On construction of transmutation operators for perturbed Bessel equations

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Abstract

A representation for the kernel of the transmutation operator relating the perturbed Bessel equation with the unperturbed one is obtained in the form of a functional series with coefficients calculated by a recurrent integration procedure. New properties of the transmutation kernel are established. A new representation of the regular solution of the perturbed Bessel equation is given presenting a remarkable feature of uniform error bound with respect to the spectral parameter for partial sums of the series. A numerical illustration of application to the solution of Dirichlet spectral problems is presented.

1 Introduction

In \cite{1} it was proved that a regular solution of the perturbed Bessel equation

\[-u'' + \left(\frac{l(l + 1)}{x^2} + q(x)\right) u = \omega^2 u, \quad x \in (0, b],\]

(1.1)

where $q \in C[0, b]$, $l \geq -\frac{1}{2}$, for all $\omega \in \mathbb{C}$ can be obtained from a regular solution of the unperturbed Bessel equation

\[-y'' + \frac{l(l + 1)}{x^2} y = \omega^2 y\]

(1.2)

with the aid of a transmutation (transformation) operator in the form of a Volterra integral operator,

\[u(\omega, x) = T[y(\omega, x)] = y(\omega, x) + \int_0^x K(x, t)y(\omega, t)dt.\]

(1.3)

Here the kernel $K$ is $\omega$-independent continuous function with respect to both arguments satisfying the Goursat condition

\[K(x, x) = \frac{1}{2} \int_0^x q(s) ds.\]

(1.4)
A regular solution of (1.2) can be chosen in the form

\[ y(\omega, x) = \sqrt{x}J_{l+\frac{1}{2}}(\omega x) \]

where \( J_\nu \) stands for the Bessel function of the first kind and of order \( \nu \). The transmutation operator (1.3) is a fundamental object in the theory of inverse problems related to (1.1) and has been studied in a number of publications (see, e.g., [2], [3], [4], [5], [6], [7], [8]).

Up to now apart from a successive approximation procedure used in [1], [4] for proving the existence of \( K \) and a series representation proposed in [3] requiring the potential to possess holomorphic extension onto the disk of radius \( 2\sqrt{1+2l} \), no other construction of the transmutation kernel \( K \) has been proposed. In this relation we mention the recent work [9] where \( K \) was approximated by a special system of functions called generalized wave polynomials.

In the present paper we obtain an exact representation of the kernel \( K \) in the form of a functional series whose coefficients are calculated following a simple recurrent integration procedure. The representation has an especially simple and attractive form in the case when \( l \) is a natural number. It revealed some new properties of the kernel \( K \).

The representation is obtained with the aid of a recent result from [10] where a Fourier-Legendre series expansion was derived for a certain kernel \( R \) related to the kernel \( K \). Here with the aid of an Erdelyi-Kober fractional derivative we express \( K \) in terms of \( R \) which leads to a series representation for the kernel \( K \).

The obtained form of the kernel \( K \) is appropriate both for studying the exact solution as well as the properties of the kernel itself, and for numerical applications.

In Section 2 some previous results are recalled and an expression for the kernel \( K \) in terms of \( R \) is derived. In Section 3 a functional series representation for \( K \) is obtained in the case of integer parameter \( l \). Its convergence properties are studied.

In Section 4 the obtained representation of the kernel \( K \) is used for deriving a new representation for regular solution of the perturbed Bessel equation enjoying the uniform (\( \omega \)-independent) approximation property (Theorem 4). Since the regular solution \( u(\omega, x) \) in this representation is the image of \( \sqrt{\omega x}J_{l+1/2}(\omega x) \) under the action of the transmutation operator \( T \), \( u(\omega, x) \) does not decay to zero as \( \omega \to \infty \). Hence partial sums of the representation provide good approximation even for arbitrarily large values of the spectral parameter \( \omega \). This is an advantage in comparison with the representation from [10] derived for a regular solution decaying as \( \omega^{-l-1} \) when \( \omega \to \infty \), making the approximation by partial sums useful for reasonably small values of \( \omega \) only.

In Section 5 a functional series representation for the kernel \( K \) is obtained in the case of a noninteger \( l \). Section 6 contains some numerical illustrations confirming the validity of the presented results.

## 2 A representation of \( K(x, t) \) using an Erdelyi-Kober operator

In this section, with the aid of a result from [10] we obtain a representation of the kernel \( K(x, t) \) in terms of an Erdelyi-Kober fractional derivative applied to a certain Fourier-Legendre series. Two transmutation operators are used, a modified Poisson transmutation operator and the transmutation operator (1.3).

### 2.1 The modified Poisson transmutation operator

The Poisson transmutation operator examined in [11] and [7] is adapted to work with the singular differential Bessel operator \( B_\gamma = \frac{d^2}{dx^2} + \frac{\gamma}{x} \frac{d}{dx} \), where \( \gamma > 0 \). We will use a slightly modified Poisson
operator defined on $C([0, b])$ of the form (see [10] and [9])

$$Y_{l,x}f(x) = \frac{x^{-l}}{2^{l+\frac{1}{2}}\Gamma\left(l + \frac{3}{2}\right)} \int_{0}^{x} (x^2 - s^2)^l f(s)ds, \quad l \geq -\frac{1}{2}. \quad (2.1)$$

The following equality is valid

$$Y_{l,x}[\cos \omega x] = \frac{\sqrt{\pi}\Gamma(l + 1)}{2\omega^{l+1}\Gamma\left(l + \frac{3}{2}\right)} \sqrt{\omega x} J_{l+\frac{1}{2}}(\omega x).$$

Moreover, $Y_{l,x}$ is an intertwining operator for $\frac{d^2}{dx^2}$ and $\frac{d^2}{dx^2} - \frac{l(l+1)}{x^2}$ in the following sense. If $v \in C^2[0, b]$ and $v'(0) = 0$ then

$$Y_{l,x} \frac{d^2v}{dx^2} = \left(\frac{d^2}{dx^2} - \frac{l(l+1)}{x^2}\right) Y_{l,x} v.$$

In particular, the regular solution of equation (1.2) satisfying the asymptotic relations $y_0(x) \sim x^{l+1}$ and $y_0'(x) \sim (l+1)x^l$, when $x \to 0$ can be written in the form

$$y(\omega, x) = \frac{2^{l+\frac{3}{2}}\Gamma\left(l + \frac{3}{2}\right)}{\sqrt{\pi}\Gamma(l + 1)} Y_{l,x}[\cos \omega x] = \Gamma\left(l + \frac{3}{2}\right) \frac{2^{l+\frac{1}{2}}\omega^{-l-\frac{1}{2}}}{\sqrt{x}} J_{l+\frac{1}{2}}(\omega x). \quad (2.2)$$

The composition of two transmutation operators (1.3) and (2.1) allows one to write a regular solution of (1.1) in the form (see [10])

$$u(\omega, x) = \frac{2^{l+\frac{3}{2}}\Gamma\left(l + \frac{3}{2}\right)}{\omega^{l+\frac{3}{2}}} \sqrt{x} J_{l+\frac{1}{2}}(\omega x) + \int_{0}^{x} R(x, t) \cos \omega t dt, \quad (2.3)$$

where the kernel $R(x, t)$ is a sufficiently regular function which admits a convergent Fourier–Legendre series expansion presented in [10]. Namely,

$$R(x, t) = \sum_{k=0}^{\infty} \frac{\beta_k(x)}{x} P_{2k}\left(\frac{t}{x}\right), \quad (2.4)$$

where $P_m(x)$ stands for the Legendre polynomial of order $m$, and the coefficients $\beta_k$ can be computed following a simple recurrent integration procedure [10].

### 2.2 A representation of $K(x, t)$ in terms of $R(x, t)$

**Theorem 1.** Let $q \in C[0, b]$ be a complex-valued function. Then the following equality is valid

$$K(x, t) = \frac{\sqrt{\pi}}{\Gamma\left(l + \frac{3}{2}\right)} \int_{0}^{x} \frac{t^{l+1}}{\Gamma(n-l-1)} \left(-\frac{d}{dt}\right)^n \int_{t}^{x} (s^2 - t^2)^{n-l-2}sR(x, s)ds, \quad (2.5)$$

where $n$ can be arbitrary integer satisfying $n > l + 1$.

**Proof.** The following relation between the kernels $K$ and $R$ was obtained in [10] (3.10) and (3.12)

$$\frac{2\Gamma\left(l + \frac{3}{2}\right)}{\sqrt{\pi}\Gamma(l + 1)} \int_{s}^{x} K(x, t)t^{-l}(t^2 - s^2)^l dt = R(x, s). \quad (2.6)$$

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Let us invert this equality using an inverse Erdelyi-Kober operator. For $\alpha > 0$ the left-sided Erdelyi-Kober operator is (see [12, (18.3)])

$$I^\alpha_{x-2;\eta} f(s) = \frac{2s^{\alpha\eta}}{\Gamma(\alpha)} \int_s^x (t^2 - s^2)^{\alpha-1} t^{2(1-\alpha-\eta)-1} f(t) dt, \quad (2.7)$$

that means we can rewrite (2.6) as

$$\frac{\Gamma \left( l + \frac{d}{2} \right)}{\sqrt{\pi}} t^{l+1} \Gamma(l+1)\Gamma(2l+5) P_m^{(l+1)}(x) P_m^{(l+1)} \left( 1 - 2 \frac{t^2}{x^2} \right), \quad (3.1)$$

where $P_m^{(\alpha,\beta)}$ stands for a Jacobi polynomial and the coefficients $\beta_k$ are those from Subsection 2.1.

The series (3.1) converges absolutely for any $x \in (0, b]$ and $t \in (0, x)$ and converges uniformly with respect to $t$ on any segment $[\varepsilon, x - \varepsilon] \subset (0, x)$. Under the additional assumption that $q \in C^{l+5}[0, b]$ the series (3.1) converges absolutely and uniformly with respect to $t$ on $[0, x]$.

**Proof.** For any nonnegative integer $l$ and $n > l + 1$ we can rewrite formula (2.5) in the form

$$K(x, t) = \frac{(-1)^{l+1} \sqrt{\pi} t^{l+1}}{\Gamma \left( l + \frac{d}{2} \right) \Gamma(n - l - 1)} \int_x^t (t^2 - s^2)^{n-l-2} s R(x, s) ds. \quad (3.2)$$

Choosing $n = l + 2$ we obtain

$$K(x, t) = \frac{(-1)^{l+1} \sqrt{\pi} t^{l+1}}{\Gamma \left( l + \frac{d}{2} \right)} \left( \frac{d}{2tdt} \right)^{l+1} \int_x^t s R(x, s) ds = \frac{(-1)^{l+1} \sqrt{\pi} t^{l+1}}{\Gamma \left( l + \frac{d}{2} \right)} \left( \frac{d}{2tdt} \right)^{l+1} R(x, t). \quad (3.3)$$

Substituting the series expansion (2.4) into (3.3) we get (we left the justification of the possibility of termwise differentiation of (2.4) to the end of the proof)

$$K(x, t) = \frac{(-1)^{l+1} \sqrt{\pi} t^{l+1}}{\Gamma \left( l + \frac{d}{2} \right)} \sum_{k=0}^{\infty} \frac{\beta_k(x)}{x} \left( \frac{d}{2tdt} \right)^{l+1} P_2k \left( \frac{t}{x} \right).$$

Now we would like to calculate the derivatives

$$\left( \frac{d}{2tdt} \right)^{l+1} P_2k \left( \frac{t}{x} \right) = \left( \frac{d}{dt^2} \right)^{l+1} P_2k \left( \frac{t}{x} \right).$$
Note that by formula 8.911 from [13] the following equality is valid
\[ P_{2k} \left( \frac{t}{x} \right) = (-1)^k \frac{(2k-1)!!}{2^k k!} \binom{1}{k} \binom{k + \frac{1}{2}}{l} \binom{l}{2} \binom{l}{2} \frac{t^2}{x^2} \binom{-k, k + \frac{1}{2}; \frac{1}{2}}{2} F_1 \left( -k, k + \frac{1}{2}; \frac{1}{2}, \frac{t^2}{x^2} \right). \]

Now application of formula 15.2.2 from [14],
\[ \frac{d^n}{dz^n} F(a, b; c; z) = \frac{(a)_n(b)_n}{(c)_n} F(a + n, b + n; c + n; z) \]
leads to the relation
\[ \left( \frac{d}{dt^2} \right)^{l+1} P_{2k} \left( \frac{t}{x} \right) = (-1)^k \frac{(\frac{1}{2})_k}{k!} \binom{k + \frac{1}{2}}{l} \binom{l}{2} \binom{l}{2} \frac{t^2}{x^2} \binom{-k, k + \frac{1}{2}; 1/2, \frac{1}{2}, \frac{1}{2}}{2} F_1 \left( l + 1 - k, k + l + \frac{3}{2}; l + 3; \frac{3}{2}, \frac{t^2}{x^2} \right). \]

Taking into account that \((-k)_{l+1} = 0\) when \(l \geq k\) and \((-k)_{l+1} = (-1)^{l+1} \frac{k!}{(k+l)!}\) when \(l < k\), we obtain
\[ \left( \frac{d}{dt^2} \right)^{l+1} P_{2k} \left( \frac{t}{x} \right) = 0 \]
when \(l \geq k\), and
\[ \left( \frac{d}{2t dt} \right)^{l+1} P_{2k} \left( \frac{t}{x} \right) = (-1)^{k+l+1} \frac{(\frac{1}{2})_k}{k!} \binom{k + \frac{1}{2}}{l} \binom{l}{2} \binom{l}{2} \frac{t^2}{x^2} \binom{-k, k + \frac{1}{2}; 1/2, \frac{1}{2}, \frac{1}{2}}{2} F_1 \left( l + 1 - k, k + l + \frac{3}{2}; l + 3; \frac{3}{2}, \frac{t^2}{x^2} \right) \]
when \(l < k\). Thus,
\[ K(x, t) = \frac{\sqrt{\pi} t^{l+1}}{x^{2l+3} \Gamma \left( l + \frac{3}{2} \right)} \sum_{k=l+1}^{\infty} (-1)^k \beta_k(x) \frac{\Gamma \left( k + \frac{3}{2} \right)}{\Gamma \left( k - l \right) \Gamma \left( l + \frac{3}{2} \right)} F_1 \left( l + 1 - k, k + l + \frac{3}{2}; l + 3; \frac{3}{2}, \frac{t^2}{x^2} \right). \]

The series for \(2F_1(a, b; c; z)\) terminates if either \(a\) or \(b\) is a nonpositive integer, in which case the function reduces to a polynomial. In particular, according to [14] formula 15.4.6,
\[ 2F_1 \left( -m, m + \alpha + 1 + \beta; \alpha + 1; x \right) = \frac{m!}{(\alpha + 1)_m} P^{(\alpha, \beta)}_{m}(1 - 2x). \]

Substitution of (3.3) into (3.4) gives (3.1).

Using the asymptotic formula [14] (6.1.40) one can check that
\[ \log \frac{\Gamma(m + 2l + 5/2)}{\Gamma(m + l + 3/2)} = (l + 1) \log(m + l + 1) + O \left( \frac{1}{m + l + 1} \right), \quad m \to \infty. \]

Theorem 7.3.2.2 from [15] states that
\[ \left| P^{(l+1/2, l+1)}_m(z) \right| \leq \frac{C_{\varepsilon}}{\sqrt{m}} \]
uniformly for $z \in [-1 + \varepsilon, 1 - \varepsilon]$. And it was shown in [10] (4.15)) that for $q \in C^1[0, b]$  

$$
|\beta_{m+l+1}(x)| \leq \frac{c_2x^{l+3}}{(m+l)^{l+2}}, \quad m \geq 2.
$$

(3.8)

Combining (3.6), (3.7) and (3.8) we obtain the absolute convergence of the series (3.1) for $x \in (0, b]$ and $t \in (0, x)$, uniform with respect to $t$ on any $[\varepsilon_1, x - \varepsilon_1]$.

For the whole segment $[0, x]$ note that the Jacobi polynomials satisfy  

$$
|P_{m+l+1/2,l+1}^l(z)| \leq Cm^{l+1}, \quad z \in [-1, 1],
$$

see [15] Theorem 7.32.4], while for $q \in C^{2l+5}[0, b]$ the coefficients $\beta_{m+l+1}$ satisfy [10] (4.15))  

$$
|\beta_{m+l+1}(x)| \leq \frac{c_2x^{2l+5}}{(m+l)^{2l+4}}.
$$

(3.10)

Combining (3.6), (3.9) with (3.10) we obtain the uniform convergence with respect to $t$ on the whole segment $[0, x]$.

The possibility of termwise differentiation of the series (2.4) follows directly from the presented results. Namely, the expressions  

$$
\left(\frac{d}{dt}\right)^j R(x,t), \quad j \leq l + 1
$$

lead to series similar to (3.1) but having $j$ instead of $l + 1$, uniformly convergent with respect to $t$ on any segment $[\varepsilon, x - \varepsilon]$. □

**Corollary 1.** Let $q \in C^{2l+5}[0, b]$. Using the formula  

$$
P_n^{(a,b)}(-1) = (-1)^n \binom{n+\beta}{n},
$$

from (3.1) and taking into account (1.4) we find the relation  

$$
\frac{1}{2} \int_0^x q(s) \, ds = \frac{(-1)^{l+1}}{x^{l+2}} \frac{\sqrt{\pi}}{\Gamma(l + \frac{3}{2})} \sum_{m=0}^\infty \beta_{m+l+1}(x) \frac{\Gamma(m+2l+\frac{5}{2})}{\Gamma(m+l+\frac{3}{2})} \binom{m+l+1}{m}.
$$

(3.11)

The representation (3.1) may be substituted termwise in (1.3) under less restrictive convergence assumption than the uniform convergence. Namely, $L_1[0, x]$ convergence with respect to $t$ is sufficient to apply the transmutation operator (1.3) with the integral kernel given by (3.1) to a bounded function. Let us rewrite the formula (3.1) as

$$
K(x,t) = \frac{(-1)^{l+1} \sqrt{\pi}}{x^{2l+3} \Gamma(l + \frac{3}{2})} \sum_{m=0}^\infty \frac{(-1)^m \Gamma(m+2l+\frac{5}{2})}{\Gamma(m+l+\frac{3}{2})} \beta_{m+l+1}(x) \frac{\Gamma(m+2l+\frac{5}{2})}{\Gamma(m+l+\frac{3}{2})} \binom{m+l+1}{m} \frac{1}{2} \int_0^x q(s) \, ds = \frac{(-1)^{l+1}}{x^{l+2}} \frac{\sqrt{\pi}}{\Gamma(l + \frac{3}{2})} \sum_{m=0}^\infty \beta_{m+l+1}(x) \frac{\Gamma(m+2l+\frac{5}{2})}{\Gamma(m+l+\frac{3}{2})} \binom{m+l+1}{m}.
$$

(3.12)

The following proposition states the $L_1$ convergence of the series (3.12) under slightly relaxed requirement on the smoothness of the potential $q$.

**Proposition 1.** Let $l \geq 1$. Under the condition $q \in C^{2l+1}[0, b]$ the series (3.12) converges for every fixed $x$ in $L_1[0, x]$ norm.

**Proof.** Consider  

$$
p_m := \int_0^x \left| P_m^{(l+1/2,l+1)} \left(1 - \frac{2t^2}{x^2}\right) \right| dt = \frac{x^{l+2}}{2l+2} \int_0^{1/2} \left| P_m^{(l+1/2,l+1)}(z) \right| dz
$$

$$
\leq \frac{x^{l+2}}{2l+2} \int_0^{1/2} \left| \frac{1}{\sqrt{1-z}} P_m^{(l+1/2,l+1)}(z) \right| dz + \frac{x^{l+2}}{4} \int_0^{1/4} \left| P_m^{(l+1,l+1/2)}(z) \right| dz.
$$

(3.13)
Theorem 7.34 from [15] states that
\[
\int_0^1 (1 - x)^\mu \left| P_n^{(\alpha, \beta)}(x) \right| \, dx \sim \begin{cases} 
\frac{n^{\alpha - 2\mu - 2}}{n^{-1/2}}, & \text{if } 2\mu < \alpha - \frac{\beta}{2}, \\
\frac{n^{-1/2}}{n^{1/2}}, & \text{if } 2\mu > \alpha - 3/2, \end{cases} \quad n \to \infty,
\]
whenever \(\alpha, \beta, \mu\) are real numbers greater than \(-1\). Using this result to estimate both integrals in (3.13) we obtain that for some constant \(C\) and all \(m \geq 1\)
\[
p_m \leq C x^{l_2} m^{l_1 - 1}.
\]
Hence \(L_1[0, x]\) norms of the functions \(\frac{\Gamma(m+2l+5/2)}{\Gamma(m+l+3/2)} P_m^{l/(l+1)l+1}(1 - 2x^2)\) grow at most as \(m^{2l}, m \to \infty\), and the estimate
\[
|\beta_{m+l+1}(x)| \leq \frac{c_3 x^{2l+3}}{(m+1)^{2l+2}},
\]
valid (see [10] (4.150)) under the condition \(q \in C^{2l+1}[0, b]\), is sufficient to assure the convergence of the series (3.1) in the \(L_1[0, x]\) norm. \(
\square\)

**Remark 1.** The smoothness requirements \(q \in C^{2l+5}[0, b]\) in Theorem 2 and Corollary 1 and \(q \in C^{2l+1}[0, b]\) in Proposition 1 may be excessive. However the minimal requirement \(q \in C^1[0, b]\) may be insufficient in general neither for the representation (3.11) to converge, nor for (3.1) to converge in \(L_1[0, x]\). Indeed, one can easily check that the factor \(\frac{\Gamma(m+2l+5/2)}{\Gamma(m+l+3/2)} \frac{m^{l+1}}{(m+l)^{2l+2}}\) grows as \((m+l+1)^{2l+2}, m \to \infty\) requiring the coefficients \(\beta_m\) to decay faster than \(m^{-2l-2}\) in order to fulfill at least the necessary convergence condition (terms of a series goes to zero as \(m \to \infty\)). Similarly, by slightly changing the reasoning in the proof of Proposition 1 one can see that the numbers \(p_m\) grow as \(m^{-l-1}\), requiring the coefficients \(\beta_m\) to decay faster than \(m^{-2l}\) for the \(L_1\) convergence of the series (3.1).

Numerical experiments similar to those from [10] Section 9.1] suggest that this may not happen. For the potential
\[
q_2(x) = \begin{cases} 
1, & x \in [0, \pi/2], \\
1 + (x - \pi/2)^2, & x \in [\pi/2, \pi], \end{cases}
\]
having its second derivative bounded on \([0, \pi]\), and for \(l = 3\), the observed decay rate of the coefficients \(\beta_m(x)\) was \(C m^{-7.5}, m \to \infty\), insufficient for the convergence of the series (3.11). While for \(l = 5\) the observed decay rate of the coefficients \(\beta_m(x)\) was \(C m^{-9.5}, m \to \infty\), insufficient for the \(L_1\) convergence of the series (3.1).

**Corollary 2.** The integral of the power function multiplied by the kernel \(K(x, t)\) has the form
\[
\int_0^x t^\alpha K(x, t) \, dt = \frac{x^{\alpha-l-1}}{2 \left(\frac{\alpha+1}{2} + 1\right)} \frac{\sqrt{\pi}}{\Gamma \left(\frac{l+3}{2}\right)} \sum_{m=0}^{\infty} (-1)^{m+l+1} \frac{\beta_{m+l+1}(x) \Gamma \left(m+2l+\frac{5}{2}\right)}{m!} \times \frac{\Gamma \left(m+2l+\frac{5}{2}\right)}{(m+l+\frac{3}{2})} \times \frac{\Gamma \left(m+l+\frac{5}{2}\right)}{(m+l+\frac{3}{2})}, \quad \text{Re} \alpha > -l - 2. \quad (3.14)
\]

**Proof.** Due to (3.14), we have
\[
\int_0^x t^\alpha K(x, t) \, dt = \frac{1}{x^{2l+3}} \frac{\sqrt{\pi}}{\Gamma \left(\frac{l+2}{2}\right)} \sum_{m=0}^{\infty} (-1)^{m+l+1} \frac{\beta_{m+l+1}(x) \Gamma \left(m+2l+\frac{5}{2}\right)}{m!} \times \frac{\Gamma \left(m+2l+\frac{5}{2}\right)}{(m+l+\frac{3}{2})} \times \frac{\Gamma \left(m+l+\frac{5}{2}\right)}{(m+l+\frac{3}{2})} \times \int_0^x t^\alpha \beta_{l+1} P_m^{l+1/2, l+1} \left(1 - \frac{t^2}{x^2}\right) \, dt. \quad (3.15)
\]
Consider the integral
\[
\int_0^x t^{\alpha+l+1} P_m^{(l+\frac{1}{2},l+1)} \left( 1 - 2 \frac{t^2}{x^2} \right) dt = 2^{-2-\frac{\alpha+l}{2}} x^{\alpha+l+2} \int_{-1}^1 (1-z)^{\alpha+l} P_m^{(l+\frac{1}{2},l+1)}(z) dz.
\]

Here we use the formula 16.4.3 from [16]
\[
\int_{-1}^1 (1-x)^{\rho}(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x) dx = \frac{2^{\rho+\sigma+1} \Gamma(\rho+1) \Gamma(\sigma+1) \Gamma(n+\alpha+1)}{\Gamma(\rho+\sigma+2) n! \Gamma(\alpha+1)}
\times \, _3F_2\left(-n, \alpha + \beta + n + 1, \rho + 1; \alpha + 1, \rho + \sigma + 2; 1\right),
\]
\[
\text{Re } \rho > -1, \quad \text{Re } \sigma > -1.
\]

In particular, we have
\[
\int_{-1}^1 (1-z)^{\alpha+l} P_m^{(l+\frac{1}{2},l+1)}(z) dz = \frac{2^{\alpha+l+1}}{m! \Gamma(l+\frac{3}{2})} \, _3F_2\left(-m,2l+m+1,\frac{\alpha+l}{2}+1;\frac{3}{2},\frac{\alpha+l}{2}+2;1\right).
\]
(3.16)

Substitution of (3.16) into (3.15) gives us (3.14).

**Remark 2.** In [18] the images of \(x^{2k+l+1}\) under the action of the operator \(T\) were obtained for any \(l \geq -1/2\) and \(k = 0, 1, 2, \ldots\). In the case of integer \(l\) they can be derived from (3.14). For example, one can see using (3.14) together with the equality [10, formula 9.1] \(\sum_{k=0}^{\infty} \beta_k(x) = 0\) that for \(\alpha = 1\) and \(l = 0\) the following relation is valid
\[
\int_0^x t K(x,t) dt = \beta_0(x).
\]

**4 Representation of the regular solution for \(l = 1, 2, \ldots\)**

Substituting (3.12) into (1.3) we obtain that the regular solution of (1.1) has the form
\[
u(\omega, x) = \sqrt{x} J_{l+1/2}(\omega x) + \frac{\sqrt{\pi}}{x^{2l+3} \Gamma(l+3/2)} \sum_{m=0}^{\infty} \frac{(-1)^{m+l+1} \Gamma(m+2l+5/2) \beta_{m+l+1}(x)}{\Gamma(m+l+3/2)} \int_0^x t^{l+3/2} J_{l+1/2}(\omega t) P_m^{(l+1/2,l+1)} \left( 1 - 2 \frac{t^2}{x^2} \right) dt.
\]

Consider the integrals
\[
I_{k,m}(\omega, x) := \int_0^x t^{k+3/2} J_{k+1/2}(\omega t) P_m^{(k+1/2,k+1)} \left( 1 - 2 \frac{t^2}{x^2} \right) dt.
\]

\footnote{Unfortunately, the formulas 7.391.2 from [13] and 2.22.2.8 from [17] for the same integral, as well as the formula 16.4.3 in the English edition of [16] contain mistakes.}
For $m = 0$ the formula 1.8.1.3 from [17] gives
\[ I_{k,0}(\omega, x) = \frac{x^{k+3/2}}{\omega}J_{k+3/2}(\omega x). \] (4.1)

Let $m > 0$. Integrating by parts, taking into account (1.1) and noting that $\frac{d}{dx}P_n^{(\alpha, \beta)}(x) = \frac{1}{2}(n + \alpha + \beta + 1)P_{n-1}^{(\alpha+1, \beta+1)}(x)$ we obtain that
\[ I_{k,m}(\omega, x) = \frac{t^{k+3/2}}{\omega}J_{k+3/2}(\omega t)P_m^{(k+1/2, k+1)} \left( 1 - \frac{t^2}{x^2} \right) \bigg|_{t=0}^x 
- \int_0^x \frac{t^{k+3/2}}{\omega}J_{k+3/2}(\omega t) \cdot \frac{1}{2} (m + 2k + 5/2)P_m^{(k+3/2, k+2)} \left( 1 - \frac{t^2}{x^2} \right) \cdot \left( -\frac{4t}{x^2} \right) dt 
= (-1)^m \left( m + k + 1 \right) I_{k,0}(\omega, x) + \frac{2m + 4k + 5}{\omega x^2} I_{k+1, m-1}(\omega, x). \]

Hence we obtain the following result.

**Theorem 3.** Let $q \in C^{2l+1}(0, b)$. Then the regular solution $u(\omega, x)$ of equation (1.1) satisfying the asymptotics $u(\omega, x) \sim \frac{\sqrt{\omega x}}{2^{l+1/2}\Gamma(l+3/2)}$ when $x \to 0$ has the form
\[ u(\omega, x) = \sqrt{\omega x}J_{l+1/2}(\omega x) + \frac{\sqrt{\pi \omega}}{x^{2l+3}\Gamma(l+3/2)} \sum_{m=0}^{\infty} \frac{(-1)^{m+l+1}\Gamma(m+2l+5/2)}{\Gamma(m+l+3/2)} \beta_{m+l+1}(x) I_{l,m}(\omega, x), \] (4.2)

where the coefficients $\beta_k$ are those from (2.4) and the functions $I_{l,m}$ are given by the following recurrent relations
\[ I_{k,0}(\omega, x) = \frac{x^{k+3/2}}{\omega}J_{k+3/2}(\omega x), \quad k = l, l+1, \ldots, \] (4.3)
\[ I_{k,m}(\omega, x) = (-1)^m \left( k + m + 1 \right) I_{k,0}(\omega, x) + \frac{2m + 4k + 5}{\omega x^2} I_{k+1, m-1}(\omega, x), \quad m \in \mathbb{N}. \] (4.4)

Since the representation (1.2) is obtained using the transmutation operator whose kernel is $\omega$-independent, partial sums of the series (4.2) satisfy the following uniform approximation property. Let
\[ K_N(x, t) = \frac{\sqrt{\pi t^{l+1}}}{x^{2l+3}\Gamma(l+3/2)} \sum_{m=0}^{N} \frac{(-1)^{m+l+1}\Gamma(m+2l+5/2)}{\Gamma(m+l+3/2)} \beta_{m+l+1}(x) P_{l+1/2}(t) \left( 1 - \frac{t^2}{x^2} \right), \] (4.5)
and
\[ u_N(\omega, x) = \sqrt{\omega x}J_{l+1/2}(\omega x) + \frac{\sqrt{\pi \omega}}{x^{2l+3}\Gamma(l+3/2)} \sum_{m=0}^{N} \frac{(-1)^{m+l+1}\Gamma(m+2l+5/2)}{\Gamma(m+l+3/2)} \beta_{m+l+1}(x) I_{l,m}(\omega, x). \] (4.6)

Due to the $L_1$ convergence of the series (3.1) with respect to $t$, for each $x \in (0, b]$ there exists
\[ \varepsilon_N(x) = \| K(x, \cdot) - K_N(x, \cdot) \|_{L_1[0, x]} \] (4.7)
and $\varepsilon_N(x) \to 0$ as $N \to \infty$.

Due to the asymptotic expansion of the Bessel function for large arguments [14, 9.2.1] there exists a constant $c_l$ such that
\[ |\sqrt{z}J_{l+1/2}(z)| \leq c_l, \quad z \in \mathbb{R}. \] (4.8)
**Theorem 4** ((Uniform approximation property)). Under the conditions of Proposition 4 the following estimate holds

\[ |u(\omega, x) - u_N(\omega, x)| \leq c_l \varepsilon_N(x) \]  

(4.9)

for any \( \omega \in \mathbb{R} \).

**Proof.** Since the functions \( u(\omega, x) \) and \( u_N(\omega, x) \) are the images of the same function \( \sqrt{\omega x} J_{l+1/2}(\omega x) \) under the action of the integral operators of the form (1.3), one with the integral kernel \( K \) and second with the integral kernel \( K_N \), we have

\[ |u(\omega, x) - u_N(\omega, x)| \leq \int_0^x |K(x, t) - K_N(x, t)| \cdot |\sqrt{\omega t} J_{l+1/2}(\omega t)| \, dt \leq c_l \varepsilon_N(x), \]

where we used (4.7) and (4.8).

**Remark 3.** Another uniform approximation property was proved in [10] for the regular solution \( \tilde{u}(\omega, x) \) of equation (1.1) satisfying the asymptotics \( \tilde{u}(\omega, x) \sim x^{l+1}, x \to 0 \). Namely, an estimate of the form

\[ |\tilde{u}(\omega, x) - \tilde{u}_N(\omega, x)| \leq \sqrt{x} \varepsilon_N(x) \]

independent on \( \omega \in \mathbb{R} \) was proven. The estimate provided by Theorem 4 is better for large values of \( \omega \) due to the following. Since

\[ \tilde{u}(\omega, x) = \frac{2^{l+1/2} \Gamma(l + 3/2)}{\omega^{l+1}} u(\omega, x), \]

and for each fixed \( x \) the function \( u(\omega, x) \) remains bounded as \( \omega \to \infty \), the function \( \tilde{u}(\omega, x) \) decays at least as \( \omega^{-l-1} \) as \( \omega \to \infty \), meaning that the uniform error estimate is useful only in some neighbourhood of \( \omega = 0 \). Meanwhile the estimate (4.9) remains useful even for large values of \( \omega \).

5 The case of a noninteger \( l \)

**Theorem 5.** Let \( q \in C^1[0, b] \). For \( l \geq -1/2 \) the following formula for the kernel \( K \) is valid

\[ K(x, t) = \frac{\sqrt{\pi}}{\Gamma(l + 3/2)} \frac{t^{l+1}}{x(x^2 - t^2)^{l+1}} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{\Gamma(k - l)} \beta_k(x) P_k^{(l+1/2, -l-1)} \left( 1 - \frac{2t^2}{x^2} \right), \]

(5.1)

where \( P_k^{(l+1/2, -l-1)} \) are polynomials given by the same formulas as the classical Jacobi polynomials [15, 4.22], [14, 22.3].

The series in (5.1) converges absolutely for any \( x \in (0, b) \) and \( t \in (0, x) \), uniformly with respect to \( t \) on any segment \( t \in [c, x] \). Under the additional assumption that \( q \in C^{2[l+1]}[0, b] \), where \([l]\) denotes the largest integer not exceeding \( l \), the convergence is uniform with respect to \( t \) on \( [0, x] \).

**Proof.** Let \( l = [l] + \{l\} \), \( p := [l] \), \( p = -1, 0, 1, \ldots \) and \( \lambda := \{l\}, 0 \leq \lambda < 1 \).

Taking in formula (2.5) \( n = p + 2 \) we obtain

\[ K(x, t) = \frac{\sqrt{\pi}}{\Gamma(l + 3/2)} \frac{t^{l+1}}{\Gamma(1 - \lambda)} \left( -\frac{d}{2tdt} \right)^{p+2} \int_t^x (s^2 - t^2)^{-\lambda} s R(x, s) \, ds. \]

(5.2)
Substituting the series expansion

\[ R(x, t) = \sum_{k=0}^{\infty} \frac{\beta_k(x)}{x} P_{2k} \left( \frac{t}{x} \right) \]

into (5.2) we get (the possibility to differentiate termwise follows similarly to the proof of Theorem 2)

\[ K(x, t) = \frac{2\sqrt{\pi}}{\Gamma \left( l + \frac{1}{2} \right) \Gamma(1 - \lambda)} \sum_{k=0}^{\infty} \frac{\beta_k(x)}{x} \left( -\frac{d}{2t dt} \right)^{p+2} \int_{t}^{x} (s^2 - t^2)^{-\lambda} s P_{2k} \left( \frac{s}{x} \right) ds. \tag{5.3} \]

Consider the integral

\[ I_k := \int_{t}^{x} (s^2 - t^2)^{-\lambda} s P_{2k} \left( \frac{s}{x} \right) ds. \]

Using formula 2.17.2.9 from [17] and noting that \( B(1 - \lambda, \lambda - 1) = 0 \) we obtain that

\[ I_k = \frac{(-1)^k (\lambda - 1/2)_k x^{2 - 2\lambda}}{2(1 - \lambda)_{k+1}} 2F_1 \left( \lambda - 1 - k, \lambda - \frac{1}{2} + k; \lambda - \frac{1}{2}; \frac{t^2}{x^2} \right). \]

Proceeding as in the proof of Theorem 2 we get

\[ \left( -\frac{d}{2t dt} \right)^{p+2} I_k = \frac{(-1)^{k+p+2} (\lambda - 1/2)_k x^{2 - 2\lambda} (\lambda - 1 - k)_{p+2} (\lambda - 1/2 + k)_{p+2}}{2(1 - \lambda)_{p+2} x^{2(p+2)}} \times 2F_1 \left( \lambda - k + p + 1, \lambda + k + p + 3/2; \lambda + p + 3 \frac{t^2}{x^2} \right). \]

Noting that \( \lambda + p = l \) and using \( (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \) we obtain

\[ \left( -\frac{d}{2t dt} \right)^{p+2} I_k = \frac{(-1)^{k+p+2} \Gamma(l+1-k) \Gamma(l+3/2+k) \Gamma(1-\lambda)}{2 \Gamma(2+k-l) \Gamma(\lambda-1-k) \Gamma(3/2) \Gamma^{2l+2}} \times 2F_1 \left( l + 1 - k, l + 3/2 + k; l + 3 \frac{t^2}{x^2} \right). \]

Using the reflection formula \( \Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z} \) we see that

\[ \frac{\Gamma(l+1-k)}{\Gamma(2+k-l) \Gamma(\lambda-1-k)} = \frac{\sin \pi (\lambda-1-k)}{\sin \pi (l+1-k) \Gamma(k-l)} = \frac{(-1)^{l-k} \sin \pi (\lambda+p) \Gamma(k-l)}{(-1)^{l-k} \sin \pi (\lambda+p) \Gamma(k-l)} = \frac{(-1)^p}{\Gamma(k-l)}, \]

and hence from the formula 15.3.3 from [14] we get

\[ \left( -\frac{d}{2t dt} \right)^{p+2} I_k = \frac{(-1)^{k+1} \Gamma(l+3/2+k) \Gamma(1-\lambda)}{2 \Gamma(k-l) \Gamma(l+3/2) x^{2l+2}} \times 2F_1 \left( k + \frac{1}{2}, -k; l + 3 \frac{t^2}{x^2} \right) \]

\[ = \frac{(-1)^k \Gamma(l+3/2+k) \Gamma(1-\lambda)}{2 \Gamma(k-l) \Gamma(l+3/2) (x^2 - t^2)^{l+1}} 2F_1 \left( -k, k + \frac{1}{2}; l + 3 \frac{t^2}{x^2} \right) \]

\[ = \frac{(-1)^k (1-\lambda) k! P_{k}^{(l+1/2,-l-1)} \Gamma(l+1/2) \Gamma(l+3/2) \Gamma(1-\lambda)}{2 \Gamma(k-l) (x^2 - t^2)^{l+1}} \times 2F_1 \left( -k, k + \frac{1}{2}; l + 3 \frac{t^2}{x^2} \right), \tag{5.4} \]
where \( P_{n}^{(\alpha,\beta)} \) stands for the Jacobi polynomials (see [14, 15.4.6]), however note that due to the second parameter equal to \(-l-1\) the polynomials \( P_{k}^{(l+1/2, -l-1)} \) are not classical orthogonal polynomials, even though they are given by the same formulas and satisfy the same recurrence relations, see [15, 4.22] for additional details.

Combining (5.3) with (5.4) finishes the proof of (5.1). Convergence of the series can be obtained similarly to the proof of Theorem 2 with the only difference that for \( q \in C^{2[l]+5}[0, b] \) the coefficients \( \beta_k \) satisfy
\[
|\beta_k(x)| \leq \frac{c_{2l+2}}{(k+1)!(x+1)^{l+1}}, \text{ see } [10, (4.17)].
\]

**Remark 4.** The representation (5.1) is also valid for integer values of \( l \) (and in such case can be simplified to those of (3.1)). Indeed, for integer \( l \geq 0 \) the terms having \( k \leq l \) in (5.1) are equal to zero due for the factor \( \Gamma(k-l) \). While for \( k \geq l+1 \) one obtains using the formula 4.22.2 from [15]
\[
P_{k}^{(l+1/2, -l-1)}(x) = (-1)^{k} P_{l}^{(l-1,l+1/2)}(-x)
\]
\[
= \frac{(-1)^{k}}{(l+1)} \binom{k + l + 1/2}{l + 1} \left(\frac{x - 1}{2}\right)^{l+1} P_{l}^{(l+1/2, l+1)}(-x)
\]
\[
= \frac{\Gamma(k + l + 3/2)(k - l - 1)!}{\Gamma(k + 1/2)k!} \left(\frac{x + 1}{2}\right)^{l+1} P_{l}^{(l+1/2, l+1)}(x).
\]
(5.5)

Applying (5.5) in (5.1) one easily arrives at (3.1).

### 6 Numerical illustration

#### 6.1 Integer \( l \): a spectral problem

The approximate solution (4.6) can be used for numerical solution of the Dirichlet spectral problem for equation (1.1), i.e., for finding those \( \omega \) for which there exists a regular solution of (1.1) satisfying
\[
u(\omega, b) = 0.
\]
The uniform approximation property (4.9) leads to a uniform error bound for both lower and higher index eigenvalues. An algorithm is straightforward: one computes coefficients \( \beta_k \), chooses \( N \) as an index where the values \( |\beta_k(b)| \) cease to decay due to machine precision limitation and looks for zeros of the analytic function \( F(\omega) := u_N(\omega, b) \). We refer the reader to [10] for implementation details regarding the computation of \( \beta_k \). We want to emphasize that the presented numerical results are only “proof of concept” and are not aimed to compete with the best existing software packages.

Consider the following spectral problem
\[
-u'' + \left(\frac{l(l + 1)}{x^2} + x^2\right) u = \omega^2 u, \quad 0 \leq x \leq \pi,
\]
(6.1)
\[
u(\omega, \pi) = 0.
\]
(6.2)
The regular solution of equation (6.1) can be written as
\[
u(\omega, x) = x^{l+1} e^{x^2/2} \, _1F_1\left(\frac{\omega^2 + 2l + 3}{4}; \frac{3}{2}; -x^2\right)
\]
allowing one to compute with any precision arbitrary sets of eigenvalues using, e.g., Wolfram Mathematica. We compare the results provided by the proposed algorithm to those of [9] and
where other methods based on the transmutation operators were implemented. The following values of \( l \) were considered: 1, 2, 5 and 10. All the computations were performed in machine precision using Matlab 2012. For each value of \( l \) we computed 200 approximate eigenvalues. In Table 1 we show absolute errors of some eigenvalues for \( l = 1 \). For \( l > 1 \) the analytic approximation proposed in [9] produced considerably worse results, for that reason we compared the approximate results only with those from [10]. We present the results on Figure 1.

| \( n \) | \( \omega_n \) (Exact) | \( \Delta \omega_n \) (1.6) | \( \Delta \omega_n \) (10) | \( \Delta \omega_n \) (9) |
|-------|-----------------|-----------------|-----------------|-----------------|
| 1     | 2.24366651120741 | 2.2 \times 10^{-12} | 1.1 \times 10^{-14} | 3.6 \times 10^{-6} |
| 2     | 3.0903600792814 | 7.4 \times 10^{-12} | 5.1 \times 10^{-14} | 4.1 \times 10^{-6} |
| 5     | 5.7818870721372 | 1.1 \times 10^{-11} | 7.6 \times 10^{-14} | 2.1 \times 10^{-5} |
| 10    | 10.647259934013  | 4.7 \times 10^{-12} | 1.5 \times 10^{-12} | 9.9 \times 10^{-6} |
| 20    | 20.5753329357456 | 1.2 \times 10^{-13} | 1.2 \times 10^{-12} | 3.1 \times 10^{-6} |
| 50    | 50.5305689586825 | 9.7 \times 10^{-12} | 1.7 \times 10^{-12} | 3.7 \times 10^{-6} |
| 100   | 100.51535963269  | 6.5 \times 10^{-12} | 1.0 \times 10^{-11} | 3.4 \times 10^{-7} |
| 200   | 200.507698855317  | 3.5 \times 10^{-12} | 2.9 \times 10^{-11} | 2.4 \times 10^{-7} |

Table 1: The eigenvalues for the spectral problem (6.1), (6.2) for \( l = 1 \) compared to those produced by the approximation (1.6) with \( N = 13 \) and to those reported in [10] and [9]. \( \Delta \omega_n \) denotes the absolute error of the computed eigenvalue \( \omega_n \).

The obtained results confirm Remark 3: the proposed method outperforms the method from [10] for large index eigenvalues. Uniform (and even decaying) absolute error of approximate eigenvalues can be appreciated. The loss of accuracy for large values of \( l \) can be easily explained from the representation (1.2). Recall that due to recurrent formulas used for computation of the coefficients \( \beta_m \) (see [10] (6.11)) the absolute errors of the computed coefficients \( \beta_m \) are slowly growing as \( m \to \infty \). The coefficients \( \Gamma(m+2l+1/2)/\Gamma(m+l+3/2) \) also grow as \( m \to \infty \). For \( l = 5 \) the first coefficient is about \( 4.8 \cdot 10^5 \), while for \( l = 10 \) the first coefficient is about \( 2 \cdot 10^{13} \), which explains the loss of accuracy.

### 6.2 Non-integer \( l \): approximate integral kernel

We illustrate the representation (5.1) constructing numerically the integral kernel \( K \). Unfortunately we are not aware of any single nontrivial potential \( q \) for which the integral kernel \( K \) is known in a closed form. In [9] an analytic approximation was proposed and revealed excellent numerical performance for the potential \( q(x) = x^2, x \in [0, \pi] \) and for \( l = -0.5 \) or \( l = 0.5 \) (the Goursat data (1.1) was satisfied with an error less than \( 10^{-12} \) and a large set of eigenvalues was calculated with absolute errors smaller than \( 10^{-11} \)). It is worth to mention that for other values of \( l \), say 1/3 or 3/2, and for other potentials, the performance of the method from [9] was considerably worse. We consider the same potential and the same values of \( l \) to illustrate the numerical behavior of the representation (5.1), using the approximate kernel \( K \) obtained with the method from [9] in the role of an exact one for all the comparisons.

It is known [10] Proposition 4.5] that for non-integer values of \( l \) the coefficients \( \beta_k \) decay as \( k^{-2l-3} \) when \( k \to \infty \), hence the series in (5.1) converges rather slow. For that reason we computed the coefficients \( \beta_k \) for \( k \leq 150 \). The necessary values of the Jacobi polynomials \( P_k^{l+1/2, -l-1} \) were calculated using the recurrent formula (4.5.1) from [15]. It is worth to mention that the whole computation took few seconds. On Figure 2 we present the kernel \( K \) for \( l = 0.5 \). On Figure 3 we show the absolute error, the value of the difference \( |K(\pi, t) - K_{150}(\pi, t)| \), \( t \in [0, \pi] \) for \( l = -0.5 \) and \( l = 0.5 \). The growth of the error as \( t \to \pi \) can be explained by the division over \((x^2 - t^2)^{l+1} \).
Figure 1: Absolute errors of the first 200 eigenvalues for the spectral problem (6.1), (6.2) for different values of \( l \) obtained using the proposed method (solid lines, marked as TO on the legends) and those produced by the method from [10] (dashed lines, marked as NSBF on the legends). The parameter \( N \) over each figure corresponds to the truncation parameter used for the approximate solution (4.6).

Nevertheless, a remarkable accuracy can be appreciated. The obtained approximation may be used, in particular, for solution of spectral problems, we leave the detailed analysis for a separate study.

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Figure 2: The integral kernel $K$ of the transmutation operator for equation (6.1) with $l = 0.5$.

Figure 3: Absolute error of the approximate integral kernel for equation (6.1) at $x = \pi$ obtained from the truncated representation (5.1) taking coefficients up to $k = 150$, i.e., the value $|K(\pi, t) - K_{150}(\pi, t)|$, $t \in [0, \pi)$. Upper (blue) line corresponds to $l = -0.5$, lower (red) line corresponds to $l = 0.5$.

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