LOCAL GRÖBNER FAN: POLYHEDRAL AND COMPUTATIONAL APPROACH

ROUCHDI BAHLOUL AND NOBUKI TAKAYAMA

Abstract. The goal of this paper is to show that the local Gröbner fan is a polyhedral fan for ideals in the ring of power series and the homogenized ring of analytic differential operators. We will also discuss relations between the local Gröbner fan and the (global) Gröbner fan for a given ideal and algorithms of computing local Gröbner fans. In rings of differential operators, finiteness and convexity of local Gröbner fans were firstly proved by Assi, Castro-Jiménez and Granger. But they did not prove that they are polyhedral fans.

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Introduction

The Gröbner fan for a (homogeneous) polynomial ideal was introduced by Mora and Robbiano [MR88]. It was also studied by Sturmfels [St95]. Given an ideal and a weight vector on the variables we are interested in the equivalence class (the Gröbner cone) of the weight vector defined by the equality of the initial ideals. In this case it has been proved that the closure of the Gröbner cones form a polyhedral fan. In the ring of algebraic differential operators, Assi, Castro and Granger [ACG00] studied the structure of the Gröbner cones but they did not prove that it gives rise
to a polyhedral fan. This last fact has been proved by Saito, Sturmfels and the second author [SST00].

Is there a similar construction for power series ideals or more generally for ideals in the homogenized ring of analytic differential operators $h_{(0,1)}(D_0)$? Assi, Castro-Jiménez and Granger [ACG01] proved that any standard cone (Gröbner cone) is open, polyhedral and convex and they are in finite number. Since the ring of analytic power series $O_0$, as a subring of $h_{(0,1)}(D_0)$, is graded, their theorem implies that any standard cone (Gröbner cone) of an ideal in $O_0$ is open, polyhedral, and convex. We note that this fact has already been remarked by Assi [As93]. However, the importance of polyhedral properties of the fan was not evoked and it will be one of our main purpose here.

The way of discussing about the ring of power series and the homogenized ring of analytic differential operators are analogous. However, we will make separate treatments, because the commutative case is interesting itself and more readers will be interested in.

Historically, analogous constructions to Gröbner fan already appeared in works by Lejeune-Jalabert and Teissier [LT73] with the notion of critical tropism. In $D$-module theory, we have the notion of slopes with works by Laurent and Mebkhout [Lau87, Meb96, LM99], see also [ACG96]. Critical tropisms or slopes are in fact contained in the trace of a Gröbner fan or a standard fan on a 2-dimensional space.

Now, as an introductive illustration, let us review the case of a principal ideal in the formal power series ring.

Let $f$ be a formal power series $\sum_{\alpha \in E} c_\alpha x^\alpha$ in $n$ variables. Here, the set $E$ is the support of the power series in the space of exponents. The convex polyhedron

$$\text{New}(f) = \text{conv}\{\alpha + N^n | \alpha \in E\}$$

is called the Newton polyhedron of $f$. We are interested in the polyhedral structure of it.

Let $u$ be a vector in $\mathbb{R}_{\leq 0}^n$, which we will call a weight vector. For a given weight vector $u$, let us consider the height of points of the Newton polyhedron with respect to the direction $u$. The set of the highest points of $\text{New}(f)$

$$\text{face}_u(\text{New}(f)) = \{\alpha \in \text{New}(f) \subset \mathbb{R}^n | u \cdot \alpha \geq u \cdot \alpha' \text{ for all } \alpha' \in \text{New}(f)\}$$

called the face of $\text{New}(f)$ with respect to $u$. It is known that there exist only a finite number of faces. For a given face $F = \text{face}_u(\text{New}(f))$, the normal cone of $F$ is defined by

$$N(F) = \{u' \in \mathbb{R}_{\leq 0}^n | \text{face}_{u'}(\text{New}(f)) = \text{face}_u(\text{New}(f))\}.$$  

It can be defined in terms of the initial term as follows

$$N(F) = \{u' \in \mathbb{R}_{\leq 0}^n | \text{in}_{u'}(f) = \text{in}_u(f), \text{supp}(u') = \text{supp}(u)\},$$

where $\text{supp}(u) = \{i | u_i \neq 0\}$. The normal cone $N(F)$ is an open rational convex polyhedral cone. The collection of the closures of the normal cones

$$\bar{E}(f) = \{\overline{N(F)} | F \text{ runs over the faces of } \text{New}(f)\}$$

is called the normal fan of the Newton polyhedron. Figure [4] illustrates the situation for $f(x, y) = x^3 - y^2$. It is well-known and fundamental that the normal fan is a polyhedral fan. Let us recall the definition.

**Definition 0.0.1.** A polyhedral fan is a finite collection of (closed) polyhedral cones satisfying the following axioms:

1. Every face of a cone is again a cone.
2. The intersection of any two cones is a face of both.
Can we generalize the results above to ideals and left ideals of differential operators? We can define an analogue of the normal cone by using an equivalence of initial ideals as in [11]. Then, it is natural to ask if the collection of the closures of the cones is a polyhedral fan or not. The collection has been called the Gröbner fan or the standard fan. The normal fan defined above is the Gröbner fan of the principal ideal generated by \( f \).

In case of non-principal ideals, we will see that the Gröbner fan is locally a common refinement of the normal fans of a nice set of generators. Sturmfels introduced this geometric picture of Gröbner fan in case of homogeneous ideals of the polynomial ring [St95]. We will apply his idea to local settings with the help of the division theorem of Assi, Castro-Jiménez and Granger [ACG01]. This division takes place in the homogenized ring of analytic (or formal) differential operators. Such an environment is necessary if we want a division adapted to a general weight vector. Indeed, a general division does not exist in the ring \( \hat{\mathcal{D}} \) of formal differential operators as illustrated with the following example by T. Oaku.

**Example 0.0.2.** Take \( n = 1 \). We want to divide \( f = x \) by \( g = x + x^k \partial \) with respect to a weight vector \( w = (u, v) \), with \( u < 0 \) and \( u + v \geq 0 \). Take \( k \geq 2 \) sufficiently large such that \( x \succ x^k \partial \). Here \( \prec \) is a refinement of the partial order defined by \( w \).

Suppose that a division theorem holds:

\[
x = q \cdot (x + x^k \partial) + r, \quad q, r \in \hat{\mathcal{D}}
\]

where \( r \) cannot be divided by \( g \), i.e. the leading monomial of \( r \) is not divisible by \( x \) the leading monomial of \( g \). But \( r = x - q \cdot (x + \partial x^k + k x^{k-1}) = (1 - q - q \partial x^{k-1} - k x^{k-2}) \cdot x \) therefore the leading monomial of \( r \) contains \( x \). It is a contradiction.

Finally, let us mention about applications of our result. The first author gave a constructive method to compute a local Bernstein-Sato polynomial for a given set of analytic functions or polynomials in [Ba05]. An essential step of the construction is the computation of an analytic standard fan, and we will give algorithms to compute it. The second application is given by N. Touda who constructs local...
tropical variety based on our result that “the local Gröbner fan is a polyhedral fan” [To05]. This is an analogous construction of tropical variety by D. Speyer and B. Sturmfels [SS04].

1. DEFINITIONS AND STATEMENT OF THE MAIN RESULTS

We fix an integer \( n \geq 1 \). We denote by \( x \) the system of variables \((x_1, \ldots, x_n)\).

The partial derivation \( \frac{\partial}{\partial x_i} \) shall be denoted by \( \partial_i \). We will use the notation \( x^\alpha \) for \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) where \( \alpha \in \mathbb{N}^n \). In the same way \( \partial^\beta \) denotes \( \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \).

The symbol \( k \) denotes a field. \( k[x] \) denotes the localization of \( k[x] \) at the point \( 0 \in k^n \). When \( k = \mathbb{C} \) this is a subring of \( \mathcal{O}_0 \) germ at 0 of analytic functions. The latter is itself a subring of the formal power series ring \( \hat{\mathcal{O}} = k[[x]] \). In this section, the symbol \( C \) shall denote one of the rings \( k[x], k[x]_0, \mathcal{O}_0, \hat{\mathcal{O}} \).

We denote by \( \mathcal{O} \) the field. We fix an integer \( \alpha, \beta, k \) such that \( \partial_{\alpha, \beta, k} \) is a subring of the formal power series ring \( \hat{\mathcal{O}} = k[[x]] \). This is an analogous construction of tropical variety by D. Speyer and B. Sturmfels [SS04].

First let us define the \((0, 1)\)-homogenization. Let us introduce the ring \( h_{(0,1)}(\hat{D}) \). It is the \( k \)-algebra generated by \( \hat{\mathcal{O}}, \partial_i \ (i = 1, \ldots, n) \) and a new variable \( h \) with

\[
\partial_i c(x) - c(x) \partial_i = \frac{\partial c(x)}{\partial x_i} \cdot h,
\]

\( i = 1, \ldots, n, \) as the only non trivial commutation relations. We can replace \( \hat{\mathcal{O}} \) by \( k[x], k[x]_0, \mathcal{O}_0 \) and we obtain the subrings \( h_{(0,1)}(D) \subset h_{(0,1)}(D_{(0)}) \subset h_{(0,1)}(D_0) \subset h_{(0,1)}(\hat{D}) \). An element \( P \) in one of these rings has a unique writing as \( P = \sum_{\alpha, \beta, k} c_{\alpha, \beta, k} x^\alpha \partial^\beta h^k \). We define the support \( \text{Supp}(P) \subset \mathbb{N}^{2n+1} \) of \( P \) as the set of \((\alpha, \beta, k)\) such that \( c_{\alpha, \beta, k} \neq 0 \). We define its degree \( \deg(P) \) as the maximum of \(|\beta|+k\). We say that \( P \) is homogeneous if for any \((\alpha, \beta, k) \) in \( \text{Supp}(P) \), \(|\beta|+k = \deg(P) \). An ideal in \( h_{(0,1)}(\mathcal{R}) \) is said to be homogeneous if it can be generated by homogeneous elements.

Let \( P = \sum_{\alpha, \beta} c_{\alpha, \beta} x^\alpha \partial^\beta \) in \( \mathcal{R} \). The support \( \text{Supp}(P) \subset \mathbb{N}^{2n} \) is defined in the same way. We define the \((0, 1)\)-degree of \( P \) as the maximum of \(|\beta| \) for \((\alpha, \beta) \) in \( \text{Supp}(P) \). Its \((0, 1)\)-homogenization is defined by \( h_{(0,1)}(P) = \sum c_{\alpha, \beta} x^\alpha \partial^\beta h^{\deg(P)-|\beta|} \). It is a homogeneous element of \( h_{(0,1)}(\mathcal{R}) \) of degree the degree of \( P \).

Now let us introduce the \((1, 1)\)-homogenized Weyl algebra \( h_{(1,1)}(D) \) [1391, CN97]. It is the \( k \)-algebra generated by the symbols \( x_i, \partial_i \) and \( h \) where the only non trivial relations are

\[
\partial_i x_i - x_i \partial_i = h^2; \ i = 1, \ldots, n.
\]
For $P \in h_{(1,1)}(D)$, its degree shall be the total degree in all the variables. It is homogeneous if all its monomials have the same degree. For $P = \sum_{\alpha, \beta} c_{\alpha, \beta} x^\alpha y^\beta$ in $D$, the $(1,1)$-degree is the total degree in the variables $x_1$ and $y_1$. We define the $(1,1)$-homogenization of $P$ as $h_{(1,1)}(P) = \sum_{\alpha, \beta} c_{\alpha, \beta} x^\alpha y^\beta h^{d-|\alpha|-\beta}$ where $d$ is the $(1,1)$-degree of $P$. This is a homogeneous element of degree $d$.

For $f \in C$ and a local weight vector $u \in U_{\text{loc}}$, we denote by $\text{ord}^u(f)$ the maximum of the scalar products $u \cdot \alpha$ for $\alpha \in \text{Supp}(f)$. In the case $f \in k[x]$, this definition is valid for $u \in U_{\text{glob}}$. The order gives rise to a filtration $F^u(C)$ given by $F^u_k = \{ f | \text{ord}^u(f) \leq k \}$ as well as to the associated graded ring $\text{gr}^u(C) = \oplus_k F^u_k / F^u_{<k}$. If $C = k[x]$ and $u \in U_{\text{glob}}$, this graded ring is equal to $k[x]$. All these rings can be seen as subrings of $\text{gr}^u(\hat{O})$. For $f \in C$, its initial form $\text{in}_u(f)$ shall be the class of $f$ in $F^u_k / F^u_{<k}$ where $k = \text{ord}^u(f)$. In literature, we also find the terminology principal symbol with the notation $\sigma^u(f)$. Now if $I$ is an ideal in $C$, the restriction of $F^u(C)$ to $I$ gives rise to a graded ideal in $\text{gr}^u(C)$ which we denote by $\text{in}_u(I)$. This ideal is in fact generated by all the $\text{in}_u(f)$ for $f \in I$.

Now all the definitions above are valid if we replace $C$ by $R$ and $U_{\text{loc}}$ (resp. $U_{\text{glob}}$) by $W_{\text{loc}}$ (resp. $W_{\text{glob}}$). For example if write $w = (u, v)$ then $\text{ord}^w(P)$ shall be the maximum of $(u, v) \cdot (\alpha, \beta)$ for $(\alpha, \beta) \in \text{Supp}(P)$. Moreover if we put the weight 0 to the variable $h$, we can extend the definitions above to $h_{(1,1)}(D)$ or $h_{(0,1)}(D)$ with $W_{\text{glob}}$ and to $h_{(0,1)}(D_1)$, $h_{(0,1)}(D_0)$, $h_{(0,1)}(\hat{D})$ with $W_{\text{loc}}$.

**Remark 1.0.3.** The subscript “loc” means that we are working with local rings. In this case, the weights $u_i$ of the $x_i$ variables need to be $\leq 0$ if we want the definition above to work. Moreover the conditions $u_i + v_i \geq 0$ are necessary to respect the non commutative structure of the rings in the differential case. Indeed we have: $\text{ord}^w(x_i \partial_i) = u_i + v_i$ and $\text{ord}^w(x_i + \partial_i) = u_i + v_i$. Similarly, we would like to have $\text{ord}^w(\partial_i x_i) = v_i + u_i$. Since $\partial_i x_i = x_i \partial_i + 1$ and $\text{ord}^w(x_i \partial_i) = u_i + v_i$, we need $u_i + v_i \geq 0$.

Let $u$ be in $U_{\text{loc}}$ and define a strata in the weight space:

$$S_C(u) = \{ u \in U_{\text{loc}} | \text{gr}^u(C) = \text{gr}^u'(C) \}.$$  

We simply write $S(u)$ when no confusion arises. Then for a given ideal $I$ in $C$, consider the equivalence relation

$$u \sim u' \iff u' \in S(u) \text{ and } \text{in}_u(I) = \text{in}_{u'}(I)$$

for which we denote by $C_I[u]$ the class of $u$. This class is called a Gröbner cone (the Gröbner cone of $I$ w.r.t. $u$). We then denote by $\mathcal{E}(I, U_{\text{loc}})$ the set the Gröbner cones $C_I[u]$ for $u \in U_{\text{loc}}$. This is the open Gröbner fan of $I$ on $U_{\text{loc}}$. If $R = k[x]$, we can do the same by replacing $U_{\text{loc}}$ with $U_{\text{glob}}$ and obtain $\mathcal{E}(I, U_{\text{glob}})$. When the context is clear we shall only write $\mathcal{E}(I)$ (for example if $C = \hat{O}$). The notation $\mathcal{E}$ comes from the word “éventail”.

Finally, the closure of $C_I[u]$ will be called the closed Gröbner cone of $I$ w.r.t. $u$. The set of the closed Gröbner cones shall be denoted by $\mathcal{E}(I, U_{\text{loc}})$ (resp. $\mathcal{E}(I, U_{\text{glob}})$) and called the closed Gröbner fan of $I$.

We can give the same definitions for an ideal $I$ in $R$, $h_{(1,1)}(D)$ or $h_{(0,1)}(R)$.

**Note.**

- In [MRS] Mora and Robbiano introduced and studied the Gröbner fan for a homogeneous ideal in $k[x]$. It has been proved by Sturmfels [St95] that the closed Gröbner fan is a polyhedral fan.
• In [ACG00], Assi, Castro-Jiménez and Granger introduced $E(h_{(1,1)}(I), \mathcal{W}_{\text{glob}})$ for an ideal $I$ in $D$ and called it the Gröbner fan of $I$. It is not strictly speaking a fan, although for this case, Saito, Sturmfels and Takayama [SST00] proved that $E(J)$ is a polyhedral fan for any homogeneous $J$ in $h_{(1,1)}(D)$.

• In [ACG01], Assi, Castro-Jiménez and Granger defined and worked with $E(h_{(0,1)}(I), \mathcal{W}_{\text{loc}})$ for $I$ in $D_0$ (or $\hat{D}$). They proved it is a finite collection of convex rational polyhedral cones and called it the (analytic) standard fan of $h_{(0,1)}(I)$, however they did not prove that $E(h_{(0,1)}(I))$ is a polyhedral fan.

Let us state the main contributions of the present paper.

(A) Let $I$ be an ideal in $O_0$ or $\hat{O}$ then the closed Gröbner fan $\hat{E}(I, \mathcal{U}_{\text{loc}})$ is a polyhedral fan.

(B) Let $I$ be a homogeneous ideal in $h_{(0,1)}(D_0)$ or $h_{(0,1)}(\hat{D})$ then the closed Gröbner fan $\hat{E}(I, \mathcal{W}_{\text{loc}})$ is a polyhedral fan.

(C) Let $I$ be an ideal in $k[x]$ then $\hat{E}(k[x]_0I, \mathcal{U}_{\text{loc}}) = E(\hat{O}I, \mathcal{U}_{\text{loc}})$ and we provide an algorithm to compute its restriction to any linear subspace.

(D) Let $I$ be an ideal in $D$ then we provide an algorithm to compute the open Gröbner fan of $\hat{D}I$ and the closed Gröbner fan of $h_{(0,1)}(\hat{D}I)$ (the latter being a fan).

(E) Comparison theorems of local and global Gröbner fans (Theorems 4.1.6, 4.1.8, 4.1.9, 4.2.3, 4.2.4, 5.1.1).

The fan in (C) (resp. (D)) shall be called the local Gröbner fan (at $x = 0$) of $I$.

The way of discussing about the ring of power series and the homogenized ring of analytic differential operators are analogous. However, we will make separate treatments; in section 2, we discuss about the ring of power series and, in section 3, we only prove the results which are specific to the case of differential operators and we refer to section 2 otherwise. In section 4 we give general results about the different kinds of Gröbner fans and we compare them (E). In section 5 we will be concerned with (C) and (D). Results (A) and (B) will be fundamental for enumerating the Gröbner cones. We end section 5 with some examples.

2. THE CLOSED GRÖBNER FAN IN $\hat{O}$ OR $O_0$ IS A POLYHEDRAL FAN

In this section we will prove the following theorem.

Theorem 2.0.4. Let $I$ be an ideal in $O_0$ or in $\hat{O}$ then the closed Gröbner fan $\hat{E}(I, \mathcal{U}_{\text{loc}})$ is a rational polyhedral fan.

2.1. Division theorem and standard bases in $\text{gr}^u(\hat{O})$ and $\text{gr}^u(O_0)$. In the following, we will state a division theorem in the graded rings $\text{gr}^u(\hat{O})$ and $\text{gr}^u(O_0)$, let us describe them more explicitly.

Lemma 2.1.1. Let $u \in \mathcal{U}_{\text{loc}}$ be a local weight vector and for some $0 \leq m \leq n$ assume that $u = (u_1, \ldots, u_m, 0, \ldots, 0)$ with $u_i \neq 0$. Then $\text{gr}^u(O_0) = C[x_{m+1}, \ldots, x_n][x_1, \ldots, x_m]$ and $\text{gr}^u(\hat{O}) = k[[x_{m+1}, \ldots, x_n]][x_1, \ldots, x_m]$.

The proof immediately follows from the definition. Let us denote by $C$ one of $\hat{O}, O_0$. Thanks to the preceding lemma, we shall see $\text{gr}^u(C)$ as a subring of $C$. An element $f \in C$ is said to be $u$-homogeneous (or simply homogeneous if the context is clear) if all its monomials have the same $u$-order.

Let $<$ be a total order on $\mathbb{N}^n$ (or equivalently on the terms $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$). It is said to be a monomial order if $\alpha < \alpha'$ implies $\delta + \alpha < \delta + \alpha'$. It is called local if moreover $\alpha < 0$ for any $\alpha$. On the opposite side, a monomial order with $\alpha \geq 0$ is called a global or a well order.
Let $f \in \mathcal{C}$ be a non zero power series then we define its leading exponent $\exp_\prec(f)$ w.r.t. $\prec$ as the maximum of $\text{Supp}(P)$ for $\prec$. We define its leading term $\text{Lt}_\prec(f) = x^{\exp_\prec(f)}$, its leading coefficient $\text{lc}_\prec(f)$ (in $\mathfrak{k}$ or $\mathbb{C}$) as the coefficient corresponding to $\text{Lt}_\prec(f)$ and its leading monomial $\text{lm}_\prec(f) = \text{lc}_\prec(f) \text{Lt}_\prec(f)$.

Let us state a division theorem in $\text{gr}^u(\mathbb{C})$ for $\prec$ with unique quotients and remainder similarly to [CG04].

Let $g_1, \ldots, g_r$ be non zero homogeneous elements in $\text{gr}^u(\mathbb{C})$. Define a partition of $\mathbb{N}^n$ associated with the $\exp_\prec(g_i)$ as $\Delta_1 = \exp_\prec(g_1) + \mathbb{N}^n$, $\Delta_j = (\exp_\prec(g_j) + \mathbb{N}^n) \setminus (\Delta_1 \cup \cdots \cup \Delta_{j-1})$ for $j > 1$ and $\Delta = \mathbb{N}^n \setminus (\cup_j \Delta_j)$.

**Theorem 2.1.2** (Division theorem in $\text{gr}^u(\mathcal{O}_0)$ and $\text{gr}^u(\hat{O})$). For any homogeneous $f \in \text{gr}^u(\mathbb{C})$, there exists a unique $(q_1, \ldots, q_r, R) \in \text{gr}^u(\mathcal{C})^{r+1}$ made of homogeneous elements such that $f = \sum_j q_j g_j + R$ and

- for any $j$, either $q_j = 0$ or $\text{Supp}(q_j) + \exp_\prec(g_j) \subset \Delta_j$,
- either $R = 0$ or $\text{Supp}(R) \subset \Delta$.

We call $R$ the remainder of the division of $f$ by the $g_j$ w.r.t. $\prec$.

**Proof.** Let us first suppose $u = (0)$. In this case $\text{gr}^u(\mathbb{C}) = \mathcal{C}$ and any element in $\mathcal{C}$ is $u$-homogeneous (of order $0$). Let us show how we can derive the result from [CG04, Theorem 1.5.1]. By Robbiano’s theorem [Ro85], the order $\prec$ can be defined as a lexicographical order with respect to $n$ weight vectors. Let $w^1$ be the first of them. Then define the order $\prec_{w^1}$ in a lexicographical way by $w^1$ and by $\prec_0$ where $\prec_0$ is the inverse of $\prec$ (and thus $\prec_0$ is a well order). The orders $\prec$ and $\prec_{w^1}$ are equivalent. Since $\prec$ is local $w^1$ has non positive components. Thus with the order $\prec_{w^1}$, we are exactly under the hypotheses of [CG04, Theorem 1.5.1]. Now suppose $u$ is not necessarily zero. Since $\text{gr}^u(\mathbb{C})$ is a subring of $\mathcal{C}$ (by lemma 2.1.1), we make the division in $\mathcal{C}$ and we remark that the division process conserves $u$-homogeneity which concludes the proof.

As a consequence we have:

$$\exp_\prec(f) = \max_{\prec} \{ \exp_\prec(q_j g_j), j = 1, \ldots, r; \exp_\prec(R) \}.$$ 

With the notations of the proof above, this implies the following:

$$\text{ord}_{w^1}(f) = \max \{ \text{ord}_{w^1}(q_j g_j), j = 1, \ldots, r; \text{ord}_{w^1}(R) \}.$$ 

Let $J$ be an ideal in $\text{gr}^u(\mathbb{C})$. We suppose $J$ to be $u$-homogeneous, i.e. generated by homogeneous elements. We define the set of the leading exponents of $J$ as

$$\text{Exp}_\prec(J) = \{ \exp_\prec(f) | f \in J, f \neq 0 \}.$$ 

This set is stable by sums in $\mathbb{N}^n$ thus by Dickson lemma:

**Definition 2.1.3.** There exists $G = \{ g_1, \ldots, g_r \} \subset J$ (made of homogeneous elements) such that $\text{Exp}_\prec(J) = \cup_j (\exp_\prec(g_j) + \mathbb{N}^n)$. Such a set $G$ is called a (homogeneous) $\prec$-standard basis of $J$.

The statement concerning homogeneity follows from the following: take $g_j \in G$, then any $u$-homogeneous part of $g$ belongs to $J$. Let $g'_j$ be the one that contains the leading term of $g$. The set of $g'_j$ is a homogeneous standard basis.

The following statements are equivalent, see [CG04, Cor. 1.5.4]:

- A set $G \subset J$ is a homogeneous standard basis of $J$
- For any homogeneous $f \in \text{gr}^u(\mathbb{C})$: $f \in J$ if and only if the remainder of the division of $f$ by $G$ is zero.

Keeping the notations of the proof above, we see that for such a standard basis, the set of $\text{in}_{w^1}(g)$ for $g \in G$ generates $\text{in}_{w^1}(J)$. In fact we have more than that: Given a local order $\prec$ and $u \in \mathcal{U}_{\text{loc}}$, we define the local order $\prec_u$ in a lexicographical way by $u$ and $\prec$. 


Lemma 2.1.4.  
(1) Given $I \subset C$, if $G$ is a $\prec_u$-standard basis of $I$ then $\text{in}_u(G)$ is a (homogeneous) standard basis of $\text{in}_u(I)$ for $\prec$.
(2) $\text{Exp}_{\prec_u}(f) = \text{Exp}_{\prec}(\text{in}_u(f))$.

Proof. Statement (2) easily follows from (1) and the fact that for any $g \in G$, $\text{exp}_{\prec_u}(g) = \text{exp}_{\prec}(\text{in}_u(g))$. Let us prove (1). Let $f'$ be in $\text{in}_u(I)$. We want to prove that $\text{exp}_{\prec_u}(f')$ is in $\text{exp}_{\prec}(\text{in}_u(g)) + \mathbb{N}^n$ for some $g \in G$. By considering the homogeneous part of $f'$ that contains the $\prec$-leading term, we may assume $f'$ to be homogeneous. Let $f \in I$ be such that $\text{in}_u(f) = f'$ and let us consider the division of $f$ by $G$ w.r.t. $\prec_u$ as in Theorem 2.1.2: \[ f = \sum q_j g_j \text{ where } G = \{g_1, \ldots, g_r\} \]

\[(\star) \quad \text{Supp}(q_j) + \text{exp}_{\prec_u}(g_j) \subset \Delta_j.\]

We have $\text{ord}_u(f) \geq \text{ord}_u(q_j g_j)$ for any $j$. Let $q'_j = \text{in}_u(g_j)$ if $\text{ord}_u(f) = \text{ord}_u(q_j g_j)$ and $q'_j = 0$ otherwise then $f' = \sum q'_j \text{in}_u(g_j)$. Since $\text{exp}_{\prec_u}(g) = \text{exp}_{\prec}(\text{in}_u(g))$, the partition of $\mathbb{N}^n$ associated with $\text{exp}_{\prec}(g_j)$ is the same as that associated with $\text{exp}_{\prec}(\text{in}_u(g_j))$. Moreover, for any $h \in C$, $\text{Supp}(\text{in}_u(h)) \subset \text{Supp}(h)$, thus the relation $(\star)$ becomes
\[ \text{Supp}(q'_j) + \text{exp}_{\prec_u}(g_j) \subset \Delta_j. \]

This means that the remainder of the division of $f'$ by the $\text{in}_u(g_j)$ w.r.t. $\prec$ is zero from which the conclusion follows. \hfill \Box

Now given a standard basis $G$ of $I \subset \text{gr}^u(C)$. We say that $G$ is minimal if for $g, g' \in G$, $\text{exp}_{\prec}(g) \in \text{exp}_{\prec}(g') + \mathbb{N}^n$ implies $g = g'$. We say that $G$ is reduced if it is minimal, unitary (i.e. $\text{lc}_{\prec}(g) = 1$ for any $g \in G$) and if for any $g \in G$, $\text{Supp}(g) \setminus \{\text{exp}_{\prec}(g)\} \subset \mathbb{N}^n \setminus \text{Exp}_{\prec}(I)$.

Lemma 2.1.5.  
(1) Given a homogeneous ideal $J$ in $\text{gr}^u(C)$ and a local order $\prec$, a reduced standard basis w.r.t. $\prec$ exists and is unique. Moreover it is made of homogeneous elements.

(2) Given $I$ in $C$, if $G$ is the reduced $\prec_u$-standard basis then $\text{in}_u(G)$ is the reduced $\prec$-standard basis of $\text{in}_u(I)$.

Proof. The proof of the first statement is classical, we omit it. The second one easily follows from the fact that for any $g \in G$, $\text{exp}_{\prec_u}(g) = \text{exp}_{\prec}(\text{in}_u(g))$ and $\text{Supp}(\text{in}_u(g)) \subset \text{Supp}(g)$.
\hfill \Box

Lemma 2.1.6. Suppose we have two local orders $\prec_1$ and $\prec_2$ and a homogeneous ideal $J \subset \text{gr}^u(C)$. Let $G$ be a (resp. the reduced) $\prec_1$-standard basis of $J$ and suppose for any $g \in G$ that $\text{exp}_{\prec_1}(g) = \text{exp}_{\prec_2}(g)$ then $G$ is a (resp. the reduced) $\prec_2$-standard basis of $J$.

Sketch of Proof. Let $f \in J$. Divide $f$ by $G$ w.r.t. $\prec_1$ for which the remainder is zero. By assumption, this division is also the division w.r.t. $\prec_2$, and since the remainder is zero, $G$ is a $\prec_2$-standard basis of $J$. The statement concerning the reducibility is trivial. \hfill \Box

2.2. Back to Gröbner fans. For a local weight vector $u \in \mathcal{U}_{\text{loc}}$, call $\text{supp}(u) = \{1 \leq i \leq n | u_i \neq 0\}$ the support of $u$. By lemma 2.1.4 we have:
\[ u' \in S_C(u) \iff \text{supp}(u) = \text{supp}(u'). \]

Proposition 2.2.1. Given an ideal $I$ in $C$ and $u \in \mathcal{U}_{\text{loc}}$, then for any local order $\prec$, the $\prec_u$-reduced standard basis $G$ of $I$ satisfies:
\[ C_I[u] = \{u' \in \mathcal{U}_{\text{loc}} | \text{supp}(u') = \text{supp}(u') \text{ and } \forall g \in G, \text{in}_u(g) = \text{in}_{u'}(g)\}. \]
Proof. We already know that \( u' \in S_C(u) \iff \text{supp}(u) = \text{supp}(u') \) then under this assumption we have to prove that \( \text{in}_{u}(g) = \text{in}_{u'}(g) \) for any \( g \in G \) if and only if \( \text{in}_{u}(I) = \text{in}_{u'}(I) \). Take \( u' \) in the RHS. By assumption, for any \( g \in G \), \( \exp_{\prec_{u}}(g) = \exp_{\prec}(\text{in}_{u}(g)) = \exp_{\prec}(\text{in}_{u'}(g)) = \exp_{\prec_{u'}}(g) \) so by lemma 2.1.6, \( G \) is also the \( \prec_{u'} \)-reduced standard basis of \( I \). It follows that \( \text{in}_{u}(I) \) and \( \text{in}_{u'}(I) \) have a common set of generators which is \( \text{in}_{u}(G) = \text{in}_{u'}(G) \).

Now take \( u' \) in the LHS. By Lemma 2.1.4(2), we have \( \exp_{\prec_{G}}(I) = \exp_{\prec}(\text{in}_{u}(I)) = \exp_{\prec}(\text{in}_{u'}(I)) = \exp_{\prec_{u'}}(I) \) which implies that \( G \) is the reduced standard basis for \( \prec_{u'} \). Thus by lemma 2.1.1(2), \( \text{in}_{u}(G) \) and \( \text{in}_{u'}(G) \) are both the \( \prec \)-reduced standard basis of \( \text{in}_{u}(I) = \text{in}_{u'}(I) \). By unicity of the reduced standard basis, \( \text{in}_{u}(G) = \text{in}_{u'}(G) \). As a consequence, \( u' \in \text{RHS} \).

Corollary 2.2.2. Let \( I \) be an ideal in \( O_0 \) and let \( \hat{I} = \hat{O} \cdot I \) then the Gröbner fans \( \mathcal{E}(I, U_{\text{loc}}) \) and \( \mathcal{E}(\hat{I}, U_{\text{loc}}) \) are equal.

Proof. Let \( u \) be in \( U_{\text{loc}} \). Let \( G \) be the \( \prec_{u} \)-reduced standard basis of \( I \) then by using the Buchberger criterion involving the \( S \)-functions (see [CG04 Prop. 1.6.2]), we obtain that \( G \) is a \( \prec_{u} \)-standard basis of \( \hat{I} \) which in turn implies that it is the reduced one. Moreover by lemma 2.2.1 we have that \( S_{\hat{O}}(u) = S_{\hat{O}_{\text{loc}}}(u) \) thus \( C_{I}[u] = C_{\hat{I}}[u] \) which completes the proof.

With this corollary, we may assume that our ideal is in \( \hat{O} \). We will work in \( \hat{O} \) for the rest of this section. Let \( g \) be in \( \hat{O} \), we define its Newton polyhedron as the following convex hull:

\[
\text{New}(g) = \text{conv}(\text{Supp}(g) + \mathbb{N}^n).
\]

The set \( \text{Supp}(g) + \mathbb{N}^n \) is by definition stable by sums so let \( E(g) \) be the minimal finite subset of \( \text{Supp}(g) + \mathbb{N}^n \) such that \( \text{Supp}(g) + \mathbb{N}^n = E(g) + \mathbb{N}^n \). As a consequence we have

\[
\text{New}(g) = \text{conv}(E(g)) + \mathbb{R}^n_0
\]

which implies that \( \text{New}(g) \) is strictly speaking a polyhedron \([Z95, p.30]\).

First we have the following result which is a part of Theorem 2.0.4.

Theorem 2.2.3. For any ideal \( I \) in \( \hat{O} \), the set \( \{\text{in}_{u}(I)\mid u \in U_{\text{loc}}\} \) is finite.

Proof. We sketch the proof by giving the main steps since it is very similar to that of [ACG01 Th. 4].

(a) Given \( g \in \hat{O} \), the set of \( \exp_{\prec}(g) \), \( \prec \) being any local order, is finite. By using the same arguments as in the proof of [ACG01 Prop. 17], we can easily show that it is contained in \( E(g) \).

(a') The set of \( \text{in}_{u}(g) \), \( u \in U_{\text{loc}} \), is finite. Indeed it is in one to one correspondence with the set of faces of \( \text{New}(g) \).

(b) Given an ideal \( I \) in \( \hat{O} \), the set \( \{\exp_{\prec}(I)\mid \prec \text{ is a local order}\} \) is finite. The proof uses the same arguments as that of [ACG01 Th. 5] and is based on (a). We can also prove it as [St95, Th. 1.2].

(c) This last step is the main part. Thanks to the preceding statement, it is enough to prove the following: Given \( E = \exp_{\prec}(I) \) for some local order \( \prec \), the set of all \( \text{in}_{u}(I) \) such that \( \exp_{\prec}(I) = E \) is finite. Let \( G \) be a \( \prec \)-standard basis of \( I \) then for any \( u \) as above, \( \text{in}_{u}(I) \) is generated by the set \( \{\text{in}_{u}(g)\mid g \in G\} \). By (a'), there is only a finite number of such sets.

\( \square \)
Now let \( u \) be in \( \mathcal{U}_{\text{loc}} \) and let \( \prec \) be any local order. Let \( G \) be the \( \prec_u \)-reduced standard basis of \( I \). Let
\[
Q = \sum_{g \in G} \text{New}(g)
\]
be the Minkowski sum of the Newton polyhedra of the \( g \) in \( G \).

**Proposition 2.2.4.**
\[
C_I[u] = N_Q(\text{face}_u(Q))
\]
where \( N_Q(\text{face}_u(\cdot)) \) denotes the normal cone of the face w.r.t. \( u \).

As a direct consequence:

**Corollary 2.2.5.** The Gröbner cone \( C_I[u] \) is a convex rational relatively open polyhedral cone.

The proof of the proposition will be decomposed into several lemmas.

For \( g \in \hat{O} \) and \( u \in \mathcal{U}_{\text{loc}} \), let \( E_u(g) \) be the elements of \( E(g) \) which are maximum for the scalar product with \( u \). Let \( e_i \in \mathbb{N}^n, i = 1, \ldots, n \) be the canonical base of the semi-group \( \mathbb{N}^n \).

**Lemma 2.2.6.**
\[
\text{face}_u(\text{New}(g)) = \text{conv}(E_u(g)) + \sum_{i \notin \text{supp}(u)} \mathbb{R}_{\geq 0} \cdot e_i.
\]

**Proof.** It is easy to see that \( \text{face}_u(\mathbb{R}^n_{\geq 0}) \) is equal to the second member of the RHS. Moreover there is a general identity for polyhedra
\[
\text{face}_u(P + P') = \text{face}_u(P) + \text{face}_u(P').
\]
Thus it suffices to show this equality: \( \text{face}_u(\text{conv}(E(g))) = \text{conv}(E_u(g)) \). Let us prove it. Take \( \alpha \) in the LHS. Written as a sum \( \alpha = \sum_j c_j \alpha_j, c_j \geq 0, \sum_j c_j = 1 \), of elements in \( E(g) \), we see that the scalar product \( u \cdot \alpha \) is equal to some \( u \cdot \alpha_j \). This implies that \( u \) evaluated on the LHS or on the RHS gives the same number. From this observation, we easily derive the desired equality. \( \square \)

Let \( g_1, \ldots, g_r \) be in \( \hat{O} \). Put \( Q_j = \text{New}(g_j) \) and let \( Q \) be the Minkowski sum of the \( Q_j \)'s.

**Lemma 2.2.7.** For any \( u \in \mathcal{U}_{\text{loc}} \), we have
\[
N_Q(\text{face}_u(Q)) = \bigcap_{j=1}^r N_Q(\text{face}_u(Q_j)).
\]

This lemma is known for polytopes (see [Ziegler, Prop. 7.12, p. 198]). We can prove it by reducing the case of polyhedra to the case of polytopes by truncating our polyhedra.

Now, here is the last lemma before the proof of proposition 2.2.4.

**Lemma 2.2.8.** Let \( g \in \hat{O} \) then for \( u, u' \in \mathcal{U}_{\text{loc}} \):
\[
[\text{in}_u(g) = \text{in}_{u'}(g) \text{ and } \text{supp}(u) = \text{supp}(u')] \iff \text{face}_u(\text{New}(g)) = \text{face}_{u'}(\text{New}(g)).
\]

**Proof.** First let us prove this claim:

**Claim.** Under the condition \( \text{supp}(u) = \text{supp}(u') \) the following equivalence holds:
\[
E_u(g) = E_{u'}(g) \iff \text{in}_{u}(g) = \text{in}_{u'}(g).
\]
Suppose we have \( E_u(g) = E_{u'}(g) \), it is enough to prove \( \text{Supp(\text{in}_{u}(g))} = \text{Supp(\text{in}_{u'}(g))} \).
Let \( \alpha \) be in \( \text{Supp(\text{in}_{u}(g))} \). Then there exists \( e \in E(g) \) and \( \alpha' \in \mathbb{N}^n \) such that \( \alpha = e + \alpha' \). We have \( u \cdot \alpha \leq u \cdot e \) but by definition of \( \text{in}_{u}(g) \), \( u \cdot \alpha \geq u \cdot e \). Thus \( e \) belongs to \( E_u \) which is equal to \( E_{u'}(g) \) by assumption. This also means that \( u \cdot \alpha' = 0 \) but since \( \text{supp}(u) = \text{supp}(u') \), this implies \( u' \cdot \alpha' = 0 \). Thus \( u' \cdot \alpha = u' \cdot e \). Now take
any $\alpha''$ in $\text{Supp}(g)$. We can write $\alpha'' = e' + \alpha'$ with $e' \in E$ and $\alpha' \in \mathbb{N}^n$. Therefore, $u' \cdot \alpha'' \leq u' \cdot e' \leq u' \cdot e = u' \cdot \alpha$. This means that $\alpha$ belongs to $\text{Supp}(\text{in}_{u'}(g))$. By symmetry the other inclusion holds. The right-left implication can be shown with similar arguments.

Let us return to the proof of the lemma. The left-right implication follows immediately from the claim and Lemma 2.2.6. For the right-left one, by using the same arguments as in the proof of the previous lemma (i.e. the boundedness of $\text{conv}(E)$), we can show that $\text{supp}(u) = \text{supp}(u')$ and $E_u(g) = E_{u'}(g)$, we then conclude by the claim above. \hfill \Box

Now we are ready to give the

Proof of Proposition 2.2.4. By Lemma 2.2.7

$$N_Q(\text{face}_u(Q)) = \bigcap_{j=1}^r N_{\text{New}(g_j)}(\text{face}_u(\text{New}(g_j))).$$

By the previous lemma,

$$N_{\text{New}(g_j)}(\text{face}_u(\text{New}(g_j))) = \{ u' \in S(u) | \text{in}_u(g_j) = \text{in}_{u'}(g_j) \}. $$

We then conclude by using Prop. 2.2.5. \hfill \Box

2.3. Proof of Theorem 2.0.4. We recall that we are given an ideal $I$ in $\hat{O}$. Let $u \in U_{\text{loc}}$.

Lemma 2.3.1. For any $u' \in \overline{C_I[u]} \setminus C_I[u]$ there exists a local order $\prec$ such that the reduced standard bases of $I$ with respect to $\prec_u$ and to $\prec_{u'}$ are the same.

Proof. The proof is inspired by [Ba05, Prop. 2.1]. We define the order $\prec$ in a lexicographical way by $u$ and by any local order $\prec$. Note that $\prec = \prec_u$. Let $G$ be the $\prec$-reduced standard basis of $I$. In order to prove the lemma, it is enough (see lemma 2.1.6) to show the following for any $g \in G$: $\exp_{\prec_u}(g) = \exp_{\prec_{u'}}(g)$.

First since $\text{Supp}(\text{in}_{u'}(g)) \subset \text{Supp}(g)$, we have $\exp_{\prec_u}(g) \geq u \exp_{\prec_u}(\text{in}_{u'}(g))$. Let us prove the reverse inequality. First we prove:

$$(\ast) \quad \text{ord}'(g) = u' \cdot \exp_{\prec_u}(g).$$

For any $\alpha \in \text{Supp}(g)$, and for any $u'' \in C_I[u]$, $u'' \cdot \exp_{\prec_u}(g) \geq u'' \cdot \alpha$. Since $u'$ is the limit of elements $u'' \in C_I[u]$, we obtain the same inequality for $u'$ which proves $\ast$. Therefore $\exp_{\prec_u}(g) \in \text{Supp}(\text{in}_{u'}(g))$. The desired inequality is then a direct consequence of the definition of $\exp_{\prec_u}(\text{in}_{u'}(g))$. The claim and the lemma are proven. \hfill \Box

Finally, we can prove Theorem 2.0.4.

Proof. The proof is exactly the same as that of [S1993, Prop. 2.4]. For the sake of completeness let us give the main arguments. For $u \in U_{\text{loc}}$, let $u'$ be in $C_I[u] \setminus C_I[u]$. Take an order $\prec$ as in the previous lemma and let $G$ be the $\prec$-reduced standard basis of $I$. Then by Prop. 2.2.7 $C_I[u] = N_Q(\text{face}_u(Q))$ and $C_I[u'] = N_Q(\text{face}_{u'}(Q))$ where $Q$ is the Minkowski sum of $\text{New}(g)$ for $g \in G$. By hypothesis on $u'$, $\text{face}_u(Q)$ is a face of the polyhedron $\text{face}_{u'}(Q)$ thus $C_I[u']$ is a face of the closed convex cone $C_I[u]$.

Now let us check the two axioms for being a fan. For axiom (1), let $F$ be a face of the closure of some $C_I[u]$. Take any $u'$ in the relative interior of $F$ then by the
arguments above, $F$ is equal to the face $\overline{C}_I[u]$. For axiom (2), let $u, u'$ be any in $U_{\text{loc}}$ and consider the closed convex cone $P = \overline{C}_I[u] \cap \overline{C}_I[u']$. We have seen that for any $u'' \in P$, $\overline{C}_I[u'']$ is a face of both $\overline{C}_I[u]$ and $\overline{C}_I[u']$. Thus $P$ is a (finite) union of common faces. By convexity of $P$ this union is a singleton.

3. THE CLOSED GRÖBNER FAN IN $\mathcal{h}_{0,1}(\mathcal{D})$ OR $\mathcal{h}_{0,1}(\mathcal{D}_0)$ IS A POLYHEDRAL FAN

**Theorem 3.0.2.** Let $I$ be a homogeneous ideal in $\mathcal{h}_{0,1}(\mathcal{D})$ or in $\mathcal{h}_{0,1}(\mathcal{D}_0)$ then its closed Gröbner fan $\tilde{\mathcal{F}}(I, U_{\text{loc}})$ is a rational polyhedral fan.

The outline of the proof for this theorem is analogous to Theorem 2.0.3 However, there are technical differences. So, in this section we shall mainly prove the results specific to the differential case.

3.1. **Division in the graded rings** $\text{gr}^w(\mathcal{h}_{0,1}(\mathcal{D}))$ AND $\text{gr}^w(\mathcal{h}_{0,1}(\mathcal{D}_0))$. Let $\prec$ be a total monomial order on $\mathbb{N}^{2n}$ (or equivalently on the terms $x^\alpha \xi^\beta$ where $\xi_i$ is a commutative variable corresponding to $\partial_i$). Such an order shall be called local admissible or simply admissible if for any $i$, $x_i \prec 1$ and $x_i \xi_i > 1$.

In the following we need a generalized version of the division theorem of [ACG01] in $\mathcal{h}_{0,1}(\mathcal{D})$. Indeed we will prove it for any admissible order. However we only treat the formal case and we will refer to [ACG01] for the analytic case.

The next step will be a division theorem in the graded ring $\text{gr}^w(\mathcal{h}_{0,1}(\mathcal{D}))$ for any local weight vector $w \in W_{\text{loc}}$.

Take an admissible order $\prec$ on $\mathbb{N}^{2n}$. Define the order $\prec^h$ on $\mathbb{N}^{2n+1}$:

$$(\alpha, \beta, k) \prec^h (\alpha', \beta', k') \iff \begin{cases} k + |\beta| < k' + |\beta'| \text{ or} \\ k + |\beta| = k' + |\beta'| \text{ and } (\alpha, \beta) \prec (\alpha', \beta'). \end{cases}$$

As before we have the notion leading exponent $\exp_{\prec^h}$, leading coefficient $\text{lc}_{\prec^h}$, leading term $\text{lt}_{\prec^h}$ and leading monomial $\text{lm}_{\prec^h} = \text{lc}_{\prec^h} \cdot \text{lt}_{\prec^h}$.

Let $P_1, \ldots, P_r \in \mathcal{h}_{0,1}(\mathcal{D})$. Consider the partition $\Delta_1 \cup \cdots \cup \Delta_r \cup \Delta$ of $\mathbb{N}^{2n+1}$ associated with the $\exp_{\prec^h}(P_j)$ as done in the previous section.

**Theorem 3.1.1** (Division theorem in $\mathcal{h}_{0,1}(\mathcal{D})$). For any $P \in \mathcal{h}_{0,1}(\mathcal{D})$, there exists a unique $(Q_1, \ldots, Q_r, R) \in (\mathcal{h}_{0,1}(\mathcal{D}))^{r+1}$ such that

- $P = Q_1 P_1 + \cdots + Q_r P_r + R$
- for any $j$, if $Q_j \neq 0$ then $\text{Supp}(Q_j) + \exp_{\prec^h}(P_j) \subset \Delta_j$
- if $R \neq 0$, $\text{Supp}(R) \subset \Delta$.

Moreover if $P$ and the $P_j$ are homogeneous then so are the $Q_j$ and $R$.

In order to compare Gröbner fans in the analytic and the formal case, we need a convergent division theorem in $\mathcal{h}_{0,1}(\mathcal{D}_0)$. This has been proved in [ACG01].

**Theorem 3.1.2** ([ACG01] Theorem 7]). Let $w \in W_{\text{loc}}$ and define the admissible order $\prec$ in a lexicographical way by $[|\beta|, \prec_0]$ where $\prec_0$ is any fixed admissible order then define the admissible order $\prec_w$ by refining $w$ by $\prec$. Then if we take $\prec = \prec_w$ in the theorem above then the following holds: if $P$ and the $P_j$ are in $\mathcal{h}_{0,1}(\mathcal{D}_0)$ then so are the $Q_j$ and $R$.

Now let us prove the division theorem in $\mathcal{h}_{0,1}(\mathcal{D})$. According to [Ro85], the order $\prec$ is defined by $2n$ independent weight vectors. Let $w = (u, v)$ be the first of them. Since $\prec$ is admissible, we have $u_i \leq 0$ and $u_i + v_i \geq 0$, that is $w \in W_{\text{loc}}$. The first step of the proof of the division theorem is to reduce to the case where $u$ has no zero components.
Lemma 3.1.3. There exists a weight vector \( w' = (u', v') \in W_{\text{loc}} \) with \( u'_i < 0 \) and \( u'_i + v'_i > 0 \) such that the order \( \prec^h_{w'} \) defined first by \( k + |\beta| \), then by \( w' \) and finally by \( \prec \) satisfies: \( \exp_{\prec^h_{w'}}(P_j) = \exp_{\prec^h}(P_j) \) for any \( j = 1, \ldots, r \).

For the proof of this lemma, we refer to \cite[Prop. 8]{ACG01}. Now once this lemma is proven then proving the division theorem for \( \prec^h_{w'} \) shall prove the theorem for \( \prec^h \).

Proof of the formal division theorem. The unicity is easy, and the statement concerning homogeneity also. Let us prove the existence. In order to simplify the notation, we will omit the subscript \( \prec^h \).

Define the following sequences \( P^{(i)}, Q^{(i)}_1, \ldots, Q^{(i)}_r, R^{(i)} \) as follows:

1. Put \( (P^{(0)}, Q^{(0)}_1, \ldots, Q^{(0)}_r, R^{(0)}) = (P, 0, 0, \ldots, 0) \).
2. For \( i \geq 0 \), if \( P^{(i)} = 0 \) then put

\[
(P^{(i+1)}, Q^{(i+1)}_1, \ldots, Q^{(i+1)}_r, R^{(i+1)}) = (P^{(i)}, Q^{(i)}_1, \ldots, Q^{(i)}_r, R^{(i)}).
\]

3. If \( \exp(P^{(i)}) \in \tilde{\Delta} \) then

\[
(P^{(i+1)}, Q^{(i+1)}_1, \ldots, Q^{(i+1)}_r, R^{(i+1)}) = (P^{(i)} - \text{Im}(P^{(i)}), Q^{(i)}_1, \ldots, Q^{(i)}_r, R^{(i)} + \text{Im}(P^{(i)})).
\]

4. If not, then \( \exists ! j \in \{1, \ldots, r\} \) such that \( \exp(P^{(i)}) \in \Delta_j \); put

\[
P^{(i+1)} = P^{(i)} - \frac{\log(P^{(i)})}{\log(P^{(j)})} \cdot (x, \partial, h)^{\exp(P^{(i)}) - \exp(P^{(j)})} \cdot P^{(j)},
\]

\[
Q^{(i+1)}_j = Q^{(i)}_j + \frac{\log(P^{(i)})}{\log(P^{(j)})} \cdot (x, \partial, h)^{\exp(P^{(i)}) - \exp(P^{(j)})},
\]

for \( j \neq j' \), \( Q^{(i+1)}_{j'} = Q^{(i)}_{j'} \),

\[
R^{(i+1)} = R^{(i)}.
\]

By construction, \( R^{(i)} \) tends to some \( R \in h_{(0,1)}(\hat{D}) \) since \( R^{(i+1)} - R^{(i)} \) is one monomial for which the exponent does not belong to \( \text{Supp}(R^{(i)}) \) and moreover the degree of \( R^{(i)} \) is bounded (by that of \( P \)). The same occurs for \( Q^{(i)}_j \) that tends to some \( Q_j \in h_{(0,1)}(\hat{D}) \). To prove theorem, it remains to prove that \( P^{(i)} \) tends to 0.

For this purpose, we see \( h_{(0,1)}(\hat{D}) \) as a free \( \hat{D} \)-module and we make it a topological space with the \((x)\)-adic topology. If we write \( P^{(i)} = \sum_{\beta, k} p^{(i)}_{\beta k}(x) \partial^\beta h^k \) then \( P^{(i)} \) tends to 0 if and only if each \( p^{(i)}_{\beta k}(x) \) tends to 0 in \( \hat{D} \).

Now, by construction, we have \( P = P^{(i)} + \sum_j Q^{(i)}_j P_j + R^{(i)} \) and for \( i > 0 \):

\[
(2) \quad \exp(P) > \exp(P^{(i)}) > \exp(P^{(i+1)}).
\]

By this relation there exists \( d \in \mathbb{N} \) such that all the \( P^{(i)} \) have degree \( \leq d \). So we can write \( P^{(i)} = \sum_{(\beta, k) \in F} p^{(i)}_{\beta k}(x) \partial^\beta h^k \) with \( F \) finite.

For each \( i \), all the \( \exp_{\prec^h}(p^{(i)}_{\beta k}(x) \partial^\beta h^k) \) with \( (\beta, k) \in F \) are pairwise distinct so \( \exp_{\prec^h}(P^{(i)}) \) equals the maximum of them, thus by relation (2) and by finiteness of \( F \), the following holds:

Fix any \( (\alpha, k) \in F \) then for any \( i_0 \) there exists \( i_1 \geq i_0 \) such that: for any \( i \geq i_1 \),

\[
\exp_{\prec^h}(p^{(i)}_{\beta k}(x) \partial^\beta h^k) < \exp_{\prec^h}(p^{(i_0)}_{\beta k}(x) \partial^\beta h^k).
\]

By the previous lemma, we may assume that \( \prec \) is defined by weight vectors where the first of them \( w = (u, v) \) is such that \( u_i < 0 \).

Now fix \( (\alpha, k) \in F \). The preceding relation implies

\[
\text{ord}^u(p^{(i)}_{\beta k}(x)) \leq \text{ord}^u(p^{(i_0)}_{\beta k}(x)).
\]

But since \( u \) has non zero components, the set of \( \alpha \) such that \( u \cdot \alpha \) equals some
constant is a finite set. So in the previous relation we cannot have an equality for all the \( i \geq i_1 \). Thus their exists \( i_2 \geq i_1 \) such that for any \( i \geq i_2 \), we have:

\[
\text{ord}^u(p^{(i)}_{h,k}(x)) < \text{ord}^v(p^{(i)}_{h,k}(x)).
\]

With this final statement, it is easy to conclude that \( p^{(i)}_{h,k}(x) \) tends to 0 in \( \hat{O} \) (again thanks to the fact that all the \( u_i \) are < 0).

Now we shall derive from the theorem above a division theorem in the graded ring \( gr^w(h_{(0,1)}(\hat{D})) \) or \( gr^w(h_{(0,1)}(D_0)) \).

In order to better understand the division in these graded algebras, let us describe the latter more precisely.

**Lemma 3.1.4.** For \( w = (u, v) \), let us assume (for simplicity) that:

- \( u_i < 0 \) for \( 1 \leq i \leq n_2 \) with \( u_i + v_i = 0 \) for \( 1 \leq i \leq n_1 \) and \( u_i + v_i > 0 \) for \( n_1 < i \leq n_2 \),
- \( u_i = 0, v_i > 0 \) for \( n_2 < i \leq n_3 \),
- \( u_i = v_i = 0 \) for \( n_3 < i \leq n \).

Then the graded ring \( gr^w(h_{(0,1)}(\hat{D})) \) is canonically isomorphic to

\[
\mathbb{K}[x_{n_2+1}, \ldots, x_n][x_1, \ldots, x_{n_2}, \xi_{n_2+1}, \ldots, \xi_{n_3}, \partial_1, \ldots, \partial_{n_1}, \partial_{n_3+1}, \ldots, \partial_n][h]
\]

where \( \xi_i \) is a commutative variable and for \( a(x) \in \mathbb{K}[x_{n_2+1}, \ldots, x_n][x_1, \ldots, x_{n_2}] \),

\[
[\partial_i, a(x)] = \frac{\partial_i(a(x))}{\partial x_i}. h
\]

Concerning \( gr^w(h_{(0,1)}(D_0)) \), we have the same result replacing \( \mathbb{K}[x_{n_2+1}, \ldots, x_n] \) with \( \mathbb{C}\{x_{n_2+1}, \ldots, x_n\} \).

**Proof.** The proof consists of a simple verification.

Let \( w \) be in \( W_{\text{loc}} \). Given an admissible order \( < \), we define the (admissible) order \( <_w \) by refining the partial order defined by \( w \).

The ring \( gr^w(h_{(0,1)}(\hat{D})) \) is a bi-graded ring. There is a graduation by the total degree in the \( \partial_i \), \( \xi_i \) and \( h \). There is another graduation associated with \( w \). So in order to avoid confusions, a homogeneous element for the second graduation shall be called \( w \)-homogeneous.

Let \( P_1, \ldots, P_r \) be bihomogeneous in \( gr^w(h_{(0,1)}(\hat{D})) \). Let \( \mathbb{N}^{2n+1} = (\cup_j \Delta_j) \cup \hat{\Delta} \) be the partition associated with the \( \exp^{-w} \) of \( P_1 \).

**Corollary 3.1.5** (Division theorem in \( gr^w(h_{(0,1)}(\hat{D})) \)). For any bihomogeneous \( P \in gr^w(h_{(0,1)}(\hat{D})) \), there exists a unique \( (Q_1, \ldots, Q_r, R) \in (gr^w(h_{(0,1)}(\hat{D})))^{r+1} \) made of bihomogeneous elements such that

- \( P = Q_1P_1 + \cdots + Q_rP_r + R \)
- for any \( j \), if \( Q_j \neq 0 \) then \( \text{Supp}(Q_j) + \exp^{-w}(P_j) \subset \Delta_j \)
- if \( R \neq 0 \), \( \text{Supp}(R) \subset \hat{\Delta} \).

**Remark 3.1.6.** The division theorem in \( h_{(0,1)}(\hat{D}) \) is a particular case of the latter if we take \( w = (0) \).

**Proof of the corollary.** There exist homogeneous elements \( P', P'_{1}, \ldots, P'_{r} \) in \( h_{(0,1)}(\hat{D}) \) such that \( \text{in}_w(P') = P \) and \( \text{in}_w(P'_j) = P_j \). Now let us consider the division of \( P' \) by the \( P'_j \) w.r.t. the order \( \prec^h \): \( P' = \sum_j Q'_jP'_j + R' \) with

\[
(Q') \quad \text{Supp}(Q'_j) + \exp^{-w}(P'_j) \subset \Delta_j \quad \text{and} \quad \text{Supp}(R') \subset \hat{\Delta},
\]

where the partition is associated with the \( \exp^{-w} \) of \( P'_j \). By homogeneity, it follows that \( \text{ord}^w(P') \geq \text{ord}^w(Q'_jP'_j) \) and \( \text{ord}^w(P') \geq \text{ord}^w(R') \). Set \( Q_j = \text{in}_w(Q'_j) \) (resp. \( R = \text{in}_w(R') \)) if the inequality above is an equality and \( Q_j = 0 \) (resp. \( R = 0 \)) otherwise. Then we have \( P = \sum_j Q_jP_j + R \).
Now, by definition of $\prec_h$, for any homogeneous $Q \in h_{(0,1)}(\hat{D})$, $\exp_{\prec_h}(Q) = \exp_{\prec_h}(\text{in}_w(Q))$. This implies that the partition of $\mathbb{N}^{2n+1}$ associated with the $\exp_{\prec_h}(P_j)$ is the same as that associated with the $\exp_{\prec_h}(P_j)$.

Moreover for any $Q \in h_{(0,1)}(\hat{D})$, $\text{Supp}(\text{in}_w(Q)) \subset \text{Supp}(Q)$ so the relation (**) becomes

$$\text{Supp}(Q_j) + \exp_{\prec_h}(P_j) \subset \Delta_j \text{ and } \text{Supp}(R) \subset \bar{\Delta}.$$ 

We remark that the $Q_j$ and $R$ are $w$-homogeneous and then bihomogeneous. The existence is proven. The unicity can be proved easily. \hfill \square

If we take $\preceq = \prec$ (see Th. 3.1.2), then with the same proof, we derive:

**Corollary 3.1.7** (Division theorem in $\text{gr}^w(h_{(0,1)}(D_0))$ for $\prec$). For any bihomogeneous $P \in \text{gr}^w(h_{(0,1)}(D_0))$, there exists a unique $(Q_1, \ldots, Q_r, R) \in (\text{gr}^w(h_{(0,1)}(D_0)))^{r+1}$ made of bihomogeneous elements such that

- $P = Q_1 P_1 + \cdots + Q_r P_r + R$
- for any $j$, if $Q_j \neq 0$ then $\text{Supp}(Q_j) + \exp_{\prec_h}(P_j) \subset \Delta_j$
- if $R \neq 0$, $\text{Supp}(R) \subset \bar{\Delta}$

3.2. **Standard bases in the graded ring** $\text{gr}^w(h_{(0,1)}(\hat{D}))$. The symbol $\prec$ still denotes any admissible order. Let $J$ be a bihomogeneous ideal in $\text{gr}^w(h_{(0,1)}(\hat{D}))$ or in $\text{gr}^w(h_{(0,1)}(D_0))$ (and in the latter case, we take $\preceq = \prec$). We define

$$\text{Exp}_{\prec_h}(J) = \{\exp_{\prec_h}(f)|0 \neq f \in J\}.$$ 

This set is stable by sums in $\mathbb{N}^{2n+1}$, thus by Dickson lemma:

**Definition 3.2.1.** There exists a finite set $G = \{g_1, \ldots, g_r\} \subset J$ such that

$$\text{Exp}_{\prec_h}(J) = \bigcup_{j=1}^r (\exp_{\prec_h}(g_j) + \mathbb{N}^{2n+1}).$$ 

Such a set is called a $\prec_h$-standard basis of $J$.

Given $G \subset J$, the following statements are equivalent:

- $G$ is a bihomogeneous $\prec_h$-standard basis of $J$.
- For any $f \in \text{gr}^w(h_{(0,1)}(\hat{D}))$, $f \in J$ if and only if the remainder of the division of $f$ by $G$ (w.r.t. $\prec_h$) is zero.

In the following lemma, we gather the results needed in the sequel. The generalizations of the division theorem in the previous subsection are used to prove the following analogous statements with Lemmas 2.4.1, 2.4.5, 2.4.6.

**Lemma 3.2.2.**
- Given a bihomogeneous ideal $J$ in $\text{gr}^w(h_{(0,1)}(\hat{D}))$ (resp. in $\text{gr}^w(h_{(0,1)}(D_0))$) and an admissible order $\prec$ (resp. $\preceq = \prec$), a reduced standard basis w.r.t. $\prec_h$ exists and is unique. Moreover it is bihomogeneous.
- Let $\prec_1$, $\prec_2$ be admissible orders. If $G$ is a homogeneous (the reduced) $\prec_1$-standard basis of $J$ and $\exp_{\preceq_h}(g) = \exp_{\prec_1}(g)$ for any $g \in G$ then $G$ is a homogeneous (the reduced) $\prec_h$-standard basis of $J$.
- Given $I$ homogeneous in $h_{(0,1)}(\hat{D})$ (resp. $h_{(0,1)}(D_0)$), if $G$ is the reduced $\prec_h$-standard basis of $I$ then $\text{in}_w(G)$ is the reduced $\prec_h$-standard basis of $\text{in}_w(I)$. Moreover we have $\text{Exp}_{\prec_h}(I) = \text{Exp}_{\prec_h}(\text{in}_w(I))$. 
3.3. Back to the Gröbner fan. For \( w = (u, v) \in W_{\text{loc}} \), we call support of \( w \) the set \( \text{supp}(w) \subset \{1, \ldots, n\}^2 \) defined as:

\[
\text{M}(w) \times \text{P}(w) = \left\{ i|u_i < 0 \right\} \times \left\{ i|u_i + v_i > 0 \right\}.
\]

Note that \( M(w) \) and \( P(w) \) are independent. A consequence of lemma 3.1.4 is: For \( w = (u, v) \in W_{\text{loc}} \),

\[
S(w) = S(w') \iff \text{supp}(w) = \text{supp}(w').
\]

**Proposition 3.3.1.** Let \( w \in W_{\text{loc}} \). Given a homogeneous ideal \( I \) in \( h_{(0,1)}(\hat{D}) \) (resp. in \( h_{(0,1)}(D_0) \)) and an arbitrary admissible order \( \prec \) (resp. the order \( \prec \bowtie \)), the \( \prec_w \)-reduced standard basis \( G \) of \( I \) satisfies:

\[
C_I[w] = \{ w' \in W_{\text{loc}}|\text{supp}(w) = \text{supp}(w') \text{ and } \forall g \in G, \text{in}_w(g) = \text{in}_{w'}(g) \}.
\]

**Proof.** Exactly the same as that of Proposition 2.2.1. \( \square \)

**Corollary 3.3.2.** Let \( I \) be a homogeneous ideal in \( h_{(0,1)}(D_0) \) then the Gröbner fans \( \mathcal{E}(I, W_{\text{loc}}) \) and \( \mathcal{E}(h_{(0,1)}(\hat{D}) \cdot I, W_{\text{loc}}) \) are equal.

**Proof.** We put \( \bowtie \preceq \) and use the same arguments as for Corollary 3.2.2. \( \square \)

After this corollary, we will work in \( h_{(0,1)}(\hat{D}) \) for the rest of the section.

Let us denote by \( \pi \) the natural projection \( \mathbb{N}^2n \times \mathbb{N} \to \mathbb{N}^2n \), \( (\alpha, \beta, k) \mapsto (\alpha, \beta) \).

For \( g \in h_{(0,1)}(\hat{D}) \) let us define the Newton polyhedron of \( g \) as the following convex hull:

\[
\text{New}(g) = \text{conv} \left( \{ (\alpha, \beta) \in \mathbb{Z}^2n|\forall (u, v) \in W_{\text{loc}}, (u, v) \cdot (\alpha, \beta) \leq 0 \} \right).
\]

If we denote by \( W^*_{\text{loc}} \) the polar dual cone of \( W_{\text{loc}} \) then the set under bracket is \( W^*_{\text{loc}} \cap \mathbb{Z}^2n \). Let us characterize \( W^*_{\text{loc}} \): for \( i = 1, \ldots, n \), let \( e_i \in \mathbb{N}^2n \) the vector having 1 in its \( i \)th component and zero for the others; and let \( e'_i \in \mathbb{Z}^2n \), \( e'_i = (0, \ldots, 0, -1, 0, \ldots, 0, -1, 0, \ldots, 0) \) with the \(-1\) placed at position \( i \) and \( n+i \). Then

\[
W^*_{\text{loc}} = \left\{ \sum_{i=1}^{n} \lambda_i e_i + \sum_{i=1}^{n} \lambda'_i e'_i | \lambda_i, \lambda'_i \geq 0 \right\}.
\]

Let \( E(g) \subset \pi(\text{Supp}(g)) \) be a finite subset such that: \( \pi(\text{Supp}(g)) + (\oplus \mathbb{N} e_i) \times (\oplus \mathbb{N} e'_i) = E(g) + (\oplus \mathbb{N} e_i) \times (\oplus \mathbb{N} e'_i) \). This is possible by Dickson lemma. We have

\[
(3) \quad \text{New}(g) = \text{conv}(E(g)) + W^*_{\text{loc}},
\]

which assures that \( \text{New}(g) \) is strictly speaking a polyhedron.

Let \( w \) be in \( W_{\text{loc}} \), \( \prec \) be any admissible order and \( G \) be the \( \prec_w \)-reduced standard basis of \( I \). Let

\[
Q = \sum_{g \in G} \text{New}(g)
\]

be the Minkowski sum of the Newton polyhedra of the \( g \) in \( G \).

**Proposition 3.3.3.**

\[
C_I[w] = \text{N}_Q(\text{face}_w(Q))
\]

where \( \text{N}_Q(\text{face}_w(\cdot)) \) denotes the normal cone of the face w.r.t. \( w \).

The proof will be based on the following lemmas.
Lemma 3.3.4. For $g \in h_{(0,1)}(\hat{D})$ and $w \in \mathcal{W}_{\text{loc}}$, let $E_w(g)$ be the set of elements in $E(g)$ which are maximum for the scalar product with $w$ then

$$\text{(4) } \text{face}_w(\text{New}(g)) = \text{conv}(E_w(g)) + \left\{ \sum_{i \notin M(w)} \lambda_i e_i + \sum_{i \notin P(w)} \lambda'_i e'_i \mid \lambda_i, \lambda'_i \geq 0 \right\}.$$  

Proof. It is easy to see that $\text{face}_w(\text{New}(g))$ is the second member of the sum in the RHS. Moreover for any polyhedra, there is a general identity $\text{face}_w(P + P') = \text{face}_w(P) + \text{face}_w(P')$ so it suffices to show the equality $\text{face}_w(\text{conv}(E(g))) = \text{conv}(E_w(g))$. This equality follows easily from the fact that the height of $\text{face}_w(\text{conv}(E(g)))$ and $\text{conv}(E_w(g))$ w.r.t. $w$ is the same. This fact can be shown as in the proof of lemma 2.2.6.

Let $g_1, \ldots, g_r$ be in $h_{(0,1)}(\hat{D})$. Put $Q_j = \text{New}(g_j)$ and let $Q = \sum_j Q_j$ be their Minkowski sum.

Lemma 3.3.5. For any $w \in \mathcal{W}_{\text{loc}}$,

$$N_Q(\text{face}_w(Q)) = \bigcap_{j=1}^r N_{Q_j}(\text{face}_w(Q_j)).$$

We can prove it by reducing to the case of polytopes of this lemma. Here is the last lemma before the proof of proposition 3.3.3.

Lemma 3.3.6. Let $g \in h_{(0,1)}(\hat{D})$ then for $w, w' \in \mathcal{W}_{\text{loc}}$:

$$[\text{in}_w(g) = \text{in}_{w'}(g) \text{ and } S(w) = S(w')] \iff \text{face}_w(\text{New}(g)) = \text{face}_{w'}(\text{New}(g)).$$

Proof. The proof is the same as that of lemma 2.3.8 where one has just to replace $\mathbb{N}^n$ with $\mathcal{W}_{\text{loc}}^* \cap \mathbb{Z}^{2n}$.

Proof of Proposition 3.3.8. By lemma 3.3.6

$$N_Q(\text{face}_w(Q)) = \bigcap_{j=1}^r N_{\text{New}(g_j)}(\text{face}_w(\text{New}(g_j))).$$

By the previous lemma,

$$N_{\text{New}(g_j)}(\text{face}_w(\text{New}(g_j))) = \{w' \in S(w) \mid \text{in}_w(g_j) = \text{in}_{w'}(g_j)\}.$$ 

We then conclude by using Prop. 3.3.4.

3.4. Proof of Theorem 3.0.2 Let $I$ be a homogeneous ideal in $h_{(0,1)}(\hat{D})$.

Lemma 3.4.1. Let $w' \in C_f[w] \setminus C_I[w]$ for some $w \in \mathcal{W}_{\text{loc}}$. Then there exists an admissible order $\prec$ such that the reduced standard bases of $I$ w.r.t. $\prec^h_w$ and to $\prec^h_{w'}$ agree.

Proof. The proof is a slight modification of [Ba05, Prop. 2.1]. Since it is similar to that of lemma 2.3.1, we sketch it. We define the order $\prec$ first by $w$ and then by any admissible order $\prec$. Let $G$ be the $\prec^h$-reduced standard basis of $I$. Note that for homogeneous elements, $\exp_{\prec^h_w}$ coincides with $\exp_{\prec^h_{w'}}$. Now, to prove the lemma, it is enough to show the following for any $g \in G$: $\exp_{\prec^h_w}(g) = \exp_{\prec^h_{w'}}(g)$.

$$\exp_{\prec^h_w}(g) = \exp_{\prec^h_{w'}}(\text{in}_w(g)) \text{ because } g \text{ is homogeneous}$$

$$= \exp_{\prec^h_w}(\text{in}_{w'}(g)) \text{ by definition of } \prec$$

$$= \exp_{\prec^h_{w'}}(g).$$

The last equality can be proved as in the proof of lemma 2.3.1.
By [ACG01, Th. 4], we know that the number of different in\(_w(I)\) is finite, so in order to finish the proof of Theorem 3.0.1 it remains to prove that \(\mathcal{E}(I, W_{\text{loc}})\) satisfies the two axioms for being a complex which can be done with the same arguments as in the proof of Theorem 2.0.3.

When considering applications or when doing calculations, we often need to consider \(\text{Gröbner fans} \) restricted to linear subspaces in the space of the weight vectors.

**Corollary 3.4.2.** Let \(L\) be a linear subspace in \(\mathbb{R}^{2n}\) then for any homogeneous ideal \(I\) in \(h_{(0,1)}(\mathcal{D})\), the restriction of \(\mathcal{E}(I, W_{\text{loc}})\) to \(L\) is a polyhedral fan in \(W_{\text{loc}} \cap L\).

**Proof.** Since we have proved that the local \(\text{Gröbner fan}\) is a polyhedral fan, the restriction is also a polyhedral fan [ZS95, p. 195]. \(\square\)

4. Existence and comparison of the different \(\text{Gröbner fans}\)

In what follows, we will discuss about the relation between the several kinds of \(\text{Gröbner fans}\). We will discuss both for polynomial rings and associate rings and the case of differential operators.

Proofs are analogous each other. However statements are different because the spaces of the weight vectors are different.

In order to clarify the difference, we present lemmas and propositions for all the cases. The proofs will be made for the case of differential rings.

In the following \(U_{\text{loc}}'\) denotes the \(u \in U_{\text{loc}}\) such that \(u_i < 0\) and \(W_{\text{loc}}'\) denotes the \(w = (u, v) \in W_{\text{loc}}\) with \(u_i < 0\) and \(u_i + v_i > 0\). In other terms, \(U_{\text{loc}}'\) and \(W_{\text{loc}}'\) are the interior of \(U_{\text{loc}}\) and \(W_{\text{loc}}\) respectively.

4.1. Without homogenization.

4.1.1. The local \(\text{Gröbner fan}.\)

In order to compare the \(\text{Gröbner fans}\), let us deal with the graded algebras first.

**Remark 4.1.1.**

- Take \(u \in U_{\text{loc}}\) and assume (for simplicity) that \(u_i < 0\) for \(1 \leq i \leq n_2\) and \(u_i = 0\) for \(n_2 < i \leq n\) then \(\text{gr}^u(\mathcal{O})\) is canonically isomorphic to \(k[[x_{n_2 + 1}, \ldots, x_n]][x_1, \ldots, x_{n_2}]\).

- Take \(w = (u, v) \in W_{\text{loc}}\) and assume that the variables \(x_i\) and \(\partial_i\) are ordered in a way that:
  - \(u_i < 0\) for \(1 \leq i \leq n_2\) with \(u_i + v_i = 0\) for \(1 \leq i \leq n_1\) and \(u_i + v_i > 0\) for \(n_1 < i \leq n_2\),
  - \(u_i = 0, v_i > 0\) for \(n_2 < i \leq n_3\),
  - \(u_i = v_i = 0\) for \(n_3 < i \leq n\).

Then \(\text{gr}^w(\mathcal{D})\) is canonically isomorphic to

\[
(5) \quad k[[x_{n_2 + 1}, \ldots, x_n]][x_1, \ldots, x_{n_2}, \xi_{n_1 + 1}, \ldots, \xi_{n_3}, \partial_1, \ldots, \partial_{n_1}, \partial_{n_3 + 1}, \ldots, \partial_n]
\]

where \(\xi_i\) is a commutative variable.

In the following result, \(k[x]_{(x_{n_2 + 1}, \ldots, x_n)}\) shall denote the localization with respect to the prime ideal generated by \(x_{n_2 + 1}, \ldots, x_n\) (or equivalently the localization along the space defined by this ideal).

**Lemma 4.1.2.** Let \(u\) be in \(U_{\text{loc}}\) and \(w = (u, v)\) be in \(W_{\text{loc}}\). Let us take the same situation as above, then:

(i) \(\text{gr}^u(\mathcal{O}) = k[[x_{n_2 + 1}, \ldots, x_n]][x_1, \ldots, x_{n_2}] \otimes_{k[x]} \text{gr}^u(k[x]_0)\).
(ii) $\text{gr}^u(k[x]_0) = k[x](x_{n_2+1}, \ldots, x_n) \otimes \text{gr}^u(k[x])$.

(iii) If $k = \mathbb{C}$, we have two similar results, one for $\text{gr}^u(\hat{D})$ as a tensor of $\text{gr}^u(\mathcal{O}_0)$ and the other one for $\text{gr}^u(\mathcal{O}_0)$ as a tensor of $\text{gr}^u(k[x]_0)$.

(1) $\text{gr}^w(\hat{D}) = k[[x_{n_2+1}, \ldots, x_n]][x_1, \ldots, x_{n_2}] \otimes \text{gr}^w(D(\{0\}))$.

(2) $\text{gr}^w(D(\{0\})) = k[x](x_{n_2+1}, \ldots, x_n) \otimes \text{gr}^w(D)$.

(3) If $k = \mathbb{C}$, we have two similar results, one for $\text{gr}^u(\hat{D})$ as a tensor of $\text{gr}^u(D(\{0\}))$ and the other one for $\text{gr}^u(D_0)$ as a tensor of $\text{gr}^u(D(\{0\}))$.

Roughly speaking, this lemma says that the local variables $x_i$ for which $u_i < 0$ become global and the other ones stay unchanged. The following is then trivial.

**Corollary 4.1.3.** For any $u \in U'_{\text{loc}}$, $\text{gr}^u(\hat{D}) = \text{gr}^u(\mathcal{O}_0) = \text{gr}^u(k[x]_0) = \text{gr}^u(k[x])$.

For any $w \in W'_{\text{loc}}$, we have $\text{gr}^w(D) = \text{gr}^w(D(\{0\})) = \text{gr}^w(D)$.

**Proof of the Lemma.** For statement (1), the equality is trivial since $\text{gr}^w(D)$ is canonically isomorphic to $k[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n, \partial_1, \ldots, \partial_n]$. We prove statement (2). First, let us remark that both rings in this statement are subrings of $\text{gr}^w(D)$ then it is enough to prove the double inclusion. For the left-right one, it is enough to show that $\text{in}_w(P)$ is in the RHS for any $P \in D(\{0\})$. Such a $P$ is equal to $1/q(x)P'$ with $P' \in D$ and $q(x) \in k[x]$ with $q(0) \neq 0$. Since $\text{in}_w(1/q(x)P') = \text{in}_w(1/q(x))\text{in}_w(P')$, let us show that $\text{in}_w(1/q(x))$ is in $k[x](x_{n_2+1}, \ldots, x_n)$.

For simplicity, assume that $q(0) = 1$ and write $q(x) = 1 - v_0 - v_1$ where $v_0 = v_0(x_{n_2+1}, \ldots, x_n)$ and $v_1$ is in the ideal generated by $x_1, \ldots, x_{n_2}$. As a series in $k[x]$, we have

$$1/q(x) = V_1 + 1 + v_0 + v_0^2 + \cdots$$

where $V_1 \in \sum_{i=1}^{n_2} k[x][x_i]$. By comparing the $w$-orders, we see that $0 = \text{ord}_w(1+v_0 + v_0^2 + \cdots) > \text{ord}_w(V_1)$. As a consequence $\text{in}_w(1/q(x)) = 1 + v_0 + v_0^2 + \cdots = 1/(1-v_0)$ and it is in $k[x](x_{n_2+1}, \ldots, x_n)$.

The right-left inclusion follows from the following: for any $P$ in the RHS, $P$ is a finite sum of elements of the form $1/q \otimes \text{in}_w(P')$ where $P' \in D$ and $q \in k[x_{n_2+1}, \ldots, x_n]$ with $q(0) \neq 0$. But for such elements, we have $\text{in}_w(1/q) = 1/q$ so $\text{in}_w(1/qP') = 1/q \cdot \text{in}_w(P')$ and then $P$ belongs to the LHS.

Finally, the proof of statement (3) uses the same arguments as that of (1) and (2).

Our main goal in this part is to present an algorithm for computing two kinds of fan, the first one is the following.

**Definition 4.1.4.** Let $I$ be an ideal in $k[x]$ (resp. in $D$). The set $\mathcal{E}(k[x]_0 I, U'_{\text{loc}})$ (resp. $\mathcal{E}(D(\{0\}) I, W'_{\text{loc}})$) shall be called the local (open) Gröbner fan of $I$ (at $0 \in k^n$).

Take $u \in U'_{\text{loc}}$ (resp. $w \in W'_{\text{loc}}$). We have seen that in this case, all the graded rings that we considered are equal to $\text{gr}^u(k[x])$ (resp. $\text{gr}^w(D)$), so the initial ideals considered in the next proposition are comparable.

**Proposition 4.1.5.** Take $u \in U'_{\text{loc}}$ and $w \in W'_{\text{loc}}$.

(i) If $I \subset k[x]$ then $\text{in}_u(k[x]_0 I) = \text{in}_u(I)$.

(ii) If $I \subset k[x]_0$ then $\text{in}_w(\mathcal{O} I) = \text{in}_w(I)$.

(iii) If $k = \mathbb{C}$ and $I \subset k[x]_0$ then $\text{in}_u(\mathcal{O} I) = \text{in}_u(I)$.

(iv) If $k = \mathbb{C}$ and $I \subset \mathcal{O}_0$ then $\text{in}_w(\mathcal{O} I) = \text{in}_w(I)$.

(1) If $I \subset D$ then $\text{in}_w(D(\{0\}) I) = \text{in}_w(I)$.
As we said, all these initial ideals are in a same ring \( \text{gr}^w(D) \). Let us first suppose that \( I \) is in \( D \) and let us prove that \( \text{in}_w(\hat{D}I) = \text{in}_w(I) \). Using the inclusions

\[
\text{in}_w(I) \subset \text{in}_w(D(0)I) \subset \text{in}_w(\hat{D}I),
\]

claim (1) follows from the inclusion \( \text{in}_w(\hat{D}I) \subset \text{in}_w(I) \), that we shall subsequently prove.

For this inclusion, it is enough to prove that for \( P \in \hat{D}I \), \( \text{in}_w(P) \in \text{in}_w(I) \). Such a \( P \) can be written as a finite sum

\[
P = \sum_j c_j(x)P_j
\]

with \( c_j(x) \in k[[x]] \) and \( P_j \in I \). Put \( m = \text{ord}_w(P) \) and for each \( j \), let us write

\[
c_j(x) = \sum_{\alpha} c_{j,\alpha} x^\alpha
\]

where \( c_{j,\alpha} \in k \). Now let us decompose \( c_j(x) = p_j(x) + s_j(x) \) where

\[
p_j(x) = \sum_{\alpha + \text{ord}_w(P_j) \geq m} c_{j,\alpha} x^\alpha.
\]

Since all the \( u_i \) are negative, each \( p_j(x) \) is a polynomial. By construction, we have \( \text{ord}_w(s_j(x)P_j) < m \). Therefore,

\[
\text{in}_w(P) = \text{in}_w \left( \sum_j p_j(x)P_j \right) \in \text{in}_w(I),
\]

and the inclusion is proved.

Finally, it is easy to remark that the method used for proving claim (1) can be adapted with few changes to prove the other claims.

As a trivial consequence, we obtain:

**Corollary 4.1.6.** For \( I \) in \( k[x] \), \( E(I, U_{\text{loc}}') = E(k[x]_0I, U_{\text{loc}}') \). For \( I \subset D \), \( E(I, W_{\text{loc}}') = E(D(0)I, W_{\text{loc}}') \).

In other words, if we restrict ourselves to the interior of the weight vector space then, global and local Gröbner fans do agree. So the obstruction to have equality should be found in the border and indeed they are not equal in general.

**Example 4.1.7.** (1) Take \( I = (1) \) in \( k[x] \). Then \( E(\hat{O}I, W_{\text{loc}}) \) is equal to \( E(I, W_{\text{loc}}) \cap S_{\hat{O}} \) where \( S_{\hat{O}} \) is the stratification given by the graded rings. After this example we may ask if such a relation always happens. We shall see in Th. 5.1.1 that this relation is true if the ideal \( I \) is homogeneous for a weight vector having positive components. In the general case, the relation does not hold:

(2) Take \( I = (1-x_3, x_1 + x_2) \) in \( k[x_1, x_2, x_3] \). For the weights, we consider the space \( U_{12} = \{(u_1, u_2, 0)|u_1 \leq 0\} \). In this space, the open global Gröbner fan has three cones \{\( u_1 < u_2 \), \( u_1 = u_2 \), \( u_1 > u_2 \)\} while the local Gröbner fan is trivial since \( 1 - x_3 \) is invertible. Here trivial means that we have four cones \{\( 0 \), \( u_1 < 0, u_2 = 0 \), \( u_1 < 0, u_2 < 0 \), \( u_1 = 0, u_2 < 0 \)\}. In this example, \( E(\hat{O}I, U_{12}) \) has 4 cones and \( E(I, U_{12}) \cap S_{\hat{O}} \) has 6 cones.

For a given algebraic ideal \( I \), we called \( E(I, U_{\text{loc}}') \) (resp. in \( W_{\text{loc}} \)) the local Gröbner fan of \( I \), let us precise this term.

**Proposition 4.1.8.**
Theorem 4.1.9. Let by Assi et al. [ACG01].

\[ \text{local Gröbner fan defined in 4.1.4 agrees with the analytic standard fan constructed agree:} \]

and for \( I \)

(1) To prove the equality, it is enough to prove the

Proof of Theorem 4.1.9.

Remark 4.1.10. By the previous theorem, for \( I \subset k[x] \), we have

\[ \mathcal{E}(I, U_{loc}) \] is a refinement of \( \mathcal{E}(k[x]) \).

and for \( I \in D \), we have:

\[ \mathcal{E}(I, W_{loc}) \]

is a refinement of \( \mathcal{E}(D_0 I, W_{loc}) \).

Proof of Theorem 4.1.9.

(1) To prove the equality, it is enough to prove the right-left inclusion, the opposite one being trivial. For this, we are reduced to prove: for \( P \in D_0 I \), \( \text{in}_w(P) \) is in the LHS. For such a \( P \), there exists \( q(x) \in k[x] \) with \( q(0) \neq 0 \) such that \( q(x)P \in I \), which implies that \( \text{in}_w(q(x))\text{in}_w(P) \in \text{in}_w(I) \). Now \( \text{in}_u(q(x)) = \text{in}_w(q(x)) \) and since the \( u_i \) are non positive, \( \text{in}_u(q(x)) \) is invertible in \( \text{gr}_w(D_0 I) \) (and not only in \( D_0 I \)).

(2) The proof for this statement is based on the homogenized version of this theorem to be proved in the next subsection (see Th. 4.2.5). As in (1), only the right-left inclusion is non trivial. Let \( P \in D \cdot I \). By lemma 4.2.2 (proved independently in the next subsection), \( h_{(0,1)}(P) \in h_{(0,1)}(D)h_{(0,1)}(I) \). By theorem 4.2.5, \( \text{in}_w(h_{(0,1)}(P)) \) belongs to \( \text{gr}_w(h_{(0,1)}(D))\text{in}_w(h_{(0,1)}(I)) \). We dehomogenize and remark that \( [\text{in}_w(h_{(0,1)}(P))]_{h=1} = \text{in}_w(P) \) to conclude.

(3) (a) is a consequence of (b) and (2) while (b) is a direct consequence of theorem 4.2.5.

□

Using the previous theorem, let us give the

Proof of Prop. 4.1.8. Statement (1) follows from the following implication: for any \( w, w' \), if \( \text{in}_w(DI) \subset \text{in}_{w'}(DI) \) then \( \text{in}_w(I) \subset \text{in}_{w'}(I) \). Let us prove it. By the previous theorem, we are reduced to prove:

\[ \text{(*)} \quad \text{gr}_w(D)\text{in}_w(I) \subset \text{gr}^{w'}(D)\text{in}_{w'}(I) \Rightarrow \text{in}_w(I) \subset \text{in}_{w'}(I) . \]

This is a direct consequence of the faithful flatness of \( k[x] \) over \( k[x]_{(x_1, \ldots, x_n)} \) [Mat89] p. 62.

Let us detail the argument. We take the notations of remark 4.1.1. Let us denote by \( \tilde{x} \) the set of variables \( x_{n_2+1}, \ldots, x_n \), and by \( k[\tilde{x}] \) the localization of \( k[\tilde{x}] \) w.r.t.
the maximal ideal generated by the set $\tilde{x}$. Now, let us first remark that the LHS of (⋆) implies that for $w$ and $w'$, graded rings are implicitly equal. The LHS of (⋆) takes place in $k[\tilde{x}] \otimes_{k[\tilde{x}]} \text{gr}^w(D)$ and the RHS in $k[\tilde{x}] \otimes_{k[\tilde{x}]} \text{gr}^{w'}(D)$. Now let us work in the $k[\tilde{x}]_{(\bar{\tilde{x}})}$-modules category. Then LHS of (⋆) is equivalent to

$$k[[\tilde{x}]] \otimes_{k[\tilde{x}]_{(\bar{\tilde{x}})}} \text{in}_w(I) \subset k[[\tilde{x}]] \otimes_{k[\tilde{x}]_{(\bar{\tilde{x}})}} \text{in}_{w'}(I)$$

does not involve $\bar{\tilde{x}}$. In this (and the next) example, we show how to construct reduced Gröbner bases using the $k[\tilde{x}]_{(\bar{\tilde{x}})}$-modules category. The notation $\text{in}_w(I)$ is defined by considering $\text{in}_w(I) = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{f \in I} x_i^w f I$. The following question is natural: is the global Gröbner fan $E$ equivalent to the exactness of $0 \to \text{in}_{w'}(I) \to \text{in}_w(I) \to 0$

which means that $\text{in}_w(I) \subset \text{in}_{w'}(I)$, and we are done.

For statement (2), the proof is similar and based on the faithful flatness of $\hat{O}$ over $O_0$ [Ma89, p. 62].

4.1.2. On the passage from local to global. We already know (see Remark 4.1.10) that the global Gröbner fan of $I$ is a refinement of the local one (at 0) so we could ask whether we have a passage from local to global.

Given an ideal $I$ in $D$, we considered the local Gröbner fan $E(D_{(0)} I, W_{\text{loc}})$ at $0 \in k^n$. Now, for any $x^0 \in k^n$, we can consider the local Gröbner fan of $I$ at the point $x^0$ which is defined by considering $(u, v) \in W_{\text{loc}}$ as weights on the variables $x_i - x_i^0$ and $\partial_i$ (note that by the affine change of coordinates $x \mapsto x' = x - x^0$, the derivations are not affected i.e. $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x'}$). Another definition is $E(D_{(0)} I_{x^0}, W_{\text{loc}})$.

The following question is natural: is the global Gröbner fan $E(I, W_{\text{loc}})$ the common refinement of the local ones? The next example shows that the answer is no.

**Example 4.1.11.** Let $f = 1 + x_1 + x_2 \in k[x_1, x_2]$. To simplify, we will talk about open Gröbner cones. The restriction to $U_{\text{loc}}$ of the global Gröbner fan is made of 4 cones: $\{0\}$, $C_1 = \{u_1 < 0, u_2 = 0\}$, $C_2 = \{u_1 < 0, u_2 < 0\}$ and $C_3 = \{u_1 = 0, u_2 < 0\}$. Now for $x_1^0, x_2^0 \in k$, write $f = 1 + x_1^0 + x_2^0 + (x_1 - x_1^0) + (x_2 - x_2^0)$. In the case $1 + x_1^0 + x_2^0 \neq 0$, $f$ is invertible in $k[x_1, x_2](x_1 - x_1^0, x_2 - x_2^0)$, thus the local Gröbner fan is trivial at such a point; that means that it is made of the 4 cones $\{0\}, C_1, C_2, C_3$. Now if $1 + x_1^0 + x_2^0 = 0$ then the local Gröbner fan at $(x_1^0, x_2^0)$ is made of 6 cones: $\{0\}, C_1, \{u_1 < 0, u_2 < 0\}$, $\{u_1 < 0, u_2 < 0, u_1 = u_2\}$, $\{u_1 < 0, u_2 < 0, u_1 > u_2\}$ and $C_3$.

Thus the common refinement of the (two) local Gröbner fans is the latter and it is different from the global one.

4.2. **With homogenization.** Gröbner fans can be described explicitly by using the notion of reduced Gröbner or standard bases. For an ideal $I$ in $k[x]$ or in $D$ it is not possible, in general, to construct reduced Gröbner bases when the order is not a global order, that is when we work with weight vectors having some negative components. That explains why we need to work with one more variable in homogenized rings. In these kind of rings we can compute reduced Gröbner bases for any orders. Another consequence of homogenization is that Gröbner fans of homogenized ideals have convex cones (see [ACG00] and [SST00]).
In this short subsection, we state homogeneous analogues to the results of the preceding section. Most of the proofs are the same and will not be written. First, here are two lemmas useful for the sequel.

**Lemma 4.2.1.** For a given ideal $I \subset D$, we can compute $h_{(1,1)}(I)$ as follows: take any admissible (global) order $\prec$ and compute a $\prec_{(1,1)}$-standard basis $G$ of $I$ then $h_{(1,1)}(I)$ will be generated by the $h_{(1,1)}(P)$ for $P \in G$.

**Sketch of proof.** Let $P$ be in $G$. By definition of $G$, we can write $P = \sum_j Q_j P_j$ with $P_j \in G$ and $Q_j \in D$ with $\text{ord}_{(1,1)}(P) \geq \text{ord}_{(1,1)}(Q_j P_j)$. The homogenization then gives: $h_{(1,1)}(P) = \sum h_{\text{ord}_{(1,1)}(P_j)} - \text{ord}_{(1,1)}(Q_j) \cdot h_{(1,1)}(Q_j) \cdot h_{(1,1)}(P_j)$. Since the set of the $h_{(1,1)}(P_j)$, $P$ running over $I$, generates $h_{(1,1)}(I)$, we are done. □

**Lemma 4.2.2.** Given $I$ in $D$. We can compute generators of $h_{(0,1)}(I)$ in $h_{(1,1)}(D)$ by homogenizing a $\prec_{(0,1)}$-standard basis of $I$ and we have

$$h_{(0,1)}(\hat{D}I) = h_{(1,1)}(\hat{D})h_{(0,1)}(I).$$

**Sketch of proof.** Let $G$ be a $\prec_{(0,1)}$-standard basis of $I$ then with the same proof as above, we prove the first statement. Now by using Buchberger S-criterion, $G$ is also a $\prec_{(0,1)}$-standard basis of $\hat{D}I$ which implies that $h_{(0,1)}(G)$ also generates $h_{(0,1)}(\hat{D}I)$. □

As it is easy to see, a homogeneous counterpart of Prop. 4.1.8 holds from which follows a counterpart of Cor. 4.1.9 (the proofs are the same and omitted).

**Corollary 4.2.3.** For a $(0,1)$-homogeneous ideal $I \subset h_{(0,1)}(D)$, $\mathcal{E}(I, W'_{\text{loc}}) = \mathcal{E}(h_{(0,1)}(D_0), h_{(0,1)}(I), W'_{\text{loc}})$.

Here is the homogeneous version of Prop. 4.1.8.

**Proposition 4.2.4.**

1. For $I \subset D_{(0)}$, $\mathcal{E}(h_{(0,1)}(I), W_{\text{loc}}) = \mathcal{E}(h_{(0,1)}(I), W_{\text{loc}})$.

2. If $k = \mathbb{C}$ and $I \subset D_0$ then $\mathcal{E}(h_{(0,1)}(I), W_{\text{loc}}) = \mathcal{E}(h_{(0,1)}(I), W_{\text{loc}})$.

Statement (2) is a direct consequence of 3.3.2 but it can be proved as Prop. 4.1.8. The proof of (1) is almost the same as that of Prop. 4.1.8 providing the fact that $h_{(0,1)}(\hat{D}I) = h_{(0,1)}(\hat{D})h_{(0,1)}(I)$ (this is Lemma 4.2.2).

**Theorem 4.2.5.** Let $w$ be in $W_{\text{loc}}$.

1. For $I \subset D$, $\text{gr}_{w}(h_{(0,1)}(D_0), \text{in}_{w}(h_{(0,1)}(I)) = \text{in}_{w}(h_{(0,1)}(D)(I))$.

2. For $I \subset D_{(0)}$, $\text{gr}_{w}(h_{(0,1)}(D), \text{in}_{w}(h_{(0,1)}(I)) = \text{in}_{w}(h_{(0,1)}(\hat{D}I))$.

3. When $k = \mathbb{C}$, we have similar results concerning (a) $I \subset D_{(0)}$ and $\text{in}_{w}(h_{(0,1)}(D_0))$ and (b) $I \subset D_0$ and $\text{in}_{w}(h_{(0,1)}(\hat{D}))$.

**Remark 4.2.6.** As a consequence of (1) of this theorem, we obtain: for $I \subset D$,

$$\mathcal{E}(h_{(0,1)}(I), W_{\text{loc}}) \text{ refines } \mathcal{E}(h_{(0,1)}(D_0), W_{\text{loc}}).$$

**Proof of the Theorem.** Concerning (1), the proof is the same as for Th. 3.1.9. Let us prove (2). By [GOT01] Cor. 3.3 and by the previous lemma, $\text{Exp}_{w}^{h}(h_{(0,1)}(I)) = \text{Exp}_{w}^{h}(h_{(0,1)}(\hat{D}I))$. Let $G$ be a homogeneous $\prec_{w}^{h}$-standard basis of $h_{(0,1)}(I)$. The equality above implies that $G$ is also a $\prec_{w}^{h}$-standard basis of $h_{(0,1)}(\hat{D}I)$. As a consequence, $\text{in}_{w}(G)$ generates $\text{in}_{w}(h_{(0,1)}(I))$ over $\text{gr}_{w}(h_{(0,1)}(D_0))$ and also generates $\text{in}_{w}(h_{(0,1)}(\hat{D}I))$ over $\text{gr}_{w}(h_{(0,1)}(\hat{D}))$. This implies the desired equality.

Statement (3)(a) is a consequence of (2) and (3)(b). For the latter, consider a standard basis of $I$ w.r.t. the order $\prec_{w}^{h}$ (see section 3) then the initial forms
in_w(·) form a system of generators of in_w(h_{(0,1)}(I)) and of in_w(h_{(0,1)}(\hat{D})h_{(0,1)}(I)). Finally, we remark that h_{(0,1)}(\hat{D})h_{(0,1)}(I) = h_{(0,1)}(\hat{D}I) which can be easily shown by homogenizing a (0,1)-standard basis of I.

\section{Algorithms for local Gröbner fans}

In this section, we will focus on the following problems:

Given an ideal I in \(k[x]\) (resp. in \(D\)), find an algorithm for computing the following local Gröbner fans \(\hat{E}(\hat{O}I,\hat{U}_{\text{loc}} \cap L)\) and \(\hat{E}(h_{(0,1)}(\hat{D}I),\hat{W}_{\text{loc}} \cap L)\). Here, \(L\) is a linear subspace in a space of weights. Our approach is based on the fact that the local Gröbner fan can be refined by the global Gröbner fan of some homogeneous ideal, the latter being computable.

Since the fans \(\hat{E}(\hat{O}I,\hat{U}_{\text{loc}})\) and \(\hat{E}(h_{(0,1)}(\hat{D}I),\hat{W}_{\text{loc}})\) are polyhedral fans, it is enough for obtaining the Gröbner fan restricted to the space \(L\) to enumerate the maximal dimensional cones in the whole weight space. All lower dimensional Gröbner cones are obtained by taking faces of the maximal dimensional cones. We proved that local and global Gröbner cones agree in maximal dimensional strata of the whole weight space. Hence, if we ignore complexity of computation, we do not need to consider the problem of enumerating the local Gröbner cones in a weight space restricted to the linear subspace \(L\). However, the cost of enumerating all the Gröbner cones in the maximal dimensional strata in the whole weight space is very high in general. For example, for an \(A\)-hypergeometric system associated to the matrix \(A = (1,2,3)\), there are more than 1500 maximal dimensional cones. Our implementation could not finish the enumeration in 3 days. However, the small Gröbner fan, which is a fan restricted to the linear subspace \(u_i + v_i = 0\), consists of only 7 maximal dimensional cones. This is the main reason why we restrict our weight space to a linear subspace. We note that local and global Gröbner cones do not agree in general in the restricted weight space \(\hat{W}_{\text{loc}} \cap L\). See Example 5.5.1 (see also Ex. 5.5.4).

Finally, let us mention that Jensen has developed a software package called Gfan \cite{Je} (see also Fukuda et al. \cite{FJST}), which can compute the (global) Gröbner fan of a polynomial ideal. In \cite{FJST}, the authors propose a theory of Gröbner fans for non-homogeneous ideals, but these fans are not “local”. Indeed it is easy to construct an example of two polynomial ideals having the same Gröbner fan in the sense of \cite{FJST} and for which the local Gröbner fans are different: in \(k[x_1, x_2]\), consider \(I_1 = (g)\) and \(I_2 = (1 + g)\) where \(g = x_1 + x_2 + x_1^2x_2 + x_1^3x_2\).

\subsection{The commutative case}

Let \(I\) be in \(k[x]\). We want an algorithm for \(E(\hat{O}I, \hat{U}_{\text{loc}})\).

\textbf{Theorem 5.1.1.} Suppose there exists a weight vector \(\alpha \in (\mathbb{N}_{>0})^n\) such that \(I\) is \(\alpha\)-homogeneous then

\[ E(\hat{O}I, \hat{U}_{\text{loc}}) = E(I, \hat{U}_{\text{loc}}) \cap S_{\hat{O}}. \]

In other terms, local and global Gröbner fans coincide up to the stratification by the ring \(\hat{O}\).

\textit{Proof.} It is easy to see that \(\hat{E}(I, \hat{U}_{\text{glob}})\) is a polyhedral fan. Indeed the proof is the same as in the known case where \(\alpha = (1, \ldots, 1)\) (see \cite{St95}). It then implies that \(\hat{E}(I, \hat{U}_{\text{loc}}) \cap \hat{E}(\hat{O}, \hat{U}_{\text{loc}})\) is a fan. Moreover, a polyhedral fan depends only on its maximal cones and by Cor. 5.1.6 local and global Gröbner fans coincide on the interior of \(\hat{U}_{\text{loc}}\) so \(\hat{E}(\hat{O}I, \hat{U}_{\text{loc}}) = \hat{E}(I, \hat{U}_{\text{loc}}) \cap \hat{E}(\hat{O}, \hat{U}_{\text{loc}})\). This last equality is equivalent to the one we had to prove. \(\square\)

Now suppose that \(I\) is not homogeneous for any positive weight vector. Choose a weight vector \(\alpha \in (\mathbb{N}_{>0})^n\). Let \(\{f_i, i\}\) be a given set of generators of \(I\). Let \(h\) be
a new variable and let \( I^{(h)} \) be the ideal of \( k[x, h] \) generated by the set of \( h_\alpha(f_i) \), where \( h_\alpha(f_i) \) is the \( \alpha \)-homogenization of \( f_i \).

**Proposition 5.1.2.** \( \mathcal{E}(I^{(h)}, U_{\text{glob}}) \) refines \( \mathcal{E}(I, U_{\text{glob}}) \).

**Proof.** Given \( u \) and \( u' \), we have to prove this implication:

\[
\text{in}_u(I^{(h)}) = \text{in}_{u'}(I^{(h)}) \Rightarrow \text{in}_u(I) = \text{in}_{u'}(I).
\]

Suppose the LHS true. Let \( f \) be in \( I \) and let us prove that \( \text{in}_u(f) \in \text{in}_{u'}(I) \). Let us write \( f = \sum_i q_i f_i \). By homogenizing, we obtain: there exists \( l \) and \( l_i \) for any \( i \) such that

\[
h^l h_\alpha(f) = \sum_i h^l h_\alpha(q_i) h_\alpha(f_i)
\]

so \( h^l h_\alpha(f) \) belongs to \( I^{(h)} \). By taking the initial form w.r.t. \( u \), we obtain by hypothesis: \( \text{in}_u(h^l h_\alpha(f)) = \sum_j r_j \text{in}_{u'}(g_j) \) with \( g_j \in I^{(h)} \). It suffices to set \( h = 1 \) to obtain the desired relation. Thus this inclusion holds: \( \text{in}_u(I) \subset \text{in}_{u'}(I) \). The reverse inclusion holds by symmetry which ends the proof. \( \square \)

Combining this proposition with Rem. 5.1.10 we obtain:

**Corollary 5.1.3.** The fan \( \mathcal{E}(I^{(h)}, U_{\text{loc}}) \) refines \( \mathcal{E}(k[x]_0 I, U_{\text{loc}}) \).

**Algorithm 5.1.4 (Computation of \( \mathcal{E}(k[x]_0 I, U_{\text{loc}} \cap L) \)).**

**Input:** an ideal \( I \) in \( k[x] \). A linear subspace \( L \) in \( \mathbb{R}^n \).

**Output:** the local Gröbner fan \( \mathcal{E}(k[x]_0 I, U_{\text{loc}} \cap L) \).

**Step 1:**

Compute the set \( \Sigma_0 \) of maximal cones of the global Gröbner fan of \( I^{(h)} \) restricted to \( U_{\text{loc}} \cap L \): \( \mathcal{E}(I^{(h)}, U_{\text{loc}} \cap L) \).

**Step 2:**

For any \( C, C' \in \Sigma_0 \), using an écarts division, compare \( \text{in}_u(k[x]_0 I) \) and \( \text{in}_{u'}(k[x]_0 I) \) in \( \text{gr}_u(k[x]_0) \) for some \( u \in C, u' \in C' \). If it is equal, glue \( C \) and \( C' \). By continuing this process, we construct the set \( \Sigma \) of maximal cones of \( \mathcal{E}(k[x]_0 I, U_{\text{loc}} \cap L) \). From \( \Sigma \), construct the set \( \mathcal{E} \) of all the cones.

**Output** \( \mathcal{E} \).

Let us add two remarks to the algorithm.

The step 1 can be performed by flipping of the maximal dimensional cones in the linear space \( L \) with respect to the facets. As to details, see Algorithm 3.6 of the book by Sturmfels [St95]. Note that the correctness of the flipping procedure for the enumeration comes from the fact that the Gröbner fan is a polyhedral fan. The flipping procedure can be accelerated by Collart-Kalkbrener-Mall’s Gröbner walk method [CKM97]. We note that the method also works in rings of differential operators.

The second remark is on the écarts division in Step 2. Suppose that \( \text{supp} u = \text{supp} u' = \{1, 2, \ldots, m\} \). Then, we have

\[
\text{gr}^u(k[x]_0) \simeq \{ f/g | f \in k[x], g \in k[x_{m+1}, \ldots, x_n], g(0) \neq 0 \}.
\]

The écarts division in literatures usually suppose that denominators are in \( k[x] \), but our denominator \( g \) lies in \( k[x_{m+1}, \ldots, x_n] \). Hence, in our case, a reducer must be chosen so that the multiplier monomial for the pseudo-division is in the ring \( k[x_{m+1}, \ldots, x_n] \) instead of \( k[x] \) as in the case of the usual écarts division [Mo82, Gr94, GP96].
5.2. The non homogeneous case.

**Proposition 5.2.1.** Let \( I \) be in \( D \) then \( \mathcal{E}(h_{(1,1)}(I), W) \) refines \( \mathcal{E}(I, W) \) for any \( W \subset W_{\text{glob}} \).

In fact, we have a more general result. Let \( G \) be any system of generators of \( I \) over \( D \) and let \( I^{(b)} \subset h_{(1,1)}(D) \) be generated by \( \{ h_{(1,1)}(P) | P \in G \} \) (of course, this ideal is not uniquely determined).

**Remark 5.2.2** (On the computation of \( \mathcal{E}(I^{(b)}, W_{\text{loc}}) \)). Since \( I^{(b)} \) is homogeneous, it is well known that its Gröbner fan can be computed by using reduced Gröbner bases w.r.t. to well-orders (see e.g. [SST00]).

**Proposition 5.2.3.** The fan \( \mathcal{E}(I^{(b)}, W) \) refines \( \mathcal{E}(I, W) \) (for \( W \subset W_{\text{glob}} \)).

**Proof.** The same as that of Prop. 5.1.2. \( \square \)

Now combining the previous proposition with Remark 4.1.10 we obtain:

**Corollary 5.2.4.** The fan \( \mathcal{E}(I^{(h)}, W_{\text{loc}}) \) is a refinement of \( \mathcal{E}(D_{(0)}I, W_{\text{loc}}) \).

By this Corollary, we obtain the following algorithm.

**Algorithm 5.2.5 (Computation of \( \mathcal{E}(D_{(0)}I, W_{\text{loc}}) \)).**

Input: an ideal \( I \) in \( D \).

Output: the local Gröbner fan \( \mathcal{E}(D_{(0)}I, W_{\text{loc}}) \).

Step 1:

- Compute one of the two (global) Gröbner fans and call it \( \mathcal{E}_0 \):
  - \( \mathcal{E}(h_{(1,1)}(I), W_{\text{loc}}) \) (see Lemma 4.2.1 for how to compute \( h_{(1,1)}(I) \))
  - \( \mathcal{E}(I^{(b)}, W_{\text{loc}}) \)

This computation can be done as in [SST00, Chapter 2].

Step 2:

Two cones \( C \) and \( C' \) of \( \mathcal{E}_0 \) are said to be in a same class if for one \( w \in C \) and one \( w' \in C' \), we have \( \text{in}_{w}(D_{(0)}I) = \text{in}_{w'}(D_{(0)}I) \).

By using Algorithm 5.2.6, compute the classes of \( \mathcal{E}_0 \).

Set \( \mathcal{E} := \) the set of the classes of \( \mathcal{E}_0 \).

Output \( \mathcal{E} \).

We note that \( \mathcal{E}(D_{(0)}I, W_{\text{loc}}) \) is not a polyhedral fan in general. Hence, we need to perform the merging procedure in Step 2 for all dimensional Gröbner cones.

**Algorithm 5.2.6.**

Input: \( I \subset D, w, w' \in W_{\text{loc}} \).

Output: \( 1 \) if \( \text{in}_{w}(D_{(0)}I) = \text{in}_{w'}(D_{(0)}I) \) and \( 0 \) if not.

1. Compute \( G_1 \) a \( w \)-standard basis of \( I \) and \( G_2 \) a \( w' \)-standard basis of \( I \).
2. By a reduction via an écart (division as in [GOT04]), compare \( G_1 \) and \( G_2 \).

Let us make some remarks on Algorithm 5.2.6. Concerning step (1), the following fact is basic: Let \( H_1 \) (resp. \( H_2 \)) be the reduced Gröbner basis of \( I^{(h)} \) w.r.t. a well order that privileges \( w \) (resp. \( w' \)). Then by the specialization \( h = 1 \) we can set \( G_1 = H_{i|h=1} \). Thus Step (1) does not require extra computations since the reduced Gröbner basis of \( I^{(h)} \) were needed in Algorithm 5.2.6.

For Step (2), we can use two methods. The first one is an écart division in \( D \) as in [GOT04] (which works although the écart division of loc. cit. is stated in \( h_{(0,1)}(D) \)). The second method is based on the following equivalence:

\[
\text{in}_{w}(D_{(0)}I) = \text{in}_{w'}(D_{(0)}I) \iff (\text{in}_{w}(D_{(0)}I))^{(h)} = (\text{in}_{w'}(D_{(0)}I))^{(h)}.
\]

The right-left implication can be proved in a similar way as that of Prop. 5.1.2 while the left-right one is trivial. Using this equivalence, we can use écart division in \( h_{(0,1)}(D) \).
5.3. The doubly homogenized Weyl algebra. In the next subsection, we present two variants for an algorithm of computing the Gröbner fan of \( h_{(0,1)}(D) \). One variant is based on the doubly homogenized Weyl algebra that we introduce here.

The doubly homogenized Weyl algebra \( h'(D) \) is generated by

\[ x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, h, h' \]

with the relations

\[ \partial_i x_i = x_i \partial_i + hh'. \]

Let \( \prec \) be a total order on the set of normally ordered monomials \( x^\alpha \partial^\beta h^p h'^q \) in \( h'(D) \). Such an order is called a multiplicative monomial order if the following two conditions hold:

1. \( x_i \partial_i \succ hh' \) for \( i = 1, 2, \ldots, n; \)
2. \( x^\alpha \partial^\beta h^k h'^l \prec x^\alpha' \partial^\beta' h^k' h'^{l'} \implies x^\alpha + a \partial^\beta + b h^k + q h'^{l'} \prec x^\alpha' + a \partial^\beta' + b h^k' + q h'^{l'}. \)

Under this definition, the theory of Gröbner basis works analogously with \( h \) and by \( t' \) a weight for \( h' \).

**Theorem 5.3.1.** Consider the weight space

\[ u_i + v_i \geq t + t' \]

and fix a homogeneous left ideal \( J \) in \( h'(D) \). The collection of closures of the Gröbner cones of \( J \) in the weight space is a polyhedral fan.

The proof is analogous to the case of \( h_{(1,1)}(D) \) (see [SST00]). Indeed with the definitions below we dispose of a well order and reduced Gröbner bases exist. As for Prop. 5.1.2 we have:

**Proposition 5.3.2.** If we restrict the fan to the linear subspace \( t = t' = 0 \), then it is a refinement of the Gröbner fan of \( J_{|h'=1} \) in \( h_{(0,1)}(D) \)

Let \( \prec_1 \) be a multiplicative monomial order in \( h_{(0,1)}(D) \) satisfying (2.1), (2.2), (2.3) of [GOT04]:

1. \( x_i \partial_i \succ_1 h. \)
2. \( |\beta| + k < |\beta'| + k' \) then \( x^\alpha \partial^\beta h^k \prec_1 x^\alpha' \partial^\beta' h'^k. \)
3. \( x^\alpha \preceq_1 1. \)

We define an order on \( h'(D) \) as a block order on \( \prec_1 \) as follows:

\[ x^\alpha \partial^\beta h^k h'^l \prec x^\alpha' \partial^\beta' h'^k' h'^{l'} \iff \begin{cases} |\alpha| + |\beta| + k + l < |\alpha'| + |\beta'| + k' + l' \text{ or} \\ \alpha \prec_1 \beta \end{cases} \]

**Lemma 5.3.3.** Let \( f, g \in h'(D) \) be homogeneous and \( \prec \) be an order satisfying (6). If \( \exp_{\prec}(f) \preceq \exp_{\prec}(g) \), then \( \exp_{\prec_1}(f_{|h'=1}) \preceq_1 \exp_{\prec_1}(g_{|h'=1}). \)

This lemma does not hold for \( h' \mapsto h. \)

**Proof.** Suppose that \( \text{lm}_{\prec}(f) \) is equal to \( cx^\alpha \partial^\beta h^k h'^l \). Since \( f \) is homogeneous and \( \prec \) is a block order, the leading monomial of \( f_{|h'=1} \) is \( \text{lm}_{\prec_1}(f_{|h'=1}) = cx^\alpha \partial^\beta h^k. \) In other words, cancelation does not happen when \( h' \mapsto h > 1 \). It completes the proof.

**Theorem 5.3.4.** Fix a homogeneous left ideal \( J \) in \( h'(D) \) generated by \( F \). Let \( \prec_1 \) be an ordering in \( h_{(0,1)}(D) \) satisfying (2.1), (2.2), (2.3) of [GOT04]: Let \( G \) be a Gröbner basis of \( I \) for the order \( \prec \) defined as in (6). Then, the dehomogenization \( G_{|h'=1} \) is a standard basis in the sense of [GOT04] Def. 3.1 of the ideal generated by \( F_{|h'=1} \) with respect to the order \( \prec_1 \).
Proof. \( G_{|h'|=1} \) generates the ideal generated by \( F_{|h'|=1} \) since \( G \) generates the ideal generated by \( F \). Denote the elements of \( G \) by \( \{g_i\} \). Since \( G \) is a Gröbner basis, we have Buchberger's S-pair criterion:
\[
\text{sp}(g_i, g_j) = \sum_k q_{ijk}g_k, \quad \exp_\prec(\text{sp}(g_i, g_j)) \geq \exp_\prec(q_{ijk}g_k).
\]

It follows from Lemma 5.3.3 that \( \text{sp}(g_i, g_j)_{|h'|=1} = \sum_k q_{ijk}_{|h'|=1}g_k_{|h'|=1} \) with
\[
\exp_\prec(\text{sp}(g_i, g_j)_{|h'|=1}) \geq \exp_\prec((q_{ijk}g_k)_{|h'|=1}).
\]

Then, by Theorem 3.2 of [GOT04], we conclude the theorem. \( \square \)

5.4. The homogeneous case. Denote by \( J_1 \) the ideal of \( h'(D) \) generated by homogenization of \( h_{(0,1)}(I) \) with the variable \( h' \) (in fact we homogenize any set of generators of \( h_{(0,1)}(I) \)). The ideal \( J_1 \) is homogeneous in the doubly homogenized Weyl algebra \( h'(D) \). Combining Proposition 5.3.2 and Remark 4.2.6 we have:

**Corollary 5.4.1.** The fan \( \mathcal{E}(J_1, \mathcal{W}_{\text{loc}}) \) is a refinement of \( \mathcal{E}(h_{(0,1)}(D_{(0)}I), \mathcal{W}_{\text{loc}}) \).

This Corollary gives a first variant for computing local Gröbner fan. Let us explain the second variant. Now, we use the notions of [GOT04]: for \( P \in h_{(0,1)}(D) \), we denote by \( P(s) \) the \((-1,1)\)-homogenization in \( h_{(0,1)}(D)[s] \) (here \( s \) is a new variable commuting with the other ones). Let \( G \) be a set of generators of \( h_{(0,1)}(I) \) over \( h_{(0,1)}(D) \), then define \( h_{(0,1)}(I)[s] \) as the ideal of \( h_{(0,1)}(D)[s] \) generated by \( P(s) \) for \( P \in G \). To simplify denote by \( J_2 \) this ideal.

Then we can consider the Gröbner fan \( \mathcal{E}(J_2, \mathcal{W}_{\text{loc}}) \) as we did before, by putting a weight 0 on \( s \).

**Remark 5.4.2** (On the computation of \( \mathcal{E}(J, \mathcal{W}_{\text{loc}}) \)). In \( h_{(0,1)}(D)[s] \) we dispose of a well-order \( \prec_s \) (notation of [GOT04]), thus the Gröbner fan of \( J \) can be computed via reduced Gröbner bases computations.

**Proposition 5.4.3.** The fan \( \mathcal{E}(J_2, \mathcal{W}_{\text{loc}}) \) is a refinement of \( \mathcal{E}(h_{(0,1)}(I), \mathcal{W}_{\text{loc}}) \).

We omit the proof since it is similar to that of Prop. 5.1.2. By Prop. 5.4.3 and Remark 4.2.6 we have:

**Corollary 5.4.4.** The fan \( \mathcal{E}(J_2, \mathcal{W}_{\text{loc}}) \) is a refinement of \( \mathcal{E}(h_{(0,1)}(D_{(0)}I), \mathcal{W}_{\text{loc}}) \).

As a consequence, we obtain the next algorithm.

**Algorithm 5.4.5** (Computation of \( \mathcal{E}(h_{(0,1)}(D_{(0)}I), \mathcal{W}_{\text{loc}} \cap L) \)).
Input: an ideal \( I \) in \( D \). A linear subspace \( L \) in \( \mathbb{R}^{2n} \).
Output: the local Gröbner fan \( \mathcal{E}(h_{(0,1)}(D_{(0)}I), \mathcal{W}_{\text{loc}} \cap L) \).

Step 1:

- Compute the set \( \Sigma_0 \) of the maximal cones of the global Gröbner fan \( \mathcal{E}(J, \mathcal{W}_{\text{loc}} \cap L) \) where \( J \) is one of \( J_1, J_2 \).

Step 2:

- As before, compute the classes of \( \Sigma_0 \), by using Algorithm 5.4.6.

- From the set of the classes of \( \Sigma_0 \), construct the set \( \mathcal{E} \) of all the cones of \( \mathcal{E}(J, \mathcal{W}_{\text{loc}} \cap L) \).

Output \( \mathcal{E} \).

**Algorithm 5.4.6.**
Input: \( I \subset D, w, w' \in \mathcal{W}_{\text{loc}} \)
Output: \( 1 \) if \( \text{in}_w(h_{(0,1)}(D_{(0)}I)) = \text{in}_{w'}(h_{(0,1)}(D_{(0)}I)) \) and 0 if not.

1. Compute \( G_1 \) a \( w \)-standard basis of \( I \) and \( G_2 \) a \( w' \)-standard basis of \( h_{(0,1)}(I) \).
2. By a reduction via an écart division in \( \text{gr}^w(h_{(0,1)}(D)) \) as in [GOT04], compare \( G_1 \) and \( G_2 \).
Let us give some remarks on this algorithm.  
- As before, we can obtain $G_1$ and $G_2$ by dehomogenizing operators computed in Algorithm 5.4.5. The justification of Step (1) lies in the fact that $h_{(0,1)}(I)$ generates $h_{(0,1)}(D(0)I)$ over $h_{(0,1)}(D(0))$. Moreover we have seen in Lemma 4.2.2 how to compute $h_{(0,1)}(I)$. 
- The écart division of [GOT04] (stated in $h_{(0,1)}(D)$) works well in $\text{gr}(h_{(0,1)}(D))$ with a modification; a reducer must be chosen so that the multiplier monomial for the pseudo division is in the ring $k[x_{m+1}, \ldots, x_n]$ instead of $k[x]$ as in the case of the usual écart division. Here, we assume that $\text{Supp}(u) = \{1, \ldots, m\}$. 
- Finally, since $\bar{\mathcal{E}}(h_{(0,1)}(D\mathcal{I}), \mathcal{W}_{\text{loc}}$ is a polyhedral fan, it is enough, in step 1, to compute only the maximal cones. 

5.5. Tips for implementation and examples. We implemented the algorithms except Algorithm 5.2.5 in this paper in a combination of Kan [Ta91] and Polymake [GJ]. The following is a list of known implementations for Gröbner fans and some efficiency studies including Gröbner walk. 

1. Macaulay [BS] command hull implemented by A. Reebs constructs state polytopes. 
2. The Singular developing team [GPS04] implements the Gröbner walk techniques [CKM97, [FGLM93]. 
3. B. Huber and R. Thomas gave an efficient algorithm and implementation specialized to getting Gröbner fan for toric ideals [HT00]. 

Our implementation is new in view of the following aspects. 

1. It is the first implementation for local Gröbner fans. 
2. Computation is split into a system for algebra (Kan) and a system for geometry (Polymake). They are connected by the OpenXM-RFC 104 protocol (OoHG, OpenXM on HTTP/GET) [OpenXM]. This design gives us a robust system, since the polymake is a strong, flexible, and robust system for polytopes developed by E. Gawrilow and M. Joswig linked with GMP and cdd by K. Fukuda. We use polymake properties DIM, FACETS, INEQUALITIES. 

We have explained our algorithms for local Gröbner fans in the previous sections. The dominant part in the computation time is Step 1’s. Let us explain some details of Step 1, which obtains the maximal dimensional Gröbner cones in a homogenized ring. 

1. Parametrize the weight space $\mathcal{W}_{\text{loc}} \cap L$ by a polyhedron $W \subseteq \mathbb{R}^{\dim(\mathcal{W}_{\text{loc}} \cap L)}$. The parameterization is given by a matrix $W_{\text{cone}}$. 
2. Find a starting weight vector $u$ such that the dimension of the (projected) Gröbner cone $C[u] \subseteq W$ is $\dim(\mathcal{W}_{\text{loc}} \cap L)$. 
3. If $C[u]$ is not pointed, then there exists a non-zero linear space in $C[u]$. Let $L_{\text{cone}}$ be the maximal linear space in $C[u]$. We construct Gröbner cones in $W/L_{\text{cone}}$. In other words we work in the orthogonal space to $L_{\text{cone}}$. It is necessary because the Polymake and polyhedral algorithms works efficiently only for pointed cones. Note that all maximal dimensional cones in the $W$ space contain the linear space $L_{\text{cone}}$ since the Gröbner fan is a polyhedral fan. 
4. Enumerate the facets of $C[u]/L_{\text{cone}}$. They are either on the border of $W/L_{\text{cone}}$ or not. 
5. Perturb the weight vector $u$ with respect to a facet $F$, which is not on the border of the weight space $W$, as explained in [St95, Chapter 3]. 
6. Lift the new weight vector in $W/L_{\text{cone}}$ to $\mathcal{W}_{\text{loc}} \cap L$. Let $u'$ be the lifted weight vector. Construct the new reduced Gröbner basis and construct
The ideal is used for computing a Bernstein-Sato polynomial for Example 5.5.1.

Let us consider the left ideal

\[ I \]

in the homogenized Weyl algebra \( \mathbb{Q}(t_1, t_2, x, y, \partial_{t_1}, \partial_{t_2}, \partial_x, \partial_y) \). We will examine several Gröbner fans in the weight space

\[ \mathbb{R}^4 \]

via a local Gröbner fan [Ba05].

We will consider several Gröbner fans in the weight space

\[ W = \{-w_1, -w_2, 0, 0, w_1, w_2, 0, 0 \} \mid w_1, w_2 \in \mathbb{R}_{\geq 0} \} \]

Here, the weight vector stands for \((t_1, t_2, x, y, \partial_{t_1}, \partial_{t_2}, \partial_x, \partial_y)\).

Gröbner fan in the homogenized Weyl algebra \( h_{(1,1)}(D) \). We consider the ideal \( I_h \) in the homogenized Weyl algebra \( h_{(1,1)}(D) \) generated by

\begin{align*}
    h_{(1,1)}(g_1) &= t_1 - y, \\
    h_{(1,1)}(g_2) &= t_2 h - (y - (x - h)^2), \\
    h_{(1,1)}(g_3) &= (-2x + 2h) \partial_{t_2} + h \partial_x, \\
    h_{(1,1)}(g_4) &= \partial_{t_1} + \partial_{t_2} + \partial_y.
\end{align*}

The ideal is used for computing a Bernstein-Sato polynomial for \( f_1 = y \) and \( f_2 = y - (x - 1)^2 \) via a local Gröbner fan [Ba05].

We will consider several Gröbner fans in the weight space

\[ W = \{-w_1, -w_2, 0, 0, w_1, w_2, 0, 0 \} \mid w_1, w_2 \in \mathbb{R}_{\geq 0} \} \]

Here, the weight vector stands for \((t_1, t_2, x, y, \partial_{t_1}, \partial_{t_2}, \partial_x, \partial_y)\).

Gröbner fan in the homogenized Weyl algebra \( h_{(1,1)}(D) \). We consider the ideal \( I_h \) in the homogenized Weyl algebra \( h_{(1,1)}(D) \) generated by

\begin{align*}
    h_{(1,1)}(g_1) &= t_1 - y, \\
    h_{(1,1)}(g_2) &= t_2 h - (y - (x - h)^2), \\
    h_{(1,1)}(g_3) &= (-2x + 2h) \partial_{t_2} + h \partial_x, \\
    h_{(1,1)}(g_4) &= \partial_{t_1} + \partial_{t_2} + \partial_y.
\end{align*}

Let \( S \) be the stratification of \( W \) by \( \text{gr}(D_{(0)}) \) or \( \text{gr}(h_{(0,1)}(D_{(0)})) \):

\begin{align*}
    S &= \{W', \tilde{W}_1, \tilde{W}_2, \{0\}\}, \\
    W' &= \{-w_1, -w_2, 0, 0, w_1, w_2, 0, 0 \} \mid w_1, w_2 > 0 \}
    W_{t_1} &= \{-w_1, 0, 0, 0, w_1, 0, 0 \} \mid w_1 > 0 \}
    W_{t_2} &= \{0, -w_2, 0, 0, w_2, 0, 0 \} \mid w_2 > 0 \}.
\end{align*}

The Gröbner fan \( \tilde{E}(h_{(1,1)}(D) \cdot I_h, W) \cap S \) consists of

\[ \{F_1, F_2, L_{12}, \tilde{W}_1, \tilde{W}_2, \{0\}\} \]

where

\begin{align*}
    F_1 &= \{-w_1, -w_2, 0, 0, w_1, w_2, 0, 0 \} \in W \mid w_1 > w_2 \} \\
    F_2 &= \{-w_1, -w_2, 0, 0, w_1, w_2, 0, 0 \} \in W \mid w_1 < w_2 \} \\
    L_{12} &= \{-w_1, -w_2, 0, 0, w_1, w_2, 0, 0 \} \in W \mid w_1 = w_2 \}.
\end{align*}

The initial ideal for weights in \( F_1 \) is

\[ \text{in}_{F_1}(I_h) = \{-y, -x^2 + 2hx - h^2, 2ht_2 \partial_{t_2} + (hx - h^2) \partial_x + 2h^3, (-2x + 2h) \partial_{t_1}, \partial_{t_1}\}. \]

Figure 2. Gröbner fan for Example 5.5.1

Example 5.5.1. Let us consider the left ideal \( I \) in \( D = \mathbb{Q}(t_1, t_2, x, y, \partial_{t_1}, \partial_{t_2}, \partial_x, \partial_y) \) generated by

(7) \( g_1 = t_1 - y, g_2 = t_2 - (y - (x - 1)^2), g_3 = (-2x + 2h) \partial_{t_2} + \partial_x, g_4 = \partial_{t_1} + \partial_{t_2} + \partial_y. \)

The ideal is used for computing a Bernstein-Sato polynomial for \( f_1 = y \) and \( f_2 = y - (x - 1)^2 \) via a local Gröbner fan [Ba05].
The initial ideal for weights in $F_2$ is
\[ \text{in}_{F_2}(I_h) = \{ -y, -x^2 + 2hx - h^2, 2ht_2 \partial t_4 + (hx - h^2) \partial x + 2h^3, (-2x + 2h) \partial t_1, \partial t_2 \}. \]
The initial ideal for weights in $L_{12}$ is
\[ \text{in}_{L_{12}}(I_h) = \{ -y, -x^2 + 2hx - h^2, 2h(t_1 \partial t_2 - t_2 \partial t_1) - (hx - h^2) \partial x - 2h^3, \]
\[ -2(x \partial t_2 - h \partial t_1), \partial t_1 + \partial t_2 \}. \]

Here, $\text{in}_{F_1}(\cdot)$ means the initial with respect to a weight vector in $F_1$. Since the initial does not depend on the choice of the weight vector, our notation does not have an ambiguity.

**Gröbner fan in the Weyl algebra $D$.** By utilizing the ideal membership algorithm, we can see that $\text{in}_{F_1}(I_h)_{h=1} \neq \text{in}_{F_2}(I_h)_{h=1} \neq \text{in}_{L_{12}}(I_h)_{h=1}$. Hence, we have
\[ \mathcal{E}(D \cdot I, W) \cap S = \{ \bar{F}_1, \bar{F}_2, \bar{L}_{12}, \bar{W}_{t_1}, \bar{W}_{t_2}, \{0\} \}. \]

**Local Gröbner fan in the local ring of differential operators $D_{(0)}$.** First, let us compare $\text{in}_{F_1}(I_h)_{h=1}$ and $\text{in}_{F_2}(I_h)_{h=1}$ in $\text{gr}_{F_1}(D_{(0)}) = \{ \ell / g | \ell \in D, g \in Q[x, y] \}$. By the tangent cone algorithm or by examining the bases, which contain the unit $-x^2 + 2x - 1$, we can see that the dehomogenizations of the three initial ideals are the same ideal
\[ \text{in}_{F_1}(I) = \text{in}_{F_2}(I) = \text{in}_{L_{12}}(I) \text{ in } \text{gr}_{F_1}(D_{(0)}) = \text{gr}_{F_2}(D_{(0)}) = \text{gr}_{L_{12}}(D_{(0)}). \]
The local Gröbner fan consists of only one maximal dimensional cone. Hence, we have
\[ \mathcal{E}(D_{(0)} \cdot I, W) = \{ \bar{F}_1 \cup \bar{F}_2, \bar{W}_{t_1}, \bar{W}_{t_2}, \{0\} \} = S. \]
This calculation shows us that either the global Gröbner fan $\mathcal{E}(D \cdot I, W_{\text{loc}}) \cap S_{D_{(0)}}$ or the local Gröbner fan $\mathcal{E}(D_{(0)} \cdot I, W_{\text{loc}})$ are not polyhedral fan. Indeed, our comparison result Cor. 4.1.6 says that maximal dimensional equivalence classes agree, and if we suppose that they are both polyhedral fans, then they must agree in the restricted weight space $W$, too. It is a contradiction.

**Global Gröbner fan in the homogenized Weyl algebra $h_{(0, 1)}(D)$.** The algebra $h_{(0, 1)}(D)$ is the Weyl algebra defined with commutation relation $\partial_i x_j = x_j \partial_i + h \delta_{i,j}$. Let us enumerate Gröbner cones of $I$ in the weight space $W$. Note that $I$ is already $(0, 1)$-homogeneous. We first homogenize $g_1, \ldots, g_4$ in the doubly homogenized Weyl algebra defined with commutation relation $\partial_i x_j = x_j \partial_i + hh' \delta_{i,j}$ as follows;
\[
\begin{align*}
  h'(g_1) &= t_1 - y, \quad h'(g_2) = t_2 h' - (y h' - (x - h')^2), \\
  h'(g_3) &= (-2x + 2h') \partial t_2 + h' \partial x, \quad h'(g_4) = \partial t_1 + \partial t_2 + \partial y.
\end{align*}
\]
By a Gröbner basis computation of the ideal $J_1$ generated by the $h'(g_i)$'s, we get two maximal dimensional cones $F_1$ and $F_2$. The initial ideals are
\[
\begin{align*}
  \text{in}_{F_1}(J_1) &= \{ -y, \partial t_2, -2h' \partial t_1 + 2x \partial t_4, -h'^2 + 2x h' - x^2, \\
  &\quad xh' \partial x - 2t_1 x \partial t_1 - x^2 \partial x - 2xh' h, -4t_1 x \partial t_1^2 - 6x^2 \partial t_1 h, \}, \\
  \text{in}_{F_2}(J_1) &= \{ -y, \partial t_1, 2h' \partial t_2 - 2x \partial t_4, -h'^2 + 2x h' - x^2, \\
  &\quad xh' \partial x - 2t_2 x \partial t_2 - x^2 \partial x - 2xh' h, -4t_2 x \partial t_2^2 - 6x^2 \partial t_2 h, \}, \\
  \text{in}_{L_{12}}(J_1) &= \{ -y, \partial t_1 + \partial t_2, 2h' \partial t_2 - 2x \partial t_4, -h'^2 + 2x h' - x^2, \\
  &\quad xh' \partial x + 2t_1 x \partial t_2 - 2t_2 x \partial t_4 - x^2 Dx - 2xh' h, \\
  &\quad 4t_1 x \partial t_2^2 - 4t_2 x \partial t_4^2 - 6x^2 \partial t_4 h \}.
\end{align*}
\]
By the ideal membership algorithm, we conclude that \( \text{in}_{F_1}(I')_{|h'=1} \neq \text{in}_{F_2}(I')_{|h'=1} \neq \text{in}_{L_{12}}(I')_{|h'=1} \), which implies that \( \text{in}_{F_1}(I) \neq \text{in}_{F_2}(I) \neq \text{in}_{L_{12}}(I) \) in \( h_{(0,1)}(D) \) and \( F_1, F_2 \) and \( L_{12} \) are distinct. Hence, we conclude that
\[
\tilde{E}(h_{(0,1)}(D) \cdot I, W) \cap S = \{ F_1, F_2, L_{12}, \tilde{W}_{t_1}, \tilde{W}_{t_2}, \{ 0 \} \}.
\]

**Local Gröbner fan in \( h_{(0,1)}(D_{(0)}) \).** First, let us compare \( \text{in}_{F_1}(I')_{|h=1} \) and \( \text{in}_{F_2}(I')_{|h=1} \) in \( \text{gr}_{F_1}(h_{(0,1)}(D_{(0)})) = \{ \ell/g | \ell \in h_{(0,1)}(D), g \in \mathbb{Q}[x, y], g(0) \neq 0 \} \). By the tangent cone algorithm or by examining the basis, we can see that the dehomogenizations of the two initial ideals are the same ideal in \( \text{gr}_{F_1}(h_{(0,1)}(D_{(0)})) = \text{gr}_{F_2}(h_{(0,1)}(D_{(0)})) \).

The local Gröbner fan consists of only one maximal dimensional cone. Hence, we have
\[
\tilde{E}(h_{(0,1)}(D_{(0)}) \cdot I, W) = \{ F_1 \cup F_2, \tilde{W}_{t_1}, \tilde{W}_{t_2}, \{ 0 \} \} = S.
\]

This calculation shows us that the global Gröbner fan \( \tilde{E}(h_{(0,1)}(D) \cdot I, W_{\text{loc}}) \cap S \) is not a polyhedral fan. Because, the local Gröbner fan \( \tilde{E}(h_{(0,1)}(D_{(0)}) \cdot I, W_{\text{loc}}) \) is a polyhedral fan and we have a comparison result Cor. \( 4.2.3 \) which says that maximal dimensional equivalence classes agree.

**Remark.** In the computations above, the local Gröbner fans are trivial: i.e. there is no slope between \( v_1 = (-1, 0, 0, 0, 1, 0, 0) \) and \( v_2 = (0, -1, 0, 0, 0, 1, 0) \). The global Gröbner fans are not trivial: there is the slope \( L_{12} = \langle (1, 1) \rangle = (1 \cdot v_1 + 1 \cdot v_2) \). This may be explained by the following: The only point where the local Bernstein-Sato is not “trivial” (in some sense) is \((1, 0)\) because for any \((x_0, y_0) \neq (1, 0)\), \( f_1 \) or \( f_2 \) becomes a unit in \( \mathcal{O}_{(x_0, y_0)} \). Now in local coordinates \((x', y') \) around \((1, 0)\), \((f_1, f_2)\) is written as \((y', y' - x'^2)\) and the Bernstein-Sato ideal is known to be generated by \((s_1 + 1)(s_2 + 1)(2s_1 + 2s_2 + 3)(2s_1 + 2s_2 + 5)\). We then clearly see the linear form (or the slope) \((1, 1)\).

**Example 5.5.2.** This example tries to show how far we can enumerate Gröbner cones with our implementation. Let \( \ell_k \) be a differential operator
\[
\ell_k = \partial_{x_k} - \left( \sum_{i=1}^{n} x_i \partial_{x_i} + 1/2 \right) (x_k \partial_{x_k} + 1/(2k + 1))
\]
We consider the ideal \( I^h \) in the homogenized Weyl algebra \( h_{(1,1)}(D) \) in \( n \) variables generated by \( h_{(1,1)}(\ell_1), \ldots, h_{(1,1)}(\ell_n) \), which is a system of differential equations for a Lauricella hypergeometric series with irregular singularities when \( h = 1 \).

| \( n \) | Number of maximal dimensional cones in \( \mathbb{R}^{2n} \) |
|---|---|
| 1 | 2 |
| 2 | 39 |
| 3 | 3246 |

**Example 5.5.3.** In our experience, computation in the doubly homogenized Weyl algebra \( h'(D) \) sometimes explodes huge memory space. For example, consider the left ideal generated by
\[
x^3 + y^2 h' - t_1 h'^2, y^3 + x^2 h' - t_2 h'^2, 3x^2 \partial t_1 + 2xh' \partial t_2 + h'^2 \partial x, 3y^2 \partial t_2 + 2yh' \partial t_1 + h'^2 \partial y
\]
in the doubly homogenized Weyl algebra \( \mathbb{Q}(h, h', x, y, t_1, t_2, \partial x, \partial y, \partial t_1, \partial t_2) \). We want to get the Gröbner fan in the restricted weight space
\[
\{(0, 0, -w_1, -w_2, 0, 0, w_1, w_2) | w_i \geq 0 \}, \text{ where } -w_i \text{ stands for } t_i
\]
to compute a Bernstein-Sato polynomial for polynomials \( x^3 + y^2 \) and \( x^2 + y^3 \) with the method given in [Ba05]. However, our implementation exhausts 2G bytes of memory in the stage of constructing the reduced Gröbner basis from a Gröbner basis of Collart-Kalkbrener-Mall’s Gröbner walk.
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Contact the authors, or download from zariski.harvard.edu via anonymous ftp.

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Institut Camille Jordan, UMR CNRS 5208, Université Claude Bernard - Lyon 1, Bât. Jean Braconnier, 43 Boulevard du 11 novembre 1918, 69622 Villeurbanne, France
E-mail address: bahloul@math.univ-lyon1.fr

Department of Mathematics, Faculty of Science, Kobe University, 1-1, Rokkodai, Nada-ku, Kobe 657-8501, Japan
E-mail address: takayama@math.kobe-u.ac.jp