Rotating charged Black Holes in Einstein-Born-Infeld theories and their ADM mass.

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Abstract

In this work, the solution of the Einstein equations for a slowly rotating black hole with Born-Infeld charge is obtained. Geometrical properties and horizons of this solution are analyzed. The conditions when the ADM mass (as in the nonlinear static cases) and the ADM angular momentum of the system have been modified by the non linear electromagnetic field of the black hole, are considered.

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I. INTRODUCTION

The four dimensional solutions with spherical symmetry of the Einstein equations coupled to Born-Infeld fields have been well studied in the literature[1]. In particular, the electromagnetic field of the Born-Infeld monopole, in constrast to Maxwell counterpart, contributes to the ADM mass of the system (i.e. the four momentum of asymptotically flat manifolds). B. Hoffmann was the first who studied such static solutions in the context of the general relativity with the idea to obtain a consistent particle-like model. Unfortunately, these static Einstein-Born-Infeld models generate conical singularities at the origin [1, 9]. This type of singularities cannot be removed as global monopoles or other non-localized topological defects of the spacetime [6, 7]. More recently, the solutions in Einstein-Born-Infeld (EBI) theory become significatives because in open superstring theory [13, 14], loops calculations lead the BI Lagrangian with \( b = \frac{1}{2\pi \alpha'} \), where \( \alpha' \)is the inverse of the string tension and \( b \) the Born Infeld parameter.

In this report, the solution for slowly rotating black hole with Born-Infeld charge is obtained. This solution, asymptotically flat, presents non linear terms that modifies asymptotically the mass and the angular momentum (ADM values) of the spacetime. Families of solutions are obtained varying \( b \). It is well known that the mass and angular momentum in the Kerr-Newman model of the spacetime appear as integration constants that correspond to the ADM values of this geometry. In our non linear model of rotating charged black hole the ADM mass and angular momentum are not zero when we put these integration constants (that are required in the Kerr Newman’s model) equal to zero. For particular values of \( b \) there are solutions with \( M^2 < a^2 + Q^2 \) without naked singularities.

In the Einstein-Born-Infeld model of a rotating black hole, one expects to find a metric with an asymptotic behaviour as the well known Kerr-Newman’s metric. There are few difficulties: the metric is non-diagonal (rotating frame) and the energy-momentum tensor of Born-Infeld includes the invariant pseudoscalar [2, 3] (the magnetic field appears because of the rotation of the compact object). The used convention is the spatial of Landau and Lifshitz (1962), with signatures of the metric, Riemann and Einstein tensors all positives (++++)[4, 5]
II. STATEMENT OF THE PROBLEM

The theory starts with a Born-Infeld field interacting with gravity and is described by the action

\[ I = \int d^4x (\sqrt{g}R - \mathcal{L}_{BI}) \]

where:

\[ \mathcal{L}_{BI} = \frac{b^2}{4\pi} \left\{ \sqrt{-g} - \sqrt{|\det(g_{\mu\nu} + b^{-1}F_{\mu\nu})|} \right\} \]

that with the expansion of the determinants in four dimensions, is easy to obtain the standard form of the Born-Infeld lagrangian:

\[ L_{BI} = \frac{b^2}{4\pi} \left\{ 1 - \sqrt{1 + \frac{1}{2}b^{-2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16}b^{-4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2} \right\} \]

In an orthonormal frame (tetrad) the above given lagrangian becomes:

\[ L_{BI} = \frac{b^2}{4\pi} \left( 1 - \sqrt{1 - \frac{2S}{b^2} - \frac{P^2}{b^4}} \right) \]

where we have defined the invariants of the electromagnetic tensor \( F \) as

\[ S \equiv -\frac{1}{4}F_{ab}F^{ab} = L_M \]

\[ P \equiv -\frac{1}{4}F_{ab}\tilde{F}^{ab} \]

with the conventions

\[ \tilde{F}^{ab} = \frac{1}{2}\varepsilon^{abcd}F_{cd} \]

\( a, b, c, \ldots \equiv Tetrad\ indexes \)

We consider the following ansatz for the line element (as the Boyer and Linquist interval [8, 9] for the Kerr-Newman’s geometry) with the expected asymptotic behaviour:

\[ ds^2 = -\frac{\Delta}{\rho^2} [dt - asin^2\varphi d\varphi]^2 + \frac{sin^2\theta}{\rho^2} [(r^2 + a^2) d\varphi - adt]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \]

(1)

where the functions to be determined \( \Delta \) and \( \rho \), are in principle depending on \( r \) and \( \theta \). To obtain the Einstein equations, the most powerful is the Cartan’s method [5, 8, 9]. This
method applies differential forms and is based on two fundamental geometric equations (structure equations). In the orthonormal basis of 1-forms the line element is written:

\[ ds^2 = -(\omega^0)^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 \]

where the association between coordinate and orthonormal frame is not trivial in the case of axially rotating symmetry and requires solving an equation system. Explicitly, the tetrad is:

\[
\begin{align*}
\omega^0 &= \frac{\sqrt{\Delta}}{\rho} (dt - a \cdot sin^2 \theta d\varphi) \\
\omega^1 &= \frac{sin \theta}{\rho} \left[ (r^2 + a^2) d\varphi - adt \right] \\
\omega^2 &= \frac{\rho}{\sqrt{\Delta}} dr \\
\omega^3 &= \rho d\theta
\end{align*}
\]

For the electromagnetic tensor \( F \), we propose a similar structure to the \( F \) of the Boyer and Lindquist generalization for the Kerr-Newman problem [8, 9]:

\[
F = F_{20} dr \wedge [dt - asin^2 \theta d\varphi] + F_{31} sin \theta d\theta \wedge [(r^2 + a^2) d\varphi - adt]
\]

\[
= F_{20} \omega^2 \wedge \omega^0 + F \omega^3 \wedge \omega^1
\]

where \( F_{20} \) and \( F_{31} \) are to be determined. We can see that \( F_{20} \) and \( F_{31} \) are the only field components in the tetrad.

Next we find the energy momentum tensor components in the rotating system (tetrad). We shall use the metric symmetrized expression of \( T^a_b \):

\[
T^a_b = \delta^a_b \mathcal{L}_{BI} - \frac{\partial \mathcal{L}_{BI}}{\partial S} F^a_l F^l_b - \frac{\partial \mathcal{L}_{BI}}{\partial P} F^a_l \tilde{F}^l_b
\]

In our case, the energy-momentum tensor takes a diagonal form

\[
-T_{00} = T_{22} = \frac{b^2}{4\pi} (1 - u) \\
T_{11} = T_{33} = \frac{b^2}{4\pi} (1 - u^{-1})
\]

(2)

where:

\[
u \equiv \sqrt{\frac{(F_{31})^2 + 1}{1 - (F_{02})^2}} ; \quad \left( F_{ab} = \frac{F_{ab}}{b} \right)
\]
As the geometrical symmetries of the Riemann tensor
\[ R_{\alpha\mu\nu\lambda} + R_{\alpha\nu\lambda\mu} + R_{\alpha\lambda\mu\nu} = 0 \]
and the well known Bianchi identities
\[ R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\nu\lambda;\mu} + R_{\alpha\beta\lambda\mu;\nu} = 0 \]
\( \rho \) immediatelly can be determinated:
\[ \rho^2 = r^2 + a^2 \cos^2 \theta = \Sigma \]

We can see without losing generality that the function \( \rho \) is the same as the \( \rho \) of Boyer and Linquist and does not depend on the axially symmetric source considered. With the function \( \rho \) found, only \( \Delta \) left to be found.

From the Einstein equations with the components (2) of the energy-momentum tensor of Born-Infeld in the tetrad [9, 12], we obtain the following expression
\[ 2 (8\pi) (T_{11} + T_{22}) = -\frac{2}{\rho^2} + \frac{\partial_r \partial_r \Delta_{(r, \theta)}}{\rho^2} \]
that must to be solved with the following boundary conditions:
\[ \lim_{r \to \infty} \Delta_{RBI} \to \Delta_{Kerr-Newman} \]
that will give us an assymptotically flat solution, with the correct Maxwellian behaviour at great distances of the non linear source of the electromagnetic fields; and
\[ \Delta \left( r_h \right) = 0 \quad \text{with} \quad R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \bigg|_{r = r_h} \neq \infty \]
It is the usual regularity condition for the horizon.

Notice that the expression (3) for the well known Kerr-Newman model takes the simply form
\[ 2 = \partial_r \partial_r \Delta_{(r)} \]
because the energy-momentum tensor in the Maxwell theory is traceless. Notice also, from the expression given above, that in the Kerr-Newman model the ADM mass \( M_{ADM} \), the angular momentum \( a_{KN} \) and the charge \( Q \) must to be appear necessarily as integration constants. We will show, that is not the case in the BI theory, and these parameters arises naturally.
The fields will be obtained from the dynamical (eulerian) equations in the tetrad form. Let us solve these equations with the following boundary condition: the fields asymptotically have the same behaviour as the electromagnetic fields of the Kerr-Newman model.

\[ \nabla_a \left[ \frac{1}{b^2 R} F^{ab} + \left( \frac{P}{b^3 R} \right) \tilde{F}^{ab} \right] = 0 \]

with the Maxwellian asymptotical behaviour of the fields, explicitly

\[ F^{20}|_{r \to \infty} \to -\frac{Q}{\rho^4} (r^2 - a^2 \cos^2 \theta) \]

\[ F^{31}|_{r \to \infty} \to \frac{2Q}{\rho^4} ar \cos \theta \]

Then we obtain \( r_o = \sqrt{\frac{Q}{b}} \):

\[ \left( \mathcal{F}_{20} \right)^2 = \frac{r_o^4 r^2}{\rho^8 (r_o^4 + r^4)} \left[ r^2 (r^2 - a^2 \cos^2 \theta)^2 - 4a^2 \cos^2 \theta r_o^4 \right] \]

\[ \left( \mathcal{F}_{31} \right)^2 = \frac{(r_o^4 + r^4)}{r_o^4 r^2} \left[ 4 \cdot \cos^2 \theta r^2 a^2 r_o^8 (r^2 - a^2 \cos^2 \theta)^2 \right] \]

Putting all the ingredients in the equation (3), we obtain the following expression:

\[ 4 \rho^2 b^2 - 2 \rho^2 b^2 \left[ 2 \sqrt{\rho^4 r^2 - 4a^2 \cos^2 \theta (r_o^4 + r^4)} \left( r^2 - a^2 \cos^2 \theta \right) \right] - \]

\[ -2 \rho^2 b^2 \left[ \sqrt{\rho^4 r^2 - 4a^2 \cos^2 \theta (r_o^4 + r^4)} \frac{r}{(r^2 - a^2 \cos^2 \theta)} \right] + 2 = \partial_r \partial_r \Delta \]

This expression, although exact, is not integrable by transcendental functions as in the static cases. One must make an expansion in power series for small (slowly rotating) \( a/r \)

\[ 4 \rho^2 b^2 - 2 \rho^2 b^2 \left\{ 2 \sqrt{\frac{r_o^4}{r^4} + r^4} \left[ 1 + \frac{a^2}{r^2} \cos^2 \theta \left( 1 + \frac{2r_o^4}{r^4} \right) - 3 \frac{a^4}{r^4} \cos^4 \theta \left[ 1 + \frac{2r_o^4}{r^4} \left( 1 + \frac{r_o^4}{r^4} \right) \right] \right] + \]

\[ -2 \sqrt{\frac{r_o^4}{r^4} + r^4} \left[ 1 + \frac{a^2}{r^2} \cos^2 \theta \left( 1 - \frac{2r_o^4}{r^4} \right) - \frac{a^4}{r^4} \cos^4 \theta \left[ 1 + \frac{2r_o^4}{r^4} \left( 1 - \frac{r_o^4}{r^4} \right) \right] \right] \right\} + 2 = \partial_r \partial_r \Delta \]

\( \left( \frac{a}{r} << 1 ; r_o \lesssim r \right) \)

This expansion does not affects the regularity condition of the horizon. Looking at the last equation, one can see that \( \Delta \) depends on the radial coordinate \( r \) and the angular coordinate
\( \theta \). The integrals are calculated in the indefinite form and the values of the two constants \( A ( \theta ) \) and \( B ( \theta ) \) of the problem are selected according to the asymptotical flat behaviour of \( \Delta \) and the metric. Is useful to see, previously, an intermediate computation

\[
\partial_r \Delta = 2r + \frac{2}{3} b_2 \left\{ 2 \left( r^3 - r \sqrt{r^4 + r_0^4} \right) + \sqrt{r} r_0^3 F \left[ i \text{ArcSinh} \left[ (-1)^{1/4} \frac{r}{r_0} \right], -1 \right] \right\} + \frac{b^2}{3} \frac{a^2 \cos^2 \theta \left[ r + \sqrt{r^4 + r_0^4} \right]}{r_0^2} \left[ \frac{4r_0^4}{5r^5} - \frac{2}{5r} - \frac{54}{11r^3} a^2 \cos^2 \theta - \frac{16}{11r^7} r_0^4 a^2 \cos^2 \theta - \frac{12}{11r^{11}} r_0^8 a^2 \cos^2 \theta \right] - \left[ \frac{18}{5} \frac{r_0^2 b_2^2 \cos^2 \theta}{r_0^4} \left( E \left[ i \text{ArcSinh} \left[ (-1)^{1/4} \frac{r}{r_0} \right], -1 \right] - F \left[ i \text{ArcSinh} \left[ (-1)^{1/4} \frac{r}{r_0} \right], -1 \right] \right) \right] - \left[ \frac{2}{11r_0} \left( 2 \sin^2 \theta + \sin^4 \theta \right) \right]
\]

The obtained solution takes the following form:

\[
\Delta (r, \theta) = r^2 + P_{ST} + \left\{ 2a^2 \frac{Q^2}{(r_0)^3} \left\{ \cos^2 \theta \frac{9}{5} \sqrt{i} E \left[ \frac{\pi}{4}, 2 \right] + 1.525 \sin^2 \theta \right\} - 2M + 2 \left( \frac{a}{r_0} \right)^4 Q^2 (2 \sin^2 \theta - \sin^4 \theta) 0.8576 + 2a^2 \frac{Q^2}{(r_0)^4} \cos^2 \theta \left\{ r^2 + \frac{4}{5} \sqrt{r_0^4 + r^4} - \frac{1}{10} \left( \frac{r_0}{r} \right)^4 \sqrt{r_0^4 + r^4} - r \left[ \frac{9}{5} \sqrt{i} E \left[ \frac{1}{2} \text{ArcCos} \left[ i \left( \frac{r}{r_0} \right)^2 \right], 2 \right] \right] + 2 \left( \frac{a}{r_0} \right)^4 Q^2 \cos^2 \theta \left\{ -2 \text{ArcSinh} \left( \frac{r^2}{r_0^4} \right) + \sqrt{1 + \left( \frac{r}{r_0} \right)^4 \left[ - \left( \frac{r_0}{r} \right)^2 \frac{163}{770} - \left( \frac{r_0}{r} \right)^6 \frac{313}{385} - \left( \frac{r_0}{r} \right)^10 \frac{21}{385} \right] \right} \right] - 2a^2 \frac{Q^2}{(r_0)^4} \cos^2 \theta \left\{ \frac{17}{11} (-1)^{1/4} \frac{r}{r_0} F \left[ i \text{ArcSinh} \left[ (-1)^{1/4} \frac{r}{r_0} \right], -1 \right] \right\} \right\} \]

where the constants have been selected to obtain asymptotically the Kerr-Newman metric and \( P_{ST} \) is identical to the similar quantity in the static case:

\[
P_{ST} = \frac{1}{3} Q^2 \left\{ \tau^4 + \tau^2 \sqrt{\tau^4 + 1} + 2\tau (-1)^{1/4} F \left[ \text{Arcsin} \left[ (-1)^{3/4} \tau \right], -1 \right] \right\} ; \quad \left( \tau \equiv \frac{r}{r_0} \right)
\]

We can see that this solution contains new terms that do not appear in the Kerr-Newman model as in Reissner-Nördstrom and the static Born-Infeld model. There are products of charge and angular momentum. The expansion is for \( a/r << 1 \) and \( r_0 \ll r \). Is useful to remark here that the \( a_{KN} \) appear as a parameter into the constant of integration \( B ( \theta ) \) (in the same manner that \(-2M \) into the constant \( A ( \theta ) \)). The parameter \( a \) is the parameter of deformation of the spherical symmetry in the Boyer and Lindquist type interval, eq. (1).
III. ANALYSIS OF THE METRIC IN THE BORN-INFELD ROTATING CASE

The general behaviour of the metric is similar to the Kerr-Newman’s model (almost globally). As one can see from the last expression for the $\Delta$, the metric has two horizons and depends strongly on $r_0$ (related to Born-Infeld parameter $b^2 \equiv Q^2 / (r_0)^4$) and its quotient with $a^2$. The asymptotical behaviour of the $\Delta (r)$ is:

$$\Delta (r) \cong r^2 - \left[ 2M + \frac{Q^2}{r_0} \left( \frac{4}{3} \cdot 1.854 - 1.525 \cdot \frac{2a^2}{r_0} - \frac{2a^4}{r_0^2} \cdot 2.8653 \right) \right] r + Q^2 + a_K^2 + a^2 \left( \frac{0.8576}{r_0^4} a^2 Q^2 \right)$$

that corresponds to have asymptotically an ADM mass:

$$M_{ADM} = M + \left[ \frac{Q^2}{r_0} \left( \frac{2}{3} \cdot 1.854 - 1.525 \cdot \frac{a^2}{r_0^2} - \frac{a^4}{r_0^2} \cdot 2.8653 \right) \right]$$

and an ADM angular momentum:

$$a_{ADM} = \sqrt{a_K^2 + 0.8576 \cdot \frac{2a^4 Q^2}{r_0^4}}$$

where $M$ and $a_K$ are the two constants of integration that appear in the Kerr-Newman model (these constants are related to the asymptotic values of mass and angular momentum in the Maxwellian-linear case of a black hole [8]). Notice that the numerical coefficient 1.854 is characteristic of the static EBI monopole and arises from the leading terms in the expansion of the hypergeometric function $F$. The numerical coefficient 1.525 is part of the integration constant $A(\theta)$ and is obtained from the asymptotic conditions imposed on the rotating EBI model.

IV. CONCLUSIONS

In this report a solution of the Einstein-Born-Infeld equations for slowly rotating black holes is presented. The general behaviour of the geometry is strongly modified according to the value that takes $r_0$ (Born-Infeld radius[1,2]) relative to $a$ value. This metric permits solutions of $M^2 < a_{ADM}^2 + Q^2$, $M = 0$ and $a_K = 0$ with a regular horizon (see Figures 1, 2 and 3). The spacetime of the Born-Infeld-rotating monopole (in contrast to Maxwell counterpart) have intrinsic or particular ADM values of mass and angular momentum. In this non-linear electromagnetic rotating model, the mass and angular momentum of the spacetime($a_{ADM}$ and $M_{ADM}$) can be driven by: the electromagnetic charge $Q$, the absolute
field of Born-Infeld $b$ (or the BI radius: $r_0$) and the $a$ parameter from the geometry. Solutions with $M < 0$ exist, strongly bounded for the $M_{ADM}$ value, this $M_{ADM}$ value cannot to be negative because naked singularities appear in the spacetime, violating in this manner the positive mass theorem for Riemannian (assymptotically flat) manifolds [10,11]. In order to clarify some points related with the electromagnetic fields and the topology of the manifolds, we can make the following comments: the problem in the integrability conditions for the metric, that carry us to make a ”slowly rotating” expansion (i.e. $a/r << 1$), there exist also in the Bianchi identity for the electromagnetic fields. For the limit in which the metric solution is integrable and remains valid(i.e. $a/r << 1$), the Bianchi identity also remains valid, obviously because it is precisely the geometrical caracter of the electromagnetic field tensor (closed form), and depends on the geometry of the space-time where these electromagnetic fields are living. All the remarks and computations are absolutely consistent with the ”slowly rotating regime”. From the point of view of the Hamilton-Jacobi equations and the Petrov classification for this type of non-linear rotating solutions, a good analysis of this problem was made in [16]. The origin of this difficulties concerning on the integrability and a cure for this problem will be given in a forthcoming work where we will analyze the exact rotating solution[17].

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