Quantum Projectors and Local Operators in Lattice Integrable Models

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Abstract

In the framework of the quantum inverse scattering method, we consider a problem of constructing local operators for two-dimensional quantum integrable models, especially for the lattice versions of the nonlinear Schrödinger and sine-Gordon models. We show that a certain class of local operators can be constructed from the matrix elements of the monodromy matrix in a simple way. They are closely related to the quantum projectors and have nice commutation relations with the half of the matrix elements of the elementary monodromy matrix. The form factors of these operators can be calculated by using the standard algebraic Bethe ansatz techniques.
1 Introduction

In two-dimensional quantum integrable models, the quantum inverse scattering method (QISM) \cite{1,2,3,4,5} provides a powerful tool for investigating physical quantities. Among them, the correlation functions have been studied extensively. In order to calculate the correlation functions, it is necessary to deal with states and local operators. At the early stages of the development of QISM, the problem of constructing states was solved by means of the Bethe ansatz.

Recently, a great progress was made for constructing local operators in a large class of spin chain models \cite{6,8,7} which contain the XXX and XXZ spin chains with spin $1/2$. Simple inverse mappings from the matrix elements of the monodromy matrix to the local spin variables were found \cite{8,7}. The inverse mappings help the calculations of form factors and correlation functions of the spin variables in the framework of QISM \cite{8,9,10}.

The XXX and XXZ spin chains with spin $1/2$ are fundamental models, i.e. the auxiliary space and the quantum space at a site are isomorphic and the elementary monodromy matrix has a special point at which it becomes the permutation operator for the auxiliary and quantum spaces. The construction of the inverse mapping depends deeply on the existence of the permutation operators. Some non-fundamental models such as higher spin XXX chains were solved by means of the fusion procedure \cite{7}. One characteristic property of these models is that the matrix elements of the elementary monodromy matrix are numerical.

But for some non-fundamental integrable models, such as the lattice nonlinear Schrödinger (LNS) models \cite{11,12,13,14,15,16,17,18,19} and the lattice sine-Gordon (LSG) model \cite{2,20,21,14}, the problem of constructing the inverse mapping is not solved. The LNS and LSG models are closely related to the XXX and XXZ spin chains respectively. Their elementary monodromy matrices are realized by quantum operators and the quantum space at each site is related to an infinite-dimensional representation of the Lie algebra $sl_2$ or its quantum deformation $U_q(sl_2)$. Only at very specific values of the coupling constant, the infinite dimensional representations are truncated into finite-dimensional ones. The approach by the fusion procedure is possible only at these special points and is very artificial. Moreover, we should take the infinite-dimensional representation limit which has many difficulties.

Therefore, it is better to consider the inverse mapping in more direct way. This paper is an attempt toward the construction of the inverse mapping.

The form factor bootstrap \cite{22} is one of approaches to obtain the correlation functions and was applied to the (continuum limit of) LNS models and LSG models. In this approach, creation operators of the states are Zamolodchikov-Faddeev (ZF) creation operators. The ZF creation-annihilation operators are constructed by using the quantum reflection operators $B(\lambda)A^{-1}(\lambda)$ and their conjugate $D^{-1}(\lambda)C(\lambda)$. The local operators are treated by means of the quantum Gel’fand-Levitan equations \cite{23,24}. The calculation procedure for the form
factors are summarized into the axioms by Smirnov [22]. (See also [25,26,27,28] for the approach by the quantum Gel’fand-Levitan equation in case of quantum nonlinear Schrödinger model).

In contrast to the Gel’fand-Levitan method, we use the reflection operators to construct local operators. The elementary monodromy matrices of LNS and LSG models have special points at which they factorize into quantum projectors. The constructed operators are closely related to these quantum projectors. In this paper, a basis of states is chosen to be the Bethe eigenstates. We show that the form factors of the operators can be calculated by using the algebraic relations in the framework of QISM.

This paper is organized as follows. In section 2, the main idea for constructing the local operators is explained. In section 3, we show that form factors of these local operators can be calculated in the framework of the standard algebraic Bethe ansatz method. Some properties of these local operators are discussed in section 4. The explicit form of the operators are given for LNS and LSG models in sections 5 and 6 respectively. Section 7 is devoted to discussion.

2 Local operators from quantum projectors

Let $L_n(\lambda) (n = 1, 2, \ldots, N)$ be an infinitesimal monodromy matrix of lattice models with the intertwining property:

$$R(\lambda, \mu)(L_n(\lambda) \otimes L_n(\mu)) = (L_n(\mu) \otimes L_n(\lambda))R(\lambda, \mu).$$

(2.1)

Here the numerical $R$-matrix has the form:

$$R(\lambda, \mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda, \mu) & c(\lambda, \mu) & 0 \\ 0 & c(\lambda, \mu) & b(\lambda, \mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

(2.2)

The functions $b(\lambda, \mu)$ and $c(\lambda, \mu)$ are rational for LNS model

$$b(\lambda, \mu) = \frac{\lambda - \mu - i\kappa}{\lambda - \mu - i\kappa}, \quad c(\lambda, \mu) = \frac{\lambda - \mu}{\lambda - \mu - i\kappa},$$

(2.3)

and are trigonometric for LSG model

$$b(\lambda, \mu) = \frac{i \sin \gamma}{\sinh(\lambda - \mu + i\gamma)}, \quad c(\lambda, \mu) = \frac{\sinh(\lambda - \mu)}{\sinh(\lambda - \mu + i\gamma)}.$$  

(2.4)

The monodromy matrix of the lattice model is given by

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = L_N(\lambda)L_{N-1}(\lambda) \ldots L_1(\lambda).$$

(2.5)
At the points $\lambda = \nu$ where the quantum determinant vanishes, the elementary monodromy matrix factorizes into quantum projectors:

$$(L_n(\lambda))_{ij}^{\lambda=\nu} = P_i(n)Q_j(n), \quad i, j = 1, 2.$$  \hfill (2.6)

Then the monodromy matrix at $\lambda = \nu$ can be written as

$$(T(\nu))_{ij} = P_i(N)WQ_j(1),$$  \hfill (2.7)

where

$$W = w(N|N-1)w(N-1|N-2)\ldots w(2|1), \quad w(n+1|n) = \sum_{i=1}^{2} Q_i(n+1)P_i(n).$$  \hfill (2.8)

So far, these quantum projectors have been used mainly for constructing the conserved quantities. A simple observation is that these quantum projectors can be used for constructing a certain class of local operators of the lattice models. For example, if $D(\nu)$ is invertible then we have two operators which depend on field variables of site 1 or site $N$ only:

$$(D(\nu))^{-1}C(\nu) = (Q_2(1))^{-1}Q_1(1), \quad B(\nu)D(\nu) = P_1(N)(P_2(N))^{-1}.$$  \hfill (2.9)

For simplicity, we impose the periodic boundary condition: $L_{n+N}(\lambda) = L_n(\lambda)$. Then the shift operator $U$ can be defined by

$$UL_n(\lambda)U^{-1} = L_{n+1}(\lambda), \quad U|\Omega\rangle = |\Omega\rangle.$$  \hfill (2.10)

Here $|\Omega\rangle$ is the reference state: $C(\lambda)|\Omega\rangle = 0$. Using this shift operator, we can relate certain local operators to the matrix elements of the monodromy matrix:

$$Q_n := (Q_2(n))^{-1}Q_1(n) = U^{n-1}D(\nu)C(\nu)U^{-n+1},$$

$$P_n := P_1(n)P_2(n))^{-1} = U^nB(\nu)D(\nu)U^{-n}.$$  \hfill (2.11)

Off-shell properties of these operators can be extracted from the form factors:

$$\langle \Omega| \left( \prod_{k=1}^{M} C(\mu_k) \right) \left( \prod_{i=1}^{M'} B(\lambda_i) \right) |\Omega\rangle, \quad C_n = Q_n \text{ or } P_n.$$  \hfill (2.12)

Here we assume that the sets of spectral parameters $\{\mu_k\}$ and $\{\lambda_i\}$ satisfy the Bethe equations respectively.

We call a state $\prod_k B(\lambda_k)|\Omega\rangle$ Bethe state for generic $\{\lambda_k\}$. When we emphasize that $\{\lambda_k\}$ satisfy the Bethe equations, we call the state Bethe eigenstate.

Let us denote the eigenvalues of diagonal part of the monodromy matrix on the reference state by

$$(L_n(\lambda))_{11} |\Omega\rangle = a_1(\lambda)|\Omega\rangle, \quad (L_n(\lambda))_{22} |\Omega\rangle = d_1(\lambda)|\Omega\rangle.$$  \hfill (2.13)
It is known that the Bethe eigenstates are also eigenstates for the shift operator with the following eigenvalues [11, Theorem 3]

\[ U \left( \prod_{l=1}^{M} B(\lambda_l) \right) |\Omega\rangle = \left( \prod_{j=1}^{M} r_1(\lambda_j) \right) \left( \prod_{l=1}^{M} B(\lambda_l) \right) |\Omega\rangle, \]

(2.14)

where \( r_1(\lambda) = a_1(\lambda)/d_1(\lambda) \).

Thus, the form factors of \( Q_n \) (resp. \( P_n \)) are easily represented by the form factors of \( D^{-1}(\nu)C(\nu) \) (resp. \( B(\nu)D^{-1}(\nu) \)). These form factors can be calculated by using the algebraic commutation relations.

The time evolution of these operators is controlled by the Hamiltonian operators of the models. The Hamiltonian operator is also diagonalized on the Bethe eigenstates. The form factors of operators at any time can be easily expressed by those of the operators at a time (e.g. at \( t = 0 \)). We do not discuss the time evolution in this paper.

Consideration for other points at which \( A(\nu) \) is invertible is quite similar. Therefore we omit these cases.

### 3 Form Factors

In this section, we calculate the form factors of \( D^{-1}(\nu)C(\nu) \) in general setting. The calculation for \( B(\nu)D^{-1}(\nu) \) is similar. So we omit the case of \( B(\nu)D^{-1}(\nu) \).

We forget the lattice structure for a while and treat the matrix elements \( A(\lambda), B(\lambda), C(\lambda) \) and \( D(\lambda) \) as abstract objects. Let

\[ A(\lambda)|\Omega\rangle = a(\lambda)|\Omega\rangle, \quad D(\lambda)|\Omega\rangle = d(\lambda)|\Omega\rangle. \]

(3.1)

We assume that there is at least one zero for \( a(\lambda) \): \( a(\nu_A) = 0 \). Also, we assume that \( D(\nu_A) \) is an invertible operator.

The action of \( A(\lambda) \) on the Bethe states is well known:

\[ A(\mu) \prod_{l=1}^{M} B(\lambda_l)|\Omega\rangle = a^{(M)}(\mu|\{\lambda_l\}) \prod_{l=1}^{M} B(\lambda_l)|\Omega\rangle + \sum_{j=1}^{M} b^{(M)}(\mu|\lambda_j|\{\lambda_l\}_{l\neq j}) B(\mu) \prod_{l=1}^{M} B(\lambda_l)|\Omega\rangle, \]

(3.2)

where

\[ a^{(M)}(\mu|\{\lambda_l\}) = a(\mu) \prod_{l=1}^{M} f(\lambda_l, \mu), \quad b^{(M)}(\mu|\lambda_j|\{\lambda_l\}_{l\neq j}) = a(\lambda_j) g(\mu, \lambda_j) \prod_{l=1}^{M} f(\lambda_l, \lambda_j). \]

(3.3)

Here \( f(\lambda, \mu) = 1/c(\lambda, \mu) \) and \( g(\lambda, \mu) = b(\lambda, \mu)/c(\lambda, \mu) \).
For generic $\mu$, $\lambda$, we have the following lemma:

$$D^{-1}(\mu)C(\mu)B(\lambda) = f(\lambda, \mu)B(\lambda)D^{-1}(\mu)C(\mu) - g(\lambda, \mu)A(\lambda) + g(\lambda, \mu)D^{-1}(\mu)D(\lambda)\left(A(\mu) - B(\mu)D^{-1}(\mu)C(\mu)\right).$$  \hspace{1cm} (3.4)

The proof is simple:

$$D^{-1}(\mu)C(\mu)B(\lambda) = D^{-1}(\mu)[C(\mu), B(\lambda)] + D^{-1}(\mu)B(\lambda)D(\mu)D^{-1}(\mu)C(\mu)$$

$$= D^{-1}(\mu)g(\lambda, \mu) (D(\lambda)A(\mu) - D(\mu)A(\lambda)) + D^{-1}(\mu) (f(\lambda, \mu)D(\mu)B(\lambda) - g(\lambda, \mu)D(\lambda)B(\mu)) D^{-1}(\mu)C(\mu)$$

$$= f(\lambda, \mu)B(\lambda)D^{-1}(\mu)C(\mu) - g(\lambda, \mu)A(\lambda) + g(\lambda, \mu)D^{-1}(\mu)D(\lambda)\left(A(\mu) - B(\mu)D^{-1}(\mu)C(\mu)\right).$$ \hspace{1cm} (3.5)

Then using this lemma and by induction, we can prove that $D^{-1}(\nu_A)C(\nu_A)$ acts on the right Bethe states as follows:

$$D^{-1}(\nu_A)C(\nu_A) \prod_{l=1}^{M} B(\lambda_l) |\Omega\rangle = \sum_{j=1}^{M} b^{(M)}(\nu_A|\lambda_j\rangle\langle\lambda_j|_{\nu_A}) \prod_{l=1}^{M} B(\lambda_l) |\Omega\rangle. \hspace{1cm} (3.6)$$

This is quite similar to the action of the nonlinear Schrödinger field $\Psi(0)$ on the Bethe states \cite{29}. Therefore, the calculation procedure for the form factors of $\Psi(0)$ \cite{30,31} can be also applied to the following form factors:

$$F_M := \langle \Omega | \left( \prod_{k=1}^{M} C(\mu_k) \right) D^{-1}(\nu_A)C(\nu_A) \left( \prod_{l=1}^{M+1} B(\lambda_l) \right) |\Omega\rangle / \langle \Omega | \Omega \rangle. \hspace{1cm} (3.7)$$

Here $\{\mu_k\}$ and $\{\lambda_l\}$ are solutions of the Bethe equations respectively.

After some calculations which are a slight modification of \cite{30,31}, we have

$$F_M = \prod_{k=1}^{M+1} \prod_{l=1}^{M+1} h(\lambda_k, \lambda_l) \prod_{1 \leq k < t \leq M} g(\mu_k, \mu_k) \prod_{1 \leq k < t \leq M+1} g(\lambda_k, \lambda_l)$$

$$\times \prod_{l=1}^{M+1} d(\mu_l) \prod_{l=1}^{M+1} d(\lambda_l) \left( \sum_{j=1}^{M+1} (-1)^{j-1} g(\nu_A, \lambda_j) \det_{M} S^{(j)} \right). \hspace{1cm} (3.8)$$

Here $S^{(j)}$ ($j = 1, 2, \ldots, M + 1$) is an $M \times M$ matrix obtained by removing $j$-th row from an $(M + 1) \times M$ matrix $S$ whose matrix elements are defined by

$$S_{kl} = \frac{\prod_{m=1}^{M} h(\lambda_k, \mu_m) \prod_{m=1}^{M} h(\mu_m, \lambda_k)}{\prod_{m=1}^{M} h(\lambda_k, \lambda_m) \prod_{m=1}^{M} h(\lambda_m, \lambda_k)}, \hspace{1cm} (3.9)$$
\[(k = 1, 2, \ldots, M + 1, l = 1, 2, \ldots, M).\] Also, \(t(\lambda, \mu) = b^2(\lambda, \mu)/c(\lambda, \mu)\) and \(h(\lambda, \mu) = 1/b(\lambda, \mu).\)

In the following, we will show that the sum in the right hand side of eq. (3.8) can be rewritten by using a single determinant.

By using the Cauchy determinant identity or by evaluating the residues, we can prove the following identity:

\[
\sum_{j=1}^{M+1} g(\eta, \lambda_j) \xi_j = \prod_{j=1}^{M+1} g(\eta, \lambda_j) \prod_{i=1}^{M} \frac{1}{g(\eta, \mu_i)},
\] (3.10)

where

\[
\xi_k := \prod_{l=1}^{M+1} g(\lambda_l, \lambda_k) \prod_{i=1}^{M} \frac{1}{g(\mu_i, \lambda_k)}.
\] (3.11)

With help of this identity, it is possible to check that the \((M + 1)\)-dimensional vector \(\xi_k\) is a left null vector of the matrix \(S\): \(\sum_{k=1}^{M+1} \xi_k S_{kl} = 0\). The substitution of \(S_{M+1,l} = -\sum_{k=1}^{M} (\xi_k/\xi_{M+1}) S_{kl}\) into \(\det S^{(j)}\) leads to

\[(-1)^{j-1} \det S^{(j)} = (-1)^{M} \frac{\xi_j}{\xi_{M+1}} \det S^{(M+1)}.\] (3.12)

In other words, the combination of \((-1)^{j-1} \det S^{(j)}/\xi_j\) is \(j\)-independent quantity:

\[
\left(\det S^{(1)}\right)/\xi_1 = -\left(\det S^{(2)}\right)/\xi_2 = \ldots
\]
\[
= (-1)^{j-1} \left(\det S^{(j)}\right)/\xi_j = \ldots = (-1)^{M} \left(\det S^{(M+1)}\right)/\xi_{M+1}.
\] (3.13)

By virtue of eq. (3.12) and eq. (3.10) for \(\eta = \nu_A\), we have the final result for the form factors:

\[
\langle \Omega | \left(\prod_{l=1}^{M} C(\mu_l)\right) D^{-1}(\nu_A) C(\nu_A) \left(\prod_{l=1}^{M+1} B(\lambda_l)\right) | \Omega\rangle / \langle \Omega | \Omega \rangle
\]
\[
= (-1)^{M} \det S_{kl1 \leq k, l \leq M} \prod_{l=1}^{M} d(\mu_l) \prod_{l=1}^{M+1} d(\lambda_l) \prod_{k=1}^{M+1} \prod_{l=1}^{M} h(\lambda_k, \lambda_l)
\]
\[
\times \prod_{1 \leq k < l \leq M} g(\mu_k, \mu_k) \prod_{1 \leq k < l \leq M} g(\lambda_k, \lambda_l) \prod_{i=1}^{M} g(\nu_A, \lambda_j) \prod_{i=1}^{M} g(\nu_A, \mu_i).
\] (3.14)

Here we have used the relation \(\det S^{(M+1)} = \det S_{kl1 \leq k, l \leq M} = \det S^{(M+1)}\).
4  Some properties of $Q_n$ and $P_n$

In this section, we discuss some properties of the local operators $Q_n$ and $P_n$.

For a spectral parameter $\mu$, let us define $\mu^\vee := \mu + i\kappa$ for the rational case and $\mu^\vee := \mu - i\gamma$ for the trigonometric case. Then

$$T(\mu)\sigma_2 T'(\mu)\sigma_2 = \text{det}_q(T(\mu))I_2. \quad (4.1)$$

Here $t$ denotes the transpose for the auxiliary space and $\text{det}_q(T(\mu))$ is a central element called quantum determinant.

The lemma \([3,4]\) can be rewritten as follows:

$$D^{-1}(\mu)C(\mu)B(\lambda) = f(\lambda, \mu)B(\lambda)D^{-1}(\mu)C(\mu) - g(\lambda, \mu)A(\lambda)$$

$$+ g(\lambda, \mu)D(\lambda)D^{-1}(\mu)D^{-1}(\mu^\vee)\text{det}_q(T(\mu)). \quad (4.2)$$

At $\mu = \nu_A$, the quantum determinant vanishes in the module constructed over the reference state. Without loss of generality, we can set $\text{det}_q(T(\nu_A)) = 0$.

The following relation comes from the intertwining property:

$$D^{-1}(\mu)C(\mu)D(\lambda) = f(\lambda, \mu)D(\lambda)D^{-1}(\mu)C(\mu) - g(\lambda, \mu)C(\lambda). \quad (4.3)$$

Now let us recall the lattice structure \([2,5]\) and the definition of the local operator $Q_1 = D^{-1}(\nu_A)C(\nu_A)$. From eqs. (4.2) and (4.3), we immediately have

$$Q_1B(\lambda) = f(\lambda, \nu_A)B(\lambda)Q_1 - g(\lambda, \nu_A)A(\lambda), \quad (4.4)$$

$$Q_1D(\lambda) = f(\lambda, \nu_A)D(\lambda)Q_1 - g(\lambda, \nu_A)C(\lambda). \quad (4.5)$$

It turns out that these relations arise as a consequence of

$$Q_1(L_1(\lambda))_{i2} = f(\lambda, \nu_A)(L_1(\lambda))_{i2}Q_1 - g(\lambda, \nu_A)(L_1(\lambda))_{i1}, \quad i = 1, 2. \quad (4.6)$$

Applying the shift operator to this equation, we have

$$Q_n(L_n(\lambda))_{i2} = f(\lambda, \nu_A)(L_n(\lambda))_{i2}Q_n - g(\lambda, \nu_A)(L_n(\lambda))_{i1}. \quad (4.7)$$

Compare to the action of $Q_1 = D^{-1}(\nu_A)C(\nu_A)$ on the right Bethe states \([3,6]\), the action on the left Bethe states are complicated. For a spectral parameter $\mu$, let $\mu^{(m)} := \mu + i\kappa$ for LNS model and $\mu^{(m)} := \mu - i\kappa$ for LSG model. (Note that $\mu^{(l)} = \mu^\vee$). From

$$\langle \Omega \prod_{k=1}^{M} C(\mu_k)D(\lambda) = d^{(M)}(\lambda|\{\mu_k\})\langle \Omega \prod_{k=1}^{M} C(\mu_k) + \sum_{j=1}^{M} c^{(M)}(\lambda|\mu_j|\{\mu_k\}_{k \neq j})\langle \Omega |C(\lambda) \prod_{k=1}^{M} C(\mu_k), \quad (4.8)$$

7
\[ d^{(M)}(\lambda|\{\mu_k\}) = d(\lambda) \prod_{k=1}^{M} f(\lambda, \mu_k), \quad c^{(M)}(\lambda|\mu_j|\{\mu_k\}_{k \neq j}) = d(\mu_j) g(\mu_j, \lambda) \prod_{k=1}^{M} f(\mu_j, \mu_k), \]

we have

\[
\langle \Omega | \left( \prod_{k=1}^{M} C(\mu_k) \right) D^{-1}(\lambda) = \left( d^{(M)}(\lambda|\{\mu_k\}) \right)^{-1} \langle \Omega | \left( \prod_{k=1}^{M} C(\mu_k) \right) - \sum_{j=1}^{M} \frac{c^{(M)}(\lambda|\mu_j|\{\mu_k\}_{k \neq j})}{d^{(M)}(\lambda|\{\mu_k\})} \langle \Omega | \left( \prod_{k=1}^{M} C(\mu_k) \right) D^{-1}(\lambda^{-1})C(\lambda^{-1}).
\]

Here we have used \( C(\lambda)D^{-1}(\lambda) = D^{-1}(\lambda^{-1})C(\lambda^{-1}) \). If we use these relations recursively, we can see that the result of the action of \( D^{-1}(\nu_A)C(\nu_A) \) on the left Bethe state yields terms which contain operators \( C(\nu_A^{(-m)}) \) for \( m = 0, 1, \ldots, M \).

The origin of these complicated action is the following commutation relation

\[ C(\mu)D^{-1}(\lambda) = c(\lambda, \mu)D^{-1}(\lambda)C(\mu) + b(\lambda, \mu)D^{-1}(\lambda)D(\mu)D^{-1}(\lambda^{-1})C(\lambda^{-1}) \]

which can be derived from the intertwining properties.

To conclude, the local operator \( Q_n \) has nice commutation relations with the half of the matrix elements of the infinitesimal monodromy matrix.

Similarly, from

\[
C(\lambda)B(\mu)D^{-1}(\mu) = f(\lambda, \mu)B(\mu)D^{-1}(\mu)C(\lambda) - g(\lambda, \mu)A(\lambda) + g(\lambda, \mu)\det_q(T(\mu))D^{-1}(\mu^\nu)D^{-1}(\mu)D(\lambda),
\]

we can derive the following property of the local operator \( P_n = U^n B(\nu_A)D^{-1}(\nu_A)U^{-n} \):

\[
(L_n(\lambda))_{2j} P_n = f(\lambda, \nu_A)P_n(L_n(\lambda))_{2j} - g(\lambda, \nu_A)(L_n(\lambda))_{1j}, \quad j = 1, 2.
\]

The operator \( P_N \) acts nicely on the left Bethe states and has complicated action on the right Bethe states.
5 Lattice Nonlinear Schrödinger model

The Hamiltonian of the quantum nonlinear Schrödinger model is given by

\[ H^{(\text{NLS})} = \int dx \left( \partial \psi^* \frac{\partial \psi}{\partial x} + \kappa \psi^* \psi \psi \right). \] (5.1)

Various types of LNS models have been proposed \[11, 12, 13, 14, 15, 16, 17, 18, 19\].

For simplicity, we use the LNS model of \[13, 14\] as an example. Let put the system in a box of length \(2L\): \((-L < x \leq L)\), and discretize it to the lattice with \(N\)-sites:

\[ x_n = -L + n \Delta, \quad (n = 1, 2, \ldots, N). \]

Here the lattice spacing is given by \(\Delta = 2L/N\). The elementary operators for this lattice model are constructed from original fields as follows:

\[ \psi_n = \int_{x_n-\Delta}^{x_n} \psi(x, 0) dx, \quad \psi^*_n = \int_{x_n-\Delta}^{x_n} \psi^*(x, 0) dx. \] (5.2)

They satisfy the canonical commutation relations: \([\psi_m, \psi^*_n] = \Delta \delta_{m,n}\).

The infinitesimal monodromy matrix is given by \[13, 14\]

\[ L_n(\lambda) = \begin{pmatrix} 1 - i(\lambda/2)\Delta + (\kappa/2)\psi_n^* \psi_n & \sqrt{\kappa} \psi_n^* (1 + (\kappa/4)\psi_n^* \psi_n)^{1/2} \\ \sqrt{\kappa} (1 + (\kappa/4)\psi_n^* \psi_n)^{1/2} \psi_n & 1 + i(\lambda/2)\Delta + (\kappa/2)\psi_n^* \psi_n \end{pmatrix}. \] (5.3)

For example, at \(\lambda = \nu_A := -2i/\Delta\), the infinitesimal monodromy matrix factorizes into the quantum projectors:

\[ (L_n(\nu_A))_{ij} = P_i(n)Q_j(n) \]

where

\[ P_1(n) = \sqrt{\kappa/2} \psi_n^*, \quad P_2(n) = \sqrt{2} (1 + (\kappa/4)\psi_n^* \psi_n)^{1/2}, \] (5.4)

\[ Q_1(n) = \sqrt{\kappa/2} \psi_n, \quad Q_2(n) = \sqrt{2} (1 + (\kappa/4)\psi_n^* \psi_n)^{1/2}. \] (5.5)

Thus, for generic coupling constant \(\kappa\),

\[ w(n+1|n) = 2 \left(1 + (\kappa/4)\psi_{n+1}^* \psi_{n+1}\right)^{1/2} \left(1 + (\kappa/4)\psi_n^* \psi_n\right)^{1/2} + (\kappa/2)\psi_{n+1}^* \psi_n^* \] (5.6)

is an invertible operator. There exists \(D^{-1}(\nu_A)\).

The corresponding local operators are

\[ Q_n = \sqrt{\kappa/2} (1 + (\kappa/4)\psi_n^* \psi_n)^{-1/2} \psi_n, \quad P_n = \sqrt{\kappa/2} \psi_n^* (1 + (\kappa/4)\psi_n^* \psi_n)^{-1/2}. \] (5.7)

In the continuum limit \(\Delta \to 0\), they become the field operators of the quantum nonlinear Schrödinger model:

\[ Q_n \to \sqrt{\kappa/2} \Delta \psi(x, 0), \quad P_n \to \sqrt{\kappa/2} \Delta \psi^*(x, 0), \quad x = -L + n\Delta, \] (5.8)

and the form factors \[3, 14\] give consistent results with \[31\].
6 Application to the lattice sine-Gordon model

The Hamiltonian of the quantum sine-Gordon model is given by

\[ H^{(SG)} = \int dx \left( \frac{1}{2} \pi^2 + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 + \frac{m^2}{\beta^2} (1 - \cos \beta u) \right). \]  

The infinitesimal monodromy matrix is given by \[ \mathcal{L}_n(\lambda) = \begin{pmatrix} \pi_n^{-1/2} \varphi(u_n) \pi_n^{-1/2} & -i(m\Delta/2) \sin((\beta/2)u_n + i\lambda) \\ -i(m\Delta/2) \sin((\beta/2)u_n - i\lambda) & \pi_n^{1/2} \varphi(u_n) \pi_n^{-1/2} \end{pmatrix}, \]  

\[ \pi_n = \exp \left( \frac{i}{4} \beta p_n \right), \quad \varphi(u_n) = (1 + 2r \cos \beta u_n)^{1/2}, \quad r = \left( \frac{m\Delta}{4} \right)^2. \]  

Here \( \gamma = \beta^2/8 \) and the lattice operators \( u_n \) and \( p_n \) \((n = 1, 2, \ldots, N)\) satisfy the canonical commutation relations: \([u_n, p_m] = i\delta_{nm}\). In the continuum limit \( \Delta \to 0 \), \( u_n \to u(x) \), \( p_n \to \pi(x)\Delta, \) \((-L < x = -L + n\Delta \leq L)\).

In order to construct the reference state \(|\Omega\rangle\), the elementary monodromy matrix should be taken as a composite of the infinitesimal monodromy matrices of two-adjacent sites \[ \mathcal{L}_k(\lambda) = \mathcal{L}_{2k}(\lambda) \mathcal{L}_{2k-1}(\lambda), \quad k = 1, 2, \ldots, N/2. \]  

Here we assume the number of sites \( N \) is even. Then

\[ (\mathcal{L}_k(\lambda))_{11}|\Omega\rangle = a_1(\lambda)|\Omega\rangle, \quad (\mathcal{L}_k(\lambda))_{22}|\Omega\rangle = d_1(\lambda)|\Omega\rangle, \]  

\[ a_1(\lambda) = 1 + 2r \cosh(2\lambda - i\gamma), \quad d_1(\lambda) = 1 + 2r \cosh(2\lambda + i\gamma). \]  

Thus, the shift operator is defined by

\[ U \mathcal{L}_k U^{-1} = \mathcal{L}_{k+1}, \quad U|\Omega\rangle = |\Omega\rangle. \]  

In other words, it acts as two-site shift for the local variables: \( Uu_n U^{-1} = u_{n+2}, \) \( Up_n U^{-1} = p_{n+2}. \) In general, the periodic boundary condition has the form: \( u_{n+N} = u_n + (2\pi/\beta)Q, \) \( p_{n+N} = p_n \) where \( Q \) is the topological charge. In this paper, we only consider the sector with \( Q = 0 \) for simplicity.

Let us introduce a positive “momentum cutoff” parameter \( \Lambda \) by \( 2r \cosh \Lambda = 1. \)

At \( \lambda = \nu_A^{(\epsilon, \epsilon')}(1/2)(i\gamma + \epsilon\Delta + i\epsilon'\pi), \) \( (\epsilon, \epsilon') = \pm 1, \) \( a_1(\lambda) \) vanishes and the infinitesimal monodromy matrix factorizes into the quantum projectors:

\[ \left( L_n(\nu_A^{(\epsilon, \epsilon')}) \right)_{ij} = P_i^{(\epsilon, \epsilon')}(n) Q_j^{(\epsilon, \epsilon')}(n). \]
These quantum projectors are proportional to unitary operators. From \((\mathcal{L}_k(\lambda))_{21}\Omega = 0\), we can see that the operator

\[
w^{(\epsilon,\epsilon')}(2k|2k-1) = \sum_{i=1}^{2} Q_i^{(\epsilon,\epsilon')}(2k) P_i^{(\epsilon,\epsilon')}(2k-1)
\]  

(6.9)

has a zero eigenvalue. So, \(w^{(\epsilon,\epsilon')}(2k|2k-1)\) is not invertible and consequently, \(D(\nu_A^{(\epsilon,\epsilon')})\) is also not invertible. The assumption in section 3 that \(D(\nu_A)\) has the inverse does not hold. We should modify the argument of section 3.

Although \(D^{-1}(\nu_A^{(\epsilon,\epsilon')})\) does not exist, we can define the following unitary operators:

\[
P_n^{(\epsilon)} := \epsilon' P_1^{(\epsilon,\epsilon')}(n) \left(P_2^{(\epsilon,\epsilon')}(n)\right)^{-1}, \quad Q_n^{(\epsilon)} := \epsilon' \left(Q_2^{(\epsilon,\epsilon')}(n)\right)^{-1} Q_1^{(\epsilon,\epsilon')}(n).
\]  

(6.10)

The left hand sides of the above equations do not depend on a choice of \(\epsilon' = \pm 1\). Without loss of generality, we set \(\epsilon' = 1\). Let \(\nu_A^{(\epsilon)} := \nu_A^{(\epsilon,1)}\).

Let us introduce the following unitary operators:

\[
\mathcal{O}_n^{(\epsilon)} := \pi_n^{-1/2} \left[\cos(1/2)(\beta u_n - i \epsilon \Lambda)\right]^{1/2} \pi_n^{-1/2}, \quad \epsilon = \pm 1.
\]  

(6.11)

Then \(P_n^{(\epsilon)} = i \mathcal{O}_n^{(-\epsilon)}\) and \(Q_n^{(\epsilon)} = -i \mathcal{O}_n^{(\epsilon)}\).

By using the explicit expressions, we can check that these operators satisfy eqs. (4.7) and (4.14).

Therefore, in place of \(D^{-1}(\nu_A)C(\nu_A)\), we can use the well-defined operator \(Q_1^{(\epsilon)}\). Because the unitary operator \(Q_1^{(\epsilon)}\) does not annihilate the reference state, the action of \(Q_1^{(\epsilon)}\) on the Bethe state has an “anomalous” term:

\[
Q_1^{(\epsilon)} \prod_{l=1}^{M+1} B(\lambda_l)|\Omega\rangle = \sum_{j=1}^{M+1} b^{(M+1)}(\nu_A^{(\epsilon)}) |\lambda_j\rangle|\{\lambda_l\}_{l\neq j}\rangle \prod_{l=1}^{M+1} B(\lambda_l)|\Omega\rangle
\]

\[
+ \left(\prod_{l=1}^{M+1} B(\lambda_l)\right) Q_1^{(\epsilon)}|\Omega\rangle.
\]  

(6.12)

But because of

\[
\langle \Omega| \left(\prod_{k=1}^{M} C(\mu_k)\right) \left(\prod_{l=1}^{M+1} B(\lambda_l)\right) = 0,
\]  

(6.13)

the anomalous term gives no contribution to the form factors

\[
\langle \Omega| \left(\prod_{k=1}^{M} C(\mu_k)\right) Q_1^{(\epsilon)} \left(\prod_{l=1}^{M+1} B(\lambda_l)\right) |\Omega\rangle.
\]  

(6.14)

The formula (3.14) gives the correct result even for \(Q_1^{(\epsilon)}\).
It may seem that the anomalous term would contribute to the form factors for the same numbers of \(C(\mu)\) and \(B(\lambda)\). If we recall
\[
\langle \Omega | \left( \prod_{k=1}^{M} C(\mu_k) \right) \left( \prod_{l=1}^{M} B(\lambda_l) \right) | \Omega \rangle = \mathcal{N}^{(M)}(\{\lambda_l\}) \delta_{\{\mu_k\},\{\lambda_l\}} \langle \Omega | \langle \Omega | \Omega \rangle, \tag{6.15}
\]
where \(\mathcal{N}^{(M)}(\{\lambda_l\})\) is the norm of the Bethe eigenstate:
\[
\mathcal{N}^{(M)}(\{\lambda_l\}) = \langle \Omega | \left( \prod_{k=1}^{M} C(\lambda_k) \right) \left( \prod_{l=1}^{M} B(\lambda_l) \right) | \Omega \rangle \langle \Omega | \Omega \rangle, \tag{6.16}
\]
(for the explicit form, see [32, 33]), then
\[
\langle \Omega | \left( \prod_{k=1}^{M} C(\mu_k) \right) Q_1^{(e)} \left( \prod_{l=1}^{M} B(\lambda_l) \right) | \Omega \rangle = \delta_{\{\mu_k\},\{\lambda_l\}} \mathcal{N}^{(M)}(\{\lambda_l\}) \left( \prod_{l=1}^{M} f(\lambda_l, \nu_A^{(e)}) \right) \langle \Omega | Q_1^{(e)} | \Omega \rangle = 0. \tag{6.17}
\]
\(\langle \Omega | Q_1^{(e)} | \Omega \rangle\) vanishes due to the factor \(\exp(-i(\beta/4)p_1)\) in \(Q_1^{(e)}\).

We conclude that although \(D^{-1}(\nu_A^{(e)})\) does not exist, the result of section 3 is still correct for the LSG model. Thus, as a mnemonic, we can write:
\[
Q_1^{(e)} = D^{-1}(\nu_A^{(e)}) C(\nu_A^{(e)}). \tag{6.18}
\]
By means of this mnemonic, we can make clear that the operators (6.11) have different character depending on whether \(n\) is even or not:
\[
O^{(e)}_{2k} = -iU^k B(\nu_A^{(-e)}) D^{-1}(\nu_A^{(-e)}) U^{-k}, \quad O^{(e)}_{2k+1} = iU^k D^{-1}(\nu_A^{(e)}) C(\nu_A^{(e)}) U^{-k}. \tag{6.19}
\]
The operators at even sites (resp. odd sites) are creation-type (resp. annihilation-type) operators.

7 Discussion

In this paper, we showed that a certain class of local operators can be constructed by using the quantum projectors. The form factors of these operators were calculated by using the techniques of the algebraic Bethe ansatz.

For LNS model, these local operators (5.7) are lattice analogues of the continuum nonlinear Schrödinger fields \(\psi(x)\) and \(\psi^*(x)\). Because the inputs are the elementary monodromy matrix \(L_n(\lambda)\) (5.3), the “dressing” of the output by a factor \((1 + (\kappa/4)\psi_n^*\psi_n)^{-1/2}\) seems unavoidable if one try to keep simplicity of the inverse mapping.
For LSG model, we considered the local operators $O^{(e)}_n$ in the sector of the zero topological charge $Q$. Consideration for the sector $Q \neq 0$ is necessary. Moreover, in order to make connection with the quantum sine-Gordon model, we should consider the thermodynamic limit. Notice that

$$\lim_{\Delta \to 0} e^{i\gamma} O^{(e)}_n (O^{(-e)}_n)^{-1} = \lim_{\Delta \to 0} e^{-i\gamma} (O^{(-e)}_n)^{-1} O^{(e)}_n = e^{i\beta u(x)}.$$  

(7.1)

Therefore, in principle, the form factors of the exponential operators $e^{\pm i\beta u(x)}$ can be evaluated using those of $O^{(e)}_n$. Also, the results may be used to consider the form factors in the finite volume.

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