UNIFORMIZATION, UNIPOTENT FLOWS AND THE Riemann Hypothesis

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Abstract. We prove equidistribution of certain multidimensional unipotent flows in the moduli space of genus $g$ principally polarized abelian varieties (ppav). This is done by studying asymptotics of $\Gamma_g \sim Sp(2g, \mathbb{Z})$-automorphic forms averaged along unipotent flows, toward the codimension-one component of the boundary of the ppav moduli space. We prove a link between the error estimate and the Riemann hypothesis. Further, we prove $\Gamma_g^{-r}$ modularity of the function obtained by iterating the unipotent average process $r$ times. This shows uniformization of modular integrals of automorphic functions via unipotent flows.

Introduction

Let $\mathcal{H}_g := \{ \tau \in Mat(g, \mathbb{C}) | \tau = \tau^t, \Im(\tau) > 0 \}$, the genus $g$ Siegel half space, i.e. the set of symmetric complex $g \times g$ matrices $\tau$, with positive definite imaginary part $\Im(\tau)$. We indicate with $\Gamma_g \sim Sp(2g, \mathbb{Z})$ the discrete group of symplectic transformations, with action on $\tau$ given by

$$\tau \rightarrow (a\tau + b)(c\tau + d)^{-1}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2g, \mathbb{Z}).$$

The coset space $\Gamma_g \backslash \mathcal{H}_g$ is the moduli space of genus $g$ principally polarized abelian varieties (ppav) $\mathcal{A}_g$, and for genera $g = 1, 2, 3$, $\mathcal{A}_g$ is isomorphic to the moduli space of compact Riemann surfaces $\mathcal{M}_g$.

$\mathcal{H}_g$ is a homogenous space, since it is isomorphic to the Lie coset $\mathcal{H}_g \simeq Sp(2g, \mathbb{R})/(Sp(2g, \mathbb{R}) \cap SO(2g, \mathbb{R}))$, the set of real symplectic matrices over the orthosymplectic ones.

By Iwasawa decomposition of a symplectic matrix in $Sp(2g, \mathbb{R})$, one can find an interesting set of coordinates for the Lie coset $\mathcal{A}_g$. Every symplectic matrix can be written as $UAK$, with $K \in Sp(2g, \mathbb{R}) \cap SO(2g, \mathbb{R})$, $A = \diag(V_g, V_g^{-1})$, diagonal $2g \times 2g$ matrix, ($V_g := \diag(\sqrt{v_1}, \ldots, \sqrt{v_g}), v_i > 0, i = 1, \ldots, g$), and

Date: January 11, 2013.
\[ U = \begin{pmatrix} U_g & W_g U_g^{-1} \\ 0 & U_g^{-1} \end{pmatrix} \]

is a \( 2g \times 2g \) real unipotent matrix, with \( U_g \) upper unitriangular \( g \times g \) real matrix, and \( W_g \) symmetric \( g \times g \) real matrix.

This leads to the following Iwasawa parametrization of the Siegel Half space

\[ \tau_g = W_g + i U_g V_g^2 U_g^{-1}. \]  

In this paper we exploit Iwasawa parametrization, in conjunction with the Rankin-Selberg method, for investigating properties of a certain class of unipotent averages of automorphic forms on \( \Gamma \backslash \mathcal{H}_g \). From the behavior of those averages in the asymptotics limit toward the codimension one component of the \( \Gamma \backslash \mathcal{H}_g \) boundary, we prove ergodicity of (multidimensional) unipotent flows. It turns out that the error estimate depends on \( \Theta := \text{Sup} \{ \Re(\rho) | \zeta^*(\rho) = 0 \} \), the superior of the real part of the non trivial zeros of the Riemann zeta function. Therefore, evaluation of the error estimate would prove or disprove the Riemann hypothesis. Our result generalizes a well known theorem by Zagier on the long horocycle average of a \( SL(2, \mathbb{Z}) \) automorphic functions of rapid decay \([Za1],[Za2]\).

In order to announce the two main results of this paper, we need to introduce further notations. Given \( \tau_g \in \mathcal{H}_g \), we use a \((g-r)\)-corank block decomposition

\[ \tau_g = \begin{pmatrix} \tau_r & \tau_2 \\ \tau_2^t & \tau_{g-r} \end{pmatrix}, \]

with \( \tau_r \in \mathcal{H}_r \), \( \tau_{g-r} \in \mathcal{H}_{g-r} \).

In the \( r = 1 \) case, the Iwasawa coordinatization \((0.1)\) gives

\[ \tau_g = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2^t & \tau_{g-1} \end{pmatrix} = \begin{pmatrix} w_{11} + i(v_1 + \bar{u} V_{g-1}^2 \bar{u}) & \bar{w} + i \bar{u} V_{g-1}^2 U_{g-1}^t \\ \bar{w}^t + i U_{g-1} V_{g-1}^2 \bar{u} & \tau_{g-1} \end{pmatrix}, \]

where \( \bar{w} : (w_{12}, \ldots, w_{1g}) \) is in the first row of the symmetric real matrix \( W_g \), \( \bar{u} := (u_{12}, \ldots, u_{1g}) \) is in the first row of the unitriangular real matrix \( U_g \). We also use the following notation \( \bar{w} := w^t, \bar{u} := u^t \) to denote column vectors.

The first main result of this paper concerns modularity under the subgroup of transformations \( \Gamma_{g-1} \) of the average of an automorphic function \( f(\tau) \) along the \((2g-1)\) unipotent directions \((w_{11}, \bar{w}, \bar{u})\):
**Theorem 1.** Given a $\Gamma_g$-invariant automorphic function $f = f(\tau)$, let us consider the unipotent average

$$< f >_{v_1}(\tau_g^{-1}) := \int_{\mathbb{R}^{2g-1}} dw_1 dw_2 dw_3 f(\tau_g),$$

where $\tau_g$ is given in Iwasawa coordinates according to the corank $(g-1)$ decomposition given in \cite{[0,2]}.

The integral function $< f >_{v_1}(\tau_g^{-1})$ on $\mathbb{R}_{>0} \times \mathcal{H}_{g-1}$ is invariant under the

$\Gamma_g^{-1}$ modular group $\Gamma_{g-1}$:

$$< f >_{v_1}((a\tau_g^{-1} + b)(c\tau_g^{-1} + d)^{-1}) = < f >_{v_1}(\tau_g^{-1}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{g-1}.$$

The $\mathcal{H}_g$ boundary is given by $g-1$ components $\mathbb{F}_{g-r}$, $r = 1, \ldots, g-1$. For the quotient space $\Gamma_g \backslash \mathcal{H}_g$ the $(g-r)$-corank boundary component is given by

$$\Gamma_g \backslash \mathbb{F}_{g-r} = \{ \begin{pmatrix} i\infty_r \\ 0 \end{pmatrix}, \tau_{g-r} \in \mathcal{H}_r \},$$

where $i\infty_r := \text{diag}(i\infty, \ldots, i\infty)$ represents $r$ copies of the $\Gamma_1 \backslash \mathcal{H}_1$ cusp. Theorem 1 shows that the unipotent average $< f >_{v_1}(\tau_g^{-1})$ is a $\Gamma_{g-1}$ invariant modular function, defined on $\mathbb{R}_{>0} \times \mathcal{H}_{g-1}$. This function can be thought to be related to the $(g-1)$-corank component $\mathbb{F}_{g-1}$ of the boundary of $\mathcal{H}_g$. The distance from this boundary component is controlled by $v_1 > 0$, and one recovers the average along the $\mathbb{F}_{g-1}$ component of the boundary in the $v_1 \to 0$ limit.

The second main result of this paper is given by theorem 2 below. Theorem 2 shows that in the limit $v_1 \to 0$, the $< f >_{v_1}(\tau_g^{-1})$ averaged on the modular domain $\Gamma_{g-1} \backslash \mathcal{H}_{g-1}$ converges to the $f$ average on the modular domain $\Gamma_g \backslash \mathcal{H}_g$. The error term is related to the non trivial zeros of the Riemann zeta function $\zeta(s)$, and an estimate of this quantity provides a proof (or a disproof) of the Riemann hypothesis:
**Theorem 2.** Let $f = f(\tau)$ a $\Gamma_g$-invariant function of rapid decay for $\tau$ going to all the components $F_{g-r}$, $r = 1, \ldots, g-1$ of the $\mathcal{H}_g$ boundary. Let $f(\tau)$ be differentiable up to second order, with Laplacian $\Delta f$ of rapid decay, then the following asymptotic holds true:

$$
\int_{\mathcal{D}_{g-1}} d\mu_{g-1} < f \rho (v_1, \tau_{g-1}) \sim \frac{Vol(\mathcal{D}_{g-1})}{2Vol(\mathcal{D}_g)} \int_{\mathcal{D}_g} d\mu_g f(\tau) + O(v_1^{g-\frac{3}{4}}), \quad v_1 \to 0.
$$

Here $\mathcal{D}_g \sim \Gamma_g \backslash \mathcal{H}_g$ is a $\Gamma_g$ fundamental domain, with volume given by the formula $Vol(\mathcal{D}_g) = 2 \prod_{k=1}^g \zeta(2k)$, and $\Theta := \sup \{ \Re(\rho) | \zeta^*(\rho) = 0 \}$ is the superior of the real part of the non trivial zeros $\rho$'s of the Riemann zeta function $\zeta(s)$.

Theorem 2 provides a quite interesting connection between asymptotic unipotent dynamics in the ppav moduli space $\Gamma_g \backslash \mathcal{H}_g$ and the Riemann hypothesis. Indeed, in the genus $g$ case, the error estimate is $O(v_1^{g-1/4})$ if and only if the Riemann hypothesis is true ($\Theta = 1/2$). Theorem 2 generalizes Zagier genus $g = 1$ result [Za1],[Za2], for modular functions of rapid decay at the cusp.

Moreover, there is an interesting corollary of theorem 1 which follows by iterating the operation of averaging along unipotent directions. Let us use the following notation $w^{(r)} := (w_{r,r+1}, \ldots w_{r,g})$, and $u^{(r)} := (u_{r,r+1}, \ldots u_{r,g})$, where $w_{ij} := (W_g)_{ij}$, $u_{ij} := (U_g)_{ij}$

**Corollary 1.** Let $f = f(\tau_g)$ a $\Gamma_g$-invariant automorphic function. For $r = 1, \ldots, g-1$, the following unipotent average

$$
<f>_{v_1,\ldots,v_r}(\tau_{g-r}) := \int_{R_{g-r}>0} \prod_{i=1}^r dw_{i1} \int_{R_{g-r}>0} \prod_{i=1}^r du_{i1} f(\tau_g),
$$

is a $\Gamma_{g-r}$-invariant function on $R_{g-r} > 0 \times \mathcal{H}_{g-r}$:

$$
<f>_{v_1,\ldots,v_r}((a\tau_{g-r}+b)(c\tau_{g-r}+d)^{-1}) = <f>_{v_1,\ldots,v_r}(\tau_{g-r}), \quad (a \ b \ c \ d) \in \Gamma_{g-r}.
$$

Results of this paper given in theorems 1 and 2 suggest interesting connections with powerful results on measure rigidity and equidistribution of unipotent flows, provided by Ratner theory [Ra], and by more recent developments, (see for example [ELPV],[EMV],[EW]).

We also would like to mention an interesting connection between unipotent dynamics in homogeneous spaces and string theory, [CC],[C1],[C2],[ACER].
In fact, results in this paper have applications for shedding light in ultraviolet/infrared dualities descending from finiteness of closed string perturbative amplitudes [CC]. For genus one, (one-loop), closed string amplitudes, equidistribution theorems for long horocycles in the modular surface $SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{R})$ connects [C1] vacuum stability with asymptotic supersymmetry [KS]. Moreover, graded spectra of closed string excitations exhibit oscillating patterns with frequencies given by the imaginary parts of the non trivial zeros of the Riemann zeta function [ACER]. In one-loop stable closed string vacua, asymptotic supersymmetry is maximal if and only if the Riemann hypothesis is true [ACER]. As suggested in [CC], equidistribution theorems for unipotent averages in the ppav moduli space when applied to higher genus closed string amplitudes produce generalizations of the one-loop result [KS].

The connection between homogenous space dynamics and string theory works also in the opposite direction, namely from string theory to the theory of automorphic forms. By using consistency conditions from string theory, it provides information on certain asymptotic averages of automorphic forms. The advantages of translating the dynamical problems in string theory terms has been shown in the specific case of the horocycle flow in [C2]. There certain asymptotics for long horocycles averages of modular invariant functions with not so mild growing conditions have been considered. It is also worth to mention that results in [CC] and those in this paper indicate an intriguing relation between ultraviolet properties of closed strings on stable backgrounds and the Riemann hypothesis. These relations thus extend beyond one loop order, where they were shown to exist in [ACER]. Moreover, results of this paper may also be useful for studying and probing non perturbative conjectures related to modularity of the effective string action [Bi], [GMRV], [GRV], [GLW], [LV], [P], [OP]. They may find also applications for genus $g = 2$ superstring amplitudes [DHP], and for testing recent proposals for genus $g \geq 3$ closed string amplitudes [CDPvG], [DPvG], [DPGC], [Gr], [GSM2], [GSM], [GKV], [MV], [MV3], [MV2], [Mo].

The organization of the rest of the paper is the following: in section §1 we present some technical facts related to Iwasawa coordinatization of $\mathcal{H}_g$, instrumental for our proofs. Section §2 contains the proofs of theorems 1 and 2 on ergodicity of unipotent flows and error estimates, and some lemmas, instrumental for those proofs.

Acknowledgements

MC thanks Gerard van der Geer for enlighting discussions. The work of MC was supported at various stages by the Superstring Marie Curie Training Network under the contract MRTN-CT-2004-512194 at the Hebrew University.
of Jerusalem, by a visiting fellowship at the ESI Schroedinger Center for Mathematical Physics in Vienna, by the Italian MIUR-PRIN contract 20075ATT78 at the University of Milano Bicocca, by the Theory Unit at CERN, and by a "Angelo Della Riccia" fellowship at the University of Amsterdam.

1. Iwasawa parametrization for $\mathcal{H}_g$ and Eisenstein series

1.1. Iwasawa parametrization for $\mathcal{H}_g$. The genus $g$ Siegel upper space $\mathcal{H}_g$ is the set of complex $g \times g$ symmetric matrices with positive definite imaginary part $\mathcal{H}_g = \{ \tau \in \text{Mat}(g, \mathbb{C}) | \tau = \tau^t, \Im(\tau) > 0 \}$. $\mathcal{H}_g$ is isomorphic to the Lie coset $\mathcal{H}_g \simeq Sp(2g, \mathbb{R})/(Sp(2g, \mathbb{R}) \cap SO(2g, \mathbb{R}))$. For a given $m \in Sp(2g, \mathbb{R})/(Sp(2g, \mathbb{R}) \cap SO(2g, \mathbb{R}))$

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the bijective map is given by

(1.1) $\tau(m) = (a\mathbb{I} + b)(c\mathbb{I} + d)^{-1}$.

The Iwasawa decomposition allows to write a symplectic matrix in $Sp(2g, \mathbb{R})$ as $UVK$, $K \in SO(2g, \mathbb{R}) \cap Sp(2g, \mathbb{R})$, $V$ positive definite diagonal matrix and $U$ unipotent matrix. It is convenient the following $g \times g$ blocks parametrization

(1.2) $V = \begin{pmatrix} V_g & 0 \\ 0 & V_g^{-1} \end{pmatrix}$, \hspace{1cm} $V_g = \text{diag} (\sqrt{v_1}, \ldots, \sqrt{v_g})$

for the abelian part with $v_i > 0$, $i = 1, \ldots, g$,

$$U = \begin{pmatrix} U_g & W_gU_g^{-t} \\ 0 & U_g^{-t} \end{pmatrix},$$

for the unipotent part, with $W_g$ symmetric real $g \times g$ matrix

(1.3) $W_g = \begin{pmatrix} w_{11} & w_{12} & \ldots & w_{1g} \\ w_{21} & w_{22} & \ldots & w_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ w_{1g} & w_{2g} & \ldots & w_{gg} \end{pmatrix}$

and $U_g$ upper unitriangular real $g \times g$ matrix

(1.4) $U_g = \begin{pmatrix} 1 & u_{12} & \ldots & u_{1g} \\ 0 & 1 & \ldots & u_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$. 
With the above parametrization, eq. (1.1) gives the following Iwasawa parametrization for $\mathcal{H}_g$

$$\tau_g(m) = W_g + iU_g V_g^2 U^t_g.$$ (1.5)

1.2. $\mathcal{H}_g$ measure in Iwasawa coordinates. The $\Gamma_g$-invariant measure $d\mu_g$ in $\mathcal{H}_g$ is given by

$$d\mu_g = \frac{1}{\det(\Im(\tau_g))^{g+1}} \prod_{i \leq j} d\Re(\tau_g)_{ij} d\Im(\tau_g)_{ij},$$

where $\tau_{ij} = \Re(\tau)_{ij} + i\Im(\tau)_{ij}$.

The following two lemmas give $d\mu_g$ in Iwasawa coordinates

Lemma 1. The following holds true

$$\det(\Im(\tau_g(m))) = \prod_{i=1}^{g} v_i,$$ (1.6)

Proof. It follows directly from (1.5).

Lemma 2. The Jacobian determinant of $\tau_g(m)$ map given in (1.5) is

$$J_g = \prod_{i=1}^{g} v^{g-1-i}_g.$$

Proof. Let us take the following block parametrization

$$U_g = \begin{pmatrix} U_{g-1} & \tilde{u} \\ 0 & 1 \end{pmatrix},$$

where upper line denotes column vectors and lower line row vectors. $U_{g-1}$ is a $(g-1)$-dimensional upper unitriangular real matrix, $\tilde{u}$ is a $(g-1)$-dimensional column vector, $\tilde{u} = u^t$. Thus, one has

$$\Im(\tau_g) = U_g V_g^2 U^t_g = \begin{pmatrix} U_{g-1} & \tilde{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V^2_{g-1} & 0 \\ 0 & v_g \end{pmatrix} \begin{pmatrix} U_{g-1}^t & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} U_{g-1} V^2_{g-1} U^t_{g-1} + \tilde{u} v_g & \tilde{u} v_g \\ \tilde{u} v_g & v_g \end{pmatrix}$$

$$= \begin{pmatrix} \Im(\tau_{g-1}) + \tilde{u} v_g & \tilde{u} v_g \\ \tilde{u} v_g & v_g \end{pmatrix},$$

and therefore

$$d\Im(\tau_g) = v^{g-1}_g dv_g \wedge d\tilde{u} \wedge d\Im(\tau_{g-1}).$$
From which it follows that

\[ J_g = v_g^{g-1} J_{g-1}, \]

and by iteration one recovers (1.6).

**Proposition 1.** The \( H \) measure \( d\mu_g \) in Iwasawa coordinates is given by

\[
d\mu_g = \prod_{i=1}^{g} dv_i v_i^{i-g-2} \prod_{i \geq j}^{g} dw_{ij} \prod_{i > j}^{g} du_{ij}.
\]

Proof.

\[
d\mu_g = \frac{J_g}{\det(\mathcal{I}(\tau_g))^{g+1}} \prod_{i=1}^{g} dv_i \prod_{i \leq j}^{g} dw_{ij} \prod_{i < j}^{g} du_{ij}
\]

\[
= \prod_{i=1}^{g} v_i^{g-i} \prod_{i \leq j}^{g} dw_{ij} \prod_{i < j}^{g} du_{ij}
\]

\[
= \prod_{i=1}^{g} v_i^{i-2} \prod_{i \leq j}^{g} dw_{ij} \prod_{i < j}^{g} du_{ij}.
\]

\[ \square \]

1.3. Eisenstein series. Let us introduce the following blocks decomposition \( \tau_g \in H \)

\[
\tau_g = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}
\]

where \( \tau_{11} \in H_r, \tau_{22} \in H_{g-r} \).

The \((g-r)\) corank Eisenstein series, associated to the \( F_{g-r} \) component of the boundary of \( H \), is defined by

(1.7) \[
E_{g,r}(\tau, s) = \sum_{\mathbf{g} \cap P_g \setminus \mathbf{g}} \left( \frac{\det(\mathcal{I}(\gamma(\tau)))}{\det(\mathcal{I}(\gamma(\tau)))} \right)^s,
\]

where \( P_{g,r} \subset Sp(2g, \mathbb{R}) \) is the parabolic subgroup which stabilizes the \( F_{g-r} \).

For our purposes it is useful the knowledge of the analytic properties of the \( r = 1 \) Eisenstein series, given by the following proposition [Ya]:

**Proposition 2.** The \( r = 1 \) Eisenstein series \( E_{g,1}(\tau, s) \) of the family given in (1.7)

(1.8) \[
E_{g,1}(\tau, s) = \sum_{\mathbf{g} \cap P_{g,1} \setminus \mathbf{g}} \left( \frac{\det(\mathcal{I}(\gamma(\tau)))}{\det(\mathcal{I}(\gamma(\tau)))} \right)^s, \quad \Re(s) > g,
\]
can be analytically continued to the full s plane to a meromorphic function with a simple pole in s = g with residue \( \frac{1}{2\zeta(2g)} \), and poles in s = \( \frac{p}{2} \), where p’s are the non trivial zeros of the Riemann zeta function, \( \zeta^*(\rho) = \pi^{-\rho/2}\Gamma(\frac{\rho}{2})\zeta(\rho) = 0 \).

1.4. \( \mathcal{H}_{g-1} \hookrightarrow \mathcal{H}_g \) embedding.

**Lemma 3.** (\( \mathcal{H}_{g-1} \hookrightarrow \mathcal{H}_g \) embedding). Given \( \tau_g \in \mathcal{H}_g \) and \( \tau_{g-1} \in \mathcal{H}_{g-1} \), the following decomposition holds

\[
\Im(\tau_g) = \begin{pmatrix} v_1 + \frac{uV_g^2}{g-1}\bar{u} & \frac{uV_g^2}{g-1}U_{g-1} \\ U_{g-1}V_{g-1} & \Im(\tau_{g-1}) \end{pmatrix}.
\]

**Proof.**

\[
\Im(\tau_g) = \begin{pmatrix} 1 & u \\ 0 & U_{g-1} \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & V_{g-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & U_{g-1} \end{pmatrix}^{-1} \begin{pmatrix} v_1 + \frac{uV_g^2}{g-1}\bar{u} & \frac{uV_g^2}{g-1}U_{g-1} \\ U_{g-1}V_{g-1} & \Im(\tau_{g-1}) \end{pmatrix}.
\]

From the previous result one has also the following blocks decomposition for \( \tau_g \) in terms of \( \mathcal{H}_1 \) and \( \mathcal{H}_{g-1} \) subspaces:

**Proposition 3.**

\[
\tau_g = \begin{pmatrix} w_{11} + i(v_1 + \frac{uV_g^2}{g-1}\bar{u}) & w + \frac{uV_g^2}{g-1}U_{g-1} \\ \bar{w} + iU_{g-1}V_{g-1} & \Im(\tau_{g-1}) \end{pmatrix}.
\]

1.5. \( \Gamma_{g-1} \hookrightarrow P_{g,1} \subset \Gamma_g \) parabolic embedding. A matrix in \( P_{g,1} \cap \Gamma_g \subset \text{Sp}(2g,\mathbb{Z}) \) has the following form

\[
\begin{pmatrix} 1 & m & q & n \\ 0 & a & n^t & b \\ 0 & 0 & 1 & 0 \\ 0 & c & -m^t & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{g-1}, \ m, n \in \text{Mat}(1 \times (g-1),\mathbb{Z}), \ q \in \mathbb{Z}.
\]

It is useful the following decomposition for the elements in \( P_{g,1} \cap \Gamma_g \) (see for example [HKW]):
Proposition 4. Every matrix in $P_{g,1} \cap \Gamma_g$ can be decomposed as follows:

\[
(1.9) \quad \begin{pmatrix}
1 & m & q & n \\
0 & a & n^t & b \\
0 & 0 & 1 & 0 \\
0 & c & -m^t & d
\end{pmatrix} = g_1 \cdot g_2 \cdot g_3,
\]

with

\[
(1.10) \quad g_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a & 0 & b \\
0 & a & 0 & 0 \\
0 & c & 0 & d
\end{pmatrix}, \quad \begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix} \in \Gamma_{g-1},
\]

\[
(1.11) \quad g_2 = \begin{pmatrix}
1 & m & 0 & n \\
0 & 1 & n^t & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -m^t & 1
\end{pmatrix}, \quad m, n \in \text{Mat}(1 \times (g-1), \mathbb{Z}),
\]

and

\[
(1.12) \quad g_3 = \begin{pmatrix}
1 & 0 & q & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad q \in \mathbb{Z}.
\]

Proposition 5. The action of $g_1, g_2, g_3$ on $\tau \in \mathcal{H}_g$

\[
\tau = \begin{pmatrix}
\tau_1 \\
\tau_2 \\
\tau_3
\end{pmatrix} \quad \tau_1 \in \mathcal{H}_1, \tau_3 \in \mathcal{H}_{g-1}, \tau_2 \in \text{Mat}(1 \times (g-1), \mathbb{C}),
\]

is given by:

\[
g_1(\tau) = \begin{pmatrix}
\tau_1 - \tau_2(c\tau_3 + d)^{-1}c\tau_2^t \\
(c\tau_3 + d)^{-1}c\tau_2^t \\
(\tau_2 - c\tau_3)(c\tau_3 + d)^{-1}c\tau_2^t \\
(\tau_2 - c\tau_3)(c\tau_3 + d)^{-1}c\tau_2^t
\end{pmatrix} \quad \tau_1' = \tau_1 + m\tau_3, \tau_2' = \tau_2 + (m^t \tau_2) + (m \tau_3)\tau_2 + (m \tau_3 \tau_2 + m^t \tau_2) + nm^t,
\]

where the entries * are given by symmetry of $\tau$. 

2. Proofs of theorem 1 and theorem 2

2.1. Unfolding of the modular integral. Given a function \( f = f(\tau) \) on \( \mathcal{H}_g \) invariant under the modular group \( \Gamma_g \sim Sp(2g, \mathbb{Z}) \), let us consider the following Rankin-Selberg type modular integral

\[
I_{g,1}(s) = \int_{\Gamma_g \backslash \mathcal{H}_g} d\mu_g \ f(\tau) E_{g,1}(\tau, s),
\]

where \( E_{g,1}(\tau, s) \) is the non-holomorphic \( g - 1 \)-corank Eisenstein series, introduced in \( \text{(1.8)} \) section \( \text{(1.3)} \).

\[ E_{g,1}(\tau, s) = \sum_{\gamma \in P_g \cap \Gamma_g \backslash \Gamma_g} \left( \frac{\det(3(\gamma(\tau)))}{\det(3((\gamma(\tau)22))} \right)^s, \quad \Re(s) > g \]
related to the \( (g - 1) \) corank component \( F_{g-1} \) of the boundary of the modular domain \( \Gamma_g \backslash \mathcal{H}_g \).

Under suitable growing conditions for \( f \) at the boundary, that are stated as sufficient conditions in theorem \( 2 \) it is possible in \( \text{(2.1)} \) to exchange integration on the modular domain with the sum over the modular transformations \( \gamma \)'s appearing in the Eisenstein series \( \text{(2.2)} \). This operation allows to unfold the original integration domain \( \Gamma_g \backslash \mathcal{H}_g \) into the larger domain \( (P_{g,1} \cap \Gamma_g) \backslash \mathcal{H}_g \). As we shall see, this latter integration domain has simplified features which becomes transparent in Iwasawa coordinates.

Whenever it is allowed to exchange the sum with the integral, for \( I_{g,1}(s) \) one finds

\[
I_{g,1}(s) = \int_{0}^{\infty} dv_1 v_1^{s-g-1} \int_{(P_{g,1} \cap \Gamma_g) \backslash \mathcal{H}_g} d\mu_{g-1} \int dw_11 dw_1 \ f \left( w_{11} + i(v_1 + \bar{w}_1 V_{g-1}^{-1} \bar{u}) \bar{w} + iV_{g-1}^{2} U_{g-1}^{-1} \right),
\]
where integration along \( \bar{w} \) and \( \bar{u} \) takes into account identifications by the parabolic subgroup \( P_{g,1} \) given in proposition \( \text{(5)} \).

Let us notice that \( I_{g,1}(s) \) involves a Mellin integral transform in the abelian Iwasawa coordinate \( v_1 \in \mathbb{R}_{>0} \). If certain conditions for the existence of the inverse Mellin transform are fulfilled, then, (by using proposition \( \text{(6)} \), one gets the following asymptotic
\[
\lim_{v_1 \to 0} \int_{(P_g, \Gamma_g) \backslash H_g} d\mu_{g-1} \int d\mathbf{w} d\mathbf{u} \left( (w_{11} + i(v_1 + \mathbf{u} V_{g-1}^{2} \bar{\mathbf{u}})) \bar{\mathbf{u}} + i \mathbf{u} V_{g-1}^{2} U_{g-1}^{t} \right)
= \frac{1}{2 \zeta^* (2g)} \int_{\Gamma_g \backslash H_g} d\mu_{g} f(\tau)
= \frac{Vol(D_{g-1})}{2Vol(D_{g-1})} \int_{\Gamma_g \backslash H_g} d\mu_{g} f(\tau).
\] (2.4)

Last line of (2.4) follows from the formula \( Vol(D_{g-1}) = 2 \prod_{k=1}^{g} \zeta^* (2k) \) for the volume of a fundamental region \( D_{g} \simeq \Gamma_g \backslash H_g \) of the modular group \( \Gamma_g \) in \( H_g \).

2.2. Proof of Theorem 1. The above discussion and eq. (2.4) are suggestive of the existence of a \( H_g \to H_{g-1} \) reduction for modular integral of automorphic functions through the operation of averaging along unipotent directions \( w_{11}, \mathbf{w}, \mathbf{u} \) defined in (0.2). The above argument is turned into a rigorous proof by the following:

**Theorem 1.** Given a \( \Gamma_g \)-invariant automorphic function \( f = f(\tau) \), let us consider the unipotent average

\[
< f >_{v_1}(\tau_{g-1}) := \int_{\mathbb{R}^{2g-1}} d\mathbf{w} d\mathbf{u} f(\tau),
\]

where \( \tau_{g} \) is given in Iwasawa coordinates according to the corank \( (g - 1) \) decomposition given by (0.2) and in proposition 3.

The integral function \( < f >_{v_1}(\tau_{g-1}) \) on \( \mathbb{R}_{>0} \times H_{g-1} \) is invariant under the genus \( (g - 1) \) modular group \( \Gamma_{g-1} \):

\[
< f >_{v_1}((a\tau_{g-1} + b)(c\tau_{g-1} + d)^{-1}) = < f >_{v_1}(\tau_{g-1}), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_{g-1}.
\]

**Proof.** The action of \( \Gamma_{g-1} \) on \( < f >_{v_1}(\tau_{g-1}) \) is provided by the embedding of \( \Gamma_{g-1} \to \Gamma_g \) defined by \( g_1 \) as in Proposition 3. As \( f \) is \( \Gamma_g \) invariant, the proof will follow from the fact that the measure \( dw_{11} \mathbf{d}w \mathbf{d}u \) is \( \Gamma_{g-1} \)-invariant, and that the action of \( \Gamma_{g-1} \) over the Siegel space lives \( v_1 \) invariant. This permits to reabsorb the transformation in a change of variables which leaves the expression of \( < f >_{v_1}(\tau_{g-1}) \) invariant in form.
Let us first consider the action of $g_1$, given by (1.10), on the coset space $H_g$.

Using the Iwasawa construction, the generic point of the coset has the form

$$x := \left( U_g V_g \begin{pmatrix} W_g U_g V_g \end{pmatrix}^{-t} \right)$$

so that, in particular, its first column is $(\sqrt{v_1}, 0, \ldots, 0)^t$

$$x = \begin{pmatrix} \sqrt{v_1} & \cdots \\ 0 & \cdots \end{pmatrix}.$$

By acting on $x$ from the left with $g_1$, one finds the following structure

$$(2.6) \quad g_1x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} \sqrt{v_1} & \cdots \\ 0 & \cdots \\ 0 & \cdots \\ 0 & \cdots \end{pmatrix} = \begin{pmatrix} \sqrt{v_1} & r^{(1)} \\ 0 & r^{(2)} \\ \vdots & \vdots \\ 0 & r^{(2g)} \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{g-1}.$$

In particular, the $(2g-1)$ vectors $r^{(j)}$, $j = 2, \ldots, 2g$ are linearly independent, since $\det(g_1x) \neq 0$. The symplectic matrix $g_1x \in Sp(2g, \mathbb{R})$ in (2.6) is no more in the quotient $Sp(2g, \mathbb{R})/(Sp(2g, \mathbb{R}) \cap SO(2g, \mathbb{R}))$, since it does not have the blocks structure (2.5).

However, by multiplying $g_1x$ from the right by an orthosymplectic matrix $K \in Sp(2g, \mathbb{R}) \cap SO(2g, \mathbb{R})$,

$$K = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad A'B = B'A, \quad A'A + B'B = I,$$

one can determine the $g_1x$ coset representative

$$g_1xK = \begin{pmatrix} \sqrt{v_1} & r^{(1)} \\ 0 & r^{(2)} \\ \vdots & \vdots \\ 0 & r^{(2g)} \end{pmatrix} \begin{pmatrix} A_{11} & \cdots \\ \bar{a} & \cdots \end{pmatrix} = \begin{pmatrix} \sqrt{v_1} & \cdots \\ 0 & \cdots \end{pmatrix}, \quad \bar{a} := (A_{21}, \ldots, A_{g1}, -B_{11}, \ldots, -B_{g1})^t.$$

In particular, equality for the elements in the first columns of the previous equation gives
and, since the \((2g - 1)\) vectors \(\mathbf{g}^{(j)}, j = 2, \ldots, 2g\) are linearly independent, one finds \(\mathbf{a} = \mathbf{0}\). This implies that

\[
A_{j1} = \delta_{j1} \sqrt{\varepsilon_1/v_1}, \quad B_{j1} = 0, \quad j = 1, \ldots, g,
\]

and by using \((A'A + B'B)_{11} = 1\) one then gets \(\tilde{\varepsilon}_1 = v_1\).

The new defined coordinates are then such that, in the notation of Proposition \(53\)

\[
g_1 \left( \begin{pmatrix} w_{11} + i(v_1 + uV_{g-1}^2 \tilde{u}) & w + iuV_{g-1}^2 U_{g-1}^t \tau_{g-1} \\ \bar{w} + iU_{g-1}^t V_{g-1}^2 \tilde{u} & \bar{w} + i\bar{u}V_{g-1}^2 \bar{U}_{g-1}^t \bar{\tau}_{g-1} \end{pmatrix} \right) = \begin{pmatrix} w_{11} + i(v_1 + \tilde{u}V_{g-1}^2 \tilde{u}) & \bar{w} + i\bar{u}V_{g-1}^2 \bar{U}_{g-1}^t \tilde{\tau}_{g-1} \\ \bar{w} + iU_{g-1}^t \bar{V}_{g-1}^2 \tilde{u} & \bar{w} + i\bar{u}\bar{V}_{g-1}^2 \bar{U}_{g-1}^t \end{pmatrix}.
\]

(2.7)

Notice that \(\tilde{V}_{g-1}\) and \(\tilde{U}_{g-1}\) are defined by \(\tilde{\tau}_{g-1}\), that does not depends on \(\tilde{u}\) and \(\tilde{v}\), so that the transformation of coordinates \((w_{11}, \tilde{u}, \tilde{v}) \mapsto (\tilde{w}_{11}, \tilde{u}, \tilde{v})\) is defined by the components \(g_1(x)_{1j}, j = 1, \ldots, g\) of relation \((2.7)\). This gives the linear transformation

\[
\begin{align*}
\tilde{w}_{11} + i(\tilde{v}_1 + \tilde{u}V_{g-1}^2 \tilde{u}) &= w_{11} + i(v_1 + uV_{g-1}^2 \tilde{u}) - (w + iuV_{g-1}^2 U_{g-1}^t (c\tau_{g-1} + d)^{-1} c(\bar{w} + iU_{g-1}^t V_{g-1}^2 \tilde{u}), \\
(\bar{w} + i\bar{V}_{g-1}^2 \bar{U}_{g-1}^t) &= (\bar{w} + iuV_{g-1}^2 U_{g-1}^t (c\tau_{g-1} + d)^{-1} iU_{g-1}^t V_{g-1}^2 \tilde{u}).
\end{align*}
\]

(2.8)

By differentiating and by taking the determinant one thus gets

\[
d\tilde{w}_{11} d\tilde{u}^{g-1} d\tilde{u}^{g-1} \det(\tilde{V}_{g-1}^2) = dw_{11} dw^{g-1} dw^{g-1} \det(V_{g-1}^2) / \det(c\tau_{g-1} + d)^2,
\]

where we have used that \(\det\tilde{U}_{g-1} = 1\). Now,

\[
\det(\tilde{V}_{g-1}^2) = \det(\tilde{U}_{g-1}^t V_{g-1}^2 \tilde{U}_{g-1}^t) = \det \Im(\tilde{\tau}_{g-1}).
\]

From
\[ 2i\Im(\tau^{-1}) = (a\tau^{-1} + b)(c\tau^{-1} + d)^{-1} - (\tau^{-1}a + b)^{-1}(\tau^{-1}a + b) \\
= (\tau^{-1}a + b)^{-1}[(c\tau^{-1} + d)(a\tau^{-1} + b) - (\tau^{-1}a + b)(c\tau^{-1} + d)](c\tau^{-1} + d)^{-1} \\
= 2i(\tau^{-1}a + b)^{-1}\Im(\tau^{-1})(c\tau^{-1} + d)^{-1}, \]

where we have used \( a'd - c'b = I, \ a'c = c'a \) and \( b'd = d'b \), one gets

\[ (2.9) \quad \det(\tilde{V}^2_{g^{-1}}) = \det(V^2_{g^{-1}})/|\det(\tau^{-1})|^2, \]

which shows the invariance of the measure. The fact that \( \tilde{v}_1 \) and \( \tilde{\tau}_{g^{-1}} \) do not depend on \( w_{11}, \tilde{w}, \tilde{u} \), implies that the range of coordinates remains unchanged.

In conclusion, we have

\[ \langle f \rangle_{v_1(\tilde{\tau}_{g^{-1}})} = \int dw_{11} dw \, du \, f \left( \left( \begin{array}{c} w_{11} + i(v_1 + \hat{u}V^2_{g^{-1}}) \, \hat{w} + i\hat{u}V^2_{g^{-1}} \, \hat{u} \end{array} \right) \right), \]

since \( f \) is \( \Gamma_g \) invariant, this implies \( \langle f \rangle_{v_1(\tilde{\tau}_{g^{-1}})} = \langle f \rangle_{v_1(\tau^{-1})}. \]

2.3. Proof of Theorem 2. We now give the proof of theorem 2. We start by recalling a standard property concerning Mellin integral transforms, (Proposition 6). In order to prove Theorem 2 we shall also need Proposition 7, whose proof is postponed to §2.4.

**Proposition 6**. Let \( \varphi = \varphi(s) \) be the following meromorphic function on the \( s \) plane

\[ \varphi(s) = \sum_{i=1}^{l} \frac{C_i}{(s - s_i)^{n_i}}, \quad n_i \in \mathbb{N}_{\geq 0}, \]

then the following identity holds

\[ \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} ds \, y^{-s} \varphi(s) = \sum_{i=1}^{l} (-)^{n_i} \frac{C_i}{n_i!} y^{-s_i} \log^{n_i} y, \quad \sigma > \max\{\Re(s_i)\}. \]

**Proof**. It is easily obtained by using residues theorem, and by closing the integration contour such that it contains the points \( s = s_i, i = 1, \ldots, l. \) \( \square \)
Theorem 2. Let \( f = f(\tau) \) a \( \Gamma_g \)-invariant function of rapid decay for \( \tau \) going to all the components \( \mathbb{H}_g \) of the \( \mathbb{H}_g \) boundary. Let \( f(\tau) \) be differentiable up to second order, with Laplacian \( \Delta f \) of rapid decay, then the following asymptotic holds true:

\[
\int_{\mathcal{D}_g} d\mu_g < f > \sim \frac{\text{Vol}(\mathcal{D}_g)}{2 \text{Vol}(\mathcal{D}_g)} \int_{\mathcal{D}_g} d\mu_g f(\tau) + O(v_1^{g-\Theta}), \quad v_1 \to 0,
\]

where \( \mathcal{D}_g \sim \Gamma_g \backslash \mathbb{H}_g \) is a \( \Gamma_g \) fundamental domain, with volume \( \text{Vol}(\mathcal{D}_g) = 2 \prod_{k=1}^{g} \zeta^*(2k) \). Integration along unipotent coordinates \( w_1, w_2, u \) takes into account the identifications by the parabolic subgroup \( P_{g,1} \), given in proposition 5, and \( \Theta := \sup \{ \Re(\rho) | \zeta^*(\rho) = 0 \} \), is the superior of the real part of the non trivial zeros \( \rho \)'s of the Riemann zeta function.

Proof. Let us consider the modular integral

\[
I_{g,1}(s) = \int_{\Gamma_g \backslash \mathbb{H}_g} d\mu_g f(\tau) E_{g,1}(\tau).
\]

The Eisenstein series \( E_{g,1}(\tau, s) \) defined in (1.8) is of polynomial growth for \( \tau \) going to each component of the \( \mathbb{H}_g \) boundary. For \( \Re(s) > g \) one can use the series representation for the Eisenstein series, and, by Lebesgue dominated convergence one can exchange the series with the modular integral

\[
I_{g,1}(s) = \int_{\Gamma_g \backslash \mathbb{H}_g} d\mu_g f(\tau) \sum_{\Gamma_g \cap P_{g,1} \backslash \Gamma_g} \left( \frac{\text{det}(\Im(\gamma(\tau)))}{\text{det}(\Im(\gamma(\tau)_{22}))} \right)^s
\]

\[
= \sum_{\Gamma_g \cap P_{g,1} \backslash \Gamma_g} \int_{\Gamma_g \backslash \mathbb{H}_g} d\mu_g f(\tau) \left( \frac{\text{det}(\Im(\gamma(\tau)))}{\text{det}(\Im(\gamma(\tau)_{22}))} \right)^s
\]

\[
= \int_{\Gamma_g \cap P_{g,1} \backslash \mathbb{H}_g} d\mu_g f(\tau) \left( \frac{\text{det}(\Im(\tau))}{\text{det}(\Im(\tau_{22}))} \right)^s.
\]

In the last line modular transformations \( \gamma \)'s in the coset \( (\Gamma_g \cap P_{g,1}) \backslash \Gamma_g \) are used to unfold the integration domain.

Since \( f(\tau) \) is of rapid decay for \( \tau \) going at the \( \mathbb{H}_g \) boundary, the modular integral is uniformly convergent with respect to the variable \( s \), and thus \( I_{g,1}(s) \) inherits analytic properties of the Eisenstein series \( E_{g,1}(s, \tau) \). Then, due to proposition 2 \( I_{g,1}(s) \) can be analytically continued to the full \( s \) plane to a meromorphic function with a simple pole in \( s = g \), and poles in \( s = \rho/2 \), where \( \rho \)'s are the non trivial zeros of the Riemann zeta function, \( \zeta^*(\rho) = 0 \). Thus one can write the following expansion
for multiple Riemann zeros \( \rho \)'s, one has to rise the denominator in the above formula by the appropriate power.

In (2.11), \( C_g \) is given by

\[
C_g = \frac{1}{2\zeta(2g)} \int_{\mathcal{D}_g} d\mu_g f(\tau),
\]

since \( E_{g,1}(\tau, s) \) has a simple pole in \( s = g \) with residue \( 1/2\zeta(2g) \).

Then, by using Iwasawa coordinates one can write \( I_{g,1}(s) \) in the following convenient form

\[
I_{g,1}(s) = \int_0^\infty dv_1 v_1^{-s-g-1} \int_{\Gamma_{g-1}\backslash \mathcal{H}_g} d\mu_{g-1} \int dw_11 dw_1 \frac{w_11 + i(v_1 + \bar{u}V_{g-1}^2 \bar{u})}{\bar{w} + iU_{g-1}V_{g-1}^2 \bar{u}},
\]

where the decomposition in terms of a modular integral over \( \Gamma_{g-1}\backslash \mathcal{H}_g \) follows from theorem [1].

Equation (2.12) states that \( I_{g,1}(s) \) is the Mellin transform of the following integral function

\[
v_{1}^{-g} F_{g,1}(v_1) := \int_{\Gamma_{g-1}\backslash \mathcal{H}_g} d\mu_{g-1} \int dw_11 dw_1 \frac{w_11 + i(v_1 + \bar{u}V_{g-1}^2 \bar{u})}{\bar{w} + iU_{g-1}V_{g-1}^2 \bar{u}}.
\]

If the following integral defining the \( I_{g,1}(s) \) inverse Mellin transform

\[
\mathcal{M}^{-1}[I_{g,1}(s)](y) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} ds y^{-s} I_{g,1}(s) = \frac{y^{-\sigma}}{2\pi i} \int_{-\infty}^{\infty} dt y^{-it} I_{g,1}(\sigma + it),
\]

is convergent, then through proposition [6] one obtains the \( v_1 \to 0 \) asymptotic for the function \( F_{g,1}(v_1) \).

Since \( f(\tau) \) is twice differentiable, we use \( \Delta E_{g,1}(\tau, s) = 2\pi i s(g-s) E_{g,1}(\tau, s) \)

(a proof of this result is given in proposition [7].

By integration by parts one then finds

\[
I_{g,1}(s) = \frac{2^{-s-g-1}}{s(g-s)} \int_{\mathcal{D}_g} d\mu_g E_{g,1}(\tau, s) \Delta f(\tau).
\]
This shows that \( I_{g,1}(\sigma + it) \) falls off as \( O(t^{-2}) \) for \( t \to \pm \infty \), for all \( \sigma \)'s where the following integral

\[ \int_{D_g} d\mu_g E_{g,1}(\tau, s)\Delta f(\tau), \]

is convergent. Under our assumption that \( \Delta f(\tau) \) is of rapid decay for \( \tau \) going to the boundary, the above integral is convergent, since \( E_{g,1}(\tau, s) \) is of polynomial growth for \( \tau \) going to each component of the \( H_g \) boundary. It follows that the integral (2.14) is convergent, and thus \( \mathcal{M}^{-1}[I_{g,1}(s)](v_1) \) exists, and

\[ \mathcal{M}^{-1}[I_{g,1}(s)](v_1) = F(v_1). \]

By analytic properties of \( I_{g,1}(s) \) given in eq. (2.11), except for a simple pole in \( s = g \), \( I_{g,1}(s) \) is analytic on \( \Re(s) > \Theta \), where \( \Theta = \text{Sup}\{\Re(\rho)|\zeta^*(\rho) = 0\} \).

Then, asymptotic eq. (2.10) including dependence of the error estimate on \( \Theta \), follows from (2.12) and by using proposition 6. Finally, the ratio appearing in eq. (2.12) between volumes of modular domains follows from the formula

\[ \text{Vol}(D_g) = 2 \prod_{k=1}^{g} \zeta^*(2k). \]

2.4. \( H_g \) Laplacian and the rank \((g - 1)\) Eisenstein series. Let \( G_{IJ} \) be the \( H_g \) \( Sp(2g, \mathbb{R}) \)-invariant metric, with infinitesimal line element \( ds^2 = G_{IJ}d(X_I dX_J) \), where \( X_I \) is a system of \( g(g + 1) \) real coordinates for \( H_g \). We indicate with \( G^{IJ} \) the inverse metric of \( G_{IJ} \), \( G^{IK} G_{KJ} = \delta^I_J \). We also use the notation \( G := \det G_{IJ} \), for the determinant of the metric. Let us consider the \( H_g \) Laplacian operator \( \Delta := -\frac{1}{\sqrt{|G|}} \partial_I \sqrt{|G|} G^{IJ} \partial_J \). In this section we prove that the \((g - 1)\) corank Eisenstein series \( E_{g,1}(\tau, s) \) defined in §1.3 is an eigenfunction of the \( H_g \) Laplacian \( \Delta \),

(2.16) \[ \Delta E_{g,1}(\tau, s) = 2^{\frac{g-1}{g+1}} s(g - s) E_{g,1}(\tau, s). \]

In order to prove this result, we first need the following lemma:

**Lemma 4.** In Iwasawa coordinates: \( G^{v_i, v_i} = 2^{\frac{g-1}{g+1}} v_i^2 \).

**Proof.** Let us consider the Iwasawa decomposition \( UVK \) of \( Sp(2g, \mathbb{R}) \). The elements of the quotient \( H_g \) are represented by the points \( h = UV \). In [CCDOS] it has been shown that the invariant metric can then be obtained as

(2.17) \[ ds^2 = \kappa \text{Tr}(J \otimes J), \quad J = \Pi((UV)^{-1} d(UV)) \]
where $\Pi$ projects orthogonally to the space tangent to $K$, and $\kappa$ is a normal-
ization constant. As $K$ is the intersection with the orthogonal group $SO(2g\mathbb{R})$, $\Pi$
 takes the symmetric part, so that

$$J = V^{-1}dV + \frac{1}{2}(V^{-1}U^{-1}dUV + V dU^tU^{-1}V^{-1}).$$

In order to compute the trace in (2.17) note that the parenthesis is the sum of nilpotent
matrices (as $U$ unipotent implies $U^{-1}dU$ nilpotent), whereas $V^{-1}dV$ is diagonal, so that mixed
products have vanishing trace and we remain with the terms

$$(2.18) \quad ds^2 = \kappa \sum_{i=1}^{g} \frac{1}{v_i} dv_i^2 + \frac{\kappa}{2} Tr(V^{-2}U^{-1}dUV^2dU^tU^{-1}).$$

Notice that there are no off-diagonal terms of the form $dV \otimes dU$, so that from (2.18) we get

$$G^{v_i v_i} = \frac{2}{\kappa} v_i^2, \quad i = 1, \ldots, g.$$ 

To compute $\kappa$ we can compute the determinant of the metric (2.18) and compare the
result with Proposition 1. We know that the determinant does not depend on the
coordinates $u_{ij}$ and $w_{ij}$ in $U$ so that we can compute it for $u_{ij} = 0$ and $w_{ij} = 0$. This gives

$$V^{-2}dU = \begin{pmatrix} V_g^{-2}dU_g & V_g^{-2}dW_g \\ 0 & -V_g^{-2}dU_g \end{pmatrix}, \quad V^2 dU^t = \begin{pmatrix} V_g^2 dU^t_g & 0 \\ V_g^{-2}dW_g & -V_g^{-2}dU_g \end{pmatrix}.$$ 

Then

$$Tr(V^{-2}U^{-1}dUV^2dU^tU^{-1}) = Tr[2V_g^2 dU^t_g V_g^{-2}dU_g + V_g^{-2}dW_g V_g^{-2}dW_g].$$ 

First, notice that

$$(V_g^{-2} dW_g V_g^{-2})_{ij} = \frac{1}{v_i v_j} dw_{ij}.$$ 

Thus
\[ A := \text{Tr}(V_g^{-2} dW_g V_g^{-2} dW_g) = \sum_{i,j} \frac{1}{v_i v_j} dw_{ij} dw_{ji} = \sum_{i=1}^{g} \frac{1}{v_i^2} dw_{ii}^2 + 2 \sum_{i<j} \frac{1}{v_i v_j} dw_{ij}^2. \]

Then, this part of the metric is diagonal and contributes to the determinant with the term
\[ \det A = 2^{g(g-1)/2} \prod_{1 \leq i < j \leq g} \frac{1}{v_i v_j} = 2^{g(g-1)/2} \prod_{i=1}^{g} \frac{1}{v_i^2}. \]

In the same way we get
\[ B := \text{Tr}[2V_g^2 dU_g V_g^{-2} dU_g] = 2 \sum_{1 \leq i < j \leq g} \frac{v_i}{v_i} dU_{ij}^2. \]

Again, this is diagonal and it contributes to the determinant with the term
\[ \det B = 2^{g(g-1)/2} \prod_{1 \leq i < j \leq g} \frac{v_j}{v_i} \prod_{1 \leq i < j \leq g} \frac{1}{v_j^2} = 2^{g(g-1)/2} \prod_{i=1}^{g} v_i^{2(i-1)} = 2^{g(g-1)/2} \prod_{i=1}^{g} v_i^{2i-2g}. \]

The term
\[ C := \sum_{i=1}^{g} \frac{1}{v_i^2} dw_i^2 \]
gives the contribution
\[ \det C = \prod_{i=1}^{g} \frac{1}{v_i^2}, \]
and by taking into account the factor \( \kappa/2 \) we finally have
\[ G = \left( \frac{\kappa}{2} \right)^{g(g+1)} \det A \det B \det C = \left( \frac{\kappa}{2} \right)^{g(g+1)} 2^{g(g-1)} \prod_{i=1}^{g} v_i^{2i-2g-2}. \]

Comparing with Proposition \( \square \) gives
\[ \frac{\kappa}{2} = 2^{-\frac{g}{g+1}}, \]
therefore one finally gets
By using lemma \[4\] we are then able to prove the following proposition.

\begin{proposition}
Let $\Delta$ be the $H_9$ Laplacian operator
\[
\Delta := -\frac{1}{\sqrt{|G|}} \partial_I \sqrt{|G|} G^{IJ} \partial_J,
\]
then
\[
(2.19) \quad \Delta E_{g,1}(\tau, s) = 2^{\frac{g+1}{2}} s (g-s) E_{g,1}(\tau, s).
\]
\end{proposition}

\begin{proof}
In Iwasawa coordinates $\sqrt{G} = \prod_{i=1}^{g} v_i^{i-g-2}$. With the help of lemma \[4\] by direct computation in Iwasawa coordinates one finds
\[
\Delta \left( \frac{\det(\mathcal{Z}(\tau))}{\det(\mathcal{Z}(\tau)_{22})} \right)^s = \Delta v_1^s = 2^{\frac{g+1}{2}} s (g-s) v_1^s,
\]
then, by $\Gamma_9$-invariance of the Laplacian operator one also has
\[
\Delta \left( \frac{\det(\mathcal{Z}(\gamma(\tau)))}{\det(\mathcal{Z}(\gamma(\tau))_{22})} \right)^s = 2^{\frac{g+1}{2}} s (g-s) \left( \frac{\det(\mathcal{Z}(\gamma(\tau)))}{\det(\mathcal{Z}(\gamma(\tau))_{22})} \right)^s, \quad \gamma \in Sp(2g, \mathbb{Z}),
\]
eq (2.19) then follows.
\end{proof}

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