1. Introduction

One way to understand symmetry of some objects is to look for what acts on them and study operations on these objects. In this way we study symmetry of groups by considering endofunctors $\phi : \text{Groups} \to \text{Groups}$. To understand how such an operation $\phi$ deforms groups we consider natural transformations $\epsilon_G : \phi(G) \to G$. A choice of a functor $\phi : \text{Groups} \to \text{Groups}$ and a natural transformation $\epsilon_G : \phi(G) \to G$ is called an augmented functor and denoted by $(\phi, \epsilon)$. By iterating the augmentation we obtain two homomorphisms $\epsilon^{\phi(G)} : \phi^2(G) \to \phi(G)$ and $\phi(\epsilon_G) : \phi^2(G) \to \phi(G)$. Among all augmented functors $(\phi, \epsilon)$ there are the idempotent ones for which this iteration process does not produce anything new and the homomorphisms $\epsilon^{\phi(G)}$ and $\phi(\epsilon_G)$ are isomorphisms for any group $G$. The universal central extension of the maximal perfect subgroup of $G$, with the natural projection as augmentation, is an example of an idempotent functor. Idempotent functors are related to the concept of cellularity which was introduced originally in homotopy theory and has been used to organize information about spaces. In recent years these functors have been considered in algebraic context of groups, chain complexes etc., see for example [A, Ca1, CaD, CaRoSce, CDFS, CFGS, E, FGS, FGSS, Fl, MP, RoSc].

The main aim of this paper is to understand how idempotent functor deform finite groups, particularly the simple ones. Our first result is (see Corollary 4.4, where preservation of nilpotency and solvability is also discussed):

**Theorem A.** Let $(\phi, \epsilon)$ be an idempotent functor. If $G$ is finite, then so is $\phi(G)$.

In this way finite groups are acted upon by idempotent functors. How complicated is this action? To measure it, we study the orbits of this action:

**Definition 1.1.** $\text{Idem}(G) := \{\text{isomorphism class of } \phi(G) \mid (\phi, \epsilon) \text{ is idempotent}\}$.

Although the collection of idempotent functors does not even form a set, the number of different values idempotent functors can take on a given finite group is finite (see Corollary 6.10):

**Theorem B.** If $G$ is a finite group, then $\text{Idem}(G)$ is a finite set.

One might then try to enumerate this set. One aim of this paper is to do that for finite simple groups for which we find that $\text{Idem}(G)$ has in general very few elements (see Corollary 7.10 and Section 11). Recall that by functoriality $\text{Aut}(G)$ acts on the Schur multiplier $H_2(G)$ of $G$. Let $\text{InvSub}(H_2(G))$ denote the set of all subgroups of $H_2(G)$ which are invariant (not necessarily pointwise fixed) under this action.

**Theorem C.** Let $G$ be a finite simple group. There is a bijection between $\text{Idem}(G)$ and the set:

$$\{0\} \coprod \text{InvSub}(H_2(G)).$$

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**Key words and phrases.** idempotent functor, cellular cover, central extensions, Schur multiplier.

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The bijection in the above theorem can be described explicitly. The value of an idempotent functor on $G$ could be the trivial group. This corresponds to the element $0$ in the above set. If the value is not trivial, then it can be constructed as follows: first take the universal central extension $H_2(G)\triangleleft E \to G$, then, for an invariant subgroup $K \subset H_2(G)$, take the quotient $E/K$. Such quotients are exactly the non-trivial values idempotent functors take on a simple group $G$.

The composition $(\phi', \epsilon', \epsilon'_{\phi(-)} \epsilon)$ of two idempotent functors $(\phi, \epsilon)$ and $(\phi', \epsilon')$ is not, in general, idempotent. Such compositions give new operations on groups. We can then study the orbits of the action of this broader collection of operations:

**Definition 1.2.** Let $n$ be a positive integer.

$$\text{Idem}^n(G) := \{\text{isomorphism classes of } \phi_1 \cdots \phi_n(G) \mid \text{for all } i, (\phi_i, \epsilon_i) \text{ is idempotent} \}$$

Since the identity functor, with the augmentation given by the identity, is idempotent, $\text{Idem}^1(G) \subset \text{Idem}^2(G) \subset \text{Idem}^3(G) \cdots$. Using this increasing sequence of inclusions, we define:

$$\text{Idem}^\infty(G) := \bigcup_{k \geq 1} \text{Idem}^k(G).$$

By applying all idempotent functors to a finite group, according to Theorem B, we get only finitely many values. We can then apply idempotent functors to these newly obtained groups to get yet again some finite set of groups. We can keep iterating this procedure. It turns out that this process eventually stabilizes, the set of values remains unchanged, and by repeating these operation arbitrary number of times we get only a finite number of different isomorphism classes of groups (see Corollary 10.1 and Proposition 10.2):

**Theorem D.** If $G$ is finite, then $\text{Idem}^\infty(G)$ is a finite set. If in addition $G$ is simple, then $\text{Idem}^\infty(G) = \text{Idem}^2(G)$.

## 2. Idempotent Functors and Cellular Covers

One well known example of an idempotent functor is given by the cellularization and many of the present results are extensions and generalizations of results and technique developed for these functors (see for example [Ca1] [CDFS] [RoSc] [Fl] [FGS].) Recall that, for any group $A$, there is a functor $\text{cell}_A: \text{Groups} \to \text{Groups}$ and a natural transformation $c_{A,G} : \text{cell}_A G \to G$. This augmentation is required to fulfill the following properties:

- $\text{Hom}(A, c_{A,G}) : \text{Hom}(A, \text{cell}_A G) \to \text{Hom}(A, G)$ is a bijection.
- For any group homomorphism $f : X \to Y$ for which $\text{Hom}(A, f)$ is a bijection, $\text{Hom}(\text{cell}_A G, f)$ is also a bijection.

Recall from [FGS] the notion of a cellular cover of a group $G$. This is a homomorphism $c : A \to G$ such that $\text{Hom}(A, c) : \text{Hom}(A, A) \to \text{Hom}(A, G)$ is a bijection. There are two facts to bare in mind when discussing cellular covers:

- if $c : A \to G$ is a cellular cover, then $A = \text{cell}_A G$, and
- for any idempotent functor of the form $(\text{cell}_A, c_A)$, $c_{A,G} : \text{cell}_A G \to G$ is a cellular cover.

The purpose of this section is to establish a bijection between $\text{Idem}(G)$ and equivalence classes of cellular covers of $G$:

**Definition 2.1.**

1. Two cellular covers $c : A \to G$ and $d : B \to G$ are defined to be equivalent if there is an isomorphism $h : A \to B$ for which $dh = c$.
2. The symbol $\text{Cov}(G)$ denotes the collection of equivalence classes of cellular covers of $G$.

**Lemma 2.2.**

1. If $c : A \to G$ and $d : B \to G$ are cellular covers for which $\text{Hom}(A, d)$ and $\text{Hom}(B, c)$ are bijections, then $c$ and $d$ are equivalent.
If $c : A \to G$ and $d : B \to G$ are cellular covers for which $A$ and $B$ are isomorphic groups, then $c$ and $d$ are equivalent.

**Proof.** For (1) note that the bijectivity of $\text{Hom}(A, d) : \text{Hom}(A, B) \to \text{Hom}(A, G)$ implies that there is a unique $h : A \to B$ for which $dh = c$. The same argument gives a unique $h' : B \to A$ for which $ch' = d$. We thus get equalities $ch'h = c$ and $dhh' = d$ which imply that $h'h = \text{id}_A$ and $hh' = \text{id}_B$, here we use again that $c$ and $d$ are cellular covers. Part (2) follows from (1) since if $A$ and $B$ are isomorphic, then the hypothesis of (1) holds. □

**Proposition 2.3.** Let $G$ be a group. The function assigning to the equivalence class of a cellular cover $c : A \to G$ the group $A$ is a bijection between $\text{Cov}(G)$ and $\text{Idem}(G)$.

**Proof.** Let $c : A \to G$ be a cellular cover. Then, by the remarks above $A$ is isomorphic to $\text{cell}_A G$ and hence it is the value of the idempotent functor $(\text{cell}_A, e_A)$. Thus our map takes an equivalence class of a cellular cover $c : A \to G$ to an element in $\text{Idem}(G)$. Also, by Lemma 2.2(2), our map is injective.

It remains to show that all elements in $\text{Idem}(G)$ are obtained in this way. Let $(\phi, \epsilon)$ be an idempotent functor. We claim that $\epsilon_G : \phi(G) \to G$ is a cellular cover, i.e., the map of sets $\text{Hom}(\phi(G), \epsilon_G) : \text{Hom}(\phi(G), \phi(G)) \to \text{Hom}(\phi(G), G)$ is a bijection.

We show the surjectivity first. Let $f : \phi(G) \to G$ be a homomorphism. Consider the following commutative square:

$$
\begin{array}{ccc}
\phi^2(G) & \phi(f) & \phi(G) \\
\epsilon_{\phi(G)} \downarrow & & \epsilon_G \\
\phi(G) & f & G
\end{array}
$$

Since $\epsilon_{\phi(G)}$ is an isomorphism, $f$ factors through $\epsilon_G$. As this happens for any $f$, $\text{Hom}(\phi(G), \epsilon_G)$ is a surjection.

It remains to show the injectivity of $\text{Hom}(\phi(G), \epsilon_G)$. Let $f, g : \phi(G) \to \phi(G)$ be two homomorphisms for which $\epsilon_G f = \epsilon_G g$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
\phi^2(G) & \phi(f) & \phi(G) \\
\epsilon_{\phi(G)} \downarrow & & \epsilon_G \\
\phi(G) & f & G
\end{array}
\begin{array}{ccc}
\phi^2(G) & \phi(g) & \phi(G) \\
\epsilon_{\phi(G)} \downarrow & & \epsilon_G \\
\phi(G) & g & G
\end{array}
$$

Since $\epsilon_G f = \epsilon_G g$, then $\phi(\epsilon_G) \phi(f) = \phi(\epsilon_G) \phi(g)$. As $\phi(\epsilon_G)$ is an isomorphism, $\phi(f) = \phi(g)$. Consequently $f \epsilon_{\phi(G)} = g \epsilon_{\phi(G)}$. Again since $\epsilon_{\phi(G)}$ is an isomorphism, $f = g$. □

According to 2.3 for any group $A$ representing an element in $\text{Idem}(G)$, there is a unique, up to an isomorphism of $A$, homomorphism $c : A \to G$ which is a cellular cover.

### 3. Nilpotent Groups

If the only group homomorphism from $G$ to $X$ is the trivial homomorphism, then we write $\text{Hom}(G, X) = 0$. The property of not having any non-trivial homomorphism into a given group is not preserved by subgroups in general. For example $\text{Hom}(Q, \mathbb{Z}/p) = 0$, however for the subgroup $\mathbb{Z} \subset Q$, $\text{Hom}(\mathbb{Z}, \mathbb{Z}/p) \neq 0$. Dually the property of not receiving any non-trivial homomorphism from a given group is not preserved by quotients in general. For example $\text{Hom}(\mathbb{Z}/p, Q) = 0$, however for the quotient $Q \to \mathbb{Z}/p^{\infty}$, $\text{Hom}(\mathbb{Z}/p, \mathbb{Z}/p^{\infty}) \neq 0$. The reason is that the subgroup $\mathbb{Z}$ and the quotient $\mathbb{Z}/p^{\infty}$ of $Q$ are too "small". The aim of this section is to show that if $G$ is nilpotent and...
Hom(G, X) = 0, then Hom(H, X) = 0 for any "big" subgroup H ⊆ G. Dually, if Hom(X, G) = 0, then Hom(X, H) = 0 for any "big" quotient G → H. The adjective big is clarified by the following proposition:

**Proposition 3.1.** Let G be a nilpotent group and X be a group.

1. If Hom(G, X) = 0, then, for any i, Hom(Γ_i(G), X) = 0.
2. If Hom(G, X) = 0, then Hom(H, X) = 0 for any normal subgroup H ≤ G which G/H is finitely generated.
3. If Hom(X, G) = 0, then Hom(X, G/H) = 0 for any finite normal subgroup H ≤ G.

This proposition can be used to show that certain groups are finite:

**Corollary 3.2.** Let K ⊆ H be a normal subgroup such that:

(a) K is nilpotent,
(b) Hom(H, K) = 0,
(c) H/K is finitely generated,
(d) for some j ≥ 1, K ∩ Γ_j(H) is finite.

Then K is finite and K ⊆ Γ_i(H) for any i ≥ 1.

**Proof.** Note that to prove the corollary it is enough to show K = K ∩ Γ_i(H) for any i ≥ j. Let G := H/Γ_i(H) and consider Q := K/(K∩Γ_i(H)). Then Q is (isomorphic to) a subgroup of G and G/Q is finitely generated. Assume that Q is non-trivial. Then, by Lemma 3.3(2) below, Hom(G, Q) ≠ 0, and then also Hom(H, Q) ≠ 0.

On the other hand, by hypothesis (b) Hom(H, K) = 0. As K ⊆ Γ_i(H) is finite and K is nilpotent Proposition 3.1(3) implies that Hom(H, Q) = 0. This contradiction completes the proof.

Our key tool to prove the following basic lemma. It provides existence of certain non-trivial homomorphisms. These are known facts whose proofs are included for self-containment.

**Lemma 3.3.** Let G be a nilpotent group.

1. For any i, there is a non-trivial homomorphism from G_{ab} to any non-trivial quotient of Γ_i(G).
2. If H ≤ G is a normal subgroup for which G/H is finitely generated, then there is a non-trivial homomorphism from G_{ab} to any non-trivial quotient of H.
3. Let H be a finite group. If Hom(G, H) = 0, then the function G ∋ g → g|H| ∈ G is a surjection.
4. If H is a finite proper normal subgroup of G, then there is a non-trivial homomorphism (G/H)_{ab} → G.

**Proof.** Recall that, for any sequence of elements x_2, ..., x_i in G, the following map of sets is a group homomorphism:

\[
G_{ab} = G/[G, G] \ni x[G, G] \mapsto [x, x_2, ..., x_i]Γ_{i+1}(G) ∈ Γ_i(G)/Γ_{i+1}(G).
\]

We will refer to it as the homomorphism given by the sequence x_2, ..., x_i and denote it by [−, x_2, ..., x_i].

(1): The proof is by reverse induction on i. Let j be maximal such that Γ_j(G) is non-trivial. For any proper subgroup K ⊆ Γ_j(G), there is an element [x_1, ..., x_j] of Γ_j(G) which does not belong to K. The desired non-trivial homomorphism is given by the composition:

\[
G_{ab} \xrightarrow{[−, x_2, ..., x_j]} Γ_j(G) \xrightarrow{\text{quotient}} Γ_j(G)/K.
\]

Assume i < j. Let K be a proper normal subgroup of Γ_i(G). There are two possibilities:
\[ \Gamma_{i+1}(G) \subset K: \text{ Let } [x_1, \ldots, x_i] \text{ be an element of } \Gamma_i(G) \text{ which does not belong to } K. \text{ The desired non-trivial homomorphism is given by the composition:} \]

\[ G_{ab} \xrightarrow{[\ldots, x_i]} \Gamma_i(G)/\Gamma_{i+1}(G) \xrightarrow{\text{quotient}} \Gamma_i(G)/K. \]

- \( \Gamma_{i+1}(G) \not\subset K: \) Let \( f : G_{ab} \rightarrow \Gamma_{i+1}(G)/(\Gamma_{i+1}(G) \cap K) \) be a non-trivial homomorphism which exists by the inductive assumption. The desired non-trivial homomorphism is given by the composition:

\[ G_{ab} \xrightarrow{f} \Gamma_{i+1}(G)/(\Gamma_{i+1}(G) \cap K) \xrightarrow{\text{quotient}} \Gamma_i(G)/K. \]

(2): Let \( K \) be a proper normal subgroup of \( H \). Our aim is to construct a non-trivial homomorphism \( G_{ab} \rightarrow H/K \). We consider two cases:

- **\( G/H \text{ is infinite:} \)** In this case, for some \( i \), \( \Gamma_i(G/H)/\Gamma_{i+1}(G/H) \) is a finitely generated infinite abelian group. For this \( i \), the group of integers \( \mathbb{Z} \) is a quotient of \( \Gamma_i(G/H)/\Gamma_{i+1}(G/H) \) and so \( \mathbb{Z} \) is a quotient of \( \Gamma_i(G) \). By (1), there is a non-trivial homomorphism \( G_{ab} \rightarrow \mathbb{Z} \). Its image must be isomorphic to \( \mathbb{Z} \) and consequently there is a surjection \( G_{ab} \rightarrow \mathbb{Z} \). The composition of this surjection with any non-trivial homomorphism \( \mathbb{Z} \rightarrow H/K \) is the desired non-trivial homomorphism \( G_{ab} \rightarrow H/K \).

- **\( G/H \text{ is finite:} \)** The desired non-trivial homomorphism \( G_{ab} \rightarrow H/K \) will be constructed by induction on the order of \( G/H \).

- \( |G/H| = 1: \) Since \( H = \Gamma_0(G) = G \), the existence of the non-trivial homomorphism \( G_{ab} \rightarrow \Gamma_0(G)/K \) is given by statement (1).

- \( |G/H| \) is a prime number \( p: \) In this case \( G/H \) is isomorphic to \( \mathbb{Z}/p \). We proceed by induction on the nilpotency class of \( G \).

  - \( G \) is abelian: The multiplication by \( p \) homomorphism \( p : G/K \rightarrow G/K \) factors as:

\[
\begin{array}{ccc}
G/K & \xrightarrow{f} & H/K \\
\downarrow & & \downarrow \\
G/K & \xrightarrow{f} & G/K.
\end{array}
\]

There are two possibilities:

- \( p : G/K \rightarrow G/K \) is non-trivial: In this case \( f \) can not be trivial either and the desired homomorphism can be taken to be the composition:

\[
\begin{array}{ccc}
G & \xrightarrow{\text{quotient}} & G/K \\
\downarrow & & \downarrow \\
G & \xrightarrow{f} & H/K.
\end{array}
\]

- \( p : G/K \rightarrow G/K \) is trivial: The abelian groups \( G/K \) and \( H/K \) are then \( \mathbb{Z}/p \)-vector spaces. As \( H/K \) is not trivial, it contains \( \mathbb{Z}/p \) as a subgroup and the desired homomorphism can be taken to be the composition:

\[
\begin{array}{ccc}
G & \xrightarrow{\text{quotient}} & G/H = \mathbb{Z}/p \\
\downarrow & & \downarrow \\
G & \xrightarrow{f} & H/K.
\end{array}
\]

- Let \( i = \max\{ j \mid \Gamma_j(G) \neq 1 \} > 0: \) Since \( G/H \) is abelian, \( \Gamma_i(G) \) is a subgroup of \( H \). There are two possibilities:

- \( \Gamma_i(G) \subset K: \) Consider the following sequence of groups:

\[
K/\Gamma_i(G) \leq H/\Gamma_i(G) \leq G/\Gamma_i(G).
\]

As the nilpotence class of \( G/\Gamma_i(G) \) is smaller than that of \( G \), by the inductive assumption there is a non-trivial homomorphism:

\[
f : (G/\Gamma_i(G))_{ab} \rightarrow (H/\Gamma_i(G))/((K/\Gamma_i(G)) = H/K
\]
The following composition gives the desired non-trivial homomorphism:
\[ G_{ab} \xrightarrow{\text{quotient}} (G/\Gamma_i(G))_{ab} \xrightarrow{f} H/K. \]

* \( \Gamma_i(G) \not\subset K \): Let \( \{x_1, \ldots, x_i\} \) be element of \( \Gamma_i(G) \) which does not belong to \( K \). The following composition gives the desired non-trivial homomorphism:
\[ G_{ab} \xrightarrow{[-x_2, \ldots, -x_i]} \Gamma_i(G) \xrightarrow{\text{quotient}} H \rightarrow H/K. \]

* \(|G/H| > 1\) and \(|G/H|\) is not a prime: Since \( G/H \) is finite and nilpotent and its order is not a prime number, there is a sequence of proper normal subgroups \( H \triangleleft L \triangleleft G \). By the inductive assumption there is a non-trivial homomorphism \( f : L_{ab} \rightarrow H/K \). Let \( Kf \) be the kernel of \( f \). By the inductive assumption applied to \( L \subseteq G \) there is also a non-trivial homomorphism \( g : G_{ab} \rightarrow L_{ab}/Kf \). The following composition gives the desired non-trivial homomorphism:
\[ G_{ab} \xrightarrow{g} L_{ab}/Kf \xrightarrow{\text{quotient}} H/K. \]

(3): We prove the statement by induction on the nilpotence class of \( G \).

- \( G \) is abelian: If \(|H| = 1\), the statement is clear. Assume \(|H| > 1\) and let \( p \) be a prime dividing \(|H|\). Since the group \( \mathbb{Z}/p \) is a subgroup of \( H \), we also have \( \text{Hom}(G, \mathbb{Z}/p) = 0 \). This means that \( G \otimes \mathbb{Z}/p = 0 \) and hence the multiplication by \( p \) homomorphism \( p : G \rightarrow G \) is a surjection. As this happens for all the primes dividing \(|H|\), same is true for the homomorphism \( G \ni g \mapsto |H|g \in G \).

- Let \( i = \max\{j \mid \Gamma_j(G) \neq 1\} > 0 \): We claim that:
\[ \text{Hom}(G/\Gamma_i(G), H) = 0, \quad \text{Hom}(\Gamma_i(G), H) = 0. \]

The first equality is clear as \( G/\Gamma_i(G) \) is a quotient of \( G \) and \( G \) has no non-trivial homomorphisms into \( H \). Let \( f : \Gamma_i(G) \rightarrow H \) be a homomorphism and \( L \subset H \) be its image. If \( L \) were non-trivial, then by statement (1), there would be a non-trivial homomorphism \( g : G_{ab} \rightarrow L \). The following composition would be then a non-trivial homomorphism from \( G \) to \( H \) which contradicts our assumption:
\[ G \xrightarrow{\text{quotient}} G_{ab} \xrightarrow{g} L \xrightarrow{\text{quotient}} H. \]

Let \( g \in G \). We need to show that there is an element whose \(|H|\)-th power is \( g \). Since the nilpotence class of \( G/\Gamma_i(G) \) is smaller than that of \( G \), by the inductive assumption, there is \( h \in G \) such that, for some \( a \in \Gamma_i(G) \), \( h^{|H|}a = g \). As \( \Gamma_i(G) \) is abelian and \( \text{Hom}(\Gamma_i(G), H) = 0 \), there is also \( b \in \Gamma_i(G) \) for which \( b^{|H|} = a \). The triviality of \( \Gamma_{i+1}(G) \) implies that \( b \) is central in \( G \). It follows that \( g = (hb)^{|H|} \).

(4): We first claim that we may assume that \( G/H \) is abelian. Indeed suppose part (4) holds in this case. Consider \( C_G(H) \). If \( C_G(H) \) is not central in \( G \), then pick \( x \in Z_2(G) \cap C_G(H) \). The map \( g \mapsto [g, x] \), for all \( g \in G \), is a non-trivial homomorphism from \( G \) to \( Z(G) \), that contains \( H \) in its kernel. Hence it induces a non-trivial homomorphism \( G/H \rightarrow G \) and we are done.

Hence \( C_G(H) = Z(G) \) and since \( G/C_G(H) \) is isomorphic to a subgroup of \( \text{Aut}(H) \) and \( H \) is finite, we conclude that \( G/Z(G) \) is finite and hence, by a theorem of Schur, \([G, G]\) is finite. But now, by our hypothesis there exists a non-trivial homomorphism \( G/H \rightarrow G \) and hence also a non-trivial homomorphism \( G/H \rightarrow G \).

It remains to prove part (4) under the hypothesis that \( G/H \) is abelian, i.e., \([G, G]\) is a subgroup of \( H \). Under this assumption we proceed by induction on the order \(|H|\) to show the existence of a non-trivial homomorphism \( G/H \rightarrow G \).

If \(|H| = 1\), then \( G \) and \( G/H \) are isomorphic and the statement is clear.
Assume \(|H| > 1\). If there is a non-trivial homomorphism \(G/H \to H\), then its composition with the inclusion \(H \subset G\) gives a non-trivial homomorphism \(G/H \to G\) and the statement is proven. Thus we can assume \(\text{Hom}(G/H, H) = 0\) and consequently, according to (3), \(G/H\) is \(|H|\)-divisible.

Consider the map
\[
h : G/H \to G/[G, G] \text{ defined by } Hg \mapsto [G, G]|H|.
\]
It is easy to check that this is a group homomorphism, its kernel \(K_h\) is annihilated by \(|H|\) and since \(G/H\) is \(|H|\)-divisible, \(\text{Hom}(G/H, K_h) = 0\).

We can use \(h\) to form the following pull-back square:

\[
\begin{array}{c}
K_h \\
\downarrow \downarrow \\
[G, G] \\
\downarrow \downarrow \\
[G, G] \\
\end{array} 
\begin{array}{c}
\leftarrow P \\
\downarrow \downarrow \\
G/\text{quotient} \\
\downarrow \downarrow \\
G/[G, G] \\
\end{array}
\begin{array}{c}
\leftarrow K_h \\
\downarrow \downarrow \\
\leftarrow \leftarrow \\
\downarrow \downarrow \\
\leftarrow \leftarrow \\
\end{array}
\]

There are two possibilities:

- \([G, G] = H\): In this case the non-trivial homomorphism \(G/H = G_{ab} \to G\) is given by statement (1).
- \([G, G] \subsetneq H\): Notice that the image \(P \to G/H\) is abelian, so we can apply the inductive assumption (with \(P\) in place of \(G\) and \([G, G]\) in place of \(H\)) to deduce that there is a non-trivial homomorphism \(\alpha : G/H \to P\). Consider the composition of \(\alpha\) with the vertical homomorphism \(h' : P \to G\) in the above diagram. If this composition were trivial then \(\alpha : G/H \to P\) would factor through \(K_h \subset P\). This however is impossible since there are no non-trivial homomorphisms from \(G/H\) to \(K_h\).

We are now ready to prove:

**Proof of (1)&(2):** Let \(H\) be either \(\Gamma_i(G)\) or a normal subgroup of \(G\) for which \(G/H\) is finitely generated. Assume \(\text{Hom}(G, X) = 0\). Let \(f : H \to X\) be a homomorphism. If \(f\) is non-trivial, then according by Lemma 3.3(1&2), there is a non-trivial homomorphism \(G_{ab} \to \text{im}(f)\). This implies the existence of a non-trivial homomorphism \(G \to X\) contradicting \(\text{Hom}(G, X) = 0\). Hence \(\text{Hom}(H, X) = 0\).

(3): Let \(f : X \to G/H\) be an arbitrary homomorphism. Consider its image \(B \subset G/H\) and the following pull-back square:

\[
\begin{array}{c}
H \\
\downarrow \\
\leftarrow \\
\downarrow \\
\leftarrow \\
\end{array} 
\begin{array}{c}
P \\
\downarrow \\
G/\text{quotient} \\
\downarrow \\
G/H \\
\end{array} 
\begin{array}{c}
\leftarrow B \\
\downarrow \\
\leftarrow \\
\downarrow \\
\leftarrow \\
\end{array}
\]

According to Lemma 3.3(4), if \(H \subset P\) were a proper subgroup, then there would be a non-trivial homomorphism \(B \to P\). The composition of this homomorphism with the injection \(P \hookrightarrow G\) would give a non-trivial homomorphism \(B \to G\). This however is impossible since the composition of this homomorphism with the \(f : X \to B\) would be also non-trivial contradicting the assumption \(\text{Hom}(X, G) = 0\). We can conclude that \(H = P\) and hence \(B = 0\). Consequently \(f\) is the trivial homomorphism and \(\text{Hom}(X, G/H) = 0\).
4. Generalized subgroups

Ultimately we would like to classify elements of $\text{Idem}(G)$ for a finite group $G$ using some classical invariants. According to [23] this is equivalent to the enumeration of $\text{Cov}(G)$. Unfortunately we are unable to enumerate $\text{Cov}(G)$. It turns out however that it is easier to give a classification for a bigger collection which is the subject of Section 6. In this section we define this bigger collection which we call generalized subgroups, and discuss some properties of its elements.

**Definition 4.1.** Let $G$ be a group.

1. A homomorphism $a : X \to G$ is called a generalized subgroup of $G$ if $\text{Hom}(X,a) : \text{Hom}(X,X) \to \text{Hom}(X,G)$ is an injection of sets (but not necessarily a bijection as it is in the case of a cellular cover of $G$).

2. Two generalized subgroups $a : X \to G$ and $b : Y \to G$ are defined to be equivalent if there is an isomorphism $h : X \to Y$ for which $bh = a$.

3. The symbol $\text{Sub}(G)$ denotes the collection of equivalence classes of generalized subgroups of $G$.

We start the study of generalized subgroups of $G$ by giving their direct characterization:

**Proposition 4.2.** A homomorphism $a : X \to G$ is a generalized subgroup of $G$ if and only if the following conditions are satisfied:

1. $\text{Hom}(X,\text{Ker}(a)) = 0$.
2. $\text{Hom}(X,\text{Ker}(a))$ is a central subgroup of $X$.

**Proof.** Assume first that $a : X \to G$ is a generalized subgroup of $G$. Let $x \in \text{Ker}(a)$. Consider the identity $\text{id} : X \to X$ and the conjugation $c_x : X \to X$. Then $ac_x = a\text{id}$. It follows that $c_x = \text{id}$ and hence $x$ is in the center of $X$. This shows (a).

Consider now the trivial homomorphism $X \to X$ and the composition of some $f : X \to \text{Ker}(a)$ with the inclusion $\text{Ker}(a) \subset X$. The compositions of these homomorphisms with $a$ are equal to the trivial homomorphism. Thus any such $f$ must be trivial and consequently $\text{Hom}(X,\text{Ker}(a)) = 0$ which is requirement (b).

Assume that conditions (a) and (b) are satisfied. We need to show injectivity of $\text{Hom}(X,a) : \text{Hom}(X,X) \to \text{Hom}(X,G)$. Let $f, g : X \to X$ be homomorphisms. Assume $af = ag$. This means that, for any $x \in X$, $f(x)g(x)^{-1}$ belongs to $\text{Ker}(a)$. We claim that the function $X \ni x \mapsto f(x)g(x)^{-1} \in \text{Ker}(a)$ is a group homomorphism. This follows from the fact that $\text{Ker}(a)$ is central in $X$:

$$f(xy)g(xy)^{-1} = f(x)f(y)g(y)^{-1}g(x)^{-1} = f(x)g(x)^{-1}f(y)g(y)^{-1}.$$ 

Since $\text{Hom}(X,\text{Ker}(a)) = 0$, we can conclude that $f(x)g(x)^{-1}$ is the identity element for any $x \in X$. Consequently $f = g$. \hfill $\square$

We can use this direct characterization to show that generalized subgroups of $G$ inherit certain properties of $G$.

**Proposition 4.3.** Let $a : X \to G$ be a generalized subgroup.

1. If $G$ is nilpotent, respectively solvable, then so is $X$.
2. If $G$ is finite, then so is $X$. Moreover $\text{Ker}(a) \subset \Gamma_i(X)$ for any $i$.
3. If $G$ is finitely generated and nilpotent, then $a : X \to G$ is an injection. In particular $X$ is also finitely generated.

**Proof.** (1): Assume $G$ is nilpotent and $\Gamma_i(G) = 0$. We claim that $\Gamma_i(X) = 0$. The assumption $\Gamma_i(G) = 0$ implies that $\Gamma_i(X)$ is in the kernel of $a$ and hence, according to [23] it is central in $X$. It follows that $\Gamma_{i+1}(X) = 0$ and $X$ is a nilpotent group. We can now use Lemma [3.3(1)]. If $\Gamma_i(X)$
were non-trivial, there would be a non-trivial homomorphism $X \to \Gamma_i(X)$. The composition of this homomorphism with the inclusion $\Gamma_i(X) \subset \text{Ker}(a)$ would be then also non-trivial. This contradicts the fact that $\text{Hom}(X, \text{Ker}(a)) = 0$ (see 11.2). Consequently $\Gamma_i(X) = 0$.

Similar argument works for solvable groups. If $G^{(i)} = 0$, then $X^{(i)} \subset \text{Ker}(a)$ and hence $X^{(i)}$ is central in $X$. This implies that $X^{(i+1)} = 0$ and consequently $X$ is solvable.

(2): Assume $G$ is finite. We apply 11.2 to the subgroup $K_a := \text{Ker}(a) \leq X$ to prove that $K_a$ is finite. It would then follow that $X$ is also finite. Since $K_a$ is central in $X$, it is abelian and hence nilpotent. This is hypothesis (a) of 3.2. Hypothesis (b) of 3.2 is condition (b) in 11.2. As $G$ is finite, then so is its subgroup $a(X) \cong X/K_a$. In particular this quotient is finitely generated and we get hypothesis (c) of 3.2. As $X/K_a$ is finite and $K_a$ is central in $X$, the quotient $X/Z(X)$ is also finite. It follows that the commutator $[X, X]$ is finite (see Robb 10.1.4, p. 287). In particular $K_a \cap [X, X]$ is finite and we get hypothesis (d) of 3.2. We can then conclude that $K_a$ is a finite group and $K_a \subset \Gamma_{i}(X)$ for any $i$.

(3): Assume $G$ is finitely generated and nilpotent. As in (2) we will apply 11.2 to the subgroup $K_a = \text{Ker}(a) \leq X$. Hypotheses (a) and (b) of 3.2 are clear. Since $G$ is finitely generated and nilpotent, then so is any of its subgroups. In particular $X/K_a$ is finitely generated. This shows that hypothesis (c) of 3.2 holds. As $G$ is nilpotent, there is $i$ for which $\Gamma_i(G) = 0$. It then follows that $\Gamma_i(X)$ is also trivial (see the proof of part (1)). In particular $K_a \cap \Gamma_i(X)$ is finite. We can conclude that $K_a \subset \Gamma_i(X) = 0$ and hence $a$ is an injection.

**Corollary 4.4.** Let $(\phi: \text{Groups} \to \text{Groups}, \epsilon: \phi \to \text{id})$ be an idempotent functor. If $G$ is $s$-nilpotent, or solvable, or finite, or finitely generated and $s$-nilpotent, then so is $\phi(G)$.

**Proof.** Recall from 2.3 that the map $\epsilon_G: \phi(G) \to G$ is a cellular cover. In particular, it is a generalized subgroup, and so the corollary follows from 4.3.

An inclusion $X \subset G$ is of course an example of a generalized subgroup of $G$. In the case $G$ is finitely generated and nilpotent all the generalized subgroups are inclusions (see 4.3(3)). In this case $\text{Sub}(G)$ is simply the collection of all the subgroups of $G$. For example the set $\text{Sub}(\mathbb{Z}/n)$, of subgroups of the cyclic group $\mathbb{Z}/n \ (n > 0)$, can be enumerated by the set $\{k \in \mathbb{Z} \mid k > 0 \text{ and } k \text{ divides } n\}$ of all positive divisors of $n$. For any such $k$, the corresponding subgroup is generated by $n/k$ and is isomorphic to $\mathbb{Z}/k$. Note that the inclusion $\mathbb{Z}/k \subset \mathbb{Z}/n$ is not only a generalized subgroup but it is also a cellular cover. Thus in this case we have an equality $\text{Cov}(\mathbb{Z}/n) = \text{Sub}(\mathbb{Z}/n)$. More generally let $A$ be a finite abelian group. Recall that, for an integer $k$, the $k$-torsion subgroup of $A$ consists of all $a \in A$ for which $ka = 0$.

**Proposition 4.5.** If $A$ is a finite abelian group, then:

$$\text{Cov}(A) = \{X \in \text{Sub}(A) \mid X \text{ is the } k \text{-torsion subgroup of } A \text{ for some } k\}$$

**Proof.** If $X \subset A$ is the $k$-torsion subgroup, then $X$ is $k$-torsion. Since any homomorphism $f: X \to A$ takes the $k$-torsion elements to the $k$-torsion elements, the image of $f$ sits in the subgroup $X \subset A$. This means that $X \subset A$ is a cellular cover.

Let $X \subset A$ be a cellular cover and $k$ be the exponent of $X$, i.e., the smallest positive integer $k$ for which $kX = 0$. Since $X$ is finite, there is a surjection $X \to \mathbb{Z}/k$. For any $k$-torsion element $x \in A$, consider the composition of this surjection $X \to \mathbb{Z}/k$ and a homomorphism $\mathbb{Z}/k \to A$ that maps some generator to the element $x$. Since $X \subset A$ is a cellular cover, the image of this composition has to lie in $X$. It follows that $X$ contains all the $k$-torsion elements of $A$. As $X$ consists of $k$-torsion elements, $X$ is the $k$-torsion subgroup of $A$. □
5. The initial cellular cover

The aim of this section is to construct an example of a cellular cover of a finite group, which we call the initial cellular cover, a generalization of the well known universal cover of a perfect group. This cellular cover will be used in our classification results in the following sections. The information about $G$ needed for our construction is contained in the first two homology groups of $G$. We therefore start with a brief recollection of some facts about the first two homology groups of finite groups and central extensions. We do it for self containment and to set up notation. We refer the reader to e.g. [G, Ka, Rob, W], for further information.

Recall that, for two finite abelian groups $A$ and $B$, the groups $A \otimes B$, $\text{Hom}(A, B)$, $\text{Hom}(B, A)$, $B \otimes A$, $\text{Ext}^1(A, B)$, and $\text{Ext}^1(B, A)$ are isomorphic. Thus all these groups are zero if and only if the orders of $A$ and $B$ are relatively prime.

For a group $G$, $H_n(G)$ denotes the $n$-th integral homology group of $G$. The first homology group $H_1(G)$ is naturally isomorphic to the abelianization $G/[G, G]$ of $G$. Via this isomorphism, for a homomorphism $f : X \to G$, $H_1(f) : X/[X, X] \to G/[G, G]$ is given by $x[X, X] \mapsto f(x)[G, G]$. The second homology group $H_2(G)$ is also called the Schur multiplier of $G$. Recall that if $G$ is finite, then, for any $n$, $H_n(G)$ is also a finite group whose exponent divides $|G|$. If $K$ is finite and cyclic, then $H_2(K) = 0$.

For an abelian group $K$ and a group $G$, a central extension of $G$ by $K$ is a group $X$ containing $K$ in its center and a surjective homomorphism $f : X \to G$ for which $\text{Ker}(f) = K$. Two such central extensions $f : X \to G$ and $f' : X' \to G$ are equivalent if there is a homomorphism $h : X \to X'$ for which $f'h = f$ and $h$ restricted to $K$ is the identity. Such $h$ necessarily has to be an isomorphism.

Let us recall that the equivalence classes of central extensions of $G$ by $K$ form a set which can be identified with the second cohomology group $H^2(G, K)$ (see [Rob, 11.1.4, p. 318]). An effective tool to study the group $H^2(G, K)$ is the universal coefficient exact sequence ([Rob, 11.4.18, p.349]):

$$
0 \to \text{Ext}^1(H_1(G), K) \to H^2(G, K) \xrightarrow{\mu} \text{Hom}(H_2(G), K) \to 0
$$

If $f : X \to G$ represents an equivalence class of a central extension of $G$ by $K$, then the homomorphism $\mu(f) : H_2(G) \to K$ is called the differential of $f$. This differential fits into the following exact sequence ([Ka 2.5.6]), called the exact sequence of $f$:

$$
H_2(X) \xrightarrow{H_2(f)} H_2(G) \xrightarrow{\mu(f)} K \xrightarrow{\alpha} H_1(X) \xrightarrow{H_1(f)} H_1(G) \to 0
$$

where the homomorphism $\alpha$ is given by $K \ni x \mapsto x[X, X] \in X/[X, X] = H_1(X)$. This sequence is functorial. This means that, for two central extensions $f : X \to G$ of $G$ by $K_f$ and $g : Y \to H$ of $H$ by $K_g$ that fit into the following commutative diagram:

\[
\begin{array}{ccc}
K_f & \xrightarrow{f} & G \\
\downarrow{h_1} & & \downarrow{h} \\
Y & \xrightarrow{g} & H \\
K_g & \xrightarrow{f'} & G \\
\end{array}
\]

the following diagram of the their exact sequences also commutes:

\[
\begin{array}{ccccccccc}
H_2(X) & \xrightarrow{H_2(f)} & H_2(G) & \xrightarrow{\mu(f)} & K_f & \xrightarrow{H_1(f)} & H_1(G) & \to 0 \\
H_2(K_f) & \downarrow{H_2(h_1)} & H_2(K) & \xrightarrow{h_1} & H_1(K_f) & \xrightarrow{H_1(h_1)} & H_1(K) & \to 0 \\
H_2(Y) & \xrightarrow{H_2(g)} & H_2(H) & \xrightarrow{\mu(g)} & K_g & \xrightarrow{H_1(g)} & H_1(H) & \to 0 \\
H_2(K_g) & \downarrow{H_2(h)} & H_2(K) & \xrightarrow{h} & H_1(K_g) & \xrightarrow{H_1(h)} & H_1(K) & \to 0 \\
\end{array}
\]
Under the assumption that $X$ is finite (it is actually enough to assume that only $K$ is finite), the exact sequence of $f$ can be extended one step further to an exact sequence, called also the exact sequence of $f$:

$$H_1(X) \otimes K \rightarrow H_2(X) \xrightarrow{H_2(f)} H_2(G) \xrightarrow{\mu(f)} K \xrightarrow{\alpha} H_1(X) \xrightarrow{H_1(f)} H_1(G) \rightarrow 0$$

**Definition 5.1.** For a finite group $G$, $H_{2\setminus 1}(G)$ denotes the localization $H_2(G)[S^{-1}]$ where $S$ is the set of primes that divide the order of $H_1(G)$.

The group $H_{2\setminus 1}(G)$ is simply the quotient of $H_2(G)$ by the $S$-torsion, and the localization homomorphism $H_2(G) \rightarrow H_2(G)[S^{-1}] = H_{2\setminus 1}(G)$ is the quotient homomorphism. Since the orders of $H_2(G)[S^{-1}]$ and $H_1(G)$ are coprime, the group $\text{Ext}^1(H_1(G), H_{2\setminus 1}(G))$ is trivial. The homomorphism $\mu : H^2(G, H_{2\setminus 1}(G)) \rightarrow \text{Hom}(H_2(G), H_{2\setminus 1}(G))$ is therefore an isomorphism. It follows that there is a unique central extension $e_G : E \rightarrow G$ of $G$ by $H_{2\setminus 1}(G)$ whose differential $\mu(e_G)$ is the localization homomorphism:

$$H_2(G) \xrightarrow{\text{localization}} H_2(G)[S^{-1}] \xrightarrow{\mu(e_G)} H_{2\setminus 1}(G).$$

We call the extension $e_G : E \rightarrow G$ the initial extension of $G$. In the case $G$ is perfect, i.e., if $H_1(G) = 0$, then $H_{2\setminus 1}(G) = H_2(G)$ and the initial extension is the universal central extension of $G$.

The key property of the initial extension of a finite group $G$ is that its differential $\mu(e_G) : H_2(G) \rightarrow H_{2\setminus 1}(G)$ is a surjection (this means that $e_G$ is a, so called, stem extension).

**Proposition 5.2.** Let $G$ be a finite group and $f : X \rightarrow G$ be a central extension of $G$ by $H_{2\setminus 1}(G)$ whose differential $\mu(f) : H_2(G) \rightarrow H_{2\setminus 1}(G)$ is a surjection. Then:

1. $H_1(f) : H_1(X) \rightarrow H_1(G)$ is an isomorphism.
2. The following is an exact sequence:
   
   $$0 \rightarrow H_2(X) \xrightarrow{H_2(f)} H_2(G) \xrightarrow{\mu(f)} H_{2\setminus 1}(G) \rightarrow 0$$
3. $H^2(X, H_{2\setminus 1}(G)) = 0$.
4. The homomorphism $f : X \rightarrow G$ is a cellular cover (i.e. $\text{Hom}(X, f)$ is a bijection).
5. The cellular covers $f : X \rightarrow G$ and $e_G : E \rightarrow G$ are equivalent.

**Proof.** Since $G$ is finite, $H_2(G)$ is finite and so is its quotient $H_{2\setminus 1}(G)$. The group $X$ is then also finite and we have the following exact sequence:

$$H_1(X) \otimes H_{2\setminus 1}(G) \rightarrow H_2(X) \xrightarrow{H_2(f)} H_2(G) \xrightarrow{\mu(f)} H_{2\setminus 1}(G) \xrightarrow{\alpha} H_1(X) \xrightarrow{H_1(f)} H_1(G) \rightarrow 0$$

(1): As $\mu(f)$ is a surjection, the homomorphism $\alpha$, in the above sequence, is trivial, and hence $H_1(f) : H_1(X) \rightarrow H_1(G)$ is an isomorphism. This is statement (1).

(2): The orders of $H_1(G)$ and $H_{2\setminus 1}(G)$ are coprime and thus $H_1(G) \otimes H_{2\setminus 1}(G) = 0$. Using statement (1), we then get $H_1(X) \otimes H_{2\setminus 1}(G) = 0$. The homomorphism $H_2(f)$ is therefore an injection which proves statement (2).

(3): By the universal coefficient exact sequence, to show the statement, we need to prove that $\text{Ext}^1(H_1(X), H_{2\setminus 1}(G)) = 0$ and $\text{Hom}(H_2(X), H_{2\setminus 1}(G)) = 0$. The triviality of $\text{Ext}^1(H_1(X), H_{2\setminus 1}(G))$ follows from the fact that the orders of $H_1(X) = H_1(G)$ and $H_{2\setminus 1}(G)$ are coprime.

Since $H_{2\setminus 1}(G)$ is the localization $H_2(G)[S^{-1}]$, where $S$ is the set of primes that divide the order of $H_1(G)$, the homomorphism $\mu(f)$ factors uniquely as:

$$H_2(G) \xrightarrow{\text{localization}} H_{2\setminus 1}(G) \xrightarrow{\mu(f)} H_{2\setminus 1}(G).$$
The surjectivity of $\mu(f)$ implies the surjectivity of $h$. As a surjective homomorphism between finite groups, $h$ is an isomorphism. The kernel of $\mu(f)$, which by (2) is given by $H_2(X)$, is therefore isomorphic to the kernel of the localization homomorphism $H_2(G) \to H_2(G)[S^{-1}]$. The primes dividing the order of $H_2(X)$ are thus among the primes dividing the order of $H_1(G)$. This means that the orders of $H_2(X)$ and $H_{2\downarrow\uparrow}(G)$ are coprime and hence the group $\text{Hom}(H_2(X), H_{2\downarrow\uparrow}(G))$ is also trivial.

(4): We need to show $\text{Hom}(X, f) : \text{Hom}(X, X) \to \text{Hom}(X, G)$ is a bijection. The kernel $H_{2\downarrow\uparrow}(G)$ of $f : X \to G$ is central in $X$. Moreover, as the orders of $H_{2\downarrow\uparrow}(G)$ and $H_1(X)$ are relatively prime, $\text{Hom}(X, H_{2\downarrow\uparrow}(G)) = \text{Hom}(H_1(X), H_{2\downarrow\uparrow}(G)) = 0$. The injectivity of $\text{Hom}(X, f)$ follows then from 1.2.

It remains to prove that $\text{Hom}(X, f) : \text{Hom}(X, X) \to \text{Hom}(X, G)$ is also surjective. Let $g : X \to G$ be an arbitrary homomorphism. Consider the following commutative diagram, where the right hand square is a pull-back square:

\[
\begin{array}{ccc}
H_{2\downarrow\uparrow}(G) & \xrightarrow{P} & X \\
\downarrow & & \downarrow \ \\
H_{2\downarrow\uparrow}(G) & \xrightarrow{X} & G
\end{array}
\]

Note that $f' : P \to X$ represents a central extension of $X$ by $H_{2\downarrow\uparrow}(G)$. According to statement (3) any such central extension is split. Let $s : X \to P$ be its section. The composition $g' s : X \to X$ is then a homomorphism for which $f g' s = g$. This shows surjectivity of $\text{Hom}(X, f)$.

(5): The argument to show that $f : X \to G$ and $e_G : E \to G$ are equivalent cellular covers is the same as in the proof of the surjectivity in the previous statement. Consider the following commutative diagram, where the bottom right square is a pull-back square:

\[
\begin{array}{ccc}
H_{2\downarrow\uparrow}(G) & \xrightarrow{P} & E \\
\downarrow & & \downarrow \ \\
H_{2\downarrow\uparrow}(G) & \xrightarrow{X} & G
\end{array}
\]

Both $e' : P \to X$ and $f' : P \to E$ represent central extensions. Statement (3) implies that these extensions are split. Using their sections we can construct homomorphisms $h : X \to E$ and $g : E \to X$ for which $e_G h = f$ and $f g = e_G$. It follows that $e_G h g = e_G$ and $f g h = f$. As $e_G$ and $f$ are cellular covers, we can conclude $h g = \text{id}_E$ and $g h = \text{id}_X$. This proves (5).

\[\square\]

**Definition 5.3.** Let $G$ be a finite group. We call the homomorphism $e_G : E \to G$ the **initial cellular cover** of $G$. We will use the same name for the equivalence class in $\text{Cov}(G)$ represented by $e_G : E \to G$.

### 6. Generalized subgroups of a finite group

The collection $\text{Cov}(G)$ is a subcollection of $\text{Sub}(G)$. Thus to show for example that $G$ has finitely many cellular covers it is enough to show that $\text{Sub}(G)$ is a finite set. The aim of this section is to do that under the assumption that $G$ is a finite group.

For a homomorphism $a : X \to G$, we use the symbol $I_a$ to denote its image $\text{im}(a)$. This is one of the two invariants we use to enumerate generalized subgroups of $G$. Note that if generalized subgroups $a : X \to G$ and $b : Y \to G$ are equivalent, then they have the same images. Thus the function $a \mapsto I_a$ is well defined on the collection $\text{Sub}(G)$ of equivalence classes of generalized subgroups. Furthermore
it is immediate from the definition that a homomorphism \( a : X \to G \) is a generalized subgroup of \( G \) if and only if \( a : X \to I_a \) is a generalized subgroup of \( I_a \). Thus any generalized subgroup is a composition of a surjective generalized subgroup and an inclusion. This is the reasons why surjective generalized subgroups are important for us.

**Definition 6.1.** \( \text{SurSub}(G) \) denotes the collection of equivalence classes of generalized subgroups of \( G \) which are represented by surjective homomorphisms.

For any subgroup \( I \) of \( G \), let \( \text{in}_I : \text{SurSub}(I) \subset \text{Sub}(G) \) be the function that assigns to an equivalence class of a surjective generalized subgroup \( a : X \to I \) of \( I \) the equivalence class of the composition \( a : X \to I \subset G \). By summing up these inclusions over all the subgroups of \( G \), it is then clear that we get a bijection:

**Proposition 6.2.** The following function is a bijection:

\[
\prod_{I \subset G} \text{in}_I : \prod_{I \subset G} \text{SurSub}(I) \to \text{Sub}(G)
\]

To enumerate \( \text{Sub}(G) \) it thus suffices to enumerate \( \text{SurSub}(I) \) for all subgroups \( I \) of \( G \). We will do that under the assumption that \( G \) is finite. Let us then assume that \( G \) is finite.

We start with defining a set used to enumerate the collections \( \text{SurSub}(I) \).

**Definition 6.3.** Let \( A \) be an abelian group.

1. Two surjections \( \sigma : A \to K \) and \( \tau : A \to L \) are defined to be equivalent, if there is an isomorphism \( h : K \to L \) such that \( h\sigma = \tau \) (such an isomorphism, if it exists, is necessary unique).

2. The symbol \( \text{Quot}(A) \) denotes the set of equivalence classes of surjections out of \( A \).

Note that the subgroup of \( A \) given by the kernel of a surjection \( \sigma : A \to K \) depends only on the equivalence class of \( \sigma \) in \( \text{Quot}(A) \). It is then clear that the function that assigns to an element \( [\sigma] \) in \( \text{Quot}(A) \) the subgroup \( \text{Ker}(\sigma) \) of \( A \) is a bijection between \( \text{Quot}(A) \) and the set of all the subgroups of \( A \) which, in the case \( A \) is finitely generated, coincides with the set \( \text{Sub}(A) \). Thus for a finitely generated abelian group \( A \), we shall identify \( \text{Quot}(A) \) with \( \text{Sub}(A) \). For example let \( k \) be a positive integer. The element of \( \text{Quot}(A) \) that corresponds to the \( k \)-torsion subgroup of \( A \) is denoted by \( q_k \).

It is represented by the surjection, denoted by the same symbol

\[ q_k : A \to A/(k\text{-torsion}), \]

that maps an element \( a \in A \) to its coset. In the case of the cyclic group \( \mathbb{Z}/n \) \( (n > 0) \), these are all the elements of \( \text{Quot}(\mathbb{Z}/n) \). For any \( k > 0 \) dividing \( n \), the \( k \)-torsion subgroup of \( \mathbb{Z}/n \) is the subgroup generated by \( n/k \). It is the unique subgroup isomorphic to \( \mathbb{Z}/k \). In this way \( \text{Quot}(\mathbb{Z}/n) \) is in bijection with the set of all positive divisors of \( n \).

From now until Definition 6.8 we fix a subgroup \( I \) of \( G \). We enumerate \( \text{SurSub}(I) \) using the set \( \text{Quot}(H_{2\setminus1}(I)) \) (recall that \( H_{2\setminus1}(I) \) denotes the localization \( H_2(I)[S^{-1}] \), where \( S \) is the set of primes dividing the order of \( H_I(1) \), see [5.1]). To do that we define two functions:

\[
\mu : \text{SurSub}(I) \to \text{Quot}(H_{2\setminus1}(I)), \quad \Psi : \text{Quot}(H_{2\setminus1}(I)) \to \text{SurSub}(I),
\]

and show that their compositions \( \mu \Psi \) and \( \Psi \mu \) are the identities. For a surjective generalized subgroup \( a : X \to I \), the value \( \mu(a) \in \text{Quot}(H_{2\setminus1}(I)) \) is called the differential of \( a \). Recall that according to [4.2] the kernel \( K_a := \text{Ker}(a) \) of \( a \) is central in \( X \). Thus the homomorphism \( a : X \to I \) represents a central extension of \( I \) by \( K_a \). We use the corresponding element in \( H^2(I, K_a) \) to define the differential. First we need:

**Proposition 6.4.** Let \( a : X \to I \) be a surjective generalized subgroup of a finite group \( I \). Then:
(1) \(H_1(a) : H_1(X) \rightarrow H_1(I)\) is an isomorphism.

(2) \(\text{Ext}^1(H_1(I), K_a) = H_1(X) \otimes K_a = 0\).

(3) \(\mu : H^2(I, K_a) \rightarrow \text{Hom}(H_2(I), K_a)\) is an isomorphism.

(4) \(0 \rightarrow H_2(X) \overset{H_2(a)}{\rightarrow} H_2(I) \overset{\mu(a)}{\rightarrow} K_a \rightarrow 0\) is an exact sequence.

(5) If \(Y \subset X\) is a subgroup such that \(a(Y) = I\), then \(Y = X\).

**Proof.** Since \(G\) is finite, by \([4.3](2)\), \(X\) is also finite and we have an exact sequence of the central extension \(a : X \rightarrow G:\)

\[
H_1(X) \otimes K_a \rightarrow H_2(X) \overset{H_2(a)}{\rightarrow} H_2(I) \overset{\mu(a)}{\rightarrow} K_a \rightarrow H_1(X) \overset{H_1(a)}{\rightarrow} H_1(I) \rightarrow 0
\]

(1): The finiteness of \(G\) implies also that \(K_a \subset [X, X]\) (see \([4.3](2)\)). The homomorphism \(\alpha : K_a \rightarrow H_1(X)\), in the above exact sequence, is then trivial and \(H_1(a)\) is an isomorphism. This proves (1).

(2): According to \([4.2](2)\), \(\text{Hom}(H_1(X), K_a) = \text{Hom}(X, K_a) = 0\). The orders of \(H_1(X)\) and \(K_a\) are therefore relatively prime numbers. As \(H_1(X)\) and \(H_1(I)\) are isomorphic (statement (1)), we get \(\text{Ext}^1(H_1(I), K_a) = H_1(X) \otimes K_a = 0\) which is statement (2).

(3): This is a consequence of the universal coefficient exact sequence and the triviality of \(\text{Ext}^1(H_1(I), K_a)\) (statement (2)).

(4): This follows from the exact sequence of the central extension \(a : X \rightarrow G\) above and the triviality of \(H_1(X) \otimes K_a\) (statement (2)).

(5): We have \(X = Y K_a\) and since \(K_a\) is central in \(X\) we get that \([X, X] = [Y, Y]\). However, as we observed earlier in the proof, \(K_a \subset [X, X]\) and it follows that \(K_a \subset Y\) and so \(Y = X\). \(\square\)

**The differential of a surjective generalized subgroup.** If \(a : X \rightarrow I\) is a surjective generalized subgroup, then according to \([6.3](3)\), the homomorphism \(\mu : H^2(I, K_a) \rightarrow \text{Hom}(H_2(I), K_a)\) is an isomorphism. The extension \(a : X \rightarrow I\), which is an element of \(H^2(I, K_a)\), can be then identified with the homomorphism \(\mu(a) : H_2(I) \rightarrow K_a\). According to \([6.3](4)\) such homomorphisms associated with generalized subgroups are surjections. Furthermore, as \(H_1(I) \otimes K_a = 0\) (see \([6.3](2)\)), the primes that divide the order of \(H_1(I)\) do not divide the order of \(K_a\). This means that the localization \(K_a \rightarrow K_a[S^{-1}]\) is an isomorphism, where \(S\) is the set of primes that divide the order of \(H_1(I)\). Consequently \(\mu(a) : H_2(I) \rightarrow K_a\) factors uniquely as:

\[
H_2(I) \xrightarrow{\text{localization}} H_2(I)[S^{-1}] = H_2\backslash_1(I_a) \xrightarrow{\mu(a)} K_a.
\]

We will use the same symbol \(\mu(a) : H_2\backslash_1(I) \rightarrow K_a\) to denote the surjection in the above factorization. We can now define:

**Definition 6.5.** Let \(a : X \rightarrow I\) be a surjective generalized subgroup of \(I\). The element in \(\text{Quot}(H_2\backslash_1(I))\) represented by the surjection \(\mu(a) : H_2\backslash_1(I) \rightarrow K_a\) is called the deferential of \(a\) and is denoted also by the same symbol \(\mu(a)\).

Assume now that \(a : X \rightarrow I\) and \(b : Y \rightarrow I\) are equivalent surjective generalized subgroups of \(I\) and \(h : X \rightarrow Y\) is an isomorphism for which \(bh = a\). By the naturality of the exact sequence of a central extension, we get a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H_2(X) & \overset{H_2(a)}{\rightarrow} & H_2(I) & \overset{\mu(a)}{\rightarrow} & K_a & \rightarrow & 0 \\
\downarrow H_2(h) & & \downarrow \text{id} & & \downarrow h & & \downarrow & & \\
0 & \rightarrow & H_2(Y) & \overset{H_2(b)}{\rightarrow} & H_2(I) & \overset{\mu(b)}{\rightarrow} & K_b & \rightarrow & 0 \\
\end{array}
\]
After localizing with respect to the set $S$ of primes that divide the order of $H_1(I)$, we get then that $\mu(b) : H_{2\setminus 1}(I) \to K_a$ is the composition of $\mu(a) : H_{2\setminus 1}(I) \to K_a$ and the isomorphism $h : K_a \to K_b$. The surjections $\mu(a)$ and $\mu(b)$ are thus equivalent and define the same element in $\text{Quot}(H_{2\setminus 1}(I))$. It follows that the differential is well defined on the collection $\text{SurSub}(I)$ of equivalence classes of generalized subgroups. In this way we get a well-defined function:

$$\text{SurSub}(I) \ni [a : X \to I] \overset{\mu}{\longrightarrow} \mu(a) \in \text{Quot}(H_{2\setminus 1}(I)),$$

which we also denote by $\mu$.

Next we define an inverse to $\mu$ (see [6, Thm.], which we denote by $\Psi : \text{Quot}(H_{2\setminus 1}(I)) \to \text{SurSub}(I)$. Let us choose a surjection $\sigma : H_{2\setminus 1}(I) \to K$ that represents a given element in $\text{Quot}(H_{2\setminus 1}(I))$. Recall that $e_I : E \to I$ denotes the initial central extension of $I$ by $H_{2\setminus 1}(I)$ (see Section 5). Define:

$$X := \text{colim}( K \xrightarrow{\sigma} H_{2\setminus 1}(I)^c \xrightarrow{=} E ).$$

This is just $E$ divided by the kernel of the map $H_{2\setminus 1}(I) \to K$. Let $a : X \to I$ be the homomorphism that fits into the following commutative diagram where $\pi : E \to X$ is the structure map of the colimit:

$$
\begin{array}{cccccc}
H_{2\setminus 1}(I)^c & \xrightarrow{=} & E & \xrightarrow{e_I} & I \\
\downarrow{\sigma} & & \downarrow{\pi} & & \downarrow{id} \\
K^c & \xrightarrow{=} & X & \xrightarrow{a} & I \\
\end{array}
$$

Note that $a : X \to I$ is a central extension of $I$ by $K$. By the naturality of the exact sequence of a central extension we get a commutative diagram of homology groups:

$$
\begin{array}{cccccc}
H_2(I) & \xrightarrow{\mu(e_I)} & H_{2\setminus 1}(I) & \xrightarrow{\sigma} & H_1(E) & \xrightarrow{H_1(e_I)} & H_1(I) & \to 0 \\
\downarrow{id} & & \downarrow{\mu(a)} & & \downarrow{\alpha} & & \downarrow{id} & \\
H_2(I) & \xrightarrow{\mu(a)} & K & \xrightarrow{\sigma} & H_1(X) & \xrightarrow{H_1(a)} & H_1(I) & \to 0 \\
\end{array}
$$

As $\mu(e_I)$ and $\sigma$ are surjections, then so is $\mu(a)$. The homomorphism $\alpha$ is therefore trivial and consequently $H_1(a) : H_1(X) \to H_1(I)$ is an isomorphism. Since $K$ is a quotient of $H_{2\setminus 1}(I)$, the primes that divide the order of $H_1(X)$ do not divide the order of $K$. It follows that $\text{Hom}(X,K) = \text{Hom}(H_1(X),K) = 0$. We define $\Psi([\sigma])$ to be the element of $\text{SurSub}(I)$ given by the equivalence class represented by this surjective generalized subgroup $a : X \to I$. It is straight forward to check that $\Psi([\sigma])$ does not depend on the choice of a surjection $\sigma : H_{2\setminus 1}(I) \to K$ representing the given element in $\text{Quot}(H_{2\setminus 1}(I))$. In this way we have a well defined function $\Psi : \text{Quot}(H_{2\setminus 1}(I)) \to \text{SurSub}(I)$. Note further that $\sigma = \mu(a)$. This means that $\mu\Psi = \text{id}$.

To show that $\Psi\mu$ is also the identity, let us choose a surjective generalized subgroup $a : X \to I$. Let $b : Y \to I$ be a surjective generalized subgroup representing $\Psi\mu(a)$. We need to show that $a$ and $b$ are equivalent. Consider the pair $\mu(a)$. Since $\mu\Psi$ is the identity:

$$\mu(b) = \mu\Psi\mu(a) = \mu(a)$$

This means that the differential $\mu(b)$ is equivalent to $\mu(a)$. As in this case the differential determines the central extension it comes from, $a$ and $b$ are indeed equivalent generalized subgroups. We just have shown:

**Theorem 6.6.** Let $I$ be a finite group. The function $\mu : \text{SurSub}(I) \to \text{Quot}(H_{2\setminus 1}(I))$ is a bijection.

For reference we record the equalities $\mu\Psi = \text{id}$ and $\Psi\mu = \text{id}$ in the form of:
Proposition 6.7. Any surjective generalized subgroup $a : X \to I$ fits into the following commutative ladder of short exact sequences with the left square being a push-out:

$$
\begin{array}{c}
H_2 \downarrow \leftarrow E \rightarrow I \\
\mu(a) \downarrow \pi \rightarrow id \\
K_a \downarrow X \rightarrow I
\end{array}
$$

Theorem 6.6 can be used to enumerate all the generalized subgroups of $G$.

Definition 6.8. $\text{In}(G)$ is defined to be the set of pairs $(I, \sigma)$ where $I$ is a subgroup of $G$ and $\sigma \in \text{Quot}(H_2(I))$.

As a corollary of 6.6 we get:

Corollary 6.9. For a finite group $G$, the following function is a bijection between $\text{Sub}(G)$ and $\text{In}(G)$:

$$\text{Sub}(G) \ni [a : X \to G] \mapsto (I_a, \mu(a : X \to I_a)) \in \text{In}(G)$$

Since $\text{Cov}(G)$ is a subcollection of $\text{Sub}(G)$ and $\text{In}(G)$ is finite, 6.9 implies:

Corollary 6.10. For a finite group $G$, the collections $\text{Sub}(G)$, $\text{Cov}(G)$, and $\text{Idem}(G)$ are finite sets.

7. Surjective cellular covers of finite groups

Recall that $G$ is assumed to be finite. According to 6.9, a generalized subgroup of such a group $G$ is determined by two invariants: its image and its differential. For a given subgroup $I \subset G$ the differential classify all possible generalized subgroups of $G$ whose image is $I$, or equivalently generalized subgroup of $I$ which are represented by surjective homomorphisms. Thus to classify cellular covers we need to determined first the subgroups of $G$ which are images of cellular covers and then, for any such subgroup $I$, identify these surjective generalized subgroups of $I$ which are cellular covers of $G$. Unfortunately we can not say much about the first step in this process. We do not know how to identify subgroups of $G$ which are images of cellular covers. However we can deal with the second step: the cellular covers of finite groups which are represented by surjective homomorphisms in several important cases. This is the aim of this section.

Definition 7.1. $\text{SurCov}(G)$ denotes the collection of cellular covers of $G$ that are represented by surjective homomorphisms.

$H_2 \downarrow \uparrow$ as a functor. To describe surjective covers we use functorial properties of $H_2$ (see 5.1). Any homomorphism $f : X \to G$ induces a homomorphism of the Schur multipliers $H_2(f) : H_2(X) \to H_2(G)$ (the Schur multiplier is a functor). In general this homomorphism $H_2(f)$ however does not induce a homomorphism between $H_2(X)$ and $H_2(G)$. For that we need an additional assumption on $X$ and $G$. We need to assume that both $X$ and $G$ are finite and that the set $S_X$ of primes that divide the order of $H_1(X)$ is a subset of the set $S_G$ of primes that divide the order of $H_1(G)$. In this case there is a unique homomorphism:

$$H_2 \downarrow \uparrow(f) : H_2(X) = H_2(X)[S_X^{-1}] \to H_2(G)[S_G^{-1}] = H_2(G)$$

for which the following diagram commutes:

$$
\begin{array}{c}
H_2(X) \xrightarrow{\text{localization}} H_2(X)[S_X^{-1}] \xrightarrow{H_2(f)} H_2(G)[S_G^{-1}] \xrightarrow{H_2(f)} H_2(G)
\end{array}
$$
This is because $H_{2\setminus 1}(G)$ is uniquely divisible by the primes in $S_X$. Observe further the uniqueness implies $H_{2\setminus 1}(fg) = H_{2\setminus 1}(f)H_{2\setminus 1}(g)$, for any two homomorphisms $g : Y \rightarrow X$ and $f : X \rightarrow G$ for which the inclusions $S_Y \subset S_X \subset S_G$ of sets of prime numbers that divide the corresponding orders of the abelianizations hold.

We conclude that $H_{2\setminus 1}$ is a functor on the full subcategory of finite groups with a fixed isomorphism type of $H_1$. For example let $X$ and $G$ be finite groups for which there is a surjective homomorphism $c : X \rightarrow G$ which is a generalized subgroup. By [6.31], $H_1(X)$ and $H_1(G)$ are isomorphic. Thus, for such groups $X$ and $G$, any homomorphism $f : X \rightarrow G$ induces $H_{2\setminus 1}(f) : H_{2\setminus 1}(X) \rightarrow H_{2\setminus 1}(G)$.

Since $G$ is finite, according to [6.34], a surjective generalized subgroup $c : X \rightarrow G$ induces an exact sequence:

$$0 \rightarrow H_2(X) \xrightarrow{H_2(c)} H_2(G) \xrightarrow{\mu(c)} K_c \rightarrow 0.$$  

After localization with respect to the set $S$ of primes that divide the order of $H_1(X) = H_1(G)$ (see [6.31]), we get again an exact sequence:

$$0 \rightarrow H_{2\setminus 1}(X) \xrightarrow{H_{2\setminus 1}(c)} H_{2\setminus 1}(G) \xrightarrow{\mu(c)} K_c \rightarrow 0.$$  

Thus the kernel of the differential $\mu(c) : H_{2\setminus 1}(G) \rightarrow K_c$ is given by the image of $H_{2\setminus 1}(c) : H_{2\setminus 1}(X) \subset H_{2\setminus 1}(G)$ which we simply denote by $H_{2\setminus 1}(X)$.

To enumerate the set $\text{SurCov}(G)$ of surjective cellular covers of $G$ we need to understand for which surjective generalized subgroups $c : X \rightarrow G$ the function $\text{Hom}(X,c) : \text{Hom}(X,X) \rightarrow \text{Hom}(X,G)$ is a surjection. We start by determining the image of $\text{Hom}(X,c)$. This image consists of homomorphisms $f : X \rightarrow G$ that can be lifted through $c$ and expressed as compositions of some $s : X \rightarrow X$ and $c : X \rightarrow G$. The following proposition describes such homomorphisms:

**Proposition 7.2.** Let $c : X \rightarrow G$ be a surjective generalized subgroup. A homomorphism $f : X \rightarrow G$ can be expressed as a composition of $s : X \rightarrow X$ and $c : X \rightarrow G$ if and only if the image of $H_{2\setminus 1}(f)$ lies in the image $H_{2\setminus 1}(c)$.

**Proof.** Clearly if $f = cs$ for some $s : X \rightarrow X$, then $H_{2\setminus 1}(f) = H_{2\setminus 1}(c)H_{2\setminus 1}(s)$ and hence the image of $H_{2\setminus 1}(f)$ is in the image of $H_{2\setminus 1}(c)$.

Assume that $H_{2\setminus 1}(f)$ is in the image of $H_{2\setminus 1}(c)$. Recall that $c : X \rightarrow G$ fits into the following commutative diagram where the left bottom square is a push-out square and $e_G : E \rightarrow G$ is the initial extension (see [6.7]):

\[
\begin{array}{c}
\xymatrix{ H_{2\setminus 1}(X) \ar[r]^{\text{id}} & H_{2\setminus 1}(X) } \\
H_{2\setminus 1}(c) \ar[u] & \\
H_{2\setminus 1}(G) \ar[u] & E \ar[r]^e & G } \\
\mu(c) \ar[u] & \pi \ar[u] & \text{id} \ar[u] \\
K_c \ar[u] & X \ar[r]^c & G } 
\end{array}
\]

To prove the lemma we will construct the following commutative diagram:

\[
\begin{array}{c}
\xymatrix{ \text{Ker}(\pi) \ar[r] & H_{2\setminus 1}(X) \ar[r]^e & E \ar[r]^{\pi} & X } \\
H_{2\setminus 1}(f) \ar[u] & \ar[r] f & H_{2\setminus 1}(G) \ar[r]^e & E \ar[r]^{\pi} & X \ar[r]^c & G } \\
\ar[u] f' & \ar[r] f & \ar[u] \text{id} & \ar[u] s & \ar[u] }
\end{array}
\]
Since $e_G : E \to G$ is a cellular cover (see [2, 4]), there is a unique $\tilde{f}$ for which $e_G \tilde{f} = f\pi$. By the naturality of the differentials we have a commutative square:

$$
\begin{array}{c}
H_2(X) \xrightarrow{\mu(\pi)} H_{2\setminus 1}(X) \\
\mu(\pi) \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{array}
$$

This implies that the restriction of $\tilde{f}$ to $H_{2\setminus 1}(X)$ is given by $H_{2\setminus 1}(f)$. By the assumption the image of $H_{2\setminus 1}(f)$ lies in the image of $H_{2\setminus 1}(c)$ which gives the homomorphism $f'$. As $\tilde{f} : E \to E$ maps the kernel $\text{Ker}(\pi) \subset E$ to itself, we get the desired homomorphism $s : X \to X$.

**Corollary 7.3.** A surjective generalized subgroup $c : X \to G$ is a cellular cover if and only if, for any homomorphism $f : X \to G$, the image of $H_{2\setminus 1}(f) : H_{2\setminus 1}(X) \to H_{2\setminus 1}(G)$ lies in the image of $H_{2\setminus 1}(c)$.

We can use this corollary to prove:

**Proposition 7.4.**

1. Let $k$ be a positive divisor of the order of $H_{2\setminus 1}(G)$. If $c : X \to G$ is a surjective generalized subgroup whose differential is given by the quotient homomorphism $q_k : H_{2\setminus 1}(G) \to H_{2\setminus 1}(G)/(k\text{-torsion})$, then $c$ is a cellular cover.

2. If $H_{2\setminus 1}(G)$ is cyclic, then all surjective generalized subgroups of $G$ are cellular covers.

**Proof.** (1): Recall that we have an exact sequence:

$$
0 \to H_{2\setminus 1}(X) \xrightarrow{H_{2\setminus 1}(c)} H_{2\setminus 1}(G) \xrightarrow{\mu(c)} H_{2\setminus 1}(G)/(k\text{-torsion}) \to 0.
$$

Thus, $H_{2\setminus 1}(X)$ is the $k$-torsion subgroup of $H_{2\setminus 1}(G)$. Let $f : X \to G$ be a homomorphism. Since $H_{2\setminus 1}(X)$ is $k$-torsion, the image of $H_{2\setminus 1}(f) : H_{2\setminus 1}(X) \to H_{2\setminus 1}(G)$ lies in the $k$-torsion subgroup of $H_{2\setminus 1}(G)$ which is the image of $H_{2\setminus 1}(c)$. According to Corollary 7.3, $a$ is a cellular cover.

(2): If $H_{2\setminus 1}(G)$ is cyclic, then any of its subgroups is the $k$-torsion subgroup for some $k$. Statement (1) follows then from statement (2).

**The action of Out(G).** According to [6, 8], a surjective generalized subgroup $c : X \to G$ is determined by its differential $\mu(c) : H_{2\setminus 1}(G) \to K_c$ which in turn is determined by its kernel $H_{2\setminus 1}(X)$. The following functions are bijections:

$$
\begin{array}{c}
\text{SurSub}(G) \longrightarrow \text{Quot(H}_{2\setminus 1}(G)) \longrightarrow \text{Sub(H}_{2\setminus 1}(G)); \\
(c : X \to G) \longmapsto [\mu(c) : H_{2\setminus 1}(G) \to K_c] \longmapsto H_{2\setminus 1}(X).
\end{array}
$$

In this way surjective generalized subgroups of $G$ are enumerated by subgroups of $H_{2\setminus 1}(G)$. To enumerate the set $\text{S}ur\text{Cov}(G)$, we need to identify these elements in $\text{Quot(H}_{2\setminus 1}(G))$, or equivalently in $\text{Sub(H}_{2\setminus 1}(G))$, which are differentials of surjective cellular covers. For that we look at the action of $\text{Out}(G)$ on these sets. Let $h : G \to G$ be an automorphism. Consider the induced isomorphism $H_2(h) : H_2(G) \to H_2(G)$ and its localization $H_{2\setminus 1}(h) : H_{2\setminus 1}(G) \to H_{2\setminus 1}(G)$ with respect to the set $S$ of primes that divide the order of $H_1(G)$. The function:

$$
\text{Sub(H}_{2\setminus 1}(G)) \times \text{Aut}(G) \ni (H, h) \longmapsto H_{2\setminus 1}(h)^{-1}(H) \in \text{Sub(H}_{2\setminus 1}(G))
$$

defines a right action of $\text{Aut}(G)$ on the set $\text{Sub(H}_{2\setminus 1}(G))$ of all subgroups of $H_{2\setminus 1}(G)$. Since inner automorphisms induce the identity on homology, this action induces an action of $\text{Out}(G)$ on $\text{Sub(H}_{2\setminus 1}(G))$. 
The corresponding right action of \( \text{Out}(G) \) on Quot\((H_{2\setminus 1}(G))\) can be described as follows. Let \( h : G \to G \) be an automorphism. For a surjective homomorphism \( \sigma : H_{2\setminus 1}(G) \to K \), the composition \( \sigma H_{2\setminus 1}(h) : H_{2\setminus 1}(G) \to K \) is also surjective. Note further if \( \sigma : H_{2\setminus 1}(G) \to K \) and \( \tau : H_{2\setminus 1}(G) \to K \) define the same element in Quot\((H_{2\setminus 1}(G))\), then so do their compositions \( \sigma H_{2\setminus 1}(h) \) and \( \tau H_{2\setminus 1}(h) \).

The following induced function defines a right action of \( \text{Out}(G) \) on Quot\((H_{2\setminus 1}(G))\):

\[
\text{Quot}(H_{2\setminus 1}(G)) \times \text{Out}(G) \ni ([\sigma], [h]) \mapsto [\sigma H_{2\setminus 1}(h)] \in \text{Quot}(H_{2\setminus 1}(G))
\]

Moreover the bijection that assigns to an element \( \sigma \) in Quot\((H_{2\setminus 1}(G))\) its kernel \( \text{Ker}(\sigma) \), which is an element in Sub\((H_{2\setminus 1}(G))\), is an equivariant isomorphism.

We will be interested in the fixed points of these actions. The reason for this is:

**Proposition 7.5.** If \( c : X \to G \) is a surjective cellular cover, then its differential \( [\mu(c) : H_{2\setminus 1}(G) \to K_c] \in \text{Quot}(H_{2\setminus 1}(G)) \) and \( H_{2\setminus 1}(X) \in \text{Sub}(H_{2\setminus 1}(G)) \) are fixed by the action of \( \text{Out}(G) \).

**Proof.** Let \( h : G \to G \) be an automorphism. Since \( c : X \to G \) is a cellular cover, there is a unique homomorphism \( h' : X \to X \) that fits into the following commutative square:

\[
\begin{array}{ccc}
X & \xrightarrow{h'} & X \\
\downarrow{c} & & \downarrow{c} \\
G & \xrightarrow{h} & G
\end{array}
\]

By the naturality of the differential we then get an induced commutative square:

\[
\begin{array}{ccc}
H_{2\setminus 1}(G) & \xrightarrow{H_{2\setminus 1}(h)} & H_{2\setminus 1}(G) \\
\downarrow{\mu(c)} & & \downarrow{\mu(c)} \\
K_c & \xrightarrow{h'} & K_c
\end{array}
\]

It follows that \( H_{2\setminus 1}(h) \) maps the kernel of \( \mu(c) \) to itself. This means that, as an element of \( \text{Sub}(H_{2\setminus 1}(G)) \), this kernel is invariant under the action of \( \text{Out}(G) \). \( \Box \)

**Corollary 7.6.**

(1) Let \( k > 0 \) be a divisor of the exponent of \( H_{2\setminus 1}(G) \). Then the \( k \)-torsion subgroup of \( H_{2\setminus 1}(G) \) is fixed by \( \text{Out}(G) \).

(2) If \( H_{2\setminus 1}(G) \) is cyclic, then the action of \( \text{Out}(G) \) on the sets \( \text{Sub}(H_{2\setminus 1}(G)) \) and \( \text{Quot}(H_{2\setminus 1}(G)) \) is trivial.

Let InvQuot\((H_{2\setminus 1}(G)) \subset \text{Quot}(H_{2\setminus 1}(G)) \) and InvSub\((H_{2\setminus 1}(G)) \subset \text{Sub}(H_{2\setminus 1}(G)) \) be the fixed points of the action of \( \text{Out}(G) \). According to what has been proven, we have the following sequence of inclusions:

\[
\{ k \in \mathbb{Z} \mid k > 0 \text{ and } k \text{ divides the order of } H_{2\setminus 1}(G) \}
\]

Let \( k \) and \( q_k \) be an automorphism. For a surjective homomorphism \( \sigma : H_{2\setminus 1}(G) \to K \), the composition \( \sigma H_{2\setminus 1}(h) : H_{2\setminus 1}(G) \to K \) is also surjective. Note further if \( \sigma : H_{2\setminus 1}(G) \to K \) and \( \tau : H_{2\setminus 1}(G) \to K \) define the same element in Quot\((H_{2\setminus 1}(G))\), then so do their compositions \( \sigma H_{2\setminus 1}(h) \) and \( \tau H_{2\setminus 1}(h) \).
Proposition 7.4 can be rephrased as:

**Corollary 7.7.** Let $G$ be a group for which $H_{2,1}(G)$ is cyclic. Then all the inclusions in the above diagram are bijections. In particular:

1. The differential $\mu : \text{SurCov}(G) \to \text{Quot}(H_{2,1}(G))$ is a bijection.
2. Any surjective generalized subgroup of $G$ is a cellular cover.
3. Let $c : X \to G$ be a surjection. Then $\text{Hom}(X,c) : \text{Hom}(X,X) \to \text{Hom}(X,G)$ is a bijection if and only if it is an injection.

8. **Cellular covers of finite simple groups**

The aim of this section is to classify cellular covers of finite simple groups. A simple group $G$ has a trivial abelianization and thus $H_{2,1}(G) = H_2(G)$. According to 7.5 the differential induces an inclusion $\mu : \text{SurCov}(G) \subseteq \text{InvQuot}(H_2(G))$. Our key result is:

**Theorem 8.1.** If $G$ is a finite simple group, then $\mu : \text{SurCov}(G) \subseteq \text{InvQuot}(H_2(G))$ is a bijection.

**Corollary 8.2.** Let $G$ be a finite simple group. Then the sets $\text{Cov}(G)$ and $\text{Idem}(G)$ are in bijection with $\{0\} \coprod \text{InvSub}(H_2(G))$.

**Proof.** Recall that according to 2.3 the sets $\text{Cov}(G)$ and $\text{Idem}(G)$ are in bijection with each other. Let $c : X \to G$ be a cellular cover. Since the image of $c$ is a normal subgroup of $G$, this image is either the trivial group or the whole $G$. In the first case $X$ has to be trivial. In the second case $c$ is a surjective cellular cover of $X$. Thus according to 8.1 the assignment that maps the trivial cellular cover to the element 0 and a surjective cellular cover $c : X \to G$ to the image of $H_2(c) : H_2(X) \subset H_2(G)$, is the desired bijection between $\text{Cov}(G)$ and $\{0\} \coprod \text{InvSub}(H_2(G))$. □

The key property of finite simple groups used to prove the above theorem is:

**Lemma 8.3.** Let $c : X \to G$ be a surjective generalized subgroup of a finite simple group $G$. Then any non-trivial homomorphism $f : X \to G$ can be expressed as a composition of $c : X \to G$ and some automorphism $G \to G$.

**Proof.** Let $K_f = \text{Ker}(f)$ and $K_c = \text{Ker}(c)$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
K_f \cap K_c & \subseteq & K_c \\
\downarrow & & \downarrow \\
K_f & \subseteq & X \\
\downarrow & & \downarrow \\
G & \xrightarrow{c} & G \\
\end{array}
$$

The image of $g$ is a normal subgroup of $G$. Since $G$ is simple, there are two possibilities. Either $g$ is a surjection or it is the trivial homomorphism.

Assume that $g$ is trivial. In this case $K_f$ is a subgroup of $K_c$ and we have the following commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & G \\
\downarrow & & \downarrow \\
G & \leftarrow & X/K_f
\end{array}
$$
Finiteness of $G$ implies that the surjection $X/K_f \to G$ and the injection $X/K_f \hookrightarrow G$ in the above diagram have to be isomorphisms. We can then use the commutativity of this diagram to conclude that $f$ can be expressed as a composition of $c : X \to G$ and some automorphism $G \to G$.

We will show that under the assumption that $f$ is non-trivial, the homomorphism $g$ can not be surjective. Assume to the contrary that $g$ is surjective. In this case we can use (6.4)(5) to get equality $K_f = X$. Consequently the surjectivity of $g$ implies the triviality of $f$. $\square$

**Corollary 8.4.** Let $G$ be a finite simple group and $X \in \text{Idem}(G)$. Then any non-trivial homomorphism $f : X \to G$ is a cellular cover of $G$.

**Proof.** Let $X \in \text{Idem}(G)$. According to (2.8) there is a homomorphism $c : X \to G$ which is a cellular cover. The image of $c$ is a normal subgroup of $G$. It is then the trivial group or the whole group $G$. In the first case $X$ is the trivial group and there is no non-trivial homomorphisms from $X$ to $G$. In the second case we can use (5.3) to conclude that $f : X \to G$ can be expressed as a composition of $c : X \to G$ and some automorphism $G \to G$. It is then clear that $\text{Hom}(X,f) : \text{Hom}(X,X) \to \text{Hom}(X,G)$ is a bijection and therefore $f$ is a cellular cover. $\square$

**Proof of (8.4)** Let $c : X \to G$ be a surjective generalized subgroup whose differential $\mu(c) : H_2(G) \to K_f$ represents an element in $\text{InvQuot}(H_2(G))$. We will use (7.3) to prove the theorem. According to (5.3) any non-trivial homomorphism $f : X \to G$ is a composition of $c : X \to G$ and an automorphism $h : G \to G$. Consequently $H_2(f) = H_2(h)H_2(c)$. As the image of $H_2(c)$ is fixed by the action of $\text{Out}(G)$ on $\text{Sub}(H_2(G))$, we have:

$$\text{image}(H_2(f)) = \text{image}(H_2(h)H_2(c)) = \text{image}(H_2(c))$$

By (7.3) $c : X \to G$ is then a cellular cover. $\square$

9. **Iterated Generalized Subgroups and Cellular Covers**

Consider the group $\text{PSL}_2(q)$ where $q$ is a power of an odd prime and distinct from 3 and 9. The initial cellular cover of $\text{PSL}_2(q)$ is represented by the universal central extension which is a surjection $c : \text{SL}_2(q) \to \text{PSL}_2(q)$ whose kernel is the center of $\text{SL}_2(q)$. This center is isomorphic to $\mathbb{Z}/2$ and it is the only subgroup of $\text{SL}_2(q)$ isomorphic to $\mathbb{Z}/2$. Consequently the inclusion $\mathbb{Z}/2 \subset \text{SL}_2(q)$ is also a cellular cover. The composition of these two cellular covers $\mathbb{Z}/2 \subset \text{SL}_2(q) \to \text{PSL}_2(q)$ is the trivial homomorphism which is not a generalized subgroup. Thus in general neither the composition of two cellular covers is a cellular cover nor is the composition of two generalized subgroups a generalized subgroup. The aim of this section is to discuss the possible homomorphisms that are obtained as compositions of several cellular covers and of repeated generalized subgroups.

**Definition 9.1.** Let $G$ be a group and $n$ a positive integer.

1. A homomorphism $a : X \to G$ is defined to be an $n$-iterated generalized subgroup of $G$ if $a$ can be expressed as a composition:

$$X \xrightarrow{a_n} X_n \xrightarrow{a_{n-1}} \cdots \xrightarrow{a_1} X_1 \xrightarrow{a_0} G$$

where all $a_i$'s are generalized subgroups, i.e., $a_1 : X_1 \to G$ is a generalized subgroup of $G$, $a_2 : X_2 \to X_1$ is a generalized subgroup of $X_1$, $a_3 : X_3 \to X_2$ is a generalized subgroup of $X_2$, etc. Two such $n$-iterated generalized subgroups $a : X \to G$ and $b : Y \to G$ are defined to be equivalent, if there is an isomorphism $h : X \to Y$ such that $bh = a$.

2. Analogously $a : X \to G$ is defined to be an $n$-iterated surjective generalized subgroup, or $n$-iterated cellular cover, or $n$-iterated surjective cellular cover of $G$ if $a$ can be expressed as a composition of respectively $n$ surjective generalized subgroups, or $n$ cellular...
covers, or \( n \) surjective cellular covers. Two such homomorphisms \( a \) and \( b \) are equivalent, if there is an isomorphism \( h \) such that \( bh = a \).

Iterated cellular covers and iterated generalized subgroups have the following two features in common with the ordinary cellular covers and generalized subgroups:

**Proposition 9.2.** Let \( G \) be a group.

1. If \( a : X \rightarrow G \) is an \( n \)-iterated generalized subgroup of \( G \), then its kernel is central in \( X \).
2. If \( c : X \rightarrow G \) is an \( n \)-iterated cellular cover of \( G \), then its image is a fully invariant subgroup of \( G \).

**Proof.** (1): First notice that if \( a : X \rightarrow Y \) is a generalized subgroup, then \( a^{-1}(Z(Y)) \subseteq Z(X) \). Indeed, for \( x \in a^{-1}(Z(Y)) \) the map \( a \circ c_x = a \), where \( c_x \) is conjugation by \( x \), hence \( c_x = id \), so \( x \in Z(X) \). Now by induction on \( n \) it is easily seen that the kernel of an \( n \)-iterated generalized subgroup \( a : X \rightarrow G \) is central in \( X \).

(2): Assume \( c : X \rightarrow G \) is a composition of \( n \) cellular covers \( X_n \xrightarrow{c_n} \cdots \xrightarrow{c_1} X_1 \xrightarrow{c_1} G \). For any homomorphism \( h : G \rightarrow G \), using the fact that \( c_i \)'s are cellular covers, there is a unique sequence of homomorphisms \( \{h_i : X_i \rightarrow X_i\}_{1 \leq i \leq n} \) for which the following diagram commutes:

\[
\begin{array}{cccc}
X & \xrightarrow{c_n} & \cdots & \xrightarrow{c_1} & G \\
\downarrow{h_n} & & \downarrow{h_1} & \parallel \parallel & \downarrow{h} \\
X & \xrightarrow{c_n} & \cdots & \xrightarrow{c_1} & G
\end{array}
\]

Commutativity of this diagram implies that \( h \) maps the image of \( c \) into itself. \qed

**Definition 9.3.** Let \( G \) be a group and \( n \) a positive integer.

1. \( \text{Sub}^n(G) \) denotes the collection of equivalence classes of \( n \)-iterated generalized subgroups of \( G \).
2. \( \text{SurSub}^n(G) \) denotes the collection of equivalence classes of \( n \)-iterated surjective generalized subgroups of \( G \).
3. \( \text{Cov}^n(G) \) denotes the collection of equivalence classes of \( n \)-iterated cellular covers of \( G \).
4. \( \text{SurCov}^n(G) \) denotes the collection of equivalence classes of \( n \)-iterated surjective cellular covers of \( G \).

According to this definition \( \text{Sub}(G) = \text{Sub}^1(G) \), \( \text{SurSub}(G) = \text{SurSub}^1(G) \), \( \text{Cov}(G) = \text{Cov}^1(G) \), and \( \text{SurCov}(G) = \text{SurCov}^1(G) \).

The identity homomorphisms are cellular covers and hence the following inclusions hold:

\[
\begin{array}{cccc}
\text{SurCov}^n(G) & \supseteq & \text{SurSub}^n(G) & \hookrightarrow \text{SurSub}^{n+1}(G) \\
\downarrow & & \downarrow & \downarrow \\
\text{Cov}^n(G) & \supseteq & \text{Sub}^n(G) & \hookrightarrow \text{Sub}^{n+1}(G)
\end{array}
\]

By summing up these inclusions we can extend Definition 9.3 to:
Definition 9.4. Let $G$ be a group. Define:
\[
\text{SurCov}^\infty(G) := \bigcup_{n \geq 1} \text{SurCov}^n(G) \quad \text{SurSub}^\infty(G) := \bigcup_{n \geq 1} \text{SurSub}^n(G),
\]
\[
\text{Cov}^\infty(G) := \bigcup_{n \geq 1} \text{Cov}^n(G) \quad \text{Sub}^\infty(G) := \bigcup_{n \geq 1} \text{Sub}^n(G)
\]

We are interested in the above collections primary in the case when $G$ is finite. Recall that a finite group $G$ has a composition series $1 = G_0 \subset \cdots \subset G_i = G$ and any such series has the same length $l$ which is called the composition length of $G$.

Theorem 9.5. Let $G$ be a finite group and $l$ its composition length. Then $\text{Sub}^{l+1}(G) = \text{Sub}^\infty(G)$.

To prove the theorem we need:

Lemma 9.6. Let $G$ be finite and $X_2 \xrightarrow{a_2} X_1 \xrightarrow{a_1} G$ be generalized subgroups.

1. If any prime that divides the order of $H_1(X_2)$ divides also the order of $H_1(X_1)$, then the composition $a_1a_2 : X_2 \to G$ is a generalized subgroup.
2. If $a_2$ is surjective, then the composition $a_1a_2 : X_2 \to G$ is a generalized subgroup.

Proof. (1): According to 9.2, the kernel $K_{a_1a_2}$ of the composition $a_1a_2 : X_2 \to G$ is central in $X_2$. Thus to show that $a_1a_2$ is a generalized subgroup, we need to prove $\text{Hom}(X_2, K_{a_1a_2}) = 0$ (see 9.2). The kernels $K_{a_1}$ and $K_{a_2}$ of $a_1$ and $a_2$ and the kernel $K_{a_1a_2}$ are abelian groups and they fit into an exact sequence $K_{a_2} \subset K_{a_1a_2} \to K_{a_1}$. Since $\text{Hom}(X_2, K_{a_2}) = 0$, we get an inclusion:
\[
\text{Hom}(X_2, K_{a_1a_2}) \subset \text{Hom}(X_2, K_{a_1}) = \text{Hom}(H_1(X_2), K_{a_1}).
\]

Since $a_1$ is a generalized subgroup $\text{Hom}(H_1(X_1), K_{a_1}) = \text{Hom}(X_1, K_{a_1}) = 0$. This means that the primes that divide the order of $K_{a_1}$ do not divide then order of $H_1(X_1)$. By the assumption the primes that divide the order of $K_{a_1}$ can not divide the order of $H_1(X_2)$ either. Consequently $\text{Hom}(H_1(X_2), K_{a_1}) = 0$.

(2): If $a_2 : X_2 \to X_1$ is surjective, then $H_1(X_2)$ and $H_1(X_1)$ are isomorphic (see 9.4(1)). Statement (2) follows thus from (1). \qed

Proof of 9.6 Assume that $a : X \to G$ can be expressed as a composition of generalized subgroups $X_n \xrightarrow{a_n} \cdots \xrightarrow{a_1} X_1 \xrightarrow{a_0} G$. By performing compositions if necessary, we can assume that none of the adjacent composition $a_ia_{i+1}$ is a generalized subgroup. To prove the theorem we need to show that $n \leq l + 1$. To do that it is enough to prove that the image of the composition $a_1 \cdots a_i : X_{i+1} \to G$ is a proper subgroup of the image of the composition $a_1 \cdots a_i : X_i \to G$ for any $1 \leq i \leq n - 1$. This is because these images will lead to a proper normal series of length $n - 1$ in $G$. Since the composition length of $G$ is $l$, we must have $n \leq l + 1$.

Assume that this is not the case. Let $i$ be an index for which $b := a_1 \cdots a_i$ and $ba_{i+1}+1$ have the same image in $G$. Let $I_{a_{i+1}} = a_{i+1}(X_{i+1})$ and let $K_b := \text{Ker} b$. By 9.6(1), $\text{Ker} b$ is central in $X_i$ and by hypothesis $X_i = I_{a_{i+1}}K_b$. Hence $[X_i, X_i] = [I_{a_{i+1}}, I_{a_{i+1}}]$ and it follows that $H_1(I_{a_{i+1}})$ is a subgroup of $H_1(X_i)$. But according to 6.4(1), $H_1(X_{i+1}) \cong H_1(I_{a_{i+1}})$. This together with 9.6(1) shows that $a_ia_{i+1}$ is a generalized subgroup, a contradiction. \qed

Corollary 9.7. Let $G$ be a finite group.

1. $\text{Sub}^\infty(G)$, $\text{Cov}^\infty(G)$, $\text{SurSub}^\infty(G)$, and $\text{SurCov}^\infty(G)$ are finite sets.
2. There is a positive integer $N$ for which $\text{Cov}^N(G) = \text{Cov}^\infty(G)$.
3. For any positive integer $n$, $\text{SurSub}(G) = \text{SurSub}^n(G) = \text{SurSub}^\infty(G)$. 

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Proof. (1): A direct consequence of \(6.3\) and the fact that a generalized subgroup of a finite group is finite (see \(4.3\)). We have the following bijections:

\[
\text{SurSub}(G) \xrightarrow{\text{bijection}} \text{Sub}^n(G) \quad \text{for any } n.
\]

Finiteness of \(\text{Sub}^\infty(G)\) is then a consequence of \(9.5\).

(2): Since \(\text{Cov}^\infty(G)\) is finite and it is a sum of an increasing sequence of sets \(\text{Cov}^1(G) \subset \text{Cov}^2(G) \subset \cdots\), we must have that, for some \(N\), \(\text{Cov}^N(G) = \text{Cov}^\infty(G)\).

(3): According to \(9.6(2)\) a composition of surjective generalized subgroups is a generalized subgroup, which is obviously surjective.

For a finite group \(G\), we have an inclusion \(\text{SurCov}^\infty(G) \subset \text{SurSub}(G)\) (see \(9.7(3)\)). To understand some constrains on elements of \(\text{SurSub}(G)\) which lie in \(\text{SurCov}^\infty(G)\), let us recall that, according to \(6.6\), we have the following bijections:

\[
\begin{align*}
\text{SurSub}(G) & \xrightarrow{\text{bijection}} \text{Quot}(H_{2\setminus1}(G)) \\
\text{Sub}(H_{2\setminus1}(G)) & \xrightarrow{\text{bijection}} H_{2\setminus1}(X).
\end{align*}
\]

**Proposition 9.8.** Let \(G\) be a finite group. If \(c : X \to G\) is an \(n\)-iterated surjective cellular cover, then its differential \(\mu(c) : H_{2\setminus1}(G) \to K_c\) represents an element in \(\text{InvQuot}(H_{2\setminus1}(G))\).

**Proof.** The argument is exactly the same as in the proof of \(4.14\). Assume that \(c : X \to G\) can be expressed as a composition of \(n\)-surjective cellular covers: \(X_n \xrightarrow{c_n} \cdots \xrightarrow{c_1} X_1 \xrightarrow{c_1} G\). For any isomorphism \(h : G \to G\), using the fact that \(c_i\)'s are cellular covers, there is a unique sequence of isomorphisms \(\{h_i : X_i \to X_i\}_{1 \leq i \leq n}\) for which the following diagram commutes:

\[
\begin{array}{cccccc}
X & \xrightarrow{X_n} & \cdots & \xrightarrow{X_1} & G \\
\downarrow{h_n} & & & \downarrow{h_1} & \\
X & \xrightarrow{X_n} & \cdots & \xrightarrow{X_1} & G
\end{array}
\]

By the naturality of the differential we get an induced commutative diagram:

\[
\begin{array}{ccc}
H_{2\setminus1}(G) & \xrightarrow{\mu(c)} & K_c \\
\downarrow{H_{2\setminus1}(h)} & & \downarrow{h_c} \\
H_{2\setminus1}(G) & \xrightarrow{\mu(c)} & K_c
\end{array}
\]

Commutativity of this diagram implies that \(H_{2\setminus1}(h)\) maps the kernel of \(\mu(c)\) onto itself. Since \(h\) was an arbitrary automorphism of \(G\), the kernel of \(\mu(c)\) belongs to \(\text{InvSub}(H_{2\setminus1}(G))\) and consequently \(\mu(c) : H_{2\setminus1}(G) \to K_c\) represents an element in \(\text{InvQuot}(H_{2\setminus1}(G))\).

**Corollary 9.9.** Let \(G\) be a finite group.

1. If \(H_{2\setminus1}(G)\) is cyclic, then, for any positive integer \(n\):

\[
\text{SurCov}(G) = \text{SurCov}^n(G) = \text{SurCov}^\infty(G) = \text{SurSub}(G).
\]

2. If \(G\) is simple, then, for any positive integer \(n\):

\[
\text{SurCov}(G) = \text{SurCov}^n(G) = \text{SurCov}^\infty(G).
\]
Proof. (1): This is a consequence of the inclusion SurCov\(^\infty\)(G) \(\subset\) SurSub(G), which follows from (9.7(3)), and the equality SurCov(G) = SurSub(G) (see (7.7(2)).

(2): By [9.8] the differential \(\mu: \text{SurCov}^{\infty}(G) \to \text{InvSub}(H_{2\setminus 1}(G))\) is an injection. Since its restriction \(\text{SurCov}(G) \subset \text{SurCov}^{\infty}(G) \xrightarrow{\mu} \text{InvSub}(H_{2\setminus 1}(G))\) is a bijection (see 8.1) the statement follows. \(\square\)

We finish this section with:

**Proposition 10.9.**

1. If \(G\) is finitely generated nilpotent, then, for any positive integer \(n\), \(\text{Sub}(G) = \text{Sub}^n(G) = \text{Sub}^{\infty}(G)\).

2. If \(A\) is finite abelian, then, for any positive integer \(n\), \(\text{Cov}(A) = \text{Cov}^n(A) = \text{Cov}^{\infty}(A)\).

3. If \(G\) is finite and simple, \(\text{Cov}^2(G) = \text{Cov}^{\infty}(G)\).

**Proof.** (1): A generalized subgroup of a finitely generated nilpotent group is an injection (see 4.3(3)). The statement follows from the fact that the composition of injections is an injection.

(2): If \(A\) is finite abelian then, according to 14.3, the cellular covers of \(A\) are given by the \(k\)-torsion subgroups of \(A\). The \(m\)-torsion subgroup of the \(k\)-torsion subgroup of \(A\) is simply the \(\text{lcm}(k,m)\)-torsion subgroup of \(A\) and hence it is also a cellular cover of \(A\). Composition of two such cellular covers is then a cellular cover which shows (2).

(3): Let \(c: X \to G\) belong to \(\text{Cov}^n(G)\). We will show by induction on \(n\) that \(c \in \text{Cov}^2(G)\). If \(n = 2\), there is nothing to prove. Let \(n > 2\). Assume that \(c\) can be expressed as a composition of cellular covers \(X_n \xrightarrow{c_n} \cdots \to X_1 \xrightarrow{c_1} G\). Since \(G\) is simple, there are two possibilities: either \(X_1 = 0\) or \(c_1\) is a surjection. In the first case any cellular cover of \(X_1\) is the trivial group too. The same holds for all \(X_i\)’s. Thus \(X = 0\) and \(c\) is the trivial homomorphism, in particular \(c \in \text{Cov}(G)\).

Assume \(c_1\) is a surjection. The image of \(c_1c_2: X_2 \to G\) is a fully invariant subgroup in \(G\) (see 9.2(2)). Thus it is either the whole group \(G\) or it is the trivial group. In the first case, according to 6.3(5), \(c_2\) is a surjection. We can then use 9.3(2) to conclude that the composition \(c_1c_2: X_2 \to G\) is a surjective cellular cover of \(G\). In this case \(c\) can be expressed as a composition of \(n - 1\) cellular covers. By the inductive assumption \(c \in \text{Cov}^2(G)\).

The last possibility is that the image of \(c_1c_2\) is the trivial group which means that the image of \(c_2\) sits in the kernel \(K_{c_1}\) of \(c_1\). We will prove that the composition \(c_2c_3 \cdots c_n : X \to X_1\) is a cellular cover. In this way we can express \(c\) as a composition of two cellular covers proving the statement.

The kernel \(K_{c_1}\) is a finite abelian group. Since \(c_2 : X_2 \to K_{c_1}\) is also a cellular cover, \(c_2\) must be an injection and consequently \(X_2\) is an abelian group. We can then use (2) to get that the composition \(c_3 \cdots c_n : X \to X_2\) is a cellular cover, which in particular means that it is an injection. Let \(f : X \to X_1\) be an arbitrary homomorphism. As \(X\) is (isomorphic to) the \(k\)-torsion subgroup (for some \(k\)) of the finite abelian group \(X_2\), there is a surjection \(\pi : X_2 \to X\). Because \(c_2 : X_2 \subset X_1\) is a cellular cover, the image of the composition \(\pi f : X_2 \to X_1\) sits in the image of \(c_2\). Thus \(f\) has to map \(X\) into the image of \(c_2\). The fact that \(c_3 \cdots c_n : X \to X_2\) is a cellular cover implies that \(f\) has to map \(X\) into the image of \(c_3 \cdots c_n\). This means that \(f\) factors through \(c_2c_3 \cdots c_n : X \to X_1\), proving that the later homomorphism is a cellular cover. \(\square\)

The statement (3) in 10.10 cannot be made stronger. For example, in the case of the group \(\text{PSL}_2(q)\) where \(q\) is a power of an odd prime different from 3 and 9, there is a proper inclusion \(\text{Cov}(\text{PSL}_2(q)) \not\subset \text{Cov}^2(\text{PSL}_2(q)) = \text{Cov}^{\infty}(\text{PSL}_2(q))\).

10. Iterating idempotent functors

Recall that according to (2.3) the function \(F\) that assigns to an equivalence class of a cellular cover, represented by \(c : X \to G\), the group \(X\) induces a bijection between \(\text{Cov}(G)\) and \(\text{Idem}(G)\). This
means that if \( X \) represents an element in \( \text{Idem}(G) \), then up to an isomorphism of \( X \), there is a unique homomorphism \( c : X \to G \) which is a cellular cover of \( G \).

For any positive integer \( n \), the function \( F \) induces a surjection (which we also call \( F \)) from \( \text{Cov}^n(G) \) onto \( \text{Idem}^n(G) \). This surjectivity and 9.7(1) gives:

**Corollary 10.1.** If \( G \) is a finite, then \( \text{Idem}^\infty(G) \) is a finite set.

In general we do not know if the function \( F \) induces a bijection between \( \text{Cov}^n(G) \) and \( \text{Idem}^n(G) \). We do not know if, for a group \( X \), that represents an element in \( \text{Idem}^n(G) \), there is a unique, up to an isomorphism of \( X \), homomorphism \( f : X \to G \) which is a composition of a sequence of \( n \) cellular covers. We can show however that this is true for finite simple groups:

**Proposition 10.2.** Let \( G \) be a finite simple group. Then the function that assigns to an element in \( \text{Cov}^2(G) \), represented by \( f : X \to G \), the element in \( \text{Idem}^\infty(G) \), represented by \( X \), is a bijection.

**Proof.** If \( G \) is abelian, then the proposition follows from 2.3 since in this case \( \text{Cov}(G) = \text{Cov}^\infty(G) \) (see 9.10(2)).

Assume \( G \) is not abelian. According to 9.10(3), \( \text{Cov}^2(G) = \text{Cov}^\infty(G) \). Thus the function \( F \) between \( \text{Cov}^2(G) \) and \( \text{Idem}^\infty(G) \) is surjective.

Let \( X \in \text{Idem}^\infty(G) \) be a non-trivial group. There are two possibilities: \( X \) is abelian or not. In the first case we claim that the trivial homomorphism from \( X \) to \( G \) is the only homomorphism that represents an element in \( \text{Cov}^2(G) \). Assume that this is not the case. Then there are two cellular covers \( c_2 : X \to X_1 \) and \( c_1 : X_1 \to G \) whose composition \( c_1c_2 : X \to G \) is non trivial. As the image of such a composition is a fully invariant subgroup of \( G \) (see 9.2), \( c_1c_2 \) has to be a surjection, which can only happen in the case \( G \) is abelian.

Assume that \( X \) is not abelian. If the composition \( c_1c_2 \) were trivial, then \( c_2 : X \to K_{c_1} \) would be a cellular cover of the kernel \( K_{c_1} \) which would require \( X \) to be abelian. The composition \( c_1c_2 \) is therefore non-trivial and hence it has to be a surjection. The homomorphism \( c_1 \) is then also a surjection. By 9.5(5) we can conclude that \( c_2 \) is a surjection too. The composition \( c_1c_2 \) is then a cellular cover (see 9.9(2)), and hence the homomorphism \( c_1c_2 \) is unique, up to an automorphism of \( X \). \( \square \)

## 11. Explicit examples

In this section we will illustrate Theorem 8.1 and its Corollary 8.2. We let \( G \) denote a finite simple group. We use the symbol \( \exp(H_2(G)) \) to denote the exponent of \( H_2(G) \) and \( \sigma_0(G) \) the number of different positive divisors of \( \exp(H_2(G)) \). Recall that the exponent of a finite abelian group \( A \) is the least positive integer \( k \) for which \( kA = 0 \). Let \( c_G : E \to G \) be the universal central extension of \( G \). The center of \( E \) is isomorphic to \( H_2(G) \).

According to 8.2 \( \text{Idem}(G) \) is in bijection with the set \( \{0\} \bigsqcup \text{InvQuot}(H_2(G)) \). Explicitly, the element 0 corresponds to the trivial group in \( \text{Idem}(G) \). Any non-trivial element in \( \text{Idem}(G) \) is the quotient of \( E \) by an \( \text{Out}(G) \)-invariant subgroup of its center \( H_2(G) \). A basic example of such a subgroup is given by the \( k \)-torsion subgroup for some \( k \) dividing the exponent \( \exp(H_2(G)) \) of \( H_2(G) \) (see 7.6(1)). The number of such basic invariant subgroups of \( H_2(G) \) is therefore given by \( \sigma_0(G) \). Thus the set \( \text{Idem}(G) \) contains at least \( \sigma_0(H_2(G)) + 1 \) elements. The question is if there are any other invariant subgroups of \( H_2(G) \)? For example in the case \( H_2(G) \) is cyclic, since the action of \( \text{Out}(G) \) on \( \text{Sub}(H_2(G)) \) is trivial (see 7.4), all the subgroups of \( H_2(G) \) are invariant. In this case the set \( \text{Idem}(G) \) has exactly \( \sigma_0(H_2(G)) + 1 \) elements. It turns out that this happens for almost all simple groups. The only exceptions are the groups \( D_n(q) \) \( (n \geq 3) \) for odd \( q \) and even \( n \). In this case the Schur multiplier is \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) and hence its exponent is 2 and consequently \( \sigma_0(D_n(q)) = 2 \). However, the number of invariant subgroups in the Schur multiplier turns out to be 3 and hence \( \text{Idem}(D_n(q)) \) has 4 elements.
Proposition 11.1. The following table lists the size of $\text{Idem}(G)$ (fifth column) for all finite simple groups $G$. In the first column the boxed entries contain the names of the groups with restrictions on relevant indices that are required for the groups to be simple or to avoid some repetition. The notation is taken from [GLS1]. The constrains below the boxes distinguish between different Schur multipliers, Schur multipliers are the content of the second column. The third column contains $\exp(H_2(G))$ and the forth $\sigma_0(G)$. In the first column, we write $\bullet G \bullet$ to denote these groups $G$ for which the Schur multiplier is not cyclic and yet $\text{Idem}(G)$ has $\sigma_0(H_2(G)) + 1$ elements. We use $\bullet\bullet G \bullet\bullet$ to denote the cases for which $\text{Idem}(G)$ has more than $\sigma_0(H_2(G)) + 1$ elements.

| $G$ | $H_2(G)$ | $\exp(H_2(G))$ | $\sigma_0(G)$ | $|\text{Idem}(G)|$ |
|-----|----------|-----------------|---------------|-----------------|
| $\mathbb{Z}/p$ | | 0 | 1 | 1 | 2 |
| Alternating groups | | | | | |
| $A_n$, $n \geq 5$ | $\mathbb{Z}/2$ | 2 | 2 | 3 |
| $n \neq 6, n \neq i$ | | | | | |
| $A_6, A_7$ | $\mathbb{Z}/6$ | 6 | 4 | 5 |
| Linear groups | | | | | |
| $A_n(q)$, $n \geq 1$ | $\mathbb{Z}/(n+1,q-1)$ | $(n+1,q-1)$ | $\sigma_0(n+1,q-1)$ | $\sigma_0 + 1$ |
| $(n,q) \neq (1,2)$ | | | | | |
| $(n,q) \neq (1,3)$ | | | | | |
| $(n,q) \neq (1,4)$ | | | | | |
| $(n,q) \neq (1,9)$ | | | | | |
| $(n,q) \neq (2,2)$ | | | | | |
| $(n,q) \neq (2,4)$ | | | | | |
| $(n,q) \neq (3,2)$ | | | | | |
| $A_3(2)$ | $\mathbb{Z}/2$ | 2 | 2 | 3 |
| $\bullet A_2(4) \bullet$ | $\mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$ | 12 | 6 | 7 |
| $G$       | $H_2(G)$            | $\exp(H_2(G))$ | $\sigma_0(G)$ | $\text{Idem}(G)$ |
|-----------|---------------------|-----------------|---------------|------------------|
| $^2A_n(q)$, $n \geq 2$  | $\mathbb{Z}((n+1,q+1))$ | $(n+1,q+1)$     | $\sigma_0(n+1,q+1)$ | $\sigma_0+1$     |
| $(n,q) \neq (2,2)$   |                     |                 |               |                  |
| $(n,q) \neq (3,2)$   |                     |                 |               |                  |
| $(n,q) \neq (3,3)$   |                     |                 |               |                  |
| $(n,q) \neq (5,2)$   |                     |                 |               |                  |
| $^2A_4(2)$          | $\mathbb{Z}/2$      | 2               | 2             | 3                |
| $^2A_5(2)$          | $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$ | 6               | 4             | 5                |
| $^2A_3(3)$          | $\mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$ | 12              | 6             | 7                |

**Orthogonal groups of type B**

| $B_n(q)$, $n \geq 2$ | $\mathbb{Z}/2$ | 2 | 2 | 3 |
|-----------------------|-----------------|---|---|---|
| $(n,q) \neq (2,2)$   |                 |   |   |   |
| $q$ odd,             |                 |   |   |   |
| $(n,q) \neq (3,3)$   |                 |   |   |   |
| $B_3(3)$             | $\mathbb{Z}/6$  | 6 | 4 | 5 |
| $B_3(2)$             | $\mathbb{Z}/2$  | 2 | 2 | 3 |

**Suzuki Groups**

| $^2B_2(2^{2n+1})$, $n \geq 1$ | 0 | 1 | 1 | 2 |
|------------------------------|---|---|---|---|
| $n > 1$                      |   |   |   |   |
| $^2B_2(8)$                   | $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ | 2 | 2 | 3 |

**Symplectic groups**

| $C_n(q)$, $n \geq 3$, $q$ odd | $\mathbb{Z}/2$ | 2 | 2 | 3 |

**Orthogonal groups of type D**

| $D_n(q)$, $n \geq 4$ | 0 | 1 | 1 | 2 |
|-----------------------|---|---|---|---|
| $q$ even, $(n,q) \neq (4,2)$ | | | | |
| $D_4(2)$              | $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ | 2 | 2 | 3 |
| $D_n(q)$, $n \geq 5$  | $\mathbb{Z}/4$ | 4 | 3 | 4 |
| $q$ odd, $n$ odd     | | | | |

**Unitary groups**

| $A_n(q)$, $n \geq 2$ | $\mathbb{Z}/((n+1,q+1))$ | $(n+1,q+1)$ | $\sigma_0((n+1,q+1))$ | $\sigma_0+1$ |
|-----------------------|-----------------------------|-------------|------------------------|--------------|
| $(n,q) \neq (2,2)$   |                             |             |                        |              |
| $(n,q) \neq (3,3)$   |                             |             |                        |              |
| $(n,q) \neq (5,2)$   |                             |             |                        |              |

$\mathbb{Z}/((n+1,q+1))$ denotes the cyclic group of order $(n+1,q+1)$. The table shows the structure of the groups for different values of $n$, $q$, and the additional conditions specified in the table.
| $G$ | $H_2(G)$ | $\exp(H_2(G))$ | $\sigma_0(G)$ | $\text{Idem}(G)$ |
|-----|---------|----------------|--------------|----------------|
| **$D_n(q)$, $n \geq 4$** | $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ | 2 | 2 | 4 |
| $q$ odd, $n$ even | | | | |
| $2D_n(q)$, $n \geq 4$ | $\mathbb{Z}/2$ | 2 | 2 | 3 |
| $q$ even | | | | |
| $2D_n(q)$, $n \geq 4$ | $\mathbb{Z}/2$ | 2 | 2 | 3 |
| $q$ odd | | | | |
\[ G \quad H_2(G) \quad \exp(H_2(G)) \quad \sigma_0(G) \quad \text{Idem}(G) \]

\[ \text{Sporadic groups} \]

|   |   |   |   |   |
|---|---|---|---|---|
| \( M_{11} \) | 0 | 1 | 1 | 2 |
| \( M_{12} \) | \( \mathbb{Z}/2 \) | 2 | 2 | 3 |
| \( M_{22} \) | \( \mathbb{Z}/12 \) | 12 | 6 | 7 |
| \( M_{23}, M_{24}, J_1 \) | 0 | 1 | 1 | 2 |
| \( J_2 \) | \( \mathbb{Z}/2 \) | 2 | 2 | 3 |
| \( J_3 \) | \( \mathbb{Z}/3 \) | 3 | 2 | 3 |
| \( J_4 \) | 0 | 1 | 1 | 2 |
| \( HS \) | \( \mathbb{Z}/2 \) | 2 | 2 | 3 |
| \( He \) | 0 | 1 | 1 | 2 |
| \( Mc \) | \( \mathbb{Z}/3 \) | 3 | 2 | 3 |
| \( Suz \) | \( \mathbb{Z}/6 \) | 6 | 4 | 5 |
| \( Ly \) | 0 | 1 | 1 | 2 |
| \( Ru \) | \( \mathbb{Z}/2 \) | 2 | 2 | 3 |
| \( O'N \) | \( \mathbb{Z}/3 \) | 3 | 2 | 3 |
| \( Co_1 \) | \( \mathbb{Z}/2 \) | 2 | 2 | 3 |
| \( Co_2, Co_3 \) | 0 | 1 | 1 | 2 |
| \( F'_{22} \) | \( \mathbb{Z}/6 \) | 6 | 4 | 5 |
| \( F'_{23} \) | 0 | 1 | 1 | 2 |
| \( F'_{24} \) | \( \mathbb{Z}/3 \) | 3 | 2 | 3 |
| \( F_5, F_3 \) | 0 | 1 | 1 | 2 |
| \( F_2 \) | \( \mathbb{Z}/2 \) | 2 | 2 | 3 |
| \( F_1 \) | 0 | 1 | 1 | 2 |

**Proof.** The proposition can be derived by elementary arguments from [GLS3, 6.3.1]. For self containment we present these elementary arguments below.

We need to explain the table only for the groups whose Schur multiplier is not cyclic. There are just seven such cases.

**Cases:** \( 2A_5(2), 2B_2(8), D_4(2), 2E_6(2) \). In all of these cases the Schur multiplier is of the form \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) or \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3 \). To prove the proposition we need to find all the invariant subgroups of the 2-torsion part. In all of these cases, according to [GLS3, 6.3.1], the group \( \text{Out}(G) \) contains \( \mathbb{Z}/3 \) which acts faithfully on the 2-torsion part \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). However if \( \psi : \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) is an automorphism such that \( \psi \neq \text{id} \) and \( \psi^3 = \text{id} \), then \( \psi \) has no eigenvectors. Consequently the only subgroups of \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) which are invariant under \( \psi \) are the trivial subgroup and the whole group.

**Case:** \( A_2(4) \). In this case the Schur multiplier is isomorphic to \( \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \). To prove the proposition we need to understand the invariant subgroups of the 4-torsion part \( V := \mathbb{Z}/4 \oplus \mathbb{Z}/4 \). Again according to [GLS3, 6.3.1], the group \( \text{Out}(A_2(4)) \) contains \( \mathbb{Z}/3 \) which acts faithfully on \( \mathbb{Z}/4 \oplus \mathbb{Z}/4 \).
Let $\psi : V \to V$ be an automorphism of order 3. We claim that the only $\psi$-invariant subgroups in $V$ are the trivial subgroup, the Frattini subgroup $\Phi \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and the whole group $V$. We have $V = [V, \psi] \oplus C_V(\psi)$ and so if $C_V(\psi)$ is non-trivial, then $[V, \psi] \cong \mathbb{Z}/4$ contradicting the fact that $\psi$ is faithful on $[V, \psi]$. Hence $C_V(\psi)$ is trivial. Let $K$ be a $\psi$-invariant subgroup. If $|K| = 2$ or 4 and $K \neq \Phi$, then $K$ is cyclic and hence it is centralized by $\psi$, a contradiction. If $|K| = 8$, then $K \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$, so $\psi$ centralizes the Frattini subgroup of $K$, a contradiction.

**Case:** $2A_3(3)$. In this case the Schur multiplier is isomorphic to $\mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$. To prove the proposition we need to understand the invariant subgroup of the 3-torsion part. According to [GLS3](#) 6.3.1, the group $\text{Out}(2A_3(3))$ contains $\mathbb{Z}/4$ which acts faithfully on $V := \mathbb{Z}/3 \oplus \mathbb{Z}/3$. Let $\psi : V \to V$ be an automorphism of order 4 acting faithfully. If there would be a proper non-trivial $\psi$-invariant subspace $W \subset V$, then $V$ would split as a direct sum $V = W \oplus U$ with $U$ $\psi$-invariant. But then $\psi^2$ would centralize $V$, a contradiction. Hence the only $\psi$-invariant subspaces are the trivial one and $V$.

**Case:** $D_n(q)$, $n \geq 4$, $q$ odd, $n$ even. In this case the Schur multiplier is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. According to [GLS3](#) 6.3.1, after an appropriate choice of a base, automorphisms of $D_n(q)$ act on the Schur multiplier $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ either as the identity or the transposition. Moreover there is an element that does act as a transposition. It follows that, with this choice of a base, the invariant subgroups are: the trivial subgroup, the diagonal, and the whole group. 

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