1$. Thus, for any $|x_1 - x_2| < \delta_0 := \min\{\delta_1, \delta\}, x_1, x_2 \in \bar{\Omega}$, we have

$$|S_{t_n}(x_1, \theta; \varphi_n) - S_{t_n}(x_2, \theta; \varphi_n)|^2 \leq \epsilon^2 + 2\theta \epsilon^2 d.$$

This proves the equicontinuity of $\{S_{t_n}(x, \theta; \varphi_n)\}_{n \geq 1}$. Similarly, we can verify that $\{I_{t_n}(x, \theta; \varphi_n)\}_{n \geq 1}$ and $\{e_{t_n}(x, \theta; \varphi_n)\}_{n \geq 1}$ are also equicontinuous in $x \in \bar{\Omega}$ for all $n \geq 1$. Consequently, $\Psi_t$ is asymptotically compact. □
2.3. Basic reproduction number and global attractivity. Obviously, $E_0(x) = (S_0(x), 0, 0) = \left( \frac{\lambda(x)}{d}, 0, 0 \right)$ is the disease-free steady state for system (3)-(5). Linearizing the system (3) at $E_0(x)$, we get the following cooperative system for the infected host cells and free virus

$$\frac{\partial I}{\partial t} = e^{-n\tau} f(x, S_0(x)) v(x, t - \tau) - aI(x, t),$$
$$\frac{\partial v}{\partial t} = D\Delta v(x, t) + k(x)I(x, t) - uv(x, t). \quad (16)$$

Suppose that the initial population distribution for the system is given by $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{R}^+$. As time evolves, the spatial distribution of the infective individuals under the synthetical influences of mortality, mobility, and transfer of individuals among the infected compartments is described by

$$\frac{\partial I}{\partial t} = -aI(x, t),$$
$$\frac{\partial v}{\partial t} = D\Delta v(x, t) - uv(x, t).$$

Let $(I(t, \varphi), v(t, \varphi))$ be the distribution of the infective individuals and virus for $t > 0$. Since there are no infective agents for $t < 0$, it is easy to see that

$$I(t, \varphi)(x) = e^{-at}\varphi_2(x) \quad (17)$$

and

$$v(t, \varphi)(x) = e^{-ut} \int_\Omega \Gamma(t, x, y, D)\varphi_3(y)dy. \quad (18)$$

Moreover, the distribution of the new infection rate of cells induced by viruses at time $t$ is

$$F_1(t, \varphi)(x) = \begin{cases} 0 & \text{if } 0 < t < \tau, \\ e^{-n\tau} f(x, S_0(x)) v(x, t - \tau) & \text{if } t \geq \tau, \end{cases}$$

and the distribution of the new fission rate of produced free virion from infected cells at time $t$ is

$$F_2(t, \varphi)(x) = k(x)I(t, \varphi)(x).$$

Consequently, the distribution of total new infections of cell is

$$\int_0^\infty F_1(t, \varphi)dt = \int_0^\infty e^{-n\tau} f(x, S_0(x)) v(x, t)dt := \hat{F}_1(\varphi), \quad (19)$$

and the distribution of total new produced free virions is

$$\int_0^\infty F_2(t, \varphi)dt := \hat{F}_2(\varphi). \quad (20)$$

Obviously, $\hat{F} = (\hat{F}_1, \hat{F}_2)$ is a continuous and positive operator, which maps the initial infection distribution $\varphi$ to the distribution of the total infective members produced during the infection period. Now, we define the spectral radius of $\hat{F}, r(\hat{F})$, as the basic reproduction number $R_0$ of model (3) (see [20, Theorem 3.1]). For $\hat{F}_2$, we have

$$\hat{F}_2(\varphi)(x) = \frac{k(x)\varphi_2(x)}{a}.$$ 

Define the operator $B_2$ by

$$B_2(\varphi_3) = D\Delta \varphi_3 - u\varphi_3, \quad (21)$$
with the Neumann boundary condition. By [16, Theorem 3.12], we have
\[ \int_0^\infty v(t, \varphi) dt = \int_0^\infty v(t, \varphi_3) dt = -B_2^{-1}\varphi_3. \] (22)

Hence, it follows from (19) that \( \hat{F}_1(\varphi)(x) = -e^{-\tau f(x, S_0(x))}B_2^{-1}\varphi_3(x) \).

We are now in a position to show that \( R_0 \) is a threshold value for disease invasion. Consider the following auxiliary system of equation (16)
\[ \begin{align*}
\frac{\partial I}{\partial t} &= e^{-\tau f(x, S_0(x))v(x, t) - aI(x, t)}, \\
\frac{\partial v}{\partial t} &= D\Delta v(x, t) + k(x)I(x, t) - w(x, t).
\end{align*} \] (23)

For \( \varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{Y}^+ \), we define the operator \( C = (C_1, C_2) \) by
\[ \begin{align*}
C_1(\varphi)(x) &= e^{-\tau f(x, S_0(x))}\varphi_3(x), \\
C_2(\varphi)(x) &= k(x)\varphi_2(x),
\end{align*} \]
and \( B = (B_1, B_2) \) by
\[ \begin{align*}
B_1(\varphi)(x) &= -a\varphi_2(x), \\
B_2(\varphi)(x) &= D\Delta \varphi_3(x) - u\varphi_3(x).
\end{align*} \]

Set \( A^H = B + C \), where \( H = f(x, S_0(x)) \). Let \( s(\hat{A}) \) be the spectral bound of an operator \( \hat{A} \), that is, \( s(\hat{A}) = \sup\{\Re \zeta : \zeta \in \sigma(\hat{A})\} \), where \( \sigma(\hat{A}) \) is the spectral set of the operator \( \hat{A} \). It is easy to see that \( s(B) \) is negative.

**Lemma 2.3.** \( R_0 - 1 \) and the spectral bound \( s(A^H) \) of \( A^H \) have the same sign.

**Proof.** By [16, Theorem 3.5], \( s(A^H) \) has the same sign as \( \max(-CB^{-1})^{-1} \). It suffices to show that \( R_0 = \max(-CB^{-1})^{-1} \). First, letting \( Q(t) \) be the solution semigroup generated by \( B \), we then have
\[ \langle \zeta - B \rangle^{-1}\varphi = \int_0^\infty e^{-\zeta t}Q(t)\varphi dt, \quad \zeta > s(B), \quad \varphi \in \mathbb{Y}^+. \]

From \( s(B) < 0 \), it follows that
\[ -B^{-1}\varphi = \int_0^\infty Q(t)\varphi dt, \quad \varphi \in \mathbb{Y}^+. \] (24)

As a consequence, we have (22) and
\[ \begin{align*}
C_1(-B^{-1}\varphi) &= e^{-\tau f(x, S_0(x))}(-B_2^{-1}\varphi) = \hat{F}_1(\varphi), \\
C_2(-B^{-1}\varphi) &= k(x)(-B_1^{-1}\varphi) = \hat{F}_2(\varphi).
\end{align*} \]

This proves \( \max(-CB^{-1})^{-1} = \max(\hat{F}) = R_0 \).

Our next goal is to show that \( s(A^H) \) is a principal eigenvalue of \( A^H \) on \( C(\bar{\Omega}, \mathbb{R}^2) \).

**Theorem 2.4.** \( s(A^H) \) is a geometrically simple eigenvalue with a positive eigenfunction.

**Proof.** For \( \varphi' = (\varphi_2, \varphi_3) \in \mathcal{N}(\zeta I - A^H) \), we have
\[ \begin{align*}
e^{-\tau f(x, S_0(x))}\varphi_3(x) - a\varphi_2 &= \zeta\varphi_2, \\
D\Delta \varphi_3 + k(x)\varphi_2 - w\varphi_3 &= \zeta\varphi_3.
\end{align*} \] (25)
For $\zeta > -a$, we obtain from the first equation of (25) that

$$\varphi_2 = \frac{e^{-n\tau}}{\zeta + a} f(x, S_0(x)) \varphi_3(x) := \xi(\zeta, x, \varphi_3).$$

(26)

Substituting it into the second equation of (25), we get

$$L_\zeta(\varphi_3) := D \Delta \varphi_3 + k(x) \xi(\zeta, x, \varphi_3) - w \varphi_3 = \zeta \varphi_3.$$  

(27)

Set

$$k := \min_{x \in \bar{\Omega}} k(x), \quad f(\hat{x}, S_0(\hat{x})) := \min_{x \in \Omega} f(x, S_0(x)),$$

and

$$\zeta^* = \frac{\sqrt{(u - a)^2 + 4e^{-n\tau} \beta f(\hat{x}, S_0(\hat{x})) k}}{2}.$$  

Then it is easy to see that $\zeta^* > -a$. If we choose $\varphi_3 = 1$, then we have

$$L_{\zeta^*}(\varphi_3) = -u + k(x) \xi(\zeta^*, x, 1)$$

$$\geq -u + \frac{k \beta f(\hat{x}, S_0(\hat{x}))}{\zeta^* + a} = \zeta^* \varphi_3.$$  

By similar arguments to those in [19, Theorem 2.3], we conclude that $s(A^H)$ is a geometrically simple eigenvalue with a positive eigenfunction.

The following result is on the persistence of system (3).

**Lemma 2.5.** For any $\varphi \in \mathbb{Y}^+$ satisfying (4), the following statements are valid for the system (3):

- (i) There exists $h(x) > 0$, $\forall x \in \bar{\Omega}$, such that
  $$\lim_{t \to \infty} \inf_{t \geq 0} S(x, t; \varphi) \geq h(x);$$

- (ii) If there exists some $i \geq 0$ such that $\varphi(\cdot, i; \varphi) \neq 0$, then $\varphi(x, t; \varphi) > 0$, $\forall x \in \bar{\Omega}$, $t > i$;

- (iii) If there exists some $\bar{i} \geq 0$ such that $I(\cdot, \bar{i}; \varphi) \neq 0$ and $\varphi(\cdot, \bar{i}; \varphi) \neq 0$, then $I(x, t; \varphi) > 0$, $\forall x \in \bar{\Omega}$, $t > \bar{i}$.

**Proof.** By the properties of $f(\cdot, S)$, it follows from the mean value theorem that

$$f'(\cdot, S)S \leq f(\cdot, S) \leq f'(\cdot, 0)S, \quad S > 0.$$

From Lemma 2.1, $\exists \bar{M} > 0$, such that $0 \leq f(\cdot, S)v \leq \bar{M} \cdot S$. Then, Eq. (3) implies

$$\frac{\partial S(x, t)}{\partial t} \geq \lambda(x) - (d + \bar{M})S(x, t),$$

for each $x \in \Omega$, $t > 0$. Consider

$$\frac{\partial S_1(x, t)}{\partial t} = \lambda(x) - (d + \bar{M})S_1(x, t).$$

(28)

We know that the system (28) admits a unique positive equilibrium $\frac{\lambda(x)}{d + \bar{M}}$, which is globally asymptotically stable in $C(\bar{\Omega}, \mathbb{R})$. Thus, by the comparison theorem, we can obtain that there exists $h(x) = \frac{\lambda(x)}{d + \bar{M}} > 0$, such that

$$\lim_{t \to \infty} \inf_{t \geq 0} S(x, t; \varphi) \geq h(x), \quad \forall x \in \bar{\Omega}.$$
Next, we prove the part (ii). First of all, $v$ satisfies the following inequality:
\[
\begin{aligned}
\frac{\partial v(x,t)}{\partial t} &\geq D\Delta v(x,t) - uv(x,t), \ x \in \Omega, \ t > 0, \\
\frac{\partial v(x,t)}{\partial \sigma} & = 0, \ x \in \partial \Omega, \ t > 0,
\end{aligned}
\]
with initial conditions (4). Let $\hat{v}(x,t)$ be the solution of the following equation
\[
\begin{aligned}
\frac{\partial \hat{v}(x,t)}{\partial t} & = D\Delta \hat{v}(x,t) - \hat{u}\hat{v}(x,t), \ x \in \Omega, \ t > 0, \\
\frac{\partial \hat{v}(x,t)}{\partial \sigma} & = 0, \ x \in \partial \Omega, \ t > 0,
\end{aligned}
\]
with initial conditions (4). We know that there exists some $\bar{t} > 0$ such that $\hat{v}(\cdot,t;\varphi) \not\equiv 0$. Now, we show that $\hat{v}$ is positive on $\bar{\Omega} \times (\bar{t},+\infty)$. If there exists a point $(x_0,\bar{t}) \in \Omega \times (\bar{t},+\infty)$ such that $v(x_0,\bar{t};\varphi) = 0$, then $v(x,t;\varphi) \equiv 0$ for all $x \in \Omega$, $t > \bar{t}$ from the strong maximum principle, contradicting to $v(\cdot,\bar{t};\varphi) \not\equiv 0$. If there exists a point $(x_0,\bar{t}) \in \partial \Omega \times (\bar{t},+\infty)$ such that $v(x_0,\bar{t};\varphi) = 0$, it follows from the Hopf boundary lemma that $\frac{\partial v(x_0,\bar{t})}{\partial \nu} < 0$. This contradicts to the boundary condition. By the comparison principle, we obtain that $v(x,t) \geq \hat{v}(x,t) > 0$ for all $(x,t) \in \Omega \times (\bar{t},+\infty)$. This completes the proof of statement (ii).

For (iii), suppose that there exist $x_0 \in \Omega$ and $\bar{t} > t$ such that $I(x_0,\bar{t};\varphi) = 0$. From the second equation of system (3), it follows that
\[
0 = \frac{\partial I(x_0,\bar{t})}{\partial t} = e^{-n\tau}f(x,S(x,\bar{t} - \tau))v(x,\bar{t} - \tau) - a(x_0)I(x_0,\bar{t}).
\]
Hence,
\[
0 = \frac{\partial I(x_0,\bar{t})}{\partial t} = e^{-n\tau}f(x,S(x,\bar{t} - \tau))v(x,\bar{t} - \tau) = 0,
\]
which implies $v(x,\bar{t} - \tau) = 0$. This contradicts to (ii).

Now, we state the main theorem in this section.

**Theorem 2.6.** Assume that $\varphi \in \mathbb{Y}^+$. Then the following statements are valid.

(i) If $R_0 < 1$, then the disease-free equilibrium $E_0(x) = (S_0(x),0,0)$ is globally attractive in $\mathbb{Y}^+$;

(ii) If $R_0 > 1$, then system (3)-(5) admits at least one positive steady state $E^*(x)$, and there exists a $\delta > 0$ such that for every $\varphi \in \mathbb{Y}^+$ with $\varphi_i(\cdot) \not\equiv 0$ for $i = 2,3$, we have
\[
\lim_{t \to \infty} \inf_{x \in \Omega} S(x,t) \geq \delta, \quad \lim_{t \to \infty} \inf_{x \in \Omega} I(x,t) \geq \delta \quad \text{and} \quad \lim_{t \to \infty} \inf_{x \in \Omega} v(x,t) \geq \delta,
\]
uniformly for all $x \in \Omega$.

**Proof.** By Theorem 2.4, there exists a positive eigenfunction $\phi$ such that $A^H\phi = s(A^H)\phi$. Let $\phi = (\phi_2, \phi_3)$. Then we have
\[
e^{-n\tau}H\phi_3 - a\phi_2 = s(A^H)\phi_2,
\]
\[
D\Delta \phi_3 + k(x)\phi_2 - u\phi_3 = s(A^H)\phi_3. \tag{30}
\]

We first consider the case that $R_0 < 1$ (or equivalently, $s(A^H) < 0$). It is easy to see that $s(A^H)$ is the principal eigenvalue of the problem (30). By Theorem 2.4 and the continuity, there exists a $\rho_0 > 0$ such that $s(A^{I(x,S_0(x)+\rho_0)})$ is still the principal eigenvalue of the eigenvalue problem (30) and
\[
s(A^{I(x,S_0(x)+\rho_0)}) < 0.
\]
From the comparison principle, it follows that there is a \( t_0 = t_0(\varphi) \) such that
\[
S(x, t - \tau; \varphi) \leq S_0(x) + \rho_0, \ \forall t \geq t_0, \ x \in \Omega.
\]

Consider
\[
\begin{aligned}
\frac{\partial I_1(x, t)}{\partial t} &\leq e^{-n \tau} f(x, S_0(x) + \rho_0)v(x, t - \tau) - aI_1(x, t), \quad x \in \Omega, \ t \geq t_0, \\
\frac{\partial I_2(x, t)}{\partial t} &\leq D \Delta v(x, t) + k(x)I_1(x, t) - uv(x, t), \quad x \in \Omega, \ t \geq t_0, \\
\frac{\partial I_2(x, t)}{\partial \tau} &\leq 0, \quad x \in \partial \Omega, \ t \geq t_0.
\end{aligned}
\tag{31}
\]

By Theorem 2.4, there is a strongly positive eigenfunction \( \hat{\phi} \) corresponding to \( s(A^{H_1}) \) with \( H_1 = f(x, S_0(x) + \rho_0) \). Let \( w(x, t; \varphi) = (I_2(x, t; \varphi), \nu_2(x, t; \varphi)) \) be the unique solution of
\[
\begin{aligned}
\frac{\partial I_2(x, t)}{\partial t} &= -aI_2(x, t) + e^{-n \tau} f(x, S_0(x) + \rho_0)\nu_2(x, t - \tau), \quad x \in \Omega, \ t \geq t_0, \\
\frac{\partial \nu_2(x, t)}{\partial t} &= D \Delta \nu_2(x, t) + k(x)I_2(x, t) - \nu_2(x, t), \quad x \in \Omega, \ t \geq t_0, \\
\frac{\partial \nu_2(x, t)}{\partial \tau} &= 0, \quad x \in \partial \Omega, \ t \geq t_0,
\end{aligned}
\tag{32}
\]

for any \( \varphi \in C([-\tau, 0], C(\bar{\Omega}, \mathbb{R}^2)) \). The comparison principle implies that
\[
(I(x, t; \varphi), \nu(x, t; \varphi)) \leq w(x, t; \varphi), \quad \forall t \geq t_0.
\]

Choose a sufficiently large number \( K > 0 \) such that \( 0 \leq \varphi(x, \theta) \leq K\tilde{\phi}(x) \) for all \( (x, \theta) \in \Omega \times [-\tau, 0] \). By the comparison principle again, we obtain
\[
0 \leq w(x, t; \varphi) \leq Kw(x, t; \hat{\phi}), \quad \forall t \geq 0, \ x \in \Omega.
\]

Let
\[
w^+(x, t) := \hat{\phi}(x), \quad \forall t \in [-\tau, \infty), \ x \in \bar{\Omega}.
\]

Since \( s(A^I(x, S_0(x) + \rho_0)) < 0 \), it is easy to see that \( w^+(x, t) \) is an upper solution of system (32). Then,
\[
0 \leq w(x, t; \hat{\phi}) \leq w^+(x, t) = \hat{\phi}(x), \quad \forall t \in [-\tau, \infty), \ x \in \Omega.
\tag{33}
\]

Let \( w_I(\varphi)(\theta) = w(\cdot, t + \theta; \varphi) \) for all \( t \geq 0, \ \theta \in [-\tau, 0] \). It then follows from (33) that \( w_I(\hat{\phi}) \leq \hat{\phi} \) for all \( t \geq 0 \). By the comparison principle, we obtain \( w_{t+s}(\hat{\phi}) = w_I(w_s(\hat{\phi})) \leq w_s(\hat{\phi}) \) for all \( t, s \geq 0 \). This implies that \( w_I(\hat{\phi}) \) is nonincreasing in \( t \in [0, \infty) \). In view of [29, Lemma 3.1], \( w_t \) is a \( \kappa \)-contraction for each \( t > 0 \), and it then follows that \( w_I \) is asymptotically smooth. Therefore, the omega limit set \( \omega(\hat{\phi}) \) of the bounded orbit \( \gamma^+(\hat{\phi}) = \{w_t(\hat{\phi}) : t \geq 0 \} \) is nonempty, compact, and invariant. Since \( w_I(\hat{\phi}) \) is nonincreasing for \( t \geq 0 \), it then follows that \( \omega(\hat{\phi}) = e(x) \), where \( e(x) \) is a nonnegative steady state of system (32). Clearly, \( e(x) \) is also an equilibrium of the auxiliary system of linear system (32) as the auxiliary system of (16). Due to \( s(A^{H_1}) < 0 \), \( e(tA^{H_1})\hat{\phi}(x) \) is a solution of the auxiliary system of system (32). It implies that \( e(x) = 0 \), and it follows from the comparison arguments that every solution of the last two equations of (3) converges to zero.

Note that the asymptotic equation of the first equation (3) of is
\[
\frac{\partial S(x, t)}{\partial t} = \lambda(x) - dS(x, t).
\tag{34}
It is easy to see \( \lim_{t \to \infty} S(x, t; \varphi) = S_0(x) \) uniformly for \( x \in \bar{\Omega} \). This proves part (i).

Next, we consider the case \( R_0 > 1 \) or \( s(A^H) > 0 \). Let

\[
\mathcal{W}_0 = \{ \varphi \in \mathcal{Y}^+ : \varphi_2(\cdot) \neq 0 \text{ and } \varphi_3(\cdot) \neq 0 \},
\]

and

\[
\partial \mathcal{W}_0 = \mathcal{Y}^+ \setminus \mathcal{W}_0 = \{ \varphi \in \mathcal{Y}^+ : \varphi_2(\cdot) \equiv 0 \text{ or } \varphi_3(\cdot) \equiv 0 \}.
\]

By Lemma 2.5, it follows that for any \( \varphi \in \mathcal{W}_0 \),

\[
I(x, t; \varphi) > 0, \ v(x, t; \varphi) > 0 \quad \text{for } x \in \Omega, \ t > 0.
\]

Let

\[
M_0 := \{ \varphi \in \partial \mathcal{W}_0 : \Psi_t \varphi \in \partial \mathcal{W}_0, \ \forall \ t > 0 \},
\]

and \( \omega(\varphi) \) be the omega limit set of the orbit \( O^+(\varphi) := \{ \Psi_t \varphi : t \geq 0 \} \).

Claim 1: \( \omega(\psi) = \{(S_0(x), 0, 0)\}, \ \forall \psi \in M_0 \).

Due to \( \psi \in M_0 \), we have \( \Psi_t \psi \in \partial \mathcal{W}_0 \), \( \forall t \geq 0 \). Then, \( I(\cdot, t; \psi) \equiv 0 \) or \( v(\cdot, t; \psi) \equiv 0 \), \( \forall t \geq 0 \). For \( v(\cdot, t; \psi) \equiv 0 \), \( \forall t \geq 0 \), \( S \) satisfies (34). Thus, we obtain \( \lim_{t \to \infty} S(x, t; \psi) = S_0(x) \) uniformly for \( x \in \Omega \). Further, we can see that \( \lim_{t \to \infty} I(x, t; \psi) = 0 \) uniformly for \( x \in \Omega \) from the second equation in (3). For \( v(\cdot, t_0; \psi) \neq 0 \) for some \( t_0 \geq 0 \), then Lemma 2.5 implies that \( v(x, t; \psi) > 0, \ \forall x \in \Omega, \ t > t_0 \). Hence, \( I(\cdot, t; \psi) \equiv 0, \ \forall t > t_0 \). In view of the \( v \) equation in (3), it is easy to see that \( \lim_{t \to \infty} v(x, t; \psi) = 0 \) uniformly for \( x \in \Omega \). Again, the equation for \( S \) is asymptotic to (34), and the theory for asymptotically autonomous semiflows implies that \( \lim_{t \to \infty} S(x, t; \psi) = S_0(x) \) for \( x \in \Omega \). Hence, \( \omega(\psi) = \{(S_0(x), 0, 0)\}, \ \forall \psi \in M_0 \).

If \( R_0 > 1 \) or \( s(A^f(x, S_0(x) - \iota)) > 0 \), there is a small \( \iota > 0 \) such that \( s(A^f(x, S_0(x) - \iota)) \) is the principal eigenvalue of the eigenvalue problem (30) and \( s(A^f(x, S_0(x) - \iota)) > 0 \). Let \( \tilde{\psi} := (\tilde{\psi}_2, \tilde{\psi}_3) \) be the strongly positive eigenfunction corresponding to \( s(A^f(x, S_0(x) - \iota)) \).

Claim 2: \( E_0(x) = (S_0(x), 0, 0) \) is a uniform weak repellor for \( \mathcal{W}_0 \) in the sense that

\[
\lim_{t \to \infty} \sup \| \Psi_t \varphi - E_0(x) \| \geq \iota, \ \forall \varphi \in \mathcal{W}_0.
\]

Suppose that there exists a \( \varphi_0 \in \mathcal{W}_0 \) such that

\[
\lim_{t \to \infty} \sup \| \Psi_t \varphi_0 - E_0(x) \| < \iota.
\]

Then, there exists a \( t_1 > 0 \) such that \( S(x, t; \varphi_0) > S_0(x) - \iota, \ \forall t \geq t_1, \ x \in \bar{\Omega} \). Hence, \( S(x, t; \varphi_0) \) satisfies

\[
\begin{cases}
\frac{\partial I(x, t)}{\partial t} \geq -aI(x, t) + e^{-\eta t} f(x, S_0(x) - \iota) v(x, t - \tau), \ x \in \Omega, \ t \geq t_1, \\
\frac{\partial v(x, t)}{\partial t} = D\Delta v(x, t) + k(x) I(x, t) - uv(x, t), \ x \in \Omega, \ t \geq t_1, \\
\frac{\partial v(x, t)}{\partial \nu} = 0, \ x \in \partial \Omega, \ t \geq t_1.
\end{cases}
\]

(35)

By Theorem 2.4, there is a strongly positive eigenfunction \( \phi_0 \) corresponding to \( s(A^{H_2}) \) with \( H_2 = f(x, S_0(x) - \iota) \). Let \( h(x, t; \varphi) = (I_3(x, t; \varphi), v_3(x, t; \varphi)) \) be the
unique solution of
\[
\begin{align*}
\frac{\partial I_3(x,t)}{\partial t} & = -a I_3(x,t) + e^{-\gamma(t)} f(x, S_0(x) - i) v_3(x,t - \tau), \ x \in \Omega, \ t \geq t_1, \\
\frac{\partial v_3(x,t)}{\partial t} & = D \Delta v_3(x,t) + k(x) I_3(x,t) - w_3(x,t), \ x \in \Omega, \ t \geq t_1,
\end{align*}
\]
(36)

with \( h(x, \theta, \varphi) = \varphi(x, \theta) \) for all \((x, \theta) \in \bar{\Omega} \times [-\tau, 0]\), for any \( \varphi \in C([-\tau, 0], C(\bar{\Omega}, \mathbb{R}^2)) \), and

\[
h^{-}(x,t) := \phi_i(x), \ \forall t \in [-\tau, \infty), \ x \in \bar{\Omega}.
\]

Due to \( s(A^{H_2}) > 0 \), it is easy to see that \( h^{-}(x,t) \) is a lower solution of system (36). The comparison principle yields

\[
( I(x,t; \phi_i, v(x,t; \phi_i)) \geq h(x,t; \phi_i) \geq h^{-}(x,t) = \phi_i(x)) \geq 0, \ \forall x \in \bar{\Omega}, \ t \geq t_1,
\]
(37)

which implies \( h_t(\phi_i) \geq \phi_i \) for all \( t \geq 0 \). By the comparison principle again, we obtain

\[
h_t(\phi_i) = h_t(h_s(\phi_i)) \geq h_s(\phi_i) \text{ for all } t, s \geq 0 \text{ and therefore, } h_t(\phi_i) \text{ is nondecreasing in } t \in [0, \infty).
\]

Recall that \( h_t \) is a \( \kappa \)-contraction for each \( t > 0 \). It then follows that \( h_t \) is asymptotically smooth. Furthermore, we conclude that the solution \( h(\cdot; x, \phi_i) \) is unbounded. If not, the omega limit set \( \omega(\phi_i) \) of the bounded orbit \( \gamma^+(\phi_i) = \{h_t(\phi_i) : t \geq 0\} \) is nonempty, compact, and invariant. Since \( h_t(\phi_i) \) is nondecreasing in \( t \in [0, \infty) \), it then follows that \( \omega(\phi_i) = \varepsilon^+(x) \), where \( \varepsilon^+(x) \) is a nonnegative steady state of system (36). Clearly, \( \varepsilon^+(x) \) is also an equilibrium of the auxiliary system of (36). Since \( s(A^{H(x, S_0(x) - i)}) > 0 \), \( e^+(\varepsilon^+(x)) )_i \phi_i(x) \) is a solution of the auxiliary system of (36). We can obtain that the solution of system (3) about \( I \) and \( v \) under given initial value \( \phi_i \) is unbounded. This contradiction proves the claim 2.

Define a continuous function \( p : \mathbb{Y}^+ \to [0, \infty) \) by

\[
p(\varphi) := \min \{ \min_{x \in \Omega} \varphi_2(x), \min_{x \in \Omega} \varphi_3(x) \}, \ \forall \varphi \in \mathbb{Y}^+.
\]

It is easy to see that \( p^{-1}(0, \infty) \subseteq \mathbb{W}_0 \). In addition, if \( p(\varphi) > 0 \) or \( \varphi \in \mathbb{W}_0 \) with \( p(\varphi) = 0 \), then \( p(\Psi_t \varphi) > 0 \), \( \forall \ t > 0 \), that is, \( p \) is a generalized distance function for the semiflow \( \Psi_t : \mathbb{Y}^+ \to \mathbb{Y}^+ \). From the above claims, it follows that any forward orbit of \( \Psi_t \) in \( M_0 \) converges to \( E_0(x) \) which is isolated in \( \mathbb{Y}^+ \), and \( W^s(E_0(x)) \cap \mathbb{W}_0 \neq \emptyset \), where \( W^s(E_0(x)) \) is the stable set of \( E_0(x) \). Thus, there is no cycle in \( M_0 \) from \( \{E_0(x)\} \) to itself. By [14, Theorem 3], it follows that there exists a \( \delta > 0 \) such that

\[
\min_{\psi \in \omega(\varphi)} p(\psi) > \delta, \ \forall \varphi \in \mathbb{W}_0.
\]

Hence,

\[
\lim_{t \to \infty} \inf_{t \to \infty} I(\cdot; t; \varphi) \geq \delta \text{ and } \lim_{t \to \infty} \inf_{t \to \infty} v(\cdot; t; \varphi) \geq \delta, \ \forall \varphi \in \mathbb{W}_0.
\]

From Lemma 2.5, there exists a \( 0 < \delta \leq \tilde{\delta} \) such that

\[
\lim_{t \to \infty} s(\cdot; t; \varphi) \geq \delta, \ \lim_{t \to \infty} \inf_{t \to \infty} I(\cdot; t; \varphi) \geq \delta \text{ and } \lim_{t \to \infty} \inf_{t \to \infty} v(\cdot; t; \varphi) \geq \delta, \ \forall \varphi \in \mathbb{W}_0.
\]

By [7, Theorem 3.7 and Remark 3.10], it follows that \( \Psi_t : \mathbb{W}_0 \to \mathbb{W}_0 \) has a global attractor \( A_0 \). From Theorem 4.7 in [7], we know that \( \Psi_t \) has an equilibrium \( \tilde{\varepsilon}(\cdot) \in \mathbb{W}_0 \). Furthermore, Lemma 2.5 implies that \( \tilde{\varepsilon}(\cdot) \) is a positive steady state of (3). This completes the proof.
3. Spatially homogeneous model. In this section, we study the global attractivity of system (3) in case that all the coefficients are positive constants.

3.1. Existence of equilibria. When all the coefficients are positive constants, system (3) becomes

\[
\begin{align*}
  \frac{\partial S(x,t)}{\partial t} &= \lambda - dS(x,t) - f(S(x,t))v(x,t), \\
  \frac{\partial I(x,t)}{\partial t} &= e^{-\tau}f(S(x,t))v(x,t-\tau) - aI(x,t), \\
  \frac{\partial v(x,t)}{\partial t} &= D\Delta v(x,t) + kI(x,t) - uv(x,t),
\end{align*}
\]

in \((x,t) \in \Omega \times [0, \infty)\). The equilibria of the model (38) satisfies

\[
\begin{align*}
  \lambda - d\bar{S} - f(\bar{S})\bar{v} &= 0, \\
  e^{-\tau}f(\bar{S})\bar{v} - a\bar{I} &= 0, \\
  k\bar{I} - u\bar{v} &= 0.
\end{align*}
\]

We know that system (38) always admits an infection-free equilibrium \(E_0 = (S_0, I_0, v_0) = (\frac{\lambda}{d}, 0, 0)\). Aside from \(E_0\), the system may have a positive equilibrium \(E^* = (S^*, I^*, v^*)\). The basic reproduction number for ODE model corresponding to the system (38) is \(R_0 = \frac{e^{-\tau}k\bar{I}(\frac{\lambda}{d})}{au}\). By the similar arguments as in [19] and [23], one can show that

**Lemma 3.1.** The basic reproduction number for system (38) is also given by \(R_0\).

**Proposition 1.** System (38) always has a disease-free equilibrium \(E_0\). Moreover, if \(R_0 > 1\), then system (38) has a unique positive equilibrium \(E^*\) satisfying (39).

3.2. Global attractivity. In this subsection, we show the global attractivity of equilibria for system (38) via Lyapunov functionals.

**Theorem 3.2.** For the system (38), the following statements are valid

(i) If \(R_0 < 1\), then the disease-free equilibrium \(E_0\) is globally attractive in \(\mathbb{Y}^+\).

(ii) If \(R_0 > 1\), then the positive equilibrium \(E^*\) is globally attractive in \(\mathbb{Y}^+_0\).

**Proof.** The statement (i) has already been shown in Theorem 2.6, when \(\lambda\) and \(k\) are spatially dependent. Of course, it is also can be proved by constructing a Lyapunov function, which is omitted here.

For the part (ii), we choose

\[
V(\varphi) = \int_\Omega \left[ \varphi_1(x,0) - S^* - \int_{\xi}^{\varphi_1(x,0)} \frac{f(S^*)}{f(\xi)} \frac{d\xi}{S^*} + e^{\tau}I^*g \left( \frac{\varphi_2(x,0)}{I^*} \right) \right. \\
+ \frac{e^{\tau}a}{k} \frac{v^*}{v^*} g \left( \frac{\varphi_3(x,0)}{v^*} \right) + f(S^*)v^* \int_{-\tau}^{0} g \left( \frac{f(\varphi_1(x,\theta))\varphi_3(x,\theta)}{f(S^*)v^*} \right) d\theta \left. \right] dx,
\]

where \(g(q) = q - 1 - \ln q\) for \(q \in \mathbb{R}_+\). The function \(V\) is non-negative with respect to positive solutions of system (38). Denote

\[
S_t = S_t(x,0), \quad S_{t,\tau} = S_t(x,-\tau), \quad I = I(x,t), \quad v_t = v_t(x,0), \quad v_{t,\tau} = v_{t,\tau}(x,-\tau).
\]
Then,
\[
\dot{V}_{(38)} = \int_{\Omega} \left[ \dot{S}_t - \frac{f(S^*) S_t}{f(S_t)} \dot{S}_t + I^* \dot{I} e^{n_\tau} - \frac{I^*}{T} \dot{I} e^{n_\tau} + \frac{ae^{n_\tau}}{k} \left( v_t - \frac{v^*}{v_t} \right) 
\right.
+ f(S^*) v^* g \left( \frac{f(S^*) v^*}{f(S_t) v^*} \right) - f(S^*) v^* g \left( \frac{f(S_{l,t}) v_{l,t}}{f(S^*) v^*} \right) \] dx.
\]

(40)

Using the fact that \( \lambda = dS^* + f(S^*) v^* \), \( e^{-n_\tau} f(S^*) v^* = aI^* \) and \( kI^* = uv^* \), we obtain
\[
\dot{V}_{(38)} = \int_{\Omega} \left[ dS^* \left( 1 - \frac{S_t}{S^*} \right) \left( 1 - \frac{f(S^*)}{f(S_t)} \right) - \frac{f^2(S^*) v^*}{f(S_t)} + f(S^*) v^* + f(S^*) v_t 
\right.
- \frac{I^*}{T} f(S_{l,t}) v_{l,t} + e^{n_\tau} aI^* - \frac{au e^{n_\tau}}{k} v_t - \frac{ae^{n_\tau} v^*}{v_t} I + \frac{au e^{n_\tau}}{k} v^* + f(S^*) v^* \ln \frac{S_{l,t} v_{l,t}}{S_t v_t} 
+ \frac{e^{n_\tau} a}{k} D \Delta v_t - \frac{e^{n_\tau} a v^*}{k v_t} D \Delta v_t \bigg] dx 
\]

that is,
\[
\dot{V}(\varphi) = \int_{\Omega} \left[ dS^* \left( 1 - \frac{\varphi_{1,0}}{S^*} \right) \left( 1 - \frac{f(S^*)}{f(\varphi_{1,0})} \right) - f(S^*) v^* g \left( \frac{I^* f(\varphi_{1,0}) \varphi_{3,0}}{f(S^*) v^* \varphi_2} \right) 
- f(S^*) v^* g \left( \frac{f(S^*) v^*}{f(\varphi_{1,0})} \right) - f(S^*) v^* g \left( \frac{v^* \varphi_2}{I^* \varphi_{3,0}} \right) + \frac{e^{n_\tau} a}{k} D \Delta \varphi_{3,0} - \frac{e^{n_\tau} a v^*}{k \varphi_{3,0}} D \Delta \varphi_{3,0} \right] dx,
\]

where
\[
\varphi_{1,0} = \varphi_1(x, 0), \varphi_{1,0} = \varphi_1(x, \theta), \varphi_{3,0} = \varphi_3(x, 0), \varphi_{3,0} = \varphi_3(x, \theta).
\]

From the Divergence Theorem and the Neumann boundary conditions (5), we can obtain
\[
0 = \int_{\partial \Omega} \frac{1}{\varphi_{3,0}} \nabla \varphi_{3,0} \cdot \mathbf{n} dS = \int_{\Omega} \text{div} \left( \frac{1}{\varphi_{3,0}} \nabla \varphi_{3,0} \right) dx
= \int_{\Omega} \left( \frac{1}{\varphi_{3,0}} \Delta \varphi_{3,0} - \frac{1}{\varphi_{3,0}^2} \| \nabla \varphi_{3,0} \|^2 \right) dx,
\]
and
\[
\int_{\Omega} \frac{1}{\varphi_{3,0}} \Delta \varphi_{3,0} dx = \int_{\Omega} \frac{1}{\varphi_{3,0}^2} \| \nabla \varphi_{3,0} \|^2 dx \geq 0.
\]

Meanwhile, we know that \( \int_{\Omega} \Delta \varphi_{3,0} dx = 0 \). Thus, we obtain
\[
\dot{V}(\varphi) = \int_{\Omega} \left[ dS^* \left( 1 - \frac{\varphi_{1,0}}{S^*} \right) \left( 1 - \frac{f(S^*)}{f(\varphi_{1,0})} \right) - f(S^*) v^* g \left( \frac{I^* f(\varphi_{1,0}) \varphi_{3,0}}{f(S^*) v^* \varphi_2} \right) 
- f(S^*) v^* g \left( \frac{f(S^*) v^*}{f(\varphi_{1,0})} \right) - f(S^*) v^* g \left( \frac{v^* \varphi_2}{I^* \varphi_{3,0}} \right) - \frac{e^{n_\tau} a v^*}{k \varphi_{3,0}^2} \| \nabla \varphi_{3,0} \|^2 \right] dx.
\]
From the properties of $f(S)$, it follows that
\[ dS^* \left( 1 - \frac{\varphi_1,0}{S^*} \right) \left( 1 - \frac{f(S^*)}{f(\varphi_1,0)} \right) \leq 0. \]
Thus, $\dot{V}(\varphi) \leq 0$ holds. Applying LaSalle Invariance Principle, we know that
the solutions of system (38) must converge to the largest positive invariant set of $M = \{ \dot{V}(\varphi) = 0 \}$.

Finally, we show that $M$ consists of only the interior equilibrium $E^*$. Indeed,
$\dot{V}(\varphi) = 0$ if and only if $S(x, t) = S^*$, $I(x, t) = I^*$, $v(x, t) = v^*$. Thus, $M = \{ E^* \}$.
By Lemma 2.2, the orbit $\gamma^+(\varphi) = z_t(\cdot, \theta; \varphi)$ is precompact in $\mathbb{Y}^+$. According to
Theorem 4.3.4 in [3], we know $E^*$ is globally attractive.

4. Simulations. In this section, we carry out some numerical simulations to illustrate our theoretical results. For the system (38), we choose $f(S) = \beta S$ and $\Omega = (0, \pi)$. Then, we have
\[
\begin{align*}
\frac{\partial S(x, t)}{\partial t} &= \lambda(x) - dS(x, t) - \beta S(x, t)v(x, t), \\
\frac{\partial I(x, t)}{\partial t} &= e^{-n\tau}\beta S(x, t - \tau)v(x, t - \tau) - aI(x, t), \\
\frac{\partial v(x, t)}{\partial t} &= D\Delta v(x, t) + kI(x, t) - uv(x, t),
\end{align*}
\]
with initial conditions
\[
\begin{align*}
S^0(x) &= \varphi_1(x, \theta) > 0, & I^0(x) &= \varphi_2(x, 0) \geq 0, \\
v^0(x) &= \varphi_3(x, \theta) \geq 0, & x &\in (0, \pi), \quad -\tau \leq \theta \leq 0,
\end{align*}
\]
and the homogeneous Neumann boundary condition
\[
\frac{\partial v}{\partial x} = 0, \quad x = 0, \pi, \quad t > 0.
\]
In the following, we choose four groups of data and present numerical simulations for the system (41)-(43).

(a) $\lambda(x) = \sin x$, $\beta = 0.03$, $a = 0.3$, $u = 0.3$, $d = 0.2$, $k = 0.14$, $D = 2$, $n = 0.3$ and $\tau = 2.8$. In this case, the disease-free equilibrium $(5 \sin x, 0, 0)$ is globally attractive, see Fig.1;
(b) $\lambda(x) = \sin x$, $\beta = 0.03$, $a = 0.3$, $u = 0.3$, $d = 0.2$, $k = 2$, $D = 2$, $n = 0.3$ and $\tau = 2.8$. In this case, the solutions are uniformly persistent, see Fig.2;
(c) $\lambda = 165$, $\beta = 0.002$, $a = 0.3$, $u = 0.3$, $d = 0.2$, $k = 0.1$, $D = 0.2$, $n = 0.3$ and $\tau = 2.8$. Under this set of parameter values, we have that $R_0 = 0.7915 < 1$. Hence, the disease-free equilibrium $E_0 = (825, 0, 0)$ is globally attractive, see Fig.3;
(d) $\lambda = 165$, $\beta = 0.002$, $a = 0.3$, $u = 0.3$, $d = 0.2$, $k = 0.14$, $D = 0.2$, $n = 0.3$ and $\tau = 2.8$. Now, we have that $R_0 = 1.1081 > 1$, and therefore, the positive equilibrium $E^* = (745, 23.1, 11.9)$ is globally attractive, see Fig.4.

5. Summary and discussion. In this paper, a diffusive virus model (3) with time delay and a general incidence function in a bounded domain, is proposed and analyzed. Heterogeneity in the production rates of uninfected target cells as well as that of new viruses, together with zero-flux boundary condition, are assumed. It is shown that model (3) admits two equilibria: a disease-free equilibrium and an endemic equilibrium. The dynamics of this model are shown to be determined by the basic reproduction number $R_0$. Precisely, it is proved that the virus dies out if
$R_0 < 1$, while uniformly persists if $R_0 > 1$ (in terms of the persistence theory). In particular, we also prove global attractivity of equilibria for system (38), provided that all parameters are constants.

Observe that the basic reproduction number $R_0$ is independent of the diffusion coefficient $D$, the results reveal that the spatial diffusion of the virus has no impact on the global dynamics of (3), subject to the Neumann boundary condition.

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