Weinberg’s approach to nucleon-nucleon scattering revisited

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Abstract

We propose a new, renormalizable approach to nucleon-nucleon scattering in chiral effective field theory based on the manifestly Lorentz invariant form of the effective Lagrangian without employing the heavy-baryon expansion. For the pion-less case and for the formulation based on perturbative pions, the new approach reproduces the known results obtained by Kaplan, Savage and Wise. Contrary to the standard formulation utilizing the heavy-baryon expansion, the nonperturbatively resummed one-pion exchange potential can be renormalized by absorbing all ultraviolet divergences into the leading S-wave contact interactions. We explain in detail the differences to the heavy-baryon formulation and present numerical results for two-nucleon phase shifts at leading order in the low-momentum expansion.

1 Introduction

The last two decades have witnessed a renewed interest in the nuclear force problem and nuclear physics thanks to the development and application of effective field theory (EFT) methods. Much of this research has been influenced by the seminal work of Weinberg [1] who was the first to apply chiral perturbation theory (ChPT) to nucleon-nucleon scattering. Using the heavy baryon (HB) formulation of ChPT, he showed that reducible time-ordered nucleon-nucleon (NN) diagrams yield enhanced contributions to the scattering amplitude as compared to naive dimensional analysis. The enhancement can be traced back to the appearance of pinch singularities emerging from the two-nucleon intermediate states. The enhanced contributions can be most easily and efficiently resummed by solving the Lippmann-Schwinger (LS) equation. The description of low-energy nucleon dynamics, therefore, naturally reduces to the conventional, quantum mechanical 4N-body problem where the nuclear forces are defined as a kernel of the corresponding dynamical equation and can be derived order-by-order in ChPT.
Starting from the pioneering work of Ref. [2], this approach has developed rapidly over the last decades and is nowadays widely employed in studies of low-energy few- and many-nucleon dynamics and nuclear structure calculations, see [3,4,5] for recent review articles. While offering many attractive features such as simplicity and the ability to use well-developed machinery to treat few- and many-body dynamics, Weinberg’s approach suffers from being rather intransparent with regard to renormalization. One issue is related to the fact that iterations of the truncated NN potential within the LS equation generate contributions to the amplitude beyond the order one is working. These higher-order terms generally involve ultraviolet (UV) divergencies which cannot be absorbed by counter terms (contact interactions) included in the truncated potential so that one needs to employ a finite UV cutoff $\Lambda$ of the order of a natural hard scale, say $\Lambda \sim \Lambda_\chi \sim M_\rho [6]$. While subleading and higher-order corrections to the potential do not have to be resummed in Weinberg’s power counting scheme and can be treated perturbatively, the LS equation for the leading-order (LO) potential already turns out to be not renormalizable (in the usual sense). In particular, infinitely many counter terms are needed to absorb UV divergences emerging from iterations of the LO long-range potential due to one-pion exchange (OPE), whose singular $1/r^3$-piece generates UV divergencies in all spin-triplet partial waves. This problem, in fact, shows up in every spin-triplet partial wave. To be specific, consider the lowest-order potential in Weinberg’s approach,

$$V^{LO} = -\frac{g_A^2}{4F^2_{\pi}} \tau_1 \cdot \tau_2 \frac{\vec{\sigma}_1 \cdot \vec{q} \cdot \vec{\sigma}_2 \cdot \vec{q}}{q^2 + M^2_{\pi}} + C_S + C_T \vec{\sigma}_1 \cdot \vec{\sigma}_2, \quad (1)$$

where $g_A$, $F_\pi$ and $M_\pi$ are the nucleon axial-vector coupling, pion decay constant and the pion mass, respectively, $\vec{\sigma}_i$ ($\tau_i$) denote the spin (isospin) Pauli matrices of the $i$-th nucleon and $\vec{q} = \vec{p}' - \vec{p}$ is the nucleon momentum transfer. It is easy to see by dimensional arguments that the $2n$-th iteration of this potential in the LS equation generates a logarithmic divergence $\propto (Qm)^{2n} [7]$, where $m$ is the nucleon mass and $Q$ denotes the generic soft scale corresponding to external three-momenta of the nucleons and $M_\pi$. In the $^1S_0$ channel, where the singular tensor part of the OPE potential vanishes, the coefficients in front of the logarithmic divergences do not involve external momenta and can be absorbed by derivative-less contact operators with multiple insertions of $M^2_\pi$. The potentially enhanced contributions of these higher-order $M^2_\pi$-dependent operators might become an issue if one is interested in the quark mass dependence of nucleon-nucleon scattering but do not affect the predictive power of the theory in terms of describing the energy dependence of the phase shift at the physical values of the quark masses. On the other hand, in spin-triplet channels, the coefficients of the logarithmic divergences do involve powers of external momenta. Their removal requires the inclusion of an infinite number of higher-derivative contact interactions. For example, calculating the diagrams of Fig. 1 in dimensional regularization with $n$ spatial dimensions one finds

1 If cutoff regularization is employed, which is normally the case in calculations with non-perturbative pions, one also needs to keep track of power-like divergences. The coefficients in front of power-like divergences generally also involve powers of external momenta.
the divergent parts proportional to \( m^2 M_\pi^2/(n-3) \) for graph (a) and, among many other divergent terms, \( m^6 (\vec{p}^6 + \vec{p}''^6)/(n-3) \) for graph (b) \cite{7}. The appearance of such divergences seems to indicate that the same enhancement, which is responsible for non-perturbativeness of the OPE potential \cite{2}, also applies to higher-order short-range operators. This feature is sometimes referred to as inconsistency of Weinberg’s approach. It should, however, be understood that this issue does not affect the predictive power of Weinberg’s approach with regard to describing the energy dependence of the scattering amplitude provided a suitably chosen finite cutoff is employed along the lines of Ref. \cite{6}. The predictive power of such a framework can be understood in terms of the modified effective range expansion \cite{8,9} and relies on the knowledge of the long-range tale of the interaction driven by the pion exchange. We also emphasize that there is no consensus on the relevance of the inconsistency issue for renormalized contributions to the scattering amplitude, see e.g. Ref. \cite{10}.

The possibility to remove the UV cutoff \( \Lambda \) from the LS equation by enforcing the limit \( \Lambda \to \infty \) (or, equivalently, \( \Lambda \gg \Lambda_\chi \)) non-perturbatively was also explored by several authors, see e.g. \cite{11}. It is possible to obtain a finite, manifestly non-perturbative solution of the LS equation with a singular \( 1/r^3 \)-potential by including one/no contact operator in each attractive/repulsive channel \cite{11}. Unless all UV divergences emerging from iterations of the LS equation are absorbed by counter terms, such a procedure is expected to violate the low-energy theorems and is incompatible with the principles of EFT \cite{12}. Presently, there is still no consensus in the community on the most consistent and efficient way to organize the chiral expansion in the few-nucleon sector, see \cite{6,11,12,13,14,15,16,17,18,19,20} for samples of different views and formulations.

Treating the exchange of pions perturbatively as suggested by Kaplan, Savage and Wise (KSW) \cite{21} obviously allows one to avoid the above-mentioned inconsistency and, at the same time, provides a transparent and analytically solvable EFT framework for nucleon-nucleon scattering. In the KSW approach, the OPE potential is shifted to next-to-leading order (NLO). The LO NN amplitude, therefore, becomes renormalizable perturbatively as well as non-perturbatively. Corrections beyond LO are treated as perturbations. While the KSW scheme is free of any inconsistencies with respect to renormalization, it was shown not to converge in low spin-triplet

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\footnote{One way to justify the need to resum the OPE and the LO contact operators is to treat the nucleon mass as a separate scale in the problem, which is assumed to be much larger than the chiral symmetry breaking scale \cite{11}.}
partial waves \cite{22,23,24}, see however Ref. \cite{25} for a modified formulation.

In this paper we identify the origin of non-renormalizability of the LO NN amplitude in Weinberg’s approach with the non-relativistic expansion of the nucleon propagators. It is by now well established in the single-baryon sector of ChPT that the chiral power counting can be maintained without relying on the non-relativistic or HB expansion, see Ref. \cite{26} and references therein. We propose here a formulation of chiral EFT for NN scattering with non-perturbative pions based on the manifestly Lorentz-invariant form of the effective Lagrangian which is consistent and renormalizable. We will refer to this formulation as the modified Weinberg approach. We demonstrate in the case of perturbative pions how renormalization is carried out in this framework and present cutoff-independent results for NN phase shifts at LO in the modified Weinberg approach. Our paper is organized as follows: In section 2 we describe the framework and specify the dynamical equation for the NN amplitude. Based on the (modified) KSW approach at NLO in the EFT expansion, we demonstrate in section 3 the consistency of this scheme with respect to the power counting. The results for non-perturbative pions at LO in the modified Weinberg approach are presented in section 4. Finally, our findings are summarized in section 5.

2 The framework

To derive the dynamical equation for NN scattering in chiral EFT we follow closely the procedure of Ref. \cite{27} (but refrain from using high-derivative regularization and performing expansion in inverse powers of the nucleon mass \( m \)). We start with the manifestly Lorentz-invariant effective Lagrangian for pions and nucleons. It is organized in a derivative and quark-mass expansion and consists of the purely mesonic, pion-nucleon and nucleon-nucleon parts, whose lowest-order contributions can be found e.g. in Refs. \cite{27,28,29}, see also \cite{26} and references therein. Following Weinberg \cite{30}, we employ time-ordered perturbation theory without performing non-relativistic expansion to calculate NN scattering amplitude. We decompose the numerator of the standard fermion propagator as

\[
\slashed{p} + m = 2 m P_+ + (\slashed{p} - m \slashed{v}),
\]

where \( P_+ \equiv (1 + \slashed{v})/2 \) with \( \slashed{v} = (1, 0, 0, 0) \), and identify the second terms as a higher-order correction (to be included perturbatively). For NN scattering the two-nucleon intermediate states generate enhanced contributions \cite{1}. Therefore, defining the two-nucleon-irreducible diagrams as the effective potential \( V \), the NN off-mass-shell scattering amplitude \( T \) satisfies the integral equation written symbolically as

\[
T = V + V G T,
\]
where $G$ is the two-nucleon propagator. Substituting the expansions of $V$, $G$ and $T$ in a small parameter (pion mass or small momenta) in Eq. (3), the leading-order contribution emerges from solving the equation non-perturbatively

$$T_0 = V_0 + V_0 G_0 T_0,$$

while corrections are calculated perturbatively using the solution to the LO equation.

The physical amplitude is obtained from the off-shell amplitude $T$ via $Z^2 \bar{\psi} \bar{u}_3 \bar{u}_4 T u_1 u_2$ where $Z_\psi$ is the residue of the nucleon propagator and $u_i, \bar{u}_i$ are Dirac spinors. To determine the physical amplitude order-by-order we expand the Dirac spinors in small quantities as

$$u = u_0 + u_1 + u_2 + \cdots, \quad \bar{u} = \bar{u}_0 + \bar{u}_1 + \bar{u}_2 + \cdots,$$

where

$$u_0(p) = P_+ u(p), \quad \bar{u}_0(p) = \bar{u}(p) P_+, \cdots.$$  

Consequently, the lowest-order on-shell amplitude requires the knowledge of the quantity $\tilde{T}_0 = P_+ P_0 P_+ P_+$ which fulfills the integral equation

$$\tilde{T}_0 = \tilde{V}_0 + \tilde{V}_0 G_0 \tilde{T}_0,$$

where $\tilde{V}_0 = P_+ P_+ V_0 P_+ P_+$ is the projected potential. In Weinberg’s approach, the projected lowest-order potential consists of derivative-less contact interactions and the OPE piece. Following Ref. [27], we choose to treat corrections to the static OPE potential perturbatively so that the LO potential takes the form of Eq. (1) after switching to the two-component Pauli spinors. In the center-of-mass frame with incoming (outgoing) three-momentum of the nucleons $\vec{p}(\vec{p}')$, the LO equation takes the form

$$\tilde{T}_0 (\vec{p}', \vec{p}) = \tilde{V}_0 (\vec{p}', \vec{p}) - \frac{m^2}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\tilde{V}_0 (\vec{p}', \vec{k}) \tilde{T}_0 (\vec{k}, \vec{p})}{\sqrt{\vec{k}^2 + m^2} \left( p_0 - \sqrt{\vec{k}^2 + m^2} + i\epsilon \right)} ,$$

where we have suppressed the spin and isospin indices. For the half-off-shell kinematics, $p_0$ is given by $p_0 = \sqrt{\vec{p}'^2 + m^2}$. This equation is, of course, not new and was proposed for the first time in Ref. [31]. This and many other equations of a similar type, all satisfying relativistic elastic unitarity, have been extensively studied in the literature in the context of three-dimensional reductions of the Bethe-Salpeter equation, see e.g. Ref. [32].

It is obvious that Eq. (8) has a milder UV behavior than the LS equation, which emerges from keeping the leading term in the $1/m$-expansion of the integrand. It is perturbatively renormalizable in the sense that all UV divergences generated by its iterations are absorbable into the low-energy constants (LECs) $C_{S,T}$ in Eq. (1). Nonperturbatively, the UV behavior of this equation will be addressed in section.
While all UV divergences emerging from iterations of Eq. (8) can be removed by the \( C_{S,T} \)-counter terms, the resulting \( \Lambda \)-independent expression for the amplitude still violates the power counting due to the explicit appearance of the nucleon mass in the integrand and requires additional, finite renormalization. This can be achieved by subtractions which remove the positive powers of \( m \) and restore the chiral power counting, see Ref. [33] for an extensive discussion. Such additional, finite subtractions in the case at hand only affect the lowest-order contact interactions of the \( C_{S,T} \) type. An explicit example of renormalization will be given in the next section for the case of perturbative pions.

At low energies, the two-nucleon scattering amplitude calculated based on the manifestly Lorentz-invariant effective Lagrangian can, of course, be expanded in inverse powers of the nucleon mass. In the HB approach, the \( 1/m \)-expansion is carried out already at the level of the Lagrangian which implies that one first expands the integrand of Eq. (8) and then calculates the scattering amplitude. This is legitimate in perturbative calculations (apart from some well-known cases such as e.g. the scalar form factor of the nucleon, in which the HB approach fails to reproduce the analytic structure of the relativistic loop diagrams for soft momenta). We will show in the next section that the two ways of doing \( 1/m \)-expansion lead to the same result for the case of perturbative pions in the KSW framework. However, care is required when resumming the pion exchange potential in the Weinberg approach. The leading nonrelativistic approximation of the two-nucleon propagator

\[
\frac{m^2}{2 \left( \vec{k}^2 + m^2 \right) \left( p_0 - \sqrt{\vec{k}^2 + m^2 + i\epsilon} \right)} \rightarrow \frac{m}{\vec{p}^2 - \vec{k}^2 + i\epsilon} \tag{9}
\]

obviously has a more singular UV behavior compared to the one appearing in Eq. (8). As a consequence, in contrast to the approach presented above and corresponding to Eq. (8), an infinite set of counter terms needs to be included in the nonrelativistic framework if one aims at removing the UV cutoff by taking the limit \( \Lambda \rightarrow \infty \).

### 3 Perturbative pions at next-to-leading order

It is instructive to apply Eq. (8) to the case of perturbative pions where all calculations can be carried out analytically. In order to facilitate the comparison with the original work by Kaplan, Savage and Wise based on the HB effective Lagrangian, we adopt here the normalization of the amplitude of Ref. [21]. The expansion for the S-wave scattering amplitude in the KSW approach has the form

\[
\mathcal{A} = \mathcal{A}_{-1} + \mathcal{A}_0 + \mathcal{A}_1 + \cdots, \tag{10}
\]
\[ \mathcal{A}_{-1} = \begin{array}{c} \times + \bigotimes + \cdots \end{array} - \begin{array}{c} \bigotimes + \times \end{array} \]

\[ \mathcal{A}_0 = \begin{array}{c} p^2 \end{array} \begin{array}{c} \mathcal{A}_0^{(I)} \mathcal{A}_0^{(II)} \mathcal{A}_0^{(III)} \mathcal{A}_0^{(IV)} \mathcal{A}_0^{(V)} \end{array} \begin{array}{c} M^2_\pi \end{array} \]

Fig. 2. The leading and subleading contributions to the NN scattering amplitude in the KSW approach. The solid dots denote the lowest-order contact operators and the leading pion-nucleon vertex \( \propto g_A \) while the filled squares refer to the subleading contact terms proportional to \( p^2 \) or \( M^2_\pi \).

where the subscript indicates the power of the soft scale \( Q \). The leading-order contribution \( \mathcal{A}_{-1} \) emerges from resummation of the LO contact interactions as shown in Fig. 2. Using the two-nucleon Green function from Eq. (8), the LO amplitude has the form

\[ \mathcal{A}_{-1} = \frac{-C_0}{1 - C_0 I(p)} , \quad (11) \]

where the dimensionally regularized (DR) integral \( I(p) \) is given in \( n \) spatial dimensions by

\[ I(p) = \frac{m^2}{2} \frac{\mu^{3-n}}{(2\pi)^n} \int \frac{d^nk}{[k^2 + m^2]} \left[ p_0 - \sqrt{k^2 + m^2 + i0^+} \right] \]

\[ = \frac{1}{8\pi^2} \left[ -\left( \bar{\lambda} + 2 - 2 \ln \frac{m}{\mu} \right) m^2 \right. \]

\[ - \left. \frac{m^2}{\sqrt{m^2 + p^2}} \left( \pi m + 2i\pi p - 2p \sinh^{-1} \left( \frac{p}{m} \right) \right) \right] + \mathcal{O}(n-3) , \quad (12) \]

with the divergent quantity \( \bar{\lambda} \) defined as \( \bar{\lambda} \equiv -1/(n-3) - \gamma - \ln(4\pi) \) and \( \mu \) being the scale parameter of DR. Further, the (bare) LEC \( C_0 \) is simply the properly normalized linear combination of \( C_{S,T} \). Here and in what follows, we use the notation \( p \equiv |\vec{p}|, k \equiv |\vec{k}| \). Renormalization of \( \mathcal{A}_{-1} \) is achieved by subtracting the loop integral at \( p^2 = -\nu^2 \) with \( \nu \) chosen to be of order \( \mathcal{O}(Q) \),

\[ I_R(p, \nu) = I(p) - I(i\nu) = -\frac{m(\nu + i\nu)}{4\pi} + \mathcal{O}(p^2, \nu^2) , \quad (13) \]

and replacing \( C_0 \) by \( C_0^R(\nu) \) which yields

\[ \mathcal{A}_{-1} = \frac{-C_0^R(\nu)}{1 - C_0^R(\nu) I_R(p, \nu)} . \quad (14) \]

Notice that while just using DR in combination with \( \overline{\text{MS}} \) would be sufficient to render the expressions finite, one additional finite subtraction would have to be
performed in order to remove from $I_{R}^{\overline{MS}}$ terms of order $\sim m^{2}$ (i.e. $O(Q^{0})$) which violate the power counting. The renormalized expression for $A_{-1}$ clearly agrees with the KSW result of Ref. [21] modulo higher-order terms emerging from the $1/m$-expansion of $I_{R}(p^{2}, \nu^{2})$.

The first corrections to the scattering amplitude are generated by diagrams shown in the second line of Fig. 2. The renormalized contributions of the dressed subleading contact operators have the form

$$A_{0}^{(I)} = A_{-1}^{2} \left[ \frac{C_{R}^{2} m^{2} \left(2m^{2} + p^{2} - 2m\sqrt{m^{2} + p^{2}}\right)}{8\pi C_{0}^{R}} - \frac{2 C_{2}^{R} p^{2}}{(C_{0}^{R})^{2}} \right],$$

$$A_{0}^{(V)} = -\frac{D_{R}^{2} M_{\pi}^{2}}{(C_{0}^{R})^{2}} A_{-1},$$

where $C_{R}^{2} \equiv C_{R}^{2}(\nu)$, $D_{R}^{2} \equiv D_{2}(\nu)$ denote the corresponding renormalized LECs and the two subtraction points in $A_{0}^{(I)}$ are set to zero for the sake of convenience. The details of the calculation will be given in a separate publication. While $A_{0}^{(V)}$ agrees with the HB result of Ref. [21] (modulo higher-order corrections from $I_{R}(p, \nu)$), the HB result for $A_{0}^{(I)}$ is entirely given by the second term in square brackets. Given the scaling of the renormalized LECs $C_{0}^{R} \sim O(Q^{-1})$ and $C_{2}^{R} \sim O(Q^{-2})$ [21], one observes that the first term in the brackets is of order $\sim Q^{3}$ while the second one is of order $\sim Q^{2}$. Both approaches, therefore, again lead to the same result modulo corrections of a higher order.

We now discuss the contributions involving pions. The second diagram in Fig. 2 simply yields the S-wave projected OPE potential,

$$A_{0}^{(II)} = \frac{g_{A}^{2}}{4F_{\pi}^{2}} \left(1 + \frac{M_{\pi}^{2}}{4p^{2}} \ln \frac{M_{\pi}^{2} + 4p^{2}}{M_{\pi}^{2}}\right).$$

Notice that at the order we are working, there is no need to distinguish between the physical and the chiral-limit values of the LECs such as $g_{A}$, $m$ and $F_{\pi}$. The renormalized contribution of the third diagram reads

$$A_{0}^{(III)} = \frac{g_{A}^{2}}{2F_{\pi}^{2}} A_{-1} \left[I_{R}(p, \nu) - M_{\pi}^{2} I_{1\text{loop}}(p)\right],$$

$$I_{1\text{loop}}(p) = \frac{m^{2}}{2} \int \frac{d^{n}k}{(2\pi)^{n}} \frac{1}{[k^{2} + m^{2}] \left[p_{0} - \sqrt{k^{2} + m^{2}} + i\epsilon\right] [(k - p)^{2} + M_{\pi}^{2}]}$$

$$= -\frac{m}{8\pi p} \left[\tan^{-1}\left(\frac{2p}{M_{\pi}}\right) + \frac{i}{2} \ln \frac{M_{\pi}^{2} + 4p^{2}}{M_{\pi}^{2}}\right] + O\left(\frac{p}{m}, \frac{M_{\pi}}{m}\right).$$

Again, this agrees with the HB KSW result modulo terms of a higher order emerging from $1/m$-expansion of $I_{R}(p, \nu)$ and $I_{1\text{loop}}(p)$. Finally and most interestingly,
for the fourth diagram we obtain

$$ A_0^{(IV)} = \frac{g_A^2}{4F_\pi^2} A_{-1}^2 \left[ M_\pi^2 I_{2\text{loop}} - I_R(p, \nu)^2 \right], \quad (18) $$

where the scalar two-loop integral has the form

$$ I_{2\text{loop}} = \frac{m^4}{4} \int \frac{d^nk_1 d^n k_2}{(2\pi)^2} \frac{1}{[k_1^2 + m^2] \left[ p_0 - \sqrt{k_1^2 + m^2} + i\epsilon \right] [k_2^2 + m^2]} \times \frac{1}{\left[ p_0 - \sqrt{k_2^2 + m^2} + i\epsilon \right] [(k_1 - k_2)^2 + M_\pi^2]}$$

$$ = \frac{m^2}{16\pi^2} \left[ \frac{\ln 8}{4} - \frac{2G}{\pi} - \frac{7\zeta(3)}{2\pi^2} - \frac{1}{2} \ln \frac{M_\pi^2 + 4p^2}{m^2} + i \tan^{-1} \left( \frac{2p}{M_\pi} \right) \right] + \ldots \quad (19) $$

where $G \approx 0.916$ is Catalan’s constant and the ellipses refer to terms of order $pm$, $M_\pi m$ and higher. The HB result of Ref. [21] for this diagram in our notation has the following form

$$ A_{0,\text{HB}}^{(IV)} = \frac{g_A^2 m^2}{64\pi^2 F_\pi^2} A_{-1}^2 \left[ M_\pi^2 \left[ -\frac{1}{2} \ln \frac{M_\pi^2 + 4p^2}{\nu^2} + i \tan^{-1} \left( \frac{2p}{M_\pi} \right) + 1 \right] - (\nu + ip)^2 \right]. \quad (20) $$

Clearly, the difference in the constant terms in the square brackets of Eqs. (19) and (20) can be compensated by a finite shift of the LEC $D_2^R$. While all relevant terms non-polynomial in $M_\pi^2$ and $p^2$ are exactly the same in both cases, the HB result features a logarithmic dependence of the renormalization scale which reflects the overall logarithmic divergence of the integral $I_{2\text{loop}}$ when the integrand is approximated by the leading term in the $1/m$-expansion. It is, therefore, necessary to include the $D_2 M_\pi^2$ counter term in the HB approach at the same level as the diagram (a) of Fig. [1] which appears at LO in the Weinberg approach, in order to absorb the corresponding logarithmic divergence. This is, in fact, the first manifestation of the above-mentioned inconsistency issue of the Weinberg approach. In contrast, the original integral $I_{2\text{loop}}$ is finite and fulfills the power counting without any additional subtractions. Consequently, from the renormalization point of view, there is no need to promote the $D_2 M_\pi^2$-term to LO if the OPE potential is treated nonperturbatively at the same footing as the $C_0$-term within the modified Weinberg approach proposed in this paper. The same arguments also apply to more complicated diagrams with higher iterations of the OPE potential.
4 Non-perturbative pions at leading order

We now turn to the case of nonperturbative pions and numerically solve Eq. (8) in the partial wave basis for the LO potential given in Eq. (1). We employ a momentum-space cutoff $\Lambda$ when integrating over $k$ in order to regularize the divergent integrals. As discussed in the previous sections, the LO equation (8) is perturbatively renormalizable so that one can safely remove the cutoff by taking the limit $\Lambda \to \infty$ in any iteration. Nonperturbatively, the UV behavior in Eq. (8) can be understood by approximating the two-nucleon propagator for $k \to \infty$ via

$$\frac{1}{(k^2 + m^2)} \left( p_0 - \sqrt{k^2 + m^2 + i\epsilon} \right) = \frac{p_0 + \sqrt{k^2 + m^2}}{(k^2 + m^2)(p^2 - k^2 + i\epsilon)} \to \frac{1}{k(p^2 - k^2 + i\epsilon)}.$$ (21)

The UV behavior of this equation in the partial-wave decomposed form coincides with the one of the LS equation in $2+1$ space-time dimensions. The OPE part of the potential therefore behaves in coordinate space for $r \to 0$ as $\sim 1/r^2$ in $2+1$ space-time dimensions. It is well known that the LS equation does not possess a unique solution if the strength of the attractive $1/r^2$ potential exceeds some critical value which depends on the partial wave, see [34] for more details. The same sort of non-uniqueness emerges in the context of the Skornyakov–Ter-Martirosyan equation [35,36] which has also been addressed from the EFT point of view [37]. In the case at hand, we found that the non-unique solutions only appear in the $^3P_0$ partial wave. This situation is visualized in Fig. 3 where we compare the dependence of the $^3P_0$ and $^3P_2$ phase shifts on the UV cutoff $\Lambda$ at the fixed energy of $E_{lab} = 100$ MeV.
Fig. 4. Phase shifts calculated at LO in the modified Weinberg approach as functions of laboratory energy in comparison with the Nijmegen [39] (filled circles) and Virginia Tech [40] (open triangles) partial wave analyses. Left panel: S- and P-waves, right panel: D-waves and the mixing angles $\epsilon_{1,2}$.

100 MeV. While the phase shift in the $^3P_2$ channel quickly approaches the $\Lambda \to \infty$ limit, the observed limit-cycle-like behavior of the $^3P_0$ phase shift reflects the non-uniqueness of solution of Eq. (8). While we still let the possibility open to fix the solution from physical principles without the need to rely on the data, see the discussion in Ref. [38], we follow here a more pragmatic approach of Ref. [37]. Specifically, we fix the solution in the $^3P_0$ partial wave by including a counter term of the form $C_{^3P_0} p' p'/\Lambda^2$ and tuning the LEC $C_{^3P_0}$ to the Nijmegen partial wave analysis (PWA). Notice that the residual $\Lambda$-dependence of $C_{^3P_0}$ is of a logarithmic type at any finite order in the loop expansion. Consequently, it is easy to see by dimensional arguments that the iterations of this contact interaction do not require the inclusion of higher-order counter terms. Therefore, the removed-cutoff limit is indeed legitimate from the EFT point of view in this case, contrary to the situation when positive powers of $\Lambda$ appear in momentum-dependent counter terms [12]. A more detailed analysis of this issue will be published elsewhere.

We are now in the position to discuss results for phase shifts. We employ the exact isospin symmetry as appropriate at LO and use the following values for the LECs entering the OPE potential

$$M_\pi = 138 \text{ MeV}, \quad F_\pi = 92.4 \text{ MeV}, \quad g_A = 1.267. \quad (22)$$

The LECs $C_S$, $C_T$ and $C_{^3P_0}$ are fitted to Nijmegen $^1S_0$, $^3S_1$ and $^3P_0$ phase shifts at energies $E_{lab} < 25$ MeV in the limit $\Lambda \to \infty$. The resulting, cutoff-independent predictions for phase shifts in $S$-, $P$- and $D$-waves and the mixing angles $\epsilon_{1,2}$ are visualized in Fig. 4. Given that the calculations are carried out at LO, the agreement with the Nijmegen PWA is rather good. The large deviation for the $^1S_0$ phase shift
In addition to the predicted energy dependence of the phase shifts, the proper inclusion of the pion-exchange physics can be tested in theoretical predictions for the coefficients in the effective range expansion

\[ p^{2l+1} \cot \delta_l(p) = -\frac{1}{a} + \frac{1}{2}rp^2 + v_2p^4 + v_3p^6 + v_4p^8 + \ldots, \]  

(23)

where \(a\), \(r\) and \(v_i\) denote the scattering length, effective range and shape parameters, respectively, and \(l\) is the orbital angular momentum. The energy dependence of the two-particle scattering amplitude near threshold is driven by the long-range tale of the interaction which imposes correlations between the coefficients in the effective range expansion [23]. These correlations are determined by the long-range interaction and may be regarded as low-energy theorems (LETs). In tables 1 and 2 the LETs in the KSW and Weinberg approaches are confronted with the results of the Nijmegen PWA for the \(^1S_0\) and \(^3S_1\) partial waves, respectively. Since in the KSW approach the LO S-wave amplitude does not involve effects due to OPE, one needs to go to at least NLO in order to test the LETs in this framework. The analytic expressions for the S-wave shape parameters at NLO in the KSW scheme can be found in Ref. [23]. Clearly, the modified version of the KSW approach discussed in section 3 yields the same results for \(v_i\) modulo terms of order \(1/m\) and higher. The LETs are known to be strongly violated in the KSW approach [23], see tables 1 and 2. The non-perturbative treatment of the OPE potential leads to an improved description of the LETs in the \(^1S_0\) channel. It is, however, still rather poor at LO which should not come as a surprise given that the long-range part of the OPE potential generates only a small contribution to the \(^1S_0\) phase shift. One may, therefore, expect that the LETs are strongly affected by the two-pion exchange contributions in this partial wave. In the \(^3S_1\) channel, in contrast, the

| \(^1S_0\) partial wave | \(a\) [fm] | \(r\) [fm] | \(v_2\) [fm\(^2\)] | \(v_3\) [fm\(^4\)] | \(v_4\) [fm\(^6\)] |
|------------------------|------------|------------|-----------------|----------------|----------------|
| NLO KSW from Ref. [23] | fit        | fit        | −3.3            | 18             | −108           |
| LO Weinberg            | fit        | 1.50       | −1.9            | 8.6(8)         | −37(10)        |
| Nijmegen PWA           | −23.7      | 2.67       | −0.5            | 4.0            | −20            |

Table 1
Predictions for the coefficients in the effective range expansion of the \(^1S_0\) phase shifts (low-energy theorems) with perturbative and non-perturbative treatment of the OPE potential in comparison with the values from the Nijmegen PWA (extracted using the Nijm II potential [41,42]).
### Table 2

| $^3S_1$ partial wave | $a$ [fm] | $r$ [fm] | $v_2$ [fm$^3$] | $v_3$ [fm$^5$] | $v_4$ [fm$^7$] |
|---------------------|----------|----------|----------------|----------------|----------------|
| NLO KSW from Ref. [23] | fit | fit | $-0.95$ | $4.6$ | $-25$ |
| LO Weinberg | fit | $1.60$ | $-0.05$ | $0.8(1)$ | $-4(1)$ |
| Nijmegen PWA | $5.42$ | $1.75$ | $0.04$ | $0.67$ | $-4.0$ |

Predictions for the coefficients in the effective range expansion of the $^3S_1$ phase shifts (low-energy theorems) with perturbative and non-perturbative treatment of the OPE potential in comparison with the values from the Nijmegen PWA [43].

LETs are well reproduced at LO in the Weinberg approach. The discrepancy for $v_2$ in the $^3S_1$ channel should not be taken too seriously given the very small value of this coefficient. We further emphasize that the errors quoted for $v_3,4$ refer to the estimated uncertainty of our numerical extraction of these parameters from the phase shifts.

### 5 Summary and conclusions

In this paper we applied the manifestly Lorentz-invariant form of the effective Lagrangian to the problem of nucleon-nucleon scattering without relying on the heavy-baryon expansion. The LO contribution to the scattering amplitude in the resulting modified Weinberg approach can be obtained by solving the LS-type of integral equation (8) with the kernel given by the OPE potential and derivative-less contact interactions. Contrary to its nonrelativistic counterpart, this equation is renormalizable, i.e. all UV divergences generated by its iterations can be absorbed by redefinition of the two LO contact interactions. The explicit appearance of the nucleon mass in the propagators, however, makes it necessary to perform additional, finite subtractions in order to restore the proper scaling of the renormalized contributions in accordance with the power counting. Such additional subtractions only affect the values of the LECs accompanying the LO contact interactions. Consequently, the LO equation is renormalizable and consistent in the EFT sense.

In the case of perturbative pions, the new approach is shown to reproduce the well-known results of the HB KSW framework modulo terms of a higher order in the $1/m$-expansion. When pions are treated non-perturbatively as suggested in the Weinberg scheme, the formulation we propose, being renormalizable, offers the appealing possibility to remove the UV cutoff in the way compatible with the principles of EFT. We have analyzed two-nucleon scattering at LO in the modified Weinberg approach. We found that the integral equation does not possess a unique solution in the $^3P_0$ partial wave similarly to the Skornyakov–Ter-Martirosyan equation for spin-doublet nucleon-deuteron scattering. One possible way to fix the solution
in this channel is to include the corresponding contact interaction whose strength is tuned to reproduce the low-energy data \[37\]. The obtained cutoff-independent results for phase shifts at LO in the modified Weinberg scheme are in a reasonably good agreement with the Nijmegen PWA. The LETs for the coefficients in the effective range expansion are shown to be fulfilled to a very good (fair) accuracy in the \(3S_1\) (\(1S_0\)) channel.

Clearly, the LO calculations presented here should be extended to higher orders in the chiral expansion. Given that the main benefit of the new formulation is its renormalizability, it is natural to treat higher-order corrections in perturbation theory. Recent studies \[18,19,20\] carried out within the nonrelativistic framework seem to suggest that a perturbative treatment of the two-pion exchange might be phenomenologically successful\(^3\). Also the fairly small deviations between the Nijmegen PWA and the LO results, see Fig. \[4\], seem to be consistent with the possibility of a perturbative treatment of higher-order corrections.

The proposed formulation offers also further advantages apart from its transparency with regard to renormalization. In particular, one may hope to benefit from removing the finite cutoff artifacts which are unavoidable in the nonrelativistic framework with non-perturbative pions. Furthermore, the avoidance of the \(1/m\)-expansion can be advantageous in situations where the momentum scale \(\sim \sqrt{M_\pi m}\) associated with radiative pions must be explicitly taken into account (such as e.g. pion production in NN collisions and the quark-mass dependence of contact interactions \[44\]).

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